Supersymmetric isolated horizons in ADS spacetime

Ivan Booth

Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John’s, Newfoundland, Canada, A1C 5S7

Tomáš Liko

Department of Physics and Physical Oceanography
Memorial University of Newfoundland
St. John’s, Newfoundland, Canada, A1B 3X7
And
Institute for Gravitation and the Cosmos
Pennsylvania State University
University Park, Pennsylvania 16802, USA

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Abstract

We discuss various physical aspects of nonextremal, extremal and supersymmetric black holes in asymptotically anti-de Sitter (ADS) spacetimes. Specifically, we discuss how the isolated horizon (IH) framework leads to an ambiguity-free description of rotating black holes in these spacetimes. We then apply this framework to investigate the properties of supersymmetric isolated horizons (SIHs) in four-dimensional $N = 2$ gauged supergravity. Among other results we find that they are necessarily extremal, that rotating SIHs must have non-trivial electromagnetic fields, and that non-rotating SIHs necessarily have constant curvature horizon cross sections and a magnetic (though not electric) charge.

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*Electronic mail: ibooth at math.mun.ca
†Electronic mail: liko at gravity.psu.edu
1 Introduction

Currently there is a lot of interest in the anti-de Sitter (ADS)/conformal field theory (CFT) correspondence [1–4]. A significant amount of effort on the gravity side has been focused on finding charged and rotating black hole solutions in five-dimensional ADS spacetime, both nonextremal in general [5–10] and supersymmetric in particular [11–14].

For these black holes, however, there is an ambiguity in how the conserved charges are defined. This was first pointed out by Caldarelli et al [15]. The ambiguity arises because for rotating black holes in ADS spacetime there are two distinct natural choices for the timelike Killing field. Defining the charges with respect to one corresponds to a frame at infinity that is non-rotating and with the other corresponds to a frame at infinity that is rotating. The original motivation for defining the conserved charges using the latter Killing field was that the corresponding boundary CFT conserved charges satisfy the first law of thermodynamics [5]; but this comes at the cost that the bulk conserved charges do not [15, 16]. This claim has by now been corrected. As was shown in [17], one can always pass from the bulk conserved charges to the boundary conserved charges in such a way that both sets separately satisfy the first law. The key to this resolution is that the conserved charges of a rotating black hole in ADS spacetime have to be measured with respect to the timelike vector which corresponds to a frame that is non-rotating at infinity.

From the above considerations, it is clear that rotation in ADS spacetime should be independent of the coordinates that are used. This is especially crucial when considering supersymmetric black holes in ADS spacetime (the extremal limit of a non-rotating ADS black hole results in a naked singularity). The purpose of this paper is two-fold: to discuss how the isolated horizon framework provides a resolution to the above pathology, and to investigate the conditions imposed by supersymmetry on the corresponding black holes (in four dimensions).

We shall consider the phase space of solutions to the equations of motion for the Einstein-Maxwell-Chern-Simons (EM-CS) action

\[ S = \frac{1}{16\pi G_D} \int_M \Sigma_{IJ} \wedge \Omega^{IJ} - 2\Lambda \epsilon - \frac{1}{4} F \wedge *F - \frac{2\lambda}{3\sqrt{3}} A \wedge F^{(D-1)/2}, \]  

where \( \lambda = 0 \) if \( D \) is even and \( \lambda = 1 \) if \( D \) is odd. In this paper, spacetime indices \( a, b, \ldots \in \{0, \ldots, D-1\} \) will be raised and lowered using the metric tensor \( g_{ab} \), while internal Lorentz indices \( I, J, \ldots \in \{0, \ldots, D-1\} \) will be raised and lowered using the Minkowski metric \( \eta_{IJ} = \text{diag}(-1, 1, \ldots, 1) \). The action (1) depends on the coframe \( e^I \), the gravitational connection \( A^I \) and the electromagnetic connection \( A \). The coframe determines the metric \( g_{ab} \), \( (D - m) \)-form \( \Sigma_{I_1 \ldots I_m} \) and spacetime volume.
element $\epsilon_{a_1...a_D}$:

$$g_{ab} = \eta_{IJ} e^I_a \otimes e^J_b$$

$$\Sigma_{I_1...I_m} = \frac{1}{(D - m)!} \epsilon_{I_1...I_m I_{m+1}...I_D} e^{I_{m+1}} \wedge \cdots \wedge e^{I_D}$$

$$\epsilon_{a_1...a_D} = \epsilon_{I_1...I_D} e^{I_1} \cdots e^{I_D}.$$  \hfill (3)

