FRONTS AND INTERFACES
IN
BISTABLE EXTENDED MAPPINGS

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Abstract

We study the interfaces’ time evolution in one-dimensional bistable extended
dynamical systems with discrete time. The dynamics is governed by the competition
between a local piece-wise affine bistable mapping and any couplings given by the
convolution with a function of bounded variation. We prove the existence of traveling
wave interfaces, namely fronts, and the uniqueness of the corresponding selected
velocity and shape. This selected velocity is shown to be the propagating velocity
for any interface, to depend continuously on the couplings and to increase with
the symmetry parameter of the local nonlinearity. We apply the results to several
examples including discrete and continuous couplings, and the planar fronts’ dy-
namics in multi-dimensional Coupled Map Lattices. We eventually emphasize on
the extension to other kinds of fronts and to a more general class of bistable ex-
tended mappings for which the couplings are allowed to be nonlinear and the local
map to be smooth.

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1 Introduction

The fronts in bistable discrete dynamical systems defined on a one-dimensional lat-
tice have been investigated in various models such as in Lattice Dynamical Systems
(LDS)\textsuperscript{3}, or in Coupled Map Lattices (CML)\textsuperscript{4, 5}. These models can be viewed
as being in the class of (extended) dynamical systems for which the time evolution
is given by

\[ u^{t+1} = \mathcal{L}u^t + \mathcal{L}'f \circ u^t, \]

where \( u^t \) is a lattice configuration at time \( t \in \mathbb{N} \) (or \( \mathbb{Z} \)), and \( \mathcal{L} \) and \( \mathcal{L}' \) are some linear,
continuous and homogeneous operators, the so-called couplings. The mapping \( F \) is
the direct product of a (local) bistable one-dimensional map \( f \), i.e. the dynamics
induced by \( f \) has two stable fixed points. In LDS \( \mathcal{L} \) is often chosen to be the
Laplacian to model a diffusive process and \( \mathcal{L}' \) is a multiple of the identity\textsuperscript{6, 7, 8}. On the opposite, in CML
the diffusion is included in \( \mathcal{L}' \) and \( \mathcal{L} \) is generally assigned
to zero\textsuperscript{9, 10, 11}. In all these models, the dynamics is governed by the competition
between the fixed points along the lattice and then results in pattern formation.
To describe the fronts in these space-time discrete systems, one has to consider bounded configurations depending on a spatio-temporal continuous variable, the so-called (front) shapes [5]. It is then useful to extend the domain of \( L \) and of \( L' \), and therefore the phase space of these extended systems, to bounded functions of a real variable. By doing so, one obtains a more general class of extended dynamical systems with discrete time for which, to each point of the real line is associated a bistable map, and these maps interact through \( L \) and \( L' \). This new class contains in particular LDS and CML.

In this work, to obtain explicit expressions, the local map is chosen to be piece-wise affine. However we shall see how the existence of fronts can be stated for some bistable models with a smooth, say \( C^\infty \), local map. Up to a linear change of variable, it is always possible to assume the two local fixed points to be 0 and 1. The resulting expression is then (see Figure 1)

\[
f(x) = ax + (1 - a)H_c(x),
\]

where \( a \in [0, 1), c \in (0, 1] \) and

\[
H_c(x) = \begin{cases} 0 & \text{if } x < c \\ 1 & \text{if } x \geq c \end{cases},
\]

is the Heaviside function at \( c \).

By combining relations (1) and (2), one obtains the general expression for a piece-wise affine bistable extended mapping

\[
u^{t+1} = L_1u^t + L_2H_c \circ u^t,
\]

The goal of this article is to describe the dynamics of fronts and of interfaces in this system. Some general conditions are imposed (below) to \( L_1 \) and to \( L_2 \) in order to ensure the model to be interpreted as a bistable mapping with homogeneous couplings. We shall also see how the results apply to the description of planar fronts in multi-dimensional CML.

The plan of the article is as follows. We begin by defining the dynamical system and particularly the couplings under consideration. Still in Section 2, a representation of couplings in terms of convolutions is given and several related properties

\[1\] In all the paper, \( H_\omega \) stands for the Heaviside function at \( \omega \).
are reminded or established. The definition and the existence of fronts are exposed in Section 2. In this section, we also investigate the set of possible fronts velocities and the set of the symmetry parameter $c$ for the (non-)existence of fronts. Furthermore by introducing a distance in the set of functions associated with the couplings, the selected velocity is shown to vary continuously with the latter. In the following section, we prove that any interface, i.e. any orbit for which the state at each time is a configuration linking two different phases, always propagates with the fronts velocity. Section 5 deals with applying this formalism to the analysis of the fronts velocity in a LDS and in the continuous purely diffusive model. Moreover we describe how the planar fronts’ dynamics in multi-dimensional CML enters in this framework. Finally the results stated in the plateaus, i.e. the intervals in $c$ for which the velocity is constant, are extended to bistable extended mappings with nonlinear couplings and smooth local maps. We also prove the existence of other fronts as the unstable ones.

2 Definitions and preliminary results

2.1 The dynamical system

First of all, the phase space is chosen to be the space, called $\mathcal{M}$, of bounded Borel measurable functions on $\mathbb{R}$, endowed with the uniform norm

$$\|u\| = \sup_{x \in \mathbb{R}} |u(x)|.$$  

In this phase space, the action of the dynamics on the orbits $\{u^t\}_{t \geq 0}$ is given by the relation (3) where $L_1$ and $L_2$ are linear continuous operators satisfying the following conditions

(i) $L_1$ and $L_2$ are s-homogeneous
(ii) $L_1$ and $L_2$ are positive
(iii) $(L_1 + L_2)1 = 1$ and $L_2 \neq 0$,

where 1 stands for the function $u(x) = 1$. For $\omega \in \mathbb{R}$ and $u \in \mathcal{M}$, let

$$\sigma^\omega u(x) = u(x - \omega) \quad \forall x \in \mathbb{R},$$

be the shift (or translation) operator. An operator $L$ with domain $\mathcal{M}$ is said to be homogeneous if

$$L\sigma^\omega = \sigma^\omega L \quad \forall \omega \in \mathbb{R},$$

and is said to be s-homogeneous if it is homogeneous and commutes with the pointwise limit of equi-bounded sequences in $\mathcal{M}$, i.e. if $\{u_n\}_{n \in \mathbb{N}}$, $u_n \in \mathcal{M}$ is such that

$$\exists m \in \mathbb{R} : \sup_{n \in \mathbb{N}} \|u_n\| < m \quad \text{and} \quad \forall x \in \mathbb{R} \quad \lim_{n \to +\infty} u_n(x) = u(x),$$

then

$$\forall x \in \mathbb{R} \quad \lim_{n \to +\infty} (Lu_n)(x) = (Lu)(x).$$

The s-homogeneity reflects the homogeneous and the punctual characteristics of the dynamics (3). An operator $L$ is positive if $\forall u \in \mathcal{M}$ such that $u \geq 0$, we have $Lu \geq 0$. The condition (ii), together with (iii), indicates that the dynamics (3) can be interpreted as the competition between a local reaction and a global diffusive process. The condition (iii) guarantees the invariance of the local fixed points by the couplings.

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2u $\geq 0$ means $u(x) \geq 0 \quad \forall x \in \mathbb{R}$. 

After having defined the dynamics using $s$-homogeneous operators, we now obtain an integral representation of the latter using the Lebesgue-Stieltjes integral. The following statement reveals the existence of a bijection between the $s$-homogeneous operators on $\mathcal{M}$ and the right continuous functions of bounded variation vanishing at $-\infty$. This set of functions will be denoted by $BV_0$.

**Proposition 2.1** A linear and continuous operator $L$ of $\mathcal{M}$ into itself is $s$-homogeneous iff it exists a unique function $h \in BV_0$, such that
\[
(6) \quad \forall u \in \mathcal{M}, \quad (Lu)(x) = \int_{\mathbb{R}} u(x-y) dh(y) \quad \forall x \in \mathbb{R}.
\]

Moreover $L$ is positive iff $h$ is increasing.

Proof: If the integral (6) defines $L$, then the Lebesgue dominated convergence theorem shows that $L$ is $s$-homogeneous. Conversely let $h(x) = LH_0(x)$. Then by relations (4) and (5), $h$ is right continuous and $\lim_{x \to -\infty} h(x) = 0$. By using (4) and the linear continuity of $L$, we conclude that $h$ is of bounded variation in $\mathbb{R}$ (similarly as in the proof of the Riesz theorem in [10]). Then (6) is true for all the functions of the form
\[
u = \sum_{n=1}^{N} s_n \left( H_{x_{n-1}} - H_{x_n} \right).
\]
Any continuous function is the point-wise limit of a sequence of functions of this form. Relation (6) then holds in the (Baire) class $B_0$ of all bounded continuous functions. By the same argument, it is also valid in the class $B^1$ of functions which are point-wise limits of functions in $B^0$. We can proceed in this way transfinitely to conclude that the proposition is true for any bounded function in a Baire class $B^{\alpha}$ (where $\alpha$ is a countable ordinal). But $\bigcup_{\alpha} B^{\alpha}$ is exactly the set of all Borel measurable functions $\mathbb{I}$. Let us recall that, for a function $h \in BV_0$, the convolution is defined by
\[
\forall u \in \mathcal{M} \quad h \ast u(x) = \int_{\mathbb{R}} u(x-y) dh(y) \quad \forall x \in \mathbb{R}.
\]
We have adopted an unusual ordering in this definition to indicate that the convolution by $h \in BV_0$ corresponds to the action of an operator in $\mathcal{M}$. Moreover notice that the convolution $h \ast u$ is defined even if $u \notin BV_0$ and is thus in general non-commutative.

We then obtain an equivalent expression for the dynamical system (3)
\[
u^{t+1} = h_1 \ast u^t + h_2 \ast H_c \circ u^t, \quad (7)
\]
where $h_1$ (resp. $h_2$) corresponds to $L_1$ (resp. $L_2$) according to Proposition 2.1 and is thus an increasing function in $BV_0$. We denote by $\mathcal{I}$ the subset of $BV_0$ and of $\mathcal{M}$ composed with right continuous increasing functions vanishing at $-\infty$. This alternative expression for the dynamical system, combined with the properties of the convolution and of the functions in $\mathcal{I}$, both described in the following section, will prove to be convenient for the analysis of fronts. To conclude this section, we mention that the interfaces’ dynamics in a similar but space-time continuous bistable extended dynamical system, i.e. a first order in time differential equation for which the interaction is given by the convolution with a $C^1$ function, has been investigated in [3].

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3Requiring the functions to be bounded is not restrictive because any unbounded Borel measurable function is a point-wise limit of bounded Borel measurable functions. Hence we can consider the classes $B^{\alpha}$ containing only bounded functions.
2.2 Basic properties

Using Fubini theorem, it is shown that the convolution is commutative and associative in $BV_0$. As a consequence, two any linear continuous and s-homogeneous operators commute.

