Flow of magnetohydrodynamic viscous fluid by curved configuration with non-linear boundary driven velocity

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ABSTRACT
This paper discusses the flow of magnetohydrodynamic (MHD) viscous fluid driven by non-linear stretching curved surface. The relevant system of the flow configuration is considered using orthogonal curvilinear geometry in presence of radially varying magnetic field. The governing partial differential equations of the flow problem are reduced into boundary layer regime which are then transformed using similarity variables into ordinary differential equations. The resulting equations of the curved case are not amenable to analytic solutions due to non-linearity of the system. Thus a computational approach through Runge-Kutta fourth order together with the shooting technique is adopted for the numerical solution. The existing solution for flat surface is recovered in validating the present models. The novelty of this work contains the correctness and analytical solution of the existing models over flat surface. The results are interesting and can be useful in polymer dynamics.

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Nomenclature

| Symbol | Description                        | Unit          |
|--------|------------------------------------|---------------|
| ν      | kinematic viscosity                | m²/s          |
| ρ      | fluid density                      | kg/m³         |
| Reₘ    | magnetic Reynolds number           | dimensionless |
| α      | linear stretching strength         | 1/t           |
| β      | Non-linear stretching strength     | 1/Lt          |
| B      | magnetic field                     | Tesla         |
| B₀     | magnetic field strength            | A/m           |
| R, λ   | radius of curvature, curvature     | m             |
| f, h   | stream function                    | dimensionless |
| M      | Hartmann number                    | dimensionless |
| Reₗ    | Reynolds number                    | dimensionless |
| ξ      | dimensionless radius of curvature  | dimensionless |
| P      | pressure                           | Pascal or N/m²|
| σ      | fluid conductivity                 | Siemens/m    |
| J      | current density                    | A/m²         |
| V(υ,ν,0)|velocity field                     | m/s           |
| U      | axial velocity                     | m/s           |
| V      | radial velocity                    | m/s           |
| U∞     | free steam velocity                | m             |
| L      | characteristics length             | m             |

1. Introduction

The linear and non-linear stretching of sheets have been broadly discussed in the literature because of its wide applications in the chemical and polymer industry where the stretching phenomena are used to improve the quality of the finished product. To avoid repetition, the relevant background literature and recent studies with applications for stretching cases have been presented in the articles [1–19]. In this present study, the subsequent discussion is confined to some relevant and fundamental studies for non-linear stretching. These papers serve as a spotlight to the current findings, which consequently extends to curved structures. Takhar et al. [20] investigated the flow over non-linear quadratic stretching of the flat plate over decades ago in which they showed that the fluid velocity increases in comparison to linear stretching. Kumaran et al. [21] explained that the velocity due to non-linear, quadratic, of the stretching sheet superposes the linear case. Kumar and Sanjayanand [22] showed in consequences with linear mass flux for viscoelastic fluid flow in the boundary layer region, the choice of quadratic stretching velocity parallel to the boundary sheet. The effect of suction and blowing on impermeable surface was addressed by Raptis and Perdikis [23]. In this study, an inverse solution is proposed that transformed the governing equations into three self-similar partial differential equations. Kelson [24] made an observation that the solution procedures given in [20,23] lack authenticity. Meaning two components of the momentum equations are addressed while finding the solution and ignored the third equation resulting in a non-physical solution. He presented an analytical solution for the problem discussed by Raptis and Perdikis but
the models remain incomplete in comparison to the current analysis. Cortell [25,26] revisited the problem using the same incomplete similarity variable and presented an alternate analytical approach to get a solution of non-linear stretching for the impermeable surface in non-Newtonian viscous-elastic MHD fluid with heat generation. The objective of this paper can be highlighted as follows: first, the discussion ensued by Kelson has been settled by considering the current transformation variables that generate complete models in generalized curvilinear coordinates. A straightforward procedure is adopted and an analytical solution is provided using the homotopy analysis method and satisfies all the three equations and boundary conditions. The non-linear part of quadratic stretching from the curved surface has been undertaken which was hitherto not known to date. All the plus points of the plane non-linear stretching are far more important for curved non-linear stretching from mathematical and practical points of view. Also, linear and non-linear stretching strength are examined and are important coefficients that have potential influences on the flow field. To invoke further applications in industrial and engineering processes, electrically conducting fluid with a magnetic field is also considered in the realm of MHD. The magnetic field is considered to be variable rather than constant to satisfy Maxwell equations. To our understanding, mentioning some background literature on curved surfaces will be another spectrum to its applications in the polymer industry. The strength of applied magnetic field. The continuity and momentum governing equations for 2-D fluid flow [27] are modified with the variable magnetic fields to satisfy the magnetic field is variable rather than constant to satisfy Maxwell equations. The present geometry gives a 3D view of the existing physical description with no flow kinetics being considered in the z-direction. The Lorentz force due to the applied magnetic field is a resistive force restraining the mutual velocity between the medium and the magnetic field; it is shown that an additional resistive force due to non-linear stretching is also there to be reckoned with. The results are found worthwhile in controlling fluid flow rate in medical apparatus made with prestressed bladder for dispensing fluids.

