Integrability of the Trapped Ionic System

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Abstract

In this paper we will explore the 3D system with Hamiltonian

\[ H = \frac{1}{2} \left( p_r^2 + \frac{p_{\theta}^2}{r^2} + p_z^2 \right) + Ar^2 + Bz^2 + Cz^3 + Dr^2 z + Ez^4 + Fr^2 z^2 + G r^4, \]

describing trapped ionic system in the quadrapole field with a superposition of rationally symmetric hexapole and octopole fields for meromorphic integrability. We use the Lyapunov and Ziglin-Morales-Ruiz-Ramis’s classical methods for the proofs.

1 Introduction

Study of the atom’s external fields influence affected significantly the atomic physics at the beginning of the 20th century. Trapping phenomena creation by applying static electric and magnetic fields is a remarkable property. The ideal ion trap is based on a pure 3D quadrapole field on which different types of the quadrapole mass spectrometer were based, and properties of ionic motion are obtained in Mathieu’s equation (see [1] for details). In cylindrical coordinates \((x = r \cos \theta, y = r \sin \theta, z = z)\) and assuming appropriate constraints for simplicity we obtain Hamiltonian

\[ H = \frac{1}{2} \left( p_r^2 + \frac{p_{\theta}^2}{r^2} + p_z^2 \right) + Ar^2 + Bz^2 + Cz^3 + Dr^2 z + Ez^4 + Fr^2 z^2 + Gr^4, \] (1.1)

where \(A, B, C, D, E, F,\) and \(G\) are appropriate constants. The Hamiltonian (1.1) describes a system with tree degree of freedom having \(Z\)-axial symmetry \(- \theta\) is a cyclic and then \(p_{\theta}\) is a constant of motion. The existence of a sufficient number of first integrals of a Hamiltonian system determines whether it is of the two possible types: quasi-periodic–integrable or chaotic–non-integrable.

Motivation for this study is to generalize the result of [1] for non-integrability using a different approach.
2 2D model

In this section we study the 2D case \( p_\theta = 0 \)

\[
H = \frac{1}{2}(p_r^2 + p_z^2) + Ar^2 + Bz^2 + Cz^3 + Dr^2z + Ez^4 + Fr^2z^2 + Gr^4,
\]  

(2.1)

for existing an additional meromorphic integral of motion.

The equations of motion are

\[
\dot{r} = p_r, \quad \dot{p}_r = -(2Ar + 2Drz + 2Fr^2z^2 + 4Gr^3),
\]

\[
\dot{z} = p_z, \quad \dot{p}_z = -(2Bz + 3Cz^2 + Dr^2 + 4Ez^3 + 2Fr^2z).
\]  

(2.2)

Now we find a proper partial solution for (2.1). Let we put \( r = p_r = 0 \) in (2.2) and we find

\[
\ddot{z} = -(2Bz + 3Cz^2 + 4Ez^3),
\]

multiplying by \( \dot{z} \) and integrating by the time \( t \) we have

\[
\dot{z}^2 = -2(Ez^4 + Cz^3 + Bz^2 + h),
\]  

(2.3)

where \( h \) is the constant. Further we follow the procedures for Ziglin-Morales-Ruiz-Ramis’s theory and we find an invariant manifold \((r, p_r, z, p_z) = (0, 0, z, \dot{z})\) here \( z \) is the solution of (2.3). According to theory, the solution of (2.3) must be a rational function of Weierstrass \( \wp \)-function, but it is not important for us right now. Finding the Variation Equations (VE) we have \( \dot{\xi}_1 = dr, \eta_1 = dp_r, \xi_2 = dz, \) and \( \eta_2 = dp_z \) and we obtain:

\[
\ddot{\xi}_1 = -2(A + Dz + Fz^2)\xi_1,
\]

\[
\ddot{\xi}_2 = -2(B + 3Cz + 6Ez^2)\xi_2.
\]  

(2.4)

