Data-Driven Inversion-Based Control: closed-loop stability analysis for MIMO systems

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Abstract—Data-Driven Inversion-Based Control (D²-IBC) is a recently introduced control design method for uncertain nonlinear systems, relying on a two degree-of-freedom architecture, with a nonlinear controller and a linear controller running in parallel. In this paper, extending to the MIMO case a previous result holding for the SISO case, we derive a finite-gain stability sufficient condition for a closed-loop system for a nonlinear MIMO plant, connected in feedback with a D²-IBC controller.

I. INTRODUCTION

Data-Driven Inversion-Based Control (D²-IBC) is a recently introduced control design method for uncertain nonlinear systems, relying on a two degree-of-freedom architecture, with a nonlinear controller and a linear controller running in parallel. [1], [2]. Despite many different approaches for joint design of identification and control have been already proposed (see [3] and the reference therein for a comprehensive overview), as far as we are aware, D²-IBC is the first “identification for control” method for nonlinear dynamical systems, where also stability guarantees are provided and enforced directly in the identification algorithm. However, the underlying stability sufficient condition is valid for Single Input Single Output (SISO) systems only. In this paper, we extend this sufficient condition to Multiple Input Multiple Output (MIMO) systems.

Notation. A column vector \( x \in \mathbb{R}^{n_x} \) is denoted as \( x = (x_1, \ldots, x_{n_x}) \). A row vector \( x \in \mathbb{R}^{1 \times n_x} \) is denoted as \( x = [x_1, \ldots, x_{n_x}]^\top \), where \( \top \) indicates the transpose.

A discrete-time signal (i.e. a sequence of vectors) is denoted with the bold style: \( x = (x_1, x_2, \ldots) \), where \( x_t \in \mathbb{R}^{n_x \times 1} \) and \( t = 1, 2, \ldots \) indicates the discrete time; \( x_{i,t} \) is the \( i \)-th component of the signal \( x \) at time \( t \).

A regressor, i.e. a vector that, at time \( t \), contains \( n \) present and past values of a variable, is indicated with the bold style and the time index: \( x_t = (x_{t,1}, x_{t-1,1}, \ldots, x_{t-n,1}) \).

The \( \ell_p \) norms of a vector \( x = (x_1, x_2, \ldots, x_{n_x}) \) are defined as

\[
\|x\|_p \equiv \begin{cases} 
\left(\sum_{i=1}^{n_x} |x_i|^p\right)^{\frac{1}{p}}, & p < \infty, \\
\max_i |x_i|, & p = \infty.
\end{cases}
\]

The \( \ell_p \) norms of a signal \( x = (x_1, x_2, \ldots) \) are defined as

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\|x\|_p \equiv \begin{cases} 
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\max_i |x_i|, & p = \infty.
\end{cases}
\]

where \( x_{i,t} \) is the \( i \)-th component of the signal \( x \) at time \( t \).

II. THE D²-IBC APPROACH

Consider a nonlinear discrete-time MIMO system in regression form:

\[
y_{t+1} = g(y_t, u_t, \xi_t) \\
y_t = (y_t, \ldots, y_{t-n+1}) \\
u_t = (u_t, \ldots, u_{t-n+1}) \\
\xi_t = (\xi_t, \ldots, \xi_{t-n+1})
\]

where \( u_t \in U = [-\bar{u}, \bar{u}]^{n_u} \subset \mathbb{R}^{n_u} \) is the saturated input, \( y_t \in \mathbb{R}^{n_y} \) is the output, \( \xi_t \in \Xi = [-\bar{\xi}, \bar{\xi}]^{n_c} \subset \mathbb{R}^{n_c} \) is a disturbance, and \( n \) is the system order. Both \( U \) and \( \Xi \) are compact sets.

Suppose that the system (1) is unknown, but a set of measurements is available:

\[
D \equiv \{\bar{u}_t, \bar{y}_t\}_{t=1}^{1-L}
\]

where \( \bar{u}_t \in U, \bar{y}_t \in Y, Y = [-\bar{\bar{y}}, \bar{\bar{y}}]^{n_y} \subset \mathbb{R}^{n_y} \) is the \( i \)-th component of \( \bar{y}_t \). The tilde is used to indicate the input and output samples of the data set, which is supposed to be available at time \( t = 0 \) when the controller needs to be designed. The input signals employed to generate (2) are also assumed to be such that the system output does not diverge.

