On the existence of functionals for the mean values of observables

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The aim of this work is to study the existence of mean values of observables for infinite-particle systems. Using solutions of the initial value problems to the BBGKY hierarchy and to its dual, we prove the local, in time, existence of the mean value functionals in the cases where either observables or states vary in time. We also discuss problems on the existence of such functionals for several different classes of observables and for an arbitrary time interval.

Key words: infinite-particle systems; BBGKY hierarchy; dual BBGKY hierarchy; cumulant (semi-invariant); mean values of observables.

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Content

1 Introduction  
2 The regularized solution of the dual BBGKY hierarchy  
3 The mean value of observables: the evolution of observables  
4 The mean value of observables: the evolution of states  
5 Conclusion
1 Introduction

A particular progress in the study of dynamics of infinitely many particles have been achieved during the last decade \([4,5,13,17]\). As is well known the evolution of states of such systems is completely determined by an initial value problem to the BBGKY hierarchy \([4]\). A solution of this initial value problem represented in the form of a series expansion as a result of integration of perturbation series with respect to the time variables was first constructed in \([12]\) for a one-dimensional system of particles interacting via short range potential with hard-core for initial data close to equilibrium states. The divergence of integrals with respect to the configuration variables in every term of the expansion is a typical problem which complicates the construction of the solution for infinite-particle systems. The method of an interaction region \([4,12,15]\) provides one of the approaches to eliminate this obstacle. A further result on the infinite-particle dynamics for three-dimensional hard sphere systems \([5,6,11,14,17]\) is based on the construction of a solution of the BBGKY hierarchy in the form of perturbation series.

Recently a solution of the initial value problem to the BBGKY hierarchy \([7,9]\) has been constructed in the form of expansion over particle clusters, the evolution of which is governed by the corresponding order cumulant of the evolution operators of finitely many particles. The above-mentioned representations for a solution of the BBGKY hierarchy are particular cases of those from \([7,9]\). The method of an interaction region for this representations is developed in \([16]\).

In this paper we develop a similar approach to the regularization of a solution of the dual BBGKY hierarchy \([7,8]\) which describes the evolution of marginal observables of many-particle systems \([2,8]\). We prove the existence of the mean values for infinite-particle systems in both cases where either the evolutions of observables or the evolution of states is considered.

We first present some preliminary facts about description of infinite-particle systems. We consider a one-dimensional system of identical particles (with unit mass) interacting via a short range pair potential \(\Phi\) (with hard core) that possesses the following properties

\[
\begin{align*}
\Phi & \in C^2(\sigma, R], \quad 0 < \sigma < R < \infty, \\
\Phi(|q|) & = \begin{cases} +\infty, & |q| \in [0, \sigma), \\
0, & |q| \in (R, \infty), \\
\end{cases} \\
\Phi'(\sigma + 0) & = 0,
\end{align*}
\]

where \(R\) is the radius of forces acting between particles with the length \(\sigma > 0\).

Every \(i\)th particle is characterized by a coordinate \(q_i\) of the center of hard-core and momentum \(p_i\). Let us denote \(x_i \equiv (q_i, p_i) \in \mathbb{R} \times \mathbb{R}, \quad i \geq 1\). For \(n\)-particle system the inequalities

\(|q_i - q_j| \geq \sigma, \quad i \neq j \in \{1, \ldots, n\}, \quad n \geq 2,\)

hold, i.e. particles can occupy only admissible configurations. The set \(W_n \equiv \{(q_1, \ldots, q_n) \in \mathbb{R}^n \mid \text{for at least one pair } (i, j), \quad i \neq j \in \{1, \ldots, n\}, \quad \text{such that } |q_i - q_j| < \sigma, \quad n \geq 2,\)

is a region of forbidden configurations.

At the initial instant \(t = 0\) observables are described by the sequences of marginal (\(s\)-particle) functions \(G(0) = \{G_0, G_1(0, x_1), \ldots, G_s(0, x_1, \ldots, x_s), \ldots\}\) and states by the sequences of marginal (\(s\)-particle) distribution functions \(F(0) = \{F_0, F_1(0, x_1), \ldots, F_s(0, x_1, \ldots, x_s), \ldots\}\). A mean value (a mathematical expectation) of observable \(G(0)\) for a system in the state \(F(0)\) is determined by the following functional \([2,4]\):

\[
(G)(0) = \langle G(0) | F(0) \rangle = \frac{1}{s!} \int_{(\mathbb{R}^n \setminus W_n) \times \mathbb{R}^e} dx_1 \ldots dx_s G_s(0, x_1, \ldots, x_s) F_s(0, x_1, \ldots, x_s),
\]

where \(F_0 = 1, \quad G_0 = 0, \quad F_0 = 1\).

To describe infinite particle systems we introduce the space \(C_\gamma\) of sequences \(g = (g_0, g_1(x_1), \ldots, g_n(x_1, \ldots, x_n), \ldots)\) of bounded (continuous) functions \(g_n(x_1, \ldots, x_n), \quad n \geq 0\) (**g** is a number), given on the phase space \((\mathbb{R}^n \setminus W_n) \times \mathbb{R}^n\), symmetric with respect to arbitrary permutations of the arguments \(x_i, \quad i = 1, \ldots, n\), and equal zero in the region of forbidden configurations \(W_n\), with the norm \([2]\)

\[
\|g\|_{C_\gamma} = \sup_{n \geq 0} \sup_{x_1, \ldots, x_n} |g_n(x_1, \ldots, x_n)|,
\]
where $0 < \gamma < 1$ is a constant. We denote by $C_{\gamma,0} \subset C_\gamma$ the subspace of finite sequences of continuously differentiable functions with compact supports on the configuration space. The sequence of functions $g \in C_{\gamma,0}$ is treated as a quasiobservable (analog an local observable [1, 4]).

States of infinite particle systems are described by sequences from the space $L_{\xi,\beta}^\infty$ of sequences $f = (f_0, f_1(x_1), \ldots, f_n(x_1, \ldots, x_n), \ldots)$ of functions $f_n(x_1, \ldots, x_n)$, $n \geq 0$ ($f_0 = 1$), given on the phase space $(\mathbb{R}^n \setminus W_n) \times \mathbb{R}^n$, symmetric with respect to arbitrary permutations of the arguments $x_i$, $i = 1, \ldots, n$, and equal zero on the forbidden configurations $W_n$, with the norm $[4, 12]$

$$
\|f\|_{L_{\xi,\beta}^\infty} = \sup_{n \geq 0} \xi^{-n} \sup_{x_1,\ldots,x_n} |f_n(x_1,\ldots,x_n)| \exp \left\{ \beta \sum_{i=1}^n \frac{p_i^2}{2} \right\},
$$

where $\xi, \beta > 0$ are constants.

