PROJECTIVE COMPACTNESS
AND CONFORMAL BOUNDARIES

ANDREAS ČAP AND A. ROD GOVER

Abstract. This article studies relations between geometric structures on the interior of a smooth manifold with boundary (at infinity) and structures on the boundary, and also the corresponding relations between asymptotics of interior quantities and boundary quantities. Examples arising from reductions of projective holonomy have recently led to a general formulation of a concept of projective compactness in terms of projective differential geometry. Basically, projective compactness of a torsion free affine connection on the interior is defined as the condition that appropriate projective modifications admit a smooth extension to the boundary. Via the Levi–Civita connection, one obtains a concept of projective compactness for pseudo–Riemannian metrics. In addition, one has to involve a real parameter $\alpha \in (0,2]$ called the order of projective compactness, which is related to volume growth.

The main result of the article is that, for a pseudo–Riemannian metric, projective compactness of order two is equivalent to a certain asymptotic form. This form is the basis of a notion of projectively compact metrics introduced by Fefferman and Graham. An important ingredient for our result is that the projective structure induces a conformal structure on the boundary, which can be related to the asymptotic form. This also leads to results on the asymptotics of curvature, including a proof that pseudo–Riemannian metrics which are projectively compact of order two always satisfy an asymptotic version of the Einstein equation.

The second main result is an explicit description of the conformal tractor bundle, associated to the conformal structure on the boundary, in terms of projective tractors on the interior. This is done for general affine connections which are projectively compact of order two and induce a non–degenerate boundary geometry, with simplifications in the pseudo–Riemannian case also indicated. Many of the results of the article are proved for general orders of projective compactness.

MSC2010: Primary 53A20, 53B21, 53B10; Secondary 35N10, 53A30, 58J60

Both authors gratefully acknowledge support from the Royal Society of New Zealand via Marsden Grant 13-UOA-018; AC gratefully acknowledges support by project P23244-N13 of the “Fonds zur Förderung der wissenschaftlichen Forschung” (FWF) and also the hospitality of the University of Auckland.

1. Introduction

Consider a smooth manifold $\overline{M}$ with boundary $\partial M$ and interior $M$. The study of geometric structures on $\partial M$ induced by complete Riemannian (or pseudo–Riemannian) metrics on $M$, and of the relation between asymptotic data on $M$ and data on $\partial M$, has a long history and interesting applications in mathematics
and physics, e.g. [11, 20, 23]. A model case for this situation is provided con-
formally compact metrics on \( M \), i.e. complete metrics for which an appropriate con-
formal rescaling extends to the boundary. Such a metric gives rise to a well-
deﬁned conformal class of metrics on \( \partial M \), which then is referred to as the confor-
mal inﬁnity of the interior metric. Originating and ﬂourishing in general relativity
(see e.g. [12, 17, 18, 26, 27]), this concept has also found important applications
in geometric scattering theory ([21, 24, 25]) and the conjectural AdS–CFT corre-
spondence in physics ([11, 13]). If one in addition requires the conformally compact
metric on \( M \) to be negative Einstein, one arrives at the notion of a Poincaré–
Einstein metric. Realizing a given conformal class on a manifold formally as the
conformal inﬁnity of a Poincaré–Einstein metric is closely related to the Fefferman–
Graham conformal ambient metric construction [15, 16], thus providing a central
tool for generating conformal invariants.

The ambient metric construction also has a deep relation to projective differ-
ential geometry. Indeed, in Chapter 4 of [16] Fefferman and Graham consider a
certain asymptotic form for pseudo–Riemannian metrics, which they call projec-
tively compact, and they observe that appropriate projective modiﬁcations of their
Levi–Civita connections admit smooth extensions to the boundary. They did not
go further into the relations to projective differential geometry, however. On the
other hand, in the classical and visionary articles [28, 29], Schoute and Haantjes
develop a construction essentially equivalent to the ambient metric, but based on
projective differential geometry.

Guided by examples arising from reductions of projective holonomy (see [7, 8]),
a conceptual approach to projective compactness was developed in our article [6].
The basic idea there was to start with a linear connection \( \nabla \) on \( M \) and use local
deﬁning functions for the boundary to deﬁne projective modiﬁcations of \( \nabla \) which
are then required to admit a smooth extension to the boundary. Applying this to
the Levi–Civita connection, the concept then is automatically deﬁned for pseudo–
Riemannian metrics. It turned out that it is natural to involve a real parameter
\( \alpha > 0 \), called the order of projective compactness. For a local deﬁning function \( \rho \)
for the boundary (see Section 2.1 for detailed deﬁnitions) projective compactness
of \( \nabla \) of order \( \alpha \) then is the requirement that if two vector ﬁelds \( \xi \) and \( \eta \) are smooth
up to the boundary, then

\[
\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \frac{1}{\alpha \rho} d\rho(\xi) \eta + \frac{1}{\alpha \rho} d\rho(\eta) \xi
\]

admits a smooth extension to the boundary. In the case of connections preserving
a volume density, this order measures the growth of that density towards the
boundary. The main cases of interest are \( \alpha = 1 \) and \( \alpha = 2 \).

In case that \( 2/\alpha \) is an integer, the article [6] describes an asymptotic form of
a pseudo–Riemannian metric on \( M \) which is sufﬁcient for projective compactness
of order \( \alpha \). This is related to the metrics studied by Fefferman and Graham as
described above, in the case \( \alpha = 2 \), which will be the case of main interest for this
article. For \( \alpha = 2 \), this asymptotic form is given by

\[
g = \frac{h}{\rho} + \frac{Cd\rho^2}{\rho^2}.
\]
Here $\rho$ is a local defining function for the boundary, $C$ is a nowhere vanishing function which is smooth up to the boundary and asymptotically constant in a certain sense, and $h$ is a symmetric $(0,2)$–tensor field, which is smooth up to the boundary and whose boundary value is non–degenerate in tangent directions.

In the case $\alpha = 2$, the projectively compact connections related to reductions of projective holonomy are exactly the Levi–Civita connections of non–Ricci–flat Einstein metrics. In this case, we have proved in [6] that an asymptotic form as above is always available, with the function $C$ being a constant related to the Einstein constant. One of the main results of this article is Theorem 5 which shows that the asymptotic form (with constant $C$) is available for any pseudo–Riemannian metric that is projectively compact of order two. Hence the asymptotic form can be used as an equivalent definition of projective compactness in this case.

An important ingredient in the proof of this theorem is the study of the boundary geometry induced by a projectively compact connection $\nabla$ on $M$. By definition, the projective structure on $M$ defined by (the projective equivalence class of) $\nabla$ admits a smooth extension to $\overline{M}$. The boundary $\partial M$ is, of course, a hypersurface in $\overline{M}$ and it is a classical fact that a the projective structure on $\overline{M}$ gives rise to a symmetric $(0,2)$–tensor field, which is well defined up to conformal rescaling, on any smoothly embedded hypersurface. This “projective second fundamental form” is a main object of study in Section 2 of this article.

We first show that the projective second fundamental form can be described in terms of the asymptotic behavior of the Schouten–tensor (or equivalently the Ricci-tensor). This also leads to results on the asymptotic behavior of the curvature of a projectively compact connection. On the other hand for metrics having an asymptotic form with $\alpha = 2$ as discussed above, the projective second fundamental form is related to the tensor field $h$ showing up in the asymptotic form. Together these results lead to a finer description of the curvature of a metric admitting the asymptotic form. In particular, we deduce that such a metric satisfies an asymptotic version of the Einstein equation, which also implies that the scalar curvature admits a smooth extension to the boundary with constant boundary value.

Based on this, the proof of Theorem 5 is given in Section 3. Apart from a rather subtle direct analysis, this uses deep tools from projective geometry and projective tractor calculus. In particular, we exploit the fact that a Levi–Civita connection in a projective class gives rise to a solution of a projectively invariant overdetermined system of PDEs, which is nicely related to tractors. The basic strategy we use is to first prove the existence of an asymptotic form locally around boundary points satisfying a technical condition (which are shown to form a dense open subset in the boundary), and then use the information provided by the asymptotic form to show that the technical condition is always satisfied.

In Section 4, we continue the study of the boundary geometry induced by a connection which is projectively compact of order two, assuming that the projective second fundamental form is non–degenerate. (By Theorem 5 this is always satisfied for Levi–Civita connections, in general it can be characterized in terms of the asymptotics of the Schouten tensor.) Under these assumptions, the boundary
inherits a pseudo–Riemannian conformal structure, which can be described in terms of (conformal) tractors. In the case of the Levi–Civita connection of a non–Ricci–flat Einstein metric, one can use the general theory of holonomy reductions to show that there is a simple relation between conformal tractors on the boundary and projective tractors in the interior. The main aim of Section 4 is to show that, although the relation is considerably more complicated in general (which is not surprising in view of the rather intricate relation between the geometries on $M$ and on $\partial M$), it can still be described explicitly as follows.

The projectively compact connection gives rise to a canonical defining density for the boundary, and applying the BGG splitting operator to this density, one obtains a bundle metric on the standard tractor bundle. We first show that this bundle metric is non–degenerate on all of $\overline{M}$ and that taking the restriction of the projective standard tractor bundle endowed with this bundle metric to $\partial M$, one obtains a standard tractor bundle for the induced conformal structure. Next, we describe how the conformal standard tractor connection on $\partial M$ can be constructed from the projective standard tractor connection on $M$ in two steps. First one can construct a torsion free tractor connection on all of $M$, which is metric for the given bundle metric. Restricting this to the boundary, a final step of normalization leads to the conformal standard tractor connection. Several simplifications in particular cases (for example for projective compact pseudo–Riemannian metrics) are discussed.

2. THE BOUNDARY GEOMETRY

A projectively compact connection on the interior of a manifold with boundary induces a projective structure on the whole manifold. As a hypersurface in a projective manifold, the boundary inherits the so–called projective second fundamental form, a conformal class of bilinear forms on the tangent bundle of the boundary. Our main aim in this section is to relate this structure on the boundary to data on the interior. We first do this for general projectively compact connections, showing that the projective second fundamental form is related to the asymptotics of the Schouten tensor, which leads to results of the asymptotic form of the curvature of a projectively compact connection. Then we turn to pseudo–Riemannian metrics admitting an asymptotic form which is sufficient for projective compactness. If the order of projective compactness is less than two, then in this case the projective second fundamental form vanishes, so the boundary is totally geodesic. In the case of order two, the projective second fundamental form is determined by one of the tensor fields showing up in the asymptotic form. The relation to the Schouten tensor then is used to prove that any such metric satisfies an asymptotic form of the Einstein equation.

2.1. Projective compactness. We start by reviewing some key concepts and results from [6]. Throughout this article, smooth means $C^\infty$, we consider a smooth manifold $\overline{M}$ with boundary $\partial M$ and interior $M$. By a local defining function for $\partial M$ we mean a smooth function $\rho : U \to [0, \infty)$ defined on an open subset of $\overline{M}$ such that $\rho^{-1}(\{0\}) = U \cap \partial M$ and $d\rho(x) \neq 0$ for all $x \in U \cap \partial M$. By $\mathcal{E}(w)$ we
will denote the bundle of densities of projective weight \( w \). Then there also is the concept of a defining density of weight \( w \), which is a local section \( \sigma \) of \( E(w) \) which is of the form \( \rho \hat{\sigma} \) for a local defining function \( \rho \) for \( \partial M \) and a section \( \hat{\sigma} \) of \( E(w) \) which is nowhere vanishing on \( U \).

Given an affine connection \( \nabla \) and a one–form \( \Upsilon \) on some manifold, we will write \( \hat{\nabla} = \nabla + \Upsilon \) for the projectively modified connection defined by

\[
\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi,
\]

for vector fields \( \xi \) and \( \eta \). Two connections are related in this way if and only if they have the same geodesics up to parameterization.

Now in the setting of a manifold with boundary, \( \overline{M} = M \cup \partial M \), a linear connection \( \nabla \) on \( TM \) is called projectively compact of order \( \alpha > 0 \) if and only if for any point \( x \in \partial M \), there is a local defining function \( \rho \) for \( \partial M \) defined on a neighborhood \( U \) of \( x \), such that the projectively modified connection \( \hat{\nabla} = \nabla + \frac{d\rho}{\alpha \rho} \) admits a smooth extension to all of \( U \). This means that for all vector fields \( \xi, \eta \) which are smooth up to the boundary, also

\[
\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \frac{1}{\alpha \rho} d\rho(\xi)\eta + \frac{1}{\alpha \rho} d\rho(\eta)\xi
\]

admits a smooth extension up to the boundary. Equivalently, the Christoffel symbols of \( \hat{\nabla} \) in some local chart have to admit such an extension.

It is easily verified that this condition is independent of the choice of the defining function \( \rho \), i.e. if the projective modification associated to \( \rho \) extends, then also the one associated to any other defining function is smooth up to the boundary. On the other hand, the parameter \( \alpha \) can not be eliminated. Indeed, it turns out that for connections which are special, i.e. preserve a volume density, \( \alpha \) controls the growth of a parallel volume density towards the boundary, see section 2.2 of [6]. The result on volume growth can be nicely reformulated in terms of defining densities. If \( \nabla \) is projectively compact of order \( \alpha \) and preserves a volume density, then for each \( w \), the density bundle \( E(w) \) admits non–zero parallel sections. However, precisely for \( w = \alpha \), such a section can be extended by zero to a defining density for \( \partial M \). It also turns out that for connections preserving a volume density, projective compactness of order \( \alpha \) is equivalent to the fact that the projective structure of \( \nabla \) extends to all of \( \overline{M} \) plus the appropriate rate of volume growth, see Proposition 2.3 of [6]. As in most of [6] we will restrict to the case \( 0 < \alpha \leq 2 \) in this article. It turns out that for these values of \( \alpha \), the boundary is at infinity, see Proposition 2.4 in [6].

### 2.2. The induced conformal structure on the boundary.

Suppose that \( \overline{M} = M \cup \partial M \) is a smooth manifold with boundary and that \( \nabla \) is an affine connection on \( M \) which is projective compact of some order \( \alpha \in (0, 2] \). Let us recall the construction of the projectively invariant second fundamental form for the extended projective structure.

Choose a local defining function \( \rho \) for \( \partial M \), let \( \hat{\nabla} \) be any connection in the projective class which is smooth up to the boundary and consider \( \nabla d\rho \in \Gamma(S^2T\overline{M}) \). Denoting the one–form \( d\rho \) by \( \rho_a \), we see that for a projectively equivalent connection \( \nabla \), we get \( \nabla_a \rho_b = \nabla_a \rho_b - \Upsilon_a \rho_b - \Upsilon_b \rho_a \), so we see that \( \nabla_a \rho_b \) and \( \nabla_a \rho_b \) have
the same restriction to \( T\partial M \times T\partial M \). On the other hand, changing the defining function \( \rho \) to \( \hat{\rho} = e^f \rho \), we get \( \hat{\rho}_a = \hat{\rho} f_a + e^f \rho_a \), where \( f_a = df \), and thus

\[
\hat{\nabla}_a \hat{\rho}_b = \hat{\rho}_a f_b + \hat{\rho} \hat{\nabla}_a f_b + e^f f_a \rho_b + e^f \hat{\nabla}_a \rho_b.
\]

Hence the restriction of \( \hat{\nabla}_a \hat{\rho}_b \) to \( T\partial M \times T\partial M \) is conformal to the restriction of \( \nabla_a \rho_b \), so the (possibly degenerate) conformal class \([\nabla_a \rho_b]\) on \( T\partial M \) is canonical. We will refer to \( \nabla_a \rho_b \) as a representative of the projective second fundamental form.

