Alpha Power Transformed Extended Bur II Distribution: Properties and Applications

A. A. Ogunde¹*, B. Ajayi² and D. O. Omosigho²

¹Department of Statistics, University of Ibadan, Ibadan, Nigeria.
²Department of Mathematics and Statistics, The Federal Polytechnic, Ado-Ekiti, Ekiti, Nigeria.

Authors’ contributions

This work was carried out in collaboration among all authors. Author AAO designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors BA and DOO managed the analyses of the study. Author AAO managed the literature searches. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2020/v16i830208

Received: 01 April 2020
Accepted: 07 June 2020
Published: 22 June 2020

Abstract

This paper presents a new generalization of the extended Bur II distribution. We redefined the Bur II distribution using the Alpha Power Transformation (APT) to obtain a new distribution called the Alpha Power Transformed Extended Bur II distribution. We derived several mathematical properties for the new model which includes moments, moment generating function, order statistics, entropy etc. and used a maximum likelihood estimation method to obtain the parameters of the distribution. Two real-world data sets were used for applications in order to illustrate the usefulness of the new distribution.

Keywords: Alpha power transformation; moments; order statistics; Bur II distribution; real data sets.

1 Introduction

Burr [1], introduced a system of distributions which contains the Burr XII (BXII) distribution as the most commonly used among different class of these distributions. If a random variable has the BXII distribution,
it is such that \( X^{-1} \) has the scaled Burr III (BIII) distribution with cumulative distribution function (cdf) defined (for \( X > 0 \)) by [2].

\[
\bar{G}(x) = (1 + \left(\frac{x^{-1}}{\alpha}\right)^{\lambda})^{-\theta}
\] (1)

Where \( x \in (0, \infty) \) and \( \lambda, \sigma, \theta \) are real-valued parameters that determine the mean, variance, kurtosis, and skewness of a distribution.

The Burr Type III and Type XII distributions attract special attention because they include several families of nonnormal probability distributions (e.g., the Gamma distribution) with varying degrees of skewness and kurtosis [3,4,5,6]. Further, these distributions have been used in a variety of applied mathematics contexts. Some examples include modeling events associated with software reliability growth [7], risk measurement [8,9], life testing [10,11], forestry [12,13], modeling crop prices [14], reliability analysis [15], meteorology [16], fracture roughness [17,18] etc.

Suppose we let \( \sigma = 1 \) in (1) we obtain another unique distribution named Bur II distribution which can also be used effectively in modeling reliability problems, meteorology, life testing etc.

The cdf of Bur II distribution is given by

\[
G(x) = (1 + x^{-\lambda})^{-\theta}
\] (2)

And the corresponding probability density function (pdf) is given by

\[
g(x) = \lambda \theta x^{-(\lambda+1)}(1 + x^{-\lambda})^{-(\theta+1)}
\] (3)

Where \( x \in (0, \infty) \) and \( \lambda, \theta \) are real-valued parameters that determine the mean, variance, kurtosis, and skewness of the distribution.

1.1 Alpha power transformed extended Bur II (APTEBII) distribution

In recent time several methods have been developed to induce flexibility into standard probability distributions in order to increase their areas of applications and also to obtain a better fits. This work focuses on the use of the method developed by [19], called the Alpha Power Transformation (APT) to obtain a new distribution named Alpha Power Extended Bur II (APTEBII) distribution. [20] Studied the properties of Alpha Power extended Exponential distribution, [21] studied the properties of Alpha power transformed generalized exponential distribution. Alpha Power transformed Weibull distribution was investigated by [22].

Let \( g(x) \) and \( G(x) \) represent the pdf and the cdf of a continuous random variable \( X \), respectively. The APT of \( G(x) \) for \( x \in R \) is given by

\[
F(x) = \begin{cases} 
\frac{\alpha G(x) - 1}{\alpha - 1}, & \text{if } \alpha > 0, \alpha \neq 1 \\
G(x), & \text{if } \alpha = 1 
\end{cases}
\] (4)

And the associated pdf is given by

\[
f(x) = \begin{cases} 
\frac{\log \alpha}{\alpha - 1} g(x)\alpha^{G(x)}, & \text{if } \alpha > 0, \alpha \neq 1 \\
g(x), & \text{if } \alpha = 1 
\end{cases}
\] (5)

Putting (2) in (4), we obtain the cdf of Alpha Power Transformed Extended Bur II (APTEBII) distribution given by
The graph of the cumulative density function of APTEBII distribution is drawn below with the value of $\alpha = 10.5$

$$F(x) = \begin{cases} 
\frac{\alpha^{1+x^{-\lambda}} - 1}{\alpha - 1}, & \text{if } \alpha > 0, \alpha \neq 1 \\
\frac{1}{(1 + x^{-\lambda})^{-\theta}}, & \text{if } \alpha = 1
\end{cases}$$ \hspace{1cm} (6)

The graph of the cumulative density function of APTEBII distribution drawn above indicates that APTEBII distribution has a proper pdf.