Here $\epsilon_{I_1...I_D}$ is the totally antisymmetric Levi-Civita tensor. The volume $D$-form $\epsilon$ is given by

$$\epsilon = e^0 \wedge \cdots \wedge e^{D-1}.$$  \hfill (4)

The gravitational connection determines the curvature two-form

$$\Omega^I_J = dA^I_J + A^I_K \wedge A^K_J = \frac{1}{2} R^I_{JKL} e^K \wedge e^L,$$  \hfill (6)

with $R^I_{JKL}$ as the Riemann tensor. The electromagnetic connection $A$ determines the curvature

$$F = dA.$$  \hfill (7)

The constants appearing in the action (1) are the $D$-dimensional Newton constant $G_D$ and the cosmological constant which in terms of the ADS radius $L$ is given by

$$\Lambda = -(D - 1)(D - 2)/(2L^2).$$

The equations of motion are derived from independently varying the action with respect to the fields $(e, A, A)$. To get the equation of motion for the coframe we note the identity

$$\delta \Sigma_{I_1...I_m} = \delta e^M \wedge \Sigma_{I_1...I_m M}.$$  \hfill (8)

This leads to

$$\Sigma_{IJK} \wedge \Omega^{JK} + \frac{3}{L^2} \Sigma_I = \mathcal{T}_I,$$  \hfill (9)

where $\mathcal{T}_I$ denotes the electromagnetic stress-energy $(D - 1)$-form. The equation of motion for the connection $A$ is

$$\mathcal{D} \Sigma_{IJ} = 0;$$  \hfill (10)

this equation says that the torsion $T^I = \mathcal{D} e^I$ is zero. The equation of motion for the connection $A$ is

$$d \star F - \frac{4(D + 1)\Lambda}{3\sqrt{3}} F^{(D-1)/2} = 0.$$  \hfill (11)

The second term in this equation is the contribution due to the CS term in the action. In even dimensions the equation reduces to the standard Maxwell equation $d \star F = 0$. 

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Figure 1: The region of the $D$-dimensional spacetime $\mathcal{M}$ being considered has an internal boundary $\Delta$ representing the event horizon, and is bounded by two $(D-1)$-dimensional spacelike hypersurfaces $M^\pm$ which extend from the inner boundary $\Delta$ to the boundary at infinity $\mathcal{I}$. $M$ is a partial Cauchy surface that intersects $\Delta$ in a compact $(D-2)$-space $S$ and $\mathcal{I}$ in a $(D-2)$-space $C$.

In four dimensions the theory describes the bosonic sector of $N=2$ gauged supergravity, and in five dimensions the theory describes the bosonic sector of $N=1$ gauged supergravity. General properties of supersymmetric black holes in ADS supergravity were recently investigated in [13, 14]. Here we investigate the important issue of rotation using a more general framework that does not require the spacetime to be globally stationary.

2 Boundary conditions

Let us begin with some general remarks concerning the geometrical setup. We shall consider a manifold $(\mathcal{M}, g_{ab})$ with boundaries. The conditions that are imposed on the inner boundary capture the notion of an isolated black hole that is in local equilibrium with its (possibly) dynamic surroundings. For details we refer the reader to [19] and the references therein. For the boundary conditions we follow [18, 20, 21].

First we give some general comments about the structure of the manifold. Specifically, $\mathcal{M}$ is a $D$-dimensional Lorentzian manifold with topology $\mathbb{R} \times M$, contains a $(D-1)$-dimensional null surface $\Delta$ as inner boundary (representing the horizon), and is bounded by $(D-1)$-dimensional spacelike manifolds $M^\pm$ that extend from $\Delta$ to infinity. Following [18], we assume that the manifold $\mathcal{M}$ can be conformally completed to an asymptotically ADS spacetime $\tilde{\mathcal{M}}$, where $\tilde{\mathcal{M}} \cong \mathcal{M} \cup \mathcal{I}$ and $\mathcal{I}$ is a timelike boundary. The null surface $\Delta \cong \partial \mathcal{M}$ is by definition the inner boundary of $\mathcal{M}$. The topology of $\Delta$ is $\mathbb{R} \times S^{D-2}$ and the topology of $\mathcal{I}$ is $\mathbb{R} \times \mathbb{C}^{D-2}$, with $S^{D-2}$ a compact $(D-2)$-space and $\mathbb{C}^{D-2}$ a $(D-2)$-space. $M$ is a partial Cauchy surface such that $S^{D-2} \cong \Delta \cap M$ and $\mathbb{C}^{D-2} \cong \mathcal{I} \cap M$. See Figure 1.

