The $p$-integrable Teichmüller space for $p \geq 1$

By Huaying WEI$^*$ and Katsuhiko MATSUZAKI$^{**}$

Abstract: We verify that the $p$-integrable Teichmüller space $T_p$ admits the canonical complex Banach manifold structure for any $p \geq 1$. Moreover, we characterize a quasisymmetric homeomorphism corresponding to an element of $T_p$ in terms of the $p$-Besov space for any $p > 1$.

Key words: universal Teichmüller space; Weil–Petersson Teichmüller space; Bers embedding; bi-Lipschitz quasiconformal extension; analytic Besov space.

1. Introduction

The universal Teichmüller space $T$ is a certain quotient space of all normalized quasiconformal homeomorphisms of the unit disk $D$ (or the upper half-plane $U$). This is the total space of all other Teichmüller spaces defined by means of quasiconformal mappings. See [11, 13, 18].

A quasiconformal map $f$ of $D$ into the complex plane $C$ is determined uniquely by its complex dilatation $\mu_f(z) = f'_z/f_z$ up to the post composition of a conformal map. Then, the space $M(D)$ of all Beltrami coefficients $\mu$, which are measurable functions with $\|\mu\|_{\infty} < 1$, is identified with the space of normalized quasiconformal maps on $D$. Let $f^\mu : D \to D$ denote the quasiconformal homeomorphism with complex dilatation $\mu \in M(D)$ normalized by fixing boundary points 1, $-1$, and $-i$.

We say that $\mu$ and $\nu$ in $M(D)$ are Teichmüller equivalent if the extensions of $f^\mu$ and $f^\nu$ to the unit circle $S$, which are quasisymmetric homeomorphisms, are the same. Then, $T$ is the quotient space by this equivalence relation; the quotient map $\pi : M(D) \to T$ ($\pi(\mu) = [\mu]$) is called the Teichmüller projection. The topology on $T$ is induced by the Teichmüller distance (see [13 Sect. III.2]); equivalently, $[\mu]$ converges to $[\nu]$ in $T$ if

$$\inf \{ \|\mu * \nu^{-1}\|_{\infty} | \mu \in [\mu], \nu \in [\nu] \} \to 0,$$

where $\mu * \nu^{-1}$ stands for the complex dilatation of $f^\mu \circ (f^\nu)^{-1}$. We call this the Teichmüller topology.

Integrable Teichmüller spaces are subspaces of $T$ given by integrable Beltrami coefficients $\mu$ with

respect to the hyperbolic metric on $D$.

**Definition 1.1.** For $\mu \in M(D)$ and $p > 0$, let

$$\|\mu\|_p = \left( \int_D |\mu(z)|^p (1 - |z|^2)^{-2} \, dx \, dy \right)^{\frac{1}{p}}.$$

The space of all Beltrami coefficients $\mu \in M(D)$ with $\|\mu\|_p < \infty$ is denoted by $M_p(D)$. The $p$-integrable Teichmüller space $T_p$ is defined by $\pi(M_p(D))$. For $p \geq 1$, we regard $M_p(D)$ as the open subset of the Banach space of norm $\|\mu\|_\infty + \|\mu\|_p$. We equip $T_p$ with the Teichmüller topology induced by this norm.

For $p = 2$, $T_2$ is also known as the Weil–Petersson Teichmüller space. This was first introduced by Cui [2] and developed by Takhtajan and Teo [25]. It is equipped with a complex Hilbert manifold structure and with an invariant Kähler metric called the Weil–Petersson metric. Shen [22] characterized the quasisymmetric extension $h$ of a quasiconformal self-homeomorphism $f$ of $D$ with $\mu_f \in M_2(D)$ by the condition that $h$ is absolutely continuous and $\log h'$ belongs to the Sobolev space $H^{1/2}(S)$.

For $p > 2$, $T_p$ was first considered by Guo [12]. The complex Banach manifold structure for $T_p$ was provided by Yanagishita [31] including the case with the Fuchsian group action. This was also done by Tang and Shen [27] where they also generalized the characterization of the quasisymmetric extension $h$ to $S$ of a quasiconformal self-homeomorphism $f$ with $\mu_f \in M_p(D)$. The $p$-Weil–Petersson metric was introduced and investigated in [16] and [33].

In contrast, the case of $0 < p \leq 1$ was studied by Alberge and Brakalova [3] and they in particular proved that the quasisymmetric extension $h$ is continuously differentiable in this case.

