Selfdual 2-form formulation of gravity and classification of energy-momentum tensors

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Abstract

It is shown how the different irreducibility classes of the energy-momentum tensor allow for a Lagrangian formulation of the gravity-matter system using a selfdual 2-form as a basic variable. It is pointed out what kind of difficulties arise when attempting to construct a pure spin-connection formulation of the gravity-matter system. Ambiguities in the formulation especially concerning the need for constraints are clarified.

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1 Introduction

Recently there has been interest in formulations of gravity using self-dual two forms as basic variable [1]. This approach allows for an almost complete elimination of the metric in the action, leaving the self-dual part of the spin connection and a scalar density as the only gravitational quantities in the theory. This formulation provides a natural covariantization of Ashtekar’s canonical formalism [2].

We briefly review the procedure for vacuum general relativity. The action as function of the basic two-form $\Sigma^{AB}$, the Weyl part of the curvature $\Psi_{ABCD} = \Psi_{(ABCD)}$, and the spin-connection one-form $\omega_{AB}$ with $R_{AB} = d\omega_{AB} + \omega_{AC} \wedge \omega^C_B$ is given as follows:

$$S[\Sigma^{AB}, \omega_{AB}, \Psi_{ABCD}] = \int \Sigma^{AB} \wedge R_{AB} - \frac{1}{2} \Psi_{ABCD} \Sigma^{AB} \wedge \Sigma^{CD}$$

(1)

Note that summation convention is used in this paper. Capital latin letters range over 0,1. They are raised and lowered according to the rules in [4]. Symmetrization and anti-symmetrization in indices are denoted by $(AB)$ and by $[AB]$ respectively. The indices coming from the second fundamental representation of $SL(2, \mathbb{C})$ are denoted by $\dot{A}, \dot{B}$ etc. However, in the present formulation care has to be taken of the reality conditions [1].

The variation of the action with respect to $\Psi_{ABCD}, \omega_{AB}$, and $\Sigma^{AB}$ yields the equations of motion:

$$\Sigma^{(AB} \wedge \Sigma^{CD)} = 0, \quad D\Sigma^{AB} = 0, \quad R_{AB} = \Psi_{ABCD} \Sigma^{CD}.$$  

(2)

$D$ denotes the covariant derivative with respect to $\omega_{AB}$. The first equation shows that the self dual two-form can be expressed in terms of basic tetrads, i.e. $\Sigma^{AB} = \theta^A_A \wedge \theta^B_A$. The fact that the covariant derivative applied to $\Sigma^{AB}$ vanishes shows that $\omega_{AB}$ is the self-dual part of the spin-connection. The last equation in (2) is just the Einstein equation in the vacuum.

The metric independent formulation can now be obtained by first solving (2) for $\Sigma^{AB}$ and then eliminating $\Psi_{ABCD}$ under the condition that the it has a vanishing trace. The latter is only possible if $\Psi_{ABCD}$ is invertible. Thus the present formulation does not allow vacuum spacetimes of Petrov type $\{31\}, \{4\}$, and $\{-\}$. Hence it is not clear if this procedure is reasonable when attempting to quantize the theory since under certain circumstances the Petrov type might change, see e.g. [3].

The final form of the action [1] then involves only $\omega_{AB}$ and a scalar density $\eta$ of weight $-1$.

$$S[\omega_{AB}, \eta] = \int \eta (\epsilon R^{AC} \wedge R^{BD})(\epsilon R_{AB} \wedge R_{CD})$$

(3)

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In this paper a systematic treatment of matter couplings to gravity using the self-dual 2-form $\Sigma^{AB}$ as basic variable will be presented. It will be shown how the different irreducibility classes of the energy-momentum tensor of matter enter in a Lagrangian formulation of the problem. Therefore this treatment is general since it does not rely on some specific examples of matter couplings.

The outline of the paper is as follows. In section 2 we recall some basic facts about Einstein theory in spinorial form and point out the problems and objectives in a Lagrangian formulation using $\Sigma^{AB}$ as basic variable. The different irreducibility classes of the energy-momentum tensor are presented in section 3. Since we want to treat $\Sigma^{AB}$ as a form the matter degrees of freedom have to be embedded in differential forms. This embedding introduces undesired degrees of freedom which have to be projected out. The constraints causing this projection are calculated in section 4. In section 5 the the matter Lagrangians are formulated. Finally a few remarks on the physically important cases of scalar- and spinorial- matter actions are made in section 6.

