On one example of a Nikishin system

Sergey P. Suetin

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Abstract

The paper puts forward an example of a Markov function $f = \text{const} + \sigma$ such that the three functions $f$, $f^2$, and $f^3$ form a Nikishin system. A conjecture is proposed that there exists a Markov function $f$ such that, for each $n \in \mathbb{N}$, the system $f, f^2, \ldots, f^n$ constitutes a Nikishin system.

Bibliography: 20 titles.

1 Introduction and statement of the problem

As distinct from Padé polynomials, which are constructed from one function $f$, a construction of the Hermite–Padé polynomials corresponding, for example, to a two-dimensional multiindex, requires at least two functions $f_1$ and $f_2$, which should be in a sense independent. Namely, in order that the definition of the Padé polynomials be meaningful it is necessary that the original function $f$ should not be a rational function. In other words, it is necessary that the pair of functions $1, f$ should be independent over the field of rational functions $\mathbb{C}(z)$. Likewise, in order that the definition of the Hermite–Padé polynomials for a pair of functions $f_1, f_2$ be meaningful it is required that the three functions $1, f_1, f_2$ be independent over the field $\mathbb{C}(z)$. For the definition of Hermite–Padé polynomials and general properties of these polynomials, see, above all, [11] and [10], and also [20].

1. More precisely, here we speak about the power series expansion defined at some fixed point $z_0$ of the Riemann sphere $\mathbb{C}$, for example, $z_0 = \infty$. 


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The family of functions involved in the construction of Hermite–Padé polynomials is usually called a system. The two best known systems in the theory of Hermite–Padé polynomials are an Angelesco system and a Nikishin system; for the formal definition of such systems and their properties, see, above all, [10], and also [6], [20], [4], [5], [13]. More general (the so-called “mixed”) systems of functions were considered in [14], [1], [12]. Below we will briefly discuss the meaningfulness of these concepts in the case of a pair of functions $f_1$ and $f_2$.

Given an arbitrary (positive Borel) measure $\sigma$ with support $\text{supp} \sigma$ on the real line $\mathbb{R}$, $\text{supp} \sigma \subset \mathbb{R}$, we denote by

$$\tilde{\sigma}(z) := \int \frac{d\sigma(x)}{z - x}, \quad z \notin \text{supp} \sigma,$$

the Cauchy transform of the measure $\sigma$.

For a pair of functions $f_1$ and $f_2$ of the form (1) the property that this pair forms an Angelesco system appears to be quite natural. Namely, in this case the functions $f_1$ and $f_2$ can be written as

$$f_1(z) := \tilde{\sigma}_1(z), \quad f_2(z) := \tilde{\sigma}_2(z),$$

where it is assumed that the supports of the measures $\sigma_1$ and $\sigma_2$ are disjoint, $\text{supp} \sigma_1 \cap \text{supp} \sigma_2 = \emptyset$, $\text{supp} \sigma_1, \text{supp} \sigma_2 \subset \mathbb{R}$.

If in (2) the supports of the measures $\sigma_1$ and $\sigma_2$ are equal, $\text{supp} \sigma_1 = \text{supp} \sigma_2 = \Delta$, and if

$$d\sigma_2(x) = \tilde{\sigma}_3(x) d\sigma_1(x), \quad x \in \Delta,$$

where the third measure $\sigma_3$, $\text{supp} \sigma_3 \subset \mathbb{R}$, is such that $\text{supp} \sigma_3 \cap \Delta = \emptyset$, then the pair of functions $f_1$ and $f_2$ of the form (1) is said to form a Nikishin system.

At first glance the definition of an Angelesco system looks more natural than that of a Nikishin system. For example, an Angelesco system is formed by the pair of functions

$$f_1(z) := \frac{1}{[(z - a_1)(z - b_1)]^{1/2}}, \quad f_2(z) := \frac{1}{[(z - a_2)(z - b_2)]^{1/2}},$$

where $a_1 < b_1 < a_2 < b_2$ and a branch of the root is chosen so that $[(z - a_j)(z - b_j)]^{1/2}/z \to 1$ as $z \to \infty$. 

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The purpose of the present note is to present an example of a Markov function $f = \hat{\sigma} + \text{const}$ such that the pair of functions $f, f^2$ forms a Nikishin system under a certain minimal extension of the original definition of such a system (see (13)–(15), and also Remark 2 below). As a result, it turns out that, from the point of view of the general problem of efficient analytic continuation of a multivalued analytic function defined by a power series (for more details, see [19]), the concept of a Nikishin system is by no means less meaningful than that of an Angelesco system. It is worth noting that this fact is also manifested in some papers on Nikishin systems; see, for example, [7], [2], [3], [9], [8], [18], [17] and the references given therein.

