A New Probe of Gaussianity and Isotropy for CMB Maps

J. Hamann†, Q. T. Le Gia‡, I. H. Sloan§, Y. G. Wang¶, R. S. Womersley∥

School of Physics, The University of New South Wales, Sydney, NSW 2052, Australia
School of Mathematics and Statistics, The University of New South Wales, Sydney, NSW 2052, Australia

ABSTRACT

We introduce a new mathematical tool (henceforth referred to as the “probe”) to analyse the randomness of purported isotropic Gaussian random fields on the sphere. We apply the probe to assess the full-sky cosmic microwave background (CMB) temperature maps produced by the Planck collaboration (PR2 2015 and PR3 2018), with special attention to the inpainted maps. To study the randomness of the fields represented by each map we use the autocorrelation of the sequence of probe coefficients (which are just the full-sky Fourier coefficients $a_{\ell,0}$ if the $z$ axis is taken in the probe direction). If the field is isotropic and Gaussian then the probe coefficients for a given direction should be realisations of uncorrelated scalar Gaussian random variables. We find that for most of the maps there are many directions for which this is not the case. We make a first attempt at justifying the features of the temperature maps that contribute to the apparent lack of randomness. In the case of Commander 2015 we mimic an aspect of the observed behaviour with a model field that is Gaussian but not isotropic. In contrast, the non-inpainted 2018 SEVEM map (which has visible equatorial pollution) is modelled by an isotropic Gaussian random field plus a non-random needlet-like structure located near the galactic centre.

Key words: cosmic microwave background – probe – spherical harmonics – autocorrelation

1 INTRODUCTION

In this paper we introduce a mathematical tool (henceforth referred to as the “probe”) for analysing the randomness of purported isotropic Gaussian random fields on the sphere. We apply the probe to assess the full-sky cosmic microwave background (CMB) temperature maps produced by the Planck consortium (Planck Collaboration et al. 2016, 2018a). This probe should be regarded as complementary to other methods of searching for non-Gaussianity (Akrami et al. 2019a) or deviations from statistical isotropy of the CMB (Akrami et al. 2019b). It relies on the Fourier coefficients, even those of high degree, being easily computed. (The computation of approximate Fourier coefficients in the case of the Planck maps is easy, even up to degree 3,000, because they are given at the HEALPix points (Górski et al. 2005), which are designed to make the harmonic analysis efficient.)

Assuming that the origin of the primordial density fluctuations lies in a phase of generic slow-roll inflation (Mukhanov 2005), the primary CMB anisotropies (i.e., the temperature fluctuations on the surface of last scattering) can be very well described as a realisation of a statistically isotropic Gaussian random field on the 2-dimensional unit sphere. Note, however, that the observed CMB is a superposition of the primary anisotropies and secondary anisotropies (Aghanim et al. 2008) generated due to the propagation through an anisotropic medium, and that the secondary anisotropies are expected to deviate from Gaussianity, most notably due to weak gravitational lensing (Lewis & Challinor 2006) and the Sunyaev-Zel’Dovich effect (Sunyaev & Zeldovich 1980).

The tool we shall introduce is a highly directional axially symmetric spherical harmonic of arbitrary degree $\ell$, with axis in the direction of an arbitrary unit vector $p$. When convolved with the field it provides sensitive information about the extent to which the field, as represented by the map, is truly Gaussian and isotropic. We shall concentrate initially on the inpainted versions of the CMB maps, because the non-inpainted versions often display visible pollution from foreground effects in the masked region near the galactic equator. The inpainted versions, in contrast, all appear good.
where \( \delta_{jk} \) is the Kronecker symbol and the positive constant \( C_\ell \) depends on \( \ell \) only. The sequence \( C_\ell \) is known as the angular power spectrum of the field. As is usual in the CMB context, we require for the monopole and dipole \( C_0 = C_1 = 0 \). The set of the \( a_{\ell,m} \) with \( \ell \geq 2 \) and \( m \geq 0 \) are in the Gaussian case not just uncorrelated but are independent Gaussian random variables. It is easily seen that the Gaussian field \( T \) has mean zero,

\[
\mathbb{E}[T(x)] = 0, \quad x \in S^2,
\]

covariance

\[
\mathbb{E}[T(x)T(z)] = \sum_{\ell=2}^{\infty} C_\ell \frac{2\ell+1}{4\pi} P_\ell(x \cdot z), \quad x, z \in S^2,
\]

where \( P_\ell \) is the Legendre polynomial of degree \( \ell \), and we used the addition theorem (Müller 1966)

\[
P_\ell(x \cdot z) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell,m}(x)Y_{\ell,m}(z), \quad x, z \in S^2.
\]

For the moment we concentrate on the coefficients \( a_{\ell,0} \) (i.e. we take \( m = 0 \)), but we consider simultaneously many different \( \ell \) values, recalling that the \( a_{\ell,0} \) are to a good approximation supposed to be instances of independent random variables. It is useful to consider exactly what the coefficients \( a_{\ell,0} \) tell us about a field \( T \). Remembering that

\[
Y_{\ell,0}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos \theta) = \sqrt{\frac{2\ell+1}{4\pi}} a_\ell \mathbf{n} \cdot \mathbf{x},
\]

where \( \mathbf{n} \) is the unit vector in the direction of the positive \( z \) axis, it follows from (2) that \( a_{\ell,0} \) is a real number given by

\[
a_{\ell,0} = \int_{S^2} \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\mathbf{x} \cdot \mathbf{n}) T(\mathbf{x}) d\sigma(\mathbf{x}).
\]

Thus \( a_{\ell,0} \) is the convolution of \( T \) with a spherical harmonic of degree \( \ell \) that is axially symmetric about the \( z \)-axis.

In Figure 3 we show for the Commander 2015 map the (real) numbers \( a_{\ell,0} \) divided by \( \sqrt{C_\ell} \), for values of \( \ell \) from 2 up to 2,500. Here the squared normalising factor \( \hat{C}_\ell \) is an empirical estimate of \( C_\ell \), defined by

\[
\hat{C}_\ell := \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell,m}|^2, \quad \ell \geq 0.
\]

If the Gaussianity assumption holds then the scaled coefficients \( a_{\ell,0}/\sqrt{\hat{C}_\ell} \) should be instances of independent mean-zero Gaussian random variables with variance close to 1. To the eye this appears satisfactorily to be the situation for the \( a_{\ell,0} \) from Commander 2015 illustrated in Figure 3.

