Padé approximant related to the Wallis formula

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Abstract

Based on the Padé approximation method, in this paper we determine the coefficients $a_j$ and $b_j$ such that

$$
\pi = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \left\{ \frac{n^k + a_1 n^{k-1} + \cdots + a_k}{n^k + b_1 n^{k-1} + \cdots + b_k} + O\left(\frac{1}{n^{2k+3}}\right) \right\}, \quad n \to \infty,
$$

where $k \geq 0$ is any given integer. Based on the obtained result, we establish a more accurate formula for approximating $\pi$, which refines some known results.

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1 Introduction

It is well known that the number $\pi$ satisfies the following inequalities:

$$
\frac{2}{2n+1} \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 < \pi < \frac{1}{n} \left(\frac{(2n)!!}{(2n-1)!!}\right)^2, \quad n \in \mathbb{N} := \{1, 2, 3, \ldots\},
$$

where

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!, \quad (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

This result is due to Wallis (see [1]).

Based on a basic theorem in mathematical statistics concerning unbiased estimators with minimum variance, Gurland [1] yielded a closer approximation to $\pi$ than that afforded by (1.1), namely,

$$
\frac{4n+3}{(2n+1)^2} \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 < \pi < \frac{4}{4n+1} \left(\frac{(2n)!!}{(2n-1)!!}\right)^2, \quad n \in \mathbb{N}.
$$

By using (1.2), Brutman [2] and Falaleev [3] established estimates of the Landau constants.
Mortici [4], Theorem 2, improved Gurland’s result (1.2) and obtained the following double inequality:

\[
\left( \frac{n + \frac{1}{4}}{n^2 + \frac{1}{2}n + \frac{3}{32}} + \frac{9}{2048n^5} - \frac{45}{8192n^6} \right) \left( \frac{(2n)!!}{(2n-1)!!} \right)^2 < \pi < \left( \frac{n + \frac{1}{4}}{n^2 + \frac{1}{2}n + \frac{3}{32}} + \frac{9}{2048n^5} \right) \left( \frac{(2n)!!}{(2n-1)!!} \right)^2, \quad n \in \mathbb{N}, \quad (1.3)
\]

We see from (1.3) that

\[
\pi = \left( \frac{(2n)!!}{(2n-1)!!} \right)^2 \left\{ \frac{n + \frac{1}{4}}{n^2 + \frac{1}{2}n + \frac{3}{32}} + \mathcal{O} \left( \frac{1}{n^5} \right) \right\}, \quad n \to \infty.
\]

(1.4)

Based on the Padé approximation method, in this paper we develop the approximation formula (1.4) to produce a general result. More precisely, we determine the coefficients \(a_j\) and \(b_j\) such that

\[
\pi = \left( \frac{(2n)!!}{(2n-1)!!} \right)^2 \left\{ \frac{n^k + a_1 n^{k-1} + \cdots + a_k}{n^{k+1} + b_1 n^k + \cdots + b_{k+1}} + \mathcal{O} \left( \frac{1}{n^{2k+3}} \right) \right\}, \quad n \to \infty,
\]

(1.5)

where \(k \geq 0\) is any given integer. Based on the obtained result, we establish a more accurate formula for approximating \(\pi\), which refines some known results.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2 Lemmas

Euler’s gamma function \(\Gamma(x)\) is one of the most important functions in mathematical analysis and has applications in diverse areas. The logarithmic derivative of \(\Gamma(x)\), denoted by \(\psi(x) = \Gamma'(x)/\Gamma(x)\), is called the psi (or digamma) function.

The following lemmas are required in the sequel.

Lemma 2.1 ([5]) \textit{Let} \(r \neq 0\) \textit{be a given real number and} \(\ell \geq 0\) \textit{be a given integer. The following asymptotic expansion holds:}

\[
\frac{\Gamma(x + 1)}{\Gamma(x + \frac{1}{2})} \sim \sqrt{x} \left( 1 + \sum_{j=1}^{\infty} \frac{p_j}{x^j} \right)^{x/r}, \quad x \to \infty,
\]

(2.1)

