Derivatives and exceptional poles of the local exterior square $L$-function for $GL_m$

Yeongseong Jo$^{1,2}$

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Abstract
Let $\pi$ be an irreducible admissible representation of $GL_m(F)$, where $F$ is a non-archimedean local field of characteristic zero. In 1990’s Jacquet and Shalika established an integral representation for the exterior square $L$-function. We complete, following the method developed by Cogdell and Piatetski-Shapiro, the computation of the local exterior square $L$-function $L(s, \pi, \wedge^2)$ via the integral representation in terms of $L$-functions of supercuspidal representations by a purely local argument. With this result, we show the equality of the local analytic $L$-functions $L(s, \pi, \wedge^2)$ via the integral representation for the irreducible admissible representation $\pi$ for $GL_m(F)$ and the local arithmetic $L$-functions $L(s, \wedge^2(\phi(\pi)))$ of its Langlands parameter $\phi(\pi)$ through local Langlands correspondence.

Keywords Jacquet–Shalika integral · Local exterior square $L$-function · Exceptional poles · Bernstein–Zelevinsky derivatives

Mathematics Subject Classification 11F70 · 11F85 · 11S23 · 11S37

1 Introduction

Let $F$ be a non-archimedean field of characteristic zero and $W'_F$ be a Weil–Deligne group. The local Langlands correspondence for $GL_m(F)$ established by Harris–Taylor [12] and also by Henniart [13] asserts a bijection between the set of isomorphism class of irreducible admissible representations of $GL_m(F)$ and the set of equivalence class of $m$-dimensional Frobenius semisimple complex representations of $W'_F$.

For a $m$-dimensional Frobenius semisimple complex representations of $\phi$ of $W'_F$, the correspondence associates an irreducible admissible representation $\pi(\phi)$ of $GL_m(F)$. Let

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Yeongseong Jo
jo.59@buckeyemail.osu.edu

$^1$ Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA
$^2$ Present Address: Department of Mathematics, The University of Iowa, Iowa City, IA 52242, USA
$\wedge^2$ be the exterior square representation of $GL_m(\mathbb{C})$. Then we can define Artin’s local exterior square $L$-function as $L(s, \wedge^2(\phi))$. There are two more ways to associated a local exterior square $L$-function to an irreducible admissible representation $\pi(\phi)$. One is by analyzing the Euler products that appear in the non-constant Fourier coefficients of Eisenstein Series. This method is called the Langlands-Shahidi method in [33] and we denote it by $L_{Sh}(s, \pi(\phi), \wedge^2)$. The second $L$-function, denoted by $L(s, \pi(\phi), \wedge^2)$, is defined as the greatest common divisor of $\mathbb{C}[q^{\pm 1}]$-fractional ideal. Each rational function in a $\mathbb{C}[q^{\pm 1}]$-fractional ideal is obtained as a Rankin-Selberg type integral representation, which was constructed by Jacquet and Shalika [18] in 1990’s. One might naturally expect that these three exterior square $L$-functions are equal $L(s, \wedge^2(\phi)) = L_{Sh}(s, \pi(\phi), \wedge^2)$. Henniart [14] proves that the Artin and Langlands-Shahidi $L$-functions are equal $L(s, \wedge^2(\phi)) = L_{Sh}(s, \pi(\phi), \wedge^2)$ for every irreducible admissible representation $\pi(\phi)$. Kewat and Raghunathan [21] recently prove that $L_{Sh}(s, \Delta, \wedge^2) = L(s, \Delta, \wedge^2)$ for $\Delta$ a discrete series representation.

In the mid 1990’s Cogdell and Piatetski-Shapiro embarked on the local analysis of the exterior square $L$-function via Jacquet and Shalika integrals to express $L(s, \pi, \wedge^2)$ in terms of $L$-functions for supercuspidal representations [8]. Although the expression for $L$-functions $L(s, \pi \times \sigma)$ of pairs $(\pi, \sigma)$ of irreducible generic representations of $GL_n(F)$ are established by Jacquet, Piatetski-Shapiro, and Shalika [15], Cogdell and Piatetski-Shapiro [9] developed a new technique to calculate the local $L$-functions defined by the Rankin-Selberg integrals in terms of $L$-functions for supercuspidal representations. It turns out that every pole is contributed by an “exceptional pole” of the Rankin-Selberg $L$-function for the pair of derivatives $(\pi^{(k)}, \sigma^{(k)})$ in the sense of Bernstein and Zelevinsky [4,5,35]. In other words, these are exceptional poles of $L$-function $L(s, \pi^{(k)} \times \sigma^{(k)})$. After all, the poles of $L(s, \pi \times \sigma)$ occur when derivative of one representation is contragredient to derivative of another representation up to some unramified twist. This beautiful idea is due to Piatetski-Shapiro who associated representation theory with analytic theory of $L$-functions. Afterwards the method of exceptional poles and derivatives are the central object of studying for the structure of Asai $L$-functions in [24] and Bump and Friedberg $L$-functions in [28].

Recently there has been a lot of interests in the study of the local exterior square $L$-functions via integral representations [3,20,21,30]. In particular, Cogdell and Matringe [7,27] prove the local functional equation of the exterior square $L$-functions attached to any irreducible admissible representations of $GL_m(F)$ based on the Bernstein-Zelevinsky theory of derivatives [4,5,35], and the theory of linear periods and Shalika periods. We expect to utilize these local functional equations to prove the stability and multiplicativity of the local $\gamma$-factor, and then complete the theory of the local exterior square $\gamma$-factors. We will return to this in the future.

In this article we complete the project stared by Cogdell and Piatetski-Shapiro [8], which was to compute the local exterior square $L$-functions for all irreducible admissible representations of $GL_m(F)$ by a purely local argument. We follow the similar lines as in Cogdell and Piatetski-Shapiro [9], which is prominently improved by Matringe in [28]. The technique relies on the theory of Shalika functionals on Bernstein–Zelevinsky derivatives and the notion of exceptional poles. Eventually this makes it possible to obtain the multiplicativity relation of the local exterior square $L$-functions.

The main results of this paper are precisely stated as blow.

**Theorem** (Theorem 5.14) Let $\phi$ be a $m$-dimensional Frobenius semisimple complex representation of Weil–Deligne group $W_f$ and $\pi = \pi(\phi)$ the irreducible admissible representation of $GL_m$ associated to $\phi$ under the local Langlands correspondence. Then we have

$$L(s, \wedge^2(\phi)) = L(s, \pi(\phi), \wedge^2).$$
The following multiplicativity of the analytic exterior square $L$-functions for irreducible generic representations is exploited to prove the equality between the exterior square $L$-function through integral representations for an irreducible admissible representation (Analytic side), and via its Langlands parameter (Arithmetic side).

**Tempered case (Corollary 5.9)**

Let $\pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ be the normalized induced representation of $GL_m(F)$ from the standard parabolic subgroup, where $\Delta_i$ are irreducible square integrable representations of $GL_{m_i}(F)$. Then

$$L(s, \pi, \wedge^2) = \prod_{1 \leq k \leq t} L(s, \Delta_k, \wedge^2) \prod_{1 \leq i < j \leq t} L(s, \Delta_i \times \Delta_j).$$

We remark that the representation $\pi$ is irreducible by [35, Theorem 4.2].

**Non-tempered case (Theorem 5.7, Corollary 5.8)**

Let $\pi$ be an irreducible admissible representation of $GL_m(F)$. Let $\Pi = \text{Ind}(\Delta_1 v^{u_1} \otimes \cdots \otimes \Delta_t v^{u_t})$ be the standard module with each $\Delta_i$ an irreducible square integrable representation, the $u_i$ real and ordered so that $u_1 \geq \cdots \geq u_t$ such that $\pi$ is the unique irreducible quotient of $\Pi$. Then

$$L(s, \pi, \wedge^2) = \prod_{k=1}^t L(s + 2u_k, \Delta_k, \wedge^2) \prod_{1 \leq i < j \leq t} L(s + u_i + u_j, \Delta_i \times \Delta_j).$$

Let us briefly describe the contents of this article. The first part of Sect. 2 is concerned with the result of Bernstein and Zelevinsky about the classification of irreducible admissible representations of $GL_m(F)$, derivatives of such representations, and Whittaker models in [4,5,35]. We then introduce the Jacquet–Shalika integrals which will define a local exterior square $L$-function and recall standard facts from [19] about exceptional poles, Shalika subgroup, Shalika functionals, and factorization formula for local exterior square $L$-functions. The end of Sect. 2 is a reminder of the classical results about their functional equations [7,27]. In Sect. 3, we employ this factorization formula in order to express the exterior square $L$-functions in the case where the representation is quasi-square integrable in terms of exceptional $L$-functions of supercuspidal representations. In Sect. 4, we show that Jacquet–Shalika integrals defining $L$-functions $L(s, \pi_u, \wedge^2)$ for deformed representation $\pi_u$ are rational by using Bernstein’s Theorem [2]. By combining the result of Sect. 2 with the deformation method of Sect. 4, and Hartogs’ Theorem, we are able to prove a weak version of the multiplicativity of $y$-factor in Sect. 5. The rest part of Sect. 5 is devoted to adapting the methods of [9,24,28] to complete the computation of $L(s, \pi, \wedge)$ for an irreducible admissible representation $\pi$. The multiplicativity formalism in Sect. 5 lets us express $L$-functions of square integrable representations in terms of exterior or symmetric square $L$-functions for supercuspidal representations which in turn replace exceptional $L$-functions in Sect. 3 and show the agreement of the local exterior square $L$-functions on the analytic side deduced from integral representations and $L$-functions on the arithmetic counter part via Langlands parameters.

### 2 Preliminary results

Let $F$ be a nonarchimedean local field of characteristic 0, with ring of integers $\mathcal{O}$, prime ideal $p$, and fix a uniformizer $\varpi$ so that $p = (\varpi)$. Let $q = |\mathcal{O}/p|$ denote the cardinality of the residue class field. We let $v : F^\times \to \mathbb{Z}$ be the associated valuation with $v(\varpi) = 1$ and normalize the
absolute value so that $|a| = q^{-v(a)}$. We let $\mathcal{M}_n(F)$ be the $n \times n$ matrices, $\mathcal{N}_n(F)$ the subspace of upper triangular matrices of $\mathcal{M}_n(F)$. We denote by $GL_n(F)$ the general linear group of invertible matrices of $\mathcal{M}_n(F)$ and $\mathcal{N}_n(F)$ the subgroup of upper triangular unipotent matrices. Let $K_n = GL_n(O)$ be the standard maximal compact subgroup of $GL_n(F)$. We denote by $Z_n(F)$ the center of $GL_n(F)$. We fix a nontrivial additive character $\psi : F \to \mathbb{C}$ and extend it to a character of $\mathcal{N}_n(F)$ by setting $\psi(n) = \psi(n_{1,2} + \cdots + n_{n-1,n})$ with $n \in \mathcal{N}_n(F)$. We shall identify $F^n$ with the space of row vectors of length $n$. We denote by $P_n(F)$ the mirabolic subgroup of $GL_n(F)$ of matrices with last row equal to $e_n = (0, 0, \ldots, 0, 1)$ and $U_n(F)$ the unipotent radical of $P_n(F)$. As a group, $P_n(F)$ has the structure of a semi-direct product $P_n(F) \simeq GL_{n-1}(F) \times U_n(F)$. We then restrict the additive character of $\mathcal{N}_n(F)$ to $U_n(F)$ by $\psi(u) = \psi(u_{n-1,n}).$ Throughout this paper, we abuse notation by letting $GL_n = GL_n(F)$, $\mathcal{N}_n = \mathcal{N}_n(F)$, etc.

Let $G$ be a locally compact group. We denote by $d_L g$ a left Haar measure on $G$. We denote $d_R g$ be a right Haar measure on $G$. We let $\delta_G$ denote the modular character on $G$ satisfying $d_L(xg) = \delta_G(g)^{-1} d_L g(x)$ for $x, g \in G$. We denote by $S(F^n)$ the space of Schwartz-Bruhat functions $\Phi : F^n \to \mathbb{C}$ which are locally constant and of compact support. The Fourier transform on $S(F^n)$ will be defined by

$$\hat{\Phi}(y) = \int_{F^n} \Phi(x) \psi(x' y) \, dx$$

for $y \in F^n$. Once $\psi$ is chosen, we assume that the measure on $F^n$ used in this integral is the corresponding self-dual Haar measure. Therefore the Fourier inversion formula takes the form $\hat{\hat{\Phi}}(x) = \Phi(-x)$.

### 2.1 Derivatives and Whittaker models

In this paper, we will often say “representation” instead of “admissible representation”. Let $(\pi, V_\pi)$ be a representation of $GL_m$. The representation $\pi$ is said to be generic if the dimension of the space $\text{Hom}_{N_m}(V_\pi, \psi)$ is not zero. In this case, a nontrivial linear functional $\lambda \in \text{Hom}_{N_m}(V_\pi, \psi)$ is called a Whittaker functional on $V_\pi$. If $\pi$ is irreducible and $m \geq 2$, then Gelfand and Kazhdan in [11] prove that $\dim(\text{Hom}_{N_m}(V_\pi, \psi)) = 0$ or 1. Let $\text{Ind}_{N_m}^{GL_m}(\psi)$ denote the full space of smooth functions $W : GL_m \to \mathbb{C}$ which satisfy $W(ng) = \psi(n) W(g)$ for all $n \in N_m$. $GL_m$ acts on this by right translation $\rho$. Frobenius reciprocity implies the isomorphism between $\text{Hom}_{N_m}(V_\pi, \psi)$ and $\text{Hom}_{GL_m}(V_\pi, \text{Ind}_{N_m}^{GL_m}(\psi))$. If $\dim(\text{Hom}_{N_m}(V_\pi, \psi)) = 1$, we denote the image of the intertwining operator $V_\pi \rightarrow \text{Ind}_{N_m}^{GL_m}(\psi)$ by $\mathcal{W}(\pi, \psi)$ and call it Whittaker model of $\pi$. If $\pi$ is irreducible and generic, $(\pi, V_\pi)$ is isomorphic to $(\rho, \mathcal{W}(\pi, \psi))$. Let $\pi'$ be the representation of $GL_m$ on the same space $V_\pi$ with the action $\pi'(g) = \pi'(g^{-1})$. We denote by $(\pi', V_{\pi'})$ the contragredient representation of $(\pi, V_\pi)$. If $\pi$ is irreducible, then $\pi' = \tilde{\pi}$ from [11]. Let $w = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \\ & & 1 \end{pmatrix}$ denote the long Weyl elements of $GL_m$. We define $\tilde{\mathcal{W}}$ by $\tilde{\mathcal{W}}(g) = W(w' g^{-1}) \in \mathcal{W}(\pi', \psi^{-1})$.

Let $\nu(g) = |\det(g)|$ be the unramified determinant character of $GL_m$ for any $m$. We say that $\pi$ is square integrable if its central character is unitary and

$$\int_{Z_m \backslash GL_m} |\langle \pi(g) v, \bar{v} \rangle|^2 \, dg < \infty,$$
for all \( v \in V_\pi \) and \( \tilde{v} \in V_\pi^\vee \). Here a function of the form \( g \mapsto \langle \pi(g)v, \tilde{v} \rangle \) for \( g \in GL_m \) is a matrix coefficient. A representation is said to be quasi-square-integrable or a discrete series if it is some twist of a square integrable representation by an unramified character \( v^s \) for some \( s \in \mathbb{C} \). Let \( Q = MN_Q \) be a proper standard parabolic subgroup of \( GL_r \), where \( M \) is the Levi subgroup of \( Q \) and \( N_Q \) the unipotent radical of \( Q \). A representation \( (\rho, V_\rho) \) of \( GL_r \) is said to be supercuspidal if \( r_M(\rho) = 0 \) for all \( M \) where \( r_M(\rho) \) is the natural representation of \( M \neq GL_r \) on \( V_\rho/V_\rho(N_Q, 1) \). If \( \rho \) is an irreducible supercuspidal representation of \( GL_r \), the normalized induced representation \( \text{Ind}^{GL_m}_Q(\rho \otimes \rho v \otimes \cdots \otimes \rho v^{1-1}) \) from standard parabolic subgroup \( Q \) has the unique irreducible quotient, that we denote by \( [\rho, \rho v, \ldots, \rho v^{1-1}] \). We call such a representation of \( GL_m \) a segment. According to Bernstein and Zelevinsky [5,35, Theorem 9.3], we have the following result.

**Theorem 2.2** (Bernstein and Zelevinsky) \( \Delta \) is an irreducible quasi-square-integrable representation of \( GL_m \) if and only if \( \Delta \simeq [\rho, \rho v, \ldots, \rho v^{1-1}] \) for some \([\rho, \rho v, \ldots, \rho v^{1-1}] \).

Let \( \pi = \text{Ind}^{GL_m}_Q(\Delta_1 \otimes \cdots \otimes \Delta_r) \) be a normalized induced representation of \( GL_m \) from the standard parabolic \( Q \) where each \( \Delta_i \) is an irreducible quasi-square integrable representation of \( GL_m \). The earlier work of Rodier in [31] shows that \( \pi \) is generic and \( \dim(\text{Hom}_{N_m}(V_\pi, \psi)) = 1 \). These are the only representations of generic representations needed for the application. From now on, we suppress the group \( GL_m \) and \( Q \) for reducing notational burden if there is no risk of confusion.

A representation \( \pi \) of \( GL_m \) is called a standard module if \( \pi \) is a normalized induced representation of the form \( \pi = \text{Ind}(\Delta_1 v^{\mu_1} \otimes \cdots \otimes \Delta_r v^{\mu_r}) \) where each \( \Delta_i \) is now an irreducible square integrable representation of \( GL_{m_i} \), each \( u_i \) is real and they are ordered so that \( u_1 \geq u_2 \geq \cdots \geq u_t \). This representation is referred to as a representation of Langlands type in [9,28]. In current literature [6], for the standard module \( \pi \) we use tempered representations in place of irreducible square integrable representations \( \Delta_i \) and \( u_1 > u_2 > \cdots > u_t \) in lieu of \( u_1 \geq u_2 \geq \cdots \geq u_t \) but those forms are equivalent according to [22, Section 2].

For a representation \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_r) \) of \( GL_m \), we present the theory of derivatives \( \pi^{(k)} \) of representations developed by Bernstein and Zelevinsky [5,35]. Derivatives \( \pi^{(k)} \) in the sense of Bernstein and Zelevinsky are representation of smaller size group \( GL_{m-k} \) and more details can be found in [4]. The last item on the following Theorem is known as the Leibnitz rule for derivatives.

**Theorem 2.2** (Bernstein and Zelevinsky) Let \( \rho \) be an irreducible supercuspidal representation of \( GL_r \). Let \( \Delta = [\rho, \ldots, \rho v^{1-1}] \) be an irreducible quasi-square integrable representation of \( GL_m \) with \( m = \ell r \).

1. \( \rho^{(0)} = \rho \), \( \rho^{(k)} = 0 \) for \( 1 \leq k \leq r - 1 \) and \( \rho^{(r)} = 1 \).
2. \( \Delta^{(k)} = 0 \) if \( k \) is not a multiple of \( r \), \( \Delta^{(0)} = \Delta = \Delta^{(r)} = [\rho v^k, \ldots, \rho v^{1-1}] \) for \( 1 \leq k \leq \ell - 1 \), and \( \Delta^{(\ell r)} = 1 \).
3. If \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_r) \) is a representation of \( GL_m \), then we have a filtration of \( \pi^{(k)} \) for \( 1 \leq k \leq n \) whose successive quotients are the representations \( \text{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_r^{(k_r)}) \) with \( k = k_1 + \cdots + k_r \).

### 2.2 Integral representation of Jacquet and Shalika for even case \( m = 2n \)

Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_r) \) be a representation of \( GL_m \). Let us denote by \( \omega_\pi \) the central character of \( \pi \). As we have seen in Theorem 2.2, \( \pi^{(k)} = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_r)^{(k)} \) has a filtration
whose successive quotients are the $\text{Ind}(\Delta^{(k_1)}_1 \otimes \cdots \otimes \Delta^{(k_t)}_t)$ with $k = k_1 + \cdots + k_t$. We let $k$ be a multi-index for $k = (k_1, k_2, \ldots, k_t)$ with $|k| = k_1 + \cdots + k_t = k$. We then denote by $\pi^\pm$ nonzero successive quotients $\text{Ind}(\Delta^{(k_1)}_1 \otimes \cdots \otimes \Delta^{(k_t)}_t)$. We call $(\omega_{\pi^\pm})$ the family of the central characters for all nonzero successive quotients $\pi^\pm = \text{Ind}(\Delta^{(k_1)}_1 \otimes \cdots \otimes \Delta^{(k_t)}_t)$. For any character $\chi$ of $F^\times$, $\chi$ can be uniquely decomposed as $\chi = \chi_0 \chi_0^\prime$, where $\chi_0$ is a unitary character and $s_0$ is a real number. To proceed from this point, let us make a convention that, we use the notion $s_0 = \text{Re}(\chi)$ for the real part of the exponent of the character $\chi$.

Let $\pi$ be an irreducible generic unitary representation of $GL_{2n}$. Let $\sigma_{2n}$ be the permutation matrix given by

$$\sigma_{2n} = \begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n \\ 1 & 3 & \cdots & 2n-1 & 2 & 4 & \cdots & 2n \end{pmatrix}.$$

In 1990, Jacquet and Shalika [18] established an integral representation for the exterior square $L$-function for $GL_{2n}$ in the following way:

$$J(s, W, \Phi) = \int_{N_n \setminus G_n} \int_{\mathcal{N}_n \setminus M_n} W(\sigma_{2n} \left( \begin{array}{cc} I_n & X \\ I_n & g \end{array} \right) \left( \begin{array}{c} g \\ g \end{array} \right)) \psi^{-1}(\text{Tr}X) dX \Phi(e_n g) |\text{det}(g)|^s \, dg,$$

where $W \in \mathcal{W}(\pi, \psi)$ and $\Phi \in \mathcal{S}(F^n)$. It is proved in Section 7 of [18] that there is $r_\pi$ in $\mathbb{R}$ such that the above integrals $J(s, W, \Phi)$ converge for $\text{Re}(s) > r_\pi$.

