COMPOSITION OPERATORS ON HARDY-SOBOLEV SPACES WITH BOUNDED REPRODUCING KERNELS

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ABSTRACT. For any real $\beta$ let $H^2_\beta$ be the Hardy-Sobolev space on the unit disc $\mathbb{D}$. $H^2_\beta$ is a reproducing kernel Hilbert space and its reproducing kernel is bounded when $\beta > 1/2$. In this paper, we characterize that for a non-constant analytic function $\varphi : \mathbb{D} \to \mathbb{D}$, when the composition operator $C_\varphi$ on $H^2_\beta$ is Fredholm. For $1/2 < \beta < 1$, we also prove that $C_\varphi$ has dense range in $H^2_\beta$ if and only if the polynomials are dense in a certain Dirichlet space of the domain $\varphi(\mathbb{D})$. It follows that if the range of $C_\varphi$ is dense in $H^2_\beta$, then $\varphi$ is a weak-star generator of $H^\infty$, although the conclusion is false for the classical Dirichlet space $\mathbb{D}$. Moreover, we study the relation between the density of the range of $C_\varphi$ and the cyclic vector of the multiplier $M_\varphi^\beta$.

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. For $f \in H(\mathbb{D})$ we use

$$\mathcal{R}f(z) = z \frac{\partial f}{\partial z}(z)$$

to denote the radial derivative of $f$ at $z$. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is the Taylor expansion of $f$, it is easy to see that

$$\mathcal{R}f(z) = \sum_{k=1}^{\infty} k a_k z^k.$$
More generally, for any real number $\beta$ and any $f \in H(\mathbb{D})$ with the Taylor expansion above, we define

$$R^\beta f(z) = \sum_{k=1}^{\infty} k^\beta a_k z^k$$

and call it the radial derivative of $f$ of order $\beta$.

It is clear that these fractional radial differential operators satisfy $R^\alpha R^\beta = R^{\alpha+\beta}$. When $\beta < 0$, the effect of $R^\beta$ on $f$ is actually “integration” instead of “differentiation”. For example, radial differentiation of order $-3$ is actually radial integration of order 3.

For $\beta \in \mathbb{R}$, the Hardy-Sobolev space $H^2_\beta$ consists of all analytic functions $f$ on $\mathbb{D}$ such that $R^\beta f$ belongs to the classical Hardy space $H^2$. It is clear that $H^2_\beta$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{\beta} = f(0)g(0) + \langle R^\beta f, R^\beta g \rangle_{H^2}.$$  

The induced norm in $H^2_\beta$ is then given by

$$\|f\|_{\beta}^2 = |f(0)|^2 + \|R^\beta f\|_{H^2}^2.$$

Recall that $H^2$ is the space of analytic functions $f$ on $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 \, d\sigma(\zeta) < \infty,$$

where $d\sigma$ is the normalized Lebesgue measure on the unit circle $\mathbb{T} = \partial \mathbb{D}$. It is well known that every function $f \in H^2$ has radial limits

$$f(\zeta) = \lim_{r \to 1^-} f(r\zeta)$$

for almost all $\zeta \in \mathbb{T}$. Moreover, the radial limit function $f(\zeta)$ above belongs to $L^2(\mathbb{T}, d\sigma)$. The inner product in $H^2$ can then be written as

$$\langle f, g \rangle_0 = \langle f, g \rangle_{H^2} = \int_{\mathbb{T}} f(\zeta) \overline{g(\zeta)} \, d\sigma(\zeta),$$

and its induced norm on $H^2$ is given by

$$\|f\|_0^2 = \|f\|_{H^2}^2 = \int_{\mathbb{T}} |f(\zeta)|^2 \, d\sigma(\zeta).$$

It is well known that a function $f \in H(\mathbb{D})$ belongs to $H^2$ if and only if

$$\int_{\mathbb{D}} |R f(z)|^2 (1 - |z|^2) \, dA(z) < \infty,$$

where $dA$ is the normalized area measure on $\mathbb{D}$. See [24, 28, 29]. More generally, for any $t > -1$, we consider the weighted area measure

$$dA_t(z) = (t + 1)(1 - |z|^2)^t \, dA(z),$$
which is a probability measure on $\mathbb{D}$. The spaces

$$A^2_t = L^2(\mathbb{D}, dA_t) \cap H(\mathbb{D})$$

are called weighted Bergman spaces (with standard weights). When $t = 0$, we simply write $A^2$ for the ordinary Bergman spaces. The following result establishes a natural connection between Hardy-Sobolev spaces and weighted Bergman spaces via fractional derivatives.

**Proposition 1** ([7]). Suppose $\beta \in \mathbb{R}$ and $f \in H(\mathbb{D})$. Then the following conditions are equivalent.

(a) $f \in H^2_{\beta}$.

(b) $\mathcal{R}^{\beta+1}f \in A^2_1$.

If $N$ is a nonnegative integer with $N > \beta$, then the conditions above are also equivalent to

(c) $\mathcal{R}^N f \in A^2_{2(N-\beta)-1}$.