\textit{with the coefficients} \(p_j \equiv p_j(\ell, r) \quad (j \in \mathbb{N})\) \textit{given by}

\[
p_j = \sum_k \frac{r^{k_1 + k_2 + \cdots + k_j}}{k_1! k_2! \cdots k_j!} \left( \frac{2^2 - 1}{1 \cdot 1 \cdot 2^2} \right)^{k_1} \left( \frac{2^4 - 1}{2 \cdot 3 \cdot 2^4} \right)^{k_2} \cdots \left( \frac{2^{2j} - 1}{j(2j - 1)2^{2j}} \right)^{k_j},
\]

(2.2)

\textit{where} \(B_i\) \textit{are the Bernoulli numbers summed over all nonnegative integers} \(k_j\) \textit{satisfying the equation}

\[
(1 + \ell) k_1 + (3 + \ell) k_2 + \cdots + (2j + \ell - 1) k_j = j.
\]
In particular, setting \((\ell, r) = (0, -2)\) in (2.1) yields
\[
x \left( \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \right)^2 \sim 1 + \sum_{j=1}^{\infty} c_j x^j, \quad x \to \infty,
\]
where the coefficients \(c_j \equiv p_j(0, -2)\) \((j \in \mathbb{N})\) are given by
\[
c_j = \sum \frac{(-2)^{k_1+k_2+\ldots+k_j}}{k_1!k_2!\ldots k_j!} \left( \frac{2^2 - 1)B_{2j}}{1 \cdot 1 \cdot 2^2} \right)^{k_1} \left( \frac{(2^4 - 1)B_{4j}}{2 \cdot 3 \cdot 2^4} \right)^{k_2} \ldots \left( \frac{(2^{2j} - 1)B_{2^j}}{j(2j-1)2^{2j}} \right)^{k_j},
\]
summed over all nonnegative integers \(k_j\) satisfying the equation
\[
k_1 + 3k_2 + \ldots + (2j-1)k_j = j.
\]

**Lemma 2.2** ([5]) Let \(m, n \in \mathbb{N}\). Then, for \(x > 0\),
\[
\sum_{j=1}^{2m} \left( 1 - \frac{1}{2^{2j}} \right) \frac{2B_{2j} (2j + n - 2)!}{x^{j+n-1}} \left( \psi^{(n-1)}(x + 1) - \psi^{(n-1)} \left( x + \frac{1}{2} \right) \right) + \frac{(n-1)!}{2x^n} < \sum_{j=1}^{2m-1} \left( 1 - \frac{1}{2^{2j}} \right) \frac{2B_{2j} (2j + n - 2)!}{x^{j+n-1}}.
\]
In particular, we have
\[
U(x) < \psi(x + 1) - \psi \left( x + \frac{1}{2} \right) < V(x),
\]
where
\[
V(x) = \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^3} - \frac{1}{128x^4} + \frac{17}{2048x^5} - \frac{31}{2048x^6} + \frac{691}{16,384x^7} - \frac{5,461}{32,768x^8} + \frac{929,569}{1,048,576x^9}.
\]
and
\[
U(x) = V(x) - \frac{3,202,291}{524,288x^8}.
\]

For our later use, we introduce Padé approximant (see [6–11]). Let \(f\) be a formal power series
\[
f(t) = c_0 + c_1 t + c_2 t^2 + \cdots.
\]
The Padé approximation of order \((p, q)\) of the function \(f\) is the rational function, denoted by
\[
[p/q]_t(t) = \frac{\sum_{j=0}^{p} a_j t^j}{1 + \sum_{j=1}^{q} b_j t^j},
\]
where \( p \geq 0 \) and \( q \geq 1 \) are two given integers, the coefficients \( a_j \) and \( b_j \) are given by (see [6–8, 10, 11])

\[
\begin{align*}
  a_0 &= c_0, \\
  a_1 &= c_0 b_1 + c_1, \\
  a_2 &= c_0 b_2 + c_1 b_1 + c_2, \\
  &\vdots \\
  a_p &= c_0 b_p + \cdots + c_{p-1} b_1 + c_p, \\
  0 &= c_{p+1} + c_p b_1 + \cdots + c_{p+1} b_q, \\
  &\vdots \\
  0 &= c_{p+q} + c_{p+q-1} b_1 + \cdots + c_p b_q,
\end{align*}
\]

(2.9)

and the following holds:

\[
[p/q] f(t) - f(t) = O(t^{p+q+1}).
\]

(2.10)

