Noncommutative Superspace, Supermatrix and Lowest Landau Level

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Abstract

By using graded (super) Lie algebras, we can construct noncommutative superspace on curved homogeneous manifolds. In this paper, we take a flat limit to obtain flat noncommutative superspace. We particularly consider $d = 2$ and $d = 4$ superspaces based on the graded Lie algebras $osp(1|2)$, $su(2|1)$ and $psu(2|2)$. Jacobi identities of supersymmetry algebras and associativities of star products are automatically satisfied. Covariant derivatives which commute with supersymmetry generators are obtained and chiral constraints can be imposed. We also discuss that these noncommutative superspaces can be understood as constrained systems analogous to the lowest Landau level system.

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1 Introduction

Deformation of superspace by introducing noncommutativity has attracted interests recently. In papers [1, 2, 3], it is discussed that the background of the RR field strengths (in the first two papers, graviphotons) in string theory gives rise to non anti-commutative fermionic coordinates. (In this paper, we use the word noncommutative for the non anti-commutativity of the fermionic coordinates unless there is any confusion.) This phenomenon is similar to the well-known case of the string theory in the NS-NS two form $B$ background, where the bosonic space-time coordinates become noncommutative [4, 5].

Noncommutative superspace has been already studied by several papers from the field theoretic approach. In the paper [6], they discussed a possibility that the anticommutators of fermionic variables are written in terms of the space-time coordinates so that the space-time can be generated as a composite of fermions. Quantum deformations of the Poincare supergroup were also considered in [7] and then it was shown that the chiral operators are not closed under star products with fermionic noncommutativity [8]. In the paper [9], general forms of the deformed superspace in $d = 4$ are discussed by imposing the covariance under supertranslations, Jacobi identity and closure of chiral superfields under the star product. $\mathcal{N} = 1/2$ supersymmetry in $d = 4$ was proposed very recently [3] and its radiative corrections are also studied [33].

There is another approach to study the noncommutative superspace from supermatrix models based on graded Lie algebras. This approach is a natural extension of constructing bosonic noncommutative space based on matrix models. In addition to an advantage that the system can be realized in terms of finite matrices, this matrix model approach is especially useful to study gauge theories on noncommutative space because noncommutative space-time and gauge fields on it are unified by single matrices. In this approach, constructions of the open Wilson lines or background independence of the noncommutative gauge theories become manifest [10, 11, 12]. Generalizations to supermatrices were first investigated in the paper by H.Grosse, C.Klimcik and P.Presnajder [13]. These authors studies supermatrix models based on the $osp(1|2)$ graded Lie algebra and constructed supersymmetric actions for scalar multiplets on two-dimensional fuzzy supersphere. They further studied noncommutative de Rham complex and forms in another paper [14]. Gauge theories on noncommutative superspace can be also constructed [15]. Recently the concept of noncommutative superspace based on a supermatrix was also introduced in proving the Dijkgraaf-Vafa conjecture as the large N reduction by H.Kawai, T.Kuroki and T.Morita [16], which motivated us to start the present work.

In this paper, we first construct noncommutative superspace based on graded (super) Lie algebras and then take their flat limits to obtain flat noncommutative superspace. Though this approach is restricted to construct noncommutative superspace with Lie algebraic structures, there is an advantage that the symmetry is manifest and that Jacobi identities of the algebras and associativity of star products are automatically satisfied. We also construct covariant derivatives and impose chiral constraints.

Noncommutative superspace can be also understood as a constrained system analogous to the lowest Landau level system of particles moving in a constant magnetic field. For
superparticles moving in a magnetic field on superspace, we can introduce two mutually commutative sets of operators, covariant derivatives and guiding center coordinates. By imposing the lowest Landau level constraint, superspace coordinates become noncommutative. Such noncommutativity on superspace was first discussed in [6]. Our approach is similar to theirs in two points; one is to consider de Sitter algebra instead of the Poincare algebra and the other is to derive noncommutativity through Dirac brackets.

We will here explain a basic idea to induce fermionic noncommutativity by considering a particle in a generalized magnetic field on superspace. We first introduce supercovariant derivatives;

\begin{align}
D_i &= D^0_i - eA_i(x, \theta) \quad (1.1) \\
D_{\alpha} &= D^0_{\alpha} - eA_{\alpha}(x, \theta). \quad (1.2)
\end{align}

\(D_i\) are bosonic covariant derivatives in space-time directions and \(D_{\alpha}\) are fermionic ones in Grassmannian directions. In the following we write them together as \(D_I = (D_i, D_{\alpha})\). These derivatives are familiar in supersymmetric gauge theories [17] or in supergravities [18]. They transform covariantly under local gauge transformations on superspace. If the derivatives without gauge fields \(D^0_I\) satisfy

\[
[D^0_I, D^0_J] = iT^K_{IJ} D^0_K, \tag{1.3}
\]

we can define field strengths as

\[
iF_{IJ} = [D_I, D_J] - iT^K_{IJ} D_K. \tag{1.4}
\]

Now we consider a superspace with a constant magnetic field, namely, with a constant \(F_{IJ}\) background. This is analogous to the two-dimensional system in a constant magnetic field. A particle moving in a strong magnetic field is restricted in the lowest Landau level and the guiding center coordinates become noncommutative. In other words, the lowest Landau level condition can be made second class by imposing \(D_x = D_y = 0\) and the Dirac brackets between coordinates become noncommutative. In a magnetic field on superspace, by generalizing the bosonic second class constraints to include fermionic parts of the covariant derivatives, we can evaluate Dirac brackets for the superspace coordinates. In general this leads to the noncommutative algebras for the superspace coordinates. In this paper, we actually show it explicitly for some examples.

This way to generate fermionic noncommutativity becomes more important if we consider gauge theories on superspace based on supermatrix models. In ordinary matrix models of the IKKT type [19], we expand the basic bosonic degrees of freedom \(A_i\) around some classical solution \(A_i^{cl}\); \(A_i = A_i^{cl} + a_i\). The classical background \(A_i^{cl}\) defines the background space-time and \(a_i\) are interpreted as gauge fields on this space-time. If the classical solutions are noncommutative, we can obtain noncommutative gauge theories [10] [11]. This unification of space-time and the gauge field into a single matrix \(A_i\) can be generalized to a unification of noncommutative superspace coordinate \(\theta_{\alpha}\) and fermionic

\(^1\)The field strengths must satisfy the (modified) Bianchi identity and the background \(F_{IJ}\) are generally dependent on superspace coordinates because of the nonvanishing \(T^K_{IJ}\).
gauge field $a_\alpha$ into a single fermionic matrix $A_\alpha = \theta_\alpha + a_\alpha$. In order to generate such a fermionic background, we need to consider supermatrix models \cite{20,15} whose components are again supermatrices. The fermionic noncommutativity can be generated dynamically as the background of the unified fermionic matrix $A_\alpha$. We want to discuss this issue in a forthcoming paper.

In section 2, as a warming up exercise, we briefly review the noncommutative (fuzzy) sphere based on $su(2)$ algebra and its flat limit by the Inönü-Wigner contractions. The noncommutativity on the fuzzy sphere can be understood as the noncommutativity of the coordinates of a particle restricted in the lowest Landau level on the commutative sphere in a monopole background. After taking a flat limit, following a general formalism of left and right $SU(2)$ multiplications on group manifold, we can construct mutually commutative set of covariant derivatives and guiding center coordinates. Though it is a well-known fact, we explain it in details to clarify the origin of noncommutativity as a constrained system.

In section 3, we first construct a noncommutative curved superspace based on a graded Lie algebra $osp(1\mid2)$ and take its flat limit. To make a flat limit together with a fermionic noncommutativity, we need to take an asymmetric scaling limit of two grassmannian coordinates. In this limit, only half of the supersymmetry generators can generate space-time supersymmetry. The other half generates non-dynamical supersymmetry. We can impose a chiral constraint to remove the latter half. After imposing the chiral constraint, the system becomes essentially a one-dimensional system. In the latter half of the section, we consider the noncommutative superspace as a constrained system. We first obtain mutually commutative sets of operators for superparticle in a constant magnetic field on superspace; super covariant derivatives and super guiding center coordinates. Then by imposing the lowest Landau level conditions we obtain the noncommutative relations for the superspace coordinates.

In section 4, we consider a noncommutative superspace based on $su(2\mid1)$ algebra. This gives again two-dimensional supersphere but with twice as many as supercharges, i.e. four. We again need to take an asymmetric scaling for grassmannian coordinates when we take a flat limit. All procedures can be performed in parallel to the $osp(1\mid2)$ case.

In section 5, we consider four dimensional superspace based on $psu(2\mid2)$ superalgebra. This gives a coset supermanifold $PSU(2\mid2)/U(1)^2$ whose bosonic part is Euclidean $AdS^2 \times S^2$. In this case there are varieties to scale grassmannian coordinates. We first give an example of a similar scaling to the two-dimensional case. In this case, we can easily perform the same calculation as the supersphere cases and obtain covariant derivatives which commute with supercharges. Among eight supersymmetries, four of them are dynamical and generate space-time supersymmetries. Space-time translations in only two directions out of four appear in the anticommutators of supersymmetries and after imposing chiral constraints this system becomes essentially two-dimensional. We also give another example where all four space-time translation generators appear in the anticommutators. In the latter half of this section, we explain briefly that the Seiberg’s $d = 4$ noncommutative superspace \cite{3} can be understood as a constrained system.
In the appendix A, we review a general method using the Cartan one-forms to obtain mutually commutative sets of generators, left and right multiplications on the group manifold. They become covariant derivatives and guiding center coordinates in our cases. We then explain a method for the generalized Inönü-Wigner contractions. In the following appendices, we give some detailed calculations for supercovariant derivatives and global charges (guiding center coordinates).

2 Fuzzy sphere and flat limit

In this section, we review the bosonic fuzzy sphere as a warming up. A fuzzy sphere and field theories on it can be constructed by matrix models based on the \( su(2) \) Lie algebra. (For more details, see [22, 23, 24], for example.) Take representation matrices \( \hat{\mathbf{l}}_i \) of \( su(2) \) with an angular momentum \( L \). The size of the matrices is \( N = 2L + 1 \). The noncommutative coordinates are defined as \( \hat{x}_i = \alpha \hat{\mathbf{l}}_i \). Then the radius of the sphere \( r = \alpha^2 L(L+1) \). Any \( N \times N \) matrix can be expanded in terms of noncommutative spherical harmonics \( \hat{Y}_{lm} \) where \( l \) runs from 0 to \( 2L \). The coordinates satisfy the noncommutative algebra:

\[
[\hat{x}_i, \hat{x}_j] = i\alpha \epsilon_{ijk} \hat{x}_k. \tag{2.1}
\]

We now take the noncommutative flat limit of the fuzzy sphere algebra (2.1). This corresponds to considering the vicinity of the north pole and scaling \((x_1, x_2)\) coordinates so that the noncommutativity is fixed. This can be achieved by redefining coordinates as

\[
\hat{x} = \sqrt{\frac{\Theta}{r}} \hat{x}_1, \quad \hat{y} = \sqrt{\frac{\Theta}{r}} \hat{x}_2. \tag{2.2}
\]

They satisfy the commutation relation \([\hat{x}, \hat{y}] = i\Theta \). \( \Theta \) is a noncommutative parameter.

