ARBITRARILY SPARSE SPECTRA FOR SELF-AFFINE SPECTRAL MEASURES

L.-X. AN¹,† and C.-K. LAI²,*

¹School of Mathematics and Statistics, and Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan 430079, P.R. China
e-mail: anlixianghai@163.com

²Department of Mathematics, San Francisco State University, 1600 Holloway Avenue, San Francisco, CA 94132, United States
e-mail: cklai@sfsu.edu

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Dedicated to the memory of Dr. Tian-You Hu‡

Abstract. Given an expansive matrix \( R \in M_d(\mathbb{Z}) \) and a finite set of digit \( B \) taken from \( \mathbb{Z}^d/R(\mathbb{Z}^d) \). It was shown previously that if we can find an \( L \) such that \( (R, B, L) \) forms a Hadamard triple, then the associated fractal self-affine measure generated by \( (R, B) \) admits an exponential orthonormal basis of certain frequency set \( \Lambda \), and hence it is termed as a spectral measure. In this paper, we show that if \#B < |\text{det}(R)|, not only it is spectral, we can also construct arbitrarily sparse spectrum \( \Lambda \) in the sense that its Beurling dimension is zero.

1. Introduction

1.1. Definitions and main results. Let \( R \in M_d(\mathbb{Z}) \) be an expansive matrix (i.e. all of its eigenvalues have modulus strictly greater than 1). Let \( B, L \subset \mathbb{Z}^d \) be finite sets of integer vectors with \( q := \#L = \#B \). We say that the system \((R, B, L)\) forms a Hadamard triple if the matrix

\[
H = \frac{1}{\sqrt{q}} \left[ e^{-2\pi i (R^{-1}b, l)} \right]_{b \in B, l \in L}
\]

is unitary, i.e., \( H^*H = I \).
Given an expansive matrix $R \in M_d(\mathbb{Z})$ and given $B \subset \mathbb{Z}^d$. By the result of Hutchinson [12], we can define the affine iterated function system (IFS) \[ \{ \tau_b(x) = R^{-1}(x + b), \; x \in \mathbb{R}^d, \; b \in B \} \] which has a unique compact attractor $T(R, B)$, called self-affine set, satisfying
\[ T(R, B) = \bigcup_{b \in B} \tau_b(T(R, B)). \]

The self-affine measure (with equal weights) is the unique probability measure $\mu = \mu(R, B)$ satisfying
\[ \mu(E) = \frac{1}{q} \sum_{b \in B} \mu(\tau_b^{-1}(E)) \]
for all Borel subsets $E$ of $\mathbb{R}^d$. This measure is supported on the attractor $T(R, B)$.

In a previous work, Dutkay, Hausserman and the second-named author proved the following theorem ([8], see [17] for the proof on $\mathbb{R}^1$).

**Theorem 1.1.** Suppose that $(R, B, L)$ forms a Hadamard triple on $\mathbb{R}^d$. Then the self-affine measure $\mu(R, B)$ admits an exponential orthonormal basis $E(\Lambda) = \{ e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda \}$ for some countable set $\Lambda \subset \mathbb{R}^d$.

We say that a Borel probability measure $\mu$ on $\mathbb{R}^d$ is called a spectral measure if we can find a countable set $\Lambda \subset \mathbb{R}^d$ such that the set of exponential functions $E(\Lambda) := \{ e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda \}$ forms an orthonormal basis for $L^2(\mu)$. If such $\Lambda$ exists, then $\Lambda$ is called a spectrum for $\mu$. The above theorem said that $\mu(R, B)$ is a spectral measure if $(R, B, L)$ forms a Hadamard triple with some $L \subset \mathbb{Z}^d$.

In this paper, we study in more detail about the sparseness of the spectrum as measured by Beurling dimension.

**Definition 1.2.** Denote $Q^d_h(x) = x + [-h, h]^d$ be the cube centered at $x$ on $\mathbb{R}^d$.

(1) Let $\Lambda$ be a discrete subset of $\mathbb{R}^d$. For $r > 0$, the upper Beurling density corresponding to $r$ (or $r$-Beurling density) is defined by
\[ D^+(\Lambda) = \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\#(\Lambda \cap Q^d_h(x))}{h^r}. \]

(2) The upper Beurling dimension (or simply the Beurling dimension) of $\Lambda$ is defined by
\[ \dim^+(\Lambda) := \sup \{ r > 0 : D^+(\Lambda) > 0 \} = \inf \{ r > 0 : D^+(\Lambda) < \infty \}. \]

Our main result is the following.
Theorem 1.3. If \((R, B, L)\) forms a Hadamard triple on \(\mathbb{R}^d\) with \(|\det R| < |\#B|\), then the spectral measure \(\mu(R, B)\) admits a spectrum \(\Lambda\) with Beurling dimension zero.

1.2. Historical overview. Historically, spectral measure was first studied by Fuglede who introduced the notion of spectral sets and explored its relationship with translational tile [9]. The first singularly continuous spectral measure was found by Jorgensen and Pedersen [15]. They showed that the standard middle-fourth Cantor measure \(\mu_4\) is spectral, while the middle-third Cantor measure \(\mu_3\) is not spectral. The measure \(\mu_4\) is generated by \(R = 4\) and \(B = \{0, 2\}\), where \((R, B, L)\) forms a Hadamard triple with \(L = \{0, 1\}\). Using \(L\), they found that one of the spectra of \(\mu_4\) is given by

\[
\Lambda_0 = \left\{ \sum_{j=0}^{n-1} 4^j \varepsilon_j : \varepsilon_j \in \{0, 1\}, n \geq 1 \right\}.
\]

This is, however, not the only spectrum for \(\mu_4\). We can also see that \((4, \{0, 2\}, L_n)\) with \(L_n = \{0, 5^n\}\) also form Hadamard triples. Indeed, one can also show that \(5^n \Lambda_0\) all are spectra of \(\mu_4\). A direct calculation shows that all these spectra have Beurling dimension \(\ln 2 / \ln 4\).

In an attempt to capture the right density condition for the spectra of \(\mu_4\), Dutkay, Han, Sun and Weber [5] proposed the notion of Beurling dimension, and they brought this notion from the study of Gabor pseudo-frame in [2]. In [5], the authors showed that all spectra of \(\mu_4\) must have Beurling dimension at most \(\ln 2 / \ln 4\) which is the Hausdorff dimension of the attractor. Under a technical condition on the spectrum \(\Lambda\), a spectrum of \(\mu\) must have a Beurling dimension \(\ln 2 / \ln 4\). It used to be a conjecture that the technical condition can be removed. However, Dai, He and the second-named author disproved the conjecture by exhibiting a spectrum of Beurling dimension zero for \(\mu_4\) [3]. The existence of sparse spectra with Beurling dimension zero is also true for other one-dimensional self-similar measures whose digit sets are consecutive \(\{0, 1, \ldots, q - 1\}\)\(^1\). Our main Theorem 1.3 now further generalizes the behavior of arbitrarily sparseness of spectra in Beurling dimension to all singular self-affine measures generated by Hadamard triples.

The arbitrarily sparseness behavior was in stark contrast with the classical cases. The classical result of Landau [19] showed that if \(\Lambda\) is a spectrum for \(L^2(\Omega)\) and \(\Omega \subset \mathbb{R}^d\) (or more generally \(E(\Lambda)\) is a Fourier frame for \(L^2(\Omega)\)), then the \(d\)-Beurling density of \(\Lambda\) must be at least the Lebesgue measure of \(\Omega\) and thus its Beurling dimension must be \(d\). Therefore arbitrarily sparse

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\(^1\)In private communication with Y. Wang and B. Strichartz, they also have noticed such arbitrarily sparse behavior of spectra of \(\mu_4\) around 2000.
The spectrum does not exist for Lebesgue measure. For a simple proof of Landau’s theorem, one can consult [21, Chapter 5]. Landau’s theorem is now fundamental in modern sampling theory (see e.g. [1,21] for details).

The arbitrarily sparseness makes us ask naturally if there is any lower bound for the Beurling dimension of the spectra of a spectral measure. Throughout the paper, the Fourier transform of a Borel probability measure $\mu$ on $\mathbb{R}^d$ is defined to be

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} d\mu(x).$$

The following proposition provides a simple but useful answer.

**Proposition 1.4.** Let $\mu$ be a finite Borel and singular measure on $\mathbb{R}^d$ such that its Fourier transform satisfies that for $|\xi|$ large enough,

$$|\hat{\mu}(\xi)|^2 \leq C|\xi|^{-\gamma} \quad (1.2)$$

and let $E(\Lambda)$ be a set of exponentials such that there exists $A > 0$ for which the following inequality holds:

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} \left| \int f(x)e^{-2\pi i \lambda \cdot x} d\mu(x) \right|^2, \text{ for all } f \in L^2(\mu). \quad (1.3)$$

Then $\gamma \leq \text{dim}^+(\Lambda)$.

The supremum of $\gamma$ such that (1.2) holds is called the Fourier dimension of $\mu$. This proposition implies that the Beurling dimension is at least the Fourier dimension of $\mu$. For self-affine measure of our consideration, it is easy to prove that all such measures have Fourier dimension zero.

