Gauge invariant approach to nonmetricity theories and the second clock effect

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In this paper we discuss on recent attempts aimed at demonstrating that, contrary to well-known results, the second clock effect (SCE) does not take place in generalized Weyl spaces – spaces with arbitrary nonmetricity – denoted here as $W_4$ spaces. These attempts include Weyl gauge theories of gravity, as well as the symmetric teleparallel theories (STTs). Our approach to this issue is based on the adoption of Weyl gauge symmetry (WGS) which is a manifest symmetry of the basic laws of Weyl geometry. We shall consistently adapt mathematical and geometrical quantities and concepts so that the resulting geometrical framework be gauge invariant. This issue is of special relevance for the fate of nonmetricity theories, including a class of the STTs which is being intensively applied in the cosmological framework. As we shall show, if realize that WGS is a manifest symmetry of generalized Weyl spaces $W_4$, and identify physical vectors and tensors with corresponding hypothetical vectors and tensors living in $W_4$, neither the Weyl gauge theories nor the nonmetricity theories are free of the SCE, unless Weyl integrable geometry (WIG) spaces are considered.

I. INTRODUCTION

In general relativity (GR) when two identical clocks, initially synchronized, are parallel transported along different paths, a certain loss of synchronization arises that is called as the “first clock effect.” In Weyl spacetimes, where the following “vectorial” nonmetricity condition is satisfied [1]

$$\nabla_{\alpha} g_{\mu\nu} = -Q_{\alpha \mu\nu} \tag{1}$$

with $Q_{\alpha}$ – the Weyl gauge vector, an additional effect arises: the two clocks not only have lost their initial synchronization, but, they go at different rates. It is known as the second clock effect \[2, 14\]. The SCE causes a serious observational issue: an unobserved broadening of spectral lines. This issue was enough to reject the original Weyl’s gauge invariant gravitational theory and its spectral lines. This issue was enough to reject the original Weyl’s gauge invariant gravitational theory and its related geometrical framework \[12\].

Despite of this well-known result, in recent papers the occurrence of the second clock effect has been challenged \[10, 11, 13, 14\]. It has been demonstrated in \[10\] a lemma (lemma 2) which basically states that, in spaces with generalized nonmetricity $Q_{\alpha \mu\nu}$, where the teleparallel condition (vanishing generalized curvature) is satisfied, the SCE does not take place. In that reference, however, the authors did not consider the WGS which is a manifest symmetry of generalized Weyl spaces and, besides, as we shall show in section \[VII\] of the present paper, they considered an statement of the parallel transport law which is not valid for tangent vectors with conformal weight $-1$. In consequence, the main equation in section 5 of \[10\] – equation (15) – is not valid when any of the vectors in the inner product has weight $-1$. In addition, the demonstration starts with a path integral along a closed timelike curve (CTC) – see equation (12) of the above reference – which may bring into consideration causality issues, as it is discussed in section \[IX\] of the present paper.

In reference \[13\] the authors demonstrated that if one takes into account the Weyl gauge symmetry, and one requires, besides, the presence of massive matter fields to represent atoms, observers and clocks, then Weyl gauge theories do not predict a SCE. This result is interesting, besides, because it was obtained in standard Weyl geometry spaces, where it is supposed that the occurrence of the SCE was demonstrated long ago. However, as we shall show below in section \[IV\] the assumptions made in \[13, 14\] amount to deny relating physical vectors and tensors with corresponding (hypothetical) vectors and tensors living in generalized Weyl space $W_4$. In other words, these assumptions amount to giving up the description of physical phenomena in $W_4$ space.

In this paper we shall show that WGS is not only a manifest symmetry of standard Weyl space $W_4$, but it is also a manifest symmetry of generalized Weyl spaces $W_4$, thus confirming previous results \[1\]. This result is possible after the development in section \[IV\] of the gauge invariant theory of parallel transport, including the related concepts of gauge derivative along given path, etc. Here we adopt a parallel transport law that differs from the one undertaken in \[16\]. Yet, similar results regarding to the occurrence of the second clock effect, are obtained in the present paper. There are other issues of current interest that we shall investigate in this paper as well. For instance: i) is the generalization of nonmetricity $Q_{\alpha \mu\nu}$ phenomenologically viable? and ii) do the matter fields interact with nonmetricity? The answers to the above questions carry important consequences for the nonmetricity theories, including the symmetric teleparallel theories of gravity.

Our main goal will be to demonstrate that, if properly take into account the WGS and, besides, identify the physical vectors and tensors with the corresponding hypothetical vectors and tensors in $W_4$, the SCE must necessarily take place in generalized Weyl spaces. These include as subclasses the standard Weyl geometry spaces

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and those spacetimes where the teleparallel condition is satisfied. Our demonstration contradicts recent results in [10] and in [13, 14] and confirms well-known (long standing) results. Although this paper is not intended as a comment on the mentioned references, here we shall show why the corresponding results should be taken with caution.

All of the results discussed here are based on the assumption of a parallel transport law that differs from the one assumed in [16] and also, in a specific consistency hypothesis that allows identification of physical vectors and tensors with related hypothetical vectors and tensors living in \( W_4 \) space. Through the paper, for simplicity, we consider only Riemann-Christoffel and nonmetricity contributions to the curvature and affinity of space. The torsion contribution is omitted not only for simplicity of the analysis but, also, because consideration of spaces with torsion is beyond the scope of our planned discussion.

We have organized this paper in the following way. Sections III and IV are introductory. In section III the basic notions of generalized Weyl space – spaces with Riemann-Christoffel curvature and arbitrary nonmetricity – as well as our conventions, are given. Then, in section IV the rudiments of generalized Weyl gauge symmetry are exposed. Sections V and VI are dedicated mostly to expose the mathematical developments behind a consistent statement of gauge invariant parallel transport which is alternative to the one assumed in [16]. The parallel transport law is required to understand the second clock effect and related subjects covered in this paper. In section VI in particular, the master equations that serve as the mathematical basis for the SCE, are derived. In the second part of the paper, which is composed of sections VII, VIII, III, and IX we deal with the geometrical and phenomenological consequences of our gauge invariant approach to generalized Weyl spaces. It is in these sections where we demonstrate the inevitability of the SCE in \( W_4 \) as a consequence of: (i) the assumed parallel transport law and (ii) the consistency requirement that allows to identify physical vectors and tensors with the related hypothetical vectors and tensors in generalized Weyl space \( W_4 \). Besides, in these sections we explore the answers to the questions stated in the former paragraphs as, for instance: Do the spinor matter fields and spinning test bodies interact with nonmetricity? which is investigated in section VIII. In section IX we discuss on the results obtained, while concluding remarks are given in section X. For completeness an appendix section has been included. In appendix A we discuss on what happens if consider gauge symmetry in a situation like the one discussed in lemma 2 of [10], while in appendix B we have included a brief reply to a comment appeared in reference [11].

In this paper, unless otherwise stated, we use the units \( h = c = 1 \) and the following signature of the metric is chosen: \((-+++)\).

### II. BACKGROUND AND CONVENTIONS

The generalized Weyl geometry [1], denoted here by \( W_4 \), is defined as the class of four-dimensional (torsionless) manifolds \( M_4 \) that are paracompact, Hausdorff, connected \( C^\infty \), endowed with a locally Lorentzian metric \( g \) that obeys the following nonmetricity condition:

\[
\nabla \mu g_{\nu\mu} = -Q_{\alpha\mu\nu},
\]

where \( Q_{\alpha\mu\nu} \) is the nonmetricity tensor and the covariant derivative \( \nabla \mu \) is defined with respect to the generalized torsion-free affine connection of the manifold:

\[
\Gamma_{\alpha\mu\nu} = \{\alpha \}_{\mu\nu} + L^\alpha_{\mu\nu} \xrightarrow{\text{symbr.}} \Gamma = \{\} + L,
\]

where

\[
\{\alpha \}_{\mu\nu} := \frac{1}{2} g^{\alpha\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}),
\]

is the Levi-Civita (LC) connection, while

\[
L^\alpha_{\mu\nu} := \frac{1}{2} (Q^\alpha_{\mu\nu} + Q^\alpha_{\nu\mu} - Q^\nu_{\alpha\mu}),
\]

is the disformation tensor. The nonmetricity tensor \( Q_{\alpha\mu\nu} \) is symmetric in the second and third indices. It measures how much the length of given vector varies during parallel transport [17].

Standard Weyl geometry, denoted here by \( \tilde{W}_4 \), is a subclass of \( W_4 \) which is defined by the choice of vectorial nonmetricity \( Q_{\alpha\mu\nu} = \tilde{Q}_{\alpha\mu\nu} g_{\mu\nu} \) and so gives rise to [1], where the operator \( \nabla \alpha \) denotes covariant derivative defined with respect to the connection [3], but with the disformation tensor given by:

\[
L^\alpha_{\mu\nu} := \frac{1}{2} (\tilde{Q}_{\mu\nu}^\alpha + \tilde{Q}_{\nu\mu}^\alpha - \tilde{Q}_{\alpha\mu\nu}),
\]

instead of [3].

In this paper we call as “generalized curvature tensor” of \( W_4 \) spacetime, the curvature of the connection, symbolically \( R(\Gamma) \), whose coordinate components are given by:

\[
R^\alpha_{\sigma\mu\nu} := \partial_\mu \Gamma^\alpha_{\nu\sigma} - \partial_\nu \Gamma^\alpha_{\mu\sigma} + \Gamma^\alpha_{\rho\mu} \Gamma^\rho_{\nu\sigma} - \Gamma^\rho_{\nu\mu} \Gamma^\rho_{\mu\sigma},
\]

or, if take into account the decomposition [3]:

\[
R^\alpha_{\sigma\mu\nu} = \tilde{R}^\alpha_{\sigma\mu\nu} + \tilde{\nabla}_\mu L^\alpha_{\nu\sigma} - \tilde{\nabla}_\nu L^\alpha_{\mu\sigma} + L^\alpha_{\mu\lambda} L^\lambda_{\nu\sigma} - L^\alpha_{\nu\lambda} L^\lambda_{\mu\sigma},
\]
where $\widehat{R}^\alpha_{\sigma\mu\nu}$ is the Riemann-Christoffel or LC curvature tensor,

$$\widehat{R}^\alpha_{\sigma\mu\nu} := \partial_\sigma \{^\alpha_{\nu\mu} - \partial_\nu \{^\alpha_{\mu\sigma} \} + \{^\alpha_{\lambda\mu} \} \{^\lambda_{\nu\sigma} \} - \{^\alpha_{\nu\lambda} \} \{^\lambda_{\mu\sigma} \},$$

and $\widehat{\nabla}_\alpha$ is the LC covariant derivative. Besides, the LC Ricci tensor $\widehat{R}_{\mu\nu} = \widehat{R}^\lambda_{\mu\lambda\nu}$, and LC curvature scalar read:

$$\hat{R}_{\mu\nu} = \partial_\lambda \{^\lambda_{\nu\mu} \} - \partial_\nu \{^\lambda_{\mu\lambda} \} + \{^\lambda_{\sigma\mu} \} \{^\sigma_{\nu\lambda} \} - \{^\lambda_{\nu\sigma} \} \{^\sigma_{\mu\lambda} \},$$

$$\hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu},$$

respectively. We call $R^\alpha_{\sigma\mu\nu}$ as generalized curvature tensor because it is contributed both by LC curvature $\widehat{R}^\alpha_{\sigma\mu\nu}$, and by nonmetricity through disformation $L^\alpha_{\mu\nu}$. We have that,

$$R_{\mu\nu} = \hat{R}_{\mu\nu} + \widehat{\nabla}_\lambda L^\lambda_{\mu\nu} - \widehat{\nabla}_\nu L^\lambda_{\mu\lambda},$$

$$+ L^\lambda_{\nu\sigma} L^\sigma_{\mu\lambda} - L^\lambda_{\mu\nu} L^\sigma_{\lambda\sigma}.$$

$$R = \hat{R} + Q + \partial Q,$$

where the nonmetricity scalar $Q$ and the boundary term $\partial Q$, are defined as it follows:

$$Q := L^\nu \tau_{\nu\lambda} L^{\lambda\kappa} - L^{\nu\kappa} L^\lambda_{\nu\tau\kappa},$$

$$\partial Q := \widehat{\nabla}_\lambda \left( L^{\nu\lambda}_{\nu\kappa} - L^{\lambda\nu}_{\nu\kappa} \right).$$

If take into consideration the teleparallel condition (see equation [22] below), equation [12] is the mathematical basis for the claimed equivalence between GR and STT.

