Excitation Spectrum of Composite Fermions

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We show that the excitation spectrum of interacting electrons at filling factor \( \nu = \nu^*/(2\nu^* + 1) \) is well described in terms of non-interacting composite fermions at filling factor \( \nu^* \), but does not have a one-to-one correspondence with the excitation spectrum of non-interacting electrons at \( \nu^* \). In particular, the collective modes of the fractional quantum Hall states are not analogous to those of the integer quantum Hall states. We also speculate on the nature of the compressible state at \( \nu = 1/2 \).

A composite fermion is an electron carrying two (in general, an even number of) vortices of the many-body wave function. The wave functions for a many body system of non-interacting composite fermions at (effective) filling factor \( \nu^* \), \( \Phi_{\nu^*} \), are obtained from the wave functions of non-interacting electrons at \( \nu^* \), \( \Phi^{n,\alpha}_{\nu^*} \), as

\[
\Phi^{n,\alpha}_{\nu^*} = \mathcal{P} D \Phi^{n,\alpha}_{\nu^*}.
\]

Here, the Jastrow factor \( D = \prod_{j<k} (z_j - z_k)^2 \), where \( z_j = x_j + iy_j \) denotes the coordinates of the \( j \)-th electron, simply attaches two vortices to each electron of \( \Phi^{n,\alpha}_{\nu^*} \) to convert it into a composite fermion. The operator \( \mathcal{P} \) projects the state on to the lowest Landau level (LL), as is appropriate in the large magnetic field limit. Due to the LL structure, the total (kinetic) energy of non-interacting electrons at \( \nu^* \) is quantized at \( n\hbar \omega_c \), measured relative to the ground state energy, where \( \hbar \omega_c \) is the cyclotron energy. The integer \( \alpha \) labels the various degenerate many-body states in the \( n \)-th ‘band’. The LL’s of electrons at \( \nu^* \) are mapped into ‘quasi-LL’s’ of composite fermions.

According to the composite fermion theory, the strongly correlated liquid of interacting electrons at filling factor \( \nu \) resembles a weakly interacting gas of composite fermions at filling factor \( \nu^* = \nu(1 - 2\nu)^{-1} \). It has been shown in the past that the spectrum of interacting electrons at \( \nu \) contains a clearly defined low-energy band which can be accurately represented as the lowest-energy band of composite fermions (i.e., by the states \( \Phi^{n,\alpha}_{\nu^*} \)). In particular, at \( \nu = p/(2p + 1) \), where \( p \) is an integer, the ground state contains \( p \) filled quasi-LL’s of composite fermions, which explains the fractional quantum Hall effect (FQHE) of electrons as the integer QHE (IQHE) of composite fermions.

The transformation from electrons to composite fermions in Eq.(1) has the unusual feature that the Hilbert spaces of electrons and composite fermions are of different sizes. For electrons at \( \nu^* \), since all higher LL’s are available, there are an infinite number of states. On the other hand, the Hilbert space of composite fermions at \( \nu^* \) is of the same size as the Hilbert space of electrons at \( \nu \) restricted to the lowest LL, which is finite (for a finite system). Therefore, any one-to-one correspondence between the electron and composite fermions systems must break down at sufficiently large energies.

This work demonstrates that this happens right beyond their lowest band. However, the higher bands of composite fermions continue to provide a good description of higher energy eigenstates of interacting electrons at \( \nu \). The lack of a one-to-one matching between the excitation spectrum of interacting electrons at \( \nu \) and non-interacting electrons at \( \nu^* \) is of relevance to two issues of current interest. The first concerns the collective modes of the FQHE states: we find that many different collective modes of the IQHE state map into the same collective mode of the FQHE state. Second, we speculate that the compressible state at \( \nu = 1/2 \) has the intriguing property that it possesses a well defined Fermi surface, but it is not a regular Landau Fermi liquid.

Our numerical calculations employ the spherical geometry in which \( N \) electrons move on the surface of a sphere under the influence of a radial magnetic field produced by a magnetic monopole of suitable strength placed at the center. The flux through the surface of the sphere is \( N_\phi \hbar c/e \), where \( N_\phi \) is an integer. We consider spinless electrons confined to their lowest LL, as appropriate in the limit of large magnetic field. The eigenstates are labeled by their total orbital angular momentum \( L \), which is analogous to the wave vector of the planar geometry; larger \( L \) corresponds to larger wave vector.

