ON THE GEOMETRY OF FLAT PSEUDO-RIEMANNIAN HOMOGENEOUS SPACES

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ABSTRACT

Let $M = \mathbb{R}^n_s/\Gamma$ be complete flat pseudo-Riemannian homogeneous manifold, $\Gamma \subset \text{Iso}(\mathbb{R}^n_s)$ its fundamental group and $G$ the Zariski closure of $\Gamma$ in $\text{Iso}(\mathbb{R}^n_s)$. We show that the $G$-orbits in $\mathbb{R}^n_s$ are affine subspaces and affinely diffeomorphic to $G$ endowed with the $(0)$-connection. If the restriction of the pseudo-scalar product on $\mathbb{R}^n_s$ to the $G$-orbits is non-degenerate, then $M$ has abelian linear holonomy. If additionally $G$ is not abelian, then $G$ contains a certain subgroup of dimension 6. In particular, for non-abelian $G$, orbits with non-degenerate metric can appear only if $\dim G \geq 6$. Moreover, we show that $\mathbb{R}^n_s$ is a trivial algebraic principal bundle $G \to M \to \mathbb{R}^{n-k}$. As a consequence, $M$ is a trivial smooth bundle $G/\Gamma \to M \to \mathbb{R}^{n-k}$ with compact fiber $G/\Gamma$.

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1. Introduction

A geodesically complete flat pseudo-Riemannian homogeneous manifold $M$ with signature $(n-s,s)$ is of the form $M = \mathbb{R}^n_s/\Gamma$, where $\mathbb{R}^n_s$ denotes $\mathbb{R}^n$ endowed with a non-degenerate symmetric bilinear form of signature $(n-s,s)$, and the fundamental group $\Gamma$ is a 2-step nilpotent unipotent group of isometries.

These spaces were first studied by Joseph Wolf in [15], and we review some of his results in Section 3. More recent studies by Oliver Baues and the author [1, 2, 5, 6] investigated among other things the holonomy groups of these spaces. For some time only $M$ with abelian linear holonomy group (given by the linear parts of $\Gamma$) were known, and it was unknown whether $M$ with non-abelian linear holonomy existed. For example, it was shown in [2] that for compact $M$ the linear holonomy group is always abelian. The first example of a non-compact $M$ with non-abelian linear holonomy in dimension $n = 14$ was given by the author in his thesis [5], see also [6].

In the present article we study the geometry of geodesically complete flat pseudo-Riemannian homogeneous spaces. This is somewhat facilitated by considering the action of the real Zariski closure $G$ of $\Gamma$ in $\text{Iso}(\mathbb{R}^n_s)$ rather than $\Gamma$ itself. The group $G$ is a unipotent algebraic group and shares many algebraic properties with $\Gamma$, as outlined in Section 3. This is essentially due to the fact that $G$ and $\Gamma$ have the same transitive centralizer.

The structure of the $G$-orbits is investigated in Section 4. These orbits are affine subspaces of $\mathbb{R}^n$ and by Proposition 4.4 are affinely diffeomorphic to $G$ endowed with the $(0)$-connection given by $\nabla_X Y = \frac{1}{2}[X,Y]$.

In Section 5 we assume the restriction of the pseudo-scalar product to the $G$-orbits to be non-degenerate. In this case, some rather strong restrictions on $G$ are imposed, as the quotients under $\Gamma$ of the $G$-orbits are themselves compact flat pseudo-Riemannian homogeneous spaces (Proposition 5.1). As aforementioned, the linear holonomy group of $M$ is then necessarily abelian. Moreover it is shown in Proposition 5.6 that if the fundamental group $\Gamma$ (or equivalently $G$) is not abelian and if the restriction of the metric to the $G$-orbits is non-degenerate, then $G$ contains $H_3 \ltimes_{\text{Ad}^*} \mathfrak{h}_3^*$ as a subgroup, where $H_3$ is the Heisenberg group and $\mathfrak{h}_3^*$ its Lie algebra dual. In particular, for non-abelian $G$, spaces with non-degenerate orbits can appear only if $\text{dim } G \geq 6$. We conclude in Corollary 5.7 that if $\mathbb{R}^n_s$ has $G$-orbits with non-degenerate metric, every flat pseudo-Riemannian homogeneous space $M = \mathbb{R}^n_s/\Gamma$ fibers into copies of $G/\Gamma$. 
which contain a pseudo-Riemannian submanifold \((H_3 \ltimes_{\Ad} h_3^*)/\Lambda\), where \(\Lambda\) is a lattice in \(H_3 \ltimes_{\Ad} h_3^*\) contained as a subgroup in \(\Gamma\). These results represent a modest first step towards a classification of those flat pseudo-Riemannian homogeneous spaces with non-degenerate orbits.

The orbits are the fibers of a principal \(G\)-bundle over the quotient \(\mathbb{R}^n/G\). As \(G\) is a unipotent algebraic group acting on the affine space \(\mathbb{R}^n\), it begs the question of whether this bundle is in fact a bundle of affine algebraic varieties.

In Section 6 we answer this question in Theorem 6.2 stating that \(\mathbb{R}^n\) is a trivial algebraic principal \(G\)-bundle \(G \to \mathbb{R}^n \to \mathbb{R}^{n-k}\), where \(k = \dim G\). As a direct consequence, \(M\) is a smooth trivial bundle \(G/\Gamma \to M \to \mathbb{R}^{n-k}\) with compact fiber \(G/\Gamma\) (Theorem 6.3). Here, the triviality is an established fact about algebraic actions of unipotent groups, outlined in Appendix A. But it is not clear a priori that the quotient \(\mathbb{R}^n/G\) exists as an affine variety. This can be seen in Proposition 6.9 by identifying \(\mathbb{R}^n\) with the algebraic homogeneous space \(U/U_p\) and relating the \(G\)-action to the action on \(U/U_p\) of a certain algebraic subgroup \(U'\) of \(U\). By a result of Rosenlicht [13], \(U/U'\) is an affine variety isomorphic to an affine space \(\mathbb{R}^{n-k}\), and it is also a quotient for the \(G\)-action on \(\mathbb{R}^n\). So \(\mathbb{R}^n/G\) indeed exists as an affine variety.

ACKNOWLEDGEMENT: I wish to thank Oliver Baues for reading an earlier version of this article and suggesting improvements to the exposition.

2. Prerequisites on Algebraic Groups and Unipotent Groups

References for this section are Borel [3, Chapters I and II] and Raghunathan [11, Preliminaries and Chapter II].

2.1. ALGEBRAIC GROUPS. A linear algebraic group \(G\) is a subgroup of \(\text{GL}_n(\mathbb{C})\) defined as the solution set of a system of polynomial equations in the matrix coefficients. In other words, \(G\) is a subgroup which is also an affine variety in \(\text{GL}_n(\mathbb{C})\). We will omit the term “linear” in the following, as all groups in question are matrix groups.