**A. Properties and identities of the curvature in W4**

For the (torsionless) connection $\nabla$ of W4 space it is verified the second Bianchi identity:

$$\nabla_\mu R^\lambda_{\nu\lambda\sigma} + \nabla_\nu R^\lambda_{\mu\lambda\sigma} + \nabla_\sigma R^\lambda_{\lambda\nu\mu} = 0.$$  

(14)

From this identity, taking into account that

$$\nabla_\alpha g_{\mu\nu} = -Q_{\alpha\mu\nu}, \nabla_\alpha g^{\mu\nu} = Q_{\alpha\mu}^{\mu\nu}, Q_{\mu\nu}^{\nu\mu} = Q_{\nu},$$

eq etc., we obtain the following equation:

$$\nabla^\nu G_{\nu\sigma} = \frac{1}{2} (Q_{\alpha\mu}^{\mu\nu} - Q_{\alpha}^{\mu\nu}) R_{\mu\nu}$$

$$+ \frac{1}{2} (Q_{\lambda}^{\mu\nu} - Q_{\lambda}^{\nu\mu}) R^\lambda_{\mu\nu},$$

where $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} R/2$ is the Einstein’s tensor of W4 space. This equation amounts to a generalization of the Bianchi identity of the Einstein’s tensor. In standard Weyl geometry space, since

$$Q_{\alpha} g^{\mu\nu} - Q_{\alpha}^{\mu\nu} = (Q_{\alpha} - Q_{\alpha}) g^{\mu\nu} = 0,$$

the well-known Bianchi identity of the generalized Einstein’s tensor: $\nabla^\nu G_{\nu\alpha} = 0$, is recovered.

The symmetries of the generalized curvature tensor in $W_4$ space differ from those in Riemann space $V_4$. For instance:

$$R^\alpha_{\sigma\mu\nu} = -R^\alpha_{\sigma\nu\mu},$$

$$R_{\alpha\sigma\mu\nu} = -R_{\sigma\alpha\mu\nu} + \nabla_\mu Q_{\nu\alpha\sigma} - \nabla_\nu Q_{\mu\alpha\sigma},$$

The last equation, known as the third Bianchi identity, in compact form can be written in the following way [10]:

$$\nabla_{[\mu} Q_{\nu\rho\sigma]} = R_{(\sigma\alpha)\mu\nu}.$$  

(18)

In standard Weyl geometry where $Q_{\alpha\mu\nu} = Q_\alpha g_{\mu\nu}$, so that $Q_{\mu\nu}^{\nu\mu} = Q_\nu$ and $Q_{\nu}^{\nu\mu} = 4Q_{\nu}$,

$$\nabla_\mu Q_{\nu\alpha\sigma} - \nabla_\nu Q_{\mu\alpha\sigma} = (\nabla_\nu Q_\nu - \nabla_\nu Q_\mu) g_{\alpha\sigma}$$

$$= (\partial_\mu Q_\nu - \partial_\nu Q_\mu) g_{\alpha\sigma},$$

the property [11] can be written in the following way:

$$\partial_\mu Q_\nu g_{\alpha\sigma} = R_{(\sigma\alpha)\mu\nu}.$$  

(19)

Besides, in the particular case when $Q_\alpha = \partial_\alpha \phi$ ($\phi$ is a scalar function), since $\partial_\mu Q_\nu - \partial_\nu Q_\mu = 0$, the generalized curvature tensor possesses the same symmetries of the tensor indices as the Riemann-Christoffel curvature tensor. This special case of Weyl geometry is known in the bibliography as Weyl integrable geometry or WIG for short.

**B. Symmetric teleparallel spacetimes**

We shall call as teleparallel Weyl space $Z_4$, a paracompact, Hausdorff, connected $C^\infty$ four-dimensional manifold $M_4$, endowed with a locally Lorentzian metric $g$ and the generalized (torsionless) connection $\Gamma$ decomposed as in [3] and satisfying [17, 18]:

$$\Gamma^\alpha_{\mu\nu} = \{^\alpha_{\mu\nu} \} + L^\alpha_{\mu\nu} = (\Lambda^{-1})^\alpha_{\lambda} \partial_{\mu} \Lambda^\lambda_{\nu},$$

(20)

where $\Lambda^\mu_{\nu}$ is an element of the general linear group $GL(4, \mathbb{R})$ [17] and $(\Lambda^{-1})^\alpha_{\lambda}$ is its inverse, so that:

$$(\Lambda^{-1})^\alpha_{\lambda} \Lambda^\lambda_{\nu} = \delta^\alpha_{\nu}.\$$

The connection $\Gamma$ is purely inertial. In the absence of torsion this form of the connection leads to the additional constraint $\partial_{[\mu} \Lambda^\alpha_{\nu]} = 0$. Hence, the general element of $GL(4, \mathbb{R})$ determining the connection can be parametrized by a set of functions $c^\mu$ so that [17]:

$$\Gamma^\alpha_{\mu\nu} = \{^\alpha_{\mu\nu} \} + L^\alpha_{\mu\nu} = (\Lambda^{-1})^\alpha_{\lambda} \partial_{\mu} \Lambda^\lambda_{\nu},$$

(20)
\[ R(\Gamma) = 0 \iff R_\sigma^{\alpha \mu \nu} = 0. \] (22)

Hence, teleparallel Weyl space \( Z_4 \) is the same as flat \( W_4 \) space:

\[ W_4 \xrightarrow{R(\Gamma) = 0} Z_4. \] (23)

The constraint \( (21) \) further restricts the fields \( \chi^\mu \), the metric and the nonmetricity to satisfy (see [17] for a similar relationship in the absence of nonmetricity):

\[ \partial_\alpha g_{\mu \nu} + Q_{\alpha \mu \nu} = 2 \frac{\partial x^\lambda}{\partial \chi^\alpha} \partial_\rho (\mu \chi^\sigma g_{\nu \lambda}). \] (24)

Under a specific choice of the field \( \chi^\mu \), the above equation determines fixed relationships between the derivative of the metric and nonmetricity tensor.

In what follows we shall explore the consequences of Weyl gauge symmetry for generalized Weyl spaces, which include the teleparallel spaces as a particular class.

### III. WEYL GAUGE SYMMETRY

WGS or invariance under local changes of scale [13], is a manifest symmetry of generalized Weyl geometry spaces \( W_4 \). In reference [1] an alternative definition of \( W_4 \) space is given, where the WGS is made evident (definition 3 of the mentioned reference): “A conformally generalized Weyl structure is a differentiable manifold \( \mathcal{M} \) endowed with a unique torsion-free affine connection and a conformally related equivalence class of metric tensors.” Here we shall give a very compact exposition of this subject.

The geometric laws that define \( W_4 \) (also \( \tilde{W}_4 \) and \( Z_4 \)), among which is the nonmetricity condition [2], are invariant under generalized (local) Weyl rescalings or, also, Weyl gauge transformations. These amount to simultaneous conformal transformations of the metric \(^1\) and gauge transformations of nonmetricity [11, 16, 19].

\(^1\) Here we assume that the conformal transformation of the metric in the Weyl gauge transformations, does not represent a diffeomorphism or, properly, a conformal isometry. Moreover, the spacetime coincidences or events, as well as the spacetime coordinates that label the points in spacetime, are not modified or altered by the conformal transformations in any way.
which follows from \( \text{[8]} \) under the teleparallel requirement \( R^\alpha_{\sigma\mu\nu} = 0 \). From \( \text{[3]} \) it follows that,

\[
\hat{R} = -Q - \partial Q.
\]  

(32)

On the other hand the teleparallel condition \( \text{[20]} \), or in the form \( R^\alpha_{\sigma\mu\nu} = 0 \), is invariant under \( \text{[25]} \). Moreover, equations \( \text{[3]} \) and \( \text{[31]} \), on which the mentioned equivalence between GR and pure nonmetricity theories of gravity is based, are Weyl gauge invariant as well. We should not forget that the dynamical equivalence between GR and STEGR takes place in flat \( W_4 \) space, i.e., in teleparallel \( Z_4 \) space, which is manifest gauge symmetric. Hence, WGS being a manifest symmetry of \( Z_4 \) space, which is cornerstone to discuss on the SCE. Also, we shall develop the theory of gauge invariant parallel transport, which is required to define gauge covariant differentiation of vectors and tensors in generalized Weyl spaces. Below we shall develop the theory of gauge invariant parallel transport, which is cornerstone to discuss on the SCE. Although the present theory contains new elements not previously considered, to a great extent it is based in the work of references \( \text{[22, 23]} \). Here we assume a statement of the parallel transport law that slightly differs from the one assumed in the latter references and in \( \text{[16]} \).