The pdf of APTEBII distribution is given by

$$f(x) = \begin{cases} 
\frac{\lambda \theta \log_{\alpha} x^{-\lambda} (1 + x^{-\lambda})^{-(\theta+1)} x^{(1+x^{-\lambda})^{-\theta}}}{\lambda \theta x^{-\lambda} (1 + x^{-\lambda})^{-\theta}}, & \text{if } \alpha > 0, \alpha \neq 1 \\
\frac{\lambda \theta x^{-\lambda} (1 + x^{-\lambda})^{-\theta}}{\lambda \theta x^{-\lambda} (1 + x^{-\lambda})^{-\theta}}, & \text{if } \alpha = 1
\end{cases}$$ \hspace{1cm} (7)

And the pdf of APTEBII distribution is given by

The graph of the pdf of APTEBII distribution is drawn below with the value of $\alpha = 0.5$

**Fig. 1. The graph of the cdf of APTEBII distribution**

**Fig. 2. The graph of the pdf of APTEBII distribution**
The graph drawn above indicates that the APTEBII distribution is unimodal and can be used to address the problem of non-monotone failure rate that is common in real life data.

The survival function of APTEBII distribution is given by

\[ S(x) = \begin{cases} \frac{\alpha}{\alpha - 1} \left( 1 - \alpha^{(1 + x^{-\lambda})^{-\theta}} \right), & \text{if } \alpha \neq 1 \\ 1 - (1 + x^{-\lambda})^{-\theta}, & \text{if } \alpha = 1 \end{cases} \]

(8)

The graph of the APTEBII distribution is drawn below with the value of taken as \( \alpha = 0.5 \)

![Survival function of APTEBII Distribution](image)

**Fig. 3.** The graph of the survival function of APTEBII distribution

And the hazard function is given by

\[ h(x) = \begin{cases} \lambda \theta x^{-\left(\lambda + 1\right)} \left(1 + x^{-\lambda}\right)^{-\left(\theta + 1\right)} \frac{\alpha^{(1 + x^{-\lambda})^{-\theta} - 1} \log \alpha}{1 - \alpha^{(1 + x^{-\lambda})^{-\theta} - 1}}, & \text{if } \alpha \neq 1 \\ \lambda \theta x^{-\left(\lambda + 1\right)} \left(1 + x^{-\lambda}\right)^{-\left(\theta + 1\right)} \frac{1}{1 - (1 + x^{-\lambda})^{-\theta}}, & \text{if } \alpha = 1 \end{cases} \]

(9)

The graph of the hazard function of APTEBII distribution is drawn below with the values \( \alpha = 0.5 \) and \( \lambda = 1.5 \)

![Hazard function of APTEBII Distribution](image)

**Fig. 4.** The graph of the hazard function of APTEBII distribution
The graph drawn above indicates that the hazard function of APTEBII distribution is increasing.

2 Statistical Properties of APTEBII Distribution

The APTEBII distribution can be simulated by inverting cdf (6) as follows: if \( u \) follows uniform distribution on \((0, 1)\), then

\[
q(u) = \left( \frac{1 + u(\alpha - 1)}{\log \alpha} \right)^{-\frac{1}{\theta}} - 1, \quad 0 \leq u \leq 1.
\]  

(10)

In particular the first three quantile, \( q_1, q_2, q_3 \) for the APTEBII distribution were obtained by setting \( u = 0.25 \) for \( q_1 \), \( u = 0.5 \) for \( q_2 \) and \( u = 0.75 \) for \( q_3 \) representing the lower, middle and the upper quartiles.

2.1 Skewness and Kurtosis

The symmetry of a distribution is measured by the coefficient of skewness of a distribution and the coefficient of kurtosis is also a measure of the heaviness of the tail of the distribution.

The Bowley’ skewness [23], is based on quartiles of a distribution as follows:

\[
S = \frac{q \left( \frac{3}{4} \right) + q \left( \frac{1}{4} \right) - 2q \left( \frac{1}{2} \right)}{q \left( \frac{3}{4} \right) - q \left( \frac{1}{4} \right)}
\]  

(11)

And the Moors ‘kurtosis [24] is based on octiles of a distribution, given by

\[
\kappa = \frac{q \left( \frac{4}{5} \right) - q \left( \frac{3}{5} \right) - q \left( \frac{3}{5} \right) + q \left( \frac{1}{5} \right)}{q \left( \frac{4}{5} \right) - q \left( \frac{1}{5} \right)}
\]  

(12)

Where \( q(\cdot) \) represent the quantile function, and can be obtained from (10).

