Recursive algorithm and branching for nonmaximal embeddings

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Abstract

Recurrent relations for branching coefficients in affine Lie algebras integrable highest weight modules are studied. The decomposition algorithm based on the injection fan technique is developed for the case of an arbitrary reductive subalgebra. In particular, we consider the situation where the Weyl denominator becomes singular with respect to the subalgebra. We demonstrate that for any reductive subalgebra it is possible to define the injection fan and the analogue of the Weyl numerator—the tools that describe explicitly the recurrent properties of branching coefficients. Possible applications of the fan technique in conformal field theory models are considered.

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1. Introduction

The branching problem for affine Lie algebras emerges in conformal field theory (CFT), for example, in the construction of modular-invariant partition functions [1]. Recently, the problem of conformal embeddings was considered in [2].

There are different approaches to deal with branching coefficients. Some of them use the BGG resolution [3] (for Kac–Moody algebras the algorithm is described in [4, 5]), the Schur function series [6], the BRST cohomology [7], Kac–Peterson formulas [4, 8] or some combinatorial methods applied in [9].

In this paper, we prove that for an arbitrary reductive subalgebra, branching coefficients are subject to a set of recurrent properties that can be explicitly formulated and that there exists an effective and simple algorithm to solve these recurrent relations step by step. The basic idea is similar to the one used in [11] for maximal embeddings. In our case, the algorithm is essentially different, and new properties of singular weights are determined to deal with an arbitrary reductive injection \( \mathfrak{a} \rightarrow \mathfrak{g} \).
The principal point is to consider the subalgebra \( \mathfrak{a} \) together with its counterpart \( \mathfrak{a}_\perp \) orthogonal to \( \mathfrak{a} \). For any reductive algebra \( \mathfrak{a} \) the subalgebra \( \mathfrak{a}_\perp \subset \mathfrak{g} \) is regular and reductive. For a highest weight module \( L^\mu(\mathfrak{g}) \) and orthogonal pair of subalgebras \( (\mathfrak{a}, \mathfrak{a}_\perp) \), we consider the so-called singular element \( \Psi^\mu_\perp(\mathfrak{a}_\perp) \) (the numerator in the Weyl character formula \( ch(L^\mu) = \frac{\Psi^\mu_\perp(\mathfrak{a}_\perp)}{\Psi^\mu(\mathfrak{a})} \), see for example [10]), the Weyl denominator \( \Psi^\mu(\mathfrak{a}) \) and the projection \( \Psi^\mu_\perp(\mathfrak{a}_\perp) = \pi_a \frac{\Psi^\mu_\perp(\mathfrak{a}_\perp)}{\Psi^\mu(\mathfrak{a})} \). We prove that for any highest weight \( h \)-diagonalizable module \( L^\mu(\mathfrak{g}) \) and orthogonal pair \( (\mathfrak{a}, \mathfrak{a}_\perp) \), the element \( \Psi^\mu_\perp(\mathfrak{a}_\perp) \) has a decomposition with respect to the set of Weyl numerators \( \Psi^\mu(\mathfrak{a})_\perp \) of \( \mathfrak{a}_\perp \). This decomposition provides the possibility of constructing a recurrent property for branching coefficients corresponding to the injection \( \mathfrak{a} \rightarrow \mathfrak{g} \). The property is formulated in terms of the specific element \( \Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}} \) of the group algebra \( \mathcal{E}(\mathfrak{g}) \) called ‘the injection fan’. Using this tool, we formulate a simple and explicit algorithm for branching coefficients computations applicable for arbitrary (maximal or nonmaximal) subalgebras of finite-dimensional or affine Lie algebras. In the case of maximal embedding, the corresponding fan is unsubtracted, the singular element becomes trivial \( \Psi^\mu_\perp(\mathfrak{a}_\perp) = \Psi^\mu(\mathfrak{g}) \) and the relations described earlier in [11] are reobtained.

We demonstrate that our algorithm is effective and can be used in studies of conformal embeddings and coset constructions in rational CFT.

The paper is organized as follows. In subsection 1.1, we fix the general notations. In section 2, we derive the decomposition formula based on recurrent properties of anomalous branching coefficients and describe the decomposition algorithm for integrable highest weight modules \( L_\mathfrak{g} \) with respect to a reductive subalgebra \( \mathfrak{a} \subset \mathfrak{g} \) (subsection 2.5). In section 3, we present several simple examples for finite-dimensional Lie algebras. Affine Lie algebras and their applications in CFT models are considered in section 4. General properties of the proposed algorithm and possible further developments are also discussed in section 5.

1.1. Notation

Consider affine Lie algebras \( \mathfrak{g} \) and \( \mathfrak{a} \) with underlying finite-dimensional subalgebras \( \mathfrak{g} \) and \( \mathfrak{a} \) and an injection \( \mathfrak{a} \rightarrow \mathfrak{g} \) such that \( \mathfrak{a} \) is a reductive subalgebra \( \mathfrak{a} \subset \mathfrak{g} \) with the correlated root spaces: \( h_\mathfrak{a}^* \subset h_\mathfrak{g}^* \) and \( h_\mathfrak{a}^* \subset h_\mathfrak{g}^* \). We use the following notations.

\( L^\mu(\mathfrak{a}) \) — the integrable module of \( \mathfrak{g} \) with the highest weight \( \mu \) (resp. integrable \( \mathfrak{a} \)-module with the highest weight \( \nu \));
\( r_\mathfrak{a}(r_\mathfrak{g}) \) — the rank of the algebra \( \mathfrak{g} \) (resp. \( \mathfrak{a} \));
\( \Delta \) (\( \Delta_\mathfrak{a} \)) — the root system; \( \Delta^+ \) (resp. \( \Delta^+_\mathfrak{a} \)) — the positive root system (of \( \mathfrak{g} \) and \( \mathfrak{a} \) respectively);
\( \text{mult}(\alpha) \) (\( \text{mult}_\mathfrak{a}(\alpha) \)) — the multiplicity of the root \( \alpha \) in \( \Delta \) (resp. in \( \Delta_\mathfrak{a} \));
\( \Delta_\mathfrak{a} \) — the finite root system of the subalgebra \( \mathfrak{a} \) (resp. \( \mathfrak{a} \));
\( N^\mu \) (\( N_\mathfrak{a}^\mu \)) — the weight diagram of \( L^\mu \) (resp. \( L^\mu_\mathfrak{a} \));
\( W_\mathfrak{a} \) (\( W_\mathfrak{g} \)) — the corresponding Weyl group;
\( C_\mathfrak{a} \) (\( C_\mathfrak{g} \)) — the fundamental Weyl chamber;
\( \bar{C}_\mathfrak{a} \) (\( \bar{C}_\mathfrak{g} \)) — the closure of the fundamental Weyl chamber;
\( \rho_\mathfrak{a} \) (\( \rho_\mathfrak{g} \)) — the Weyl vector;
\( \epsilon(\omega) := \det(\omega) \);
\( \alpha_i, (\beta_j) \) — the \( i \)th (resp. \( j \)th) basic root for \( \mathfrak{g} \) (resp. \( \mathfrak{a} \)); \( i = 0, \ldots, r \) (\( j = 0, \ldots, r_\mathfrak{a} \));
\( \delta \) — the imaginary root of \( \mathfrak{g} \) (and of \( \mathfrak{a} \) if any);
\( \alpha_i^\vee, (\beta_j^\vee) \) — the basic co-root for \( \mathfrak{g} \) (resp. \( \mathfrak{a} \)), \( i = 0, \ldots, r \) (\( j = 0, \ldots, r_\mathfrak{a} \));
\( \bar{\xi}, \xi(\alpha) \) — the finite (classical) part of the weight \( \xi \in P \) (resp. \( \xi(\alpha) \in P_\mathfrak{a} \));
\[ \lambda = (\lambda; k; n) \]—decomposition of the affine weight \( \lambda \) indicating the finite part \( \hat{\lambda} \), the level \( k \) and the grade \( n \);