By construction, the projective second fundamental form only depends on the extended projective structure on the manifold \( \overline{M} \) with boundary, and not on the specific projectively compact connection \( \nabla \) on \( M \). It turns out, however, that there is a nice relation to the projectively compact connection on the inside.

**Proposition 1.** Let \( \nabla \) be a linear connection on \( TM \) which is projectively compact of some order \( \alpha \in (0, 2] \), and let \( P_{ab} \) be the Schouten tensor of \( \nabla \).

Then for any local defining function \( \rho \) for \( \partial M \) the smooth section \( \rho P_{ab} + \frac{\alpha - 1}{\alpha^2} \rho_a \rho_b \) admits a smooth extension to the boundary and its boundary value restricts to a representative of the projective second fundamental form on \( T\partial M \).

**Proof.** Let \( \hat{\nabla} \) be the projective modification of \( \nabla \) associated to \( \rho \), which extends to the boundary, i.e. \( \hat{\nabla}_a = \nabla_a + \Upsilon_a \) with \( \Upsilon_a = \frac{\alpha}{\alpha^2} \). Then of course the Schouten tensor \( \hat{P}_{ab} \) of \( \hat{\nabla} \) is smooth up to the boundary. The relation between \( P_{ab} \) and \( \hat{P}_{ab} \) from [2] reads as

\[
P_{ab} = \hat{P}_{ab} + \hat{\nabla}_a \Upsilon_b + \Upsilon_a \Upsilon_b.
\]

Now \( \hat{\nabla}_a (\frac{1}{\alpha} \rho_b) = -\frac{1}{\alpha^2} \rho_a \rho_b + \frac{1}{\alpha} \hat{\nabla}_a \rho_b \). On the other hand, \( \Upsilon_a \Upsilon_b = \frac{1}{\alpha^2} \rho_a \rho_b \), and inserting this, we get that

\[
P_{ab} = \hat{P}_{ab} + \frac{1}{\alpha} \hat{\nabla}_a \rho_b - \frac{\alpha - 1}{\alpha^2} \rho_a \rho_b
\]

and thus

\[
\rho P_{ab} + \frac{\alpha - 1}{\alpha^2 \rho} \rho_a \rho_b = \frac{1}{\alpha} \hat{\nabla}_a \rho_b + \rho \hat{P}_{ab}.
\]

Since the right hand side is evidently smooth up to the boundary with boundary value \( \frac{1}{\alpha} \hat{\nabla}_a \rho_b \), the result follows. \( \square \)

**2.3. Curvature asymptotics.** We next prove a general result on the asymptotic behavior of the curvature of a connection which is projectively compact of some order \( \alpha \in (0, 2] \). This is similar to the fact that conformally compact pseudo–Riemannian metrics are asymptotically hyperbolic, see [19].

To formulate the result, recall that from each symmetric \( \binom{0}{2} \)-tensor field, one can build up a tensor having curvature symmetries by putting \( R_{ab} := \delta^c_{[a} \varphi_{b]d} - \delta^c_{[a} \varphi_{d]} \).

In particular, we can apply this to \( \binom{0}{2} \)-tensor fields which have rank one, i.e. are of the form \( \varphi_{ab} = \psi_a \psi_b \) for a one–form \( \psi = \psi_a \). In this case we call the corresponding curvature tensor the rank–one curvature tensor determined by \( \psi \).

**Proposition 2.** Let \( \nabla \) be a linear connection on \( TM \) which is projectively compact of some order \( \alpha \in (0, 2] \), let \( R = R_{ab} := \psi_a \psi_b \) be the curvature tensor of \( \nabla \). Let \( \rho \) be a
local defining function for $\partial M$ and let $\nabla = \nabla + \frac{d\rho}{\rho^2}$ be the associated connection in the projective class.

(i) If $\alpha = 1$, then $\rho R$ admits a smooth extension to the boundary with boundary value

$$\delta^c_a \nabla_b \rho d - \delta^c_b \nabla_a \rho d.$$ 

(ii) If $\alpha \neq 1$, then $\rho^2 R$ admits a smooth extension to the boundary with boundary value equal to $\frac{1-\alpha}{\alpha^2}$ times the rank–one curvature tensor determined by the one–form $dp$.

Proof. The decomposition of the curvature tensor used in projective geometry, see section 3.1 of [2], reads as

$$R_{ab}^c d = C_{ab}^c d + \delta^c_a P_{bd} - \delta^c_b P_{ad} + \beta_{ab} \delta^c_d.$$ 

Here $C_{ab}^c d$ is the projective Weyl curvature, $P_{ab}$ is the projective Schouten tensor and $\beta_{ab} = P_{ba} - P_{ab}$ (so this vanishes for connections preserving a volume density). Now the projective Weyl curvature is projectively invariant, so since the projective structure extends smoothly to $\overline{M}$, $C_{ab}^c d$ admits a smooth extension to the boundary.

We have analyzed the behavior of $P_{ab}$ in Proposition 1. If $\alpha = 1$, then $\rho P_{ab} = \nabla_a \rho b + \rho P_{ab}$, where $P_{ab}$ is the Schouten tensor of $\nabla_a$. Of course $\rho P_{ab}$ is smooth up to the boundary, and we conclude that, $\beta_{ab} = \beta_{ab}$, so $\beta_{ab}$ is smooth up to the boundary. This completes the proof of (i).

(ii) If $\alpha \neq 1$, then Proposition 1 shows that $\rho^2 P_{ab}$ admits a smooth extension to the boundary with boundary value $\frac{1-\alpha}{\alpha^2} \rho_a \rho_b$, so again the result follows. 

2.4. Projectively compact pseudo–Riemannian metrics and asymptotic forms. Projective compactness of a pseudo–Riemannian metric $g$ on $M$ is defined as projective compactness of its Levi–Civita connection $\nabla$. In section 2.4 of [1], we have proved that a certain asymptotic form for $g$ implies projective compactness of order $\alpha$ for any fixed $\alpha \in (0, 2]$ such that $\frac{2}{\alpha}$ is an integer. We next want to specialize the results on the boundary conformal structure and on curvature asymptotics to metrics admitting such an asymptotic form. These will play an important technical role (in the case $\alpha = 2$) in the next section.

The assumptions for this asymptotic form is that locally around each boundary point, we find a defining function $\rho$ and a nowhere vanishing smooth function $C$ with additional properties specified below, such that the $\binom{2}{\alpha}$–tensor field

$$h := \rho^{2/\alpha} g - C \frac{d\rho^2}{\rho^{2/\alpha}}$$

admits a smooth extension to the boundary with the boundary value being non–degenerate on $T\partial M$. The additional property required from $C$ is that for each vector field $\zeta$ which is smooth up to the boundary and satisfies $d\rho(\zeta) = 0$, the function $\rho^{-2/\alpha} \zeta \cdot C$ is smooth up to the boundary. Theorem 2.6 of [6] then states that under these assumptions (including $2/\alpha \in \mathbb{Z}$), the Levi–Civita connection of $g$ is projectively compact of order $\alpha$. 

If one assumes that a projectively compact metric of order 2 is a non–Ricci–flat Einstein metric, then it defines a reduction of projective holonomy. This important special case is discussed in detail in [6], and in particular Proposition 3.6 of that article shows that for an Einstein–metric with (non–zero) scalar curvature $R$, one always obtains an asymptotic form as above for which $C$ can be taken to be the constant $-\frac{n(n+1)}{4R}$. Indeed as we shall prove in section 3 below, an asymptotic form with constant $C$ is always available for pseudo–Riemannian metrics which are projectively compact of order 2 (using the results obtained here). So for $\alpha = 2$, we will pay special attention to the case that $C$ is constant.

**Proposition 3.** Suppose that we are in the setting of Theorem 2.6 of [6], i.e. $\frac{2}{\alpha} \in \mathbb{Z}$ and the tensor field $h$ defined in (3) is smooth up to the boundary with the boundary value being non–degenerate on $\partial \mathcal{M}$.

(i) If $\alpha < 2$, then the projective second fundamental form for $\partial \mathcal{M}$ vanishes identically, so $\partial \mathcal{M}$ is totally geodesic.

(ii) If $\alpha = 2$, then the restriction of $h$ to boundary directions is a representative of the projective second fundamental form for $\partial \mathcal{M}$. If in addition $C$ is constant, then, more precisely, the boundary value of $h$ coincides with $-2C\nabla d\rho$, where $\hat{\nabla} = \nabla + \frac{d\rho}{2\rho}$ is the projective modification of the Levi–Civita connection $\nabla$ of $g$ associated to $\rho$.

**Proof.** We use ideas from the proof of Theorem 2.6 of [6] and also the notation introduced there. In the proof of that theorem, one first constructs a vector field $\zeta_0$ such that $d\rho(\zeta_0) \equiv 1$ and $\zeta_0$ is orthogonal with respect to $h$ to all vector fields in the kernel of $d\rho$. In particular, as observed there, one can compute the boundary value of $-d\rho(\nabla_\zeta \eta)$ as the boundary value of $\frac{1}{\alpha} C \rho^{4/\alpha} g(\nabla_\xi \eta, \zeta_0)$.

A key ingredient in the proof of Theorem 2.6 of [6] is the modified Koszul formula, which says that $2g(\nabla_\xi \eta, \zeta_0)$ can be computed as

\begin{equation}
\xi \cdot g(\eta, \zeta_0) - \zeta_0 \cdot g(\xi, \eta) + \eta \cdot g(\xi, \zeta_0) + g([\xi, \eta], \zeta_0) - g([\xi, \zeta_0], \eta) - g([\eta, \zeta_0], \xi) + \frac{2d\rho(\xi)}{\alpha\rho} g(\eta, \zeta_0) + \frac{2d\rho(\eta)}{\alpha\rho} g(\xi, \zeta_0).
\end{equation}

Let us first assume that $d\rho(\xi) = d\rho(\eta) = 0$. Then $g(\xi, \zeta_0)$ and $g(\eta, \zeta_0)$ vanish identically. Next, $g([\xi, \zeta_0], \eta) = \frac{1}{\rho^2} h([\xi, \zeta_0], \eta)$, so after multiplication by $\rho^{4/\alpha}$ this extends smoothly to the boundary by zero, and the same holds for the corresponding term with $\xi$ and $\eta$ exchanged. Finally,

\[ g([\xi, \eta], \zeta_0) = d\rho([\xi, \eta])(\frac{C}{\rho^{4/\alpha}} + \frac{h(\zeta_0, \zeta_0)}{\rho^{2/\alpha}}) , \]

but by the assumption on $\xi$ and $\eta$, $d\rho([\xi, \eta]) = -dd\rho(\xi, \eta) = 0$, so this term does not contribute at all. In conclusion, we see that we can compute the boundary value of $-d\rho(\nabla_\zeta \eta)$ as the boundary value of \[ \frac{1}{\alpha} C \rho^{4/\alpha} \zeta_0 \cdot g(\xi, \eta) = \frac{1}{\alpha C} \rho^{2-\alpha} \zeta_0 \cdot h(\xi, \eta). \]

Up to terms vanishing along the boundary, this equals $\frac{1}{\alpha C} \rho^{2-\alpha} h(\xi, \eta)$. But since $d\rho(\eta) = 0$, we get $-d\rho(\nabla_\xi \eta) = (\nabla_\xi d\rho)(\eta)$, so we get (i) and the first part of (ii).
To obtain the second statement in (ii) we have to analyze (in the case $\alpha = 2$ and for $C$ being constant) the modified Koszul formula (4) for general vector fields $\xi$ and $\eta$, which needs much more care. From the proof of Theorem 2.6 in [6] we see that (always taking into account that $\alpha = 2$)

\[
g(\eta, \zeta_0) = d\rho(\eta) (C\rho^2 + \frac{1}{\rho} h(\zeta_0, \zeta_0))
g(\xi, \eta) = C\rho^2 d\rho(\xi) d\rho(\eta) + \frac{1}{\rho} h(\xi, \eta).
\]

Now if we plug the appropriate versions of these into the modified Koszul formula (4) and carry out the differentiations, we can sort the terms according to powers of $\rho$. In the proof of Theorem 2.6 of [6] it is shown that the terms containing $\frac{1}{\rho^3}$ add up to zero. We have to determine the terms containing $\frac{1}{\rho^2}$ while we may ignore terms containing $\frac{1}{\rho}$ or no negative power of $\rho$. The first and third term in (4) together contribute

\[
(5) \quad C\xi \cdot d\rho(\eta) + C\eta \cdot d\rho(\xi) - 2d\rho(\xi)d\rho(\eta)h(\zeta_0, \zeta_0)
\]

to the coefficient of $\frac{1}{\rho^2}$. Now the last part of this cancels with the contribution of the last two summands in (4). On the other hand, the only contribution of the fourth summand in (4) to the coefficient of $\frac{1}{\rho}$ is $C\rho^2 d\rho(\xi, \eta)$. Expanding $0 = dd\rho(\xi, \eta)$ we see that this adds up with the second term in (5) to $C\xi \cdot d\rho(\eta)$, so the overall contribution of all terms we have considered so far is $2C\xi \cdot d\rho(\eta)$.

Next, the contribution of the second summand of (4) to the coefficient of $\frac{1}{\rho^2}$ is given by

\[
h(\xi, \eta) - C\zeta_0 \cdot (d\rho(\xi)d\rho(\eta)),
\]

while the fifth and sixth summands contribute

\[
-Cd\rho([\xi, \zeta_0])d\rho(\eta) - Cd\rho([\eta, \zeta_0])d\rho(\xi).
\]

But since $d\rho(\zeta_0) \equiv 1$, the fact that $0 = dd\rho(\xi, \zeta_0)$ implies that $\zeta_0 \cdot d\rho(\xi) = d\rho([\xi, \zeta_0])$ and likewise for $\eta$, so these terms together only contribute $h(\xi, \eta)$.

Collecting the results, we see that the boundary value of $-d\rho(\nabla_\xi \eta)$ can be computed as the boundary value of $\frac{1}{2C}(2C\xi \cdot d\rho(\eta) + h(\xi, \eta))$. Bringing the first term to the other side, we obtain the boundary value of $(\nabla d\rho)(\xi, \eta)$ which implies the result. \hfill $\square$

Next, we describe the curvature for pseudo–Riemannian metrics admitting an asymptotic form (with constant $C$) as above for $\alpha = 2$. In particular, we can prove that such metrics satisfy an asymptotic version of the Einstein equation. This also provides an interpretation of the constant $C$.

**Theorem 4.** Let $g = g_{ab}$ be a pseudo–Riemannian metric on $M$ with inverse $g^{ab}$, such that for a constant $C$ and $\alpha = 2$, the tensor field $h = h_{ab}$ defined in (3) admits a smooth extension to the boundary. Let $R_{abcd}$ be the Riemann curvature of $g$, $R_{ab} = R_{dadb}$ its Ricci curvature and $R = g^{ab}R_{ab}$ its scalar curvature.

(i) The scalar curvature $R$ admits a smooth extension to the boundary with boundary value the constant $\frac{-n(n+1)}{4C}$.

