On observational signatures of multi-fractional theory

Mahnaz Asghari\textsuperscript{a,b}, Ahmad Sheykhi\textsuperscript{a,b}
\textsuperscript{a}Department of Physics, College of Sciences, Shiraz University, Shiraz 71454, Iran
\textsuperscript{b}Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran

Abstract
We study the multi-fractional theory with \( q \)-derivatives, where the multi-fractional measure is considered to be in the time direction. The evolution of power spectra and also the expansion history of the universe are investigated in the \( q \)-derivatives theory. According to the matter power spectra diagrams, the structure growth would increase in the multi-fractional model, expressing incompatibility with low redshift measurements of large scale structures. Furthermore, concerning the diagrams of Hubble parameter evolution, there is a reduction in the value of Hubble constant which conflicts with local cosmological constraints. Thus, primary numerical investigations imply that \( q \)-derivatives theory has no potential to relieve observational tensions. We also explore the multi-fractional model with current observational data, principally Planck 2018, weak lensing, supernovae, baryon acoustic oscillations (BAO), and redshift-space distortions (RSD) measurements. Numerical analysis reveals that the degeneracy between multi-fractional parameters makes them remain unconstrained under observations. Furthermore, observational constraints on \( H_0 \) and \( \sigma_8 \), detect no significant departure from standard model of cosmology.

Keywords: Cosmology, Multi-fractional theory

1. Introduction
In light of considerable astrophysical and cosmological observations, it is certain that \( \Lambda \)CDM model affords the most appropriate explanation of the universe. The concordance \( \Lambda \)CDM model described by dark matter and also the cosmological constant \( \Lambda \) as dark energy, is introduced in the framework of general relativity (GR) after the discovery of accelerated expansion of the universe \cite{1, 2}. The success of standard cosmological model has been also confirmed by the majority of observational measurements including the cosmic microwave background (CMB) anisotropies \cite{3, 8}, large scale structures \cite{9, 11}, and baryon acoustic oscillations (BAO) \cite{12, 14}. However, despite such accomplishments, it is known that there are some tensions between the inferred values of cosmological parameters from local and global observational measurements. In detail, results concerning direct determinations of Hubble constant indicate significant discrepancies from CMB data based on \( \Lambda \)CDM model \cite{15, 18}. The most recent local measurement performed by Hubble Space Telescope (HST) and the SH0ES collaboration reports \( H_0 = 73.30 \pm 1.04 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} \) \cite{19}, which is in 5\( \sigma \) difference with Planck 2018 results \cite{8}. Also there is a less significant discrepancy known as \( \sigma_8 \) tension, where the value of structure growth parameter \( \sigma_8 \) predicted from low redshift observations is inconsistent with Planck data \cite{20}. Thus, these observational tensions might imply that there is a possibility for new physics beyond the standard \( \Lambda \)CDM model. In this direction, one can consider multi-fractional scenarios which are categorized to four independent types of theories, mainly ordinary, weighted, \( q \) and fractional derivatives \cite{21, 23}. Formerly, R. A. El-Nabulsi proposed a formalism known as the fractional action-like variational approach to discuss the application of fractional calculus in cosmology \cite{24, 26}. More investigations on implications of fractional approach in GR and cosmology are also performed in literature \cite{27, 36, 37}. Thereafter, multi-fractional theories in which the geometry is characterized by a fundamental scale, were proposed to improve the physical interpretation of quantum gravity \cite{38, 40}. For some related studies on multi-fractional theories refer to \cite{41, 47}. In the present study, we are interested in multi-fractional theory with \( q \)-derivatives, which describes a more intuitive multifractal spacetime \cite{48, 53}. We revise the observational tensions in the framework of multi-fractional theory, and also investigate the \( q \)-derivatives theory with cosmological probes.

The organization of our paper is as follows. In section 2 we explain modified field equations in multi-fractional spacetime. Section 3 contains numerical results, as well as derived observational constraints on cosmological parameters. We present our conclusions in section 4.