More precisely, we will give an example of a function $f$ of the form
\begin{equation}
 f(z) = C + \hat{\sigma}(z) \tag{5}
\end{equation}
(cf. (1)), where $C \neq 0$ is some real constant, $\sigma$ is a measure supported on the interval $[-1, 1]$, supp $\sigma = [-1, 1]$, such that the pair of functions
\begin{equation}
 f_1(z) := f(z), \quad f_2(z) := f^2(z) \tag{6}
\end{equation}
forms a Nikishin system. Furthermore, it will be shown that, for the function $f(z)$ considered below (see (11)) of the form (5), the three functions $f, f^2, f^3$ also form a Nikishin system.

**Remark 1.** One consequence of the presence of the term $C \neq 0$ in representation (5) is that the function $f_2(z) = f^2(z)$ can no longer be written in the form (2)–(3). Nevertheless, somewhat more involved representations will be shown to hold. Namely, the following representations are valid
\begin{equation}
 f(z) - C = \hat{\sigma}(z), \quad f^2(z) - Cf(z) = \hat{s}_1(z), \quad f^3(z) - Cf^2(z) = \hat{s}_2(z), \tag{7}
\end{equation}
where supp $s_j = [-1, 1], j = 1, 2$; for more details, see § 2 and Remark 2 below.

The possibility of the existence of a Markov function $f$ for which similar representations would hold for an arbitrary power $f^n$ will be discussed below (see Conjecture [11]).

## 2 Definitions and statement of the main result

Let $\Delta_1 := [-1, 1],
\varphi(z) := z + (z^2 - 1)^{1/2}, \quad z \notin \Delta_1, \tag{8}$
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be the function inverse to the Zhukovskii function. Recall that we have chosen and fixed a branch of the square root such that \((z^2 - 1)^{1/2}/z \to 1\) as \(z \to \infty\). So, \(|\varphi(z)| > 1\) for \(z \notin \Delta_1\). Hence, for any complex number \(A\) such that \(|A| > 1\), the multivalued analytic function

\[
f(z) := f(z; A, \alpha) := \left( A - \frac{1}{\varphi(z)} \right)^{\alpha}, \quad \text{where} \quad \alpha \in \mathbb{C} \setminus \mathbb{Z}, \tag{9}
\]

admits a holomorphic (i.e., a single-valued analytic) branch in the domain \(D_1 := \mathbb{C} \setminus \Delta_1\). However, in the domain \(\mathbb{C} \setminus \{-1, 1\}\) the function \(f(z)\) is already a multivalued analytic function, the set of branch points \(\Sigma\) of this function consisting of three points: \(\Sigma = \{\pm 1, a\}\), where \(a = (A + 1/A)/2\) and, hence, \(|a| > 1\). Note that \(1/\varphi(z) = z - (z^2 - 1)^{1/2}\) in accordance with the above choice of the branch of the root in (9).

The class of multivalued analytic functions \(\mathcal{Z}\) consisting of all functions obtained by multiplication of a finite number of functions of the form (9)

\[
f(z) := \prod_{j=1}^{m} \left( A_j - \frac{1}{\varphi(z)} \right)^{\alpha_j}, \tag{10}
\]

where \(|A_j| > 1\), \(\alpha_j \in \mathbb{C} \setminus \mathbb{Z}\) for all \(j = 1, \ldots, m\), and \(\sum_{j=1}^{m} \alpha_j \in \mathbb{Z}\), was introduced and studied in [15] (see also [16], [19]). In the present paper, we shall be concerned only with the case when in (10) \(m = 2\), \(A_1, A_2\) are real, and \(\alpha_1 = \alpha_2 = -1/2\). We shall also assume that \(1 < A_1 < A_2\). So, the functions to be considered are of the form

\[
f(z) := \left[ \left( A_1 - \frac{1}{\varphi(z)} \right) \left( A_2 - \frac{1}{\varphi(z)} \right) \right]^{-1/2}, \tag{11}
\]

or, in other words, \(f(z) = f_1(z)f_2(z)\), where

\[
f_1(z) := f(z; A_1, -1/2) := \left( A_1 - \frac{1}{\varphi(z)} \right)^{-1/2} = \frac{1}{(A_1 - 1/\varphi(z))^{1/2}}, \tag{12}
\]

\[
f_2(z) := f(z; A_2, -1/2) := \left( A_2 - \frac{1}{\varphi(z)} \right)^{-1/2} = \frac{1}{(A_2 - 1/\varphi(z))^{1/2}},
\]

\(z \in D_1, 1 < A_1 < A_2\).