In this paper, we consider the case of $1 \leq p < 2$ and prove that $T_p$ is also endowed with the com-
plex Banach manifold structure (Theorem 1.1). The results formulated on $D$ and $S$ can be suitably translated into the upper half-plane $U$ and the real line $R$. We also prove that a quasisymmetric homeomorphism $h : R \to R$ has a quasiconformal extension $f$ to $U$ with $\mu_f$ in $M_p(U)$ for $p > 1$ if and only if $h$ is locally absolutely continuous and $\log h'$ is in the $p$-Besov space $B_p(R)$ (Theorem 5.5). The claim for $M_p(D)$ and $B_p(S)$ is also verified. These results improve those in the recent papers [14, 15].

2. The Bers embedding. For $\mu \in M(D)$, let $f_\mu$ be the quasiconformal self-homeomorphism of the extended complex plane $\hat{C}$ such that its complex dilatation is $\mu$ on $D$ and 0 on the exterior disk $D^* = \{ z \in C \mid |z| > 1 \} \cup \{ \infty \}$. We take the Schwarzian derivative $S_{f_\mu|D^*}$ of the conformal homeomorphism $f_\mu$ on $D^*$. This belongs to the complex Banach space $A_\infty(D^*)$ of all holomorphic maps $\varphi$ on $D^*$ with $\| \varphi \|_{A_\infty} = \sup_{z \in D^*} |(z^2 - 1)^2| \varphi(z) | < \infty$.

A map $\Phi : M(D) \to A_\infty(D^*)$ is defined by $\mu \mapsto S_{f_\mu|D^*}$, which we call the Bers Schwarzian derivative map. Then, $\pi(\mu_1) = \pi(\mu_2)$ if and only if $\Phi(\mu_1) = \Phi(\mu_2)$ for $\mu_1, \mu_2 \in M(D)$. Hence, there is a well-defined injection $\beta : T \to A_\infty(D^*)$ such that $\beta \circ \pi = \Phi$, which is called the Bers embedding.

The following property of $\Phi$ is well known. See [13 Th.V.5.3], [18 Sect.3.4, 3.5].

Proposition 2.1. $\Phi : M(D) \to A_\infty(D^*)$ is a holomorphic split submerison onto the image.

This proposition in particular implies (a) $\Phi$ is continuous; (b) $\Phi$ has a local continuous right inverse at every point in the image. By the formula

$$ (1) \quad \mu \ast \nu^{-1}(f'(z)) = \frac{\mu(z) - \nu(z)}{1 - \nu(z)\mu(z)} \cdot \frac{df'(z)}{f'(z)}, $$

we see that the quotient topology on $T$ induced by $\pi$ coincides with the Teichmüller topology. Then, (a) is equivalent to saying that $\beta$ is continuous, and (b) implies that $\beta^{-1}$ is continuous. Hence, we have:

Proposition 2.2. The Bers embedding $\beta : T \to A_\infty(D^*)$ is a homeomorphism onto the open subset $\Phi(M(D))$. Thus, $T$ is endowed with the complex structure modeled on $A_\infty(D^*)$. The Teichmüller projection $\pi : M(D) \to T$ is holomorphic with a local holomorphic right inverse at every point of $T$.

We restrict $\Phi$ to $M_p(D)$. Let

$$ A_p(D^*) = \{ \varphi \in A_\infty(D^*) \mid \| \varphi \|_{A_p} < \infty \}; $$

$$ \| \varphi \|_{A_p} = \left( \int_{D^*} |\varphi(z)|^p (|z|^2 - 1)^{2p-2} dxdy \right)^{\frac{1}{p}}. $$

For $p \geq 1$, $A_p(D^*)$ is a complex Banach space with this norm. There is a constant $c_p > 0$ depending only on $p$ such that $\| \varphi \|_{A_p} \leq c_p \| \varphi \|_{A_p}$. We will see in Lemma 5.2 below that $\Phi(M_p(D)) \subset A_p(D^*)$.

