Critical Behavior of the Sandpile Model as a Self-Organized Branching Process

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Kinetic equations, which explicitly take into account the branching nature of sandpile avalanches, are derived. The dynamics of the sandpile model is described by the generating functions of a branching process. Having used the results obtained the renormalization group approach to the critical behavior of the sandpile model is generalized in order to calculate both critical exponents and height probabilities.

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As it was realized a few years ago, almost all fractal structures we can see in nature are "self-organized" in a sense that they spontaneously grow, obeying rather simple dynamical rules [1]. The concept of Self-Organized Criticality (SOC) has recently been introduced by Bak et al. [2] to emphasize the fact that the production of self-similar structures is not the only possible result of the irreversible dynamics of an extended system. They have shown that such a system can also evolve stochastically into a certain critical state similar to that of second order phase transition. It lacks therein any characteristic length as well as timescale and obeys power-law distributions. The critical state is independent of the initial configuration of the system and, unlike ordinary critical phenomena, no fine tuning of control parameters is necessary to arrive at this state.

To illustrate the basic ideas of SOC, Bak et al. used a cellular automaton now commonly known as "sandpile" because of the crude analogy between its dynamical rules and the way sand topples when building a real sand pile. The formulation of this model is given in terms of integer height variables $z_i$ at each site of a square lattice $L$. Particles are added randomly and the addition of a particle increases the height at that site by one. If this height exceeds the critical value $z_c = 4$, then the site topples, and on toppling its height decreases by 4 and the heights at each of its nearest neighbors increases by 1. They may become unstable in their turn and the dynamical process continues. Open boundary conditions are usually assumed, so that particles can leave the system.

Dhar [3] has shown that this model is exactly solvable due to an Abelian group structure hidden in its dynamics. The fact that almost all characteristics of the Abelian sandpile model are calculable analytically is the reason for considering this model as a perfect proving ground for various approaches to SOC. Among others, the Renormalization Group (RG) approach proposed recently by Pietronero et al. [4,5] seems to be the most promising one because it explicitly describes the self-similar dynamics of the SOC models at the critical state.

The purpose of this Letter is to generalize this renormalization scheme by exploiting the analogy between large scale dynamics of the sandpile model and chain reactions of a special kind. At first, we should define coarse grained variables proper for describing the behavior of the system on a sublattice $L_b$ which consists of the cells of size $b$ on the initial lattice $L$. To keep the connection with the original formulation of the sandpile model, we will characterize the static properties of a cell by four quantities

$$n^{(b)} = (n_A, n_B, n_C, n_D), \quad n_A + n_B + n_C + n_D = 1, \quad (1)$$

which are nothing but the probabilities for a cell to behave like a site on the initial lattice with a height 1, 2, 3 or 4 respectively in the coarse grained dynamics, i.e. the addition of a "coarse grained particle" to the cell transforms it to the next one in the alphabet. For example, the cell B characterized by the vector (0,1,0,0) will be transformed to the cell C with the vector (0,0,1,0). The last variable $n_D$ is the probability for the cell to behave like a critical one in a sense that the addition of a "coarse grained particle" to the cell induces relaxations into some neighboring cells or, in other words, subrelaxation processes on a minimal scale span the cell and transfer energy to some of its neighbors.

According to Pietronero et al. [4], independently of the dynamics of the model at the minimal scale such a relaxation process leads to four possible situations for a coarse grained cell of size $b$. Namely, after the relaxation of the critical cell "coarse grained particles" can be transferred to one, two, three or four neighboring cells with the probabilities

$$p^{(b)} = (p_1, p_2, p_3, p_4), \quad p_1 + p_2 + p_3 + p_4 = 1. \quad (2)$$

The probabilities $p_i$ are normalized sums of all the processes that ignite the corresponding number of neighboring cells, independently of their position. The distribution of particles after toppling at the minimal scale is characterized by the vector $p^{(1)} = (0,0,0,1)$.

While the first set of the coarse grained variables $n^{(b)}$ describes the height configuration at the scale $b$, the second set $p^{(b)}$ defines the toppling rules of the model and, in some sense, characterizes the phase space for the relaxation dynamics at the same scale.

In this framework the coarse grained dynamics of the sandpile model can be represented as the following branching process on the sublattice $L_b$. 

1
A + ϕ → B,
B + ϕ → C,
C + ϕ → D,
D + ϕ → \begin{cases} p_1 : & D + \hat{ϕ} \\
p_2 : & C + 2\hat{ϕ} \\
p_3 : & B + 3\hat{ϕ} \\
p_4 : & A + 4\hat{ϕ}. \end{cases} (3)

Here ϕ and \(\hat{ϕ}\) denote the ”coarse grained particles” obtained by the cell and the particles transferred to the neighboring cells, respectively.

