Carleman estimate and an inverse source problem for the Kelvin–Voigt model for viscoelasticity

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Abstract

We consider the Kelvin–Voigt model for the viscoelasticity, and prove a Carleman estimate for functions without compact supports. Then we apply this Carleman estimate to prove the Lipschitz stability in determining a spatial varying function in an external source term of Kelvin–Voigt model by a single measurement.

Keywords: Carleman estimate, inverse source problem, Kelvin–Voigt model

1. Introduction and main results

Let $T$ be a positive constant, $x' = (x_1, ..., x_n) \in \mathbb{R}^n$, and $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $\partial \Omega \in C^\infty$, let $\bar{v} = \bar{v}(x')$ be the unit outward normal vector at $x'$ to $\partial \Omega$. Here we understand $x' \in \mathbb{R}^n$ as the spatial variable and $x_0$ as the time variable. We set $x = (x_0, x') = (x_0, x_1, ..., x_n)$, and $Q := (-T, T) \times \Omega$, $\Sigma := (-T, T) \times \partial \Omega$.

Here and henceforth $i = \sqrt{-1}$ and $^T$ denotes the transposes of vectors and matrices under consideration, and $D = (D_0, D')$, $D_0 = \frac{1}{i} \partial_{x_0}$, $D' = (\frac{1}{i} \partial_{x_1}, ..., \frac{1}{i} \partial_{x_n})$, $\nabla' = (\partial_{x_1}, ..., \partial_{x_n})$, $\nabla = (\partial_{x_0}, \nabla')$. 

In the cylinder domain $Q$, we consider the Kelvin–Voigt model:

$$
\rho \ddot{w} = L_{\lambda,\mu}(x, D') \partial_{\lambda,\mu} w + L_{\lambda,\mu}(x, D') w + F \quad \text{in} \ Q,
$$

(1.1)

$$
\left. w \right|_{\Sigma} = 0,
$$

(1.2)

$$
\left. w(0, x') \right| = 0, \quad x' \in \Omega.
$$

(1.3)

Here $w(x) = (w_1(x), \ldots, w_n(x))^T$ is the displacement and $F(x) = (F_1(x), \ldots, F_n(x))^T$ is an external force. For Lamé coefficients $\lambda(x)$ and $\mu(x)$, the partial differential operator $L_{\lambda,\mu}(x, D')$ is defined by

$$
L_{\lambda,\mu}(x, D') w = \mu(x) \Delta w + (\mu(x) + \lambda(x)) \nabla' \text{div} w + (\text{div} w) \nabla' \lambda + (\nabla' w + (\nabla' w)^T) \nabla' \mu,
$$

(1.4)

and the operator $L_{\lambda,\mu}(x, D')$ is defined similarly.

The equation (1.1) is called the Kelvin–Voigt model and is one model equation for the viscoelasticity. The viscoelasticity indicates a mixed physical property of the viscosity and the elasticity, and is frequently observed in human tissues. Thus a mathematical model equation for the viscoelasticity is important for example for the medical diagnosis. Here we do not provide a comprehensive list of references on medical applications and, as for related works we refer for example, to Catheline, Gennisson, Delon, Fink, Sinkus, Abouelkaram and Cutillo [7], de Buhan [10], Royston, Mansy and Sandler [39], Sinkus, Tanter, Xydeas, Catheline, Bercoff and Fink [40] and the references therein. As for theoretical treatments of the viscoelasticity, the readers can consult Lakes [33], and Renardi, Hrusa and Noel [37].

On the basis of model (1.1), we can state that in the diagnosis, the main task is to detect some anomalies in the spatially varying coefficients in (1.1) which may indicate some organizational abnormalities such as tumors. Thus inverse coefficient problems of determining $\lambda(x')$, $\mu(x')$, $\tilde{\lambda}(x')$, $\tilde{\mu}(x')$ by available boundary data, are required from medical points of view. Such inverse coefficient problems can be reduced to a system given by (1.1). In this article, as a boundary data for the inverse problem we consider derivatives of $w_{\mu}$, $\text{div} u$ and $u$ on $(0, T) \times \tilde{\Gamma}$ where $\tilde{\Gamma}$ is an arbitrarily chosen subboundary of $\partial \Omega$. Here $w_{\mu}$ is defined later in section 1 and is the rotation of a vector-valued function $u$ for the three dimensional case $n = 3$.

As more formulation of the inverse problem, let $w_k$, $k = 1, 2$, satisfy (1.1) with $F = 0$ and the coefficients $\rho_2, \lambda_k, \mu_k, \tilde{\lambda}_k, \tilde{\mu}_k$. Setting $u = w_1 - w_2$, $f = \rho_1 - \rho_2$, $\tilde{g} = \tilde{\lambda}_1 - \tilde{\lambda}_2$, $\tilde{h} = \tilde{\mu}_1 - \tilde{\mu}_2$ and $R = -\partial_{\lambda,\mu} w_2$, we have

$$
\rho_1 \partial_{\lambda,\mu}^2 u = L_{\lambda_1,\mu_1}(x, D') \partial_{\lambda,\mu} u + L_{\lambda_2,\mu_2}(x, D') u + R(x) f(x') + S(g, h, \nabla g, \nabla h, g, h, \nabla g, \nabla h),
$$

where $S$ is some linear combination of $\nabla g, \nabla h, \nabla^2 g, \nabla^2 h$, $j = 0, 1$. Then the primary theoretical subject is the stability: estimate some or all of $\rho_1 - \rho_2$, $\lambda_1 - \lambda_2$, $\mu_1 - \mu_2$, $\tilde{\lambda}_1 - \tilde{\lambda}_2$ and $\tilde{\mu}_1 - \tilde{\mu}_2$ by extra boundary data of $w_{\mu}(t, x)$, $\text{div} w_1 - w_2$ and $(w_1 - w_2)$ on $(0, T) \times \tilde{\Gamma}$ with a finite number of suitable chosen initial values.

In this article, as a simple case, assuming that all the coefficients are known except for $\rho_1, \rho_2$, we discuss system (1.1) where $F(x) = R(x) f(x')$. This is the first step to the theoretical study for the inverse coefficient problems related to the medical diagnosis, and we mainly discuss an inverse problem of determining an $x'$-depending factor $f(x')$ in (1.1) by boundary data of $w_{\mu}$, $\text{div} u$ and $u$ on $(0, T) \times \tilde{\Gamma}$ with arbitrarily chosen subboundary $\tilde{\Gamma} \subset \partial \Omega$.

This is called an inverse source problem. Theorem 1.2 asserts the Lipschitz stability for our
inverse source problem in the case where in measuring boundary data, we adopt Sobolev norms of them on \((0, T) \times \Gamma\) of high order. On the other hand, in corollary 1.1, assuming some \textit{a priori} bound for the solutions \(u\) in a Sobolev space on \((0, T) \times \Gamma\) of higher order, we can adopt the \(L^2\)-norm of the normal derivative \(\partial_{\vec{n}} u\) on \((0, T) \times \Gamma\) to obtain a H"older stability estimate.

In both cases of theorem 1.2 and corollary 1.1, our boundary data for the inverse problem are not directly measured data by equipments for medical practice, but for establishing the uniqueness or the stability within spatial varying coefficients, such non-discrete data are indispensable and theoretically consistent. Moreover we expect to approximate our data by interpolating real discrete data in practice, and our main result theorem 1.2 on the inverse problem can qualify such a data-interpolation procedure, that is, we can know rates of the improvements of the precision of approximating solutions when noises in measurements are small and discrete errors become more accurate. We state this issue more after the statement of theorem 1.2.

The uniqueness and the stability are not only the main theoretical issues for our inverse problem, but also useful in applying the Tikhonov regularization. In particular, see Cheng and Yamamoto [9] which proves that the stability such as in theorem 1.2 guarantees the same rate of the stability for the convergence of Tikhonov regularized approximations under adequate choice of regularizing parameters.

For the uniqueness and the stability for the inverse problem, Bukhgeim and Klibanov [6] establishes one fundamental methodology on the basis of Carleman estimate. Their methodology assumes a relevant Carleman estimate. Therefore for our inverse problem, first we have to establish a relevant Carleman estimate for system (1.1).

Therefore the main achievements of this article is to establish a Carleman estimate (theorem 1.1) for (1.1) and an application to an inverse source problem of determining a spatially varying factor \(f(x')\) (theorem 1.2 and corollary 1.1).

Also as is seen later, the principal part of (1.1) is a strongly coupled parabolic Lamé system \(\rho \partial_{\alpha_0} u - L_{\lambda, \mu}(x, D')\), which causes a serious difficulty for proving the Carleman estimate for (1.1). On the other hand, for the inverse coefficient problems, our first main result theorem 1.1 is the fundamental Carleman estimate, but after application of the key Carleman estimate, we still need non-trivial arguments, and so we postpone the inverse coefficient problems for the Kelvin–Voigt model to our forthcoming work.

Now we reduce (1.1) to an integro-parabolic system. Setting
\[
\mathbf{u} = \partial_{\alpha_0} \mathbf{w},
\]
from (1.1)–(1.3), we readily obtain
\[
P(x, D)\mathbf{u} \equiv \rho \partial_{\alpha_0} \mathbf{u} - L_{\lambda, \mu}(x, D')\mathbf{u} - \int_0^\infty L_{\lambda, \mu}(x, \tilde{x}_0, D')\mathbf{u}(\tilde{x}_0, x')d\tilde{x}_0 = \mathbf{F} \quad \text{in } Q,
\]
\[
\mathbf{u}|_\Sigma = 0.
\]
(1.6) (1.7)

Here and henceforth, for the sake of more generality, assuming in (1.6) that \(\tilde{\lambda}, \tilde{\mu}\) depend on \(x = (x_0, x')\in Q\) and \(\tilde{x}_0 \in (-T, T)\), we set
\[
L_{\lambda, \mu}(x, \tilde{x}_0, D')\mathbf{u} = \tilde{\mu}(x, \tilde{x}_0)\Delta \mathbf{u} + (\tilde{\mu}(x, \tilde{x}_0) + \tilde{\lambda}(x, \tilde{x}_0))\nabla' \text{div } \mathbf{u}
\]
\[
+ (\text{div } \mathbf{u})\nabla' \tilde{\lambda} + (\nabla' \mathbf{u} + (\nabla' \mathbf{u})^T)\nabla' \tilde{\mu}, \quad (x, \tilde{x}_0) \in Q \times [-T, T].
\]

We mainly consider system (1.6) with boundary condition (1.7). Henceforth the coefficients \(\rho, \lambda, \mu, \tilde{\lambda}, \tilde{\mu}\) are assumed to satisfy
\[
\rho, \lambda, \mu \in C^2(\overline{Q}), \quad \rho(x) > 0, \mu(x) > 0, \lambda(x) + \mu(x) > 0 \quad \text{for } x \in \overline{Q}
\]
(1.8)
and
\[ \tilde{\lambda}, \tilde{\mu} \in C^2([-T, T] \times \overline{\Omega} \times [-T, T]). \] (1.9)

More precisely, the main purposes of this article are to establish

1. a Carleman estimate for functions without compact supports;
2. the Lipschitz stability in an inverse source problem of determining spatially varying factor $f(x')$ of $F$, where we can interpret $F$ also as a force acted to the system after choosing suitable physical units for (1.6).

A Carleman estimate is an $L^2$-weighted estimate of solution $u$ to (1.6) which holds uniformly in large parameter. Carleman estimates have been well studied for single equations (e.g. Hörmander [13], Isakov [27]). A Carleman estimate yields several important results such as the unique continuation, the energy estimate called an observability inequality and the Hörmander [13], Isakov [27]). A Carleman estimate yields several important results such as the unique continuation, the energy estimate called an observability inequality and the stability in inverse problems. However for systems whose principal part is coupled, even for isotropic Lamé system (that is, (1.6) with $\tilde{\lambda} = \tilde{\mu} \equiv 0$), the Carleman estimate is difficult to be proved for functions $u$ whose supports are not compact in $Q$. In this article, we first establish theorem 1.1, a Carleman estimate for $u$ satisfying (1.6) and having no compact support.

In order to prove the stability global in $\Omega$ for the inverse problem with lateral data limited on some subboundary, we need a Carleman estimate for functions without compact supports. Otherwise we have to observe the extra data on the whole lateral subboundary $(0, T) \times \partial\Omega$.

In establishing a Carleman estimate for our system (1.6) for non-compactly supported $u$, we should emphasize the two main difficulties:

- The principal part $\rho \partial_{x_0} - L_{\lambda, \mu}(x, D')$ is strongly coupled.
- The Lamé operator $L_{\lambda, \mu}(x, \tilde{\lambda}, \tilde{\mu}, D')$ appears as an integral.

As for Carleman estimates for functions without compact supports or applications to inverse problems for the Lamé system with $L_{\lambda, \mu}(x, D') = 0$, we refer to Bellassoued, Imanuvilov and Yamamoto [3], Bellassoued and Yamamoto [4], Imanuvilov, Isakov and Yamamoto [15], Imanuvilov and Yamamoto [21–24]. In this article, we modify the arguments in those papers and establish a Carleman estimate for system (1.6) in the case where $u$ does not have compact support. Then we apply the Carleman estimate to an inverse source problem by modifying the method in Imanuvilov and Yamamoto [17–19] and Beilina, Cristofol, Li and M. Yamamoto [1] discussing scalar hyperbolic equations. As for the methodology for applying Carleman estimates to inverse problems, we refer to a pioneering paper Bukhgeim and Klibanov [6], and also Beilina and Klibanov [2], Bellassoued and Yamamoto [5], Klibanov [30, 31], Klibanov and Timonov [32].

There are works on inverse problems related to other model equations of the viscoelasticity and we refer to Cavaterra, Lorenzi and Yamamoto [8], de Buhan [10], de Buhan and Osses [11], Grasselli [12], Imanuvilov and Yamamoto [25], Janno [28], Janno and von Wolfersdorf [29], Lorenzi, Messina and Romanov [34], Lorenzi and Romanov [35], Loreti, Sforza and Yamamoto [36], Romanov and Yamamoto [38], von Wolfersdorf [43].

For the statement of the Carleman estimate for (1.6), we introduce notations and a definition, according to the classical source book Hörmander [13]. Throughout the article, let $\tilde{\xi}$ denote the complex conjugate of $\xi \in \mathbb{C}$. We set $\langle a, b \rangle = \sum_{k=0}^{n} a_k \overline{b_k}$ for $a = (a_0, \ldots, a_n), b = (b_0, \ldots, b_n) \in \mathbb{C}^n$, $\xi = (\xi_0, \ldots, \xi_n), \xi' = (\xi_1, \ldots, \xi_n), \xi = (\xi_0, \ldots, \xi_{n-1}), \tilde{\xi} = (\partial_{x_0}, \ldots, \partial_{x_{n-1}}), \xi = (\xi_0, \ldots, \xi_{n-1}, \tilde{s}), D = (D_0, \ldots, D_{n-1})$. Moreover let the double signs correspond.
For $\beta \in C^2(\overline{U})$, we introduce the symbol:
\[ p_{\rho,\beta}(x, \xi) = i\rho(x)\xi_0 + \beta(x)|\xi'|^2. \]

Let $\Gamma_0$ be some relatively open subset on $\partial \Omega$. We set $\tilde{\Gamma} = \partial \Omega \setminus \Gamma_0$ and $\Sigma_0 = (-T, T) \times \Gamma_0$.

In order to prove the Carleman estimate for the viscoelastic Lamé system, we assume the existence of a real-valued function $\psi$ which is pseudoconvex with respect to the symbols $p_{\rho,\beta}(x, \xi)$ and $p_{\rho,\lambda+2\mu}(x, \xi)$. More precisely, we can state as follows.

For functions $f(x, \xi)$ and $g(x, \xi)$, we introduce the Poisson bracket
\[ \{f, g\} = \sum_{j=0}^n \left( \partial_{\xi_j} f \partial_{\xi_j} g - \partial_{\xi_j} g \partial_{\xi_j} f \right). \]

Denote
\[ \widetilde{\psi}(x_0) = \frac{1}{(x_0 + T)^3(T - x_0)^3}. \]

We introduce

**Condition 1.1.** We say that a function $\varphi = \tilde{\psi}^{-1} \in C^{0,2}(\overline{U})$ with $\partial_{\xi_j} \varphi \tilde{\psi}^{-1} \in L^\infty(Q)$, is pseudoconvex with respect to the symbol $p_{\rho,\beta}(x, \xi)$ if there exists a constant $C_1 > 0$ such that
\[ \frac{\text{Im}\{\overline{p}_{\rho,\beta}(x, \xi_0, \tilde{\zeta}), p_{\rho,\beta}(x, \xi_0, \tilde{\zeta})\}}{|s|} > C_1 \widetilde{\psi}(x_0)M(\xi, \varphi(x_0)s)^2 \quad \forall (x, \xi, s) \in S, \quad (1.11) \]

where
\[ S = \{(x, \xi, s); x \in \overline{U}, M(\xi, \varphi(x_0)s) = 1, \quad p_{\rho,\beta}(x, \xi_0, \tilde{\zeta}) = 0\}, \]

\[ \tilde{\zeta} = (\xi_1 + i|s|\partial_{\xi_1} \varphi, \ldots, \xi_n + i|s|\partial_{\xi_n} \varphi) \]

and
\[ M(\xi, s) = (\xi_0^2 + \sum_{i=1}^n \xi_i^2 + s^4)^{\frac{1}{2}}. \]

We assume that there exists a positive constant $C_2$ such that
\[ \partial_{\xi_0} \varphi(x)|_{\Sigma_0} < 0 \quad \text{and} \quad |\partial_{\xi_0} \varphi(x)| \geq C_2 \widetilde{\psi}(x_0) \quad \forall x \in \Sigma_0 \quad (1.13) \]

and
\[ -\partial_{\xi_0} \varphi(x) > \frac{1}{\sqrt{2}} \sqrt{\frac{\mu(x)}{(\lambda + 2\mu)(x)}|\partial_{\xi_0} \varphi(x)|} \quad \forall x \in \Sigma_0, \quad \forall \tilde{\tau}: \text{tangential vector satisfying } |\tilde{\tau}| = 1. \]

If the pair $(\bar{\mu}, \bar{\lambda})$ is not identically equal to zero in $Q$, then we assume that
\[ \partial_{\xi_0} \varphi(x) < 0 \quad \text{on } (0, T) \times \overline{\Omega} \quad \text{and} \quad \partial_{\xi_0} \varphi(x) > 0 \quad \text{on } (-T, 0) \times \overline{\Omega}. \]

Let us assume
\[ \varphi(x) < 0 \quad \text{on } \overline{\Omega}, \quad \nabla^* \varphi(x) \neq 0 \quad \text{on } \overline{U}. \]

(1.16)
and
\[ \frac{\partial_y \varphi}{\varphi} \in C^4(\mathcal{D}) \quad \forall j \in \{1, \ldots, n\}. \]

(1.17)

Let \( a_\beta(x, \xi') = \beta(x)|\xi'|^2 \). Furthermore we introduce

**Condition 1.2.** We say that a function \( \varphi \hat{\varphi}^{-1} \in C^{0,2}(\mathcal{D}) \) is pseudoconvex with respect to the symbol \( a_\beta(x, \xi') \) if there exists a constant \( C_3 > 0 \) independents of \( x_0, s, \xi' \) such that

\[ \text{Im}[\hat{\varphi}_\beta(x, \xi' - i|\nabla' \varphi), a_\beta(x, \xi' + i|\nabla' \varphi) - \hat{\varphi}_\beta(x_0, s)] > C_3 |\varphi'(x_0) - \hat{\varphi}_\beta(x_0, s)|^2 \quad \forall (x, \xi', s) \in \mathcal{K}, \]

where
\[ \mathcal{K} = \{(x, \xi', s); x \in \mathcal{D}, |(\xi', s)| = 1, a_\beta(x, \xi' + i|\nabla' \varphi) = 0\} \]

Throughout this article, let \( N = \{1, 2, 3, \ldots\} \), \( \alpha = (\alpha_0, \ldots, \alpha_n) =: (\alpha_0, \alpha') \), \( \alpha' = (\alpha_1, \ldots, \alpha_n) \) with \( \alpha_0, \ldots, \alpha_n \in \mathbb{N} \cup \{0\} \) and \( |\alpha| = 2 \alpha_0 + \sum_{j=1}^n \alpha_j \). Define the norms

\[ \|\alpha\| = \sum_{\alpha=0}^{\infty} \sum_{|\alpha'| \leq 2} |\partial_x^\alpha \varphi(x)| \leq \frac{C_4}{(x_0 + T)^3(T - x_0)^3} \sum_{\alpha=0}^{\infty} |\partial_x^\alpha \varphi(x)| \leq \frac{C_5}{(x_0 + T)^3(T - x_0)^3} \]

for \( x \in \mathcal{Q} \), and

\[ \|\partial_{\alpha} \varphi(x)\| \leq \frac{C_6}{(x_0 + T)^3(T - x_0)^2} \quad x \in ((0, T) \times \partial \Omega) \cup ((-T, 0) \times \partial \Omega). \]

(1.20)

For any function \( f = (f_1, \ldots, f_n) \), we introduce the differential form \( \omega_\xi = \sum_{j=1}^n f_j \partial x_j. \) We identify the differential form \( \omega_\xi \) with the vector-function:

\[ \omega_\xi = (\partial_{\alpha_1} f_1 - \partial_{\alpha_2} f_2, \ldots, \partial_{\alpha_n} f_n - \partial_{\alpha_{n-1}} f_{n-1} - \partial_{\alpha_0} f_0) \]

Denote
\[ \|u\|_{\mathcal{B}(\varphi, \xi, 0)}^2 = \int_{\mathcal{Q}} \left( \sum_{|\alpha| = 2|\alpha'| \leq 1} (s \varphi')^{4-2|\alpha|} |\partial_x^\alpha u|^2 + s \varphi |\nabla \omega_\xi|^2 \right) e^{2\nu \varphi} dx, \]
\[ \|u\|_{\mathcal{A}(\varphi, \xi, 0)}^2 = \int_{\Omega} \left( \sum_{|\alpha'| = 0} (s \varphi')^{4-2|\alpha'|} |\partial_x^\alpha u|^2 + s \varphi |\nabla \omega_\xi|^2 \right) e^{2\nu \varphi} dx' \]

(1.21)

Finally we introduce norms
\[ \|\mathbf{F}^{\nu \varphi}\|_{\mathcal{B}(\varphi, \xi, 0)}^2 = \|\text{div} \mathbf{F}^{\nu \varphi}\|_{L^2(\mathcal{Q})}^2 + \||\omega_\xi| \mathbf{F}^{\nu \varphi}\|_{L^2(\mathcal{Q})}^2 + \|(s \varphi)^{3/2} \mathbf{F}^{\nu \varphi}\|_{H^{1/2}(\Sigma)}^2 + \|\mathbf{F}^{\nu \varphi}\|_{L^2(\mathcal{Q})}^2, \]

where
and for any positive $p$ we introduce the norm
\[
\|u\|_{H^{p/2}r(\Sigma)} = (\|u\|_{H^{p/2}_r(\Sigma)}^2 + (\langle \Delta \hat{u} \rangle^{1/2} u_{L(\Sigma)}^2)^{1/2}).
\]

Now we are ready to state our first main result, a Carleman estimate:

**Theorem 1.1.** Let $F, \text{div} F, d\omega, \Psi \in L^2(Q)$. $u \in L^2(0, T; H^2(\Omega))$, $\partial_\omega u \in L^2(0, T; H^1(\Omega))$ satisfy (1.6), (1.7). Let (1.8) and (1.9) hold true, and let a function $\varphi$ satisfy (1.13) to (1.20), conditions 1.1 and 1.2 with $\beta = \mu$ and $\beta = \lambda + 2\mu$. Then there exists $s_0 > 0$ such that for any $s > s_0$ the following estimate holds true:

\[
\left\| u \right\|_{B(\varphi, s \xi)} + \|\Delta^2 \varphi u_{F}^\alpha \|_{H^{\frac{p}{2}}r(\Sigma)} + \|\partial_\alpha u_{F}^\alpha \|_{H^{\frac{p}{2}}r(\Sigma)} \\
\leq C_5 (\|\varphi_{F}^\alpha \|_{L^1(\xi)} + \|\partial_\alpha u_{F}^\alpha \|_{H^{\frac{p}{2}}r(\Sigma)} + \|\Delta^2 \varphi u_{F}^\alpha \|_{H^{\frac{p}{2}}r(\Sigma)} + \|\Delta^2 \varphi \|_{L^1(\xi)}). \tag{1.22}
\]

Here the constant $C_7 > 0$ is independent of $s$.

