A CHARACTERIZATION RELATED TO SCHRÖDINGER EQUATIONS ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper we consider the following problem
\[
\begin{aligned}
-\Delta_g u + V(x)u &= \lambda \alpha(x) f(u), &\text{in } M \\
u &\geq 0, &\text{in } M \\
u &\to 0, &\text{as } d(x_0, x) \to \infty
\end{aligned}
\tag{P_\lambda}
\]
where \((M, g)\) is a \(N\)-dimensional \((N \geq 3)\), non-compact Riemannian manifold with asymptotically non-negative Ricci curvature, \(\lambda\) is a real parameter, \(V\) is a positive coercive potential, \(\alpha\) is a bounded function and \(f\) is a suitable nonlinearity. By using variational methods we prove a characterization result for existence of solutions for \((P_\lambda)\).

1. Introduction

The existence of standing waves solutions for the nonlinear Schrödinger equation
\[
ih\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(x)\psi - f(x, |\psi|), \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+ \setminus \{0\},
\]
has been intensively studied in the last decades. The Schrödinger equation plays a central role in quantum mechanic as it predicts the future behavior of a dynamic system. Indeed, the wave function \(\psi(x, t)\) represents the quantum mechanical probability amplitude for a given unit-mass particle to have position \(x\) at time \(t\). Such equation appears in several fields of physics, from Bose–Einstein condensates and nonlinear optics, to plasma physics (see for instance [BW02, CNY08] and reference therein).

A Lyapunov-Schmidt type reduction, i.e. a separation of variables of the type \(\psi(x, t) = u(x)e^{-\frac{iE}{\hbar}t}\), leads to the following semilinear elliptic equation
\[-\Delta u + V(x)u = f(x, u), \quad \text{in } \mathbb{R}^N.\]

With the aid of variational methods, the existence and multiplicity of nontrivial solutions for such problems have been extensively studied in the literature over the last decades. For instance, the existence of positive solutions when the potential \(V\) is coercive and \(f\) satisfies standard mountain pass assumptions, are well known after the seminal paper of Rabinowitz [Rab92]. Moreover, in the class of bounded from below potentials, several attempts have been made to find general assumptions on \(V\) in order to obtain existence and multiplicity results (see for instance [BPW01, BW95, BF78, Wil96, Str77]). In such papers the nonlinearity \(f\) is required to satisfy the well-know Ambrosetti-Rabinowitz condition, thus it is superlinear at infinity. For a sublinear growth of \(f\) see also [Kri07].

Most of the aforementioned papers provide sufficient conditions on the nonlinear term \(f\) in order to prove existence/multiplicity type results. The novelty of the present paper is to establish a characterization result for stationary Schrödinger equations on unbounded domains; even more, our arguments work on not necessarily linear structures. Indeed, our results fit the research direction where the solutions of certain PDEs are influenced by the geometry of the ambient structure (see for instance [FKV15, FK16, Kri09, Kri12, LY86, Ma06] and reference thereon).
therein). Accordingly, we deal with a Riemannian setting, the results on $\mathbb{R}^N$ being a particular consequence of our general achievements.

In order to give the precise statement of our result, let us denote by $(M,g)$ a $N$-dimensional $(N \geq 3)$, complete, non-compact Riemannian manifold with asymptotically non-negative Ricci curvature with a base point $\bar{x}_0 \in M$, i.e.,

$$(C) \quad \text{Ric}_{(M,g)}(x) \geq -(N-1)H(d_g(\bar{x}_0, x)), \text{ for all } x \in M,$$

for all $H \in C^1([0, \infty))$ is a non-negative bounded function satisfying $\int_0^\infty tH(t)dt = b_0 < +\infty$,

(here and in the sequel $d_g$ is the distance function associated to the Riemannian metric $g$). For an overview on such property see [AX10, PRS08].

Let $x_0 \in M$ be a fixed point, $\alpha : M \to \mathbb{R}_+ \\setminus \{0\}$ a bounded function and $f : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function with $f(0) = 0$ such that there exist two constants $C > 0$ and $q \in (1,2^*)$ (being $2^*$ the Sobolev critical exponent) such that

$$f(\xi) \leq k(1 + \xi^{q-1}) \text{ for all } \xi \geq 0. \quad (1.1)$$

Denote by $F : \mathbb{R}_+ \to \mathbb{R}_+$ the function $F(\xi) = \int_0^\xi f(t)dt$.

We assume that $V : M \to \mathbb{R}$ is a measurable function satisfying the following conditions:

$(V_1)$ $V_0 = \text{essinf}_{x \in M} V(x) > 0$;

$(V_2)$ $\lim_{d_g(x_0,x) \to \infty} V(x) = +\infty$, for some $x_0 \in M$.

The problem we deal with is written as:

$$\begin{cases}
-\Delta_g u + V(x)u = \lambda_\alpha(x)f(u), & \text{in } M \\
u \geq 0, & \text{in } M \\
u \to 0, & \text{as } d_g(x_0, x) \to \infty.
\end{cases} \quad (P_\lambda)$$

Our result reads as follows:

**Theorem 1.1.** Let $N \geq 3$ and $(M,g)$ be a complete, non-compact $N$-dimensional Riemannian manifold satisfying the curvature condition $(C)$, and $\inf_{x \in M} \text{Vol}_g(B_x(1)) > 0$. Let also $\alpha : M \to \mathbb{R}_+ \\setminus \{0\}$ be in $L^\infty(M) \cap L^1(M)$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function with $f(0) = 0$ verifying $(1.1)$ and $V : M \to \mathbb{R}$ be a potential verifying $(V_1), (V_2)$. Assume that for some $a > 0$, the function $\xi \to \frac{F(\xi)}{\xi^2}$ is non-increasing in $(0,a]$. Then, the following conditions are equivalent:

(i) for each $b > 0$, the function $\xi \to \frac{F(\xi)}{\xi^2}$ is not constant in $(0,b]$;

(ii) for each $r > 0$, there exists an open interval $I_r \subseteq (0, +\infty)$ such that for every $\lambda \in I_r$, problem $(P_\lambda)$ has a nontrivial solution $u_\lambda \in H^1_g(M)$ satisfying

$$\int_M (|\nabla u_\lambda(x)|^2 + V(x)u_\lambda^2) \, dv_g < r.$$ 

**Remark 1.2.** (a) One can replace the assumption $\inf_{x \in M} \text{Vol}_g(B_x(1)) > 0$ with a curvature restriction, requiring that the sectional curvature is bounded from above. Indeed, using the Bishop-Gromov theorem one can easily get that $\inf_{x \in M} \text{Vol}_g(B_x(1)) > 0$.

(b) A more familiar form of Theorem 1.1 can be obtained when $\text{Ric}_{(M,g)} \geq 0$; it suffices to put $H \equiv 0$ in $(C)$.

The following potentials $V$ fulfills assumptions $(V_1)$ and $(V_2)$:

(i) Let $V(x) = d^\theta_g(x, x_0) + 1$, where $x_0 \in M$ and $\theta > 0$.

(ii) More generally, if $z : [0, +\infty) \to [0, +\infty)$ is a bijective function, with $z(0) = 0$, let $V(x) = z(d^\theta_g(x, x_0)) + c$, where $x_0 \in M$ and $c > 0$. 
The work is motivated by a result of Ricceri ([Ric15]) where a similar theorem is stated for one-dimensional Dirichlet problem; more precisely, \((i)\) from Theorem 1.1 characterizes the existence of the solutions for the following problem
\[
\begin{cases}
-u'' = \lambda \alpha(x)f(u), & \text{in } (0, 1) \\
u > 0, & \text{in } (0, 1) \\
u(1) = u(0) = 0.
\end{cases}
\]

In the above theorem it is crucial the embedding of the Sobolev space \(H^1_0((0, 1)) \subset C^0([0, 1])\).

Recently, this result has been extended by Anello to higher dimension, i.e. when the interval \((0, 1)\) is replaced by a bounded domain \(\Omega \subset \mathbb{R}^N \ (N \in \mathbb{N})\) with smooth boundary ([Ane16]).

The generalization follows by direct minimization procedures and contains a more precise information on the interval of parameters \(I\). See also [MBR15] for a similar characterization in the framework of fractal sets.

Let us note that in our setting the situation is much more delicate with respect to those treated in the papers [Ane16, Ric15]. Indeed, the Riemannian framework produces several technical difficulties that we overcome by using an appropriate variational formulation.

One of the main tools in our investigation is a recent result by Ricceri [Ric14] (see Theorem C in Section 2). The main difficulty in the implication \((i) \Rightarrow (ii)\) in Theorem 1.1, consists in proving the boundedness of the solutions. To overcome this difficulty we use the Nash-Moser iteration method adapted to the Riemannian setting.

In proving \((ii) \Rightarrow (i)\), we make use of a recent result by Poupaud [Pou05] (see Theorem D in Section 2) concerning the discretness of the spectrum of the operator \(u \mapsto -\Delta_g u + V(x)u\).

It is worth mentioning that such result was first obtained by Kondrat’ev and Shubin ([KS99]) for manifolds with bounded geometry and relies on the generalization of Molchanov’s criterion.

However, since the bounded geometry property is a strong assumption and implies the positivity of the radius of injectivity, many efforts have been made for improvement and generalizations.

Later, Shen [She03] characterized the discretness of the spectrum by using the basic length scale function and the effective potential function. For further recent studies in this topic, we invite the reader to consult the papers [CM11, CM13, BKT16].

The outline of the paper is as follows. In §2 we present a series of preparatory definitions and results which are used throughout the paper. In §3 we prove our main result.