2. Field equations in multi-fractional theory with \( q \)-derivatives

The theory with \( q \)-derivatives is characterized by replacing the coordinates \( x^i \) by the multi-fractional profile \( q^i(x^i) \) given by \cite{21, 53}

\[
q^i(x^i) = \int \, \text{d}x^{(\mu)} \nu_{(\mu)}(x^i),
\]

(1)
where there is no contraction on index $\mu$ in the parenthesis. Correspondingly, by using $\frac{\partial}{\partial \nu(x)} = \frac{1}{v(x)} \frac{\partial}{\partial \nu}$, the metric connection and the Riemann tensor in fractional frame are defined as [21][53]

$$g_{\mu\nu} = \frac{1}{2} \delta^{\alpha\beta}(\frac{1}{v(x)} \frac{\partial}{\partial \nu} g_{\mu\nu} + \frac{1}{v(x)} \frac{\partial}{\partial \nu} g_{\mu\nu} - \frac{1}{v(x)} \frac{\partial}{\partial \nu} g_{\mu\nu})$$

(2)

$$R_{\mu\nu\rho\sigma} = \frac{1}{v(x)} \frac{\partial}{\partial \nu} g_{\mu\rho} - \frac{1}{v(x)} \frac{\partial}{\partial \nu} g_{\mu\sigma} + \frac{\partial^2}{\partial \nu \partial \nu} g_{\mu\rho} - R_{\mu\nu} g_{\rho\sigma}$$

(3)

Also, the total action in the fractional frame takes the form [21][46][53]

$$S = \frac{1}{16\pi G} \int d^4 x \sqrt{-g_R} (R - 2\Lambda) + S_m$$

(4)

where the action measure is [46][49]

$$d^4 q(x) = \Pi_{\rho=0} d\xi^\rho (x^\nu) = d^4 x \Pi_{\rho=0} \nu(x^\nu) = d^4 x v(x)$$

(5)

so we have

$$S = \frac{1}{16\pi G} \int d^4 x v(x) \sqrt{-g_R} (R - 2\Lambda) + S_m$$

(6)

where $S = \Pi_{\rho=0} \nu(x^\nu) = v_0(x^0) v_1(x^1) v_2(x^2) v_3(x^3)$ [21][22].

Action (6) results in the following field equations in multi-fractal cosmology [21]

$$q R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (q R - 2\Lambda) = 8\pi G q T_{\mu\nu}$$

(7)

where $q R_{\mu\nu}$ is $q R_{\mu\nu}$ and $q R = g_{\mu\nu} q R_{\mu\nu}$ [53].

It is more convenient to contemplate multi-fractal structure along each direction $x^\nu$ as described in the literature [51], however for the sake of simplicity, in this work we consider it only in the time direction where $v(x) $ defined as

$$v(\tau) = 1 + (\beta_{\alpha} a^\nu_0)$$

(8)

in which $\alpha > 0$ is the fractional exponent and $\beta_{\alpha} \geq 0$ is a dimensionless constant (for more information on the range of the fractional exponent $\alpha$ refer to [53]). According to equation (8) it is evident that multi-fractal effects are more significant in late time, as also illustrated in figure [51] obtained from the modified version of the CLASS code [53].

In the following, we investigate multi-fractal theory in a flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe, described by the metric

$$ds^2 = a^2(\tau) [-d\tau^2 + (\delta_{ij} + h_{ij}) d\chi^i d\chi^j]$$

(9)

in the synchronous gauge, where

$$h_{ij}(\chi, \tau) = \int d^3 k e^{ik\cdot\chi} [\delta_{ij} + 3\partial_i h_{ij}] 6\pi(\kappa, \tau)$$

(10)

with $\kappa = kk$, in which $h$ and $\eta$ are scalar perturbations [55]. Likewise, conformal Newtonian gauge is represented by the metric

$$\frac{d^2}{d\tau^2} = a^2(\tau) (-(1 + 2\Psi) r^2 + (1 - 2\Phi) d\tilde{z}^2)$$

(11)

with gravitational potentials $\Psi$ and $\Phi$ [55]. We also assume the energy content of the universe as a perfect fluid with the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + g_{\mu\nu} p$$

(12)

which is similar to GR, since considering multi-fractal only in the time direction. So, field equations in the theory with $q$-derivatives in background level take the form

$$\frac{1}{v^2} H^2 = \frac{8\pi G}{3} \sum_i \frac{\rho_i}{v}$$

(13)

$$\frac{1}{v^2} \left[ \frac{H}{v} \frac{v'}{v} - 2 \frac{H^2}{v} - 3H^2 \right] = 8\pi G \frac{\sum_i \rho_i}{v}$$

(14)

in which a prime indicates a deviation with respect to the conformal time, and $H = a'/a^2$ is the Hubble parameter. Then, from equation (13), the total density parameter for a universe consists of radiation (R), baryons (B), dark matter (DM) and cosmological constant ($\Lambda$) can be written as

$$\Omega_{eq} = \frac{1}{v^2}$$

(15)

Moreover, it is possible to write modified field equations to linear order of perturbations in synchronous gauge (syn), given by

$$\frac{1}{v^2} \frac{a'}{a} h'' - 2k^2 \eta = 8\pi G a^3 \sum_i \delta \rho_{i(syn)}$$