2.1 Fuzzy sphere as the lowest Landau level

The noncommutative coordinates on the fuzzy sphere can be understood as the guiding center coordinates on an ordinary sphere in a magnetic monopole at the origin. Let us see this explicitly. We consider a particle moving on a sphere with radius \( r \) in the field of a monopole put at the origin. The Hamiltonian is given by

\[
H = \frac{1}{2m} D_i^2 \tag{2.3}
\]

where \( D_i \) is the covariant derivative \( D_i = -i(\partial_i - ieA_i) \). In a monopole magnetic field, the commutator becomes

\[
[D_i, D_j] = -ie\epsilon_{ijk} \frac{x_k}{r^2}. \tag{2.4}
\]

The Hamiltonian is written in terms of the following deformed \( su(2) \) generators [25]

\[
K_i = \epsilon_{ijk} x_j D_k + eg \frac{x_i}{r}. \tag{2.5}
\]
as
\[ H = \frac{K_i^2 - e^2g^2}{2mr^2}. \] (2.6)

A consistency condition requires that the allowed values of the total angular momentum generated by \( K_i \) are
\[ L = |eg|, |eg| + 1, \ldots. \] (2.7)

Each state with a fixed \( L \) corresponds to each Landau level state. The lowest Landau level states have degeneracy \( 2|eg| + 1 \). Such degeneracies are described by the eigenvalue of \( K_3 \) and they are generated by acting the \( K_+ \) operator on the highest weight state. To describe the degeneracy in each Landau level, we can define the guiding center coordinates \( \hat{X}_i \) as
\[ \hat{X}_i = \alpha K_i \] (2.8)
where \( \alpha \) is defined so that the radius of \( \hat{X}_i \) becomes \( r \). Then \( r^2 = \alpha^2 L(L + 1) \). Since they commute with the Hamiltonian, they are constants of motion and can be interpreted as the guiding center coordinates of the cyclotron motion on the sphere. In the large \( g \) limit, \( \alpha = r/eg \) and the first term in \( K_i \) can be neglected compared to the second term. Then \( \hat{X}_i \) is identified with the commutative coordinate \( x_i \). This means that the cyclotron radius becomes small in the large magnetic field limit. These guiding center coordinates are noncommutative;
\[ [\hat{X}_i, \hat{X}_j] = i\epsilon_{ijk} \alpha \hat{X}_k. \] (2.9)

This is nothing but the commutation relation (2.1) on the fuzzy sphere.

### 2.2 Flat limit of fuzzy sphere

In this subsection, we apply the general method (see Appendix A) to obtain covariant derivatives and guiding center coordinates to the simplest case. We first parametrize the group manifold of \( SU(2) \) by \( g = \exp(iL \cdot x) \). Following the method of the (generalized) İnönü-Wigner contractions, we can obtain a flat limit by taking the following scaling of the parameter space. We first take the scaling
\[ x \rightarrow sx, \ y \rightarrow sy, \ z \rightarrow z. \] (2.10)

Then if we take up to the second order of \( s^2 \), we can obtain the algebra
\[ \begin{align*}
[ L_{x[1]}, L_{y[1]} ] &= iL_{z[2]} , \quad [ L_{x[1]}, L_{z[2]} ] = [ L_{y[1]}, L_{z[2]} ] = 0 \\
[ L_{z[0]}, L_{x[1]} ] &= iL_{y[1]} , \quad [ L_{y[1]}, L_{z[0]} ] = iL_{x[1]}.
\end{align*} \] (2.11)

The generator \( L_{z[2]} \) is a center and can be considered as a constant. The generator \( L_{z[0]} \) is a rotation generator on the two dimensional plane. If we instead take the scaling of \( z \) as
\[ z \rightarrow s^2 z, \] (2.12)
the generator \( L_{z[0]} \) disappears and the algebra of the noncommutative plane which is given by the first line of (2.11) can be obtained. In the following we consider this case.
The covariant derivatives (right multiplication generators) of the algebra generated by $L_{x[1]}, L_{y[1]}$ and the center $L_{z[2]}$ can be constructed by the Cartan 1-form $g^{-1}dg = dx^m e_m a T_a$ as $D_a = -i(e^{-1})_a^m \partial_m$. We take the parametrization $(x, y, \phi)$ for $T_a = (L_{x[1]}, L_{y[1]}, L_{z[2]})$. Then the covariant derivatives are given as

\begin{align}
D_x &= \frac{\partial}{i\partial x} + \frac{y}{2} \frac{\partial}{i\partial \phi} \\
D_y &= \frac{\partial}{i\partial y} - \frac{x}{2} \frac{\partial}{i\partial \phi} \\
D_z &= \frac{\partial}{i\partial \phi}.
\end{align}

They satisfy the algebra

\[ [D_x, D_y] = iD_z \].

The global charges (left multiplication generators) are constructed similarly and given by

\begin{align}
\hat{K}_x &= \frac{\partial}{i\partial x} - \frac{y}{2} \frac{\partial}{\partial \phi} \\
\hat{K}_y &= \frac{\partial}{i\partial y} + \frac{x}{2} \frac{\partial}{\partial \phi} \\
\hat{K}_z &= \frac{\partial}{i\partial \phi}.
\end{align}

These global charges commute with the covariant derivatives.

Since $\phi$ is a coordinate conjugate to the center $L_{z[2]}$ and $\partial/\partial \phi$ is a generator to multiply a constant $(L_{z[2]})$ on $g$, we can fix it as a constant: $D_z = \frac{\partial}{i\partial \phi} = -\Theta^{-1} \neq 0$. Then the commutator becomes

\[ [D_x, D_y] = -i\Theta^{-1}. \]

We define the noncommutative coordinates $X, Y$ (the guiding center coordinates) as

\begin{align}
X &= \frac{1}{2} \left( x - 2\Theta \frac{\partial}{i\partial y} \right) = -\Theta \hat{K}_y = x - \Theta D_y \\
Y &= \frac{1}{2} \left( y + 2\Theta \frac{\partial}{i\partial x} \right) = \Theta \hat{K}_x = y + \Theta D_x .
\end{align}

They satisfy the algebra

\[ [X, Y] = i\Theta . \]

The transformation from $(x, y, p_x, p_y)$ to $(X, Y, D_x, D_y)$ is familiar in the two dimensional system in a constant magnetic field. If we consider a particle constrained in the lowest Landau level, we impose the lowest Landau level condition (for $\Theta > 0$)

\[ \langle D_x - iD_y | LLL \rangle = 0. \]
This constraint can be made as the second class by imposing \( D_x = 0, \ D_y = 0 \). The Dirac bracket for these second class constraints is given by

\[
[O_1, O_2]_D = [O_1, O_2] + i\Theta [O_1, D_x] [D_y, O_2] - i\Theta [O_1, D_y] [D_x, O_2].
\] (2.21)

The Dirac bracket of the original coordinates becomes

\[
[x, y]_D = i\Theta.
\] (2.22)

On the other hand using facts that \((X, Y)\) are equal to \((x, y)\) up to the second class constraints and they commute with \((D_x, D_y)\), we have

\[
[x, y]_D = [X, Y]_D = [X, Y] = i\Theta.
\] (2.23)

Therefore the Dirac bracket of the original coordinates gives the noncommutative coordinate algebra \([2,19]\). In this picture, the noncommutative space is considered as a space whose phase space degrees of freedom is reduced by the constraint.

3 Noncommutative flat superspace from \(osp(1|2)\)

In the following sections we generalize the method in the previous section to superspaces. A construction of noncommutative space is performed by generalization of ordinary Lie algebras to super Lie algebras. In this way, we can systematically construct supermatrix models and field theories on noncommutative homogeneous superspaces. This was first studied in \([13]\) for the case of scalar multiplets with \(osp(1|2)\) symmetry. A gauge theory can be similarly constructed \([15]\). We then take the flat limit with an appropriate scaling of operators. We can also understand the noncommutative superspace as a constrained system whose phase space dimension is reduced by the second class constraints. This gives an interpretation that the noncommutative superspace is a supersymmetric analog of the lowest Landau level system. In this section, we take the \(osp(1|2)\) super Lie algebra. This gives a two dimensional noncommutative supersphere with two real supercharges on the fuzzy sphere.

3.1 \(osp(1|2)\) algebra and fuzzy supersphere

The graded commutation relations of \(osp(1|2)\) algebra are given by

\[
\begin{align*}
\hat{l}_i, \hat{l}_j & = i\epsilon_{ijk} \hat{l}_k, \\
\hat{l}_i, \hat{v}_\alpha & = \frac{1}{2} (\sigma_i)_{\beta\alpha} \hat{v}_\beta, \\
\{\hat{v}_\alpha, \hat{v}_\beta\} & = \frac{1}{2} (C\sigma_i)_{\alpha\beta} \hat{l}_i,
\end{align*}
\] (3.1)

where \(C = i\sigma_2\). The even part of this algebra is \(su(2)\) which is generated by \(\hat{l}_i\) \((i = 1, 2, 3)\) and the odd generators \(\hat{v}_\alpha\) \((\alpha = 1, 2)\) are \(su(2)\) spinors. In this paper, we also write \(\hat{v}_1 = \hat{v}_+\)
and \( \hat{v}_2 = \hat{v}_- \). The irreducible representations of \( osp(1|2) \) algebra \( 21 \) are characterized by the values of the Casimir operator \( \hat{K}_2 = \hat{l}_i \hat{l}_i + C_{\alpha \beta} \hat{v}_\alpha \hat{v}_\beta = L(L + \frac{1}{2}) \) where quantum number \( L \) is called super spin and \( L \in \mathbb{Z}_{\geq 0}/2 \). Each representation consists of spin \( L \) and \( L - \frac{1}{2} \) representations of \( su(2) \), \( |L, l_3 \rangle \), \( |L - \frac{1}{2}, l_3 \rangle \) and its dimension is \( N = (2L+1) + 2L = 4L + 1 \).

