CONTRACTIONS, COGENERATORS, AND WEAK STABILITY

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Abstract. A $C_0$ contraction semigroup $T$ on a Hilbert space $H$ and its cogenerator $D$ define a $W^*$ - algebra, $\mathcal{M}^T$ - the limit algebra - which determines the structure of the subspace of weakly Poisson recurrent (wPr) vectors and gives a necessary and sufficient condition for $T$ and $D$ to be weakly stable equivalent.

1. Introduction and Summary

1.1. Introduction. Let $T = \{T_t : t \geq 0\}$ be a $C_0$ contraction semigroup on a Hilbert space $H$, $T$ its cogenerator, $D = \{T^n : 0 \leq n < +\infty\}$. $T$ and $D$ have a common space of flight( or almost weakly stable) vectors $H_0$. Assume $H = H_0$.

A commutative contraction semigroup $S$ splits $H_0$, $H_0 = H_m(S) \oplus H_w(S)$ [9, Theorem 2.5] where $H_m(S)$ is the space of weakly Poisson recurrent (wPr) vectors and $H_w(S)$ is the weakly - stable subspace. $T$ and $D$ define dynamical systems on $H$. We shall examine the interaction of the limit operators of $T$ and $D$ and the limit states of these systems. These limit operators generate a $W^*$ - algebra, $\mathcal{M}^T$ - the limit algebra - which determines the structure of the subspace $H_m$ and the interaction of the limit states. $\mathcal{M}^T$ also determines a necessary and sufficient condition[Section 5] for equivalence of weak stability of $T$ and $D$ i.e., $H_w(T) = H_w(D)$. Reference [2] motivated this research with the open question [2, 2.23].

1.2. Preliminaries. We use the definitions and notation of [9, 10]. $T$ and $D$ have a common unitary subspace $H_u \subset H_0$. $H_u$ is closed and reducing and $H_m \subset H_u$ for both $T$ and $D$. $U$ is the unitary group defined on $H_u$ by $T$ and $C$ is the corresponding group on $H_u$ for $D$ [9, 10]. $T$ and $D$ each split $H_0$, $H_m(T) = H_m(U)$ and $H_m(D) = H_m(C)$. $T = UP_w \oplus TP_w$ and similarly for $D$.

1.3. Example: A Weakly Stable Equivalent $C_0$ Contraction Semigroup and Cogenerator. [12, 1.3]. Take the Hilbert space $H$ to be $H = L^2([0, 1] \oplus \mathbb{Z}^+)$. [9, 10] define for a strictly increasing, continuous-singular function $F$ on $[0, 1]$ a spectral family $\{F_\theta : 0 \leq \theta < 1\}$ with unitary operator $U = \int_0^1 e^{2\pi i \theta} dF_\theta$. $U$ has purely continuous spectrum and hence $L^2[0, 1] = H_0(U)$. [Jacobs-Glicksburg-Deleeuw Theorem [3]]. $U$ is the Cayley transform of the self-adjoint operator $A = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda$, and

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1Dynamical system - the action of a commutative contraction semigroup $S$ on a Hilbert space $H$. 
the cogenerator of the $C_0$ unitary group $U : U_t = \int_{-\infty}^{+\infty} e^{it\lambda} dE_\lambda$, $-\infty < t < +\infty$ on $L^2[0,1]$. Since \( H_0 \) is reducing, \( H_0(C) = H_0(U) \), and from [9, 10, 6.2], \( H_m(C) = H_0(C) \). Similarly \( H_m(U) = H_0(U) \). Hence \( U \) and \( C \) are weakly Poisson recurrent and $L^2[0,1] = H_m(U) = H_m(C)$.

Let \( W \{ z_0, z_1, \ldots \} = \{ 0, z_0, z_1, \ldots \} \) be the unilateral shift on $l^2(\mathbb{Z}^+)$. $W$ is an isometry and completely non-unitary(cnu) [7] and hence weakly stable. Since 1 is not an eigenvalue of $W$ and $W$ is isometric, $W$ is the cogenerator of a $C_0$ isometric semigroup \( W = \{ W_t \} \). Since $W$ is cnu $W$ cannot have a closed invariant subspace on which each \( W_t \) is unitary, hence it is also cnu and therefore weakly-stable. [7, 9] [16, IX.9],[2, 3.2]

Define a $C_0$ contraction semigroup on \( H \) by \( T : T_t = U_t \oplus W_t \) for $t \geq 0$. The cogenerator of $T$ is $T = U \oplus W$, \( D = \{ T^n : n \geq 0 \} \) its semigroup. Then $H_p(T) = H_p(D) = \{ 0 \}$, i.e. \( H = H_0(T) = H_0(D) \) consists of flight vectors[9]. The equivalence of weak stability will be proved in [Example 5.6].

