CHARACTER FORMULAS AND TAME REPRESENTATIONS FOR $\mathfrak{gl}(m|n)$

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Abstract. In 1994, Kac and Wakimoto suggested a generalization of Bernstein and Leites character formula for basic Lie superalgebras, and the natural question was raised: to which simple highest weight modules does it apply? They called modules that satisfy this character formula tame, and proved the formula in some special cases. In this paper, we prove a similar formula for a larger class of finite dimensional simple modules for the Lie superalgebra $\mathfrak{gl}(m|n)$.

1. Introduction

It has been long known that character formulas for simple finite dimensional representations of Lie superalgebras are a nontrivial extension of the classical case. The problem originates from the existence of the so called atypical roots. In the absence of these roots, Kac proved in 1997 that the Weyl character formula generalizes in a straightforward fashion $[K2, K3]$. In 1980, an elegant Weyl-type character formula was proven by Bernstein and Leites $[BL]$ for representations of atypicality 1 (see Section 2.4). Let $L(\lambda)$ be a finite dimensional simple representation of highest weight $\lambda$ and atypical root $\beta$, then

$$e^\rho R \cdot \text{ch} L(\lambda) = \sum_{w \in W} (-1)^{l(w)} w \left( \frac{e^{\lambda+\rho}}{1 + e^{-\beta}} \right).$$

Great efforts were made to generalize this formula to all finite dimensional modules of $\mathfrak{gl}(m|n)$. It was shown in $[VHKT]$ that such a formula does not hold in general but does hold for important families of modules, such as the covariant and contravariant modules. In $[KW1]$, Kac and Wakimoto stated a similar formula for the case when all of the atypical roots are simple, which was proven by the authors in $[CHR]$. Modules satisfying the Kac-Wakimoto character formula were called tame in $[KW1]$, however the term tame was used differently in $[KW2]$.

In $[S1, S2]$, Serganova proved an algorithmic character formula in terms of generalized Kazhdan-Lusztig polynomials. Brundan gave an explicit algorithm for computing these Kazhdan-Lusztig polynomials $[B]$ by using techniques from the theory of quantum groups. Using Brundan’s algorithm, Su and Zhang proved a closed formula that consists of an alternating sum of Bernstein-Leites characters. A new approach using super duality was pioneered by Cheng, Wang and Zhang in $[CWZ]$.

There are two classes of representations for which the Su-Zhang formula consists of one Bernstein-Leites term, namely the totally connected and the totally disconnected ones, (where the former contains the covariant and contravariant modules $[MV$, Corollary 3.5$]$). In this paper, we generalize these two classes to the class of piecewise disconnected modules (see Definition 16). Roughly speaking, these are the modules whose highest weight splits into components, each of which resembles a totally connected module while the relation between these components resembles a totally disconnected module.

Let $L(\lambda)$ be a piecewise disconnected module of highest weight $\lambda$ with respect to the standard choice of simple roots. We prove the following 1-term character formula for $L(\lambda)$:

$$e^\rho R \cdot \text{ch} L(\lambda) = \frac{(-1)^{l(\lambda_\rho)-\lambda_\rho|S_\lambda}}{t_\lambda} \sum_{w \in W} (-1)^{l(w)} w \left( \frac{e^{(\lambda_\rho)_\rho}}{\prod_{\beta \in S_\lambda} (1 + e^{-\beta})} \right).$$

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where $S_{\lambda}$ is a maximal orthogonal set of atypical roots; the weight $(\lambda^{\rho})^\parallel$ is obtained by adding certain atypical roots to $\lambda + \rho$; $|(\lambda^{\rho})^\parallel - \lambda^\parallel|_{S_{\lambda}}$ is the number of such roots added; and $t_{\lambda}$ is a positive integer determined by the lengths of the atypical components $\lambda$ (see Definitions [15, 23] and [24]).

Our proof uses Brundan’s algorithm [B] and is based on ideas from [SZ]. Unlike the totally connected and totally disconnected cases, for a general piecewise disconnected weight $\lambda$, the weight $(\lambda^{\rho})^\parallel$ appearing in formula (1.1) does not correspond to a highest weight vector for any choice of simple roots.

2. Preliminaries

2.1. The general linear Lie superalgebra. Let $\mathfrak{g}$ denote the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ over the complex field $\mathbb{C}$. As a vector space, $\mathfrak{g}$ can be identified with the endomorphism algebra $\text{End}(V_0 \oplus V_1)$ of a $\mathbb{Z}_2$-graded vector space $V_0 \oplus V_1$ with $\dim V_0 = m$ and $\dim V_1 = n$. Then $g = g_0 \oplus g_1$, where

\[ g_0 = \text{End}(V_0) \oplus \text{End}(V_1) \quad \text{and} \quad g_1 = \text{Hom}(V_0, V_1) \oplus \text{Hom}(V_1, V_0). \]

A homogeneous element $x \in g_0$ has degree 0, denoted $\text{deg}(x) = 0$, while $x \in g_1$ has degree 1, denoted $\text{deg}(x) = 1$. We define a bilinear operation on $\mathfrak{g}$ by letting

\[ [x, y] = xy - (-1)^{\text{deg}(x)\text{deg}(y)}yx \]

on homogeneous elements and then extending linearly to all of $\mathfrak{g}$.

By fixing a basis of $V_0$ and $V_1$, we can realize $\mathfrak{g}$ as the set of $(m + n) \times (m + n)$ matrices, where

\[ g_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in M_{m,m}, \ B \in M_{n,n} \right\} \quad \text{and} \quad g_1 = \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \mid C \in M_{m,n}, \ D \in M_{n,m} \right\}, \]

and $M_{r,s}$ denotes the set of $r \times s$ matrices.

2.2. Root space decomposition and choice of simple roots. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the set of diagonal matrices, and it has a natural basis

\[ \{E_{1,1}, \ldots, E_{m,m}; E_{m+1,m+1}, \ldots, E_{m+n,m+n}\}, \]

where $E_{ij}$ denotes the matrix whose $ij$-entry is 1 and all other entries are 0. Fix the dual basis $\{\varepsilon_1, \ldots, \varepsilon_m; \delta_1, \ldots, \delta_n\}$ for $\mathfrak{h}^*$. We define a bilinear form on $\mathfrak{h}^*$ by $(\varepsilon_i, \varepsilon_j) = \delta_{ij} = - (\delta_i, \delta_j)$ and $(\varepsilon_i, \delta_j) = 0$.

Then $\mathfrak{g}$ has a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_\alpha \right) \oplus \left( \bigoplus_{\alpha \in \Delta_1} \mathfrak{g}_\alpha \right)$, where the set of roots of $\mathfrak{g}$ is $\Delta = \Delta_0 \cup \Delta_1$, with

\[ \Delta_0 = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq m \} \cup \{ \delta_k - \delta_l \mid 1 \leq k \neq l \leq n \}, \]

\[ \Delta_1 = \{ \pm(\varepsilon_i - \delta_k) \mid 1 \leq i \leq m, \ 1 \leq k \leq n \}, \]

and $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}$, $\mathfrak{g}_{\delta_k - \delta_l} = \mathbb{C}E_{m+k,m+l}$, $\mathfrak{g}_{\varepsilon_i - \delta_k} = \mathbb{C}E_{i,m+k}$, $\mathfrak{g}_{\delta_k - \varepsilon_i} = \mathbb{C}E_{m+k,i}$.