We also consider the Bers embedding $\beta$ on the $p$-integrable Teichmüller space $T_p = \pi(M_p(D))$. Then, $\beta : T_p \to A_p(D^*)$ is an injection onto $\Phi(M_p(D))$. The topology on $T_p$ is the Teichmüller topology induced from $M_p(D)$ in which $[\mu]$ converges to $[\nu]$ in $T_p$ if

$$ \inf \{ \| \mu \ast \nu^{-1} \|_p + \| \mu \ast \nu^{-1} \|_\infty \mid \mu \in [\mu], \nu \in [\nu] \} \to 0. $$

This turns out to be the same as the topology defined by replacing $\| \mu \ast \nu^{-1} \|_p + \| \mu \ast \nu^{-1} \|_\infty$ with $\| \mu \ast \nu^{-1} \|_p$.

### 3. Bi-Lipschitz quasi-conformal forms

We show that there is some $\nu \in M_p(D)$ in every Teichmüller class $[\mu] \in T_p$ such that $f^\nu$ is a bi-Lipschitz self-diffeomorphism of $D$ in the hyperbolic metric. We adapt several claims in Takhtajan and Teo [28] for $p = 2$ to the general case $p \geq 1$.

#### Lemma 3.1.

Let $\mu, \nu \in M_p(D)$ for $p \geq 1$ and assume that $f^\nu$ is a bi-Lipschitz self-homeomorphism of $D$ in the hyperbolic metric. Then, $\mu \ast \nu^{-1}$ belongs to $M_p(D)$. In particular, the complex dilatation $\nu^{-1}$ of $(f^\nu)^{-1}$ is in $M_p(D)$. Moreover, $\mu$ converges to $\nu$ in $M_p(D)$ if and only if $\mu \ast \nu^{-1}$ converges to $0$.

**Proof.** By formula (1) and change of variables,

$$ \| \mu \ast \nu^{-1} \|_p = \int_D \left| \frac{\mu - \nu}{1 - \nu \mu} \circ (f^\nu)^{-1}(z) \right|^p \frac{dxdy}{(1 - |z|^2)^2} \leq C \int_D |\mu(z) - \nu(z)|^p \frac{dxdy}{(1 - |z|^2)^2} = C \| \mu - \nu \|_p^p, $$

and $\| \mu \ast \nu^{-1} \|_p \leq C^{-1} \| \mu - \nu \|_p$ for a constant $C \geq 1$ depending only on $\| \nu \|_\infty$ and the bi-Lipschitz constant of $f^\nu$.

By this lemma, any $\nu \in M_p(D)$ such that $f^\nu$ is bi-Lipschitz defines a map $r_\nu : M_p(D) \to M_p(D)$ by $r_\nu(\mu) = \mu \ast \nu^{-1}$ for $\mu \in M_p(D)$. A similar computation shows that there is a constant $C > 0$ with the same dependence as above such that

$$ \| r_\nu(\mu_1) - r_\nu(\mu_2) \|_p \leq C \| \mu_1 - \mu_2 \|_p $$

for any $\mu_1, \mu_2 \in M_p(D)$. The same inequality is true for $|| \cdot ||_\infty$. Hence, $r_\nu$ is continuous. By considering $(r_\nu)^{-1} = r_{\nu^{-1}}$, we see that $r_\nu$ is a homeomorphic (in fact biholomorphic) automorphism of $M_p(D)$.

#### Lemma 3.2.

For $p \geq 1$, there exists a constant $C > 0$ such that $\| \Phi(\mu) \|_{A_p} \leq C \| \mu \|_p$ for every $\mu \in M_p(D)$. Moreover, $\Phi : M_p(D) \to A_p(D^*)$ is holomorphic.
Proposition 2.1. see that $\Phi$ is Gateaux holomorphic. As in [13, section 3.2, $\nu$ equivalent to $\mu$] is also true for $\Phi$. This gives an estimate of the derivative of $\Phi$, and thus the first claim of our lemma follows. This in particular implies that $\Phi$ is locally bounded.

To show that $\Phi$ is holomorphic, it suffices to see that $\Phi$ is Gâteaux holomorphic. As in Chap.1, Lem.2.9], we can verify this by a usual argument (see Remark 1.4 below), for which we may rely on Proposition 2.4.

Remark 3.3. In other papers [7 12 22], the claim that $\Phi : M_p(D) \to A_p(D^*)$ is continuous (and hence holomorphic) was proved for $p \geq 2$. This is due to the integral representation of the Schwarzian derivative formulated by Astala and Zinsmeister [4 Form.(4.4)]:

\[
(|\zeta|^2 - 1)^2 |\mu(\zeta)|^2 \leq C \int_D \frac{|\mu(z)|^2}{|z - \zeta|^4} \, dx \, dy \quad (\zeta \in D^*)
\]

for $\mu \in M_2(D)$, where $C > 0$ is a constant depending only on $||\mu||_\infty$. We can modify this formula for the estimate of the difference $\Phi(\mu_1) - \Phi(\mu_2)$ (see 2.4 Form.(5.4))). Moreover, the Hölder inequality yields the corresponding claim for $p \geq 2$.