These processes can formally be reinterpreted as an irreversible chemical reaction which takes place at each cell of the sublattice \(L_b\). Now the coarse grained variables \(n_A, n_B, n_C, n_D\) and \(n_\varphi\) denote the concentrations of the respective species A, B, C, D, and \(\varphi\). Following to standard prescriptions of the chemical physics we can write kinetic equations corresponding to this scheme of chemical reactions

\[\dot{n}_A = n_\varphi (p_4 n_D - n_A), \quad \dot{n}_B = n_\varphi (p_3 n_D + n_A - n_B), \quad \dot{n}_C = n_\varphi (p_2 n_D + n_B - n_C), \quad \dot{n}_D = n_\varphi (p_1 n_D + n_C - n_D), \quad \dot{n}_\varphi = n_\varphi (\bar{p} n_D - 1) + \bar{p} \nabla^2 (n_\varphi n_D) + \eta(r,t) \] (4e)

where \(\bar{p} = p_1 + 2p_2 + 3p_3 + 4p_4\) is equal to the average number of particles leaving the cell on toppling and \(r\) is the position vector of the cell in the 2D space. The noise term \(\eta(r,t)\), being non-negative, mimics the random addition of particles to the system. The diffusion term \(\nabla^2 (n_\varphi n_D)\) describes the transfer of particles into the neighboring cells, and the diffusion coefficient \(\nu\) for the discrete Laplacian on the square lattice is equal to 1/4.

The only mobile specie in this scheme of reactions is \(\varphi\) and it is the field \(n_\varphi\) which describes the dynamics of avalanches. When it is equal to zero, all toppling processes die. Then, due to the noise term \(\eta(r,t)\), particles are added randomly into the system initiating a branching process directed to the open boundary of the system. This process mutates species in the cells it has visited and topples the critical ones. Finally, the system will reach the steady state where the probability that the activity will die is on average balanced by the probability that the activity will branch. Thus, the chain reaction maintains this stationary state and all further avalanches cannot change the concentrations of species A, B, C, and D. Therefore, the steady state is characterized by the conditions

\[\dot{n}_A = \dot{n}_B = \dot{n}_C = \dot{n}_D = 0 \] (5)

and Eqs. (4a) lead to the following relationships between concentrations of species \(n^{(b)}\) at the stationary state and branching probabilities \(p^{(b)}\)

\[n_A^* = p_4/\bar{p}, \quad n_B^* = (p_3 + p_4)/\bar{p}, \quad n_C^* = (p_2 + p_3 + p_4)/\bar{p}, \quad n_D^* = (p_1 + p_2 + p_3 + p_4)/\bar{p} = 1/\bar{p}. \] (6d)

The relation (4d) between the probability \(n_A^*\) and branching probabilities \(p^{(b)}\) has already appeared in the paper [3]. It was derived there from the assumption that at the stationary state the flow of particles in a cell was on average balanced by the flow of particles out of the cell.

If we neglect the fluctuations of \(n^{(b)}\) at the steady state, Eq. (4e), describing the propagation of an active process, becomes simply the diffusion equation

\[\dot{n}_\varphi = \nu \nabla^2 (n_\varphi) + \eta(r,t) \] (7)

and coincides with that for the flow of particles at the critical state obtained by Zhang [3].

To describe in detail the branching process underlying the large scale behavior of the sandpile model, let us consider the following generating function:

\[\sigma(N, E, S, W) = \frac{p_1}{4} (N + E + S + W) \]
\[+ \frac{p_2}{6} (NE + NS + NW + ES + EW + SW) \]
\[+ \frac{p_3}{4} (NES + NEW + NSW + ESW) + p_4 \text{ NESW} \] (8)

where symbols N, E, S and W correspond to the north, east, south and west directions on the square lattice, respectively. The coefficient of each term of this polynomial, say \(p_{2NE}/6\), is equal to the probability that after relaxation of a critical cell particles will go only northward and eastward.

It is easy to check directly that this function has the following properties: (a) if any letter, for example N, is replaced by zero, the function \(\sigma(0, E, S, W)\) counts all the processes that do not send a particle northward; (b) on the contrary, the difference \(\sigma(1, E, S, W) - \sigma(0, E, S, W)\) corresponds to the sum over all the processes that definitely send a particle to the north and, possibly, to some other directions; (c) this generating function is normalized so that \(\sigma(1, 1, 1, 1) = 1\) due to the normalization condition on branching probabilities, Eq. (3); (d) if the particle that has been sent northward after the relaxation of a critical cell ignites the northern neighboring cell, the corresponding letter N should be replaced by another \(\sigma\)-function and each term of the generating function \(\sigma(\sigma(N, E, S, W), E, S, W) - \sigma(0, E, S, W)\) corresponds to the process that consists of the two successive topplings.

