$\mathcal{PT}$-symmetric non-Hermitian quantum field theories with supersymmetry

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We formulate supersymmetric non-Hermitian quantum field theories with $\mathcal{PT}$ symmetry, starting with free chiral boson/fermion models and then including trilinear superpotential interactions. We consider models with both Dirac and Majorana fermions, analyzing them in terms of superfields and at the component level. We also discuss the relation between the equations of motion, the (non)invariance of the Lagrangian and the (non)conservation of the supercurrents in the two models. We exhibit a similarity transformation that maps the free-field supersymmetric $\mathcal{PT}$-symmetric Dirac model to a supersymmetric Hermitian theory, but there is, in general, no corresponding similarity transformation for the Majorana model. In this model, we find generically a mass splitting between bosons and fermions, even though its construction is explicitly supersymmetric, offering a novel non-Hermitian mechanism for soft supersymmetry breaking.

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I. INTRODUCTION

Conventional quantum mechanics and quantum field theory are formulated using Hermitian Hamiltonians and Lagrangians, respectively. However, in recent years there has been increasing interest in extensions to non-Hermitian quantum theories [1], particularly those with $\mathcal{PT}$ symmetry [2,3], which have real spectra and find applications in many areas such as optonics [4,5] and phase transitions [6,7]. It has also been suggested that non-Hermitian quantum field theory might also have applications in fundamental physics, e.g., to neutrino physics [8–11], dark matter [12], Higgs decays [13], and particle mixing [14]. It has been shown that it is possible to carry over to $\mathcal{PT}$-symmetric non-Hermitian theorems familiar concepts from Hermitian quantum field theory such as the spontaneous breaking of global symmetries [15–17] and the Englert-Brout-Higgs mechanism in gauge theories [16,18–21], despite the appearance of subtleties [22,23] in the relationship between current conservation, Lagrangian symmetries, and Noether’s theorem [24] in non-Hermitian $\mathcal{PT}$-symmetric theories.

Supersymmetry [25] is a very attractive framework within the conventional Hermitian quantum field theory paradigm, as it plays a key role in string theory and may play interesting phenomenological roles by stabilizing the hierarchies of mass scales [26], providing a candidate for dark matter [27], aiding the grand unification of gauge couplings [28], and stabilizing the electroweak vacuum [29]. Moreover, approximate supersymmetry emerges in a number of less fundamental physical systems in optonics [30], condensed-matter physics [31], atomic physics, and nuclear physics [32]. Hence, it is interesting to explore whether and how the framework of supersymmetry can be extended to $\mathcal{PT}$-symmetric non-Hermitian quantum field theories, as we do here for the first time in 3 + 1 dimensions.1

We start by considering $\mathcal{PT}$-symmetric non-Hermitian theories with free bosons and fermions, studying whether they accommodate supersymmetry, as is the case for free Hermitian theories. We recall that a necessary condition for supersymmetry is that the fermionic and bosonic mass spectra coincide. This is not a trivial issue, since a non-Hermitian fermionic mass term $\propto \bar{\psi} \gamma^5 \psi$ is possible for a

1See Ref. [33] for a pioneering discussion in 1 + 1 dimensions. We note also that the appearance of supersymmetry in a $\mathcal{PT}$-symmetric quantum-mechanical model was discovered in Refs. [34,35]. Relations between Hermitian theories and $\mathcal{PT}$-symmetric non-Hermitian theories have been derived in the framework of supersymmetric quantum mechanics [36], see Refs. [37,38] and references therein.
single species of fermion, whereas a non-Hermitian bosonic squared-mass term is possible only if there are at least two complex bosons: $\propto \phi_i^\dagger \phi_b - \phi_i^\dagger \phi_a$. We discuss the construction of $\mathcal{PT}$-symmetric supersymmetric theories with a pair of chiral superfields, using the superfield representation and an appropriate superpotential, examining the conditions for the mass spectra to be real and identical, and discussing the extension to interacting theories. We present our discussion in two formulations of the fermionic sector, one in terms of Dirac fermions and the other in terms of Majorana fermions.

We discuss the model with Dirac fermions in Sec. II, constructing the $\mathcal{PT}$-symmetric non-Hermitian free-particle model with Dirac fermions in Sec. II A and showing in Sec. II B how it can be related by a similarity transformation [39] to a free-particle Hermitian supersymmetric model. We then discuss supersymmetry transformations in Sec. II C, introducing four possible definitions of the supercurrent and discussing the corresponding (non) invariance properties of the Lagrangian and the (non) conservation of the corresponding supercurrents. Non-Hermitian dimension-3 bosonic interactions are introduced in Sec. II D. The free-particle model with Majorana fermions is discussed in Sec. III, initially in its component representation in Sec. III A and then in its superfield representation in Sec. III B, after which we discuss the particle spectrum of the Majorana model in Sec. III C. We look for a similarity transformation to a Hermitian model in Sec. III D, finding that it is not possible, in general, to map the non-Hermitian Majorana model to a supersymmetric Hermitian one. Supersymmetry transformations and the supercurrent are discussed in Sec. III E. Finally, in Sec. IV, we discuss our conclusions and mention some directions for future work.

