ANALYSIS OF A SCALAR NONLOCAL PERIDYNAMIC MODEL WITH A SIGN CHANGING KERNEL

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Abstract. In this paper, a scalar peridynamic model is analyzed. The study extends earlier works in the literature on scalar nonlocal diffusion and nonlocal peridynamic models to include a sign changing kernel. We prove the well-posedness of both variational problems with nonlocal constraints and time-dependent equations with or without damping. The analysis is based on some nonlocal Poincaré type inequalities and compactness of the associated nonlocal operators. It also offers careful characterizations of the associated solution spaces such as compact embedding, separability and completeness along with regularity properties of solutions for different types of kernels.

1. Introduction. This work is motivated by the bond-based peridynamic model (PD) of continuum mechanics proposed by S. Silling [20]. The model asserts that in a material body of mass density \( m = m(x) \) a pair of particles interact through a vector valued force density function \( f \) so that the nonlocal peridynamic equation of motion for the displacement field \( u \)

\[
m(x)u_{tt}(x,t) = \int_{\Omega} f(u(x',t) - u(x,t), x' - x) \, dx' - b(x,t),
\]

is satisfied in a bounded domain \( \Omega \), along with suitable initial and nonlocal boundary conditions to be specified later. On the right hand side of (1), the first expression represents the force per unit reference volume at a particle due to interaction with other particles, while \( b \) denotes a given loading force density function. The focus of this work is on the linear bond-based PD model for isotropic and microelastic materials, where \( f \) is a linear function of the displacement field given by

\[
f(u(x',t) - u(x,t), x' - x) = C(x' - x)u(x',t) - u(x,t)).
\]

The kernel \( C = C(\xi) \), for homogeneous and isotropic system, is of the form

\[
C(|\xi|) = -2\rho(|\xi|)\xi \otimes \xi - 2\lambda g(|\xi|)I,
\]

where \( \lambda \in \mathbb{R} \) is a scalar parameter, \( \rho \) and \( g \) are functions of \( |\xi| \).

Analytical and numerical aspects of the linear bond-based PD model have been studied by [1, 9, 11, 12, 13, 14, 16, 24], in the case when \( \rho \) is a positive function and
The latter corresponds to the situation when we assume the force between any two particles vanishes in the reference configuration. Under this assumption, the material is said to be pairwise equilibrated and the model describes a network of springs with positive spring constants [20]. Silling argued in [20] that this assumption is too restrictive to impose in modeling real materials, but rather it is reasonable to expect a nonzero force interaction between particles but the body to be unstressed (in the sense of Silling) in the reference configuration. An implication is that the distribution of forces between particles is repulsive for some interactions and attractive for others which places a restriction on the force density function \( g \) that it needs to be sign changing. This motivates our current analysis of the linear peridynamic model for scalar fields when \( g \) is not identically 0 but in fact changes sign. In this case the scalar PD model is given by

\[
m(x) u_{tt}(x, t) = \mathcal{P}_\lambda u(x, t) + b(x, t),
\]

where the PD linear operator \( \mathcal{P}_\lambda \) is formally defined by

\[
\mathcal{P}_\lambda u(x) := -\int_\Omega 2\rho(|x - x'|) + \lambda g(|x - x'|)(u(x') - u(x))dx'.
\]

Although motivated by the study of peridynamics, our work is of interests in a broader context as well as the results obtained are equally applicable to other nonlocal models. Indeed, PD belongs to the class of nonlocal mathematical models that arise naturally in many important fields. The monograph [3] presents results on the nonlocal diffusion models that approximate, for instance, classical equations of porous media flow. The work of [19] models self-organized dynamics using nonlocal equations. A study of nonlocal operators with application to image processing is given in [15]. Other areas of application include, to name a few, modeling wave propagation, pattern formation and population aggregation.

Let us highlight the main findings in this work. Using variational techniques, the well posedness is established for a scalar linear PD (nonlocal diffusion) model with a sign-changing kernel, effectively extending results obtained in [1, 9, 12, 13, 14, 16, 24]. This presents not only a theoretical generalization but also a justification to some of the concepts originally studied by Silling. Indeed, as shown later, the solvability of PD equilibrium equation can be related to a material stability condition considered in [20, 23]. To implement the variational approach, in Section 2, we define a solution space which is a function space closely associated with the PD model. For a broad range of kernels satisfying minimal conditions, we prove further structural properties of the space such as completeness in an appropriate norm, compactness in \( L^2(\Omega) \) and separability. These results have not been established in such a general form before in the literature, although they are known to hold for fractional Sobolev spaces which are particular cases of our study. Our analysis is also direct and it neither uses Fourier transform (as in [12, 10]) for special domains nor attempts to relate the solution space to well known Hilbert spaces for special kernels (as in [11]). In Section 3, the PD operator \( \mathcal{P}_\lambda \) and its leading nonlocal operator are analyzed further. For the convenience of readers with applied background, we first present some analysis which help clarify the meaning of nonlocal operators under consideration. Then, necessary inequalities such as coercivity are established through a nonlocal Poincaré-type inequality, which is also proved under very general conditions. The inequality established here is tighter since it holds for kernels with weaker singularities as opposed to the strongly singular kernels necessary in fractional Sobolev spaces. In Section 4 we provide the well-posedness
of the steady state nonlocal scalar model as a nonlocal variational problem with volume constraints. Finally in Section 5, we state a well-posedness result of the scalar time-dependent peridynamic equation as an initial value problem defined over proper closed subspaces of $L^2(\Omega)$.

2. A solution space. In this section, a solution space for the nonlocal equations is studied which is the natural energy space corresponding to the energy functional associated with the leading operator of $\mathcal{P}_\lambda$. We establish properties of the solution space such as the completeness, compactness, and separability for more general kernels. In previous studies, these results have been shown only for special kernels, by utilizing the equivalence with well known function spaces like fractional Sobolev spaces. While our studies proceed in parallel to the analysis of local second order elliptic equations, we highlight the necessary modifications to the traditional techniques that are used in the analysis here.

2.1. Definition. We think of $\mathcal{P}_\lambda$ as a perturbation of the nonlocal operator

$$u \mapsto \mathcal{L}u := -2 \int_{\Omega} \rho(|x' - x|) (u(x') - u(x)) \, dx'.$$

(3)

The energy functional associated with the operator $\mathcal{L}$ is

$$(\mathcal{L}u, u) = \int_{\Omega} \int_{\Omega} \rho(|x' - x|) (u(x') - u(x))^2 \, dx' \, dx.$$

(4)

Observe that $\mathcal{L}$ is essentially a nonlocal operator studied for many of the linear nonlocal problems (see, e.g., [3]) when $\rho(|\xi|)$ is locally integrable. As we will see shortly, any $u \in L^2(\Omega)$ makes the energy functional in (4) finite for such a special case, but this is not so in general. The generality of the kernels we consider makes the analysis undertaken here different from that of the nonlocal linear analysis in [3]. Note that in the terminology of nonlocal calculus [10], we can write $\mathcal{L} = -D(\rho D^s)$ where $D$ and $D^s$ represent some nonlocal divergence operator and its dual nonlocal gradient operator. The space of interest to us, denoted by $\mathcal{S}(\Omega)$, is a subspace of $L^2(\Omega)$ given by

$$\mathcal{S}(\Omega) = \{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \rho(|x - x'|)|u(x') - u(x)|^2 \, dx' \, dx < \infty \}$$

which can be conveniently written as $\mathcal{S}(\Omega) = \{ u \in L^2(\Omega) : \sqrt{\rho} D^s(u) \in L^2(\Omega \times \Omega) \}$ (see [10, 11] for detailed definitions). Formally, we may define a bilinear form $(\cdot, \cdot) : \mathcal{S}(\Omega) \times \mathcal{S}(\Omega) \to \mathbb{R}$ by

$$(u, v) = \int_{\Omega} \int_{\Omega} \rho(|x - x'|)(u(x') - u(x))(v(x') - v(x)) \, dx' \, dx.$$

Denoting the $L^2$ inner product by $(\cdot, \cdot)$, $(\mathcal{S}(\Omega), (\cdot, \cdot)_{s})$ is a real inner product space with the inner product $(\cdot, \cdot)_{s}$ defined as

$$(u, w)_{s} = (u, w) + ((u, w)).$$

We use $\|u\|$ to denote the $L^2$ norm of $u$, $|u|_{s}$ to denote the seminorm $\sqrt{(u, u)}$ of $u$ in $\mathcal{S}(\Omega)$ and $\| \cdot \|_{s}$ to denote the norm on $\mathcal{S}(\Omega)$ defined by $\|u\|_{s}^2 = \|u\|_{L^2}^2 + |u|_{s}^2$.

Later on we may need to deal with space of functions defined over different domains; say $\mathcal{S}(\Omega)$ and $\mathcal{S}(\Omega')$. In this case, we will explicitly write $| \cdot |_{\mathcal{S}(\Omega)}$ to denote the seminorm corresponding to the space $\mathcal{S}(\Omega)$. Where there is no ambiguity we will just write $\mathcal{S}$ instead of $\mathcal{S}(\Omega)$. 

Throughout the paper we assume $\Omega$ is a connected bounded domain with sufficiently smooth boundary. We work under this condition to simplify our exposition, and all of our results hold true under a relaxed condition on the regularity of the boundary as done in [11]. The condition we impose on $\rho$ is:

$$\begin{align*}
\rho = \rho(|\xi|) \text{ is nonnegative, compactly supported}, \ |\xi|^2 \rho(|\xi|) \in L^1_{loc}(\mathbb{R}^d), \\
\text{and there exists a constant } \sigma > 0, \text{ such that } (0, \sigma) \subset \text{supp}(\rho).
\end{align*}$$

\text{(H)}

The assumption that $\rho = \rho(|\xi|)$ is not necessarily locally integrable but rather has a finite second moment is a general condition to give, in the context of PD, a well-defined elastic modulus [20]. As we proceed, other necessary conditions on $\rho$ may be provided in order to establish some specialized analytic properties. We next give characterizations of $\mathcal{S}(\Omega)$ for which there are no corresponding results available in the literature. As noted before, if in addition $\rho \in L^1_{loc}(\mathbb{R}^d)$, then $\mathcal{S} = L^2(\Omega)$. In general, $\mathcal{S}(\Omega)$ is a proper subspace of $L^2(\Omega)$, but with many desired properties.