- \( P \) (resp. \( P_a \))—the weight lattice;
- \( m(\xi) \; (m(\xi)^a) \)—the multiplicity of the weight \( \xi \in P \) (resp. \( \xi \in P_a \)) in the module \( L^\mu \) (resp. \( L^\mu_a \));
- \( ch(L^\mu) \) (resp. \( ch(L^\mu_a) \))—the formal character of \( L^\mu \) (resp. \( L^\mu_a \));
- \( R := \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{mult}(\alpha)} \) (resp. \( R_a := \prod_{\beta \in \Delta^+_a} (1 - e^{-\beta})^{\text{mult}_a(\beta)} \))—the Weyl denominator.

### 2. Recurrent relations for branching coefficients

Consider an integrable module \( L^\mu \) of \( \mathfrak{g} \) with the highest weight \( \mu \) and let \( a \subset \mathfrak{g} \) be a reductive subalgebra of \( \mathfrak{g} \). With respect to \( a \), the module \( L^\mu \) is completely reducible:

\[ L^\mu \bigotimes \mathfrak{g} \Downarrow a = \bigoplus_{\nu \in P^+ a} b(\mu) \nu L^\nu_a. \]

Using the projection operator \( \pi_a \) (to the weight space \( \mathfrak{h}^* a \)), one can rewrite this decomposition in terms of formal characters:

\[ \pi_a \circ ch(L^\mu) = \sum_{\nu \in P^+ a} b(\mu) \nu ch(L^\nu_a). \] (1)

We are interested in branching coefficients \( b(\mu) \nu \).

#### 2.1. Orthogonal subalgebra and injection fan

In this subsection, we shall introduce some simple constructions that will be used in our studies of branching and in particular the ‘orthogonal partner’ \( a^\perp \) for a reductive subalgebra \( a \) in \( \mathfrak{g} \).

In the Weyl–Kac formula, both numerator and denominator can be considered as formal elements containing the singular weights of the Verma modules \( V^\xi \) with the highest weights \( \xi = \mu \) and \( \xi = 0 \) [10]. We attribute singular elements to the corresponding integrable modules \( L^\mu \) and \( L^\nu_a \):

\[ \Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w(\mu + \rho) - \rho}, \]

\[ \Psi^{(\nu)}_a := \sum_{w \in W_a} \epsilon(w) e^{w(v + \rho_a) - \rho_a}, \]

and use the Weyl–Kac formula in the form

\[ ch(L^\mu) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}} = \frac{\Psi^{(\mu)}}{R}. \] (2)

Applying formula (2) to the branching rule (1), we obtain the relation connecting the singular elements \( \Psi^{(\mu)} \) and \( \Psi^{(\nu)}_a \):

\[ \pi_a \left( \frac{\sum_{w \in W} \epsilon(w) e^{w(\mu + \rho) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})} \right) = \sum_{\nu \in P^+ a} b(\mu) \nu \frac{\sum_{w \in W_a} \epsilon(w) e^{w(v + \rho_a) - \rho_a}}{\prod_{\beta \in \Delta^+_a} (1 - e^{-\beta})}, \]

\[ \pi_a \left( \frac{\Psi^{(\mu)}}{R} \right) = \sum_{\nu \in P^+ a} b(\mu) \nu \frac{\Psi^{(\nu)}_a}{R_a}. \] (3)
Here, $\Delta^+_a$ is the set of positive roots of the subalgebra $a$. (Without loss of generality, we consider them as vectors from the positive root space $\mathfrak h^+$ of $\mathfrak g$.)

Consider the root subspace $\mathfrak h^+_{\perp a}$ orthogonal to $a$,

$$\mathfrak h^+_{\perp a} := \{ \eta \in \mathfrak h^+ | \forall h \in \mathfrak h_a; \eta(h) = 0 \},$$

and the roots (correspondingly—positive roots) of $g$ orthogonal to $a$,

$$\Delta_{a\perp} := \{ \beta \in \Delta_g | \forall h \in \mathfrak h_a; \beta(h) = 0 \},$$

$$\Delta^+_{a\perp} := \{ \beta^+ \in \Delta^+_g | \forall h \in \mathfrak h_a; \beta^+(h) = 0 \}. $$

Let $W_{a\perp}$ be the subgroup of $W$ generated by the reflections $w_{\beta}$ for the roots $\beta \in \Delta^+_{a\perp}$. The subsystem $\Delta_{a\perp}$ determines the subalgebra $a_{\perp}$ with the Cartan subalgebra $\mathfrak h_{a\perp}$. Let

$$\mathfrak h^+_{\perp} := \{ \eta \in \mathfrak h^+_{\perp a} | \forall h \in \mathfrak h_{a\perp}; \eta(h) = 0 \}$$

and consider the subalgebras

$$\tilde{a}_{\perp} := a_{\perp} \oplus \mathfrak h_{\perp}$$

$$\tilde{a} := a \oplus \mathfrak h_{\perp}.$$ 

Algebras $a$ and $a_{\perp}$ form the ‘orthogonal pair’ $(a, a_{\perp})$ of subalgebras in $\mathfrak g$.

For the Cartan subalgebra, we have the decomposition

$$\mathfrak h = \mathfrak h_a \oplus \mathfrak h_{a\perp} \oplus \mathfrak h_{\perp} = \mathfrak h_a \oplus \mathfrak h_{a\perp} = \mathfrak h_{\perp} \oplus \mathfrak h_a.$$ 

(4)

For the subalgebras of an orthogonal pair $(a, a_{\perp})$, we consider the corresponding Weyl vectors, $\rho_{a}$ and $\rho_{a\perp}$, and form the so-called ‘defects’ $D_a$ and $D_{a\perp}$ of the injection:

$$D_a := \rho_a - \pi_a \rho,$$ 

$$D_{a\perp} := \rho_{a\perp} - \pi_{a\perp} \circ \rho.$$ 

(5)

(6)

For the highest weight module $L_\mu^\rho$ consider the singular weights $\{(w(\mu + \rho) - \rho) | w \in W\}$ and their projections to $\mathfrak h^+_{\perp a\perp}$ (additionally shifted by the defect $-D_{a\perp}$):

$$\mu_{\tilde{a}_{\perp}}(w) := \pi_{\tilde{a}_{\perp}} \circ [w(\mu + \rho) - \rho] - D_{a\perp}, \quad w \in W.$$ 

Among the weights $\{\mu_{\tilde{a}_{\perp}}(w) | w \in W\}$ choose those located in the fundamental chamber $\mathfrak C_{\tilde{a}_{\perp}}$ and let $U$ be the set of representatives $u$ for classes $W/W_{a\perp}$ such that

$$U := \{ u \in W | \mu_{\tilde{a}_{\perp}}(u) \in \mathfrak C_{\tilde{a}_{\perp}} \}.$$ 

(7)

For the same set $U$ introduce the weights

$$\mu_{a}(u) := \pi_a \circ [u(\mu + \rho) - \rho] + D_{a\perp},$$ 

To simplify the form of relations we shall now on omit the sign ‘$\circ$’ in projected weights.