$\Delta$ is a weakly isolated horizon (WIH). That is, $\Delta$ is a null surface and has a degenerate metric $g_{ab}$ with signature $0+\ldots+$ (with $D-2$ nondegenerate spatial direc-
tions) along with an equivalence class of null normals $[\ell]$ (defined by $\ell \sim \ell' \iff \ell' = k\ell$ for some constant $k$) such that the following conditions hold: (a) the expansion $\theta(\ell)$ of $\ell_a$ vanishes on $\Delta$; (b) the field equations hold on $\Delta$; (c) the stress-energy tensor is such that the vector $-T^{ab}_a\ell^b$ is a future-directed and causal vector; (d) $\mathcal{L}_\ell \mathcal{A} = 0$ and $\ell^\flat \mathcal{A} = 0$ for all $\ell \in [\ell]$ (see below).

The first three conditions determine the intrinsic geometry of $\Delta$. Since $\ell$ is normal to $\Delta$ the associated null congruence is necessarily twist-free and geodesic. By condition (a) that congruence is non-expanding. Then the Raychaudhuri equation implies that $T_{ab}^a\ell^b = -\sigma_{ab}\sigma^{ab}$, with $\sigma_{ab}$ the shear tensor, and applying the energy condition (c) we find that $\sigma_{ab} = 0$. Thus, together these conditions tell us that the intrinsic geometry of $\Delta$ is “time-independent” in the sense that all of its (two-dimensional) cross sections have identical intrinsic geometries.

Next, the vanishing of the expansion, twist and shear imply that [20]

$$\nabla_\ell \ell_b \approx \omega_a \ell_b,$$

with “$\approx$” denoting equality restricted to $\Delta$ and the underarrow indicating pull-back to $\Delta$. Thus the one-form $\omega$ is the natural connection (in the normal bundle) induced on the horizon. These conditions also imply that [20]

$$\ell^\flat \mathcal{F} = 0.$$  (13)

With the field equations (11) and the Bianchi identity $d\mathcal{F} = 0$, it then follows that

$$\mathcal{L}_\ell \mathcal{F} \approx \ell^\flat d\mathcal{F} + d(\ell^\flat \mathcal{F}) = 0.$$  (14)

This implies that the electric charge is independent of the choice of cross sections $S^{D-2}$ [22]. Similarly (in four-dimensions) the magnetic charge is also a constant.

From (12) we find that

$$\ell^a \nabla_a \ell^b = (\ell^\flat \omega) \ell^b,$$

and define the surface gravity $\kappa(\ell) = \ell^\flat \omega$ as the inaffinity of this geodesic congruence. Note that it is certainly dependent on specific element of $[\ell]$ as under the transformation $\ell \rightarrow k\ell$:

$$\kappa(\ell) \rightarrow k\kappa(\ell).$$  (16)

In addition to the surface gravity, we also define the electromagnetic scalar potential $\Phi(\ell) = -\ell^\flat A$ for each $\ell \in [\ell]$ and this has a similar dependence.

Now, it turns out that if the first three conditions hold, then one can always find an equivalence class $[\ell]$ such that (d) also holds. Hence this last condition does not further restrict the geometries under discussion, but only the scalings of the null normal. However, making such a choice ensures that [20]:

$$d\kappa(\ell) = d(\ell^\flat \omega) = 0 \quad \text{and} \quad d\Phi(\ell) = d(\ell^\flat A) = 0.$$  (17)

This establishes the zeroth law of WIH mechanics: the surface gravity and scalar potential are constant on $\Delta$. 


3 Conserved charges: non-rotating reference frame

The derivation of the conserved charges involves first finding the symplectic structure on the covariant phase space $\Gamma$ consisting of solutions $(e, A, A)$ to the field equations (9), (10) and (11) on $\mathcal{M}$. Once we have a suitable (closed and conserved) symplectic two-form

$$\Omega \equiv \Omega(\delta_1, \delta_2),$$

the conserved charges are obtained by evolving the system with respect to appropriate vector fields (symmetries). Two sets of conserved charges arise this way: those at $I$ corresponding to a non-rotating frame at infinity and those at $\Delta$ corresponding to the horizon charges that satisfy the first law of black-hole mechanics.