2.2. Completeness of $\mathcal{S}(\Omega)$. We first show that $\mathcal{S}(\Omega)$ is a Hilbert space.

**Theorem 2.1.** $(\mathcal{S}(\Omega), (\cdot, \cdot)_{\Omega})$ is a Hilbert space for any nonnegative $\rho = \rho(|\xi|)$.

**Proof.** It suffices to check that the space $\mathcal{S}$ is complete. Let $u_n \in \mathcal{S}$ be a Cauchy sequence in $\mathcal{S}$. Then it is also Cauchy in $L^2(\Omega)$ and converges to some $u$ strongly in $L^2(\Omega)$. We now show that $|u_n - u|_{\Omega} \to 0$ as $n \to \infty$. Let $\epsilon > 0$, choose $K$ large such that for all $n, m \geq K$, $|u_n - u_m|_{\Omega}^2 \leq \epsilon$. For $\delta > 0$, we introduce a modified operator with a truncated kernel

$$u \mapsto \mathcal{L}_\delta u := -2 \int_{\Omega} \rho_\delta(|x' - x|) (u(x') - u(x)) \, dx', \quad \text{where } \rho_\delta = \rho(\chi_{(0,\infty)}).$$

Then, for all $n, m$,

$$\int_{\Omega} \int_{\Omega} \rho_\delta(|x' - x|) \left| (u_n(x') - u_n(x)) - (u_m(x') - u_m(x)) \right|^2 \, dx' \, dx \leq |u_n - u_m|_{\Omega}^2.$$

Since the modified kernel $\rho_\delta$ is integrable, the left hand side can be written as the integral of $\mathcal{L}_\delta(u_n - u_m)(x)$ on $\Omega$. Note that for a fixed $n$,

$$\lim_{m \to \infty} \mathcal{L}_\delta(u_n - u_m)(x) = \mathcal{L}_\delta(u - u_n)(x), \quad \forall x \in \Omega,$$

and by dominated convergence theorem, for all $\delta > 0$ and for all $n \geq K$,

$$\int_{\Omega} \mathcal{L}_\delta(u - u_n)(x) \, dx = \lim_{m \to \infty} \int_{\Omega} \mathcal{L}_\delta(u_m - u_n)(x) \, dx.$$

That is, for all $\delta > 0$ and for all $n \geq K$,

$$\int_{\Omega} \mathcal{L}_\delta(u - u_n)(x) \, dx \leq \epsilon.$$

Rewriting the left hand expression for the kernel $\rho_\delta$ which is integrable, we have

$$\int_{\Omega} \int_{\Omega} \rho_\delta(|x' - x|) \left| (u_n(x') - u_n(x)) - (u_n - u)(x) \right|^2 \leq \epsilon, \quad \forall \delta > 0, \ n \geq K.$$

Now applying Fatou’s lemma, we obtain that

$$\int_{\Omega} \int_{\Omega} \rho(|x' - x'|) \left| (u_n(x') - u_n(x)) - (u_n - u)(x) \right|^2 \leq \epsilon, \quad \forall n \geq K,$$

thus, for any $\epsilon > 0$, there exists $K$ large such that $|u_n - u|_{\Omega}^2 \leq \epsilon$ for $n \geq K$. \qed
2.3. Embeddings. We now explore further the relationship between \( S(\Omega) \) and the standard function spaces \( L^2(\Omega) \), \( H^1(\Omega) \) and \( C^\infty(\Omega) \). We will present a sufficient condition that guarantee us compact embedding of \( S(\Omega) \) into \( L^2(\Omega) \), a result that is crucial in establishing well posedness of the PD model. We will show that functions in \( S(\Omega) \) can be extended in a controlled way so that they can become functions in \( S(\mathbb{R}^d) \). A consequence of the extension result is the density of \( C^\infty(\Omega) \), and in turn the separability of \( S(\Omega) \). We notice that these properties have been shown for fractional Sobolev spaces (i.e., for \( S(\Omega) \) with special kernels). We provide a treatment of these results in a unified way for the nonstandard function spaces \( S(\Omega) \) corresponding to a broad class of kernels.

2.3.1. Compact embedding. While \( S(\Omega) = L^2(\Omega) \) for any locally integrable \( \rho(|\xi|) \), the next lemma presents a sufficient condition for compactness that is applicable to spaces \( S(\Omega) \) associated with a larger class of kernels.

**Lemma 2.2.** Suppose that in addition to \((H)\), \( \rho(|\xi|)|\xi|^2 \) is nonincreasing in \(|\xi|\) and satisfying the density condition

\[
\frac{\delta^2}{\int_{B_\delta(0)} \rho(|\xi|)|\xi|^2 d\xi} \to 0 \quad \text{as} \quad \delta \to 0. \tag{6}
\]

Then \( S \) is properly contained in \( L^2(\Omega) \) and that the embedding \( S \hookrightarrow L^2(\Omega) \) is compact. That is, if the sequence \( u_n \) is bounded in \( S \), then it is precompact in \( L^2(\Omega) \).

Before we give the proof of the lemma, let us make a couple of points. The first one is that if \( \rho \) satisfies the density condition \((6)\), then necessarily \( \rho(|\xi|) \) cannot be integrable in any neighborhood of the origin. The second point is that there are functions that satisfy \((6)\), the primary examples being \( \rho(|\xi|) \) that are comparable to \(|\xi|^{-d-2s}\) for any \( s \in (0, 1) \), near the origin. The space \( S(\Omega) \) corresponding to such \( \rho \) is equivalent to the standard fractional Sobolev space \( H^s(\Omega) \) whose compact embedding into \( L^2(\Omega) \) is well known. There are also other classes of kernels satisfying \((6)\) and yet the corresponding \( S(\Omega) \) not necessarily coinciding with known function space. For example, let \( d \geq 2 \), and consider the radial function \( \rho_1 \) defined by

\[
\rho_1(|\xi|) := \begin{cases} 
-\ln(|\xi|)/|\xi|^d & \text{when} \ |\xi| \leq 1, \\
0 & \text{otherwise}.
\end{cases} \tag{7}
\]

Then \( \rho_1 \) is not integrable but \( \rho_1(|\xi|)|\xi|^2 \) is locally integrable and nonincreasing. Moreover, for any small \( \delta > 0 \) and \( |S^{d-1}| \) being the volume of the \( d-1 \) dimensional unit ball,

\[
\int_{B_\delta(0)} |\xi|^2 \rho_1(|\xi|) d\xi = -|S^{d-1}| \int_0^\delta \ln(r) r dr = -|S^{d-1}| \frac{\delta^2}{2} (\ln(\delta) - 1/2),
\]

which says that indeed satisfies \((6)\). On the other hand, for any \( \epsilon > 0 \) small,

\[
\rho_1(|\xi|)|\xi|^{d+\epsilon} = -\ln(|\xi|)|\xi|^{\epsilon} \to 0 \quad \text{as} \ |\xi| \to 0,
\]

implying that we cannot bound \( \rho(|\xi|) \) from below by a power function of negative exponent. Thus the density condition \((6)\) is weaker than the one given in [11].

**Remark 1.** Unfortunately Lemma 2.2 does not apply for some kernels that are of interest to us. Take, for example, \( \rho_0(|\xi|) := \frac{1}{|\xi|^7} \), which does not satisfy \((6)\) and yet
we expect that the corresponding space
\[ \{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x') - u(x)|^2}{|x' - x|^d} dx' dx < \infty \} \]
remains compactly embedded in \( L^2(\Omega) \).

We now come to the proof of the Lemma 2.2 which uses the following extension result [4, pp. 11], which is a generalization of a similar result for fractional Sobolev spaces, see [21] for a detailed statement and proof.

**Lemma 2.3.** (Extension Lemma) If in addition to (H), \( \rho(|\xi|)|\xi|^2 \) is nonincreasing. Then there exists a linear extension operator \( E : S(\Omega) \to S(\mathbb{R}^d) \) such that
1. \( Eu(x) = u(x) \) when \( x \in \Omega \) and \( Eu \) is compactly supported.
2. There exists a universal constant \( C \), depending only on \( \rho \) and \( \Omega \), such that
   \[ |Eu|_{S(\mathbb{R}^d)} \leq C \| u \|_{S(\Omega)}. \]

We should mention that Lemma 2.2 is a variant of the compactness result [4, Theorem 4] but differs from the result stated in [4, Theorem 4] for the latter applies to a delta-sequence of kernels while the former applies to a fixed kernel satisfying the precisely formulated condition (6). The proof of Lemma 2.2 we sketch below is a modification of the argument used in [4, Theorem 4], taking into account the above mentioned difference.