The Weyl group of $\mathfrak{g}$ is $W = \text{Sym}(m) \times \text{Sym}(n)$, and $W$ acts on $\mathfrak{h}^*$ by permuting the indices of the $\varepsilon$’s and by permuting the indices of the $\delta$’s. In particular, the even reflection $s_{\varepsilon_i - \varepsilon_j}$ interchanges the $i$ and $j$ indices of the $\varepsilon$’s and fixes all other indices, while $s_{\delta_k - \delta_l}$ interchanges the $k$ and $l$ indices of the $\delta$’s and fixes all other indices.

A set of simple roots $\pi \subset \Delta$ determines a decomposition of $\Delta$ into positive and negative roots, $\Delta = \Delta^+ \cup \Delta^-$. There is a corresponding triangular decomposition of $\mathfrak{g}$ given by $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$. Let $\Delta^+_d = \Delta_d \cap \Delta^+$ for $d \in \{0, 1\}$. For the rest of the paper, we fix the standard choice of simple roots

\[ \pi = \{ \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n \}. \]

The corresponding decomposition $\Delta = \Delta^+ \cup \Delta^-$ is given by

\[ (2.1) \quad \Delta^+_0 = \{ \varepsilon_i - \varepsilon_j \}_{1 \leq i < j \leq m} \cup \{ \delta_k - \delta_l \}_{1 \leq k < l \leq n} \quad \text{and} \quad \Delta^+_1 = \{ \varepsilon_i - \delta_k \}_{1 \leq i \leq m, \ 1 \leq k \leq n}. \]
The standard choice of simple roots has the unique property that $W$ fixes $\Delta^+_1$. Moreover, it contains a basis for $\Delta^+_0$, which we denote by $\pi_0$.

Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+_1} \alpha - \frac{1}{2} \sum_{\alpha \in \Delta^+_0} \alpha$. Then for $\alpha \in \pi$, we have $(\rho, \alpha) = (\alpha, \alpha)/2$.

We define the root lattice as $Q = \sum_{\alpha \in \pi} \mathbb{Z}\alpha$ and the positive root lattice as $Q^+ = \sum_{\alpha \in \pi} \mathbb{N}\alpha$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. A partial order is defined on $h^*$ by $\mu > \nu$ when $\mu - \nu \in Q^+$.

2.3. Finite dimensional modules for $g = gl(m|n)$. For each weight $\lambda \in h^*$, the Verma module of highest weight $\lambda$ is the induced module

$$M(\lambda) := \text{Ind}_{n^+}^{\mathbb{C}^+} C_{h^*},$$

where $C_{h^*}$ is the one-dimensional module such that $h \in h^*$ acts by scalar multiplication of $\lambda(h)$ and $n^+$ acts trivially. The Verma module $M(\lambda)$ has a unique simple quotient, which we denote by $L(\lambda)$. Given $\lambda \in h^*$, we use the following abbreviation

$$\lambda^\rho := \lambda + \rho. $$

For each $\lambda \in h^*$, let $L_0(\lambda)$ denote the simple highest weight $g_0$-module with respect to $\pi_0$. The Kac module of highest weight $\lambda$ with respect to $\pi$ is the induced module

$$\overline{\mathcal{T}}(\lambda) := \text{Ind}_{g_0 \uparrow n^+}^{\mathcal{T}(\lambda)} L_0(\lambda)$$

defined by letting $n^+_1 := \oplus_{\alpha \in \Delta^+_1} g_0 \alpha$ act trivially on the $g_0$-module $L_0(\lambda)$. The unique simple quotient of $\overline{\mathcal{T}}(\lambda)$ is $L(\lambda)$.

Let $h^*_R = \sum_{\alpha \in \pi} \mathbb{R}\alpha$. A weight $\nu \in h^*_R$ is called integral (resp. dominant; strictly dominant) if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ (resp. $\langle \lambda, \alpha \rangle \geq 0$; $\langle \lambda, \alpha \rangle > 0$) for all $\alpha \in \Delta^+_0$, where $\langle \lambda, \alpha \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$.

For a proof of the following proposition see for example [M, 14.1.1].

**Proposition 1.** Let $g = gl(m|n)$ and $\lambda \in h^*$. Then, $L(\lambda)$ is a finite dimensional $g$-module iff $L_0(\lambda)$ is finite dimensional $g_0$-module iff the Kac module $\overline{\mathcal{T}}(\lambda)$ is finite dimensional $\lambda$ is a dominant integral weight iff $\lambda^\rho$ is a strictly dominant integral weight.

An element $\lambda \in h^*_R$ is called regular if $(\nu, \varepsilon_i) \neq (\nu, \varepsilon_j)$ and $(\nu, \delta_i) \neq (\nu, \delta_j)$ for all $i \neq j$. An element $\nu \in h^*_R$ is regular if and only if there exists $w \in W$ such that $w(\nu)$ is strictly dominant.

2.4. Atypical modules. Let $L(\lambda)$ be a finite dimensional $g$-module. We call $\beta \in \Delta_T$ atypical if $(\lambda^\rho, \beta) = (\beta, \beta) = 0$. The atypicality of $L(\lambda)$ is the maximal number of linearly independent roots $\beta_1, \ldots, \beta_r$ such that $(\beta_i, \beta_j) = 0$ and $(\lambda^\rho, \beta_i) = 0$ for $i, j = 1, \ldots, r$. Such a set $S_\lambda = \{\beta_1, \ldots, \beta_r\}$ is called a $\lambda^\rho$-maximal isotropic set, and we assume the elements of $S_\lambda$ are ordered so that $\beta_i = \varepsilon_{p_i} - \delta_{q_i}$ and $q_i < q_{i+1}$. As in [KW1], we denote the atypicality of $L(\lambda)$ by $\text{atp}(\lambda^\rho) = r$. The module $L(\lambda)$ is called typical if this set is empty, and atypical otherwise. For the standard choice of simple roots the set $S_\lambda$ is uniquely determined.

Let $P$ denote the set of integer weights, $P^+$ the set of dominant integral weights, and define

$$P^+ = \{\mu \in P^+ \mid (\mu^\rho, \varepsilon_i) \in \mathbb{Z}, (\mu^\rho, \delta_j) \in \mathbb{Z}\}.$$  

**Remark 2.** When studying the characters of simple finite dimensional atypical modules, we may restrict without loss of generality to the case that $\lambda \in P^+$. See Remark 8 in [CHR].

2.5. Weight diagrams. The weight diagrams studied in this paper were introduced by Brundan and Stroppel in [BS1]. They were used by Grusson and Serganova in [GS] to give algorithmic character formulas for basic classical Lie superalgebras.

Let $\lambda \in P^+$ and write

$$\lambda^\rho = \sum_{i=1}^{m} a_i \varepsilon_i - \sum_{j=1}^{n} b_j \delta_j.$$
On the \( \mathbb{Z} \)-lattice, put \( \times \) above \( t \) if \( t \in \{a_i\} \cap \{b_j\} \), put \( \ast \) above \( t \) if \( t \in \{a_i\} \setminus \{b_j\} \), and put \( < \) above \( t \) if \( t \in \{b_j\} \setminus \{a_i\} \). If \( t \notin \{a_i\} \cup \{b_j\} \), then we refer to the placeholder above \( t \) as an "empty spot."

