On The Spezialschar Of Maass

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Abstract

Let $M_k^{(n)}$ be the space of Siegel modular forms of degree $n$ and even weight $k$. In this paper firstly a certain subspace $\text{Spez}(M_k^{(2n)})$ the Spezialschar of $M_k^{(2n)}$ is introduced. In the setting of the Siegel three-fold it is proven that this Spezialschar is the Maass Spezialschar. Secondly an embedding of $M_k^{(2)}$ into a direct sum $\oplus_{\nu=0}^{\lfloor \frac{k}{10} \rfloor} \text{Sym}^2 M_{k+2\nu}$ is given. This leads to a basic characterization of the Spezialschar property. The results of this paper are directly related to the non-vanishing of certain special values of L-functions related to the Gross-Prasad conjecture. This is illustrated by a significant example in the paper.

Introduction

Hans Maass introduced and applied in a series of papers [Ma79], [Ma79II] and [Ma79III] the concept of a Spezialschar to prove the Saito-Kurokawa conjecture [Za80]. Let $M_k^{(2)}$ be the space of Siegel modular forms of degree 2 and weight $k$. Let $A$ be the set of positive semidefinite half-integral matrices of degree 2. Hence $T \in A$ can be identified with the quadratic form $T = [n, r, m]$. A modular form $F \in M_k^{(2)}$ is in the Spezialschar if the Fourier coefficients $A(T)$ of $F$ satisfy the relation

$$A([n, r, m]) = \sum_{d|(n, r, m)} d^{k-1} A(\frac{nm}{d^2}, \frac{r}{d}, 1)$$

for all $T \in A$. The space of such special forms is nowadays called the Maass Spezialschar $M_{k, \text{Maass}}$.

The purpose of this paper is twofold. First we introduce the concept of the Spezialschar $\text{Spez}(M_k^{(2n)})$ for Siegel modular forms of even degree $2n$. This is done in terms of the Hecke algebra $\mathcal{H}^n$ attached to Siegel modular forms of degree $n$. Let us fix the embedding

$$Sp_n \times Sp_m \rightarrow Sp_{n+m} \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \times \left( \begin{array}{cc} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{array} \right) \mapsto \left( \begin{array}{cccc} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{array} \right).$$

(2)
Let $|k$ be the Petersson slash operator and let $\tilde{T}$ be the normalized Hecke operator $T \in \mathcal{H}^n$ (see (54)). Let $\rtimes_T = (\tilde{T} \times 1_{2n}) - (1_{2n} \times \tilde{T})$ and

$$\text{Spez} \left( M_k^{(2n)} \right) := \left\{ F \in M_k^{(2n)} \mid F|_k \rtimes_T = 0 \text{ for all } T \in \mathcal{H}^n \right\}. \quad (3)$$

Then we have

**Theorem 0.1** The Spezialschar introduced in this paper is the Maass Spezialschar in the case of the Siegel three-fold.

$$\text{Spez}(M_k^{(2)}) = M_k^{\text{Maass}}. \quad (4)$$

The second topic of this paper is the characterization of the space of Siegel modular forms of degree two and the corresponding Spezialschar in terms of Taylor coefficients and certain differential operators:

$$D_{k,2\nu} : M_k^{(2)} \longrightarrow M_k^{\text{Sym}} \oplus S_{k+2\nu} \oplus \ldots \oplus S_{k+2\lfloor \frac{k}{10} \rfloor}. \quad (5)$$

Here $\nu \in \mathbb{N}_0$ and $M_k^{\text{Sym}} = \text{Sym}^2(M_{k+2\nu})$. Before we summarize the main results we give an example which also serves as an application. Let $F_1, F_2, F_3$ be a Hecke eigenbasis of the space of Siegel cusp forms $S_2^{(20)}$ of weight 20 and degree 2. Let $F_1$ and $F_2$ generate the Maass Spezialschar. Let $f_1$ and $f_2$ be the normalized Hecke eigenbasis of $S_2^{(1)}$. Then we have:

$$D_{20,4} F_j = \alpha_j f_1 \otimes f_1 + \beta_j (f_1 \otimes f_2 + f_2 \otimes f_1) + \gamma_j f_2 \otimes f_2. \quad (6)$$

It is conjectured by Gross and Prasad [G-P92] that the coefficients $\alpha_j, \beta_j, \gamma_j$ are related to special values of certain automorphic $L$-functions. Recently the Gross-Prasad conjecture has been proven by Ikeda [Ike05] for the Maass Speizschar and $\nu = 0$. Moreover we show in this paper that the vanishing at such special values has interesting consequences. We have $F_j \in S_{20}^{\text{Maass}}$ if and only if the special value $\beta_j$ is zero. More generally:

**Theorem 0.2** Let $k \in \mathbb{N}_0$ be even. Then we have the embedding

$$D_k = \bigoplus_{\nu=0}^{\lfloor \frac{k}{10} \rfloor} \bigoplus_{\nu=0}^{\lfloor \frac{k}{10} \rfloor} D_{k,2\nu} : M_k^{(2)} \longrightarrow M_k^{\text{Sym}} \oplus S_{k+2}^{\text{Sym}} \oplus \ldots \oplus S_{k+2\lfloor \frac{k}{10} \rfloor}^{\text{Sym}}. \quad (7)$$

For $F \in S_k^{(2)}$ we have $D_k F \in S_k^{\text{Sym}}$. 

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Surprisingly the Maass Spezialschar property can be recovered in $M_k^{\text{Sym}} \oplus S_{k+2}^{\text{Sym}} \oplus \ldots \oplus S_{k+2 \lfloor \frac{k}{10} \rfloor}^{\text{Sym}}$ in the following transparent way. Let $(f_j)$ be the normalized Hecke eigenbasis of $M_k$. Let us define the diagonal subspaces $M_k^D = \{ \sum_j \alpha_j f_j \otimes f_j \in M_k^{\text{Sym}} \}$ and $S_k^D = S_k^{\text{Sym}} \cap M_k^D$. Then we can state

**Theorem 0.3** Let $F \in M_k^{(2)}$. Then we have

$$F \in M_k^{\text{Maass}} \iff D_k F \in M_k^D \oplus S_{k+2}^D \oplus \ldots \oplus S_{k+2 \lfloor \frac{k}{10} \rfloor}^D \quad (8)$$

and similarly

$$F \in S_k^{\text{Maass}} \iff D_k F \in \oplus S_{k+2 \nu} \quad (9)$$

These two theorems give a transparent explanation of our example from a general point of view.

