Approximation of Entropy Numbers

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Received: 16 March 2017 / Accepted: 26 July 2019 / Published online: 3 August 2019
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Abstract

The purpose of this article is to develop a technique to estimate certain bounds for entropy numbers of diagonal operator on $\ell^p$ spaces for $1 < p < \infty$, which improves the existing bounds. The approximation method we develop in this direction works for a very general class of operators between Banach spaces, in particular, separable Hilbert spaces. As a consequence of this technique we also obtain an alternative proof for the following result for a bounded linear operator $T$ between two separable Hilbert spaces:

$$\epsilon_k(T) = \epsilon_k(T^*) = \epsilon_k(|T|)$$

for each $k \in \mathbb{N}$,

where $\epsilon_k(T)$ is the $k$th entropy number of $T$.

Keywords Entropy numbers · Truncation method · Operators on Banach spaces

Mathematics Subject Classification Primary 47B06; Secondary 47A58

1 Introduction and Preliminaries

Let $X$, $Y$ be normed linear spaces and $T : X \to Y$ be a bounded linear operator. We use the notation $BL(X, Y)$ for the space of all bounded linear operators from $X$ to $Y$, $BL(X)$ for $BL(X, X)$ and $X'$ for the dual space $BL(X, \mathbb{K})$ of $X$, where $\mathbb{K}$ stands for the scalar fields $\mathbb{R}$ or $\mathbb{C}$. Further, the range space of $T$ is denoted by $R(T)$, the closed ball with center $x_0$ and radius $r$ by $U(x_0, r)$ and the closed unit ball $\{x \in X : \|x\| \leq 1\}$.
by $U_X$. For $k \in \mathbb{N}$, the $k$th entropy number of $T$ is defined by

$$
\epsilon_k(T) := \inf \{ \epsilon > 0 : T(U_X) \subseteq \bigcup_{i=1}^{m} U(y_i, \epsilon) \text{for some } \{y_i\}_{i=1}^{m} \subseteq Y, 1 \leq m \leq k \}.
$$

Among various measures of compactness, the entropy numbers are well known in studying the compactness of bounded linear operators between Banach spaces. However, not much have been known about the entropy numbers of general bounded linear operators between Banach spaces even though numerous works have been made on estimating these quantities (see [2,4,7,9,10]).

Note that the first entropy number of $T$ is equal to $\|T\|$ and $(\epsilon_k(T))$ is a non-increasing sequence. The entropy numbers of a bounded linear operator $T$ between Banach spaces measure the degree of compactness of $T$ and it is well known that

$$
T \text{ is compact if and only if } \lim_{k \to \infty} \epsilon_k(T) = 0.
$$

The entropy numbers of an operator satisfy most of the properties of $s$-numbers (See [13]). Hence in literature, entropy numbers are called pseudo $s$-numbers and they play an important role in the theory of operator ideals and are also used in statistical learning theory.

Consider the problem of approximating a quantity (in our case, the entropy numbers) assigned to $T \in BL(X, Y)$, using a sequence of operators $(T_n)$ in $BL(X, Y)$ that converges to $T$ in some sense. There is a rich literature on such approximation techniques for the quantities like norm, eigenvalues, approximation numbers etc. Projection methods, and more generally finite section methods, mainly deal with such kind of problems in which one tries to make use of finite rank operators or truncated projections of an operator between infinite dimensional spaces to study about the operator. It has been shown that the norms, eigenvalues, singular values and approximation numbers [1,5,11], solutions of systems of linear equations etc. can be estimated by such kind of methods, under various assumptions on the spaces or the operators and on the sense of convergence used.

Here we develop such a method to estimate the entropy numbers of operators acting between infinite dimensional Banach spaces. To be more precise, for $T \in BL(X, Y)$, we consider the sequence $(T_n := Q_n T P_n)$ of operators (usually each of them of finite rank) which converges to $T$ in the Strong Operator Topology (SOT) or Weak Operator Topology (WOT). We will show that the sequence $(\epsilon_k(T_n))$ of entropy numbers of $(T_n)$ converges to $\epsilon_k(T)$ for each $k \in \mathbb{N}$, under some assumptions on the space $Y$ and on the sequences of operators $(Q_n)$ and $(P_n)$.