The proof of Lemma 2.2. Suppose that \( u_n \) is a bounded sequence in \( S \) and \( \rho \) satisfies the density condition. Without loss of generality we assume that the mean value of \( u_n \) is 0. By Lemma 2.3, we may even assume that \( \Omega = \mathbb{R}^d \) and that \( \text{Supp}(u_n) \subset B \), the unit ball. For any \( \delta > 0 \), let \( P^\delta \) be the mollifier
\[ P^\delta(x) = \frac{1}{|B_\delta(0)|} \chi_{B_\delta}(0). \]

We claim that the boundedness of the sequence in \( S \) and the density condition on the kernel imply that
\[ \lim_{\delta \to 0} \limsup_{n \to \infty} \| u_n - P^\delta * u_n \|_{L^2(\mathbb{R}^d)} = 0, \]
from which the lemma follows, as in the proof of [4, Theorem 4]. Following the exact sequence of steps in [4] we obtain that
\[
\int_{\mathbb{R}^d} |u_n(x) - P^\delta * u_n(x)|^2 dx \\
= \frac{d^2}{|B_\delta|} \int_0^\delta \left[ \int_{S^{d-1}} \int_{\mathbb{R}^d} |(u_n(x + \tau s) - u_n(x))|^2 dx d\sigma(s) \right] \tau^{d-1} d\tau \\
\leq \frac{d^2}{|B_\delta|} \int_0^\delta \tau^{d-1} F_n(\tau) d\tau,
\]
where
\[ F_n(\tau) = \int_{S^{d-1}} \int_{\mathbb{R}^d} |(u_n(x + \tau s) - u_n(x))|^2 dx d\sigma(s). \]
Observe that the boundedness of \( u_n \) in \( S \) implies that
\[ \int_0^1 \tau^{d-1} F_n(\tau) \rho(\tau) d\tau \leq C. \]
Now applying \cite[Lemma 2]{4} we obtain that, with \( g(t) = F_n(t)/\tau^2 \) and \( h(t) = \rho(t)|t|^2 \),
\[
|B_\delta|^{-1} \int_0^\delta \frac{\tau^{d-1}F_n(\tau)}{\tau^2} d\tau \leq C \left( \int_0^\delta \tau^{d-1}F_n(\tau)\rho(\tau)d\tau \right) \left( \int_{B_\delta} \rho(|\xi|)|\xi|^2d\xi \right)^{-1}.
\]
Then by (8) we obtain that for any \( n \) and any \( \delta > 0 \),
\[
|B_\delta|^{-1} \int_0^\delta \frac{\tau^{d-1}F_n(\tau)}{\tau^2} d\tau \leq C \left( \int_{B_\delta} \rho(|\xi|)|\xi|^2d\xi \right)^{-1}.
\]
In particular we have for all \( n \) and any \( \delta > 0 \),
\[
\frac{1}{|B_\delta|} \int_0^\delta \tau^{d-1}F_n(\tau)d\tau \leq C \frac{\delta^2}{|B_\delta|} \int_0^\delta \tau^{d-1}F_n(\tau)\frac{d\tau}{\tau^2} \leq C\delta^2 \left( \int_{B_\delta} \rho(|\xi|)|\xi|^2d\xi \right)^{-1}.
\]
First taking the limsup as \( n \to \infty \), then taking the limit \( \delta \to 0 \) and recalling the density condition (6), the right hand side of the above inequality converges to 0. That completes the sketch of the proof.

2.3.2. Density of smooth functions and separability. Our next aim is to prove the density of smooth functions and use it to prove the separability of \( S(\Omega) \). We begin with Theorem 1 of \cite{4} that states that \( H^1(\Omega) \) is continuously embedded into \( S(\Omega) \), a result that we will utilize shortly.

**Lemma 2.4.** \cite[Theorem 1]{4} Assume \((H)\) holds, then \( H^1(\Omega) \) is continuously embedded in \( S(\Omega) \) and there exists a positive constant \( C = C(\Omega, \rho) \) such that
\[
|u|_s \leq C|u|_{H^1} \quad \text{for all } u \in H^1(\Omega).
\]

It is well known that \( C^\infty(\Omega) \) is dense in \( H^s(\Omega) \), for all \( s \in (0, 1) \). Lemma 2.5 below says so is it in \( S(\Omega) \), corresponding to any nonincreasing \( \rho \). Note that we have assumed \( \partial\Omega \) is sufficiently smooth.

**Lemma 2.5.** Assume that \((H)\) holds and \( \rho(r)|r|^2 \) is nonincreasing. Then \( C^\infty(\Omega) \) is dense in \( S \).

**Proof.** **Step 1.** Let \( \phi_\epsilon \) be the standard mollifiers. Then for \( u \in L^2(D) \) define \( u_\epsilon = u \ast \phi_\epsilon \), the standard mollification. Then \( u_\epsilon \) is infinitely differentiable in the compact subset of \( D_\epsilon = \{ x \in D : \text{dist}(x, \partial D) > \epsilon \} \). Then we have that for \( \epsilon > 0 \),
\[
\int_{D_\epsilon} \int_{D_\epsilon} \rho(|x' - x|)|u_\epsilon(x') - u_\epsilon(x)|^2 \, dx' \, dx
= \int_{D_\epsilon} \int_{D_\epsilon} \rho(|x' - x|)|u_\epsilon(x') - u_\epsilon(x)|^2 \, dx' \, dx
= \int_{D_\epsilon} \int_{D_\epsilon} \rho(|x' - x|) \left| \int_{B(0, \epsilon)} (u(x' - z) - u(x - z)) \phi_\epsilon(z) \, dz \right|^2 \, dx' \, dx
\leq \int_{B(0, \epsilon)} \phi_\epsilon(z) \int_{D_\epsilon} \int_{D_\epsilon} \rho(|x' - x|)|u(x' - z) - u(x - z)|^2 \, dx' \, dz \, dz
\leq \int_{D} \int_{D} \rho(|x' - x|)|u(x') - u(x)|^2 \, dx' \, dx.
\]
where we have applied Jensen’s inequality and interchanged the integrals.

**Step 2.** Now let \( u \in S(\Omega) \). Let \( U \in S(\mathbb{R}^d) \) be its extension guaranteed by Lemma 2.3. Assume that \( \text{supp}(U) \subset \Omega' \) where \( \Omega \subset \Omega' \), and \( \Omega' \) is a smooth bounded domain.
Let $D$ be a bounded domain such that $\Omega' \subset D$ and $\text{dist}(\partial\Omega', D) \geq 1$. Let $U_\epsilon$ be the mollification of $U$. Then clearly $U_\epsilon \to U$ strongly in $L^2(\Omega')$, and by Step 1, $|U_\epsilon - U|_{S(\Omega')} \leq |U|_{S(D)}$. We claim that $U_\epsilon \to U$ in $S(\Omega')$. It suffices to check that $|U_\epsilon - U|_{S(\Omega')} \to 0$ as $\epsilon \to 0$. We first note that for each $\delta > 0$,

$$|U_\epsilon - U|_{S(\Omega')} = I_1(\delta) + I_2(\delta)$$

where

$$I_1(\delta) = \int_{\Omega'} \int_{\Omega'} \rho_\delta(|x' - x|)(U_\epsilon - U)(x') - (U_\epsilon - U)(x))^2 \, dx' \, dx$$

with $\rho_\delta$ given in (5) and

$$I_2(\delta) = \int_{\Omega'} \int_{\Omega'} \rho(|x' - x|)\chi(|x' - x|)\delta(x - x')(U_\epsilon - U)(x') - (U_\epsilon - U)(x))^2 \, dx' \, dx.$$

Let us estimate each of these quantities. On the one hand, because the truncated kernel function $\rho_\delta \in L^1_{\text{loc}}(\mathbb{R}^d)$, we have the estimate

$$I_1(\delta) \leq C(\delta)\|U_\epsilon - U\|_{L^2(\Omega')}.$$

On the other hand,

$$I_2(\delta) \leq 2 \int_{\Omega'} \int_{\Omega'} \rho(|x' - x|)\chi(|x' - x|)\delta(x - x')(U_\epsilon - U)(x') - (U_\epsilon - U)(x))^2 \, dx' \, dx + 2 \int_{\Omega'} \int_{\Omega'} \rho(|x' - x|)\chi(|x' - x|)\delta(x - x')(U(x') - U(x))^2 \, dx' \, dx.$$

Apply Step 1, we get

$$I_2(\delta) \leq 4 \int_{\Omega} \int_{\Omega} \rho(|x' - x|)\chi(|x' - x|)\delta(x - x')(U(x') - U(x))^2 \, dx' \, dx.$$

Application of Lebesgue dominated convergence theorem, we obtain that

$$I_2(\delta) \to 0 \quad \text{as} \quad \delta \to 0.$$

Now let $\tau > 0$ be arbitrary. Then there exists $\delta_0$ such that $\forall 0 < \delta \leq \delta_0$, $I_2(\delta) < \tau$. Then

$$|U_\epsilon - U|_{S(\Omega')} \leq C(\delta_0)\|U_\epsilon - U\|_{L^2(\Omega')} + \tau,$$

and letting $\epsilon \to 0$, we have

$$\lim_{\epsilon \to 0} |U_\epsilon - U|_{S(\Omega')} \leq \tau,$$

proving our claim.

Finally, we prove the separability of $S(\Omega)$, which is needed to apply standard variational techniques in establishing the well posedness of PD equations of motion. Indeed, our application of results in [7] requires us to consider the “space triple” $S(\Omega) \subset L^2(\Omega) \subset S'(\Omega)$; where $S$ needs to be a separable Hilbert space that is densely embedded in $L^2(\Omega)$.

**Theorem 2.6.** Assume that (H) holds and $\rho(r)r^2$ is nonincreasing. Then the space $S(\Omega)$ is separable as stated below.

