Environment-averaged Lévy-Lorentz gas

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Abstract

We consider an environment-averaged Lévy-Lorentz (LL) gas, yielding a non-homogeneous persistent random walk. We point out how the lack of translational invariance yields a highly non-trivial model, for which however long time transport property may be investigated by appropriate continuum limits. The original LL gas refers to a very diluted distribution of scatterers: here we extend the model to the opposite localized case, and unveil the parameters range in which subdiffusion arises.

Keywords: persistent random walk, Lévy-Lorentz gas, anomalous transport

1. Introduction

The Lorentz gas was introduced in 1905 \cite{1}, to model transport properties of electrons in metals: in its standard form it consists in a point particle moving in a periodic array of scatterers, with which it collides elastically. Translational symmetry allows to infer many properties of this extended system from the reduced dynamics in an elementary cell (see for instance \cite{2}). In particular dynamical and transport properties are deeply influenced by the shape of scatterers and the geometry of the lattice: circular obstacles lead to a Sinai billiard for the reduced dynamics, and the chaotic properties of such a system (see \cite{3}) are crucial in dealing with the extended case. The geometry of the lattice, and eventually the size of the scatterers, determine whether particles may travel for arbitrarily long times without experiencing any collision: the so called infinite horizon case, which leads, in two dimensions, to a logarithmic correction to the variance of the particle position. For a recent review of (transport) properties of different types of Lorentz gases see \cite{4}.

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Much less is known about dynamical and transport properties when the scatterers are placed randomly, breaking translational symmetry [5]. This motivates the introduction of simplified models, which however still present, as we will see, considerable complexity: a major role is played by persistent random walks in one dimension. Persistency consists in assigning to each site a transmission and a reflection coefficient in such a way that each step the walker undertakes at time \( n \) not only depends on the position at the same time, but also on where the walker was one time step earlier. It is interesting to observe that (homogeneous) persistent random walks have been introduced nearly a century ago, as a model of diffusion by discontinuous movements [6, 7]: many of the relevant results have been obtained (for different time regimes) by introducing appropriate continuum limits [8, 9, 10, 11]. Inhomogeneous persistency arises naturally when we distribute randomly scatterers in the lattice (empty sites being given a null reflection coefficient): models of this kind have been introduced in [12] as a variant of Sinai diffusion (see [13]).

In recent years a great interest emerged for the case of a diluted distribution of scatterers, whose mutual distance is characterized by a heavy tailed Lévy distribution [14, 15, 16, 18, 17]: besides theoretical interest the model (LL-gas) is tightly connected to experimentally fabricated Lévy glasses [19]. In particular in [18] we have introduced a non-homogeneous persistent random walk which can be viewed as an environment-averaged LL gas.

The subject of the present paper is to study the environment averaged model where we reverse the role of transmission and reflection w.r.t. the model introduced in [18]. The paper is organized as follows: in section (2) we provide the general setting, in the framework of persistent random walks, and the different probabilistic points of view that may be defined, then in section (3) we discuss the different continuum limits, and mention how the standard picture for homogeneous persistent walks becomes considerably more complex when translational invariance is broken, then in section (4) we give analytical estimates for the “localized” case, and compare them with numerical experiments; finally in section (5) we present our conclusions.

2. Lévy-Lorentz gas: the quenched, the annealed and the averaged

We start with a general setting: a persistent 1-d random walk with position dependent transmission and reflection coefficients. In the canonical persistent random walk scheme a particle starts moving from the position \( x = 0 \) in a random direction and then at each site it is reflected or transmitted according to a certain constant probability. In the LL gas scheme instead reflection may occur only at certain positions, where the scatterers are present. Scattering sites are placed at random positions in such a way that the relative distances are drawn from a Lévy-like probability distribution function (PDF), i.e. a distribution that decays with a heavy tail for large values of the argument:

\[
\mu(\xi) \sim \xi^{-(1+\alpha)}, \quad 0 < \alpha < 2
\] (1)
Notice that in this range the variance of the distance diverges, and in the restricted range $0 < \alpha \leq 1$ also the mean distance is infinite. The set of the positions of the scatterers through which transmittance and reflection are assigned among the lattice is called the environment, namely
\[
\Omega = \{ j \mid j\text{-th site is a scatterer} \}
\]