Also notice that in the setting of many Banach spaces with Schauder basis, we can choose the sequence $(T_n)$ to be the truncations of $T$ (that is $T_n = P_n T P_n$, where $P_n$’s are projections onto finite dimensional subspace spanned by first $n$ elements of the basis with $\|P_n\| \leq 1$ for each $n \in \mathbb{N}$). Hence it helps us to use the finite dimensional linear algebraic techniques in the computation of entropy numbers of an infinite dimensional operator. As an application of this technique in the Banach space setting, we obtain a better estimate for the entropy numbers of certain diagonal operators between $\ell_p$ spaces, $1 < p < \infty$. 
As a second application of our main result, we obtain a different proof for the following result.

**Theorem 1.1** Let $H_1, H_2$ be two complex separable Hilbert spaces and $T \in BL(H_1, H_2)$. Then for each $k \in \mathbb{N}$,

$$
\epsilon_k(T) = \epsilon_k(T^*) = \epsilon_k(|T|),
$$

where $T^*$ denotes the adjoint of $T$ and $|T|$ is the unique positive square root of $T^*T$.

We remark that the above result for compact operators on a Hilbert space was proved by Edmunds and Edmunds [6], making use of the spectral theorem. In a more general setting, namely for bounded linear operators defined on a Hilbert space, another method using the polar decomposition was suggested by Brent Carl in the Mathscinet Review [3] of the article [6].

The article is organized as follows. In the next section, we establish the approximation technique for entropy numbers of a bounded linear operator between two Banach spaces, under the assumption that the co-domain is a reflexive space. In the third section, we obtain a better estimate for the entropy numbers of certain diagonal operators between $\ell_p$ spaces, $1 < p < \infty$, as an application of the approximation results to the non-Hilbert space settings. In the final section, we present two different proofs of Theorem 1.1; first one using the approximation techniques developed in this article and the second one given by B. Carl in the Mathscinet Review [3] of the article [6]. We also discuss some of the future possibilities of the considered problems at the end of the article.

### 2 Approximation of Entropy Numbers

Let $S, T \in BL(X, Y)$. It is a well known property of the entropy numbers that

$$
\epsilon_k(T + S) \leq \epsilon_k(T) + \|S\|, \ k \in \mathbb{N}.
$$

From this, it follows that

$$
\epsilon_k(T_n) = \epsilon_k(T_n - T + T) \leq \epsilon_k(T) + \|T_n - T\|.
$$

Replacing the role of $T_n$ and $T$ in the above inequality, we obtain

$$
|\epsilon_k(T_n) - \epsilon_k(T)| \leq \|T_n - T\|.
$$

This implies that $\epsilon_k(T_n) \to \epsilon_k(T)$ as $n \to \infty$ if $T_n \to T$ in the norm sense. In other words, it says that the entropy number function $\epsilon_k$ (for a fixed $k$) is a continuous real valued function on $BL(X, Y)$, with respect to the operator norm topology on $BL(X, Y)$. In finite section methods and applications, we approximate $T$ by sequences of finite rank operators $(T_n)$ in $BL(X, Y)$. Hence the convergence of $(T_n)$ to $T$ in the norm topology will force us to restrict our attention to compact operators. Therefore
we are interested in approximating $\epsilon_k(T)$ by $(\epsilon_k(T_n))$ when $(T_n)$ converges to $T$ in a weaker senses of convergence (weaker than the norm convergence).

In this regard, for $T \in BL(X, Y)$ and for each $n \in \mathbb{N}$, we consider

$$T_n := Q_nTP_n$$

where $Q_n \in BL(Y)$, $P_n \in BL(X)$ with $\|Q_n\|\|P_n\| \leq 1$.

