Zhu’s algebra, the $C_2$ algebra, and twisted modules

Matthias R. Gaberdiel and Terry Gannon

Abstract. In his landmark paper, Zhu associated two associative algebras to a VOA: what are now called Zhu’s algebra and the $C_2$-algebra. The former has a nice interpretation in terms of the representation theory of the VOA, while the latter only serves as a finiteness condition. In this paper we undertake first steps to unravel the interpretation of the $C_2$ vector space. In particular we suggest that it sees and controls the twisted representations of the VOA.

I. Introduction

We study the modules of a finite group $G$ through an associative algebra $\mathbb{C}G$ (its group algebra) with identical modules, and we study the modules of a Lie algebra $\mathfrak{g}$ through an associative algebra $U\mathfrak{g}$ (its universal enveloping algebra), again with the same modules. Likewise, the modules of a vertex operator algebra (VOA) $\mathcal{V}$ are in natural one-to-one correspondence with those of an associative algebra $\mathcal{A}(\mathcal{V})$ called Zhu’s algebra.

The most privileged class of Lie algebras are the finite-dimensional semi-simple ones. The analogous notion for VOAs is termed rationality: any sufficiently nice module of a rational VOA is completely reducible into a direct sum of simple modules. For a Lie algebra $\mathfrak{g}$, semi-simplicity is equivalent to the structural property that the radical of $\mathfrak{g}$ vanishes. For a VOA $\mathcal{V}$, the structural characterisation of rationality is conjectured to be the finite-dimensionality and semi-simplicity of the algebra $\mathcal{A}(\mathcal{V})$.

Closely related to Zhu’s algebra is the $C_2$-algebra $\mathcal{A}_{[2]}(\mathcal{V}) = \mathcal{V}/C_2(\mathcal{V})$. The hypothesis of its finite-dimensionality was first introduced and heavily used in [20]. It implies for instance that $\mathcal{V}$ and its modules are finitely generated, that $\mathcal{V}$ has only finitely many irreducible modules, that the fusion coefficients are finite, and that the $\mathcal{V}$-modules $M$ have characters $\chi_M(\tau)$ holomorphic in $\mathbb{H}$ and obeying a weak modular invariance. It had been often conjectured that the finite-dimensionality of $\mathcal{A}_{[2]}(\mathcal{V})$ is equivalent to rationality of $\mathcal{V}$, but a counterexample is the so-called triplet

2000 Mathematics Subject Classification. Primary 17B69; Secondary 81T40.

This paper is based on work done during an extended visit by TG at ETH funded by the CTS – he thanks them warmly for their generous hospitality. The research of MRG is supported in part by the Swiss National Science Foundation, while that of TG is supported in part by NSERC. We thank Andy Neitzke, who collaborated in an early stage of this work.
algebra \[\mathcal{A}_2(V)\] finite-dimensionality to characterise the \textit{finite logarithmic} VOAs \[\mathcal{A}(V)\]. To prove the finite-dimensionality of \(\mathcal{A}_2(V)\) from the semi-simplicity and finite-dimensionality of \(\mathcal{A}(V)\) is a fundamental problem of VOA theory (first conjectured in \([20]\)), and one of the motivations for our work.

For the remainder of this paper, let \(V\) be a rational VOA. For concreteness, we can take this to mean:

1. \(V\) is a \textit{simple} VOA (so \(V\) is an irreducible \(V\)-module for itself);
2. as a \(V\)-module, \(V\) is isomorphic to its contragredient;
3. \(V\) is of \textit{CFT-type}: \(V_n = 0\) for \(n < 0\), \(V_0 = \mathbb{C}|0\rangle\); and
4. \(V\) is \textit{regular}: every weak \(V\)-module is completely reducible (these hypotheses imply then semi-simplicity of \(\mathcal{A}(V)\) as well as finite-dimensionality of \(\mathcal{A}_2(V)\) \([17]\)). Under these conditions, Zhu’s theorem \([20]\) applies, and the \(\mathbb{C}\)-span of the characters corresponding to the irreducible modules is \(\text{SL}_2(\mathbb{Z})\)-invariant. Also, Verlinde’s formula holds, and the \(V\)-modules form a modular tensor category \([15]\).

In this paper we shall mainly deal with very specific examples: most importantly, \(V\) associated to even lattices, and to affine algebras at integral level. For the construction and basic properties of these VOAs, see \textit{e.g.} \([16]\); for other general surveys of VOAs \etc. see \textit{e.g.} \([5,13]\).

The authors have been exploring the intimate relation between Zhu’s algebra and the \(C_2\)-algebra in rational VOAs \([11]\); in the present paper we motivate and review some initial results and early speculations. We describe in Section II Zhu’s algebra, the \(C_2\)-algebra, and the notion of twisted modules. Zhu’s definitions \([20]\) of \(\mathcal{A}(V)\) and \(\mathcal{A}_2(V)\) are very technical and unmotivated; we include a motivation due to Gaberdiel and Goddard \([10]\) which underlies our approach. We review how Zhu’s algebra sees the \(V\)-modules, and explain how the closely related \(\mathcal{A}_2(V)\) sees the twisted \(V\)-modules. Section III presents some of the general theory: \textit{e.g.} the dual space \(\mathcal{A}(V)^*\) embeds naturally in \(\mathcal{A}_2(V)^*\); what meaning can be ascribed to the ‘discrepancy’ \(\mathcal{A}_2(V)^*/\mathcal{A}(V)^*\) and how often is it zero? Section IV summarises some of our calculations. We collect in Section V some speculations and open questions.

It is an honour to dedicate this paper to Geoff Mason – his work has been an inspiration to both of us.

\section*{II. Background}

\subsection*{II.1. Zhu’s algebra: the what and the why.} Let us begin by describing informally the ideas behind Zhu’s algebra (surely at least some of this fed Zhu’s intuition), because it is crucial to the rest of this paper. As with much of VOA theory, the basic ideas come from conformal field theory (CFT) or, equivalently, perturbative string theory.

The correlation functions are how a quantum field theory makes contact with experiment. All physical content is there. In the case of a CFT, the correlation functions are (essentially) the conformal blocks. Consider the Riemann sphere \(\mathbb{P}^1\), and choose \(n \geq 2\) distinct points \(w_i \in \mathbb{P}^1\); to each of these points we select an irreducible \(V\)-module \(M_i\), and a vector \(a_i \in M_i\). The corresponding conformal blocks will be a span of formal objects of the form

\[\langle \mathcal{Y}_1 \cdots \mathcal{Y}_{n-2} \rangle ,\]  

(2.1)
where each \( \mathcal{Y}_i \) is an intertwining operator \( \text{e.g. } \mathcal{Y}_1 \) will be a linear map sending \( a_1 \otimes a_2 \in M^1 \otimes M^2 \) to a formal series with coefficients in some irreducible \( \mathcal{V} \)-module \( N^1 \) contained in the fusion product of \( M^1 \) and \( M^2 \). The details are not important: the conformal block will be a complex-valued function, multilinear in the \( a_i \in M^i \), and locally meromorphic (as a density or differential form) in the configuration space \( (w_1, \ldots, w_n) \in (\mathbb{P}^1)^n \setminus \Delta \), where \( \Delta \) consists of the diagonals, where some \( w_i \) coincides with some \( w_j \). The space of all such conformal blocks, has dimension given by a Verlinde formula and depends only on \( M^i \). It suffices to consider only highest-weight states \( a_i \in (M^i)_0 \), i.e. the vectors in \( M^i \) of lowest \( L_0 \)-eigenvalue, as the other conformal blocks are obtained from these by standard differential operators.