If the equations defining \(G\) consist of polynomials over a subfield \(K\) of \(\mathbb{C}\), the \(G\) is a \(K\)-defined algebraic group. The \(K\)-points of \(G\) are \(G(K) = G \cap \text{GL}_n(K)\). Here, we are interested in the case \(K = \mathbb{R}\) and refer to the \(\mathbb{R}\)-points of some algebraic group as a real algebraic group.
Example 2.1: If \( b \) is a symmetric bilinear form on \( \mathbb{R}^n \) represented by a matrix \( B \), then its linear isometry group \( O(b) \) is a real algebraic group given by \( g^\top B g = B \) which componentwise consists of real polynomial equations in the matrix coefficients of \( g \). Similarly, the affine isometry group \( \text{Iso}(b) = O(b) \ltimes \mathbb{R}^n \) is a real algebraic group, albeit as a linear subgroup of \( \text{GL}_{n+1}(\mathbb{R}) \) to accommodate the translations.

Example 2.2: If \( b \) is a symmetric bilinear form and \( H \) is any subgroup of \( \text{Iso}(b) \), then its centralizer \( Z_{\text{Iso}(b)}(H) \) is a real algebraic group. Its elements \( g \) must satisfy the equations \( gh = hg \) for all \( h \in H \), and these are componentwise real polynomial equations in the matrix coefficients of \( g \) (as a matrix in \( \text{GL}_{n+1}(\mathbb{R}) \)). Note that this does not require \( H \) to be an algebraic group itself.

For an arbitrary subgroup \( G \subset \text{GL}_n(K) \) we call the smallest algebraic group \( \overline{G} \) containing \( G \) the Zariski closure of \( G \). Then \( G \) is dense in \( \overline{G} \) in the Zariski topology. This fact often allows to extend properties of \( G \) to \( \overline{G} \) by continuity in the Zariski topology. If \( G \subset \text{GL}_n(\mathbb{R}) \), then its real Zariski closure \( \overline{G}(\mathbb{R}) \) is a real algebraic group.

In Section 6 we employ some properties of algebraic group actions: A free action of an algebraic group on an affine variety \( V \) is principal if the map \( \theta : G \times V \rightarrow V \times V \), \( (g, x) \mapsto (g.x, x) \) is an algebraic isomorphism. This amounts to saying that the map \( \beta : V \times V \rightarrow G \), \( (y, x) \mapsto g_{yx} \) with \( g_{yx}.x = y \) is a morphism (it is well-defined because the action is free).

Let \( V \) be an affine \( K \)-variety and \( G \) a \( K \)-defined algebraic group acting by \( K \)-morphisms on \( V \). An affine variety \( W \) is called a geometric quotient if there exists a quotient morphism \( \pi : V \rightarrow W \), that is, \( \pi \) is a surjective and open \( K \)-morphism, its fibers are the orbits of the \( G \)-action and the pullback by \( \pi \) induces an algebraic isomorphism \( K[\pi(U)] \cong K[U]^G \) for each open subset \( U \subseteq V \), where \( K[U]^G \) is the ring of \( G \)-invariant regular functions on \( U \) and \( K \) is the algebraic closure of \( K \) (these quotients are called “geometric” to distinguish them from a different concept of quotients called “algebraic” which is of no interest here).

2.2. Unipotent Groups. In this article we are mostly concerned with unipotent groups. These are matrix groups \( G \) in which every element \( g \in G \) is a unipotent matrix, that is, \( (g - I)^k = 0 \) holds for some \( k > 0 \), where \( I \) denotes the identity matrix. If \( G \) is unipotent, its Zariski closure and real Zariski closure (if applicable) are also unipotent by continuity.
Example 2.3: Let $G$ be a unipotent Lie group with Lie algebra $\mathfrak{g}$. Then the exponential map $\exp : \mathfrak{g} \to G$ is a polynomial diffeomorphism and thus $G$ is an algebraic group. In particular, $G$ coincides with its Zariski closure.

If $G$ is a unipotent Lie group, then there exists a connected normal Lie subgroup $H$ of codimension 1 and a subgroup $A \cong \mathbb{G}_a$ (the additive group of the field of definition) such that $G$ is the semidirect product

$$G = A \cdot H.$$  

See Onishchik and Vinberg [9, Chapter 2, Section 3.1] for details.

Applying the decomposition (2.1) repeatedly and exploiting the fact that $\exp : \mathfrak{g} \to G$ is a diffeomorphism, we find the existence of a Malcev basis $X_1, \ldots, X_k$ of $\mathfrak{g}$, which is a basis such that every $g \in G$ can be written as

$$g = g(t_1, \ldots, t_k) = \exp(t_1 X_1 + \ldots + t_k X_k)$$

for unique real parameters $t_1, \ldots, t_k$, the exponential coordinates.

If $\Gamma$ is a discrete subgroup of $G$, then $G$ is the Zariski closure of $\Gamma$ if and only if $\Gamma$ is a lattice (meaning $G/\Gamma$ is compact). In this case, the dimension of $G$ equals the rank (or Hirsch length) of $\Gamma$ (see Raghunathan [11, Theorem 2.10]).

3. Prerequisites on Flat Homogeneous Spaces

Let $\mathbb{R}_s^n$ denote the space $\mathbb{R}^n$ endowed with a non-degenerate symmetric bilinear form of signature $(n - s, s)$ and $\text{Iso}(\mathbb{R}_s^n)$ its group of isometries. We will assume $n - s \geq s$ throughout.

Affine maps of $\mathbb{R}^n$ are written as $\gamma = (I + A, v)$, where $I + A$ is the linear part ($I$ the identity matrix), and $v$ the translation part. The image space of $A$ is denoted by $im A$.

Let $M$ denote a complete flat pseudo-Riemannian homogeneous manifold. Then $M$ is of the form $M = \mathbb{R}_s^n/\Gamma$ with fundamental group $\Gamma \subset \text{Iso}(\mathbb{R}_s^n)$. In particular, $\Gamma$ acts without fixed points on $\mathbb{R}_s^n$. Homogeneity is determined by the action of the centralizer $Z_{\text{Iso}(\mathbb{R}_s^n)}(\Gamma)$ of $\Gamma$ in $\text{Iso}(\mathbb{R}_s^n)$ (see [16, Theorem 2.4.17]):

**Theorem 3.1:** Let $p : \tilde{M} \to M$ be the universal pseudo-Riemannian covering of $M$ and let $\Gamma$ be the group of deck transformations. Then $M$ is homogeneous if and only if $Z_{\text{Iso}(\tilde{M})}(\Gamma)$ acts transitively on $\tilde{M}$. 
Wolf studied subgroups \( G \subseteq \text{Iso}(\mathbb{R}^n_s) \) such that \( Z_{\text{Iso}(\mathbb{R}^n_s)}(G) \) acts transitively on \( \mathbb{R}^n_s \). To avoid repetitions we shall call them \textit{Wolf groups} throughout this article.

**Remark 3.2:** By continuity, the real Zariski closure of a Wolf group \( G \) is also a Wolf group. Moreover, \( G \) is abelian if and only if its Zariski closure is.

**Proposition 3.3:** \( G \) acts freely and properly on \( \mathbb{R}^n_s \).

**Proof.** \( G \) acts freely because its centralizer acts transitively. Properness was proved in a previous article by Baues and Globke [2, Proposition 7.2].

