ON THE PREHISTORY OF GROWTH OF GROUPS

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Abstract. The subject of growth of groups has been active in the former Soviet Union since the early 50's and in the West since 1968, when articles of Švarc and Milnor have been published, independently. The purpose of this note is to quote a few articles showing that, before 1968 and at least retrospectively, growth of groups has already played some role in various subjects.

The notion of growth for finitely generated groups appears in articles published independently by Efremovich and Švarc in the early 50's, and by Milnor in 1968 [Efre–53, Svar–55, Miln–68a, Miln–68b]. (Švarc left Soviet Union in 1989, and now his name is rather written Schwarz.) Very soon after his first paper, Milnor in [Miln–68c] called attention to the fact that [Svar–55] “contains many ideas utilized in [3]” (where [3] = [Miln–68b]).

Before 1968, the paper [Svar–55], written by Švarc during his undergraduate years, was essentially ignored outside the former Soviet Union. Concerning the period between 1955 and 1968, we quote two extracts, from [Avez–76, Definition I.6] and [Grom–93, Item 0.5]. After having defined exponential growth and non-exponential growth, Avez writes: “This notion is due to V. Arnold (oral communication, 1965), Švarc [Svar–55], and Milnor [Miln–68b]. Finite extensions of finitely generated nilpotent groups are the only known examples of groups of non-exponential growth [Wolf–68].” In what he calls some random historical remarks, Gromov writes: “The ideas of the growth of balls, Folner sets and sets of conjugacy classes in groups (especially in fundamental groups of manifolds of negative curvature, see [Marg–67] [Marg–69]) were quite popular in the sixties among ergodic theorists in Moskow and Leningrad. (Much of these ideas I learned at the time from A. Vershik, D. Kazhdan and G. Margulis.) Then the geometers took a part in the story and related the growth to curvature. The first results here for non-negative curvature are due to A. Švarc [Svar–55]. Similar results were obtained independently by J. Milnor [Miln–68b].”

For a description of the results of Efremovich and Švarc, we quote the following lines from [Svar–08]. ”My first serious work was inspired by Efremovich’s remark that the ‘volume invariant’ of universal covering of a compact manifold is a topological invariant of the manifold. (If two compact manifolds are homeomorphic, then the natural homeomorphism between universal coverings is uniformly
Efremovich proved that under certain conditions the growth of the volume of a ball with radius tending to infinity is an invariant of uniformly continuous homeomorphisms. I proved that the volume invariant of universal covering can be expressed in terms of the fundamental group of the original manifold; in modern language it is determined by the growth of the fundamental group. I also gave estimates for volume invariants of manifolds with non-positive and with negative curvature. Thirteen years later J. Milnor published a paper containing the same results with the only difference that Milnor was able to use in his proofs some theorems derived after the appearance of my paper. At the moment of writing his first paper in this direction Milnor did not know about my work, but his second paper contained corresponding references. The notion of growth of a group (volume invariant of a group in my terminology) was studied later in numerous papers (one should mention, in particular, the results by Gromov and Grigorchuk). A new interesting field — geometric group theory — was born from these papers."

For a short description of the work of Efremovich, see also [Efremovich].

The importance of the subject of group growth was largely recognized with the results of Gromov, showing that a finitely generated group has polynomial growth if and only if it has a nilpotent subgroup of finite index [Grom–81], and Grigorchuk, showing the existence of groups of intermediate growth [Grig–83, Grig–84] (but it is still unknown whether there exist any finitely presented group of intermediate growth). Several reviews of the subject have appeared, of which we mention [GrHa–97] and [Grig–14], and there is a nice exposition of the theory in the book by Mann [Mann–12]. Growth of groups extends naturally to the setting of locally compact groups; in particular Guivarc’h and Jenkins [Guiv–73, Jenk–73] have characterized connected Lie groups of polynomial growth as those of Type (R), i.e., as those for which $\text{ad}(x)$ has purely imaginary eigenvalues for all $x$ in the Lie algebra of the group (this is considerably easier to prove than Gromov’s characterization of finitely generated groups of polynomial growth). More generally, Losert has characterized locally compact groups (not necessarily Lie groups) of polynomial growth in a series of papers, the first being [Lose–87].

The purpose of this note is to mention a few articles published before 1968, and for some even before 1955. It can be seen retrospectively how the notion of group growth has been used early for various purposes.

