The Tate spectrum of $ER(n)$

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Abstract

The $2^{n+2}(2^n - 1)$-periodic spectrum $ER(n)$ is obtained as the $\mathbb{Z}/2$ homotopy fixed points the real Johnson-Wilson spectrum $ER(n)$. In this paper we show that the Tate spectrum associated to $ER(n)$ is trivial.

1 Introduction

The $\mathbb{Z}/2$-equivariant Johnson-Wilson spectrum $ER(n)$ was first constructed by Hu and Kriz in \cite{6}. Kitchloo and Wilson \cite{8} have used the homotopy fixed point spectrum of this to solve certain non-immersion problems of real projective spaces. The homotopy fixed point spectrum $ER(n)$ is $2^{n+2}(2^n - 1)$-periodic compared to the $2(2^n - 1)$-periodic $E(1)$. The spectrum $ER(1)$ is $KO(2)$ and $E(1)$ is $KU(2)$.

Our main theorem is

**Theorem 1.1.** The Tate spectrum $\tau(ER(n))$ is trivial.

The triviality of the Tate spectrum for Atiyah’s $K$-theory with reality and subsequently the spectrum $KR$ was proved by Fajstrup in \cite{2}. Results in similar directions can be found in \cite{4}, \cite{5} and \cite{6}. In \cite{4} Hu shows the fixed points of the Tate spectrum for $BP_R$ is $H\mathbb{Z}/2$. In \cite{5} Hu computes the coefficient ring of $t(BP_R < n >)$. The triviality of the Tate spectrum also means that we can define $ER(n)$ as the homotopy orbit spectrum $ER(n)_{h\mathbb{Z}/2}$.

The main result is restated subsequently as Theorem 3.1.

2 $G$-spectra and $G$-cohomology theories

We will be working in the equivariant stable homotopy category of Lewis-May-Steinberger. Let us begin by recalling some definitions. Let $G$ be a finite group.

A complete universe $U$ is an infinite dimensional real inner product space with $G$ acting through isometries such that $U$ contains a countably infinite sum of all irreducible representations of $G$ as subspace.

A $G$-spectrum $k_G$ indexed on $U$ associates a based $G$-space $k_G(V)$ to each finite dimensional $G$-subspace $V \subset U$ such that for any two $G$ subspaces $V$ and $W$ of $U$ with $V \subset W$ the structure maps $k_G(V) \xrightarrow{\Omega^W-V} k_G(W)$ are $G$-homeomorphisms. Here, $W-V$ denotes the orthogonal complement of $V$ in $W$. For any $V$, $S(V)$ denotes the orthogonal complement of $V$. Then for a $G$-space $Y$, $\Omega^V Y$ denotes the $G$-space of all based maps $S^V \to Y$.

Let $U = \bigoplus(V_i)$ for a complete set of distinct irreducible representations $V_i$. Then $RO(G, U)$ is the free abelian group generated by the $V_i$. We define the $RO(G, U)$-graded homology and cohomology theory associated to any $G$-spectrum $k_G$. For any virtual representation $a = V-W$ there are sphere $G$-spectra $S^a = \Sigma^{-W}S^V$, and we let

$$k_G^a(X) = [X \wedge S^a, k_G]_G$$

and

$$k_a^G(X) = [S^a, X \wedge k_G]_G$$
for any $G$-spectrum $X$. For a $G$-space $Y$ let $k^a_G(Y) = k^a_G(\Sigma^\infty Y)$ and similarly for homology; here $\Sigma^\infty$ is the functor from $G$-spaces to $G$-spectra that is left adjoint to the 0th space functor.

A map of $G$-spectra $f : E \to F$ is a stable equivalence when $f^*_n : \pi_n(E^H) \to \pi_n(F^H)$ is an isomorphism for all $n$ and all $H \leq G$.

