Construction of discrete analogs of differential equations for bending of plates and shells with discontinuous parameters based on spline approximation method

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Abstract. A new approach to the construction of discrete analogs of the differential equations of bending of plates with discontinuous parameters based on the two-dimensional spline approximation method is proposed. At its core, it is close to the Finite Element Method, since all constructions at the beginning are carried out for a single cell of a finite element, and then applied to the entire area surrounding the base node. The use of such a technique makes it possible to reduce the number of unknowns in each node of the grid area, and the introduction of parameter gaps into the resolving system makes it possible to take into account their effect on the stress-strain state of the structures. This approach to the construction of discrete analogs of partial differential equations allows developing a universal method for constructing discrete analogs of these equations. This technique will significantly reduce the amount of computation in the strength analysis of buildings and structures of various functional purposes using lamellar and shell elements.

1. Introduction

Thin plates and shells are widely used in various areas of construction. They are the main elements of interfloor ceilings, pavement structures, pavements for roads and airfields, etc., which have various shapes of contours and openings. Stress challenges arise in construction and other areas where strength issues are crucial [1,2]. Since the issues of economics and weight in some types of structures play a primary role, the safety margins of all elements must be fully utilized and, therefore, in these cases, the first place is occupied by studies to determine the design characteristics of elements with discontinuous parameters [3,4].

2. Main part

Consider the general differential equation in matrix form (1) used in the calculation of bent plates [5, 6]:

$$\varphi_i, \varphi , \varphi , \varphi , \varphi , \varphi = 0$$

In the formula (1) $\varphi$ is the row vector of constant coefficients within the considered area and

$$\varphi = \{\varphi , \varphi , \varphi , \varphi \}$$
\( \omega \) – the generalized parameter, which is a function taken as an unknown (deflection, bending moment, etc.), has the form of a vector-column of derivatives of the function

\[
\mathbf{o} = \begin{bmatrix}
\frac{\partial \omega}{\partial \xi} \\
\frac{\partial^2 \omega}{\partial \xi^2} \\
\frac{\partial^3 \omega}{\partial \xi^3} \\
\frac{\partial^4 \omega}{\partial \xi^4}
\end{bmatrix}, \quad \mathbf{o}^\prime = \begin{bmatrix}
\frac{\partial \omega}{\partial \eta} \\
\frac{\partial^2 \omega}{\partial \eta^2}
\end{bmatrix}, \quad \mathbf{o}^{\prime\prime} = \begin{bmatrix}
\frac{\partial^2 \omega}{\partial \xi^2} \\
\frac{\partial^3 \omega}{\partial \xi^3} \\
\frac{\partial^4 \omega}{\partial \xi^4}
\end{bmatrix}, \quad \mathbf{o}^{\prime\prime\prime} = \begin{bmatrix}
\frac{\partial^3 \omega}{\partial \xi^3} \\
\frac{\partial^4 \omega}{\partial \xi^4}
\end{bmatrix};
\]

where \( u \) and \( \zeta \) – loading functions (force, deformation, mixed).

We use the dimensionless coordinate system \( \xi, \eta \) with the center at the point \( ij \). At point \( ij \) is given rectangular area consisting of four rectangular elements adjacent to each other (fig.1). On this area we define some functions \( \omega, u, \zeta \), continuous within each element and having a finite discontinuity of the first kind within the boundaries of the elements [7].

![Figure 1. Item node numbering.](image)

We introduce new constant coefficients \( \alpha, \alpha_2, \alpha_3, \delta_1, \delta_2 \) as some combination of \( \phi_i \) values:

\[
\phi_1 = \alpha_1^2, \quad \phi_2 = 2\alpha_1 \delta_1, \quad \phi_3 = \alpha_1 + \delta_1^2, \quad \phi_4 = \delta_1, \quad \phi_5 = 2\alpha_1 \alpha_2, \quad \phi_6 = 2(\alpha_1 \delta_2 + \alpha_2 \delta_1), \quad \phi_7 = \alpha_2 + 2\delta_1 \delta_2,
\]

\[
\phi_8 = \alpha_2 + 2\delta_1 \delta_2, \quad \phi_9 = 2\alpha_1 \alpha_3 + \alpha_2^2, \quad \phi_10 = 2(\alpha_1 \delta_1 + \alpha_2 \delta_2), \quad \phi_11 = 2\alpha_2 \alpha_3 + \alpha_3^2, \quad \phi_12 = \alpha_3 + \delta_2^2, \quad \phi_13 = 2\alpha_1 \delta_1, \quad \phi_14 = \alpha_3^2.
\]

We replace the fourth-order differential equation with two second-order differential equations so that the unknowns of the second equation that are being sought are partial derivatives of the right side of the first equation

\[
\alpha \mathbf{o}^{\prime\prime} + \delta \mathbf{o}^{\prime} = -u, \quad \alpha u^{\prime\prime} + \delta u^{\prime} = -\zeta,
\]

where \( \alpha = \{\alpha_1, \alpha_2, \alpha_3\}, \quad \delta = \{\delta_1, \delta_2\} \).

