Adversarially Robust Learning of Real-Valued Functions

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Abstract

We study robustness to test-time adversarial attacks in the regression setting with $\ell_p$ losses and arbitrary perturbation sets. We address the question of which function classes are PAC learnable in this setting. We show that classes of finite fat-shattering dimension are learnable. Moreover, for convex function classes, they are even properly learnable. In contrast, some non-convex function classes provably require improper learning algorithms. We also discuss extensions to agnostic learning. Our main technique is based on a construction of an adversarially robust sample compression scheme of a size determined by the fat-shattering dimension.

1 Introduction

Learning a predictor that is resilient to test-time adversarial attacks is a fundamental problem in contemporary machine learning. A long line of research has studied the robustness of deep learning-based models (e.g., [43, 14, 22, 31]). From the theoretical standpoint, there has been a lot of effort to provide provable guarantees of such methods (e.g., [19, 42, 27, 45, 15, 7, 10, 38, 35, 36, 37, 6, 11, 16, 9, 13, 44, 8]), which is the focus of this work.

In the robust PAC learning framework, the problem of learning binary function classes was studied by Montasser et al. [34]. They showed that uniform convergence does not hold in this setting, and as a result, robust empirical risk minimization is not sufficient to ensure learnability. Yet, they showed that VC classes are learnable, by considering an improper learning rule (that is, the algorithm outputs a function that is not in the function class).

In this work, we aim to provide a theoretical understanding of the robustness of real-valued predictors in the PAC learning model, with an arbitrary set of perturbations. The work of Attias et al. [7] considered this question for a finite set of perturbations. They obtained sample complexity guarantees based on uniform convergence, which is no longer true for an arbitrary set of perturbations. We address the fundamental question, which real-valued function classes are robustly learnable? We also ask, are real-valued convex classes properly robustly learnable? On the one hand, some non-convex function classes provably require improper learning due to Montasser et al. [34]. On the other hand, Mendelson [33] showed that for non-robust regression with the mean squared error, proper learning is sufficient.

Before presenting our contributions, we formalize the learning model. Let $\mathcal{H} \subseteq [0, 1]^X$ be a hypothesis class. We formalize the adversarial attack by a perturbation function $\mathcal{U} : X \rightarrow 2^X$, where $U(x)$ is the set of possible perturbations (attacks) on $x$. In practice, we usually consider $\mathcal{U}(x)$ to be the $\ell_p$ ball centered at $x$. In this work, we have no restriction on $\mathcal{U}$, besides $x \in U(x)$. Let $D$ be an unknown distribution over the instance space $X$, and $f : X \rightarrow [0, 1]$ a target function that we wish to learn. We consider two loss functions. First, we define the $\ell_p$ robust loss of a function $h$ on $x$ by $\sup_{z \in U(x)} |h(z) - f(x)|^p$, where $1 \leq p < \infty$. Then, we define the $\eta$-ball robust loss by $\int \sup_{z \in U(x)} |h(z) - f(x)| \geq \eta$ for some predefined parameter $\eta > 0$. For example, this loss function was studied by Anthony et al. [5, 21,4], Anthony and Bartlett [3]. Learning with this loss function facilitates learning with the $\ell_p$ robust loss.

Our contributions. We denote the $\gamma$-fat-shattering dimension of $\mathcal{H}$ by $\text{fat}(\mathcal{H}, \gamma)$. Denote the dual $\gamma$-fat-shattering dimension by $\text{fat}^*(\mathcal{H}, \gamma)$, which is the dimension of the dual class. This dimension is finite as long as the $\gamma$-fat-shattering is finite (see [28], and Eq. (5)).

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• We start with a positive result regarding convex function classes, which are typical real-valued classes. In Section 3, we present a proper learning algorithm for these classes. This is in contrast to non-convex function classes, which are provably require improper learning algorithms, even in the binary-valued case [34]. For the $\eta$-ball robust loss, in the realizable setting, our algorithm requires a sample of size $\tilde{O}\left(\frac{\text{fat}(H, \eta)}{\epsilon^3} + \frac{1}{\epsilon^3} \log \frac{1}{\delta}\right)$.

This result holds for the $\ell_p$ robust loss for $\epsilon = \eta^{1/p}$.

• We present an improper learning algorithm with a substantial sample complexity improvement. In Section 4, we study the $\eta$-ball robust loss, in the realizable and agnostic settings. The sample complexity of our algorithm is

\[
\begin{align*}
\text{Realizable: } & \tilde{O}\left(\frac{\text{fat}(H, \eta)}{\epsilon} \frac{\text{fat}^*(H, \eta)}{\epsilon^2}\right), \\
\text{Agnostic: } & \tilde{O}\left(\frac{\text{fat}(H, \eta)}{\epsilon} \frac{\text{fat}^*(H, \eta)}{\epsilon^2}\right).
\end{align*}
\]

• In Section 5, we study the $\ell_p$ robust loss. In the realizable setting we show learnability with sample of size

\[
\tilde{O}\left(\frac{\text{fat}(H_\epsilon)}{\epsilon} \frac{\text{fat}^*(H_\epsilon)}{\epsilon^2}\right).
\]

For agnostic regression, we provide an algorithm with error of $\sqrt{\text{OPT}_H + \epsilon}$, where $\text{OPT}_H$ is the optimal error obtained by a function in $H$. Our algorithm requires a sample of size

\[
\tilde{O}\left(\frac{\text{fat}(H_\epsilon)}{\epsilon} \frac{\text{fat}^*(H_\epsilon)}{\epsilon^2}\right).
\]

Related work. Robust PAC learning. The setting of agnostic adversarially robust regression with finite perturbation sets was studied in [7]. Subsequently, improved bounds appeared in [29]. The setting of adversarially robust PAC learning of binary function classes, with an arbitrary perturbation sets, was studied in [34]. We study the more general setting of real-valued classes, and show that the behaviour of convex classes is different from non-convex classes in terms of proper learnability.

Boosting real-valued functions. Our main techniques are based on constructions of adversarially robust sample compression schemes of a size determined by the fat-shattering dimension of the function class. In the non-robust setting, Hanneke et al. [24] showed how to convert a boosting algorithm (originally introduced by Kégl [26]), into a sample compression scheme.

Learning from approximate interpolation. A closely related model of (non-robust) learning with real-valued functions, is learning from approximate interpolation. Learning with the $\eta$-ball loss was considered in [4], where they showed uniform convergence of classes with finite pseudo-dimension. A relaxed model was studied by [3]. For more details, see [5, Section 21.4].

2 Problem setup and preliminaries

Let $\mathcal{X}$ be the instance space, and $\mathcal{H} \subseteq [0, 1]^\mathcal{X}$ a hypothesis class. We implicitly assume that all hypothesis classes are satisfying mild measure-theoretic conditions (see e.g., [18, section 10.3.1] and [39, appendix C]). A perturbation function $U : \mathcal{X} \to 2^\mathcal{X}$ maps an input to a set $U(x) \subseteq \mathcal{X}$. Let $D$ be a distribution over $\mathcal{X}$ and $f : \mathcal{X} \to [0, 1]$ be the target function.

We consider the following loss functions. Define the $\eta$-ball robust loss function of $h$ with respect to $f$ on point $x$, and with respect to a perturbation function $U$, by

\[
\ell_U^\eta(h, f; x) = \mathbb{I}\left\{ \sup_{z \in U(x)} |h(z) - f(x)| \geq \eta \right\}.
\]

\[\tilde{O}(\cdot)\) stands for omitting poly-logarithmic factors of $(\text{fat}(H, \gamma), \text{fat}^*(H, \gamma), 1/\epsilon, 1/\delta, 1/\eta)$.}
The non-robust version of this loss function is also known as \( \eta \)-ball or \( \eta \)-tube loss (see for example \cite[Section 21.4]{5}). We define also the \( \ell_p \) robust loss function by

\[
\ell_{p,\ell}(h,f;x) = \sup_{z \in \mathcal{U}(x)} |h(z) - f(x)|^p.
\]

Denote the error of a function \( h \) with respect to the target function \( f \), loss function \( \ell \), and distribution \( \mathcal{D} \), by

\[
\text{Err}(h,f;\mathcal{D}) = \mathbb{E}_{x \sim \mathcal{D}}[\ell(h,f;x)].
\]

For notational simplicity we denote \( \text{Err}_\eta(\cdot) = \text{Err}_{\eta,\ell}(\cdot) \) and \( \text{Err}(\cdot) = \text{Err}_{\ell}(\cdot) \).

Let a sample \( S = \{(x_i, f(x_i))\}_{i=1}^m \), consists of \( m \) i.i.d. points drawn from \( \mathcal{D} \), and labeled by \( f \). Define an \( \eta \)-robust empirical minimizer, \( \eta\text{-RERM} : (\mathcal{X} \times \mathcal{Y})^m \to \mathcal{H} \), such that

\[
h \in \eta\text{-RERM}_{\mathcal{H}}(S) \Rightarrow \forall (x, f(x)) \in S : \sup_{z \in \mathcal{U}(x)} |h(z) - f(x)| \leq \eta.
\]

Note that having an approximate robust ERM with empirical robust error at most \( \eta/m \), is equivalent to \( \eta\text{-RERM}_{\mathcal{H}} \) optimizer in the aforementioned sense.

**Models.** We precisely define the models for robustly learn real-valued functions in the realizable and agnostic settings. We start with the \( \eta \)-ball robust loss (Eq. (1)), and refer to this model by robust \( \eta \)-regression.

We say that a distribution \( \mathcal{D} \) is \( \eta \)-realizable with respect to \( \mathcal{H}, \mathcal{U} \) and \( f \), if for all \( x \) in the support of \( \mathcal{D} \), it holds that

\[
\inf_{h \in \mathcal{H}} \left\{ \sup_{z \in \mathcal{U}(x)} |h(z) - f(x)| \geq \eta \right\} = 0.
\]

In short, we say the \( \mathcal{D} \) is realizable where \( \eta \) is clear from the context.

**Definition 2.1 (Robust \( \eta \)-regression)** For any \( \eta, \epsilon, \delta \in (0, 1) \), the sample complexity of realizable robust \( (\eta, \epsilon, \delta)\)-PAC learning for a class \( \mathcal{H} \subseteq [0, 1]^X \) and the \( \ell_p \) loss function, denoted by \( \mathcal{M}_{\text{RE}}(\eta, \epsilon, \delta, \mathcal{H}, \mathcal{U}, \ell_p) \), is the smallest integer \( m \) for which there exists a learning algorithm \( \mathcal{A} : (\mathcal{X} \times \mathcal{Y})^m \to [0, 1]^X \), such that for every distribution \( \mathcal{D} \) over \( \mathcal{X} \) that is \( \eta \)-realizable w.r.t. \( \mathcal{H}, \mathcal{U} \) and \( f \), for a random sample \( S \sim \mathcal{D}^m \), with probability at least \( 1 - \delta \) over \( S \), it holds that

\[
\text{Err}_{\eta}(\mathcal{A}(S), f; \mathcal{D}) = \mathbb{E}_{x \sim \mathcal{D}}\left[ \left\{ \sup_{z \in \mathcal{U}(x)} |\mathcal{A}(S)(z) - f(x)| \geq \eta \right\} \right] \leq \epsilon.
\]

If no such \( m \) exists, define \( \mathcal{M}_{\text{RE}}(\eta, \epsilon, \delta, \mathcal{H}, \mathcal{U}, \ell_p) = \infty \), and \( \mathcal{H} \) is not robustly \( (\eta, \epsilon, \delta)\)-PAC learnable w.r.t. the \( \ell_p \) loss function.