2 Matter Couplings

In this section facts about the spinorial formulation [4] of Einstein theory are recalled since they will be needed in the remainder of the paper.

The curvature 2-form $R_{AB}$ which is the (anti-) self-dual part of the Riemann curvature can be regarded as a Lorentz or $SL(2, \mathbb{C})$ tensor respectively. Denoting curved space-time indices by greek letters we can give the decomposition of this tensor into $SL(2, \mathbb{C})$ irreducible parts. Denoting by $\varepsilon$ the $SL(2, \mathbb{C})$ invariant skew symbol we get (cf. [4]):

$$R_{AB \gamma \delta} = R_{AB \, CC \, DD} = \Psi_{(ABCD)}\varepsilon_{CD} + \Phi_{(AB)(CD)}\varepsilon_{CD} + \Lambda(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC})\varepsilon_{CD} \quad (4)$$

Here $\Psi_{(ABCD)}$ corresponds as mentioned above to the Weyl curvature, $\Phi_{(AB)(CD)}$ is the traceless Ricci tensor, and $\Lambda$ contains the scalar parts of the Riemann tensor. Making use of the decomposition of the product of the tetrad 1-forms cf. [3]

$$\theta^{A\hat{A}} \wedge \theta^{B\hat{B}} = \epsilon^{AB}\hat{\Sigma}^{A\hat{B}} + \epsilon^{A\hat{B}}\Sigma^{AB}, \quad (5)$$

we get:
\[ R_{AB} = \Psi_{(ABCD)} \Sigma^{CD} + \Phi_{(AB)(CD)} \hat{\Sigma}^{CD} + 2\Lambda \Sigma_{AB}. \] 

For later purposes we rewrite the Einstein equations in the $SL(2, \mathbb{C})$ version. The matter part entering these equations is described by the energy-momentum tensor $T_{\alpha\beta} = T_{\beta\alpha}$. Due to its symmetries this tensor admits the following decomposition:

\[ T_{\alpha\beta} = T_{(AB)(\dot{A}\dot{B})} + \epsilon_{AB}\epsilon_{\dot{A}\dot{B}} \frac{1}{4} T^a_a \]

Obviously the first summand is the traceless part whereas the second one corresponds to the scalar part of $T_{\alpha\beta}$.

The Einstein equations with cosmological constant contribution $\lambda$ and gravitational coupling $\gamma$ are $R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \lambda g_{\alpha\beta} = -8\pi \gamma T_{\alpha\beta}$. Completely equivalent equations are obtained using the $SL(2, \mathbb{C})$ decomposition by a straightforward calculation:

\[ \Phi_{(AB)(CD)} = 4\pi \gamma T_{(AB)(\dot{A}\dot{B})}, \quad \Lambda = \frac{1}{6} \lambda + \frac{1}{3} \pi \gamma T^a_a \]

To be complete we mention that the Bianchi identities establish differential relations among the irreducible parts of the Riemann tensor. Denoting by $\nabla_{\hat{A}\hat{A}}$ the covariant derivative these relations are simply $\nabla_{\hat{A}\hat{B}} \Psi_{(ABCD)} = \nabla_{\dot{A}\dot{B}} \Phi_{(CD)(\dot{A}\dot{B})}$ and $\nabla_{\hat{A}\dot{A}} \Phi_{(AB)(\dot{A}\dot{B})} + 3\nabla_{\dot{B}\dot{B}} \Lambda = 0$.

The problem to be considered in this paper is how to couple matter to gravity in an action formulation which uses the 2-form $\Sigma_{AB}$ as a basic variable not only in the gravity part (as outlined in the introduction) but also in the matter part.

We will consider actions of the form $S_{total} = S_{gravity} + S_{matter}$. In the remainder of the paper we do not mention the exterior product explicitly. Up to constants this action takes the form with $\phi$ denoting matter degrees of freedom:

\[ S_{total} = \int \Sigma^{AB} R_{AB} - \frac{1}{2} \Psi_{(ABCD)} \Sigma^{AB} \Sigma^{CD} - \frac{1}{6} \lambda \Sigma^{AB} \Sigma_{AB} + S_{matter}[\Sigma^{AB}, \phi] \] 

$S_{matter}$ has to be formulated in such a way that the variation by $\Sigma^{AB}$ and maybe having used the matter equations of motion the Einstein equations are obtained in the form \(8\) after expanding $R_{AB}$ \(8\). For the case of the Yang-Mills-gravity system this has been done in \(8\). A detailed treatment can be found in section 5.