Note that each \( \times \) corresponds to some atypical root \( \beta_i \). We number the \( \times \)'s left to right, which is consistent with the chosen ordering of \( S_\lambda \).

**Example 3.** If 

\[
\lambda^\rho = 10e_1 + 9e_2 + 8e_3 + 5e_4 + 4e_5 \in \mathfrak{h}^*,
\]

then the corresponding weight diagram \( \Delta_\lambda \) is

\[
(2.3) \quad \begin{array}{cccccccccccc}
-1 & 0 & < & 1 & 2 & 3 & \times & 4 & \times & 5 & \times & 6 & < & 7 & \times & 8 & \times & 9 & 10 & 11 & 12.
\end{array}
\]

2.6. **Characters and category** \( \mathcal{O} \). Let \( M \) be a module from the BGG category \( \mathcal{O} \) \([M] 8.2.3\). Then \( M \) has a weight space decomposition \( M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu \), where \( M_\mu = \{ x \in M \mid h.x = \mu(h)x \text{ for all } h \in \mathfrak{h}^* \} \), and the character of \( M \) is by definition \( \chi M = \sum_{\mu \in \mathfrak{h}^*} \dim M_\mu e^\mu \).

Denote by \( \mathcal{E} \) the algebra of rational functions \( \mathbb{Q}(e^\nu, \nu \in \mathfrak{h}^*) \). The group \( W \) acts on \( \mathcal{E} \) by mapping \( e^\nu \) to \( e^{w(\nu)} \). For \( \beta \in \Delta^+_1 \), we identify elements of the form \( \frac{1}{1+e^\beta} \) with their expansion as geometric series in the domain \( |e^{-\beta}| < 1 \). Since \( \Delta^+_1 \) is fixed by \( W \), expanding commutes with the action of \( W \).

The Weyl denominator of \( \mathfrak{g} \) is defined to be

\[
R = \frac{\prod_{\alpha \in \Delta^+_1} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta^+_1} (1 + e^{-\alpha})}.
\]

Then \( e^\rho R \) is \( W \)-skew-invariant, i.e., \( w(e^\rho R) = (-1)^{l(w)} e^\rho R \), and \( \chi L(\lambda) \) is \( W \)-invariant for \( \lambda \in \mathbb{P}^+ \). The character of a Verma module \( M(\lambda) \) with \( \lambda \in \mathfrak{h}^* \) is \( \chi M(\lambda) = e^{\lambda} R^{-1} \). The character of the Kac module \( \overline{T}(\lambda) \) with \( \lambda \in \mathbb{P}^+ \) is

\[
(2.4) \quad \chi \overline{T}(\lambda) = \frac{1}{e^\rho R} \sum_{w \in W} (-1)^{l(w)} w(e^{\lambda^\rho}).
\]

For \( X \in \mathcal{E} \), we define

\[
\mathcal{F}_W(X) := \sum_{w \in W} (-1)^{l(w)} w(X).
\]

**Lemma 4.** If \( \nu \in \mathfrak{h}^*_R \) is not regular, then \( \mathcal{F}_W(e^\nu) = 0 \).

**Proof.** If \( \nu \in \mathfrak{h}^*_R \) is not regular then \( \nu \) has a non-trivial stabilizer in \( W \). So the stabilizer of \( \nu \) in \( W \) must contain a reflection \( \sigma \) \([G] 4.1.1\). Then \( \mathcal{F}_W(e^\nu) = \mathcal{F}_W(e^{\sigma(\nu)}) = (-1)^{l(\sigma)} \mathcal{F}_W(e^\nu) = -\mathcal{F}_W(e^\nu) \).

2.7. **Character formulas and Kazhdan-Lusztig polynomials.** Serganova introduced the generalized Kazhdan-Lusztig polynomials \( K_{\lambda,\mu}(q) \) in \([S1]\) to give an algorithmic character formula for finite dimensional irreducible representations of \( \mathfrak{gl}(m|n) \). Brundan gave a new algorithm in \([B]\) for computing the generalized Kazhdan-Lusztig polynomials for \( \mathfrak{gl}(m|n) \) which can be described in terms of paths, (see Section 2.8).

**Theorem 5** (Serganova \([S1]\), Brundan \([B]\)). For each \( \lambda, \mu \in \mathbb{P}^+ \),

\[
\chi L(\lambda) = \sum_{\mu \in \mathfrak{h}^*} K_{\lambda,\mu} (-1) \chi \overline{T}(\mu).
\]

where

\[
K_{\lambda,\mu}(q) = \sum_{\theta \in P_{\lambda,\mu}} q^{l(\theta)}
\]

and \( P_{\lambda,\mu} \) is the set of paths from \( D_\mu \) to \( D_\lambda \) and \( l(\theta) \) denotes the length of the path \( \theta \).
2.8. Paths. We recall Brundan’s algorithm [3] to compute \( K_{\lambda,\mu}(q) \) using weight diagrams.

We define a right move map from the set of (labeled) weight diagrams to itself in two steps.

Definition 6. Let \( D_{\mu} \) be a weight diagram for \( \mu \in \mathbb{P}^+ \), and choose a labeling of the \( \times \)'s with indexing set \( \{1, \ldots, r\} \). Then for each \( \times \), starting with the rightmost \( \times \), “mark” the next empty spot to the right of it (which is unmarked). The right move \( R_i \) is then defined by moving \( \times_i \) to the empty spot it marked.

Definition 7. Let \( \lambda, \mu \in \mathbb{P}^+ \). Label the \( \times \)'s in the diagram \( D_{\mu} \) from left to right with \( 1, \ldots, r \). A right path from \( D_{\mu} \) to \( D_{\lambda} \) is a sequence of right moves \( \theta = R_{i_1} \circ \cdots \circ R_{i_k} \) where \( i_1 \leq \cdots \leq i_k \) and \( \theta(D_{\mu}) = D_{\lambda} \). The length of the path is \( l(\theta) := k \).

Define a partial order on \( P \) by \( \mu^0 \preceq \lambda^0 \) if and only if \( \lambda^0 \) and \( \mu^0 \) have the same typical entries, \( \text{atp}(\lambda^0) = \text{atp}(\mu^0) \) and the \( i \)-th atypical entry of \( \mu^0 \) is less than or equal to the \( i \)-th atypical entry of \( \lambda^0 \).

Remark 8. For each \( \mu, \lambda \in \mathbb{P}^+ \), there exists a path from \( D_{\mu} \) to \( D_{\lambda} \) if and only if \( \mu^0 \preceq \lambda^0 \) [3].

Let \( P_{\lambda,\mu} \) denote the set of paths from \( D_{\mu} \) to \( D_{\lambda} \). If \( P_{\lambda,\mu} \) is non-empty, it contains a unique longest path, which sends the \( i \)-th \( \times \) of \( \mu^0 \) to the location of the \( i \)-th \( \times \) of \( \lambda^0 \). We call this path the trivial path from \( D_{\mu} \) to \( D_{\lambda} \) and denote its length by \( l_{\lambda,\mu} \).