We make note if $F(0) \in L_{\xi,\beta}^\infty$ and $G(0) \in C_{\gamma,0}$, the following estimate

$$
|\langle G(0)F(0) \rangle| \leq \|G(0)\|_{C_\gamma} \|F(0)\|_{L_{\xi,\beta}^\infty} \sum_{s=0}^{\infty} \left( \frac{C \xi}{\gamma} \sqrt{\frac{2\pi}{\beta}} \right)^s
$$

holds, therefore functional (1.2) is well-defined under the condition

$$
\xi < \frac{\gamma}{C} \sqrt{\frac{\beta}{2\pi}},
$$

where $C = \max_{i=1,\ldots,s} |l_i(0)|$ and $|l_i(0)|$ is a length of the interval $l_i(0)$ such that $\Omega_s(0) = l_1(0) \times \ldots \times l_s(0)$ is a support of function $G_s(0)$ in the configuration space.

Let us note that for observable of the additive type $G^{(1)}(0) = (0, a_1(0, x_1), 0, \ldots, 0, \ldots)$ functional (1.2) has the form

$$
\langle G^{(1)}(0) \rangle = \langle G^{(1)}(0)F(0) \rangle = \int_{\mathbb{R} \times \mathbb{R}} dx_1 a_1(0, x_1) F_1(0, x_1).
$$

(1.3)

If $F(0) \in L_{\xi,\beta}^\infty$ and $G(0) \in C_{\gamma,0}$, then functional (1.3) is well-defined for an arbitrary value of the parameter $\xi > 0$ since

$$
|\langle G^{(1)}(0)F(0) \rangle| \leq \|G^{(1)}(0)\|_{C_\gamma} \|F(0)\|_{L_{\xi,\beta}^\infty} \frac{C \xi}{\gamma} \sqrt{\frac{2\pi}{\beta}} < \infty.
$$

At an arbitrary instant of time $t \in \mathbb{R}^1$ the mean value of observable is determined by the following functional [4]:

$$
\langle G(t) \rangle = \langle G(t)F(0) \rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \int_{(\mathbb{R} \setminus W_s) \times \mathbb{R}^s} dx_1 \ldots dx_s G_s(t, x_1, \ldots, x_s) F_s(0, x_1, \ldots, x_s),
$$

(1.4)

where $G(t) = (0, G_1(t, x_1), \ldots, G_s(t, x_1, \ldots, x_s), \ldots)$ is a solution of the initial value problem to the dual BBGKY hierarchy [2, 8] with the initial data $G(0)$, or by the other functional:

$$
\langle G(t) \rangle = \langle G(0)F(t) \rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \int_{(\mathbb{R} \setminus W_s) \times \mathbb{R}^s} dx_1 \ldots dx_s G_s(0, x_1, \ldots, x_s) F_s(t, x_1, \ldots, x_s),
$$

(1.5)

where $F(t) = (1, F_1(t, x_1), \ldots, F_s(t, x_1, \ldots, x_s), \ldots)$ is a solution of the initial value problem to the BBGKY hierarchy [1, 4, 13] with the initial data $F(0)$.

Let us outline the structure of the paper. In the second section we apply the procedure of the regularization suggested in [16] to a solution of the dual BBGKY hierarchy. Then, in Sections 3 and 4, we prove the existence of functionals (1.4) and (1.5) using results of [16]. In the final section we give some concluding remarks.
2 The regularized solution of the dual BBGKY hierarchy

Before proving the existence of functional (2.1) we cite preliminary facts about a solution of the dual BBGKY hierarchy and construct a regularized representation for the solution which in the next section allows us to compensate the divergent terms appearing in functional (2.1) for infinite-particle systems.

Let \((x_1, \ldots, x_s) = Y, (x_{j_1}, \ldots, x_{j_{s-n}}) = Y \setminus X, \{j_1, \ldots, j_{s-n}\} \subseteq \{1, \ldots, s\}\), i.e. \(X = (x_1, \ldots, x_{j_{s-n}}, x_s)\), where \((x_1, \ldots, x_{j_{s-n}}) \equiv (x_1, \ldots, x_{j_{s-n}}, x_{j_{s-n}+1}, \ldots, x_s)\). We denote by \(|X|\) a number of elements of the set \(X\), \(0 \leq |X| = n \leq s\).

We introduce the evolution operator \(S_\delta(t, Y), s \geq 1\), defined on the space \(C_\gamma\) of sequences of continuous functions by the following formula [4]

\[
S_{\gamma_Y}(t, Y) f_{\gamma_Y}(Y) = \begin{cases} 
0, & Y \in (\{0\} \times \mathbb{R}^{|Y|}) \setminus \mathcal{M}_0^{|Y|}, \\
\mathcal{M}_0^{|Y|} & Y \in (\mathbb{R}^{|Y|} \times \mathbb{R}^{|Y|}) \cup \mathcal{M}_0^{|Y|}, 
\end{cases}
\]

(2.1)

where \(X_j(t, Y), j = 1, \ldots, |Y|\), is the phase trajectory [4] of a system of \(|Y| = s\) particles with the initial data \(X_j(0, Y) = x_j\) and \(S_{\gamma_Y}(0) = I\) is a unit operator. We note that the phase trajectory of a system [4] with the interaction potential \(\Pi\) is defined not for all initial data \((x_1, \ldots, x_n) \in (\mathbb{R}^n \setminus W_n) \times \mathbb{R}^n\), it is defined almost everywhere on the phase space \((\mathbb{R}^n \setminus W_n) \times \mathbb{R}^n\), namely, exterior to a certain set \(\mathcal{M}_0^n\) of the Lebesgue measure zero (the set \(\mathcal{M}_0^n\) contains the initial data \((x_1, \ldots, x_n) \in (\mathbb{R}^n \setminus W_n) \times \mathbb{R}^n\) for which: i) triple and more order particle collisions occur at the instant \(t \in (-\infty, +\infty)\); ii) the infinite number of collisions occurs within a finite time interval [4,14]). The evolution operator (2.1) is well defined as \(t \in (-\infty, +\infty)\) [4] under the conditions \(\Pi\) on the interaction potential \(\Phi\).