(ii) The tensor field $R_{ab} + \frac{1}{4C}g_{ab}$ admits a smooth extension to the boundary.
(iii) Up to terms which admit a smooth extension to the boundary, the curvature of $g_{ab}$ is given by

$$R_{ab}{}^c{}_d = -\frac{1}{4\rho} \delta^c_{[a} \rho_{b]} \rho_d - \frac{1}{4\rho} \delta^c_{[a} h_{b]} d.$$

Proof. By Proposition 1 and formula (2) from its proof, $\rho P_{ab} + \frac{1}{4\rho} \rho a \rho_b$ admits a smooth extension to the boundary with boundary value $\frac{1}{2} \nabla_a \rho_b$. On the other hand, we have assumed that $-\frac{1}{4C} \rho g_{ab} + \frac{1}{4\rho} \rho a \rho_b$ admits a smooth extension to the boundary, and by Proposition 3, the boundary value $-\frac{1}{4C} h_{ab}$ also equals $\frac{1}{2} \nabla_a \rho_b$. Forming the difference, we conclude that $\rho (P_{ab} + \frac{1}{4C} g_{ab})$ admits a smooth extension to the boundary with boundary value zero, so $P_{ab} + \frac{1}{4C} g_{ab}$ admits a smooth extension to the boundary. Now in dimension $n + 1$, we have $R_{ab} = \frac{n}{4} P_{ab}$, which immediately implies (ii).

Next, in the beginning of the proof of Theorem 12, we will see (without reference to the current developments) that denoting by $g^{ab}$ the inverse of $g_{ab}$ the tensor field $\rho^{-1} g^{ab}$ admits a smooth extension to the boundary. Consequently, $g^{ab}$ itself admits a smooth extension to the boundary with boundary value 0. Together with the above, this implies that

$$g^{ab} (R_{ab} + \frac{n}{4C} g_{ab}) = g^{ab} R_{ab} + \frac{n(n+1)}{4C}$$

admits a smooth extension to the boundary with boundary value zero, so we get (i).

To prove (iii) we use the formula for the curvature from the proof of Proposition 2, taking into account that $\beta_{ab} = 0$. Since we know from above, that $P_{ab} + \frac{1}{4C} g_{ab}$ admits a smooth extension to the boundary, we may replace $P_{ab}$ by $-\frac{1}{4C} g_{ab}$, and then the claim follows from inserting the asymptotic form

$$g_{ab} = \frac{1}{\rho} h_{ab} + \frac{C}{\rho^2} \rho a \rho_b$$

for $g$.

\[\square\]

3. Projective compactness and asymptotic form

In this section, we prove the main result of this article, namely that a pseudo–Riemannian metric $g$ which is projectively compact of order $\alpha = 2$ admits an asymptotic form as in (6) locally around each boundary point. Together with Theorem 2.6 of [6], this provides an equivalent description of such metrics. Explicitly, we want to prove the following.

**Theorem 5.** Let $g$ be a pseudo–Riemannian metric on $M$, which is projectively compact of order $\alpha = 2$. Then there is a nowhere–vanishing locally constant function $C$ on a neighborhood of $\partial M$ such that for each local defining function $\rho$ for the boundary, the \(\binom{n}{2}\)-tensor–field

$$h = \rho g - C d\rho \odot d\rho$$

extends smoothly to the boundary and its boundary value is non–degenerate in directions tangent to the boundary.

In particular, part (ii) of Proposition 3 and Theorem 4 apply to $g$. 
After developing some tools, we will prove this theorem locally around points which satisfy an additional condition (see Theorem [11]). These points are shown to form a dense open subset in the boundary in Lemma [10]. Using this, an analysis of the projective metricity equation in Theorem [12] will lead to a proof that the additional condition is satisfied in all boundary points.

3.1. Geodetic transversals. The first technical step we need is a construction of adapted coordinates which should be of independent interest. This can partly be done in the general setting of affine connections which are projectively compact of arbitrary order.

So we suppose that $\nabla$ is a linear connection on $TM$, which is projectively compact of some order $\alpha$, and that $\rho$ is a local defining function for the boundary $\partial M$ defined on an open set $U \subset \overline{M}$. Then by definition the affine connection $\nabla + \frac{d\rho}{\rho}$ defined on $U \cap M$ extends smoothly to all of $U$, and we temporarily denote this connection by $\overset{\rho}{\nabla}$.

**Definition 6.** A geodetic transversal for $\rho$ is a smooth vector field $\mu \in \mathfrak{X}(U)$ such that $\overset{\rho}{\nabla}_u \mu = 0$ (i.e. the flow lines of $\mu$ are geodesics for $\overset{\rho}{\nabla}$) and such that $d\rho(\mu)$ is identically one on $U \cap \partial M$.

If $d\rho(\mu) \equiv 1$ on all of $U$, then $\mu$ is called a strict geodetic transversal for $\rho$.

**Lemma 7.** (i) Given $U$ and $\rho$ and a vector field $\mu_0$ along $U \cap \partial M$ such that $d\rho(\mu_0) = 1$ on $U \cap \partial M$, we can (possibly shrinking $U$) extend $\mu_0$ uniquely to a geodetic transversal for $\rho$.

(ii) If $\mu$ is any geodetic transversal for $\rho$ then for each point $x \in U \cap \partial M$ there is an open neighborhood $\tilde{V}$ of $x$ in $\overline{M}$, a positive $\epsilon \in \mathbb{R}$, and a diffeomorphism $\Phi: \tilde{V} \rightarrow [0, \epsilon) \times V$ where $V = \tilde{V} \cap \partial M$ such that $\Phi(y) = (0, y)$ for all $y \in V$ and such that denoting by $t$ the coordinate in $[0, \epsilon)$ we have $\Phi^*(\partial_t) = \mu$.

(iii) Suppose that $\alpha = 2$ and that we have given a geodetic transversal $\mu$ for $\rho$ on $U$. Then (possibly shrinking $U$) we can find a defining function $\hat{\rho}$ and a smooth function $f$ which is identically one along the boundary such that $\hat{\mu} := f \mu$ is a strict geodetic transversal for $\hat{\rho}$.

**Proof.** Extend $\mu_0$ to a local smooth frame for $TM|_{\partial M}$. Denoting by $p: \overline{P \overline{M}} \rightarrow \overline{M}$ the linear frame bundle of $\overline{M}$ and by $\theta \in \Omega^1(\overline{P \overline{M}}, \mathbb{R}^{n+1})$ its soldering form, the frame defines a smooth map $s: U \cap \partial M \rightarrow \overline{P \overline{M}}$ such that $p \circ s = \text{id}$.

Now $\overset{\rho}{\nabla}$ defines a principal connection on $\overline{P \overline{M}}$, so we can talk about horizontal vector fields on $\overline{P \overline{M}}$ and such a field is uniquely determined by its value under $\theta$. In particular, let $X \in \mathfrak{X}(\overline{P \overline{M}})$ be the horizontal vector field whose value under $\theta$ is always the first vector in the standard basis of $\mathbb{R}^{n+1}$. This means that for any frame $u \in \overline{P \overline{M}}$, $T_u p \cdot X(u)$ is the first element in the frame $u$, so in particular $T_{s(y)p} X(s(y)) = \mu_0(y)$ for all $y \in U \cap \partial M$.

Let us denote by $\text{Fl}^X_t$ the flow of the vector field $X$, and consider the map $(y, t) \mapsto p(\text{Fl}^X_t(s(y))))$, which is defined and smooth on an open neighborhood of $(U \cap \partial M) \times \{0\}$ in $(U \cap \partial M) \times [0, \infty)$. Evidently, its tangent map in $(y, 0)$ restricts to the identity on $T_y \partial M$ and maps $\partial_t$ to $\mu_0(y)$ so it is a linear isomorphism. Hence for
any \( y \in U \cap \partial M \), it restricts to a diffeomorphism on a set of the form \( V \times [0, \varepsilon) \) where \( V \subset \partial M \) is an open neighborhood of \( y \). Since the flow lines of \( X \) in \( \mathring{P}M \) project to geodesics in \( M \), we can define \( \mu \) as the image of \( \partial_t \) under this diffeomorphism to complete the proof of (i), and use the inverse of the diffeomorphism to complete the proof of (ii).

(iii) Starting from \( \rho \), any smooth local defining function \( \hat{\rho} \) for the boundary can be written as \( \hat{\rho} = e^h \rho \) for some smooth function \( h \). This implies that \( \hat{d}\hat{\rho} = \hat{\rho}dh + e^h d\rho \), and the one–form \( dh \) is smooth up to the boundary. This implies that \( \frac{d\hat{\rho}}{\hat{\rho}} = \frac{1}{2} dh + \frac{d\rho}{\rho} \) and thus \( \hat{\nabla} = \nabla + \frac{1}{2} dh \). Of course, the geodetic extension of \( \mu_0 \) with respect to \( \hat{\nabla} \) must be of the form \( f\hat{\mu} \) for some smooth function \( f \). This satisfies a differential equation, which we obtain via

\[
0 = \hat{\nabla}_{\hat{f}\hat{\mu}} = \nabla_{\hat{f}\mu} + f^2 dh(\mu)\mu.
\]

Expanding the covariant derivative using the fact that \( \nabla_{\hat{\mu}} = 0 \), we arrive at

\[
0 = f\hat{d}(\mu) + f^2 dh(\mu).
\]

We can initially assume that \( f|_{U \cap \partial M} = 1 \) which implies \( h|_{U \cap \partial M} = 0 \). Possibly shrinking \( U \), \( f \) will be nowhere vanishing, so we may divide by \( f^2 \) to arrive at the equation \( \frac{d\hat{\rho}}{\hat{\rho}} = dh(\mu) \). The left hand side is a logarithmic derivative, which shows that \( \log(f) + h \) is constant along flow lines of \( \mu \), and since it vanishes along \( U \cap \partial M \), it is identically zero. Thus we conclude that \( f = e^{-h} \), and hence it remains to solve the equation

\[
1 = f\hat{d}(\mu) = e^{-h}(e^h \rho dh(\mu) + e^h d\rho(\mu)),
\]

and thus \( 1 - d\rho(\mu) = \rho dh(\mu) \). But by assumption \( d\rho(\mu) \) is smooth and identically one along \( \partial M \), so the left hand side is smooth and vanishes along the boundary. Thus it can be written as \( \rho \tilde{f} \) for some smooth function \( \tilde{f} \) and of course \( dh(\mu) = \tilde{f} \) has a unique solution which satisfies the initial condition \( h|_{\partial M} = 0 \).

This has an immediate application to projectively compact metrics:

**Proposition 8.** Suppose that \( g \) is a pseudo–Riemannian metric on \( M \), which is projectively compact of order \( \alpha = 2 \). Suppose further that \( \rho \) is a local defining function for \( \partial M \), and that \( \mu \) is a geodetic transversal for \( \rho \).

Then the function \( \rho^2 g(\mu, \mu) \) is constant along flow lines of \( \mu \) and thus admits a smooth extension to the boundary.

**Proof.** Denoting by \( \nabla \) the (projectively compact) Levi–Civita connection of \( g \), we have \( \mu \cdot g(\mu, \mu) = 2g(\nabla_{\mu}\mu, \mu) \). On the other hand, we get \( 0 = \hat{\nabla}_{\mu}\mu = \nabla_{\mu}\mu + \hat{\rho} d\rho(\mu)\mu \), whence \( \nabla_{\mu}\mu = -\frac{\hat{\rho}}{\rho} d\rho(\mu)\mu \). Inserting this in the previous equation, we get \( \mu \cdot g(\mu, \mu) = -\frac{\hat{\rho}}{\rho} d\rho(\mu) g(\mu, \mu) \) which immediately implies that \( \mu \cdot (\rho^2 g(\mu, \mu)) = 0 \). □
3.2. A technical condition. We are now in a position to prove Theorem 5 locally around boundary points, which satisfy an additional technical condition. We will prove later on that this condition is always satisfied.

To formulate the condition, suppose that \( x \in \partial M \) is a boundary point and \( \mu_0 \in T_x M \setminus T_x \partial M \) is an inward pointing tangent vector. Then there is a well defined geodesic path emanating from \( x \) in direction \( \mu_0 \), and we choose a regular parameterization \( c : [0, \epsilon) \to \overline{M} \) of this path. In view of Proposition 8 we see that the fact that \( g(c(t))(c'(t), c'(t)) \neq 0 \) for sufficiently small \( t > 0 \) depends only on \( \mu_0 \) and not on the choice of parameterization.

Definition 9. We say that \( x \in \partial M \) admits a non-null transversal if \( \mu_0 \) can be chosen in such a way that, for a regular parameterization \( c \) of the geodesic path emanating from \( x_0 \) in direction \( \mu_0 \), we have \( g(c(t))(c'(t), c'(t)) \neq 0 \) for sufficiently small \( t > 0 \).

This condition is trivially satisfied in each point in the case that \( g \) is definite, in general we can prove right away that it is satisfied on an open dense subset.

Lemma 10. For any pseudo–Riemannian metric \( g \) on \( M \) which is projectively compact of order two, the set of all boundary points admitting a non–null transversal is open and dense in \( \partial M \).

Proof. If \( \mu_0 \) is is a non–null transversal in \( x_0 \in \partial M \), then we extend it to a neighborhood of \( x \), then construct a geodetic transversal \( \mu \) from this extension and consider the induced chart as in part (ii) of Lemma 7. By continuity of the expression for \( g \) in the chart, \( \mu \) gives rise to non–null transversals locally around \( x \).

To complete the proof, let us suppose that \( x_0 \in \partial M \) has an open neighborhood \( U \) such that no point in \( U \) admits a non–null transversal. Then we take some connection \( \tilde{\nabla} \) in the projective class which is smooth up to the boundary on a neighborhood of \( x_0 \). By [30], we can find a neighborhood \( V \) of \( x_0 \) which is geodesically convex for \( \tilde{\nabla} \), i.e. any two points in \( V \) can be joined by a unique geodesic and this geodesic is contained in \( V \). Now take a point \( x_1 \in M \cap V \) and the geodesic connecting it to \( x_0 \). Looking at the initial direction of this geodesic in the space of rays in \( T_{x_1} M \), it is evident that there is an open neighborhood consisting of directions corresponding to geodesics reaching boundary points contained in \( U \). But by assumption, none of these geodesics can correspond to a non–null transversal, whence each of these directions must be null for \( g(x_1) \), which is a contradiction. \( \square \)

Now we can deduce existence of the asymptotic form.

Theorem 11. The conclusion of Theorem 5 holds locally around each boundary point \( x_0 \) which admits a non–null transversal.

Proof. Take a tangent vector \( \mu_0(x_0) \) at \( x_0 \) as in the definition of admitting a non–null transversal, extend it to a vector field \( \mu_0 \) along \( \partial M \). Then choose a defining function for which this gives rise to a geodetic transversal. Next, apply Lemma 7 to obtain a neighborhood \( U \) of \( x_0 \) in \( \overline{M} \), a local defining function \( \rho \) for \( \partial M \) and a strict geodetic transversal \( \mu \) for \( \rho \) defined on \( U \). Using part (2) of Lemma 7 and
choosing a chart with coordinates \((x^1, \ldots, x^n)\) along the boundary, we can work in a chart associated to \(\mu\) from now on. The condition that \(\Phi^*\partial_t = \mu\) from Lemma 8 together with the fact that \(\mu\) is a strict transversal implies that \(\rho = t\) on the domain of our chart. Thus we work in the chart \((t, x^1, \ldots, x^n)\) with the coordinate \(t\) as our defining function and with \(\partial_t\) as the geodetic transversal.