Lemma 9 (Brundan, [3] Lemma 3.42)). For all \( \lambda, \mu \in \mathbb{P}^+ \) and \( \theta \in P_{\lambda,\mu} \), \( l(\theta) = l_{\lambda,\mu} \) (mod 2).

The following is a corollary of Theorem 5 Lemma 9 and Equation (2.4).

Corollary 10. Let \( \lambda \in \mathbb{P}^+ \), and let \( P_{\lambda} = \{ \mu \in \mathbb{P}^+ \mid P_{\lambda,\mu} \text{ is non-empty} \} \). Then

\[
e^\rho R \cdot \text{ch} \ L_\pi(\lambda) = \sum_{\mu \in P_{\lambda}} d_{\lambda,\mu} \cdot (-1)^{l_{\lambda,\mu}} \frac{1}{\pi(W)} (e^{\mu^0})
\]

where \( d_{\lambda,\mu} \) is the number of paths from \( D_{\mu} \) to \( D_{\lambda} \).

3. PIECEWISE DISCONNECTED WEIGHTS

3.1. Piecewise disconnected weights. We will see that some simple highest weight modules have particularly nice character formulas. In this section we characterize their highest weights.

Definition 11. A weight \( \lambda \in \mathbb{P}^+ \) is called totally connected if in the weight diagram \( D_{\lambda} \) there are no empty spots between the \( \times \)'s.

Definition 12. A weight \( \lambda \in \mathbb{P}^+ \) is called piecewise disconnected if the diagram \( D_{\lambda} \) contains at least one empty spot between every two \( \times \)'s.

Remark 13. Definitions 11 and 12 are equivalent to those given in [SZ] Section 3.7.

Definition 14. Let \( \lambda \in \mathbb{P}^+ \). We call a nonempty contiguous subsection of the weight diagram \( D_{\lambda} \) an atypical component if it contains an \( \times \), but does not contain an empty spot and is maximal with this property. If \( \times_j \) and \( \times_k \) belong to the same atypical component then we write \( j \sim k \).

Definition 15. Let \( \lambda \in \mathbb{P}^+ \). Enumerate the atypical components of \( D_{\lambda} \) left to right \( T_1, \ldots, T_N \), and let \( t_i \) be the number of \( \times \)'s contained in \( T_i \) for \( i = 1, \ldots, N \). We define \( t_\lambda = t_1! t_2! \cdots t_N! \).

Definition 16. We call a weight \( \lambda \in \mathbb{P}^+ \) and the corresponding weight diagram \( D_{\lambda} \) piecewise disconnected if \( t_i \leq s_i \), where \( s_i \) is the number of empty spots between \( T_i \) and \( T_{i+1} \), for \( i = 1, \ldots, N - 1 \).

Remark 17. A totally connected weight \( \lambda \) is piecewise disconnected with \( N = 1 \) and \( t_\lambda = r! \). A totally disconnected weight \( \lambda \) is piecewise disconnected with \( N = r \) and \( t_\lambda = 1 \). Here \( r = \text{atp}(\lambda^0) \).

Example 18. The weight diagram \( D_{\lambda} \) in Example 3 is piecewise disconnected, but is neither totally connected nor totally disconnected. It has two atypical components, namely, \( T_1 = \{4,5,6\}, T_2 = \{8,9,10\} \), and \( t_1 = 1, t_2 = 2, s_1 = 1 \).
Example 19. If
\[ \lambda^0 = 7\varepsilon_1 + 5\varepsilon_2 + 4\varepsilon_3 - 4\delta_1 - 5\delta_2 - 7\delta_3 \]
then the corresponding weight diagram \( D_\lambda \) is not piecewise disconnected.

... \(-1\ 0\ 1\ 2\ \underleftarrow{x}\ \underleftarrow{x}\ 4\ 5\ 6\ \underleftarrow{x}\ 7\ ...\)

Remark 20. A weight \( \lambda \in \mathbb{P}^+ \) is totally connected if and only if for every \( \mu \in \mathbb{P}^+ \) the only possible path from \( D_\mu \) to \( D_\lambda \) is the trivial path, whereas it is totally disconnected if and only if there exists \( \mu \in \mathbb{P}^+ \) with \( r! \) paths from \( D_\mu \) to \( D_\lambda \), where \( r = \text{atp}(\lambda^0) \).

3.2. Definition of \((\lambda^0)^\dagger\). In this section, we define the integral weight \((\lambda^0)^\dagger\) which appears in the statement of the main theorem (Theorem 25).

Let \( \lambda \in \mathbb{P}^+ \) and write \( \lambda^0 \) as in (2.2). We refer to the coefficient \( a_i \) (resp. \( b_j \)) as the \( \varepsilon_i \)-entry (resp. \( \delta_j \)-entry). If \( \pm(\varepsilon_k - \delta_l) \in S_\lambda \), then we call the \( \varepsilon_k \) and \( \delta_l \) entries atypical. Otherwise, an entry is called typical.

Definition 21. If \( \lambda \in \mathbb{P}^+ \) is piecewise disconnected, we denote by \((\lambda^0)^\dagger\) the element obtained from \( \lambda^0 \) by replacing each atypical entry with the maximal atypical entry in the atypical component to which it belongs.

Remark 22. If \( \lambda \in \mathbb{P}^+ \) is totally disconnected then \((\lambda^0)^\dagger = \lambda^0 \), whereas if \( \lambda \in \mathbb{P}^+ \) is totally connected then all the atypical entries of \((\lambda^0)^\dagger\) equal the maximal atypical entry of \( \lambda^0 \).

Example 23. If \( \lambda^0 \) is as in Example 3 then
\[ (\lambda^0)^\dagger = 10\varepsilon_1 + 9\varepsilon_2 + 10\varepsilon_3 + 5\varepsilon_4 + 4\varepsilon_5 - \delta_1 - 4\delta_2 - 6\delta_3 - 10\delta_4 - 10\delta_5. \]

Definition 24. If \( \nu \in \mathfrak{h}^* \) can be written as \( \nu = \sum_{\alpha \in S_\lambda} k_\alpha \alpha \), then we define
\[ |\nu|_{S_\lambda} := \sum_{\alpha \in S_\lambda} k_\alpha. \]

Observe that \(|(\lambda^0)^\dagger - \lambda^0|_{S_\lambda} \) is non-negative integer.

4. MAIN THEOREM

The main theorem of this paper is as follows.

Theorem 25. Let \( \lambda \in \mathbb{P}^+ \) be a piecewise disconnected weight. Then
\[ e^\rho R \cdot \text{ch} \ L(\lambda) = \frac{(-1)^{|(\lambda^0)^\dagger - \lambda^0|_{S_\lambda}}}{t_\lambda} \sum_{w \in W} (-1)^{l(w)} w \left( \frac{e^{(\lambda^0)^\dagger}}{\prod_{\beta \in S_\lambda} (1 + e^{-\beta})} \right), \]
where \( t_\lambda = t_1! t_2! \cdots t_N! \) (see Definition 13) and \( S_\lambda \) is the (unique) \( \lambda^0 \)-maximal isotropic set of roots.

Equation (4.1) can be written in a form that is more consistent with the approach of [VHKT] Section 3.

