On Star–Wheel Ramsey Numbers

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Abstract For two given graphs $G_1$ and $G_2$, the Ramsey number $R(G_1, G_2)$ is the least integer $r$ such that for every graph $G$ on $r$ vertices, either $G$ contains a $G_1$ or $\overline{G}$ contains a $G_2$. In this note, we determined the Ramsey number $R(K_{1,n}, W_m)$ for even $m$ with $n + 2 \leq m \leq 2n - 2$, where $W_m$ is the wheel on $m + 1$ vertices, i.e., the graph obtained from a cycle $C_m$ by adding a vertex $v$ adjacent to all vertices of the $C_m$.

Keywords Ramsey number · Star · Wheel

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1 Introduction

Throughout this paper, all graphs are finite and simple. For a pair of graphs $G_1$ and $G_2$, the Ramsey number $R(G_1, G_2)$, is defined as the smallest integer $r$ such that for
every graph $G$ on $r$ vertices, either $G$ contains a $G_1$ or $\overline{G}$ contains a $G_2$, where $\overline{G}$ is the complement of $G$. Note that $R(G_1, G_2) = R(G_2, G_1)$. We denote by $P_n$ ($n \geq 1$) and $C_n$ ($n \geq 3$) the path and cycle on $n$ vertices, respectively. The bipartite graph $K_{1,n}$ ($n \geq 2$) is called a \textit{star}. The \textit{wheel} $W_n$ ($n \geq 3$) is the graph obtained from a cycle $C_m$ by adding a vertex $v$ adjacent to all vertices of the $C_m$. For two graphs $G_1$ and $G_2$, the union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is defined as $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

In this note we consider the Ramsey numbers for stars versus wheels. There are many results on this area. Hasmawati [4] determined the Ramsey number $R(K_{1,n}, W_m)$ for $m \geq 2n$.

\textbf{Theorem 1} (Hasmawati [4]) If $n \geq 2$ and $m \geq 2n$, then

$$R(K_{1,n}, W_m) = \begin{cases} n + m - 1, & \text{if both } n \text{ and } m \text{ are even;} \\ n + m, & \text{otherwise.} \end{cases}$$

So from now on we consider the case that $m \leq 2n - 1$. For odd $m$, Chen et al. [2] showed that if $m \leq n + 2$, then $R(K_{1,n}, W_m) = 3n + 1$. Hasmawati et al. [5] proved that the values remain the same even if $m \leq 2n - 1$.

\textbf{Theorem 2} (Hasmawati et al. [5]) If $3 \leq m \leq 2n - 1$ and $m$ is odd, then

$$R(K_{1,n}, W_m) = 3n + 1.$$ 

So it remains the case $m \leq 2n - 2$ and $m$ is even. Surahmat and Baskoro [7] determined the Ramsey numbers of stars versus $W_4$.

\textbf{Theorem 3} (Surahmat and Baskoro [7]) If $n \geq 2$, then

$$R(K_{1,n}, W_4) = \begin{cases} 2n + 1, & \text{if } n \text{ is even;} \\ 2n + 3, & \text{if } n \text{ is odd.} \end{cases}$$

Chen et al. [2] established $R(K_{1,n}, W_6)$, and Zhang et al. [8, 9] established $R(K_{1,n}, W_8)$.

In this note we first give a lower bound on $R(K_{1,n}, W_m)$ for even $m \leq 2n - 2$. One can check that when $m = 6, 8$, the lower bound on $R(K_{1,n}, W_m)$ in Theorem 4 is the exact value, see [2,8,9].

\textbf{Theorem 4} If $6 \leq m \leq 2n - 2$ and $m$ is even, then

$$R(K_{1,n}, W_m) \geq \begin{cases} 2n + m/2 - 1, & \text{if both } n \text{ and } m/2 \text{ are even;} \\ 2n + m/2, & \text{otherwise.} \end{cases}$$

Moreover, we establish the exact values when $n + 2 \leq m \leq 2n - 2$. We will show that the lower bound in Theorem 4 is the exact value if $m \geq n + 2$.