(16)

$$\frac{1}{v^2} k^2 \eta' = 4\pi G a^2 \sum_i (\rho_i + p_i) \theta_i(syn)$$

(17)

$$\frac{1}{v^2} \left( \frac{1}{2} h'' + 3\eta'' + \left( - \frac{1}{2} \frac{v'}{v} + \frac{a'}{a} h' + 6\eta' \right) - k^2 \eta = 0 \right.$$

(18)

while in conformal Newtonian gauge (con) we have

$$\frac{3}{v^2} \left( \frac{a'}{a} \Phi' + \left( \frac{a'}{a} \right)^2 \Psi \right) + k^2 \Phi = -4\pi G a^2 \sum_i \delta \rho_{i(con)}$$

(20)

$$\frac{1}{v^2} \left( k^2 \Phi' + \frac{a'}{a} k^2 \Psi \right) = 4\pi G a^2 \sum_i (\rho_i + p_i) \theta_i(con)$$

(21)

$$\Phi = \Psi$$

(22)

$$\frac{1}{v^2} \left( \frac{v'}{v} - 2\Psi + \frac{a'}{a} \Phi' + \Psi \left( \frac{2\eta''}{a} - \left( \frac{a'}{a} \right)^2 \right) + \frac{a'}{a} (\Psi' + 2\Phi) + \Phi'' \right)$$

$$+ \frac{1}{v^2} (\Phi - \Psi) = 4\pi G a^2 \sum_i \delta \rho_{i(con)}$$

(23)

$^2$Cosmic Linear Anisotropy Solving System
Furthermore, conservation equations in the theory with $q$-derivatives for $i$th component of the universe in background and perturbation levels are

\[
\rho_i' + 3\frac{a'}{a}\rho_i(1 + w_i) = 0,
\]

\[
\delta_i^{(\text{syn})} = -3\frac{a'}{a}(c_i^2 - w_i)\delta_{i(\text{syn})} - \frac{1}{2}(1 + w_i)h'
- (1 + w_i)\left[9\frac{(a')^2}{a^2}(c_i^2 - c_i^2)\frac{1}{k^2} + v\right]\theta_{i(\text{syn})},
\]

\[
\theta_i^{(\text{syn})} = \frac{a'}{a} \left[3(w_i + c_i^2 - c_i^2) - 1\right]\theta_{i(\text{syn})} + \frac{k^2 c_i^2}{1 + w_i}\delta_{i(\text{syn})}.
\]

Also, it is clear that choosing $v = 1$ (or correspondingly $\beta = 0$) recovers field equations in standard cosmology. It should be mentioned that in the rest of the paper, we will investigate multi-fractional theory in the synchronous gauge.

### 3. Results

This section is devoted to numerical study of the theory with $q$-derivatives, by employing a modified version of the CLASS code\cite{54} according to field equations in multi-fractional cosmology. Furthermore, we use multi-fractional theory with observations by using the MCMC\cite{55} package MONTE PYTHON\cite{56,57}.

Figure 1: Left panel shows $v(\tau)$ in term of conformal time for different values of $\beta$ compared to standard model of cosmology, where $\alpha = 1$, and right panel shows analogous diagrams for different values of $\alpha$, while $\beta = 0.08$.

Here, we are interested to modify the publicly available code CLASS to accommodate the multi-fractional field equations explained in section\cite{2}. We consider Planck 2018 data\cite{3} for cosmological parameters, in numerical study.

The CMB temperature anisotropy diagrams in theory with $q$-derivatives compared to standard cosmological model, are displayed in figure\cite{2}. In upper panels we can see the $TT$ component of CMB power spectra for different values of $\beta$, while $\alpha = 1$; and correspondingly, lower panels show power spectra for different values of $\alpha$, where $\beta = 0.08$. According to this figure, it can be understood that there is more deviation from $\Lambda$CDM model for larger values of $\beta$, or equivalently, smaller values of $\alpha$, when the other parameter is fixed. This feature is also predictable from modified field equations described in section\cite{2}.

In figure\cite{3} we illustrate the matter power spectra diagrams in multi-fractional theory comparing with $\Lambda$CDM model. Upper panels depict power spectra for different values of $\beta$, in which $\alpha = 1$; while lower panels demonstrate power spectra diagrams considering different values of $\alpha$, where $\beta = 0.08$. This figure expresses an enhancement in the growth of structure for theory with $q$-derivatives, considering larger values of $\beta$, or correspondingly, smaller values of $\alpha$ (when the other parameter is fixed), which is in contrast with low redshift structure formation\cite{20}.