1. Carl Friedrich Gauss and the growth of $\mathbb{Z}^2$ (1834). The free abelian group of rank two, $\mathbb{Z}^2$, has to be seen as the lattice of integer points in the Euclidean plane; this has been so even before the concept of group was made precise in its present form. Consider the length function on $\mathbb{Z}^2$ given by the Euclidean norm, and the growth of $\mathbb{Z}^2$ as the function $R$ defined by

$$R(t) = |\{(a, b) \in \mathbb{Z}^2 | a^2 + b^2 \leq t\}| \quad \text{for all } t \geq 0,$$
i.e., \( R(t) \) is the number of points of \( \mathbb{Z}^2 \) in the disc of radius \( \sqrt{t} \) centred at the origin. The function \( R(t) \) is interesting in number theory, more precisely in the study of integers which are sums of two squares; but we like to view also \( R(t) \) as a function describing the growth of \( \mathbb{Z}^2 \). In 1843, Gauss showed that

\[
| R(t) - \pi t | \leq 2\pi(1 + \sqrt{2t}) = O(\sqrt{t}).
\]

In [Gauss, Pages 271 and 280], Gauss wrote thirty values of \( R(k) \), including \( R(10000) = 31417 \) and \( R(100000) = 314197 \). After Gauss, it has been shown that \( | R(t) - \pi t | = O(t^{\alpha}) \) for values \( \alpha < 1/2 \), in particular for \( \alpha = 1/3 \) (Sierpinski, 1906); the best estimate today seems to be \( | R(t) - \pi t | = O(t^{\alpha+\varepsilon}) \) for \( \alpha = 517/1648 = 0.31371... \) and for all \( \varepsilon > 0 \); see [BoWa], as well as [BeKZ–18]. It is conjectured that the estimate holds for every \( \alpha > 1/4 \).

For \( k \) a non-negative integer, set \( r_2(k) = \{ (a, b) \in \mathbb{Z}^2 \mid \sqrt{a^2 + b^2} = k \} \), so that \( R(k) = \sum_{j=0}^k r_2(j) \). Values of \( r_2(k) \) and \( R(k) \) for small \( k \) and relevant references are given in [OEIS, A004018 and A057655]. The series \( \sum_{k=0}^{\infty} r_2(k) z^k \) is \((\theta_3(z))^2\), where \( \theta_3 \) is the third Jacobi theta function [CoSl–99, Chapter IV, Section 5].

2. Word length and growth type of a finitely generated group. Let \( \Gamma \) be a finitely generated group and \( S \) a finite generating set of \( \Gamma \). The word length function \( \ell_S : \Gamma \to \mathbb{N} \) is defined by \( \ell_S(\gamma) = \min \{ k \geq 0 \mid \gamma \in (S \cup S^{-1})^k \} \). Let \( \sigma(\Gamma, S; k) \) denote the cardinal of the sphere \( \{ \gamma \in \Gamma \mid \ell_S(\gamma) = k \} \) and \( \beta(\Gamma, S; k) \) denote the cardinal of the ball \( \{ \gamma \in \Gamma \mid \ell_S(\gamma) \leq k \} \). It is straightforward to check that \( \sigma(\Gamma, S; k) \leq |S \cup S^{-1}|((|S \cup S^{-1}| - 1)^{k-1} \) for all \( k \geq 1 \); in particular, \( \beta(\Gamma, S; k) \leq e^{bk} \) for an appropriate constant \( b > 0 \) and for all \( k \geq 0 \). The group \( \Gamma \) is said to be of exponential growth if there exist constants \( c > 1 \), \( R \geq 0 \) such that \( \beta(\Gamma, S; k) \geq e^{ck} \) for all \( k \geq R \), of subexponential growth otherwise, of polynomial growth if there exist constants \( C > 0 \) and \( d \in \mathbb{N} \) such that \( \beta(\Gamma, S; k) \leq Ck^d \) for all \( k \geq 0 \), and of intermediate growth if it is of subexponential growth and not of polynomial growth. The definitions do not depend on the choice of \( S \), because the inequalities hold for one finite generating set \( S \) if and only if they hold for all finite generating sets.

