BPS Solutions in D=5
Dilaton-Axion Gravity

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Abstract

We show that the D=5 dilaton-axion gravity compactified on a 2-torus possesses the
SL(4,R)/SO(4) matrix formulation. It is used for construction of the SO(2,2)-invariant
BPS solution depended on the one harmonic function.

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1 Introduction

Low energy limit of superstring theories provides some modifications of general relativity possessing non-trivial symmetry groups and different matrix formulations. So, the bosonic sector of heterotic string theory, which describes by the $D = d + 3$ dilaton-axion gravity coupled to the $n$ Abelian vector fields, allows the $SO(d+1,d+n+1)$ symmetry after the compactification on a $d$-torus [1]-[2]. Its $SO(d+1,d+n+1)/[SO(d+1) \times O(d+n+1)]$ chiral formulation was extensively used for construction of exact solutions and for general analysis of the low energy heterotic string states, especially in the critical case of $d=7$ and $n=16$.

In this letter we consider the $d=2, n=0$ theory, which also arises in the frames of the $D=5$ supergravity [3], and can be considered as its special electromagnetic-free case [4]-[5]. We show that this theory allows the $SL(4,R)/SO(4)$ formulation in addition to the known one $SO(3,3)/[SO(3) \times O(3)]$. The similar situation takes place for the model with with $d=n=1$, when both the $SO(2,3)/[SO(2) \times O(3)]$ and $Sp(4,R)/U(2)$ representations become possible.

Next, following the Kramer-Neugebauer approach [6], developed in [7] for the $D=5$ Kaluza-Klein theory and in [8]-[9] for the $d=n=1$ system, we study the backgrounds trivial at space infinity and depended on the one harmonic function. It is shown that the corresponding solutions of motion equations can be obtained from the trivial one (i.e. from the solution describing the empty and flat $D=5$ space-time) using some matrix operator of the coset $SL(4,R)/SO(2,2)$. This operator is parametrized by the full set of physical charges; the group $SO(2,2)$, being the maximal isometry subgroup preserving a vacuum state, form the general symmetry group of the solutions under consideration.

The constructed matrix operator defines the general BPS solution in a parametric form. We present explicit formulae in the case of Coulomb dependence of this operator on the space coordinates. The corresponding solution describes the BPS interacting point sources and contains the black hole case as well the naked singularity one.

Some classes of $D=5$ BPS solutions with non-trivial values of electromagnetic charges were obtained in [10].
2 New Chiral Formulation

In this letter we study the D=5 dilaton-axion gravity described by the action

\[ S^{(5)} = \int |G^{(5)}|^2 e^{-\phi^{(5)}} \left\{ R^{(5)} + \phi^{(5)}_{,M} \phi^{(5)}_{,M} - \frac{1}{12} H^{(5)}_{M N K} H^{(5) M N K} \right\}, \]  

(1)

where \( \phi^{(5)} \) is dilaton and the axion \( H^{(5)}_{M N K} \) is related with the antisymmetric Kalb-Ramond field \( B_{N K} \) (M, N, K = 1, ..., 5) as follows:

\[ H^{(5)}_{M N K} = \partial_M B_{N K} + \partial_K B_{M N} + \partial_N B_{K M}. \]  

The compactification of this theory on a 2-torus was performed in [1]-[2]. The complete set of the 3-dimensional variables includes

two 2 \times 2 matrices \( G = [G_{pq}] \) and \( B = [B_{pq}] \) \( (p, q = 1,2) \),

\[ G_{pq} = G^{(5)}_{p+3,q+3}, \quad B_{pq} = B^{(5)}_{p+3,q+3}; \]  

(2)

two 2 \times 3 matrices \( U = [U_{p\mu}] \) and \( V = [V_{p\mu}] \) \( (\mu, \nu = 1,2,3) \),

\[ U_{p\mu} = (G^{-1})_{pq} G^{(5)}_{p+3,\mu}, \quad V_{p\mu} = B^{(5)}_{p+3,\mu} - B_{pq} U_{q\mu}; \]  

