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Brunn-Minkowski inequality for the 1-Riesz capacity and level set convexity for the 1/2-Laplacian

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Abstract
We prove that the 1-Riesz capacity satisfies a Brunn-Minkowski inequality, and that the capacitary function of the 1/2-Laplacian is level set convex.

Keywords: fractional Laplacian; Brunn-Minkowski inequality; level set convexity; Riesz capacity.

1 Introduction
In this paper we consider the following problem

\[
\begin{cases}
(-\Delta)^s u = 0 & \text{on } \mathbb{R}^N \setminus K \\
u = 1 & \text{on } K \\
\lim_{|x| \to +\infty} u(x) = 0
\end{cases}
\]

where \( N \geq 2, s \in (0, N/2), \) and \((-\Delta)^s\) stands for the \( s \)-fractional Laplacian, defined as the unique pseudo-differential operator \((-\Delta)^s : \mathcal{S} \mapsto L^2(\mathbb{R}^N)\), being \( \mathcal{S} \) the Schwartz space of functions with fast decay to 0 at infinity, such that

\[
\mathcal{F}(-\Delta)^s f = |\xi|^{2s} \mathcal{F}(f)(\xi),
\]

where \( \mathcal{F} \) denotes the Fourier transform. We refer to the guide [12, Section 3] for more details on the subject. A quantity strictly related to Problem (1) is the so-called Riesz potential energy of a set \( E \), defined as

\[
I_\alpha(E) = \inf_{\mu(E)=1} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{d\mu(x) d\mu(y)}{|x-y|^{N-\alpha}}, \quad \alpha \in (0, N).
\]

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It is possible to prove (see [18]) that if $E$ is a compact set, then the infimum in the definition of $I_\alpha(E)$ is achieved by a Radon measure $\mu$ supported on the boundary of $E$ if $\alpha \leq N - 2$, and with support equal to the whole $E$ if $\alpha \in (N - 2, N)$. If $\mu$ is the optimal measure for the set $E$, we define the Riesz potential $v$ of $E$ as

$$v(x) = \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x - y|^{N-\alpha}},$$

so that

$$I_\alpha(E) = \int_{\mathbb{R}^N} v(x)d\mu(x).$$

It is not difficult to check (see [18, 15]) that the potential $v$ satisfies

$$(-\Delta)\frac{2}{\alpha} v = c(\alpha, N) \mu,$$

where $c(\alpha, N)$ is a positive constant, and that $v = I_\alpha(E)$ on $E$. In particular, if $s = \alpha/2$, then $v_K = v/I_{2s}(K)$ is the unique solution of Problem (1).

Following [18], we define the $\alpha$-Riesz capacity of a set $E$ as

$$\text{Cap}_\alpha(E) := \frac{1}{I_\alpha(E)}.$$  

(4)

We point out that this is not the only concept of capacity present in literature. Indeed, another one is given by the 2-capacity of a set $E$, defined by

$$C_2(E) = \min \left\{ \int_{\mathbb{R}^N} |
abla \varphi|^2 : \varphi \in C^1(\mathbb{R}^N, [0, 1]), \varphi \geq \chi_E \right\}$$

(5)

where $\chi_A$ is the characteristic function of the set $A$. It is possible to prove that, if $E$ is a compact set, then the minimum in (5) is achieved by a function $u$ satisfying

$$\begin{cases} 
\Delta u = 0 & \text{on } \mathbb{R}^N \setminus E \\
u = 1 & \text{on } E \\
\lim_{|x| \to +\infty} u(x) = 0.
\end{cases}$$

(6)

It is worth stressing that the 2-capacity and the $\alpha$-Riesz capacity share several properties, and coincide if $\alpha = 2$. We refer the reader to [19, Chapter 8] for a discussion of this topic.

