Forced convection heat transfer of power law non-Newtonian fluids between two semi-infinite plates with variable thermal conductivity

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Abstract. This paper presents an investigation of forced convection heat transfer in power-law non-Newtonian fluids between two semi-infinite plates with variable thermal conductivity. Three cases of different thermal conductivity models are considered: (i) thermal conductivity is a constant, (ii) thermal conductivity is a linear function of temperature, (iii) thermal conductivity is a power-law function of temperature gradient (Zheng’s model). Governing equations are solved using the finite element method with the ‘ghost’ time introduced to the control equations, which does not affect the results because the velocity and temperature will remain unchanged when the steady state is reached. Results for the solutions of different variable models are presented as well as the analysis of the associated heat transfer characteristics. It is shown that the heat transfer behaviours are strongly dependent on the power-law index \( n \) in all models. For example, when \( n < 1 \), the temperature in model (iii) is higher than that in model (i) and (ii), while the situation is reversed when \( n > 1 \).

Keywords. Power-law fluid, variable thermal conductivity, heat transfer

1. Introduction

In recent years, considerable attention has been devoted to the issue of predicting the behaviours of non-Newtonian fluids. Fluids (such as molten plastics, pulps, slurries, emulsions, etc.), which do not obey the assumption of classical Newtonian fluids that the stress tensor is directly proportional to the deformation tensor, are exploited in various engineering applications. Many constitutive equations have been proposed to describe the flow and heat transfer mechanisms of non-Newtonian fluids among which the Ostwald-de Waele model \[1\] (the power-law model) has been being the most successful one so far \[2\].

Difficulties of understanding the rheological behaviours of non-Newtonian fluids impose great challenge for the investigation on non-Newtonian transport with varying thermal conductivity. Some of the significant works are introduced briefly below. In 1966, Kays \[3\] firstly reported that for a
liquid, the thermal conductivity $k$ varied with temperature in an approximately linear manner in the range from 0 to 204 °C. Following Kays’s work, a semi-empirical formula of the thermal conductivity was introduced by Arunachalam et al. [4]. Later, Chaim [5] studied the effects of thermal conductivity, which was treated as a linear function of temperature over a linearly stretching sheet. Hossain et al. [6] observed the natural convection behaviour past a truncated cone with the fluid having a linear temperature-dependent thermal conductivity. They also investigated the free convection boundary layer over a vertical wavy cone maintained at uniform surface temperature [7]. With a variable linear thermal conductivity, flow and heat transfer of an electrically conducting viscoelastic fluid over a continuously stretching sheet was analysed by Salem [8] in the presence of a uniform magnetic field. Moreover, Abel et al. [9] considered the flow and heat transfer of a power-law fluid past a vertical stretching sheet when effects of variable thermal conductivity and non-uniform heat source/sink were addressed. Recently, Zheng et al. [10] performed a mathematical model by taking the effects of power-law kinematic viscosity into account. Based on Fourier’s law and the analogy between velocity boundary layer and thermal one, Zheng [10] defined the thermal conductivity as a function of temperature gradient. Based on this thermal conductivity model, Li et al. [11] conducted a numerical investigation on steady convection of heat and diffusion of power-law fluids in a circular duct, considering the effects of power-law viscosity on the temperature field.

Motivated by the works mentioned above, we herein present a research on steady heat transfer of power-law fluids by using varying specific heat capacity and thermal conductivity and results are compared to each other. Three different physical property models: (i) thermal conductivity is a constant (traditional model); (ii) thermal conductivity is a linear function of temperature [3-9]; (iii) thermal conductivity is a function of temperature gradient (Zheng’s model [10]).

2. Model description and mathematical formulation

The forced convection between two semi-infinite plates is studied under the assumptions of laminar flow and incompressible fluids. The internal fluid is considered as a non-Newtonian liquid, i.e. power-law fluid. The coupled partial differential equations for continuity, momentum and energy are:

\[ \nabla \cdot \vec{u} = 0 \]
\[ \rho ((\vec{u} \cdot \nabla) \vec{u}) = -\nabla p + \nabla \cdot \tau \]
\[ \rho c_p ((\vec{u} \cdot \nabla) T) = \nabla \cdot (k \nabla T) \]

In above equations, $\vec{u}$ is the velocity; $\rho$ is the density; $p$ is the pressure; $c_p$ is the fluid specific heat; and $k$ is the fluid thermal conductivity, which follows three different forms below.

\[ k = \lambda k_0 = \lambda = \text{constant} \]
\( k = \lambda k_0 = \lambda \left( 1 + \chi(T - T_w) \right) \) \hspace{1cm} (5)

\( k = \lambda k_0 = \lambda \left( \frac{\partial \bar{T}}{\partial \bar{x}} \right)^2 + \left( \frac{\partial \bar{T}}{\partial \bar{y}} \right)^2 \frac{(n+1)}{2} \) \hspace{1cm} (6)