Indeed, a stronger result was proved in [14] in which if (1.2) holds for a spectral measure, then its spectrum must satisfy

$$\sum_{\lambda \in \Lambda \setminus \{0\}} |\lambda|^{-\gamma} = \infty. \quad (1.4)$$

One can show that (1.4) implies that $\gamma \leq \text{dim}^+(\Lambda)$. We here will provide another independent proof of Proposition 1.4.

Using (1.4), authors in [14] showed that the surface measure of any convex body with everywhere positive Gaussian curvature does not admit any Fourier frame, and is therefore not spectral. In view of this result, an interesting problem that arises but not yet appeared to have a simple answer is that:

**Question.** Does there exist a singular spectral measure whose Fourier dimension is positive?
For some more results about Beurling dimension, Fourier decay and spectral measures, one can also refer to [13,20,22]. In [22], the relationship of different dimensions were studied and the above question was also mentioned.

It is also worth mentioning that a widely open problem is to determine if $\mu_3$ admits a Fourier frame or Riesz basis. Beurling dimension has been an indicator to see if such frame is possible to exist [6]. It was recently found that it is possible to construct an exponential Riesz sequence (note that a complete Riesz sequence will be a Riesz basis) with maximal Beurling dimension $\log 2 / \log 3$ [4].

1.3. Sketch of the proof. We now sketch the proof of Theorem 1.3. First, it is known that in the Hadamard triple, $B$ must be a distinct representative in the group $\mathbb{Z}^d / R(\mathbb{Z}^d)$. Therefore, $\# B \leq |\det (R)|$. When $\# B = |\det R|$, then $B$ is a distinct representative in the group $\mathbb{Z}^d / R(\mathbb{Z}^d)$. The measure $\mu$ is just the Lebesgue measure supported on the fundamental domain $T(R, B)$. Hence the spectra of spectral measure $\mu$ has Beurling dimension $d$. Therefore, $\# B < |\det R|$ is necessary in the assumption. In this case, $\mu$ is singular to the Lebesgue measure.

Throughout the paper, we will assume, without loss of generality, $0 \in B \cap L$. Otherwise, we do a translation of the measure. Similar to the strategy of the proof in [8], for any singular to the Lebesgue measure $\mu$ on $\mathbb{R}^d$, its periodic zero set is defined as follows:

$$Z(\mu) = \{ \xi \in \mathbb{R}^d : \hat{\mu}(\xi + k) = 0, \text{ for all } k \in \mathbb{Z}^d \}.$$ 

Our proof of Theorem 1.3 is divided into two cases $Z(\mu) = \emptyset$ or $Z(\mu) \neq \emptyset$.

Definition 1.5. We say that a countable set $\Lambda = \{ \lambda_n \}_{n=0}^{\infty} \subset \mathbb{R}^d$ is called $b$-lacunary, if $\lambda_0 = 0$, $|\lambda_1| \geq b$ and for all $n \geq 1$,

$$|\lambda_{n+1}| \geq b|\lambda_n|.$$

As we will see, lacunary sequences must have Beurling dimension zero (see Proposition 2.3). Our theorem in this case is as follows:

Theorem 1.6. If $(R, B, L)$ forms a Hadamard triple with $\# B < |\det R|$ and $Z(\mu(R, B)) = \emptyset$, then for all $b > 1$, the spectral measure $\mu(R, B)$ admits a $b$-lacunary spectrum $\Lambda$.

The case $Z(\mu(R, B)) \neq \emptyset$ is more complicated. Our strategy is to reduce the self-affine pair $(R, B)$ to a pair $(\tilde{R}, \tilde{B})$ which has quasi product-form structure on $\mathbb{R}^d = \mathbb{R}^r \times \mathbb{R}^{d-r}$. The self-affine measure $\mu(\tilde{R}, \tilde{B})$ projections on $\mathbb{R}^r$ is a self-affine measure $\mu(\tilde{R}_1, \tilde{B}_1)$ satisfies $Z(\mu(\tilde{R}_1, \tilde{B}_1)) = \emptyset$. Then we can construct a spectrum of $\mu(R, B)$ has zero Beurling dimension.

We organize our paper as follows: In Section 2, we present some preliminaries. We will review the property of Beurling dimension of a discrete
set and the condition \( \mathcal{Z}(\mu) = \emptyset \). In Section 3, we will prove Theorem 1.6. In Section 4, we will conjugate with some matrix so that \((R, B)\) are of the quasi-product form and complete the proof of Theorem 1.3. We will finally prove Proposition 1.4 in the last section.

2. Preliminaries

In this section, we will set up some basic propositions for the rest of our paper. These results that serve as the basis for our proofs.

2.1. Beurling dimension. We will establish some basic properties of Beurling dimension in this subsection.

**Proposition 2.1.** Let \( R \) be an invertible matrix in \( M_d(\mathbb{R}) \) and \( \Lambda \subset \mathbb{R}^d \) is a discrete set. Then
\[
\dim^+(\Lambda) = \dim^+(R\Lambda).
\]

**Proof.** For the invertible matrix \( R \), there exist constants \( c_1 > c_2 > 0 \) such that
\[
Q^d_{c_2}(0) \subset R^{-1}Q^d_1(0) \subset Q^d_{c_1}(0).
\]
As \( hQ^d_1(0) = Q^d_h(0) \), it implies that for all \( h > 0 \),
\[
Q^d_{c_2h}(0) \subset R^{-1}Q^d_h(0) \subset Q^d_{c_1h}(0).
\]
Note that \( \#(\Lambda \cap Q^d_{c_2h}(y)) = \#((\Lambda - y) \cap Q^d_{c_2h}(0)) \).
\[
\#(\Lambda \cap Q^d_{c_2h}(R^{-1}x)) = \#((\Lambda - R^{-1}x) \cap Q^d_{c_2h}(0))
\]
\[
\leq \#((\Lambda - R^{-1}x) \cap R^{-1}Q^d_h(0)) \leq \#(\Lambda \cap Q^d_{c_1h}(R^{-1}x)).
\]
As \( \#(R\Lambda \cap Q) = \#(\Lambda \cap R^{-1}Q) \), we have
\[
\#((\Lambda - R^{-1}x) \cap R^{-1}Q^d_h(0)) = \#(R\Lambda \cap Q^d_h(x)).
\]
We have thus obtained
\[
\#(\Lambda \cap Q^d_{c_2h}(R^{-1}x)) \leq \#(R\Lambda \cap Q^d_h(x)) \leq \#(\Lambda \cap Q^d_{c_1h}(R^{-1}x)).
\]
Dividing by \( h^r \) and taking supremum and lim sup, we have
\[
c_2^r D^+_r(\Lambda) \leq D^+_r(R\Lambda) \leq c_1^r D^+_r(\Lambda).
\]
Therefore, \( \dim^+(\Lambda) = \dim^+(R\Lambda) \) follows. \( \Box \)

**Lemma 2.2.** Let \( b > 1 \) and \( \Lambda \) is a \( b \)-lacunary set. For any cubes \( Q^d_h(x) \) with \( h > 1 \) and \( x \in \mathbb{R}^d \), we have
\[
\#(\Lambda \cap Q^d_h(x)) \leq \log_b(4\sqrt{dh}) + 2.
\]
Proof. Denote $B(x, r)$ be the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$. Let $A_1 = B(0, b)$ and $A_n = B(0, b^n) \setminus B(0, b^{n-1})$ be the annuli regions centered at the origin for any $n \geq 2$. Then

$$\mathbb{R}^d = \bigcup_{n=1}^{\infty} A_n,$$

and the union in the right-hand side is pairwise disjoint. For any $h > 1$ and $x \in \mathbb{R}^d$, we can find integers $n_{x, h} \geq 0$ and $k_{x, h} \geq 1$ such that

$$(2.1) \quad Q_h^d(x) \subset \bigcup_{i=1}^{k_{x, h}} A_{n_{x, h} + i} \quad \text{and} \quad Q_h^d(x) \cap A_{n_{x, h} + i} \neq \emptyset,$$

for all $i = 1, \ldots, k_{x, h}$.

If $k_{x, h} > 2$, then the second equation in (2.1) implies that $\text{diam}(Q_h^d(x)) \geq \text{dist}(A_{n_{x, h} + 1}, A_{n_{x, h} + k_{x, h}})$. So

$$2\sqrt{dh} \geq b^{n_{x, h} + k_{x, h} - 1} - b^{n_{x, h} + 1} \geq b^{k_{x, h} - 2} - 1,$$

which implies that

$$(2.2) \quad k_{x, h} \leq \log_b(2\sqrt{dh} + 1) + 2 \leq \log_b(4\sqrt{dh}) + 2.$$

It is clear that the inequality (2.2) also holds for $k_{x, h} \leq 2$. Since $\Lambda$ is $b$-lacunary, $(\Lambda \setminus \{0\}) \cap A_n$ has at most one element. Combining (2.1) and (2.2), we have

$$\#(\Lambda \cap Q_h^d(x)) \leq \#(\Lambda \cap \bigcup_{i=1}^{k_{x, h}} A_{n_{x, h} + i}) \leq k_{x, h} \leq \log_b(4\sqrt{dh}) + 2. \quad \square$$

As one can imagine that a $b$-lacunary set must be very sparse in $\mathbb{R}^d$, the following proposition gives us the affirmative answer. We will use this frequently in the rest of the paper.