Thus, the first \( p + q + 1 \) coefficients of the series expansion of \([p/q] f\) are identical to those of \( f \). Moreover, we have (see [9])

\[
[p/q] f(t) = \left[ \begin{array}{cccc}
  \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} & \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} & \cdots & \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \\
  \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} & \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} & \cdots & \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} & \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} & \cdots & \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \\
  \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} & \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} & \cdots & \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \\
\end{array} \right] \left[ \begin{array}{c}
  f_0(t) \\
  f_1(t) \\
  \vdots \\
  f_p(t) \\
\end{array} \right],
\]

(2.11)

with \( f_0(x) = c_0 + c_1 x + \cdots + c_n x^n \), the \( n \)th partial sum of the series \( f \) in (2.7).

### 3 Main results

Let

\[
f(x) = x \left( \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \right)^2.
\]

(3.1)

It follows from (2.3) that, as \( x \to \infty \),

\[
f(x) \sim \sum_{j=0}^{\infty} \frac{c_j}{x^j} = \frac{1}{4x} + \frac{1}{32x^2} + \frac{1}{128x^3} - \frac{5}{2,048x^4} - \frac{23}{8,192x^5} + \frac{53}{65,536x^6} - \cdots,
\]

(3.2)

with the coefficients \( c_j \) given by (2.4). In what follows, the function \( f \) is given in (3.1).
Based on the Padé approximation method, we now give a derivation of formula (1.4). To this end, we consider

\[
[1/2]_f(x) = \frac{\sum_{j=0}^{1} a_j x^{-j}}{1 + \sum_{j=1}^{2} b_j x^{-j}}.
\]

Noting that

\[
c_0 = 1, \quad c_1 = -\frac{1}{4}, \quad c_2 = \frac{1}{32}, \quad c_3 = \frac{1}{128}
\]

holds, we have, by (2.9),

\[
\begin{align*}
a_0 &= 1, \\
a_1 &= b_1 - \frac{1}{8}, \\
0 &= \frac{1}{32} - \frac{1}{4} b_1 + b_2, \\
0 &= \frac{1}{128} + \frac{1}{32} b_1 - \frac{1}{4} b_2,
\end{align*}
\]

that is,

\[
a_0 = 1, \quad a_1 = \frac{1}{4}, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{3}{32}.
\]

We thus obtain that

\[
[1/2]_f(x) = \frac{1 + \frac{1}{4 x}}{1 + \frac{1}{2 x} + \frac{3}{32 x^2}},
\]

and we have, by (2.10),

\[

x \left( \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \right)^2 = \frac{1 + \frac{1}{4 x}}{1 + \frac{1}{2 x} + \frac{3}{32 x^2}} = O \left( \frac{1}{x^3} \right), \quad x \to \infty.
\]

Noting that

\[
\frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)} = \sqrt{\pi} \cdot \frac{(2n-1)!!}{(2n)!!}, \quad n \in \mathbb{N} \text{ (the Wallis ratio)}
\]

holds, replacing \( x \) by \( n \) in (3.4) yields (1.4).

From the Padé approximation method introduced in Section 2 and the asymptotic expansion (3.2), we obtain a general result given by Theorem 3.1. As a consequence, we obtain (1.5).

**Theorem 3.1** The Padé approximation of order \((p, q)\) of the asymptotic formula of the function \( f(x) = x (\Gamma(x + \frac{1}{2})^2) \) (at the point \( x = \infty \)) is the following rational function:

\[
[p/q]_f(x) = \frac{1 + \sum_{j=1}^{p} a_j x^{-j}}{1 + \sum_{j=1}^{q} b_j x^{-j}} = x \left( \frac{x^p + a_1 x^{p-1} + \cdots + a_p}{x^q + b_1 x^{q-1} + \cdots + b_q} \right),
\]

(3.6)
where \( p \geq 0 \) and \( q \geq 1 \) are two given integers and \( q = p + 1 \) (an empty sum is understood to be zero), the coefficients \( a_i \) and \( b_i \) are given by

\[
\begin{align*}
  a_1 &= b_1 + c_1, \\
  a_2 &= b_2 + c_1 b_1 + c_2, \\
  &\vdots \\
  a_p &= b_p + \cdots + c_{p-1} b_1 + c_p, \\
  0 &= c_{p+1} + c_p b_1 + \cdots + c_{p-q+1} b_q, \\
  &\vdots \\
  0 &= c_{p+q} + c_p b_1 + \cdots + c_p b_q,
\end{align*}
\]

and \( c_j \) is given in (2.4), and the following holds:

\[
f(x) - \left[ \frac{p}{q} \right] f(x) = O \left( \frac{1}{x^{p+q+1}} \right), \quad x \to \infty. \tag{3.8}
\]