Next we change the variable in equations (2.4) by \( \xi_i(t) = \xi_i(z(t)) \), where \( z(t) \) is a solution of (2.3) and we have \( \frac{d\xi_i(t)}{dt} = \frac{d\xi_i}{dz} \frac{dz(t)}{dt} \), for \( i = 1, 2 \) and

\[
\frac{d^2\xi_1}{dt^2} = \frac{d^2\xi_1}{dz^2} \left( \frac{dz}{dt} \right)^2 + \frac{d\xi_1}{dz} \frac{d^2z}{dt^2}
\]

\[
= -(2Bz + 3Cz^2 + 4Ez^3) \frac{d^2\xi_1}{dz^2} - 2(Ez^4 + Cz^3 + Bz^2 + h) \frac{d\xi_1}{dz}
\]

\[
+ 2(A + Dz + Fz^2)\xi_1 = 0,
\]

and

\[
\frac{d^2\xi_2}{dt^2} = \frac{d^2\xi_2}{dz^2} \left( \frac{dz}{dt} \right)^2 + \frac{d\xi_2}{dz} \frac{d^2z}{dt^2}
\]

\[
= -(2Bz + 3Cz^2 + 4Ez^3) \frac{d^2\xi_2}{dz^2} - 2(Ez^4 + Cz^3 + Bz^2 + h) \frac{d\xi_2}{dz}
\]

\[
+ 2(B + 3Cz + 6Ez^2)\xi_2 = 0.
\]
If we denote with \( \frac{d}{dz} \) we obtain for the VE two Fuchsian linear differential equations with four singularities:

\[
\begin{align*}
\xi_1'' + \frac{4Ez^3 + 3Cz^2 + 2Bz}{2(Ez^4 + Cz^3 + Bz^2 + h)} \xi_1' - \frac{A + Dz + Fz^2}{Ez^4 + Cz^3 + Bz^2 + h} \xi_1 &= 0, \\
\xi_2'' + \frac{4Ez^3 + 3Cz^2 + 2Bz}{2(Ez^4 + Cz^3 + Bz^2 + h)} \xi_2' - \frac{B + 3Cz + 6Ez^2}{Ez^4 + Cz^3 + Bz^2 + h} \xi_2 &= 0.
\end{align*}
\tag{2.5}
\]

For the Normal Variation Equations (NVE) we suppose that \( h = 0 \) (with suitable initial conditions for example) and we obtain:

\[
\begin{align*}
\xi_1'' + \frac{4Ez^3 + 3Cz^2 + 2Bz}{2(Ez^4 + Cz^3 + Bz^2 + h)} \xi_1' - \frac{A + Dz + Fz^2}{Ez^4 + Cz^3 + Bz^2 + h} \xi_1 &= 0. \\
\xi_2'' + \frac{4Ez^3 + 3Cz^2 + 2Bz}{2(Ez^4 + Cz^3 + Bz^2 + h)} \xi_2' - \frac{B + 3Cz + 6Ez^2}{Ez^4 + Cz^3 + Bz^2 + h} \xi_2 &= 0.
\end{align*}
\tag{2.6}
\]

The equation (2.6) is a Fuchsian and has four singularities \( z = 0, z = \infty \) and \( z = z_i \), where \( z_{1,2} = \frac{-C \pm \sqrt{C^2 - 4BC}}{2E} \). We will miss the index 1 of \( \xi_1 \).

\[
\begin{align*}
\xi'' + \frac{4Ez^2 + 3Cz + 2B}{2z(Ez^2 + Cz + B)} \xi' - \frac{A + Dz + Fz^2}{z^2(Ez^2 + Cz + B)} \xi &= 0.
\end{align*}
\tag{2.7}
\]

The Fuchsian equations with four singularities of rank 2 like (2.7) are Heun equations.

## 3 The Integrability and Non-Integrability

In this section we will recall some classical and more advanced results that we will need for our researches.