Using these simple rules we can write the generating function for any chain of relaxation processes on the sublattice \(L_b\).

Now, following to the general ideas of the paper [3], we are ready to define a renormalization transformation for the relaxation dynamics of the sandpile model. The
standard real space RG approach consists of considering a block $2 \times 2$ of cells of the lattice $\mathcal{L}_b$ to be a single cell of the size $2b$ on the lattice $\mathcal{L}_{2b}$. Thus, given branching probabilities $p^{(6)}$ we want to find an analogous set of probabilities $p^{(2b)}$ on the lattice $\mathcal{L}_{2b}$. To this end, we should count up all the possible toppling processes that span the starting block of cells and transfer particles to some neighboring blocks. The spanning rule implies that we have to consider only those connected chains of processes that span the block from left to right or from top to bottom neglecting the processes not extending over the resulting scale $2b$. On Fig. 1 all the different types of toppling processes that satisfy this spanning condition are shown. The blocks that have no, or have only one critical cell inside are not included in the renormalization of the dynamics because they do not lead to a relaxation process that spans the cell of size $2b$.

We amplify the set of toppling processes considered in the RG scheme of Pietronero et al. by taking into account also the processes, Fig. 1 (c), of relaxation of the cell $C$ provided it has got two particles during the toppling process inside the blocks. As it is clear from the dashed box, we have to consider only those connected chains of processes that span the starting block of cells and transfer particles to some neighboring blocks. The spanning rule implies that we have to consider only those connected chains of processes that span the block from left to right or from top to bottom neglecting the processes not extending over the resulting scale $2b$. On Fig. 1 all the different types of toppling processes that satisfy this spanning condition are shown. The blocks that have no, or have only one critical cell inside are not included in the renormalization of the dynamics because they do not lead to a relaxation process that spans the cell of size $2b$.

At the stationary state each type of blocks has an additional weight given by the product of the probabilities $n^{(6)}$ of the cells inside the block and the numerical factor equal to the numbers of different blocks with the same relaxation schemes. Thus, to the blocks on Fig. 1 the following relative weights should be ascribed

\[
W_a = 4n_3^2(n_A + n_B + n_C)^2, \quad (9a)
\]
\[
W_b = 4n_3^2(n_A + n_B), \quad (9b)
\]
\[
W_c = 4n_3^3 n_C, \quad (9c)
\]
\[
W_d = n_3^3. \quad (9d)
\]

The generating function corresponding to the relaxation processes inside the blocks with the weight $W_a$ can be written with the use of the $\sigma$-function, Eq. (8),

\[
\Sigma_a(N, E, S, W) = \{ \sigma(\sigma(N, 1, 1, W), 1, S, W) - \sigma(0, 1, S, W) + c.p. \}/N_a, \quad (10)
\]

where the notation $c.p.$ implies all possible cyclic permutations of the symbols $N, E, S$ and $W$ and $N_a$ is the normalization factor such that $\Sigma_a(1, 1, 1, 1) = 1$. To write this function we start from the left down cell and define the arguments of the $\sigma$-function corresponding to the toppling of this cell. For the process definitely spans the block, the left up cell should topple and we write another $\sigma$-function instead of the symbol $N$. By going eastward the process will terminate inside the block and this branch of the toppling process cannot affect the neighboring blocks. Hence, we should write the number 1 instead of the symbol $E$. The other symbols $S$ and $W$ correspond to the branches of the toppling process which immediately get out the initial block of cells.

Analogously, we can write the generating function of the blocks with the weight $W_b$

\[
\Sigma_b(N, E, S, W) = \{ \sigma(\sigma(N, E, 1, 1), 1, S, W) - \sigma(0, 1, S, W)
+ \sigma(\sigma(N, 1, 1, W), \sigma(1, E, S, 1), S, W) - \sigma(0, 0, S, W)
+ \sigma(1, \sigma(\sigma(N, E, 1, 1), E, S, 1), S, W) - \sigma(1, 0, S, W)
+ c.p. \}/N_b, \quad (11)
\]

The generating functions $\Sigma_c$ and $\Sigma_d$ for the blocks with the weights $W_c$ and $W_d$ in Fig. 1 are quite similar in principle but the expressions are rather long and will be published in a complete paper.