2. Preliminaries

2.1. Elements from Riemannian geometry.

In the sequel, let \(N \geq 3\) and \((M, g)\) be an \(N\)-dimensional Riemannian manifold. Set also \(T_x M\) its tangent space at \(x \in M\), \(TM = \bigcup_{x \in M} T_x M\) the tangent bundle, and \(d_g : M \times M \to [0, +\infty)\) the distance function associated to the Riemannian metric \(g\).

Let \(B_x(\rho) = \{y \in M : d_g(x, y) < \rho\}\) be the open metric ball with center \(x\) and radius \(\rho > 0\). If \(dv_g\) is the canonical volume element on \((M, g)\), the volume of an open bounded set \(\Omega \subset M\) is \(\text{Vol}_g(\Omega) = \int_{\Omega} dv_g = H^N(\Omega)\), where \(H^N(S)\) denotes the \(N\)-dimensional Hausdorff measure of \(\Omega\) with respect to the metric \(d_g\).

The manifold \((M, g)\) has Ricci curvature bounded from below if there exists \(h \in \mathbb{R}\) such that \(\text{Ric}_{(M, g)} \geq hg\) in the sense of bilinear forms, i.e., \(\text{Ric}_{(M, g)}(X, X) \geq h|X|^2\) for every \(X \in T_x M\) and \(x \in M\), where \(\text{Ric}_{(M, g)}\) is the Ricci curvature, and \(|X|\) denotes the norm of \(X\) with respect to the metric \(g\) at the point \(x\).

The behavior of the volume of geodesic balls is given by the following theorem (see [GHL87, PRS08]):

**Theorem A.** ([PRS08, Corollary 2.17], [AX10]). Let \((M, g)\) be an \(N\)-dimensional complete Riemannian manifold. If \((M, g)\) satisfies the curvature condition \((C)\), then the following volume growth property holds true:

\[
\frac{\text{Vol}_g(B_x(R))}{\text{Vol}_g(B_x(r))} \leq e^{(N-1)h_0} \left(\frac{R}{r}\right)^N, \quad 0 < r < R,
\]

\(\blacksquare\)
The Laplace-Beltrami operator is given by
\[ ∆(u) = \sum_{ij} \frac{1}{g^{ij}} \frac{∂^2 u}{∂x_i ∂x_j} \]
where \( g^{ij} \) are the local components of \( g \) in associated coordinates \((x^i)\).

Let \( p > 1 \). The norm of \( L^p(M) \) is given by
\[ ∥u∥_{L^p(M)} = \left( \int_M |u|^p dv_g \right)^{1/p}. \]

Let \( u : M \to ℝ \) be a function of class \( C^1 \). If \((x^i)\) denotes the local coordinate system on a coordinate neighbourhood of \( x \in M \), and the local components of the differential of \( u \) are denoted by \( u_i = \frac{∂u}{∂x_i} \), then the local components of the gradient \( ∇_g u \) are \( u^i = g^{ij} u_j \). Here, \( g^{ij} \) are the local components of \( g^{-1} = (g_{ij})^{-1} \). In particular, for every \( x_0 \in M \) one has the eikonal equation
\[ |∇_g d_g(x_0, ·)| = 1 \text{ on } M \setminus \{x_0\}. \]
The Laplace-Beltrami operator is given by \( ∆_g u = \text{div}(∇_g u) \) whose expression in a local chart of associated coordinates \((x^i)\) is
\[ ∆_g u = g^{ij} \left( \frac{∂^2 u}{∂x_i ∂x_j} - \Gamma^k_{ij} \frac{∂u}{∂x_k} \right), \]
where \( \Gamma^k_{ij} \) are the coefficients of the Levi-Civita connection. The \( L^p(M) \) norm of \( ∇_g u(x) ∈ T_x M \) is given by
\[ ∥∇_g u∥_{L^p(M)} = \left( \int_M |∇_g u|^p dv_g \right)^{1/p}. \]
The space \( H^1_g(M) \) is the completion of \( C^∞_0(M) \) with respect to the norm
\[ ∥u∥_{H^1_g(M)} = √{∥u∥^2_{L^2(M)} + ∥∇_g u∥^2_{L^2(M)}}. \]

2.2. Variational tools. Let us consider the functional space\( H^1_V(M) = \left\{ u ∈ H^1_g(M) : \int_M (|∇_g u|^2 + V(x)u^2) dv_g < +∞ \right\} \)endowed with the norm
\[ ∥u∥_V = \left( \int_M |∇_g u|^2 dv_g + \int_M V(x)u^2 dv_g \right)^{1/2}. \]

It was proved by Aubin [Aub75] and independently by Cantor [Can74] that the Sobolev embedding \( H^1_g(M) → L^2^*(M) \) is continuous for complete manifolds with bounded sectional curvature and positive injectivity radius. The above result was generalized ([Heb99]) for manifolds with Ricci curvature bounded from below and positive injectivity radius. Taking into account that, if \((M, g)\) is an \( N \)-dimensional complete non-compact Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius, then \( \inf \{ Vol_g(B_x(1)) > 0 \} \), we have the following result:

**Theorem B.** [Heb99, Var89] Let \((M, g)\) be a complete, non-compact \( N \)-dimensional Riemannian manifold such that its Ricci curvature is bounded from below and \( \inf \{ Vol_g(B_x(1)) > 0 \} \). Then the embedding \( H^1_g(M) → L^p(M) \) is continuous for \( p ∈ [2, 2^*]. \)
It is clear that if \((M, g)\) is a Riemannian manifold satisfying the curvature condition \((C)\), and \(\inf_{x \in M} \text{Vol}_g(B_x(1)) > 0\) then the above theorem holds true. If \(V\) is bounded from below by a positive constant, it is clear that the embedding \(H^1_V(M) \hookrightarrow H^1(M)\) is continuous and thus, the above result is still true replacing \(H^1(V)\) with \(H^1(V)\).

In order to employ a variational approach we need the next Rabinowitz-type compactness result (see Rabinowitz [Rab92]):

**Lemma 2.1.** Let \((M, g)\) be a complete, non-compact \(N\)-dimensional Riemannian manifold satisfying the curvature condition \((C)\), and \(\inf_{x \in M} \text{Vol}_g(B_x(1)) > 0\). If \(V\) satisfies \((V_1)\) and \((V_2)\), the embedding \(H^1_V(M) \hookrightarrow L^p(M)\) is compact for all \(p \in [2, 2^*]\).

**Proof.** Let \(\{u_k\}_k \subset H^1_V(M)\) be a bounded sequence, i.e., \(\|u_k\|_V \leq \eta\) for some \(\eta > 0\). Since \(H^1_V(M) \hookrightarrow H^1_g(M)\) is continuous and \(H^1_g(M) \hookrightarrow L^2_{\text{loc}}(M)\) is compact, we can find \(u \in H^1_V(M)\) such that \(u_k \rightharpoonup u\) in \(H^1_g(M)\) and \(u_k \to u\) in \(L^2_{\text{loc}}(M)\) (up to a subsequence). Let \(\varepsilon > 0\) and choose \(q = q(\varepsilon) > 0\) big enough. By \((V_2)\), there exists \(R > 0\) such that \(V(x) \geq q\) for every \(x \in M \setminus B_R(x_0)\). Thus,

\[
\int_{M \setminus B_R(x_0)} |u_k - u|^2 \, dv_g \leq \frac{1}{q} \int_{M \setminus B_R(x_0)} V(x) |u_k - u|^2 \, dv_g \leq \frac{(\eta + \|u\|_V)^2}{q} < \frac{\varepsilon}{2}.
\]

On the other hand, for \(k\) big enough

\[
\int_{B_R(x_0)} |u_k - u|^2 \, dv_g < \frac{\varepsilon}{2},
\]

and we deduce at once that \(u_k \rightharpoonup u\) in \(L^2(M)\). Now, let \(p \in (2, 2^*)\) and \(\theta = \frac{N}{p} (1 - \frac{2}{2^*})\). It is clear that \(\theta \in (0, 1)\), then using H"older inequality one can see that for \(u \in H^1_g(M)\),

\[
\int_M |u|^p \, dv_g = \int_M |u|^\theta p \cdot |u|^{(1-\theta)p} \, dv_g \leq \left( \int_M (|u|^\theta p)^{\frac{\theta}{2^*}} \, dv_g \right)^{\frac{2^*}{\theta}} \cdot \left( \int_M (|u|^{(1-\theta)p})^{\frac{1}{1-\theta} \frac{2^*}{p}} \, dv_g \right)^{(1-\theta) \frac{p}{2^*}},
\]

or

\[
\|u\|_{L^p(M)} \leq \|u\|_{L^2(M)}^{\theta} \cdot \|u\|_{L^{2^*}(M)}^{1-\theta}.
\]

Thus,

\[
\|u_k - u\|_{L^p(M)} \leq \|u_k - u\|_{L^{2^*}(M)}^{1-\theta} \|u_k - u\|_{L^2(M)}^{\theta} \leq C \|\nabla_g (u_k - u)\|_{L^2(M)} \|u_k - u\|_{L^2(M)},
\]

being \(C > 0\) the embedding constant of \(H^1_g(M) \hookrightarrow L^{2^*}(M)\). Therefore, \(u_k \rightharpoonup u\) in \(L^p(M)\). □

To prove our main results we use the following abstract result due to Ricceri (the same exploited in [Ric15] for the study of the one-dimensional case):

**Theorem C.** [Ric14, Theorem A] Let \((X, \langle \cdot, \cdot \rangle)\) be a real Hilbert space, \(J : X \to \mathbb{R}\) a sequentially weakly upper semicontinuous and Gâteaux differentiable functional, with \(J(0) = 0\). Assume that, for some \(r > 0\), there exists a global maximum \(\hat{x}\) of the restriction of \(J\) to \(B_r = \{x \in X : \|x\|^2 \leq r\}\) such that

\[
J'(\hat{x}) < 2J(\hat{x}).
\]

Then, there exists an open interval \(I \subset (0, +\infty)\) such that, for each \(\lambda \in I\), the equation \(x = \lambda J'(x)\) has a non-zero solution with norm less than \(r\).