To describe the recurrent properties for branching coefficients $h^{(\mu)}_\gamma$, we shall use the technique elaborated in [11]. One of the main tools is the set of weights $\Gamma_{a\rightarrow \perp}$ called the injection fan. As far as we consider the general situation (where the injection is not necessarily maximal) the notion of the injection fan is modified.

**Definition 1.** For the product

$$\prod_{\alpha \in \Delta \setminus \Delta^+_{a\perp}} (1 - e^{-\pi_a \alpha})^{\text{mult}(\alpha) - \text{mult}_a(\pi_a \alpha)} = \sum_{\gamma \in P_a} s(\gamma) e^{-\gamma}$$

(8)

consider the carrier $\Phi_{a\perp \subset a} \subset P_a$ of the function $s(\gamma) = \det(\gamma)$:

$$\Phi_{a\perp \subset a} = \{ \gamma \in P_a | s(\gamma) \neq 0 \}.$$ 

(9)
The ordering of roots in $\Delta_a$ induces the natural ordering of the weights in $P_a$. Denote by $\gamma_0$ the lowest vector of $\Phi_{a \subset \mathfrak{g}}$. The set

$$\Gamma_{a \subset \mathfrak{g}} = \{ \xi - \gamma_0 | \xi \in \Phi_{a \subset \mathfrak{g}} \} \setminus \{0\}$$

is called the injection fan.

In the next subsection, we shall see how the injection fan defines the recurrent properties of branching coefficients. It must be noted that the injection fan is the universal instrument that depends only on the injection.

2.2. Decomposing the singular element

Now, we shall prove that the Weyl–Kac character formula (in terms of singular elements) describes the particular case of a more general relation.

**Lemma 1.** Let $(a, a_1)$ be the orthogonal pair of reductive subalgebras in $\mathfrak{g}$, with $\widehat{a}_1 = a_1 \oplus \mathfrak{h}_1$ and $\widehat{a} = a \oplus \mathfrak{h}_1$. $L^\mu$ be the highest weight module with the singular element $\Psi^\mu$, and $R_{a_1}$ be the Weyl denominator for $a_1$.

Then, the element $\Psi^\mu(a_1, a_1) = \pi_a(\frac{\Psi^\mu}{R_{a_1}})$ can be decomposed into the sum over $u \in U$ (see (7)) of singular weights $\omega^\mu(a_1)$ with the coefficients $\epsilon(u)\dim(L^\mu_{\widehat{a}_1}(a_1))$:

$$\Psi^\mu(a_1, a_1) = \pi_a \left( \frac{\Psi^\mu}{R_{a_1}} \right) = \sum_{u \in U} \epsilon(u)\dim(L^\mu_{\widehat{a}_1}(a_1)) \omega^\mu(a_1).$$

**Proof.** With $u \in U$ and $v \in W_{a_1}$ apply the decomposition

$$u(\mu + \rho) = \pi_a u(\mu + \rho) + \pi_{\widehat{a}} u(\mu + \rho)$$

to the singular weight

$$vu(\mu + \rho) - \rho = \pi_a(u(\mu + \rho)) - \rho + \rho_{a_1} + \pi_{\mathfrak{h}_1} \rho$$

$$+ v(\pi_{\widehat{a}} u(\mu + \rho) - \rho_{a_1} - \rho_{a_1} - \pi_{\mathfrak{h}_1} \rho).$$

(12)

Use the defect $D_{a_1}$ (6) to simplify the first summand in (12):

$$\pi_a(u(\mu + \rho)) - \rho + \rho_{a_1} + \pi_{\mathfrak{h}_1} \rho$$

$$= \pi_a(u(\mu + \rho)) - \pi_a \rho - \pi_{a_1} \rho + \rho_{a_1}$$

$$= \pi_a(u(\mu + \rho)) - \rho + D_{a_1},$$

and the second one

$$v(\pi_{\widehat{a}} u(\mu + \rho) - \rho_{a_1} + \rho_{a_1} - \pi_{\mathfrak{h}_1} \rho$$

$$= v(\pi_{\widehat{a}} u(\mu + \rho) - D_{a_1} - \pi_{a_1} \rho - \pi_{\mathfrak{h}_1} \rho + \rho_{a_1} - \rho_{a_1}$$

$$= v(\pi_{\widehat{a}} [u(\mu + \rho) - \rho] - D_{a_1} - \rho_{a_1} - \rho_{a_1}).$$

These expressions provide a kind of a factorization in the anomalous element $\Psi^\mu$, and we find in it the combination of anomalous elements $\Psi^\mu_{\widehat{a}_1}$ of the subalgebra $\widehat{a}_1$ and modules $L^\mu_{\widehat{a}_1}$:

$$\Psi^\mu = \sum_{u \in U} \sum_{v \in W_{a_1}} \epsilon(v)\epsilon(u) e^{\pi_a[u(\mu + \rho) - \rho]}$$

$$= \sum_{u \in U} \epsilon(u) e^{\pi_a[u(\mu + \rho) - \rho] + D_{a_1}} \sum_{v \in W_{a_1}} \epsilon(v) e^{(\pi_{\widehat{a}_1} u(\mu + \rho) - \rho_{a_1} + \rho_{a_1} - \rho_{a_1})}$$

$$= \sum_{u \in U} \epsilon(u) e^{\pi_a[u(\mu + \rho) - \rho] + D_{a_1}} \Psi^\mu_{\widehat{a}_1}.$$
Dividing both sides by the Weyl element $R_a = \prod_{\beta \in \Delta_{a_\perp}} (1 - e^{-\beta})^{\text{mult}(\beta)}$ and projecting them to the weight space $h_a^\perp$, we obtain the desired relation

$$
\Psi_{(a, a_\perp)}^{(\mu)} = \sum_{u \in W/W_{a_\perp}} \epsilon(u) e^{\varphi(u(\mu + \rho) - \rho)} \prod_{\beta \in \Delta_{a_\perp}} (1 - e^{-\beta})^{\text{mult}(\beta)}
\prod_{\beta \in \Delta_{a_\perp}} (1 - e^{-\beta})^{\text{mult}(\beta)}
$$

$$
= \sum_{u \in U} \epsilon(u) \dim \left( L_{a_\perp}^{\varphi(u(\mu + \rho) - \rho)} \right) e^{\varphi(u(\mu + \rho) - \rho)}.
\tag{13}
\]"
Then, expand the sum in parentheses (with respect to the formal basis in $\mathcal{E}$):

$$
\Psi_{(a, a_1)}^{(\mu)} = - \sum_{\gamma \in \Phi_{a \subset g}} s(\gamma) e^{-\gamma} \sum_{\lambda \in P_a} k^{(\mu)}_{\lambda+\gamma} e^\lambda = - \sum_{\gamma \in \Phi_{a \subset g}} s(\gamma) k^{(\mu)}_{\gamma+\gamma} e^\gamma.
$$