## 2.1 The Tate Spectrum

We recall necessary definitions and facts from [3]. The various equivariant spectra associated to any given $G$-spectrum $k_G$ are displayed in the Tate diagram given below. Let $X_+$ be the disjoint union of the $G$-space $X$ with a fixed base point and let $EG$ be a contractible free $G$-space. Let $\overline{EG}$ be the unreduced suspension of $EG$. Then there is a cofibering

$$EG_+ \to S^0 \to \overline{EG} \to \Sigma EG_+.$$ (1)

Let $F(EG_+, k_G)$ be the function $G$-spectrum of maps from $EG_+$ to $k_G$ with diagonal $G$ action. The projection $EG_+ \to S^0$ induces a map of $G$-spectra

$$\epsilon : k_G = F(S^0, k_G) \to F(EG_+, k_G).$$

Smashing $\epsilon$ with the previous cofibering gives the following map of cofiberings of $G$-spectra, known as the Tate diagram.

$$
\begin{array}{ccc}
  k_G \wedge EG_+ & \longrightarrow & k_G \\
  \downarrow & & \downarrow \\
  F(EG_+, k_G) \wedge EG_+ & \longrightarrow & F(EG_+, k_G) \\
\end{array}
$$

Notice that the leftmost vertical arrow is always an equivalence. We define the Borel, geometric and Tate spectra associated to $k_G$ respectively as follows:

- $c(k_G) = F(EG_+, k_G)$
- $g(k_G) = k_G \wedge \overline{EG}$
- $t(k_G) = F(EG_+, k_G) \wedge \overline{EG}$
- $f(k_G) = k_G \wedge EG_+$

It follows that, up to equivalence, $t(k_G)$ is the cofiber of the composite map

$$k_G \wedge EG_+ \to k_G \to F(EG_+, k_G).$$

The canonical smash product pairing

$$F(X, Y) \wedge F(X', Y') \to F(X \wedge X', Y \wedge Y')$$

and the equivalences

$$EG_+ \wedge EG_+ \simeq EG_+ \quad \overline{EG} \wedge \overline{EG} \simeq \overline{EG}$$

give the following proposition.

**Proposition 2.1.** If $k_G$ is a ring $G$-spectrum then $c(k_G)$ and $t(k_G)$ are ring $G$-spectra and the following part of the Tate diagram is a commutative diagram of ring $G$-spectra:

$$
\begin{array}{ccc}
  k_G & \longrightarrow & k_G \wedge \overline{EG} \\
  \downarrow & & \downarrow \\
  c(k_G) & \longrightarrow & t(k_G) \\
\end{array}
$$

The unit of $t(k_G)$ is the smash product of the unit of $c(k_G)$ and the canonical map $S^0 \to \overline{EG}$. 

2
3 \(ER(n)\)

First we describe the \(\mathbb{Z}/2\)-spectrum \(M\mathbb{R}\) that represents Real complex cobordism. Let \(MU(n)\) denote the Thom space of the universal bundle \(\gamma_n\) over \(BU(n)\). Complex conjugation induces an action of \(\mathbb{Z}/2\) on \(MU(n)\). The space \(MU(n)\) (with conjugation action) is placed in dimension \(n(1+\alpha)\) where 1 and \(\alpha\) denote the trivial and sign representations of \(\mathbb{Z}/2\), respectively. The canonical Real bundle \(\gamma_n\) of dimension \(n\) over \(BU(n)\) gives maps between Thom spaces,

\[
\Sigma^{1+\alpha}BU(n)^\gamma_n \rightarrow BU(n+1)^{\gamma_{n+1}}
\]

which in gives our required structure maps. Spectrification makes \(M\mathbb{R}\) a \(\mathbb{Z}/2\)-spectrum. Recall that \(RO(\mathbb{Z}/2) = \mathbb{Z} \oplus \mathbb{Z}\alpha\) where \(\alpha\) represents the sign representation. For \(V\) in \(RO(\mathbb{Z}/2)\) define \(M\mathbb{R}(V)\) as \(\colim \Omega^{n(1+\alpha)-V}MU(n)\). \(M\mathbb{R}\) is a multiplicative real-oriented spectrum in the sense of \([6]\). We shall write the coefficient ring \(M\mathbb{R}_*\), the \(\ast\) denoting the \(RO(\mathbb{Z}/2)\)-grading.