Word lengths, spheres and balls can be found in the literature much before the theory of group growth. For example \( \ell_S(\gamma) \) appears as the “exponent of the substitution \( \gamma \)” in [Poin–82, Page 11], the paper in which Poincaré shows a presentation of a Fuchsian group in terms of one of its fundamental polygons in the hyperbolic plane. The word metric on \( \Gamma \), defined by \( d_S(\gamma, \gamma') = \ell_S(\gamma^{-1}\gamma') \), has been used systematically by Dehn in his first paper on decision problems in group theory; see [Dehn–11] and [DeSt–87, Pages 130 and 143]. Spheres an balls, noted respectively \( \Gamma_k \) and \( \bigcup_{j=k}^{\infty} \Gamma_k \) appear in [ArKr–63], where the authors establish equidistribution in the 2-sphere of the points of the orbit of a semigroup generated by two appropriate rotations.
3. Waclaw Sierpinski (1946), Georgii Adel’son-Vel’skii and Yuli Anatoljevitch Shreider (1957), Joseph Rosenblatt (1974), and the supramenability of groups of subexponential growth. In the 1929 paper which created the subject of amenability [vNeu–29], John von Neumann considers actions of a group \( \Gamma \) on a set \( X \) given with a non-empty subset \( E \). Such an action is amenable if there exists a finitely additive positive measure \( \mu \) on \( X \) normalized by \( \mu(E) = 1 \) and invariant by \( \Gamma \) (the measure need not be finite, except of course when \( E = X \)). The group \( \Gamma \) itself is amenable (eine messbare Gruppe in [vNeu–29]) if every action of \( \Gamma \) on every set \( X \) given with \( E = X \) is amenable, and this holds as soon as the left action of \( \Gamma \) on itself is amenable (with \( E = X = \Gamma \)). The group \( \Gamma \) is supramenable (a terminology due to Rosenblatt [Rose–74]) if every action of \( \Gamma \) on a set \( X \) given with any subset \( E \neq \emptyset \) is amenable, and this holds as soon as the left action of \( \Gamma \) on itself, with any \( E \), is amenable. The \( \Gamma \)-set \( E \) has a paradoxical decomposition if there exist a partition of \( E \) in disjoint sets \( A_1, \ldots, A_k, B_1, \ldots, B_l \) and elements \( g_1, \ldots, g_k, h_1, \ldots, h_l \) in \( \Gamma \) such that \( E \) is equal to both the disjoint unions \( \bigsqcup_{i=1}^k g_i A_i \) and \( \bigsqcup_{j=1}^l h_j B_j \). A paradoxical decomposition of \( E \) is an obstruction to the existence of \( \mu \) as above [vNeu–29, Page 82]; remarkably it is the only obstruction: either \( E \) has a paradoxical decomposition or there exists a \( \Gamma \)-invariant finitely additive positive measure \( \mu \) on \( X \) normalized by \( \mu(E) = 1 \) [Tars–36].

For example, the action on \( X = \mathbb{R}^d \) of the isometry group of the Euclidean space \( \mathbb{R}^d \) given with the unit ball \( E \) is amenable when \( d = 1 \) and \( d = 2 \), and is not when \( d \geq 3 \). In dimension 3, Hausdorff and Banach & Tarski have obtained famous results which express non-amenability in a spectacular way: the action of the rotation group \( \text{SO}(3) \) on the unit ball in \( \mathbb{R}^3 \) is non-amenable, see [Haus–14, Appendix to Chapter X, Page 469], and two bounded subsets \( A \) and \( B \) of \( \mathbb{R}^3 \) with non-empty interiors are equidecomposable (this means that there exist partitions \( A = \bigsqcup_{i=1}^k A_i, B = \bigsqcup_{i=1}^k B_i \), and isometries \( g_1, \ldots, g_k \) of \( \mathbb{R}^k \) such that \( g_1 A_1 = B_1, \ldots, g_k A_k = B_k \)), see [BaTa–24].

In [Sier–46], Sierpinski saw that any finitely generated subgroup of the isometry group of \( \mathbb{R} \) is of subexponential growth (indeed of polynomial growth), and that this implies that the action of this isometry group on \( \mathbb{R} \) is not paradoxical. The argument shows essentially that the isometry group of \( \mathbb{R} \) is supramenable, and much more (see below).

In [AdSr–57, Theorem 2], it is shown that a finitely generated group of subexponential growth is amenable.

Later, Rosenblatt showed much more. He introduced the terminology “supramenable”, as defined above; moreover he defined a group to be exponentially bounded if all its finitely generated subgroups are of subexponential growth. He showed that exponentially bounded groups are supramenable. Moreover a finitely generated solvable group either has a nilpotent subgroup of finite index, and thus is of polynomial growth and supramenable, or is not supramenable and
contains a free semigroup on two generators, and thus is of exponential growth [Rose–74].

It is unknown whether there exist finitely generated groups of exponential growth which are supramenable.