2. Flow description

Consider 2D boundary layer flow of an incompressible viscous fluid by a curved surface. The surface is driven by non-linear velocity \( u(s) = \alpha s + \beta s^2 \) in the \( s \)-direction. The build-up of the flow in \( r \)-direction determines the boundary layer thickness. The fluid is conducted under the application of a variable magnetic field acting in the transverse direction to the fluid motion. It is given by

\[
B(r) = \lambda B_0 \hat{e}_r
\]

where \( \lambda = \frac{1}{r+1} \) is the curvature, \( R \) the radius of curvature, \( B_0 \) denotes the strength of applied magnetic field parameter where induced magnetic field is neglected for small magnetic Reynolds number and \( \hat{e}_r \) is the unit vector in the radial direction. If \( u, v \) are \( x \)- and \( r \)-components of velocity \( \mathbf{V} \), then the flow kinetics \( \mathbf{V} = (u(r,x,0), v(r,x,0), 0) \) is explicit for configuration in Figure 1.

The current density and Lorentz force \( \mathbf{F} = \mathbf{J} \times \mathbf{B} \) can be expressed in the absence of electrical field \( (\mathbf{E} = 0) \) as

\[
\mathbf{J} = (\mathbf{V} \times \mathbf{B})
\]

\[
\mathbf{F} = (-\sigma \lambda^2 B_0^2 u, 0, 0)
\]

where \( \sigma \) and \( B_0 \) are the conductivity of the fluid and the strength of applied magnetic field. The continuity and momentum governing equations for 2-D fluid flow [27] are modified with the variable magnetic fields to satisfy the polynomial power law as investigated for \( m = 1 \) and 2. The novelty of this paper can be described as follows. Curved linear stretching exists, but quadratic curved stretching was not available in the literature. It is observed that the non-linear stretching enhances the fluid flow, which has great significance regarding its applications in the polymer industry. The strength of the non-linear stretching velocity is found to matter a lot. This point has been investigated for the first time showing that the strength of the non-linear part, on the flow field, is significantly higher than the linear part of the stretching velocity. The boundary layer equations governing curved structure are a 2D problem represented by the three scalar components of the momentum equations. The present geometry gives a 3D view of the existing physical description with no flow kinetics being considered in the z-direction. The Lorentz force due to the applied magnetic field is a resistive force restraining the mutual velocity between the medium and the magnetic field; it is shown that an additional resistive force due to non-linear stretching is also there to be reckoned with. The results are found worthwhile in controlling fluid flow rate in medical apparatus made with prestressed bladder for dispensing fluids.
the solenoidal nature of magnetism ($\nabla \cdot \mathbf{B} = 0$).

\begin{align}
R\lambda \partial_r u + \partial_v v + \lambda v &= 0 \tag{4} \\
\nu \partial_r u + R\lambda u \partial_r u + \lambda u v
&= -\frac{1}{\rho} R\lambda \partial_r p + \nu \left( \partial_t u + R^2 \lambda^2 \partial_r u + \lambda \partial_r u \right)
+ 2R\lambda \partial_r v - \lambda^2 u \tag{5} \\
\nu \partial_v v + R\lambda u \partial_v v - \lambda u^2
&= -\frac{1}{\rho} \partial_r p + \nu \left( \partial_t v + R^2 \lambda^2 \partial_r v + \lambda \partial_r v \right)
- 2R\lambda \partial_r u - \lambda^2 v \tag{6}
\end{align}