In general if a cosmological constant is present or matter is coupled to gravity the self-dual formulation of section 1 and the elimination of the metric degrees of freedom become difficult \[1\], see also \[6,8\].
The origin of the problem is due to the need for a constraint that ensures the tracelessness of the Weyl tensor in the presence of a cosmological constant or of matter fields. We review briefly the first case (see the last ref. in [1] and [7]) while the problems in the matter coupled case are addressed at the end of section 5. For the moment we adopt a kind of index free notation.

One can rewrite the action (9) without matter contributions using \(X := \Psi + \lambda/3\) and a Lagrange multiplier \(\mu\):

\[
S = \int \Sigma R - \frac{1}{2} X \Sigma \Sigma + \mu (\text{tr}X - \lambda)
\]  

(10)

The last term is needed to ensure the tracelessness of the Weyl tensor.

Applying an identity [1] which holds for any 3x3-matrix to \(X\) and solving the \(\Sigma\) equations of motion we arrive at an action of the form:

\[
S = \int \frac{1}{2} \text{tr}(X^{-1}M) + \rho ((\text{tr}X^{-1})^2 - (\text{tr}X^{-2}) - 2(\text{det}X)^{-1} \lambda)
\]

(11)

We define \(M_{ABCD} := R_{AB} \wedge R_{CD}\) and \(\rho = \mu \text{det}X/2\). To obtain a pure spin-connection formulation one has to eliminate \(X\) which can be done by considering the variation of (14) by \(X^{-1}\). This yields:

\[
M = 4\rho (X^{-1} - \text{tr}X^{-1} 1 + \lambda(\text{det}X)^{-1} X)
\]

(12)

It has been pointed out in [1] that (12) is difficult to solve for \(X^{-1} = X^{-1}(M, \rho)\) but in [7] and in the last ref. of [1] a partially satisfactory solution to this problem is proposed. One assumes the existence of a solution \(Y(M, \rho) = X^{-1}(M, \rho)\). Inserting (12) with \(Y\) into (14) one can show that the resulting action is a functional of \(\text{tr}M, \text{tr}(M^2), \text{det}M,\) and \(\rho\). These constituents can be related to \(Y\). In the course one has to solve a quadratic equation which leads to the following actions, \(\chi := \lambda/(8\rho)\):

\[
S = \frac{1}{2\lambda} \int \chi^{-1}((1 + \chi \text{tr}M) \pm ((1 + \chi \text{tr}M)^2 - 2\chi^2(\text{tr}M^2 - \frac{1}{2}(\text{tr}M)^2) + 8\chi^3(\text{det}M))^\frac{1}{2})
\]

(13)

However, the so obtained action is not unique.

For certain types of matter couplings and in the presence of a cosmological constant the action (11) has been modified in [1] to account for the additional physical information:

\[
\int \Sigma^{AB} \wedge \Gamma_{AB} - \frac{1}{2} \Xi_{ABCD} \Sigma^{AB} \Sigma^{CD} + \frac{1}{2} (\Xi_{AB} \wedge \Delta) \Sigma_{CD} \Sigma^{CD}
\]

(14)
In this expression $\Gamma_{AB} = R_{AB} + M_{AB}$ with $M_{AB}$ describing one part of the matter action. $\Xi_{ABCD}$ contains the Weyl part of the curvature and another part of the matter action. This object may not be traceless and therefore causes problems (see Erratum in [1]). $\Delta$ denotes additional trace parts coming from the matter action or the cosmological constant. In section 5 we will present a general framework for an action comparable to (9) and (14) corresponding to the irreducibility classes of the energy-momentum tensor of matter.

### 3 Classification of the energy-momentum tensor

The coupling of matter to gravity leads to Einstein equations with a nonvanishing energy-momentum tensor. It is possible to give an algebraic classification [4, 9] of both the energy-momentum tensor and the Ricci tensor. The latter contains direct information of the matter part by the field equations (8).