As it was already pointed out in [Ric15], the following remark adds some crucial information about the interval \(I\):

**Remark 2.2.** Set \(\beta_r = \sup_{B_r} J\), \(\delta_r = \sup_{x \in B_r \setminus \{0\}} \frac{J(x)}{\|x\|^2}\) and \(\eta(s) = \sup_{y \in B_r} \frac{r - \|y\|^2}{s - J(y)}\), for all \(s \in (\beta_r, +\infty)\). Then, \(\eta\) is convex and decreasing in \(]\beta_r, +\infty[\). Moreover, \(I = \frac{1}{2} \eta(\beta_r, r \delta_r)\).
2.3. On the spectrum of $-\Delta_g + V(x)$. In this subsection we recall a key tool on the discreteness of the spectrum of the operator $u \mapsto -\Delta_g u + V(x)u$ which we state in a convenient form for our purposes:

**Theorem D.** [Pou05, Corollary 0.1] Let $(M, g)$ be a complete, non-compact $N$-dimensional Riemannian manifold. Let $V : M \to \mathbb{R}$ be a potential verifying $(V_1), (V_2)$. Assume the following on the manifold $M$:

(A1) there exists $r_0 > 0$ and $C_1 > 0$ such that for any $0 < r \leq \frac{r_0}{2}$, one has $\text{Vol}_g(B_x(2r)) \leq C_1 \text{Vol}_g(B_x(r))$ (doubling property);

(A2) there exists $q > 2$ and $C_2 > 0$ such that for all balls $B_x(r)$, with $r \leq \frac{r_0}{2}$ and for all $u \in H^1_0(B_x(r))$

$$\left( \int_{B_x(r)} |u - u_{B_x(r)}|^2 \text{d}v_g \right)^{\frac{1}{2}} \leq C_2 r \text{Vol}_g(B_x(r))^{\frac{q-2}{q}} \left( \int_{B_x(r)} |\nabla_g u|^2 \text{d}v_g \right)^{\frac{1}{2}},$$

where $u_{B_x(r)} = \frac{1}{\text{Vol}_g(B_x(r))} \int_{B_x(r)} u \text{d}v_g$ (Sobolev-Poincaré inequality).

Then the spectrum of the operator $-\Delta_g + V(x)$ is discrete.

It is clear that in our setting condition (A1) holds (see Theorem A). It was proved by Maheux and Saloff-Coste (see for instance [MSC95, HK95]) that the Sobolev-Poincaré inequality is true for complete non-compact Riemannian manifolds with Ricci curvature bounded from below, thus Theorem D is valid for Riemannian manifolds satisfying the curvature condition (C).

3. Proof of the main result

The energy functional associated to problem $(\mathcal{P}_\lambda)$ is the functional $\mathcal{E} : H^1_V \to \mathbb{R}$ defined by

$$\mathcal{E}(u) = \frac{1}{2} \|u\|^2_V - \lambda \int_M \alpha(x) F(u) \text{d}v_g,$$

which is of class $C^1$ in $H^1_V$ with derivative, at any $u \in H^1_V$, given by

$$\mathcal{E}'(u)(v) = \int_M (\nabla_g u \nabla_g v + V(x)uv) \text{d}v_g - \lambda \int_M \alpha(x) f(u)v \text{d}v_g, \text{ for all } v \in H^1_V.$$

Weak solutions of problem $(\mathcal{P}_\lambda)$ are precisely critical points of $\mathcal{E}$.

Because of the sign of $f$, it is clear that critical points of $\mathcal{E}$ are non negative functions. More properties of critical points of $\mathcal{E}$ can be deduced by the following regularity theorem which is crucial in the proof of the Theorem 1.1. We adapt to our setting the classical Nash Moser iteration techniques.

**Theorem 3.1.** Let $N \geq 3$ and $(M, g)$ be a complete, non-compact $N$-dimensional Riemannian manifold satisfying the curvature condition (C), and $\inf_{x \in M} \text{Vol}_g(B_x(1)) > 0$. Let also $\varphi : M \times \mathbb{R}_+ \to \mathbb{R}$ be a continuous function with primitive $\Phi(x, t) = \int_0^t \varphi(x, \xi) d\xi$ such that, for some constants $k > 0$ and $q \in (2, 2^*)$ one has

$$|\varphi(x, \xi)| \leq k(\xi + \xi^{q-1}), \text{ for all } \xi \geq 0, \text{ uniformly in } x \in M.$$

Let $u \in H^1_V(M)$ be a non negative critical point of the functional $\mathcal{G} : H_V \to \mathbb{R}$

$$\mathcal{G}(u) = \frac{1}{2} \|u\|^2_V - \int_{\mathbb{R}^N} \Phi(x, u) \text{d}v_g.$$

and $x_0 \in M$. Then,

(i) for every $\rho > 0$, $u \in L^\infty(B_{x_0}(\rho))$;

(ii) $u \in L^\infty(M)$ and $\lim_{d_g(x_0, x) \to \infty} u(x) = 0$. 



Proof. Let \( u \) be a critical point of \( G \). Then,
\[
\int_M (\nabla_g u \nabla_g v + V(x) uv) \, dv_g = \int_M \varphi(x,u) \, dv_g \quad \text{for all } v \in H_V. \tag{3.1}
\]
For each \( L > 0 \), define
\[
 u_L(x) = \begin{cases} 
 u(x) & \text{if } u(x) \leq L, \\
 L & \text{if } u(x) > L. 
\end{cases}
\]
Let also \( \tau \in C^\infty(M) \) with \( 0 \leq \tau \leq 1 \).

For \( \beta > 1 \), set \( v_L = \tau^2 u_L \beta^{-1} \) and \( w_L = \tau u_L \beta^{-1} \) which are in \( H_V^1(M) \).
Thus, plugging \( v_L \) into (3.1), we get
\[
\int_M (\nabla_g u \nabla_g v_L + V(x) uv_L) \, dv_g = \int_M \varphi(x,u) \, dv_g, \tag{3.2}
\]
A direct calculation yields that
\[
\nabla_g v_L = 2 \tau \ u_L \beta^{-1} \nabla_g \tau + \tau^2 \ u_L \beta^{-1} \nabla_g u + 2(\beta - 1) \tau \ u_L \beta^{-3} \nabla_g u_L,
\]
and
\[
\int_M \nabla_g u \nabla_g v_L \, dv_g = \int_M \left[ 2 \tau \ u_L \beta^{-1} \nabla_g u \nabla_g \tau + \tau^2 \ u_L \beta^{-1} \nabla_g u \right] ^2 \, dv_g \\
+ \int_M 2(\beta - 1) \tau \ u_L \beta^{-3} \nabla_g u \nabla_g u_L \, dv_g \tag{3.3}
\]
\[
geq \int_M \left[ 2 \tau \ u_L \beta^{-1} \nabla_g u \nabla_g \tau + \tau^2 \ u_L \beta^{-1} \nabla_g u \right] ^2 \, dv_g,
\]
since
\[
2(\beta - 1) \int_M \tau \ u_L \beta^{-3} \nabla_g u \nabla_g u_L \, dv_g = \int_{\{u \leq L\}} \tau \ u_L \beta^{-1} \nabla_g u \, dv_g \geq 0.
\]
Notice that
\[
|\nabla_g u_L|^2 = u^2 \ u_L \beta^{-1} |\nabla_g \tau|^2 + \tau^2 \ u_L \beta^{-1} |\nabla_g u|^2 + (\beta - 1)^2 \tau^2 \ u_L \beta^{-2} |\nabla_g u_L|^2 \\
+ 2 \tau \ u_L \beta^{-1} \nabla_g \tau \nabla_g u + 2(\beta - 1) \tau \ u^2 \ u_L \beta^{-3} \nabla_g \tau \nabla_g u_L + 2(\beta - 1) \tau \ u_L \beta^{-3} \nabla_g u \nabla_g u_L.
\]
Then, one can observe that
\[
\int_M \tau \ u^2 \ u_L \beta^{-2} |\nabla_g u_L|^2 \, dv_g = \int_{\{u \leq L\}} \tau \ u_L \beta^{-1} |\nabla_g u|^2 \, dv_g \leq \int_M \tau \ u_L \beta^{-1} |\nabla_g u|^2 \, dv_g,
\]
and
\[
\int_M \tau \ u^2 \ u_L \beta^{-3} \nabla_g u \nabla_g u_L \, dv_g = \int_{\{u \leq L\}} \tau \ u_L \beta^{-1} |\nabla_g u|^2 \, dv_g \leq \int_M \tau \ u_L \beta^{-1} |\nabla_g u|^2 \, dv_g,
\]
and also that
\[
2 \int_M \tau \ u^2 \ u_L \beta^{-3} \nabla_g \tau \nabla_g u_L \, dv_g \leq 2 \int_M \tau \ u^2 \ u_L \beta^{-3} |\nabla_g \tau| \cdot |\nabla_g u_L| \, dv_g \\
= 2 \int_M (\tau \ u L \beta^{-2} |\nabla_g u_L|) \cdot (u \ u L \beta^{-1} |\nabla_g \tau|) \, dv_g \\
\leq \int_M \tau \ u^2 \ u_L \beta^{-2} |\nabla_g u_L|^2 \, dv_g + \int_M u^2 \ u_L \beta^{-1} |\nabla_g \tau|^2 \, dv_g \\
\leq \int_M \tau \ u^2 \ u_L \beta^{-1} |\nabla_g u|^2 \, dv_g + \int_M u^2 \ u_L \beta^{-1} |\nabla_g \tau|^2 \, dv_g,
\]
Therefore
\[
\int_M |\nabla_g w_L|^2 dv_g \leq \int_M u^2 u_L^{2(\beta-1)} |\nabla_g \tau|^2 dv_g + \beta^2 \int_M \tau u^2 u_L^{2(\beta-1)} |\nabla_g u|^2 dv_g + 2 \int_M \tau u^2 u_L^{2(\beta-1)} |\nabla_g u|^2 dv_g + (2(\beta + 1) - 1) \int_M \tau u^2 u_L^{2(\beta-1)} |\nabla_g u|^2 dv_g + 2 \int_M \tau u^2 u_L^{2(\beta-1)} |\nabla_g \tau|^2 dv_g.
\]

(3.4)

In the sequel we will need the constant \( \gamma = \frac{2-2^s}{2-g+2} \). It is clear that \( 2 < \gamma < 2^* \).

