On the tautological ring of Humbert curves

Received: 17 November 2021 / Accepted: 1 November 2022 / Published online: 14 November 2022

Abstract. We exhibit a 2-dimensional family of non-hyperelliptic curves of genus 5, called Humbert curves, for which the tautological ring injects into cohomology. In particular, Humbert curves have a multiplicative Chow–Künneth decomposition (in the sense of Shen–Vial), and their Ceresa cycle is torsion.

1. Introduction

Given a smooth projective variety $Y$ over $\mathbb{C}$, let $A^i(Y) := CH^i(Y)_{\mathbb{Q}}$ denote the Chow groups of $Y$ (i.e. the groups of codimension $i$ algebraic cycles on $Y$ with $\mathbb{Q}$-coefficients, modulo rational equivalence [18,47]). The intersection product defines a ring structure on $A^*(Y) = \bigoplus_i A^i(Y)$, the Chow ring of $Y$.

Motivated by the particular behaviour of Chow rings of K3 surfaces [6] and of abelian varieties [2], Beauville [3] has famously conjectured that for certain special varieties, the Chow ring should admit a multiplicative splitting. To make concrete sense of Beauville’s elusive “splitting property” conjecture, Shen–Vial [41] introduced the concept of multiplicative Chow–Künneth decomposition (we will abbreviate to “MCK decomposition”). In short, this is a graded decomposition of the Chow motive of a smooth projective variety, such that the intersection product respects the grading (cf. Sect. 2.3 for details).

It is something of a challenge to understand the class of varieties admitting an MCK decomposition: abelian varieties, K3 surfaces and cubic hypersurfaces are in this class. As for curves, it is known that hyperelliptic curves are in this class, whereas not all curves are in this class.

The main result of the present paper is about Humbert curves; these are certain non-hyperelliptic curves of genus 5 that were studied as far back as 1894 (cf. [21] and Sect. 2.1 below).

Theorem. (=Theorems 3.1 and 3.2) Let $C$ be a Humbert curve, and $m \in \mathbb{N}$. Let

$$R^*(C^m) := \left\langle (p_{ij})^*(\Delta_C), (p_k)^*(K_C) \right\rangle \subset A^*(C^m)$$

Supported by ANR grant ANR-20-CE40-0023

R. Laterveer: Institut de Recherche Mathématique Avancée, CNRS – Université de Strasbourg, 7 Rue René Descartes, 67084 Strasbourg Cedex, France
robert.laterveer@math.unistra.fr

Mathematics Subject Classification: 14C15 · 14C25 · 14C30

https://doi.org/10.1007/s00229-022-01445-4
be the $\mathbb{Q}$-subalgebra generated by (pullbacks of) the diagonal $\Delta_C \subset C \times C$ and (pullbacks of) the canonical divisor $K_C$. The cycle class map induces injections

$$R^*(C^m) \hookrightarrow H^*(C^m, \mathbb{Q})$$

for all $m \in \mathbb{N}$.

In particular, $C$ has an MCK decomposition, and the Faber–Pandharipande cycle

$$\Delta_C \cdot (p_j)^*(K_C) - \frac{1}{8} K_C \times K_C \in A^2(C \times C) \quad (j = 1, 2)$$

and the Ceresa cycle

$$[C] - (-1_{\text{Jac}(C)})_*[C] \in A_1(\text{Jac}(C))$$

(for an appropriate choice of embedding of $C$ in its Jacobian $\text{Jac}(C)$) are zero.

The statement about the tautological ring $R^*(C^m)$ is inspired by Tavakol’s work [43,44], where the same injectivity statement is proven for hyperelliptic curves.

Theorem 3.1 provides the first examples of non-hyperelliptic curves with an MCK decomposition. These are also the first examples of non-hyperelliptic curves with torsion Ceresa cycle (non-hyperelliptic examples where the class of the Ceresa cycle in the intermediate Jacobian is torsion are given in [7] and [4], cf. also [5] for a result modulo algebraic equivalence). The Ceresa cycle is famously known to be torsion for hyperelliptic curves, but non-torsion in general (cf. [10] and Sect. 2.4 below). As for the Faber–Pandharipande cycle, this is known to be torsion for plane curves and for hyperelliptic curves, but non-torsion in general (cf. [19] and Remark 2.14 below).

**Conventions.** In this article, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups are with rational coefficients: we will denote by $A_j(Y)$ the Chow group of $j$-dimensional cycles on $Y$ with $\mathbb{Q}$-coefficients; for $Y$ smooth of dimension $n$ the notations $A_j(Y)$ and $A^{n-j}(Y)$ are used interchangeably. For a morphism $f : X \to Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of $f$.

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [38,40]) will be denoted $\mathcal{M}_{\text{rat}}$.

(Added in revision) Cf. also the recent work [39].
2. Preliminaries

2.1. Humbert curves

**Definition 2.1.** A smooth projective curve \( C \) is called a Humbert–Edge curve of type \( n \) if \( C \subset \mathbb{P}^n \) can be defined as a complete intersection of \( n-1 \) quadrics

\[
\begin{align*}
    x_0^2 + x_1^2 + \cdots + x_n^2 &= 0, \\
    \lambda_0 x_0^2 + \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2 &= 0, \\
    \lambda_0^2 x_0^2 + \lambda_1^2 x_1^2 + \cdots + \lambda_n^2 x_n^2 &= 0, \\
    \vdots & \notag \\
    \lambda_0^{n-2} x_0^2 + \lambda_1^{n-2} x_1^2 + \cdots + \lambda_n^{n-2} x_n^2 &= 0,
\end{align*}
\]

where \( \lambda_i \in \mathbb{C}, i = 0, \ldots, n \), are pairwise distinct complex numbers.