Table 1 displays the percentage points of some specific choices of the parameters taken \((\lambda = 2.5, \theta = 5.0)\) and varying the value of parameter \( \alpha \). It contains the lower quartile \((u = 0.25)\), median \(u = 0.5\) and the upper quartile \((u = 0.75)\), Bowley’ skewness \( (s) \) and Moors ‘kurtosis.

| \( \alpha \) | 0.25 | 0.5 | 0.75 | 1/8 | 3/8 | 5/8 | 7/8 | \( S \) | \( k \) |
|-------------|------|-----|------|-----|-----|-----|-----|------|------|
| 1.5         | 0.0335 | 0.0235 | 0.0171 | 0.0405 | 0.0279 | 0.0191 | 0.01474 | -0.2152 | -0.4511 |
| 2.5         | 0.2839 | 0.1188 | 0.0612 | 0.5214 | 0.0177 | 0.0837 | 0.04510 | -0.4824 | -2.0918 |
| 15          | 0.1943 | 0.0271 | 0.0081 | 5.3901 | 0.0622 | 0.0131 | 0.00503 | -0.7956 | -15.4411 |
| 20          | 0.1168 | 0.0153 | 0.0044 | 1.1462 | 0.0361 | 0.0077 | 0.00272 | -0.8060 | -9.8323 |
| 25          | 0.0761 | 0.0095 | 0.0027 | 0.6378 | 0.0229 | 0.0048 | 0.00165 | -0.8135 | -8.3333 |

The skewness, kurtosis, median, lower quartile and the upper quartile of the APTEBII distribution for several values of parameters are listed in Table 1. The values indicate that skewness and kurtosis are negative for all values of the parameters considered. This indicates that the distribution is negatively skewed and increased for the increase in the values \( \alpha \) keeping the values of \( \lambda \) and \( \theta \) fixed at 2.5 and 1.5 respectively.
Table 2 displays the percentage points of some specific choices of the parameters taken \((\lambda = 2.5, \theta = -1.5)\) and varying the value of parameter \(\alpha\). It contains the lower quartile \((u = 0.25)\), median \((u = 0.5)\) and the upper quartile \((u = 0.75)\), Bowley’ skewness \((s)\) and Moors ’kurtosis.

**Table 2. Skewness and Kurtosis of the APTEBII distribution for different values of parameters**

| \(\alpha\) | 0.25 | 0.5 | 0.75 | 1/8 | 3/8 | 5/8 | 7/8 | 5 | \(\kappa\) |
|-----------|------|-----|-----|-----|-----|-----|-----|----|--------|
| 1.5       | 29.8926 | 42.5415 | 58.5038 | 24.6736 | 35.8306 | 50.0807 | 67.8678 | 0.1158 | 0.2317 |
| 2.5       | 3.5219  | 8.4151  | 16.3457  | 1.9179  | 5.6469  | 11.9425 | 21.7450 | 0.2369 | 0.4734 |
| 15        | 5.1450  | 36.8826  | 123.954  | 0.1855  | 16.0801  | 71.4519 | 198.805 | 0.4657 | 0.9381 |
| 20        | 8.5615  | 65.3241  | 226.8845  | 0.8725  | 27.6963  | 129.008 | 367.603 | 0.4800 | 0.9700 |
| 25        | 13.1411 | 104.8401 | 371.6415 | 1.5679 | 43.6410 | 209.5513 | 605.8644 | 0.4884 | 0.9881 |

The skewness, kurtosis, median, lower quartile and the upper quartile of the APTEBII distribution for several values of parameters are listed in Table 2. The values indicate that skewness and kurtosis are positive for all values of the parameters and increased for the increase in the values \(\alpha\) keeping the values of \(\lambda\) and \(\theta\) fixed at 2.5 and -1.5 respectively. The APTEBII distribution is positively skewed and leptokurtic for all values of parameters considered.

### 2.2 Random number generation

The random variate \(Q\) from (APTEBII) distribution can be generated as \(q_u\) according to (10), where \(u \sim U(0,1)\).

### 2.3 Moments

In this subsection, the \(r^{th}\) moment and moment generating function (mgf) of APTEBII distribution are derived. Using the following series representation in pdf (7)

\[
a^k = \sum_{p=0}^{\infty} \frac{(\log \alpha)^p}{p!} k^p
\]

(13)

Using the above expression, we can re-write the pdf in (7) as

\[
f(x) = \frac{\lambda \theta}{\alpha - 1} \sum_{p=0}^{\infty} \frac{(\log \alpha)^p}{p!} x^{-(\alpha+1)}(1 + x^{-\lambda})^{-\theta(p+1)+1}
\]

(14)

And an expression for the \(r^{th}\) moment is given by

\[
E(X^r) = \frac{\lambda \theta}{\alpha - 1} \sum_{p=0}^{\infty} \frac{(\log \alpha)^p}{p!} \int_{-\infty}^{\infty} x^{r-(\alpha+1)}(1 + x^{-\lambda})^{-\theta(p+1)+1} dx
\]

(15)

Letting, \(m=[\theta(p + 1) + 1]\) and \(Z = (1 + x^{-\lambda})\) in (15), we have

\[
E(X^r) = \frac{\theta}{m(\alpha - 1)} \sum_{p=0}^{\infty} \frac{(\log \alpha)^p}{p!} \int_{-\infty}^{\infty} Z^{\frac{1}{m}}(Z^{\frac{1}{m}} - 1)^{r} dz
\]