In what follows, \(\sqrt{\cdot}\) denotes the positive square root of a nonnegative real number; i.e., \(\sqrt{a^2} = |a|\) for \(a \in \mathbb{R}\).

The main result of the present paper is as follows.
Proposition 1. Let \( f(z) \) be the function defined by representation (11), where \( 1 < A_1 < A_2 \), and let \( a_j = (A_j + 1/A_j)/2, j = 1, 2 \). Then, for \( z \in D \),

\[
f(z) = \frac{1}{\sqrt{A_1 A_2}} + \hat{\sigma}(z), \quad (13)
\]

\[
f^2(z) = \frac{1}{A_1 A_2} + \frac{1}{\sqrt{A_1 A_2}} \hat{\sigma}(z) + \hat{s}_1(z), \quad (14)
\]

\[
f^3(z) = \frac{1}{(A_1 A_2)^3} + \frac{1}{A_1 A_2} \hat{\sigma}(z) + \frac{1}{\sqrt{A_1 A_2}} \hat{s}_1(z) + \hat{s}_2(z), \quad (15)
\]

where \( \sigma \) is the measure supported on the interval \([-1, 1]\), the measures \( s_1 \) and \( s_2 \) are defined by the representations \( s_1 = \langle \sigma, \sigma_2 \rangle \) and \( s_2 = \langle \sigma, \sigma_2, \sigma \rangle \). Moreover, supp \( s_1 = \text{supp} \ s_2 = [-1, 1] \), supp \( \sigma_2 = [a_1, a_2] \subset \mathbb{R} \setminus [-1, 1] \), and the measures \( \sigma \) and \( \sigma_2 \) have the following explicit representations

\[
d\sigma(x_1) = \frac{\sqrt{1 - x_1^2}}{4\pi \sqrt{A_1 A_2}(a_1 - x_1)(a_2 - x_1)} \left[ \frac{h_2(x_1)}{h_1(x_1)} + \frac{h_1(x_1)}{h_2(x_1)} \right] dx_1, \quad x_1 \in [-1, 1], \quad (16)
\]

\[
d\sigma_2(x_2) = \frac{dx_2}{\pi \sqrt{(\varphi(x_2) - A_1)(A_2 - \varphi(x_2))}}, \quad x_2 \in (a_1, a_2), \quad (17)
\]

where

\[
h_j(x_1) := \left( A_j - (x_1 + i\sqrt{1 - x_1^2}) \right)^{1/2} + \left( A_j - (x_1 - i\sqrt{1 - x_1^2}) \right)^{1/2} > 0
\]

for \( x_1 \in [-1, 1], j = 1, 2 \).

Following [6], in Proposition 1 we used the following notation for the measure \( s_1 \): \( d \langle \sigma, \sigma_2 \rangle (x_1) := \hat{s}_2(x_1) d\sigma(x_1), x_1 \in \Delta_1 := (-1, 1) \), which is legitimate under our assumption that \( \Delta_1 \cap \Delta_2 = \emptyset \), where \( \Delta_2 := [a_1, a_2] \). In the definition of the measure \( s_2 \) we follow the standard convention to the effect that \( d \langle \sigma, \sigma_2, \sigma \rangle := d \langle \sigma, \langle \sigma_2, \sigma \rangle \rangle \) (for more details, see [6], and also [1], [4], [5]). According to what has been said, the three functions \( \sigma(z), s_1(z) \) and \( s_2(z) \) form a (classical) Nikishin system. This being so, in view of (13)–(15), it is also natural to regard the system of functions \( f, f^2, f^3 \) as a Nikishin system, because this system is generated by a linear combination of three functions, \( \hat{\sigma}, \hat{s}_1 \) and \( \hat{s}_2 \), which forms a Nikishin system.
3 Proof of Proposition 1