2. Dynamical Systems on \( H_0 \)

The limit states of $T$ and $D$ are determined by their action on $H_\omega$ - the unitary space - so we consider the groups $U$ and $C$. In the following $S$ will denote one of the above: $T$, $D$, $U$, or $C$ which will be apparent from the context. To be self contained we collect some basic notions from [9,10]. Definitions and results will be stated for $S = T$ or $U$, the extension to $S = D$ or $C$ will be clear. We examine the dynamics of $S$ with a classical eye.

2.1. Dynamical Systems. The semigroup $S$ defines a dynamical system on $H$:

\[
x(t) = T_t x_0, \quad x(-t) = T^*_t x_0 \quad \text{for} \quad t \geq 0, \quad \text{for all} \quad x_0 \in H.
\]  

(2.1)

For (2.1) $S$ has $\omega$ and $\alpha$ - limit operators, $\Omega = \{ V : V = \omega - \lim T_{t_k}, \quad t_k \uparrow +\infty \} \quad \text{and} \quad A = \Omega^\ast$. These operators define the $\omega$ and $\alpha$ - limit states of $x_0$, $\Omega(x_0) = \{ y : y = \omega - \lim T_{t_k} x_0, \quad t_k \uparrow +\infty \} = \Omega \ast x_0$ and $A(x_0) = \Omega^\ast \ast x_0$. [8, 9, 10]. These sets are $S, S^\ast$ invariant.[9, 10]. The system (2.1) also has limit cycles as in the finite dimensional case of the Poincare- Bendixson theorem,

\[
y(t, x_0) = T_t V x_0, \quad y(-t, x_0) = T^*_t V^* x_0 \quad \text{for} \quad t \geq 0 \quad \text{and} \quad V \in \Omega.
\]  

(2.2)

The trajectories of (2.2) converge weakly and pointwise to the limit cycles, i.e. given $x_0$ and $V = \omega - \lim T_{t_k}$ then

\[
\omega - \lim_{n \to \infty} x(s + t_n, x_0) = y(s, x_0).
\]  

(2.3)

2.2. Subspaces and Recurrence. For $x \in H_0$, $M(x, S) = sp \Omega(x, S)$ is the limit subspace for $x$. Note that $\Omega \ast x = \Omega(x, S) = \cap \{ \Omega(T x) : s \geq 0 \}$ and $T_t \Omega(x) = \Omega(x)$ for all $t \geq 0, [9, 10]$

$H_\omega(S)$ is the collection of weakly stable states - $\omega - \lim T_s x = 0$. They form a a closed reducing subspace $H_\omega(S) = \bigcap \{ ker V : V \in \Omega S \} = \bigcap \{ ker V^* : V \in \Omega S \} = H_\omega(S^\ast) \quad [9, 10]$.

A vector $x$ in $H_0$ is Poisson recurrent($Pr$) if and only if $x \in \Omega(x, S) = \Omega \ast x$, i.e. $x = V x$ for some $V \in \Omega$. If $x$ is $Pr$ then $x \in H_\omega$ and $\omega - \lim T_{t_n} x = x$ implies $s - \lim T_{t_n} x = x$.

A vector $x$ in $H_0$ is weakly Poisson recurrent (wPr) if and only if $x \in M(x, S)$. $H_m$ is the collection of wPr vectors. Note that $x \in H_0$ is (wPr) if there is a net for this $x$ in $\mathfrak{S}$, $\{(\sum_{k=1}^n a_k T_{s_k}) : \alpha \in \Delta \}$ with $\omega - \lim \Delta (a_k T_{s_k}) x = x$. Since $H_m = H_0 \ominus H_\omega$, $H_m$ is a closed and reducing subspace and $H_m(S) = H_m(S^\ast)$. If $x$ is wPr then $M(x, S)$ is reducing and
\[ M(x, S) = \overline{sp \Omega \cdot x} = \overline{sp \Omega^* \cdot x} = M(x, S^*) \] [9, 3.4 - 5], i.e. the future of a wPr vector coincides with its past.