By the density of $C^\infty(\Omega)$ in $S(\Omega)$, Theorem 2.6 follows from standard arguments that use the separability and continuous embedding of $\mathcal{H}^1(\Omega)$ in $S(\Omega)$. Indeed, any countable dense subset of $\mathcal{H}^1(\Omega)$ can be shown to be dense $S(\Omega)$ as well.
3. The nonlocal operator. The objective of this section is twofold. First, we make clarifications on the definition of the operators $\mathcal{L}$ and $\mathcal{P}_\lambda$. It is not difficult to see that when $\rho(|\xi|)$ is locally integrable, the pointwise definition of $\mathcal{L}u$, (3), makes sense almost everywhere in $\Omega$ for any $u \in L^2(\Omega)$ and that the operator $\mathcal{L}$ is a bounded linear map from $L^2(\Omega) \to L^2(\Omega)$. In this case, $\mathcal{S} = L^2(\Omega)$, and that for any $u$ and $w$ in $L^2(\Omega)$

$$(\mathcal{L}u, w) = \int_\Omega w(x) \left(-2\int_\Omega \rho(|x' - x|)(u(x') - u(x))dx'\right)dx = ((u, w)).$$

In the absence of additional assumptions on $\rho$, however, the operator $\mathcal{L}$ is an unbounded operator, and the integral in (3) may even diverge for some $u \in L^2(\Omega)$. In this case the operator $\mathcal{L}$ has to be understood in a generalized sense which is made precise here. Second, we will obtain some basic results such as the coercivity of the operator $\mathcal{L}$ which leads to its invertibility. The coercivity property is precisely a consequence of a nonlocal Poincaré-type inequality which holds over subspaces of $\mathcal{S}(\Omega)$ that avoid nonzero scalars. As a perturbation of $\mathcal{L}$, we will also establish Gårding-type inequality (weak coercivity) for $\mathcal{P}_\lambda$.

3.1. Definition. We may define $\mathcal{L}$ as a linear operator from $\mathcal{S}$ to $\mathcal{S}'$ by

$$(\mathcal{L}u, w) = ((u, w)),$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing of $(\mathcal{S}, \mathcal{S}')$. This definition of $\mathcal{L}$ agrees with the one defined in (3) in the principal value sense; that is, for any $u \in \mathcal{S}$,

$$\mathcal{L}u(x) = P.V. - 2\int_{\Omega} \rho(|x' - x|)(u(x') - u(x))dx' \text{ in } \mathcal{S}'(\Omega). \quad (9)$$

To make this precise, for each small $\delta > 0$ and each $u \in \mathcal{S}(\Omega)$, we consider $\mathcal{L}_\delta$ and $\rho_\delta$ given in (5). For $w \in \mathcal{S}(\Omega)$,

$$(\mathcal{L}_\delta(u), w) = (\mathcal{L}_\delta(u), w) = \int_\Omega \int_\Omega \rho_\delta(|x' - x|)(u(x') - u(x))(w(x') - w(x))dx'dx.$$

Using the pointwise convergence $\rho_\delta \to \rho$, and dominated convergence theorem, we see that the last double integral converges to $((u, w)) = (\mathcal{L}u, w)$ which is precisely the convergence in (9).

For smooth compactly supported function $\phi$, $\mathcal{L}\phi$ in (9) can actually makes sense pointwise. Indeed, it follows from the definition that $\mathcal{L}_\delta(\phi)(x)$ is a continuous function and that

$$\sup_{0 < \delta < d/2x \in \Omega} |\mathcal{L}_\delta(\phi)| \leq C, \quad C = C(\phi), \text{ constant}, \quad (10)$$

where $d = \text{dist}(\text{Supp}(\phi), \partial \Omega)$. For fixed $x \in \Omega$, we can now let $\delta \to 0$ to obtain

$$\mathcal{L}\phi(x) = \lim_{\delta \to 0} \mathcal{L}_\delta(\phi)(x).$$

Using (10) and the definition of $\mathcal{L}\phi$ we can even conclude that

$$\mathcal{L}_\delta(\phi) \to \mathcal{L}\phi \quad \text{strongly in } L^2(\Omega), \quad \forall \phi \in C_c^\infty(\Omega).$$

As we discussed in the previous section we interpret our peridynamic operator $\mathcal{P}_\lambda$ as a perturbation of the operator $\mathcal{L}$. We assume first that $g$ is a radial function such that $g(|\xi|)$ is locally integrable and supported in a small ball around 0. The nonlocal operator

$$\mathcal{L}_g u(x) = -2\int_\Omega g(|x' - x|)(u(x') - u(x))dx'.$$
is, thus, a linear bounded operator on $L^2(\Omega)$. For later use we sometimes write $L_g$ as
\[ L_g u(x) = -2g * \overline{u}(x) + 2G(x)u(x), \quad x \in \Omega \tag{11} \]
where $\ast$ represents the standard convolution, $\overline{u}(x)$ is the zero extension of $u$ and $G(x) = \int_\Omega g(|x' - x|)dx$. Note that the function $G$ is continuous and bounded in $\Omega$, since $g$ is locally integrable. Moreover, for any $u, w \in L^2(\Omega)$
\[ (L_g u, w) = \int_\Omega \int_\Omega g(|x' - x|)(u(x') - u(x))(w(x') - w(x))dx'dx. \]
In terms of the operators defined so far, the peridynamic operator $P_\lambda$ is the sum
\[ P_\lambda = L + \lambda L_g. \tag{12} \]
Observe that we are not imposing any condition on the sign of $g$. As a consequence, the perturbation of $L$ by $\lambda L_g$ may result in a nonlocal integral operator $P_\lambda$ with a sign changing kernel function. This also justifies our use of the phrase “leading operator” for $L$ in relation to $P_\lambda$.

3.2. Invertibility of the nonlocal operator $L$. We aim to prove the invertibility of the operator $L$ over vector subspaces of $S(\Omega)$ that avoid its zero set. The key in achieving this is the following nonlocal Poincaré-type inequality that holds over all subspaces of functions in $L^2(\Omega)$ satisfying certain volume constrains. Imposing volume constrains, as opposed to boundary conditions, on functions is necessitated by the fact that the operator $L$ is nonlocal. An important subspace that is motivated by the PD model is the subspace $V$ of functions $u \in L^2(\Omega)$ such that $u(x) = 0$ for a.e. $x \in \omega \subset \Omega$. Another is the subspace of mean zero functions in $L^2(\Omega)$.

**Proposition 1.** Suppose that $\rho$ satisfies $(H)$ and $V$ is a closed subspace of $L^2(\Omega)$ that intersects $\mathbb{R}$ trivially. Then there exists $C = C(\rho, V, \Omega)$ such that
\[ \|u\|^2 \leq C \int_\Omega \int_\Omega \rho(|x' - x|) \|x' - x\|^2(u(x') - u(x))^2dx'dx \quad \forall u \in V. \]

We should note that the inequality is an improvement of those given in [3, 11, 1, 22]. Indeed, in Proposition 1 we only need that $\rho$ has a finite second moment. The argument used in the work of [3, 11] takes advantage of the fact that the operators associated with the class of kernels $\rho$ considered in their work are Hilbert-Schmidt. Moreover, in [11] by choosing $\rho$ appropriately the function space $S$ is made to be equivalent to the standard fractional Sobolev spaces from which the nonlocal Poincaré inequality is deduced using the fact that fractional spaces are compactly contained in $L^2(\Omega)$. In [22] a nonlocal Poincaré inequality is given that holds for functions in $L^p$ spaces $(1 \leq p < \infty)$ with mean zero. The proof given here slightly differs in that the kernel is assumed to be just integrable. Moreover, our argument does not seem to depend on properties of local function spaces.

An easy consequence of the nonlocal Poincaré-type inequality is that the operator $L$ is positive definite, whose proof follows from the fact that
\[ \int_\Omega \int_\Omega \rho(|x' - x|) \|x' - x\|^2(u(x') - u(x))^2dx'dx \leq |\text{diam}(\Omega)|^2 \langle Lu, u \rangle. \]
Define the constant $\kappa := C|\text{diam}(\Omega)|^2$, where $C$ is as in (1).

**Corollary 1.** Suppose that $\rho$ satisfies $(H)$ and $V$ is a closed subspace of $L^2(\Omega)$ that intersects $\mathbb{R}$ trivially. Then there exists $\kappa = \kappa(\rho, V, \Omega)$ such that
\[ \|u\|^2 \leq \kappa \langle Lu, u \rangle, \quad \text{for all } u \in V \cap S(\Omega). \]
Proof of Proposition 1. The proof adapts the proof of the standard Poincaré inequality. For this reason, we highlight those crucial steps only. Suppose the conclusion of the proposition is false. Then there exist \( u_n \in V \) such that for all \( n, \| u_n \| = 1 \), and as \( n \to \infty \),

\[
\int_\Omega \int_\Omega \rho(|x' - x|)|x' - x|^2 (u_n(x') - u_n(x))^2 dx' dx \to 0.
\]

We claim that \( \| u_n \|_{L^2} \to 0 \), as \( n \to \infty \), resulting in a contradiction that for all \( n, \| u_n \| = 1 \). To prove the claim, suppose that \( u \) is the weak limit in \( L^2 \) of the bounded sequence \( u_n \). Since \( V \) is a closed subspace of \( L^2(\Omega) \), we have \( u \in V \).