Let us probe the (vacuum-to-vacuum) conformal block \( (2.1) \) by ‘inserting’ a state \( v \in \mathcal{V} \). This amounts to adding an \((n+1)\)-st point \( w_{n+1} \), with module \( M^{n+1} = \mathcal{V} \). Conventionally we shall take, without loss of generality, \( w_{n+1} = 0 \) (which we think of as time \( t = -\infty \)), so the other \( w_i \) are now required to avoid 0. The resulting conformal block is a function

\[
\eta_{\{(w_1, M^1), \ldots, (w_n, M^n)\}} \in \text{Hom}\left( \mathcal{V}, (M^1 \otimes \cdots \otimes M^n)^* \right),
\]

where ‘\( \otimes \)’ denotes fusion product, ‘\( * \)’ denotes a restricted dual, and ‘\( \text{Hom} \)’ is the space of \( \mathcal{V} \)-module maps (intertwines). This looks fancier than it really is: ‘\( \text{Hom} \)’ chooses the specific conformal block (from among the space of these); the conformal block is a complex-valued function of \( v \in \mathcal{V} \) and the \( a_i \in M^i \). Choosing \( v \) to be the vacuum \( |0\rangle \) in the VOA recovers the conformal block \( (2.1) \).

There is a large subspace \( \mathcal{O}_{\{w_1, \ldots, w_n\}} \) of \( \mathcal{V} \), independent of the choice of modules \( M^i \), for which \( (2.2) \) must vanish for elementary reasons. For example, if say \( w_1 = \infty \) (and the \( a_i \) are indeed highest-weight), then \( \mathcal{O}_{\{w_1, \ldots, w_n\}} \) is spanned by

\[
\text{Res}_z \left( Y(a, z)z^{1-m-(2-n)|a|} \prod_{j=2}^n (z-w_j)^{|a|b} \right)
\]

for all choices of integer \( m > 0 \), and vectors \( a, b \in \mathcal{V} \), where \( a \) is homogeneous with respect to \( L_0 \): \( L_0 a = |a| a \). Indeed, such a vector can be (formally) written as a contour integral, and it is easy to see that the integrand does not have any poles apart from at \( 0 \).

Let us define \( \mathcal{A}_{\{w_1, \ldots, w_n\}}(\mathcal{V}) = \mathcal{V}/\mathcal{O}_{\{w_1, \ldots, w_n\}} \). Then for any \( a_i \in (M^i)_0 \), we can regard \( \eta_{\{(w_1, M^1), \ldots, (w_n, M^n)\}} \) as a linear functional in \( \mathcal{A}_{\{w_1, \ldots, w_n\}}(\mathcal{V}) \). The converse is also true but much deeper: any \( \eta \in \mathcal{A}_{\{w_1, \ldots, w_n\}}(\mathcal{V}) \) is a correlation function, for some choice of modules \( M^i \) and highest weight states \( a_i \in (M^i)_0 \) attached to the points \( w_i \in \mathbb{P}^1 \). For a rational VOA, all these spaces will be finite-dimensional.

An elementary observation [19] is that this space \( \mathcal{A}_{\{w_1, \ldots, w_n\}}(\mathcal{V}) \) is independent of the order of the \( w_i \)’s. Moreover, by an analytic continuation argument based on a Knizhnik–Zamolodchikov-like connection, for each homotopy class of paths in the configuration space \( (\mathbb{P}^1 \setminus \{0\}) \setminus \Delta \) linking two \( n \)-tuples \( (w_1, \ldots, w_n) \) and \( (w'_1, \ldots, w'_n) \), we get a natural isomorphism \( \mathcal{A}_{\{w_1, \ldots, w_n\}}(\mathcal{V}) \cong \mathcal{A}_{\{w'_1, \ldots, w'_n\}}(\mathcal{V}) \). This means we get an action of the braid group \( B_n \) on \( \mathcal{A}_{\{w_1, \ldots, w_n\}}(\mathcal{V}) \) by automorphisms, although this is not important for us.
Comparing (2.3) with [20], we see that Zhu’s algebra $\mathcal{A}(\mathcal{V})$ is $\mathcal{A}_{(\infty,-1)}(\mathcal{V})$. In this case, the correlation functions will vanish unless $M^1$ and $M^2$ are duals (contragredients) of each other. So what we find from the above is that Zhu’s algebra carries information on all the irreducible $\mathcal{V}$-modules: its dual $\mathcal{A}(\mathcal{V})^*$ naturally decomposes into a sum of nontrivial subspaces, one for each pair $(M, M^*)$. We will make this more precise shortly.

The algebraic structure of Zhu’s algebra comes from the action of the zero-modes $o(a) = a_{|a| - 1}$ on the highest weight states. Indeed, $o(a)$ commutes with $L_0$ and thus maps each homogeneous subspace $M_n$ of any (irreducible) $\mathcal{V}$-module $M = \bigoplus_{n \geq 0} M_n$ to itself, in particular the highest weight space $M_0$. We can describe this zero mode action in terms of a product in the $(n + 1)$-st module $M^{n+1} = \mathcal{V}$; for example, for the case considered by Zhu where we just have two points $w_1 = \infty$ and $w_2 = -1$, the relevant product is [20]

$$a \ast b = \text{Res}_{z} \left( Y(a, z) z^{-1} (z + 1)^{|a|b} \right)$$

for any $a, b \in \mathcal{V}$. By similar contour integral arguments as above it is easy to see

$$\eta_{(\infty, M^1), (-1, M^2)}(a \ast b)[m_1 \otimes m_2] = \eta_{(\infty, M^1), (-1, M^2)}(b)[o(a)m_1 \otimes m_2],$$

where $m_i \in (M^i)_0, \ i = 1, 2$. As is expected from (2.5) $\mathcal{O}(\mathcal{V}) = \mathcal{O}_{(\infty,-1)}(\mathcal{V})$ is an ideal for this action. Thus we obtain an action of $\mathcal{V}$ on $\mathcal{A}(\mathcal{V})$, and this turns $\mathcal{A}(\mathcal{V})$ into an associative algebra. In fact, the product structure simply describes the product of the zero modes: i.e. on highest weight states one finds $o(a \ast b) = o(a)o(b)$.

In general, the other spaces $\mathcal{A}_{(w_1, \ldots, w_n)}(\mathcal{V})$ do not seem to naturally be algebras. However, their duals $\mathcal{A}_{(w_1, \ldots, w_n)}(\mathcal{V})^*$ always have $n$ commuting actions of $\mathcal{A}(\mathcal{V})$ (one for each point $w_i$) – see [9] for details.

For rational VOAs, Zhu [20] proved that his associative algebra $\mathcal{A}(\mathcal{V})$ is finite-dimensional and semi-simple, and hence by Wedderburn’s Theorem a direct sum of matrix algebras: in fact we have

$$\mathcal{A}(\mathcal{V}) \cong \bigoplus_{M} \text{End}(M_0) = \bigoplus_{M} M_0^* \otimes M_0,$$

where $M$ runs over all irreducible $\mathcal{V}$-modules. This is the precise sense in which Zhu’s algebra sees all $\mathcal{V}$-modules. The 2-point correlation functions are parametrised by a choice of module $M$ and states $a_1 \in M_0$ and $a_2 \in (M^*)_0 = (M_0)^*$, together filling out the summand $M_0 \otimes (M^*_0)$ in the dual of (2.6).

