MOSER INEQUALITIES IN GAUSS SPACE

ANDREA CIANCHI\textsuperscript{1}, VÍT MUSIL\textsuperscript{1,2,3}, AND LUBOŠ PICK\textsuperscript{3}

Abstract. The sharp constants in a family of exponential Sobolev type inequalities in Gauss space are exhibited. They constitute the Gaussian analogues of the Moser inequality in the borderline case of the Sobolev embedding in the Euclidean space. Interestingly, the Gaussian results have features in common with the Euclidean ones, but also reveal marked diversities.

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1. Introduction and main results
The present paper deals with a family of exponential type Sobolev inequalities in Gauss space \((\mathbb{R}^n, \gamma_n)\), namely the space \(\mathbb{R}^n\) endowed with the Gauss probability measure \(\gamma_n\) given by
\[
d\gamma_n(x) = \left(2\pi\right)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} \, dx \quad \text{for} \, x \in \mathbb{R}^n.
\]
The inequalities to be considered admit diverse variants. All of them concern, for a given \(\beta > 0\), the uniform bound
\[
\int_{\mathbb{R}^n} e^{(\kappa|u|)^{\frac{2\beta}{2+\beta}}} \, d\gamma_n \leq C
\]
for suitable positive constants \(\kappa\) and \(C\), and for every weakly differentiable function \(u\) in \(\mathbb{R}^n\) subject to a constraint on some kind of exponential integrability for \(|\nabla u|^\beta\), and to the normalization
\[
m(u) = 0.
\]
Here, and in what follows, \(m(u)\) denotes either the mean value \(mv(u)\) or the median \(med(u)\) of \(u\) over \((\mathbb{R}^n, \gamma_n)\).

The most straightforward version of the relevant gradient constraint reads
\[
\int_{\mathbb{R}^n} e^{|
abla u|^\beta} \, d\gamma_n \leq M
\]
for some constant \(M > 1\). This assumption on \(M\) is made since the integral in (1.3) cannot be smaller than 1, and equals 1 if and only if \(u\) is constant.

Inequalities of this form go back to [3, 11, 22, 38]. They can be equivalently stated as embeddings of the Gaussian Orlicz-Sobolev spaces \(W^1 \exp L^\beta(\mathbb{R}^n, \gamma_n)\) into the Orlicz spaces \(\exp L^{\frac{2\beta}{2+\beta}}(\mathbb{R}^n, \gamma_n)\), associated with Young functions equivalent near infinity to \(e^{\alpha|u|^\beta}\) and \(e^{\frac{2\beta}{2+\beta}|u|^\beta}\), respectively. In particular, in [22] it is shown that the exponent \(\frac{2\beta}{2+\beta}\) in (1.1) is the largest possible that makes these embeddings true. Interestingly, since \(\frac{2\beta}{2+\beta} < \beta\), there is a loss in the degree of integrability between \(|\nabla u|\) and \(u\) in the exponential scale. In fact, results of [22] ensure that \(\exp L^{\frac{2\beta}{2+\beta}}(\mathbb{R}^n, \gamma_n)\) is the optimal target space

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for embeddings of $W^1 \exp L^\beta (\mathbb{R}^n, \gamma_n)$ within the class of all Orlicz spaces on $(\mathbb{R}^n, \gamma_n)$, and even in the larger class of all rearrangement-invariant spaces.

The exponential embeddings in question extend, at a different scale, a family of Gaussian embeddings for the Sobolev spaces $W^{1,p}(\mathbb{R}^n, \gamma_n)$, with $p \in [1, \infty)$, into the Orlicz spaces $L^p(\log L)^{1/2} (\mathbb{R}^n, \gamma_n)$, where at least a minimal gain of integrability between $|\nabla u|$ and $u$ is guaranteed. This result for $p = 2$ was established in the seminal paper by Gross [32], whose researches were also motivated by applications to quantum field theory and to inequalities on infinite-dimensional spaces. The generalization to the case when $p \neq 2$ is contained in [2].

In [22], a borderline Gaussian Orlicz space, in the region between power and exponential type spaces, is exhibited with the property that membership of $u$ to the same space, this piece of information about its degree of integrability being sharp. The Orlicz space in question is denoted by $\exp(\frac{1}{4} \log^2 L)(\mathbb{R}^n, \gamma_n)$, and is built upon any Young function equivalent to $e^{\frac{1}{4} \log^2 t}$ near infinity. The relevant Sobolev embedding thus tells us that $W^1 \exp(\frac{1}{4} \log^2 L)(\mathbb{R}^n, \gamma_n)$ is embedded into $\exp(\frac{1}{4} \log^2 L)(\mathbb{R}^n, \gamma_n)$, and that the target space is optimal among all rearrangement-invariant spaces.

Further results about Gaussian Sobolev type inequalities are the subject of a rich literature in the areas of convexity in high dimensions, isoperimetric inequalities, spectral theory, probability, hypercontractive semigroups. Besides those mentioned above, contributions in this connection include [7, 8, 12, 13, 15, 17, 23, 26, 31, 42, 44, 47].

Our focus is on a sharp form of inequality (1.1). Specifically, we investigate the optimal—largest possible—constant $\kappa$ for which inequality (1.1) holds under the normalization condition (1.2), and either (1.3) or some alternate closely related assumption.

This can be regarded as a Gaussian counterpart of the question addressed in the celebrated paper by Moser [43], dealing with the optimal constant in an exponential inequality established in [45, 50, 52]. The latter arises in the borderline case of the Sobolev embedding theorem in the Euclidean setting, namely in (subsets of) $\mathbb{R}^n$ equipped with the Lebesgue measure. Moser’s inequality tells us that there exists a constant $C = C(n)$ such that

\[
(1.4) \quad \int_{\mathbb{R}^n} \left( e^{(n \omega_n^{1/n}|u|)^{n'}} - 1 \right) \, dx \leq C |\text{sprt}(u)|
\]

for every weakly differentiable function $u$ in $\mathbb{R}^n$, with support of finite Lebesgue measure, fulfilling

\[
(1.5) \quad \int_{\mathbb{R}^n} |\nabla u|^n \, dx \leq 1.
\]

Here, $\omega_n$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$, $|\text{sprt}(u)|$ stands for the measure of the support of $u$, and $n' = \frac{n}{n-1}$. Moreover, the constant $n \omega_n^{1/n}$ is sharp in inequality (1.4), since the integral on the left-hand side fails to be uniformly bounded under constraint (1.5) and under an upper bound for $|\text{sprt}(u)|$, if $n \omega_n^{1/n}$ is replaced by any larger constant.

Such a result has paved the way to numerous investigations on exponential inequalities for limiting Sobolev embeddings, including versions for higher-order derivatives [1, 4, 27], unrestricted supports [30, 34, 36, 40, 41, 48], more general measures in (1.4) [20, 28], subsets of $\mathbb{R}^n$ and arbitrary boundary values [19, 29, 37], Riemannian manifolds [9, 16, 27, 35, 39, 51], the Heisenberg group [24] or more general Carnot groups [6], perturbations of the space $W^{1,\nu}(\mathbb{R}^n)$ [4, 5, 18, 33].

The conclusions that will be derived on the Gaussian inequality (1.1) share some traits with the Euclidean ones, but also exhibit sharp dissimilarities. This is not only due to the presence of a measure that decays exponentially fast near infinity but also to an exponential integrand in the gradient constraint.

Our results can be stated with a gradient constraint either in integral form, as in (1.3), or in a norm form. The two formulations are not completely equivalent, because of the nature of norms in Orlicz spaces. Also, weak type norms of the gradient in exponential spaces—also called Marcinkiewicz norms—are included in our discussion. In all these variants, the sharp constant $\kappa$ in inequality (1.1),
namely the supremum among all values of \( \kappa \) that render it true, turns out to depend only on \( \beta \) and agrees with

\[
\kappa_\beta = \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{\beta}.
\]

Differences arise in connection with the central property that such a supremum be attained or not, namely with the validity of inequality (1.1) with \( \kappa = \kappa_\beta \). Notice that the fact that \( \kappa_\beta \) is independent of the dimension \( n \) is consistent with the whole theory of Gaussian Sobolev inequalities.

Here, we state the result about problem (1.1)-(1.3). The picture is completed in Section 3, where variations on constraint (1.3) are analyzed. In all cases, the conclusions take a different form depending on whether \( \beta \in (0, 2] \) or \( \beta \in (2, \infty) \). The limiting situation when, instead of (1.3), a bound on \( \| \nabla u \|_{L^\infty(R^n, \gamma_n)} \) is imposed, is considered as well.

**Theorem 1.1 [Integral form].** Let \( n \geq 1 \).

**Part 1.** Assume that \( \beta \in (0, 2] \).

1. (i) If \( 0 < \kappa \leq \kappa_\beta \), then for every \( M > 1 \) there exists a constant \( C = C(\beta, M) \) such that inequality (1.1) holds for every function \( u \) obeying (1.2) and (1.3).

1. (ii) If \( \kappa > \kappa_\beta \), then for any \( M > 1 \) there exists a function \( u \) obeying (1.2) and (1.3) that makes the integral in (1.1) diverge.

**Part 2.** Assume that \( \beta \in (2, \infty) \).

2. (i) If \( 0 < \kappa < \kappa_\beta \), then for every \( M > 1 \) there exists a constant \( C = C(\beta, M) \) such that inequality (1.1) holds for every function \( u \) obeying (1.2) and (1.3).

2. (ii) If \( \kappa = \kappa_\beta \), then there exist \( M > 1 \) and \( C > 0 \) such that inequality (1.1) holds for every function \( u \) obeying (1.2) and (1.3), and there exists \( M > 1 \) such that (1.1) fails, whatever \( C \) is, as \( u \) ranges over all functions obeying (1.2) and (1.3).

2. (iii) If \( \kappa > \kappa_\beta \), then for any \( M > 1 \) there exists a function \( u \) obeying (1.2) and (1.3) that makes the integral in (1.1) diverge.

Let us briefly comment on some peculiarities of Theorem 1.1. The appearance of a threshold value \( \beta = 2 \), which dictates the form of the result, is a new phenomenon in the frames of Moser and Gaussian type inequalities. In particular, it is striking that the value of the constant \( M \) appearing in condition (1.3) is immaterial when \( \beta \in (0, 2] \), but affects the conclusions if \( \beta \in (2, \infty) \). By contrast, the value 1 appearing on the right-hand side of (1.5) is critical, inasmuch as inequality (1.4) fails if 1 is replaced by any larger constant.

One more unexpected assertion of Theorem 1.1 is that, if \( \kappa > \kappa_\beta \), then just single functions \( u \) can be exhibited, for which the integral in (1.1) diverges, to demonstrate the failure of inequality (1.1). Instead, the integral in (1.4) is finite for each function \( u \in W^{1,n}(R^n) \) whose support has finite measure, even if \( n \omega_1^{1/n} \) is replaced by any larger constant. Inequality (1.4) fails in this case just because its left-hand side is not uniformly bounded by some constant depending only on \( |\text{sprt}(u)| \). An analogue in the Gaussian case holds in the subspace \( W^1 \exp E^\beta(R^n, \gamma_n) \) of those functions in \( W^1 \exp L^\beta(R^n, \gamma_n) \) such that

\[
\int_{R^n} e^{\lambda |\nabla u|^2} \, d\gamma_n < \infty
\]

for every \( \lambda > 0 \).

**Theorem 1.2 [Single functions in \( W^1 \exp E^\beta(R^n, \gamma_n) \)].** Let \( \beta > 0 \) and let \( u \in W^1 \exp E^\beta(R^n, \gamma_n) \). Then

\[
\int_{R^n} e^{(\kappa |u|)^{2/\beta}} \, d\gamma_n < \infty
\]

for every \( \kappa > 0 \).

Like that of Moser’s paper, and those of most of the related contributions mentioned above, our approach rests upon a suitable symmetrization argument, which reduces the Gaussian inequalities in
question to inequalities for one-variable functions. The symmetrization of use in the present framework, called Ehrhard symmetrization in what follows, was introduced in [25] and is in its turn related to the isoperimetric inequality in Gauss space [14, 49]. Basic properties of Ehrhard symmetrization are recalled in Section 4, where they are exploited in the proof of some key inequalities for our method.

The one-dimensional problems to be faced after symmetrization present specific difficulties compared with those arising in the Euclidean setting. A distinctive complication is that both the isoperimetric function in Gauss space and its norms in exponential Orlicz spaces do not admit expressions in closed form. This calls for precise asymptotic estimates for the relevant expressions, that are established in Section 5. Let us add that the choice of appropriate norms in the Orlicz spaces is also critical for certain inequalities to hold with exact constants. With this material at disposal, our proofs of Theorems 1.1 and 1.2, as well as those of the other main results stated in Section 3, are accomplished in Section 6. The necessary function-space background is recalled in Section 2 below.

2. Function spaces

Basic definitions and properties concerning function spaces involved in our discussion are collected in this section. For more details and proofs we refer to the monographs [10] and [46].

Let \((\Omega, \mathcal{F}, \mu)\) be a probability space, namely a measure space \(\Omega\) endowed with a probability measure \(\mu\). Assume that \((\Omega, \mathcal{F}, \mu)\) is non-atomic. In fact, we shall just be concerned with the case when \(\Omega\) is either \(\mathbb{R}^n\) endowed with the Gauss measure \(\gamma_n\), or \((0, 1)\) endowed with the Lebesgue measure. In the latter case, the measure will always be omitted in the notation. More generally, we shall simply write \(\mathbb{R}\) instead of \((\Omega, \mathcal{F}, \mu)\) when no ambiguity can arise. The notation \(\mathcal{M}(\mathbb{R})\) is employed for the space of real-valued, \(\nu\)-measurable functions on \(\mathbb{R}\).

Let \(\phi \in \mathcal{M}(\mathbb{R})\). The decreasing rearrangement \(\phi^*: [0, 1] \to [0, \infty]\) of \(\phi\) is given by

\[
\phi^*(s) = \inf \{ t \geq 0 : \nu(\{ x \in \mathbb{R} : |\phi(x)| > t \}) \leq s \} \quad \text{for } s \in [0, 1].
\]

Similarly, the signed decreasing rearrangement \(\phi^\#: [0, 1] \to [-\infty, \infty]\) of \(\phi\) is defined as

\[
\phi^\#(s) = \inf \{ t \in \mathbb{R} : \nu(\{ x \in \mathbb{R} : \phi(x) > t \}) \leq s \} \quad \text{for } s \in [0, 1].
\]

If \(\phi\) is integrable on \(\mathbb{R}\), we also define the maximal function \(\phi^{**} : (0, 1) \to [0, \infty]\) associated with \(\phi^*\) as

\[
\phi^{**}(s) = \frac{1}{s} \int_0^s \phi^*(r) \, dr \quad \text{for } s \in (0, 1).
\]

The functions \(\phi^*\) and \(\phi^{**}\) are non-increasing and \(\phi^* \leq \phi^{**}\).

The Hardy-Littlewood inequality implies that, if \(\phi, \psi \in \mathcal{M}(\mathbb{R})\), then

\[
\int_E |\phi \psi| \, d\nu \leq \int_0^{\|E\|} \phi^*(s)\psi^*(s) \, ds
\]

for every measurable set \(E \subset \mathbb{R}\).

The median \(\text{med}(\phi)\) and the mean value \(\text{mv}(\phi)\) of \(\phi\) are defined as

\[
\text{med}(\phi) = \phi^\#(\frac{1}{2}) \quad \text{and} \quad \text{mv}(\phi) = \int_\mathbb{R} \phi \, d\nu.
\]

Of course, \(\text{mv}(\phi)\) is well defined only if \(\phi\) is integrable over \(\mathbb{R}\).

A Young function \(A : [0, \infty) \to [0, \infty]\) is a left-continuous convex function vanishing at \(0\), which is not constant in \((0, \infty)\). Any Young function \(A\) admits the representation

\[
A(t) = \int_0^t a(\tau) \, d\tau \quad \text{for } t \geq 0,
\]

for some non-decreasing left-continuous function \(a : [0, \infty) \to [0, \infty]\). Of course any finite-valued, nonnegative convex function on \([0, \infty)\) vanishing at \(0\) is continuous. Thus, the additional assumption about left-continuity of a Young function is only relevant in the case when it attains the value \(\infty\).

By \(A : [0, \infty) \to [0, \infty]\) we denote the Young conjugate of \(A\), defined as

\[
\tilde{A}(t) = \sup \{ \tau t - A(\tau) : \tau \geq 0 \} \quad \text{for } t \geq 0.
\]
Young functions equivalent near infinity, in the sense that \( A \) is finite. We also define \( m \) for \( \phi \).

The space \( L^A(\mathcal{R}) \) built upon a Young function \( A \) is defined as
\[
L^A(\mathcal{R}) = \left\{ \phi \in \mathcal{M}(\mathcal{R}) : \int_{\mathcal{R}} A\left(\frac{\phi}{\lambda}\right) \, d\nu < \infty \text{ for some } \lambda > 0 \right\}.
\]
The space \( L^A(\mathcal{R}) \) is a Banach space equipped with the Luxemburg norm given by
\[
\|\phi\|_{L^A(\mathcal{R})} = \inf \left\{ \lambda > 0 : \int_{\mathcal{R}} A\left(\frac{\phi}{\lambda}\right) \, d\nu \leq 1 \right\}
\]
for \( \phi \in L^A(\mathcal{R}) \). One has that \( L^A(\mathcal{R}) = L^B(\mathcal{R}) \) (up to equivalent norms) if and only if \( A \) and \( B \) are Young functions equivalent near infinity, in the sense that \( A(c_1t) \leq B(t) \leq A(c_2t) \) for some constants \( c_1 \) and \( c_2 \), and for sufficiently large \( t \).