IV. GAUGE SYMMETRY AND PARALLEL TRANSPORT

Parallel transport consistent with gauge symmetry is required to define gauge covariant differentiation of vectors and tensors in generalized Weyl spaces. Below we shall develop the theory of gauge invariant parallel transport, which is cornerstone to discuss on the SCE. Although the present theory contains new elements not previously considered, to a great extent it is based in the work of references \( \text{[22, 23]} \). Here we assume a statement of the parallel transport law that slightly differs from the one assumed in the latter references and in \( \text{[16]} \).

\[\Gamma^\alpha_{\mu\nu} = 0 \Rightarrow \{^\alpha_{\mu\nu}\} = -L^\alpha_{\mu\nu}, \]  

(33)
i.e., we have that,

\[\partial_\alpha g_{\mu\nu} = -Q_{\alpha\mu\nu}. \]  

(34)

Equations \( \text{[33]} \) and \( \text{[34]} \) are obviously invariant under \( \text{[25]} \). Hence, WGS should play an important role in the development of the STTs as well. In this regard, in \( \text{[19]} \), a family of conformal – thus gauge invariant – theories with second-order field equations and having the metric tensor as the fundamental variable, was formulated within the symmetric teleparallel framework. Our discussion in the present paper must be of importance, at least, for this family of theories. The \( f(Q) \) theories, on the other hand, are not gauge invariant in general, so that, in these cases, the gravitational Lagrangian does not share gauge invariance of \( Z_4 \) space and one may dispense with gauge symmetry.

A. Gauge derivative operators

In order to make the gauge symmetry compatible with well-known derivation rules and with the inclusion of fields into \( W_4 \), it is necessary to introduce the Weyl gauge derivative operators in a way that is equivalent to the one appearing in \( \text{[22, 23]} \). Let \( T \) be a tensor in \( W_4 \), with coordinate components \( T^{\alpha\beta\gamma}_{\mu\nu} \) and with conformal weight \( w(T) = w \), so that under \( \text{[25]} \): \( T \rightarrow \Omega^w T \). Then, the Weyl gauge differential of the tensor, its Weyl gauge derivative and Weyl gauge covariant derivative, respectively, are defined as it follows:

\[d^* T := dT + \frac{w}{2} Q^*_\alpha dx^\alpha T,\]

\[\partial^*_\alpha T := \partial_\alpha T + \frac{w}{2} Q^*_\alpha T,\]

\[\nabla^*_\alpha := \nabla_\alpha + \frac{w}{2} Q^*_\alpha,\]  

(35)

where

\[d^* T = dx^\mu \partial^*_\mu T,\]  

(36)

and

\[Q^*_\alpha = \frac{a}{s} Q_\alpha + \frac{b}{4s} Q^\mu_{\alpha \mu},\]  

(37)

is a linear combination of contributions \( Q_\alpha \equiv Q^\mu_{\alpha \mu} \) and \( Q^\mu_{\alpha \mu} \), with arbitrary constants \( a, b \) and \( s = a + b \). Notice that in \( \tilde{W}_4 \), where \( Q_{\alpha\mu\nu} = Q_\alpha g_{\mu\nu} \), we have that \( Q^*_\alpha \) coincides with the Weyl gauge vector: \( Q^*_\alpha = Q_\alpha \).

The above definitions warrant that the gauge differential, the gauge derivative and the gauge covariant derivative, transform like the geometrical object itself, i.e., under \( \text{[25]} \):

\[d^* T \rightarrow \Omega^w d^* T,\]

\[\partial^*_\alpha T \rightarrow \Omega^w \partial^*_\alpha T,\]

\[\nabla^*_\alpha T \rightarrow \Omega^w \nabla^*_\alpha T.\]

Since it will be useful in the subsequent analysis, as an illustration, let us write the gauge covariant derivative of the metric tensor (the conformal weight of the metric \( w(g) = 2 \)):

\[\nabla^*_\alpha g_{\mu\nu} = -Q^\alpha_{\mu\nu} + Q^*_\alpha g_{\mu\nu}.\]  

(38)

Notice that in \( \tilde{W}_4 \), since \( Q_{\alpha\mu\nu} = Q_\alpha g_{\mu\nu} \) and \( Q^*_\alpha = Q_\alpha \), the gauge covariant derivative of the metric vanishes:

\[\nabla^*_\alpha g_{\mu\nu} = 0.\]
B. Parallel transport in $\tilde{W}_4$ space

In order to discuss on the notion of gauge invariant parallel transport in nonmetricity spaces, we start focusing the discussion in standard Weyl space $W_4$, where the analysis is simpler and then we shall aim at its generalization to $W_4$.

Let $C$ be a curve in $\tilde{W}_4$ that is parametrized by the affine parameter $\xi$: $x^\mu(\xi)$. We can define the gauge covariant derivative along the path $x^\mu(\xi)$ to be given by the following operator:

$$D^* := \frac{d x^\mu}{d \xi} \nabla_\mu^*, \quad (39)$$

where the gauge covariant derivative $\nabla_\mu^*$ is given by $\nabla_\mu^*$ with $Q_\alpha = Q_\alpha - \text{the Weyl gauge vector}$. Then, the parallel transport of given tensor $T$ with coordinate components $T_{\beta_1 \beta_2 \cdots \beta_j}$, along the path $x^\mu(\xi)$, is defined by the following requirement (this definition coincides with the one in [22, 23]):

$$D^* T := \frac{d x^\mu}{d \xi} \nabla_\mu^* T = 0 \iff D^* T_{\beta_1 \beta_2 \cdots \beta_j} = 0. \quad (40)$$

Let us show that (41) is valid only in standard Weyl space $\tilde{W}_4$. For this purpose let us consider the parallel transport law (40) applied to a space-like unit vector $t$ along the path $C$ in generalized Weyl space $W_4$ (arbitrary nonmetricity), with coordinates $x^\mu(\xi)$. The unit vector has coordinate components $t^\mu$, such that:

$$(t, t) = g_{\mu\nu} t^\mu t^\nu = 1, \quad (41)$$

and its weight is $w(t) = -1$. Applying (39) to both sides of (41) and taking into account (35), we get that:

$$D^* t^\alpha := J^\alpha_{\mu\nu} \frac{d x^\mu}{d \xi} t^\nu, \quad (42)$$

where for compactness of writing we have introduced the gauge invariant tensor:

$$J^\alpha_{\mu\nu} := \frac{1}{2} (Q^{\alpha}_{\mu\nu} - Q_{\mu\nu}^\alpha \delta_\nu^\alpha), \quad J_{\mu\alpha\nu} = \frac{1}{2} (Q_{\mu\alpha\nu} - Q_{\alpha\mu\nu}). \quad (43)$$

Notice that this tensor is symmetric in its first and third indices: $J_{\mu\alpha\nu} = J_{\nu\alpha\mu}$.

Hence, in generalized Weyl space $W_4$ with arbitrary nonmetricity $Q_{\alpha\mu\nu} \neq Q_\alpha g_{\mu\nu}$, the unit vector $t$ is not parallel transported along $C$ in the sense of (10). Only in the particular subclass of standard Weyl space $\tilde{W}_4 \subset W_4$, since $Q_{\alpha\mu\nu} = Q_\alpha g_{\mu\nu}$, the tensor (43) vanishes and the unit vector is parallel transported according to (10). From equations (35), (41) and (42) it follows:

$$\frac{D^* t_\alpha}{d \xi} = \frac{d x^\mu}{d \xi} \nabla_\mu t_\alpha = -J^{\mu}_{\alpha\mu} \frac{d x^\mu}{d \xi} t_\nu. \quad (44)$$

C. Dagger derivative

It is convenient to define the gauge covariant “dagger” derivative (or just dagger derivative for short) of the unit tangent vector with components $s^\mu$:

$$\nabla_\alpha^\dagger s^\beta := \nabla_\alpha^\dagger s^\beta - J^{\beta}_{\alpha\mu} s^\mu, \quad (45)$$

and

$$\nabla_\alpha^\dagger t_\beta := \nabla_\alpha^\dagger t_\beta + J^{\mu}_{\alpha\beta} t_\mu. \quad (46)$$

In terms of the dagger derivative equations (42) and (44) can be written in the form of parallel transport equations along the worldline $x^\alpha(\xi)$:

$$\frac{D^\dagger t_\alpha}{d \xi} := \frac{d x^\mu}{d \xi} \nabla_\mu^\dagger t_\alpha = 0, \quad (47)$$

respectively.

For an arbitrary vector $v$ with coordinate components $v^\alpha$ and conformal weight $w(v) = w$, its dagger derivative is defined as it follows:

$$\nabla_\alpha^\dagger v^\beta := \nabla_\alpha^\dagger v^\beta + w J^{\beta}_{\alpha\mu} v^\mu, \quad (48)$$

while

$$\nabla_\alpha^\dagger v_\beta := \nabla_\alpha^\dagger v_\beta + \bar{w} J^{\mu}_{\alpha\beta} v_\mu, \quad (49)$$

where $w(v_\alpha) = \bar{w} = 2 + w$ is the conformal weight of the covariant components of the vector $v$.

In general, for an arbitrary $(p, q)$-tensor $T$, with coordinate components $T_{\alpha_1 \alpha_2 \cdots \alpha_p}$ and with conformal weight $w(T) = w$, its dagger derivative is defined as:

$$\nabla_\mu^\dagger T_{\beta_1 \beta_2 \cdots \beta_q} = \nabla_\mu^\dagger T_{\beta_1 \beta_2 \cdots \beta_q} + \frac{w}{p + q} \left( \left( T_{\alpha_1 \alpha_2 \cdots \alpha_p} - \frac{1}{\alpha_1} \delta_{\alpha_1}^\alpha \right) \frac{1}{\alpha_1} \alpha_1 \delta_{\alpha_1}^\alpha \right)$$

$$+ \cdots + J_{\mu\alpha\beta}^\nu T_{\beta_1 \beta_2 \cdots \beta_q} + \frac{w}{p + q} \left( \left( T_{\alpha_1 \alpha_2 \cdots \alpha_p} - \frac{1}{\alpha_1} \delta_{\alpha_1}^\alpha \right) \frac{1}{\alpha_1} \alpha_1 \delta_{\alpha_1}^\alpha \right)$$

$$+ \cdots + J_{\mu\beta_1 \beta_2 \cdots \beta_q}^\alpha T_{\alpha_1 \alpha_2 \cdots \alpha_p} \right). \quad (50)$$

3 Since under the gauge transformations $Q_\alpha$ transforms like $Q_\alpha$, the tensor $J^\alpha_{\mu\nu}$ is gauge invariant.
D. Parallel transport law in $W_4$ space

In order to consistently define parallel transport in $W_4$, in a such a way that any vector or tensor can be parallel transported in the usual way: constant coordinate components along parallel transport curve, we must introduce what we call as dagger derivative along the path of parallel transport $C$ with coordinates $x^\alpha(\xi)$.

Consider an arbitrary vector $v$ with coordinate components $v^\alpha$ and conformal weight $w(v) = w$. The dagger derivative of this vector along $C$ is defined in the following way:

$$D^\dagger v^\alpha := \frac{dx^\mu}{d\xi} \nabla^\dagger_\mu v^\alpha. \quad (51)$$

We require that under an infinitesimal parallel transport along the curve $C$, the components of this vector do not change:

$$D^\dagger v^\alpha = 0.$$  

In general, for an arbitrary $(p, q)$-tensor $T$, with coordinate components $T^\alpha_1\alpha_2\cdots^\alpha_p_\beta_1\beta_2\cdots^\beta_q$ and with conformal weight $w(T) = w$, its dagger derivative is given by $[50]$. Hence, the parallel transport law requires that the dagger derivative of the tensor $T$ along the path of parallel transport $C$, vanishes:

$$D^\dagger T := \frac{dx^\mu}{d\xi} \nabla^\dagger_\mu T = 0. \quad (52)$$

This equation drives parallel transport of tensor $T$ along $x^\alpha(\xi)$ in generalized Weyl space $W_4$. Worth noting that in standard Weyl space $W_4$, since $Q_{\alpha\mu\nu} = Q_{\alpha} g_{\mu\nu}$, then $J^\dagger_{\mu\nu} = 0$, so that $\nabla^\dagger_\alpha \rightarrow \nabla_\alpha$, $D^\dagger/d\xi \rightarrow D^*/d\xi$, and $[52]$ transforms into $[40]$.