We reproduce now Sierpinski’s argument, cast in the more general situation of a group $\Gamma$ acting on a set $X$ (instead of the affine group of $\mathbb{R}$ acting on $\mathbb{R}$), and a nonempty subset $E$ of $X$. Suppose that there exists a paradoxical decomposition of $E$: there exist as above subsets $A_1, \ldots, A_k, B_1, \ldots, B_\ell$ of $E$ and a subset $S$ of elements $g_1, \ldots, g_k, h_1, \ldots, h_\ell$ of $\Gamma$ (not necessarily distinct from each other) such that

$$E = \left( \bigsqcup_{i=1}^k A_i \right) \sqcup \left( \bigsqcup_{j=1}^\ell B_j \right) = \bigsqcup_{i=1}^k g_i A_i = \bigsqcup_{j=1}^\ell h_j B_j.$$

The following argument shows that the subgroup of $\Gamma$ generated by $S$ has exponential growth.

Set $A = \bigsqcup_{i=1}^k A_i$, $B = \bigsqcup_{j=1}^\ell B_j$. Define bijections $\varphi : E \to A$ and $\psi : E \to B$ by $\varphi(x) = g_i^{-1}(x)$ when $x \in g_i A_i$ and $\psi(x) = h_j^{-1}(x)$ when $x \in h_j B_j$. Choose $x_0 \in E$. Observe first that $\varphi(x_0) \neq \psi(x_0)$, because $A$ and $B$ are disjoint, then that $\varphi\varphi(x_0), \varphi\psi(x_0), \psi\varphi(x_0), \psi\psi(x_0)$ are also distinct, because $\varphi$ and $\psi$ are injective and $A \cap B = \emptyset$, and so on. This shows that, for any positive integer $k$, the $2^k$ words of length $k$ in $\varphi$ and $\psi$ are maps $E \to E$ with distinct values at $x_0$. For any of these words, say $\chi$, the value $\chi(x_0)$ is of the form $s_1^{-1}s_2^{-1}\cdots s_k^{-1}(x_0)$, for $s_1, s_2, \ldots, s_k \in S$. It follows that the subgroup of $\Gamma$ generated by $S$ has at least $2^k$ distinct elements $\gamma$ of word length $\ell_S(\gamma) \leq k$, and this ends the argument.

Sierpinski’s argument shows that a group $\Gamma$ which can act on a pair $X \supset E$ such that $E$ has a paradoxical decomposition has a finitely generated subgroup of exponential growth. By contraposition, it follows that a finitely generated group of subexponential growth is supramenable, a result much stronger than the one in [AdSr–57], and a proof much more direct than the one in [Rose–74].

4. **Hans Ulrich Krause and finitely generated abelian groups with isomorphic Cayley graphs (1953).** In his thesis [Krau–53, Satz 16.1], Krause shows that two finitely generated abelian groups have isomorphic Cayley graphs with respect to well-chosen generating sets if and only if the two following conditions are satisfied: (i) the two groups have the same rank, and (ii) their torsion groups have the same order. In the proof, it is shown that the rank of a finitely generated abelian group $\Gamma$ is the polynomial growth rate $\lim_{k \to \infty} (\ln |S^k|)/\ln k$, where $S$ is a symmetric generating set of $\Gamma$.

5. **Jacques Dixmier and polynomial growth of nilpotent connected Lie groups (1960, 1966).** Lemma 3 of [Dixm–60] is the following. Let $G$ be a nilpotent connected Lie group, $\mu$ a Haar measure on $G$, and $H$ a compact subset of $G$. Then there exists an integer $N$ (which depends on $G$ but not on $H$) such
that $\mu(H^k) = O(k^N)$ when $k \to \infty$; in other words, $G$ is a group of polynomial growth.

The lemma is used by Dixmier in the proof of the following result. Consider a locally compact group $G$, the group algebra $L^1(G)$, and the two-sided ideal $I$ of those elements $f \in L^1(G)$ such that, for every irreducible unitary representation $\pi$ of $G$, the operator $\pi(f)$ is of finite rank. If $G$ is a nilpotent connected Lie group, then $I$ is dense in $L^1(G)$. (The same property of $I$ was established earlier for semisimple Lie groups by Harish–Chandra.)

Polynomial growth has been established later for solvable connected Lie groups of type (R), in [Dixm–66].

6. Henri Dye and orbital equivalence (1963). Theorem 1 of [Dye–63] establishes the following. Let $\Gamma$ be a finitely generated group, generated by a finite subset $F$. The notation of Dye is $h_1 = |F|$ and $h_k = |F^k \setminus F^{k-1}|$ for $k \geq 2$. If

$$\inf_{k \geq 1} \frac{h_{2k}}{h_1 + \cdots + h_k} = 0,$$

then $\Gamma$ is approximately finite. In particular, finitely generated groups of polynomial growth are approximately finite.