Proof of \( i \). Putting together (3.3), (3.4), with (3.2), recalling that \( \beta > 1 \), and bearing in mind the growth of the function \( \varphi \), we obtain that
\[
\|w_L\|^2_{L^\gamma} = \int_M (|\nabla_g w_L|^2 + V(x)w_L^2) dv_g
\]
\[
\leq \beta \int_M u^2 u_L^{2(\beta-1)} |\nabla_g \tau|^2 dv_g + 2 \beta^2 \int_M \left( |\nabla_g u|^2 + V(x)\tau u^2 u_L^{2(\beta-1)} \right) dv_g
\]
\[
= \beta \int_M u^2 u_L^{2(\beta-1)} |\nabla_g \tau|^2 dv_g + 2 \beta^2 \int_M \left( |\nabla_g u|^2 + V(x)uvL \right) dv_g
\]
\[
= \beta \int_M u^2 u_L^{2(\beta-1)} |\nabla_g \tau|^2 dv_g + 2 \beta^2 \int_M \varphi(x,u)uL dv_g
\]
\[
\leq \beta \int_M u^2 u_L^{2(\beta-1)} |\nabla_g \tau|^2 dv_g + 2 \beta^2 k \int_M \left( \tau u^2 u_L^{2(\beta-1)} + u^2 u_L^{2(\beta-1)} \right) dv_g
\]
\[
= \beta \int_M u^2 u_L^{2(\beta-1)} |\nabla_g \tau|^2 dv_g + 2 \beta^2 k \int_M w_L^2 dv_g + 2 \beta^2 k \int_M u^2 u_L^{2(\beta-1)} dv_g.
\]

Let \( R, r > 0 \). In the proof of case \( i \), \( \tau \) verifies the further following properties: \(|\nabla \tau| \leq \frac{2}{\tau} \) and
\[
\tau(x) = \begin{cases} 
1 & \text{if } d_g(x_0, x) \leq R, \\
0 & \text{if } d_g(x_0, x) > R + r.
\end{cases}
\]

Then, applying Hölder inequality yields that
\[
I_1 \leq \frac{4}{r^2} \int_{R \leq d_g(x_0, x) \leq R + r} u^2 u_L^{2(\beta-1)} dv_g
\]
\[
\leq \frac{4}{r^2} (\text{Vol}_g (A[R, R + r]))^{1-\frac{2}{7}} \left( \int_{A[R, R + r]} u^\gamma u_L^{(\beta-1)} dv_g \right)^{\frac{2}{7}},
\]
where \( A[R, R + r] = \{ x \in M : R \leq d_g(x_0, x) \leq R + r \} \). Then, from Theorem A, we have that
\[
I_1 \leq 4\omega_N^{1-\frac{2}{7}} e^{(N-1)\beta_0 (1-\frac{2}{7})} \left( \frac{R + r}{r^2} \right)^{N(1-\frac{2}{7})} \left( \int_{d_g(x_0, x) \leq R + r} u^\gamma u_L^{(\beta-1)} dv_g \right)^{\frac{2}{7}}.
\]

In a similar way, we obtain that
\[
I_2 \leq \int_{d_g(x_0, x) \leq R + r} u^2 u_L^{2(\beta-1)} dv_g
\]
\[
\leq \omega_N^{1-\frac{2}{7}} e^{(N-1)\beta_0 (1-\frac{2}{7})} (R + r)^{N(1-\frac{2}{7})} \left( \int_{d_g(x_0, x) \leq R + r} u^\gamma u_L^{(\beta-1)} dv_g \right)^{\frac{2}{7}},
\]
and also that
\[
I_3 = \int_M u^{q-2} w_L^2 d\nu_g \leq \left( \int_M u^2 d\nu_g \right)^{\frac{q-2}{q}} \left( \int_M w_L^2 d\nu_g \right)^{\frac{2}{q}}.
\]

In the sequel we will use the notation \( \mathcal{J} = \left( \int_{d_g(x_0, x) \leq R+r} u^{\gamma(L-1)} d\nu_g \right)^{\frac{2}{\gamma}} \). Therefore, summing up the above computations, we obtain that
\[
\|w_L\|_V^2 \leq 4\beta \omega_N^{1-\frac{2}{r}} e^{(N-1)\rho_0(1-\frac{2}{r})} \frac{(R+r)^{N(1-\frac{2}{r})}}{r^2} \mathcal{J} + 2\beta^2 \kappa \|u\|_{L^2(M)}^{q-2} \mathcal{J}
\]
Moreover, if \( C_* \) denotes the embedding constant of \( H^1_0(M) \), one has into \( L^{2^*}(M) \),
\[
\|w_L\|_V^2 \geq C_* \|w_L\|_{L^{2^*}(M)}^2 \geq C_* \left( \int_M (\tau u u^{\beta-1})^2 d\nu_g \right)^{\frac{2}{\gamma}} \geq C_* \left( \int_{d_g(x_0, x) \leq R} (u u^{\beta-1})^2 d\nu_g \right)^{\frac{2}{\gamma}}.
\]
Combining the above computations with (3.5), and bearing in mind that \( \beta > 1 \), we get
\[
\left( \int_{d_g(x_0, x) \leq R} (u u^{\beta-1})^2 d\nu_g \right)^{\frac{2}{\gamma}} \leq 4C_*^{-1} \beta^2 \omega_N^{1-\frac{2}{r}} e^{(N-1)\rho_0(1-\frac{2}{r})} \frac{(R+r)^{N(1-\frac{2}{r})}}{r^2} \mathcal{J} + 2kC_*^{-1} \beta^2 \|u\|_{L^2(M)}^{q-2} \mathcal{J}
\]
Taking the limit as \( L \to +\infty \) in (3.6), we obtain
\[
\left( \int_{d_g(x_0, x) \leq R} u^{2\gamma} d\nu_g \right)^{\frac{2}{\gamma}} \leq 4C_*^{-1} \beta^2 \omega_N^{1-\frac{2}{r}} e^{(N-1)\rho_0(1-\frac{2}{r})} \frac{(R+r)^{N(1-\frac{2}{r})}}{r^2} \left( \int_{d_g(x_0, x) \leq R+r} u^{\gamma\beta} \right)^{\frac{2}{\gamma}} + 2kC_*^{-1} \beta^2 \omega_N^{1-\frac{2}{r}} e^{(N-1)\rho_0(1-\frac{2}{r})} \frac{(R+r)^{N(1-\frac{2}{r})}}{r^2} \left( \int_{d_g(x_0, x) \leq R+r} u^{\gamma\beta} \right)^{\frac{2}{\gamma}} + 2C_*^{-1} \beta^2 k \|u\|_{L^2(M)}^{q-2} \left( \int_{d_g(x_0, x) \leq R+r} u^{\gamma\beta} \right)^{\frac{2}{\gamma}}.
\]
Thus, for every \( R > 0, \ r > 0, \ \beta > 1 \) one has
\[
\|u\|_{L^{2^*}(d_g(x_0, x) \leq R)} \leq (C_*^{-1})^{\frac{1}{2}} \beta^\frac{1}{\gamma} \left( C_1 \frac{(R+r)^{N(1-\frac{2}{r})}}{r^2} + C_2 (R+r)^{N(1-\frac{2}{r})} + C_3 \right)^{\frac{1}{2}} \|u\|_{L^{2^*}(d_g(x_0, x) \leq R+r)},
\]
where \( C_1 = 4\omega_N^{1-\frac{2}{r}} e^{(N-1)\rho_0(1-\frac{2}{r})}, \ C_2 = 2k\omega_N^{1-\frac{2}{r}} e^{(N-1)\rho_0(1-\frac{2}{r})}, \ C_3 = 2k \|u\|_{L^2(M)}^{q-2} \).