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To be entered later.

**Notation**
Let $Z \in \mathbb{C}^{n,n}$ and $\text{tr}$ the trace of a matrix then we put $e\{Z\} = e^{2\pi i (\text{tr} Z)}$. For $l \in \mathbb{Z}$ we define $\pi_l = (2\pi i)^l$. Let $x \in \mathbb{R}$ then we use Knuth’s notation $\lfloor x \rfloor$ to denote the greatest integer smaller or equal to $x$. Let $A_2$ denote the set of half-integral positive-semidefinite matrices. We parametrize the elements $T = \left( \begin{smallmatrix} n & \frac{k}{2} \\ r & m \end{smallmatrix} \right)$ with $T = [n, r, m]$. The subset of positive-definit matrices we denote with $A_2^+$.

**1 Ultraspherical Differential Operators**

Let us start with the notation of the ultraspherical polynomial $p_{k,2\nu}$. Let $k$ and $\nu$ be elements of $\mathbb{N}_0$. Let $a$ and $b$ be elements of a commutative ring. Then we put

$$p_{k,2\nu}(a, b) = \sum_{\mu=0}^\nu (-1)^\mu \frac{(2\nu)!}{\mu!(2\nu - 2\mu)!} \frac{(k + 2\nu - \mu - 2)!}{(k + \nu - 2)!} a^{2\nu - 2\mu} b^\mu. \quad (10)$$

If we specialize the parameters we have $p_{k,0}(a, b) = 1$ and $p_{k,2\nu}(0, 0) = 0$ for $\nu \in \mathbb{N}$. 

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Let $\mathbb{H}_n$ be the Siegel upper half-space of degree $n$. Let $M_k^{(n)}$ the vector space of Siegel modular forms on $\mathbb{H}_n$ with respect to the full modular group $\Gamma_n = \text{Sp}_n(\mathbb{Z})$. Moreover let $S_k^{(n)}$ denote the subspace of cusp forms. If $n = 1$ we drop the index to simplify notation. We denote the coordinates of the three-fold $\mathbb{H}_2$ by $(\tau, z, \tilde{\tau})$ for $(\tilde{\tau}, \tilde{z}) \in \mathbb{H}_2$ and put $q = e\{\tau\}, \xi = e\{z\}$ and $\tilde{q} = e\{\tilde{\tau}\}$. Let $\hat{d}_k$ be the dimension of $S_k$.

**Definition 1.1** Let $k, \nu \in \mathbb{N}_0$ and let $k$ be even. Then we define the ultraspherical differential operator $D$ on the space of holomorphic functions $F$ on $\mathbb{H}_2$ in the following way:

$$D_{k,2\nu} F(\tau, \tilde{\tau}) = p_{k,2\nu} \left( \frac{1}{2\pi i} \frac{\partial}{\partial z}, \left( \frac{1}{2\pi i} \right)^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tilde{\tau}} \right) F \bigg|_{z=0} (\tau, \tilde{\tau}). \quad (11)$$

In the case $\nu = 0$ we get the pullback $F(\tau, 0, \tilde{\tau})$ of $F$ on $\mathbb{H} \times \mathbb{H}$.

Let $F \in M_k^{(2)}$ with $T$-th Fourier coefficient $A_F(n, r, m)$ for $T = [n, r, m] \in \mathbb{A}_2$. Then we have

$$D_{k,2\nu} F(\tau, \tilde{\tau}) = \sum_{n,m=0}^{\infty} A^F_{k,2\nu}(n,m)q^n \tilde{q}^m \quad \text{with} \quad A^F_{k,2\nu}(n,m) = \sum_{r \in \mathbb{Z}, r^2 \leq 4nm} p_{k,2\nu}(r, nm) A_F(n, r, m). \quad (12)$$

Let $M_k^{\text{Sym}} = \text{Sym}^2 M_k$ and $S_k^{\text{Sym}} = \text{Sym}^2 S_k$. $S_k^{\text{Sym}} = (S_k \otimes S_k)^{\text{Sym}}$. Let us further introduce a related Jacobi differential operator $D^\text{J}_{k,2\nu}$. This is given by exchanging $\pi_1 \frac{\partial}{\partial \tau}$ with $m$ in the definition of the ultraspherical differential operator given in (11). Applying the operator $D^\text{J}_{k,2\nu}$ on Jacobiforms $\Phi \in J_{k,m}$ of weight $k$ and index $m$ on $\mathbb{H} \times \mathbb{C}$ matches with the effect of the operator $D_{2\nu}$ introduced in [E-Z85] (§3, formula (2)) on $\Phi$.

Since $F \in M_k^{(2)}$ has a Fourier-Jacobi expansion of the form

$$F(\tau, z, \tilde{\tau}) = \sum_{m=0}^{\infty} \Phi^F_m(\tau, z) \tilde{q}^m, \quad \text{with} \quad \Phi^F_m \in J_{k,m} \quad (13)$$
it makes sense to consider $D_{k,2\nu}$ with respect to this decomposition in a Fourier-
Jacobi expansion

$$D_{k,2\nu} = \bigoplus_{m=0}^{\infty} D_{k,2\nu}^{1,m}. \quad (14)$$

**Lemma 1.2** Let $k, \nu \in \mathbb{N}_0$ and let $k$ be even. Then $D_{k,2\nu}$ maps $M^{(2)}_k$ to $M^{\text{Sym}}_{k+2\nu}$ and to $S^{\text{Sym}}_{k+2\nu}$ if $\nu \neq 0$. Moreover the subspace $S^{(2)}_k$ of cusp forms is always mapped to $S^{\text{Sym}}_{k+2\nu}$.