Each of the equations (2) differs only in the designation of variables. We construct a discrete analogue of the differential equation:

\[
\alpha \mathbf{o}^{\prime\prime} + \delta \mathbf{o}^{\prime} = -\theta. \quad (3)
\]

Then, by replacing the variable \( \theta \), we can construct a discrete analogue of equation (1) as the result of solving a system of two equations (2).
The formation of a discrete analog of equation (3) is performed first for the area occupied by cell IV (fig.1). The solution of this differential equation at the nonzero right part can be approximated by a two-dimensional spline (incomplete cubic polynomial) as:

$$\omega(\xi, \eta) = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_7 \\ a_3 & a_5 & a_8 & a_{11} \\ a_6 & a_9 & 0 & 0 \\ a_{10} & a_{12} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \xi \\ \xi^2 \\ \xi^3 \end{bmatrix}. \quad (4)$$

To find the coefficients $a_i$ of the polynomial (4), we define all their derivatives and then we find the values of the functions $\omega(\xi, \eta)$ with their derivatives at the origin $\xi=0, \eta=0$. Using an abridged notation of partial derivatives of the function $\omega$ of the following form:

$$\alpha \omega^{ii} + \delta \omega^i = -\theta,$$

$$\omega^{xxxx} = \partial^4 \omega/\partial \xi^4, \quad \omega^{xxy} = \partial^3 \omega/\partial \xi^3, \quad \omega^{xx} = \partial^2 \omega/\partial \xi^2,$$

and so on, we obtain:

$$\omega(\xi, \eta) = \begin{bmatrix} a_0 & a_0^\xi & \frac{1}{2} a_6^\xi & \frac{1}{6} a_0^{xxxx} \\ a_0^\eta & a_0^{\eta} & \frac{1}{2} a_6^{\eta} & \frac{1}{6} a_0^{xxxx} \\ \frac{1}{2} a_0^{xxx} & \frac{1}{2} a_0^{xxy} & 0 & 0 \\ \frac{1}{6} a_0^{xx} & \frac{1}{6} a_0^{xy} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \xi \\ \xi^2 \\ \xi^3 \end{bmatrix}. \quad (5)$$

Similar to (5) we write the expressions for the derivatives of the function $\omega$:

$$\omega^\eta(\xi, \eta) = \begin{bmatrix} a_0^\eta & a_0^{\eta\eta} & \frac{1}{2} a_6^{\eta} & \frac{1}{6} a_0^{xxxx} \\ a_0^{\eta} & a_0^{\eta\eta} & \frac{1}{2} a_6^{\eta\eta} & \frac{1}{6} a_0^{xxxx} \\ \frac{1}{2} a_0^{\eta\eta\eta} & \frac{1}{2} a_0^{\eta\eta\eta} & 0 & 0 \\ \frac{1}{6} a_0^{\eta\eta} & \frac{1}{6} a_0^{\eta\eta\eta} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \xi \eta \\ \xi^2 \eta^2 \\ \xi^3 \end{bmatrix}, \quad (6)$$

$$\omega^{\eta\eta}(\xi, \eta) = \begin{bmatrix} \frac{1}{2} a_0^{\eta\eta} & \frac{1}{2} a_0^{\eta\eta\eta} & \frac{1}{6} a_0^{xxxx} \\ \frac{1}{2} a_0^{\eta\eta} & \frac{1}{2} a_0^{\eta\eta\eta} & \frac{1}{6} a_0^{xxxx} \\ \frac{1}{6} a_0^{\eta\eta\eta} & \frac{1}{6} a_0^{\eta\eta\eta} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \xi \eta^2 \\ \xi^2 \eta^3 \end{bmatrix}, \quad (7)$$