Substituting the expression obtained in lemma 1 into the left-hand side, we obtain

$$
\Psi_{(a, a_1)}^{(\mu)} = \sum_{u \in U} \varepsilon(u) e^{s_a(u(\mu+\rho))} \dim \left( L^\mu_{\Delta_1} \right) 
= \sum_{u \in U} \varepsilon(u) e^{s_a(u(\mu+\rho)-\rho)} \dim \left( L^\mu_{\Delta_1} \right) 
= - \sum_{\gamma \in \Phi_{a \subset g}} \sum_{\lambda \in P_a} s(\gamma) k^{(\mu)}_{\gamma+\gamma} e^\gamma.
$$

The immediate consequence of this equality is

$$
\sum_{u \in U} \varepsilon(u) \dim \left( L^\mu_{\Delta_1} \right) \delta_{\xi, \pi_a(u(\mu+\rho)-\rho)} + \sum_{\gamma \in \Phi_{a \subset g}} s(\gamma) k^{(\mu)}_{\gamma+\gamma} = 0, \quad \xi \in P_a. \tag{16}
$$

The obtained formula means that the coefficients $k^{(\mu)}_{\gamma+\gamma}$ for $\gamma \in \Phi_{a \subset g}$ are not independent, they are subject to linear relations, and the form of these relations changes when the tested weight $\xi$ coincides with one of the ‘singular weights’ $[\pi_a[u(\mu+\rho)] | u \in U]$. To conclude the proof, we extract the lowest weight $\gamma_0 \in \Phi_{a \subset g}$ and pass to a summation over the vectors of the injection fan $\Gamma_{a \to g}$ (see definition 1). Thus, we obtain the desired recurrent relation (15).

\[\square\]

### 2.4. Embeddings and orthogonal pairs in simple Lie algebras

In this subsection, we discuss some properties of ‘orthogonal pairs’ of subalgebras in simple Lie algebras of classical series.

When both $\mathfrak{g}$ and $\mathfrak{a}$ are finite dimensional, all regular embeddings can be obtained by a successive elimination of nodes in the extended Dynkin diagram of $\mathfrak{g}$ (and $\Delta_{a_1}^\perp = \emptyset$ if $\mathfrak{a}$ is maximal). For the classical series $A$, $C$ and $D$ when the regular injection $\mathfrak{a} \to \mathfrak{g}$ is thus fixed, the Dynkin diagram for $\mathfrak{a}_1$ is obtained from the extended diagram of $\mathfrak{g}$ by eliminating the subdiagram of $\mathfrak{a}$ and the adjacent nodes (see table 1).

In the case of $B$ series, the situation is different. The reason is that here the subalgebra $\mathfrak{a}_1$ may be larger than the one obtained by the elimination of the subdiagram of $\mathfrak{a}$ and the adjacent nodes. The subalgebras of the orthogonal pair, $\mathfrak{a}$ and $\mathfrak{a}_1$, must not form a direct sum in $\mathfrak{g}$. It can be directly checked that when $\mathfrak{g} = B_r$ and $\mathfrak{a} = B_{r_a}$, the orthogonal subalgebra is $\mathfrak{a}_1 = B_{r-r_a}$. Consider the injection $B_{r_a} \to B_r, \quad 1 < r_a < r$. By eliminating the simple root $\alpha_{r_a-1} = e_{r_a-1} - e_{r_a}$, one splits the extended Dynkin diagram of $B_r$ into the disjoint diagrams for $\mathfrak{a} = B_{r_a}$ and $D_{r-r_a}$. But the system $\Delta_{a_1}$ contains not only the simple roots $\{e_1 - e_2, e_2 - e_3, \ldots, e_{r_a-2} - e_{r_a-1}, e_1 + e_2\}$ but also the root $e_{r_a-1}$. Thus, $\Delta_{a_1}$ forms the subsystem of the type $B_{r-r_a}$ and the orthogonal pair for the injection $B_{r_a} \to B_r$ is $(B_{r_a}, B_{r-r_a})$.

In the next section, a particular case of such an orthogonal pair is presented for the injection $B_2 \to B_4$ (see figure 3).

The complete classification of regular subalgebras for affine Lie algebras can be found in the recent paper [12]. From the complete classification of maximal special subalgebras in
Table 1. Subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$ for the classical series.

| $\mathfrak{g}$ | Extended diagram of $\mathfrak{g}$ | Diagrams of the subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$ |
|----------------|-----------------------------------|----------------------------------|
| $A_n$          | ![Diagrams](image1)               | ![Diagrams](image2)              |
| $C_n$          | ![Diagrams](image3)               | ![Diagrams](image4)              |
| $D_n$          | ![Diagrams](image5)               | ![Diagrams](image6)              |

classical Lie algebras [13], we can deduce the following list of pairs of orthogonal subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$:

- $\text{su}(p) \oplus \text{su}(q) \subset \text{su}(pq)$
- $\text{so}(p) \oplus \text{so}(q) \subset \text{so}(pq)$
- $\text{sp}(2p) \oplus \text{sp}(2q) \subset \text{so}(4pq)$
- $\text{sp}(2p) \oplus \text{so}(q) \subset \text{sp}(2pq)$
- $\text{so}(p) \oplus \text{so}(q) \subset \text{so}(p+q)$ for $p$ and $q$ odd.

2.5. Algorithm for recursive computation of branching coefficients

The recurrent relation (15) allows us to formulate an algorithm for recursive computation of branching coefficients. In this algorithm, there is no need to construct the module $L^{(\mu)}_g$ or any of the modules $L^{(\mu)}_a$.

It contains the following steps.

i. Construct the root system $\Delta_\mathfrak{a}$ for the embedding $\mathfrak{a} \rightarrow \mathfrak{g}$.

ii. Select all positive roots $\alpha \in \Delta^+$ orthogonal to $\mathfrak{a}$, i.e. form the set $\Delta_\mathfrak{a}^\perp$.

iii. Construct the set $\Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}}$. Relation (8) defines the sign function $s(\gamma)$ and the set $\Phi_{a \subset g}$, where the lowest weight $\gamma_0$ is to be subtracted to obtain the fan (10): $\Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}} = \{ \xi - \gamma_0 | \xi \in \Phi_{a \subset g} \} \setminus \{0\}$.

iv. Construct the set $\hat{\Psi}^{(\mu)} = \{ w(\mu + \rho) - \rho; \; w \in W \}$ of singular weights for the $\mathfrak{g}$-module $L^{(\mu)}_g$.

v. Select the weights $\{ \mu_{\hat{\mathfrak{a}}}^{(\mu)}(w) = \pi_{\hat{\mathfrak{a}}}^{-1}[w(\mu + \rho) - \rho] - D_{\mathfrak{a}_\perp} \in \mathcal{C}_{\hat{\mathfrak{a}}}^\perp \}$. Since the set $\Delta_\mathfrak{a}^\perp$ is fixed, we can easily check whether the weight $\mu_{\hat{\mathfrak{a}}}^{(\mu)}(w)$ belongs to the main Weyl chamber $\mathcal{C}_{\hat{\mathfrak{a}}}^\perp$ (by computing its scalar product with the fundamental weights of $\mathfrak{a}_\perp^+$).

vi. For the weights $\mu_{\hat{\mathfrak{a}}}^{(\mu)}(w)$, calculate dimensions of the corresponding modules, $\dim(L^{(\mu_{\hat{\mathfrak{a}}}^{(\mu)}(w)})$, using the Weyl dimension formula, and construct the singular element $\psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)}$.

vii. Calculate the anomalous branching coefficients using the recurrent relation (15) and select among them those corresponding to the weights in the main Weyl chamber $\mathcal{C}_{\hat{\mathfrak{a}}}$.