The antisymmetrized second variation of the surface term gives the symplectic current. Integrating this current over a spacelike hypersurface $M$ gives the bulk symplectic structure (with the choice of $M$ being arbitrary). This two-form, however, is generally not conserved. This is due to the fact that the symplectic current can “leak” across the horizon. In order to obtain a symplectic structure that is conserved on $\Gamma$ we need to find the pull-back of the current to $\Delta$ and add the integral of this term to the symplectic structure.

To find the conserved charges for any system, one examines the canonical transformations that are generated by the corresponding Hamiltonians. For some smooth vector field $\xi$ that preserves the boundary conditions of Section 2 and any vector field $\delta$ that is tangent to $\Gamma$, it follows that the necessary and sufficient condition for $\delta \xi$ to be a phase space symmetry (i.e. that $\mathcal{L}_{\delta} \Omega = 0$ on $\Gamma$) is that

$$\Omega(\delta, \delta \xi) = \delta \mathcal{H}_\xi,$$

where $\mathcal{H}_\xi$ is the Hamiltonian generating the infinitesimal diffeomorphism and is given by

$$\mathcal{H}_\xi = \mathcal{D}_\xi^{(I)} - \mathcal{D}_\xi^{(\Delta)}.$$

The conserved charges for WIHs in asymptotically ADS spacetimes with no matter fields were derived in [18]. Inclusion of matter fields does not involve any significant modifications to the conserved charges.

As was shown in Appendix B of [23], inclusion of antisymmetric tensor fields in the action does not contribute anything to the charges at $I$ because the fields fall off too quickly. Therefore the charges at infinity for EM-CS theory are precisely the ones that were derived in [18]; these are the Ashtekar-Magnon-Das (AMD) charges [24, 25]:

$$\mathcal{D}_\xi^{(I)} = \frac{L}{8\pi G_D} \oint_{C_{D-2}} \tilde{E}_{ab} \xi^a \tilde{u}^b \tilde{\xi},$$

where $L$ is the ADM mass of the black hole.
with $\tilde{n}^a$ the unit timelike normal to $\mathbb{C}^{D-2}$, $\tilde{\varepsilon}$ the area form on $\mathbb{C}^{D-2}$ and $\tilde{E}_{ab}$ the leading-order electric part of the Weyl tensor. Explicitly we have that
\begin{equation}
\tilde{E}_{ab} = \frac{1}{D-3} \tilde{\Omega}^{3-D} \tilde{C}_{abcd} \tilde{n}^c \tilde{n}^d ,
\end{equation}
where $\tilde{n}^a = \tilde{\nabla}^a \Omega$, and $\Omega$ is the conformal factor defined via $\tilde{g}_{ab} = \tilde{\Omega}^2 g_{ab}$ which relates the unphysical metric $\tilde{g}_{ab}$ on $\tilde{\mathcal{M}}$ and the physical metric $g_{ab}$ on $\mathcal{M}$.

Gibbons et al [16] showed that the asymptotic time translation Killing field for an exact solution has to be chosen in such a way that the frame at infinity is non-rotating. If this is done then the AMD charge evaluated for the solution will result in an expression for mass that satisfies the first law. Moreover, Gibbons et al [17] showed that using this definition for the asymptotic time translation has to be used for a consistent transition to the conserved charges of the boundary CFT.

At the horizon, inclusion of Maxwell fields gives rise to an electric charge [26]
\begin{equation}
Q = \frac{1}{8\pi G_D} \oint_{S^{D-2}} \Phi .
\end{equation}
where $\Phi$ is the electromagnetic charge density:
\begin{equation}
\Phi = \star F - \frac{4(D+1)\lambda}{3\sqrt{3}} A \wedge F^{(D-3)/2} ,
\end{equation}
(not to be confused with the Coulomb potential $\Phi$). Due to the presence of the CS term (when $\lambda = 1$), the charge $Q$ may fail to be gauge invariant if the horizon has a non-trivial topology. Other conserved quantities at the horizon include the horizon entropy and angular momenta [26]
\begin{align}
S &= \frac{1}{4G_D} \oint_{S^{D-2}} \tilde{\varepsilon} ,
\end{align}
\begin{align}
\mathcal{J}_\iota &= \frac{1}{8\pi G_D} \oint_{S^{D-2}} [(\phi_\iota \omega) \tilde{\varepsilon} + (\phi_\iota A) \Phi] ,
\end{align}
where $\phi_\iota$ are rotational Killing fields, and we define the area element of the cross section $S^{D-2}$ of the horizon
\begin{equation}
\tilde{\varepsilon} = \vartheta^{(1)} \wedge \cdots \wedge \vartheta^{(D-2)} .
\end{equation}
The index $\iota \in \{1, \ldots, [(D-1)/2]\}$ is a rotation index and corresponds to $[(D-1)/2]$ independent rotation parameters in $D$ dimensions and are given by the Casimirs of the rotation group $SO(D-1)$, where $\lfloor \cdot \rfloor$ denotes “integer value of”.