**Proposition 2.3.** Let $b > 1$ and $\Lambda$ is a $b$-lacunary set. Then

$$\dim^+(\Lambda) = 0.$$

**Proof.** For any $h > 1$ and $x \in \mathbb{R}^d$, Lemma 2.2 implies that

$$\#(\Lambda \cap Q_h^d(x)) \leq \log_b(4\sqrt{dh}) + 2.$$

Hence for any $r > 0$, we have

$$(2.3) \quad \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\#(\Lambda \cap Q_h^d(x))}{h^r} \leq \lim_{h \to \infty} \frac{\log_b(4\sqrt{dh}) + 2}{h^r} = 0.$$
That is to say $D^+_r(\Lambda) = 0$. From the definition of Beurling dimension, we have $\dim^+(\Lambda) = 0$. □

2.2. Periodic zero set $\mathcal{Z}(\mu) = \emptyset$. In this subsection, we will be devoted to understanding the condition

$$\mathcal{Z}(\mu) = \{ \xi \in \mathbb{R}^d : \hat{\mu}(\xi + k) = 0, \forall k \in \mathbb{Z}^d \} = \emptyset$$

when $\mu$ is a singular measure. The following is an important observation and we will strength the conclusion further when $\mu$ is a self-affine measures in the next section.

**Proposition 2.4.** Suppose the periodic zero set $\mathcal{Z}(\mu)$ is empty and $\mu$ is singular. Then for all $\xi \in \mathbb{R}^d$, the set

$$K_\xi = \{ k \in \mathbb{Z}^d : \hat{\mu}(\xi + k) \neq 0 \}$$

is infinite.

**Proof.** Note that $K_{\xi+k_0} = K_\xi + k_0$ for any $\xi \in \mathbb{R}^d$ and $k_0 \in \mathbb{Z}^d$. So we just need to consider the set $K_\xi$ for $\xi \in [0,1)^d$. Let $\mu_\xi$ be the complex measure $e^{-2\pi i \xi \cdot x} d\mu(x)$. Consider the complex measure on $\mathbb{T}^d$, which we identify as $[0,1)^d$,

$$\nu_{\mathbb{T}^d, \xi}(E) = \sum_{n \in \mathbb{Z}^d} \mu_\xi(E + n)$$

for all Borel set $E \in \mathbb{T}^d$. Then $\nu_{\mathbb{T}^d, \xi}$ is a measure on $\mathbb{T}^d$ and its Fourier coefficients equal to

$$\hat{\nu}_{\mathbb{T}^d, \xi}(n) = \hat{\mu}_\xi(n) = \hat{\mu}(\xi + n)$$

(for details, see [16]). Since $\mathcal{Z}(\mu)$ is empty, $K_\xi$ is not an empty set and therefore, $\nu_{\mathbb{T}^d, \xi}$ is not a zero measure on $\mathbb{T}^d$. We establish the following claim:

**Claim.** $\nu_{\mathbb{T}^d, \xi}$ is singular to the Lebesgue measure on $\mathbb{T}^d$.

**Proof of claim.** We first note that $\nu_{\mathbb{T}^d, \xi}$ is absolutely continuous with respect to $\nu_{\mathbb{T}^d, 0}$. Indeed, if $\nu_{\mathbb{T}^d, 0}(E) = 0$, then $\mu(E + n) = 0$ for all $n \in \mathbb{Z}^d$. Hence, $\mu_\xi(E + n) = 0$ and $\nu_{\mathbb{T}^d, \xi}(E) = 0$ follows. Therefore, we just need to show that $\nu_{\mathbb{T}^d, 0}$ is singular with respect to the Lebesgue measure.

Let $m_{\mathbb{R}^d}$ and $m_{\mathbb{T}^d}$ be the Lebesgue measure on $\mathbb{R}^d$ and $\mathbb{T}^d$ respectively. Since $\mu$ is singular to $m_{\mathbb{R}^d}$, we can find a set $A$ such that $m_{\mathbb{R}^d}(A) = 0$ and $\mu$ is supported on $A$. For each $n \in \mathbb{Z}^d$, let

$$A_n = (A \cap ([0,1)^d + n)) - n,$$
so that

\[ A = \bigcup_{n \in \mathbb{Z}^d} (A_n + n). \]

Let \( E = \bigcup_{n \in \mathbb{Z}^d} A_n \) and let \( \mathbb{T}^d \setminus E \) be its complement. Then

\[ \nu_{\mathbb{T}^d,0}(\mathbb{T}^d \setminus E) = \sum_{n \in \mathbb{Z}^d} \mu \left( \bigcap_{m \in \mathbb{Z}^d} (\mathbb{T}^d \setminus A_m) + n \right) \leq \sum_{n \in \mathbb{Z}^d} \mu((\mathbb{T}^d \setminus A_n) + n) = 0 \]

since

\[(\text{supp } \mu) \cap ([0,1)^d + n) \subset A_n + n.\]

Hence, we know \( \nu_{\mathbb{T}^d,0} \) is supported on \( E \). But we know that

\[ m_{\mathbb{T}^d}(E) \leq \sum_{n \in \mathbb{Z}^d} m_{\mathbb{T}^d}(A_n) = \sum_{n \in \mathbb{Z}^d} m_{\mathbb{R}^d}(A_n + n) = 0. \]

This shows that \( \nu_{\mathbb{T}^d,0} \) is a singular measure with respect to \( m_{\mathbb{T}^d} \). This justifies the claim. \( \square \)

Finally we argue by contradiction. Suppose that \( \#K_\xi < \infty \). Then we can find some \( N_0 \) such that the measure \( \nu_{\mathbb{T}^d,\xi}(n) = 0 \) for all \( |n| > N_0 \). Then we know that the Fourier coefficients \( \{ \nu_{\mathbb{T}^d,\xi}(n) : n \in \mathbb{Z}^d \} \) is square-summable. By the unitary isomorphism of \( L^2(\mathbb{T}^d) \) and \( \ell^2(\mathbb{Z}^d) \), we can find \( f \in L^2(\mathbb{T}^d) \), in fact a trigonometric polynomial, such that for every \( n \in \mathbb{Z}^d \),

\[ \nu_{\mathbb{T}^d,\xi}(n) = \hat{f}(n). \]

This means that \( \nu_{\mathbb{T}^d,\xi} = f(x)dx \), which is a contradiction since \( \nu_{\mathbb{T}^d,\xi} \) is singular to the Lebesgue measure. \( \square \)

We remark that the above proposition is clearly false if \( \mu \) is not singular. For example, if \( \mu \) is the Lebesgue measure on \([0,1]\). Then \( K_0 = \{0\} \) only since the Fourier transform of the characteristic function of the unit interval is equal to zero on all non-zero integers.

**Proposition 2.5.** Suppose that the periodic zero set \( \mathbb{Z}(\mu) \) is empty. Then there exists \( \varepsilon > 0 \) and \( \delta > 0 \) such that for all \( \xi \in [0,1]^d \), there exists \( k_\xi \in \mathbb{Z}^d \) such that

\[ |\hat{\mu}(\xi + y + k_\xi)| \geq \varepsilon \]

whenever \( |y| < \delta \). Especially, for \( \xi = 0 \), we can take \( k_0 = 0 \).

**Proof.** As \( \mathbb{Z}(\mu) \) is empty, for any \( \xi \in [0,1]^d \), we can find \( k_\xi \in \mathbb{Z}^d \) and \( \varepsilon_\xi > 0 \) such that

\[ |\hat{\mu}(\xi + k_\xi)| \geq \varepsilon_\xi > 0. \]
By the continuity of $\hat{\mu}$, we can find $\delta_\xi > 0$ such that for all $|y| \leq \delta_\xi$, we have

$$|\hat{\mu}(\xi + y + k_\xi)| \geq \frac{\varepsilon_\xi}{2}.$$ 

As $[0, 1]^d \subset \bigcup_{\xi \in [0, 1]^d} B(\xi, \delta_\xi/2)$, by the compactness of $[0, 1]^d$, we can find $\xi_1, \ldots, \xi_N \in [0, 1]^d$ such that $[0, 1]^d \subset B(\xi_1, \delta_{\xi_1}/2) \cup \ldots \cup B(\xi_N, \delta_{\xi_N}/2)$. We now take

$$\delta = \min \left\{ \frac{\delta_{\xi_j}}{2} : j = 1, \ldots, N \right\}, \quad \varepsilon = \min \left\{ \frac{\varepsilon_{\xi_j}}{2} : j = 1, \ldots, N \right\}.$$

Now, $\delta$ and $\varepsilon$ are positive and independent of $\xi \in [0, 1]^d$. We claim that the stated property holds. Indeed, for any $\xi \in [0, 1]^d$, $\xi \in B(\xi_j, \delta_{\xi_j}/2)$ for some $j = 1, \ldots, N$. Hence,

$$|\hat{\mu}(\xi + k_{\xi_j})| = |\hat{\mu}(\xi + (\xi - \xi_j) + k_{\xi_j})| \geq \frac{\varepsilon_{\xi_j}}{2} \geq \varepsilon. \quad \square$$

### 3. Proof of Theorem 1.6

In this section, we first outline how one can construct a Fourier basis for the self-affine measure $\mu(R, B)$ and then prove Theorem 1.6 that we can find $b$-lacunary spectra if the periodic zero set is empty.