\textbf{Theorem 5} If $n + 2 \leq m \leq 2n - 2$ and $m$ is even, then

$$R(K_{1,n}, W_m) = \begin{cases} 2n + m/2 - 1, & \text{if both } n \text{ and } m/2 \text{ are even;} \\ 2n + m/2, & \text{otherwise.} \end{cases}$$
2 Preliminaries

We denote by $v(G)$ the order of $G$, by $\delta(G)$ the minimum degree of $G$, $c(G)$ the circumference of $G$, and $g(G)$ the girth of $G$, respectively. The graph $G$ is said to be panyclic if $G$ contains cycles of every length between 3 and $v(G)$, and weakly panyclic if $G$ contains cycles of every length between $g(G)$ and $c(G)$.

We will use the following results.

**Theorem 6** (Dirac [3]) Every 2-connected graph $G$ has circumference $c(G) \geq \min \{2\delta(G), v(G)\}$.

**Theorem 7** (Brandt et al. [1]) Every non-bipartite graph $G$ with $\delta(G) \geq (v(G)+2)/3$ is weakly panyclic and has girth 3 or 4.

**Theorem 8** (Jackson [6]) Let $G$ be a bipartite graph with partition sets $X$ and $Y$, $2 \leq |X| \leq |Y|$. If for every vertex $x \in X$, $d(x) \geq \max(|X|, |Y|/2 + 1)$, then $G$ has a cycle containing all vertices in $X$, (i.e., of length $2|X|$).

A graph $G$ is said to be $k$-regular if every vertex of $G$ has degree $k$.

**Lemma 1** Let $k$ and $n$ be two integers with $n \geq k + 1$ and $k$ or $n$ is even. Then there is a $k$-regular graph of order $n$ each component of which is of order at most $2k+1$.

**Proof** We first assume that $k + 1 \leq n \leq 2k + 1$. If $k$ is even, then let $G$ be the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and every vertex $v_i$ is adjacent to the $k$ vertices in $\{v_i \pm 1, v_i \pm 2, \ldots, v_i \pm k/2\}$, where the subscripts are taken modulo $n$. Then $G$ is a $k$-regular graph of order $n$. If $k$ is odd, then $n$ is even and $n - 1 - k$ is even. Similarly as above we can get an $(n - 1 - k)$-regular graph $H$ of order $n$. Then $G = \overline{H}$ is a $k$-regular graph of order $n$. Since $n \leq 2k + 1$, every component of $G$ has order at most $2k+1$.

Now we assume that $n \geq 2k + 2$.

If $k$ is even, then let

$$n = q(2k + 1) + r, \quad 0 \leq r \leq 2k.$$ 

Note that $q \geq 1$. If $r = 0$, then the union of $q$ copies of a $k$-regular graph of order $2k + 1$ is a required graph. If $k + 1 \leq r \leq 2k$, then the union of $q$ copies of a $k$-regular graph of order $2k + 1$ and one copy of a $k$-regular graph of order $r$ is a required graph. Now we assume that $1 \leq r \leq k$. Note that $k + 1 \leq k + r \leq 2k$. Then the union of $q - 1$ copies of a $k$-regular graph of order $2k + 1$, one copy of a $k$-regular graph of order $k + 1$, and one copy of a $k$-regular graph of order $k + r$, is a required graph.

If $k$ is odd, then $n$ is even. Let

$$n = 2kq + r, \quad 0 \leq r < 2k.$$ 

Clearly $r$ is even. If $r = 0$ then the union of $q$ copies of a $k$-regular graph of order $2k$ is a required graph. If $k + 1 \leq r < 2k$, then the union of $q$ copies of a $k$-regular graph of order $2k$ and one copy of a $k$-regular graph of order $r$ is a required graph. Now we
assume that $2 \leq r \leq k - 1$. Note that $k + 1 \leq k + r - 1 \leq 2k$. Then the union of $q - 1$ copies of a $k$-regular graph of order $2k$, one copy of a $k$-regular graph of order $k + 1$, and one copy of a $k$-regular graph of order $k + r - 1$, is a required graph. □

3 Proof of Theorem 4

For convenience we define a constant $\theta$ such that $\theta = 1$ if both $n$ and $m/2$ are even, and $\theta = 0$ otherwise. We will construct a graph $G$ of order $2n + m/2 - \theta$ such that $G$ contains no $K_{1,n}$ and $\overline{G}$ contains no $W_m$.