Then due to the property $\epsilon_k(SRT) \leq \|S\|\epsilon_k(R)\|T\|$ (see [4] for eg.), we have

$$\epsilon_k(T_n) \leq \epsilon_k(T)$$

and hence

$$\sup_n \epsilon_k(T_n) \leq \epsilon_k(T).$$

Thus if the limit of $\epsilon_k(T_n)$ exists, it can not be greater than $\epsilon_k(T)$. To show the existence of the limit, we start with a simple lemma. First we define entropy numbers of a bounded subset of a metric space.

**Definition 2.1** Let $S$ be a bounded subset of a metric space $X$. For $k \in \mathbb{N}$, the $k$th entropy number of $S$ is defined by

$$\epsilon_k(S) = \inf\{\epsilon > 0 : S \subseteq \bigcup_{i=1}^{m} U(x_i, \epsilon), \text{ for some } \{x_i\}_{i=1}^{m} \subseteq X, 1 \leq m \leq k\}.$$

The collection $\{x_1, x_2, \ldots, x_m\} \subseteq X$ is called an $\epsilon$-net for $S$. Without loss of generality, we assume that in an $\epsilon$-net for $S$, each $U(x_i, \epsilon)$ intersects with $S$.

Note that the $k$th entropy number $\epsilon_k(T)$ of an operator $T \in BL(X, Y)$ is the $k$th entropy number of $T(U_X)$.

**Lemma 2.2** Let $A$ be a bounded non empty subset of a normed linear space $X$ with $M = \sup_{x \in A} \|x\|$, and let $\epsilon > \epsilon_0(A)$. Then there exists an $\epsilon-$net $\{x_i\}_{i=1}^{k}$ for $A$ with $\|x_i\| \leq 2M$ for each $i = 1, 2, \ldots, k$. In particular if $T : Y \rightarrow X$ is a bounded linear map, then for every $\epsilon > \epsilon_k(T)$, there exists an $\epsilon-$net $\{x_i\}_{i=1}^{k}$ for $T(U_Y)$, such that for each $i = 1, 2, \ldots, k$, $\|x_i\| \leq 2\|T\|$.

**Proof** Recall that $\epsilon_k(A) \leq \epsilon_1(A) \leq M$.

Now if $\epsilon_k(A) = M$, then since $A \subset U(0, M)$, we can choose $U(x_i, \epsilon) = U(0, \epsilon)$ for each $i = 1, 2, \ldots, k$.

If $\epsilon_k(A) < M$ and $\epsilon_k(A) < \epsilon < M$, then by the definition of $\epsilon_k(A)$, there exists $\{x_1, x_2, \ldots, x_k\}$ in $X$ such that $A \subseteq \bigcup_{i=1}^{k} U(x_i, \epsilon)$. Now for each $i = 1, 2, \ldots, k$, take $z_i \in A \cap U(x_i, \epsilon)$, and then we have

$$\|x_i\| \leq \|x_i - z_i\| + \|z_i\| < 2M.$$

The particular case follows if we take $A = T(U_Y)$ and $M = \|T\|$. $\square$
Now we show that \( \lim_{n \to \infty} \epsilon_k(T_n) = \epsilon_k(T) \) whenever \( T_n = Q_nT P_n \to T \) in the Strong Operator Topology, where \( \|Q_n\|\|P_n\| \leq 1 \), provided that the co-domain space is reflexive.

**Theorem 2.3** Let \( X \) be a normed linear space, \( Y \) be a reflexive Banach space and \( T \in BL(X, Y) \). Let \( (P_n) \) and \( (Q_n) \) be sequences of operators in \( BL(X) \) and \( BL(Y) \) respectively such that \( \|Q_n\|\|P_n\| \leq 1 \) for each \( n \in \mathbb{N} \) and let \( T_n := Q_nT P_n \). If \( T_n \) converges to \( T \) in the pointwise sense of convergence, then for each \( k \in \mathbb{N} \),

\[
\lim_{n \to \infty} \epsilon_k(T_n) = \epsilon_k(T).
\]

**Proof** Fix \( k \in \mathbb{N} \). Let us denote \( d_n := \epsilon_k(T_n) \) and \( d = \epsilon_k(T) \) for \( n \in \mathbb{N} \).