**Proof:**
We know from the work of Eichler and Zagier \[E-Z85\] since $D_{k,2\nu}^{1,m} \Phi_m \in M_{k+2\nu}$. Let $\nu > 0$ then $D_{k,2\nu}^{1,m} \Phi_m \in S_{k+2\nu}$ and for $F \in S^{(2)}_k$ we have $D_{k,2\nu}^{1,m} \Phi_m \in S_{k+2\nu}$ for all $\nu \in \mathbb{N}_0$. We are now ready to act with the ultraspherical differential operator with respect to its Fourier-Jacobi expansion directly on the Fourier-Jacobi expansion of $F$ in a canonical way

$$D_{k,2\nu} F(\tau, \tilde{\tau}) = \sum_{m=0}^{\infty} \left( D_{k,2\nu}^{1,m} \Phi_m \right)(\tau) \tilde{q}^m, \quad (15)$$

where all "coefficients" $a^F_m(\tau) = D_{k,2\nu}^{1,m} \Phi_m(\tau)$ are modular forms. This shows us, that if we apply the Peterson slash operator $|_{k+2\nu} \gamma$ here $\gamma \in \Gamma$ to this function with respect to the variable $\tau$, the function is invariant. The same argument also works for the Fourier-Jacobi expansion with respect to $\tau$. From this we deduce that

$$D_{k,2\nu} F(\tau, \tilde{\tau}) = \sum_{i,j} \alpha_{i,j} f_i(\tau) f_j(\tilde{\tau}).$$

Here $(f_i)_i$ is a basis of $M_{k+2\nu}$. Finally the cuspidal conditions in the lemma also follow from symmetry arguments.

**Remark 1.3** Let $F : \mathbb{H}_2 \longrightarrow \mathbb{C}$ be holomorphic. Let $g \in \text{Sl}_2(\mathbb{R})$ and let $J = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$. Then we have:

$$D_{k,2\nu}(F|_{k}(g \times 1_2)) = (D_{k,2\nu} F)|_{k+2\nu}(g \times 1_2) \quad (16)$$

$$D_{k,2\nu}(F|_{k}(1_2 \times g)) = (D_{k,2\nu} F)|_{k+2\nu}(1_2 \times g) \quad (17)$$

$$D_{k,2\nu}(F|_{k}J) = (D_{k,2\nu} F)|_{k+2\nu}J. \quad (18)$$

**Remark 1.4** There are other possibilities for construction of differential operators as used in this section (see Ibukiyama for a overview \[Ibu99\]). But since
the connection between our approach and the theory developed by Eichler and Zagier \cite{E-Z85} is so useful we decided to do it this way. We also wanted to introduce the concept of Fourier-Jacobi expansion of differential operators, which is interesting in its own right.

2 Taylor Expansion Of Siegel Modular Forms

The operators $D_{k,2\nu}$ can be seen at this point as somewhat artificial. If we apply $D_{k,2\nu}$ to Siegel modular forms $F$ we lose information. For example we know that $\dim S_{20}^{(2)} = 3$ and contains a two dimensional subspace of Saito-Kurokawa lifts. Since $\dim S_{20}^{\text{Sym}} = 1$ we obviously lose informations if we apply $D_{20,0}$. But even worse let $F_1$ and $F_2$ be a Hecke eigenbasis of the space of Saito-Kurokawa lifts and $F_3$ a Hecke eigenform of the orthogonal complement then we have $D_{20,0}F_j \neq 0$ for $j = 1, 2, 3$. The general case seems to be even worse, since for example $\dim M_{k}^{(2)} \sim k^3$ and $\dim M_{k}^{\Sym} \sim k^2$. On the other hand from an optimistic viewpoint we may find about $k$ pieces $D_{k,2\nu}F$ which code all the relevant information needed to characterize the Siegel modular forms $F$.

Paul Garrett in his fundamental papers \cite{Ga84} and \cite{Ga87} introduced the method of calculating pullbacks of modular forms to study automorphic L-functions. We also would like to mention the work of Piatetski-Shapiro, Rallis and Gelbart at this point (see also \cite{GPR87}). And recently Ichino in his paper: Pullbacks of Saito-Kurokawa lifts \cite{Ich05} extended Garrett’s ideas in a brilliant way to prove the Gross-Prasad conjecture \cite{G-P92} for Saito-Kurokawa lifts. In the new language we have introduced, it is obvious to consider Garretts pullbacks as the 0-th Taylor coefficients of $F$ around $z = 0$. Hence it seems to be very lucrative to study also the higher Taylor coefficients and hopefully get some transparent link.

Let $k \in \mathbb{N}_0$ be even. Let $F \in M_{k}^{(2)}$ and $\Phi \in J_{k,m}$. Then we denote by

$$F(\tau, z, \tilde{\tau}) = \sum_{\nu=0}^{\infty} \chi_{2\nu}(\tau) \ z^{2\nu} \quad \text{and} \quad \Phi(\tau, z) = \sum_{\nu=0}^{\infty} \chi_{2\nu}(\tau) \ z^{2\nu} \quad (19)$$

the corresponding Taylor expansions with respect to $z$ around $z = 0$. Here we already used the invariance of $F$ and $\Phi$ with respect to the transformation $z \mapsto (\tau) \ z$ since $k$ is even. Suppose $\chi_{2\nu_0}$ is the first non-vanishing Taylor coefficient,
then we denote $2\nu_0$ the vanishing order of the underlying form. If the form is identically zero we define the vanishing order to be $\infty$. To simplify our notation we introduce normalizing factor

$$
\gamma_{k,2\nu} = \left( \frac{1}{2\pi i} \right)^{2\nu} \frac{(k+2\nu-2)! \cdot (2\nu)!}{(k+2\nu-2)!}.
$$

Further we put

$$
\chi_{\mu,\mu}^{2\nu} = \frac{\partial^{2\mu}}{\partial^\mu \partial \tilde{\tau}^\mu} \chi_{2\nu}^F
$$
and

$$
\xi_{\mu,\mu}^{2\nu} = \left( \gamma_{k,2\nu} \right)^{-1} \frac{\partial^{2\mu}}{\partial^\mu \partial \tilde{\tau}^\mu} D_{k,2\nu} F.
$$

Then a straightforward calculation leads to the following useful formula.