The boundary conditions satisfied by the sheet are [20]

\begin{align}
u w &= \alpha s + \beta s^2, v w = 0 & \text{at } r = 0, \tag{7} \\
u &= 0, \partial_r u = 0 & \text{as } r \to \infty \tag{8}
\end{align}

In Equations (4)–(8), $p$ is the pressure, $\alpha (1/t)$ and $\beta (1/L)$ are the stretching strengths for $\alpha$, $\beta \in [0, 1]$. $\nu$ is the kinematic viscosity of the fluid, $\rho$ is the density of the fluid and $L$ is the characteristic length. Through appropriate scaling, the following non-dimensional variables are defined.

\begin{equation}
\begin{aligned}
\hat{t} &= \frac{t}{\delta}, \hat{R} &= \frac{R}{\delta}; \\
\hat{\rho} &= \frac{\partial}{\rho U^2}, \hat{u} = \frac{u}{U}, \hat{v} = \frac{v l}{U \delta}, \hat{\nu} = \frac{\nu l}{U \delta} \tag{9}
\end{aligned}
\end{equation}

After applying Equation (9) in Equations (4)–(6), the continuity expression is satisfied identically and momentum boundary layer equations are

\begin{align}
\nu \partial_t u + R\lambda u \partial_r u + \lambda u v
&= -\frac{1}{\rho} R\lambda \partial_r p + \nu \left( \partial_t u + R^2 \lambda^2 \partial_r u + \lambda \partial_r u \right)
- \frac{\sigma B_0^2 R^2 \lambda^2}{\rho} u \tag{10} \\
\lambda u^2
&= \frac{1}{\rho} \partial_r p \tag{11}
\end{align}

Introducing the following dimensionless variables due to the geometry of the physical problem

\begin{equation}
\begin{aligned}
\eta &= t \left( \frac{\alpha + \beta s}{v} \right)^\frac{1}{2}, u &= \alpha sf'(\eta) + \beta s^2 h'(\eta), \tag{12} \\
R &= \xi \left( \frac{v}{\alpha + \beta s} \right)^\frac{1}{2}, \rho &= \rho u_n^2 P(\eta), \\
\nu &= -\frac{R}{R + \tau} \left( \frac{v}{\alpha + \beta s} \right)^\frac{1}{2} \left\{ \left( 2\alpha f(\eta) + \frac{\beta}{2(\alpha + \beta s)} \right) \right\} \tag{13}
\end{aligned}
\end{equation}

where $f(\eta)$ and $h(\eta)$ represent the stream functions and the primes denote the differentials. Using Equations (12) and (13) in the above boundary layer equations, we get

\begin{equation}
\begin{aligned}
\alpha^2 s \left[ \hat{\xi} + \frac{1}{\hat{\xi}} \left( f'' - \left( f' \right)^2 \right) - \frac{\hat{\xi}}{\hat{\xi} + \eta^2} \left( f'' - \left( f' \right)^2 \right) \right]
&= -\frac{\hat{\xi}}{\hat{\xi} + \eta^2} \left( h'' - 2\hat{\xi} f' \right)
+ 2\lambda s \left[ -\frac{\hat{\xi}}{\hat{\xi} + \eta^2} \left( h' - \frac{1}{\hat{\xi}^2} \right) \right]
= 0 \tag{14}
\end{aligned}
\end{equation}

\begin{equation}
\frac{1}{\hat{\xi} + \eta^2} \left( f' \right)^2 \left( \frac{2}{\hat{\xi} + \eta^2} \right)
+ \frac{1}{\hat{\xi} + \eta^2} \left( -p f' \right) \tag{15}
\end{equation}

subject to the relevant conditions which take the form.