**Example of a function $\varphi$ which satisfies the conditions of Theorem 1.1.** Let $\varphi \in C^2(\Omega)$ satisfy $\varphi_{\mid \partial \Omega} = 0$, $|\nabla \varphi(x)| > 0$ on $x' \in \Omega$, $\partial_\nu \varphi < 0$ on $\partial \Omega \setminus \Gamma$. We set $\varphi(x) = \frac{1}{\xi} - \frac{1}{\xi} e^{(x, \cdot)}$. $e^{(x, \cdot)} > 0$ on $(-T, T)$, $\partial_\nu e^{(x, \cdot)} > 0$ on $(0, T)$ and $\partial_\nu e^{(x, \cdot)} > 0$ on $(-T, 0)$, $\partial_\nu e^{(x, \cdot)} = 0$ for all $j \in \{0, 1, 2\}$, $\partial_\nu e^{(x, \cdot)}(\pm T) \neq 0$, and $\tilde{\ell}$ is smooth on $[-T, T]$.

Provided that parameter $\lambda > 0$ is sufficiently large, we can prove that conditions 1.1 and 1.2 hold true (e.g., Imanuvilov, Puel and Yamamoto [16]). The normal derivative of the function $\varphi$ on $\Sigma_0$ is strictly negative and so (1.14) holds true. Inequality (1.16) follows from the fact that the function $\varphi$ does not have critical points on $\overline{\Omega}$. The properties of the function $\tilde{\ell}$ imply (1.18) to (1.20).

Next we apply Carleman estimate (1.22) to an inverse source problem of determining a spatially varying factor $f(x')$ of a source term of the form $F(x) := R(x) f(x')$. Now we assume that $\rho, \lambda, \mu$ are independent of $x$: $\rho(x) = \rho(x')$, $\lambda(x) = \lambda(x')$, $\mu(x) = \mu(x')$ for $x \in (0, T) \times \Omega$. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $\eta \in (0, 1)$ be a fixed time moment. We consider

\[
\rho(x') \partial_\alpha^2 u = L_{\lambda, \mu}(x', D'') \partial_\omega u + L_{\lambda, \mu}(x', D') u(x) + R(x) f(x') \quad \text{in} \quad (0, T) \times \Omega, \tag{1.23}
\]

\[
u(x, x') = a(x'), \quad \partial_\alpha u(x, x') = b(x') \quad x' \in \Omega \tag{1.24}
\]

and

\[
\nu|_{(0, T) \times \partial \Omega} = 0 \tag{1.25}
\]

Here $R(x)$ is an $n \times n$ matrix function and $f(x')$ is an $\mathbb{R}^n$-valued function.

We further assume

\[
\lambda, \mu \in C^2([0, T] \times \overline{\Omega}). \tag{1.26}
\]

We consider

**Inverse source problem.** Let functions $a, b, R$ be given and $\tilde{\Gamma}$ be an arbitrary fixed open subset of $\partial \Omega$. Determine $f(x') \in \Omega$ by $\partial_\nu u|_{(0, T) \times \tilde{\Gamma}}$ and $u(\eta, \cdot), \partial_\omega u(\eta, \cdot)$ in $\Omega$.

We state our main result on the inverse source problem.

**Theorem 1.2.** Let $a, b \in H^1(\Omega)$, $f \in H^1(\Omega)$, $u \in L^2(0, T; H^3(\Omega))$, $\partial_\omega u \in L^2(0, T; H^1(\Omega))$, $\partial_\alpha u \in L^2(0, T; H^1(\Omega))$, $\partial_\alpha^2 u \in L^2(0, T; H^1(\Omega))$.
\( k \in \{1, 2, 3\} \) satisfy (1.23)–(1.25). We assume that there exists a constant \( \delta_0 > 0 \) such that
\[
|\det R(\eta, x')| \geq \delta_0 > 0, \quad x' \in \Omega, \quad R \in C^1([0, T] \times \Omega).
\] (1.27)
Moreover we assume that the Lamé coefficients \( \lambda, \mu, \rho, \tilde{\mu}, \tilde{\lambda} \) satisfy (1.26) and (1.8). Then there exists a constant \( C_8 > 0 \) such that
\[
\|[\mathbf{f}]\|_{H^1(\Omega)} \leq C_8 \left( \|a\|_{H^1(\Omega)} + \|b\|_{H^1(\Omega)} + \sum_{j=0}^{2} \left( \|\partial_\tau^j (\mathbf{d} \mathbf{u}, \nabla \mathbf{u})\|_{H^{1/2}((0,T) \times \tilde{\Gamma})} + \|\partial_\tau^j \partial_\nu (\mathbf{d} \mathbf{u}, \mathbf{u})\|_{L^2((0,T) \times \tilde{\Gamma})} \right) \right).
\] (1.28)
Here for \( p, q \geq 0 \), we set
\[
\|\mathbf{w}\|_{H^{p+q}((0,T) \times \tilde{\Gamma})} = \left( \|\mathbf{w}\|_{H^{p+q}(0,T;L^2(\tilde{\Gamma}))}^2 + \|\mathbf{w}\|_{L^2((0,T) \times H^p(\tilde{\Gamma}))}^2 \right)^{1/2}.
\]
The boundary data of \( \mathbf{u} \) for the inverse problem are described by the third, the fourth and the fifth terms on the right-hand side of (1.28).

The theorem asserts the global Lipschitz stability, which is the best possible stability for the inverse problem. Moreover this means that the precision of numerical solutions of \( \mathbf{f} \) is improved correspondingly in a linear order, if the accuracy of the used discretization is improved and measurement errors are smaller. This is not always the case, and in many inverse problems, the stability rate is very weak (e.g. a logarithmic rate), so that even high accurate numerical methods or very precise measurements may not make contribution for better numerical solutions of inverse problems, and we have to be satisfied if numerical solutions for inverse problems can catch only rough profiles of solutions. Theorem 1.2 implies that such improvements in data and methods can make sense for better numerical approaches.

The norm on the right-hand side of (1.28) is rather strong and the following corollary allows us to choose a weaker norm for the boundary data.

**Corollary 1.1.** In addition to the assumptions of theorem 1.2, we further assume
\[
\|\partial_\tau^j \mathbf{u}\|_{H^{1+\frac{j}{2}+\frac{j+1}{2}}((0,T) \times \tilde{\Gamma})} \leq M
\]
with arbitrarily fixed constants \( \delta > 0 \) and \( M > 0 \). Then there exist constants \( C_9 = C_9(M, \delta) > 0 \) and \( \theta = \theta(\delta) \in (0,1) \) such that
\[
\|[\mathbf{f}]\|_{H^1(\Omega)} \leq C_9 \left( \|a\|_{H^1(\Omega)} + \|b\|_{H^1(\Omega)} + \|\partial_\tau^j \mathbf{u}\|_{L^2((0,T) \times \tilde{\Gamma})}^\theta \right).
\]
In the corollary, we omit the dependency of \( \theta(\delta) \) on \( \delta \), but we note that \( \lim_{\delta \to 0} \theta(\delta) = 0 \).

The corollary asserts the stability by the weaker norm under a priori bound \( M \) on the data. The corollary can be easily derived from theorem 1.2. Indeed \( \nabla' \mathbf{u} \) can be given on \((0,T) \times \tilde{\Gamma}\) by linear combinations of the components of \( \partial_\tau \mathbf{u} \) because \( \nabla' \mathbf{u} = 0 \) on \((0,T) \times \tilde{\Gamma}\) by (1.25). Therefore \( \mathbf{d} \mathbf{u} \) and \( \nabla \mathbf{u} \) are given by \( \partial_\tau \mathbf{u} \) on \((0,T) \times \tilde{\Gamma}\). Hence the interpolation inequality in the Sobolev spaces \( H^{p,q}(0, T) \times \tilde{\Gamma}\) with \( p, q \geq 0 \) yields
\[
\|\partial_\tau^j \mathbf{u}\|_{H^{1+\frac{j}{2}+\frac{j+1}{2}}((0,T) \times \tilde{\Gamma})} \leq C_{10} \|\partial_\tau^j \mathbf{u}\|_{H^{1+\frac{j}{2}+\frac{j+1}{2}}((0,T) \times \tilde{\Gamma})}
\leq C_{11} \|\partial_\tau^j \mathbf{u}\|_{H^{1+\frac{j}{2}+\frac{j+1}{2}}((0,T) \times \tilde{\Gamma})} \|\partial_\tau^j \mathbf{u}\|_{H^{p,q}(0, T) \times \tilde{\Gamma}}^\theta \quad j = 0, 1, 2,
\]
with some \( \theta_1 \in (0, 1) \). As for the norms of the rest boundary terms, we can argue similarly, so that we can verify the corollary.

As is already remarked, this theorem directly yields the uniqueness and the stability for an inverse coefficient problem of determining the density \( \rho(x') \), \( x' \in \Omega \) in (1.1) by data \( \partial_t^j u(\eta, \cdot) \), \( j = 0, 1 \) and \( u \) in \( (0, T) \times \Gamma \) provided that other coefficients are known. The inverse problems of determining \( \lambda(x'), \mu(x') \), etc., can be considered based on theorem 1.1. However for keeping a reasonable size of the article, we postpone the arguments to a succeeding paper.

Contents.

The article is composed of six sections. In section 1, we stated the two main results theorems 1.1 and 1.2 in a self-contained way. The proof of theorem 1.1 is quite technical, divided into three steps and each step is given in sections 2–4. The proofs of propositions 2.3–2.5 stated in section 2 are postponed to Appendix. In section 5, we derive theorem 1.2 from theorem 1.1, and the derivation is based on [6], which can be understood independently of sections 2–4.

2. First step of proof of theorem 1.1

Consider the following boundary value problem

\[
P(x, D)v \triangleq (P_{\rho, \mu}(x, D)v_1, P_{\rho, \lambda+2\mu}(x, D)v_2) = q \quad \text{in} \quad Q.
\]

(2.1)

Here \( P_{\rho, \beta}(x, D) = \rho \partial_{\nu} - \beta \Delta \) and \( q \) is a given vector-valued function with \( \frac{n^2 - n}{2} + 1 \) components which belong to \( L^2(Q) \). Moreover a vector-valued function \( v_1 \) with \( (n^2 - n)/2 \) components is defined by

\[
v_1 = (v_{1,2}, \ldots, v_{n-1,n}) = (v_{k,j})_{1 \leq k < j \leq n}.
\]

We set \( v = (v_1, v_2) \). Let \( v \) satisfy the boundary condition

\[
B(x, D')v = g \quad \text{on} \quad \Sigma,
\]

(2.2)

where \( g = (g_1, \ldots, g_{\frac{n^2 - n}{2} + 1}) \) is a given vector-valued function and the boundary operator \( B(x, D') \) is constructed in the following way

\[
B(x, D') = (B_1(x, D'), B_2(x')) \quad \text{and} \quad B_1(x, D') = (b_1(x, D'), \ldots, b_n(x, D'))
\]

and

\[
b_k(x, D')v = -\sum_{j=1, j \neq k}^{n} \text{sign}(k - j) \partial_{\nu} v_{j,k} \frac{(\lambda + 2\mu)(x)}{\mu(x)} \partial_{\nu} v_2
\]

for \( 1 \leq k \leq n \), and \( B_2(x') \) is a smooth matrix-valued function constructed in the following way: Consider an \( n \times n \) matrix such that on the main diagonal we have \( \nu_k(x') \), the \( n \)th row is \( (\nu_1(x'), \ldots, \nu_n(x')) \), the first \( n - 1 \) elements of the last column are \(-\nu_1(x'), \ldots, -\nu_{n-1}(x')\) and all the rest elements are zero:

\[
B_2(x') = \begin{pmatrix}
\nu_1(x') & 0 & \ldots & 0 & \ldots & -\nu_1(x') \\
0 & \nu_2(x') & \ldots & \ldots & \ldots & \\
0 & 0 & \nu_3(x') & \ldots & -\nu_2(x') & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \\
\nu_1(x') & \ldots & \ldots & \nu_{n-1}(x') & \nu_n(x')
\end{pmatrix}.
\]
If \( \nu(x') \neq 0 \), then the determinant of such a matrix is not equal to zero. Denote the inverse to this matrix by \( B_2(x') \) and set \( w := (w_1, \ldots, w_n) = B_2\tilde{v} \) where \( \tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_n) \), \( \tilde{v}_j = v_{j,a} \) for \( j \in \{1, \ldots, n-1\} \) and \( \tilde{v}_n = \tilde{v}_2 \). Then

\[
B_2(x')\tilde{v} = \left( \nu_2 w_1 - \nu_1 w_2, \ldots, \nu_n w_1 - \nu_1 w_n, \nu_{n-1} w_{n-1} - \nu_{n-1} w_n, \sum_{j=1}^n \nu_j w_j \right).
\]

We are not investigating the existence of solution for problem (2.1)–(2.2), but assume the existence of function \( \tilde{v} \) which satisfies (2.1)–(2.2) with some \( g \). Our current goal is to establish the Carleman estimate for solution of this problem.

Without loss of generality, we can assume that

\[
\tilde{v}(0) = -\tilde{v}_n = (0, \ldots, 0, -1).
\]

We have

**Proposition 2.1.** Let \( v \in H^{1,2}(Q) \) satisfy (2.1), (2.2). There exists \( s_0 > 1 \) such that

\[
\sum_{|\alpha| \leq 2} \|\nabla^{\alpha} v \phi\|_{L^2(Q)} + \|\nabla^2 v \phi\|_{L^2(Q)} + \|\nabla\nabla v \phi\|_{L^2(Q)} + \|\nabla^2 \nabla v \phi\|_{L^2(Q)} + \|\nabla^2 \nabla v \phi\|_{L^2(Q)} + \|\nabla^2 \nabla v \phi\|_{L^2(Q)}
\]

for all \( s \geq s_0 \). Here \( C_1 \) is independent of \( s \).

The proof of proposition 2.1 is complete at the beginning of section 4.

First, by an argument based on the partition of unity (e.g. lemma 8.3.1 in [13]), it suffices to prove inequality (2.4) locally, by assuming that

\[
\text{supp } v \subset B(y^*, \delta),
\]

where \( B(y^*, \delta) \) is the ball of the radius \( \delta > 0 \) centered at some point \( y^* \).

Otherwise, without loss of generality, we can assume that \( y^* = (y_0^*, 0, \ldots, 0) \). Let \( \theta \in C_2^\infty(1, 2) \) be nonnegative and

\[
\sum_{t = -\infty}^{\infty} \theta(2^{-t} t) = 1 \quad \text{for all } t \in \mathbb{R}.
\]

Set \( v_\ell(x) = v(x) \nu(\ell x_0) \) and \( g_\ell(x) = g(x) \nu(\ell x_0) \), where

\[
\nu(\ell x_0) = \theta \left( 2^{-\ell} 2^{-2|y_0^*|} \right),
\]

and

\[
\theta \in C^\infty[-T, T], \quad \theta|_{[T, -T]} = T + x_0, \quad \theta|_{[T/2, T]} = T - x_0, \quad \theta|_{[-T, -T/2]} = T - x_0,
\]

\[
\theta_0 < 0 \text{ on } (-T, 0), \quad \theta_0 < 0 \text{ on } (0, T), \quad \theta_0 < 0 \text{ on } (-T, 0).
\]

Observe that it suffices to prove the Carleman estimate (2.4) for the function \( v_\ell \) instead of \( v \), provided that the constants \( C_1 \) and \( s_0 \) are independent of \( \ell \). Observe that if \( G \subset \mathbb{R}^n \) is a bounded domain and \( g \in L^2(G) \), then there exist constants \( C_2 \) and \( C_3 \) (see e.g. [41]) such that

\[
C_2 \sum_{\ell = -\infty}^{\infty} \|\nu g\|^2_{L^2(G)} \leq \|g\|^2_{L^2(G)} \leq C_3 \sum_{\ell = -\infty}^{\infty} \|\nu g\|^2_{L^2(G)}.
\]
Denote the norm on the left-hand side of (2.4) as $\| \cdot \|_*$. Suppose that the estimate (2.4) is true for any function $v_\ell$ with constants $C_1$ and $s_0$ which are independent of $\ell$. By (2.9) for some constant $C_4$ independent of $s$ we have

\[ \|ve^{(\ell)}\|_* = \| \sum_{\ell = -\infty}^{+\infty} v_\ell e^{(\ell)} \|_* \leq \sum_{\ell = -\infty}^{+\infty} \|v_\ell e^{(\ell)}\|_* \leq C_4 \sum_{\ell = -\infty}^{+\infty} (\|\kappa_\ell P(x, D)ve^{(\ell)}\|_{L^2(\Omega)}) \]

+ \|e^{(\ell)}[\kappa_\ell, P(x, D)]v\|_{L^2(\Omega)} + \|\|ve^{(\ell)}\|_{H^\frac{1}{2}, \frac{1}{4} (\Sigma)} + \|\partial_x ve^{(\ell)}\|_{H^\frac{1}{2}, \frac{1}{4} (\Sigma)} + \|\kappa_\ell ge^{(\ell)}\|_{H^\frac{1}{2}, \frac{1}{4} (\Sigma)} \].

(2.10)

Assume that near $(0, \ldots, 0)$, the boundary $\partial \Omega$ is locally given by an equation $x_n - \theta(x_1, \ldots, x_{n-1}) = 0$ and if $(x_1, \ldots, x_n) \in \Omega$, then $x_n - \theta(x_1, \ldots, x_{n-1}) > 0$, where $\theta \in C^3$ and $\theta(0) = 0$. Since $\theta(0) = -\theta_n$, we have

\[ (\partial_{x_1} \theta(0), \ldots, \partial_{x_{n-1}} \theta(0)) = 0. \]

(2.11)

Denote

\[ F(x) = (x_0, \ldots, x_{n-1}, x_n - \theta(x_1, \ldots, x_{n-1})). \]

(2.12)

By lemma A.7, from (2.10) we obtain

\[ \|ve^{(\ell)}\|_* = \| \sum_{\ell = -\infty}^{+\infty} v_\ell e^{(\ell)} \|_* \leq \sum_{\ell = -\infty}^{+\infty} \|v_\ell e^{(\ell)}\|_* \leq C_3 (\|P(x, D)ve^{(\ell)}\|_{L^2(\Omega)}) \]

+ \|e^{(\ell)}[\kappa_\ell, P(x, D)]v\|_{L^2(\Omega)} + \|\|ve^{(\ell)}\|_{H^\frac{1}{2}, \frac{1}{4} (\Sigma)} + \|\partial_x ve^{(\ell)}\|_{H^\frac{1}{2}, \frac{1}{4} (\Sigma)} + \|\kappa_\ell ge^{(\ell)}\|_{H^\frac{1}{2}, \frac{1}{4} (\Sigma)} \].

(2.13)

Using (2.7) and (2.8), we estimate the norm of the commutator $[\kappa_\ell, P(x, D)]$ and obtain

\[ \sum_{\ell = -\infty}^{+\infty} \|e^{(\ell)}[\kappa_\ell, P(x, D)]v\|_{L^2(\Omega)} \leq C_6 \sum_{\ell = -\infty}^{+\infty} (\|\partial_{x_1} \kappa_\ell \nabla ve^{(\ell)}\|_{L^2(\Omega)} + \|\partial_x^2 \kappa_\ell ve^{(\ell)}\|_{L^2(\Omega)}) \]

\[ \leq C_7 \sum_{\ell = -\infty}^{+\infty} (\|\nabla ve^{(\ell)}\|_{L^2(\Omega)} + \|\kappa_\ell ve^{(\ell)}\|_{L^2(\Omega)}) \]

\[ \leq C_8 \|\kappa_\ell ve^{(\ell)}\|_{L^2(\Omega)} + \|\kappa_\ell ve^{(\ell)}\|_{L^2(\Omega)}. \]

(2.14)

From (2.13) and (2.14), we obtain (2.4).

Now, without loss of generality we assume that

\[ \text{supp } v \subset B(y^\sharp, \delta) \cap \text{supp } \kappa_\ell, y^\sharp \in \text{supp } \kappa_\ell \]

(2.15)

where $B(y^\sharp, \delta)$ is the ball of the radius $\delta > 0$ centered at some point $y^\sharp = (y^\sharp_0, 0, \ldots, 0)$.

We set

\[ \Delta_\rho u = \sum_{j=1}^{n-1} (\partial_{x_j}^2 u - 2\partial_{x_j} \theta \circ F^{-1}(y) \partial_{x_j}^2 u) + (1 + |\nabla' \theta|^2 \circ F^{-1}(y)) \partial_{x_n}^2 u. \]

Henceforth we set

\[ y = (y_0, y') = (y_0, y_1, \ldots, y_n), \quad Q := \mathbb{R}^n \times [0, \gamma]. \]

After the change of variables, the equations (2.1) have the form

\[ P(y, D)v = (\rho \partial_{x_0} v_1 - \mu \Delta_\rho v_1, \rho \partial_{x_0} v_2 - (\lambda + 2\mu) \Delta_\rho v_2) = q, \quad \text{on } Q. \]

(2.16)
\[
\bar{B}(y, D')v = g.
\]  
(2.17)

where \(\gamma\) is some positive constant. Without loss of generality, we can assume \(\gamma = 1\). Here for functions \(\rho \circ F^{-1}(y), \mu \circ F^{-1}(y)\) and \(\lambda \circ F^{-1}(y)\), we used the notations \(\rho, \mu, \lambda\). Similarly by \(q_1, q_2\), we denote the functions \(q_1 \circ F^{-1}(y), q_2 \circ F^{-1}(y)\).

The operator \(\bar{B}(y, D')\) is obtained from \(B(x, D')\) in the following way

\[
\bar{B}(y, D') = (\bar{B}_1(y, D'), \bar{B}_2(F^{-1}(y))), \quad \bar{B}_1(y, D') = (\bar{b}_1(y, D'), \ldots, \bar{b}_n(y, D'))
\]

and

\[
\bar{b}_k(y, D')v = -\sum_{j=1, j \neq k}^{n} \sum_{m=1}^{n} \text{sign}(k-j)\partial_{x_j} v_{j,k} F^{-1} \circ F^{-1}(y)
- \frac{(\lambda + 2\mu)}{\mu} \sum_{m=1}^{n} \partial_{x_j} v_{2} F^{-1} \circ F^{-1}(y).
\]

Since \(F'\) is the unit matrix, from the above equality we have

\[
\bar{b}_k(y, D')v = -\sum_{j=1, j \neq k}^{n} \sum_{m=1}^{n} \text{sign}(k-j)\partial_{x_j} v_{j,k} - \frac{(\lambda + 2\mu)}{\mu} \partial_{x_j} v_{2}.
\]

(2.18)

Now we introduce operators

\[
P_{\rho, \mu}(y, D, \bar{s}) = e^{i|\varphi|} P_{\rho, \mu}(y, D) e^{-i|\varphi|}, \quad P_{\rho, \lambda + 2\mu}(y, D, \bar{s}) = e^{i|\varphi|} P_{\rho, \lambda + 2\mu}(y, D) e^{-i|\varphi|},
\]

where \(\bar{s} = s\bar{\varphi}(y^*_0)\).

