The fermion density operator in the droplet bosonization picture

Alberto Enciso\textsuperscript{a} and Alexios P. Polychronakos\textsuperscript{b}

\textsuperscript{a}Departamento de Física Teórica II, Universidad Complutense
28040 Madrid, Spain
aenciso@fis.ucm.es

\textsuperscript{b}Physics Department, City College of the CUNY
New York, NY 10031, USA
alexios@sci.ccny.cuny.edu

Abstract

We derive the phase space particle density operator in the 'droplet' picture of bosonization in terms of the boundary operator. We demonstrate that it satisfies the correct algebra and acts on the proper Hilbert space describing the underlying fermion system, and therefore it can be used to bosonize any Hamiltonian or related operator. As a demonstration we show that it reproduces the correct excitation energies for a system of free fermions with arbitrary dispersion relations.
1 Introduction

The description of a one-dimensional fermion system in terms of bosonic degrees of freedom, called bosonization, is a well-established and useful technique [1,2]. The advantage of such a transmutation lies in the fact that the dynamics of the equivalent bosonic theory encodes many-body excitations of the fermion system, with large coherent (classical) bosonic excitations representing large collective states of the fermions. It provides, therefore, a convenient setting to study strongly correlated fermions.

It is intuitively obvious that such bosonic descriptions must trivially exist in any number of dimensions, at least within each fermion number sector: any excitation of the system that does not change the number of fermions (or changes them by an even number) is inherently bosonic and can be obtained with the action of a bosonic operator on the ground state. The full theory, therefore, can be described in terms of a collection of these bosonic operators. Examples of such operators are the “hydrodynamic” current and particle densities, or the more general phase space density operator that will be examined in this paper.

Such descriptions, however, do not qualify as proper bosonizations. The set of bosonic operators describing the system would form, in general, an overcomplete set and would make the identification of the proper features of the fermion system difficult. More formally, the algebra of the bosonic operators would, in general, admit several irreducible representations, of which only one would be the fermion Hilbert space. Consequently, the classical action for the bosonic fields could not be used to identify proper collective excitations of the fermions.

The above-mentioned hydrodynamic or phase space density variables provide examples of such a redundancy: their algebra is the same for all systems, and admits representations that describe fermions, bosons, as well as a host of other systems (parastatistics, fractional filling states etc.) A successful bosonization of a system should realize the fermion states as its complete, irreducible Hilbert space.

Bosonization works successfully for one spatial dimension, where special properties of line-like spaces come into play. In particular, statistics and spin are essentially irrelevant notions in one dimension and thus fermionic and bosonic theories could a priori have the same degrees of freedom. On a more intuitive level, fermion dynamics around the Fermi (or Dirac) sea become tantamount to one-dimensional wave propagation, with the corresponding phonon states encoding quantum excitations [1]. Further, exponentials in the bosonic field (‘vertex operators’) can be properly defined to obey fermionic commutation relations. Overall, we have an exact mapping of states between the two systems both at the many-body and the field theory level.

Bosonization can be motivated or formulated from various starting points (operator, field theory, many-body etc.). A particularly intuitive approach is to consider a semiclassical description in terms of a dense collection of fermions forming a ‘droplet’ in their phase space and study the dynamics of the Fermi surface of the droplet [4]. In one dimension, this leads to a chiral theory that fully reproduces the standard bosonization results. In particular, for nonrelativistic
fermions, it reproduces the results of the collective field method [7].

The basic advantage of the phase space approach is that, in principle, it
works in any number of dimensions and it seems to capture the relevant degrees
of freedom for a successful bosonization. In fact, an adaptation of this method
properly taking into account the quantum nature of the phase space was pro-
posed by one of the authors as the starting point for an exact (that is, complete
and irreducible) bosonization in arbitrary dimensions [5]. This could become a
useful tool in the study of correlated fermion systems in dimensions higher that
one. (For other approaches to higher dimensional bosonization see [6].)

In paving the way to such applications, it is important to establish the
full description of the Fermi system in the phase space droplet picture. In
particular, it is important to have an exact mapping of all fermion operators in
terms of the (bosonic) field describing the droplet boundary. This would provide
the bosonized form of any many-body Hamiltonian, including interactions, and
requires deriving the boundary field expression of a complete set of fermion
operators. Such a set is encoded by the phase space density operator, in terms
of which any other operator can be obtained.

The purpose of this paper is to do this in the controllable case of one spatial
dimension, in which standard bosonization works. Specifically, the expression
for the quantum density operator will be given as a function of the droplet
boundary field. It will be shown that this operator obeys the correct algebra
and acts on the correct Hilbert space provided by the unique irreducible repre-
sentation of the boundary field. It therefore realizes a bosonization of any
fermion system in the droplet picture and can be used as the starting point for
higher-dimensional generalizations.

2 Review of the droplet method

The starting point is a semiclassical description of the many-body fermion state
in terms of its phase space density. For a full discussion of this method in any
number of dimensions we refer the reader to [8]. For our specific case of one-
dimensional systems, the phase space density is a function of two phase space
variables \( \rho(x, p) \) that gives the number of particles \( dN \) inside a small phase
space area \( dx \, dp \):

\[
dN = \rho(x, p) \, dx \, dp.
\]

In terms of a (large) number of individual particles, this can be thought as the
smoothed-out version of the phase space point density

\[
\rho(x, p) = \sum_{i=1}^{N} \delta(x - x_i) \delta(p - p_i)
\]  

(1)

where \((x_i, p_i)\) are the phase space positions of the particles.