(3)

and also the 3-dimensional dilaton, metric and Kalb-Ramond fields

\[ \phi = \phi^{(5)} - \frac{1}{2} \ln \det G, \]  

(4)

\[ g_{\mu\nu} = e^{-2\phi} \left[ G^{(5)} - U^T G U \right]_{\mu\nu}, \]  

(5)

\[ B_{\mu\nu} = \left[ B^{(5)} - U^T B U - \frac{1}{2} \left( U^T V - V^T U \right) \right]_{\mu\nu} \]  

(6)

(The quantity \( B_{\mu\nu} \) can be taken equal to zero in view of its non-dynamic properties in three dimensions).

This system allows the dualization on shell. Namely, one can introduce the 3-dimensional pseudoscalar columns \( u = [u_\mu] \) and \( v = [v_\mu] \) accordingly to

\[ \nabla u = e^{-2\phi} \left[ G \nabla \times \vec{U} - BG^{-1} (\nabla \times \vec{V} + B \nabla \times \vec{U}) \right], \]  

(7)
\[ \nabla v = e^{-2\phi} G^{-1} \left[ \nabla \times \vec{V} + B \nabla \times \vec{U} \right]. \]  

(8)

Here the $2 \times 1$ vector columns $\vec{U}$ and $\vec{V}$ are constructed of the components $\vec{U}_p = (U_{p\mu})$ and $\vec{V}_p = (V_{p\mu})$ correspondingly and all $3$-dimensional vector operations are related with the metric $g_{\mu\nu}$.

The resulting model becomes chiral one with the action

\[ S = \int d^3 x g^{\frac{3}{2}} \left\{ -R + \frac{1}{8} \text{Tr}(J^M)^2 \right\}, \]

(9)

where $J^M = \nabla M M^{-1}$. The symmetric matrix $M$

\[ M = \begin{pmatrix} G^{-1} & G^{-1}B \\ -BG^{-1} & G - BG^{-1}B \end{pmatrix} \]

(10)

has the following $3 \times 3$ block components [12]:

\[ G = \begin{pmatrix} -e^{-2\phi} + v^T G v \\ v^T G \\ G v \\ G \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -(u + Bv)^T \\ u + Bv \end{pmatrix}. \]

(11)

It possesses the $SO(3,3)$ group property

\[ M^T \mathcal{L} M = \mathcal{L}, \quad \text{where} \quad \mathcal{L} = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}, \]

(12)

where $I_3$ is the $3 \times 3$ unit matrix, so $M \in SO(3,3)/SO(3) \times SO(3)$. The other form of this matrix was presented by Sen in [2] for the case of $d=7$, $n=16$.

Two group isomorphisms, $SO(3,3) \sim SL(4,R)$ and $SO(3) \times SO(3) \sim SO(4)$, provide an existence of the new $SL(4, R)/SO(4)$ chiral formulation of a problem. Actually, let us introduce the $3 \times 1$ column $H$

\[ B_{mn} = \epsilon_{mnk} H_k, \]

(13)

where $m, n, k = 1, 2, 3$ and $\epsilon_{mnk}$ is the antisymmetric tensor with $\epsilon_{123} = 1$. Then, after some algebraic manipulations one obtains from Eq. (9) that

\[ S = \int d^3 x g^{\frac{3}{2}} \left\{ -R + \frac{1}{4} \text{Tr}(J^N)^2 \right\}, \]

(14)

where $J^N = \nabla N N^{-1}$ and $N$ is given by

\[ N = (\det G)^{\frac{1}{2}} \begin{pmatrix} G & G H \\ \mathcal{H}^T G & \det G + \mathcal{H}^T \mathcal{H} \end{pmatrix}. \]

(15)
It is easy to prove that the symmetric matrix $N$ is unimodular, so this matrix actually belongs to $\text{SL}(4,\mathbb{R})/\text{SO}(4)$. This new chiral formulation is the simplest one in view of its trivial group property and low matrix dimension. We use it in the next section for the analysis of asymptoticaly flat heterotic string backgrounds.