If the distribution \( \mathcal{D} \) is not \( \eta \)-realizable, the agnostic sample complexity, denoted by \( \mathcal{M}_{\text{AG}}(\eta, \epsilon, \delta, \mathcal{H}, \mathcal{U}, \ell_p) \), is defined similarly with the following difference. We require from the learning algorithm to output a function, such that with probability at least \( 1 - \delta \),

\[
\text{Err}_{\eta}(\mathcal{A}(S), f; \mathcal{D}) \leq \inf_{h \in \mathcal{H}} \text{Err}(h, f; \mathcal{D}) + \epsilon = \text{OPT}_{\mathcal{H}} + \epsilon.
\]

We denote \( \mathcal{M}_{\text{RE}}^\eta = \mathcal{M}_{\text{RE}}(\eta, \epsilon, \delta, \mathcal{H}, \mathcal{U}, \ell_p) \) and \( \mathcal{M}_{\text{AG}}^\eta = \mathcal{M}_{\text{AG}}(\eta, \epsilon, \delta, \mathcal{H}, \mathcal{U}, \ell_p) \) for notational simplicity.

Our second model is learning with the \( \ell_p \) robust loss (Eq. (2)), we refer to this model by robust regression.

**Definition 2.2 (Robust regression)** For any \( \epsilon, \delta \in (0, 1) \), the sample complexity of realizable robust \( (\epsilon, \delta)\)-PAC learning for a class \( \mathcal{H} \subseteq [0, 1]^X \) and the \( \ell_p \) loss function, denoted by \( \mathcal{M}_{\text{RE}}(\epsilon, \delta, \mathcal{H}, \mathcal{U}, \ell_p) \), is the smallest integer \( m \) for which there exists a learning algorithm \( \mathcal{A} : (\mathcal{X} \times \mathcal{Y})^m \to [0, 1]^X \), such that for every distribution \( \mathcal{D} \) over \( \mathcal{X} \) that is \( \epsilon \)-realizable w.r.t. \( \mathcal{H}, \mathcal{U} \) and \( f \), for a random sample \( S \sim \mathcal{D}^m \), with probability at least \( 1 - \delta \) over \( S \), it holds that

\[
\text{Err}(\mathcal{A}(S), f; \mathcal{D}) = \mathbb{E}_{x \sim \mathcal{D}}\left[ \sup_{z \in \mathcal{U}(x)} |\mathcal{A}(S)(z) - f(x)|^p \right] \leq \epsilon.
\]
If no such \( m \) exists, define \( \mathcal{M}_{\text{RE}}(\varepsilon, \delta, \mathcal{H}, \mathcal{U}, \ell_{\mathcal{PL}}) = \infty \), and \( \mathcal{H} \) is not robustly \((\varepsilon, \delta)\)-PAC learnable w.r.t. the \( \ell_{\mathcal{PL}} \) loss function.

If the distribution \( \mathcal{D} \) is not \( \varepsilon \)-realizable, the agnostic sample complexity, denoted by \( \mathcal{M}_{\text{AG}}(\varepsilon, \delta, \mathcal{H}, \mathcal{U}, \ell_{\mathcal{PL}}) \), is defined similarly with the following difference. We require from the learning algorithm to output a function, such that with probability at least \( 1 - \delta \),

\[
\text{Err}(A(S), f; \mathcal{D}) \leq \inf_{h \in \mathcal{H}} \text{Err}(h, f; \mathcal{D}) + \varepsilon = \text{OPT} + \varepsilon.
\]

We denote \( \mathcal{M}_{\text{RE}} = \mathcal{M}_{\text{RE}}(\varepsilon, \delta, \mathcal{H}, \mathcal{U}, \ell_{\mathcal{PL}}) \) and \( \mathcal{M}_{\text{AG}} = \mathcal{M}_{\text{AG}}(\varepsilon, \delta, \mathcal{H}, \mathcal{U}, \ell_{\mathcal{PL}}) \) for notational simplicity.

Note that there is a fundamental difference between the models. In the robust \( \eta \)-regression, we demand from the learning algorithm to find a function that is almost everywhere within \( \eta \) from the target function. That is, on \( 1 - \varepsilon \) mass of points in the support of \( \mathcal{D} \), we find an approximation up to \( \eta \). On the other hand, in the robust regression model, we aim to be close to the target function on average.

**Complexity measures.** **Fat-shattering dimension.** Let \( \mathcal{F} \subseteq [0, 1]^\mathcal{X} \) and \( \gamma > 0 \). We say that \( S = \{x_1, \ldots, x_m\} \subseteq \mathcal{X} \) is \( \gamma \)-shattered by \( \mathcal{F} \) if there exists a witness \( r = (r_1, \ldots, r_m) \in [0, 1]^m \) such that for each \( \sigma = (\sigma_1, \ldots, \sigma_m) \in \{-1, 1\}^m \) there is a function \( f_{\sigma} \in \mathcal{F} \) such that

\[
\forall i \in [m] \left\{ \begin{array}{ll}
f_{\sigma}(x_i) \geq r_i + \gamma, & \text{if } \sigma_i = 1 \\
f_{\sigma}(x_i) \leq r_i - \gamma, & \text{if } \sigma_i = -1. \end{array} \right.
\]

The fat-shattering dimension of \( \mathcal{F} \) at scale \( \gamma \), denoted by \( \text{fat}(\mathcal{F}, \gamma) \), is the cardinality of the largest set of points in \( \mathcal{X} \) that can be \( \gamma \)-shattered by \( \mathcal{F} \). This parametrized variant of the Pseudo-dimension [1] was first proposed by Kearns and Schapire [25]. Its key role in learning theory lies in characterizing the PAC learnability of real-valued function classes [1, 12].

**Dual fat-shattering dimension.** Define the dual class \( \mathcal{F}^* \subseteq [0, 1]^{\mathcal{H}} \) of \( \mathcal{F} \) as the set of all functions \( g_w : \mathcal{F} \rightarrow [0, 1] \) defined by \( g_w(f) = f(w) \). If we think of a function class as a matrix whose rows and columns are indexed by functions and points, respectively, then the dual class is given by the transpose of the matrix. The dual fat-shattering at scale \( \gamma \), is defined as the fat-shattering at scale \( \gamma \) of the dual class, and denoted by \( \text{fat}^*(\mathcal{F}, \gamma) \). We have the following bound due to Kleer and Simon [28, Corollary 3.8 and inequality 3.1],

\[
\text{fat}^*(\mathcal{F}, \gamma) \lesssim \frac{1}{\sqrt{\gamma}}^{\text{fat}(\mathcal{F}, \gamma/2) + 1}. \tag{5}
\]

**Covering numbers.** We say that \( \mathcal{G} \subseteq [0, 1]^{\mathcal{H}} \) is \( \varepsilon \)-cover for \( \mathcal{F} \subseteq [0, 1]^{\mathcal{H}} \) in \( d_\infty \) norm, if for any \( f \in \mathcal{F} \) the exists \( g \in \mathcal{G} \) such that for any \( x \in \mathcal{H} \), \( |f(x) - g(x)| \leq \varepsilon \). The \( \varepsilon \)-covering number of \( \mathcal{F} \) is the minimal cardinality of any \( \varepsilon \)-cover, and denoted by \( \mathcal{N}(\varepsilon, \mathcal{F}, d_\infty) \).

**Sample compression scheme.** We say that a pair of functions \((\kappa, \rho)\) is Uniformly \( \eta \)-approximate sample compression scheme of size \( k \) for class \( \mathcal{H} \subseteq [0, 1]^\mathcal{X} \), if for any \( m \in \mathbb{N}, h \in \mathcal{H} \), and sample \( S = \{(x_i, h(x_i))\}_{i=1}^m \), it holds for the compression function that \( \kappa(S) \subseteq S, |\kappa(S)| \leq k \), and the reconstruction function \( \rho(\kappa(S)) = \hat{h} \) satisfies

\[
\max_{1 \leq i \leq m} |\hat{h}(x_i) - h(x_i)| \leq \eta.
\]

Moreover, we say that \((\kappa, \rho)\) is \((\eta, \varepsilon)\)-approximate sample compression scheme, if

\[
\frac{1}{m} \sum_{i=1}^m \mathbb{I}\left[|\hat{h}(x_i) - h(x_i)| \geq \eta\right] \leq \varepsilon.
\]

**Notation.** We use the notation \( \tilde{O}(\cdot) \) for omitting poly-logarithmic factors of \( \text{fat}(\mathcal{H}, \gamma), \text{fat}^*(\mathcal{H}, \gamma), 1/\varepsilon, 1/\delta, 1/\eta \). We denote \([n] = \{1, \ldots, n\}\), and \( \exp(\cdot) = e^{\cdot} \). \( \lesssim \) and \( \gtrsim \) denote inequalities up to a constant factor, and \( \approx \) denotes equality up to a constant factor.

### 3 Convex function classes are properly learnable

In this section, we present an algorithm for the \( \eta \)-ball robust loss. In the case of convex function class, this algorithm is proper. Recall that uniform convergence does not necessarily hold. Instead, the proof is based on a construction of a sample compression scheme.
**Theorem 3.1** For any $\mathcal{H} \subseteq [0,1]^X$ with finite $\gamma$-fat-shattering for all $\gamma > 0$, any $U : X \rightarrow 2^X$, and any $\eta, \epsilon, \delta \in (0,1)$. Assuming $\eta$-realizability (Eq. (4)), there exists a numerical constant $c \in (0, \infty)$, such that Algorithm 1 ($\eta, \epsilon, \delta$)-PAC learns $\mathcal{H}$ in the robust $\eta$-regression setting with sample of size

$$
\mathcal{M}_{\text{RE}}^\eta = \tilde{O}\left(\frac{\text{fat}(\mathcal{H}, cn)}{\epsilon^4} + \frac{1}{\epsilon^3} \log \frac{1}{\delta}\right).
$$

Using Eq. (5), we conclude,

$$
\mathcal{M}_{\text{RE}}^\eta = \tilde{O}\left(\frac{\text{fat}(\mathcal{H}, cn)}{\eta^4} + \frac{1}{\eta^3} \log \frac{1}{\delta}\right).
$$

Moreover, Algorithm 1 is a proper learning rule for convex classes.

**Proof overview.** The complete proof is in Appendix B. We are given a labeled sample $S = \{(x_i, f(x_i))\}_{i=1}^m$, and access to $\eta$-robust empirical risk minimizer (see Eq. (3)). We follow the steps in Algorithm 1.

1. We start with inflating the training set by including all possible perturbations (whenever the same perturbation is mapped to more than one input, we assign the label of the the input with the smallest index). We call this set $S_U$.