By Proposition 8, the function \(t^2 g(\partial_t, \partial_t)\) is constant in \(t\), and by construction the values are non–zero around \(x_0\), so shrinking \(U\) if necessary, we conclude that

\[
g(\partial_t, \partial_t) = t^{-2} C(x)
\]

for some smooth nowhere vanishing function \(C: V \rightarrow \mathbb{R}\) (which will be shown to be constant later on). Moreover, from the proof of Proposition 8 we see that the Levi–Civita connection \(\nabla\) be constant later on). Moreover, from the proof of Proposition 8 we see that the Levi–Civita connection \(\nabla\) of \(g\) satisfies \(\nabla_{\partial_t} \partial_t = -\frac{1}{t} \partial_t\).

Now we can take the equation \(t^2 g(\partial_t, \partial_t) = C(x)\) and apply the vector field \(\partial_t = \frac{\partial}{\partial t}\) to it, to obtain

\[
2t^2 g(\nabla_{\partial_t} \partial_t, \partial_t) = \partial_t \cdot C.
\]

Dividing by two, using that \(\nabla_{\partial_t} \partial_t = \nabla_{\partial_t} \partial_t\), and bringing the derivative out, we arrive at

\[
\frac{1}{2} \partial_t \cdot C = t^2 \partial_t \cdot g(\partial_t, \partial_t) - t^2 g(\partial_t, \nabla_{\partial_t} \partial_t).
\]

Rewriting the last term (including the sign) as \(tg(\partial_t, \partial_t)\) we conclude that \((9)\) can be rewritten as

\[
\frac{1}{2} \partial_t \cdot C = t(\partial_t \cdot tg(\partial_t, \partial_t)).
\]

This can be interpreted as an ODE on \(tg(\partial_t, \partial_t)\) whose general solution (for \(t > 0\)) is

\[
tg(\partial_t, \partial_t) = b_t(x) + \log(t) \frac{1}{2}(\partial_t \cdot C)(x).
\]

Now returning to \((9)\), we conclude that

\[
0 = \partial_t \cdot (t^2 g(\nabla_{\partial_t} \partial_t, \partial_t)).
\]

Denoting by \(\tilde{\nabla} = \nabla + \frac{\partial t}{\partial t}\) the connection associated to the defining function \(t\), which by definition extends to the boundary, we obtain \(\nabla_{\partial_t} \partial_t = \nabla_{\partial_t} \partial_t - \frac{1}{t} \partial_t\). Denote by \(\tilde{\Gamma}\) the Christoffel symbols of \(\tilde{\nabla}\) (with index 0 indicating the \(t\)-coordinate and \(i, j, k\) indicating other coordinates). Since \(\tilde{\nabla}\) extends to the boundary, all these functions extend smoothly to the boundary and

\[
\tilde{\nabla}_{\partial_t} \partial_t = \tilde{\Gamma}^0_{i0} \partial_t + \sum_j \tilde{\Gamma}^j_{i0} \partial_j.
\]

Inserting all that into \((12)\) and using \((10)\) and \((8)\), we obtain

\[
0 = \partial_t \cdot \left( - \frac{1}{2} tg(\partial_t, \partial_t) + \tilde{\Gamma}^0_{i0} t^2 g(\partial_t, \partial_t) + \sum_j \tilde{\Gamma}^j_{i0} t^2 g(\partial_j, \partial_t) \right)
\]

\[
= \frac{1}{4t} \partial_t (\partial_t \cdot C) + (\partial_t \cdot \tilde{\Gamma}^0_{i0}) C + \sum_j \left( (\partial_t \cdot (t \tilde{\Gamma}^j_{i0})) tg(\partial_j, \partial_t) + \frac{1}{2} \tilde{\Gamma}^j_{i0} (\partial_j \cdot C) \right),
\]

which holds for all nonzero values of \(t\). Multiplying by \(t\), we obtain an equation for \(\frac{1}{t}(\partial_t \cdot C)(x)\) with the right hand side depending on \(t\). But going through the terms in the right hand side of the equation we see that they are obtained from
functions extending smoothly to the boundary by either a product with \( t \) or with \( t^2 g(\partial_i, \partial_j) \). While the former clearly are smooth up to the boundary with boundary value zero, \( \partial_i \cdot C = 0 \) shows that the latter extend continuously to the boundary with boundary value zero. Hence we conclude that \( \partial_i \cdot C = 0 \) for all \( i \), so \( C \) is indeed a non–zero constant.

Inserting this into (11), we conclude that \( tg(\partial_i, \partial_j) = b_i(x) \) for some smooth function \( b_i \) on \( V \), so again this admits a smooth extension to the boundary. Hitting this equation with the vector field \( \partial_j \) we obtain

\[
tg(\nabla_{\partial_j} \partial_i, \partial_j) + tg(\partial_i, \nabla_{\partial_j} \partial_j) = (\partial_j \cdot b_i)(x).
\]

In the first term in the left hand side, we can simply replace \( \nabla \) by \( \nabla \), while the second term can be rewritten as \( tg(\partial_i, \nabla_{\partial_j} \partial_j) - \frac{1}{2}tg(\partial_i, \partial_j) \). Then we can expand the covariant derivatives using Christoffel symbols. Multiplying by \( t \) and rearranging terms, we obtain the following expression for \( tg(\partial_i, \frac{1}{2} \partial_j - t \sum_k \hat{\Gamma}^k_{ij} \partial_k) \)

\[
\hat{\Gamma}^0_{ij} t^2 g(\partial_i, \partial_j) + \sum_k \hat{\Gamma}^k_{ij} t^2 g(\partial_k, \partial_j) + t^2 \hat{\Gamma}^0_{ij} g(\partial_i, \partial_j) - t(\partial_j \cdot b_i)
\]

This expression is smooth up to the boundary with boundary value \( C \hat{\Gamma}^0_{ij} \). On the other hand, the vector fields \( \frac{1}{2} \partial_j - t \sum_k \hat{\Gamma}^k_{ij} \partial_k \) form a linearly independent family in a neighborhood of the boundary which lie in \( \ker(\partial \rho) \) everywhere and span this subspace at the boundary, and thus locally around the boundary. Hence for vector fields \( \xi, \eta \) which are in the kernel of \( \partial \rho \) and smooth up to the boundary, we first conclude that \( tg(\partial_i, \eta) \) extends smoothly to the boundary and then that \( tg(\xi, \eta) \) has the same property. Finally it is also clear that the boundary value of \( tg(\partial_i, \eta) \) is given by \( 2C \hat{\Gamma}^0_{ij} \).

Now consider the symmetric \( (0,2) \)–tensor field \( h := tg - C \hat{\Gamma}^{ij} \). By construction, this vanishes on \( (\partial_i, \partial_j) \) and its values on \( (\partial_i, \partial_i) \) and \( (\partial_i, \partial_j) \) coincide with those of \( tg \) and hence are smooth up to the boundary. To complete the proof, it thus remains to show that the boundary value is non–degenerate on \( T\partial M \) i.e. that the boundary value of the matrix \( tg(\partial_i, \partial_j) \) has non–zero determinant. By Proposition 2.3 of [5], we know that \( g \) has volume asymptotics of order \( \frac{n+2}{2} \) (where \( \dim(M) = n + 1 \)), which by definition means that volume density of \( g \) can be written as \( t^{\frac{n+2}{2}} \nu \) for some nowhere vanishing density \( \nu \). Using the well–known formula for the volume density of a metric, this says that the determinant of the matrix of the metric in a local coordinate frame has the form \( t^{\frac{n+2}{2}} f \) for some function which is smooth up to the boundary and nowhere vanishing. Let us write this matrix as \( g_{\alpha \beta} \) with \( \alpha, \beta = 0, \ldots, n \). Then we know that \( g_{00} = f^2 \), \( g_{ii} = g_{i0} = \frac{b_i}{t^2} \) and \( g_{ij} = g(\partial_i, \partial_j) \) for \( i, j > 0 \). If we multiply the first row of this matrix by \( t^2 \) and all other rows by \( t \), we obtain a matrix for whose entries we have proved smoothness up to the boundary already. The determinant of this matrix is given by \( t^{n+2} \det(g_{\alpha \beta}) = f \), so this is nowhere vanishing. But expanding this determinant with respect to the first row we get

\[
C \det(tg(\partial_i, \partial_j)) + \sum_i t b_i \det(A_i),
\]
Cap, Gover

for matrices $A_i$ of functions whose entries are smooth up to the boundary. So only the first summand contributes to the (non–zero) boundary value, which shows that the boundary value of $\det(tg(\partial_i, \partial_j))$ is non–zero. □

3.3. Relation to tractor metrics. To complete the proof of Theorem 5, it remains to prove that each boundary point admits a non–null transversal. Intuitively, this sounds like a very plausible (or even trivial) statement, but it turns out to be fairly subtle. This can be seen from the example of the projective compactification of Minkowksi space which is discussed in detail in [6] as an example of a pseudo–Riemannian metric which is projectively compact of order one. In this case, the boundary is a sphere, which inherits a projective structure together with a reduction of projective holonomy to a Lorentz group. In particular, there is a smooth hypersurface in the boundary corresponding to limits of null lines. In this example, it is easy to see from the description of geodesics as projectivizations of planes that any geodesic in the interior approaching a boundary point in this hypersurface must be null. So in that case, non–null transversals are available exactly on the complement of this hypersurface. (And indeed existence of an asymptotic form in this case is proved only locally around boundary points in this complement in [6].)

The additional key ingredient is the fact that a pseudo–Riemannian metric in a projective class gives rise to a solution of the so–called metricity equation. This solution was used for metrics which are projectively compact of order 1 in [6] and we can take some facts and computations from there, but we also obtain new ingredients, which should allow for much broader applications. These heavily rely on interpretations of the metricity equation in terms of projective tractor bundles. Since a similar correspondence will play an important role in section 4 below, we briefly review the basic ingredients here.

Given a smooth manifold of dimension $n+1$ endowed with a projective structure, one can construct a vector bundle $\mathcal{T}^*$ of rank $n+2$, which contains the bundle $\mathcal{E}_a(1)$ of weighted one–forms as a smooth subbundle such that the quotient is isomorphic to $\mathcal{E}(1)$. This so–called standard cotractor bundle can be canonically endowed with a linear connection $\nabla^{\mathcal{T}^*}$ determined by the projective structure. Together, the configuration of bundle, subbundle and connection is uniquely determined up to isomorphism. One can then apply constructions with vector bundles and induced connections to obtain general tractor bundles, each of which is endowed with a canonical tractor connection. In particular, the standard tractor bundle $\mathcal{T}$ is the dual bundle to $\mathcal{T}^*$. We will mainly need the bundles $S^2\mathcal{T}$ and $S^2\mathcal{T}^*$ of symmetric bilinear forms on $\mathcal{T}$ respectively $\mathcal{T}^*$.

Writing the composition series for $\mathcal{T}^*$ from above as $\mathcal{T}^* = \mathcal{E}_a(1) \oplus \mathcal{E}(1)$, one can describe the induced composition series for the other tractor bundles mentioned above as

$$\mathcal{T} = \mathcal{E}(-1) \oplus \mathcal{E}^a(-1)$$

(14)

$$S^2\mathcal{T} = \mathcal{E}(-2) \oplus \mathcal{E}^a(-2) \oplus \mathcal{E}^{(ab)}(-2)$$

$$S^2\mathcal{T}^* = \mathcal{E}_{(ab)}(2) \oplus \mathcal{E}_a(2) \oplus \mathcal{E}(2).$$
A choice of connection in the projective class gives rise to an isomorphism $\mathcal{T}^* \cong \mathcal{E}_a(1) \oplus \mathcal{E}(1)$ and likewise for the other tractor bundles. Given such a choice, we write sections of a tractor bundle as column vectors with the component describing the canonical quotient of the tractor bundle on top and the component in the canonical subbundle in the bottom. Changing the connection projectively by a one–form $\Upsilon_a$, there are explicit formulae for the changes of these identifications. For the bundles $S^2\mathcal{T}$ and $S^2\mathcal{T}^*$ we follow the conventions from [6], and the corresponding formulae for these cases are given as equations (3.5) and (3.11) in that reference.

Via the so–called BGG–machinery (see [10], [3], and the sketch in [6]), each tractor bundle induces a natural differential operator acting on sections of its canonical quotient bundle, which defines an overdetermined system of PDEs (“first BGG–equation”) on that bundle. Closely related to this is the so–called splitting operator $L$, which maps sections of the quotient bundle to sections of the tractor bundle.

We will need two instances of this construction. On the one hand, suppose that in our usual setting $\overline{M} = M \cup \partial M$, one has given a special affine connection $\nabla$ on $M$ which is projectively compact of order two. Since $\nabla$ is special, each density bundle $\mathcal{E}(w)$ admits non–zero sections which are parallel for $\nabla$ and these are unique up to multiplication by a locally constant function. Denoting by $\tau$ such a section of $\mathcal{E}(2)$, it is proved in Proposition 2.3 of [6] that extending $\tau$ by zero to $\partial M$, one obtains a smooth section of $\mathcal{E}(2)$ on all of $\overline{M}$, which is a defining density for $\partial M$. Applying the splitting operator, we obtain a section $L(\tau)$ of $S^2\mathcal{T}^*$ and thus a (possibly degenerate) bundle metric on the standard tractor bundle, which will play a crucial role in section 4.

The second application is related to the metricity equation, see [14], which corresponds to the tractor bundle $S^2\mathcal{T}$. Take a projective manifold $N$ and a pseudo–Riemannian metric $g_{ab}$ on $N$ with inverse $g^{ab}$. Then $g$ canonically determines a volume density on $N$, and forming an appropriate power of this density one obtains a nowhere–vanishing section $\sigma \in \mathcal{E}(1)$ which is parallel for the Levi–Civita connection of $g$. It then turns out that the Levi–Civita connection of $g$ lies in the given projective class if and only if $\sigma^{-2}g^{ab}$ is a solution of the metricity equation. The crucial fact for what follows is that this can be characterized in terms of the image under the splitting operator. Indeed, in [14], the authors construct a modification of the tractor connection on $S^2\mathcal{T}$ such that $\sigma^{-2}g^{ab}$ solves the metricity equation if and only if $L(\sigma^{-2}g^{ab})$ is parallel for this modified connection. In fact, such connections can be constructed for any tractor bundle (associated to any parabolic geometry), see [22]. An explicit formula for the splitting operator $L$ is derived in Proposition 3.1 of [9] (unfortunately with a sign error in the printed version, that is easily corrected). Given a connection $\mathring{\nabla}$ in the projective class, the formula for $L(\sigma^{ab})$ in the splitting determined by $\mathring{\nabla}$ on a manifold of dimension
n + 1 is given by

\[
\left( \sigma^{ab} - \frac{1}{n+2} \nabla_d \sigma^{dc} - \frac{1}{n+2} \nabla_d \nabla_e \sigma^{de} + \frac{1}{n+1} P_{de} \sigma^{de} \right).
\]

3.4. Analysis of the metricity equation. Let us now return to the setting of a smooth manifold with boundary, \( M = M \cup \partial M \), and suppose that we have given a pseudo–Riemannian metric \( g \) on \( M \) which is projectively compact of order 2. Then there is a natural defining density \( \tau \in \Gamma(E(2)) \) for \( \partial M \), which is obtained by extending \( \text{vol}(g)^{-2/(n+2)} \) by 0 to the boundary. Hence we can write the solution of the metricity equation determined by \( g \) as \( \tau^{-1} g^{ab} \). Observe that \( \tau \) is nowhere vanishing on \( M \) and parallel for the Levi–Civita connection \( \nabla \) of \( g \).