For the following properties see Wolf’s book [16, Chapter 3]:

**Lemma 3.4:** \( G \) consists of affine transformations \( g = (I + A, v) \), where \( A^2 = 0 \), \( v \perp \text{im} \, A \) and \( \text{im} \, A \) is totally isotropic.

**Lemma 3.5:** For \( g_i = (I + A_i, v_i) \in G \), we have \( A_1 A_2 = -A_2 A_1, \ A_1 v_2 = -A_2 v_1, \ A_1 A_2 A_3 = 0 \) and \( [g_1, g_2] = (I + 2A_1 A_2, 2A_1 v_2) \).

**Lemma 3.6:** If \( g = (I + A, v) \in G \), then \( \langle Ax, y \rangle = -\langle x, Ay \rangle \), \( \text{im} \, A = (\ker \, A)^\perp \), \( \ker \, A = (\text{im} \, A)^\perp \) and \( Av = 0 \).

**Remark 3.7:** Let \( g = (I + A, v) \in G \) and \( X = (A, v) \). It follows from \( A^2 = 0 \) and \( Av = 0 \) that \( X^2 = (A, v)^2 = 0 \), so the elements of \( G \) are unipotent. Moreover,

\[
\exp(X) = \exp(A, v) = (I + A, v) = g.
\]

**Theorem 3.8:** \( G \) is 2-step nilpotent (meaning \( [G, [G, G]] = \{\text{id}\} \)) and unipotent.

If \( \Gamma \) is the fundamental group of \( M \), it also determines the holonomy of \( M \): For \( \gamma = (I + A, v) \in \Gamma \), set \( \text{Hol}(\gamma) = I + A \) (the linear component of \( \gamma \)). We write \( A = \log(\text{Hol}(\gamma)) \). The \textit{linear holonomy group} of \( \Gamma \) (or of \( M \)) is

\[
\text{Hol}(\Gamma) = \{\text{Hol}(\gamma) \mid \gamma \in \Gamma\}.
\]

This name is justified by the following observation: Let \( x \in M \) and \( \gamma \in \pi_1(M, x) \) be a loop. Then \( \text{Hol}(\gamma) \) corresponds to the parallel transport \( \tau_x(\gamma) : T_xM \to T_xM \) in a natural way, see [16, Lemma 3.4.4].
4. The Affine Structure on the Orbits

Let $G \subset \text{Iso}(\mathbb{R}_+^n)$ be a Zariski-closed Wolf group of dimension $k$ and $\mathfrak{g}$ its Lie algebra. We study the affine structure on the orbits $F_p = G.p$ of the $G$-action on $\mathbb{R}^n$. The orbit map at $p$ is $\theta : G \to F_p, g \mapsto g.p$.

**Proposition 4.1:** Let $X_1, \ldots, X_k$ be a Malcev basis of $\mathfrak{g}$, with $X_i = (A_i, v_i)$. For every $p \in \mathbb{R}^n$, set $b_i(p) = A_i p + v_i$. Then the orbit $F_p$ is the affine subspace

\[
F_p = p + \text{span}\{b_1(p), \ldots, b_k(p)\}
\]
of dimension $\dim F_p = k$.

**Proof.** By (3.1), $\exp(X) = \exp(A, v) = (I + A, v)$ for all $X = (A, v) \in \mathfrak{g}$.

Now let $t_1, \ldots, t_k$ be the exponential coordinates (2.2) for $G$. Then

\[
g(t_1, \ldots, t_k).p = p + t_1(A_1 p + v_1) + \ldots + t_k(A_k p + v_k)
= p + t_1 b_1(p) + \ldots + t_k b_k(p)
\]
parameterizes an affine subspace in $\mathbb{R}^n$. The assertion on the dimension is standard, taking into account that $G$ acts freely. 

Since $G$ acts freely, the natural affine connection $\nabla$ on the affine space $F_p$ pulls back to a flat affine connection $\nabla$ on $G$ through the orbit map.

**Remark 4.2:** Because $X^2 = 0$ for all $X \in \mathfrak{g} \subset \text{Mat}_{n+1}(\mathbb{R})$, $\exp(X) = I + X$. So $G = I + \mathfrak{g}$ is an affine subspace of $\text{Mat}_{n+1}(\mathbb{R})$ which therefore has a natural affine connection $\nabla^G$. This connection is left-invariant because left-multiplication is linear on $\text{Mat}_{n+1}(\mathbb{R})$. The orbit map $\theta : G \to F_p, I + X \mapsto (I + X).p$ is an affine map (if one chooses $I \in G$ and $p \in F_p$ as origins, the linear part of $\theta$ is $X \mapsto X \cdot p$ and the translation part is $+p$). It is also a diffeomorphism onto $F_p$ because the action is free and $\exp$ is a diffeomorphism.

From the above we immediately obtain:

**Corollary 4.3:** $(G, \nabla^G)$ is affinely diffeomorphic to $(G, \nabla)$.

It follows from Lemma 3.5 and (3.1) that $XY = \frac{1}{2}[X, Y]$ for all $X, Y \in \mathfrak{g}$. There exists a bi-invariant flat affine connection $\tilde{\nabla}$ on $G$ given by

\[
(\tilde{\nabla}_X Y)_g = \frac{1}{2}[X, Y]_g = gX Y,
\]

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\]

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(\tilde{\nabla}_X Y)_g = \frac{1}{2}[X, Y]_g = gX Y,
\]
where $X, Y$ are left-invariant vector fields on $G$ and $X_I, Y_I \in \mathfrak{g}$ their respective values at the identity $I$. In fact, $\tilde{\nabla}$ is bi-invariant because $[X, Y]$ is Ad($G$)-invariant, and it is flat because $\mathfrak{g}$ is 2-step nilpotent. The connection $\tilde{\nabla}$ is called the $(0)$-connection on $G$.

**Proposition 4.4:** The $(0)$-connection $\tilde{\nabla}$ on $G$ coincides with the flat affine connection $\nabla$ on $G$.

**Proof.** As both connections are left-invariant, it suffices to show that they coincide on left-invariant vector fields. Expressed in matrix terms, left-invariance for vector fields means $X_g = gX_I$ for all $X \in \mathfrak{g}, g \in G$. So for all $X, Y \in \mathfrak{g}$,

$$(\nabla_X Y)_g = \lim_{t \to 0} \frac{Y_{g \exp(tX)} - Y_g}{t} = \lim_{t \to 0} \frac{g(I + tX_I)Y_I - gY_I}{t} = \lim_{t \to 0} tX_IY_I = gX_IY_I$$

where the first and last equality hold by definition. \hfill \blacksquare

The metric $\langle \cdot, \cdot \rangle$ on the orbit $F_p$ pulls back to a field $(\cdot, \cdot)$ of (possibly degenerate) left-invariant symmetric bilinear forms on $G$ which is parallel with respect to $\nabla$. By abuse of language we call $(\cdot, \cdot)$ the orbit metric on $G$. Since all orbits are isometric, the pair $(\nabla, (\cdot, \cdot))$ does not depend on $p$ and is an invariant of $G \subset \text{Iso}(\mathbb{R}_x^n)$.