Let $\pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ be a representation of $GL_{2n}$. We introduce the existence Theorem 2.13 and Theorem 2.16 in [19] due to Jacquet and Shalika [18].

**Theorem 2.3** For every $0 \leq k \leq 2n-1$, let $(\omega_{\pi^\pm})$ be the family of the central characters for all nonzero successive quotients of the form $\pi^\pm = \text{Ind}(\Delta^{(k_1)}_1 \otimes \cdots \otimes \Delta^{(k_t)}_t)$ with $|k| = k_1 + \cdots + k_t$ appearing in the composition series of $\pi^{(k)}$. Let $W \in \mathcal{W}(\pi, \psi)$ and $\Phi \in \mathcal{S}(F^n)$.

(i) If for all $1 \leq k \leq n$, we have $\text{Re}(s) > -\frac{1}{k} \text{Re}(\omega_{\pi^\pm 2n-2k})$, then each local integral $J(s, W, \Phi)$ converges absolutely.

(ii) Each $J(s, W, \Phi) \in \mathbb{C}(q^{-s})$ is a rational function of $q^{-s}$ hence $J(s, W, \Phi)$ as a function of $s$ extends meromorphically to all of $\mathbb{C}$.

(iii) Each $J(s, W, \Phi)$ can be written with a common denominator determined only by the representation $\pi$. Hence the family has “bounded denominators”.

(iv) Let \[ \mathcal{J}(\pi) = \{ J(s, W, \Phi) \mid W \in \mathcal{W}(\pi, \psi), \Phi \in \mathcal{S}(F^n) \} \] denote the $\mathbb{C}$-linear span of the local integrals $J(s, W, \Phi)$. Then the family of local integrals $\mathcal{J}(\pi)$ is a $\mathbb{C}[q^s, q^{-s}]$-ideal of $\mathbb{C}(q^{-s})$ containing the constant 1.

We can now define the local exterior square $L$-function $L(s, \pi, \wedge^2)$ attached to the representation $\pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ of $GL_{2n}$. To do this, note that $\mathbb{C}[q^s, q^{-s}]$ is a Principal Ideal Domain (PID) hence $\mathcal{J}(\pi)$ is a principal fractional ideal. Since this ideal contains 1, we may choose a unique generator of $\mathcal{J}(\pi)$ of the form $P(q^{-s})^{-1}$ with $P(X) \in \mathbb{C}[X]$ having $P(0) = 1$.

**Definition** We define \[ L(s, \pi, \wedge^2) = \frac{1}{P(q^{-s})} \] to be the unique normalized generator of $\mathcal{J}(\pi)$. 

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A function of the form $P(q^{-s})^{-1}$, where $P(X) \in \mathbb{C}[X]$ is a polynomial satisfying $P(0) = 1$, is called an Euler factor. We define Shalika subgroup $S_{2n}$ of $GL_{2n}$ by

$$S_{2n} = \left\{ \begin{pmatrix} I_n & Z \\ I_n & g \end{pmatrix} \mid Z \in \mathcal{M}_n, \ g \in GL_n \right\}.$$  

Let us define an action of the Shalika group $S_{2n}$ on $S(F^n)$ by

$$R \left( \begin{pmatrix} I_n & Z \\ I_n & g \end{pmatrix} \right) \Phi(x) = \Phi(xg)$$

for $\Phi \in S(F^n)$. Let $\Theta$ be the character of $S_{2n}$ given by

$$\Theta \left( \begin{pmatrix} I_n & Z \\ I_n & g \end{pmatrix} \right) = \psi(TrZ).$$

Suppose there is a function $J(s, W, \Phi)$ in $\mathcal{J}(\pi)$ having a pole of order $d$ at $s = s_0$ and that this is the highest order pole of the family at $s = s_0$. Then the Laurent expansion at $s = s_0$ will be of the form

$$J(s, W, \Phi) = \frac{B_{s_0}(W, \Phi)}{(q^s - q^{s_0})^d} + \text{higher order terms.}$$

The coefficient of the leading term, $B_{s_0}(W, \Phi)$, defines a non-trivial bilinear form on $\mathcal{W}(\pi, \psi) \times S(F^n)$ satisfying $B_{s_0}(\rho(h)W, R(h)\Phi) = |\det(h)|^{-\frac{q_0}{2}} \Theta(h)B_{s_0}(W, \Phi)$ with $h \in S_{2n}$. The space $S(F^n)$ has a small filtration $\{0\} \subset S_0(F^n) \subset S(F^n)$, where $S_0(F^n) = \{ \Phi \in S(F^n) \mid \Phi(0) = 0 \}$. This filtration is stable under the action $R$.

**Definition** A pole $s = s_0$ of $L(s, \pi, \wedge^2)$ is called exceptional if, for the associated bilinear form $B_{s_0}$, the coefficient of the highest order pole of the family $\mathcal{J}(\pi)$ vanishes identically on $\mathcal{W}(\pi, \psi) \times S_0(F^n)$.

If $s_0$ is an exceptional pole of $\mathcal{J}(\pi)$, then the bilinear form $B_{s_0}$ factors to a non-zero bilinear form on $\mathcal{W}(\pi, \psi) \times (S(F^n)/S_0(F^n))$. The quotient $S(F^n)/S_0(F^n)$ is isomorphic to $\mathbb{C}$ via the map $\Phi \rightarrow \Phi(0)$.

**Definition** A nonzero linear functional $\Lambda$ on $V_\pi$ is called a Shalika functional if it satisfies $\Lambda(\pi(h)v) = \psi(TrZ)\Lambda(v)$ for all $h = \left( \begin{pmatrix} I_n & Z \\ I_n & g \end{pmatrix} \right) \in S_{2n}$ and $v \in V_\pi$.

We say that a nonzero linear functional $\Lambda_s$ on $V_\pi$ is a twisted Shalika functional if $\Lambda_s(\pi(h)v) = |\det(h)|^{-\frac{q_0}{2}} \psi(TrZ)\Lambda_s(v)$ for $h = \left( \begin{pmatrix} I_n & Z \\ I_n & g \end{pmatrix} \right) \in S_{2n}$ and $v \in V_\pi$.

If $s_0$ is an exceptional pole of $\mathcal{J}(\pi)$, the bilinear form $B_{s_0}$ can be written as $B_{s_0}(W, \Phi) = \Lambda_{s_0}(W)\Phi(0)$ with $\Lambda_{s_0} \in \text{Hom}_{S_{2n}}(\mathcal{W}(\pi, \psi), \mathbb{C})$ a twisted Shalika functional. If the ideal $\mathcal{J}(\pi)$ has an exceptional pole of order $d_{s_0}$ at $s = s_0$ and this is the highest order pole of the family at $s = s_0$, then this pole contributes by a factor of $(1 - q^{s_0}q^{-s})^{d_{s_0}}$ to $L(s, \pi, \wedge^2)^{-1}$.

As these factors $(1 - q^{s_0}q^{-s})^{d_{s_0}}$ are relatively prime in $\mathbb{C}[q^{\pm s}]$ for distinct exceptional poles $s = s_0$, this leads us to the following Definition.

**Definition** Let $L_{ex}(s, \pi, \wedge^2)^{-1} = \prod_{s_0}(1 - q^{s_0}q^{-s})^{d_{s_0}}$ where $s_0$ runs over the exceptional poles of $\mathcal{J}(\pi)$ with $d_{s_0}$ the maximal order of the pole at $s = s_0$.  

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Applying the partial Iwasawa decomposition $GL_n = Z_n P_n K_n$, we decompose the integral $J(s, W, \Phi)$ as:

$$J(s, W, \Phi) = \int_{K_n} \int_{N_n \backslash P_n} \int_{N_n \backslash M_n} W \left( \sigma_{2n} \left( \begin{array}{c} I_n \ X \\ I_n \end{array} \right) \left( \begin{array}{c} p_k \\ p_k \end{array} \right) \right) |\det(p)|^{s-1} \psi^{-1}(TrX) dX \times \int_{\mathbb{C}^\times} \omega(a) |a|^s \Phi(e_n ak) d^\times d\nu dk.$$

Focusing on integration over $N_n \backslash P_n$ and $\mathbb{C} \backslash M_n$, we define the following integral.

**Definition** We define the following integral:

$$J(0)(s, W) = \int_{N_n \backslash P_n} \int_{N_n \backslash M_n} W \left( \sigma_{2n} \left( \begin{array}{c} I_n \ X \\ I_n \end{array} \right) \left( \begin{array}{c} p \\ p \end{array} \right) \right) |\det(p)|^{s-1} \psi^{-1}(TrX) dX dp.$$

Let $J(0)(\pi)$ denote the span of the rational functions defined by the integrals $J(0)(s, W)$. As in [19, Section 3.1], $J(0)(\pi)$ is a fractional ideal of $\mathbb{C}[q^{\pm s}]$. To obtain the unique normalized generator, we need the following Lemma 2.15 [19].

**Lemma 2.4** (Belt [3]) There exists $W \in \mathcal{W}(\pi, \psi)$ such that $J(0)(s, W)$ is a non-zero constant which only depends on $\psi$.

**Remark 2.5** Non-vanishing result of the local exterior square $L$-function at the $p$-adic places is carried out as a part of the Ph.D thesis by Dustin Belt [3] supervised by Shahidi. However this result was never published. Hence we use this occasion to record this non-vanishing explicitly.

By virtue of Lemma 2.4, we let $L(0)(s, \pi, \wedge^2)$ be the Euler factor which generates the $\mathbb{C}[q^{\pm s}]$-fractional ideal $J(0)(\pi)$ in $\mathbb{C}(q^{-s})$. Following the notation from [24,25,28] about $L$-function, we denote

$$L(0)(s, \pi, \wedge^2) = \frac{L(s, \pi, \wedge^2)}{L(0)(s, \pi, \wedge^2)}.$$

**Proposition 2.6** The fractional ideal $J(0)(\pi)$ is spanned by the integral $J(s, W, \Phi)$ for $W$ in $\mathcal{W}(\pi, \psi)$ and $\Phi \in S_0(F^n)$. As $\mathbb{C}[q^{\pm s}]$-fractional ideals in $\mathbb{C}(q^{-s})$, we have the inclusion $J(0)(\pi) \subset J(\pi)$. The Euler factor $L(0)(s, \pi, \wedge^2)^{-1}$ divides $L(s, \pi, \wedge^2)^{-1}$, and the quotient

$$L(0)(s, \pi, \wedge^2) = \frac{L(s, \pi, \wedge^2)}{L(0)(s, \pi, \wedge^2)}$$

has simple poles, which are exactly the exceptional poles of $L(s, \pi, \wedge^2)$.

We know that $L_{\text{ex}}(s, \pi, \wedge^2)^{-1} = \prod_{i}(1 - q^{s_i - s})d_i$, where the $s_i$’s are the exceptional poles of $L(s, \pi, \wedge^2)$ and the $d_i$’s their orders in $L(s, \pi, \wedge^2)$. Therefore $L(0)(s, \pi, \wedge^2)$ can be expressed as $L(0)(s, \pi, \wedge^2)^{-1} = \prod_{i}(1 - q^{s_i - s})$.

To proceed further, we need to make a simplifying assumption. We assume that $\pi = \Ind(\Delta_1 \otimes \cdots \otimes \Delta_r)$ is an irreducible generic representation of $GL_{2n}$ such that all derivatives $\pi^{(m)}$ of $\pi$ are completely reducible with irreducible generic constituents. We remark that we need the assumption to utilize the induction on size $m$ of $GL_{2m}$ in the proof of Theorem 2.7. As we shall see in Sect. 4, this assumption is satisfied in all $\pi$ in “general position”. Now we introduce the factorization of the $L$-function in terms of exceptional poles of even derivatives of $\pi$ in [19, Proposition 4.1, Theorem 4.7].
Theorem 2.7 Let \( \pi \) be an irreducible generic representation of \( GL_2 \). Then

\[
L(s, \pi, \lambda^2) = L(s, \omega_\pi).
\]

For \( n > 1 \), let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be an irreducible generic representation of \( GL_{2n} \) such that all derivatives \( \pi^{(m)} \) of \( \pi \) are completely reducible with irreducible generic constituents of the form \( \pi^{2m} = \text{Ind}(\Delta_1^{(m_1)} \otimes \cdots \otimes \Delta_t^{(m_t)}) \) where \( m \) is a partition of \( m = m_1 + \cdots + m_t \). Then

(i) \( L(0)(s, \pi, \lambda^2)^{-1} = \text{l.c.m}_{2m}(L_{\text{ext}}(s, \pi^{2m}, \lambda^2)^{-1}) \) with \( 0 < m < n \)

(ii) \( L(s, \pi, \lambda^2)^{-1} = \text{l.c.m}_{2m}(L_{\text{ext}}(s, \pi^{2m}, \lambda^2)^{-1}) \) with \( 0 \leq m < n \)

where the least common multiple is with respect to divisibility in \( \mathbb{C}[q^{\pm s}] \) and is taken over all constituents \( \pi^{2m} \) of \( \pi^{(2m)} \) for each \( m \).

2.3 The functional equation of exterior square \( L \)-functions for even case \( m = 2n \)

In this Section, we review the local functional equation for exterior square \( L \)-functions from Matringe and Cogdell [7,27]. We denote by \( M_{2n} \) the standard Levi of \( GL_{2n} \) associated to the partition \((n, n)\) of \( 2n \). Let \( H_{2n} = \sigma_{2n} M_{2n} \sigma_{2n}^{-1} \). We let \( \delta_{2n} \) denote the character of \( H_{2n} \) given by \( \delta_{2n} : \sigma_{2n}(g_1, g_2) \sigma_{2n}^{-1} \mapsto \left| \frac{\det(g_1)}{\det(g_2)} \right| \). Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be a representation of \( GL_{2n} \). Let \( \tau_{2n} \) be the matrix \( \begin{pmatrix} I_n \end{pmatrix} \). As in [20], the bilinear form \( B_{s, \pi, \psi} : (W, \Phi) \mapsto J(s, W, \Phi) \frac{L(s, \pi, \lambda^2)}{L(s, \pi, \lambda^2)} \) on \( \mathcal{W}(\pi, \psi) \times S(F^n) \) satisfies \( B_{s, \pi, \psi}(\rho(h)W, R(h)\Phi) = |h|^{-\frac{n}{2}} \Theta(h)B_{s, \pi, \psi}(W, \Phi) \) for \( h \in S_{2n} \), which belongs to the space \( \text{Hom}_{S_{2n}}(\mathcal{W}(\pi, \psi) \otimes S(F^n), | \cdot |^{-\frac{n}{2}} \Theta) \). The local functional equation for irreducible generic representations in the even case is given in Matringe [27]. Cogdell and Matringe only use the assumption that \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) is the generic representation in Remark of Section 3.4 of [7] to construct the uniqueness of functionals in the space \( \text{Hom}_{S_{2n}}(\pi \otimes S(F^n), | \cdot |^{-\frac{n}{2}} \Theta) \) for almost all \( s \). As the surjection from \( \pi \) to its Whittaker model \( \mathcal{W}(\pi, \psi) \) induces an injection of \( \text{Hom}_{S_{2n}}(\mathcal{W}(\pi, \psi) \otimes S(F^n), | \cdot |^{-\frac{n}{2}} \Theta) \) to \( \text{Hom}_{S_{2n}}(\pi \otimes S(F^n), | \cdot |^{-\frac{n}{2}} \Theta) \), we obtain the following one dimensionality result.

Proposition 2.8 (Matringe) Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be a representation of \( GL_{2n} \). For all values of \( q^{-s} \) except a finite number, the space \( \text{Hom}_{S_{2n}}(\mathcal{W}(\pi, \psi) \otimes S(F^n), | \cdot |^{-\frac{n}{2}} \Theta) \) is of dimension at most 1.

The reader should notice that our integral differs from the one in [27, Theorem 4.1]. In order to incorporate the Weyl element \( \tau_{2n} \) to our functional equation, we need to check the following invariance property of bilinear form corresponding to the dual representation \( \pi^t \).

Lemma 2.9 The bilinear form on \( \mathcal{W}(\pi, \psi) \times S(F^n) \) which is defined by \( C_{s, \pi, \psi} : (W, \Phi) \mapsto B_{1-s, \pi^t, \psi^{-1}}(\rho(\tau_{2n} W, \Phi) \) belongs to \( \text{Hom}_{S_{2n}}(\mathcal{W}(\pi, \psi) \otimes S(F^n), | \cdot |^{-\frac{n}{2}} \Theta) \).

From the quasi-invariance of two bilinear forms \( B_{s, \pi, \psi} \) and \( C_{s, \pi, \psi} \), we have the following functional equation associated with the Weyl element \( \tau_{2n} \).
Theorem 2.10 Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be a representation of \( GL_{2n} \). There exists an invertible element \( \varepsilon(s, \pi, \lambda^2, \psi) \) of \( \mathbb{C}[q^{\pm s}] \), such that for every \( W \in \mathcal{V}(\pi, \psi) \), and every \( \Phi \) in \( S(F^n) \), we have the following functional equation
\[
\varepsilon(s, \pi, \lambda^2, \psi) J(s, W, \Phi) L(s, \pi, \lambda^2) = \frac{J(1-s, \rho(\tau_{2n})\tilde{W}, \tilde{\Phi})}{L(1-s, \pi', \lambda^2)}.
\]
We then define the local exterior square \( \gamma \)-factor:
\[
\gamma(s, \pi, \lambda^2, \psi) = \frac{\varepsilon(s, \pi, \lambda^2, \psi)L(1-s, \pi', \lambda^2)}{L(s, \pi, \lambda^2)}.
\]
Let \( \pi \) be an irreducible generic representation of \( GL_{2n} \). Upon the analysis of proof of [27, Theorem 4.1], we will determine the exceptional points \( s \) in \( \mathbb{C} \) on which the uniqueness of functionals in \( \text{Hom}_{S_{2n}}(\pi \otimes S(F^n), | \cdot |^{-\frac{s}{2}} \Theta) \) potentially fails. These points play a role in deformation argument, which will be detailed in Sect. 4.

Proposition 2.11 Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be an irreducible generic representation of \( GL_{2n} \). For every \( 1 \leq k \leq 2n \), suppose that all the nonzero successive quotients \( \pi^{2n-k} = \text{Ind}(\Delta_1^{(a_1)} \otimes \cdots \otimes \Delta_t^{(a_t)}) \) of the derivatives \( \pi^{(2n-k)} \) are irreducible and generic. The space \( \text{Hom}_{S_{2n}}(\pi \otimes S(F^n), | \cdot |^{-\frac{s}{2}} \Theta) \) is of dimension at most 1, except possibly when there exist \( 1 \leq k \leq n \) and one of the central characters of the nonzero constituents \( \pi^{2n-2k} \) such that
\[
\omega_{\pi^{2n-2k}}(\sigma) = q^{ks}.
\]

The proof of Proposition 2.11 is standard and is due to the argument of Matringe in [27,28] by adapting Proposition 2.12 below to our setting. This Proposition 2.12 is immediately deduced from Proposition 4.3 and Theorem 4.1 in [27].

Proposition 2.12 (Matringe) Let \( \pi \) be an irreducible generic representation of \( GL_{2n} \). There exists a real number \( r_{\pi} \) depending on \( \pi \) such that for all \( s_0 > r_{\pi} \) we have a linear injection \( \text{Hom}_{P_{2n} \cap S_{2n}}(\pi, \Theta) \) into the space \( \text{Hom}_{P_{2n} \cap H_{2n}}(\pi, \delta_{2n}^{s_0}) \).

Before we end this Section, we remark that we need the technical assumption that every nonzero constituents are irreducible and generic in Proposition 2.11 to apply Proposition 2.12 to all nonzero constituents in the proof.

2.4 The odd case \( m = 2n+1 \)

Let \( \pi \) be an irreducible generic unitary representations of \( GL_{2n+1} \). We let \( \sigma_{2n+1} \) be the permutation matrix associated with
\[
\sigma_{2n+1} = \begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n & 2n+1 \\ 1 & 3 & \cdots & 2n-1 & 2 & 4 & \cdots & 2n & 2n+1 \end{pmatrix}.
\]
In 1990, Jacquet and Shalika [18] established an integral representation for the exterior square \( L \)-function for \( GL_{2n+1} \) and Cogdell and Matringe [7] introduce the following integral to relate it with the local functional equation. For \( W \in \mathcal{V}(\pi, \psi) \) and \( \Phi \in S(F^n) \), we define the integrals:
\[
J(s, W, \Phi) = \int_{N_n \backslash GL_n} \int_{N_n \backslash M_n} \int_{F^n} \Phi(x)dx \psi^{-1}(\text{Tr}X)dX|\text{det}(g)|^{s-1}dg.
\]
It was proven in Section 9 of [18] that there is \( r_\pi \) in \( \mathbb{R} \) such that the above integrals \( J(s, W, \Phi) \) converge for \( \text{Re}(s) > r_\pi \).

Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be a representation of \( GL_{2n+1} \). There is an exact analogous of Theorem 2.3 for \( m = 2n + 1 \), see [19] for more details. The integral \( J(s, W, \Phi) \) satisfies the following functional equation [7, Proposition 3.5]

\[
\varepsilon(s, \pi, \Delta^2, \psi) \frac{J(s, W, \Phi)}{L(s, \pi, \Delta^2)} = \frac{J(1-s, \rho(\tau_{2n+1})\widehat{W}, \widehat{\Phi})}{L(1-s, \pi^t, \Delta^2)},
\]

where \( \tau_{2n+1} = \begin{pmatrix} I_n & \\ \end{pmatrix} \). We then define the local exterior square \( \gamma \)-factor:

\[
\gamma(s, \pi, \Delta^2, \psi) = \frac{\varepsilon(s, \pi, \Delta^2, \psi)L(1-s, \pi^t, \Delta^2)}{L(s, \pi, \Delta^2)}.
\]

Finally we also have the following analogue of the factorization Theorem for the odd case in [19, Proposition 5.1, Theorem 5.5].