Due to the norm assumptions on \( Q_n \) and \( P_n \) it follows that \( d_n \leq d \) for all \( n \), and hence

\[
\lim \inf d_n \leq \lim \sup d_n \leq d.
\]

If \( d = 0 \), the result is obvious. Hence we assume \( d \neq 0 \). Assume, if possible, that \( \lim \inf d_n \neq d \). Then there exists an \( \epsilon > 0 \), and a subsequence \( (d_m)_{m \in \mathbb{N}_1} \) of \( (d_n) \) such that

\[
d_m < d - \epsilon \quad \text{for all} \quad m \in \mathbb{N}_1,
\]

where \( \mathbb{N}_1 \) is an infinite subset of \( \mathbb{N} \). Then for each \( m \in \mathbb{N}_1 \), \( T_m(U_X) \) can be covered by \( k \) (or less number of) balls of radii \( d_1 - \epsilon \). Without loss of generality, we assume that there are \( k \) balls. Let the corresponding \( (d - \epsilon) \) - net be \( \{y_1^m, y_2^m, \ldots, y_k^m\} \) for each \( m \in \mathbb{N}_1 \). That is,

\[
T_m(U_X) \subseteq \bigcup_{i=1}^k B(y_i^m, d - \epsilon).
\]

By Lemma 2.2, we can choose \( (y_i^m) \) with \( \|y_i^m\| \leq 2\|T_m\| \leq 2\|T\| \) for each \( i \in \{1, 2, \ldots, k\} \). Therefore \( (y_i^m) \) is a bounded sequence in \( Y \). Since \( Y \) is a reflexive space and \( (y_i^m) \) is a bounded sequence, by Eberly - Schmulyan theorem [12], \( (y_i^m) \) has a weakly convergent subsequence, say \( (y_i^{m_j}) \) which converges to some \( y_1 \) in the weak sense of convergence. Now since \( (y_2^{m_j}) \) is also bounded in \( Y \), we get a subsequence \( (y_2^{m_{j_2}}) \) which is weakly convergent to some \( y_2 \). Taking subsequence of subsequences, we can find an infinite set \( \mathbb{N}_2 \subseteq \mathbb{N}_1 \) such that for each \( i = 1, 2, \ldots, k \), \( y_i^m \to y_i \) in the weak sense of convergence, as \( m \to \infty \) in \( \mathbb{N}_2 \).

That is, for each \( f \in Y' \), there exists an \( m(f) \in \mathbb{N}_2 \) such that for each \( i \in \{1, 2, \ldots, k\} \),

\[
|f(y_i^m) - f(y_i)| < \frac{\epsilon}{4}, \quad \text{for all} \quad m \geq m(f), \ m \in \mathbb{N}_2.
\]

Now, we claim that \( T(U_X) \subseteq \bigcup_{i=1}^k B(y_i, d - \frac{\epsilon}{2}) \).
To prove this claim, we take any $x \in U_X$ and $f \in Y'$ such that $\|f\| \leq 1$. Since $T x \in T(U_X)$ and $T_m x \to T x$ as $m \to \infty$ in $\mathbb{N}^2$, we can find an $m^* \in \mathbb{N}^2$ such that $m^* \geq m(f)$ and

$$\|T_{m^*} x - T x\| < \frac{\epsilon}{4}.$$ 

Since $T_{m^*}(U_X)$ is covered by $k$ balls of radii $d - \epsilon$, we have

$$\|T_{m^*} x - y^m_l\| < d - \epsilon,$$

for some $l \in \{1, 2, \ldots, k\}$ and $y^m_l \in Y$. Now $y^m_l \to y_l$ weakly. Therefore

$$|f(T x) - f(y_l)| \leq |f(T x) - f(T_{m^*} x)| + |f(T_{m^*} x) - f(y^m_l)| + |f(y^m_l) - f(y_l)|$$

$$< \frac{\epsilon}{4} + d - \epsilon + \frac{\epsilon}{4} = d - \frac{\epsilon}{2},$$

proving our claim by Hahn-Banach Theorem. But then $d = \epsilon_k(T) \leq d - \frac{\epsilon}{2} < d$, a contradiction. Thus $\lim_{n \to \infty} d_n = d$. \qed

**Remark 2.4** It can be observed that in the above theorem, if $P_n$ and $Q_n$ are such that $T_n$ converges to $T$ in WOT, then the same proof given above will work. Hence the theorem could be restated under the weaker assumption that $T_n \overset{WOT}{\to} T$. Also, it can be seen that the reflexivity assumption on the co-domain is redundant, provided we assume the co-domain to be the dual space of some separable space. In particular, all the above mentioned conditions hold when $X$ and $Y$ are separable Hilbert spaces.