We consider the traceless part of the hermitian energy-momentum tensor: $T_{ABCD} = T_{(AB)(CD)}$. The first step in the classification is the reducibility of this object.

| Case | Real dimension |
|------|----------------|
| A    | (2, 2)         | $T_{ABC\dot{D}}$ irreducible 9 |
| B1   | (1, 1)(1, 1)   | $T_{ABC\dot{D}} = \Gamma_{(A\Lambda B)}^{(C\dot{D})}$ 7 |
| B2   | (1, 1)         | $T_{ABC\dot{D}} = \Gamma_{(A\Lambda B)}^{(C\dot{D})}$ 7 |
| C    | (1, 1)         | $T_{ABC\dot{D}} = \pm \Lambda_{(A\Lambda B)}^{(C\dot{D})}$ 4 |
| D    | (1, 1)         | $T_{ABC\dot{D}} = \rho_{(A\Lambda B)(C\dot{D})}$ 6 |
| E    | (1, 0)         | $T_{ABC\dot{D}} = \pm \rho_{(A\sigma B)(C\dot{D})}$ 5 |
| F    | (1, 0)         | $T_{ABC\dot{D}} = \pm \rho_{A\rho B\rho C\dot{D}}$ 3 |
| G    | (0)            | $T_{ABC\dot{D}} = 0$ 0 |

Obviously this table shows that some cases can be obtained by specification of more general ones. This means for instance that the classes B1 and B2 can be obtained from A; or class F follows either from E or from B2.

A further classification can be obtained by considering the eigenspaces of the tensor. For the purpose of this paper this refinement is not important. Details can be found in [4, 9].
For example the energy-momentum tensor of a massless real scalar field \( \phi \) is \( T_{\dot{A}\dot{B}\dot{C}\dot{D}} = (\nabla_{\dot{A}}\phi)(\nabla_{\dot{B}}\phi) \) and therefore in class C; in the massless spinor case we have \( T_{\dot{A}\dot{B}\dot{C}\dot{D}} = \psi_A \nabla_B \tilde{\psi}_D \) in class D.

The use of this classification to formulate matter actions in the sense of section 2 enables to discuss in full generality all possible matter sources for gravity and not only the ones treated in \([1]\).

4 Description of the constraints

Since our aim is to formulate matter actions using the 2-form \( \Sigma^{AB} \) as basic variable the entities of the Lagrangian (fields, derivatives, etc.) have to be written in the language of differential forms. The differential form may contain more physical degrees of freedom than one actually wants to describe. In this section we will present constraints that project out the undesired degrees of freedom. It is one of the ideas of this work to show how far one can come using only the tensor structure. Therefore the explicit dependence of the projection valued constraints on the physical fields needs not to be considered.

For example a 1-form valued spinor field \( \rho_A \) contains a spin 3/2 and a spin 1/2 part:

\[
\rho_A = \rho_{AM} M^{\dot{M}} = \left( \rho_{(AM)} + \epsilon_{AM} \tilde{\rho}_{\dot{M}} \right) \theta^{M\dot{M}}, \quad \rho_A \in (1/2) \oplus (1/2). \tag{15}
\]

We denote by \((i, j)\) in the usual way the finite-dimensional representations corresponding to \( SL(2, \mathbb{C}) \).

The undesired degrees of freedom have to be projected out by the use of constraints. The choice of the constraints should still allow a formulation of the action of the form \((14)\).

The sufficient cases for the present purposes are discussed. In what follows the object \( \tau \) denotes the Lagrange multiplier.

A 2-form \( \rho^A \) contains components in the representations \((3/2, 0)\), \((1/2, 0)\), and \((1/2, 1)\). The following constraint can be considered:

\[
\tau_{(AB)C} \Sigma^{AB} \rho^C; \quad \tau_{(AB)C} : 0 - \text{form}. \tag{16}
\]

By direct inspection and using \([3]\) the orthogonality relation \( \Sigma^{AB} \wedge \Sigma^{CD} = 0 \) it follows that the index symmetries of \( \tau_{(AB)C} \) decide which components can be projected out.

\[
\begin{align*}
\tau_{(AB)C} &= \tau_{(ABC)} & \text{eliminates} & (3/2, 0) \\
\tau_{(AB)C} &= \tau_{([AB]C)} & \text{eliminates} & (1/2, 0)
\end{align*} \tag{17}
\]
There is no constraint of the form (16) which can eliminate the \((1, 1)\) component. This result is in contrast to the supergravity case considered in [1]. There the first constraint of (17) has been taken. This leaves besides of the desired Rarita-Schwinger field a Weyl-spinor component in the action.