II. $\mathcal{PT}$-SYMMETRIC NON-HERMITIAN SUPERSYMMETRIC MODEL WITH DIRAC FERMIONS

A. Free-particle model construction

The minimal model we consider contains two $N = 1$ scalar chiral superfields $\Phi_a : a = 1, 2$, which can be written as follows in conventional notation:

$$\Phi_a = \phi_a + \sqrt{2} \theta X_a + \theta \Phi_a - i \theta \sigma^a \theta^i \sigma^i \Phi_a + \frac{i}{\sqrt{2}} \theta \sigma^a \theta^i \sigma^i \Phi_a,$$

where the $\phi_{1,2}$ are the complex scalar components of the superfields, the $X_{1,2}$ are two-component Weyl fermions, the $F_{1,2}$ are complex auxiliary fields, and $\theta^a$ and $\theta^i_a$ are Grassmann variables. Assuming a minimal Kähler potential, the kinetic part $\mathcal{L}_K$ of the corresponding free Lagrangian can be written in the usual way as

$$\mathcal{L}_K = \int \frac{d^2 \theta^i d^2 \bar{\theta}}{4} (|\Phi_1|^2 + |\Phi_2|^2) = \partial_a \bar{\Phi}^a \partial_a \Phi_a + i \bar{\Phi}^a \sigma^a \phi_a,$$

$$\mathcal{L}_W = \int \frac{d^2 \theta^i d^2 \bar{\theta}}{4} (|\Phi_1|^2 + |\Phi_2|^2) = \partial_a \bar{\Phi}^a \partial_a \Phi_a + i \bar{\Phi}^a \sigma^a \phi_a,$$

up to surface terms. One can construct a free-field $\mathcal{PT}$-symmetric model by postulating the following non-Hermitian combination of superpotential terms:

$$\mathcal{L}_W = \int \frac{d^2 \theta^i d^2 \bar{\theta}}{4} (|\Phi_1|^2 + |\Phi_2|^2) = \partial_a \bar{\Phi}^a \partial_a \Phi_a + i \bar{\Phi}^a \sigma^a \phi_a,$$

where $\xi$ is a real parameter, with

$$W = m \Phi_1 \Phi_2,$$

which yields

$$\mathcal{L}_W = \int \frac{d^2 \theta^i d^2 \bar{\theta}}{4} (|\Phi_1|^2 + |\Phi_2|^2) = \partial_a \bar{\Phi}^a \partial_a \Phi_a + i \bar{\Phi}^a \sigma^a \phi_a,$$

Due to the non-Hermiticity of the Lagrangian,

$$\mathcal{L}_W = \mathcal{L}_K + \mathcal{L}_W,$$

we have that

$$\frac{\partial \mathcal{L}_W}{\partial F_a} = F_a + m(1 + \xi) \phi_a = 0$$

except for trivial solutions, where we use the notation $\xi \equiv 2$ and $\xi \equiv 1$. It would therefore appear that there is a four-fold ambiguity in the choice of on-shell condition for the auxiliary fields $F_1$ and $F_2$. However, as first identified in Ref. [22], we are, in fact, free to choose any one of the Euler-Lagrange equations to define the equations of motion; each choice leads to the same physics. In the present case, we can readily convince ourselves that any choice leads to the same Lagrangian for the remaining scalar and fermionic fields:

$$\mathcal{L}^{\text{OS}}_{\text{Dirac}} = \partial_a \phi_a \bar{\partial} \phi_a - m^2 (1 - \xi^2) |\phi_a|^2$$

Choosing the equations of motion for the scalar and fermionic fields by varying with respect to $\phi_a$ and $\phi_a$, respectively, we have

$$\mathcal{L}^{\text{OS}}_{\text{Dirac}} = \partial_a \phi_a \bar{\partial} \phi_a - m^2 (1 - \xi^2) |\phi_a|^2$$

where the superscript "OS" indicates that the auxiliary fields have been evaluated on-shell. Alternatively, we could have arrived at Eq. (8) directly and unambiguously via the path integral by functionally integrating over the auxiliary field, as shown in the Appendix.
rely on the existence of a discrete antilinear symmetry of the same squared mass eigenvalues scalar and four fermion degrees of freedom all have the Hermitian mass term is then odd under both manifesting supersymmetry at the level of the mass in terms of which the Lagrangian takes the form
\[ \mathcal{L}^{\text{OS}}_{\text{Dirac}} = \partial_{\mu} \phi_{\mu} \partial^{\mu} \phi_{\mu} - (m^2 - \mu^2)|\phi_{\mu}|^2 + \bar{\psi} i \gamma \psi - m\bar{\psi}\psi - \mu\bar{\gamma}_{3}\psi, \]
where we have defined \( \mu \equiv m\xi \) and the gamma matrices are understood in the Weyl basis:
\[ \gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \]
in which \( \sigma^i (i = 1, 2, 3) \) are the Pauli matrices. The four scalar and four fermion degrees of freedom all have the same squared mass eigenvalues
\[ M^2 = m^2 - \mu^2, \]
manifesting supersymmetry at the level of the mass spectrum.

Considering only the transformations of the c-number fields,\(^\text{2}\) the Lagrangian is \( PT \)-symmetric if we take
\[ \mathcal{P} : \psi(t, \mathbf{x}) \to \psi^*(t, -\mathbf{x}) = P\psi(t, \mathbf{x}), \]
\[ \tilde{\psi}(t, \mathbf{x}) \to \tilde{\psi}^*(t, -\mathbf{x}) = \tilde{\psi}(t, \mathbf{x})P, \]
\[ \mathcal{T} : \psi(t, \mathbf{x}) \to \psi^*(-t, \mathbf{x}) = T\psi^*(t, \mathbf{x}), \]
\[ \tilde{\psi}(t, \mathbf{x}) \to \tilde{\psi}^*(-t, \mathbf{x}) = \tilde{\psi}^*(t, \mathbf{x})T, \]
where \( P = \gamma^0 \) and \( T = i\gamma^1\gamma^3 \) in \( 3 + 1 \) dimensions. The anti-Hermitian mass term is then odd under both \( \mathcal{P} \) and \( \mathcal{T} \). We note that the eigenvalues are independent of the sign of \( \mu \), and that the eigenvalues are real when \( |\mu| < |m| \), in which case the model is in the unbroken phase of \( PT \) symmetry. Exceptional points occur at \( \mu = \pm m \), corresponding to \( \xi = \pm 1 \). In these cases, the theory becomes massless and we lose either the left- or the right-chiral Weyl fermion