**Theorem 2.13** Let \( \pi \) be a character of \( GL_1 \). Then

\[
L(s, \pi, \Delta^2) = 1.
\]

For \( n > 0 \), let \( \pi \) be an irreducible generic representation of \( GL_{2n+1} \) such that all derivatives \( \pi^{(m)} \) of \( \pi \) are completely reducible with irreducible generic constituents of the form \( \pi^m = \text{Ind}(\Delta_1^{(m_1)} \otimes \cdots \otimes \Delta_t^{(m_t)}) \) where \( m \) is a partition of \( m = m_1 + \cdots + m_t \). Then

\[
L(s, \pi, \Delta^2)^{-1} = \left\{ \text{l.c.m}_{m+1}[L_{ex}(s, \pi^{2m+1}, \Delta^2)]^{-1} \right\}
\]

where the least common multiple is with respect to divisibility in \( \mathbb{C}[q^{\pm 1}] \) and is taken over all \( m \) with \( m = 0, 1, \cdots, n-1 \) and for each \( m \) all constituents \( \pi^{2m+1} \) of \( \pi^{2m+1} \).

### 3 Computation of exterior square L-functions for discrete series

#### 3.1 Exterior square L-functions for supercuspidal representations

Let \( \rho \) be an irreducible supercuspidal representation of \( GL_r \). In this Section, we compute the \( L \)-functions \( L(s, \rho, \Delta^2) \) using a purely local argument. In fact, we also characterize the poles at \( s = 0 \) of the exterior square \( L \)-functions \( L(s, \Delta, \Delta^2) \) for an irreducible quasi-square integrable representation \( \Delta \) by \( \text{Hom}_{GL_{2n}}(\Delta, \Theta) \neq 0 \) in Proposition 3.4. We recall the following result about Shalika functionals of Jacquet and Rallis in [17, Proposition 6.1] plays a crucial role in our study of poles and representations.

**Theorem 3.1** (Jacquet and Rallis) Let \( \pi \) be an irreducible admissible representation of \( GL_{2n} \). Then the dimension of the space of Shalika functionals is at most one. If \( \pi \) has a non-zero Shalika functional, then \( \pi \cong \overline{\pi} \).

We would like to give a characterization of the existence of the local twisted Shalika functional in terms of the occurrence of exceptional poles of local exterior square \( L \)-functions. More generally, this property is true under the following condition that irreducible generic unitary representations satisfy.
Lemma 3.2 Let $\pi$ be an irreducible generic representation of $GL_{2n}$. Suppose that the space $Hom_{S_{2n}}(\pi, \Theta) \neq 0$ and $Hom_{P_{2n}\cap S_{2n}}(\pi, \Theta)$ is of dimension at most 1. Then $L_{ex}(s, \pi, \wedge^2)$ has a pole at $s = 0$.

Proof The proof is completely analogous to that of Theorem 1.4 in [1], Theorem 3.1 in [25], or Proposition 4.7 in [28], so we only sketch the proof. Since the space $Hom_{S_{2n}}(\pi, \Theta)$ is non-zero, $\pi$ has a trivial central character, because $\omega_\pi(z)A(W) = A(\pi(zI_{2n})W) = \Theta(zI_{2n})A(W) = A(W)$ for all $zI_{2n} \in Z_{2n}$, and $W \in \mathcal{W}(\pi, \psi)$. For $Re(s) \ll 0$, by applying the partial Iwasawa decomposition $GL_n = Z_n P_n K_n$,

$$J(1 - s, \rho(\tau_{2n})\hat{W}, \hat{\Phi}) = \int_{K_n} \int_{\mathbb{R}_n \setminus P_n} \int_{\mathbb{R}_n \setminus \mathcal{M}_n} \rho(\tau_{2n})\hat{W} \left( \sigma_{2n} \begin{pmatrix} I_n & X \\ I_n & p k \end{pmatrix} \right) \times \int_{F^x} |a|^n(1-s)\hat{\Phi}(e_nak)d^x a |\det(p)|^{-s}\psi(TrX)dXdk.$$

(3.1)

We define a functional on $\mathcal{W}(\pi, \psi)$

$$\Lambda_{1-s, \hat{\pi}} : W \mapsto \frac{J_{(0)}(1 - s, \rho(\tau_{2n})\hat{W})}{L(1 - s, \hat{\pi}, \wedge^2)},$$

which is an element of $Hom_{P_{2n}\cap S_{2n}}(\pi, | \cdot |^{-\frac{1}{2}} \Theta)$. For a fixed $W$ in $\mathcal{W}(\pi, \psi)$, the functions $s \mapsto \Lambda_{1-s, \hat{\pi}}(W)$ is a polynomial in $\mathbb{C}[q^{\pm \frac{1}{2}}]$ and therefore entire function in $s$. We may write (3.1) as

$$\frac{J(1 - s, \rho(\tau_{2n})\hat{W}, \hat{\Phi})}{L(1 - s, \hat{\pi}, \wedge^2)} = \int_{K_n} \rho(\tau_{2n})\hat{W} \left( \rho \begin{pmatrix} k \\ i \end{pmatrix} \right) \int_{F^x} |a|^n(1-s)\hat{\Phi}(e_nak)d^x a dk.$$ 

The above equality is true for all $s$ with $Re(s) < 1$ because of Tate integral. For $s = 0$ we arrive at

$$\frac{J(1, \rho(\tau_{2n})\hat{W}, \hat{\Phi})}{L(1, \hat{\pi}, \wedge^2)} = \Lambda_{1, \hat{\pi}}(W) \int_{K_n} \hat{\Phi}(e_nak)|a|^n d^x a dk.$$ 

Since any non-trivial $S_{2n}$-invariant form in $Hom_{S_{2n}}(\pi, \Theta)$ is, in particular, a non-trivial $P_{2n} \cap S_{2n}$-quasi invariant form in $Hom_{P_{2n}\cap S_{2n}}(\pi, \Theta)$ by restriction, one-dimensionality implies that $Hom_{S_{2n}}(\pi, \Theta) = Hom_{P_{2n}\cap S_{2n}}(\pi, \Theta)$. This gives

$$\frac{J(1, \rho(\tau_{2n})\hat{W}, \hat{\Phi})}{L(1, \hat{\pi}, \wedge^2)} = \Lambda_{1, \hat{\pi}}(W) \int_{K_n} \hat{\Phi}(e_nak)|a|^n d^x a dk = \Lambda_{1, \hat{\pi}}(W)\Phi(0)$$

where the last equality is from Fourier inversion. Now let us write down the functional equation of exterior square $L$-function at $s = 0$ in Theorem 2.10:

$$\Lambda_{1, \hat{\pi}}(W)\Phi(0) = \frac{J(1, \rho(\tau_{2n})\hat{W}, \hat{\Phi})}{L(1, \hat{\pi}, \wedge^2)} = \varepsilon(0, \pi, \wedge^2, \psi) \frac{J(0, W, \Phi)}{L(0, \pi, \wedge^2)}.$$

By Lemma 2.4, there exists $W_0 \in \mathcal{W}(\pi, \psi)$ such that $J_{(0)}(s, W_0)$ is a non-zero constant. Let $K^o \subset K_n$ be a sufficiently small compact open subgroup which stabilizes $W_0$. Now choose $\Phi^o \in S_0(F^n)$ to be the characteristic function of $e_nK^o$. With these choices of $W_0$ and $\Phi^o$, the integral reduces to $J(0, W_0, \Phi^o) = Vol(e_nK^o)J_{(0)}(0, W_0) \neq 0$. In addition $\varepsilon(0, \pi, \wedge^2, \psi)$ is a non-zero constants. Since $\Lambda_{1, \hat{\pi}}(W_0)\Phi^o(0)$ is 0 for $\Phi^o \in S_0(F^n)$, $L(s, \pi, \wedge^2)$ has an exceptional pole at $s = 0$. \qed

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Let $\rho$ be an irreducible supercuspidal representation of $GL_{2n}$. We have already seen in Sect. 2.2 that if $L(s, \rho, \wedge^2)$ has a pole at $s = 0$, then $\text{Hom}_{S_{2n}}(\rho, \Theta) \neq 0$. We would like to prove the converse of this statement. We introduce the following uniqueness result for unitary representations, obtained from the argument in the proof of Corollary 4.18 of [28].

Proposition 3.3 (Matringe) We let $\pi$ be an irreducible unitary generic representation of $GL_{2n}$. Then the vector space $\text{Hom}_{P_{2n} \cap H_{2n}}(\pi, \delta_{2n}^{s_0})$ is of dimension at most one for any real number $s_0 > 0$.

The following Proposition is a direct consequence of Lemma 3.2 and Proposition 3.3.

Proposition 3.4 Let $\Delta$ be an irreducible quasi-square integrable representation of $GL_{2n}$. Then $\text{Hom}_{S_{2n}}(\Delta, \Theta) \neq 0$ if and only if $L_{ex}(s, \Delta, \wedge^2)$ has a pole at $s = 0$.

Proof We have already seen in Sect. 2.2 that if $L_{ex}(s, \Delta, \wedge^2)$ has a pole at $s = 0$, then $\text{Hom}_{S_{2n}}(\Delta, \Theta) \neq 0$. Conversely suppose that $\text{Hom}_{S_{2n}}(\Delta, \Theta) \neq 0$. Since $\Delta$ has a central character and $\text{Hom}_{S_{2n}}(\Delta, \Theta)$ is non-zero, $\Delta$ has necessarily a trivial central character, and so $\Delta$ is an irreducible square integrable representation. From Proposition 2.12, the vector space $\text{Hom}_{P_{2n} \cap S_{2n}}(\Delta, \Theta)$ embeds as a subspace of $\text{Hom}_{P_{2n} \cap H_{2n}}(\Delta, \delta_{2n}^{s_0})$ for some positive real number $s_0$. Then the later space $\text{Hom}_{P_{2n} \cap H_{2n}}(\Delta, \delta_{2n}^{s_0})$ is one dimensional by Proposition 3.3. The results follows immediately from Lemma 3.2. \qed

Our aim is to obtain analogue result of Proposition 3.6 in [25] for $L(s, \rho, \wedge^2)$, where $\rho$ is an irreducible supercuspidal representation of $GL_r$. We use the Remark below to establish our main Theorem.

Remark 3.5 Let $\pi$ be an irreducible generic representation of $GL_{2n}$. We set $(W \otimes |\cdot|^{s_1})(g) = W(g) |\det(g)|^{s_1}$ for $g \in GL_{2n}$. Since $J(s + 2s_1, W, \Phi) = J(s, W \otimes |\cdot|^{s_1}, \Phi)$, it can be seen that $L(s + 2s_1, \pi, \wedge^2) = L(s, \pi^{|s_1|}, \wedge^2)$ for all $s_1 \in \mathbb{C}$.

Applying Theorem 2.7 and Theorem 2.13, $L(s, \rho, \wedge^2)$ equals to $L^{(0)}(s, \rho, \wedge^2)$ having simple poles for $r = 2n$ and $L(s, \rho, \wedge^2) = 1$ for $r = 2n + 1$. Hence poles of $L(s, \rho, \wedge^2)$ are necessarily simple if they exist. For $s_0 \in \mathbb{C}$, $\rho^{s_0} \pi$ is still a supercuspidal representation. Taking Remark 3.5 into account, replacing $\rho$ by $\rho^{s_0} \pi$ in Proposition 3.4 implies that the exceptional pole at $s = s_0$ exists if and only if $\text{Hom}_{S_{2n}}(\rho^{s_0} \pi, \Theta) \neq 0$. Since these are the only possible poles of the exterior square $L$-function, we therefore arrive at the following conclusion.

Theorem 3.6 Let $\rho$ be an irreducible supercuspidal representation of $GL_r$.

(i) If $r = 2n$, $L(s, \rho, \wedge^2)$ has simple poles and we have

$$L(s, \rho, \wedge^2) = L_{ex}(s, \rho, \wedge^2) = \prod (1 - aq^{-s})^{-1}$$

with the product over all $a = q^{s_0}$ such that $\text{Hom}_{S_{2n}}(\rho^{s_0} \pi, \Theta) \neq 0$.

(ii) If $r = 2n + 1$, we have $L(s, \rho, \wedge^2) = 1$. 
3.2 Exterior square $L$-function for quasi-square-integrable representations.

In this paragraph, we recall the results from [9,15] about $L$-function for Rankin-Selberg convolution. Let $(\rho, V_\rho)$ be a supercuspidal representation of $GL_r$ and $(\rho', V_{\rho'})$ a supercuspidal representation of $GL_r$. If $r \neq r'$, then $L(s, \rho \times \rho') = 1$. Otherwise, it is proven in [9,15] that the poles of $L(s, \rho \times \rho')$ can only occur at those $s = s_0$ where $\tilde{\rho} \simeq \rho' v^{s_0}$ and those are necessarily simple. Therefore, we have

$$L(s, \rho \times \rho') = \prod (1 - \alpha q^{-s})^{-1},$$

where $\alpha$ runs over all $\alpha = q^{s_0}$ with $\tilde{\rho} \simeq \rho' v^{s_0}$. Let $(\pi, V_\pi)$ and $(\sigma, V_\sigma)$ be irreducible quasi-square-integrable representations for $GL_n$. Let $\pi$ be the segment $\Delta = [\rho, \rho v, \ldots, \rho v^{l-1}]$. $n = \ell r$. Likewise let us take $\sigma$ with the segment $\Delta' = [\rho', \rho' v, \ldots, \rho' v'^{-1}]$ with $n = \ell' r'$. We assume that $\ell \geq \ell'$. It is proven in [9] that exceptional poles can only occur at those $s = s_0$ where $(\Delta(n-k)) \simeq \Delta(n-k)v^{s_0}$ and those are necessarily again simple. Hence we have

$$L_{ex}(s, \Delta(n-k) \times \Delta'(n-k)) = \prod (1 - \alpha q^{-s})^{-1}$$

where $\alpha$ accounts for all $\alpha = q^{s_0}$ with $(\Delta(n-k)) \simeq \Delta(n-k)v^{s_0}$ and exceptional $L$-functions $L_{ex}(s, \Delta(n-k) \times \Delta'(n-k))$ are defined in Sect. 2 of [9]. By considering the derivatives of the associated segments, we can conclude that

$$L_{ex}(s, \Delta(n-k) \times \Delta'(n-k)) = L(\ell - 1 + j + s, \rho \times \rho')$$

for appropriate $i$ and $j$ such that $\ell - i = \ell' - j$. Then we arrive at the following result.

$$L(s, \Delta \times \Delta') = \prod_{j=0}^{\ell'-1} L(\ell - 1 + j + s, \rho \times \rho').$$

Furthermore, $L(s, \Delta \times \Delta')$ has simple poles. Let us investigate $L(0)(s, \Delta, \chi^2)$ defined in Proposition 2.6 to see how $L(0)(s, \Delta, \chi^2)$ can be used to establish the divisibility of $L$-functions.

**Lemma 3.7** Let $\Delta$ be an irreducible quasi-square-integrable representation of $GL_{2n}$ with the segment $\Delta = [\rho, \rho v, \ldots, \rho v^{l-1}]$ and $\rho$ an irreducible supercuspidal representation of $GL_r$ with $2n = \ell r$. Then $L(0)(s, \Delta, \chi^2)$ divides $L(\ell - 1 + s, \rho \times \rho^{-1})$ in $\mathbb{C}[q^{\pm s}]$.

**Proof** We assume that $L(s, \Delta, \chi^2)$ has a pole at $s = s_0$. As $L(0)(s + s_0, \Delta, \chi^2)$ is equal to $L(0)(s, \Delta v^{\frac{\pi}{2}}, \chi^2), s = 0$ is an exceptional pole of $L(0)(s, \Delta v^{\frac{\pi}{2}}, \chi^2)$. Then we have seen in Sect. 2.2 that there is a non-zero twisted Shalika functional which belong to Hom$(\Delta v^{\frac{\pi}{2}}, \Theta)$. Theorem 3.1 of Jacquet and Rallis asserts that $\Delta v^{\frac{\pi}{2}}$ is self-contragredient. On the other hand, the segment of $\Delta v^{\frac{\pi}{2}}$ can be written as $[\rho v^{\frac{\pi}{2}}, \ldots, \rho v^{l-1+\frac{\pi}{2}}]$ and $(\Delta v^{\frac{\pi}{2}}) \sim$ a quasi-square-integrable representation associated to $[\tilde{\rho} v^{-\ell+1-\frac{\pi}{2}}, \ldots, \tilde{\rho} v^{-2-\frac{\pi}{2}}]$. Hence we see that $(\Delta v^{\frac{\pi}{2}}) \sim \Delta v^{\frac{\pi}{2}}$ if and only if $\tilde{\rho} v^{-\ell+1-\frac{\pi}{2}} \simeq \rho v^{2+\frac{\pi}{2}}$. The last condition is exactly the same condition that the $L$-function $L(\ell - 1 + s, \rho \times \rho)$ has a pole at $s = s_0$, and this pole will be simple. Since both of the $L$-functions $L(0)(s, \Delta, \chi^2)$ and $L(\ell - 1 + s, \rho \times \rho)$ have simple poles, $L(0)(s, \Delta, \chi^2)$ divides $L(\ell - 1 + s, \rho \times \rho^{-1})$ in $\mathbb{C}[q^{\pm s}]$.

$L(0)(s, \Delta, \chi^2)$ in Lemma 3.7 can be replaced by the exceptional $L$-function $L_{ex}(s, \Delta, \chi^2)$. We utilize the derivative theory of Bernstein and Zelevinsky in Theorem 2.2 for irreducible
We have since \( L \) common poles, we arrives at \( k \) different

\[ L_{\text{ex}}(s, \Delta, \wedge^2) = L^{(0)}(s, \Delta, \wedge^2). \]

Moreover \( L_{\text{ex}}(s, \Delta, \wedge^2)^{-1} \) divides \( L(\ell - 1 + s, \rho \times \rho)^{-1} \) in \( \mathbb{C}[q^{\pm s}] \).

**Proof** We investigate the case that \( r \) is even. Suppose that \( \ell = 1 \). Then \( \Delta = \rho \) is an irreducible supercuspidal representation. By Theorem 3.6, we can conclude that \( L_{\text{ex}}(s, \rho, \wedge^2) = L^{(0)}(s, \rho, \wedge^2) \). We show the statement of Lemma by induction on \( \ell \). We now assume that the following statement is satisfied for all positive integers \( 1 \leq k \leq \ell \)

\[ L_{\text{ex}}(s, [\rho, \ldots, \rho v^{k-1}], \wedge^2) = L^{(0)}(s, [\rho, \ldots, \rho v^{k-1}], \wedge^2). \]

Throughout this proof the least common multiple is always taken in terms of divisibility in \( \mathbb{C}[q^{\pm s}] \). According to Theorem 2.7, we have

\[ L^{(0)}(s, [\rho, \ldots, \rho v^\ell], \wedge^2)^{-1} = \text{l.c.m.}_{1 \leq k \leq \ell} \left\{ L_{\text{ex}}(s, [\rho v^k, \ldots, \rho v^\ell], \wedge^2)^{-1} \right\}. \]

We can rewrite \([\rho', \ldots, \rho' v^{\ell-k}]\) for \([\rho v^k, \ldots, \rho v^\ell]\) with \(\rho' = \rho v^k\) an irreducible supercuspidal representation. Induction hypothesis on the length of \([\rho', \ldots, \rho' v^{\ell-k}]\) is applicable and we have

\[ L^{(0)}(s, [\rho, \ldots, \rho v^\ell], \wedge^2)^{-1} = \text{l.c.m.}_{1 \leq k \leq \ell} \left\{ L^{(0)}(s, [\rho v^k, \ldots, \rho v^\ell], \wedge^2)^{-1} \right\}. \]

We deduce from Lemma 3.7 that \( L^{(0)}(s, [\rho v^k, \ldots, \rho v^\ell], \wedge^2)^{-1} \) divides \( L(s + \ell - k, \rho v^k \times \rho v^k)\) for \(0 \leq k \leq \ell\). As \( L(s + \ell - k, \rho v^k \times \rho v^k)^{-1} = L(s + \ell + k, \rho \times \rho)^{-1}\) for different \( k \) with \(0 \leq k \leq \ell\) are relative prime in \( \mathbb{C}[q^{\pm s}] \), so are \( L^{(0)}(s, [\rho, \ldots, \rho v^\ell], \wedge^2)^{-1}\) and \( L^{(0)}(s, [\rho, \ldots, \rho v^\ell], \wedge^2)^{-1}\).

We notice from Proposition 2.6 that

\[ L(s, [\rho, \ldots, \rho v^\ell], \wedge^2) = L^{(0)}(s, [\rho, \ldots, \rho v^\ell], \wedge^2) L^{(0)}(s, [\rho, \ldots, \rho v^\ell], \wedge^2). \]

Since \( L \)-functions \( L^{(0)}(s, [\rho, \ldots, \rho v^\ell], \wedge^2) \) and \( L^{(0)}(s, [\rho, \ldots, \rho v^\ell], \wedge^2) \) do not share any common poles, we arrives at

\[ L_{\text{ex}}(s, [\rho, \ldots, \rho v^\ell], \wedge^2) = L^{(0)}(s, [\rho, \ldots, \rho v^\ell], \wedge^2). \]

For \( r \) odd, we can argue the same way.

We begin with computing local exterior square \( L \)-functions for irreducible quasi-square-integrable representations for \( GL_{2n} \).

**Proposition 3.9** Let \( \Delta \) be an irreducible quasi-square-integrable representation of \( GL_{2n} \) with the segment \( \Delta = [\rho, \rho v, \ldots, \rho v^{\ell-1}] \) and \( \rho \) an irreducible supercuspidal representation of \( GL_r \) with \( 2n = tr \).