To study the randomness of a sequence, the one-sample Kolmogorov-Smirnov test gives a quantitative measure of the null hypothesis, which in this case is that the scaled coefficients \( a_{\ell,0}/\sqrt{\hat{C}_\ell} \) are independent samples from a standard normal distribution. It gives a \( p \)-value of 0.80 for the Commander 2015 data in Figure 3, indicating that the null hypothesis cannot be rejected for this data set.

To find if a sequence of similarly scaled real numbers is correlated or uncorrelated, a standard device in the world of time series is to compute the autocorrelation for lags of 1, 2, . . . (For a definition of autocorrelation and full information on the computation, see Section 2.) In Figure 4 we compute the autocorrelations for the data in Figure 3 for all lags

\[\text{Figure 1.} \text{ Inpainted CMB map, Commander 2015, } N_{\text{Side}} = 2048\]

\[\text{Figure 2.} \text{ Inpainted CMB map, SEVEM 2018, } N_{\text{Side}} = 2048\]
from $1$ up to $50$, using all the data from $\ell = 2$ to $2,500 =: L$. (The autocorrelation for lag $0$ by definition always has the value $1$.) The two horizontal blue lines in Figure 4 are at $\pm t_L$, where

$$t_L = 2/\sqrt{L -1} = 0.04$$

is the $95.45\%$ confidence interval if the input data consist of iid standard normal random variables. If the hypothesis holds that the input data are iid normal random variables, then the autocorrelations should themselves be uncorrelated, and almost all of the autocorrelations should lie between the blue lines. That indeed seems to be qualitatively the case for the data from Commander 2015 in Figure 4.

The coefficients $a_{\ell,0}$ are seen in (7) to be associated with the direction $\mathbf{n}$ of the positive $z$ axis. But the $z$-axis is not the only interesting direction. We now define the probe, which will allow us to test the field for any direction $\mathbf{p}$, with $\mathbf{p}$ an arbitrary unit vector. The probe is a real-valued mathematical function on the sphere of the simple form

$$P_{\ell,\mathbf{p}}(x) := \sqrt{2\ell + 1}/4\pi P_\ell(x \cdot \mathbf{p}), \quad \ell = 0, 1, \ldots$$

which is a spherical harmonic of degree $\ell$, rotationally symmetric about an axis in the direction of $\mathbf{p}$. The probe coefficient for the direction $\mathbf{p}$ is the inner product of $P_{\ell,\mathbf{p}}$ with the given real scalar field $T(x)$,

$$T_{\ell,\mathbf{p}} := \int_{S^2} P_{\ell,\mathbf{p}}(x)T(x)d\sigma(x), \quad \ell = 0, 1, \ldots$$

In the special case when $\mathbf{p} = \mathbf{n} = (0,0,1)$ we have

$$P_{\ell,\mathbf{p}}(x) = \sqrt{2\ell + 1}/4\pi P_\ell(\cos \theta) = Y_{\ell,0}(\theta, \phi)$$

and $T_{\ell,\mathbf{p}} = a_{\ell,0}$.

To make the same point differently, if the $z$-axis is chosen in the direction of $\mathbf{p}$ then the probe $P_{\ell,\mathbf{p}}$ is just the spherical harmonic of degree $\ell$ with $m = 0$, and the probe coefficient $T_{\ell,\mathbf{p}}$ is just $a_{\ell,0}$.

The autocorrelation for probe direction $\mathbf{p}$ is intimately connected to that at its antipode $-\mathbf{p}$: for we shall see in the next section that the probe coefficient has the symmetry property

$$T_{\ell,-\mathbf{p}} = (-1)^\ell T_{\ell,\mathbf{p}}.$$  

An example of a direction $\mathbf{p}$ for which the probe for Commander 2015 reveals a significant departure from randomness.
is, in galactic coordinates, \((l, b) = (35.20, 24.30)\). For this direction the autocorrelations of the scaled probe coefficients given in Figure 5 show strong correlation even up to a lag of 50. For completeness, the underlying scaled probe coefficients are shown in Figure 6. (We explain below how this direction was discovered.)

To allow us to explore the Planck temperature maps for all directions \(p\), so seeking regions where the autocorrelations of scaled values of \(T_{\ell,p}\) depart significantly from the expected behaviour, we compute for each such temperature map the autocorrelation discrepancy (defined in Section 2) of the numbers \(T_{\ell,p}\), with \(p\) varying over the sphere. This is a measure of extreme departure of the autocorrelation from that of a Gaussian distribution, designed to allow us to detect regions (or the antipodes of such regions) in which the autocorrelation departs most strongly from the Gaussian assumption. In the case of Commander 2015 the AC discrepancy reaches a maximum value of 2.075 over 12,582,912 HEALPix points (with \(N_{\text{side}} = 1024\)) on the sphere. The corresponding autocorrelation is shown in Figure 5.

In Figures 7, 8, 9, 10 and 11 we show the AC discrepancy maps for Commander 2015 and all four of the inpainted 2018 maps, all with the same colour map. We observe that NILC appears almost perfect, that Commander 2018, SEVEM, and SMICA have visible pollution in the inpainted region; while Commander 2015 has small but interesting pollution in two regions (and their antipodes) well away from the inpainted region.

A general observation about all of the AC discrepancy maps in Figures 7–11 is that the larger values (that is, the larger departures from the background blue) appear to be instances of white noise, with no continuity between one pixel and the next. (We shall see later, in Section 6, that the corresponding AC discrepancies of the non-inpainted maps have a completely different character — they are larger, more wild, but essentially continuous.) That being the case, it follows that (contrary to our expectation) there is no special significance attached to the point at which the AC discrepancy happens to reach its maximum value — the large value may be just a consequence of a random fluctuation. On the other hand, the visibly interesting regions in the AC discrepancy maps in Figures 7–11 do appear to be systemic, especially as (in all but Commander 2015) the biggest anomalies are in the inpainted mask regions, where one would expect maximum distortion.

We are aware of other work on directional statistics applied to CMB maps, collected from WMAP seven year Internal Linear Combination (ILC7) data by Naselsky et al. (2012), and from Planck 2015 data by Zhao (2014); Cheng et al. (2016), but their purposes and methods are quite different from the current study.

The paper is organised as follows. In Section 2 we review necessary mathematical background for random fields on the unit sphere, and define autocorrelations and AC discrepancy of scaled probe coefficients. In Section 3 we report empirical results for autocorrelations of the CMB maps. In Section 4 we generate realisations of isotropic Gaussian random fields with best-fit angular power spectrum from the Planck data, in order to demonstrate that these realisations exhibit nothing like the localised artifacts seen in the Planck data. In Section 5 we show that a key feature of the Commander 2015 AC discrepancy map can be mimicked by an input.
field that is Gaussian but not quite isotropic. In Section 6 we consider the non-inpainted 2018 maps, and then in Section 7 present a model to explain the most significant feature of the AC discrepancy map in the case of non-inpainted SMICA 2018. Finally, in Section 8 we give brief conclusions.