Lemma 3.4. Each Teichmüller class $[\mu] \in T_p (p \geq 1)$ contains $\nu \in M_p(D)$ such that $f' = \Phi(\nu)$ is a bi-Lipschitz diffeomorphism in the hyperbolic metric with the bi-Lipschitz constant depending only on $||\nu||_\infty$.

Proof. If $||\nu||_\infty < 1/3$, then the Ahlfors–Weill section $\nu = (\sigma(\varphi) / M(D))$ defined by

\[
(2) \quad \sigma(\varphi)(z^*) = -\frac{1}{2} (z^*)^2 (1 - |z|^2)^2 \varphi(z) \quad (z^* = 1/z)
\]

for $\varphi = \Phi(\mu)$ satisfying $||\varphi||_\infty < 2$ is Teichmüller equivalent to $\mu$. See [13 Th.II.5.1]. By Lemma 3.2, $\varphi \in A_p(D^*)$. Then, we see that $\nu$ belongs to $M_p(D)$ by formula 2. Moreover, as in [28 Chap.1, Lem.2.5], $f'$ is a bi-Lipschitz self-diffeomorphism of $D$ if $||\mu||_\infty < \delta$ for some $\delta \leq 1/3$.

For an arbitrary $\mu \in M_p(D)$, let $\mu_k = k\mu / n \in M_p(D)$ ($k = 1, \ldots, n$), where $n \in \mathbf{N}$ is chosen so that $||\mu_k + \mu_{k+1}||_\infty < \delta$. Suppose we obtain $\nu_k \in M_p(D)$ with $[\nu_k] = [\mu_k]$ and $f^{\nu_k}$ is bi-Lipschitz. Lemma 3.1 implies $\nu_{k+1} - \nu_k \in M_p(D)$. Hence, we have

\[
\sigma(\Phi(\mu_{k+1} * \nu_k^{-1})) = \sigma(\Phi(\mu_{k+1} * \mu_k^{-1})) \in M_p(D)
\]
as above, which gives a bi-Lipschitz self-diffeomorphism of $D$. Its composition with $f^{\nu_k}$ is also bi-Lipschitz, and let $\nu_{k+1}$ be the complex dilatation of this composition. We have $\nu_{k+1} \in M_p(D)$ by Lemma 3.1 and $[\nu_{k+1}] = [\mu_k+1]$. By induction, $\nu = \nu_n$ satisfies $||\nu||_\infty = ||\mu||_\infty$ and $f^{\nu}$ is bi-Lipschitz.

Remark 3.5. For $p \geq 2$, to obtain the bi-Lipschitz diffeomorphism as in the above lemma, the barycentric extension due to Douady and Earle (see [8 Th.2]) was used in [7 27 53], and others.

4. The complex structure of integrable Teichmüller spaces We endow $T_p$ for $p \geq 1$ with the complex Banach manifold structure. The corresponding result to Proposition 2.2 will be proved. For $p \geq 2$, this was proved in [27 Th.2.1] as a consequence of the fact that $\Phi : M_p(D) \to A_p(D^*)$ is continuous by Tang [28 Th.3.1]. This also implies that the Teichmüller topology and the quotient topology induced by $\pi$ on $T_p$ are the same for $p \geq 2$. We can extend this to $p \geq 1$ as the following theorem shows.

Theorem 4.1. For $p \geq 1$, the Bers embedding $\beta : T_p \to A_p(D^*)$ is a homeomorphism onto an open subset $\Phi(M_p(D))$. Thus, $T_p$ is endowed with the complex structure modeled on $A_p(D^*)$. The Teichmüller projection $\pi : M_p(D) \to T_p$ is holomorphic and has a local holomorphic right inverse at every point of $T_p$.

