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DOI: https://doi.org/10.1515/crelle-2014-0100

Posted at the Zurich Open Repository and Archive, University of Zurich
ZORA URL: https://doi.org/10.5167/uzh-207536
Journal Article
Published Version

Originally published at:
De Philippis, Guido; Maggi, Francesco (2017). Dimensional estimates for singular sets in geometric variational problems with free boundaries. Journal für die Reine und Angewandte Mathematik, 2017(725):217-234.
DOI: https://doi.org/10.1515/crelle-2014-0100
Dimensional estimates for singular sets in geometric variational problems with free boundaries

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Abstract. We show that singular sets of free boundaries arising in codimension one anisotropic geometric variational problems are $\mathcal{H}^{n-3}$-negligible, where $n$ is the ambient space dimension. In particular our results apply to capillarity type problems, and establish everywhere regularity in the three-dimensional case.

1. Introduction

In [1], having in mind applications to capillarity problems and to relative isoperimetric problems, we studied the regularity of free boundaries in anisotropic geometric variational problems. The main result contained in [1] asserts that free boundaries are regular outside closed sets of vanishing $\mathcal{H}^{n-2}$-measure. In this paper we improve upon this result by showing $\mathcal{H}^{n-3}$-negligibility of singular sets, see Theorem 1.5 below.

The “interior part” of this statement dates back to [6]. The boundary case is addressed here by combining the set of ideas introduced in [6] with the $\mathcal{H}^{n-2}$-negligibility we have obtained in [1] (see, in particular, Lemma 2.7 below).

We note that singular sets must necessarily be smaller than merely $\mathcal{H}^{n-3}$-negligible. Indeed, a general argument due to Almgren (and appeared in [9, Lemma 5.1]) implies that the set of $s > 0$ such that singular sets of minimizers of a given elliptic functional are $\mathcal{H}^s$-negligible is open. At the same time, the cone over $S^1 \times S^1 \subset \mathbb{R}^4$ minimizes a suitable elliptic anisotropic functional [5]. This example may lead to conjecture that singular sets of arbitrary anisotropic functionals have Hausdorff dimension at most $n - 4$, although we are not aware of further evidence supporting this possibility.

The $\mathcal{H}^{n-3}$-negligibility of the singular set, although not optimal, has two interesting consequences. Firstly, and obviously, it implies everywhere regularity in $\mathbb{R}^3$; secondly, it provides the needed regularity in order to exploit second variation arguments in the study of geometric

Francesco Maggi was supported by the NSF Grant DMS-1265910.
properties of minimizers; see for example [8] and Lemma 2.5 below (actually \(\mathcal{H}^{n-3}\)-locally finiteness of the singular set would be enough for this, see for instance [2, Section 4.7.2]).

We now define the class of functionals and the notion of minimizers that we shall use.

**Definition 1.1 (Regular elliptic integrands).** Given an open set \(A \subset \mathbb{R}^n\), \(\lambda \geq 1, \ell \geq 0\), we consider the family \(\mathcal{E}(A, \lambda, \ell)\) of functions \(\Phi : \text{cl}(A) \times \mathbb{R}^n \to [0, \infty]\) such that \(\Phi(x, \cdot)\) is convex and positively one-homogeneous on \(\mathbb{R}^n\) with \(\Phi(x, \cdot) \in C^{2,1}(S^{n-1})\) for every \(x \in \text{cl}(A)\), and such that the following properties hold for every \(x, y \in \text{cl}(A), \nu, \nu' \in S^{n-1}\), and \(e \in \mathbb{R}^n\):

\[
\frac{1}{\lambda} \leq \Phi(x, \nu) \leq \lambda, \\
|\Phi(x, \nu) - \Phi(y, \nu)| + |\nabla \Phi(x, \nu) - \nabla \Phi(y, \nu)| \leq \ell |x - y|, \\
|\nabla \Phi(x, \nu)| + \|\nabla^2 \Phi(x, \nu)\| + \frac{\|\nabla^2 \Phi(x, \nu) - \nabla^2 \Phi(x, \nu')\|}{|\nu - \nu'|} \leq \lambda, \\
\text{and}
\]

\[\nabla^2 \Phi(x, \nu)[e] \cdot e \geq \frac{|e - (e \cdot \nu)\nu|^2}{\lambda}.
\]

In the above definition \(\nabla \Phi\) and \(\nabla^2 \Phi\) stand for the gradient and Hessian of \(\Phi\) in the \(\nu\)-variable. \(|L| = \sup\{|Le : |e| = 1\}\) is the operator norm of a linear map \(L : \mathbb{R}^n \to \mathbb{R}^n\), \(L[e]\) is the action of \(L\) on \(e \in \mathbb{R}^n\), and \(\text{cl}(A)\) is the closure of \(A\). We also set

\[
\mathcal{E}_*(\lambda) = \mathcal{E}(\mathbb{R}^n, \lambda, 0)
\]

for the class of regular autonomous elliptic integrand (indeed, \(\ell = 0\) forces \(\Phi(x, \nu) = \Phi(\nu)\)). We shall regard \(\mathcal{E}_*(\lambda)\) as a subset of \(C^{2,1}(S^{n-1})\) by the obvious identification of a one-homogeneous function with its trace on the sphere. With this identification it is immediate to check that \(\mathcal{E}_*(\lambda)\) is a compact subset with respect to uniform convergence on \(S^{n-1}\). Finally, if \(\Phi \in \mathcal{E}(A, \lambda, \ell)\) and \(E\) is a set of locally finite perimeter in \(A\), then we set

\[
\Phi(E; G) = \int_{G \cap \partial^* E} \Phi(x, \nu_E(x)) \, d\mathcal{H}^{n-1}(x) \in [0, \infty] \quad \text{for all } G \subset A.
\]

Here \(\partial^* E\) denotes the reduced boundary of \(E\) in \(A\) and \(\nu_E\) is the measure-theoretic outer unit normal to \(E\); see [4, Chapter 15].

**Definition 1.2 (Almost-minimizers).** Let an open set \(A\) and an open half-space \(H\) in \(\mathbb{R}^n\) be given (possibly \(H = \mathbb{R}^n\)), together with \(r_0 \in (0, \infty]\) and \(\Lambda \geq 0\). Given \(\Phi \in \mathcal{E}(A \cap H, \lambda, \ell)\) and a set \(E \subset H\) of locally finite perimeter in \(A\), one says that \(E\) is an \((\Lambda, r_0)\)-minimizer of \(\Phi\) in \((A, H)\) if

\[
\Phi(E; H \cap W) \leq \Phi(F; H \cap W) + \Lambda |E \Delta F|
\]

whenever \(F \subset H, E \Delta F \subset \subset W, \) and \(W \subset \subset A\) is open with \(\text{diam}(W) < 2r_0\); see Figure 1.1. When \(\Lambda = 0\), and \(r_0 = +\infty\), one simply says that \(E\) is a minimizer of \(\Phi\) in \((A, H)\).

**Remark 1.3.** As proved in [1, Lemma 6.1], up to local diffeomorphisms, minimizers of capillarity-type problems fall in the framework of Definition 1.2. Other applications include relative isoperimetric problems in Riemannian and Finsler geometry.
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Figure 1.1. The situation in Definition 1.2: roughly speaking, $E$ minimizes $\Phi$ with respect to perturbations $F$ which agree with $E$ on $H \cap \partial B_{x,r}$ and are allowed to freely move the boundary of $E$ close to $B_{x,r} \cap \partial H$. In other words, we impose a Dirichlet condition on $H \cap \partial B_{x,r}$ and a Neumann condition of $B_{x,r} \cap \partial H$.

Remark 1.4. Since $\mathcal{E}(A \cap H, \lambda, \ell)$ is invariant by isometries of $\mathbb{R}^n$ (in the sense that, if $f(x) = x_0 + R[x], R \in O(n)$, then $E$ is a $(\Lambda, r_0)$-minimizer of $\Phi$ in $(A, H)$ if and only if $f(E)$ is a $(\Lambda, r_0)$-minimizer of $\Phi^f$ in $(f(A), f(H))$ where $\Phi^f(x, v) = \Phi(f^{-1}(x), R^{-1}v)$ belongs to $\mathcal{E}(f(A) \cap f(H), \lambda, \ell)$, see [1, Lemma 2.18]) and we are interested in boundary regularity, in the sequel we can and do assume that $H$ is a fixed half-space with $0 \in \partial H$.