We can speed up the algorithm by one-time computation of the representatives of the conjugate classes $W/W_{\mathfrak{a}_\perp}$.

The next section contains examples illustrating the application of this algorithm.
Figure 1. Regular embedding of $A_1$ into $B_2$. Simple roots $\alpha_1$ and $\alpha_2$ of $B_2$ are presented as dashed vectors. The simple root $\beta = \alpha_1 + 2\alpha_2$ of $A_1$ is grey. The highest weight of the fundamental representation $L_{B_2}^{(1,0)}$ is black. The weights of the singular element $\Psi_1(\omega_1)$ are marked by circles with superscripts indicating the corresponding determinants $\epsilon(w)$.

3. Branching for finite-dimensional Lie algebras

3.1. Regular embedding of $A_1$ into $B_2$

Consider the regular embedding $A_1 \rightarrow B_2$. Simple roots $\alpha_1$ and $\alpha_2$ of $B_2$ are presented as dashed vectors in figure 1. We denote the corresponding Weyl reflections by $w_1$ and $w_2$. The simple root $\beta = \alpha_1 + 2\alpha_2$ of $A_1$ is grey.

Let us perform the reduction of the fundamental representation $L_{B_2}^{(1,0)}$ (\omega_1 is the black vector in figure 1) according to the steps of the algorithm. The root $\alpha_1$ is orthogonal to $\beta$, so we have $\Delta_{\alpha_1}^+ = \{\alpha_1\}$ (step (ii)). According to definition 1, the fan $\Gamma_{A_1 \rightarrow B_2}$ (step (iii)) consists of two weights:

$$\Gamma_{A_1 \rightarrow B_2} = \{(1; 2), (2; -1)\},$$

where the second component is the value of the sign function $s(\gamma)$. Singular weights $\{w(\omega_1 + \rho) - \rho; \ w \in W\}$ (step (iv)) are indicated by circles with the superscript $\epsilon(w)$. The space $U$ is the factor $W/W_{\alpha_1}$, where $W_{\alpha_1} = \{e, w_1\}$. This means that singular weights located above the $\beta$-line belong to the Weyl chamber $C_{\beta-}$. According to formula (6), we have $D_{\alpha_1} = 0$ and $b_\perp = 0$; thus, $[\mu_{\alpha_1}(w) = \pi_{\alpha_1}[w(\mu + \rho) - \rho]]$. We obtain four highest weights for $\alpha_\perp$-modules. In terms of $\alpha_\perp$-fundamental weight $\frac{1}{2}\alpha_1$, these highest weights $[\mu_{\alpha_\perp}(u) = \pi_{\alpha_\perp}[u(\mu + \rho) - \rho]] u \in U$ are $\{(1)(2)(1)\}$ (step v). In figure 2, the corresponding weight diagrams $\{N_{\alpha_\perp}^{\mu_{\alpha_\perp}(u)}\}$ are attached to the set of $\alpha$-weights $[\mu_{\alpha}(u)] = \{\pi_{\alpha}[u(\mu + \rho) - \rho]\} = \{(1) (0) (-4) (-5)\}$. In fact, we do not need the weight diagrams but only the dimensions of modules $L_{\alpha_\perp}^{\mu_{\alpha_\perp}(u)}$, multiplied by $\epsilon(u)$ (step vi). Obtained
values are to be attributed to the points \{(1) (0) (-4) (-5)\} in \(P_a\). The singular element \(\Psi^{(\mu)}_{(0,0,-)}\) has the set of weights with anomalous multiplicities:

\[
\{ (1; 2), (0; -3), (-4; 3), (-5; -2) \}.
\]

(17)

Applying formula (15) with the fan \(\Gamma_{A_1 \rightarrow B_1}\) to the set (17) (step vii), we obtain zeros for the weights greater than the highest anomalous vector \((1; 2)\) and \(k_1^{(1,0)} = 2\) for the vector \((1; 2)\) itself. For the anomalous weight \((0; -3)\) on the boundary of \(\bar{C}_{a(0)}\), the recurrent relation gives

\[
k_0^{(1,0)} = -1 \cdot k_2^{(1,0)} + 2 \cdot k_1^{(1,0)} - 3 \cdot \delta_{0,0} = 1,
\]

and the branching is completed: \(L_{a_2 \downarrow A_1} = 2L_{A_1}^{\mu(A_1)} \bigoplus L_{A_1}^{2\mu(A_1)}\).

3.2. Embedding \(B_2\) into \(B_4\)

Consider the regular embedding \(B_2 \rightarrow B_4\). The corresponding Dynkin diagrams are presented in the figure 3.
Figure 4. The singular element $e^\gamma \Psi(\mu)_{e_{a}, e_{a\perp}}$ displayed in the weight subspace $P_a$ for $a = B_2$ with the basis $\{e_1, e_2\}$. We see the projected singular weights $\{\pi_{a\perp} [u(\mu + \rho) - \rho] + \gamma_0 | u \in U\}$ shifted by $\gamma_0$ and supplied by multipliers $\epsilon(u) \dim(L_{a\perp}^{\mu, \rho}(u))$.

In the orthonormal basis $\{e_1, \ldots, e_4\}$, simple roots and positive roots of $B_4$ are

$S_{B_4} = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4\}$,

$\Delta^+_{B_4} = \{(e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4, e_1 - e_2, e_2 - e_3, e_3, e_4, e_2 + e_4, e_2 + e_3, e_1 + e_3, e_1 + e_2)\}$.

The subalgebra $a = B_2$ is fixed by the simple roots

$S_{B_2} = \{e_3 - e_4, e_4\}$.

Its orthogonal counterpart $a_\perp = B_2$ has

$S_{a_\perp} = \{e_1 - e_2, e_2\}$,

$\Delta^+_{a_\perp} = \{e_1 - e_2, e_1 + e_2, e_1\}$.

As far as the set $\Delta^+_{B_4} \setminus \Delta^+_{a_\perp}$ is fixed, the injection fan $\Gamma_{B_4 \rightarrow B_2}$ can be constructed using definition 1. As far as for this injection $s(\gamma_0) = -1$ in the recursion formula, we need only the factor $s(\gamma + \gamma_0)$. The result is presented in figure 5.

Consider the $B_4$-module $L^\mu$ with the highest weight $\mu = 2e_1 + 2e_2 + e_3 + e_4$; $\dim(L^{0,1,0,2}) = 2772$. Here, the defect is nontrivial, $D_{a_\perp} = -2(e_1 + e_2)$, while $h_\perp = 0$. Among the singular weights, there are 48 vectors with the property $\{\mu_{a_\perp}(u) = \pi_{a_\perp} [u(\mu + \rho) - \rho] - D_{a_\perp} \in C_{a_\perp}\}$. The set $U = \{u\}$ is thus fixed. Compute the dimensions of the corresponding $a_\perp$-modules with the highest weights $\mu_{a_\perp}(u)$ (using the Weyl dimension formula) and multiply them by $\epsilon(u)$. The result is the singular element $\Psi(\mu)_{e_{a_\perp}, e_{a_\perp}}$ shown in figure 4.