A Humbert curve is a Humbert–Edge curve of type 4.

**Theorem 2.2.** A Humbert–Edge curve of type \( n \geq 4 \) is not hyperelliptic, and not trigonal.

**Proof.** This is [8, Theorems 2.1 and 2.2]. \( \square \)

**Remark 2.3.** A Humbert–Edge curve \( C \) of type \( n \) has genus

\[ g(C) = 2^{n-2}(n - 3) + 1. \]

In particular, Humbert curves are of genus 5, and Humbert–Edge curves of type 2 and type 3 are quadrics resp. elliptic curves.

An alternative coordinate-free definition is as follows: a smooth curve \( C \) of genus \( g(C) = 2^{n-2}(n - 3) + 1 \) is a Humbert–Edge curve of type \( n \) if and only if \( C \) admits an action of a group \( G \cong (\mathbb{Z}/2\mathbb{Z})^n \) with generators \( \sigma_0, \ldots, \sigma_n \) such that \( \sigma_0 \cdots \sigma_n = 1 \) and the fixed loci of \( \sigma_i \) are pairwise disjoint of cardinality \( 2^{n-1} \) [8,13].

The Jacobian of Humbert–Edge curves is studied in [13] and [1].

Humbert curves are named after Humbert [21], who first described them as certain curves in \( \mathbb{P}^3 \): a Humbert curve is the locus of points of tangency of lines through a fixed point in \( \mathbb{P}^3 \) and tangent to twisted cubics through 5 general points. Subsequently, this was generalized by Edge [11,12]. Yet another characterization is as follows: a genus 5 curve is a Humbert curve if and only if it has 5 bi-elliptic involutions (i.e., involutions yielding a quotient curve of genus 1) [23, Sections 2.1 and 2.2]. A remarkable relation between Humbert curves and Weddle surfaces via Prym varieties is exhibited in [45].

The moduli space of Humbert curves is 2-dimensional and is explicitly described in [20, Section 2]. For \( n \geq 5 \) odd, the moduli space of Humbert–Edge curves of type \( n \) is isomorphic to the moduli space of hyperelliptic curves of genus \( \frac{n+1}{2} \) [9, Section 3.3].
Remark 2.4. As explained and exploited in [1, Section 2], Humbert–Edge curves have a nice inductive structure. Given a Humbert–Edge curve $C \subset \mathbb{P}^n$ of type $n$, let $\sigma_i (i = 0, \ldots, n)$ be the automorphism of $C$ obtained by sending $x_i$ to $-x_i$. Then the quotient $C/\langle \sigma_i \rangle$ is a Humbert–Edge curve of type $n - 1$.

2.2. The motive of Humbert curves

Notation 2.5. Let $C \subset \mathbb{P}^4$ be a Humbert curve, defined by equations

$$\begin{cases}
x_0^2 + x_1^2 + \cdots + x_4^2 = 0, \\
\lambda_0 x_0^2 + \lambda_1 x_1^2 + \cdots + \lambda_4 x_4^2 = 0, \\
\lambda_0^2 x_0^2 + \lambda_1^2 x_1^2 + \cdots + \lambda_4^2 x_4^2 = 0,
\end{cases}$$

where $\lambda_0, \ldots, \lambda_4$, are pairwise distinct complex numbers.

For $i = 0, \ldots, 4$, let $\sigma_i \in \text{Aut}(C)$ be the involution of $C$ obtained by sending $x_i$ to $-x_i$. Let $G \cong (\mathbb{Z}/2\mathbb{Z})^4 \subset \text{Aut}(C)$ be the group with generators $\sigma_0, \ldots, \sigma_4$ and relation $\sigma_0 \cdots \sigma_4 = 1$.

Proposition 2.6. Let $C \subset \mathbb{P}^4$ be a Humbert curve. There is a splitting of the Chow motive

$$h(C) = h^0(C) \oplus h^2(C) \oplus \bigoplus_{i=0}^4 h^1_{(i)}(C) \quad \text{in} \quad \mathcal{M}_{\text{rat}},$$

with the property that $H^*(h^0(C), \mathbb{Q}) = H^0(C, \mathbb{Q})$, $H^*(h^2(C), \mathbb{Q}) = H^2(C, \mathbb{Q})$, and

$$H^*(h^1_{(i)}(C), \mathbb{Q}) = \{ a \in H^1(C, \mathbb{Q}) \mid (\sigma_i)_*(a) = a \text{ and } (\sigma_j)_*(a) = -a \forall j \neq i \} \quad (i = 0, \ldots, 4).$$

Moreover, writing $\rho_i : C \to C_i := C/\langle \sigma_i \rangle$ for the quotient morphism, there are isomorphisms of Chow motives

$$^1\Gamma_{\rho_i} : h(C_i) \cong h^0(C) \oplus h^1_{(i)}(C) \oplus h^2(C) \quad \text{in} \quad \mathcal{M}_{\text{rat}} \quad (i = 0, \ldots, 4),$$

and the $C_i$ are elliptic curves.