We shall consider a collection of non-interacting spinless fermions with a
single-particle Hamiltonian \( H_{sp}(x, p) \). For simplicity, we assume that we have
identified Darboux coordinates in phase space, with single-particle Poisson brackets
\[ \{x, p\}_\text{sp} = 1. \]

Ordinary particle position and momentum are such coordinates, but in principle we can use any other canonical pair.

Semiclassically there is one quantum state per area $2\pi\hbar$ in phase space. The exclusion principle mandates that we can put at most one particle per state, and implies a maximal value for the fermion density $\rho_0 = 1/2\pi\hbar$, corresponding to a dense collection of fermions each occupying one quantum of phase space area.

The ground state of the system would consist of fermions occupying all available single-particle states up to a maximal energy $E_F$. Semiclassically, this corresponds to a density $\rho = \rho_0$ for all points in phase space with energy $E = H_{\text{sp}}(x, p) < E_F$ and $\rho = 0$ for points with $E = H_{\text{sp}}(x, p) > E_F$. This is the ‘phase space droplet’ picture. This state is parametrized by its one-dimensional boundary, in this case the set of phase space points at energy $E_F$:

\[ \{(x_b, p_b) : H_{\text{sp}}(x_b, p_b) = E_F\}. \]

We shall assume that this is a single connected curve, although situations where the boundary is disconnected, corresponding to topologically nontrivial or disconnected droplets do arise and can be easily treated. The density is given as

\[ \rho(x, p) = \rho_0 \vartheta(E_F - H_{\text{sp}}(x, p)), \]

where $\vartheta(y) = \frac{1}{2}(1 + \text{sgn}(y))$ is the step function. The Fermi level $E_F$ and the total number of particles $N$ are related as

\[ N = \int \rho(x, p) \, dx \, dp = \frac{A_F}{2\pi\hbar}, \]

where $A_F$ is the phase space area of the ground state droplet.

Excitations of the system can be visualized as deformations of the droplet, in which its boundary deviates from the Fermi level $E_F$. The density function remains constant and equal to $\rho_0$ inside the droplet and zero outside. Such states are determined by the (arbitrary) shape of the boundary. A convenient way to parametrize the boundary is to give the value of one of the phase space coordinates, say $p$, on the boundary as a function of the other:

\[ p_b = R(x). \]

We shall call the function $R(x)$ the boundary field. The density of the droplet is given in terms of $R(x)$ as

\[ \rho(x, p) = \rho_0 \vartheta(R(x) - p). \quad (2) \]

The phase space density description of the system has a hamiltonian structure, inherited from the single-particle canonical structure. The density function
satisfies the field theory Poisson brackets
\[ \{ \rho(x, p), \rho(x', p') \}_\text{FT} = \partial_x \rho(x, p) \delta(x - x') \delta'(p - p') - \partial_p \rho(x, p) \delta'(x - x') \delta(p - p'). \] (3)

These are the standard Poisson field brackets, implying the canonical structure for test functions \( \rho[f] = \int dx \, dp \, f(x, p) \rho(x, p) \):
\[ \{ \rho[f], \rho[g] \}_\text{FT} = \rho[\{ f, g \}_\text{sp}] \]

The hamiltonian is the total energy \( H \) of the state, found by summing the energies of all particles:
\[ H = \int \Delta_H \rho(x, p) \, dx \, dp = \rho[H_{\text{sp}}] \] (4)

The above reproduce the evolution of the field \( \rho \) as the canonical equation of motion
\[ \dot{\rho} = \{ \rho, H \}_\text{FT} = \{ \rho, H_{\text{sp}} \}_\text{sp} \]
This is the same as the equation implied by the evolution of the particles making up the state according to their single-particle equations of motion.

For a droplet state, the density inside the droplet remains constant, by Liouville’s theorem. Its evolution is determined by the evolution of the boundary, that is, the boundary field \( R(x) \). The hamiltonian becomes
\[ H = \rho_o \int \Delta_H \rho(x, p) \, dx \, dp \] (5)

The canonical structure for the field \( R(x) \) can be determined through a hamiltonian reduction of the original Poisson brackets to the submanifold of functions \( \rho(x, p) \) of the droplet form. The result is
\[ \{ R(x), R(x') \}_\text{FT} = -\frac{1}{\rho_o} \delta'(x - x') = -2\pi \hbar \delta'(x - x') \]

Conversely, the above Poisson structure for \( R(x) \) implies the Poisson brackets for the density function \( \rho(x, p) \) as expressed in (2).

The advantage of the above semiclassical construction is that its main features survive quantization. Specifically, the boundary field’s Poisson brackets are elevated to quantum commutators (from now on we use the convention \( \hbar = 1 \) and denote quantum operators with hatted symbols):
\[ [\hat{R}(x), \hat{R}(x')] = -2\pi \hbar \delta'(x - x') \] (6)
We recognize the above commutation relations as a chiral abelian current algebra; the field \( \hat{R}(x) \) is the chiral field that bosonizes the fermion theory around the Fermi level \( E_F \). If the fermion system has more than one Fermi points, the values of the boundary field at each point become independent commuting chiral fields.
The above chiral algebra has a unique irreducible unitary representation. Its states are into one-to-one correspondence with the excitations of the fermion system around the Fermi level. Furthermore, the Hamiltonian \( \hat{H}(x) \), diagonalized in the states of this representation, reproduces the correct excitation energies of the fermion system, at least for the cases of free relativistic or nonrelativistic fermions. We recover the results of standard (abelian) bosonization, with the boundary field \( \hat{R}(x) \) becoming the chiral part of the many-body particle current operator in the bosonic description. For free fermions on a compact space there are generically two Fermi points and the corresponding boundary fields constitute the chiral components of the derivative of the boson field.