The following example is an explicit illustration of Theorem 2.3.

**Example** Consider the identity map $I : \ell^p \to \ell^p$, $1 < p < \infty$ over the field $\mathbb{R}$. For each fixed $k \in \mathbb{N}$,

$$\epsilon_k(I) = 1.$$
Now consider the sequence of standard projection operators \( (P_n) \) on \( \ell_p \), which converges to the identity operator \( I \) point wise. It follows from [4, Proposition 1.3.2] that

\[
1 = \| P_n \| \geq \epsilon_k(P_n) \geq k^{-\frac{1}{n}}.
\]

Hence it follows that

\[
\epsilon_k(P_n) \to \epsilon_k(I) \text{ as } n \to \infty.
\]

**Remark 2.6** It can be seen that if we do not assume \( T_n = Q_n T P_n \) with the norm conditions, the conclusion is no longer true. For example, if we take \( T_n \) on \( \ell_2 \) as

\[
(T_n(x))(j) = 0 \text{ for } j \neq n \text{ and } (T_n(x))(n) = x_n, \text{ for each } n \in \mathbb{N},
\]

then \( T_n \) converges to 0 in SOT; whereas \( \lim_{n \to \infty} \epsilon_1(T_n) = \lim_{n \to \infty} \| T_n \| = 1 \neq 0 = \epsilon_1(0) \).

**Remark 2.7** If we choose \( X = Y \) to be a Banach space with a Schauder basis and \( P_n = Q_n \) to be projections of norm 1 onto the finite dimensional subspace spanned by first \( n \) elements of the Schauder basis of \( X \), then the sequence \( T_n \) shall be identified as the first \( n \times n \) block of the infinite matrix that represents \( T \). Hence Theorem 2.3 brings out the possibility of using linear algebra techniques in computing \( \epsilon_k(T) \).

### 3 A Banach Space Application: A Better Estimate for Entropy Numbers

Finding estimates for entropy numbers is a difficult task and it has been an interesting problem in the case of many concrete operators (See [7,9,10,14]). However, not much have been achieved in this regard to the best of our knowledge. In [7], the authors obtained certain estimates for entropy numbers of some special types of diagonal operators on real Banach spaces with a 1-unconditional basis (See Proposition 1.7 of [7]). This result was modified for similar diagonal operators on \( \ell_p \) spaces (real and complex), \( 1 \leq p \leq \infty \) in [4].

Here, as an application of our approximation techniques to operators on Banach spaces, we sharpen these estimates thereby strengthening Proposition 1.3.2 of [4]. The major ingredients in the proof are the techniques used to prove Proposition 1.3.2 [4] and Theorem 2.3.

**Theorem 3.1** Let \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \geq \ldots \geq 0 \) and let \( D \) be the operator defined on \( \ell_p \), \( 1 < p < \infty \) by

\[
D((\zeta_1, \zeta_2, \ldots, \zeta_k, \ldots)) = (\sigma_1 \zeta_1, \sigma_2 \zeta_2, \ldots, \sigma_k \zeta_k, \ldots), \text{ for } (\zeta_n) \in \ell_p.
\]

Then for each \( k \in \mathbb{N} \),

\[
\sup_{1 \leq n < \infty} k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}} \leq \epsilon_k(D) \leq 4 \cdot \sup_{1 \leq n < \infty} k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}}
\]
in the case of real \( \ell_p \) spaces and

\[
\sup_{1 \leq n < \infty} k^{-\frac{1}{2n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}} \leq \epsilon_k(D) \leq 4 \cdot \sup_{1 \leq n < \infty} k^{-\frac{1}{2n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}}
\]

in the case of complex \( \ell_p \) spaces.