Let \( \bar{Y} = [-\bar{\bar{y}}, \bar{\bar{y}}]^{n_y} \), with \( 0 \leq \bar{\bar{y}} \leq \bar{\bar{y}} \), be a domain of interest for the trajectories of the system (1). The aim is to control the system (1) in such a way that, starting from any initial condition \( y_0 \in \mathcal{Y}^0 \equiv \mathbb{R}^{n_y}, \) the system output sequence \( y = (y_1, y_2, \ldots) \) tracks any reference sequence \( r = (r_1, r_2, \ldots) \in R \subseteq \mathbb{R}^{\infty} \subset \ell_{\infty} \).

The set of all possible disturbance sequences is defined as \( \Xi \equiv \{\xi_t = (\xi_1, \xi_2, \ldots) : \xi_t \in \Xi, \forall t\} \).

To accomplish this task, we consider a feedback control structure with two controllers, \( K^{nl} \) and \( K^{lin} \), working in parallel; \( K^{nl} \) is a nonlinear controller used to guide the system (1) along the trajectories of interest, while \( K^{lin} \) is a linear controller aimed to enhance the tracking precision.
A. Nonlinear controller design

The first step needed to design the nonlinear controller is to identify from the data a model for the system of the form

$$\hat{y}_{t+1} = f(y_t, u_t) \equiv f(q_t, u_t)$$

$$q_t = (y_t, \ldots, y_{t-n+1}, u_t-1, \ldots, u_{t-n+1})$$

where $u_t$ and $y_t$ are the system input and output, and $\hat{y}_t$ is the model output. A parametric structure is taken for the function $f$:

$$f(q_t, u_t) = \sum_{i=1}^{N} \alpha_i \phi_i(q_t, u_t)$$

where $\phi_i : \mathbb{R}^{n_y+n_u} \to \mathbb{R}$ are polynomial basis functions and $\alpha_i \in \mathbb{R}^{n_y \times 1}$ are parameter vectors that can be identified by means of convex optimization.

Once a model of the form has been identified, the current command action $u_{t+1}^{nl}$ of the nonlinear controller $K^{nl}$ is computed by the on-line inversion of the obtained models, given the available regressor $q_t$. This inversion is performed by solving the following optimization problem:

$$u_{t+1}^{nl} = \arg \min_{u} J_t(\hat{u})$$

where the current objective function $J_t$ is

$$J_t(\hat{u}) = \sum_{i=1}^{n} \frac{\rho_{y_i}}{\rho_{u_i}} \left( r_{t+1,i} - f_i(q_t, \hat{u}) \right)^2 + \frac{\mu_i}{\rho_{u_i}} \hat{u}^2 + \frac{\lambda_i}{\rho_{u_i}} \delta \hat{u}^2$$

where $r_{t+1,i}$ is the $i$th component of $r_{t+1}$ and $f_i$ is the $i$th component of $f$; $\rho_{y_i} = \frac{||\hat{y}_{i-1} - \ldots - \hat{y}_{0}||}{\rho_{u_i}}$, and $\rho_{u_i}$ are parameter vectors that can be identified by means of convex optimization.

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B. Linear controller design

The linear controller $K^{lin}$ is defined by the centralized extended PID (Proportional Integrative Derivative) control law

$$u_t^{lin} = u_{i-1}^{lin} + \sum_{i=0}^{n_\theta} B_i e_{t-i}$$

where $e_t = r_t - y_t$ is the tracking error, $n_\theta$ is the controller order and $B_i \in \mathbb{R}^{n_u \times n_\theta}$ are the controller parameter matrices. All the entries of these matrices are contained in a vector $\theta = (\theta_1, \ldots, \theta_{n_\theta \times n_u \times n_\theta})$. Note that, for $n_\theta = 1$ and $n_\theta = 2$, the standard PI and PID controller are selected, respectively. The goal of $K^{lin}$ in the proposed architecture is to compensate for model-inversion errors and boost the control performance by assigning a desired dynamics to the resulting nonlinearly-compensated system.

In the considered setting, where most of the information about the system is that inferrable from data, finding a good control-oriented model of the error system, i.e. the system describing the relationship between $u_t^{lin}$ and $y_t$, is not an easy task. Therefore, in this paper, the Virtual Reference Feedback Tuning (VRFT) method, originally developed in [3] and extended to the MIMO case in [4], is employed and adapted to the present setting. Its rationale is briefly recalled next for self-consistency of the paper.