The mapping of states fails only when the number of excited fermions reaches \( N \), in which case we deplete the Fermi sea. This is a large-deformation effect in the droplet picture, expressed by a condition of the form \( \hat{R}(x) > 0 \), which would introduce nontrivial constraints in the states of the Hilbert space of \( \hat{R}(x) \) and would complicate the picture. Such effects are purely nonperturbatively in the number of fermions \( N \) and can be ignored for large \( N \). In the sequel we shall talk about the bosonization scheme being ‘exact’ and will always mean up to nonperturbative effects.

### 3 The algebra of density operators

The quantum density operator \( \hat{\rho}(x,p) \) would naively satisfy the commutation relations implied by elevating the Poisson brackets (3) into quantum commutators. This, however, is not the case. To see this, start with the quantum version of the many-body density expression (1):

\[
\hat{\rho}(x,p) = \sum_{i=1}^{N} \delta(x - \hat{x}_i)\delta(p - \hat{p}_i).
\]

Since \( \hat{x}_i \) and \( \hat{p}_i \) do not commute, the above expression is not even hermitian and requires the specification of a quantum ordering.

It is convenient to adopt the Weyl (symmetric) ordering between \( \hat{x}_i \) and \( \hat{p}_i \). In terms of the Fourier transform \( \hat{\rho}(l, k) \) of \( \hat{\rho}(x, p) \) this is expressed as

\[
\hat{\rho}(l, k) = \sum_{i=1}^{N} e^{-il\hat{x}_i - ik\hat{p}_i}.
\]

Using the Baker–Campbell–Hausdorff formula, it is now a simple matter to derive the commutators

\[
[\hat{\rho}(l, k), \hat{\rho}(l', k')] = 2i \sin \left( \frac{l'k - lk}{2} \right) \hat{\rho}(l + l', k + k').
\] (7)

This is the famous ‘sine’ algebra that deforms the classical field theory Poisson brackets. It is realized, for instance, by the spatial density of planar particles in the lowest Landau level (modulo a factor arising from the different,
anti-normal ordering in that case) \cite{11}. In a different basis, it is known as the \( W_\infty \) algebra \cite{10}. We shall simply call it the (quantum) density algebra.

The use of the above Weyl-ordered operator \( \hat{\rho}(x,p) \) is that, if the single-particle quantum hamiltonian \( \hat{H}_{sp} \) is expressed in a Weyl-ordered form in terms of \( \hat{x} \) and \( \hat{p} \) (which is always possible), then the exact quantum many-body hamiltonian can be expressed as

\[
\hat{H} = \sum_{i=1}^{N} \hat{H}_{sp}(\hat{x}_i, \hat{p}_i) = \int H_{sp}(x,p) \hat{\rho}(x,p) \, dx \, dp ,
\]

with \( H_{sp}(x,p) \) the Weyl ‘symbol’ of \( \hat{H}_{sp} \), that is, the Weyl-ordered classical expression of \( \hat{H}_{sp} \) in terms of \( x \) and \( p \). In effect, the many-body hamiltonian is obtained via the (properly ordered) classical expression by the use of \( \hat{\rho}(x,p) \). All other many-body operators can similarly be obtained as appropriate expressions involving \( \hat{\rho}(x,p) \), which becomes a ‘universal’ many-body operator.

In the context of the phase space bosonization, what remains to be shown is that the droplet expression (2) for the quantum density operator \( \hat{\rho}(x,p) \) in terms of \( \hat{R}(x) \) satisfies the proper quantum mechanical commutation relations. If this is the case, then the bosonization of any many-body fermion system has been achieved. Indeed, as argued before, \( \hat{R}(x) \) reproduces the proper many-body fermion Hilbert space and \( \hat{\rho}(x,p) \), acting on this Hilbert space and satisfying the proper algebra, would be the correct quantum mechanical expression for the density operator. In particular, the expression (3) for the hamiltonian would be exact and would reproduce the correct energy states.

It is possible to start with the expression (2) in terms of \( \hat{R}(x) \) and, using the operator commutation relations \cite{11} for \( \hat{R}(x) \), show that it formally obeys the correct operator algebra \cite{12}. This proof, however, suffers from shortcomings related to the presence of singularities. Specifically, the expression (2) for \( \hat{\rho}(x,p) \) is formal and contains infinities. A proper definition requires normal ordering and the normal-ordered expression for \( \hat{\rho}(x,p) \) fails to reproduce the correct algebra.

In the next sections we show that a proper quantum mechanical definition of the droplet density \( \hat{\rho}(x,p) \) requires a nontrivial modification; once this is done, the correct operator algebra is recovered.

4 The semiclassical droplet density operator

Although the semiclassical droplet density presented above is not the appropriate expression for the quantum density operator \( \hat{\rho}(x,p) \), it will nevertheless be used as the starting point for an exact bosonization of the system.

For concreteness, we consider fermions on a periodic space of period \( L = 2\pi \), which allows for discrete integer Fourier modes for functions of \( x \). We shall use Darboux coordinates \( (x,p) \in [-\pi, \pi] \times [0, \infty) \) where the momentum is positive. (This amounts to considering only ‘half’ the Fermi sea states, and thus working around one Fermi level with one boundary field operator.) So \( (x,p) \) can be
thought of as angular and radial variables describing the droplet. The quantum boundary field $\hat{R}(x)$ obeys the commutation relations (8), or, in terms of modes,

$$[\hat{R}_n, \hat{R}_m] = n\delta_{n+m},$$


(8)

where

$$\hat{R}_n = \frac{1}{2\pi} \int \hat{R}(x) e^{-inx} dx \quad (n \in \mathbb{Z})$$

denote the Fourier components of the field $\hat{R}(x)$.