To define approximate finiteness, consider actions of countable groups on non-atomic standard probability spaces by measure preserving transformations. Two such actions of $\Gamma_1$ on $X_1$ and $\Gamma_2$ on $X_2$ are orbit equivalent if there exists a measure preserving Borel isomorphism $f : X_1 \to X_2$ such that $f(\Gamma_1 x)$ coincides with the orbit $\Gamma_2 f(x)$ for almost all $x$ in $X_1$. Consider some ergodic measure preserving action of the infinite cyclic group $\mathbb{Z}$ on a non-atomic standard probability space; a basic example is the Bernoulli shift action $\beta$ of $\mathbb{Z}$ on $(\mathbb{Z}/2\mathbb{Z})^\mathbb{Z}$. A countable group $\Gamma$ is approximately finite in the sense of Dye if, for every ergodic measure preserving action $\alpha$ of $\Gamma$ on a non-atomic probability space $X$, the actions $\alpha$ and $\beta$ are orbit equivalent.

It is now known that an infinite countable group is approximately finite if and only if it is amenable [OrWe–80, Hjor–05].

7. Gregori Margulis, growth of fundamental group and existence of Anosov flows (1967). On a compact Riemannian smooth manifold $M$, an Anosov flow is a smooth flow $\Phi = \{\Phi_t\}_{t \in \mathbb{R}}$ which satisfies the following conditions. There exists a $\Phi$-invariant continuous splitting $TM = E_u \oplus E^T \oplus E_s$ of the tangent bundle of $M$, where the three terms are respectively the unstable (or expanding) subbundle of $TM$, the line bundle tangent to $\Phi$, and the stable (or contracting) subbundle of $TM$, and there exist constants $\nu > 0$, $c > 0$, such that

$$\| (\Phi_t)_* (v) \| \geq ce^{\nu t} \| v \| \text{ and } \| (\Phi_{-t})_* (v) \| \leq ce^{-\nu t} \| v \| \text{ for all } v \in E^u \text{ and } t \geq 0$$

and

$$\| (\Phi_t)_* (v) \| \leq ce^{\nu t} \| v \| \text{ and } \| (\Phi_{-t})_* (v) \| \geq ce^{-\nu t} \| v \| \text{ for all } v \in E^s \text{ and } t \geq 0$$

(the two conditions with $(\Phi_{-t})_* (v)$ follow from the two conditions with $(\Phi_t)_* (v)$, see [AnSi–67, Page 121]).
In one of his first published papers, Margulis shows that, if a 3-dimensional manifold $M$ has an Anosov flow, then the fundamental group of $M$ has exponential growth [Marg–67]. This has been generalized to manifolds of higher dimensions and Anosov flows with one of the subbundles $E^u, E^s$ of rank one [PlTh–72].

For the contrast, let us quote the following result of Franks. On a compact Riemannian smooth manifold $M$, a $C^1$ map $f : M \to M$ is expanding if there are constants $\lambda > 1$ and $c > 0$ such that $\|T^m f^m v\| \geq c\lambda^m \|v\|$ for all $v \in TM$ and $m \geq 1$. Here is the result: If a compact manifold admits an expanding self-map, then its fundamental group has polynomial growth [Fran–70, Theorem 8.3].

8. **Harry Kesten and recurrent random walks on groups (1967).** Let $\Gamma$ be a finitely generated group. A symmetric probability measure $\mu$ on $\Gamma$ such that $\{\gamma \in \Gamma \mid \mu(\gamma) > 0\}$ is a finite generating set gives rise to a random walk on $\Gamma$. The group $\Gamma$ is recurrent if there exists such a measure such that the associated random walk is recurrent (equivalently if for any such measure the associated random walk is recurrent). It is a classical theorem of Pólya that the simple random walk on $\mathbb{Z}^d$ is recurrent if $d \leq 2$ and transient if $d \geq 3$ [Poly–21], and it has been known since at least 1962 that an infinite finitely generated abelian group is recurrent if and only if it is a finite extension of $\mathbb{Z}$ or a finite extension of $\mathbb{Z}^2$ [Dudl–62].

Kesten has conjectured that the recurrence of a group depends on its growth type. It is conjectured more precisely in [Kest–67, Conjecture 4] that a finitely generated group which is recurrent cannot be of exponential growth. The conjecture was made more precise (the growth of a recurrent group is at most quadratic) and generalized to second countable locally compact groups; see the introduction of [GuRa–12]. For discrete groups, the final result is due to Varopoulos (1986) : a finitely generated group is recurrent if and only if it is of at most quadratic growth, if and only it is either finite, or a finite extension of $\mathbb{Z}$, or a finite extension of $\mathbb{Z}^2$; see [VaSC–92]. (A finitely generated group $\Gamma$ is of quadratic growth if, with the notation of Section 2, there exists a constant $C > 0$ such that $\beta(\Gamma,S;k) \leq Ck^2$ for all $k \geq 0$.)