The condition \( \hat{K}_2 = L(L + \frac{1}{2}) \) defines the two-dimensional super sphere. Consider polynomials \( \Phi(l_i, v_\alpha) \) of the representation matrices \( l_i \) and \( v_\alpha \) with super spin \( L \). Let us denote the space spanned by \( \Phi(l_i, v_\alpha) \) as \( A_L \). The \( osp(1|2) \) algebra acts on \( A_L \) by the three kinds of action, the left action (\( \hat{l}_i \), \( \hat{v}_\alpha^L \)), the right action (\( \hat{l}_i^R \), \( \hat{v}_\alpha^R \)) and the adjoint action (\( \hat{L}_i \equiv \hat{l}_i^L - \hat{l}_i^R, \hat{V}_\alpha = \hat{v}_\alpha^L - \hat{v}_\alpha^R \)),

\[
\begin{align*}
\hat{l}_i^L \Phi &= l_i \Phi, & \hat{v}_\alpha^L \Phi &= v_\alpha \Phi, \\
\hat{l}_i^R \Phi &= \Phi l_i, & \hat{v}_\alpha^R \Phi &= \Phi v_\alpha, \\
\hat{L}_i \Phi &= [l_i, \Phi], & \hat{Q}_\alpha \Phi &= [v_\alpha, \Phi].
\end{align*}
\]

We can define supersymmetrized spherical harmonics \( \hat{Y}_{km}^S \) which are generalization of the ordinary spherical harmonics to the super sphere (see \( 14 \) for the details). \( k \) can take either an integer or half an integer value and they are bosonic or fermionic functions respectively. Any \( N \times N \) supermatrix can be expanded in terms of the super spherical harmonics as

\[
\Phi(l_i, v_\alpha) = \sum_{k=0,1/2,1,...}^{2L} \phi_{km} \hat{Y}_{km}^S,
\]

where the coefficient \( \phi_{km} \) for the even (odd) spherical harmonics is Grassmann even (odd). We can map the supermatrix \( \Phi(l_i, v_\alpha) \) to a function on the superspace \( (x_i, \theta_\alpha) \) by

\[
\Phi(l_i, v_\alpha) \rightarrow \phi(x_i, \theta_\alpha) = \sum_{k,m} \phi_{km} y_{km}^S(x_i, \theta_\alpha),
\]

where \( y_{km}^S(x_i, \theta_\alpha) \) are ordinary superspherical functions. A product of supermatrices is mapped to a noncommutative star product of functions. An explicit form of the star product is given in \( 20 \).

In addition to the \( osp(1|2) \) generators \( (\hat{l}_i, \hat{v}_\alpha) \), we can define additional generators with which they form bigger algebra \( osp(2|2) \). These additional generators are

\[
\hat{\gamma} = -\frac{1}{L + 1/4} (C_{\alpha \beta} \hat{v}_\alpha \hat{v}_\beta + 2L(L + 1/2)), \quad \hat{d}_\alpha = [\hat{\gamma}, \hat{v}_\alpha] = \frac{1}{2(L + 1/4)} (\sigma_i)_{\beta \alpha} (\hat{v}_\beta \hat{l}_i + \hat{l}_i \hat{v}_\beta).
\]

Commutation relations for the additional generators are given by

\[
\begin{bmatrix}
[\hat{\gamma}, \hat{v}_\alpha] &= \hat{d}_\alpha, \\
[\hat{\gamma}, \hat{d}_\alpha] &= \hat{v}_\alpha,
\end{bmatrix}
\]

\[
[\hat{\gamma}, \hat{d}_\beta] = \hat{0}, \quad [\hat{d}_\alpha, \hat{d}_\beta] = \hat{0}.
\]
The adjoint action of the fermionic generators $D_\alpha = \text{adj} \hat{d}_\alpha$ plays a role of the covariant derivatives. On the other hand, the adjoint action of the original fermionic generators $Q_\alpha = \text{adj} \hat{v}_\alpha$ are interpreted as super symmetry generators. We will show that they commute in the flat limit. These additional generators also play an important role in constructing kinetic terms for a scalar multiplet on the super sphere \cite{13}.

The commutative limit is discussed in \cite{13} and the fuzzy super sphere becomes the ordinary two-dimensional supersphere with two real grassmannian coordinates. This limit can be taken by keeping the radius $r$ of the sphere fixed and taking the large $L$ limit.

### 3.2 Flat noncommutative superspace

We now take a flat limit. Namely we consider the vicinity of the north pole. For the bosonic generators, we perform the same scaling as (2.2) to obtain the flat noncommutative coordinates. For the fermionic generators, in order to keep the noncommutativity, one possible choice is to scale both of $v_\alpha$ as $\sqrt{L}$. Then from the $osp(1|2)$ algebra \cite{32} we can read that the anticommutator of $v_+$ and $v_-$ survives to be a constant. The other anticommutators among $v$ vanish. But with this choice we can easily see that the supersymmetry algebras acting on the noncommutative superspace become trivial, i.e. the anticommutators between supercharges do not generate translation of the space. All supersymmetries become non-dynamical symmetries. So we need to take another scaling limit where $v_+$ and $v_-$ are scaled asymmetrically. We define new superspace coordinates as

\[
\hat{x}_i = \sqrt{\frac{\Theta}{L}} \hat{l}_i \quad \text{for } i = 1, 2
\]
\[
\hat{\theta}_+ = \sqrt{2} \left( \frac{\Theta}{L} \right)^{1/4} \hat{v}_+
\]
\[
\hat{\theta}_- = \sqrt{2} \left( \frac{\Theta}{L} \right)^{3/4} \hat{v}_-.
\]

We multiply a constant on $\hat{\theta}_\alpha$ for convenience. Since we are considering around the north pole, the $\hat{l}_3$ is scaled as $L$. With the above scaling, the algebra among the coordinates become

\[
[\hat{x}_+, \hat{x}_-] = 2\Theta \quad \text{or} \quad [\hat{x}_1, \hat{x}_2] = i\Theta
\]

\[
[\hat{x}_1, \hat{x}_2] = i\Theta
\]
\[ \{ \hat{\theta}^+, \hat{\theta}^+ \} = \hat{x}^+ \]
\[ \{ \hat{\theta}^+, \hat{\theta}^- \} = -\Theta \]
\[ [\hat{x}^-, \hat{\theta}^+] = \hat{\theta}^- . \quad (3.11) \]

where we have defined \( \hat{x}^\pm = \hat{x}_1 \pm i\hat{x}_2 \). All the other commutators vanish. By redefining the coordinates as

\[ \hat{\varphi}^+ = \hat{\theta}^+ + \frac{1}{2\Theta} \hat{x}_+ \hat{\theta}^- \]
\[ \hat{\varphi}^- = \hat{\theta}^- , \quad (3.12) \]

the noncommutativity of the coordinates is written simply as a canonical form;

\[ [\hat{x}^+, \hat{x}^-] = 2 \Theta, \quad \{ \hat{\varphi}^+, \hat{\varphi}^- \} = -\Theta . \quad (3.14) \]

The scaling of the additional generators \( \hat{d}_\alpha \) can be automatically determined from the scaling (3.11) because they are written in terms of the \( osp(1|2) \) generators. The scaled generators are defined by

\[ \hat{d}_+ = \sqrt{2} \left( \frac{\Theta}{L} \right)^{-1/4} \hat{d}_+ = \hat{\theta}^+ + \frac{1}{2\Theta} \hat{x}_+ \hat{\theta}^- = \hat{\varphi}^+ + \frac{1}{2\Theta} \hat{\varphi}_- \hat{x}_+, \quad (3.15) \]
\[ \hat{d}^- = \sqrt{2} \left( \frac{\Theta}{L} \right)^{-3/4} \hat{d}^- = -\hat{\theta}^- = -\hat{\varphi}_-. \quad (3.16) \]

The anticommutator with \( \hat{\theta} \) is given by

\[ \{ \hat{d}_\alpha, \hat{\theta}_\beta \} = -C_{\alpha\beta} \Theta , \quad (3.17) \]

which is consistent with the scaling of \( \hat{\gamma} \). \( \hat{\gamma} \) becomes a constant and commutes with all the other generators.

Now we define generators of supersymmetry transformations and covariant derivatives which mutually commute by

\[ P_\pm = \pm \frac{1}{2} \text{adj} \hat{x}_\pm \quad (3.18) \]
\[ Q_\alpha = \text{adj} \hat{\theta}_\alpha . \quad (3.19) \]

They generate supersymmetry transformations on the superspace coordinates \((\hat{x}, \hat{\varphi})\). They satisfy the supersymmetry algebra

\[ [P_+, P_-] = 0 \quad (3.20) \]
\[ [P_-, Q_+] = -\frac{1}{2} Q_- \quad (3.21) \]
\[ \{ Q_+, Q_- \} = 0 \quad (3.22) \]
\[ \{ Q_+, Q_+ \} = 2 P_+ \quad (3.23) \]
\[ \{ Q_-, Q_- \} = 0 . \quad (3.24) \]
$Q_+$ is a dynamical supersymmetry and generates space-time translation into $x_-$ direction. But the $Q_-$ is a non-dynamical supersymmetry and its anticommutator vanishes. This is caused by the asymmetric scaling of the coordinates. Because of this asymmetric scaling, we cannot take a further limit to obtain an ordinary two-dimensional superspace with two dynamical supersymmetries.