When defining the environment, one can follow two different procedures: in the equilibrium case (see [14]) a scatter is placed at position $x = -L$ and then random independent length intervals are generated according to (1), determining the locations of the other scatterers, until the sum of the intervals does not exceed $2L$ (eventually $L \to \infty$); in the nonequilibrium case instead a scatterer is always placed at $x = 0$, and then the intervals are generated in both directions until the positions $x = L$ and $x = -L$ are reached. In the rest of the paper we will only deal with the nonequilibrium case, since it is the closest to experimental realizations [15].

Given a realization of the environment, transmittance and reflection among the lattice are assigned according to
\[
r_j^\Omega = k \cdot \delta_j^\Omega \\
t_j^\Omega = 1 - r_j^\Omega
\]
where
\[
\delta_j^\Omega = \begin{cases} 
1 & \text{if } j \in \Omega \\
0 & \text{if } j \notin \Omega
\end{cases}
\]
and $0 < k < 1$. Notice that in the nonequilibrium case we always have $\delta_0^\Omega = 1$. Then time evolution is written in its simplest form once we introduce the quantities $R$ and $L$ in the following way:
\[
R_j^\Omega(n) = \text{Prob} (\text{The walker is at site } j \text{ after } n \text{ steps and leaves to the right}) \\
L_j^\Omega(n) = \text{Prob} (\text{The walker is at site } j \text{ after } n \text{ steps and leaves to the left})
\]

These are evolved through the following Chapman-Kolmogorov equations:
\[
\begin{align*}
R_j^\Omega(n+1) &= t_j^\Omega \cdot R_{j-1}^\Omega(n) + r_j^\Omega \cdot L_{j+1}^\Omega(n) \\
L_j^\Omega(n+1) &= t_j^\Omega \cdot L_{j+1}^\Omega(n) + r_j^\Omega \cdot R_{j-1}^\Omega(n)
\end{align*}
\]
with the initial conditions
\[
\begin{align*}
R_0^\Omega(0) &= L_0^\Omega(0) = \frac{1}{2} \\
R_j^\Omega(0) &= L_j^\Omega(0) = 0 \quad \forall j \neq 0
\end{align*}
\]
From (2) and (3) one can derive the equations and the initial conditions for the probability of the displacement of the walker $P_j^\Omega(n) = R_j^\Omega(n) + L_j^\Omega(n)$ and the probability current $M_j^\Omega(n) = R_j^\Omega(n) - L_j^\Omega(n)$:
\[
\begin{align*}
P_j^\Omega(n+1) &= R_j^\Omega(n) + L_j^\Omega(n) \\
M_j^\Omega(n+1) &= (t_j^\Omega - r_j^\Omega) \cdot (R_{j-1}^\Omega(n) - L_{j+1}^\Omega(n))
\end{align*}
\]
with
\[ P_0^Ω(0) = 1 \]
\[ M_0^Ω(0) = 0 \]

The object of study in the quenched version of the LL gas is \( P^Ω \), the PDF of the process for a typical environment \( Ω \), from which the mean square displacement (MSD) can be derived in order to characterize the transport properties of the system. In rough terms, this means that one is interested in evolving the probabilities once a typical realization of the environment has been fixed \([12, 16]\). In this case very few results have been established: the most remarkable one is the validity of the central limit theorem in the range \( α > 1 \) \([16]\), even if convergence of the moments has not been proven and it still remains an open question.