3.1

Given \( x_1 \in \Delta_1^\circ \), we let \( f^+_j(x_1) \) and \( f^-_j(x_1) \), \( j = 1, 2 \), denote the limiting values of the function \( f_j(z) \) as \( z = x_1 + i\varepsilon \to x_1 \in \Delta_1^\circ \), \( \varepsilon \to 0 \), assuming that \( z \) lies, respectively, in the upper half-plane \( (\varepsilon > 0) \) and in the lower half-plane \( (\varepsilon < 0) \). It is easily seen that

\[
f^+_j(x_1) = \left( A_j - (x_1 - i\sqrt{1-x_1^2}) \right)^{-1/2}, \quad f^-_j(x_1) = \left( A_j - (x_1 + i\sqrt{1-x_1^2}) \right)^{-1/2}.
\]

A direct consequence of (18) is that, for \( x_1 \in \Delta_1^\circ \),

\[
\Delta f_j(x_1) := (f^+_j - f^-_j)(x_1)
\]

\[
= \frac{\left( A_j - (x_1 + i\sqrt{1-x_1^2}) \right)^{1/2} - \left( A_j - (x_1 - i\sqrt{1-x_1^2}) \right)^{1/2}}{\left[ \left( A_j - (x_1 + i\sqrt{1-x_1^2}) \right) \left( A_j - (x_1 - i\sqrt{1-x_1^2}) \right) \right]^{1/2}}
\]

\[
= -\frac{2i\sqrt{1-x_1^2}}{\sqrt{(A_j - x_1)^2 + (1-x_1^2)}h_j(x_1)}
\]

\[
= -\frac{2i\sqrt{1-x_1^2}}{\sqrt{2A_j(a_j - x_1)}h_j(x_1)}, \tag{19}
\]

where

\[
h_j(x_1) := \left( A_j - (x_1 + i\sqrt{1-x_1^2}) \right)^{1/2} + \left( A_j - (x_1 - i\sqrt{1-x_1^2}) \right)^{1/2} \tag{20}
\]

for \( x_1 \in \Delta_1^\circ \), \( j = 1, 2 \). Moreover, we have

\[
(f^+_j + f^-_j)(x_1) = \left( A_j - (x_1 + i\sqrt{1-x_1^2}) \right)^{-1/2} + \left( A_j - (x_1 - i\sqrt{1-x_1^2}) \right)^{-1/2}
\]

\[
= \frac{h_j(x_1)}{\sqrt{2A_j(a_j - x_1)}}. \tag{21}
\]

It is easily checked that each function \( h_j \), which is holomorphic on the interval \( \Delta_1^\circ \), extends holomorphically from this interval to some neighborhood
of $\Delta_1$. Moreover, $h_j(x_1) \neq 0$ for $x_1 \in \Delta_1$, and therefore, for $x_1$ from some neighborhood of $\Delta_1$. It is also worth noting that the function $f_j^+ + f_j^-$, which is holomorphic on the interval $\Delta_1^0$, extends holomorphically to some neighborhood of the interval $\Delta_1$.

We have $f = f_1 f_2$, and hence, for $x_1 \in \Delta_1^0$, using the identity

$$2\Delta f(x_1) := 2(f^+ - f^-)(x_1) = \Delta f_1(x_1)(f_2^+ + f_2^-)(x_1) + \Delta f_2(x_1)(f_1^+ + f_1^-)(x_1)$$

and employing relations (19) and (21), we get

$$2\Delta f(x_1) = -\frac{i\sqrt{1 - x_1^2}}{\sqrt{A_1 A_2}(a_1 - x_1)(a_2 - x_1)} \left[ \frac{h_2(x_1)}{h_1(x_1)} + \frac{h_1(x_1)}{h_2(x_1)} \right], \quad x_1 \in \Delta_1. \quad (22)$$

Moreover, it is also immediate that

$$\frac{h_1(x_1)}{h_2(x_1)} + \frac{h_2(x_1)}{h_1(x_1)} > 0 \quad \text{for} \quad x_1 \in \Delta_1.$$

We have $f(\infty) = 1/\sqrt{A_1 A_2}$ by definition (11) of the function $f$, and hence, applying Cauchy’s theorem to the function $f$, we get the following representation

$$f(z) - \frac{1}{\sqrt{A_1 A_2}} = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(t)}{t-z} \, dt, \quad z \in \text{ext } \gamma_1, \quad (23)$$

where $\gamma_1$ is an arbitrary closed Jordan curve separating the interval $\Delta_1$ from the infinity point and containing the point $z$ in the unbounded component $\text{ext } \gamma_1$ of its complement $\mathbb{C} \setminus \gamma_1$; we assume that the curve $\gamma_1$ has positive orientation relative to $\text{ext } \gamma_1$. From (23) it easily follows that