For the initial data \(G(0) \in C_\gamma\) in case of \(\gamma < e^{-1}\) a solution of the initial value problem to the dual BBGKY hierarchy is represented by the formula [7,9]

\[
G_{\gamma_Y}(t, Y) = \sum_{n=0}^s \sum_{1=j_1<\ldots<j_{s-n}}^s \mathcal{A}_{X_{j_1}+1}(t, Y \setminus X, X) G_{\gamma_{Y \setminus X_i}}(0, Y \setminus X), \quad 1 \leq |X| = n \leq s,
\]

(2.2)

where the operator \(\mathcal{A}_{|X|+1}(t, Y \setminus X, X)\) is a cumulant of \(|X| + 1\)th order for the evolution operators \(S_{\gamma_Y}(t, Y_i)\) [2.2], \(|Y_i| \geq 1\), determined by the formula [7,9]

\[
\mathcal{A}_{|X|+1}(t, Y \setminus X, X) = \sum_{Y_i \subseteq \mathcal{P}} (-1)^{|P|-1} (|P| - 1)! \prod_{Y_i \subseteq \mathcal{P}} S_{\gamma_{Y_i}}(t, Y_i),
\]

(2.3)

where \(\sum_{Y_i \subseteq \mathcal{P}}\) is a sum over all possible partitions \(\mathcal{P}\) of the set \(Y = \{Y \setminus X, X\}\) into \(|P|\) nonempty mutually disjoint subsets \(Y_i \subset Y, Y_i \cap Y_j = \emptyset, i \neq j\), and the subset \(Y \setminus X\) is treated as one element, i.e. \(|Y| = |X| + 1 = n + 1\). Let us define \(\mathcal{A}_{|X|+1}(t, Y \setminus X, X) = 0\) as \(Y = X\). For example, the 1st and 2nd order cumulants have the form, correspondingly:

\[
\mathcal{A}_1(t, Y) = S_1(t, Y),
\]

\[
\mathcal{A}_2(t, Y \setminus x_s, x_s) = S_1(t, Y) - S_{s-1}(t, Y \setminus x_s) S_1(t, x_s),
\]

(2.3a)

(2.3b)

where \(Y \setminus x_s \equiv \{x_1 \cup \ldots \cup x_{s-1}\}\) is treated as one element.

We note the cumulant can be defined on the set with elements being finitely many-particle disjoint sets. For instance, the cumulant of 2nd order defined on the two-element set with two elements as disjoint sets \(Y \setminus X\) and \(Z\) such that \(Y \setminus X \cap Z = \emptyset\) has the form

\[
\mathcal{A}_2(t, Y \setminus X, Z) = S_{|Y \setminus X|+|Z|}(t, Y \setminus X, Z) - S_{|Y \setminus X|}(t, Y \setminus X) S_{|Z|}(t, Z).
\]

(2.4)

Lemma 2.1. For the cumulant of \((n+1)\)th order \(\mathcal{A}_{|X|+1}(t, Y \setminus X, X)\) (2.3) the following representation is true [7]:

\[
\mathcal{A}_{|X|+1}(t, Y \setminus X, X) = \sum_{Z \subset X \setminus \emptyset \setminus X_{|Y \setminus X|}} \mathcal{A}_2(t, Y \setminus X, Z) \sum_{Q \setminus X \setminus Z = \bigcup X_i} (-1)^{|Q|} |Q|! \prod_{X_i \subseteq Q} \mathcal{A}_1(t, X_i),
\]
where $\sum_{Z}$ is a sum over all nonempty subsets $Z$ of the set $X$, $\sum_{Q}$ is a sum over all partitions $Q$ of the set $X \setminus Z$ into $|Q|$ nonempty disjoint subsets $X_i \subset X \setminus Z$, $X_k \cap X_l = \emptyset$, $k \neq l$.

We apply Lemma 2.1 in constructing a new representation for the solution of the initial value problem to the dual BBGKY hierarchy.

With representation (2.5) for solution (2.2) in mind the function $G_{|Y \setminus X|}(0, Y \setminus X)$ proved to not depend on the set of variables $X_i \subset X \setminus Z$. Since $X_i \not\subset Y \setminus X$, then the cumulants of the 1st order $\mathfrak{A}_1(t, X_i)$ do not act on the variables of the function $G_{|Y \setminus X|}(0, Y \setminus X)$. Therefore an identical expression to (2.2) is as follows:

$$G_{|Y|}(t, Y) = \mathfrak{A}_1(t, Y) G_{|Y|}(0, Y) +$$
$$+ \sum_{n=1}^{s} \sum_{1=j_1<\ldots<j_n}^{s} \sum_{\substack{Z \subset X \setminus X_i \not= \emptyset}} \mathfrak{A}_2(t, Y \setminus X, Z) G_{|Y \setminus X|}(0, Y \setminus X) \sum_{Q: X \setminus Z = \bigcup_i X_i} (-1)^{|Q|} |Q|!,$$

where $1 \leq |X| = n \leq s$.

From last one, due to the equalities

$$\sum_{Q: X \setminus Z = \bigcup_i X_i} (-1)^{|Q|} |Q|! \equiv \sum_{k=1}^{|X \setminus Z|} (-1)^{k} s(|X \setminus Z|, k),$$

and

$$\sum_{k=1}^{m} (-1)^{k} s(m, k) = (-1)^m, \quad m \geq 1,$$

where $s(|X \setminus Z|, k) \equiv s(m, k)$ are Stirling numbers of the second kind, it follows that an equivalent representation to solution (2.2) of the initial value problem to the dual BBGKY hierarchy is an expansion (a regularized solution)

$$G_{|Y|}(t, Y) = \mathfrak{A}_1(t, Y) G_{|Y|}(0, Y) + \sum_{n=1}^{s} \sum_{1=j_1<\ldots<j_n}^{s} \sum_{\substack{Z \subset X \setminus X_i \not= \emptyset}} (-1)^{|X \setminus Z|} \mathfrak{A}_2(t, Y \setminus X, Z) G_{|Y \setminus X|}(0, Y \setminus X),$$

where

$$1 \leq |X| = n \leq |Y| = s. \quad (2.6)$$

Thus, for $G(0) \in C_{\gamma,0}$ cumulant representation (2.2) - (2.3) for a solution of the initial value problem to the BBGKY hierarchy is equivalent to regularized one (2.6) and its structure allows to compensate divergent terms appearing in functional (1.3) under $F(0) \in L_{\infty, \beta}$.

**Lemma 2.2.** If $G(0) \in C_{\gamma}$, then under condition $0 < \gamma < 1$ for expansion (2.6) the estimate

$$|G_{|Y|}(t, Y)| \leq 2 c^2 \|G(0)\| c_{\gamma} \frac{s!}{\gamma^s} \quad (2.7)$$

is valid.