**Step 1.** We show that the weak limit \( u \) is in fact 0. Introducing the linear bounded operator on \( L^2(\Omega) \),

\[
u \mapsto Lu(x) = -2 \int_\Omega \rho(|x - x'|)|x' - x|^2 (u(x') - u(x)) dx' .
\]

Let \( \psi \in C_c^\infty(\Omega) \), and define the sequence

\[
J_n(\psi) = \int_\Omega L\psi(x) u_n(x) dx .
\]

Since \( \rho(|x' - x|)|x' - x|^2 \psi(x')||u_n(x)|| \) and \( \rho(|x' - x|)|x' - x|^2 \psi(x)||u_n(x)|| \) are integrable in \( \Omega \times \Omega \), we can iterate the integrals in \( J_n(\psi) \) and change the order of integration to obtain that

\[
J_n(\psi) = 2 \int_\Omega \int_\Omega \rho(|x' - x|)|x' - x|^2 (u_n(x') - u_n(x)) \psi(x') dx' dx = \int_\Omega Lu_n(x') \psi(x') dx' .
\]

Now applying Hölder’s inequality we obtain that,

\[
J_n(\psi) \leq 2 \left[ \int_\Omega \int_\Omega |(u_n(x') - u_n(x))|^2 \rho(|x' - x|)|x' - x|^2 dx' dx \right]^{\frac{1}{2}} .
\]

\[
\left[ \int_\Omega \int_\Omega |\psi(x')|^2 \rho(|x' - x|)|x' - x|^2 dy dx \right]^{\frac{1}{2}}
\]

\[
\leq C \left[ \int_\Omega \int_\Omega |(u_n(x') - u_n(x))|^2 \rho(|x' - x|)|x' - x|^2 dx' dx \right]^{\frac{1}{2}} \left[ \int_\Omega |\psi(x')|^2 dx' \right]^{\frac{1}{2}}
\]

Note that the right hand side goes to 0, as \( n \to \infty \) by assumption. So, for all smooth and compactly supported \( \psi \), we have

\[
\int_\Omega L\psi(x) u_n(x) dx \to 0 \quad \text{as} \quad n \to \infty .
\]

On the other hand, because \( u_n \to u \) as \( n \to \infty \) weakly in \( L^2 \), we have

\[
0 = \lim_{n \to \infty} L\psi(x) u_n(x) dx = \int_\Omega L\psi(x) u(x) dx .
\]

Again iterating the integrals and changing variables we obtain that for all \( \psi \)

\[
\int_\Omega Lu(x) \psi(x) dx = \int_\Omega L\psi(x) u(x) dx = 0 .
\]

From this it follows that \( Lu(x) = 0 \) for almost all \( x \in \Omega \). Now multiplying by \( u(x) \) and integrating in \( x \), then iterating the integrals, we obtain

\[
\int_\Omega \int_\Omega \rho(|x' - x|)|x' - x|^2 (u(x') - u(x))^2 dx' dx = 0 ,
\]
and thus, \( u(x') = u(x) \) for almost all \( x, x' \in \Omega \) such that \( \rho(x' - x) > 0 \) except a measure zero set. Using the facts that \( \rho(|\xi|) > 0 \) if \( 0 < |\xi| < \sigma \), and \( \Omega \) is compact and connected, a covering argument shows that \( u \) is a constant function. However the only constant function in \( V \) is the zero function. Therefore, \( u \) must be identically zero.

**Step 2.** We show next that, as \( n \to \infty \), \( u_n \to 0 \) strongly in \( L^2 \). We note that

\[
(Lu_n, u_n) = \int_{\Omega} \int_{\Omega} \rho(|x - x'|)|x' - x|^2 |u_n(x') - u_n(x)|^2 \, dx' \, dx \to 0, \quad \text{as } n \to \infty.
\]

Now we can write \( L \) as a sum of two operators

\[
(Lu_n)(x) = (K * \pi_n)(x) + a(x)u_n(x) \quad (\pi = u(x) \text{ if } x \in \Omega, \ 0 \text{ otherwise})
\]

where

\[
K * \pi_n(x) = \int_{\mathbb{R}^d} -2\rho(|x' - x|)|x' - x|^2 \pi_n(x') \, dx' \quad \text{and } a(x)
\]

\[
= 2 \int_{\Omega} \rho(|x' - x|)|x' - x|^2 \, dx'.
\]

It follows then that \( K * \pi_n \to 0 \) strongly in \( L^2(\Omega) \) as \( n \to \infty \), since the convolution operator is a compact operator and \( u_n \to 0 \) in \( L^2(\mathbb{R}^d) \) by Step 1. Note also that by assumption on \( \rho \) and the absolute continuity of the integral, \( a(x) \) is a positive continuous function on \( \overline{\Omega} \). Therefore there exists a positive constant \( c \) such that \( a(x) \geq c > 0 \) for all \( x \in \Omega \). Putting together the above observations we obtain that

\[
0 = \lim_{n \to \infty} (Lu_n, u_n) = \lim_{n \to \infty} (K * u_n, u_n) + \int_{\Omega} a(x)|u_n(x)|^2 \, dx' \geq c \int_{\Omega} |u_n(x)|^2 \, dx
\]

as asserted.

Another consequence of the nonlocal Poincaré-type inequality is that if \( V \) is a closed subspace of \( L^2(\Omega) \) such that \( V \cap \mathbb{R} = \{0\} \), then the inner product space \( \langle V \cap S, \langle \cdot, \cdot \rangle \rangle \) is a separable Hilbert space with the seminorm \( |\cdot|_s \) being an equivalent norm. Hereafter, \( \mathcal{V}_s \) denote the generic subspace \( V \cap S \) with its dual denoted by \( \mathcal{V}'_s \).

It is possible to pose variational problems in \( \mathcal{V}_s \). A standard application of Riesz representation theorem yields the following result.

**Lemma 3.1.** Assume that \( \rho \) satisfies (H). For a given \( b \in \mathcal{V}'_s \), there exists a unique \( u \in \mathcal{V}_s \) such that

\[
\int_{\Omega} \int_{\Omega} \rho(x' - x)(u(x') - u(x))(w(x') - w(x)) \, dx' \, dx = \langle b, w \rangle,
\]

for all \( w \in \mathcal{V}_s \). Moreover, \( |u|_s = |b|_{\mathcal{V}'_s} \).

It follows then that the operator \( \mathcal{L} \) restricted on \( \mathcal{V}_s \), that is, \( \mathcal{L} : \mathcal{V}_s \to \mathcal{V}'_s \) is an isometry. In addition, the restriction of inverse operator \( \mathcal{L}^{-1} \) to \( L^2(\Omega) \), that is, \( \mathcal{L}^{-1} : L^2(\Omega) \to \mathcal{V}_s \), is linear and bounded, and satisfies the inequality \( |\mathcal{L}^{-1}b|_s \leq C|b|_{L^2} \).

This follows from the continuous embedding of \( L^2(\Omega) \) into \( S' \) and \( |b|_{S'} \leq \|b\|_{L^2(\Omega)} \) for any \( b \in L^2(\Omega) \).

The PD operator \( \mathcal{P}_\lambda \), as a perturbation of the positive definite operator \( \mathcal{L} \), satisfies a Gårding-type inequality, whose proof is just a combination of the properties of the operators \( \mathcal{L} \) and \( \mathcal{L}_g \).
Lemma 3.2. The operator $P_\lambda : S \rightarrow S'$ is a linear, bounded, self-adjoint operator. Moreover, it is weakly coercive in the sense that we can find positive constants $c$ and $C$, depending on $\lambda$ such that
\[
(P_\lambda u, u) + c\|u\|^2 \geq C\|u\|^2_s, \quad \forall u \in S.
\]

4. The scalar peridynamic equilibrium equation. In this section we will finally tackle the question of well-posedness of the scalar peridynamic equation:

\[
P_\lambda u = b, \quad u \in V_s. \tag{13}
\]

Here $V_s$ represents a generic subspace of $V \cap S$, where $V$ is a closed subspace of $L^2(\Omega)$ trivially intersecting $\mathbb{R}$. The necessary volume constraints of the problem are encoded in the space $V_s$. Let us define precisely what we mean by a solution to (13).

Definition 4.1. Let $b \in V'_s$. We say $u \in V_s$ is a solution to (13) if
\[
\langle Lu, w \rangle + \lambda \langle L_g u, w \rangle = \langle b, w \rangle \quad \text{for all } w \in V_s.
\]

4.1. Well-posedness via minimization. Given $u, w \in V_s$ and $b \in L^2(\Omega)$, we begin by introducing the function
\[
B_\lambda(u, w) = \langle Lu, w \rangle + \lambda \langle L_g u, w \rangle - \langle b, w \rangle,
\]
and the associated quadratic energy functional $\Pi_\lambda$
\[
\Pi_\lambda(u) = B_\lambda(u, u) \quad \text{for } u \in V_s.
\]

This functional is precisely the scalar potential energy functional studied in [20]. We show that for $|\lambda|$ small enough, $\Pi_\lambda$ has a minimizer.

The following lemma estimates the growth of $\Pi_\lambda(u)$ for large $u$ and the estimate depends on $g$ and the subspace $V_s$ through its nonlocal Poincaré constant.

Lemma 4.2. Suppose that $\rho$ satisfies $(H)$ and $g$ is a compactly supported locally integrable radial function. Then for all $u \in V_s$,
\[
\frac{\Pi_\lambda(u)}{\|u\|_{L^2_s}} \geq \frac{1}{2} \kappa (1 - 2\kappa |\lambda| (\|g\|_{L^1(\mathbb{R}^d)} + \|G\|_{L^\infty(\Omega)}) - \|b\| \|u\|_{L^2_s})^2,
\]
with $\kappa = \kappa(V_s)$ being the constant in Corollary 1. In particular, if
\[
|\lambda| < \frac{1}{2\kappa (\|g\|_{L^1(\mathbb{R}^d)} + \|G\|_{L^\infty(\Omega)})}, \tag{14}
\]
the functional is coercive on $V_s$ in the sense that
\[
\Pi_\lambda(u) \to \infty, \quad \text{when } \|u\|_{L^2_s} \to \infty \text{ and } u \in V_s.
\]

Proof. By definition of the seminorm and application of the Cauchy-Schwarz inequality, we have
\[
\Pi_\lambda(u) \geq |u|^2_s - |\lambda| \|L_g u\| \|u\| - \|b\| \|u\|.
\]

Using (11), we can estimate $\|L_g u\|$, as
\[
\|L_g u\| \leq 2\|g \ast u\| + \|G\|_{L^\infty(\Omega)} \|u\| \leq 2(\|g\|_{L^1(\mathbb{R}^d)} + \|G\|_{L^\infty(\Omega)}) \|u\|.
\]

By the nonlocal Poincaré inequality we have a constant $\kappa = \kappa(V_s)$ such that
\[
|u|^2_s \geq \frac{1}{(1 + \kappa)} \|u\|^2_s, \quad \forall u \in V_s.
\]

The combination of the above two inequalities proves the lemma. \qed
Given that the functional $\Pi_\lambda$ is quadratic, application of the direct method of calculus of variation yields the following result.