\[^{2}\text{More generally, the viability of non-Hermitian theories may rely on the existence of a discrete antilinear symmetry of the Hamiltonian [40].}\]
on-shell [9,10]. We note that, by virtue of the supersymmetry, the scalar sector inherits the masslessness in spite of having an entirely Hermitian Lagrangian. In addition, beyond the exceptional point in the \( PT \)-broken phase, where \( |\xi| > 1 \) and \( |\mu| > |m| \), both the scalar and fermion mass eigenspectra become complex. The scalar sector inherits the \( PT \) phase transition from the fermion sector by virtue of the supersymmetry.

**B. Mapping to a Hermitian theory via a similarity transformation**

The Lagrangian in Eq. (8) can be mapped to that of a Hermitian theory by the following similarity transformation [39]:
\[ \mathcal{L}^{\text{OS}}_{\text{Dirac}} \to \mathcal{L}^{\text{OS}}_{\text{Dirac}}' = S\mathcal{L}^{\text{OS}}_{\text{Dirac}}S^{-1}, \quad \mathcal{L}^{\text{OS}}_{\text{Dirac}} = (\mathcal{L}^{\text{OS}}_{\text{Dirac}}')^\dagger, \]
with
\[ S = \exp\left[-\text{arctanh}\xi \int \mathcal{d}^3x (\gamma_1^+(t, \mathbf{x})\chi_1(t, \mathbf{x})
+ \chi_1(t, \mathbf{x})\chi_1^+(t, \mathbf{x}))\right]. \]
Noting that (wherein there is no summation over \( a \) and \( b \))
\[ \int \mathcal{d}^3y [\chi_a^+(t, \mathbf{y})\chi_a(t, \mathbf{y}), \chi_a(t, \mathbf{x})\chi_b(t, \mathbf{x})] = -(1 + \delta_{ab})\chi_a(t, \mathbf{x})\chi_b(t, \mathbf{x}), \]
\[ \int \mathcal{d}^3y [\chi_a^+(t, \mathbf{y})\chi_a(t, \mathbf{y}), \chi_a^+(t, \mathbf{x})\chi_a(t, \mathbf{x})] = (1 + \delta_{ab})\chi_a^+(t, \mathbf{x})\chi_a(t, \mathbf{x}), \]
and using the identities
\[ \sum_{n=0}^{\infty} \frac{1}{n!} (\pm \text{arctanh}\mu)^n = \exp(\pm \text{arctanh}\mu) = \left(1 \pm \frac{\mu}{\xi}\right)^{1/2}, \]
we then find
\[ \mathcal{L}^{\text{OS}}_{\text{Dirac}} = \partial_{\mu} \phi_{\mu} \partial^{\mu} \phi_{\mu} - m^2(1 - \xi^2)|\phi_{\mu}|^2
+ i\gamma^1_{\alpha\beta}\partial_{\alpha} \phi_{\mu} \partial_{\beta} \phi_{\mu} - m\sqrt{1 - \xi^2} (\chi_1^\dagger \chi_2 + \chi_2 \chi_1^\dagger), \]
which is Hermitian, as required. We note that this Lagrangian is isospectral to the original non-Hermitian one.
The Lagrangian in Eq. (19) can be expressed (off-shell) in terms of chiral superfields as
\[ \mathcal{L}'_{\text{Dirac}} = \mathcal{L} + \sqrt{1 - \frac{2}{\xi}} \left[ \int d^2 \theta W + \int d^2 \theta W' \right], \] (20)
and we obtain
\[ \mathcal{L}'_{\text{Dirac}} = \partial \bar{\phi}_\alpha \partial \phi_\alpha + i \phi_\alpha \bar{\sigma}_{\alpha \beta} \partial \chi_{\alpha \beta} + F^\dagger_a F_a 
+ m \sqrt{1 - \frac{2}{\xi}} (\bar{\phi}_a \chi_{2 \alpha} + F^\dagger_a \phi_\alpha - \chi^\dagger_{2 \alpha} \chi_{1 \alpha}). \] (21)

For the Hermitian Lagrangian, there is no ambiguity in choosing the on-shell condition for the auxiliary fields, which are
\[ F_a = -m \sqrt{1 - \frac{2}{\xi}} \phi_\alpha, \] (22)
and we immediately recover the Lagrangian in Eq. (19).

C. Supersymmetry transformations and supercurrents

Turning to the supersymmetry transformations, we can readily confirm that the Lagrangian given by Eqs. (2) and (5) is invariant under the following transformations, up to total derivatives:
\[ \delta \phi_\alpha = \sqrt{2} e^{\alpha \beta} \chi_{\alpha \beta}, \quad \delta \phi_\alpha = \sqrt{2} e^{\alpha \beta} \chi_{\alpha \beta}, \] (23a)
\[ \delta \chi_{\alpha \beta} = \sqrt{2} e_a \phi_\alpha - \sqrt{2} i (\sigma^a \epsilon^\dagger \partial \phi_\dagger), \]
\[ \delta \chi_{\alpha \beta} = \sqrt{2} e^a \phi_\dagger + \sqrt{2} i (\sigma^a \epsilon) \partial \phi_\dagger. \] (23b)
\[ \delta F_a = -\sqrt{2} i (\sigma^a \epsilon^\dagger \partial \chi_{\alpha \beta}), \quad \delta F_a = -\sqrt{2} i (\sigma^a \epsilon) \partial \chi_{\alpha \beta}. \] (23c)