Recall that
\[
L^\infty(\mathcal{R}) \to L^A(\mathcal{R}) \to L^1(\mathcal{R})
\]
for every Young function \( A \). Here, the arrow “\( \to \)” denotes continuous embedding.

The Orlicz norm \( \|\cdot\|_{L^A(\mathcal{R})} \), given by
\[
\|\phi\|_{L^A(\mathcal{R})} = \sup \left\{ \int_{\mathcal{R}} \phi \psi \, d\nu : \int_{\mathcal{R}} \tilde{A}(\psi) \, d\nu \leq 1 \right\}
\]
for \( \phi \in L^A(\mathcal{R}) \), is equivalent to the Luxemburg norm.

If \( \phi \in L^A(\mathcal{R}) \) and \( E \subset \mathcal{R} \) is a measurable set, we use the abridged notations
\[
\|\phi\|_{L^A(E)} = \|\phi \chi_E\|_{L^A(\mathcal{R})} \quad \text{and} \quad \|\phi\|_{L^A(\mathcal{R})} = \|\phi \chi_E\|_{L^A(\mathcal{R})}.
\]
In particular,
\[
\|1\|_{L^A(E)} = \nu(E)\tilde{A}^{-1}(1/\nu(E)).
\]
Here, \( \tilde{A}^{-1} \) denotes the (generalized) right-continuous inverse of \( \tilde{A} \). A sharp form of the H"older inequality in Orlicz spaces tells us that
\[
\int_{\mathcal{R}} \phi \psi \, d\nu \leq \|\phi\|_{L^A(\mathcal{R})} \|\psi\|_{L^\tilde{A}(\mathcal{R})}
\]
for every \( \phi \in L^A(\mathcal{R}) \) and \( \psi \in L^\tilde{A}(\mathcal{R}) \).

The Marcinkiewicz space \( M^A(\mathcal{R}) \), associated with a Young function \( A \), is defined as the space of all functions \( \phi \in \mathcal{M}(\mathcal{R}) \) for which the norm
\[
\|\phi\|_{M^A(\mathcal{R})} = \sup_{s \in (0,1)} \frac{\phi^*(s)}{s A^{-1}(1/s)}
\]
is finite. We also define \( m^A(\mathcal{R}) \) as the collection of all functions \( \phi \in \mathcal{M}(\mathcal{R}) \) for which the quantity
\[
\|\phi\|_{m^A(\mathcal{R})} = \sup_{s \in (0,1)} \frac{\phi^*(s)}{s A^{-1}(1/s)}
\]
is finite.
is finite. Note that the functional \( \| \cdot \|_{m^A(\mathcal{R})} \) is a quasi-norm, in the sense that it enjoys the same properties of a norm, save that the triangle inequality holds up to a multiplicative constant. The embeddings \( L^A(\mathcal{R}) \to M^A(\mathcal{R}) \to m^A(\mathcal{R}) \) hold for every Young function \( A \), and

\[
\| \phi \|_{m^A(\mathcal{R})} \leq \| \phi \|_{M^A(\mathcal{R})} \leq \| \phi \|_{L^A(\mathcal{R})}
\]

for every \( \phi \in L^A(\mathcal{R}) \).

We denote by \( E^A(\mathcal{R}) \) the subspace of \( L^A(\mathcal{R}) \) defined by

\[
E^A(\mathcal{R}) = \left\{ \phi \in M(\mathcal{R}) : \int_{\mathcal{R}} A\left( \frac{|\phi|}{\lambda} \right) \, d\nu < \infty \text{ for every } \lambda > 0 \right\}.
\]

The space \( E^A(\mathcal{R}) \) coincides with the subspace of functions in \( L^A(\mathcal{R}) \) having an absolutely continuous norm. Recall that a function \( \phi \in L^A(\mathcal{R}) \) is said to have an absolutely continuous norm if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\| \phi \|_{L^A(G)} < \varepsilon
\]

for every measurable set \( G \subset \mathcal{R} \) with \( \nu(G) \leq \delta \).

Let \( \phi \in \mathcal{M}(\mathcal{R}) \). One has that \( \phi \in L^A(\mathcal{R}) \) if and only if \( \phi^* \in L^A(0, 1) \). Furthermore,

\[
\| \phi \|_{L^A(\mathcal{R})} = \| \phi^* \|_{L^A(0, 1)} \quad \text{and} \quad \| \phi \|_{L^A(\mathcal{R})} = \| \phi^* \|_{L^A(0, 1)}
\]

for every \( \phi \in \mathcal{M}(\mathcal{R}) \). Similarly, \( \phi \in E^A(\mathcal{R}) \) if and only if \( \phi^* \in E^A(0, 1) \). Also, \( \phi \in M^A(\mathcal{R}) \) if and only if \( \phi^* \in M^A(0, 1) \), and

\[
\| \phi \|_{M^A(\mathcal{R})} = \| \phi^* \|_{M^A(0, 1)}
\]

for every \( \phi \in \mathcal{M}(\mathcal{R}) \). Obviously, one also has that

\[
\| \phi \|_{m^A(\mathcal{R})} = \| \phi^* \|_{m^A(0, 1)}.
\]

Hardy’s lemma tells us that, given any nonnegative functions \( \phi, \psi \in \mathcal{M}(0, 1) \) and any non-increasing function \( \zeta : (0, 1) \to [0, \infty) \),

\[
\int_0^s \phi(r) \, dr \leq \int_0^s \psi(r) \, dr \quad \text{for } s \in (0, 1), \quad \text{then} \quad \int_0^s \phi(r) \zeta(r) \, dr \leq \int_0^s \psi(r) \zeta(r) \, dr \quad \text{for } s \in (0, 1).
\]

Given \( \beta \in (0, \infty) \), we denote by \( \exp L^\beta(\mathcal{R}) \) the Orlicz space associated with any Young function \( B(t) \) equivalent to \( e^{\beta t} \) near infinity. Its subspace \( E^\beta(\mathcal{R}) \) is defined according to definition (2.10). Notice that, for this choice of \( B \), one has that \( L^B(\mathcal{R}) = M^B(\mathcal{R}) = \exp L^B(\mathcal{R}) \), up to equivalent norms. By \( \exp L^\infty(\log L)^\alpha(\mathcal{R}) \) we denote the Orlicz space associated with any Young function \( A(t) \) equivalent to \( t^p \log^a t \) near infinity, where either \( p > 1 \) and \( a \in \mathbb{R} \), or \( p = 1 \) and \( a \geq 0 \). The space \( L^\infty(\mathcal{R}) \) is also an Orlicz space corresponding to the choice \( A(t) = \infty(1, \infty)(t) \).

The Orlicz-Sobolev space \( W^1 L^A(\mathbb{R}^n, \gamma_n) \) associated with a Young function \( A \) is defined as

\[
W^1 L^A(\mathbb{R}^n, \gamma_n) = \{ u : u \text{ is weakly differentiable in} \ \mathbb{R}^n, \ \text{and} \ |\nabla u| \in L^A(\mathbb{R}^n, \gamma_n) \}.
\]

Owing to the second embedding in (2.4) and to the inclusion \( W^{1,1}(\mathbb{R}^n, \gamma_n) \subset L(\log L)^{1/2}(\mathbb{R}^n, \gamma_n) \), any function \( u \in W^1 L^A(\mathbb{R}^n, \gamma_n) \) belongs to \( L^1(\mathbb{R}^n, \gamma_n) \). The space \( W^1 L^A(\mathbb{R}^n, \gamma_n) \), equipped with the norm given by

\[
\| u \|_{W^1 L^A(\mathbb{R}^n, \gamma_n)} = \| u \|_{L^1(\mathbb{R}^n, \gamma_n)} + \| \nabla u \|_{L^A(\mathbb{R}^n, \gamma_n)}
\]

for \( u \in W^1 L^A(\mathbb{R}^n, \gamma_n) \), is a Banach space.

The space \( W^1 \exp E^\beta(\mathbb{R}^n, \gamma_n) \) is defined analogously, on replacing \( L^A(\mathbb{R}^n, \gamma_n) \) by \( E^A(\mathbb{R}^n, \gamma_n) \) on the right-hand side of equation (2.12).

The Orlicz-Sobolev spaces \( W^1 \exp L^\beta(\mathbb{R}^n, \gamma_n) \) and \( W^1 \exp E^\beta(\mathbb{R}^n, \gamma_n) \) are associated with the spaces of exponential type \( \exp L^\beta(\mathbb{R}^n, \gamma_n) \) and \( \exp E^\beta(\mathbb{R}^n, \gamma_n) \).
3. Main results, continued

The results of this section complement Theorem 1.1, and describe the conclusions that can be derived about the attainability of the threshold constant $\kappa_\beta$ in inequality (1.1) when a counterpart of condition (1.3) is prescribed in norm form. Although the norms in question are equivalent, inequality (1.1) turns out to be sensitive to the chosen norm, and hence the conclusions in its connection may differ. The borderline case when the $L^\infty$ norm of the gradient replaces its norm in an exponential space is also considered.

We begin by considering the case of the Luxemburg norm. Namely, we address the validity of inequality (1.1) for functions $u$ fulfilling the condition

$$\|\nabla u\|_{L^B(R^n,\gamma_n)} \leq 1,$$

where $B$ is any Young function such that

$$B(t) = Ne^{\beta t} \text{ for } t > t_0,$$

for some $N > 0$ and $t_0 > 0$.

For norms of this type, the situation is analogous to that stated in Theorem 1.1 under a constraint in integral form.

**Theorem 3.1 [Luxemburg norms].** Let $n \geq 1$.

Part 1. Assume that $\beta \in (0, 2]$.

1. (i) If $\kappa < \kappa_\beta$, then for every $N > 0$ and for every Young function $B$ as in (3.2), there exists a constant $C = C(\beta, N, t_0)$ such that inequality (1.1) holds for every function $u$ obeying (1.2) and (3.3).

1. (ii) If $\kappa > \kappa_\beta$, then for any $N > 0$ and for any Young function $B$ as in (3.2), there exists a function $u$ obeying (1.2) and (3.1) that makes the integral in (1.1) diverge.

Part 2. Assume that $\beta \in (2, \infty)$.

2. (i) If $0 < \kappa < \kappa_\beta$, then for every $N > 0$, and for every Young function $B$ as in (3.2) there exists a constant $C = C(\beta, N, t_0)$ such that inequality (1.1) holds for every function $u$ obeying (1.2) and (3.1).

2. (ii) If $\kappa = \kappa_\beta$, then for every $N > 0$, there exists a Young function $B$ as in (3.2) and a constant $C > 0$ such that inequality (1.1) holds for every function $u$ obeying (1.2) and (3.1), and there exists a Young function $B$ as in (3.2) such that inequality (1.1) fails, whatever $C$ is, as $u$ ranges over all functions obeying (1.2) and (3.1).

2. (iii) If $\kappa > \kappa_\beta$, then for any $N > 0$ and for any Young function $B$ as in (3.2), there exists a function $u$ obeying (1.2) and (3.1) that makes the integral in (1.1) diverge.

Let us next examine constraints on trial functions in (1.1) imposed in terms of an exponential Marcinkiewicz norm defined as in (2.7), or a quasi-norm given as in (2.8). Specifically, we take into account functions $u$ subject to (1.2) and either condition

$$\|\nabla u\|_{M^B(R^n,\gamma_n)} \leq 1,$$

or

$$\|\nabla u\|_{m^B(R^n,\gamma_n)} \leq 1,$$

where $B$ is as in (3.2). Interestingly, the result differs from that of Theorem 3.1, but is the same in both cases (3.3) and (3.4).

**Theorem 3.2 [Marcinkiewicz norms].** Let $n \geq 1$.

Part 1. Assume that $\beta \in (0, 2]$.

1. (i) If $0 < \kappa < \kappa_\beta$, then for every $N > 0$ and for every Young function $B$ as in (3.2), there exists a constant $C = C(\beta, \kappa, N, t_0)$ such that inequality (1.1) holds for every function $u$ obeying (1.2) and (3.3).
(1.ii) If $\kappa \geq \kappa_\beta$, then for any $N > 0$ and for any Young function $B$ as in (3.2), there exists a function $u$ obeying (1.2) and (3.3) that makes the integral in (1.1) diverge.

Part 2. Assume that $\beta \in (2, \infty)$.

(2.i) If $0 < \kappa < \kappa_\beta$, then for every $N > 0$ and for every Young function $B$ as in (3.2), there exists a constant $C = C(\beta, \kappa, N, t_0)$ such that inequality (1.1) holds for every function $u$ obeying (1.2) and (3.3).

(2.ii) If $\kappa = \kappa_\beta$, then for every $N > 0$, there exist a Young function $B$ as in (3.2) and a constant $C > 0$ such that inequality (1.1) holds for every function $u$ obeying (1.2) and (3.3), and there exists a Young function $B$ as in (3.2) such that inequality (1.1) fails, whatever $C$ is, as $u$ ranges over all functions obeying (1.2) and (3.3).

(2.iii) If $\kappa > \kappa_\beta$, then for any $N > 0$ and for any Young function $B$ as in (3.2), there exists a function $u$ obeying (1.2) and (3.3) that makes the integral in (1.1) diverge.

The same statement holds if condition (3.3) is replaced by (3.4) throughout.

Remark 3.3. Theorem 3.2 shows one more diversity between Gaussian and Euclidean Moser type inequalities. Indeed, part (2.ii) tells us that the threshold value $\kappa_\beta$ is admissible in inequality (1.1), at least if $\beta > 2$, under the gradient constraint of Marcinkiewicz type (3.3) or (3.4), for suitable Young functions $B$ fulfilling condition (3.2). This is never the case in the corresponding Euclidean results when Marcinkiewicz type norms of the gradient are employed [4, 5].

Remark 3.4. In view of the inequalities in (2.9), the conclusions in the negative direction contained in Theorem 3.1 imply those of Theorem 3.2 about the norm $\| \cdot \|_{M^B(\mathbb{R}^n, \gamma_n)}$, and the latter imply those about the quasi-norm $\| \cdot \|_{m^B(\mathbb{R}^n, \gamma_n)}$. Of course, reverse implications hold about the conclusions in the positive direction.

Remark 3.5. Condition (3.2) can be relaxed by requiring that there exist constants $N_2 > N_1 > 0$ and $t_0 > 0$ such that

$$N_1 e^{\beta t} \leq B(t) \leq N_2 e^{\beta t} \quad \text{for } t > t_0.$$  

Properly modified statements of Theorems 3.1 and 3.2 hold under assumption (3.5), with $N$ replaced by $N_1$ in the assertions in the positive direction, and by $N_2$ in those in the negative direction.

Our last main result deals with a limiting version, as $\beta \to \infty$, of inequality (1.1) for functions subject to condition (3.1). The resulting inequality is

$$\int_{\mathbb{R}^n} e^{(\kappa |u|)^2} \, d\gamma_n \leq C,$$

under conditions (1.2) and

$$\| \nabla u \|_{L^\infty(\mathbb{R}^n, \gamma_n)} \leq 1.$$  

The exponent 2 is the largest admissible for $|u|$ in (3.6) under assumption (3.7). Also, the threshold value of $\kappa$ in (3.6) is $\frac{1}{\sqrt{2}}$, namely $\lim_{\beta \to \infty} \kappa_\beta$.

Theorem 3.6 [$L^\infty$ norm]. Let $n \geq 1$.

(i) If $0 < \kappa < \frac{1}{\sqrt{2}}$, then there exists a constant $C = C(\kappa)$ such that inequality (3.6) holds for every function $u$ obeying (1.2) and (3.7)

(ii) If $\kappa \geq \frac{1}{\sqrt{2}}$, then there exists a function $u$ obeying (1.2) and (3.7), that makes the integral in (3.6) diverge.

4. Ehrhard symmetrization and ensuing inequalities

Key tools in our approach are some rearrangement inequalities for the gradient of Sobolev functions on Gauss space. These inequalities in their turn rely upon the isoperimetric inequality that links the
Gauss measure of a set $E \subset \mathbb{R}^n$ to its Gauss perimeter. Recall that the Gauss perimeter $P_{\gamma_n}(E)$ of a measurable set $E$ can be defined as

$$P_{\gamma_n}(E) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\partial^M E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x),$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and $\partial^M E$ the essential boundary of $E$ in the sense of geometric measure theory, namely the set of points of $\mathbb{R}^n$ at which the density of $E$ is neither 0 nor 1. The Gaussian isoperimetric inequality asserts that half-spaces minimize Gauss perimeter among all measurable subsets of $\mathbb{R}^n$ with prescribed Gauss measure $[14, 49]$. Note that

$$\gamma_n(\{x \in \mathbb{R}^n : x_1 \geq t\}) = \Phi(t) \text{ for } t \in \mathbb{R},$$

where $\Phi : \mathbb{R} \to (0, 1)$ is the function defined as

$$(4.1) \quad \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-\frac{x^2}{2}} dx \text{ for } t \in \mathbb{R}. $$

Moreover,

$$P_{\gamma_n}(\{x \in \mathbb{R}^n : x_1 \geq t\}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \text{ for } t \in \mathbb{R}. $$

Here, $x_1$ denotes the first component of the point $x \in \mathbb{R}^n$. Thereby, on defining the function $I : [0, 1] \to [0, \infty)$ as

$$I(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\Phi^{-1}(s)^2}{2}} \text{ for } s \in (0, 1),$$

and $I(0) = I(1) = 0$, the Gaussian isoperimetric inequality takes the analytic form

$$I(\gamma_n(E)) \leq P_{\gamma_n}(E)$$

for every measurable set $E \subset \mathbb{R}^n$. The function $I$ is accordingly called the isoperimetric function (or isoperimetric profile) of Gauss space. Note that it is symmetric about $\frac{1}{2}$, namely

$$(4.2) \quad I(s) = I(1-s) \text{ for } s \in [0, 1].$$

Also,

$$(4.3) \quad -\Phi'(t) = I(\Phi(t)) \text{ for } t \in \mathbb{R}. $$

An Ehrhard symmetral of a function $u \in \mathcal{M}(\mathbb{R}^n, \gamma_n)$ is a function, equimeasurable with $u$, whose level sets are half-spaces. Thus, the function $u^\bullet : \mathbb{R}^n \to \mathbb{R}$ defined as

$$u^\bullet(x) = u^\circ(\Phi(x_1)) \text{ for } x \in \mathbb{R}^n,$$

is an Ehrhard symmetral of $u$.