\(ER(n)\) is the real-oriented spectrum analogue of \(E(n)\) as defined by \([6]\).

3.1 The Borel Spectral Sequence for \(ER(n)\)

In order to compute \(c(ER(n))_*\), one uses the Borel Spectral sequence,

\[
E_2 = H^*(\mathbb{Z}/2, E(n)_*[\sigma, \sigma^{-1}]) \Rightarrow F(\mathbb{E}Z/2_+, ER(n))_*,
\]

where

\[
F(\mathbb{E}Z/2_+, ER(n))_* = E(n)_*[\sigma^{\pm 1}].
\]

The Tate spectral sequence is

\[
\hat{H}^*(\mathbb{Z}/2, E(n)_*[\sigma^{\pm 1}]) \Rightarrow t(ER(n))_*,
\]

where \(\hat{H}^*\) denotes the Tate cohomology.

From an anlogue of \([Theorem\ 2.2b\ [6]]\) we have

\[
ER(n)_{*((1+\alpha)} \cong E(n)_* \cong \mathbb{Z}(2)[v_0, v_1, \ldots, v_n^{\pm 1}]
\]

where \(v_0 = 2\), and degree of \(v_k\) is \((2^k - 1)(1 + \alpha)\). The element \(a\) is given by the cofiber sequence

\[
\begin{array}{ccc}
\mathbb{Z}/2_+ & \xrightarrow{a} & S^0
\end{array}
\]

Consider the Tate spectral sequence. The \(E_1\)-term is

\[
E(n)_*[a^{\pm 1}, \sigma^{\pm 1}].
\]

In order get to the \(E_2\)-term we have to know the action of \(\mathbb{Z}/2\) on \(E(n)_*[\sigma^{\pm 1}]\). We shall write \(v_R^C\) for the generators of \(E(n)_*\). For a sequence non-negative integers \(R = (r_0, r_1, \ldots)\), with \(r_i = 0\) for \(i > n\), we write the monomial

\[
v_R^C = \prod_{i \geq 0} (v_R^C)^{r_i}.
\]

The degree of \(v_R^C\) is \(\Sigma_{i \geq 0} 2r_i(2^i - 1)\). The generator of \(\mathbb{Z}/2\) acts on \(v_k\) by 1, and on \(\sigma\) by \(-1\). We also have

\[
v_R^C = v_k^C \sigma^{-2^{k+1}}.
\]

Therefore \(\mathbb{Z}/2\) acts on \(v_R^C\sigma^l\) by \((-1)^{\frac{2^{k+1}}{a}}\). For the Tate spectral sequence this gives

\[
E_2 = E(n)_*[\sigma^{\pm 2}, a^{\pm 1}]/(2a)
\]

and for the Borel SS this gives

\[
E_2 = E(n)_*[\sigma^{\pm 2}, a]/(2a),
\]

where \(E(n)_* = \mathbb{Z}(2)[v_1, v_2, \ldots, v_n^{\pm 1}]\). Every element has an \(RO(\mathbb{Z}/2)\)-degree \(k + l\alpha\). We call \(k + l\) it’s total degree.
Proposition 3.1. \[ \text{The } E_\infty \text{-term of the Borel spectral sequence for } E\mathbb{R}(n) \text{ is given by} \]

\[
\mathbb{Z}_2\langle v_k \sigma^{j_2^{k+1}}, a, v_n, \sigma^{\pm 2^{n+1}} \rangle/I, n > k \geq 0, \text{le } \mathbb{Z}
\]

where \( I \) is the ideal generated by the relations:

\[
v_0 = 2, \\
v_k \sigma^{j_2^{k+1}} = 0, \\
v_m \sigma^{j_2^{m+1}} v_k \sigma^{\pm 2^{m-k}k+1} = v_k v_m \sigma^{(l+s)2^{m+1}} m \geq k.
\]

The bidegrees of the generators are given by

\[
|a| = -\alpha, |v_n| = (2^k - 1)(1 + \alpha), |\sigma| = (\alpha - 1).
\]

Define the element \( y(n) = v_n^{2^\lambda - 1} \sigma^{2^{n+1 - 2^{n+1 - 1}} - 1} \) in degree \( \lambda(n) - \alpha \) where \( \lambda(n) = 2^{2n+1} - 2^{n+2} + 1 \).