In the annealed version instead one is interested in \( \langle P^Ω \rangle \), the average over all possible environments of the respective \( P^Ω \). In this setting it is possible to derive analytically the asymptotic behaviour of the MSD, which is expected to grow linearly in time only for \( α > 3/2 \), while for smaller values of the exponent anomalous diffusion is predicted \([15]\):

\[ \langle x_t^2 \rangle \sim \begin{cases} t^{2\alpha/3-\alpha^2} & 0 < \alpha < 1 \\ t^{4-\alpha} & 1 \leq \alpha \leq 3/2 \end{cases} \] (6)

These results have been obtained with a scaling hypothesis for the probability distribution of the walker, which is decomposed into a central part and a subleading term describing the behaviour at large distances:

\[ \langle p^Ω(x, t) \rangle = \frac{1}{\ell(t)} F \left( \frac{|x|}{\ell(t)} \right) + \mathcal{H}(|x|, t) \] (7)

where the correlation length \( \ell(t) \) is determined by using estimates according to the related resistance model treated in \([20]\), which give

\[ \ell(t) \sim \begin{cases} t^{\frac{1}{1+\alpha}} & 0 < \alpha < 1 \\ t^{\frac{2}{2-\alpha}} & 1 \leq \alpha \end{cases} \] (8)

We remark that as for the quenched version, the CLT is valid in the restricted range \( α > 1 \), even though for \( 1 \leq \alpha < 3/2 \) the MSD doesn’t converge to the one of a normal distribution. Regarding the case \( 0 < \alpha < 1 \), it has been proven recently the validity of a generalized CLT for the finite-dimensional distributions of the continuous-time process, with the same scaling exponent \( 1/(1+\alpha) \) appearing in the correlation length \([17]\) characterizing the PDF.

In \([18]\) we proposed the environment-averaged version of the LL gas where we studied the process on an “averaged” environment: here \( Ω = \mathbb{Z} \), but the reflection coefficients among the lattice are taken equal to \( k\pi_j \), where \( 0 < k < 1 \) is a constant and \( \pi_j \) is the asymptotic probability of finding a scatterer at site \( j \).
on a typical environment. Depending on the value of the characteristic exponent of the distribution of distances, $\pi_j$ has the form

$$\pi_j = \begin{cases} \\ \frac{\alpha \sin(\pi \alpha)}{\zeta(\alpha)} \left( \frac{\pi}{\zeta(1+\alpha)} \right)^{j+1-\alpha} & 0 < \alpha < 1 \\ \frac{\pi \zeta(1+\alpha)}{\zeta(1+\alpha)} & 1 < \alpha \end{cases}$$

for $j \neq 0$, while $\pi_0 = 1$ since we are dealing with the nonequilibrium case. We call $P^{(\Omega)}$ the law of the process and we observe that in the regime $0 < \alpha < 1$ the model corresponds to a non-homogeneous persistent random walk with a power-law decay of the reflection coefficient. The system displays normal diffusion in the range $1 < \alpha < 2$, while for $0 < \alpha < 1$ the whole moments spectrum can be described as follows:

$$\langle |x_t|^{q} \rangle \sim t^{-\frac{1}{\alpha}}$$

which indicates that the model is weakly anomalous, meaning that there is a single scale ruling the behaviour of the whole anomalous moments spectrum [21].

In the next section we will show and analytically prove with the use of a continuum limit that, although disorder is removed, the hypothesis [7] and the form of the scaling length [8] of the annealed PDF are still valid for $P^{(\Omega)}$, and that the environment-averaged PDF is also able to capture the form [9] of the annealed MSD in the range $0 < \alpha < 1$.

3. Continuum limits

3.1. The diffusion approximation for the averaged environment

The continuum limit for a persistent random walk on a lattice has been considered in many of the classical papers, as [8, 9, 10, 11]. We call $\delta x$ the lattice spacing, $\delta t$ the time step and set $x = j \delta x$ and $t = n \delta t$. We use the short-hand notation $r^{(\Omega)} = r(x,t;\langle \Omega \rangle)$ and $l^{(\Omega)} = l(x,t;\langle \Omega \rangle)$ to denote the probability densities of being at position $x$ at time $t$ and leaving to the right or left respectively on the averaged environment, and write

$$R_j^{(\Omega)}(n) = \delta x \cdot r^{(\Omega)}$$
$$L_j^{(\Omega)}(n) = \delta x \cdot l^{(\Omega)}$$

while for the quantities $P^{(\Omega)} = R^{(\Omega)} + L^{(\Omega)}$ and $M^{(\Omega)} = R^{(\Omega)} - L^{(\Omega)}$ we set