Corollary 26. Let \( \lambda \in \mathbb{P}^+ \) be a piecewise disconnected weight. Then
\[ e^\rho R \cdot \text{ch} \ L(\lambda) = \frac{(-1)^{\sum_{i=1}^r k_i}}{t_\lambda} \sum_{w \in W} (-1)^{l(w)} w \left( \frac{e^{\lambda + \rho}}{\prod_{i=1}^r (1 + e^{-\beta_i}) e^{-k_i \beta_i}} \right), \]
where \( t_\lambda = t_1! t_2! \cdots t_N! \) and \( k_1, \ldots, k_r \in \mathbb{N} \) such that \( (\lambda^0)^\dagger - \lambda^0 = \sum_{i=1}^r k_i \beta_i \) for \( S_\lambda = \{ \beta_1, \ldots, \beta_r \} \).
4.1. A map from the set of paths to $\text{Sym}(r)$. In this section, we define for each $\lambda, \mu \in \mathbb{P}^+$, an injective map from the set of paths $P_{\lambda, \mu}$ to $\text{Sym}(r)$, where $r$ is the atypicality of $\lambda$, and describe the image of this map when $\lambda$ is piecewise disconnected. The image of such a map for general $\lambda$ was described by Su and Zhang in [SZ, Section 3.8].

For $\lambda, \mu \in \mathbb{P}^+$, number the $x$'s of $D_{\mu}$ left to right $x_1, \ldots, x_r$ and number the $\tilde{x}$'s of $D_{\lambda}$ left to right $\tilde{x}_1, \ldots, \tilde{x}_r$. Then a path $\theta \in P_{\lambda, \mu}$ determines uniquely an element of $\text{Sym}(r)$ given by the ordering

$$x_k \mapsto \tilde{x}_{\sigma(k)}.$$ 

In this way, we define the map $\Theta_{\lambda, \mu} : P_{\lambda, \mu} \to \text{Sym}(r)$. The map $\Theta_{\lambda, \mu}$ is injective, since a path is determined by this ordering. The image of the trivial path is the identity element of $\text{Sym}(r)$.

Example 27. Let $D_{\lambda}$ be as in Example 19 and let $D_{\mu}$ be

(4.3) $$\ldots -1 \ 0 \ \tilde{x}_1 \ \tilde{x}_2 \ \tilde{x}_3 \ 4 \ 5 \ 6 \ 7 \ldots.$$ 

There are two paths from $D_{\mu}$ to $D_{\lambda}$, namely, the trivial path and the path $R_1 R_1 R_2 R_2 R_2 R_3 R_3$ which can be computed as follows.

$$R_3 R_3 (D_{\mu}) = \ldots -1 \ 0 \ \tilde{x}_1 \ \tilde{x}_2 \ 3 \ 4 \ \tilde{x}_3 \ 6 \ 7 \ldots$$

$$R_2 R_2 R_2 R_3 R_3 (D_{\mu}) = \ldots -1 \ 0 \ \tilde{x}_1 \ 2 \ 3 \ 4 \ \tilde{x}_3 \ 6 \ \tilde{x}_2 \ 7 \ldots$$

$$D_{\lambda} = R_1 R_1 R_1 R_2 R_2 R_2 R_3 R_3 (D_{\mu}) = \ldots -1 \ 0 \ 1 \ 2 \ 3 \ \tilde{x}_1 \ \tilde{x}_3 \ 4 \ 5 \ 6 \ \tilde{x}_2 \ 7 \ldots.$$ 

The image of this non-trivial path under the map $\Theta_{\lambda, \mu}$ is the cycle (23). There are no other paths, because if the 4 and 5 positions were filled before the 7 position then the 7 position would be held, making the path impossible to complete.

For an element $\nu \in P$ with atp $(\nu) = r$ let $S_\nu = \{\varepsilon_{n_1} - \delta_{n_1}, \ldots, \varepsilon_{n_r} - \delta_{n_r}\}$ be such that $n_1 < \ldots < n_r$. We denote $\nu_i := (\nu, \sigma_{n_i})$. Then $x_k = (\mu^\nu)_k$ and that $\tilde{x}_k = (\lambda^\nu)_k$.

In the following lemma we describe the image of $\Theta_{\lambda, \mu}$ for an arbitrary piecewise disconnected weight.

Lemma 28. If $\lambda \in \mathbb{P}^+$ is piecewise disconnected, then

$$\text{Im} \ \Theta_{\lambda, \mu} = \{ \sigma \in \text{Sym}(r) \mid \sigma(\mu^\nu) \leq \lambda^\nu, \text{ and } \sigma^{-1}(j) < \sigma^{-1}(k) \text{ if } j < k \text{ and } j \sim k \},$$

where $j \sim k$ when $j$ and $k$ label $\tilde{x}$'s from the same atypical component of $\lambda$.

Proof. Let $\theta \in P_{\lambda, \mu}$. Since the $x$'s move in order from left to right to their respective destinations, we have that $x_k \leq \tilde{x}_{\sigma(k)}$. This ensures that $\sigma(\mu^\nu) \leq \lambda^\nu$. When an $x$ reaches its destination, it holds the next empty spot after it. Hence, the $x$'s must go in order into each atypical component so that every spot can be filled, that is, if $j < k$ and $j \sim k$ then $\sigma_{\tilde{x}}^{-1}(j) < \sigma_{\tilde{x}}^{-1}(k)$. Hence, we always have inclusion. When $\lambda$ is piecewise disconnected, these conditions on $\sigma \in \text{Sym}(r)$ are sufficient to define a path $\theta$ from $D_{\mu}$ to $D_{\lambda}$ which satisfies $x_k \mapsto \tilde{x}_{\sigma(k)}$. Indeed, the number of empty spots following the next is greater than or equal to the number of $x$'s in a given atypical component, so an $x$ does not hold an $\tilde{x}$ spot. \hfill \Box

Remark 29. If $\lambda$ is not piecewise disconnected then Lemma 28 does not hold. See [SZ, Section 3.8] for a description of the image in the general case.

In the following lemma we change the defining conditions of the set from Lemma 28 by replacing $\lambda^\nu$ with $(\lambda^\nu)^\triangleright$, and then we show that this does not change the set.

Lemma 30. If $\lambda \in \mathbb{P}^+$ is piecewise disconnected, then

(4.4) $$\text{Im} \ \Theta_{\lambda, \mu} = \{ \sigma \in \text{Sym}(r) \mid \sigma(\mu^\nu) \leq (\lambda^\nu)^\triangleright, \text{ and } \sigma^{-1}(j) < \sigma^{-1}(k) \text{ if } j < k \text{ and } j \sim k \}.$$
Proof. Let $A_{\lambda,\mu} = LHS$ and $B_{\lambda,\mu} = RHS$. By Lemma 28, $A_{\lambda,\mu} \subseteq B_{\lambda,\mu}$. Now suppose towards a contradiction that $\sigma \in B_{\lambda,\mu} \setminus A_{\lambda,\mu}$. Choose $s$ maximal such that $(\lambda^\rho)_{\sigma(s)} < (\mu^\rho)_s \leq (\lambda^\rho)^\circ_{\sigma(s)}$. By definition $(\lambda^\rho)^\circ_{\sigma(s)} = (\lambda^\rho)_k$, where $k$ is the index of the maximal atypical entry in the atypical component containing $(\lambda^\rho)_{\sigma(s)}$. Thus $(\mu^\rho)_s = (\lambda^\rho)_j$ for some $s < j < k$, since the atypical components of $\lambda^\rho$ are connected and $\mu^\rho$ is regular with the same typical entries as $\lambda^\rho$. Thus $s < \sigma^{-1}(j)$ since $\sigma(s) \sim j$. Then since $\mu^\rho$ is strictly dominant we have that $(\lambda^\rho)_j = (\mu^\rho)_s < (\lambda^\rho)^\circ_{\sigma^{-1}(j)}$. Note that we also have $(\mu^\rho)_{\sigma^{-1}(j)} \leq (\lambda^\rho)^\circ$ since $\sigma \in B_{\lambda,\mu}$. This contradicts the maximality of $s$, since $\sigma^{-1}(j)$ is larger and satisfies the required properties. Hence $A_{\lambda,\mu} = B_{\lambda,\mu}$. \hfill \Box