$$f(z) - \frac{1}{\sqrt{A_1 A_2}} = \frac{1}{2\pi i} \int_{\Delta_1} \frac{\Delta f(x_1)}{x_1-z} \, dx = \int_{\Delta_1} \frac{d\sigma(x_1)}{z-x_1} = \tilde{\sigma}(z), \quad (24)$$

where, for $x_1 \in \Delta_1$,

$$d\sigma(x_1) = -\frac{1}{2\pi i} \Delta f(x_1) \, dx = \frac{\sqrt{1 - x_1^2}}{4\pi \sqrt{A_1 A_2}(a_1 - x_1)(a_2 - x_1)} \left[ \frac{h_2(x_1)}{h_1(x_1)} + \frac{h_1(x_1)}{h_2(x_1)} \right] \, dx_1. \quad (25)$$

Using (24) and (25), this establishes

$$f(z) = \frac{1}{\sqrt{A_1 A_2}} + \tilde{\sigma}(z), \quad z \in D_1,$$

thereby proving representations (13) and (16).
3.2

We set \( \rho_1(x_1) := f^+(x_1) + f^-(x_1) = (f_1^+ f_2^+ + f_1^- f_2^-)(x_1) \), \( x_1 \in \Delta_1^\circ \). It is easily seen (see (11) and (21)) that the function \( \rho_1 \in \mathcal{H}(\Delta_1^\circ) \) extends holomorphically from the interval \( \Delta_1^\circ \) to some neighborhood of the interval \( \Delta_1 \). Moreover, the function \( \rho_1 \) is holomorphic on the domain \( D_2 := \mathbb{C} \setminus \Delta_2 \) and can be represented in this domain as

\[
\rho_1(z) = \left[ (A_1 - (z - (z^2 - 1)^{1/2})) (A_2 - (z - (z^2 - 1)^{1/2})) \right]^{-1/2} + \left[ (A_1 - (z + (z^2 - 1)^{1/2})) (A_2 - (z + (z^2 - 1)^{1/2})) \right]^{-1/2} = \left[ \left( A_1 - \frac{1}{\varphi(z)} \right) \left( A_2 - \frac{1}{\varphi(z)} \right) \right]^{-1/2} + \left[ (A_1 - \varphi(z)) (A_2 - \varphi(z)) \right]^{-1/2}.
\]

(26)

Given \( x_2 \in \Delta_2^\circ \), we denote by \( \rho_1^+(x_2) \) the limiting values of the function \( \rho_1(z) \) as \( z \to x_2 \) assuming that \( z \) lies in the upper half-plane, and denote by \( \rho_1^-(x_2) \) the limiting values of \( \rho_1(z) \) as \( z \to x_2 \) assuming that \( z \) lies in the lower half-plane. Using (26),

\[
\Delta \rho_1(x_2) := (\rho_1^+ - \rho_1^-)(x_2) = \frac{-2i}{\sqrt{(\varphi(x_2) - A_1)(A_2 - \varphi(x_2))}}, \quad x_2 \in \Delta_2^\circ.
\]

(27)

We have \( \rho_1(\infty) = 1/\sqrt{A_1 A_2} \). Hence, by (26)

\[
\rho_1(z) - \frac{1}{\sqrt{A_1 A_2}} = \frac{1}{2\pi i} \int_{\gamma_2} \frac{\rho_1(t) dt}{t - z} = \frac{1}{2\pi i} \int_{\Delta_2} \frac{\Delta \rho_1(x_2) dx_2}{x_2 - z},
\]

(28)

where \( \gamma_2 \) is an arbitrary negatively oriented closed Jordan curve separating the interval \( \Delta_2 \) from the infinity point; the point \( z \) lies in that connected component of \( \mathbb{C} \setminus \gamma_2 \) which contains the infinity point.