9. **Generating functions.** To encode a sequence $(a_k)_{k \geq 0}$ of integral numbers, several types of series or functions can be used, and the best choice depends on the subject. One choice is the **ordinary generating function** of the sequence $(a_k)_{k \geq 0}$:

$$\Sigma(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{Z}[[z]].$$

When $\Sigma(z)$ converges for $z$ small enough and whenever possible, we like to identify the “simple function of analysis” of which $\Sigma(z)$ is the Taylor series at the origin. The book [FlSe–09] is a very rich source of examples and theorems on these generating functions.
An early example occurs in a letter of Euler to Goldbach dated September 4, 1751. The letter is reproduced partly in [FlSe–99, Section I.1], and in full in [Euler, Letter 154, Pages 489–491]. (In [Knut–97, Section 1.2.9], Knuth mentions three earlier appearances of generating functions by de Moivre, by Stirling, and by Euler in connection with numbers of partitions of integers.) In his letter, Euler considers the number \( c_k \) of decompositions of a convex \((k + 2)\)-gon in triangles; set moreover \( c_0 = 1 \). The generating function of \((c_k)_{k \geq 0}\)

\[
\sum_{k \geq 0} c_k z^k = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7 + \cdots
\]

\[
= 1 + \sum_{k=1}^{\infty} \frac{2 \cdot 6 \cdot 10 \cdots (4k - 2)}{2 \cdot 3 \cdot 4 \cdots (k + 1)} z^k = \frac{1 - \sqrt{1 - 4z}}{2z}
\]

is algebraic. (Euler does not consider our \( c_0 \), and writes \( a \) instead of our \( z \), so that his series sums up to \( \frac{1 - 2a - \sqrt{1 - 4a}}{2a} \).) The numbers \( c_k \) are now known as Catalan numbers, and are often written in terms of binomial coefficients: \( c_k = \frac{1}{k+1} \binom{2k}{k} \). For 214 different kinds of objects that are counted using Catalan numbers and for a historical survey, see [Stan–15].

The simplest sequences are those which satisfy a linear recurrence relation; they correspond precisely to rational generating functions. More precisely, consider a positive integer \( d \) and complex numbers \( q_1, q_2, \ldots, q_d \) with \( q_d \neq 0 \). Set \( Q(z) = 1 + q_1 z + q_2 z^2 + \cdots + q_d z^d = \prod_{j=1}^{e} (1 - \gamma_j z)^{d_j} \), where \( \gamma_1, \ldots, \gamma_e \in \mathbb{C} \) are the distinct roots of \( Q \) and \( d_1, \ldots, d_e \) their multiplicities; note that \( \sum_{j=1}^{e} d_j = d \). Then, for a sequence \((a_k)_{k \geq 0}\), the following conditions are equivalent

- (R1) \( \sum_{k \geq 0} a_k z^k = P(z)/Q(z) \) for some polynomial \( P(z) \) of degree less than \( d \),
- (R2) \( a_{k+d} + q_1 a_{k+d-1} + q_2 a_{k+d-2} + \cdots + q_d a_k = 0 \) for all \( k \geq 0 \),
- (R3) \( a_k = \sum_{j=1}^{e} P_j(k) \gamma_j^k \) for all \( k \geq 0 \), for some polynomials \( P_j(z) \) of degree less than \( d_j \) (with \( j = 1, \ldots, e \)).

For this, and for variations (when \( \deg P \geq d \) or when \( Q(z) = (1 - z)^d \)), see [Stan–98, Chapter 0].

The Fibonacci sequence \((F_k)_{k \geq 0} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots)\) is a notorious example:

1. generating function \( \sum_{k=0}^{\infty} F_k z^k = \frac{z}{1-z-z^2} \),
2. linear recursion \( F_{k+2} - F_{k+1} - F_k = 0 \) for all \( k \geq 0 \),
3. and Binet formula \( F_k = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^k - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^k \) (published by Binet in 1845, but already in [Ber–1728, § 7] and [Eul–1767, Page 128]).