Covariant derivatives can be defined similarly by the adjoint action of $\hat{d}_\alpha'$:

$$D_\alpha = \text{adj} \, \hat{d}_\alpha'. \tag{3.25}$$

They anticommute with $Q_\alpha$; $\{D_\alpha, Q_\beta\} = 0$. Their commutation relations are

$$\{D_+, D_+\} = -2P_+ \tag{3.26}$$
$$\{D_+, D_-\} = \{D_-, D_-\} = 0 \tag{3.27}$$
$$[P_-, D_+] = -\frac{1}{2} D_- \tag{3.28}$$

Functions on the superspace $(\hat{x}, \hat{\phi})$ are given as $N \times N$ supermatrices. Generically they depend on full set of supercoordinates: $\Phi(\hat{x}_+, \hat{x}_-, \hat{\phi}_+, \hat{\phi}_-)$. We can consistently constrain them by imposing the chiral constraint as

$$D_- \Phi = -[\hat{\phi}_-, \Phi] = 0. \tag{3.29}$$

This automatically leads to

$$Q_- \Phi = 0 \tag{3.30}$$

and the superfield $\Phi$ depends only on $(\hat{x}_+, \hat{x}_-, \hat{\phi}_-)$. This is the chiral superfield and supersymmetry is generated by $Q_+$ whose anticommutator becomes $P_+$. To summarize, on the chiral superfields $\Phi(\hat{x}_+, \hat{x}_-, \hat{\phi}_-)$, the algebra of supersymmetry $Q_+$, translations $P_\pm$ and the covariant derivative $D_+$ is given by

$$\{Q_+, Q_+\} = -\{D_+, D_+\} = 2P_+ \tag{3.31}$$
$$[P_\pm, Q_+] = [P_\pm, D_+] = 0 \tag{3.32}$$
$$[P_+, P_-] = 0. \tag{3.33}$$

They are written as differential operators on chiral superfields $\Phi(\hat{x}_+, \hat{x}_-, \hat{\phi}_-)$:

$$P_\pm = \Theta \frac{\partial}{\partial \hat{x}_\pm} \tag{3.34}$$
$$Q_+ = -\Theta \frac{\partial}{\partial \hat{\phi}_-} - \hat{\phi}_- \frac{\partial}{\partial \hat{x}_-} \tag{3.35}$$
$$D_+ = -\Theta \frac{\partial}{\partial \hat{\phi}_-} + \hat{\phi}_- \frac{\partial}{\partial \hat{x}_-} \tag{3.36}$$

We can further constrain the superfield by $P_-$:

$$P_- \Phi = 0. \tag{3.37}$$

Then the superfield becomes independent of the $\hat{x}_+$ coordinate: $\Phi(\hat{x}_-, \hat{\phi}_-)$, and the system becomes essentially one-dimensional.
3.3 Magnetic field in 2d superspace

The noncommutative superspace can be understood as a system restricted by some constraints analogous to the lowest Landau level states in the bosonic case. In this subsection, we first derive mutually commutative set of differential operators acting on superparticles in the commutative superspace. They correspond to the covariant derivatives in a magnetic field and the guiding center coordinates discussed in the previous section.

In order to obtain mutually commutative set of generators on the flat space, we again begin with the $\text{osp}(1|2)$ algebra (3.2), where we denote the generators as $\hat{L}_i = L_i$, $\sqrt{2}\hat{v}_\alpha = Q_\alpha$, and take the Inönü-Wigner contraction. We parametrize the group manifold of $\text{osp}(1|2)$ by $(x,y,\phi,\theta^\pm)$ as

$$ g = \exp(i x \cdot L) \exp(i \theta^\alpha Q_\alpha). $$

We then take the scaling as

$$ x, y \to sx, sy, \phi \to s^2\phi, \theta^+ \to \sqrt{s}\theta^+, \theta^- \to \sqrt{s}^3\theta^- $$

and take $s \to 0$ limit.

If we take up to the second order of $s$, we have the algebra

$$ [L_{x[1]}, L_{y[1]}] = iL_{z[2]} $$
$$ [L_{-[1]}, Q_{+[1/2]}] = Q_{-[3/2]} $$
$$ \{Q_{+[1/2]}, Q_{+[1/2]}\} = L_{+[1]} $$
$$ \{Q_{+[1/2]}, Q_{-[3/2]}\} = -L_{z[2]} $$

Covariant derivatives are defined as the right multiplication on the group manifold generated by this algebra. They are calculated in the appendix resulting as

$$ D_x = \frac{\partial}{\partial x} + \frac{y}{2i} \frac{\partial}{\partial \phi} - \frac{1}{2} \theta^+ \frac{\partial}{\partial \theta^-} $$
$$ D_y = \frac{\partial}{\partial y} - \frac{x}{2i} \frac{\partial}{\partial \phi} + \frac{1}{2} \theta^+ \frac{\partial}{\partial \theta^-} $$
$$ D_z = \frac{\partial}{\partial \phi} $$
$$ D_+ = \frac{\partial}{\partial \theta^+} - \frac{1}{2} \theta^- \frac{\partial}{\partial \phi} + \frac{x + iy}{4i} \theta^+ \frac{\partial}{\partial \phi} + \frac{1}{2} \theta^+ (\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}) $$
$$ D_- = \frac{\partial}{\partial \theta^-} - \frac{1}{2} \theta^+ \frac{\partial}{\partial \phi} $$

satisfying

$$ [D_x, D_y] = iD_z $$
$$ [D_-, D_+] = D_- $$
$$ \{D_+, D_+\} = D_+ $$
$$ \{D_+, D_-\} = -D_-, \text{ others } = 0 $$

These commutators are supersymmetric generalization of (2.15). The terms in the r.h.s. with a $D_z$ term are interpreted as the effect of the background magnetic field.
The global charges (left multiplications) are similarly calculated in the appendix resulting as

\[
\hat{L}_x = \frac{\partial}{i\partial x} - \frac{y}{2i} \frac{\partial}{\partial \phi},
\]
\[
\hat{L}_y = \frac{\partial}{i\partial y} + \frac{x}{2i} \frac{\partial}{\partial \phi},
\]
\[
\hat{L}_z = \frac{\partial}{i\partial \phi},
\]
\[
\hat{Q}_+ = \frac{\partial}{i\partial \theta^+} + \frac{1}{2} \theta^\pm \frac{\partial}{\partial \phi} - \frac{x + iy}{2} \frac{\partial}{\partial \theta^-} - \frac{1}{2} \theta^+(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})
\]
\[
\hat{Q}_- = \frac{\partial}{i\partial \theta^-} + \frac{1}{2} \theta^+ \frac{\partial}{\partial \phi}
\]

satisfying

\[
\left[ \hat{L}_x, \hat{L}_y \right] = -i \hat{L}_z
\]
\[
\left[ \hat{L}_-, \hat{Q}_+ \right] = -\hat{Q}_-
\]
\[
\left\{ \hat{Q}_+, \hat{Q}_+ \right\} = -\hat{L}_+
\]
\[
\left\{ \hat{Q}_+, \hat{Q}_- \right\} = \hat{L}_z, \quad \text{others} = 0 .
\]

These global charges commute/anticommutate with covariant derivatives including \(\{Q,D\} = 0\). These operators are interpreted as the guiding center coordinates in a constant magnetic field on the two dimensional superspace.

The covariant derivatives (3.40) can be considered as the supercovariant derivatives (1.1), (1.2) in a constant magnetic field. In (3.40), the terms containing derivatives with respect to \(\phi\) are contributions from the gauge fields. Applying the definition of the field strength given in the introduction (1.3), the above system has a constant magnetic field in \((x,y)\) and \((\theta_+, \theta_-)\) directions;

\[
F_{x,y} = B, \quad F_{\theta_+, \theta_-} = iB
\]

where we write \(D_z = B\). In the next subsection we show that this induces the noncommutativity on superspace.

### 3.4 Noncommutative superspace as the lowest Landau level system

We now calculate the Dirac bracket of the super coordinates by imposing the 'lowest Landau level' constraints. Since \(\phi\) is a coordinate conjugate to the center \(L_z[2]\) we can fix it as a constant; \(D_z = -\Theta^{-1} \neq 0\). This leads to the fermionic noncommutative algebra as well as the bosonic noncommutative algebra (2.17),

\[
[D_x, D_y] = -i\Theta^{-1} , \quad \{D_+, D_-\} = \Theta^{-1} .
\]

13
These algebras are shown to induce noncommutative fermionic coordinates as well as noncommutative bosonic coordinates;

\[
[X, Y] = i\Theta, \quad \{\psi^+, \psi^-\} = \Theta.
\]  

The 'lowest Landau level' conditions are

\[
(D_x - iD_y)|\text{LLL}\rangle = 0, \quad \tag{3.47}
\]

\[
D_-|\text{LLL}\rangle = 0. \quad \tag{3.48}
\]

Analogous to the bosonic case the canonical analysis can be performed. Second class constraints are given by

\[
D_x = D_y = D_+ = D_- = 0 \quad \text{and the Dirac bracket for the system is calculated as (we have dropped terms up to the second class constraints)}
\]

\[
[O_1, O_2]_D = [O_1, O_2] + \Theta (i [O_1, D_x] [D_y, O_2] - i [O_1, D_y] [D_x, O_2] - [O_1, D_+] [D_-, O_2] - [O_1, D_-] [D_+, O_2]). \quad \tag{3.49}
\]

Then the Dirac brackets of the original coordinates become

\[
[x, y]_D = i\Theta, \quad \{\theta^+, \theta^-\}_D = \Theta, \quad \text{others} = 0. \quad \tag{3.50}
\]

We can introduce noncommutative guiding center coordinates which are written in terms of the global charges \((L_i, Q_\alpha)\) in such a way that they are equal to the original coordinates up to the second class constraints. The results are

\[
X = \frac{1}{2} \left( x - 2\Theta \frac{\partial}{\partial y} \right) = -\Theta \hat{L}_y = x - \Theta D_y - \frac{i}{2} \Theta \hat{\theta}^+ D_-
\]

\[
Y = \frac{1}{2} \left( y + 2\Theta \frac{\partial}{\partial x} \right) = \Theta \hat{L}_x = y + \Theta D_x + \frac{i}{2} \Theta \hat{\theta}^+ D_- \n\]

\[
\psi^+ = \frac{1}{2} \hat{\theta}^+ + \Theta \frac{\partial}{\partial \theta^-} = i\Theta \hat{Q}_- = \theta^+ + i\Theta D_-
\]

\[
\psi^- = \frac{1}{2} \theta^+ + \Theta \frac{\partial}{\partial \theta^-} = i\Theta \hat{Q}_+ = \frac{\Theta}{2} \psi^+ \hat{L}_+ = \theta^- + i\Theta D_+ + \frac{\Theta}{2} D_+ \theta^+. \quad \tag{3.51}
\]

They satisfy

\[
[X, Y] = [x, y]_D = i\Theta, \quad \{\psi^+, \psi^-\} = \{\theta^+, \theta^-\}_D = \Theta. \quad \tag{3.52}
\]

and all the other commutators vanish. This algebra is nothing but the canonical commutation relation \((3.14)\) for the redefined coordinates. The fermionic coordinates \(\psi^\alpha\) are related to \(\hat{\varphi}\) as

\[
\hat{\varphi}_\alpha = \epsilon_{\alpha\beta} \psi^\beta. \quad \tag{3.53}
\]
4 Noncommutative superspace from su(2|1)

In this section, we consider noncommutative superspace based on su(2|1) super algebra (or equivalently osp(2|2) algebra). This gives two-dimensional supersphere with four real supercharges. This type of noncommutative superspace was studied in [31] though an explicit relation to our case is not manifest.