Furthermore, it is exciting to contemplate the expansion history of the universe in multi-fractional theory. Figure\cite{4} depicts the evolution of Hubble parameter in theory with $q$-derivatives, which represents a suppression in the current value of $H$ compared to standard cosmological model (considering larger values of $\beta$, or equivalently, smaller values of $\alpha$). Thus, we can see that Hubble tension might become more serious in multi-fractional cosmology.

### 3.2. Observational constraints

Now we turn our attention to confronting the multi-fractional theory with current observations by making use of the public MCMC package MONTE PYTHON. Accordingly, the baseline parameter set we consider in MCMC method includes $\{100\Omega_{m0}, H_0^2\}$.
Figure 2: Upper panels show the CMB power spectra diagrams (left) and their relative ratio with respect to ΛCDM model (right) for different values of β, considering α = 1. Lower panels show analogous diagrams for different values of α, regarding β = 0.08.

Figure 3: Upper panels show the matter power spectra diagrams (left) and their relative ratio with respect to standard cosmological model (right) for different values of β, considering α = 1. Lower panels show analogous diagrams for different values of α, where β = 0.08.

Ω_{DM,0}h^2, 100 θ_s, ln(10^{10} A_s), n_s, τ_{reio}, β, α], where Ω_{B,0}h^2 and Ω_{DM,0}h^2 indicate the baryon and cold dark matter densities re-
respectively, $\theta_s$ is the ratio of the sound horizon to the angular diameter distance at decoupling, $A_s$ represents the amplitude of the primordial scalar perturbation spectrum, $n_s$ indicates the scalar spectral index, $\tau_{\text{reion}}$ stands for the optical depth to reionization, and multi-fractional parameters $\beta$ and $\alpha$ measure deviations from standard cosmology. Additionally, there are four derived parameters consistent with reionization redshift ($\tau_{\text{reion}}$), the matter density parameter ($\Omega_{\text{m,0}}$), the Hubble constant ($H_0$), and the root-mean-square mass fluctuations on scales of 8 h$^{-1}$ Mpc ($\sigma_8$). Also, based on preliminary numerical analysis, we choose the prior range $[0, 0.1]$ for $\beta$, and the prior range $[1, 5]$ for fractional exponent $\alpha$.

In order to put constraints on cosmological parameters, we consider the following likelihoods in MCMC analysis: the Planck likelihood with Planck 2018 data (containing high-$l$ TT,TE,EE, low-$l$ EE, low-$l$ TT, and lensing) [8], the Planck-SZ likelihood for the Sunyaev-Zeldovich (SZ) effect measured by Planck [58, 59], the CFHTLenS likelihood with the weak lensing data [60, 61], the Pantheon likelihood with the supernovae data [62], the BAO likelihood with the baryon acoustic oscillations data [63, 64], and the BAORSD likelihood for BAO and redshift-space distortions (RSD) measurements [65, 66].

It is worth mentioning that using different likelihoods to investigate cosmological models beyond the standard $\Lambda$CDM paradigm, can introduce unknown bias and inconsistency which should be considered and evaluated carefully. So it is important to exercise caution and test the reliability and suitability of these likelihoods in case of studying non-standard cosmological models. It is known that the Planck 2018 data is the most reliable and powerful probe to test cosmological models. Moreover, the BAO and supernovae datasets are appropriate probes to constrain the expansion history of the universe. Thus, it is convenient in the literature to apply the Planck likelihood combined with BAO and/or supernovae datasets to investigate beyond $\Lambda$CDM model. For some recent related investigations refer to e.g. [67-70]. On the other hand, the Planck SZ cluster counts data, weak lensing data, and RSD measurements could prove useful in constraining the structure growth parameter $\sigma_8$. The Planck SZ and weak lensing measurements constrain a combination of cosmological parameters $S_8 \equiv \sigma_8 (\Omega_m/\Omega_m,\text{fiducial})^{\alpha}$ (where $\alpha$ indicates the degeneracy direction), in which $\Lambda$CDM is assumed as the fiducial model. So it is expected that the obtained data to be dependent on the fiducial $\Lambda$CDM model. However, the cluster counts and weak lensing measurements usually consider some non-linear effects to constrain $S_8$, and since we are interested in linear perturbations in our study, we can assume that there is no need to modify the mass bias in this paper. Furthermore, the RSD measurements constrain the combination $f\sigma_8$, where $f$ is the growth rate. The combination of $f\sigma_8$ is independent of bias [71], so there is no need to consider modifications on bias parameter in using this likelihood. There are also some studies in the literature which have been used the combined Planck, BAO, supernovae, and RSD dataset to explore non-standard cosmological models [72-74]. Overall, while we acknowledge that using different likelihoods to test non-standard cosmological models can introduce bias or inconsistency, we believe that it is convenient to apply a combination of the above mentioned likelihoods in our MCMC analysis, without considering any substantial modifications on bias parameter.