Proof. Let $z \in A^1(C)$ be the unique 0-cycle of degree 1 that is $\sigma_i$-invariant for all $i$. Concretely, this 0-cycle can be constructed as follows: let $\rho : C \to C/G \cong \mathbb{P}^1$ be the quotient morphism. Then $\rho$ has degree $2^4$ and one defines

$$z := \frac{1}{2^4} \rho^*(p) \in A^1(C),$$

where $p \in C/G$ is an arbitrary point. This choice of 0-cycle defines a decomposition

$$h(C) = h^0(C) + h^1(C) + h^2(C) \quad \text{in} \quad \mathcal{M}_{\text{rat}},$$

where $h^0(C)$ and $h^2(C)$ are defined by the projectors $\pi_C^0 := z \times C$ resp. $\pi_C^2 := C \times z$. 
Let us write $\rho_i: C \to C_i := C/\langle \sigma_i \rangle$ for the quotient morphism, so that $(\rho_i)^* H^1(C_i, \mathbb{Q})$ is the $\sigma_i$-invariant part.

The curve $C_i$ is (a Humbert–Edge curve of type 3 and so) an elliptic curve. Moreover, for $i \neq j$ the intersection $(\rho_i^*)^* H^1(C_i, \mathbb{Q}) \cap (\rho_j^*)^* H^1(C_j, \mathbb{Q})$ is empty, since the quotient $C/\langle \sigma_i, \sigma_j \rangle$ is (a Humbert–Edge curve of type 2 and so) a rational curve. Since $\dim H^1(C, \mathbb{Q}) = 10$ and $\dim H^1(C_i, \mathbb{Q}) = 2$, it follows that there is a decomposition of cohomology

$$H^1(C, \mathbb{Q}) = \bigoplus_{i=0}^{4} (\rho_i)^* H^1(C_i, \mathbb{Q}).$$

To upgrade this decomposition to a motivic decomposition, one writes

$$\Delta_C = \frac{1}{2} (\Delta_C + \Gamma_{\sigma_i} + \Delta_C - \Gamma_{\sigma_i}) \quad (i = 0, \ldots, 4),$$

which gives

$$\pi^1_C = \frac{1}{2^5} \sum_{r_0=0}^{1} \sum_{r_1=0}^{1} \cdots \sum_{r_4=0}^{1} \pi^1_C \circ (\Delta_C + (-1)^{r_0} \Gamma_{\sigma_0}) \circ (\Delta_C + (-1)^{r_1} \Gamma_{\sigma_1})$$

$$\circ \cdots \circ (\Delta_C + (-1)^{r_4} \Gamma_{\sigma_4})$$

$$\in A^1(C \times C). \quad (1)$$

Each summand in (1) with more than one $r_i$ equal to 0 vanishes (because the quotient $C/\langle \sigma_i, \sigma_j \rangle$ is a rational curve which has no $h^1$ part in its motive). Also, the summand in (1) with all $r_i$ equal to 1 vanishes (because $\sigma_0 \circ \sigma_1 \circ \cdots \sigma_4$ is the identity, which does not have an anti-invariant part). It follows that (1) simplifies to a 5-term decomposition

$$\pi^1_C = \pi^1_{C,(0)} + \pi^1_{C,(1)} + \cdots + \pi^1_{C,(4)} \in A^1(C \times C),$$

where $\pi^1_{C,(i)}$ is defined as the summand in (1) with $r_i = 0$ (and hence all $r_j, j \neq i$ equal to 1).

The elliptic curves $C_i$ have a decomposition $h(C_i) = h^0(C_i) \oplus h^1(C_i) \oplus h^2(C_i)$, induced by the degree 1 0-cycle $(\rho_i)_*(z) \in A^1(C_i)$. For $k = 0, 2$, there are isomorphisms

$$\Gamma_{\rho_i} : h^k(C_i) \cong h^k(C) \quad \text{in } \mathcal{M}_{\text{rat}} \quad (i = 0, \ldots, 4). \quad (2)$$

The above construction gives an isomorphism

$$\Gamma_{\rho_i} : h^1(C_i) \cong h^1(C) \quad \text{in } \mathcal{M}_{\text{rat}} \quad (i = 0, \ldots, 4) \quad (3)$$

(with inverse given by $\frac{1}{2} \Gamma_{\rho_i}$). Combining (2) and (3) proves the “moreover” part. □
Remark 2.7. As in [30], let us write $h(C)+++++, h(C)++++−$ etc. for the sub-
motive of $h(C)$ that is $σ_i$-invariant for $i = 0, . . . , 4$, resp. that is $σ_4$-anti-invariant
and $σ_i$-invariant for $i = 0, . . . , 3$. The argument of Proposition 2.6 then gives the
decomposition

$$h(C) = h(C)+++++ ⊕ h(C)^{+−−−} ⊕ h(C)^{−+−−} ⊕ h(C)^{−−+−} ⊕ h(C)^{−−−+},$$

where $h^0(C) ⊕ h^2(C)$ is equal to the first summand, and $h^1_{(0)}(C), . . . , h^1_{(4)}(C)$ are
equal to the other summands.

2.3. MCK decompositions

Definition 2.8. (Murre [37]) Let $X$ be a smooth projective variety of dimension $n$. We say that $X$ has a CK decomposition if there exists a decomposition of the diagonal

$$\Delta_X = \pi_X^0 + \pi_X^1 + \cdots + \pi_X^{2n} \text{ in } A^n(X \times X),$$

such that the $\pi_X^i$ are mutually orthogonal idempotents and $(\pi_X^i)_*H^*(X, \mathbb{Q}) = H^i(X, \mathbb{Q}).$

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.)