**Proof.** Let $G(0) \in C_{\gamma}$. According to formulae (2.1) and (2.3), (2.5) the following inequalities hold:

$$|\mathfrak{A}_1(t, Y) G_{|Y|}(0, Y)| \leq \|G(0)\| c_{\gamma} \frac{|Y|!}{\gamma^{|Y|}} \quad (2.8)$$

and

$$|\mathfrak{A}_2(t, Y \setminus X, Z) G_{|Y \setminus X|}(0, Y \setminus X)| \leq 2 \|G(0)\| c_{\gamma} \frac{|Y \setminus X|!}{\gamma^{|Y \setminus X|}}. \quad (2.9)$$
From (2.8) and (2.9) for expansion (2.6) we get the following estimate

\[ |G(t_Y)(t, Y)| \leq 2 \|G(0)\| c_2 \sum_{n=0}^{\infty} \sum_{1=j_1 < \ldots < j_{n-1}}^{s} \frac{2^n (s-n)!}{\gamma^{s-n}}. \]  

(2.10)

Since \(0 < \gamma < 1\), \(\sum_{n=0}^{s} \frac{2^n}{n!} \leq e^2\), estimate (2.10) takes form (2.7) of Lemma 2.2.

\[ \square \]

3 The mean value of observables: the evolution of observables

We establish the existence of functional (1.4) in the case of the evolution of the observable determined by the regularized expansion (2.6).

**Proposition 3.1.** For a system of particles with a pair interaction potential \(\Phi\) satisfying (1.7) if \(F(0) \in L_{\xi, \beta}^\infty\), \(G(0) \in C_{\gamma,0}\) and observable \(G(t)\) is determined by expansions (2.6), then for

\[ \xi < \frac{\gamma}{e \max (C; 2C_1)} \sqrt{\frac{2\pi}{\beta'}} \quad \text{and} \quad 0 \leq t < t_0, \quad t_0 = \frac{1}{C_2} \left( -\tilde{C}_1 + \frac{\gamma}{2e\xi} \sqrt{\frac{2\pi}{\beta'}} \right), \]  

(3.1)

functional (1.4) is well-defined and the following estimate holds:

\[ \left| \langle G(t)|F(0) \rangle \right| \leq 2 e^C \|G(0)\| \|G(0)\| \|c_2 \sum_{s=0}^{\infty} \left( \frac{e C - \xi}{\gamma} \sqrt{\frac{2\pi}{\beta'}} \right)^s \sum_{n=0}^{\infty} \left( \frac{e C - \xi}{\gamma} \sqrt{\frac{2\pi}{\beta'}} \right)^n (\tilde{C}_1 + \tilde{C}_2 t)^n, \]  

(3.2)

where \(C = \max_{i=1, \ldots, j_{n-1}} |l_i(0)|\) and \(|l_i(0)|\) is a length of the interval \(l_i(0)\) from compact \(\Omega_{|Y \setminus X|}(0) = l_1(0) \times \ldots \times l_{i+1}(0)\) on which the function \(G_{s-n}(0)\) is supported, \(\tilde{C}_1 = \max(2R, 1)\), \(\tilde{C}_2 = \max(2(4b+1), \frac{2}{\beta'})\), \(\beta = \beta' + \beta''\), \(b \equiv \sup_{q \in [r,R]} |\Phi(q)| (\left\lfloor \frac{R}{\sigma} \right\rfloor)\) and \(\left\lfloor \frac{R}{\sigma} \right\rfloor\) is an integer part of the number \(\frac{R}{\sigma}\).

**Proof.** Let \(F(0) \in L_{\xi, \beta}^\infty\), \(G(0) \in C_{\gamma,0}\), then the observable \(G_{|Y \setminus X|}(0, Y \setminus X)\) as \(Y \setminus X = (x_{j_1}, \ldots, x_{j_{n-1}})\) is supported in the configuration space on the compact set which we denote by \(\Omega_{|Y \setminus X|}(0) = l_1(0) \times \ldots \times l_{i+1}(0)\), where \(l_i(0)\) segment with the length \(|l_i(0)|\) such that \(q_i \in l_i(0)\), \(i = 1, \ldots, j_{n-1}\). If \(n = 0\) the compact comes as \(\Omega_{|Y \setminus X|}(0) = l_1(0) \times \ldots \times l_Y(0)\) (see p. 4).

If \(G_2(t, x_1, \ldots, x_s) = G_{|Y \setminus X|}(t, Y, Y)\) is a solution of the dual BBGKY hierarchy [16] determined by expansion (2.6), then the expression for functional (1.4) has the form

\[ \langle G(t)|F(0) \rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \int_{(\mathbb{R}^s \setminus W_s) \times \mathbb{R}^s} dY \left( \mathfrak{A}_1(t, Y) G_{|Y|}(0, Y) + \right. \]  

\[ + \sum_{n=1}^{s} \sum_{1=j_1 < \ldots < j_{n-1}}^{s} \sum_{\mathcal{Z} \subset \mathbb{R}^s \setminus \emptyset} (-1)^{|\mathcal{Z}|} \mathfrak{A}_2(t, Y \setminus \mathcal{Z}, Z) G_{|Y \setminus \mathcal{Z}|}(0, Y \setminus \mathcal{Z}, X) F_{|Y|}(0, Y), \]  

(3.3)

1 \leq |X| = n \leq |Y| = s.

In function (3.3) the region \(\mathbb{R}^s \setminus W_s\) of integration with respect to the configuration variables is restricted to a domain within which the integrand is finite and nonzero. Since the 1st order cumulant \(\mathfrak{A}_1(t, Y)\) is the operator \(S_{|Y|}(t, Y)\) (2.1), therefore in expansion (3.3) the function \(\mathfrak{A}_1(t, Y) G_{|Y|}(0, Y) F_{|Y|}(0, Y)\) is integrated not over the whole configuration space, and only with respect to variables of such a compact \(\Omega_{|Y|}(0) \subset \mathbb{R}^s \setminus W_s\), that is shifted along the configuration trajectory with the finite volume \(V_{\Omega_{|Y|}}(t) = l_Y(t)^s\).