The graded commutation relations of su(2|1) algebra are given by

\[
\begin{align*}
[\hat{l}_i, \hat{l}_j] &= i\epsilon_{ijk}\hat{l}_k, \\
[\hat{l}_i, \hat{q}_\alpha] &= -\frac{1}{2}(\sigma_i)_\alpha^\beta \hat{q}_\beta, \\
[\hat{l}_i, \hat{q}_\alpha] &= \frac{1}{2}(\sigma_i)_\alpha^\beta \hat{q}_\beta, \\
[\hat{B}, \hat{q}_\alpha] &= \frac{1}{2}\hat{q}_\alpha, \\
[\hat{B}, \hat{\bar{q}}_\alpha] &= -\frac{1}{2}\hat{\bar{q}}_\alpha, \\
\{\hat{q}_\alpha, \hat{q}_\beta\} &= (\sigma_i)_\alpha^\beta \hat{l}_i + \delta_\alpha^\beta \hat{B}, \quad \text{others} = 0.
\end{align*}
\]

This contains an osp(1|2) super algebra as a subalgebra. There are two Casimir operators. The second Casimir operator of su(2|1) algebra is given by

\[\hat{K}_2 = \hat{l}_i\hat{l}_i - \hat{B}^2 + \frac{1}{2}(\hat{q}_\alpha\hat{q}_\alpha - \hat{\bar{q}}_\alpha\hat{\bar{q}}_\alpha).\]

The third Casimir operator is \(\hat{K}_3 \sim \hat{K}_2\hat{B} + \cdots\). We will consider a coset space \(SU(2|1)/U(1)^2\) and this defines a super sphere with four supercharges.

Typical irreducible representations are characterized by two quantum numbers, \((b, L)\). The eigenvalues of the two Casimir operators are given by \(K_2 = L^2 - b^2\) and \(K_3 = b(L^2 - b^2)\). (There are other types of irreducible representations but we do not consider them here.) In terms of the osp(1|2) subalgebra, this representation is decomposed into two representations with superspin \(L\) and \(L - 1/2\). Hence the dimension of the irreducible representation is \(N = 8L\). Any supermatrix with this size can be expanded in terms of polynomials generated by \(\hat{l}_i, \hat{q}_\alpha\) and \(\hat{\bar{q}}_\alpha\). \(\hat{B}\) can be solved by the third Casimir \(\hat{K}_3\) and the polynomials do not depend on \(\hat{B}\). Since three \(\hat{l}_i\) satisfy the constraint given by the second Casimir, this defines two-dimensional supersphere with four grassmannian coordinates.

In order to take a flat limit, we introduce the following superspace coordinates,

\[
\begin{align*}
\hat{x}_i &= \left(\frac{\Theta}{L}\right)^{\frac{1}{4}} \hat{l}_i, \quad \text{for } i = 1, 2 \\
\hat{\theta}_1 &= \left(\frac{\Theta}{L}\right)^{\frac{1}{4}} \hat{q}_1, \quad \hat{\theta}_2 = \left(\frac{\Theta}{L}\right)^{\frac{1}{4}} \hat{\bar{q}}_2, \\
\hat{\theta}_1 &= \left(\frac{\Theta}{L}\right)^{\frac{1}{4}} \hat{q}_1, \quad \hat{\theta}_2 = \left(\frac{\Theta}{L}\right)^{\frac{1}{4}} \hat{\bar{q}}_2, \\
\hat{b} &= \left(\frac{1}{L}\right) \hat{B}.
\end{align*}
\]
Again we need to take an asymmetric scaling for the fermionic coordinates. The $\hat{l}_3$ is scaled as $L$ since we are considering the vicinity of the north pole on the sphere. In the large $L$ limit, the algebra among the coordinates becomes

$$
\begin{align*}
[\hat{x}_+, \hat{x}_-] &= 2\Theta, \\
[\hat{x}_+, \hat{\theta}_1] &= -\hat{\theta}_2, \\
[\hat{x}_+, \hat{\theta}^2] &= \hat{\theta}^1, \\
\{\hat{\theta}_1, \hat{\theta}^1\} &= (\hat{b} + 1)\Theta, \\
\{\hat{\theta}_1, \hat{\theta}^2\} &= \hat{x}_-, \\
\{\hat{\theta}_2, \hat{\theta}^2\} &= (\hat{b} - 1)\Theta, \quad \text{others} = 0,
\end{align*}
$$
(4.4)

where $\hat{x}_\pm = \hat{x}_1 \pm i\hat{x}_2$. Since $\hat{b}$ is a center of the algebra, we set it as a constant $b$. Furthermore by the transformations

$$
\begin{align*}
\hat{\varphi}_1 &= \hat{\theta}_1 + \frac{1}{2\Theta}\hat{x}_-\hat{\theta}_2, \quad \hat{\varphi}_2 = \hat{\theta}_2, \\
\hat{\varphi}^1 &= \hat{\theta}^1, \quad \hat{\varphi}^2 = \hat{\theta}^2 - \frac{1}{2\Theta}\hat{x}_-\hat{\theta}^1,
\end{align*}
$$
(4.5)

this algebra reduces to a simpler noncommutative algebra for the superspace coordinates;

$$
\begin{align*}
[\hat{x}_+, \hat{x}_-] &= 2\Theta, \\
\{\hat{\varphi}_1, \hat{\varphi}^1\} &= (b + 1)\Theta, \\
\{\hat{\varphi}_2, \hat{\varphi}^2\} &= (b - 1)\Theta, \quad \text{others} = 0.
\end{align*}
$$
(4.6)

Now we define the generators of supersymmetry in the flat limit as

$$
\begin{align*}
P_\pm &= \pm\frac{1}{2}\text{adj}(\hat{x}_\pm), \\
Q_\alpha &= \text{adj} \left(\hat{\theta}_\alpha\right), \\
\bar{Q}^\alpha &= \text{adj} \left(\hat{\theta}^\alpha\right).
\end{align*}
$$
(4.7)

Then the following supersymmetry algebra holds

$$
\begin{align*}
[P_+, Q_1] &= -\frac{1}{2}Q_2, \\
[P_+, \bar{Q}^2] &= \frac{1}{2}\bar{Q}^1, \\
\{Q_1, \bar{Q}^2\} &= -2P_-, \quad \text{others} = 0.
\end{align*}
$$
(4.8)

There are four supercharges but only half of them generated by $Q_1$ and $\bar{Q}^2$ are dynamical supersymmetries. The other half are non-dynamical supersymmetries and do not generate space-time translation.
We next introduce additional operators \( \hat{d}_\alpha \) corresponding to the generators eq. (3.8) in the \( \text{osp}(1|2) \) case,

\[
\hat{d}_\alpha \equiv \frac{1}{2L} \left\{ (\sigma_i)_\alpha^\beta \left( \hat{l}_i \hat{q}_\beta + \hat{q}_\beta \hat{l}_i \right) - \left( \hat{B} \hat{q}_\alpha + \hat{q}_\alpha \hat{B} \right) \right\},
\]

\[
\hat{d}_\alpha \equiv \frac{1}{2L} \left\{ (\sigma_i)_\beta^\alpha \left( \hat{l}_i \hat{q}_\beta + \hat{q}_\beta \hat{l}_i \right) - \left( \hat{B} \hat{q}_\alpha + \hat{q}_\alpha \hat{B} \right) \right\}.
\]

We can obtain covariant derivatives by taking the scaling limit of these generators,

\[
D_\alpha = \text{adj} \left( \hat{d}_\alpha' \right),
\]

\[
\bar{D}_\alpha = \text{adj} \left( \hat{d}_\alpha'' \right),
\]

\[
\hat{d}_1' = \left( \frac{\Theta}{L} \right)^{\frac{1}{4}} d_1 = (1 - b) \hat{\theta}_1 + \frac{1}{\Theta} x_- \hat{\theta}_2 = (1 - b) \hat{\phi}_1 + \frac{1 + b}{2\Theta} x_- \hat{\phi}_2,
\]

\[
\hat{d}_2' = \left( \frac{\Theta}{L} \right)^{\frac{1}{4}} d_2 = (1 + b) \hat{\theta}_2 = -(1 + b) \hat{\phi}_2,
\]

\[
\hat{d}_1'' = \left( \frac{\Theta}{L} \right)^{\frac{3}{4}} d_1 = (1 - b) \hat{\theta}_1 = (1 - b) \hat{\phi}_1,
\]

\[
\hat{d}_2'' = \left( \frac{\Theta}{L} \right)^{\frac{3}{4}} d_2 = -(1 + b) \hat{\theta}_2 + \frac{1}{\Theta} x_- \hat{\theta}_1 = -(1 + b) \hat{\phi}_2 + \frac{1 - b}{2\Theta} x_- \hat{\phi}_1.
\]

The covariant derivatives anticommute with \( \hat{Q}_\alpha \) and \( \hat{\bar{Q}}_\alpha \) and satisfy the following algebra,

\[
[P_+, D_1] = -\frac{1}{2} D_2,
\]

\[
[P_+, \bar{D}_2] = \frac{1}{2} \bar{D}_1,
\]

\[
\{ D_1, \bar{D}_2 \} = 2(b^2 - 1) P_-.
\]

Functions on the superspace \((\hat{x}, \hat{\phi}, \hat{\bar{\phi}})\) are supermatrices and generally written as functions of supercoordinates: \( \Phi (\hat{x}_\pm, \hat{\phi}_\alpha, \hat{\bar{\phi}}^\alpha) \). We can constrain the function by imposing the following constraints,

\[
D_2 \Phi = -(1 + b) [\hat{\phi}_2, \Phi] = 0,
\]

\[
\bar{D}_1 \Phi = (1 - b) [\hat{\bar{\phi}}^1, \Phi] = 0.
\]

These conditions automatically mean

\[
Q_2 \Phi = \bar{Q}^1 \Phi = 0,
\]

\( ^2 \)These operators are introduced so that they anticommute with the supersymmetry generators in the flat limit. There might be possible \( 1/L \) corrections to them before taking the flat limit.
and the superfield $\Phi$ depends only on $(\hat{x}_\pm, \hat{\varphi}_2, \hat{\varphi}_1)$. This is the chiral superfield and supersymmetries are generated by $Q_1$ and $\bar{Q}^2$ whose anticommutator becomes $P_-$. As a result, on the chiral superfields, the algebra among generators for supersymmetries $Q_1$ and $\bar{Q}^2$, translations $P_\pm$, and covariant derivatives $D_1$ and $\bar{D}^2$ is given by

$$\{Q_1, \bar{Q}^2\} = -2P_-, \quad \{D_1, \bar{D}^2\} = 2(b^2 - 1)P_-,$$

others $= 0 \quad (4.14)$

They are written as differential operators;

$$P_\pm = \Theta \frac{\partial}{\partial x_\mp},$$

$$Q_1 = (1 + b)\Theta \frac{\partial}{\partial \varphi_1^1} + \varphi_2 \frac{\partial}{\partial x_+},$$

$$\bar{Q}^2 = -(1 - b)\Theta \frac{\partial}{\partial \varphi_2} - \varphi_1^1 \frac{\partial}{\partial x_+},$$

$$D_1 = (1 + b) \left[ (1 - b)\Theta \frac{\partial}{\partial \varphi_1^1} - \varphi_2 \frac{\partial}{\partial x_+} \right],$$

$$\bar{D}^2 = (1 - b) \left[ (1 + b)\Theta \frac{\partial}{\partial \varphi_2} - \varphi_1^1 \frac{\partial}{\partial x_+} \right].$$

We can further constrain the superfield by $P_+$:

$$P_+ \Phi = 0 \quad (4.15)$$

Then the superfield becomes independent of the $\hat{x}_-$ coordinate and the system becomes essentially one-dimensional system with two supersymmetries.