Remark 2.9. The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [22,37].

Definition 2.10. (Shen–Vial [41]) Let $X$ be a smooth projective variety of dimension $n$. Let $\Delta_X^{sm} \in A^{2n}(X \times X \times X)$ be the class of the small diagonal

$$\Delta_X^{sm} := \{(x, x, x) \mid x \in X\} \subset X \times X \times X.$$

An MCK decomposition is a CK decomposition $\{\pi_X^i\}$ of $X$ that is multiplicative, i.e. it satisfies

$$\pi_X^k \circ \Delta_X^{sm} \circ (\pi_X^i \times \pi_X^j) = 0 \text{ in } A^{2n}(X \times X \times X) \text{ for all } i + j \neq k. \quad (4)$$

Here $\pi_X^i \times \pi_X^j$ is by definition $(p_{13})^*(\pi_X^i) \cdot (p_{24})^*(\pi_X^j) \in A^{2n}(X^4)$, where $p_{rs} : X^4 \to X^2$ denotes projection on $r$th and $s$th factors.

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition”.)

Remark 2.11. Note that the vanishing (4) is always true modulo homological equivalence; this is because the cup product in cohomology respects the grading.

The small diagonal (seen as a correspondence from $X \times X$ to $X$) induces the multiplication morphism

$$\Delta_X^{sm} : h(X) \otimes h(X) \to h(X) \text{ in } \mathcal{M}_{\text{rat}}.$$
Suppose $X$ has a CK decomposition

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \text{ in } \mathcal{M}_{\text{rat}}.$$

By definition, this decomposition is multiplicative if for any $i, j$ the composition

$$h^i(X) \otimes h^j(X) \to h(X) \otimes h(X) \xrightarrow{\Delta_X^m} h(X) \text{ in } \mathcal{M}_{\text{rat}}$$

factors through $h^{i+j}(X)$.

If $X$ has an MCK decomposition, then setting

$$A^i_{(j)}(X) := (\pi_X^{2i-j})_* A^i(\mathcal{X}),$$

one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends $A^i_{(j)}(X) \otimes A^{i'}_{(j')}(X)$ to $A^{i+i'}_{(j+j')}(X)$.

It is expected that for any $X$ with an MCK decomposition, one has

$$A^i_{(j)}(X) \equiv 0 \text{ for } j < 0, \quad A^i_{(0)}(X) \cap A^{i}_{\text{hom}}(X) \equiv 0;$$

this is related to Murre’s conjectures B and D, that have been formulated for any CK decomposition [37].

The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville’s “splitting property” conjecture [3]. To give some idea: hyperelliptic curves have an MCK decomposition [41, Example 8.16], but the very general curve of genus $\geq 3$ does not have an MCK decomposition [15, Example 2.3]. As for surfaces: a smooth quartic in $\mathbb{P}^3$ has an MCK decomposition, but a very general surface of degree $\geq 7$ in $\mathbb{P}^3$ should not have an MCK decomposition [15, Proposition 3.4]. For a more detailed discussion, as well as examples of varieties with an MCK decomposition, we refer to [41, Section 8], as well as [15, 16, 25–36, 42, 46].

2.4. MCK decomposition for curves

**Proposition 2.12.** Let $C$ be a curve of genus $g$, and $z \in A^1(C)$ a zero-cycle of degree 1. The following are equivalent:

1. The CK decomposition

$$\pi^0_C = z \times C, \quad \pi^2_C = C \times z, \quad \pi^1_C = \Delta_C - \pi^0_C - \pi^2_C$$

is MCK;

2. The modified small diagonal

$$\Gamma_3(C, z) := \Delta_C^m - p^*_{12}(\Delta_X) p^*_5(z) - p^*_2(\Delta_C) p^*_1(z) - p^*_1(\Delta_C) p^*_2(z) + p^*_1(z) p^*_2(z) + p^*_1(z) p^*_3(z) + p^*_2(z) p^*_3(z) \text{ in } A^2(C^3)$$

is zero;
(3) the class $[C] \in A^{g-1}(\text{Jac}(C))$ is in $A^{g-1}_{(0)}(\text{Jac}(C))$, where $C$ is embedded in the Jacobian $\text{Jac}(C)$ using the 0-cycle $z$, and $A^*_a(\text{Jac}(C))$ refers to the Beauville decomposition [2].

**Proof.** This is [15, Proposition 3.1].

This leads to non-existence results:

**Corollary 2.13.** Let $C$ be a very general curve of genus $> 2$. Then $C$ does not admit an MCK decomposition.

**Proof.** If $C$ admits an MCK decomposition, then Proposition 2.12 implies that the Ceresa cycle

$$[C] - (-1_{\text{Jac}(C)})_*[C] \in A_1(\text{Jac}(C))$$

must vanish. But Ceresa has shown that the Ceresa cycle of a very general curve of genus $> 2$ is non-torsion modulo algebraic equivalence [10].