(16)

Also, let \(u = Z^{\frac{1}{m}}\), then (16) will transform to
\[ E(X^r) = \frac{\theta}{(\alpha - 1)} \sum_{p=0}^{\infty} \frac{(\log \alpha)^p}{p!} \int_{-\infty}^{\infty} Z^{\frac{1}{m}} (Z^{\frac{1}{m}} - 1)^{r-2} \, dz \]  

(17)

Substitute for \( U = Z^{\frac{1}{m}} \) in (17), then we have

\[ E(X^r) = \frac{-\theta}{(\alpha - 1)} \sum_{p=0}^{\infty} \frac{(\log \alpha)^p}{p!} \int_{-\infty}^{\infty} U^{m+2}(U - 1)^{-2} \, dz \]  

(18)

Since,

\[- \int_{0}^{1} U^{m+2}(U - 1)^{-2} \, dz = \int_{-\infty}^{\infty} U^{m+2}(1-U)^{-2} \, dz\]

Finally we have

\[ E(X^r) = \mu'_r = \frac{\lambda}{\alpha - 1} \sum_{p=0}^{\infty} \frac{(\log \alpha)^p}{p!} B \left\{ [\theta(p + 1) + 4], 1 - \frac{r}{\lambda} \right\}, \quad \lambda > r \]  

(19)

Where

\[ B(y,z) = \int_{0}^{1} x^{y-1} (1-x)^{z-1} \, dx \]

Table 3: The mean and variance of APTEBII distribution

| \( \alpha, \lambda, \theta \) | \( \mu_1' \)   | \( \mu_2' \)   | \( \mu_4' \)   | \( \mu_3' \)   | \( \mu_5' \)   | \( \text{Var} \) |
|-----------------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 5.0, 1.5, 5.5              | 0.0713          | 0.1374          | 0.4383          | 2.8919          | 0.1323          |                |
| 10.0, 1.5, 5.5             | 0.0409          | 0.1834          | 0.0695          | 0.0420          | 0.1817          |                |
| 10.0, 2.5, 8.5             | 0.0081          | 0.0012          | 0.0180          | 0.0324          | 0.0011          |                |
| 12.5, 5.5, 10.5            | 0.0409          | 0.0627          | 0.0999          | 0.1654          | 0.0610          |                |
2.4 Moment generating function of APTEBII distribution

Table 3 contains values of mean ($\mu'$), variance($\sigma^2$) and the coefficient of variation (CV) of APTEBII distribution for some certain values of parameters.

In this sub-section we derived the moment generating function of the APTEBII distribution.

The moment generating function of a distribution defined by

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$$  \hspace{1cm} (22)

Putting (20) in (22), we have an expression for the moment generating function of APTEBII distribution given by

$$M_X(t) = \frac{\lambda}{\alpha - 1} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r \theta^p}{r!} \log^p \left[ \frac{\theta (p + 1) + 4}{\alpha} \right]$$  \hspace{1cm} (23)

3 Renyl Entropy

The entropy of a random variable provides the basis for estimating the amount of information (or uncertainty) contained in a random observation in preference to its parent distribution (population). A large value of entropy implies the greater uncertainty in the data, [25]. The concept of entropy found applications in various field of life such as in science, engineering and economics. The Renyi entropy of a random variable $X$, for $\lambda \neq 0$ and $\lambda \neq 1$, is defined by

$$I_\lambda(X) = (1 - \lambda)^{-1} \log \left( \int_{-\infty}^{\infty} f(x)^\lambda \, dx \right)$$  \hspace{1cm} (24)

The Renyl entropy of APTEBII distribution can be obtained by inserting (7) in (24) as follows

$$I_\lambda(X) = (1 - \lambda)^{-1} \log \left[ \frac{\log \theta}{1 - \alpha} \theta^{\lambda \alpha - 1} \sum_{i=0}^{\infty} \frac{\log \theta}{i!} B \left[ 1 - \lambda \theta (i + 1) + 1, \frac{\theta (i + 1) + 4}{\alpha} \right] \right]$$  \hspace{1cm} (25)

By making appropriate substitution, the Renyl entropy of APTEBII distribution is given by

$$I_\lambda(X) = -(1 - \lambda)^{-1} \log \left[ \frac{\log \theta}{1 - \alpha} \theta^{\lambda \alpha - 1} \sum_{i=0}^{\infty} \frac{\log \theta}{i!} B \left[ 1 - \lambda \theta (i + 1) + 1, \frac{\theta (i + 1) + 4}{\alpha} \right] \right]$$  \hspace{1cm} (26)

Taking the value of $\alpha = 1.5, \lambda = 1.5, \theta = 0.5$ and $\lambda = 5$ the Renyl entropy was observed to be $-0.6029$

4 Order Statistics

The order statistics found applications in reliability and life testing and it also plays an important role in statistical inference.