10. **Growth series for finitely generated groups.** Let \( \Gamma \) be a finitely generated group and \( S \) a finite generating set of \( \Gamma \). For each \( k \geq 0 \), let \( \sigma(\Gamma, S; k) \) be the cardinal of the sphere of radius \( k \) and let \( \beta(\Gamma, S; k) \) the cardinal of the ball of radius \( k \), as defined in Section 2. The **growth series** of the pair \((\Gamma, S)\) is the
The radius of convergence of this series is strictly positive and is $1/\omega(G, S)$, where \(\omega(G, S) = \lim_{k \to \infty} \sigma(\Gamma, S; k)^{1/k}\) is the exponential growth rate of the pair \((G, S)\). It is sometimes better to consider

\[
B(\Gamma, S; z) = \sum_{k=0}^{\infty} \beta(\Gamma, S; k) z^k = \frac{\Sigma(\Gamma, S; z)}{1 - z}.
\]

For example, for the infinite cyclic group \(\Gamma = \mathbb{Z}\) generated by \(S = \{1\}\), we have

\[
\Sigma(\mathbb{Z}, \{1\}; z) = 1 + 2z + 2z^2 + 2z^3 + 2z^4 + 2z^5 + \cdots = \frac{1 + z}{1 - z}.
\]

More generally, for the free abelian group \(\mathbb{Z}^n\) generated by a basis \(S_n\), we have

\[
\Sigma(\mathbb{Z}^n, S_n; z) = \sum_{k=0}^{\infty} \left( \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{(k + n - \ell - 1)!}{k - \ell}\right) z^k = \left(1 + \frac{z}{1 - z}\right)^n.
\]

The infinite sum simplifies to \(\Sigma_{k=0}^{\infty} 4kz^k, \Sigma_{k=1}^{\infty} (4k^2 + 2)z^k, \Sigma_{k=1}^{\infty} \frac{8k(k^2+2)}{3} z^k\), when \(n = 2, 3, 4\), respectively (sequences A008574, A005899, A008412 in [OEIS]).

Growth series have been studied for several other classes of groups. For a Coxeter system \((\Gamma, S)\) with \(S\) finite, the growth series \(\Sigma(\Gamma, S; z)\) is a rational function. See exercise 26 of Chap. IV §1 and exercise 10 of Chap. VI §4 in [Bour–68]. This function has interesting values; for example, its value at 1 is rational and is the inverse of the Euler–Poincaré characteristic of the group \(\Gamma\) [Serr–71, Proposition 17, Page 112].

For a Gromov hyperbolic group \(\Gamma\) and an arbitrary generating set \(S\), Gromov has shown that \(\Sigma(\Gamma, S; z)\) is a rational function [Grom–87, Corollary 5.2.A']. This generalizes a result of Cannon [Cann–84, Theorem 7].

There are some groups \(\Gamma\) with generating sets \(S\) such that \(\Sigma(\Gamma, S; z)\) is an irrational algebraic function [Parr–92]. The growth series of a pair \((\Gamma, S)\) can also be transcendental. Stoll showed that there are groups \(\Gamma\) with two finite generating sets \(S, T\) such that \(\Sigma(\Gamma, S; z)\) is rational and \(\Sigma(\Gamma, T; z)\) transcendental [Stol–96].

11. Hilbert series. Consider again the group \(\Gamma = \mathbb{Z}^n\) for some \(n \geq 1\) and an arbitrary finite generating set \(S\). Then there exists a polynomial \(P \in \mathbb{Z}[z]\) such that

\[
\Sigma(\mathbb{Z}^n, S; z) = \frac{P(z)}{(1 - z)^n}.
\]

Here is one way to show this: the group algebra \(\mathbb{C}[\Gamma]\), with linear basis \(\delta_{\gamma}\gamma'\in\Gamma\) and multiplication defined by \(\delta_{\gamma}\delta_{\gamma'} = \delta_{\gamma\gamma'}\), has a filtration \(\mathbb{C}[\Gamma] = \bigcup_{k \geq 0} B_k\) where \(B_k\) is
the linear subspace generated by \( \{ \delta_\gamma \mid \ell_\Sigma(\gamma) \leq k \} \); set moreover \( B_{-1} = \{0\} \). The associated graded algebra \( A = \bigoplus_{k \geq 0} (B_k/B_{k-1}) \) is commutative and generated by a finite set of elements of degree 1. It is a theorem of Hilbert that the Hilbert series
\[
\sum_{k \geq 0} \dim_{\mathbb{C}}(B_k/B_{k-1}) z^k = \Sigma(Z, S; z)
\]
of such an algebra is rational of the form \( \frac{P(z)}{(1-z)^n} \); for a proof, see for example [AtMa–69, Theorem 11.1]. The observation that the growth series of \((\Gamma, S)\) is the Hilbert series of an appropriate graded algebra, and thus in particular a rational function, is due to several authors, including [Wagr–82].