**Remark 2.14.** In case $Y \subset \mathbb{P}^m$ is a smooth hypersurface (of any degree), there is equality

$$\Delta_Y \cdot (p_i)^*(h) = \sum a_k (p_1)^*(h^k) \cdot (p_2)^*(h^{m-k}) \quad \text{in} \quad A^m(Y \times Y),$$

with $a_k \in \mathbb{Q}$, as follows from the excess intersection formula. On the other hand, in case $Y \subset \mathbb{P}^m$ is a complete intersection of codimension at least 2, in general there is no equality of the form (5). Indeed, let $C$ be a very general curve of genus $g \geq 4$. The Faber–Pandharipande cycle

$$FP(C) := \Delta_C \cdot (p_j)^*(K_C) - \frac{1}{2g-2}K_C \times K_C \quad \text{in} \quad A^2(C \times C) \quad (j = 1, 2)$$

is homologically trivial but non-zero in $A^2(C \times C)$ [19,48] (this cycle $FP(C)$ is the “interesting 0-cycle” in the title of [19]). In particular, for the very general complete intersection $Y \subset \mathbb{P}^3$ of bidegree $(2, 3)$, the cycle

$$FP(Y) := \Delta_Y \cdot (p_j)^*(h) - \frac{1}{6}h \times h \quad \text{in} \quad A^2(Y \times Y)$$

is homologically trivial but non-zero, and so there cannot exist an equality of the form (5) for $Y$.

This is intimately related to MCK decompositions. Indeed, if the curve $Y$ had an MCK decomposition which is generically defined, then $K_Y \in A^1_{(0)}(Y)$ and hence $FP(Y) \in A^2_{(0)}(Y \times Y) \cong \mathbb{Q}$, and so $FP(Y)$ would be zero.
3. The main result

Theorem 3.1. Any Humbert curve has an MCK decomposition.

Proof. We consider the decomposition

$$h^0(C), h^2(C), h^1(C) = h^1_{(0)}(C) \oplus \cdots \oplus h^1_{(4)}(C)$$

of Proposition 2.6, and we are going to verify this is an MCK decomposition.

By definition, this means the following: given three motives

$$M_1, M_2, M_3 \in \left\{ h^0(C), h^2(C), h^1_{(0)}(C), \ldots, h^1_{(4)}(C) \right\}$$

of degree say \(m_1\) resp. \(m_2\) resp. \(m_3 \neq m_1 + m_2\), we need to ascertain that the composition of the “multiplication map” with the projection to \(M_3\)

$$M_1 \otimes M_2 \xrightarrow{\Delta_{C}^{sm}} h(C) \rightarrow M_3$$

is zero.

As a first step, let us assume that there exists one \(i\) such that

$$M_1, M_2, M_3 \in \left\{ h^0(C), h^2(C), h^1_{(i)}(C) \right\} .$$

In this case, we have a commutative diagram

$$M_1 \otimes M_2 \xrightarrow{\Delta_{C}^{sm}} h(C) \rightarrow M_3$$

$$\uparrow \left\{ \Gamma_{\rho_i} \times \Gamma_{\rho_i} \right\} \uparrow \left\{ \Gamma_{\rho_i} \right\} \downarrow \Gamma_{\rho_i}$$

$$h^{m_1}(C_i) \otimes h^{m_2}(C_i) \xrightarrow{\Delta_{C_i}^{sm}} h(C_i) \rightarrow h^{m_3}(C_i)$$

where \(C_i = C/\langle \sigma_i \rangle\) is an elliptic curve. The left square commutes because of contravariance of the intersection product; the right square commutes because of Lieberman’s lemma and the fact that \((\rho_i \times \rho_i)_*(p) = \pi_{C_i}^{m_3}\), where \(p\) is the projector on \(M_3 \in \{h^0(C), h^2(C), h^1_{(i)}(C)\}\). Now the decomposition \(h(C_i) = h^0(C_i) \oplus h^1(C_i) \oplus h^2(C_i)\) of Proposition 2.6 is MCK (indeed, the 0-cycle used to define this decomposition is supported on the Weierstrass points), and so the composition of the two lower horizontal arrows is zero (since by assumption \(m_3 \neq m_1 + m_2\)). But the outer two vertical arrows are isomorphisms (Proposition 2.6), and so the composition of the top horizontal arrows is zero as well.

It remains to consider all the cases where there are at least two different \(h^1_{(i)}(C)\) among \(M_1, M_2, M_3\). For notational ease, we will assume

$$h^1_{(0)}(C), h^1_{(1)}(C) \in \{M_1, M_2, M_3\} .$$

Let us assume \(M_1 = h^1_{(0)}(C)\) and \(M_2 = h^1_{(1)}(C)\). In terms of the notation of Remark 2.7, the “multiplication map”

$$\Delta_{C}^{sm} : h^1_{(0)}(C) \otimes h^1_{(1)}(C) = h(C)^+ \otimes h(C)^- \rightarrow h(C)$$
factors over $h^{+++}(C)$, which is zero (cf. Remark 2.7).

Next, let us assume $M_1 = M_2 = h^1_{(0)}(C)$ (and so $M_3 = h^1_{(1)}(C)$). Then

$$\Delta^m_C : h^1_{(0)}(C) \otimes h^1_{(0)}(C) = h(C)^{+++} \otimes h(C)^{+++} \to h(C)$$

factors over $h^{++++}(C)$, and so

$$\Delta^m_C : M_1 \otimes M_2 \to h(C) \to h(C)^{+++} = M_3$$

is zero.

