CHAIN HOMOTOPY MAPS AND A UNIVERSAL DIFFERENTIAL FOR
KHOVANOV-TYPE HOMOLOGY

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Abstract. We give chain homotopy maps of Khovanov-type link homology of a universal differential. The universal differential, discussed by Mikhail Khovanov, Marco Mackaay, Paul Turner and Pedro Vaz, contains the original Khovanov's differential and Lee's differential. We also consider the conditions of any differential ensuring the Reidemeister invariance for the chain homotopy maps.

1. Introduction.

1.1. Motivation. The aim of this note is to show that there exists certain homotopy maps ensuring the Reidemeister invariance of Khovanov-type homology of a universal differential. By using Viro’s construction [10, 3] of the Khovanov homology $H_{i,j}$ [4], a universal differential [9, 7] $C^i \rightarrow C^{i+1}$ on Khovanov complex $C^i := \oplus_{j} C^{i,j}$ [3] over the ring $\mathbb{Z}[s,t]$ can be defined as

\begin{equation}
\delta_{s,t}(S \otimes [x]) := \sum_{T} T \otimes [xa]
\end{equation}

such that

\begin{equation}
\begin{array}{c}
p \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ q \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ a \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ [x] 
\end{array} \quad \rightarrow \quad \begin{array}{c}
q \colon p \\
p \colon q \\
a \ \ \ \ \ \ \ \ \ \ \ \ \ [xa]
\end{array}
\end{equation}

where $a$ is a crossing of a link diagram, $p$ and $q$ denote signs (+ or −) and $p: q$ and $q: p$ are defined by Figure 1.

This differential follows a universal Frobenius algebra [5]. As in [9, 7], the differential possess the following property. When $s = t = 0$, we get the original Khovanov theory [4] and when $(s, t) = (0, 1)$, Lee theory [6] introducing Rasmussen’s invariants [8]. When we choose the coefficient $\mathbb{Z}/2[s]$ (take $t = 0$), the theory is known as Bar-Natan theory [1]. On the other hand, there exist the explicit chain homotopy maps [10, 2] for the original Khovanov homology. The chain homotopy maps ensuring Reidemeister invariance of Khovanov homology are given by [10] for the first Reidemeister move, by [2] for the other Reidemeister moves.

Here, the following question naturally arise. Replacing the differential of the original Khovanov homology [4] by the universal differential, what are chain homotopy maps and retractions needed for ensuring the Reidemeister invariance? When the homotopy maps [10, 2] are given, is there any explicit condition for a universal differential on the Khovanov complex to induce the Reidemeister invariance of a family of homology?

In this note, we discuss these problem in detail. We arrive at the result (Theorem 2.1 and 2.2) as mentioned at the end of this note.

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1.2. Definition and Notation. We use the definition of Khovanov homology defined in [3, Section 2] except for replacing the Frobenius calculus [3, Figure 3] in the definition of the differential $d$ by Figure 1. Thus let us use the same symbols $p : q$ and $q : p$ as [3, Figure 3] in a generalised meaning as follows: for a given signs $p$ and $q$, the signs $p : q$ and $q : p$ of circles are defined by Figure 1. We denote the generalised differential by $\delta_{s,t}$. $m(p, q)$ denotes $p : q$ in (3–4) for signs $p, q$.

Remark 1.1. Note that the symbols $p : q$ and $q : p$ are generalised. In fact, considering the case $s = t = 0$ (resp. $s = 0$ and $t = 1$), the differential give the original Khovanov homology (resp. Lee’s differential).

Figure 1. A generalised Frobenius calculus of signed circles.
2.1. **On the first Reidemeister invariance.** The first Reidemeister move is $D' = \overleftarrow{\bigtriangleup} \overset{\sim}{\rightarrow} D$, we consider the composition

\[
C(D') = C \oplus C_{\text{contr}} \overset{\rho_1}{\longrightarrow} C\overset{\text{isom}}{\rightarrow} C(D)
\]

where $a$ is a crossing and $C, C_{\text{contr}}, \rho_1$ and the isomorphism are defined in the following formulas (11)–(14).