II.2. When $2$ become $1$: the $C_2$ algebra. The conformal blocks (2.1) live on the moduli space of an $n$-punctured sphere. For example, for $n = 4$, this moduli space can be identified with the sphere minus 3 points (the Möbius transformations send the 3 points to $0, 1, \infty$, and the fourth point is then free provided it avoids those). Now in string theory, it is meaningful (‘amplitude factorisation’) to move towards boundary points in these moduli spaces, e.g. to send two of these $n$ points together. This is also meaningful in number theory (‘cusps’), and the Deligne-Mumford compactification of moduli space tells us to interpret those boundary points as surfaces with nodes. For example, for $n = 4$, the three boundary components correspond to the three ways to partition the 4 points into 2 pairs; the corresponding surface is two tangential spheres, each containing two of the points.

In any case, we are led to consider what happens when some of the $w_i$ coincide. Most of the treatment of the previous subsection goes through without change. We can formally speak of conformal blocks, and define the subspaces $\mathcal{O}_{(w_1, \ldots, w_n)}(\mathcal{V})$ and
the corresponding quotient spaces $A_{(w_1,\ldots, w_n)}(V)$ as before. To understand more explicitly what happens consider Zhu’s algebra $A(V)$ which is naturally isomorphic to $A_{(\infty, w_2)}(V)$ for any $w_2 \neq \infty$. For any such $w_2$, the elements spanning $O_{(\infty, w_2)}(V)$ are of the form (after appropriate rescaling)

$$\text{Res}_z \left( Y(a, z) z^{-2} (1 - \frac{z}{w_2})^{|a|} b \right) = \sum_{l=0}^{|a|} |q| \binom{|a|}{l} (-w_2)^{-l} a_{-2+l} b .$$

In the limit $w_2 \to \infty$, only the leading term survives, and thus the space $O_{(\infty, \infty)}(V)$ is spanned by the elements of the form $a_{-2} b$; this is precisely the $C_2(V)$ space of Zhu, and hence $A_{(\infty, \infty)}(V)$ is the $C_2$ quotient space of Zhu. Similarly, the product $a * b$ in $A_{(\infty, w_2)}(V)$ (appropriately rescaled) can be written as

$$a *_{w_2} b = \text{Res}_z \left( Y(a, z) z^{-1} (1 - \frac{z}{w_2})^{|a|} b \right) = \sum_{l=0}^{|a|} |q| \binom{|a|}{l} (-w_2)^{-l} a_{-1+l} b .$$

This tends to the product $ab := a_{-1} b$ of Zhu for the $C_2$-space $A_{(\infty, \infty)}(V)$. The resulting algebra is both commutative and associative, for elementary reasons. In fact, Zhu noticed that this quotient also has a Poisson structure: the Lie bracket

$$\{a, b\} := a_{-1} b.$$

We shall call this (commutative Poisson) algebra the $C_2$-algebra $A_2(V)$.

We can thus think of Zhu’s algebra as being a deformation of the $C_2$-algebra. However, this deformation picture is certainly false if taken too literally (as it has in the literature – see e.g. the last paragraph of Section 4 of [1]). In particular, although the ideal $O_{(\infty, w_2)}(V)$ tends to $O_{(\infty, \infty)}(V)$, elements that were non-trivial in the former can tend to 0 in the limit. The correct statement is that there is a natural surjection $O_{(\infty, w_2)}(V) \to O_{(\infty, \infty)}(V)$ (which may not be an injection). Thus the $\dim A(V)$ of Zhu’s algebra is bounded above by that of the $C_2$-algebra $A_2(V)$, and the dual $A(V)^*$ can be regarded as a subspace of $A_2(V)^*$. We return to this in Section III.1.

As before we can reorder the $n$ points $w_i$ so that the first $n_1$ are identical, the next $n_2$ are identical (but different from the first $n_1$), etc, where $n_1 \geq n_2 \geq n_k > 0$ is some partition of $n$; then the analytic continuation argument shows that the spaces $A_{(w_1, \ldots, w_n)}(V)$ and $A_{(w'_1, \ldots, w'_n)}(V)$ are isomorphic whenever the $w_i$ and the $w'_i$ correspond to the same partition of $n$. Thus we may speak of the space $A_{(n_1, \ldots, n_k)}(V)$. This explains our notation $A_2(V)$ for the $C_2$-algebra; likewise, Zhu’s algebra $A(V)$ is $A_{[1,1]}(V)$ in this notation.

It is elementary that Zhu’s algebra $A(V)$ sees two commuting actions of the automorphism group $\text{Aut}(V)$ of the VOA, one attached to each point $w_i$. As these points are brought together to form $A_2(V)$, what survives is the diagonal action. So the $C_2$-algebra carries an adjoint action of $\text{Aut}(V)$, helping significantly to organise $A_2(V)$, which in specific calculations can get quite large. For lattice VOAs $V_L$, $\text{Aut}(V_L)$ contains the automorphism group of the lattice $L$; for affine algebra VOAs $V_{g,k}$, $\text{Aut}(V_{g,k})$ contains the simply connected Lie group corresponding to $g$.

The importance of Zhu’s algebra is that its representation theory is isomorphic to that of the VOA. On the other hand, it is hard to imagine any useful direct relation between the $A_2(V)$-modules and the $A(V)$- or $V$-modules. As an algebra, $A_2(V)$ is isomorphic to the $d \times d$ diagonal matrices, where $d = \dim A_2(V)$. Hence there are exactly $d$ irreducible $A_2(V)$-modules, all one-dimensional: the $i$th one
is the projection to the $i$th diagonal entry of the matrices. Nevertheless, we will explain next subsection that $A_{[2]}$, or rather its dual space, is intimately connected to the representation theory of the VOA.

II.3. Twisted modules for lazy people. We shall assume the reader is familiar with the usual notion of a VOA module – see e.g. [16,5,13] for more details. Twisted modules are a natural generalisation, and a central part of the whole VOA story. Indeed, they are key to the orbifold construction. They are at least as important for VOAs, as projective representations are to groups. In fact they are sort of a dual concept to projective representation: to unprojectify a projective representation, you take a central extension of the group; to untwist a twisted module, you restrict to a subalgebra of the VOA.

Probably the easiest path to twisted modules is through the loop algebra. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra (over $\mathbb{C}$). By the loop algebra $L\mathfrak{g}$ we mean the space of all combinations $\sum_{n \in \mathbb{Z}} a_n t^n$, where $a_n \in \mathfrak{g}$ and all but finitely many $a_n$ are 0 ($t$ is a formal variable). This inherits a Lie algebra structure from $\mathfrak{g}$. The nontwisted affine Kac-Moody algebra $\mathfrak{g}^{(1)}$ is just the extension of $\mathfrak{g}$ by a central element $c$ and a derivation $\ell_0$.