The Eq.(1) can be easily generalized to spherical geometry: the Jastrow factor \( D \) is now the square of the wave function of the lowest filled LL, and the interacting electron system at \( N_\phi \) is related to the non-interacting composite fermion system at \( N_\phi^* = N_\phi - 2(N - 1) \). Due to the symmetry of the problem, it is sufficient to work in the sector in which the \( z \)-component of the angular momentum \( L_z \) is 0, with the understanding that each state in this sector represents a degenerate multiplet of \( 2L + 1 \) states.

We consider in detail the spectrum of seven electrons at \( N_\phi = 18 \). The low-energy part of the exact Coulomb spectrum of this system is shown in Fig.1. This system corresponds to non-interacting composite fermions at \( N_\phi^* = 6 \). The lowest composite fermion band contains only one state, with the lowest filled quasi-LL. The wave function of this state is identical to the Laughlin
wave function for the 1/3 ground state $\nu_3$, which has been tested in detail in the past $[6]$. The second band of composite fermions is obtained by exciting a single composite fermion to the next quasi-LL, which is related to the second band of electrons at $\nu^* = 1$. The latter has states at $L = 1, 2, ..., 7$, but, as found in the past $[3]$, the $L = 1$ state does not produce any composite fermion state. The other composite fermion states have a good overlap with the exact eigenstates of Fig.1, as shown in Table II. The disappearance of the $L = 1$ state is the first indication that the non-interacting electrons and composite fermions have different excitation spectra.

Next we consider the third band of composite fermions. This is related to the third ($2h\omega_c$) band of non-interacting electrons at $\nu^* = 1$, in which either one electron has been excited across two LL’s, leaving a hole in the lowest LL, or two electrons have been excited by one LL each, leaving two holes in the lowest LL. Table I gives the number of independent electron states in this band for all $L$, denoted by $N_2(L)$. We construct the corresponding composite fermion states according to Eq.(1), and find, surprisingly, that they are not all linearly independent. They are also not orthogonal to the composite fermion states of the lower two bands $\nu_1$. Table I also gives the number, $N_2^*(L)$, of the new (i.e., orthogonal to the states of the lower two bands) linearly independent composite fermion states in the third band. Clearly, the third band of composite fermions contains significantly fewer states than the third band of non-interacting electrons. In the present example, the total number of states in the third band reduces from 75 to 51 upon the composite fermion transformation. These results dramatically underscore the lack of a one-to-one correspondence between states in higher bands of non-interacting composite fermions and non-interacting electrons.

Encouragingly, the $N_2^*(L)$ states above the second band in Fig.1 seem to form a more or less well defined band. (This would not be true if we took $N_2(L)$ states instead, as the analogy to non-interacting electrons would suggest.) Furthermore, the $N_2^*(L)$ linearly independent composite fermion states have significant overlap only with states in this band, but not with states outside this band $\nu_1$ – i.e., they provide a reasonable basis for the interacting electron states in this band. For each $L$, we have constructed an orthogonal basis of the composite fermion states which is expected to be close to the exact eigenstates in this band $\nu_3$. The overlaps between the composite fermion states in this basis and the exact electron states of the second band are shown in Table II. These are reasonably high, and indicate the validity of the composite fermion description for higher bands. (The relatively poor overlaps for the highest energy states of this band are to be expected, since they mix most strongly with states in the higher bands). We believe that a similar construction of higher and higher composite fermion bands will lead to a systematic description of higher and higher energy eigenstates of Fig.1.

Barring rare coincidences, one does not expect seemingly different trial wave functions to be mathematically linearly dependent (unless, of course, their number exceeds the size of the Hilbert space). The frequent linear dependence of the composite fermion trial wave functions provides an indication that they possess certain non-trivial mathematical symmetries. An appreciation of the formal structure of the composite fermion states will require a better understanding of the projection operator.