\begin{equation}
f(0) = 0, \quad f'(0) = 1, \quad h(0) = 0, \quad h'(0) = 1 \quad \text{at } \eta = 0 \tag{16}
\end{equation}

\begin{equation}
h'(\infty) = 0, h''(\infty) = 0, f'(\infty) = 0, \quad f''(\infty) = 0 \quad \text{as } \eta \to \infty \tag{17}
\end{equation}

in which $M = \sqrt{\sigma B_0^2 \alpha^2 / \mu}$ denotes the Hartmann number. Total pressure $P(\eta)$ inside the boundary layer
In which $f''(\eta) = 0$ reduce to

\begin{equation}
22) \quad f'''(\eta) + 2f''(\eta) - 3f'(\eta) - 2M^2f(\eta) = 0
\end{equation}

subject to

\begin{equation}
(25) \quad \begin{cases}
f(0) = 0 \\
f'(0) = 1 \\
f'(\infty) = 0 \\
h(0) = 0 \\
h'(0) = 1 \\
h'(\infty) = 0
\end{cases}
\end{equation}

By comparing Equations (23) and (24) with the existing problems [20–23], the terms $f'''(\eta)$ and $h'''(\eta)$ are missing as a result of incorrect similarity variable $\eta = y \sqrt{\alpha/\nu}$ taken earlier in the literature. In other words, the physical behaviour only captures the linear strength $\alpha$ whereas the non-linear $\beta$ is omitted which void the transformations. The accurate models are given in Equations (22)–(24) which can be obtained using the variable $\eta = y \sqrt{(\alpha + \beta x)/\nu}$. Subhas and Mahesha [6] have provided the solution to Equation (22)

\begin{equation}
(26) \quad f(\eta) = \left(1 - e^{-\sqrt{1+M^2} \eta}\right) / \left(\sqrt{1+M^2}\right)
\end{equation}

for $\eta \leq \sqrt{1+M^2} < \infty$.

Equations (23) and (24) can be reduced to

\begin{equation}
(27) \quad \dot{h}^\nu + f'h'' + 2hf'' - h'f'' = \lambda \omega \eta - 2e^{-\omega} M^2h'' = 0
\end{equation}

After using Equation (26) and Equation (27), we get

\begin{equation}
(28) \quad \dot{h}^\nu - \frac{1 - e^{-\omega}}{\lambda} \eta - h'' - (2e^{-\omega} + M^2)h'' + \omega e^{-\omega} h' + 2ae^2 e^{-\omega} = 0
\end{equation}

where $\omega = \sqrt{1+M^2}$. Now, interest here is to solve (28) with (25) for function $h(\eta)$.

### 3. Analytic solution

Consider homotopy analysis method (HAM) in which the convergence region is given as $-c \leq h \leq c$. The series approximations for the unknown is expressed in the form

\begin{equation}
(29) \quad h(\eta) = h_0(\eta) + \sum_{m=1}^{\infty} h_m(\eta)
\end{equation}

#### 3.1. First-order solution

The zeroth-order deformation through auxiliary function $H_0 = e^{-\eta}$, linear operator $L_h$, and initial guess $h_0(\eta)$ are given as follows:

\begin{equation}
(30) \quad h_0 = \frac{1 - e^{-b\eta}}{b}
\end{equation}

\begin{equation}
(31) \quad L_h = \frac{2h''}{b^2} + 2b^2 e^{-b\eta} h + 2b^2 e^{-2b\eta}
\end{equation}

such that Equation (28) satisfies the property in the form

\begin{equation}
(32) \quad L_h = C_1 + C_2 e^{-\eta} + C_3 e^\eta + C_4 e^{-2\eta}
\end{equation}

where $C_j$ $(j = 1 - 4)$ are arbitrary constants, $h_n$ represents the auxiliary parameter, and $H_n$ the convergence control parameter. Homotopy parameter is bounded in the form $0 \leq q \leq 1$ and the required equations with the boundary conditions can now be written as

\begin{equation}
(33) \quad (1-q)L_h[h(\eta); q) - h_0(\eta); q)] = qh_0H_0N_0h(\eta); q)
\end{equation}

\begin{equation}
(34) \quad h(\eta); q = 0, h'(\eta); q = 1, h'(\infty); q) = h'(\infty); q) = 0
\end{equation}

In this analysis, the embedding auxiliary parameter provides us the liberty to select, choose and control the convergence region of this solution. The acceptable range $0 \leq h_n \leq 1$ is taken for simplicity thereby the solution $h(\eta)$ is obtained when $h_n$ becomes unity.
3.2. Non-linear operator