First,

\[
C := C \left( p \bigtriangleup \bigtriangledown \otimes [x] - m(p : +) \bigtriangledown \bigtriangleup \otimes [x] \right),
\]

\[
C_{\text{contr}} := C \left( p \bigtriangledown \bigtriangleup \otimes [x], p \bigtriangledown \bigtriangleup \otimes [xa] \right).
\]

Second, the retraction $\rho_1 : C \left( \bigtriangleup \bigtriangledown \right) \rightarrow C \left( p \bigtriangleup \bigtriangledown \otimes [x] - m(p : +) \bigtriangledown \bigtriangleup \otimes [x] \right)$ is defined by the formulas

\[
p \bigtriangleup \bigtriangledown \otimes [x] \mapsto p \bigtriangleup \bigtriangledown \otimes [x] - m(p : +) \bigtriangledown \bigtriangleup \otimes [x],
\]

\[
p \bigtriangledown \bigtriangleup \otimes [x], p \bigtriangledown \bigtriangledown \otimes [xa] \mapsto 0.
\]

It is easy to see that $\delta_{s,t} \circ \rho_1 = \rho_1 \circ \delta_{s,t}$. Then $\rho_1$ is certainly a chain map. Note that we have $\delta_{s,t} \circ \rho_1 = \rho_1 \circ \delta_{s,t}$ using only (2) and not using the property (3)–(8).

Third, the isomorphism

\[
C \left( p \bigtriangledown \bigtriangledown \otimes [x] - m(p : +) \bigtriangledown \bigtriangleup \otimes [x] \right) \rightarrow C \left( \bigtriangledown \bigtriangleup \otimes [x] \right)
\]

is defined by the formulas

\[
p \bigtriangledown \bigtriangledown \otimes [x] - m(p : +) \bigtriangledown \bigtriangleup \otimes [x] \mapsto \bigtriangledown \bigtriangleup \otimes [x].
\]

The homotopy connecting $\circ \rho_1$ to the identity : $C \left( \bigtriangledown \bigtriangledown \right) \rightarrow C \left( \bigtriangledown \bigtriangledown \right)$ such that $\delta_{s,t} \circ h_1 + h_1 \circ \delta_{s,t} = \text{id} - \circ \rho_1$, is defined by the formulas:

\[
p \bigtriangledown \bigtriangledown \otimes [xa] \mapsto p \bigtriangledown \bigtriangleup \otimes [x], \text{ otherwise} \mapsto 0.
\]

**Remark 2.1.** The explicit formula (15) of the homotopy map $h_1$ in the case $(s = t = 0)$ of the original Khovanov homology is given by Oleg Viro [10, Subsection 5.5].
We can verify $\delta_{s,t} \circ h_1 + h_1 \circ \delta_{s,t} = \text{id} - \text{in} \circ \rho_1$ by a direct computation as follows.

\[
(h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1) \left( p \begin{array}{c} + \\ \end{array} \otimes [x] \right) = h_1 \left( m(p : +) \begin{array}{c} \leftarrow \\ \end{array} \otimes [xa] \right)
\]

\[
= m(p : +) \begin{array}{c} \leftarrow \\ \end{array} \otimes [x]
\]

\[
= (\text{id} - \rho_1) \left( p \begin{array}{c} + \\ \end{array} \otimes [x] \right).
\]

Similarly,

\[
(h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1) \left( p \begin{array}{c} - \\ \end{array} \otimes [x] \right) = h_1 \left( m(p : -) \begin{array}{c} \rightarrow \\ \end{array} \otimes [xa] \right)
\]

\[
= p \begin{array}{c} - \\ \end{array} \otimes [x]
\]

\[
= (\text{id} - \rho_1) \left( p \begin{array}{c} - \\ \end{array} \otimes [x] \right),
\]

\[
(h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1) \left( \begin{array}{c} \rightarrow \\ \end{array} \otimes [xa] \right) = \begin{array}{c} \rightarrow \\ \end{array} \otimes [xa] = (\text{id} - \rho_1) \left( \begin{array}{c} \rightarrow \\ \end{array} \otimes [xa] \right).
\]

2.2. On the second Reidemeister invariance. In this section, consider the second Reidemeister move of Khovanov homology for $\delta_{s,t}$. In [2, Subsection 2.2], a chain homotopy map, a retraction and an isomorphism for the second Reidemeister invariance of Khovanov homology for $\delta_{0,0}$ were given.