**Proposition 4.5:** The orbit metric $(\cdot, \cdot)$ is bi-invariant on $G$, that is

$$([X, Y], Z) = -(Y, [X, Z])$$

for all left-invariant vector fields $X, Y, Z$.

If $(\cdot, \cdot)$ is non-degenerate this is well-known (O’Neill [8, Proposition 11.9]).

**Proof.** Let $X, Y, Z$ be a left-invariant vector fields on $G$. Fix $g \in G$ and let $X^+, Y^+, Z^+$ be the (right-invariant) Killing fields on $G$ such that $X_g^+ = X_g, Y_g^+ = Y_g, Z_g^+ = Z_g$. The flow $\psi_t(h)$ of $X^+$ at $h \in G$ is given by

$$\psi_t(h) = \exp(tgX_Ig^{-1})h = L_{\exp(tgX_Ig^{-1})}(h),$$

in particular $\psi_t(g) = g \exp(tX_I)$. Because $\exp(-gX_Ig^{-1})g \exp(X_I) = g$ and $Y$ is left-invariant,

$$[X^+, Y]_g = \frac{d}{dt} \bigg|_{t=0} d\psi_t Y_{\psi_t(g)} = \frac{d}{dt} \bigg|_{t=0} dL_{\exp(-tX_Ig^{-1})} Y_g \exp(tX_I) = \frac{d}{dt} \bigg|_{t=0} Y_g = 0.$$
This implies
\[(\nabla_X + Y)_g = (\nabla_Y X^+)_g.\]
\(\nabla_X Y\) is tensorial in \(X\), so
\[(\nabla_X Y)_g = (\nabla_X Y)_g = (\nabla_Y X^+)_g = (\nabla_Y + X^+)_g.\]
\(X^+, Y^+\) are pullbacks of Killing fields on \(\mathbb{R}_n^a\) restricted to \(F\), so [1, Proposition 3.10 (1)] gives \(\nabla_{Y^+}X^+ = -\nabla_{X^+}Y^+\). Then
\[(\nabla_X Y)_g = -(\nabla_X Y)_g.\]
Now it follows from (4.2) and the computations above that
\[(\frac{1}{2}[X, Y], Z)_g + (Y, \frac{1}{2}[X, Z])_g = (\nabla_X Y, Z^+)_g + (Y^+, \nabla_X Z)_g\]
\[= (-\nabla_Y X, Z^+)_g + (Y^+, -\nabla_Z X)_g\]
\[= (\nabla_Y + X^+, Z^+)_g + (Y^+, \nabla_Z + X^+_X)_g\]
\[= 0.\]
\((\cdot, \cdot)\) is a tensor, so we can replace \(Z\) by \(Z^+\) and \(Y\) by \(Y^+\) in the first line. The last equality holds because \(\nabla X^+\) is skew-symmetric with respect to \((\cdot, \cdot)\) for a Killing field \(X^+\) (see [1, Subsection 3.4.1]).

As \(g\) was arbitrary, \([(X, Y], Z) = -(Y, [X, Z])\) holds everywhere. So \((\cdot, \cdot)\) is bi-invariant.

Remark 4.6: The orbit metric \((\cdot, \cdot)\) on \(G\) induces a symmetric bilinear form \((\cdot, \cdot)\) on \(\mathfrak{g}\) which satisfies \([(X, Y], Z) = -(Y, [X, Z])\) for all \(X, Y, Z \in \mathfrak{g}\). Such a form is called invariant. The radical \((\cdot, \cdot)\) in \(\mathfrak{g}\) is the subspace
\[\mathfrak{r} = \{X \in \mathfrak{g} \mid (X, g) = \{0\}\}.\]
The radical \(\mathfrak{r}\) is an ideal in \(\mathfrak{g}\) due to the invariance of \((\cdot, \cdot)\).

Lemma 4.7: The commutator subalgebra \([\mathfrak{g}, \mathfrak{g}]\) is a totally isotropic subspace of \(\mathfrak{g}\) with respect to \((\cdot, \cdot)\). The center \(\mathfrak{z}(\mathfrak{g})\) is orthogonal to \([\mathfrak{g}, \mathfrak{g}]\).

Proof. \(\mathfrak{g}\) is 2-step nilpotent. So
\[([X_1, X_2], [X_3, X_4]) = -(X_2, [X_1, [X_3, X_4]]) = -(X_2, 0) = 0\]
for all \(X_i \in \mathfrak{g}\). If \(Z \in \mathfrak{z}(\mathfrak{g})\), then \((Z, [X_1, X_2]) = -([X_1, Z], X_2) = 0.\]

Corollary 4.8: Assume there exists \(Z \in [\mathfrak{g}, \mathfrak{g}]\) and \(Z^* \in \mathfrak{g}\) such that \((Z, Z^*) \neq 0\). Then \(Z^* \not\in \mathfrak{z}(\mathfrak{g})\).
Lemma 4.9: If \( Z = [X,Y] \), then \( Z \perp \text{span}\{X,Y,Z\} \).

Proof. Use invariance and 2-step nilpotency.

Example 4.10: Let \( \Gamma \cong H_3(\mathbb{Z}) \) with Zariski closure \( G \cong H_3 \) (the discrete and real Heisenberg groups, respectively). Assume \( M = \mathbb{R}^n_s/\Gamma \) is a flat pseudo-Riemannian homogeneous manifold. The orbit metric induced on \( G \) is degenerate, and \( \mathfrak{z}(\mathfrak{g}) \subset \mathfrak{r} \): Let \( X,Y \) denote the Lie algebra generators of \( \mathfrak{h}_3 \) and \( Z = [X,Y] \). By Lemma 4.9 \( Z \in \mathfrak{r} \). So the non-degenerate case is excluded.

The possible signatures are \((0,0,3), (1,0,2), (1,1,1)\) and \((2,0,1)\).

Remark 4.11: One easily realizes the signatures in the above example by modifying the metric in [2, Example 6.4] accordingly.

5. Non-Degenerate Orbits

As before, let \( M = \mathbb{R}^n_s/\Gamma \) and \( G \) the Zariski closure of \( \Gamma \) with Lie algebra \( \mathfrak{g} \). If \( \mathfrak{g} \) is not abelian and the orbit metric \((\cdot,\cdot)\) is non-degenerate then there are some strong constraints on the structure of \( \mathfrak{g} \).

Proposition 5.1: If the orbit metric on \( G \) is non-degenerate, then the quotient \( F_p/\Gamma \) of each orbit is itself a compact flat pseudo-Riemannian homogeneous space. In particular, the linear holonomy group of \( M \) is abelian.

Proof. The orbits \( F_p \) are affine subspaces of \( \mathbb{R}^n \) and isometric to \( G \). So \( F_p/\Gamma = G/\Gamma \) is a compact flat pseudo-Riemannian space. Moreover, in Proposition 6.6 we will see that a group centralizing \( \Gamma \) acts transitively on \( F_p \), so \( F_p/\Gamma \) is homogeneous. It was shown in [2, Corollary 3.3] that the linear holonomy of \( G \) is abelian.