Fix \( \rho > 0 \). We are going to apply (3.7) choosing first \( \beta = \frac{2}{\gamma}, \ R = \rho + \frac{\rho}{2}, \ r = \frac{\rho}{2} \), to get
\[
\|u\|_{L^{2^*}(d_g(x_0, x) \leq \rho+\frac{\rho}{2})} \leq (C_*^{-1})^{\frac{1}{2}} \beta^\frac{1}{\gamma} \left( C_1 2^{N(1-\frac{2}{r})} \rho^{N(1-\frac{2}{r})-2} + C_2 (2\rho)^{N(1-\frac{2}{r})} + C_3 \right)^{\frac{1}{2}} \|u\|_{L^{2^*}(d_g(x_0, x) \leq 2\rho)}
\]
Noticing that \( \gamma^2 = 2^* \beta \), we can apply (3.7) with \( \beta^2 \) in place of \( \beta \) and \( R = \rho + \frac{\rho}{2}, \ r = \frac{\rho}{2} \). We obtain
\[ \| u \|_{L^2, \beta^2(d_g(x_0,x) \leq \rho + \frac{\beta}{2})} \leq \left( C_*^{\beta^{-1}} \right)^{\frac{1}{2\beta}} \rho^{\frac{1}{2\beta}} e^{\frac{1}{\beta^2} \log \left( C_1 2^{N(1-\frac{2}{\beta})} \rho^{N(1-\frac{2}{\beta}) - 2} + C_2 (2\rho)^{N(1-\frac{2}{\beta})} + C_3 \right)} . \]

Iterating this procedure, for every integer \( n \) we obtain
\[
\| u \|_{L^2, \beta^n(d_g(x_0,x) \leq \rho)} \leq \| u \|_{L^2, \beta^n(d_g(x_0,x) \leq \rho + \frac{\beta}{2})} \leq \left( C_*^{\beta^{-1}} \right)^{\sum_{i=1}^n \frac{1}{2\beta^i} \rho^{\sum_{i=1}^n \frac{1}{2\beta^i}}} e^{\sum_{i=1}^n \frac{1}{2\beta^i} \log \left( C_1 2^{N(1-\frac{2}{\beta})} \rho^{N(1-\frac{2}{\beta}) - 2} + C_2 (2\rho)^{N(1-\frac{2}{\beta})} + C_3 \right)} \| u \|_{L^2, (d_g(x_0,x) \leq 2\rho)}.\]

If
\[ \sigma = \frac{1}{2} \sum_{n=1}^\infty \frac{1}{\beta^n} = \frac{1}{2(\beta - 1)}, \quad \vartheta = \sum_{n=1}^\infty \frac{n}{\beta^n}, \quad \eta = \sum_{n=1}^\infty \log \left( C_1 2^{N(1-\frac{2}{\beta})} \rho^{N(1-\frac{2}{\beta}) - 2} + C_2 (2\rho)^{N(1-\frac{2}{\beta})} + C_3 \right) \]

Passing to the limit as \( n \to \infty \), we obtain
\[ \| u \|_{L^2, \beta^n(d_g(x_0,x) \leq \rho)} \leq \left( C_*^{\beta^{-1}} \right)^{\sigma \beta^\vartheta \eta} \| u \|_{L^2, (d_g(x_0,x) \leq 2\rho)}.\]

Since \( u \in L^2(M) \), claim (i) follows at once. Notice that \( \eta \) depends on \( \rho \).

**Proof of (ii).** Since \( V \) is coercive, we can find \( \bar{R} > 0 \) such that
\[ V(x) \geq 2k \quad \text{for} \quad d_g(x_0,x) \geq \bar{R} \]
(where \( k \) is from the growth of \( \varphi \)). Without loss of generality we can assume that \( k \geq 1 \).

Let \( R = \max\{\bar{R}, 1\}, 0 < r \leq \frac{R}{2} \). In the proof of case (ii), \( \tau \) verifies the following properties: \( |\nabla \tau| \leq \frac{2}{r} \) and \( \tau \) is such that
\[ \tau(x) = \begin{cases} 0 & \text{if} \ d_g(x_0,x) \leq R, \\ 1 & \text{if} \ d_g(x_0,x) > R + r. \end{cases} \]

From (3.2), we get
\[
\int_M (\nabla_g u \nabla_g v_L + 2kuv_L) dv_g = \int_{d_g(x_0,x) \geq R} (\nabla_g u \nabla_g v_L + 2kuv_L) dv_g \\
\leq \int_{d_g(x_0,x) \geq R} (\nabla_g u \nabla_g v_L + V(x) uv_L) dv_g \\
= \int_M (\nabla_g u \nabla_g v_L + V(x) uv_L) dv_g = \int_M \varphi(x,u) v_L dv_g \\
\leq k \int_M (uv_L + u^{r-1} v_L) dv_g, 
\]
thus
\[
\int_M (\nabla_g u \nabla_g v_L + uv_L) dv_g \leq \int_M (\nabla_g u \nabla_g v_L + ku v_L) dv_g \leq k \int_M u^{r-1} v_L dv_g. 
\]

From (3.3) and (3.4), and since \( w_L^2 = u \cdot v_L \),
\[
\int_M (\nabla_g w_L)^2 + w_L^2) dv_g \leq \beta \int_M u^2 u^{2(\beta^{-1})} \nabla_g \tau^2 dv_g + 2\beta^2 \int_M \nabla_g u \nabla_g v_L dv_g + \int_M uv_L dv_g \\
\leq \beta \int_M u^2 u^{2(\beta^{-1})} \nabla_g \tau^2 dv_g + 2\beta^2 \int_M (\nabla_g u \nabla_g v_L + uv_L) dv_g \\
\leq \beta \int_M u^2 u^{2(\beta^{-1})} \nabla_g \tau^2 dv_g + 2\beta^2 k \int_M u^{r-1} v_L dv_g. 
\]
Thus,
\[
\|w_L\|_{H^1_\gamma(M)}^2 \leq \beta \int_M u^2 \left( u_L^{2(\beta-1)} |\nabla g|^2 \right) \, dv_g + 2 \beta^2 k \int_M u^{q-2} u_L^2 \, dv_g.
\]
As in the proof of i) one has
\[
I_1 \leq 4 \omega_N \frac{1}{2} e^{(N-1)b_0(1-\frac{\gamma}{2})} \frac{(R+r)^{N(1-\frac{\gamma}{2})}}{r^2} \left( \int_{d_g(x_0,x) \geq R} u^\gamma u_L^{(\beta-1)} \, dv_g \right)^{\frac{2}{\gamma}},
\]
and
\[
I_2 \leq \|u\|_{L^2(M)}^{q-2} \left( \int_{d_g(x_0,x) \geq R} u^\gamma u_L^{(\beta-1)} \, dv_g \right)^{\frac{2}{\gamma}}.
\]
Since,
\[
\|w_L\|_{H^1_\gamma(M)}^2 \geq C^* \|w_L\|_{L^2(M)}^2 = C^* \left( \int_M (\tau u u_L^{(\beta-1)} \, dv_g \right)^{\frac{2}{\gamma}} \geq C^* \left( \int_{d_g(x_0,x) \geq R+r} (u u_L^{(\beta-1)} \, dv_g \right)^{\frac{2}{\gamma}},
\]
where \( C^* \) denotes the embedding constant of \( H^1_\gamma(M) \) into \( L^{2\gamma} (M) \), we obtain
\[
\left( \int_{d_g(x_0,x) \geq R+r} (u u_L^{(\beta-1)} \, dv_g \right)^{\frac{2}{\gamma}} \leq 4(C^*)^{-1} \beta^2 \omega_N^{1-\frac{\gamma}{2}} e^{(N-1)b_0(1-\frac{\gamma}{2})} \frac{(R+r)^{N(1-\frac{\gamma}{2})}}{r^2} \left( \int_{d_g(x_0,x) \geq R} u^\gamma u_L^{(\beta-1)} \, dv_g \right)^{\frac{2}{\gamma}} + 2(C^*)^{-1} \beta^2 k \|u\|_{L^2(M)}^{q-2} \left( \int_{d_g(x_0,x) \geq R} u^\gamma u_L^{(\beta-1)} \, dv_g \right)^{\frac{2}{\gamma}}.
\]
Taking the limit as \( L \to +\infty \) in the above inequality, we obtain
\[
\left( \int_{d_g(x_0,x) \geq R+r} u^{2\beta} \, dv_g \right)^{\frac{2}{\gamma}} \leq 4(C^*)^{-1} \beta^2 \omega_N^{1-\frac{\gamma}{2}} e^{(N-1)b_0(1-\frac{\gamma}{2})} \frac{(R+r)^{N(1-\frac{\gamma}{2})}}{r^2} \left( \int_{d_g(x_0,x) \geq R} u^\gamma u_L^{(\beta-1)} \, dv_g \right)^{\frac{2}{\gamma}} + 2(C^*)^{-1} \beta^2 k \|u\|_{L^2(M)}^{q-2} \left( \int_{d_g(x_0,x) \geq R} u^\gamma u_L^{(\beta-1)} \, dv_g \right)^{\frac{2}{\gamma}}.
\]
Thus, for every \( R > \max \{ \bar{R}, 1 \} \), \( 0 < r \leq \frac{R}{2} \), \( \beta > 1 \) one has
\[
\|u\|_{L^{2\beta}(d_g(x_0,x) \geq R+r)} \leq ((C^*)^{-1})^{\frac{1}{\gamma}} \beta^{\frac{1}{\gamma}} \left( C_1 \frac{(R+r)^{N(1-\frac{\gamma}{2})}}{r^2} + C_2 \right)^{\frac{1}{\gamma}} \|u\|_{L^{\gamma\beta}(d_g(x_0,x) \geq R)}, \tag{3.8}
\]
where \( C_1 = 4 \omega_N^{1-\frac{\gamma}{2}} e^{(N-1)b_0(1-\frac{\gamma}{2})} \), \( C_2 = 2k \|u\|_{L^2(M)}^{q-2} \). Fix \( \rho > \max \{ \bar{R}, 1 \} \). We are going to apply (3.8) choosing first \( \beta = \frac{2^*}{\gamma} \), \( R = \rho + \frac{\rho}{2} \), \( r = \frac{\rho}{2} \) to get

\[
\|u\|_{L^{2\beta}(d_g(x_0,x) \geq 2\rho)} \leq ((C^*)^{-1})^{\frac{1}{\gamma}} \beta^{\frac{1}{\gamma}} \left( C_1 2^{N(1-\frac{\gamma}{2})} \rho^{N(1-\frac{\gamma}{2})-2\gamma} + C_2 \right)^{\frac{1}{\gamma}} \|u\|_{L^{2\beta}(d_g(x_0,x) \geq \rho+\frac{\rho}{2})}.
\]
Noticing that \( \gamma^2 = 2^* \beta \), let us apply (3.8) with \( \beta^2 \) in place of \( \beta \) and \( R = \rho + \frac{\rho}{2} \), \( r = \frac{\rho}{2} \), to obtain
and combining the above inequality with claim i

Thus, combining the previous two inequalities we get

Iterating this procedure, for every integer $n$ we obtain

Since $N(1 - \frac{1}{2}) < 2$ and $\rho > 1$, one has $\rho^{N(1 - \frac{1}{2}) - 2} < 1$, and the previous estimate implies

If

passing to the limit as $n \to \infty$, we obtain

where $C_0 = (C^*)^{-\sigma} \beta^\vartheta e^\zeta$ does not depend on $\rho$. Taking into account that $u \in L^2(M)$, and combining the above inequality with claim i), we obtain that $u \in L^\infty(M)$. Moreover, as $\lim_{\rho \to \infty} \|u\|_{L^2((d_\rho(x_0,x)\geq \rho))} = 0$, we deduce also that $\lim_{d_\rho(x_0,x) \to \infty} u(x) = 0$.