We extend the chain homotopy map, the retraction and the isomorphism for $\delta_{0,0}$ to maps $h_2, \rho_2$ and isom$_2$ for $\delta_{s,t}$ by replacing $p : q$ and $q : p$ defined [3, Figure 3] with the “generalised” $p : q$ and $q : p$ defined Figure 1. We will show that these extended maps become the chain homotopy map, the retraction and the isomorphism for the second Reidemeister invariance of Khovanov homology for $\delta_{s,t}$.

Let $a, b$ be crossings, $x$ sequence of crossings with negative markers and $p, q$ be signs. For a crossing with no markers or no signs in the following formulas, any markers or signs are allowed. Let $\bar{p}$ be $p$ unless the upper arc is connected to one of the other arcs in the picture.

For a link diagram $D' = a \stackrel{b}{\longrightarrow}$, let $S_{+-}(p,q)$ be $p \begin{array}{c} + \\ q \end{array}$, $S_{-+}(p,q)$ be $p \begin{array}{c} \leftarrow \\ q \end{array}$, $S_{+-}(p,q)$ be $p \begin{array}{c} - \\ q \end{array}$ and $S_{-+}(p,q)$ be $p \begin{array}{c} + \\ q \end{array}$.

The second Reidemeister move is $D' \xrightarrow{\delta} D$, we consider the composition

\[
\mathcal{C}(D') = \mathcal{C} \oplus \mathcal{C}_{\text{contr}} \xrightarrow{\rho_2} \mathcal{C} \xrightarrow{\text{isom}_2} \mathcal{C}(D)
\]

where $\mathcal{C} \oplus \mathcal{C}_{\text{contr}}, \rho_2$ and isom$_2$ are defined in [2, Section 2.1] by replacing $p : q$ and $q : p$ in [2, Section 2.1] with the “generalised” $p : q$ and $q : p$ defined Figure 1.

Let $h_2$ be the chain homotopy maps given by [2, Section 2.1, Equation (7)]. To verify the second Reidemeister invariance of $H^i(D)$, it is sufficient to show that $\delta_{s,t} \circ \rho_2 = \rho_2 \circ \delta_{s,t}$ and $\delta_{s,t} \circ h_2 + h_2 \circ \delta_{s,t} = \text{id} - \text{in} \circ \rho_2$. In the following, we reveal where are non-trivial parts in the proof of these two equation and figure out a good way to avoid the “hard” calculation.
• On the proof of the \( \delta_{s,t} \circ \rho_2 = \rho_2 \circ \delta_{s,t} \).

We denote a big sum of the states \( \sum_u S \otimes [xu] \) by \( S^u \) where \( u \) ranges over crossings of \( D' \) except for \( a \) and \( b \).

First, it is easy to see that we have \( \delta_{s,t} \circ \rho_2(S_{++,+}(p, q)) = (\rho_2 \circ \delta_{s,t})(S_{++,+}(p, q)) = 0 \) depending only \( (2) \): \( \delta_{s,t} \circ \rho_2(S_{++,+}(p, q)) = (\rho_2 \circ \delta_{s,t})(S_{++,+}(p, q)) = \rho_2(S^u_{++,+}) \) and \( \delta_{s,t} \circ \rho_2(S_{++,+}(p, q)) = (\rho_2 \circ \delta_{s,t})(S_{++,+}(p, q)) = \rho_2(S^u_{++,+}) \) follows from \( (1) \)–\( (6) \).