The expression

\[ \sum_{n=1}^{s} \sum_{1=j_1 < \ldots < j_{n-1}}^{s} \sum_{\mathcal{Z} \subset \mathbb{R}^s \setminus \emptyset} (-1)^{|\mathcal{Z}|} \mathfrak{A}_2(t, Y \setminus \mathcal{Z}, Z) G_{|Y \setminus \mathcal{Z}|}(0, Y \setminus \mathcal{Z}, X) F_{|Y|}(0, Y) \]
equals zero if within time interval $[0, t)$ particles with arbitrary initial data $X$ from $\mathbb{R}^n \setminus W_n$ don’t interact with particles with fixed initial data $Y \setminus X$, since in this case for the 2nd order cumulant $\mathfrak{A}_2(t, Y \setminus X, Z)$ (3.4) as $Z \subset X$, $Z \neq \emptyset$, the following property

$$\mathfrak{A}_2(t, Y \setminus X, Z) G_{|Y \setminus X|(0, Y \setminus X)} = 0$$

is true. Owing to the finiteness of potential $\Phi$ (1.1) within time interval $[0, t)$ interaction (elastic collision) can not occur between particles at a distance more than can be accomplished by these particles in total. By this means, in expansion (3.3) the region of integration over the configuration variables is bounded by an interaction region $\Omega_i |X|(t, \Omega_{|Y \setminus X|}(0))$ of the particles with arbitrary initial data $X$ the particles with fixed initial data $Y \setminus X \in \Omega_{|Y \setminus X|}(0)$, within time interval $[0, t]$ and has a finite volume $V_{\Omega_i |X|}(t)$. We note that for all $Z \subset X$ it is true $\Omega_{|Z|}(t) \subset \Omega_i |X|.(t)$. Indeed, since at the initial moment $t = 0$ an arbitrary $i$th particle with the phase coordinate from $Y \setminus X$, $i = j_1, \ldots, j_{|Y \setminus X|}$, has a fixed values of momentum $p_i$ and the localized configuration coordinate $q_i$, then at an arbitrary instant $\tau$, $\tau \in [0, t)$, momentum $p_i(\tau, Y \setminus X)$ is determined from the initial value problem to the Hamilton equations [4] for a system of the finite number $|Y \setminus X|$ of particles. Therefore within $[0, t)$ such $i$th particle accomplishes a distance $\int_0^t d\tau |p_i(\tau, Y \setminus X)|$. In total $|Y \setminus X|$ particles accomplish the following one

$$\sum_{i,j=1}^s \int_0^t d\tau |p_i(\tau, Y \setminus X)|,$$

and conditions (3.6) imply upper boundedness of the sum of momenta at the instant $\tau$:

$$2 \sum_{i=1}^s p_i(\tau) \leq p_i^2(\tau) + 1,$$

for an arbitrary value of momentum $p_i(\tau) \equiv p_i(\tau, Y)$ the law of conservation of energy

$$\sum_{i=1}^s \frac{p_i^2(\tau)}{2} + \sum_{i<j=1}^s \Phi(q_i - q_j) = \sum_{i=1}^s \frac{p_i^2(\tau)}{2} + \sum_{i<j=1}^s \Phi(q_i(\tau) - q_j(\tau))$$

and conditions (3.6) imply upper boundedness of the sum of momenta at the instant $\tau$:

$$2 \sum_{i=1}^s p_i(\tau) \leq \sum_{i=1}^s p_i^2 + (4b + 1) s,$$

where $b$ is determined by condition (3.3).
In consideration of (3.7), from (3.5) we derive
\[ |l_k(X)(t)| \leq |l_{kY\setminus X}(0)| + 2s(R + (4b + 1)t) + t \sum_{i=1}^{s} p_i^2. \tag{3.8} \]

Let us assume \( C = \max_{i=j_1,\ldots,j_{-n}} |l_i(0)|, C_1 \equiv 2R, C_2 \equiv 2(4b + 1) \) and \( b \) is determined by condition (3.6) then volume \( V_{\Omega, X}(t) \) of the interaction region \( \Omega_{X}(t, \Omega_{Y\setminus X}(0)) \) is finite, namely,
\[ V_{\Omega, X}(t) \leq (C + (C_1 + C_2t)s + t \sum_{i=1}^{s} p_i^2)^n C^{s-n}. \tag{3.9} \]

Thus equality (3.4) is satisfied in \( s \)th term of series (3.3) over integration with respect to the configuration variables located exterior to the interaction region \( \Omega_{X}(t, \Omega_{Y\setminus X}(0)) \). As to the structure \( \Omega_{XF}(0) = \Omega_{XF}(t, \Omega_{Y\setminus X}(0)) \) as \( |X| = n = 0 \).

Taking into account the cumulant property (3.2), by Lemma 2.2 and inequality (3.5) for functional \( \mathcal{K} \) the following estimate holds:
\[ \left| \langle G(t) | F(0) \rangle \right| \leq 2 \| F(0) \|_{L^\infty_{\mathcal{K}, \gamma}} \| G(0) \|_{C_n} \sum_{s=0}^{\infty} \frac{\xi_s}{\gamma^s} \int dp_1 \cdots dp_s \exp \left\{ -\beta \sum_{i=1}^{s} \frac{p_i^2}{2} \right\} \sum_{n=0}^{2s} \frac{2^n}{n!} \times \left( C + (C_1 + C_2t)s + t \sum_{i=1}^{s} p_i^2 \right)^n C^{s-n}. \tag{3.10} \]

In view of the relation
\[ \left( C + (C_1 + C_2t)s + t \sum_{i=1}^{s} p_i^2 \right)^n = \sum_{k=0}^{n} \frac{n! C^{k}}{k!} \sum_{r=0}^{n-k} \frac{s^r}{r!} (C_1 + C_2t)^r \frac{t^{n-k-r}}{(n-k-r)!} \left( \sum_{i=1}^{s} p_i^2 \right)^{n-k-r} \]
and the inequality
\[ \left( \sum_{i=1}^{s} p_i^2 \right)^{n-k-r} \exp \left\{ -\beta \sum_{i=1}^{s} \frac{p_i^2}{2} \right\} \leq (n-k-r)! \left( \frac{2}{\beta^r} \right)^{n-k-r} \tag{3.11} \]
calculating the integrals over momentum variables in expression (3.10) we get
\[ \left| \langle G(t) | F(0) \rangle \right| \leq 2 \| F(0) \|_{L^\infty_{\mathcal{K}, \gamma}} \| G(0) \|_{C_n} \sum_{s=0}^{\infty} \frac{\xi_s}{\gamma^s} \left( \frac{2^n}{\beta^r} \right)^{\frac{1}{2}} \times \sum_{n=0}^{\infty} \frac{C^n}{k!} \sum_{k=0}^{n} C_k^{k} \sum_{r=0}^{n-k} \frac{s^r}{r!} (C_1 + C_2t)^r \left( \frac{2t}{\beta^r} \right)^{n-k-r}. \tag{3.12} \]

where \( \beta = \beta' + \beta'' \).