2. Then, we would like to discretize the set in the following way. (a) Construct a set of functions $\hat{\mathcal{H}}$, such that each function is the output or $\eta/4$-robust empirical risk minimizer for $\mathcal{H}$, performing on $d_{\eta, \epsilon} = \tilde{O}(\text{fat}(\mathcal{H}, n/32)/\epsilon)$ points from the original sample $S$. The size of $\hat{\mathcal{H}}$ is bounded by $\lesssim (\frac{m}{d_{\eta, \epsilon}})^{d_{\eta, \epsilon}}$, up to log factors. (b) Define a discretization $\hat{S}_U \subseteq S_U$ as following. We look at the dual space, where each $(z, y) \in \hat{S}_U$ defines a function over $\hat{\mathcal{H}}$, $\hat{f}(z, y) : \hat{\mathcal{H}} \rightarrow [0,1]$ such that $f(z, y)(h) = |h(z) - y|$. We find a minimal $\eta/2$ covering for $\hat{S}_U$ in $d_\infty$ norm, which is of size $N(\eta/2, \hat{S}_U, d_\infty)$. Denote this cover by $\hat{S}_U$.

3. Finally, we execute a variant of Multiplicative Weights algorithm on $\hat{S}_U$ for $\approx \log |\hat{S}_U|$ rounds. In each round, we compute a predictor using $d_{\eta, \epsilon}$ samples. The average of these predictors has $\epsilon$ empirical loss on $\hat{S}_U$ for the $\eta/2$-ball loss. By the covering argument, this implies an empirical loss of $\epsilon$ on $\hat{S}_U$ for the $\eta$-ball loss. By the construction of $\hat{S}_U$, this also implies an empirical robust loss of $\epsilon$ on $S$ for the $\eta$-ball robust loss. That is,

$$
\frac{1}{m} \sum_{i=1}^m \sup_{z \in \hat{U}(x_i)} |\hat{h}(z) - f(x_i)| \geq \eta \leq \epsilon,
$$

where $\hat{h}(\cdot)$ is the output of our algorithm (the average of the sequence of predictors returned in step 3). Moreover, this sequence of predictors can be represented by an approximate sample compression scheme of size $\approx \text{fat}(\mathcal{H}, cn)\text{fat}(\mathcal{H}, \eta/64)/\epsilon^2$, for some numerical constant $c \in (0, \infty)$. The proof follows by applying a sample compression generalization bound. Note that for convex function classes, the average of such functions is in the class.

**Algorithm 1** Robust real-valued learner (high-level)

**Input:** Hypothesis class $\mathcal{H} \subseteq [0,1]^X$, labeled sample $S = \{(x_i, f(x_i))\}_{i=1}^m$, $x_i \sim D$.

**Parameters:** $\eta, \epsilon$.

**Algorithms used:** Approximate robust empirical risk minimizer $\eta$-RERM$_\mathcal{H}$ (Eq. (3)), a variant of Multiplicative Weights (Algorithm 4).

1. **Inflate** $S$ to $S_U$ to include all perturbed points.

2. **Discretize** $\hat{S}_U \subseteq S_U$:

   (a) Construct a function class $\hat{\mathcal{H}}$, where each $\hat{h} \in \hat{\mathcal{H}}$ defined by $\eta/4$-RERM$_\mathcal{H}$ optimizer on $\tilde{O}(\text{fat}(\mathcal{H}, \eta/32)/\epsilon)$ points from $S$.

   (b) Define $\hat{S}_U$ to be the minimal cover of $S_U$ under $d_\infty$ norm at scale $\eta/2$, which is of size $N(\eta/2, \hat{S}_U, d_\infty)$, where each $(z, y) \in \hat{S}_U$ defines a function over $\hat{\mathcal{H}}$, $\hat{f}(z, y) : \hat{\mathcal{H}} \rightarrow [0,1]$ such that $f(z, y)(h) = |h(z) - y|$.

3. **Execute** a variant of Multiplicative Weights algorithm on $\hat{S}_U$.

**Output:** Approximate sample compression scheme with $\epsilon$ empirical robust error on $S$. 
2.1 Remark. We show in the next sections that results in the realizable case can be translated to the agnostic case, and also to the robust regression setting. In the next sections we present our results given the improved improper learner described in the next section.

4 Realizable and agnostic robust \( \eta \)-regression

In this section, we study robust \( \eta \)-regression (see Definition 2.1) in the realizable and agnostic settings. We propose an algorithm with a substantial sample complexity improvement. The main difference from Algorithm 1, is that each base predictor is a weak learner, in the sense of learning real-valued functions with the \( \eta \)-ball loss function (see Definition 4.2 below). These weak learners are chosen carefully by a modified variant of a boosting algorithm for real-valued learners, suggested by Kégl [26], which is called MedBoost. The output of this algorithm is a median of the chosen weak learners, which is an improper rule for convex classes. Hanneke et al. [24] showed how to convert the boosting algorithm MedBoost into a sample compression scheme that is independent of the accuracy parameter \( \epsilon \). We first present our result for the realizable case.

Theorem 4.1 For any \( \mathcal{H} \subseteq [0, 1]^X \) with finite \( \gamma \)-fat-shattering for all \( \gamma > 0 \), any \( \mathcal{U} : \mathcal{X} \rightarrow 2^X \), and any \( \eta, \epsilon, \delta \in (0, 1) \). Assuming \( \eta \)-realizability (Eq. (4)), there exists a numerical constant \( c \in (0, \infty) \), such that Algorithm 2 \((\eta, \epsilon, \delta)\)-PAC learns \( \mathcal{H} \) in the robust \( \eta \)-regression setting with sample of size

\[
\mathcal{M}_\text{RE}^\eta = \tilde{O}\left(\frac{\text{fat} (\mathcal{H}, c\eta)}{\epsilon} + \frac{1}{\epsilon} \log \frac{1}{\delta}\right).
\]

Using Eq. (5), we conclude,

\[
\mathcal{M}_\text{RE}^\eta = \tilde{O}\left(\frac{\text{fat} (\mathcal{H}, c\eta) \cdot \text{fat}^\gamma (\mathcal{H}, c\eta)}{\eta \epsilon} + \frac{1}{\epsilon} \log \frac{1}{\delta}\right).
\]

The following definition is central in our proof.

Definition 4.2 (Weak real-valued learner) Let \( \eta \in [0, 1], \beta \in [0, \frac{1}{2}] \). We say that \( h : \mathcal{X} \rightarrow [0, 1] \) is \((\eta, \beta)\)-weak learner with respect to distribution \( \mathcal{D} \) and target hypothesis \( f \in \mathcal{H} \subseteq [0, 1]^X \), if

\[
P_{x \sim \mathcal{D}}(\lvert h(x) - f(x) \rvert > \eta) < \frac{1}{2} - \beta.
\]

This notion of a weak learner must be formulated carefully. For example, taking a learner guaranteeing absolute loss \( \frac{1}{2} - \beta \) is known to not be strong enough for boosting to work. On the other hand, by making the requirement too strong (for example, AdaBoost.R in [21]), then the sample complexity of weak learning will be high that weak learners cannot be expected to exist for certain function classes. We can now present the overview of the proof.

Proof overview. The complete proof is in Appendix C. We are given a labeled sample \( S = \{(x_i, f(x_i))\}_{i=1}^m \), and access to \( \eta \)-robust empirical risk minimizer (see Eq. (3)). We follow the steps in Algorithm 2.

1) Inflating the training set. This is identical to Algorithm 1.

2) The discretization is similar to Algorithm 1 with a crucial difference. In step (a), we construct a set of functions \( \mathcal{H} \), such that each function is the output of \( \eta/8 \)-robust empirical risk minimizer for \( \mathcal{H} \) performing on \( d= \tilde{O}(\text{fat}(\mathcal{H}, \eta/64)) \) points from the original sample \( S \), which is independent of \( \epsilon \). Finding a minimal cover \( \tilde{S}_\eta \) in step (b) is the same.

3) We execute a modified variant of the real-valued boosting algorithm MedBoost, on the set \( \tilde{S}_\eta \). The output of the algorithm is a uniformly \( \eta/4 \)-approximate sample compression scheme for the set \( \tilde{S}_\eta \), when having \( \approx \log (\lvert \tilde{S}_\eta \rvert) \) boosting rounds [24, Corollary 6]. In the full proof, we carefully explain the existence of a weak learner for every distribution over \( \tilde{S}_\eta \).

4) Finally, we sparsify the the sequence of predictors returned by the modified MedBoost, using a method suggested by Hanneke et al. [24]. The idea is that we can sample functions from the ensemble, and with high probability we are guaranteed that roughly \( \text{fat}^\gamma (\mathcal{H}, c\eta) \) predictors are sufficient (\( c \in (0, \infty) \) is a numerical constant), due to a uniform convergence property in the dual space. This step returns (with high probability) a sequence predictors that is a uniformly \( \eta/2 \)-approximate sample compression.
scheme for $S_d$. By the covering argument, this implies a uniformly $\eta$-approximate sample compression scheme for $S_d$, and finally, a robust uniformly $\eta$-approximate sample compression scheme for $S$, that is,

$$\forall (x, f(x)) \in S : \sup_{z \in \tilde{U}(x)} \left| \hat{h}(z) - f(x) \right| \leq \eta,$$

where $\hat{h}(\cdot)$ is output of our algorithm. The size of sample compression scheme is roughly $\approx \fat(H, \eta/64) \fat^*(H, c\eta)$, for some numerical constant $c \in (0, \infty)$. Note that the sample compression size in independent of $\epsilon$. The proof follows by applying a sample compression generalization bound.

**Algorithm 2** Improper robust real-valued learner with median boosting (high-level)

**Input:** Hypothesis class $H \subseteq [0, 1]^X$, labeled sample $S = \{(x_i, f(x_i))\}_{i=1}^m$, $x_i \sim D$.

**Parameters:** $\eta$.

**Algorithms used:** Approximate robust empirical risk minimizer $\eta$-RERM$_H$ (Eq. (3)), a variant of MedBoost (Algorithm 5), sparsification method (Algorithm 6).

1. **Inflate** $S$ to $S_d$ to include all perturbed points.
2. **Discretize** $\tilde{S}_d \subseteq S_d$:
   (a) **Construct** a function class $\tilde{H}$, where each $\hat{h} \in \tilde{H}$ defined by $\eta/8$-RERM optimizer on $\tilde{O}(\fat(H, \eta/64))$ points from $S$.
   (b) **Define** $\tilde{S}_d$ to be the minimal cover of $S_d$ under $d_\infty$ norm at scale $\eta/4$, which is of order $N(\eta/4, S_d, d_\infty)$. Each $(z, y) \in S_d$ defines a function over $\tilde{H}$, $f_{(z, y)} : \tilde{H} \rightarrow [0, 1]$ such that $f_{(z, y)}(\hat{h}) = |\hat{h}(z) - y|$.
3. **Execute** modified MedBoost algorithm on $\tilde{S}_d$, where $\tilde{H}$ consists of weak learners for any distribution over $\tilde{S}_d$.
4. **Sparsify** the sequence of predictors returned by step 3.

**Output:** Uniformly $\eta$-approximate sample compression scheme for $S$, for the $\eta$-ball robust loss.

### 4.1 Agnostic setting

We establish an upper bound on the sample complexity of the agnostic setting, by using a reduction to the realizable case. The main argument was originally suggested in [17]. The full proof is in Appendix C.