(i) Suppose that \( r \) is even. Then

\[ L(s, \Delta, \wedge^2) = \prod_{i=1}^{n} L_{\text{ex}}(s, \Delta^{(2n-2i)}, \wedge^2) = \prod_{k=0}^{\ell-1} L_{\text{ex}}(s, [\rho v^k, \rho v^{k+1}, \ldots, \rho v^{\ell-1}], \wedge^2). \]
(ii) Suppose that \( r \) is odd. Then
\[
L(s, \Delta, \wedge^2) = \prod_{i=1}^{n} L_{ex}(s, \Delta^{(2n-2i)}, \wedge^2) = \prod_{k=0}^{\ell - 1} L_{ex}(s, [\rho v^{2k}, \rho v^{2k+1}, \ldots, \rho v^{\ell - 1}], \wedge^2).
\]

**Proof** Throughout this proof, the least common multiple is always taken in terms of divisibility in \( \mathbb{C}[q^{\pm s}] \). As the proof for \( r \) odd is almost identical, we will produce only the case for \( r \) even. We have seen in Theorem 2.7 that the poles of \( L \)-function \( L(s, \Delta, \wedge^2) \) are precisely the poles of the exceptional contribution by the \( L \)-functions for even derivatives. In fact,
\[
L(s, \Delta, \wedge^2)^{-1} = \frac{1}{l.c.m.} \{L_{ex}(s, [\rho v^k, \ldots, \rho v^{\ell - 1}], \wedge^2)^{-1}\}.
\]

By Lemma 3.8, \( L_{ex}(s, [\rho v^k, \ldots, \rho v^{\ell - 1}], \wedge^2)^{-1} \) divides \( L(s + \ell - 1 - k, \rho v^k \times \rho v^k)^{-1} = L(s + \ell - 1 + k, \rho \times \rho)^{-1} \). As \( L(s + \ell - 1 + k, \rho \times \rho)^{-1} \) for different \( k \) are relatives prime in \( \mathbb{C}[q^{\pm s}] \), the least common multiple in \( L(s, \Delta, \wedge^2) \) can be reduced to
\[
L(s, \Delta, \wedge^2)^{-1} = \prod_{k=0}^{\ell - 1} L_{ex}(s, [\rho v^k, \ldots, \rho v^{\ell - 1}], \wedge^2)^{-1}.
\]

This completes the proof. \( \square \)

We provide the analogous results of Proposition 3.9 for \( GL_{2n+1} \).

**Proposition 3.10** Let \( \Delta \) be an irreducible quasi-square integrable representation of \( GL_{2n+1} \) with the segment \( \Delta = \{\rho, \rho v, \ldots, \rho v^{\ell - 1}\} \) and \( \rho \) an irreducible supercuspidal representation of \( GL_r \) with \( 2n + 1 = r \ell \) and \( \ell > 1 \). Then
\[
L(s, \Delta, \wedge^2) = \prod_{i=1}^{n} L_{ex}(s, \pi^{(2n+1-2i)}, \wedge^2) = \prod_{k=1}^{\ell - 1} L_{ex}(s, [\rho v^{2k-1}, \rho v^{2k}, \ldots, \rho v^{\ell - 1}], \wedge^2).
\]

**Proof** From Theorem 2.13, we know that
\[
L(s, \Delta, \wedge^2)^{-1} = \frac{1}{l.c.m.} \{L_{ex}(s, [\rho v^{2k-1}, \rho v^{2k}, \ldots, \rho v^{\ell - 1}], \wedge^2)^{-1}\},
\]
where the least common multiple is taken in terms of divisibility in \( \mathbb{C}[q^{\pm s}] \). According to Lemma 3.8, \( L_{ex}(s, [\rho v^{2k-1}, \rho v^{2k}, \ldots, \rho v^{\ell - 1}], \wedge^2)^{-1} \) divides \( L(s + \ell - 2k, \rho v^{2k-1} \times \rho v^{2k-1})^{-1} = L(s + \ell - 2k, \rho \times \rho)^{-1} \) in \( \mathbb{C}[q^{\pm s}] \) for \( 1 \leq k \leq \frac{\ell - 1}{2} \). As \( L(s + \ell - 2k, \rho \times \rho)^{-1} \) for different \( k \) are relatives prime in \( \mathbb{C}[q^{\pm s}] \), we have
\[
L(s, \Delta, \wedge^2)^{-1} = \prod_{k=1}^{\frac{\ell - 1}{2}} L_{ex}(s, [\rho v^{2k-1}, \rho v^{2k}, \ldots, \rho v^{\ell - 1}], \wedge^2).
\]

\( \square \)

Let \( \Delta \) be an irreducible square integrable representation of \( GL_{2n} \) with the segment \( \Delta = \{\rho v^{-\ell - 1}, \ldots, \rho v^{\ell - 1}\} \) and \( \rho \) an irreducible unitary supercuspidal representation of \( GL_r \) with \( 2n = \ell r \). Then
\[
L(s, \Delta \times \Delta) = \prod_{k=0}^{\ell - 1} L \left( s + \ell - 1 + k, \rho v^{-\ell - 1} \times \rho v^{\ell - 1} \right) = \prod_{k=0}^{\ell - 1} L(s + k, \rho \times \rho). \tag{3.2}
\]
Applying Proposition 3.9 to $L(s, \Delta, \wedge^2)$, we have

$$L(s, \Delta, \wedge^2) = \prod_{k=0}^{\ell-1} L_{ex} \left( s, \left[ \rho v^{2k+\frac{1-\ell}{2}}, \ldots, \rho v^{\frac{\ell-1}{2}} \right], \wedge^2 \right)$$

if $r$ is even or

$$L(s, \Delta, \wedge^2) = \prod_{k=0}^{\frac{\ell-1}{2}} L_{ex} \left( s, \left[ \rho v^{2k+\frac{1-\ell}{2}}, \ldots, \rho v^{\frac{\ell-1}{2}} \right], \wedge^2 \right)$$

if $r$ is odd. We employ Lemma 3.8 to deduce that $L_{ex}(s, [\rho v^{j+\frac{1-\ell}{2}}, \ldots, \rho v^{\frac{\ell-1}{2}}], \wedge^2)^{-1}$ divides $L(s + \ell - 1 - j, \rho v^{j+\frac{1-\ell}{2}} \times \rho v^{\frac{\ell-1}{2}})^{-1} = L(s + j, \rho \times \rho)^{-1}$ for $j = k$ or $2k$. Hence $L(s, \Delta, \wedge^2)^{-1}$ divides $L(s, \Delta \times \Delta)^{-1} = \prod_{k=0}^{\ell-1} L(s + k, \rho \times \rho)^{-1}$ in $\mathbb{C}[q^{\pm s}]$. $L(s, \Delta, \wedge^2)$ has simple poles because $L(s, \Delta \times \Delta)$ has simple poles. Since $\rho$ is unitary, the pole of $L(s, \rho \times \rho)$ must lie on the line $\text{Re}(s) = 0$. Thus the pole of $L(s, \Delta, \wedge^2)$ will lie on the line $\text{Re}(s + k) = 0$ or $\text{Re}(s) = -k$ for $k = 0, \ldots, \ell - 1$. Since these are the only possible poles of the exterior square $L$-function, we get as a Proposition the main result of [20].

**Proposition 3.11** Let $\Delta$ be an irreducible square integrable representation of $GL_{2n}$ with the segment $\Delta = [\rho v^{-\frac{1}{2}}, \ldots, \rho v^{\frac{1}{2}}]$ and $\rho$ a supercuspidal representation of $GL_r$ with $2n = \ell r$. Then $L(s, \Delta, \wedge^2)$ has no poles in $\mathbb{R}(s) > 0$ and has simple poles.

(i) Suppose $r$ is even. Then

$$L(s, \Delta, \wedge^2) = \prod_{i=1}^{n} L_{ex} \left( s, \Delta^{(2n-2i)}, \wedge^2 \right) = \prod_{k=0}^{\ell-1} L_{ex} \left( s, \left[ \rho v^{2k+\frac{1-\ell}{2}}, \ldots, \rho v^{\frac{\ell-1}{2}} \right], \wedge^2 \right).$$

(ii) Suppose $r$ is odd. Then

$$L(s, \Delta, \wedge^2) = \prod_{i=1}^{n} L_{ex} \left( s, \Delta^{(2n-2i)}, \wedge^2 \right) = \prod_{k=0}^{\frac{\ell-1}{2}} L_{ex} \left( s, \left[ \rho v^{2k+\frac{1-\ell}{2}}, \ldots, \rho v^{\frac{\ell-1}{2}} \right], \wedge^2 \right).$$

Moreover $L(s, \Delta, \wedge^2)^{-1}$ divides $L(s, \Delta \times \Delta)^{-1}$ in $\mathbb{C}[q^{\pm s}]$.

The following Proposition characterizes the irreducible square integrable representation of $GL_{2n}$ admitting a Shalika functional in terms of poles of exterior square $L$-function. This result is also presented in Section 6 of [27].

**Proposition 3.12** Let $\Delta$ be an irreducible square integrable representation of $GL_{2n}$. Then $\text{Hom}_{S_{2n}}(\Delta, \Theta) \neq 0$ if and only if $L_{ex}(s, \Delta, \wedge^2)$ has a pole at $s = 0$, or equivalently if and only if $L(s, \Delta, \wedge^2)$ has a pole at zero.

**Proof** The first equivalence is proved in Proposition 3.4. Now $L_{ex}(s, \Delta, \wedge^2)$ is the only factor having pole at $\text{Re}(s) = 0$, because we can apply Lemma 3.8 and Proposition 3.11 so that $L_{ex}(s, [\rho v^{j+\frac{1-\ell}{2}}, \ldots, \rho v^{\frac{\ell-1}{2}}], \wedge^2)^{-1}$ divides the factor $L(s + k, \rho \times \rho)^{-1}$. This asserts the second equivalence condition. □
Let $\Delta$ be an irreducible square integrable representation of $GL_{2n+1}$ with the segment $\Delta = [\rho \nu^{-\frac{c_1}{2}}, \ldots, \rho \nu^{-\frac{c_1}{2}}]$ and $\rho$ an unitary supercuspidal representation of $GL_r$ with $2n+1 = r \ell$ and $\ell > 1$. We deduce from Proposition 3.8 and Proposition 3.10 that

$$L(s, \Delta, \wedge^2)^{-1} = \prod_{k=1}^{\ell-1} L_{ex}(s, \left[\rho \nu^{2k-1+\frac{1-\ell}{2}}, \ldots, \rho \nu^{\frac{1-\ell}{2}}\right], \wedge^2)^{-1}$$

divides products of Rankin-Selberg $L$-function

$$\prod_{k=1}^{\ell-1} L(s + 2k - 1, \rho \times \rho)^{-1} \equiv \prod_{k=1}^{\ell-1} L(\rho Ls - 2k, \rho \nu^{1+\frac{1-\ell}{2}} \times \rho \nu^{\frac{1-\ell}{2}})^{-1}$$

in $\mathbb{C}[q^{\pm s}]$. $L(s, \Delta, \wedge^2)^{-1}$ in turn divides $L(s, \Delta \times \Delta)^{-1}$ in $\mathbb{C}[q^{\pm s}]$ from (3.2) and $L(s, \Delta, \wedge^2)$ has simple poles. Since $\rho$ is unitary, the pole of $L(s, \rho \times \rho)$ must lie on the line $\text{Re}(s) = 0$. Hence the only possible poles of the exterior square $L$-function will lie on the line $\text{Re}(s + 2k - 1) = 0$ or $\text{Re}(s) = -2k + 1$ for $k = 1, \ldots, \ell - 1$. Thus we get as a Proposition the main result of [20].

**Proposition 3.13** We let $\Delta$ be an irreducible square integrable representation of $GL_{2n+1}$ with the segment $\Delta = [\rho \nu^{-\frac{c_1}{2}}, \ldots, \rho \nu^{-\frac{c_1}{2}}]$ and $\rho$ an irreducible unitary supercuspidal representation of $GL_r$ with $2n+1 = r \ell$ and $\ell > 1$. Then $L(s, \Delta, \wedge^2)$ has simple poles and has no poles in $\text{Re}(s) \geq 0$. Moreover,

$$L(s, \Delta, \wedge^2) = \prod_{i=1}^{n} L_{ex}(s, \Delta^{2n+1-2i}, \wedge^2) = \prod_{k=1}^{\ell-1} L_{ex}(s, \left[\rho \nu^{2k-1+\frac{1-\ell}{2}}, \ldots, \rho \nu^{\frac{1-\ell}{2}}\right], \wedge^2).$$

Moreover $L(s, \Delta, \wedge^2)^{-1}$ divides $L(s, \Delta \times \Delta)^{-1}$ in $\mathbb{C}[q^{\pm s}]$.

### 4 Deformation and Local L-Functions

#### 4.1 Deformation and general position

The aim of this Section is to define what we mean by “in general position” for a representation of $GL_m$. Let $\pi = \text{Ind}_{Q}^{GL_m}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ be a representation of $GL_m$, where each $\Delta_t$ is a quasi-square-integrable representation of $GL_m$, $m = m_1 + \cdots + m_t$, and the induction is normalized parabolic induction from the standard parabolic subgroup $Q$ of $GL_m$ associated to the partition $(m_1, \ldots, m_t)$. In general, the constituents of the derivatives of these representations are not quasi-square-integrable thus we do not have a good way of explicitly analyzing the occurrence of given possible poles. To resolve these difficulties we deform the representation. We recall a few facts from [9, Section 3] about deformations of representations and behavior of the derivatives. Let $M \simeq GL_{m_1} \times \cdots \times GL_{m_t}$ denote the Levi subgroup of $Q$. If $u = (u_1, \ldots, u_t) \in \mathbb{C}^t$ then $u$ defines an unramified character $v^u$ of $M$ via $v^u(g_1, \ldots, g_t) = v(g_1)^{u_1} \cdots v(g_t)^{u_t}$ for $(g_1, \ldots, g_t) \in M$. Let $D_{\pi}$ denote the complex manifold $(\mathbb{C}^{2^m})^{2^m}$. The map $u \mapsto (q^{u_1}, \ldots, q^{u_t})$ induces a group isomorphism between $D_{\pi}$ and $(\mathbb{C}^{2^m})^{2^m}$. For convenience, we will let $q^u$ denote $(q^{u_1}, \ldots, q^{u_t})$. We will use $q^{\pm u}$ as short for $(q^{\pm u_1}, \ldots, q^{\pm u_t})$. For each $u \in D_{\pi}$, we may define the representation $\pi_u$ by

$$\pi_u = \text{Ind}(\Delta_1 v^{u_1} \otimes \cdots \otimes \Delta_t v^{u_t}).$$
We note that each representation $\pi_u$ is a generic representation of $GL_m$.

Suppose that the quasi-square-integrable representation $\Delta_i$ corresponds to the segment $[\rho_i, \rho_i v, \ldots, \rho_i v^{\ell_i-1}]$ where $\rho_i$ is a supercuspidal representation of $GL_{\ell_i}$. Then $\Delta_i$ is a representation of $GL_{m_i}$ where $m_i = r_i \ell_i$ and $m = \sum m_i = \sum r_i \ell_i$. We say that two segments $\Delta_i = [\rho_i, \rho_i v, \ldots, \rho_i v^{\ell_i-1}]$ and $\Delta_j = [\rho_j, \rho_j v, \ldots, \rho_j v^{\ell_j-1}]$ are linked if as sets $\Delta_i \not\subseteq \Delta_j$, $\Delta_j \not\subseteq \Delta_i$, and $\Delta_i \cup \Delta_j$ is again a segment. Zelevinsky in [35] establishes that the representations $\pi$ is irreducible as long as the segments corresponding to the $\Delta_i$ are unlinked.

Now consider the derivative of $\pi$ with the determinantal character $v^u_i$ then, setting $\Delta_i, u_i = \Delta_i v^u_i$, we see that $\Delta_i, u_i$ is again quasi-square-integrable representation associated to $[(\rho_i v^u_i), (\rho_j v^u_i) v, \ldots, (\rho_j v^u_i) v^{\ell_i-1}]$. By the result of Bernstein and Zelevinsky in Theorem 2.2, the $k^{th}$ derivatives $\pi_u^{(k)}$ will be glued from the representation $\text{Ind}(\Delta_{1, u_1}^{(k_1)} \otimes \cdots \otimes \Delta_{t, u_t}^{(k_t)})$ for all possible partitions $k = k_1 + \cdots + k_t$ with $0 \leq k_j \leq m_i$. We let set

$$\pi_u^{(k_1, \ldots, k_t)} = \text{Ind}(\Delta_{1, u_1}^{(k_1)} \otimes \cdots \otimes \Delta_{t, u_t}^{(k_t)}) \circ \text{Ind}(\Delta_{1, u_1}^{(k_1)} \otimes \cdots \otimes \Delta_{t, u_t}^{(k_t)})^t.$$

If we consider $\Delta_i, u_i$ which is associated to $[(\rho_i v^u_i), \ldots, (\rho_i v^u_i) v^{\ell_i-1}]$ we see that $\Delta_i, u_i$ is zero unless $k_i = a_i r_i$ with $0 \leq a_i \leq \ell_i$ and $\Delta_{a_i, u_i}^{(a_i r_i)}$ is the quasi-square-integrable representation attached to the segment $[(\rho_i v^a_i) v, \ldots, (\rho_i v^a_i) v^{\ell_i-1}]$. So we have that $\pi_u^{(k_1, \ldots, k_t)} = 0$ unless $(k_1, \ldots, k_t) = (a_1 r_1, \ldots, a_t r_t)$ with $0 \leq a_i \leq \ell_i$.

**Definition** Let $\pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ be a representation of $GL_m$. We assume that $\Delta_i$ is an irreducible quasi-square integrable representation of $GL_{m_i}$ where $m = \sum m_i = \sum r_i \ell_i$. We say that $u = (u_1, \ldots, u_t) \in \mathcal{D}_\pi$ is in general position if it satisfies the following properties:

1. For every sequence of nonnegative integers $(k_1, \ldots, k_t)$ such that the representation

$$\pi_u^{(k_1, \ldots, k_t)} = \text{Ind}(\Delta_{1, u_1}^{(k_1)} v^{u_1} \otimes \cdots \otimes \Delta_{t, u_t}^{(k_t)} v^{u_t})$$

this representation is irreducible.

2. If $(a_1 r_1, \ldots, a_t r_t)$ and $(b_1 r_1, \ldots, b_t r_t)$ are two different sequences such that

$$\sum_{i=1}^t a_i r_i = \sum_{i=1}^t b_i r_i,$$

then the two representations

$$\text{Ind}(\Delta_1^{(a_1 r_1)} v^{u_1} \otimes \cdots \otimes \Delta_t^{(a_t r_t)} v^{u_t}) \quad \text{and} \quad \text{Ind}(\Delta_1^{(b_1 r_1)} v^{u_1} \otimes \cdots \otimes \Delta_t^{(b_t r_t)} v^{u_t})$$

have distinct central characters.

3. If $(i, j) \in \{1, \ldots, t\}$ with $i \neq j$, then $L(s, \Delta_i v^{u_i}, \lambda^2)$ and $L(s, \Delta_j v^{u_j}, \lambda^2)$ have no common poles.

4. If $(i, j, k, l) \in \{1, \ldots, t\}$ with $\{i, j\} \neq \{k, l\}$, then $L(s, \Delta_i v^{u_i} \times \Delta_j v^{u_j})$ and $L(s, \Delta_k v^{u_k} \times \Delta_l v^{u_l})$ have no common poles.

5. If $(i, j, k) \in \{1, \ldots, t\}$ with $i \neq j$, then $L(s, \Delta_i v^{u_i} \times \Delta_j v^{u_j})$ and $L(s, \Delta_k v^{u_k}, \lambda^2)$ have no common poles.

6. If $1 \leq i \neq j \leq t$ and $(\Delta_i^{(a_i r_i)})^e \simeq \Delta_i^{(a_j r_j)} v^e$ for some complex number $e$, then the space

$$\text{Hom}_{p(2 \inj_{-a_i r_i})} S_{2 \inj_{-a_i r_i}} (\text{Ind}(\Delta_i^{(a_i r_i)} v^{u_i-a_i r_i+e} \otimes \Delta_i^{(a_j r_j)} v^{u_j-a_j r_j+e})), \Theta)$$

is of dimension at most 1.
The important point is that the notion \( u \in D_\pi \) in general position depends only on the representation \( \pi \). The purpose of (2) is that outside of a finite number on hyperplanes in the \( u \in D_\pi \) the central characters of the \( \pi_u^{(a_1r_1, \ldots, a_lr_l)} \) will be distinct and so there are only trivial extension among these representations. Therefore off these hyperplanes, the derivatives \( \pi_u^{(k)} = \oplus \pi_u^{(a_1r_1, \ldots, a_lr_l)} \) are completely reducible, where \( k = \sum a_ir_i \) and each \( \pi_u^{(a_1r_1, \ldots, a_lr_l)} \) is irreducible. The reason for condition (6) is that the occurrence of an exceptional pole of \( L(s, \pi, \wedge^2) \) at \( s = 0 \) is related to the occurrence of Shalika functional for \( \pi \) from Lemma 3.2 under this condition (6).

We now check that outside a finite number of hyperplanes in \( u \), the deformed representation \( \pi_u \) is in general position.

**Proposition 4.1** Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_l) \) be a representation of \( GL_m \). We assume that \( \Delta_i = [\rho_i, \rho_i \pi, \ldots, \rho_i \pi^{l_i-1}] \) is a representation of \( GL_{m_i} \) where \( \rho_i \) is an irreducible supercuspidal representation of \( GL_{r_i}, m_i = r_i \ell_i \) and \( m = \sum m_i = \sum r_i \ell_i \). The elements \( u \in D_\pi \) which is not in general position belong to a finite number of affine hyperplanes.