Now, we consider the following minimization problem:

(M) $\min \left\{ \|u\|_{H^1}^2 : u \in H^1_\psi(M), \|\alpha^\frac{1}{2} u\|_{L^2(M)} = 1 \right\}$.

Lemma 3.2. Problem (M) has a non negative solution $\varphi_\alpha \in L^\infty(M)$ such that for every $x_0 \in M$, $\lim_{d_\rho(x_0,x) \to \infty} \varphi_\alpha(x) = 0$. Moreover, $\varphi_\alpha$ is an eigenfunction of the equation

$-\Delta_\rho u + V(x)u = \lambda \alpha(x)u, \quad u \in H^1_\psi(M)$.
corresponding to the eigenvalue $\|\varphi_\alpha\|_V^2$.

Proof. Notice first that $\alpha^{\frac{1}{2}}u \in L^2(M)$ for any $u \in H^1_V(M)$. Fix a minimizing sequence $\{u_n\}$ for problem (M), that is $\|u_n\|_V^2 \to \lambda_\alpha$, being

$$\lambda_\alpha = \inf \left\{ \|u\|_V^2 : u \in H^1_V(M), \|\alpha^{\frac{1}{2}}u\|_{L^2(M)} = 1 \right\}.$$ 

Then, there exists a subsequence (still denoted by $\{u_n\}$) weakly converging in $H^1_V(M)$ to some $\varphi_\alpha \in H^1_V(M)$. By the weak lower semicontinuity of the norm, we obtain that

$$\|\varphi_\alpha\|_V^2 \leq \liminf_n \|u_n\|_V^2 = \lambda_\alpha.$$ 

In order to conclude, it is enough to prove that $\|\alpha^{\frac{1}{2}}\varphi_\alpha\|_{L^2(M)} = 1$. Since $\{u_n\}$ converges strongly to $\varphi_\alpha$ in $L^2(M)$ and $\alpha \in L^\infty(M)$,

$$\alpha^{\frac{1}{2}}u_n \to \alpha^{\frac{1}{2}}\varphi_\alpha \quad \text{in} \quad L^2(M),$$

thus, by the continuity of the norm, $\|\alpha^{\frac{1}{2}}\varphi_\alpha\|_{L^2(M)} = 1$ and the claim is proved. Clearly, $\varphi_\alpha \neq 0$. Replacing eventually $\varphi_\alpha$ with $|\varphi_\alpha|$ we can assume that $\varphi_\alpha$ is non negative. Equivalently, we can write

$$\lambda_\alpha = \inf_{u \in H^1_V(M) \setminus \{0\}} \frac{\|u\|_V^2}{\|\alpha^{\frac{1}{2}}u\|_{L^2(M)}^2}.$$ 

This means that $\varphi_\alpha$ is a global minimum of the function $u \to \frac{\|u\|_V^2}{\|\alpha^{\frac{1}{2}}u\|_{L^2(M)}^2}$, hence its derivative at $\varphi_\alpha$ is zero, i.e.

$$\int_M (\nabla g \varphi_\alpha \nabla g v + V(x) \varphi_\alpha v) dv_g - \|\varphi_\alpha\|_V^2 \int_M \alpha(x) \varphi_\alpha v dv_g = 0 \quad \text{for any} \quad v \in H^1_V(M)$$

(recall that $\|\alpha^{\frac{1}{2}}\varphi_\alpha\|_{L^2(M)} = 1$). The above equality implies that $\varphi_\alpha$ is an eigenfunction of the problem

$$-\Delta_g u + V(x)u = \lambda_\alpha(x)u, \quad u \in H^1_V(M)$$

corresponding to the eigenvalue $\|\varphi_\alpha\|_V^2$. From Theorem 3.1 we also have that $\varphi_\alpha$ is a bounded function and $\lim_{d_\rho(x,x_0) \to \infty} \varphi_\alpha(x) = 0$. \hfill $\Box$

Now we are in the position to prove our main theorem.

3.1. **Proof of Theorem 1.1.** $(i) \Rightarrow (ii)$.  

From the assumption, we deduce the existence of $\sigma_1 \in (0, +\infty]$ defined as

$$\sigma_1 \equiv \lim_{\xi \to 0} \frac{F(\xi)}{\xi^2}.$$ 

Assume first that $\sigma_1 < \infty$. 

Define the following continuous truncation of $f$,

$$\tilde{f}(\xi) = \begin{cases} 0, & \text{if } \xi \in (-\infty, 0] \\ f(\xi), & \text{if } \xi \in (0, a] \\ f(a), & \text{if } \xi \in (a, +\infty) \end{cases}$$

Now we are in the position to prove our main theorem.
and let $\hat{F}$ its primitive, that is $\hat{F}(\xi) = \int_{0}^{\xi} \hat{f}(t)dt$, i.e.

$$
\begin{aligned}
\hat{F}(\xi) &= \begin{cases} 
F(\xi), & \text{if } \xi \in (-\infty, a] \\
F(a) + f(a)(\xi - a), & \text{if } \xi \in (a, +\infty).
\end{cases}
\end{aligned}
$$

Observe that, from the monotonicity assumption on the function $\xi \mapsto \frac{F(\xi)}{\xi^2}$, the derivative of the latter is non-positive, that is

$$
f(\xi)\xi \leq 2F(\xi) \quad \text{for all } \xi \in [0, a].$$

This implies

$$
\hat{f}(\xi)\xi \leq 2\hat{F}(\xi) \quad \text{for all } \xi \in \mathbb{R},
$$

or that the function $\xi \mapsto \frac{\hat{F}(\xi)}{\xi^2}$ is not increasing in $(0, +\infty)$. Then,

$$
\sigma_1 = \lim_{\xi \to 0} \frac{F(\xi)}{\xi^2} = \lim_{\xi \to 0} \frac{\hat{F}(\xi)}{\xi^2} = \sup_{\xi > 0} \frac{\hat{F}(\xi)}{\xi^2}.
$$

Moreover,

$$
\hat{F}(\xi) \leq \sigma_1 \xi^2 \quad \text{and} \quad \hat{f}(\xi) \leq 2\sigma_1, \quad \text{for all } \xi \in \mathbb{R}
$$

Define now the functional

$$
J : H^1_{\nu}(M) \to \mathbb{R}, \quad J(u) = \int_{M} \alpha(x)\hat{F}(u)dv_g,
$$

which is well defined, sequentially weakly continuous, Gâteaux differentiable with derivative given by

$$
J'(u)(v) = \int_{M} \alpha(x)\hat{f}(u)v dv_g \quad \text{for all } v \in H^1_{\nu}(M).
$$

Moreover, $J(0) = 0$ and

$$
\sup_{u \in H^1_{\nu}(M) \setminus \{0\}} \frac{J(u)}{\|u\|_{T}^2} = \frac{\sigma_1}{\lambda_1}.
$$

Indeed, from (3.11) immediately follows that

$$
\frac{J(u)}{\|u\|_{T}^2} \leq \frac{\sigma_1}{\lambda_1} \quad \text{for every } u \in H^1_{\nu}(M) \setminus \{0\}.
$$

Also, using the monotonicity assumption, for every $t > 0$, and for every $x \in M$, such that $\varphi_\alpha(x) > 0$

$$
\frac{\hat{F}(t\varphi_\alpha(x))}{(t\varphi_\alpha(x))^2} \geq \frac{\hat{F}(t\|\varphi_\alpha\|_{L^\infty(M)})}{t^2\|\varphi_\alpha\|^2_{L^\infty(M)}},
$$

thus

$$
J(t\varphi_\alpha) = \int_{\{\varphi_\alpha > 0\}} \alpha(x)\hat{F}(t\varphi_\alpha)^2(t\varphi_\alpha)^2 dv_g \geq \frac{\hat{F}(t\|\varphi_\alpha\|_{L^\infty(M)})}{\|\varphi_\alpha\|^2_{L^\infty(M)}} \int_{M} \alpha(x)\varphi_\alpha^2 dv_g
$$

$$
= \frac{\hat{F}(t\|\varphi_\alpha\|_{L^\infty(M)})}{\|\varphi_\alpha\|^2_{L^\infty(M)}} > 0.
$$

Thus,

$$
\frac{J(t\varphi_\alpha)}{\|t\varphi_\alpha\|^2_{T}} = \frac{J(t\varphi_\alpha)}{t^2\lambda_1} \geq \frac{\hat{F}(t\|\varphi_\alpha\|_{L^\infty(M)})}{t^2\|\varphi_\alpha\|^2_{L^\infty(M)}} \frac{1}{\lambda_1}.
$$