$$\omega^\xi(\xi, \eta) = \begin{bmatrix} a_0^\xi & a_0^{\xi\xi} & \frac{1}{2} a_6^\xi & \frac{1}{6} a_0^{xxxx} \\ a_0^{\xi} & a_0^{\xi\xi} & \frac{1}{2} a_6^{\xi\xi} & \frac{1}{6} a_0^{xxxx} \\ \frac{1}{2} a_0^{\xi\xi\xi} & \frac{1}{2} a_0^{\xi\xi\xi} & 0 & 0 \\ \frac{1}{6} a_0^{\xi\xi} & \frac{1}{6} a_0^{\xi\xi\xi} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2\xi \\ 3\xi^2 \end{bmatrix}, \quad (8)$$
\[
\omega^{xx}(\xi, \eta) = \left[1 \quad \eta \right] \begin{bmatrix}
\frac{1}{2} \alpha_{00}^{xx} & \frac{1}{6} \alpha_{00}^{xyz} \\
\frac{1}{2} \alpha_{00}^{yy} & \frac{1}{6} \alpha_{00}^{zzy}
\end{bmatrix} \begin{bmatrix}
2 \\
6 \xi
\end{bmatrix},
\] (9)

\[
\omega^{zz}(\xi, \eta) = \left[1 \quad 2 \eta \quad 3 \eta^2 \right] \begin{bmatrix}
\frac{1}{2} \alpha_{00}^{zz} & 0 & 0 \\
\frac{1}{6} \alpha_{00}^{xxx} & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 \\
2 \xi \\
3 \xi^2
\end{bmatrix},
\] (10)

Since the equation (3) includes derivatives of function 1, not higher than the second order, and from the geometric parameters, these derivatives are associated with derivatives of higher orders [8,9], it is necessary to differentiate equation (3) by \(\xi\) and \(\eta\):

\[
\alpha (\omega^{\mu})^\xi + \delta (\omega^{\nu})^\eta,
\] (11)

\[
\alpha (\omega^{\nu})^\eta + \delta (\omega^{\mu})^\xi.
\] (12)

Using the expressions (5)-(10) and placing the origin of the local coordinate system at point \(ij\), we determine the values of the function \(\omega\) and its derivatives at the angular points of the element under consideration at \(h_{i+1} = h\) and \(\tau_{i+1} = \tau\). The values of the coordinates \((\xi, \eta)\) of the angular vertices of the rectangular area IV for the points 00, 01, 10 and 11 will be equal, respectively \((0,0), (0,\tau), (h,0)\) and \((h,\tau)\) [10-12].

We write equations (3), (11) and (12) for each of them:

\[
\alpha \omega_0^{00} + \delta \omega_0^{00} = -\theta_0;
\]

\[
\alpha (\omega_0^{00})^\xi + \delta (\omega_0^{00})^\eta = -\theta_0^\xi;
\]

\[
\alpha (\omega_0^{00})^\eta + \delta (\omega_0^{00})^\xi = -\theta_0^\eta;
\]

\[
\alpha \omega_{01}^{00} + \delta \omega_{01}^{00} = -\theta_{01};
\]

\[
\alpha (\omega_{01}^{00})^\xi + \delta (\omega_{01}^{00})^\eta = -\theta_{01}^\xi;
\]

\[
\alpha (\omega_{01}^{00})^\eta + \delta (\omega_{01}^{00})^\xi = -\theta_{01}^\eta;
\]

\[
\alpha \omega_{10}^{00} + \delta \omega_{10}^{00} = -\theta_{10};
\]

\[
\alpha (\omega_{10}^{00})^\xi + \delta (\omega_{10}^{00})^\eta = -\theta_{10}^\xi;
\]

\[
\alpha (\omega_{10}^{00})^\eta + \delta (\omega_{10}^{00})^\xi = -\theta_{10}^\eta;
\]

To formation a discrete analog [13-15] of differential equation (3) for the area occupied by a rectangular cell IV, we make a linear combination of the equations of the system (13). In the corner points we substitute the values of the functions \(\omega\) and \(\theta\) and their derivatives calculated on the basis of the approximating polynomial. As a result of such a linear combination we get:
The discrete analog of equation (3) for the combination of elements I-IV is obtained by summing the left and right sides of equation (16) [19,20]. Taking into account formulas (17-20), we obtain in a general form the final form of the discrete analogue of equation (3):

\[
\sum_{k=1}^{n} a_k \Delta \omega_k = -\tau h \Delta \omega^6 - c \Delta \omega^9 + \beta_1 \Delta \omega^{62} + \beta_2 \Delta \omega^{99} + 4 \alpha_1 \tau \Delta \omega_{61} + 4 \alpha_2 \tau \Delta \omega_{91} = \frac{\tau h \Theta^9}{3} + \frac{\tau^2 h \Delta \Theta^9}{6} + \frac{\tau^2 h^2 \Delta \Theta^9}{2}.
\]
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3. Conclusions

As a result of the research, the authors propose a technology for the formation of discrete analogues (finite-difference operators) of differential equations describing the stress-strain state of various plate elements. On the basis of this technique, it is possible to create a unified calculation algorithm for modeling the stress-strain state of elements with discontinuous parameters.

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