We assume \( \tilde{C}_1 = \max(C_1, 1), \tilde{C}_2 = \max(C_2, \frac{2}{\beta'}) \). Then for arbitrary \( t > 0 \) the following inequalities
\[ \tilde{C}_1 + \tilde{C}_2t \geq 1 \quad \text{and} \quad (\tilde{C}_1 + \tilde{C}_2t) \frac{\beta'}{2t} \geq 1 \]
are satisfied. Because of these we have
\[ (C_1 + C_2t)^r \left( \frac{2t}{\beta'} \right)^{n-k-r} \leq (\tilde{C}_1 + \tilde{C}_2t)^n. \tag{3.13} \]

Considering (3.13) and inequalities
\[ \sum_{k=0}^{n} \frac{C_k^k}{k!} \leq e^C, \quad \sum_{r=0}^{n-k} \frac{s^r}{r!} \leq e^s \tag{3.14} \]
estimate (3.12) takes the form

\[ |\langle G(t)|F(0)\rangle| \leq 2e^{C} \|F(0)\|_{L_{\xi,\beta}^{\infty}} \|G(0)\|_{C_{\gamma}} \sum_{s=0}^{\infty} \left( \frac{eC \xi}{\gamma} \right)^{s} \left( \frac{2\pi}{\beta^{r}} \right)^{s} \sum_{n=0}^{s} \left( \frac{2}{C} \right)^{n} (\tilde{C}_{t} + \tilde{C}_{2t})^{s}. \]  

(3.15)

After inverting the order of summation in (3.15) we definitely obtain estimate (3.2).

Thus far, from inequality (3.2) it follows under condition (3.1) functional (3.3) for the mean value of observables is well-defined, with corresponding to functional (3.4) in the case of the regularized representation for a solution of the initial value problem to the dual BBGKY hierarchy.

\[ \text{Estimate (3.12) takes the form} \]

We note that regularized expansion (2.0) for initial observables of additive type \( G^{(1)}(0) = (0, a_1(0, x_1), 0, \ldots, 0, \ldots) \) has the form

\[ G^{(1)}(t) = (0, G^{(1)}_{1}(t, x_1), G^{(1)}_{2}(t, x_1, x_2), \ldots, G^{(1)}_{s}(t, x_1, \ldots, x_s), \ldots) \]

where

\[ G^{(1)}_{1}(t, x_1) = \mathcal{A}_{1}(t, x_1)a_1(0, x_1), \]

and

\[ G^{(1)}_{s}(t, x_1, \ldots, x_s) = \sum_{j=1}^{s} \sum_{Z \subset \mathcal{Y} \setminus x_j, Z \neq \emptyset} (-1)^{\mathcal{Y} \setminus Z} \mathcal{A}_{2}(t, x_j, Z) \ a_1(0, x_j), \quad s \geq 2. \]

(3.16b)

If \( F(0) \in L_{\xi,\beta}^{\infty}, G^{(1)}(0) \in C_{\gamma,0} \) and \( G^{(1)}_{s}(t, x_1, \ldots, x_s) \) is determined by formulae (3.16a), (3.16b) then functional (3.1) takes the form

\[ \langle G^{(1)}(t) | F(0) \rangle = \int_{\mathbb{R}^{1} \times \mathbb{R}^{1}} dx_1 \mathcal{A}_{1}(t, x_1) a_1(0, x_1) F_{s}(0, x_1, \ldots, x_s) + \]

\[ + \sum_{s=2}^{\infty} \frac{1}{s!} \int_{(\mathbb{R} \setminus W_{s}) \times \mathbb{R}^{s}} dx_1 \ldots dx_s \sum_{j=1}^{s} \sum_{Z \subset \mathcal{Y} \setminus x_j, Z \neq \emptyset} (-1)^{\mathcal{Y} \setminus Z} \mathcal{A}_{2}(t, x_j, Z) a_1(0, x_j) F_{s}(0, x_1, \ldots, x_s) \]

(3.17)

and the following estimate holds:

\[ \left| \langle G^{(1)}(0) | F(t) \rangle \right| \leq 2e^{C+1} \frac{C \xi}{\gamma} \sqrt{\frac{2\pi}{\beta^{r}}} \|F(0)\|_{L_{\xi,\beta}^{\infty}} \|G^{(1)}(0)\|_{C_{\gamma}} \sum_{s=0}^{\infty} \left( \frac{eC \xi}{\gamma} \right)^{s} \left( \frac{2\pi}{\beta^{r}} \right)^{s} (\tilde{C}_{t} + \tilde{C}_{2t})^{s}. \]  

(3.18)

Inequality (3.18) is similar to (3.15). Indeed, let \( F(0) \in L_{\xi,\beta}^{\infty} \) and \( G^{(1)}(0) \in C_{\gamma,0} \). For (3.17) we get

\[ \int_{\mathbb{R} \times \mathbb{R}} dx_1 |\mathcal{A}^{+}_{1}(t, x_1) a_1(0, x_1)| |F_{1}(0, x_1)| \leq \frac{\xi C(t)}{\gamma} \|F(0)\|_{L_{\xi,\beta}^{\infty}} \|G^{(1)}(0)\|_{C_{\gamma}}, \]

(3.19)

where \( C(t) \equiv |l_{1}(0)| \sqrt{\frac{2\pi}{\beta^{r}}} + \frac{2}{\beta^{r}} \), and

\[ \sum_{s=2}^{\infty} \frac{1}{s!} \int_{(\Omega_{1}(0) \times \Omega_{1}^{-1}(t)) \times \mathbb{R}^{s}} dx_1 \ldots dx_s \sum_{j=1}^{s} \sum_{Z \subset \{x_1, \ldots, x_s \} \setminus x_j, Z \neq \emptyset} |\mathcal{A}_{2}(t, x_j, Z) a_1(0, x_j)| |F_{s}(0, x_1, \ldots, x_s)| \leq \]

\[ \leq \frac{C \xi}{\gamma} e^{C+1} \sqrt{\frac{2\pi}{\beta^{r}}} \|F(0)\|_{L_{\xi,\beta}^{\infty}} \|G^{(1)}(0)\|_{C_{\gamma}} \sum_{s=0}^{\infty} \left( \frac{2eC \xi}{\gamma} \right)^{s} \left( \frac{2\pi}{\beta^{r}} \right)^{s} (\tilde{C}_{t} + \tilde{C}_{2t})^{s}, \]

(3.20)

where notations of estimate (3.12) are in service. According to (3.19) and (3.20) we obtain estimate (3.18).

It is notable that the existence of functionals for observables of s-fold class has also been considered for another representation of the solution to the dual BBGKY hierarchy formulated in [2].


4 The mean value of observables: the evolution of states

We establish the existence of functional (1.5), if the evolution of states is described by the regularized representation [16] for a solution of the initial value problem to the BBGKY hierarchy.