2 MATHEMATICAL BACKGROUND

In this section we recall some necessary mathematical background for random and non-random fields on the unit sphere $S^2$, and define autocorrelation and autocorrelation discrepancy.

A real-valued field $T$ on the unit sphere can be expanded in terms of its Fourier series,

$$T(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_{\ell,m}(\theta, \phi),$$

$$a_{\ell,m} = \int_{S^2} Y_{\ell,m}(\theta, \phi) T(x) \, d\theta \, d\phi,$$  \hspace{1cm} (13)

where the $Y_{\ell,m}(\theta, \phi) \equiv Y_{\ell,m}(x)$ for $\ell = 0, 1, 2, \ldots$; $m = -\ell, \ldots, \ell$ are the orthonormal complex-valued spherical harmonics (Varshalovich et al. 1988) of degree $\ell$, and $\theta \in [0, \pi]$ is the polar angle of $x$ and $\phi \in [0, 2\pi]$ is the azimuthal angle. More explicitly,

$$Y_{\ell,m}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} \frac{\sin m\theta}{\sin \theta} P_{\ell,m}(\cos \theta) \exp(im\phi), \quad m \geq 0,$$

$$Y_{\ell,m}(\theta, \phi) = (-1)^m Y_{\ell,-m}(\theta, \phi), \quad m < 0,$$

where $P_{\ell,m}$ denotes the associated Legendre function, defined in terms of the Legendre polynomial $P_{\ell}$ by

$$P_{\ell,m}(\mu) = (-1)^m (1 - \mu^2)^{m/2} \frac{d^m}{d\mu^m} P_{\ell}(\mu), \quad m = 0, 1, \ldots, \ell$$

for $\ell = 0, 1, 2, \ldots$ and $\mu \in [-1, 1]$.

As stated before, the probe coefficient for the field $T$ at degree $\ell$ and direction $p$ is given by (see (11) and (10))

$$T_{\ell,p} = \sqrt{\frac{2\ell + 1}{4\pi}} \int_{S^2} T(x) P_{\ell}(x \cdot p) \, d\theta \, d\phi.$$  \hspace{1cm} (14)

The probe coefficient then has the symmetry shown in (12), following immediately from $P_{\ell}(\tau) = (-1)^\ell P_{\ell}(\tau)$, $\ell \in [-1, 1]$.

The probe coefficients are easily computed once the coefficients $a_{\ell,m}$ are known, since from (14) and (13) we have

$$T_{\ell,p} = \sqrt{\frac{4\pi}{2\ell + 1}} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_{\ell,m}(p).$$  \hspace{1cm} (15)

When $T$ is a random field, following Marinucci & Peccati (2011, Chapter 5), the field $T$ is said to be strongly isotropic if, for every $N \in \mathbb{N}$, every $x_1, \ldots, x_N \in S^2$ and every $p \in SO(3)$ (the group of rotations in $\mathbb{R}^3$) the multivariate random vectors $(T(x_1), \ldots, T(x_N))$ and $(T(p x_1), \ldots, T(p x_N))$ have the same law.

Furthermore, $T$ is said to be Gaussian if for all $N \in \mathbb{N}$ and for all $x_1, \ldots, x_N \in S^2$ the random vector $(T(x_1), \ldots, T(x_N))$ has a Gaussian distribution, i.e. $\Sigma_{j=1}^{N} \alpha_j T(x_j)$ is a normally distributed random variable for all $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, N$.

The expansion in (13) is understood to converge in the sense that

$$\lim_{L \to \infty} \mathbb{E} \left[ \int S^2 \left( T(y) - \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_{\ell,m}(y) \right)^2 \, d\sigma(y) \right] = 0,$$

and also for every fixed $x \in S^2$,

$$\lim_{L \to \infty} \mathbb{E} \left[ \left( T(x) - \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_{\ell,m}(x) \right)^2 \right] = 0.$$

For an isotropic zero-mean random field $T$, the covariance function, defined by $a_T(x \cdot y) := \mathbb{E} \left[ T(x) T(y) \right]$, is a zonal function on $S^2$, i.e. it depends only on the dot product $x \cdot y$. It is given in terms of the angular power spectrum $C_\ell$, since from (15) and (3) we have

$$\mathbb{E} \left[ T_{\ell,p} \right] = 0,

$$\mathbb{E} \left[ T_{\ell,p} T_{\ell',p'} \right] = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell'}^{\ell'} \mathbb{E} \left[ a_{\ell,m} a_{\ell',m'} \right] Y_{\ell,m}(p) Y_{\ell',m'}(p)$$

$$= \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell'}^{\ell'} C_\ell \delta_{\ell,\ell'} \delta_{m,m'} Y_{\ell,m}(p) Y_{\ell',m'}(p)$$

$$= \begin{cases} 0, & \text{if } \ell \neq \ell', \\ C_\ell, & \text{if } \ell = \ell', \end{cases}$$

where in the last step we used the addition theorem (5).

For a given map and for fixed $p$, we explore the correlations between the successive values of the scaled $T_{\ell,p}$ for varying $\ell$ by making use of the notion of autocorrelation. Autocorrelation plots (Box et al. 2015) are a commonly-used tool for studying correlation in a time series, in which case the autocorrelations are computed for different time lags. In the present case the autocorrelation is between scaled values of $T_{\ell,p}$ for a fixed direction $p$ and variable $\ell$, and the lags are with respect to values of $\ell$, not time. (Scaling is necessary so that each supposedly independent random variable has

Figure 11. AC discrepancy map, inpainted SMICA 2018, $N_{\text{Side}} = 1024$, $k_{\max} = 10$, max = 0.203 at $(i, b) = (261.21, -2.57)$

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the same variance.) If the scaled coefficients are iid standard
ormal random variables then such autocorrelations
should mostly lie within the 95.45% confidence intervals (i.e.
the blue boundary lines) for any \( p \) and all positive lags, as in
Figure 4. Autocorrelations which display distinct patterns,
as in Figure 5, are certainly not compatible with realisations of
iid random variables.