We will define first the integrability in the sense of Liouville. Let \( (M, \omega) \) be a symplectic manifold on dimension \( 2n \) (real): \( M \) is smooth and \( \omega \) is closed \( 2n \)-form. We denote \( X_H \) hamiltonian vector field corresponding to the \( H \). Let also \( f_1 = H, \ldots f_n \) be functions in involution, i. e.

\[ \{f_i, f_j\} = X_{f_i} f_j = 0, \quad i, j = 1, \ldots, n. \]

We suppose that the differentials \( df_1, \ldots df_n \) are linearly independent on the level set

\[ L_c := \{ m, f_i(m) = c_i ; \quad i = 1, \ldots, n \}. \]

**Theorem 1. (Liouville–Arnold)** We suppose that \( \tilde{L}_c \) is a compact component of the level set \( L_c \). Then

a) \( \tilde{L}_c \) is invariant under the flows generated by \( X_{f_i}, i = 1, \ldots, n; \)

b) There exists a neighbourhood \( U \) of \( \tilde{L}_c \) and diffeomorphism \( J : f(U) \to V \), so that we have \( I = J \circ f \), and the symplectic form \( \omega \) in the coordinates \( (I, \phi) \) takes the form

\[ \omega = \sum df \wedge dI; \]

c) \( \tilde{L}_c \) is diffeomorphic to the \( n \)-dimensional torus.
Unfortunately, the above Theorem is valid only for real manifolds and functions. In the complex case, some specifications are required. We say that a system (with \( n \) degree of freedom) is integrable in the sense of Liouville, if it has a complete set of \( n \) independent first integrals in involution. Here the smoothness does not have to be considered above \( \mathbb{R} \). It is possible to consider analytic or meromorphic integrals on complex manifold instead of real smooth functions on real–\( M \). We recall some notions and facts about integrability of Hamiltonian systems in the complex domain, the Ziglin–Morales-Ruiz–Ramis’s theory and its relations with differential Galois groups of linear equations. We will follow [9] and [8].

We consider a Hamiltonian system
\[
\dot{x} = X_H(x), \quad t \in \mathbb{C}, \quad x \in M
\]
(3.1)
corresponding to an analytic Hamiltonian \( H \), defined on the complex \( 2n \)-dimensional manifold \( M \). If we suppose the system (3.1) has a non-equilibrium solution \( \Psi(t) \). We denote by \( \Gamma \) its phase curve. We can write the equation in variation (VE) near this solution
\[
\dot{\xi} = DX_H(\Psi(t))\xi, \quad \xi \in T\Gamma M.
\]
(3.2)

Further, we consider the normal bundle of \( \Gamma \), \( F := T\Gamma M/TM \) and let \( \pi : T\Gamma M \to F \) be the natural projection. The equation (3.2) induces an equation on \( F \)
\[
\dot{\eta} = \pi_*(DX_H(\Psi(t))(\pi^{-1}\eta)), \quad \eta \in F.
\]
(3.3)
which is called a normal variational equation (NVE) around \( \Gamma \). The (NVE) (3.3) admits a first integral \( dH \), linear on the fibers of \( F \). The level set \( F_r := \{\eta \in F|dH(\eta) = r\}, r \in \mathbb{C} \), is \( (2n-2) \)-dimensional affine bundle over \( \Gamma \). We will call \( F_r \) the reduced phase space of (3.3) and the restriction of the (NVE) on \( F_r \) is called the reduced normal variational equation.

Then the main result of the Ziglin–Morales-Ruiz–Ramis’s [9] theory is:

**Theorem 2.** If we suppose that the Hamiltonian system (3.1) has \( n \) meromorphic first integrals in involution. Then the identity component \( G^0 \) of the Galois group of the variational equation is abelian.

Next we consider a linear system
\[
y' = A(x)y, \quad y \in \mathbb{C}^n,
\]
(3.4)
or an equivalently linear homogeneous differential equation
\[
y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0,
\]
(3.5)
with \( x \in \mathbb{C}P^1 \) and \( A \in \text{gl}(n, \mathbb{C}(x)) \), \( (a_j(x) \in \mathbb{C}(x)) \). Let \( S := \{x_1, \ldots, x_s\} \) be the set of singular points of (3.4) (or (3.5)) and let \( Y(x) \) be a fundamental solution of (3.4) (or (3.5)) at \( x_0 \in \mathbb{C}\setminus S \). By the existence theorem this solution is analytic near of \( x_0 \). The continuation of \( Y(x) \) along
a nontrivial loop on $\mathbb{CP}^1$ defines a linear automorphism of the space of solutions, called the monodromy. Analytically this transformation can be presented as a follows: the linear automorphism $\Delta_\gamma$, associated with a loop $\gamma \in \pi_1(\mathbb{CP}^1 \setminus S, x_0)$ corresponds to multiplication of $Y(x)$ from the right by a constant matrix $M_\gamma$, called monodromy matrix

$$\Delta_\gamma Y(x) = Y(x)M_\gamma.$$ 

The set of these matrices forms the monodromy group.