The $\text{SL}(4,\mathbb{R})/\text{SO}(4)$ coset formulation arose previously in the frame s of the $D=6$ Kaluza-Klein theory compactified on a 3-torus $[12]$. Such isomorphism of two different theories has not any analogy for other values of $d$ and $n$.

### 3 Harmonic Solution

The equations of motion

$$\nabla \left( J^N \right) = 0, \quad R_{\mu\nu} = \frac{1}{4} \text{Tr} \left( J^N J^N_{\mu\nu} \right), \quad (16)$$

allow the solution anzats $[ \mathcal{N} = \mathcal{N}(\lambda)$ where the coordinate function $\lambda(x)$ satisfies to the Laplace equation

$$\nabla^2 \lambda = 0. \quad (17)$$

From the matter part of Eq. (16) it follows that

$$\mathcal{N} = e^{\lambda T} \mathcal{N}_0 \equiv \mathcal{S} \mathcal{N}_0, \quad (18)$$

where $\mathcal{N}_0 = \mathcal{N}|_{\lambda=0}$ and $T$ is the constant matrix. The Einstein part of Eq. (16) transforms to

$$R_{\mu\nu} = \frac{1}{4} \text{Tr} \left( T^2 \right) \nabla_\mu \lambda \nabla_\nu \lambda, \quad (19)$$

so both the equations (17) and (19) correspond to the action

$$S = \int d^3x g^\frac{1}{2} \left\{-R + \frac{1}{4} \text{Tr}(T^2) \left( \nabla \lambda \right)^2 \right\}. \quad (20)$$

We define

$$\mathcal{N}_0 = \begin{pmatrix}
-I \\
0
\end{pmatrix}, \quad (21)$$

where block matrices are $2 \times 2$ ones. The corresponding quantities $\phi_0, u_0, v_0, B_0$ are trivial and $G_0 = \text{diag} (-1, 1)$, so this field configuration describes the empty and flat 5-dimensional space-time.
To provide the SL(4,R)/SO(4) coset property of $N$, one must restrict the operator $S$ by the relations $\det S = 1$ and $S^T = N_0 S N_0$. Then for the matrix $T$ one obtains:

$$\text{Tr} \, T = 0, \quad \text{and} \quad T^T = N_0 T N_0.$$ (22)

Using these formulae one can establish after some algebraic calculations that the matrix $T$ satisfies to the relation

$$T^4 = \frac{1}{4} \text{Tr}(T^4) + \frac{1}{2} \text{Tr}(T^2) \left[ T^2 - \frac{1}{4} \text{Tr}(T^2) \right] + \frac{1}{3} \text{Tr}(T^3) T.$$ (23)

This property is crucial for the explicit calculation of the exponential $S = e^{\lambda T}$. Namely, let us denote four roots of the equation

$$k^4 = \frac{1}{4} \text{Tr}(T^4) + \frac{1}{2} \text{Tr}(T^2) \left[ k^2 - \frac{1}{4} \text{Tr}(T^2) \right] + \frac{1}{3} \text{Tr}(T^3) k,$$ (24)

as $k_\alpha$, where $\alpha = 1, 2, 3, 4$. Then for the different roots the exponentiation result is given by

$$S = \sum_{\alpha} e^{k_\alpha \lambda} \left\{ k_\alpha^3 - \frac{1}{2} \text{Tr}(T^2) k_\alpha - \frac{1}{3} \text{Tr}(T^3) + [k_\alpha^2 - \frac{1}{2} \text{Tr}(T^2)]T + k_\alpha T^2 + T^3 \right\} \prod_{\beta \neq \alpha} (k_\alpha - k_\beta)^{-1},$$ (25)

while the formulae for various cases of coinciding roots can be obtained from Eq. (25) using corresponding limes procedure. The exceptional case of zero roots is related with zero traces, $\text{Tr}(T^2) = \text{Tr}(T^3) = \text{Tr}(T^4) = 0$. This leads to the vanishing value of $T^4$ (see Eq.(23)), so for the operator under consideration one has $S = 1 + \lambda T + \lambda^2 T^2 / 2 + \lambda^3 T^3 / 6$.