**Theorem 12.** Let \( g^{ab} \) be a pseudo–Riemannian metric on \( M \) which is projectively compact of order two. Then we have

(i) The corresponding solution \( \tau^{-1} g^{ab} \) of the metricity equation admits a smooth extension to all of \( \overline{M} \).

(ii) The section \( L(\tau^{-1} g^{ab}) \) of \( S^2 T \) admits a smooth extension to all of \( \overline{M} \) and it defines a non–degenerate bundle metric on \( T^* \) on an open neighborhood of \( \partial M \).

(iii) Each point \( x \in \partial M \) admits a non–null transversal.

**Proof.** In [14], the authors construct a linear connection \( \overline{\nabla} \) on the tractor bundle \( S^2 T \), such that \( \sigma^{ab} \in \Gamma(E^{(ab)(-2)}) \) is a solution of the metricity equation if and only if \( L(\sigma^{ab}) \) is parallel for the connection \( \overline{\nabla} \). In particular, \( L(\tau^{-1} g^{ab}) \) defines a section of \( S^2 T \) over the interior \( M \subset \overline{M} \) which is parallel for \( \overline{\nabla} \), so we can extend it by parallel transport to a parallel section over all of \( \overline{M} \). Projecting this extension to the quotient bundle \( E^{(ab)(-2)} \), we obtain a smooth extension of \( \tau^{-1} g^{ab} \) to all of \( \overline{M} \).

To understand the properties of these extensions, we start working over \( M \). There we can use the Levi–Civita connection \( \nabla \) of \( g \), which by definition lies in (the restriction of the) projective class, to split the tractor bundle \( S^2 T \). Since both \( \tau \) and \( g^{ab} \) are parallel for \( \nabla \), formula (15) shows that, in this splitting,

\[
L(\tau^{-1} g^{ab}) = \begin{pmatrix}
\tau^{-1} g^{ab} \\
0 \\
\frac{1}{n+1} \tau^{-1} g^{cd} P_{cd}
\end{pmatrix}.
\]

Now the tractor connection \( \nabla^T \) on the standard cotractor bundle is volume preserving, so there is a section of \( \Lambda^{n+2} T^* \) which is parallel for the induced tractor connection. This means that there is a well defined (up to a non–zero constant multiple) determinant of the bundle metric \( L(\tau^{-1} g^{ab}) \), and this is a smooth function on \( \overline{M} \). From the formula in slots above, it is clear that over \( M \) we have

\[
\det(L(\tau^{-1} g^{ab})) = \tau^{-n-2} \det(g^{ab}) \frac{1}{n+1} g^{cd} P_{cd}.
\]

But since \( \det(g^{ab}) = \text{vol}(g)^{-2} \), we see that \( \tau^{-n-2} \det(g^{ab}) = 1 \), so that, over \( M \), \( \det(L(\tau^{-1} g^{ab})) = \frac{1}{n+1} g^{cd} P_{cd} \). Then Theorems [11] and [3] imply that, locally around any point \( x_0 \in \partial M \) which admits a non–null transversal, the boundary value of
Corollary 13. Let $\overline{M} = M \cup \partial M$ be a manifold with boundary and let $g$ be a pseudo–Riemannian metric on $M$ which is projectively compact of order $\alpha = 2$. Then the scalar curvature $R$ of $g$ admits a smooth extension to the boundary and the boundary value is a nowhere vanishing locally constant function $\hat{C}$ on $\partial M$. Moreover, locally around each boundary point there are local coordinates $(t, x^1, \ldots, x^n)$ for $M$ such that $\partial M$ is given by $t = 0$ and such that for all $t \neq 0$ we have
\[ g = h(t, x^1, \ldots, x^n) \frac{dt^2}{t} - \frac{n(n + 1)}{4C} \frac{dt^2}{t^2}. \]
Here $\hat{C}$ is extended to the interior as a locally constant function and $h$ is a \( \binom{n}{2} \)-tensor field, which is smooth up to the boundary with the boundary value being non–degenerate on $T\partial M$. 

\det (L(\tau^{-1} g^{ab}))$ is a non–zero constant. By Lemma 10 these points form a dense open subset in the boundary, and the boundary value is a locally constant function with non–zero values on this subset. But this implies that $\det (L(\tau^{-1} g^{ab}))|_{\partial M}$ is locally constant with non–zero values everywhere (and in particular the constants have to match up within connected components of $\partial M$). Hence $L(\tau^{-1} g^{ab})$ is a non–degenerate bundle metric on an open neighborhood of $\partial M$.

To complete the proof, we have to compute the expression for $L(\tau^{-1} g^{ab})$ in a scale that is smooth up to the boundary. Consider a boundary point $x \in \partial M$ and a defining function $\rho$ for $\partial M$ defined on a neighborhood of $x$, and let $\nabla = \nabla + \frac{d \rho}{\rho}$ be the corresponding connection which extends to the boundary. In Proposition 2.3 of [6] it is shown that then $\hat{\tau} = \frac{2}{\rho}$ is parallel for $\nabla$ and hence smooth and nowhere vanishing on the domain of definition of $\rho$. Moreover, from formula (3.11) in [6], we see that in the splitting determined by $\nabla$ and over $M$, we get
\[
L(\tau^{-1} g^{ab}) = \begin{pmatrix}
\tau^{-1} g^{ab} \\
-\tau^{-1} g^{ac} \Upsilon_c \\
\frac{1}{n+1} \tau^{-1} g^{cd} P_{cd} + \tau^{-1} g^{cd} \Upsilon_c \Upsilon_d
\end{pmatrix}.
\]
Here $\Upsilon$ is the one–form describing the projective change from $\nabla$ to $\hat{\nabla}$, i.e. $\Upsilon_a = \frac{d \rho}{\rho}$. Inserting this, and using $\tau^{-1} = \rho^{-1} \hat{\tau}^{-1}$, we conclude form the first two lines that $\rho^{-1} g^{ab}$ and $\rho^{-2} g^{ac} \rho_c$ admit smooth extensions to the boundary. Denoting by $\lambda^{ab}$ the boundary value of $\rho^{-1} g^{ab}$, see that $\lambda^{ac} \rho_c \neq 0$ would contradict smoothness of $\rho^{-2} g^{ac} \rho_c$ up to the boundary. Hence the line spanned by $\rho_a$ lies in the null–space of the boundary value $\lambda^{ab}$, so this bilinear form has rank at most $n$. But the fact that $L(\tau^{-1} g^{ab})$ extends to a non–degenerate bilinear form on the boundary implies that the rank of $\lambda^{ab}$ must be at least $n$, and if it equals $n$, then the boundary value of $\psi^a = \rho^{-2} g^{ac} \rho_c$ must be injective on the nullspace of $\lambda^{ab}$. This implies that $\psi^c \rho_c \neq 0$ along the boundary, so $\psi^c(x)$ is a transversal at each $x$. But then $\rho^2 g^{cd} \psi^c \psi^d = \psi^c \rho_c$ implies that this transversal is not null. 

Together with Theorem 11, this completes the proof of Theorem 3. Moreover, together with Theorem 4, the proof of Theorem 11 gives us the following result in local coordinates.
Conversely, a pseudo–Riemannian metric admitting such a coordinate form with locally constant $C$ is projectively compact of order two.

4. Boundary Tractors

For the last part of this article, we assume that we have given a special affine connection on $M$ which is projectively compact of order two and has the property that the projective second fundamental form is non–degenerate (in directions tangent to the boundary) at each boundary point. (Observe that by Theorem 5 and Proposition 3, this condition is always satisfied in the case of a pseudo–Riemannian metric which is projectively compact of order two.) Then we obtain a well defined conformal structure on the boundary $\partial M$, which can be conceptually described by the associated tractor bundle and tractor connection. In this section we give a description of these boundary tractors in terms of the projective structure in the interior. We derive formulae for the ingredients used in this description both in terms of asymptotics of data associated to the projectively compact connection in the interior and in terms of data which are manifestly smooth up to the boundary. In contrast to the usual presentation of conformal tractors, our description is entirely based on connections from the projective class, we do not choose a connection on the boundary which is compatible with the conformal structure.

4.1. The tractor bundle and its metric. In spite of the rather complicated relation between a projectively compact connection on $M$ and the induced conformal structure on $\partial M$, one can relate the tractor bundles associated to these structures fairly easily. As we have observed in 3.3, a special affine connection $\nabla$ on $M$ gives rise to a defining density $\tau \in \Gamma(\mathcal{E}(2))$ for $\partial M$ which is unique up to a non–zero constant factor. The main property of $\tau$ is that, over $M$, it is parallel for $\nabla$. Via the BGG splitting operator, we obtain a section $L(\tau)$ of the tractor bundle $S^2\mathcal{T}^* \otimes \overline{M}$.

The motivation for the developments in this section comes from the special case of Levi–Civita connections of non–Ricci–flat Einstein metrics. In this case, the section $L(\tau)$ of $S^2\mathcal{T}^*$ is parallel for the tractor connection, thus defining a reduction of projective holonomy to a pseudo–orthogonal group. Via the general theory of holonomy reductions developed in [8], one obtains an induced conformal structure on the boundary, which by Proposition 3 coincides with the one discussed in this article. The general theory further implies that one can obtain the conformal standard tractor bundle restricting the projective standard tractor bundle to the boundary, endowing it with the bundle metric $L(\tau)$ and that the restriction of the projective standard tractor connection to this bundle is the conformal standard tractor connection, see Sections 3.1 and 3.2 of [8].

Surprisingly, the first part of this works in complete generality.

Proposition 14. Let $\overline{M} = M \cup \partial M$ be a smooth manifold with boundary and suppose that $\nabla$ is a linear connection on $TM$ which is projectively compact of order two and such that the projective second fundamental form on $\partial M$ is non–degenerate.
Then endowing the restriction \( \mathcal{T}|_{\partial M} \) of the projective standard tractor bundle with the line subbundle \( \mathcal{T}^1|_{\partial M} \) and the bundle metric \( L(\tau)|_{\partial M} \), one obtains a standard tractor bundle for the induced conformal structure on \( \partial M \).

Explicitly, this means that \( \mathcal{T}^1|_{\partial M} \) is isomorphic to the conformal density bundle \( \mathcal{E}[-1] \) and isotropic for \( L(\tau)|_{\partial M} \), the quotient \( (\mathcal{T}^1)^+ / \mathcal{T}^1 \) is isomorphic to \( T\partial M \otimes \mathcal{E}[-1] \) and the metric on this quotient induced by \( L(\tau) \) coincides with the conformal metric defined by the projective second fundamental form.

**Proof.** In Section 3.3 of [6] it is shown that in the splitting \( S^2T^* \cong \mathcal{E}(2) \oplus \mathcal{E}_a(2) \oplus \mathcal{E}_{ab}(2) \) determined by \( \nabla \) (which is only defined over \( M \)) we have

\[
L(\tau) = \begin{pmatrix}
\tau \\
0 \\
P_{ab}\tau
\end{pmatrix},
\]

where we use that the Schouten tensor of a special affine connection is symmetric. Now we can easily analyze the boundary behavior \( L(\tau) \). Consider a local defining function \( \rho \) for \( \partial M \) and let \( \tilde{\nabla} = \nabla + \frac{d\rho}{4\rho} \) be the corresponding projectively rescaled connection which admits a smooth extension to the boundary. Then this is again a special affine connection and denoting by \( \hat{\tau} \in \Gamma(\mathcal{E}(2)) \) the corresponding parallel density (which is smooth up to the boundary and nowhere vanishing), we get \( \tau = \rho\hat{\tau} \). Using the formulae from Section 3.1 of [6], we see that in the splitting corresponding to \( \tilde{\nabla} \) (which is defined up to the boundary), we get

\[
L(\tau) = \begin{pmatrix}
\frac{\rho\hat{\tau}}{2} \\
\frac{\rho a\hat{\tau}}{P_{ab}\rho} + \frac{\rho a\rho b}{4\rho} \hat{\tau}
\end{pmatrix}.
\]

Along the boundary, the top slot vanishes, while the middle slot is evidently nowhere vanishing with pointwise kernel isomorphic to \( T\partial M \subset T\mathcal{M}|_{\partial M} \). Finally, by formula (2) from the proof of Proposition 1, the boundary value of the bottom slot is \( \frac{1}{2}\hat{\tau} \tilde{\nabla}_a\rho b \), so the restriction of this bilinear form to boundary directions is non-degenerate by the assumptions.

Together, this shows that \( L(\tau)|_{\partial M} \) defines a non-degenerate bundle metric on the restriction \( \mathcal{T}|_{\partial M} \) and that \( \mathcal{T}^1 \subset \mathcal{T} \) is isotropic for this bundle metric along the boundary. Moreover, the form of the middle slot of \( L(\tau) \) in (17) implies that the quotient \( (\mathcal{T}^1)^+ / \mathcal{T}^1 \) can be identified with \( T\partial M(-1) \subset T\mathcal{M}(-1)|_{\partial M} \).

Finally, recall that there is the canonical conormal bundle \( \mathcal{N} \subset T^*\mathcal{M}|_{\partial M} \), which is defined as the annihilator of \( T\partial M \). Now for the top exterior powers, we get \( (\Lambda^nT^*\mathcal{M})|_{\partial M} \cong \mathcal{N} \otimes (\Lambda^nT^*\partial M) \). In terms of the usual conventions for projective and conformal density bundles (see [2]) this reads as \( \mathcal{E}(-n - 2)|_{\partial M} \cong \mathcal{N} \otimes \mathcal{E}[-n] \). Now since the top slot of \( L(\tau) \) vanishes along \( \partial M \), its middle slot \( \hat{\tau} \rho a \) is actually independent of all choices, thus defining a nowhere vanishing section of \( \mathcal{N}\mathcal{E}(2) \cong \mathcal{E}(-n)|_{\partial M} \otimes \mathcal{E}[n] \). In particular, this induces a canonical isomorphism \( \mathcal{E}(n)|_{\partial M} \cong \mathcal{E}[n] \) and hence also an identification \( \mathcal{E}(-1)|_{\partial M} \cong \mathcal{E}[-1] \).

This shows that we obtain the claimed composition series for \( \mathcal{T}|_{\partial M} \). Since the bundle metric on \( (\mathcal{T}^1)^+ / \mathcal{T}^1 \) induced by \( L(\tau) \) clearly comes from the restriction of
\[ \frac{1}{2} \tau \nabla_a \rho_b \] to tangential directions, we also get the correct conformal metric on the quotient. \[ \Box \]

4.2. The asymptotically parallel case. Without further assumptions, one can certainly not follow the developments in the Einstein case discussed in 4.1 directly, since the projective standard tractor connection is not compatible with the bundle metric \( L(\tau) \). Indeed, the covariant derivative of \( L(\tau) \) with respect to the canonical tractor connection on \( S^2 T^* \) can be computed explicitly, see Section 3.3 of \([6]\). There it is shown that, in the splitting on \( M \) determined by the projectively compact connection \( \nabla a \), this derivative is given by putting \( \tau \nabla_a P_{bc} \) into the bottom slot of the tractor, while the other two slots are identical zero. Since the bottom slot is the injecting slot, it has the same form in any other splitting, so in particular, this section has to admit a smooth extension to the boundary. We next give a direct proof for the fact that \( \tau \nabla_a P_{bc} \) admits a smooth extension and derive a formulae for this tensor in terms of objects which are manifestly smooth up to the boundary as well as an alternative description, which is valid for Levi–Civita connections.