First, define the desired behavior for the closed loop system by means of a discrete transfer function $M$. In the present setting, this function is used to assign a specific desired dynamic to the nonlinearly-compensated resulting system. Typically, for $n_y = n_u = m$, $M$ is a diagonal $m \times m$ transfer function composed by $m$ asymptotically stable low-pass filters, synthesizing the desired closed loop behavior for each output. The virtual reference rationale permits the design of $K^{lin}$ without identifying any model of the system, based on the following observation: in a virtual operating condition, where the closed-loop system behaves exactly as $M$, the virtual reference signal $r_v^t$ would be given by the filtering of the output $y_t$ by the model $M^{-1}$. Since the inverse model $M^{-1}$ is non-causal, the filtering task must be accomplished off-line using a set of available data. The optimal controller is the one giving the measured $u_t^{lin}$ as output when fed by the virtual error $e_t^v = r_v^t - y_t$, as to minimize the cost function

$$J_{VR} = \sum_{t} ||u_t^{lin} - K^{lin}(\theta) e_t^v||^2_2.$$ (8)

Two versions of the VRFT method in the D^2-IBC setting are available, corresponding to two cases, related to bijectivity of the function $f$ in [3]. The details of these two versions are not discussed here for the sake of brevity.

III. Closed-loop stability analysis

The closed-loop system formed by the plant, controlled in feedback by the parallel connection of $K^{nl}$ and $K^{lin}$, is described by the following equations:

$$y_{t+1} = g(y_t, u_t, \xi_t)$$

$$u_t = u_t^{nl} + u_t^{lin}$$

$$u_t^{nl} = K^{nl}(r_{t+1}, y_t, u_{t-1}^{nl})$$

$$u_t^{lin} = K^{lin}(r_{t-1} - y_t, u_{t-1}^{lin})$$

where $K^{nl}$ and $K^{lin}$ are defined in [5] and [4], respectively, and $u_t \in U$, $\forall t$. The reference initial condition is chosen as $r_0 = y_0$.

In the following, we study the stability properties of such a close-loop system. The following stability notion is adopted.
Definition 1: A nonlinear system (possibly time-varying), with inputs \( r_t \) and \( \xi_t \), and output \( y_t \), is finite-gain \( \ell_\infty \) stable on \( (\mathcal{Y}^0, \mathcal{R}, \Xi) \) if finite and non-negative constants \( \Gamma_r \), \( \Gamma_\xi \) and \( \Lambda \) exist such that
\[
\|y\|_\infty \leq \Gamma_r \|r\|_\infty + \Gamma_\xi \|\xi\|_\infty + \Lambda
\]
for any \((y_0, r, \xi) \in \mathcal{Y}^0 \times \mathcal{R} \times \Xi\). □

Note that this finite-gain stability definition is more general than the standard one, which is obtained for \( \mathcal{R} = E_\infty \) and \( \Xi = \ell_\infty \), see e.g. [6].

In order to study how to guarantee finite-gain stability of the feedback system (9), some additional assumptions are introduced.

Assumption 1 (Lipschitzianity): The function \( g \) in (1) and (2) is Lipschitz continuous on \( Y^\infty \times U^\infty \times \Xi^\infty \). Without loss of generality, it is also assumed that \( Y^\infty \times U^\infty \times \Xi^\infty \) contains the origin. □

This assumption is mild, since most real-world dynamic systems are described by functions that are Lipschitz continuous on a compact set.

From Assumption 1 it follows that \( g \) can be written as
\[
g(y_t, u_t, \xi_t) = g^0(y_t, u_t) + g^\xi_t \xi_t
\]
where \( g^0(y_t, u_t) \) is a time-varying parameter (dependent on \( y_t \), \( u_t \) and \( \xi_t \) ) bounded on \( Y^\infty \times U^\infty \times \Xi^\infty \) as \( \|g^\xi_t\|_{\ell_\infty} \leq \gamma_\xi \), for some \( \gamma_\xi < \infty \). Assumption 2 together with (4), implies that the residue function
\[
\Delta(y_t, u_t) \triangleq g^0(y_t, u_t) - f(y_t, u_t)
\]
is Lipschitz continuous on \( Y^\infty \times U^\infty \). Hence, a finite and non-negative constant \( \gamma_y \) exists, such that
\[
\|\Delta(y, u) - \Delta(y', u)\|_\infty \leq \gamma_y \|y - y'\|_\infty
\]
for all \( y, y' \in Y^\infty \) and all \( u \in U^\infty \).

Assumption 2 (Model accuracy): The following inequality holds: \( \gamma_y < 1 \). □

The meaning of this assumption is clear: it requires \( \gamma_y \) to accurately describe the variability of \( g \) with respect to \( y_t \).