**Proof** For \( n \in \mathbb{N} \), define \( D_n := P_n D P_n \), where \( P_n \) are the standard \( n \)th projections defined by \( P_n(x) = (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots) \), for \( x = (\xi_1, \xi_2, \ldots, \xi_j, \ldots) \) in \( \ell_p \), and let \( D_n = D_n |_{R(P_n)} \).

Let \( \epsilon > \epsilon_k(D_n) \). Then there exists \( \{x_1, x_2, \ldots, x_k\} \subset R(P_n) \) such that

\[
\tilde{D}_n(U_{R(P_n)}) \subseteq \bigcup_{i=1}^{k} (x_i + \epsilon U_{R(P_n)}).
\]

Now, since \( \text{Vol} \left( \tilde{D}_n(U_{R(P_n)}) \right) = \sigma_1 \sigma_2 \ldots \sigma_n \text{Vol} \left( U_{R(P_n)} \right) \), we have by comparison of volumes, \( \sigma_1 \sigma_2 \ldots \sigma_n \text{Vol} \left( U_{R(P_n)} \right) \leq k \epsilon^n \text{Vol} \left( U_{R(P_n)} \right) \) which implies \( \epsilon \geq k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}} \). Since this is true for every \( \epsilon > \epsilon_k(D_n) \), we get \( \epsilon_k(D_n) \geq k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}} \).

Now, since \( \epsilon_k(D_n) \leq \epsilon_k(D) \), we get \( \epsilon_k(D) \geq k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}} \). Taking supremum over \( n \in \mathbb{N} \), we get \( \epsilon_k(D) \geq \sup_{1 \leq n < \infty} k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}} \), which proves one side of the required inequality in the real case.

For the upperbound, we define \( \delta(k) = \sup_{1 \leq n < \infty} k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}} \) and claim that for each \( k \in \mathbb{N} \) there exists an index \( r \) with \( \sigma_{r+1} \leq 2 \delta(k) \). Let \( k \in \mathbb{N} \). Since \( 2^m \rightarrow \infty \) as \( m \rightarrow \infty \), there exists an \( r \) with \( k \leq 2^{r+1} \) and so \( 1 \leq 2 k^{-\frac{1}{r+1}} \). Then due to monotonicity of \( (\sigma_i) \),

\[
\sigma_{r+1} \leq (\sigma_1 \sigma_2 \ldots \sigma_{r+1})^{\frac{1}{r+1}} \leq 2 k^{-\frac{1}{r+1}} (\sigma_1 \sigma_2 \ldots \sigma_{r+1})^{\frac{1}{r+1}} \leq 2 \sup_{1 \leq n < \infty} k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}}.
\]

Thus \( \sigma_{r+1} \leq 2 \delta(k) \). Now suppose that \( \sigma_1 \leq 2 \delta(k) \). Then \( \epsilon_k(D) \leq \|D\| = \sigma_1 \leq 2 \delta(k) \), which gives

\[
\sup_{1 \leq n < \infty} k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}} \leq \epsilon_k(D) \leq 2 \sup_{1 \leq n < \infty} k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}}.
\]

Now, if \( \sigma_1 > 2 \delta(k) \), there exists an \( m \) with \( \sigma_{m+1} \leq 2 \delta(k) < \sigma_m \). Now \( D_m : \ell_p \rightarrow \ell_p \) is of rank \( m \). Let \( y_1, y_2, \ldots, y_N \) be a maximal system of elements in \( D_m(U) \) with \( \|y_i - y_j\| > 4 \delta(k) \) for \( i \neq j \). Then \( D_m(U) \subseteq \bigcup_{i=1}^{N} (y_i + 1 + 4 \delta(k) U) \) and \( \epsilon_N(D_m) \leq 4 \delta(k) \). Now using Theorem 2.3 as \( m \rightarrow \infty \), we get \( \epsilon_N(D) \leq 4 \delta(k) = 4 \sup_{1 \leq n < \infty} k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}} \). It can be shown that \( k \geq N \) (see [4]). Hence

\[
\epsilon_k(D) \leq 4 \cdot \sup_{1 \leq n < \infty} k^{-\frac{1}{n}} (\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}}.
\]
The proof for the complex $\ell_p$ case follows in a similar way. □

**Remark 3.2** Theorem 3.1 is valid for $p = 1$ also, since $\ell_1$ is the dual of $c_0$ and therefore Corollary 2.5 is applicable. For $\ell_\infty$, even though it is the dual of $\ell_1$ and other assumptions of Corollary 2.5 are applicable, we lack useful projections in this case.