From (27) and (28) we see that

\[
\rho_1(z) = \frac{1}{\sqrt{A_1 A_2}} + \hat{\sigma}_2(z), \quad z \in D_2,
\]

(29)

where

\[
d\sigma_2(x_2) := -\frac{1}{2\pi i} \Delta \rho_1(x_2) dy = \frac{1}{\pi} \frac{dx_2}{\sqrt{(\varphi(x_2) - A_1)(A_2 - \varphi(x_2))}}, \quad x_2 \in \Delta_2^\circ.
\]

(30)
So, we have
\[ \rho_1(z) := (f^+ + f^-)(z) = \frac{1}{\sqrt{A_1 A_2}} + \tilde{\sigma}(z), \]
where \( \sigma_2 \) is the positive measure with support in \( \Delta_2 \) defined by representation (30). Hence, for \( x_1 \in \Delta_1 \),
\[ \frac{\Delta f^2}{\Delta f}(x_1) = (f^+ + f^-)(x_1) = \frac{1}{\sqrt{A_1 A_2}} + \tilde{\sigma}(x_1). \tag{31} \]
As a result (see (25)), we have, for \( x_1 \in \Delta_1 \),
\[ \Delta f^2(x_1) \, dx_1 = \left( \frac{1}{\sqrt{A_1 A_2}} + \tilde{\sigma}(x_1) \right) \Delta f(x_1) \, dx_1 \]
\[ = - \left( \frac{1}{\sqrt{A_1 A_2}} + \tilde{\sigma}(x_1) \right) 2\pi i \, d\sigma(x_1). \tag{32} \]
Since \( f^2(\infty) = 1/(A_1 A_2) \), it follows from (32) that
\[ f^2(z) - \frac{1}{A_1 A_2} = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f^2(t)}{t - z} \, dt = -\frac{1}{2\pi i} \int_{\Delta_1} \frac{\Delta f^2(x_1)}{z - x_1} \, dx_1 \]
\[ = \frac{1}{\sqrt{A_1 A_2}} \tilde{\sigma}(z) + \int_{\Delta_1} \frac{\tilde{\sigma}(x_1) \, d\sigma(x_1)}{z - x_1} = \frac{1}{\sqrt{A_1 A_2}} \tilde{\sigma}(z) + \tilde{s}_1(z), \]
where \( s_1 = \langle \sigma, \sigma_2 \rangle, \) \( \text{supp} \, s_1 = \Delta_1. \) Therefore,
\[ f^2(z) = \frac{1}{A_1 A_2} + \frac{1}{\sqrt{A_1 A_2}} \tilde{\sigma}(z) - \tilde{s}_1(z), \quad z \in D_1. \]
This completes the proof of representations (14) and (17).

3.3

We now set
\[ \rho_2(x_1) := \frac{\Delta f^3(x_1)}{\Delta f(x_1)}, \quad x_1 \in \Delta_1^\circ. \tag{33} \]
Given \( x_1 \in \Delta_1^\circ \), we have
\[ f^+(x_1) = \left[ \left( A_1 - (x_1 - i \sqrt{1 - x_1^2}) \right) \left( A_2 - (x_1 - i \sqrt{1 - x_1^2}) \right) \right]^{-1/2}, \]
\[ f^-(x_1) = \left[ \left( A_1 - (x_1 + i \sqrt{1 - x_1^2}) \right) \left( A_2 - (x_1 + i \sqrt{1 - x_1^2}) \right) \right]^{-1/2}, \]
and hence,
\[
(f^+)^2(x_1) = \frac{1}{(A_1 - (x_1 - i\sqrt{1-x_1^2})) (A_2 - (x_1 - i\sqrt{1-x_1^2}))},
\]
\[
(f^-)^2(x_1) = \frac{1}{(A_1 - (x_1 + i\sqrt{1-x_1^2})) (A_2 - (x_1 + i\sqrt{1-x_1^2}))}.
\]

It follows that the functions \( f^+ f^- \) and \( (f^+)^2 + (f^-)^2 \) extend analytically from the interval \( \Delta \) to the domain \( D \). Consequently, the function \( \rho \), which is given by representation (33), extends holomorphically to the domain \( D \). Moreover,
\[
f^+(x_1) f^-(x_1) = \frac{1}{\sqrt{2A_1(a_1 - x_1)} \sqrt{2A_2(a_2 - x_1)}} = \frac{1}{2\sqrt{A_1(a_1 - x_1)A_2(a_2 - x_1)}},
\]
\[
(f^+)^2(x_1) + (f^-)^2(x_1) = \left( \frac{A_1 - (x_1 + i\sqrt{1-x_1^2})}{A_2 - (x_1 + i\sqrt{1-x_1^2})} \right) + \left( \frac{A_1 - (x_1 - i\sqrt{1-x_1^2})}{A_2 - (x_1 - i\sqrt{1-x_1^2})} \right)
\]
\[
= \frac{4A_1(a_1 - x_1)A_2(a_2 - x_1)}{4A_1(a_1 - x_1)A_2(a_2 - x_1)}.
\]