Let \( q = (q_t)_{t=2}^L, \quad L \in \mathbb{N}, \) be a finite real-valued se-
quence. (We start from \( t = 2 \) because the CMB maps have the
monopole and dipole subtracted.) Let

\[
\hat{q}_L := \frac{1}{L-1} \sum_{t=2}^{L} q_t
\]

be the empirical mean of \( q \). For \( k = 0, 1, \ldots, k_{\text{max}}, \) let

\[
\beta_k := \frac{1}{L-k} \sum_{t=2}^{L-k} (q_t - \hat{q}_L)(q_{t+k} - \hat{q}_L).
\]

(16)

The autocorrelation of \( q \) for lag \( k \) is

\[
\alpha_k := \frac{\beta_k}{\beta_0}
\]

(17)

See e.g., Box et al. (2015) and Hamilton (1994).

To apply this to the probe coefficients \( T_{f,p} \), we first normalise
them by dividing by the square root of the (empirical)
variance \( \hat{C}_t \) given by (8).

Given a probe direction \( p \in S^2 \) and \( L \in \mathbb{N} \) and the probe
coefficients \( \{T_{f,p}\}_{t=2}^{L} \) of \( T \), let

\[
\tilde{T}_{f,p} := T_{f,p}/\sqrt{\hat{C}_t}, \quad t = 2, \ldots, L,
\]

(19)

be the scaled probe coefficients. With this normalisation the
\( \tilde{T}_{f,p} \) should to a good approximation be instances of inde-
sible standard normal random variables, and their auto-
correlations should be uncorrelated.

To understand the close relation between the autocorre-
lations at \( p \) and \( -p \), as foreshadowed in the Introduction, we
note that if (as is very often the case) the mean defined by
(16) is small enough to be neglected, then from (17) using
(12) we see that \( \beta_k(-p) = (-1)^k \beta_k(p) \), from which it follows
using (18) that the autocorrelations at \( p \) and \( -p \) differ by
the same factor \((-1)^k\).

We have already seen in Figures 4 and 5 the graphs of
autocorrelations of scaled probe coefficients \( \tilde{T}_{f,p} \) for Comman-
der 2015, for two different directions \( p \). In order to find other
directions \( p \) in which the scaled probe coefficients have sig-
nificant departures from randomness, we introduce a mea-
sure of that departure, the autocorrelation discrepancy, or
AC discrepancy:

**Definition 1.** For a field \( T \) on the sphere, the autocorrela-
tion discrepancy with maximum lag \( k_{\text{max}} \) is

\[
D_{k_{\text{max}}}(p) := \sum_{k=1}^{k_{\text{max}}} \max \left| \alpha_k(\tilde{T}_{f,p}) - t_L, 0 \right|,
\]

where the autocorrelation \( \alpha_k \) is given by (18), the sequence
\( \tilde{T}_{f,p} \) is given by (19), and \( t_L \) is the threshold constant given by (9).

The function \( D_{k_{\text{max}}}(p) \) with maximum lag \( k_{\text{max}} \) and trun-
cation degree \( L \) and varying direction \( p \) is called for short the
AC discrepancy map for the field \( T \).

For iid data with finite variance (as would be expected
for a Gaussian CMB map), the autocorrelations \( \alpha_k \),
\( k > 0 \), are approximately iid Gaussian random
variables with mean 0 and a lag-dependent variance of approxi-
mately \((L-k-1)/(L-1)^2\) (Pfeifer & Deutsch 1981). Given a lag-
dependent threshold constant \( t_L = 2/\sqrt{L-1} \), the probability
of an autocorrelation of lag \( k \) falling within this range is
thus given by

\[
\mathbb{P}(|\alpha_k| \leq t_L) \approx \text{erf}\left(\sqrt{\frac{2\sqrt{L}}{L-1}} - k - 1 \right).
\]

Hence, for a Gaussian map, the probability of the AC dis-
crepancy with maximum lag \( k_{\text{max}} \) being zero in a given direc-
tion is

\[
\mathbb{P}(D_{k_{\text{max}}} = 0) \approx \prod_{k=1}^{k_{\text{max}}} \left(1 - \text{erf}\left(\sqrt{\frac{2\sqrt{L}}{L-1}} - k - 1 \right)\right).
\]

(20)

### 3 Empirical CMB Autocorrelations

In this section, we will apply the probe to the (mainly in-
painted) CMB maps produced by the Planck consortium
to obtain the AC discrepancy results shown in Figures 7, 8, 9, 10
and 11.

The temperature maps themselves are obtained by dif-
ferent principles:

- **Commander** (Eriksen et al. 2006, 2008) is a Bayesian
  parametric method that works in the map domain.

- **NILC** (Delabrouille et al. 2009) is an implementation of an internal linear combination (ILC) that works in the needle
domain.

- **SEVEM** (Fernández-Cobos et al. 2012) is an implementation of the template-cleaning approach to component separa-
tion that works in the map domain. Foreground templates are
constructed by differencing pairs of maps from the low-
and high-frequency channels.

- **SMICA** (Cardoso et al. 2008) is a non-parametric method
  that works in the spherical harmonic domain.

In the experiments we take \( L = 2500 \) and use the data at
the 50,311,648 healpix points \((N_{\text{side}} = 2048)\), as in the
previous Figure 3 for Commander 2015. The healpy
package (Górski et al. 2005) is used to calculate the Fourier coef-
ficients \( a_{f,m} \). In Figures 7, 8, 9, 10 and 11 we probe over
12,582,912 healpix points \( p \) (corresponding to \( N_{\text{side}} = 1024 \))
to produce the AC discrepancy maps \( D_k(p) \) with maximum
lag \( k_{\text{max}} = 10 \).

A reasonable conclusion from the five AC discrepancy
maps is that there are significant departures from the as-
sumed model of an isotropic Gaussian random field, espe-
ically in the masked regions of the CMB maps (assuming,
that is, that the anomalies in the AC discrepancy maps arise
from local departures in the masked regions, as seems a rea-
sonable supposition).

### 4 Random fields

In this section we study the properties of isotropic Gaussian
random fields on the sphere, so we can exclude the possi-
bility that the AC discrepancy seen in Figures 7 to 11 are accidental realisations of such a random process.

\[ \text{http://healpix.sourceforge.net} \]
First, in Figure 12 we show one realisation of an isotropic Gaussian random field with the best-fit angular power spectrum provided by Planck Collaboration et al. (2018b) at $N_{\text{side}} = 2048$ using the healpy package. For this realisation map, in Figure 13 we show its AC discrepancy map, and in Figure 14 we show the angular power spectrum. We observe in the AC discrepancy map a pattern of apparently uniformly distributed random small discrepancies over the whole sphere, as is expected.

To strengthen that conclusion we computed in total 10 independent realisations similar to Figure 12, and in every case found a similarly uninteresting AC discrepancy map.