The following properties are a direct consequence of the definition

(i) If $h_1, h_2 \in \mathcal{I}$, then $h_1 * h_2 \in \mathcal{I}$ and $\|h_1 * h_2\| = \|h_1\| \|h_2\|.$

(ii) Let $h \in \mathcal{I}$ and $u_1, u_2 \in \mathcal{M}$. If $u_1 \leq u_2,$ then $h * u_1 \leq h * u_2.$

(iii) $\forall u \in \mathcal{M} \ : \ \sigma(u) = H_\omega * u,$

(iv) $\forall \omega, \omega' \in \mathbb{R} \ : \ H_\omega * H_{\omega'} = H_{\omega + \omega'}.$

Moreover let the left closure of a real subset $A$ be defined by

$$\overline{A} = \bigcap_{\delta > 0} A + [0, \delta).$$

Using the diagonal process, one shows two basic properties of the left closure.

(v) $\overline{A + B} = \overline{A} + \overline{B}$.

(vi) $\bigcup_{i \in \mathbb{N}} \overline{A}_i = \bigcup_{i \in \mathbb{N}} A_i.$

We now introduce two sets which (partly) characterize the right continuous increasing functions. For the interpretation in terms of coupling, they determine the sets of points coupled to 0 and thus, using the homogeneity, the generalized lattice on which the coupling acts. For $h \in \mathcal{I}$ the set of increase points is defined as follows.

$$E(h) = \{ x \in \mathbb{R} : \forall \delta > 0 \ : \ h(x) > h(x - \delta) \}.$$  

The set of discontinuity points, a subset of $E(h)$, is given by

$$D(h) = \{ x \in \mathbb{R} : \ h(x) > h(x - 0) \}.$$  

where $h(x - 0) = \lim_{\delta \to 0, \delta > 0} h(x - \delta).$ As a first result, one has a sufficient condition for $E(h)$ to reduce to $D(h)$.

**Lemma 2.2** If $h \in \mathcal{I}$ and $E(h)$ is countable then $E(h) = D(h)$.

**Proof:** If $h \in \mathcal{I}$, then it is equal to the sum of a continuous function $h_c$ and a step function $h_d$ which are such that $h_c, h_d \in \mathcal{I}$ and $D(h) = E(h_d) = E(h)$. Hence $E(h) \setminus D(h) \subset E(h_c)$. Now by assumption let $\{a_n\}_{n \in \mathbb{N}}$ be an enumeration of $E(h_c)$. We obtain

$$\int_{E(h_c)} dh_c = \sum_{n \in \mathbb{N}} h_c(a_n) - h_c(a_n - 0) = 0,$$

because $\mu_h(E(h)^c) = 0$, where $\mu_h$ is the Lebesgue-Stieltjes measure associated with $h \in \mathcal{I}$, and where $A^c$ denotes the complement of $A$ in $\mathbb{R}$. As a consequence $h_c = 0$ and $h = h_d$.

Moreover, the evolution of these two sets under the convolution and the summation is given in the following statement.

**Proposition 2.3** (1) Let $h, h' \in \mathcal{I}$. Then $h * h' \in \mathcal{I}$ and

$$(1a) \quad E(h * h') = E(h) + E(h'),
(1b) \quad D(h * h') = D(h) + D(h').$$

\footnote{i.e. if $u_1(x) \leq u_2(x) \ \forall x \in \mathbb{R}$}

\footnote{$A + B = \{x + y : x \in A \text{ and } y \in B\}$}
(2) Let \( \{h_i\}_{i \in \mathbb{N}} \) be a uniformly summable sequence of functions in \( \mathcal{I} \). Then \( h = \sum_{i \in \mathbb{N}} h_i \in \mathcal{I} \) and
\begin{align*}
(2a) \quad E(h) & = \bigcup_{i \in \mathbb{N}} E(h_i), \\
(2b) \quad D(h) & = \bigcup_{i \in \mathbb{N}} D(h_i).
\end{align*}

Proof: We only show relation (1a) here. Relation (1b) can be proved similarly. Statement (2b) is a direct consequence of uniform convergence and (2a) follows from (8) below.

For an increasing function, the definition of \( E(h) \) implies that for a given \( \delta > 0 \),
\[ h(x) - h(x - \delta) > 0 \iff (x - \delta, x] \cap E(h) \neq \emptyset. \tag{8} \]

Consequently given \( h, h' \in \mathcal{I} \) we have for any \( x \in \mathbb{R} \) and \( \delta > 0 \)
\[ h \ast h'(x) - h \ast h'(x - \delta) = \int_{(-E(h)+(x-\delta,x]) \cap E(h')} h(x-y) - h(x-y-\delta) dh'(y), \]
from which we deduce the relation
\[ \forall \delta > 0 \quad h \ast h'(x) - h \ast h'(x - \delta) > 0 \iff \forall \delta > 0 \ x \in E(h) + E(h') + [0, \delta). \]

This equivalence is nothing but (1a). \( \square \)

3 The fronts

With the previous definitions and properties provided, we are now able to analyse the dynamics of the simplest structures linking two different phases, namely the fronts. In this section the latter are firstly defined. Then we state their existence, in particular by considering various conditions on the couplings \( h_1 \) and \( h_2 \) and on the parameter \( c \). We also prove the continuous dependence of their velocity on the couplings using an adequate topology. We finally consider the existence of anti-fronts by relating it to the fronts’ existence.

3.1 The existence of fronts

Definition 3.1 A front of velocity \( v \) is an orbit of (7) defined by
\[ u^t = H_{c t + x_0} \ast \phi \quad \forall t \in \mathbb{Z}, \]
where \( x_0 \in \mathbb{R} \) and the shape \( \phi \in \mathcal{M} \) is such that
\[ H_c \circ \phi = H_o. \tag{9} \]

This definition is somehow restrictive because, for the sake of simplicity in this section, it imposes \( \phi(0) \geq c \). However, we shall see in Section 6 how to relax this condition by imposing the shape to satisfy, instead of (9), the condition \( \phi(x) < c \) if \( x < 0 \) and \( \phi(x) \geq c \) if \( x > 0 \), and therefore by letting \( \phi(0) \) arbitrary.

Because of the translational invariance in (7), if it exists a front, then there is also an infinite number of fronts with the same velocity and the same shape, one for each value of \( x_0 \).

The phases of the system are the fixed points 0 and 1 which compete together when coupled.
As we shall see below in the proof of Theorem 3.2, this definition implies that the shape $\phi$ is a right continuous increasing function which depends on the velocity. Moreover it behaves asymptotically as

$$\lim_{z \to -\infty} \phi(z) = 0 \quad \text{and} \quad \lim_{z \to +\infty} \phi(z) = 1.$$ 

The fronts thus defined are actually the travelling waves resulting from the motion of an interface between the two different phases of the dynamical system under consideration.

The natural questions about the fronts deal with their existence, their uniqueness and the selected velocity given two any couplings $h_1$ and $h_2$ and a value of the symmetry parameter $c$. It is also of interest to determine the set $P$ of possible velocities when $c$ varies in $(0, 1]$, and the couplings are kept fixed.

**Theorem 3.2** Given two couplings $h_1$ and $h_2$, there is a Baire first category set $G$ such that for any $c \in (0, 1) \setminus G$, the dynamical system $\mathcal{H}$ has a unique front shape. The corresponding fronts have a velocity $\bar{v}(c)$ which increases with $c$ and which can be extended to a left continuous function on $(0, 1)$. The set $P$ of possible velocities is a dense subset of the range of the function $\bar{v}$ and has the following bounds

$$\inf P = v_{\min} \quad \text{and} \quad \sup P = v_{\max},$$

where $v_{\min} = \min \{\inf E(h_1), \inf E(h_2)\}$ and $v_{\max} = \max \{\sup E(h_1), \sup E(h_2)\}$.

Assuming some additional (general) conditions on $h_1$ and $h_2$, the set $G$ is shown to be countable and nowhere dense (see section 3.4). If $v_{\min} = \inf E(h_1)$ and $v_{\max} = \sup E(h_1)$, then $P$ is dense in $(v_{\min}, v_{\max})$ and therefore $\bar{v}$ is a continuous function of $c$ (see Section 3.3). If the function $h_2$ is continuous, then $G$ is empty, $P$ contains $(v_{\min}, v_{\max})$, $\bar{v}$ is continuous and the shape $\phi$ also varies continuously with its argument.

For $c \in G$ no front exists. We have instead a “ghost front”, i.e. a sequence of configurations in $\mathcal{I}$ which is not an orbit of $\mathcal{H}$ but which attracts a set of initial conditions $\mathcal{I}$. Ghost fronts are due to the discontinuous character of the local map possibly combined with the couplings. The ghost front’s velocity $\bar{v}(c)$ may belong to $P$ but it may not since, in general, $P$ and the range of $\bar{v}(\cdot)$ are different. If $\bar{v}(c)$ belongs to $P$ this ghost orbit can be avoided by changing the value of the local map at $c$ for fixed couplings.

### 3.2 Proof of the fronts’ existence

The proof of Theorem 3.2 consists in investigating the function $\phi$, which can be defined independently of the front existence, to deduce the conditions for it to be a front shape.

Assuming the existence of a front of velocity $v$, we obtain from $\mathcal{H}$

$$H_v \ast \phi = h_1 \ast \phi + h_2 \ast H_0.$$

Then by using the convolution properties, we solve this equation to obtain the following expression, where the dependence on $v$ has been added

$$\phi(z, v) = \sum_{k=0}^{\infty} h_2 \ast h_1^k(z + (k + 1)v) \quad \forall z \in \mathbb{R},$$

(10)

where for any $h \in BV_0$ we denote

$$h^{\ast 0} = H_0 \quad \text{and} \quad h^{\ast (k+1)} = h^{\ast k} \ast h \quad \text{for} \ k \geq 0.$$
Consequently a front of velocity \( v \) exists iff the function defined by (10) satisfies condition (9). From relation (10), the convolution properties, the condition (iii) on the couplings and Proposition 2.3, it turns out that

\[
\forall v \in \mathbb{R} \quad \phi(., v) \in \mathcal{I},
\]

and

\[
E(\phi(., v)) = \bigcup_{k=0}^{\infty} E(h_2) + +_k E(h_1) + \{-(k + 1)v\},
\]

where for any real subset \( A \) we use the notation

\[
+_0 A = \{0\} \quad \text{and} \quad +_{(k+1)} A = +_k A + A \quad \text{for} \ k \geq 0.
\]

Similarly we have

\[
\forall z \in \mathbb{R} \quad \phi(z, .) \in \mathcal{I},
\]

and

\[
E(\phi(0, .)) = \bigcup_{k=0}^{\infty} \frac{E(h_2) + +_k E(h_1)}{k + 1},
\]

where for any \( y \in \mathbb{R} \setminus \{0\} \)

\[
\frac{A}{y} = \{ \frac{x}{y} : x \in A \}.
\]

Property (11) gives the following necessary condition for a front of velocity \( v \) to exist for some \( c \in (0, 1] \)

\[
0 \in E(\phi(., v)).
\]

Actually if \( 0 \notin E(\phi(., v)) \), then

\[
\exists \delta > 0 : \forall \epsilon \in (0, \delta) \quad \phi(-\epsilon, v) = \phi(0, v),
\]

and the condition (11) cannot be satisfied for any \( c \). Conversely by choosing \( c = \phi(0, v) \), we conclude that for all \( v \) for which relation (11) holds, there is (at least) a value of \( c \) for which the fronts of velocity \( v \) exist. The set of possible velocities is then given by

\[
P = \{ v \in \mathbb{R} : 0 \in E(\phi(., v)) \}.
\]