It was shown in [26] that these charges satisfy the first law. The angular momenta contain contributions from gravitational as well as electromagnetic fields, referred to here as $\mathcal{J}_{Grav}$ and $\mathcal{J}_{EM}$ respectively. If $\phi$ is the restriction to $\Delta$ of a global rotational Killing field $\varphi$ contained in $\mathcal{M}$, then the electromagnetic contribution to (26) can be interpreted as angular momentum of the electromagnetic radiation in the bulk [21]. This is because the bulk integral can be written as the
sum of two surface terms – one at \( \Delta \) and one at \( \mathcal{I} \); the latter surface term is zero due to the fall off conditions at \( \mathcal{I} \). Therefore the condition for a WIH to be non-rotating is that \( \mathcal{J}_{\text{Grav}} = 0 \) for all rotational Killing fields. This will be discussed in greater detail in the next section.

4 Supersymmetric isolated horizons

Until now we have discussed the mechanics of WIHs in arbitrary dimensions. We now specialize to supersymmetric horizons in ADS spacetime and in particular we focus on the bosonic sector of four-dimensional \( N = 2 \) gauged supergravity. In this case, black holes are solutions to the bosonic equations of motion and so the fermion fields vanish. By definition, supersymmetric solutions are invariant under the full supersymmetry transformations. This means that for black hole solutions, these transformations should leave the fermion fields unchanged (and vanishing). Therefore any such black hole solutions must admit a Killing spinor field.

For full stationary black hole solutions such as those discussed in [27, 28], the Killing spinor gives rise to a (timelike) time-translation Killing vector field in the region outside of the black hole horizon. However, in the quasilocal spirit of the isolated horizon programme we will only assume the existence of a Killing spinor on the horizon itself. In this case the spinor will generate a null geodesic vector field that has vanishing twist, shear, and expansion and this is an allowed \( \ell \) on the WIH.

In order to proceed we now restrict our attention to fully isolated horizons (IHs). These are WIHs for which there is a scaling of the null normals for which the commutator \([\mathcal{L}_\ell, \mathcal{D}] = 0\), where \( \mathcal{D} \) is the intrinsic covariant derivative on the horizon. In contrast to condition (d) for WIHs, this condition cannot always be met and geometrically such horizons not only have time-invariant intrinsic geometry, they also have time-invariant extrinsic geometry. That said it is clear that this condition similarly fixes \( \ell \) only up to a constant scaling. As such it does not uniquely determine the value of the surface gravity \( \kappa(\ell) \) but does fix its sign. In particular this allows us to invariantly say whether or not \( \kappa(\ell) \) vanishes. This then gives rise to an invariant characterization of extremality that is intrinsic to the horizon: an extremal isolated horizon is an IH on which \( \kappa(\ell) = 0 \). Further discussion of this notion of extremality and how it relates to other characterizations may be found in [30].

Finally we define a supersymmetric isolated horizon (SIH) as an IH on which the null vector generated by the Killing spinor coincides (up to a free constant) with the preferred null vector field arising from the IH structure. As we shall now see these are necessarily extremal as well as having restricted geometry, rotation, and matter fields.