**Proof** The condition (1), (2) and (4) are explained in [28, Proposition 5.2] and conditions (3) and (5) can be checked as in [28, Proposition 5.2]. (6) is a consequence of Proposition 2.12.

\[ \square \]

The crucial aspect of considering deformed representations \( \pi_u \) in general position is that \( L(s, \pi_u, \wedge^2) \) are easy to analyze for those representations. In Section 5, we eventually specialize the result to \( u = 0 \) which may not be in general position.

### 4.2 Rationality of Jacquet–Shalika integrals

In this Section, we closely follow the standard argument appear in [9, Section 3.2], so we adopt the terminologies from [9, Section 3.2]. Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_l) \) be a representation of \( GL_{2n} \) and for each \( u \in D_\pi \) we define the deformed representation by \( \pi_u = \text{Ind}(\Delta_1 \pi^{a_1} \otimes \cdots \otimes \Delta_l \pi^{a_l}) \). We realize each of these representations on the common vector space \( F_\pi = \text{Ind}_Q^{K_{2n}}(\Delta_1 \otimes \cdots \otimes \Delta_l) \). Let \( W^{(0)}_\pi \) be the Whittaker space attached to \( \pi_u \). The reader should consult [9,28] for a complete account. However as our integrals are not identical to those of \( GL_n \times GL_m \) treated in [9], we repeat the essential points by making clear how the proofs have to be modified and filling in some of the missing arguments. For \( W_u \in W^{(0)}_\pi \), and \( \Phi \in S(F^n) \), the local integral for \( GL_{2n} \) are

\[
J(s, W_u, \Phi) = \int_{N_n \backslash GL_n} \int_{N_n \backslash M_n} W_u \left( \sigma_{2n} \left( \begin{array}{cc} I_n & X \\ I_n & I_n \end{array} \right) \left( \begin{array}{c} g \\ g \end{array} \right) \right) \psi^{-1}(\text{Tr}X)dX \Phi(e_n g) |\det(g)|^s dg.
\]

We exploit Bernstein’s Theorem to show that \( J(s, W_u, \Phi) \) for \( W_u \in W^{(0)}_\pi \) and \( \Phi \in S(F^n) \) are rational functions in \( q^{\pm u} \) and \( q^{\pm s} \). In general, Bernstein’s Theorem is applied to outside an undermined complement of a countable proper subvarieties of \( D \) whereas we take general position, which is outside of a very specific specific set (Proposition 4.1), into account.

**Proposition 4.2** Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_l) \) be a representation of \( GL_{2n} \). With the above notation, for \( W_u \in W^{(0)}_\pi \) and \( \Phi \in S(F^n) \) the integral \( J(s, W_u, \Phi) \) is a rational function of \( q^{-u} \) and \( q^{-s} \).
The local exterior square $L$-functions

Proof As in [9,28], our method will be to employ the Bernstein’s Theorem regarding solutions of a polynomial family of affine equations. Let $V = \mathcal{F}_\pi \otimes_\mathbb{C} S(F^n)$ be the underlying vector space whose dimension is countable over $\mathbb{C}$ and let $V^* = \text{Hom}_\mathbb{C}(V, \mathbb{C})$ denote its algebraic dual. We let $D = D_\pi \times D_s$, where $D_s = (\mathbb{C}/ \frac{2\pi i}{\log(q)}) \simeq \mathbb{C}^\times$. Let us choose a basis $\{ f_i \}$ of $\mathcal{F}_\pi$ and a basis $\{ \Phi_k \}$ of $S(F^n)$. For a fixed $d = (u, s) \in D$, we are given a system of equation $\mathcal{E}'_d$:

$$\left\{ \left( \begin{array}{c} \pi_u(g) \\ g \end{array} \right) \pi_u(g_i) f_i \otimes R \left( \begin{array}{c} g \\ g \end{array} \right) \Phi_k - |\text{det}(g)|^{-\delta} \pi_u(g_i) f_i \otimes \Phi_k, 0 \right\}$$

$$\left| g \in GL_n, \ g_i \in GL_{2n} \right\}$$

$$\bigcup \left\{ \left( \begin{array}{c} I_n \\ Z \end{array} \right) \pi_u(g_i) f_i \otimes R \left( \begin{array}{c} I_n \\ Z \end{array} \right) \Phi_k - \psi(\text{Tr}Z) \pi_u(g_i) f_i \otimes \Phi_k, 0 \right\}$$

According to Theorem 2.3 along with Proposition 2.11 and Proposition 4.1, we define $\Omega \subset D$ by the conditions that $\Omega$ is the intersection of, the complements of the hyperplanes on which each deformed representation $\pi_u(a_i r_1, \ldots, a_i r_t)$ is reducible, and some half spaces $\text{Re}(L(a_1, \ldots, a_t)(u, s)) > 0$ on which our integrals converge absolutely and the uniqueness up to scalar holds. Here the linear forms $L(a_1, \ldots, a_t)(u, s)$ are given by

$$q^{L(a_1, \ldots, a_t)(u, s)} = q^{k_s + \sum_{i=1}^t (\ell_i - a_i) r_i u_i} \prod_{i=1}^t \omega_{\Delta_{0}}^{1}(\sigma)$$

(4.1)

for any $0 \leq a_i \leq \ell_i$ satisfying $\sum a_i r_i = 2n - 2k$ for some $1 \leq k \leq n$. In light of proof of Lemma 2.4 or [3, Lemma 2.3] together with Proposition [9, Proposition 3.1], we add the single normalization equation

$$\left( \sum_i \pi_u(g_i) h_i \otimes \Phi', cP(q^{\frac{1}{2}u}) \right)$$

(4.1)

for appropriate $g_i \in GL_{2n}$, $h_i \in \mathcal{F}_\pi$, $\Phi' \in S(F^n)$ and $c > 0$ a volume. We then remove the scalar ambiguity in our system of equations on $\Omega$. Let $\mathcal{E}_d$ be the system $\mathcal{E}'_d$ with the equation (Nd) adjoined. A family of system $\{ E_d \ | \ d \in D \}$ is a polynomial family of system of equations. For each $d \in \Omega$, the system $\mathcal{E}_d$ has a unique solution $\lambda_d \in V^*$ which denote the linear functional $\lambda_d : f \otimes \Phi \mapsto J(s, W_{f,u}, \Phi)$.

Let $\mathcal{M} = \mathbb{C}(D)$ be the field of fractions of $\mathbb{C}[D]$. We denote by $V_{\mathcal{M}}$ the space $\mathcal{M} \otimes_\mathbb{C} V$ and by $V_{\mathcal{M}}^*$ the space $\text{Hom}_\mathcal{M}(V_{\mathcal{M}}, \mathcal{M})$. Bernstein’s Theorem appeared first in [9] or [2,28, Theorem 2.11] asserts that there is a unique solution $\lambda \in V_{\mathcal{M}}^*$ such that $\lambda(d) = \lambda_d$ on $\Omega$. Putting another way, for $d = (u, s) \in \mathcal{E}_d$, we have

$$\lambda(u, s)(f, \Phi) = J(s, W_{f,u}, \Phi)$$

and the left hand side $\lambda(u, s)(f, \Phi)$ defines a rational function in $\mathbb{C}(q^{-u}, q^{-s})$.

We fix any $u \in D_\pi$ and then define an open set $\mathcal{U}$ of $\mathcal{D}_s$ by the condition that

$$\max\{ \text{Re}(L(a_1, \ldots, a_t)(u, s)) \} > 0$$

on which the maximum runs over all $0 \leq a_i \leq \ell_i$ satisfying $\sum a_i r_i = 2n - 2k$ for some $1 \leq k \leq n$. From (4.1), $\mathcal{U}$ is not empty. If $s$ lies in $\mathcal{U}$, we obtain
\[ \lambda(u, s)(f, \Phi) = J(s, W_{f,u}, \Phi), \]  
\text{(4.2)}

and both are rational functions in \( \mathbb{C}(q^{-s}) \). Hence they are equal on this open set \( \mathcal{U} \). We then extend the equality in (4.2) meromorphically to the whole space \( \mathcal{D}_s \). This concludes that

\[ \lambda(u, s)(f, \Phi) = J(s, W_{f,u}, \Phi) \]
on \( \mathcal{D}_\pi \times \mathcal{D}_\gamma \) and hence \( J(s, W_u, \Phi) \) defines a rational function in \( \mathbb{C}(q^{-u}, q^{-\gamma}) \).

If we now look at the local functional equation for the even \( GL_{2n} \) case from Theorem 2.10, it reads

\[ J(1 - s, \rho(\tau_{2n})\hat{W}_u, \hat{\Phi}) = \gamma(s, \pi_u, \wedge^2, \psi)J(s, W_u, \Phi) \]

where

\[ \gamma(s, \pi_u, \wedge^2, \psi) = \frac{\varepsilon(s, \pi_u, \wedge^2, \psi)L(1 - s, (\pi_u)^t, \wedge^2)}{L(s, \pi_u, \wedge^2)} \]

for \( W_u \in \mathcal{W}(\pi_u, \psi) \) and \( \Phi \in \mathcal{S}(F^n) \). If \( W_u \in \mathcal{W}_\pi^{(0)} \), then \( \rho(\tau_{2n})\hat{W}_u \in \mathcal{W}_\pi^{(0)} \). Under deformation, \( (\pi_u)^t = \pi^t_{u^t} \), if \( u = (u_1, \ldots, u_t) \) then \( u^t = (-u_t, \ldots, -u_1) \). Note that in the Whittaker model, \( W^{(0)}_u \in \mathcal{W}_\pi^{(0)} \) belongs to \( \mathcal{W}(\pi_u, \psi) \) and both are rational functions in \( \mathbb{C}(q^{-u}) \).

From Proposition 4.2 the integrals \( J(1 - s, \rho(\tau_{2n})\hat{W}_u, \hat{\Phi}) \) appearing in the functional equation are rational functions of \( q^{-u} \) and \( q^{-\gamma} \) for \( W_u \in \mathcal{W}_\pi^{(0)} \). Therefore \( \gamma(s, \pi_u, \wedge^2, \psi) \) must also be rational.

**Corollary 4.3** Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be a representation of \( GL_{2n} \). Then the gamma factor \( \gamma(s, \pi_u, \wedge^2, \psi) \) belongs to \( \mathbb{C}(q^{-u}, q^{-\gamma}) \).

Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be a representation of \( GL_{2n+1} \) and for each \( u \in \mathcal{D}_\pi \) we define the deformed representation by \( \pi_u = \text{Ind}(\Delta_1 v^{\nu_1} \otimes \cdots \otimes \Delta_t v^{\nu_t}) \). The odd case \( 2n+1 \) runs along the same lines. For \( W_u \in \mathcal{W}_\pi^{(0)} \) and \( \Phi \in \mathcal{S}(F^n) \), the local integral, which occurs in the functional equation for \( GL_{2n+1} \), is defined by

\[ J(s, W_u, \Phi) = \int_{N_0 \backslash GL_1} \int_{N_0 \backslash M_n} \int_{F^n} W_u \left( \sigma_{2n+1} \begin{pmatrix} I_n & X \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \right) \left( \begin{pmatrix} I_n \\ x \end{pmatrix} \right) \]

\[ \times \Phi(x)dx \psi^{-1}(\text{Tr}X)dX|\det(g)|^{s-1}dg. \]

We apply Bernstein’s Theorem to prove the rationality of Jacquet–Shalika integrals. The proof is the same as in Proposition 4.2, so we omit it. Refer to the proof of [19, Proposition 7.6] for more details.

**Proposition 4.4** Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be a representation of \( GL_{2n+1} \). For \( W_u \in \mathcal{W}_\pi^{(0)} \) and \( \Phi \in \mathcal{S}(F^n) \), the integral \( J(s, W_u, \Phi) \) is a rational function of \( q^{-u} \) and \( q^{-\gamma} \).

From the local functional equation for the odd \( GL_{2n+1} \) case in (2.1), we obtain

\[ J(1 - s, \rho(\tau_{2n+1})\hat{W}_u, \hat{\Phi}) = \gamma(s, \pi_u, \wedge^2, \psi)J(s, W_u, \Phi), \]

where

\[ \gamma(s, \pi_u, \wedge^2, \psi) = \frac{\varepsilon(s, \pi_u, \wedge^2, \psi)L(1 - s, (\pi_u)^t, \wedge^2)}{L(s, \pi_u, \wedge^2)} \]

for \( W_u \in \mathcal{W}(\pi_u, \psi) \) and \( \Phi \in \mathcal{S}(F^n) \). If \( W_u \in \mathcal{W}_\pi^{(0)} \), then \( \rho(\tau_{2n+1})\hat{W}_u \in \mathcal{W}_\pi^{(0)} \). For \( W_u \in \mathcal{W}_\pi^{(0)} \) and \( \Phi \in \mathcal{S}(F^n) \), the integrals \( J(1 - s, \rho(\tau_{2n+1})\hat{W}_u, \hat{\Phi}) \) involved in the functional...
equation are again rational functions in $\mathbb{C}(q^{-u}, q^{-s})$ by proceeding Proposition 4.4 and hence so must $\gamma(s, \pi_u, \wedge^2, \psi)$ be.

**Corollary 4.5** Let $\pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_i)$ be a representation of $GL_{2n+1}$. Then the gamma factor $\gamma(s, \pi_u, \wedge^2, \psi)$ belongs to $\mathbb{C}(q^{-u}, q^{-s})$.

### 4.3 $L$-functions for deformed representations in general position

Let $\text{Ind}(\Delta_1 \nu^{u_1} \otimes \Delta_2 \nu^{u_2})$ be the normalized induced representation from two segments $\Delta_1 \nu^{u_1}$ and $\Delta_2 \nu^{u_2}$ in general position under deformation. The purpose of this Section is to provide the equality of exceptional $L$-functions $L_{ex}(s, \text{Ind}(\Delta_1^{(k_1)} \nu^{u_1} \otimes \Delta_2^{(k_2)} \nu^{u_2}), \wedge^2)$ and Rankin-Selberg $L$-functions $L_{ex}(s, \Delta_1^{(k_1)} \nu^{u_1} \times \Delta_2^{(k_2)} \nu^{u_2})$ defined in Section 2 of [9]. If $\pi_u = \text{Ind}(\Delta_1 \nu^{u_1} \otimes \cdots \otimes \Delta_i \nu^{u_i})$ is a deformed representation of $\pi$ in general position, we only need to consider those exceptional $L$-functions $L_{ex}(s, \text{Ind}(\Delta_1^{(k_1)} \nu^{u_1} \otimes \Delta_2^{(k_2)} \nu^{u_2}), \wedge^2)$ or $L_{ex}(s, \Delta_1^{(k_1)} \nu^{u_1}, \wedge^2)$. We will explain this claim in Section 5. Hence we are only holding at two segments $\text{Ind}(\Delta_1^{(k_1)} \nu^{u_1} \otimes \Delta_2^{(k_2)} \nu^{u_2})$ to compute the exceptional $L$-functions. The following Hartogs’ Theorem plays an important role to determine poles of $L_{ex}(s, \text{Ind}(\Delta_1^{(k_1)} \nu^{u_1} \otimes \Delta_2^{(k_2)} \nu^{u_2}), \wedge^2)$. We quote a statement in [23], Chapter III, Section 4.3.

**Theorem 4.6** (Hartogs) Let $\mathcal{M}$ be an $n$-dimensional complex manifold with $n \geq 2$. If $N \subset \mathcal{M}$ is an analytic subset of codimension 2 or more, then every holomorphic function on $\mathcal{M} – N$ extends to a holomorphic function on $\mathcal{M}$.

The following Lemma was originally stated in [8] and is proved by Matringe in [29, Proposition 5.3] to understand the Shalika functional on the representation of the form $\text{Ind}(\Delta_i \otimes \Delta_j)$ with $(\Delta_i) \sim \Delta_j$ up to a twist.

**Lemma 4.7** (Matringe) Let $\Delta$ be a square-integrable representation of $GL_m$. The normalized induced representation of the form $\text{Ind}(\Delta \nu^s \otimes \Delta \nu^{-s})$ has a nontrivial Shalika functional for any complex number $s \in \mathbb{C}$.

As a consequence of this Lemma, we obtain the following Proposition that will be important for what follows.

**Proposition 4.8** Let $\pi = \text{Ind}(\Delta_1 \otimes \Delta_2)$ be a representation of $GL_m$. We let $\nu = (u_1, u_2) \in \mathcal{D}_\pi$ be in general position and $\pi_u = \text{Ind}(\Delta_1^{u_1} \otimes \Delta_2^{u_2})$ an irreducible generic representation of $GL_m$. We assume that the segment $\Delta_i = [\rho_i, \rho_i \nu, \ldots, \rho_i \nu^{e_i-1}]$ is an irreducible quasi-square-integrable representation of $GL_{m_i}$, where $\rho_i$ is an irreducible supercuspidal representation of $GL_{r_i}$, $m_i = r_i e_i$ for $i = 1, 2$ and $m = m_1 + m_2$. Then we have

$$L^{(0)}(s, \text{Ind}(\Delta_1^{u_1} \otimes \Delta_2^{u_2}), \wedge^2) = L_{ex}(s, \Delta_1^{u_1} \times \Delta_2^{u_2}).$$

**Proof** The exceptional $L$-function $L_{ex}(s, \Delta_1^{u_1} \times \Delta_2^{u_2})$ can have a pole only at those $s_0$ for which $(\Delta_1^{u_1}) \sim \Delta_2^{u_2+s_0}$, that is, $(\Delta_1) \sim \Delta_2^e$ with $e = u_1 + u_2 + s_0$. We write $\Delta_1^{u_1 + \frac{s_0}{2}} = \Delta_0 \otimes \nu^{\frac{s_0}{2}}$ where $\Delta_0$ is a square integrable representation with $r$ a real number. With replaced $\Delta_1^{u_1 + \frac{s_0}{2}}$ by $\Delta_0 \otimes \nu^{\frac{s_0}{2}}$, according to Lemma 4.7, $\text{Ind}(\Delta^{u_1 + \frac{s_0}{2}} \otimes \Delta_2^{u_2 + \frac{s_0}{2}}) \sim \text{Ind}(\Delta_0 \nu^{s} \otimes \nu^{\frac{s_0}{2}})$ has a nontrivial Shalika functional. The condition (6) of general position says that $\text{Hom}_{S_{2m_1} \cap S_{2m_2}}(\text{Ind}(\Delta_1^{u_1 + \frac{s_0}{2}} \otimes \Delta_2^{u_2 + \frac{s_0}{2}}), \Theta)$ is of dimension at most one. The induced representation $\text{Ind}(\Delta_1^{u_1} \otimes \Delta_2^{u_2})$ being irreducible and...
generic, Lemma 3.2 implies that \(L^{(0)}(s, v^{m_1} \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2)\) has a pole at \(s = 0\), and so \(L^{(0)}(s, \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2)\) has a pole at \(s = s_0\). As both \(L\)-functions \(L^{(0)}(s, \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2)\) and \(L_{\text{ex}}(s, \Delta_1 v^{m_1} \times \Delta_2 v^{m_2})\) have the same simple poles at \(s = s_0\), we obtain that

\[
L_{\text{ex}}(s, \Delta_1 v^{m_1} \times \Delta_2 v^{m_2})^{-1} \text{ divides } L^{(0)}(s, \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2)^{-1},
\]

where we take the divisibility in \(\mathbb{C}[q^{\pm s}]\).

We claim that in fact

\[
L^{(0)}(s, \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2) = L_{\text{ex}}(s, \Delta_1 v^{m_1} \times \Delta_2 v^{m_2}).
\]

To end this we suppose that \(L^{(0)}(s, \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2)\) has a pole at \(s = s_0\). The existence of exceptional pole of \(L(s, \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2)\) at \(s = s_0\) ensures a nontrivial Shalika functional. According to Theorem 3.1 of Jacquet and Rallis, the existence of the nontrivial Shalika functional imposes the twisted self contragredient condition

\[
\text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}) \sim \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}) v^{s_0}.
\]

As the induced representation \(\text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2})\) is irreducible, the way that this is possible is that \((\Delta_1 v^{m_1})^{\sim} \simeq \Delta_1 v^{m_1+s_0}\) and \((\Delta_2 v^{m_2})^{\sim} \simeq \Delta_2 v^{m_2+s_0}\), or \((\Delta_1 v^{m_1})^{\sim} \sim \Delta_1 v^{m_1+s_0}\) and \((\Delta_2 v^{m_2})^{\sim} \sim \Delta_2 v^{m_2+s_0}\).