An ortho-normal set \( \{ x_\tau : \tau \in \Pi \} \subset \mathcal{H}_m(S) \) is a \textit{recurrent spanning set} [9, 10] for \( S \) if \( \mathcal{H}_m(S) = \bigoplus_{\Pi} M_x \) for \( M_x = M(x_\tau, S) \). From [9, Theorem 2.5] if \( \mathcal{H}_m(S) \neq 0 \) then \( S \) has an ortho-normal \textit{recurrent spanning set} \( \{ x_\tau : \tau \in \Pi \} \subset \mathcal{H}_m(S) \).

3. Limit Algebras

Prompted by (2.1 - 2.2) the composition of the limit-cycles and the structure of the space \( \mathcal{H}_m \) we consider the limit operators \( \Omega_S \) of \( S = \mathcal{U} \) or \( C \) and the algebra they generate in \( \mathcal{L}(\mathcal{H}_u) \).

3.1. Algebras. For any commutative collection of operators \( \mathcal{A} \subset L(\mathcal{H}_u) \), let \( \mathcal{M}^A \) be the least *-closed, weakly closed sub-algebra of \( L(\mathcal{H}_u) \) containing \( \mathcal{A} \). \( \mathcal{A} \) is the generating set of \( \mathcal{M}^A \) and \( \mathcal{P}_A \) is its unit.[14, 1.7],

In particular, for \( \mathcal{A} = \Omega_S \), \( \mathcal{M}^{\Omega_S} \) is the \textit{limit algebra} generated by the limit operators \( \Omega_S \) of \( S \). Note that for all \( x \in \mathcal{H}, V \in \Omega_S, V x = V x_m \in \mathcal{H}_m \) and hence \( \mathcal{M}^{\Omega_S} \) is a subalgebra of \( L(\mathcal{H}_m) \).

3.2. The Generator Sets \( \mathcal{A} \). On the unitary space \( \mathcal{H}_u \) of \( \mathcal{U} \) and \( C \) let \( \mathcal{E} = \{ E_\lambda : -\infty < \lambda < +\infty \} \) and \( \mathcal{F} = \{ F_\theta : 0 \leq \theta < 1 \} \) be their respective spectral families with \( A \) the self-adjoint operator generating the unitary group \( \mathcal{U} \). Note that \( \mathcal{E} \) and \( \mathcal{F} \) satisfy \( E_\lambda = F_{-2\arccot(\lambda)}, -\infty < \lambda < +\infty \) [13, § 121] and hence \( \mathcal{E} = \mathcal{F} \).

The spaces of Borel functions essentially bounded \( \mathcal{E}, \mathcal{F} \) a.e., \( \mathcal{L}_\mathcal{E}, \mathcal{L}_\mathcal{F} \) have corresponding algebras[13, § 109,129]

\[ \mathcal{L}_\mathcal{E} = \{ u(A) = \int_{-\infty}^{+\infty} u(\lambda) dE_\lambda : u \in \mathcal{L}_\mathcal{E} \}, \]

and

\[ \mathcal{L}_\mathcal{F} = \{ w(U) = \int_{0}^{1} w(e^{2\pi i \theta}) dF_\theta : w \in \mathcal{L}_\mathcal{F} \}. \]

Since \( \mathcal{E} = \mathcal{F} \) we have \( \overline{\mathcal{L}_\mathcal{E}} = \overline{\mathcal{L}_\mathcal{F}} \) and \( \mathcal{M}^\mathcal{E} = \overline{\mathcal{L}_\mathcal{E}} = \mathcal{L}_\mathcal{E} = \mathcal{L}_\mathcal{F} = \overline{\mathcal{L}_\mathcal{F}} = \mathcal{M}^\mathcal{F} \)

with unit \( \mathcal{P}_u \) the orthogonal projection on \( \mathcal{H}_u \).

The algebras defining the dynamics of \( S \) have the following generating sets: \( \mathcal{E} = \mathcal{F} \), the groups \( \mathcal{U} \) and \( C \), and the sets of limit operators and closures - \( \Omega_C, \Omega_\mathcal{U}, \overline{\Omega_\mathcal{U}}, \overline{\Omega_C} \).

\( \mathcal{P}_u \) is the unit of \( \mathcal{M}^C \) and \( \mathcal{M}^\mathcal{U} \) while the identities for \( \mathcal{H}_m(C) \) and \( \mathcal{H}_m(U) \) are the units \( \mathcal{P}_{m_\mathcal{C}} \in \mathcal{M}^{\Omega_C} \) and \( \mathcal{P}_{m_\mathcal{U}} \in \mathcal{M}^{\Omega_\mathcal{U}} \) respectively.