For four-dimensional \( N = 2 \) gauged supergravity, we shall employ the conven-
The corresponding (bosonic) action is
\[ S = \frac{1}{16\pi G_4} \int_{\mathcal{M}} \Sigma_{IJ} \wedge \Omega^{IJ} + \frac{6}{L^2} \epsilon - \frac{1}{4} F \wedge \star F . \] (28)

The necessary and sufficient condition for supersymmetry with vanishing fermion fields is that there exists a Killing spinor \( \epsilon^a \) such that
\[ \left[ \nabla_a + i \frac{1}{4} F_{bc} \gamma^{bc} \gamma_a + \frac{1}{L} \gamma_a \right] \epsilon = 0 . \] (29)

Here, \( \gamma^a \) are a set of gamma matrices that satisfy the anticommutation rule
\[ \gamma^a \gamma^b + \gamma^b \gamma^a = 2 \eta^{ab} \] (30)
and the antisymmetry product
\[ \gamma_{abcd} = \epsilon_{abcd} . \] (31)

\( \gamma_{a_1...a_D} \) denotes the antisymmetrized product of \( D \) gamma matrices. The spinor \( \epsilon \) satisfies the reality condition
\[ \bar{\epsilon} = i(\epsilon)^\dagger \gamma_0 ; \] (32)

overbar denotes complex conjugation and \( \dagger \) denotes Hermitian conjugation.

From \( \epsilon \) one can construct five bosonic bilinears \( f, g, V^a, W^a \) and \( \Psi^{ab} = \Psi^{[ab]} \) where
\[ f = \bar{\epsilon} \epsilon , \quad g = i\bar{\epsilon} \gamma_5 \epsilon , \quad V^a = \bar{\epsilon} \gamma^a \epsilon , \quad W^a = i\bar{\epsilon} \gamma_5 \gamma^a \epsilon , \quad \Psi^{ab} = \bar{\epsilon} \gamma^{ab} \epsilon . \] (33)

These are inter-related by several algebraic relations (from the Fierz identities) and differential equations (from the Killing equation (29)) [29]. For our purposes the significant ones are:
\[ V_a V^a = -W_a W^a = -(f^2 + g^2) , \] (34)
\[ V^a W_a = 0 , \] (35)
\[ g W_a = \Psi_{ab} V^b , \] (36)
\[ f \Psi_{ab} = -\epsilon_{abcd} V^c W^d + \frac{1}{2} \eta_{abcd} \Psi^{cd} , \] (37)
\[ \nabla_a f = F_{ab} V^b , \] (38)
\[ \nabla_a g = -\frac{1}{L} W_a - \frac{1}{2} \epsilon_{abcd} V^b F^{cd} , \] (39)
\[ \nabla_a V_b = \frac{1}{L} \Psi_{ab} - f F_{ab} + \frac{g}{2} \epsilon_{abcd} F^{cd} , \] (40)
\[ \nabla_a A_b = -\frac{g}{L} g_{ab} - F^c (a \epsilon_{b)cd} \Psi^{de} + \frac{1}{4} \eta_{abcd} F^{cd} \Psi^{ef} \] and
\[ \nabla_c \Psi_{ab} = 2 L g_{cl} (a V_b) + 2 F^d (a \epsilon_{b)de} W^e + F^d (a \epsilon_{b)def} W^d F^{ef} . \] (42)

These are general relations for the existence of a Killing spinor in spacetime. Although the Killing spinor may exist in a neighbourhood of the horizon, we only
require that it exist on the horizon itself. Henceforth we specialize by setting \( f = g = 0 \) and at the same time require that the relations hold on \( \Delta \). Thus, the differential equations (38)-(42) are only required to hold when the derivatives are pulled-back onto the horizon.

With \( f = g = 0 \), equation (34) implies that \( V^a \) and \( W^a \) are both null. On an SIH we identify \( \ell^a = V^a \) and so condition (12) together with the differential constraint (40) implies that

\[
\nabla_a \ell_b = \omega_a \ell_b = \frac{1}{L} \Psi_{ab} ,
\]

and using the skew-symmetry of \( \Psi_{ab} \) we can write

\[
\Psi_{ab} = L(\omega_a \ell_b - \omega_b \ell_a) .
\]

Then by equation (36)

\[
\ell \omega = 0 \iff \kappa(\ell) = 0 .
\]

Thus, an SIH is necessarily extremal.