Additionally, \( \mathfrak{g} \) must contain a subalgebra of a certain type.

Definition 5.2: A butterfly algebra \( \mathfrak{b}_6 \) is a 2-step nilpotent Lie algebra of dimension 6 endowed with an invariant bilinear form \((\cdot,\cdot)\) such that there exists \( Z \in [\mathfrak{b}_6,\mathfrak{b}_6] \) with \( Z \notin \mathfrak{r} \).

The naming in Definition 5.2 will become clear after the proof of the following proposition:
**Proposition 5.3:** A butterfly algebra $\mathfrak{b}_6$ admits a vector space decomposition

$$\mathfrak{b}_6 = \mathfrak{v} \oplus [\mathfrak{b}_6, \mathfrak{b}_6],$$

where the subspaces $\mathfrak{v}$ and $[\mathfrak{b}_6, \mathfrak{b}_6]$ are totally isotropic and dual to each other. In particular, $(\cdot, \cdot)$ is non-degenerate of signature $(3,3)$.

**Proof.** Let $X, Y \in \mathfrak{b}_6$ such that $Z = [X, Y] \neq 0$. By Lemma 4.7 $[\mathfrak{b}_6, \mathfrak{b}_6]$ is totally isotropic. By assumption there exists $Z^* \in \mathfrak{b}_6 \setminus [\mathfrak{b}_6, \mathfrak{b}_6]$ such that

$$(Z, Z^*) = 1.$$ 

As a consequence of Lemma 4.9, $X, Y, Z^*$ are linearly independent, so they span a 3-dimensional subspace $\mathfrak{v}$. Since $(\cdot, \cdot)$ is invariant,

$$1 = ([X, Y], Z^*) = (Y, [Z^*, X]),$$

$$1 = ([X, Y], Z^*) = (X, [Z^*, Y]).$$

Set $X^* = [Z^*, Y]$ and $Y^* = [Z^*, X]$. Lemma 4.9 further implies that $X^*, Y^*, Z$ are linearly independent, hence span a 3-dimensional subspace $\mathfrak{w}$ of $[\mathfrak{b}_6, \mathfrak{b}_6]$.

Since $\mathfrak{b}_6$ is 2-step nilpotent $[\mathfrak{b}_6, \mathfrak{b}_6] \subset \mathfrak{z}(\mathfrak{b}_6)$. But $\mathfrak{v} \cap \mathfrak{z}(\mathfrak{b}_6) = \{0\}$, so it follows from dimension reasons that $\mathfrak{w} = [\mathfrak{b}_6, \mathfrak{b}_6] = \mathfrak{z}(\mathfrak{b}_6)$. By construction also

$$X \perp \text{span}\{Y^*, Z\}, \quad Y \perp \text{span}\{X^*, Z\}.$$ 

After a base change we may assume that $\mathfrak{v}$ is a dual space to $[\mathfrak{b}_6, \mathfrak{b}_6]$. 

The bases $\{X, Y, Z^*\}$ and $\{X^*, Y^*, Z\}$ from the proof above are dual bases to each other. The following diagram describes the relations between these bases, where solid lines from two elements indicate a commutator and dashed lines indicate duality between the corresponding elements:

![Diagram](https://via.placeholder.com/150)

This explains the name. In particular, the following corollary justifies to speak of “the” butterfly algebra.\footnote{Some people misguided believe that the name bat algebra would be more apt.}
Corollary 5.4: Any two butterfly algebras are isometric and isomorphic as Lie algebras.

Proof. In every butterfly algebra one can find a basis as in the proof of Proposition 5.3. Mapping such a basis of one butterfly algebra to the corresponding elements of another yields an isometric Lie algebra isomorphism.

From the above we conclude:

Corollary 5.5: Let $h_3$ denote the Heisenberg algebra and $\text{ad}^*$ its coadjoint representation. Up to isometric isomorphism, the butterfly algebra is

\begin{equation}
b_6 = h_3 \oplus \text{ad}^* h_3^*
\end{equation}

with $(\cdot, \cdot)$ given by

\begin{equation}
(X + X^*, Y + Y^*) = X^*(Y) + Y^*(X)
\end{equation}

for $X, Y \in h_3$, $X^*, Y^* \in h_3$.

So $b_6$ is the Lie algebra of the Lie group $B_6 = H_3 \ltimes \text{Ad}^* h_3^*$, where $H_3$ denotes the Heisenberg group. For a lattice $\Lambda$ in $B_6$ denote the space $B_6/\Lambda$ by $M_6(\Lambda)$. Baues [1, Example 4.3] gave the spaces $M_6(\Lambda)$ as a class of compact homogeneous pseudo-Riemannian manifolds of dimension 6 with non-abelian fundamental group and noted that these are the only possible examples. This is the minimal dimension for such examples with non-abelian fundamental group according to [1, Corollary 4.9]. We can rederive this result as a consequence of the following:

Proposition 5.6: If $G$ is not abelian and the orbit metric on $G$ is non-degenerate, then $G$ contains a butterfly subgroup $B \cong B_6$. In particular, $\dim G \geq 6$. Moreover, $B$ can be chosen such that it contains a lattice $\Lambda = B \cap \Gamma$.

Proof. Let $\mathfrak{g}$ denote the Lie algebra of $G$ and let $\mathfrak{g}_\Gamma = \log(\Gamma)$ denote the discrete subset of $\mathfrak{g}$ which maps to $\Gamma$ under the exponential map.

There exists a Malcev basis of $\mathfrak{g}$ contained in $\mathfrak{g}_\Gamma$ and in particular, as $\mathfrak{g}$ is not abelian, we can find $X, Y, Z \in \mathfrak{g}_\Gamma$ such that $[X, Y] = Z \neq 0$. This is evident from the construction of a Malcev basis strongly based on $\Gamma$ in Corwin and Greenleaf’s proof of [1, Theorem 5.1.6].

Recall that $\mathfrak{g}$ is isometric to the $G$-orbits when endowed with the metric $(\cdot, \cdot)$ induced by the orbit metric (Remark 4.6). Thus there exists $Z^* \in \mathfrak{g}_\Gamma$ such that
(Z, Z*) \neq 0$, otherwise the Malcev basis would span a space orthogonal to $Z$, which contradicts the non-degeneracy of $(\cdot, \cdot)$. By Corollary 4.8, $Z^* \not\in \mathfrak{z}(\mathfrak{g})$.

We can then complete this to a basis $\{X, Y, Z, X^*, Y^*, Z^*\}$ of a butterfly subalgebra as in the proof of Proposition 5.3. Here, $X^*, Y^*$ are elements of $\mathfrak{g}_\Gamma$ because they arise as commutators of elements of $\mathfrak{g}_\Gamma$ and for 2-step nilpotent Lie groups $\exp([X, Y]) = [\exp(X), \exp(Y)]$ holds by the Baker-Campbell-Hausdorff formula, so $\mathfrak{g}_\Gamma$ is closed under commutators.

Now define $\Lambda$ as the discrete subgroup generated by the exponentials of the basis above. By construction $\Lambda \subseteq \Gamma$ and the Zariski closure of $\Lambda$ is a butterfly subgroup $B \subseteq G$.