Hardy-Sobolev spaces contain many classical analytic function spaces as special cases. For example, $H^{2-1/2}$ is the Bergman space $A^2$, $H^2_0$ is the Hardy space $H^2$, and $H^2_{1/2}$ is the Dirichlet space $\mathcal{D}$ consisting of analytic functions $f$ on $\mathbb{D}$ such that

$$\|f\|^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$ 

More generally, for any domain $G \subset \mathbb{C}$ and any positive measure $d\omega$ on $G$, we will use $A^2(G, d\omega)$ to denote the weighted Bergman space of analytic functions $f$ on $G$ such that

$$\int_{G} |f(z)|^2 d\omega(z) < \infty.$$ 

Similarly, we use $\mathcal{D}(G, d\omega)$ for the weighted Dirichlet space of analytic functions $f$ on $G$ with

$$\int_{G} |f'(z)|^2 d\omega(z) < \infty.$$ 

When $d\omega$ is ordinary area measure, we will simply write $A^2(G)$ and $\mathcal{D}(G)$.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map $\mathbb{D}$. For any Hilbert space $H$ of analytic functions on $\mathbb{D}$ we consider the composition operator $C_\varphi : H \to H$ defined by $C_\varphi f = f \circ \varphi$. For $\beta < 1/2$, every composition operator is bounded on $H^2_\beta$. However, this is not so for $\beta \geq 1/2$. For example, not every composition operator is bounded on the Dirichlet space. There are conditions (in terms of Carleson type measures, for example) that tell us exactly when $C_\varphi$ is bounded on $\mathcal{D}$. See [13, 22, 31] for example.
A widely-studied problem is to characterize the Fredholmness of the composition operators on various function spaces. We mention the following result from [9].

**Theorem 2.** Suppose $\varphi : G \to G$ is an analytic self-map on a domain $G \subset \mathbb{C}$ and $H$ is a Hilbert space of analytic functions on $G$. If the reproducing kernel of $H$ has the property that $K(w, w) \to \infty$ as $w$ approaches the boundary of $G$, then the following conditions are equivalent.

(i) $C_{\varphi}$ is a Fredholm operator on $H$.
(ii) $C_{\varphi}$ is an invertible operator on $H$.
(iii) $\varphi \in \text{Aut}(\Omega)$.

This covers several previous results in the literature. For example, Fredholm composition operators on the Hardy space were characterized in [3, 10, 12], Fredholm composition operators on the Bergman space were characterized in [2, 3], and Fredholm composition operators on the Dirichlet space were characterized in [11]. For the study of Fredholm composition operators on other spaces of analytic functions on domains in $\mathbb{C}$ and $\mathbb{C}^n$, see [3, 12, 13, 15, 17, 18, 20, 32].

Although the assumption $K(w, w) \to \infty$ as $w \to \partial \Omega$ is a mild and natural condition, we suspect that the result above may still be true without this assumption. It is easy to construct Hilbert spaces of analytic functions whose reproducing kernel does not satisfy $K(w, w) \to \infty$ as $w \to \partial \Omega$. For example, if $\beta > 1/2$, then $H^2_\beta$ has a bounded reproducing kernel $K(z, w)$, so the proof in [9] fails to work in this case. In section 2, we will give a different proof to show that the result above still holds when $\beta > 1/2$.

The density of the range of a composition operator is another interesting problem. Bourdon and Roan studied the problem for the Hardy space (see [3, 23]) and Cima raised the problem for the Dirichlet space in [11]. In [8], we settled Cima's problem completely:

**Theorem 3.** Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, non-constant, and $G = \varphi(\mathbb{D})$. Then the following two conditions are equivalent.

(i) $C_{\varphi} : \mathbb{D} \to \mathbb{D}$ is bounded and has dense range.
(ii) $\varphi$ is univalent and the polynomials are dense in $A^2(G)$.

In [3], Bourdon proved the following result.

**Theorem 4.** If $G = \varphi(\mathbb{D})$, where $\varphi$ is a weak-star generator of $H^\infty$, then the polynomials are dense in $A^2(G)$.

It is thus natural for us to consider the following problem.

**Question 5.** Does the density of polynomials in $A^2(G)$ imply that $\varphi$ is a weak-star generator of $H^\infty$?
In general, the answer is no. In fact, Sarason gave a condition in [25] for \( \phi \) to be a weak-star generator of \( H^\infty \), which combined with the Corollary 2 in that paper yields a bounded simply connected domain \( G \) such that the polynomials are dense in \( A^2(G) \) but any Riemann map \( \varphi : \mathbb{D} \to G \) is not a weak-star generator of \( H^\infty \); see [3] and [19].

In section 3, we will give a necessary and sufficient condition for composition operators to have dense range on Hardy-Sobolev spaces. Our result shows that if \( \varphi \) is a univalent self-map of \( \mathbb{D} \), then the density of polynomials in the weighted Dirichlet spaces

\[
\mathcal{D}(\varphi(\mathbb{D})), (1 - |\varphi^{-1}|^2)^{1-2\beta} dA, \quad \frac{1}{2} < \beta < 1,
\]

implies that \( \varphi \) is a weak-star generator of \( H^\infty \).

The density of the range of the composition operator \( C_\varphi \) is relative to the cyclic vectors of the multiplier \( M^{\beta}_\varphi \) with symbol \( \varphi \) defined as \( M^{\beta}_\varphi f = \varphi f \) for any \( f \in H^2_\beta \). In the last part of this paper, we discuss the relations between the density of \( C_\varphi \) on \( H^2_\beta \) and the cyclic vectors of \( M^{\beta}_\varphi \) for \( \frac{1}{2} < \beta < 1 \).

2. Fredholm Composition Operators

We know that for any \( \beta \in \mathbb{R} \), \( C_\varphi \) is bounded on \( H^2_\beta \) if \( \varphi \in Aut(\mathbb{D}) \) (see [11]). For \( \beta > 1/2 \) we have \( H^2_\beta = H^\infty_\beta \subset A(\mathbb{D}) \), where \( H^\infty_\beta \) is the multiplier algebra of \( H^2_\beta \) (see [21]), and \( A(\mathbb{D}) \) is the disc algebra. Since the reproducing kernel of \( H^2_\beta \) is bounded in this case, the proof in [9] fails. However, we still have the following result.

**Theorem 6.** Suppose \( \beta > 1/2 \) and \( \varphi : \mathbb{D} \to \mathbb{D} \) is analytic. Then \( C_\varphi \) is Fredholm on \( H^2_\beta \) if and only if \( \varphi \in Aut(\mathbb{D}) \), the automorphism group of \( \mathbb{D} \).