Moreover, we have

\[
[p/q]f(x) = \left| \begin{array}{cccc}
  \frac{1}{n!} f(x) & \frac{1}{n!} f_1(x) & \cdots & \frac{1}{n!} f_q(x) \\
  \frac{1}{n!} f_{p-q+1} & \frac{1}{n!} f_{p-q+2} & \cdots & \frac{1}{n!} f_{p+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{1}{n!} f_{q-p+1} & \frac{1}{n!} f_{q-p+2} & \cdots & \frac{1}{n!} f_{q+1} \\
  \frac{1}{n!} f_{r-p+1} & \frac{1}{n!} f_{r-p+2} & \cdots & \frac{1}{n!} f_{r+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{1}{n!} f_{r-q+1} & \frac{1}{n!} f_{r-q+2} & \cdots & \frac{1}{n!} f_{r+q} \\
\end{array} \right|,
\]

with \( f_n(x) = \sum_{j=0}^{n} \frac{c_j}{x^j} \), the \( n \)th partial sum of the asymptotic series (3.2).

**Remark 3.1** Using (3.9), we can also derive (3.3). Indeed, we have

\[
[1/2]f(x) = \left| \begin{array}{cccc}
  \frac{1}{n!} f(x) & \frac{1}{n!} f_1(x) & \cdots & \frac{1}{n!} f_q(x) \\
  \frac{1}{n!} f_{p-q+1} & \frac{1}{n!} f_{p-q+2} & \cdots & \frac{1}{n!} f_{p+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{1}{n!} f_{q-p+1} & \frac{1}{n!} f_{q-p+2} & \cdots & \frac{1}{n!} f_{q+1} \\
  \frac{1}{n!} f_{r-p+1} & \frac{1}{n!} f_{r-p+2} & \cdots & \frac{1}{n!} f_{r+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{1}{n!} f_{r-q+1} & \frac{1}{n!} f_{r-q+2} & \cdots & \frac{1}{n!} f_{r+q} \\
\end{array} \right| = 1 + \frac{1}{2x} + \frac{3}{32x^2}.
\]

Replacing \( x \) by \( n \) in (3.8) applying (3.5), we obtain the following corollary.

**Corollary 3.1** As \( n \to \infty \),

\[
\pi = \left( \frac{(2n)!}{(2n-1)!} \right)^2 \left\{ \frac{n^p + \sum_{j=1}^{p} a_j n^{p-j} + O\left( \frac{1}{n^{p+q+2}} \right)}{n^q + \sum_{j=1}^{q} b_j n^{q-j}} \right\}, \quad n \to \infty, \tag{3.10}
\]

where \( p \geq 0 \) and \( q \geq 1 \) are two given integers and \( q = p + 1 \), and the coefficients \( a_i \) and \( b_i \) are given by (3.7).
Remark 3.2 Setting \((p, q) = (k, k + 1)\) in (3.10) yields (1.5).

Setting

\[(p, q) = (4, 5) \quad \text{and} \quad (p, q) = (5, 6)\]

in (3.10), respectively, we find

\[
\pi = \left( \frac{(2n)!!}{(2n - 1)!!} \right)^2 \left\{ \frac{n^4 + n^3 + \frac{107}{64} n^2 + \frac{91}{128} n + \frac{789}{4096}}{n^5 + \frac{5}{2} n^4 + \frac{125}{64} n^3 + \frac{295}{256} n^2 + \frac{1689}{4096} n + \frac{945}{16384}} + O\left( \frac{1}{n^{11}} \right) \right\} \quad (3.11)
\]

and

\[
\pi = \left( \frac{(2n)!!}{(2n - 1)!!} \right)^2 \left\{ \frac{n^5 + \frac{5}{2} n^4 + \frac{125}{64} n^3 + \frac{295}{256} n^2 + \frac{1689}{4096} n + \frac{945}{16384}}{n^6 + \frac{3}{2} n^5 + \frac{131}{32} n^4 + \frac{93}{32} n^3 + \frac{729}{4096} n^2 + \frac{4881}{8192} n + \frac{10395}{131072}} + O\left( \frac{1}{n^{13}} \right) \right\} \quad (3.12)
\]

as \(n \to \infty\).