Specifically, we obtain
\[ \delta \mathcal{L}_{\text{Dirac}} = -i \sqrt{2} \partial_\alpha \{ e^{\alpha \beta} \sigma^a \chi^\dagger_{\alpha \beta} [F_a + m (1 + \xi) \phi_\alpha] + e^a \chi^\dagger_{\alpha \beta} \partial_\beta \phi_\alpha + m (1 - \xi) \sigma^\alpha \chi_{2 \alpha} \phi_\dagger [F_a + m (1 + \xi) \phi_\alpha] \}. \] (24)

This is as we would expect, given that Eqs. (2) and (5) are constructed, respectively, from D and F terms. The corresponding supercurrent is
\[ J_{\text{Dirac}} = \sqrt{2} e^{\alpha \beta} \sigma^a \chi^\dagger_{\alpha \beta} \partial_\beta \phi_\alpha + im (1 + \xi) \sigma^a \chi^\dagger_{\alpha \beta} \phi_\alpha + m (1 + \xi) \sigma^\alpha \chi_{2 \alpha} \phi_\dagger [F_a + m (1 + \xi) \phi_\alpha]. \] (25)

This current is not Hermitian, and is not conserved, except in the Hermitian limit \( \xi \to 0 \). Specifically, using the equations of motion in Eq. (9), we find
\[ \partial_a J_{\text{Dirac}}^a = \sqrt{2} e^{a \beta} (1 + \xi) \chi_{a \beta} \partial_\alpha \phi_\dagger + \sqrt{2} e^a (1 - \xi) \partial_\alpha \chi^\dagger_{a \beta} \phi_\dagger [F_a + m (1 + \xi) \phi_\alpha] \neq 0. \] (26)

The latter is, however, not unexpected, since we know that conserved currents are not related to transformations that leave the Lagrangian invariant in the case of non-Hermitian theories, see Ref. [22].

We have seen already that there is a four-fold freedom in choosing the on-shell condition for the auxiliary fields \( F_a \). While each choice leads to the same Lagrangian, it is clear from Eq. (23) that these choices lead to distinct supersymmetry transformations. In general, and as we will show, there are 16 possible sets of supersymmetry transformations, which we summarize as follows by introducing the independent parameters \( s_a, \bar{s}_a = \pm 1 \) for \( a = 1, 2 \):
\[ \delta \phi_\alpha = \sqrt{2} e^{\alpha \beta} \chi_{a \beta}, \quad \delta \phi_\alpha = \sqrt{2} e^{\alpha \beta} \chi_{a \beta}, \] (27a)
\[ \delta \chi_{a \beta} = -\sqrt{2} e_a \phi_\alpha + i (\sigma^a \epsilon) \partial_\alpha \phi_\dagger, \]
\[ \delta \chi_{a \beta} = \sqrt{2} e^a \phi_\dagger - i (\sigma^a \epsilon) \partial_\alpha \phi_\dagger. \] (27b)

The variation of the Lagrangian under these transformations is
\[ \delta \mathcal{L}^{\text{Dirac}} = \sqrt{2} e^{a \beta} \{ -im \chi_{a \beta} \partial_\alpha \phi_\dagger [F_a + m (1 + \xi) \phi_\alpha] 
- m (1 - \xi) \chi_{a \beta} \phi_\dagger \}
+ \sqrt{2} e^a \{ \partial_\alpha \chi^\dagger_{a \beta} \partial_\beta \phi_\dagger - im \sigma^a \chi_{2 \alpha} \phi_\dagger \}
+ im \xi \sigma^a \chi_{2 \alpha} \phi_\dagger \}
+ m^2 \xi (1 + \xi) (1 + \xi) \chi_{a \beta} \phi_\dagger \}. \] (28)

We see that this reduces to a total derivative: (i) in the Hermitian limit \( \xi \to 0 \) and (ii) for \( s_a = +1 \) and \( \bar{s}_a = -1 \). The latter case corresponds to making the following replacements in the off-shell transformations in Eq. (23):
\[ F_a \to \langle F_a \rangle = -m (1 + \xi) \phi_\dagger, \] (29a)
\[ F^\dagger_a \to \langle F^\dagger_a \rangle = -m (1 - \xi) \phi_\dagger. \] (29b)

where we reiterate that \( \langle F_a \rangle \neq \langle F^\dagger_a \rangle^\dagger \), see the Appendix.

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3We remark that we are working here with the field variables and not their expectation values; since, as illustrated in the Appendix, we have that \( \langle 0 \rangle \neq \langle 0 \rangle \) in general for non-Hermitian theories, the divergence of the expectation value of the current may still vanish. We leave further study of this, and the subtleties of the classical limit/background-field method for non-Hermitian quantum field theories, to future work.
parameters undergo wave-function renormalization only. In this case, the non-Hermitian parameters could be naturally small, by analogy with soft supersymmetry-breaking parameters in a Hermitian supersymmetric model.

Finally, we remark that the similarity transformation in Sec. II B does not map this theory to a Hermitian one. The reasons are two-fold: first, we have that

\[ S_{\lambda}^{l} \phi_{2,\alpha} S^{-1} \rightarrow \left( \frac{1 + \xi}{1 - \xi} \right)^{1/2} \phi_{2,\alpha}, \]

which leaves the Yukawa interactions non-Hermitian; and second, this similarity transformation acts only on the fermion fields and therefore leaves the dimension-3 bosonic interactions non-Hermitian. Any similarity transformation of the interacting theory to a Hermitian one would depend on the specific form of the interactions,\(^5\) and we leave further investigation to future work.

