GROUND STATE SOLUTIONS OF NEHARI-POHOZAEV TYPE FOR SCHRÖDINGER-POISSON PROBLEMS WITH GENERAL POTENTIALS

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Abstract. This paper is dedicated to studying the following Schrödinger-Poisson problem
\[
\begin{aligned}
-\triangle u + V(x)u + \phi u &= f(u), \quad x \in \mathbb{R}^3, \\
-\triangle \phi &= u^2, \quad x \in \mathbb{R}^3,
\end{aligned}
\]
where \( V(x) \) is weakly differentiable and \( f \in C(\mathbb{R}, \mathbb{R}) \). By introducing some new tricks, we prove the above problem admits a ground state solution of Nehari-Pohozaev type and a least energy solution under mild assumptions on \( V \) and \( f \). Our results generalize and improve the ones in [D. Ruiz, J. Funct. Anal. 237 (2006) 655-674], [J.J. Sun, S.W. Ma, J. Differential Equations 260 (2016) 2119-2149] and some related literature.

1. Introduction. In this paper, we are concerned with the existence of ground state solutions for the Schrödinger-Poisson problem
\[
\begin{aligned}
-\triangle u + V(x)u + \phi u &= f(u), \quad x \in \mathbb{R}^3, \\
-\triangle \phi &= u^2, \quad x \in \mathbb{R}^3,
\end{aligned}
\]  
(1.1)

where \( V \) and \( f \) satisfy
(V1) \( V \in C(\mathbb{R}^3, \mathbb{R}) \) and the spectrum \( \sigma(-\triangle + V) > 0 \) of the self-adjoint operator
\[-\triangle + V;\]
(F1) \( f \in C(\mathbb{R}, \mathbb{R}), \) and there exist constants \( C_0 > 0 \) and \( p \in (2,6) \) such that
\[|f(t)| \leq C_0 \left(1 + |t|^{p-1}\right), \quad \forall \ t \in \mathbb{R};\]
(F2) \( f(t) = o(t) \) as \( t \to 0.\)

Problem (1.1), also known as the nonlinear Schrödinger-Maxwell problem, was first introduced in [1] as a model describing solitary waves for the nonlinear stationary Schrödinger equations interacting with the electrostatic field. It has a strong physical meaning because it appears in quantum mechanics models (see e.g. [6,7,22])

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and in semiconductor theory [5,25,26]. For more details in the physical aspects, we refer the readers to [4,5].

It is well known that Problem (1.1) can be reduced to a nonlinear Schrödinger equation with a nonlocal term. Indeed, as we shall see in Section 2, for any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta \phi = u^2$ by using the Lax-Milgram theorem, then inserted into the first equation, gives

$$-\Delta u + V(x)u + \phi_u(x)u = f(u).$$  \hspace{1cm} (1.2)

Moreover, under our assumptions it is standard to see that (1.2) is variational and its solutions are the critical points of the functional defined in $H^1(\mathbb{R}^3)$ by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx. \hspace{1cm} (1.3)$$

Hence if $u \in H^1(\mathbb{R}^3)$ is a critical point of $I$, then the pair $(u, \phi_u)$ is a solution of (1.1) (see [12] for more details). For the sake of simplicity, in many cases we just say $u \in H^1(\mathbb{R}^3)$, instead of $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, is a weak solution of (1.1).

In recent years, there has been increasing attention to problems like (1.1) on the existence of positive solutions, ground state solutions, multiple solutions and semiclassical states, see e.g. [2,3,8,10,15,19,27,32,34,35]. The greatest part of the literature focuses on the study of Problem (1.1) with $V(x) \equiv 1$ or $V(x) = V(|x|)$, and $f$ satisfying the following assumptions of Ambrosetti-Rabinowitz type and 4-superlinear

\begin{align*}
\text{(AR)} & \quad f(t)t \geq 4F(t) \geq 0, \ \forall \ t \in \mathbb{R}, \text{ where } F(t) = \int_0^t f(s) \, ds; \\
\text{(SF)} & \quad \lim_{|t| \to \infty} \frac{F(t)}{t} = \infty.
\end{align*}

In fact, under (AR) and (SF), it is easy to verify the Mountain Pass geometry and the boundedness of (PS) or $(C)_c$ sequences for $I$.

For the following special form of (1.1)

$$\begin{cases}
-\Delta u + u + \phi u = |u|^{p-2} u, & x \in \mathbb{R}^3, \\
-\Delta \phi = u^2, & x \in \mathbb{R}^3,
\end{cases} \hspace{1cm} (1.4)$$

its functional defined in $H^1(\mathbb{R}^3)$ by

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx, \hspace{1cm} (1.5)$$

where $F(u) = |u|^p/p$ is associated with (1.4), there are more results on the existence of solutions. For example, in [11,12] a radial positive solution of (1.4) is found for $4 < p < 6$. To do that they use the mountain pass theorem. It is easy to show that $\Phi$ attains a local minimum at zero. Moreover, in [12] it is pointed out that $\Phi$ is unbounded below even for $p > 3$. Furthermore, in [13] a related Pohozaev equality is found. With this equality in hand, the authors can prove that there does not exist nontrivial solutions of (1.4) for $p \leq 2$ or $p \geq 6$.

In the case $2 < p \leq 4$, it is very difficult to verify the Mountain Pass geometry and the boundedness of (PS) or $(C)_c$ sequences for $\Phi$. By introducing Nehari-Pohozaev manifold and establishing a key inequality, Ruiz [27] proved that Problem (1.4) admits a positive radial solution if $3 < p \leq 4$, but does not have a nontrivial solution for $2 < p \leq 3$. Obviously, this result fills the gap $p \in (2, 4]$ which is left in the previous results. Ruiz’ approach in [27] consists of minimizing $\Phi$ on the Nehari-Pohozaev manifold $M$, which is defined by a condition which is a combination of the Nehari equation and the Pohozaev equality, since the usual method of Nehari
manifold becomes invalid in this case. To the best of our knowledge, this approach is entirely new in the literature.

In fact, the positive radial solution obtained in [27] is not a minimizer of $\Phi$ on $\mathcal{M}$, but on $\mathcal{M}' \subset \mathcal{M}$ which consists of radially symmetric functions in $\mathcal{M}$. Based on Ruiz’ approach in [27] and a concentration-compactness argument on suitable measures, Azzollini and Pomponio [3] obtained a minimizer of $\Phi$ on $\mathcal{M}$ for (1.4) (it will be called a ground solution of Nehari-Pohozaev type) under the same assumption $3 < p \leq 4$ as in [27].

The approaches used in [3,27] are successful for (1.4), however, they are no longer applicable for the following problem with more general nonlinearity

$$\begin{align*}
-\Delta u + u + \phi u &= f(u), \quad x \in \mathbb{R}^3, \\
-\Delta \phi &= u^2, \quad x \in \mathbb{R}^3.
\end{align*}$$

(1.6)

Even for the case when $f(u) = |u|^{p_1-2}u + |u|^{p_2-2}u$ with $3 < p_1 < p_2 \leq 4$, it is difficult to obtain a ground solution of (1.6) by using the approaches in [3, 27].

Applying an alternative method, Sun and Ma [31] proved that (1.6) admits a least energy solution if $f$ satisfies (F1), (F2) and the following assumption of Ambrosetti-Rabinowitz type

\((\text{AR}^{'})\) there exists $\mu > 3$ such that $f(t)t \geq \mu F(t) > 0$ for $t \in \mathbb{R} \setminus \{0\}$.

To do that, Sun and Ma [31] used Jeanjeans monotonicity trick [20] to construct a special (PS) sequence, and by using a Pohozaev identity and a global compactness lemma, they proved the boundedness of the special (PS) sequence and hence, got a nontrivial critical point.

In [29], Seok used more general Jeanjeans monotonicity trick [21] to prove that (1.6) admits infinitely many finite energy radial solutions if $f$ is an odd function and satisfies (F1), (F2) and the following assumption

\((\text{SC})\) the function $[2f(t)t - 3F(t)]/t^3$ is monotone increasing to $\infty$ on $(0, \infty)$.

Since the radial solutions are critical points of $\Phi$ in $H^1(\mathbb{R}^3)$ which is the set of radially symmetric functions in $H^1(\mathbb{R}^3)$, and the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is compact for $2 < q < 6$. In this case, it is easy to verify the following condition

$$u_n \rightharpoonup u_0 \text{ in } H^1(\mathbb{R}^3) \Rightarrow u_n \to u_0 \text{ in } H^1(\mathbb{R}^3).$$

Hence, the global compactness lemma used in [31] is not necessary in [29].

We point out that the global compactness lemma is very crucial to prove the above result in [31]. The first global compactness lemma was proved by Struwe [30] in 1984. In the proof, it is essential the following equality

$$\Phi'(u_n) = \Phi'(u_0) + \Phi'(u_n - u_0) + o(1) \quad (1.7)$$

reasoning from condition $u_n \to u_0$ in $H^1(\mathbb{R}^3)$. In order to show (1.7), it is necessary to show

$$\lim_{n \to \infty} \sup_{\|\varphi\| \leq 1} \left| \int_{\mathbb{R}^3} [f(u_n) - f(u_n - u_0) - f(u_0)] \varphi dx \right| = 0. \quad (1.8)$$

It is standard to show (1.8) under (F2) and the following assumption

\((\text{F1}')\) $f \in C^1(\mathbb{R}, \mathbb{R})$ and there exist constants $C_0 > 0$ and $p \in (2, 2^*)$ such that

$$|f'(t)| \leq C_0(1 + |t|^{p-2}), \quad \forall \ t \in \mathbb{R},$$
which is assumed in [30]. However, it is difficult to show (1.8) if one uses (F1) with \( p \in (2, 2^*) \) instead of (F1'). It is worth pointing out that the crucial equality (1.7) is not presented in the proof of the global compactness lemma of [31], see [31] Lemma 5.6.

In [34], the authors proved that the following nonlinear Schrödinger-Poisson problem

\[
\begin{aligned}
-\Delta u + V(x)u + \phi u &= |u|^{p-2}u, \quad x \in \mathbb{R}^3, \\
-\Delta \phi &= u^2, \quad x \in \mathbb{R}^3
\end{aligned}
\]  

(1.9)

admits a least energy solution if \( p \in (3, 6) \) and \( V \) satisfies (V1) and the following assumptions

(V2) \( V(\infty) := \liminf_{|y| \to \infty} V(y) \geq (\neq) V(x); \)

(V3) \( V(x) \) is weakly differentiable, and satisfies \( (\nabla V(x), x) \in L^\infty(\mathbb{R}^3) \cup L^{3/2}(\mathbb{R}^3) \), and

\[ 2V(x) + (\nabla V(x), x) \geq 0 \quad \text{a.e. } x \in \mathbb{R}^3. \]

To prove the above result, the authors in [34] first applied Ruiz' result to the “limit problem” associated with (1.9), and got a minimizer \( u_0^\infty \) on the Nehari-Pohozaev manifold \( M_\lambda^\infty \) for \( \lambda \in [1/2, 1] \), see Section 4. Then they used Jeanjeans monotonicity trick [20] to show that there exists a bounded (PS) sequence \( \{u_n(\lambda)\} \) at the level \( c_\lambda \) for almost every \( \lambda \in [1/2, 1] \). Then by comparing \( c_\lambda \) with the energy \( m_\lambda^\infty \) of the minimizer \( u_0^\infty \) and using a global compactness lemma, they can get a nontrivial critical point \( u_0^\infty \) which possesses energy \( c_\lambda \). Finally, with a Pohozaev identity, they proved that (1.9) admits a least energy solution by a standard argument. It is worth pointing out that it seems to be not sufficient the proof of the inequality \( c_\lambda < m_\lambda^\infty \) for \( \lambda \in [1/2, 1] \) in [34], see Remark 4.6 in present paper.

Now, a natural question is whether Ruiz' result (or the one in [3]) on the existence of ground solution for (1.4) with \( 3 < p \leq 4 \) can be generalized to (1.1)?

To answer the above question, we must overcome three main difficulties: 1) verifying the boundedness of the minimizing sequence of \( I \) on the Nehari-Pohozaev manifold \( M \); 2) showing \( \inf_M I \) can be achieved (due to the lack of compactness of the Sobolev spaces embeddings in the unbounded domain \( \mathbb{R}^3 \)); 3) proving the minimizer of \( I \) on \( M \) is a critical point (because it is not assumed that \( f \) is differentiable, \( M \) may not be a C^1-manifold in \( H^1(\mathbb{R}^3) \)).

Motivated by the above works, in the present paper, we will introduce some new tricks to obtain a ground solution of Nehari-Pohozaev type for (1.1) (i.e. a minimizer of \( I \) on the Nehari-Pohozaev manifold \( M \)). This answers the above question affirmatively. In addition, we also prove the existence of the least energy solutions of (1.1) and (1.6), which make a substantial improvement to the main results in [31, 34]. In particular, we give a proof of (1.8) under (F1) and (F2), see Lemma 2.7.

To state our results, we make the following assumptions on the potential \( V \) and the nonlinearity \( f \).