**Lemma 2.1** Let $k, \nu \in \mathbb{N}_0$ and let $k$ be even. Let $F \in M_k^{(2)}$. Then we have

$$
(D_{k,2\nu} F)(\tau, \tilde{\tau}) = \gamma_{k,2\nu} \sum_{\mu=0}^{\nu} (-1)^\mu \frac{(k+2\nu-\mu-2)!}{(k+2\nu-2)! \cdot \mu!} \left( \frac{\partial^{2\mu} \chi_{2\nu-2\mu}^F}{\partial^\mu \partial \tilde{\tau}^\mu} \right). \tag{22}
$$

A similar formula is valid for Jacobiforms with normalizing factor $\gamma_{k,2\nu}^J = \gamma_{k,2\nu}$.

**Corollary 2.2** Let $2\nu_0$ be the vanishing order of $F \in M_k^{(2)}$. Then we have $D_{k,2\nu} F = 0$ for $\nu < \nu_0$ and

$$
D_{k,2\nu_0} F(\tau, \tilde{\tau}) = \gamma_{k,2\nu_0} \chi_{2\nu_0}^F(\tau, \tilde{\tau}) \in M_{k+2\nu_0}^{\text{Sym}} \setminus \{0\}. \tag{23}
$$

Similiarly we have for $\Phi \in J_{k,m}$ with vanishing order $2\nu_0$ the properties $D_{k,2\nu}^{J,m} \Phi = 0$ for $\nu < \nu_0$ and $D_{k,2\nu_0}^{J,m} \Phi = \gamma_{k,2\nu_0} \chi_{2\nu_0} \Phi \in M_{k+2\nu_0}$.

**EXAMPLE:** It is well known that $\dim S_{10}^{(2)} = 1$. Let $\Phi = \Phi_{10} \in S_{10}^{(2)}$ be normalized in such a way that $A^\Phi(1,1,1) = 1$. Then it follows from $D_{10,0}^{J,m} \Phi = 0$ that $A^\Phi(1,0,1) = -2$ since $\dim S_{10}^{\text{Sym}} = 0$. Then $\Phi$ has the Taylor expansion

$$
\Phi_{10}(\tau, z, \tilde{\tau}) = \frac{3}{5} \pi_2 \Delta(\tau) \Delta(\tilde{\tau}) \ z^2 + \Delta'(\tau) \Delta'(\tilde{\tau}) \ z^4 + O(z^6). \tag{24}
$$

We can also express the Taylor coefficients $\chi_{2\nu}^F$ in terms of the modular forms $D_{k,2\nu} F$. This can be done by inverting the formula (22). Finally we get

$$
\chi_{2\nu} = \sum_{\mu=0}^{\nu} \frac{(k+2\nu-2\mu-1)!}{(k+2\nu-\mu-1)! \cdot \mu!} \xi_{\mu,\mu}^{2\nu}. \tag{25}
$$
Before we state our first main result about the entropy of the family $D_{k,0}F$, $D_{k,2}F$, $D_{k,4}F$ . . . we introduce some further notation.

$$W_k = M_k^{\text{Sym}} \oplus \bigoplus_{j=1}^{\lfloor \frac{k}{10} \rfloor} S_{k+2j}^{\text{Sym}} \quad \text{and} \quad (26)$$

$$W_k^{\text{cusp}} = S_k^{\text{Sym}} \oplus \bigoplus_{j=1}^{\lfloor \frac{k}{10} \rfloor} S_{k+2j}^{\text{Sym}}. \quad (27)$$

These spaces will be the target of our next consideration. More precisely we define a linear map from the space of Siegel modular forms of degree 2 into these spaces with remarkable properties.

**Theorem 2.3** Let $k \in \mathbb{N}_0$ be even. Then we have the linear embedding

$$\mathbb{D}_k : \begin{cases} M_k^{(2)} & \hookrightarrow W_k \\ F & \mapsto \bigoplus_{\nu=0}^{\lfloor \frac{k}{10} \rfloor} D_{k,2\nu}F. \end{cases} \quad (28)$$

Since $D_{k,0}S_k^{(2)}$ is cuspidal we have the embedding of $S_k^{(2)}$ into $W_k^{\text{cusp}}$.

**Remark 2.4** It can be deduced from [Hei06] that $D_{k,0} \oplus D_{k,2}$ is surjective. Hence for $k < 20$ we have:

- $M_k^{(2)}$ is isomorphic to $M_k$ for $k < 10$ and
- $M_k^{(2)}$ is isomorphic to $M_k^{\text{Sym}} \oplus S_k^{\text{Sym}}$ for $10 \leq k < 20$ and $S_k^{(2)} \simeq S_k \oplus S_{k+2}$.

**Proof:**

First of all we recall that we have already shown that $D_{k,0}M_k^{(2)} \subseteq M_k^{\text{Sym}}$ and $D_{k,2\nu}M_k^{(2)} \subseteq S_{k+2\nu}^{\text{Sym}}$ for $\nu > 0$. Let $F \in M_k^{(2)}$ and suppose that $\mathbb{D}_kF$ is identically zero. Then it would follow from our inversion formula (25) that

$$F(\tau, z, \tilde{\tau}) = \sum_{\nu=\lfloor \frac{k}{10} \rfloor + 1}^{\infty} \chi_{2\nu}(\tau, \tilde{\tau}) z^{2\nu}. \quad (29)$$

For such $F$ the general theory of Siegel modular forms of degree 2 says that the special function $\Phi_{10} \in S_k^{(2)}$, which we already studied in one of our examples, divides $F$ in the $\mathbb{C}$–algebra of modular forms. And this is fulfilled at least with a
power of \(\lfloor \frac{k}{10} \rfloor + 1 = t_k > 0\). Hence there exists a Siegel modular form \(G\) of weight \(k - 10t_k\). But since this weight is negative and non-trivial Siegel modular forms of negative weight do not exist the form \(G\) has to be identically zero. Hence we have shown that if \(D_k F \equiv 0\) then \(F \equiv 0\). And this proves the statement of the theorem.

\[\square\]

**Remark 2.5** The number \(\lfloor \frac{k}{10} \rfloor\) in the Theorem is optimal. This follows directly from properties of \(\Phi_{10}\).

**Remark 2.6** Let \(E_k^{2,1}(f)\) be a Klingen Eisenstein series attached to \(f \in S_k\). Let \(E_k\) denote an elliptic Eisenstein series of weight \(k\). Then it can be deduced from [Ga87] that \(D_{k,2} E_k^{2,1}(f) = f \otimes E_k + E_k \otimes f \mod S_k^{\text{Sym}}\).