Now let $\alpha$ be any automorphism of $\mathfrak{g}$, of order $N < \infty$. We can diagonalise $\alpha$: for $0 \leq j < N$ let $\mathfrak{g}_j$ be the eigenspace of $\alpha$ in $\mathfrak{g}$ with eigenvalue $\xi^j_N$, where we write $\xi_N = e^{2\pi i / N}$. Of course $\alpha$ extends to an automorphism of $L\mathfrak{g}$ by sending $t^n$ to $\xi_N^n t^n$, and to the affine algebra $\mathfrak{g}^{(1)}$ by fixing $c$ and $\ell_0$. By the twisted affine algebra $\mathfrak{g}^{(N)}$ we mean the subalgebra of $\mathfrak{g}^{(1)}$ fixed by $\alpha$. The twisted affine algebras behave very similarly to the more familiar nontwisted ones.

Now let $\rho$ be any integrable highest weight representation of $\mathfrak{g}^{(N)}$. We can lift $\rho$ to $\mathfrak{g}^{(1)}$ by defining $\rho(at^n) = \xi^{i+n}_N \rho(at^{-j}) t^{j+n}$ for $a \in \mathfrak{g}_j$. This will not be a true representation of the nontwisted algebra $\mathfrak{g}^{(1)}$, as it obeys

$$[\rho(at^n), \rho(bt^m)] = \xi^{i+k+m+n}_N \rho([at^n, bt^m])$$

when $a \in \mathfrak{g}_j$ and $b \in \mathfrak{g}_k$. We call such a $\rho$ a twisted representation of $\mathfrak{g}^{(1)}$. Thus a true representation of a twisted affine algebra lifts to a twisted representation of a nontwisted affine algebra.

The definition for VOAs is very analogous (see e.g. [5,13]). Incidentally, it is possible to generalise the spaces $A_{[w_1, \ldots, w_k]}(\mathcal{V})$ of Section II.1 to the case where now at some (or all) of the $w_i$ states from a twisted $\mathcal{V}$-module $M'$ are inserted– see e.g. [3].

Twisted modules are a crucial, though unexplored, part of the $C_2$-algebra story. We explained at the end of Section II.1 how, for any $\mathcal{V}$-module $M$, any choice $u \in M_0, v \in M_0'$ yields a unique vector $u \otimes v \in A(\mathcal{V})^*$. Since $A(\mathcal{V})^*$ embeds in $A_{[2]}(\mathcal{V})^*$, $u \otimes v$ can also be regarded as a vector in $A_{[2]}(\mathcal{V})^*$. If instead $M$ is a twisted $\mathcal{V}$-module, then $u \otimes v$ maps into the appropriate twisted Zhu’s algebra $A_{\rho}(\mathcal{V})$, defined in [3]. Implicit in the above treatment is that twisted modules are characterised by monodromy properties about the point $w$ they have been inserted; as the two points $w_i$ are brought together, we cannot tell any more whether $u \otimes v$ came from twisted or untwisted modules. This means that each $A_{\rho}(\mathcal{V})^*$ also embeds into $A_{[2]}(\mathcal{V})^*$. Clearly, the images for different automorphisms $\rho$ can overlap, and we do not yet understand the relation between these different images. But it should be clear that the $C_2$-algebra must be large enough to contain every $A_{\rho}(\mathcal{V})$. This accounts for some, and perhaps all, instances where the $C_2$-algebra is
larger than Zhu’s algebra. It also provides a partial, and perhaps complete, answer to the question of the direct relevance of the $C_2$-algebra (or rather its dual) to the representation theory of $\mathcal{V}$.

III. Abstract nonsense

In this section we collect some general comments about the $C_2$-algebra and its relation with Zhu’s algebra.

III.1. Zhu’s algebra as a deformation of the $C_2$-algebra. As we have explained before in Section II.2, Zhu’s algebra $A(V)$ is a ‘deformation’ of the $C_2$ algebra $A_{[2]}(V)$. As we have also explained there, the dimension of $A(V)$ may be smaller than that of the $C_2$ space $A_{[2]}(V)$. The situation is vaguely reminiscent of deformation quantisation, where a commutative Poisson algebra (describing the classical world) is deformed into a noncommutative algebra (describing the quantum world). For this reason we suggest calling a VOA anomalous if the dimension of $A_{[2]}(V)$ is strictly larger than that of $A(V)$.

Note that $A(V_1 \otimes V_2) = A(V_1) \otimes A(V_2)$ and $A_{[2]}(V_1 \otimes V_2) = A_{[2]}(V_1) \otimes A_{[2]}(V_2)$, so the $C_2$-algebra and Zhu’s algebra of the tensor product $V_1 \otimes V_2$ of VOAs will have equal dimension iff the same holds for both $V_1$ and $V_2$.

As explained in Section II.2, we can think of the dual $A(V)^*$ as being a subspace of $A_{[2]}(V)^*$. Let us call the quotient $A_{[2]}(V)^*/A(V)^*$ the deficiency, for want of a better name. This finite-dimensional space is then nontrivial iff $V$ is anomalous. Is there a cohomological interpretation for the deficiency? Of course there is a rich relation of Hochschild cohomology to the deformation theory of algebras [14]. For example, the group $H^i(A_{[2]}(V); A_{[2]}(V))$ for $i = 1, 2$ respectively, equals the space of infinitesimal automorphisms, respectively the space of infinitesimal deformations, of the $C_2$-algebra, and this group for $i = 3$ controls whether these infinitesimal deformations can be ‘integrated’. Hence whenever that second cohomology group vanishes, the VOA will either be anomalous, or $\dim M_0 = 1$ for all irreducible $M$. However, this remark is too naive to be of any value, because the $C_2$-algebra is too uninteresting. A proper cohomological treatment of deficiency etc. would have to involve more of the structure of $\mathcal{V}$.

III.2. Filtrations versus gradings. An algebra $A$ is called graded if $A$ is the direct sum $\oplus_{n=0}^{\infty} A^n$ of subspaces $A^n$, such that $A^mA^n \subseteq A^{m+n}$. For example, the polynomials $A = \mathbb{C}[x]$ are graded by degree, so each $A^n = \mathbb{C}x^n$ here is one-dimensional.

An algebra $A$ is called filtered if $A$ is the union $\bigcup_{n=0}^{\infty} A_n$ of an increasing sequence $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ of subspaces, such that $A_mA_n \subseteq A_{m+n}$. Any graded algebra is filtered: just take $A_n = \oplus_{m=0}^{n} A^m$. A filtered algebra which is not graded, is the universal enveloping algebra $U\mathfrak{g}$: assign degree 1 to every element of $\mathfrak{g}$, and let $U\mathfrak{g}_n$ consist of all polynomials in $\mathfrak{g}$, each term in which has total degree $\leq n$. Degree does not define a grading on $U\mathfrak{g}$ (unless $\mathfrak{g}$ is abelian): for any noncommuting $x, y \in \mathfrak{g}$, $xy$ and $yx$ both have degree 2 but their difference $[x, y]$ has degree 1.

The $C_2$-algebra is graded by $L_0$-eigenvalue, since its ideal $C_2(V)$ is spanned by homogeneous elements $a_{-2}b$, and the product $a_{-1}b$ respects $L_0$-grading. On the other hand, Zhu’s algebra is only filtered by $L_0$, since the elements (2.3) spanning its ideal are not homogeneous.

There is a standard way to go from a filtered algebra $A = \bigcup_n A_n$ to a graded algebra $A_{\gr}$: define $(A_{\gr})^n = A_n/A_{n-1}$. If $A$ is in fact graded, then $A_{\gr} \cong A$. 