**Proof.** Obviously, we only need to prove that the Fredholmness of \( C_\varphi \) on \( H^2_\beta \) implies that \( \varphi \in Aut(\mathbb{D}) \). If \( C_\varphi \) is Fredholm, then \( \varphi \) must be univalent on \( \mathbb{D} \) (see the proof of Theorem 4 in [6]). Note \( \varphi \in H^2_\beta \) implies \( \varphi \in A(\mathbb{D}) \), for \( \beta > 1/2 \). Thus, \( \varphi \) is univalent except for a limited number of points on \( \partial \mathbb{D} \). Otherwise, we can find two sequences \( \{z_n\} \) and \( \{w_n\} \) in \( \partial \mathbb{D} \) such that \( \varphi(z_n) = \varphi(w_n) \). For \( \beta > 1/2 \) the kernel function \( K_z(w) \) is also well-defined for \( z \in \partial \mathbb{D} \) and we can easily prove that \( C_\varphi^{*} K_z = K_{\varphi(z)} \). Thus

\[
C_\varphi^{*}(K_{z_n} - K_{w_n}) = K_{\varphi(z_n)} - K_{\varphi(w_n)} = 0.
\]

However, the functions \( K_{z_n} - K_{w_n} \) are clearly linearly independent, which contradicts to the Fredholmness of \( C_\varphi \). This shows that \( \varphi \) must be univalent except for a limited number of points on \( \partial \mathbb{D} \).

Now \( \varphi(\mathbb{D}) \) is a simply connected domain. If \( \varphi \) is not a surjection, then there are two cases for us to consider.
The first case is when $\partial \mathbb{D} \setminus \partial \varphi(\mathbb{D}) \neq \emptyset$. Then for any $\zeta \in \partial \mathbb{D} \setminus \partial \varphi(\mathbb{D})$ there is a neighborhood $U(\zeta, \delta) = \{z||z - \zeta| < \delta\}$ of $\zeta$ such that

$$U(\zeta, \delta) \cap \varphi(\mathbb{D}) = \emptyset.$$ 

Let $f_\zeta(z) = (1 + z\bar{\zeta})/2$ be the peak function at $\zeta$ and consider $f_n(\zeta) = f_n^\zeta(z)$. According to [7], the sequence $\{f_n\}$ converges to 0 weakly in $H^2_\beta$. Since $C_{\varphi}f_n = f_n(\varphi(z)) = \frac{(1 + \varphi(z)\bar{\zeta})^n}{\|f_n^\zeta\|_\beta}$, it is not difficult to check that $\|C_{\varphi}f_n\|_\beta \to 0$, which implies that $C_{\varphi}$ cannot be Fredholm, a contradiction.

The second case is when $\partial \mathbb{D} \subsetneq \partial \varphi(\mathbb{D})$. Since $\varphi(\mathbb{D})$ is simply connected, we see that $\partial \varphi(\mathbb{D}) \setminus \partial \mathbb{D}$ contains an arc $L$ from a boundary point of $\mathbb{D}$ into the inner of $\mathbb{D}$. As $\partial \varphi(\mathbb{D}) = \varphi(\partial \mathbb{D})$ is a closed curve, $L$ must be an overlapping arc of two arcs with opposite directions. Since $\varphi$ is univalent except for a limited number of points on $\partial \mathbb{D}$, we see that the pre-image of $L$ under $\varphi$ has two non-intersecting arcs in $\partial \mathbb{D}$. Let $l_1$ and $l_2$ be two arcs such that $\varphi(l_1) = \varphi(l_2) = L$ as a subset of $\partial \mathbb{D}$ but $\varphi(l_1)$ and $\varphi(l_2)$ have opposite directions in $\partial \varphi(\mathbb{D})$. Choose sequences $\{z_n\} \subset l_1$ and $\{w_n\} \subset l_2$ such that $\varphi(z_n) = \varphi(w_n)$. Then

$$C_{\varphi}^\ast(K_{z_n} - K_{w_n}) = K_{\varphi(z_n)} - K_{\varphi(w_n)} = 0.$$ 

However, the functions $K_{z_n} - K_{w_n}$ are clearly linear independent, which contradicts to the Fredholmness of $C_{\varphi}$ again.

Therefore, $\varphi$ must be a surjection, and hence an automorphism of the unit disc. 

\[\Box\]

3. Weak-star generators and composition operators

In [19], S. N. Mergeljan and A. P. Talmadjan showed that if sufficiently many slits are put in the unit disc then we can obtain a domain $G$ such that the polynomials are dense in $A^2(G)$. By the Riemann mapping theorem, there is an analytic homeomorphism $\varphi : \mathbb{D} \to G$, so $C_{\varphi}$ has dense range in $\mathbb{D}$ by Theorem [3] but $\varphi$ is not a weak-star generator of $H^\infty$ by Corollary 2 of [25]. However, the boundary of the above domain is not a Jordan curve, the Riemann map may not be continuous up to the boundary, and $\varphi$ does not belong to the disc algebra $A(\mathbb{D})$. Furthermore, $\varphi \notin \mathcal{D}_1 - 2\beta$ for $1/2 < \beta < 1$, where

$$\mathcal{D}_1 - 2\beta = \{f \in H(\mathbb{D})|f' \in A^2_{1-2\beta}\}$$
is the weighted Dirichlet space with the norm
\[ \|f\|_{D_{1-\beta}} = \left[ |f(0)|^2 + \int_D |f'(z)|^2 (1 - |z|^2)^{-2\beta} dA \right]^{\frac{1}{2}}. \]
Thus, for \( \beta < 1 \), proposition 1 shows that \( f \in H^2_\beta \) if and only if \( Rf \in A^2_{1-\beta} \) and hence \( H^2_\beta = D_{1-\beta} \), see [7] for more details.

The following result is due to P. Bourdon.