The non-linear operator for Equation (28) is expressed as

\[ N_h = h''(\eta; q) - \left(1 - e^{-b\eta}\right)\frac{h(\eta; q)}{b}h''(\eta; q) \]

\[ - 2e^{-b\eta}h''(\eta; q) - M^2h''(\eta; q) + \beta e^{-b\eta}h(\eta; q) \]

\[ + 2b^2e^{-b\eta}h(\eta; q) - b^3e^{-b\eta} \]

(35)

3.3. mth-order deformation

The deformation of mth-order can be written as

\[ L_h(h_m - \chi_mh_{m-1}) = h_R^m \]

(36)

\[ \chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \]

(37)

\[ h_R^m = h_{m-1}(\eta; q) - \left(1 - e^{-b\eta}\right) \]

\[ \times \sum_{j=0}^{m-1}[h_j(\eta; q) h''_{m-j-1}(\eta; q) - 2e^{-b\eta}h_{m-j-1}(\eta; q)] \]

\[ - M^2h''_{m-1}(\eta; q) + \beta e^{-b\eta}h''_{m-1}(\eta; q) \]

\[ + 2b^2e^{-b\eta}h_{m-1}(\eta; q) - b^3e^{-b\eta} \]

(38)

With the above analysis, Equation (28) and Equation (25) give

\[ h(\eta) = \frac{1 - e^{-b\eta}}{b} + \beta e^{-b\eta} - be^{-\eta}(1 + 2b) \]

\[ + 2b^2e^{-2\eta} - 2b^2e^{-\eta} \]

(39)

Thus the velocity component becomes

\[ u = \alpha se^{-b\eta} + \beta s^2 \left(\frac{e^{-b\eta} - be^{-\eta}}{b}\right) \]

\[ + (1 + 2b)be^{-\eta}(1 + 2b) \]

\[ - 4b^2e^{-2\eta} + 2b^2e^{-\eta} \]

(40)

and the corresponding vertical component velocity can be recovered by inserting Equations (26) and (40) into \( v \), i.e.

\[ v = \frac{\sqrt{v}}{\alpha + \beta s} \left[ \frac{1 - e^{-b\eta}}{b} \right] \]

\[ + 2\beta s \left(\frac{1 - e^{-b\eta}}{b} + \beta e^{-b\eta} - \beta e^{-\eta}(1 + 2b) \right) \]

(41)

This gives the complete solution of MHD viscous flow past a quadratic stretching of flat surface. In the subsequent analysis, our focus is to address non-linear (quadratic) in curved coordinates. After eliminating the pressure term, Equations (19)–(21) yield

\[ f'' + \frac{1}{(\xi + \eta)^{2}} \left(\frac{2f'' + \xi f'''}{-\xi f''}\right) \]

\[ + \frac{1}{(\xi + \eta)^{2}} \left(\frac{\xi f'''}{-\xi f'' - f''}'\right) \]

\[ - \frac{M^2\xi^2}{(\xi + \eta)^2} \left(\frac{f''}{-\xi f'' - f''}'\right) = 0 \]

(42)

\[ h'' + f'' + \frac{2}{(\xi + \eta)^{2}}(h'' + f'') - \frac{1}{(\xi + \eta)^2}(h'' + f'') \]

\[ + \frac{\xi}{(\xi + \eta)^2} \left(\frac{h''f + 2hf''}{-2h''f - h''f}'\right) \]

\[ - \frac{1}{(\xi + \eta)^{2}} \left(\frac{2\xi hf' + \xi hf'}{-h' - f'}\right) \]

\[ + \frac{\xi}{(\xi + \eta)^2} \left(\frac{h'f - 3hf'}{+2hf''}\right) \]

\[ - \frac{M^2\xi^2}{(\xi + \eta)^2} \left(\frac{h'' - \frac{1}{(\xi + \eta)^{2}}h'}{0} \right) \]

(43)

\[ h'' + \frac{2}{(\xi + \eta)^{2}}h'' - \frac{1}{(\xi + \eta)^2}h'' + \frac{1}{(\xi + \eta)^2}h' \]

\[ + \frac{2\xi}{(\xi + \eta)^2} \left(\frac{hh''}{-h'h} + \frac{2\xi}{(\xi + \eta)^2} \left(\frac{hh''}{-(h')^2}\right) \right) \]

\[ - \frac{2\xi}{(\xi + \eta)^2} h' \]