**Remark 2.7** It would be interesting to have a different proof of the Theorem independent of the special properties of \(\Phi_{10}\).

**Remark 2.8** The asymptotic limit of the dimension of the quotient of \(W_k / M_k^{(2)}\) is equal to \(\frac{91}{25}\). Let us put \(d_k = \dim M_k\).

- The dimension of the target space \(W_k\):
  \[
  \dim W_k \sim \frac{1}{288} \int_0^{10} (k + 2x)^2 dx \sim \frac{1}{288} \cdot \frac{1}{2} \cdot \frac{91}{5^3} k^3
  \]

- The asymptotic dimension formula of \(M_k^{(2)}\) is given by
  \[
  \dim M_k^{(2)} \sim \frac{1}{288} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{91}{5^3} k^3 \] (see [Ma79], Introduction).

3 **The Spezialschar**

In this section we first recall some basic facts on the Maass Spezialschar [Za80]. Then we determine the image of the Spezialschar in the space \(W_k\) for all even weights \(k\). Then finally we introduce a Spezialschar as a certain subspace of the space of Siegel modular forms of degree \(2n\) and weight \(k\). Then we show that in the case \(n = 1\) this Spezialschar coincides with the Maass Spezialschar.
3.1 Basics of the Maass Spezialschar

Let $J_{k,m}$ be the space of Jacobi forms of weight $k$ and index $m$. We denote the subspace of cusp forms with $J_{k,m}^{\text{cusp}}$. Let $|_{k,m}$ the slash operator for Jacobi forms and $V_l \ (l \in \mathbb{N}_0)$ be the operator, which maps $J_{k,m}$ to $J_{k,ml}$. More precisely, let

$$\Phi(\tau, z) = \sum c(n, r) q^n \xi^r \in J_{k,m}.$$ 

Then $(\Phi |_{k,m} V_l)(\tau, z) = \sum c^*(n, r) q^n \xi^r$ with

$$c^*(n, r) = \sum_{a | (n, r, l)} a^{k-1} c\left(\frac{nl}{a^2} \frac{r}{a}\right) \text{ for } l \in \mathbb{N}$$ \hspace{1cm} (30)

and for $l = 0$, we have $c^*(0, 0) = c(0, 0) \left(\frac{-2k}{B_{2k}}\right)$ and for $l = 0$ and $n > 0$ we have $c^*(n, r) = c(0, 0) \sigma_{k-1}(n)$. This includes the theory of Eisenstein series in a nice way \cite{E-Z85}.

**Definition 3.1** The lifting $V$ is given by the linear map

$$\mathcal{V} : \begin{cases} J_{k,1} \rightarrow M_k^{(2)} \\ \Phi \mapsto \sum_{l=0}^\infty (\Phi |_{k,1} V_l) q^l. \end{cases}$$ \hspace{1cm} (31)

The image of this lifting is the Maass Spezialschar $M_k^{\text{Maass}}$ of weight $k$. The subspace of cusp forms we denote with $S_k^{\text{Maass}}$.

**Remark 3.2**

- The lifting is invariant by the Klingen parabolic of $Sp_2(\mathbb{Z})$. Since the Fourier coefficients satisfy $A(n, r, m) = A(m, r, n)$ the map $\mathcal{V}$ is well-defined.

- If we restrict the Saito-Kurokawa lifting to Jacobi cusp forms we get Siegel cusp forms.

- Let $\Phi \in J_{k,m}$ and $l, \mu \in \mathbb{N}_0$. Then we have

$$D^{J_{k,m}l}_{k,2\mu}(\Phi |_{k,m} V_l) = \left(D^{J_{k,2\mu}}_{k,l} \Phi\right) |_{k} T_l.$$ \hspace{1cm} (32)

Here $T_l$ is the Hecke operator on the space of elliptic modular forms.

- Let $F \in M_k^{\text{Maass}}$ be the lift of $\Phi \in J_{k,1}$. Then $F$ is a Hecke eigenform if and only if $\Phi$ is a Hecke-Jacobi eigenform.
From this consideration we conclude [E-Z85]:

**Proposition 3.3** Let $F \in M^{(2)}_k$ be a Siegel modular form. Then the following properties are equivalent

- **ARITHMETIC** Let $A(n, r, m)$ denote the Fourier coefficients of $F$ then
  \[ A(n, r, m) = \sum_{d | (n, r, m)} d^{k-1} A\left(\frac{n}{d^2}, \frac{r}{d}, 1\right) \]  
  \[ (33) \]

- **LIFTING** Let $\Phi_1^F$ be the first Fourier-Jacobi coefficient of $F$. Then all other Fourier-Jacobi coefficients satisfy the identity
  \[ \Phi_m^F = \Phi_1^F |_{k, 1} V_m. \]  
  \[ (34) \]

Let $F \in S^{(2)}_k$ be a Hecke eigenform. Then $F$ is a Saito-Kurokawa lift if and only if the spinor $L$-function $Z(F, s)$ of degree 4 has a pole ([Ev80]).

### 3.2 The Diagonal of $\mathbb{W}_k$

Let $(f_j)$ be the normalized Hecke eigenbasis of $M_k$. With this notation we introduce the diagonal space

\[ M_k^D = \{ \sum_j \alpha_j f_j \otimes f_j \in M_k^{\text{Sym}} \} \]  
\[ (35) \]

and the corresponding cuspidal subspace $S_k^D$. Now we are ready to distinguish the Maass Spezialschar in the vector spaces $\mathbb{W}_k$ and $\mathbb{W}_k^{\text{cusp}}$.

**Theorem 3.4** Let $k$ be a natural even number. Let $F$ be a Siegel modular form of degree two and weight $k$. Then we have

\[ F \in M_k^{\text{Maass}} \iff \mathbb{D}_k F \in M_k^D \oplus S_{k+2}^D \oplus \ldots \oplus S_{k+2\lfloor \frac{k}{2} \rfloor}^D. \]  
\[ (36) \]

Let $F$ be a cuspform. Then we have

\[ F \in S_k^{\text{Maass}} \iff \mathbb{D}_k F \in \bigoplus_{\nu=0}^{\lfloor \frac{k}{2} \rfloor} S_{k+2\nu}^D. \]  
\[ (37) \]
Remark 3.5

- The Theorem 3.4 describes a link between Siegel modular forms and elliptic Hecke eigenforms.