It is easy to check that $m/2 - 1$ or $n + m/2 - \theta - 1$ is even. By Lemma 1, there exists an $(m/2 - 1)$-regular graph $H$ of order $n + m/2 - \theta - 1$ such that each component of $H$ has order at most $m - 1$. Let $G = H \cup K_n$. Then $\nu(G) = 2n + m/2 - \theta - 1$.

We first show that $G$ contains no $K_{1,n}$. Clearly $K_n$ contains no $K_{1,n}$. Note that every vertex in $H$ has degree $m/2 - 1$, and then every vertex in $\overline{H}$ has degree $\nu(H) - 1 - m/2 + 1 = n - \theta - 1$. Thus $\overline{H}$ contains no $K_{1,n}$.

Second we show that $\overline{G}$ contains no $W_m$. Suppose to the contrary that $\overline{G}$ contains a $W_m$. Let $x$ be the hub of the $W_m$. If $x$ is contained in $K_n$, then all vertices of the wheel other than $x$ are in $V(H)$. This implies that $H$ has a cycle $C_m$. But every component of $H$ has order less than $m$, a contradiction. So we assume that $x \in V(H)$. Note that $x$ has $m/2 - 1$ neighbors in $H$. At least $m/2 + 1$ vertices of the wheel are in the $K_n$. This implies that there are two vertices in the $K_n$ such that they are adjacent in $\overline{G}$, a contradiction.

This implies that $R(K_{1,n}, W_m) \geq 2n + m/2 - \theta$. □

4 Proof of Theorem 5

Note that by our assumption $n \geq 4$ and $m \geq 6$. We already showed $R(K_{1,n}, W_m) \geq 2n + m/2 - \theta$ in Theorem 4. Now we prove that $R(K_{1,n}, W_m) \leq 2n + m/2 - \theta$ when $n + 2 \leq m \leq 2n - 2$. Let $G$ be a graph of order

$$\nu(G) = 2n + m/2 - \theta.$$ 

Suppose that $\overline{G}$ has no $K_{1,n}$, i.e.,

$$\delta(G) \geq n + m/2 - \theta. \hspace{1cm} (1)$$

We will prove that $G$ has a $W_m$. We assume to the contrary that $G$ contains no $W_m$. We choose such a graph $G$ with minimum size.

Let $u$ be a vertex of $G$ with maximum degree. Set

$$H = G[N(u)] \quad \text{and} \quad I = V(G) \setminus (\{u\} \cup N(u)).$$

Note that $\nu(H) = d(u)$. 

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Claim 1. $d(u) \geq n + m/2$; and for every $v \in V(H)$, $d(v) = n + m/2 - \theta$.

Proof If $\theta = 0$, then by (1), $d(u) \geq n + m/2$. If $\theta = 1$, then $n$ and $m/2$ are both even. Thus $v(G) = 2n + m/2 - 1$ is odd. If every vertex of $G$ has degree $n + m/2 - 1$, then $G$ has an even order, a contradiction. This implies $d(u) \geq n + m/2$.

Let $v$ be a vertex in $H$. Clearly $d(v) \geq \delta(G) \geq n + m/2 - \theta$. If $d(v) \geq n + m/2 - \theta + 1$, then $d(u) \geq d(v) \geq n + m/2 - \theta + 1$. Thus $G' = G - uv$ has size less than $G$ with $\delta(G') \geq n + m/2 - \theta$. Since $G'$ is a subgraph of $G$, it contains no $W_m$, a contradiction.

By Claim 1, we assume that

$$v(H) = n + m/2 + \tau, \quad \text{where} \quad \tau \geq 0. \quad (2)$$

Claim 2. $\delta(H) \geq m/2 + \tau$.

Proof Let $v$ be an arbitrary vertex of $H$. By Claim 1, $d(v) = n + m/2 - \theta$. Note that $v(G - H) = (2n + m/2 - \theta) - (n + m/2 + \tau) = n - \theta - \tau$. Thus

$$d_H(v) \geq d(v) - v(G - H) = (n + m/2 - \theta) - (n - \theta - \tau) = m/2 + \tau.$$

Thus the claim holds.