This set has the following properties

\[
\bigcup_{k=0}^{\infty} \frac{E(h_2) + +_k E(h_1)}{k + 1} \subset P \subset E(\phi(0, .)),
\]

from which the bounds on \( P \) are deduced.\(^7\)

Now one has to determine which fronts of possible velocity are selected given a value of the parameter \( c \). From relation (10) it turns out that if \( v_1 < v_2 \) then \( \phi(0, v_1) \leq \phi(- (v_2 - v_1), v_2) \), and thus if \( c \leq \phi(0, v_1) \) then the existence condition (11) shows that \( v_2 \) cannot be the fronts velocity for this value of \( c \). According to this property, the velocity of the (assumed) existing fronts is uniquely given by

\[
\bar{v}(c) = \min \{ v \in \mathbb{R} : \phi(0, v) \geq c \}.
\]

\(^8\)\( \bar{v}(c) \) is defined at \( c = 1 \) iff \( v_{\max} < \infty \). Indeed, if \( v_{\max} = \infty \), the set \( \{ v \in \mathbb{R} : \phi(0, v) \geq 1 \} \) is empty.
The corresponding fronts exist iff

$$\forall \delta > 0 \quad \phi(-\delta, \bar{v}(c)) < c,$$

and the non-existence set is given by

$$G = \{ c : \exists \delta > 0 \quad c = \phi(-\delta, \bar{v}(c)) \}.$$

From its definition and from relation (12) the function $\bar{v}$ is clearly increasing. To show that it is left continuous, we notice that given $\delta > 0$ one has $\phi(0, \bar{v}(c) - \delta) < c$. Let then $c'$ be such that $\phi(0, \bar{v}(c) - \delta) < c' < c$. Again from the definition, we obtain $\bar{v}(c) - \delta < \bar{v}(c')$ and consequently $\bar{v}$ is left continuous since it is increasing. Moreover its range is $E(\phi(0,..))$ and thus relation (14) shows that $P$ is a dense subset of this range.

To achieve the proof we have to show that $G$ is of Baire first category. To that goal let

$$G^{(n)} = \{ c : c = \phi\left(-\frac{1}{n} - 0, \bar{v}(c)\right) \}.$$

We have

$$G = \bigcup_{n=1}^{\infty} G^{(n)},$$

and it then remains to show that each $G^{(n)}$ is nowhere dense. By contradiction, we assume that $G^{(n)}$ is dense in the (non-degenerated) interval $(c_1, c_2)$. Then $(c_1, c_2] \subset G^{(n)}$ since both the functions $\bar{v}$ and $\phi(-\frac{1}{n} - 0,..)$ are left continuous and increasing. It follows that $(c_1, c_2] \subset G$. Now for any $c \in (c_1, c_2)$ we have

$$\phi(0 - 0, \bar{v}(c_1)) \leq c_1 < c = \phi(0 - 0, \bar{v}(c)) < c_2 = \phi(0 - 0, \bar{v}(c_2)),$$

that is to say $\bar{v}(c) \in (\bar{v}(c_1), \bar{v}(c_2))$ and $\bar{v}(c) \notin P$. This is impossible since $\bar{v}$ is increasing and $P$ is a dense subset of the range of $\bar{v}$.

\[\Box\]

### 3.3 Characterization of the velocity set $P$

We now proceed to the analysis of $P$ and to its consequences. In the following lemma we firstly describe $P$ inside and outside the convex hull of $E(h_1)$. Let us define

$$i_j = \inf E(h_j) \quad \text{and} \quad s_j = \sup E(h_j) \quad \text{for} \quad j = 1, 2.$$ 

**Lemma 3.3** (i) If $\#E(h_1) > 1$ then for any $\theta \in (0, 1)$, $\theta$ irrational, we have

$$\forall a, b \in E(h_1), \quad a + \theta(b - a) \in P.$$ 

(ii) If $s_1 < s_2 < +\infty$ then

$$P \cap (s_1, s_2] = E(\phi(0,..)) \cap (s_1, s_2].$$

(iii) If $\#E(h_1) > 1$ and $s_2 = +\infty$ then for any $\theta > 1$, $\theta$ irrational, we have

$$\forall a, b \in E(h_1), \quad a < b : \quad a + \theta(b - a) \in P.$$
The proof is given in Appendix A. Naturally one has statements similar to (ii) when \(-\infty < i_2 < i_1\) and to (iii) when \(i_2 = -\infty\). Namely 

(iii') If \(-\infty < i_2 < i_1\) then 

\[ P \cap [i_2, i_1) = E(\phi(0,.)) \cap [i_2, i_1). \]

(iii”) If \#E(h_1) > 1 and \(i_2 = -\infty\) then for any \(\theta > 1, \theta\) irrational, we have 

\[ \forall a, b \in E(h_1), a < b : b + \theta(a - b) \in P. \]

Moreover since \(\bar{v}\) is a left continuous increasing function, a discontinuity of \(\bar{v}\) means the absence of an interval in its range. As \(P\) is contained in this range, statement (i) of the previous lemma implies the following.

**Corollary 3.4** In the interval \(\bar{v}^{-1}((i_1, s_1))\), the selected velocity \(\bar{v}(.)\) is a continuous function of \(c\).

Furthermore still in statement (i) of Lemma 3.3, we have seen that all the irrational velocities in \((i_1, s_1)\) are fronts velocities. We now give a sufficient condition for all the rational velocities in this interval to be fronts velocities. To that goal the lattice generated by a real subset \(A\) is introduced

\[ R(A) = A + \bigcup_{k=0}^{\infty} k(A - A), \]

where \(\overline{A}\) denotes the (usual) closure of \(A\). Some characterizations of the lattices generated by real subsets are listed in the following remark.

**Remark 3.5** (i) If \(R(A) \neq \mathbb{R}\) and \(\#A > 1\), then it exists \(\alpha \in \mathbb{R}^+\) such that \(R(A) = a + \alpha \mathbb{Z}\) for any \(a \in A\).

(ii) If \(A\) contains an infinite bounded set, then \(R(A) = \mathbb{R}\).

(iii) If there are \(x_0, x_1, x_2 \in A\) such that \(\frac{x_2 - x_0}{x_2 - x_0} \in \mathbb{R} \setminus \mathbb{Q}\), then \(R(A) = \mathbb{R}\).

(iv) If \(E(h) \neq D(h)\), then \(R(E(h)) = \mathbb{R}\) (see Lemma 2.2).

**Proposition 3.6** If \(E(h_2) \cap R(E(h_1)) \neq \emptyset\), then \((i_1, s_1) \subset P\).

In particular, if one of the last conditions in Remark 3.3 holds for \(E(h_1)\) then \((i_1, s_1) \subset P\) independently of \(h_2\).

**Proof:** Let \(d \in E(h_2) \cap R(E(h_1))\) and \(\delta > 0\). There are \(l \geq 1\), \(\{a_n\}_{0 \leq n \leq l}, a_n \in E(h_1), a_n < a_{n+1}\), and \(\{m_i\}_{0 \leq i < l}, m_i \in \mathbb{Z}\), such that

\[ d \leq a_0 + \sum_{i=0}^{l-1} m_i(a_{i+1} - a_i) < d + \frac{\delta}{2}. \tag{16} \]

We can suppose that \(a_0\) is arbitrarily close to \(i_1\) and \(a_l = s_1\) by choosing, if necessary, \(m_0 = 1\) and \(m_{l-1} = 0\). Now let \(J = 2 \max_{0 \leq i \leq l-1} |m_i|\) and if \(l > 1\) let \(M, N \in \mathbb{Z}^+\) be such that

\[ 0 \leq N(a_l - a_{l-1}) - M \sum_{i=0}^{l-2} (l-i)J(a_{i+1} - a_i) < \frac{\delta}{2}. \tag{17} \]

If \(l = 1\) we set \(N = M = 0\). Let us choose any \(0 < p < q\) co-prime integers and define

\[ m_{l-1} = -m_{l-1} - N + jp \quad \text{and} \quad n_i = -m_i + M(l-i)J + jp \quad \text{for} \quad 0 \leq i \leq l-2, \]
where the integer \( j \) is chosen large enough so that
\[
j q - 1 \geq n_0 \geq n_1 \geq \cdots \geq n_{l-1} \geq 0.
\]
By adding (16) and (17) we obtain
\[
d \leq a_0 - \sum_{i=0}^{l-1} (n_i - (k+1)\frac{p}{q})(a_{i+1} - a_i) < d + \delta, \tag{18}
\]
where \( k = jq - 1 \). It is easy to see that the latter is equivalent to
\[-\delta < d + (k - n_0)a_0 + (n_0 - n_1)a_1 + \cdots + (n_{l-2} - n_{l-1})a_{l-1} + n_{l-1}a_l - (k+1)v \leq 0,
\]
where \( v = a_0 + \frac{p}{q}(a_l - a_0) \). Since \( \delta \) is arbitrary, we deduce that \( v \in P \) and the proof is achieved by using Lemma 3.3 (i).

\[\square\]

### 3.4 Characterization of the non-existence set \( G \)

Phrased differently, Theorem 3.2 claims to each velocity \( v \in P \) corresponds a set of the parameter \( c \) for which the only existing fronts are those of velocity \( \bar{v}(c) \) and their shape is unique, the sets corresponding to different velocities being disjoint. We now describe both these sets and the complement of their union, i.e. the set \( G \).

Firstly if \( \bar{v}(c) \in P \setminus D(\phi(0, .)) \) then
\[
\phi(x, \bar{v}) < \phi(0, \bar{v}) = c \quad \forall x < 0,
\]
and the fronts exist. If \( \bar{v}(c) \in D(\phi(0, .)) \) since we have
\[
\phi(0 - 0, \bar{v}) = \phi(0, \bar{v} - 0),
\]
then either \( \phi(0 - 0, \bar{v}) < c \) and the fronts exist, or \( \phi(0 - 0, \bar{v}) = c \). In the latter case, condition (3) is satisfied or not according to \( \bar{v}(c) \in P_1 \) or \( \bar{v}(c) \notin P_1 \) where
\[
P_1 = \left\{ v \in \mathbb{R} : 0 \in E(\phi(., v)) \setminus \{0\} \right\}.
\]
In this way, we conclude that for \( h_1, h_2 \) and \( v \in P \) fixed, there exists a set of the parameter \( c \) given by
\[
c = \phi(0, v) \quad \text{if} \quad v \in P \setminus D(\phi(0, .))
\]
\[
\phi(0, v - 0) < c \leq \phi(0, v) \quad \text{if} \quad v \in D(\phi(0, .))
\]
\[
c = \phi(0, v - 0) \quad \text{if} \quad v \in D(\phi(0, .)) \cap P_1
\]
for which (only) the fronts of velocity \( \bar{v}(c) = v \) exist.