If $A$ is finite-dimensional, then $\dim A_{gr} = \dim A$. For example, $U_{\mathfrak{g}_{gr}}$ is naturally isomorphic to the symmetric (polynomial) algebra $S\mathfrak{g}$, obtained by identifying $\mathfrak{g}$ with $U\mathfrak{g}_1/U\mathfrak{g}_0$. $U\mathfrak{g}$ carries two commuting $\mathfrak{g}$-actions: the left- and right-regular actions $gu$ and $-ug$; $S\mathfrak{g}$ carries the adjoint $\mathfrak{g}$-action $gu - ug$.

It is elementary to verify that the ‘gradification’ $\mathcal{A}(V)_{gr}$ can be identified (though not canonically) with a subspace of $\mathcal{A}[2](V)$, and hence with all of $\mathcal{A}[2](V)$ if their dimensions match. What role in the general story this gradification plays, is not yet clear to us. But as we shall discuss in Section III.4, for the VOAs associated to affine algebras, this point of view could be very important.

### III.3. Zhu’s algebra and the $C_2$-algebra for lattices.

Let $L$ be any even positive-definite lattice (so $\alpha \cdot \alpha \in 2\mathbb{Z}_{\geq 0}$ for any $\alpha \in L$). Let $n$ be its dimension. Fix a basis $\{\beta_1, \ldots, \beta_n\}$ of $L$. See e.g. [16] for the construction of $\mathcal{V}_L$. As a vector space, $\mathcal{V}_L$ is spanned by terms of the form

$$\beta_{i_1}(-k_1) \beta_{i_2}(-k_2) \cdots \beta_{i_m}(-k_m) e^\alpha,$$

where $m \geq 0$, each $k_i \in \mathbb{Z}_{\geq 0}$, and $\alpha \in L$. The oscillators $\beta_i(-k)$ commute with each other – apart from that, the vectors in (3.1) are linearly independent.

It can be shown [11] that the $C_2$-algebra ideal $C_2(\mathcal{V}_L)$ is spanned by all terms of the form (3.1), provided at least one $k_i$ is $\geq 2$, together with all vectors of the form

$$\beta_{i_1}(-1) \cdots \beta_{i_m}(-1) \gamma(-1)^{\max\{0,1+\gamma \cdot \gamma - |\gamma \cdot \alpha|\}} e^\alpha.$$

Thus a basis for $\mathcal{A}[2](\mathcal{V}_L)$ can be found with coset representatives of the form

$$\beta_{i_1}(-1) \cdots \beta_{i_m}(-1) e^\alpha,$$

where $\alpha$ belongs to the finite set

$$SL = \{\alpha \in L \mid \gamma \cdot \gamma \geq \gamma \cdot \alpha \ \forall \gamma \in L\}$$

of ‘small’ lattice vectors; of course which oscillators $\beta_i(-1)$ to choose in (3.3) depends very much on the choice of $\alpha \in SL$.

It is easy to see from this description of $C_2(\mathcal{V}_L)$ that $\mathcal{A}[2](\mathcal{V}_L)$ is finite-dimensional for any $L$ (first proved in [4]). Next section we explain how to use the preceding paragraph to find $\mathcal{A}[2](\mathcal{V}_L)$, or at least lower bounds for $\dim \mathcal{A}[2](\mathcal{V}_L)$, for explicit $L$.

The irreducible modules for $\mathcal{V}_L$ are in natural one-to-one correspondence with the cosets $[t] \in L^*/L$, where $L^*$ is the dual lattice of $L$. The character of the module corresponding to $[t]$ is the theta series of the shifted lattice $[t]$, divided by $\eta(\tau)^n$. Its leading term is the number $N_{[t]}$ of vectors in $[t]$ of smallest norm. The dimension of Zhu’s algebra is then

$$\dim \mathcal{A}(\mathcal{V}_L) = \sum_{[t] \in L^*/L} N_{[t]}^2.$$  

A priori, there seems little relation between (3.4) and $\dim \mathcal{A}[2](\mathcal{V}_L)$ – a reason for this is implicit in Section IV.4.

### III.4. Affine Lie algebras.

An important and nontrivial class of rational VOAs are associated to a choice of finite-dimensional simple Lie algebra $\mathfrak{g}$, and a positive integer $k$ (the ‘level’). The associated rational VOA was constructed in [8] and will be denoted $\mathcal{V}_{\mathfrak{g},k}$. Its homogeneous space $(\mathcal{V}_{\mathfrak{g},k})_1$ is canonically identified
with $\mathfrak{g}$. This VOA is intimately connected to the affine non-twisted algebra $\mathfrak{g}^{(1)}$; in particular, as spaces $V_{\mathfrak{g},k}$ is the integrable $\mathfrak{g}^{(1)}$-module $L(k\Lambda_0)$, and the irreducible $\mathfrak{V}_{\mathfrak{g},k}$-modules are the level $k$ integrable highest weight $\mathfrak{g}^{(1)}$-modules $L(\lambda)$. Write $\lambda = \sum_{i=0}^r \lambda_i \Lambda_i$.

Zhu’s algebra here can be identified [8] with the quotient $U\mathfrak{g}/\langle e^{k+1}_\theta \rangle$, where $\langle e^{k+1}_\theta \rangle$ is the 2-sided ideal of $U\mathfrak{g}$ generated by $e^{k+1}_\theta$ ($\theta$ is the highest root of $\mathfrak{g}$). The space $M_\lambda$ for the $\mathfrak{V}_{\mathfrak{g},k}$-module associated to $\lambda$, can be identified with the irreducible $\mathfrak{g}$-module with highest weight $\lambda = \sum_{i=1}^r \lambda_i \Lambda_i$, so the dimension of Zhu’s algebra then follows from e.g. Weyl’s dimension formula.

The $C_2$-algebra arises naturally as a quotient $S\mathfrak{g}/I(k)$. Here, $S\mathfrak{g}$ is generated by the $-1$-modes of $(V_{\mathfrak{g},k})_1 \cong \mathfrak{g}$, and the $\mathfrak{g}$-action on it comes from the zero-modes of $(V_{\mathfrak{g},k})_1 \cong \mathfrak{g}$. The $m$'th graded piece of $S\mathfrak{g}$ can be identified with the $m$'th symmetric power of the adjoint module of $\mathfrak{g}$. The ideal $I(k)$ is generated from $e^{k+1}_\theta$ using the $\mathfrak{g}$-action on $S\mathfrak{g}$ described earlier.

Zhu’s algebra inherits the filtration of $U\mathfrak{g}$. Put $I_n = \langle e^{k+1}_\theta \rangle \cap U\mathfrak{g}_n$ and write $I_{gr} = \oplus_n I_n/I_{n-1}$ as usual. Then the ‘gradation’ $\mathcal{A}(\mathcal{V})_{gr}$ is canonically isomorphic to $S\mathfrak{g}/I_{gr}$. We would like to understand better the relation between the ideals $I_{gr}$ and $I(k)$ of $S\mathfrak{g}$, as this seems a very promising approach to the question of anomalous $V_{\mathfrak{g},k}$. The former ideal contains the latter, and this defines the surjection $\mathcal{A}_2[2](V_{\mathfrak{g},k}) \to \mathcal{A}(V_{\mathfrak{g},k})$. For most pairs $\mathfrak{g}, k$ it seems, these ideals are identical (see Section IV.5 below).