**Theorem 4.3** For any $H \subseteq [0, 1]^X$ with finite $\gamma$-fat-shattering for all $\gamma > 0$, any $U : X \rightarrow 2^X$, and any $\eta, \epsilon, \delta \in (0, 1)$, for some numerical constant $c \in (0, \infty)$, the sample complexity of agnostic robust $\eta$-regression is

$$\mathcal{M}_\text{AG}^\eta = \tilde{O}\left( \frac{\fat(H, c\eta) \fat^*(H, c\eta)}{\epsilon^2} + \frac{1}{\epsilon^2} \log \frac{1}{\delta} \right).$$

Using Eq. (5), we conclude

$$\mathcal{M}_\text{AG}^\eta = \tilde{O}\left( \frac{\fat(H, c\eta) \fat^*(H, c\eta)}{\eta \epsilon^2} + \frac{1}{\epsilon^2} \log \frac{1}{\delta} \right).$$

**Proof overview.** We reduce the problem to the realizable case. Denote by $\mathcal{M}_\text{RE}^\eta$ the sample complexity of $(\eta, 1/3, 1/3)$-PAC learning for $H$, with respect to a perturbation function $U$, in the realizable robust case.

Using a $\eta$-approximate RERM, find the maximal subset $S' \subseteq S$ such that $\inf_{h \in H} \widehat{\text{Err}}_U^\eta(h, f; S') = 0$. That is, we have zero empirical robust loss for the $\eta$-ball robust loss on $S'$. Now, $\mathcal{M}_\text{RE}^\eta$ samples suffice for weak robust learning for any distribution $D$ on $S'$ (see Appendix C, proof of Theorem 4.1). We can now execute the MedBoost algorithm for finding a sample compression scheme. Finally, the result follows from generalization of a sample compression scheme.
5 Realizable and agnostic robust regression

In this section, we study robust regression (see Definition 2.2) in the realizable and agnostic settings. The main challenge is that the boosting approach for the $\eta$-ball robust loss, does not apply on the $\ell_p$ robust loss. We start with an upper bound on the sample complexity for the realizable case. This follows from Theorem 4.1 and by using (reverse) Markov inequality. Complete proofs for this section are in Appendix D.

**Theorem 5.1** Assuming $\eta$-realizability (Eq. (4)), a learning algorithm for robust $\eta$-regression also applies for robust regression by taking $\eta = \epsilon$. As a result, following Theorem 4.1, for any $\mathcal{H}, \mathcal{U}, \epsilon, \delta \in (0, 1)$, there exists a numerical constant $c \in (0, \infty)$, such that Algorithm 2 ($\epsilon, \delta$)-PAC learns $\mathcal{H}$ in the robust regression setting with $\ell_p, \mathcal{U} (\cdot)$ loss ($1 \leq p < \infty$), with sample of size

$$M_{RE} = \tilde{O} \left( \frac{\text{fat}(\mathcal{H}, ce^{1/p}) \text{fat}^* (\mathcal{H}, ce^{1/p})}{\epsilon} + \frac{1}{\epsilon} \log \frac{1}{\delta} \right).$$

Using Eq. (5), we conclude,

$$M_{RE} = \tilde{O} \left( \frac{\text{fat}(\mathcal{H}, ce^{1/p}) 2^{\text{fat}(\mathcal{H}, ce^{1/p})}}{\epsilon^{1+1/p}} + \frac{1}{\epsilon} \log \frac{1}{\delta} \right).$$

5.1 Agnostic setting

We proceed to the more challenging problem of agnostic robust regression. Note that in the agnostic case, having an agnostic learner for robust $\eta$-regression does not apply to the robust regression setting. The reason is that the optimal function in $\mathcal{H}$ has different scales of robustness on different points. We obtain the following result.

**Theorem 5.2** For any $\mathcal{H} \subseteq [0, 1]^X$ with finite $\gamma$-fat-shattering for all $\gamma > 0$, any $\mathcal{U} : \mathcal{X} \rightarrow 2^X$, and any $\eta, \epsilon, \delta \in (0, 1)$, for some numerical constant $c \in (0, \infty)$, with probability $1 - \delta$, Algorithm 3 outputs a function with error at most $\sqrt{\text{OPT}_\mathcal{H}} + \epsilon$, for the $\ell_p, \mathcal{U} (\cdot)$ robust loss ($1 \leq p < \infty$), and using a sample of size

$$\tilde{O} \left( \frac{\text{fat}(\mathcal{H}, ce^{1/p}) \text{fat}^* (\mathcal{H}, ce^{1/p})}{\epsilon^2} + \frac{1}{\epsilon^2} \log \frac{1}{\delta} \right).$$

Using Eq. (5), we have

$$\tilde{O} \left( \frac{\text{fat}(\mathcal{H}, ce^{1/p}) 2^{\text{fat}(\mathcal{H}, ce^{1/p})}}{\epsilon^{2+1/p}} + \frac{1}{\epsilon^2} \log \frac{1}{\delta} \right).$$

**Proof overview.** We would like to use the agnostic regressor for the $\eta$-ball robust loss. By Markov inequality, we know that by taking $\eta = \sqrt{\text{OPT}_\mathcal{H}}$, we have

$$P_{x \sim \mathcal{D}} \left( \sup_{z \in \mathcal{U} (x)} |h^*(z) - f(x)| > \sqrt{\text{OPT}_\mathcal{H}} \right) \leq \frac{\text{OPT}_\mathcal{H}}{\sqrt{\text{OPT}_\mathcal{H}}} = \sqrt{\text{OPT}_\mathcal{H}},$$

where $\text{OPT}_\mathcal{H} = \inf_{h \in \mathcal{H}} \mathbb{E}_{x \sim \mathcal{D}} \left[ \sup_{z \in \mathcal{U} (x)} |h(z) - f(x)| \right]$, and $h^*$ is the optimal function. The problem is that $\text{OPT}_\mathcal{H}$ is not known to us. Therefore, we have the following steps.

1. We define a grid, such that one of its elements satisfies $\sqrt{\text{OPT}_\mathcal{H}} < \hat{\theta} < 2 \sqrt{\text{OPT}_\mathcal{H}}$.
2. For each element in the grid, we execute the agnostic regressor for the $\eta$-robust loss.
3. We choose the optimal function on a holdout labeled set.

With high probability, the algorithm outputs a function with error at most $\sqrt{\text{OPT}_\mathcal{H}} + \epsilon$. 
Algorithm 3: Agnostic robust regression

Input: Hypothesis class $\mathcal{H} \subseteq [0, 1]^X$, labeled sample $S = \{(x_i, f(x_i))\}_{i=1}^m$, $x_i \sim D$, holdout labeled sample $\tilde{S} = \{(x_i, f(x_i))\}_{i=1}^n$, $x_i \sim D$.

Algorithms used: Agnostic learner for robust $\eta$-regression (see Theorem 4.3): Agnostic-$\eta$-Regressor.

1. Define a grid $\Theta = \{\frac{1}{m}, \frac{2}{m}, \frac{4}{m}, \frac{8}{m}, \ldots, 1\}$.  
2. Define $\mathcal{H}_\Theta = \{h_\theta = \text{Agnostic-}\theta\text{-Regressor}(S) : \theta \in \Theta\}$.  
3. Find an optimal function on the holdout set

$$\hat{h}_\theta = \arg\min_{h_\theta \in \mathcal{H}_\Theta} \frac{1}{|\tilde{S}|} \sum_{(x, f(x)) \in \tilde{S}} \mathbb{1}_{\left[\sup_{z \in U(x)} |h_\theta(z) - f(x)| \geq \theta\right]}$$

Output: $\hat{h}_\theta$.

6 Discussion

In this paper, we studied the robustness of real-valued functions to test time attacks. We showed that finite fat-shattering is sufficient for learnability. We proved upper bounds for the $\eta$-ball robust loss and the robust $\ell_p$ loss in the realizable and agnostic settings. We also showed that convex classes are properly learnable. We leave several interesting open questions for a future research.

- It would be interesting to study whether there is an inherent sample complexity gap between proper and improper learning algorithms for learning convex classes. We showed that our improper learner Algorithm 2 requires much less samples than Algorithm 1.

- An important future directions is to understand the sample complexity for agnostic regression. The main challenge is that for the $\ell_p$ loss we cannot have boosting algorithm, and so a different approach should be taken.

- We showed that the fat-shattering dimension is sufficient. What is a necessary condition? In the binary-valued case, we know that having a finite VC is not necessary.

- To what extent can we benefit from unlabeled samples for learning real-valued functions? This question was considered by Attias et al. [8] for binary function classes, where they showed that the labeled sample complexity can be arbitrarily smaller compared to the fully-supervised setting.

- In this work we focused on the statistical aspect of robustly learn real-valued functions. It would be interesting to explore the computational aspect as well.
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References

[1] Noga Alon, Shai Ben-David, Nicolo Cesa-Bianchi, and David Haussler. Scale-sensitive dimensions, uniform convergence, and learnability. *Journal of the ACM (JACM)*, 44(4):615–631, 1997.

[2] Noga Alon, Steve Hanneke, Ron Holzman, and Shay Moran. A theory of pac learnability of partial concept classes. In *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 658–671. IEEE, 2022.

[3] Martin Anthony and Peter L Bartlett. Function learning from interpolation. *Combinatorics, Probability and Computing*, 9(3):213–225, 2000.

[4] Martin Anthony, Peter Bartlett, Yuval Ishai, and John Shawe-Taylor. Valid generalisation from approximate interpolation. *Combinatorics, Probability and Computing*, 5(3):191–214, 1996.

[5] Martin Anthony, Peter L Bartlett, Peter L Bartlett, et al. Neural network learning: Theoretical foundations, volume 9. cambridge university press Cambridge, 1999.

[6] Hassan Ashtiani, Vinayak Pathak, and Ruth Urner. Black-box certification and learning under adversarial perturbations. In *International Conference on Machine Learning*, pages 388–398. PMLR, 2020.

[7] Idan Attias, Aryeh Kontorovich, and Yishay Mansour. Improved generalization bounds for robust learning. In *Algorithmic Learning Theory*, pages 162–183. PMLR, 2019.

[8] Idan Attias, Steve Hanneke, and Yishay Mansour. A characterization of semi-supervised adversarially-robust pac learnability. *arXiv preprint arXiv:2202.05420*, 2022.

[9] Pranjal Awasthi, Natalie Frank, and Mehryar Mohri. Adversarial learning guarantees for linear hypotheses and neural networks. In *International Conference on Machine Learning*, pages 431–441. PMLR, 2020.

[10] Pranjal Awasthi, Natalie Frank, Anqi Mao, Mehryar Mohri, and Yutao Zhong. Calibration and consistency of adversarial surrogate losses. *Advances in Neural Information Processing Systems*, 34, 2021.

[11] Pranjal Awasthi, Natalie Frank, and Mehryar Mohri. On the existence of the adversarial bayes classifier. *Advances in Neural Information Processing Systems*, 34, 2021.

[12] Peter L Bartlett and Philip M Long. Prediction, learning, uniform convergence, and scale-sensitive dimensions. *Journal of Computer and System Sciences*, 56(2):174–190, 1998.

[13] Robi Bhattacharjee, Somesh Jha, and Kamalika Chaudhuri. Sample complexity of robust linear classification on separated data. In *International Conference on Machine Learning*, pages 884–893. PMLR, 2021.

[14] Battista Biggio, Igino Corona, Davide Maiorca, Blaine Nelson, Nedi Tomić, Pavel Laskov, Giorgio Giacinto, and Fabio Roli. Evasion attacks against machine learning at test time. In *Joint European conference on machine learning and knowledge discovery in databases*, pages 387–402. Springer, 2013.
[15] Daniel Cullina, Arjun Nitin Bhagoji, and Prateek Mittal. Pac-learning in the presence of adversaries. In Advances in Neural Information Processing Systems, pages 230–241, 2018.