### 4 An Application to Hilbert Space Operators: Entropy Numbers of Operators and Their Adjoints

In this section we discuss about the entropy numbers of Hilbert space operators. Let $H, H_1, H_2$ denote complex Hilbert spaces and $T : H_1 \to H_2$ be a bounded linear operator. The adjoint operator of $T$ is denoted by $T^*$. Note that $T$ has the polar decomposition $T = V|T|$, where $V$ is a partial isometry and $|T|$ is the unique positive square root of the operator $T^*T$. This decomposition is unique provided the null space of $V$ and the null space of $|T|$ are the same. We refer [8] for more details. The question of the identity $\varepsilon_k(T) = \varepsilon_k(T^*)$ was motivated by the problem posed by Carl whether $\varepsilon_k(T^*) = O(n^{-\alpha})$ if $\varepsilon_k(T)$ is so. Some positive partial answers to this problem are available in the literature [6,7]. The following theorem was proved in [6] settling the problem for compact operators acting on a Hilbert space.

**Theorem 4.1** ([6]) Let $H$ be a Hilbert space and $T \in BL(H)$ be compact. Then for each $k \in \mathbb{N},$

$$
\varepsilon_k(T) = \varepsilon_k(T^*) = \varepsilon_k(|T|).
$$

The proof given in [6] makes use of the spectral theorem for compact operators between Hilbert spaces. It is well known that every bounded linear operator between separable Hilbert spaces can be approximated by compact operators in the strong sense of convergence, with the help of orthonormal projections. This observation along with Theorem 2.3 renders a direct extension of Theorem 4.1 to all bounded linear operators between separable Hilbert spaces, providing an alternate proof for Theorem 1.1.

By a different approach, a simple proof of the result in Theorem 1.1 was given by B. Carl in the Mathscinet Review [3] of the paper [6], where separability of the space is not required. Since this proof is not explicitly available in any article in the literature to the best of our knowledge (other than in the Mathscinet Review [3]), for the sake of completeness, we provide it as Theorem 4.3 at the end. We remark that Carl’s technique works also for operators defined between two different Hilbert spaces.

**An Alternate Proof for Theorem 1.1:**

For $n \in \mathbb{N}$, let $P_n \in B(H_1)$ and $Q_n \in B(H_2)$ denote the $n$th standard orthonormal projections, which converge to the identity operator on the respective spaces in the Strong Operator Topology. Then, $Q_nTP_n$ and $P_nT^*Q_n$ are compact operators satisfying

$$
Q_nTP_n \xrightarrow{SOT} T \quad \text{and} \quad P_nT^*Q_n \xrightarrow{SOT} T^*, \quad \text{as} \quad n \to \infty.
$$
Also \( \|P_n\| \leq 1, \|Q_n\| \leq 1 \). Hence by Theorem 2.3, for each \( k \in \mathbb{N} \),

\[
\epsilon_k(Q_n T P_n) \rightarrow \epsilon_k(T) \quad \text{and} \quad \epsilon_k(P_n T^* Q_n) \rightarrow \epsilon_k(T^*), \quad \text{as} \ n \rightarrow \infty.
\]

Since \( Q_n T P_n \) and \( P_n T^* Q_n \) are compact operators, by Theorem 4.1,

\[
\epsilon_k(Q_n T P_n) = \epsilon_k(P_n T^* Q_n) \quad \text{for each} \ k \in \mathbb{N}, \ n \in \mathbb{N},
\]

we get the required conclusion, \( \epsilon_k(T) = \epsilon_k(T^*) \).