The following result is established in [22, Lemma 3.3], and is the point of departure in the proof of fundamental properties of $u^\bullet$.

**Proposition 4.1.** Assume that $u \in W^{1,1}(\mathbb{R}^n, \gamma_n)$. Then the function $u^\circ$ is locally absolutely continuous in $(0, 1)$, the function $u^\bullet \in W^{1,1}(\mathbb{R}^n, \gamma_n)$, and

$$(4.4) \quad \int_0^s (-u^\circ I)^*(r) \, dr = \int_0^s |\nabla u^\bullet|^*(r) \, dr \leq \int_0^s |\nabla u|^*(r) \, dr \quad \text{for } s \in [0, 1].$$

A Gaussian Pólya-Szegő principle on the non-increase of Lebesgue [25], and more generally Orlicz [22], gradient norms under Ehrhard symmetrization, can immediately be derived from Proposition 4.1, via Hardy’s lemma (2.11).

**Proposition 4.2.** Let $A$ be a Young function. Assume that $u \in W^{1,A}(\mathbb{R}^n, \gamma_n)$. Then $u^\bullet \in W^{1,A}(\mathbb{R}^n, \gamma_n)$, and

$$(4.5) \quad \|u^\circ I\|_{L^A(0,1)} = \|\nabla u^\bullet\|_{L^A(\mathbb{R}^n, \gamma_n)} \leq \|\nabla u\|_{L^A(\mathbb{R}^n, \gamma_n)}.$$

The next Proposition can serve as a replacement for the Pólya-Szegő inequality (4.5) in dealing with certain functionals that depend on the gradient, but are not norms.
Proposition 4.3. Assume that the function $u \in W^{1,1}(\mathbb{R}^n, \gamma_n)$ satisfies $\text{med}(u) = 0$. Then

\begin{equation}
0 \leq u^\circ(s) \leq \frac{1}{I(s)} \int_0^s |\nabla u|^*(r) \, dr + \int_s^r \frac{1}{I(r)} |\nabla u|^*(r) \, dr \quad \text{for } s \in (0, \frac{1}{2}]
\end{equation}

and

\begin{equation}
0 \leq -u^\circ(1-s) \leq \frac{1}{I(s)} \int_0^s |\nabla u|^*(r) \, dr + \int_s^r \frac{1}{I(r)} |\nabla u|^*(r) \, dr \quad \text{for } s \in (0, \frac{1}{2}].
\end{equation}

Proof. Proposition 4.1, combined with Hardy’s lemma (2.11), implies that

\begin{equation}
\int_0^s (-u^\circ I)^*(r) \zeta(r) \, dr \leq \int_0^s |\nabla u|^*(r) \zeta(r) \, dr \quad \text{for } s \in (0, 1),
\end{equation}

for any non-increasing function $\zeta : (0, 1) \to [0, \infty)$. Since we are assuming that $\text{med}(u) = u^\circ(\frac{1}{2}) = 0$, we have that $u^\circ(s) \geq 0$ for $s \in (0, \frac{1}{2}]$, and

\begin{equation}
u^\circ(s) = \int_s^r -u^\circ(r) \, dr = \int_0^1 \chi(s, \frac{r}{2}) \frac{1}{I(r)} (-u^\circ(r) I(r)) \, dr
\end{equation}

\begin{equation}
\leq \int_0^1 \left( \chi(s, \frac{r}{2}) \right)^* I(r) (-u^\circ I)^*(r) \, dr \leq \int_0^1 \left( \chi(s, \frac{r}{2}) \right)^* |\nabla u|^*(r) \, dr \quad \text{for } s \in (0, \frac{1}{2}],
\end{equation}

where the first inequality follows from Hardy-Littlewood inequality (2.1) and the second one is due to (4.8). Furthermore, inasmuch as $I$ is increasing in $(0, \frac{1}{2}]$, if $s \in (0, \frac{1}{2}]$, then

\begin{equation}
\left( \frac{\chi(s, \frac{r}{2})}{I} \right)^*(r) = \frac{\chi(0, \frac{r}{2}-s)}{I(r+s)} \quad \text{for } r \in [0, 1].
\end{equation}

On the right-hand side of equality (4.10), and in similar equalities below, there is a slight abuse of notation, since $I$ is only defined in $[0, 1]$. However, this is immaterial, since $\chi(0, \frac{r}{2}-s)(r) = 0$ if $r+s > \frac{1}{2}$.

From equations (4.9) and (4.10), one deduces that

\begin{equation}
0 \leq u^\circ(s) \leq \int_0^{\frac{1}{2}-s} |\nabla u|^*(r) \, dr \quad \text{for } s \in (0, \frac{1}{2}].
\end{equation}

Similarly, $u^\circ(s) \leq 0$ for $s \in [\frac{1}{2}, 1]$ and

\begin{equation}
-u^\circ(1-s) = \int_0^{1-s} -u^\circ(r) \, dr \leq \int_0^1 \left( \frac{\chi(s, 1-s, \frac{r}{2})}{I} \right)^* |\nabla u|^*(r) \, dr \quad \text{for } s \in (0, \frac{1}{2}].
\end{equation}

Also, owing to equation (4.2) and to the monotonicity of $I$ on $(0, \frac{1}{2}]$, if $s \in (0, \frac{1}{2}]$, then

\begin{equation}
\left( \frac{\chi(s, 1-s)}{I} \right)^*(r) = \left( \frac{\chi(s, \frac{r}{2})}{I} \right)^*(r) = \frac{\chi(0, \frac{r}{2}-s)}{I(r+s)} \quad \text{for } r \in [0, 1].
\end{equation}

Hence,

\begin{equation}
0 \leq -u^\circ(1-s) \leq \int_0^{\frac{1}{2}-s} |\nabla u|^*(r) \, dr \quad \text{for } s \in (0, \frac{1}{2}].
\end{equation}

Now, define the function $\hat{I} : [0, 1] \to [0, \infty)$ as

\begin{equation}
\hat{I}(s) = \begin{cases} I(s) & \text{for } s \in [0, \frac{1}{2}] \\
I(\frac{1}{2}) & \text{for } s \in (\frac{1}{2}, 1].
\end{cases}
\end{equation}
Then,
\[
\int_0^{\frac{1}{2}} \frac{|\nabla u|^s(r)}{I(r+s)} \, dr \leq \int_0^{\frac{1}{2}} \frac{|\nabla u|^s(r)}{I(r+s)} \, dr = \int_0^1 \frac{|\nabla u|^s(r)}{I(r+s)} \, dr + \int_1^\infty \frac{|\nabla u|^s(r)}{I(r+s)} \, dr
\]
\[
\leq \frac{1}{I(s)} \int_0^s |\nabla u|^s(r) \, dr + \int_1^s \frac{|\nabla u|^s(r)}{I(r)} \, dr
\]
\[
\leq \frac{1}{I(s)} \int_0^s |\nabla u|^s(r) \, dr + \int_1^s \frac{|\nabla u|^s(r)}{I(r)} \, dr
\quad \text{for } s \in (0, \frac{1}{2}],
\]
where the first and the last inequalities hold since \( I = \hat{I} \) on \( (0, \frac{1}{2}] \), and the second is due to the monotonicity of \( \hat{I} \). Inequality (4.6) now follows from (4.11) and (4.13), and inequality (4.7) from (4.12) and (4.13).

A sharp estimate for the difference between the median and the mean value of any Sobolev function in terms of the \( L^1(\mathbb{R}^n, \gamma_n) \) norm of its gradient is the subject of the following proposition.

**Proposition 4.4.** Let \( u \in W^{1,1}(\mathbb{R}^n, \gamma_n) \). Then
\[
|\med(u) - \mv(u)| \leq \sqrt{\frac{\pi}{2}} \|\nabla u\|_{L^1(\mathbb{R}^n, \gamma_n)}.
\]
The constant \( \sqrt{\frac{\pi}{2}} \) in inequality (4.14) is sharp.

**Proof.** Owing to Proposition 4.1, the function \( u^o \) is locally absolutely continuous in \((0, 1)\). Hence, by Fubini’s theorem,
\[
u^o(s) - \mv(u) = u^o(s) - \int_0^1 u^o(r) \, dr = \int_0^1 (u^o(s) - u^o(r)) \, dr = \int_0^1 \int_r^1 u^{o'}(\eta) \, d\eta \, dr
\]
\[
= \int_0^1 r u^{o'}(r) \, dr - \int_0^1 (1 - r) u^{o'}(r) \, dr = \int_0^1 \left( \chi_{(s,1)}(r) - r \right) (-u^{o'}(r)) \, dr
\quad \text{for } s \in (0, 1).
\]
Therefore
\[
|\med(u) - \mv(u)| = |u^o(\frac{1}{2}) - \mv(u)| = \left| \int_0^1 \chi_{(\frac{s}{2},1)}(s) - s \right| \frac{1}{I(s)}(-u^{o'}(s)I(s)) \, ds \right|
\]
\[
\leq \sup_{s \in (0,1)} \left| \chi_{(\frac{s}{2},1)}(s) - s \right| \frac{1}{I(s)} \int_0^1 (-u^{o'}(s)I(s)) \, ds.
\]
Now, notice that the function \( s \mapsto s/I(s) \) is increasing on \((0, \frac{1}{2}] \). Indeed, this is equivalent to the fact that the function \( t \mapsto \Phi(t)e^{\frac{t^2}{2}} \) is decreasing on \((0, \infty)\), a property that can be easily verified via differentiation and by the inequality
\[
\int_t^\infty e^{-\frac{\tau^2}{2}} \, d\tau \leq e^{-\frac{t^2}{2}} \quad \text{for } t > 0,
\]
which is shown e.g. in [21, Lemma 3.4]. From the monotonicity of \( s/I(s) \) in \((0, \frac{1}{2}] \) and property (4.2) we have that
\[
\sup_{s \in (0,1)} \left| \chi_{(\frac{s}{2},1)}(s) - s \right| \frac{1}{I(s)} \leq \max\left\{ \sup_{s \in (0,1)} \frac{s}{I(s)}, \sup_{s \in (\frac{1}{2},1)} \frac{1 - s}{1 - I(s)} \right\} = \sup_{s \in (0,1)} \frac{s}{I(s)} = \frac{1}{2I(\frac{1}{2})} = \sqrt{\frac{\pi}{2}}.
\]
Furthermore, by Proposition 4.1,
\[
\int_0^1 (-u^{o'}(s)I(s)) \, ds = \int_0^1 (-u^{o'}I)^+(s) \, ds \leq \int_0^1 |\nabla u|^s(s) \, ds = \|\nabla u\|_{L^1(\mathbb{R}^n, \gamma_n)}.
\]
On combining estimates (4.16), (4.17) and (4.18), one obtains (4.14).
Lemma 4.5. Next, from Proposition 4.4, Hölder’s inequality (2.6) and equation (2.5) we infer that

\[
F := \begin{cases} 
0 & \text{for } x_1 \in (-\infty, 0] \\
kx_1 & \text{for } x_1 \in (0, \frac{1}{2}] \\
1 & \text{for } x_1 \in (\frac{1}{2}, \infty). 
\end{cases}
\]

Indeed, med\(u_k\) = 0 for \(k \in \mathbb{N}\), \(\lim_{k \to \infty} \text{mv}(u_k) = \frac{1}{2}\), and \(\lim_{k \to \infty} \|\nabla u_k\|_{L^1(\mathbb{R}^n, \gamma_n)} = \frac{1}{2\pi} \).

Given a Young function \(B\), define the functions \(F_{L^B}\) and \(F_{m^B}\) from \((0, \infty)\) into \((0, \infty)\) as

\[
F_{L^B}(t) = \left\| \frac{1}{\sqrt{t}} \right\|_{L^B(\Phi(t), \frac{1}{2})} + \sqrt{\frac{\pi}{2}} B^{-1}(1)
\]

and

\[
F_{m^B}(t) = e^{\frac{1}{2}} \int_t^\infty \left( \frac{1}{\Phi(t)} \right)^{\frac{1}{2}} dt + \int_0^t B^{-1}(1) \frac{1}{\Phi(t)} dt + \sqrt{\frac{\pi}{2}} \int_0^1 B^{-1}(1) \frac{1}{s} ds
\]

for \(t > 0\).

The next lemma provides us with a bound for the integral in (1.1) for any function \(u\) satisfying either condition (3.1) or (3.4). Such a bound amounts to an integral depending on either the function \(F_{L^B}\) or \(F_{m^B}\), respectively.

Lemma 4.5. Let \(\beta > 0\) and \(\kappa > 0\), and let \(B\) be a Young function. Let \(X\) denote either \(L^B\) or \(m^B\), and let \(F_X\) be defined as in (4.19) or (4.20), respectively. Then

\[
\int_{\mathbb{R}^n} e^{(\kappa |u|)^{\frac{2\beta}{2+\beta}}} d\gamma \leq \sqrt{\frac{\pi}{2}} \int_0^\infty e^{(\kappa |u_x(t)|)^{\frac{2\beta}{2+\beta}}} dt
\]

for every weakly differentiable function \(u\) in \(\mathbb{R}^n\) satisfying (1.2) and such that

\[
\|\nabla u\|_{X(\mathbb{R}^n, \gamma_n)} \leq 1.
\]

Proof. Assume that \(u\) obeys \(\text{mv}(u) = 0\), the case when \(\text{med}(u) = 0\) being even simpler. Let us set \(v = u - \text{med}(u)\). Then \(v\) is weakly differentiable, \(\text{med}(v) = 0\) and \(\nabla u = \nabla v\).

Let us begin by considering the case when \(X = L^B\). By Hölder’s inequality (2.6), Proposition 4.2 and (4.22), we have that

\[
0 \leq v^s(s) = \int_s^\infty -v^o(r)I(r) \frac{dr}{I(r)} \leq \|\nabla v\|_{L^B(\mathbb{R}^n, \gamma_n)} \left\| \frac{1}{I} \right\|_{L^B(s_1, \frac{1}{2})}
\]

for any \(s \in (0, \frac{1}{2}]\). On the other hand, owing to equation (4.2),

\[
0 \leq -v^s(s) = \int_s^\infty -v^o(r)I(r) \frac{dr}{I(r)} \leq \|\nabla v\|_{L^B(\mathbb{R}^n, \gamma_n)} \left\| \frac{1}{I} \right\|_{L^B(s_1, \frac{1}{2})} \leq \left\| \frac{1}{I} \right\|_{L^B(\frac{1}{2}, \frac{1}{4})} \tag{4.24}
\]

for \(s \in (\frac{1}{2}, 1)\).

Next, from Proposition 4.4, Hölder’s inequality (2.6) and equation (2.5) we infer that

\[
|\text{med}(u)| \leq \sqrt{\frac{\pi}{2}} \|\nabla u\|_{L^1(\mathbb{R}^n, \gamma_n)} \leq \sqrt{\frac{\pi}{2}} \|\nabla u\|_{L^B(\mathbb{R}^n, \gamma_n)} \|1\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq \sqrt{\frac{\pi}{2}} B^{-1}(1),
\]

whence, on setting \(C = \sqrt{\frac{\pi}{2}} B^{-1}(1)\),

\[
|u(x)| \leq |v(x)| + |\text{med}(u)| \leq |v(x)| + C \quad \text{for } x \in \mathbb{R}^n.
\]
By (4.25), since med\( (v) = 0 \),

\[
\int_{\mathbb{R}^n} e^{(\kappa|u|)^{2/n}} \, d\gamma_n \leq \int_{\mathbb{R}^n} e^{(\kappa|v|+\kappa C)^{2/n}} \, d\gamma_n
\]

(4.26)

\[
= \int_0^1 \frac{1}{2} e^{(\kappa v^o(s)+\kappa C)^{2/n}} \, ds + \int_0^1 \frac{1}{2} e^{(-\kappa v^o(1-s)+\kappa C)^{2/n}} \, ds.
\]

Hence, via inequalities (4.23) and (4.24), one deduces that

\[
\int_{\mathbb{R}^n} e^{(\kappa|u|)^{2/n}} \, d\gamma_n \leq 2 \int_0^1 \frac{1}{2} \exp \left\{ \left( \kappa \left\| \frac{1}{r} \right\|_{L^2(B(s,\frac{1}{2}))} + \kappa C \right)^{\frac{2}{n}} \right\} \, ds.
\]

Inequality (4.21) thus follows by the change of variables \( \Phi(t) = s \).