The “theorem of Hilbert” refers to Theorem IV in [Hilb–90, Page 512]. In fact, Hilbert shows that the series satisfies a condition like (R3) of our Section 9, rather than (R1). But it was already standard in this time to write “Hilbert series” which are rational functions for the dimensions of the homogeneous components of a graded algebra; I am grateful to Hanspeter Kraft for showing me that this can be found in the work of Sylvester on the theory of invariants, around 1880; see for example papers 38, 40, and 59, in [Sylvester].

Hilbert series are also called Poincaré series, especially when they encode dimensions of homology spaces; see [Babe–86].

More generally, when \( \Gamma \) is a virtually abelian finitely generated group and \( S \) an arbitrary finite generating set, the series \( \Sigma(\Gamma, S; z) \) is rational [Bens–83].

12. Eugène Ehrhart and the number of integral points in the multiples of a polytope (1962). Consider an Euclidean space \( V \) of dimension \( n \), a lattice \( \Gamma \) in \( V \), i.e., a subgroup of \( V \) isomorphic to \( \mathbb{Z}^n \) and generated by a basis of \( V \), a polytope \( P \) which is the convex hull of a finite subset of \( \Gamma \), and for each non-negative integer \( k \) the number \( E_P(k) \) of points in \( kP \cap \Gamma \). In 1962, Ehrhart published a note on the numbers \( E_P(k) \) and the series \( \sum_{k=0}^{\infty} E_P(k) z^k \) [Ehrh–62, Brio–95]. For a polytope of non-empty interior, this series is a growth series of the group \( \Gamma \approx \mathbb{Z}^n \) for a particular choice of generating sets.

Note that, for the lattice \( \mathbb{Z}^n \) in \( \mathbb{R}^n \) and the convex hull \( P = \text{Conv}(\pm e_1, \ldots, \pm e_n) \), where \( \{e_1, \ldots, e_n\} \) is the standard basis of \( \mathbb{R}^n \), we have, with the notation of Section 10,
\[
\sum_{k=0}^{\infty} E_P(k) z^k = B(\mathbb{Z}^n, \mathbb{Z}^n \cap P; z) = \frac{1}{1-z} \left( \frac{1+z}{1-z} \right)^n.
\]

Other cases are studied from this point of view in [BaHV–99]. For example, when \( V = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{j=1}^{n+1} x_i = 0\} \), \( \Gamma = \mathbb{Z}^{n+1} \cap V \approx \mathbb{Z}^n \), and \( P \) is the convex hull of \( \{\pm (e_i - e_j) \mid 1 \leq i < j \leq n + 1\} \),
\[
\sum_{k=0}^{\infty} E_P(k) z^k = B(\Gamma, \Gamma \cap P; z) = \frac{1}{(1-z)^{n+1}} \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) z^j = \frac{1}{1-z} P_n \left( \frac{1+z}{1-z} \right),
\]
where \( P_n \) is the Legendre polynomial of degree \( n \).
13. **Theta functions.** Consider a Euclidean vector space $V$ of dimension $n$, with scalar product denoted by $\langle \cdot | \cdot \rangle$, and a lattice $\Gamma$ in $V$. For elements of $\Gamma$, consider no longer the word length as above, but rather the norm $\Gamma \to \mathbb{R}_+$, $x \mapsto \| x \| = \sqrt{\langle x | x \rangle}$. The **theta function** of $\Gamma$ is defined by

$$\Theta_\Gamma(\tau) = \sum_{x \in \Gamma} e^{i\pi \tau \| x \|^2},$$

so that $\Theta_\Gamma$ is a holomorphic function on the upper half-plane $\{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \}$. When $\Gamma$ is an integral lattice, namely when $\langle x | y \rangle \in \mathbb{Z}$ for all $x, y \in \Gamma$, the theta series is alternatively viewed as a power series in $q = e^{i\pi \tau}$:

$$\Theta_\Gamma(q) = \sum_{x \in \Gamma} q^{|x|^2} = \sum_{r=0}^{\infty} |\{x \in \Gamma \mid \langle x | x \rangle = r\}| q^r.$$

For example, when $\Gamma = \mathbb{Z}$ is embedded the standard way in the real line $V = \mathbb{R}$, the series is

$$\Theta_\mathbb{Z}(q) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + \cdots = \theta_3(q)$$

where $\theta_3$ is as above the third Jacobi theta function. More generally, for $\mathbb{Z}^n$ embedded the standard way in the standard Euclidean space $\mathbb{R}^d$, we have $\Theta_{\mathbb{Z}^n}(q) = (\theta_3(q))^n$ [CoSl–99, op. cit.].

It is tempting to compare the two boxed formulas of this paper related to $\mathbb{Z}$, and more generally to speculate whether theta functions could be of some interest for other groups than lattices in Euclidean spaces.

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