Let \( T = V|T| \) be the polar decomposition of \( T \). Then \( |T| = V^* T \). So

\[
\epsilon_k(P_n |T| P_n) = \epsilon_k(P_n V^* T P_n) \leq \epsilon_k(T), \quad \text{since} \ \|P_n\|\|V^*\||P_n\| \leq 1.
\]

By taking limit \( n \rightarrow \infty \), we have \( \epsilon_k(|T|) \leq \epsilon_k(T) \).

To prove the other way inequality, consider

\[
\epsilon_k(Q_n T P_n) = \epsilon_k(Q_n V|T| P_n) \leq \epsilon_k(|T|)
\]

Taking limit, we obtain \( \epsilon_k(T) \leq \epsilon_k(|T|) \). Thus \( \epsilon_k(T) = \epsilon_k(|T|) \) for each \( k \in \mathbb{N} \).

**Remark 4.2** Note that, in the above theorem we have used the separability of Hilbert spaces to get finite rank projections converge to the identity operator on the corresponding Hilbert spaces with respect to the strong operator topology. If the separability condition is dropped we may get finite rank projections but they may not converge to the identity operator in the Strong Operator Topology.

Now we give the proof by Carl in the Mathscinet Review [3] of the paper [6], in which separability of the space is not required.

**Theorem 4.3** (See [3]) Let \( T \in BL(H_1, H_2) \). Then for each \( k \in \mathbb{N} \),

\[
\epsilon_k(T) = \epsilon_k(T^*) = \epsilon_k(|T|).
\]

**Proof** Let \( T = V|T| \) be the polar decomposition of \( T \). Then

\[
\epsilon_k(T) \leq \|V\|\epsilon_k(|T|) = \epsilon_k(|T|),
\]

for each \( k \in \mathbb{N} \). Also, as \( |T| = V^* T \), the other inequality also holds true. Since \( T^* = |T| V^* \), the equality \( \epsilon_k(T^*) = \epsilon_k(|T|) \) holds true. \( \square \)

**Remark 4.4** We would like to remark that if one requires to prove the equality \( \epsilon_k(T) = \epsilon_k(T^*) \) for bounded linear operators \( T \) between certain special Banach spaces (like \( \ell_p \), \( 1 \leq p < \infty \), which is reflexive/dual of a separable space and where one has finite rank projections of norm 1 which converges to identity operator in SOT/WOT) and its transpose operator \( T^* \), it is sufficient to prove the result for compact operators or for finite rank operators, in view of Theorem 2.3. However, it is still an open problem to prove the equality \( \epsilon_k(T) = \epsilon_k(T^*) \) for finite rank/compact operators between Banach spaces.
4.1 Some of the Important Problems

1. Consider the original problem by B. Carl. It is still an open problem to show the identity $\epsilon_k(T) = \epsilon_k(T^*)$ for a bounded operator $T$ on an arbitrary Banach space. Even it is not clear whether $\epsilon_k(T^*) = O(k^{-\alpha})$ if $\epsilon_k(T)$ is so, in the general setting. However, since our techniques are applicable for reflexive Banach spaces with suitable projections, it suffices to show these for compact operators on such spaces.

2. It is not known to us whether the conclusion of Theorem 2.3 holds for operators whose co-domains are not isometric to dual of separable spaces. In other words, an example to show that the duality assumption on the co-domain space is mandatory will be worth finding.

3. It is of interest to check whether the estimate obtained in Theorem 3.1 is strict or can be improved further.

4. To study the behavior of entropy numbers under a random perturbation is another important problem.

Acknowledgements The authors wish to thank Prof. M.N.N. Namboodiri of CUSAT, Dr. G. Ramesh and Dr. D. Venku Naidu of IIT Hyderabad, for their valuable suggestions. Also we are thankful to Kerala School Of Mathematics (KSOM), Kozhikode, for the local hospitality provided during our discussion meetings. V.B. Kiran Kumar is supported by Start-up Research Grant of the University Grant Commission, India. The valuable suggestions made by the referees are also gratefully acknowledged.

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