We denote the principal symbols of the operators \(P_{\rho, \mu}(y, D, \bar{s})\) and \(P_{\rho, \lambda + 2\mu}(y, D, \bar{s})\) by \(p_{\rho, \mu}(y, \xi, \bar{s}) = p_{\rho, \mu}(y, \xi + i|s|\nabla \varphi)\) and \(p_{\rho, \lambda + 2\mu}(y, \xi, \bar{s}) = p_{\rho, \lambda + 2\mu}(y, \xi + i|s|\nabla \varphi)\) respectively.

The principal symbol of the operator \(P_{\rho, \beta}(y, D, \bar{s})\) has the form

\[
p_{\rho, \beta}(y, \xi, \bar{s}) = ip(y)(\xi_0 + i|s|\varphi_{y_0}) + \beta \sum_{j=1}^{n-1} (\xi_j + i|s|\varphi_{y_j})^2
- 2(\nabla^j \varphi_j, (\xi' + i|s|\nabla^j \varphi_j) (\xi_n + i|s|\varphi_{y_n}) + (\xi_n + i|s|\varphi_{y_n})^2 G).
\]

(2.20)

where \(G(y_1, \ldots, y_{n-1}) = 1 + |\nabla \varphi(y_1, \ldots, y_{n-1})|^2\). The zeros of the polynomial \(p_\beta(y, \xi, \bar{s})\) with respect to the variable \(\xi_n\) for \(M(\xi, \bar{s}) \geq 1\), \(\xi = (\xi_1, \ldots, \xi_{n-1})\) and \(y \in B(y^*, \delta) \cap \text{supp } \kappa\), are

\[
\Gamma_{\beta}^\pm(y, \xi, \bar{s}) = (-i|\bar{s}| \mu \kappa \varphi_0 \kappa(\xi, \bar{s}) + \alpha_{\beta}^\pm(y, \xi, \bar{s})).
\]

(2.21)

Here we recall that the function \(\kappa\) is given by (2.7), and we set \(\varphi = (\varphi_0, \ldots, \varphi_n), \varphi(y) = \frac{\varphi_0(y)}{\varphi_0(y^*)}\), \(M = \{(\xi, \bar{s}); M(\xi, \bar{s}) = 1\}\), \(\tilde{\kappa}_\ell(y) = \eta_\ell(y) \sum_{k=-40}^{\ell+40} \kappa_\ell(y_0), \eta_\ell \in C_0^\infty(B(y^*, 2\delta)), \eta_\ell|_{B(y^*, \frac{\|\mu_\ell\|}{\|\mu_\ell\|})} = 1\).

(2.22)
\[ \alpha^\gamma_\delta(y, \tilde{\xi}, \tilde{s}) = \bar{\mu}_\epsilon(y) \left( -\sum_{j=1}^{n-1} \left( \xi_j + i|\varphi| \right) \bar{\partial}_\gamma \partial(y_1, \ldots, y_{n-1}) \nu(\tilde{\xi}, \tilde{s}) \right) \]
\[ \pm \sqrt{r_\beta(y, \tilde{\xi}, \tilde{s}) \pm i(1 - \kappa(\tilde{\xi}, \tilde{s})) M(\tilde{\xi}, \tilde{s})} \right). \]
\[ (2.23) \]
\[ r_\beta(y, \tilde{\xi}, \tilde{s}) = \kappa^2(\tilde{\xi}, \tilde{s}) \frac{(-i\rho_\alpha - \beta \sum_{j=1}^{n-1} \left( \xi_j + i|\varphi| \right)) G + \beta(\xi + i|\varphi|, \nabla \theta)^2}{\beta G^2}. \]
\[ (2.24) \]

where \( \chi_\nu \in C_0^\infty(\mathbb{M}) \) is identically equal to 1 in some neighborhood of \((\tilde{\xi}', \tilde{s}') \in \mathbb{M} \) and \( \text{supp} \chi_\nu \subset \mathcal{O}(\tilde{\zeta}', \delta_1). \)

Assume that
\[ \kappa(\tilde{\xi}, \tilde{s})|_{\text{supp} \chi_\nu} = 1, \quad \text{supp} \kappa(\tilde{\xi}, \tilde{s}) \subset \mathcal{O}(\tilde{\zeta}', 2\delta_1), \quad 1 \geq \kappa(\tilde{\xi}, \tilde{s}) \geq 0 \quad \text{on} \quad \mathbb{M}. \]
\[ (2.25) \]

We extend the function \( \chi_\nu \) on \( \mathbb{R}^{n+1} \) as follows:
\[ \chi_\nu := \chi_\nu(\xi_0/M^2(\tilde{\xi}, \tilde{s}), \xi_1/M(\tilde{\xi}, \tilde{s}), \ldots, \xi_{n-1}/M(\tilde{\xi}, \tilde{s})) \quad \text{for} \quad M(\tilde{\xi}, \tilde{s}) > 1 \] and \( \chi_\nu := \chi_\nu(\xi_0/M^2(\tilde{\xi}, \tilde{s}), \xi_1/M(\tilde{\xi}, \tilde{s}), \ldots, \xi_{n-1}/M(\tilde{\xi}, \tilde{s})) \kappa^*(M(\tilde{\xi}, \tilde{s})) \]
for \( M(\tilde{\xi}, \tilde{s}) < 1, \) where \( \kappa^* \in C_0^\infty(\mathbb{R}^1), \kappa^*(t) \geq 0, \kappa^*(t) = 1 \) for \( t \geq 1 \) and \( \kappa^*(t) = 0 \) for \( t \in [0, 1/2]. \) In a similar way, we extend the function \( \kappa(\tilde{\xi}, \tilde{s}) \) on \( \mathbb{R}^{n+1}. \)

Denote by \( \tilde{\chi}_\nu(y, \tilde{D}, \tilde{s}) \) the pseudodifferential operator with the symbol \( \nu(y) \chi_\nu(\xi, \tilde{s}) \) and \( \eta(\tilde{y}) = \eta_\nu(\tilde{y}) \sum_{k=10}^{10} \kappa_{t=\tilde{s}}. \) Here \( \eta_\nu \in C_0^\infty(B(y^*, \delta^*)) \) and \( \eta_\nu B(y^*, \delta^*) = 1. \) We set \( w_\nu = \tilde{\chi}_\nu(y, \tilde{D}, \tilde{s})w, w_{1, \nu} = \tilde{\chi}_\nu(y, \tilde{D}, \tilde{s})w_1, \)
\[ w_{k, \nu} = \tilde{\chi}_\nu(y, \tilde{D}, \tilde{s})w_{k, \nu} \] and \( \nu = \nu\nu|_{\varphi}. \)

**Definition.** Let \( \mathcal{O} \subset \mathbb{R}^n \) be a domain and \( \tilde{k} \in \mathbb{N}, \kappa > 0. \) Then, we say that the symbol \( a(y, \xi, \tau) \in W^{k, \infty}(\mathcal{O} \times \mathbb{R}^{n+1}) \) belongs to the class \( W^{m, \infty}_{\psi, \kappa}(\mathcal{O}) \) if:

(A) There exists a compact set \( K \subset \subset \mathcal{O} \) such that \( a(y, \xi, \tau)|_{\mathcal{O} \setminus K} = 0; \)

(B) For any \( \alpha = (\alpha_0, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^{n+1}, \) there exists a constant \( C_\alpha \)
\[ \left\| \partial_\xi^{\alpha_0} \partial_\tau^{\alpha_n-1} \partial_\tau^\alpha a(y, \xi, \tau) \right\|_{W^{k, \infty}(\mathcal{O})} \leq C_\alpha \left( \tau^2 + \epsilon_0 + \sum_{i=1}^{n-1} \tau_i^2 \right)^{\frac{k-|\alpha|}{2}}, \]

where \( |\alpha| = 2\alpha_0 + \sum_{i=1}^{n-1} \alpha_i \) and \( M(\tilde{\xi}, \tilde{s}) \geq 1; \)

(C) For any \( N \in \mathbb{N}, \) the symbol \( a \) can be represented as
\[ a(y, \xi, \tau) = \sum_{j=0}^{N-1} a_j(y, \xi, \tau) + R_N(y, \xi, \tau). \]

Here the functions \( a_j \) have the following properties:
\[ a_j(y, \lambda^{\xi_0}, \lambda^{\xi_1}, \ldots, \lambda^{\xi_{n-1}}, \lambda^{\tau}) = \lambda^{n-j} a_j(y, \xi, \tau) \]
for any \( \lambda > 1 \) and all \( (y, \xi, \tau) \in \{ (y, \xi, \tau); \ y \in K, M(\xi, \tau) > 1 \}, \) and there exists a constant \( C_\alpha \) such that
\[ \left\| \partial_\xi^{\alpha_0} \partial_\tau^{\alpha_n-1} \partial_\tau^\alpha a_j(y, \xi, \tau) \right\|_{W^{k, \infty}(\mathcal{O})} \leq C_\alpha \left( \tau^2 + \epsilon_0 + \sum_{i=1}^{n-1} \tau_i^2 \right)^{\frac{k-|\alpha|}{2}}, \]
for any multi-index $\alpha$ and any $(\tilde{\zeta}, \tilde{\tau})$ satisfying $M(\tilde{\zeta}, \tilde{\tau}) \geq 1$, and the term $R_N$ satisfies the estimate

$$
\|R_N(\cdot, \tilde{\zeta}, \tilde{\tau})\|_{W^{k, \infty}(\mathcal{O})} \leq C_N \left( \tilde{\tau}^2 + |\xi| + \sum_{j=1}^{n-1} \xi_j^2 \right)^{-\frac{n-2}{2}}
$$

for all $(\tilde{\zeta}, \tilde{\tau})$ satisfying $M(\tilde{\zeta}, \tilde{\tau}) \geq 1$. With each symbol $a(\tilde{y}, \tilde{\zeta}, \tilde{\tau})$, we associate the pseudodifferential operator. See [42] for details.

Let $X^2(\mathcal{O}) = W^{1, \infty}(\mathcal{O})$ or $X^2(\mathcal{O}) = C^2(\mathcal{O})$. We introduce the semi-norm of a symbol $a$ by

$$
\pi_{X^2}(a) = \sum_{j=0}^{4n+4} \sup_{|\alpha| \leq 4n+4} \sup_{|\xi|, |\tilde{\tau}| \geq 1} |a^{(\alpha)}_{y, \xi, \tilde{\tau}}(\tilde{\zeta}, \tilde{\tau})|_{X^2(\mathcal{O})} / (1 + |\tilde{\zeta}, \tilde{\tau}|)^{n-j-|\alpha|}
$$

and the pseudodifferential operators with the symbols $\Gamma_\beta^1$ belong to the class $W^{k, \infty}_{cl} \mathcal{S}^1(\mathcal{O})$ for any $k \in \{0, 1\}$ and

$$
\pi_{W^{k, \infty}_{cl}(\mathcal{O})}(\Gamma_\beta^1) \leq C_{10} \pi(\tilde{\zeta}, \tilde{\tau}).
$$

Denote $a_\beta(y, \xi, \eta) = \beta(y) \sum_{k=1}^{n-1} \xi_k^2 - 2\xi_\eta \sum_{k=1}^{n-1} \partial_{\xi_k} \tilde{\zeta} + \xi_\eta^2 (1 + G)$ and $a_\beta(y, \xi, \eta) = \beta(y) \sum_{k=1}^{n-1} \xi_k \eta_k - (\xi_\eta \sum_{k=1}^{n-1} \partial_{\xi_k} \tilde{\tau} + \eta_\eta \sum_{k=1}^{n-1} \partial_{\xi_k} \tilde{\tau} + \xi_\eta^2 (1 + G)$. Then we have

**Proposition 2.2.** Let $w \in H^{1,2}(\mathcal{Q})$, supp $w \subset B(y^*, \delta) \cap \text{supp } \eta$ and $P_\beta(y, D, \tilde{\zeta})w_\nu \in L^2(\mathcal{Q})$. Then there exist positive constants $\delta(y^*)$, $C_{11}$, $C_{12}$ independent of $s$, such that

$$
\begin{align*}
C_{11} \int_{\mathcal{Q}} \left( |\tilde{\zeta}| \varphi \sum_{k=1}^{n} |\partial_{\zeta_k} \tilde{\chi}_w|^2 + |\tilde{\tau}|^2 \varphi^2 |\tilde{\chi}_w|^2 \right) dy + \Xi_\beta(\tilde{\chi}_w) & \leq \|P_\beta(y, D, \tilde{\zeta}) \tilde{\chi}_w w_\nu\|_{L^2(\mathcal{Q})} + C_{12} \delta(\tilde{\tau}) \| (\partial_{\zeta_\nu} \tilde{\chi}_w, \tilde{\chi}_w w_\nu) (\cdot, 0) \|_{H^{1,2}(\mathcal{R}^n \times H^{1,2}(\mathcal{R}^n))}.
\end{align*}
$$

(2.28)

for all $s \geq s_0$, where $\epsilon(\delta) \to +0$ as $\delta \to +0$ and

$$
\Xi_\beta(w) = \sum_{j=1}^{3} \mathcal{J}_j(\beta, w).
$$

(2.29)

The proof of the proposition is given in [26].
In some cases, we can represent the operator $P_{\rho,\beta}(y, D, \bar{s})$ as a product of two first order pseudodifferential operators.

**Proposition 2.3.** Let $w \in H^{1,2}(Q)$ satisfy supp $w \subset B(y^*, \delta) \cap$ supp $\eta_\ell$, and $P_{\beta}(y, D, \bar{s})w_\nu \in L^2(Q), w_\nu = \tilde{\chi}_\nu(y, D, \bar{s})w$. We assume that $r_\beta(y^*, \zeta^*) \neq 0$ with $y^* \in$ supp $\eta_\ell$ and supp $\chi_\nu \subset O(\zeta^*, \delta_1)$. Then we can factorize the operator $P_{\rho,\beta}(y, D, \bar{s})$ into a product of two first order pseudodifferential operators:

$$
P_{\rho,\beta}(y, D, \bar{s})w_\nu = \beta G\left(\frac{1}{1} \partial_{\lambda^*} - \Gamma_\beta^L (y, D, \bar{s})\right) \left(\frac{1}{1} \partial_{\lambda^*} - \Gamma_\beta^L (y, D, \bar{s})\right) w_\nu + T_\beta w_\nu,$$

(2.31)

where $T_\beta : H^{1,2}(\mathbb{R}^{n+1}) \to L^2(0, \gamma; L^2(\mathbb{R}^n))$ satisfies the estimate

$$
\|T_\beta w_\nu\|_{L^2(0, \gamma; L^2(\mathbb{R}^n))} \leq C_{13} \tilde{\varphi}^\gamma(y_0^*) \|w\|_{H^{1,2}(\mathbb{R}^{n+1})},
$$

(2.32)

The proof of proposition 2.3 is given in the appendix.

Let $y^* \in$ supp $\eta_\ell$.

We have

**Proposition 2.4.** Let $-\infty < a_1 < a < b < b_1 < +\infty, p \in \mathbb{N}$ and supp $v \subset I_1 = [a, b] \times \mathbb{R}^{n-1}$. Then there exists a constant $C_{23} > 0$ such that

$$
\|M^p(D, s)v\|_{L^2(0, \gamma; [a_1, b_1] \times \mathbb{R}^{n-1})} \leq \frac{C_{23}}{\min\{a-a_1, b_1-b\}^p} \|v\|_{L^1(\mathbb{R}^n)}.
$$

The proof of proposition 2.4 is given in appendix.

We apply proposition 2.4 in order to estimate the $H^{1,2}$-norm of $(1 - \eta_\ell)\chi_\nu(D, \bar{s})w$. For all sufficiently large $\ell$, we have

$${\text{supp}}_w \subset [-T + (\ell + 2)^{-4}, -T + (\ell - 2)^{-4}] \cup [T - (\ell - 2)^{-4}, T - (\ell + 2)^{-4}] \times \mathbb{R}^n$$

and

$${\text{supp}}(1 - \eta_\ell) \subset [-T, T] \setminus [-T + (\ell + 11)^{-4}, -T + (\ell - 11)^{-4}] \cup [T - (\ell - 11)^{-4}, T - (\ell + 11)^{-4}].$$

Therefore

$$
\|(1 - \eta_\ell)\chi_\nu(D, \bar{s})w(\cdot, y_\nu)\|_{H^{1,2}(\mathbb{R})} \leq C_{30} \tilde{\varphi}^\gamma(y_0^*) \|w(\cdot, y_\nu)\|_{L^2(\mathbb{R}^n)}.
$$

(2.33)

By the same argument as in lemma A.5, we obtain

$$
\|\tilde{s}\|\|(1 - \eta_\ell)\chi_\nu(D, \bar{s})w(\cdot, y_\nu)\|_{H^{1,2}(\mathbb{R} \setminus \mathbb{R})} \leq C_{32} \|w(\cdot, y_\nu)\|_{L^2(\mathbb{R}^n)}.
$$

(2.34)

By (2.33) and (2.34) we have

$$
\|(1 - \eta_\ell)\chi_\nu(D, \bar{s})w(\cdot, y_\nu)\|_{H^{1,2}(\mathbb{R}^n)} \leq C_{33} \tilde{\varphi}^\gamma(y_0^*) \|w(\cdot, y_\nu)\|_{L^2(\mathbb{R}^n)}.
$$

(2.35)

Denote
Let us consider the equation
\[
\left( \frac{1}{\nu} \partial_\nu - \Gamma^\nu_\mu(y, D, \bar{s}) \right) V = p \quad \text{on } Q, \quad V|_{\nu=0} = 0.
\] (2.36)
For solutions of this problem, we can prove an a priori estimate.

**Proposition 2.5.** Let \( r_\beta(y^*, \zeta^*) \neq 0 \), \( V = V_+^\nu(k, j) \) if \( \beta = \mu \) and \( V = V_{\lambda+2\mu}^\nu \) if \( \beta = \lambda + 2\mu \).

There exists a constant \( C_{34} > 0 \) such that
\[
\| V(\cdot, 0) \|_{H^\frac{1}{2} + \gamma(R^\nu)} \leq C_{34} \gamma_0^\nu \| w \|_{H^\frac{1}{2} + \gamma(Q)} + \| p \|_{L^2(Q)).
\] (2.37)

### 3. Second step of proof of theorem 1.1

We argue separately in the three cases where

- \( r_\mu(y^*, \zeta^*) = 0 \)
- \( r_{\lambda+2\mu}(y^*, \zeta^*) = 0 \)
- \( r_\mu(y^*, \zeta^*) \neq 0 \) and \( r_{\lambda+2\mu}(y^*, \zeta^*) \neq 0 \),

where \( r_\mu \) and \( r_{\lambda+2\mu} \) are defined by (2.24).

#### 3.1. Case \( r_\nu(y^*, \zeta^*) = 0 \)

Let \( \text{supp} \chi_\nu \subset O(y^*, \delta_1(y^*)) \), and \( (y^*, \zeta^*) \) be a point on \( \mathbb{R}^{n+1} \times \mathbb{M} \) such that \( r_\mu(y^*, \zeta^*) = 0 \). By (2.3) and (2.24), this equality implies
\[
\sum_{j=1}^{n-1} (\xi_j^*)^2 = (\tilde{s})^2 \sum_{j=1}^{n-1} |\varphi_j(y^*)|^2 \quad \text{and} \quad \frac{\rho(y^*) \xi_0^*}{\mu(y^*)} + \tilde{s} \sum_{j=1}^{n-1} \xi_j^* \varphi_j(y^*) = 0. \] (3.1)

By (3.1) and (1.13), we obtain
\[
\tilde{s}^* \neq 0.
\] (3.2)

By (3.1) there exists \( C_1 > 0 \) such that
\[
\sum_{j=1}^{n-1} \xi_j^* \leq \tilde{s} \sum_{j=1}^{n-1} |\varphi_j(y^*)|^2 \quad \text{and} \quad \frac{\rho(y^*) \xi_0^*}{\mu(y^*)} + \sum_{j=1}^{n-1} \xi_j^* \varphi_j(y^*) \leq \delta_1 C_1 M(\tilde{s}, \tilde{s}) \] (3.3)
for all \( (y^*, \zeta) \in O(y^*, \delta_1(y^*)) \). Hence, by (3.3) and (2.35), we can choose some constants \( C_2, C_3 \) such that
\[
\\| A_3(\mu, w_{k, j, \nu}) \| \leq C_2 \delta_1 \| (\partial_{\nu} \varphi_{\nu}(\cdot, 0), \varphi_{\nu}(\cdot, 0)) \|_{L^2(R^n)} + C_3 \| (\partial_\nu \varphi_{\nu}(\cdot, 0), \varphi_{\nu}(\cdot, 0)) \|_{L^2(R^n) \times H^{1, 1}(R^n)}. \] (3.4)

We recall that by (2.28) there exist \( C_4 > 0 \) and \( C_5 > 0 \) such that
\[
C_4 \| \tilde{s} \|_{H^{\frac{1}{2}}(Q)}^2 + \| \tilde{s} \|_{H^{\frac{1}{2}}(Q)}^2 + C_5 \| \tilde{s} \|_{H^{\frac{1}{2}}(Q)}^2 + \| \tilde{s} \|_{H^{\frac{1}{2}}(Q)}^2 + \| \tilde{s} \|_{H^{\frac{1}{2}}(Q)}^2 + \| \tilde{s} \|_{H^{\frac{1}{2}}(Q)}^2.
\] (3.5)
where $\epsilon(\delta) \to 0$ as $\delta \to +0$.

By (3.2), (2.35) and $\tilde{r}^* \neq 0$, inequality (3.3) yields

$$\|\xi_1(\mu, w_{1,\nu})\| \leq C \frac{\delta_1 \mu(\ast)}{|\tilde{r}^*|} \left(\|\partial_{\nu} w_{\nu}(\cdot, 0)\|_2^2 + |\tilde{r}^*|^2 \|w_{1,\nu}\|^2\right)(\gamma, 0) d\gamma$$

Hence from (3.6) and (3.4), we obtain

$$\Xi_\mu(w_{1,\nu}) \geq C_8 \int_{\mathbb{R}^*} \left(\|\partial_{\nu} w_{1,\nu}\|^2 + |\tilde{r}^*|^3 \|w_{1,\nu}\|^2\right)(\gamma, 0) d\gamma$$

(3.7)

$$- \epsilon(\delta)|\tilde{r}^*|\|\partial_{\nu} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)\|_2^2 \leq \Xi_\lambda(\gamma, 0)$$

(3.8)

where $\epsilon(\delta) \to 0$ as $\delta \to +0$.

Now we consider two subcases:

Subcase A. Let $r_{\lambda+2\mu}(y^*, \zeta^*) = 0$.

Similarly to (3.7), we obtain

$$\Xi_{\lambda+2\mu}(w_{2,\nu}) \geq C_{10} \int_{\mathbb{R}^*} \left(\|\partial_{\nu} w_{2,\nu}\|^2 + |\tilde{r}^*|^3 \|w_{2,\nu}\|^2\right)(\gamma, 0) d\gamma$$

$$- \epsilon(\delta)|\tilde{r}^*|\|\partial_{\nu} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)\|_2^2 \leq \Xi_{\lambda+2\mu}(w_{2,\nu}, \nu, (\gamma, 0))$$

(3.9)

Subcase B. Let $r_{\lambda+2\mu}(y^*, \zeta^*) \neq 0$. Then by propositions 2.3 and 2.5, there exists a constant $C_{14}$ independent of $s$ such that

$$\left\|\frac{1}{2} \partial_{\nu} w_{2,\nu} - \Gamma_{\lambda+2\mu}(y, \tilde{D}, \tilde{\zeta})\right\|_{H^{\frac{1}{2}}(\mathbb{R}^*)}$$

(3.10)
\[ C_{14} (\| P_{\nu, \lambda + 2\mu} (y, D, \tilde{s}) w_{2, \nu} \|_{L^2(\mathcal{Q})} + \tilde{\varphi}^{\tilde{s}} (y_0^\nu) \| w_2 \|_{H^{1, 1/2}(\mathcal{Q})}) \].