In a series of works (see for instance [5, 10, 17] and the monography [16]) it has been proved that the solutions of (6) are level set convex provided $E$ is a convex body, that is, a compact convex set with non-empty interior. Moreover, in [1] (and later in [9] in a more general setting and in [8] for the logarithmic capacity in 2 dimensions) it
has been proved that the 2-capacity satisfies a suitable version of the Brunn-Minkowski inequality: given two convex bodies $K_0$ and $K_1$ in $\mathbb{R}^N$, for any $\lambda \in [0,1]$ it holds
\[
C_2(\lambda K_1 + (1-\lambda)K_0)^{\frac{1}{N-2}} \geq \lambda C_2(K_1)^{\frac{1}{N-2}} + (1-\lambda)C_2(K_0)^{\frac{1}{N-2}}.
\]
We refer to [20, 14] for a comprehensive survey on the Brunn-Minkowski inequality.

The main purpose of this paper is to show the analogous of these results in the fractional setting $\alpha = 1$, that is, $s = 1/2$ in Problem (1). More precisely, we shall prove the following result.

**Theorem 1.1.** Let $K \subset \mathbb{R}^N$ be a convex body and let $u$ be the solution of Problem (1) with $s = 1/2$. Then

(i) $u$ is level set convex, that is, for every $c \in \mathbb{R}$ the set $\{u > c\}$ is convex;

(ii) the 1-Riesz capacity $\text{Cap}_1(K)$ satisfies the following Brunn-Minkowski inequality: for any couple of convex bodies $K_0$ and $K_1$ and for any $\lambda \in [0,1]$ we have
\[
\text{Cap}_1(\lambda K_1 + (1-\lambda)K_0)^{\frac{1}{N-1}} \geq \lambda\text{Cap}_1(K_1)^{\frac{1}{N-1}} + (1-\lambda)\text{Cap}_1(K_0)^{\frac{1}{N-1}}.
\]

The proof of the Theorem 1.1 will be given in Section 2, and relies on the results in [11, 9] and on the following observation due to L. Caffarelli and L. Silvestre.

**Proposition 1.2 ([7]).** Let $f : \mathbb{R}^N \to \mathbb{R}$ be a measurable function and let $U : \mathbb{R}^N \times [0,+\infty)$ be the solution of
\[
\Delta (x,t)U(x,t) = 0, \quad \text{on } \mathbb{R}^N \times (0, +\infty) \quad U(x,0) = f(x).
\]
Then, for any $x \in \mathbb{R}^N$ there holds
\[
\lim_{t \to 0^+} \partial_t U(x,t) = (-\Delta)^{\frac{1}{2}} f(x).
\]

Eventually, in Section 3 we provide an application of Theorem 1.1 and we state some open problems.

## 2 Proof of the main result

This section is devoted to the proof of Theorem 1.1.

**Lemma 2.1.** Let $K$ be a compact convex set with positive 2-capacity and let $(K_\varepsilon)_{\varepsilon > 0}$ be a family of compact convex sets with positive 2-capacity such that $K_\varepsilon \to K$ in the Hausdorff distance, as $\varepsilon \to 0$. Letting $u_\varepsilon$ and $u$ be the capacitary functions of $K_\varepsilon$ and $K$ respectively, we have that $u_\varepsilon$ converges uniformly on $\mathbb{R}^N$ to $u$ as $\varepsilon \to 0$. As a consequence, we have that the sequence $C_2(K_\varepsilon)$ converges to $C_2(K)$, and that the sets $\{u_\varepsilon > s\}$ converge to $\{u > s\}$ for any $s > 0$, with respect to the Hausdorff distance.
Proof. We only prove that \( u_\varepsilon \to u \) uniformly as \( \varepsilon \to 0 \) since this immediately implies the other claims. Let \( \Omega_\varepsilon = K \cup K_\varepsilon \). Since \( u_\varepsilon - u \) is a harmonic function on \( \mathbb{R}^N \setminus \Omega_\varepsilon \), we have that

\[
\sup_{\mathbb{R}^N \setminus \Omega_\varepsilon} |u_\varepsilon - u| \leq \sup_{\partial \Omega_\varepsilon} |u_\varepsilon - u| \leq \max \left\{ 1 - \min_{\partial \Omega_\varepsilon} u, 1 - \min_{\partial \Omega_\varepsilon} u_\varepsilon \right\}.
\]