### III. PT-SYMMETRIC NON-HERMITIAN SUPERSYMmetric MODEL WITH MAJORANA FERMIONS

#### A. Component representation

We consider first a minimal free-particle model containing two complex scalar fields \( \phi_1, \phi_2 \) and two Majorana fermions \( \psi_1, \psi_2 \), with mass terms that include both Hermitian and anti-Hermitian mixing [9,10,39,43]. Notice that this amounts to four bosonic and four fermionic degrees of freedom.

The \( PT \)-symmetric, non-Hermitian free-boson Lagrangian is (\( a = 1, 2 \))

\[ L_{\text{Dirac}}^{\text{OS}} = \partial_{\mu} \phi_{a}^{\dagger} \partial_{\mu} \phi_{a} - m^{2}(1 - \xi^{2})|\phi_{a}|^{2} + \partial_{\mu} \chi_{a}^{\dagger} \partial_{\mu} \chi_{a} - m(1 - \xi) \chi_{a}^{\dagger} \chi_{2,a} - m(1 + \xi) \chi_{2,a} \chi_{a}^{\dagger} \]

leading to the following on-shell Lagrangian:

\[ L_{\text{Dirac}} = \partial_{\mu} \phi_{a}^{\dagger} \partial_{\mu} \phi_{a} - m^{2}(1 - \xi^{2})|\phi_{a}|^{2} + \partial_{\mu} \chi_{a}^{\dagger} \partial_{\mu} \chi_{a} - m(1 - \xi) \chi_{a}^{\dagger} \chi_{2,a} - m(1 + \xi) \chi_{2,a} \chi_{a}^{\dagger} \]

where \( W_{l} \) is the same arbitrary third-order polynomial function of the complex scalar fields \( \phi_1, \phi_2 \), and summations over the indices \( a, b \) are to be understood. The two equivalent extremum conditions for the auxiliary fields become

\[ \frac{\partial L}{\partial F_{a}^{\dagger}} = F_{a} + m(1 \pm \xi) \phi_{a}^{\dagger} + \frac{\partial W_{l}}{\partial \phi_{a}^{\dagger}} = 0, \]

leading to the following on-shell Lagrangian:

\[ L_{\text{osc}}^{\text{Dirac}} = \partial_{\mu} \phi_{a}^{\dagger} \partial_{\mu} \phi_{a} - m^{2}(1 - \xi^{2})|\phi_{a}|^{2} + \partial_{\mu} \chi_{a}^{\dagger} \partial_{\mu} \chi_{a} - m(1 - \xi) \chi_{a}^{\dagger} \chi_{2,a} - m(1 + \xi) \chi_{2,a} \chi_{a}^{\dagger} \]

leading to the following on-shell Lagrangian:

\[ L_{\text{osc}}^{\text{Dirac}} = \partial_{\mu} \phi_{a}^{\dagger} \partial_{\mu} \phi_{a} - m^{2}(1 - \xi^{2})|\phi_{a}|^{2} + \partial_{\mu} \chi_{a}^{\dagger} \partial_{\mu} \chi_{a} - m(1 - \xi) \chi_{a}^{\dagger} \chi_{2,a} - m(1 + \xi) \chi_{2,a} \chi_{a}^{\dagger} \]

\[ - \frac{\partial^{2} W_{l}}{\partial \phi_{a}^{\dagger} \partial \phi_{b}^{\dagger} \chi_{a} \chi_{b}^{\dagger}} - \frac{\partial^{2} W_{l}}{\partial \phi_{a}^{\dagger} \partial \phi_{b}^{\dagger} \chi_{a} \chi_{b}^{\dagger}} \]

We make two key observations: first, the interacting Lagrangian remains independent of the choice of extremum condition for the auxiliary fields, as in the free case; and second, the non-Hermiticity of the free part of the Lagrangian has metastasized into the interactions.\(^4\)

Specifically, the Lagrangian (33) contains non-Hermitian bosonic interactions of dimension 3, whereas the dimension-4 bosonic interactions are Hermitian, as are the dimension-4 fermion-boson Yukawa interactions in Eq. (31), by virtue of the assumption that \( \Delta L_{W,\text{Dirac}}^{\text{AO}} \) is Hermitian. We expect that the renormalization properties of this softly-non-Hermitian model are similar to those of a Hermitian supersymmetric model, i.e., the Lagrangian

\[ m_{1}^{2}, m_{2}^{2} \text{ and } \mu_{2}^{2} \text{ are real. The eigenvalues of the mass matrix are} \]

\[ M_{s,\pm}^{2} = \frac{1}{2} (m_{1}^{2} + m_{2}^{2}) \pm \frac{1}{2} \sqrt{(m_{1}^{2} - m_{2}^{2})^{2} - 4\mu_{2}^{4}}, \]

and these are real as long as

\[ (m_{1}^{2} - m_{2}^{2})^{2} \geq 4\mu_{2}^{4}. \]

\(^4\)The corollary of this observation is that, unlike the case of a purely scalar field theory, where non-Hermiticity may be restricted to dimension-2 mass terms, non-Hermiticity in such a supersymmetric field theory cannot be limited to mass terms alone, but must include also dimension-3 terms.