In particular, if for a given velocity \( v \in P \), it exists an interval such that the selected velocity is \( \bar{v}(c) = v \) for \( c \) in this interval, \( v \) is said to correspond to a plateau. According to the previous conclusion, a velocity \( v \) corresponds to a plateau iff \( v \in D(\phi(0, .)) \). Notice that Proposition 2.3 applied to relation (14) leads to
\[
D(\phi(0, .)) = \bigcup_{k=0}^{\infty} \frac{D(h_2) + k D(h_1)}{k + 1}.
\]

Let us now give some general conditions for the couplings to satisfy in order for \( G \) (the set of \( c \) values for which no front exist) to be countable and/or nowhere dense. According to the previous reasoning, this set is given by \( G = G_1 \cup G_2 \) where
\[
G_1 = \{ \phi(0, v) : v \in E(\phi(0, .)) \setminus P \} \quad \text{and} \quad G_2 = \{ \phi(0 - 0, v) : v \in D(\phi(0, .)) \setminus P_1 \}.
\]
\( G_2 \) is countable since the discontinuity set of an increasing function is \( \mathbb{Q} \). It is nowhere dense by the inequalities (19). Hence it remains to investigate these properties for \( G_1 \).
Proposition 3.7  $G$ is countable if one of the following conditions holds

(i) $\#E(h_1) \neq 1$
(ii) $\#E(h_1) = 1$ and $E(h_2)$ is bounded.

It is possible to find some examples in which $G$ is not countable.

Proof: Firstly if $h_1 = 0$ then $\phi(x, v) = h_2(x + v)$. In this case we obviously have $E(\phi(0, .)) = P$ and thus $G_1 = \emptyset$. Assume now that $\#E(h_1) = 1$ and $E(h_2)$ is bounded. Then statement (ii) of Lemma 3.4 implies that $\#G_1 \leq 1$. Finally if $\#E(h_1) > 1$ then $(i_1, s_1) \setminus P$ is countable as follows from statement (i) of Lemma 3.3. So is $(E(\phi(0, .)) \setminus P) \setminus [i_1, s_1]$ as it can be deduced from the assertions (ii) and (iii) of the same lemma.

To state conditions under which $G$ is nowhere dense in $(0, 1)$, it is useful to introduce a notation and some definitions. Given two real subsets $A$ and $B$, the notation $A \subset^\infty B$ has the following meaning

$$A \subset^\infty B \iff \begin{cases} \sup A = +\infty \Rightarrow \sup B = +\infty \\ \inf A = -\infty \Rightarrow \inf B = -\infty \end{cases}$$

In particular, if $A$ is bounded no restriction is imposed on $B$ and thus $A \subset^\infty B$ for any $B \subset \mathbb{R}$. The second concept that will be of use is the following.

Definition 3.8  A subset $A$ of $\mathbb{R}$ is commensurable if

$$\forall x_0, x_1, x_2 \in A \quad x_2 \neq x_0 \Rightarrow \frac{x_1 - x_0}{x_2 - x_0} \in \mathbb{Q}.$$  

A is said to be incommensurable otherwise.

Proposition 3.9  $G$ is nowhere dense if one of the following conditions is satisfied

(i) $E(h_1)$ is incommensurable
(ii) $h_1 = 0$
(iii) $\#E(h_1) = 1$ and $E(h_2) \subset^\infty D(h_2)$
(iv) $E(h_2) \cap R(E(h_1)) \neq \emptyset$ and $E(h_2) \subset^\infty D(h_2) \cup E(h_1)$
(v) $D(h_2) \neq \emptyset$ and $E(h_2) \subset^\infty D(h_2) \cup E(h_1)$.

Although many cases do not fit in these propositions, we can conclude that $G$ is countable and nowhere dense in most of the physical examples of piece-wise affine bistable extended mappings. Actually, Propositions 3.8 and 3.9 cover the case of cellular automata (i.e. $h_1 = 0$), the case of lattice dynamics (i.e. $D(h_2) = E(h_2)$) (with finite range for $G$ to be countable) from which belong LDS and CML, and more generally the case $E(h_2) \subset C(h_1)$ that is expected to occur in the model (1).

Since we have

$$\mathcal{L}u + \mathcal{L}'F \circ u = (\mathcal{L} + a\mathcal{L}') u + (1 - a)\mathcal{L}'H_e \circ u,$$

and thus $\mathcal{L} + a\mathcal{L}'$ is expected to act at all the points where $\mathcal{L}'$ acts.

Proof of Proposition 3.9: (i) If $E(h_1)$ is incommensurable then Proposition 3.6 implies $(i_1, s_1) \subset P$. Now if $s_2 = +\infty$ let $x_0, x_1, x_2 \in E(h_1)$ be such that $\frac{x_1 - x_0}{x_2 - x_0} \in \mathbb{R} \setminus \mathbb{Q}$. By using statement (iii) of Lemma 3.3 with $a = x_0$, $b = x_1$ and with $a = x_0$, $b = x_2$ respectively, we conclude that $(s_1, +\infty) \subset P$. If $s_2 < +\infty$ then Lemma 3.3 (ii) results in $(s_1, s_2) \cap E(\phi(0, .)) \setminus P = \emptyset$. One can proceed similarly for the cases $i_2 = -\infty$ and $i_2 > -\infty$ to conclude that $E(\phi(0, .)) \setminus P$ has at most two points ($i_1$ and $s_1$) and consequently $\#G_1 \leq 2$.  

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Firstly we prove that the condition \( E(v) \) is nowhere dense in \( \bar{v}^{-1}((i_1, s_1)) \).

**(ii)** If \( \sup D(h_2) = +\infty \) and \( a \in D(h_1) \), then \( G \) is nowhere dense in \( \bar{v}^{-1}((a, +\infty)) \).

**(iii)** If \( \inf D(h_2) = -\infty \) and \( a \in D(h_1) \), then \( G \) is nowhere dense in \( \bar{v}^{-1}((-\infty, a)) \).

To achieve the proof of Proposition 3.9, we are going to show that under the conditions in (iii), (iv) and (v), \( G_1 \) is nowhere dense in each of the following intervals \( \bar{v}^{-1}((i_2, i_1)) \), \( \bar{v}^{-1}((i_1, s_1)) \) and \( \bar{v}^{-1}((s_1, s_2)) \), concluding that \( G_1 \) and consequently \( G \) is nowhere dense in \( (0, 1) \).

Firstly we prove that the condition \( E(h_2) \supset D(h_2) \cup E(h_1) \) implies that \( G \) is nowhere dense in \( \bar{v}^{-1}((s_1, s_2)) \) by considering the following cases. If \( \sup D(h_2) = +\infty \), then we use Lemma 3.10 (ii). If \( \sup E(h_1) = +\infty \), then there is nothing to prove because \( \bar{v}^{-1}((s_1, s_2)) = \bar{v}^{-1}(\emptyset) = \emptyset \). If \( \inf E(h_2) < +\infty \), Lemma 3.3 statement (ii) results in \( \bar{v}^{-1}((s_1, s_2)) \cap G_1 = \emptyset \).

Similarly \( G \) is nowhere dense in \( \bar{v}^{-1}((i_2, i_1)) \).

Now if \( \#E(h_1) = 1 \) then \( \bar{v}^{-1}((i_1, s_1)) \) is empty, proving (iii). If \( E(h_2) \cap R(E(h_1)) \neq \emptyset \) then by Proposition 3.6, \( \bar{v}^{-1}((i_1, s_1)) \cap G_1 = \emptyset \) and (iv) is proved. Finally if \( D(h_2) \neq \emptyset \) then by Lemma 3.10 (i), \( G \) is nowhere dense in \( \bar{v}^{-1}((i_1, s_1)) \) and (v) holds.

**3.5 Continuity of the fronts velocity**

We now consider the dependence of the fronts velocity on the couplings. In particular, we are going to show that small variations of the couplings induce small variations of the selected velocity. As couplings variations, we also would like to allow for changes in the sets of increase points of the couplings \( h_1 \) and \( h_2 \). For instance, this is the case when the planar fronts’ direction varies in multi-dimensional CML (see Section 5.3). We start with a definition of a distance in \( \mathcal{I} \). For \( h, h' \in \mathcal{I} \) let

\[
d(h, h') = \inf \{ \epsilon > 0 : h(x - \epsilon) - \epsilon \leq h'(x) \leq h(x + \epsilon) + \epsilon \quad \forall x \in \mathbb{R} \}.
\]

One easily checks that \( d(\cdot, \cdot) \) is a distance. For this distance, the ball of radius \( \epsilon \) centered at \( h \) is the set of functions for which the graph lies in the band of width \( \epsilon \) in the direction of the line \( y = -x \) around the graph of \( h \). Similarly in the set of left continuous increasing functions defined on \( (0, 1) \), we define \( \bar{d}(v, v') \) as the infimum of the positive numbers \( \epsilon \) such that

\[
\begin{align*}
&\left\{ \begin{array}{ll}
v(x - \epsilon) - \epsilon \leq v'(x) & \text{if} \quad x \in (\epsilon, 1) \\
v'(x) \leq v(x + \epsilon) + \epsilon & \text{if} \quad x \in (0, 1 - \epsilon)
\end{array} \right.
\end{align*}
\]

From this definition one concludes that \( \bar{d}(\cdot, \cdot) \) is a distance and \( \bar{d}(\cdot, \cdot) \leq 1 \). The continuity of the selected velocity in the subsequent topology is an immediate consequence of the following statement. Let \( \bar{v} \) (resp. \( \bar{v}' \)) be given by \( (14) \) for the couplings \( h_1 \) and \( h_2 \) (resp. \( h_1' \) and \( h_2' \)).
**Proposition 3.11** For the couplings \( h_1, h'_1, h_2, h'_2 \), let
\[
m = \max\{d(h_1, h'_1), d(h_2, h'_2)\}.
\]
Then
\[
\tilde{d}(\bar{v}, \bar{v}') \leq \frac{2m}{1 - \|h'_1\|}.
\]
As a consequence, for any fixed \( c \in \bar{v}^{-1}((i_1, s_1)) \), \( \bar{v} \) is a continuous function of the couplings in the Euclidean topology.

**Proof:** Let us firstly show the continuous dependence of the shape in the topology induced by \( d(.,.) \). We notice that for any \( \epsilon \geq d(h, h') \), the following holds
\[
H_\epsilon \ast h' - \epsilon \leq h \leq H_{-\epsilon} \ast h' + \epsilon.
\]
Hence by induction and by using the convolution properties, we obtain for any \( k \geq 1 \)
\[
H_{ke} \ast h'^{sk} - \epsilon \sum_{i=0}^{k-1} \|h^i\|\|h'^{k-1-i}\| \leq h'^{sk} \leq H_{-ke} \ast h'^{sk} + \epsilon \sum_{i=0}^{k-1} \|h^i\|\|h'^{k-1-i}\|.
\]
Replacing these inequalities in the expression of \( \phi \) and using again the convolution properties result in the following
\[
\forall \epsilon > m \quad \phi'(x, v - \epsilon) - \frac{2\epsilon}{1 - \|h'_1\|} \leq \phi(x, v) \leq \phi'(x, v + \epsilon) + \frac{2\epsilon}{1 - \|h'_1\|},
\]
where \( \phi' \) is computed with \( h'_1 \) and \( h'_2 \). From the right inequality we obtain
\[
c - \frac{2\epsilon}{1 - \|h'_1\|} \leq \phi(0, \bar{v}(c)) - \frac{2\epsilon}{1 - \|h'_1\|} \leq \phi'(0, \bar{v}(c) + \epsilon),
\]
and hence by the definition of \( \bar{v} \), it follows
\[
\bar{v}' \left( c - \frac{2\epsilon}{1 - \|h'_1\|} \right) \leq \bar{v}(c) + \epsilon.
\]
Similarly we have
\[
c - \frac{2\epsilon}{1 - \|h'_1\|} \leq \phi'(0, \bar{v}'(c)) - \frac{2\epsilon}{1 - \|h'_1\|} \leq \phi(0, \bar{v}'(c) + \epsilon),
\]
and by the definition of \( \bar{v} \) and by replacing \( c \) by \( c + \frac{2\epsilon}{1 - \|h'_1\|} \), we obtain
\[
\bar{v}(c) \leq \bar{v}' \left( c + \frac{2\epsilon}{1 - \|h'_1\|} \right) + \epsilon.
\]
Hence \( \tilde{d}(\bar{v}, \bar{v}') \leq \frac{2\epsilon}{1 - \|h'_1\|} \) and the statement is obtained by choosing \( \epsilon \) arbitrarily close to \( m \).