(8)

Moreover, by Hausdorff convergence, we know that there exists a sequence \( (r_\varepsilon)_\varepsilon \) infinitesimal as \( \varepsilon \to 0 \) such that \( K_\varepsilon \subset K + B_{r_\varepsilon} \), where \( B(r) \) indicates the ball of radius \( r \) centred at the origin. Thus

\[
\min \left\{ \min_{\partial \Omega_\varepsilon} u, \min_{\partial \Omega_\varepsilon} u_\varepsilon \right\} \geq \min \left\{ \min_{K + B(2r_\varepsilon)} u, \min_{K_\varepsilon + B(2r_\varepsilon)} u_\varepsilon \right\}.
\]

(9)

Since the right-hand side of (9) converges to 1 as \( \varepsilon \to 0 \), from (8) we obtain

\[
\lim_{\varepsilon \to 0} \sup_{\mathbb{R}^N \setminus \Omega_\varepsilon} |u_\varepsilon - u| = 0,
\]

which brings to the conclusion.

Remark 2.2. Notice that a compact convex set has positive 2-capacity if and only if its \( \mathcal{H}^{N-1} \)-measure is non-zero (see [13]).

Proof of Theorem 1.1. We start by proving claim (i). Let us consider the problem

\[
\begin{cases}
-\Delta(x,t)U(x,t) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\
U(x,0) = 1 & x \in K \\
U_t(x,0) = 0 & x \in \mathbb{R}^N \setminus K \\
\lim_{|x(t)| \to \infty} U(x,t) = 0
\end{cases}
\]

(10)

By Proposition 1.2 we have that \( U(x,0) = u(x) \) for every \( x \in \mathbb{R}^N \). Notice also that, for any \( c \in \mathbb{R} \), we have

\[
\{ u \geq c \} = \{ (x,t) : U(x,t) \geq c \} \cap \{ t = 0 \}
\]

which entails that \( u \) is level set convex, provided that \( U \) is level set convex. In order to prove this, we introduce the problem

\[
\begin{cases}
\Delta(x,t)V(x,t) = 0 & \text{in } \mathbb{R}^{N+1} \setminus K \\
V = 1 & x \in K \\
\lim_{|x(t)| \to \infty} V(x,t) = 0
\end{cases}
\]

(11)

whose solution is given by the capacitary function of the set \( K \) in \( \mathbb{R}^{N+1} \), that is, the function which achieves the minimum in Problem (5).
Since $K$ is symmetric with respect to the hyperplane $\{t = 0\}$ (where it is contained), it follows, for instance by applying a suitable version of the Pólya-Szegő inequality for the Steiner symmetrization (see for instance [2, 4]), that $V$ is symmetric as well with respect to the same hyperplane. In particular we have that $\partial_t V(x, 0) = 0$ for all $x \in \mathbb{R}^N \setminus K$. This implies that $V(x, t) = U(x, t)$ for every $t \geq 0$. To conclude the proof, we are left to check that $V$ is level set convex. To prove this we recall that the capacitary function of a convex body is level set convex, as proved in [9]. Moreover, by Lemma 2.1 applied to the sequence of convex bodies $K_\varepsilon = K + B(\varepsilon)$ we get that $V$ is level set convex as well. This concludes the proof of $(i)$.

To prove $(ii)$ we start by noticing that the 1-Riesz capacity is a $(1 - N)$-homogeneous functional, hence inequality (7) can be equivalently stated (see for instance [1]) by requiring that, for any couple of convex sets $K_0$ and $K_1$ and for any $\lambda \in [0, 1]$, the inequality

$$\text{Cap}_1(\lambda K_1 + (1 - \lambda)K_0) \geq \min\{\text{Cap}_1(K_0), \text{Cap}_1(K_1)\} \quad (12)$$

holds true.

We divide the proof of (12) into two steps.