Now one can place the fan $\Gamma$ (see figure 5) in the highest of the weights presented in figure 4 and start the recursive determination of the branching coefficients (using
4. Applications to conformal field theory

4.1. Conformal embeddings

Branching coefficients for an embedding of affine Lie algebra into affine Lie algebra can be used to construct modular-invariant partition functions for Wess–Zumino–Novikov–Witten models in CFT ([1, 14–16]). In these models current algebras are affine Lie algebras.

The modular-invariant partition function is crucial for the conformal theory to be valid on the torus and higher genus Riemann surfaces. It is important for the applications of CFT to string theory and to critical phenomena description.

The simplest modular-invariant partition function has the diagonal form:

$$Z(\tau) = \sum_{\mu \in \mathcal{P}_+} \chi_{\mu}(\tau) \bar{\chi}_{\mu}(\bar{\tau}).$$

(18)

Here, the sum is over the set of highest weights for integrable modules in a Wess–Zumino–Witten (WZW) model and \(\chi_{\mu}(\tau)\) are the normalized characters (see [1]) of these modules.

To construct nondiagonal modular invariants is not an easy problem, although for some models the complete classification of modular invariants is known [17, 18].

Consider the WZW model with the affine Lie algebra \(\mathfrak{a}\). Nondiagonal modular invariants for this model can be constructed from the diagonal invariant if there exists an affine algebra \(\mathfrak{g}\) such that \(\mathfrak{a} \subset \mathfrak{g}\). Then, we can replace the characters of the \(\mathfrak{g}\)-modules in the diagonal modular-invariant partition function (18) by the decompositions

$$\sum_{\nu \in \mathcal{P}_+} b_{\nu}^{(\mathfrak{a})} \chi_{\nu},$$
containing the normalized characters $\chi_\nu$ of the corresponding $\mathfrak{a}$-modules. Thus, we obtain a nondiagonal modular-invariant partition function for the theory with the current algebra $\mathfrak{a}$:

$$Z_\mathfrak{a}(\tau) = \sum_{\nu, \lambda \in P^+_\mathfrak{a}} \chi_\nu(\tau) M_{\nu\lambda} \overline{\chi}_\lambda(\bar{\tau}). \quad (19)$$

The effective reduction procedure is crucial for this construction. The embedding is required to preserve the conformal invariance. Let $X^j_{-n}$ and $\tilde{X}^{j'}_{-n}$ be the lowering generators for $\mathfrak{g}$ and for $\mathfrak{a} \subset \mathfrak{g}$, correspondingly. Let $\pi_\mathfrak{a}$ be the projection operator of $\pi_\mathfrak{a}: \mathfrak{g} \rightarrow \mathfrak{a}$. In the theory attributed to $\mathfrak{g}$ with the vacuum $|\lambda\rangle$, the states can be described as

$$X^j_{-n_1} X^j_{-n_2} \cdots |\lambda\rangle, \; n_1 \geq n_2 \geq \cdots > 0.$$  

And for the sub-algebra $\mathfrak{a}$, the corresponding states are

$$\tilde{X}^{j'}_{-n_1} \tilde{X}^{j'}_{-n_2} \cdots |\pi_\mathfrak{a}(\lambda)\rangle.$$  

The $\mathfrak{g}$-invariance of the vacuum entails its $\mathfrak{a}$-invariance, but this is not the case for the energy–momentum tensor. So the energy–momentum tensor of the larger theory should contain only the generators $\tilde{X}$. Then, the relation

$$T_\mathfrak{g}(z) = T_\mathfrak{a}(z) \quad (20)$$

leads to the equality of central charges

$$c(\mathfrak{g}) = c(\mathfrak{a})$$

and to the relation

$$\frac{k \dim \mathfrak{g}}{k + \dim \mathfrak{a}} = \frac{x_\mathfrak{g} \dim \mathfrak{a}}{x_\mathfrak{a} k + \dim \mathfrak{a}}. \quad (21)$$

Here, $x_\mathfrak{g}$ is the so-called ‘embedding index’: $x_\mathfrak{g} = \frac{|\pi_\mathfrak{g}/\Theta_1|^2}{|\pi_\mathfrak{a}/\Theta_1|^2}$ with $\Theta$ and $\Theta_2$ being the highest roots of $\mathfrak{g}$ and $\mathfrak{a}$, while $\mathfrak{g}$ and $\mathfrak{a}$ are the corresponding dual Coxeter numbers.

It can be demonstrated that the solutions of equation (21) exist only for the level $k = 1$ [1].

The complete classification of conformal embeddings is given in [16]. Relation (21) and the asymptotics of the branching functions can be used to prove the finite reducibility theorem [19]. It states that for a conformal embedding $\mathfrak{a} \hookrightarrow \mathfrak{g}$, only a finite number of branching coefficients have nonzero values.

**Note 4.1.** The orthogonal subalgebra $\mathfrak{a}_\perp$ is always trivial for conformal embeddings $\mathfrak{a} \hookrightarrow \mathfrak{g}$.

**Proof.** Consider the modes expansion of the energy–momentum tensor

$$T(z) = \frac{1}{2(k + h)} \sum_n z^{-n-1} L_n.$$  

The modes $L_n$ are constructed as the combinations of normally-ordered products of $\mathfrak{g}$-algebra generators:

$$L_n = \frac{1}{2(k + h)} \sum_{\alpha} \sum_m :X^\alpha_m X^\alpha_m:.$$  

In the case of a conformal embedding, the energy–momentum tensors $T_\mathfrak{g}(z)$ and $T_\mathfrak{a}(z)$ are equal (see (20)).
4.1.1. Special embedding $\hat{A}_1 \rightarrow \hat{A}_2$. Consider the case where both $g$ and $a$ are affine Lie algebras: $\hat{A}_1 \rightarrow \hat{A}_2$ and the injection is the affine extension of the special injection $A_1 \rightarrow A_2$ with the embedding index $x_\gamma = 4$. As far as $g$-modules to be considered are of level one, the necessary $a$-modules will be of level $\bar{k} = kx_\gamma = 4$.

There exist three-level one-fundamental weights of $\hat{A}_2$. It is easy to see that the set $\Delta_{a_\perp}$ is empty and the subalgebra $a_\perp = 0$. Then, $D_{a_\perp} = 0$, $h_\perp$ is one-dimensional Abelian subalgebra, and the dimension of $\bar{a}_{\perp} = a_{\perp} \oplus h_\perp$ is also 1. It is convenient to choose the classical root for $\hat{A}_1$ to be $\beta = \frac{1}{2}(\alpha_1 + \alpha_2)$.

Using definition (1), we construct the fan $\Gamma_{\hat{A}_1 \rightarrow \hat{A}_2}$. In this case $\gamma_0 = 0$ and its sign $s(0) = -1$; thus, we are to use the sign function $s(\gamma)$ (see figure 6).

Consider the module $L^{(0,0;0)}_{\omega_0}$ of $\hat{A}_2$. Here, we use the (finite part; level; grade) presentation of the highest weight, and the finite part coordinates are the Dynkin indices (see section (1.1)).

The set $\Psi^{(a_\perp)}(\hat{A}_1, a_{\perp} = 0)$ is displayed in figure 7 up to the sixth grade.

The next step is to project the anomalous weights to $P_{\hat{A}_1}$. The result is the element $\Psi^{(a_{\perp})}(\hat{A}_1, a_{\perp} = 0)$ presented in figure 8 up to the 12th grade.