**Corollary 5.7:** Under the assumptions of Proposition 5.6, $M$ is the disjoint union of compact manifolds $G/\Gamma$, each of which contains a submanifold isometric to $M_6(\Lambda)$ for a certain lattice $\Lambda$ in $B_6$.

**Proof.** $\mathbb{R}^n$ is the union of disjoint $G$-orbits $F_p$, each isometric to $G$. Hence $M$ is the disjoint union of the $F_p/\Gamma$ which are isometric to $G/\Gamma$. They are compact, as $G$ is the Zariski closure of $\Gamma$ (see Raghunathan [11, Theorem 2.1]).

If $\Lambda$ is the lattice in the butterfly subgroup $B$ from Proposition 5.6 then $B$ is isometric to $B_6$ and

$$B/(\Gamma \cap B) = B/\Lambda = M_6(\Lambda).$$

As $B$ is isometric to an affine subspace $B.p \subseteq F_p$, it follows that $F_p/\Gamma$ contains a submanifold (isometric to) $M_6(\Lambda)$.

In Section 6 we investigate the structure of $M$ as a fiber bundle with compact fiber $G/\Gamma$.

**Remark 5.8:** The concept of a butterfly algebra can be generalized to higher dimensions by setting

$$\mathfrak{b}_{n,\omega} = \mathfrak{n} \oplus_\omega \mathfrak{n}^*,$$

where $\mathfrak{n}$ is a $k$-dimensional 2-step nilpotent Lie algebra and $\omega$ a 2-cocycle for the coadjoint action ([11 Section 5.3]), and defining a pseudo-scalar product as in (5.2). Let $B_{n,\omega}$ denote a simply connected Lie group with Lie algebra $\mathfrak{b}_{n,\omega}$. It is known that every compact flat pseudo-Riemannian homogeneous space $M$ of split signature $(k, k)$ can be realized as a quotient $B_{a,\omega}/\Gamma$, where $a$ is an abelian Lie algebra (see Baues and Globke [2 Section 3]). An interesting open question along these lines is whether a compact flat pseudo-Riemannian homogeneous
space $M$ with split signature $(k, k)$ and non-abelian fundamental group $\Gamma$ can be realized as $B_{n,\omega}/\Gamma$, where $n$ is some non-abelian 2-step nilpotent Lie algebra.

6. A Trivial Bundle

Let $G$ be an algebraic Wolf group and let $L$ denote its centralizer in $\text{Iso}(\mathbb{R}^n_s)$. In this section we will be mostly concerned with the properties of $G$ as an algebraic group acting on the affine space $\mathbb{R}^n$, so we drop the index $s$ from $\mathbb{R}^n_s$.

**Lemma 6.1:** The $G$-action on $\mathbb{R}^n$ is principal (as defined in Section 2).

**Proof.** As outlined in Remark 4.2, the orbit map $\theta_p : G \to F_p$, $g \mapsto g.p$, is an affine map in $g$.

If we express $g = g(t_1, \ldots, t_k)$ in exponential coordinates, then for fixed $p, q$ in the same $G$-orbit the equation $g(t_1, \ldots, t_k).p = q$ forms an inhomogeneous system of linear equations with unknowns $t_1, \ldots, t_k$. It has a unique solution because the action is free. It is well-known from linear algebra that the solution to such a system can be expressed by expressions polynomial in the components of $p$ and $q$. Hence the map $\beta(q, p) = gqp$ with $gqp.p = q$ is a morphism and the $G$-action is principal.

**Theorem 6.2:** Assume that $L = Z_{\text{Iso}(\mathbb{R}^n_s)}(G)$ acts transitively on $\mathbb{R}^n$. The orbit space $\mathbb{R}^n/G$ is isomorphic to $\mathbb{R}^{n-k}$ as an affine algebraic variety, and there exists an algebraic cross section $\sigma : \mathbb{R}^n/G \to \mathbb{R}^n$. In particular, $\mathbb{R}^n$ is a trivial algebraic principal $G$-bundle

\begin{equation}
G \to \mathbb{R}^n \to \mathbb{R}^{n-k}.
\end{equation}

**Proof.** $\mathbb{R}^n/G$ exists as an affine variety and as such is isomorphic to $\mathbb{R}^{n-k}$ by Proposition 6.9 below. We show $\pi : \mathbb{R}^n \to R^n/G \cong \mathbb{R}^{n-k}$ is a principal bundle for $G$: $\mathbb{R}^n$ and $\mathbb{R}^{n-k}$ are smooth (hence normal) varieties. As a consequence of homogeneity and Rosenlicht [12 Theorem 10], $\mathbb{R}^{n-k}$ can be covered by Zariski-open sets $W$ such that on each $W$ there exists a local algebraic cross section $\sigma_W : W \to \mathbb{R}^n$, and $\pi$ is a locally trivial fibration.

The $G$-action is principal, so for any $p \in \mathbb{R}^n$ and $g \in G$, the map $\beta(g.p, p) = g$ defined on the graph of the action is a morphism. Thus the bundle’s coordinate changes by $G$ are morphisms and the bundle is algebraic. The existence of an algebraic cross section $\sigma : \mathbb{R}^n/G \to \mathbb{R}^n$ now follows from Theorem A.3. 


If \( G \) is the real Zariski closure of the fundamental group \( \Gamma \) of a flat pseudo-Riemannian homogeneous space \( M \), then \( G \) is an algebraic Wolf group and we can apply Theorem 6.2 to its action on \( \mathbb{R}^n \). We can then take the quotient for the action of \( \Gamma \) and obtain the following:

**Theorem 6.3:** Let \( M = \mathbb{R}_s^n / \Gamma \) be a complete flat pseudo-Riemannian homogeneous manifold and \( k = \text{rank} \Gamma \). Let \( G \) denote the real Zariski closure of \( \Gamma \). Then \( M \) is a trivial fiber bundle

\[
(6.2) \quad G / \Gamma \rightarrow M \rightarrow \mathbb{R}^{n-k}
\]

with structure group \( G / \Gamma_n \), where \( \Gamma_n \) is the largest subgroup of \( \Gamma \) which is a normal subgroup of \( G \).

For the assertion on the structure group see Steenrod [14, Theorem 7.4].

In order to complete the proof of Theorem 6.2 it remains to show that \( \mathbb{R}^n / G \) is in fact isomorphic to \( \mathbb{R}^{n-k} \) as an affine variety (Proposition 6.9). To this end, we use a result due to Rosenlicht [13, Theorem 5] which states that any algebraic homogeneous space for \( \mathbb{R} \)-defined unipotent groups is isomorphic to some \( \mathbb{R}^m \) as an affine variety. So we need to identify \( \mathbb{R}^n / G \) with a homogeneous space \( U / U' \) for some unipotent algebraic group \( U \) with algebraic subgroup \( U' \) of \( \mathbb{R}_s^n \). Choose \( U \) to be the unipotent radical of \( L \). The subgroup \( U' \) will be the group defined in (6.3) below.