Of course, different statements of the parallel transport law lead to different results. In $[10]$, for instance, the parallel transport law $[40]$ is applied to generalized Weyl space $W_4$, which led to the conclusion that unit tangent vectors with weight $w = -1$ do not obey the parallel transport equation $[40]$ in $W_4$. What we have shown in this section is that it is possible to postulate the parallel transport law $[52]$, which is satisfied by any vectors and tensors in $W_4$ space.\footnote{A similar situation to the one described above occurs in $[10]$ during the demonstration of lemma 2, where the assumed statement of parallel transport law is not satisfied by unit tangent vectors of weight $w = -1$ (see section VII).}

E. Variation of length during parallel transport

It has been well established in the bibliography that in Weyl geometry spaces, during parallel transport, the length of vectors (and tensors) varies and depends on followed path. Given that there has been a renaissance of old discussions about this issue $[10]$, in this section, for sake of generality, we aim at explaining this effect in spaces with arbitrary nonmetricity $Q_{\alpha\mu\nu}$.

Let us assume that the vector $v$ with components $v^\alpha$ and conformal weight $w(v) = w$, is submitted to parallel transport along the path $C$, which, as before, is parametrized by the affine parameter $\xi$: $x^\mu = x^\mu(\xi)$. The length $v \equiv \|v\|$ of the vector is defined as,

$$v^2 = g_{\mu\nu} v^\mu v^\nu, \quad (53)$$

where for space-like vector $v^2 > 0$, while for timelike vector $v^2 < 0$ ($v$ is an imaginary quantity). Taking the gauge covariant derivative along the path $C$ in both sides of $[53]$, we obtain that,

$$\frac{D^* v^2}{d\xi} = \frac{D^* g_{\mu\nu}}{d\xi} v^\mu v^\nu + 2 g_{\mu\nu} v^\mu D^* v^\nu = 0. \quad (54)$$

If realize that along the path of parallel transport:

$$\frac{D^* g_{\mu\nu}}{d\xi} = -2 J_{\mu\lambda\nu} \frac{dx^\lambda}{d\xi}, \quad (55)$$

and if further take into account $[48]$, then equation $[54]$ can be written in the form of the parallel transport law $[52]$.

$$\frac{D^\dagger v^2}{d\xi} = \frac{dx^\lambda}{d\xi} \nabla^\dagger_\lambda v^2 = 0, \quad (56)$$

where we have defined the dagger covariant derivative of the length squared

$$\nabla^\dagger_\alpha v^2 := \nabla^\alpha_\alpha v^2 + 2(1+w) J_{\mu\lambda\nu} v^\mu v^\nu. \quad (57)$$

Besides, if define the spacelike unit vector $t$:

$$t := \frac{v}{v} \Leftrightarrow t^\alpha = v^\alpha/v, \ (t, t) = g_{\mu\lambda\nu} t^\mu t^\nu = 1, \quad (58)$$

where the conformal weight $w(t) = -1$, and recall the definition of gauge differential in $[35]$, then the parallel transport law $[56]$ can be finally written in the form:

$$\frac{D^\dagger v^2}{d\xi} = 0 \Rightarrow \frac{d\ln v}{d\xi} = \frac{w + 1}{2} Q_{\lambda\mu\nu} t^\mu t^\nu \frac{dx^\lambda}{d\xi}. \quad (59)$$

Formal integration of $[59]$.\footnote{A similar situation to the one described above occurs in $[10]$ during the demonstration of lemma 2, where the assumed statement of parallel transport law is not satisfied by unit tangent vectors of weight $w = -1$ (see section VII).}
\[ \Delta \ln v = -\frac{w + 1}{2} \int_{C} Q_{\lambda\mu\nu} t^{\mu} t^{\nu} dx^{\lambda}, \tag{60} \]

leads to the equation of variation of the length of vector \( v \) during parallel transport along path \( C \), from the origin \( x = 0 \) to the spacetime point \( x \):

\[ v(x) = v(0) \exp \left( -\frac{w + 1}{2} \int_{C} Q_{\mu} dx^{\mu} \right), \tag{65} \]

where \( v(0) = C \) is an integration constant which we identify with the magnitude of the length of vector \( v \), evaluated at the starting point of the worldline \( C \).

Let us apply equation (61) to the four-momentum vector \( p \), which is at the core of the SCE. Let the path of the particle be parameterized by the proper time \( \tau \). The coordinate components of the four-velocity vector \( u \) are \( u^\mu := dx^\mu/d\tau \), where \( d\tau = ds \) (\( s \) is the arc-length and the imaginary unit \( i \) arises due to our metric signature choice). Consequently, the length of the four-velocity vector \( u \) is the magnitude of the length of vector \( v(u) = -1 \). After the above specifications we have that,

\[ p := mu \Rightarrow p \equiv ||p|| = \pm im, \tag{62} \]

where \( m \) is the mass of a point particle and the conformal weight \( w(p) = -2 \) (the conformal weight of the mass parameter \( w(m) = -1 \)). If in (61) we make the replacement \( p \rightarrow v \), we obtain the following equation:

\[ m(x) = m(0) \exp \left( -\frac{1}{2} \int_{C} Q_{\lambda\mu\nu} w^\mu w^\nu dx^{\lambda} \right), \tag{63} \]

where we have taken into consideration that the components of the spacelike unit vector \( t \), that is tangent to \( C \), are given by:

\[ t^\mu = \frac{p^\mu}{p} = \frac{mdx^\mu/d\tau}{\pm im} = \pm i dx^\mu/d\tau = \pm iw^\mu. \tag{64} \]

Equation (63) quantifies the variation of the mass of a particle which is moving with speed \( u^\mu = dx^\mu/d\tau \) along the path \( C \). The dependence of the mass parameter on the speed pattern of the particle during its motion, which is apparent in (63), is a new effect that does not arise in \( \tilde{W}_4 \) space.

1. **Standard Weyl space \( \tilde{W}_4 \)**

In \( \tilde{W}_4 \), since \( Q_{\alpha\mu\nu} = Q_{\alpha}g_{\mu\nu} \) and, consequently the vector \( Q^\mu \) coincides with the Weyl gauge vector, equation (61) simplifies to:

\[ v(x) = v(0) \exp \left( -\frac{w + 1}{2} \int_{C} Q_{\mu} dx^{\mu} \right), \]

while the equation (63) transforms into the following equation:

\[ m(x) = m(0) \exp \left( \frac{1}{2} \int_{C} Q_{\mu} dx^{\mu} \right). \tag{66} \]

Equation (63) is the basis for the explanation of the SCE in \( W_4 \), while (66) drives the SCE in \( \tilde{W}_4 \).

F. **Variation of the inner product of vectors during parallel transport**

For generality, let us consider also the parallel transport of the inner product of two vectors. Take two vector fields \( v \) and \( w \) with coordinate components \( v^\mu \), \( w^\mu \) and conformal weights \( w(v) = w_v \) and \( w(w) = w_w \), respectively. Let these vector fields be parallel transported along the curve \( C \), that is parametrized by \( \xi \):

\[ D^\xi v^\mu/d\xi = 0 \quad \text{and} \quad D^\xi w^\mu/d\xi = 0. \]

Using the same procedure that we followed above, we get that:

\[ \frac{d(v, w)}{d\xi} = -(2 + w_v + w_w)Q_{\lambda\mu\nu} \frac{dx^\lambda}{d\xi} v^\mu w^\nu. \tag{67} \]

Next, take into account the following chain of equalities:

\[ (v, w) = vw \cos \theta \quad \Rightarrow \quad v^\mu w^\nu = t_v^\mu t_w^\nu \cos \theta = \frac{1}{\kappa} t_v^\mu t_w^\nu (v, w), \tag{68} \]

where \( t_v^\mu \) and \( t_w^\mu \) are spacelike unit vectors, and \( \kappa \equiv \cos \theta \) (\( \theta \) is the angle between vectors \( v \) and \( w \)) is a real constant taking values in the interval \([-1, 1]\). If substitute (68) into (67), we get that,

\[ \frac{d\ln(v, w)}{d\xi} = -\frac{2 + w_v + w_w}{\kappa} Q_{\lambda\mu\nu} t_v^\mu t_w^\nu \frac{dx^\lambda}{d\xi}, \tag{69} \]

whose integration along the closed path \( C \) yields to:

\[ \Delta \ln(v, w) = -\frac{2 + w_v + w_w}{\kappa} \int_{C} Q_{\lambda\mu\nu} t_v^\mu t_w^\nu dx^\lambda. \tag{70} \]

In the latter equation the quantity \( \kappa \) has been taken out of the integral since, as it can be straightforwardly

\footnote{Recall that \( v := ||v|| \) and \( w := ||w|| \) are the lengths of vectors \( v \) and \( w \), respectively.}
demonstrated, it does not depend on path: \( ds/d\xi = 0 \). Equation (70) can be written in the following way:

\[
(v, w) = (v, w)_0 e^{-\frac{2}{\kappa} \int \xi \, \mathcal{L}_{\gamma_{\mu_\nu} t_{\mu_\nu}^* w} d\xi}.
\]

(71)

It should be emphasized that equations (65), (66), (69), (81), (82), (85) are a direct consequence of the nonmetricity law (2) of generalized Weyl space \( W_4 \), while (52) is our assumed parallel transport law. Equations (60), (61), (63) and related equations (70) and (71) above, are also consequence of the nonmetricity law (2) and of (52). This means that if (2) is valid in \( W_4 \), then the mentioned and related equations are valid in \( W_4 \) as well. This conclusion is important because the adoption of \( W_4 \) as the underlying geometric background space, entails that the SCE is inevitable. Otherwise, if one denies the physical occurrence of the SCE, one renounces to equations like (63) and to the parallel transport law (52), i.e., one renounces to identify physical vectors and tensors with corresponding hypothetical vectors and tensors in \( W_4 \). Consequently, one gives up the description of physical phenomena in \( W_4 \) space (see the related discussion in [16]).