One might worry that the reduction in the number of states in such low-energy bands in going from non-interacting electrons to non-interacting composite fermions might be a finite size effect. We do not believe that this is the case, since, for any given $L$, the number of states in the first three bands is only a small fraction of the total number of states even for $N = 7$ (Table I).

We believe that the qualitative features discovered above in the context of $\nu = 1/3$ are true in general; i.e., the system of interacting electrons at $\nu = \nu^*/(2\nu^* + 1)$ is well described as a system of non-interacting composite fermions at $\nu^*$, but, except for the lowest band, does not have any simple relation with non-interacting electrons at $\nu^*$. Collective modes: The branch in the second band of Fig.1 is interpreted as a collective mode at small $L$ (i.e., small wave vectors), as a roton mode at intermediate $L$, and as a quasiparticle-phonon excitation at large $L$. Soon after Laughlin’s theory of the ground state at $\nu = 1/(2p+1)$, Girvin, MacDonald and Platzman (GMP) [11] developed a single mode approximation, which provides an excellent description of this branch at small and intermediate $L$. Recently, a Chern-Simons (CS) field theoretical formulation of the composite fermion theory has been used by Simon and Halperin [12] and by Lopez and Fradkin [13] to investigate the collective modes of the general $\nu = p/(2p + 1)$ FQHE states. These studies predict a series of intra-Landau-level collective modes at $\nu$, analogous to the inter-LL collective modes of electrons at $\nu^* = p$ [14]. In our language, the intra-LL collective modes of electrons at $\nu$ are equivalent to the inter-quasi-LL collective modes of composite fermions at $\nu^*$. In analogy to the inter-LL electron modes, their wave functions are given by states in which a single composite fermion is excited from one of the filled quasi-LL’s to an empty quasi-LL.

We consider the issue of the number of collective modes for the 1/3 state. This question is of experimental interest, since the observation of a collective mode of the 1/3 state has recently been reported [15]. To this end, we denote states, in which one composite fermion is excited from the 0th quasi-LL to the qth quasi-LL, by $\overline{\psi}_L(0 \to q) = \mathcal{P}D\psi_L(0 \to q)$, (2) where $\psi_L(0 \to q)$ is wave function of electrons (at $\nu^* = 1$)
in which one electron has been excited to the \( q \)th LL, leaving a hole in the 0th LL. We have already seen that the states \( \psi_L(0 \to 1) \) give a good description of the \textit{entire} low-energy branch in the second band of Fig.1. We have also studied the \( 0 \to 2, 0 \to 3, \) and \( 0 \to 4 \) excitations. We find the surprising result that
\[
\Psi_2(0 \to 2) = \bar{\Psi}_2(0 \to 1)
\]
(3)
and
\[
\Psi_3(0 \to 3) = \bar{\Psi}_3(0 \to 2) = \bar{\Psi}_3(0 \to 1).
\]
(4)
Thus, all composite fermion collective mode states with \( L = 2 \) and 3 are \textit{mathematically} identical \cite{16}. The exact equality of these states also holds for the \( \nu = 1/3 \) systems of 4, 5, and 6 electrons, and, we believe, is true for arbitrary \( N \). Note that \( N_{\text{tot}}(L) \) is fairly large for \( L = 2 \) and 3, and therefore the fact that the various single-composite-fermion-excitation states are identical is rather non-trivial. For \( L = 4 \) and 5, the various composite fermion modes are not identical, but they still have a reasonably good overlap with \( \Psi_L(0 \to 1) \). These results show that for small \( L \), i.e., for small wave vectors, different electron collective modes at \( \nu^* = 1 \) produce a single collective mode at \( \nu = 1/3 \) \cite{17}. This again emphasizes that the excitations of the FQHE system at \( \nu \) do not resemble the excitations of the non-interacting electron system at \( \nu^* \).