**Proposition 15.** Let \( \nabla \) be a special affine connection on \( M \), which is projectively compact of order 2 and induces a non–degenerate boundary geometry on \( \partial M \) and let \( P_{ab} \) be its Schouten tensor. Let \( \rho \) be a local defining function for the boundary and let \( \hat{\nabla} = \nabla + 2 \rho \frac{\partial}{\partial \rho} \) be the corresponding connection in the projective class. Then we have

(i) \( \rho \nabla_a P_{bc} = \frac{1}{2} \tau \nabla_a \nabla_b \rho_c + \rho_a \hat{P}_{bc} + \frac{1}{2} \rho_b \hat{P}_{ac} + \frac{1}{2} \rho_c \hat{P}_{ab} + \rho \nabla_a \hat{P}_{bc} \), and the right hand side provides a smooth extension of the left hand side to the boundary.

(ii) If \( \nabla \) is the Levi–Civita connection of a pseudo–Riemannian metric \( g_{ab} \), then for \( \Phi_{ab} := P_{ab} + \frac{1}{4C} g_{ab} \), where \( C \) is the constant occurring in the asymptotic form of \( g \), we get

\[ \rho \nabla_a P_{bc} = \rho_a \Phi_{bc} + \frac{1}{2} \rho_b \Phi_{ac} + \frac{1}{2} \rho_c \Phi_{ba} + \rho \hat{\nabla}_a \Phi_{bc}. \]

All terms in the right hand side admit smooth extensions to the boundary and the last summand does not contribute to the boundary value.

**Proof.** (i) For \( \alpha = 2 \), equation (2) from the proof of Proposition 1 reads as

\[ (18) \quad \rho P_{bc} + \frac{1}{4p} \rho_b \rho_c = \frac{1}{2} \nabla_b \rho_c + \rho \hat{P}_{bc}. \]

Applying \( \hat{\nabla}_a \) to this equation, the second term on the left hand side gives

\[ (19) \quad \frac{1}{4p} \rho_a \rho_b \rho_c + \frac{1}{4p} \rho_b \nabla_a \rho_c + \frac{1}{4p} \rho_c \nabla_a \rho_b. \]

Now we can collect half of the first summand in this expression with the second summand to obtain

\[ \frac{1}{4p} \rho_b (\nabla_a \rho_c - \frac{1}{4p} \rho_a \rho_c). \]

Now from (18) we see that we can replace the bracket by \( 2 \rho (P_{ac} - \hat{P}_{ac}) \) and thus obtain

\[ \frac{1}{2} \rho_b P_{ac} - \frac{1}{2} \rho_b \hat{P}_{ac}. \]

Likewise the second half of the first term in (19) adds up with the last term in this formula to the same term with \( b \) and \( c \) exchanged.
To compute $\hat{\nabla}_a$ of the first term in the left hand side of (18) we use the standard formulae for the action of projectively related connections on tensor fields to obtain

$$\hat{\nabla}_a P_{bc} = \nabla_a P_{bc} - 2\Upsilon_a P_{bc} - \Upsilon_b P_{ac} - \Upsilon_c P_{ba}.$$  

Here $\Upsilon$ describes the change from $\nabla$ to $\hat{\nabla}$, i.e. $\Upsilon_a = \frac{\dot{\varrho}}{2\varrho}$. We have to multiply all that by $\varrho$ and add $\varrho P_{bc}$ to obtain the contribution of the first term on the left hand side. Hence we conclude that applying $\hat{\nabla}_a$ to the left hand side of (18) we obtain

$$\varrho \nabla_a P_{bc} - \frac{1}{2} \varrho P_{ac} - \frac{1}{2} \varrho P_{ab}.$$  

Applying $\hat{\nabla}_a$ to the right hand side of (18) directly leads to the claimed formula.

(ii) Observe first that $\Phi_{ab}$ admits a smooth extension to the boundary by the proof of Theorem 4, so the last statement is evident. Moreover on $M$, we obtain

$$\hat{\nabla}_a \Phi_{bc} = \nabla_a \Phi_{bc} - 2\Upsilon_a \Phi_{bc} - \Upsilon_b \Phi_{ac} - \Upsilon_c \Phi_{ba},$$

as in the proof of part (i) with $\Upsilon_a = \frac{\dot{\varrho}}{2\varrho}$. From this the claimed formula follows immediately by multiplying by $\varrho$ and rearranging terms. \hfill $\Box$

As mentioned in 4.1, in the case of the Levi–Civita connection of an Einstein metric, the bundle metric $L(\tau)$ is parallel over all of $\overline{M}$, and one obtains the conformal standard tractor connection on the boundary as a restriction of the projective standard tractor connection. The argument which was used to prove this in Proposition 3.2 of \cite{8} actually can be applied in a significantly more general situation as we will show next.

Surprisingly, it suffices to assume that $\nabla^{S^2T^*} L(\tau)$ vanishes along the boundary (although this is not enough to ensure compatibility of the tractor curvature with $L(\tau)$ along the boundary). In view of Proposition 15, $\nabla^{S^2T^*} L(\tau)$ vanishes on $\partial M$ if and only if $\nabla_a P_{bc}$ admits a smooth extension to all of $\overline{M}$. Moreover, for a pseudo–Riemannian metric $g_{ab}$ which is projectively compact of order two, vanishing of $\nabla^{S^2T^*} L(\tau)$ along $\partial M$ follows from the fact that the boundary value of $R_{ab} + \frac{n}{4c} g_{ab}$ vanishes identically. The last condition is a (by one order) stronger asymptotic form of the Einstein equation than the one that $g_{ab}$ satisfies by Theorem 4.

**Theorem 16.** Let $\overline{M} = M \cup \partial M$ be a smooth manifold of dimension $n+1 \geq 4$ with boundary and suppose that $\nabla$ is a linear connection on $TM$ which is projectively compact of order two and such that the projective second fundamental form on $\partial M$ is non–degenerate. Assume further that the canonical defining density $\tau \in \Gamma(E(2))$ for $\partial M$ determined by $\nabla$ has the property that $\nabla^{S^2T^*} L(\tau)|_{\partial M} = 0$.

Then one can restrict the projective standard tractor connection to the conformal standard tractor bundle on $\partial M$ constructed in Proposition 14 and the result is the canonical normal conformal tractor connection.

**Proof.** It is no problem to restrict the tractor connection on $\mathcal{T} \to \overline{M}$ to a linear connection on $\mathcal{T}|_{\partial M} \to \partial M$. Since we have assumed that $\nabla^{S^2T^*} L(\tau)|_{\partial M} = 0$, this produces a tractor connection, which is compatible with the bundle metric $L(\tau)|_{\partial M}$. To complete the proof, it remains to verify that the curvature of this
tractor connection satisfies the normalization condition imposed on a conformal standard tractor connection.

This normalization condition is best described in two steps. The first requirement on the curvature is that it maps the distinguished subbundle $T^1|_{\partial M}$ to itself. Skew symmetry of the curvature then implies that it also preserves the orthocomplement of this subbundle, so there is an induced endomorphism on the quotient space, which is isomorphic to $T\partial M \otimes E(-1)$. Since the on endomorphisms, including the projective weight causes no difference, one can view the result as a section of $\Lambda^2 T^* \partial M \otimes \text{End}(T\partial M)$, and the second part of the normalization condition is that the Ricci–type contraction of this tensor field vanishes.

Now the curvature of the restricted connection is just the restriction of the curvature of the projective standard tractor connection. This means that one only inserts vectors tangent to the boundary into the two–form part of the curvature, but the endomorphism part still acts on the full bundle. It is well known (see [2]) that the curvature of the projective standard tractor connection satisfies similar normalization conditions. In particular, this curvature vanishes identically on the distinguished subbundle $T^1$. Similarly as above, this implies that the values of the curvature descend to endomorphisms of the quotient $T/T^1 \cong TM(-1)$. So one obtains a section of $\Lambda^2 T^* M \otimes \text{End}(TM)$ and the Ricci–type contraction of this vanishes (and the tensor itself coincides with the projective Weyl curvature of any connection in the projective class).

Now the fact that the subbundle $T^1$ is annihilated of course carries over to the restriction, so the first part of the conformal normalization condition is satisfied. Now suppose that we can further show that values of the endomorphisms obtained from the projective Weyl curvature along the boundary always lie in $T\partial M \subset T\overline{M}|_{\partial M}$. Then using a basis of $T_x\overline{M}$ consisting of a basis of $T_x\partial M$ for $x \in \partial M$ and one transversal vector, one immediately concludes that the Ricci type contraction of the projective Weyl curvature coincides with the Ricci–type contraction over the subspaces $T\partial M$, so that latter vanishes. Hence we can complete the proof by verifying this property of the projective Weyl curvature. This can be done by taking a locally non–vanishing section $\sigma$ of $T^1$ and proving that, denoting by $\kappa$ the curvature of the projective tractor connection $\nabla^T$, we get

$$L(\tau)(\kappa(\xi, \eta)(t), \sigma)|_{\partial M} = 0$$

for all $\xi, \eta \in \mathfrak{X}(M)$ and any section $t \in \Gamma(T)$. (We could actually assume in addition that $\xi$ is tangent to $\partial M$ and that $L(\tau)(t, \sigma) = 0$, but these assumptions are not needed.) Note that this would follow immediately under the assumption that the one–jet of $\nabla^{s2T^*} L(\tau)$ vanishes along $\partial M$, since this implies skew symmetry of $\kappa(\xi, \eta)$ with respect to $L(\tau)$ along the boundary.

Under the weaker assumptions we have made, we have to supply a direct argument which uses the additional information on $\nabla^{s2T^*} L(\tau)$ we have available. We start with the defining equation

$$(\nabla^{s2T^*}_\xi L(\tau))(t_1, t_2) = \xi \cdot (L(\tau)(t_1, t_2)) - L(\tau)(\nabla^T_\xi t_1, t_2) - L(\tau)(t_1, \nabla^T_\xi t_2)$$
for \( t_1, t_2 \in \Gamma(\mathcal{T}) \). Using this, one directly computes that

\[
L(\tau)(\nabla^T_\xi \nabla^T_{\eta} t_1, t_2) - L(\tau)(t_1, \nabla^T_{\eta} \nabla^T_\xi t_2)
= \xi \cdot (L(\tau)(\nabla^T_{\eta} t_1, t_2)) - (\nabla^s_{\xi} T^* L(\tau))(\nabla^T_{\eta} t_1, t_2)
- \eta \cdot (L(\tau)(t_1, \nabla^T_\xi t_2)) + (\nabla^s_{\eta} T^* L(\tau))(t_1, \nabla^T_\xi t_2).
\]

Observe that the terms involving a covariant derivative of \( L(\tau) \) by assumption vanish along the boundary, so we can drop them for the further considerations. Subtracting the same term with \( \xi \) and \( \eta \) exchanged we obtain

\[
\xi \cdot (L(\tau)(\nabla^T_{\eta} t_1, t_2)) + L(\tau)(t_1, \nabla^T_{\eta} t_2))
= \xi \cdot \eta \cdot (L(\tau)(t_1, t_2)) - \xi \cdot \left( \nabla^s_{\eta} T^* L(\tau)(t_1, t_2) \right),
\]

minus the same expression with \( \xi \) and \( \eta \) exchanged. Now the second term in the right hand side does not vanish along the boundary in general. However, we only have to consider this in the case that \( t_2 = \sigma \in \Gamma(T^1) \). But the fact that \( \nabla^s_{\eta} T^* L(\tau) \) is concentrated in the bottom slot (over all of \( \overline{M} \)) which we have noted in the beginning of Section 4.2 exactly means that any covariant derivative of \( L(\tau) \) vanishes identically provided that one of its entries is form the subbundle \( T^1 \). So the only potential contribution to the boundary value coming from these two terms is

\[
(20) \quad \xi \cdot \eta \cdot (L(\tau)(t_1, t_2)) - \eta \cdot \xi \cdot (L(\tau)(t_1, t_2)).
\]

To arrive at

\[
L(\tau)(\kappa(\xi, \eta)(t_1), t_2) + L(\tau)(t_1, \kappa(\xi, \eta)(t_2)),
\]

we further have to subtract

\[
L(\tau)(\nabla^T_{[\xi, \eta]} t_1, t_2) + L(\tau)(t_1, \nabla^T_{[\xi, \eta]} t_2)
= [\xi, \eta] \cdot (L(\tau)(t_1, t_2)) - (\nabla^s_{[\xi, \eta]} T^* L(\tau))(t_1, t_2).
\]

Now the first term on the right hand side cancels with (20), while the second one vanishes along the boundary by assumption. Now the claim follows since \( \kappa(\xi, \eta) \) vanishes on the subbundle \( T^1 \). \( \square \)

4.3. The inverse of the tractor metric. Before we can proceed towards the description of the normal tractor connection on the boundary in the case that \( L(\tau) \) is not parallel along the boundary, we have to derive some further properties of the Schouten–tensor \( P_{ab} \) of \( \nabla \). In Proposition 14 we have seen that non–degeneracy of the boundary geometry implies that the bundle metric \( L(\tau) \) is non–degenerate on \( \partial M \). By continuity, it is non–degenerate on some open neighborhood of the boundary and we will from now on restrict to this neighborhood, i.e. assume that \( L(\tau) \) is non–degenerate on all of \( \overline{M} \). On \( M \) we can return to the scale determined by \( \tau \), and there, in view of (10), non–degeneracy of \( L(\tau) \) is equivalent to non–degeneracy of the Schouten–tensor \( P_{ab} \). This means that we can use \( P_{ab} \) as a Riemannian metric on \( M \), but of course, the Levi–Civita connection of this metric is not in the projective class in general.
By non-degeneracy, we can also form the inverse $P^{ab}$ of $P_{ab}$ as a bilinear form. We can derive asymptotic properties of $P^{ab}$ using the inverse $L(\tau)^{-1}$ of the tractor metric, which is a smooth section of $S^2T$ over all of $\overline{M}$. This could be done similarly to the proof of Theorem 12 but at this point, there is a more efficient way to proceed.

**Proposition 17.** Let $\nabla$ be a special affine connection on $M$, which is projectively compact of order 2 and induces a non-degenerate boundary geometry on $\partial M$. Let $\rho$ be a local defining function for the boundary and let $\nabla = \nabla + \frac{\partial \rho}{\partial t}$ be the corresponding connection in the projective class. Then in the splitting of $S^2T$ defined by the connection $\hat{\nabla}$, the inverse $L(\tau)^{-1}$ of the tractor metric is given by

$$L(\tau)^{-1} = \begin{pmatrix} \hat{\tau}^{-1} P^{ab} \\ 2\hat{\tau}^{-1}t^b \\ \hat{\tau}^{-1}\psi \end{pmatrix}$$

where $\tau = \rho \hat{\tau}$, $t^a = -\frac{1}{4\rho^2}P^{ab}\rho_b$, and $\psi$ is a function which is smooth up to the boundary. Moreover, we obtain

$$t^a\rho_a = 1 - \rho \psi \\ t^a(\rho P_{ab} + \frac{1}{4\rho^2}\rho_a\rho_b) = -\frac{1}{4}\psi \rho_b$$

$$\rho^{-1}P^{ac}(\rho P_{cb} + \frac{1}{4\rho^2}\rho_c\rho_b) + t^a\rho_b = \delta^a_b$$

In particular, the tensor fields $\rho^{-1}P^{ab}$ and $\rho^{-2}P^{ab}\rho_b$ on $M$ admit smooth extensions to all of $\overline{M}$.