If \(\rho^A\) is taken as in (15) a good constraint is again of the form \(\tau_{(AB)C} \Sigma^{AB} \rho^C\). Here \(\tau_{(AB)C}\) has to be a 1-form. Like in (17) one gets:

\[
\begin{align*}
\tau_{(AB)C} &= \tau_{(ABC)} \quad \text{eliminates} \quad (1, 1/2) \\
\tau_{(AB)C} &= \tau_{[(AB)C]} \quad \text{eliminates} \quad (0, 1/2)
\end{align*}
\]  

(18)

In this case the advantage is that each component of \(\rho^A\) can be projected out.

Given a 2-form \(\rho^{(AB)} \in (2, 0) \oplus (1, 1) \oplus (1, 0) \oplus (0, 0)\) the constraint \(\tau \Sigma_{AB} \rho^{AB}\) eliminates uniquely the \((0, 0)\) component whereas \(\tau_{(AB)(CD)} \Sigma^{AB} \rho^{CD}\) projects out exactly one of the components \((2, 0)\), \((1, 0)\), or \((0, 0)\) obviously depending on the symmetries of \(\tau\) among the pairs \((AB)\) and \((CD)\).

The last case important for this paper is a 1-form \(\rho^{AB} \in (3/2, 1/2) \oplus (1/2, 1/2)\). Here the condition \(\tau \Sigma_{AB} \rho^{AB}\) allows for getting rid of the \((1/2, 1/2)\) part. Another possible constraint is \(\tau_{(AB)(CD)} \Sigma^{AB} \rho^{CD}\) with \(\tau\) being a 1-form. Here we get:

\[
\begin{align*}
\tau_{(AB)(CD)} &= \tau_{(ABCD)} \quad \text{eliminates} \quad (3/2, 1/2) \\
\tau_{(AB)(CD)} &\neq \tau_{(ABCD)} \quad \text{eliminates} \quad (1/2, 1/2)
\end{align*}
\]  

(19)

5 Matter action in the self-dual 2-form formulation

Now the question of how to construct matter actions in the scheme of section 2 is addressed. Notice that in this paper only trace-free energy-momentum tensors are being considered. Thus mass terms etc. are not treated but this extension is straightforward, see [1, 7].

We consider as an ansatz an action of the form:

\[
S_{\text{ansatz}} = \int \Sigma^{AB} \Omega_{AB}
\]

(20)

\(\Omega_{AB}\) should contain the matter degrees of freedom. However, its explicit dependence on physical fields is not needed for the discussion. Due to its tensor structure \(\Omega_{AB}\) contains
the SL(2, C) irreducible components (2,0), (1,0), (0,0), and (1,1). Only the last term can be related to the traceless part of the energy-momentum tensor.

The reducible energy-momentum tensors of section 3 can be obtained if we have:

\[ \Omega_{AB} = \rho_A \sigma_B, \quad \rho_A F \sigma_B, \quad \Gamma_{AB}, \quad L_{AC} K^C_B. \]  

(21)

Since \( \Omega_{AB} \) is a 2-form the differentials can be assigned in various ways to its constituents in (21). One first decomposes the forms into their irreducible parts according to (15). Then the result is expanded with respect to the fundamental 2-forms using (5). In general one then gets from (20) the expression:

\[ S_{\text{ansatz}} = \int \Sigma_{AB} \Omega_{(AB)(CD)} \Sigma_{CD} \]

\[ + \sum \left( (\Omega_{(ABCD)} + \epsilon_{AC} \Omega_{X(BC)} X) \Sigma^{AB} \Sigma^{CD} + \Omega_{AB} \Sigma_{RS} \Sigma^{RS} \right) \]

(22)

The sum refers to the fact that the constituents of \( \Omega \) may split into components which contribute to (22) separately. As above the same symbols are used for the forms itself and its irreducible components.

In general one has to be careful about the completely symmetric terms. If \( \Omega_{AB} \) can be split, the following subtlety in the (2,0) component might occur.

\[ \Omega_{(AB)(MN)} = X_{(MN)C} \epsilon^{CD} Y_{(DAB)} = a X_{(MNC)} \epsilon^{CD} Y_{(DAB)} + b X_M Y_{(NAB)} \]

(23)

where \( a X \) and \( b X \) denote the irreducible components of the constituent \( X \). The last term is completely symmetric and yields an additional term in (22) which cannot be seen directly by the symmetries of \( \Omega_{AB} \) itself.