Finally, we assume that \( \bar{v}(c) \in (i_1, s_1) \). We have shown that
\[
\bar{v} \left( c - \frac{2\epsilon}{1 - \|h'_1\|} \right) \leq \bar{v}'(c) \leq \bar{v} \left( c + \frac{2\epsilon}{1 - \|h'_1\|} \right) + \frac{2\epsilon}{1 - \|h'_1\|},
\]
for \( \epsilon > m \). Thus from the continuity of \( \bar{v}(,) \) in \( \bar{v}^{-1}((i_1, s_1)) \), we obtain for \( c \) fixed
\[
\lim_{m \to 0} \bar{v}'(c) = \bar{v}(c).
\]
\[\square\]
3.6 The anti-fronts

To conclude the study of the existence of travelling wave interfaces in our dynamical system, we mention that the results about the fronts can be employed to deduce the existence of anti-fronts and the uniqueness of their shape. An anti-front is a travelling wave for which the shape $\phi^{\text{anti}}$ obeys the condition

$$H_c \circ \phi^{\text{anti}}(x) = H_0(-x) \quad \forall x \in \mathbb{R}.$$

By defining $\phi^a(x) = \phi^{\text{anti}}(-x)$ and by using the shape equation, one can easily prove the following statement.

**Proposition 3.12** The dynamical system (7) has anti-fronts of velocity $v$ and shape $\phi^{\text{anti}}$ iff there exist fronts of velocity $-v$ and shape $\phi^a$ for the dynamics given by

$$u^{t+1} = h_1^a * u^t + h_2^a * H_c \circ u^t,$$

where $h_i^a(x) = \|h_i\| - h_i(-x - 0)$ for $i = 1, 2$.

Moreover it is possible to relate $\bar{v}^{\text{anti}}(c)$ the velocity of anti-fronts to the fronts velocity for the same couplings, i.e. those represented by $h_1$ and $h_2$, in the following way

$$\bar{v}^{\text{anti}}(c) = \bar{v}(1 - c + 0).$$

Hence in general the selected velocities of fronts and of anti-fronts, as well as the corresponding non-existence sets $G$ and $G^{\text{anti}}$, are different.

4 Velocity of interfaces

In this section, an also important result, when one has in mind physical applications of bistable extended mappings, is presented. In fact we are going to prove that the fronts velocity $\bar{v}$ is actually the propagation velocity for any orbit of (7) composed at each time of a configuration linking two different phases, namely a $c$-interface.

**Definition 4.1** A $c$-interface is a function $u \in \mathcal{M}$ such that

$$\exists J_i, J_s \in \mathbb{R} : \begin{cases} u(x) < c & \text{if } x < J_i, \\ u(x) \geq c & \text{if } x > J_s. \end{cases}$$

The set of $c$-interfaces is denoted by $\mathcal{J}^c$ and $J_i(u)$ and $J_s(u)$ are the functions from $\mathcal{J}^c$ to $\mathbb{R}$ defined by

$$J_i(u) = \inf \{ x : u(x) \geq c \} \quad \text{and} \quad J_s(u) = \sup \{ x : u(x) < c \}.$$

The following proposition guarantees that the image under the dynamics (5) of a $c$-interface is a $c$-interface.

**Proposition 4.2** If $c \in (0, 1)$ and $u \in \mathcal{J}^c$ then $h_1 * u + h_2 * H_c \circ u \in \mathcal{J}^c$.

This statement is still true for $c = 1$ if $\bar{v}_{\text{max}} < +\infty$.

**Proof:** Let $u^1 = h_1 * u + h_2 * H_c \circ u$ and $\tilde{u}$ be defined by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \leq J_s(u) \\ c & \text{if } x > J_s(u) \end{cases}$$

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We have \( \tilde{a} \leq a \). By using the convolution properties, we then obtain

\[
\liminf_{x \to +\infty} u^1(x) \geq c\|h_1\| + \|h_2\| > c,
\]

since \( \|h_1\| = 1 - \|h_2\| \) and \( c < 1 \). Similarly, one proves

\[
\limsup_{x \to -\infty} u^1(x) \leq c\|h_1\| < c.
\]

Hence the terminology \( c \)-interface orbit \( \{u^i\}_{i \in \mathbb{N}} \) is justified. As it has been pointed out in the case of CML \([5]\), these orbits can evolve towards different attractors. So a general common (asymptotic) stability result (in the sense of Lyapunov) in \( J^c \) cannot hold. However the \( c \)-interface orbits all have a common dynamical characteristic which is the same propagation velocity.

**Theorem 4.3** If \( c \) is a point of continuity of the function \( \bar{v} \) and \( \{u^i\}_{i \in \mathbb{N}} \) is a \( c \)-interface orbit, then its velocity exists and is given by \( \bar{v}(c) \), that is to say

\[
\lim_{t \to +\infty} \frac{J_i(u^t)}{t} = \lim_{t \to +\infty} \frac{J_s(u^t)}{t} = \bar{v}(c).
\]

The result cannot hold if \( c \) is a discontinuity point of \( \bar{v} \) because in this situation, the dynamical system has (also) ghost fronts of velocity \( \bar{v}(c + 0) \) which attract some initial conditions. The proof of Theorem 4.3 is based on the following proposition.

**Proposition 4.4** For any fixed \( c \in (0, 1) \) we have \( \forall u^0 \in J^c \) \( \forall \epsilon > 0 \exists \sigma_0, \sigma_1 \in \mathbb{R} \) such that

\[
\sigma_0 + t(\bar{v}(c) - \epsilon) \leq J_i(u^t) \leq J_s(u^t) \leq \sigma_1 + t\bar{v}(c + \epsilon).
\]

Clearly if \( v_{\max} < +\infty \), we have a similar claim for \( c = 1 \).

**Proof:** Given \( u^0 \in J^c \), by iterating the dynamics \([3]\) we obtain for \( t \geq 1 \)

\[
u^t = h_1^t * u^0 + \sum_{k=0}^{t-1} h_2 * h_1^k * H_c \circ u^{t-1-k}.
\]

Since \( \{u^i\}_{i \in \mathbb{N}} \) is composed of \( c \)-interfaces, the convolution properties and the definitions of \( J_i \) and of \( J_s \) lead to the following inequalities

\[
\sum_{k=0}^{t-1} h_2 * h_1^k * H_0^c(u^{t-1-k}) \leq u^t \leq \sum_{k=0}^{t-1} h_2 * h_1^k * H_{J_i(u^{t-1-k})},
\]

where \( H_0^c(x) = H_\omega(x - 0) \forall \omega, x \in \mathbb{R} \). Now given \( \epsilon > 0 \) let

\[
\Delta_0 = c - \phi(0, \bar{v}(c) - \epsilon) \quad \text{and} \quad \Delta_1 = \phi(0, \bar{v}(c + \epsilon)) - c.
\]

By the definition of \( \bar{v}(c) \), we have \( \Delta_0 > 0 \) and \( \Delta_1 > \epsilon > 0 \). Let then \( t_0, t_1 \in \mathbb{N} \) be such that

\[
\|h_1\|t_0\|u^0\| < \Delta_0 \quad \text{and} \quad \|h_1\|t_1(\|u^0\| + 1) < \Delta_1,
\]

and \( \sigma_0, \sigma_1 \in \mathbb{R} \) be defined by

\[
\sigma_0 = \min_{0 \leq k \leq t_0-1} \left( J_i(u^k) - k(\bar{v}(c) - \epsilon) \right) \quad \text{and} \quad \sigma_1 = \max_{0 \leq k \leq t_1-1} \left( J_s(u^k) - k\bar{v}(c + \epsilon) \right).
\]

We are going to prove by induction that

\[
\sigma_0 + k(\bar{v}(c) - \epsilon) \leq J_i(u^k) \leq J_s(u^k) \leq \sigma_1 + k\bar{v}(c + \epsilon) \quad \forall k \in \mathbb{N}.
\]
By construction the left (resp. right) inequality is true for $0 \leq k \leq t_0 - 1$ (resp. $0 \leq k \leq t_1-1$). Assume the left (resp. right) inequality in \eqref{eq:1} holds for $0 \leq k \leq t-1$ with $t \geq t_0$ (resp. $t \geq t_1$). It comes out that

$$u^t(x) \leq \|h_1\|^{t_0} u^0 \| + \sum_{k=0}^{t-1} h_2 \ast h_1^{\ast k} \ast H_0(x - \sigma_0 - (t-1-k)(\bar{v}(c) - \epsilon)) \quad \forall x \in \mathbb{R},$$

and

$$u^t(x) \geq -\|h_1\|^{t_1} \|u^0\| + \sum_{k=0}^{t-1} h_2 \ast h_1^{\ast k} \ast H_0^1(x - \sigma_1 - (t-1-k)\bar{v}(c + \epsilon)) \quad \forall x \in \mathbb{R}.$$

Hence using the definition of the function $\phi(z, v)$ we have

$$u^t(x) \leq \|h_1\|^{t_0} \|u^0\| + \phi(x - \sigma_0 - t(\bar{v}(c) - \epsilon), \bar{v}(c) - \epsilon) \quad \forall x \in \mathbb{R},$$

and for any $\delta > 0$

$$u^t(x) \geq -\|h_1\|^{t_1} (\|u^0\| + 1) + \phi(x - \sigma_1 - \delta - t\bar{v}(c + \epsilon), \bar{v}(c + \epsilon)) \quad \forall x \in \mathbb{R},$$

the latter being obtained by using the relation $\left\| \sum_{k=t_1}^{\infty} h_2 \ast h_1^{\ast k} \right\| = \|h_1\|^{t_1}$. Consequently if $x \leq \sigma_0 + t\bar{v}(c) - \epsilon)$ we have

$$u^t(x) \leq \|h_1\|^{t_0} \|u^0\| + \phi(0, \bar{v}(c) - \epsilon) < c,$$

in other terms $\sigma_0 + t\bar{v}(c) - \epsilon) \leq J_1(u^t)$. If $x \geq \sigma_1 + \delta + t\bar{v}(c + \epsilon)$ then

$$u^t(x) \geq -\|h_1\|^{t_1} (\|u^0\| + 1) + \phi(0, \bar{v}(c + \epsilon)) > c,$$

proving the right inequality $J_1(u^t) \leq \sigma_1 + t\bar{v}(c + \epsilon).$

\textit{Proof of theorem 4.3.} From the last proposition, it is deduced the following inequalities

$$\forall \epsilon > 0 \quad \bar{v}(c) - \epsilon \leq \liminf_{t \to +\infty} \frac{J_1(u^t)}{t} \leq \limsup_{t \to +\infty} \frac{J_1(u^t)}{t} \leq \bar{v}(c + \epsilon),$$

and thus

$$\bar{v}(c) \leq \liminf_{t \to +\infty} \frac{J_1(u^t)}{t} \leq \limsup_{t \to +\infty} \frac{J_1(u^t)}{t} \leq \bar{v}(c + 0).$$

Since $J_1(u^t) \leq J_1(u^t)$ we obtain the existence of the limits if $\bar{v}(c) = \bar{v}(c + 0).$ 

\section{Examples}

In order to illustrate the results obtained in the general framework, we now present three examples. One focuses on discrete couplings, the second deals with continuous couplings, and we thirdly show how planar fronts in multi-dimensional CML enter in the general framework.\footnote{In this section, we will refer both to the set $P$ of possible velocities and to $G$ the set of values of $c$ for which no front exists. (See proof of Theorem 3.4.)}
5.1 One-dimensional Lattice Dynamical System

In this first example, we choose the couplings as acting on a one-dimensional lattice and as being of finite range. The interest here resides in emphasizing on two characteristics. One is the absence of some rational velocities in \((i_1, s_1)\) and the other is the structure of \(P\) in \((i_2, s_2) \setminus (i_1, s_1)\) when the set of increase points of \(h_2\) is bounded.

