The Fermion–Boson Transformation in Fractional Quantum Hall Systems

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A Fermion to Boson transformation is accomplished by attaching to each Fermion a single flux tube carrying an even number of flux quanta. This has led to a simple intuitive picture of the fractional quantum Hall effect (FQHE). Shortly after the introduction of the CF and CB transformations, Xie et al. introduced a Fermion→Boson (F→B) mapping connecting a 2D Fermion system at filling \( \nu_F \) with a 2D Boson system at filling \( \nu_B \), where \( \nu_F = \nu_B + 1 \). These authors stated that the sizes of the many body Hilbert spaces for the Boson and Fermion systems were identical, and that their numerical calculations verified that the mapping accurately transformed the ground state of the Fermion system to that of the Boson system both for incompressible FQH states and for other low lying states. The F→B transformation leads to identical energy spectra if and only if these ground states were incompressible FQH states. In this paper we show that the F→B transformation leads to identical energy spectra if and only if the pseudopotential \( V^* (L_{12}) \) describing the interactions among the particles is of the “harmonic” form \( V_H(L_{12}) = A + B L_{12}(L_{12} + 1) \), where \( A \) and \( B \) are constants, and \( L_{12} \) is the total angular momentum of the interacting pair. Anharmonic effects (due to \( \Delta V(L_{12}) = V(L_{12}) - V_H(L_{12}) \)) cause the interacting Fermion and interacting Boson spectra to differ for every value of the filling factor \( \nu_F = \nu_B(1 + \nu_B)^{-1} \). However, for appropriately chosen (short range) model pseudopotentials, the F→B mapping accurately transforms the ground state of the Fermion system to that of the Boson system both for incompressible FQH states and for other low lying states. The F→B mapping is also very useful in understanding the relation between the Haldane hierarchy of Boson quasiparticle (QP) condensates and the CF hierarchy of Fermion QP condensed states.

I. INTRODUCTION

The transformation of electrons into composite Fermions (CF) by attaching to each particle a flux tube carrying an even number of flux quanta has led to a simple intuitive picture of the fractional quantum Hall effect (FQHE). Shortly after the introduction of the CF picture, Xie et al. introduced a Fermion→Boson (F→B) mapping connecting a 2D Fermion system at filling \( \nu_F \) with a 2D Boson system at filling \( \nu_B \), where \( \nu_F = \nu_B + 1 \). These authors stated that the sizes of the many body Hilbert spaces for the Boson and Fermion systems were identical, and that their numerical calculations verified that the mapping accurately transformed the ground state of the Fermion system to that of the Boson system both for incompressible FQH states and for other low lying states. The F→B transformation leads to identical energy spectra if and only if these ground states were incompressible FQH states. In this paper we show that the F→B transformation leads to identical energy spectra if and only if the pseudopotential \( V^* (L_{12}) \) describing the interactions among the particles is of the “harmonic” form \( V_H(L_{12}) = A + B L_{12}(L_{12} + 1) \), where \( A \) and \( B \) are constants, and \( L_{12} \) is the total angular momentum of the interacting pair. Anharmonic effects (due to \( \Delta V(L_{12}) = V(L_{12}) - V_H(L_{12}) \)) cause the interacting Fermion and interacting Boson spectra to differ for every value of the filling factor \( \nu_F = \nu_B(1 + \nu_B)^{-1} \). However, for appropriately chosen (short range) model pseudopotentials, the F→B mapping accurately transforms the ground state of the Fermion system to that of the Boson system both for incompressible FQH states and for other low lying states. The F→B mapping is also very useful in understanding the relation between the Haldane hierarchy of Boson quasiparticle (QP) condensates and the CF hierarchy of Fermion QP condensed states.