\( \chi \) is the thermal conductivity parameter. The density of fluid is assumed constant. The power-law rheological model of Ostwald-de Waele [12] is used, where \( n = 1 \) represents Newtonian fluid (thermal conductivity is a constant), \( n < 1 \) describes pseudoplastic fluid and \( n > 1 \) depicts dilatant fluid. The constitutive relations for \( \bar{\tau}_u \) are given as (\( \mu \) is the kinematic viscosity):

\[
\bar{\tau}_u = \mu \left( 2 \left( \frac{\partial \bar{u}}{\partial \bar{x}} \right)^2 + 2 \left( \frac{\partial \bar{v}}{\partial \bar{y}} \right)^2 + \left( \frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right)^2 \right)^{\frac{n-1}{2}} \left( \nabla \bar{u} + \nabla \bar{v}^T \right)
\] \hspace{1cm} (7)

To nondimensionalize the governing equations, \( \bar{x} \) and \( \bar{y} \) are scaled by \( H_m (1 \text{ m/s}) \); \( \bar{u} \) and \( \bar{v} \) by \( U_0 (\text{m/s}) \); \( \bar{p} \) by \( \mu U_0^n / H^n \); and \( \bar{T} \) by \( T_r \) (room temperature). For the sake of simplicity, the same symbols will be used for all dimensionless variables, only without dash above.

The fluid flow boundary conditions include a uniform horizontal velocity at the inlet and no-slip at walls (\( \bar{u} = \bar{v} = 0 \)). The thermal boundary conditions consist of uniform hot inlet temperature and cold temperatures at walls.

The inlet conditions:

\[
\bar{u} = 1, \hspace{0.5cm} \bar{v} = 0, \hspace{0.5cm} T = 1
\] \hspace{1cm} (8)

Wall conditions:

\[
\bar{u} = \bar{v} = 0, \hspace{0.5cm} T = 0
\] \hspace{1cm} (9)

3. Numerical scheme

These governing equations, with boundary conditions, are solved numerically using the finite element method by the same means in [13]. A uniform triangular mesh on a 100×40 grid is performed calculating the domain which is 5×2 (\( x \times y \)) in dimensionless form. Computations are accomplished in an Intel Core 2 Duo personal computer with 2.0 GB of RAM.

The domain of our interest is defined as \( \Omega \subset R^2 \). The boundary of \( \Omega \) is \( \Gamma \) which is sufficiently smooth. Consider \( \bar{u} \) in space \( Z \), which is defined as \( H^1(\Omega) \) if \( n \leq 1 \) or \( W^{1,n+1}(\Omega) \) if \( n > 1 \); \( T \) in \( W \), which is defined as \( W^{1,4}(\Omega) \) if \( n \leq 3 \) or \( W^{1,n+1}(\Omega) \) if \( n > 3 \), and \( p \) in \( L^2(\Omega) \) (see Ref. [13]). A weak formulation of the equations (1)-(4) is to find \( \bar{u}, T \) and \( p \) such that

\[
\int_\Omega ((\nabla \cdot \bar{u}) q) d\bar{x} = 0, \forall q \in L^2(\Omega)
\] \hspace{1cm} (10)

\[
\int_\Omega (\text{Re}(\bar{u} \cdot \nabla \bar{v}) - p(\nabla \cdot \bar{v}) + \tau_u \cdot \nabla \bar{v}) d\bar{x} = 0, \forall \bar{v} = (v_1, v_2) \in Z
\] \hspace{1cm} (11)

\[
\int_\Omega (\text{Re} \cdot \text{Pr}(\bar{u} \cdot \nabla T) v_j) + k_0 \nabla T \cdot \nabla v_j) d\bar{x} = 0, \forall v_j \in W
\] \hspace{1cm} (12)

In the weak form of this power-law fluid system, \( q, \bar{v}, v_j \) are the so-called trial functions. Besides, \( \text{Re} = \rho H^n U_0^{3-n} / \mu \), \( \text{Pr} = c_p \mu H^{1-n} / 2 \mu U_0^{1-n} \). For all calculated cases, \( \text{Re} = 10 \), \( \text{Pr} = 2 \), unless otherwise specified.

In Eqs. (11-12), \( \tau_u \) and \( k \) are super-nonlinear. A ‘ghost’ time is introduced into the equations as an iterative method. Thus, Eqs. (11-12) becomes:
\[
\int_{\Omega} \left( \frac{\mathbf{u}^i - \mathbf{u}^{i+1}}{dt} \cdot \mathbf{v} + \text{Re}((\mathbf{u}^i \cdot \nabla \mathbf{u}^{i+1}) \cdot \mathbf{v}) - p(\nabla \cdot \mathbf{v}) + \tau_u^{i+1} \nabla \mathbf{v} \right) dx = 0 \tag{13}
\]

\[
\int_{\Omega} \left( \frac{T^i - T^{i+1}}{dt} \cdot \mathbf{v} + \text{Re}((\mathbf{u}^i \cdot \nabla T^{i+1}) \mathbf{v}) + k_0^{i-1} \nabla T^{i+1} \cdot \nabla \mathbf{v} \right) dx = 0, \forall \mathbf{v} \in W \tag{14}
\]

where

\[
\tau_u^{i+1} = \mu \left( 2 \left( \frac{\partial \mathbf{u}^i}{\partial x} \right)^2 + 2 \left( \frac{\partial \mathbf{v}^i}{\partial y} \right)^2 + \left( \frac{\partial \mathbf{u}^i}{\partial y} + \frac{\partial \mathbf{v}^i}{\partial x} \right)^2 \right)^{\frac{(n-1)}{2}} (\nabla \mathbf{u}^i + \nabla \mathbf{u}^{T,i-i}) \tag{15}
\]