On the other hand, from (2.17) and (2.18), we have
\[ (\lambda + 2\mu)(y^*) (\partial_{x_i} w_{2, \nu} - [\tilde{s}] \varphi_{\nu}(y^*) w_{2, \nu}) - \mu(y^*) \sum_{j=1}^{n-1} (\partial_{x_j} w_{j, \nu} - [\tilde{s}] \varphi_{\nu}(y^*) w_{j, \nu}) \right) (\cdot, 0) = r \text{ in } \mathbb{R}^n, \tag{3.12} \]

where the function \( r \) satisfies
\[ \| r \|^2_{L^2(\mathbb{R}^n)} \leq \epsilon(\delta) \| (\partial_{x_i} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|^2_{L^2(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n)} + C_{15} \| H^{1/2}(\mathbb{R}^n) + C_{16} \| \tilde{\varphi}^{y^*} \|^2_{L^2(\mathbb{R}^n)} \tag{3.13} \]

with some constants \( C_{15}, C_{16} \) which are independent of \( \tilde{s} \).

From (3.12), (3.13) and (3.7), it follows that
\[ \| [\tilde{s}] (\lambda + 2\mu)(y^*) (\partial_{x_i} w_{2, \nu} - [\tilde{s}] \varphi_{\nu}(y^*) w_{2, \nu}) (\cdot, 0) \|^2_{L^2(\mathbb{R}^n)} \]
\[ \leq C_{17} \sum_{j=1}^{n-1} \Xi_{\nu}(w_{j, \nu}) + \epsilon(\delta) \| [\tilde{s}] \|^2_{L^2(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n)} \]
\[ + C_{18} \| \tilde{\varphi}^{y_0^\nu} \| (\partial_{x_i} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|^2_{L^2(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n)} + \| \tilde{\varphi}^{y^*} \| \tilde{\varphi}^{y^*} \|^2_{L^2(\mathbb{R}^n)} \tag{3.14} \]

Then this estimate, Gårding’s inequality (A.2) in lemma A.6 and (3.11) imply
\[ \| [\tilde{s}] \|^3_{L^1(\mathbb{R}^n)} \leq C_{19} \sum_{j=1}^{n-1} \Xi_{\nu}(w_{j, \nu}) + \epsilon(\delta) \| [\tilde{s}] \|^2_{L^2(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n)} \]
\[ + C_{20} \| \tilde{\varphi}^{y^*} \|^2_{L^2(\mathbb{R}^n)} + \tilde{\varphi}^{y_0^\nu} \| (\partial_{x_i} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|^2_{L^2(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n)} \]
\[ + \| P_{\nu, \lambda + 2\mu} (y, D, \tilde{s}) w_{2, \nu} \|^2_{L^2(\mathcal{Q})} + \tilde{\varphi}^{y^*} \| w_2 \|_{H^{1, 1/2}(\mathcal{Q})}. \tag{3.15} \]

Inequalities (3.7), (3.15) and (3.14) imply (3.10).

3.2. Case \( r_{\lambda + 2\mu}(y^*, \zeta^*) = 0 \)

Let \( (y^*, \zeta^*) \) be a point on \( \mathbb{R}^{n+1} \times \bar{M} \) such that \( r_{\lambda + 2\mu}(y^*, \zeta^*) = 0 \) and \( \text{supp } \chi_{\nu} \subset \mathcal{O}(y^*, \delta_1(y^*)) \).

Since the case \( r_\mu(y^*, \zeta^*) = r_{\lambda + 2\mu}(y^*, \zeta^*) = 0 \) was treated in the previous section, one can assume that
\[ r_\mu(y^*, \zeta^*) \neq 0. \tag{3.16} \]
By (2.3) and (2.24), the equality \( r_{\lambda+2\mu}(y^*, \zeta^*) = 0 \) implies that
\[
\sum_{j=1}^{n-1} (\xi_j^*)^2 = (\tilde{s}^*)^2 \sum_{j=1}^{n-1} |\varphi_j(y^*)|^2 \quad \text{and} \quad \frac{\rho(y^*)\xi_0^*}{(\lambda + 2\mu)(y^*)} - \tilde{s}^* \sum_{j=1}^{n-1} \xi_j \varphi_j(y^*) = 0. \tag{3.17}
\]

Form (3.17) and (1.13) we immediately obtain \( \tilde{s}^* \neq 0 \).

By (3.17) there exists \( C_{21} > 0 \) such that for all \((y^*, \zeta)\) from \( \mathcal{O}(y^*, \delta_1(y^*))\) we have
\[
\left| \sum_{j=1}^{n-1} \xi_j^* - \tilde{s}^* \sum_{j=1}^{n-1} |\varphi_j(y^*)|^2 \right| + \frac{\rho(y^*)\xi_0^*}{(\lambda + 2\mu)(y^*)} + \tilde{s}^* \sum_{j=1}^{n-1} \xi_j \varphi_j(y^*) \leq \delta_1 C_{21} M(\tilde{\zeta}, \tilde{\xi}). \tag{3.18}
\]

By (2.28) there exists \( C_{22} > 0 \) independent of \( s \) such that
\[
\Xi_{\lambda+2\mu}(w_{2,\nu}) + C_{22} (|\tilde{s}|^2|w_{2,\nu}|^2_{L^2(\Omega)}) + (|\tilde{\xi}|^2|w_{2,\nu}|^2_{L^2(\Omega)}) \leq C_{23} \|P_{\lambda+2\mu}(y, D, \tilde{s}) w_{2,\nu}|^2_{L^2(\Omega)} + \|\|\partial_{\mu} w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)\|\|^2_{H^\frac{1}{2} + \frac{1}{2} \gamma(R^*) \times H^\frac{1}{2} + \frac{1}{2} \gamma(R^*)}
\]
where \( \epsilon(\delta) \to 0 \) as \( \delta \to +0 \).

By (2.29), (2.30), (3.18) and (2.19), we have
\[
\left| \tilde{g}_2(\lambda + 2\mu, w_{2,\nu}) + \tilde{g}_3(\lambda + 2\mu, w_{2,\nu}) \right| \leq C_{24} \delta_1 \|\|\partial_{\mu} w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)\|\|^2_{H^\frac{1}{2} + \frac{1}{2} \gamma(R^*) \times H^\frac{1}{2} + \frac{1}{2} \gamma(R^*)}
\]
\[
+ C_{25} \tilde{\nu}(y^*) \|\|\partial_{\mu} w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)\|\|^2_{L^2(\Omega) \times H^{1/2}(R^*)}
\]
where \( \epsilon(\delta) \to 0 \) as \( \delta \to +0 \). Since \( \tilde{s}^* \neq 0 \), we have
\[
\Xi_{\lambda+2\mu}(w_{2,\nu}) \geq C_{28} \int_{\mathbb{R}^n} (|\tilde{s}|^2|\partial_{\mu} w_{2,\nu}|^2 + |\tilde{\xi}|^2|w_{2,\nu}|^2) (y, 0) dy
\]
\[
- \epsilon \|\|\partial_{\mu} w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)\|\|^2_{H^\frac{1}{2} + \frac{1}{2} \gamma(R^*) \times H^\frac{1}{2} + \frac{1}{2} \gamma(R^*)}
\]
\[
- C_{29} \tilde{\nu}(y^*) \|\|\partial_{\mu} w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)\|\|^2_{L^2(\Omega) \times H^{1/2}(R^*)}
\]
where \( \epsilon(\delta) \to 0 \) as \( \delta \to +0 \). Since \( \tilde{s}^* \neq 0 \), we have
\[
\Xi_{\lambda+2\mu}(w_{2,\nu}) \geq C_{28} \sum_{j=1}^{n-1} (|\tilde{s}|^2|\partial_{\mu} w_{2,\nu}|^2 + |\tilde{\xi}|^2|w_{2,\nu}|^2) (y, 0) dy
\]
\[
- \epsilon \|\|\partial_{\mu} w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)\|\|^2_{H^\frac{1}{2} + \frac{1}{2} \gamma(R^*) \times H^\frac{1}{2} + \frac{1}{2} \gamma(R^*)}
\]
\[
- C_{29} \tilde{\nu}(y^*) \|\|\partial_{\mu} w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)\|\|^2_{L^2(\Omega) \times H^{1/2}(R^*)}
\]
(3.21)

Since \( r_{\mu}(y^*, \zeta^*) \neq 0 \), by propositions 2.3 and 2.5, there exists a constant \( C_{30} \) independent of \( s \) such that
\[
\left\| \left( \frac{1}{i} \partial_{\mu} w_{1,\nu} - \Gamma_{\mu}^+(y, D, \tilde{s}) w_{1,\nu} \right) \right\|_{H^\frac{1}{2} + \frac{1}{2} \gamma(R^*)} \leq C_{30} \|P_{\mu}(y, D, \tilde{s}) w_{1,\nu}\|_{L^2(\Omega)} + \tilde{\nu}(y^*) \|w_{1}\|_{H^\frac{1}{2} + \frac{1}{2} \gamma(\Omega)}. \tag{3.22}
\]
By (2.17) and (2.18), for any $k \in \{1, \ldots, n-1\}$, we obtain
\[
\mu(\gamma^*) (\partial_n w_{k,n} - [\bar{\gamma}] \varphi_n(\gamma^*) w_{k,n}) (\gamma, 0)
\]
\[
= (\lambda + \mu)(\gamma^*) (\partial_n w_{2,n} - [\bar{\gamma}] \varphi_n(\gamma^*) w_{2,n}) (\gamma, 0) + r(\gamma) \quad \text{in } \mathbb{R}^n,
\]
where the function $r$ satisfies estimate (3.13). From (3.23), (3.13) and (3.22), we obtain
\[
|\bar{s}| \|w_{1,\nu}(\cdot, 0) - [\bar{s}] \varphi_n(\gamma^*) w_{1,\nu}(\cdot, 0)\|_{L^2(\mathbb{R})}^2 \\
\leq C_{31} \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{n-1} |\partial_n w_{2,\nu}(\gamma, 0)|^2 + [\bar{s}]^2 |w_{2,\nu}(\gamma, 0)|^2 \right) d\gamma \\
+ |P_{\rho,\mu}(\gamma, D, s)w_{1,\nu}\|_{L^2(\mathbb{R})}^2 + [\varphi^2(\gamma^*)] \|w\|_{H^{1,1}(\mathbb{R})}^2 \\
+ \epsilon(\delta)|\bar{s}| \|\varphi_n(\cdot, 0), w_{\nu}(\cdot, 0))\|_{L^2(\mathbb{R}^n) \times H^{1,1}(\mathbb{R}^n)}^2 \\
+ [\varphi^2(\gamma^*)] \|\varphi_n(\cdot, 0), w_{\nu}(\cdot, 0))\|_{L^2(\mathbb{R}^n) \times H^{1,1}(\mathbb{R}^n)}^2 + |\bar{s}| \|\mathbf{g}^{(|\gamma^*|)}\|_{L^2(\mathbb{R})}^2 \right).
\]

Gårding’s inequality (lemma A.6) and (3.24), (3.22) yield
\[
|\bar{s}| \|w_{1,\nu}(\cdot, 0)\|_{H^{1,1}(\mathbb{R}^n)}^2 \\
\leq C_{32} \left( \|P_{\rho,\mu}(\gamma, D, s)w_{1,\nu}\|_{L^2(\mathbb{R})}^2 + [\varphi^2(\gamma^*)] \|w\|_{H^{1,1}(\mathbb{R})}^2 \\
+ \int_{\mathbb{R}^n} \left( \sum_{j=1}^{n-1} |\partial_n w_{2,\nu}(\gamma, 0)|^2 + [\bar{s}]^2 |w_{2,\nu}(\gamma, 0)|^2 \right) d\gamma \\
+ \epsilon(\delta)|\bar{s}| \|\varphi_n(\cdot, 0), w_{\nu}(\cdot, 0))\|_{L^2(\mathbb{R}^n) \times H^{1,1}(\mathbb{R}^n)}^2 \\
+ [\varphi^2(\gamma^*)] \|\varphi_n(\cdot, 0), w_{\nu}(\cdot, 0))\|_{L^2(\mathbb{R}^n) \times H^{1,1}(\mathbb{R}^n)}^2 + |\bar{s}| \|\mathbf{g}^{(|\gamma^*|)}\|_{L^2(\mathbb{R})}^2 \right).
\]

Inequalities (3.21), (3.24) and (3.25) imply
\[
\Xi_{\lambda+2\mu}(w_{2,\nu}) \geq C_{33}|\bar{s}| \|\varphi_n(\cdot, 0), w_{\nu}(\cdot, 0))\|_{L^2(\mathbb{R}^n) \times H^{1,1}(\mathbb{R}^n)}^2 \\
- C_{34}(\|P(\gamma, D, s)w_{\nu}\|_{L^2(\mathbb{R})}^2 + [\varphi^2(\gamma^*)] \|w\|_{H^{1,1}(\mathbb{R})}^2) \\
- C_{35}[\varphi^2(\gamma^*)] \|\varphi_n(\cdot, 0), w_{\nu}(\cdot, 0))\|_{L^2(\mathbb{R}^n) \times H^{1,1}(\mathbb{R}^n)}^2.
\]

From (3.19) and (3.26), we obtain (3.10).

3.3. Case $r_\mu(\gamma^*, \zeta^*) \neq 0$ and $r_{\lambda+2\mu}(\gamma^*, \zeta^*) \neq 0$

In this section we consider the conic neighborhood $\mathcal{O}(\gamma^*, \delta_1(\gamma^*))$ of the point $(\gamma^*, \zeta^*)$ such that
\[
r_\mu(\gamma^*, \zeta^*) \neq 0, \quad r_{\lambda+2\mu}(\gamma^*, \zeta^*) \neq 0.
\]

In that case, thanks to (3.27) and proposition 2.3, factorization (2.31) holds true for $\beta = \mu$ and $\beta = \lambda + 2\mu$. Then proposition 2.5 yields the \textit{a priori} estimate.
\[
\sum_{k,j=1,k<j}^n \| V^+_{\mu}(k,j) (\cdot,0) \|_{H^{\frac{\mu}{2}}(\mathbb{R}^n)} + \| V^+_{\lambda+2\mu}(\cdot,0) \|_{H^{\frac{\lambda}{2}}(\mathbb{R}^n)}
\]
\[
\leq C_{36} \left( \| P(y,D,\bar{s})w_\nu \|_{L^2(\Omega)} + \varphi_{\bar{s}}^{\frac{\mu}{2}}(y_0^*) \| w \|_{H^{\mu+\frac{\nu}{2}}(\Omega)} \right). \tag{3.28}
\]

Using (3.11) and (3.22), we rewrite (3.12) and (3.23) as
\[
\left( \frac{\lambda + 2\mu}{\mu} (y^*) \left( \partial_{y_i} w_{2,\nu} - [\bar{s}] \varphi w_{2,\nu} \right) - i\alpha^+_{\mu}(\bar{y},0,\tilde{D},\bar{s})w_{j,\nu,\nu} \right) (\cdot,0) \tag{3.29}
\]
\[
= V^+_{\mu}(j,n)(\cdot,0) - r_{j,n,\nu,\nu} \quad \text{in} \quad \mathbb{R}^n,
\]
where \( k \in \{1, \ldots, n-1\} \) and
\[
\left( \sum_{k=1}^{n-1} \lambda \lambda + 2\mu(y^*) \left( - \partial_{y_k} w_{k,\nu,\nu} + [\bar{s}] \varphi w_{k,\nu,\nu} \right) - i\alpha^+_{\lambda+2\mu}(\bar{y},0,\tilde{D},\bar{s})w_{j,\nu,\nu} \right) (\cdot,0)
\]
\[
= V^+_{\lambda+2\mu}(\cdot,0) - r_{2,\nu,\nu}, \tag{3.30}
\]
where the function \( r = (r_{1,\nu,\nu}, \ldots, r_{n-1,\nu,\nu}, r_{2,\nu,\nu}) \) satisfies estimate (3.13). Let \( B(y,\tilde{D},\bar{s}) \) be the matrix pseudodifferential operator with the symbol
\[
B(y,\tilde{D},\bar{s}) = \begin{pmatrix}
-\alpha^+_{\mu}(\bar{y},0,\tilde{s},\bar{s}) & 0 & \cdots & \frac{\lambda + 2\mu}{\mu}(\tilde{s}_{\nu} - [\bar{s}] \varphi) \\
0 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\mu}{\lambda + 2\mu}(-i\xi_{\nu} + [\bar{s}] \varphi_1) & \cdots & \frac{\mu}{\lambda + 2\mu}(-i\xi_{\nu-1} + [\bar{s}] \varphi_{\nu-1}) & -\alpha^+_{\lambda+2\mu}(\bar{y},0,\tilde{s},\bar{s})
\end{pmatrix}. \tag{3.31}
\]

We have

**Proposition 3.1.** Let \( \zeta = (\xi_0, \xi_1 + i[\bar{s}] \varphi_1, \ldots, \xi_{n-1} + i[\bar{s}] \varphi_{n-1}) \). The following formula is true:

\[
\det B(y^*,\tilde{s},\bar{s}) = (-i)^n (\alpha^+_{\mu}(y^*,\tilde{s},\bar{s}))^{n-1} \alpha^+_{\lambda+2\mu}(y^*,\tilde{s},\bar{s})
\]
\[
+ (-1)^{n-1} (-i)^{n-2} (\alpha^+_{\mu}(y^*,\tilde{s},\bar{s}))^{n-2} \sum_{j=1}^{n-1} (-i\xi_{j} + [\bar{s}] \varphi_{j}(y^*))^2. \tag{3.32}
\]

**Proof.** By \( B_{(y,\tilde{s},\bar{s})} \) we denote the \( n \times n \) matrix determined by (3.31) and \( B_{k,j,n}(y,\tilde{s},\bar{s}) \) be the minor obtained from the matrix \( B_{(y,\tilde{s},\bar{s})} \) by crossing out the \( k \)th row and the \( j \)th column. Our proof is based on the induction method. Except the formula (3.32), we claim
\[
|B_{1,n-1,n}(y^*,\tilde{s},\bar{s})| = (-i)^n (\alpha^+_{\mu}(y^*,\tilde{s},\bar{s}))^{n-2} (-i\xi_{1} + [\bar{s}] \varphi_{1}(y^*))^2. \tag{3.33}
\]

For \( n = 2, 3 \), we can easily verify the formulae by direct computations. Suppose that (3.32) and (3.33) are true for \( n - 1 \). Then
\[
\det B_{n-1}(y^*,\tilde{s},\bar{s}) = (-i)^{n-1} (\alpha^+_{\mu}(y^*,\tilde{s},\bar{s}))^{n-2} \alpha^+_{\lambda+2\mu}(y^*,\tilde{s},\bar{s})
\]
\[
+ (-1)^{n-1} (-i)^{n-3} (\alpha^+_{\mu}(y^*,\tilde{s},\bar{s}))^{n-3} \sum_{j=1}^{n-2} (-i\xi_{j} + [\bar{s}] \varphi_{j}(y^*))^2 \tag{3.34}
\]
and
\[
|B_{1,n-1,n}(y^*, \tilde{\xi}, \tilde{x})| = (-i)^{n-1} \frac{\mu}{\lambda + 2\mu} (y^*)(\alpha_{\mu}^+(y^*, \tilde{\xi}, \tilde{x}))^{n-2} (-i\xi_1 + |\tilde{x}|\varphi_1(y^*))^2.
\] (3.35)

Since \( \det B_n(y^*, \tilde{\xi}, \tilde{x}) = -i\alpha_{\mu}^+(y^*, \tilde{\xi}, \tilde{x})|B_{1,n-1,n}(y^*, \tilde{\xi}, \tilde{x})| \)
\[+(-1)^{1+n} \frac{\lambda + 2\mu}{\mu} (y^*)(i\xi_1 - |\tilde{x}|\varphi_1(y^*))|B_{1,n-1,n}(y^*, \tilde{\xi}, \tilde{x})|, \]
by (3.32) and (3.35) we have
\[
\det B_n(y^*, \tilde{\xi}, \tilde{x}) = -i\alpha_{\mu}^+(y^*, \tilde{\xi}, \tilde{x})((-i)^{n-1}(\alpha_{\mu}^+(y^*, \tilde{\xi}, \tilde{x}))^{n-2}\alpha_{\lambda + 2\mu}^+(y^*, \tilde{\xi}, \tilde{x})
\]
\[+(-1)^{n-1}(-i)^{n-3}(\alpha_{\mu}^+(y^*, \tilde{\xi}, \tilde{x}))^{n-3} \sum_{j=2}^{n-1} (-i\xi_j + |\tilde{x}|\varphi_j(y^*))^2
\]
\[+(-1)^{1+n}(-i\alpha_{\mu}^+(y^*, \tilde{\xi}, \tilde{x}))^{n-2}(-i\xi_1 + |\tilde{x}|\varphi_1(y^*))^2.
\]

The proof of the proposition is complete.

If \( (\tilde{\xi}, \tilde{\eta}) \not\in \{ (\tilde{\xi}, \tilde{x}) \mid \det B(y^*, \tilde{\xi}, \tilde{x}) = 0 \} \), then by (3.29) and (3.30) we have
\[
\| (\partial_{\nu} w_\nu (\cdot, 0), w_\nu (\cdot, 0)) \|_{L^1/2^{\tilde{\eta}}(\mathbb{R}^n) \times H^{1/2}_0(\mathbb{R}^n)} \leq C_{37} \left( \frac{\tilde{\eta}}{\eta} \right) \| w \|_{H^{1/2}(\mathbb{R}^n)} + \| g e^{i|\tilde{\eta}|} \|_{H^{1/2}(\mathbb{R}^n)}
\]
+ \[\| P(y, D, \tilde{x}) w_\nu (\cdot, 0) \|_{L^1(\mathbb{R}^n)} + \frac{C_{38}}{1 + |\tilde{x}|} \| (\partial_{\nu} w_\nu (\cdot, 0), w_\nu (\cdot, 0)) \|_{L^1(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n)} \right).
\] (3.36)

From this inequality and proposition 2.2, we obtain (3.10).