$$P_j^{(\Omega)}(n) = \delta x \cdot p^{(\Omega)} = \delta x \cdot p(x,t;\langle \Omega \rangle)$$
$$M_j^{(\Omega)}(n) = \delta t \cdot m^{(\Omega)} = \delta t \cdot m(x,t;\langle \Omega \rangle)$$

where $p^{(\Omega)}$ and $m^{(\Omega)}$ are defined in terms of $r^{(\Omega)}$ and $l^{(\Omega)}$ as [18]:

$$p^{(\Omega)} = \frac{\delta x}{\delta t} \cdot \left( r^{(\Omega)} - l^{(\Omega)} \right)$$

$$m^{(\Omega)} = \frac{\delta x}{\delta t} \cdot \left( r^{(\Omega)} + l^{(\Omega)} \right)$$

5
Inserting (11) and (12) into the Chapman-Kolmogorov equations (4) and expanding the functions \( r^{(\Omega)} \), \( l^{(\Omega)} \), \( p^{(\Omega)} \), \( m^{(\Omega)} \) up to second order in both \( \delta x \) and \( \delta t \), one gets the following pair of coupled equations (we drop the superscript \( \langle \Omega \rangle \) for the functions):

\[
\begin{cases}
\dot{p}\delta t + \frac{1}{2}\ddot{p}\delta t^2 = -m'\delta t + \frac{1}{2}p''\delta x^2 \\
m\dot{t} + \dot{m}\delta t^2 + \frac{1}{2}\ddot{m}\delta t^3 = \left( t^{(\Omega)} - r^{(\Omega)} \right) \cdot \left( m\delta t - p'\delta x^2 + \frac{1}{2}m''\delta x^2\delta t \right)
\end{cases}
\] (14)

where \( r^{(\Omega)} \) is defined according to (9) (we take \( k = 1/2 \)):

\[
r^{(\Omega)} = r(x; \langle \Omega \rangle) = \begin{cases} 
\alpha \sin(\pi\alpha) & \text{if } x < \alpha \\
\zeta(1+\alpha) & \text{if } x > \alpha
\end{cases}
\] (15)

We can get a closed equation for \( p \) employing the diffusion approximation: we consider the limit \( \delta x, \delta t \to 0 \) but with \( \delta x^2/\delta t = D_0 \) kept constant. By dropping higher order terms we get the following set of equations:

\[
\begin{cases}
\frac{\partial p}{\partial t} = \frac{\partial m}{\partial x} + D_0 \frac{\partial^2 p}{\partial x^2} \\
m = \frac{D_0}{2} \frac{t^{(\Omega)} - r^{(\Omega)}}{r^{(\Omega)}} \frac{\partial p}{\partial x}
\end{cases}
\] (16)

Inserting the second one into the first, we finally get

\[
\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left( D_\alpha \frac{\partial p}{\partial x} \right)
\] (17)

where we set \( D_0 = 1 \) and \( D_\alpha \) is

\[
D_\alpha(x) = \frac{t^{(\Omega)}}{2r^{(\Omega)}} = \begin{cases} 
\Lambda|x|^{1-\alpha} - \frac{1}{2} & \text{if } 0 < \alpha < 1 \\
\zeta(1+\alpha) & \text{if } 1 < \alpha
\end{cases}
\] (18)

with

\[
\Lambda = \frac{\pi}{\alpha \sin(\pi\alpha)\zeta(1+\alpha)}
\] (19)

### 3.2. Solutions to the diffusion equation and evaluation of the mean square displacement

Given the form of the diffusion coefficient (18), the diffusion equation (17) can be solved exactly. In the regime \( \alpha > 1 \) - where, for the quenched LL gas, the average distance between scatterers is finite and the central limit theorem holds - we have a constant diffusion coefficient written as the ratio of transmission and reflection, in agreement with the result deduced in (10). The solution to the diffusion equation is then

\[
p(x, t; \langle \Omega \rangle) = \frac{1}{\sqrt{4\pi D_\alpha t}} e^{-\frac{x^2}{4D_\alpha t}}
\] (20)
with the moments described by
\[ \langle |x_t|^q \rangle \sim \ell(t)^q = t^{q/2} \] (21)
displaying normal transport, as anticipated.