A useful statistic to test the assumption of isotropic Gaussian random fields is the proportion of zero pixels in the AC discrepancy maps. As explained already, the probe coefficients $(\mathcal{P}_L)^{T_{\ell, p}}$ for a given direction $p$ should be realisations of independent Gaussian random variables with variance $C_T$. The scaled probe coefficients $(T_{\ell, p}/\sqrt{C_T})^{L_{\ell, p}^2}$ should therefore be independent samples from a standard normal distribution, at least to the extent that the empirical variances are close to the true variances. It is well known (see for example Brockwell & Davis (2016, p.16)) that their autocorrelations $(\alpha_{k, l} = 1, 10)$ should also be approximately independent for large $L$. More precisely, it is easily seen that the autocorrelations for different lags $k$ are themselves uncorrelated (since in every product of the mean-zero random variables $q_c, q_{c+k}, q_{c}, q_{c+k}$, there is always at least one of the subscripts different from all the others). From the uncorrelation and the Gaussianity of the underlying variables it is known (see for example Peccati & Tudor (2005)) that the autocorrelations are asymptotically independent. Given the large number $L - 1 = 2,499$ of variables in this case, it seems reasonable to assume independence of the autocorrelations to sufficient accuracy. As per Equation (20), the probability of the AC discrepancy with maximum lag $k_{\text{max}}$ = 10 being zero is $P(D_{10} = 0) = 0.6293$.

For comparison, using the 10 random field realisations, we find the fraction of the pixels having the value zero is $0.6301 \pm 0.0009$, which is consistent with the predicted value of 0.6293, bearing in mind that the independence is only approximate, and the fact that we are using empirical rather than true variances for the probe coefficients. The AC discrepancy maps of the Planck maps, on the other hand, have significantly lower fractions of zeros, ranging from 0.4569 (SEVEM 2018) to 0.5168 (Commander 2015), to 0.6038 (SHICA 2018).

We show the distribution of non-zero AC discrepancy values for two of the Planck AC discrepancy maps in Figure 15: the excess over the observation for an isotropic Gaussian random field is clearly evident.

We think we have established adequately that the five inpainted Planck maps are (with the exception of NILC) not pure realisations of isotropic Gaussian random fields. In the next section we develop a possible model for a dominant aspect of the Commander 2015 discrepancy map.

5 A MODEL FOR NON-ISOTROPIC RANDOM FIELD

The five AC discrepancy maps in Figures 7 to 11 for the inpainted Planck temperature data all appear to come from fields that may be random, and even possibly Gaussian, but are surely not isotropic, as is clear when they are compared with the AC discrepancy map derived from an instance of an isotropic Gaussian random field in the preceding section.
are firstly much larger: note in the case of shows the corresponding model field for the inpainted maps. The magnitudes in the 2015 map the anomalies, in the form of extended clouds in with the inpainted masked regions, but for the average over 2015 AC discrepancies in Figure 7, and by trial and error, with $\delta = 1$. Figure 16 shows the corresponding model field $T^{\text{mod}}$ at $N_{\text{side}} = 2048$, with $\eta = 0.2$ and $\eta^{\text{rand}}$ being the instance 1 of the Gaussian random field as shown in Figure 12. It seems fair to say that the non-isotropy of the model field (which reaches 20% at the point $x_c$) is not apparent to the eye.

In Figure 17, we show the AC discrepancy map of the model field, using the same colour map as for Commander 2015 in Figure 7. There seems to be qualitative agreement with the two blobs in the latter. The proportion of the pixels with the value zero is 0.5665, which as expected lies between the purely random value 0.6293 and the figure of 0.5168 for Commander 2015. Figure 18 shows the corresponding angular power spectrum. It is clear that the added mild anisotropy has no significant effect on the angular power spectrum.

6 THE NON-INPAINTED 2018 MAPS

To this point we have considered only the inpainted Planck temperature maps, and have suggested that they can be modelled as non-isotropic but still Gaussian random fields. In this section, in contrast, we consider the non-inpainted Planck 2018 temperature maps. At face value they have less interest to us because they generally exhibit obvious pollution near the galactic equatorial plane; for the case of non-inpainted SEVEM 2018 see Figure 19. Clearly such equatorial pollution is not consistent with the field being a realisation of a Gaussian random field. Nevertheless, there is value in applying our probe in such cases because (as we see in Figure 20 in the case of SEVEM 2018) the AC discrepancy maps have a very different character from that we have seen in Figures 7–11 for the inpainted maps. The magnitudes in the AC discrepancy map Figure 20 are firstly much larger: note that the colour map is 60 times larger in scale than those in earlier maps, and even then is saturated. Secondly, the large values occur globally, especially on the great circle through the galactic centre and the poles. Thirdly, in contrast to the apparent white noise character of the AC discrepancies for the inpainted maps, the AC discrepancies now seem to be continuous.

In the following section we present a model field with the same principal characteristics as Figure 20, yet which consists simply of an isotropic Gaussian random field plus

\[
\begin{align*}
\mathbb{E}[T^{\text{mod}}(x)T^{\text{mod}}(x')] &= \left[1 + \eta T^{\text{wend}}(x)\right] \left[1 + \eta T^{\text{wend}}(x')\right] \mathbb{E}[T^{\text{rand}}(x)T^{\text{rand}}(x')] \\
&= \left[1 + \eta T^{\text{wend}}(x)\right] \left[1 + \eta T^{\text{wend}}(x')\right] \sum_{\ell \geq 2} C_\ell \frac{2\ell + 1}{4\ell} P_\ell(x \cdot x'),
\end{align*}
\]

which according to Section 2 makes $T^{\text{mod}}$ a Gaussian random field. The covariance of the model field is

\[
\begin{align*}
\mathbb{E}[T^{\text{mod}}(x)^2] &= \left[1 + \eta T^{\text{wend}}(x)\right]^2 \sum_{\ell \geq 2} C_\ell \frac{2\ell + 1}{4\ell}
\end{align*}
\]

which now depends on $x$.

In the following experiment we take the Wendland function $T^{\text{wend}}$ to be centred at the point $x_c$ with galactic coordinates $(l, b) = (90, 30)$, which corresponds roughly to the centre of the upper left blob in Figure 7, and by trial and error, with $\delta = 1$. Figure 16 shows the corresponding model field $T^{\text{mod}}$ at $N_{\text{side}} = 2048$, with $\eta = 0.2$ and $T^{\text{rand}}$ being the instance 1 of the Gaussian random field as shown in Figure 12. It seems fair to say that the non-isotropy of the model field (which reaches 20% at the point $x_c$) is not apparent to the eye.