The report on observational constraints imposed by the "Planck + Planck-SZ + CFHTLenS + Pantheon + BAO + BAORSD" dataset is presented in table [1]. The fitting results correspond to selected parameters of multi-fractional model are also shown in figure [5]. Concerning numerical results, there is a degeneracy between multi-fractional parameters $\beta$ and $\alpha$, which makes them remain unconstrained under observational data. Accordingly, in the direction of obtaining better constraints on these parameters, it is recommended that we fix one of them in MCMC analysis. Thereupon, table [2] presents corresponding results, where we have fixed $\beta$ to 0.01, and then $\alpha$ to 4, which are close to derived best fit values displayed in table [1]. However, MCMC results show that multi-fractional parameters are degenerate with other cosmological parameters, and so still remain unconstrained. The results are more apparent in figure [6], which demonstrates marginalized 1σ and 2σ confidence limit contours. On the other hand, observational constraints on $H_0$ and $\sigma_8$ represent no significant deviation from standard cosmological model.
Table 1: Best fit values of cosmological parameters with the 1σ and 2σ confidence levels from "Planck + Planck-SZ + CFHTLenS + Pantheon + BAO + BAORSD" dataset for ΛCDM and the theory with q-derivatives.

| parameter | 1σ values | 2σ values | 1σ values | 2σ values |
|-----------|-----------|-----------|-----------|-----------|
| $100\Omega_{k,0}h^2$ | 2.264 | 2.264 | 2.264 | 2.264 |
| $\Omega_{DM,0}h^2$ | 0.1166 | 0.1166 | 0.1165 | 0.1165 |
| $100\theta_s$ | 1.042 | 1.042 | 1.042 | 1.042 |
| $\ln(10^{10}A_s)$ | 3.034 | 3.034 | 3.034 | 3.034 |
| $n_s$ | 0.9712 | 0.9712 | 0.9708 | 0.9708 |
| $\tau_{reio}$ | 0.05358 | 0.05358 | 0.04628 | 0.04628 |
| $\beta$ | — | — | 0.01029 | 0.01029 |
| $\alpha$ | — | — | 4.524 | 4.524 |
| $\Omega_{k,0}$ | 7.502 | 7.502 | 7.502 | 7.502 |
| $\Omega_{k,0}h^2$ | 0.02871 | 0.02871 | 0.0287 | 0.0287 |
| $H_0$ (km/s/Mpc) | 69.56 | 69.54 | 69.54 | 69.54 |
| $\sigma_8$ | 0.8079 | 0.8044 | 0.8022 | 0.8022 |

Figure 5: The 1σ and 2σ constraints on some selected cosmological parameters of multi-fractal model compared to ΛCDM.

4. Conclusions

This work is devoted to explore the multi-fractal theory with q-derivatives by cosmological probes. Multi-fractal theories are considered to improve the renormalization properties of perturbative quantum gravity. In q-derivatives theory the coordinates are replaced by the multi-fractal profile $q^t(x^t)$, which results in modified field equations described in section 2. We concentrate on the multi-fractal measure in the time direction $t(\tau)$ as defined in equation (5), which is more effective in late time. Considering numerical analysis based on the modified version of the CLASS code, according to the q-derivatives theory, it is found that observational tensions would not be relieved in multi-fractal theory with q-derivatives. Actually, matter power spectra diagrams report larger structure formation compared to standard cosmological model, which exhibits inconsistency with local measurements of structure growth. On the other hand there is a suppression in the value of Hubble constant in q-derivatives theory depicted in figure 7, disclosing more tensions with low redshift estimations of this parameter. Hence, according to primary numerical results, multi-fractal theory with q-derivatives is not effective in addressing observed tensions between low-redshift measurements and CMB data. To be more precise, we also perform an MCMC calculation using CMB, weak lensing, supernovae, BAO, and RSD data, to constrain cosmological parameters. Numerical results indicate that because of the degeneracy between multi-fractal parameters $\beta$ and $\alpha$ with each other and also with other cosmological parameters, it is not possible to put constraints on them by observational data. Moreover, concerning MCMC analysis, obtained constraints on $H_0$ and $\sigma_8$ report no considerable departure from ΛCDM model.
Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No new data were generated or analysed during the current study.

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Figure 6: The 1σ and 2σ constraints on some selected cosmological parameters of multi-fractional model, while β is fixed to 0.01 (left panel) and α is fixed to 4 (right panel).
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