**Proposition 7** (Corollary 3.7 in [3]). Let \( \varphi \) map \( \mathbb{D} \) univalently onto \( G \subset \mathbb{D} \). If the polynomials are dense in \( A^2(G, (1 - |\varphi^{-1}|^2) dA) \), then \( C_\varphi : H^2 \to H^2 \) has dense range.

Proposition 7 extends a result of Roan [23] and supplies additional examples of composition operators with dense range. As a special case of our next result, we see that the density of polynomials in \( A^2(G, (1 - |\varphi^{-1}|^2) dA) \) is also a necessary condition for the density of the range of \( C_\varphi \) in \( H^2_\beta \), that is, the converse of Bourdon’s result above is also true.

We will use the notion \( R(C_\varphi) \) to denote the range of a composition operator. The space on which \( C_\varphi \) acts is usually obvious from the context, or it will be specified whenever there is a possibility for confusion.

**Theorem 8.** Suppose \( \beta < 1 \) and \( \varphi \) is a non-constant analytic self-map of \( \mathbb{D} \). Then \( C_\varphi \) has dense range in \( H^2_\beta = D_{1-\beta} \) if and only if \( \varphi \) is univalent and the polynomials are dense in \( D(G, (1 - |\varphi^{-1}|^2)^{1-2\beta} dA) \), where \( G = \varphi(\mathbb{D}) \).

**Proof.** First assume that \( C_\varphi \) has dense range in \( D_{1-\beta} \). It is easy to see that \( \varphi \) must be univalent. In fact, if there are \( z_1, z_2 \in \mathbb{D}, z_1 \neq z_2 \), such that \( \varphi(z_1) = \varphi(z_2) \), then for any \( f \in D_{1-\beta} \) we have \( C_\varphi f(z_1) = C_\varphi f(z_2) \), which clearly contradicts the assumption that the range of \( C_\varphi \) is dense in \( D_{1-\beta} \). To prove that the polynomials are dense in \( D(G, (1 - |\varphi^{-1}|^2)^{1-2\beta} dA) \), fix any \( g_0 \in D(G, (1 - |\varphi^{-1}|^2)^{1-2\beta} dA) \). Since \( C_\varphi g_0 \in D_{1-\beta} \) and \( C_\varphi \) has dense range in \( D_{1-\beta} \), we can find a sequence \( \{p_k\} \) of polynomials such that \( \| C_\varphi p_k - C_\varphi g_0 \|_{D_{1-\beta}} \to 0 \) in \( D_{1-\beta} \). This, by a change of variables, is equivalent to \( \| p_k - g_0 \| \to 0 \) in \( D(G, (1 - |\varphi^{-1}|^2)^{1-2\beta} dA) \).

Conversely, assume \( \varphi \) is univalent and the polynomials are dense in the space \( D(G, (1 - |\varphi^{-1}|^2)^{1-2\beta} dA) \). It is clear that \( C_\varphi \) is an invertible operator from \( D(G, (1 - |\varphi^{-1}|^2)^{1-2\beta} dA) \) onto \( D_{1-\beta} \), with the inverse being \( C_{\varphi^{-1}} \). Thus for any \( g \in D_{1-\beta} \) there is an \( f \in D(G, (1 - |\varphi^{-1}|^2)^{1-2\beta} dA) \) such that \( C_\varphi f = g \). Let \( \{p_k\} \) be a sequence of polynomials such that \( p_k \to f \) in \( D(G, (1 - |\varphi^{-1}|^2)^{1-2\beta} dA) \). Then, by a change of variables again, \( \| C_\varphi p_k - g \|_{D_{1-\beta}} = \| C_\varphi p_k - C_\varphi f \|_{D_{1-\beta}} \to 0 \) in \( D_{1-\beta} \). This shows that the range of \( C_\varphi \) is dense in \( D_{1-\beta} \). \( \square \)
However, if the image $\varphi(\mathbb{D})$ has infinite area, even if $\varphi \in A^2(\mathbb{D})$, then the polynomials may not be dense in $A^2(\varphi(\mathbb{D}))$. Here is an example.

Let $f(z) = 1/\sqrt{z}$ be the principal branch of $1/\sqrt{z}$ on $\mathbb{C} \setminus [0, +\infty)$. Then the function

$$
\varphi(z) = f(1 + z) = \frac{1}{\sqrt{1 + z}}
$$

is analytic function on $\mathbb{D}$. It is obvious that $\varphi$ belongs to $A^2$ and is univalent in the open unit disc, but $\varphi' \notin A^2$, that is, the region $\varphi(\mathbb{D})$ has infinite area. This implies that the polynomials are not dense in $A^2(\varphi(\mathbb{D}))$. In fact, if

$$
g(w) = \varphi^{-1}(w) = \frac{1}{w^3} - 1,
$$

then $g \notin A^2(\varphi(\mathbb{D}))$, but $g' \in A^2(\varphi(\mathbb{D}))$. However, $g'$ cannot be approximated by polynomials in $A^2(\varphi(\mathbb{D}))$.

This example also implies that the Dirichlet space is not necessarily contained in the Bergman space on a general domain in the complex plane. See [8] and additional references there.

**Proposition 9.** Suppose $\beta < 1$ and $\varphi \in D_{1-2\beta}$ is univalent. Then $C_{\varphi}$ is an invertible operator from $D(\varphi(\mathbb{D})), (1 - |\varphi^{-1}|^2)^{1-2\beta}dA)$ onto $D_{1-2\beta}$ with the inverse being $C_{\varphi^{-1}}$. Moreover, $C_{\varphi}$ preserves the Dirichlet semi-norms.

**Proof.** This follows from an easy change of variables. We leave the routine details to the interested reader. $\square$

To further characterize the dense range of $C_{\varphi}$ on $D_{1-2\beta}$ and its relation to weak-star generator of $H^\infty$, we still need the following lemmas.