We add another object to the system (3.4) (or (3.5)) - a differential Galois group. We have differential field $K$ is a field with a derivation $\partial = \frac{d}{dx}$, i.e. an additive mapping satisfying derivation rule. A differential automorphism of $K$ is an automorphism commuting with the derivation.

The coefficient field in (3.4) (and (3.5)) is $K = \mathbb{C}(x)$. Let $y_{ij}$ be elements of the fundamental matrix $Y(x)$. Let $L(y_{ij})$ be the extension of $K$ generated by $K$ and $y_{ij}$ - a differential field. This extension is called a Picard–Vessiot’s extension. Similarly to classical Galois Theory we define the Galois group $G := Gal_K(L) = Gal(L/K)$ to be the group of all differential automorphisms of $L$ leaving the elements of $K$ fixed. The Galois group is in fact an algebraic group. It has a unique connected component $G^0$ which contains the identity and is a normal subgroup of finite index. The Galois group $G$ can be represented as an algebraic linear subgroup of $GL(n, \mathbb{C})$ by

$$\sigma(Y(x)) = Y(x)R_\sigma,$$

where $\sigma \in G$ and $R_\sigma \in GL(n, \mathbb{C})$.

We can do the same locally at $a \in \mathbb{CP}^1$, replacing $\mathbb{C}(x)$ by the field of germs of meromorphic functions at $a$. In this way we can speak of a local differential Galois group $G_a$ of (3.4) at $a \in \mathbb{CP}^1$, defined in the same way for Picard-Vessiot extensions of the field $\mathbb{C}\{x-a\}[(x-a)^{-1}]$.

One should note that by its definition the monodromy group is contained in the differential Galois group of the corresponding system.

Next, we review some facts from the theory of linear systems with singularities. We call a singular point $x_i$ regular if any of the solutions of (3.4) (or of (3.5)) has at most polynomial growth in arbitrary sector with a vertex at $x_i$. Otherwise the singular point is called irregular.

We say that the system (3.4) has a singularity of the Fuchs type at $x_i$ if $A(x)$ has a simple pole at $x = x_i$. For the equation (3.5) the Fuchs type singularity at $x_i$ means that the functions $(x - x_i)^2a_j(x)$ are holomorphic in a neighborhood of $x_i$.

If the system (3.4) has a singularity of the Fuchs type, then this singularity is regular. The opposite is not true. However, for the equation (3.5) the regular singularities coincide with the singularities of the Fuchs type.

A system with only regular singularities is called Fuchsian system. For such systems we have:

**Theorem 3.** (Schlesinger) The differential Galois group coincides with the Zariski closure in $GL(n, \mathbb{C})$ of the monodromy group.
4 Main result

In this section we study the equation (2.7) for a Liouvillian solutions. First we find conditions for branching for solutions of this equation. We will use Lyapunov’s classical idea to prove non-integrability using the branching of the solutions of the variational equations around an appropriate partial solution. If the solutions of the VE branching then the Hamiltonian system has not additional first integral. (See [2] for details.) We find the indicial equations to the singular points $z = 0$ and $z = \infty$. We have