(V4) \( V(x) \) is weakly differentiable, and satisfies \( \lim_{|x| \to \infty} (\nabla V(x), x) = 0 \) or \( (\nabla V(x), x) \in L^\nu(\mathbb{R}^3) \) with \( \nu \geq 3/2 \). For almost every \( x \in \mathbb{R}^3 \), the function \( t \mapsto t^2[V(tx) - (\nabla V(tx), tx)] \) is increasing on \( (0, +\infty) \), and for some \( \varrho_0 > 0 \)

\[ 2V(x) + (\nabla V(x), x) \geq \varrho_0 \quad \text{a.e. } x \in \mathbb{R}^3; \]
(V5) $V(x)$ is weakly differentiable, and satisfies \((\nabla V(x), x) \in L^\infty(\mathbb{R}^3)\), and for some $\theta_0 > 0$
\[ 2V(x) + (\nabla V(x), x) \geq \theta_0 \text{ a.e. } x \in \mathbb{R}^3; \]
(F3) $\lim_{|t| \to \infty} \frac{F(t)}{|t|^3} = \infty$;
(F4) there exists $\theta_0 \in [0, 1)$ such that
\[ \frac{2f(\tau)\tau - 3F(\tau)}{|\tau|^3} - \frac{2f(t^2\tau)t^2\tau - 3F(t^2\tau)}{(t^2|\tau|)^3} \text{ sign}(1 - t) + \theta_0 \frac{1 - t^2}{2t^2|\tau|} \geq 0, \]
\[ \forall x \in \mathbb{R}^3, t > 0, \tau \neq 0; \]
(F4') the function $[2f(t)\tau - 3F(t)]/t^3$ is nondecreasing on $\mathbb{R}$;
(F5) $f(t)\tau \geq 3F(t) \geq 0$, $\forall t \in \mathbb{R}$, and there exist $\kappa > 3/2$ and $C_1 > 0$ such that
\[ \frac{f(t)}{t} > \frac{\gamma_0}{2} \Rightarrow \left| \frac{f(t)}{t}\right|^{\kappa} \leq C_1|f(t)\tau - 3F(t)|, \]
\[ \gamma_0 \text{ is Sobolev imbedding constant such that } \gamma_0\|u\|^2_2 \leq \|u\|^2 \text{ for } u \in H^1(\mathbb{R}^3). \]

**Remark 1.1.** Note that
\[ V(x) = \int_0^1 \frac{d}{dt}[t^2V(tx)]dt = \int_0^1 t[2V(tx) + (\nabla V(tx), tx)]dt, \quad \forall x \in \mathbb{R}^3, \]
then it follows from (V2) and (V3) that $V(x) \geq 0$ for all $x \in \mathbb{R}^3$, and
\[ \left[ \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \, dx \right]^{1/2} \]
is a norm which is equivalent to the usual norm of $H^1(\mathbb{R}^3)$. Hence, the second part of (V1) holds.

**Remark 1.2.** There are indeed functions which satisfy (V1)-(V5). An example is given by $V(x) = V_1 - \frac{1}{|x|+1}$, where $V_1 > 1$ is a positive constant.

**Remark 1.3.** It is easy to see that (F4') implies (F4) with $\theta_0 = 0$. Furthermore, there are many functions satisfying (F4), (F4') or (F5). For example,
\[ f(t) = a|t|^{3/2}t + b|t|^{1/2}t \]
satisfies (F4) with $\theta_0 = 2/3$ and $15\sqrt{10a} \geq 14b^{3/2} > 0$, but does not satisfy (F4'), (F5), (AR') or (SC). The function
\[ f(t) = 3|t|t \ln(1 + |t|^2) + \frac{2|t|^3}{1 + t^2} \]
satisfies (F4') and (F5), but does not satisfy (AR').

To state our results, we define two functionals on $H^1(\mathbb{R}^3)$ as follows:
\[ J(u) = \frac{3}{2}\|\nabla u\|^2_2 + \frac{1}{2} \int_{\mathbb{R}^3} [V(x) - (\nabla V(x), x)]u^2 \, dx \]
\[ + \frac{3}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 \, dx - \int_{\mathbb{R}^3} [2f(u)u - 3F(u)] \, dx \quad (1.10) \]
and
\[
\tilde{J}(u) = \frac{3}{2} \|\nabla u\|_2^2 + \frac{1}{2} \|u\|_2^2 + \frac{3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 \, dx - \int_{\mathbb{R}^3} [2f(u) - 3F(u)] \, dx.
\]
(1.11)

Let
\[
\mathcal{M} := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : J(u) = 0 \}
\]
and
\[
\tilde{\mathcal{M}} := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : \tilde{J}(u) = 0 \}.
\]
(1.12) (1.13)

Now, we state our results of this paper.

**Theorem 1.4.** Assume that \( f \) satisfies (F1)-(F4). Then Problem (1.6) has a solution \( u_0 \in H^1(\mathbb{R}^3) \) such that \( \Phi(u_0) = \inf_{\mathcal{M}} \Phi > 0 \).

**Theorem 1.5.** Assume that \( V \) and \( f \) satisfy (V1), (V2), (V4), (F1)-(F3) and (F4'). Then Problem (1.1) has a solution \( u_0 \in H^1(\mathbb{R}^3) \) such that \( I(u_0) = \inf_{\mathcal{M}} I > 0 \).

**Theorem 1.6.** Assume that \( V \) and \( f \) satisfy (V1)-(V3), (F1)-(F3), (F4') and (F5). Then Problem (1.1) has a least energy solution \( u_0 \in H^1(\mathbb{R}^3) \setminus \{0\} \).

**Theorem 1.7.** Assume that \( V \) and \( f \) satisfy (V1), (V2), (V5), (F1)-(F3) and (F4'). Then Problem (1.1) has a least energy solution \( u_0 \in H^1(\mathbb{R}^3) \setminus \{0\} \).

**Theorem 1.8.** Assume that \( f \) satisfies (F1)-(F3) and (F5). Then Problem (1.6) has a least energy solution \( u_0 \in H^1(\mathbb{R}^3) \setminus \{0\} \).

**Remark 1.9.** Theorems 1.4 and 1.5 answer the question mentioned above. Theorems 1.7 and 1.8 make a substantial improvement to the main results in [31,34].

The paper is organized as follows. In Section 2, we give some notation and preliminaries, and give the proof of Theorem 1.4. In Section 3, we complete the proof of Theorems 1.5. Section 4 is devoted to finding a least energy solution for (1.1) and (1.6). Theorems 1.6-1.8 will be proved in this section.

Throughout this paper, we let \( u_t(x) := u(tx) \) for \( t > 0 \), and denote the norm of \( L^s(\mathbb{R}^3) \) by \( \|u\|_s = (\int_{\mathbb{R}^3} |u|^s \, dx)^{1/s} \) for \( s \geq 2 \), \( B_r(x) = \{ y \in \mathbb{R}^3 : |y - x| < r \} \), and positive constants possibly different in different places, by \( C_1, C_2, \ldots \).

2. **Ground state solutions for (1.6).** Hereafter, \( H^1(\mathbb{R}^3) \) is the usual Sobolev space with the standard scalar product and norm
\[
(u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) \, dx, \quad \|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx,
\]
and
\[
D^{1,2}(\mathbb{R}^3) = \{ u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3) \}
\]
equipped with the norm defined by
\[
\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx.
\]
It is easy to show that (1.1) can be reduced to a single equation (1.2) with a non-local term. Namely, for any \( u^2 \in L^1_{loc}(\mathbb{R}^3) \) such that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \, dx \, dy < \infty,
\]
the distributional solution
\[
\phi_u(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \, dy = \frac{1}{|x|} * u^2
\]
(2.1)
of the Poisson equation
\[-\Delta \phi = u^2, \quad x \in \mathbb{R}^3\]
belongs to \( D^{1,2}(\mathbb{R}^3) \) and is the unique weak solution in \( D^{1,2}(\mathbb{R}^3) \) (see e.g. [28] for more details), and
\[
\int_{\mathbb{R}^3} \nabla \phi_u \cdot \nabla v \, dx = \int_{\mathbb{R}^3} u^2 v \, dx, \quad \forall \ v \in H^1(\mathbb{R}^3),
\]
(2.2)
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \, dx \, dy = \int_{\mathbb{R}^3} \phi_u(x)u^2 \, dx.
\]
(2.3)
Moreover, \( \phi_u(x) > 0 \) when \( u \neq 0 \). By Hardy-Littlewood-Sobolev inequality (see [23]), we have the following inequality:
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)v(y)|}{|x-y|} \, dx \, dy \leq \frac{8 \sqrt{2}}{3 \sqrt{\pi}} \|u\|_{6/5} \|v\|_{6/5}, \quad u, v \in L^{6/5}(\mathbb{R}^3).
\]
(2.4)
Formally, the solutions of (1.1) are then the critical points of the reduced functional (1.5). Indeed, (V1), (F1), (F2) and (2.4) imply that \( I \) and \( \Phi \) are two well-defined classes of \( C^1 \) functional, and that
\[
\langle I'(u), v \rangle = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + V(x)uv \, dx + \int_{\mathbb{R}^3} [\phi_u(x)u - f(u)] v \, dx
\]
(2.5)
and
\[
\langle \Phi'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) \, dx + \int_{\mathbb{R}^3} [\phi_u(x)u - f(u)] v \, dx.
\]
(2.6)

**Lemma 2.1.** Assume that (F1) and (F4) hold. Then
\[
\frac{2(1-t^3)}{3} F(\tau) + (t^3-2)F(\tau) + \frac{1}{t^3} F(t^2 \tau) + \frac{\theta_0}{6} (1-t)^2 (2+t) \tau^2 \geq 0, \quad \forall \ t > 0, \ \tau \in \mathbb{R}.
\]
(2.7)

**Proof.** It is evident that (2.7) holds for \( \tau = 0 \). For \( \tau \neq 0 \), let
\[
g(t) = \frac{2(1-t^3)}{3} f(\tau) + (t^3-2)F(\tau) + \frac{1}{t^3} F(t^2 \tau) + \frac{\theta_0}{6} (1-t)^2 (2+t) \tau^2, \quad t > 0.
\]
(2.8)
Then from (F4), one has
\[
g'(t) = t^2 |\tau|^3 \left[ \frac{2f(t^2 \tau) t^2 \tau - 3F(t^2 \tau)}{t^6 |\tau|^3} - \frac{2f(\tau) \tau - 3F(\tau)}{|\tau|^3} - \frac{\theta_0(1-t^2)}{2t^2 |\tau|} \right]
\]
which together with (2.8), implies (2.7) holds.
Lemma 2.2. Assume that (F1), (F2) and (F4) hold. Then
\[ \Phi(u) \geq \Phi(t^2u_t) + \frac{1-t^3}{3} \bar{J}(u) + \frac{(1-\theta_0)(1-t)^2(2+t)}{6} \|u\|_2^2, \quad \forall u \in H^1(\mathbb{R}^3), \quad t \geq 0. \] \tag{2.9}

Proof. Note that
\[ \Phi(t^2u_t) = \frac{1}{2} \int_{\mathbb{R}^3} (t^3|\nabla u|^2 + tu^2) \, dx + \frac{t^3}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 \, dx - \frac{1}{t^3} \int_{\mathbb{R}^3} F(t^2u) \, dx. \] \tag{2.10}

Thus, by (1.11), (2.7) and (2.10), one has
\[ \Phi(u) - \Phi(t^2u_t) = \frac{1}{2} \int_{\mathbb{R}^3} \left[(1-t^3)|\nabla u|^2 + (1-t)u^2\right] \, dx + \frac{1-t^3}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 \, dx + \int_{\mathbb{R}^3} \left[ \frac{1}{t^3} F(t^2u) - F(u) \right] \, dx \]
\[ = \frac{1-t^3}{3} \bar{J}(u) + \frac{(1-t^3)(2+t)}{6} \int_{\mathbb{R}^3} u^2 \, dx \]
\[ + \int_{\mathbb{R}^3} \left[ \frac{2(1-t^3)}{3} f(u) + (t^3-2)F(u) + \frac{1}{t^3} F(t^2u) \right] \, dx \]
\[ \geq \frac{1-t^3}{3} \bar{J}(u) + \frac{(1-\theta_0)(1-t^3)(2+t)}{6} \|u\|_2^2. \]

This shows that (2.9) holds. \qed

From Lemma 2.2, we have the following corollary immediately.

Corollary 2.3. Assume that (F1), (F2) and (F4) hold. Then for \( u \in \bar{M} \)
\[ \Phi(u) = \max_{t \geq 0} \Phi(t^2u_t). \] \tag{2.11}

Lemma 2.4. Assume that (F1)-(F4) hold. Then for any \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \), there exists a unique \( t(u) > 0 \) such that \( t(u)^2u_{t(u)} \in \bar{M} \).

Proof. Let \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \) be fixed and define a function \( \zeta(t) := \Phi(t^2u_t) \) on \([0, \infty)\). Clearly, by (2.10), we have
\[ \zeta'(t) = 0 \Leftrightarrow \frac{1}{2} \int_{\mathbb{R}^3} (3t^3|\nabla u|^2 + tu^2) \, dx + \frac{3t^3}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 \, dx + \frac{3}{t^3} \int_{\mathbb{R}^3} F(t^2u) \, dx - \frac{2}{t} \int_{\mathbb{R}^3} f(t^2u)u \, dx = 0 \]
\[ \Leftrightarrow \bar{J}(t^2u_t) = 0 \Leftrightarrow t^2u_t \in \bar{M}. \]

It is easy to verify, using (F1), (F2) and (F3), that \( \zeta(0) = 0, \zeta(t) > 0 \) for \( t > 0 \) small and \( \zeta(t) < 0 \) for \( t \) large. Therefore \( \max_{t \in [0, \infty)} \zeta(t) \) is achieved at \( t_0 = t(u) > 0 \) so that \( \zeta'(t_0) = 0 \) and \( t_0^2u_{t_0} \in \bar{M} \).

Next we claim that \( t(u) \) is unique for any \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \). In fact, for any given \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \), let \( t_1, t_2 > 0 \) such that \( \zeta'(t_1) = \zeta'(t_2) = 0 \). Then \( \bar{J}(t_1^2u_{t_1}) = \bar{J}(t_2^2u_{t_2}) = \bar{J}(t_0^2u_{t_0}) \) for some \( t_0 \).
\( \bar{J}(t_2^2 u_{t_2}) = 0 \). Jointly with (2.9), we have
\[
\Phi(t_1^2 u_{t_1}) \geq \Phi(t_2^2 u_{t_2}) + \frac{t_1^2 - t_2^2}{3t_1^3} \bar{J}(t_1^2 u_{t_1}) + \frac{(1 - \theta_0)(t_1 - t_2)^2(2t_1 + t_2)}{6t_1^2} \|u\|_2^2 \\
= \Phi(t_1^2 u_{t_1}) + \frac{(1 - \theta_0)(t_1 - t_2)^2(2t_1 + t_2)}{6t_1^2} \|u\|_2^2
d(2.12)
\]
and
\[
\Phi(t_1^2 u_{t_1}) \geq \Phi(t_2^2 u_{t_2}) + \frac{t_1^2 - t_2^2}{3t_2^3} \bar{J}(t_2^2 u_{t_2}) + \frac{(1 - \theta_0)(t_2 - t_1)^2(2t_2 + t_1)}{6t_2^2} \|u\|_2^2 \\
= \Phi(t_2^2 u_{t_2}) + \frac{(1 - \theta_0)(t_2 - t_1)^2(2t_2 + t_1)}{6t_2^2} \|u\|_2^2.
d(2.13)
\]
(2.12) and (2.13) imply \( t_1 = t_2 \). Therefore, \( t(u) > 0 \) is unique for any \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \).