In order to have these both characteristics, we set for \(a \in (0, 1)\) and \(\epsilon \in (0, 1)\)

\[
L_1 = \frac{a}{2} (\sigma^1 + \sigma^{-1}) \quad \text{and} \quad L_2 = (1 - a) \left( (1 - \epsilon) \text{Id} + \frac{\epsilon}{2} (\sigma^2 + \sigma^{-2}) \right),
\]

where \(\text{Id}\) is the identity on \(\mathcal{M}\). It is immediate to verify that these obey the linearity, continuity, positivity, \(s\)-homogeneity and normalization conditions. It is also easy to see that \(D(h_1) = E(h_1) = (-1, 1)\) and \(D(h_2) = E(h_2) = (-2, 0, 2)\) and consequently \(E(h_2) \cap R(E(h_1)) = \emptyset\). Therefore, Proposition 3.4 does not apply and some rational velocities are expected to be absent from \(P\). The results are contained in the following statement. In the remaining of this section, \(p\) and \(q\) denote two co-prime integers.

**Proposition 5.1** For the couplings given by (21) we have \(P \subset [-2, 2]\) and

(i) the velocity \(v \in [-1, 1]\) belongs to \(P\) iff either \(v\) is irrational or \(v = \frac{p}{q}\) with \(p + q\) odd,

(ii) the velocity \(v \in [1, 2]\) (resp. \([-2, -1]\)) belongs to \(P\) iff it exists \(k \in \mathbb{N}\) such that

\[
v = 1 + \frac{1}{k+1} \quad \text{(resp. } v = -1 - \frac{1}{k+1}).
\]

Notice that the set

\[-1, 1] \setminus P = \left\{ v \in [-1, 1] : v = \frac{p}{q}, p + q \text{ even} \right\}

is dense in \([-1, 1]\) but \(G\) is nowhere dense.

We have plotted on Figure 2 the selected velocity \(\bar{v}(c)\) versus \(c \in \left[\frac{1}{2}, 1\right]\), the function being symmetric, i.e. \(\bar{v}(c) = -\bar{v}(1-c+0)\). This function is a Devil’s staircase in \([\bar{v}^{-1}(-1), \bar{v}^{-1}(1)]\) and a classical staircase in the complementary intervals.

**Proof of Proposition 5.1** (i) By a direct calculation, we obtain

\[
\phi(x, v) = (1 - a) \sum_{k=0}^{\infty} \sum_{n \in \mathbb{Z}} \left( (1 - \epsilon) l_n^k + \frac{\epsilon}{2} (l_{n-2}^k + l_{n+2}^k) \right) H_{n-v(k+1)}(x) \quad \forall x \in \mathbb{R},
\]

where the coefficients \(l_n^k\) are the entries of the powers \(L_1^k\). These coefficients are known explicitly \(\mathbb{R}\). In particular they obey the following property

\[
l_n^k \neq 0 \quad \text{iff} \quad |n| \leq k \text{ and } \frac{n + k}{2} \in \mathbb{Z}.
\]

On one hand all the irrational velocities \([-1, 1]\) are possible as given by Lemma 3.3 (i). On the other hand we have

\[
\phi(x, \frac{p}{q}) = \phi(0 - 0, \frac{p}{q}) = \phi(0, \frac{p}{q} - 0) \quad \text{for} \quad x \in \left[ -\frac{1}{q}, 0 \right],
\]

and thus the proof of Theorem 3.2 tells that \(\frac{p}{q} \in [-1, 1]\) belongs to \(P\) iff \(\phi(0, \frac{p}{q} - 0) < \phi(0, \frac{q}{q})\). Using the definition of the Heaviside function, we have

\[
\phi(0, \frac{p}{q}) - \phi(0, \frac{p}{q} - 0) = (1 - a) \sum_{k=1}^{\infty} \left( (1 - \epsilon) l_{kp}^{kq-1} + \frac{\epsilon}{2} (l_{kp-2}^{kq-1} + l_{kp+2}^{kq-1}) \right)
\]

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Figure 2: The selected velocity \( \bar{v}(c) \) versus \( c \) for the one-dimensional lattice dynamical system example \((a = \epsilon = 0.6)\).

Taking into account relation (22), it turns out that
\[
\phi(0, \frac{p}{q}) = \phi(0, \frac{p}{q} - 0) \quad \text{iff} \quad \forall k \geq 1, \; \exists j \in \mathbb{N} : k(p + q) = 2j,
\]
from which we conclude that \( \frac{p}{q} \in P \) iff \( p + q \) is odd.

(ii) is easily deduced from the proof of statement (ii) Lemma 3.3. \( \Box \)

5.2 Continuous example

As a continuous couplings example, we consider for \( a \in (0, 1) \), \( L_1 = aL \) and \( L_2 = (1 - a)L \) where \( L \) is now the continuous diffusion operator expressed in the integral formulation, i.e. the corresponding function \( h \) is defined by
\[
h(x) = \int_{-\infty}^{x} e^{-\pi y^2} \, dy,
\]
so that the couplings obey the linearity, continuity, positivity, \( s \)-homogeneity and normalization conditions. In this case we have \( E(h_1) = E(h_2) = \mathbb{R} \) and \( D(h_1) = D(h_2) = \emptyset \) thus \( P = \mathbb{R} \) and \( G \) is empty. We conclude that for any \( a, c \in (0, 1) \) it exists a unique front shape. The corresponding velocity is a continuous function of \( (a, c) \) with range \( \mathbb{R} \).

The front velocity \( \bar{v}(c) \) in this continuous model, restricted to the values in \([0, 1]\), is shown on Figure 3. Notice that \( \bar{v}(.) \) is an odd function.

5.3 Multi-dimensional Coupled Map Lattices

We now turn to the application of the previous results to the dynamics of planar fronts in discrete bistable extended systems defined on a \( d \)-dimensional lattice, by considering a simple situation, that of homogeneous CML. However a similar
Figure 3: The selected velocity \( \bar{v}(c) \) versus \( c \) for the continuous coupling \( (a = 0.4) \).

analysis can be applied, without supplementary difficulties, to more general multi-dimensional extended mappings. After defining these systems, we shall see how the homogeneity implies the reduction of the dynamics, for the orbits having equal components on \((d-1)\)-dimensional planes, to a one-dimensional extended mapping. The existence of planar fronts in these multi-dimensional models then directly follows from this reduction and from Theorem 3.2. Finally, the monotonicity of the dynamics allows to conclude about the velocity of interfaces, even when their dynamics does not reduce to a one-dimensional extended mapping.

5.3.1 Definitions

Multi-dimensional CML are introduced following the ideas of [5]. For \( d \in \mathbb{N}, d > 0 \), let

\[
\mathcal{M}' = \{ u = \{ u_s \}_{s \in \mathbb{Z}^d}, \ u_s \in \mathbb{R} : \| u \|_\infty < +\infty \},
\]

be the phase space under consideration. A homogeneous \( d \)-dimensional CML is a discrete time dynamical system for which each time iteration of an orbit \( \{ u^t \} \) of elements of \( \mathcal{M}' \) is given by the following relation

\[
u^{t+1} = LF \circ u^t.\] (23)

Here the mapping \( F \) is a direct product of one-dimensional identical mappings, i.e.

\[
(F \circ u)_s = f(u_s) \quad \forall s \in \mathbb{Z}^d,
\]

where \( f \) maps \( \mathbb{R} \) into itself. The coupling \( L \) is required to be a linear, continuous, positive and \( s \)-homogeneous operator of \( \mathcal{M}' \) into itself. Its explicit expression is then

\[
(Lu)_s = \sum_{n \in \mathbb{Z}^d} l_n u_{s-n} \quad \forall s \in \mathbb{Z}^d,
\]

where the real coefficients \( l_n \geq 0 \) obey the normalization condition

\[
\sum_{n \in \mathbb{Z}^d} l_n = 1.
\]
A point \( s \) in the multi-dimensional lattice \( \mathbb{Z}^d \) is denoted by \( \{s_i\}_{1 \leq i \leq d} \). The difference of two elements in \( \mathbb{Z}^d \) is to be understood as the component-wise difference. Moreover we are going to use the Euclidean inner product in \( \mathbb{R}^d \), i.e.

\[
 r, s = \sum_{i=1}^{d} r_is_i \quad \forall r, s \in \mathbb{R}^d. 
\]

### 5.3.2 One-dimensional orbits

Multi-dimensional homogeneous CML satisfy any translational invariance symmetry along each lattice axis. To each of these symmetries corresponds a reduction of the dimension of the lattice supporting the dynamics. Hence by combining \( d - 1 \) linearly independent translations, the resulting orbit’s dynamics is governed by a one-dimensional extended mapping, as we describe now.