For the four 2018 maps the faults appear to be associated with the inpainted masked regions, but for the Commander 2015 map the anomalies, in the form of extended clouds in the AC discrepancy map in Figure 7, are well away from the masked region.

Because the temperature field in the unmasked region is more reliable than that in the masked region, we concentrate in this section on an attempt to model in an approximate way the main “blobs” (at two antipodes) of the Commander 2015 AC discrepancies in Figure 7. Our model field takes the form

\[
T^{\text{mod}} := \left(1 + \eta T^{\text{wend}}\right) T^{\text{rand}},
\]

where $T^{\text{rand}}$ is an isotropic Gaussian random field as in the preceding section, $\eta$ is a small positive parameter, and $T^{\text{wend}}$ is a deterministic field that is rotationally symmetric about a point $x_c \in \mathbb{S}^2$. Specifically, we take the function $T^{\text{wend}}$ to be one of a family of locally supported “Wendland functions” (Wendland 1995; Le Gia et al. 2010) defined by

\[
T^{\text{wend}}(x) := \varphi(||x - x_c||)/\delta) + \varphi(t) = \left(1 - t^2\right)/(4t + 1),
\]

where $\delta > 0$ is a variable scale factor. We note that the support of $T^{\text{wend}}(x)$ is a spherical cap of angular radius $\delta$.

From (21) the model field is a mean-zero field. It is Gaussian because for all $N \geq 0$ and all $x_1, \ldots, x_N \in \mathbb{S}^2$ and all $a_j \in \mathbb{R}$ we have

\[
\sum_{j=1}^N a_j T^{\text{mod}}(x_j) = \sum_{j=1}^N b_j T^{\text{rand}}(x_j) \quad \text{with} \quad b_j := a_j \left[1 + \eta T^{\text{wend}}(x_j)\right],
\]
A New Probe of Gaussianity and Isotropy

Figure 16. Model field \((1 + \eta T_{\text{wend}}) T_{\text{rand}}\), where \(T_{\text{wend}}\) is the Wendland function given by (22) with scale \(\delta = 1\) and centre at \((l, b) = (90, 30)\), and the factor \(\eta = 0.2\) for the map \(T_{\text{rand}}\) which is instance 1 of Gaussian random field generated from the best-fit angular power spectrum, \(N_{\text{Side}} = 2048\).

Figure 17. AC discrepancy map for the model field \(T_{\text{mod}}\) in Figure 16, \(N_{\text{Side}} = 1024\), \(k_{\text{max}} = 10\), max = 0.189 at \((l, b) = (282.96, -28.72)\).

Figure 18. Angular power spectrum, the model field in Figure 16.

Figure 19. Non-inpainted CMB map of SEVEM 2018.

Figure 20. AC discrepancy map for non-inpainted CMB map of SEVEM 2018, \(N_{\text{Side}} = 1024\), \(k_{\text{max}} = 10\), max = 9.37 at \((l, b) = (180.62, 0.04)\).

Figure 21. Plot of the function \(h\) defined in (26).

7 A MODEL FOR NON-INPAINTED SEVEM 2018

Our model in this case takes the form

\[ T_{\text{total}}(x) = T_{\text{rand}}(x) + \gamma T_{\text{need}}(x), \]

where \(\gamma\) is an adjustable parameter, \(T_{\text{rand}}\) is a mean-zero, strongly isotropic Gaussian random field with angular power spectrum \((C_\ell)_{\ell \geq 2}\) as in Section 4, and \(T_{\text{need}}\) is a narrow “needlet-like” spherical polynomial of (high) degree \(L_0\), of
the general form: for \( x_0, x \in S^2 \),

\[
T_{\text{need}}(x) := \sum_{\ell=0}^{L_0} H_{\ell} \sum_{m=-\ell}^{\ell} G_{\ell,m} Y_{\ell,m}(\rho^{-1} x) Y_{\ell,m}(\rho^{-1} x_0),
\]

where \( \rho \) is an arbitrary rotation matrix in \( SO(3) \), and \( H_{\ell} \) and \( G_{\ell,m} \) are “filters” satisfying

\[
0 \leq H_{\ell} \leq 1, \quad 0 \leq G_{\ell,m} = G_{\ell,-m} \leq 1 \quad \text{for } |m| \leq \ell \leq L_0.
\]

We shall say that the function defined by (24) is a needlet-like field centred at \( x_0 \). In our applications, \( x_0 \) is the galactic centre with \((l, b) = (0, 0)\).

The simplest case is that in which \( G_{\ell,m} = 1 \) for all \( \ell, m \). In that case, the addition theorem (5) reduces the right-hand side of (24) to

\[
\frac{1}{4\pi} \sum_{\ell=2}^{L_0} \left( 2(\ell + 1) H_{\ell} P_{\ell}(\rho^{-1} x_0) \right)
= \frac{1}{4\pi} \sum_{\ell=2}^{L_0} \left( 2(\ell + 1) H_{\ell} P_{\ell}(x : x_0) \right),
\]

since the dot product is invariant under rotation. Thus in the case \( G_{\ell,m} = 1 \) for all \( \ell, m \) and \( H_{\ell} \) is a smooth function with support in \((0, 1)\). The smoothness and the compactness of the support ensures that the needlet decays rapidly for \( x \) away from \( x_0 \). In the present work we take \( H_{\ell} \) to be of the form (25), with \( L_0 = \frac{3}{2} L \) and

\[
H_{\ell} = h \left( \frac{\ell}{L_0} \right),
\]

where \( h \) is a smooth function with support in \([0, 1)\). The smoothness and the compactness of the support ensures that the needlet decays rapidly for \( x \) away from \( x_0 \). In the present work we take \( H_{\ell} \) to be of the form (25), with \( L_0 = \frac{3}{2} L \) and

\[
h(t) = \begin{cases} 
2(6t - 5) & \text{for } t \in [1/6, 1/3], \\
1 & \text{for } t \in [1/3, 5/6], \\
2(6t - 5) & \text{for } t \in [5/6, 1], \\
0 & \text{otherwise},
\end{cases}
\]

with \( p(t) = 924(1-t)^6 - 4752(1-t)^7 + 10395(1-t)^8 - 12320(1-t)^9 + 8316(1-t)^{10} - 3024(1-t)^{11} + 462(1-t)^{12}. \) See Narcowich et al. (2006) and Le Gia et al. (2017). We note that \( h \), illustrated in Figure 21, and its first five derivatives, are continuous since \( p^{(k)}(0) = 0, k = 1, \ldots, 6 \) and \( p^{(7)}(1) = 0, k = 1, \ldots, 5 \), and \( p(0) = 1, p(1) = 0 \).