**Lemma 2.5.** Assume that (F1)-(F4) hold. Then
\[
\inf_{u \in \mathcal{M}} \Phi(u) := m_0 = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t \geq 0} \Phi(t^2 u_t).
\]
Proof. Both Corollary 2.3 and Lemma 2.4 imply the above lemma.

**Lemma 2.6.** Assume that (F1)-(F4) hold. Then
i). there exists \( \rho_0 > 0 \) such that \( \|u\| \geq \rho_0, \quad \forall u \in \mathcal{M}; \)
ii). \( m_0 = \inf_{u \in \mathcal{M}} \Phi(u) > 0. \)

**Proof.** Since \( \bar{J}(u) = 0, \quad \forall u \in \mathcal{M}, \) by (F1), (F2), (1.11) and Sobolev embedding theorem, it has
\[
\frac{1}{2} \|u\|^2 \leq \frac{1}{2} \int_{\mathbb{R}^3} (3|\nabla u|^2 + u^2) \, dx + \frac{3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 \, dx \\
= \int_{\mathbb{R}^3} [2f(u)u - 3F(u)] \, dx \\
\leq \frac{1}{4} \|u\|^2 + C_1 \|u\|^p,
d(2.14)
\]
where \( C_1 \) is a positive constant. This implies
\[
\|u\| \geq \rho_0 := \frac{1}{(4C_1)^{1/(p-2)}}, \quad \forall u \in \mathcal{M}.
d(2.15)
\]
Since \( \|\nabla u\|^2 + \|u\|^2 \geq \rho_0^2 \) for all \( u \in \mathcal{M}, \) there are two possible cases:
1) inf \( u \in \mathcal{M} \|u\|_2 := \rho_1 > 0 \) and 2) inf \( u \in \mathcal{M} \|\nabla u\|_2 := \rho_2 > 0. \)

**Case 1.** \( \inf_{u \in \mathcal{M}} \|u\|_2 := \rho_1 > 0. \) In this case, from (2.9) (\( t = 0 \)), one has
\[
\Phi(u) \geq \frac{1 - \theta_0}{3} \|u\|_2^2 \geq \frac{1 - \theta_0}{3} \rho_1^2, \quad \forall u \in \mathcal{M}.
d(2.16)
\]

**Case 2.** \( \inf_{u \in \mathcal{M}} \|\nabla u\|_2 := \rho_2 > 0. \) From (F1) and (F2), there exists \( C_2 > 0 \) such that
\[
|F(t)| \leq \frac{1}{2} |t|^2 + C_2 |t|^6, \quad \forall t \in \mathbb{R}.
d(2.17)
\]
For $u \in \tilde{\mathcal{M}}$, by the Sobolev inequality, one has $S\|u\|_0^2 \leq \|\nabla u\|_2^2$. Let $t_u = \left( \frac{s^3}{\mathbb{R}^2} \right)^{1/6} \|\nabla u\|_2^{-2/3}$. From (2.9), (2.10), (2.17) and the Sobolev inequality, we have

$$
\Phi(u) \geq \Phi(t_u^2 u_n) = \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{t_u^3}{2} \|\nabla u\|_2^2 + t_u u^2 \right) dx + \frac{t_u^3}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 dx - \frac{1}{t_u} \int_{\mathbb{R}^3} F(t_u^2 u)dx
$$

$$
\geq \frac{t_u^3}{2} \|\nabla u\|_2^2 - C_2 t_u^6 \|u\|_6^6
$$

$$
\geq \frac{t_u^3}{2} \|\nabla u\|_2^2 - C_2 S^{-3} t_u^6 \|\nabla u\|_2^2
$$

$$
= \frac{1}{3\sqrt{6C_2}S^{-3}} \quad \forall u \in \tilde{\mathcal{M}}.
$$

Cases 1) and 2) show that $m_0 = \inf_{u \in \tilde{\mathcal{M}}} \Phi(u) > 0$. 

**Lemma 2.7.** Assume that (F1) and (F2) hold. If $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$, then along a subsequence of $\{u_n\}$,

$$
\lim_{n \to \infty} \sup_{v \in H^1(\mathbb{R}^3), \|v\| \leq 1} \left| \int_{\mathbb{R}^3} \left[ f(u_n) - f(u_n - \bar{u}) - f(\bar{u}) \right] \phi dx \right| = 0. \quad (2.18)
$$

**Proof.** This lemma has been proved in [1, Appendix] and [14, Page 77-80]. Here, we give a simpler and more direct proof.

Set $v_n = u_n - \bar{u}$. Then $v_n \to 0$ in $H^1(\mathbb{R}^3)$, $v_n \to 0$ in $L^s_{\text{loc}}(\mathbb{R}^3)$ for $2 \leq s \leq 6$ and $v_n \to 0$ a.e. $x \in \mathbb{R}^3$. For any $a > 0$, we set

$$
A_n^a = \{ x \in \mathbb{R}^3 : |v_n(x)| \leq a \}, \quad B_n^a = \mathbb{R}^3 \setminus A_n^a.
$$

Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$, then there exists a constant $C_1 > 0$ such that

$$
|B_n^a| \leq \frac{1}{a^2} \int_{B_n^a} |v_n|^2 dx \leq \frac{C_1}{a^2} \to 0 \quad \text{as } a \to \infty. \quad (2.19)
$$

Define

$$
g(t) = \begin{cases} 
\frac{L(t)}{t}, & t \neq 0, \\
0, & t = 0,
\end{cases}
$$

then (F1) and (F2) imply $g \in C(\mathbb{R}, \mathbb{R})$, and there exists $C_2 > 0$ such that

$$
|g(t)| \leq C_2(1 + |t|^{p-2}), \quad \forall t \in \mathbb{R}. \quad (2.20)
$$

Note that

$$
f(u_n) - f(u_n - \bar{u}) = f(v_n + \bar{u}) - f(v_n)
$$

$$
= g(v_n + \bar{u})\bar{u} + [g(v_n + \bar{u}) - g(v_n)]v_n. \quad (2.21)
$$

By (2.20), the Hölder inequality and the Sobolev inequality, one has

$$
\left| \int_{B_n^a} [g(v_n + \bar{u}) - g(v_n)]v_n \phi dx \right|
$$

$$
\leq C_3 \int_{B_n^a} \left( 1 + |v_n|^{p-2} + |\bar{u}|^{p-2} \right) |v_n||\phi| dx
$$

$$
\leq C_3 \left[ |B_n^a|^{2/3} \|v_n\|_6 \|\phi\|_6 + |B_n^a|^{1-p/6} \left( \|v_n\|_{6}^{p-1} \|\phi\|_6 + \|\bar{u}\|_{6}^{p-2} \|v_n\|_6 \|\phi\|_6 \right) \right]
$$
\begin{equation}
\leq C_4 \left( |B_n^{a_2/3}| + |B_n^{a_{1-p/6}}| \right) \|\varphi\|.
\tag{2.22}
\end{equation}

For any \( \varepsilon > 0 \), (2.22), together with (2.19), implies that there exists \( \hat{a} > 0 \) such that
\begin{equation}
\sup_{\varphi \in H^1(\mathbb{R}^3), \|\varphi\| \leq 1} \left| \int_{B_n^\delta} \left[ g(v_n + \hat{u}) - g(v_n) \right] v_n \varphi \, dx \right| \leq \varepsilon, \quad \forall \ n \in \mathbb{N}.
\tag{2.23}
\end{equation}

By the uniformly continuity of \( g \) on \([-\hat{a}, \hat{a}]\), there exists \( \delta > 0 \) such that
\begin{equation}
|g(t+h) - g(t)| < \varepsilon, \quad \text{for all } (t, h) \in [-\hat{a}, \hat{a}] \times [-\delta, \delta].
\tag{2.24}
\end{equation}

Set
\begin{equation}
C^\delta = \{ x \in \mathbb{R}^3 : |\tilde{u}(x)| \leq \delta \}, \quad D^\delta = \mathbb{R}^3 \setminus C^\delta.
\end{equation}

Clearly,
\begin{equation}
|D^\delta| \leq \frac{1}{\delta^2} \int_{D^\delta} |\tilde{u}|^2 \, dx \leq \frac{\|\tilde{u}\|^2}{\delta^2}.
\tag{2.25}
\end{equation}

From (2.20), (2.24), (2.25), the Hölder inequality and the Sobolev inequality, one has
\begin{align*}
&\left| \int_{A_n^\delta} \left[ g(v_n + \hat{u}) - g(v_n) \right] v_n \varphi \, dx \right| \\
= &\left| \int_{A_n^\delta \cap C^\delta} \left[ g(v_n + \hat{u}) - g(v_n) \right] v_n \varphi \, dx + \int_{A_n^\delta \cap D^\delta} \left[ g(v_n + \hat{u}) - g(v_n) \right] v_n \varphi \, dx \right| \\
\leq &\varepsilon \|v_n\| \|\varphi\| + C_5 \int_{A_n^\delta \cap D^\delta} \left( 1 + |v_n|^{p-2} + |\hat{u}|^{p-2} \right) |v_n| \|\varphi\| \, dx \\
\leq &\varepsilon \|v_n\| \|\varphi\| + C_5 \left( \int_{D^\delta} |v_n|^2 \, dx \right)^{1/p} \|\varphi\| + C_5 \left( \int_{D^\delta} |v_n|^p \, dx \right)^{(p-1)/p} \\
+ &C_7 \left( \int_{D^\delta} |v_n|^2 \, dx \right)^{1/p} \|\varphi\| \|\varphi\|^{(p-1)/p} \\
\leq &\left\{ C_6 \varepsilon + C_7 \left[ \left( \int_{D^\delta} |v_n|^2 \, dx \right)^{1/p} + \left( \int_{D^\delta} |v_n|^p \, dx \right)^{(p-1)/p} \right] \right\} \|\varphi\|,
\end{align*}

which, together with \( v_n \to 0 \), implies
\begin{equation}
\sup_{\varphi \in H^1(\mathbb{R}^3), \|\varphi\| \leq 1} \left| \int_{A_n^\delta} \left[ g(v_n + \hat{u}) - g(v_n) \right] v_n \varphi \, dx \right| \leq C_6 \varepsilon + o(1), \quad n \to \infty.
\tag{2.26}
\end{equation}

By a standard argument, it is easy to prove that there exists \( R > 0 \) such that
\begin{equation}
\sup_{\varphi \in H^1(\mathbb{R}^3), \|\varphi\| \leq 1} \left| \int_{\mathbb{R}^3 \setminus B_R(0)} g(v_n + \hat{u}) \hat{u} \varphi \, dx \right| \leq \varepsilon, \quad \forall \ n \in \mathbb{N}
\tag{2.27}
\end{equation}

and
\begin{equation}
\sup_{\varphi \in H^1(\mathbb{R}^3), \|\varphi\| \leq 1} \left| \int_{\mathbb{R}^3 \setminus B_R(0)} f(\hat{u}) \varphi \, dx \right| \leq \varepsilon.
\tag{2.28}
\end{equation}
Since \( u_n \to \bar{u} \) in \( L^2(B_R(0)) \cap L^p(B_R(0)) \), then \[33\] Lemma A.1 implies that there exists a subsequence \( \{u_{j_n}\} \) of \( \{u_n\} \) and \( w \in L^2(B_R(0)) \cap L^p(B_R(0)) \) such that

\[
|u_{j_n}(x)| \leq w(x), \quad \text{a.e. } x \in B_R(0). \tag{2.29}
\]

Then by (F1), (F2) and \[2.29\], we have

\[
|f(u_{j_n}) - f(u_n - \bar{u}) - f(\bar{u})|^{p/(p-1)} \leq C_8 \left( |u_{j_n}| + |\bar{u}| + |u_n|^{p-1} + |\bar{u}|^{p-1} \right)^{p/(p-1)} \\
\leq C_9 \left( |u_{j_n}|^{p/(p-1)} + |\bar{u}|^{p/(p-1)} + |u_n|^p + |\bar{u}|^p \right) \\
\leq 2C_9 \left( |w|^{p/(p-1)} + |\bar{w}|^p \right), \quad \text{a.e. } x \in B_R(0). \tag{2.30}
\]

Since \( |w|^{p/(p-1)} + |\bar{w}|^p \in L^1(B_R(0)) \) and

\[
f(u_{j_n}) - f(u_n - \bar{u}) - f(\bar{u}) \to 0, \quad \text{a.e. } x \in B_R(0),
\]

then it follows from \[2.30\] and the Lebesgue dominated convergence theorem that

\[
\int_{B_R(0)} |f(u_{j_n}) - f(u_n - \bar{u}) - f(\bar{u})|^{p/(p-1)} \, dx = o(1). \tag{2.31}
\]

Note that

\[
\left| \int_{B_R(0)} [f(u_{j_n}) - f(u_n - \bar{u}) - f(\bar{u})] \varphi \, dx \right| \\
\leq \left( \int_{B_R(0)} |f(u_{j_n}) - f(u_n - \bar{u}) - f(\bar{u})|^{p/(p-1)} \, dx \right)^{(p-1)/p} \|\varphi\|_p \\
\leq C_{10} \left( \int_{B_R(0)} |f(u_{j_n}) - f(u_n - \bar{u}) - f(\bar{u})|^{p/(p-1)} \, dx \right)^{(p-1)/p} \|\varphi\|. \tag{2.32}
\]

From \[2.31\] and \[2.32\], one has

\[
\sup_{\varphi \in H^1(\mathbb{R}^3), \|\varphi\| \leq 1} \left| \int_{B_R(0)} [f(u_{j_n}) - f(u_n - \bar{u}) - f(\bar{u})] \varphi \, dx \right| \\
\leq C_{10} \left( \int_{B_R(0)} |f(u_{j_n}) - f(u_n - \bar{u}) - f(\bar{u})|^{p/(p-1)} \, dx \right)^{(p-1)/p} = o(1). \tag{2.33}
\]