Claim 3. $H$ is separable.

Proof By (2) and $2n - 2 \geq m$, $v(H) \geq m \geq 3$. Suppose to the contrary that $H$ is 2-connected. By Claim 2 and Theorem 6, $c(H) \geq m$. Also note that

$$3\delta(H) \geq 3m/2 + 3\tau \geq n + m/2 + 3\tau + 2 \geq v(H) + 2,$$

i.e., $\delta(H) \geq (v(H) + 2)/3$.

If $H$ is non-bipartite, then by Theorem 7, $H$ is weakly pancyclic and of girth 3 or 4. Thus $H$ contains $C_m$. Note that $u$ is adjacent to every vertex of the $C_m$, hence $G$ contains a $W_m$, a contradiction.

If $H$ is bipartite, say with partition sets $X$ and $Y$, then $|X| \geq m/2 + \tau$ and

$$|Y| = v(H) - |X| \leq (n + m/2 + \tau) - (m/2 + \tau) = n,$$

since $\delta(H) \geq m/2 + \tau$. Let $X'$ be a subset of $X$ with $|X'| = m/2$. Note that for every vertex $x$ of $X'$,

$$d_Y(x) = d_H(x) \geq m/2 \geq n/2 + 1 \geq |Y|/2 + 1.$$

By Theorem 8, the subgraph of $H$ induced by $X' \cup Y$ contains a $C_m$. Thus $G$ contains a $W_m$, a contradiction.
If $H$ is disconnected, then $H$ has at least two components; if $H$ is connected, then $H$ has at least two end-blocks. Now let $D$ be a component or an end-block of $H$ such that $\nu(D)$ is as small as possible. We define a constant $\varepsilon$ such that $\varepsilon = 1$ if $D$ is an end-block of $H$, and $\varepsilon = 0$ otherwise. Thus

$$\nu(D) \leq (\nu(H) + \varepsilon)/2.$$ \hfill (3)

If $D$ is an end-block of $H$, then let $z$ be the cut-vertex of $H$ contained in $D$.

**Claim 4.** For every two vertices $v, w \in V(D)$ which are not cut-vertices of $H$, $|N_I(v) \cap N_I(w)| \geq m/2 - 1$.

**Proof** Note that $d_I(v) = d(v) - 1 - d_H(v) \geq d(v) - \nu(D)$, and $d_I(w) \geq d(w) - \nu(D)$.

$$|N_I(v) \cap N_I(w)| \geq d_I(v) + d_I(w) - |I| \geq d(v) + d(w) - 2\nu(D) - |I|$$

$$\geq 2\delta(G) - (\nu(H) + \varepsilon) - |I| = 2\delta(G) - \nu(G) + 1 - \varepsilon$$

$$= 2(n + m/2 - \theta) - (2n + m/2 - \theta) + 1 - \varepsilon$$

$$= m/2 + 1 - \theta - \varepsilon \geq m/2 - 1.$$

Thus the claim holds. \hfill \Box

Suppose that there is a vertex $v \in V(D)$ which is not a cut-vertex of $H$ such that $v$ has $m/2$ neighbors in $V(D)$ each of which is not a cut-vertex of $H$. Then let $X$ be the set of such $m/2$ neighbors of $v$ and $Y = \{u\} \cup N_I(v)$. Let $B$ be the bipartite subgraph of $G$ with partition sets $X$ and $Y$, and for any two vertices $x \in X$ and $y \in Y$, $xy \in E(B)$ if and only if $xy \in E(G)$.

Note that $|X| = m/2$. By Claim 4, every vertex of $X$ has at least $m/2$ neighbors in $Y$. By Claim 1 and Claim 2, $d(v) = n + m/2 - \theta$ and $d_H(v) \geq m/2 + \tau$. Thus $|Y| = d(v) - d_H(v) \leq n - \theta - \tau$. Since $m \geq n + 2, m/2 \geq |Y|/2 + 1$. By Theorem 8, $B$ contains a $C_m$. Note that $v$ is adjacent to every vertex of the $C_m$, hence $G$ has a $W_m$, a contradiction.