V. GAUGE SYMMETRY, AUTO-PARALLELS AND GEODESICS

In general auto-parallels – “straightest curves” of the geometry – do not coincide with the geodesics, which are the “shortest curves” [26, 27]. There goes a discussion on whether auto-parallels or geodesics describe the motion of test particles [26, 27]. However, there are particular cases when auto-parallels and geodesics coincide as, for instance, in GR. As we shall see, these coincide as well in \( W_4 \). Anyway, geodesics and auto-parallels can be associated exclusively with the motion of spinless test (point) particles. Spinor fields like the fermions obey the Dirac equation in curved background, while extended spinning test bodies obey the Mathisson-Papapetrou-Dixon equations [28, 31] (see section VIII).

[6] We have that \( \kappa = (v_\mu, t_{\nu_\mu} \equiv g_{\mu_\nu} t_{\mu_\nu}^* w_\nu) \). Take the gauge covariant derivative along the path \( x^\alpha(\xi) \), in both sides of this equation:

\[
\frac{dx}{d\xi} = D^\alpha g_{\mu_\nu} t_{\mu_\nu}^* w_\alpha + g_{\mu_\nu} D^\alpha t_{\mu_\nu}^* w_\alpha + \kappa t_{\mu_\nu}^* t_{\mu_\nu}^* w_\alpha.
\]

Hence, according to (52), we get that,

\[
\frac{dx}{d\xi} = D^\alpha g_{\mu_\nu} t_{\mu_\nu}^* w_\alpha + (J_{\mu_\alpha} t_{\mu_\nu}^* w_\nu + J_{\nu_\alpha} t_{\mu_\nu}^* w_\mu) \frac{dx^\alpha}{d\xi},
\]

so that, if take into account (55) and the symmetry of tensor \( J_{\mu_\alpha} \) in its first and third indices, we finally obtain that \( dx/d\xi = 0 \).

Here we obtain a bit different results compared with those that were obtained in [16], on the basis of a different transport law (40) instead of (52).

A. Auto-parallels

In generalized Weyl space \( W_4 \) the “timelike” auto-parallels are those curves along which the gauge covariant derivative of the tangent four-velocity vector \( v_\alpha \) vanishes. Here \( u_\mu = dx^\mu/d\tau \) are the coordinate components of \( u_\alpha \) and, as long as this does not cause loss of generality, we chose the proper time \( \tau \) to be the affine parameter along the auto-parallel curve. The conformal weight of the four-velocity vector \( w_\alpha \) is −1. In other words, the auto-parallel curves satisfy:

\[
D^\alpha v_\alpha = u_\mu \nabla^\mu u_\alpha = 0 \Rightarrow \frac{d u_\alpha}{d\tau} + \Gamma^\alpha_{\mu_\nu} u_{\mu} u_{\nu} - \frac{1}{2} Q^\alpha_{\mu_\nu} u_{\mu} u_{\nu} = 0,
\]

or, in explicit form, in terms of the arc-length \( d\tau \rightarrow ids \):

\[
\frac{d^2 x_\alpha}{ds^2} + \Gamma^\alpha_{\mu_\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - \frac{1}{2} Q^\alpha_{\mu_\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \iff \frac{d^2 x_\alpha}{ds^2} + \{\alpha_{\mu_\nu}\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.
\]

(72)

In the same fashion, in \( W_4 \) the “null” auto-parallels are those curves along which the gauge covariant derivative of the wave vector \( \kappa_\alpha := dx^\alpha/\lambda \) (\( \lambda \) is a parameter along the null auto-parallel), vanishes:

\[
D^\alpha \kappa_\alpha = k_\mu \nabla_\mu \kappa_\alpha = 0 \Rightarrow \frac{d k_\alpha}{d\lambda} + \Gamma^\alpha_{\mu_\nu} k_\mu k_\nu - Q^\alpha_{\mu_\nu} k_\mu k_\nu = 0,
\]

where we have taken into account that the conformal weight of the wave vector \( w_\alpha = -2 \), i.e., it coincides with the weight of the four-momentum since, in the quantum limit both should be related by \( p = \hbar k \). Then, the auto-parallel null curves satisfy the following equations:

\[
\frac{d k_\alpha}{d\lambda} + \Gamma^\alpha_{\mu_\nu} k_\mu k_\nu - Q^\alpha_{\mu_\nu} k_\mu k_\nu = 0 \iff \frac{d k_\alpha}{d\lambda} + \{\alpha_{\mu_\nu}\} k_\mu k_\nu = 0.
\]

(73)

[7] Here it is implicitly assumed that the universal constant \( \hbar \) is not transformed by the Weyl gauge transformations.
In standard Weyl space \( \tilde{W}_4 \), since \( Q_{\alpha\mu} = Q_\alpha g_{\mu\nu} \), then, according to [37],

\[
Q^\alpha = \frac{a}{s} Q_\alpha + \frac{b}{4} Q_\alpha \delta^\mu_\alpha = \frac{a + b}{s} Q_\alpha = Q_\alpha.
\]

In this case for the timelike autoparallels we have that:

\[
\frac{d^2 x^\alpha}{d\lambda^2} + \left\{ \alpha_{\mu\nu} \right\} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - \frac{1}{2} Q_{\mu\nu} h^{\mu\alpha} = 0,
\]

(74)

where

\[
h^{\mu\alpha} := g^{\mu\alpha} + u^\mu u^\alpha = g^{\mu\alpha} - \frac{dx^\mu}{d\lambda} \frac{dx^\alpha}{d\lambda},
\]

(75)

is the orthogonal projection tensor, which projects any vector or tensor onto the hypersurface orthogonal to the four-velocity vector \( u^\mu = dx^\mu/d\tau \). Meanwhile, for null autoparallels, since

\[
Q^\alpha_{\mu\nu} k^{\mu} k^{\nu} = Q^\alpha g_{\mu\nu} k^{\mu} k^{\nu} = 0,
\]

one obtains the standard GR result:

\[
\frac{dk^\alpha}{d\lambda} + \left\{ \alpha_{\mu\nu} \right\} k^{\mu} k^{\nu} = 0.
\]

(76)

**B. Geodesic equations**

The geodesic equations are equations of motion in the sense that these are the result of applying the variational principle of least action. Time-like and null particles follow geodesics. When these are compared with the corresponding auto-parallels one can measure how much the motion paths depart from the straightest curves of the geometry.

In the GR context the action of timelike particles reads

\[
S = m \int ds,
\]

where \( m \) is the constant mass parameter. In generalized Weyl space \( W_4 \), since the mass, being the squared length of the four-momentum of the particle, varies in spacetime, then \( m \) can not be taken out of the action integral. The action integral in \( W_4 \) reads:

\[
S = \int m ds.
\]

From this action the following equations of motion – geodesic equations – can be derived (see [16] for details of the derivation):

\[
\frac{d^2 x^\alpha}{d\lambda^2} + \left\{ \alpha_{\mu\nu} \right\} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - \frac{1}{2} \frac{\delta m}{m \delta x^\mu} h^{\mu\alpha} = 0,
\]

(77)

where the non-Riemannian term \( \delta m/m \delta x^\mu \) accounts for the variation of mass during parallel transport. If one writes equation (63) in variational form, one obtains:

\[
\frac{1}{m} \frac{\delta m}{\delta x^\mu} = -\frac{1}{2} Q_{\mu\nu} h^{\mu\alpha} w^\nu = \frac{1}{2} Q_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}.
\]

(78)

In order to determine the term \( \delta m/m \delta x^\mu \) in (77) we assume a “consistency” hypothesis [16]. This hypothesis (or postulate if you want) allows to identify hypothetical vectors and tensors living in \( W_4 \) (for instance the four-momentum (62)) with the corresponding physical vectors and tensors. Otherwise one can not describe given physical phenomenon in \( W_4 \) space. In the present case the most reasonable consistency hypothesis is to identify the physical mass parameter in (77) with the hypothetical mass parameter \( m = ip \) which obeys (63). In other words, we substitute equation (78) back into (77). We get that:

\[
\frac{d^2 x^\alpha}{d\lambda^2} + \left\{ \alpha_{\mu\nu} \right\} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - \frac{1}{2} \frac{\delta m}{m \delta x^\mu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0.
\]

Worth noting that, in standard Weyl space \( \tilde{W}_4 \), since \( Q_{\alpha\mu} = Q_\alpha g_{\mu\nu} \), equation (79) transforms into the time-like autoparallel (74) of \( W_4 \). Hence, the autoparallels of \( \tilde{W}_4 \) and the corresponding geodesics, coincide.

This is true as well of the null autoparallels and geodesics. Actually, the null geodesic equations can be derived from the following action:

\[
S_{null} = \frac{1}{2} \int g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu d\xi,
\]

(80)

where the dot accounts for derivative with respect to the parameter \( \lambda \) of the path \( x^\mu(\lambda) \) followed by photons (by radiation in general). From (80) the GR null geodesic equations (75) are obtained. These coincide with the null autoparallels.

The null geodesic equations do not depend on \( Q_{\alpha\mu} \). This means that photons and radiation probe the Riemann affine structure of spacetime. In other words, photons and radiation interact only with the metric field, i.e., with the LC curvature of spacetime. These do not interact with nonmetricity.

**VI. GAUGE SYMMETRY AND THE SECOND CLOCK EFFECT**

Let us base the physical analysis of the SCE on the functioning of an atomic clock which measures the International Atomic Time. The principle of operation of

\footnote{Notice that we keep using variation instead of differentiation to underline that, in general, \( \delta m \) is not a perfect differential.}
an atomic clock is based on atomic physics: it measures the electromagnetic signal that electrons in atoms emit when they change energy levels. For instance, the energy of each energy level in the hydrogen atom, labeled by \( n \), is given by: \( E_n \approx -\frac{m\alpha^2}{2n^2} \), where \( m \) is the mass of the electron and \( \alpha \approx 1/137 \) is the fine-structure constant. Any changes in the mass \( m \) over spacetime will cause changes in the energy levels and, consequently, in the energy of the atomic transitions

\[
\omega_{if} = |E_{n_f} - E_{n_i}| = \frac{ma^2}{2} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right). 
\]

Hence, the functioning of atomic clocks will be affected by the variation of masses over spacetime, according to equation (63).