By (2.21)–(2.24), if \( \det B(y^*, \tilde{\xi}, \tilde{x}) = 0 \) and (3.27) holds true, then

\[
(\tilde{\xi}, \tilde{\eta}) \in \mathcal{U} = \left\{ (\tilde{\xi}, \tilde{x}) \in \mathbb{R}^n; \sum_{j=1}^{n-1} (\xi_j + i|\tilde{x}|\varphi_j(y^*)) = \frac{-\rho(y^*)i\xi_0}{(\lambda + 3\mu)(y^*)} \right\}
\] (3.37)

If \( (\tilde{\xi}, \tilde{x}) \in \mathcal{U} \), then
\[
\sum_{j=1}^{n-1} \xi_j^2 = \Re \sum_{j=1}^{n-1} \varphi_j^2(y^*).
\] (3.37)

By (2.21), (2.23) and (2.11), we obtain
\[
\Gamma^\pm_\beta (y^*, \tilde{\xi}, \tilde{\eta}) = -i|\tilde{\eta}|\varphi_n(y^*) \pm e^{i\tilde{x}} \sqrt{\frac{(\lambda + 3\mu - \beta)(y^*)}{\beta(y^*)}} \sqrt{|\tilde{x}| \sum_{j=1}^{n-1} \xi_j^2 \varphi_j(y^*)}.
\]

Therefore
Here, in order to obtain the last inequality in (3.38), we used (3.37). Condition (1.15) yields
\[-\text{Im } \Gamma_{\lambda + 2\mu}^+ (y^*, \tilde{r}^*, \tilde{s}^*) > 0. \tag{3.39}\]

Then by proposition 2.5 there exists a constant $C_{39}$ independent of $\tilde{s}$ such that
\[
\left\| \left( \frac{1}{2} \partial_{\nu_n} w_{2,\nu} - \Gamma_{\lambda + 2\mu}^+ (y, \tilde{D}, \tilde{s}) \right) w_{2,\nu} \big|_{y_0=0} \right\|_{H^{\frac{1}{2}, \frac{3}{2}(\mathbb{R}^n)}} \leq C_{39} (\|P_{\rho,\lambda + 2\mu} (y, D, \tilde{s}) w_{2,\nu} \|_{L^2(Q)} + \tilde{\varphi}^\beta (y_0^*) \|w_2\|_{H^{\frac{1}{2}, 1}_{\nu_j}(Q)}).
\tag{3.40}\]

Inequality (3.40) implies
\[
\|\alpha_{\lambda + 2\mu} (y, \tilde{D}, \tilde{s}) w_{2,\nu} \|_{H^{\frac{1}{2}, \frac{3}{2}(\mathbb{R}^n)}} \leq C_{40} (\|P_{\rho,\lambda + 2\mu} (y, D, \tilde{s}) w_{2,\nu} \|_{L^2(Q)} + \tilde{\varphi}^\beta (y_0^*) \|w_2\|_{H^{\frac{1}{2}, 1}_{\nu_j}(Q)}).
\tag{3.41}\]

By Gårding’s inequality, from (3.41) we obtain
\[
\|w_{2,\nu} (\cdot, 0)\|_{H^{\frac{1}{2}, \frac{3}{2}(\mathbb{R}^n)}} \leq C_{41} (\|P_{\rho,\lambda + 2\mu} (y, D, \tilde{s}) w_{2,\nu} \|_{L^2(Q)} + \tilde{\varphi}^\beta (y_0^*) \|w_2\|_{H^{\frac{1}{2}, 1}_{\nu_j}(Q)}).
\tag{3.42}\]

From (3.40) and (3.42), we obtain
\[
\| (\partial_{\nu_n} w_{2,\nu}, w_{2,\nu}) (\cdot, 0) \|_{L^2(\mathbb{R}^n) \times H^{\frac{1}{2}, \frac{3}{2}(\mathbb{R}^n)}} \leq C_{42} (\|P_{\rho,\lambda + 2\mu} (y, D, \tilde{s}) w_{2,\nu} \|_{L^2(Q)}
\tag{3.43}\]
\[
+ \tilde{\varphi}^\beta (y_0^*) \|w_2\|_{H^{\frac{1}{2}, 1}_{\nu_j}(Q)}).
\]

On the other hand, (3.11) and (3.23) hold true. In terms of (3.43), we obtain from (3.23)
\[
\mu (y^*) (\partial_{\nu_n} w_{k,\nu,\nu} - \tilde{r} |\varphi_{\nu} (y^*) w_{k,\nu,\nu}) (\cdot, 0) = \tilde{\mathbf{r}} \text{ in } \mathbb{R}^n, \tag{3.44}\]

where the function $\tilde{\mathbf{r}}$ satisfies the estimate
\[
\|\tilde{\mathbf{r}}\|_{L^2(\mathbb{R}^n)} \leq \epsilon (\delta) \| (\partial_{\nu_n} w_{\nu,\nu} (\cdot, 0), w_{\nu,\nu} (\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^{\frac{1}{2}, \frac{3}{2}(\mathbb{R}^n)}}^2 + C_{43} \left( \| (\partial_{\nu_n} w_{\nu,\nu} (\cdot, 0), w_{\nu,\nu} (\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^{\frac{1}{2}, \frac{3}{2}(\mathbb{R}^n)}} + \|P(y, D, \tilde{s}) w_{\nu,\nu} \|_{L^2(\mathbb{R}^n)} \right)^2
\tag{3.45}\]
\[
+ C_{43} \| \mathbf{g} \|_{L^2(\mathbb{R}^n)}^2.
\]

By (3.45), (3.42) and (3.28), we have
\[
\|\alpha_{\nu} (\tilde{s}, 0, D, \tilde{s}) w_{\nu,\nu} (\cdot, 0) \|_{H^{\frac{1}{2}, \frac{3}{2}(\mathbb{R}^n)}} \leq C_{45} (\|P(y, D, \tilde{s}) w_{\nu,\nu} \|_{L^2(\mathbb{R}^n)}
\tag{3.46}\]
\[
+ \tilde{\varphi}^\beta (y_0^*) \|w_2\|_{H^{\frac{1}{2}, 1}_{\nu_j}(Q)} + \| (\partial_{\nu_n} w_{\nu,\nu} (\cdot, 0), w_{\nu,\nu} (\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^{\frac{1}{2}, \frac{3}{2}(\mathbb{R}^n)}} + \| \mathbf{g} \|_{L^2(\mathbb{R}^n)}^2) \right).
By Gårding’s inequality, from (3.46) we have
\[
\|w_{1,\nu}(\cdot,0)\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)} \leq C_{46}(\|P(y,D,\tilde{\gamma})w_{\nu}\|_{L^2(\Omega)}) + \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)}
\] (3.47)
\[
+\sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)} + \sqrt{\nu^2}\|\partial_{\nu}^{(\cdot)}(w(\cdot,0),w(\cdot,0))\|_{L^2(\mathbb{R}^n)} + \|\partial_{\nu}^{(\cdot)}w(\cdot,0)\|_{L^2(\mathbb{R}^n)} + \|\partial_{\nu}^{(\cdot)}w(\cdot,0)\|_{L^2(\mathbb{R}^n)}.
\]

From (3.47) and (3.42), we obtain (3.36). From (3.36) and proposition 2.2, we reach (3.10).

4. Completion of proof of theorem 1.1

Without lost of generality we can assume that dist (supp \(w\), \(-T\) \(\times\) \(\Omega\)) > 0. Let us take the covering of the surface \(M = \{(\xi_0,\tilde{x})\}; M(\xi_0,\tilde{x}) = 1\) by conical neighborhoods \(O(\zeta^*,\delta_1(\zeta^*))\).

From this covering, we take a finite subcovering \(\bigcup_{\nu=1}^N O(\zeta^*,\delta_1(\zeta^*))\). By (1.17) such a subcovering can be taken independently of parameter \(\ell > \ell_0\) where \(\ell_0\) is sufficiently large.

For index \(\ell \in \{1, \ldots, \ell_0\}\), we consider an individual subcovering of the set \(M\). Let \(\chi_\ell\) be a partition of unity associated to one of above chosen subcoverings. Hence \(\sum_{\nu=1}^N \chi_\ell(\xi,\tilde{x}) \equiv 1\) for all \((\xi,\tilde{x})\) such that \(M(\xi,\tilde{x}) > 1\).

Let \(\chi_0(\xi,\tilde{x}) \in C_0^\infty(\mathbb{R}^{n+1})\) be a nonnegative function which is identically equal to one if \(M(\xi,\tilde{x}) \leq 1\). Then estimate (3.10) yields

\[
\|\partial_{\nu}^{(\cdot)}w(\cdot,0),w(\cdot,0))\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}) \times H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R})} + \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)}
\]
\[
\leq C_4 \left(\sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)} + \|\partial_{\nu}^{(\cdot)}w(\cdot,0)\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}) \times H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R})} + \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)}\right)
\]
\[
+ C_1 \sum_{\nu=0}^N \left(\|\partial_{\nu}^{(\cdot)}w(\cdot,0),w(\cdot,0))\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}) \times H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R})} + \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)}\right)
\]
\[
+ C_3 \left(\|\partial_{\nu}^{(\cdot)}w(\cdot,0),w(\cdot,0))\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}) \times H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R})} + \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)}\right)
\]
\[
+ C_4 \left(\sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)} + \|\partial_{\nu}^{(\cdot)}w(\cdot,0)\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}) \times H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R})} + \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)}\right).
\]

By (2.35), there exists a constant \(C_4\) independent of \(\tilde{x}, \ell\) and \(\nu\) such that
\[
\sum_{\nu=0}^N \left(\|\partial_{\nu}^{(\cdot)}w(\cdot,0),w(\cdot,0))\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}) \times H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R})} + \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)}\right)
\]
\[
\leq C_4 \left(\sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)} + \|\partial_{\nu}^{(\cdot)}w(\cdot,0)\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}) \times H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R})} + \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)}\right).
\]

In order to estimate the last terms in (4.1), using inequality (4.2), we obtain
\[
\|\partial_{\nu}^{(\cdot)}w(\cdot,0),w(\cdot,0))\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}) \times H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R})} + \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)}
\]
\[
\leq C_5 \left(\sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)} + \|\partial_{\nu}^{(\cdot)}w(\cdot,0)\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}) \times H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R})} + \|\partial_{\nu}^{(\cdot)}w(\cdot,0)\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}) \times H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R})} + \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)}\right)
\]
\[
+ \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)} + \|\partial_{\nu}^{(\cdot)}w(\cdot,0)\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}) \times H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R})} + \sqrt{\nu^2}\|w_{\nu}\|_{H^\frac{1}{2}+\frac{\nu}{2}(\mathbb{R}^n)}\right).
\]
\[
\begin{align*}
\lesssim C_1\left(\sqrt{\varphi(y_0)}\|w\|_{H^{1/2}(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)} + \|P(y, D, \tilde{z})\|_{L^2(\Omega)}\right) \\
+ C_8 \left(\sqrt{\varphi(y_0)}\|\partial_y w(\cdot, 0), w(\cdot, 0)\|_{L^2(\Omega)} + \sqrt{\varphi(y_0)}\|w\|_{H^{1/2}(\Omega)}\right).
\end{align*}
\]

Hence there exists \( s_0 > 1 \) such that for all \( s \geq s_0 \) we see
\[
\|\partial_y w(\cdot, 0), w(\cdot, 0)\|_{L^2(\Omega)} + \sqrt{\varphi(y_0)}\|w\|_{H^{1/2}(\Omega)} 
\leq C_9 \left(\sqrt{\varphi(y_0)}\|w\|_{H^{1/2}(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)} + \|P(y, D, \tilde{z})\|_{L^2(\Omega)}\right).
\]

The proof of proposition 2.1 is complete.

Now we proceed to the completion of the proof of theorem 1.1.

Short calculations imply that the functions \( d_\omega, \text{div} u \) satisfy the equations
\[
P_{\rho, \mu}(x, D) d_\omega - \int_0^\tau \tau(\tilde{x}, \tilde{x}_0) \Delta d\omega \, d\tilde{x}_0 = q_1 \quad \text{in } Q := (-T, T) \times \Omega,
\]
\[
P_{\rho, \lambda+2\mu}(x, D) \text{div} u - \int_0^\tau (\lambda + 2\tilde{\mu})(\tilde{x}, \tilde{x}_0) \Delta \text{div} u \, d\tilde{x}_0 = q_2 \quad \text{in } Q,
\]
where
\[
q_1 = K_1(x, D) \text{div} u + \int_0^\tau (\tilde{K}_1(x, \tilde{x}_0, D) d\omega_u + \tilde{K}_2(x, \tilde{x}_0, D) d\omega) d\tilde{x}_0 + \rho \text{div} \theta_{/\rho},
\]
\[
q_2 = K_3(x, D) \text{div} u + \int_0^\tau (\tilde{K}_3(x, \tilde{x}_0, D) d\omega_u + \tilde{K}_4(x, \tilde{x}_0, D) d\omega) d\tilde{x}_0 + \rho \text{div} (F/\rho),
\]
where \( K_j(x, D), \tilde{K}_j(x, \tilde{x}_0, D) \) are first order differential operators with \( C^1 \)-coefficients.

Now we introduce new unknown function \( v = (v_1, v_2) \) by formulae
\[
v_1 = d_\omega + \int_0^\tau \frac{\mu(x, \tilde{x}_0)}{\mu(x)} d\omega_u \, d\tilde{x}_0, \quad v_2 = \text{div} u + \int_0^\tau \frac{\lambda + 2\tilde{\mu}(x, \tilde{x}_0)}{\lambda + 2\mu}(\tilde{x}, \tilde{x}_0) d\tilde{x}_0.
\]

More specifically, we can write
\[
v_1 = (v_{1,2}, \ldots, v_{n-1,n}), \quad v_{k,k} = \partial_{x_k} u_k - \partial_{x_k} u_j + \int_0^\tau \frac{\mu(x, \tilde{x}_0)}{\mu(x)} (\partial_{x_k} u_k - \partial_{x_k} u_j)(\tilde{x}_0, x') d\tilde{x}_0.
\]

Then from (4.5) we have
\[
P(x, D) v = (P_{\rho, \mu}(x, D) v_1, P_{\rho, \lambda+2\mu}(x, D) v_2) = q \quad \text{in } Q,
\]
where \( q = (q_1, q_2) = (\tilde{q}_1, q_2 + \tilde{q}_2) : \)
\[
\tilde{q}_1 = -\mu \int_0^\tau \left(2\nabla \frac{\mu(x, \tilde{x}_0)}{\mu(x)} \nabla(d_\omega) + \Delta \left(\frac{\mu(x, \tilde{x}_0)}{\mu(x)} d_\omega\right)\right) d\tilde{x}_0 + \rho \frac{\partial_2}{\partial_1} \int_0^\tau \frac{\mu(x, \tilde{x}_0)}{\mu(x)} d\omega_u \, d\tilde{x}_0.
\]
\begin{equation}
\tilde{q}_2 = - (\lambda + 2\mu) \int_0^{\alpha_0} \left( 2(\nabla' (\tilde{\lambda} + 2\tilde{\mu})(x, \tilde{x}_0), \nabla' \text{div} u) + \Delta \left( \frac{\tilde{\mu}(x, \tilde{x}_0)}{(\lambda + 2\mu)(x)} \right) \text{div} u \right) \, d\tilde{x}_0 \\
+ \rho \partial_{\tilde{x}_0} \int_0^{\alpha_0} \frac{\tilde{\lambda} + 2\tilde{\mu}(x, \tilde{x}_0)}{(\lambda + 2\mu)(x)} \text{div} u \, d\tilde{x}_0.
\end{equation}

Next we show that the boundary condition (2.2) holds true for some function \(g\).

We start with the following proposition:

**Proposition 4.1.** Let \(R(x, \tilde{x}_0, D')\) be a partial differential operator of order \(\tilde{j}\) with \(C^0\)-smooth coefficients and let \(\kappa \geq 0\). Then there exists a constant \(C_{10}\) independent of \(s\) such that

\begin{equation}
\left\| e^{\phi \tilde{\varphi}^\kappa} \int_0^{\alpha_0} R(x, \tilde{x}_0, D') u(\tilde{x}_0, \cdot) \, d\tilde{x}_0 \right\|_{L^2(-T, T)} \leq \frac{C_{10}}{\sqrt{s}} \sum_{|\alpha| \leq \tilde{j}, \alpha_0 = 0} \| e^{\phi \tilde{\varphi}^\kappa} \partial_\xi^\alpha u \|_{L^2(-T, T)}
\end{equation}

for all \(s \geq 1\). If \(r_0 \in C^1(\Sigma)\) and \(p \in (0, 1)\), then there exists a constant \(C_{11}\) independent of \(s\) such that

\begin{equation}
\left\| e^{\phi \tilde{\varphi}^\kappa} \int_0^{\alpha_0} r_0 u(\tilde{x}_0, \cdot) \, d\tilde{x}_0 \right\|_{H^{\frac{1}{2}}(\Sigma)} \leq \frac{C_{11}}{\sqrt{s}} \left\| e^{\phi \tilde{\varphi}^\kappa} u \right\|_{H^{\frac{1}{2}, p}(\Sigma)}.
\end{equation}

**Proof.** Instead of inequality (4.12), it suffices to prove

\begin{equation}
\left\| e^{\phi \tilde{\varphi}^\kappa} \int_0^{\alpha_0} |u(\tilde{x}_0, \cdot)| \, d\tilde{x}_0 \right\|_{L^2(-T, T)} \leq \frac{C_{12}}{\sqrt{s}} \left\| e^{\phi \tilde{\varphi}^\kappa} u \right\|_{L^2(-T, T)} \forall x' \in \Omega.
\end{equation}

The inequality (4.12) is equivalent to

\begin{equation}
\left\| e^{\phi \tilde{\varphi}^\kappa} \int_0^{\alpha_0} |g(\tilde{x}_0)| \, d\tilde{x}_0 \right\|_{L^2(0, T)} \leq \frac{C_{13}}{\sqrt{s}} \left\| e^{\phi \tilde{\varphi}^\kappa} g \right\|_{L^2(0, T)} \forall s \geq 1,
\end{equation}

where \(g\) is an arbitrary function from \(L^2(0, T)\). Applying the Cauchy inequality, we have

\begin{equation}
\left\| e^{\phi \tilde{\varphi}^\kappa} \int_0^{\alpha_0} |g(\tilde{x}_0)| \, d\tilde{x}_0 \right\|_{L^2(0, T)} \leq \int_0^T \int_0^{\alpha_0} g^2(\tilde{x}_0) \, d\tilde{x}_0 \, dx_0 dx_0
\end{equation}

\begin{equation}
= \int_0^T \left( \int_0^{\alpha_0} x_0 \tilde{\varphi}^\kappa \left( \frac{\partial_x^2 \tilde{\phi}}{2\tilde{\varphi}} \right) \right) \int_0^{\alpha_0} g^2(\tilde{x}_0) \, d\tilde{x}_0 \, dx_0 dx_0
\end{equation}

\begin{equation}
= - \int_0^T \int_0^{\alpha_0} g^2(\tilde{x}_0) \, d\tilde{x}_0 \, dx_0 dx_0
\end{equation}

By (1.15) we see that \(\frac{x_0 \tilde{\varphi}^\kappa}{\partial_x^2 \tilde{\varphi}} \in W^{1, \infty}(0, T/2)\) and \(\left| \frac{x_0 \tilde{\varphi}^\kappa}{\partial_x^2 \tilde{\varphi}} \right| \leq C_{14} \tilde{\varphi}^\kappa\). Therefore

\begin{equation}
\left\| e^{\phi \tilde{\varphi}^\kappa} \int_0^{\alpha_0} |g(\tilde{x}_0)| \, d\tilde{x}_0 \right\|_{L^2(0, T)} \leq \frac{C_{15}}{\sqrt{s}} \left\| e^{\phi \tilde{\varphi}^\kappa} g \right\|_{L^2(0, T)} \forall s \geq 1.
\end{equation}
Here in order to obtain the last inequality, we used (1.15) and (1.16).

In order to prove the estimate (4.13), we first show that

\[
\left\| e^{\varphi} \int_0^{\tau_0} r_0 u(\tilde{x}_0, \cdot) d\tilde{x}_0 \right\|_{H^{1/2}(\Sigma)} \leq C_{17} \left\| e^{\varphi} u \right\|_{H^{1/2}(\Sigma)}.
\]

(4.16)

Consider the operator \( K_w = e^{\varphi(r) } \int_0^{\tau_0} e^{-\varphi(\tilde{x}_0, \cdot)} r_0(\tilde{x}_0) w(\tilde{x}_0, \cdot) d\tilde{x}_0 \). By (1.15), we have

\[
\left\| K \right\|_{L(\mathcal{L}(\Sigma), L^1(\Sigma))} \leq C_{18} \left\| r_0 \right\|_{L^\infty(\Sigma)} \]

(4.17)

and (4.15) yields

\[
\left\| K \right\|_{L(H^{1/2}(\Sigma), H^{1/2}(\Sigma))} \leq C_{19} \left\| r_0 \right\|_{H^1(\Sigma)}.
\]

(4.18)

From (4.17), (4.18) and the interpolation argument, we obtain (4.16). By (4.15) we see

\[
\left\| K \right\|_{L(H^{s_1}(\Sigma), H^{s_1}(\Sigma))} \leq C_{20} \left\| r_0 \right\|_{H^{1/2}(\Sigma)}.
\]

(4.19)

From (4.17), (4.19) and the interpolation argument, we have

\[
\left\| e^{\varphi} \int_0^{\tau_0} r_0 u(\tilde{x}_0, \cdot) d\tilde{x}_0 \right\|_{H^{s_1}(\Sigma)} \leq C_{21} \left\| e^{\varphi} u \right\|_{H^{s_1}(\Sigma)}.
\]

(4.20)

Estimates (4.16) and (4.20) imply (4.13). The proof of the proposition is complete. \( \blacksquare \)

**Proposition 4.2.** Let \( v \) be given by (4.8) and \( u \) satisfy (1.6). Then \( v \) satisfies problem (2.1) - (2.2) with functions \( g \) and \( q \) and we obtain

\[
\left\| q e^{\varphi} \right\|_{L^2(\Omega)} \leq C_{22} \left( \left\| \nabla (\text{div } u) e^{\varphi} \right\|_{L^2(\Omega)} + \left\| (\nabla \text{div } u) e^{\varphi} \right\|_{L^2(\Omega)} + \left\| (\text{div } u) e^{\varphi} \right\|_{L^2(\Omega)} + \left\| \text{div } F e^{\varphi} \right\|_{L^2(\Omega)} \right)
\]

(4.21)

and

\[
\left\| g e^{\varphi} \right\|_{H^{\frac{1}{2}}(\Sigma)} \leq C_{23} \left( \left\| \text{Fe}^{\varphi} \right\|_{H^{\frac{1}{2}}(\Sigma)} + \left\| (\partial_z u) e^{\varphi} \right\|_{H^{\frac{1}{2}}(\Sigma)} \right).
\]

(4.22)

**Proof.** Let \( q = (q_1, q_2) \) be given by (4.6)-(4.11). Applying (4.12) to estimate \( q \), we obtain (4.21). Next we show that \( v \) satisfies the boundary conditions (2.2). We set all the components of the function \( g \) starting from \( n + 1 \) be equal to zero:

\[
g_k = 0 \quad \text{for all} \quad k \geq n + 1.
\]

(4.23)
By (1.6) and the zero Dirichlet boundary conditions for the function $u$, we have

$$-L_{\lambda,\mu}(x, D')u + \int_0^{\xi_0} L_{\lambda,\mu}(x, \tilde{x}_0, D')u(\tilde{x}_0, x')d\tilde{x}_0 = F \quad \text{on } \Sigma. \quad (4.24)$$

Next we move all the terms containing the first derivatives of the function $u$ into the right-hand side, divide both sides by $\mu$ and denote the right-hand side of obtained equality as $g_i$. Then the first $n$ components of the function $g$ are defined by formula:

$$(g_1, \ldots, g_n) = \frac{1}{\mu} (F - (\div u)\nabla' \lambda - (\nabla' u + (\nabla' u)^T)\nabla' \mu)$$

$$-\int_0^{\xi_0} ((\div u)\nabla' \lambda + (\nabla' u + (\nabla' u)^T)\nabla' \mu)d\tilde{x}_0) \quad \text{on } \Sigma. \quad (4.25)$$

Then by (1.1) and (1.2), we have

$$-\Delta u - \left(\frac{\lambda + \mu}{\mu}\nabla' \div u - \int_0^{\xi_0} \left(\frac{\tilde{\lambda}(x, \tilde{x}_0)}{\tilde{\mu}(x)}\Delta u(\tilde{x}_0, x') + \frac{\tilde{\lambda}(x, \tilde{x}_0) + \tilde{\mu}(x, \tilde{x}_0)}{\mu(x)}\nabla' \div u \right)d\tilde{x}_0 \right) = (g_1, \ldots, g_n) \quad \text{on } \Sigma. \quad (4.26)$$

By (4.25) and $u|_\Sigma = 0$, we can choose some functions $p_{k,m} \in C^1([-T, T]^2 \times \Omega)$ such that

$$\sum_{j=1}^n \|g_j e^{\nu_j}\|_{H^{\frac{1}{2}}(\Sigma)} \leq C_24 \left( \|Fe^{\nu_j}\|_{H^{\frac{1}{2}}(\Sigma)} + \|\div e^{\nu_j}\|_{H^{\frac{1}{2}}(\Sigma)} \right)$$

$$+ \sum_{k,m=1}^n \left\| \int_0^{\xi_0} p_{k,m}(\tilde{x}_0, x)\partial_{x} u_m(\tilde{x}_0, x')d\tilde{x}_0 e^{\nu_j} \right\|_{H^{\frac{1}{2}}(\Sigma)}. \quad (4.27)$$

Estimating the last term on the right-hand side of (4.27) and using (4.13), we obtain (4.22).