II. GAUGE TRANSFORMATIONS IN TWO-DIMENSIONAL SYSTEMS

By attaching to each Fermion or Boson of charge \(-e\) a fictitious flux tube carrying an even number \(2p\) of flux quanta oriented opposite to the applied magnetic field, the eigenstates and particle statistics are unchanged. The “gauge field” interactions between the charge on one particle and the vector potential due to the flux quanta on every other particle make the Hamiltonian more complicated. Only when the mean field approximation is made does the problem simplify. In addition to these (CF and CB) transformations, a F→B transformation can be made by attaching to each Fermion an odd number \(2p + 1\) of flux quanta (one flux quantum changes the statistics; other \(2p\) flux quanta describe an additional CF or CB transformation). If the particles are confined to the surface of a sphere containing at its center a magnetic monopole of strength \(2e\) the lowest shell of mean field composite particles has angular momentum \(l_{FB} = |l_F - p(N - 1)|\) where \(l_F = S_F\) or \(l_B = |l_B - p(N - 1)|\) where \(l_B = S_B\). In the F→B transformation (with \(p = 0\), \(l_F\) is replaced by \(l_B = |l_B - \frac{1}{2}(N - 1)|\).

III. SOME USEFUL THEOREMS

When a shell of angular momentum \(l\) contains \(N\) identical particles (Fermions or Bosons), the resulting \(N\) particle states can be classified by eigenvectors \(|L, M, \alpha\rangle\), where \(L\) is the total angular momentum, \(M\) its \(z\)-component, and \(\alpha\) a label which distinguishes independent multiplets with the same total angular momentum \(L\). In the mean field CF (CB) transformation \(l_F\) (\(l_B\)) is transformed to \(l_{FB}\) (\(l_B\)). In trying to understand why the mean field CF picture correctly predicted the low lying
band of states in the interacting electron spectrum, the following theorem was important.

**Theorem 1.** The set of allowed total angular momentum multiplets of \( N \) Fermions each with angular momentum \( l_F \) is a subset of the set of allowed multiplets of \( N \) Fermions each with angular momentum \( l_F = l_F + (N - 1) \).

Thus, if we define \( g_{NL}(L) \) as the number of independent multiplets of total angular momentum \( L \) formed by addition of the angular momenta of \( N \) Fermions, each with angular momentum \( l \), then \( g_{NL}(L) \leq g_{NL}(L) \) for every value of \( L \). A few examples for small systems suggest that the theorem is correct, but a general mathematical proof is non-trivial. A proof using the methods of combinatorics and the Kohn theorem has been given recently. The same method allows the proof of a second theorem.

**Theorem 2.** The set of allowed total angular momentum multiplets of \( N \) Bosons each with angular momentum \( l_B \) is identical to the set of multiplets for \( N \) Fermions each with angular momentum \( l_F = l_F + \frac{1}{2} (N - 1) \).

From Theorem 2 it follows immediately that Theorem 1 also applies to Bosons. Theorem 2 is a stronger statement than a simple equality of the sizes of the many body Hilbert spaces.

### IV. INTERACTION EFFECTS

In studying why the mean field CF picture correctly predicts the low lying states of a 2D electron system in a magnetic field, the “harmonic pseudopotential”

\[
V_H(L) = A + B \hat{L}^2
\]

was introduced. Here \( A \) and \( B \) are constants and \( \hat{L}^2 \) is the total angular momentum operator of the pair of particles. It was shown that for the harmonic pseudopotential the energy of any multiplet of angular momentum \( L \) was given by

\[
E_{LA} = A \cdot \frac{1}{2} N (N - 1) + B \cdot N (N - 2) l(l + 1) + B \cdot L (L + 1).
\]

The energy is independent of \( \alpha \), so that every multiplet with the same value of \( L \) has the same energy. Equation (3) holds both for Fermions and for Bosons. If \( B_F = B_B = B \), then the spectrum of \( N \) Bosons each with angular momentum \( l_B \) is identical (up to a constant) to that of \( N \) Fermions each with angular momentum \( l_F = l_B + \frac{1}{2} (N - 1) \). This is a rather surprising result because Fermions and Bosons sample different sets of values of the pair angular momentum. For example, for \( N = 9 \) and \( l_F = 12 \) (corresponding to \( \nu_F = \frac{4}{3} \)) the allowed values of the Fermion pair angular momentum consist of all odd integers between 1 and 23; for the corresponding Boson system with \( l_B = 8 \) (\( \nu_B = \frac{1}{3} \)), the allowed values of \( L_{12} \) are all even integers between 0 and 16. Despite the totally different set of pseudopotential coefficients sampled, up to a constant, the spectra of the Boson and Fermion systems interacting through a harmonic pseudopotential are the same.