$$\lambda^2 - A = 0 \quad (B \neq 0),$$

roots are $\lambda_j = \pm \sqrt{A}/B$, $j = 1, 2$ for $z = 0$, second one is $\rho^2 - \rho - F_E = 0$, with roots $\rho_k = \frac{1 \pm \sqrt{1 + 4}}{2}$, $k = 1, 2$ for $z = \infty$. The solutions around $z = 0$ are branching when $\lambda_j, \rho_k \notin \mathbb{Z}$, $j = 1, 2$ and near $z = \infty$ we have branching - when $\rho_k, \notin \mathbb{Z}$, $k = 1, 2$. This is an application of the Frobenius’s method ([7] p. 70): the fundamental system of solutions around of the singular points are presented in the form $\Phi_j(z) = x^{\lambda_j} \Psi_j(z)$, where $\Psi_j(z)$ are holomorphic functions (locally) for $j = 1, 2$. Then the solutions of (2.7) are branching for $\lambda_j, \rho_k \notin \mathbb{Z}$, $j = 1, 2$. For $z = \infty$ we do the same and obtain that $\rho_k, \notin \mathbb{Z}$, $k = 1, 2$. We proved the following theorem.

Theorem 4. Let $p = \sqrt{\frac{A}{B}} \notin \mathbb{Z}$ and $q = \sqrt{1 + \frac{4}{E}}$ with roots $\lambda_j \notin \mathbb{Z}, j = 1, 2$. Then 2D system (2.7) has no additional meromorphic first integral.

We are finished with a rough first estimate. Now we take in (2.7) $F = \frac{p^2 - 1}{4} E$ and $A = q^2 B$, where $p, q \in \mathbb{Z}$, $p$ is odd. (This is the case when the solutions are not branching i.e. $\lambda_j, \rho_k \in \mathbb{Z}$.)

We will change the variables in (2.7) with $x = \frac{1}{z}$ (we change singularities $0 \leftrightarrow \infty$) and - $s_i = \frac{1}{s_i}$ we will follow [9] (p. 144) and we have

$$\frac{d^2 \xi}{dx^2} + \left( \frac{1/2}{x - s_1} + \frac{1/2}{x - s_2} \right) \frac{d \xi}{dx} - \left( \frac{1/4 - p^2/4}{x^2} + \frac{p^2 - 1 - 4q^2 E}{4x(x - s_1)(x - s_2)} \right) \xi = 0. \quad (4.1)$$

The next reduction is obtained as in [7] p. 78 we put $\xi(x) = x^{\frac{p+1}{2}} \zeta(x)$, and we get $x = \tau s_1$ after that. Next we have (here $\alpha = \frac{s_2}{s_1}$)

$$\frac{d^2 \zeta}{d\tau^2} + \left( \frac{p + 1}{\tau} + \frac{1/2}{\tau - 1} + \frac{1/2}{\tau - \alpha} \right) \frac{d \zeta}{d\tau} + \left( \frac{((p+1)(p+3)}{4} - q^2 E) \tau + \frac{p^2 - 1}{4}(1 + \alpha) + Ds_2}{\tau(\tau - 1)(\tau - \alpha)} \right) \zeta = 0. \quad (4.2)$$

The equation (4.2) is Heun equation with singularities $\tau = 0, 1, \alpha, \infty$ in standard form.
We need some basic information about the Heun equations. We will follow [3], [4], [5], and [6]. The Heun equation

\[
\frac{d^2\Lambda}{dx^2} + \left( \frac{c}{x} + \frac{d}{x-1} + \frac{a+b-c-d+1}{x-\alpha} \right) \frac{d\Lambda}{dx} + \left( \frac{abx-Q}{x(x-1)(x-\alpha)} \right) \Lambda = 0
\] (4.3)

is a second order Fuchsian differential equation on the Riemann sphere \( \mathbb{P}^1 \) with four regular singular points. The local solution at \( x = 0 \) we denote with \( H(\alpha, Q, a, b, c, d, x) \) - the Heun function. If \( \Lambda_1(x) = H(\alpha, Q, a, b, c, d, x) \) is the solution of (4.3), the other local solution around \( x = 0 \) we obtain from the formula of d’Alambert \( \Lambda_2(x) = \Lambda_1(x) \int \frac{dx}{\Lambda_1'(x)} \).