Next, assume that \( X = m^B \). Assumption (4.22) and the very definition of Marcinkiewicz quasi-norm implies that

\[
|\nabla u|^s = |\nabla v|^s \leq B^{-1} \left( \frac{1}{s} \right) \text{ for } s \in (0, 1).
\]

By Proposition 4.4,

\[
|\text{med}(u) - \text{mv}(u)| \leq \sqrt{\frac{\pi}{2}} \|\nabla u\|_{L^1(\mathbb{R}^n, \gamma_n)} \leq \sqrt{\frac{\pi}{2}} \|B^{-1}(1/s)\|_{L^1(0, 1)} = \sqrt{\frac{\pi}{2}} \int_0^1 B^{-1} \left( \frac{1}{s} \right) \, ds.
\]

Thus, inequalities (4.25) and (4.26) continue to hold, with \( C = \sqrt{\pi} \int_0^1 B^{-1} \left( \frac{1}{s} \right) \, ds \). Hence, by Proposition 4.3 and inequality (4.27),

\[
\int_{\mathbb{R}^n} e^{(\kappa|u|)^{2/n}} \, d\gamma_n \leq 2 \int_0^1 \frac{1}{2} \exp \left\{ \left( \frac{\kappa}{I(s)} \int_s^\Phi B^{-1} \left( \frac{1}{r} \right) \, dr + \kappa \int_s^\infty \frac{B^{-1}(1/\tau)}{I(\tau)} \, d\tau + \kappa C \right)^{\frac{2}{n}} \right\} \, ds.
\]

Thereby, inequality (4.21) follows via the change of variables \( r = \Phi(t) \) and \( t = \Phi(s) \). \( \square \)

An analogue of Lemma (4.5) under condition (3.7) is provided by the last result of this section.

**Lemma 4.6.** Let \( \kappa > 0 \). Then

\[
\int_{\mathbb{R}^n} e^{(\kappa|u|^2)} \, d\gamma_n \leq \sqrt{\frac{\pi}{2}} \int_0^\infty e^{\kappa F_{L^\infty}(t)^{2/n^2}} \, dt
\]

for every weakly differentiable function \( u \) obeying (1.2) and such that \( \|\nabla u\|_{\infty} \leq 1 \). Here, \( F_{L^\infty} : (0, \infty) \rightarrow (0, \infty) \) denotes the function defined as

\[
F_{L^\infty}(t) = \begin{cases} t & \text{if med}(u) = 0 \\ t - 2\Phi'(t) - 2\Phi(t) & \text{if } \text{mv}(u) = 0 \end{cases} \text{ for } t > 0.
\]

**Proof.** Assume first that med\( (u) = 0 \). We have that

\[
0 \leq u^o(s) = \int_s^\frac{1}{2} -u^o(r) I(r) \frac{dr}{I(r)} \leq \|\nabla u^o\|_{L^\infty(0, 1)} \int_s^\frac{1}{2} \frac{dr}{I(r)} \leq \|\nabla u\|_{L^\infty(\mathbb{R}^n, \gamma_n)} \int_0^{\Phi^{-1}(s)} \frac{\Phi'(t)}{I(\Phi(t))} \, dt \leq \Phi^{-1}(s) \text{ for } s \in (0, \frac{1}{2}),
\]

where we have made use of the change of variables \( r = \Phi(t) \), of Proposition 4.2 and of equation (4.3). Similarly, thanks to equation (4.2),

\[
0 \leq -u^o(s) = \int_s^{\frac{1}{2}} -u^o(r) I(r) \frac{dr}{I(r)} \leq \|\nabla u^o\|_{L^\infty(0, 1)} \int_s^\frac{1}{2} \frac{dr}{I(r)} \leq \int_s^\frac{1}{2} \frac{dr}{I(r)} \leq \Phi^{-1}(1-s) \text{ for } s \in (\frac{1}{2}, 1).
\]
Therefore,
\[ \int_{\mathbb{R}^n} e^{(\kappa |u|)^2} d\gamma_n \leq \int_0^1 e^{(\kappa u^2(s))^2} ds + \int_0^1 e^{(\kappa u^2(1-s))^2} ds \leq 2 \int_0^1 e^{(\kappa \Phi^{-1}(s))^2} ds = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{(\kappa s)^2 - \frac{s^2}{2}} ds, \]
namely (4.28).

Next, assume that \( \nu v(u) = 0 \). By (4.15),
\[ u^0(s) = \int_0^1 \frac{\chi(s,1)(r) - r}{I(r)} (-u^0(r)I(r)) dr \quad \text{for } s \in (0,1), \]
whence, by Proposition 4.2 and equation (4.2),
\[ |u^0(s)| \leq \|u^0 I\|_{L^\infty(0,1)} \int_0^1 \frac{|\chi(s,1)(r) - r|}{I(r)} dr \leq \|\nabla u\|_{L^\infty(\mathbb{R}^n, \gamma_n)} \left( \int_0^s \frac{r}{I(r)} dr + \int_s^1 \frac{1-r}{I(r)} dr \right) \]
(4.30)
By a change of variables, equation (4.3) and Fubini’s theorem,
\[ \int_0^s \frac{r}{I(r)} dr = \int_{\Phi^{-1}(s)}^{-\Phi^{-1}(s)} \Phi(t) dt = \int_{\Phi^{-1}(s)}^{-\Phi^{-1}(s)} \frac{\Phi(t) dt}{I(\Phi(t))} = \int_{\Phi^{-1}(s)}^{-\Phi^{-1}(s)} \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{s^2}{2}} ds dt \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{\Phi^{-1}(s)}^{-\Phi^{-1}(s)} e^{-\frac{s^2}{2}} ds \int_t^\infty e^{-\frac{s^2}{2}} dt \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{\Phi^{-1}(s)}^{-\Phi^{-1}(s)} e^{-\frac{s^2}{2}} ds - \Phi^{-1}(s) \sqrt{\frac{2}{\pi}} \int_{\Phi^{-1}(s)}^{-\Phi^{-1}(s)} e^{-\frac{s^2}{2}} ds \]
\[ = I(s) - \Phi^{-1}(s)s \quad \text{for } s \in (0,1). \]
Now, observe that \( \Phi(-t) = 1 - \Phi(t) \) for \( t \in \mathbb{R} \), whence
\[ \Phi^{-1}(1-s) = -\Phi^{-1}(s) \quad \text{for } s \in (0,1). \]
Thus, owing to equation (4.2), inequality (4.30) yields
(4.31) \[ |u^0(s)| \leq I(s) - \Phi^{-1}(s)s + I(s) + \Phi^{-1}(s)(1-s) = \Phi^{-1}(s) + 2I(s) - 2\Phi^{-1}(s)s \quad \text{for } s \in (0,1). \]
Altogether, by the symmetry of the rightmost side of (4.31) about \( \frac{1}{2} \) and a change of variables,
\[ \int_{\mathbb{R}^n} e^{(\kappa |u|)^2} d\gamma_n = \int_0^1 e^{(\kappa |u^0(s)|)^2} ds \leq \int_0^1 e^{\kappa^2(\Phi^{-1}(s)+2I(s)-2\Phi^{-1}(s)s)^2} ds \]
\[ = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{\kappa^2(t+2I(\Phi(t)) - t\Phi(t))^2 - \frac{t^2}{2}} dt. \]
Equation (4.28) hence follows via (4.3). \( \square \)

5. Asymptotic expansions

We are concerned here with various delicate asymptotic estimates for norm and integral functionals, of exponential type, evaluated at the function \( \Phi \) introduced in (4.1). Specifically, we deal with the functions \( F_{\alpha,\beta} \) and \( F_{\beta,\alpha} \) defined by (4.20) and (4.19).

Given a function \( F \) defined in some neighborhood of infinity, and \( k \in \mathbb{N} \), the notation
\[ F(t) = \mathcal{E}_1(t) + \cdots + \mathcal{E}_k(t) + \cdots \quad \text{as } t \to \infty \]
means that
\[ \lim_{t \to \infty} \frac{F(t)}{\mathcal{E}_1(t)} = 1 \quad \text{if } k = 1, \quad \text{and} \quad \lim_{t \to \infty} \frac{F(t) - \left[ \mathcal{E}_1(t) + \cdots + \mathcal{E}_j(t) \right]}{\mathcal{E}_{j+1}(t)} = 1 \quad \text{for } 1 \leq j \leq k-1, \text{ otherwise.} \]

Clearly, if
\[ F(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t) + \cdots \quad \text{as } t \to \infty, \]
and \( \sigma > 0 \), then
(5.1) \[ [F(t)]^\sigma = \mathcal{E}_1^\sigma(t) + \sigma \mathcal{E}_1^{\sigma-1}(t) \mathcal{E}_2(t) + \cdots \quad \text{as } t \to \infty. \]
Parallel notations will be used for asymptotic formulas as $t \to t_0$ and $t \to t_0^+$, for some $t_0 \in \mathbb{R}$.

We begin with two basic asymptotic expansions contained in Lemmas 5.1 and 5.2 below. They easily follow from elementary considerations, via applications of L’Hôpital’s rule. Their proofs are omitted, for brevity.

**Lemma 5.1.** Let $\Phi$ be given by (4.1). Then

\begin{equation}
\tag{5.2}
- \log \Phi(t) = \frac{t^2}{2} + \log t + \cdots \quad \text{as } t \to \infty
\end{equation}

and

\begin{equation}
\tag{5.3}
- \Phi'(t) = t\Phi(t) + \frac{\Phi(t)}{t} + \cdots \quad \text{as } t \to \infty.
\end{equation}

**Lemma 5.2.** Let $\beta > 0$ and let $\Phi$ be given by (4.1). Assume that $B$ is any Young function satisfying condition (3.2) for some $N > 0$. Then

\begin{equation}
\tag{5.4}
e\frac{t^2}{2} \int_1^\infty B^{-1} \left( \frac{1}{\Phi'(t)} \right) e^{-\frac{t^2}{2}} \, dt = 2 - \frac{1}{2} t^2 + \cdots \quad \text{as } t \to \infty.
\end{equation}

The next result provides us with an expansion for the function $F_m^\nu$ defined by (4.20), which holds for every function $B$ fulfilling assumption (3.2) and for every $\beta > 0$.

**Lemma 5.3.** Let $\beta > 0$ and let $\Phi$ be given by (4.1). Assume that $B$ is a Young function satisfying condition (3.2) for some $N > 0$. Then

\begin{equation}
\tag{5.5}
\left[ \kappa_\beta F_m^\nu(t) \right]_{\frac{2\nu}{2\rho+\nu}} = \frac{t^2}{2} + \begin{cases}
\frac{\beta}{2} \log t + \cdots & \text{if } \beta \in (0, 2)
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
\frac{1}{2} \log (t^2) + \cdots & \text{if } \beta = 2 \\
\frac{1}{2} t + \cdots & \text{if } \beta \in (2, \infty)
\end{cases}
\end{equation}

where $F_m^\nu$ is defined by (4.20), $\kappa_\beta$ is given by (1.6), and $c' = c'(B) \in \mathbb{R}$ is a constant depending on the global behavior of $B$. Consequently,

\begin{equation}
\tag{5.6}
B^{-1} \left( \frac{1}{\Phi(t)} \right) = (\log \Phi(t) - \log N)^{\frac{1}{\beta}} = \left( \frac{t^2}{2} + \log t + \cdots \right)^{\frac{1}{\beta}} = 2 - \frac{1}{2} t^2 + \frac{2}{\beta} \frac{1}{2} t^{2-\frac{2}{2}} \log t + \cdots \quad \text{as } t \to \infty,
\end{equation}

for some $M > 0$.

**Proof.** Denote the integral on the left-hand side of (5.4) by $J(t)$ and define the function $g: (0, \infty) \to (0, \infty)$ as

$$g(t) = 2 - \frac{1}{2} \frac{\beta}{2} t^2 + \cdots \quad \text{for } t > 0.$$ 

Clearly $B^{-1}(r) = (\log r - \log N)^{\frac{1}{\beta}}$ near infinity. By L’Hôpital’s rule and Lemma 5.1,

\begin{align*}
\lim_{t \to \infty} \frac{J(t)}{g(t)} &= \lim_{t \to \infty} \frac{B^{-1} \left( \frac{1}{\Phi'(t)} \right)}{2 - \frac{1}{2} t^2} \\
&= \lim_{t \to \infty} \left( \frac{-\log \Phi(t) - \log N}{t^2} \right)^{\frac{1}{\beta}} = 1.
\end{align*}

In order to compute the second term in expansion (5.4), let us begin by observing that, thanks to (5.2) and (5.1),

\begin{equation}
\tag{5.6}
B^{-1} \left( \frac{1}{\Phi(t)} \right) = (\log \Phi(t) - \log N)^{\frac{1}{\beta}} = \left( \frac{t^2}{2} + \log t + \cdots \right)^{\frac{1}{\beta}}
\end{equation}

\begin{equation}
= 2 - \frac{1}{2} t^2 + \frac{2}{\beta} \frac{1}{2} t^{2-\frac{2}{2}} \log t + \cdots \quad \text{as } t \to \infty,
\end{equation}

for some $M > 0$.
for every $\beta > 0$. Let us now distinguish the relevant three cases in equation (5.4). Assume first that $\beta \in (0, 2)$. Then $t^{2-\beta} \to 1$ as $t \to \infty$ and, by L'Hôpital's rule and equation (5.6),

$$\lim_{t \to \infty} \frac{J(t) - g(t)}{2-\beta t^{2}} \log t = \lim_{t \to \infty} \frac{B^{-1} \left( \frac{1}{\Phi(t)} \right) - 2^{-\frac{1}{2}} t^{\frac{\beta}{2}}}{t^{\frac{2}{\beta} - 1}} \log t + \frac{\beta}{2-\beta} \frac{t^{2}}{t^{\frac{2}{\beta} - 2}} = \frac{2}{\beta} 2^{-\frac{1}{2}} \log t.$$

If $\beta = 2$, then similarly, by (5.6),

$$\lim_{t \to \infty} \frac{J(t) - 2^{-\frac{1}{2}} t^{2}}{2 \log(\log t)^{2}} = \lim_{t \to \infty} \frac{B^{-1} \left( \frac{1}{\Phi(t)} \right) - \frac{1}{\sqrt{2}}}{t^{-1} \log t} = \frac{1}{\sqrt{2}}.$$

Finally, when $\beta \in (2, \infty)$,

$$\lim_{t \to \infty} \left( J(t) - 2^{-\frac{1}{2}} \frac{\beta}{2 + \beta} t^{\frac{\beta}{2} + 1} \right) = \lim_{t \to \infty} \left( \int_{0}^{t} B^{-1} \left( \frac{1}{\Phi(t)} \right) \, d\tau - 2^{-\frac{1}{2}} \int_{0}^{t} \tau^{\frac{\beta}{2}} \, d\tau \right)$$

$$= \int_{0}^{\infty} \left( B^{-1} \left( \frac{1}{\Phi(t)} \right) - 2^{-\frac{1}{2}} \tau^{\frac{\beta}{2}} \right) \, d\tau,$$

where the latter integral converges thanks to (5.6).

Let us now focus on (5.5). By equation (4.20), the functional $F_{m,n}$ can be written as the sum of three terms. The first one is, thanks to Lemma 5.2, of a lower order than the second one. The third term is just a constant. Therefore

$$[\kappa_{\beta} F_{m,n}(t)]^{\frac{2\alpha}{2\pi\beta}} = \left( \kappa_{\beta} 2^{-\frac{1}{2}} \frac{\beta}{2 + \beta} t^{\frac{\beta}{2} + 1} + \kappa_{\beta} \left( \frac{2^{-\frac{1}{2}} \frac{\beta}{2 + \beta} t^{\frac{\beta}{2} + 1}}{\sqrt{2}} \left( \log t \right)^{2} + \cdots \right) \right)^{\frac{2\alpha}{2\pi\beta}}$$

for some constant $c$, where the three cases in the brace correspond to $\beta \in (0, 2), \beta = 2$ and $\beta \in (2, \infty)$, respectively. The conclusion follows by (5.1). □

The following lemma tells us that, if $\beta \in (2, \infty)$, then a function $B$ as in Lemma 5.3 can be chosen in such a way that the constant $c'$ appearing on the right-hand side of equation (5.5) attains prescribed negative values.

**Lemma 5.4.** Assume that $\beta \in (2, \infty)$ and $\lambda > 0$. Then, given any $N > 0$, there exists a Young function $B$ satisfying condition (3.2) and such that

$$\int_{0}^{t} B^{-1} \left( \frac{1}{\Phi(t)} \right) \, d\tau \leq 2^{-\frac{1}{2}} \frac{\beta}{2 + \beta} t^{\frac{\beta}{2} + 1} - \lambda + \cdots \text{ as } t \to \infty.$$

Consequently, given any $\mu > 0$, the function $B$ can be chosen in such a way that

$$[\kappa_{\beta} F_{m,n}(t)]^{\frac{2\alpha}{2\pi\beta}} \leq \frac{t^{\beta}}{2} - \mu t^{\frac{\beta}{2}} + \cdots \text{ as } t \to \infty,$$

where $F_{m,n}$ is defined by (4.20) and $\kappa_{\beta}$ is given by (1.6).