- Let $F \in M_{20}^{(2)}$ and let $(f_j)$ be a Hecke eigenbasis of $S_{24}$. Then $F \in M_k^{Maass}$ if and only if

$$\mathcal{D}_{20,4} F = \alpha_0 E_{24} \otimes E_{24} + \alpha f_1 \otimes f_1 + \gamma f_2 \otimes f_2$$

(38)

here $\alpha_0, \alpha, \gamma \in \mathbb{C}$.

Proof:
We first show that if $F$ is in the Maass Spezialschar then $\mathcal{D}_{k,2\nu} F$ is an element of the diagonal space. Let $\nu \in \mathbb{N}_0$ and $\Phi^I_1$ be the first Fourier-Jacobi coefficient of $F$. Then we have

$$(\mathcal{D}_{k,2\nu}(V\Phi)) (\tau, \bar{\tau}) = \sum_{l=0}^{\infty} \left( \mathcal{D}_{k,2\nu}^{IJ}(\Phi|_{k,1}V_l) \right) (\tau) \tilde{q}^l.$$  

(39)

Here we applied the Fourier-Jacobi expansion of the differential operator $\mathcal{D}_{k,2\nu}$ acting on Siegel modular forms. Then we used the formula (32) to interchange the operators $\mathcal{D}_{k,2\nu}^{IJ}$ and $V_l$ to get

$$(\mathcal{D}_{k,2\nu} F) (\tau, \bar{\tau}) = \sum_{l=0}^{\infty} \left( \mathcal{D}_{k,2\nu}^{IJ}\Phi \right) |_{kT_l} \tilde{q}^l.$$  

(40)

Now let $\left( f_{j_1}^{k+2\nu} f_{j_2}^{k+2\nu} \right)_{j=1}^{\tilde{d}_{k+2\nu}}$ be a normalized Hecke eigenbasis of $S_{k+2\nu}$. Let $1 \leq j_1, j_2 \leq \tilde{d}_{k+2\nu}$. Then we have

$$\langle (\mathcal{D}_{k,2\nu} F) (\ast, \bar{\tau}), f_j^{k+2\nu} \rangle = \langle \left( \mathcal{D}_{k,2\nu}^{IJ}\Phi \right), f_j^{k+2\nu} \rangle f_j^{k+2\nu},$$

(41)

which leads to the desired result

$$\langle (\mathcal{D}_{k,2\nu} F), f_j^{k+2\nu} \otimes f_{j_2}^{k+2\nu} \rangle = 0 \text{ for } j_1 \neq j_2.$$  

(42)

It remains to look at the Eisenstein part if $\nu = 0$. Since the space of Eisenstein series has the basis $E_k$ and is orthogonal to the functions given in (41) we have proven that the Spezialschar property of $F$ implies that $\mathbb{D}_k F \in \mathcal{W}_k^D$.  

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Now let us assume that $F \notin M^\text{Maass}_k$. Then we show that $\mathbb{D}_k F \notin \mathbb{W}_k^D$. Since the map
\[(\mathcal{D}_{k,0} \oplus \mathcal{D}_{k,2}) : M^\text{Maass}_k \rightarrow M^D_k \oplus S^D_{k+2}\]
is an isomorphism, we can assume that $(\mathcal{D}_{k,0} \oplus \mathcal{D}_{k,2})(F)$ projected on $M^D_k \oplus S^D_{k+2}$ is identically zero. Altering $F$ by an element of the Maass Spezialschar does not change the property we have to prove. If $\mathcal{D}_{k,0} F \notin M^D_k$ or $\mathcal{D}_{k,2} F \notin S^D_{k+2}$ we are done otherwise we can assume that
\[(\mathcal{D}_{k,0} \oplus \mathcal{D}_{k,2})(F) \equiv 0.\]
(44)
Then we have the order $F = 2\nu_0 \geq 4$ and $k \geq 20$, since $F \notin M^\text{Maass}_k$. Let $\Phi_{10} \in S^{(2)}_{10}$ be the Siegel cusp form (24) of weight 10 and degree 2. It has the properties that $\chi_{\Phi_{10}} \equiv 0$ and $\chi_{\Phi_{10}}^2(\tau, \bar{\tau}) = c \Delta(\tau) \Delta(\bar{\tau})$ with $c \neq 0$. Since order $F = 2\nu_0$ we also have
\[\Phi_{10} \parallel F.\]
(46)
This means that there exists a $G \in S_{k-10\nu_0}$ such that $\chi^G_0$ is non-trivial and
\[F = (\Phi_{10})^{\nu_0} G.\]
(47)
Hence we have for the first nontrivial Taylor coefficient of $F$ the formula
\[\chi_{2\nu_0}^F(\tau, \bar{\tau}) = \left(\chi_{2}^{\Phi_{10}}(\tau, \bar{\tau})\right)^{\nu_0} \chi_{0}^{G}(\tau, \bar{\tau}) \]
\[= c^{\nu_0} \Delta(\tau)^{\nu_0} \Delta(\bar{\tau})^{\nu_0} \chi_{0}^{G}(\tau, \bar{\tau}).\]
(48)
(49)
And the coefficient $a_1(\bar{\tau})$ of $q$ is identically zero. Now let us assume for a moment that $\chi^F_{2\nu_0} \in S^D_{k+2\nu_0}$. Then we have
\[\chi_{2\nu_0}^F(\tau, \bar{\tau}) = \sum_{l=1}^{d_{k+2\nu_0}} \alpha_l \int_{I}^{k+2\nu_0}(\tau) \int_{I}^{k+2\nu_0}(\bar{\tau})\]
(50)
and the coefficient of $q$ is given by $\sum_{l=1}^{d_{k+2\nu_0}} \alpha_l \int_{I}^{k+2\nu_0}(\bar{\tau})$. Since $\left(\int_{I}^{k+2\nu_0}\right)_{l=1}^{d_{k+2\nu_0}}$ is a basis we have $\alpha_1 = \ldots = \alpha_{d_{k+2\nu_0}} = 0$. But since we assumed that order $F =$
2\nu_0 we have a reductio ad absurdum. Hence we have shown that \chi_{2\nu_0}^F \notin S_{k+2\nu_0}
which proves our theorem.