In earlier work it was emphasized that the harmonic pseudopotential led to an “anti-Hund’s rule” with the lowest energy state having the lowest allowed value of \( L \). It is the positive anharmonicity \( \Delta V(L) > 0 \) that causes Laughlin correlations. It is useful to introduce the “relative” angular momentum \( R = 2l - L \). For Bosons \( R_B = 0, 2, 4, \ldots \) while for Fermions \( R_F = 1, 3, 5, \ldots \); in both cases, \( R \leq 2l \). We can write the pseudopotential and its harmonic and anharmonic parts in terms of \( R \), and call them \( V(R), V_H(R), \) and \( \Delta V(R) \), respectively. It is more reasonable to make simple models for \( \Delta V(R) \) (e.g. assume that it vanishes for all \( R \) greater than some value) than for \( V(R) \) itself. From equation (3) and the equation for the total energy,

\[
E_{LA} = \frac{1}{2} N (N - 1) \sum_R \mathcal{G}_{LA}(R) V(R),
\]

where \( \mathcal{G}_{LA}(R) \) is the coefficient of fractional grandparentage (CFGP) it is readily ascertained that the interacting Boson and interacting Fermion systems cannot have identical spectra when \( \Delta V(R) \) is non-zero.

Xie et al. determined the Boson and Fermion eigenfunctions by exact numerical diagonalization for six particle systems connected through the F-B transformation. They then transformed the Boson eigenfunctions into Fermion eigenfunctions by multiplying them by \( \prod_{i<j}(z_i - z_j) \), as required by the B-F transformation. The overlap of these transformed Boson eigenfunctions with the exact Fermion eigenfunctions was then evaluated. The overlap was quite close to unity for incompressible quantum fluid states when the full Coulomb interaction was used. A similar result was obtained for a model short range interaction \( H_1 \) for which \( V(R) \) vanished for \( R > 1 \) and was equal to the Coulomb values at \( R = 0 \) (for Bosons) or at \( R = 1 \) (for Fermions). However, when the interaction was approximated by \( H_3 \) for which \( V(R) \) vanished for \( R > 3 \) and was equal to the Coulomb values at \( R = 0 \) and 2 (for Bosons) or at \( R = 1 \) and 3 (for Fermions), the overlap was considerably smaller. The reason appears to be that for Fermions \( H_3 \) is subharmonic at \( R = 3 \), while for Bosons it is (marginally) superharmonic in the entire range of \( R \), and that for a subharmonic pseudopotential Laughlin correlations are not expected to occur. By a subharmonic (superharmonic) behavior of \( V(R) \) at a certain value \( R_0 \) we mean that \( V(R_0) \) is larger (smaller) than a value \( V_H(R_0) \) for which \( V(R) \) would be harmonic (i.e., linear in \( L_{12}(L_{12} + 1) \)) in the range \( R_0 - 2 \leq R \leq R_0 + 2 \). We will later use an anharmonicity parameter \( x \) defined as \( x(R_0) = V(R_0)/V_H(R_0) \) for the \( H_3 \) interaction, \( x(3) = 1.3 \) (for Fermions) and \( x(2) = 0.8 \) (for Bosons).

We have evaluated numerically the eigenstates of an eight electron system at \( 2S_F = 19 \) to 23 (these states correspond to Laughlin \( \nu_F = \frac{1}{3} \) states with zero, one,
We have used the full Coulomb pseudopotential, or two QP’s) for a number of different pseudopotentials. We have used the full Coulomb pseudopotential, in which \( V_x(1) = 1 \), \( V_x(R \geq 5) = 0 \), and \( V_x(3) = x \cdot V_H(3) \) is an arbitrary fraction \( x \) of the “harmonic” value. We perform the same calculations for eight boson systems at \( 2S_B = 12 \) to 16 (here, \( V_x(0) = 1 \), \( V_x(R \geq 4) = 0 \), and \( V_x(2) = x \cdot V_H(2) \)).