Generally, the Heun functions are transcendental, and their monodromy is not well known. For special values of parameters \( a, b, c, d, Q, x \) they can be Liouvillian. The Heun equation has Liouvillian solutions if and only if its monodromy is one of the following options, reducible, or infinite dihedral, or finite [6]. If the parameters \( a \) and \( b \) are \( \in \mathbb{Z}_- \) - non-negative integers, we obtain Heun’s polynomials i. e. this is the Liouvillian solutions. Necessary conditions for reducibility are \( a, b, c-a, c-b, d-a, d-b, c+d-a, \) or \( c+d-b \in \mathbb{Z} \) (for details see [4]). There are more accurate estimates for reducibility [6], but they are also not complete. These estimates are made at one free parameter. The one of main results of the paper [6] and [5] is the following Theorem:

**Theorem 5.** The Heun functions \( H(\alpha, Q, a, b, c, d, x) \) are reducible in in the following cases for \( k, l \in \mathbb{Z}, \alpha \in \mathbb{C} \):

a) \( a = k + \frac{1}{2}, b = k + \frac{1}{2}, c = 2l + 1 \) and \( d = 2\alpha \), or \( a = 2k + 1, b = 2l + 1, c = \alpha, d = \alpha \);

b) \( a = 2k + 1, b = 2k + 1, c = 2\alpha, d = 2\alpha \);

c) \( a = k + \frac{1}{2}, b = k + \frac{1}{2}, c = 2k + 1 \) and \( d = 4\alpha \), or \( a = 4k + 2, b = \alpha, c = \alpha, d = 2\alpha \);

d) \( a = \frac{1}{2}, b = k + \frac{1}{2}, c = 2k + 1 \) and \( d = 3\alpha \), or \( a = \frac{1}{2}, b = 3k + \frac{3}{2}, c = \alpha, d = 2\alpha \);

e) \( a = k + \frac{1}{2}, b = 3k + \frac{3}{2}, c = \alpha, d = 3\alpha \);

f) \( a = 2k, b = \alpha, c = \alpha \) and \( d = 2\alpha + 2l \), or \( a = k, b = k, c = 2\alpha, d = 2\alpha + 2l \).

Now we are ready to return to our equation (4.2). We have, using Heun’s equation terminology for (4.3), \( c = p + 1, d = \frac{1}{2}, a + b - c - d + 1 = \frac{1}{2}, \) we obtain \( a + b = p + 1 \) and \( ab = \frac{(p+1)(p+3)}{4} \). For \( a \) and \( b \) we have \( a = \frac{p+1+\sqrt{\Delta}}{2}, b = \frac{p+1-\sqrt{\Delta}}{2} \), where \( \Delta = 4q^2E - 2(p+1) \), \( p, q \in \mathbb{Z} \). It is not difficult to check that one of the options for solvable monodromy, i. e. \( a, b, c-a, c-b, d-a, d-b, c+d-a, \) or \( c+d-b \in \mathbb{Z} \) is impossible for (4.2). The conditions for the Heun equation (4.3) to have trivial solutions - Heun polynomials are \( a = 0 \) or \( b = 0 \) - for equation (4.2) this is

\[
M_0 := \{(E, p, q) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\text{odd}} : 4q^2E = (p + 1)(p + 3), p \in \mathbb{Z}, q \in \mathbb{Z}_{\text{odd}}\},
\]

and \( a, \) and \( b \in \mathbb{Z}_- \) i. e.

\[
M_{\mathbb{Z}_-} := \{(E, p, q) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\text{odd}} : 2q^2E > p + 1, \frac{(p + 1)(p + 3)}{4} - q^2E \in \mathbb{Z}_+, p \leq -2\}
\]
In the case of the equation (4.2), the following set satisfy the conditions of Theorem (5) for the solvability:

$$M_a := \{(E, p, q) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{odd} : 2q^2E = 2l + 1, l \in \mathbb{Z}, q \in \mathbb{Z}_{odd}\}.$$ 

The main result of our exposition is the following theorem:

**Theorem 6.** Let $\{(E, p, q) \in M_a \setminus (M_0 \cup M_{Z_{even}}) \subset \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{odd}, \text{ where } p = \sqrt{\frac{A}{B}} \text{ and } q = \sqrt{1 + \frac{4E}{B}}, \text{ then 2D system (2.2) has no additional meromorphic first integral.}\}$