4.1 Some distribution of order statistics

Suppose we let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ be an ordered observation in a random sample of size $n$ drawn from APTEBII distribution with cdf, $F(x)$ given by (6) and pdf $f(x)$, given by (7). The pdf of $X_{(i)}$, $i = 1, 2, \ldots, n$ is given by
Where, $h = 1 + x^{-\lambda}$

Then the pdf of the largest order statistics $X_{(n)} (i = n)$, the smallest order statistics $X_{(1)} (i = 1)$ and the distribution of the median order $X_{(m+1)}$, when $n = 2m + 1$ ($n$ is odd number) are respectively given by

$$f_{X_{(n)}} = n\lambda \theta (\alpha - 1)^{-n}(\log \alpha) x^{-(\lambda+1)}h^{-\theta+1}\alpha^{-\theta}(\alpha^{h^{-\theta} - 1})^{n-1}$$

(28)

And

$$f_{X_{(1)}} = n\lambda \theta (\alpha - 1)^{-n}x^{-(\lambda+1)}h^{-\theta+1}\alpha^{-\theta}(\alpha^{h^{-\theta} - 1})^{n-1}$$

(29)

Then the joint pdf of the two order statistics $X_{ij}$ for $i, j = 1, 2, \ldots, n$ of the APTEBII distribution is given by

$$f_{X_{(i)}, X_{(j)}}(i, j, t, j) = Z\left[\alpha^{h^{-\theta} - 1} - \frac{\alpha^{h^{-\theta} - \theta + 1}}{\alpha - 1}\right]^{i-1}\left[\alpha^{h^{-\theta} - \theta + 1} - \frac{\alpha^{h^{-\theta} - 1}}{\alpha - 1}\right]^{j-1}\left[\alpha^{h^{-\theta} - 1} - \frac{\alpha^{h^{-\theta} - \theta + 1}}{\alpha - 1}\right]^{n-j}$$

$$\times \left\{\frac{\lambda \theta \log \alpha}{\alpha - 1} x_{i}^{-(\lambda+1)}h_{i}^{-\theta+1}\alpha^{-\theta}\right\} \left\{\frac{\lambda \theta \log \alpha}{\alpha - 1} x_{j}^{-(\lambda+1)}h_{j}^{-\theta+1}\alpha^{-\theta}\right\}$$

(31)

Where $Z = \frac{n!}{(i-1)!(j-1)!(n-i-j)}$, $h_{i} = 1 + x_{i}^{-\lambda}$ and $h_{j} = 1 + x_{j}^{-\lambda}$

4.2 Joint distribution of $i^{th}$ and $j^{th}$ order statistics

5 Estimation of Parameters

Here we derive the maximum likelihood estimators (MLEs) for the parameters of the APTEBII distribution. Let $(X_{1}, X_{2}, \ldots, X_{n})$ be a random sample of size $n$ from an APTEBII $(\alpha, \lambda, \theta)$ distribution then the likelihood function can be written as

$$L = \prod_{i=1}^{n} \frac{\lambda \theta \log \alpha}{\alpha - 1} x^{-(\lambda+1)}(1 + x^{-\lambda})^{-(\theta+1)}\alpha(1 + x^{-\lambda})^{-\theta}$$

(32)

Then, the log-likelihood function, $l$, is given by

$$l = n \log(\alpha) + \log(\theta) + \log\left(\frac{\log \alpha}{\alpha - 1}\right) + \log \alpha \sum_{i=1}^{n} (1 + x_{i}^{-\lambda})^{-\theta} - (\theta + 1) \sum_{i=1}^{n} \log x_{i}$$

$$- (\theta + 1) \sum_{i=1}^{n} \log (1 + x_{i}^{-\lambda})$$

(33)
The log-likelihood function can be maximized either directly or by solving the nonlinear equations obtained by differentiating (33). Thus the components of the score vector are given by

\[
\frac{\partial l}{\partial \alpha} = n(\alpha - 1 - a \log \alpha) + \frac{1}{\alpha} \sum_{i=1}^{n} (1 + x_i^{-\lambda})^{-\theta}
\]

\[
\frac{\partial l}{\partial \lambda} = \lambda \theta \log(\alpha) \sum_{i=1}^{n} \frac{(1 + x_i^{-\lambda})^{-\theta} x_i^{-\lambda} \log(x)}{1 - x_i^{-\lambda}} + (\theta + 1) \sum_{i=1}^{n} \frac{\lambda x_i^{-\lambda} \log(x)}{1 + x_i^{-\lambda}}
\]

\[
\frac{\partial l}{\partial \theta} = - \sum_{i=1}^{n} \frac{(1 + x_i^{-\lambda})^{-\theta} \log(1 - x_i^{-\lambda}) \log(\alpha) - \sum_{i=1}^{n} \log(1 - x_i^{-\lambda})}{\theta}
\]