Let \( k \in \mathbb{R}^d \) be such that \( k.k = 1 \), a fixed direction. By induction and by using the coupling homogeneity, one shows that for any orbit \( \{u^t\}_{t \in \mathbb{N}} \) generated by the dynamics (23), if the property

\[
 \forall r, s \in \mathbb{Z}^d: \quad k.r = k.s \Rightarrow u^t_r = u^t_s, \quad (24)
\]

holds for \( t = 0 \), then it is satisfied for any \( t \geq 0 \). In this case the time evolution reduces to that of a one-dimensional discrete extended mapping. Actually by introducing the reduced coordinate \( \omega = k.s \) and the corresponding component \( z_{\omega}^t = u^t_s \), we obtain the component-wise expression for the reduced dynamics

\[
 z_{\omega}^{t+1} = \sum_{\nu \in Z(k)} \lambda_{\nu}f(z_{\omega-\nu}^t) \quad \forall \omega \in Z(k), \quad (25)
\]

where

\[
 Z(k) = \{ \omega \in \mathbb{R} : \omega = k.s, \ s \in \mathbb{Z}^d \} \quad \text{and} \quad \lambda_{\omega} = \sum_{n : k.n = \omega} l_n \quad \forall \omega \in Z(k).
\]

Similarly as in Remark 3.3, if \( \forall i, j \frac{k_i}{k_j} \in \mathbb{Q} \) then it exists \( m \in \mathbb{R}^+ \) such that \( Z(k) = m\mathbb{Z} \) and if \( \exists i, j \) such that \( \frac{k_i}{k_j} \in \mathbb{R} \setminus \mathbb{Q} \) then \( Z(k) \) is dense in \( \mathbb{R} \). Furthermore for a given multi-dimensional coupling \( L \), the one-dimensional (discrete) reduced coupling \( \Lambda \) defined by

\[
 (\Lambda z)_\omega = \sum_{\nu \in Z(k)} \lambda_{\nu} z_{\omega-\nu},
\]

and thus the dynamical properties of the corresponding one-dimensional orbits of (23), depend on the direction \( k \). From now on we choose as a local map, the piecewise affine model (4). The reduced system (25) then turns out to be a particular case of the system (4) with the coupling functions given by \( h_1 = ah_k \) and \( h_2 = (1-a)h_k \) where

\[
 h_k(\omega) = \sum_{\nu \in Z(k)} \lambda_{\nu}H_0(\omega - \nu) = \sum_{n \in \mathbb{Z}^d} l_n H_0(\omega - k.n). \quad (26)
\]

### 5.3.3 The planar fronts

**Definition 5.2** A planar front of velocity \( v \) in the direction \( k \) is an orbit of (23) defined by

\[
 u^t_s = \phi(k.s - vt - x_0) \quad \forall t \in \mathbb{Z}, \ s \in \mathbb{Z}^d,
\]

where \( x_0 \in \mathbb{R} \) and the shape \( \phi \in \mathcal{M} \) and obeys the condition (4).

---

\[\text{10This restriction is not necessary if } k \text{ is an irrational direction, i.e. if } k.s = 0 \text{ implies } s = 0.\]
As for fronts, the existence of planar fronts only depend on the existence of their shape, and is therefore independent of the translation parameter $x_0$.

Since the planar fronts are invariant under translations orthogonal to their direction, we can apply the previous reduction to obtain the equivalence between the planar fronts of (23) and the fronts of the corresponding reduced system (25). The planar front existence in piece-wise affine bistable $d$-dimensional CML then follows from Theorem 3.2 and the propositions of Section 8. Let us denote by $i_k = \inf \{ \omega \in Z(k) : \lambda_\omega > 0 \}$ and by $s_k = \sup \{ \omega \in Z(k) : \lambda_\omega > 0 \}$.

**Corollary 5.3** Given $a, L$ and $k$, there is a countable nowhere dense set $G$ such that, for any $c \in (0, 1] \setminus G$, there exist planar fronts in the system (21) with a unique shape. The corresponding velocity is a continuous function of $(a, c, k, L)$ with range $[i_k, s_k]$.

In particular for a given coupling, the selected velocity as well as the extreme velocities and the plateaus, depend continuously on the planar fronts’ orientation.

As a sketch of proof, one firstly notices that from (25) it follows the relations $E(h_1) = D(h_1) = E(h_2) = D(h_2) \subset Z(k) \cap [i_k, s_k]$ and hence $P = E(\phi(0, \cdot)) = [i_k, s_k]$ and $G = G_2$. This allows to state the existence following the existence’s proof in the general case. The continuity versus $k$ and $L$ is a consequence of Proposition 3.1 and of the fact that small variations of $k$ and of $L$ induce small variations of the reduced coupling $A$ for the distance $d(\cdot, \cdot)$ as can be easily seen from (24).

### 5.3.4 Velocity of interfaces

Similarly as in the one-dimensional case, the subsequent question to the existence of fronts is the velocity of interfaces in multi-dimensional CML. Naturally their definition must take into account the orientation.

**Definition 5.4** A $(k, c)$-interface is a configuration $u \in \mathcal{M}'$ such that

$$\exists J_{k_1}, J_{k_2} \in \mathbb{R} : \left\{ \begin{array}{l} u_s < c \quad \text{if} \quad s.k < J_{k_1} \\ u_s \geq c \quad \text{if} \quad s.k > J_{k_2}. \end{array} \right.$$  

The set of $(k, c)$-interfaces is denoted by $\mathcal{J}_k^c$ and $J_{k_1}(u)$ and $J_{k_2}(u)$ are the functions from $\mathcal{J}_k^c$ to $\mathbb{R}$ defined by

$$J_{k_1}(u) = \inf \{ s.k : u_s \geq c \} \quad \text{and} \quad J_{k_2}(u) = \sup \{ s.k : u_s < c \}.$$  

To each $(k, c)$-interface $u$ we can associate two $(k, c)$-interfaces $\underline{u}$ and $\overline{u}$ satisfying relation (24) and defined for any $s \in \mathbb{Z}^d$ by

$$\underline{u_s} = \inf \{ u_n : n.k = s.k \} \quad \text{and} \quad \overline{u_s} = \sup \{ u_n : n.k = s.k \}.$$  

We have $\underline{u_s} \leq u_s \leq \overline{u_s}$ $\forall s \in \mathbb{Z}^d$ and then $J_{k_1}(\overline{u}) \leq J_{k_1}(u) \leq J_{k_1}(\underline{u})$ and $J_{k_2}(\overline{u}) \leq J_{k_2}(u) \leq J_{k_2}(\underline{u})$. For the configurations $\underline{u}$ and $\overline{u}$ the dynamics can be reduced to a one-dimensional extended mapping as was done for planar fronts. Consequently we can apply Proposition 1.2, Theorem 1.3 and the order preservation of the dynamics to conclude that a $(k, c)$-interface initial condition $u^0$ generates a $(k, c)$-interface orbit $\{u^t\}_{t \in \mathbb{N}}$ and the following statement holds.

**Corollary 5.5** For any $u^0 \in \mathcal{J}_k^c$ the following limits exist and do not depend on the particular choice of $u^0$

$$\lim_{t \to +\infty} \frac{J_{k_2}(u^t)}{t} = \lim_{t \to +\infty} \frac{J_{k_2}(u^t)}{t} = \bar{v}_k,$$

where $\bar{v}_k$ is the velocity of the planar front in the direction $k$.  

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6 Other fronts and general bistable extended mappings

Starting from the knowledge of the fronts dynamics in piece-wise affine bistable extended mappings with homogeneous linear couplings, one can extend the results to other kinds of fronts and to more general systems in which one or many of the previous conditions are relaxed. Here we firstly consider changes in the local map and then we allow for nonlinear couplings.

Let us recall that a fronts velocity \( v \) corresponds to a plateau iff it is a discontinuity point of the function \( \phi(0,.) \). In this case according to (19), the interval corresponding to this plateau is given by \( I_v = (\phi(0-0,v),\phi(0,v)) \), the left bound being included if \( v \in I_1 \). In the following \( I_0^v \) denotes the interior of this interval.

In the interior of a plateau and without changing the couplings, the existence of fronts is claimed in systems defined by (1) but with now a \( C^\infty \) bistable local map. Actually inside the plateaus, the distance from the front shape to the discontinuity \( c \) is positive. In this case one can modify the local map in a neighborhood \( V \) of \( c \) where the fronts never go, i.e. \( V \subset I_0^v \), without modifying these orbits. The existence of fronts in some bistable extended mappings with a \( C^\infty \) local map then directly follows from this observation. Similarly it is possible to extend the result on the velocity of interfaces to these mappings.

Condition (9) in the front definition imposes the shape to be a right continuous function as it is claimed in relation (11). However one can ask about the existence of fronts of a different kind, namely fronts having a left continuous shape satisfying instead of (9), the condition

\[
\begin{align*}
\dot{\phi}(x) &< c \quad \text{if} \quad x \leq 0 \\
\dot{\phi}(x) &\geq c \quad \text{if} \quad x > 0
\end{align*}
\]

Obviously, if such fronts exist, we have

\[
\phi(0) = \phi(x-0),
\]

where \( \phi \) is given by (10). Consequently according to Theorem 3.2 the dynamical system (1) admits fronts with a left continuous shape iff it has fronts with a right continuous shape \( \phi \) which does not touch \( c \) (i.e. \( \phi(0) \neq c \)). The latter occurs when the velocity belongs to the interior of a plateau. Moreover following the previous argument, fronts with a left continuous shape also exist in a larger class of bistable extended mappings for which smooth local maps are allowed.

Still in \( I_0^v \) for \( v \in D(\phi(0,.)) \), the local map can be modified (now only) at \( c \) in order to obtain a third kind of fronts still with the same velocity \( v \) but which will be unstable (in the sense of Lyapunov)\footnote{Notice that Theorem 4.3 imposes all the fronts defined here to propagate with the same velocity.}. One has to consider the following definition of the Heaviside function instead of the one used up to now.

\[
H_r^v(x) = \begin{cases} 
0 & \text{if } x < 0 \\
r & \text{if } x = 0 \\
1 & \text{if } x > 0
\end{cases}
\]

for 0 \( \leq r \leq 1 \). The corresponding front shape follows from the introduction of this function in condition (3) and from solving the subsequent shape equation. The solution \( \phi' \) now depends continuously on \( r \) and we have, writing explicitly the dependence on \( r \),

\[
\phi'(0-0,v,1) = \phi'(0,v,0) < \phi'(0,v,r) < \phi'(0,v,1) = \phi(0,v).
\]
Hence given \( c \) inside a plateau, the front shape \( \phi' \) exists if \( r \) is such that
\[
\phi'(0, v, r) = c \quad \text{and} \quad f(c) = ac + r(1 - a).
\]
It is unstable since any small enough positive (resp. negative) perturbation of \( \phi' \) at 0 belongs to the basin of attraction of the front with shape \( \phi \) (resp. \( \tilde{\phi} \). Actually one can show the linear stability of the latter fronts inside the plateaus. For \( r = 1 \), \( \phi' \) coincide with \( \phi \) whereas for \( r = 0 \) it is nothing but \( \tilde{\phi} \). Moreover the extension from discontinuous to \( C^\infty \) local map can still be done inside a plateau by requiring \( f(c) = ac + r(1 - a) \). The front shape built with the present definition of the Heaviside function will be stable if \( |f'(c)| < 1 \) and unstable if \( |f'(c)| > 1 \).

According to these results, we conjecture that in the dynamical system (1), if the local map is continuous and bistable, there are fronts with right continuous shape, fronts with left continuous shape and unstable fronts, the various shapes only differing at their discontinuity points.

Keeping on the assumption \( c \in I_0 \), one can modify the couplings in the following way. We perturb \( \mathcal{L} \) and \( \mathcal{L}' \) with two homogeneous operators in \( \mathcal{M} \), namely \( \tilde{\mathcal{L}} \) and \( \tilde{\mathcal{L}}' \), of class \( C^1 \). Instead of (1) we now consider the following mapping on \( \mathcal{M} \)
\[
\dot{u}_{t+1} = \mathcal{F}(u_t, \epsilon),
\tag{27}
\]
where
\[
\mathcal{F}(u, \epsilon) = (1 - \epsilon)\mathcal{L} + \epsilon\tilde{\mathcal{L}} + (1 - \epsilon)\mathcal{L}' + \epsilon\tilde{\mathcal{L}}' \mathcal{F} \circ u.
\]
A front of velocity \( v \) in this dynamical system is a travelling wave for which the shape obeys the equation
\[
\mathcal{G}(u, \epsilon) \equiv \mathcal{F}(u, \epsilon) - \sigma^v u = 0
\]
with the condition \( H_c \circ u = H_0 \) (where the Heaviside function is the right continuous one). Its existence can be proved using a continuation of the shape \( \phi \) at \( \epsilon = 0 \) (similarly as for the continuation of solutions from the uncoupled limit in [11]).