Using the recurrent relation (15) with the fan $\Gamma_{\hat{A}_1 \rightarrow \hat{A}_2}$ and the singular weights in $\Psi^{(a_{\perp})}(\hat{A}_1, a_{\perp} = 0)$, we obtain the anomalous branching coefficients presented in figure 9. Inside the Weyl chamber $\tilde{C}_{\hat{A}_1}$ (its boundaries are indicated in figure 9) there are only two nonzero anomalous weights and both have multiplicity 1. These are the highest weights of $a$-submodules and the multiplicities are their branching coefficients. Thus, we obtain the decomposition

$$L_{\hat{A}_1 \rightarrow \hat{A}_2}^{(0,0;1;3)} = L_{\hat{A}_1}^{(0,0;4;0)} \oplus L_{\hat{A}_1}^{(4,4;0)}.$$ 

Note that the finite reducibility theorem holds.

In these combinations, we are to substitute $a$-generators in terms of $g$-generators and obtain the energy–momentum tensor $T_g$. But if the set of generators attributed to $\Delta_{a_{\perp}}$ is not empty, this is not possible, since $T_g$ contains generators $X_\alpha^n$ for $\alpha \in \Delta_{a_{\perp}}$. $\square$

Figure 6. The fan $\Gamma_{\hat{A}_1 \rightarrow \hat{A}_2}$ for $\hat{A}_1 \rightarrow \hat{A}_2$ in the basis $\{\beta, \delta\}$. Note that $\gamma_0 = 0$, so the values of $s(\gamma)$ are prescribed to the weights $\gamma \in \Gamma_{\hat{A}_1 \rightarrow \hat{A}_2}$. 

...
Figure 7. Singular weights of the module $L_{\omega} = L^{(0,0,1;0)}$. The classical (grade zero) cross-section of the diagram is shown separately in the right part of the figure. We use the orthogonal basis with the unit vector equal to $\alpha_1$. The weights $w(\omega_0 + \rho) - \rho$ are marked by crosses when $\epsilon(w) = 1$ and by box when $\epsilon(w) = -1$. Simple roots of the classical subalgebra $A_2$ are grey, and the grey diagonal plane corresponds to the Cartan subalgebra of the embedded algebra $A_1$.

Figure 8. The singular element $\Psi_h^{(0)}\Gamma_{A_1} \rightarrow \hat{A}_2$ displayed in $P_{\hat{A}_1}$ with the basis $\{\beta, \delta\}$.

The same fan $\Gamma_{A_1} \rightarrow \hat{A}_2$ can be used for any other highest weight module $L^{\mu}_{\hat{A}_2}$. In particular, for irreducible modules of level one we obtain the trivial branching:

$$L^{(1,0;1;0)}_{\hat{A}_2 \rightarrow \hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}, \quad L^{(0,1;1;1)}_{\hat{A}_2 \rightarrow \hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}.$$
Figure 9. Anomalous branching coefficients for $\hat{A}_1 \subset \hat{A}_2$. The boundaries of the main Weyl chamber $\bar{C}_{\hat{A}_1}$ are indicated by black lines. Two anomalous highest weights located in the main Weyl chamber are marked by stars. Both have multiplicity 1, so the branching coefficients for them are equal to 1.

Using these results, the modular-invariant partition function is easily found:

$$Z = \left| \chi(4;4;0) + \chi(0;4;0) \right|^2 + 2\chi^2((2;4;0)).$$

4.2. Coset models

Coset models [20] tightly connected with the gauged WZW models are actively studied in string theory, especially, in string models on anti-de Sitter space [21–25]. The characters in coset models are proportional to branching functions:

$$\chi_{\nu}(\tau) = e^{2\pi i r(m_{\mu} - m_{\nu})} b_{\nu}(\tau),$$

with

$$m_{\mu} = \frac{|\mu + \rho|^2}{2(k + g)} - \frac{|\rho|^2}{2g}.$$  \hfill (22)

The problem of branching functions construction in the coset models was considered in [7, 26, 27].

Let us return to our example 3.1 and consider the affine extension of the injection $A_1 \rightarrow B_2$. Since this embedding is regular and $x_e = 1$, the subalgebra modules and the initial module are of the same level. The set of positive roots with zero projection on the root space of the subalgebra $\hat{A}_1$ is the same as in the finite-dimensional case: $\Delta_{\hat{A}_1}^{+} = \{\alpha_1\}$ and $\mathfrak{a}_\perp = A_1$. It is easy to see that here $\mathfrak{h}_\perp$ is trivial and $D_{\mathfrak{a}_\perp} = 0$.

Using definition (1), we obtain the fan $\Gamma_{\hat{A}_1 \rightarrow \hat{B}_2}$. Note that here the lowest weight $\gamma_0$ of the fan is zero and $s(\gamma_0) = -1$. Values of the sign function $s(\gamma)$ for $\gamma \in \Gamma_{\hat{A}_1 \rightarrow \hat{B}_2}$ are presented in figure 10. We restricted the computation to the 12th grade.

Consider the level one module $L^{(1,0;1,0)}_{\hat{B}_2}$ with the highest weight $\omega_1 = (1, 0; 1; 0)$, where the finite part coordinates are in the orthogonal basis $e_1, e_2$. The set of anomalous weights for this module up to the sixth grade is presented in figure 11.
Figure 10. The fan $\Gamma_{\hat{A}_1 \rightarrow \hat{B}_2}$ for $\hat{A}_1 \rightarrow \hat{B}_2$ in the basis $\{\beta, \delta\}$. Values of $s(\gamma)$ are shown for the weights $\gamma \in \Gamma_{\hat{A}_1 \rightarrow \hat{B}_2}$.

Figure 11. Singular weights for $L(1,0;1,0;\hat{B}_2)$. The standard basis $\{e_1, e_2\}$ is used for the classical cross-section. The weights in the zero grade are the same as in figure 1. The weights $w(\omega_1 + \rho) - \rho$ are marked by crosses if $\epsilon(w) = 1$ and by boxes for $\epsilon(w) = -1$. Simple roots of the classical subalgebra $\hat{B}_2$ are grey, and grey diagonal plane corresponds to the Cartan subalgebra of the embedded algebra $\hat{A}_1$.

According to the recursive algorithm (section 2.5), we project these anomalous weights to $P_{\hat{A}_1}$ and find the dimensions of the corresponding $a_{\perp}$-modules $L^{\rho_{\perp}}_{a_{\perp}}(w(\mu + \rho)) - \rho_{\perp}$. In the grade zero, this projection gives exactly the set $\Psi^{(\mu)}_{(A_1; A_1)}$ for the embedding of the classical Lie
algebra $A_1 \to B_2$. To see this compare figure 1 with figure 12, where the singular element $\Psi^{(\mu)}_{(A_1, A_1)}$ for the affine embedding $\hat{A}_1$ is presented up to the 12th grade.