Using Eq. (24) it is easy to express three independent traces in terms of the roots $k_\alpha$:

$$\text{Tr}(T^4) - \frac{1}{2} [\text{Tr}(T^2)]^2 = 4 \prod_{\alpha} k_\alpha, \quad \text{Tr}(T^3) = 3 \prod_{\alpha} k_\alpha \sum_{\beta} k_\beta^{-1},$$

$$\text{Tr}(T^2) = - \sum_{\beta \neq \alpha} k_\alpha k_\beta.$$ (26)

The reversed relations (roots as functions of traces) are more complicated. These ones can be obtained using the Cardano and Ferrari formulae; we will not write them here.
Thus, original theory reduces to the system (20) in the frames of the anzats under consideration. This means that any solution of this system, which coincides with the 3-dimensional Einstein-Klein-Gordon model, can be transformed to the solution of the theory (1) using the operator $S$ defined by Eq. (25).

Now let us establish symmetries of the discussing backgrounds. It is easy to see that the matrix $\Gamma$, defined by the relation $\Gamma^T = -N_0 \Gamma N_0$, is the generator of symmetry transformation preserving the vacuum value $N_0$. Actually, the matrix $\mathcal{C} = e^{\Gamma}$ belongs to $SL(4, R)$ and can be considered as the operator of symmetry transformation $N \rightarrow \mathcal{C}^T N \mathcal{C}$. Next, it satisfies to the relation $\mathcal{C}^T N_0 \mathcal{C} = N_0$, i.e. moreover $\mathcal{C} \in SO(2, 2)$. For the anzats under consideration this transformation is equivalent to the map $T \rightarrow \mathcal{C}^T T \mathcal{C}^{T^{-1}}$ which preserves the algebraic property (12). Thus, it defines the $SO(2, 2)$ reparametrization of the matrix $T$. Finally, one can see that $T \in \text{sl}(4, R)/\text{so}(2, 2)$, thus $S \in \text{SL}(4, R)/\text{SO}(2, 2)$.

## 4 BPS States

The effective 3-dimensional system (20) is equivalent to the static D=4 Einstein theory if $\text{Tr}(T^2) > 0$; the Einstein’s metric element $g_{tt}$ is related with our harmonic function $\lambda$ as $g_{tt} = \sqrt{\text{Tr}(T^2)/2\ln|\lambda|}$. Below we consider backgrounds saturated to the Bogomol’nyi-Gibbons-Hull bound $\text{Tr}(T^2) = 0$ which are related with BPS states of the original heterotic string theory. In this case the 3-metric $g_{\mu\nu}$ becomes flat, so one can put $ds^2(3) = dr^2$ without loss of generality. Next, the trivial at space infinity solution of Eq. (17) reads $\lambda = \sum_k \lambda_k / |\vec{r} - \vec{r}_k|$, where $\lambda_k = \text{const}$. The resulting field configuration describes the system of BPS sources located at the points $\vec{r}_k$ (see [] for the details in the case of $d=n=1$ theory).

Let us define the background charges as ($r \to \infty$)

$$
\phi \to \frac{D}{r}, \quad B \to \frac{2K}{r} \sigma_2, \quad u \to \frac{2N_u}{r}, \quad v \to \frac{2N_v}{r}, \quad G \to -\left(1 - \frac{2M}{r}\right) \sigma_3.
$$

(27)

Then, comparing these formulae with the equation $\mathcal{N} \to (1 + T/r)\mathcal{N}_0$, one can find the following relationship between block components of $T$ and introduced charges:

$$
T = \begin{pmatrix}
-D + \text{Tr}M & 2N_v^T & 2K \\
2\sigma_3 N_v & D + \text{Tr}M - 2M & -2\sigma_1 N_u \\
-2K & 2N_u \sigma_2 & -D - \text{Tr}M
\end{pmatrix}.
$$

(28)