We have also carried out a preliminary study of the collective modes of other FQHE states. Here, the situation is more complicated, since several quasi-LL’s are occupied. Our findings are as follows. (i) The low-energy branch (in the second band) exhausts (to a good approximation) the spectral weight of the GMP collective mode excitation at small \( L \), suggesting only one intra-LL collective mode at small wave vectors. (ii) \( \bar{\Psi}_L(p \to 1) \) always provides a good description of the entire low-energy branch. (iii) For non-interacting electrons, there are in general several degenerate inter-LL collective modes, which may hybridize. For \( 2/5 \), one linear combination of \( \bar{\psi}_L(1 \to 3) \) and \( \bar{\psi}_L(0 \to 2) \) is almost identical to the \( \psi_L(1 \to 2) \) for small \( L \), and does not produce any new collective mode. The orthogonal combination is not close to any single electron eigenstate. Further work is in progress.

It might be argued that our conclusions might not remain true in the presence of a small amount of LL mixing, always present in experiment. In this case, it would seem natural \textit{not} to project the composite fermion states on to the lowest LL, so they would not be identical. However, clearly, the unprojected composite fermion states will simply provide \textit{different approximations} to the \textit{same} collective mode. The essential point, quite obvious on physical grounds, is that the \textit{intra-LL} spectrum of interacting electrons, or the \textit{number of intra-LL collective modes}, cannot change in any essential manner when a small amount of LL mixing is allowed. Therefore, our conclusions, obtained with the lowest LL approximation, should remain valid for the more realistic situation of a large though not infinite \( B \).

\( \nu = 1/2 \) state: Before closing, we comment on the possible implications of our results on another issue of current interest, namely on the nature of the compressible state at \( \nu = 1/2 \). Consider the \( p/(2p+1) \) sequence of incompressible FQHE states. At low temperatures, the thermodynamic properties of the incompressible FQHE states are dominated by the excitation gap, and the above considerations, which concern the spectrum \textit{above} the gap, are not of much relevance. However, the gap vanishes in the limit \( p \to \infty \), as the sequence approaches \( \nu = 1/2 \), and the excited states become relevant to the thermodynamics. Two comments pertaining to this limit may be made, \textit{provided} it is assumed that the composite fermion description remains valid for arbitrarily large \( p \) (which is unfortunately not testable in finite system studies \cite{15}). (i) The \textit{ground state} of interacting electrons at \( \nu = 1/2 \) resembles a Fermi sea of composite fermions with a well defined Fermi surface, as proposed by Halperin, Lee, and Read \cite{14}. Recent experiments lend strong support to the existence of a Fermi sea at \( \nu = 1/2 \) \cite{20}. (ii) However, the \textit{excitations} of the 1/2 system do \textit{not} have a one-to-one correspondence with the excitations of electrons at zero magnetic field. Therefore, despite the existence of the Fermi surface, the \( \nu = 1/2 \) state is not a regular Landau Fermi liquid (it may be termed “composite fermion liquid”). Nevertheless, since it is \textit{related} to the electron Fermi liquid, a Fermi-liquid-like description with renormalized parameters might be valid.

In conclusion, while the composite fermion trial wave functions provide a good description of the excitation spectrum of interacting electrons in the FQHE regime, a mean-field treatment, which implies a one-to-one correspondence between the states of interacting electrons at \( \nu = \nu^*/(2\nu^* + 1) \) and non-interacting electrons at \( \nu^* \), is not valid beyond the lowest band. This work was supported in part by NSF under Grant No. DMR93-18739.

\begin{thebibliography}{10}
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\bibitem{} The term “quasi-LL” is used for the energy levels of composite fermions to emphasize that these are different from the \textit{real} LL’s of electrons. While real LL’s of electrons occur due to a quantization of the kinetic energy, quasi-LL’s of composite fermions occur as a result of an “effective” quantization of the interaction energy. Composite fermions can occupy several quasi-LL’s even when the
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electrons are completely confined to their lowest real LL.

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[7] See, for example, G. Fano, F. Ortolani and E. Colombo, Phys. Rev. B 34, 2670 (1986).

[8] A curious feature is that the third band contains the states of the lower two bands; i.e., the composite fermion states of the lower bands can be exactly expressed as linear combinations of the third band states.
[9] Remember that they are already orthogonal to the states of the lower bands.