Passing to the limit as $t \to 0^+$, from (3.10), condition (3.12) follows at once. Let us now apply Theorem C with $X = H^1_{\nu}(M)$ and $J$ as above. Let $r > 0$ and denote by $\hat{u}$ the global maximum
of \( J_{|B_{\sigma_1}(r)} \). We observe that \( \hat{u} \neq 0 \) as \( J(t\varphi_n) > 0 \) for every \( t \) small enough, thus \( J(\hat{u}) > 0 \). If \( \hat{u} \in \text{int}(B_{\sigma_0}(r)) \), then, it turns out to be a critical point of \( J \), that is \( J'(\hat{u}) = 0 \) and (2.1) is satisfied. If \( \|\hat{u}\|_V = r \), then, from the Lagrange multiplier rule, there exists \( \mu > 0 \) such that \( J'(\hat{u}) = \mu \hat{u} \), that is, \( \hat{u} \) is a solution of the equation

\[-\Delta_y u + V(x)u = \frac{1}{\mu} \alpha(x)\tilde{f}(u), \quad \text{in} \ M.\]

Also, by Theorem 3.1, \( \hat{u} \in L^\infty(M) \) and \( \lim_{d_g(x_0,x) \to \infty} \hat{u}(x) = 0 \). Condition (3.9) implies in addition that

\[J'(\hat{u})(\hat{u}) - 2J(\hat{u}) = \int_M \alpha(x)\tilde{f}(\hat{u})\hat{u} - 2\tilde{F}(\hat{u})]dv_g \leq 0.\]

If the latter integral is zero, then, being \( \alpha > 0 \), \( \tilde{f}(\hat{u}(x))\hat{u}(x) - 2\tilde{F}(\hat{u}(x)) = 0 \) for all \( x \in M \), which in turn implies that \( \tilde{f}(\xi)\xi - 2\tilde{F}(\xi) = 0 \) for all \( \xi \in [0,\|\hat{u}\|_{L^\infty(M)}] \), that is, the function \( \xi \to \frac{\tilde{F}(\xi)}{\xi} \) is constant in the interval \([0,\|\hat{u}\|_{L^\infty(M)}] \). In particular it would be constant in a small neighborhood of zero which is in contradiction with the assumption (i). This means that (2.1) is fulfilled and the thesis applies: there exists an interval \( I \subseteq (0, +\infty) \) such that for every \( \lambda \in I \) the functional

\[u \to \frac{\|u\|_V^2}{2} - \lambda J(u)\]

has a non-zero critical point \( u_\lambda \) with \( \int_M (|\nabla u_\lambda|^2 + V(x)u_\lambda^2)dv_g < r \). In particular, \( u_\lambda \) turns out to be a nontrivial solution of the problem

\[
\begin{aligned}
-\Delta_y u + V(x)u &= \lambda \alpha(x)\tilde{f}(u), &\text{in} & M \\
u &\geq 0, &\text{in} & M \\
u &\to 0, &\text{as} & d_g(x, x_0) \to \infty.
\end{aligned}
\]

(\( \mathcal{P}_\lambda \))

From Remark 2.2, we know that \( I = \frac{1}{2}\left\{ \eta(r\delta_r), \lim \eta(s) \right\} \). It is clear that

\[\eta(r\delta_r) = \sup_{y \in B_r} \frac{r - \|y\|^2_\nu}{r\delta_r - J(y)} \geq \frac{1}{\delta_r}\]

and by the definition of \( \delta_r \),

\[\frac{r - \|y\|^2}{r\delta_r - J(y)} \leq \frac{r - \|y\|^2_\nu}{r\delta_r - \delta_r\|y\|^2_\nu} = \frac{1}{\delta_r}\]

for every \( y \in B_r \). Thus, recalling (3.12),

\[\eta(r\delta_r) = \frac{1}{\delta_r} = \frac{\lambda_\alpha}{\sigma_1} \]

Notice also that from Theorem 3.1, \( u_\lambda \in L^\infty(M) \). Let us prove that

\[\lim_{\lambda \to \frac{1}{2\sigma_1}} \|u_\lambda\|_{L^\infty(M)} = 0.\]

Fix a sequence \( \lambda_n \to \left( \frac{1}{2\sigma_1} \right) \). Since \( \|u_{\lambda_n}\|^2_\nu \leq r \), \( \{u_{\lambda_n}\} \) admits a subsequence still denoted by \( \{u_{\lambda_n}\} \) which is weakly convergent to some \( u_0 \in B_{\sigma_0}(r) \). Moreover, from the compact embedding of \( H^1_V(M) \) in \( L^2(M) \), \( \{u_{\lambda_n}\} \) converges (up to a subsequence) strongly to \( u_0 \) in \( L^2(M) \). Thus, being \( u_{\lambda_n} \) a solution of (\( \mathcal{P}_{\lambda_n} \)),

\[
\int_M (\nabla_g u_{\lambda_n} \nabla_g v + V(x)u_{\lambda_n}v)dv_g = \lambda_n \int_M \alpha(x)\tilde{f}(u_{\lambda_n})vdv_g \quad \text{for all} \ v \in H^1_V(M), \quad (3.13)
\]
passing to the limit we obtain that $u_0$ is a solution of the equation

$$-\Delta_g u + V(x)u = \frac{\lambda_n}{2\sigma_1} \alpha(x)\tilde{f}(u) \text{ in } M.$$ 

Assume $u_0 \neq 0$. Thus, testing (3.13) with $v = u_{\lambda_n}$,

$$\|u_{\lambda_n}\|^2_V = \lambda_n \int_M \alpha(x)\tilde{f}(u_{\lambda_n})u_{\lambda_n}dv_g,$$

and passing to the limit,

$$\|u_0\|^2_V \leq \liminf_{n \to \infty} \|u_{\lambda_n}\|^2_V = \frac{\lambda_n}{2\sigma_1} \int_M \alpha(x)\tilde{f}(u_0)u_0dv_g$$

$$< \frac{\lambda_n}{\sigma_1} \int_M \alpha(x)\tilde{F}(u_0)dv_g \leq \lambda_n \int_M \alpha(x)u_0^2dv_g$$

$$\leq \|u_0\|^2_V.$$

The above contradiction implies that $u_0 = 0$, and that $\lim_{n \to \infty} \|u_{\lambda_n}\|_V = 0$. Thus, in particular, because of the embedding into $L^{2^*}(M)$, we deduce that $\lim_{n \to \infty} \|u_{\lambda_n}\|_{L^{2^*}(M)} = 0$ and from Theorem 3.1, $\lim_{n \to \infty} \|u_{\lambda_n}\|_{L^\infty(M)} = 0$. Therefore,

$$\lim_{\lambda \to \frac{\lambda_n}{\sigma_1} +} \|u_{\lambda}\|_{L^\infty(M)} = 0.$$

This implies that there exists a number $\varepsilon_r > 0$ such that for every $\lambda \in \left(\frac{\lambda_n}{2\sigma_1}, \frac{\lambda_n}{\sigma_1} + \varepsilon_r\right)$, $\|u_{\lambda}\|_{L^\infty(M)} \leq a$. Hence, $u_{\lambda}$ turns out to be a solution of the original problem $(\mathcal{P}_\lambda)$ and the proof of this first case is concluded.

Assume now $\sigma_1 = +\infty$. The functional

$$K : H^1_v(M) \to \mathbb{R}, \quad K(u) = \int_M \alpha(x)F(u)dv_g,$$

is well defined and sequentially weakly continuous. Let $r > 0$ and fix $\lambda \in I = \left\{\frac{1}{\lambda^*}, \frac{1}{\lambda^*} + \varepsilon_r\right\}$ where

$$\lambda^* = \inf_{\|y\|_V \leq r} \sup_{\|y\|_V \leq r} \frac{K(u) - K(y)}{r - \|y\|_V^2}$$

(with the convention $\frac{1}{\lambda^*} = +\infty$ if $\lambda^* = 0$). Denote by $u_{\lambda}$ the global minimum of the restriction of the functional $\mathcal{E}$ to $B_r$. Then, since

$$\lim_{t \to 0} \frac{K(t\varphi_{\lambda})}{\|t\varphi_{\lambda}\|^2_V} = +\infty,$$

it is easily seen that $\mathcal{E}(u_{\lambda}) < 0$, therefore, $u_{\lambda} \neq 0$. The choice of $\lambda$ implies, via easy computations, that $\|u_{\lambda}\|_V^2 < r$. So, $u_{\lambda}$ is a critical point of $\mathcal{E}$, thus a weak solution of $(\mathcal{P}_\lambda)$.