The above is a set of mutually commuting harmonic oscillators. In standard current algebra representation fashion, we consider the Fock space of the oscillators consisting on the vacuum state $|0\rangle$, which is annihilated by all the positive modes $\hat{R}_n$ ($n > 0$), and all the excited states generated by the action of creation operators $\hat{R}_n^\dagger$. Since the zero mode $\hat{R}_0$ is a Casimir operator, its action on the vacuum $\hat{R}_0|0\rangle = \frac{N}{2}|0\rangle$ defines a conserved quantity $N$, which corresponds to the number of fermions. For concreteness, let us assume that $N$ is an odd integer.

We shall use the notation

$$|n_1, \ldots, n_j\rangle = \hat{R}_{-n_1} \cdots \hat{R}_{-n_j} |0\rangle$$

to represent the action of the negative modes; one should observe that, since negative modes commute among themselves, any permutation of $(n_1, \ldots, n_j)$ defines the same vector, so the state $|n_1, \ldots, n_j\rangle$ is only defined up to permutations.

A first attempt to define the quantum density operator would be to normal order the expression for $\vartheta(\hat{R}(x) - p)$, so as to have an operator with finite matrix elements between excited states; that is,

$$:e^{i\sum_{n\in\mathbb{Z}} c_n \hat{R}_n} : = e^{i\sum_{n<0} c_n \hat{R}_n} e^{\sum_{m\geq 0} c_m \hat{R}_m},$$

which is tantamount to moving all the positive (resp. negative) modes to the right (resp. left) in any polynomial in the Fourier components of $\hat{R}(x)$. As the positive and negative modes commute among themselves, the above expression is meaningful. We can thus introduce the quantum counterpart

$$\hat{\varrho}(x, p) = \frac{1}{2\pi} :\vartheta(\hat{R}(x) - p) :$$

(9)

of the semiclassical density.

It can be seen, however, that this is not the correct quantum expression, as it does not satisfy the proper algebra. In fact, it is instructive to study the spectrum of the quantum hamiltonian operator

$$\hat{H}_0 = \int H_{sp}(x, p) \hat{\varrho}(x, p) dx dp - E_0,$$

(10)

and compare with the exact many-body fermion spectrum. Here the integral is taken over the domain $[-\pi, \pi] \times [0, \infty)$ and we have subtracted the inessential central term

$$E_0 = \frac{1}{2\pi} \int H_{sp}(x, p) \vartheta(\hat{R}_0 - p) dx dp$$

7
to shift to zero the ground state energy. Let us henceforth assume that the
single particle Hamiltonian is

$$H_{sp}(x, p) = p^s$$

(11)

for some positive integer $s$. With this assumption, one can easily find the
expression

$$\hat{H}_v = \frac{1}{2\pi(s + 1)} \int : \hat{R}(x)^{s+1} : dx - \frac{1}{s + 1} \hat{R}_0^{s+1}$$

$$= \frac{1}{2\pi(s + 1)} \sum_{n \in \mathbb{Z}^{s+1}} \hat{R}_{n_1} \cdots \hat{R}_{n_{s+1}} \int e^{i(n_1 + \cdots + n_{s+1})x} dx - \frac{1}{s + 1} \hat{R}_0^{s+1}$$

$$= \frac{1}{s + 1} \sum_{n \in \mathbb{Z}^{s+1} - \{0\}, \sum n_i = 0} \hat{R}_{n_1} \cdots \hat{R}_{n_s}$$

(12)

for the integral (10). Since $\hat{R}_0 = \frac{N}{2}$, one has the series expansion

$$\hat{H}_v = s\left(\frac{N}{2}\right)^{s-1} \hat{R}_{-n} \hat{R}_n$$

$$+ \frac{1}{2}s(s - 1)\left(\frac{N}{2}\right)^{s-2} (\hat{R}_{-n} \hat{R}_{-m} \hat{R}_{n+m} + \hat{R}_{-n-m} \hat{R}_n \hat{R}_m) + O(N^{s-3}\hat{R}^4).$$

(13)

Here and in what follows, all the dummy indices $n, m$ range from 1 to $\infty$ unless
otherwise stated, and summation over repeated indices is understood.

In order to study the low energy modes of $\hat{H}_v$, it is convenient to define the
order of a state $|n_1, \ldots, n_j\rangle$ of the Fock space to be the integer $n_1 + \cdots + n_j$.
From Eq. (12) it is clear that the subspace of states of any fixed order $r$ is
invariant under $\hat{H}_v$. Hence both $|0\rangle$ (ground state) and $|1\rangle$ (first excited state)
must be eigenstates of $\hat{H}_v$, whose corresponding energies

$$E_0^v = 0, \quad E_1^v = s\left(\frac{N}{2}\right)^{s-1}$$

(14)

can be easily obtained from Eq. (13).