**Lemma 10** ([25]). A sequence $\{\psi_n\}^\infty_1$ in $H^\infty$ converges weak-star to the function $\psi$ if and only if it is uniformly bounded and converges pointwise to $\psi$ on $\mathbb{D}$.

**Lemma 11** (Mergelyan’s Theorem [26]). If $K$ is a compact subset of the plane whose complement is connected, then every complex function that is continuous on $K$ and analytic on its (topological) interior can be uniformly approximated on $K$ by polynomials.

It follows from Proposition 9 that if $1/2 < \beta < 1$ and $\varphi \in D_{1-2\beta}$ is univalent, then

$$
\varphi^{-1} \in D(\varphi(\mathbb{D})), (1 - |\varphi^{-1}(z)|^2)^{1-2\beta}dA).
$$

A standard argument shows that the operators from Proposition 9 satisfy

$$
C_{\varphi^{-1}}^*K_w = K_{\varphi^{-1}(w)}, \quad C_{\varphi}^*K_z = K_{\varphi(z)},
$$

where $K_w(u) = \tilde{K}(u, w)$ and $K_z(v) = K(v, z)$ are the reproducing kernels of $D(\varphi(\mathbb{D})), (1 - |\varphi^{-1}|^2)^{1-2\beta}dA)$ at $w \in \varphi(\mathbb{D})$ and of $D_{1-2\beta}$ at $z \in$...
Since $K(u,z)$ is continuous on $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$, we know that $K(u,v)$ is also continuous on $\varphi(\mathbb{D}) \times \varphi(\mathbb{D})$. Hence each function $f$ in $\mathcal{D}(\varphi(\mathbb{D})), (1 - |\varphi^{-1}|^2)^{1-2\beta} dA$ is continuous on $\varphi(\mathbb{D})$ by properties of the reproducing kernel. In particular, $\varphi^{-1}$ is continuous on $\varphi(\mathbb{D})$. Furthermore, by Lemma 11, $\varphi^{-1}$ can be uniformly approximated on $\varphi(\mathbb{D})$ by polynomials.

**Proposition 12.** Suppose $1/2 < \beta < 1$ and $\varphi$ is a univalent analytic self-map of $\mathbb{D}$ with $\varphi \in \mathcal{D}_{1-2\beta}$. Then $\text{Lat}(M^\beta_\varphi) = \text{Lat}(M^\beta_z)$, where $M^\beta_\varphi$ and $M^\beta_z$ are multiplication operators on the weighted Bergman space $A^2_{1-2\beta}$, and $\text{Lat}(M^\beta_\varphi)$ and $\text{Lat}(M^\beta_z)$ are their invariant subspace lattices.

**Proof.** Since $\varphi \in \mathcal{D}_{1-2\beta}$, it is clear that $M^\beta_\varphi$ is bounded on $A^2_{1-2\beta}$. Lemma 11 implies that there is a sequence $\{p_k\}$ of polynomials such that $p_k(z) \to \varphi^{-1}(z)$ uniformly on $\mathbb{D}$, and this implies that $p_k(\varphi(z)) \to z$ uniformly on $\mathbb{D}$. Thus

$$\int_{\mathbb{D}} |(p_k(\varphi)(z) - z)g(z)|^2 (1 - |z|^2)^{1-2\beta} dA(z) \to 0, \quad g \in A^2_{1-2\beta}.$$ 

This shows that $M^\beta_{p_k(\varphi)}$ converges to $M^\beta_z$ in the weak operator topology. Hence, $\text{Lat}(M^\beta_\varphi) \subset \text{Lat}(M^\beta_z)$. The reversed inclusion is obvious, so we have $\text{Lat}(M^\beta_\varphi) = \text{Lat}(M^\beta_z)$. \qed

**Corollary 13.** Suppose $1/2 < \beta < 1$ and $\varphi$ is a univalent analytic self-map of $\mathbb{D}$ with $\varphi \in \mathcal{D}_{1-2\beta}$. Then $C_\varphi$ has dense range in $\mathcal{D}_{1-2\beta}$ if and only if $H^\infty(\varphi(\mathbb{D}))$ is dense in $A^2(\varphi(\mathbb{D})), (1 - |\varphi^{-1}|^2)^{1-2\beta} dA$.

**Proof.** This is a direct consequence of Theorem 8, because every bounded analytic function can be approximated by polynomials in the norm topology of $A^2(\varphi(\mathbb{D})), (1 - |\varphi^{-1}|^2)^{1-2\beta} dA$. \qed

**Theorem 14.** Suppose $1/2 < \beta < 1$ and $\varphi$ is an analytic self-map of $\mathbb{D}$ such that $C_\varphi$ is bounded on $\mathcal{D}_{1-2\beta}$. If $R(C_\varphi)$ is dense in $\mathcal{D}_{1-2\beta}$, then $\varphi$ is a weak-star generator of $H^\infty$.

**Proof.** For any $f \in \mathcal{D}_{1-2\beta}$ there is a sequence $\{p_k\}$ of polynomials such that $$\|C_\varphi p_k - f\|_{\mathcal{D}_{1-2\beta}} \to 0.$$ 

Note that

$$|p_k(\varphi)(z) - f(z)| = |\langle p_k(\varphi) - f, K_z \rangle| \leq \|p_k(\varphi) - f\|_{\mathcal{D}_{1-2\beta}} \|K_z\|_{\mathcal{D}_{1-2\beta}},$$

where $K_z$ is the reproducing kernel of $\mathcal{D}_{1-2\beta}$ at $z$. Since $1/2 < \beta < 1$, the function $z \mapsto \|K_z\|_{\mathcal{D}_{1-2\beta}} = \sqrt{|K(z, z)|}$ is bounded on $\mathbb{D}$. Thus $p_k(\varphi)(z)$ converges uniformly to $f(z)$. Furthermore, $\|p_k(\varphi) - f\|_{\infty} \to 0$ as $k \to \infty$. 