For ease of presentation we now assume that the SIH is foliated into spacelike two-surfaces \( \Delta_v \). One can always construct such a foliation (and its labelling) so that the associated null normal \( n \equiv dv \) satisfies \( \ell \cdot n = -1 \) [31]. Then the two-metric and area form on \( \Delta_v \) can be written as

\[
\tilde{q}_{ab} = g_{ab} + \ell_a n_b + \ell_b n_a \quad \text{and} \quad \tilde{\epsilon}_{cd} = -\ell^a n^b \epsilon_{abcd} \]

respectively. Now we note that \( \omega \) can be written as

\[
\omega_a = -\kappa(\ell) n_a + \tilde{\omega}_a ,
\]

with \( \tilde{\omega} \) the pull-back to \( \Delta_v \) of \( \omega \). Then with \( \kappa(\ell) = 0 \) it follows that \( \omega_a = \tilde{\omega}_a \) and hence \( \omega_a \in T^*(\Delta_v) \). Finally, with respect to this foliation, the usual restriction (13) and (redundantly) equation (38) implies that the electromagnetic field takes the form

\[
F_{ab} = E_{\perp}(\ell_a n_b - n_a \ell_b) + B_{\perp} \tilde{\epsilon}_{ab} + (\tilde{X}_a \ell_b - \tilde{X}_b \ell_a) ,
\]

on \( \Delta \). Here, \( E_{\perp} \) and \( B_{\perp} \) are the electric and magnetic fluxes through the surface and \( \tilde{X}^a \in T(\Delta_v) \) describes flows of electromagnetic radiation along (but not through) the horizon.

With these preliminaries in hand we can consider the properties of SIHs in asymptotically ADS spacetimes. In particular, it was previously shown [26] that in the absence of a cosmological constant, SIHs are necessarily non-rotating with \( \omega = 0 \) and so it is natural to consider how the addition of a negative cosmological constant affects the rotation properties. First, relations (35) and (37) tell us that

\[
W^a = L\beta V^a
\]
for some function $\beta$ (the factor of $L$ has been included for later convenience). Then the pull-back of (39) trivially vanishes without giving us any new information but (41) provides a differential equation for $\beta$ on each $\Delta_v$.

$$d_\alpha \beta + \beta \tilde{\omega}_a = B_\perp \tilde{\omega}_a - E_\perp \tilde{\epsilon}_a^\beta \tilde{\omega}_b ,$$  \hspace{1cm} (50)

where $d_\alpha$ is the intrinsic covariant derivative on $\Delta_v$, along with its time-invariance: $\mathcal{L}_\ell \beta = 0$.

Next applying the various properties of extremal IHs, one can show that the pull-back of (42) is

$$\nabla_\mathcal{E} \Psi_{ab} = 2L \left( \frac{1}{L^2} - \beta B_\perp \right) \tilde{q}_{[a} \ell_{b]} + 2L \beta E_\perp \tilde{\epsilon}_{[a} \ell_{b]} ,$$  \hspace{1cm} (51)

and combining this with (44) we find that

$$d_\alpha \tilde{\omega}_b + \tilde{\omega}_a \tilde{\omega}_b = \left( \frac{1}{L^2} - \beta B_\perp \right) \tilde{q}_{ab} + \beta E_\perp \tilde{\epsilon}_{ab} .$$  \hspace{1cm} (52)

Now as was seen in (26), the gravitational angular momentum associated with a rotational Killing field $\phi^a$ is

$$J_{\text{Grav}} = \frac{1}{8\pi G_4} \oint_{\Delta_v} \tilde{\epsilon}_{\phi} \tilde{\omega} ,$$  \hspace{1cm} (53)

and so a necessary condition for non-zero angular momentum is a non-vanishing rotation one-form $\tilde{\omega}_a$. That said, this is not quite sufficient as it is possible for a non-vanishing $\phi \cdot \tilde{\omega}$ to integrate to zero. For example, consider the case where $\Delta_v$ has topology $S^2$ and $\phi^a$ is a Killing field (and so divergence-free). Then for some function $\zeta$ we can write $\phi^a = \tilde{\epsilon}^{ab} d_b \zeta$ and

$$\oint_{\Delta_v} \tilde{\epsilon}_{\phi} \tilde{\omega} = \oint_{\Delta_v} \zeta d\tilde{\omega} .$$  \hspace{1cm} (54)

Thus, for all closed rotational one-forms ($d\tilde{\omega} = 0$) the associated gravitational angular momentum will vanish. As such, it is standard in the isolated horizon literature (see e.g. [21]) to take $d\tilde{\omega} \neq 0$ as the defining characteristic of a rotating isolated horizon. In our case

$$d_{[a} \tilde{\omega}_{b]} = \beta E_\perp \tilde{\epsilon}_{ab} ,$$  \hspace{1cm} (55)

and so an SIH is rotating if and only if $\beta E_\perp \neq 0$. Thus, a rotating horizon must have a non-trivial electromagnetic field. This is in agreement with known exact solutions: rotating supersymmetric Kerr-Newmann-AdS black holes as well as those with cylindrical or higher genus horizons all have non-trivial EM fields [29].
5 Summary and discussion

Let us summarize the role that IHs play in ADS spacetime, and the resulting conclusions that we can draw from them regarding the generic properties of nonextremal, extremal, and supersymmetric black holes.