Let now $E$ be a $(\Lambda, r_0)$-minimizer of $\Phi$ in $(A, H)$ of some $\hat{\phi} \in \mathcal{E}(A \cap H, \lambda, \ell)$, and set

$$M_A(E) = A \cap \text{cl}(H \cap \partial E).$$

The regular set $R_A(E)$ of $E$ in $A$ is defined by

$$R_A(E) = \{x \in M_A(E) : \text{there exists an } r_x > 0 \text{ such that } M_A(E) \cap B_{x,r_x} \text{ is a } C^1 \text{-manifold with boundary contained in } \partial H \},$$

while $\Sigma_A(E) = M_A(E) \setminus R_A(E)$ is called the singular set $\Sigma_A(E)$ of $E$ in $A$. In this way, $\Sigma_A(E)$ is relatively closed in $A$. We shall also set

$$R_G(E) = R_A(E) \cap G, \quad \Sigma_G(E) = \Sigma_A(E) \cap G \quad \text{for all } G \subset A.$$

By combining the results of [6] for the interior situation with the ones of [1] for the boundary situation, one sees that $E \cap A$ is (equivalent to) an open set, that $A \cap \partial E \cap \partial H$ is a set of finite perimeter in $\partial H$, and that

\begin{align*}
\mathcal{H}^{n-3}(\Sigma_A \cap H(E)) &= 0 \quad \text{by [6]}, \tag{1.2} \\
\mathcal{H}^{n-2}(\Sigma_A \cap \partial H(E)) &= 0 \quad \text{by [1]}, \tag{1.3}
\end{align*}

with $\nabla \Phi(x, v_E) \cdot v_H = 0$ at every $x \in R_A \cap \partial H(E)$. Moreover, one has a characterization of the regular and singular sets in terms of the following notion of excess: given $x \in A$ and $r < \text{dist}(x, \partial A)$ and denoting by $B_{x,r}$ the open ball centered at $x$ and with radius $r$, we define spherical excess of $E$ at the point $x$, at scale $r$, relative to $H$ as

$$\text{exc}^H(E, x, r) = \inf \left\{ \frac{1}{r^{n-1}} \int_{B_{x,r} \cap H \cap \partial^* E} \frac{|v - v_E|^2}{2} \, d\mathcal{H}^{n-1} : v \in S^{n-1} \right\},$$

where $\partial^* E = \partial E \setminus \partial H$. The excess vanishes at the scale of the Hausdorff measure $\mathcal{H}^{n-1}$ of the blow-up of $E$ at $x$, and it measures the deviation from the spherical geometry at the scale $r$.
Then, for positive constants $\epsilon = \epsilon(n, \lambda)$ and $c = c(n, \lambda)$, we have that

\[(1.4) \quad \text{exc}^H(E, x, r) < \epsilon \implies M_A(E) \cap B_{x, cr} \subset R_A(E),\]

see [1, Theorem 3.1]. In particular

\[(1.5) \quad \Sigma_A(E) = \left\{ x \in M_A(E) : \liminf_{r \to 0^+} \text{exc}^H(E, x, r) \geq \epsilon(n, \lambda) \right\}.

**Theorem 1.5.** If $\Phi \in \mathcal{E}(A, \lambda, \ell)$ and $E$ is a $(\Lambda, r_0)$-minimizer of $\Phi$ in $(A, H)$, then

\[\mathcal{H}^{n-3}(\Sigma_A \cap \partial H(E)) = 0.\]

We now describe the proof of Theorem 1.5. First of all, by a blow-up argument, Theorem 1.5 is seen to be equivalent to the following theorem.

**Theorem 1.6.** If $\Phi \in \mathcal{E}_*(\lambda)$, $B = B_{0,1}$, and $E$ is a minimizer of $\Phi$ in $(B, H)$, then

\[\mathcal{H}^{n-3}(\Sigma_B \cap \partial H(E)) = 0.\]

We deduce Theorem 1.6 from the following two propositions, where we set

$\mathcal{E}_{**}(\lambda) = \{ \Phi \in \mathcal{E}_*(\lambda) : \text{equation (1.6) holds true for every minimizer } E \text{ of } \Phi \text{ in } (B, H) \}.$

**Proposition 1.7.** The set $\mathcal{E}_{**}(\lambda)$ is open in $\mathcal{E}_*(\lambda)$ in the uniform convergence on $S^{n-1}$.

**Proposition 1.8.** The set $\mathcal{E}_{**}(\lambda)$ is closed in $\mathcal{E}_*(\lambda)$ in the uniform convergence on $S^{n-1}$.

**Proof of Theorem 1.6.** Obviously, $\mathcal{E}_*(\lambda)$ is convex, thus connected. By [3] (or, alternatively, by [1, Corollary 1.4]), the isotropic functional $\Phi(v) = |v|$ belongs to $\mathcal{E}_{**}(\lambda)$ for all $\lambda \geq 1$. Propositions 1.7 and 1.8 thus imply $\mathcal{E}_{**}(\lambda) = \mathcal{E}_*(\lambda)$.

In Section 2 we prove Propositions 1.7 and 1.8 and show that Theorem 1.6 implies Theorem 1.5. Second variation formulas used in these arguments are collected in the appendix.

We close this introduction by describing the main ideas behind the two key propositions. Proposition 1.7 is based on the idea that, roughly speaking, for every $s > 0$ the map

$\Phi \mapsto \sup \{ \mathcal{H}^s(\Sigma_B \cap \partial H(E)) : E \text{ is a minimizer of } \Phi \text{ in } (B, H) \}$

is upper semi-continuous on $\mathcal{E}_*(\lambda)$ with respect to the uniform convergence on $S^{n-1}$. Concerning Proposition 1.8, one starts by observing that, if $\Phi \in \mathcal{E}_*(\lambda)$, then $R_A(E)$ is a $C^2$-manifold with boundary. Denoting by $\Pi_E$ the second fundamental form of $R_A(E)$, we set

\[(1.7) \quad ||\Pi_E||^2(G) = \int_{G \cap R_E(A)} ||\Pi_E||^2 d\mathcal{H}^{n-1} \in [0, \infty] \quad \text{for all } G \subset \mathbb{R}^n,\]

where $||\Pi_E||^2$ is the squared Hilbert–Schmidt norm of the tensor $\Pi_E$, which equals the sum of the squared principal curvatures of $R_A(E)$. One then shows that $\Phi \in \mathcal{E}_{**}(\lambda)$ if and only if

$||\Pi_E||^2(B) \leq C$ for every minimizer $E$ of $\Phi$ in $(B, H)$,

for some $C = C(n, \lambda)$, and hence concludes by proving that the map

$\Phi \mapsto \sup \{ ||\Pi_E||^2(B) : E \text{ is a minimizer of } \Phi \text{ in } (B, H) \}$

is lower-semicontinuous on $\mathcal{E}_*(\lambda)$ with respect to the uniform convergence on $S^{n-1}$. 


2. Proofs

Here and in the following we say that $E_h \to E$ in $A$ as $h \to \infty$ if $|E_h \Delta E| \cap A \to 0$ as $h \to \infty$, and that $E_h \to E$ locally in $A$ as $h \to \infty$ if, for every $K \subset A$, we have $E_h \to E$ in $K$ as $h \to \infty$. Moreover, we set $I_\varepsilon(S)$ for the $\varepsilon$-neighborhood of $S \subset \mathbb{R}^n$.

We begin with a classical lemma concerning convergence of minimizers and of singular sets, see for instance [4, Lemma 28.14]

**Lemma 2.1.** Let $\{ \Phi_h \}_{h \in \mathbb{N}} \subset \mathcal{E}_*(\lambda)$ with $\Phi_h \to \Phi$ in $C^0(S^{n-1})$ as $h \to \infty$, and let $\{ E_h \}_{h \in \mathbb{N}}$ be such that $E_h$ is a $(\Lambda_h, r_h)$-minimizer of $\Phi_h$ in $(A, H)$ with $\Lambda_h \to \Lambda < \infty$ and $r_h \to r_0 > 0$ as $h \to \infty$. Then there exists a $(\Lambda, r_0)$-minimizer $E$ of $\Phi$ in $(A, H)$ such that, up to subsequences, $E_h \to E$ locally in $A$ as $h \to \infty$. Moreover, for every $\varepsilon > 0$ and $K \subset A$ there exists an $h_0 > 0$ such that

\[ (2.1) \quad \Sigma_K(E_h) \subset I_\varepsilon(\Sigma_K(E)) \quad \text{for all } h \geq h_0. \]

In particular,

\[ (2.2) \quad J^s_\infty(\Sigma_K(E)) \geq \limsup_{h \to \infty} J^s_\infty(\Sigma_K(E_h)) \quad \text{for all } s \in [0, n], \]

where $J^s_\infty$ is defined for every $G \subset \mathbb{R}^n$ as

\[ J^s_\infty(G) = \inf \left\{ \sum_{i \in \mathbb{N}} \omega_s \left( \frac{\text{diam}(G_i)}{2} \right)^s : G \subset \bigcup_{i \in \mathbb{N}} G_i, \ G_i \text{ open} \right\}, \quad \omega_s = \frac{\pi^{s/2}}{\int_0^\infty t^{s/2} e^{-t} \, dt}. \]

**Proof.** The local convergence in $A$ to a minimizer $E$ of $\Phi$ follows by [1, Theorem 2.9]. Since $\text{exc}^H(E_h, x, r) \to \text{exc}^H(E, x, r)$ for a.e. $r > 0$ and for every $x \in A$ (cf. with [1, equation (3.10)]) and by (1.4) and (1.5), one proves (2.1). Finally, if $\{ G_i \}_{i \in \mathbb{N}}$ is an open covering of $\Sigma_K(E)$, then there exists an $\varepsilon > 0$ such that $\{ G_i \}_{i \in \mathbb{N}}$ is a covering of $I_\varepsilon(\Sigma_K(E))$, and thus of $\Sigma_K(E_h)$ too, provided $h \geq h_0$; by minimizing on all the open coverings we obtain (2.2). \qed