All these generating functions are polynomials describing in detail toppling processes inside the blocks. As it is easy to check directly, besides the terms with the first powers of the symbols $N, E, S$ and $W$ these generating functions have also the terms with their squares. The first terms correspond to the processes when after the relaxation of the block only one particle goes in a given direction, while the last ones describe the processes with two particles going in the same direction. According to the RG ideology, we should consider both of them as the transfer of the only new "coarse grained particle". To this end, we can simply replace all the squares of the
symbols (N, E, S, W) by their first powers. Finally, we will get polynomials of the same form as the original one, Eq. 8, but with the new branching probabilities.

Now, to write the complete generating function corresponding to the relaxation of the block 2 × 2 of the cells on the lattice L, we should average the Σ-functions with the corresponding weights of blocks defined by the height probabilities n(b) at the scale b, Eqs. (9a-9d).

\[ \Sigma(N, E, S, W) = \{W_a \Sigma_a(N, E, S, W) + W_b \Sigma_b(N, E, S, W) + W_c \Sigma_c(N, E, S, W) + W_d \Sigma_d(N, E, S, W)\}/N. \] (12)

To finish the whole RG scheme we have to define an analogous renormalization equations for the height probabilities n(2b). Due to the nonlocal properties of the dynamics of the sandpile model, the direct renormalization of these quantities seems to be very difficult. Instead, we will use the stationary state relations, Eqs. (6a-6d), to define the renormalized height probabilities.

Given this RG transformation we can study how the system evolve under the successive doubling of the scale. The final result is that independent of the vectors p(1) and n(1) at the minimal scale the system flows at the fixed point parameters. Below, we briefly repeat their arguments. By using the discrete length scale b(k) = 2^k and the avalanche distribution in the form \( P(r)dr = r^{1-\tau}dr \) we can define the probability that the relaxation process spans the cell of size b(k) and dies at the neighboring cells not extending over the scale b(k+1)

\[ K = \int_{b(k+1)}^{b(k)} P(r)dr / \int_{b(k)}^{\infty} P(r)dr = 1 - 2^{(1-\tau)} \] (13)

Asymptotically (k → ∞) we can express K in terms of fixed point parameters in the following way:

\[ K = p_1^*(1 - n_D^*) + p_2^*(1 - n_D^*)^2 + p_3^*(1 - n_D^*)^3 + p_4^*(1 - n_D^*)^4. \] (14)

Using these two expressions, Eqs. (13, 14), the exponent \( \tau \) is given by the formula

\[ \tau = 1 - \frac{1}{2} \frac{\ln(1 - K)}{\ln 2} = 1.248 \] (15)
in excellent agreement with the proposed value \( \tau = 5/4 \).

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These results obtained with the use of the RG approach can be compared with the exact ones for the sandpile model. As it has been shown in the paper 8, an avalanche can be considered as a sequence of waves of topplings each consisting of sites that toppled only once in that wave. Being more simple objects, waves admit a representation in terms of spanning trees covering the lattice sites. The branching probabilities of the spanning trees have been calculated exactly by Manna et al. 8. Their results are presented in Table I. The hypothesis that branching probabilities for spanning trees should coincide with that for the topping process seems to be quite plausible, but it has yet to be proved. The exact height probabilities for the sandpile model presented in Table I were calculated by Priezzhev 8.

Recently, Priezzhev et al. 8 have used the known exponents for spanning trees to argue that in the sandpile model the probability for the number of sites to involve in the avalanche equal to s varies as \( P(s) \approx s^{-\tau} \), for large s, where \( \tau = 5/4 \).

As it has been noted by Pietronero et al. 4 the avalanche exponent \( \tau \) can be obtained directly from the fixed point parameters. Below, we briefly repeat their arguments. By using the discrete length scale b(k) = 2^k and the avalanche distribution in the form \( P(r)dr \approx r^{(1-\tau)}dr \) we can define the probability that the relaxation process spans the cell of size b(k) and dies at the neighboring cells not extending over the scale b(k+1)

\[ K = \int_{b(k+1)}^{b(k)} P(r)dr / \int_{b(k)}^{\infty} P(r)dr = 1 - 2^{(1-\tau)} \] (13)

Asymptotically (k → ∞) we can express K in terms of fixed point parameters in the following way:

\[ K = p_1(1 - n_D^*) + p_2(1 - n_D^*)^2 + p_3(1 - n_D^*)^3 + p_4(1 - n_D^*)^4. \] (14)

Using these two expressions, Eqs. (13, 14), the exponent \( \tau \) is given by the formula

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