From (24), the Fourier coefficient \( a_{\ell,m} \) of \( T_{\text{need}} \) with \( \rho = 1 \) and general \( G_{\ell,m} \) is given by

\[
a_{\ell,m} = H_{\ell} G_{\ell,m} Y_{\ell,m}(x_0).
\]

From formula (15), when \( G_{\ell,m} = 1 \), the corresponding probe coefficient \( T_{\ell,p}^{\text{need}} \) for any direction \( p \in S^2 \) is given by

\[
T_{\ell,p}^{\text{need}} := \sqrt{\frac{4\pi}{2\ell + 1}} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_{\ell,m}(p) 
= \sqrt{\frac{4\pi}{2\ell + 1}} \sum_{m=-\ell}^{\ell} H_{\ell} Y_{\ell,m}(x_0) Y_{\ell,m}(p) 
= \sqrt{\frac{2\ell + 1}{4\pi}} H_{\ell} P_{\ell}(x_0 : p),
\]

where in the last line, we again used the addition theorem (5).

To determine the autocorrelation of the probe coefficients of the needlet they must first be normalised. We do this by dividing the non-zero probe coefficients by \( \sqrt{C_{\ell}} \), where

\[
C_{\ell} := \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |a_{\ell,m}|^2 
= \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} H_{\ell}^2 |Y_{\ell,m}(x_0)|^2 = \frac{1}{4\pi} H_{\ell}^2. \tag{28}
\]

Thus our input to the autocorrelation algorithm is in this case

\[
q_{\ell} := \frac{T_{\ell,p}^{\text{need}}}{\sqrt{C_{\ell}}} = \sqrt{2\ell + 1} P_{\ell}(x_0 : p), \quad L_0/6 \leq \ell \leq L. \tag{29}
\]

We now use the (Laplace) asymptotic expression for the Legendre polynomial (Szegö 1975, p.194): for \( \lambda \in (0, \pi) \),

\[
P_{\ell}(\cos \lambda) = \sqrt{2\ell + 1} (\pi \sin \lambda)^{-1/2} \cos \left( \ell \left( \frac{\pi}{4} \right) - \lambda - \pi/4 \right) + O\left( \ell^{-3/2} \right)
\]

Then from (16) we have, with \( \cos \lambda = p : x_0 \in (0, 1) \)

\[
q_{\ell}^{\text{need}} := \frac{1}{L - 1} \sum_{\ell=L_0/6}^{L} q_{\ell},
\]

with an implied constant that depends on \( \lambda \). To a sufficient approximation we may therefore neglect \( q_{\ell}^{\text{need}} \) compared to \( q_{\ell}^{\text{need}, \text{need}} \), and so obtain, using (17) and (18)

\[
a_k = \frac{\beta_k}{\beta_0} \approx \frac{1}{L - 1} \sum_{\ell=L_0/6}^{L-k} q_{\ell}^{\text{need}, \text{need}}<
\]

where

\[
\beta_k \approx \frac{1}{L - 1} \sum_{\ell=L_0/6}^{L-k} q_{\ell}^{\text{need}, \text{need}}<
\]

\[
= \frac{4}{\sqrt{\pi} \sin \lambda \left( L - 1 \right)} \sum_{\ell=L_0/6}^{L-k} \cos((\ell + 1/2)\lambda - \pi/4) \times \cos((\ell + k + 1/2)\lambda - \pi/4)
\]

\[
= \frac{2}{\sqrt{\pi} \sin \lambda \left( L - 1 \right)} \times \sum_{\ell=L_0/6}^{L-k} \left[ \cos(k\lambda) + \cos((2\ell + k + 1)\lambda - \pi/2) \right]
\]

\[
= \frac{2}{(L - 1)\sqrt{\pi} \sin \lambda \left( L - 1 \right)} \times (L - 2) \cos(k\lambda) + O(1)
\]

\[
= \frac{2(L - 2)}{(L - 1)\sqrt{\pi} \sin \lambda \left( L - 1 \right)} \cos(k\lambda),
\]

from which it follows that for small \( k \) and large \( L \),

\[
a_k = \frac{\beta_k}{\beta_0} \approx \cos(k\lambda),
\]

\[
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\]
where $\cos \lambda = \mathbf{p} \cdot \mathbf{x}_0$. Thus in the case of an axially symmetric needlelet the autocorrelation has a distinctive oscillation as the lag $k$ varies.

But clearly the case $G_{\ell,m} = 1$, with its resulting axial symmetry, is not suitable for the present scenario, so we now consider other choices for $G_{\ell,m}$, initially with rotation $\rho$ set equal to the identity matrix $I$. To gain insight into how to choose $G_{\ell,m}$, it is useful to remember that the spherical harmonic $Y_{\ell,0}$ is concentrated at the two poles, see (6), whereas, as is well known, $Y_{\ell,m}$ for $|m| \approx \ell$ is concentrated near the equator. With this in mind, we define

$$G_{\ell,m} := \begin{cases} |m| - \nu \ell, & |m| \geq \nu \ell, \ell \geq 0, \\ 0, & \text{otherwise}, \end{cases} \quad (31)$$

for a factor $\nu \in (0, 1]$. This choice eliminates all values of $|m|$ smaller than $\nu \ell$, and has the value 1 for $|m| = \ell$ and in between interpolates linearly. The result is a field that is concentrated near the galactic equator. For our present probes (where we need a field concentrated on the plane $y = 0$), we need also a rotation $\rho$ of angle $\pi/2$ about the $x$ axis to change the phase of concentration from the plane $z = 0$ to $y = 0$.

In summary, our model of the non-random contribution to the field takes the form (24), together with (31) and the rotation $\rho$ by $\pi/2$ as described in the last paragraph. While we have not proved that our asymmetric needlelet has the same approximate scaled autocorrelation as in (30), it does appear to be the case.

We need to do Fourier analysis on (24), so that we construct the corresponding probe coefficients and compute the autocorrelations and AC discrepancies. To handle the rotation in (24), we could use the machinery of Wigner D-matrices, but a simpler strategy is available, namely to carry out the computations of probe coefficients and autocorrelation in an unrotated frame, i.e. with $\rho = I$, and then simply rotate the AC discrepancy map.