We now prove Proposition 1.7 by using Lemma 2.1. To this end we recall some properties of $J^s_\infty$. First of all, $J^s_\infty \geq J^s$, with

\[ (2.3) \quad J^s(G) = 0 \quad \text{if and only if} \quad J^s_\infty(G) = 0. \]

Moreover, for every $G \subset \mathbb{R}^n$ and $s \in [0, n]$ we have

\[ (2.4) \quad \limsup_{r \to 0} \frac{J^s_\infty(G \cap B(x, r))}{r^s} \geq c(s) > 0 \quad \text{for } J^s\text{-a.e. } x \in G, \]

see [7, Theorem 3.26 (2)]. We now set

\[ E^{x,r} = \frac{E - x}{r} \quad \text{for all } x \in \mathbb{R}^n, \ r > 0, \]

and we notice that, if $\Phi \in \mathcal{B}(A, \lambda, \ell)$ and $E$ is a $(\Lambda, r_0)$-minimizer of $\Phi$ in $(A, H)$, then $E^{x,r}$ is a $(\Lambda r, r_0 r)$-minimizer of $\Phi^{x,r}$ in $(A^{x,r}, H^{x,r})$, where

\[ \Phi^{x,r}(y, v) = \Phi(x + r y, v) \quad \text{for all } y \in A^{x,r}, \ v \in S^{n-1}. \]

We shall also frequently use the facts that if $x \in A \cap \partial H$ and $0 \in \partial H$ (see Remark 1.4), then $H^{x,r} = H$ for every $r > 0$ and $A^{x,r}$ eventually contains every compact set of $\mathbb{R}^n$ as $r \to 0$; and that if $\Phi \in \mathcal{B}_*(\lambda)$, then $\Phi^{x,r} = \Phi$. 
Proof of Proposition 1.7. Let \( \Phi \in \mathcal{E}_\ast(\lambda) \). Furthermore, assume there exists a sequence \( \{ \Phi_h \}_{h \in \mathbb{N}} \subset \mathcal{E}_\ast(\lambda) \setminus \mathcal{E}_\ast^\ast(\lambda) \) such that \( \Phi_h \to \Phi \) in \( C^0(\mathbb{S}^{n-1}) \) as \( h \to \infty \). In particular, for every \( h \in \mathbb{N} \) there exists a minimizer \( E_h \) of \( \Phi_h \) in \( (B, H) \) such that \( \mathcal{H}^{n-3}(\Sigma_{B \cap \partial H}(E_h)) > 0 \). By (2.4), there exist \( x_h \in \Sigma_{B \cap \partial H}(E_h) \) and \( r_h \to 0 \) with

\[
(2.5) \quad \frac{r_h}{\text{dist}(x_h, \partial B)} \to 0 \quad \text{as} \quad h \to \infty
\]
such that

\[
\mathcal{H}_{\infty}^{n-3}(\Sigma_{B \cap \partial H}(E_h) \cap B_{x_h, r_h}) \geq c(n)r_h^{n-3}.
\]

Let us set \( F_h = (E_h)^{x_h, r_h} \). Then \( F_h \) is a minimizer of \( \Phi_h \) in \( (B^{x_h, r_h}, H) \) and

\[
\mathcal{H}_{\infty}^{n-3}(\Sigma_{B \cap \partial H}(F_h)) = \frac{\mathcal{H}_{\infty}^{n-3}(\Sigma_{B \cap \partial H}(E_h) \cap B_{x_h, r_h})}{r_h^{n-3}} \geq c(n) > 0.
\]

By Lemma 2.1, there exists a minimizer \( F \) of \( \Phi \) in \( (\mathbb{R}^n, H) \) (since \( B^{x_h, r_h} \to \mathbb{R}^n \) by (2.5)) such that \( \mathcal{H}_{\infty}^{n-3}(\Sigma_{B \cap \partial H}(F)) > 0 \). By (2.3), this contradicts the fact that \( \Phi \in \mathcal{E}_\ast^\ast(\lambda) \). \( \Box \)

The same argument gives the following lemma.

Lemma 2.2. If \( A \) is an open set, \( \Phi \in \mathcal{E}_\ast^\ast(\lambda) \) and \( E \) is a \((\Lambda, r_0)\)-minimizer of \( \Phi \) in \((A, H)\), then

\[
\mathcal{H}^{n-3}(\Sigma_{A \cap \partial H}(E)) = 0.
\]

Proof. If \( E \) is a \((\Lambda, r_0)\)-minimizer of \( \Phi \) in \((A, H)\) with \( \mathcal{H}^{n-3}(\Sigma_{A \cap \partial H}(E)) > 0 \), then by arguing as in the proof of Proposition 1.7 we can find \( r_h \to 0 \) as \( h \to \infty \) and \( x \in \Sigma_{A \cap \partial H}(E) \) such that

\[
\mathcal{H}_{\infty}^{n-3}(\Sigma_{A \cap \partial H}(E) \cap B_{x, r_h}) \geq c(n)r_h^{n-3}.
\]

Hence \( E_h = E^{x, r_h} \) is a \((\Lambda r_h, r_0/r_h)\)-minimizer of \( \Phi \) in \((B^{x, r}, H^{x, r})\). By Lemma 2.1, there exists a minimizer \( F \) of \( \Phi \) in \((\mathbb{R}^n, H)\) such that

\[
\mathcal{H}_{\infty}^{n-3}(\Sigma_{B \cap \partial H}(F)) \geq c(n),
\]

against \( \Phi \in \mathcal{E}_\ast^\ast(\lambda) \). \( \Box \)

We now come to the proof of Lemma 1.8. Given \( \Phi_h \to \Phi \) and a minimizer \( E \) of \( \Phi \), we shall need to approximate \( E \) by minimizers of \( \Phi_h \). This will be done by minimizing \( \Phi_h \) plus a suitable lower order perturbation.

Definition 2.3. Given \( g \in L_{\text{loc}}^\infty(A) \) one says that \( E \) is a minimizer of \( \Phi + \int g \) on \((A, H)\) if \( E \subset H \) is a set of locally finite perimeter in \( A \), and

\[
(2.6) \quad \Phi(E; W \cap H) + \int_{W \cap H \cap E} g(x) \, dx \leq \Phi(F; W \cap H) + \int_{W \cap H \cap F} g(x) \, dx
\]

whenever \( F \subset H \) and \( E \Delta F \subset W \) with \( W \subset A \) open.

Note that if \( E \) is a minimizer of \( \Phi + \int g \) on \((A, H)\), then for every \( A' \subset A \) one has that \( E \) is a \((\Lambda, \infty)\)-minimizer of \( \Phi \) in \((A', H)\) with \( \Lambda = \|g\|_{L^\infty(A')} \). In particular, \( R_A(E) \) is always a \( C^1 \)-manifold with boundary. Moreover, by exploiting the Euler–Lagrange equation
We deduce (2.7) as

\[ |\mathbf{II}_E|^2 = |\mathbf{II}_E|^2 \mathcal{H}^{n-1} \mathcal{L} R_A (E), \]

compare with (1.7). The continuity of \( \mathbf{II}_E \) on \( R_A (E) \) guarantees that \( |\mathbf{II}_E|^2 \) is a Radon measure on \( A \setminus \Sigma_A (E) \).

**Lemma 2.4.** Let \( \{ \Phi_h \}_{h \in \mathbb{N}} \subset \mathcal{E}_* (\lambda) \) be given with \( \Phi_h \to \Phi \in C^0 (S^{n-1}) \) as \( h \to \infty \), let \( \{ g_h \}_{h \in \mathbb{N}} \subset \text{Lip} (\mathbb{R}^n) \) with \( \text{Lip} g_h \leq C \) and \( g_h \to g \) locally uniformly on \( \mathbb{R}^n \) as \( h \to \infty \), and let \( E_h \) (resp., \( E \)) be a minimizer of \( \Phi_h + \int g_h \) (resp., \( \Phi + \int g \)) on \( (A, H) \) with \( E_h \to E \) locally in \( A \) as \( h \to \infty \). Then,

\[ |\mathbf{II}_E|^2 (A^c) \leq \liminf_{h \to \infty} |\mathbf{II}_{E_h}|^2 (A^c) \]

for every open set \( A^c \subset A \).