**Theorem 4.3.** Suppose that $\rho$ satisfies (H) and $g$ is a compactly supported locally integrable radial function. Under the condition (14), the energy functional $\Pi_\lambda$ is bounded from below on $V_s$ and attains its infimum in $V_s$. Moreover the minimizer of $\Pi_\lambda$ is unique.

As a necessary condition the minimizer in the above theorem solves the scalar peridynamic equilibrium equation.

**Corollary 2.** Suppose that $\rho$ satisfies (H) and $g$ is a compactly supported locally integrable radial function. Suppose (14) holds, and that $b \in L^2(\Omega)$. Then the unique minimizer in Theorem 4.3 solves the equilibrium equation

$$P_\lambda u = b.$$

Moreover, there exists a constant $C$ such that $|u|_s \leq C||b||$.

The second variation of the functional $\Pi_\lambda$ at the minimizer gives the following necessary condition:

**Corollary 3.** Let $\rho$ satisfies (H) and $g$ be a compactly supported locally integrable radial function. If (14) holds, then there exists $\alpha > 0$ such that

$$|v|_s^2 + \lambda (L_g v, v) \geq \alpha \|v\|_s^2 \quad \forall v \in V_s.$$  \hfill (15)

**Proof.** The second variation of the functional at the minimizer is given by $|v|_s^2 + \lambda (L_g v, v)$ and is nonnegative. Moreover, application of Lemma 4.2 for $b \equiv 0$, we obtain that (15) where $\alpha$ is taken to be

$$\alpha = \frac{1}{1 + \kappa} \left(1 - 2\kappa|\lambda|(\|g\|_{L^1(\mathbb{R}^d)} + \|G\|_{L^\infty})\right).$$

The constant $\alpha$ is positive since (14) holds. \qed

If in addition $\rho(|\xi|)$ is locally integrable, we may rewrite (15) to read as

$$(-2\rho(|\cdot|) * v - 2g * v, v) + (\Lambda v, v) \geq \alpha \|v\|_s^2, \quad \forall v \in V_s.$$  \hfill (16)

where

$$\Lambda(x) = 2 \int_{\Omega} (\rho(|x' - x|) + \lambda g(|x' - x|))dx'.$$

Note that $\Lambda(x)$ is continuous and bounded in $\Omega$. For some subspaces of $L^2(\Omega)$, we may even get a pointwise inequality.

**Corollary 4.** Suppose that $\rho$ satisfies (H) and is locally integrable. Suppose also $g$ is a compactly supported locally integrable radial function, and that $\omega \subset \Omega$ be open such that $|\omega| > 0$. Define the closed subspace of $L^2(\Omega)$

$$V_\omega = \{u \in L^2(\Omega) : u(x) = 0, \quad x \in \omega\}.$$

If $\lambda$ satisfies (14), corresponding to $V_\omega$, then

$$\Lambda(x) \geq \alpha \left(2 \int_{\Omega} \rho(|x' - x|)dx' + 1 \right) > \alpha, \quad \forall x \in \Omega \setminus \overline{\omega}.$$
Proof. Let $x_0 \in \Omega \setminus \omega$ and for each $\epsilon > 0$ small such that $B(x_0, \epsilon) \subset \Omega \setminus \omega$ define the sequence of functions
\[
v_\epsilon(x) = \frac{1}{\sqrt{|B(x_0, \epsilon)|}} \chi_{B(x_0, \epsilon)}(x).
\]
Then $v_\epsilon \in V_\omega$. Moreover $v_\epsilon$ weakly converges to 0 in $L^2(\Omega)$. Indeed, for any $\psi \in L^2(\Omega)$,
\[
(v_\epsilon, \psi) = \frac{1}{\sqrt{|B(x_0, \epsilon)|}} \int_{B(x_0, \epsilon)} \psi(x) dx = \sqrt{|B(x_0, \epsilon)|} \int_{B(x_0, \epsilon)} \psi(x) dx \to 0 \quad \text{as} \quad \epsilon \to 0 \quad \text{for a.e} \quad x_0 \in \Omega.
\]
where the convergence is possible by Lebesgue Differentiation Theorem. As a consequence, for any $\phi \in L^1(\mathbb{R}^d)$,
\[
\phi * v_\epsilon \to 0, \quad \text{strongly in} \quad L^2(\Omega) \quad \text{as} \quad \epsilon \to 0.
\]
After simple calculation we also have
\[
\lim_{\epsilon \to 0} \|v_\epsilon\|_s^2 = 2 \int_{\Omega} \rho(|x' - x_0|) dx' + 1.
\]
It follows from the inequality (16) and the continuity of $\Lambda(x)$ that
\[
\Lambda(x_0) = \lim_{\epsilon \to 0} \int_{B(x_0, \epsilon)} \Lambda(x) dx = \lim_{\epsilon \to 0} (\Lambda v_\epsilon, v_\epsilon) \geq \alpha(2 \int_{\Omega} \rho(|x' - x_0|) dx' + 1),
\]
proving the corollary.

The same pointwise inequality holds for $L^2$ functions with zero mean, that is, functions in a closed subspace of $L^2(\Omega)$,
\[
V = \{u \in L^2(\Omega) : \int_{\Omega} u(x) dx = 0\}.
\]

Corollary 5. Let $\rho$ satisfy (H) and be locally integrable. Let also $g$ be a compactly supported locally integrable radial function. If $\lambda$ satisfies (14), corresponding to this subspace $V$, then
\[
\Lambda(x) \geq \alpha \left(2 \int_{\Omega} \rho(|x' - x_0|) dx' + 1\right) > \alpha, \quad \forall x \in \Omega.
\]

Proof. The proof is similar to that of the previous corollary using the sequence
\[
v_\epsilon(x) = \frac{1}{\sqrt{|B(x_0, \epsilon)|}} \left(\chi_{B(x_0, \epsilon)}(x) - \frac{|B(x_0, \epsilon)|}{\Omega}\right).
\]

Remark 2. We note that the above result is a generalization of the pointwise stability condition given in [20, equation (88)] as a characterization of material stability. Indeed, if $\omega$ is a layer of points in $\Omega$ that are not more than $\delta$ away from the boundary, and $\rho$ and $g$ have the same horizon $\delta$, then the function $\Lambda(x)$ is a constant in $\Omega \setminus \omega$, given by
\[
\Lambda_0 = \int_{B(0, \delta)} \rho(|\xi|) + \lambda g(|\xi|) d\xi
\]
and from the corollary, $\Lambda_0 > 0$, even though the integrand potentially changes sign. Further discussions on how the quantity $\Lambda(x)$ affects the propagation of discontinuity can be found in [23].

4.2. Well-posedness via Fredholm Alternative. The existence result proved in the previous subsection applies when $\lambda$ is sufficiently small. For $|\lambda|$ large the potential energy function may not grow at infinity or the necessary condition (15) may not hold. This implies that a minimizer may not exist. However we may still have a well-posed equilibrium equation (13) by imposing conditions on $g$, $\lambda$ and $V_s$. We first study the solvability when $S$ is compactly contained in $L^2(\Omega)$, and then study the problem when $S = L^2(\Omega)$. The main tool we are going to use is the Fredholm Alternative Theorem.

Given $b \in L^2(\Omega)$, a solution $u \in V_s$ solves the linear scalar peridynamic equilibrium equation

$$\mathcal{P}_\lambda u = b,$$

if and only if,

$$\langle \mathcal{L}u, v \rangle + \lambda \mathcal{L}_g u, v \rangle = \langle b, v \rangle \quad \text{for all } v \in V_s. \tag{17}$$

The later in turn is true if and only if

$$\langle \mathcal{L}u, v \rangle = \langle b - \lambda \mathcal{L}_g u, v \rangle \quad \text{for all } v \in V_s.$$

That is, if and only if

$$u = \mathcal{L}^{-1}(b - \lambda \mathcal{L}_g u).$$

The last equation can be rewritten to read as the variational equation

$$(\mathcal{I} + \lambda \mathcal{L}^{-1} \mathcal{L}_g)u = \mathcal{L}^{-1} b. \tag{18}$$

Denote the composite operator $K := \mathcal{L}^{-1} \mathcal{L}_g$. Then $K : L^2(\Omega) \to L^2(\Omega)$ is a bounded linear operator and that

$$v = Ku \text{ if and only if } \langle \mathcal{L}v, w \rangle = \langle \mathcal{L}_g u, w \rangle, \quad \text{for all } w \in V_s.$$

Observe that the range of $K$ is contained in $V_s$, with the estimate that

$$|Ku|_s \leq C \|b\|_{L^2}.$$

To apply Fredholm Alternative theorem to prove the well posedness of (18), we need to show the compactness of $K$ which depends on the integrability of $\rho$.

4.2.1. The case when $\rho$ satisfies (6). In this case, since $S$ is compactly embedded in $L^2(\Omega)$, $K$ is a compact operator. We denote that the spectrum of the compact operator $K$ by $\Gamma$ and is countable, $\Gamma = \{\mu_k\}_{k=1}^\infty$. Moreover, if $\Gamma$ is countably infinite, then $|\mu_k| \to 0$ as $k \to \infty$. Thus we have the following well-posedness result.

**Theorem 4.4** (Existence theorem). Suppose that in addition to (H) and (6), $\rho(r)r^2$ is nonincreasing. Suppose also $g$ is a compactly supported locally integrable radial function. Then there exists the countable set $\Sigma = \{\lambda_k\}$, $\lambda_k = \frac{1}{\mu_k}$, and $\mu_k \neq 0$, such that the variational equation

$$\mathcal{P}_\lambda u = b, \quad u \in V_s \tag{19}$$

has a unique solution for $b \in L^2(\Omega)$ if and only if $\lambda \notin \Sigma$. Moreover the following holds.