For any index $\hat{j} \in \{1, \ldots, n\}$, short computations imply

$$-\Delta u_{\hat{j}} = \sum_{j=1, j \neq \hat{j}}^n -\partial_{x_j} (\partial_{x_j} u_{\hat{j}} - \partial_{x_{\hat{j}}} u_j) - \partial_{x_{\hat{j}}}^2 u_{\hat{j}} - \sum_{j=1, j \neq \hat{j}}^n \partial_{x_j} \partial_{x_{\hat{j}}} u_j$$

$$= -\sum_{j=1, j \neq \hat{j}}^n \partial_{x_j} (\partial_{x_j} u_{\hat{j}} - \partial_{x_{\hat{j}}} u_j) - \partial_{x_{\hat{j}}} \div u \quad \text{in } \Omega. \quad (4.28)$$

Using the equality (4.28), we rewrite $\hat{j}$th equation in (4.26) as

$$b_{\hat{j}}(x, D)v = -\sum_{j=1, j \neq \hat{j}}^n \text{sign}(j - \hat{j})\partial_{x_j} v_{\hat{j}} - \frac{\lambda + 2\mu}{\mu}\partial_{x_{\hat{j}}} v_{\hat{j}} = g_{\hat{j}} \quad \text{on } \Sigma. \quad (4.29)$$

The construction of the operator $B_1(x, D)$ is complete.

Now we construct the matrix $B_2(x')$. By (1.7) and (2.3), we have

$$\tau_{kj}(x) = \nu_k(x')\partial_{x_k} u_j - \nu_j(x')\partial_{x_k} u_k, \quad 1 \leq k < j \leq n$$

Using the equality (4.28), we rewrite $\hat{j}$th equation in (4.26) as

$$b_{\hat{j}}(x, D)v = -\sum_{j=1, j \neq \hat{j}}^n \text{sign}(j - \hat{j})\partial_{x_j} v_{\hat{j}} - \frac{\lambda + 2\mu}{\mu}\partial_{x_{\hat{j}}} v_{\hat{j}} = g_{\hat{j}} \quad \text{on } \Sigma. \quad (4.29)$$

The construction of the operator $B_1(x, D)$ is complete.
and
\[ v_{k,j}(y^*) = 0 \quad \text{for} \quad 1 \leq k < j < n, \quad v_{j,j}(y^*) = -\partial_{x_j}u_j(y^*), \quad v_2 = -\partial_{x_1}u_1(y^*). \]

Set \( \tilde{v} = (v_{1,1}, \ldots, v_{2,n}, \ldots, v_{n-1,n}, v_2) \). Obviously in a small neighborhood of \( y^* \), there exists a smooth matrix \( B_3(x') \) such that
\[
(\partial_{x_i}u_1, \ldots, \partial_{x_n}u_n) = B_3(x') \tilde{v} \quad \forall x \in \Sigma \cap B(y^*, \delta).
\]

Then \( B_2(x')v = v - (\nu_2 \partial_{x_1}u_1 - \nu_1 \partial_{x_2}u_2, \ldots, \nu_n \partial_{x_n}u_1 - \nu_1 \partial_{x_n}u_n, \ldots, \nu_n \partial_{x_n}u_{n-1} - \nu_{n-1} \partial_{x_n}u_n, \sum_{j=1}^n \nu_j \partial_{x_j}u_j) = 0 \). The proof of proposition 4.2 is complete. □

By proposition 4.2, the Carleman estimate (2.4) holds true for \( v \) given by formulae (4.8). Estimating the right-hand side of (2.4) and using the inequalities (4.21) and (4.22), we have
\[
\left\| (\partial_\nu v, v)e^{\nu \varphi} \right\|_{L^2(\Sigma)} + \sqrt{\int_Q \sum_{|\alpha| \leq 2}(s \varphi)_{2\alpha} \left| \partial^\nu_{\alpha} \omega \right|^2 e^{\nu \varphi} dx} \leq C_{25} \left( \left\| F e^{\nu \varphi} \right\|_{L^2(\Sigma)} + \left\| (\partial_\nu u) e^{\nu \varphi} \right\|_{L^2(\Sigma)} + \left\| (\partial_\nu v, v) e^{\nu \varphi} \right\|_{L^2(\Sigma)} + \left\| (\partial_\nu v, v) e^{\nu \varphi} \right\|_{L^2(\Sigma)} \right)
\]
\[
+ \int_Q \sum_{|\alpha| \leq 2}(s \varphi)_{2\alpha} \left| \partial^\nu_{\alpha} \omega \right|^2 e^{\nu \varphi} dx \leq C_{26} \int_Q \sum_{|\alpha| \leq 2} \left( s \varphi \right)_{2\alpha} \left| \partial^\nu_{\alpha} v \right|^2 e^{\nu \varphi} dx
\]
(4.29)

for all sufficiently large \( s \).

Next we prove

**Proposition 4.3.** Let \( \delta > 0 \) be sufficiently small. There exists \( s_0 > 0 \) such that there exists a constant \( C_{27} = C_{27}(s_0) > 0 \) independent of \( s \) such that
\[
\left\| \partial_\nu (\omega u) e^{\nu \varphi} \right\|_{L^2(\Sigma)} + \left\| \partial_\nu (\omega u) e^{\nu \varphi} \right\|_{L^2(\Sigma)} \leq C_{27} \left( \left\| \omega e^{\nu \varphi} \right\|_{L^2(\Sigma)} + \left\| F e^{\nu \varphi} \right\|_{L^2(\Sigma)} \right)
\]
(4.31)

for all \( s \geq s_0 \).

**Proof.** By (4.12) there exists a constant \( C_{28} \) independent of \( s \) such that
\[
\left\| (\omega \omega - \omega \delta u, \omega u) e^{\nu \varphi} \right\|_{L^2(\Sigma)} \leq C_{28} \left\| \omega e^{\nu \varphi} \right\|_{L^2(\Sigma)}.
\]
(4.32)

Thanks to the zero Dirichlet boundary conditions on \( \Sigma \), there exists a smooth matrix \( V(x) \) such that \( \partial_\nu u = V(x)(\omega \omega - \omega \delta u, \omega u) \). Therefore
\[
\left\| \partial_\nu (\omega u) e^{\nu \varphi} \right\|_{L^2(\Sigma)} \leq C_{29} \left\| (\omega \omega - \omega \delta u, \omega u) e^{\nu \varphi} \right\|_{L^2(\Sigma)}.
\]
(4.33)

From equation (1.6) on \( \Sigma \), we have
\[
\partial_\nu^2 u = \tilde{A}(x, D') \partial_\nu u + \tilde{B}(x) F + \int_0^{\delta} \tilde{L}_{\lambda, \tilde{\nu}}(x, \tilde{x}_0, D') u(x, \tilde{x}_0, x') dx_0,
\]
(4.34)
where $\tilde{A}(x, D')$ is a first order differential operator on $\partial\Omega$ and $\tilde{B}$ is a $C^1$-matrix valued function. From (4.34) and (4.33), we have

$$\|\dot{\partial}_2 \text{ue}^\sigma\|_{H^{\frac{1}{2}}(\Sigma)} \leq \|\tilde{A}(x, D')\partial_2 \text{u}\|_{H^{\frac{1}{2}}(\Sigma)} + \|\tilde{B}\|_{H^{\frac{1}{2}}(\Sigma)}$$

$$+ \left\| \tilde{B} \int_0^s L(x, \tilde{x}, D') \text{u}(\tilde{x}_0, \tilde{x}'_0) d\tilde{x}_0 \right\|_{H^{\frac{1}{2}}(\Sigma)} \leq C_{30} \left( (\|\text{d}_{\Sigma}\text{u}\), \text{div} \text{u}\|_{H^{\frac{1}{2}}(\Sigma)} + \right.$$

$$+ \sum_{j=1}^n \left\| \tilde{B}_j \int_0^s \tilde{B}_j(\tilde{x}_0, x) \partial_2^2 \text{u}(\tilde{x}_0, x') d\tilde{x}_0 \right\|_{H^{\frac{1}{2}}(\Sigma)} + \sum_{j=1}^n \left\| \tilde{B}_j \int_0^s \tilde{B}_j(\tilde{x}_0, x) \partial_2^2 \text{u}(\tilde{x}_0, x') d\tilde{x}_0 \right\|_{H^{\frac{1}{2}}(\Sigma)}$$

$$+ \left\| \text{Fe}^\sigma \|_{H^{\frac{1}{2}}(\Sigma)} \right).$$

(4.35)

In order to obtain the last inequality, we used (4.33). By (4.13), for any $s \geq 1$ we have

$$\sum_{j=1}^n \left\| \tilde{B}_j \int_0^s \tilde{B}_j(\tilde{x}_0, x) \partial_2^2 \text{u}(\tilde{x}_0, x') d\tilde{x}_0 \right\|_{H^{\frac{1}{2}}(\Sigma)} + \sum_{j=1}^n \left\| \tilde{B}_j \int_0^s \tilde{B}_j(\tilde{x}_0, x) \partial_2^2 \text{u}(\tilde{x}_0, x') d\tilde{x}_0 \right\|_{H^{\frac{1}{2}}(\Sigma)}$$

$$\leq \frac{C_{32}}{s} \left( \|\dot{\partial}_2 \text{ue}^\sigma\|_{H^{\frac{1}{2}}(\Sigma)} + \|\partial_2 \text{ue}^\sigma\|_{H^{\frac{1}{2}}(\Sigma)} \right).$$

(4.36)

From (4.35), (4.36) and (4.33), we obtain (4.31). The proof of proposition 4.3 is complete.

Next we prove

**Proposition 4.4.** Let $\text{u} \in H^1(\Omega)$ and $\text{u}|_{\Sigma} = 0$. There exists $s_0 > 1$ and constant $C_{33}$ independent of $s$, such that

$$\sum_{|\alpha| \leq 2} \int_{\Omega} (s\tilde{\varphi})^{4-2|\alpha|} |\partial_\alpha \text{u}|^2 e^{2s\varphi} dx \leq C_{33} (||s\tilde{\varphi})^{2} \nabla' \text{d}\text{u} e^{\varphi}||^2_{L^2(\Omega)} + ||s\tilde{\varphi})^{2} (\text{div} \text{u} e^{\varphi}||^2_{L^2(\Omega)}$$

$$+ ||s\tilde{\varphi})^{2} \partial_\alpha \text{d}\text{u} e^{\varphi}||^2_{L^2(\Omega)} + ||s\tilde{\varphi})^{2} (\nabla' \text{d}\text{u} e^{\varphi}||^2_{L^2(\Omega)} + \partial_\alpha \text{div} \text{u} e^{\varphi}||^2_{L^2(\Omega)})$$

$$+ s^2 \partial_\alpha \partial_\alpha \text{u} e^{\varphi}||^2_{L^2(\Omega)}$$

(4.37)

for all $s \geq s_0$.

**Proof.** Let $x_0 \in (-T, T)$ be an arbitrary but fixed. For any index $j \in \{1, \ldots, n\}$, short computations imply

$$-\Delta u_j = \sum_{j=1, j \neq j}^n -\partial_\alpha (\partial_\beta u_j - \partial_\alpha u_j) - \partial_\alpha^2 u_j - \sum_{j=1, j \neq j}^n \partial_\alpha \partial_\alpha u_j$$

$$= -\sum_{j=1, j \neq j}^n \partial_\alpha (\partial_\beta u_j - \partial_\alpha u_j) - \partial_\alpha \text{div} \text{u} \text{ in } \Omega.$$

(4.38)
Then the Carleman estimate with non-zero boundary values for the Laplace operator implies
\[
\int_0^s \left( \sum_{j,k=1}^n |\partial_{x_j}^2 u|^2 + s^2 \partial_t^2 |\nabla' u|^2 + s^4 \partial_t^4 |u|^2 \right) e^{2s^2} dx \leq C_{35}(s^2 \partial_t^{3/2} \nabla' \text{div} u e^{2s^2})^2_{L^2(\Omega)} + \|s^4 \partial_t^2 \nabla' d\omega \|_{L^2(\Omega)}^2 + \int_\Sigma s^2 \partial_t^2 |\partial_\nu u|^2 e^{2s^2} d\Sigma. \tag{4.39}
\]
We differentiate both sides of equation (4.38) with respect to the variable \(s\) and take the \(H^{-1}\)-Carleman estimate in [20]:
\[
\int_0^s s^2 \partial_t^2 |\partial_\nu u|^2 e^{2s^2} dx \leq C_{35}(s^2 \partial_t^{3/2} \text{div} \partial_\nu u e^{2s^2})^2_{L^2(\Omega)} + \|s^4 \partial_t^2 \partial_\nu d\omega u e^{2s^2}||_{L^2(\Omega)}^2 + \int_\Sigma s^2 \partial_t^2 |\partial_\nu \partial_\nu u|^2 e^{2s^2} d\Sigma. \tag{4.40}
\]
Combination of (4.39) and (4.40) implies (4.37). The proof of the proposition is complete.  

By propositions 4.3 and 4.4, from (4.29) and (4.30), we obtain
\[
\|\partial_\nu^2 u e^{\varphi_s}\|_{H^{1/2}+\frac{1}{2}\Sigma}^2 + \|\partial_\nu u e^{\varphi_s}\|_{H^{1/2}+\frac{1}{2}\Sigma}^2 \leq C_{36} \left( \|\partial_\nu^2 u e^{\varphi_s}\|_{H^{1/2}+\frac{1}{2}\Sigma}^2 + \|\partial_\nu u e^{\varphi_s}\|_{H^{1/2}+\frac{1}{2}\Sigma}^2 \right) + \|\partial_\nu (d\omega u, \text{div} u) e^{\varphi_s}\|_{H^{1/2}+\frac{1}{2}\Sigma}^2 + \|\partial_\nu \partial_\nu u e^{\varphi_s}\|_{L^2(\Sigma)}^2 + \|\partial_\nu \partial_\nu \partial_\nu u e^{\varphi_s}\|_{L^2(\Sigma)}^2 \tag{4.41}
\]
for all \(s \geq s_0\). By (4.21) and (4.41), we obtain the estimate (1.22). Thus the proof of theorem 1.1 is finished.

5. Proof of theorem 1.2

We differentiate equations (1.23)–(1.25) with respect to \(x_0\):
\[
\rho \partial_{x_0}^2 u = L_{\lambda, \mu}(x', D') \partial_{x_0}^2 u + L_{\lambda, \mu}(x, D') \partial_\nu u + L_{\lambda, \lambda, \lambda, \lambda, \mu}(x, D') u + \partial_\lambda R(x) f(x') \text{ in } (0, T) \times \Omega. \tag{5.1}
\]
From (1.23) and (1.24), we obtain
\[
\partial_{x_0}^2 u(\eta, \cdot) = \frac{1}{\rho} (L_{\lambda, \mu}(x', D') b + L_{\mu(\eta), \lambda}(x', D') a + R(\eta, \cdot) f), \quad \partial_{x_0} u(\eta, \cdot) = b, \tag{5.2}
\]
\[
\partial_{x_0}^2 u|_{(0, T) \times \partial \Omega} = 0. \tag{5.3}
\]
We set \(y = \partial_{x_0}^2 u\). Then
\[
\partial_{x_0} u(x_0, x') = \int_\eta y(\bar{x}, x') d\bar{x}_0 + b(x').
\]
Using this equality, we rewrite (5.1)–(5.3) in terms of the unknown function $\mathbf{y}$:

$$
\rho \partial_\eta \mathbf{y} = L_{\lambda, \mu}(x', D') \mathbf{y} + L_{\lambda, \mu}(x, D') \int_0^\eta \mathbf{y}(x_0, x') dx_0 + \mathbf{F}, \quad \text{in } \Omega \times (0, T),
$$

where $\mathbf{F}(x) = \rho \partial_\eta L_{\lambda, \mu}(x, D') \mathbf{u} + L_{\lambda, \mu}(x, D') \eta' \mathbf{u} + \partial_\eta R(x) \mathbf{f}(x')$.

Let $\tilde{\mathbf{y}}(x) = \mathbf{y}(x_0 + \eta, x', \tilde{\mathbf{u}}(x)) = \mathbf{u}(x_0 + \eta, x', \tilde{\mathbf{u}}(x))$ and $\tilde{\mathbf{F}} = \mathbf{F}(x_0 + \eta, x', \tilde{\mathbf{u}}(x))$. Then

$$
\rho \partial_\eta \tilde{\mathbf{y}} = L_{\lambda, \mu}(x', D') \tilde{\mathbf{y}} + L_{\lambda, \mu}(x, D') \int_0^\eta \tilde{\mathbf{y}}(x_0, x') dx_0 + \tilde{\mathbf{F}}, \quad \text{in } \Omega \times (-\eta, T - \eta),
$$

(5.4)

Denote $Q_\eta = (-\hat{T}, \hat{T}) \times \Omega, \hat{\Sigma}_\eta = (-\hat{T}, \hat{T}) \times \Gamma$, where $\hat{T} = \frac{1}{2} \max \{\eta, T - \eta\}$. Let

$$
\varphi(x) = e^{\lambda x} - e^{2\lambda |x|},
$$

where the function $\psi$ is constructed in [14] and $\lambda$ is a sufficiently large positive parameter. We apply Carleman estimate (1.22) to the system (5.4) and (5.5):

$$
\|\tilde{\mathbf{y}}\|_{\mathcal{B}(\hat{T}, \hat{T}; \hat{\Sigma}_\eta)} \leq C_{\lambda} \left( \|\mathbf{f} e^{\varphi}\|_{\mathcal{B}(\hat{T}, \hat{T}; \hat{\Sigma}_\eta)} + \|\mathbf{d} \omega \mathbf{y}, \mathbf{y} e^{\varphi}\|_{\mathcal{H}^2(\hat{\Sigma}_\eta)} \right) + \|\mathbf{d} \omega \tilde{\mathbf{y}}, \mathbf{y} e^{\varphi}\|_{\mathcal{H}^2(\hat{\Sigma}_\eta)} + \|\tilde{\omega} \partial_\eta \mathbf{y} e^{\varphi}\|_{\mathcal{L}(\hat{\Sigma}_\eta)} \forall s \geq s_0.
$$

(5.6)

The stationary phase argument (see e.g. [41]) yields

$$
\|\mathbf{F} e^{\varphi}\|_{\mathcal{B}(\hat{T}, \hat{T}; \hat{\Sigma}_\eta)} \leq \frac{C_{\lambda}}{\sqrt{s}} \left( \sum_{|\alpha| \leq 2} \|\partial^\alpha \mathbf{b} e^{\varphi}\|_{\mathcal{H}^{s}(\hat{T}; \hat{\Sigma}_\eta)} \right) + \|\mathbf{d} \omega \mathbf{y}, \mathbf{y} e^{\varphi}\|_{\mathcal{L}(\hat{T}; \hat{T}; \hat{\Sigma}_\eta)} + \|\tilde{\omega} \partial_\eta \tilde{\mathbf{y}} e^{\varphi}\|_{\mathcal{L}(\hat{T}; \hat{T}; \hat{\Sigma}_\eta)} \forall s \geq s_1.
$$

(5.7)

for all $s \geq 1$.

On the other hand, by (5.5) and (1.27), we have

$$
\|\mathbf{f} e^{\varphi}\|_{\mathcal{H}^{s}(\hat{T}; \hat{\Sigma}_\eta)} \leq C_{\lambda} \|\mathbf{f} e^{\varphi}\|_{\mathcal{H}^{s}(\hat{T}; \hat{\Sigma}_\eta)} \leq C_{\lambda} \|\mathbf{f} e^{\varphi}\|_{\mathcal{H}^{s}(\hat{T}; \hat{\Sigma}_\eta)} + \|\mathbf{b} e^{\varphi}\|_{\mathcal{H}^{s}(\hat{T}; \hat{\Sigma}_\eta)} \forall s \geq s_0.
$$

(5.8)

for all $s > 0$. Observe that by (1.21) there exists a constant $C_5$ independent of $s$ such that

$$
\|\tilde{\mathbf{y}}(0, \cdot) e^{\varphi(0, \cdot)}\|_{\mathcal{H}^{s}(\hat{T}; \hat{\Sigma}_\eta)} \leq C_{\lambda} \|\tilde{\mathbf{y}}(0, \cdot) e^{\varphi(0, \cdot)}\|_{\mathcal{H}^{s}(\hat{T}; \hat{\Sigma}_\eta)} \forall s \geq 1.
$$

(5.9)

Using (5.9) to estimate the first term on the right-hand side of (5.8) and applying (5.6), we obtain

$$
\|\mathbf{f} e^{\varphi}\|_{\mathcal{H}^{s}(\hat{T}; \hat{\Sigma}_\eta)} \leq C_{\lambda} \|\mathbf{f} e^{\varphi}\|_{\mathcal{H}^{s}(\hat{T}; \hat{\Sigma}_\eta)} \leq C_{\lambda} \|\mathbf{f} e^{\varphi}\|_{\mathcal{H}^{s}(\hat{T}; \hat{\Sigma}_\eta)} + \|\mathbf{b} e^{\varphi}\|_{\mathcal{H}^{s}(\hat{T}; \hat{\Sigma}_\eta)} \forall s \geq s_1.
$$

(5.10)
From (5.10) and (5.7), we have
\[
\left\| fe^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} \leq C_0 \left( \left\| be^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} + \left\| ae^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} \right) + \frac{1}{\sqrt{s}} \sum_{|\alpha| \leq 2} \left\| \partial^\alpha_x b e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} + \left\| f e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} + \left\| (d\omega_\ast \mathbf{u}, \text{div} \tilde{\mathbf{u}}) e^{\nu_0(\cdot)} \right\|_{L^2(\Omega)} + \left\| \tilde{\mathbf{u}} e^{\nu_0(\cdot)} \right\|_{L^2(\Omega)}
\]
\[
+ \left\| \partial_\ast \left( d\omega_\ast \text{div} \tilde{\mathbf{y}} \right) e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} + \left\| \tilde{\mathbf{u}} e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} + \left\| \text{div} \tilde{\mathbf{y}} e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} \quad \forall s \geq s_1.
\]  
(5.11)

We introduce the functions \( \varphi_\ast(x) = \frac{x - \ell_\ast(x_0)}{\ell_\ast(x_0)} \) and \( \tilde{\varphi}_\ast(x_0) = \frac{1}{(T - x_0)(T + x_0)} e_m \), where \( \ell_\ast \in C^1([-2\bar{T}, 2\bar{T}]) \), \( \ell_\ast(\pm 2\bar{T}) = 0 \), \( \ell_\ast(x_0) = (\bar{T} - x_0)(\bar{T} + x_0)^3 \) on \([-\bar{T}/4, \bar{T}/4] \) and \( \ell_\ast(x_0) \geq (\bar{T} - x_0)^3(\bar{T} + x_0)^3 \) on \([-\bar{T}, \bar{T}] \). Therefore
\[
\varphi_\ast(x) = \varphi(x) \quad \forall x \in Q_{\bar{T}}.
\]  
(5.12)

In order to estimate the norm of \( \partial_\ast \tilde{\mathbf{u}} \) on the right-hand side of (5.11), we apply Carleman estimate (1.22) to \( \tilde{\mathbf{u}} \) satisfying (1.23)−(1.25) and using the stationary phase argument, we obtain:
\[
\left\| \tilde{\mathbf{u}} \right\|_{\mathcal{B}(\varphi_\ast, s, Q_{\bar{T}})} \leq C_{10} (\| Rfe^{\nu_0(\cdot)} \|_{Y(\varphi_\ast, s, Q_{\bar{T}})} + \left\| \tilde{\mathbf{u}} \right\|_{L^2(\Omega)} + \left\| \text{div} \tilde{\mathbf{u}} e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} + \left\| \text{div} \tilde{\mathbf{u}} e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)})
\]
\[
+ \left\| \partial_\ast \left( d\omega_\ast \text{div} \tilde{\mathbf{u}} \right) e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} + \left\| \text{div} \tilde{\mathbf{u}} e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} + \left\| \text{div} \tilde{\mathbf{u}} e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} \quad \forall s \geq s_2.
\]  
(5.13)

By (1.21), we see
\[
\left( \int_{Q_{\bar{T}}} \sum_{|\alpha| \leq 2, \alpha_0 = 0} \left( \frac{1}{s \varphi_\ast} |\partial^\alpha_x (d\omega_\ast \text{div} \tilde{\mathbf{u}})|^2 + |\partial^\alpha_x \tilde{\mathbf{u}}|^2 \right) e^{2\nu_0(\cdot)} \right)^{\frac{1}{2}} \leq C_{12} \left\| \tilde{\mathbf{u}} \right\|_{\mathcal{B}(\varphi_\ast, s, Q_{\bar{T}})}.
\]  
(5.14)

Inequalities (5.6) and (5.14) imply
\[
\left( \int_{Q_{\bar{T}}} \sum_{|\alpha| \leq 2, \alpha_0 = 0} \left( \frac{1}{s \varphi_\ast} |\partial^\alpha_x (d\omega_\ast \text{div} \tilde{\mathbf{u}})|^2 + |\partial^\alpha_x \tilde{\mathbf{u}}|^2 \right) e^{2\nu_0(\cdot)} \right)^{\frac{1}{2}} \leq C_{13} (s^{-\frac{1}{2}} \left\| fe^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} + \left\| (d\omega_\ast \text{div} \tilde{\mathbf{u}}) e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} + \left\| \text{div} \tilde{\mathbf{u}} e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} + \left\| \tilde{\mathbf{u}} e^{\nu_0(\cdot)} \right\|_{H^s(\Omega)} \quad \forall s \geq s_2.
\]  
(5.15)

By (5.11), (5.12) and (5.15) for sufficiently large \( s \), we reach (1.28). The proof of theorem 1.2 is completed.