If we vary in \(u_1\) and \(u_2\), the integrals \(J(s, W_u, \Phi)\) with \(W_u \in \mathcal{W}_u^{(0)}\), \(\Phi \in \mathcal{S}(F^n)\) for either \(m = 2n\) or \(m = 2n + 1\) define rational functions in \(\mathbb{C}(q^{-u_1}, q^{-u_2}, q^{-s})\) due to Bernstein’s Theorem. For \(u = (u_1, u_2)\) in the Zariski open subset of general position, these rational functions \(J(s, W_u, \Phi)\) can have poles coming from \(L^{(0)}(s, \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2)\). The poles of \(L^{(0)}(s, \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2)\) can lie along the locus defined by one equation \(q^{-(u_1+u_2+s+m)} \omega_{\Delta_1}(\sigma) \omega_{\Delta_2}(\sigma) = 1\) from \((\Delta_1 v^{m_1})^{\sim} \sim \Delta_2 v^{m_2+s}\), or two independent equations \(q^{-(u_1+u_2+s+m)} \omega_{\Delta_1}(\sigma) = 1\) and \(q^{-(2u_2+s+m)} \omega_{\Delta_2}(\sigma) = 1\) from \((\Delta_1 v^{m_1})^{\sim} \simeq \Delta_1 v^{m_1+s}\) and \((\Delta_2 v^{m_2})^{\sim} \sim \Delta_2 v^{m_2+s}\), respectively. By Theorem 4.6 of Hartogs, viewing the local integrals \(J(s, W_u, \Phi)\) as meromorphic functions of \(u_1, u_2\) and \(s\), every singularities of our integrals cannot be accounted for the form \((\Delta_1 v^{m_1})^{\sim} \sim \Delta_1 v^{m_1+s}\) and \((\Delta_2 v^{m_2})^{\sim} \sim \Delta_2 v^{m_2+s}\). So if \(L^{(0)}(s, \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2)\) has a pole at \(s = s_0\), then \((\Delta_1 v^{m_1})^{\sim} \sim \Delta_2 v^{m_2+s}\) which implies that \(L_{\text{ex}}(s, \Delta_1 v^{m_1} \times \Delta_2 v^{m_2})\) has a pole at \(s = s_0\). As both \(L\)-functions \(L^{(0)}(s, \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2)\) and \(L_{\text{ex}}(s, \Delta_1 v^{m_1} \times \Delta_2 v^{m_2})\) have the same simple poles at \(s = s_0\), we obtain that

\[
L^{(0)}(s, \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2}), \lambda^2)^{-1} \text{ divides } L_{\text{ex}}(s, \Delta_1 v^{m_1} \times \Delta_2 v^{m_2})^{-1},
\]

where we take the divisibility in \(\mathbb{C}[q^{\pm s}]\). \(\square\)

If \(\pi_u = \text{Ind}(\Delta_1 v^{m_1} \otimes \Delta_2 v^{m_2})\) is an irreducible generic representation of \(GL_m\) in general position, so are various derivatives \(\text{Ind}(\Delta_1^{(a_1 r_i)} v^{m_1} \otimes \Delta_2^{(a_2 r_j)} v^{m_2})\). Moreover each segment \(\Delta_i^{(a_1 r_i)} = [\rho_1 v^{a_1}, \ldots, \rho_1 v^{\ell_i-1}]\) can be written as \(\Delta_i^{(a_1 r_i)} = [\rho_1^i v^{\ell_i-a_1-1}]\) for \(\rho_1^i = \rho_1 v^{a_1}\) and \(0 \leq a_1 < \ell_i - 1\). Hence we have the following result.

**Corollary 4.9** With the notation in Proposition 4.8, we have

\[
L^{(0)}(s, \text{Ind}(\Delta_1^{(a_1 r_i)} v^{m_1} \otimes \Delta_2^{(a_2 r_j)} v^{m_2}), \lambda^2) = L_{\text{ex}}(s, \Delta_1^{(a_1 r_i)} v^{m_1} \times \Delta_2^{(a_2 r_j)} v^{m_2})
\]

for \(0 \leq a_1 < \ell_1\) and \(0 \leq a_2 < \ell_2\).

We furthermore can translate exceptional poles of \(L\)-functions \(L(s, \text{Ind}(\Delta_1^{(k_1)} \otimes \Delta_2^{(k_2)}), \lambda^2)\) for the derivative of a pair of fixed segments \(\Delta_1\) and \(\Delta_2\) in terms of exceptional poles of Rankin-Selberg \(L\)-functions \(L(s, \Delta_1^{(k_1)} \times \Delta_2^{(k_2)})\).
Proposition 4.10  With the notation in Proposition 4.8, we have

\[ L_{ex}(s, \text{Ind}(\Delta_1 v^{\mu_1} \otimes \Delta_2 v^{\mu_2}), \wedge^2) = L_{ex}(s, \Delta_1 v^{\mu_1} \times \Delta_2 v^{\mu_2}). \]

**Proof** We begin with the case when \( \ell_1 = \ell_2 = 1 \), that is, \( \Delta_1 v^{\mu_1} = \rho_1 v^{\mu_1} \) and \( \Delta_2 v^{\mu_2} = \rho_2 v^{\mu_2} \) are also irreducible supercuspidal representations. If \( r_1 \neq r_2 \), then \( L_{ex}(s, \text{Ind}(\Delta_1 v^{\mu_1} \otimes \Delta_2 v^{\mu_2}), \wedge^2) = L^{(0)}(s, \text{Ind}(\Delta_1 v^{\mu_1} \otimes \Delta_2 v^{\mu_2}), \wedge^2) = L_{ex}(s, \Delta_1 v^{\mu_1} \times \Delta_2 v^{\mu_2}) = 1 \). As in Proposition 2.6,

\[
L(s, \text{Ind}(\rho_1 v^{\mu_1} \otimes \rho_2 v^{\mu_2}), \wedge^2) = L^{(0)}(s, \text{Ind}(\rho_1 v^{\mu_1} \otimes \rho_2 v^{\mu_2}), \wedge^2) L(s, \text{Ind}(\rho_1 v^{\mu_1} \otimes \rho_2 v^{\mu_2}), \wedge^2).
\]

We first treat the derivatives for \( \text{Ind}(\rho_1 v^{\mu_1} \otimes \rho_2 v^{\mu_2}) \). As \( u \) is in general position, and because of Proposition 2.7, we have

\[
L^{(0)}(s, \text{Ind}(\rho_1 v^{\mu_1} \otimes \rho_2 v^{\mu_2}), \wedge^2) = L(s, \rho_1 v^{\mu_1}, \wedge^2) L(s, \rho_2 v^{\mu_2}, \wedge^2).
\]

If \( r_1 = r_2 \) is an odd number, Theorem 3.6 ensures that

\[
L^{(0)}(s, \text{Ind}(\rho_1 v^{\mu_1} \otimes \rho_2 v^{\mu_2}), \wedge^2) = 1.
\]

According to Proposition 4.8, \( L^{(0)}(s, \text{Ind}(\rho_1 v^{\mu_1} \otimes \rho_2 v^{\mu_2}), \wedge^2) = L^{(0)}(s, \text{Ind}(\rho_1 v^{\mu_1} \times \rho_2 v^{\mu_2}), \wedge^2) \). The condition (5) of general position in Section 4.1 implies that two \( L \)-functions \( L^{(0)}(s, \text{Ind}(\rho_1 v^{\mu_1} \otimes \rho_2 v^{\mu_2}), \wedge^2) = L(s, \rho_1 v^{\mu_1} \times \rho_2 v^{\mu_2}) \) and \( L^{(0)}(s, \text{Ind}(\rho_1 v^{\mu_1} \otimes \rho_2 v^{\mu_2}), \wedge^2) = L(s, \rho_1 v^{\mu_1}, \wedge^2) L(s, \rho_2 v^{\mu_2}, \wedge^2) \) do not share common poles. Hence we conclude

\[
L_{ex}(s, \Delta_1 v^{\mu_1} \times \Delta_2 v^{\mu_2}) = L_{ex}(s, \text{Ind}(\Delta_1 v^{\mu_1} \otimes \Delta_2 v^{\mu_2}), \wedge^2).
\]

Now we assume that \( \ell_1, \ell_2 > 1 \). It is sufficient to prove the case when \( r = r_1 = r_2 \) and \( \ell = \ell_1 = \ell_2 \). Otherwise \( L_{ex}(s, \text{Ind}(\Delta_1 v^{\mu_1} \otimes \Delta_2 v^{\mu_2}), \wedge^2) = L^{(0)}(s, \text{Ind}(\Delta_1 v^{\mu_1} \otimes \Delta_2 v^{\mu_2}), \wedge^2) = L_{ex}(s, \Delta_1 v^{\mu_1} \times \Delta_2 v^{\mu_2}) = 1 \) according to Corollary 4.9. We assume that the following statement is satisfied for all integers \( k \) with \( 1 \leq k \leq \ell - 1 \)

\[
L_{ex}(s, \text{Ind}(\rho_1, \ldots, \rho_1 v^k) v^{\mu_1} \otimes [\rho_2, \ldots, \rho_2 v^k] v^{\mu_2}), \wedge^2) = L_{ex}(s, [\rho_1, \ldots, \rho_1 v^k] v^{\mu_1} \times [\rho_2, \ldots, \rho_2 v^k] v^{\mu_2}).
\]

We show this by induction on the length of segment \( \ell \). Let \( \Delta_i = [\rho_i, \ldots, \rho_i v^k] \) be an irreducible quasi-square-integrable representation of \( GL_m \) with \( m = (\ell + 1)r \) for \( i = 1, 2 \). Since \( u \in D_\pi \) for \( \text{Ind}(\Delta_1 v^{\mu_1} \otimes \Delta_2 v^{\mu_2}) \) is in general position, according to Proposition 2.8 we find

\[
L^{(0)}(s, \text{Ind}(\Delta_1 v^{\mu_1} \otimes \Delta_2 v^{\mu_2}), \wedge^2) = 1, \quad \text{l.c.m.} \{L_{ex}(s, \text{Ind}(\Delta_1^{(a_1 r)} v^{\mu_1} \otimes \Delta_2^{(a_2 r)} v^{\mu_2}), \wedge^2)\}^{-1},
\]

where the least common multiple is taken over all \( 0 \leq a_1, a_2 \leq \ell + 1 \) and \( a_1 + a_2 > 0 \) such that \( 2m = (a_1 + a_2)r \) is an even number. We consider \( L_{ex}(s, \text{Ind}(\Delta_1^{(a_1 r)} v^{\mu_1} \otimes \Delta_2^{(a_2 r)} v^{\mu_2}), \wedge^2) \) for three different cases. If \( 0 \leq a_1 \neq a_2 < \ell + 1, \)

\[
L_{ex}(s, \text{Ind}(\Delta_1^{(a_1 r)} v^{\mu_1} \otimes \Delta_2^{(a_2 r)} v^{\mu_2}), \wedge^2) = L_{ex}(s, \Delta_1^{(a_1 r)} v^{\mu_1} \times \Delta_2^{(a_2 r)} v^{\mu_2}) = 1
\]

according to Corollary 4.9. For \( a_1 = \ell + 1 \) or \( a_2 = \ell + 1 \), as the order of the derivatives varies, by Propositions 3.9 and 3.10, we see that the least common multiple of the factors \( L_{ex}(s, \Delta_i^{(a r)} v^{\mu_i}, \wedge^2) \) will precisely contribute to

\[
L(s, \Delta_i v^{\mu_i}, \wedge^2)^{-1} = \text{l.c.m.} \{L_{ex}(s, \Delta_i^{(a r)} v^{\mu_i}, \wedge^2)\}^{-1},
\]
where the least common multiple is taken over all $0 \leq a_i \leq \ell + 1$ such that $m - a_i r$ is the even number. We are left with the case when $0 < a_1 = a_2 < \ell + 1$. The length of the segment $\Delta_j^{(a_i r)} = [\rho_1 v^a_1, \ldots, \rho_i v^\ell]$ is $\ell - a_i + 1$ and $\text{Ind}(\Delta_1^{(a_1 r)} v^u_1 \otimes \Delta_2^{(a_2 r)} v^u_2)$ is still an irreducible generic representation in general position. The induction hypothesis on the length $k$ of segments for $1 \leq k \leq \ell$ then enable us to obtain the least common multiple

$$l.c.m. \left\{ L_{ex}(s, \text{Ind}(\Delta_1^{(a_1 r)} v^u_1 \otimes \Delta_2^{(a_2 r)} v^u_2), \lambda^2)^{-1} \right\}$$
$$= \prod_{0 < a_1 = a_2 < \ell + 1} L_{ex}(s, \Delta_1^{(a_1 r)} v^u_1 \otimes \Delta_2^{(a_2 r)} v^u_2)^{-1}.$$ 

By taking all three contributions by $L(0)(s, \text{Ind}(\Delta_1 v^u_1 \otimes \Delta_2 v^u_2), \lambda^2)$ into account, we write the $L$-function as

$$L(0)(s, \text{Ind}(\Delta_1 v^u_1 \otimes \Delta_2 v^u_2), \lambda^2) = L(s, \Delta_1 v^u_1, \lambda^2) L(s, \Delta_2 v^u_2, \lambda^2) \prod_{0 < a_1 = a_2 < \ell + 1} L_{ex}(s, \Delta_1^{(a_1 r)} v^u_1 \otimes \Delta_2^{(a_2 r)} v^u_2),$$

which does not have any common poles with the $L$-function $L(0)(s, \text{Ind}(\Delta_1 v^u_1 \otimes \Delta_2 v^u_2), \lambda^2) = L_{ex}(s, \Delta_1 v^u_1 \times \Delta_2 v^u_2)$ because $u$ is in general position. As a result, we have

$$L_{ex}(s, \text{Ind}(\Delta_1 v^u_1 \otimes \Delta_2 v^u_2), \lambda^2) = L(0)(s, \text{Ind}(\Delta_1 v^u_1 \otimes \Delta_2 v^u_2), \lambda^2),$$

from Proposition 2.6, hence $L_{ex}(s, \text{Ind}(\Delta_1 v^u_1 \otimes \Delta_2 v^u_2), \lambda^2) = L_{ex}(s, \Delta_1 v^u_1 \times \Delta_2 v^u_2)$ which is what we want. \hfill $\square$

As we mention in the proof, $\text{Ind}(\Delta_1^{(a_1 r)} v^u_1 \otimes \Delta_2^{(a_2 r)} v^u_2)$ is an irreducible generic representation in general position with the segment $\Delta_j^{(a_i r)} = [\rho_1 v^a_1, \ldots, \rho_i v^\ell]$ of length $\ell_i - a_i$. Thus the previous Proposition 4.10 is still applicable for various derivatives $\text{Ind}(\Delta_1^{(a_1 r)} v^u_1 \otimes \Delta_2^{(a_2 r)} v^u_2)$ and we have the following Corollary.

**Corollary 4.11** With the notation in Proposition 4.8, we have

$$L_{ex}(s, \text{Ind}(\Delta_1^{(a_1 r)} v^u_1 \otimes \Delta_2^{(a_2 r)} v^u_2), \lambda^2) = L_{ex}(s, \Delta_1^{(a_1 r)} v^u_1 \otimes \Delta_2^{(a_2 r)} v^u_2)$$

for $0 \leq a_1 < \ell_1$ and $0 \leq a_2 < \ell_2$.

In Section 5, this will be the first step to prove multiplicity result of exterior square $L$-function for deformed representations in general position.

## 5 Multiplicativity of $L$-Functions for the Formation of Langland Quotients

### 5.1 Exterior square $L$-functions for irreducible admissible representations

Let us begin with representations $\pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ of $GL_m$ with the $\Delta_i$ irreducible quasi-square-integrable. For now, $\pi$ need not to be irreducible. We deform the representation to families $\pi_u$ with $u = (u_1, \ldots, u_t) \in D_\pi$. For $u$ in general position, $\pi_u$ is irreducible, and its higher derivatives are completely reducible. In this Section, we take the approach of Cogdell and Piatetski-Shapiro in Sect. 4 of [9] to compute the local exterior square $L$-functions for all.
irreducible admissible representations of $GL_m$. Many of results are available in the literature [9,24,28] henceforth we will only a sketch of the proof and reference the arguments most of those articles. However we repeat the crucial points by filling in some of the missing proof of those articles.

**Theorem 5.1** Let $\pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_r)$ be a representation of $GL_m$. Let $u = (u_1, \ldots, u_r) \in \mathcal{D}_{\pi}$ be in general position and $\pi_u = \text{Ind}(\Delta_1 v^{\mu_1} \otimes \cdots \otimes \Delta_r v^{\mu_i})$ the deformed representation of $\pi$ on $GL_m$. Then we have

$$L(s, \pi_u, \wedge^2) = \prod_{1 \leq k \leq t} L(s + 2u_k, \Delta_k, \wedge^2) \prod_{1 \leq i < j \leq t} L(s + u_i + u_j, \Delta_i \times \Delta_j).$$

**Proof** Let us fix points $u$ in general position. Consider $J(s, W_u, \Phi)$ if $m = 2n$ or $m = 2n + 1$ with $W_u \in \mathcal{W}_n^{(0)}$, and $\Phi \in \mathcal{S}(F^n)$, and the fractional ideals $\mathcal{J}(\pi_u) \subset \mathcal{C}(q^{-s})$ they generate. Then by Theorem 2.13, we know

$$L(s, \pi_u, \wedge^2)^{-1} = \text{l.c.m.} \{L_{\text{ex}}(s, \pi_u^{(a_1 r_1, \ldots, a_r r_i)}), \wedge^2)^{-1}\} \quad (5.1)$$

where $0 \leq a_j \leq \ell_i$, $m - \sum a_i r_i$ is an even number, and the least common multiple is taken in terms of divisibility in $\mathbb{C}[q^{\pm s}]$. Each such exceptional $L$-function can have poles which lie along the twisted self-contragrediance requirement

$$(\pi_u^{(a_1 r_1, \ldots, a_r r_i)})^\sim \simeq \pi_u^{(a_1 r_1, \ldots, a_r r_i)} v^s$$

or equivalently

$$\text{Ind}(\Delta_{1, u_1}^{(a_1 r_1)} \otimes \cdots \otimes \Delta_{r_i, u_i}^{(a_r r_i)})^\sim \simeq \text{Ind}(\Delta_{1, u_1}^{(a_1 r_1)} \otimes \cdots \otimes \Delta_{r_i, u_i}^{(a_r r_i)}) v^s. \quad (5.2)$$

For $x$ a positive real number, let $\lfloor x \rfloor$ be the greatest integer less than or equal to $x$. Since the induced representations are irreducible, the only way that (5.2) is possible is that there is a reordering of those indices $i$ for which $a_i \neq \ell_i$ and under this reordering there exists an integer $p$ between 0 and $\lfloor t/2 \rfloor$ such that $(\Delta_{ij}^{(a_i r_i)}, v^{u_{ij}})^\sim \simeq \Delta_{ij+1}^{(a_i r_i)} v^{u_{ij+1}} v^s$ or equivalently $(\Delta_{ij}^{(a_i r_i)}, v^{u_{ij}})^\sim \simeq \Delta_{(i+1)j}^{(a_i r_i)} v^{u_{ij}+u_{ij+1}} v^s$ for $i < j$ and $j = 1, 3, \ldots, 2p - 1$ if $p \neq 0$, and $(\Delta_{ij}^{(a_i r_i)}, v^{u_{ij}})^\sim \simeq \Delta_{ij}^{(a_i r_i)} v^{u_{ij}} v^s$ or equivalently $(\Delta_{ij}^{(a_i r_i)}, v^{u_{ij}})^\sim \simeq \Delta_{ij}^{(a_i r_i)} v^{u_{ij}+u_{ij+1}} v^s$ for $j > 2p$.

Now let us see how the condition (5.2) varies in $u = (u_1, \ldots, u_r) \in \mathcal{D}_{\pi}$. Then $J(s, W_u, \Phi)$ define rational functions in $\mathcal{C}(q^{-s}, q^{-u})$ by Bernstein’s Theorem. For $u$ in the Zariski open subset of general position, these rational functions can have poles coming from the exceptional contributions by the exterior square $L$-functions in (5.1). Each such exceptional $L$-function can have poles which lie along the locus defined by a finite number of hyperplanes, where there is one equation

$$q^{-(u_i + u_j + s)(m_i - a_i r_i)} \Delta_{i}^{(a_i r_i)}(\mathfrak{m}) \Delta_{j}^{(a_j r_j)}(\mathfrak{m}) = 1$$

for every pair $(i, j)$ of indices $i < j$ such that $r_i \neq \ell_i$, $r_j \neq \ell_j$, and $(\Delta_{i}^{(a_i r_i)})^\sim \simeq \Delta_{j}^{(a_j r_j)} v^{u_i + u_j + s}$, or one equation

$$q^{-(2u_i + s)(m_i - a_i r_i)} \Delta_{i}^{(a_i r_i)}(\mathfrak{m}) = 1$$

for every indices $i$ such that $r_i \neq \ell_i$ and $(\Delta_{i}^{(a_i r_i)})^\sim \simeq \Delta_{i}^{(a_i r_i)} v^{2u_i + s}$. So if there is more than one pair of indices $i < j$ such that $r_i \neq \ell_i$, $r_j \neq \ell_j$, and $(\Delta_{i}^{(a_i r_i)})^\sim \simeq \Delta_{j}^{(a_j r_j)} v^{u_i + u_j + s}$,
or more than one index $i$ satisfying $(\Delta^{(a_ir_j)}_i) \sim \Delta^{(a_ir_i)}_i v^{2u_i+s}$ and $r_i \neq \ell_i$, as we are assuming, then this singular locus will be defined by 2 or more independent equations and hence will be of codimension greater than or equal to 2. By Hartogs’s (Theorem 4.6), we know that every singularity of our local integrals must be accounted by an exceptional contribution of the form $L_{ex}(s, \text{Ind}(\Delta^{(a_ir_i)}_i v^{u_i} \otimes \Delta^{(a_ir_j)}_j v^{u_j}), \lambda^2)$ for $i < j$ or of the form $L_{ex}(s, \Delta^{(a_ir_i)}_i v^{u_i}, \lambda^2)$. According to Proposition 4.11, the first type is the same contribution as Rankin-Selberg $L$-function

$$L_{ex}(s, \text{Ind}(\Delta^{(a_ir_i)}_i v^{u_i} \otimes \Delta^{(a_ir_j)}_j v^{u_j}), \lambda^2) = L_{ex}(s, \Delta^{(a_ir_i)}_i v^{u_i} \times \Delta^{(a_ir_j)}_j v^{u_j})$$

$$= L_{ex}(s + u_i + u_j, \Delta^{(a_ir_i)}_i \times \Delta^{(a_ir_j)}_j).$$

For fixed $i$ and $j$ with $i < j$, we see that the least common multiple of the inverse of these factors $L_{ex}(s + u_i + u_j, \Delta^{(a_ir_i)}_i \times \Delta^{(a_ir_j)}_j)$ will contribute a factor of $L(s + u_i + u_j, \Delta_i \times \Delta_j)^{-1}$ to $L(s, \pi_u, \lambda^2)^{-1}$. For fixed $i$, by the analysis in Proposition 3.9 and Proposition 3.10, the least common multiple of the inverse of the second types of factors $L_{ex}(s, \Delta^{(a_ir_i)}_i v^{u_i}, \lambda^2)$ contributes a factor of $L(s, \Delta_i v^{u_i}, \lambda^2)^{-1} = L(s + 2u_i, \Delta_i, \lambda^2)^{-1}$ to $L(s, \pi_u, \lambda^2)^{-1}$.