4.2. A bijection of indexing sets. In this section, we change the indexing set of the character formula in (2.5) from $P_\lambda$ to a particular subset of $(\lambda^\rho - NS_\lambda)$.

Fix $\lambda \in \mathbb{P}^+$. For each $\mu \in P_\lambda$, the $W$ orbit of $\mu^\rho$ intersects $(\lambda^\rho - NS_\lambda)$. We denote by $\overline{\mu}$ the unique maximal element of this intersection with respect to the standard order on $\overline{\mu}$. We define

$$C^{\text{Levi}}_{\lambda, \text{reg}} := \{\overline{\mu} \in \lambda^\rho - NS_\lambda \mid \mu \in P_\lambda\}.$$ 

Since $P_\lambda \subseteq \mathbb{P}^+$, this defines a bijection between the sets $P_\lambda$ and $C^{\text{Levi}}_{\lambda, \text{reg}}$. Recall that $S_\lambda = \{\beta_1, \ldots, \beta_r\}$ is ordered so that $\beta_i = \varepsilon_{p_i} - \delta_{q_i}$ and $q_i < q_{i+1}$. For $\nu \in (\lambda^\rho)^\circ - NS_\lambda$ and $i = 1, \ldots, r$, define

$$\nu_{\beta_i} = (\nu, \delta_{q_i}).$$

Lemma 31. One has

$$C^{\text{Levi}}_{\lambda, \text{reg}} = \{\nu \in \lambda^\rho - NS_\lambda \mid \nu_{\beta_1} < \nu_{\beta_2} < \ldots < \nu_{\beta_r} \text{ and } \nu \text{ is regular}\}.$$ 

Proof. Clearly we have $\subseteq$, since $\mu^\rho$ is strictly dominant. The reverse inclusion follows from Remark 8 since for regular $\nu \in \lambda^\rho - NS_\lambda$ and $w \in W$ with $w(\nu)$ strictly dominant, $w(\nu) \leq \lambda^\rho$ by definition. \hfill \Box

Definition 32. For $\overline{\mu} \in C^{\text{Levi}}_{\lambda, \text{reg}}$, define $\tilde{d}_{\lambda,\overline{\mu}}$ to be the number of paths from $D_\mu$ to $D_\lambda$, where $\mu$ is the unique dominant element in the $W$ orbit of $\overline{\mu}$.

The following lemma is proven using techniques from [SZ, Section 4.1].

Lemma 33. One has

$$e^R \cdot \text{ch} L(\lambda) = \sum_{\overline{\mu} \in C^{\text{Levi}}_{\lambda, \text{reg}}} \tilde{d}_{\lambda,\overline{\mu}} (-1)^{\lambda^\rho - \overline{\mu}} |S_\lambda| F_W (e^{\overline{\mu}}).$$

Proof. By Corollary 10 it suffices to show that for each $\mu \in P_\lambda$,

$$(-1)^{\lambda^\rho - \overline{\mu}} |S_\lambda| F_W (e^{\overline{\mu}}) = (-1)^{\lambda^\rho - \overline{\mu}} |S_\lambda| F_W (e^{\overline{\mu}}).$$

Let $w' \in W$ such that $w'(\mu^\rho) = \overline{\mu}$. To complete the proof it is sufficient to show that $|\lambda^\rho - \overline{\mu}|_{S_\lambda} = l_{\lambda,\mu} + l(w')$. The number $|\lambda^\rho - \overline{\mu}|_{S_\lambda}$ is the sum of the differences between the atypical entries of $\lambda^\rho$ and $\overline{\mu}$. This is equal to the number of moves in the trivial path $l_{\lambda,\mu}$ plus the number of spots being skipped. We will show that $l(w')$ is exactly the number of spots skipped in the trivial path.

The element $w' \in W$ for which $w'(\mu^\rho) = \overline{\mu}$ can be described explicitly in terms of the trivial path $\theta$. Denote $\theta = R_{j_1} \circ \cdots \circ R_{j_N}$, then $w' = w_1 \cdots w_N$ where each $w_j$ is defined as follows. Suppose that the move $R_{j_i}$ moved the $x$ at $n_j$ to an empty spot at $n_j + k_j + 1$, namely, it skipped over $k_j$ spots with $>$ and $<$, then $w_j = s_{j_1} \cdots s_{j_{k_j-1}}$ where $s_i$ is of the form $s_{\varepsilon_i - \varepsilon_{i+1}}$ if the $i$-th skip is over the $>$ of $\varepsilon_i$ and is of the form $s_{\delta_{j_i} - \delta_{j_{i+1}}}$ if it is over the $<$ of $\delta_j$. It is easy to see that the expression is reduced, so $l(w_j) = k_j$ is the number of spots skipped in the move $R_{j_i}$. Also $l(w') = \sum l(w_i)$, so $l(w')$ is exactly the number of spots skipped in the trivial path. \hfill \Box
4.3. Paths and permutations for piecewise disconnected weights. In this section, we show that if \( \lambda \in \mathbb{P}^+ \) is a piecewise disconnected weight, then for each \( \mu \in P_\lambda \) there exists a \( t_\lambda \) to 1 map from the set of paths from \( \mu \) to \( \lambda \) to a certain subset of the Weyl group. This is a crucial step in the proof of the main theorem.

Let \( W_r \) be the subgroup of \( W \) that permutes \( S_\lambda \). Then \( W_r \cong Sym(r) \) and is generated by elements of the form \( s_{\epsilon_1-\epsilon_j}s_{\delta_j-\delta_{\rho}} \) where \( \epsilon_1 - \delta_{\tau}, \epsilon_j - \delta_{\rho} \in S_\lambda \). So \( |W_r| = r! \) and all \( w \in W_r \) have positive sign.

Fix \( \lambda \in \mathbb{P}^+ \), and recall the notation of Section 3.1. We define a subgroup of \( W_r \) that preserves the atypical components of \( \lambda^\rho \), that is,

\[
W_r(t_\lambda) = \langle s_{\epsilon_1-\epsilon_j}s_{\delta_j-\delta_{\rho}} \mid i \sim j > .
\]

So \( w \in W_r(t_\lambda) \) and \( \lambda_\beta \in T_i \) imply that \( \lambda_\beta(w(\beta)) \in T_i \). Clearly,

\[
W_r(t_\lambda) \cong Sym(t_1) \times \cdots \times Sym(t_N)
\]

and hence \( W_r(t_\lambda) \) has cardinality \( t_\lambda \).