By iterating the invariance identity (1.1), the Fourier transform of the self-affine measure $\mu = \mu(R, B)$ can be expressed as an infinite product

$$\hat{\mu}(\xi) = \prod_{n=1}^{\infty} \delta_B((R^n)^{-1}\xi) = \prod_{n=1}^{\infty} \delta_{R^{-n}B}(\xi).$$

Here $\delta_A$ denotes the equal-weighted Dirac mass supported on the finite set $A$. From this infinite product, we obtain another expression of the self-affine measure through an infinite convolution of atomic measures

$$\mu(R, B) = \delta_{R^{-1}B} \ast \delta_{R^{-2}B} \ast \cdots = w - \lim_{n \to \infty} (\delta_{R^{-1}B} \ast \delta_{R^{-2}B} \ast \cdots \ast \delta_{R^{-n}B})$$

where $w$-lim is the weak limit of the probability measures.

Given a subsequence of positive integers $\{n_k\}$, we define

$$B_{n_k} = B + RB + \cdots + R^{n_k-1}B \quad \text{and} \quad L_{n_k} = L + R^tL + \cdots + (R^t)^{n_k-1}L.$$  

Let $m_k = n_1 + \cdots + n_k$. Then the self-affine measure $\mu = \mu(R, B)$ can be factorized along this subsequence as

$$(3.1) \quad \mu = \delta_{R^{-m_1}B_{n_1}} \ast \delta_{R^{-m_2}B_{n_2}} \ast \cdots.$$
Define also
\[ \mu_{>k} = \delta_{R^{-m_{k+1}}B_{n_{k+1}}} \ast \delta_{R^{-m_{k+2}}B_{n_{k+2}}} \ast \cdots. \]

The following lemma is known, whose proof can be found in [8, Corollary 3.3].

**Lemma 3.1.** Suppose \((R, B, L)\) forms a Hadamard triple. For any \(n \geq 1\),
(i) \((R^n, B_n, L_n)\) is also a Hadamard triple;
(ii) if \(\tilde{L}_n \equiv L_n \pmod{(R^l)^n}\), then \((R^n, B_n, \tilde{L}_n)\) is also a Hadamard triple.

We note that \((R, B, L)\) forms a Hadamard triple if and only if \(\{e^{2\pi i \ell \cdot x} : \ell \in L\}\) will form an orthonormal basis for \(L^2(\delta_{R^{-1}B})\). Hence, since we know \(\{(R^n, B_n, L_n)\}\) form Hadamard triples, we define

\[ (3.2) \quad \Lambda_k = L_{n_1} + (R^l)^{m_1}L_{n_2} + \cdots + (R^l)^{m_{k-1}}L_{n_k}, \text{ and } \Lambda = \bigcup_{k=1}^{\infty} \Lambda_k, \]

and \(E(\Lambda)\) forms a mutually orthogonal set for \(L^2(\mu(R, B))\) [8, Lemma 4.2]. The following is the main theorem giving a sufficient condition for an orthogonal set to be complete.

**Theorem 3.2** [8,18]. Let \((R, B, L)\) be a Hadamard triple. Let \(\Lambda_k\) and \(\Lambda\) be defined as in (3.2). Suppose that

\[ (3.3) \quad \delta(\Lambda) := \inf_{k \geq 1} \inf_{\lambda \in \Lambda_k} |\hat{\mu}_{>k}(\lambda)|^2 > 0. \]

Then the self-affine measure \(\mu\) is a spectral measure with a spectrum \(\Lambda\) in \(\mathbb{Z}^d\).

Condition (3.3) is a sufficient condition guaranteeing the mutually orthogonal sets to be complete. This condition was first proposed by Strichartz [23,24]. In general, it cannot be removed. On the other hand, this condition is also not necessary [3]. The following theorem provides a strengthened result of Proposition 2.4 in the case of self-affine measures.

**Theorem 3.3.** Suppose a self-affine measure \(\mu := \mu(R, B)\) satisfies \(\mathcal{Z}(\mu) = \emptyset\). Then there is an \(\varepsilon_0 > 0\) such that for any \(\xi \in [0, 1]^d\),

\[ \mathcal{K}_{\xi, \varepsilon_0} = \{k \in \mathbb{Z}^d : |\hat{\mu}(\xi + k)| \geq \varepsilon_0\} \]

is an infinite set.

**Proof.** Take \(\varepsilon > 0\) and \(\delta > 0\) be the constants defined as in Proposition 2.5. For \(\xi = 0\), from Proposition 2.4, \(\mathcal{K}_0\) is an infinite set, we can choose a \(t_0 \in \mathcal{K}_0 \setminus \{0\}\). Then there exists \(\varepsilon' > 0\) and \(\delta' > 0\) such that

\[ (3.4) \quad |\hat{\mu}(y + t_0)| \geq \varepsilon' \quad \text{for all } |y| < \delta'. \]
Let $\varepsilon_0 = \varepsilon \cdot \varepsilon'$ and $\delta_0 = \min\{\delta, \delta'\}$ which are positive constants independent of $\xi \in [0, 1]^d$.

From Proposition 2.5, for any $\xi \in [0, 1]^d$, there is a $k_\xi \in \mathbb{Z}^d$ such that

\begin{equation}
\hat{\mu}(\xi + k_\xi) \geq \varepsilon.
\end{equation}

Fix $\xi \in [0, 1]^d$ and $k_\xi$, we can find an integer $n_0 \geq 1$ such that

\begin{equation}
| (R_t)^{-n_0}(\xi + k_\xi) | < \delta_0.
\end{equation}

When $n \geq n_0$, since $| (R_t)^{-n}(\xi + k_\xi) | < \delta_0$, the inequality (3.4) implies that

\begin{equation}
| \hat{\mu}((R_t)^{-n}(\xi + k_\xi) + t_0) | \geq \varepsilon'.
\end{equation}

It together with inequality (3.5), we have

\begin{align*}
| \hat{\mu}(\xi + k_\xi + (R_t)^n t_0) | &= \prod_{k=1}^{\infty} | \hat{\delta}_{R-kB}(\xi + k_\xi + (R_t)^n t_0) | \\
&= \prod_{k=1}^{n} | \hat{\delta}_{R-kB}(\xi + k_\xi) | \cdot \prod_{k=1}^{\infty} | \hat{\delta}_{R-kB}((R_t)^{-n}(\xi + k_\xi) + t_0) | \\
&\geq | \hat{\mu}(\xi + k_\xi) | \cdot | \hat{\mu}((R_t)^{-n}(\xi + k_\xi) + t_0) | \geq \varepsilon \cdot \varepsilon' = \varepsilon_0,
\end{align*}

where the second last inequality follows from

\begin{equation}
| \hat{\delta}_{R^{-n}B}(\xi) | = \frac{1}{\text{card}(B)} \left| \sum_{b \in B} e^{-2\pi i R^{-n} b \cdot \xi} \right| \leq 1 \quad \text{for all } \xi \in \mathbb{R}^d, \; n \geq 1.
\end{equation}

So

\begin{equation}
\{ \{ k_\xi + (R_t)^n t_0 \}_{n=n_0}^{\infty} \} \subset \mathcal{K}_{\xi, \varepsilon_0}.
\end{equation}

As $(R_t)^k$ is an expanding matrix, 1 is not an eigenvalue of $(R_t)^k$. So $(R_t)^n t_0 \neq (R_t)^m t_0$ when $n \neq m$. Hence $\mathcal{K}_{\xi, \varepsilon_0}$ is an infinite set. □

**Proof of Theorem 1.6.** Let $\varepsilon_0 > 0$ be a constant defined as in Theorem 3.3. Then the uniform continuity of $\hat{\mu}$ implies that there exists a $\delta_0 > 0$ such that for all $\xi \in [0, 1]^d$ and $|y| < \delta_0$, we have

\begin{equation}
| \hat{\mu}(\xi + y + k) | \geq \varepsilon_0/2
\end{equation}

holds for $k \in \mathcal{K}_{\xi, \varepsilon_0}$.