Proof. The continuity of $\beta$ follows from Lemmas 3.1 3.2 and 3.4. The fact that $\Phi(M_p(D))$ is open in $A_p(D^*)$ is included in the claim that there is a local continuous right inverse $s_\beta$ of $\Phi$ at every point $\psi \in \Phi(M_p(D))$, which is proved in Lemma 3.3 below. Then by Lemma 1.5, $s_\beta$ is in fact holomorphic. By Lemma 3.1 $\pi : M_p(D) \to T_p$ is continuous at $\nu = s_\beta(\psi)$ because we choose $f^\nu$ to be bi-Lipschitz. Since $\beta^{-1}(\psi) = \pi \circ s_\beta(\psi)$, we see that $\beta^{-1}$ is continuous at $\psi$. The homeomorphism $\beta$ transfers the claim for $\Phi$ to that for $\pi$.

Remark 4.2. In the argument of [28] showing the above theorem for $T_2$, the continuity of $\beta$ is similarly obtained as above, but the continuity of $\beta^{-1}$ requires $p = 2$ in the second part of the proof of [28 Chap.1, Th.2.3].

Lemma 4.3. For every $\psi \in \Phi(M_p(D))$ and for every $\nu \in M_p(D)$ with $\Phi(\nu) = \psi$ such that $f^{\nu}$ is bi-Lipschitz, there is a continuous map $s_\nu : V_\psi \to M_p(D)$ on some neighborhood $V_\psi \subset A_p(D^*)$ of $\psi$ such that $\Phi \circ s_\nu$ is the identity on $V_\psi$, and $s_\nu(\psi) = \nu$.

Proof. By Lemma 3.3, for any $\psi \in \Phi(M_p(D))$, we can take $\nu \in B_p(D)$ such that $\Phi(\nu) = \psi$ and...
$f_\nu\big|_\Omega$ is a bi-Lipschitz diffeomorphism. The quasiconformal reflection $j : f_\nu(D) \to f_\nu(D^*)$ over the quasicircle $f_\nu(S)$ is defined by $j(\zeta) = f_\nu(f_\nu^{-1}(\zeta^*))$ for $\zeta \in f_\nu(D)$. Then, there is a constant $c_1 > 0$ such that
\[
|j_2(\zeta)|\rho_\Omega^*\langle j(\zeta) \rangle \leq c_1 \rho_\Omega(\zeta),
\]
where $\rho_\Omega$ and $\rho_\Omega^*$ denote the hyperbolic densities of $\Omega = f_\nu(D)$ and $\Omega^* = f_\nu(D^*)$. Moreover, by [3] Lem.3, there is $c_2 > 0$ such that
\[
|\zeta - j(\zeta)|^2 \rho_\Omega(\zeta) \rho_\Omega^*(j(\zeta)) \leq c_2.
\]
The constants $c_1$ and $c_2$ depend only on $\|\nu\|_\infty$. Hence, for $c = c_1c_2$, we have
\[
|\zeta - j(\zeta)|^2 \leq c \rho_\Omega^2(j(\zeta)).
\]

As in [9] Sect.6 and [13] Sect.11.4.2, there is a constant $\varepsilon \in (0, 1/c)$ depending only on $\|\nu\|_\infty$ such that if $\varphi \in A_\nu(D^*)$ satisfies $\|\varphi\|_\infty < \varepsilon$, then there is a quasiconformal self-homeomorphism $g$ of $\mathcal{C}$ conformal on $\Omega^*$ such that $S_{g\circ f_\nu,\Omega^*} = \varphi + \varphi'$. In this manner, $g$ is given so that its complex dilatation is
\[
\mu_g(\zeta) = \frac{S_g(j(\zeta))(\zeta - j(\zeta))^2 j_2(z)}{2 + S_g(j(\zeta))(\zeta - j(\zeta))^2 j_2(z)} (\zeta \in \Omega).
\]
By setting $\zeta = f_\nu(z)$, we obtain from (3) that
\[
|S_g(j(\zeta))(\zeta - j(\zeta))^2 j_2(z)| \leq \frac{1}{c} |\varphi(z^*)|\rho_\Omega^2(z^*) < 1.
\]
Hence, $|\mu_g(f_\nu(z))| \leq \frac{1}{c} |\varphi(z^*)|\rho_\Omega^2(z^*)$ for every $z \in D$, and thus $\mu_g \circ f_\nu \in M_\rho(D)$. We denote the complex dilatation of $g \circ f_\nu$ by $\nu_\varphi$. Formula (11) yields
\[
|\nu_\varphi(z) - \nu_\varphi(z)| \leq \frac{1}{\sqrt{1 - \|\mu_g\|_\infty}} \frac{|\mu_g \circ f_\nu(z) - \mu_g \circ f_\nu(z)|}{\sqrt{1 - \|\mu_g\|_\infty}}
\]
for any $\varphi$ and $\varphi'$ in $A_\rho(D^*)$ with $\|\varphi\|_\infty, \|\varphi'\|_\infty < \varepsilon/c_\rho$, where $g$ and $g'$ are the corresponding quasiconformal homeomorphisms. As both $\mu_g \circ f_\nu$ and $\nu = \nu_\varphi$ belong to $M_\rho(D)$, so does $\nu_\varphi$.