**Proposition 6.1.** Given \( \mathcal{L}, \mathcal{L}', \tilde{\mathcal{L}} \) and \( \tilde{\mathcal{L}}' \), \( v \in D(\phi(0,.)) \) and \( c \in I_0 \), it exists \( \epsilon_0 \) such that for all \( \epsilon \in [0, \epsilon_0) \), the system (27) has fronts of velocity \( v \); their shape being a continuous function of \( \epsilon \).

**Proof:** For \( \epsilon = 0 \) let \( \phi \) be given by Theorem 3.2. By assumption on \( v \) and \( c \) we have
\[
\inf_x |\phi(x) - c| = \delta,
\]
where \( \delta \) is given by
\[
\delta = \min \{c - \phi(0 - 0, v), \phi(0, v) - c\} > 0.
\]
Let \( U \) be the neighbourhood of \( \phi \) in \( \mathcal{M} \) defined as the set of functions \( u \) such that
\[
\exists x_0 \in \mathbb{R} : \|\phi - \sigma^{x_0} u\| < \delta,
\]
hence for \( u \in U \), we have
\[
\begin{cases}
   u(x) < c & \text{if } x < x^0 \\
   u(x) > c & \text{if } x \geq x^0.
\end{cases}
\]
In \( U \), \( \mathcal{G} \) is \( C^1 \) since \( \tilde{\mathcal{L}}, \tilde{\mathcal{L}}' \) and \( \mathcal{F} \) are. Moreover \( D_u \mathcal{G}(\phi, 0) \) is invertible and its inverse is bounded as follows from solving the front shape equation in [11]. By the
implicit function theorem it exists $\epsilon_0$ and a unique continuous function $u(\epsilon)$ such that $u(0) = \phi$ and for any $\epsilon \in [0, \epsilon_0)$, we have $u(\epsilon) \in U$ and $\mathcal{G}(u(\epsilon), \epsilon) = 0$. \hfill \Box$

Finally we notice that no supplementary difficulties are found when one combines the two previous extensions, that is to say when one both perturbs the local map and the couplings.

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A Proof of Lemma 3.3

In all the proof, we will employ the sequences \( \{k_n\}_{n \in \mathbb{N}}, k_n \in \mathbb{N} \) and \( \{\epsilon_n\}_{n \in \mathbb{N}}, \epsilon_n \in \mathbb{R}, \epsilon_n > 0 \) so that
\[
\lim_{n \to +\infty} k_n = +\infty \quad \text{and} \quad \lim_{n \to +\infty} \epsilon_n = 0.
\]

(i) By assumption let \( a, b \in E(h_1) \) be such that \( a < b \) and let \( d \in E(h_2) \). Since \( \theta \) is irrational the sequences \( k_n \) and \( \epsilon_n \) can be chosen so that
\[
-\epsilon_n < - \left( (k_n + 1)\theta - \frac{d-a}{b-a} \right) \leq 0 \quad \forall n \in \mathbb{N},
\]
where \( \{.\} \) stands for the fractional part of a real number \( \mathbb{R} \). Moreover since \( 0 < \theta < 1 \) the sequence
\[
j_n = \lfloor (k_n + 1)\theta - \frac{d-a}{b-a} \rfloor,
\]
satisfies the inequalities \( 0 \leq j_n \leq k_n \) for \( n \) sufficiently large, say \( n \geq n_0 \). Here \( \lfloor x \rfloor \) denotes for the floor function \( \mathbb{R} \). Using the equality \( x = \lfloor x \rfloor + \{x\} \), we obtain
\[
-(b-a)\epsilon_n < d + (k_n - j_n)a + j_nb - (k_n + 1)(a + \theta(b-a)) \leq 0 \quad \forall n \geq n_0,
\]
that is to say \( a + \theta(b-a) \in P \).

(ii) From relation (14) let us assume that
\[
v \in P \setminus \bigcup_{k=0}^{\infty} \frac{E(h_2) + \epsilon_k E(h_1)}{k+1}.
\]
Then there exist the sequences \( \{a_{i,n}\}_{n \in \mathbb{N}, 1 \leq i \leq k_n}, a_{i,n} \in E(h_1) \), and \( \{d_n\}_{n \in \mathbb{N}}, d_n \in E(h_2) \) such that
\[
-\epsilon_n < \frac{1}{k_n + 1} \left( d_n + \sum_{i=1}^{k_n} a_{i,n} \right) - v \leq 0 \quad \forall n \in \mathbb{N}.
\]
Using the definition of \( i_j \) and \( s_j \), we have
\[
\frac{i_2 + k_n i_1}{k_n + 1} \leq \frac{1}{k_n + 1} \left( d_n + \sum_{i=1}^{k_n} a_{i,n} \right) \leq \frac{s_2 + k_n s_1}{k_n + 1} \quad \forall n \in \mathbb{N}.
\]
Consequently if \( E(h_1) \) and \( E(h_2) \) are bounded then \( i_1 \leq v \leq s_1 \). In this case one has
\[
E(\phi(0,\cdot)) \cap [i_1, s_1]^c = \bigcup_{k=0}^{\infty} \frac{E(h_2) + \epsilon_k E(h_1)}{k+1} \cap [i_1, s_1]^c,
\]
and (ii) is proved.

(iii) For \( \theta > 1, \theta \) irrational and \( v = a + \theta(b-a) \) where \( a, b \in E(h_1) \) are such that \( a < b \), let \( \{d_n\}_{n \in \mathbb{N}} \) be an increasing sequence of elements of \( E(h_2) \) such that
\[
\lim_{n \to +\infty} d_n = +\infty.
\]
Let us define
\[
\beta_n = \frac{d_n - v}{b-a} \quad \text{and} \quad \tilde{k}_n = \lceil \frac{b-a}{\theta} \rceil,
\]
where \( \lceil x \rceil \) stands for the ceiling function \( \mathbb{R} \). By compactness it exists \( \alpha \in [0, \theta] \) and a strictly increasing sequence of integers \( \{n_i\}_{i \in \mathbb{N}} \) such that
\[
\alpha = \lim_{i \to +\infty} \tilde{k}_n, \theta - \beta_n.
\]
Let also \( \{m_j\}_{j \in \mathbb{N}}, m_j \in \mathbb{N} \) be such that
\[
\frac{\epsilon_j}{2} \leq \{m_j \theta + \alpha\} < \frac{2\epsilon_j}{3}.
\]
The increase of \( \beta_n \) results in the following. For all \( j \) it exists \( n_j \) such that
\[
-1 + \frac{\beta_{n_j} + 1}{\theta - 1} - \frac{\beta_{n_j}}{\theta} \geq m_j \quad \text{and} \quad |\tilde{k}_{n_j} \theta - \beta_{n_j} - \alpha| < \frac{\epsilon_j}{3}.
\]
Finally let \( k_j = \tilde{k}_{n_j} + m_j \). From the previous inequalities, we obtain the following
\[
0 \leq |k_j \theta - \beta_{n_j}| \leq k_j
\]
and
\[
0 \leq \{k_j \theta - \beta_{n_j}\} < \epsilon_j.
\]
Now let \( p_j = \lfloor k_j \theta - \beta_{n_j} \rfloor \) to obtain
\[
-(b - a)\epsilon_j < d_{n_j} + (k_j - p_j)a + p_j b - (k_j + 1)(a + \theta(b - a)) \leq 0,
\]
and then to conclude that \( v \in P \).

**B Proof of Lemma 3.10**

Lemma 3.10 is a consequence of the combination of the following statements.

**Lemma B.1**

(i) If \( D(h_2) \neq \emptyset \) and \( E(h_1) = D(h_1) \) then \( D(\phi(0, .)) \) is dense in \((i_1, s_1)\).

(ii) If \( \sup D(h_2) = +\infty \) and \( a \in D(h_1) \) then \( D(\phi(0, .)) \) is dense in \((a, +\infty)\).

(iii) If \( \inf D(h_2) = -\infty \) and \( a \in D(h_1) \) then \( D(\phi(0, .)) \) is dense in \((-\infty, a)\).

**Lemma B.2** If \( D(\phi(0, .)) \) is dense in \((v_1, v_2)\) then \( G \) is nowhere dense in \( \bar{v}^{-1}(v_1, v_2) \).

**Proof of Lemma B.1** Let \( a, b \in D(h_1) \) and \( d \in D(h_2) \). We have
\[
\forall 0 \leq j \leq k \quad \frac{d + (k - j)a + jb}{k + 1} \in D(\phi(0, .)).
\]
Hence for any \( \theta \in [0, 1] \) by choosing \( j_k = \lfloor k \theta \rfloor \) we obtain
\[
a + \theta(b - a) = \lim_{k \to \infty} \frac{d + (k - j_k)a + j_k b}{k + 1},
\]
which proves (i).

Now let \( \{d_n\}_{n \in \mathbb{N}}, d_n \in D(h_2) \) and \( \lim_{n \to \infty} d_n = +\infty \). For \( \theta > 0 \) by choosing \( k_n = \lfloor \frac{d_n}{\theta} \rfloor \) we have
\[
\frac{d_n + k_n a}{k_n + 1} \in D(\phi(0, .)),
\]
and thus
\[
a + \theta = \lim_{k \to \infty} \frac{d_n + k_n a}{k_n + 1},
\]
(ii) is proved and one can proceed similarly for (iii). \(\square\)

To prove Lemma B.2 we need the following auxiliary lemma.
Lemma B.3 If $\bar{v}(c_1) = \bar{v}(c_2)$ then $(c_1, c_2) \cap G = \emptyset$.

Proof: Let $c_1 < c_2$ be such that $\bar{v}(c_1) = \bar{v}(c_2)$. For any $c \in (c_1, c_2)$ we obtain, by using the definition of the function $\bar{v}$, $\bar{v}(c) = \bar{v}(c_1) = \bar{v}(c_2)$ and

$$\phi(0, \bar{v}(c)) \geq c_2 > c > c_1 \geq \phi(0, \bar{v}(c) - 0) = \phi(0 - 0, \bar{v}(c)),$$

and hence $c \notin G$. \hfill \Box

Now let $c_1 < c_2$ be such that $c_1, c_2 \in G \cap \bar{v}^{-1}((v_1, v_2))$. Then either $\bar{v}(c_1) = \bar{v}(c_2)$ and by Lemma B.3 $(c_1, c_2) \cap G = \emptyset$, or $\bar{v}(c_1) < \bar{v}(c_2)$. In the latter case by density of $D(\phi(0, .))$ it exists $v^* \in (\bar{v}(c_1), \bar{v}(c_2)) \cap D(\phi(0, .))$. Let us choose $c_1^*, c_2^* \in (\phi(0, v^* - 0), \phi(0, v^*))$ such that $c_1^* < c_2^*$. We have $(c_1^*, c_2^*) \subset (c_1, c_2)$ and $\bar{v}(c_1^*) = \bar{v}(c_2^*)$. So again Lemma B.3 implies that $(c_1^*, c_2^*) \cap G = \emptyset$. In both cases $G$ is not dense in $(c_1, c_2)$. \hfill \Box