**Definition 34.** For each \( \nu \in C^{\text{Lexi}}_{\lambda, \text{reg}} \), let

\[
W_r(\lambda, \nu) := \left\{ w \in W_r \mid w(\nu) \in (\lambda^\rho)^\# - NS_\lambda \right\},
\]

and let \( c_{\lambda, \nu} = |W_r(\lambda, \nu)| \).

Then

\[
\mathcal{F}_W \left( \sum_{w \in W_r(\lambda, \nu)} e^{w(\nu)} \right) = c_{\lambda, \nu} \cdot \mathcal{F}_W (e^\nu).
\]

**Proposition 35.** Let \( \lambda \in \mathbb{P}^+ \) be a piecewise disconnected weight. Then for every \( \mu \in P_\lambda \), the number of paths from \( D_\mu \) to \( D_\lambda \) equals \( \frac{1}{t_\lambda} |W_r(\lambda, \mu)| \). Hence, for each \( \nu \in C^{\text{Lexi}}_{\lambda, \text{reg}} \), we have that \( \frac{d_{\lambda, \nu}}{c_{\lambda, \nu}} = \frac{1}{t_\lambda} \).

**Proof.** First, we observe that there is a natural bijection between the sets \( W_r(\lambda, \pi) \) and

\[
\tilde{B}_{\lambda, \mu} = \left\{ \sigma \in Sym(r) \mid \sigma(\mu^\rho) \preceq (\lambda^\rho)^\# \right\},
\]

since the bijective map \( P_\lambda \to C^{\text{Lexi}}_{\lambda, \text{reg}} \), defined by \( \mu^\rho \mapsto \pi \) preserves the relative order of the atypical roots. So we may in fact identify \( W_r(\lambda, \pi) \) with \( \tilde{B}_{\lambda, \mu} \) under this correspondence.

Now by Lemma 30 \( d_{\lambda, \mu} := |P_\lambda, \mu| \) equals the cardinality of the set in \( (\ref{eq:examples}) \), which we denote by \( B_{\lambda, \mu} \). We claim that there is a bijection of sets \( W_r(t_\lambda) \times B_{\lambda, \mu} \cong \tilde{B}_{\lambda, \mu} \) defined by \( (w, \sigma) \mapsto w\sigma \). Now by definition, \( (\lambda^\rho)^\# = (\lambda^\rho)^\# \) when \( \lambda_j^\rho \) belong to the same atypical component, that is, when \( j \sim k \).

Since \( W_r(t_\lambda) \) preserves each atypical component, the map is well-defined, that is, \( \sigma(\mu^\rho) \preceq (\lambda^\rho)^\# \) implies that \( w\sigma(\mu^\rho) \preceq (\lambda^\rho)^\# \) for any \( w \in W_r(t_\lambda) \).

If \( \sigma \in B_{\lambda, \mu} \), then the atypical entries of each atypical component of \( \sigma(\mu^\rho) \) are in increasing order and distinct, since \( \sigma \in B_{\lambda, \mu} \) satisfies: \( \sigma^{-1}(j) < \sigma^{-1}(k) \) when \( j < k \) and \( j \sim k \). It is not difficult to show that the map defined above is bijective. Indeed, given \( \sigma' \in \tilde{B}_{\lambda, \mu} \) there exists a unique \( w \in W_r(t_\lambda) \) such that the atypical entries of each atypical component of \( w^{-1}\sigma'(\mu^\rho) \) are in increasing order, that is, such that \( w^{-1}\sigma' \in B_{\lambda, \mu} \). Therefore, \( W_r(t_\lambda) \times B_{\lambda, \mu} \cong \tilde{B}_{\lambda, \mu} \) and \( t_\lambda \cdot d_{\lambda, \mu} = c_{\lambda, \mu} \).

**Example 36.** If \( \lambda \in \mathbb{P}^+ \) is not piecewise disconnected, then the ratio \( \frac{d_{\lambda, \mu}}{c_{\lambda, \nu}} \) is not necessarily constant. Consider the weight \( \lambda \) from Example 19. If \( \mu \) is the weight from Example 27 then \( d_{\lambda, \mu} = 2 \) and \( c_{\lambda, \mu} = 6 \), whereas, if \( \mu = \lambda \) then \( d_{\lambda, \mu} = 1 \) and \( c_{\lambda, \mu} = 2 \).
4.4. Enlarging the indexing set. In this section, we enlarge the indexing set $C_{\lambda, \text{reg}}^{\text{Lexi}}$ by adding non-regular elements, namely, we define

$$C_{\lambda}^{\text{Lexi}} = \left\{ \nu \in (\lambda^\triangledown)^\sharp - NS_\lambda \mid \nu_{\beta_1} < \nu_{\beta_2} < \ldots < \nu_{\beta_r} \right\}.$$  

Lemma 37. If $\nu \in C_{\lambda}^{\text{Lexi}} \setminus C_{\lambda, \text{reg}}^{\text{Lexi}}$, then $\nu$ is not regular.

Proof. Let $j$ be such that $\lambda^\triangledown_{\beta_j} < \nu_{\beta_j} \leq (\lambda^\triangledown)^\sharp_{\beta_j}$ and $\nu_{\beta_i} \leq \lambda^\triangledown_{\beta_i}$ for all $i > j$. By definition of $(\lambda^\triangledown)^\sharp$, all the integers between $\lambda^\triangledown_{\beta_j} + 1$ and $(\lambda^\triangledown)^\sharp_{\beta_j}$ are entries of $\lambda^\triangledown$. The typical entries of $\nu$ are the same as of $\lambda^\triangledown$ and there are $r - j + 1$ atypical entries which are strictly greater than $\lambda^\triangledown_{\beta_j}$. This implies that there must be equal entries of the same type, and hence $\nu$ is not regular. $\square$

Lemma 38. Let $c_\lambda = \left\{ w(\nu) \in (\lambda^\triangledown)^\sharp - NS_\lambda \mid w \in W_r, \nu \in C_{\lambda}^{\text{Lexi}} \right\}$ and

$$D_{\lambda} = \left\{ \nu \in (\lambda^\triangledown)^\sharp - NS_\lambda \mid \nu_{\beta_i} \neq \nu_{\beta_j} \text{ for all } i \neq j \right\}.$$  

Then $c_\lambda \subseteq D_{\lambda}$ as multisets, and hence elements of $((\lambda^\triangledown)^\sharp - NS_\lambda) \setminus c_\lambda$ are not regular.