3.3. Limit Algebras and Limit Spaces. For all \( x \in \mathcal{H}, M(x, S) = M(x_m, S) \) and from (2.2) \( M(x, S) = S^* \) invariant, \( \mathcal{H}_m(S) = \mathcal{H}_m(S^*) \), and \( \mathcal{P}_m(S) = P_m(S^*) \) [9, 3.4 - 5].

\textbf{Theorem 3.1.} For the groups \( S = \mathcal{U} \) and \( C \), and \( x \in \mathcal{H} \), then

\( a) \ M(x, S) = \overline{sp \ \Omega_S \cdot x} = \mathcal{M}^{\Omega_S} \cdot x = \mathcal{M}^{\Omega_S^*} \cdot x = \overline{sp \ \Omega_S^* \cdot x} = \mathcal{M}^S \mathcal{P}_{m_S} \cdot x, \)

and

\( b) \ \mathcal{M}^{\Omega_S^*} = \mathcal{M}^{\Omega_S} = \mathcal{M}^S \mathcal{P}_{m_S}. \)

The unit of \( \mathcal{M}^{\Omega_S} \) is \( \mathcal{P}_{m_S} \) the orthogonal projection onto \( \mathcal{H}_m(S) \).
Proof. Statement a) follows from the above remarks, [Section 2], and [9,10]. Note that if $P_m$ is the projection on $H_m$, $T_1 P_m x$ and $T_1 T_i P_m x \in M(x, S)$ [9, 3.4-5] and therefore for all $x \in H_0$, $M^{10} : x \subset M_S \cdot P_m x \subset M(x, S)$ and hence $M^{10} : x = M_S P_m : x = M(x, S)$.

For b), fix $x \in H_0$ and $T \in M^S$. For this $x$ and any $\epsilon > 0$, since $M(x, S) = sp \Omega_S : x$, there exists $\sum a_i V_k \in sp \Omega_S \subset M^{10}$ such that $\| \sum a_i V_k x - T P_m x \| < \epsilon$. Since $x \in H_0$ and $\epsilon > 0$ were arbitrary $T P_m \in (M^{10})^a = M^{10}$ (the $W^*$ algebra $M^{10}$ is strong-operator closed [14, §1.15.1]). □

Theorem 3.2. For the subsets of $\mathcal{L}(H_\omega)$ of [Section 3.2], $\mathcal{C}$, $\mathcal{U}$, $\mathcal{E}$, $\mathcal{F}$,

$$M^U = \overline{\mathcal{C}} = M^E = M^F = \overline{\mathcal{F}} = M^C,$$

is a $W^*$ subalgebra of $\mathcal{L}(H_\omega)$ with unit $P_a$ the identity of $\mathcal{L}(H_\omega)$.

Proof.

1) Since $\mathcal{U} \subset \overline{\mathcal{C}} \subset \mathcal{C}$ then $M^\mathcal{U} \subset \overline{M^\mathcal{C}} = M^\mathcal{C}$ and hence $\overline{M^\mathcal{U}} = \overline{M^\mathcal{C}} = \overline{M^\mathcal{F}}$ since $\mathcal{F} = \mathcal{E}$, i.e.

$$\overline{M^\mathcal{U}} = M^\mathcal{C} \subset \overline{\mathcal{C}} = \overline{\mathcal{F}} = M^F = \overline{M^F} = \overline{M^\mathcal{U}} = M^\mathcal{C}.$$

2) To show : $M^\mathcal{F} \subset M^\mathcal{C}$;

Let $A$ be the self-adjoint generator of $\mathcal{U}$. As in the von Neumann construction of the spectral theorem for $A$ from the Cayley transform $U$ [13, § 109 and § 125], the unique spectral family $\mathcal{F} = \{ F_\phi : 0 < \phi < 1 \}$ for $U$ is the strong operator limit of polynomials in $U$ and $U^*$. Hence $\mathcal{F} \subset \mathcal{C}$ and therefore $M^\mathcal{F} = \overline{M^\mathcal{F}} \subset M^\mathcal{C}$.