[16] Chen Dan, Yuting Wei, and Pradeep Ravikumar. Sharp statistical guarantees for adversarially robust gaussian classification. In International Conference on Machine Learning, pages 2345–2355. PMLR, 2020.

[17] Ofir David, Shay Moran, and Amir Yehudayoff. Supervised learning through the lens of compression. Advances in Neural Information Processing Systems, 29:2784–2792, 2016.

[18] Richard M Dudley. A course on empirical processes. In Ecole d’été de Probabilités de Saint-Flour XII-1982, pages 1–142. Springer, 1984.

[19] Uriel Feige, Yishay Mansour, and Robert Schapire. Learning and inference in the presence of corrupted inputs. In Conference on Learning Theory, pages 637–657, 2015.

[20] Sally Floyd and Manfred Warmuth. Sample compression, learnability, and the vapnik-chervonenkis dimension. Machine learning, 21(3):269–304, 1995.

[21] Yoav Freund and Robert E Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. Journal of computer and system sciences, 55(1):119–139, 1997.

[22] Ian J Goodfellow, Jonathon Shlens, and Christian Szegedy. Explaining and harnessing adversarial examples. arXiv preprint arXiv:1412.6572, 2014.

[23] Thore Graepel, Ralf Herbrich, and John Shawe-Taylor. Pac-bayesian compression bounds on the prediction error of learning algorithms for classification. Machine Learning, 59(1-2):55–76, 2005.

[24] Steve Hanneke, Aryeh Kontorovich, and Menachem Sadigurschi. Sample compression for real-valued learners. In Algorithmic Learning Theory, pages 466–488. PMLR, 2019.

[25] Michael J Kearns and Robert E Schapire. Efficient distribution-free learning of probabilistic concepts. Journal of Computer and System Sciences, 48(3):464–497, 1994.

[26] Balázs Kégl. Robust regression by boosting the median. In Learning Theory and Kernel Machines, pages 258–272. Springer, 2003.

[27] Justin Khim and Po-Ling Loh. Adversarial risk bounds via function transformation. arXiv preprint arXiv:1810.09519, 2018.

[28] Pieter Kleer and Hans Simon. Primal and dual combinatorial dimensions. arXiv preprint arXiv:2108.10037, 2021.

[29] Aryeh Kontorovich and Idan Attias. Fat-shattering dimension of k-fold maxima. arXiv preprint arXiv:2110.04763, 2021.

[30] Nick Littlestone and Manfred Warmuth. Relating data compression and learnability. 1986.

[31] Aleksander Madry, Aleksandar Makelov, Ludwig Schmidt, Dimitris Tsipras, and Adrian Vladu. Towards deep learning models resistant to adversarial attacks. arXiv preprint arXiv:1706.06083, 2017.

[32] Andreas Maurer and Massimiliano Pontil. Empirical bernstein bounds and sample variance penalization. arXiv preprint arXiv:0907.3740, 2009.

[33] Shahar Mendelson. An optimal unrestricted learning procedure. arXiv preprint arXiv:1707.05342, 2017.

[34] Omar Montasser, Steve Hanneke, and Nathan Srebro. Vc classes are adversarially robustly learnable, but only improperly. arXiv preprint arXiv:1902.04217, 2019.

[35] Omar Montasser, Surbhi Goel, Ilias Diakonikolas, and Nathan Srebro. Efficiently learning adversarially robust halfspaces with noise. In International Conference on Machine Learning, pages 7010–7021. PMLR, 2020.
A Auxiliary results and preliminaries

The following claims will be used in the next sections. We start with the following generalization result.

**Theorem A.1 (Generalization from approximate interpolation)** [5, Theorems 21.13 and 21.14] Let $\mathcal{H} \subseteq [0,1]^X$ with a finite fat-shattering dimension (at any scale). For any $\eta, \alpha, \epsilon, \delta \in (0,1)$, $h^* \in \mathcal{H}$, distribution $D$ over $\mathcal{X}$, and random sample $S \sim D^m$, if

$$m(\eta, \alpha, \epsilon, \delta) = O\left(\frac{\log^2 \left(\frac{\text{fat}(\mathcal{H}, \alpha/8)}{\alpha \epsilon} \right) + \log \frac{1}{\delta}}{\epsilon}ight),$$

then with probability at least $1 - \delta$ over $S$, for every $h \in \mathcal{H}$ satisfying: $\forall 1 \leq i \leq m : |h(x_i) - h^*(x_i)| \leq \eta$, it holds that $\mathbb{P}_{x \sim D} (x : |h(x) - h^*(x)| > \eta + \alpha) \leq \epsilon$.

The following is a bound on the covering numbers in $d_\infty$.

**Lemma A.2** [40, Theorem 4.4] Let $\mathcal{F} \subseteq [0,1]^\Omega$ be a class of functions and $|\Omega| = n$. Then for any $0 < a < 1$ and $0 < t < 1/2$,

$$\log N(t, \mathcal{F}, d_\infty) \leq C v \log(n/vt) \cdot \log^a(2n/v),$$

where $v = \text{fat}_{\text{cat}}(\mathcal{F})$, and $C, c$ are universal constants.
A.1 Sample compression bounds

The following generalization for sample compression in the realizable case was proven by Littlestone and Warmuth [30], Floyd and Warmuth [20]. Their proof is for the 0-1 loss, but it applies similarly for bounded loss functions. We use it with the $\eta$-ball robust loss.

Lemma A.3 (Sample compression generalization bound) Let a sample compression scheme $(\kappa, \rho)$, and a loss function $\ell : \mathbb{R} \times \mathbb{R} \to [0, 1]$. In the realizable case, for any $\kappa(S) \leq m$, any $\delta \in (0, 1)$, and any distribution $\mathcal{D}$ over $\mathcal{X} \times \{0, 1\}$, for $S \sim \mathcal{D}^m$, with probability $1 - \delta$,

$$\left| \text{Err}(\rho(\kappa(S)); \mathcal{D}) - \overline{\text{Err}}(\rho(\kappa(S)); S) \right| = \mathcal{O} \left( \frac{|\kappa(S)| \log(m) + \log \frac{1}{\delta}}{m} \right).$$

The following generalization for sample compression in the agnostic case was proven by Graepel et al. [23]. Their proof is for the 0-1 loss, but it applies similarly for bounded loss functions. We use it with the $\eta$-ball robust loss.

Lemma A.4 (Agnostic sample compression generalization bound) Let a sample compression scheme $(\kappa, \rho)$, and a loss function $\ell : \mathbb{R} \times \mathbb{R} \to [0, 1]$. In the agnostic case, for any $\kappa(S) \leq m$, any $\delta \in (0, 1)$, and any distribution $\mathcal{D}$ over $\mathcal{X} \times \{0, 1\}$, for $S \sim \mathcal{D}^m$, with probability $1 - \delta$,

$$\left| \text{Err}(\rho(\kappa(S)); \mathcal{D}) - \overline{\text{Err}}(\rho(\kappa(S)); S) \right| \leq \mathcal{O} \left( \sqrt{\frac{|\kappa(S)| \log(m) + \log \frac{1}{\delta}}{m}} \right).$$

The following data-dependent compression based generalization bound in the agnostic case, is a variation of the classic bound by Graepel et al. [23]. It follows the same arguments while using the empirical Bernstein bound instead of Hoeffding’s inequality. A variation of this bound, with respect to the 0-1 loss, appears in Alon et al. [2, Lemma 42], and Maurer and Pontil [32, Section 5]. The exact same arguments follows for a bounded loss function, and we use it for the $\eta$-ball robust loss.

Lemma A.5 (Data-dependent agnostic sample compression generalization bound) Let a sample compression scheme $(\kappa, \rho)$, and a loss function $\ell : \mathbb{R} \times \mathbb{R} \to [0, 1]$. In the agnostic case, for any $\kappa(S) \leq m$, any $\delta \in (0, 1)$, and any distribution $\mathcal{D}$ over $\mathcal{X} \times \{0, 1\}$, for $S \sim \mathcal{D}^m$, with probability $1 - \delta$,

$$\left| \text{Err}(\rho(\kappa(S)); \mathcal{D}) - \overline{\text{Err}}(\rho(\kappa(S)); S) \right| \leq \mathcal{O} \left( \sqrt{\overline{\text{Err}}(\rho(\kappa(S)); S)} \left( \frac{|\kappa(S)| \log(m) + \log \frac{1}{\delta}}{m} \right) + \frac{|\kappa(S)| \log(m) + \log \frac{1}{\delta}}{m} \right).$$

B Proofs for Section 3

Proof of Theorem 3.1 Fix $\eta, \epsilon, \delta \in (0, 1)$. Let $\mathcal{H} \subseteq [0, 1]^X$, fix a distribution $\mathcal{D}$ over the instance space $\mathcal{X}$. Let a sample $S = \{(x_1, f(x_1)), \ldots, (x_m, f(x_m))\}$ such that $x_1, \ldots, x_m$ are drawn i.i.d. from $\mathcal{D}$, and labeled with $f(x_i) = y_i$. We assume that the distribution $\mathcal{D}$ is $\eta$-realizable with respect to $\mathcal{H}, \mathcal{U}$ and $f$. That is, for all $x$ in the support of $\mathcal{D}$, it holds that

$$\inf_{h \in \mathcal{H}} \left\{ \sup_{x \in \mathcal{U}(x)} |h(z) - f(x)| \geq \eta \right\} = 0.$$

Some of the ideas are similar to the binary-valued case [34, Theorem 4]. Inflating the training set is the same. The discretization step is different, in that we are taking strong learners for any distribution over $\mathcal{S}_\delta$, instead of weak learners, and that we are doing that for real-valued functions. Crucially, these strong learners for $\mathcal{S}_\delta$ can be found using generalization from interpolation (Theorem A.1), due to [3]. In the last step, we are running a variant of Multiplicative Weights updates, and taking the average of the returned mixture of functions. This yields a proper learner for convex classes. We remark that running a boosting algorithm (using weak learners) as in [34], and as we do in Algorithm 2, will result in an improper learner.