Let us compare energies (14) with the actual ones. The exact energies can be obtained by quantizing the single-particle momentum, which is restricted to
take integer values, and then filling the energy levels with $N$ fermions. Hence,
the ground state energy is obtained by filling all the momenta up to $p = \frac{N-1}{2}$
with fermions. The first excited level corresponds to lifting the top fermion (at
$p = \frac{N-1}{2}$) to the next available level, i.e., $p = \frac{N+1}{2}$. The second excitations are
obtained by promoting either the top fermion to $p = \frac{N+3}{2}$ or the second highest
$p = \frac{N-3}{2}$ to $p = \frac{N-1}{2}$. Subtracting the ground state energy, this yields the
values

$$E_0^{\text{exact}} = 0,$$

(15a)

$$E_1^{\text{exact}} = 2^{-s}[(N + 1)^s - (N - 1)^s],$$

(15b)

$$E_2^{\text{exact}} = 2^{-s}[\pm (N \pm 3)^s \mp (N \mp 1)^s].$$

(15c)
for the lowest eigenenergies of the system. Therefore, the Hamiltonian solely provides the leading semiclassical term of the exact energy.

We therefore conclude that the normal-ordered semiclassical density is, as expected, exact only in the semiclassical limit \( N \to \infty \). A further modification is needed for the exact operator.

## 5 The exact quantum density operator

The main result of this paper is the construction of a modified density operator \( \hat{\rho}(x, p) \) which satisfies the sine algebra

\[
[\hat{\rho}(l, k), \hat{\rho}(l', k')] = 2i \sin \left( \frac{lk'}{2} \right) \hat{\rho}(l + l', k + k')
\]

and provides an exact bosonization of the system via

\[
\hat{H} = \int H_{sp}(x, p) \hat{\rho}(x, p) \, dx \, dp.
\]

As before, \( \hat{\rho}(l, k) \) denotes the Fourier transform of the density operator, in terms of which one can recover \( \hat{\rho}(x, p) \) as

\[
\hat{\rho}(x, p) = \frac{1}{4\pi^2} \sum_{l=-\infty}^{\infty} e^{i lx} \int \hat{\rho}(l, k) e^{-ikp} \, dk
\]

We claim that the modified density operator is given by

\[
\hat{\rho}(l, k) = \int \frac{2i}{4\pi \sin \frac{xml}{2}} a(x, k) b(x, k) \hat{R}_{0} e^{-i k x} \, dx,
\]

where

\[
a(x, k) = \frac{2i}{n} \sin \frac{nk}{2} \hat{R}_{-n} e^{-inx}, \quad b(x, k) = \frac{2i}{n} \sin \frac{nk}{2} \hat{R}_{n} e^{inx}.
\]

Since the classical limit amounts to \( k, l \ll 1 \), and the Fourier transform of the step function is \( \frac{1}{2\pi} \), it is clear that we recover \( \hat{\rho}(x, p) \) in the classical limit from Eq. (18).

Let us prove that the modified density operator satisfies the sine algebra. We shall multiply each term of the Fourier series in Eq. (19) by \( e^{-\epsilon n} \) to render the sum (strongly) convergent, and take the limit \( \epsilon \downarrow 0 \) at the end of the calculation. This is equivalent to acting with the operators \( \hat{\rho}(l, k) \) on the vacuum, or states generated from the vacuum through the action of a finite number of creation operators \( \hat{R}_{-n} \), which is the proper physical Hilbert space. On such states only a finite number of terms in the expansion of \( \hat{\rho}(l, k) \) survive and all expressions are finite and convergent. To simplify the notation, from now on we shall write \( a, a' \) instead of \( a(x, k), a(x', k') \) (and similarly for \( b \)).
Since the commutator
\[
[a, b'] = -4n \delta_{nm} \frac{\sin \frac{nk}{2} \sin \frac{mk}{2}}{\sin \frac{k}{2} \sin \frac{k'}{2}} e^{i\frac{\pi}{2}(x - x')n} e^{i(x - x')m} \\
= \log \left( \frac{1 - e^{i\left(\frac{\pi}{2}x + x' + i\epsilon\right)}}{1 - e^{i\left(-\frac{\pi}{2}x + x' + i\epsilon\right)}} \right) \\
= \log \left( 1 + \frac{2 \sin \frac{k}{2} \sin \frac{k'}{2}}{\cos \frac{k+k'}{2} - \cos(x - x' + i\epsilon)} \right)
\]
is a central element, we can use the Baker–Campbell–Hausdorff formula to write
\[
[\hat{\rho}(l, k), \hat{\rho}(l', k')] = -\int \left[ e^{a'b}, e^{a'b'} \right] \frac{e^{i(k+k')\hat{R}_0 - i(lx + l'x')}}{16\pi^2 \sin \frac{k}{2} \sin \frac{k'}{2}} dx \, dx' \\
= \int e^{a+a'} e^{b+b'} f(x - x', k, k', l') e^{i(k+k')\hat{R}_0 - i(x + l'x')} dx \, dx',
\]
where the function \( f = f(x - x', k, k', l') \) is given by
\[
f = -\frac{e^{i\rho'(x-x')}}{16\pi^2 \sin \frac{k}{2} \sin \frac{k'}{2}} (e^{[a, b']} - e^{[a', b']}) \\
= e^{i\rho'(x-x')} \left( \frac{1}{\cos \frac{k+k'}{2} - \cos(x - x' + i\epsilon)} - \frac{1}{\cos \frac{k+k'}{2} - \cos(x - x' + i\epsilon)} \right).
\]