$(ii) \Rightarrow (i)$. We follow the idea of [Ane16]. For the sake of completeness we give the details. Assume by contradiction that there exist two positive constants $b, c$ such that

$$\frac{F(\xi)}{\xi^2} = c \quad \text{for all } \xi \in (0, b].$$

Thus,

$$f(\xi) = 2c\xi \quad \text{for all } \xi \in [0, b]. \quad (3.14)$$

Let $\{r_n\}$ be a sequence of positive numbers such that $r_n \to 0^+$. Then, for every $n \in \mathbb{N}$ there exists an interval $I_n$ such that for every $\lambda \in I_n$, $(\mathcal{P}_\lambda)$ has a solution $u_{\lambda,n}$ with $\|u_{\lambda,n}\|_V^2 < r_n$. Thus,

$$\limsup_{n \to \infty} \sup_{\lambda \in I_n} \|u_{\lambda,n}\|_V = 0.$$
Since $f(\xi) \leq k(\xi + \xi^{q-1})$ for all $\xi \geq 0$ (this follows from the growth assumption (1.1) and equality (3.14)), and being $u_{\lambda,n}$ a critical point of $\mathcal{E}$, from the continuous embedding of $H^1_0(M)$ into $L^2(M)$ and by Theorem 3.1 we obtain that
\[
\limsup_{n} \sup_{\lambda \in I_n} ||u_{\lambda,n}||_{\infty} = 0.
\]
Let us fix $n_0$ big enough, such that $\sup_{\lambda \in I_n} ||u_{\lambda,n}||_{\infty} < b$. We deduce that for every $\lambda \in I_{n_0}$, $u_{\lambda,n_0}$ is a solution of the equation
\[-\Delta u + V(x)u = 2\lambda c_0(x)u, \text{ in } M,
\]
against the discreteness of the spectrum of the Schrödinger operator $-\Delta u + V(x)$ established in Theorem D. $\square$

\textbf{Remark 3.3.} Notice that without the growth assumption (1.1) the result holds true replacing the norm of the solutions $u_\lambda$ in the Sobolev space with the norm in $L^\infty(M)$.

We conclude the section with a corollary of the main result in the euclidean setting. We propose a more general set of assumption on $V$ which implies both the compactness of the embedding of $H^1_0(\mathbb{R}^N)$ into and the discreteness of the spectrum of the Schrödinger operator [BF78]. Namely, let $N \geq 3$, $\alpha : \mathbb{R}^N \to \mathbb{R}_+ \setminus \{0\}$ be in $L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function with $f(0) = 0$ such that there exist two constants $k > 0$ and $q \in (1,2^*)$ such that
\[f(\xi) \leq k(1 + \xi^{q-1}) \text{ for all } \xi \geq 0.
\]
Let also $V : \mathbb{R}^N \to \mathbb{R}$ be in $L^\infty_{\text{loc}}(\mathbb{R}^N)$, such that $\text{essinf}_{\mathbb{R}^N} V \equiv V_0 > 0$ and
\[
\int_{B(x)} \frac{1}{V(y)} dy \to 0 \quad \text{as } |x| \to \infty,
\]
where $B(x)$ denotes the unit ball in $\mathbb{R}^N$ centered at $x$. In particular, if $V$ is a strictly positive ($\text{essinf}_{\mathbb{R}^N} V > 0$), continuous and coercive function, the above conditions hold true.

\textbf{Corollary 3.4.} Assume that for some $a > 0$ the function $\xi \to \frac{f(\xi)}{\xi^2}$ is non-increasing in $(0,a]$. Then, the following conditions are equivalent:

(i) for each $b > 0$, the function $\xi \to \frac{f(\xi)}{\xi^2}$ is not constant in $(0,b]$;

(ii) for each $r > 0$, there exists an open interval $I \subseteq (0, +\infty)$ such that for every $\lambda \in I$, problem

\[
\begin{cases}
-\Delta u + V(x)u = \lambda \alpha(x)f(u), & \text{in } \mathbb{R}^N \\
 u \geq 0, & \text{in } \mathbb{R}^N \\
u \to 0, & \text{as } |x| \to \infty
\end{cases}
\]

has a nontrivial solution $u_\lambda \in H^1(\mathbb{R}^N)$ satisfying $\int_{\mathbb{R}^N} (|\nabla u_\lambda|^2 + V(x)u_\lambda^2) \, dx < r$.

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\textbf{REFERENCES}

[AX10] L. Adriano and C. Xia. Sobolev type inequalities on Riemannian manifolds. \textit{J. Math. Anal. Appl.}, 371(1):372–383, 2010.

[Ane16] G. Anello. A characterization related to the Dirichlet problem for an elliptic equation. \textit{Funkcial. Ekvac.}, 59(1):113–122, 2016.

[Aub75] T. Aubin. Problèmes isopérimétriques et espaces de Sobolev. \textit{C. R. Acad. Sci. Paris Sér. A-B}, 280(5):Aii, A279–A281, 1975.
[BPW01] T. Bartsch, A. Pankov, and Z.-Q. Wang. Nonlinear Schrödinger equations with steep potential well. *Commun. Contemp. Math.*, 3(4):549–569, 2001.

[BW95] T. Bartsch and Z. Q. Wang. Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$. *Comm. Partial Differential Equations*, 20(9-10):1725–1741, 1995.

[BF78] V. Benci and D. Fortunato. Discreteness conditions of the spectrum of Schrödinger operators. *J. Math. Anal. Appl.*, 64(3):695–700, 1978.

[BW02] J. Byeon and Z.-Q. Wang. Standing waves with a critical frequency for nonlinear Schrödinger equations. *Arch. Ration. Mech. Anal.*, 165(4):295–316, 2002.

[BKT16] L. P. Bonorino, P. K. Klaser, and M. Telechevsky. Boundedness of Laplacian eigenfunctions on manifolds of infinite volume. *Comm. Anal. Geom.*, 24(4):753–768, 2016.

[Can74] M. Cantor. Sobolev inequalities for Riemannian bundles. *Bull. Amer. Math. Soc.*, 80:239–243, 1974.

[CNY08] D. Cao, E. S. Noussair, and S. Yan. Multiscale-bump standing waves with a critical frequency for nonlinear Schrödinger equations. *Trans. Amer. Math. Soc.*, 360(7):3813–3837, 2008.

[CM11] A. Cianchi and V. G. Maz'ya. On the discreteness of the spectrum of the Laplacian on noncompact Riemannian manifolds. *J. Differential Geom.*, 87(3):469–491, 2011.

[CM13] A. Cianchi and V. G. Maz'ya. Bounds for eigenfunctions of the Laplacian on noncompact Riemannian manifolds. *Amer. J. Math.*, 135(3):579–635, 2013.

[Cro80] C. B. Croke. Some isoperimetric inequalities and eigenvalue estimates. *Ann. Sci. École Norm. Sup.* (4), 13(4):419–435, 1980.

[FKV15] C. Farkas, A. Kristály, and C. Varga. Singular Poisson equations on Finsler-Hadamard manifolds, *Calc. Var. Partial Differential Equations* 54 (2015), no. 2, 1219–1241. MR 3396410

[FK16] C. Farkas and A. Kristály, Schrödinger-Maxwell systems on non-compact Riemannian manifolds, *Nonlinear Anal. Real World Appl.* 31 (2016), 473–491. MR 3490853

[GHL87] S. Gallot, D. Hulin, and J. Lafontaine. Riemannian geometry. Universitext. Springer-Verlag, Berlin, 1987.

[Heb99] E. Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, volume 5 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[HK95] P. Hajłasz and P. Koskela. Sobolev meets Poincaré. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(10):1211–1215, 1995.

[KS99] V. Kondrat’ev and M. Shubin. Discreteness of spectrum for the Schrödinger operators on manifolds of bounded geometry. In *The Maz’ya anniversary collection, Vol. 2 (Rostock, 1998)*, volume 110 of *Oper. Theory Adv. Appl.*, pages 185–226. Birkhäuser, Basel, 1999.

[Kri07] A. Kristály. Multiple solutions of a sublinear Schrödinger equation. *NoDEA Nonlinear Differential Equations Appl.*, 14(3-4):291–301, 2007.

[Kri09] A. Kristály. Asymptotically critical problems on higher-dimensional spheres, *Discrete Contin. Dyn. Syst. Syst.* 23 (2009), no. 3, 919–935. MR 2461832

[Kri12] A. Kristály. Bifurcations effects in sublinear elliptic problems on compact Riemannian manifolds, *J. Math. Anal. Appl.* 385 (2012), no. 1, 179–184. MR 2832084

[LY86] P. Li and S.T. Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(3-4):153–201, 1986.

[Ma06] L. Ma. Gradient estimates for a simple elliptic equation on complete non-compact Riemannian manifolds. *J. Funct. Anal.*, 241(1):374–382, 2006.

[MSC95] P. Maheux and L. Saloff-Coste. Analyse sur les boules d’un opérateur sous-elliptique. *Math. Ann.*, 303(4):713–740, 1995.

[MBR15] G. Molica Bisci and V. D. Rădulescu. A characterization for elliptic problems on fractal sets. *Proc. Amer. Math. Soc.*, 143(7):2957–2968, 2015.

[PRS08] S. Pigola, M. Rigoli, and A. G. Setti. *Vanishing and finiteness results in geometric analysis. A generalization of the Bochner technique*. Volume 266 of Progress in Mathematics. Birkhäuser Verlag, Basel, 2008.

[Pou05] C. Poupaud. On the essential spectrum of Schrödinger operators on riemannian manifolds. *Mathematische Zeitschrift*, 251(1):1–20, 2005.

[Rab92] P. H. Rabinowitz. On a class of nonlinear Schrödinger equations. *Z. Angew. Math. Phys.*, 43(2):270–291, 1992.

[Ric14] B. Ricceri. A note on spherical maxima sharing the same Lagrange multiplier. *Fixed Point Theory Appl.*, pages 2014:25, 2014.

[Ric15] B. Ricceri. A characterization related to a two-point boundary value problem. *J. Nonlinear Convex Anal.*, 16(1):79–82, 2015.
[She03] Z. Shen. The spectrum of Schrödinger operators with positive potentials in Riemannian manifolds. Proc. Amer. Math. Soc., 131(11):3447–3456, 2003.

[Str77] W. A. Strauss. Existence of solitary waves in higher dimensions. Comm. Math. Phys., 55(2):149–162, 1977.

[Var89] N. Th. Varopoulos. Small time Gaussian estimates of heat diffusion kernels. I. The semigroup technique. Bull. Sci. Math., 113(3):253–277, 1989.

[Wil96] M. Willem. Minimax theorems. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.

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