**Proof.** The regularity, in particular [1, Lemma 3.4] theory ensures that if \( x \in R_{A \cap H} (E) \), then there exist \( h_x \in \mathbb{N} \), \( r_x > 0 \) and \( v_x \in S^{n-1} \) such that, if we set

\[
C_x = x + \{ y \in \mathbb{R}^n : |y \cdot v_x| < r_x, |y - (y \cdot v_x)v_x | < r_x \}, \]
\[
D_x = x + \{ y \in \mathbb{R}^n : y \cdot v_x = 0, |y - (y \cdot v_x)v_x | < r_x \}, \]

then \( C_x \subset A \cap H \) and there exist \( u_h, u \in C^{2,\alpha} (D_x) \) with \( u_h \to u \) in \( C^{2,\alpha} (D_x) \) as \( h \to \infty \) and

\[
C_x \cap \partial E = C_x \cap R_A (E) = \{ z + u(z)v_x : z \in D_x \}, \]
\[
C_x \cap \partial E_h = C_x \cap R_A (E_h) = \{ z + u_h(z)v_x : z \in D_x \} \]

for every \( h \geq h_x \). In particular, if \( \varphi \in C^0 (C_x) \), then, as \( h \to \infty \),

\[
\varphi(z, u_h) \sqrt{1 + |\nabla u_h|^2} |\mathbf{II}_{E_h}(z + u_h v_x)|^2 \to \varphi(z, u) \sqrt{1 + |\nabla u|^2} |\mathbf{II}_E(z + u v_x)|^2 \]

for every \( z \in D_x \), and, actually, locally uniformly on \( z \in D_x \). Thus, by the area formula for graphs one finds

\[
\int_{\mathbb{R}^n} \varphi \; d|\mathbf{II}_E|^2 = \lim_{h \to \infty} \int_{\mathbb{R}^n} \varphi \; d|\mathbf{II}_{E_h}|^2 \quad \text{for all } \varphi \in C^0 (C_x). \]

By a covering argument, we conclude that

\[ (2.8) \quad \int_{\mathbb{R}^n} \varphi \; d|\mathbf{II}_E|^2 = \lim_{h \to \infty} \int_{\mathbb{R}^n} \varphi \; d|\mathbf{II}_{E_h}|^2 \quad \text{for all } \varphi \in C^0 ((A \cap H) \setminus \Sigma_A (E)). \]

If now \( A^c \subset A \) is open, then by (2.8),

\[
|\mathbf{II}_E|^2 ((A^c \cap H) \setminus \Sigma_A (E)) \leq \liminf_{h \to \infty} |\mathbf{II}_{E_h}|^2 ((A^c \cap H) \setminus \Sigma_A (E)) \leq \liminf_{h \to \infty} |\mathbf{II}_{E_h}|^2 (A^c). \]

We deduce (2.7) as \( |\mathbf{II}_E|^2 (A \cap \partial H) = 0 \) and \( |\mathbf{II}_E|^2 (\Sigma_A (E)) = 0 \).
We now exploit a second variation argument to show that the $\mathcal{H}^{n-3}$-negligibility of singular sets implies uniform $L^2$-estimates on second fundamental forms.

Lemma 2.5. Let $\Phi \in \mathcal{E}_{**}(\lambda)$, $g \in C^2(\mathbb{R}^n)$, $A$ be a bounded open set, and $E$ be a minimizer of $\Phi + \int g$ on $(A, H)$. Then,

$$\frac{||II_E||^2(B_{x,r})}{r^{n-3}} \leq C_0(n, \lambda, \text{Lip}(g)) \quad \text{for all } B_{x,2r} \subset A.$$

Proof. By Lemma A.5 in the appendix, there exists a constant $C = C(n, \lambda, \text{Lip}(g))$ such that

$$(2.9) \quad \int_{R_A(E)} ||II_E||^2 \zeta^2 \, d\mathcal{H}^{n-1} \leq C \int_{R_A(E)} |\nabla \zeta|^2 + \zeta^2 \, d\mathcal{H}^{n-1}$$

whenever $\zeta \in C^1_c(A)$ with $\text{spt} \, \zeta \cap \Sigma_A(E) = \emptyset$. We shall now exploit $\Phi \in \mathcal{E}_{**}(\lambda)$ to deduce that (2.9) holds true for every $\zeta \in C^1_c(A)$. To this end let us fix such a $\zeta \in C^1_c(A)$, and let us assume without loss of generality that $|\zeta| \leq 1$ on $\mathbb{R}^n$. Since $E$ is a $(\Lambda, \infty)$-minimizer of $\Phi$ in $(A, H)$, by Lemma 2.2 and by (1.2) one has $\mathcal{H}^{n-3}(\Sigma_A(E)) = 0$. In particular, given $\varepsilon > 0$ we can find a countable cover $\{F_k\}_{k \in \mathbb{N}}$ of $\Sigma_A(E)$ such that

$$(2.10) \quad \text{diam}(F_k) < \varepsilon_k, \quad \sum_{k \in \mathbb{N}} \varepsilon_k^{n-3} < \varepsilon.$$

By (2.10), for every $k \in \mathbb{N}$ we choose $x_k \in F_k$ so that $F_k \subset B_{x_k,2\varepsilon_k}$. Since $\{B_{x_k,2\varepsilon_k}\}_{k \in \mathbb{N}}$ is an open covering of $\Sigma_A(E)$, by compactness $\{B_{x_k,2\varepsilon_k}\}_{k=1}^N$ is an open covering of $\Sigma_A(E) \cap \text{spt} \, \zeta$ for some $N \in \mathbb{N}$, and thus of $I_{\delta}(\Sigma_A(E) \cap \text{spt} \, \zeta)$ for some $\delta > 0$ such that $\delta \to 0^+$ as $\varepsilon \to 0^+$. Correspondingly we consider $\psi_k \in C^1_c(B_{x_k,3\varepsilon_k} ; [0, 1])$ such that

$$(2.11) \quad \psi_k = 1 \quad \text{on } B_{x_k,2\varepsilon_k}, \quad |\nabla \psi_k| \leq \frac{2}{\varepsilon_k},$$

and set $\psi = \max\{\psi_k : 1 \leq k \leq N\}$. In this way,

$$\psi = 1 \quad \text{on } I_{\delta}(\Sigma_A(E) \cap \text{spt} \, \zeta).$$

This implies that $\zeta_0 = (1 - \psi) \zeta$ is a Lipschitz function with $\text{spt} \, \zeta_0 \cap \Sigma_A(E) = \emptyset$. By approximation, we can apply (2.9) to $\zeta_0$ in order to find

$$(2.12) \quad \int_{R_A(E) \setminus I_{\delta}(\Sigma_A(E))} ||II_E||^2 \zeta^2 \, d\mathcal{H}^{n-1} \leq C \int_{R_A(E)} |\nabla \zeta|^2 + |\nabla \psi|^2 + \zeta^2 \, d\mathcal{H}^{n-1}$$

with $C = C(n, \lambda, \text{Lip}(g))$. By the second conditions in (2.10) and (2.11), we easily find

$$\int_{R_A(E)} |\nabla \psi|^2 \leq \sum_{k=1}^N \int_{R_A(E) \cap B_{x_k,3\varepsilon_k}} |\nabla \psi_k|^2 \leq 4 \sum_{k=1}^N \frac{P(E;B_{x_k,3\varepsilon_k})}{\varepsilon_k^2} \leq C \sum_{k \in \mathbb{N}} \varepsilon_k^{n-3} < C \varepsilon,$$

where we have used the upper density estimate

$$P(E;B_{x,r}) \leq C(n, \lambda) r^{n-1},$$
We start with the following lemma. By plugging this last estimate into (2.12), and then letting \( \varepsilon \to 0^+ \), we conclude as desired that (2.9) holds for every \( \zeta \in C^1_c(A) \). Finally, for \( B_{x,2r} \subseteq A \) and \( \zeta \in C^1_c(B_{x,2r}) \) with \( \zeta = 1 \) on \( B_{x,r} \) and \( \left| \nabla \zeta \right| \leq C/r \), equation (2.9) gives

\[
\left| \Pi_E \right|^2(B_{x,r}) \leq C \frac{P(E; B_{x,r})}{r^2} \leq Cr^{n-3},
\]
	hanks{We finally prove that if \( \left| \Pi_E \right|^2 \) is a finite measure, the singular set is \( \mathcal{H}^{n-3} \)-negligible. We start with the following lemma.}

**Lemma 2.6.** There exists a constant \( \delta = \delta(n, \lambda) \) such that if \( \Phi \in \mathcal{E}_\ast(\lambda) \), \( E \) is a minimizer of \( \Phi \) in \( (B, H) \), \( 0 \in \partial H \), and

\[
\left| \Pi_E \right|^2(B) \leq \delta,
\]

then \( 0 \in R_E(B) \).