Formulas (3.11) and (3.12) motivate us to establish the following theorem.

Theorem 3.2 The following inequality holds:

\[
\frac{x^5 + \frac{5}{2} x^4 + \frac{51}{16} x^3 + \frac{133}{64} x^2 + \frac{5243}{4096} x + \frac{3867}{16384}}{x^6 + \frac{3}{2} x^5 + \frac{113}{32} x^4 + \frac{93}{32} x^3 + \frac{729}{4096} x^2 + \frac{4881}{8192} x + \frac{10395}{131072}} \times \left( \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \right)^2
\]

\[
< \frac{x^4 + x^3 + \frac{107}{64} x^2 + \frac{91}{128} x + \frac{789}{4096}}{x^5 + \frac{5}{2} x^4 + \frac{125}{64} x^3 + \frac{295}{256} x^2 + \frac{1689}{4096} x + \frac{945}{16384}}.
\]  

(3.13)

The left-hand side inequality holds for \(x \geq 4\), while the right-hand side inequality is valid for \(x \geq 3\).

Proof It suffices to show that

\[F(x) > 0 \quad \text{for} \quad x \geq 4 \quad \text{and} \quad G(x) < 0 \quad \text{for} \quad x \geq 3,\]

where

\[F(x) = 2 \ln \left( \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \right) - \ln \frac{x^5 + \frac{5}{2} x^4 + \frac{51}{16} x^3 + \frac{133}{64} x^2 + \frac{5243}{4096} x + \frac{3867}{16384}}{x^6 + \frac{3}{2} x^5 + \frac{113}{32} x^4 + \frac{93}{32} x^3 + \frac{729}{4096} x^2 + \frac{4881}{8192} x + \frac{10395}{131072}}\]

and

\[G(x) = 2 \ln \left( \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \right) - \ln \frac{x^4 + x^3 + \frac{107}{64} x^2 + \frac{91}{128} x + \frac{789}{4096}}{x^5 + \frac{5}{2} x^4 + \frac{125}{64} x^3 + \frac{295}{256} x^2 + \frac{1689}{4096} x + \frac{945}{16384}}.\]
Using the following asymptotic expansion (see \[12\]):

\[
\left[ \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \right]^2 \sim \frac{1}{x} \exp \left( -\frac{1}{4x} + \frac{1}{96x^3} - \frac{1}{320x^5} + \frac{17}{7,168x^7} - \frac{31}{9,216x^9} \right. \\
+ \frac{691}{90,112x^{11}} - \frac{5,461}{212,992x^{13}} + \frac{929,569}{7,864,320x^{15}} - \cdots, \quad x \to \infty, \quad (3.14)
\]

we obtain that

\[
\lim_{x \to \infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} G(x) = 0.
\]

Differentiating \(F(x)\) and applying the first inequality in (2.6), we find

\[
F'(x) = -2 \left[ \psi(x + 1) - \psi \left( x + \frac{1}{2} \right) \right] + \frac{P_{10}(x)}{P_{11}(x)}
\]

\[
< -2U(x) + \frac{P_{10}(x)}{P_{11}(x)} = -\frac{P_{16}(x - 4)}{524,288x^{18}P_{11}(x)},
\]

where

\[
P_{10}(x) = 4\left( 20,998,323 + 301,244,208x + 1,329,622,624x^2 + 3,532,111,872x^3 \\
+ 6,831,390,720x^4 + 8,950,906,880x^5 + 9,510,060,032x^6 \\
+ 6,476,005,376x^7 + 4,244,635,648x^8 + 1,342,177,280x^9 + 536,870,912x^{10} \right),
\]

\[
P_{11}(x) = \left( 16,384x^5 + 20,480x^4 + 52,224x^3 + 34,048x^2 + 20,972x + 3,867 \right) \\
\times \left( 131,072x^6 + 196,608x^5 + 462,848x^4 + 380,928x^3 + 247,328x^2 \\
+ 78,096x + 10,395 \right)
\]