In Fig. 1 we contrast the energy spectra for the fermion and boson systems at \( \nu_F = \frac{1}{2} (\nu_B = \frac{1}{2}) \) for the Coulomb pseudopotential appropriate for the lowest Landau level \( (a-a') \), and for the model pseudopotentials \( H_1 (b-b') \) and \( H_3 (c-c') \). In Fig. 2 we do the same for the state containing two Laughlin quasielectrons (QE). The lowest states in \( (a-a') \) and \( (b-b') \) are quite similar consisting of Language \( L = 0 \) ground state in Fig. 1 and two-QE states with \( l_{QE} = \frac{1}{2} (N - 1) = \frac{2}{3} \) giving \( L = N - 2, N - 4, \ldots = 0, 2, 4, \ldots \). The magnetoroton band (at \( 2 \leq L \leq 8 \)) is apparent in Fig. 2 although the gaps and band widths are different for different pseudopotentials. The pseudopotential used in \( (c-c') \) gives very different results both in Fig. 1 and Fig. 2. As mentioned before, this results because \( V(3) \) used in fermion pseudopotential \( H_3 \) is too large to lead to Laughlin correlations.

To illustrate this point we have calculated the energy spectra using pseudopotential \( V_x \) with different values of \( x \). In Fig. 3 we show the spectra at \( \nu_F = \frac{1}{2} (\nu_B = \frac{1}{2}) \) for \( x = \frac{1}{2}, 1, \text{ and } \frac{3}{2} \). For \( x < 1 \), \( V_x(R) \) is superharmonic at \( R = 3 \) (for fermions; \( x = x(3) \)) or at \( R = 2 \) (for bosons; \( x = x(2) \)), and Laughlin correlations with an \( L = 0 \) ground state occur. For \( x \geq 1 \) there is little resemblance between the numerical spectra and that associated with the full Coulomb interaction. Furthermore, the fermion and boson systems are quite different from one another.

From the eigenfunctions we can determine CFGP’s \( G_{L_0}(R) \) for each state \( |L, \alpha \rangle \). In Fig. 4 we plot the \( x \)-dependence of the CFP’s \( G_{L_0}(R) \) from pair states at three smallest values of \( R \) calculated for the lowest energy \( L = 0 \) state of eight fermions at \( 2S_F = 21 (\nu_F = \frac{1}{3}) \) and eight bosons at \( 2S_B = 14 (\nu_B = \frac{1}{2}) \). In both systems, a Laughlin incompressible state with vanishing \( G(1) \) (for fermions) or \( G(0) \) (for bosons) occurs at small \( x \), and a rather abrupt transition occurs at \( x \approx 1 \), implying a change of the nature of the correlations when the pseudopotential \( V_x(R) \) changes from super- to subharmonic. At \( x > 1 \), the correlations in the two systems are quite different and, for example, another abrupt transition occurs in the boson system at \( x \approx 4 \) (not shown in the figure), which is absent in the fermion system.
angular momentum $l$ having $\nu = \frac{1}{2}$ and the allowed values of the pair angular momentum of the two QP’s are $N, N-2, N-4, \ldots$. For a Fermion system with $I_B = \frac{1}{2}(N-1)-2$, the allowed values of the QP pair angular momentum are $N-2, N-4, \ldots$. The two sets can be made identical only if a hard core repulsion forbids the Boson QP pair from having the largest allowed pair angular momentum $L_{12}^{\text{MAX}} = N - 3$. This behavior is observed in Fig. 3 where the Boson treatment of two QE’s (i.e., the CB transformation) would predict states at $L = 0, 2, 4, 6,$ and $8$, but the $L = 8$ state does not occur in the low energy band.

Since the description of CB’s (with hard core QE interaction) and CF’s give identical sets of QP states, filled QP levels (implying daughter states) occur at identical values of the applied magnetic field. In earlier work we have emphasized that both the Haldane hierarchy and CF hierarchy schemes assume the validity of the mean field approximation, and we have shown that this approximation is expected to fail when the QP–QP interaction is subharmonic. Numerical results show when the mean field approximation is valid and when it fails.