The element \( y(n) \) is clearly invertible in the \( RO(\mathbb{Z}/2) \)-graded ring \( c(E\mathbb{R}(n))_* \). Define \( x(n) = y(n) a \).

3.2 Strong completion and cofibrations

Assume given a representation \( V \) of \( G \) whose unit sphere \( S(V) \) is a free \( G \)-space. The union \( S(\infty V) \) of the spheres \( S(qV) \) is a model for \( EG \), the union \( D(\infty V) \) of unit discs \( D(qV) \) is \( G \)-contractible, and the quotient \( D(\infty V)/S(\infty V) \simeq S^{\infty V} \) is a model for \( EG \).

In homology and cohomology, the description of \( \tilde{E}G \) as \( S^{\infty V} \) implies the following canonical isomorphisms for a \( G \)-spectrum \( k_G \), an integer \( n \) (or more general element of \( RO(G) \)), and a \( G \)-space \( X \), where \( X \) is finite in the case of cohomology:

\[
(k_G \wedge \tilde{E}G)_n(X) = \text{colim}_q (k_G \wedge q^V)_n(X) = \text{colim}_q k_G^{q_n-qV}(X)
\]

and

\[
(k_G \wedge \tilde{E}G)^n(X) = \text{colim}_q (k_G \wedge q^V)^n(X) = \text{colim}_q k_G^{n+qV}(X)
\]

The colimits can be interpreted algebraically. Let \( V \) be any representation of \( G \), let \( e : S^0 \to S^V \) be induced by \( 0 \subset V \), and let \( a_V \in k_G^V(S^0) \) be the image of the identity element of \( k_G^0(S^0) \) under the map \( e^* : k_G^0(S^0) = k_G^V(S^V) \to k_G^V(S^0) \). Therefore given a ring spectrum \( k_G \) and any \( G \)-representation \( V \), \( (k_G \wedge S^{\infty V})_*(X) \) and, if \( X \) is finite, \( (k_G \wedge S^{\infty V})^*(X) \), are the localizations of \( k_G^* \) and \( k_G^* \) away from \( a_V \). Thus multiplication by \( a_V \) provides a periodicity isomorphism with period \( V \) on \( (k_G \wedge S^{\infty V})_*(X) \) and \( (k_G \wedge S^{\infty V})^*(X) \).

Following Proposition 2.1 this has the following implication.

Proposition 3.2. Let \( k_G \) be a ring \( G \)-spectrum. If \( G \) acts freely on the unit sphere \( S(V) \), then \( t(k_G)_*(X) \) and, if \( X \) is finite, \( t(k_G)^*(X) \) are localizations of \( c(k_G)_*(X) \) and \( c(k_G)^*(X) \) away from \( a_V \). Therefore, multiplication by \( a_V \) provides a periodicity isomorphism with period \( V \) on \( t(k_G)_*(X) \) and \( t(k_G)^*(X) \).

In the case of \( E\mathbb{R}(n) \), \( G = \mathbb{Z}/2, V = \alpha, a_V = a \), and there is a commutative square of \( \mathbb{Z}/2 \)-equivariant ring spectra:

\[
\begin{array}{ccc}
E\mathbb{R}(n) & \xrightarrow{g} & E\mathbb{R}(n) = \widetilde{E\mathbb{Z}/2} \wedge E\mathbb{R}(n) \\
\downarrow & & \downarrow \\
c(\mathbb{R}(n)) & \xrightarrow{t} & t(\mathbb{R}(n))
\end{array}
\]

The element \( a : S^0 \subset S^\alpha \) acts

1. nilpotently on \( c(\mathbb{R}(n))_* \).
2. invertibly on $g(ER(n))_*$ and $t(ER(n))_*$.