We elaborate on each one of the steps as described in Algorithm 1.
1. Define the inflated training data set

\[ S_\mathcal{U} = \bigcup_{i \in [n]} \{(z, y_i(z)) : z \in \mathcal{U}(x_i)\}, \]

where \( I(z) = \min\{i \in [m] : z \in \mathcal{U}(x_i)\}. \)

2. Discretize \( S_\mathcal{U} \) to a finite set \( \hat{S}_\mathcal{U} \) as following.

(a) Denote \( d_{\eta, \epsilon} = O(\fat(\mathcal{H}, \eta/32)/\epsilon) \). Define a set of functions, such that each function is accurate on \( d_{\eta, \epsilon} \cdot \log^2 \frac{1}{\eta} \) points in \( \mathcal{S} \), w.r.t. to the \( \eta/4 \)-ball loss,

\[ \hat{\mathcal{H}} = \left\{ \eta/4\text{-RERM}_H(S') : S' \subseteq S, |S'| = d_{\eta, \epsilon} \cdot \log^2 \frac{1}{\eta} \right\}. \]

Recall the definition of \( \eta\text{-RERM}_H \) (see Eq. (3)),

\[ h \in \eta\text{-RERM}_H(S') \rightarrow \forall x \in S': |h(x) - f(x)| \leq \eta. \]

The cardinality of this class is bounded as following

\[ |\hat{\mathcal{H}}| = \left( d_{\eta, \epsilon} \cdot \log^2 \frac{m}{\eta} \right) \lesssim \left( \frac{m}{d_{\eta, \epsilon} \cdot \log^2 \frac{1}{\eta}} \right)^{d_{\eta, \epsilon} \cdot \log^2 \frac{1}{\eta}}. \] (6)

(b) A discretization \( \hat{S}_\mathcal{U} \subseteq S_\mathcal{U} \) will be defined by the dual class of \( \hat{\mathcal{H}} \), as the covering numbers of \( S_\mathcal{U} \) in \( d_\infty \) norm. The dual class of \( \hat{\mathcal{H}} \), \( \hat{\mathcal{H}}^* \subseteq [0, 1]^{\hat{\mathcal{H}}} \), is defined as the set of all functions \( f_{(z, y)} : \hat{\mathcal{H}} \rightarrow [0, 1] \) such that \( f_{(z, y)}(h) = |h(z) - y| \), for any \( (z, y) \in S_\mathcal{U} \). Formally, \( \hat{\mathcal{H}}^* = \{ f_{(z, y)} : (z, y) \in S_\mathcal{U} \} \), where \( f_{(z, y)} = (f_{(z, y)}(h_1), \ldots, f_{(z, y)}(h_{|\hat{\mathcal{H}}|})) \). We take \( \hat{S}_\mathcal{U} \subseteq S_\mathcal{U} \) to be the minimal \( \eta/2 \)-covering for \( S_\mathcal{U} \) in \( d_\infty \),

\[ \quad \sup_{(z, y) \in S_\mathcal{U}} \inf_{(z, y) \in \hat{S}_\mathcal{U}} \| f_{(z, y)} - f_{(z, y)} \|_\infty \leq \eta/2. \] (7)

Denote the dual \( c\eta \)-fat-shattering by \( d_{\eta}^* = \fat^*(\mathcal{H}, c\eta) \), where \( c \in (0, \infty) \) is a numerical constant.

By applying Lemma A.2 on the dual space, for any \( a > 0 \), we have the following bound

\[ |\hat{S}_\mathcal{U}| \leq \mathcal{N}(\eta/2, S_\mathcal{U}, d_{\eta}^*) \]

\[ \lesssim \exp \left( \frac{d_{\eta}^* \log \frac{|\hat{\mathcal{H}}|}{\eta \cdot d_{\eta}^*}}{\log^a \left( \frac{|\hat{\mathcal{H}}|}{d_{\eta}^*} \right)} \right), \] (8)

and we can take \( a \) arbitrarily small such that \( |\hat{S}_\mathcal{U}| \lesssim \exp \left( \frac{d_{\eta}^* \log \frac{|\hat{\mathcal{H}}|}{\eta \cdot d_{\eta}^*}}{2} \right) \).

3. We execute the following variant of Multiplicative Wights (MW) algorithm, on the set \( \hat{S}_\mathcal{U} \).

**Algorithm 4 Modified Multiplicative Weights**

**Input:** \( \mathcal{H}, S, \hat{S}_\mathcal{U}, \eta\text{-RERM}_H \).

**Parameters:** \( \xi, T \).

**Initialize** \( P_1 = \text{Uniform}(\hat{S}_\mathcal{U}) \).

For \( t = 1, \ldots, T \):

i. Set \( d_{\eta, \epsilon} = O(\fat(\mathcal{H}, \eta/32)/\epsilon) \).

Find \( O \left( d_{\eta, \epsilon} \cdot \log^2 \frac{1}{\eta} \right) \) points \( S_t \subseteq \hat{S}_\mathcal{U} \) such that any \( h \in \mathcal{H} \) satisfying \( \forall (x, y) \in S_t : |h(x) - y| \leq \eta/4 \), it holds that \( \mathbb{E}_{x \sim P_t} \left[ \sum_{z \in \mathcal{Z}} \mathbb{I} \left( |h(z) - y| \geq \eta/2 \right) \right] \leq \epsilon \). (This set exists due due Theorem A.1).

ii. Let \( S'_t \) be the original \( d_{\eta, \epsilon} \cdot \log^2 \frac{1}{\eta} \) points in \( S \) with \( S_t \subseteq \bigcup_{(x, y) \in S'_t} \bigcup \{(z, y) : z \in \mathcal{U}(x)\} \).

iii. Let \( \hat{h}_t = \eta/4\text{-RERM}_H(S'_t) \).

iv. For each \( (x, y) \in \hat{S}_\mathcal{U} \):

\[ P_{t+1}(x, y) \propto P_t(x, y) e^{-\xi \{ |h(x) - y| \leq \eta/2 \} } \]

**Output:** classifiers \( \hat{h}_1, \ldots, \hat{h}_T \) and sets \( S'_1, \ldots, S'_T \).
An approximate sample compression scheme. The output of the algorithm is a sequence of functions \( \hat{h}_1, \ldots, \hat{h}_T \), and their corresponding sets that encodes them \( S'_1, \ldots, S'_T \). We take the output function to be

\[
\hat{h}(x) = \frac{1}{T} \sum_{i=1}^{T} \hat{h}_i(x).
\]

As a result, we have an \((\eta/2, \epsilon)\)-approximate sample compression scheme, \( \rho(\kappa(S_u)) = \hat{h} \), for \( S_u \). Formally,

\[
\frac{1}{|S_u|} \sum_{(x, f(x)) \in S_u} I \left\{ \left| \hat{h}(x) - f(x) \right| \geq \eta/2 \right\} \leq \epsilon. \tag{9}
\]

In order to guarantee Eq. (9), we follow standard analysis of MW / \( \alpha \)-Boost, and setting the number of predictors to be \( T \approx \log |S_u| \). For any distribution \( \mathcal{P}_T \) over \( S_u \), we have a base learner \( \hat{h} \), satisfying

\[
\mathbb{E}_{x \sim \mathcal{P}_T} \left[ \mathbb{I} \left\{ \left| \hat{h}(x) - y \right| \geq \eta/2 \right\} \right] \leq \epsilon, \tag{10}
\]

due to Theorem A.1. Note that we can find these base learners in \( \hat{H} \), as defined in step 2(a). Crucially, we use strong base learners in order to ensure low empirical loss of the average base learners. See [41, Section 6] for the full details.

From the covering argument (Eq. (8)) and Eq. (9), we have

\[
\frac{1}{|S_u|} \sum_{(x, f(x)) \in S_u} I \left\{ \left| \hat{h}(x) - f(x) \right| \geq \eta \right\} \leq \epsilon. \tag{10}
\]

Finally, we conclude that \( \hat{h} \) has an empirical robust error for the \( \eta \)-ball robust loss,

\[
\frac{1}{|S|} \sum_{(x, f(x)) \in S} I \left\{ \sup_{x \in \mathcal{U}(x)} \left| \hat{h}(x) - f(x) \right| \geq \eta \right\} \leq \epsilon. \tag{11}
\]

We summarize the sample compression compression size for representing \( \hat{h} \). We have \( T = \mathcal{O} \left( \log |S_u| \right) \) predictors, where each one of them requires \( d_{n, \epsilon} \cdot \log^2 \frac{1}{\eta} \) points. By counting the number of predictors using Eq. (8), we get

\[
\log(|S_u|) \lesssim \log \left( \exp \left( d_{n}^* \log \left( \frac{|\hat{H}|}{\eta \cdot d_{n}^*} \right) \right) \right)
\]

\[
\lesssim d_{n}^* \log \left( \frac{|\hat{H}|}{\eta \cdot d_{n}^*} \right)
\]

\[
\lesssim d_{n}^* \log \left( \frac{1}{\eta \cdot d_{n}^*} \left( \frac{m}{d_{n, \epsilon} \log^2 \frac{1}{\eta}} \right)^{d_{n, \epsilon} \log^2 \frac{1}{\eta}} \right)
\]

\[
\lesssim d_{n}^* d_{n, \epsilon} \log^3 \frac{1}{\eta} \log \left( \frac{m}{d_{n, \epsilon} \log^2 \frac{1}{\eta}} \right) + d_{n}^* \log \left( \frac{1}{\eta \cdot d_{n}^*} \right)
\]

\[
\lesssim d_{n}^* d_{n, \epsilon} \log^3 \frac{1}{\eta} \log \left( \frac{m}{d_{n, \epsilon} \log^2 \frac{1}{\eta}} \right)
\]

All together we have a compression of size

\[
\mathcal{O} \left( d_{n, \epsilon} \cdot \log^2 \frac{1}{\eta} \log(|S_u|) \right) \approx d_{n, \epsilon}^* d_{n}^* \log^3 \frac{1}{\eta} \log \left( \frac{m}{d_{n, \epsilon} \log^2 \frac{1}{\eta}} \right).
\]

We can now obtain generalization via Lemma A.5,

\[
\left| \text{Err}(\rho(\kappa(S)); D) - \hat{\text{Err}}(\rho(\kappa(S)); S) \right| \lesssim \sqrt{\text{Err}(\rho(\kappa(S)); S) \frac{|\kappa(S)| \log(m) + \log \frac{1}{\epsilon}}{m}} + \frac{|\kappa(S)| \log(m) + \log \frac{1}{\epsilon}}{m}
\]

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We can now plug in the compression size. We have the following learning rate,

\[
\sqrt{\text{Err}(\rho(\epsilon(S)); S)} \approx \frac{1}{m} \left( (\text{fat}(\mathcal{H}, \eta/32) / \epsilon)^2 \text{fat}^*(\mathcal{H}, c \eta) \log^{\frac{1}{\eta}} \left( \frac{m}{d_{\eta,c} \log^{\frac{1}{\eta}} \frac{2}{\eta}} \right) \log(m) + \log \frac{1}{\delta} \right).
\]

By computing \( m \) for having an error at most \( \epsilon \), we conclude the proof. 

\[\Box\]

C Proofs for Section 4

Proof of Theorem 4.1 Fix \( \eta, \epsilon, \delta \in (0, 1) \). Let \( \mathcal{H} \subseteq [0, 1]^X \), fix a distribution \( \mathcal{D} \) over the instance space \( X \). Let a sample \( S = \{(x_1, f(x_1)), \ldots, (x_m, f(x_m))\} \) such that \( x_1, \ldots, x_m \) are drawn i.i.d. from \( \mathcal{D} \), and labeled with \( f(x_i) = y_i \), for some \( f \in \mathcal{H} \).

The main improvement of this algorithm, is that we are able to incorporate the median boosting algorithm. The base learners in this construction are weak learners, which requires less samples. We are able to combine them into sample compression scheme with low robust empirical error, by taking a median, which is robust measure to errors.

We elaborate on each one of the steps as described in Algorithm 2.

1. Define the inflated training data set

\[
S_\mathcal{U} = \bigcup_{i \in [n]} \{(z, y_{I(z)}): z \in \mathcal{U}(x_i)\},
\]

where \( I(z) = \min\{i \in [m]: z \in \mathcal{U}(x_i)\} \).