for \( x_1 \in \Delta \). Since \( 1 < a_1 < a_2 \), it can be easily shown that \((f^+)^2 + (f^-)^2 + f^+ f^-)(x_1) > 0 \) for \( x_1 \in \Delta \). So, using the definition of the function \( \varphi(z) \) and employing the identity
\[
\frac{\Delta f^3(x_1)}{\Delta f(x_1)} = ((f^+)^2 + (f^-)^2)(x_1) + (f^+ f^-)(x_1),
\]
where \( x \in \Delta \), we arrive at the explicit representation
\[
\rho_2(z) = \frac{1}{(A_1 - 1/\varphi(z))(A_2 - 1/\varphi(z))} + \frac{1}{(A_1 - \varphi(z))(A_2 - \varphi(z))} + \left[ (A_1 - 1/\varphi(z))(A_2 - 1/\varphi(z))(A_1 - \varphi(z))(A_2 - \varphi(z)) \right]^{-1/2} \tag{34}
\]
for the function \( \rho_2 \in \mathcal{H}(D) \), where \( z \in D \). Moreover, \( \rho_2(x_1) > 0 \) for \( x_1 \in \Delta \), \( \rho_2(\infty) = 1/(A_1A_2) \), and for \( x_2 \in \Delta \), we have
\[
\Delta \rho_2(x_2) := \rho_2^+(x_2) - \rho_2^-(x_2)
\]
\[
= \frac{-2i}{\sqrt{(A_1 - 1/\varphi(x_2))(A_2 - 1/\varphi(x_2))(\varphi(x_2) - A_1)(A_2 - \varphi(x_2))}} \tag{35}
\]
Therefore,
\[
\rho_2(z) = \rho_2(\infty) + \frac{1}{2\pi i} \int_{\gamma_2} \frac{\rho_2(t)}{t-z} \, dt \\
= \frac{1}{A_1 A_2} + \frac{1}{2\pi i} \int_{a_1}^{a_2} \frac{\Delta \rho_2(x_2)}{x_2 - z} \, dx_2,
\]
where \( z \in D_2, \gamma_2 \) is an arbitrary closed Jordan curve separating the interval \( \Delta_2 \) from the point \( z \) and from the infinity point and which is positively oriented with respect to the domain containing the point \( z \). The following representation for the function \( \rho_2(z) \) is a direct consequence of (35) and (36).

We have
\[
\rho_2(z) = \frac{1}{A_1 A_2} + \tilde{\sigma}_3(z),
\]
where \( \sigma_3 \) is a positive measure supported on the interval \( \Delta_2 \), \( \text{supp} \sigma_3 = \Delta_2 \), and moreover,
\[
d\sigma_3(x_2) = \frac{1}{\pi} \frac{dx_2}{\sqrt{(A_1 - 1/\varphi(x_2))(A_2 - 1/\varphi(x_2))((\varphi(x_2) - A_1)(A_2 - \varphi(x_2)))}},
\]
\( x_2 \in \Delta_2^c \).