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Appendix

In the appendix, we prove propositions 2.3–2.5. For it, we first show lemmata A.1–A.7 whose proofs are the same as in e.g. [16] or [25].

The following lemma allows us to extend the definition of the operator $A$ in Sobolev spaces.

**Lemma A.1.** Let $a(\tilde{y}, \tilde{z}, s) \in L^\infty_{\infty} S^{1,s}(O)$ be the symbol of $A$. Then $A \in \mathcal{L}(H^{1,1}_{0}(O); L^2(O))$ and

$$
\|A\|_{\mathcal{L}(H^{1,1}_{0}(O); L^2(O))} \leq C_1(\pi_{L^\infty}(a)).
$$

**Lemma A.2.** Let $a(\tilde{y}, \tilde{z}, s) \in W^{\infty}_{cl} S^{1,s}(O)$. Then $A(\tilde{y}, \tilde{z}, s)^* = A^*(\tilde{y}, \tilde{z}, s) + R$, where $A^*$ is the pseudodifferential operator with symbol $a(\tilde{y}, \tilde{z}, s)$ and $R \in \mathcal{L}(H^{1,1}_{0}(O); L^2(O))$ satisfies

$$
\|R\|_{\mathcal{L}(H^{1,1}_{0}(O); L^2(O))} \leq C_2\pi_{W^{\infty}_{\infty}(O)}(a).
$$

**Lemma A.3.** Let $a(\tilde{y}, \tilde{z}, s) \in W^{\infty}_{cl} S^{1,s}(O)$ and $b(\tilde{y}, \tilde{z}, s) \in W^{\infty}_{cl} S^{0,s}(O)$. Then

$$
A(\tilde{y}, \tilde{z}, s)^* B(\tilde{y}, \tilde{z}, s) = C(\tilde{y}, \tilde{z}, s) + R_0
$$

where $C(\tilde{y}, \tilde{z}, s)$ is the operator with symbol $a(\tilde{y}, \tilde{z}, s)$ and $R_0 \in \mathcal{L}(H^{1,1}_{0}(O); L^2(O))$. Moreover we have

$$
\|R_0\|_{\mathcal{L}(H^{1,1}_{0}(O); L^2(O))} \leq C_3(\pi_{W^{\infty}_{\infty}(O)}(a)\pi_{W^{\infty}_{\infty}(O)}(b) + \pi_{W^{\infty}_{\infty}(O)}(a)\pi_{W^{\infty}_{\infty}(O)}(b)).
$$

The direct consequence of lemma A.3 is the following commutator estimate.

**Lemma A.4.** Let $a(\tilde{y}, \tilde{z}, s) \in W^{\infty}_{cl} S^{1,s}(O)$ and $b(\tilde{y}, \tilde{z}, s) \in W^{\infty}_{cl} S^{1,s}(O)$. Then the commutator $[A, B]$ belongs to the space $\mathcal{L}(H^{1,1}_{0}(O); L^2(O))$ and

$$
\|[A, B]\|_{\mathcal{L}(H^{1,1}_{0}(O); L^2(O))} \leq C_4(\pi_{W^{\infty}_{\infty}(O)}(a)\pi_{W^{\infty}_{\infty}(O)}(b) + \pi_{W^{\infty}_{\infty}(O)}(a)\pi_{W^{\infty}_{\infty}(O)}(b) + \pi_{W^{\infty}_{\infty}(O)}(a)\pi_{W^{\infty}_{\infty}(O)}(b)).
$$

Next we show

**Lemma A.5.** Let $a(\tilde{y}, \tilde{z}, s) \in W^{\infty}_{cl} S^{1,s}(O)$ be a symbol with compact support in $O$. Let $O_1, O_2 \subset O$ and $\overline{O_2} \cap \overline{O_3} = \emptyset$. Suppose that $u \in H^{1,1}_{0}(O)$ and $\sup u < O_1$. Then there exists a constant $C_5$ such that

$$
\|A(\tilde{y}, \tilde{z}, s)u\|_{H^{1,1}_{0}(O_2)} \leq C_5\pi_{W^{\infty}_{\infty}(O)}(a)\|u\|_{H^{1,1}_{0}(O_1)}.
$$

(A.1)

We shall use the following variant of Gårding’s inequality.

**Lemma A.6.** Let $p(\tilde{y}, \tilde{z}, s) \in W^{\infty}_{cl} S^{2,s}(O)$ be a symbol with compact support in $O$. Let $u \in H^{1,1}_{0}(O)$ and $\sup u < O_1$. Let $O_1 \subset O_2 \subset O_3 \subset O$ and $\tilde{\gamma} \in C^{\infty}(O_3)$ satisfy $\tilde{\gamma} |_{O_2} = 1$ and $\text{Re} \ p(\tilde{y}, \tilde{z}, s) > C\tilde{M}^{\infty}(\tilde{y}, s)$ for any $\tilde{y} \in O_3$. Then
\[
\text{Re}(P(\bar{\gamma}, \bar{D}, s)u, u)_{L^2(\mathcal{O})} \geq \frac{\hat{C}}{2} \|u\|_{H^{\frac{1}{2}, \perp}(\mathcal{O})}^2
\]

\[-C_6 \left( (C_{w*}(\mathcal{O}, p) + 1)\pi\mathcal{C}((\mathcal{O}, p) + 1)\pi\mathcal{C}(\mathcal{O}, (\overline{\gamma})) + \sum_{k=0}^{1} (C_{w*}(\mathcal{O}, p) + 1)\pi\mathcal{C}(\mathcal{O}, (\overline{\gamma})) \right) \|u\|_{L^2(\mathcal{O})}^2. \tag{A.2}\]

**Proof.** Consider the pseudodifferential operator \(A(\bar{\gamma}, \bar{D}, s)\) with symbol \(A(\bar{\gamma}, \bar{\xi}, s) = (\gamma \text{Re} p(\bar{\gamma}, \bar{\xi}, s) - \frac{\tilde{C}}{2} M^2(s, s)) \in W^{1, \infty}_0(\mathcal{O})\). Then, according to lemmata A.1–A.3, we see

\[
A(\bar{\gamma}, \bar{D}, s)^* A(\bar{\gamma}, \bar{D}, s) = \gamma \text{Re} p(\bar{\gamma}, \bar{D}, s) - \frac{\tilde{C}}{2} M^2(\bar{D}, s) + R,
\]

where \(R \in \mathcal{L}(H^{\frac{1}{2}, \perp}(\mathcal{O}); L^2(\mathcal{O}))\) and

\[
\|R\|_{\mathcal{L}(H^{\frac{1}{2}, \perp}(\mathcal{O}); L^2(\mathcal{O}))} \leq C_7 (C_{w*}(\mathcal{O}, p)\pi\mathcal{C}(\mathcal{O}, a)) \]

\[
\leq C_8 (C_{w*}(\mathcal{O}, p) + 1)\pi\mathcal{C}(\mathcal{O}, (\overline{\gamma})) + \sum_{k=0}^{1} (C_{w*}(\mathcal{O}, p) + 1)\pi\mathcal{C}(\mathcal{O}, (\overline{\gamma})). \tag{A.3}\]

Therefore

\[
\text{Re}(P(\bar{\gamma}, \bar{D}, s)u, u)_{L^2(\mathcal{O})} = \|A(\bar{\gamma}, \bar{D}, s)\|_{L^2(\mathcal{O})}^2 - ((1 - \overline{\gamma})M^2(\bar{D}, s)u, u)_{L^2(\mathbb{R}^n)}
\]

\[
+ \frac{\tilde{C}}{2} \|u\|_{H^{\frac{1}{2}, \perp}(\mathcal{O})}^2 + (Ru, u)_{L^2(\mathcal{O})}. \tag{A.3}\]

By (A.3) we observe

\[
|(Ru, u)_{L^2(\mathcal{O})}|
\]

\[
\leq C_9 (C_{w*}(\mathcal{O}, p) + 1)\pi\mathcal{C}(\mathcal{O}, (\overline{\gamma})) + \sum_{k=0}^{1} (C_{w*}(\mathcal{O}, p) + 1)\pi\mathcal{C}(\mathcal{O}, (\overline{\gamma})) \|u\|_{L^2(\mathcal{O})} \|u\|_{H^{\frac{1}{2}, \perp}(\mathcal{O})}. \tag{A.4}\]

By lemma A.5, we have

\[
|((1 - \overline{\gamma})M^2(\bar{D}, s)u, u)_{L^2(\mathbb{R}^n)}| \leq C_{10} \pi\mathcal{C}(\mathcal{O}, (\overline{\gamma})) \|u\|_{L^2(\mathcal{O})} \|u\|_{H^{\frac{1}{2}, \perp}(\mathcal{O})},
\]

so that we obtain the statement of the lemma.

**Lemma A.7.** Let \(p \in \left\{ \frac{1}{2}, \frac{3}{2} \right\} \). Then there exists a constant \(C_{11}\) independent of \(s\) such that

\[
\sum_{\ell = -\infty}^{\infty} \|z\|_{H^{\ell} \mathcal{P}^s(\Sigma)}^2 \leq C_{11} \|z\|_{H^{\frac{3}{2}} \mathcal{P}^s(\Sigma)}^2. \tag{A.5}\]

**Proof.** We note that

\[
\|z\|_{H^{\frac{3}{2}} \mathcal{P}^s(\Sigma)}^2 = \|z\|_{H^{\frac{3}{2}} \mathcal{P}^s(\Sigma)}^2 + \|z\|_{H^{\frac{3}{2}} \mathcal{P}^s(\Sigma)}^2. \tag{A.6}\]
Since $\kappa_\ell$ depends only on $x_0$, by (2.9) we have
\[
\sum_{\ell = -\infty}^{\infty} \|z_\ell\|^2_{H^0(\Sigma)} \leq C_{12} \|z\|^2_{H^0(\Sigma)}. \tag{A.7}
\]

Now we estimate the first term on the right-hand side of (A.6). Let $\tilde{z} = z \circ F^{-1}$ and the mapping $F$ be given by (2.12). By the definition of the norm in the Sobolev–Slobodetskii space, we have
\[
\sum_{\ell = -\infty}^{\infty} \|\kappa_\ell z\|^2_{H^0(\Sigma)} = \sum_{\ell = -\infty}^{\infty} \|\kappa_\ell \tilde{z}\|^2_{H^0([-T,T] \times \mathbb{R}^{n-1})}
\]
\[
= \sum_{\ell = -\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{[-T,T] \times [-T,T]} \frac{|(\kappa_\ell \tilde{z})(y) - (\kappa_\ell \tilde{z})(x_0)|^2}{|y - x_0|^{p+1}} \, dx_0 \, dy_0 \, dy'
\]
\[
\leq 4 \sum_{\ell = -\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{[-T,T] \times [-T,T]} \frac{|\kappa_\ell'(y) - \kappa_\ell(x_0)|^2 |\tilde{z}(y_0,\cdot')|^2}{|y - x_0|^{p+1}} \, dx_0 \, dy_0 \, dy'
\]
\[
+ 4 \sum_{\ell = -\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{[-T,T] \times [-T,T]} \frac{|\tilde{z}(y_0,\cdot') - \tilde{z}(x_0,\cdot')|^2 |\kappa_\ell(y_0)|^2}{|y - x_0|^{p+1}} \, dx_0 \, dy_0 \, dy' =: I_1 + I_2.
\]

We estimate the terms $I_j$ separately. By (2.9), we have
\[
I_2 = \sum_{\ell = -\infty}^{\infty} \int_{[-T,T]} |\kappa_\ell(y_0)|^2 \int_{\mathbb{R}^{n-1}} \int_{[-T,T]} \frac{|\tilde{z}(y_0,\cdot') - \tilde{z}(x_0,\cdot')|^2}{|y - x_0|^{p+1}} \, dx_0 \, dy_0 \, dy'
\]
\[
\leq C_{13} \int_{[-T,T]} \int_{\mathbb{R}^{n-1}} \int_{[-T,T]} \frac{|\tilde{z}(y_0,\cdot') - \tilde{z}(x_0,\cdot')|^2}{|y - x_0|^{p+1}} \, dx_0 \, dy_0 = C_{14} \|\tilde{z}\|^2_{H^0((-T,T) \times \mathbb{R}^{n-1})},
\]

We set $Z_4 = \left\{ (x_0, y_0) \in [-T, T] \times [-T, T]; \frac{1}{|x_0 - y_0|} \leq \lambda \tilde{\gamma}(y_0) \right\}$ with large positive parameter $\lambda$. Short computations imply
\[
I_1 = \sum_{\ell = -\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{[-T,T] \times [-T,T]} \frac{|\kappa_\ell(y_0) - \kappa_\ell(x_0)|^2 |\tilde{z}(y_0,\cdot')|^2}{|y - x_0|^{p+1}} \, dx_0 \, dy_0 \, dy'
\]
\[
= \sum_{\ell = -\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{Z_4} \frac{|\kappa_\ell(y_0) - \kappa_\ell(x_0)|^2 |\tilde{z}(y_0,\cdot')|^2}{|y - x_0|^{p+1}} \, dx_0 \, dy_0 \, dy'
\]
\[
+ \sum_{\ell = -\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{[-T,T] \times [-T,T] \setminus Z_4} \frac{|\kappa_\ell(y_0) - \kappa_\ell(x_0)|^2 |\tilde{z}(y_0,\cdot')|^2}{|y - x_0|^{p+1}} \, dx_0 \, dy_0 \, dy' =: P_1 + P_2.
\]

Using the definition of the set $Z_4$, we have
\[
P_2 \leq C_{15}(\lambda) \sum_{\ell = -\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{Z_4} \frac{|\kappa_\ell(y_0) - \kappa_\ell(x_0)|^2 |\tilde{z}(y_0,\cdot')|^2}{|y - x_0|^{1 - \delta \rho^2}} \, dx_0 \, dy_0 \, dy'
\]

36
\[ \leq 2C_{15}(\lambda) \sum_{\ell=-\infty}^{\infty} \int_{\mathbb{R}^{2\ell+1}} \int_{[-T,T] \times [-T,T]} \frac{(|\kappa_{\ell}(y_0)|^2 + |\kappa_{\ell}(x_0)|^2)|\mathcal{Z}(y_0, x')|^2}{|y_0 - x_0|^{1-\delta}} \frac{d\mathcal{Z}d\mathcal{Y}d\mathcal{X}'}{\mathcal{Z}^2(y_0)} \]

\[ \leq C_{16}(\lambda) \int_{\mathbb{R}^{2\ell+1}} \int_{[-T,T]} |\mathcal{Z}(y_0, x')|^2 \frac{d\mathcal{Z}d\mathcal{Y}d\mathcal{X}'}{\mathcal{Z}^2(y_0)} \sup_{y_0 \in [-T,T]} \|\mathcal{Y} - x\|_2, \]

\[
+ C_{17}(\lambda) \sum_{\ell=-\infty}^{\infty} \int_{\mathbb{R}^{2\ell+1}} \int_{[-T,T]} |\kappa_{\ell}(y_0)|^2 |\mathcal{Z}(y_0, x')|^2 \frac{d\mathcal{Z}d\mathcal{Y}d\mathcal{X}'}{\mathcal{Z}^2(y_0)} \sup_{y_0 \in [-T,T]} \|\mathcal{Y} - x\|_2, \]

\[ \leq C_{18} \|\hat{\mathcal{Z}}\|_{L^2((-T,T) \times \mathbb{R}^{2\ell+1})}. \]

Next we estimate \( P_1 \). For any positive \( \epsilon_1 \), there exists a constant \( C_{19}(\epsilon) \) such that

\[ P_1 \leq C_{19} \left( \sum_{\ell=-\infty}^{\infty} \int_{|T-T_{-1}| > |T-T_{-1}||z_1|} |\kappa_{\ell}(y_0) - \kappa_{\ell}(x_0)|^2 |\mathcal{Z}(y_0, x')|^2 \frac{d\mathcal{Z}d\mathcal{Y}d\mathcal{X}'}{|y_0 - x_0|^{1+p}} \right. \]

\[ + \left. \sum_{\ell=-\infty}^{\infty} \int_{|T-T_{-1}| > |T-T_{-1}||z_1|} |\kappa_{\ell}(y_0) - \kappa_{\ell}(x_0)|^2 |\mathcal{Z}(y_0, x')|^2 \frac{d\mathcal{Z}d\mathcal{Y}d\mathcal{X}'}{|y_0 - x_0|^{1+p}} \right). \]

We estimate the first term in this inequality. The estimate of the second one is the same.

If \((x_0, y_0) \in [T, T - \epsilon_1] \times [T, T - \epsilon_1] \setminus \mathcal{Z}_1\), then we have

\[ |x_0 - y_0| \leq 2|T - y_0|^2. \]

Therefore

\[ |\kappa(x_0) - \kappa(y_0)| \leq \sup_{\lambda \in [0, 1]} \frac{|y_0 - x_0|}{|T - \lambda y_0 - (1 - \lambda)y_0|^2} \leq \frac{C_{20}|y_0 - x_0|}{(|T - y_0| - |x_0 - y_0|)^2} \leq \frac{C_{21}|y_0 - x_0|}{|T - y_0|^2}. \]

Using this inequality, we obtain

\[ \leq C_{22} \sum_{\ell=-\infty}^{\infty} \int_{|T-T_{-1}| > |T-T_{-1}||z_1|} |\kappa_{\ell}(y_0) - \kappa_{\ell}(x_0)|^2 |\mathcal{Z}(y_0, x')|^2 \frac{d\mathcal{Z}d\mathcal{Y}d\mathcal{X}'}{|y_0 - x_0|^{1+p}} \]

\[ \leq C_{23} \sum_{\ell=-\infty}^{\infty} \int_{|T-T_{-1}| > |T-T_{-1}||z_1|} (|\kappa_{\ell}(y_0)|^{2-p-\delta} + |\kappa_{\ell}(x_0)|^{2-p-\delta}) |\mathcal{Z}(y_0, x')|^2 \frac{d\mathcal{Z}d\mathcal{Y}d\mathcal{X}'}{|y_0 - x_0|^{1-\delta}} \]

\[ \leq C_{24} \left\| \frac{\hat{\mathcal{Z}}}{\mathcal{Z}^{(p+\delta)}} \right\|_{L^2((-T,T) \times \mathbb{R}^{2\ell+1})}^2. \]
Taking parameter $\delta$ sufficiently small, we obtain (A.5). The proof of the proposition is complete.