Note that

\[
\left| \int_{\mathbb{R}^3} [f(u_{j_n}) - f(u_n - \bar{u}) - f(\bar{u})] \varphi \, dx \right| \\
\leq \left| \int_{B_R(0)} [f(u_{j_n}) - f(u_n - \bar{u}) - f(\bar{u})] \varphi \, dx \right| + \left| \int_{\mathbb{R}^3 \setminus B_R(0)} f(\bar{u}) \varphi \, dx \right| \\
+ \left| \int_{\mathbb{R}^3 \setminus B_R(0)} [f(u_{j_n}) - f(u_n - \bar{u})] \varphi \, dx \right|
\]
Then it follows from (2.23), (2.26), (2.27), (2.28), (2.33) and (2.34) that

\[
\int_{B_R(0)} [f(u_{jn}) - f(u_{jn} - \bar{u}) - f(\bar{u})] \varphi dx + \int_{\mathbb{R}^3 \setminus B_R(0)} f(\bar{u})\varphi dx \\
+ \int_{A^+_n} [g(v_{jn} + \bar{u}) - g(v_{jn})] v_{jn} \varphi dx + \int_{B_R^c} [g(v_{jn} + \bar{u}) - g(v_{jn})] v_{jn} \varphi dx \\
+ \int_{\mathbb{R}^3 \setminus B_R(0)} g(v_{jn} + \bar{u})\bar{u} \varphi dx.
\]

(2.34)

Then it follows from (2.23), (2.26), (2.27), (2.28), (2.33) and (2.34) that

\[
\sup_{\varphi \in H^1(\mathbb{R}^3), \|\varphi\| \leq 1} \left| \int_{\mathbb{R}^3} [f(u_{jn}) - f(u_{jn} - \bar{u}) - f(\bar{u})] \varphi dx \right| \leq (C_0 + 3)\varepsilon + o(1).
\]

Since \(\varepsilon > 0\) is arbitrary, now the conclusion of Lemma 2.7 follows from the above inequality.

\[\square\]

**Lemma 2.8.** Assume that (V1), (V3), (F1) and (F2) hold. If \(u_n \to \bar{u}\) in \(H^1(\mathbb{R}^3)\), then

\[
I(u_n) = I(\bar{u}) + I(u_n - \bar{u}) + o(1), \quad (2.35)
\]

\[
I'(u_n) = I'(\bar{u}) + I'(u_n - \bar{u}) + o(1), \quad (2.36)
\]

\[
\langle I'(u_n), u_n \rangle = \langle I'(\bar{u}), \bar{u} \rangle + \langle I'(u_n - \bar{u}), u_n - \bar{u} \rangle + o(1) \quad (2.37)
\]

and

\[
J(u_n) = J(\bar{u}) + J(u_n - \bar{u}) + o(1). \quad (2.38)
\]

**Proof.** Set

\[
I_1(u) = \int_{\mathbb{R}^3} \|\nabla u\|^2 + V(x)u^2 dx, \quad I_2(u) = \int_{\mathbb{R}^3} \phi_u(x)u^2 dx, \quad I_3(u) = \int_{\mathbb{R}^3} F(u) dx.
\]

(2.39)

Let \(w_n = u_n - \bar{u}\). Then \(w_n \to 0\) in \(H^1(\mathbb{R}^3)\), \(w_n \to 0\) in \(L^s_{\text{loc}}(\mathbb{R}^3)\), \(2 \leq s < 6\) and \(w_n \to 0\) a.e. on \(\mathbb{R}^3\). By some standard arguments (see [34, Lemma 2.2]), one can prove that

\[
I(u_n) = I(\bar{u}) + I(w_n) + o(1) \quad (2.40)
\]

and

\[
I'_1(u_n) = I'_1(\bar{u}) + I'_1(w_n) + o(1), \quad I'_2(u_n) = I'_2(\bar{u}) + I'_2(w_n) + o(1). \quad (2.41)
\]

Employing Lemma 2.7, we also have

\[
I'_3(u_n) = I'_3(\bar{u}) + I'_3(w_n) + o(1).
\]

(2.42)

From (2.39), (2.41) and (2.42), we have

\[
I'(u_n) = I'(\bar{u}) + I'(w_n) + o(1). \quad (2.43)
\]

Since \(w_n \to 0\) in \(H^1(\mathbb{R}^3)\) and \(w_n \to 0\) in \(L^s_{\text{loc}}(\mathbb{R}^3)\) for \(s \in [2, 6)\), it follows from (F1), (F2), (2.1) and (2.4) that

\[
\langle I'(w_n), \bar{u} \rangle = \int_{\mathbb{R}^3} [\nabla w_n \cdot \nabla \bar{u} + V(x)w_n \bar{u}] dx + \int_{\mathbb{R}^3} \phi_{w_n}(x)w_n \bar{u} dx - \int_{\mathbb{R}^3} f(w_n) \bar{u} dx
\]

\[
= o(1). \quad (2.44)
\]

(2.43) and (2.44) imply

\[
\langle I'(u_n), u_n \rangle = \langle I'(w_n), w_n \rangle + \langle I'(\bar{u}), \bar{u} \rangle + \langle I'(w_n), u_n - \bar{u} \rangle + o(1)
\]

\[
= \langle I'(w_n), w_n \rangle + \langle I'(\bar{u}), \bar{u} \rangle + o(1).
\]
Noting that
\[ J(u) = 2\langle f'(u), u \rangle - 3J(u) + \|\nabla u\|^2 - \frac{1}{2} \int_{\mathbb{R}^3} (V(x), x) u^2 \, dx - \frac{1}{2} I_2(u), \quad (2.45) \]
then from (2.35), (2.37) and (2.45), we can prove that (2.38) holds. □

**Lemma 2.9.** Assume that (F1)-(F4) hold. Then \( m_0 \) is achieved.

**Proof.** We introduce a new functional \( \Psi : H^1(\mathbb{R}^3) \to \mathbb{R} \) defined as follows:

\[ \Psi(u) = \frac{1}{3} \|u\|^2 + \frac{2}{3} \int_{\mathbb{R}^3} [f(u)u - 3F(u)] \, dx, \quad \forall u \in H^1(\mathbb{R}^3). \quad (2.46) \]

For any \( u \in \mathcal{M} \), we have \( \Psi(u) = \Phi(u) \geq m_0 \). Let \( \{u_n\} \subset \mathcal{M} \) be such that \( \Phi(u_n) \to m_0 \). Since \( \bar{J}(u_n) = 0 \), then it follows from (2.9) with \( t = 0 \) that

\[ m_0 + o(1) = \Phi(u_n) \geq \frac{1 - \theta_0}{3} \|u_n\|^2. \quad (2.47) \]

This shows that \( \{\|u_n\|\} \) is bounded. Next, we prove that \( \{\|\nabla u_n\|\} \) is also bounded. Arguing by contradiction, suppose that \( \|\nabla u_n\| \to \infty \). Let \( t_n = (\sqrt{6m_0} / \|\nabla u_n\|)^{2/3} \). Hence, using (F1), (F2), (2.9), (2.10), (2.47) and \( \bar{J}(u_n) = 0 \), one has

\[ m_0 + o(1) = \Phi(u_n) \geq \Phi(t_n^2(u_n)x_n) \]
\[ = \frac{1}{2} \int_{\mathbb{R}^3} (t_n^3|\nabla u_n|^2 + t_n u_n^2) \, dx + \frac{t_n^3}{4} \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 \, dx \]
\[ - \frac{t_n^3}{4} \int_{\mathbb{R}^3} F(t_n^3 u_n) \, dx \]
\[ \geq \frac{t_n^3}{2} \|\nabla u_n\|^2 - C_1 t_n \|u_n\|^2 - \frac{S^3}{216m_0^3} t_n^6 \|u_n\|^6 \]
\[ \geq \frac{t_n^3}{2} \|\nabla u_n\|^2 - C_1 t_n \|u_n\|^2 - \frac{1}{216m_0^3} t_n^6 \|\nabla u_n\|^6 \]
\[ = 2m_0 - \frac{(6m_0)^{1/3} C_1}{\|\nabla u_n\|^{2/3}} \|u_n\|^2 \]
\[ = 2m_0 + o(1). \quad (2.48) \]

This contradiction implies that \( \{\|\nabla u_n\|\} \) is also bounded, and so \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \). From Lemma 2.6 (i) and (1.11), one has

\[ \frac{1}{2} \theta_0 \leq \frac{1}{2} ||u_n||^2 \leq \frac{1}{2} \int_{\mathbb{R}^3} (3|\nabla u_n|^2 + u_n^2) \, dx + \frac{3}{4} \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 \, dx \]
\[ = \int_{\mathbb{R}^3} [2f(u_n)u_n - 3F(u_n)] \, dx. \quad (2.49) \]

Using (F1), (F2) and (2.49), Lions’ concentration compactness principle [33, Lemma 1.21], we can prove that there exist \( \delta > 0 \) and a sequence \( y_n \in \mathbb{R}^3 \) such that \( \int_{B_1(y_n)} |u_n|^2 \, dx > \delta \). Let \( \hat{u}_n(x) = u_n(x + y_n) \). Then we have \( \|\hat{u}_n\| = \|u_n\| \) and

\[ \bar{J}(\hat{u}_n) = 0, \quad \Phi(\hat{u}_n) \to m_0, \quad \int_{B_1(0)} |\hat{u}_n|^2 \, dx > \delta. \quad (2.50) \]
Therefore, there exists \( \bar{u} \in H^1(\mathbb{R}^3) \setminus \{0\} \) such that, passing to a subsequence,

\[
\begin{align*}
\hat{u}_n & \to \bar{u}, \quad \text{in } H^1(\mathbb{R}^3); \\
\bar{u}_n & \to \bar{u}, \quad \text{in } L^s(\mathbb{R}^3), \quad \forall \ s \in [1, 6); \\
\hat{u}_n & \to \bar{u}, \quad \text{a.e. on } \mathbb{R}^3.
\end{align*}
\]

(2.51)

Let \( w_n = \hat{u}_n - \bar{u} \). Then (2.51) and Lemma 2.8 yield

\[
\Phi(\hat{u}_n) = \Phi(\bar{u}) + \Phi(w_n) + o(1)
\]

(2.52)

and

\[
\bar{J}(\hat{u}_n) = \bar{J}(\bar{u}) + \bar{J}(w_n) + o(1).
\]

(2.53)

From (1.5), (1.11), (2.50), (2.52) and (2.53), one has

\[
\Psi(w_n) = m_0 - \Psi(\bar{u}) + o(1), \quad \bar{J}(w_n) = -\bar{J}(\bar{u}) + o(1).
\]

(2.54)

If there exists a subsequence \( \{w_n\} \) of \( \{w_n\} \) such that \( w_n = 0 \), then going to this subsequence, we have

\[
\Phi(\bar{u}) = m_0, \quad \bar{J}(\bar{u}) = 0,
\]

(2.55)

which implies the conclusion of Lemma 2.9 holds. Next, we assume that \( w_n \neq 0 \).

In view of Lemma 2.4, there exists \( t_n > 0 \) such that \( t_n^2(w_n)_{t_n} \in \mathcal{M} \). We claim that \( \bar{J}(\bar{u}) \leq 0 \). Otherwise, if \( \bar{J}(\bar{u}) > 0 \), then (2.54) implies \( \bar{J}(w_n) < 0 \) for large \( n \). From (1.5), (1.11), (2.59), (2.46) and (2.45), we obtain

\[
\begin{align*}
m_0 - \Psi(\bar{u}) + o(1) &= \Psi(w_n) \\
&= \frac{1}{3} \|w_n\|^2_2 + \frac{2}{3} \int_{\mathbb{R}^3} [f(w_n)w_n - 3F(w_n)] dx \\
&= \Phi(w_n) - \frac{1}{3} \bar{J}(w_n) \\
&\geq \Phi(t_n^2(w_n)_{t_n}) - \frac{t_n^3}{3} \bar{J}(w_n) + \frac{(1 - \theta_0)(1 - t_n^2)(2 + t_n)}{6} \|w_n\|^2_2 \\
&\geq m_0 - \frac{t_n^3}{3} \bar{J}(w_n) + \frac{(1 - \theta_0)(1 - t_n^2)(2 + t_n)}{6} \|w_n\|^2_2 \\
&\geq m_0, \quad \text{for large } n \in \mathbb{N},
\end{align*}
\]

which implies \( \bar{J}(\bar{u}) \leq 0 \) due to \( \Psi(\bar{u}) > 0 \). Since \( \bar{u} \in H^1(\mathbb{R}^3) \setminus \{0\} \), in view of Lemma 2.4, there exists \( \bar{t} > 0 \) such that \( \bar{t}^2\bar{u}_{\bar{t}} \in \mathcal{M} \). Let \( t \to 0 \) in (2.7), then we have

\[
f(\tau)\tau - 3F(\tau) + \frac{\theta_0}{2} \tau^2 \geq 0, \quad \forall \ \tau \in \mathbb{R}.
\]

(2.56)

From (1.5), (1.11), (2.9), (2.46) and Fatou’s lemma, one has

\[
m_0 = \lim_{n \to \infty} \left[ \Phi(\hat{u}_n) - \frac{1}{3} \bar{J}(\hat{u}_n) \right] \\
\]

\[
= \lim_{n \to \infty} \left[ \frac{1}{3} \|\hat{u}_n\|^2_2 + \frac{2}{3} \int_{\mathbb{R}^3} [f(\hat{u}_n)\hat{u}_n - 3F(\hat{u}_n)] dx \right] \\
\geq \frac{1}{3} \|ar{u}\|^2_2 + \frac{2}{3} \int_{\mathbb{R}^3} [f(\bar{u})\bar{u} - 3F(\bar{u})] dx \\
\]

\[
= \Phi(\bar{u}) - \frac{1}{3} \bar{J}(\bar{u}) \\
\]

\[
\geq \Phi(\bar{t}^2\bar{u}_{\bar{t}}) - \frac{\bar{t}^3}{3} \bar{J}(\bar{u}) + \frac{(1 - \theta_0)(1 - \bar{t})^2(2 + \bar{t})}{6} \|\bar{u}\|^2_2 \\
\geq m_0 - \frac{\bar{t}^3}{3} \bar{J}(\bar{u}) + \frac{(1 - \theta_0)(1 - \bar{t})^2(2 + \bar{t})}{6} \|\bar{u}\|^2_2 \geq m_0,
\]
which implies (2.55) also holds.

**Lemma 2.10.** Assume that (F1)-(F4) hold. If \( \tilde{u} \in \tilde{M} \) and \( \Phi(\tilde{u}) = m_0 \), then \( \tilde{u} \) is a critical point of \( \Phi \).