To evaluate the commutator (21), it is convenient to change variables in the integral from \( x \) and \( x' \) to \( x \) and \( y = x - x' \), and use periodicity to evaluate the \( y \)-integral over the boundary of the rectangle
\[
\Omega = \{ y \in \mathbb{C} : -\pi < \text{Re} y < \frac{k+k'}{2}, (\text{sgn} l') \text{ Im } y > 0 \}.
\]
For concreteness, let us suppose that \( l' \) is positive. Let us denote the operator-valued function \( e^{a+a'} e^{b+b'} \) by \( F \) and set \( K = \frac{k+k'}{2} \). The function \( fF \) has two poles in \( \Omega \), namely \( y_{\pm} = \pm K + i\epsilon \), and its residues are
\[
\text{Res}(fF; y_{\pm}) = \pm \frac{e^{i\rho'(y_{\pm})}}{8\pi^2 \sin K} F \big|_{y=y_{\pm}}.
\]
In the limit \( \epsilon \downarrow 0 \), the poles tend to \( \pm K \) and one can write
\[
\left. F \right|_{y=\pm K} = e^{\frac{\pi}{2} \hat{R}_n e^{-i\pi x} (\sin \frac{nk}{2} + \sin \frac{mk}{2})} e^{\frac{\pi}{2} \hat{R}_m e^{i\pi x} (\sin \frac{nk}{2} + \sin \frac{mk}{2}) e^{i\pi m K}} \\
= e^{\frac{\pi}{2} \hat{R}_n e^{-i\pi x (\pm \frac{\pi}{2})} \sin nK} e^{\frac{\pi}{2} \hat{R}_m e^{i\pi x (\pm \frac{\pi}{2})} \sin nK} \\
= e^{a(x+\pm \frac{k'}{2}, k+k')} e^{b(x+\pm \frac{k'}{2}, k+k')}.
\]
(23)
Combining Eqs. (22) and (23) one can evaluate the integral (21) by residues as
\[
[\hat{\rho}(l, k), \hat{\rho}(l', k')] = 2\pi i \lim_{\epsilon \downarrow 0} \int \left[ \text{Res}(fF; y+) + \text{Res}(fF; y-) \right] e^{i(k+k')\hat{R}_0 - i(l+l')x} dx
\]
\[
= \frac{e^{ilK}}{4\pi} \int \frac{e^{i(k+k')\hat{R}_0 - i(l+l')x} e^{a(x-k/2)} e^{b(x-k/2)}}{\sin K} dx
\]
\[
- \frac{e^{-ilK}}{4\pi} \int \frac{e^{i(k+k')\hat{R}_0 - i(l+l')x} e^{a(x+k/2)} e^{b(x+k/2)}}{\sin K} dx
\]
\[
= 2i \sin \left( \frac{lk - l'k}{2} \right) \hat{\rho}(l + l', k + k').
\]

This completes the proof of Eq. (7).

6 The energy spectrum

We shall now demonstrate that the spectrum of the Hamiltonian (17) expressed in terms of the above density operator encodes the full low-energy dynamics of the \(N\)-fermion system.

By an appropriate canonical transformation, almost any single-particle hamiltonian can be brought to the form
\[
H_{sp}(x, p) = h(p),
\]
where \(x\) is periodic and \(p > 0\). This is true for hamiltonians with compact, simply laced equal-energy lines \(H_{sp} = E\), where \(x\) is the angular variable around equal-energy lines and \(p\) an appropriate conjugate variable. Corresponding quantum hamiltonians can also be brought to this form upon proper reordering.

It is easy to show that all such bosonized many-body hamiltonians commute with each other. Indeed, the expression
\[
\hat{H}_s = \int h(p) \hat{\rho}(x, p) dx dp
\]
will involve only the Fourier modes \(\hat{\rho}(0, k)\). All such modes commute with each other, and so will the \(\hat{H}_s\). They constitute a space of commuting 'charges', with the hamiltonians \(\hat{H}_s\) arising from the single-particle hamiltonian \(h(p) = p^s\) forming a basis for this space. All such hamiltonians have the same set of eigenstates, which are, then, independent of \(h\) or \(s\).

We shall, therefore, consider such a hamiltonian \(\hat{H}_s\), and, for simplicity, will also subtract the zero-mode contribution to the energy:
\[
\hat{H}_s = \int p^s \hat{\rho}(x, p) dx dp - E_0,
\]
\[
E_0 = \int p^s \hat{\rho} (\hat{R}_0 - p) dp.
\]
A more manageable formula for \( \hat{H}_s \) can be obtained performing the sum over \( l \) and the \( p \)-integral in Eq. (26), yielding

\[
\hat{H} = \sum_{l=-\infty}^{\infty} \int \frac{e^{ia} e^{ib} e^{ik(h_0 - p) - il(x-x')}}{16\pi^3 i \sin \frac{k}{2}} \, dx \, dx' \, dk \, dp - E_0
\]

\[
= \sum_{l=-\infty}^{\infty} \int \frac{p^s e^{ik(h_0 + ily)}}{8\pi^2 i l! j! \sin \frac{k}{2}} a^i b^j \, dx \, dy \, dk \, dp - E_0
\]

\[
= (-i)^s \frac{1}{4\pi i} \frac{\partial^s}{\partial k^s} \bigg|_{k=0} \left[ \frac{e^{iNk/2}}{\sin \frac{k}{2}} \int \left( \sum \frac{a^i b^j}{i!} - 1 \right) \, dx \right].
\]  

(26)

Despite the presence of a sine in the denominator, it is not difficult to check that the integrand is in fact nonsingular at \( k = 0 \).