Second,

\[
(\rho_2 \circ \delta_{s,t})(S_{++,+}(p, q) \otimes [x]) = \rho_2\left( S_{++,+}(p : q, q : p) \otimes [xa] + q : q \otimes [xb] \right) \\
= -\rho_2(S_{++,+}(p, q) \otimes [xb]) + \rho_2(S_{++,+}(p, q) \otimes [xb]) \\
= 0.
\]

The second equality of \( (20) \) follows from

\[
f_{---}(\delta_{s,t}(S_{++,+}(p, q) \otimes [x])) = S_{++,+}(p, q) \otimes [xb]
\]

where \( f_{---} \) is the homomorphism defined by \( f_{---}(S) = S \) if \( S = S_{++,+}(p, q) \otimes [xb] \) of any \( p, q \), \( f_{---}(S) = 0 \) otherwise. We have \( (21) \) because \( (6) \) and \( (7) \) implies that \( \delta_{s,t}(S_{++,+}(p, q) \otimes [x]) = S_{++,+}(p, q) \otimes [xb] + \sum_u S_u \) where \( u \) is neither \( S_{++,+}(p, q) \otimes [xb] \) of any \( p \) and \( q \) nor \( S_{++,+}(p, q) \otimes [xb] \) of any \( p \) and \( q \). Then \( \rho_2 \) is certainly a chain map.

• On the proof of the \( \delta_{s,t} \circ \rho_2 + \rho_2 \circ \delta_{s,t} = \text{id} - \in \circ \rho_2 \).

In the beginning, let us show the equation depending only on \( (4) \)–\( (6) \) of Figure

\[
(h_2 \circ \delta_{s,t} + \delta_{s,t} \circ h_2)(S_{--,+}(p, q) \otimes [xab]) = S_{--,+}(p, q) \otimes [xab] \\
= (\text{id} - \rho_2)(S_{--,+}(p, q) \otimes [xab]),
\]

\[
(h_2 \circ \delta_{s,t} + \delta_{s,t} \circ h_2)(S_{++,--}(p, q) \otimes [xa]) = h_2(S_{--,+}(p : q, q : p) \otimes [xab]) \\
= -S_{--,+}(p : q, q : p) \otimes [xb] \\
= (\text{id} - \rho_2)(S_{++,--}(p, q) \otimes [xa]),
\]

\[
(h_2 \circ \delta_{s,t} + \delta_{s,t} \circ h_2)(S_{++,--}(p, q) \otimes [xb]) = h_2(-S_{--,+}(p, q) \otimes [xab]) \\
= S_{++,--}(p, q) \otimes [xb] \\
= (\text{id} - \rho_2)(S_{++,--}(p, q) \otimes [xb]).
\]

Second, let us show the equation depending only on \( (7) \)–\( (8) \) of Figure

\[
(h_2 \circ \delta_{s,t} + \delta_{s,t} \circ h_2)(S_{++}(p, q) \otimes [x]) = h_2\left( q : q \otimes [xb] \right) \\
= S_{++}(p, q) \otimes [x] \\
= (\text{id} - \rho_2)(S_{++}(p, q) \otimes [x]).
\]

The first equality of \( (25) \) follows from Lemma \( (2) \).

**Lemma 2.1.** \( h_2\left( q : q \otimes [xb] \right) = h_2(S_{++}(p, q) \otimes [xb]) \).

_Proof._ \( h_2 : S_{++}(p, q) \otimes [xb] \to S_{++}(p, q) \otimes [x], S_{--}(p, q) \otimes [xab] \to -S_{--}(p, q) \otimes [xb] \) and otherwise \( \to 0 \) \( (2) \). On the other hand, we have \( (21) \). \( \square \)

Third, let us show the equation depending on all the relation in Figure \( \text{I} \). We use the homomorphism \( f_{++,--} \) defined by \( f_{++,--}(S) = S \) if \( S = S_{++,--}(p, q) \otimes [xb] \) of any \( p \) and \( q \), \( f_{++,--}(S) = 0 \) otherwise.
(h_2 \circ \delta_{s,t} + \delta_{s,t} \circ h_2)(S_+ -(p, q) \otimes [xb]) = S_+ - (m(p : +), q) \otimes [xb] \\
+ S_+ - (p, q) \otimes [xb] \quad (: \quad \text{21}) \\
+ f_+ - - (\delta_{s,t}(S_+ + (p, q) \otimes [x])) \\
+ S_+ - (p : q, q : p) \otimes [xa] \\
= S_+ - (p, q) \otimes [xb] \\
+ S_+ - - ((p : q) : (q : p), (q : p) : (p : q)) \\
\otimes [xb] \\
+ S_+ - (p : q, q : p) \otimes [xa] \\
= (\text{id} - \rho_2)(S_+ - (p, q) \otimes [xb]). \\
(26)

The third equality of \text{26} follows from Lemma 2.2.