**Proof.** Assume that \( \Phi'(\tilde{u}) \neq 0 \). Then there exist \( \delta > 0 \) and \( \rho > 0 \) such that
\[
\|u - \tilde{u}\| \leq 3\delta \Rightarrow \|\Phi'(u)\| \geq \rho. \tag{2.57}
\]
First, we prove that
\[
\lim_{t \to 1} \|t^2 \tilde{u}_t - \tilde{u}\| = 0. \tag{2.58}
\]
Arguing by contradiction, suppose that there exist \( \varepsilon_0 > 0 \) and a sequence \( \{t_n\} \) such that
\[
\lim_{n \to \infty} t_n = 1, \quad \|t_n^2 \tilde{u}_{t_n} - \tilde{u}\|^2 \geq \varepsilon_0. \tag{2.59}
\]
Since \( \tilde{u} \in H^1(\mathbb{R}^3) \), there exist \( U \in C_0(\mathbb{R}^3, \mathbb{R}^3) \) and \( v \in C_0(\mathbb{R}^3, \mathbb{R}) \) such that
\[
\int_{\mathbb{R}^3} |\nabla \tilde{u} - U|^2 < \frac{\varepsilon_0}{20}, \quad \int_{\mathbb{R}^3} |\tilde{u} - v|^2 < \frac{\varepsilon_0}{20}. \tag{2.60}
\]
From (2.59) and (2.60), one has
\[
\|\nabla (t_n^2 \tilde{u}_{t_n}) - \nabla \tilde{u}\|^2 \leq 2 \int_{\mathbb{R}^3} |\nabla (t_n^2 \tilde{u}_{t_n}) - \nabla u|^2 \leq 2 \int_{\mathbb{R}^3} |\nabla (t_n \tilde{u}_{t_n}) - \nabla u|^2 + \int_{\mathbb{R}^3} |\nabla \tilde{u} - U|^2 dx + 2 \int_{\mathbb{R}^3} |\tilde{u} - U|^2 dx
\]
\[
\leq 2 \int_{\mathbb{R}^3} |t_n \nabla \tilde{u} - U|^2 dx + 2 \int_{\mathbb{R}^3} |\tilde{u} - U|^2 dx
\]
\[
\leq 6t_n^6 \int_{\mathbb{R}^3} |U(t_n x) - U(x)|^2 dx + 6|t_n^3 - 1|^2 \int_{\mathbb{R}^3} |U|^2 dx + (1 + 3t_n^3) \varepsilon_0 \]
\[
= \frac{2}{5} \varepsilon_0 + o(1) \tag{2.61}
\]
and
\[
\|t_n^2 \tilde{u}_{t_n} - \tilde{u}\|^2 \leq 2 \int_{\mathbb{R}^3} |t_n^2 \tilde{u}_{t_n} - v|^2 dx + 2 \int_{\mathbb{R}^3} |\tilde{u} - v|^2 dx
\]
\[
\leq 6t_n^4 \int_{\mathbb{R}^3} |v(t_n x) - v(x)|^2 dx + 6|t_n^2 - 1|^2 \int_{\mathbb{R}^3} v^2 dx + \frac{(1 + 3t_n^3) \varepsilon_0}{10}
\]
\[
= \frac{2}{5} \varepsilon_0 + o(1). \tag{2.62}
\]
Combining (2.61) with (2.62), one has
\[
\|t_n^2 \tilde{u}_{t_n} - \tilde{u}\|^2 = \|\nabla (t_n^2 \tilde{u}_{t_n}) - \nabla \tilde{u}\|^2 + \|t_n^2 \tilde{u}_{t_n} - \tilde{u}\|^2 \leq \frac{4}{5} \varepsilon_0 + o(1). \tag{2.63}
\]
(2.63) contradicts with (2.59). Therefore, (2.58) holds. Thus, there exists \( \delta_1 > 0 \) such that
\[
|t - 1| < \delta_1 \Rightarrow \|t^2 \tilde{u}_t - \tilde{u}\| < \delta. \tag{2.64}
\]
In view of Lemma 2.2, one has
\[
\Phi(t^2\bar{u}_t) \leq \Phi(\bar{u}) - \frac{(1 - \theta_0)(1 - t)^2(2 + t)}{6} \|\bar{u}\|_2^2
\]
\[
= m_0 - \frac{(1 - \theta_0)(1 - t)^2(2 + t)}{6} \|\bar{u}\|_2^2, \quad \forall \ t \geq 0.
\] (2.65)

Let \(\varepsilon := \min\{1 - \theta_0, \|\bar{u}\|_2/24, 1, 9\delta/8\}\) and \(S := B(\bar{u}, \delta)\). Then \[33\] Lemma 2.3 yields a deformation \(\eta \in C([0, 1] \times H^1(\mathbb{R}^3), H^1(\mathbb{R}^3))\) such that

i). \(\eta(1, u) = u\) if \(\Phi(u) < m_0 - 2\varepsilon\) or \(\Phi(u) > m_0 + 2\varepsilon\);

ii). \(\eta(1, \Phi^{m_0 + \varepsilon} \cap B(\bar{u}, \delta)) \subset \Phi^{m_0 - \varepsilon}\);

iii). \(\Phi(\eta(1, u)) = \Phi(u), \quad \forall u \in H^1(\mathbb{R}^3)\);

iv). \(\eta(1, u)\) is a homeomorphism of \(H^1(\mathbb{R}^3)\).

By Corollary 2.3, \(\Phi(t^2\bar{u}_t) \leq \Phi(\bar{u}) = m_0\) for \(t \geq 0\), then it follows from (2.64) and ii) that
\[
\Phi(\eta(1, t^2\bar{u}_t)) \leq m_0 - \varepsilon, \quad \forall \ t \geq 0, \ |t - 1| < \delta_1.
\] (2.66)

On the other hand, by iii) and (2.65), one has
\[
\Phi(\eta(1, t^2\bar{u}_t)) \leq \Phi(t^2\bar{u}_t)
\]
\[
\leq m_0 - \frac{(1 - \theta_0)(1 - t)^2(2 + t)}{6} \|\bar{u}\|_2^2
\]
\[
\leq m_0 - \frac{(1 - \theta_0)\delta_1^2}{3} \|\bar{u}\|_2^2, \quad \forall \ t \geq 0, \ |t - 1| \geq \delta_1.
\] (2.67)

Combining (2.66) with (2.67), we have
\[
\max_{t \in [0.5, 1.5]} \Phi(\eta(1, t^2\bar{u}_t)) < m_0.
\] (2.68)

We prove that \(\eta(1, t^2\bar{u}_t) \cap \mathcal{M} \neq \emptyset\) for some \(t \in [0.5, 1.5]\), contradicting to the definition of \(m_0\). Define
\[
\Psi_0(t) := \tilde{J}(t^2\bar{u}_t), \quad \Psi_1(t) := \tilde{J}(\eta(1, t^2\bar{u}_t)), \quad \forall \ t \geq 0.
\]

By Lemma 2.4 and the degree theory, one can derive that \(\text{deg}(\Psi_0, (0.5, 1.5), 0) = 1\). It follows from (2.65) and i) that \(\eta(1, t^2\bar{u}_t) = t^2\bar{u}_t\) for \(t = 0.5\) and \(t = 1.5\). Thus, \(\text{deg}(\Psi_1, (0.5, 1.5), 0) = \text{deg}(\Psi_0, (0.5, 1.5), 0) = 1\). Hence, \(\Psi_1(t_0) = 0\) for some \(t_0 \in (0.5, 1.5)\), that is \(\eta(1, t_0^2\bar{u}_0) \in \mathcal{M}\), which is a contradiction. \(\square\)

Theorem 1.4 is a direct corollary of Lemmas 2.6, 2.9 and 2.10.

Remark 2.11. Under the assumptions of Theorem 1.4, let \(\tilde{f}(t) = 0\) for \(t < 0\) and \(\tilde{f}(t) = f(t)\) for \(t \geq 0\). It is easy to see that \(\tilde{f}\) satisfies (F1)-(F4). Using \(\tilde{f}\) instead of \(f\) in (1.5), we can prove that \(\bar{u} > 0\) which is obtained in Theorem 1.4 by a standard argument.

3. Ground solutions for (1.1). In this section, we give the proof of Theorem 1.5. To this end, we define two new functionals as follows:
\[
I^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\infty)u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx, \quad u \in H^1(\mathbb{R}^3)
\] (3.1)
and
\[ J^\infty(u) = \frac{3}{2} \|\nabla u\|^2 + \frac{V(\infty)}{2} \|u\|^2 + \frac{3}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 \, dx - \int_{\mathbb{R}^3} [2f(u)u - 3F(u)] \, dx. \tag{3.2} \]

Set
\[ \mathcal{M}^\infty := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : J^\infty(u) = 0 \} \]
and
\[ m := \inf_{u \in \mathcal{M}} I(u), \quad m^\infty := \inf_{u \in \mathcal{M}^\infty} I^\infty(u). \]

**Lemma 3.1.** Assume that (V1) and (V4) hold. Then
\[ \beta(x, t) := 3 \left[ V(x) - tV(t^{-1}x) \right] - (1 - t^3) \left[ V(x) - (\nabla V(x), x) \right] > 0, \quad \forall \ t \in [0, 1) \cup (1, +\infty). \tag{3.3} \]

**Proof.** Note that
\[ \frac{d}{ds} [sV(s^{-1}x)] = V(s^{-1}x) - (\nabla V(s^{-1}x), s^{-1}x), \]
which, together with (V4), implies
\[
\begin{align*}
3 \left[ V(x) - tV(t^{-1}x) \right] - (1 - t^3) \left[ V(x) - (\nabla V(x), x) \right] \\
= 3 \int_t^1 \left\{ \frac{d}{ds} [sV(s^{-1}x)] - s^2 [V(x) - (\nabla V(x), x)] \right\} \, ds \\
= 3 \int_t^1 s^2 \left\{ s^{-2} [V(s^{-1}x) - (\nabla V(s^{-1}x), s^{-1}x)] - [V(x) - (\nabla V(x), x)] \right\} \, ds \\
> 0, \quad \forall \ t \in [0, 1) \cup (1, +\infty). 
\end{align*}
\]
\[ \square \]

**Lemma 3.2.** Assume that (V1), (V4), (F1), (F2) and (F4') hold. Then
\[ I(u) \geq I(t^2u_0) + \frac{1 - t^3}{3} J(u) + \frac{1}{6} \int_{\mathbb{R}^3} \beta(x, t) u^2 \, dx, \quad \forall \ u \in H^1(\mathbb{R}^3), \quad t \geq 0. \tag{3.4} \]

**Proof.** Since (F4') implies (F4) with \( \theta_0 = 0 \), then it follows from Lemma 2.1 that
\[ \frac{2(1 - t^3)}{3} f(\tau) \tau + (t^3 - 2) F(\tau) + \frac{1}{t^3} F(t^2 \tau) \geq 0, \quad \forall \ t > 0, \ \tau \in \mathbb{R}. \tag{3.5} \]

Note that
\[ I(t^2u) = \frac{t^3}{2} \|\nabla u\|^2 + \frac{t}{2} \int_{\mathbb{R}^3} V(t^{-1}x) u^2 \, dx + \frac{t^3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 \, dx - \frac{1}{t^3} \int_{\mathbb{R}^3} F(t^2u) \, dx. \tag{3.6} \]

Thus, by (1.3), (1.10), (3.3), (3.5) and (3.6), one has
\[
\begin{align*}
I(u) - I(t^2u_0) \\
= \frac{1 - t^3}{2} \|\nabla u\|^2 + \frac{t}{2} \int_{\mathbb{R}^3} [V(x) - tV(t^{-1}x)] u^2 \, dx + \frac{1 - t^3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 \, dx \\
\quad + \int_{\mathbb{R}^3} \left[ \frac{1}{t^3} F(t^2u) - F(u) \right] \, dx \\
= \frac{1 - t^3}{3} J(u) + \frac{1}{6} \int_{\mathbb{R}^3} \{3 [V(x) - tV(t^{-1}x)] - (1 - t^3) [V(x) - (\nabla V(x), x)] \} u^2 \, dx. 
\end{align*}
\]
\[ - \langle \nabla V(x), x \rangle \rangle u^2 dx + \int_{\mathbb{R}^3} \left[ \frac{2(1-t^3)}{3} f(u)u + (t^3 - 2)F(u) + \frac{1}{t^3} F(t^2u) \right] dx \geq \frac{1-t^3}{3} J(u) + \frac{1}{6} \int_{\mathbb{R}^3} \beta(x,t) u^2 dx. \]

This shows that (3.4) holds. \[ \square \]

From Lemma 3.2, we have the following two corollaries.

**Corollary 3.3.** Assume that (F1), (F2) and (F4') hold. Then
\[ I^\infty(u) \geq I^\infty(t^2 u_t) + \frac{1-t^3}{3} J^\infty(u) + \frac{(1-t)t^2(2+t)V(\infty)}{6} \| u \|_{L^2}^2, \quad \forall \, u \in H^1(\mathbb{R}^3), \, t \geq 0. \] (3.7)

**Corollary 3.4.** Assume that (V1), (V4), (F1), (F2) and (F4') hold. Then for \( u \in \mathcal{M} \)
\[ I(u) = \max_{t \geq 0} I(t^2 u_t). \] (3.8)

Similar to the proof of Lemma 2.4, we can prove the following lemma by using Lemma 3.2 instead of Lemma 2.2.

**Lemma 3.5.** Assume that (V1), (V2), (V4), (F1)-(F3) and (F4') hold. Then for any \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \), there exists a unique \( t(u) > 0 \) such that \( t(u)^2 u_t \in \mathcal{M} \).

Combining Corollary 3.4 and Lemma 3.5, we have the following lemma.