Within the framework of section 2 the Einstein equations are recovered by variation of the action with respect to \( \Sigma^{AB} \). Therefore only the first term in (22) should contribute since it contains the energy-momentum tensor of matter and in the sense of (8) it is the correct source for gravity. Hence when really formulating an action for matter fields in the formalism of [1] the remaining terms in (22) have to be subtracted consistently:

\[ S_{\text{matter}} = \int \Sigma^{AB} \Omega_{AB} \]

\[ - \frac{1}{2} \sum \left( (\Omega_{(ABCD)} + \epsilon_{AC} \Omega_{X(BC)} X) \Sigma^{AB} \Sigma^{CD} + \Omega_{AB} \Sigma_{RS} \Sigma^{RS} \right) \]

(24)

This form of the matter action allows for implementing the different irreducibility classes of the energy-momentum tensor.

After having established the equations for \( \Sigma^{AB} \) in the gravity-matter system the (1,0) component of \( \Omega_{AB} \) drops out and (24) takes a form comparable to (14) and (3).
Now it can be shown how the different irreducibility classes of the energy-momentum tensor can be brought in an action of the form (24).

There are two possibilities for the classes B1, B2, and C which may e.g. describe the massless scalar field. The first is given by $\Omega = F \Gamma^A_B$ with both $F$ and $\Gamma^A_B$ being 1-forms. According to (19) the action is given by:

$$S = \int \Sigma^{AB} F \Gamma^A_B + \frac{1}{4} F^C_C \Sigma^{AB} \Sigma^{AB} + \tau_{(ABCD)} \Sigma^{AB} \Gamma_{CD}$$

The second possibility is provided by the decomposition into two 1-forms: $\Omega_{AB} = L^{AC} K^C_B$. In this case two constraints of kind (19) are required. These may coincide depending on the actual class.

$$S = \int \Sigma^{AB} L_{AC} K^C_B - \frac{1}{4} L^{AC} K^C_C \Sigma_{AB} \Sigma^{AB} + \Sigma^{AB} \left( \tau_{(ABCD)} L^{CD} + \nu_{(ABCD)} K^{CD} \right)$$

Since the actions for the classes B1, B2, and C do contain trace parts a pure connection formulation [1] for the gravity-matter system becomes difficult even if one does not include explicitly traces of the energy-momentum tensor (see below).

Next the class D is considered. The energy-momentum tensor of a massless spin 1/2 particle is within this class. However, the occurrence of torsion will not be treated. This class can be described by $\Omega_{AB} = \rho_A \sigma_B$ with either $\rho_A$ a 2-form and $\sigma_B$ a 0-form or both being 1-forms. In both cases two constraints for projecting onto the desired physical degrees of freedom are necessary. The result for the first possibility is:

$$S = \int \Sigma^{AB} \rho_A \sigma_B + \Sigma^{AB} \left( \tau_{(ABC)} \rho^C + \nu_{(AB)} \sigma^C \right)$$

Notice that no additional subtractions have to be performed. One gets a similar result for the above mentioned second possibility.

A third description for the class D similar to the one in [1] is possible. One takes $\Omega_{AB} = \rho_A F \sigma_B$. Here $\rho$ and $F$ are 1-forms. The resulting action is:

$$S = \int \Sigma^{AB} \rho_A F \sigma_B + \frac{1}{4} \rho_C F^D \sigma^D \Sigma^{AB} \Sigma_{AB} + \tau_{(ABC)} \Sigma^{AB} \rho^C$$

It is interesting to note that while (27) permits straightforwardly a pure connection formulation the action (28) causes problems.
For the classes E and F the description has already been given in [1]. The Yang-Mills field is within these classes. One has to choose $\Omega_{AB} = \rho_A \sigma_B F$. $F$ has to be a 2-form. According to the decomposition given in (22) one gets $\Omega_{AB} = \rho_A \sigma_B (F_{(CD)} \Sigma^{CD} + \tilde{F}_{(CD)} \tilde{\Sigma}^{CD})$. The two components of $F$ correspond to its $(1, 0)$ and $(0, 1)$ parts respectively.