## 5 Noncommutative superspace from $\text{psu}(2|2)$

In this section we try to construct a noncommutative superspace in four dimensions based on $\text{psu}(2|2)$ graded algebra. There are various possibilities for the scaling of Grassmannian coordinates when we take a flat limit. We show two examples. More details will be discussed in a separate paper.

The $\text{psu}(2|2)$ graded algebra is given by

$$\begin{align*}
\{\hat{l}_i, \hat{l}_j\} &= i\epsilon_{ijk}\hat{l}_k, \\
\{\hat{l}_i, \hat{q}^\beta_{\alpha}\} &= -\frac{1}{2}(\sigma_i)_{\alpha}^\gamma \hat{q}^\beta_{\gamma}, \\
\{\hat{l}_i, \hat{\bar{q}}^\gamma_{\delta}\} &= \frac{1}{2}(\sigma_i)_{\gamma}^\beta \hat{\bar{q}}^\delta_{\beta}, \\
\{\hat{q}^\gamma_{\alpha}, \hat{\bar{q}}^\delta_{\beta}\} &= \delta^\gamma_{\alpha}^\delta \hat{l}_i - \delta^\delta_{\alpha}^\gamma \delta^\beta \hat{l}_i. \quad (5.1)
\end{align*}$$

It would have been appropriate to parametrize the superspace so that both of the two constraints are anti-chiral and the constrained superfield becomes apparently chiral. It is merely a problem of notation.
The bosonic part of $su(2|2)$ consists of two sets of $su(2)$ algebra generated by $\hat{l}_i$ and $\hat{\bar{l}}_i$ and eight odd generators transform as spinors both under two $su(2)$'s. Based on this graded algebra we can construct a four-dimensional superspace $PSU(2|2)/U(1)^2$.

### 5.1 $d = 4$ flat noncommutative superspace

In order to take a flat limit we first consider the following scaling of the superspace coordinates,

$$\hat{x}_i = \left(\frac{\Theta}{L}\right)^{\frac{1}{2}} \hat{l}_i, \quad \hat{\bar{x}}_i = \left(\frac{\Theta}{L}\right)^{\frac{1}{2}} \hat{\bar{l}}_i, \quad \text{for } i = 1, 2$$

$$\left(\hat{\theta}^1_2, \hat{\theta}^1_1, \hat{\theta}^2_2, \hat{\theta}^2_1\right) = \left(\frac{\Theta}{L}\right)^{\frac{1}{2}} \left(\hat{q}^1_2, \hat{\bar{q}}^1_1, \hat{q}^1_1, \hat{\bar{q}}^1_2\right),$$

$$\left(\hat{\bar{\theta}}^1_1, \hat{\bar{\theta}}^2_2, \hat{\bar{\theta}}^2_1, \hat{\bar{\theta}}^1_2\right) = \left(\frac{\Theta}{L}\right)^{\frac{1}{4}} \left(\hat{\bar{q}}^2_1, \hat{\bar{q}}^2_1, \hat{\bar{q}}^2_2, \hat{\bar{q}}^2_2\right).$$

(5.2)

We use a similar asymmetric scaling of the fermionic coordinates to the two-dimensional cases. The $\hat{l}_i$ and $\hat{\bar{l}}_i$ are scaled as $L$ and $cL$ respectively where $c$ is an arbitrary constant.

Taking the large $L$ limit the algebra among the coordinates becomes

$$[\hat{x}_+, \hat{x}_-] = 2\Theta,$$

$$[\hat{\bar{x}}_+, \hat{\bar{x}}_-] = 2c\Theta,$$

$$[\hat{x}_-, \hat{\theta}^1_2] = -\hat{\theta}^1_1,$$

$$[\hat{\bar{x}}_-, \hat{\bar{\theta}}^1_1] = \hat{\bar{\theta}}^2_1,$$

$$[\hat{x}_-, \hat{\theta}^2_2] = \hat{\theta}^2_2,$$

$$[\hat{\bar{x}}_-, \hat{\bar{\theta}}^2_2] = -\hat{\bar{\theta}}^2_2,$$

$$\{\hat{\theta}^1_2, \hat{\theta}^1_1\} = \hat{x}_+, \quad \{\hat{\theta}^1_2, \hat{\theta}^2_2\} = \hat{\bar{x}}_+, \quad \{\hat{\bar{\theta}}^1_1, \hat{\theta}^1_1\} = (1 - c)\Theta,$$

$$\{\hat{\bar{\theta}}^1_1, \hat{\bar{\theta}}^2_2\} = (1 + c)\Theta,$$

$$\{\hat{\theta}^1_2, \hat{\theta}^2_2\} = -(1 + c)\Theta,$$

$$\{\hat{\bar{\theta}}^2_1, \hat{\theta}^1_1\} = -(1 - c)\Theta,$$

$$\{\hat{\bar{\theta}}^2_1, \hat{\bar{\theta}}^2_2\} = c(1 - c)\Theta,$$

$$\{\text{others}\} = 0.$$

(5.3)

By introducing the following fermionic coordinates

$$\hat{\phi}^1_2 = \hat{\theta}^1_2 - \frac{1}{2\Theta} \hat{x}_+ \hat{\theta}^1_1 + \frac{1}{2c\Theta} \hat{x}_+ \hat{\theta}^2_2,$$

$$\hat{\phi}^1_1 = \hat{\theta}^1_1 + \frac{1}{2\Theta} \hat{x}_+ \hat{\theta}^1_1,$$

$$\hat{\phi}^2_2 = \hat{\theta}^2_2 - \frac{1}{2c\Theta} \hat{x}_+ \hat{\theta}^2_2,$$

$$\hat{\phi}^2_2 = \hat{\theta}^2_2 + \frac{1}{2c\Theta} \hat{x}_+ \hat{\theta}^2_2,$$

$$\hat{\phi}^{\dot{\alpha}}_\alpha = \hat{\bar{\theta}}^{\dot{\alpha}}_\alpha, \quad \hat{\phi}^{\dot{\beta}}_\alpha = \hat{\bar{\theta}}^{\dot{\beta}}_\alpha, \quad \text{for others}$$

(5.4)
this algebra is much simplified to satisfy the canonical forms;

$$\left[\hat{\mathcal{E}}_+, \hat{\mathcal{E}}_-\right] = 2\Theta, \quad \left[\hat{\mathcal{E}}_+, \hat{\mathcal{E}}_-\right] = 2c\Theta,$$

$$\left\{\hat{\varphi}_1, \hat{\varphi}_1^1\right\} = (1 - c)\Theta, \quad \left\{\hat{\varphi}_1^2, \hat{\varphi}_1^1\right\} = (1 + c)\Theta,$$

$$\left\{\hat{\varphi}_2, \hat{\varphi}_1\right\} = -(1 + c)\Theta, \quad \left\{\hat{\varphi}_2, \hat{\varphi}_2^2\right\} = -(1 - c)\Theta.$$

(5.5)

We define the generators of supersymmetry in the flat limit as

$$P_\pm = \pm \frac{1}{2}\text{adj } x_\pm, \quad \bar{P}_\pm = \pm \frac{1}{2}\text{adj } \bar{x}_\pm,$$

$$Q_\alpha^\beta = \text{adj } \theta_\alpha^\beta, \quad \bar{Q}_\alpha^\beta = \text{adj } \bar{\theta}_\alpha^\beta.$$ (5.6)

Then the supersymmetry algebra has the following forms,

$$\left[P_-, Q_2^1\right] = \frac{1}{2} Q_2^1, \quad \left[P_-, Q_1^1\right] = -\frac{1}{2} Q_1^2,$$

$$\left[\bar{P}_-, Q_2^1\right] = -\frac{1}{2} Q_2^2, \quad \left[\bar{P}_-, Q_2^2\right] = \frac{1}{2} Q_1^1,$$

$$\left\{Q_2^1, Q_1^1\right\} = 2P_+, \quad \left\{Q_2^1, Q_2^2\right\} = -2\bar{P}_+,$$

(5.7)

others = 0.

Only three supercharges $Q_1^1, \bar{Q}_1^1$ and $\bar{Q}_2^2$ generate dynamical supersymmetries, i.e., anticommutators among them become the generators of the space-time translations. The other supercharges are generators of non-dynamical supersymmetries. In order to construct covariant derivatives, it is useful to consider the following operators,

$$\hat{d}_\alpha^\beta = -\frac{1}{2L} \left\{ (\sigma_\alpha)_{\gamma}^\beta \left( \hat{\mathcal{E}}_\gamma \bar{q}_\alpha^\gamma + \bar{q}_\alpha^\gamma \hat{\mathcal{E}}_\gamma \right) + (\sigma_\gamma)_{\beta}^\alpha \left( \hat{\mathcal{E}}_\alpha \bar{q}_\gamma^\alpha + \bar{q}_\gamma^\alpha \hat{\mathcal{E}}_\alpha \right) \right\},$$

$$\hat{d}_\alpha^\beta = \frac{1}{2L} \left\{ (\sigma_\gamma)_{\alpha}^\beta \left( \hat{\mathcal{E}}_\alpha \bar{q}_\gamma^\alpha + \bar{q}_\gamma^\alpha \hat{\mathcal{E}}_\alpha \right) + (\sigma_\gamma)_{\beta}^\alpha \left( \hat{\mathcal{E}}_\alpha \bar{q}_\gamma^\alpha + \bar{q}_\gamma^\alpha \hat{\mathcal{E}}_\alpha \right) \right\}.$$ (5.8) (5.9)