In the regime \(0 < \alpha < 1\) instead the diffusion coefficient is space-dependent,
with a power-law of the form \(D_\alpha \sim \Lambda|x|^{1-\alpha}\). The solution of equation (17) in
this case is [22] (see also [23, 24] for further discussion):
\[ p(x, t; \langle \Omega \rangle) = \left[ 1 + \alpha \right]^{-1/\alpha} \frac{\exp \left[ -\frac{|x|^{1+\alpha}}{(1+\alpha)^2 \Lambda t} \right]}{\Gamma\left(\frac{1}{1+\alpha}\right)} \] (22)
which is of the form [22], with \(\mathcal{H}(|x|, t) = 0\) and \(\ell(t) = t^{1/(1+\alpha)}\). The behaviour
of the whole moments spectrum is ruled by a single scale, therefore we have once again
\[ \langle |x_t|^q \rangle \sim \ell(t)^q = t^{q/2} \] (23)

This shows that in both regimes the characteristic length of the PDF describing
the environment-averaged LL gas is the same as the one derived in the annealed model [8]. We will prove now that also the behaviour of the MSD [6] in the regime \(0 < \alpha < 1\) can be recovered: indeed, if we consider that for a time \(t\) the walker can cover at most a distance \(ct\), where \(c\) is the speed (see also [15]),
then an integration cut off must be imposed when evaluating the MSD, namely
\[ \langle x^2 \rangle_t = 2 \int_0^{ct} x^2 p(x, t; \langle \Omega \rangle) dx \] (24)
Once we evaluate the integral using [22] we get (setting \(c = 1\))
\[ \langle x^2 \rangle_t = K_\alpha \gamma \left( \frac{3}{1+\alpha}, \frac{t^\alpha}{(1+\alpha)^2 \Lambda} \right) t^{2/(1+\alpha)} \] (25)
where the constant \(K_\alpha\) is
\[ K_\alpha = \frac{[(1+\alpha)^2 \Lambda]^{2/(1+\alpha)}}{\Gamma(1/(1+\alpha))} \] (26)
and \(\gamma(\mu, z)\) denotes the lower incomplete gamma function:
\[ \gamma(\mu, z) = \int_0^z t^{\mu-1} e^{-t} dt \] (27)
Using the first term of the asymptotic expansion for large \(z\) of \(\gamma(\mu, z)\) - see [25] - we get
\[ \langle x^2 \rangle_t \sim C_\alpha t^{2/(1+\alpha)} - B_\alpha t^{2+2\alpha-\alpha^2/(1+\alpha)} \exp \left[ -\frac{t^\alpha}{(1+\alpha)^2 \Lambda} \right] \] (28)
with

\[ C_\alpha = \frac{\Gamma(3/(1 + \alpha))}{\Gamma(1/(1 + \alpha))} \left[ (1 + \alpha)^2 \Lambda \right]^{\frac{3}{1 + \alpha}} \]  

\[ B_\alpha = \frac{[(1 + \alpha)^2 \Lambda]^{\alpha/(1 + \alpha)}}{\Gamma(1/(1 + \alpha))} \]  

(29)  

(30)

The first term of equation (28) is the leading contribution to the MSD we anticipated, while the second one represents a correction due to the fact that the domain of integration is cut off, keeping track of the finite speed of the particle. For long times such a contribution is negligible for the presence of the exponential factor, nevertheless the power-law is the same as the one appearing in (9). This suggests that, although the annealed and the environment-averaged settings are quite different and the evaluations of the MSD don’t agree, the resulting PDF’s share some relevant features, showing a nontrivial connection between the two models.