If \( f \in H^\infty \), then for any \( 0 < r < 1 \), \( f_r(z) = f(rz) \in \mathcal{D}_{1-2\beta} \). Choose \( r_n \in (0, 1) \) such that \( r_n \to 1 \) as \( n \to \infty \), then \( f_r \xrightarrow{w^*} f \) in \( H^\infty \) by the dominated convergence theorem. For any \( n \), there is a sequence of polynomials \( \{p_k(n)\} \) such that \( \|p_k(n)(\varphi) - f_r\|_\infty \to 0 \) as \( k \to \infty \). Hence, we may find subsequence \( \{k_n\} \) such that \( p_{k_n}(n)(\varphi) \xrightarrow{w^*} f \) in \( H^\infty \). It follows that

\[ \{C_\varphi p_k : p_k \text{ is a polynomial}\} = \{p_k(\varphi) : p_k \text{ is a polynomial}\} \]

is weak-star dense in \( H^\infty \).

It is clear that if \( C_\varphi \) maps \( H^2_\beta \) to itself and \( 1/2 < \beta < 1 \), then \( \varphi \in H^2_\beta \subset A(\mathbb{D}) \). If \( \beta \geq 1 \), then \( z \notin \mathcal{D}_{1-2\beta} \), so the polynomials cannot be dense in \( \mathcal{D}_{1-2\beta} \). In this case, we need to consider higher order derivatives.

From the discussion above, we see that the density of polynomials in \( \mathcal{D}(\varphi(\mathbb{D})), (1-|\varphi^{-1}|^2)^{-2\beta}dA \) for \( 1/2 < \beta < 1 \) implies that \( \varphi \) is a weak-star generator of \( H^\infty \). On the other hand, \( \varphi \) being a weak-star generator of \( H^\infty \) implies that the polynomials are dense in the Dirichlet spaces \( \mathcal{D} \) and \( \mathcal{D}(\varphi(\mathbb{D})), (1-|\varphi^{-1}|^2)^{-2\beta}dA \) for all \( \beta \leq 1/2 \).

It is intriguing for us to find some relationship between the density of \( R(C_\varphi) \) on two different spaces \( \mathcal{D}_{1-2\beta_1} \) and \( \mathcal{D}_{1-2\beta_2} \) for \( 1/2 < \beta_1, \beta_2 < 1 \). We already know that if \( C_\varphi \) has dense range in \( \mathcal{D}_{1-2\beta_1} \) for some \( 1/2 < \beta_1 < 1 \), then \( \varphi \) must be a weak-star generator of \( H^\infty \), which implies that \( \varphi \) is univalent on the closed unit disc \( \overline{\mathbb{D}} \). However, this does not imply that the polynomials are dense in \( \mathcal{D}(\varphi(\mathbb{D})), (1-|\varphi^{-1}|^2)^{-2\beta}dA \) for all \( 1/2 < \beta < 1 \). In fact, for any given \( 1/2 < \beta_1 < \beta_2 < 1 \) we can find an analytic self-map of \( \mathbb{D} \) such that \( \varphi \in \mathcal{D}_{1-2\beta_1} \setminus \mathcal{D}_{1-2\beta_2} \). Then the polynomials are not dense in \( \mathcal{D}(\varphi(\mathbb{D})), (1-|\varphi^{-1}|^2)^{-2\beta_2}dA \) but they are dense in \( \mathcal{D}(\varphi(\mathbb{D})), (1-|\varphi^{-1}|^2)^{-2\beta_1}dA \). Hence, there exists an analytic self-map \( \varphi \) of \( \mathbb{D} \) such that \( C_\varphi \) has dense range in \( \mathcal{D}_{1-2\beta_1} \), but does not have dense range in \( \mathcal{D}_{1-2\beta_2} \). This also shows that \( \varphi \) being a weak-star generator of \( H^\infty \) does not imply that polynomials are dense in \( \mathcal{D}(\varphi(\mathbb{D})), (1-|\varphi^{-1}|^2)^{-2\beta_1}dA \) for all \( 1/2 < \beta < 1 \).

It is well-known that if \( \varphi \) is a weak-star generator of \( H^\infty \), then the polynomials are dense in the Bergman space \( A^2(\varphi(\mathbb{D})) \), but the converse is not true in general. The following theorem gives a condition for the converse to hold for certain analytic self-maps of \( \mathbb{D} \).

**Theorem 15.** Suppose \( 1/2 < \beta < 1 \) and \( \varphi \in \mathcal{D}_{1-2\beta} \) is an analytic map-self of \( \mathbb{D} \) such that the polynomials are dense in \( A^2(\varphi(\mathbb{D})) \). Then the following statements are equivalent to each other.

(i) \( \{C_\varphi p : p \text{ is a polynomial}\} \) is dense in \( A^2_{1-2\beta} \).

(ii) \( \varphi \) is a weak-star generator of \( H^\infty \).

(iii) \( \varphi \) is univalent on the open unit disc.
Proof. If \( \{ C \varphi p : p \text{ is a polynomial} \} \) is dense in \( A^2_{1-2\beta} \), then \( \varphi \) is clearly univalent on \( \mathbb{D} \) by the beginning of the proof of Theorem 8. This shows that (i) implies (iii).

To prove (iii) implies (ii), assume that \( \varphi \in \mathcal{D}_{1-2\beta} \) is univalent on the open unit disc. Then \( \varphi \) is also univalent on the closed unit disc by Corollary 3.5 in [3] and the continuity of \( \varphi \) on \( \mathbb{D} \). Thus, \( \varphi^{-1} \) is continuous on \( \overline{\varphi(\mathbb{D})} \). By Lemma 11, there is a sequence \( \{ p_k \} \) of polynomials such that \( p_k \) converges uniformly to \( \varphi^{-1} \). Then \( p_k \circ \varphi \) converges uniformly to \( f(z) = z \). This implies that \( \varphi \) is a weak-star generator of \( H_\infty \) since \( z \) is the weak-star generator of \( H_\infty \).