\[\square\]

**Corollary 3.6** Klingen Eisenstein series are not in the Maass Spezialschar.

**Remark 3.7** Let \( k \) be a natural even number. Let \( F \) be a Siegel modular form of degree two and weight \( k \). Then we have

\[ F \in M_k^{\text{Maass}} \iff D_{k, 2\nu} F \in M_{k+2\nu}^D \text{ for all } \nu \in \mathbb{N}_0. \quad (51) \]

### 3.3 The Spezialschar

Let \( G^+Sp_n(\mathbb{Q}) \) be the rational symplectic group with positive similitude \( \mu \). In the sense of Shimura we attach to Hecke pairs the corresponding Hecke algebras

\[
\mathcal{H}^n = (\Gamma_n, G^+Sp_n(\mathbb{Q})) \quad (52) \\
\mathcal{H}_0^n = (\Gamma_n, Sp_n(\mathbb{Q})). \quad (53)
\]

We also would like to mention that in the setting of elliptic modular forms the classical Hecke operator \( T(p) \) can be normalized such that it is an element of the full Hecke algebra \( \mathcal{H}^1 \), but not of the even one \( \mathcal{H}_0^1 \). Let \( g \in G^+Sp_n(\mathbb{Q}) \) with similitude \( \mu(g) \). Then we put

\[
\widetilde{g} = \mu(g)^{-\frac{1}{2}} g \quad (54)
\]

to obtain an element of \( Sp_n(\mathbb{R}) \). We further extend this to \( \mathcal{H}^n \).

**Definition 3.8** Let \( T \in \mathcal{H}^n \). Then we define

\[
\ltimes_T = (\widetilde{T} \times 1_{2n}) - (1_{2n} \times \widetilde{T}). \quad (55)
\]

Here \( \ltimes \) is the standard embedding of \( (Sp_n, Sp_n) \) into \( Sp_{2n} \).

Now we study the action \( |_k \ltimes_T \) on the space of modular forms of degree \( 2n \) for all \( T \in \mathcal{H}^n \) or \( T \in \mathcal{H}_0^n \). The first thing we would like to mention is that for \( F \in M_k^{(2n)} \) the function \( F|_k \ltimes_T \) is in general not an element of \( M_k^{(2n)} \) anymore. Anyway at the moment we are much more interested in the properties of the kernel of a certain map related to this action. In particular in the case \( n = 1 \) we get a new description of the Maass Spezialschar.
Definition 3.9 Let $n$ and $k$ be natural numbers. Let $M^{(2n)}_k$ be the space of Siegel modular forms of degree $2n$ and weight $k$. Then we introduce the Spezialschar corresponding to the Hecke algebras $\mathcal{H}^n$ and $\mathcal{H}_0^n$.

$$\text{Spez} \left( M^{(2n)}_k \right) = \left\{ F \in M^{(2n)}_k \mid F|_k \triangleright T = 0 \text{ for all } T \in \mathcal{H}^n \right\}$$  \hspace{1cm} (56)

$$\text{Spez}_0 \left( M^{(2n)}_k \right) = \left\{ F \in M^{(2n)}_k \mid F|_k \triangleright T = 0 \text{ for all } T \in \mathcal{H}_0^n \right\}.$$  \hspace{1cm} (57)

Moreover $\text{Spez} \left( S^{(2n)}_k \right)$ and $\text{Spez}_0 \left( S^{(2n)}_k \right)$ are the cuspidal part of the corresponding Spezialschar.

It is obvious that these subspaces of $M^{(2n)}_k$ are candidates for finding spaces of modular forms with distinguished Fourier coefficients. Further it turns that these spaces are related to the Maass Spezialschar and the Ikeda lift [Ike01]. More precisely in the first interesting case we have:

Theorem 3.10 The Spezialschar $\text{Spez} \left( M^{(2)}_k \right)$ is equal to the Spezialschar of Maass.

Proof:
Let $F \in M^{(2)}_k$. Then we have $F \in M^{\text{Maass}}_k$ if and only if $D_{k,2\nu}F \in M^D_{k+2\nu}$ for all $\nu \in \mathbb{N}_0$. This follows from Remark 3.7. On the other side the property $D_{k,2\nu}F \in M^D_{k+2\nu}$ is equivalent to the identity

$$(D_{k,2\nu}F)|_{k+2\nu} \triangleright T = 0 \text{ for all } T \in \mathcal{H}.$$  \hspace{1cm} (58)

This follows from the fact that the Hecke operators are self adjoint and that the space of elliptic modular forms has multiplicity one. To make the operator well-defined we used the embedding $\mathbb{H} \times \mathbb{H}$ into the diagonal of $\mathbb{H}_2$. We can now interchange the differential operators $D_{k,2\nu}$ and the Petersson slash operator $|_*$. This leads to

$$D_{k,2\nu}F \in M^D_{k+2\nu} \iff D_{k,2\nu}(F|_k \triangleright T) = 0.$$  \hspace{1cm} (59)

So finally it remains to show that if $D_{k,2\nu}(F|_k \triangleright T) = 0$ for all $\nu \in \mathbb{N}_0$ then it follows $F|_k \triangleright T = 0$. By looking at the Taylor expansion of the function $F|_k \triangleright T \left( \frac{\tau}{z}, \frac{z}{\tau} \right)$ with respect to $z$ around 0 we get with the same argument as given in the proof of Theorem 2.3 the desired result. 