1. For each $\lambda \notin \Sigma$, there exists a constant $C$, such that if $u$ is a solution, then

$$|u|_s \leq C \|b\|.$$

2. If (14) holds, the unique solution $u$ minimizes the potential energy $\Pi_\lambda$. 
Proof. The only part that needs proof is part 2). If $\lambda$ is in the interval determined by (14), then, by Theorem 4.3, $\Pi_\lambda$ has a unique minimizer, and the minimizer solves the linear scalar peridynamic equilibrium equation. From the property of $\mu_k$, the sequence $\{\lambda_k\}$ does not accumulate in any bounded interval. As a consequence, by uniqueness the solution guaranteed by the theorem except for a finite number of $\lambda$’s in the interval minimizes $\Pi_\lambda$. \qed

4.2.2. The case when $\rho$ is locally integrable. When $\rho$ is locally integrable the operator $\mathcal{K}$ is not compact in general. In fact we have the following lemma giving us a necessary condition for $\mathcal{K}$ to be compact.

Lemma 4.5. Suppose that $\rho$ satisfies (H) and is locally integrable. Suppose also that $\dim(V) = \infty$. Then if $\mathcal{K}$ is compact, then $\rho(r) - g(r)$ must change sign.

Proof. Suppose $\rho(r) - g(r)$ does not change sign, say $\rho(r) - g(r) \geq 0$. Let $u_n \in V$ be bounded and $v_n = \mathcal{K}u_n \in V$. Then by definition of the operator we have

$$(\mathcal{L}v_n, w) = (L_g u_n, w) \quad \text{for all } w \in V.$$ 

There are two cases. Suppose for all small neighborhood of 0, $\rho(r) - g(r) \equiv 0$, then (up to a modification of the operators) we see from the above that for all $n$, $v_n - u_n$ is a solution of the nonlocal equation $\mathcal{L}u = 0$, whose only solutions are constant functions: $v_n - u_n \in V$ for all $n$ is a constant. That is $v_n = u_n$ for all $n$. On the other hand, if there exists a neighborhood of 0 where $\rho(r) - g(r) > 0$ for some $r$, then by plugging $w = v_n - u_n$ in the above equation we obtain that

$$((\mathcal{L} - L_g)(v_n - u_n), v_n - u_n) = 0.$$ 

Again only constant functions $v_n - u_n \in V$ satisfy the above nonlocal equation and therefore for all $n$, $v_n = u_n$ as well. With this it is impossible for $\mathcal{K}$ to be compact, for one could start with a noncompact sequence $u_n \in V$. \qed

Even when we allow $\rho(r) - g(r)$ to change sign, it is unclear if $\mathcal{K}$ is compact without additional assumption. Next we present a closed subspace of $L^2(\Omega)$ and a condition on $g$ such that the compactness of $\mathcal{K} = \mathcal{L}^{-1}L_g$ holds. The main idea used here is that the operator $L_g$ is made to act like the convolution operator on elements of the subspace. Often, the subspace of interest is the set of functions that vanish in a boundary layer of $\Omega$ of positive thickness, so we define

$$V_0 = \{u \in L^2(\Omega) : u|_{\Omega \setminus \Omega_\delta} = 0\},$$

where $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial \Omega) \geq \delta\}$. Clearly $V_0$ is a closed subspace of $L^2(\Omega)$ that intersects $\mathbb{R}$ trivially.

Theorem 4.6. Suppose that $\rho$ satisfies (H) and is locally integrable. Suppose also that $g(r)$ is a function supported on $(0, \delta)$ for $\delta > 0$ small. Denote

$$m_g = 2 \int_{B_{\delta}(0)} g(|\xi|)d\xi.$$ \hspace{1cm} (20)

Then the operator $\mathcal{K} : L^2(\Omega) \to V_0 \subset L^2(\Omega)$ is compact if and only if $m_g = 0$.

Proof. Let us prove the sufficiency first. Suppose $v_n \in L^2$ is a bounded sequence. We show that the sequence $v_n = \mathcal{K}u_n$ has a strongly converging subsequence in $L^2(\Omega)$. Observe that $v_n$ is a bounded sequence in $V$ and by definition of $\mathcal{K}$ it follows that

$$(\mathcal{L}v_n, w) = (L_g u_n, w), \quad \forall w \in V_0.$$ \hspace{1cm} (21)
Finally we use the nonlocal Poincaré’s inequality to conclude that
\[ G(x) = \int_{\Omega} g(|x| - x)dx = 0 \quad \forall x \in \Omega_\delta, \tag{22} \]

Now from the definition of the subspace \( V_0 \) and (11), we see that
\[ (L_g u_n, w)_{L^2} = (-2g * u_n, w). \]

As a result we obtain by plugging \( w = v_n \) that for all \( n \),
\[ (\mathcal{L}v_n, v_n) = (-2g * \mathfrak{r}_n, v_n) \tag{23} \]

Now let \( u \) and \( v \) be the weak limits of the sequences \( u_n \) and \( v_n \) in \( L^2(\Omega) \), and \( V_0 \) respectively. Then from compactness of the convolution operator we obtain that
\[ -2g * \mathfrak{r}_n \rightarrow -2g * \mathfrak{r} \quad \text{strongly in } L^2(\Omega). \]

Moreover, letting \( n \rightarrow \infty \) in the equation (21) we obtain that for all \( w \in V_0 \),
\[ (\mathcal{L}v, w) = (-2g * \mathfrak{r}, w). \]

Again, in particular, because \( v \in V_0 \)
\[ (\mathcal{L}v, v) = (-2g * \mathfrak{r}, v). \tag{24} \]

Now we claim that
\[ \lim_{n \rightarrow \infty} (\mathcal{L}(v_n - v), v_n - v) = 0. \]

Indeed using the self-adjointness of \( \mathcal{L} \) and (23)
\[ (\mathcal{L}(v_n - v), v_n - v) = (\mathcal{L}v_n, v_n) - 2(\mathcal{L}v_n, v) + (\mathcal{L}v, v) \]
\[ = (L_g u_n, v_n) - 2(\mathcal{L}v_n, v) + (\mathcal{L}v, v). \]

The first term on in the right hand side converges to \((-2g * u, v)\) as a product of weakly and strongly converging sequences and the second term converges to \(2(\mathcal{L}v, v)\) as \( v_n \) weakly converges to \( v \). Therefore, as \( n \rightarrow \infty \)
\[ (\mathcal{L}(v_n - v), v_n - v) \rightarrow (-2g * \mathfrak{r}, v) - (\mathcal{L}v, v) = 0. \]

Finally we use the nonlocal Poincaré’s inequality to conclude that
\[ \vartheta \|v_n - v\|_{L^2(\Omega)} \leq (\mathcal{L}(v_n - v), v_n - v) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \]

That is \( v \) is in fact the strong limit of \( v_n \). That completes the proof of the sufficiency.

Let us prove the necessity. Suppose that \( \mathcal{K} \) is compact. We prove the implication by contradiction. Assume that \( m_g \neq 0 \). Choose a sequence \( u_n \) bounded in \( V_0 \) but not precompact in \( L^2(\Omega) \) (and so, is not precompact in \( L^2(\Omega_\delta) \)). Such a sequence exists since \( V_0 \) is an infinite dimensional Hilbert space.

Now by compactness, \( v_n = \mathcal{K} u_n \) is precompact in \( L^2(\Omega) \). Again rewriting the equation we have
\[ v_n = \mathcal{K} u_n \quad \text{if and only if} \quad (\mathcal{L}v_n, w) = (L_g u_n, w) \quad \forall w \in V_0. \]

Note that the set \( C_c^\infty(\Omega_\delta) \subset V_0 \) and therefore,
\[ \int_{\Omega} \mathcal{L}v_n(x)\phi(x)dx = \int_{\Omega} L_g u_n(x)\phi(x)dx \quad \text{for all } \phi \in C_c^\infty(\Omega_\delta). \]

The last equation implies that
\[ \mathcal{L}v_n(x) = L_g u_n(x) \quad \text{for almost every } x \in \Omega_\delta. \]

Thus, on \( \Omega_\delta \) we have that
\[ \mathcal{L}v_n(x) = -2g * \mathfrak{r}_n + m_g u_n. \]
This implies that on $\Omega_3$ we can write $u_n$ as a sum of two sequences. Indeed, 
\[ u_n(x) = m_g^{-1}(L_v(x)) + 2g * \pi_n, \quad \text{for almost all } x \in \Omega_3. \]
Note that the sequence on the right hand side is precompact in $V_0$, since $v_n$ is compact in $L^2(\Omega)$ and that the convolution operator is a compact operator. Therefore, $u_n|_{\Omega_3}$ is compact in $L^2(\Omega)$, which is impossible by our choice of the sequence. That completes the proof of the theorem.

For $g$ not compactly supported but locally integrable, one may consider the following subspace. Assume that $G(x)$, as defined in (22), is not identically 0, let us define a closed subspace of $L^2(\Omega)$ by 
\[ V_g = \{ v \in L^2(\Omega) : \int_{\Omega} G(x)u(x)v(x)dx = 0 \quad \forall u \in L^2(\Omega) \}. \]
Using the assumption on $g$, we infer that $V_g \cap \mathbb{R}^d = \{0\}$. By the same argument as in the above proof, one can prove the following.

**Lemma 4.7.** Suppose that $\rho$ satisfies (H) and is locally integrable. Under the above assumptions on $g$, when $V = V_g$, the operator $K : V \to L^2(\Omega)$ is compact.

Now that we have determined subspaces where $K$ is a compact operator we are ready to state the well posedness of the PD model over those subspaces.