[10] We construct this composite fermion basis as follows. Call the initial composite fermion states $\phi_1, \ldots, \phi_M$ and the exact Coulomb eigenstates in the third band of Fig. 1 $\psi_1, \ldots, \psi_N$. We construct $\phi'_i = \sum_j c_{ij} \phi_j$, where $c_{ij} = \langle \psi_i | \phi_j \rangle$. If the two bases were related by an orthogonal transformation, then we would have $\phi'_i = \psi_i$. This, however, is not the case, and $\phi'_i$ are only approximately orthogonal. We Gramm-Schmid orthogonalize them to obtain $\phi''_i$ used to calculate the overlaps in Table II.

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[16] These are the only collective mode states at $L = 2$ and $L = 3$.

[17] Another feature of the random phase approximation in the CS approach is that the “fundamental” inter-quasi-LL mode of the composite fermions, $\psi_L(p - 1 \rightarrow p)$, is pushed up to the cyclotron energy $[3]$. This is actually crucial in recovering the Kohn’s theorem at $\nu$. Thus, the lowest energy electron collective mode at $\nu$ is expected to correspond to the “next” composite fermion collective mode $[\psi_L(p - 2 \rightarrow p) \text{ or } \psi_L(p - 1 \rightarrow p + 1)]$. Similar conclusion was reached by D.H. Lee and X.G. Wen (unpublished) for somewhat different reasons. However, no evidence of such behavior has been found either in Ref. [4] or in the present work.

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Fig.1 Some low-energy lowest-LL eigenstates of seven electrons at $N_\phi = 18$. For full spectrum, see [5].

Table I. The number of independent many-body electron and composite fermion states in the $r$th band, denoted by $N_q(L)$ and $N_q^*(L)$, respectively, for the first three bands $n = 0, 1, 2$. $N_{tot}(L)$ is the total number of states at $L$ for the system of Fig.1.

| $L$ | $N_0$ | $N_1$ | $N_2$ | $N_{tot}$ |
|-----|-------|-------|-------|-----------|
| 0   | 1(1)  | 0 (0) | 3 (2) | 10        |
| 1   | 0(0)  | 1(1)  | 9 (5) | 29        |
| 2   | 0(0)  | 1 (1) | 8 (4) | 33        |
| 3   | 0(0)  | 1 (1) | 11 (7)| 48        |
| 4   | 0(0)  | 1 (1) | 9 (5) | 49        |
| 5   | 0(0)  | 1 (1) | 10 (7)| 65        |
| 6   | 0(0)  | 1 (1) | 7 (5) | 64        |
| 7   | 0(0)  | 0 (0) | 7 (6) | 75        |
| 8   | 0(0)  | 0 (0) | 3 (3) | 74        |
| 9   | 0(0)  | 0 (0) | 3 (3) | 83        |
| 10  | 0(0)  | 0 (0) | 1 (1) | 78        |
| 11  | 0(0)  | 0 (0) | 1 (1) | 86        |

Table II. The overlaps of a suitably chosen orthogonal linear combination of composite fermion states in the second (in parentheses) and third bands with the corresponding exact states of Fig.1 (in order of increasing energy).

| $L$ | $overlap$ |
|-----|-----------|
| 0   | 0.9929, 0.7909 |
| 1   | 0.9977, 0.8506 |
| 2   | (0.9031), 0.9531, 0.9550, 0.9819, 0.9224, 0.7993 |
| 3   | (0.9793), 0.9793, 0.9909, 0.9543, 0.9527 |
| 4   | (0.9970), 0.9949, 0.9970, 0.9941, 0.9784, 0.9516, 0.9962, 0.7901 |
| 5   | (0.9906), 0.9925, 0.9796, 0.9788, 0.9486, 0.9229 |
| 6   | (0.9822), 0.9914, 0.9841, 0.9861, 0.9887, 0.9685, 0.9516, 0.8735 |
| 7   | (0.9853), 0.9842, 0.9872, 0.9798, 0.9287, 0.9070 |
| 8   | 0.9943, 0.9931, 0.9921, 0.9867, 0.9839, 0.8735 |
| 9   | 0.9911, 0.9947, 0.9684 |
| 10  | 0.9947, 0.9915, 0.9702 |
| 11  | 0.9876 |
| 12  | 0.9765 |