Now we shall compute the lowest energy levels of this Hamiltonian. From Eqs. (20) and (26) it stems that \( \hat{H} \) preserves the subspace of states of a given order because of the \( x \)-integral. (In fact, the order of the state is simply the eigenvalue of the \( \hat{H}_1 \) operator.) One should also notice that the action of a term \( \int a^i b^j \, dx \) over a state of order \( r \) vanishes whenever \( i \) or \( j \) are greater than \( r \). As a consequence of these observations, \(|0\rangle\) and \(|1\rangle\) must be eigenfunctions of \( \hat{H}_s \) and \(|0\rangle\) lies in its kernel.

The above observation also implies that the sole summand in Eq. (26) whose action on the order-one state can be nonzero is

\[
\int dx \, ab \, |1\rangle = -8\pi \sin^2 \frac{k}{2} |1\rangle.
\]

This easily leads to the formula

\[
E_1 = 2(-i)^{s-1} \frac{\partial^s}{\partial k^s} \bigg|_{k=0} \left( \sin \frac{k}{2} e^{iNk/2} \right)
\]

\[
= 2^{1-s} \sum_{r=1}^{s} \frac{s}{2r-1} (-1)^{r-1} N^r - 1
\]

\[
= 2^{-s} \left[ (N+1)^s - (N-1)^s \right]
\]

for the first excited energy level. This result coincides with the exact energy (15).

It can be verified that higher energy levels are also exactly recovered. For simplicity we shall concentrate on the second excitations of the system, which can be obtained by diagonalizing the action of the Hamiltonian on the linear span of \(|2\rangle\) and \(|11\rangle\). We shall prove that they do coincide with the exact energies (15).

By the preceding remark, the summands in Eq. (26) with \( i, j > 2 \) yield a vanishing contribution. In fact, a short calculation using the commutation
relations (8) shows that the only terms which contribute to the integral are

\[ \int dx \, ab |2\rangle = -4\pi \sin^2 k |2\rangle , \]  
\[ \int dx \, a^2 b/2 |2\rangle = -8\pi i \sin^2 k \sin k |11\rangle , \]  
\[ \int dx \, ab |11\rangle = -16\pi \sin^2 k/2 |11\rangle , \]  
\[ \int dx \, a^2 b^2/4 |11\rangle = 16\pi \sin^4 k/2 |11\rangle , \]  
\[ \int dx \, ab^2/2 |11\rangle = -8\pi i \sin^2 k/2 \sin k |2\rangle . \]  

Using Eqs. (26) and (27) one can write the action of the Hamiltonian on the state \(|2\rangle\) as

\[ \hat{H} |2\rangle = (-1)^s \sum_{r=0}^{[s]} \left( \frac{s}{2r-1} \right) N^{s-2r+1} (1 + 3^{2r-1}) |2\rangle + 2^{-s} \sum_{r=0}^{[s]} \left( \frac{s}{2r} \right) N^{s-2r} (3^{2r} - 1) |11\rangle . \]  

The analogous expression

\[ \hat{H} |11\rangle = 2^{-s} \sum_{r=0}^{[s]} \left( \frac{s}{2r-1} \right) N^{s-2r+1} (1 + 3^{2r-1}) |11\rangle + 2^{-s} \sum_{r=0}^{[s]} \left( \frac{s}{2r} \right) N^{s-2r} (3^{2r} - 1) |2\rangle , \]  

can be obtained directly from the above equations using the elementary identity

\[ \sin \frac{k}{2} - \sin^3 \frac{k}{2} = \frac{\sin^2 k/2}{4\sin k/2} . \]  

Therefore the action of \(\hat{H}\) is diagonal in the basis \(\frac{1}{\sqrt{2}} (|2\rangle \pm |11\rangle)\) and the corresponding eigenvalues are

\[ \hat{H} |2\rangle = 2^{-s} \sum_{r=0}^{[s]} \left( \frac{s}{2r-1} \right) N^{s-2r+1} (1 + 3^{2r-1}) |2\rangle + 2^{-s} \sum_{r=0}^{[s]} \left( \frac{s}{2r} \right) N^{s-2r} (3^{2r} - 1) |2\rangle , \]  

The analogous expression

\[ \hat{H} |11\rangle = 2^{-s} \sum_{r=0}^{[s]} \left( \frac{s}{2r-1} \right) N^{s-2r+1} (1 + 3^{2r-1}) |11\rangle + 2^{-s} \sum_{r=0}^{[s]} \left( \frac{s}{2r} \right) N^{s-2r} (3^{2r} - 1) |2\rangle , \]  

can be obtained directly from the above equations using the elementary identity

\[ \sin \frac{k}{2} - \sin^3 \frac{k}{2} = \frac{\sin^2 k/2}{4\sin k/2} . \]  

Therefore the action of \(\hat{H}\) is diagonal in the basis \(\frac{1}{\sqrt{2}} (|2\rangle \pm |11\rangle)\) and the corresponding eigenvalues are

\[ E_{2,\pm} = 2^{-s} \sum_{r=0}^{[s]} \left( \frac{s}{2r-1} \right) N^{s-2r+1} (1 + 3^{2r-1}) \pm 2^{-s} \sum_{r=0}^{[s]} \left( \frac{s}{2r} \right) N^{s-2r} (3^{2r} - 1) \]  

\[ = 2^{-s} \sum_{r=1}^{s} \left( \frac{s}{r} \right) N^{s-r} [\pm (\mp 3)^r \mp (\mp 1)^r] \]  

\[ \mp 2^{-s} [\pm (N \mp 3)^s \mp (N \mp 1)^s] . \]  

Thus the exact result (15c) is reobtained, as we wanted to prove.