**Theorem 4.8.** Suppose that $\rho$ satisfies (H) and is locally integrable. Under the assumption that either
- the function $g(r)$ has a support contained in $[0, \delta)$, $m_g = 0$ with $m_g$ given by (20), and $V = V_0$ is the set of functions in $L^2$ that vanish on a boundary layer of thickness $\delta$; or
- when the function $G(x)$ defined in (22) is not identically zero and the subspace $V = V_g$,
there exists a countable set $\Sigma \subset \mathbb{R}$ such that the variational equation
\[ P_\lambda u = b, \quad u \in V \]
has a unique solution for $b \in L^2(\Omega)$ if and only if $\lambda \notin \Sigma$. Moreover the following holds.
1. For any $\lambda \notin \Sigma$, there exists $C$ such that if $u \in V$ is the unique solution, then 
\[ \|u\| \leq C\|b\|. \]
2. If (14) holds, then the unique solution $u$ minimizes $\Pi_\lambda$.

In the previous subsection $P_\lambda$ is taken to be the perturbation of the main nonlocal operator $L$ by the operator $L_g$ that essentially act the same way as $L$. A slight variant is when the perturbation is made by another nonlocal operator that may act differently. For example, a similar existence result as in the above theorem can be obtained if the operator $L$ is perturbed by a scalar multiple of the convolution operator $L_g$ where $L_gu = g * u$. This is possible since $L_g$, being a convolution operator, is compact and therefore the operator $K = L^{-1}L_g$ is a compact operator on $L^2(\Omega)$ as a composition of bounded and compact operators. We can then state the following theorem.

**Theorem 4.9.** Suppose that $\rho$ satisfies (H) and is locally integrable. Let $g$ be a compactly supported locally integrable function and that $V$ be a closed subspace of
Theorem 5.2. Suppose for all \( v \in V \) \( 5 \). Then there exists a countable set \( \Sigma \subset \mathbb{R} \) such that the variational equation
\[
Lu + \lambda g \ast u = b, \quad u \in V
\]
has a unique solution for each \( b \in L^2(\Omega) \) if and only if \( \lambda \notin \Sigma \). Moreover,
1. for any \( \lambda \notin \Sigma \), there exists \( C \) such that if \( u \in V \) is the unique solution, then
\[
\|u\| \leq C\|b\|.
\]
2. if \( |\lambda| < \frac{1}{2\kappa (\|g\|_{L^1(\omega \setminus \gamma)})} \), then the unique solution \( u \) minimizes a potential energy.

5. The time dependent problem. In this section we define an appropriate notion of a solution to the nonlocal equation
\[
m(x)u_t(x, t) + B(t)u(x, t) + \mathcal{P}_\lambda u(x, t) = b(x, t) \quad t \in (0, T)
\]  \( (25) \)
with initial value \( u(\cdot, 0) = u_0 \) and \( u_t(\cdot, 0) = v_0 \). In \( (25) \), \( m(x) \) is mass density and assumed to be uniformly bounded from above and below: \( C \geq m(x) \geq c > 0 \) for any \( x \in \Omega \). We assume that, for each \( t \in [0, T] \), \( B(t) \) is a bounded linear operator on \( L^2(\Omega) \) such that the map \( t \mapsto (B(t)u, v) \) is continuous on \( [0, T] \), for any \( u \), and \( v \) in \( L^2(\Omega) \). This implies that there exists a constant \( \beta > 0 \) such that
\[
|\langle B(t)u, v \rangle| \leq \beta \|u\| \|v\| \quad \forall u, v \in L^2(\Omega).
\]

An example of such a nonlocal operator used in continuum mechanics is
\[
B(t)w(x) = \int_\Omega \frac{h(x' - x)}{|x' - x|} (w(x') - w(x)) dx'
\]  \( (26) \)
which appears in [8]. Another would be to take the local operator
\[
B(t)w(x) = \beta(x, t)w(x) \quad \text{for a smooth function } \beta.
\]

To analyze the initial value problem \( (25) \), let us define some relevant function spaces to state our main result. Given a Banach space \( X \), the space of function \( L^2((0, T); X) \) is defined as
\[
L^2((0, T); X) = \{u : (0, T) \to X \text{ is Bochner measurable and }
\|u\|_{L^2((0, T); X)}^2 = \int_0^T \|u(s)\|^2_X ds < \infty\}.
\]
with \( L^2(\Omega) \) being separable Hilbert spaces, \( S \) being embedded continuously into \( L^2(\Omega) \) and \( S \) being a dense subspace of \( L^2(\Omega) \). The meaning of a solution of \( (25) \) is defined below.

Definition 5.1. Suppose that \( b \in L^2((0, T); L^2(\Omega)) \), \( u_0 \in S \), and \( v_0 \in L^2(\Omega) \). We say \( u \) is a solution to \( (25) \) if \( u \in L^2((0, T); S) \), \( u_t \in L^2((0, T); L^2(\Omega)) \) and \( u_{tt} \in L^2((0, T); S') \) such that \( u(0) = u_0 \), \( u_t(0) = v_0 \) and
\[
\frac{d}{dt}(mu_t, v) + \langle B(t)u_t, v \rangle + \langle \mathcal{P}_\lambda u, v \rangle = \langle b, v \rangle
\]
for all \( v \in S \) in the sense of distribution in \( [0, T] \).

Using standard variational techniques (see [7]), we can prove the following.

Theorem 5.2. Suppose \( b \in L^2((0, T); L^2(\Omega)) \), \( u_0 \in S \) and \( v_0 \in L^2(\Omega) \). Then the initial value problem \( (25) \) has a unique solution \( u \) in the sense of Definition 5.1. Moreover, the solution satisfies the following properties.
1. \( u \in C([0, T]; S) \) and \( u_t \in C([0, T]; L^2(\Omega)) \).
2. For all $\tau \in [0, T]$, define the energy

$$E(\tau) = \int_{\Omega} m(x)u_1^2(x, \tau)dx + (\mathcal{P}_\lambda u(\tau), u(\tau)).$$

Then $E(\tau)$ is continuous over $[0, T]$ and satisfies

$$E(\tau) + 2\int_0^\tau (B(t)u_t, u_t)dt = E(0) + 2\int_0^\tau (b(t), u_t(t))dt \quad \forall \tau \in [0, T].$$

3. There exists a positive constant $C = C(T)$ such that if $u$ solves (25), then

$$\|u\|_{L^\infty((0,T);S)} + \|u_t\|_{L^\infty((0,T);L^2(\Omega))} \leq C\left(\|u_0\|_S + \|v_0\|_{L^2(\Omega)} + \|b\|_{L^2((0,T);L^2(\Omega))}\right).$$

Let us interpret the problem we solved. Define $\Omega_T = \Omega \times (0, T)$. Then the solution found in the above theorem satisfies the nonlocal Cauchy-Neumann-type equation in the sense of distribution $D'(\Omega_T)$:

\[
\begin{aligned}
\partial_t (mu_t) + B(t)u_t + \mathcal{P}_\lambda u(x, t) &= b \quad \text{in } \Omega_T, \\
\partial_t (mu_t) + B(t)u_t + \mathcal{P}_\lambda u(x, t) &= b \quad \text{in } \Omega_T, \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = v_0(x), \quad \text{for almost all } x \in \Omega, \\
u(\cdot, t) &\in S \quad \text{for all } t \in [0, T],
\end{aligned}
\]

where the integral operator $\mathcal{P}_\lambda$

$$\mathcal{P}_\lambda u(x, t) = -2\int_{\Omega} (\rho(|x'-x|) + \lambda g(|x'-x|)) (u(x', t) - u(x, t))dx'.$$

is understood as in Section 3, and the operator $B(t)$ is either of (26) or (27).

When the operator $B(t) \equiv 0$, the equation (28) is precisely the scalar linear peridynamic equation of motion (1) for given initial displacement and velocity.

We can obtain some information about the solution from (28). For example, the solution $u$ satisfies

$$\partial_t (mu_t) + \mathcal{P}_\lambda u(x, t) = -B(t)u_t + b \in L^2(\Omega_T).$$

Moreover, taking $v \equiv 1$ as a test function in (28) we get the equality

$$\frac{d}{dt} \int_{\Omega} m(x)u_t(x, t)dx = \int_{\Omega} b(x, t)dx \quad \text{a.e } t \in (0, T),$$

which is precisely balance of linear momentum. This equality is easy to obtain for the standard local wave equation with Neumann boundary condition.

We should mention that the result in Theorem 5.2 is proved in [13, 14, 2] for the case when $S = L^p(\Omega), 1 < p < \infty$ using semigroup methods for integrable kernels. The articles [12, 24] also proved the same result in the cases when $\Omega = \mathbb{R}^d$ and for bounded domains when function spaces with special periodic boundary conditions.

6. Conclusion. In this work we have analyzed a scalar linear peridynamic model associated with a sign changing kernel. The model is posed as either a nonlocal boundary value problem with a volume constraint or a time dependent problem with both initial conditions and nonlocal volume constraints.

Our main contribution is proving well-posedness for both equilibrium and time dependent nonlocal equations of motion for this more general class of PD models that are of close relevance to materials modeling. We used standard variational methods to obtain the existence result for the time-dependent equation, while Fredholm Alternative Theorem was used to prove well-posedness for the equilibrium equation. We have presented sign changing kernels that give rise to equilibrium solutions that minimize an energy functional. Additional necessary conditions for
minimizers are obtained that hold in more general subspaces. These results are obtained as a consequence of an extensive analysis on the energy space and the nonlocal operator. The energy space is shown to be a Hilbert space with a nonlocal Poincaré-type inequality holds in the space. Conditions on the kernel is provided for the space to be separable and to be compactly embedded in $L^2(\Omega)$. Density of smooth functions as well extension theorems are also given. In turn, properties of the associated nonlocal operators such as invertibility can be established. In future works we plan to present results on the issue of regularity of solutions relative to given data. Nonlinear nonlocal problems will also be investigated. We plan to extend the results obtained in this work to the vector linear peridynamic model such as the bond-based and state-based systems [17, 18] with possibly nonpositive definite kernel functions.

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