Finally, let us assume $M_1 = h^1_{(0)}(C)$, $M_3 = h^1_{(1)}(C)$, and $M_2 \in \{h^0(C), h^2(C)\}$. Then $M_1 = h(C)^{+++}$ and $M_2$ is a submotive of $h(C)^{++++}$, and so

$$\Delta^m_C : M_1 \otimes M_2 \to h(C)$$

factors over $h(C)^{+++}$, and we conclude that composing with the projection to $M_3 = h(C)^{+++}$ yields zero. This ends the proof. \qed

**Theorem 3.2.** Let $C$ be a Humbert curve, and $m \in \mathbb{N}$. Let

$$R^*(C^m) := \langle (p_{ij})^*(\Delta_C), (p_k)^*(K_C) \rangle \subset A^*(C^m)$$

be the $\mathbb{Q}$-subalgebra generated by (pullbacks of) the diagonal $\Delta_C \subset C \times C$ and (pullbacks of) the canonical divisor $K_C$. The cycle class map induces injections

$$R^*(C^m) \hookrightarrow H^*(C^m, \mathbb{Q})$$

for all $m \in \mathbb{N}$.

**Proof.** The result is actually true for any curve with an MCK decomposition; this is the content of [15, Proposition 3.5]. Let us briefly recap the argument, which is essentially due to Tavakol [43,44], and formally similar to analogous results for cubic hypersurfaces [14, Section 2.3] and for K3 surfaces [49].

As in [14, Section 2.3], let us write $o \in A^1(C)$ for the degree 1 0-cycle of Proposition 2.6, and

$$\tau := \Delta_C - o \times C - C \times o \in A^1(C \times C)$$

(this cycle $\tau$ is just the projector $\pi_1^1$ on the motive $h^1(C)$ considered above). Moreover, let us write

$$o_i := (p_i)^*(o) \in A^1(C^m),$$

$$\tau_{i,j} := (p_{ij})^*(\tau) \in A^1(C^m).$$

We now define the $\mathbb{Q}$-subalgebra

$$\bar{R}^*(C^m) := \langle o_i, \tau_{i,j} \rangle \subset H^*(C^m, \mathbb{Q}) \quad (1 \leq i \leq m, \ 1 \leq i < j \leq m);$$
this is the image of \( R^*(C^m) \) in cohomology. One can prove (just as \([14, \text{Lemma 2.12}]\) and \([49, \text{Lemma 2.3}]\)) that the \( \mathbb{Q} \)-algebra \( \bar{R}^*(C^m) \) is isomorphic to the free graded \( \mathbb{Q} \)-algebra generated by \( o_i, \tau_{i,j} \), modulo the following relations:

\[
\begin{align*}
o_i \cdot o_i &= 0, \\
o_i \cdot o_j &= \tau_{i,j} \cdot o_i, \\
\tau_{i,j} \cdot \tau_{i,k} &= -b \cdot o_i \cdot o_j; \\
\sum_{\sigma \in S_{b+2}} \prod_{i=1}^{b/2+1} \tau_{\sigma(2i-1), \sigma(2i)} &= 0,
\end{align*}
\]

where \( b := \dim H^1(C, \mathbb{Q}) = 10 \).

To prove Theorem 3.2, it suffices to check that these relations are verified modulo rational equivalence. Relation (6) is clearly true, for reasons of dimension.

The relations (7) are pullbacks of relations taking place in \( R^*(C^2) \). The first relation of (7) concerns the Faber–Pandharipande cycle of Remark 2.14. To prove this relation modulo rational equivalence, we note that \( \tau \) is in \( A^1(0)(C^2) \) (where \( C^2 \) is given the product MCK decomposition), and \( o \) is in \( A^1(0)(C) \). It follows that

\[
\tau \cdot (p_1)^*(0) - (p_1)^*(o) \cdot (p_2)^*(o) \in A^2(0)(C^2),
\]

and it is readily checked that \( A^2(0)(C^2) \) injects into cohomology. This argument also applies to the second relation of (7).

Relation (8) is the pullback of a relation taking place in \( R^*(C^3) \) and also follows from the MCK decomposition. Indeed, we have

\[
\Delta_C^\text{sm} \circ (\pi_C^1 \times \pi_C^1) = \pi_C^2 \circ \Delta_C^\text{sm} \circ (\pi_C^1 \times \pi_C^1) \quad \text{in} \ A^2(C^3),
\]

and (using Lieberman’s lemma) this translates into

\[
(\pi_C^1 \times \pi_C^1 \times \Delta_C)^* \Delta_C^\text{sm} = (\pi_C^1 \times \pi_C^1 \times \pi_C^2)^* \Delta_C^\text{sm} \quad \text{in} \ A^2(C^3),
\]

which means that

\[
\tau_{1,3} \cdot \tau_{2,3} = \tau_{1,2} \cdot o_3 \quad \text{in} \ A^2(C^3).
\]

Finally, relation (9), which takes place in \( R^*(C^{b+2}) \) follows from the Kimura finite dimensionality relation \([24]\): relation (9) expresses the vanishing

\[
\text{Sym}^{b+2} H^1(C, \mathbb{Q}) = 0,
\]

where \( H^1(C, \mathbb{Q}) \) is considered as a super vector space. This relation is also verified modulo rational equivalence (i.e., relation (9) is also true in \( A^{(b+2)}(C^{b+2}) \)): indeed, relation (9) involves a cycle in

\[
A^*(\text{Sym}^{b+2} H^1(C))
\]

but (\( \text{Sym}^{b+1} H^1(C) \) and so a fortiori) \( \text{Sym}^{b+2} H^1(C) \) is 0 because curves are Kimura finite-dimensional \([24]\).