Proof. Clearly we have $c_\lambda \subseteq D_{\lambda}$ as sets. Since there is a unique element in the $W_r$ orbit of any $\nu \in C_{\lambda}^{\text{Lexi}}$ that satisfies $\nu_{\beta_1} < \nu_{\beta_2} < \ldots < \nu_{\beta_r}$, the orbits of distinct elements from $C_{\lambda, \text{reg}}^{\text{Lexi}}$ do not intersect. Hence, we have an inclusion of multisets. For the reverse inclusion, suppose that $\nu \in D_{\lambda}$. Take $\sigma \in W_r$ such that $\sigma^{-1}(\nu)$ satisfies $\nu_{\beta_{\sigma(1)}} < \nu_{\beta_{\sigma(2)}} < \ldots < \nu_{\beta_{\sigma(r)}}$. Since $\nu_{\beta_{\sigma(i)}} \leq \max\{\nu_{\beta_1}, \ldots, \nu_{\beta_r}\} \leq (\lambda^\triangledown)^\sharp_{\beta_i}$ we have that $\sigma^{-1}(\nu) \in (\lambda^\triangledown)^\sharp - NS_\lambda$. Hence $\sigma^{-1}(\nu) \in C_{\lambda}^{\text{Lexi}}$ and $\nu = \sigma(\sigma^{-1}(\nu)) \in c_\lambda$. $\square$

4.5. Proof of the main theorem.

Proof of Theorem 25 By Lemma 33 we have that

$$e^R \cdot \text{ch} L(\lambda) = \sum_{\nu \in C_{\lambda, \text{reg}}^{\text{Lexi}}} \tilde{d}_{\lambda, \nu} \cdot (-1)^{|\lambda^\triangledown - \nu|_{S_\lambda}} F_W(e^\nu)$$

which by (4.6) equals

$$= (-1)^{|(\lambda^\triangledown)^\sharp - (\lambda^\triangledown)|_{S_\lambda}} \sum_{\nu \in C_{\lambda, \text{reg}}^{\text{Lexi}}} \frac{\tilde{d}_{\lambda, \nu}}{c_{\lambda, \nu}} (-1)^{|(\lambda^\triangledown)^\sharp - \nu|_{S_\lambda}} F_W\left( \sum_{w \in W_r(\lambda, \nu)} e^{w(\nu)} \right).$$

Then by Proposition 35 we have

$$= (-1)^{|(\lambda^\triangledown)^\sharp - (\lambda^\triangledown)|_{S_\lambda}} \sum_{\nu \in C_{\lambda, \text{reg}}^{\text{Lexi}}} \frac{1}{t_\lambda} (-1)^{|(\lambda^\triangledown)^\sharp - \nu|_{S_\lambda}} F_W\left( \sum_{w \in W_r(\lambda, \nu)} e^{w(\nu)} \right)$$

and so by Lemma 37 and Lemma 4 we have

$$= \frac{(-1)^{|(\lambda^\triangledown)^\sharp - (\lambda^\triangledown)|_{S_\lambda}}}{t_\lambda} \sum_{\nu \in C_{\lambda, \text{reg}}^{\text{Lexi}}} (-1)^{|(\lambda^\triangledown)^\sharp - \nu|_{S_\lambda}} F_W\left( \sum_{w \in W_r(\lambda, \nu)} e^{w(\nu)} \right).$$

Then Lemma 38 and Lemma 4 yields
\[
\frac{(-1)^{(|\lambda^0| - |\lambda^\rho|) t_\lambda}}{t_\lambda} \sum_{\nu \in (|\lambda^0| - NS_\lambda)} (-1)^{(|\lambda^0| - \nu) t_\lambda} F_{\nu} (e^{\nu})
\]
which can be rewritten as follows

\[
\frac{(-1)^{(|\lambda^0| - |\lambda^\rho|) t_\lambda}}{t_\lambda} F_{W} \left( \sum_{\nu \in (|\lambda^0| - NS_\lambda)} (-1)^{(|\lambda^0| - \nu) t_\lambda} e^{\nu} \right)
\]

\[
\frac{(-1)^{(|\lambda^0| - |\lambda^\rho|) t_\lambda}}{t_\lambda} F_{W} \left( \prod_{\beta \in S_\lambda} (1 + e^{-\beta}) \right).
\]

\[\square\]

REFERENCES

[BL] I.N. Bernstein, D.A. Leites, A formula for the characters of the irreducible finite-dimensional representations of Lie superalgebras of series \(\mathfrak{gl}\) and \(\mathfrak{sl}\), C. R. Acad. Bulgare Sci. 33 (1980) 1049–1051.

[B] J. Brundan, Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra \(\mathfrak{gl}(m|n)\), J. Amer. Math. Soc. 16 (2003), no. 1, 185–231.

[BS1] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra I: Cellularity, Mosc. Math. J. 11 (2011), no. 4, 685–722

[CHR] M. Chmutov, C. Hoyt, S. Reif, Kac-Wakimoto character formula for the general linear Lie superalgebra, http://arxiv.org/abs/1310.3798

[CWZ] S.-J. Cheng, W. Wang, R.B. Zhang, Super duality and Kazhdan-Lusztig polynomials, Trans. AMS 360 (2008), 5883–5924.

[G] M. Gorelik, Weyl denominator identity for finite-dimensional Lie superalgebras, Highlights in Lie algebraic methods. New York: Birkhäuser/Springer; Progr. Math. 295, (2012) 167–188.

[GS] C. Gruson, V. Serganova, Cohomology of generalized supergrassmannians and character formulae for basic classical Lie superalgebras, Proc. Lond. Math. Soc. (3) 101 (2010), no. 3, 852–892.

[K1] V.G. Kac, Lie superalgebras, Adv. Math. 26 (1977) 8–96.

[K2] V.G. Kac, Characters of typical representations of classical Lie superalgebras, Comm. Algebra 5 (1977) 889–897.

[K3] V.G. Kac, Representations of classical Lie superalgebras, Lecture Notes in Math., vol. 676, (1978) 597–626.

[KW1] V.G. Kac and M. Wakimoto, Integrable highest weight modules over affine superalgebras and number theory, Lie theory and geometry, edited by J.-L. Brylinski et al., Progr. Math. 123, Birkhäuser, Boston, MA, (1994) 415–458.

[KW2] V.G. Kac and M. Wakimoto, Representations of affine superalgebras and mock theta functions, arXiv:1308.1261

[M] I. Musson, Lie superalgebras and enveloping algebras, Graduate Studies in Mathematics, vol. 131, 2012.

[MS] I. Musson, V. Serganova, Combinatorics of Character Formulas for the Lie Superalgebra \(\mathfrak{gl}(m|n)\), Transform. Groups 16 (2011), 555–578.

[MV] E.M. Moens, J. Van der Jeugt, A character formula for atypical critical \(\mathfrak{gl}(m|n)\) representations labeled by composite partitions, J. Phys. A 37 (2004), no. 50, 12019–12039.

[S1] V. Serganova, Kazhdan-Lusztig polynomials and character formula for the Lie superalgebra \(\mathfrak{gl}(m|n)\), Selecta Math. (N.S.) 2 (1996), no. 4, 607–651.

[S2] V. Serganova, Characters of irreducible representations of simple Lie superalgebras, Proceedings of the International Congress of Mathematicians, vol. II, 1998, Berlin, Doc. Math., J. Deutsch. Math.-Verein. (1998) 583–593.

[SZ] Y. Su, R.B. Zhang, Character and dimension formulae for general linear superalgebra, Advances in Mathematics 211 (2007) 1–33.

[WHKT] J. Van der Jeugt, J.W.B. Hughes, R.C. King, J. Thierry-Mieg, Character formulas for irreducible modules of the Lie superalgebra \(\mathfrak{gl}(m|n)\), J. Math. Phys. 31 (1990) 2278–2304.

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