**Proof:**

In this case (4.2) is solvable. We will proof that the unity of Galois group is not commutative. Let we denote with $K := \mathbb{C}[[\tau]]$ the coefficient field with standard derivation. The field $K$ is a field of rational functions in $\mathbb{C}$ and $\zeta_1(\tau)$ and $\zeta_2(\tau)$ are the local solution of (4.2) near $\tau = 0$ and let $F$ is extension generated by $K$ and $\zeta_1$, i. e. $F := K[\zeta_1(\tau)]$. The extension $F$ is Picard–Vessiot’s of $K$. The Galois group of this we denote with $G := G(F/K)$. If $Y(\tau)$ is fundamental matrix of (4.2) then

$$\sigma(Y(\tau)) = Y(\tau)R_\sigma,$$

where $\sigma \in G$ and $R_\sigma \in GL(2, \mathbb{C})$ is a natural representation of the Galois group as an algebraic linear group. We know that det $|R_\sigma| = 1$ and that is why we have

$$R_\sigma = \begin{pmatrix} c & 0 \\ \mu & c^{-1} \end{pmatrix}.$$

We will show that $\mu \neq 0$ under the assumptions of the theorem. First, we have to exclude the solution of (1.2) which are in $K$. This is the Heun’s polynomials $(E, p, q) \in M_0 \cup M_{Z_{even}}$. Second, for the solutions of $(\zeta_1(\tau), \zeta_2(\tau))$ of (4.2) we have $\zeta_2(\tau) \notin F$. If we assume the opposite $\zeta_2(\tau) \in F$, we will obtain that the solutions $(\zeta_1(\tau)$ and $\zeta_2(\tau))$ are not linearly independent over $\mathbb{C}$. Then we conclude that $\mu \neq 0$, that is why the identity component of the Galois group is not commutative.

We obtain that the Galois group is represented by the matrix group $\{ \begin{pmatrix} c & 0 \\ \mu & c^{-1} \end{pmatrix}, c \neq 0, \mu \neq 0 \}$ which is connected, solvable, but non-commutative. This finishes the proof of Theorem 6

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References

[1] Benkhali M., Kharbach J. El Fakkousy I., Chatar W., Rezzouk A., Ouazzani-Jamil M. (2018), Painleve analysis and integrability of the trapped ionic system, Phys. Lett. A, V. 382, issue 36, doi.org/10.1016/j.physleta.2018.06.034, 2515–2525.

[2] Lyapunov, A., (1954), On certain property of the differential equations of the problem of motion of a heavy rigid body having a fixed point, Soobsch. Kharkov Math. Obscht., Ser. 2, 4, 1894; 123–140. (in Russian).

[3] Maier R. S, (2007), P-symbols, Heun identities, and 3F2 identities, American Mathematical Society. Special Session on Special Functions and Orthogonal Polynomials, Contemporary Math. Ser. N 471, AMS, dx.doi.org/10.1090/conm/471/09211, 139–159.

[4] Maier R. S, (2007), The 192 solutions of the Heun equation, Math. Comp., 76, doi.org/10.1090/S0025-5718-06-01939-9, 811–843.

[5] Vidunas R., Filipuk G., (2013), Parametric Transformations between Heun and Gauss Hypergeometric Functions, Funkcialaj Ekvacioj, 56, doi.org/10.1619/fesi.56.271, 271–321.

[6] Vidunas R., (2016), Degenerate and dihedral Heun functions with parameters, Hokkaido Mathematical Journal, 45,11, doi:10.14492/hokmj/1470080750, 93–108.

[7] Poole E. G. C., (1936), Introduction to the Theory of Linear Differential Equations, Oxford, At The Clarendon Press.

[8] Christov O., Georgiev G., (2015), Non-Integrability of Some Higher-Order Painleve Equations in the Sense of Liouville, SIGMA 11, 045, doi.org/10.3842/SIGMA.2015.045.

[9] Morales-Ruiz J., (1999), Differential Galois Theory and Non-integrability of Hamiltonian Systems, Birkhäuser.