**Step 1.**
We characterize the 1-Riesz capacity of a convex set $K$ as the behaviour at infinity of the solution of the following PDE

$$\begin{cases} 
(-\Delta)^{1/2} v_K = 0 & \text{in } \mathbb{R}^N \setminus K \\
v_K = 1 & \text{in } K \\
\lim_{|x| \to \infty} |x|^{N-1} v_K(x) = \text{Cap}_1(K)
\end{cases}$$

We recall that, if $\mu_K$ is the optimal measure for the minimum problem in (2), then the function

$$v(x) = \int_{\mathbb{R}^N} \frac{d\mu_K(y)}{|x-y|^{N-1}}$$

is harmonic on $\mathbb{R}^N \setminus K$ and is constantly equal to $I_1(K)$ on $K$ (see for instance [15]). Moreover the optimal measure $\mu_K$ is supported on $K$, so that $|x|^{N-1} v(x) \to \mu_K(K) = 1$ as $|x| \to \infty$. The claim follows by letting $v_K = v/\text{I}_1(K)$.

**Step 2.**
Let $K_\lambda = \lambda K_1 + (1 - \lambda)K_0$ and $v_\lambda = v_{K_\lambda}$. We want to prove that

$$v_\lambda(x) \geq \min\{v_0(x), v_1(x)\}$$

for any $x \in \mathbb{R}^N$. To this aim we introduce the auxiliary function

$$\tilde{v}_\lambda(x) = \sup \{ \min\{v_0(x_0), v_1(x_1)\} : x = \lambda x_1 + (1 - \lambda)x_0 \},$$

5
and we notice that Step 2 follows if we show that \( v_\lambda \geq \tilde{v}_\lambda \). An equivalent formulation of this statement is to require that for any \( s > 0 \) we have
\[
\{ \tilde{v}_\lambda > s \} \subseteq \{ v_\lambda > s \}. \tag{13}
\]
A direct consequence of the definition of \( \tilde{v}_\lambda \) is that
\[
\{ \tilde{v}_\lambda > s \} = \lambda \{ v_1 > s \} + (1 - \lambda) \{ v_0 > s \}.
\]
For all \( \lambda \in [0, 1] \), we let \( V_\lambda \) be the harmonic extension of \( v_\lambda \) on \( \mathbb{R}^N \times [0, \infty) \), which solves
\[
\begin{cases}
-\Delta_{(x,t)} V_\lambda(x,t) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\
V_\lambda(x,0) = v_\lambda(x) & \text{in } \mathbb{R}^N \times \{0\} \\
\lim_{(x,t) \to \infty} V_\lambda(x,t) = 0.
\end{cases} \tag{14}
\]
Notice that \( V_\lambda \) is the capacitary function of \( K_\lambda \) in \( \mathbb{R}^{N+1} \), restricted to \( \mathbb{R}^N \times [0, +\infty) \).

Letting \( \tilde{V}_\lambda(x,t) = \sup \{ \min \{ V_0(x_0,t_0), V_1(x_1,t_1) \} : (x,t) = \lambda(x_1,t_1) + (1 - \lambda)(x_0,t_0) \} \), as above we have that
\[
\{ \tilde{v}_\lambda > s \} = \lambda \{ v_1 > s \} + (1 - \lambda) \{ v_0 > s \}.
\]
By applying again Lemma 2.1 to the sequences \( K_0^\varepsilon = K_0 + B(\varepsilon) \) and \( K_1^\varepsilon = K_1 + B(\varepsilon) \), we get that the corresponding capacitary functions, denoted respectively as \( V_0^\varepsilon \) and \( V_1^\varepsilon \), converge uniformly to \( V_0 \) and \( V_1 \) in \( \mathbb{R}^N \), and that \( \tilde{V}_\lambda^\varepsilon \), defined as in (15), converges uniformly to \( \tilde{V}_\lambda \) on \( \mathbb{R}^N \times [0, +\infty) \).