and

\[
P_{16}(x) = 73,399,302,245,132,658,732,474 + 401,687,666,421,636,714,876,048x \\
+ 882,663,824,965,187,436,960,169x^2 \\
+ 1,129,813,735,156,766,429,414,420x^3 \\
+ 975,385,167,000,268,446,720,384x^4 \\
+ 611,802,531,654,753,268,270,848x^5 \\
+ 290,696,674,545,996,984,221,376x^6 \\
+ 107,149,026,028,490,487,475,968x^7 \\
+ 31,018,031,026,615,120,693,760x^8 \\
+ 7,080,024,048,117,231,228,928x^9 \\
+ 1,270,066,473,244,063,756,800x^{10} + 177,136,978,237,041,715,200x^{11} \\
+ 18,824,726,793,935,462,400x^{12} + 1,473,208,721,923,276,800x^{13}
\]
Hence, $F'(x) < 0$ for $x \geq 4$, and we have

$$F(x) > \lim_{t \to \infty} F(t) = 0, \quad x \geq 4.$$ 

Differentiating $G(x)$ and applying the second inequality in (2.6), we find

$$G'(x) = -2 \left[ \psi(x + 1) - \psi \left( x + \frac{1}{2} \right) \right] + \frac{4P_8(x)}{P_9(x)} > -2V(x) + \frac{4P_8(x)}{P_9(x)}$$

$$= \frac{P_{14}(x-3)}{524,288x^{16}P_9(x)},$$

where

$$P_8(x) = 16,777,216x^8 + 33,554,432x^7 + 72,351,744x^6 + 79,167,488x^5 + 75,583,488x^4$$

$$+ 45,043,712x^3 + 18,211,328x^2 + 4,212,480x + 644,661,$$

$$P_9(x) = (4,096x^4 + 4,096x^3 + 6,848x^2 + 2,912x + 789)$$

$$\times (16,384x^7 + 20,480x^6 + 32,000x^5 + 18,880x^4 + 6,756x + 945)$$

and

$$P_{14}(x) = 427,884,340,806,856,575 + 5,508,337,280,234,438,700x$$

$$+ 16,278,641,070,340,979,232x^2$$

$$+ 25,110,186,749,213,013,376x^3 + 25,009,399,125,661,680,960x^4$$

$$+ 17,642,792,222,808,253,696x^5$$

$$+ 9,230,356,959,310,493,184x^6 + 3,661,094,552,739,530,752x^7$$

$$+ 1,108,535,832,992,448,000x^8$$

$$+ 255,024,028,762,675,200x^9 + 43,854,087,132,979,200x^{10}$$

$$+ 5,462,018,666,496,000x^{11}$$

$$+ 465,495,496,704,000x^{12} + 24,287,993,856,000x^{13}$$

$$+ 585,252,864,000x^{14}.$$ 

Hence, $G'(x) > 0$ for $x \geq 3$, and we have

$$G(x) < \lim_{t \to \infty} G(t) = 0, \quad x \geq 3.$$ 

The proof is complete. \qed
Corollary 3.2 For \( n \in \mathbb{N} \),

\[ a_n < \pi < b_n, \quad (3.15) \]

where

\[ a_n = \frac{n^5 + \frac{5}{2} n^4 + \frac{51}{16} n^3 + \frac{133}{64} n^2 + \frac{5.243}{0.096} n + \frac{3.867}{0.384}}{n^6 + \frac{5}{2} n^5 + \frac{111}{32} n^4 + \frac{93}{32} n^3 + \frac{7.757}{0.096} n^2 + \frac{4.881}{0.384} n + \frac{10.395}{0.384}} \left( \frac{(2n)!!}{(2n-1)!!} \right)^2, \quad (3.16) \]

and

\[ b_n = \frac{n^4 + n^3 + \frac{107}{64} n^2 + \frac{91}{128} n + \frac{789}{4.096} n + \frac{945}{0.384}}{n^5 + \frac{5}{2} n^4 + \frac{125}{64} n^3 + \frac{297}{256} n^2 + \frac{1.689}{0.096} n + \frac{945}{0.384}} \left( \frac{(2n)!!}{(2n-1)!!} \right)^2. \quad (3.17) \]