2. Discretize \( S_\mathcal{U} \) to a finite set \( \hat{S}_\mathcal{U} \) as following.

(a) Denote \( d_\eta = O(\text{fat}(\mathcal{H}, \eta/64)) \). Define a set of functions, such that each function is accurate on \( d_\eta \cdot \log^{\frac{1}{\eta}} \frac{1}{\eta} \) points in \( S \), w.r.t. to the \( \eta/8 \)-ball loss,

\[
\hat{\mathcal{H}} = \left\{ \eta/8\text{-RERM}_\mathcal{H}(S'): S' \subseteq S, |S'| = O\left(d_\eta \log^{\frac{1}{\eta}} \frac{1}{\eta}\right) \right\}.
\]

Recall the definition of \( \eta\text{-RERM}_\mathcal{H} \) (see Eq. (3)),

\[
h \in \eta\text{-RERM}_\mathcal{H}(S') \rightarrow \forall x \in S': |h(x) - f(x)| \leq \eta.
\]

In order to understand what this definition of \( \hat{\mathcal{H}} \) serves for, see step 3 below. The cardinality of this class is bounded as following

\[
|\hat{\mathcal{H}}| = \left( \frac{m}{d_\eta \log^{\frac{1}{\eta}} \frac{1}{\eta}} \right) \lesssim \left( \frac{m}{d_\eta \log^{\frac{1}{\eta}} \frac{1}{\eta}} \right)^{d_\eta \log^{\frac{1}{\eta}} \frac{1}{\eta}}.
\]

(b) A discretization \( \hat{S}_\mathcal{U} \subseteq S_\mathcal{U} \) will be defined by the dual class of \( \hat{\mathcal{H}} \), as the covering numbers of \( S_\mathcal{U} \) in \( d_\infty \) norm. The dual class of \( \hat{\mathcal{H}} \), \( \hat{\mathcal{H}}^* \subseteq [0, 1]^{\hat{\mathcal{H}}} \), is defined as the set of all functions \( f_{(z, y)}: \hat{\mathcal{H}} \rightarrow [0, 1] \) such that \( f_{(z, y)}(h) = |h(z) - y| \), for any \((z, y) \in S_\mathcal{U}\). Formally, \( \hat{\mathcal{H}}^* = \{ f_{(z, y)}: (z, y) \in S_\mathcal{U} \} \), where \( f_{(z, y)} = (f_{(z, y)}(h_1), \ldots, f_{(z, y)}(h_{|\hat{\mathcal{H}}|})) \). We take \( \hat{S}_\mathcal{U} \subseteq S_\mathcal{U} \) to be the minimal \( \eta/4 \)-covering for \( S_\mathcal{U} \) in \( d_\infty \),

\[
\sup_{(z, y) \in \hat{S}_\mathcal{U}} \inf_{(z, y) \in S_\mathcal{U}} \| f_{(z, y)} - f_{(z, y)} \|_\infty \leq \eta/4.
\]
Denote the dual \( c\eta\)-fat-shattering by \( d_{\eta}^c = \text{fat}^*(\mathcal{H}, c\eta) \), where is \( c \in (0, \infty) \) is a numerical constant.

By applying Lemma A.2 (taking \( a = 1 \)) on the dual space, we have the following bound
\[
|S_u| = \mathcal{N}(\eta/4, S_u, d_{\infty}) \\
\lesssim \exp \left( d_{\eta}^c \log \left( \frac{|\mathcal{H}|}{\eta \cdot d_{\eta}^c} \right) \log \left( \frac{|\mathcal{H}|}{d_{\eta}^c} \right) \right) \\
\lesssim \exp \left( d_{\eta}^c \log^2 \left( \frac{|\mathcal{H}|}{\eta \cdot d_{\eta}^c} \right) \right) \quad (14)
\]

3. We execute the following modified variant of the real-valued boosting algorithm MedBoost [26, 24], on the set \( S_u \). The output of the algorithm is a uniformly \( \eta/4 \)-approximate sample compression scheme for the set \( S_u \), for \( \approx \log(|S_u|) \) boosting rounds ([24, Corollary 6]). We will explain first why we have a weak learner for every distribution over \( S_u \).

**The existence of weak learners in \( \hat{\mathcal{H}} \).** By Theorem A.1, taking \( \epsilon = \delta = 1/3 \), we know that for any distribution \( D \) over \( S_u \), upon receiving an i.i.d. sample \( S'' \) from \( D \) of size \( O \left( d_{\eta}^c \log^{2} \frac{1}{\eta} \right) \), with probability \( 2/3 \) over sampling \( S'' \) from \( D \), every \( h \in \eta/8\text{-RERM}_{\mathcal{H}}(S'') \) is a \( (\eta/4, 1/6) \)-weak learner for \( D \) (see Definition 4.2). We can conclude that for any distribution \( D \) over \( S_u \), there exists set of points \( S'' \subseteq S_u \) of size \( \mathcal{O} \left( d_{\eta}^c \log^{2} \frac{1}{\eta} \right) \) that defines a weak learner for \( D \).

Moreover, we can find this weak learner in \( \hat{\mathcal{H}} \). Let \( S' \) be the original \( \mathcal{O}(d_{\eta}^c \log^{2} \frac{1}{\eta}) \) points in \( S \) that the perturbed points \( S'' \) stemmed from. That is, \( S'' \subseteq \bigcup_{(x,y) \in S'} \bigcup \{(z,y) : z \in \mathcal{U}(x)\} \). Therefore, \( \hat{h} = \eta/8\text{-RERM}_{\mathcal{H}}(S') \) is a weak learner, and can be found in \( \hat{\mathcal{H}} \). So, we can think of \( \hat{\mathcal{H}} \) as a pool of weak learners for every possible distribution over the discretized set \( \hat{S}_u \).

---

**Algorithm 5** Modified MedBoost

**Input:** \( \mathcal{H}, S, S_u, \eta\text{-RERM}_{\mathcal{H}} \).

**Parameters:** \( \eta, T \).

**Initialize** \( \mathcal{P}_1 = \text{Uniform}(\hat{S}_u) \).

For \( t = 1, \ldots, T \):

i. Set \( d_{\eta}^c = \tilde{O}(\text{fat}(\mathcal{H}, \eta/64)) \).

Find a \( (\eta/4, 1/6) \)-weak learner w.r.t. \( \mathcal{P}_t \) by finding \( \mathcal{O}(d_{\eta}^c \log^{2} \frac{1}{\eta}) \) points \( S'' \subseteq S_u \) such that any \( h \in \mathcal{H} \) satisfying: \( \forall (x,y) \in S'' : |h(x) - y| \leq \eta/8 \), it holds that \( \mathbb{E}_{x \sim \mathcal{P}_t} \left[ \mathbb{I} \left\{ |h(x) - y| \geq \eta/4 \right\} \right] \leq \epsilon \).

(This set exists due due Theorem A.1).

ii. Let \( S'_t \) be the original \( \mathcal{O}(d_{\eta}^c \log^{2} \frac{1}{\eta}) \) points in \( S \) with \( S'' \subseteq \bigcup_{(x,y) \in S'_t} \bigcup \{(z,y) : z \in \mathcal{U}(x)\} \).

iii. Let \( \hat{h}_t = \eta/8\text{-RERM}_{\mathcal{H}}(S'_t) \).

iv. For \( i = 1, \ldots, n = |S_u| \):

A. Set \( w_i^{(t)} = 1 - 2\mathbb{I} \left[ |\hat{h}_t(x_i) - y_i| > \frac{\eta}{4} \right] \).

B. Set
\[
\alpha_t = \frac{1}{2} \log \left( \frac{\left( 1 - 1/6 \right) \sum_{i=1}^{n} \mathcal{P}_t(i) \mathbb{I} \left[ w_i^{(t)} = 1 \right]}{\left( 1 + 1/6 \right) \sum_{i=1}^{n} \mathcal{P}_t(i) \mathbb{I} \left[ w_i^{(t)} = -1 \right]} \right).
\]

C. • If \( \alpha_t = \infty \): return \( T \) copies of \( h_t \), \( (\alpha_1 = 1, \ldots, \alpha_T = 1) \), and \( S'_t \).

• Else:
\[
P_{t+1}(x_i, y_i) = P_t(x_i, y_i) \frac{\exp(-\alpha_t w_i^{(t)})}{\sum_{j=1}^{n} P_t(j) \exp(-\alpha_t w_j^{(t)})}.
\]

**Output:** Classifiers \( \hat{h}_1, \ldots, \hat{h}_T \), coefficients \( \alpha_1, \ldots, \alpha_T \) and sets \( S'_1, \ldots, S'_T \).

---

A uniformly \( \eta/4 \)-approximate sample compression scheme. We argue that by taking \( T \approx
\[
\log |\mathcal{S}_U| \quad \text{(see [24, Corollary 6]), we would get}
\]

\[
\forall (x, y) \in \mathcal{S}_U : \quad \left| \text{Med}\left( \hat{h}_1(x), \ldots, \hat{h}_T(x); \alpha_1, \ldots, \alpha_T \right) - y \right| \leq \eta/4, 
\]

where Med\left( \hat{h}_1(x), \ldots, \hat{h}_T(x); \alpha_1, \ldots, \alpha_T \right) is the weighted median of \( \hat{h}_1, \ldots, \hat{h}_T \) with weights \( \alpha_1, \ldots, \alpha_T \).

From the covering argument (Eq. (14)), this implies that

\[
\forall (x, y) \in \mathcal{S}_U : \quad \left| \text{Med}\left( \hat{h}_1(x), \ldots, \hat{h}_T(x); \alpha_1, \ldots, \alpha_T \right) - y \right| \leq \eta/2. 
\]

Indeed, from the covering number argument, for any \((x, y) \in \mathcal{S}_U\) there exists \((\bar{x}, \bar{y}) \in \mathcal{S}_U\), such that for any \(\hat{h}_1 \in \{ \hat{h}_1, \ldots, \hat{h}_T \}\),

\[
\left| \hat{h}_1(x) - y \right| - \left| \hat{h}_1(\bar{x}) - \bar{y} \right| \leq \eta/4.
\]

So,

\[
\left| \text{Med}\left( \hat{h}_1(x), \ldots, \hat{h}_T(x); \alpha_1, \ldots, \alpha_T \right) - y \right| \overset{(i)}{=} \left| \text{Med}\left( \hat{h}_1(x) - y, \ldots, \hat{h}_T(x) - y; \alpha_1, \ldots, \alpha_T \right) \right| 
\]

\[
\overset{(ii)}{\leq} \left| \text{Med}\left( \hat{h}_1(\bar{x}) - \bar{y}, \ldots, \hat{h}_T(\bar{x}) - \bar{y}; \alpha_1, \ldots, \alpha_T \right) \right| + \eta/4 
\]

\[
\overset{(iii)}{\leq} \eta/2,
\]

(i) follows since the median is translation invariant. (ii) follows from the the covering argument. (iii) holds since the returned function by MedBoost is a uniformly \( \eta/4 \)-approximate sample compression for \( \mathcal{S}_U \).