We can obtain the covariant derivatives which anticommute with the supercharges in the flat limit,

$$D_1^1 = \left( \frac{\Theta}{L} \right)^{\beta} \text{adj } d_1^1 = -(1 + c)\text{adj } \theta_1^1,$$

$$D_1^2 = \left( \frac{\Theta}{L} \right)^{\beta} \text{adj } d_1^\beta = -(1 - c)\text{adj } \theta_1^2,$$

$$D_2^1 = \left( \frac{\Theta}{L} \right)^{\frac{1}{2}} \text{adj } d_2^1 = \text{adj } \left[ (1 - c)\theta_2^1 - \frac{1}{\Theta} \hat{\mathcal{E}}_+ \theta_1^1 - \frac{1}{\Theta} \hat{\mathcal{E}}_+ \theta_1^2 \right]$$

$$= \text{adj } \left[ (1 - c)\varphi_2^1 - \frac{1 + c}{2\Theta} \hat{\mathcal{E}}_+ \varphi_1^1 - \frac{1 + c}{2\Theta c} \hat{\mathcal{E}}_+ \varphi_2^2 \right],$$

20
\[ D^2_2 = \left( \frac{\Theta}{L} \right)^{\frac{3}{4}} \text{adj} \, d^2_2 = (1 + c)\text{adj} \, \hat{\theta}^2_2, \]
\[ D^1_1 = \left( \frac{\Theta}{L} \right)^{\frac{1}{4}} \text{adj} \, d^1_1 = \text{adj} \left[ (1 + c)\hat{\theta}^1_1 + \frac{1}{\Theta} \hat{x}_+ \hat{\theta}^2_1 \right], \]
\[ = \text{adj} \left[ (1 + c)\hat{\phi}^1_1 + \frac{1 - c}{2\Theta} \hat{x}_+ \hat{\phi}^2_1 \right], \]
\[ D^2_1 = \left( \frac{\Theta}{L} \right)^{\frac{3}{4}} \text{adj} \, d^2_1 = -(1 - c)\text{adj} \, \hat{\theta}^2_1 = -(1 - c)\text{adj} \, \hat{\phi}^2_1, \]
\[ D^1_2 = \left( \frac{\Theta}{L} \right)^{\frac{1}{4}} \text{adj} \, d^1_2 = (1 - c)\text{adj} \, \hat{\theta}^1_2 = (1 - c)\text{adj} \, \hat{\phi}^1_2, \]
\[ D^2_2 = \left( \frac{\Theta}{L} \right)^{\frac{3}{4}} \text{adj} \, d^2_2 = \text{adj} \left[ -(1 + c)\hat{\theta}^2_2 + \frac{1}{\Theta} \hat{x}_+ \hat{\theta}^1_1 \right], \]
\[ = \text{adj} \left[ -(1 + c)\hat{\phi}^2_2 - \frac{1 - c}{2\Theta c} \hat{x}_+ \hat{\phi}^1_1 \right]. \]

The covariant derivatives and the generators of space-time translations satisfy the following algebra,
\[
\begin{align*}
\left[ P_-, D^1_2 \right] &= \frac{1}{2} D^1_1, & \left[ P_-, D^1_1 \right] &= -\frac{1}{2} D^2_2, \\
\left[ P_-, D^2_2 \right] &= -\frac{1}{2} D^2_1, & \left[ P_-, D^2_2 \right] &= \frac{1}{2} D^1_1, \\
\left\{ D^1_2, D^1_1 \right\} &= -2(1 - c^2)P_+, & \left\{ D^2_2, D^2_1 \right\} &= -2(1 - c^2)\bar{P}_+, \\
\text{others} &= 0.
\end{align*}
\]

Field theories based on this algebra in the flat limit have two-dimensional like supersymmetries because only \( P_+ \) and \( \bar{P}_+ \) appear in the right hand sides of the algebra (5.7) and (5.11).

Next we consider another choice of scaling for the fermionic superspace coordinates,
\[
\begin{align*}
\left( \hat{\theta}^1_2, \hat{\theta}^2_2, \hat{\theta}^1_1, \hat{\theta}^2_1 \right) &= \left( \frac{\Theta}{L} \right)^{\frac{3}{4}} \left( \hat{q}^1_2, \hat{q}^2_2, \hat{q}^1_1, \hat{q}^2_1 \right), \\
\left( \hat{\theta}^1_1, \hat{\theta}^2_1, \hat{\theta}^1_2, \hat{\theta}^2_2 \right) &= \left( \frac{\Theta}{L} \right)^{\frac{3}{4}} \left( \hat{q}^1_1, \hat{q}^2_1, \hat{q}^1_2, \hat{q}^2_2 \right).
\end{align*}
\]
Then in the large $L$ limit the coordinates of the superspace satisfy

\[
\begin{align*}
[\hat{x}_+, \hat{x}_-] &= 2\Theta, & [\hat{x}_+, \hat{x}_-] &= 2c\Theta, \\
[\hat{x}_+, \hat{\theta}^1_2] &= -\hat{\theta}^2_2, & [\hat{x}_-, \hat{\theta}^1_2] &= -\hat{\theta}^1_1, \\
[\hat{x}_+, \hat{\theta}^2_2] &= \hat{\theta}^1_2, & [\hat{x}_-, \hat{\theta}^2_2] &= \hat{\theta}^2_1, \\
[\hat{\bar{x}}_+, \hat{\theta}^1_1] &= \hat{\theta}^1_1, & [\hat{\bar{x}}_-, \hat{\theta}^1_2] &= \hat{\theta}^2_2, \\
[\hat{\bar{x}}_+, \hat{\theta}^2_1] &= -\hat{\theta}^1_2, & [\hat{\bar{x}}_-, \hat{\theta}^2_1] &= -\hat{\theta}^2_1, \\
\{\hat{\bar{x}}^1_1, \hat{\bar{x}}^1_2\} &= -\hat{x}_-, & \{\hat{\bar{x}}^2_1, \hat{\bar{x}}^2_2\} &= \hat{x}_-, \\
\{\hat{\bar{x}}^1_2, \hat{\bar{x}}^2_1\} &= \hat{x}_+, & \{\hat{\bar{x}}^2_2, \hat{\bar{x}}^2_1\} &= -\hat{x}_+, \\
\{\hat{\bar{x}}^1_1, \hat{\bar{x}}^1_2\} &= \Theta(1 - c), & \{\hat{\bar{x}}^2_1, \hat{\bar{x}}^2_2\} &= -\Theta(1 + c), \\
\{\hat{\bar{x}}^2_1, \hat{\bar{x}}^2_2\} &= \Theta(1 + c), & \{\hat{\bar{x}}^2_2, \hat{\bar{x}}^2_1\} &= -\Theta(1 - c).
\end{align*}
\]

We define the generators of supersymmetries and space-time translations as

\[
\begin{align*}
Q^\beta_\beta &= \text{adj} \hat{\theta}^\beta_\beta, \\
Q^\beta_\alpha &= \text{adj} \hat{\theta}^\beta_\alpha, \\
P^\pm &= \pm \frac{1}{2} \text{adj} \hat{x}^\pm, & \bar{P}^\pm &= \pm \frac{1}{2} \text{adj} \hat{\bar{x}}^\pm.
\end{align*}
\]

In this case, it can be easily seen that the anticommutators among four supercharges $Q^\beta_\alpha$, $Q^\beta_\alpha$, $\bar{Q}^\beta_\alpha$, and $\bar{Q}^\beta_\alpha$ become four space-time translations $P^\pm$ and $\bar{P}^\pm$. Therefore noncommutative theories based on this algebra have four dynamical supersymmetries.

### 5.2 Seiberg’s noncommutative superspace as a constrained system

Here we briefly explain that the Seiberg’s noncommutative superspace can be understood as a constrained system and the noncommutative algebra for the coordinates is realized by the Dirac bracket under the constraints.

We begin with a little more general setting. We consider supercovariant derivatives with the following gauge field backgrounds,

\[
\begin{align*}
D_\mu &= \partial_\mu - \frac{i}{2} f_{\mu\nu} x^\nu, \\
D_\alpha &= \frac{\partial}{\partial \theta^\alpha} - i (\sigma^\mu \bar{\theta})_\alpha \partial_\mu + \frac{i}{2} f_{\alpha\beta} \theta^\beta, \\
\bar{D}_{\dot{\alpha}} &= - \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i (\sigma^\mu \bar{\theta})^{\dot{\alpha}} \partial_\mu - \frac{i}{2} f_{\dot{\alpha}\dot{\beta}} \theta^{\dot{\beta}}.
\end{align*}
\]

22
They are natural generalization of the bosonic covariant derivatives in a constant magnetic field, but the field strengths depend on the superspace coordinates:

\[
F_{\mu \nu} = f_{\mu \nu} \quad F_{\mu \alpha} = -\frac{i}{2} f_{\mu \nu} (\sigma^\nu \bar{\theta})_\alpha, \quad F_{\mu \dot{\alpha}} = \frac{i}{2} f_{\mu \nu} (\theta \sigma^\nu)_{\dot{\alpha}},
\]

\[F_{\alpha \beta} = f_{\alpha \beta}, \quad F_{\alpha \dot{\beta}} = i f_{\mu \nu}(\sigma^\mu)_{\alpha \dot{\beta}} x^\nu, \quad F_{\dot{\alpha} \dot{\beta}} = f_{\dot{\alpha} \dot{\beta}}.\] (5.16)

Imposing \(D_\alpha = \bar{D}_{\dot{\alpha}} = D_\mu = 0\) as the second class constraints, we can calculate the Dirac brackets of the superspace coordinates and obtain the noncommutative algebra similar to that given in [9].