**Proof.** Fix any $N > 0$ and define the function $A: [0, \infty) \to [0, \infty)$ as $A(t) = e^{\lambda t}$ for $t \geq 0$. Given $t_{0} > 0$, define the function $A: [0, \infty) \to [0, \infty)$ by

$$A(t) = \begin{cases} 
\frac{t^{2} A(t_{0})}{A(t_{0})} & \text{for } t \in [0, t_{0}) \\
A(t) & \text{for } t \in [t_{0}, \infty).
\end{cases}$$

Thus,

$$A^{-1}(\tau) = \begin{cases} 
\frac{\tau}{A(t_{0})} & \text{for } \tau \in [0, A(t_{0})] \\
A^{-1}(t) & \text{for } \tau \in [A(t_{0}), \infty).
\end{cases}$$

Also set $B = N A$. Then $B$ is a Young function, provided that $t_{0}$ is large enough, and

$$B^{-1}(\tau) = A^{-1}(\tau/N) \quad \text{for } \tau > 0.$$
We may assume that $t_0$ is so large that $NA(t_0) > 2$. Therefore,

$$\frac{1}{N\Phi(\tau)} \in [0, A(t_0)) \text{ for } \tau \in [0, \tau(t_0)), \$$

where we have set

$$\tau(t_0) = \Phi^{-1}\left(\frac{1}{NA(t_0)}\right). \tag{5.9}$$

Now, set $\tau_0 = 0$ if $N \in (0, 2]$ and $\tau_0 = \Phi^{-1}(1/N)$ if $N > 2$. Then,

$$\int_0^t B^{-1}\left(\frac{1}{\Phi(\tau)}\right) d\tau = \frac{t_0}{NA(t_0)} \int_0^{\tau(t_0)} \frac{d\tau}{\Phi(\tau)} + \int_{\tau(t_0)}^t \left(\log \frac{1}{N\Phi(\tau)}\right)^{\frac{1}{\beta}} d\tau$$

$$= \frac{t_0}{NA(t_0)} \int_0^{\tau(t_0)} \frac{d\tau}{\Phi(\tau)} + \int_{\tau_0}^t \left(\log \frac{1}{N\Phi(\tau)}\right)^{\frac{1}{\beta}} d\tau - \int_{\tau_0}^{\tau(t_0)} \left(\log \frac{1}{N\Phi(\tau)}\right)^{\frac{1}{\beta}} d\tau$$

for $t > \tau(t_0)$. Next, by Lemma 5.3,

$$\int_0^t \left(\log \frac{1}{N\Phi(\tau)}\right)^{\frac{1}{\beta}} d\tau = 2\frac{\beta}{2 + \beta} t^{2+1} + c(t), \tag{5.10}$$

where the function $c(t) \to c(\beta, N)$ as $t \to \infty$ and $c(\beta, N)$ is a constant depending only on $\beta$ and $N$. Thereby,

$$\int_0^t B^{-1}\left(\frac{1}{\Phi(\tau)}\right) d\tau = 2\frac{\beta}{2 + \beta} t^{2+1} + \lambda(t_0) + c(t), \tag{5.11}$$

where

$$\lambda(t_0) = \frac{t_0}{NA(t_0)} \int_0^{\tau(t_0)} \frac{d\tau}{\Phi(\tau)} - \int_{\tau_0}^{\tau(t_0)} \left(\log \frac{1}{N\Phi(\tau)}\right)^{\frac{1}{\beta}} d\tau. \tag{5.12}$$

Let us now analyze the asymptotic behavior of $\lambda(t_0)$ as $t_0 \to \infty$. By L’Hôpital’s rule,

$$\lim_{t_0 \to \infty} \frac{\int_0^{\tau(t_0)} \frac{d\tau}{\Phi(t)}}{\Phi(t_0)} = \lim_{t_0 \to \infty} \frac{1}{\Phi(t)} \frac{\Phi(t)}{\Phi(t_0)} = \lim_{t_0 \to \infty} \frac{1}{\frac{d}{dt}} \frac{\Phi(t)}{\Phi(t_0)} = 1,$$

where the last limit holds thanks to equation (5.3). Thus, by (5.9),

$$\frac{t_0}{NA(t_0)} \int_0^{\tau(t_0)} \frac{d\tau}{\Phi(\tau)} = \frac{t_0}{\tau(t_0)} + \cdots \text{ as } t_0 \to \infty. \tag{5.13}$$

Also, by expansion (5.2),

$$\Phi^{-1}(s) = \sqrt{2 \log \frac{1}{s} + \cdots} \text{ as } s \to 0^+ \tag{5.14}.$$

Therefore, by (5.9) and (5.14),

$$\tau(t_0) = \Phi^{-1}\left(\frac{1}{N e^{-t_0^2}}\right) = \sqrt{2t_0^2} + \cdots \text{ as } t_0 \to \infty. \tag{5.15}$$

Coupling equations (5.13) and (5.15) tells us that

$$\frac{t_0}{NA(t_0)} \int_0^{\tau(t_0)} \frac{d\tau}{\Phi(\tau)} = \frac{1}{\sqrt{2}} t_0^{1-\frac{\beta}{2}} + \cdots \text{ as } t_0 \to \infty. \tag{5.16}$$

Next, by equations (5.15) and (5.10),

$$\int_{t_0}^{\tau(t_0)} \left(\log \frac{1}{N\Phi(\tau)}\right)^{\frac{1}{\beta}} d\tau = 2\frac{\beta}{2 + \beta} \tau(t_0)^{2+1} + \cdots = \sqrt{2\frac{\beta}{2 + \beta}} t_0^{2+1} + \cdots \text{ as } t_0 \to \infty. \tag{5.17}$$
Finally, on combining equations (5.12), (5.16) and (5.17) one deduces that
\[ \lambda(t_0) = -\sqrt{2} \frac{\beta}{2 + \beta} \frac{2}{t_0^{\frac{\beta}{2}} + 1} + \cdots \text{ as } t_0 \to \infty. \]
This shows that \( \lambda(t_0) \to -\infty \) as \( t_0 \to \infty. \) Now, according to (5.11), given \( \lambda > 0, \) we may choose \( t_0 \) so large that \( \lambda(t_0) < -\lambda - c(\beta, N). \) Hence, equation (5.7) follows.

Let us now prove equation (5.8). The function \( F_{\sigma} \) agrees with the sum of the integral (5.7) with two more terms. The first additional term obeys
\[ e^{\frac{t^2}{2}} \int_{t}^{\infty} B^{-1} \left( \frac{1}{\Phi(\tau)} \right) e^{-\frac{\tau^2}{2}} d\tau \to 0 \text{ as } t \to \infty, \]
by Lemma 5.2, and the second term satisfies
\[ \int_{0}^{1} B^{-1} \left( \frac{1}{s} \right) \frac{ds}{s} = \int_{0}^{\frac{1}{A(t_0)}} A^{-1} \left( \frac{1}{sN} \right) \frac{ds}{s} + \frac{t_0}{NA(t_0)} \int_{\frac{1}{A(t_0)}}^{1} \frac{ds}{s} \]
\[ = \int_{0}^{\frac{1}{A(t_0)}} \log \left( \frac{1}{sN} \right) \frac{1}{s} ds + \frac{1}{N} t_0 e^{-\beta t_0^{\beta}} \log \left( Ne^{\beta t_0^{\beta}} \right). \]
Note that both addends on the rightmost side of the last equation approach 0 as \( t_0 \to \infty. \) Equation (5.8) then follows via (5.7) and (5.1). \( \square \)

In the remaining part of this section, we focus on an asymptotic estimate for the function \( F_{\sigma} \) given by (4.19). This is the content of Lemma 5.9. Its proof in its turn requires some preliminary asymptotic expansions that are the objective of a few lemmas.

Lemmas 5.5–5.7 below are stated without proofs. They can be derived via simple arguments relying upon L’Hôpital’s rule.

**Lemma 5.5.** Let \( \sigma \in \left[ -\frac{1}{2}, \infty \right) \) and \( d > 1. \) Define the function \( \Psi_{\sigma} : (d, \infty) \to [0, \infty) \) as
\[ \Psi_{\sigma}(t) = \int_{d}^{t} (\tau^2 - 1)^{\sigma} d\tau \text{ for } t > d. \]
If \( \sigma \in \left( -\frac{1}{2}, \infty \right), \) then
\[ \Psi_{\sigma}(t) = \frac{1}{2\sigma + 1} t^{2\sigma + 1} - \begin{cases} c + \cdots & \text{if } \sigma \in \left( -\frac{1}{2}, \frac{1}{2} \right) \\ \frac{1}{2} \log t + \cdots & \text{if } \sigma = \frac{1}{2} \end{cases} \text{ as } t \to \infty, \]
where \( c \in \mathbb{R} \) is a constant depending on \( \sigma \) and \( d. \)
If \( \sigma = -\frac{1}{2}, \) then
\[ \Psi_{\sigma}(t) = \log t + \cdots \text{ as } t \to \infty. \]

**Lemma 5.6.** Let \( \sigma \in \left[ -\frac{1}{2}, \infty \right) \) and let \( d > 1. \) Define the function \( \Upsilon_{\sigma} : (d, \infty) \to [0, \infty) \) as
\[ \Upsilon_{\sigma}(t) = \int_{d}^{t} (\tau^2 - 1)^{\sigma} \log(\tau^2 - 1) d\tau \text{ for } t > d. \]
Then
\[ \Upsilon_{\sigma}(t) = \begin{cases} \frac{2}{2\sigma + 1} t^{2\sigma + 1} \log t + \cdots & \text{if } \sigma \in \left( -\frac{1}{2}, \infty \right) \\ (\log t)^2 + \cdots & \text{if } \sigma = -\frac{1}{2} \end{cases} \text{ as } t \to \infty. \]

**Lemma 5.7.** Let \( B \) be a Young function obeying (3.2) for some \( N > 0, \) and let \( b : [0, \infty) \to [0, \infty) \) be the left-continuous function such that \( B(t) = \int_{0}^{t} b(\tau) d\tau \) for \( t > 0. \) Then
\[ b^{-1}(t) = (\log t)^{\frac{1}{2}} + \frac{1 - \beta}{\beta^2} (\log t)^{\frac{1}{2} - 1} \log t + \cdots \text{ as } t \to \infty. \]
Lemma 5.8. Let $A$ be a Young function and let $(\mathcal{R}, \nu)$ be a probability space. Then

\begin{equation}
\|\phi\|_{L^A} \leq \inf_{k > 0} \left\{ \frac{1}{k} + \frac{1}{k} \int_{\mathcal{R}} A(k|\phi|) \, d\nu \right\}.
\end{equation}

\textbf{Proof.} By Young’s inequality (2.3),

\begin{equation}
\int_{\mathcal{R}} |\phi\psi| \, d\nu \leq \frac{1}{k} \int_{\mathcal{R}} \tilde{A}(|\psi|) \, d\nu + \frac{1}{k} \int_{\mathcal{R}} A(k|\phi|) \, d\nu \quad \text{for } k > 0.
\end{equation}

Therefore

\begin{equation}
\|\phi\|_{L^A} = \sup \left\{ \int_{\mathcal{R}} |\phi\psi| \, d\nu : \int_{\mathcal{R}} \tilde{A}(|\psi|) \, d\nu \leq 1 \right\} \leq \frac{1}{k} + \frac{1}{k} \int_{\mathcal{R}} A(k|\phi|) \, d\nu \quad \text{for } k > 0,
\end{equation}

whence (5.22) follows. \hfill \Box

Lemma 5.9. Let $B$ be a Young function satisfying condition (3.2) for some constant $N > 0$. Then

\begin{equation}
\left\| \frac{1}{I} \right\|_{L^B(\Phi(t), \frac{1}{2})} \leq 2 - \frac{\beta}{2 + \beta} t^{\pi + 1} - 2^{-\frac{\beta}{2}} \left\{ \begin{array}{ll}
\frac{2}{\beta \beta} t^{\beta - 1} \log t + \cdots & \text{if } \beta \in (0, 2) \\
\frac{1}{2} (\log t)^2 + \cdots & \text{if } \beta = 2 \\
\frac{1}{\beta} \log \log t + \cdots & \text{if } \beta \in (2, \infty)
\end{array} \right\} \text{ as } t \to \infty,
\end{equation}

for a suitable constant $c = c(B) \in \mathbb{R}$. Consequently,

\begin{equation}
\left[ k_t \mathcal{F}_{LB}(t) \right]_{\frac{2\pi}{\beta}} \leq \frac{t^2}{2} \left\{ \begin{array}{ll}
\frac{2}{\beta \beta} \log t + \cdots & \text{if } \beta \in (0, 2) \\
\frac{1}{2} (\log t)^2 + \cdots & \text{if } \beta = 2 \\
\frac{1}{\beta} \log \log t + \cdots & \text{if } \beta \in (2, \infty)
\end{array} \right\} \text{ as } t \to \infty,
\end{equation}

where $\mathcal{F}_{LB}$ is defined by (4.19) and $c' = c'(B) \in \mathbb{R}$ is a suitable constant.

\textbf{Proof.} By Lemma 5.8,

\begin{equation}
\left\| \frac{1}{I} \right\|_{L^B(\Phi(t), \frac{1}{2})} \leq \inf_{k > 0} \left\{ \frac{1}{k} + \frac{1}{k} \int_{\Phi(t)}^{\frac{1}{2}} \tilde{B} \left( k \frac{1}{I(s)} \right) \, ds \right\},
\end{equation}

whence, by the change of variables $s = \Phi(\tau),$

\begin{equation}
\left\| \frac{1}{I} \right\|_{L^B(\Phi(t), \frac{1}{2})} \leq \inf_{k > 0} \left\{ \frac{1}{k} + \frac{1}{k} \sqrt{2\pi} \int_{0}^{t} \tilde{B} \left( k \sqrt{2\pi e^2} e^{-\tau} \right) e^{-\tau} \, d\tau \right\}.
\end{equation}

Choose $k = \frac{1}{\sqrt{2\pi} e^{-\frac{\sigma(t)^2}{2}}} \in \text{the expression in braces on the right-hand side of inequality (5.25), where } \sigma: (e, \infty) \to (0, \infty) \text{ is the function defined as}

\begin{equation}
\sigma(t) = \left\{ \begin{array}{ll}
\sqrt{2} \left( \frac{2}{\beta} - 1 \right) \log t & \text{if } \beta \in (0, 2) \\
\sqrt{2} \log \log t & \text{if } \beta = 2 \\
1 & \text{if } \beta \in (2, \infty)
\end{array} \right. \text{ for } t > e.
\end{equation}

Notice that

\begin{equation}
\lim_{t \to \infty} \frac{t}{\sigma(t)} = \infty.
\end{equation}

This choice of $k$ in (5.25) yields

\begin{equation}
\left\| \frac{1}{I} \right\|_{L^B(\Phi(t), \frac{1}{2})} \leq \sqrt{2\pi} e^{\frac{\sigma(t)^2}{2}} + e^{\frac{\sigma(t)^2}{2}} \int_{0}^{t} \tilde{B} \left( e^{-\frac{\sigma(t)^2}{2}} \right) e^{-\tau} \, d\tau \quad \text{for } t > e.
\end{equation}

Owing to Lemma 5.7 and the fact that $\tilde{B}(t) \leq t^{-1}(t)$ for $t > 0$, one has that