The IH framework provides a coherent physical picture whereby two sets of conserved charges arise in ADS spacetime: the charges measured at infinity and the local charges measured at the horizon. The local conserved charges at the horizon then satisfy the first law. When evaluated on exact solutions to the field equations, the charges at infinity correspond to asymptotic symmetries that are measured with respect to a non-rotating frame at infinity.

As was the case for spacetimes with no cosmological constant, supersymmetric isolated horizons in ADS spacetime have vanishing surface gravity and so are always extremal. However, in contrast to the asymptotically flat case, we found that ADS SIHs in four dimensions can be either rotating or non-rotating with strong constraints linking the rotation to the electromagnetic and Killing spinor fields. The use of these constraints to classify SIHs in ADS spacetime will appear in a future work; here we will give a taste of their application by considering the case when $\tilde{\omega} = 0$. Then, the Maxwell equations along with the extremal IH conditions tell us that $E_\perp$ and $B_\perp$ are both constant in time ($\mathcal{L}_\ell E_\perp = \mathcal{L}_\ell B_\perp = 0$) and

$$d_a B_\perp + \tilde{\epsilon}_a^b d_b E_\perp = 0. \quad (56)$$

Hence $E_\perp$ and $B_\perp$ are also constant on each $\Delta_v$. Next the supersymmetry constraint (52) says that

$$\beta B_\perp = \frac{1}{L^2} \quad \text{and} \quad \beta E_\perp = 0. \quad (57)$$

Thus, $E_\perp = 0$ while $B_\perp \neq 0$ – that is, these SIHs necessarily have magnetic, but not electric, charges. Further, applying the extremality condition from [26, 30]:

$$\frac{1}{2} \mathcal{R} = d_a \tilde{\omega}^a + \tilde{\omega}_a \tilde{\omega}^a + T_{ab} \ell^a n^b - \frac{3}{L^2}$$

$$= B_\perp^2 - \frac{3}{L^2}. \quad (58)$$

It is clear that the two-dimensional Ricci curvature $\mathcal{R}$ of the $\Delta_v$ is constant in this case – unfortunately the sign of that curvature does not seem to be determined by the equations. Consulting a listing of exact supersymmetric black hole solutions [29] we see that such solutions are known: specifically there is a supersymmetric asymptotically ADS black hole in four dimensions which can be non-rotating if the horizon cross sections have genus $g > 1$. As prescribed by our formalism, these solutions have magnetic but not electric charge.

The extremality condition (58) also gives us information in the case of rotating SIHs. Specifically, integrating and applying the Gauss-Bonnet theorem we find that:

$$\mathcal{A}_{\Delta_v} = \frac{L^2}{3} \left[ 4\pi (g - 1) + \int_{\Delta_v} \tilde{\epsilon} (E_\perp^2 + B_\perp^2 + \|\tilde{\omega}\|^2) \right], \quad (60)$$
where $A_{\Delta v} \equiv \oint_{\Delta v} \tilde{\epsilon}$ is the area of the horizon cross sections. More generally for non-extremal horizons the equality above becomes a “$\geq$” and so this equality becomes a bound. Then the maximum allowed angular momentum is bound by the genus and area of the horizon; see [30,32] for discussions of the corresponding result for asymptotically flat spacetimes and appendix B of [30] for a particular discussion of Kerr-ADS.

Alternatively, reversing the inequality, one can view it as bounding the allowed area of isolated horizons from below by the scale of the cosmological curvature and the genus of the horizon: higher genus horizons necessarily have larger areas. Similar bounds have previously been discovered for stationary ADS black holes [33–35].

The description of ADS black holes presented here is somewhat different from the description of black holes in globally stationary spacetimes where an ambiguity appears that manifests itself as a choice of whether the conserved charges are measured with respect to a frame at infinity that is rotating or non-rotating. This ambiguity does not appear in the IH framework essentially because the conserved charges of the black hole are measured at the horizon, and the corresponding first law is intrinsic to the horizon with no mixture of quantities there and at infinity!

Note added. After this work was completed, it was brought to our attention that equation (58) has been solved recently in [36] for vacuum gravity in the context of near-horizon geometries.

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