In the following we consider an easier case for simplicity, namely, a case where only \(f_{\alpha \beta}\) are nonvanishing. We then impose the second class constrain by only \(D_\alpha\)’s and calculate the Dirac bracket

\[\{A, B\}_D = \{A, B\} + i[A, D_\alpha] f_{\alpha \beta}^{-1} [D_\beta, B]\] (5.17)
as

\[
[x^\mu, x^\nu]_D = i f_{\alpha \beta}^{-1} (\sigma^\mu \bar{\theta})_\alpha (\sigma^\nu \bar{\theta})_\beta, \\
[x^\mu, \theta_\alpha]_D = -f_{\alpha \beta}^{-1} (\sigma^\mu \bar{\theta})_\beta, \\
\{\theta_\alpha, \theta_\beta\}_D = i f_{\alpha \beta}^{-1}.
\] (5.18)

This is nothing but the noncommutative algebra given in the paper by Seiberg [3]. The noncommutative parameter \(C_{\alpha \beta}\) in [3] is related to our \(f\) as \(C_{\alpha \beta} = i f_{\alpha \beta}^{-1}\). If we redefine the coordinate as

\[y^\mu = x^\mu + i \theta \sigma^\mu \bar{\theta},\] (5.19)

they become commutable with the other coordinates

\[[y^\mu, y^\nu]_D = [y^\mu, \theta^\alpha]_D = [y^\mu, \bar{\theta}^\dot{\alpha}]_D = 0\] (5.20)

In this case where only \(f_{\alpha \beta}\) is nonvanishing, we can more easily obtain the canonical pairs on the reduced superspace by a similarity transformation

\[\mathcal{O} \to \exp(-i \theta \sigma^\mu \bar{\theta} \partial_\mu) \mathcal{O} \exp(i \theta \sigma^\mu \bar{\theta} \partial_\mu).\] (5.21)

Then the mutually commutative set of supercovariant derivatives \(D_\alpha\) and global charges \(q_\alpha\) are given by

\[
D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{i}{2} f_{\alpha \beta} \theta^\beta, \\
q_\alpha = i \frac{\partial}{\partial \theta^\alpha} - 2(\sigma^\mu \bar{\theta})_\alpha \partial_\mu + \frac{1}{2} f_{\alpha \beta} \theta^\beta.
\] (5.22)

The remaining coordinates on the phase space which (anti)commute with \(D_\alpha\) are

\[\psi_\alpha = f_{\alpha \beta}^{-1}(q_\beta + 2(\sigma^\mu \bar{\theta})_{\beta \dot{\beta}} \partial_\mu) = i f_{\alpha \beta}^{-1} D_\beta + \theta_\alpha\] (5.24)
and \((x^\mu, \partial/\partial x^\mu, \bar{\theta}^\alpha, \partial/\partial \bar{\theta}^\dot{\alpha})\). They satisfy the canonical algebra with \(\psi\)

\[
\{\psi_\alpha, \psi_\beta\} = i f^{-1}_{\alpha\beta}. \tag{5.25}
\]

The set \((\psi_\alpha, x^\mu, \partial/\partial x^\mu, \bar{\theta}^\alpha, \partial/\partial \bar{\theta}^\dot{\alpha})\) gives the phase space coordinates for the constrained system we are considering now. When we construct a field theory on the noncommutative space \((x^\mu, \psi_\alpha, \bar{\theta}^\alpha)\), we can introduce the canonical conjugate to \(\psi\) as the adjoint action

\[
\partial/\partial \psi_\alpha = -i \text{adj } q_\alpha.
\]

Then the covariant derivatives\(^4\) and the supercharges can be defined in the same way as that given in \([3]\).

### 6 Conclusions and Discussions

In this paper, we have constructed noncommutative superspaces based on graded (super) Lie algebras. In particular, we consider fuzzy supersphere based on \(osp(1|2)\) and \(su(2|1)\) algebras. They give two-dimensional supersphere with two and four real supercharges. We then consider flat limits. In order to take flat limits with the fermionic noncommutativity, we needed to take an asymmetric scaling limit for fermionic coordinates on superspace. We also obtained covariant derivatives and imposed chiral constraints to remove half degrees of freedom. This method was generalized to four dimensional noncommutative superspaces based on \(psu(2|2)\) algebra \([15]\). In this case, there are varieties to assign scalings to the fermionic coordinates when we take a flat limit. We showed two examples. One is similar to the two-dimensional cases and we have obtained supersymmetry generators and covariant derivatives. This system is two-dimensional like in a sense that only two generators of space-time translation appear in the anticommutators of the supersymmetry generators. The other example is more nontrivial. With this scaling, space-time translation generators into the all four directions appear. More details are left for future investigations. It would be also interesting to investigate other scaling limits of the \(psu(2|2)\) or \(su(2|2)\) algebras which can give the noncommutative superspace given in the paper \([34]\).

We have also investigated these noncommutative superspaces as constrained systems. This is an analogue of the lowest Landau level system of particles moving in a constant magnetic field. We obtained two sets of operators, super covariant derivatives and super guiding center coordinates. They are obtained by the right and the left multiplications on the group manifolds. So they are commutative to each other. The lowest Landau level conditions are generalized by adding fermionic constrains in addition to the ordinary

\(^4\)It is confusing to use the same word as the operator defined in \((5.22)\). The covariant derivative in \((5.22)\) is an operator used to define the noncommutative superspace as a constrained system. The covariant derivative here is an operator acting within the constrained space that anticommutes with the supercharges.
bosonic condition for the lowest Landau level states. Imposing gauge fixing conditions, we calculated the Dirac brackets for superspace coordinates to obtain noncommutative superspace coordinates. This method can be extended to more general cases to obtain more general noncommutative superspaces. We want to report it in a future publication. Along this line, it will also be interesting to investigate a supersymmetric generalization of $W_\infty$ algebraic structure, which plays an important role to study the physics of the lowest Landau level systems [32].

Our construction has an advantage that Jacobi identities of the supersymmetries and the associativities of the star products are manifest, but it is restricted to the noncommutative superspaces with Lie algebraic noncommutativity. Namely the supersymmetry algebras satisfy Lie algebras. In the papers [11, 13], the anticommutators for supersymmetries contain the second order derivative operators. In order to construct these structures based on a supermatrix approach, we may need to investigate supermatrix models without the graded Lie algebraic structures.

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A Cartan one-forms and the generalized In"on"u-Wigner contraction

In this appendix, we briefly explain the method of Cartan one-forms to obtain the left and right multiplications. Then we explain the generalized In"on"u-Wigner contraction proposed in [29].

Suppose we have a Lie algebra $[T_a, T_b] = f_{abc} T_c$. A group manifold generated by this algebra can be parametrized as $g = e^{aT_a}$. Cartan one-forms $e^a$ are defined by $g^{-1}dg = ie^aT_a$ and satisfy the Maure-Cartan (MC) equation $de^a = f_{be}^c e^b e^c / (2i)$. If we write them as $e^a = dx^m e_m^a(x)$, covariant derivatives $D_a = (e^{-1})^m_n \partial_m / i$ generate the right multiplication $g \to gh$ and obey the Lie algebra $[D_a, D_b] = f_{ab}^c D_c$. The left multiplication generators are similarly obtained from $dgg^{-1}$ and commute with the right multiplications.

The generalized IW contraction can be obtained as follows. We rescale the parameter $x^a$ on the group manifold as $x^a \to s^n x^a$ and take $s \to 0$ limit. Since the Cartan one-form $e^a$ is written in terms of a polynomial of $x$’s, it can be expanded by $s$ as

$$e_a = \sum_n s^n e_{a[n]} \quad .$$  

(A.1)

Here we interpret that each $e_{a[n]}$ is a different Cartan one-form corresponding to different generators $T_{a[n]}$. In this sense, this is an expansion [30] rather than a contraction [27, 28].
MC equations are satisfied order by order

\[ de_{a[n]} = \frac{1}{2i} f^{bc} e_{b[l]} e_{c[n-l]} \]  

(A.2)

and they determine commutation relations between the generators \( T_{a[n]} \). It is obvious from the MC equations (A.2) that the commutation relations are closed among generators with weights \([n]\) less than some fixed number. Jacobi identities are automatically satisfied.

## B Supercovariant derivatives

In this appendix, we give a derivation of (3.40). An group element of the noncommutative super-translation group generated by \( T_A = \{ L_x, L_y, L_z, Q \pm \} \) satisfying (3.40) is parameterized as \( g(x, y, \phi, \theta^+, \theta^-) \). Left invariant Cartan 1-forms are obtained by \( g^{-1}dg = dz^M E_M^A T_A \) as

\[
E^x = dx - \frac{i}{2} d\theta^+ \theta^+ \\
E^y = dy + \frac{1}{2} d\theta^+ \theta^+ \\
E^z = \left( d\phi - \frac{1}{2}(dx - dy) \right) + \frac{i}{2}(d\theta^+ \theta^- + d\theta^- \theta^+) \\
E^+ = d\theta^+ \\
E^- = d\theta^- + \frac{i}{2}(dx + idy) \theta^+ .
\]  

(B.1)

The coefficients of the Cartan 1-forms are give as

\[
E_M^A = \begin{pmatrix}
1 & 0 & -\frac{y}{2} & 0 & \frac{i}{2} \theta^+ \\
0 & 1 & \frac{i}{2} & 0 & -\frac{1}{2} \theta^+ \\
0 & 0 & 1 & 0 & 0 \\
-\frac{i}{2} \theta^+ & \frac{1}{2} \theta^+ & \frac{i}{2} \theta^- & 1 & 0 \\
0 & 0 & \frac{i}{2} \theta^+ & 0 & 1
\end{pmatrix}
\]  

(B.2)
and whose inverse is given as

\[
(E^{-1})_A^M = \begin{pmatrix}
1 & 0 & \frac{y}{2} & 0 & -\frac{i}{2}\theta^+
\end{pmatrix}.
\]

Therefore the supercovariant derivatives are given as

\[
D_A = (E^{-1})_A^M \left(\frac{1}{\ell}\right) \partial_M
\]

whose components are \(\text{(B.4)}\).

### C Global charges

In this appendix, we give a derivation of \(\text{(3.42)}\). Under the global transformations a group element \(g\) is transformed into \(g \rightarrow Gg\) with infinitesimal parameters \(\varepsilon\)

\[
g^{-1}\delta_\varepsilon g = g^{-1}(G - 1)g \equiv \Delta_\varepsilon E^A T_A = \delta_\varepsilon z^M E^A_M A T_A.
\]

Expression of \(\Delta_\varepsilon E^A\)'s are obtained as

\[
\begin{align*}
g^{-1}(e^{i\varepsilon x} Q_+ - 1)g &= i\varepsilon^+[\theta^+ L_+ + i\left(\theta^- - \frac{i}{2}(x + iy)\theta^+ight) L_z + Q_+ - \frac{i}{2}(x + iy)Q_-] \\
g^{-1}(e^{i\varepsilon -} Q_- - 1)g &= i\varepsilon^-[\theta^+ L_z + Q_-] \\
g^{-1}(e^{i\varepsilon x} L_x - 1)g &= i\varepsilon^x[L_x - yL_z + \frac{i}{2}\theta^+ Q_-] \\
g^{-1}(e^{i\varepsilon y} L_y - 1)g &= i\varepsilon^y[L_y + xL_z - \frac{1}{2}\theta^+ Q_-] \\
g^{-1}(e^{i\varepsilon z} L_z - 1)g &= i\varepsilon^z L_z.
\end{align*}
\]

The global charges are given by

\[
\varepsilon \hat{Q} = \delta_\varepsilon z^M \left(\frac{1}{\ell}\right) \partial_M = \Delta_\varepsilon E^A (E^{-1})_A^M \left(\frac{1}{\ell}\right) \partial_M.
\]
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