\begin{equation}
\tilde{B}(t) \leq t(\log t)^{\frac{1}{2}} + \frac{1 - \beta}{\beta^2} t(\log t)^{\frac{1}{2} - 1} \log t + \cdots \text{ as } t \to \infty.
\end{equation}
Consequently, given \( \varepsilon > 0 \), there exists a constant \( C > 0 \) such that
\[
\tilde{B}(t) \leq t(\log t)^{\frac{1}{2}} + K_\varepsilon \frac{1 - \beta}{\beta^2} t(\log t)^{\frac{1}{2}-1} \log \log t + C \quad \text{for } t > e,
\]
where \( K_\varepsilon = 1 + \varepsilon \) if \( \beta \in (0, 1] \) and \( K_\varepsilon = 1 - \varepsilon \) if \( \beta \in (1, \infty) \). On enlarging, if necessary, the value of \( C \), we may also assume that \( \tilde{B}(t) \leq C \) for \( t \in (0, e) \). From inequalities (5.28) and (5.29), we infer that
\[
\left\| \frac{1}{I} \right\|_{L^\infty(\Phi(t), \frac{1}{t})} \leq \sqrt{2\pi}(1 + C)e^{\frac{\sigma(t)^2}{2}} + J_1(t) + K_\varepsilon \frac{1 - \beta}{\beta^2} J_2(t),
\]
where we have set
\[
J_1(t) = \int_{\sigma(t)}^t \left( \frac{\tau^2}{2} - \frac{\sigma(t)^2}{2} \right)^{\frac{1}{2}} d\tau
\]
and
\[
J_2(t) = \int_{\sigma(t)}^t \left( \frac{\tau^2}{2} - \frac{\sigma(t)^2}{2} \right)^{\frac{1}{2}-1} \log \left( \frac{\tau^2}{2} - \frac{\sigma(t)^2}{2} \right) d\tau
\]
for \( t > e \). Let us estimate the term \( J_1 \). By a change of variables, we obtain that
\[
J_1(t) = \int_{\frac{\sigma(t)^2}{2}}^{\frac{t^2}{2}} \frac{1}{\sigma(t)} \left( \frac{t^2}{2} - \frac{\sigma(t)^2}{2} \right)^{\frac{1}{2}-1} d\sigma(t) = 2^{-\frac{1}{2}} \sigma(t) \int_{1}^{\frac{t^2}{2}} (\tau^2 - 1)^{\frac{1}{2}} d\tau
\]
where \( \Psi_{\frac{1}{2}} \) is defined as in (5.18). From Lemma 5.5 and equation (5.27) one can infer that
\[
\Psi_{\frac{1}{2}} \left( \frac{t}{\sigma(t)} \right) = \frac{\beta}{2 + \beta} \left( \frac{t}{\sigma(t)} \right)^{\frac{2}{2}+1} - \left\{ \begin{array}{ll}
\frac{1}{2 - \beta} \left( \frac{t}{\sigma(t)} \right)^{\frac{2}{2}-1} + \cdots & \text{if } \beta \in (0, 2) \\
\frac{1}{2} \log \left( \frac{t}{\sigma(t)} \right) + \cdots & \text{if } \beta = 2, \text{ as } t \to \infty.
\end{array} \right.
\]
Coupling (5.31) with (5.32) yields
\[
J_1(t) = 2^{-\frac{1}{2}} \frac{\beta}{2 + \beta} t^{\frac{2}{2}+1} - 2^{-\frac{1}{2}} \left\{ \begin{array}{ll}
\frac{1}{2 - \beta} t^{\frac{2}{2}-1} \sigma(t)^2 + \cdots & \text{if } \beta \in (0, 2) \\
\frac{1}{2} \sigma(t)^2 \log \left( \frac{t}{\sigma(t)} \right) + \cdots & \text{if } \beta = 2, \text{ as } t \to \infty.
\end{array} \right.
\]
Equation (5.26) and estimates (5.33) tell us that
\[
J_1(t) = 2^{-\frac{1}{2}} \frac{\beta}{2 + \beta} t^{\frac{2}{2}+1} - 2^{-\frac{1}{2}} \left\{ \begin{array}{ll}
\frac{1}{2 - \beta} t^{\frac{2}{2}-1} \log t + \cdots & \text{if } \beta \in (0, 2) \\
\frac{1}{2} \log t \log \log t + \cdots & \text{if } \beta = 2, \text{ as } t \to \infty.
\end{array} \right.
\]
Let us next consider \( J_2 \). If \( \beta \in (0, 2] \), then, by a change of variables,
\[
J_2(t) = \sigma(t) \left( \frac{\sigma(t)^2}{2} \right)^{\frac{1}{2}-1} \int_{\frac{\sigma(t)^2}{2}}^{\frac{t^2}{2}} \frac{1}{\sigma(t)} (\tau^2 - 1)^{\frac{1}{2}-1} \log \left( \frac{\sigma(t)^2}{2} + \log(\tau^2 - 1) \right) d\tau
\]
\[
= 2^{\frac{1}{2} - \frac{1}{2}} \sigma(t)^{\frac{2}{2}-1} \left( \log \left( \frac{\sigma(t)^2}{2} \right) \right) \left( \log \left( \frac{t}{\sigma(t)} \right) \right) + \Psi_{\frac{1}{2}} \left( \frac{t}{\sigma(t)} \right) \text{ for } t > e.
\]
Here, $\Upsilon_{\frac{1}{\beta}-1}$ is defined according to (5.20). Thanks to Lemmas 5.5 and 5.6 and equation (5.27) one has that
\[
\Psi_{\frac{1}{\beta}-1}\left(\frac{t}{\sigma(t)}\right) = \begin{cases} 
\frac{\beta}{2-\beta} \left(\frac{t}{\sigma(t)}\right)^{\frac{2}{2-\beta}} + \cdots & \text{if } \beta \in (0, 2) \\
\log \left(\frac{t}{\sigma(t)}\right) + \cdots & \text{if } \beta = 2 
\end{cases} \quad \text{as } t \to \infty
\]
and
\[
\Upsilon_{\frac{1}{\beta}-1}\left(\frac{t}{\sigma(t)}\right) = \begin{cases} 
\frac{2\beta}{2-\beta} \left(\frac{t}{\sigma(t)}\right)^{\frac{2}{2-\beta}} \log \left(\frac{t}{\sigma(t)}\right) + \cdots & \text{if } \beta \in (0, 2) \\
\left(\frac{\log \left(\frac{t}{\sigma(t)}\right)}{2}\right)^2 + \cdots & \text{if } \beta = 2 
\end{cases} \quad \text{as } t \to \infty.
\]
Consequently, on making use of (5.26), one deduces that
\[
J_2(t) = \begin{cases} 
2^{-\frac{\beta}{4}} \frac{\beta}{\pi \beta} t^{\frac{2}{2-\beta}} - 1 \log t + \cdots & \text{if } \beta \in (0, 2) \\
2 \frac{t}{(\log t)^2} + \cdots & \text{if } \beta = 2 
\end{cases} \quad \text{as } t \to \infty.
\]
If $\beta \in (2, \infty)$, one can verify that $J_2(t)$ has a finite limit as $t \to \infty$. Furthermore,
\[
e^{-\frac{\sigma(t)^2}{2}} = \begin{cases} 
t^{\frac{2}{2-\beta}} - 1 & \text{if } \beta \in (0, 2) \\
\log t & \text{if } \beta = 2 \\
\frac{1}{\sigma}\log^t & \text{if } \beta \in (2, \infty)
\end{cases} \quad \text{for } t > e.
\]
Therefore, equations (5.30), (5.34), (5.35) and (5.36) enable us to conclude that, if $\beta \in (0, 1]$, then
\[
\left\| \frac{1}{T} \right\|_{L^B(\Phi(t), \frac{1}{2})} \leq 2^{-\frac{\beta}{2}} \frac{\beta}{2+\beta} t^{\frac{2}{2+\beta}} - 2^{-\frac{1}{2}} \frac{2}{\beta} t^{\frac{2}{2-\beta}} - 1 \log t + (1 + \varepsilon) \frac{1}{\beta^2} 2^{-\frac{1}{2}} \frac{4\beta}{2-\beta} t^{\frac{2}{2-\beta}} \log t + \cdots
\]
\[
= 2^{-\frac{\beta}{2}} \frac{\beta}{2+\beta} t^{\frac{2}{2+\beta}} - 2^{-\frac{1}{2}} \frac{2}{\beta} t^{\frac{2}{2-\beta}} - 1 \log t + (1 + \varepsilon) \frac{1}{\beta^2} 2^{-\frac{1}{2}} \frac{4\beta}{2-\beta} t^{\frac{2}{2-\beta}} \log t + \cdots \quad \text{as } t \to \infty,
\]
and that, if $\beta \in (1, 2)$, then
\[
\left\| \frac{1}{T} \right\|_{L^B(\Phi(t), \frac{1}{2})} \leq 2^{-\frac{\beta}{2}} \frac{\beta}{2+\beta} t^{\frac{2}{2+\beta}} - 2^{-\frac{1}{2}} \frac{2}{\beta} t^{\frac{2}{2-\beta}} - 1 \log t + (1 + \varepsilon) \frac{1}{\beta^2} 2^{-\frac{1}{2}} \frac{4\beta}{2-\beta} t^{\frac{2}{2-\beta}} \log t + \cdots \quad \text{as } t \to \infty.
\]
Thus, thanks to the arbitrariness of $\varepsilon$, equation (5.23) follows in the case when $\beta \in (0, 2)$. If $\beta = 2$, then, by (5.30), (5.34), (5.35) and (5.36),
\[
\left\| \frac{1}{T} \right\|_{L^B(\Phi(t), \frac{1}{2})} \leq 2^{-\frac{\beta}{2}} t^2 - 1 + \varepsilon \frac{1}{4} 2^{\frac{1}{2}} \log t + \cdots \quad \text{as } t \to \infty.
\]
Hence, the arbitrariness of $\varepsilon$ enables us to deduce (5.23) for $\beta = 2$. Finally, If $\beta \in (2, \infty)$, then equation (5.23) holds by (5.30), (5.34), and by the boundedness of $J_2$.

By definition (4.19), estimate (5.24) follows from equations (5.23) and (5.1).

6. Proofs of the main results

The proofs of our main results exploit relations between the assumption in integral form (1.3) and that in norm form (3.1), and the relations in (2.9) between strong and weak norms. Some steps of such proofs are stated as separate intermediate results, in order to avoid unnecessary repetitions.

The next three lemmas provide us with links between conditions (1.3) and (3.1).

Lemma 6.1. Let $\beta > 0$. Assume that $B$ is a Young function satisfying condition (3.2) for some $N \in (0, 1)$. Then there exists a constant $M > 1$ such that
\[
\|\phi\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq 1
\]
for every function \( \phi \in \mathcal{M}(\mathbb{R}^n) \) fulfilling

\[
(6.2) \quad \int_{\mathbb{R}^n} e^{\|\phi\|} \, d\gamma_n \leq M.
\]

**Proof.** Denote by \( E : [0, \infty) \to [0, \infty) \) the convex envelope of the function \( e^{\|\phi\|} - 1 \). Namely, \( E \) is the largest convex function not exceeding \( e^{\|\phi\|} - 1 \) on \( [0, \infty) \). Given \( M > 1 \), define the function \( B_M : [0, \infty) \to [0, \infty) \) as

\[
B_M(t) = \frac{E(t)}{M - 1} \quad \text{for } t \in [0, \infty).
\]

Observe, that \( B_M \) is a Young function. Now, if \( \phi \) fulfills (6.2), then

\[
\int_{\mathbb{R}^n} B_M(\|\phi\|) \, d\gamma_n \leq \frac{1}{M - 1} \int_{\mathbb{R}^n} (e^{\|\phi\|} - 1) \, d\gamma_n \leq 1,
\]

whence, by the definition of Luxemburg norm, \( \|\phi\|_{\mathcal{L}\mathcal{B}M(\mathbb{R}^n, \gamma_n)} \leq 1 \). Owing to Hölder’s inequality and to (2.5),

\[
(6.3) \quad \int_{\mathbb{R}^n} |\phi| \, d\gamma_n \leq \|\phi\|_{\mathcal{L}\mathcal{B}M(\mathbb{R}^n, \gamma_n)} \|1\|_{\mathcal{L}\mathcal{B}M(\mathbb{R}^n, \gamma_n)} \leq B^{-1}_M(1).
\]

Notice that

\[
(6.4) \quad \lim_{M \to 1^+} B^{-1}_M(1) = 0,
\]

inasmuch as \( E(t) \to 0 \) as \( t \to 0^+ \). Now, assume that \( B \) obeys (3.2) for \( t \in (t_0, \infty) \). Since \( B \) is convex and vanishes at 0,

\[
B(t) \leq N \begin{cases} t^{\frac{e^{t_0}}{t_0}} & \text{for } t \in [0, t_0) \\ e^{t_0} & \text{for } t \in [t_0, \infty). \end{cases}
\]

Therefore, by (6.2) and (6.3),

\[
\int_{\mathbb{R}^n} B(\|\phi\|) \, d\gamma_n \leq N \frac{e^{t_0}}{t_0} \int_{\{\|\phi\| \leq t_0\}} |\phi| \, d\gamma_n + N \int_{\{\|\phi\| > t_0\}} e^{\|\phi\|} \, d\gamma_n
\]

\[
\leq N \frac{e^{t_0}}{t_0} \int_{\mathbb{R}^n} |\phi| \, d\gamma_n + N \int_{\mathbb{R}^n} e^{\|\phi\|} \, d\gamma_n = N \left( \frac{e^{t_0}}{t_0} B^{-1}_M(1) + M \right).
\]

Observe, that, owing to (6.4), the expression in brackets on the rightmost side converges to 1 as \( M \) tends to \( 1^+ \). Consequently, since \( N \in (0, 1) \), \( M \) can be chosen so close to 1 that

\[
N \left( \frac{e^{t_0}}{t_0} (\log M)^{\frac{1}{\beta}} + M \right) \leq 1,
\]

whence (6.1) follows. \( \square \)

**Lemma 6.2.** Let \( \beta > 0 \). Assume that \( B \) is a Young function satisfying condition (3.2) for some \( N > 0 \). Then there exists a constant \( M > 1 \) such that inequality (6.2) holds for every function \( \phi \in \mathcal{M}(\mathbb{R}^n) \) fulfilling condition (6.1).

**Proof.** Let \( t_0 \in (0, \infty) \) be such that \( B \) fulfills condition (3.2) for \( t \in (t_0, \infty) \). By the definition of Luxemburg norm, assumption (6.1) is equivalent to

\[
\int_{\mathbb{R}^n} B(\|\phi\|) \, d\gamma_n \leq 1.
\]

Thereby

\[
\int_{\mathbb{R}^n} e^{\|\phi\|} \, d\gamma_n \leq \int_{\mathbb{R}^n} e^{t_0} \, d\gamma_n + \int_{\{\|\phi\| \geq t_0\}} e^{\|\phi\|} \, d\gamma_n \leq e^{t_0} + \frac{1}{N} \int_{\mathbb{R}^n} B(\|\phi\|) \, d\gamma_n = e^{t_0} + \frac{1}{N}.
\]

The conclusion follows by choosing \( M = e^{t_0} + 1/N \). \( \square \)
Lemma 6.3. Let $\beta > 0$ and $M > 1$. Then there exists a Young function $B$ satisfying condition (3.2) for some $N > 0$, and such that inequality (6.1) holds for every function $\phi \in M(\mathbb{R}^n)$ obeying condition (6.2).

Proof. Given $t_0 > 0$, define the function $A: [0, \infty) \to [0, \infty)$ by

$$A(t) = \begin{cases} \frac{t e^{\beta t}}{e^{\beta t_0}} & \text{for } t \in [0, t_0) \\ t e^{\beta t} & \text{for } t \in [t_0, \infty). \end{cases}$$

Clearly, $t_0$ can be chosen large enough for $A$ to be convex. Set $N = 1/(M + e^{\beta})$ and let $B = NA$. We claim that $B$ is a Young function with the required properties. Indeed, if $\phi$ is any function obeying (6.2), then

$$\int_{\mathbb{R}^n} B(|\phi|) d\gamma_n \leq N \int_{\{|\phi| \geq t_0\}} |\phi| d\gamma_n + N \int_{\{|\phi| < t_0\}} e^{\beta t_0} d\gamma_n \leq N \left(M + e^{\beta}t_0\right) = 1.$$ 

Hence, inequality (6.1) follows by the very definition of Luxemburg norm. \hfill \square

Propositions 6.4 and 6.5 below are of use in the proof of part (2.ii) of Theorems 3.1 and 3.2.

Proposition 6.4. Let $\beta \in (2, \infty)$. Then, given any $N > 0$, there exist a Young function $B$ satisfying condition (3.2) for some $t_0$, and a constant $C = C(\beta, N, t_0)$ such that

$$\int_{\mathbb{R}^n} e^{(\kappa_{\beta} |u|)^{2/\beta}} d\gamma_n \leq C$$

for every function $u \in W^{1, \exp} L^\beta(\mathbb{R}^n, \gamma_n)$ fulfilling (1.2) and (3.4).

Proof. Fix any $\mu > 0$ and let $B$ be a Young function for which equation (5.8) holds. By Lemma 4.5, for any $\kappa_{\beta}, t$, we have

$$\left[\kappa_{\beta} F_{mB}(t)\right]^{2/\beta} - \frac{t^2}{2} \leq -\mu t^{1-\frac{2}{\beta}} + \cdots \quad \text{as } t \to \infty.$$

This ensures that the integral on the right-hand side of (6.6) converges. Inequality (6.5) thus follows from (6.6). \hfill \square

Proposition 6.5. Let $\beta \in (2, \infty)$. Then given any $N > 0$, there exist a Young function $B$ satisfying condition (3.2) and a sequence of functions $\{u_k\} \subset W^{1, \exp} L^\beta(\mathbb{R}^n, \gamma_n)$, such that $\text{med}(u_k) = \text{mv}(u_k) = 0$ and

$$\|\nabla u_k\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq 1$$

for $k \in \mathbb{N}$, satisfying

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} e^{(\kappa_{\beta} |u_k|)^{2/\beta}} d\gamma_n = \infty.$$

Proof. Set $A(t) = e^{\beta t}$ for $t \geq 0$ and let $t_0 > 0$. Define the function $A: [0, \infty) \to [0, \infty)$ by

$$A(t) = \begin{cases} 0 & \text{for } t \in [0, t_0) \\ a(t_0)(t - t_0) + A(t_0) & \text{for } t \in [t_0, t_0) \\ A(t) & \text{for } t \in [t_0, \infty), \end{cases}$$

where $a$ denotes the derivative of $A$ and

$$t_0' = t_0 - \frac{A(t_0)}{a(t_0)}.$$
Notice that \( t'_0 \in (0, t_0) \). Given \( N > 0 \), set \( B = N A \). Clearly \( B(t) = N e^{t^\beta} \) for \( t > t_0 \). On denoting by \( a \) the left-continuous function such that \( A(t) = \int_0^t a(\tau) \, d\tau \) for \( t \geq 0 \), one has that

\[
a(t) = \begin{cases} 0 & \text{for } t \in [0, t'_0) \\ a(t_0) & \text{for } t \in [t'_0, t_0) \\ a(t) & \text{for } t \in [t_0, \infty) \end{cases}
\]
and \( a^{-1}(\tau) = \begin{cases} 0 & \text{for } \tau = 0 \\ t'_0 & \text{for } \tau \in (0, a(t_0)] \\ a^{-1}(\tau) & \text{for } \tau \in (a(t_0), \infty) \end{cases} \).