Since only the second term will give a correct contribution to the energy-momentum tensor the action in this case is

$$S = \int \Sigma^{AB} \rho_A \sigma_B F - \frac{1}{2} \rho_A \sigma_B F_{(CD)} \Sigma^{AB} \Sigma^{CD}$$  \hspace{1cm} (29)$$

The hermiticity requirement for the energy momentum tensor relates $F_{(CD)}$ to $\rho_A \sigma_B$. A recent treatment of the Einstein-Yang-Mills system [6] is equivalent to the one given here or in [1]. However, it can be read off (29) that after establishing the equations of motion for $\Sigma^{AB}$ the second terms becomes a trace. Therefore a pure connection formulation for classes E and F is difficult.

Formally one could obtain for all the above mentioned matter actions a pure spin-connection formulation. This could be achieved by simply preforming the following replacements in (29):

$$R_{AB} \rightarrow R_{AB} + \text{matter terms linear in } \Sigma$$

$$X \rightarrow X + \text{matter terms quadratic in } \Sigma$$ \hspace{1cm} (30)$$

After having done these substitutions one could in principle perform the procedure at the end of section 2 to obtain the pure spin-connection formulation. But in doing this one faces in addition to the pathological points in that procedure the serious problem that the matter terms are of local character. It is therefore not guaranteed globally that the operations involving $X$ in section 2 are possible everywhere in space time!

An almost unique description for the reducible classes of the energy-momentum tensor has been obtained. The irreducible ones involve fields of at least spin 3/2. For physical reasons these higher spin systems coupled to gravity are problematic although formal actions can easily be formulated within the presented scheme.

Only a formulation for the spin 3/2 field is given as a final example. One combines a 2-form $\rho_C$ and a 0-form $\sigma_{(CAB)}$ to $\Omega_{AB} = \rho_C \sigma^C_{AB}$. The object $\sigma_{(CAB)}$ then consists of the contracted product of a derivative and one component of the spin 3/2 field.

In (17) it was shown that two constraints are needed for $\rho_A$ in order to describe a Rarita-Schwinger field. The action in contrast to [1] is:

$$S = \int \Sigma^{AB} \rho_C \sigma^C_{AB} + \Sigma^{AB} (\tau_{(ABC)} + v_{(AB)C}) \rho^C$$  \hspace{1cm} (31)$$
6 Remarks on scalar- and spinor-field actions

It is necessary to make a remark about the actions for a scalar field and a spin 1/2 field. This is because in comparing the spinor-field actions of [1] and [7] one might wonder why they differ by constraint terms. It will be argued that the use of actions involving constraints like in section 5 is not really necessary.

One can rewrite the standard action of a scalar field coupled to gravity using [10, 1]:

\[ \sqrt{|g|} g_{mn} = \frac{1}{3} \epsilon^{abcd} \Sigma^{AB} \Sigma_{bc}^{B} \Sigma_{d}^{C} \Sigma^{CA} \]  

(32)

Using (26) one can check that

\[ S = \int \Sigma^{AB} L_{BC} K^{CA} - \frac{1}{2} \Sigma^{RS} \Sigma^{RS} \tilde{L}_{AB} \tilde{K}^{AB} \]  

(33)

reproduces with the help of (29) the standard scalar-field action in curved space. In the sense of (15) we denote by \( \tilde{L}_{AB} \) and \( \tilde{K}^{AB} \) the vectorial (1/2,1/2) components of the 1-forms \( L_{BC} \) and \( K^{CA} \) respectively.

(26). The constraints needed in (26) drop out because the use of formula (32) restricts the 1-forms \( K \) and \( L \):

\[ K^{AB} = \Sigma^{AB} \partial_{m} \phi dx^{n} = \tilde{K}^{(A} \theta^{B)A} \]

\[ L^{AB} = \Sigma^{AB} \partial_{m} \psi dx^{n} = \tilde{L}^{(A} \theta^{B)A} \]  

(34)

\( \phi \) and \( \psi \) are scalar fields which may coincide, i.e. \( \tilde{K}_{A\bar{A}} = \nabla_{A\bar{A}} \phi \) and \( \tilde{L}_{BB} = \nabla_{BB} \psi \)

If one now applies the obvious restriction from class B1 to class D of the energy-momentum tensor in the action it is clear that also the action for the spin 1/2 field can be formulated without using specific constraints as it has been done in [7]. The trace-term plays no role in this case due to the equations of motion for the spinor field.

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