Now, consider that
\[
\tilde{e}_{t+1} = r_{t+1} - \hat{y}_{t+1} = r_{t+1} - f(y_t, u_t)
\]
\[
= r_{t+1} - f(y_t, u_t, (r_{t+1}, r_t, y_t, u_{t-1}^{lin}))
\]
\[
= F(y_t, r_t, v_t),
\]
where \( v_t \triangleq (r_{t+1}, u_{t-1}^{lin}, u_{t-1}^{lin}) \in V \) and \( V \) is a compact set. Then, for any \((y_t, r_t, v_t) \in Y^\infty \times R^\infty \times V^\infty \),
\[
|\tilde{e}_{t+1}| \leq \Gamma_y \|y_t\|_\infty + \Gamma_r \|r_t\|_\infty + \Lambda_e
\]
(10)
where \( \Gamma_y, \Gamma_r, \Lambda_e < \infty \). This inequality directly follows from the fact that the model function \( f \) is Lipschitz continuous on \( Y^\infty \times U^\infty \). Note that (10) does not imply that \( y \in Y^\infty \).

Assumption 3 (Effective model inversion): The following inequality holds: \( \Gamma_y \leq 1 - \gamma_y \). □

This assumption is not restrictive: it is certainly satisfied if \( \mu = 0 \) and the reference \( r = (r_1, r_2, \ldots) \) is a model solution (i.e. \( r_{t+1} \) is in the range of \( f(y_t, \cdot) \) for all \( t \)). Indeed, in this case, \( \tilde{y}_{t+1} = r_{t+1}, \forall t \), since \( \mathcal{K}^{lin} \) performs an exact inversion of the model, see again (3) \( \mathcal{K}^{lin} \) gives a null input signal in this case. This implies that \( \Gamma_y = 0 \), \( \Gamma_r = 0 \) and \( \Lambda_e = 0 \). Hence, if a sufficiently small \( \mu \) is chosen and the reference is sufficiently close to a system solution, supposing that inequality (10) holds with a sufficiently small \( \Gamma_y \) is reasonable.

To formulate our last assumption, define
\[
\bar{e} \triangleq \frac{1}{1 - \lambda_y} (\lambda_r + \gamma_\xi \xi + \lambda_y)
\]
(11)
where \( \lambda_y \equiv \Gamma_y + \gamma_y < 1 \), \( \lambda_r \equiv \lambda_y + \Gamma_r \) and \( \lambda_e \equiv \Lambda_e + \max_{u \in U^\infty} \|\Delta(0, u)\|_\infty \). Note that \( \bar{e} \) is bounded, being the sum of bounded quantities. In particular, for null disturbances \( (\xi = 0) \), exact modeling \( (f = g, \Delta = 0, \gamma_y = 0) \) and reference signals properly chosen \( (\Gamma_y = 0, \Gamma_r = 0, \Lambda_e = 0) \), we have \( \bar{e} = 0 \). In realistic situations, with reasonable disturbances, sufficiently accurate modeling and reference signals properly chosen, \( \bar{e} \) can be reasonably small (that is, \( \bar{e} \ll \bar{r} \)).

Assumption 4 (Output domain exploration): The following inequality holds: \( \bar{g} \geq \bar{r} + \bar{e} \). □

This assumption requires that the set \( Y \) explored by the output data is somewhat larger than the set \( R \) where the trajectory of interest are defined. Note that it can always be met just collecting data that sufficiently enlarge the set \( Y \).

Closed-loop stability of the system (9) is stated by the following result, which also provides a bound on the tracking error.

Theorem 1: Consider the system (9) and let Assumptions 1-4 hold. Then:
(i) The feedback system (9), having inputs \( r_t \) and \( \xi_t \) and output \( y_t \), is finite-gain \( \ell_\infty \) stable on \( (\mathcal{Y}^0, \mathcal{R}, \Xi) \).
(ii) The tracking error signal \( e \equiv r - y \) is bounded as
\[
\|e\|_\infty \leq \bar{e}.
\]
(12)

Proof. The proof of the theorem is structured as follows. Firstly, the tracking error is proven to be upper bounded by a suitable combination of the norms of the output and the reference. Secondly, it is shown that, under the assumption of an effective model inversion, such a bound is equivalent to a bound on the tracking error, whatever the output is (claim (ii)). Claim (i) is derived as a straightforward consequence of claim (ii).