Lemma 2.2.

S_+ - - (m(p : +), q) \otimes [xb] + f_+ - - (\delta_{s,t}(S_+ + (p, q) \otimes [x])) \\
= S_+ - - ((p : q) : (q : p), (q : p) : (p : q)) \otimes [xb]. \\
(27)

Proof. Let us consider separately the cases where the component with \(p\) is that of \(q\) or not.

Consider the cases where the component with \(p\) is that of \(q\) (two cases).

\[
\begin{align*}
\text{LHS} &= S_+ - - (m(- : +), -) \otimes [xb] + f_+ - - (\delta_{s,t}(S_+ + (-, -) \otimes [x])) \\
&= 2S_+ - - (+, +) \otimes [xb] - sS_+ - - (-, -) \otimes [xb] \\
&= \text{RHS}, \\
\end{align*}
\]

\[
\begin{align*}
\text{LHS} &= S_+ - - (m(+ : +), +) \otimes [xb] + f_+ - - (\delta_{s,t}(S_+ + (+, +) \otimes [x])) \\
&= sS_+ - - (+, +) \otimes [xb] + 2tS_+ - - (-, -) \otimes [xb] \\
&= \text{RHS}. \\
\end{align*}
\]

Consider the cases where the component with \(p\) is not that of \(q\) (four cases).

\[
\begin{align*}
\text{LHS} &= S_+ - - (m(- : +), -) \otimes [xb] + f_+ - - (\delta_{s,t}(S_+ + (-, -) \otimes [x])) \\
&= S_+ - - (+, -) \otimes [xb] + S_+ - - (-, +) \otimes [xb] - sS_+ - - (-, -) \otimes [xb] \\
&= \text{RHS}, \\
\end{align*}
\]

\[
\begin{align*}
\text{LHS} &= S_+ - - (m(+ : +), +) \otimes [xb] + f_+ - - (\delta_{s,t}(S_+ + (+, +) \otimes [x])) \\
&= sS_+ - - (+, +) \otimes [xb] + tS_+ - - (-, +) \otimes [xb] + tS_+ - - (+, -) \otimes [xb] \\
&= \text{RHS}, \\
\end{align*}
\]

\[
\begin{align*}
\text{LHS} &= S_+ - - (m(+ : +), -) \otimes [xb] + f_+ - - (\delta_{s,t}(S_+ + (-, -) \otimes [x])) \\
&= tS_+ - - (-, -) \otimes [xb] + S_+ - - (+, +) \otimes [xb] \\
&= \text{RHS}, \\
\end{align*}
\]

\[
\begin{align*}
\text{LHS} &= S_+ - - (m(- : +), +) \otimes [xb] + f_+ - - (\delta_{s,t}(S_+ + (-, +) \otimes [x])) \\
&= S_+ - - (+, +) \otimes [xb] + tS_+ - - (-, -) \otimes [xb]. \\
&= \text{RHS}. \\
\end{align*}
\]

\[
\square
\]

2.3. On the third Reidemeister invariance. In this section, consider the third Reidemeister move of Khovanov homology for \(\delta_{s,t}\). In [2 Subsection 2.2], a chain homotopy map, a retraction and an isomorphism for the third Reidemeister invariance of Khovanov homology for \(\delta_{0,0}\) were given.
We extend the chain homotopy map, the retraction and the isomorphism for $\delta_{0,0}$ to maps $h_3$, $\rho_3$ and $\text{isom}_3$ for $\delta_{s,t}$ by replacing $p : q$ and $q : p$ defined [2] Figure 3 with the “generalised” $p : q$ and $q : p$ defined Figure 4. We will show that these extended maps become the chain homotopy map and the retraction and the isomorphism for the third Reidemeister invariance of Khovanov homology for $\delta_{s,t}$.