For each \( k \in \mathbb{N} \), define the function \( u_k : \mathbb{R}^n \to \mathbb{R} \) by

\[
u_k(x) = \text{sgn} \, x_1 \left\{
\begin{array}{ll}
\int_0^{\frac{|x_1|}{k}} a^{-1}\left( e^{\frac{\tau^2}{2}} \right) \, d\tau & \text{for } |x_1| < k \\
0 & \text{for } |x_1| \geq k.
\end{array}
\right.
\]

Clearly \( \text{med}(u_k) = \text{mv}(u_k) = 0 \) for \( k \in \mathbb{N} \). Next, set

\[
\tau(t_0) = \sqrt{2} \log a(t_0).
\]

If \( k > \tau(t_0) \), then

\[
|\nabla u_k(x)| = \begin{cases} a^{-1}\left( e^{\frac{|x_1|^2}{2}} \right) & \text{for } |x_1| < k \\
0 & \text{for } |x_1| \geq k
\end{cases} = \begin{cases} t'_0 & \text{for } |x_1| < \tau(t_0) \\
a^{-1}\left( e^{\frac{|x_1|^2}{2}} \right) & \text{for } \tau(t_0) < |x_1| < k \\
0 & \text{for } k < |x_1|.
\end{cases}
\]

Therefore,

\[
\int_{\mathbb{R}^n} B(|\nabla u_k|) \, d\gamma_n = \frac{2N}{\sqrt{2\pi}} \int_0^{t'_0} A\left( a^{-1}\left( e^{\frac{\tau^2}{2}} \right) \right) e^{-\frac{\tau^2}{2}} \, d\tau
\]

\[
= N \sqrt{2/\pi} \int_0^{\tau(t_0)} A(t_0) e^{-\frac{\tau^2}{2}} \, d\tau + N \sqrt{2/\pi} \int_{\tau(t_0)}^{t'_0} A\left( a^{-1}\left( e^{\frac{\tau^2}{2}} \right) \right) e^{-\frac{\tau^2}{2}} \, d\tau.
\]

By the definition of \( A \), one has that \( A(t'_0) = 0 \). Thus the first integral on the rightmost side of the last equation vanishes. On the other hand, we have that

\[
A(t) = Ne^{t^\beta} = \frac{1}{\beta} t^{1-\beta} a(t) \quad \text{for } t > 0,
\]

whence, owing to Lemma 5.7,

\[
A(a^{-1}(t)) = \frac{1}{\beta} t \left( a^{-1}(t) \right)^{1-\beta} = \frac{1}{\beta} t (\log t)^{\frac{1}{2}-1} \cdots \quad \text{as } t \to \infty
\]

and the second integral converges as \( k \to \infty \) since \( \beta \in (2, \infty) \). Thus, if \( k > \tau(t_0) \), then

\[
\int_{\mathbb{R}^n} B(|\nabla u_k|) \, d\gamma_n \leq M(t_0),
\]

where we have set

\[
M(t_0) = N \sqrt{2/\pi} \int_{\tau(t_0)}^{\infty} A\left( a^{-1}\left( e^{\frac{\tau^2}{2}} \right) \right) e^{-\frac{\tau^2}{2}} \, d\tau.
\]

Since \( M(t_0) \) tends to 0 as \( t_0 \to \infty \), we may choose \( t_0 \) sufficiently large that \( \int_{\mathbb{R}^n} B(|\nabla u_k|) \, d\gamma_n \leq 1 \). Hence, inequality (6.7) holds for \( k > \tau(t_0) \).

Let us now focus on equation (6.8). We have that

\[
\int_{\mathbb{R}^n} e^{(\kappa_\beta |u_k|)} \frac{2\beta}{2 + \beta} \, d\gamma_n \geq \int_{\{ |x_1| > k \}} e^{(\kappa_\beta |x_1|)} \frac{2\beta}{2 + \beta} \, d\gamma_n = 2\Phi(k) \exp \left\{ \left( \kappa_\beta \int_0^{t'_0} a^{-1}\left( e^{\frac{\tau^2}{2}} \right) \, d\tau \right)^{\frac{2\beta}{2 + \beta}} \right\}.
\]
Next,
\[ \int_0^k a^{-1}(e^{\tau^2}) \, d\tau = \left( t_0 - \frac{A(t_0)}{a(t_0)} \right) \int_{\tau(t_0)}^t \, d\tau + \int_{\tau(t_0)}^k a^{-1}(e^{\tau^2}) \, d\tau. \]

By Lemma 5.7, there exists \( \tau_0 > 0 \) such that
\[ a^{-1}(\tau) \geq (\log \tau)^{\frac{1}{\beta}} - (\log \tau)^{\frac{1}{\beta} - 1} \log \log \tau \quad \text{for} \quad \tau > \tau_0. \]

Assume, in addition, that \( t_0 \) obeys \( \tau(t_0) > \tau_0. \) Then
\[
\int_{\tau(t_0)}^k a^{-1}(e^{\tau^2}) \, d\tau \geq \int_{\tau(t_0)}^k \left( \frac{\tau^2}{2} \right)^{\frac{1}{\beta}} \, d\tau - \int_{\tau(t_0)}^k \left( \frac{\tau^2}{2} \right)^{\frac{1}{\beta} - 1} \log \frac{\tau^2}{2} \, d\tau
\]
\[
\geq 2 - \frac{\beta}{2 + \beta} k^{\frac{2}{\beta} + 1} - \sqrt{2} \frac{\beta}{2 + \beta} (\log a(t_0)) \frac{1}{\beta} + \frac{1}{\beta} - \int_{\tau(t_0)}^\infty \left( \frac{\tau^2}{2} \right)^{\frac{1}{\beta} - 1} \log \frac{\tau^2}{2} \, d\tau,
\]
since the last integral converges. Consequently, if \( k > \tau(t_0) \), then
\[
\int_{0}^{k} a^{-1}(e^{\tau^2}) \, d\tau \geq 2 - \frac{\beta}{2 + \beta} k^{\frac{2}{\beta} + 1} + \lambda(t_0),
\]
where we have set
\[ (6.10) \quad \lambda(t_0) = \sqrt{2} \left( t_0 - \frac{A(t_0)}{a(t_0)} \right) (\log a(t_0))^{\frac{1}{\beta}} - \sqrt{2} \frac{\beta}{2 + \beta} (\log a(t_0))^{\frac{1}{\beta} + \frac{1}{\beta} + 1} - \int_{\tau(t_0)}^\infty \left( \frac{\tau^2}{2} \right)^{\frac{1}{\beta} - 1} \log \frac{\tau^2}{2} \, d\tau. \]

Let us analyze the behavior of \( \lambda(t_0) \) as \( t_0 \to \infty \). One has that
\[ t_0 (\log a(t_0))^{\frac{1}{\beta}} = t_0^\beta + (\beta - 1) \log t_0 + \log \beta)^{\frac{1}{\beta} + 1} \]
\[ \quad \text{as} \quad t_0 \to \infty \]
and
\[ (\log a(t_0))^{\frac{1}{\beta} + \frac{1}{\beta} + 1} = (t_0^\beta + (\beta - 1) \log t_0 + \log \beta)^{\frac{1}{\beta} + \frac{1}{\beta} + 1} \]
\[ \quad \text{as} \quad t_0 \to \infty. \]

The remaining terms on the right-hand side of (6.10) are of a lower order, since both \( A(t_0)/a(t_0) \) and the integral approach 0 as \( t_0 \to \infty \). Thus,
\[ \lambda(t_0) = \sqrt{2} \frac{2}{2 + \beta} k^{\frac{2}{\beta} + 1} + \cdots \quad \text{as} \quad t_0 \to \infty. \]

As a consequence, given any \( \lambda > 0 \), we may choose \( t_0 \) so large that \( \lambda(t_0) > \lambda \). This choice ensures that
\[ \left( \kappa \int_0^k a^{-1}(e^{\tau^2}) \, d\tau \right)^{\frac{2\beta}{\beta + 1}} \geq \left( \kappa \beta^2 - \frac{\beta}{2 + \beta} k^{\frac{2}{\beta} + 1} + \kappa \lambda \right)^{\frac{2\beta}{\beta + 1}} = \frac{k^2}{2} + \frac{\lambda}{\sqrt{\beta}} + \cdots \quad \text{as} \quad k \to \infty. \]

Therefore, by inequality (6.9) and relation (5.2),
\[ \int_{\mathbb{R}^n} e(\kappa\beta |x_k| \frac{2\beta}{\beta + 1}) d\gamma_n \geq \exp \left\{ \log \Phi(k) + \left( \kappa \beta \int_0^k a^{-1}(e^{\tau^2}) \, d\tau \right)^{\frac{2\beta}{\beta + 1}} \right\} \]
\[ \geq \exp \left\{ \left( -\frac{k^2}{2} - \log k + \cdots \right) + \left( \frac{k^2}{2} + \frac{\lambda}{\sqrt{\beta}} k^{\frac{1}{\beta} - \frac{1}{\beta}} + \cdots \right) \right\} \]
\[ = \exp \left\{ 2^{\frac{3}{\beta}} \lambda k^{\frac{1}{\beta} - \frac{1}{\beta}} + \cdots \right\}. \]

Equation (6.8) follows, since the rightmost side of equation (6.11) tends to infinity as \( k \to \infty. \)

The next proposition implies part (1.ii) of Theorem 3.2.
Proposition 6.6. Let $\beta \in (0, 2]$. Assume that $B$ is a Young function satisfying condition (3.2) for some $N > 0$. Then there exists a function $u \in W^1 \exp L^\beta(\mathbb{R}^n, \gamma_n)$, fulfilling condition (3.3), such that $\text{med}(u) = \text{mv}(u) = 0$ and

$$
(6.12) \quad \int_{\mathbb{R}^n} e^{(\kappa s|u|)^{2\beta} \sqrt{n}} \, d\gamma_n = \infty.
$$

Proof. Let $\tau_0$ be such that $B(\tau) = Ne^{\tau^\beta}$ for $\tau \in (\tau_0, \infty)$. Set

$$
(6.13) \quad s_0 = \min \left\{ \frac{1}{2}, \frac{1}{N} e^{-\tau_0^\beta} \right\}
$$

and $t_0 = \Phi^{-1}(s_0)$. Define the function $u : \mathbb{R}^n \to \mathbb{R}$ by

$$
(6.13) \quad u(x) = \text{sgn} x_1 \begin{cases} 
\int_{\Phi(t_0)}^{\Phi(0)} \frac{g(s)}{I(s)} \, ds & \text{for } |x_1| \geq t_0 \\
0 & \text{for } |x_1| < t_0,
\end{cases}
$$

where $g : (0, s_0) \to [0, \infty)$ is the function given by

$$
g(s) = \left( \log \frac{1}{Ns} \right)^{\frac{1}{\beta}} - \frac{1}{\beta} \left( \log \frac{1}{Ns} \right)^{\frac{1}{\beta} - 1} \quad \text{for } s \in (0, s_0).
$$

Observe that $g$ is decreasing, provided that $N$ is chosen sufficiently large. Clearly $\text{med}(u) = \text{mv}(u) = 0$. Furthermore,

$$
|\nabla u(x)| = \begin{cases} 
g(\Phi(|x_1|)) & \text{for } |x_1| \geq t_0 \\
0 & \text{for } |x_1| < t_0.
\end{cases}
$$

Hence,

$$
|\nabla u|^+(s) = \begin{cases} 
g(s) & \text{for } s \in (0, s_0) \\
0 & \text{for } s \in (s_0, 1),
\end{cases}
$$

and

$$
|\nabla u|^+(s) = \begin{cases} 
(\log \frac{1}{Ns})^{\frac{1}{\beta}} & \text{for } s \in (0, s_0) \\
\frac{s_0}{s} \left( \log \frac{1}{Ns_0} \right)^{\frac{1}{\beta}} & \text{for } s \in (s_0, 1).
\end{cases}
$$

Therefore

$$
\|\nabla u\|_{MB(\mathbb{R}^n, \gamma_n)} = \sup_{s \in (0, 1)} \frac{|\nabla u|^+(s)}{B^{-1}(\frac{1}{s})} = \max \left\{ \sup_{s \in (0, s_0)} (\log \frac{1}{Ns})^{\frac{1}{\beta}}, s_0 \left( \log \frac{1}{Ns_0} \right)^{\frac{1}{\beta}} \sup_{s \in (s_0, 1)} \frac{1}{sB^{-1}(\frac{1}{s})} \right\} = 1,
$$

where the last equality holds since the function $\frac{1}{sB^{-1}(\frac{1}{s})}$ is non-increasing, inasmuch as $B$ is a Young function.

It remains to prove (6.12). We have that

$$
(6.14) \quad \int_{\mathbb{R}^n} e^{(\kappa_s|u|)^{2\beta} \sqrt{n}} \, d\gamma_n \geq \sqrt{\frac{2}{\pi}} \int_{t_0}^{\infty} e^{(\kappa_s f(t_0) g[s(\tau)])} \frac{2\beta}{2\pi} - \frac{\tau^2}{2} \, dt
$$

$$
= \sqrt{\frac{2}{\pi}} \int_{t_0}^{\infty} e^{(\kappa_s f(t_0) g[\Phi(\tau)])} \frac{2\beta}{2\pi} - \frac{\tau^2}{2} \, dt,
$$
where we have made use of the change of variables $s = \Phi(\tau)$. On the other hand, by the definition of $g$,

$$
(6.15) \quad \int_{t_0}^{t} g(\Phi(\tau)) \, d\tau = \int_{t_0}^{t} \left( \log \frac{1}{\sqrt{N} \Phi(\tau)} \right)^{\frac{1}{2}} \, d\tau - \frac{1}{\beta} \int_{t_0}^{t} \left( \log \frac{1}{\sqrt{N} \Phi(\tau)} \right)^{\frac{1}{\beta} - 1} \, d\tau \quad \text{for } t \in (t_0, \infty).
$$

Lemma 5.3 implies that

$$
\int_{t_0}^{t} \left( \log \frac{1}{\sqrt{N} \Phi(\tau)} \right)^{\frac{1}{2}} \, d\tau = 2^{-\frac{1}{2}} \beta t^{\frac{2\beta}{2} + 1} + \frac{2^{-\frac{1}{2}} \beta}{2\sqrt{2}} \int_{t_0}^{t} \left( \log t \right)^{2} \, d\tau + \cdots
$$

as $t \to \infty$, and, by analogous computations,

$$
\int_{t_0}^{t} \left( \log \frac{1}{\sqrt{N} \Phi(\tau)} \right)^{\frac{1}{\beta} - 1} \, d\tau = \left( \frac{2^{1-\frac{1}{\beta}} \beta}{2\sqrt{2}} \log t + \cdots \right) \quad \text{as } t \to \infty.
$$

Thereby, the second integral on the right-hand side of (6.15) is of a lower order than the first one as $t \to \infty$. From (5.3) we thus infer that

$$
\left( \kappa \beta \int_{t_0}^{t} g(\Phi(\tau)) \, d\tau \right)^{\frac{1}{2} \beta} - \frac{t^2}{2} = \left\{ \begin{array}{ll} 
2^{-\frac{1}{2}} \beta \log t + \cdots & \text{if } \beta \in (0, 2) \\
\frac{1}{2} \beta \log t^2 + \cdots & \text{if } \beta = 2
\end{array} \right. \quad \text{as } t \to \infty.
$$

Equation (6.12) hence follows via (6.14).

Our last preparatory result is contained in the following proposition. It is a key step in the proofs of parts (1.ii) and (2.iii) of Theorems 1.1 and 3.1, and of part (2.iii) of Theorem 3.2.

**Proposition 6.7.** Let $\beta > 0$, $M > 1$ and $\kappa > \kappa_\beta$. Assume that $B$ is a Young function satisfying condition (3.2) for some $N > 0$. Then there exists a function $u \in W^1 \exp L^\beta(\mathbb{R}^n, \gamma_n)$ such that $\text{med}(u) = \text{mv}(u) = 0$,

$$
(6.16) \quad \|\nabla u\|_{L^\beta(\mathbb{R}^n, \gamma_n)} \leq 1, \quad \int_{\mathbb{R}^n} e^{\|\nabla u\|^\beta} \, d\gamma_n \leq M
$$

and

$$
(6.17) \quad \int_{\mathbb{R}^n} e^{\kappa \|u\|^\beta} \, d\gamma_n = \infty.
$$

**Proof.** Let $u$ be the function defined as in (6.13), where

$$
g(s) = \left( \lambda \log \frac{\Phi(t_0)}{s} \right)^{\frac{1}{\beta}} \quad \text{for } s \in (0, \Phi(t_0)),
$$

for some $t_0 > 0$ and $\lambda \in (0, 1)$ to be specified later. Clearly $\text{med}(u) = \text{mv}(u) = 0$. Also,

$$
|\nabla u(x)| = \left\{ \begin{array}{ll} 
0 & \text{for } |x_1| < t_0 \\
\left( \lambda \log \frac{\Phi(t_0)}{\Phi(|x_1|)} \right)^{\frac{1}{\beta}} & \text{for } |x_1| \geq t_0.
\end{array} \right.
$$

Let $\tau_0 > 0$ be such that $B(\tau) = N e^{\tau^\beta}$ for $\tau \geq \tau_0$. Then,

$$
(6.18) \quad \int_{\mathbb{R}^n} B(|\nabla u|) \, d\gamma_n \leq \int_{\{0 < |\nabla u| \leq \tau_0\}} B(\tau_0) \, d\gamma_n + N \int_{\{x_1 > t_0\}} e^{\|\nabla u\|^\beta} \, d\gamma_n.
$$

Since the support of $\nabla u$ agrees with the union of the two half-spaces $\{x_1 > t_0\}$ and $\{x_1 < -t_0\}$,

$$
\int_{\{0 < |\nabla u| \leq \tau_0\}} B(\tau_0) \, d\gamma_n \leq 2B(\tau_0) \Phi(t_0).
$$
Furthermore, 
\[
\int_{|x_1|>t_0} e^{|\nabla u|^\beta} \, d\gamma_n = 2 \int_0^\infty \int_{\mathbb{R}^{n-1}} e^{\lambda \log \frac{\Phi(t_0)}{\Phi(x_1)}} \, d\gamma_n(x) = 2 \int_0^\infty e^{\frac{2}{\sqrt{2\pi}} t_0} \int_0^\infty e^{\lambda \log \frac{\Phi(t_0)}{\Phi(x_1)}} e^{-\frac{x_1^2}{2}} \, dx_1
\]
(6.19)
\[
= 2 \int_0^{\Phi(t_0)} \Phi(t_0) \frac{\lambda}{s} \, ds = \Phi(t_0) \int_0^1 \frac{ds}{s^{\lambda}} = \frac{2}{1-\lambda} \Phi(t_0).
\]
Combining inequalities (6.18)–(6.19) yields
\[
\int_{\mathbb{R}^n} B(|\nabla u|) \, d\gamma_n \leq 2B(\tau_0)\Phi(t_0) + \frac{2N}{1-\lambda} \Phi(t_0).
\]
Therefore, if the constants \(t_0\) and \(\lambda\) are chosen in such a way that
\[
2\Phi(t_0) \left( B(\tau_0) + \frac{N}{1-\lambda} \right) \leq 1,
\]
then \(\int_{\mathbb{R}^n} B(|\nabla u|) \, d\gamma_n \leq 1\), and by the definition of Luxemburg norm, we have that \(\|\nabla u\|_{L^p(\mathbb{R}^n, \gamma_n)} \leq 1\), namely the first inequality in (6.16) holds. As far as the second one is concerned, we infer from (6.19) that
\[
\int_{\mathbb{R}^n} e^{|\nabla u|^\beta} \, d\gamma_n \leq \int_{\mathbb{R}^n} e^0 \, d\gamma_n + \int_{|x_1|>t_0} e^{|\nabla u|^\beta} \, d\gamma_n \leq 1 + \frac{2}{1-\lambda} \Phi(t_0).
\]
Hence, if \(\lambda\) and \(t_0\) also obey
\[
\frac{2}{1-\lambda} \Phi(t_0) \leq M - 1,
\]
then \(\int_{\mathbb{R}^n} e^{|\nabla u|^\beta} \, d\gamma_n \leq M\). Thus, the second inequality in (6.16) is fulfilled as well.