Because the reductive part of \( L \) always has a fixed point (Baues [1, Lemma 2.2]), transitivity of \( L \) depends on \( U \):

**Proposition 6.4:** The centralizer \( L \) acts transitively on \( \mathbb{R}^n \) if and only if its unipotent radical \( U \subset L \) acts transitively.

In the following, we assume that \( L \) (hence \( U \)) acts transitively. For \( p \in \mathbb{R}^n \), its stabilizer \( U_p \) is an algebraic subgroup of \( U \).

**Proposition 6.5:** The quotient \( U / U_p \) is isomorphic to \( \mathbb{R}^n \) as an affine variety.

**Proof.** The quotient \( U / U_p \) is an affine variety because \( U \) is unipotent (see Borel [3, Corollary 6.9]). As \( \mathbb{R}^n \) is a geometric quotient for the action of \( U_p \) on \( U \), it is isomorphic to \( U / U_p \) as an affine variety. \( \blacksquare \)
Let \( F_p \) denote the \( G \)-orbit through \( p \). It is helpful to relate the action of \( G \) on \( \mathbb{R}^n \) to the right-action on \( U/U_p \) by the group

\[
U_{F_p} = \{ u \in U \mid u.F_p \subseteq F_p \}.
\]

**Proposition 6.6:** \( U_{F_p} \) is an algebraic subgroup of \( U \) acting transitively on \( F_p \), and \( U_p \) is a normal subgroup of \( U_{F_p} \).

**Proof.** \( U_{F_p} \) acts transitively on \( F_p \) because \( U \) acts transitively. The orbit \( F_p \) is Zariski-closed because \( G \) is unipotent by Lemma 3.4. So \( U_{F_p} \) is an algebraic group as it is the preimage of \( F_p \) under the orbit map \( U \to \mathbb{R}^n, u \mapsto u.p \).

As \( U_p \) commutes with \( G \), it fixes every point of \( F_p \). Hence it is invariant under conjugation with \( U_{F_p} \).

Fix an element \( p \in \mathbb{R}^n \) and let \( \tilde{U} = U_{F_p}/U_p \). The \( G \)-action is free, so for \( uU_p \in \tilde{U} \) there exists a unique element \( g_u \in G \) satisfying \( u.p = g_u.p \). Then the map

\[
\Phi : \tilde{U} \to G, \quad uU_p \mapsto g_u^{-1}
\]

is an isomorphism of algebraic groups.

By Proposition 6.5 there exists an isomorphism of affine varieties

\[
\Psi : U/U_p \to \mathbb{R}^n, \quad uU_p \mapsto u.p.
\]

**Lemma 6.7:** \( \Psi \) induces a bijection from the orbits of the right-action of \( \tilde{U} \) on \( U/U_p \) to the orbits of \( G \) on \( \mathbb{R}^n \).

**Proof.** Let \( \tilde{u} \in \tilde{U} \) and \( g = \Phi(\tilde{u}) \). For any \( u \in U \), \( \Psi(uU_p) = u.p \in \mathbb{R}^n \), and \( uU_p.\tilde{u} = u\tilde{u}U_p \) maps to \( \Psi(u\tilde{u}U_p) = u\tilde{u}.p \). By definition of \( \Phi \) and because \( U \) centralizes \( G \),

\[
u\tilde{u}.p = ug^{-1}.p = g^{-1}u.p.
\]

So \( \Psi \) maps the orbit \( uU_p.\tilde{U} \) to the orbit \( G.(u.p) \).

**Remark 6.8:** Note that \( \Psi \) is not equivariant with respect to the action of \( \tilde{U} = G \), but rather anti-equivariant. By this we mean that \( \Psi(\overline{u}.\tilde{u}) = \Phi(\tilde{u})^{-1}.\Psi(\overline{u}) \) holds for all \( \tilde{u} \in \tilde{U} \) and \( \overline{u} \in U/U_p \).

**Proposition 6.9:** \( U/U_{F_p} \) is a geometric quotient for the action of \( G \) on \( \mathbb{R}^n \) and isomorphic to \( \mathbb{R}^{n-k} \) as an affine algebraic variety.

We will write \( \mathbb{R}^n/G \) for the quotient \( U/U_{F_p} \).
Proof. $U/U_{F_p}$ is an algebraic homogeneous space for a unipotent group. By a theorem of Rosenlicht [13, Theorem 5], the quotient $U/U_{F_p}$ of $\mathbb{R}$-defined unipotent groups is algebraically isomorphic to an affine space $\mathbb{R}^m$. Moreover, $U/U_{F_p} = (U/U_p)/(U_{F_p}/U_p) = (U/U_p)/\tilde{U}$, so

$$\dim U/U_{F_p} = \dim U/U_p - \dim \tilde{U} = \dim \mathbb{R}^n - \dim G = n - k.$$ 

Let $\pi_0 : U/U_p \to U/U_{F_p}$ denote the quotient map. So we have morphisms

$$\begin{array}{cccc}
\Psi^{-1} & \mathbb{R}^n & \downarrow \pi & \to U/U_{F_p} \\
\pi_0 & \downarrow \pi & \quad & \quad \\
U/U_p & \quad & \quad & \quad \\
\mathbb{R}^{n-k} & \quad & \quad & \quad
\end{array}$$

where we define $\pi = \pi_0 \circ \Psi^{-1}$. Since $\Psi$ is an isomorphism and $\pi_0$ a quotient map, the map $\pi$ is a surjective open morphism. So $\pi$ is a quotient map by [3, Lemma 6.2]. Let $\bar{q} = uU_{F_p} \in U/U_{F_p}$, where $q = u.p$. Then the fiber of $\pi$ over $\bar{q}$ is

$$\pi^{-1}(\bar{q}) = \Psi(\pi_0^{-1}(\bar{q})) = \Psi(uU_pU_{F_p}) = G.(u.p) = F_q,$$

the orbit of $G$ through $q$ (use Lemma 6.7 for the third equality). Hence $U/U_{F_p}$ is a geometric quotient for the $G$-action. 

\appendix

\section*{Appendix A. Cross Sections for Unipotent Group Actions}

In this appendix we show that an affine algebraic principal bundle for a unipotent algebraic group is a trivial bundle. This result is known, see for example Kraft and Schwarz [7, Proposition IV.3.4], but we give a proof for the reader’s convenience and to ensure we can take all cross sections to be defined over the real numbers.

In the following, an \textit{algebraic principal bundle} will mean a $G$-principal bundle $\pi : V \to W$ where $V$ and $W$ are smooth affine varieties, $G$ is an algebraic group acting principally on $V$ (as defined in Section 2) such that the bundle’s coordinate changes are algebraic maps. For the applications in this article, we assume all varieties and morphisms to be defined over $\mathbb{R}$ (see also Remark A.4).
Lemma A.1: Let \( V, W \) be smooth affine varieties and \( \pi : V \to W \) an algebraic principal bundle for a unipotent action of the additive group \( G_a \). Then there exists an algebraic cross section \( \sigma : W \to V \).

Proof. Because \( W \) is affine, it can be covered by a finite system \( (U_i)_{i=1}^m \) of dense open subsets admitting local cross sections \( \sigma_i : U_i \to V \).