Notice that the eigenstates corresponding to the eigenvalues \((29)\) just obtained are, up to a normalizing factor, \(|2\rangle \pm |11\rangle\). The coefficients are not incidental: they can, in fact, be read off from the irreducible representations
of the symmetric group. Higher excited states of order $n$ can similarly be constructed as characters of representations with $n$ boxes in their Young tableaux. A complete proof of this statement is possible but will not be given here.

The simplest examples of physical systems of the form (11) are those with linear ($s = 1$) and quadratic ($s = 2$) dispersion relations. The $s = 1$ system can be interpreted as a harmonic oscillator in the action-angle basis, or as a massless relativistic particle on a circle. The $s = 2$ system would describe a free nonrelativistic particle on a circle.

Interestingly, Eq. (14) shows that the semiclassical approximation is exact for these systems. This is related to the fact that the expression for $\hat{H}_s$ obtained from (25) is of the form

$$\hat{H}_s = \frac{1}{(s+1)!} \int : \hat{R}(x)^{s+1} : \, dx + \text{lower order terms} \quad (30)$$

where the extra terms include powers $\hat{R}(x)^{s-1}$ or lower, as well as terms with derivatives of $\hat{R}(x)$. Such terms are all trivial for $s = 1, 2$ and therefore the semiclassical result is essentially exact.

7 Conclusions

We have derived the droplet expression for the fermion phase space density satisfying the proper operator algebra. We also verified that it reproduces the correct many-body fermion energies for non-interacting hamiltonians and that it leads to the standard relativistic boson field theory and cubic collective field theory for relativistic and nonrelativistic free fermions respectively.

The main thrust of this work, of course, is the higher dimensional situation. For this, the corresponding expression for the higher dimensional phase space density should be obtained. Again, a naïve (normal ordered) extension of the corresponding semiclassical expression is not appropriate and a nontrivial quantum modification would have to be performed.

This problem is presumably related, although not equivalent in any obvious way, to the problem of identifying the Fermi operator. In ordinary (abelian) one-dimensional bosonization this is given by the exponential of the boundary droplet field. This expression is not directly motivated by any droplet considerations, and does not generalize to the nonabelian case. Similarly, its generalization to higher dimensions is an open issue. This and other questions are the subject of further investigation.

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References

[1] F. Bloch, Z. Phys. 81, 363 (1933); S. Tomonaga, Prog. Theor. Phys. 5, 544 (1950).

[2] W. Thirring, Ann. Phys. (N.Y.) 3, 91 (1958); J. M. Luttinger, J. Math. Phys. 4, 1154 (1963); D. Mattis and E. Lieb, J. Math. Phys. 6, 304 (1965); S. R. Coleman, Phys. Rev. D 11, 2088 (1975); S. Mandelstam, Phys. Rept. 23 (1976) 307.

[3] E. Witten, Commun. Math. Phys. 92, 455 (1984).

[4] J. Polchinski, Nucl. Phys. B 362, 125 (1991); S. Iso, D. Karabali and B. Sakita, Phys. Lett. B 296, 143 (1992) [arXiv:hep-th/9209003]. B. Sakita, Phys. Lett. B 387, 118 (1996); D. Karabali and V. P. Nair, Nucl. Phys. B 641, 533 (2002); 679, 427 (2004); 697, 513 (2004).

[5] A.P. Polychronakos, “Bosonization in higher dimensions via noncommutative field theory”, Phys. Rev. Lett., in press [arXiv:hep-th/0502150]

[6] A. Luther, Phys. Rev. B 19, 320 (1979); F. D. M. Haldane, Varenna 1992 Lectures and cond-mat/0505529 A. Houghton and J. B. Marston, Phys. Rev. B 48, 7790 (1993); A. H. Castro Neto and E. Fradkin, Phys. Rev. Lett. 72, 1393 (1994); D. Schmeltzer and A.R. Bishop, Phys. Rev. B 50, 12733 (1994); D. V. Khveshchenko, Phys. Rev. B 52, 4833 (1995); D. Schmeltzer, Phys. Rev. 54, 10269 (1996).

[7] A. Jevicki and B. Sakita, Nucl. Phys. B 165, 511 (1980); D. Karabali and B. Sakita, Int. J. Mod. Phys. A 6, 5079 (1991); S. R. Das, A. Dhar, G. Mandal and S. R. Wadia, Mod. Phys. Lett. A 7, 71 (1992) [arXiv:hep-th/9111021]. A. Houghton and J. B. Marston, Phys. Rev. B 48, 7790 (1993); A. H. Castro Neto and E. Fradkin, Phys. Rev. Lett. 72, 1393 (1994); D. Schmeltzer and A.R. Bishop, Phys. Rev. B 50, 12733 (1994); D. V. Khveshchenko, Phys. Rev. B 52, 4833 (1995); D. Schmeltzer, Phys. Rev. 54, 10269 (1996).

[8] A. P. Polychronakos, Nucl. Phys. B 705, 457 (2005) [arXiv:hep-th/0408194]. Nucl. Phys. B 711, 505 (2005) [arXiv:hep-th/0411065].

[9] D. B. Fairlie and C. K. Zachos, Phys. Lett. B 224, 101 (1989).

[10] I. Bakas, Phys. Lett. B 228, 57 (1989); A. Cappelli, C. A. Trugenberger and G. R. Zemba, Nucl. Phys. B 396, 465 (1993) [arXiv:hep-th/9206027].

[11] S. M. Girvin, A. H. MacDonald and P. M. Platzman, Phys. Rev. B33, 2481 (1986).

[12] D. Karabali, private communication.