**Lemma 3.6.** Assume that (V1), (V2), (V4), (F1)-(F3) and (F4') hold. Then
\[ \inf_{u \in \mathcal{M}} I(u) := m = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t \geq 0} I(t^2 u_t). \]

**Lemma 3.7.** Assume that (V1), (V2), (V4), (F1)-(F3) and (F4') hold. Then \( m < m^\infty \).

**Proof.** In view of Theorem 1.4 and Remark 2.11, \( I^\infty \) has a minimizer \( u^\infty > 0 \) on \( \mathcal{M}^\infty \), i.e.
\[ u^\infty \in \mathcal{M}^\infty \text{ and } m^\infty = I^\infty(u^\infty). \]

In view of Lemma 3.5, there exists \( t_0 > 0 \) such that \( t_0^2(u^\infty)_{t_0} \in \mathcal{M} \). Thus, it follows from (V2), (3.13), (3.1) and (3.7) that
\[ m^\infty = I^\infty(u^\infty) \geq I^\infty(t_0^2(u^\infty)_{t_0}) > I(t_0^2(u^\infty)_{t_0}) \geq m. \] \[ \square \]

**Lemma 3.8.** Assume that (V1), (V2) and (V4) hold. Then there exist two constants \( \gamma_1, \gamma_2 > 0 \) such that
\[ \gamma_1 \| u \|^2 \leq 3 \| \nabla u \|^2_2 + \int_{\mathbb{R}^3} |V(x) - (\nabla V(x), x)| u^2 dx \leq \gamma_2 \| u \|^2, \quad \forall \, u \in H^1(\mathbb{R}^3). \] (3.9)

**Proof.** By (V4), one has
\[ -2V(x) + \varphi_0 \leq (\nabla V(x), x) \leq V(x), \quad \forall \, x \in \mathbb{R}^3. \] (3.10)
It follows from (V2) and (3.10) that there exists a constant $\gamma_2 > 0$ such that
\[
3\|\nabla u\|^2_2 + \int_{\mathbb{R}^3} [V(x) - (\nabla V(x), x)] u^2 \, dx \leq \gamma_2 \|u\|^2, \quad \forall \, u \in H^1(\mathbb{R}^3). \tag{3.11}
\]

Next, we prove the first inequality in (3.9). Arguing by contradiction, suppose that there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that
\[
\|u_n\| = 1, \quad 3\|\nabla u_n\|^2_2 + \int_{\mathbb{R}^3} [V(x) - (\nabla V(x), x)] u_n^2 \, dx = o(1). \tag{3.12}
\]
Thus there exists $\bar{u} \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$. Then $u_n \to \bar{u}$ in $L^s_{\text{loc}}(\mathbb{R}^3)$ for $2 \leq s < 6$ and $u_n \to \bar{u}$ a.e. in $\mathbb{R}^3$. Since $V(x) - (\nabla V(x), x) > 0$, then it follows from (3.12), the weak semicontinuity of norm and Fatou’s Lemma that
\[
0 = \lim_{n \to \infty} \left[3\|\nabla u_n\|^2_2 + \int_{\mathbb{R}^3} [V(x) - (\nabla V(x), x)] u_n^2 \, dx \right] \\
\geq 3\|\nabla \bar{u}\|^2_2 + \int_{\mathbb{R}^3} [V(x) - (\nabla V(x), x)] \bar{u}^2 \, dx,
\]
which implies $\bar{u} = 0$. Thus, from (V2) and (V4), one has
\[
\int_{\mathbb{R}^3} [V(x) - V(\infty) - (\nabla V(x), x)] u_n^2 \, dx = o(1). \tag{3.13}
\]
Both (3.12) and (3.13) imply
\[
o(1) = 3\|\nabla u_n\|^2_2 + \int_{\mathbb{R}^3} [V(x) - (\nabla V(x), x)] u_n^2 \, dx \\
= 3\|\nabla u_n\|^2_2 + V(\infty)\|u_n\|^2_2 + o(1) \\
\geq \min\{3, V(\infty)\}\|u\|^2 + o(1) \\
= \min\{3, V(\infty)\} + o(1).
\]
This contradiction shows that there exists $\gamma_1$ such that the first inequality in (3.9) holds. \hfill \square

Analogous to the proof of Lemma 2.6, we can prove the following lemma by using Lemmas 3.2 and 3.8 instead of Lemma 2.2.

**Lemma 3.9.** Assume that (V1), (V2), (V4), (F1)-(F3) and (F4) hold. Then
i). there exists $\rho_0 > 0$ such that $\|u\| \geq \rho_0$, $\forall \, u \in \mathcal{M}$; 
ii). $m = \inf_{u \in \mathcal{M}} I(u) > 0$.

**Lemma 3.10.** Assume that (V1), (V2), (V4), (F1)-(F3) and (F4) hold. Then $m$ is achieved.

**Proof.** Let $\{u_n\} \subset \mathcal{M}$ be such that $I(u_n) \to m$. Since $J(u_n) = 0$, then it follows from (3.14) with $t = 0$ and $\beta(x,0) = 2V(x) + (\nabla V(x), x) \geq \rho_0$ that
\[
m + o(1) = I(u_n) \geq \frac{1}{6} \int_{\mathbb{R}^3} \beta(x,0)u_n^2 \, dx \geq \frac{\rho_0}{6}\|u_n\|^2_2. \tag{3.14}
\]
This shows that $\{\|u_n\|_2\}$ is bounded. Next, we prove that $\{\|
abla u_n\|_2\}$ is also bounded. Arguing by contradiction, suppose that $\|
abla u_n\|_2 \to \infty$. Let $t_n = (\sqrt{6m}$
Hence, using (V2), (F1), (F2), \((3.4), (3.6), (3.14)\) and \(J(u_n) = 0\), one has

\[
m + o(1) = I(u_n) = I(t_n^2 u_n)
= \frac{t_n^4}{2} \|\nabla u_n\|^2 + \frac{t_n}{2} \int_{\mathbb{R}^3} V(t_n^{-1} x) u_n^2 \, dx + \frac{t_n^3}{4} \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 \, dx
- \frac{1}{t_n^3} \int_{\mathbb{R}^3} F(t_n^2 u_n) \, dx
\geq \frac{t_n^4}{2} \|\nabla u_n\|^2 - C_1 t_n \|u_n\|^2 - \frac{S^3}{216 m^2} t_n^6 \|u_n\|^6
\geq \frac{t_n^4}{2} \|\nabla u_n\|^2 - C_1 t_n \|u_n\|^2 - \frac{1}{216 m^2} t_n^6 \|\nabla u_n\|^6
= 2m - \frac{(6m)^{1/3} C_1}{\|\nabla u_n\|^2/2} \|u_n\|^2
= 2m + o(1).
\]

This contradiction shows that \(\{\|\nabla u_n\|_2^2\}\) is also bounded, and so \(\{u_n\}\) is bounded in \(H^1(\mathbb{R}^3)\). Passing to a subsequence, we have \(u_n \to \bar{u}\) in \(H^1(\mathbb{R}^3)\). Then \(u_n \to \bar{u}\) in \(L_{loc}^s(\mathbb{R}^3)\) for \(2 \leq s < 6\) and \(u_n \to \bar{u}\) a.e. in \(\mathbb{R}^3\). Next, we prove \(\bar{u} \neq 0\).

Arguing by contradiction, suppose that \(\bar{u} = 0\), i.e. \(u_n \to 0\) in \(H^1(\mathbb{R}^3)\). Then \(u_n \to 0\) in \(L_{loc}^s(\mathbb{R}^3)\) for \(2 \leq s < 6\) and \(u_n \to 0\) a.e. in \(\mathbb{R}^3\). By (V2) and (V4), it is easy to show that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} [V(\infty) - V(x)] u_n^2 \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^3} (\nabla V(x), x) u_n^2 \, dx = 0.
\]
From \((1.3), (1.10), (3.1), (3.2)\) and \((3.15)\), one can get

\[
I^\infty(u_n) \to m, \quad J^\infty(u_n) \to 0.
\]

From Lemma 3.9 (i), \((3.2)\) and \((3.16)\), one has

\[
\min\{3, V(\infty)\} \rho_0^2 \leq \frac{\min\{3, V(\infty)\}}{2} \|u_n\|^2
\leq \frac{3}{2} \|\nabla u_n\|^2 + \frac{V(\infty)}{2} \|u_n\|^2 + \frac{3}{4} \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 \, dx
= \int_{\mathbb{R}^3} [2f(u_n) u_n - 3F(u_n)] \, dx + o(1).
\]

Using (F1), (F2), \((3.17)\) and Lions’ concentration compactness principle \([33, Lemma 1.21]\), we can prove that there exist \(\delta > 0\) and a sequence \(\{y_n\} \subset \mathbb{R}^3\) such that \(\int_{B_1(y_n)} |u_n|^2 \, dx > \delta\). Let \(\hat{u}_n(x) = u_n(x + y_n)\). Then we have \(\|\hat{u}_n\| = \|u_n\|\) and

\[
J^\infty(\hat{u}_n) = o(1), \quad I^\infty(\hat{u}_n) \to m, \quad \int_{B_1(0)} |\hat{u}_n|^2 \, dx > \delta.
\]

Therefore, there exists \(\bar{v} \in H^1(\mathbb{R}^3) \setminus \{0\}\) such that, passing to a subsequence,

\[
\begin{cases}
\hat{u}_n \to \bar{v}, & \text{in } H^1(\mathbb{R}^3); \\
\hat{u}_n \to \bar{v}, & \text{in } L_{loc}^s(\mathbb{R}^3), \forall s \in [1, 6); \\
\hat{u}_n \to \bar{v}, & \text{a.e. on } \mathbb{R}^3.
\end{cases}
\]

Let \(w_n = \hat{u}_n - \bar{v}\). Then \((3.19)\) and Lemma 2.8 yield

\[
I^\infty(\hat{u}_n) = I^\infty(\bar{v}) + I^\infty(w_n) + o(1)
\]
and
\[ J^\infty(\hat{u}_n) = J^\infty(\hat{v}) + J^\infty(w_n) + o(1). \] (3.21)

Let \( \theta_0 = 0 \) and \( t \to 0 \) in (2.7), then we have
\[ f(\tau) - 3F(\tau) \geq 0, \quad \forall \tau \in \mathbb{R}. \] (3.22)

We introduce a new functional \( \Psi^\infty : H^1(\mathbb{R}^3) \to \mathbb{R} \) defined as follows:
\[ \Psi^\infty(u) = \frac{V(\infty)}{3} \|u\|^2 + \frac{2}{3} \int_{\mathbb{R}^3} [f(u)u - 3F(u)] dx, \quad \forall u \in H^1(\mathbb{R}^3). \] (3.23)

From (3.1), (3.2), (3.7), (3.18), (3.20), (3.21) and (3.23), one has
\[ \Psi^\infty(w_n) = m - \Psi^\infty(\hat{v}) + o(1), \quad J^\infty(w_n) = -J^\infty(\hat{v}) + o(1). \] (3.24)

If there exists a subsequence \( \{w_n\} \) of \( \{w_n\} \) such that \( w_n \to 0 \), then going to this subsequence, we have \( I^\infty(\hat{v}) = m \) and \( J^\infty(\hat{v}) = 0 \), which implies
\[ m \geq m^\infty. \] (3.25)

Next, we assume that \( w_n \neq 0 \). In view of Lemma 2.4, there exists \( t_n > 0 \) such that \( t_n^2(w_n) \in M^\infty \). We claim that \( J^\infty(\hat{v}) \leq 0 \). Otherwise, if \( J^\infty(\hat{v}) > 0 \), then (3.24) implies \( J^\infty(w_n) < 0 \) for large \( n \). From (3.1), (3.2), (3.7), (3.18), (3.19), (3.20), (3.21), (3.22) and Fatou’s lemma, one has
\[ m = \lim_{n \to \infty} \left[ I^\infty(\hat{u}_n) - \frac{1}{3} J^\infty(\hat{u}_n) \right] \]
\[ = \lim_{n \to \infty} \left[ \frac{V(\infty)}{3} \|\hat{u}_n\|^2 + \frac{2}{3} \int_{\mathbb{R}^3} [f(\hat{u}_n)\hat{u}_n - 3F(\hat{u}_n)] dx \right] \]
\[ \geq \frac{V(\infty)}{3} \|\hat{v}\|^2 + \frac{2}{3} \int_{\mathbb{R}^3} [f(\hat{v})\hat{v} - 3F(\hat{v})] dx \]
\[ = I^\infty(\hat{v}) - \frac{1}{3} J^\infty(\hat{v}) \]
\[ \geq I^\infty(\hat{v}) - \frac{1}{3} J^\infty(\hat{v}) + \frac{(1 - \hat{v})^2(2 + \hat{v}) V(\infty)}{6} \|\hat{v}\|^2 \]
\[ \geq m^\infty - \frac{1}{3} J^\infty(\hat{v}) + \frac{(1 - \hat{v})^2(2 + \hat{v}) V(\infty)}{6} \|\hat{v}\|^2 \geq m^\infty, \]
which implies (3.25) also holds. Clearly, (3.25) contradicts with Lemma 3.7, therefore, \( \hat{u} \neq 0 \). Let \( v_n = u_n - \hat{u} \). Then Lemma 2.8 yields
\[ I(u_n) = I(\hat{u}) + I(v_n) + o(1) \] (3.26)
and
\[ J(u_n) = J(\bar{u}) + J(v_n) + o(1). \] (3.27)

Set
\[ \tilde{\Psi}(u) = \frac{1}{6} \int_{\mathbb{R}^3} [2V(x) + (\nabla V(x), x)]u^2 dx + \frac{2}{3} \int_{\mathbb{R}^3} [f(u)u - 3F(u)]dx. \] (3.28)

From (3.26), (3.27) and (3.28), one has
\[ \tilde{\Psi}(v_n) = m - \tilde{\Psi}(\bar{u}) + o(1), \quad J(v_n) = -J(\bar{u}) + o(1). \] (3.29)

The rest of the proof is the same as the one of Lemma 2.9, so we omit it. □

In the same way as the proof of Lemma 2.10, we can prove the following lemma by using Lemmas 3.2 and 3.5 instead of Lemmas 2.2 and 2.4, respectively.

**Lemma 3.11.** Assume that (V1), (V2), (V4), (F1)-(F3) and \( (F4') \) hold. If \( \bar{u} \in M \) and \( I(\bar{u}) = m \), then \( \bar{u} \) is a critical point of \( I \).

Theorem 1.5 is a direct corollary of Lemmas 3.9, 3.10 and 3.11.

4. **The least energy solutions.** In this section, we give the proof of Theorems 1.6-1.8.

**Proposition 4.1.** [20] Let \( X \) be a Banach space and let \( J \subset \mathbb{R}^+ \) be an interval. We consider a family \( \{ \Phi_\lambda \}_{\lambda \in J} \) of \( C^1 \)-functional on \( X \) of the form
\[ \Phi_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in J, \]
where \( B(u) \geq 0, \forall u \in X \), and such that either \( A(u) \to +\infty \) or \( B(u) \to +\infty \), as \( \|u\| \to \infty \). We assume that there are two points \( v_1, v_2 \) in \( X \) such that
\[ c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\}, \] (4.1)
where
\[ \Gamma = \{ \gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2 \}. \]

Then, for almost every \( \lambda \in J \), there is a bounded (PS)\( c_\lambda \) sequence for \( \Phi_\lambda \), that is, there exists a sequence such that
(i). \( \{u_n(\lambda)\} \) is bounded in \( X \);
(ii). \( \Phi_\lambda(u_n(\lambda)) \to c_\lambda \);
(iii). \( \Phi'_\lambda(u_n(\lambda)) \to 0 \) in \( X^* \), where \( X^* \) is the dual of \( X \).