So we conclude that $D$ is an end-block of $H$ (i.e., $\varepsilon = 1$), and every vertex $v \in V(D) \setminus \{z\}$ has at most $m/2 - 1$ neighbors in $V(D) \setminus \{z\}$. By Claim 2, we can see that $z$ is adjacent to every vertex in $V(D) \setminus \{z\}$ and every vertex in $V(D) \setminus \{z\}$ has degree in $H$ exactly $m/2$ and $\tau = 0$.

**Claim 5.** Every vertex in $V(D) \setminus \{z\}$ is adjacent to every vertex in $I$.

**Proof** Let $v$ be a vertex in $V(D) \setminus \{z\}$. Since $d(v) = n + m/2 - \theta$ and $d_H(v) = m/2$, we have

$$d_I(v) = d(v) - 1 - d_H(v) = n - 1 - \theta.$$ 

Also note that

$$|I| = \nu(G) - 1 = \nu(H) = (2n + m/2 - \theta) - 1 - (n + m/2) = n - 1 - \theta.$$

This implies that $v$ is adjacent to every vertex in $I$. \hfill \Box
Case 1 \( N_I(z) \neq \emptyset \).

Note that \(|I| = n - 1 - \theta \geq m/2 - 1\). Let \( v \in V(D) \setminus \{z\} \) and \( u_1, u_2, \ldots, u_{m/2-1} \) be \( m/2 - 1 \) vertices in \( I \) such that \( z u_1 \in E(G) \), and let \( v_1, v_2, \ldots, v_{m/2-1} \) be \( m/2 - 1 \) vertices in \( N_D(v) \setminus \{z\} \). Then \( uz u_1 v_1 u_2 v_2 \cdots u_{m/2-1} v_{m/2-1} u \) is a \( C_m \). Since \( v \) is adjacent to every vertex of the \( C_m \), \( G \) contains a \( C_m \), a contradiction.

Case 2 \( N_I(z) = \emptyset \) and \( G[I] \) is not empty.

Let \( v \in V(D) \setminus \{z\} \) and \( u_1, u_2, \ldots, u_{m/2-1} \) be \( m/2 - 1 \) vertices in \( I \) such that \( u_1 u_2 \in E(G) \), and let \( v_1, v_2, \ldots, v_{m/2-1} \) be \( m/2 - 1 \) vertices in \( N_D(v) \setminus \{z\} \). Then \( uz v_1 u_1 v_2 u_2 v_3 \cdots u_{m/2-1} v_{m/2-1} u \) is a \( C_m \). Since \( v \) is adjacent to every vertex of the \( C_m \), \( G \) contains a \( C_m \), a contradiction.

Case 3 \( N_I(z) = \emptyset \) and \( G[I] \) is empty.

Let \( w \) be an arbitrary vertex in \( I \). Note that \( w \) is nonadjacent to every vertex in \( \{u, z\} \cup I \). Hence
\[
d(w) \leq v(G) - 2 - |I| = (2n + m/2 - 2) - (n - 1 - \theta) = n + m/2 - 1.
\]
Since \( d(w) \geq \delta(G) = n + m/2 - \theta \), we can see that \( \theta = 1 \) and \( w \) is adjacent to every vertex of \( V(H) \setminus \{z\} \). Moreover, every vertex in \( I \) is adjacent to every vertex in \( V(H) \setminus \{z\} \).

Since \( \theta = 1 \), by Claim 1, \( d(u) = n + m/2 \) and \( d(z) = n + m/2 - 1 \). Thus there is a vertex \( x \in V(H) \setminus \{z\} \) such that \( x z \notin E(G) \). By Claim 2, let \( v_1, v_2, \ldots, v_{m/2} \) be \( m/2 \) vertices in \( N_H(x) \) and \( u_1, u_2, \ldots, u_{m/2} \) be \( m/2 \) vertices in \( \{u\} \cup I \). Then \( u_1 v_1 u_2 v_2 \cdots u_{m/2} v_{m/2} u_1 \) is a \( C_m \). Since \( x \) is adjacent to every vertex of the \( C_m \), \( G \) contains a \( W_m \), a contradiction.

The proof is complete.

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