**Proof.** We argue by contradiction. Let \( \{\Phi_h\}_{h \in \mathbb{N}} \subset \mathcal{E}_\ast(\lambda) \) be such that for each \( h \in \mathbb{N} \) there exists a minimizer \( E_h \) of \( \Phi_h \) in \( (B, H) \) with \( \left| \Pi_{E_h} \right|^2(B) \to 0 \) as \( h \to \infty \) and \( 0 \in \Sigma_B(E_h) \) for every \( h \in \mathbb{N} \). By the compactness of \( \mathcal{E}_\ast(\lambda) \) and Lemma 2.1, there exist \( \Phi \in \mathcal{E}_\ast(\lambda) \) and \( E \) a minimizer of \( \Phi \) in \( (B, H) \) such that, up to subsequences, \( E_h \to E \) locally in \( B \) as \( h \to \infty \). Moreover, by (1.4), (1.5) and the continuity of the excess, \( 0 \in \Sigma_B(E) \). By (2.1), for every \( \varepsilon > 0 \) and \( r < 1 \) there exists an \( h_0 \) such that \( \Sigma_{B_r}(E_h) \subset \mathcal{I}_\varepsilon(\Sigma_{B_r}(E)) \) provided \( h \geq h_0 \). By Lemma 2.4,

\[
\left| \Pi_E \right|^2(B \setminus \text{cl}(I_\varepsilon(\Sigma_{B_r}(E)))) \leq \liminf_{h \to \infty} \left| \Pi_{E_h} \right|^2(B \setminus \text{cl}(I_\varepsilon(\Sigma_{B_r}(E))))
\]

\[
\leq \liminf_{h \to \infty} \left| \Pi_{E_h} \right|^2(B \setminus \text{cl}(\Sigma_{B_r}(E_h))) = 0.
\]

By the arbitrariness of \( \varepsilon \) and \( r \), \( \left| \Pi_E \right|^2(B) = 0 \). We now show that this last fact implies the existence of finitely many hyperplanes \( L_i \) such that

\[
(2.13) \quad M_{B_{1/2}}(E) \cap H = \bigcup_{i} L_i \cap B_{1/2} \cap H, \quad L_i \cap L_j \cap B_{1/2} \cap H = \emptyset, \quad i \neq j.
\]

Indeed, by \( \left| \Pi_E \right|^2(B) = 0 \) we have that \( R_B(E) \) is contained into the union of at most countably many hyperplanes \( L_i \). Let us set \( A_i = B \cap H \cap L_i \) and \( R_i = R_B \cap H(E) \cap L_i \). We claim that

\[
(2.14) \quad A_i \cap \partial L_i R_i \subset \Sigma_B \cap H(E),
\]

where \( \partial L_i R_i \) denotes the boundary of \( R_i \) as a subset of \( L_i \). Indeed, \( A_i \cap \partial L_i R_i \subset M_B(E) \cap H \), so that if (2.14) fails, then there exists an \( x \in A_i \cap \partial L_i R_i \) such that \( x \in \Sigma_{B_r}(E) \), contradicting (1.2). By the local \( C^1 \)-graphicality of \( R_B \cap H(E) \) at \( x \), we immediately see that \( x \) belongs to the interior of \( R_i \). By (2.14) and (1.2), we find that \( \mathcal{H}^{n-3}(A_i \cap \partial L_i R_i) = 0 \), thus that \( \mathcal{H}^{n-2}(A_i \cap \partial L_i R_i) = 0 \). This implies that the distributional derivative of \( 1_{R_i} \in L^{1,0}_\text{loc}(L_i) \) vanishes on the connected open set \( A_i \) in other words, since \( R_i \cap A_i \neq \emptyset \), it must be \( R_i = A_i \). By the upper density estimate [1, equation (2.47)], there are finitely many hyperplanes \( L_i \) such that \( L_i \cap B_{1/2} \neq \emptyset \). This proves (2.13). As \( 0 \in \Sigma_{B \cap \partial H}(E) \), there must be \( i \neq j \) such that \( 0 \in L_i \cap L_j \cap \partial H \); but then, by (2.13), \( L_i \cap L_j \subset \Sigma_{B \cap \partial H}(E) \), against (1.3).
Lemma 2.7. If $\Phi \in \mathcal{E}_*(\lambda)$, $E$ is a minimizer of $\Phi$ in $(B, H)$, and
\[ |\Pi_E|^2(B) < \infty, \]
then $\mathcal{H}^{n-3}(\Sigma_B(E)) = 0$.

Proof. By Lemma 2.6 and by scaling,
\[ |\Pi_E|^2(B_{\chi,r}) \geq \delta r^{n-3} \quad \text{for all } x \in \Sigma_B \cap \partial H(E), r < \text{dist}(x, \partial B). \tag{2.15} \]
We now prove that, if we fix $s \in (0, 1)$ and set $\Sigma_s = \Sigma_{B_s} \cap \partial H(E)$ for the sake of brevity, then
\[ \lim_{r \to 0^+} \frac{|I_r(\Sigma_s)|}{r^3} = 0. \tag{2.16} \]
Let $r < 1 - s$ and let $\{x_i\}_{i=1}^{N(r)} \subset \Sigma_s$ be such that
\[ |x_i - x_j| > 2r \quad \text{for every } i \neq j, \quad \inf_i |x - x_i| \leq 2r \quad \text{for every } x \in \Sigma_s, \]
i.e. $\{x_i\}_{i=1}^{N(r)}$ is a maximal $2r$-net on $\Sigma_s$. In this way, $\{B_{x_i,r}\}_{i=1}^{N(r)}$ is a finite disjoint family of balls to which we can apply (2.15), and such that $I_r(\Sigma_s)$ is covered by $B_{x_i,3r}$. Hence,
\[ |I_r(\Sigma_s)| \leq 3^n N(r) r^n \leq \frac{3^n r^3}{\delta} \sum_{i=1}^{N(r)} |\Pi_E|^2(B_{x_i,r}) \leq \frac{3^n r^3}{\delta} |\Pi_E|^2(I_r(\Sigma_s)). \]
Since, by assumption, $|\Pi_E|^2(B) < \infty$, we have
\[ \lim_{r \to 0^+} |\Pi_E|^2(I_r(\Sigma_s)) = |\Pi_E|^2(\Sigma_s) = 0, \]
where in the last identity we have used the fact that $|\Pi_E|^2$ is concentrated on $R_B(E)$. This proves (2.16), which immediately implies $\mathcal{H}^{n-3}(\Sigma_s) = 0$ (note that this could be directly inferred by the previous proof, however (2.16) provides a slightly stronger information). By the arbitrariness of $s$, we complete the proof. \hfill \Box

Proof of Proposition 1.8. Consider a sequence $\{\Phi_h\}_{h \in N} \subset \mathcal{E}_*(\lambda)$ such that $\Phi_h \to \Phi$ in $C^0(S^{n-1})$ as $h \to \infty$ for some $\Phi \in \mathcal{E}_*(\lambda)$, and let $E$ be a minimizer of $\Phi$ in $(B, H)$. We fix $s \in (0, 1)$ and consider the variational problems
\[ \inf \left\{ \Phi_h(F; H \cap B) + \int_F g_h(x) \, dx : F \subset H, F \Delta E \subset B_s \right\}. \tag{2.17} \]
where we have set
\[ g_h = \varphi_h \circ (\text{dist}(\cdot, E) - \text{dist}(\cdot, E^c)) \]
for a sequence of smooth mollifiers $\{\varphi_h\}_h$; in particular, $g_h \in C^\infty(\mathbb{R}^n)$ with $\text{Lip} g_h \leq 1$ for every $h \in N$. Let now $E_h$ be a minimizer in (2.17): we claim that $E_h \to E$ in $B$ as $h \to \infty$. Indeed, by [1, Theorem 2.9] there exists a set $G \subset H$ such that, up to subsequences, $E_h \to G$ locally in $B_s$ as $h \to \infty$. By comparing $E_h$ with $E$ in (2.17), by lower semicontinuity (see [1, equation (2.64)]), and setting $g = \text{dist}(\cdot, E) - \text{dist}(\cdot, E^c)$, one has
\[ \Phi(G; H \cap B) + \int_G g \leq \liminf_{h \to \infty} \Phi_h(E_h; H \cap B) + \int_{E_h} g_h \leq \Phi(E; H \cap B_s) + \int_E g. \]
By minimality of $E$ (note that $G \Delta E \subset B_s \subset B$),

$$\Phi(E; H \cap B) \leq \Phi(G; H \cap B),$$

and thus

$$0 \geq \int_G g - \int_E g = \int_{G \setminus E} \text{dist}(x, E) \, dx + \int_{E \setminus G} \text{dist}(x, E^c) \, dx.$$ 

In particular, we have $|E \Delta G| = 0$, that is, $E_h \to E$ locally in $B_s$, thus in $B$ by $E_h \Delta E \subset B_s$, as $h \to \infty$.

Since $E_h$ is a minimizer for $\Phi_h + \int g_h$ on $(B_s, H)$, by Lemma 2.5 (and Lip $g_h \leq 1$) we find

$$\frac{||\Pi E_h||^2(B_{x,r})}{r^{n-3}} \leq C(n, \lambda) \quad \text{for all } B_{x,2r} \subset B_s.$$ 

Hence, by Lemma 2.4, one finds

$$||\Pi E||^2(B_{x,r}) < \infty \quad \text{for all } B_{x,2r} \subset B_s.$$ 

By Lemma 2.7, we have $\mathcal{H}^{n-3}(\Sigma_{B_{x,r} \cap \partial H}(E)) = 0$ for every $B_{x,2r} \subset B_s$. By covering and by the arbitrariness of $s$, we find $\mathcal{H}^{n-3}(\Sigma_{B \cap \partial H}(E)) = 0$. This shows that $\Phi \in \mathcal{E}_*(\lambda)$. $\square$

As explained in the introduction, Propositions 1.7 and 1.8 imply Theorem 1.6. We finally deduce Theorem 1.5 from this last result.