VI. SUMMARY

We have shown that the F→B transformation replaces the single Fermion angular momentum $l_F$ by $I_B = l_F - \frac{1}{2}(N-1)$, and that this transformation leads to an identical set of total angular momentum multiplets. The Fermion and Boson systems have identical spectra in the presence of many body interactions only when the pseudopotential is harmonic, i.e. linear in squared places the single Fermion angular momentum $l^+_F = S^+_F = S - p(N - 1) = \frac{1}{2}(N - 1)$ when $2p$ flux quanta are attached to each electron and oriented opposite to the applied magnetic field. Thus the $N$ CF’s fill the $2l^+_F + 1$ states of the lowest CF shell giving an $L = 0$ incompressible ground state.

The F→B transformation gives $2S_B = 2S_F - (N - 1) = 2p(N-1)$ and a Boson filling factor of $\nu_B = (2p)^{-1}$. Making a CB transformation gives $I_B^c = S_B^c = S_B - p(N-1) = 0$. This also gives an $L = 0$ incompressible ground state because each CB has $I_B^c = 0$. Thus the CF description of a Laughlin state has one filled CF shell of angular momentum $l^+_F = \frac{1}{2}(N - 1)$, while the CB description has $N$ CB’s each with angular momentum $l_B^c = 0$.

For $2S_B = 2n(N-1) \pm \eta_{\text{QP}}$, where the + and – occur for quasiholes (QH) and quasielectrons (QE), respectively, we define $2l_B^c = \pm |2S_B^c| = \eta_{\text{QP}}$. This gives exactly the same set of angular momentum multiplets as obtained in the CF picture with $2S_F = (2n+1)(N-1) + \eta_{\text{QP}}$. However it gives a larger set of multiplets than are allowed by $2S_F = (2n+1)(N-1) - \eta_{\text{QP}}$. For example, for $\theta_{\text{QE}} = 2$, $l_B^c = 1$ and the allowed values of the pair angular momentum of the two QP’s are $N, N-2, N-4, \ldots$. For a Fermion system with $I_B = \frac{1}{2}(N-1)-2$, the allowed values of the QP pair angular momentum are $N-2, N-4, \ldots$. The two sets can be made identical only if a hard core repulsion forbids the Boson QP pair from having the largest allowed pair angular momentum $L_{12}^{\text{MAX}} = N - 3$. This behavior is observed in Fig. 3 where the Boson treatment of two QE’s (i.e., the CB transformation) would predict states at $L = 0, 2, 4, 6,$ and $8$, but the $L = 8$ state does not occur in the low energy band.

Since the description of CB’s (with hard core QE interaction) and CF’s give identical sets of QP states, filled QP levels (implying daughter states) occur at identical values of the applied magnetic field. In earlier work we have emphasized that both the Haldane hierarchy and CF hierarchy schemes assume the validity of the mean field approximation, and we have shown that this approximation is expected to fail when the QP–QP interaction is subharmonic. Numerical results show when the mean field approximation is valid and when it fails.

FIG. 3. The energy spectra (energy $E$ vs. angular momentum $L$) of the corresponding eight Fermion (left) and eight Boson (right) systems at the monopole strengths $2S_F = 21$ and $2S_B = 14$ (filling factors $n_F = \frac{1}{2}$ and $n_B = \frac{1}{2}$) for model interaction pseudopotentials $V_x(R)$ with $x = \frac{1}{2}$ (a–a’), $x = 1$ (b–b’), and $x = \frac{1}{2}$ (c–c’).

FIG. 4. The coefficients of fractional grandparentage $\mathcal{G}(R)$ from the pair states at three smallest values of $R$ calculated for the lowest energy $L = 0$ state of the corresponding eight Fermion (a) and eight Boson (a’) systems at the monopole strengths $2S_F = 21$ ($n_F = \frac{1}{4}$) and $2S_B = 14$ ($n_B = \frac{1}{4}$) for the model interaction $V_x$, as a function of $x$.
model interactions and shown the relation between the spectra and coefficients of fractional grandparentage for the Fermion and Boson systems. Finally, we have used the F→B transformation to clarify the relation between the Haldane Boson picture and the CF picture of the hierarchy of condensed states.

VII. ACKNOWLEDGMENT

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