\(^5\)For a discussion of similarity transformations linking quantum mechanical systems with different physical properties, see, e.g., Refs. [41,42].
The scalar Lagrangian is $PT$ symmetric with respect to transformations of the $c$-number fields, if these transform as [22]

$$\mathcal{P}: \phi_1(t, x) \rightarrow \phi_2^+(t, -x) = +\phi_1(t, x),$$
$$\phi_2(t, x) \rightarrow \phi_2^+(t, -x) = -\phi_2(t, x).$$

$$\mathcal{T}: \phi_1(t, x) \rightarrow \phi_1^+(t, -x) = +\phi_1(t, x),$$
$$\phi_2(t, x) \rightarrow \phi_2^+(t, -x) = +\phi_2(t, x).$$

The corresponding $c$-number Lagrangian is $PT$ symmetric with respect to the transformations in Eq. (14). The fermion mass terms can be written in terms of the conjugate variables $\psi^\dagger_a, \psi^\dagger_\sigma$ as

$$\mathcal{L}_{\text{ferm}} = \frac{1}{2} \bar{\psi} \gamma^\mu \gamma^\nu \psi - \frac{1}{2} m_{aa} \bar{\psi} \gamma^\mu \psi - \frac{1}{2} m_{a\sigma} \bar{\psi} \gamma^\nu \psi$$

and the $PT$-symmetric, non-Hermitian free-fermion Lagrangian is

$$\mathcal{L}_{\text{ferm}} = \frac{1}{2} \bar{\psi} \gamma^\mu \gamma^\nu \psi - \frac{1}{2} m_{aa} \bar{\psi} \gamma^\mu \psi - \frac{1}{2} m_{a\sigma} \bar{\psi} \gamma^\nu \psi$$

The mass eigenvalues are

$$M_{f, \pm} = \frac{1}{2} (m_1 + m_2) \pm \frac{1}{2} \sqrt{(m_1 - m_2)^2 + 4 \mu^2},$$

up to an overall minus sign. These are real as long as

$$(m_1 - m_2)^2 \geq 4 \mu^2.$$

**B. Superfield representation**

We use the same two $\mathcal{N} = 1$ scalar chiral superfields ($\Phi_a = 1, 2$) as in the Dirac model. In order to incorporate mass terms for the fields in the Majorana model, we introduce the following two superpotentials:

$$W = \frac{1}{2} m_1 \Phi_1^2 + \frac{1}{2} (m_{12} + m_{21}) \Phi_1 \Phi_2 + \frac{1}{2} m_{22} \Phi_2^2,$$

where $m_{ab}$ are real and symmetric, and we consider the following non-Hermitian Lagrangian:

$$\mathcal{L}_{\text{Majorana}} = \mathcal{L}_{\text{K}} + \int d^2 \theta W_+ + \int d^2 \theta W_-. $$

The scalar sector derived from the expressions (2) and (45) is

$$\mathcal{L}_{\text{scalar}} = \partial_\mu \phi_1 \partial^\mu \phi_2 + F_\mu^2 F_\mu + m_{aa} (\phi_a \bar{\phi}_a + \phi_\sigma \bar{\phi}_\sigma),$$

and the fermion sector derived from the same is given by

$$\mathcal{L}_{\text{ferm}} = i \bar{\psi}_a \Gamma_{\alpha\beta} \partial_\beta \psi_a - \frac{1}{2} m_{aa} (\bar{\psi}_a \gamma^\mu \psi_a + \bar{\psi}_\sigma \gamma^\mu \psi_\sigma)$$

As in the case of the Dirac model, we have a four-fold freedom in choosing the on-shell conditions for the auxiliary fields, e.g., we might take

$$\frac{\partial \mathcal{L}_{\text{Majorana}}}{\partial F_\mu} = F_\mu + m_{aa} \phi_\mu + m_{a\sigma} \Phi_\mu = 0.$$
where the charge-conjugate spinor is $\chi_0^a \equiv -i\sigma^2 \chi^*_a$, and $\sigma^2$ is the second Pauli matrix. Making use of the following dictionary between the Weyl and Majorana fermion bilinears:

$$\chi^a_{\alpha} \chi_{\alpha} = \frac{1}{2}(\bar{\psi}_a \psi_b - \bar{\psi}_a \gamma^5 \psi_b), \quad (a, b = 1, 2),$$

we obtain

$$\mathcal{L}_{\text{term}} = \frac{i}{2} \bar{\psi}_a i\partial \psi_a - \frac{1}{2} m_{12} \bar{\psi}_a \gamma^5 \psi_2 + \bar{\psi}_2 \gamma^5 \psi_1.$$  \hfill (54)

and identifying the latter expression with the fermionic mass terms in the Lagrangian (40), we associate

$$\mu_f = m_{12}. \quad \text{(55)}$$

### C. (Non)supersymmetric spectrum

Given the expressions (50) for the scalar mass parameters and (55) for the fermion mass parameter, one can check that the eigenvalues (36) and (42) can be written as

$$M_{s,\pm}^2 = \frac{1}{2}(m_{11}^2 + m_{22}^2) - m_{12}$$

$$\pm \frac{1}{2} \sqrt{(m_{11}^2 - m_{22}^2)^2 - 4m_{12}^2(m_{11} - m_{22})^2}, \quad \text{and}$$

$$M_{f,\pm}^2 = \frac{1}{2}(m_{11}^2 + m_{22}^2) - m_{12}$$

$$\pm \sqrt{(m_{11}^2 - m_{22}^2)^2 - 4m_{12}^2(m_{11} + m_{22})^2}.$$  \hfill (56)

We see immediately that

$$M_{s,\pm}^2(m_{11}, -m_{22}) = M_{s,\pm}^2(-m_{11}, m_{22}) = M_{f,\pm}^2(m_{11}, m_{22}).$$  \hfill (57)

Hence, although the non-Hermitian Lagrangian itself was written entirely in terms of chiral superfields, the spectrum is not supersymmetric, except in the limiting cases $m_{11} = 0$ or $m_{22} = 0$ (or the Hermitian limit $m_{12} = 0$).