4. The localized case

The environment-averaged model can give nontrivial results even when the related quenched model is instead trivial. As an example, let us consider the equivalent model of a LL gas in which every empty site is filled with a perfectly reflecting barrier. The quenched version of this model - which we will call the localized model - is indeed trivial, since the particle will be confined between two barriers and there will be no diffusion at all. In the environment-averaged version instead we keep track of all the possible configurations of disorder assigning to each site of the lattice a reflection coefficient corresponding to the expected value of reflection at that site. This means that, calling \( \pi_j \) the probability of finding a scatterer at site \( j \), we have

\[ \tau_j^{(\Omega)} = \mathbb{E}(\tau_j^\Omega) = k\pi_j + k'(1 - \pi_j) \]  

(31)

where \( k \) is the reflection probability assigned to scatterers and \( k' \) is the one assigned to “empty” sites. The environment-averaged LL gas consists in putting \( k = 1/2 \) and \( k' = 0 \), and we replaced \( \pi_j \) with its asymptotic behaviour in order to work with explicit expressions; in the localized model instead \( k' = 1 \), therefore

\[ \tau_j^{(\Omega)} = 1 - \frac{1}{2}\pi_j \]  

(32)

and we can use the same continuum limit we introduced in section 3. The resulting diffusion coefficient is

\[ D_\alpha(x) = \begin{cases} 
\frac{1}{2} \frac{\Gamma(\alpha)}{\Gamma(1 + \alpha)} & 0 < \alpha < 1 \\
\frac{1}{2} \frac{\zeta(\alpha)}{\zeta(1 + \alpha)} & 1 < \alpha < 2
\end{cases} \]  

(33)

8
displaying once again two different regimes. For $1 < \alpha < 2$, when the mean distance between scatterers is finite, $D_\alpha(x)$ is constant and we have normal diffusion. In particular the MSD can be evaluated as follows:

$$\langle x_t^2 \rangle = \frac{t}{2\zeta(\alpha)/\zeta(1 + \alpha) - 1}$$

in excellent agreement with numerical simulations (figure 1).

![Figure 1: Slope of linear growth of the second moment as obtained by numerically evolving the forward Kolmogorov equations (circles) and the analytic prediction in terms of the diffusion constant (circles). Each numerical slope has been obtained by evolving the system up to time $t = 2^{15}$.](image)

When $0 < \alpha < 1$ the diffusion coefficient is $D_\alpha(x) \sim \Sigma|x|^{\alpha-1}$, with $\Sigma = 1/4\Lambda$; the solution of the diffusion equation reads

$$p(x,t; \Omega) = \frac{[\Sigma |x|^{3-\alpha}]^{-1/(3-\alpha)}}{2\Gamma(1/(3-\alpha))} \exp \left[ - \frac{|x|^{3-\alpha}}{(3-\alpha)^2 \Sigma t} \right]$$

The system displays once again weak anomalous diffusion, with the moments spectrum behaving as

$$\langle |x_t|^q \rangle \sim t^{\gamma q \beta(\alpha)}$$
where $\beta(\alpha) = 1/(3 - \alpha)$ is the scaling exponent, such that the characteristic length of the distribution is

$$\ell(t) = t^{\beta(\alpha)}$$

as displayed by numerical simulations (figure 2). In particular the second moment is

$$\langle x_t^2 \rangle \sim t^{2/\alpha}$$

so that transport is subdiffusive.

Figure 2: Asymptotic growth exponents of qth order moments $\langle |x_t|^q \rangle \sim t^{q \beta(\alpha)}$, for $q = 0.5$ (blue crosses), $q = 2$ (squares) and $q = 4$ (green circles). The behaviour for all values of $q$ is in agreement with the analytical prediction for the scaling exponent. Data are obtained evolving the system up to time $t = 2^{18}$.

5. Conclusions

We have presented the environment-averaged version for the study of the Lvy-Lorentz gas, consisting in a non-homogeneous persistent random walk on a lattice where the effects of all the possible configurations of disorder are considered locally, in such a way that at each site the reflection coefficient is equal
to the expected value of reflection at that position on a typical structure. We have shown that this model is nontrivially connected to the annealed version of the LL gas, through the scaling exponent of the correlation length and the power-law of the MSD in the range $0 < \alpha < 1$.

Moreover we have presented and analysed the localized model, which is trivial in its quenched version, since the motion is confined and there is not diffusion. In the related environment-averaged version instead, the model displays normal diffusion for $1 < \alpha < 2$, while for $0 < \alpha < 1$ the system displays weak anomalous diffusion. In particular we have subdiffusion, with an analytically computed MSD of the form $\langle x_t^2 \rangle \sim t^{2/(3-\alpha)}$.

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