3) $M^\mathcal{C} \subset M^\mathcal{U}$:

On the reducing subspace $H_u$, $I = P_u$. Therefore as in [16, XI, 4] $(I - iA)^{-1} = \frac{1}{\pi} i \int_0^\infty e^{-t} U_i dt \in M^\mathcal{U}$. The closed operator $A(I - iA)^{-1}$ has domain $H_u$ and hence is bounded by the Closed Graph Theorem. Moreover for $x \in H_u$,

$$A(I - iA)^{-1} x = s - \lim_{t \to 0} \frac{1}{t} (U_i - I)(I - iA)^{-1} x \in M^\mathcal{U}. \quad (3.1)$$

Hence $(I + iA)(I - iA)^{-1} = (iI + A)(iI + A)^{-1} = U \in M^\mathcal{U}$. Since $M^\mathcal{U}$ is *-closed, $U^*$ is also in $M^\mathcal{U}$ and hence $M^\mathcal{C} \subset M^\mathcal{U}$.

4) Combining the previous paragraphs

From 1) and 2): $M^\mathcal{C} \subset \overline{M^\mathcal{U}} = M^\mathcal{F} \subset M^\mathcal{C}$ and hence $M^\mathcal{C} = \overline{M^\mathcal{F}} = M^\mathcal{F}$.

From 1) and 3): $M^\mathcal{C} \subset \overline{M^\mathcal{U}} = M^\mathcal{F} \subset M^\mathcal{C}$ implies

$$M^\mathcal{C} = M^\mathcal{F} = \overline{M^\mathcal{F}} = \overline{M^\mathcal{C}} \supset M^\mathcal{U} \supset M^\mathcal{C}.$$  

Theorem 3.2 follows. □

Theorem 3.3. For all $x$, $M(x, C) = M^{10} : x = M^C P_m : x$ is a separable Hilbert space and hence

$$M(x, C) = M(x_m, C) = M^{10} : x_m = \overline{M^{10}} : x_m = \overline{\mathcal{E}} : x_m = \mathcal{E} \cdot x_m = \mathcal{F} \cdot x_m.$$

A similar statement holds for $\mathcal{U}$.

Proof. The group $C$ is separable and from [Theorem 3.2] $M(x, C) = M(x_m, C) = M^{10} : x_m = \overline{\mathcal{E}} : x_m$. Hence $M(x_m, C) = M^C : x_m$ is separable. From [13, § 106], $\mathcal{E} P_m = \overline{\mathcal{E}} P_m$ in $\mathcal{L}(M_r)$ and therefore

$$M(x, C) = M^{10} : x_m = \overline{\mathcal{E}} : x_m = \mathcal{E} \cdot x_m = \mathcal{F} \cdot x_m.$$  □
4. Structure of $\mathcal{H}_m$

By [2.2 and Theorem 3.1] each $M(x, \mathcal{C}) = \mathcal{M}^{\Omega_c} \cdot x$ is $U, U^*$ invariant. Hence if $M(x, \mathcal{C}) \neq \mathcal{H}$ the closed reducing subspace $M(x, \mathcal{C})^\perp$ has a weakly wandering vector for $U$ by Krengel’s Theorem [5]. Using the construction of [9, 10] an orthonormal set $\{x_\tau\}$ can be chosen weakly wandering for $U$ such that

$$\mathcal{H}_m(\mathcal{C}) = \sum_{\tau} M(x, \mathcal{C}) = \sum_{\tau} \mathcal{M}^{\Omega_c} \cdot x_\tau.$$ (4.1)

Hence the limit cycles for $\mathcal{D}$ and its dynamical system

$$x(n, x_0) = T^n x_0, \ x(-n, x_0) = T^{-n} x_0 \ n \geq 0$$ (4.2)

are defined by $U$ and $\mathcal{M}^{\Omega_c}$:

$$y(n, x_0) = T^n V x_0 = T^n V x_m = \sum T^n V T_\tau x_\tau.$$ (4.3)

for $T_\tau \in \mathcal{M}^{\Omega_c}$ and for all $V \in \Omega_c$. For a semigroup $S = T$ or $\mathcal{D}$ (4.1) characterizes the flight vectors $[1, 2, 9, 10]$.

$$x_0 = \sum T_\tau x_\tau + x_w, T_\tau \in \mathcal{M}^{\Omega_c}, x_w \in \mathcal{H}_w.$$ (4.4)

5. Entangled Systems and Weak Stability

The results of [4.0] lead us to ask when do the limit cycles of $\mathcal{U}$ and the cogenerator group $\mathcal{C}$ approximate each other? We formalize this question:

**Definition 5.1.** $\mathcal{U}$ and $\mathcal{C}$ are **entangled** on $\mathcal{H}$ if for all $x \in \mathcal{H}$, $\Omega_\mathcal{U} \cdot x \subset M(x, \mathcal{C})$ and conversely $\Omega_\mathcal{C} \cdot x \subset M(x, \mathcal{U})$. They are **decoupled** if $\mathcal{H}_m(\mathcal{S}) \cap \mathcal{H}_m(\mathcal{U}) = \{0\}$.