This nonsupersymmetric spectrum was to be expected, in view of the different signs in the superpotentials (44). Indeed, mass terms mix coefficients appearing in the superpotentials $W_+$ and $W_-^\dagger$: the equation of motion for the auxiliary fields $F_a$ are obtained from taking functional derivatives with respect to $F_a^\dagger$, and therefore involve coefficients from $W_-^\dagger$. The resulting expression for $F_a^\dagger$ is then inserted in terms arising from $W_+$, hence mixing coefficients from $W_-$ and $W_-^\dagger$, and so failing to ensure a supersymmetric spectrum when $W_+ \neq W_-$. However, if we assume that one of the diagonal mass terms vanishes, say $m_{22} = 0$, we can understand why a supersymmetric spectrum is recovered. For a quadratic superpotential, the mass terms do not depend on the overall sign of the superpotential, and physical quantities do not depend on the sign of $m_{12}$. As a consequence all of the combinations $(m_{11}, m_{12}), (m_{11}, -m_{12}), (-m_{11}, m_{12})$ and $(-m_{11}, -m_{12})$ lead to the same spectrum, and Eq. (57) shows that we recover identical scalar and fermionic masses. If one switches on the mass term $m_{22}$, though, the above properties are still valid but we are left with an additional relative physical sign between $m_{11}$ and $m_{22}$, and we cannot expect a supersymmetric spectrum anymore.

In order to give another interpretation of the non-supersymmetric spectrum, we can make the supersymmetry breaking explicit by implementing a phase rotation of the fermion sector via the unitary transformation

$$\chi_2 \rightarrow \tilde{\chi}_2 = -i\chi_2, \quad \chi_2^\dagger \rightarrow \tilde{\chi}_2^\dagger = +i\chi_2^\dagger, \quad \text{(58)}$$

which gives the fermionic Lagrangian

$$\mathcal{L}_{\text{term}} = i\bar{\chi}_a^\dagger \gamma^\mu \partial_\mu \chi_{a,\beta} + \frac{1}{2} m_{12}(\chi_1^\dagger \chi_{1,\alpha} + \chi_{1,\alpha}^\dagger \chi_1^\dagger)$$

$$+ \frac{1}{2} m_{22}(\chi_2^\dagger \chi_{2,\alpha} + \chi_{2,\alpha}^\dagger \chi_2^\dagger)$$

$$+ m_{12}(\chi_{1,\alpha}^\dagger \chi_{1,\alpha} + \chi_{1,\alpha}^\dagger \chi_{2,\alpha}^\dagger)$$

$$= \frac{i}{2} \bar{\psi}_a i\partial \psi_a - \frac{1}{2} m_{11} \bar{\psi}_1 \psi_1 + \frac{1}{2} m_{22} \bar{\psi}_2 \psi_2$$

$$- \frac{1}{2} m_{12}(\bar{\psi}_1 i\gamma^5 \psi_2 + \bar{\psi}_2 i\gamma^5 \psi_1).$$  \hfill (59)

Putting back the scalar sector, the spectrum is now supersymmetric, but the Lagrangian itself can no longer be written entirely in terms of chiral superfields.

This supersymmetry breaking is entirely a consequence of the non-Hermiticity. In contrast, the supersymmetry remains unbroken in spite of the non-Hermiticity in the $1+1$ dimensional model of Ref. [33]. Had we taken a model with an analogous Hermitian mass mixing, arising from either of the superpotentials,\textsuperscript{6} i.e., taking

$$\mathcal{L}_{\text{Maj,Herm}} = \mathcal{L}_K + \int d^2 \theta W_+ + \int d^2 \theta^* W_-^\dagger, \quad \text{(60)}$$

\textsuperscript{6}The sign of $m_{12}$ is irrelevant.

085015-7
we would have found that the spectrum was fully supersymmetric, with squared masses given by
\[ M^2_{\text{Herm}} = \frac{1}{2} (m_1^2 + m_2^2) + m_1^2 \]
\[ \pm \frac{1}{2} \sqrt{(m_1^2 - m_2^2)^2 + 4m_1^2 (m_1^2 + m_2^2)}, \]
(61)
cf. Eqs. (36) and (42). We note that the mass spectrum remains sensitive to the relative sign of the diagonal fermion mass terms also for the Hermitian mass mixing, such that the fermion phase rotation in Eq. (58) again leads to a nonsupersymmetric model, but where this is manifest in both the Lagrangian and the spectrum.