IV. Calculations

IV.1. The Virasoro minimal models. Perhaps the easiest examples to work out are the Virasoro minimal models $\mathcal{V}_{p,q}^\text{Vir}$, where $p, q \in \{2, 3, 4, \ldots \}$ are coprime (see e.g. [6]). In this case there are $(p-1)(q-1)/2$ irreducible modules $M$, all with 1-dimensional $M_0$, so $\mathcal{A}$ here is commutative, of dimension $(p-1)(q-1)/2$. $\mathcal{A}_2$ is easy to identify because one null vector is $L_{-1}|0\rangle$, so $\mathcal{A}_2$ has a basis of the form $L_{-2}^i|0\rangle + C_2$: the other null vector, whose leading term is $L_{-2}^{(p-1)(q-1)/2}|0\rangle$, then forces $0 \leq i < (p-1)(q-1)/2$. Thus the minimal models are non-anomalous.

IV.2. Affine $sl(2)$ at level $k$. This is again very easy, and we know of at least 4 independent ways to prove that the VOA is non-anomalous. For reasons of space we shall give only one.

Let $k$, the level, be any positive integer. The rational VOA $V_{sl(2),k}$, as a space, is given by the highest weight $sl(2)^{(1)}$-module $L_{k\Lambda_0} = U(sl(2)^{(1)})|0\rangle$, and so inherits the filtration from the universal enveloping algebra. This permits us to refine the character of $V_{sl(2),k}$, to be a function not only of the usual $q$ (which keeps track of the $L_0$-eigenvalue, what we are calling the grade) and $z$ (which lies in the $SL(2)$ maximal torus so is the argument for $SL(2)$-characters), but another parameter $t$ (which will keep track of this degree). More precisely, each creation operator $x_{-n}$ will contribute 1 to the degree but $n$ to the grade.

The result is [7]:

$$
\chi_{V_{sl(2),k}}(q, z, t) = \sum_{\vec{\mathfrak{h}}, \vec{\mathfrak{e}}, \vec{\mathfrak{f}} \in \mathbb{Z}^{k}_{\geq 0}} q^{\vec{\mathfrak{h}} \cdot \vec{E} + \vec{\mathfrak{f}} \cdot \vec{F} + \vec{\mathfrak{e}} \cdot \vec{H}} z^{2(\vec{\mathfrak{e}} \cdot \vec{\mathfrak{f}})} \frac{q^{\vec{E}AE^T + \vec{H}AH^T + \vec{F}AF^T + \vec{E}BE^T + \vec{F}BF^T}}{(q)(q)(q)}.
$$

(4.1)
where for \( \vec{n} \in \mathbb{Z}^k_{\geq 0} \) we set \(|\vec{n}| = \sum_{i=1}^k in_i \), and \((q)\vec{n} = \prod_{i=1}^k (q)^{n_i} \) where \((q)_n = \prod_{i=1}^k (1 - q^i)^{-1}\). The \( k \times k \) matrices \( A \) and \( B \) are defined by \( A_{ij} = \min\{i, j\} \) and \( B_{ij} = \max\{i + j - k, 0\} \).

\( \mathcal{A}_2 \) here is the part of \( \mathcal{V}_{sl(2), q} \) built up from the creation operators \( x_{-1} \) only, i.e. the terms whose grade equals its degree. So its \((q, z)\)-character is recovered by substituting \( uq^{-1} \) for \( t \) in (4.1) and retaining only the constant term in \( u \). We find that

\[
ch_{\mathcal{A}_2(\mathcal{V}_{sl(2), q})}(q, z) = \sum_{m=0}^{2k} q^m \sum_{a=0}^{\min\{m, 2k-m\}} (-1)^{m+a} \chi_{L(a)}(z)^2, \tag{4.2}
\]

writing \( L(a) \) for the irreducible \( a + 1 \)-dimensional \( sl(2) \)-module, and hence \( \mathcal{A}_2 \) and \( \mathcal{A} \) are isomorphic as \( sl(2) \)-modules.

**IV.3. The root lattices.** Consider first the \( A_{N-1} \) root lattice, which can be identified with the integer points \( \vec{n} \in \mathbb{Z}^N \) with \( \sum_n n_i = 0 \). Its automorphism group is the symmetric group \( Sym(N) \), together with \( \vec{n} \mapsto -\vec{n} \), so this will act on \( \mathcal{A}_2 \). Recall Section III.3. The ‘short’ lattice vectors \( \vec{n} \in S_{A_{N-1}} \) are those whose coordinates \( n_i \) all lie in \( \{\pm 1, 0\} \); up to the \( Sym(N) \) symmetry, we can take these to be \( \Lambda_\ell + \Lambda_{N-\ell} \), where \( \ell \leq [N/2] \) is the number of components equal to \( +1 \) and \( \Lambda_\ell \) are the fundamental weights (the natural basis for the dual lattice). There are \( \binom{N}{2\ell} \binom{2\ell}{\ell} \) short vectors for a given \( \ell \).

The number of basis vectors (3.3) with \( \alpha = 0 \) and grade \( m \) is \( \binom{N}{m} - \delta_{m,1} \), for a total (over all \( m \)) of \( 2^N - 1 \). This number for \( \alpha = \Lambda_\ell + \Lambda_{N-\ell} \) and grade \( m \) is \( \binom{N-2\ell}{m} \), for a total of \( 2^{N-2\ell} \). Therefore the total dimension of the \( C_2 \)-algebra is

\[
\dim \mathcal{A}_2(\mathcal{V}_{A_{N-1}}) = 2^N - 1 + \sum_{\ell=0}^{[N/2]} 2^{N-2\ell} \binom{N}{2\ell} \binom{2\ell}{\ell} = \binom{2N}{N} - 1 .
\]

By comparison, (2.6) tells us that Zhu’s algebra has dimension

\[
\dim \mathcal{A} = \sum_{j=0}^{N-1} \binom{N}{j}^2 = \binom{2N}{N} - 1 .
\]

So the \( A_{N-1} \) root lattice is non-anomalous. (The ‘\(-1\)’s here, suggesting a missing term, has an analogue in any affine \( A \)-series VOA, and is explained in Section IV.5 below.)

The other root lattices can be handled similarly (in fact somewhat more easily), with the result that only \( E_8 \) is anomalous. The short vectors for \( E_8 \) are \( 0 \), a root, or the sum of \( 2 \) orthogonal roots. The \( E_8 \) Weyl group \( W(E_8) \) acts transitively on each of those \( 3 \) sets, yielding \( 1 \)-, \( 240 \)-, and \( 2160 \)-dimensional \( W(E_8) \)-representations, respectively. \( \mathcal{A}_2(\mathcal{V}_{E_8}) \) is the direct sum of the \( 2160 \)-dimensional one, with \( 8 \) copies of the \( 240 \)-dimensional one, and \( 45 \) singlets, so is \( 4125 \)-dimensional. But \( \mathcal{A}_2(\mathcal{V}_{E_8}) \) also carries an action of the \( E_8 \) Lie group (this is because the lattice VOA \( \mathcal{V}_{E_8} \) is isomorphic to the affine algebra VOA \( \mathcal{V}_{E_8,1} \), and in terms of this it decomposes into \( L(\Lambda_1) \oplus L(\Lambda_8) \oplus 2L(0) \), using the node numbering conventions of Bourbaki/LiE (where \( L(\Lambda_8) \) is the \( 248 \)-dimensional adjoint). By comparison, Zhu’s algebra is \( 1 \)-dimensional.
More generally, any (nontrivial) rational VOA with only 1 irreducible module (these can be called self-dual VOAs) will be anomalous: Zhu’s algebra will be only 1-dimensional, because of (2.6), and \(A_2\) will always be larger.