Finally, let us assume that (ii) holds. Then for any \( f \in H_\infty \) there exists a sequence \( \{ p_k \} \) of polynomials such that \( p_k(\varphi(z))\varphi(z) \to f(z) \) pointwise on \( \mathbb{D} \) and \( \{ \| p_k \|_\infty \} \) is bounded. By the dominated convergence theorem, we have \( \| C_\varphi p_k - f \|_{A^2_{1-2\beta}} \to 0 \) as \( k \to \infty \). This shows (ii) implies (i) and completes the proof of the theorem. \( \square \)

4. CYCLIC VECTORS AND COMPOSITION OPERATORS

Choosing \( \beta = 0 \) and \( \beta = \pm 1/2 \) in Theorem 8 we see that, for univalent functions \( \varphi : \mathbb{D} \to \mathbb{D} \), \( R(C_\varphi) \) is dense in \( A^2(\mathbb{D}) \) if and only if the polynomials are dense in \( A^2(\varphi(\mathbb{D})), (1 - |\varphi^{-1}|^2) dA \), \( R(C_\varphi) \) is dense in \( H^2(\mathbb{D}) \) if and only if the polynomials are dense in \( A^2(\varphi(\mathbb{D})), (1 - |\varphi^{-1}|^2) dA \) (see [3]), and \( R(C_\varphi) \) is dense in \( \mathcal{D} \) if and only if the polynomials are dense in \( A^2(\varphi(\mathbb{D})) \) (see [8]).

Closely related to these discussions, we mention the following result of Hedberg’s from [27].

Theorem 16. If \( f \) is in the Bergman space \( A^2 \) and if \( f \) is the derivative of a univalent function, then \( f \) is a cyclic vector for \( A^2 \). Equivalently, if \( \varphi \in \mathcal{D} \) is univalent, then \( H_\infty(\varphi(\mathbb{D})) \) is dense in \( A^2(\varphi(\mathbb{D})) \).

The proof of Theorem 16 in [27] is quite technical. If \( \varphi \) is univalent and \( (\varphi^{-1})' \) can be approximated by polynomials on \( \varphi(\mathbb{D}) \), we will give a simpler proof for the density of \( H_\infty(\varphi(\mathbb{D})) \) in \( A^2(\varphi(\mathbb{D})) \). The above condition about \( (\varphi^{-1})' \) seems natural because, as the (normalized) area of \( \mathbb{D} = \varphi^{-1}(\varphi(\mathbb{D})) \), we have

\[
\int_{\varphi(\mathbb{D})} |(\varphi^{-1})'|^2 dA = 1.
\]

Thus \( (\varphi^{-1})' \in A^2(\varphi(\mathbb{D})) \).

Proposition 17. Suppose \( \varphi \) is an analytic self-map of \( \mathbb{D} \) and \( \varphi \in \mathcal{D} \). Then the function \( z \) belongs to \( \overline{R(C_\varphi)} \) in \( \mathcal{D} \) if and only if \( \varphi \) is univalent and \( (\varphi^{-1})' \) can be approximated by polynomials in \( A^2(\varphi(\mathbb{D})) \).
Proof. If \( \varphi \) is univalent and there is a sequence \( \{p_k\} \) of polynomials such that
\[
\int_{\varphi(\mathbb{D})} |(p_k - (\varphi^{-1})')(w)|^2 dA(w) \to 0,
\]
then
\[
\int_{\varphi(\mathbb{D})} |(p_k - (\varphi^{-1})')(w)|^2 dA(w) \\
= \int_{\mathbb{D}} |p_k(\varphi(z)) - (\varphi^{-1})'(\varphi(z))|^2 |\varphi'(z)|^2 dA(z) \\
= \int_{\mathbb{D}} |p_k(\varphi(z))\varphi'(z) - (\varphi^{-1})'(\varphi(z))\varphi'(z)|^2 dA(z) \\
= \int_{\mathbb{D}} |p_k(\varphi(z))\varphi'(z) - 1|^2 dA(z) \to 0
\]
as \( k \to \infty \). Write
\[
q_k(z) = \int_0^z p_k(u) du, \quad k \geq 1.
\]
Then \( q_k \) is also a polynomial for each \( k \) and
\[
(C_{\varphi}q_k)'(z) = \left( \int_0^{\varphi(z)} p_k(u) du \right)' = p_k(\varphi(z))\varphi'(z).
\]
Thus
\[
\int_{\mathbb{D}} |(C_{\varphi}q_k)' - 1|^2 dA(z) \to 0, \quad k \to \infty,
\]
so the function \( z \) belongs to \( \overline{R(C_{\varphi})} \) in \( \mathbb{D} \).

Conversely, if the function \( z \) is in the closure of \( R(C_{\varphi}) \) in \( \mathbb{D} \), then \( \varphi \) is obviously univalent (see the beginning of the proof of Theorem 8), and reversing the calculations above implies that there is a sequence \( \{p_k\} \) of polynomials such that
\[
\int_{\varphi(\mathbb{D})} |(p_k - (\varphi^{-1})')(w)|^2 dA(w) \to 0
\]
as \( k \to \infty \). This ends the proof. \( \square \)

The following result gives a simpler proof for Hedberg’s theorem (i.e. Theorem 16) under an additional assumption.