(34), (35) and (36) cannot be solved analytically. Therefore, statistical software such as R, MATLAB etc can be employed in obtaining the numerical solution to the non-linear equations. For the three parameter Alpha power transform extended Bur II distribution pdf, all the second order derivatives can be obtained. Thus the distribution of the random vector is given by

\[
\left(\begin{array}{c}
\hat{\alpha} \\
\hat{\lambda} \\
\hat{\theta}
\end{array}\right) \sim N\left(\left(\begin{array}{c}
\bar{f}_{aa} \\
\bar{f}_{a\lambda} \\
\bar{f}_{a\theta}
\end{array}\right), \left(\begin{array}{ccc}
\bar{f}_{\lambda\lambda} & \bar{f}_{\lambda\theta} & \bar{f}_{\theta\theta} \\
\bar{f}_{\lambda\theta} & \bar{f}_{\theta\theta}
\end{array}\right)\right)
\]

Where \( \bar{f}_{ij} = f_{ij} \big| v = v_i = (\alpha, \lambda, \theta) \) with \( \{i, j\} = [-l_{ij}]^{-1} = \left[ \frac{\partial^2 l}{\partial v_i \partial v_j} \right] \). This gives the approximate variance covariance matrix. By solving for the inverse of the dispersion matrix, the solution will give the asymptotic variance and covariance of the MLs for \( \hat{\alpha}, \hat{\lambda}, \text{ and } \hat{\theta} \). The approximate 100(1 - c)% confidence intervals for \( \alpha, \lambda \text{ and } \theta \) can be obtained respectively as

\[
\hat{\alpha} \pm z_c \sqrt{\bar{f}_{\alpha\alpha}}, \quad \hat{\lambda} \pm z_c \sqrt{\bar{f}_{\lambda\lambda}} \text{ and } \hat{\theta} \pm z_c \sqrt{\bar{f}_{\theta\theta}}.
\]

Where \( z_c \) is the upper c\textsuperscript{th} percentile of the standard normal distribution.

### 6 Applications

To illustrate the flexibility and tractability of the APTEBII model, we provide analysis to two real data sets. The first data set taken from Aarset [26] represents the failure times of 50 devices, while second data set taken from [27], represents the failure times of air-conditioned system of an airplane. We fit the APTEBII distribution and other five competing models namely; Alpha power transformed extended exponential (APTEE), alpha power transformed Weibull (APTW) [14], APTL, APTE, exponential (E) distributions and its sub-model. The two data sets are given in Table 4.

| Table 4. The failure times of air-conditioned system of an airplane and Aarset |
|---------------------------------|
| Data 1 | 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82.83, 84, 84, 85, 85, 85, 85, 85, 85, 86, 86 |
| Data 2 | 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95 |

Descriptive statistics for the two data sets considered are presented in Table 5, which includes mean, median, variance, skewness, among others. The graphs of total test time (TTT curves) to these data are presented in Fig. 5.
Table 5. Descriptive statistics for the data sets

| Statistic    | Data set I | Data set II |
|--------------|------------|-------------|
| N            | 50         | 44          |
| Mean         | 45.69      | 223.50      |
| Median       | 48.50      | 128.50      |
| Variance     | 1078.15    | 93286.41    |
| Skewness     | -0.1421    | 3.5044      |
| Kurtosis     | -1.6267    | 15.8667     |
| Minimum      | 0.1        | 12.20       |
| Maximum      | 86         | 1776        |
| Lower quartile | 13.50    | 67.21       |
| Upper quartile | 81.25    | 219.0       |

(a) TTT plot curve to data set I
(b) TTT plot curve to data set II

Fig. 5. The graphs of total test time (TTT curves)

For all the fitted models, we compute the MLEs of the model parameters (with their corresponding standard errors in parentheses) and also the values of the Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Bayesian Information Criterion (BIC) and Kolmogorov-Smirnoff (KS) statistic used as methods of comparing fits of distributions to data. In general, it is considered that the smaller the values of this statistic the better the model fit to the data.

Table 6 present the results related to the first data set which lists the MLEs of the model parameters (with the corresponding standard errors and the confidence intervals in parentheses) and the values of the values of the AIC, HQIC and the KS test statistics. These figures in this Tables reveals that the APTEBII model has the lowest AIC, HQIC, BIC and in terms of the KS statistics the APTEE distribution has the lowest.