The action of \( G_a \) is principal, which means the map \( \beta(g.p, p) = g \) defined on the graph of the action is a morphism (Borel [3, 1.8]). But \( G_a = (\mathbb{R}, +) \), so \( \beta \) is in fact a regular real function on its domain of definition. Hence we can define regular real functions \( \beta_{ij} \) on each \( U_{ij} = U_i \cap U_j \) by

\[
\beta_{ij} : U_i \cap U_j \to \mathbb{R}, \quad p \mapsto \beta(\sigma_i(p), \sigma_j(p))
\]

satisfying

\[
\sigma_i|_{U_{ij}}(p) = \beta_{ij}(p).\sigma_j|_{U_{ij}}(p).
\]

These \( \beta_{ij} \) form a 1-cocycle in the Čech cohomology of the sheaf \( \mathcal{O}_W \) of \( \mathbb{C} \)-valued regular functions on \( W \). As \( W \) is affine, its first Čech cohomology group for \( \mathcal{O}_W \) vanishes (see for example Perrin [10, Chapter VII, Theorem 2.5]). So there exist \( \mathbb{C} \)-valued regular functions \( \alpha_i \) defined on \( U_i \) such that

\[
\beta_{ij} = \alpha_i|_{U_{ij}} - \alpha_j|_{U_{ij}}.
\]

The maps

\[
p \mapsto -\alpha_i(p).\sigma_i(p), \quad p \mapsto -\alpha_j(p).\sigma_j(p)
\]

are local cross sections defined on \( U_i, U_j \), respectively, which coincide on the open set \( U_{ij} \). By continuity, they define a morphism

\[
\sigma^{ij} : U_i \cup U_j \to V
\]

such that \( \pi \circ \sigma^{ij} \) coincides with the identity on the open subset \( U_i \subset U_i \cup U_j \). Hence \( \pi \circ \sigma^{ij} = \text{id}_{U_i \cup U_j} \) and \( \sigma^{ij} \) is a local cross section defined on \( U_i \cup U_j \).

Finitely many repetitions of this yield a global cross section \( \sigma : W \to V \).

Lemma A.2: Let \( V, W \) be smooth affine varieties and \( \pi : V \to W \) an algebraic principal \( G \)-bundle for a unipotent algebraic group \( G \). Let \( H, A \subset G \) be as in (2.1). Then \( V \) is also an algebraic principal \( H \)-bundle with base \( W \times A \).

Proof. Recall that \( A \cong G_a \) and \( A \cong G/H \) as algebraic groups. Thus there is a cross section \( G/H \to A \hookrightarrow G \). It then follows that the quotient \( V/H \) exists as an affine variety, \( V \to V/H \) is an algebraic principal \( H \)-bundle, and that
V/H → W is a bundle with structure group A. By Lemma A.1, V/H ≅ W × A as an algebraic principal A-bundle.

**Theorem A.3:** Let V, W be smooth affine varieties and π : V → W an algebraic principal bundle for a unipotent algebraic group G. Then W = V/G and there exists an algebraic cross section σ : V/G → V.

**Proof.** Let k = dim G. The case k = 1 is Lemma A.1. The theorem follows by induction on k: Let H, A denote the subgroups from (2.1).

By Lemma A.2 we may apply the induction hypothesis to H, and together with Lemma A.1 we have global cross sections:

\[ \sigma_H : V/H \to V, \]
\[ \sigma_A : (V/H)/A \to V/H. \]

(V/H)/A = (V/H)/(G/H) = V/G as affine varieties (Borel [3, Corollary 6.10]). Define a morphism σ = σ_H ◦ σ_A : V/G → V. For any orbit G.p we have

\[ (\pi \circ \sigma)(G.p) = (\pi \circ \sigma_H \circ \sigma_A)(G.p) = (\pi \circ \sigma_H \circ \sigma_A)(A.(H.p)) = G.p. \]

So \( \pi \circ \sigma = \text{id}_{V/G} \), that is σ is a global cross section for the action of G.

**Remark A.4:** As we assumed all varieties and all morphisms to be defined over \( \mathbb{R} \), the cross section σ may be taken to be defined over \( \mathbb{R} \). In the proof of Theorem A.3 we may assume σ_H to be \( \mathbb{R} \)-defined by the induction hypothesis. Further, σ_A may be assumed to be \( \mathbb{R} \)-defined, because in the proof of Lemma A.1 the local cross sections σ_i can be assumed to be \( \mathbb{R} \)-defined by Rosenlicht’s results on solvable algebraic groups [12, Theorem 10], and the 1-cocycles α_i may be replaced by their real parts and still yield \( \alpha_i - \alpha_j = \beta_{ij} \), because the latter is an \( \mathbb{R} \)-valued regular function which is defined over \( \mathbb{R} \) if the action of A is.

**References**

[1] O. Baues, *Flat pseudo-Riemannian manifolds and prehomogeneous affine representations*, in 'Handbook of Pseudo-Riemannian Geometry and Supersymmetry', EMS, IRMA Lect. Math. Theor. Phys. 16, 2010, pp. 731-817 (also arXiv:0809.0824v1)

[2] O. Baues, W. Globke, *Flat Pseudo-Riemannian Homogeneous Spaces With Non-Abelian Holonomy Group*, Proc. Amer. Math. Soc. 140, 2012, pp. 2479-2488 (also arXiv:1009.3383)

[3] A. Borel, *Linear Algebraic Groups*, 2nd edition, Springer, 1991
[4] L. Corwin, F.P. Greenleaf, *Representations of nilpotent Lie groups and their applications*, Cambridge University Press, 1990

[5] W. Globke, *Holonomy Groups of Flat Pseudo-Riemannian Homogeneous Manifolds*, Dissertation, Karlsruhe Institute of Technology, 2011

[6] W. Globke, *Holonomy Groups of Complete Flat Pseudo-Riemannian Homogeneous Spaces*, Adv. in Math. 240, 2013, pp. 88-105 (also arXiv:1205.3285)

[7] H. Kraft, G.W. Schwarz, *Reductive Group Actions with One-Dimensional Quotient*, Publ. Math. Inst. Hautes Études Sci. 76, 1992, pp. 1-97

[8] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, 1983

[9] A. Onishchik, E. Vinberg, *Lie Groups and Lie Algebras III*, Springer, 1994

[10] D. Perrin *Algebraic Geometry*, Springer, 2008

[11] M.S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer, 1972

[12] M. Rosenlicht, *Some Basic Theorems on Algebraic Groups*, Amer. J. Math. 78, no. 2, 1956, pp. 401-443

[13] M. Rosenlicht, *Questions of Rationality for Solvable Algebraic Groups over Nonperfect Fields*, Ann. Mat. Pura Appl. 62, no. 1, 1963, pp. 97-120

[14] N. Steenrod, *The Topology of Fibre Bundles*, Princeton Univ. Press, 1951

[15] J.A. Wolf, *Homogeneous manifolds of zero curvature*, Trans. Amer. Math. Soc. 104, 1962, pp. 462-469

[16] J.A. Wolf, *Spaces of Constant Curvature*, 6th edition, Amer. Math. Soc., 2011