To apply Proposition 4.1, we introduce two families of functional defined by
\[ I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 dx - \lambda \int_{\mathbb{R}^3} F(u)dx \] (4.2)
and
\[ I_\infty(\lambda) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\infty)u^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 dx - \lambda \int_{\mathbb{R}^3} F(u)dx, \] (4.3)
for \( \lambda \in [1/2, 1] \).
Lemma 4.2. Assume that (V1)-(V3), (F1) and (F2) hold. Let \( u \) be a critical point of \( I_\lambda \) in \( H^1(\mathbb{R}^3) \), then we have the following Pohozaev type identity
\[
P_\lambda(u) := \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \int_{\mathbb{R}^3} [3V(x) + \langle \nabla V(x), x \rangle] u^2 \, dx \\
+ \frac{5}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 \, dx - 3\lambda \int_{\mathbb{R}^3} F(u) \, dx = 0. \tag{4.4}
\]
We set \( J_\lambda(u) := 2(I_\lambda'(u), u) - P_\lambda(u) \), then
\[
J_\lambda(u) = \frac{3}{2} \|\nabla u\|^2 + \frac{1}{2} \int_{\mathbb{R}^3} [V(x) - \langle \nabla V(x), x \rangle] u^2 \, dx \\
+ \frac{3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 \, dx - \lambda \int_{\mathbb{R}^3} [2f(u) - 3F(u)] \, dx, \tag{4.5}
\]
for \( \lambda \in [0.5, 1] \). Correspondingly, we also let
\[
J_\lambda^\infty(u) = \frac{3}{2} \|\nabla u\|^2 + \frac{V(\infty)}{2} \|u\|^2 \\
+ \frac{3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 \, dx - \lambda \int_{\mathbb{R}^3} [2f(u) - 3F(u)] \, dx, \tag{4.6}
\]
for \( \lambda \in [0.5, 1] \). Set
\[
\mathcal{M}_\lambda^\infty := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : J_\lambda^\infty(u) = 0 \}, \quad m_\lambda^\infty := \inf_{u \in \mathcal{M}_\lambda^\infty} I_\lambda^\infty(u).
\]
By Corollary 3.3, we have the following lemma.

Lemma 4.3. Assume that (F1), (F2) and (F4') hold. Then
\[
I_\lambda^\infty(u) \geq I_\lambda^\infty(t^2 u_1^\infty) + \frac{1 - t^3}{3} J_\lambda^\infty(u) + \frac{\lambda(1 - t^2)(2 + t) V(\infty)}{6} \|u\|_2^2,
\]
\[
\forall u \in H^1(\mathbb{R}^3), \quad t \geq 0, \quad 0 \leq \lambda \leq 1. \tag{4.7}
\]
In view of Theorem 1.4 and Remark 2.11, \( I_\lambda^\infty \) has a minimizer \( u_1^\infty > 0 \) on \( \mathcal{M}_1^\infty \), i.e.
\[
u_1^\infty \in \mathcal{M}_1^\infty, \quad (I_1^\infty)'(u_1^\infty) = 0 \quad \text{and} \quad m_1^\infty = I_1^\infty(u_1^\infty). \tag{4.8}
\]

Lemma 4.4. Assume that (V1)-(V3) and (F1)-(F3) hold. Then
(i) there exists \( T > 0 \) independent of \( \lambda \) such that \( I_\lambda(T^2(u_1^\infty)_T) < 0 \) for all \( \lambda \in [0.5, 1] \);
(ii) there exists a positive constant \( \kappa_0 \) independent of \( \lambda \) such that for all \( \lambda \in [0.5, 1] \),
\[
c_\lambda := \inf_{\gamma \in C([0, 1])} \max_{t \in [0, 1]} I_\lambda(\gamma(t)) \geq \kappa_0 > \max\{I_\lambda(0), I_\lambda(T^2(u_1^\infty)_T)\},
\]
where
\[
\Gamma = \{ \gamma \in C([0, 1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = T^2(u_1^\infty)_T \};
\]
(iii) if \( 2f(\tau)\tau \geq 3F(\tau) \geq 0 \) for \( \tau \geq 0 \), then \( c_\lambda \) and \( m_\lambda^\infty \) are non-increasing on \( \lambda \in [0.5, 1] \).

The proof of Lemma 4.4 is standard, so we omit it.

Lemma 4.5. Assume that (V1)-(V3), (F1)-(F3) and (F4') hold. Then there exists \( \lambda \in [1/2, 1] \) such that \( c_\lambda < m_\lambda^\infty \) for \( \lambda \in [\lambda, 1] \).
Proof. It is easy to see that $I_{\lambda}(t^2(u_1^\infty))$ is continuous on $t \in [0, \infty)$. Hence for any $\lambda \in [1/2, 1)$, we can choose $t_\lambda \in (0, T)$ such that

$$I_{\lambda}(t_\lambda^2(u_1^\infty)) = \max_{t \in (0, T)} I_{\lambda}(t^2(u_1^\infty)).$$

Note that $I_{1/2}(t^2(u_1^\infty)) \to -\infty$ as $t \to \infty$, thus there exists $T_0 > 0$ such that

$$I_{1/2}(t^2(u_1^\infty)) \leq I_1(u_1^\infty) - 1, \quad \forall \ t \geq T_0. \quad (4.9)$$

By (4.2) and the definition of $t_\lambda$, one has

$$I_1(u_1^\infty) \leq I_{\lambda}(u_1^\infty) \leq I_{1/2}(t_\lambda^2(u_1^\infty)) \leq I_{1/2}(t^2(u_1^\infty)),$$

which, together with (4.9), implies $t_\lambda < T_0$ for $\lambda \in [1/2, 1]$. Let $\beta_0 = \inf_{\lambda \in [1/2, 1]} t_\lambda$.

If $\beta_0 = 0$, then there exists a sequence $\{\lambda_n\} \subset [1/2, 1]$ such that

$$\lambda_n \to \lambda_0 \in [1/2, 1], \quad t_{\lambda_n} \to 0.$$ 

It follows that

$$0 < c_1 \leq c_{\lambda_n} \leq I_{\lambda_n}(t_{\lambda_n}^2(u_1^\infty)) = o(1),$$

which implies $\beta_0 > 0$. Thus

$$0 < \beta_0 \leq t_\lambda < T_0, \quad \forall \lambda \in [1/2, 1]. \quad (4.10)$$

Let

$$\bar{\lambda} := \max \left\{ \frac{1}{2}, 1 - \frac{\beta_0^2 \min_{\beta_0 \leq s \leq T_0} \int_{R^3} [V(\infty) - V(s^{-1}x)] |u_1^\infty|^2dx}{2 \int_{R^3} F(T_0^2 u_1^\infty)dx} \right\}. \quad (4.11)$$

Then $1/2 \leq \bar{\lambda} < 1$. From (4.2), (4.3), (4.7), (4.10), (4.11) and Lemma 4.4 (iii), we have

$$m_{\bar{\lambda}}^\infty \geq m_1^\infty = I_{\bar{\lambda}}(u_1^\infty) \geq I_{\bar{\lambda}}(t_{\bar{\lambda}}^2(u_1^\infty)),$$

$$= I_{\lambda}(t_{\lambda}^2(u_1^\infty)) - \frac{1 - \lambda}{t_{\lambda}^2} \int_{R^3} F(t_{\lambda}^2 u_1^\infty)dx$$

$$+ \frac{t_{\lambda}}{2} \int_{R^3} [V(\infty) - V(t_{\lambda}^{-1}x)] |u_1^\infty|^2dx$$

$$> c_{\lambda} - \frac{1 - \lambda}{\beta_0^2} \int_{R^3} F(T_0^2 u_1^\infty)dx + \frac{\beta_0}{2} \min_{\beta_0 \leq s \leq T_0} \int_{R^3} [V(\infty) - V(s^{-1}x)] |u_1^\infty|^2dx$$

$$\geq c_{\lambda}, \quad \forall \lambda \in [\bar{\lambda}, 1].$$

\[ \square \]

Remark 4.6. In [34, Lemma 3.5], the inequality $c_{\lambda} < m_{\bar{\lambda}}^\infty$ is proved for $\lambda \in [1/2, 1]$. The proof is given as follows: Let $u_{\lambda}^\infty$ be the minimizer of $m_{\lambda}^\infty$, by [34, Lemma 3.3], we have

$$I_{\lambda}^\infty(u_{\lambda}^\infty) = \max_{t \geq 0} I_{\lambda}(t^2 u_{\lambda}^\infty(tx)).$$

Thus by choosing $v = t^2 u_{\lambda}^\infty(tx)$ for $t$ large in [34, Lemma 3.2], we have

$$c_{\lambda} \leq \max_{t \geq 0} I_{\lambda}(t^2 u_{\lambda}^\infty(tx)) < \max_{t \geq 0} I_{\lambda}^\infty(t^2 u_{\lambda}^\infty(tx)) = I_{\lambda}^\infty(u_{\lambda}^\infty) = m_{\lambda}^\infty.$$ 

However, the above proof is not sufficient. In fact, in [34, Lemma 3.2],

$$c_{\lambda} := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \Phi_{\lambda}(\gamma(t)),$$

where

$$\Gamma = \{ \gamma \in C([0, 1], H^1(R^3)) : \gamma(0) = 0, \gamma(1) = v \}.$$
The $v$ in $\Gamma$ should be independent of $\lambda \in [1/2, 1]$, but $t^2 u_\lambda^\infty(tx)$ is dependent of $\lambda \in [1/2, 1]$. Therefore, it is not true that $t^2 u_\lambda^\infty(tx) \in \Gamma$ for all $\lambda \in [1/2, 1]$.

**Lemma 4.7.** Assume that (V1)-(V3) and (F1)-(F3) hold. Let $\{u_n\}$ be a bounded (PS) sequence for $I_\lambda$, for $\lambda \in [1/2, 1]$. Then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, an integer $l \in \mathbb{N} \cup \{0\}$, $w^k \in H^1(\mathbb{R}^3)$ for $1 \leq k \leq l$, such that

(i). $u_n \to u_0$ with $I_\lambda'(u_0) = 0$;

(ii). $w^k \neq 0$ and $\langle (I_\lambda^\infty)'w^k, w^k \rangle = 0$ for $1 \leq k \leq l$;

(iii). $I_\lambda(u_n) \to I_\lambda(u_0) + \sum_{i=1}^l I_\lambda^\infty(w^i)$;

where we agree that in the case $l = 0$ the above holds without $w^k$.

Since (2.35) and (2.36) of Lemma 2.8 hold, so we can prove Lemma 4.7 in a standard way. We omit it here.

**Lemma 4.8.** Assume that (V1)-(V3) and (F1)-(F3) hold. Then for almost every $\lambda \in [\bar{\lambda}, 1]$, there exists $u_\lambda \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that

$$I_\lambda'(u_\lambda) = 0, \quad I_\lambda(u_\lambda) = c_\lambda. \tag{4.12}$$

**Proof.** Under (F1) and (F3), Lemma 4.4 implies that $I_\lambda(u)$ satisfies the assumptions of Proposition 4.1 with $X = H^1(\mathbb{R}^3)$ and $\Phi_\lambda = I_\lambda$. So for almost every $\lambda \in [0, 1]$, there exists a bounded sequence $\{u_n(\lambda)\} \subset H^1(\mathbb{R}^3)$ (for simplicity, we denote $\{u_n\}$ instead of $\{u_n(\lambda)\}$ such that

$$I_\lambda(u_n) \to c_\lambda > 0, \quad \|I_\lambda'(u_n)\| \to 0. \tag{4.13}$$

By Lemma 4.7, there exist $l \in \mathbb{N} \cup \{0\}$ and $u_\lambda \in H^1(\mathbb{R}^3)$ such that $I_\lambda'(u_\lambda) = 0$ and

$$u_n \to u_\lambda \quad \text{in} \quad H^1(\mathbb{R}^3), \quad I_\lambda(u_n) \to I_\lambda(u_\lambda) + \sum_{i=1}^l I_\lambda^\infty(w^i),$$

where $\{w^i\}_{i=1}^l$ are the critical points of $I_\lambda^\infty$. Since $I_\lambda'(u_\lambda) = 0$, then $J_\lambda(u_\lambda) = 0$. It follows from (4.2) and (4.5) that

$$I_\lambda(u_\lambda) = I_\lambda(u_\lambda) - \frac{1}{3} J_\lambda(u_\lambda)$$

$$= \frac{1}{6} \int_{\mathbb{R}^3} [2V(x) + (\nabla V(x), x)]u_\lambda^2 dx + \frac{2\lambda}{3} \int_{\mathbb{R}^3} [f(u_\lambda)u_\lambda - 3F(u_\lambda)] dx$$

$$\geq 0.$$

If $l \neq 0$, then

$$c_\lambda = \lim_{n \to \infty} I_\lambda(u_n) = I_\lambda(u_\lambda) + \sum_{i=1}^l I_\lambda^\infty(w^i) \geq m_\lambda^\infty, \quad \forall \lambda \in [\bar{\lambda}, 1],$$

which is a contradiction by Lemma 4.5. Thus $l = 0$ and then Lemma 4.7 implies that $u_n \to u_\lambda$ in $H^1(\mathbb{R}^3)$ and $I_\lambda(u_\lambda) = c_\lambda$. \qed