**Proof of Theorem 1.5.** The proof is essentially the same as that of Lemma 2.2. Let us briefly sketch it: assume by contradiction that there exist constants $\lambda \geq 1$, $\ell \geq 0$, $\Lambda \geq 0$, $r_0 > 0$, an open set $A$, $\Phi \in \mathcal{E}(A \cap H, \lambda, \ell)$ and $E$ a $(\Lambda, r_0)$-minimizer of $\Phi$ in $(A, H)$ such that

$$\mathcal{H}^{n-3}(\Sigma_{A \cap \partial H}(E)) > 0.$$ 

According to (2.4) we can find $x_0 \in \Sigma_{A \cap \partial H}(E)$ and $r_h \to 0$ as $h \to \infty$ such that

$$(2.18) \quad \mathcal{H}_\infty^{n-3}(\Sigma_{A \cap \partial H}(E) \cap B_{x_0,r_h}) > c(n)r_h^{n-3}.$$ 

Let us set $F_h = E^{x_0,r_h}$ and notice that $F_h$ are $(\Lambda r_h, r_0/r_h)$-minimizer of $\Phi_h$ in $(A^{x_0,r_h}, H)$ where $\Phi_h(x, \nu) = \Phi(x_0 + r_h x, \nu) \in \mathcal{E}(A^{x_0,r_h} \cap H, \lambda, \ell r_h)$. According to Lemma 2.1 and arguing as in the proof of Lemma 2.2 one finds $E_\infty$ a minimizer of $\Phi_\infty$ in $(\mathbb{R}^n, H)$ where $\Phi_\infty(\nu) = \Phi(x_0, \nu) \in \mathcal{E}_*(\lambda)$. However, by (2.18), (2.2) and (2.3), we find

$$\mathcal{H}^{n-3}(\Sigma_{B \cap \partial H}(E_\infty)) > 0,$$

a contradiction to Theorem 1.6. $\square$

**A. First and second variations of anisotropic functionals**

Lemma 2.5 relies on the second variation formulas for anisotropic functionals. For the reader’s convenience, and since this kind of computation is not so easily accessible in the literature, we include a derivation of these formulas.
We consider an open set with smooth boundary $\Omega$ in $\mathbb{R}^n$, a bounded open set $A$ with $A \cap \Omega \neq \emptyset$, and a set $E \subset \Omega$ of finite perimeter in $A$. Given $\Phi \in \mathcal{E}_*(\lambda)$ and $g \in C^2(\mathbb{R}^n)$, we compute the first and second variation of

$$(\Phi + \int g)(f_t(E)) = \int_{A \cap \Omega \cap \partial^* f_t(E)} \Phi(v_{f_t(E)}) \, d\mathcal{H}^{n-1} + \int_{A \cap f_t(E)} g,$$

where $\{f_t\}_{t \leq \varepsilon_0}$ is such that:

(i) $(x, t) \mapsto f_t(x)$ of class $C^1(\Omega \times (-\varepsilon_0, \varepsilon_0); \mathbb{R}^n)$ with $f_0 = \text{Id}$, $f_t(\Omega) = \Omega$ for every $|t| < \varepsilon_0$, and $t \in (-\varepsilon_0, \varepsilon_0)$ \mapsto $f_t(x)$ of class $C^3((-\varepsilon_0, \varepsilon_0); \mathbb{R}^n)$ uniformly with respect to $x \in \Omega$.

(ii) $\text{spt}(f_t - \text{Id}) \subset A$.

These conditions imply that

$$(A.1) \quad \frac{d}{dt} f_t(x) \cdot v_{f_t(x)} = 0, \quad x \in \partial \Omega \cap A, \quad |t| < \varepsilon_0.$$ 

We also notice that, if we define $T, Z \in C^1_c(\Omega; \mathbb{R}^n)$ by setting

$$(A.2) \quad T(x) = \frac{d}{dt} \bigg|_{t=0} f_t(x) \quad \text{and} \quad Z(x) = \frac{d^2}{dt^2} \bigg|_{t=0} f_t(x),$$

then we have, uniformly on $x \in \mathbb{R}^n$ as $t \to 0^+$,

$$(A.3) \quad f_t = \text{Id} + tT + \frac{t^2}{2} Z + O(t^3).$$

By (A.1), we find

$$(A.4) \quad T \cdot v_{f_t} = 0 \quad \text{for all} \quad x \in \partial \Omega.$$ 

By differentiating (A.1) with respect to $t$, we obtain that

$$(A.5) \quad Z \cdot v_{f_t} = -T \cdot \Pi_{f_t}[T] \quad \text{for all} \quad x \in \partial \Omega,$$

where $\Pi_{f_t} : T_x \partial \Omega \to T_x \partial f_t(\Omega)$ is the second fundamental form of $\partial \Omega$. (Note that $T(x)$ is a tangent vector to $\partial \Omega$ at $x \in \partial \Omega$ exactly by (A.4).) We now recall two basic facts. Lemma A.1 is a consequence of the classical area formula, see for example [4, Proposition 17.1], while Lemma A.2 is a standard Taylor expansion, see [4, Lemma 17.4].

**Lemma A.1.** If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz diffeomorphism with $\text{det}(\nabla f) > 0$ on $\mathbb{R}^n$, then $f(E)$ is a set of finite perimeter in $f(A)$, with $f(\partial^* E) = \mathcal{H}^{n-1} \partial^* (f(E))$ and

$$v_{f(E)}(f(x)) = \frac{\text{cof}(\nabla f(x))[v_E(x)]}{\text{cof}(\nabla f(x))[v_E(x)]} \quad \text{for} \quad \mathcal{H}^{n-1}\text{-a.e.} \quad x \in \partial^* f(E),$$

where for any invertible linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ one defines $\text{cof} L = (\text{det} L)(L^{-1})^*$. Moreover, for every $G \subset A$, one has

$$(A.6) \quad \int_{f(G \cap \partial^* E)} \Phi(v_{f(E)}(y)) \, d\mathcal{H}^{n-1}(y) = \int_{G \cap \partial^* E} \Phi(\text{cof}(\nabla f(x))[v_E(x)]) \, d\mathcal{H}^{n-1}(x).$$
Lemma A.2. If $X, Y : \mathbb{R}^n \to \mathbb{R}^n$ are linear maps, then

$$\det \left( \text{Id} + tX + \frac{t^2}{2} Y + O(t^3) \right) = 1 + t \text{tr} X \right.$$ 

$$+ \frac{t^2}{2} \left( (\text{tr} X)^2 - \text{tr}(X^2) + \text{tr} Y \right) + O(t^3),$$

and thus

$$\text{cof} \left( \text{Id} + tX + \frac{t^2}{2} Y + O(t^3) \right) = \text{Id} + t(\text{tr} X)\text{Id} - X^* \right.$$ 

$$+ \frac{t^2}{2} \left[ (\text{tr} X)^2 - \text{tr}(X^2) + \text{tr}(Y) \right] \text{Id} + 2(X^*)^2 \right.$$ 

$$- 2 \text{tr}(X) X^* - Y^* \right] + O(t^3).$$

We are now ready to compute the first and second variation of $\Phi + \int g$.

Lemma A.3. If $g \in C^2(A)$, then

$$\frac{d}{dt} \left|_{t=0} \int_{A \cap f_t(E)} g = \int_{A \cap \Omega \cap \partial^* E} g(T \cdot v_E) d\mathcal{H}^{n-1}, \right.$$ 

and

$$\frac{d^2}{dt^2} \left|_{t=0} \int_{A \cap f_t(E)} g = \int_{A \cap \Omega \cap \partial^* E} g(Z \cdot v_E) d\mathcal{H}^{n-1} \right.$$ 

$$+ \int_{A \cap \Omega \cap \partial^* E} \text{div}(g T)(T \cdot v_E) - g(\nabla T[T] \cdot v_E) d\mathcal{H}^{n-1}. \right.$$ 

Proof. Step one. We notice the validity of the following formula: if $S \in C^1_c(A; \mathbb{R}^n)$ and $E \subset \Omega$, then

$$\int_{A \cap E} g[(\text{div} S)^2 - \text{tr}(\nabla S)^2] + 2 \text{div} S \nabla g \cdot S + \nabla^2 g[S] \cdot S \right.$$ 

$$= \int_{\Omega \cap A \cap \partial^* E} \text{div}(g S)(S \cdot v_E) - g \nabla S[S] \cdot v_E d\mathcal{H}^{n-1} \right.$$ 

$$+ \int_{A \cap \Omega \cap \partial^* E} \text{div}(g S)(S \cdot v_\Omega) - g \nabla S[S] \cdot v_\Omega d\mathcal{H}^{n-1}. \right.$$ 