Let $a$, $b$ and $c$ be crossings, $x$ sequence of crossings with negative markers and $p$, $q$, $r$ be signs. For a crossing with no markers or no signs in the following formulas, any markers or signs are allowed. Let $\tilde{r}$ be $r$ unless the upper left arc is connected to one of the other arcs in the picture and let $\tilde{q}$ be $q$ unless the lower left arc is connected to one of the other arcs in the picture.

For a link diagram $D' = \begin{array}{c}
\includegraphics{diagram.png}
\end{array}$, let $S_{++}$ be $\begin{array}{c}
\includegraphics{diagram.png}
\end{array}$, $S_{+-}(p,q,r)$ be $\begin{array}{c}
\includegraphics{diagram.png}
\end{array}$, $S_{--}(p,q,r)$ be $\begin{array}{c}
\includegraphics{diagram.png}
\end{array}$.

The third Reidemeister move is $D' \sim D$, we consider the composition

\begin{equation}
\mathcal{C}(D') = \mathcal{C} \oplus \mathcal{C} \text{ contr} \xrightarrow{\rho_3} \mathcal{C} \text{ isom}_3 \xrightarrow{\text{in}} \mathcal{C}(D)
\end{equation}

where $\mathcal{C}'$, $\mathcal{C}' \text{ contr}$, $\rho_3$, $\mathcal{C}$ and the isom$_3$ are defined in the following formulas in [2] Section 2.2 by replacing $p : q$ and $q : p$ in [2] Section 2.1 with the “generalised” $p : q$ and $q : p$ defined Figure 1.

Let $h_3$ be the chain homotopy maps given by [2] Section 2.2, Equation (12)]. Similarly to the previous section, to verify the third Reidemeister invariance of $\mathcal{H}^i(D)$, it is sufficient to show that $\delta_{s,t} \circ \rho_3 = \rho_3 \circ \delta_{s,t}$ and $\delta_{s,t} \circ h_3 + h_3 \circ \delta_{s,t} = \text{id} - \text{in} \circ \rho_3$.

- On the proof of the $\delta_{s,t} \circ \rho_3 = \rho_3 \circ \delta_{s,t}$.

We denote a big sum of the states $\sum_u S \otimes [xu]$ by $S^u$ where $u$ ranges over crossings of $D'$ except for $a$, $b$, and $c$.

First, we can verify that we have the following relations: $(\rho_3 \circ \delta_{s,t})(S_{--}(p,q,r) \otimes [xab]) = (\delta_{s,t} \circ \rho_3)(S_{--}(p,q,r) \otimes [xab]) = S_{--}(p,q,r) \otimes [xabc] + S^u_{--}; (\rho_3 \circ \delta_{s,t})(S_{++}(p,q,r) \otimes [x]) = (\delta_{s,t} \circ \rho_3)(S_{++}(p,q,r) \otimes [x]) = 0;$ $(\rho_3 \circ \delta_{s,t})(S_{+-}(p,q,r) \otimes [xb]) = (\delta_{s,t} \circ \rho_3)(S_{+-}(p,q,r) \otimes [xb]) = 0;$ $(\rho_3 \circ \delta_{s,t})(S_{++} \otimes [x]) = (\delta_{s,t} \circ \rho_3)(S_{++} \otimes [x]) = \delta_{s,t}(S_{++} \otimes [x]).$
Second, \((\delta_{s,t} \circ \rho_3)(S_{+-+}(p, q, r) \otimes [xa]) = (\delta_{s,t} \circ \rho_3)(S_{+-+}(p, q, r) \otimes [xa]) = S_{+-+}(p : q, q : p, r) \otimes [x] + S_{+-+}(p : r, q, r : p) \otimes [xc] + \rho_3(S^u_{+-+}) \) follows from \([31] \)–\([33]\).