**Remark 5.2.** Some observations:

1) Assume $\mathcal{U}$ and $\mathcal{C}$ are entangled. For all $x \in \mathcal{H}$, $M(x, \mathcal{U}) = M(x, \mathcal{C})$ since from [3.1]:

$$M(x, \mathcal{C}) = \text{sp} \ \Omega_c \cdot x \subset M(x, \mathcal{U}) \ \text{and} \ M(x, \mathcal{U}) = \text{sp} \ \Omega_\mathcal{U} \cdot x \subset M(x, \mathcal{C}).$$

2) It follows from 1) that if $\mathcal{U}$ and $\mathcal{C}$ are entangled they have a common wPr subspace $\mathcal{H}_m = \mathcal{H}_m(\mathcal{U}) = \mathcal{H}_m(\mathcal{C})$ with orthogonal projection $P_m$.

3) Lemma: If $\mathcal{U}$ and $\mathcal{C}$ are entangled they have a common limit-algebra $\mathcal{M}^\Omega$.

Proof: Fix $\varepsilon > 0$ and $x \in \mathcal{H}_m$ - the common wPr subspace of 2). Let $T \in \mathcal{M}^{\Omega_\mathcal{U}}$ and use the argument of [Theorem 3.1]. $\mathcal{M}^{\Omega_\mathcal{U}} \cdot x = M(x, \mathcal{U}) = M(x, \mathcal{C}) = \mathcal{M}^{\Omega_c} \cdot x$. For this $x$ and any $\varepsilon > 0$, since $M(x, \mathcal{C}) = \text{sp} \ \Omega_c \cdot x$, there exists $\sum a_k V_k \in \text{sp} \ \Omega_c \subset \mathcal{M}^{\Omega_c}$ such that $|| \sum a_k V_k x - T x || < \varepsilon$. Since $x \in \mathcal{H}_m$ and $\varepsilon > 0$ were arbitrary $T \in (\mathcal{M}^{\Omega_c})^\Delta = \mathcal{M}^{\Omega_c}$ (the W* algebra $\mathcal{M}^{\Omega_c}$ is strong-operator closed) and $\mathcal{M}^{\Omega_\mathcal{U}} \subset \mathcal{M}^{\Omega_c}$. Interchanging $\mathcal{U}$ and $\mathcal{C}$ in the above argument yields the common limit-algebra $\mathcal{M}^\Omega = \mathcal{M}^{\Omega_\mathcal{U}} = \mathcal{M}^{\Omega_c}$ with unit $P_m$.

4) Conversely, suppose $\mathcal{U}$ and $\mathcal{C}$ have a common limit-algebra $\mathcal{M}^\Omega = \mathcal{M}^{\Omega_\mathcal{U}} = \mathcal{M}^{\Omega_c}$ with unit $P = P_m$. Then $\mathcal{H}_m(\mathcal{C}) = P_m \mathcal{H} = P_m \mathcal{H} = \mathcal{H}_m(\mathcal{U})$. By [Theorem 3.1] for all $x \in \mathcal{H}$,

$$M(x, \mathcal{C}) = \mathcal{M}^{\Omega_c} x = \mathcal{M}^{\Omega_\mathcal{U}} x = M(x, \mathcal{U})$$

and therefore $\mathcal{U}$ and $\mathcal{C}$ are entangled.

**Conclusion:**

**Theorem 5.3.** For the groups $\mathcal{U}$ and $\mathcal{C}$ T.F.A.E.

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2 There is a sequence $0 < k_0 < k_1 < \ldots \in \mathbb{Z}^+$ for which $< U^{k_i} x, U^{k_j} x > = 0$ for $k_j \neq k_i$. 

• a) $\mathcal{U}$ and $\mathcal{C}$ are entangled,
• b) $\mathcal{U}$ and $\mathcal{C}$ have a common limit-algebra $\mathcal{M}^{\Omega} = \mathcal{M}^{\Omega_{\mathcal{C}}} = \mathcal{M}^{\Omega_{\mathcal{U}}}$ with unit $P_m$,
• c) $\mathcal{H}_m(\mathcal{C}) = \mathcal{H}_m(\mathcal{U})$.