**Proof.** Over $M$, we can easily compute $L(\tau)^{-1}$ in the splitting determined by $\nabla$. There $L(\tau)$ has the simple diagonal form (16), so it is clear that, in this splitting, we have

$$L(\tau)^{-1} = \begin{pmatrix} \tau^{-1}P^{ab} \\ 0 \\ \tau^{-1} \end{pmatrix}.$$ 

The top slot of this is independent of the choice of splitting, so we see that we can use (21) to define $t^a$ and $\psi$. But then we can use formula (17) for $L(\tau)$ in the splitting determined by $\nabla$ to compute the consequences of $L(\tau)$ and $L(\tau)^{-1}$ being inverses of each other. This is most easily done by interpreting $L(\tau)$ as a map $T \rightarrow T^*$ and $L(\tau)^{-1}$ as a map $T^* \rightarrow T$. By (17) in the splitting determined by $\hat{\nabla}$, we have

$$L(\tau) \begin{pmatrix} \nu^a_1 \\ \sigma_1 \end{pmatrix}, \begin{pmatrix} \nu^b_2 \\ \sigma_2 \end{pmatrix} = \hat{\tau} \left( \rho \sigma_1 \sigma_2 + \frac{1}{2} \sigma_1 \rho_a \nu^a_2 + \frac{1}{2} \sigma_2 \rho_a \nu^a_1 + (\rho P_{ab} + \frac{1}{4\rho^2}\rho_a\rho_b)\nu^a_1\nu^b_2 \right).$$

Hence the associated map is given by

$$\begin{pmatrix} \nu^a \\ \sigma \end{pmatrix} \mapsto \begin{pmatrix} \hat{\tau}\rho \sigma + \frac{1}{2}\rho_a\nu^a \\ \hat{\tau}(\frac{\rho}{\tau}\sigma \rho_a + (\rho P_{ab} + \frac{1}{4\rho^2}\rho_a\rho_b)\nu^a_b) \end{pmatrix}.$$ 

In the same way, one verifies that (21) corresponds to the map $T^* \rightarrow T$ given by

$$\begin{pmatrix} \beta \\ \mu_a \end{pmatrix} \mapsto \begin{pmatrix} \hat{\tau}^{-1}(\rho^{-1}P^{ab}\mu_b + 2\beta t^a) \\ \hat{\tau}^{-1}(2t^a\mu_a + \psi \beta) \end{pmatrix}.$$
The fact that the composition of this with the above is the identity immediately leads to the claimed formula for \( t^a \) as well as to \( (22) \). The last claim then follows since the slots in \( (21) \) must admit smooth extensions to the boundary. □

4.4. The metric tractor connection. Now we can proceed towards a description of the normal tractor connection on the conformal standard tractor bundle obtained in Proposition 17 in the general case. We will do this in two steps, the first of which can be done on all of \( \mathcal{M} \) (assuming that \( L(\tau) \) is non–degenerate on all of \( \mathcal{M} \)). In this first step, we modify the projective standard tractor connection on \( T \) to a connection which is compatible with the bundle metric \( L(\tau) \) and torsion free (in the sense of tractor connections). In the second step, we have to restrict to the boundary, where we can then normalize this metric tractor connection to obtain the conformal standard tractor connection.

A modification of the standard tractor connection \( \nabla^T \) is determined by a con-torsion, which is an element of \( \Omega^1(\mathcal{M}, \text{End}(T)) \). Choosing a connection in the projective class, one obtains an isomorphism \( T \cong \mathcal{E}(-1) \oplus \mathcal{E}^a(-1) \) and correspondingly we get an isomorphism \( \text{End}(T) \cong \mathcal{E}_b \oplus (\mathcal{E}_b^a \oplus \mathcal{E}^a) \). We write this in a matrix form, with the action given by

\[
\begin{pmatrix}
A^a_b & \xi^a \\
\psi_b & \lambda
\end{pmatrix}
\begin{pmatrix}
\nu^b \\
\sigma
\end{pmatrix}
=
\begin{pmatrix}
A^a_b \nu^b + \sigma \xi^a \\
\lambda \sigma + \psi_b \nu^b
\end{pmatrix}.
\]

From this definition and the change of splitting on standard tractors as described in [2], one readily concludes that a change of connection described by a one–form \( \Upsilon^a \) changes this splitting as

\[
\begin{align*}
\hat{\xi}^a &= \xi^a \\
\hat{A}^a_b &= A^a_b + \xi^a \Upsilon_b \\
\hat{\lambda} &= \lambda - (c \varphi^c \\
\hat{\psi}^a &= \psi^a - A^a_b \Upsilon_c + \lambda \Upsilon_b - \Upsilon_c \varphi^c \Upsilon_b.
\end{align*}
\]

Analogously, we can describe one–forms with values in \( \text{End}(T) \) by simply adding an additional lower index to each slot. It is also straightforward to describe the linear connection on \( \text{End}(T) \) induced be the standard tractor connection. In terms of any connection \( \nabla_a \) in a projective class with Schouten–tensor \( \tilde{\mathcal{P}}_{ab} \) the standard tractor connection is, in the splitting determined by \( \nabla_a \), given by

\[
\nabla_a^T \begin{pmatrix}
\nu^b \\
\sigma
\end{pmatrix}
=
\begin{pmatrix}
\tilde{\nabla}_a \nu^b + \sigma \delta^a_b \\
\tilde{\nabla}_a \sigma - \tilde{\mathcal{P}}_{ab} \nu^b
\end{pmatrix},
\]

see [2]. From this, one deduces by a straightforward computation that the induced linear connection on \( \text{End}(T) \) is, in that splitting, given by

\[
\nabla^{\text{End}(T)}_a \begin{pmatrix}
A^b_c & \xi^b \\
\psi^c & \lambda
\end{pmatrix}
=
\begin{pmatrix}
\tilde{\nabla}_a A^b_c + \psi^c \delta^b_a + \tilde{\mathcal{P}}_{ac} \xi^b \\
\tilde{\nabla}_a \psi^c - \tilde{\mathcal{P}}_{ac} A^d_c - \lambda \tilde{\mathcal{P}}_{ac} \\
\tilde{\nabla}_a \lambda - \tilde{\mathcal{P}}_{ad} \xi^d - \psi^a
\end{pmatrix}.
\]

Now we can compute the torsion free metric connection and its curvature.

**Theorem 18.** Given \( \nabla_a \) as before, consider the \( \text{End}(T) \)–valued one–form \( \Psi \), which on \( M \) is defined in the splitting corresponding to \( \nabla_a \) as having all entries equal to zero, except for

\[
A^b_c := \frac{1}{2} \mathcal{P}^{bd}_{ac} (\nabla_a P_{dc} - \nabla_c P_{da} + \nabla_d P_{ac}).
\]
Then we have:

(i) $\Psi$ admits a smooth extension to all of $\overline{M}$ and defining a modification of the tractor connection as $\nabla^\tau_\xi s := \nabla^\tau_\xi s + \Psi(\xi)(s)$, the resulting connection is metric for $L(\tau)$.

(ii) Consider a local defining function $\rho$ for $\partial M$, let $\tilde{\nabla}_a$ be the corresponding connection in the projective class, $C_{\alpha\beta\gamma\delta}$ its projective Weyl curvature and $Y_{\alpha\beta\gamma\delta}$ its projective Cotton tensor. Further, let $t^a$ be the vector field from Proposition 17 and put

$$\psi_{ac} := t^d(\nabla_a P_{dc} - \nabla_c P_{da} + \nabla_d P_{ac}).$$

Then as a two–form with values in $\text{End}(T)$, the curvature of $\nabla^\tau$ is, in the splitting determined by $\tilde{\nabla}$, given by

$$\begin{pmatrix}
C_{\alpha\beta\gamma\delta} + 2\tilde{\nabla}_{[\alpha} A_{\beta\gamma\delta]} - 2\psi_{d[a} \delta_{\beta]} + 2A_{\gamma[a} A_{\beta\delta]} - 0 \\
Y_{\alpha\beta\gamma\delta} + 2\tilde{\nabla}_{[\alpha} \psi_{\beta\gamma\delta]} - 2\tilde{P}_{[\alpha} A_{\beta\gamma\delta]} + 2\psi_{[\alpha} A_{\beta\gamma\delta]} - 0
\end{pmatrix},$$

so in particular, $\nabla^\tau$ is a torsion free tractor connection.

Proof. Take a local defining function $\rho$ for $\partial M$ and let $\tilde{\nabla}$ be the corresponding connection in the projective class which is smooth up to the boundary. Then from (23) we see that writing $\Psi$ over $M$ in the splitting corresponding to $\tilde{\nabla}$, there are two non–zero entries, namely $\hat{A}_a^{\alpha\beta} = A_a^{\alpha\beta}$ and $\hat{\psi}_{ac} = -A_a^{\gamma\delta} Y_{\gamma\delta}$, and we will omit the hats in the notation for $A$ and $\psi$ from now on. From Propositions 17 and 15 we know that $\rho^{-1}P_{ab}$ and $\rho \nabla_a P_{bc}$ admit smooth extensions to all of $M$, whence the same is true for $A_a^{\alpha\beta}$. On the other hand $Y_a = \tilde{Y}_a$, so again by Proposition 17 $\psi_{ac}$ admits a smooth extension to the boundary and has the claimed form.

Knowing that $\nabla^\tau$ is well defined on all of $\overline{M}$, it suffices to prove that it is metric on the dense open subset $M$, where we can compute in the scale determined by $\nabla$. In that scale, formula (16) for $L(\tau)$ shows that

$$L(\tau) \begin{pmatrix} \nu^a_1 \\ \sigma_1 \end{pmatrix}, \begin{pmatrix} \nu^a_2 \\ \sigma_2 \end{pmatrix} = \tau \sigma_1 \sigma_2 + \tau P_{ab} \nu^a_1 \nu^b_2.$$

On the other hand,

$$\tau P_{ab} A_a^{\alpha\beta} \nu^a_1 \nu^a_2 = \frac{1}{2} \tau \nu^a_1 \nu^a_2 (-\nabla_a P_{cd} - \nabla_d P_{ca} + \nabla_c P_{ad}),$$

and adding the same term with $\nu_1$ and $\nu_2$ exchanged, we arrive at $-\nu^a_1 \nu^a_2 \tau \nabla_a P_{cd}$. Using this, and formula (25) for the standard tractor connection it is easy to verify by a direct computation that $\nabla_T^a$ is metric for $L(\tau)$.

For the description of the curvature, we use the description of $\Psi$ in the splitting corresponding to $\tilde{\nabla}$ from above. Now the curvature of the standard tractor connection is well known to be given in a splitting by the Weyl–curvature and the Cotton tensor, see [2]. On the other hand, it is also well known that the definition of $\nabla_T^a$ implies that its curvature is related to the one of $\nabla_T$ by

$$\tilde{R}(\xi, \eta) = R(\xi, \eta) + \nabla^\text{End}(T)\xi(\psi(\eta)) - \nabla^\text{End}(T)\eta(\psi(\xi)) - \psi(\xi, \eta) + \psi(\eta, \xi),$$

where in the last term we use the commutator of endomorphisms. The second to forth term in the right hand side are the covariant exterior derivative of the
Projective compactness 29

End(\mathcal{T})-valued one-form \Psi with respect to the connection induced by \nabla^T. This can be computed by coupling \nabla^T to the (torsion free) connection \tilde{\nabla} on T^*M, differentiating the one-form \Psi with this coupled connection and then take the alternation in the form–indices and multiply by two. Using all that, the fact that both A and \psi are symmetric in the lower indices, and formula (26) for \nabla^{\text{End}(\mathcal{T})}, the claimed formula for the curvature follows by a direction computation, and torsion freeness just means that the top right entry in the resulting matrix vanishes. □

Observe that inserting the descriptions of \rho \nabla_a P_{bc} from Proposition 15 into the formulae for \Lambda_{abc} and \psi_{ac} from the theorem, there are some cancellations. For example, in the case of a Levi–Civita connection, we obtain

\[ A_{a b c} \big|_{\partial M} = -\frac{1}{2} \rho^{-1} P^{bd} (\rho_a \Phi_{dc} + \rho_c \Phi_{da}) \big|_{\partial M} , \]

where \Phi_{ab} is the tensor from Proposition 15. A similar expression holds for \psi_{ac}.

4.5. Restricting to the boundary. Over M, the connection \tilde{\nabla}^T constructed in Theorem 4.4 is essentially uniquely determined by compatibility with the bundle metric \text{L}(\tau) and torsion freeness. (This is closely related to the proof of existence and uniqueness of the Levi–Civita connection in the Cartan picture. Likewise, the proof of Theorem 4.4 is closely related to the construction of the Levi–Civita connection.) However, if we restrict to the boundary and differentiate only in boundary directions a further normalization is possible, and this will lead to a description of the conformal standard tractor connection. We do not provide complete formulae in general, but only describe how they can be obtained. The problem is that formulae are getting quite involved without simplifying assumptions (which is not surprising in view of the rather complicated relation between the geometries in the interior and on the boundary).

In what follows, we have to distinguish between directions tangent to the boundary and transversal directions, and we will adapt the abstract index notation accordingly. We use indices \(, i, j, k, \) and so on to specify boundary directions, while indices \(a, b, c, \) and so on will be used for directions which are not necessarily tangent to the boundary. A certain amount of care is needed here and also upper and lower indices have to be distinguished. For a lower index, it is no problem to replace a “general” index by a “tangential” one; this simply corresponds to restricting a linear functional to a hyperplane. On the other hand, there is no canonical extension of a functional defined on a hyperplane to the whole space, so “tangential” lower indices cannot be replaced by “general” ones without further choices. In contrast to this, for upper indices, a “tangential” index can always be considered as a general one (corresponding to the inclusion of a hyperplane into a vector space). One can recognize tangential upper indices by the fact that they hook trivially into \(\rho_a\).

From now on, let us fix a local defining function \(\rho\) for the boundary and the corresponding connection \(\nabla_a\) in the projective class. Then consider the quantity

\[ \gamma_{ab} := \rho \hat{P}_{ab} + \frac{1}{2} \rho_a \rho_b \]

which occurs in (17). This admits a smooth extension to the boundary, and indeed by formula (18) from the proof of Proposition 15, we get

\[ \gamma_{ab} = \frac{1}{2} \nabla_a \rho_b + \rho \hat{P}_{ab} . \]

In particular, restricting to the boundary and tangential
directions, we can form $\gamma_{ij}$, and this is a representative of the projective second fundamental form. On the other hand, consider the quantity $\rho^{b} - 1 P^{ab} t^{a}$, which shows up in (21). From the fact that the vector field $t^{a}$ showing up in this proposition is smooth up to the boundary, we see that $\rho^{b} P^{ab} t^{b}$ vanishes along $\partial M$, so its restriction to $\partial M$ is actually tangential. Then the last equation in (22) shows that on tangential vectors, this restriction is actually inverse to $\gamma_{ij}$, so we denote it by $\gamma_{ij}$.