Let \((x_1, \ldots , x_s) \equiv Y, (y_{x+1}, \ldots , x_{s+n}) \equiv X, i.e. X \setminus Y = (x_{s+1}, \ldots , x_{s+n}), dx_{s+1} \ldots dx_{s+n} \equiv = d(X \setminus Y).\) We denote by \(|X|\) the number of elements of the set \(X, i.e. |X| = |Y| + |X \setminus Y| = s+n.\)

Let us consider functional (1.5) with the state \(F^t(x_1, \ldots , x_s) \equiv F_{\Omega Y}(t, Y)\) determined by a solution of the initial value problem to the BBGKY hierarchy (a regularized solution) [16]

\[
F_{\Omega Y}(t, Y) = \mathfrak{A}_1(-t, Y) F_{\Omega Y}(0, Y) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^N \setminus W_{n}) \times \mathbb{R}^n} d(X \setminus Y) \sum_{Z \subseteq X \setminus Y, Z \neq \emptyset} (-1)^{|X \setminus (Y \cup Z)|} \mathfrak{A}_2(-t, Y, Z) F_{|X|}(0, X), \quad |X \setminus Y| \geq 1, \quad (4.1)
\]

where \(\sum\) is a sum over all nonempty subsets \(Z\) of the set \(X \setminus Y\), the evolution operator \(\mathfrak{A}_1(-t, Y)\) is a 1st order cumulant of the evolution operators (2.1):

\[
\mathfrak{A}_1(-t, Y) = S_{\Omega Y}(-t, Y),
\]

the evolution operator \(\mathfrak{A}_2(-t, Y, Z)\) is a 2nd order cumulant:

\[
\mathfrak{A}_2(-t, Y, Z) = S_{\Omega Y \cup Z}(-t, Y, Z) - S_{\Omega Y}(-t, Y) S_{\Omega Z}(-t, Z).
\]

**Proposition 4.1.** For a system of particles with a pair interaction potential \(\Phi\) satisfying (1.1), if \(F(0) \in L_{\xi,\gamma}^\infty, G(0) \in C_{\gamma,0}\) and state is determined by expansion (4.1), then for

\[
\xi < \min \left( \frac{\gamma}{C}, \frac{2}{(2C_1 + 1)^2 - 1} \right) e^{-2\beta b - 1} \sqrt{\frac{\beta''}{2\pi}} \quad (4.2)
\]

and

\[
0 < t < t_0 \equiv \frac{1}{2C_2} \left( -2\tilde{C}_1 - 1 + \left( 1 + \frac{2e^{-2\beta b - 1}}{\xi} \sqrt{\frac{\beta''}{2\pi}} \right)^{\frac{1}{2}} \right), \quad (4.3)
\]

functional (1.5) is well-defined and the following estimate holds:

\[
\left| \langle G(0)|F(t) \rangle \right| \leq 2 e^C \|F(0)\|_{L_{\xi,\gamma}^\infty} \|G(0)\|_{C_{\gamma,0}} \sum_{n=0}^{\infty} \left( \frac{C_1 \xi e^{2\beta b + 1}}{\gamma} \sqrt{\frac{2\pi}{\beta''}} \right)^s \times \sum_{n=0}^{\infty} \left( 2 \xi e^{2\beta b + 1} (\tilde{C}_1 + \tilde{C}_2) (1 + \tilde{C}_1 + \tilde{C}_2) \sqrt{\frac{2\pi}{\beta''}} \right)^n, \quad (4.4)
\]

where \(C = \max_{t = 1, \ldots , n} |l_t(0)|, \tilde{C}_1 = \max(2R, 1), \tilde{C}_2 = \max(2(4b + 1), \frac{2}{\beta'})\) and \(\beta = \beta' + \beta''.\)

**Proof.** Let \(G(0) \in C_{\gamma,0}\) and \(F(t) \in L_{\xi,\gamma}^\infty\). We suppose at the initial instant \(t = 0\) data of the set \(Y\) are fixed in such a way that observable \(G_{\Omega Y}(0, Y)\) is located in the configuration space on the compact \(\Omega_{\Omega Y}(0) = l_1(0) \times \ldots \times l_Y(0)\) (see p. 4).

If the marginal distribution functions \(F_s(t, x_1, \ldots , x_s) \equiv F_{\Omega Y}(t, Y)\) in functional (1.5) are determined by expansion (4.1). Then functional (1.5) takes the form

\[
\langle G(0)|F(t) \rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \int_{(\mathbb{R}^N \setminus W_{s}) \times \mathbb{R}^n} dY \left( \mathfrak{A}_1(-t, Y) F_{\Omega Y}(0, Y) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^N \setminus W_{n}) \times \mathbb{R}^n} d(X \setminus Y) \sum_{Z \subseteq X \setminus Y, Z \neq \emptyset} (-1)^{|X \setminus (Y \cup Z)|} \mathfrak{A}_2(-t, Y, Z) F_{|X|}(0, X) \right) G_{\Omega Y}(0, Y). \quad (4.5)
\]
In expansion (4.5) we restrict the region \(\mathbb{R}^s \setminus W_n\) of integration with respect to the configuration variables since the integrand is finite and nonzero. Indeed, taking into account the 1st order cumulant \(\mathfrak{A}_1(-t, Y)\) is equivalent to operator \(S_1(t, Y)\) (2.4), therefore in expansion (4.5) the expression \(\mathfrak{A}_1(-t, Y) F_{Y|C}(0, Y) G_{Y|C}(0, Y)\) is integrated not over the whole configuration space, and only with respect to variables of the compact \(\Omega_{Y|C}(0)\), \(\Omega_{Y|C}(0) \subset \mathbb{R}^s \setminus W_n\), shifted along the configuration trajectory with a finite volume \(V_{\Omega_{Y|C}(0)} = l |Y|^s(0)\).

The expression
\[
\sum_{Z \subset X \setminus Y, Z \neq \emptyset} (-1)^{|X \setminus (Y \cup Z)|} \mathfrak{A}_2(-t, Y, Z) F_{|X|}(0, X)
\]
equals zero exterior to the region \(\Omega_{X \setminus Y}(t, \Omega_{Y|C}(0))\) of interaction of the particles with arbitrary initial data \(X \setminus Y\) with the particles with fixed initial data \(Y, Y \in \Omega_{Y|C}(0)\) within the time interval \([0, t]\) as a consequence of the equality
\[
\mathfrak{A}_2(-t, Y, Z) F_{|X|}(0, X) = 0, \quad Z \subset X \setminus Y, Z \neq \emptyset.
\]

The interaction region \(\Omega_{X \setminus Y}(t, \Omega_{Y|C}(0))\) has a finite volume \(V_{\Omega_{X \setminus Y}(t)}\) obeying the following estimate [16]
\[
V_{\Omega_{X \setminus Y}(t)}(t) \leq \left( C + (C_1 + C_2 t)(s + n) + t \sum_{i=1}^{s+n} p_i^2 \right)^n C^s, \tag{4.6}
\]
where \(C = \max_{i=1, \ldots, s} \left| l_i(0) \right|, C_1 \equiv 2R, C_2 \equiv 2(4b + 1)\) and \(b\) is determined by condition (3.6).