Thus, the matrix $T$ is the charge matrix of the theory.
Now let us consider the special solution corresponded to the case of single BPS source with $T^2 = 0$. Then $\lambda = 1/r$ and

$$ds^2_{(3)} = dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\varphi^2).$$

(29)

For the potentials $u$ and $v$ one obtains:

$$u = 2 \left(1 - \frac{2M}{R}\right) \frac{RN_u}{\Delta}, \quad v = 2 \left(1 + \frac{2\sigma_2 M}{R}\right) \frac{RN_v}{\Delta},$$

(30)

where $R = r + D + \text{Tr}M$ and $\Delta = (R - \text{Tr}M)^2 - (\text{Tr}M)^2 + 4\det M$, while the extra components of the Kalb-Ramond field read

$$B^{(5)}_{p+3,q+3} = \frac{2\sigma_2}{(R - D - \text{Tr}M)^2} \left\{ \frac{K\Delta}{R} + 2N^T_v \sigma_1 \left(1 - \frac{2M}{R}\right) N_u \right\}.$$  

(31)

Next, the 3-dimensional dilaton function and extra metric components are

$$e^{2\phi} = \frac{\Delta}{(R - D - \text{Tr}M)^2}, \quad G = - \left(1 - \frac{2M}{R}\right) \sigma_3,$$

(32)

so the 5-dimensional dilaton takes the form

$$e^{\phi^{(5)}} = \frac{\Delta}{R(R - D - \text{Tr}M)}.$$  

(33)

Thus, the physical 5-dimensional dilaton charge is $D^{(5)} = D - \text{Tr}M$. Finally, from Eqs. (2), (3), (5) and (7) it follows that the 5-dimensional line element reads

$$ds^2_{(5)} = G_{pq}(dx - \sigma_3 N_u \cos \theta d\varphi)^{3+p}(dx - \sigma_3 N_u \cos \theta d\varphi)^{3+q} + e^{2\phi} ds^2_{(3)}.$$  

(34)

Charges of the solution (29)-(34) are not independent: the matrix relation $T^2 = 0$ puts the following five scalar constraints on the set of nine charge parameters:

$$D\text{Tr}(M\sigma_3) + N^T_u \sigma_3 N_u - N^T_v \sigma_3 N_v = 0, \quad D\text{Tr}M + N^T_u N_u - N^T_v N_v = 0,$$

$$2N_v \sigma_3 N_v + 2N^T_u \sigma_3 N_u + D^2 - 2K^2 + \text{Tr}M^2 = 0, \quad \det M - K^2 = 0,$$

$$D \left[\text{Tr}(M\sigma_2) - 2K\right] + (N_v + N_u)^T \sigma_1 (N_v + N_u) = 0.$$

(35)

These constraints can be solved explicitly; the resulting solution is four-parametric and contains both the black hole (with horizon located at $R_H = D + \text{Tr}M$) and massless naked singularity branches. It is easy to establish that black-hole backgrounds are two-parametric and correspond to the additional constraint $N_u = 0$. 

7
5 Conclusion

Thus, the low-energy D=5 heterotic string theory allows the SL(4,R)/SO(4) formulation after compactification on a 2-torus. We have used this one for the general analysis of backgrounds trivial at the space infinity and depended on the one harmonic function.

It is shown that the corresponding solutions of motion equations are invariant under the action of the SO(2,2) group of transformations which is the maximal subgroup preserving the asymptotic flatness property of backgrounds. The obtained formulae represent the class of BPS solutions in a parametric form; the explicit formulae of all potentials are given in the special case of Coulomb dependence of the chiral matrix on the space coordinates.

Using the technique developed in this letter it is possible to construct rotating solutions defined by two harmonic functions, as it had been done for the D=5 Kaluza-Klein [7] and d=n=1 low energy heterotic string theories [9].

The D=5 system without electromagnetic fields can be mapped into the theory with non-trivial Maxwell sector using the “charging” matrix Harrison transformation. To perform it one can use the Ernst matrix potential technique developed in [11].

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