(44)

Equations (42)–(44) are self-similar. The physical quantity which is the frictional drag coefficient \( C_f \) is expressed as

\[ C_f = \frac{\tau_{\nu}|_{r=0}}{\frac{1}{2} \rho u_w^2} \]

(45)

where

\[ \tau_{\nu}|_{r=0} = \mu \left(\frac{1}{\partial \nu / \partial r - \frac{u}{R + r}}\right) |_{r=0} \]

(46)

Using Equations. (12) and (46) in Equation (45), one can get

\[ - \frac{1}{2} R_s^2 C_f = \left(\frac{f''(0) + h''(0) - \frac{2}{\xi}}{\xi}\right) \]

(47)

in which

\[ R_s^2 = \left[\left(\frac{\alpha + \beta s}{\nu}\right)^2\right]^{\frac{1}{2}} \]

(48)

4. Computational methodology

In this section, our focus is to present physical and plausible solutions of the three momentum equations in response to the curved structure by employing numerical approach. In the existing solution of the problem involved flat surface, the third momentum equation has been ignored without any physical justification (see Refs. [20–23]). This omission was first pointed out by Kel- son [24]. He mentioned that their choice of stream function and solutions are not able to satisfy the required
conditions at the surface. Therefore, such expressions do not represent the physical solutions of the system. Here we take up all three equations simultaneously in the subsequent analysis. Substituting Equation (42) into Equation (43), differentiating the resulting equation and considering Equation (44), we arrive at

\[
\begin{align*}
\frac{h'''}{\xi + \eta} + \frac{5}{\xi + \eta} h'' - \frac{5}{\xi + \eta} h' &= - \frac{2\xi}{\xi + \eta} \left( -2h'' + h'''' \right) - \frac{\xi}{\xi + \eta} \left( 4h'''' - 4(h')^2 \right) \\
&+ \frac{\xi}{\xi + \eta} (3h'' + 6h''') + \frac{\xi}{\xi + \eta} (6h'' + 3h''') - \frac{3M^2t^2}{\xi + \eta} (h'' + f'') = 0
\end{align*}
\]

Equations (42) and (49) are higher non-linear coupled partial differential equations and are only amenable to numerical solution. The initial expression for the above higher-order system is converted into an initial value problem using \((f,f',f'',h,h',h'',h''')^T = (z_1,z_2,z_3,z_4,z_5,z_6,z_7,z_8,z_9)^T\). The procedure of our numerical technique of the above system of equations including the boundary conditions gives the following expression:

\[
\begin{align*}
\begin{pmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4 \\
z_5 \\
z_6 \\
z_7 \\
z_8 \\
z_9
\end{pmatrix}
= \begin{pmatrix}
7\lambda z_8 - 5\lambda^2 z_5 - \lambda^3 z_2 - 2\xi^2 z_1 z_4 \\
5\lambda^2 z_5 - 4\lambda^3 z_6 - 4\xi^2 z_2 z_7 \\
4\lambda^3 z_6 - 4\xi^2 z_3 z_7 \\
+3\xi^2 z_3 z_7 + \xi^3 z_1 z_9 \\
+\xi^3 z_1 z_9 + \xi^2 z_1 z_9 \\
-\xi^2 z_7 - 3\lambda z_7 - 3\lambda z_9 \\
+\xi^2 z_7 - 3\lambda z_7 - 3\lambda z_9 \\
-3\lambda^2 z_6 + 3\lambda z_7 - 3\lambda z_9 \\
+M^2 \xi^2 \lambda \xi z_5 + z_4 + 3\lambda z_7 + z_3 \xi^2 (z_5 - z_2)
\end{pmatrix}
\end{align*}
\]

where \(\lambda\) is now in dimensionless form as \((= 1/\xi + \eta)\). At this point, the shooting technique is employed with Runge–Kutta (RK) algorithm in MATLAB. The missing initial conditions remain the challenge and are determined using Newton’s method until the boundary conditions in equations (16) and (17) are satisfied. An extra condition is specified as \(z_9 = 0 (h''(\infty) = 0)\). The initial guesses are chosen in the following form:

\[
\begin{align*}
f'' &= w_1 \\
f'' &= w_2 \\
h'' &= w_3 \\
h'' &= w_4 \\
h'' &= w_5
\end{align*}
\]