Next, we introduce a finer decomposition of $T |_{\partial M}$, which resembles the usual picture of conformal standard tractors in slots. The necessary information is basically contained in Proposition 17. In particular, we can use the transversal $t^{a}$ from there to identify $T |_{\partial M}$ along $\partial M$ with $E(1) \oplus T |_{\partial M}(-1)$ according to (27)

$$\nu^{a} \mapsto \left( \begin{array}{c} \hat{\nu}^{b} \rho_{b} \\ \nu^{a} - \nu^{b} \rho_{b} t^{b} \end{array} \right) \mapsto \left( \begin{array}{c} \beta \\ \xi_{2} \end{array} \right) \mapsto \xi^{a} + \hat{\gamma}^{-1} \beta t^{a}.$$ 

These are inverse to each other since by the first formula in (22), we have $t^{b} \rho_{b} = 1$ along $\partial M$. Now we combine this with the splitting of $T$ determined by $\hat{\nabla}_{a}$ to identify $T |_{\partial M}$ along $\partial M$ with $E(1) \oplus T |_{\partial M}(-1) \oplus E(-1)$, and we will use this splitting from now on. The second formula in (22) says that $t^{a} \gamma_{ab} = -\frac{1}{4} \psi \rho_{b}$. Combining this with the formula for $L(\tau)$ from the proof of Proposition 17, we get

$$L(\tau) \left( \begin{array}{c} \beta_{1} \\ \xi_{1} \\ \beta_{2} \\ \xi_{2} \end{array} \right) = \frac{1}{2} \beta_{1} \sigma_{2} + \frac{1}{2} \beta_{2} \sigma_{1} + \hat{\gamma}_{ij} \xi_{i} \xi_{j} - \frac{1}{4} \psi \hat{\gamma}^{-1} \beta_{1} \beta_{2}.$$ 

This is slightly different from the usual splitting of conformal standard tractors, since the line spanned by the first basis vector (corresponding to the $E(1)$–component) is not isotropic. One could correct this by passing to a line in $T$ which is not contained in the subspace $E^{a}(-1)$ (in the splitting determined by $\nabla$), but we do not do this at this stage.

**Theorem 19.** Consider the restriction of the linear connection $\tilde{\nabla}_{T}$ from Theorem 18 to the boundary (i.e. we differentiate in boundary directions only). Then the curvature of this restriction is given by restricting the two–form indices $a$ and $b$ in the formula for the curvature in Theorem 18 to tangential directions. Moreover, in our splitting, the curvature takes the form

$$\left( \begin{array}{ccc} 0 & 0 & 0 \\ V_{ij}^{k} & W_{ij}^{k \ell} & 0 \\ 0 & 0 & 0 \end{array} \right)$$ 

where $V_{ij}^{k} = V_{[ij]}^{k}$, $W_{ij}^{k \ell} = W_{[ij]}^{k \ell}$ and $W_{ij}^{r \ell} \gamma_{kr} = -W_{ij}^{r} \gamma_{kr}$. Suppose further that $n = \dim(\partial M) \geq 3$. Then putting

$$\varphi_{ij} := -\frac{1}{n-2} W_{ki}^{k} j + \frac{1}{2(n-1)(n-2)} W_{kr}^{k} s \gamma^{rs} \gamma_{ij}$$

and defining an $\text{End}(T)$–valued one–form $\Psi$ on $\partial M$ in our splitting by

$$\left( \begin{array}{ccc} 0 & 0 & 0 \\ -\frac{1}{2} \hat{\gamma}^{-1} \varphi_{ij} \gamma^{k \ell} & 0 & 0 \\ 0 & \varphi_{ij} \end{array} \right)$$
the linear connection $\nabla^0$ defined by $\nabla^0 s = \tilde{\nabla}_T s + \tilde{\Psi}(\xi)(s)$ is the normal conformal tractor connection on the conformal tractor bundle $\mathcal{T}|_{\partial M} \to \partial M$.

**Proof.** The facts that $\tilde{\nabla}_T$ can be restricted to the boundary and that the curvature is obtained by restriction follows as in the proof of Theorem 16. Writing the resulting curvature in a matrix according to our splitting, we see from the formula in Theorem 18 that the last column has to consist of zeros only. Moreover, since $\tilde{\nabla}_T$ is metric for $L(\tau)$ all the values of its curvature are skew symmetric with respect to $L(\tau)$. Knowing that there are some zero blocks already, the claimed form of the curvature is established by a simple direct computation using formula (28).

For the second part of the proof, recall that for $n \geq 3$ it follows from the general theory (see [4] and [5]) that the canonical tractor connection on a conformal standard tractor bundle is characterized by the fact that it is metric and its curvature is normal. As described in the proof of Theorem 16, normality first requires that the curvature preserves the canonical line subbundle $\mathcal{T}^1$. If this is satisfied, one obtains a tensor field describing the induced action of the curvature on $$(\mathcal{T}^1)^\perp / \mathcal{T}^1$$. (For the tractor connection $\tilde{\nabla}_T$ this is described by the component $W_{ij}^{k\ell}$ from above.) The second part of the normality condition is that the Ricci–type contraction of this component vanishes identically.

Next, it is a standard result on induced connections that the definition of $\tilde{\nabla}_T$ in terms of $\nabla_T$ and the $\text{End}(\mathcal{T})$–valued one–form $\Psi$ in Theorem 18 implies that

$$R^0(\xi, \eta) = \tilde{\nabla}_T (\xi, \eta) + \nabla_T (\xi, \eta) - \tilde{\Psi}(\xi) \tilde{\Psi}(\eta) - \tilde{\Psi}(\xi, \eta) + [\tilde{\Psi}(\xi), \tilde{\Psi}(\eta)],$$

where $\tilde{\nabla}_T$ is the connection on $\text{End}(\mathcal{T})$ induced by $\tilde{\nabla}_T$ and the last bracket denotes the commutator of endomorphisms. Now the fact that $\tilde{\Psi}$ is concentrated in the block–lower–triangular part of the matrix says that inserting any vector field $\xi$ into $\tilde{\Psi}$ one obtains a map which vanishes on $\mathcal{T}^1$, and maps $(\mathcal{T}^1)^\perp$ to $\mathcal{T}^1$ (and also maps $\mathcal{T}$ to $(\mathcal{T}^1)^\perp$). This shows that the last two terms in the right hand side of (29) do not contribute to the induced map on $(\mathcal{T}^1)^\perp / \mathcal{T}^1$ (and it also implies that the last one always vanishes).

Now as in the proof of Theorem 18, the curvature $R^0$ of $\nabla^0$ is related to the curvature $\tilde{R}$ of $\tilde{\nabla}_T$ by

$$R^0(\xi, \eta) = \tilde{R}(\xi, \eta) + \nabla_T (\xi, \eta) - \tilde{\Psi}(\xi) \tilde{\Psi}(\eta) - \tilde{\Psi}(\xi, \eta) + [\tilde{\Psi}(\xi), \tilde{\Psi}(\eta)],$$

where $\nabla_T$ is the connection on $\text{End}(\mathcal{T})$ induced by $\tilde{\nabla}_T$ and $\tilde{\Psi}$ is the connection on $\text{End}(\mathcal{T})$ induced by $\tilde{\nabla}_T$ and the last bracket denotes the commutator of endomorphisms. Now the fact that $\tilde{\Psi}$ is concentrated in the block–lower–triangular part of the matrix says that inserting any vector field $\xi$ into $\tilde{\Psi}$ one obtains a map which vanishes on $\mathcal{T}^1$, and maps $(\mathcal{T}^1)^\perp$ to $\mathcal{T}^1$ (and also maps $\mathcal{T}$ to $(\mathcal{T}^1)^\perp$). This shows that the last two terms in the right hand side of (29) do not contribute to the induced map on $(\mathcal{T}^1)^\perp / \mathcal{T}^1$ (and it also implies that the last one always vanishes).
of $\mathcal{T}^1$, which is described by the tensor $A_{a}^{\ b\ c}$ from Theorem 18. As we have noted in the proof of that theorem $A_{a}^{\ b\ c}$ vanishes along the boundary. This shows that $\Psi(\xi)$ has values in $(\mathcal{T}^1)^\perp$, so $\Psi(\eta) \circ \Psi(\xi)$ has values in $\mathcal{T}^1$ and does not contribute to the action on $(\mathcal{T}^1)^\perp/\mathcal{T}^1$ either.

Collecting the information, we see that the difference $R^0(\xi,\eta) - \tilde{R}(\xi,\eta)$ is given by
\[
\nabla^\text{End}(\mathcal{T}) \tilde{\Psi}(\eta) - \nabla^\text{End}(\mathcal{T}) \tilde{\Psi}(\xi).
\]
The first summand in this expression maps $s \in \Gamma(\mathcal{T})$ to
\[
\nabla_{\xi} \tilde{\Psi}(\eta)(s) - \nabla_{\eta} \tilde{\Psi}(\xi)(\nabla_{\xi} s).
\]
Now we can directly compute the induced action of this on $(\mathcal{T}^1)^\perp/\mathcal{T}^1$ by applying this to an element of the form $s = (0, \nu^\ell, 0)$ and computing the middle slot of the result. We do this in abstract index notation with the index $i$ corresponding to $\xi$ and $j$ corresponding to $\eta$. For the first term, applying $\tilde{\Psi}$ the result is $\phi_{ji} \nu^\ell$ in the bottom slot, and zero in the two other slots, so differentiating by $\nabla_T^i$ according to (23), this produces $\delta^k_i \phi_{ji} \nu^\ell$ in the middle slot. The middle slot in the second term is given (including the sign) by multiplying $\hat{\tau}^j - 1 \phi_{jr} \gamma^{kr}$ by the top slot of the derivative in the bracket. By (22), the latter is given by
\[
\hat{\tau} \rho_b \nabla_i \nu^b = -\hat{\tau} \rho^b \nabla_i \nu_b = -\hat{\tau} 2 \gamma_{i\ell} \nu^\ell.
\]
Collecting our results, we see that the tensor describing the action of the curvature $\nabla^0$ on $(\mathcal{T}^1)^\perp/\mathcal{T}^1$ is given by
\[
W_{ij}^{\ k\ \ell} + \delta^k_i \phi_{ji} - \delta^k_j \phi_{i\ell} - \phi_{jr} \gamma^{kr} \gamma_{i\ell} + \phi_{ir} \gamma^{kr} \gamma_{j\ell}.
\]
Forming the Ricci–type contraction, we get
\[
W_{kj}^{\ k\ \ell} + (n - 2) \phi_{jl} + \phi_{kr} \gamma^{kr} \gamma_{jl},
\]
and inserting the definition of $\phi_{jl}$ one immediately verifies that this vanishes. □

References

[1] O. Aharony, S.S. Gubser, J.M. Maldacena et al., Large N field theories, string theory and gravity, Phys. Rept. 323 (2000), 183-386.
[2] T.N. Bailey, M.G. Eastwood, and A.R. Gover, Thomas’s structure bundle for conformal, projective and related structures, Rocky Mountain J. Math. 24 (1994), 1191–1217.
[3] D.M.J. Calderbank and T. Diemer, Differential invariants and curved Bernstein-Gelfand-Gelfand sequences, J. Reine Angew. Math. 537 (2001) 67–103.
[4] A. Čap, A.R. Gover, Tractor calculi for parabolic geometries, Trans. Amer. Math. Soc. 354 (2002), no. 4, 1511–1548.
[5] A. Čap, A.R. Gover, Standard tractors and the conformal ambient metric construction, Ann. Global Anal. Geom. 24, 3 (2003) 231-259.
[6] A. Čap, A.R. Gover, Projective compactifications and Einstein metrics, to appear in J. reine angew. Math., DOI 10.1515/crelle-2014-0036, arXiv:1304.1869
[7] A. Čap, A. R. Gover, and M. Hammerl, Projective BGG equations, algebraic sets, and compactifications of Einstein geometries, J. London Math. Soc., 86 (2012), 433–454.
[8] A. Čap, A.R. Gover, and M. Hammerl, Holonomy reductions of Cartan geometries and curved orbit decompositions. Duke Math. J. 163, no. 5, 1035–1070.
[9] A. Čap, A.R. Gover, and H. Macbeth, Einstein metrics in projective geometry, Geom. Dedicata 168 (2014) 235–244.
[10] A. Čap, J. Slovák, and V. Souček, Bernstein-Gelfand-Gelfand sequences, Ann. of Math. 154 (2001), 97–113.
[11] S.Y. Cheng, and S.T. Yau, On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman’s equation, Comm. Pure Appl. Math., 33 (1980), 507–544.
[12] P. Chruściel, E. Delay, J.M. Lee, D.N. Skinner, Boundary regularity of conformally compact Einstein metrics, J. Differential Geom. 69 (2005), 111–136.
[13] S. de Haro, K. Skenderis, and S.N. Solodukhin, Holographic Reconstruction of Spacetime and Renormalization in the AdS/CFT Correspondence, Commun. Math. Phys. 217 (2001), 595–622.
[14] M.G. Eastwood, and V. Matveev, Metric connections in projective differential geometry in “Symmetries and overdetermined systems of partial differential equations”, 339–350, IMA Vol. Math. Appl., 144, Springer, New York, 2008.
[15] C. Fefferman, and C.R. Graham. Conformal invariants, in: The mathematical heritage of Élie Cartan (Lyon, 1984). Astérisque 1985, Numero Hors Serie, 95–116.
[16] C. Fefferman, and C.R. Graham. The Ambient Metric. Annals of Mathematics Studies, 178. Princeton University Press 2012.
[17] Jörg Frauendiener, Conformal Infinity, Living Rev. Relativ., 7 (2004), 2004-1, 82pp.
[18] H. Friedrich, Conformal Einstein evolution, The conformal structure of space-time, 1–50, Lecture Notes in Phys., 604, Springer, Berlin, 2002.
[19] C.R. Graham, Volume and area renormalizations for conformally compact Einstein metrics, Rend. Circ. Mat. Palermo Suppl. No. 63 (2000), 31–42.
[20] C.R. Graham, and J.M. Lee, Einstein metrics with prescribed conformal infinity on the ball. Adv. Math. 87 (1991), 186–225.
[21] C.R. Graham, and M. Zworski, Scattering matrix in conformal geometry, Invent. Math. 152 (2003), 89–118.
[22] M. Hammerl, P. Somberg, V. Souček, J. Silhan, On a new normalization for tractor covariant derivatives, J. Europ. Math. Soc. 14 no. 6 (2012) 1859–1883
[23] M. Henningson and K. Skenderis, The holographic Weyl anomaly, J. High Energy Phys. 1998, no. 7, Paper 23, 12 pp.
[24] R. Mazzeo. The Hodge cohomology of a conformally compact metric. J. Differential Geom., 28 (1988), 309–339.
[25] R.B. Melrose, Geometric scattering theory. Stanford Lectures. Cambridge University Press, Cambridge, 1995.
[26] R. Penrose, Asymptotic properties of fields and space-times, Phys. Rev. Lett. 10 (1963), 66–68.
[27] R. Penrose, Zero rest-mass fields including gravitation: asymptotic behaviour, R. Soc. London, Ser. A, 284 (1965), 159–203.
[28] J.A. Schouten, J. Haantjes, Beiträge zur allgemeinen (gekrümmten) konformen Differential-geometrie, Math. Ann. 112 (1936), 594–629.
[29] J.A. Schouten, J. Haantjes, Beiträge zur allgemeinen (gekrümmten) konformen Differential-geometrie, II, Math. Ann. 113 (1937), 568–583.
[30] J.H.C. Whitehead, Convex regions in the geometry of paths, Quart. J. Math. os-3, no. 1 (1932) 33–42.

A.Č.: Faculty of Mathematics, University of Vienna, Oskar–Morgenstern–Platz 1, 1090 Wien, Austria, A.R.G.: Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland 1142, New Zealand; Mathematical Sciences Institute, Australian National University, ACT 0200, Australia
E-mail address: Andreas.Cap@univie.ac.at
E-mail address: r.gover@auckland.ac.nz