This ends the proof. \( \square \)
4. Closing remarks

Remark 4.1. The argument of Theorem 3.1, using a motivic decomposition coming from a large automorphism group, is formally similar to that of [30], where an MCK decomposition was established for special Horikawa surfaces. In its turn, [30] was inspired by the argument of [15, Section 7.1], where MCK was established for certain Todorov surfaces. In practice, this type of argument tends to be applicable over certain special loci in the moduli space (where the automorphism group gets large), and not for the general element of some locally complete family.

Remark 4.2. It seems likely that Humbert–Edge curves (of any type $n$) admit an MCK decomposition. However, the naive “$-$ $-$ $+$” argument proving Theorem 3.1 does not suffice to settle this for $n > 4$. For example, let $C$ be a Humbert–Edge curve of type 5 (the genus of $C$ is 17), with commuting involutions $\sigma_0, \ldots, \sigma_5$. Using [1, Theorem 2.1], one can show there is a decomposition

$$h^1(C) = h(C)^{-}-+ \oplus h(C)^{+}+-\oplus \text{perm. in } M_{\text{rat}},$$

where “perm.” means all possible permutations of 2 plus signs and 4 minus signs. Now it is not clear whether the map

$$\Delta_C^{sm} : h(C)^{-}-+ \otimes h(C)^{+}+- \to h(C)^{-}-++$$

is the zero map (which is a necessary condition for the decomposition to be MCK).

Some more sophisticated reasoning is called for here; perhaps it is possible to show that $C$ satisfies condition $(\ast)$ of [17]?

Remark 4.3. In [7] and [5], examples are given of non-hyperelliptic curves for which the Ceresa cycle is torsion modulo algebraic equivalence. In view of [15, Remark 3.4], these curves hence have an MCK decomposition modulo algebraic equivalence (and the argument of Theorem 3.2 gives that the tautological ring modulo algebraic equivalence of these curves injects into cohomology). It would be interesting to settle whether these results can be lifted to rational equivalence.

Remark 4.4. (Added in revision) The recent preprint [39] contains many examples of non-hyperelliptic curves with vanishing Ceresa cycle and vanishing modified small diagonal modulo rational equivalence. In view of Proposition 2.12, these examples also have an MCK decomposition.

Acknowledgements The author thanks Yoyo for a wonderful Japanese dinner on June 18 2021.

Declarations

Data Availability Statement The author states that this is not applicable, as there are no associated data.

Conflict of interest statement. The author states that there is no conflict of interest.
References