Proof. We need only show c) implies a). If $\mathcal{H}_m(\mathcal{C}) = \mathcal{H}_m(\mathcal{U})$ then they have a common orthogonal projection $P_{mc} = P_{mu}$. Hence from [Theorem 3.1, 2] for all $x \in \mathcal{H}$,

$$M(x, \mathcal{C}) = M(x, \mathcal{C}) = M^{\Omega_{\mathcal{C}}} \cdot x = \mathcal{L}_x P_{mc} \cdot x = \mathcal{L}_x P_{mu} \cdot x = M^{\Omega_{\mathcal{U}}} P_{mu} \cdot x = M(x, \mathcal{U}),$$

and hence a) follows. □

Corollary 5.4. Since $\mathcal{U}$ and $\mathcal{C}$ are separable they are entangled if and only if

$$\mathcal{M}^{\Omega_{\mathcal{U}}} = \mathcal{L}_x P_m = \mathcal{L}_x P_m = \mathcal{M}^{\Omega_{\mathcal{C}}}.$$

Remark 5.5. By the $\mathcal{H}_0$ splitting theorem of [9 Theorem 2.5] $\mathcal{H}_0 = \mathcal{H}_m \oplus \mathcal{H}_w$. Hence as a result of [Theorem 5.3]: The weak-stability of a contraction semigroup $\mathcal{T}$ is equivalent to that of its cogenerator $T (\mathcal{H}_w(T) = \mathcal{H}_w(D))$ if and only if their unitary parts $\mathcal{U}$ and $\mathcal{C}$ have a common limit algebra (are entangled). This addresses Open Question 2.23 of T. Eisner [2, 2.23, p176].

Remark 5.6. When $\mathcal{U}$ and $\mathcal{C}$ are entangled their limit cycles can be expressed in terms of each other. For example if $\mathcal{U}$ and $\mathcal{C}$ are entangled and $\mathcal{H}_m(\mathcal{C}) = \sum \mathcal{M}(x, \mathcal{C}) = \sum \mathcal{M}^{\Omega_{\mathcal{C}}} \cdot x$ is the expansion of [4.0], then for $x_0 \in \mathcal{H}$, $V \in \mathcal{Omega}$, the limit cycles for (2.1) have the form

$$y(t, x_0) = T_0 V x_0 = \sum T_0 V T_0 x_0$$

for $T_0 \in \mathcal{M}^{\Omega}$ the common limit algebra.

Example 5.7. The Limit Algebras of [1.3]

a) Consider the semigroup of [1.3] $\mathcal{T} : T_t = U_t \oplus W_t$ for $t \geq 0$ with cogenerator $T = U \oplus W$ on $\mathcal{H} = L^2([0, 1]) \oplus L^2([\mathbb{Z}^+])$.

$U$ is the unitary operator of [9, 10] with defining spectral family $\mathcal{F} = \{F_t\}$ on $L^2([0, 1])$ with group $\mathcal{C}$. It is the cogenerator of the group $\mathcal{U} = \{U_t\}$ of [9, 10]. Using the argument of [9, Theorem 6.2] there exist subsequences $\{2^{m_r}\}$ and $\{2^{n_r}\}$ with limit operators $V \in \mathcal{Omega}$ and $W \in \mathcal{Omega}$ for which $\omega - \lim U^{2^{m_r}} = V$ and $\omega - \lim U^{2^{n_r}} = W$ and $W = V = I$.

b) The above implies $U = U I = UV \in \mathcal{Omega}$ and $U_t = U_t I = U_t V \in \mathcal{Omega}$. From the construction of [Theorem 3.5] $U_t \in \mathcal{M}^{\Omega_{\mathcal{C}}}$ and $U \in \mathcal{M}^{\Omega_{\mathcal{U}}}$. These observations imply:

$$\mathcal{M}^{\Omega_{\mathcal{C}}} \subset \mathcal{M}^{\Omega_{\mathcal{C}}} \subset \mathcal{M}^{\Omega_{\mathcal{U}}} \subset \mathcal{M}^{\Omega_{\mathcal{U}}} \subset \mathcal{M}^{\Omega_{\mathcal{C}}}$$

and hence there is a common limit algebra and by [Theorem 5.3] $\mathcal{H}_m(\mathcal{C}) = \mathcal{H}_m(\mathcal{U})$, i.e. $\mathcal{U}$ and $\mathcal{C}$ are entangled.