Indeed, if $S \in C^2_c(A; \mathbb{R}^n)$, then the assertion follows by the Divergence Theorem and by the identity

$$g[(\text{div} S)^2 - \text{tr}(\nabla S)^2] + 2 \text{div} S \nabla g \cdot S + \nabla^2 g[S] \cdot S = \text{div}(g S)S - \text{div}(g \nabla S[S]).$$

The case when $S \in C^1_c(A; \mathbb{R}^n)$ is then obtained by approximation.
Step two. Since \( f_t(A) = A \), we find \( f_t(E) \cap A = f_t(E \cap A) \). Hence by the area formula,

\[
\int_{A \cap f_t(E)} g(y) \, dy = \int_{A \cap E} g(f_t(x)) \det \nabla f_t(x) \, dx.
\]

By (A.3), (A.7) and the Taylor expansion of \( g \), we get

\[
\int_{A \cap f_t(E)} g(y) \, dy = \int_{A \cap E} g + t \int_{A \cap E} \nabla g \cdot T + g \, \text{div} \, T \\
+ \frac{t^2}{2} \int_{A \cap E} \left[ (g \, \text{div} \, Z + (\text{div} \, T)^2 - \text{tr} \,(\nabla T)^2) \\
+ 2 \, \text{div} \, T \nabla g \cdot T + \nabla^2 g[T] \cdot T + \nabla g \cdot Z \right] + O(t^3).
\]

Inasmuch, \( \text{div}(g \, T) = \nabla g \cdot T + g \, \text{div} \, T \) and \( \text{div}(g \, Z) = \nabla g \cdot Z + g \, \text{div} \, Z \), by step one and by (A.4), one finds (A.8) and

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \int_{A \cap f_t(E)} g = \int_{A \cap \Omega \cap \partial^* E} g(Z \cdot v_E) \, d\mathcal{H}^{n-1} \\
+ \int_{A \cap \Omega \cap \partial^* E} \text{div}(g \, T) \, (T \cdot v_E) \, d\mathcal{H}^{n-1} \\
- \int_{A \cap \Omega \cap \partial^* E} g(\nabla T[T] \cdot v_E) \, d\mathcal{H}^{n-1} \\
+ \int_{A \cap \Omega \cap \partial^* E} g(Z \cdot v_\Omega - \nabla T[T] \cdot v_\Omega) \, d\mathcal{H}^{n-1}.
\]

We now complete the proof of (A.9) by showing that

\( \nabla T[T] \cdot v_\Omega = Z \cdot v_\Omega \).

Indeed, by differentiating (A.4) along \( T \) one finds \( 0 = \nabla T[T] \cdot v_\Omega + T \cdot \Pi_\Omega[T] \), and then we conclude by (A.5). \( \Box \)

Lemma A.4. We have

\[\begin{align*}
(A.10) \quad & \frac{d}{dt} \bigg|_{t=0} \int_{A \cap \Omega \cap \partial^* f_t(E)} \Phi(v_{f_t(E)}) \, d\mathcal{H}^{n-1} \\
& = \int_{A \cap \Omega \cap \partial^* f_t(E)} \Phi(v_E) \, \text{div} \, T - \nabla T^*[v_E] \cdot \nabla \Phi(v_E) \, d\mathcal{H}^{n-1}.
\end{align*}\]

and

\[\begin{align*}
(A.11) \quad & \frac{d^2}{dt^2} \bigg|_{t=0} \int_{A \cap \Omega \cap \partial^* f_t(E)} \Phi(v_{f_t(E)}) \, d\mathcal{H}^{n-1} \\
& = \int_{A \cap \Omega \cap \partial^* f_t(E)} \Phi(v_E) \, \text{div} \, Z - \nabla Z^*[v_E] \cdot \nabla \Phi(v_E) \, d\mathcal{H}^{n-1} \\
& + \int_{A \cap \Omega \cap \partial^* f_t(E)} \Phi(v_E) \{(\text{div} \, T)^2 - \text{tr} \,(\nabla T)^2\} \, d\mathcal{H}^{n-1} \\
& + 2 \int_{A \cap \Omega \cap \partial^* f_t(E)} ((\nabla T)^2[v_E] \cdot \nabla \Phi(v_E) \\
& \quad - \text{div} \, T \nabla T^*[v_E] \cdot \nabla \Phi(v_E)) \, d\mathcal{H}^{n-1} \\
& + \int_{A \cap \Omega \cap \partial^* f_t(E)} \nabla^2 \Phi(v_E)[\nabla T^*[v_E]] \cdot \nabla T^*[v_E] \, d\mathcal{H}^{n-1}.
\end{align*}\]
for every function

\[ a \text{ constant } C \]

a family of admissible variations for \( j \in E \) half-space and whenever

\[ T \]

We set

\[ \tilde{f} \]

the tensor \( II \)

Note that, by one-homogeneity of \( \tilde{f} \) and by \( 2 \), we define \( II \)

where we have also used \( \Phi(v_E) = \nabla \Phi(v_E) \cdot v_E \) and \( \nabla^2 \Phi(v_E)[v_E] = 0 \). By equation (A.6) and by \( f_t(A) = A \), we find (A.10) and (A.11).

\[ \Box \]

We now come to the lemma that was used in the proof of Lemma 2.5. In the following we define \( \Pi_E^0 \) by setting

\[ \Pi_E^0(x) = \nabla^2 \Phi(v_E(x)) \Pi_E(x) \quad \text{for all } x \in R_A(E). \]

Note that, by one-homogeneity of \( \Phi \), \( \nabla^2 \Phi(v_E)[v_E] = 0 \); therefore, by symmetry of \( \nabla^2 \Phi(v_E) \), the tensor \( \Pi_E^0(x) \) is a well-defined operator from \( T_x R_A(E) \) into itself.

**Lemma A.5.** Let \( \Phi \in \mathcal{E}_* (\lambda) \), \( g \in C^2(\mathbb{R}^n) \), \( A \) be a bounded open set, \( H \) be an open half-space and \( E \) be a minimizer of \( \Phi + \int g \) on \((A, H)\). Then

\[ (A.12) \quad \int_{R_A(E)} \zeta^2 \Phi(v_E) \text{tr}[(\Pi_E^0)^2] \, d\mathcal{H}^{n-1} \]

\[ \leq \int_{R_A(E)} \Phi(v_E)^2 \nabla^2 \Phi(v_E)[\nabla \zeta] \cdot \nabla \zeta + \zeta^2 \Phi(v_E)(\nabla g \cdot \nabla \Phi(v_E)) \, d\mathcal{H}^{n-1} \]

for every function \( \zeta \in C^1_c(A) \) with the property that \( \text{spt } \zeta \cap \Sigma_A(E) = \emptyset \). Moreover, there exists a constant \( C = C(n, \lambda, \text{Lip}(g)) \) such that

\[ (A.13) \quad \int_{R_A(E)} |\Pi_E^0|^2 \zeta^2 \, d\mathcal{H}^{n-1} \leq C \int_{R_A(E)} |\nabla \zeta|^2 + \zeta^2 \, d\mathcal{H}^{n-1} \]

whenever \( \zeta \in C^1_c(A) \) with \( \text{spt } \zeta \cap \Sigma_A(E) = \emptyset \).

**Proof.** As proved in [1, Section 2.4] we have

\[ \nabla \Phi(v_E(x)) \cdot v_H = 0 \quad \text{for all } x \in R_A(E) \cap \partial H. \]

If \( \zeta \in C^1_c(A \setminus \Sigma_A(E)) \), then there exists an \( N \in C^1(\mathbb{R}^n; \mathbb{R}^n) \) such that

\[ (A.14) \quad N = v_E \quad \text{on } R_A(E) \cap \text{spt } \zeta, \]

\[ (A.15) \quad \nabla \Phi(N) \cdot v_H = 0 \quad \text{on } R_A(E) \cap \partial H \cap \text{spt } \zeta. \]

We set \( T = \zeta \nabla \Phi(N) \in C^1_c(A; \mathbb{R}^n) \) and we note that, by (A.15), \( f_t(x) = x + tT(x) \) defines a family of admissible variations for \(|t| \leq \varepsilon_0 \) and \( \varepsilon_0 \) suitably small. Since \( f_t \) is affine in \( t \),
by (A.2), one has \( Z = 0 \). In particular, by Lemma A.1, Lemma A.2, and by minimality of \( E \),

\[
(A.16) \quad 0 = \frac{d}{dt} \bigg|_{t=0} (\Phi + f g)(f_t(E))
\]

\[
= \int_{A \cap H \cap \partial^* E} g (T \cdot v_E) + \Phi \div (\nabla T)^* [v_E] \cdot \nabla \Phi \, d\mathcal{H}^{n-1},
\]

\[
0 \leq \frac{d^2}{dt^2} \bigg|_{t=0} (\Phi + f g)(f_t(E))
\]

\[
= \int_{A \cap H \cap \partial^* E} \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \, d\mathcal{H}^{n-1},
\]

where, setting for simplicity \( \Phi = \Phi(v_E), \nabla \Phi = \nabla \Phi(v_E), \) and \( \nabla^2 \Phi = \nabla^2 \Phi(v_E) \), one has

\[
\Gamma_1 = \text{div}(g T) (T \cdot v_E) - g T [T] \cdot v_E,
\]

\[
\Gamma_2 = (\text{div} T)^2 - \text{tr}((\nabla T)^2) \Phi,
\]

\[
\Gamma_3 = 2((\nabla T)^*)^2 [v_E] \cdot \nabla \Phi - \text{div} T \nabla T^* [v_E] \cdot \nabla \Phi,
\]

\[
\Gamma_4 = \nabla^2 \Phi [\nabla T^* [v_E]] \cdot \nabla T^* [v_E].
\]