In the following, the final case follows from \([27]\). Let \(f_\ast \) be the homomorphism such that \(f_\ast(S) = S \) if \(S = S_\ast(p, q, r) \otimes [x] \) of any \(p, q, r \), and \(x \), \(f_\ast(S) = 0 \) otherwise.

\[
(\rho_3 \circ \delta_{s,t})(S_{+-+}(p, q, r) \otimes [xb]) = \rho_3(S_{+-+}(p, q, m(r, +)) \otimes [xbc])
\]

\[
\begin{align*}
&= \rho_3(S_{+-+}(p, q, m(+)), r) + \rho_3(S^u_{+-+}) \\
&= S_{+-+}(p, q, m(r, +) \otimes [xbc]) \\
&\quad - S_{+-+}(p, m(q, +), r) \otimes [xbc])) + \rho_3(S^u_{+-+}).
\end{align*}
\]

On the other hand,

\[
(\delta_{s,t} \circ \rho_3)(S_{+-+}(p, q, r) \otimes [xb]) = \delta_{s,t}(-S_{+-+}(q : p, p : q, r) \otimes [xa]
\]

\[
\begin{align*}
&- S_{+-+,-}(p : q, (q : p), (q : p) \otimes [xb] \\
&- S_{+-+,-}(p : r, q, r : p) \otimes [xc]) \\
&= \delta_{s,t}(-S_{+-+,-}(p : q, (q : p), (q : p) \otimes [xb]) \\
&\quad \otimes [xb] \\
&\quad - \delta_{s,t}(S_{+-+}(p, q, r) \otimes [x]) + p : p \\
&\quad \otimes [xb] \quad (: S^u_{+-+} = 0 \) on \( C' \)
\end{align*}
\]

\[
= \delta_{s,t}(-S_{+-+,-}(q : p, m(+), q, r) \otimes [xb]
\]

\[
= \delta_{s,t}(-S_{+-+,-}(p : m(+), q, r) \otimes [xb])
\]

\[
= \rho_3(S_{+-+}(p, q, r) \otimes [x]) \quad (: S_{+-+}(p, q, r) \otimes [x])
\]

\[
\begin{align*}
&= \delta_{s,t}(-S_{+-+,-}(p, m(+), q, r) \otimes [xb] \\
&\quad + S_{+-+}(p, q, r) \otimes [xb]) \\
&= -S_{+-+,-}(p, m(+), q, r) \otimes [xbc] \\
&\quad + S_{+-+,-}(p, m(r, +)) \otimes [xbc] + \rho_3(S^u_{+-+}) \\
&\quad (: \rho_3(S_{+-+}(p, q, r) \otimes [xb]) = S_{+-+}(p, q, r) \otimes [xb] - S_{+-+,-}(p, m(+), q, r) \otimes [xb]) \quad \text{on } C'
\end{align*}
\]

where

\[
S_{+-+,-}(p : q, (q : p), (q : p), (q : p) \otimes [xb] = S_{+-+,-}(p, m(+), q, r) \otimes [xb]
\]

\[
+ f_{+-+,-}(\delta_{s,t}(S_{+-+}(p, q, r) \otimes [x])).
\]

\([33]\) follows from \([27]\). When we localise the problem to two signed circles concerning with \(p, q \) and exchange \(p, q \). \([27]\) implies \([33]\).

- On the proof of the \(\delta_{s,t} \circ h_3 + h_3 \circ \delta_{s,t} = \text{id} - \circ \rho_3 \).

- In the beginning, let us show the equation not depending on Figure 1.
Theorem 2.1. For the differential \( \delta_{s,t} \), \( \mathcal{H}^i(D') \cong \mathcal{H}^i(D) \) for \( D' \cong D \) is given by the chain homotopy map \( h_{s,t} \) and the retraction \( \rho_{s,t} \).

Theorem 2.2. Let \( \delta \) be a given differential : \( C^i \to C^{i+1} \) defined in [3, Section 2] except for replacing the Frobenius calculus [3, Figure 3] in the definition of the differential by any calculus satisfying
If (2) satisfies (4)–(6), there is a cohomology $H^i$ derived from $\delta$ preserving Reidemeister I. If (2) satisfies (4)–(6), (21) and (27), there is a cohomology $H^i$ preserving Reidemeister II and III.

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