Using the Taylor expansion in Equation (52) about \(\eta = \infty\), we obtain

\[
\begin{align*}
f''(\eta) &= w_1 + \Delta w_1 + \Delta w_2, w_3 + \Delta w_4, w_5 + \Delta w_5 \\
\frac{\partial f''}{\partial w_1} &= \Delta w_1 + \Delta w_2 \\
\frac{\partial f''}{\partial w_3} &= \Delta w_3 + \Delta w_4 + \Delta w_5
\end{align*}
\]

In which subsequent conditions \((f'', h'', h''', h''')\) are expanded in the form (53). After lengthy steps, the system handling \(\eta = \infty\) in Jacobian matrix to enhance the convergence region while using Newton’s algorithm is given as

\[
\hat{x} = \hat{A} - \hat{D}
\]

where \(\hat{x} = (w_1, w_2, w_3, w_4, w_5)^T\) and the iterations define below

\[
\begin{align*}
\hat{A}(i,j) = \begin{pmatrix}
z(9i + 3) \times z(9j + 3) + z(9i + 4) \\
+ z(9i + 4) + z(9i + 7) \times z(9j + 7) + z(9i + 8) \times z(9j + 8) + z(9i + 9) \\
\times z(9j + 9)
\end{pmatrix}
\end{align*}
\]

\[
\hat{D}(i,j) = -z(3) \times z(9i + 3) + z(9i + 4) \times z(9i + 4) + z(7) \\
\times z(9i + 7) + z(8) \times z(9i + 8) + z(9) \\
\times z(9i + 9)
\]

and ensuring that the condition \(z = 1\) remains unchanged all through the loops to a fixed tolerance value less
than $10^{-8}$. The obtained numerical results are analysed under the effect of various characterizing parameters such as linear and non-linear stretching velocity factors $\alpha$ and $\beta$, dimensionless radius of curvature $\xi$, and magnetic parameter $M$ (Hartmann number). The values of the surface drag force (skin friction coefficients) are presented in tabular form for different curvature parameters. The results are compared with the existing literature in the limit $\beta = 0$. It is clearly shown in Table 1 that our results are in good agreement with results obtained by Abbas [28], Sanni et al. [29] and Rosca and Pop [31] and our numerical values are presented for different magnetic parameter $M$.

5. Results and discussion

In this section, the flow kinetics due to curved stretching of boundary surface is seen for velocity field $u(\eta)$, the pressure distribution $P(\eta)$ and the surface drag coefficient $-R_s C_f$, in particular, the effects of

Figure 2. Effect of varying dimensionless radius of curvature on stream functions.

Figure 3. Effects of $\alpha$, $\beta$, $\xi$ and $M$ on velocity.
Figure 4. Effects of $M$, $\alpha$, $\beta$, and $\xi$ on pressure distributions.