Let us consider a collection of identical atoms that are parallel transported along neighboring paths from the origin \( x = 0 \) to a given point \( x \). Let us take the larger difference arising between the masses of any two atoms in the collection at \( x \): \( \Delta m = m(x) - \bar{m}(x) \). Then, according to (63) one gets the following gauge invariant ratio:

\[
\frac{\Delta \omega_{if}}{\omega_{if}} = \frac{\Delta m}{m} = 1 - \exp \left[ Q_C'(x) - Q_C(x) \right], 
\]

where \( \Delta \omega_{if} \) quantifies the broadening of the given spectral line and we have adopted the following notation (recall that \( u^\mu = dx^\mu/d\tau \) is the four-velocity):

\[
Q_C^\mu(x) = -\frac{1}{2} \int_C Q_{\mu\nu\sigma} u^\nu u^\sigma dx^\mu. 
\]

The novel feature in \( W_4 \) is that the broadening of the spectral lines depends not only on the followed path, but also on the speed pattern. Hence, even if two identical atoms at the origin, are parallel transported along a same path to the distant point \( x \), but following different speed patterns, at \( x \) the emission/absorption lines of one of the atoms will be shifted with respect to the same spectral lines of the other atom. In order to illustrate this novel feature, let us for brevity make the following identification:

\[
\Delta Q_C^\mu = Q_C^\mu(x) - Q_C^\mu(x) = \frac{1}{2} \int_C Q_{\lambda\mu\nu} (u^\mu u^\nu - u'^\mu u'^\nu) dx^\lambda, 
\]

where \( Q_C^\mu \) and \( Q_C'^\mu \) are given by (82). In the above equation \( u' \) stands for the four-velocity of one of the atoms, while \( u \) represents the four-velocity vector of the second atom. Then, as long as \( u' \neq u \), there is a non-vanishing relative shift of given spectral lines:

\[
\frac{\Delta \omega_{if}}{\omega_{if}} = 1 - \exp \Delta Q_C^\mu. 
\]

When instead of just two atoms, a sample of atoms of given substance is considered, the above discussed shift leads to an effective broadening of spectral lines that may be quantified by the largest possible shift \( \Delta \omega_{if} \). This shift is not to be confused with the GR shift of frequencies which is due to the propagation of photons in a curved background space, and is the same for any frequency. In Weyl space \( W_4 \) the first clock effect arises as well due to the Riemann-Christoffel (Levi-Civita) curvature of space, leading to the same shift of frequencies for any spectral line. The SCE, on the contrary, is due to the above described shift of frequencies which is different for different frequencies: \( \Delta \omega_{if} \propto \omega_{if} \), as it can be seen from equations (81) and (84).

In \( W_4 \) for the gauge-invariant ratio (81), according to (66), one gets:

\[
\frac{\Delta \omega_{if}}{\omega_{if}} = 1 - \exp \left( \int_C Q_\mu dx^\mu - \int_C Q'_\mu dx^\mu \right). 
\]

Notice that in this case the given spectral line is sharp: \( \Delta \omega_{if} = 0 \), only if either \( C' = C \), or if \( Q_\mu = \partial_\mu \phi \), where \( \phi \) is the Weyl gauge scalar. In this last case \( \int_C \partial_\mu \phi dx^\mu = \phi(x) - \phi(0) \), independent of the path joining the starting and final points. WIG is the resulting geometric structure.

\section{VII. CHALLENGING THE SCE}

There are in the bibliography points of view that are contrary to the occurrence of the SCE \cite{10,13,14,26}. Let us first comment on \cite{10} and on \cite{26} which deal with the SCE in generalized Weyl space \( W_4 \) (specifically in its subset \( Z_4 \)), and then we shall discuss on the point of view developed in \cite{13,14} where the SCE in \( W_4 \) is challenged.

In \cite{10} it is demonstrated that the SCE does not arise in symmetric teleparallel theories. This is done through lemma 2 of the mentioned reference. Their result is obtained by ignoring the gauge symmetry which is a manifest symmetry of generalized Weyl space \( W_4 \). Besides, among the most important assumptions in the demonstration of lemma 2. is that the covariant derivative of vectors fields – take, for instance, the four-velocity vector \( u \) – vanishes during parallel transport along a given curve \( x^\mu(\xi) \): \( \nabla^\mu u^\mu = 0 \). Hence, even if renounce to gauge symmetry, the demonstration is not valid for tangent vectors of weight \( w = -1 \). Actually, consider a tangent vector field \( u \) to the curve \( C \) that is parametrized by

\section{VIII. CONCLUSION}

As seen in section \cite{10} photons and radiation interact only with the LC curvature of spacetime.

Although in \cite{10} the torsion contribution is considered, here we omit it for simplicity and because it is behind the scope of the present paper.
\[\xi.\] Its conformal weight is \(w(u) = -1\), while its length \(g_{\mu\nu}u^\mu u^\nu = -1\). By taking the covariant derivative of this last equation one gets:
\[\nabla_\alpha g_{\mu\nu} u^\mu u^\nu + 2 g_{\mu\nu} u^\nu \nabla_\alpha u^\mu = 0,
\]
from where it follows that:
\[\nabla_\alpha u^\mu = \frac{1}{2} Q_{\alpha \nu} u^\nu. \tag{86}\]
This result is valid for any tangent vector whose weight \(w = -1\), so that its length has vanishing weight. This means that the result holds true even if consider gauge symmetry.

The demonstration of lemma 2 in [10] starts with the following equation for the variation of the inner product of two vectors \(u\) and \(v\) parallel transported along a closed path \(\bar{C}\) (equation (12) of the mentioned reference):
\[\Delta (u, v) = - \int_{\bar{C}} Q_{\mu\alpha\beta} u^\alpha v^\beta dx^\mu. \tag{87}\]

After applying Stoke’s theorem we have that:
\[
\begin{align*}
\int_{\bar{C}} Q_{\mu\alpha\beta} u^\alpha v^\beta dx^\mu &= \int_S \left[ \nabla_\mu \left( Q_{\nu\alpha\beta} u^\alpha v^\beta \right) \right] dx^\mu \wedge dx^\nu \\
&= \int_S \left[ R(\alpha\beta)_{\mu\nu} u^\alpha v^\beta + \nabla_\nu \left( Q_{\nu\alpha\beta} u^\beta \right) \right] dx^\mu \wedge dx^\nu,
\end{align*}
\]
where, in the last line we have taken into account the third Bianchi identity [15]. Then in [10] it is assumed that, since vectors \(u\) and \(v\) are both parallel transported (their inner product to be specific), then both \(\nabla_\mu u^\alpha = 0\) and \(\nabla_\mu v^\beta = 0\), and the second term within the surface integral vanishes. As we shall see, this assumption is correct only if, besides ignoring gauge symmetry, none of the vectors \(u\) and \(v\) had weight \(w = -1\). On the contrary, if one of the vectors in the inner product, say vector \(u\), has weight \(w(u) = -1\), then equation (87) is to be satisfied. In this case
\[
\begin{align*}
\int_{\bar{C}} Q_{\mu\alpha\beta} u^\alpha v^\beta dx^\mu &= \int_S \left[ R(\alpha\beta)_{\mu\nu} + \frac{1}{2} g_{\mu\lambda} Q_{\nu\lambda\beta} \right] u^\alpha v^\beta dx^\mu \wedge dx^\nu.
\end{align*}
\]
This means that the demonstration of lemma 2 in [10] is valid only if neither the weight of vector \(u\) nor that of vector \(v\) equal \(-1\). If one of the vectors has weight \(w = -1\), then instead of equation (15) of that reference, one obtains (see appendix A) for a gauge invariant version of the demonstration:
\[
\Delta (u, v) = - \frac{1}{2} \int_S Q_{\mu\lambda} (u Q_{\nu\lambda\beta}) u^\alpha v^\beta dx^\mu \wedge dx^\nu. \tag{88}\]

where we have set \(R(\alpha\beta)_{\mu\nu} = 0\) in order to satisfy the teleparallel condition. Hence, even under the teleparallel condition there is a net (nonvanishing) variation of the inner product of vectors during parallel transport in a closed path, contrary to the result of [10]. Yet, our main argument against demonstrations of the kind found in [10]—and in many other bibliographic references, including textbooks like [2]—is in the closed path \(\bar{C}\) required [87]. In section IX we shall comment on this.

Let us underline that the demonstration of lemma 2 of [10], is made on the basis of equation (87) for parallel transport of a scalar product of vectors—equation (12) of the mentioned reference—which is not a gauge covariant equation (compare with the related gauge covariant expression (70)). In appendix A we show how gauge symmetry modifies the above result.

In reference [26] the authors proposed a novel prescription for the parallel transport of tangent vectors that allows keeping its length the same during parallel transport along a given curve. This may be an interesting possibility that does not contradict our result that the SCE is inevitable in spaces with arbitrary nonmetricity. As we have shown the length of tangent vectors with conformal weight \(w = -1\) does not change during parallel transport. However, the length of vectors with weight \(w \neq -1\) inevitably changes along the path of parallel transport. This is particularly true for the four-momentum vector \(p = m u\), which is central to the explanation of the SCE. Additionally, the authors of [26] do not pay attention to the gauge symmetry, which is our guiding principle.

In references [13, 14] it is claimed that the nonoccurrence of the SCE is generic in \(W_4\). In order to show that it is so, an elaborate explanation is given, which is based on the assumption that there are vectors such as the four-velocity and four-momentum of an atom, that are not parallel transported along atom’s world line even if it is in free fall [14]. The above scenario is physically implemented by assuming that certain scalar (compensator) field \(\phi\) of non-geometric origin, gives masses to point-particles \(m \propto \phi(x)\). Statements like “there are physical vectors that are not parallel transported along given world line”, represent a very strong hypothesis, affecting the geometrical description of involved phenomenon. This statement rules out, for instance, generalized Weyl geometry space \(W_4\), as a potential arena for the geometrical description of given gravitational phenomena.

As it is discussed at the end of section IV, the adoption of \(W_4\) (and its subclass \(W_4\)) as the underlying geometric background space entails that the SCE is inevitable. Otherwise, renouncing to identify hypothetical vectors in \(W_4\), such as the four-momentum [22], with the corresponding physical vectors, such as the four-momentum of an atom, means that one is renouncing to the description of physical phenomena in \(W_4\) space (see the discussion on
VIII. CONSISTENCY HYPOTHESIS AND MATTER COUPLING TO NONMETRICITY

There is yet another counterargument that, according to 
\cite{32, 33}, allows avoiding the issue associated with the SCE in standard Weyl space \( \hat{W}_4 \) (vectorial nonmetricity: \( Q_{\alpha \mu \nu} = Q_{\alpha} g_{\mu \nu} \)). Although in the mentioned bibliographic references the argument was demonstrated in \( \hat{W}_4 \) space exclusively, it may be extended to generalized Weyl space \( W_4 \) as well \cite{11, 34, 33, 32, 33}.