We now construct inductively $\Lambda_k$ as in (3.2) so that $\Lambda_k$ is $b$-lacunary and $\delta(\Lambda) \geq \varepsilon_0/2 > 0$. From Lemma 3.1(ii), without loss of generality, we assume $L = \{l_0 = 0, l_1, \ldots, l_{q-1}\} \subset R^t[0, 1)^d \cap \mathbb{Z}^d$. Then

\begin{equation}
(R_t)^{-n}L_n \subset [0, 1)^d.
\end{equation}
Denote \( n_1 = 1 \). Let
\[
\lambda_1 = l_1 + (R^t)k_{(R^t)^{-1}l_1}, \quad k_{(R^t)^{-1}l_1} \in \mathcal{K}_{(R^t)^{-1}l_1, \varepsilon_0}.
\]
For \( 2 \leq i \leq q - 1 \), since \( \mathcal{K}_{(R^t)^{-1}l_i, \varepsilon_0} \) is infinite, we can take a \( k_{(R^t)^{-1}l_i} \in \mathcal{K}_{(R^t)^{-1}l_i, \varepsilon_0} \) such that
\[
\lambda_i = l_i + R^t k_{(R^t)^{-1}l_i} \quad \text{and} \quad |\lambda_i| \geq b|\lambda_{i-1}|.
\]
Then \( \Lambda_1 = \{ \lambda_0 = 0, \lambda_1, \lambda_2, \ldots, \lambda_{q-1} \} \) is \( b \)-lacunary and from (3.6)
\[
|\hat{\mu}_{>1}(\lambda_i)| = |\hat{\mu}((R^t)^{-1}l_i + k_{(R^t)^{-1}l_i})| \geq \varepsilon_0/2, \quad 1 \leq i \leq q - 1.
\]
Suppose that \( \Lambda_{k-1} \) has been constructed which is a \( b \)-lacunary set and
\[
\inf_{\lambda \in \Lambda_{k-1}} |\hat{\mu}_{>(k-1)}(\lambda)| \geq \varepsilon_0/2.
\]
We can take a large enough \( n_k \) in the subsequence with the following happen:
\[
\sup_{\lambda \in \Lambda_{k-1}} \| (R^t)^{-m_k} \lambda \| < \delta_0
\]
(recall that \( m_k = n_1 + \cdots + n_k \)). We now define
\[
\Lambda_k = \Lambda_{k-1} + \{(R^t)^{m_k-1}l_k + (R^t)^{m_k}k_{x_{l_k}} : l_k \in \mathbb{L}_{n_k} \},
\]
where \( x_{l_k} = (R^t)^{-n_k}l_k \in [0, 1)^d \) and \( k_0 = 0 \) (Notice that the above is a Minkowski sum of two sets). Then \( \Lambda_{k-1} \subset \Lambda_k \). As \( \mathcal{K}_{x_{l_i}, \varepsilon_0} \) is an infinite set, by choosing \( k_{x_{l_i}} \in \mathcal{K}_{x_{l_i}, \varepsilon_0} \) as large as we wanted, we can ensure that \( \Lambda_k \) is \( b \)-lacunary. Now writing
\[
\lambda = \lambda' + (R^t)^{m_k-1}l_k + (R^t)^{m_k}k_{x_{l_k}},
\]
for some \( \lambda' \in \Lambda_{k-1} \), we have
\[
|\hat{\mu}_{>k}(\lambda)| = |\hat{\mu}((R^t)^{-m_k} \lambda)|
\]
\[
= |\hat{\mu}((R^t)^{-m_k} \lambda' + x_{l_k} + k_{x_{l_k}})| \quad \text{(using (3.6))}
\]
\[
\geq \varepsilon_0/2 > 0.
\]
Hence, \( \delta(\Lambda) > 0 \) is now satisfied. The \( b \)-lacunary set \( \Lambda = \bigcup_{k=1}^{\infty} \Lambda_k \) is a spectrum of \( \mu \) according to Theorem 3.2. \( \square \)

As a corollary, we settle the case for the self-similar measure on \( \mathbb{R}^1 \).

**Corollary 3.4.** Let \( R > 1 \) be an integer and \( B \subset \mathbb{Z} \) be a digit set with \( \#B < R \) and \( \gcd(B) = 1 \). Suppose that \( (R, B, L) \) forms a Hadamard triple. Then \( \mathcal{Z}(\mu(R, B)) = \emptyset \) and \( \mu(R, B) \) admits a spectrum \( \Lambda \) with \( \dim^+(\Lambda) = 0. \)
Proof. It has been proved in [8, Section 5] that if \( \gcd(B) = 1 \), then the periodic zero set of \( \mu(R, B) \) is empty and therefore it has a \( b \)-lacunary spectrum in \( \mathbb{Z} \) by Theorem 1.6, which has Beurling dimension zero by Proposition 2.3. \( \square \)

4. Proof of Theorem 1.3

We first discuss some preliminary reduction that we can perform in order to prove our main theorem.

Definition 4.1. Let \( R_1, R_2 \) be \( d \times d \) integer matrices, and the finite sets \( B_1, B_2, L_1, L_2 \) be in \( \mathbb{Z}^d \). We say that two triples \((R_1, B_1, L_1)\) and \((R_2, B_2, L_2)\) are conjugate (through the matrix \( Q \)) if there exists an integer unimodular matrix \( Q \) such that \( R_2 = QR_1Q^{-1} \), \( B_2 = QB_1 \) and \( L_2 = (Q^t)^{-1}L_1 \). (Here, unimodular matrix means that its determinant is 1).

The following proposition is obtained from simple computations.

Proposition 4.2. Suppose that \((R_1, B_1, L_1)\) and \((R_2, B_2, L_2)\) are two conjugate triples, through the matrix \( Q \). Then

(i) If \((R_1, B_1, L_1)\) is a Hadamard triple then so is \((R_2, B_2, L_2)\).

(ii) The measure \( \mu(R_1, B_1) \) is spectral with spectrum \( \Lambda \) if and only if \( \mu(R_2, B_2) \) is spectral with spectrum \( (Q^t)^{-1}\Lambda \).

(iii) Spectral measures \( \mu(R_1, B_1) \) and \( \mu(R_2, B_2) \) have spectrum \( \Lambda \) with \( \dim^+(\Lambda) = 0 \) simultaneously.

Proof. The proof of (i) and (ii) can be found in, e.g., [7, Proposition 3.4]. Statement (iii) follows from the fact that \( \dim^+(\Lambda) = \dim^+(\Lambda^\perp) \) which is proved in Proposition 2.1. \( \square \)

We use \( E \times F \) to denote the Cartesian product of \( E \) and \( F \) so that \( E \times F = \{(e, f)^t : e \in E, f \in F\} \). We first introduce the following notation.

Definition 4.3. For a vector \( x \in \mathbb{R}^d \), we write it as \( x = (x_1, x_2)^t \) with \( x_1 \in \mathbb{R}^r \) and \( x_2 \in \mathbb{R}^{d-r} \). We denote by \( \pi_1(x) = x_1, \pi_2(x) = x_2 \). For a subset \( E \) of \( \mathbb{R}^d \), and \( x_1 \in \mathbb{R}^r, x_2 \in \mathbb{R}^{d-r} \), we denote by

\[
E_2(x_1) := \{y \in \mathbb{R}^{d-r} : (x_1, y)^t \in E\}, \quad E_1(x_2) := \{x \in \mathbb{R}^r : (x, x_2)^t \in E\}.
\]

We define \( \mathbb{Z}[R, B] \) to be the smallest \( R \)-invariant lattice containing all \( \sum_{j=0}^{n-1} R^j B \). To prove Theorem 1.3, there is no loss of generality to assume that \( \mathbb{Z}[R, B] = \mathbb{Z}^d \) since we can always conjugate the Hadamard triple to produce a Hadamard triple with \( \mathbb{Z}[R, B] = \mathbb{Z}^d \). If \((R, B)\) and \((\tilde{R}, \tilde{B})\) are conjugate through an integer unimodular matrix \( Q \), then

\[
\mathbb{Z}[\tilde{R}, \tilde{B}] = Q\mathbb{Z}[R, B] = Q\mathbb{Z}^d = \mathbb{Z}^d.
\]
By studying the dynamical system underlying the self-affine system, the following decomposition was proved in [8, Sections 6 and 7]. If \((R,B,L)\) is a Hadamard triple such that \(\mathbb{Z}[R,B] = \mathbb{Z}^d\) and \(\mathcal{Z}(\mu) \neq \emptyset\), we can always conjugate with some integer unimodular matrix so that \((R,B)\) are of the following quasi-product form:

\[
R = \begin{pmatrix} R_1 & 0 \\ C_0 & G_1 \end{pmatrix},
\]

\[(4.1)\]

\[
B = \left\{ (u_i, d_j(u_i))^t : 1 \leq i \leq N_1, \ 1 \leq j \leq N_2 := |\det G_1| \right\},
\]

\[(4.2)\]

where \(R_1 \in M_r(\mathbb{Z}), G_1 \in M_{d-r}(\mathbb{Z})\) are expansive matrices and \(\{d_j(u_i) : 1 \leq j \leq N_2\}\) is a complete set of representatives (mod \(G_1\mathbb{Z}^{d-r}\)) for all \(i, 1 \leq i \leq N_1\). Note that \(R^{-k}\) can be written in the following form

\[
R^{-k} = \begin{pmatrix} R_1^{-k} & 0 \\ C_k & G_1^{-k} \end{pmatrix}
\]

for some \(C_k \in M_{d-r,r}(\mathbb{R})\). For the self-affine set \(T(R,B)\), we can express it as a set of infinite sums

\[
T(R,B) = \left\{ \sum_{k=1}^{\infty} R^{-k}b_k : b_k \in B \right\}.
\]