Now, still for $u$ in general position, the $L(s + u_i + u_j, \Delta_i \times \Delta_j)^{-1}$ will be relatively prime as $i, j$ and $k$ vary. So their contribution to the exterior square $L$-functions will be their product. As $L(s, \pi_u, \lambda^2)^{-1}$ is the least common multiple of $\{L_{ex}(s, \Delta^k v^{u_k}, \lambda^2)^{-1} \mid 1 \leq k \leq t\}$ and $\{L_{ex}(s, \text{Ind}(\Delta^{(a_ir_i)}_i v^{u_i} \otimes \Delta^{(a_ir_j)}_j v^{u_j}), \lambda^2)^{-1} \mid 1 \leq i < j \leq t\}$, we obtain that

$$L(s, \pi_u, \lambda^2) = \prod_{1 \leq k \leq t} L(s + 2u_k, \Delta_k, \lambda^2) \prod_{1 \leq i < j \leq t} L(s + u_i + u_j, \Delta_i \times \Delta_j).$$

\[ \square \]

We recall some results from complex analytic geometry in [10]. Let $\mathcal{M}$ be a complex manifold and $\mathcal{X}$ an analytic hypersurface of $\mathcal{M}$. We denote by $\text{Reg}(\mathcal{X})$ the set of regular points of $\mathcal{X}$. We say that $\mathcal{X}$ is reducible if we can find analytic subsets $\mathcal{Y}, \mathcal{Z}$ of $\mathcal{M}$, neither of which equals to $\mathcal{X}$, such that $\mathcal{X} = \mathcal{Y} \cup \mathcal{Z}$. If $\mathcal{X}$ is not reducible, we say $\mathcal{X}$ is irreducible. We present the following results from [10].

**Theorem 5.2** ([10], Chapter IV, Section 4, Theorem 4.6.7) *Let $\mathcal{X}$ be an analytic hypersurface of the complex manifold $\mathcal{M}$ and $\{\mathcal{X}_i \mid i \in I\}$ denote the set of connected components of $\text{Reg}(\mathcal{X})$. Then $\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$ and this decomposition of $\mathcal{X}$ as a countable union of irreducible analytic hypersurfaces is unique up to order.*

We call the hypersurfaces $\mathcal{X}_i$ in Theorem 5.2 the irreducible component of $\mathcal{X}$. We would like to specialize the result in Theorem 5.1 to $u$ not in general position.

**Proposition 5.3** *Let $\pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ be a representation of $GL_m$. Then $L(s, \pi, \lambda^2) \in \prod_{1 \leq k \leq t} L(s, \Delta_k, \lambda^2) \prod_{1 \leq i < j \leq t} L(s, \Delta_i \times \Delta_j)\mathbb{C}[q^s, q^{-s}]$.\]

**Proof** We know from Proposition 4.2 and Proposition 4.4 that for $W_u \in \mathcal{W}_\pi^{(0)}$ and $\phi \in \mathcal{S}(F^n)$ with $m = 2n$ or $m = 2n + 1$, the ratio

$$J(s, W_u, \phi) \prod L(s + 2u_k, \Delta_k, \lambda^2) \prod L(s + u_i + u_j, \Delta_i \times \Delta_j)$$

(5.3)
is a rational functions in \( \mathbb{C}(q^{-u}, q^{-s}) \). Let \( P(q^\pm u, q^\pm s) \in \mathbb{C}[q^\pm u, q^\pm s] \) be the denominator of the rational function in (5.3). We let \( \mathcal{D}_s = (\mathbb{C}/2\pi i \log(q)) \simeq \mathbb{C}^\times \). Knowing that the exterior square \( L \)-function for the deformed representation in general position \( \pi_u \) is given by the product in Theorem 5.1, which is the inverse of a Laurent polynomial in \( \mathbb{C}[q^\pm u, q^\pm s] \), the rational function in (5.3) has no poles on the Zariski open set of \( u \) in general position. Proposition 4.1 asserts that the removed hyperplanes defining general position, \( H_1, \ldots, H_p \in \mathcal{D}_\pi = (\mathbb{C}/2\pi i \log(q))^l \), do not depend on \( s \), but only on \( u \). If \( P(q^\pm u, q^\pm s) \) is not unit in \( \mathbb{C}[q^\pm u, q^\pm s] \), then the set \( Z(\mathcal{P}) \) of zeroes of exponential polynomial \( P(q^\pm u, q^\pm s) \) which defines the hypersurface in \( \mathcal{D}_s \times \mathcal{D}_s \) is contained in the union of \( H_i \times \mathcal{D}_s \). By Theorem 5.2, one of the irreducible components of \( Z(\mathcal{P}) \) is \( H \times \mathcal{D}_s \) for \( H \) an affine hyperplane of \( \mathcal{D}_\pi \). However \( H \times \mathcal{D}_s \) cannot lie entirely in \( Z(\mathcal{P}) \), as for any fixed \( u \in H \), for \( s \) large enough, the rational functions \( J(s, W_u, \Phi) \) are defined by absolutely convergent integrals according to (4.1), hence is holomorphic, and the inverse of \( \prod L(s + 2u_k, \Delta_k, \lambda^2) \prod L(s + u_i + u_j, \Delta_i \times \Delta_j) \) is polynomial in \( \mathbb{C}[q^\pm u, q^\pm s] \), thus without poles. Therefore the ratios in (5.3) is entire and hence lie in \( \mathbb{C}[q^\pm u, q^\pm s] \). If we now specialize \( u = 0 \), we find that

\[
\frac{J(s, W, \Phi)}{\prod L(s, \Delta_k, \lambda^2) \prod L(s, \Delta_i \times \Delta_j)}
\]

have no poles for all \( W \in \mathcal{W}(\pi, \psi) \). This completes the proof. \( \square \)

Let us consider again the behavior of the gamma factor for the representation \( \pi_u = \text{Ind}(\Delta_1, v^u_1 \otimes \cdots \otimes \Delta_t, v^u_t) \) on \( GL_m \) under deformation. From the previous Proposition 5.3, we cannot conclude multiplicativity of the local \( \gamma \)-factor, but we only have the following weak version of multiplicativity of the \( \gamma \)-factor.

**Proposition 5.4** Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be a representation of \( GL_m \). We let \( u = (u_1, \ldots, u_t) \) be an element of \( \mathcal{D}_\pi \) and \( \pi_u = \text{Ind}(\Delta_1 v^{u_1} \otimes \cdots \otimes \Delta_t v^{u_t}) \) the deformed representation of \( \pi \). Then \( \gamma(s, \pi_u, \lambda^2, \psi) \) and

\[
\prod_{1 \leq k \leq t} \gamma(s + 2u_k, \Delta_k, \lambda^2, \psi) \prod_{1 \leq i < j \leq t} \gamma(s + u_i + u_j, \Delta_i \times \Delta_j, \psi)
\]

are equal up to a unit in \( \mathbb{C}[q^\pm u, q^\pm s] \).

**Proof** We know from Corollary 4.3 and Corollary 4.5 that \( \gamma(s, \pi_u, \lambda^2, \psi) \) is a rational function in \( \mathbb{C}(q^{-u}, q^{-s}) \). The local \( \epsilon \)-factor satisfies

\[
\gamma(s, \pi_u, \lambda^2, \psi) = \frac{\epsilon(s, \pi_u, \lambda^2, \psi) L(1 - s, (\pi_u)^t, \lambda^2)}{L(s, \pi_u, \lambda^2)}.
\]

For fixed \( u \) we know, by applying the functional equations twice, that \( \epsilon \)-factor \( \epsilon(s, \pi_u, \lambda^2, \psi) \) is of the form \( a q^{-b s} \), that is, it is a unit in \( \mathbb{C}[q^\pm s] \). If we now define a variant of the \( \epsilon \)-factor by

\[
\gamma(s, \pi_u, \lambda^2, \psi) = \epsilon^0(s, \pi_u, \lambda^2, \psi) \times \frac{\prod_{1 \leq k \leq t} L(1 - s - 2u_k, \Delta_k, \lambda^2) \prod_{1 \leq i < j \leq t} L(1 - s - u_i - u_j, \Delta_i \times \Delta_j)}{\prod_{1 \leq k \leq t} L(s + 2u_k, \Delta_k, \lambda^2) \prod_{1 \leq i < j \leq t} L(s + u_i + u_j, \Delta_i \times \Delta_j)}
\]

then \( \epsilon^0(s, \pi_u, \lambda^2, \psi) \in \mathbb{C}(q^{-u}, q^{-s}) \) and we have \( \epsilon^0(s, \pi_u, \lambda^2, \psi) = \epsilon(s, \pi_u, \lambda^2, \psi) \) for \( u \) in general position. The proof is almost identical to the one given by [9, Proposition 4.3]
except a subtle issue about \( e^\circ(s, \pi_u, \wedge^2, \psi) \) belonging to \( \mathbb{C}[q^{\pm u}, q^{\pm s}] \), thus we only resolve this problem here.

Since \( e^\circ \)-factor is the elements of \( \mathbb{C}(q^{-u}, q^{-s}) \), we let \( P(q^{\pm u}, q^{\pm s}) \in \mathbb{C}[q^{\pm u}, q^{\pm s}] \) be the denominator of \( e^\circ(s, \pi_u, \wedge^2, \psi) \). As the \( \varepsilon \)-factor \( \varepsilon(s, \pi_u, \wedge^2, \psi) \) is a unit in \( \mathbb{C}[q^{\pm s}] \), this implies that \( e^\circ(s, \pi_u, \wedge^2, \psi) \) has no poles on the Zariski open set of \( u \) in general position. Proposition 4.1 asserts that the removed hypersurfaces defining general position, \( H_1, \ldots, H_p \in \mathcal{D}_\pi \), do not depend on \( s \), but only on \( u \). If \( P(q^{\pm u}, q^{\pm s}) \) is not unit in \( \mathbb{C}[q^{\pm u}, q^{\pm s}] \), then the set \( Z(P) \) of zeroes of exponential polynomial \( P(q^{\pm u}, q^{\pm s}) \) which defines the hypersurface in \( \mathcal{D}_\pi \times \mathcal{D}_s \) is in the union \( \cup_i H_i \times \mathcal{D}_s \). By Theorem 5.2, one of the irreducible components of \( Z(P) \) is \( H \times \mathcal{D}_s \) for \( H \) an affine hypersurface of \( \mathcal{D}_\pi \). Thus the hypersurface \( Z(P) \) necessarily contains a set of the from \( H \times \mathcal{D}_s \), for \( H \) an affine hyperplane of \( \mathcal{D}_\pi \). Since we know from the local functional equation in Theorem 2.10 and (2.1) that \( J(1 - s, \rho(\tau_m)W_u, \Phi) = \gamma(s, \pi_u, \wedge^2, \psi)J(s, W_u, \Phi) \), (5.4) can be rewritten as

\[
J(1 - s, \rho(\tau_m)\tilde{W}_u, \Phi) = \frac{J(s, W_u, \Phi)}{\prod_{1 \leq k \leq L}(1 - s - 2u_k, \Delta_k, \wedge^2)\prod_{1 \leq i < j \leq L}(1 - s - u_i - u_j, \Delta_i \times \Delta_j)}
\]

(5.5)

Because of Proposition 5.3, the left hand side of the equality (5.5) is entire and so \( H \times \mathcal{D}_s \) is a subset of the zeros of the polynomials

\[
\prod_{1 \leq k \leq L}(s + 2u_k, \Delta_k, \wedge^2)\prod_{1 \leq i < j \leq L}(s + u_i + u_j, \Delta_i \times \Delta_j)
\]

for all \( W_u \in \mathcal{W}_\pi^{(0)} \) and \( \Phi \in S(F^n) \). Let be \( P_0 \) be the vector subspace of \( \mathbb{C}[q^{\pm u}] \) defined by \( P_0 := \{ W_u(L_m) \mid W_u \in \mathcal{W}_\pi^{(0)} \} \). Let \( \text{ind}_{N_m}^P(\psi) \) denote the space of smooth functions compactly supported modulo \( N_m \), \( \varphi : P_m \to \mathbb{C} \) which satisfies \( \varphi(np) = \psi(n)\varphi(p) \) for all \( n \in N_m \) and \( p \in P_m \). Reasoning as finding the normalized equation in proof of Proposition 4.2 or Proposition 4.4, for any \( P \in P_0 \), the hyperplane \( H \times \mathcal{D}_s \) is a subset of the zeros of the polynomials

\[
P(q^{\pm u}) \prod_{1 \leq k \leq L}(s + 2u_k, \Delta_k, \wedge^2)\prod_{1 \leq i < j \leq L}(s + u_i + u_j, \Delta_i \times \Delta_j).
\]

As we consider Gelfand and Kazdan Theorem F in [11], there exists \( P \in P_0 \) such that \( P(q^{\pm u}) = 1 \) for each fixed \( u \in \mathcal{D}_\pi \), because \( \mathcal{W}_\pi^{(0)} := \{ W_u(p) \mid W_u \in \mathcal{W}_\pi^{(0)} \} \) contains \( \text{ind}_{N_m}^P(\psi) \) from [9, Proposition 3.1]. For fixed \( u \in H \), this would imply that \( \{ u \} \times \mathcal{D}_s \) is a subset of the zeros of

\[
\prod_{1 \leq k \leq L}(s + 2u_k, \Delta_k, \wedge^2)\prod_{1 \leq i < j \leq L}(s + u_i + u_j, \Delta_i \times \Delta_j).
\]

(5.6)

This is absurd because as for any fixed \( u \in H \), (5.6) is the nonzero polynomial in \( \mathbb{C}[q^{\pm s}] \). Therefore \( e^\circ(s, \pi_u, \wedge^2, \psi) \) has no poles and hence lies in \( \mathbb{C}[q^{\pm u}, q^{\pm s}] \).

Under deformation the \( \gamma \)-factor is multiplicative up to a monomial factor. If we now specialize \( u = 0 \), then we have the following Corollary.

\[ \square \] Springer
Corollary 5.5 Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be a representation of \( GL_m \). Then \( \gamma(s, \pi, \wedge^2, \psi) \) and \( \prod_{1 \leq i < j \leq t} \gamma(s, \Delta_i \times \Delta_j, \psi) \) are equal up to a unit in \( \mathbb{C}[q^{\pm s}] \).

From this point on, we recall that every irreducible admissible representation occurs as the unique quotient module of a standard module. We define the local exterior square \( L \)-function for non-generic representation.

**Definition** Let \( \pi \) be an irreducible admissible representation of \( GL_m \) realized as an unique Langlands quotient of the standard module \( \Pi = \text{Ind}(\Delta_1 \psi^{\mu_1} \otimes \cdots \otimes \Delta_t \psi^{\mu_t}) \). Then the \( L \)-function \( L(s, \pi, \wedge^2) \) is defined by

\[
L(s, \pi, \wedge^2) = L(s, \Pi, \wedge^2).
\]

We need the following Lemma, which is analogue of Lemma 9.3 of [15] or Lemma 5.10 of [28].

**Lemma 5.6** Suppose that \( \pi = \text{Ind}(\pi_1 \otimes \pi_2) \) is a standard module of \( GL_{n_1+n_2} \) with each \( \pi_i \) a standard module of \( GL_{n_i} \), and the induction is the normalized parabolic induced representation from a standard upper maximal parabolic. Then \( L(s, \pi_2, \wedge^2)^{-1} \) divides \( L(s, \pi, \wedge^2)^{-1} \), that is, \( L(s, \pi_2, \wedge^2) = Q(q^{-s})L(s, \pi, \wedge^2) \) with \( Q(X) \in \mathbb{C}[X] \).

**Proof** In Proposition 9.1 of [15] they establish that if \( \pi = \text{Ind}(\pi_1 \otimes \pi_2) \) is a representation of \( GL_{n_1+n_2} \) with each \( \pi_i \) a generic representation of \( GL_{n_i} \), then for every \( W_2 \in \mathcal{W}(\pi_2, \psi) \) and \( \Phi \in \mathcal{S}(F^{n_2}) \) there is a \( W \in \mathcal{W}(\pi, \psi) \) such that

\[
W(h) = W_2(h)\Phi(e_{n_2}h)|\det(h)|^{\frac{n_1}{2}}.
\]

for \( h \in GL_{n_2} \). In the case that \( 2n = n_1+n_2 \) is an even number, we know from Proposition 4.6 of [19] that \( \mathbb{C}[q^{\pm s}] \)-fractional ideal \( J(\pi) \) contains the \( \mathbb{C}[q^{\pm s}] \)-fractional ideal \( J(2m-1)(\pi) \) generated by the local integrals

\[
J(2m-1)(s, W) = \int_{N_{n-m}\setminus GL_{n-m}} \int_{N_{n-m}\setminus M_{n-m}} W \left( \sigma_{2n-2m+1} \begin{pmatrix} I_{n-m} & X \\ I_{n-m} & 1 \end{pmatrix} \begin{pmatrix} g \\ g \end{pmatrix} \right) \psi^{-1}(\text{Tr}X) dX |\det(g)|^{s-2m} dg.
\]

with \( W \in \mathcal{W}(\pi, \psi) \). In the case of \( n_1 = 2m \), if we let \( W_2 \in \mathcal{W}(\pi_2, \psi) \), \( \Phi \in \mathcal{S}(F^{n-m}) \) and take \( W \) to be associated element of \( \mathcal{W}(\pi, \psi) \), we see from \( \sigma_{2n-2m+1} = \left( \sigma_{2n-2m} \right) \) that this integral turns out to be

\[
J(2m-1)(s, W) = \int_{N_{n-m}\setminus GL_{n-m}} \int_{N_{n-m}\setminus M_{n-m}} W_2 \left( \sigma_{2n-2m} \begin{pmatrix} I_{n-m} & X \\ I_{n-m} & 1 \end{pmatrix} \begin{pmatrix} g \\ g \end{pmatrix} \right) \psi^{-1}(\text{Tr}X) dX \Phi(e_{n-m}g)|\det(g)|^{s} dg.
\]

In the case of \( n_1 = 2m-1 \), we can choose any \( \Phi \in \mathcal{S}(F^{n_2}) \) such that \( \Phi(e_{n_2}) = 1 \). For every \( W_2 \in \mathcal{W}(\pi_2, \psi) \) and the associated elements \( W \) of \( \mathcal{W}(\pi, \psi) \), our integral \( J(2m-1)(s, W) \) becomes
Let \( \pi \) be an irreducible admissible \( \pi \) be an irreducible generic representation of \( GL_m \), so each \( \Delta_i \) is quasi-square-integrable and no two of segments \( \Delta_1, \cdots, \Delta_t \) are linked. Then

\[
L(s, \pi, \wedge^2) = \prod_{k=1}^{t} L(s + 2u_k, \Delta_k, \wedge^2) \prod_{1 \leq i < j \leq t} L(s + u_i + u_j, \Delta_i \times \Delta_j).
\]

**Proof** Since \( \pi \) is irreducible and generic, two of segments \( \Delta_1, \cdots, \Delta_t \) are unlinked, and so the quasi-square-integrable representations \( \Delta_1, \cdots, \Delta_t \) are rearranged to be in Langlands order without changing \( \pi \). Then the result is just a restatement of the Theorem above.

We pass to the case of a tempered representation, which is defined in [15, Section 8]. A tempered representation is of the form

\[
\pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)
\]

where \( \Delta_i \) are irreducible square-integrable. As a tempered representation is automatically irreducible by [35, Theorem 4.2] and generic, we have the following Corollary.

**Corollary 5.9** Let \( \pi = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t) \) be a tempered representation of \( GL_m \). Then we obtain

\[
L(s, \pi, \wedge^2) = \prod_{1 \leq k \leq t} L(s, \Delta_k, \wedge^2) \prod_{1 \leq i < j \leq t} L(s, \Delta_i \times \Delta_j).
\]

### 5.2 Exceptional and symmetric square \( L \)-functions

We introduce symmetric square \( L \)-functions defined by the ratio of Rankin-Selberg \( L \)-functions for \( GL_m \) by exterior square \( L \)-functions for \( GL_m \). Let \( \pi \) be an irreducible admissible representation of \( GL_m \). We let \( \Pi = \text{Ind}(\Delta_1 ^{u_1} \otimes \cdots \otimes \Delta_t ^{u_t}) \) be standard module such that \( \pi \) is Langlands quotient of \( \Pi \). We let denote the symmetric square \( L \)-function of \( \pi \) by

\[
L(s, \pi, \text{Sym}^2) = \frac{L(s, \pi \times \pi)}{L(s, \pi, \wedge^2)}.
\]
We emphasize that this definition of $L$-function $L(s, \pi, \text{Sym}^2)$ agrees with Artin’s symmetric square $L$-function on the arithmetic side. We will be concerned with the proof of this result in the next Section.

Let $\rho$ be an irreducible unitary supercuspidal representation of $GL_r$. We denote by $\overline{x}$ the complex conjugate of $x \in \mathbb{C}$. Jacquet, Piatetski-Shapiro, and Shalika establish

$$L(s, \tilde{\rho} \times \tilde{\rho}) = \overline{L(\overline{s}, \rho \times \rho)}$$

in the proof of Theorem 8.2 of [15] (page 446). The following is the analogue of this result for our $L$-functions $L(s, \rho, \wedge^2)$.