**Theorem 3.1.** The Tate spectrum $t(ER(n))$ is trivial.

**Proof.** It is the localization of $c(ER(n))$ away from $a$, so on $t(ER(n))_*$, $a$ acts invertibly as well as nilpotently.

Since the left vertical arrow of the Tate diagram is an equivalence, we see that, up to equivalence, $t(k_G)$ can be seen as the cofiber of the composite

$$k_G \wedge EG_+ \to k_G \to F(EG_+, k_G)$$

also known as the norm map. This gives us the following result.

**Proposition 3.3.** $f(ER(n)) \simeq c(ER(n))$.

**Remark 3.1.** The equivalence of the norm map in proposition 3.6 implies an equivalence between homotopy orbits and homotopy fixed points $N: ER(n)_{h\mathbb{Z}/2} \simeq ER(n)^{h\mathbb{Z}/2}$. We hope to pursue this point further in a future paper by expressing the norm map as derived Grothendieck trace.

We will construct a cofibration of spectra connecting $E(n)$ and $ER(n)$. We have the following equivalence as part of the Tate diagram.

$$\begin{array}{c}
  f(ER(n)) \\
  \simeq \\
  \downarrow \\
  E(\mathbb{R})(n) \\
  \downarrow \\
  g(ER(n)) \\
  \simeq \\
  \downarrow \\
  c(ER(n))
\end{array}$$

This implies a splitting of $\mathbb{Z}/2$-ring spectra

$$E(\mathbb{R})(n) \simeq c(ER(n)) \vee g(ER(n)).$$

There is a fibration

$$\begin{array}{c}
  \mathbb{Z}/2_+ \\
  \rightarrow \\
  S^0 \\
  a \\
  \downarrow \\
  S^\infty \\
  \rightarrow
\end{array}$$

inducing a fibration

$$\begin{array}{c}
  F(S^\infty, g(ER(n))) \\
  \simeq \\
  \downarrow \\
  F(S^0, g(ER(n))) \\
  \simeq \\
  \downarrow \\
  F(\mathbb{Z}/2_+, g(ER(n)))
\end{array}$$

This equivalence induced by $a$ implies that $F(\mathbb{Z}/2_+, g(ER(n)))$ is trivial. We have the analogous fibration

$$\begin{array}{c}
  \Sigma^{-a}c(ER(n)) \\
  \simeq \\
  \downarrow \\
  c(ER(n)) \\
  \simeq \\
  \downarrow \\
  F(\mathbb{Z}/2_+, c(ER(n)))
\end{array}$$

On fixed points we deduce the fibration of non-equivariant spectra appearing in (2).

$$\begin{array}{c}
  \Sigma^{\lambda(n)}c(ER(n)) \\
  \simeq \\
  \downarrow \\
  x(n) \\
  \simeq \\
  \downarrow \\
  F(\mathbb{Z}/2_+, ER(n))
\end{array}$$

where $ER(n) := ER(n)^{h\mathbb{Z}/2}$, the homotopy fixed points of $ER(n)$, which is just the ordinary fixed points of the Borel spectrum $c(ER(n))$. The element $x(n)$ has degree $\lambda(n) = 2^{n+1} - 2^{n+2} + 1$.

**Remark 3.2.** A proof of (2) also appears in [7]. Their idea is to break down a $\mathbb{Z}/2$-equivariant spectrum $E$ into its unstable spaces $E_\Sigma^a$, which are indexed by $V \in RO(\mathbb{Z}/2)$ and consider ordinary spectra $\{E_{n+V}, n \in \mathbb{Z}\}$ constructed out of those individual spaces. Our proof differs from theirs in that we are working directly with the genuine equivariant spectra.
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