Table 7 present the results related to the second data set which lists the MLEs of the model parameters (with the corresponding standard errors and the confidence intervals in parentheses) and the values of the values of the AIC, HQIC and the KS test statistics. These figures in this Table reveal that the APTEBII model has the lowest AIC, HQIC, BIC and in terms of the KS statistics the APTEE distribution has the lowest.
Table 6. MLEs, standard errors (in parentheses), AIC, HQIC and BIC values for the data set 1

| Model       | MLE’s                      | -2l     | AIC     | BIC     | HQIC    | KS       |
|-------------|----------------------------|---------|---------|---------|---------|----------|
| APTEBII     | $\hat{\alpha} = 0.164(0.031)$ | 169.81  | 345.62  | 351.356 | 347.804 | 0.8115   |
|             | $\beta = 2.803(0.582)$    |         |         |         |         |          |
|             | $\hat{\lambda} = 3.596(0.722)$ |         |         |         |         |          |
| APTE        | $\hat{\alpha} = 2.11(1.643)$ | 281.447 | 568.893 | 567.990 | 571.078 | 0.17419  |
|             | $\beta = 0.021 \times 10^{-4}(0.024)$ |         |         |         |         |          |
|             | $\hat{\lambda} = 0.035(6.552 \times 10^{-3})$ |         |         |         |         |          |
| APTW        | $\hat{\alpha} = 8.911(9.1374)$ | 281.962 | 569.928 | 569.026 | 572.113 | 0.17423  |
|             | $\beta = 0.685(0.128)$    |         |         |         |         |          |
|             | $\hat{\lambda} = 0.121(0.078)$ |         |         |         |         |          |
| APTE        | $\hat{\alpha} = 4.822 \times 10^{-3}$ | 283.583 | 571.167 | 570.565 | 572.623 | 0.193111 |
|             | (3.528 \times 10^{-6})    |         |         |         |         |          |
|             | $\hat{\lambda} = 1.361 \times 10^{-3}$ |         |         |         |         |          |
|             | (6.19 \times 10^{-3})     |         |         |         |         |          |
| APTL        | $\hat{\alpha} = 4.359 \times 10^{-6}$ | 267.747 | 574.495 | 569.892 | 598.951 | 0.2186   |
|             | (6.762 \times 10^{-3})    |         |         |         |         |          |
|             | $\hat{\lambda} = 8.754 \times 10^{-3}$ |         |         |         |         |          |
|             | (6.886 \times 10^{-3})    |         |         |         |         |          |
| E           | $\hat{\lambda} = 4.204 \times 10^{-4}$ | 389.68  | 781.360 | 789.059 | 697.422 | 0.9645   |
|             | (5.9448 \times 110^{-3})  |         |         |         |         |          |
| BII         | $\hat{\beta} = 3.23(0.630)$ | 1446.292| 1450.122| 1447.755| 0.8107  |          |
|             | $\hat{\lambda} = 3.081(0.629)$ |         |         |         |         |          |

Table 7. MLEs, standard errors (in parentheses), AIC, HQIC and BIC values for the data set 2

| Model       | MLE’s                      | -2l     | AIC     | BIC     | HQIC    | KS       |
|-------------|----------------------------|---------|---------|---------|---------|----------|
| APTEBII     | $\hat{\alpha} = 0.187(0.0417)$ | -91.972 | -85.973 | -81.769 | -84.628 | 0.9667   |
|             | $\beta = 45.210(8.270)$    |         |         |         |         |          |
|             | $\hat{\lambda} = 24.210(4.404)$ |         |         |         |         |          |
| APTE        | $\hat{\alpha} = 0.161(0.282)$ | 176.631 | 359.262 | 357.694 | 360.607 | 0.14683  |
|             | $\beta = 2.01 \times 10^{-4} (0.024)$ |         |         |         |         |          |
|             | $\hat{\lambda} = 0.011(0.022)$ |         |         |         |         |          |
| APTW        | $\hat{\alpha} = 6.257 \times 10^{-10}$ | 182.718 | 371.436 | 369.867 | 372.78 | 0.26544  |
|             | (9.854 \times 10^{-8})     |         |         |         |         |          |
| Apte        | $\hat{\beta} = 5.094(0.095)$ |         |         |         |         |          |
|             | $\hat{\lambda} = 7.168 \times 10^{-3} (0.057)$ |         |         |         |         |          |
| APTL        | $\hat{\alpha} = 8.688 \times 10^{-10}$ | 177.388 | 358.775 | 357.729 | 359.220 | 0.19989  |
|             | (5.998 \times 10^{-8})     |         |         |         |         |          |
|             | $\lambda = 8.536 \times 10^{-4} (2.844 \times 10^{-3})$ |         |         |         |         |          |
| E           | $\hat{\lambda} = 4.339 \times 10^{-4} (7.922 \times 10^{-5})$ | 233.056 | 468.111 | 467.589 | 468.560 | 0.8929   |
| BII         | $\hat{\beta} = 45.514(8.310)$ | 308.422 | 620.8439| 623.646 | 621.7404| 0.9667   |
|             | $\hat{\lambda} = 30.684(5.602)$ |         |         |         |         |          |
7 Conclusion

In this paper we have proposed a new three-parameter family of distribution, called the APTEBII distribution. The proposed APTEBII model has two shape parameters and one scale parameter. The APTEBII density function can take various forms depending on its shape parameters. We fit the APTEBII distribution and other five competing models namely; Alpha power transformed extended exponential distribution, alpha power transformed Weibull distribution, Alpha power transformed Lomax distribution, Alpha power exponential distribution, exponential (E) distributions and its sub-model. In modelling the two life data set presented in this work the Alpha power transformed extended Bur II distribution can also be used because of it flexibility.