Finally, from Eq. (16) we have

\[
\forall (x, y) \in S : \quad \sup_{z \in \mathcal{U}(x)} \left| \text{Med}\left( \hat{h}_1(z), \ldots, \hat{h}_T(z); \alpha_1, \ldots, \alpha_T \right) - y \right| \leq \eta/2. \tag{17}
\]

We summarize the compression size. We have have \( T = O\left( \log |\mathcal{S}_U| \right) \) predictors, where each one of them requires \( O\left( d_n \log^2 \frac{1}{\eta} \right) \) points. By counting the number of predictors using Eq. (14), we get

\[
\log(|\mathcal{S}_U|) \lesssim \log \left( \exp\left( d_n^* \log \left( \frac{\| \mathcal{H} \|}{\eta \cdot d_n^*} \log \left( \frac{\| \mathcal{H} \|}{d_n^*} \right) \right) \right) \right)
\]

\[
\lesssim d_n^* \log^2 \left( \frac{\| \mathcal{H} \|}{\eta \cdot d_n^*} \right)
\]

\[
\lesssim d_n^* \log^2 \left( \frac{1}{\eta \cdot d_n^*} \log \left( \frac{m}{d_n \log^2 \frac{1}{\eta}} \right) \right)
\]

\[
\lesssim d_n^* \log \left( \frac{1}{\eta \cdot d_n^*} \right) + d_n \log^2 \frac{1}{\eta} \log \left( \frac{m}{d_n \log^2 \frac{1}{\eta}} \right)
\]

\[
\lesssim d_n^* \log^2 \left( \frac{1}{\eta \cdot d_n^*} \right) + d_n \log \left( \frac{1}{\eta \cdot d_n^*} \right) \log \left( \frac{m}{d_n \log^2 \frac{1}{\eta}} \right) + d_n^4 \frac{1}{\eta} \log^2 \left( \frac{m}{d_n \log^2 \frac{1}{\eta}} \right)
\]

\[
\lesssim d_n^2 d_n^* \log^2 \left( \frac{m}{d_n \log^2 \frac{1}{\eta}} \right) \log^2 \left( \frac{1}{\eta \cdot d_n^*} \right) \log^4 \left( \frac{1}{\eta} \right)
\]

All together we have a compression of size

\[
O\left( d_n \cdot \log \frac{1}{\eta} \log(|\mathcal{S}_U|) \right) \approx d_n^3 d_n^* \log^2 \left( \frac{m}{d_n \log^2 \frac{1}{\eta}} \right) \log^2 \left( \frac{1}{\eta \cdot d_n^*} \right) \log^6 \left( \frac{1}{\eta} \right).
\]

We now show how we can reduce the number of predictors (and as a result reducing the sample compression size).
4. We follow the sparsification method from [24]. The idea is that by sampling functions from the ensemble, we can guarantee via a uniform convergence for the dual space, that with high probability it is sufficient to have roughly \(\approx \text{fat}^*(\mathcal{H}, \eta)\) predictors.

For \(\alpha_1, \ldots, \alpha_T \in [0, 1]\) with \(\sum_{t=1}^T \alpha_t = 1\), we denote the categorical distribution by \(\text{Cat}(\alpha_1, \ldots, \alpha_T)\), which is a discrete distribution on the set \([T]\) with probability of \(\alpha_t\) on \(t \in [T]\).

**Algorithm 6** Sparsify

**Input:** Classifiers \(h_1, \ldots, h_T\) and coefficients \(\alpha_1, \ldots, \alpha_T\), and labeled set \(S_U\).

**Parameter:** \(k\).

i. Let \(\alpha_t' = \alpha_t / \sum_{s=1}^T \alpha_s\).

ii. **Repeat:**
   A. Let \(J_i = \text{Cat}(\alpha_1', \ldots, \alpha_T')^k\).
   B. Let \(F = \{h_{J_1}, \ldots, h_{J_k}\}\).
   C. **Until** \(\forall (x, y) \in S_U : |\text{Med}(f_1(x), \ldots, f_k(x)) - y| \leq \eta\).

**Output:** Classifiers \(h_{J_1}, \ldots, h_{J_k}\).

Applying [24, Theorem 10], by taking \(k = O\left(\text{fat}^*(\mathcal{H}, \eta) \log^2(\text{fat}^*(\mathcal{H}, \eta) / \eta)\right)\), where \(c \in (0, \infty)\) is a numerical constant, the sparsification method returns with high probability function \(\{f_1, \ldots, f_k\}\), such that

\[
\forall (x, y) \in S_U : |\text{Med}(f_1(x), \ldots, f_k(x)) - y| \leq \eta.
\]

Therefore, we have,

\[
\forall (x, y) \in S : \sup_{z \in S_U(x)} |\text{Med}(f_1(z), \ldots, f_k(z)) - y| \leq \eta,
\]

with sample compression of size

\[
\text{fat}(\mathcal{H}, \eta/64) \text{fat}^*(\mathcal{H}, \eta) \log^2(\text{fat}^*(\mathcal{H}, \eta) / \eta) \log^2 \frac{1}{\eta}.
\]

By plugging it in Lemma A.3 with the \(\eta\)-ball robust loss, we have an upper bound on the sample complexity of size

\[
O\left(\frac{\text{fat}(\mathcal{H}, \eta) \text{fat}^*(\mathcal{H}, \eta)}{\epsilon}\right),
\]

for some numerical constant \(c \in (0, \infty)\).

**Proof of Theorem 4.3** The construction follows a reduction to the realizable case similar to [17], which is for the non-robust zero-one loss. Moreover, we use a margin-based analysis of MedBoost algorithm (see [26, Theorem 1]), and overcome some technical challenges.

Denote \(\Lambda_{RE} = \Lambda_{RE}(\eta, 1/3, 1/3, \mathcal{H}, \mathcal{U}, \ell^*_q)\), the sample complexity of robust \(\eta\)-regression for a class \(\mathcal{H}\) with respect to a perturbation function \(U\), taking \(\epsilon = \delta = 1/3\).

Using a robust ERM, find the maximal subset \(S' \subseteq S\) with zero empirical robust loss (for the \(\eta\)-ball loss), such that \(\inf_{h \in \mathcal{H}} \bar{\text{Err}}_\eta(h, f; S') = 0\). Now, \(\Lambda_{RE}\) samples suffice for weak robust learning for any distribution \(\mathcal{D}\) on \(S'\).

Execute the MedBoost on \(S'\), with \(T \approx \log(|S'|)\) boosting rounds, where each weak robust learner is trained on \(\approx \Lambda_{RE}\) samples. The returned weighted median \(\hat{h} = \text{Med}(h_1(z), \ldots, h_T(z) ; \alpha_1, \ldots, \alpha_T)\) satisfies \(\bar{\text{Err}}_\eta(h, f; S') = 0\), and each hypothesis \(\hat{h}_t \in \{\hat{h}_1, \ldots, \hat{h}_T\}\) is representable as set of size \(O(\Lambda_{RE})\). This defines a compression scheme of size \(\Lambda_{RE}T\), and \(\hat{h}_t\) can be reconstructed from a compression set of points from \(S\) of size \(\Lambda_{RE}T\).
Recall that \( S' \subseteq S \) is a maximal subset such that \( \inf_{h \in \mathcal{H}} \mathbb{E} \tilde{\eta}(h, f; S') = 0 \) which implies that \( \inf_{h \in \mathcal{H}} \mathbb{E} \tilde{\eta}(h, f; S') \leq \inf_{h \in \mathcal{H}} \mathbb{E} \eta(h, f; S'). \) Plugging it into an agnostic sample compression bound Lemma A.4, we have a sample complexity of \( \mathcal{O}(\frac{\log(\lambda_{\mathcal{H},c\eta})}{\epsilon^2(\mathcal{H},c\eta)}) \), for some numerical constant \( c \in (0, \infty) \).

## D Proofs for Section 5

**Proof of Theorem 5.1** Both proofs follow from combining the results for the robust \( \eta \)-regression and using reverse Markov inequality. By Theorem 4.1, for any \( \epsilon \in (0, 1) \) it holds that

\[
P_{x \sim D} \left( \sup_{z \in \mathbb{U}(x)} |h(z) - f(x)| > \eta \right) \leq \epsilon.
\]

By reverse Markov inequality, any \([0, 1]\)-valued random variable \( Z \) satisfies

\[
\mathbb{E} Z - \eta \leq \mathbb{P}(Z > \eta) \frac{1}{1 - \eta},
\]

for any \( 0 \leq \eta < 1 \). Combining Eqs. (18) and (19), we have

\[
\mathbb{E}_{x \sim D} \left[ \sup_{z \in \mathbb{U}(x)} |h(z) - f(x)| \right] \leq \epsilon(1 - \eta) + \eta = \epsilon + \eta.
\]

By taking \( \eta = \epsilon \) we conclude the result.

**Proof of Theorem 5.2** Let

\[
\text{OPT}_\mathcal{H} = \inf_{h \in \mathcal{H}} \mathbb{E}_{x \sim D} \left[ \sup_{z \in \mathbb{U}(x)} |h(z) - f(x)| \right],
\]

which is obtained by \( h^* \in \mathcal{H} \). By Markov Inequality we have

\[
P_{x \sim D} \left( \sup_{z \in \mathbb{U}(x)} |h^*(z) - f(x)| > \eta \right) \leq \frac{\mathbb{E}_{x \sim D} \left[ \sup_{z \in \mathbb{U}(x)} |h^*(z) - f(x)| \right]}{\eta} \leq \frac{\text{OPT}_\mathcal{H}}{\sqrt{\text{OPT}_\mathcal{H}}} = \sqrt{\text{OPT}_\mathcal{H}}.
\]

This means that we can apply the algorithm for agnostic robust uniform \( \eta \)-regression with \( \eta = \sqrt{\text{OPT}_\mathcal{H}} \), and obtain error of \( \sqrt{\text{OPT}_\mathcal{H}} + \epsilon \). The problem is that \( \text{OPT}_\mathcal{H} \) is not known in advance. To overcome this issue, we can have a grid search on the scale of \( \eta \), and then verify our choice using an holdout training set.

We define a grid \( \Theta = \left\{ \frac{1}{m}, \frac{2}{m}, \frac{4}{m}, \frac{8}{m}, \ldots, 1 \right\} \), such that one of its elements satisfies \( \sqrt{\text{OPT}_\mathcal{H}} < \hat{\theta} < 2\sqrt{\text{OPT}_\mathcal{H}} \).

For each element in the grid, we execute the agnostic regressor for the \( \eta \)-robust loss. That is, we define \( \mathcal{H}_\theta = \{ h_\theta = \text{Agnostic-}\theta\text{-Regressor}(|S|) : \theta \in \Theta \} \).

We choose the optimal function on a holdout labeled set \( \tilde{S} \) of size \( \approx \frac{1}{\epsilon^2} \log \frac{1}{\epsilon} \),

\[
\hat{h}_\theta = \arg\min_{h_\theta \in \mathcal{H}_\theta} \frac{1}{|\tilde{S}|} \sum_{(x, f(x)) \in \tilde{S}} \mathbb{I}\left[ \sup_{z \in \mathbb{U}(x)} |h_\theta(z) - f(x)| \geq \theta \right].
\]
With high probability, the algorithm outputs a function with error at most $\sqrt{OPT_H} + \epsilon$ for the $\ell_p$ robust loss, using a sample of size

$$\tilde{O}\left(\frac{\text{fat}(\mathcal{H}, c\epsilon^{1/p}) \text{ fat}^*(\mathcal{H}, c\epsilon^{1/p})}{\epsilon^2}\right),$$

which is the sample complexity of agnostic robust $\epsilon^{1/p}$-regression.

$\blacksquare$