D. Similarity transformation

The scalar part of the Lagrangian can be mapped to that of a Hermitian theory via the following similarity transformation [16]:
\[ \mathcal{L}_{\text{scal}} \rightarrow \mathcal{L}'_{\text{scal}} = S_{\phi} \mathcal{L}_{\text{scal}} S^{-1}_{\phi}, \]
(62)
with
\[ S_{\phi} = \exp \left[ \frac{\pi}{2} \int d^4x \left( \pi_2(t,x) \phi_2(t,x) + \pi_2^{\dagger}(t,x) \phi_2^{\dagger}(t,x) \right) \right], \]
(63)
where \( \pi_2(t,x) = \dot{\phi}_2^{\dagger}(t,x) \) is the conjugate momentum operator. This transforms
\[ \phi_2 \rightarrow -i \phi_2 \quad \text{and} \quad \phi_2^{\dagger} \rightarrow -i \phi_2^{\dagger}, \]
(64)
leading to
\[ \mathcal{L}'_{\text{scal}} = \partial_\mu \phi_1 \partial^\mu \phi_1 - m_1^2 |\phi_1|^2 - \partial_\mu \phi_2 \partial^\mu \phi_2 + m_2^2 |\phi_2|^2 + i \mu \left( \phi_1^2 \phi_2 - \phi_2^2 \phi_1 \right). \]
(65)
This Lagrangian can be obtained from
\[ \mathcal{L}'_{\text{scal}} = \partial_\mu \phi_1 \partial^\mu \phi_1 + F_1 \partial^\mu F_1 - \partial_\mu \phi_2 \partial^\mu \phi_2 - F_2 \partial^\mu F_2 \]
\[ + m_1 (\phi_1 F_1 + \phi_2^{\dagger} F_1^{\dagger}) - m_2 (\phi_2 F_2 + F_2^{\dagger} \phi_1^{\dagger}) + im_1 (\phi_1 F_2 + \phi_2 F_1 - F_2 \phi_1^{\dagger} - F_2^{\dagger} \phi_2^{\dagger}), \]
(66)
which itself arises from the supersymmetric Lagrangian with the Kähler potential
\[ \mathcal{L}'_{\text{K}} = \int d^2 \theta^* d^2 \theta (|\Phi_1|^2 - |\Phi_2|^2) \]
and the superpotential
\[ \mathcal{L}_{W, \text{Maj}} = \int d^2 \theta W' + \int d^2 \theta^* W'^*, \]
(68)
where
\[ W' = \frac{1}{2} m_1 \Phi_1^2 + \frac{1}{2} i (m_{12} + m_{21}) \Phi_1 \Phi_2 - \frac{1}{2} m_{22} \Phi_2^2. \]
(69)
However, the resulting fermionic Lagrangian is
\[ \mathcal{L}'_{\text{ferm}} = i \bar{\chi}_1 \gamma^\mu \partial_\mu \chi_1 - i \bar{\chi}_2 \gamma^\mu \partial_\mu \chi_2 \]
\[ - m_1 \left( \chi^\dagger_1 \chi_1 + \chi^\dagger_2 \chi_2 \right) + m_{22} \left( \chi^\dagger_2 \chi_2 + \chi^\dagger_1 \chi_1 \right) - i m_1 \left( \chi^\dagger_2 \chi_1 - \chi^\dagger_1 \chi_2 \right), \]
(70)
which cannot be reached by a similarity transformation of the non-Hermitian Lagrangian in Eq. (47).

Before concluding this section, we remark on the wrong sign of the kinetic term in Eq. (67). While in the context of Hermitian quantization this would lead to negative-norm modes, the presence of \( PT \) symmetry is sufficient to ensure that one can always construct a positive-definite inner product consistent with unitary evolution [1].

E. Supersymmetry transformations and supercurrents

We can readily confirm that the Lagrangian composed of Eqs. (46) and (47) is invariant under the supersymmetry transformations given in Eq. (23) up to total derivatives. Specifically, we find
\[ \delta \mathcal{L}_{\text{Maj}} = \sqrt{2} \epsilon^\alpha \left\{ -i \partial_\nu [\sigma^\alpha \bar{\chi}_1 \chi_1 (F_a + m_{aa} \phi_a^* + m_{aa} \phi_a)] \right\} \]
\[ + \sqrt{2} \epsilon^\alpha \left\{ -i \partial_\nu [\bar{\chi}_2 \chi_2 (F_a + m_{aa} \phi_a^* + m_{aa} \phi_a)] \right\}. \]
(71)
Analogously to the Dirac model, the on-shell Lagrangian is invariant under the transformations in Eq. (23), again up to total derivatives, as long as we make the replacement
\[ F_a \rightarrow \langle F_a \rangle = -m_{aa} \phi_a^* - m_{aa} \phi_a, \]
(72a)
\[ F_a^\dagger \rightarrow \langle F_a^\dagger \rangle = -m_{aa} \phi_a^* + m_{aa} \phi_a. \]
(72b)
The corresponding supercurrent is
\[ J_{\text{Maj}}^\alpha = \sqrt{2} \epsilon^\alpha \left( [\sigma^\alpha \bar{\chi}_1 \chi_1 (\partial_\mu \phi_a^* + i \sigma^\alpha \bar{\chi}_1 \chi_1 (m_{aa} \phi_a^* + m_{aa} \phi_a))] \right) \]
\[ + \sqrt{2} \epsilon^\alpha \left( [\sigma^\alpha \bar{\chi}_2 \chi_2 (\partial_\mu \phi_a^* + i \sigma^\alpha \bar{\chi}_2 \chi_2 (m_{aa} \phi_a^* - m_{aa} \phi_a))]. \]
(73)
\[ \frac{\partial F_a}{\partial F_a} = \frac{1}{2} \int D\phi^a D\phi^a \exp[-\mathcal{V}(F_a F_a + m(1 - \xi)\phi^a F_a + m(1 + \xi)\bar{\phi}^a F_a)] = -m(1 + \xi)\phi^a. \] 

(A4a)

and

\[ \langle F_a \rangle = \int D\phi D\phi D\phi \exp[-\mathcal{V}(F_a F_a + m(1 - \xi)\phi^a F_a + m(1 + \xi)\bar{\phi}^a F_a)] = -m(1 - \xi)\phi^a. \] 

(A4b)

We see immediately that

\[ \langle F_a \rangle \neq \langle F_a \rangle^*, \quad \text{(A5)} \]
that is, the vacuum state is not invariant under complex conjugation, i.e.,
\[
\langle F_a \rangle = \langle \Omega | F_a | \Omega \rangle \neq \langle \Omega | F_a^* | \Omega \rangle = \langle \Omega | F_a^* | \Omega \rangle^*.
\]

In this way, choosing the on-shell condition for \(F_a\) is equivalent to choosing whether we work with the vacuum \(\Omega\) or \(\Omega^*\), that is whether we choose
\[
\langle F_a \rangle = \langle \Omega | F_a | \Omega \rangle = -m(1 + \xi)\phi_d^* \quad \text{or} \quad \langle F_a \rangle = \langle \Omega | F_a | \Omega \rangle^* = -m(1 - \xi)\phi_d^*.
\]

Since these differ only in the sign of the non-Hermitian terms, this choice is irrelevant, as we have seen previously.

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