The argument exposed in \cite{32, 33} goes like this. The Lagrangian density of the fermion coupled with the gravitational field reads (below, for simplicity, we omit \( SU(2) \otimes U(1) \) gauge terms):

\[
\mathcal{L}_{\text{fermion}} = i \bar{\psi} \slashed{D} \psi, \tag{89}
\]

where \( \psi \) is the Dirac spinor (\( \bar{\psi} \) is its adjoint) and the slash gauge derivative is defined as:

\[
\slashed{D} := \gamma^\mu \hat{D}_\mu = \gamma^a e^a_\mu \left( \partial_\mu - \frac{1}{2} \sigma_{ab} \hat{\omega}^{ab} + \cdots \right). \tag{90}
\]

In this equation \( \gamma^a \) are the (flat) Dirac gamma matrices, \( e^a_\mu \) is the tetrad and the ellipsis stands for the missing terms corresponding to the gauge fields \( W_\mu^{(i)} \), \( B_\mu \) of the gauge group \( SU(2) \otimes U(1) \), which are not transformed by the conformal transformation of the metric. In \( \hat{W}_4 \), the Riemannian spin connection \( \hat{\omega}^{ab} \) and the commutator of the gamma matrices \( \sigma^{ab} \) (the generators of the Lorentz group in the spin representation), read:

\[
\hat{\omega}^{ab} = \epsilon^{bv} (\partial_\mu e^a_v - \lambda^a_{\lambda \mu}), \quad \sigma^{ab} = \frac{1}{4} (\gamma^{a b} - \gamma^{b a}), \tag{91}
\]

respectively. Let us consider the following gauge transformations:

\[
\begin{align*}
g_{\mu \nu} &\rightarrow \Omega^2 g_{\mu \nu}, \quad Q_\mu &\rightarrow Q_\mu - 2 \partial_\mu \ln \Omega, \quad \psi &\rightarrow \Omega^{-3/2} \psi, \\
e^a_\mu &\rightarrow \Omega^{-1} e^a_\mu, \quad \gamma^a &\rightarrow \gamma^a \leftrightarrow \sigma^{ab} \rightarrow \sigma^{ab}.
\end{align*} \tag{92}
\]

Under the above gauge transformations:

\[
\bar{\psi} \slashed{D} \psi \rightarrow \Omega^{-4} \bar{\psi} \slashed{D} \psi.
\]

Hence, the Lagrangian \( \sqrt{-g} \mathcal{L}_{\text{fermion}} \) in its Riemannian form, is already invariant under \( \hat{W}_4 \). In other words, it is not required to make the replacements (the weight \( w \) depends on whether the derivative acts on the spinor field or on the tetrad):

\[
\partial_\mu \rightarrow \partial'_\mu = \partial_\mu + \frac{w}{2} Q^*_\mu, \quad \Gamma^\alpha_{\mu \nu} \rightarrow \Gamma^\alpha_{\mu \nu},
\]

in equations \( \hat{W}_4 \), \( \hat{W}_3 \) and \( \hat{W}_2 \), in order for the Lagrangian \( \sqrt{-g} \mathcal{L}_{\text{fermion}} \) to be gauge invariant. This is true also when the \( SU(2) \otimes U(1) \) gauge fields \( W_\mu^{(i)} \) and \( B_\mu \) are included in the Lagrangian density. This means that the Weyl gauge vector \( Q_\alpha \) does not couple neither to fermions nor to other gauge fields including the electromagnetic radiation. This statement has been taken as the basis to avoid the SCE \cite{11, 32, 33, 34}. The argument may be extended to generalized nonmetricity, as shown in \cite{34}. In this reference it has been pointed out that, if take into account an appropriate minimal coupling prescription, the correct Lagrangian density should read:\footnote{In the second term within square brackets the action of the slash operator \( \slashed{D} \) should be understood in the following way: \( \bar{\psi} \slashed{D} = (D_\mu \bar{\psi}) \gamma^\mu \), where the Riemannian gauge derivative \( D_\mu \) is defined in \( \hat{W}_3 \).}

\[
\mathcal{L}_{\text{fermion}} = \frac{i}{2} \left[ \bar{\psi} (\hat{\slashed{D}} \psi) - \left( \bar{\psi} \gamma^\mu \slashed{D}_\mu \right) \psi \right]. \tag{93}
\]

Let us point out that the above argument is strictly correct only if the mass of the fields \( m_\psi \) is assumed vanishing, i.e., if consider the Lagrangian density \( \hat{W}_4 \). In this case the exposed argument is just a confirmation of the result discussed at the end of section V, that photons and radiation interact only with the metric field, i.e., with the LC curvature of spacetime. In other words, that these do not interact with nonmetricity.

Consideration of a point-dependent mass term may radically change the coupling to the nometricity, as anticipated by the form of the geodesic equation \( \hat{W}_4 \) in \( W_4 \), where a term \( \propto \delta m/\delta x^a \) arises. The hypothesis on the identification of vectors and tensors living in \( W_4 \) space with physical vectors and tensors, allows to identify any mass parameter with one of the kind \( \hat{W}_3 \). As we shall see, this recipe can be applied to Dirac’s equation for spinor matter and to the Mathisson-Papapetrou-Dixon equation for the motion of spinning test bodies as well.

If in place of \( \hat{W}_4 \) consider the following Lagrangian:

\[
\mathcal{L}_{\text{fermion}} = \left\{ \frac{i}{2} \left[ \bar{\psi} (\hat{\slashed{D}} \psi) - \left( \bar{\psi} \gamma^\mu \slashed{D}_\mu \right) \psi \right] - \bar{\psi} m_\psi \psi \right\}, \tag{94}
\]

where \( m_\psi \neq 0 \), the conclusion of references \cite{32, 33} and related works \cite{11}, may be incorrect in general. Actually, if assume spaces with arbitrary nonmetricity \( W_4 \), which means that we must identify the hypothetical vectors and tensors living in \( W_4 \) with the corresponding physical vectors and tensors, then equation \( \hat{W}_4 \) must be satisfied. In consequence, the mass of the fermion field \( m_\psi \) in \( \hat{W}_4 \) must obey:

\[
m_\psi(x) = m_\psi(0) \exp \left\{ -\frac{1}{2} \int_C Q_{\lambda \mu \nu} u^\mu u^\nu dx^\lambda \right\}, \tag{95}
\]

11 In the second term within square brackets the action of the slash operator \( \slashed{D} \) should be understood in the following way: \( \bar{\psi} \slashed{D} = (D_\mu \bar{\psi}) \gamma^\mu \), where the Riemannian gauge derivative \( D_\mu \) is defined in \( \hat{W}_3 \).
which means that there is a non-negligible (under integral) dependence of the mass \( m_\psi \) on nonmetricity in (74).

It seems appropriate to underline, once more, that the identification of the mass of the fermion in (74) with the quantity has been made in section V when the factor was identified with the quantity given by (73). This hypothesis is what makes possible to describe the motion of timelike point particles, including fermion fields, in a gravitational field depicted by the curvature and nonmetricity of \( W_4 \) space. This is why we call it as “consistency” hypothesis.

This consistency hypothesis may be applied as well to the Mathisson-Papapetrou-Dixon equation, which is the one driving the dynamics of extended, spinning test bodies in curved backgrounds [28–31]:

\[
\frac{D^* p^\alpha}{ds} = -\frac{1}{2} R^{\alpha \mu \nu \lambda} v^\mu S^\nu \lambda, \\
\frac{D^* S^{\alpha \beta}}{ds} = 2p^{[\alpha} v^{\beta]},
\]

(96)

where \( p^\alpha \) and \( v^\alpha \) are the coordinate components of the four-momentum of the spinning test body and of the unit tangent vector to the worldline \( x^\alpha(s) \), respectively. Meanwhile, \( S^{\alpha \beta} \) are the components of the spin tensor [31]. Along the worldline the following condition is satisfied:

\[
g_{\mu \nu} p^\mu S^{\nu \alpha} = 0.
\]

where the magnitude \( S \) of the spin satisfies: \( S^2 = S_{\mu \nu} S^{\mu \nu} / 2 \). Besides, we have that \( m^2 = -g_{\mu \nu} p^\mu p^\nu \) is the mass (squared) of the spinning body. Hence, if assume the consistency hypothesis, which amounts to identifying the hypothetical four-momentum \( p \) in (62) with the four-momentum of the spinning test body, its mass will satisfy (63), which is the basis for the SCE.

We have shown that, under the assumption of the parallel transport law (52) and of the consistency hypothesis that allows to identify hypothetical vectors and tensors living in \( W_4 \), with corresponding physical vectors and tensors, timelike test particles with the mass – no matter whether these are point particles, spinor fields or spinning bodies – interact with nonmetricity \( Q_{\mu \nu \alpha \beta} \). This means, in turn, that the SCE is inevitable in \( W_4 \) space. A similar result was obtained in [16] under the assumption of a different law of parallel transport.

Our point of view is in clear contradiction with statements found in the bibliography according to which “the second clock effect has nothing to do with geometry but is entirely determined by the matter coupling” [11–13–14]. Further investigation of this topic will be the subject of a forthcoming publication [35].

### IX. CLOSED TIMELIKE WORLDLINES AND THE SECOND CLOCK EFFECT

The demonstration of the absence/occurrence of the SCE in [10] (see appendix A for the gauge invariant demonstration), as well as in several textbooks [2], heavily relies on the choice of a closed path which allowed to further apply the Stoke’s theorem. The usual argument to justify closed path is that it is necessary in order to check the second clock effect, since the observers have to compare “notes” [10]. However, as we have shown above, the SCE arises even if consider paths that are not closed. Observers endowed with identical clocks at the coordinate origin \( x = 0 \), can compare the ticks of their clocks when they coincide again at some distant point \( x \), after following different trajectories that joint 0 and the point \( x \). It is not required that the spacetime origin of coordinates and the distant spacetime point coincided as it is for closed paths. As we shall see, given worldline is not closed even if it starting and ending spatial points coincide. For this worldline to be a closed one it is required, besides, that the starting and ending time coordinates coincide as well.

In general, closed paths in spacetime carry causality issues. Timelike worldlines of observers with clocks, aimed at the check of the second clock effect, are not the exception. In this regard we should differentiate the timelike worldlines \( \mathcal{C} \) with coordinates \( x^\alpha(\xi) \) (\( \xi \) is an affine parameter along the worldline), which start and end up at a same spatial point:

\[
x^0(\xi_{\text{start}}) \neq x^0(\xi_{\text{end}}), \quad x^i(\xi_{\text{start}}) = x^i(\xi_{\text{end}}) \Rightarrow x^\mu(\xi_{\text{start}}) \neq x^\mu(\xi_{\text{end}}),
\]

(98)

from those worldlines \( \mathcal{C} \), which start and end up at the same spacetime point:

\[
x^\mu(\xi_{\text{start}}) = x^\mu(\xi_{\text{end}}) \Rightarrow x^0(\xi_{\text{start}}) = x^0(\xi_{\text{end}}), \quad x^i(\xi_{\text{start}}) = x^i(\xi_{\text{end}}).
\]

(99)

While timelike worldlines of type \( \mathcal{C} \) can be associated with real (classical) motions, timelike worldlines of type \( \mathcal{C} \) are usually called as closed timelike curves (CTCs) and are plagued by causality issues as long as a CTC represents time travel [36–41]. But integrals of the kind [87]...
It is a well-known fact that Weyl geometry spaces are equipped with a conformal structure thanks, precisely, to Weyl gauge symmetry, which is a manifest symmetry of generalized Weyl spaces, is either just ignored or it is assumed not to occur in physical situations. The basis for neglecting the SCE is the gauge invariance of the Lagrangian for massless Fermions and gauge fields even in its Riemannian version (see, for instance, equation (89)), which, strictly speaking, means that the nonmetricity does not interact with massless fields (radiation).