Therefore any element \((x_1, x_2)^t \in T(R,B)\) can be written as

\[
x_1 = \sum_{k=1}^{\infty} R_1^{-k}u_{i_k}, \ x_2 = \sum_{k=1}^{\infty} C_k u_{i_k} + \sum_{k=1}^{\infty} G_1^{-k}d_{j_k}(u_{i_k}).
\]

Hence

\[
\pi_1(T(R,B)) = T(R_1, \pi_1(B))
\]

is a self-affine set where \(\pi_1(B) = \{u_i : 1 \leq i \leq N_1\}\). For each

\[
x_1 = \sum_{k=1}^{\infty} R_1^{-k}u_{i_k} \in T(R_1, \pi_1(B)),
\]

we have

\[
(T(R,B))_2(x_1) = \sum_{k=1}^{\infty} C_k u_{i_k} + \left\{ \sum_{k=1}^{\infty} G_1^{-k}d_{j_k}(u_{i_k}) : 1 \leq j_k \leq N_2 \right\}.
\]

Moreover, we can find a \(L' \equiv L \mod R^t\) such that \((R_1, \pi_1(B), L'_1(l_2))\) is a Hadamard triple on \(\mathbb{R}^r\) for all \(l_2 \in \pi_2(L')\). Let \(\mu(R_1, \pi_1(B))\) be the self-affine
measure supported on $T(R_1, \pi_1(B))$ and $\mu_2^{(x_1)}$ be the infinite convolution product

$$\delta_{G_1^{-1}B_2(i_1)} \ast \delta_{G_1^{-2}B_2(i_2)} \ast \cdots,$$

where $B_2(i_k) := \{d_j(u_{i_k}) : 1 \leq j \leq N_2\}$. We call $\mu_2^{(x_1)}$ the Cantor–Moran measure supported on

$$(T(R, B))_2(x_1) - \sum_{k=1}^{\infty} C_k u_{i_k}.$$

The measures $\mu(R_1, \pi_1(B))$ and $\mu_2^{(x_1)}$ are called a quasi-product form decomposition of $\mu(R, B)$.

**Theorem 4.4.** Let $(R, B, L)$ be a Hadamard triple on $\mathbb{R}^d$ such that $\mathbb{Z}[R, B] = \mathbb{Z}^d$ and $\mathbb{Z}(\mu) \neq \emptyset$. Then we can conjugate with an integer unimodular matrix $Q$ so that $\mu(QRQ^{-1}, QB)$ has a quasi-product form $\mu_1$ and $\mu_2^{(x_1)}$ on $\mathbb{R}^r \times \mathbb{R}^{d-r}$ with $\mathbb{Z}(\mu_1) = \emptyset$.

**Proof.** We will prove the result by using induction on dimension $d$. When $d = 1$, the assumption $\mathbb{Z}[R, B] = \mathbb{Z}$ forces $\gcd(B) = 1$. In this case, it has been proved in [8, Section 5] that $\mathbb{Z}(\mu) = \emptyset$. Hence, if $\mathbb{Z}(\mu) \neq \emptyset$, then $d \geq 2$.

For $d = 2$, as we have discussed, we can conjugate with an integer unimodular matrix $Q \in M_2(\mathbb{Z})$ so that $(QRQ^{-1}, QB)$ are of the quasi-product form on $\mathbb{R} \times \mathbb{R}$ as following:

$$QRQ^{-1} = \begin{pmatrix} R_1 & 0 \\ C_1 & G_1 \end{pmatrix}, \quad QB = \{(u, d_j(u))^t : u \in B_1, 1 \leq j \leq |G_1|\} \subset \mathbb{Z}^2,$$

and $\{d_j(u) : 1 \leq j \leq |G_1|\} \subset \mathbb{Z}$ is a complete set of representatives (mod $|G_1|$) for all $u \in B_1$ (Here $|G_1| \geq 2$ is a positive integer). Moreover, $\gcd(B_1) = 1$ since $\mathbb{Z}[QRQ^{-1}, QB] = \mathbb{Z}^2$. Then $\mu(QRQ^{-1}, QB)$ has a quasi-product form $\mu_1$ and $\mu_2^{(x_1)}$ on $\mathbb{R} \times \mathbb{R}$ where $\mu_1 = \mu(R_1, B_1)$ is the self-similar measure supported on $T(R_1, B_1) \subset \mathbb{R}$. Since $\gcd(B_1) = 1$ and $(R_1, \pi_1(B), L'(l_2))$ forms a Hadamard triple on $\mathbb{R}$ for some $L', l_2 \in \pi_2(L')$, we have $\mathbb{Z}(\mu_1) = \emptyset$. This justifies the result on dimension two.

Assume our statement is true for any dimensions less than $d$. On dimension $d$, we can conjugate with an integer unimodular matrix $Q_1 \in M_d(\mathbb{Z})$ so that $(Q_1RQ_1^{-1}, Q_1B)$ has quasi-product form

$$Q_1RQ_1^{-1} = \begin{pmatrix} R_1 & 0 \\ C_1 & G_1 \end{pmatrix}, \quad Q_1B = \{(u, d_j(u))^t : u \in B_1, 1 \leq j \leq |\det G_1|\} \subset \mathbb{Z}^d,$$
where $\mathbb{Z}[R_1, B_1] = \mathbb{Z}^r$ and $\{d_j(u) : 1 \leq j \leq |\det G_1|\} \subset \mathbb{Z}^{d-r}$ is a complete set of representatives (mod $G_1\mathbb{Z}^{d-r}$). If $\mathcal{Z}(\mu(R_1, B_1)) = \emptyset$, then $\mu_1 = \mu(R_1, B_1)$ and $\mu_2^{(x_1)}$ is the desired quasi-product form on $\mathbb{R}^r \times \mathbb{R}^{d-r}$.

If $\mathcal{Z}(\mu(R_1, B_1)) \neq \emptyset$, by the assumption on $r < d$, we can conjugate with an unimodular matrix $Q_2 \in M_r(\mathbb{Z})$ such that

$$Q_2R_1Q_2^{-1} = \begin{pmatrix} R_2 & 0 \\ C_2 & G_2 \end{pmatrix},$$

$$Q_2B_1 = \{Q_2u : u \in B_1\} = \{(v, w_j(v))^t : v \in B_2, 1 \leq j \leq |\det G_2|\} \subset \mathbb{Z}^r,$$

where $\{w_j(v) : 1 \leq j \leq |G_2|\} \subset \mathbb{Z}^{r-1}$ is a complete set of representatives (mod $G_2\mathbb{Z}^{r-1}$) and $\mathcal{Z}(\mu(R_2, B_2)) = \emptyset$. Denote

$$Q = \begin{pmatrix} Q_2 & 0 \\ 0 & I_{d-r} \end{pmatrix} Q_1,$$

where $I_{d-r}$ is the identity matrix and rewrite $d_i(u)$ as $d_i(v, w_j(v))$ if $Q_2u = (v, w_j(v))^t$. Then $(QRQ^{-1}, QB)$ has quasi-product form on $\mathbb{R}^{r_1} \times \mathbb{R}^{d-r_1}$ as

$$QRQ^{-1} = \begin{pmatrix} R_2 & 0 \\ C_2' & G_2' \end{pmatrix},$$

where

$$G_2' = \begin{pmatrix} G_2 & 0 \\ C_2 & G_1 \end{pmatrix}$$

for some matrix $C_2' \in M_{d-r, r-1}(\mathbb{Z})$ and

$$QB = \{(v, w_j(v), d_{j_1}(v, w_{j_2}(v))^t : v \in B_2, 1 \leq j_2 \leq |\det G_2|, 1 \leq j_1 \leq |\det G_1|\},$$

where

$$\{(w_{j_2}(v), d_{j_1}(v, w_{j_2}(v))^t : 1 \leq j_2 \leq |\det G_2|, 1 \leq j_1 \leq |\det G_1|\} \subset \mathbb{Z}^{d-r_1}$$

is a complete set of representatives mod $G_2'\mathbb{Z}^{d-r_1}$. As $\mathcal{Z}(\mu(R_2, B_2)) = \emptyset$, $\mu_1 = \mu(R_2, B_2)$ and $\mu_2^{(x_1)}$ is the desired quasi-product form decomposition of the self-affine measure $\mu(R, B)$ on $\mathbb{R}^{r_1} \times \mathbb{R}^{d-r_1}$. □

The spectrality for $\mu_1$ and $\mu_2^{(x_1)}$ in the quasi-product-form decomposition was proved in [8]

**Theorem 4.5** [8, Proposition 8.4]. Suppose $(R, B, L)$ is a Hadamard triple and $\mathcal{Z}(\mu) \neq \emptyset$. Let $\mu_1, \mu_2^{(x_1)}$ be a quasi-product form of $\mu$ on $\mathbb{R}^r \times \mathbb{R}^{d-r}$, then $\mu_1$ is spectral and for $\mu_1$-almost every $x_1 \in T(R_1, B_1)$, $\mu_2^{(x_1)}$ admits a spectrum $\Gamma$ which is a full-rank lattice in $\mathbb{R}^{d-r}$.
Lemma 4.6 [7, Lemma 4.5]. If $\Lambda_1$ is a spectrum for the measure $\mu_1$, then for all $x_2 \in \mathbb{R}^{d-r}$ we have
$$\sum_{\lambda_1 \in \Lambda_1} |\hat{\mu}(x_1 + \lambda_1, x_2)|^2 = \int_{T(R_1,\pi_1(B))} |\hat{\mu}_2(s)(x_2)|^2 \, d\mu_1(s).$$

We recall also the Jorgensen–Pedersen lemma for checking when a countable set is a spectrum for a measure.