Proof Noting that (3.5) holds, we see by (3.13) that the left-hand side of (3.15) holds for \( n \geq 4 \), while the right-hand side of (3.15) is valid for \( n \geq 3 \). Elementary calculations show that the left-hand side of (3.15) is also valid for \( n = 1, 2 \) and 3, and the right-hand side of (3.15) is valid for \( n = 1 \) and 2. The proof is complete. \( \square \)

4 Comparison

Recently, Lin [12] improved Mortici’s result (1.3) and obtained the following inequalities:

\[ \lambda_n < \pi < \mu_n \quad (4.1) \]

and

\[ \delta_n < \pi < \omega_n, \quad (4.2) \]

where

\[ \lambda_n = \left( 1 + \frac{1}{4n} - \frac{3}{32n^2} + \frac{3}{128n^3} + \frac{3}{2.048n^4} - \frac{33}{8.192n^5} - \frac{39}{65.536n^6} \right) \times \left( \frac{2}{2n+1} \left( \frac{(2n)!!}{(2n-1)!!} \right)^2 \right), \quad (4.3) \]

\[ \mu_n = \left( 1 + \frac{1}{4n} - \frac{3}{32n^2} + \frac{3}{128n^3} + \frac{3}{2.048n^4} \right) \frac{2}{2n+1} \left( \frac{(2n)!!}{(2n-1)!!} \right)^2, \quad (4.4) \]

\[ \delta_n = \frac{(2n)!!}{(2n-1)!!} \left( \frac{1}{n} \exp \left( -\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{-17}{7.168n^7} - \frac{-31}{9.216n^9} \right) \right), \quad (4.5) \]

\[ \omega_n = \frac{(2n)!!}{(2n-1)!!} \left( \frac{1}{n} \exp \left( -\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7.168n^7} \right) \right). \quad (4.6) \]

Direct computation yields

\[ a_n - \lambda_n = \frac{3(7,634,944n^5 + 12,928,000n^4 + 18,895,616n^3 + 9,755,072n^2 + 1,930,008n + 135,135)}{32,768n(2n+1)(131,072n^6 + 196,608n^5 + 462,848n^4 + 380,928n^3 + 247,328n^2 + 78,096n + 10,395)} \times \left( \frac{(2n)!!}{(2n-1)!!} \right)^2 > 0 \]
Table 1  Comparison between inequalities (3.15) and (4.2)

| n    | \( a_n - \delta_n \)    | \( \omega_n - b_n \)    |
|------|-------------------------|-------------------------|
| 1    | \( 6.673798 \times 10^{-3} \) | \( 3.789512 \times 10^{-3} \) |
| 10   | \( 2.264856 \times 10^{-13} \) | \( 9.947343 \times 10^{-12} \) |
| 100  | \( 2.398663 \times 10^{-24} \) | \( 1.051407 \times 10^{-20} \) |
| 1,000| \( 2.408054 \times 10^{-35} \) | \( 1.056218 \times 10^{-29} \) |
| 10,000 | \( 2.408948 \times 10^{-46} \) | \( 1.056690 \times 10^{-38} \) |

and

\[
b_n - \mu_n = \frac{3(45,056n^4 + 62,976n^3 + 66,496n^2 + 21,876n + 945)}{1,024n^4(2n + 1)(16,384n^2 + 20,480n^2 + 32,000n^2 + 18,880n^2 + 6,756n + 945)} \left( \frac{(2n)!}{(2n-1)!} \right)^2 < 0.
\]

Hence, (3.15) improves (4.1).

The following numerical computations (see Table 1) would show that \( \delta_n < a_n \) and \( b_n < \omega_n \) for \( n \in \mathbb{N} \). That is to say, inequalities (3.15) are sharper than inequalities (4.2).

In fact, we have

\[
\lambda_n = \pi + O\left(\frac{1}{n^7}\right), \quad \mu_n = \pi + O\left(\frac{1}{n^7}\right),
\]

\[
\delta_n = \pi + O\left(\frac{1}{n^{11}}\right), \quad \omega_n = \pi + O\left(\frac{1}{n^{11}}\right),
\]

\[
a_n = \pi + O\left(\frac{1}{n^{12}}\right), \quad b_n = \pi + O\left(\frac{1}{n^{10}}\right).
\]

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The authors declare that they have no competing interests.

Authors’ contributions
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