**Proof of Theorem 1.6.** In view of Lemma 4.8, there exist two sequences of $\{\lambda_n\} \subset [\bar{\lambda}, 1]$ and $\{u_{\lambda_n}\} \subset H^1(\mathbb{R}^3)$, denoted by $\{u_n\}$ for simplicity, such that

$$\lambda_n \to 1, \quad I_{\lambda_n}'(u_n) = 0, \quad I_{\lambda_n}(u_n) = c_{\lambda_n}. \tag{4.14}$$
From (V3), (4.2), (4.5) and (4.14), one has
\[
c_{\lambda_n} = I_{\lambda_n}(u_n) - \frac{1}{3} J_{\lambda_n}(u_n)
\]
\[
= \frac{1}{6} \int_{\mathbb{R}^3} [2V(x) + (\nabla V(x), x)]u_n^2 dx + \frac{2\lambda_n}{3} \int_{\mathbb{R}^3} [f(u_n)u_n - 3F(u_n)]dx
\]
\[
\geq \frac{2\lambda_n}{3} \int_{\mathbb{R}^3} [f(u_n)u_n - 3F(u_n)]dx. \tag{4.15}
\]
Next, we show that \(\{u_n\}\) is bounded in \(H^1(\mathbb{R}^3)\). Arguing by contradiction, suppose that \(\|u_n\| \to \infty\). Set \(v_n = u_n/\|u_n\|\), then \(\|v_n\| = 1\). If
\[
\delta := \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |v_n|^2 dx = 0,
\]
then by Lions’ concentration compactness principle \([33], \text{Lemma 1.21}\), \(v_n \to 0\) in \(L^s(\mathbb{R}^3)\) for \(2 < s < 6\). Set \(\kappa' = \kappa/(\kappa - 1)\) and
\[
\Omega_n := \left\{ x \in \mathbb{R}^3 : \frac{f(u_n)}{u_n} \leq \frac{\gamma_0}{2} \right\}. \tag{4.16}
\]
Then one has
\[
\int_{\Omega_n} \frac{f(u_n)}{u_n} v_n^2 dx \leq \frac{\gamma_0}{2} \|v_n\|^2 \leq \frac{1}{2}. \tag{4.17}
\]
On the other hand, by virtue of (F5), (4.15) and the Hölder inequality, one can get
\[
\int_{\mathbb{R}^3 \setminus \Omega_n} \frac{f(u_n)}{u_n} v_n^2 dx \leq \left[ \int_{\mathbb{R}^3 \setminus \Omega_n} \left| \frac{f(u_n)}{u_n} \right|^\kappa dx \right]^{1/\kappa} \|v_n\|^\frac{2}{2\kappa'}
\]
\[
\leq C_1 \left( \int_{\mathbb{R}^3 \setminus \Omega_n} [f(u_n)u_n - 3F(u_n)]dx \right)^{1/\kappa} \|v_n\|^\frac{2}{2\kappa'}
\]
\[
\leq C_2 \|v_n\|^\frac{2}{2\kappa'} = o(1). \tag{4.18}
\]
Combining (4.17) with (4.18) and making use of \(I'_{\lambda_n}(u_n) = 0\), we have
\[
1 \leq \frac{1}{\|u_n\|^2} \left[ \int_{\mathbb{R}^3} |\nabla u_n|^2 + V(x)u_n^2 dx + \int_{\mathbb{R}^3} \phi_{u_n}(x)u_n^2 dx \right]
\]
\[
= \lambda_n \int_{\mathbb{R}^3} \frac{f(u_n)}{u_n} v_n^2 dx
\]
\[
= \lambda_n \int_{\Omega_n} \frac{f(u_n)}{u_n} v_n^2 dx + \lambda_n \int_{\mathbb{R}^3 \setminus \Omega_n} \frac{f(u_n)}{u_n} v_n^2 dx
\]
\[
= \frac{1}{2} + o(1).
\]
This contradiction shows that \(\delta := \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |v_n|^2 dx > 0\).

Going if necessary to a subsequence, we may assume the existence of \(y_n \in \mathbb{R}^3\) such that \(\int_{B_1(y_n)} |v_n|^2 dx > \frac{\delta}{2}\). Let \(w_n(x) = v_n(x + y_n)\). Then \(\|w_n\| = \|v_n\| = 1\), and
\[
\int_{B_1(0)} |w_n|^2 dx > \frac{\delta}{2}. \tag{4.19}
\]
Passing to a subsequence, we have \(w_n \rightharpoonup w\) in \(H^1(\mathbb{R}^3)\), \(w_n \to w\) in \(L^s_{\text{loc}}(\mathbb{R}^3)\), \(2 \leq s < 6\), \(w_n \to w\) a.e. on \(\mathbb{R}^3\). Obviously, (4.19) implies that \(w \neq 0\).
Now we define \( \tilde{u}_n(x) = u_n(x + y_n) \), then \( \tilde{u}_n/\|u_n\| = w_n \to w \) a.e. on \( \mathbb{R}^3 \), \( w \neq 0 \). For \( x \in \{ y \in \mathbb{R}^3 : w(y) \neq 0 \} \), we have \( \lim_{n \to \infty} |\tilde{u}_n(x)| = \infty \). From (4.2), (4.5) and (4.14), we have
\[
\|\nabla u_n\|^2 \leq -\frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x) u_n^2 dx - I_{\lambda_n}(u_n) = \lambda_n \int_{\mathbb{R}^3} [f(u_n)u_n - 2F(u_n)] dx. \tag{4.20}
\]
It follows from (F3), (3.22) and (4.20) that
\[
o(1) \geq \frac{1}{\|u_n\|^3} \left[ \|\nabla u_n\|^2 - \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x) u_n^2 dx - I_{\lambda_n}(u_n) \right]
= \frac{\lambda_n}{\|u_n\|^3} \int_{\mathbb{R}^3} [f(u_n)u_n - 2F(u_n)] dx
= \frac{\lambda_n}{\|u_n\|^3} \int_{\mathbb{R}^3} [f(\tilde{u}_n)\tilde{u}_n - 2F(\tilde{u}_n)] dx
\geq \lambda_n \int_{\mathbb{R}^3} \frac{F(\tilde{u}_n)}{|\tilde{u}_n|^3} \tilde{u}_n^2 dx \to \infty. \tag{4.21}
\]
This shows that \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \). The rest of the proof is standard, so we omit it.

**Proof of Theorem 1.7.** In view of Lemma 4.8, there exist two sequences of \( \{\lambda_n\} \subset [\bar{\lambda}, 1] \) and \( \{u_{\lambda_n}\} \subset H^1(\mathbb{R}^3) \), denoted by \( \{u_n\} \) for simplicity, such that (4.14) holds. From (V5), (4.2), (4.5) and (4.14), one has
\[
c_{1/2} \geq c_{\lambda_n} = I_{\lambda_n}(u_n) - \frac{1}{3} J_{\lambda_n}(u_n)
= \frac{1}{6} \int_{\mathbb{R}^3} [2V(x) + (\nabla V(x), x)] u_n^2 dx + \frac{2\lambda_n}{3} \int_{\mathbb{R}^3} [f(u_n)u_n - 3F(u_n)] dx
\geq \frac{6}{6} \|u_n\|^2 + \frac{2\lambda_n}{3} \int_{\mathbb{R}^3} [f(u_n)u_n - 3F(u_n)] dx. \tag{4.22}
\]
which, together with (3.22), implies that \( \{\|u_n\|_2\} \) is bounded. Next, we prove that \( \{\|\nabla u_n\|_2\} \) is also bounded. Arguing by contradiction, suppose that \( \|\nabla u_n\|_2 \to \infty \). Choose \( M_0 > 1 \) such that
\[
c_{\lambda_n} + \int_{\mathbb{R}^3} |V(\infty) - V(x)| + |(\nabla V(x), x)| u_n^2 dx \leq M_0. \tag{4.23}
\]
Let \( t_n = \min \{1, (\sqrt{6}M_0/\|\nabla u_n\|_2)^{2/3}\} \). Hence, using (4.2), (4.3), (4.5), (4.6) and (4.23), one has
\[
I_{\lambda_n}(t_n^2(u_n)u_n) \leq I_{\lambda_n}(u_n) - \frac{1 - t_n^2}{3} J_{\lambda_n}(u_n)
= I_{\lambda_n}(u_n) + \frac{1}{2} \int_{\mathbb{R}^3} |V(\infty) - V(x)| u_n^2 dx
- \frac{1 - t_n^2}{3} \left[ J_{\lambda_n}(u_n) + \frac{1}{2} \int_{\mathbb{R}^3} |V(\infty) - V(x) + (\nabla V(x), x)| u_n^2 dx \right]
\leq c_{\lambda_n} + \int_{\mathbb{R}^3} |V(\infty) - V(x)| + |(\nabla V(x), x)| u_n^2 dx \leq M_0. \tag{4.24}
\]
Analogous to the proof of (2.48), we can deduce a contradiction by using (4.24). Thus \( \{\|\nabla u_n\|_2\} \) is also bounded, and so \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \). The rest proof is standard, so we omit it. \qed
Proof of Theorem 1.8. (1.6) is a special form of (1.1) with \( V(x) \equiv V(\infty) = 1 \). Making use of Lemma 4.7 and by a standard argument, we can prove that there exist a positive constant \( \kappa_1 > 0 \) and two sequences of \( \{ \lambda_n \} \subset [1/2, 1] \) and \( \{ \lambda_n \} \subset H^1(\mathbb{R}^3) \), denoted by \( \{ u_n \} \) for simplicity, such that
\[
\lambda_n \to 1, \quad I'_{\lambda_n}(u_n) = 0, \quad \kappa_1 \leq I_{\lambda_n}(u_n) \leq c_{\lambda_n}.
\]
(4.25)
The rest of the proof is the same as one of Theorem 1.6, so we omit it. \( \square \)

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REFERENCES

[1] N. Ackermann and T. Weth, Multibump solutions of nonlinear periodic Schrödinger equations in a degenerate setting, *Commun. Contemp. Math.*, 7 (2005), 269–298.

[2] A. Ambrosetti and D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, *Commun. Contemp. Math.*, 10 (2008), 391–404.

[3] I. Azzollini and A. Pomponio, Ground state solutions for the nonlinear Schrödinger-Maxwell equations, *J. Math. Anal. Appl.*, 345 (2008), 90–108.

[4] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, *Topol. Methods Nonlinear Anal.*, 11 (1998), 283–293.

[5] V. Benci and D. Fortunato, Solitary waves of the nonlinear Klein-Gordon equation coupled with Maxwell equations, *Rev. Math. Phys.*, 14 (2002), 409–420.

[6] R. Benguria, H. Brezis and E. H. Lieb, The Thomas-Fermi-von Weizsäcker theory of atoms and molecules, *Comm. Math. Phys.*, 79 (1981), 167–180.

[7] I. Catto and P. L. Lions, Binding of atoms and stability of molecules in Hartree and Thomas-Fermi type theories, Part 1: A necessary and sufficient condition for the stability of general molecular system, *Comm. Partial Differential Equations*, 17 (1992), 1051–1110.

[8] G. Cerami and G. Vaira, Positive solutions for some non-autonomous Schrödinger-Poisson systems, *J. Differential Equations*, 248 (2010), 521–543.

[9] S. T. Chen and X. H. Tang, Ground state sign-changing solutions for a class of Schrödinger-Poisson type problems in \( \mathbb{R}^3 \), *Z. Angew. Math. Phys.*, 67 (2016), Art. 102, 18 pp.

[10] S. T. Chen and X. H. Tang, Nehari type ground state solutions for asymptotically periodic Schrödinger-Poisson systems, *Taiwan. J. Math.*, 21 (2017), 363–383.

[11] G. M. Coclite, A multiplicity result for the nonlinear Schrödinger-Maxwell equations, *Comm. Appl. Anal.*, 7 (2003), 417–423.

[12] T. D'Aprile and D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, *Proc. Roy. Soc. Edinburgh Sect. -A*, 134 (2004), 893–906.

[13] T. D'Aprile and D. Mugnai, Non-existence results for the coupled Klein-Gordon-Maxwell equations, *Adv. Nonlinear Stud.*, 4 (2004), 307–322.

[14] Y. H. Ding, *Variational Methods for Strongly Indefinite Problems*, World Scientific, Singapore, 2007.

[15] X. M. He, Multiplicity and concentration of positive solutions for the Schrödinger-Poisson equations, *Z. Angew. Math. Phys.*, 62 (2011), 869–889.

[16] X. M. He and W. M. Zou, Existence and concentration of ground states for Schrödinger-Poisson equations with critical growth, *J. Math. Phys.*, 53 (2012), 023702, 19pp.

[17] L. R. Huang, E. M. Rocha and J. Q. Chen, Two positive solutions of a class of Schrödinger-Poisson system with indefinite nonlinearity, *J. Differential Equations*, 255 (2013), 2463–2483.

[18] W. N. Huang and X. H. Tang, Semiclassical solutions for the nonlinear Schrödinger-Maxwell equations, *J. Math. Anal. Appl.*, 415 (2014), 791–802.

[19] W. N. Huang and X. H. Tang, Semiclassical solutions for the nonlinear Schrödinger-Maxwell equations with critical nonlinearity, *Taiwan. J. Math.*, 18 (2014), 1203–1217.

[20] L. Jeanjean, On the existence of bounded Palais-Smale sequence and application to a Landesman-Lazer type problem set on \( \mathbb{R}^N \), *Proc. Roy. Soc. Edinburgh Sect. -A*, 129 (1999), 787–809.

[21] L. Jeanjean and J. Toland, Bounded Palais-Smale mountain-pass sequences, *C. R. Acad. Sci. Paris Sér. I Math.*, 327 (1998), 23–28.
[22] E. H. Lieb, Thomas-Fermi and related theories and molecules, Rev. Modern Phys., 53 (1981), 603–641.
[23] E. H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev inequality and related inequalities, Ann. of Math., 118 (1983), 349–374.
[24] E. H. Lieb and M. Loss, Analysis. Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, 2001.
[25] P. L. Lions, Solutions of Hartree-Fock equations for Coulomb systems, Comm. Math. Phys., 109 (1987), 33–97.
[26] P. Markowich, C. Ringhofer and C. Schmeiser, Semiconductor Equations, Springer-Verlag, New York, 1990.
[27] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Funct. Anal., 237 (2006), 655–674.
[28] D. Ruiz, On the Schrödinger-Poisson-Slater system: Behavior of minimizers, radial and nonradial cases, Arch. Ration. Mech. Anal., 198 (2010), 349–368.
[29] J. Seok, On nonlinear Schrödinger-Poisson equations with general potentials, J. Math. Anal. Appl., 401 (2013), 672–681.
[30] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z., 187 (1984), 511–517.
[31] J. J. Sun and S. W. Ma, Ground state solutions for some Schrödinger-Poisson systems with periodic potentials, J. Differential Equations, 260 (2016), 2119–2149.
[32] X. H. Tang and B. T. Cheng, Ground state sign-changing solutions for Kirchhoff type problems in bounded domains, J. Differential Equations, 261 (2016), 2384–2402.
[33] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.
[34] L. G. Zhao and F. K. Zhao, On the existence of solutions for the Schrödinger-Poisson equations, J. Math. Anal. Appl., 346 (2008), 155–169.
[35] L. G. Zhao and F. K. Zhao, Positive solutions for Schrödinger-Poisson equations with a critical exponent, Nonlinear Anal., 70 (2009), 2150–2164.

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