Lemma 4.7 [15]. Let $\mu$ be a compactly supported probability measure on $\mathbb{R}^d$. Then a countable set $\Lambda$ is a spectrum for $L^2(\mu)$ if and only if
$$\sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2 \equiv 1, \quad \xi \in \mathbb{R}^d.$$

We now state a general class of spectrum for the quasi product-form.

Theorem 4.8. Suppose $\Gamma$ is a spectrum of $\mu_2^{(x_1)}$ for $\mu_1$-almost every $x_1$ and $\{\Lambda_\gamma\}_{\gamma \in \Gamma}$ is a sequence of spectra of $\mu_1$. Then $\bigcup_{\gamma \in \Gamma} (\Lambda_\gamma \times \{\gamma\})$ is a spectrum of $\mu$.

Proof. Since $\Lambda_\gamma$ is a spectrum of $\mu_1$, by Lemma 4.6, we have
$$\sum_{\gamma \in \Gamma} \sum_{\lambda \in \Lambda_\gamma} |\hat{\mu}(x_1 + \lambda, x_2 + \gamma)|^2 = \int_{T(R_1,\pi_1(B))} \sum_{\gamma \in \Gamma} |\hat{\mu}_2(s)(x_2 + \gamma)|^2 \, d\mu_1(s)$$
$$= \int_{T(R_1,\pi_1(B))} 1 \, d\mu_1(s) = 1.$$ 

This means that $\bigcup_{\gamma \in \Gamma} (\Lambda_\gamma \times \{\gamma\})$ is a spectrum of $\mu$ by Lemma 4.7. \qed

In [8], only the case that all $\Lambda_\gamma$ are the same was considered. However, to construct zero Beurling dimension spectra, $\Lambda \times \Gamma$ is not enough since $\{0\} \times \Gamma$ is contained in the spectrum and this will contribute to the Beurling dimension $d - r$ since $\Gamma$ is lattice on $\mathbb{R}^{d-r}$. To overcome this problem, we will need different sparse spectra for each $\gamma$. Now we can prove our main result.

Proof of Theorem 1.3. If $Z(\mu) = \emptyset$, then the result follows from Theorem 1.6. Suppose now that $Z(\mu) \neq \emptyset$. Since conjugation maintains the Beurling dimension of $\Lambda$, without loss of generality, we assume $\mu$ has quasi-product form $\mu_1 = \mu(R_1, B_1), \mu_2^{(x_1)}$ on $\mathbb{R}^r \times \mathbb{R}^{d-r}$ with $Z(\mu_1) = \emptyset$ from Theorem 4.4. Moreover, $(R_1, B_1, L_1)$ forms a Hadamard triple. From Theorem 1.6, for any $b > 1$, we can construct a $b$-lacunary spectrum of $\mu_1$.

Let the full-rank lattice $\Gamma = AZ^{d-r}$ is a spectrum of $\mu_2^{(x_1)}$ for $\mu_1$-almost every $x_1 \in T(R_1, B_1)$. Let $N_{0,r} = 0$ and
$$N_{n,r} = \# \left\{ m \in \mathbb{Z}^{d-r} : 2^{n-1} \leq |m| < 2^n \right\} \quad \text{for all } n \geq 1.$$
For any $n \geq 0$, let $0 \in \mathcal{T}_n$ be a spectrum of $\mu_1$ which is $8^{N_{n,r}}$-lacunary. By a reverse triangle inequality, it implies that

$$(4.3) \quad \min \{ |\lambda - \lambda'| : \lambda \neq \lambda' \in \mathcal{T}_n \} \geq 4^{N_{n,r}}.$$  

We take $\Lambda_0 = \mathcal{T}_0$, which is a $8$-lacunary spectrum of $\mu_1$. For any $m \in \mathbb{Z}^{d-r} \setminus \{0\}$, there is an unique integer $n \geq 1$ such that $2^{n-1} \leq |m| < 2^n$. We now enumerate the set

$$\{ m \in \mathbb{Z}^{d-r} : 2^{n-1} \leq |m| < 2^n \} = \{ m_{n,1}, \ldots, m_{n,N_{n,r}} \}.$$  

We then take spectrum of $\mu_1$ on each of integers $m_{n,j}$ as follows:

$$\Lambda_{m_{n,j}} = \mathcal{T}_n + j \cdot 2^{N_{n,r}} e_1, \quad j = 1, \ldots, N_{n,r},$$  

where $e_1$ is an unit vector in the $\mathbb{R}^{d-r}$. Therefore, by Theorem 4.8,

$$\Lambda = \bigcup_{m \in \mathbb{Z}^{d-r}} (\Lambda_m \times \{ Am \}) =: \left( I_r \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix} \right) \Lambda'$$  

is a spectrum of $\mu$ where $I_r$ is the $r \times r$ identity matrix and

$$\Lambda' = \bigcup_{m \in \mathbb{Z}^{d-r}} (\Lambda_m \times \{ m \}).$$  

By Proposition 2.1, $\dim^+(\Lambda) = \dim^+(\Lambda')$. Next we will prove that

$$\dim^+(\Lambda') = 0.$$  

**Claim 1.** For any $n \geq 1$ and distinct integers $j_1, j_2$ with $1 \leq j_1, j_2 \leq N_{n,r}$,

$$\Lambda_{m_{n,j_1}} \cap \Lambda_{m_{n,j_2}} = \emptyset.$$  

**Proof.** Suppose there is an positive integer $n$ and $1 \leq j_1, j_2 \leq N_{n,r}$ such that

$$\Lambda_{m_{n,j_1}} \cap \Lambda_{m_{n,j_2}} \neq \emptyset,$$  

and choose an element $\lambda$ in the intersection. Then it has two expressions

$$\lambda = t_1 + j_1 \cdot 2^{N_{n,r}} e_1 = t_2 + j_2 \cdot 2^{N_{n,r}} e_1$$  

where $t_1, t_2$ are distinct elements in $\mathcal{T}_n$ as $j_1 \neq j_2$. It implies that

$$t_1 - t_2 = (j_2 - j_1)2^{N_{n,r}} e_1.$$  

The right-hand side of the equation implies

$$(4.4) \quad |t_1 - t_2| < N_{n,r} \cdot 2^{N_{n,r}} \leq 2^{N_{n,r}} \cdot 2^{N_{n,r}} = 4^{N_{n,r}}.$$  

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But from (4.3) we have
\[ |t_1 - t_2| \geq 4^{N_{n,r}+1}, \]
a contradiction. Hence, the claim is proved. □

**Claim 2.** We have

\[ \delta_n := \inf \left\{ |\lambda_1 - \lambda_2| : \lambda_1 \neq \lambda_2 \in \bigcup_{j=1}^{N_{n,r}} \Lambda_{m_{n,j}} \right\} = 2^{N_{n,r}}. \]

**Proof.** Note that \( 2^{N_{n,r}} e_1 \in \Lambda_{m_{n,1}}, 2 \cdot 2^{N_{n,r}} e_1 \in \Lambda_{m_{n,2}} \) and

\[ |2^{N_{n,r}} e_1 - 2 \cdot 2^{N_{n,r}} e_1| = 2^{N_{n,r}}, \]

so \( \delta_n \leq 2^{N_{n,r}}. \) On the other hand, for any distinct elements \( \lambda_1 \in \Lambda_{m_{n,j_1}}, \lambda_2 \in \Lambda_{m_{n,j_2}}, \) we can write them as

\[ \lambda_1 = t_1 + j_1 \cdot 2^{N_{n,r}} e_1, \quad \lambda_2 = t_2 + j_2 \cdot 2^{N_{n,r}} e_1 \]

for some \( t_1, t_2 \in \mathcal{T}_n. \) If \( t_1 = t_2, \) then \( j_1 \neq j_2 \) and

\[ |\lambda_1 - \lambda_2| = |(j_1 - j_2) \cdot 2^{N_{n,r}} e_1| \geq 2^{N_{n,r}}. \]

If \( t_1 \neq t_2, \) then from (4.3), \( |t_1 - t_2| \geq 4^{N_{n,r}+1}. \) It implies that

\[ |\lambda_1 - \lambda_2| \geq |t_1 - t_2| - 2^{N_{n,r}} \cdot |j_1 - j_2| \geq 4^{N_{n,r}+1} - N_{n,r} \cdot 2^{N_{n,r}} \geq 2^{N_{n,r}}. \]

This justifies the claim. □

Now we return to the proof of theorem. For any \( h > 1 \) and \( (y_1, y_2)^t \in \mathbb{R}^r \times \mathbb{R}^{d-r}, \) choose \( n_1 \geq 0 \) be the largest integer and \( n_2 > n_1 \) be the smallest integer such that

\[ Q_h^{d-r}(y_2) \cap \mathbb{Z}^{d-r} \subset B(0, 2^{n_2}) \setminus B(0, 2^{n_1}) \cup \{0\}. \]