To start with, consider that
\[
e_{t+1} = r_{t+1} - y_{t+1} = \tilde{e}_{t+1} - \delta y_t
\]
where
\[
\dot{e}_{t+1} = r_{t+1} - \dot{y}_{t+1} = F(y_t, r_t, v_t)
\]
\[
\delta y_t = \Delta(y_t, u_t) + g^T_t \xi_t.
\]
The term \(\dot{e}_{t+1}\) is bounded according to (10). Note that \(\|y_t\|_{\infty}\) could be unbounded. In order to derive a bound on \(\delta y_t\), we can use Assumption 2 and observe that, for any \(y \in Y^n\),
\[
\|\Delta(y_t, u_t)\|_{\infty} - \|\Delta(0, u_t)\|_{\infty} \leq \gamma_y \|y_t\|_{\infty}.
\]
The following inequality thus holds for any \(y \in Y^n\):
\[
\|\delta y_t\|_{\infty} \leq \|\Delta(y_t, u_t)\|_{\infty} + \gamma_y \|y_t\|_{\infty} + \gamma_y \|\xi_t\|_{\infty} + \Delta,
\]
where \(\Delta = \max_{u \in U^n} \|\Delta(0, u)\|_{\infty} < \infty\). Hence,
\[
\|e_{t+1}\|_{\infty} \leq \lambda_y \|y_t\|_{\infty} + \gamma_y \|r_t\|_{\infty} + \Gamma_y \|r_t\|_{\infty} + \Lambda_e + \gamma_y \|\xi_t\|_{\infty} + \Lambda_y,
\]
which proves that the tracking error \(e\) is bounded by a suitable combination of the norms of \(y_t\) and \(r_t\), for any \(y_t \in Y^n\). Note that (13) in this form is of no use, since \(\|y_t\|_{\infty}\) could be unbounded and thus the condition \(y_t \in Y^n\) may not hold. However, (13) can be rewritten as
\[
\|e_{t+1}\|_{\infty} \leq \lambda_y \|e_t\|_{\infty} + \gamma_y \|r_t\|_{\infty} + \sum_{k=0}^{\infty} \lambda_y^k \|e_0\|_\infty + \Lambda_e + \gamma_y \|\xi_t\|_{\infty} + \Lambda_y,
\]
that means,
\[
\|e_{t+1}\|_{\infty} \leq \lambda_y \|e_t\|_{\infty} + w,
\]
where \(w = \lambda_y \bar{r} + \gamma_y \bar{\xi} + \Lambda_y\). Inequality (15), again, holds only if \(y_t \in Y^n\).

Consider now that, by assumption, \(y_0 \in R^n \subseteq Y^n\). This implies that, at time \(t = 0\), inequality (15) holds. Being \(e_0 = r_0 - y_0 = 0\) for the selected initialization of \(r_0\), we have
\[
\|e_1\|_{\infty} \leq \lambda_y \|e_0\|_{\infty} + w = w \leq \bar{e}.
\]
Since \(\|e_1\|_{\infty} \leq \bar{e}\) and \(\|r_1\| \leq \bar{r}\), it follows from Assumption 4 that \(y_1 \in Y\). Consequently, the Lipschitzianity assumption holds and (15) can be used also for \(t = 1\), giving
\[
\|e_2\|_{\infty} \leq \lambda_y \|e_1\|_{\infty} + w \leq \lambda_y w + w
\]
\[
\leq w \sum_{k=0}^{\infty} \lambda_y^k \leq \frac{w}{1-\lambda_y} = \bar{e},
\]
where the geometric series sum has been obtained thanks to the fact that, by Assumption 3, \(\lambda_y < 1\). It follows that \(y_2 \in Y\) and, consequently, (15) can be used also for \(t = 2\). Iterating the above reasoning,
\[
\|e_3\|_{\infty} \leq \lambda_y \max \{\|e_2\|_{\infty}, \|e_1\|_{\infty}\} + w
\]
\[
\leq \lambda_y \|e_2\|_{\infty} + w \leq \lambda_y w + \lambda_y w + w \leq \bar{e}
\]
\[
\|e_{t+1}\|_{\infty} \leq w \sum_{k=0}^{t-1} \lambda_y^k \leq w \sum_{k=0}^{t-1} \lambda_y^k \leq \frac{w}{1-\lambda_y} = \bar{e}.
\]

Then, \(y_t \in Y\), \(\forall t \geq 0\) and (12) holds (claim (ii)). Claim (i) is a direct consequence of claim (ii) and the relation \(y = r - e\).

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