We start by noticing that (A.14) gives

\[
\nabla N(x) = \Pi_E(x) + a(x) \otimes v_E(x) \quad \text{for all } x \in R_A(E) \cap \text{spt } \zeta,
\]

where \( \Pi_E(x) \) is extended to be zero on \((T_x R_A(E))^\perp\) and \( a : R_A(E) \to \mathbb{R}^n \) is a continuous vector field. Hence

\[
\nabla T = \nabla \Phi \otimes \nabla \zeta + \zeta \Pi_E^\Phi + \zeta \nabla^2 \Phi[a] \otimes v_E \quad \text{on } R_A(E).
\]

By \( \nabla^2 \Phi[v_E] = 0 \) and the symmetry of \( \nabla^2 \Phi \), one finds \( \text{tr}(\nabla^2 \Phi[a] \otimes v_E) = 0 \), so that

\[
(A.17) \quad \div T = \nabla \Phi \cdot \nabla \zeta + \zeta H_E^\Phi \quad \text{on } R_A(E),
\]

where we have set

\[
H_E^\Phi = \text{tr}(\Pi_E^\Phi) = \text{tr}(\nabla^2 \Phi \Pi_E).
\]

Moreover, by \( \nabla \Phi \cdot v_E = \Phi \) and again by \( \nabla^2 \Phi[v_E] = 0 \), we find

\[
(\nabla T)^* [v_E] = \Phi \nabla \zeta \quad \text{and } T \cdot v_E = \zeta \Phi,
\]

so that (A.16) gives

\[
0 = \int_{A \cap H \cap \partial^* E} (g + H_E^\Phi) \zeta \, d\mathcal{H}^{n-1}.
\]

The validity of this condition for every \( \zeta \in C_c^1 (A \setminus \Sigma_A(E)) \) gives the well-know stationarity condition

\[
(A.18) \quad H_E^\Phi + g = 0 \quad \text{for all } x \in R_A(E).
\]

We now compute \( \Gamma_1 \). By \( \nabla \Phi \cdot v_E = \Phi \), we find

\[
\nabla T[T] = \zeta (\nabla \zeta \cdot \nabla \Phi) \nabla \Phi + \zeta^2 \Pi_E^\Phi [\nabla \nabla \Phi] + \zeta^2 \Phi \nabla^2 \Phi[a],
\]

so that, by \( \Pi_E^\Phi [\nabla \Phi] \cdot v_E = 0 \) and by \( \nabla^2 \Phi[a] \cdot v_E = 0 \) (which follow by the symmetry of \( \nabla^2 \Phi \) and by \( \nabla^2 \Phi[v] = 0 \)), we find

\[
\nabla T[T] \cdot v_E = \zeta \Phi (\nabla \zeta \cdot \nabla \Phi).
\]
By (A.17), (A.18) and a simple computation, one gets

\[ \Gamma_1 = ((\nabla \Phi \cdot \nabla g) + g \Phi^2 \xi^2 \Phi = ((\nabla \Phi \cdot \nabla g) - (H_E^\Phi)^2) \xi^2 \Phi. \]

We now start computing \( \Gamma_2 \). By (A.17), we have

\[
(\text{div } T)^2 = (\nabla \Phi \cdot \nabla \xi)^2 + \xi^2 (H_E^\Phi)^2 + 2 \xi H_E^\Phi (\nabla \Phi \cdot \nabla \xi);
\]

at the same time, writing \( \nabla T = X + Y \) where \( X = \nabla \Phi \otimes \nabla \xi + \xi \Pi_E^\Phi \) and \( Y = \xi \nabla^2 \Phi[a] \otimes \nu_E \), and noticing that \( Y^2 = 0 \), while

\[
\text{tr}(Y^2) = \text{tr}(X Y)
= \text{tr}(\xi (\nabla \xi \cdot \nabla^2 \Phi[a]) \nabla \Phi \otimes \nu_E + \xi^2 \Pi_E^\Phi \nabla^2 \Phi[a] \otimes \nu_E)
= (\nabla \xi \cdot \nabla \Phi) \nabla \Phi \otimes \nu_E + \xi^2 \Pi_E^\Phi \nabla^2 \Phi[a] \otimes \nu_E
= (\nabla \xi \cdot \nabla \Phi) \nabla \Phi \otimes \nu_E + \xi^2 \Pi_E^\Phi \nabla^2 \Phi[a] \otimes \nu_E
\]

we find that,

\[
\text{tr}((\nabla T)^2) = (\nabla \xi \cdot \nabla \Phi)^2 + \xi^2 \text{tr}((\Pi_E^\Phi)^2) + 2 \xi (\nabla \xi \cdot \Pi_E^\Phi[\nabla \Phi]) + 2 (\nabla \xi \cdot \nabla^2 \Phi[a] \nabla \Phi).
\]

Hence,

\[
\Gamma_2 = \xi^2 (H_E^\Phi)^2 \Phi + 2 \xi (\nabla \xi \cdot \nabla \Phi) H_E^\Phi \Phi - \xi^2 \text{tr}((\Pi_E^\Phi)^2) \Phi
- 2 \xi (\nabla \xi \cdot \Pi_E^\Phi[\nabla \Phi]) \Phi - 2 (\nabla \xi \cdot \nabla^2 \Phi[a] \nabla \Phi) \Phi.
\]

We now compute \( \Gamma_3 \). By (A.17) and \( (\nabla T)^*[\nu_E] = \Phi \nabla \xi \), we find

\[
\text{div } T \nabla T^*[\nu_E] \cdot \nabla \Phi = (\nabla \xi \cdot \nabla \Phi)^2 \Phi + \xi H_E^\Phi (\nabla \xi \cdot \nabla \Phi) \Phi.
\]

At the same time, writing \( \nabla T = X + Y \) with \( X \) and \( Y \) as above, we find

\[
(X^* X^*)^2 = (\nabla \xi \cdot \nabla \Phi) \nabla \xi \otimes \nabla \Phi + \xi^2 \Pi_E^\Phi[\nabla \Phi] + (\Pi_E^\Phi)^*[\nabla \xi] \otimes \nabla \Phi.
\]

\[
Y^* X^* = \xi (\nabla \xi \cdot \nabla^2 \Phi[a]) \nu_E \otimes \nabla \Phi + \xi^2 (\nu_E \otimes \nabla^2 \Phi[a]) \Pi_E^\Phi.
\]

\[
X^* Y^* = \xi \Phi \nabla \xi \otimes \nabla \Phi[a].
\]

By taking into account the fact that \( (Y^*)^2 = 0 \) (as \( Y^2 = 0 \)) and by exploiting once more that \( \nabla^2 \Phi[\nu_E] = 0 \) and \( \Pi_E^\Phi[\nu_E] = 0 \), we find that

\[
[(\nabla T)^*[\nu_E] \nabla \Phi = (\nabla \xi \cdot \nabla \Phi)^2 \Phi + \xi (\nabla \xi \cdot \nabla^2 \Phi[a]) \Phi \nu_E,
\]

so that

\[
[(\nabla T)^*[\nu_E] \cdot \nabla \Phi = (\nabla \xi \cdot \nabla \Phi)^2 \Phi + \xi (\nabla \xi \cdot \nabla^2 \Phi[a]) \Phi + \xi (\nabla \xi \cdot \nabla^2 \Phi[a]) \Phi^2.
\]

In conclusion,

\[
\Gamma_3 = 2 (\xi \nabla \xi \cdot \Pi_E^\Phi[\nabla \Phi] \Phi + (\nabla \Phi \cdot \nabla^2 \Phi[a]) \Phi^2 - \xi H_E^\Phi (\nabla \xi \cdot \nabla \Phi) \Phi),
\]

so that

\[
\Gamma_1 + \Gamma_2 + \Gamma_3 = (\nabla \Phi \cdot \nabla g - \text{tr}((\Pi_E^\Phi)^2)) \xi^2 \Phi.
\]

On noticing that \( \Gamma_4 = \Phi^2 \nabla^2 \Phi \nabla \xi \cdot \nabla \xi \), we conclude the proof of (A.12). By (1.1), one has

\[
\nabla^2 \Phi \geq \lambda^{-1} \text{Id}_{T_x(R_A(E))} \quad \text{for all } x \in R_A(E),
\]

and thus

\[
\text{tr}((\Pi_E^\Phi)^2) \geq \lambda^{-2} |\Pi_E|^2.
\]

Hence, (A.12) implies (A.13).
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Eingegangen 22. Juli 2014, in revidierter Fassung 29. Juli 2014