While the above conclusion is in perfect agreement with our results in this paper: null geodesics in $W_4$ coincide with null geodesics in Riemann space $V_4$, it is not clear why the same conclusion is straightforwardly applied to the case when one adds the mass term as in equation (101). As we have shown, a mass term changes everything. Concluding, for instance, that from Lagrangian density (94) and related equations of motion:

$$\left[i\tilde{\nabla} - m\right] \psi = 0, \quad \tilde{\psi} \left[i\tilde{\nabla} + m\right] = 0, \quad (100)$$

it follows that fermions and gauge fields do not interact with nonmetricity, amounts to assuming that the mass parameter can not be function (or functional) of the nonmetricity $Q$ with coordinate components $Q_{\alpha\mu\nu}$. However, equation (63) is an evidence that for the hypothetical four-momentum living in $W_4$, the mass is a functional which depends on followed path and also on the nonmetricity: $m = m[C, Q]$.

As we have shown in section VIII, equations like the geodesics of timelike particles (77), the Dirac equations for spinor fields (100) and the Mathisson-Papapetrou-Dixon equations (96), which drive the dynamics of spinning test bodies, for non vanishing mass $m \neq 0$, are undetermined until specific postulate or hypothesis on the geometrical nature of the mass is assumed. In the present paper, for instance, the assumed postulate on the nature of the mass parameter is that, the physical quantity and the hypothetical one: the one that appears in the definition of the four-momentum (62) living in $W_4$ space, are to be identified. This identification amounts to equiparate physical vectors and tensors with the related hypothetical vectors and tensors, living in the generalized Weyl space $W_4$. But there are other possibilities.

As an illustration let us assume, as in [13, 14], that the parallel transport law of given tensor $T$, along the worldline $x^\mu(\xi)$, reads:

$$\frac{D^\ast T}{d\xi} = \frac{dx^\mu}{d\xi} \nabla^\ast \mu T = 0, \quad (101)$$

which means that Weyl gauge symmetry is being considered as a manifest symmetry of generalized Weyl spaces $W_4$, and that the following hypothesis on the nature of mass holds:

$$m = m_0 \varphi, \quad (102)$$

where $m_0$ is a constant and $\varphi$ is a scalar field.

Let us first assume that the mass (102) is the one that appears in the definition of the hypothetical four-momentum (62). Then, since $-m^2 = g_{\mu\nu}p^\mu p^\nu$, the following chain of equations takes place:

$$-\frac{D^\ast m^2}{d\xi} = \frac{dx^\lambda}{d\xi} \nabla^\ast \lambda g_{\mu\nu}p^\mu p^\nu 
\Rightarrow \frac{d\ln m}{d\xi} = \frac{dx^\lambda}{d\xi}\left(Q^\ast_\lambda + \frac{1}{2}Q_{\alpha\mu\nu}u^\mu u^\nu\right) 
\Rightarrow \partial_\alpha \ln \varphi - \left(Q^\ast_\alpha + \frac{1}{2}Q_{\alpha\mu\nu}u^\mu u^\nu\right) = 0.$$
This case corresponds to vectorial nonmetricity with gauge vector \( Q_\alpha = 2 (Q^*_\alpha - \partial_\alpha \ln \varphi) g_{\mu\nu} \).

XI. CONCLUSION

In this paper we have demonstrated that, if assume that: (i) gauge symmetry is a manifest symmetry of generalized Weyl space, (ii) the parallel transport law \([62]\) holds in \( W_4 \) and (iii) identification of physical vectors and tensors with the related hypothetical vectors and tensors living in \( W_4 \) takes place, the second clock effect is inevitable. In consequence, under the above assumptions, gauge invariant theories that are based in \( W_4 \) background spaces, are phenomenologically ruled out. Only in the subclass of \( W_4 \) known as Weyl integrable geometry, the SCE does not take place. Hence WIG is the only phenomenologically viable non-Riemannian gauge invariant background space from the classical perspective.

Although generalizations of nonmetricity recently investigated within the framework of gauge invariant teleparallel theories of gravity, are phenomenologically ruled out in the classical context, in the domain of quantum gravity these may play a fundamental role. This and related issues are the subject of our current work.

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Appendix A: Gauge symmetry, closed paths and second clock effect

In this appendix section we shall discuss on what happens if consider gauge symmetry in a situation like the one discussed in lemma 2 of \([10]\). For simplicity, instead of parallel transport of the inner product of two vectors we shall consider parallel transport of the length of a given vector.

Consider parallel transport of a vector \( v \), with coordinate components \( v^\alpha \) and conformal weight \( w(v) = w \), along a closed path \( \bar{C} \). In this case equation \([60]\) is rewritten in the following form:

\[
\Delta \ln v(x) = -w + \frac{1}{2} \int_\mathcal{C} Q_{\lambda\mu\nu} t^\mu t^\nu dx^\lambda, \quad (A1)
\]

where \( v \equiv ||v|| \) is the length of vector \( v \). Following \([10]\), we apply the Stoke’s theorem to the path integral \((A1)\), we get:

\[
\int_\mathcal{C} Q_{\lambda\mu\nu} t^\mu t^\nu dx^\lambda = \left[ \int_{\mathcal{S}} \nabla_\lambda \left( Q_{\sigma\mu\nu} t^\mu t^\nu \right) dx^\lambda \right] \wedge dx^\sigma
\]

where \( S \) is any surface with boundary \( \bar{C} \) and we have taken into account the third Bianchi identity \([18]\). Consider further the dagger derivative \([51], [50]\):

\[
\nabla^\dagger_\lambda (t^\mu t^\nu) = \nabla_\lambda (t^\mu t^\nu) - \frac{1}{2} Q^{\mu}_{\lambda\kappa} t^\kappa t^\nu - \frac{1}{2} Q^{\nu}_{\lambda\kappa} t^\kappa t^\mu,
\]

and take into account that the unit vector \( t \) is parallel transported along the closed path \( \bar{C} \), then:

\[
\nabla^\dagger_\lambda (t^\mu t^\nu) = 0 \Rightarrow \nabla_\lambda (t^\mu t^\nu) = Q^{\mu}_{\lambda\kappa} (t^\nu t^\kappa). \quad (A2)
\]
Hence, the above path integral along the closed worldline $\bar{C}$ can be written as it follows:

$$\int_{\bar{C}} \int_{S} \left\{ R_{(\mu\nu)\lambda \sigma} + Q_{\lambda} \rho_{(\mu} Q_{\sigma\nu)} \right\} t^\mu t^\nu dx^\lambda \wedge dx^\sigma.$$  \(\text{(A3)}\)

Equation \((A1)\) can then be written in the following way (compare with \((88)\)):

$$\int_{S} \left\{ R_{(\mu\nu)\lambda \sigma} + Q_{\lambda} \rho_{(\mu} Q_{\sigma\nu)} \right\} t^\mu t^\nu dx^\lambda \wedge dx^\sigma.$$  \(\text{(A4)}\)

where, we recall, $w$ is the conformal weight of vector $v$.

We have demonstrated that the length of vectors is path dependent even if we consider the teleparallel condition $R_{\mu\nu\lambda\sigma} = 0$. This result is contrary to the statement of lemma 2 of \cite{10}. Notice that if we ignore the teleparallel condition in $\bar{W}_4$, since $Q_{\alpha\mu\nu} = Q_{\alpha} g_{\mu\nu}$, equation \((A1)\) transforms into the well-known equation:

$$\Delta \ln v(x) = -\frac{w+1}{2} \int_{S} \partial_{[\lambda} Q_{\sigma]} dx^\lambda \wedge dx^\sigma.$$  \(\text{(A5)}\)

Appendix B: Short reply to a comment in reference \cite{11}

In section IV.A of reference \cite{11} it is stated that:

- “...then modifies the theory in order to implement the “manifest symmetry”, according to which spacetime vectors $V^\mu$ transform with some weight $\alpha$ as $V^\mu \rightarrow e^{\phi} V^\mu$. The basic error in this argument is that the conformal transformation is explicit for the tangent space vectors $V^\mu \rightarrow e^{\phi} V^\mu$, and therefore spacetime vectors $\nu^\mu = e^{\phi} V^\mu$ should consistently have the zero weight $\alpha = 0$.”

The comment refers to our paper \cite{16} and also to a previous version of the present work. Before replying to this comment, in order to unify notations, here we make the following replacements: the conformal factor $\Omega \rightarrow e^\phi$, while the conformal weight of given tensor $w \rightarrow \alpha$.

According to the above comment it is a basic error to consider vectors $V^\mu$ with conformal weight $\alpha \neq 0$. Hence, our straightforward reply is to show that, indeed, there are vectors with conformal weight $\alpha \neq 0$. Take, for instance, the timelike four-velocity: $u^\mu = dx^\mu/ds$ (where $s$ is the arc-length), which is tangent to the worldline $x^\mu(s)$. Under a conformal transformation of the kind we consider in our papers \cite{16} and in the present work (see footnotes 1 in the main text of this paper):

$$g_{\mu\nu} \rightarrow e^{2\phi} g_{\mu\nu}, \quad dx^\mu \rightarrow e^{\phi} dx^\mu \Rightarrow ds \rightarrow e^{\phi} ds,$$  \(\text{(B1)}\)

the components of the four-velocity transform like:

$$u^\mu \rightarrow e^{-\phi} u^\mu,$$  \(\text{(B2)}\)

so that its weight $\alpha(u^\mu) = -1$. In consequence, the timelike four-momentum $p^\mu = mu^\mu$, where the mass $m$ of a given point-particle has weight $\alpha(m) = -1$, has conformal weight $\alpha(p^\mu) = -2$.

In general, given any vector $V^\mu$ with length

$$V = \sqrt{g_{\mu\nu} V^\mu V^\nu},$$  \(\text{(B3)}\)

and conformal weight $\alpha$, one can define the spacelike unit vector with coordinate components:

$$t^\mu = \frac{V^\mu}{V} \Rightarrow g_{\mu\nu} t^\mu t^\nu = 1.$$  \(\text{(B4)}\)

Since $\alpha(V^\mu) = \alpha$, while $\alpha(g_{\mu\nu}) = 2$, then $\alpha(V) = \alpha + 1$. Means that $\alpha(t^\mu) = -1$ independent of the conformal weight of vector $V^\mu$. As an illustration let us assume, as stated in the mentioned comment in \cite{11}, that $\alpha(V^\mu) = \alpha = 0$. Then, from \text{[B3]} it follows that $\alpha(V) = 1$, so that, taking into account \text{[B3]}, one gets that, $\alpha(t^\mu) = -1$. This example shows that, no matter whether we have vector fields with vanishing conformal weight, one can always construct spacelike unit vectors with weight $\alpha = -1$. Additionally, one can find in the bibliography quite the contrary statement regarding tangent space vectors $V^\mu$, whose conformal weight is taken to be vanishing \cite{13, 14}: $\alpha(V^\mu) = 0$.

The above examples: the timelike four-velocity and four-momentum vectors, as well as the spacelike unit vector \((B4)\), show that the comment in section IV.A of reference \cite{11} is either simply incorrect, or the authors of that reference are considering conformal transformations of a different kind to the one considered in \cite{16} and in the present paper.

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