In order to prove property (6.17), observe that, similarly to (6.14),
\[
\int_{\mathbb{R}^n} e^{(\kappa|u|)^{\frac{2\beta}{1+\beta}}} \, d\gamma_n \geq \sqrt{\frac{2}{\pi}} \int_0^\infty e \left( \kappa \lambda \beta \int_0^\tau \frac{1}{\Phi(t_0)} \left( \log \frac{\Phi(t_0)}{\Phi(t)} \right)^{\frac{1}{\beta}} \, dt \right)^{\frac{2\beta}{1+\beta}} - \frac{\tau^2}{2} \, d\tau.
\]
(6.22)
Now, let \(A\) be any Young function such that \(A(t) = e^{\tau^\beta}\) near infinity. By L’Hôpital’s rule,
\[
\lim_{t \to \infty} \int_0^t \left( \log \frac{\Phi(t_0)}{\Phi(\tau)} \right)^{\frac{1}{\beta}} \, d\tau = \lim_{t \to \infty} \int_0^t \frac{1}{\Phi(t)} \frac{d}{d\tau} \frac{\Phi(t_0)}{\Phi(\tau)} \, d\tau = \frac{1}{\beta - 1}.
\]
Thereby, thanks to Lemma 5.3,
\[
\left( \kappa \beta \int_0^t \left( \log \frac{\Phi(t_0)}{\Phi(\tau)} \right)^{\frac{1}{\beta}} \, d\tau \right)^{\frac{2\beta}{1+\beta}} = \left( \kappa \beta \int_0^t A^{-1} \left( \frac{1}{\Phi(\tau)} \right) \, d\tau + \cdots \right)^{\frac{2\beta}{1+\beta}} = t^2 \frac{2}{2} + \cdots \text{ as } t \to \infty.
\]
As a consequence,
\[
\left( \frac{\kappa \beta}{2} \int_0^t \left( \log \frac{\Phi(t_0)}{\Phi(\tau)} \right)^{\frac{1}{\beta}} \, d\tau \right)^{\frac{2\beta}{1+\beta}} - t^2 = \left( \frac{\kappa \beta}{2} \int_0^t \left( \frac{1}{\Phi(\tau)} \right)^{\frac{1}{\beta}} \, d\tau + \cdots \right)^{\frac{2\beta}{1+\beta}} = \frac{t^2}{2} + \cdots \text{ as } t \to \infty.
\]
If \(\kappa > \kappa_0\), one can choose \(\lambda \in (0,1)\) sufficiently close to 1 in such a way that the constant multiplying \(\frac{\ell^2}{2}\) on the right-hand side of (6.23) is positive. With this choice of \(\lambda\), property (6.17) follows from (6.22). Then, we choose \(t_0\) large enough for (6.20) and (6.21) to hold, whence the inequalities in (6.16) hold as well.

We are now in a position to accomplish the proofs of our main results.

Proof of Theorem 3.1. Assume that \( \kappa > 0 \). By Lemma 4.5,

\[
\int_{\mathbb{R}^n} e^{(\kappa |u|) \frac{2\beta}{\kappa + \beta}} \, d\gamma_n \leq \sqrt{\frac{2}{\pi}} \int_0^\infty e^{[\kappa \mathcal{F}_L B (t)] \frac{2\beta}{\kappa + \beta} - \frac{t^2}{2}} \, dt
\]

for any weakly differentiable \( u \) obeying (3.1) and (1.2), where \( \mathcal{F}_L B \) is given by (4.19). On the other hand, Lemma 5.9 tells us that

\[
\mathop{\text{Proof of Theorem 3.2.}}
\]

We begin by showing assertions (1.i) and (2.i). Let \( M > 1 \) such that inequality (3.1) is satisfied for any weakly differentiable function obeying (3.1) and (1.2). By Lemma 6.1, there exists a constant \( \kappa \) satisfying condition (3.2) and inequality (6.25) fails whatever \( C \) is, as \( u \) ranges over all weakly differentiable functions obeying (3.1) and (1.2). By Lemma 6.2, there exists a constant \( M > 1 \) such that inequality (3.1) is satisfied for any \( u \) obeying (1.3). This proves the positive part of (2.i).

Assertions (1.i) and (2.i) are covered by Proposition 6.7.

Let us now deal with the critical case when \( \kappa = \kappa_\beta \). Fix \( N \in (0, 1) \). By Theorem 3.1, part (2.ii), there exists a Young function \( B \) satisfying property (3.2) such that

\[
\int_{\mathbb{R}^n} e^{(\kappa_\beta |u|) \frac{2\beta}{\kappa_\beta + \beta}} \, d\gamma_n \leq C
\]

for some \( C = C(\beta, B) \) and every weakly differentiable \( u \) obeying (1.2) and (3.1). By Lemma 6.1, there exists a constant \( M > 1 \) such that condition (1.3) implies (3.1). This proves the positive part of (2.ii).

Analogously, Theorem 3.1, part (2.ii), ensures that for any \( N > 0 \) there exists a Young function satisfying condition (3.2) for which inequality (6.25) fails whatever \( C \) is, as \( u \) ranges over all weakly differentiable functions obeying (3.1) and (1.2). By Lemma 6.2, there exists a constant \( M > 1 \) such that inequality (3.1) is satisfied for any \( u \) obeying (1.3). The proof is now complete.

Proof of Theorem 3.2. We begin by showing assertions (1.i) and (2.i). Let \( \kappa > 0 \). By Lemma 4.5,

\[
\int_{\mathbb{R}^n} e^{(\kappa |u|) \frac{2\beta}{\kappa + \beta}} \, d\gamma_n \leq \sqrt{\frac{2}{\pi}} \int_0^\infty e^{[\kappa \mathcal{F}_m B (t)] \frac{2\beta}{\kappa + \beta} - \frac{t^2}{2}} \, dt
\]

for any weakly differentiable function \( u \) obeying (3.4) and (1.2). Here, \( \mathcal{F}_m B \) is the function given by (4.20). Next, Lemma 5.3 tells us that

\[
\mathop{\text{Proof of Theorem 3.1. Assume that } \kappa > 0. \text{ By Lemma 4.5},}
\]

\[
\left[ \kappa \mathcal{F}_m B (t) \right] \frac{2\beta}{\kappa + \beta} = \left( \frac{\kappa}{\kappa_\beta} \right) \frac{2\beta}{\kappa + \beta} \frac{t^2}{2} + \cdots \text{ as } t \to \infty.
\]

Hence, the integral on the right-hand side of (6.26) converges whenever \( \kappa < \kappa_\beta \). This proves properties (1.ii) and (2.ii) under condition (3.4). If \( u \) satisfies condition (3.3), then (3.4) also holds just owing to (2.9). Hence, properties (1.ii) and (2.ii) follow also in this case.

Assertion (1.ii) for the \( M^B \) norm is a straightforward consequence of Proposition 6.6, and for the \( m^B \) quasi-norm it requires the additional use of inequality (2.9).

Property (2.ii) follows via Proposition 6.5 and Proposition 6.4, combined with inequalities (2.9).
Finally, assertion (2.iii) is treated in Proposition 6.7. Inequality (2.9) has to be exploited here as well. \( \square \)

**Proof of Theorem 3.6.** Assume that the function \( u \in W^1L^\infty(\mathbb{R}^n, \gamma_n) \) fulfills conditions (1.2) and (3.7). By Lemma 4.6,

\[
\int_{\mathbb{R}^n} e^{(\kappa u)^2} \, d\gamma_n \leq \sqrt{2\pi} \int_0^\infty e^{[\kappa F_{L^\infty}(t)]^2 - \frac{t^2}{2}} \, dt,
\]

where the function \( F_{L^\infty} \) is given by (4.29). By equation (5.3),

\[
-\Phi'(t) - t\Phi(t) \to 0 \quad \text{as } t \to \infty.
\]

Consequently,

\[
[\kappa F_{L^\infty}(t)]^2 - \frac{t^2}{2} = \left( \kappa^2 - \frac{1}{2} \right) t^2 + \cdots \quad \text{as } t \to \infty,
\]

under either assumption \( \text{med}(u) = 0 \), or \( \text{mv}(u) = 0 \). Thanks to equation (6.28), the integral on the right-hand side of inequality (6.27) converges for every \( \kappa \in (0, \frac{1}{\sqrt{2}}) \). Part (i) of the statement is thus established.

In order to show part (ii), it clearly suffices to assume that \( \kappa = \frac{1}{\sqrt{2}} \). Consider the function \( u: \mathbb{R}^n \to \mathbb{R} \) defined as \( u(x) = x_1 \) for \( x \in \mathbb{R}^n \). Trivially, \( u \in W^1L^\infty(\mathbb{R}^n, \gamma_n) \), and since \( |\nabla u(x)| = 1 \) for every \( x \in \mathbb{R}^n \), the function \( u \) fulfills assumption (3.7). Moreover, \( \text{med}(u) = \text{mv}(u) = 0 \), and therefore condition (1.2) is fulfilled as well in both its variants. Notice that \( u^o(s) = \Phi^{-1}(s) \) for \( s \in (0, 1) \). Thereby,

\[
\int_{\mathbb{R}^n} e^{\frac{1}{2}u^2} \, d\gamma_n = \int_0^1 e^{\frac{1}{2}u^o(s)^2} \, ds = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2}} \, dt = \infty,
\]

where the last but one equality holds by the change of variables \( s = \Phi(t) \). The proof is complete. \( \square \)

**Proof of Theorem 1.2.** Let \( u \in W^1 \exp E^\beta(\mathbb{R}^n, \gamma_n) \) and let \( B \) be a Young function satisfying (3.2). We may assume, without loss of generality, that

\[
\|\nabla u\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq 1.
\]

Let us also assume, for the time being, that \( \text{med}(u) = 0 \). Let \( s_0 \in (0, \frac{1}{2}) \). By the local absolute continuity of \( u^o \) and Hölder’s inequality (2.6),

\[
u^o(s) = \int_s^1 -u^o(r) \, dr = \int_s^{s_0} -u^o(r)I(r) \, dr + \int_{s_0}^1 -u^o(r)I(r) \, dr \leq \|u^o I\|_{L^B(0,s_0)} \left| \frac{1}{2} \right|_{L^B(s,s_0)} + \|u^o I\|_{L^B(0,\frac{1}{2})} \left| \frac{1}{2} \right|_{L^B(s_0, \frac{1}{2})} \quad \text{for } s \in (0, s_0).
\]

Observe that

\[
\|u^o I\|_{L^B(0,s_0)} \leq \|(u^o I)^*\|_{L^B(0,s_0)} \leq \|\nabla u^o\|_{L^B(0,s_0)} = \|\nabla u\|_{L^B(E)}
\]

for a suitable measurable set \( E \subseteq \mathbb{R}^n \) such that \( \gamma_n(E) = s_0 \). Notice that the first inequality in (6.29) holds as a consequence of the Hardy-Littlewood principle [10, Chapter 2, Theorem 4.6], since, owing to inequality (2.1),

\[
\int_0^s (u^o I)_+(r) \, dr \leq \int_0^s (u^o I)_+ \chi_{(0,s)}(r) \, dr \quad \text{for } s \in (0, 1).
\]

On the other hand, the second inequality in (6.29) is a consequence of inequality (4.4). Fix \( \varrho > 0 \). Since \( |\nabla u| \) has an absolutely continuous norm, by (6.29) one can choose \( s_0 \) so small that

\[
\|u^o I\|_{L^B(0,s_0)} \leq \|\nabla u\|_{L^B(E)} < \varrho.
\]

Next, owing to Proposition 4.2,

\[
\|u^o I\|_{L^B(0,\frac{1}{2})} \leq \|\nabla u\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq 1,
\]

Finally, assertion (2.iii) is treated in Proposition 6.7. Inequality (2.9) has to be exploited here as well. \( \square \)
and
\[
\left\| \frac{1}{T} \right\|_{L^B(s_0, s)} \leq \frac{1}{I(s_0)} \left\| 1 \right\|_{L^B(0, \frac{1}{2})} = \frac{c}{I(s_0)},
\]
where we have set \( c = \left\| 1 \right\|_{L^B(0, \frac{1}{2})} \). Thanks to Lemma 5.9 and equation (5.14), there exists a constant \( C \) such that
\[
\left\| \frac{1}{T} \right\|_{L^B(s, s_0)} \leq C \left( \log \frac{1}{s} \right)^{2+\frac{\beta}{2\theta}} \text{ for } s \in (0, s_0).
\]
 Altogether, we obtain that
\[
(6.30) \quad 0 \leq u^\circ(s) \leq C \varrho \left( \log \frac{1}{s} \right)^{2+\frac{\beta}{2\theta}} + \frac{c}{I(s_0)} \text{ for } s \in (0, s_0).
\]
If the assumption that \( \text{med}(u) = 0 \) is dropped, from an application of inequality (6.30) to the function \( u - \text{med}(u) \) one infers that
\[
|u^\circ(s)| \leq C \varrho \left( \log \frac{1}{s} \right)^{2+\frac{\beta}{2\theta}} + \frac{c}{I(s_0)} + |\text{med}(u)| \text{ for } s \in (0, s_0).
\]
Now,
\[
(6.31) \quad \int_{\mathbb{R}^n} e^{(\kappa|u|)^{2\beta}} \varphi_n = \int_0^{s_0} e^{(\kappa|u^\circ(s)|)^{2\beta}} ds = \int_0^{\frac{1}{2}} e^{(\kappa|u^\circ(s)|)^{2\beta}} ds + \int_{\frac{1}{2}}^{s_0} e^{(\kappa|u^\circ(s)|)^{2\beta}} ds,
\]
and
\[
(6.32) \quad \int_{\frac{1}{2}}^{s_0} e^{(\kappa|u^\circ(s)|)^{2\beta}} ds = \int_0^{s_0} e^{(\kappa|u^\circ(s)|)^{2\beta}} ds + \int_{s_0}^{\frac{1}{2}} e^{(\kappa|u^\circ(s)|)^{2\beta}} ds.
\]
The second integral on the right-hand side of equation (6.32) is finite, since \( u^\circ \) is bounded in \( (s_0, \frac{1}{2}) \). As for the first one,
\[
\int_0^{s_0} e^{(\kappa|u^\circ(s)|)^{2\beta}} ds \leq \int_0^{s_0} e^{(\kappa C \varrho \left( \log \frac{1}{s} \right)^{2+\frac{\beta}{2\theta}} + \kappa \varphi \left( \frac{s}{I(s_0)} + |\text{med}(u)| \right)^{2\beta}} ds
\]
\[
\leq \int_0^{s_0} C_1 e^{(\kappa \varrho \left( \log \frac{1}{s} \right)^{2+\frac{\beta}{2\theta}} + \left( \frac{s}{I(s_0)} + |\text{med}(u)| \right)^{2\beta}} ds
\]
\[
= e^{C_1 \left( \frac{s}{I(s_0)} + |\text{med}(u)| \right)^{2\beta}} \int_0^{s_0} \left( \frac{1}{s} \right) C_1 \varrho \left( \log \frac{1}{s} \right)^{2+\frac{\beta}{2\theta}} ds
\]
for some constant \( C_1 \). Clearly, the last integral is convergent, provided \( \varrho \) is chosen small enough. Thus,
\[
(6.33) \quad \int_{\frac{1}{2}}^{s_0} e^{(\kappa|u^\circ(s)|)^{2\beta}} ds < \infty.
\]
An analogous argument, exploiting equation (4.2), shows that
\[
(6.34) \quad \int_{\frac{1}{2}}^{1} e^{(\kappa|u^\circ(s)|)^{2\beta}} ds < \infty.
\]
Property (1.7) follows via (6.31), (6.33) and (6.34).

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1Dipartimento di Matematica e Informatica “Ulisse Dini”, University of Florence, Viale Morgagni 67/A, 50134 Firenze, Italy
E-mail address: andrea.cianchi@unifi.it
ORCID: 0000-0002-1198-8718

2The Czech Academy of Sciences, Institute of Mathematics, Žitná 25, 115 67, Prague 1, Czech Republic
E-mail address: musil@math.cas.cz
ORCID: 0000-0001-6083-227X

3Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail address: pick@karlin.mff.cuni.cz
ORCID: 0000-0002-3584-1454