Let \( dt > 0 \) represents a time step size. \( \mathbf{u}^i = u(i \cdot dt, x, y) \) and \( T^i = T(i \cdot dt, x, y) \) are obtained at time \( i \cdot dt \). Note that we obtain thermal conductivity by using the temperature obtained in time \( i - 1 \). It does not have any effect to the equations above although we add the 'ghost' time to them because the velocity and temperature will not change any more when the steady state is reached. Grid independence and validation of the code can be found in [13].

4. Results and discussion

Figures 2-3 reveal the effects of \( \chi \) on the temperature profiles \( T \) with the model \( k = \lambda[1 + \chi (T - T_w)] \). Thermal conductivities of some fluids become bigger (\( \chi > 0 \)) while some get smaller (\( \chi < 0 \)) as temperature raises. Form these figures, we can figure out obvious differences away from the plates with varying \( \chi \) in both pseudoplastic and dilatant fluids.

![Figure 2. Temperature profiles of non-Newtonian power-law fluid with \( k = \lambda[1 + \chi (T - T_w)] \) when \( n = 1.2, x = 1.0 \)](image-url)
Figure 3. Temperature profiles of non-Newtonian power-law fluid with \( k = \lambda [1 + \chi (T - T_w) ] \) when \( n = 0.8, x = 1.0 \)

Figure 4-5 show the dimensionless temperature profiles with the mixed varying thermal conductivity model. It is also found that the solutions are strictly affected by the power-law index.

Figure 4. Effects of power-law index on temperature profiles of non-Newtonian power-law fluid with \( k = \lambda [1 + \chi (T - T_w) ] \) when \( \chi = -0.5, x = 1.0 \)
Figure 5. Effects of power-law index on temperature profiles of non-Newtonian power-law fluid with 
\[ k = \lambda [1 + \chi (T - T_w)] \] when \( \chi = 0.5, x = 1.0 \)

Figure 6-7 compare the different solutions obtained by using different thermal conductivity models. The curves of Model (i) are classical results with constant thermal conductivity (ref. Eq. (4)); also shown are the profiles of Model (ii), with a thermal conductivity assumed to be a linear function of temperature (ref. Eq.(5)) and the results of Model (iii), with a power-law temperature-dependent thermal conductivity (ref. Eq.(6)), which is proposed by Zheng [10-11]. We divide the three models into two groups—first group: Model (i) and Model (ii), second group: Model (iii). It is noticed that when \( n < 1 \), the temperatures in the second group is higher than that in the first group, and when \( n < 1 \), the situation is reversed.

Figure 6. Comparison of temperature profiles between different thermal conductivity models when \( n = 0.8, x = 1.0 \)

Figure 7. Comparison of temperature profiles between different thermal conductivity models when \( n = 1.2, x = 1.0 \)
Figure 8. Comparison of local Nusselt number ratio between different thermal conductivity models when $x = 1.0$

Local Nusselt number characterizes the thermal performance of different models. However, in Fig.8, the local Nusselt number ratio $Nu/Nu_0$ is obtained instead of the local Nusselt number because it is more natural and clear in the comparison. Note that $Nu_0$ refers to the value of traditional model when $n = 0.8$. In Fig.8, the local Nusselt number ratio is decreasing as $n$ increases for all three models. Furthermore, the local Nusselt number ratio of Zheng’s model is much higher than those of the other two models whose Nusselt number ratios are very close.

5. Conclusions

This paper presents a numerical study on the problem of steady heat transfer of power-law fluids between two semi-infinite plates with varying thermal conductivity. Three different thermal conductivity models (as a constant, as a linear function of temperature, or as a function of temperature gradient) are used and results are compared with each other. Some of the considerable findings of the paper are:

(i) The increasing values of thermal conductivity parameter $\chi$ lead to decreasing temperature in the linear model used in this paper.

(ii) The heat transfer behaviours strongly depend on the value of the power-law index in all cases considered here.

(iii) The differences of temperature obtained with varying thermal conductivity models are obvious. When $n < 1$, the temperatures obtained by Zheng’s model are higher than the constant model and the linear temperature-dependent model; opposite situation emerges when $n > 1$.

(iv) The local Nusselt number ratio decreases with $n$ increasing for all three models, and the local Nusselt number ratio of Zheng’s model is significantly higher than those the other two models, while the traditional model has the lowest local Nusselt number ratio.
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