Incidentally, one-dimensional lattices are easily shown to be non-anomalous. Another simple fact: \(V_{L \oplus L'} = V_L \otimes V_{L'}\), so \(V_{L \oplus L'}\) will be anomalous iff either \(V_L\) or \(V_{L'}\) are. Thus it suffices to consider indecomposable lattices.

IV.4. Anomalous lattices. A lesson of the previous subsections is that among the most accessible VOAs at least, the only anomalous ones are anomalous for an elementary reason (namely, that they are self-dual). Because of this, it would be tempting to guess that anomalous VOAs are rare.

However, in this subsection and the next we shall give several VOAs which are anomalous for subtle reasons. We suspect that in fact anomalous VOAs are typical, for the following reason.

The paper \([2]\) lists the indecomposable integral positive-definite lattices of small dimension and determinant (the determinant will equal the number of irreducible \(V_L\)-modules), and so can be regarded as providing some sort of random sample of lattices. What we find is that, once we cross off from their list root lattices at \(\alpha = 0\); each of the \(M\) vectors with length-squared \(\mu\) will also be ‘small’, and each of these will have at least \(1 + (n - 1)\) vectors in (3.3).

There is no need to consider the determinant-1 lattices: they are all anomalous. The 3 smallest indecomposable even lattices of determinant 2 are the root lattices \(A_1\) and \(E_7\), and the 15-dimensional lattice called \(D_{14}A_1\)\([11]\). Consider the latter. Its \(\dim A\) is readily found to be \(1^2 + 56^2 = 3137\). It has \(\mu = 2\) and \(M = 366\), so (4.3) tells us its \(A_2\) is at least 6123-dimensional. Therefore \(D_{14}A_1\)\([11]\) is anomalous. This is typical for the lattices collected in \([2]\).

To get a clue as to what is special about the anomalous lattices, we should ask what properties distinguish the \(E_8\) root lattice from the other root lattices. Of course, it is self-dual, but from our point of view this is the wrong answer, as we now see there are plenty of non-self-dual anomalous lattices. The most intriguing answer we have found is that \(E_8\) is the only root lattice whose holes do not lie in its dual. The holes of a lattice \(L\) are the points \(\bar{x}\) in the ambient space \(\mathbb{R} \otimes \mathbb{Z} L\) whose distance to any lattice point is a local maximum. If the hole is a global maximum, it is called a deep hole. For example, \(D_8 = \{ \bar{n} \in \mathbb{Z}^8 \mid \sum_i n_i \in 2\mathbb{Z} \}\) has deep holes at \((\frac{1}{2}, \frac{1}{2}, \ldots, \pm \frac{1}{2})\) and a shallow hole at \((1, 0, \ldots, 0)\), and these all lie in the weight lattice \(D_8^*\). On the other hand, \(E_8 = \langle D_8, (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \rangle\) has a deep hole at \((\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})\) and a shallow one at \((\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}, -\frac{1}{3})\), and of course neither lie in \(E_8^* = E_8\).
Now, a vector in $\mathbb{Q} \otimes \mathbb{Z} L$ (such as the holes of the integral lattice $L$) will be a dual vector in some sublattice $L_0$ of $L$ of full dimension, and thus will define an irreducible module of $\mathcal{V}_{L_0}$ and (lifting it to $\mathcal{V}_L$) a (generally) twisted module of $\mathcal{V}_L$. The holes of a lattice $L$ should define special (perhaps twisted) $\mathcal{V}_L$-modules. For example, the holes of $D_8$ all correspond to true $D_8$ modules, while the holes of $E_8$ are twisted, coming from $D_8$ and $A_8$ sublattices (those twisted modules have highest-weight spaces of dimension 16 and 9, respectively, corresponding geometrically to the 16 and 9 vectors, respectively, of $E_8$ that are closest to the given hole). Recall the discussion at the end of Section II.3, where we explain that $\mathcal{A}_2(\mathcal{V})$ should see the twisted $\mathcal{V}$-modules $M$, in the sense that there will be an embedding $M_0 \otimes M_0^* \rightarrow \mathcal{A}_2$. The image of this map may lie in the subspace of $\mathcal{A}_2$ coming from the true $\mathcal{V}$-modules, but we would guess that the twisted modules associated to holes would have an especially good chance at landing in a new part of $\mathcal{A}_2$.

IV.5. Zhu’s algebra and $C_2$-algebra for affine Lie algebras. We understand the $A$-series quite well, at arbitrary rank and level, with a conjectural description of $\mathcal{A}_2(\mathcal{V}_{sl(N),k})$ grade-by-grade as an $sl(N)$-module. In particular, $\mathcal{A}_2$ at grade $m$ seems to be given by

$$\mathcal{A}_2(\mathcal{V})^{(m)} = \oplus_{\mu \in P^+_m} L(\mu) \otimes L(\mu)^* - \oplus_{\nu \in P^{k/2}_m} L(\nu) \otimes L(\nu)^* ,$$

where we define

$$P^+_m = \{ \mu \in P_+ \mid \mu_0 \geq 0, \ t(\mu) \equiv m \ (mod \ N), \ t(\mu) \leq m, \ N\mu_0 + t(\mu) \geq m \} \quad (4.5)$$

$$P^{k/2}_m = \{ \nu \in P_+ \mid \nu_0 \geq 1, \ t(\nu) \equiv m - 1 \ (mod \ N), \ t(\nu) \leq m - 1, \ N\nu_0 + t(\nu) \geq m \} ,$$

using $N$-ality $t(\mu) = \sum_{i=1}^{N-1} i\mu_i$, writing $P_+$ for the $sl(N)$-weights with nonnegative Dynkin labels, and setting e.g. $\mu_0 = k - \sum_{i=1}^{N-1} \mu_i$.

This difference of modules appears because we are using $sl(N)$ rather than $gl(N)$. The combinations $L(\mu) \otimes L(\mu)^*$ etc. arise ultimately because of Peter-Weyl. Our conjecture is manifestly correct for grade $m \leq k$, as the null vector does not come in until $m = k + 1$. If our conjecture is correct, then the final nontrivial part of $\mathcal{A}_2$ will appear at grade $m = Nk$, where it will be a singlet. It is easy to verify that our conjecture works for $sl(2)$, and that our conjecture implies $\mathcal{V}_{sl(N),k}$ is not anomalous, for any $N$ and $k$. In fact, not only do the dimensions match, but $A$ and $\mathcal{A}_2$ here are isomorphic as $sl(N)$-modules.

The few checks we have done suggest (although it is far too early to call this even a conjecture) that likewise, $\mathcal{V}_{E_8,k}$ is not anomalous for any simple $g$, except for $g = E_8$. Of course $\mathcal{V}_{E_8,1}$ is isomorphic to the self-dual lattice VOA $\mathcal{V}_{E_8}$, which being self-dual is anomalous for elementary reasons. Remarkably, the $E_8$ VOAs are anomalous for all levels except possibly $k = 2$ [11].