Multiplicities of the highest weights inside the Weyl chamber $\mathcal{C}_{\hat{A}_1}$ (see figure 13) define the following branching coefficients (up to the 12th grade):

$$L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{B_1 \to A_1} = 2L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 1L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 4L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 2L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 8L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 8L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 15L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 15L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 12L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 12L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 26L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 26L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 51L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1} + 51L^{(\mu)\mathcal{C}_{\hat{A}_1}}_{A_1}.$$
weights is more interesting. We have found out that in the new singular element
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be solved step by step.

\[
\frac{\omega}{\Delta_1} \quad \text{reductive subalgebras (maximal as well as nonmaximal). It was shown that the branching}
\]
We have demonstrated that the injection fan technique can be used to deal with arbitrary
problem for
reductive subalgebra
\( g \subset a \perp \) to the roots of
\( a \perp \) and its submodules. For the fan
\( g \subset a \perp \) of the corresponding
\( a \perp \)-modules depend substantially on the structure of
\( a \perp \) and its submodules. For the fan
\( \Gamma_{a \rightarrow g} \), this dependence is almost obvious: in the element \( \Phi_{a \rightarrow g} \), the factors corresponding
to the roots of \( \Delta_{a \perp} \) are eliminated. The transformation in the set of projected singular
weights is more interesting. We have found out that in the new singular element \( \Psi_{[\mu]}^{[\lambda_1, \lambda_2]} \), the coefficients depend on the \( a_{\perp} \)-submodules (their highest weights \( \mu_{\perp} (u) \) are fixed by
the injection and by the weights of the initial element \( \Psi^{[\mu]} \)). Fortunately, no more information on
\( L_{[\mu_1]}^{[\lambda_1]} (u) \)-submodules is necessary than their dimensions. In the new singular element \( \Psi_{[\mu]}^{[\lambda_1, \lambda_2]} \)
weight multiplicities are equal to dimensions \( \dim (L_{[\mu_1]}^{[\lambda_1]} (u)) \) of the corresponding \( a_{\perp} \)-modules multiplied by the values \( \epsilon (u) \). As a result the highest weights of \( a \)-submodules and their
multiplicities are subject to the set of linear equations (16). These properties are valid for any
reductive subalgebra \( a \rightarrow g \) and the set can be redressed to the form of recurrent relations to be
solved step by step.

The efficiency of the obtained algorithm was illustrated in various examples. In particular,
we considered the construction of modular-invariant partition functions in the framework of
the conformal embedding method and the coset construction in rational CFT. This construction

\[
\begin{align*}
\Theta_{\omega}^{[n]} & = 42 L_{A_1}^{\omega_0 - 8} \oplus 78 L_{A_1}^{\omega_0 - 6} \oplus 85 L_{A_1}^{\omega_0 - 6} \oplus 120 L_{A_1}^{\omega_0 - 7} \\
& \oplus 139 L_{A_1}^{\omega_0 - 9} \oplus 202 L_{A_1}^{\omega_0 - 8} \oplus 222 L_{A_1}^{\omega_0 - 8} \oplus 306 L_{A_1}^{\omega_0 - 9} \\
& \oplus 346 L_{A_1}^{\omega_0 - 9} \oplus 530 L_{A_1}^{\omega_0 - 10} \oplus 482 L_{A_1}^{\omega_0 - 10} \oplus 714 L_{A_1}^{\omega_0 - 11} \\
& \oplus 797 L_{A_1}^{\omega_0 - 11} \oplus 1080 L_{A_1}^{\omega_0 - 12} \oplus 1180 L_{A_1}^{\omega_0 - 12} \oplus \cdots .
\end{align*}
\]

This result can be presented as the set of branching functions:

\[
b_0^{(\omega)} = 1 + 4q^1 + 8q^2 + 15q^3 + 29q^4 + 51q^5 + 85q^6 + 139q^7 + 222q^8 + 346q^9 + 530q^{10} + 797q^{11} + 1180q^{12} + \cdots ,
\]

\[
b_1^{(\omega)} = 2 + 2q^1 + 8q^2 + 12q^3 + 26q^4 + 42q^5 + 78q^6 + 120q^7 + 202q^8 + 306q^9 + 482q^{10} + 714q^{11} + 1080q^{12} + \cdots .
\]

Here, \( q = \exp(2\pi i r) \) and the lower index enumerates the branching functions according to
their highest weights in \( P_{\Delta_1}^+ \). These are the fundamental weights \( \omega_0 = \lambda_0 = (0, 1, 0) \) and \( \omega_0 = \alpha/2 = (1, 1, 0) \).

Now we can return to (22),

\[
X_{[\mu_1]}^{[\lambda_1]} (q) = q^{\frac{\mu_1}{2} (2 + 2q^1 + 8q^2 + 15q^3 + 29q^4 + 51q^5 + 85q^6 + 120q^7 + 202q^8 + 306q^9 + 482q^{10} + 714q^{11} + 1080q^{12} + \cdots )},
\]

\[
X_{[\mu]}^{[\lambda]} (q) = q^{\frac{\mu}{2} (1 + 4q^1 + 8q^2 + 15q^3 + 29q^4 + 51q^5 + 85q^6 + 139q^7 + 222q^8 + 346q^9 + 530q^{10} + 797q^{11} + 1180q^{12} + \cdots )},
\]

and finally obtain expansions for the \( B_2/A_1 \)-coset characters.

5. Conclusion

We have demonstrated that the injection fan technique can be used to deal with arbitrary
reductive subalgebras (maximal as well as nonmaximal). It was shown that the branching
problem for \( a \subset g \) is tightly connected with the properties of the orthogonal partner \( a_{\perp} \) of
\( a \). The subalgebra \( a_{\perp} \) corresponds to the subset \( \Delta_{a_{\perp}}^+ \) of positive roots in \( \Delta_a^+ \) that trivialize
the Cartan subalgebra \( h_{a_{\perp}} \). Both the injection fan and the sets of singular weights for highest
weight \( g \)-modules depend substantially on the structure of \( a_{\perp} \) and its submodules. For the fan
\( \Gamma_{a \rightarrow g} \), this dependence is almost obvious: in the element \( \Phi_{a \rightarrow g} \), the factors corresponding
to the roots of \( \Delta_{a_{\perp}}^+ \) are eliminated. The transformation in the set of projected singular
weights is more interesting. We have found out that in the new singular element \( \Psi_{[\mu]}^{[\lambda_1, \lambda_2]} \), the coefficients depend on the \( a_{\perp} \)-submodules (their highest weights \( \mu_{\perp} (u) \) are fixed by
the injection and by the weights of the initial element \( \Psi^{[\mu]} \)). Fortunately, no more information on
\( L_{[\mu_1]}^{[\lambda_1]} (u) \)-submodules is necessary than their dimensions. In the new singular element \( \Psi_{[\mu]}^{[\lambda_1, \lambda_2]} \)
weight multiplicities are equal to dimensions \( \dim (L_{[\mu_1]}^{[\lambda_1]} (u)) \) of the corresponding \( a_{\perp} \)-modules multiplied by the values \( \epsilon (u) \). As a result the highest weights of \( a \)-submodules and their
multiplicities are subject to the set of linear equations (16). These properties are valid for any
reductive subalgebra \( a \rightarrow g \) and the set can be redressed to the form of recurrent relations to be
solved step by step.
is useful in the study of WZW models emerging in the context of the AdS/CFT correspondence [21–23].

Further amelioration of the algorithm can be achieved by using the folded fan technique [28]. It must be mentioned that even in the case of string functions, the explicit solution of the corresponding recurrent relations is a difficult problem (see [28] for details). Nevertheless, we hope that by developing the procedure of folding, one could obtain explicit solutions for at least some of branching functions and the corresponding coset characters.

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