Then we have

\[ 2^{n_2-1} - 2^{n_1+1} \leq 2 h \sqrt{2} \leq 2^{n_2} - 2^{n_1}, \]

and so

\[ (4.5) \quad n_2 - n_1 \leq 4 \log_2(4 h \sqrt{2}). \]

Note that \( Q_h^d((y_1, y_2)^t) = Q_h^r(y_1) \times Q_h^{d-r}(y_2). \) This allows us to have the following:

\[ (4.6) \quad \#(\Lambda' \cap Q_h^d((y_1, y_2)^t)) = \sum_{m \in Q_h^{d-r}(y_2) \cap \mathbb{Z}^{d-r}} \#(\Lambda_m \cap Q_h^r(y_1)) \]

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\[
\leq \sum_{n=n_1+1}^{n_2} \sum_{j=1}^{N_{n,r}} \#(\Lambda_{m_{n,j}} \cap Q_h^r(y_1)) + \#(\Lambda_0 \cap Q_h^r(y_1)).
\]

We now decompose the first summand into two parts:

\[
\sum_{n=n_1+1}^{n_2} \ldots = \sum_{\substack{n_1 < n \leq n_2, \\ 2^{N_{n,r}} > 2h\sqrt{r}}} \ldots + \sum_{\substack{n_1 < n \leq n_2, \\ 2^{N_{n,r}} \leq 2h\sqrt{r}}} \ldots
\]

In the first part,

\[
2^{N_{n,r}} > 2h\sqrt{r} = \text{diam}(Q_h^r(y_1)).
\]

From the Claims 1 and 2, we have \(\bigcup_{j=1}^{N_{n,r}} \Lambda_{m_{n,j}}\) has at most one element in \(Q_h^r(y_1)\). So

\[
\text{(4.7)} \quad \sum_{n_1 < n \leq n_2} \sum_{j=1}^{N_{n,r}} \#(\Lambda_{m_{n,j}} \cap Q_h^r(y_1))
\leq \sum_{\substack{n_1 < n \leq n_2, \\ 2^{N_{n,r}} > 2h\sqrt{r}}} 1 \leq n_2 - n_1 \leq 4\log_2(4h\sqrt{r}).
\]

In the second case, for the integer \(n\) with \(n_1 < n \leq n_2\) with \(2^{N_{n,r}} \leq 2h\sqrt{r}\), we have

\[
N_{n,r} \leq \log_2(2h\sqrt{r}).
\]

For any \(1 \leq j \leq N_{n,j}\), \(\Lambda_{m_{n,j}}\) is a \(8^{N_{n,r}+1}\)-lacunary set, it follows from Lemma 2.2 that,

\[
\#(\Lambda_{m_{n,j}} \cap Q_h^r(y_1)) \leq \log_8 8^{N_{n,r}+1}(2h\sqrt{r}) + 2 \leq 3\log_2(2h\sqrt{r}).
\]

For \(\Lambda_0\), it is a 8-lacunary set,

\[
\text{(4.8)} \quad \#(\Lambda_0 \cap Q_h^r(y_1)) \leq \log_8(2h\sqrt{r}) + 2 \leq 3\log_2(2h\sqrt{r}).
\]

We have

\[
\text{(4.9)} \quad \sum_{\{n: n_1 < n \leq n_2, 2^{N_{n,r}} \leq 2h\sqrt{r}\}} \sum_{j=1}^{N_{n,r}} \#(\Lambda_{m_{n,j}} \cap Q_h^r(y_1))
\leq \sum_{\{n: n_1 < n \leq n_2, 2^{N_{n,r}} \leq 2h\sqrt{r}\}} N_{n,r} (3\log_2(2h\sqrt{d}))
\]
\[ \leq (n_2 - n_1) \cdot \log_2(2h\sqrt{r}) \cdot (3 \log_2(2h\sqrt{r})) \]
\[ \leq 12 \cdot \left( \log_2(4h\sqrt{r}) \right)^3 \quad \text{(using (4.5))}. \]

Putting (4.7), (4.8) and (4.9) into (4.6), we get
\[ \#(\Lambda' \cap Q^d_h((y_1, y_2)^t)) \leq C \cdot \left( \log_2(4h\sqrt{r}) \right)^3 \]
for some constant \( C > 0 \). As for any \( \gamma > 0 \),
\[ \lim_{h \to +\infty} \frac{\left( \log_2(4h\sqrt{r}) \right)^3}{h^\gamma} = 0, \]
we have
\[ D^+_\gamma(\Lambda') = \limsup_{h \to +\infty} \sup_{(y_1, y_2) \in \mathbb{R}^d} \frac{\#(\Lambda' \cap Q^d_h((y_1, y_2)^t))}{h^\gamma} = 0. \]
This shows \( \dim^+(\Lambda') = 0 \). \qed

5. Fourier decay

In this section, we will prove Proposition 1.4 and some results about Fourier decay of self-affine measures.

**Proof of Proposition 1.4.** As \( \mu \) is a singular measure, by [10, Proposition 2.1], we have that the lower (d-)Beurling density of \( \Lambda \) is zero. i.e.
\[ D^-(\Lambda) = \lim_{h \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\#(\Lambda \cap Q^d_h(x))}{h^d} = 0. \]
Note that this implies that for all \( h > 0 \), there exists \( \xi_h \) such that
\[ \#(Q^d_h(\xi_h) \cap \Lambda) \neq 0. \]
Indeed, if (5.1) is not true, then one can find \( h_0 > 0 \) such that \( Q^d_{h_0}(x) \cap \Lambda \neq \emptyset \) for all \( x \in \mathbb{R}^d \). But then we partition \( \mathbb{R}^d \) into disjoint union of cubes with side length \( h_0 \) and each cube has at least one element in \( \Lambda \). This implies that \( D^-(\Lambda) \geq h_0^{-1} > 0 \), a contradiction.

Now, using (5.1), for any \( k > 0 \), we can find \( \xi_k \) such that
\[ \#(Q^d_{2^k}(\xi_k) \cap \Lambda) = 0. \]
Suppose that \( D^+_\alpha(\Lambda) < \infty \). Then there exists a constant \( C' > 0 \) such that
\[ \#(\Lambda \cap Q^d_h(x)) \leq C'h^\alpha \quad \text{for all} \ x \in \mathbb{R}^d. \]
We now take \( f = e^{2\pi i (\xi_k, x)} \) into (1.3),
\[
A \leq \sum_{\lambda \in \Lambda} |\hat{\mu}(\lambda - \xi_k)|^2 = \sum_{\lambda \in \Lambda \cap (Q_{2^d_k}(\xi_k))^c} |\hat{\mu}(\lambda - \xi_k)|^2 \quad \text{(by (5.2))}
\]
\[
= \sum_{j=k}^{\infty} \sum_{\lambda \in (Q_{2^d_{j+1}}(\xi_k) \setminus Q_{2^d_j}(\xi_k)) \cap \Lambda} |\hat{\mu}(\lambda - \xi_k)|^2
\]
\[
\leq C \sum_{j=k}^{\infty} \#(\Lambda \cap (Q_{2^d_{j+1}}(\xi_k) \setminus Q_{2^d_j}(\xi_k))) 2^{-j \gamma} \quad \text{(by (1.2))}
\]
\[
\leq C' \sum_{j=k}^{\infty} (2^j + 1) 2^{-j \gamma} = C' \sum_{j=k}^{\infty} 2^{j(\alpha - \gamma)}.
\]
Suppose that \( \alpha < \gamma \). Then the right-hand side above will tend to zero as \( k \) tends to infinity which means that \( \Lambda \) cannot satisfy (1.3). Hence, \( \alpha \geq \gamma \). In particular, this implies that \( \gamma \leq \dim^+(\Lambda) \) by taking infimum of \( \alpha \) such that \( D_\alpha^+(\Lambda) < \infty \). □

Because of Proposition 1.4 and Theorem 1.3, we can show that all self-affine spectral measures we considered have Fourier dimension zero. However, much stronger can be proved easily as follows.

**Proposition 5.1.** Let \( R \in M_d(\mathbb{Z}) \) be an expansive matrix and \( B \in \mathbb{Z}^d \) be a finite set. Suppose that \( \#B < |\det(R)| \). Then the Fourier transform of self-affine measure \( \mu(R, B) \) does not decay to zero.

**Proof.** We think this result is probably well-known. We just present here for completeness. Note that if \( \#B < |\det(R)| \). The measure \( \mu = \mu(R, B) \) must be singular. By Proposition 2.4, we can find \( k \neq 0 \) and \( k \in \mathbb{Z}^d \) such that \( \hat{\mu}(k) \neq 0 \). For all integers \( n > 0 \), noting that \( B \) are all integer vectors,
\[
\hat{\mu}((R^t)^n k) = \prod_{j=1}^{\infty} \delta_B((-R^t)^{-j}((R^t)^n k)) = \prod_{j=n+1}^{\infty} \delta_B((-R^t)^{n-j} k) = \hat{\mu}(k) \neq 0.
\]
This shows that all such self-affine measures do not decay. □

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