dimensionless radius of curvature parameter $\xi$, the linear and non-linear stretching strengths “$\alpha$” and “$\beta$”, and the magnetic parameter $M$. The solutions are computed and graphical analysis is carried out within the range $0 \leq \eta \leq 5$. Figures 2(A) and (B) show the behaviours of stream functions $f(\eta)$, $h(\eta)$ and the corresponding momentum boundary layer thickness with dimensionless radius of curvature parameter $\xi$. Clearly it can be seen in both graphs that increasing $\xi \geq 5$ (as the curvature becomes small) decreases the boundary layer thickness. Figures 3(A) and (B) explain the strength of linear and non-linear stretching parameters $\alpha$ and $\beta$ on flow velocity. It is observed that in Figure 3(A), fixing the parameter, $\beta = 0.3$, non-linear stretching strength and varying the linear, $\alpha$, enhances the velocity and the boundary layer thickness. Similar behaviour can be seen in Figure 3(B) while fixing the parameter, $\alpha = 0.3$, linear stretching strength and varying the non-linear $\beta$. The physical attributes of these observations infer that the velocity and boundary layer can be controlled by any of the stretching coefficients. However, the effect in the presence of a non-linear term from Figure 3(A) is more significant in comparison with Figure 3(B) where linear term prevails. The influence of dimensionless radius of curvature on flow field is illustrated in Figure 3(C). It shows that the velocity and momentum boundary layer are found to decrease slightly with an increase in dimensionless radius of curvature. This observation can be seen as potential consequences of small curvature (large radius of curvature) in decreasing the centrifugal force and hence reduces the velocity. This observation has been emphasized in the introduction that the curvature plays an important role in improving the flow rate for the curved structures through the generation of secondary flow. Thus the effect of radius of curvature is significant along the surface due to centrifugal force that keeps the flow along the curved path. The effects of applied magnetic field on the flow field are presented in Figure 3(D) and (E). It can be seen from Figure 3(D) that for linear stretching ($\beta = 0$), the velocity decreases against the magnetic field. It is due to the opposing nature of the Lorentz force. Remarkably, a further decrease in the velocity field occurs by the imposition of non-linear stretching velocity through Figure
Figure 3(C). shows substantial support for the observation made in curvature parameter hand, it decreases for the higher dimensionless radius of as we move towards the curved surface. On the other presents the impact of magnetic field on skin friction (see the upper part). Whereas the lower section of Table comparing the numerical results with the published articles for increasing magnetic parameter coefficient. It is noticed that surface drag force enhances these results are well substantiated in Table1 by com-
try, to enhance and control the flow generated by the stretching velocities at hand in the curved structure. We remember that for the linear stretching, the velocity and pressure in the boundary layer region. This explains the reduction of pressure to zero (agrees with the boundary layer flow conditions) as we move towards the flat surface. Finally, we make a comparative study of the stretching velocities at hand in the curved structure. We remember that for the linear stretching, the velocity and pressure in the boundary layer decrease, whereas both of these quantities increase for non-linear stretching. It is further observed that of the two strengths of linear and non-linear parts, the non-linear part has a dominating role. These conclusions provide further insight to improve the quality of finished products in the polymer industry, to enhance and control the flow generated by the stretching of sheets specifically for the curved sheets. These results are well substantiated in Table 1 by comparing the numerical results with the published articles (see the upper part). Whereas the lower section of Table 1 presents the impact of magnetic field on skin friction coefficient. It is noticed that surface drag force enhances for increasing magnetic parameter Macross the table as we move towards the curved surface. On the other hand, it decreases for the higher dimensionless radius of curvature parameter ξ when the surface becomes flat. It shows substantial support for the observation made in Figure 3(C).

6. Conclusion

This study examines the steady MHD viscous flow by non-linear quadratic stretching surface. The problem is characterized by physically important parameters like the radius of curvature, magnetic field and stretching strength. The fifth-order coupled boundary layer equations are solved numerically by implementing shooting techniques with Runge–Kutta (RK) fourth-order algorithm. The significant results of these findings can be highlighted as follows:

- Impact of the radius of curvature ξ has a positive effect on the velocity u(η) and pressure p(η).
- Effect of magnetic parameter M decreases the velocity, the momentum boundary layer and the pressure. It is also noticed that the pressure reduces further due to higher resistive force not only because of the magnetic parameter but also with non-linear stretching strength β.
- The effects of varying non-linear terms “α” and “β” on the velocity and pressure inside the boundary layer are examined intuitively.
- An analytical solution of non-linear, quadratic stretching velocity for the plane surface is provided with appropriate similarity transformation variables.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Table 1. Comparison and variation of surface drag force.

| Radius of curvature | Abbas et al. [28] | Rosca and Pop [31] | Sanni et al. [29] | Present |
|---------------------|------------------|-------------------|------------------|---------|
| ξ                   | u = αs           | u = αs            | u = αs           | u = αs  |
| 5                   | 1.1576           | 1.15076           | 1.1576           | 1.1576  |
| 10                  | 1.0734           | 1.07172           | 1.0735           | 1.0734  |
| 20                  | 1.0355           | 1.03501           | 1.0356           | 1.0356  |
| 50                  | 1.0140           | 1.01380           | 1.0141           | 1.0141  |
| 100                 | 1.0070           | 1.00687           | 1.0070           | 1.0071  |
| 200                 | 1.0036           | 1.00342           | 1.0036           | 1.0036  |
| 1000                | 1.0008           | 1.00068           | 1.0008           | 1.0010  |

Present numerical results variation of the surface drag force for different values of M and ξ.