So, for \( z \in D_2 \) we have
\[
\frac{\Delta f^3}{\Delta f}(z) = \rho_2(z) = \frac{1}{A_1 A_2} + \tilde{\sigma}_3(z).
\]
Hence, in view of (30) it follows from (38) that
\[
d\sigma_3(x_2) = \rho_3(x_2) \, d\sigma_2(x_2), \quad x_2 \in \Delta_2^c,
\]
where
\[
\rho_3(x_2) := \frac{1}{\sqrt{(A_1 - 1/\varphi(x_2))(A_2 - 1/\varphi(x_2))}}, \quad x_2 \in \Delta_2^c.
\]
The function \( \rho_3 \) extends holomorphically from the interval \( \Delta_2^c \) to the domain \( D_1 \). Furthermore, it is clear that \( \rho_3(z) \equiv f(z), \ z \in D_1 \). So, by (13)
\[
\rho_3(z) = \frac{1}{\sqrt{A_1 A_2}} + \tilde{\sigma}(z), \quad z \in D_1,
\]
where the measure \( \sigma \) is given by representation (25).
From (39), (40) and (41) it follows that, for \( x_1 \in \Delta_1 \),
\[
\frac{\Delta f^3(x_1)}{\Delta f(x_1)} = \frac{1}{A_1 A_2} + \int_{a_1}^{a_2} \frac{\rho_3(x_2) \, d\sigma_2(x_2)}{x_1 - x_2} \\
= \frac{1}{A_1 A_2} + \frac{1}{\sqrt{A_1 A_2}} \tilde{\sigma}_2(x_1) + \int_{a_1}^{a_2} \frac{\tilde{\sigma}(x_2) \, d\sigma_2(x_2)}{x_1 - x_2} \\
= \frac{1}{A_1 A_2} + \frac{1}{\sqrt{A_1 A_2}} \tilde{\sigma}_2(x_1) + \hat{s}(x_1),
\]
(42)
where the measure \( s \) is defined as \( s := \langle \sigma_2, \sigma \rangle \), \( \text{supp } s = \text{supp } \sigma_2 = \Delta_2 \). We have \( f^3(\infty) = 1/\sqrt{(A_1 A_2)^3} \), and hence, by Cauchy’s formula,
\[
f^3(z) - \frac{1}{(A_1 A_2)^3} = \frac{1}{2\pi i} \int_{\eta} \frac{f^3(t) \, dt}{t - z} = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\Delta f^3(x_1) \, dx_1}{x_1 - z}.
\]
(43)
So, using (39), (42) and (43),
\[
f^3(z) - \frac{1}{\sqrt{A_1 A_2}} = \frac{1}{A_1 A_2} \cdot \frac{1}{2\pi i} \int_{-1}^{1} \frac{\Delta f(x_1) \, dx_1}{x_1 - z} + \frac{1}{\sqrt{A_1 A_2}} \cdot \frac{1}{2\pi i} \int_{-1}^{1} \frac{\tilde{\sigma}_2(x_1) \Delta f(x_1) \, dx_1}{x_1 - z} \\
- \frac{1}{2\pi i} \int_{-1}^{1} \frac{\tilde{s}(x_1) \Delta f(x_1) \, dx_1}{x_1 - z} \\
= \frac{1}{A_1 A_2} \tilde{\sigma}(z) + \frac{1}{\sqrt{A_1 A_2}} \int_{-1}^{1} \frac{\tilde{\sigma}_2(x_1) \, d\sigma_2(x_1)}{z - x_1} + \int_{-1}^{1} \frac{\tilde{s}(x_1) \, d\sigma_2(x_1)}{z - x_1}.
\]
(44)
Finally, from (44) and the definition of the measure \( s := \langle \sigma_2, \sigma \rangle \) we have the representation
\[
f^3(z) = \frac{1}{\sqrt{(A_1 A_2)^3}} + \frac{1}{A_1 A_2} \tilde{\sigma}(z) + \frac{1}{\sqrt{A_1 A_2}} \hat{s}_1(z) + \hat{s}_2(z),
\]
where \( s_1 = \langle \sigma, \sigma_2 \rangle, s_2 = \langle \sigma, \sigma_2, \sigma \rangle, \text{supp } s_j = [-1, 1], j = 1, 2 \).

This proves (15), and therefore, Proposition 1.

**Remark 2.** The relations
\[
f(z) - \frac{1}{\sqrt{A_1 A_2}} = \tilde{\sigma}(z), \quad f^2(z) - \frac{1}{\sqrt{A_1 A_2}} f(z) = \hat{s}_1(z), \quad f^3(z) - \frac{1}{\sqrt{A_1 A_2}} f^2(z) = \hat{s}_2(z)
\]
(45)
are immediate consequences of (13)–(15).
Conjecture 1. Let $f$ be a function from the class $\mathcal{Z}$ of the form

$$f(z) = \left( \frac{A_1 - 1/\varphi(z)}{A_2 - 1/\varphi(z)} \right)^\alpha,$$

where $1 < A_1 < A_2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $f$ is a Markov function and, for any $n \in \mathbb{N}$, the system $f, f^2, \ldots, f^n$ is a Nikishin system.

Remark 3. In accordance with representation (11) all branch points of the function $f$ are of second order, and hence in view of the above Proposition the results of [2] and [8] on the asymptotics of Hermite–Padé polynomials apply to the system of functions $f, f^2$, and the results of [7], to the system of functions $f, f^2, f^3$. It is very likely that by appropriately transforming the independent variable (see, for example, [19, § 5]), which was carried out in representation (11), and multiplying some resulting functions it might be possible to obtain those exotic, as they may seem, Nikishin systems on star-like sets which have been considered in [3] and [9]. In other words, there is a hope that examples of Nikishin systems of such kind can be found in the form $f, f^2, \ldots, f^n$.

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