**Lemma 5.10** Let $\rho$ be an irreducible unitary supercuspidal representation of $GL_r$. Then we have

$$L(s, \tilde{\rho}, \wedge^2) = \overline{L(\overline{s}, \rho, \wedge^2)} \quad \text{and} \quad L(s, \overline{\rho} \times \overline{\rho}) = \overline{L(\overline{s}, \rho \times \rho)}.$$

**Proof** Let $\overline{W(\rho, \psi)}$ be the complex conjugate of Whittaker model $W(\rho, \psi)$ defined by

$$\overline{W(\rho, \psi)} = \{ W(g) \mid W \in W(\rho, \psi), g \in GL_r \}.$$

and $\overline{\rho}$ right translation of $GL_r$ on the space $W(\rho, \psi)$. As observed in [32, Section 1], $\overline{W(\rho, \psi)}$ is Whittaker model $W(\overline{\rho}, \psi^{-1})$ which is isomorphic to $W(\overline{\rho}, \psi^{-1})$. Suppose that $r = 2n$ is even. We recall from Theorem 3.6 that $L(s, \rho, \wedge^2) = \prod_j (1 - \alpha_j q^{-s})^{-1}$, where the product runs over all $\alpha_j = q^{s_0}$ such that $\text{Hom}_{S_{2n}}(\rho v^{q_0}, \Theta) \neq 0$. In particular, $\omega_\rho v^{n_0} = 1$. Since $\rho$ is unitary and supercuspidal, the central character $\omega_\rho$ is unitary. Then the poles are purely imaginary, because $v^{n_0} = \omega_\rho^{-1}$ is also a unitary character. Hence $\overline{s_0} = -s_0$. Applying complex conjugation on twisted Shalika functionals induces the isomorphism

$$\text{Hom}_{S_{2n}}(W(\rho, \psi)v^{q_0}, \Theta) \simeq \text{Hom}_{S_{2n}}(W(\overline{\rho}, \psi^{-1})v^{-q_0}, \Theta^{-1}).$$

Hence $L(s, \rho, \wedge^2)$ has a simple pole at $s = s_0$ if and only if $L(s, \overline{\rho}, \wedge^2)$ has a simple pole at $s = -s_0$. Since these are the only possible poles of $L(s, \rho, \wedge^2)$ and $|\alpha_j| = 1$, we take the product over all such $\alpha^{-1}_j = q^{-s_0}$, which establishes

$$L(s, \tilde{\rho}, \wedge^2) = L(s, \overline{\rho}, \wedge^2) = \prod_j (1 - \alpha^{-1}_j q^{-s})^{-1} = \prod_j (1 - \overline{\alpha}_jq^{-s})^{-1} = L(\overline{s}, \rho, \wedge^2).$$

If $r = 2n + 1$ is odd, then by Theorem 3.6 we obtain $L(s, \tilde{\rho}, \wedge^2) = L(\overline{s}, \rho, \wedge^2) = 1$. \hfill $\square$

As a consequence of Lemma 5.10, we have the following result.

**Lemma 5.11** Let $\rho$ be an irreducible unitary supercuspidal representation of $GL_r$. Up to multiplication by a monomial in $q^{-s}$, $L(-s, \rho, \wedge^2)$ and $L(s, \tilde{\rho}, \wedge^2)$ are equal and similarly for the two functions $L(-s, \rho \times \rho)$ and $L(s, \overline{\rho} \times \overline{\rho})$.

**Proof** Suppose that $L(s, \rho, \wedge^2) = \prod_j (1 - \alpha_j q^{-s})^{-1}$. Since $\rho$ is unitary and supercuspidal, $|\alpha_j| = 1$ for all $j$. Then $L$-function

$$L(s, \tilde{\rho}, \wedge^2) = \prod_j (1 - \alpha^{-1}_j q^{-s})^{-1} = \prod_j (\overline{\alpha}_jq^{-s})(1 - \alpha_j q^{-s})^{-1}$$

$$= \left[ \prod_j (-\alpha^{-1}_j q^{-s}) \right] L(-s, \rho, \wedge^2).$$
is equal to \( L(-s, \rho, \wedge^2) \) up to a unit \( \prod_j (-\alpha_j^{-1}q^i) \) in \( \mathbb{C}[q^\pm] \). The case of \( L\text{-function} \)
\( L(-s, \rho \times \rho) \) can be treated in the same way.

We are now in the position to complete the proof of the main Theorem of this Section.

**Theorem 5.12** Let \( \Delta \) be an irreducible square integrable representation of \( GL_\ell \) with the segment \( \Delta = [\rho v_{\ell}^{-\frac{\ell-1}{2}}, \ldots, \rho v_{\ell}^{\frac{\ell-1}{2}}] \) and \( \rho \) an irreducible unitary supercuspidal representation of \( GL_\ell \).

(i) Suppose that \( \ell \) is even. Then we have

\[
L(s, \Delta, \wedge^2) = \prod_{i=0}^{\ell-1} L(s + 2i + 1, \rho, \wedge^2) L(s + 2i, \rho, \text{Sym}^2),
\]

\[
L(s, \Delta, \text{Sym}^2) = \prod_{i=0}^{\ell-1} L(s + 2i, \rho, \wedge^2) L(s + 2i + 1, \rho, \text{Sym}^2).
\]

(ii) Suppose that \( \ell \) is odd. Then we obtain

\[
L(s, \Delta, \wedge^2) = \prod_{i=1}^{\frac{\ell-1}{2}} L(s + 2i - 1, \rho, \wedge^2) \prod_{i=1}^{\frac{\ell-1}{2}} L(s + 2i - 1, \rho, \text{Sym}^2),
\]

\[
L(s, \Delta, \text{Sym}^2) = \prod_{i=0}^{\frac{\ell-1}{2}} L(s + 2i, \rho, \wedge^2) \prod_{i=0}^{\frac{\ell-1}{2}} L(s + 2i, \rho, \text{Sym}^2).
\]

**Proof** We start with the Whittaker model of \( \Delta \). As in 9.1 of Zelevinsky [35], \( \Delta \) is the unique irreducible quotient of the normalized induced representation \( \text{Ind}(\rho v_{\ell}^{-\frac{\ell-1}{2}} \otimes \rho v_{\ell}^{-\frac{\ell-3}{2}} \otimes \cdots \otimes \rho v_{\ell}^{\frac{\ell-1}{2}}) \). This induces a short exact sequence of smooth representation of \( GL_\ell \), namely

\[
0 \to V_\psi \to \text{Ind}(\rho v_{\ell}^{-\frac{\ell-1}{2}} \otimes \rho v_{\ell}^{-\frac{\ell-3}{2}} \otimes \cdots \otimes \rho v_{\ell}^{\frac{\ell-1}{2}}) \to \Delta \to 0
\]

where \( V_\psi \) denote the kernel. Since \( \Delta \) is irreducible and generic, there is essentially unique Whittaker functional on \( \Delta \). This will then induce a non-zero Whittaker functional on \( \text{Ind}(\rho \otimes \rho v \otimes \cdots \otimes \rho v_{\ell-1}) \). Since this representation also has a unique Whittaker functional, this must be it and we can conclude that

\[
\mathcal{W}(\Delta, \psi) = \mathcal{W}(\text{Ind}(\rho v_{\ell}^{-\frac{\ell-1}{2}} \otimes \rho v_{\ell}^{-\frac{\ell-3}{2}} \otimes \cdots \otimes \rho v_{\ell}^{\frac{\ell-1}{2}}), \psi).
\]

Our result is adopted from the proof of Proposition 8.1 in [34]. Applying Corollary 5.5 to the Whittaker model \( \mathcal{W}(\text{Ind}(\rho v_{\ell}^{-\frac{\ell-1}{2}} \otimes \cdots \otimes \rho v_{\ell}^{\frac{\ell-1}{2}}), \psi) \) implies that

\[
\gamma(s, \Delta, \wedge^2, \psi) \sim \prod_{i=0}^{\ell-1} \gamma(s + 1 - \ell + 2i, \rho, \wedge^2, \psi) \prod_{0 \leq i < j \leq \ell-1} \gamma(s + 1 - \ell + i + j, \rho \times \rho, \psi).
\]

In terms of \( L\text{-function} \), we restate it as

\[
\gamma(s, \Delta, \wedge^2, \psi) \sim \prod_{i=0}^{\ell-1} \frac{L(-s + \ell - 2i, \tilde{\rho}, \wedge^2)}{L(s + 1 - \ell + 2i, \rho, \wedge^2)} \prod_{0 \leq i < j \leq \ell-1} \frac{L(-s + \ell - i - j, \tilde{\rho} \times \tilde{\rho})}{L(s + 1 - \ell + i + j, \rho \times \rho)}.
\]
After we apply Lemma 5.11, we find

\[
\gamma(s, \Delta, \wedge^2, \psi) \sim \prod_{i=0}^{\ell-1} \frac{L(s - \ell + 2i, \rho, \wedge^2)}{L(s + 1 - \ell + 2i, \rho, \wedge^2)} \prod_{0 \leq i < j \leq \ell - 1} \frac{L(s - \ell + i + j, \rho \times \rho)}{L(s + 1 - \ell + i + j, \rho \times \rho)}.
\]

(5.7)

Then

\[
\prod_{0 \leq i < j \leq \ell - 1} \frac{L(s - \ell + i + j, \rho \times \rho)}{L(s + 1 - \ell + i + j, \rho \times \rho)} = \prod_{0 \leq i < \ell - 1} \frac{L(s - \ell + 2i + 1, \rho \times \rho)}{L(s + i, \rho \times \rho)}
\]

(5.8)

Inserting all the way to the right term in (5.8) into (5.7), we derive

\[
\gamma(s, \Delta, \wedge^2, \psi) \sim \prod_{i=0}^{\ell-1} \frac{L(s - \ell + 2i, \rho, \wedge^2)}{L(s + 1 - \ell + 2i, \rho, \wedge^2)} \prod_{i=1}^{\ell-1} \frac{L(s - \ell + i, \rho \times \rho)}{L(s - \ell + 2i, \rho \times \rho)}
\]

where we have replaced \(L(s, \rho \times \rho)\) by the product \(L(s, \rho, \wedge^2)L(s, \rho, \text{Sym}^2)\) by definition.

First assume \(\ell\) is even. We do the case \(\ell\) even, the case \(\ell\) odd being similar. Let us now cancel common factors. Then our quotient simplifies to

\[
\frac{L(1 - s, \widetilde{\Delta}, \wedge^2)}{L(s, \Delta, \wedge^2)}
\]

\[
\sim \gamma(s, \Delta, \wedge^2, \psi) \sim \prod_{i=0}^{\frac{\ell}{2}-1} \frac{L(s - \ell + 2i, \rho, \wedge^2)}{L(s + 2i + 1, \rho, \wedge^2)} \prod_{i=0}^{\frac{\ell}{2}-1} \frac{L(s - \ell + 2i + 1, \rho, \text{Sym}^2)}{L(s + 2i, \rho, \text{Sym}^2)}
\]

where we have used the equality of \(L(1 - s, \widetilde{\Delta}, \wedge^2)/L(s, \Delta, \wedge^2)\) and \(\gamma(s, \Delta, \wedge^2, \psi)\) up to units in \(\mathbb{C}[q^{\pm s}]\) according to the functional equation. Propositions 3.11 and 3.13 imply that \(L(1 - s, \widetilde{\Delta}, \wedge^2)^{-1}\) has zeros in the half plane \(\text{Re}(s) \geq 1\) while \(L(s, \Delta, \wedge^2)^{-1}\) has its zeros contained in the region \(\text{Re}(s) \leq 0\). Since the half planes \(\text{Re}(s) \geq 1\) and \(\text{Re}(s) \leq 0\) are disjoint, they do not have common factors in \(\mathbb{C}[q^{\pm s}]\). As \(\rho\) is unitary, the poles of \(L\)-function \(\prod_{i=0}^{\frac{\ell}{2}-1} L(s - \ell + 2i, \rho, \wedge^2)L(s - \ell + 2i + 1, \rho, \text{Sym}^2)\) must lie on the line \(\text{Re}(s) = \ell - i\) for \(i = -\ell + 1, \ldots, 0\), while the poles of \(L\)-function \(\prod_{i=0}^{\frac{\ell}{2}-1} L(s + 2i + 1, \rho, \wedge^2)L(s + 2i, \rho, \text{Sym}^2)\) will lie on the line \(\text{Re}(s) = -i\) for \(i = -\ell + 1, \ldots, 0\). Therefore they have no factors in common. We conclude that

\[
L(s, \Delta, \wedge^2) \sim \prod_{i=0}^{\frac{\ell}{2}-1} L(s + 2i + 1, \rho, \wedge^2)L(s + 2i, \rho, \text{Sym}^2).
\]

Both sides of the above \(L\)-functions are exactly the inverse of the normalized polynomial and hence are equal. For symmetric square \(L\)-function \(L(s, \Delta, \text{Sym}^2)\), we know from [9,15] that
According to Theorem 5.12, we know

\[ L(s, \Delta \times \Delta) = \prod_{i=0}^{s-1} L(i + s, \rho \times \rho) = \prod_{i=0}^{s-1} L(s + 2i + 1, \rho \times \rho) L(s + 2i, \rho \times \rho) . \]

As symmetric square \( L \)-function is defined by the ratio of Rankin-Selberg \( L \)-functions by exterior square \( L \)-function, we obtain

\[
L(s, \Delta, \text{Sym}^2) = \prod_{i=0}^{s-1} \frac{L(s + 2i, \rho \times \rho) L(s + 2i + 1, \rho \times \rho)}{L(s + 2i, \rho, \text{Sym}^2) L(s + 2i + 1, \rho, \wedge^2)} \\
= \prod_{i=0}^{s-1} L(s + 2i, \rho, \wedge^2) L(s + 2i + 1, \rho, \text{Sym}^2).
\]

\[ \square \]

We remark that the product formula in Theorem 5.12 agrees with that of Langland-Shahidi exterior or symmetric square \( L \)-function for square integrable representation in [34, Proposition 8.1]. On the arithmetic side, the formula is also consistent with the expression of Artin’s exterior square \( L \)-function for the Langlands parameter of square integrable representation in [27, Proposition 6.2]. We also note that the result easily follows from the agreement of exterior square \( L \)-function by means of integral representation and the Langland-Shahidi exterior square \( L \)-function for square integrable representation in Kewat and Raghunathan of [21], assuming that symmetric square \( L \)-functions are defined in our way. However our proof is completely local as opposed to the global argument in [21]. We now wish to express the exceptional \( L \)-function \( L_{ex}(s, \Delta, \wedge^2) \) in terms of \( L \)-functions of irreducible supercuspidal representations. Theorem 5.12 leads quickly to the following Theorem which is originally conjectured in [8].

**Theorem 5.13** Let \( \Delta \) be an irreducible square integrable representation of \( GL_r \) with the segment \( \Delta = [\rho v^{-\frac{t-1}{2}}, \ldots, \rho v^{\frac{t-1}{2}}] \) and \( \rho \) an irreducible unitary supercuspidal representation of \( GL_r \).

(i) Suppose that \( r \) is even. Then
\[
L_{ex}(s, [\rho v^{-\frac{t-1}{2}}, \ldots, \rho v^{\frac{t-1}{2}}], \wedge^2) = \begin{cases} 
L(s, \rho, \wedge^2) & \text{if } \ell \text{ is odd} \\
L(s, \rho, \text{Sym}^2) & \text{if } \ell \text{ is even}.
\end{cases}
\]

(ii) Suppose that \( r \) is odd. Then
\[
L_{ex}(s, [\rho v^{-\frac{t-1}{2}}, \ldots, \rho v^{\frac{t-1}{2}}], \wedge^2) = L(s, \rho, \text{Sym}^2) & \text{if } \ell \text{ is even}.
\]

**Proof** For \( \ell = 1 \), \( L_{ex}(s, \rho, \wedge^2) = L(s, \rho, \wedge^2) \), which follows from Proposition 3.6. We now begin with the simplest non-supercuspidal square integrable representation \( \Delta = [\rho v^{-\frac{1}{2}}, \rho v^{\frac{1}{2}}] \) of \( GL_2 \) with \( \rho \) an irreducible unitary supercuspidal representation of \( GL_2 \) and \( r \) an even number. The derivatives of \( \Delta \) are \( \Delta^{(0)} = \Delta \) and \( \Delta^{(r)} = \rho v^{rac{r}{2}} \). Then Proposition 3.11 gives
\[
L(s, \Delta, \wedge^2) = L_{ex}(s, [\rho v^{-\frac{1}{2}}, \rho v^{\frac{1}{2}}], \wedge^2)L(s + 1, \rho, \wedge^2).
\]

According to Theorem 5.12, we know
\[
L(s, \Delta, \wedge^2) = L(s, \rho, \text{Sym}^2)L(s + 1, \rho, \wedge^2).
\]

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The proof for the base case $\ell = 2$.

We are going to prove by induction on $\ell$. We assume that the following statement is satisfied for all integers $k$ with $1 \leq k \leq \ell$

$$L_{\text{ex}}(s, [\rho \nu^{-\frac{k}{2}}, \ldots, \rho \nu^{\frac{k}{2}}], \wedge^2) = \begin{cases} L(s, \rho, \wedge^2) & \text{if } k \text{ is odd} \\ L(s, \rho, \text{Sym}^2) & \text{if } k \text{ is even.} \end{cases}$$

The proof for $\ell$ odd is the same as $\ell$ even. Thus we only produce the case for $\ell$ even. On the one hand, Theorem 5.12 gives the formula

$$L(s, [\nu^{-\frac{1}{2}}, \ldots, \nu^{\frac{1}{2}}], \wedge^2) = L(s, \nu^2(\nu), \wedge^2) \prod_{i=1}^{\ell} L(s + 2i, \nu^{2i}, \wedge^2) \prod_{i=1}^{\ell} L(s + 2i - 1, \nu^{2i}, \text{Sym}^2).$$

On the other hand, if we combine the first part of Proposition 3.11 with our induction hypothesis, we can obtain the following equality of $L$-functions

$$L(s, [\nu^{-\frac{1}{2}}, \ldots, \nu^{\frac{1}{2}}], \wedge^2) = L_{\text{ex}}(s, [\nu^{-\frac{1}{2}}, \ldots, \nu^{\frac{1}{2}}], \wedge^2) \times \prod_{i=1}^{\ell} L_{\text{ex}}(s, [\nu^{2i-\frac{1}{2}}, \ldots, \nu^{\frac{1}{2}}], \wedge^2) L_{\text{ex}}(s, [\nu^{2i}, \nu^{2i}, \wedge^2], \wedge^2)$$

$$= L_{\text{ex}}(s, [\nu^{-\frac{1}{2}}, \ldots, \nu^{\frac{1}{2}}], \wedge^2) \prod_{i=1}^{\ell} L(s + 2i, \nu^{2i}, \wedge^2) \prod_{i=1}^{\ell} L(s + 2i - 1, \nu^{2i}, \text{Sym}^2).$$

By taking all these into account, we conclude the that

$$L_{\text{ex}}(s, [\nu^{-\frac{1}{2}}, \ldots, \nu^{\frac{1}{2}}], \wedge^2) = \begin{cases} L(s, \rho, \wedge^2) & \text{if } \ell \text{ is even} \\ L(s, \rho, \text{Sym}^2) & \text{if } \ell \text{ is odd.} \end{cases}$$

The proof for $r$ odd and $\ell$ even is a straightforward modification of the previous computation.

From Theorem 2.13, $L(s, \Delta, \wedge^2)$ are completely determined by the exceptional $L$-functions $L_{\text{ex}}(s, \Delta^{(i)}, \wedge^2)$ for the derivatives $\Delta^{(i)}$ which are representations of $GL_{m-i}$ with $m - i$ even numbers. Hence we do not consider the case when $\ell$ and $r$ are odd because $\Delta = [\nu^{-\frac{1}{2}}, \ldots, \nu^{\frac{1}{2}}]$ is a representation of odd $GL_{\ell r}$.

5.3 The relation with the local Langlands correspondence

Let $\Phi_F$ denote a choice of geometric Frobenius element of $\text{Gal}(\overline{F} / F)$. Let $\phi$ be an $m$-dimensional $\Phi_F$-semisimple representation of $W_F$, the Weil–Deligne group for $\overline{F} / F$. Let $\pi = \pi(\phi)$ be the irreducible admissible representation of $GL_m$ associated to $\phi$ by the local Langlands correspondence by Harris-Taylor [12], Henniart [13]. Let $\wedge^2$ and $\text{Sym}^2$ denote the exterior and symmetric square representations of $GL_m(\mathbb{C})$, respectively. Along the line of the proof in [21, Theorem 7.2] which in fact extends to all irreducible admissible representation of $GL_m$, we derive the equality of the exterior square arithmetic (Artin) and analytic (Jacquet–Shalika) $L$-functions for $GL_m$ through the local Langlands correspondence.
**Theorem 5.14** Let $\phi$ be a $\Phi_F$-semisimple $m$-dimensional complex representation of Weil–Deligne group $W'_F$ and $\pi = \pi(\phi)$ an irreducible admissible representation of $GL_m$ associated to $\phi$ under the local Langlands correspondence. Then we have

$$L(s, \wedge^2(\phi)) = L(s, \pi(\phi), \wedge^2).$$

The analogous results for the Asai $L$-functions [24,26] or the Bump-Friedberg $L$-function [28] are established by Matringe. For the symmetric square $L$-function $L(s, \pi, \text{Sym}^2)$, we begin with

$$L(s, \pi(\phi) \times \pi(\phi)) = L(s, \pi(\phi), \wedge^2)L(s, \pi(\phi), \text{Sym}^2).$$

On the arithmetic side, we have

$$L(s, \phi \times \phi) = L(s, \wedge^2(\phi))L(s, \text{Sym}^2(\phi)).$$

The local Langlands correspondence asserts that $L(s, \pi(\phi) \times \pi(\phi)) = L(s, \phi \times \phi)$. Then we obtain the following results from the previous Theorem 5.14.

**Corollary 5.15** Let $\phi$ be a $\Phi_F$-semisimple $m$-dimensional complex representation of Weil–Deligne group $W'_F$ and $\pi = \pi(\phi)$ an irreducible admissible representation of $GL_m$ associated to $\phi$ under the local Langlands correspondence. Then we have

$$L(s, \text{Sym}^2(\phi)) = L(s, \pi(\phi), \text{Sym}^2).$$

This Corollary explains that our definition of symmetric $L$-function is consistent with Artin’s symmetric $L$-function in arithmetic side.

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