$$\mathcal{M}^{\Omega_{\mathcal{C}}} = \mathcal{M}^{\Omega} = \mathcal{L} \mathcal{L}_x = \mathcal{L}_x \mathcal{L}_x = \mathcal{M}^{\Omega_{\mathcal{U}}} = \mathcal{M}^{\Omega}.$$  

c) Since $\mathcal{H} = L^2([0, 1])$ is separable:

$$\mathcal{M} = \mathcal{L} \mathcal{L}_x = \mathcal{L}_x \mathcal{L}_x = \mathcal{L}_x \mathcal{L}_x = \mathcal{M}^{\Omega_{\mathcal{U}}} = \mathcal{M}^{\Omega}.$$  

Moreover for each $f \in L^2([0, 1])$, $f = I f = V f = W f \in \mathcal{H}(\mathcal{C}) \cap \mathcal{H}(\mathcal{U})$, i.e. $\mathcal{H}_m(\mathcal{C}) = \mathcal{H}_m(\mathcal{U}) = L^2([0, 1])$.

d) Suppose for $x = (f, \tilde{x}) \in \mathcal{H} \omega - \lim_{t \to \infty} T_t x = 0$. Then each of its components converge weakly to 0. But since $\mathcal{H}_w(\mathcal{U}) = \mathcal{H}_w(\mathcal{C}) = 0$ from c), $f = 0$ and hence $\mathcal{H}(\mathcal{T}) = \{0\} \oplus I^+ = \mathcal{H}_w(D)$ and $\mathcal{H}_m(\mathcal{T}) = \mathcal{H}_m((D))$, i.e. $\mathcal{T}$ and $D$ of [1.3] are entangled and hence weakly stable equivalent. The
common limit algebra of $\mathcal{T} : T_t = U_t \oplus W_t$ and cogenerator $T = U \oplus W$ in $\mathcal{L}(\mathcal{H})$ is $\mathcal{M}^{\Omega_1} = \mathcal{M}^{\Omega_2} = \mathcal{L}_E \oplus \{0\}$.

6. Bibliography

1 T. Eisner, Stability of Operators and Operator Semigroups, Preprint, Mathematisches Institut, Universitat Tubingen, Tuingen, Germany.
2 T. Eisner, Stability of Operators and Operator Semigroups, Operator Theory: Advances and Applications 209, 1st Edition. 2010. Buch. vIII, 204 S. Hardcover ISBN 978 3 0346 0194 8.
3 I. Glicksberg and K. Deleuw, Applications of Almost Periodic Compactifications, Acta. Math., 105 (1961) 63-97.
4 E. Hille, R. Phillips, Functional Analysis and Semigroups, American Mathematical Society, (1946).
5 U. Krengel, Weakly wandering vectors and weakly independent partitions, Amer. Math. Soc. Trans. 164 (1972), 199-226.
6 B. Nagy and C. Foias, Harmonic Analysis of Operators on Hilbert Space, North Holland Publishing Company, Amsterdam, 1970.
7 B. Nagy and C. Foias, Functional Analysis, Frederick Ungar Publishing Company, New York, 1971.
8 V.V. Nemitsky and V.V. Stepanov (1960) Qualitative Theory of Differential Equations, Princeton University Press, Princeton, New Jersey.
9 R. E. O’Brien, Semigroup Dynamics for Flight Vectors, Inter. J. Dyn. Sys. Diff. Eq., to appear Fall 2020.
10 R. E. O’Brien, Almost Weakly Stable Contraction Semigroups are Weakly Poisson Recurrent, Preprint, DOI:10.13140/RG.2.1.4850.6725.
11 R. E. O’Brien, Flight Vectors and Limit Operators-an Example ,Research Gate Preprint, August 2018 DOI: 10.13140/RG.2.2.28268.62084.
12 R. E. O’Brien, The Limit Algebra And Weak Stability For C 0 Contraction Semigroups And Cogenerators ,Research Gate Preprint, November 2019 DOI: 10.13140/RG.2.2.33374.79686.
13 F. Riesz, B. Sz-Nagy, Functional Analysis, Frederick Ungar Publishing Co. New York 1955.
14 S. Sakai, $C^*$ - Algebras and $W^*$ - Algebras, Springer Verlag, New York, 1971.
15 I. Segal, R. Kunze, Integrals and Operators, McGraw-Hill Book Company, New York, 1968.
16 K. Yosida, Functional Analysis, Second Edition, Springer Verlag, New York, 1968.