**Proof of proposition 2.3.** Let

$$\tilde{R}(y, \tilde{D}, \tilde{s}) = \frac{i \rho (\frac{1}{\beta} \partial_{y_\alpha} + i |\tilde{s}| \varphi_\alpha)}{\beta G} + \sum_{j=1}^{n-1} \left( \frac{i \rho (\xi_j + i |\tilde{s}| \varphi_j)}{\beta G} - |\tilde{s}|^2 \varphi_j \cdot \right)$$

and $\Gamma(y, \tilde{D}, \tilde{s})$ be the operator with symbol $\Gamma_\beta^\pm (y, \tilde{\xi}, \tilde{s}) \Gamma_\beta^\pm (y, \tilde{\xi}, \tilde{s})$:

$$\Gamma(y, \tilde{\xi}, \tilde{s}) = (- |\tilde{s}|^2 (\tilde{\mu} \varphi_{\alpha})^2 + \alpha_{\beta}^2 i |\tilde{s}| \varphi_{\alpha} + \alpha_{\beta}^2 i |\tilde{s}| \varphi_{\alpha} + \alpha_{\beta}^2 \alpha_{\beta}^2)$$

$$= - |\tilde{s}|^2 (\tilde{\mu} \varphi_{\alpha})^2 + \frac{i \rho (\xi_0 + i |\tilde{s}| \varphi_0)}{\beta G} + \sum_{j=1}^{n-1} \left( \frac{i \rho (\xi_j + i |\tilde{s}| \varphi_j)}{\beta G} - |\tilde{s}|^2 \varphi_j \cdot \right)$$

on supp $\chi_\nu$. We set

$$\Upsilon_\ell = B(y^*, 2\delta) \cap \text{supp } \tilde{\mu}_\ell. \quad (A.8)$$

Then

$$\Gamma(y, \tilde{D}, \tilde{s}) w_\nu = [\Gamma, \eta_\ell] \chi_{\nu}(\tilde{D}, \tilde{s}) w + \eta_\ell \Gamma(y, \tilde{D}, \tilde{s}) \chi_\nu(\tilde{D}, \tilde{s}) w$$

$$= [\Gamma, \eta_\ell] \chi_\nu(\tilde{D}, \tilde{s}) w + \eta_\ell \tilde{R}(y, \tilde{D}, \tilde{s}) \chi_\nu(\tilde{D}, \tilde{s}) w$$

in (A.9). Direct computations imply

$$\left( \frac{1}{i} \partial_{y_\alpha} - \Gamma_\beta^-(y, \tilde{D}, \tilde{s}) \right) \left( \frac{1}{i} \partial_{y_\alpha} - \Gamma_\beta^-(y, \tilde{D}, \tilde{s}) \right)$$

$$= - \partial_{y_\alpha}^2 - \frac{1}{i} [\partial_{y_\alpha}, \Gamma_\beta^+(y, \tilde{D}, \tilde{s})] + \Gamma_\beta^-(y, \tilde{D}, \tilde{s}) \Gamma_\beta^+(y, \tilde{D}, \tilde{s})$$

$$+ i \Gamma_\beta^+(y, \tilde{D}, \tilde{s}) \partial_{y_\alpha} + i \Gamma_\beta^+(y, \tilde{D}, \tilde{s}) \partial_{y_\alpha}. \quad (A.10)$$

By lemma A.3, we have

$$\Gamma_\beta^-(y, \tilde{D}, \tilde{s}) \Gamma_\beta^+(y, \tilde{D}, \tilde{s}) = \Gamma(y, \tilde{D}, \tilde{s}) + R_0,$$

where

$$\|R_0\|_{\mathcal{L}(H_{n-3}^1(\Upsilon_\ell), L^2(\Upsilon_\ell))}$$

$$\leq C_{25} (\pi_{w, n}(\Upsilon_\ell)) (\Gamma_\beta^+ \pi_{w, \infty}(\Upsilon_\ell)) (\Gamma_\beta^- \pi_{w, \infty}(\Upsilon_\ell)) \leq C_{25} \bar{\phi} \frac{\bar{\phi}}{\psi}(y_0).$$

The commutator $[\partial_{y_\alpha}, \Gamma_\beta^+(y, \tilde{D}, \tilde{s})]$ is the pseudodifferential operator with the symbol $\partial_{y_\alpha} \Gamma_\beta^+(y, \tilde{\xi}, \tilde{s})$. Lemma A.1 yields

$$||[\partial_{y_\alpha}, \Gamma_\beta^+(y, \tilde{D}, \tilde{s})]||_{\mathcal{L}(H_{n-3}^1(\Upsilon_\ell), L^2(\Upsilon_\ell))} \leq C_{27} \bar{\phi} \frac{\bar{\phi}}{\psi}(y_0).$$
Denote
\[ R(y, \tilde{D}, \tilde{s}) = \left( 2\tilde{s} \varphi_n + \sum_{j=1}^{n-1} \frac{\partial_j \vartheta(y_1, \ldots, y_{n-1}) (\tilde{s} - \varphi_j)}{G} \right). \]

By (2.20)–(2.23), (2.25) and the fact that \( \tilde{\mu}(y)\eta(y) = \eta(y) \), the following is true:
\[
(i\Gamma^\beta_0(y, \tilde{D}, \tilde{s})\partial_{\eta_0} + i\Gamma^\beta_0(y, \tilde{D}, \tilde{s})\partial_{\eta_0})w_\nu = \tilde{\mu}_0 R(y, \tilde{D}, \tilde{s}) \partial_{\eta_0} \kappa(\tilde{D}, \tilde{s})w_\nu + \tilde{\mu}_0 \eta_0 R(y, \tilde{D}, \tilde{s}) \partial_{\eta_0} \kappa(\tilde{D}, \tilde{s})w_\nu
= \tilde{\mu}_0 R(y, \tilde{D}, \tilde{s}) \partial_{\eta_0} \kappa, \eta_0 \chi_\nu(\tilde{D}, \tilde{s})w_\nu + \eta_0 \tilde{\mu}_0 R(y, \tilde{D}, \tilde{s}) \partial_{\eta_0} \kappa(\tilde{D}, \tilde{s})w_\nu + \eta_0 R(y, \tilde{D}, \tilde{s}) \partial_{\eta_0} \kappa(\tilde{D}, \tilde{s})w_\nu.
\]

(A.11)

Since \(-\partial^2_{\eta_0}w_\nu + R(y, \tilde{D}, \tilde{s}) \partial_{\eta_0}w_\nu + \tilde{R}(y, \tilde{D}, \tilde{s})w_\nu = \frac{1}{\Gamma_0} P_{\kappa, \beta}(y, \tilde{D}, \tilde{s})w_\nu \), setting
\[ T_\beta = -R_0 + [\partial_{\eta_0}, \Gamma]_0(y, \tilde{D}, \tilde{s}) - [\Gamma, \eta_0] \chi_\nu(\tilde{D}, \tilde{s}) - [\eta_0, R(y, \tilde{D}, \tilde{s})] \chi_\nu(\tilde{D}, \tilde{s}) \]
and using (A.9)–(A.11), we obtain (2.31).

Now we prove estimate (2.32). Lemma A.4 yields
\[
\|\Gamma, \eta_0\|_{L^2(\{\Gamma, \eta_0\}^n, L^2(\gamma))} \leq C_2(\pi_\gamma(\gamma, \gamma) \eta_0 + \pi_\gamma(\gamma, \gamma) \pi_{\kappa, \rho}(\gamma, \gamma))
+ \pi_{\kappa, \rho}(\gamma, \gamma) \pi_{\kappa, \rho}(\gamma, \gamma)) \leq C_{29}\tilde{\gamma}(\gamma_0^2).
\]

(A.12)

For the differential operators \( R \) and \( \tilde{R} \), we obtain the estimates
\[
\|\mu_\beta, \tilde{R}(y, \tilde{D}, \tilde{s})\|_{L^2(\gamma, L^2(\gamma))} \leq C_{30}\tilde{\gamma}(\gamma_0^2),
\]
\[
\|\mu_\beta, R(y, \tilde{D}, \tilde{s})\partial_{\eta_0} \kappa, \eta_0\|_{L^2(\gamma, L^2(\gamma))} \leq C_{31}\tilde{\gamma}(\gamma_0^2),
\]
\[
\|\eta_0, R(y, \tilde{D}, \tilde{s})\partial_{\eta_0} \kappa, \eta_0\|_{L^2(\gamma, L^2(\gamma))} \leq C_{32}\|\gamma_0\|_{L^2(\gamma)}, \leq C_{33}\tilde{\gamma}(\gamma_0^2).
\]

(A.13)

(A.14)

(A.15)

Form (A.12)–(A.15), we obtain (2.32). The proof of the proposition is complete.

Proof of proposition 2.4. Let \( \tilde{y} \in B(0, R) \setminus (a_1, b_1) \times \mathbb{R}^{n-1} \). Consider two cases.

Case 1. Let \( |y_0| > \max \{|a|, |b|\} = \tilde{c} \). Integrating by parts, we have
\[
M^\beta(\tilde{D}, s)v = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \{\text{supp} p\}} \frac{\partial^p \tilde{M}^\beta(\tilde{\zeta}, s)}{(i(x_0 - y_0))^p} e^{i(\tilde{y} - \tilde{\zeta})} v(\tilde{\zeta}) d\tilde{\zeta} d\tilde{\xi}
= -\frac{1}{(2\pi)^n y_0^p} \int_{\mathbb{R}^n \times \{\text{supp} p\}} \frac{\partial^p \tilde{M}^\beta(\tilde{\zeta}, s)}{(1 - \frac{\tilde{y}}{y_0})^p} e^{i(\tilde{y} - \tilde{\zeta})} v(\tilde{\zeta}) d\tilde{\zeta} d\tilde{\xi}
\]

39
\[ \frac{i^p}{y_0} \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} c_k(p) \left( \frac{x_0}{y_0} \right)^k \partial_{\bar{\xi}_0}^p M^p(\bar{\xi}, s) e^{i<\bar{\xi}-\bar{\xi}_0, v>} \bar{v}(\bar{\xi}) d\bar{\xi} d\bar{\eta} \]

\[ \frac{i^p}{y_0} \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} c_k(p) \mathcal{K}(\tilde{y}, \tilde{D}, s) \left( \left( \frac{x_0}{y_0} \right)^k v \right), \]

where \( c_k(P) = \frac{c(p_0) \cdots c(p_{k-1})}{k!} \). Therefore

\[ \| M^p(\tilde{D}, s) u \|_{L^2(\mathbb{R}^n)} \leq \sup_{y \in \mathbb{R}^n} \left\{ \frac{C_{34}}{|y_0|^p} \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k=1}^{\infty} c_k(p) \left( \frac{x_0}{y_0} \right)^k v \right\} \]

\[ \leq C_{35} \frac{\|v\|_{L^2(\mathbb{R}^n)}}{|y_0|^{p-1}} \leq C_{36} \left( \min \{a - a_1, b_1 - b_1\} \right)^p. \]

**Case 2.** Let \( |y_0| \leq \max \{|a|, |b|\} \). If \( |a| > |b| \), then \( y_0 \in (b, |a|) \) and we set \( \tilde{c} = b \). On the other hand, if \( |a| < |b| \), then \( y_0 \in (|b|, a) \) and we set \( \tilde{c} = a \). In both cases, \( |x_0 - y_0| > 1 \) on \([a, b] \). Short computations imply

\[ M^p(\tilde{D}, s) v = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\partial_{\bar{\xi}_0}^p M^p(\bar{\xi}, s)}{p^p(x_0 - y_0)^p} e^{i<\bar{\xi}-\bar{\xi}_0, v>} \bar{v}(\bar{\xi}) d\bar{\xi} d\bar{\eta} \]

\[ = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\partial_{\bar{\xi}_0}^p M^p(\bar{\xi}, s)}{p^p(x_0 - y_0 + 1)^p} e^{i<\bar{\xi}-\bar{\xi}_0, v>} \bar{v}(\bar{\xi}) d\bar{\xi} d\bar{\eta} \]

\[ = (-1)^{p+1} \sum_{k=1}^{\infty} \frac{c_k(p)}{|x_0 - y_0 + 1|^p} \left( \frac{1}{x_0 - y_0 + 1} \right)^{k+1} \partial_{\bar{\xi}_0}^p M^p(\bar{\xi}, s) e^{i<\bar{\xi}-\bar{\xi}_0, v>} \bar{v}(\bar{\xi}) d\bar{\xi} d\bar{\eta} \]

\[ = \frac{i}{(2\pi)^{n/2}} \sum_{k=1}^{\infty} c_k(p) \mathcal{K}(\tilde{y}, \tilde{D}, s) \left( \frac{1}{|x_0 - y_0 + 1|^p} \left( \frac{1}{x_0 - y_0 + 1} \right)^{k+1} v \right). \]

Therefore

\[ \| M^p(\tilde{D}, s) u \|_{L^2(\mathbb{R}^n)} \]

\[ \leq \sup_{y \in \mathbb{R}^n} \left\{ \frac{C_{37}}{(2\pi)^{n/2}} \sum_{k=1}^{\infty} c_k(p) \left( \frac{1}{|x_0 - y_0 + 1|^p} \left( \frac{1}{x_0 - y_0 + 1} \right)^{k+1} v \right) \right\} \]
\[
\begin{align*}
\left\langle \nabla \psi, \Delta \psi \right\rangle &\leq C_{38} \left| c - y_0 + 1 \right|^p \sum_{k=1}^{\infty} c_k(p) \frac{1}{|c - y_0 + 1|^{k+1}} \| \psi \|_{L^2(\mathbb{R}^n)} \leq C_{39} \gamma \| \psi \|_{L^2(\mathbb{R}^n)} \left( \min \{a - y_0, y_0 - b\} \right)^p.
\end{align*}
\]

The proof of the proposition is complete.

**Proof of proposition 2.5.** Let \( M(\tilde{D}, \tilde{s}) \) be the pseudodifferential operator with the symbol \( M(\tilde{z}, \tilde{s}) \). Taking the scalar product of the equation (2.36) and the function \(-iM(\tilde{D}, \tilde{s})V\) in \( L^2(\mathbb{R}^n) \) and integrating by parts, we obtain
\[
-\text{Im} \int_0^\gamma (\Gamma_{\tilde{\beta}}(-y, \tilde{D}, \tilde{s})V, \nabla (\tilde{M}(\tilde{D}, \tilde{s})V)) dy_n = \text{Im} \int_0^\gamma (p, \nabla (\tilde{M}(\tilde{D}, \tilde{s})V)) dy_n.
\]

From (2.21)–(2.24) and assumption \( r_\beta(y^*, \zeta^*) \neq 0 \) for some positive constant \( C_{40} \), it follows that
\[
-\text{Im} \Gamma_{\tilde{\beta}}(y, \tilde{z}, \tilde{s}) \geq C_{40} M(\tilde{z}, \tilde{s}) \quad \forall (y, \tilde{z}, \tilde{s}) \in \text{supp} \sum_{k=\ell-37}^{\ell+37} \kappa_k(\gamma_0) \cap B(y^*, 1997 \delta_{1000}) \times O(y^*, \delta_1(y^*)).
\]

We set \( \tilde{V}(y) = \eta\tilde{^t}(y) \sum_{k=\ell-17}^{\ell+17} \kappa_k(\gamma_0) V(y) \) and \( \tilde{\mu}_k^t(y) = \eta_k^t(y) \sum_{k=\ell-50}^{\ell+50} \kappa_k(\gamma_0) \), where \( \eta_k^t \in C_0^\infty(B(y^*, \eta_9/20 \delta)) \), \( \eta_k^t \in C_0^\infty(B(y^*, 2 \delta)) \) and \( \eta_k^t |_{\text{supp } \eta_k^t} = 1 \). We recall that the function \( \eta_k^t \) is defined in (2.22). Short computations imply
\[
-\int_0^\gamma \text{Im} \Gamma_{\tilde{\beta}}(y, \tilde{D}, \tilde{s})V, \nabla (\tilde{M}(\tilde{D}, \tilde{s})V)) dy_n = -\int_0^\gamma \text{Im} (\tilde{\mu}_k^t \Gamma_{\tilde{\beta}}(y, \tilde{D}, \tilde{s})V, \nabla (\tilde{M}(\tilde{D}, \tilde{s})V)) dy_n
\]
\[
= -\int_0^\gamma \text{Im} (\Gamma_{\tilde{\beta}}(y, \tilde{D}, \tilde{s})V, \tilde{\mu}_k^t \tilde{M}(\tilde{D}, \tilde{s})V)) dy_n
\]
\[
-\int_0^\gamma \text{Im} (\Gamma_{\tilde{\beta}}(y, \tilde{D}, \tilde{s})(V - \tilde{V}), \tilde{\mu}_k^t \tilde{M}(\tilde{D}, \tilde{s})(V - \tilde{V})) dy_n
\]
\[
-\int_0^\gamma \text{Im} (\Gamma_{\tilde{\beta}}(y, \tilde{D}, \tilde{s})V, \tilde{\mu}_k^t \tilde{M}(\tilde{D}, \tilde{s})(V - \tilde{V}))) dy_n = \sum_{j=1}^3 J_j.
\]

We estimate each term in the above inequality separately.

By Gårding’s inequality (A.2), there exists a positive constant \( C_{43} \) such that
\[- \int_0^\gamma \text{Im}(\Gamma^{-}_\beta(y, \bar{D}, \bar{s}) \tilde{V}, \mu_3^{-}M(\bar{D}, \bar{s}) \tilde{V}))_{L^2(\mathbb{R}^\nu)} \, dy_n \]
\[- \int_0^\gamma \text{Im}(M(\bar{D}, \bar{s}) \mu_3^{-} \Gamma^{-}_\beta(y, \bar{D}, \bar{s}) \tilde{V}, \tilde{V})_{L^2(\mathbb{R}^\nu)} \, dy_n \]
\[- \int_0^\gamma \text{Im}(\tilde{\mu}_3^{-} M(\bar{D}, \bar{s}) \mu_3^{-} \Gamma^{-}_\beta(y, \bar{D}, \bar{s}) \tilde{V}, \tilde{V})_{L^2(\mathbb{R}^\nu)} \, dy_n \]
\[\geq \int_0^\gamma \text{Re}(p(y, \bar{D}, \bar{s}) \tilde{V}, \tilde{V})_{L^2(\mathbb{R}^\nu)} \, dy_n \]
\[\geq C_{41} \int_0^\gamma \| \tilde{V}(\cdot, y_n) \|^2_{H^{\frac{1}{2} + \nu}(\mathbb{R}^\nu)} \, dy_n - C_{42} \epsilon^{10} \| \tilde{V} \|^2_{L^2(\mathcal{Q})} \]
\[\geq C_{43} \int_0^\gamma \| \tilde{V}(\cdot, y_n) \|^2_{H^{\frac{1}{2} + \nu}(\mathbb{R}^\nu)} \, dy_n - C_{44} \epsilon^{\frac{2}{3}} (y'_0) \| \tilde{V} \|^2_{L^2(\mathcal{Q})}. \quad (A.18)\]

Here \( p(y, \bar{D}, \bar{s}) \) is the operator with symbol \( p(y, \bar{\xi}, \bar{s}) = i \tilde{\mu}_3^{-} M(\bar{\xi}, \bar{s}) \mu_3^{-} \Gamma^{-}_\beta(y, \bar{\xi}, \bar{s}) \). Then, by (2.20)–(2.25), there exists a positive constant \( C_{45} \) such that

\[\text{Re} p(y, \bar{\xi}, \bar{s}) = -(\tilde{\mu}_3^{-})^2 M(\bar{\xi}, \bar{s}) \text{Im} \Gamma^{-}_\beta(y, \bar{\xi}, \bar{s}) \geq C_{45} (\tilde{\mu}_3^{-})^2 \tilde{\mu}_3^{-} M(\bar{\xi}, \bar{s}) \quad \forall y \in B(y^*, 2\delta(y^*))\]

and

\[|\pi_{w^{(3)}(\varepsilon)}(\cdot) + 1| \pi_{C^{(3)}(\varepsilon)}(\gamma) + \sum_{k=0}^1 (\pi_{w^{(3)}(\varepsilon)}(\cdot) + 1)(\pi_{C^{(4)}(\varepsilon)}(\gamma) + \pi_{C^{(5)}(\varepsilon)}(\gamma))^2 \leq C_{60} \varphi^2 (y'_0),\]

and

\[O = B(y^*, 2\delta(y^*)), \quad O_1 = B\left(y^*, \frac{93}{50} \delta\right) \cap \text{supp} \sum_{k=\ell = -17}^{\ell = 17} \kappa_k, \]
\[O_2 = B\left(y^*, \frac{97}{50} \delta\right) \cap \text{supp} \sum_{k=\ell = -19}^{\ell = 19} \kappa_k, \quad O_3 = B(y^*, \delta) \cap \text{supp} \sum_{k=\ell = -36}^{\ell = 36} \kappa_k. \]
\[\tilde{\gamma}(y) = \eta_3(y) \sum_{k=\ell = -23}^{\ell = 23} \kappa_k(y_0), \quad \eta_3 \big|_{B(y^*, \frac{99}{50} \delta)} = 1, \quad \eta_3 \in C_0^{\infty} \left(B \left(y^*, \frac{99}{50} \delta\right)\right).\]

Using lemmata A.1 and A.2 and proposition 2.4, we obtain

\[|I_3| \leq \frac{C_{43}}{4} \int_0^\gamma \| \tilde{V}(\cdot, y_n) \|^2_{H^{\frac{1}{2} + \nu}(\mathbb{R}^\nu)} \, dy_n - C_{47} \| \tilde{\mu}_3^{-} M(V - \tilde{V}) \|^2_{L^2(\mathcal{Q})}. \quad (A.19)\]

We set \( \theta(y) = 1 - \eta^*_1(y) \sum_{k=\ell = -17}^{\ell = 17} \kappa_k(y_0) \). On the other hand, proposition 2.4 and lemma A.4 yield
\[ \| \tilde{\mu}_t^* M (V - \tilde{V}) \|_{L^2(Q)} \leq \| \tilde{\mu}_t^* M \left( \vartheta \left( \frac{1}{i} \partial_y w - \Gamma_{\beta}^* w \right) \right) \|_{L^2(Q)} + \| M [\Gamma_{\beta}^*, \vartheta] w \|_{L^2(Q)} \]

\[ \leq \| \vartheta M \left( \frac{1}{i} \partial_y w - \Gamma_{\beta}^* w \right) \|_{L^2(Q)} + \| [M, \vartheta] \left( \frac{1}{i} \partial_y w - \Gamma_{\beta}^* w \right) \|_{L^2(Q)} + C_{48} \| [\Gamma_{\beta}^*, \vartheta] w \|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \]

\[ \leq C_{49} \left( \varphi_t^* (y^*) \left\| \frac{1}{i} \partial_y w - \Gamma_{\beta}^* w \right\|_{L^2(Q)} + \varphi_t^* (y^*) \| w \|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \right). \]  

(A.20)

Therefore

\[ |Z_3| \leq C_{49} \int_0^\gamma \| \tilde{V}(\cdot, y_n) \|_{H^{\frac{3}{2}}(\mathbb{R}^3)}^2 \, dy_n - C_{49} \varphi_t^* (y^*_0) \| w \|_{H^{\frac{3}{2}}(\mathbb{R}^3)}^2. \]  

(A.21)

In terms of lemma A.2, direct calculations imply

\[ |Z_2| = \left| \int_0^\gamma \text{Im}((V - \tilde{V}) \cdot (\Gamma_{\beta}^* \tilde{\mu}_t^* M (\tilde{D}, \tilde{\gamma}) \tilde{V})_{L^2(\mathbb{R}^3)}) \, dy_n \right| \]

\[ + \int_0^\gamma \text{Im}((V - \tilde{V}) \cdot \tilde{\mu}_t^* M (\tilde{D}, \tilde{\gamma}) \tilde{V})_{L^2(\mathbb{R}^3)}) \, dy_n \]

\[ \leq \left| \int_0^\gamma \text{Im}(\tilde{\mu}_t^* M (\tilde{D}, \tilde{\gamma}) (V - \tilde{\gamma}) \cdot \tilde{\mu}_t^* M (\tilde{D}, \tilde{\gamma}) \tilde{V})_{L^2(\mathbb{R}^3)}) \, dy_n \right| \]

\[ + \left| \int_0^\gamma \text{Im}((V - \tilde{V}) \cdot (\Gamma_{\beta}^* \tilde{\mu}_t^* M)(\tilde{V}) \, dy_n \right| \]

\[ + \left| \int_0^\gamma \text{Im}((V - \tilde{V}) \cdot \tilde{\mu}_t^* M (\tilde{V}) \, dy_n \right| =: \sum_{k=1}^3 J_k. \]

Lemmata A.1–A.2 and (A.20) yield

\[ J_1 \leq C_{49} \int_0^\gamma \| \tilde{V}(\cdot, y_n) \|_{H^{\frac{3}{2}}(\mathbb{R}^3)}^2 \, dy_n - C_{49} \varphi_t^* (y^*_0) \| w \|_{L^2(Q)}^2. \]  

(A.22)

Lemma A.4 implies

\[ J_2 \leq C_{49} \| \tilde{V} \|_{L^2(Q)}^2 - C_{49} \varphi_t^* (y^*_0) \| w \|_{L^2(Q)}^2. \]  

(A.23)

By lemma A.2 we have

\[ J_3 \leq C_{49} \| \tilde{V} \|_{L^2(Q)}^2 - C_{49} \| w \|_{L^2(Q)}^2. \]  

(A.24)

By (A.18), (A.19), (A.21), (A.22)–(A.24), we obtain

\[ - \int_0^\gamma \text{Im}(\Gamma_{\beta}^* (y, \tilde{D}, \tilde{\gamma}) V, M(\tilde{D}, \tilde{\gamma}) V)_{L^2(\mathbb{R}^3)}) \, dy_n \geq - C_{53} \| w \|_{H^{\frac{3}{2}}(\mathbb{R}^3)}^2. \]

This inequality, (A.16) and proposition 2.3 imply (2.37).
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