On setting $\rho = I$ in (24) we then find the Fourier coefficients of $T_{\text{need}}$:

$$a_{\ell,m}^{\text{need}} = \begin{cases} H_{\ell} G_{\ell,m} Y_{\ell,m}(\mathbf{x}_0) & \text{if } |m| \geq \nu \ell, \ell \geq 0, \\ 0, & \text{otherwise}. \end{cases} \quad (32)$$

The probe coefficient $T_{\ell,p}^{\text{need}}$ for any direction $\mathbf{p} \in \mathbb{S}^2$ is then given by

$$T_{\ell,p}^{\text{need}} = \begin{cases} \sqrt{\frac{4\pi}{2\ell + 1}} \sum_{|m| \geq \nu \ell} a_{\ell,m}^{\text{need}} Y_{\ell,m}(\mathbf{p}) & \text{if } |m| \geq \nu \ell, \ell \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

To compute the autocorrelation of the combined field, we first generate a realisation of a random field and then the Fourier coefficients as in Section 4. Meanwhile, the Fourier coefficients of the needlelet-like field is given by (32). By (23) and (13), we then obtain the Fourier coefficients of the combined field $T_{\ell,p}^{\text{total}}$ as

$$a_{\ell,m}^{\text{total}} = a_{\ell,m}^{\text{rand}} + \gamma a_{\ell,m}^{\text{need}}. \quad (33)$$

Using (14) and (33),

$$T_{\ell,p}^{\text{total}} = T_{\ell,p}^{\text{rand}} + \gamma T_{\ell,p}^{\text{need}} = \sqrt{\frac{4\pi}{2\ell + 1}} \sum_{m=-\ell}^{\ell} (a_{\ell,m}^{\text{rand}} + \gamma a_{\ell,m}^{\text{need}}) Y_{\ell,m}(\mathbf{p}),$$

and then $d_{\ell,\gamma}^{\text{total}} = T_{\ell}^{\text{total}}/\sqrt{\ell}$ for $\ell = 2, 3, \ldots, L$, where $L \leq L_0$, and where the angular power spectrum is

$$C_{\ell}^{\text{total}} = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} (a_{\ell,m}^{\text{rand}} + \gamma a_{\ell,m}^{\text{need}})^2.$$

We now present the numerical results for this model with the following parameters determined by trial and error: we take $\gamma = -1.1, \nu = 0.8$ and $H_\nu$ as in equations (25) and (26) for all $\ell, m$. In Figure 22, we show the additive model field $T_{\ell,\nu}^{\text{total}}$ with these parameters, using instance 1 of the Gaussian random field illustrated in Figure 12. Note that the needlelet-like structure at the galactic centre is clearly visible.

We give in Figure 23 the AC discrepancy map for the additive model field in Figure 22. $N_{\text{Side}} = 1024, k_{\text{max}} = 10, \max = 9.55$ at $(l, b) = (180.00, -0.04)$.
galactic centre. Note that the scales in the two colour maps are the same. The maximum value of the model AC discrepancy in Figure 23 is 9.55, compared to the maximum AC discrepancy 9.37 of SEVEM 2018 in Figure 20.

In a more detailed comparison of non-inpainted SEVEM 2018 and the model, we examined the autocorrelations on the great circle through the galactic centre and the poles. There turned out to be strikingly good agreement with the oscillatory behaviour predicted by (30). (The prediction is admittedly for an isotropic needlet, but is empirically present also for our directional needlet.)

As an example of the oscillatory behaviour, in Figure 24 we show the observed scaled autocorrelations for non-inpainted SEVEM 2018 at the North Pole. For comparison, we show in Figure 25 the scaled autocorrelation at $p = n$, the North Pole, which exhibits the same oscillation. In more details, for the case $p = n$ in Figure 25, for which $\lambda = \pi/2$, the autocorrelations for odd lags $k$ vanish, except for the expected small perturbations. And for even $k$ the values alternate, exactly as expected for the function $\cos(k\pi/2)$. For small lags $k$ one observes almost the same dependence on $k$ in the case of SEVEM in Figure 24. The agreement is not as good for the amplitudes of the oscillations, but is perhaps fair, given the simplicity of the model.

Similarly, the non-inpainted SEVEM 2018 autocorrelations showed the oscillatory behaviour everywhere on the great circle through the galactic centre and the poles.

Finally, given that the model has a pronounced needlet-like structure at the galactic centre, it is natural to ask if a similar structure is present in the SEVEM 2018 temperature field. The answer is apparently yes: the maximum value of the SEVEM 2018 temperature field has the enormous value of 18161.53 ($\mu K$), occurring close to the galactic centre, at $(l, b) = (0.51, -0.04)$.

8 CONCLUSIONS

In this paper we introduce a ‘probe’ to assess the randomness of purported isotropic Gaussian random fields, and use the probe to test the hypothesis that the all-sky CMB temperature anisotropy maps from the Planck collaboration are realisations of an isotropic Gaussian field. The probe coefficients for a direction $p$ are just the coefficients $a_{\ell p}$ of the field if the $z$-axis is rotated to the direction $p$. Under the assumption that the field is isotropic and Gaussian the probe coefficients for a given direction $p$ should be independent Gaussian random variables.

Comparing with simulated statistically isotropic Gaussian full-sky maps, we find clear evidence for a departure from this assumption for all inpainted Planck maps, with the NILC 2018 map closest to the Gaussian expectation. The deviations can be made visible by a global computation of the “AC discrepancy”. Interestingly, we find the excess to lie mostly in the masked region of the maps rather than in the parts dominated by the CMB signal, but to be different for each map. This could reflect the variations in the inpainting processes used for the different maps, as well as varying degrees of success in replicating the statistical properties of the unmasked regions and thereby inadvertently introducing a violation of statistical isotropy in the full maps. A notable exception is the Commander 2015 map, which displays prominent localised anomalies well away from the masked region. We show that a similar pattern of anomalies can be obtained from a model random field which is Gaussian but not completely isotropic.

We also applied the probe to the non-inpainted Planck 2018 maps, obtaining AC discrepancies of a very different kind: instead of the apparently localised “white noise” appearance of the AC discrepancies, the AC discrepancies now appear to be continuous and (at least on the evidence of SEVEM 2018), globally dispersed, with a similar AC discrepancy obtained in this case by a model in which in addition to the isotropic random field we add a single narrow needlet-like structure located at the galactic centre. An intuitive explanation of the different behaviour is that the needlet-like structure has a very slowly decaying Fourier spectrum, making the high degree Fourier coefficients of the added deterministic field globally dominant in the probe coefficients.

We emphasise that even for perfectly Gaussian primordial perturbations, the observed temperature fluctuations of the CMB are not expected to be Gaussian, since secondary anisotropies generated by non-linear physics intro-
duce a small degree of non-Gaussianity, which would manifest itself as an evenly distributed excess in the AC discrepancy map. It will be interesting to investigate to what extent these effects contribute to the observed AC discrepancy excess. A quantitative analysis of this issue will be the topic of future work by the authors.

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