$\Box$
4 Maass relations revised

We introduced two Hecke algebras $\mathcal{H}$ and $\mathcal{H}_0$ related to elliptic modular forms. For the corresponding Spezialschar $\text{Spez}(M_k^{(2)})$ and $\text{Spez}_0(M_k^{(2)})$ we obtain:

**Theorem 4.1** Let $k$ be an even natural number. Then the even Spezialschar $\text{Spez}_0(M_k^{(2)})$ related to the Hecke algebra $\mathcal{H}_0$ which is locally generated by $T(p^2)$ is equal to the Spezialschar $\text{Spez}(M_k^{(2)})$ related to the Hecke algebra $\mathcal{H}$ which is locally generated by $T(p)$.

$$\text{Spez}_0(M_k^{(2)}) = \text{Spez}(M_k^{(2)}).$$ (60)

**Proof:**

Let $F \in M_k^{(2)}$. We proceed as follows. In the proof of Theorem 3.10 it has been shown that

$$F \in \text{Spez}(M_k^{(2)}) \iff (D_{k,2\nu}F)|_{k+2\nu} \rtimes T = 0 \text{ for all } T \in \mathcal{H} \text{ and } \nu \in \mathbb{N}_0.$$ (61)

Now we show that

$$(D_{k,2\nu}F)|_{k+2\nu} \rtimes T(p) = 0 \iff (D_{k,2\nu}F)|_{k+2\nu} \rtimes T(p^2) = 0$$ (62)

for all $\nu \in \mathbb{N}_0$ and prime numbers $p$. This would finish the proof since

$$F \in \text{Spez}_0(M_k^{(2)}) \iff (D_{k,2\nu}F)|_{k+2\nu} \rtimes T = 0 \text{ for all } T \in \mathcal{H}_0 \text{ and } \nu \in \mathbb{N}_0.$$ (63)

(\text{this can also be obtained by following the procedure of the proof of Theorem 3.10)}.

To verify the equation (62), we show that to being an element of the kernel of the operator $| \rtimes T(p^2)$ implies already to be an element of the kernel of $| \rtimes T(p)$.

To see this we give a more general proof. Let $\phi \in M_k^{\text{Sym}}$ and let $\phi|_{k \rtimes T(p^2)} = 0$. Let $(f_j)$ be a normalized Hecke eigenbasis of $M_k$. Then we have

$$\phi = \sum_{i,j} \alpha_{i,j} f_i \otimes f_j$$ (64)
Let us assume that there exists a $\alpha_{i_0, j_0} \neq 0$ with $i_0 \neq j_0$. Let us denote $\lambda_i(p^2)$ to be the eigenvalue of $f_i$ with respect to the Hecke operator $T(p^2)$. Then we have

$$0 = \phi|_k \otimes_{T(p^2)} \sum_{i,j} \alpha_{i,j} (\lambda_i(p^2) - \lambda_j(p^2)) f_i \otimes f_j.$$  \hfill (65)

From this follows that $\lambda_{i_0}(p^2) = \lambda_{j_0}(p^2)$ for all prime numbers $p$. It is easy to see at this point that then $f_{i_0}$ and $f_{j_0}$ have to be cusp forms. In the setting of cusp forms we can apply a result on multiplicity one for $SL_2$ of D. Ramakrishnan [Ra00](section 4.1) and other people to obtain $f_{i_0} = f_{j_0}$. Since this is a contradiction we have $\phi \in M^D_k$. In other words we have $\phi|_k \otimes_{T(p)} = 0$. \hfill \qed

References

[E-Z85] M. Eichler and D. Zagier, The theory of Jacobi forms, Progress in Mathematics Vol. 55, Birkhäuser Verlag (1985)

[Ev80] S. Evdokimov, Une caracterisation de l’espace de Maass de formes paraboliques modulaires de Siegel de genre 2 (en russe), Mat. Sbornik (154) 112 (1980), 133-142

[Ga84] P. Garrett, Pullbacks of Eisenstein series; Applications. In: Automorphic forms of several variables. Taniguchi symposium 1983, Birkhäuser (1984)

[Ga87] P. Garrett, Decomposition of Eisenstein series: triple product L-functions, Ann. math. 125 (1987), 209-235

[GPR87] S. Gelbart, I. Piatetski-Shapiro, S. Rallis, Explicit Constructions of Automorphic L-functions, Springer Lecture Notes in Math. no. 1254 (1987)

[G-P92] Gross and Prasad, On the decomposition of a representation of $SO_n$ when restricted to $SO_{n-1}$, Canad. J. Math. 44 (1992), 974-1002

[Hei98] B. Heim, Über Poincare Reihen und Restriktionsabbildungen, Abh. Math. Sem. Univ. Hamburg 68 (1998), 79-89

[Hei05] B. Heim, On the injectivity of the Satoh lifting of modular forms and the Taylor coefficients of Jacobi forms, Preprint MPI Bonn (68) 2005

[Hei06] B. Heim, Period integrals and the global Gross-Prasad conjecture Preprint 2005

17
[Ibu99] T. Ibukiyama, On differential operators on automorphic forms and invariant pluri-harmonic polynomials, Comment. Math. Sancti Pauli vol. 48 no.1 (1999) 103-117

[Ich05] A. Ichino, Pullbacks of Saito-Kurokawa Lifts, Invent. math. 162 (2005), 551-647

[Ike01] T. Ikeda, On the lifting of elliptic cusp forms to Siegel cusp forms of degree 2n, Ann. of Math (2) 154 no. 3 (2001), 641-681

[Ike05] T. Ikeda, Pullback of the lifting of elliptic cusp forms and Miyawakis conjecture, Preprint 2005

[Kl90] H. Klingen, Introductory lectures on Siegel modular forms, Cambridge: Cambridge University Press 1990

[K-K05] W. Kohnen, H. Kojima, A Maass space in higher genus Compos. Math 141 no.2 (2005), 313-322

[Ma79I] H. Maass, Über eine Spezialschar von Modulformen zweiten Grades I, Invent. Math. 52, (1979), 95-104

[Ma79II] H. Maass, Über eine Spezialschar von Modulformen zweiten Grades II, Invent. Math. 53, (1979), 249-253

[Ma79III] H. Maass, Über eine Spezialschar von Modulformen zweiten Grades III, Invent. Math. 53 (1979), 255-265

[Ra00] D. Ramakrishnan, Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2), Ann. math. 152 (2000), 45-111

[Wa85] J.L. Waldspurger, Sur les valeurs de certaines fonctions L automorphes en leur centre de symmetrie, Compositio Math. 54 (1985) 173-242

[Za80] D. Zagier, Sur la conjecture de Saito-Kurokawa (d'après H. Maass), Sém. Delange-Pisot-Poitou 1979/1980, Progress in Math. 12 (1980), 371-394

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