Recall that both $A(\mathcal{V}_{g,k})$ and $\mathcal{A}_2(\mathcal{V}_{g,k})$ carry an adjoint action of $g$. For odd $k \geq 1$ the $E_8$-module $L(k\Lambda_1)$ (again we follow the node numbering conventions of Lie/Bourbaki) does not appear in Zhu’s algebra as an irreducible summand, but appears in the $C_2$-algebra. One can understand this in terms of $E_8$ twisted modules lifted from $D_8$: if we decompose the above module with respect to $D_8$ we get $L(k\Lambda_1)^{E_8} = L(2k\Lambda_1)^{D_8} \oplus \cdots$. Furthermore, none of the other $E_8$-modules that appear in Zhu’s algebra can produce this $D_8$-module. On the other hand, in $D_8$ we have

$$L(k\Lambda_1)^{D_8} \otimes L(k\Lambda_1)^{D_8} = L(2k\Lambda_1)^{D_8} \oplus \cdots .$$
The module $L(k\Lambda_1)^{d_8}$ is the highest-weight space of a level $k$ twisted $E_8$-module (restricted to $D_8$). This is why $L(2k\Lambda_1)^{d_8}$ must appear in the $C_2$-algebra of $E_8$, and implies the $C_2$-algebra must be bigger than Zhu’s algebra (in fact it will strictly contain it as an $E_8$-submodule).

For even levels $k > 2$ the $E_8$-module $L((k-3)\Lambda_1 + 2\Lambda_2)$ is not in Zhu’s algebra but appears in the $C_2$-algebra. However, we do not yet know how to obtain it from twisted modules.

We do not yet know whether level 2 is also anomalous. Curiously, $E_8$ at level 2 has the only exceptional simple current (i.e. a simple current not arising from an extended Dynkin diagram symmetry) among all the affine algebras.

V. Conjectures and questions

1. Clarify the role of holes in the lattice $L$ and $A_{[2]}(V_L)$. We would guess that a lattice VOA $V_L$ is anomalous whenever $L$ has a hole not in its dual $L^*$. Lattice VOAs are simple enough that we should be able to completely characterise anomalous lattices.

2. What is $A_{[2]}(V_{g,k})$, grade by grade? In Section IV.5 we give a very satisfactory conjectural description of $A_{[2]}(V_{sl(2),k})$. We have at present no idea what $A_{[2]}(V_{g,k})$ looks like, grade by grade, for the other simple $g$.

3. Clarify the relation between $A_{[2]}$ and twisted modules. Do twisted modules suffice to span $A_{[2]}$? Can anything be said about how the images of the $g$-twisted Zhu algebras $A_g$ in $A_{[2]}$ fit together, as the automorphism $g$ varies?

4. Cohomological interpretations of $A_{[2]}^*/A^*$. See Section III.1.

5. The ‘gradation’ of Zhu’s algebra versus $C_2$-algebra. For the Lie algebra VOAs $V_{g,k}$, we give in Section III.4 an especially clean description of the graded algebra associated to Zhu’s algebra; this should permit a direct comparison of it with $A_{[2]}$ for these VOAs, and perhaps a deeper understanding of $A(V_{g,k})$ versus $A_{[2]}(V_{g,k})$.

6. Comparing related spaces. Instead of considering the vacuum module $V$, we can also study the analogous question, i.e. whether $\dim A_{[2]}(M) = \dim A_{[1,1]}(M)$ for arbitrary modules $M$. At least for the Virasoro minimal models with $(p,q) = (5,2), (7,2), (9,2), (4,3), (5,3), (7,3)$ this seems to be the case for all modules $M$. On the other hand, the dimensions of e.g. $A_{[3]}(V)$ and $A_{[1,1,1]}(V)$ seem to already differ for the minimal models. (These calculations were performed by Andy Neitzke.) It seems that comparing $A_{[2]}(V)$ and $A(V)$ is the most fundamental question here.

7. Natural maps between $A^*$ and $A_{[2]}^*$. The enveloping algebra $U\mathfrak{g}$ is a co-commutative Hopf algebra, and the polynomial algebra $S\mathfrak{g}$ is its Hopf dual. Of course the algebras $A(V_{g,k})$ and $A_{[2]}(V_{g,k})$ are naturally quotients of $U\mathfrak{g}$ and $S\mathfrak{g}$, respectively. Does something like this happen for general $V$, and does this have any significance?

References

[1] D. Brungs, W. Nahm, The associative algebras of conformal field theory, Lett. Math. Phys. 47 (1999), 379–383.

[2] J. H. Conway, N. J. A. Sloane, Low-dimensional lattices. I. Quadratic forms of small determinant, Proc. R. Soc. Lond. A418 (1988), 17–41.

[3] C. Dong, H. Li, G. Mason, Twisted representations of vertex operator algebras, Math. Annalen. 310 (1998), 571–600.
[4] C. Dong, H. Li, G. Mason, *Modular invariance of trace functions in orbifold theory and generalized moonshine*, Commun. Math. Phys. **214** (2000), 1–56.

[5] C. Dong, G. Mason, *Vertex operator algebras and moonshine: A survey*, Progress in Algebraic Combinatorics, Adv. Stud. Pure Math. 24 Math. Soc. Japan, Tokyo, 1996, pp. 101–136.

[6] C. Dong, G. Mason, Y. Zhu, *Discrete series of the Virasoro algebra and the Moonshine module*, Proc. Symp. Pure Math. 56, Amer. Math. Soc., Providence, 1994, pp. 295–316.

[7] E. Feigin, *The PBW filtration*, arXiv: math/0702279.

[8] I. Frenkel, Y. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J. **66** (1992), 123–168.

[9] M. R. Gaberdiel, A. Neitzke, *Rationality, quasirationality and finite W-algebras*, Commun. Math. Phys. **238** (2003), 305–331.

[10] M. R. Gaberdiel, P. Goddard, *Axiomatic conformal field theory*, Commun. Math. Phys. **209** (2000), 549–594.

[11] M. R. Gaberdiel, T. Gannon, (work in progress).

[12] M. R. Gaberdiel, H. G. Kausch, *A rational logarithmic conformal field theory*, Phys. Lett. **B386** (1996), 131–137.

[13] T. Gannon, *Moonshine beyond the Monster*, Cambridge University Press, Cambridge, 2006.

[14] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math. **79** (1964), 59–103.

[15] Y.-Z. Huang, *Rigidity and modularity of vertex tensor categories*, Commun. Contemp. Math. (to appear).

[16] J. Lepowsky, H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Birkhäuser, Boston, 2004.

[17] H. Li, *Some finiteness properties of regular vertex operator algebras*, J. Algebra **212** (1999), 495–514.

[18] M. Miyamoto, *Modular invariance of vertex operator algebras satisfying $C_2$-cofiniteness*, Duke Math. J. **122** (2004), 51–91.

[19] A. Neitzke, *Zhu’s theorem and an algebraic characterization of chiral blocks*, arXiv: hep-th/0005144.

[20] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc. **9** (1996), 237–302.

**Institut für Theoretische Physik, ETH Zürich, CH-8093 Zürich, Switzerland**

**Math Department, University of Alberta, Edmonton, Canada T6G 2G1**

*E-mail address:* gaberdiel@itp.phys.ethz.ch tgannon@math.ualberta.ca