Proposition 18. Suppose \( \varphi \) is an analytic self-map of \( \mathbb{D} \) and \( \varphi \in \mathbb{D} \). If the function \( z \) belongs to \( \overline{R(C_{\varphi})} \) in \( \mathbb{D} \), then \( H^\infty(\varphi(\mathbb{D})) \) is dense in \( A^2(\varphi(\mathbb{D})) \).
Proof. Assume \( \tilde{f} \in A^2(\varphi(\mathbb{D})) \). Once again, \( z \in \overline{R(C_{\varphi})} \) implies that \( \varphi \) is univalent. Thus there is an \( f \in A^2(\mathbb{D}) \) such that
\[
\tilde{f}(w) = f(\varphi^{-1}(w))(\varphi^{-1})'(w).
\]
Let \( p_k \) be the \( k \)-th partial sum of the Taylor series of \( f \). Then
\[
\|p_k - f\|_{A^2} \to 0, \quad k \to \infty.
\]
By the formula of changing variables,
\[
\|(p_k \circ \varphi^{-1})(\varphi^{-1})' - \tilde{f}\|_{A^2(\varphi(\mathbb{D}))} \to 0, \quad k \to \infty.
\]
Since \( z \in \overline{R(C_{\varphi})} \), it follows from Proposition \([17]\) that there is a sequence \( \{q_n\} \) of polynomials such that \( q_n \) converges to \((\varphi^{-1})'\) in \( A^2(\varphi(\mathbb{D})) \). For any \( \epsilon > 0 \) choose \( K_0 \) such that
\[
\|(p_k \circ \varphi^{-1})(\varphi^{-1})' - q_n\|_{A^2(\varphi(\mathbb{D}))} < \epsilon \quad \text{for} \quad k \geq K_0.
\]
Choose a positive integer \( N \) such that
\[
\|(p_{K_0} \circ \varphi^{-1})(q_n - (\varphi^{-1})')\|_{A^2(\varphi(\mathbb{D}))} < \epsilon \quad \text{for} \quad n \geq N.
\]
Then for \( n \geq N \) we have
\[
\|(p_{K_0} \circ \varphi^{-1})q_n - \tilde{f}\|_{A^2(\varphi(\mathbb{D}))} \\
\leq \|(p_{K_0} \circ \varphi^{-1})q_n - p_{K_0} \circ \varphi^{-1}(\varphi^{-1})'\|_{A^2(\varphi(\mathbb{D}))} \\
+ \|p_{K_0} \circ \varphi^{-1}(\varphi^{-1})' - \tilde{f}\|_{A^2(\varphi(\mathbb{D}))} < \epsilon.
\]
This shows that \( H^\infty(\varphi(\mathbb{D})) \) is dense in \( A^2(\varphi(\mathbb{D})) \). \( \square \)

**Theorem 19.** Suppose \( 1/2 < \beta < 1 \) and \( \varphi \) is a univalent analytic self-map of \( \mathbb{D} \) with \( \varphi \in \mathcal{D}_{1-2\beta} \). If \( C_{\varphi} \) has dense range in \( \mathcal{D}_{1-2\beta} \), then \( \varphi' \) is a cyclic vector for both \( M^2_\beta \) and \( M^3_\beta \) on \( \mathcal{D}_{1-2\beta} \).

**Proof.** Define
\[
E_\varphi : A^2(\varphi(\mathbb{D})), (1 - |\varphi^{-1}|^2)^{1-2\beta}dA) \to A^2_{1-2\beta}
\]
by
\[
E_\varphi(f)(z) = (f \circ \varphi)(z)\varphi'(z).
\]
Similarly, define
\[
E_{\varphi^{-1}} : A^2_{1-2\beta} \to A^2(\varphi(\mathbb{D})), (1 - |\varphi^{-1}|^2)^{1-2\beta}dA)
\]
by
\[
E_{\varphi^{-1}}(f)(w) = (f \circ \varphi^{-1})(w)(\varphi^{-1})'(w).
\]
Direct calculation shows that both \( E_\varphi \) and \( E_{\varphi^{-1}} \) are isometric operators and
\[
E_\varphi E_{\varphi^{-1}} = I_{A^2_{1-2\beta}}, \quad E_{\varphi^{-1}} E_\varphi = I_{A^2(\varphi(\mathbb{D})), (1-|\varphi^{-1}|^2)^{1-2\beta}dA)}.
\]
are identity operators. Thus for any function \( f \in A_{1-2\beta}^2 \) there is a function \( \tilde{f} \in A^2(\varphi(\mathbb{D}), (1 - |\varphi^{-1}|^2)^{1-2\beta}dA) \) such that \( f(z) = \tilde{f}(\varphi(z))\varphi'(z) \).

Assume \( \{p_k\} \) is a sequence of polynomials such that
\[
\int_{\varphi(D)} |p_k(w) - \tilde{f}(w)|^2(1 - |\varphi^{-1}|^2)^{1-2\beta}dA(w) \to 0, \quad k \to \infty.
\]
Then
\[
\int_{\mathbb{D}} |p_k(\varphi(z))\varphi'(z) - f(z)|^2(1 - |z|^2)^{1-2\beta}dA(z) \\
= \int_{\mathbb{D}} |p_k(\varphi(z))\varphi'(z) - \tilde{f}(\varphi(z))\varphi'(z)|^2(1 - |z|^2)^{1-2\beta}dA(z) \\
= \int_{\mathbb{D}} |p_k(\varphi(z)) - \tilde{f}(\varphi(z))|^2|\varphi'(z)|^2(1 - |z|^2)^{1-2\beta}dA(z) \\
= \int_{\varphi(D)} |p_k(w) - \tilde{f}(w)|^2(1 - |\varphi^{-1}|^2)^{1-2\beta}dA(w) \to 0
\]
as \( k \to \infty \). Note \( p_k(\varphi(z))\varphi'(z) = p_k(M_{\varphi})(\varphi'(z)) \), this shows that \( \varphi' \) is a cyclic vector of \( M_{\varphi} \) on \( \mathcal{D}_{1-2\beta} \). By Proposition[12] \( \varphi' \) is also a cyclic vector of \( M_z \) on \( \mathcal{D}_{1-2\beta} \). \( \square \)

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