Since \( \tilde{V}_\lambda^\varepsilon(x,t) \leq V_\lambda^\varepsilon(x,t) \) for any \( (x,t) \in \mathbb{R}^N \times [0, +\infty) \), as shown in [9, pages 474 - 476], we have that \( \tilde{V}_\lambda(x,t) \leq V_\lambda(x,t) \). As a consequence, we get
\[
\{ v_\lambda > s \} = \{ V_\lambda > s \} \cap H \supseteq \{ \tilde{V}_\lambda > s \} \cap H = \left[ \lambda \{ V_1 > s \} + (1 - \lambda) \{ V_0 > s \} \right] \cap H \supseteq \lambda \{ V_1 > s \} \cap H + (1 - \lambda) \{ V_0 > s \} \cap H = \lambda \{ v_1 > s \} + (1 - \lambda) \{ v_0 > s \}
\]
for any \( s > 0 \), which is the claim of Step 2.

We conclude by observing that inequality (12) follows immediately, by putting together Step 1 and Step 2. This concludes the proof of (ii), and of the theorem.

**Remark 2.3.** The equality case in the Brunn-Minkowski inequality (7) is not easy to address by means of our techniques. The problem is not immediate even in the case of the 2-capacity, for which it has been studied in [6, 9].
3 Applications and open problems

In this section we state a corollary of Theorem 1.1. To do this we introduce some tools which arise in the study of convex bodies. The support function of a convex body \( K \subset \mathbb{R}^N \) is defined on the unit sphere centred at the origin \( \partial B(1) \) as
\[
h_K(\nu) = \sup_{x \in \partial K} \langle x, \nu \rangle.
\]
The mean width of a convex body \( K \) is
\[
M(K) = \frac{2}{\mathcal{H}^{N-1}(\partial B(1))} \int_{\partial B(1)} h_K(\nu) d\mathcal{H}^{N-1}(\nu).
\]
We refer to [20] for a complete reference on the subject. We observe that, if \( N = 2 \), then \( M(K) \) coincides up to a constant with the perimeter \( P(K) \) of \( K \) (see [3]).

We denote by \( K_N \) the set of convex bodies of \( \mathbb{R}^N \) and we set
\[
K_{N,c} = \{ K \in K_N, M(K) = c \}.
\]
The following result has been proved in [3].

**Theorem 3.1.** Let \( F : K_N \to [0, \infty) \) be a q-homogeneous functional which satisfies the Brunn-Minkowski inequality, that is, such that \( F(K + L)^{1/q} \geq F(K)^{1/q} + F(L)^{1/q} \) for any \( K, L \in K_N \). Then the ball is the unique solution of the problem
\[
\min_{K \in K_N} \frac{M(K)}{F^{1/q}(K)}.
\]
(16)

An immediate consequence of Theorem 3.1, Theorem 1.1 and Definition 4 is the following result.

**Corollary 3.2.** The minimum of \( I_1 \) on the set \( K_{N,c} \) is achieved by the ball of measure \( c \). In particular, if \( N = 2 \), the ball of radius \( r \) solves the isoperimetric type problem
\[
\min_{K \in K_2, P(K) = 2\pi r} I_1(K).
\]
(17)

Motivated by Theorem 1.1 and Corollary 3.2 we conclude the paper with the following conjectures:

**Conjecture 3.3.** For any \( N \geq 2 \) and \( \alpha \in (0, N) \), the \( \alpha \)-Riesz capacity \( \text{Cap}_\alpha(K) \) satisfies the following Brunn-Minkowski inequality:
for any couple of convex bodies \( K_0 \) and \( K_1 \) and for any \( \lambda \in [0, 1] \) we have
\[
\text{Cap}_\alpha(\lambda K_1 + (1 - \lambda) K_0) \gtrless \lambda \text{Cap}_\alpha(K_1)^{\frac{\alpha}{N}} + (1 - \lambda) \text{Cap}_\alpha(K_0)^{\frac{\alpha}{N}}.
\]
(18)

**Conjecture 3.4.** For any \( N \geq 2 \) and \( \alpha \in (0, N) \), the ball of radius \( r \) is the unique solution of the problem
\[
\min_{K \in K_N, P(K) = N\omega_N r^{N-1}} I_\alpha(K).
\]
(19)
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