By this means, for (4.5) an inequality
\[
\left| \langle G(0) | F(t) \rangle \right| \leq 2 \|F(0)\|_{\mathcal{L}^\infty} \|G(0)\|_{\mathcal{C}} \sum_{s=0}^{\infty} \frac{\left( \xi e^{2\beta} \right)^s}{\gamma^s} \int_{\Omega_{Y|C}(0) \times \mathbb{R}^s} dY \exp \left\{- \beta \sum_{i=1}^{s} \frac{p_i^2}{2} \right\} \times \int_{\Omega_{X \setminus Y}(t) \times \mathbb{R}^n} d(X \setminus Y) \exp \left\{- \beta \sum_{i=s+1}^{s+n} \frac{p_i^2}{2} \right\} \tag{4.7}
\]
is satisfied. In (4.7) the following estimate
\[
\left( \mathfrak{A}_1(-t, Y) + \sum_{Z \subset X \setminus Y, Z \neq \emptyset} \mathfrak{A}_2(-t, Y, Z) \right) \exp \left\{- \beta \sum_{i=1}^{s+n} \frac{p_i^2}{2} \right\} \leq 2 |X \setminus Y| e^{2\beta |X|} \exp \left\{- \beta \sum_{i=s+1}^{s+n} \frac{p_i^2}{2} \right\}
\]
is taken into account. It is a consequence of invariance of Hamiltonian of \(n\)-particle system after acting of the evolution operator \(S_n(-t)\) (2.4), boundedness of the interaction region and conditions (3.6) on the interaction potential \(\Phi\) [4].

The expression for estimate (4.6) is represented in the following form
\[
\left( C + (C_1 + C_2 t) s + t \sum_{i=1}^{s} p_i^2 + (C_1 + C_2 t) n + t \sum_{i=s+1}^{s+n} p_i^2 \right)^n = \\
= \sum_{k=0}^{n} n! \sum_{l=0}^{\infty} \frac{C_l^k}{l!} \sum_{m=0}^{l-k-l-m} \frac{m!}{m!} (C_1 + C_2 t)^m \frac{t^{k-l-m}}{(k-l-m)!} \left( \sum_{i=1}^{s} p_i^2 \right)^{k-l-m} \times \\
\times \sum_{r=0}^{n-k} \frac{n!}{r!} (C_1 + C_2 t)^r \frac{t^{n-k-r}}{(n-k-r)!} \left( \sum_{i=s+1}^{s+n} p_i^2 \right)^{n-k-r}. \tag{4.8}
\]

After integration with respect to the configuration variables in each term of series on the right side of inequality (4.7), allowing for equality (4.8) and an inequality as (3.11) we calculate the integrals over momentum variables. As a result we obtain
\[ |\langle G(0)|F(t)\rangle| \leq 2 \|F(0)\|_{L^\infty_{\xi,\beta}} \|G(0)\|_{C_*} \sum_{s=0}^{\infty} \left( \frac{C\xi e^{2\beta b}}{\gamma} \right)^s \left( \frac{2\pi}{\beta'} \right)^{\frac{s}{2}} \sum_{n=0}^{\infty} (2\xi e^{2\beta b})^n \left( \frac{2\pi}{\beta'} \right)^{\frac{n}{2}} \times \]

\[ \times \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{C^l}{l!} \sum_{m=0}^{k-l} \frac{e^m}{m!} (C_1 + C_2 t)^m \left( \frac{2t}{\beta'} \right)^{k-l-m} \sum_{r=0}^{n-k} \frac{n^r}{r!} (C_1 + C_2 t)^r \left( \frac{2t}{\beta'} \right)^{n-k-r}, \quad (4.9) \]

where \( \beta = \beta' + \beta'' \).

Let us set the notation \( \tilde{C}_1 = \max(C_1, 1) \), \( \tilde{C}_2 = \max(C_2, \frac{2}{\beta'}) \). Applying inequalities (3.13), (3.14) and the relation

\[ \sum_{k=0}^{n} (\tilde{C}_1 + \tilde{C}_2 t)^k \leq (1 + \tilde{C}_1 + \tilde{C}_2 t)^n, \]

we see estimate (4.9) takes the form (4.4).

From (4.4) it follows that if \( \xi \) complies with condition (4.2), then expansion (4.5) converges under (4.3).

For an additive type of observables \( G^{(1)}(0) = (0, a_1(0, x_1), 0, \ldots, 0, \ldots) \) functional (4.5) becomes

\[ \langle G^{(1)}(t) \rangle = \langle G^{(1)}(0)|F(t)\rangle = \int_{\mathbb{R} \times \mathbb{R}} dx_1 a_1(0, x_1) F_1(t, x_1), \quad (4.10) \]

where \( F_1(t, x_1) \) is determined by formula (4.1) for \( |Y| = 1 \).

Functional (4.10) is well-defined in the case of

\[ \xi < \frac{2e^{-2\beta b - 1}}{(2C_1 + 1)^2 - 1} \sqrt{\frac{\beta''}{2\pi}} \quad \text{as} \quad 0 \leq t < t_0, \]

where \( t_0 \) is determined by (1.3).

5 Conclusion

We prove the local, in time, existence of functionals (1.4) and (1.5) in the cases of evolution of observables and evolution of states for the initial data \( F(0) \in L^\infty_{\xi,\beta} \) describing infinite-particle systems for the specific class of observables \( G(0) \in C_{\gamma,0} \) defined in Section 1.

The results obtained allow us to establish the existence of the mean values for classes of observables \( G(0) \in C_{\gamma,0} \) wider than we considered earlier (e.g., the local observables [1] the mean values of which are governed by the hydrodynamic equations).

Our results can also be applied in the case of the so-called intensive thermodynamic observables (e.g., the number of particles). For the corresponding mean values to be well-defined one must then introduce the densities of these observables. One can prove the existence of mean values of the intensive thermodynamic observables by using the results obtained above in the thermodynamic limit [4].

Applying the method of continuation of the BBGKY hierarchy solution developed in [4, 12], one can prove the global in time existence of functionals (1.4) and (1.5) for the initial data close to equilibrium.

Furthermore, the existence of functionals for the characteristics of a deviation from the mean values of observables (e.g., the dispersion) describing fluctuations in non-equilibrium statistical systems can be proved similarly to Propositions 3.1 and 4.1.

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