[1] Auffarth, R., Arteche, G. Rojas, A.: A decomposition of the Jacobian of a Humbert–Edge curve, arXiv:1905.12690v2
[2] Beauville, A.: Sur l’anneau de Chow d’une variété abélienne. Math. Ann. 273, 647–651 (1986)
[3] Beauville, A.: On the splitting of the Bloch–Beilinson filtration. In: Nagel, J., Peters, C. (eds.) Algebraic Cycles and Motives. London Mathematical Society. Lecture Notes, vol. 344. Cambridge University Press, Cambridge (2007)
[4] Beauville, A.: A non-hyperelliptic curve with torsion Ceresa class. C. R. Math. Acad. Sci. Paris 359(7), 871–872 (2021)
[5] Beauville, A., Schoen, C.: A non-hyperelliptic curve with torsion Ceresa cycle modulo algebraic equivalence, arXiv:2106.08390, to appear in IMRN
[6] Beauville, A., Voisin, C.: On the Chow ring of a K3 surface. J. Alg. Geom. 13, 417–426 (2004)
[7] Bisogno, D., Li, W., Litt, D., Srinivasan, P. Group-theoretic Johnson classes and non-hyperelliptic curves with torsion Ceresa class, arXiv:2004.06146
[8] Carocca, A., González-Aguilera, V., Hidalgo, R., Rodriguez, R.: Generalized Humbert curves. Israel J. Math. 164, 165–192 (2008)
[9] Castorena, A. Bosco Frías-Medina, J.: Geometric aspects of Humbert–Edge’s curves of type 5, Kummer surfaces and hyperelliptic curves of genus 2, arXiv:2106.00813
[10] Ceresa, G.: C is not algebraically equivalent to C− in its Jacobian. Ann. Math. (2) 117(2), 285–291 (1983)
[11] Edge, W.: Humbert’s plane sextics of genus 5. Math. Proc. Camb. Philos. Soc. 47(3), 483–495 (1951)
[12] Edge, W.: The common curve of quadrics sharing a self-polar simplex. Ann. Mat. Pura Appl. 114, 241–270 (1977)
[13] Frias-Medina, J., Zamora, A.: Some remarks on Humbert–Edge’s curves. Eur. J. Math. 4(3), 988–999 (2018)
[14] Fu, L., Laterveer, R., Vial, Ch.: The generalized Franchetta conjecture for some hyper-Kähler varieties, II. J. Ecol. Polytech. Math. 8, 1065–1097 (2021)
[15] Fu, L., Laterveer, R., Vial, Ch.: Multiplicative Chow–Künneth decompositions and varieties of cohomological K3 type. Ann. Mat. Pura Appl. 200, 2085–2126 (2021)
[16] Fu, L., Tian, Z., Vial, Ch.: Motivic hyperkähler resolution conjecture, I: generalized Kummer varieties. Geom. Topol. 23, 427–492 (2019)
[17] Fu, L., Vial, Ch.: Distinguished cycles on varieties with motive of abelian type and the section property. J. Alg. Geom. 29, 53–107 (2020)
[18] Fulton, W.: Intersection theory. Springer-Verlag Ergebnisse der Mathematik, Berlin (1984)
[19] Green, M., Griffiths, P.: An interesting 0-cycle. Duke Math. J. 119(2), 261–313 (2003)
[20] Hidalgo, R., Reyes, S.: Fields of moduli of classical Humbert curves. Q. J. Math. 63(4), 919–930 (2012)
[21] Humbert, G.: Sur un complexe remarquable de coniques et sur la surface de troisième ordre. J. Ecole Polytech. 64, 123–149 (1894)
[22] Jannsen, U.: On finite-dimensional motives and Murre’s conjecture. In: Nagel, J., Peters, C. (eds.) Algebraic Cycles and Motives. Cambridge University Press, Cambridge (2007)
[23] Kato, T., Magaard, K., Völklein, H.: Bi-elliptic Weierstrass points on curves of genus 5. Indag. Math. 22, 116–130 (2011)
[24] Kimura, S.-I.: Chow groups are finite dimensional, in some sense. Math. Ann. 331(1), 173–201 (2005)
[25] Laterveer, R.: A remark on the Chow ring of Küchle fourfolds of type $d3$. Bull. Aust. Math. Soc. 100(3), 410–418 (2019)
[26] Laterveer, R., Vial, Ch.: On the Chow ring of Cynk–Hulek Calabi–Yau varieties and Schreieder varieties. Can. J. Math. 72(2), 505–536 (2020)
[27] Laterveer, R.: Algebraic cycles and Verra fourfolds. Tohoku Math. J. 72(3), 451–485 (2020)
[28] Laterveer, R.: On the Chow ring of certain Fano fourfolds. Ann. Univ. Paedagog. Crac. Stud. Math. 19, 39–52 (2020)
[29] Laterveer, R.: On the Chow ring of Fano varieties of type $S2$. Abh. Math. Semin. Univ. Hambg. 90, 17–28 (2020)
[30] Laterveer, R.: Algebraic cycles and special Horikawa surfaces. Acta Math. Vietnamica 46, 483–497 (2021)
[31] Laterveer, R.: Algebraic cycles and Gushel–Mukai fivefolds. J. Pure Appl. Algebra 225(5), 106582 (2021)
[32] Laterveer, R.: Algebraic cycles and intersections of 2 quadrics. Mediterranean J. Math. 18(4), 146 (2021)
[33] Laterveer, R.: Algebraic cycles and intersections of a quadric and a cubic. Forum Math. 33(3), 845–855 (2021)
[34] Laterveer, R.: Algebraic cycles and Fano threefolds of genus 8. Portugaliae Math. (N.S.) 78 (Fasc. 3–4), 255–280 (2021)
[35] Laterveer, R.: On the Chow ring of Fano varieties on the Fatighenti–Mongardi list. Commun. Algebra 50(1), 131–145 (2022)
[36] Laterveer, R.: Algebraic cycles and intersections of three quadrics. Math. Proc. Camb. Philos. Soc. 173(2), 349–367 (2022)
[37] Murre, J.: On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II. Indag. Math. 4, 177–201 (1993)
[38] Murre, J., Nagel, J., Peters, C. Lectures on the theory of pure motives, vol. 61. American Mathematical Society University Lecture Series, Providence (2013)
[39] Qiu, C., Zhang, W.: Vanishing results in Chow groups for the modified diagonal cycles, arXiv:2209.09736
[40] Scholl, T.: Classical motives. In: Jannsen, U et al. (editors), Motives, Proceedings of Symposia in Pure Mathematics, vol. 55, Part 1 (1994)
[41] Shen, M., Vial, Ch. The Fourier transform for certain hyperKähler fourfolds, Memoirs of the AMS 240, no. 1139 (2016)
[42] Shen, M., Vial, Ch.: The motive of the Hilbert cube $X^{[3]}$. Forum Math. Sigma 4, 55 (2016)
[43] Tavakol, M.: The tautological ring of the moduli space $M_{2,2}$. Int. Math. Res. Not. 2014(24), 6661–6683 (2014)
[44] Tavakol, M.: Tautological classes on the moduli space of hyperelliptic curves with rational tails. J. Pure Appl. Algebra 222(8), 2040–2062 (2018)
[45] Varley, R.: Weddle’s surfaces, Humbert’s curves, and a certain 4-dimensional abelian variety. Am. J. Math. 108, 931–951 (1986)
[46] Vial, Ch.: On the motive of some hyperkähler varieties. J. Reine Angew. Math. 725, 235–247 (2017)
[47] Voisin, C.: Chow Rings, Decomposition of the Diagonal, and the Topology of Families. Princeton University Press, Princeton (2014)
[48] Yin, Q.: The generic nontriviality of the Faber–Pandharipande cycle. Int. Math. Res. Not. 2015(5), 1263–1277 (2015)
[49] Yin, Q.: Finite-dimensionality and cycles on powers of K3 surfaces. Comment. Math. Helv. 90, 503–511 (2015)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.