Disclaimer

The products used for this research are commonly and predominantly use products in our area of research and country. There is absolutely no conflict of interest between the authors and producers of the products because we do not intend to use these products as an avenue for any litigation but for the advancement of knowledge. Also, the research was not funded by the producing company rather it was funded by personal efforts of the authors.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Burr IW. Cumulative frequency functions, Annals of Mathematical Statistics. 1942;13:215–232.
[2] Antonio EG, da-Silva CQ. The Beta Burr III Model for Lifetime Data; 2014.
[3] Burr IW. Parameters for a general system of distributions to match a grid o \( \alpha_3 \) and \( \alpha_4 \).”, Communications in Statistics.1973; 2:1–21.
[4] Tadikamalla PR. A look at the Burr and related distributions, International Statistical Review. 1980;48(3):337–344.
[5] Rodriguez RN. A guide to the Burr Type XII distributions,” Biometrika. 1977;64(1):129–134.
[6] Headrick TC, Pant MD, Sheng Y. On simulating univariate and multivariate Burr Type III and Type XII distributions,” Applied Mathematical Sciences. 2010;4(45–48):2207–2240.
[7] Abdel-Ghaly AA, Al-Dayian GR, Al-Kashkari FH. The use of burr type XII distribution on sofware reliability growth modelling.” Microelectronics Reliability. 1997;37(2):305–313.
[8] Chernobai AS, Fabozzi FJ, Rachev ST, Operational risk: A guide to base II Capital Requirements, Models, and Analysis, John Wiley & Sons, New York, NY, USA; 2007.
[9] Sherrick BJ, Garcia P, Tirupattur V. Recovering probabilistic information from option markets: tests of distributional assumptions. Journal of Futures Markets. 1996;16(5):545–560.
[10] Wingo DR. Maximum likelihood methods for fitting the Burr type XII distribution to life test data, Biometrical Journal. 1983;25(1):77–84.
[11] Wingo DR. Maximum likelihood methods for fitting the Burr type XII distribution to multiply (progressively) censored life test data, Metrika. 1993;40(3–4):203–210.
[12] Gove JH, Ducey MJ, Leak WB, Zhang L. Rotated sigmoid structures in managed uneven-aged northern hardwood stands: A look at the Burr Type III distribution, Forestry. 2008;81(2):161–176.
Lindsay SR, Wood GR, Woollons RC. Modelling the diameter distribution of forest stands using the Burr distribution," Journal of Applied Statistics. 1996;23(6):609–619.

Nassar M, Alzaatreh A, Mead M, Abo-Kasem O. Alpha power Weibull distribution: Properties and applications, Comm. Statist. Theory Methods. 2017;46:10236–10252.

Mokhlis NA. Reliability of a stress-strength model with Burr Type III distributions, Communications in Statistics. 2005;34(7):1643–1657.

Mielke PW. Another family of distributions for describing and analyzing precipitation data. Journal of Applied Meteorology. 1973;12:275–280.

Nadarajah S, Kotz S. “q exponential is a Burr distribution," Physics Letters A. 2006;359(6):577–579.

Nadarajah S, Kotz S. On the alternative to the Weibull function. Engineering Fracture Mechanics. 2007;74(3):451–456.

Mahdavi A, Kundu D. A new method for generating distributions with an application to exponential distribution”, Communications in Statistics - Theory and Methods. 2017;46(13):6543-6557.

Amal S. Hassana, Rokaya E. Mohamda, M. Elgarhyb, Aisha Fayomi. Alpha power transformed extended exponential distribution: properties and applications.J. Nonlinear Sci. Appl. 2019;12:239–251.

Dey S, Alzaatreh A, Zhang C, Kumar D. A new extension of generalized exponential distribution with application to ozone data”, Ozone: Science & Engineering. 2017;39(4):273-285.

Dey S, Sharma VK, Mesfioui M. A new extension of weibull distribution with application to lifetime data”, Annals of Data Science. 20147;4(1):31-61.

Kenney JF, Keeping ES. Mathematics of Statistics, Part 1, 3rd edition. Van Nostrand, New Jersey; 1962.

Moors JJA. A quantile alternative for kurtosis. Journal of the Royal Statistical Society Ser. D., The Statistician. 1998;37:25-32.

Renyil AL. On Measure on Entropy and Information. In fourth Berkeley symposium on mathematical statistics and probability; 1961.

Aarset MV. How to identify a bathtub hazard rate, IEEE Trans. Reliab. 1987;36:106–108.

Linhart H, Zucchini W. Model Selection, John Wiley & Sons, New York. 1986:5.

© 2020 Ogunde et al.; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar).
http://www.sdiarticle4.com/review-history/57185