Because $\Phi(\nu_\varphi) = S_{g_\nu f_\nu,\Omega^*} = \varphi + \varphi'$, we have a local right inverse $s_\varphi$ of $\Phi$ on the neighborhood
\[
V_\varphi = \{\psi + \varphi : \|\varphi\|_\infty < \varepsilon/c_\rho \} \subset A_\rho(D^*)
\]
by the correspondence $s_\varphi : \psi + \varphi \mapsto \nu_\varphi$. By the above inequalities for $\mu_g \circ f_\nu$ and $\nu_\varphi$, we see that there is a constant $C > 0$ such that
\[
|\nu_\varphi(z) - \nu_\varphi(z)| \leq C|\varphi(z^*) - \varphi'(z^*)|\rho_\Omega^2(z^*)
\]
for $z \in D$. This implies that $s_\varphi$ is continuous. \[\square\]

**Remark 4.4.** A similar proof is in [17] Lem.7.5 for a different kind of Teichmüller space. The necessity of the bi-Lipschitz quasiconformal extension is also pointed out in [23] p.68).

**Lemma 4.5.** Let $\Psi$ be a holomorphic map on an open subset $V$ of $A_\infty(D^*)$ into $M(D)$. If $\Psi$ is continuous on the open subset $V \cap A_\rho(D^*)$ into $M_\rho(D)$ in the stronger topologies, then it is holomorphic.

**Proof.** We apply a claim on infinite-dimensional holomorphy in [5] p.28 (see also [13] Lem.V.5.1). For $\mu \in M_\rho(D)$, we consider $\alpha_E(\mu) = \int_E \mu(z)dx\,dy$ for each measurable subset $E \subset D$ as an element of the dual space of the Banach space of all measurable functions $\mu$ on $D$ with norm $\|\mu\|_\infty + \|\mu\|_p$ finite. If $\alpha_E(\mu) = 0$ for every $E \subset D$, then $\mu = 0$. Moreover, $\alpha_E(\Psi(\varphi)) \in C$ for each $E$ depends holomorphically on $\varphi \in V$. Then, by the claim cited above, we see that $\Psi : V \cap A_\rho(D^*) \to M_\rho(D)$ is holomorphic. \[\square\]

**Remark 4.6.** Similar applications of this argument can be found in [10] p.591, [25] p.141, and [33] p.964. Moreover, we can also deduce the holomorphy of $\Psi$ by showing that they are locally bounded and Gâteaux holomorphic. This is also based on [5] p.28 and formulated in [31] Lem.6.1.

For every $[\nu] \in T_p$, we define a map $R_{[\nu]} : T_p \to T_p$ by $R_{[\nu]}([\mu]) = [\pi(r_\nu(\mu))]$ for $[\mu] \in T_p$, where $r_\nu$ is the homeomorphic automorphism of $M_\rho(D)$ given by a bi-Lipschitz representative $\nu$. This is well defined as the right translation in the group structure of $T_p$. By taking the local right inverse of the projection $\pi$ at each $[\mu] \in T_p$ as in Theorem 4.1, we see that $R_{[\nu]}$ is continuous, and hence it is a homeomorphic automorphism of $T_p$. A similar argument as in Lemma 5.5 shows that $R_{[\nu]}$ is in fact a biholomorphic automorphism of $T_p$ identified with the open subset $\beta(T_p)$ in $A_\rho(D^*)$ for $p > 1$.

**5. Characterization by Besov spaces**

We intrinsically characterize the quasisymmetric extension $f_\nu|_S$ of $f_\nu : D \to D$ with $\mu \in M_\rho(D)$ for $p > 1$. Hereafter, we change the roles of $D$ and $D^*$ for the sake of the definition of function spaces.

For a holomorphic function $\phi$ on $D$, we define
\[
\|\phi\|_{B_p} = \left(\int_D |\phi'(z)|^p(1 - |z|^2)^{p-2}\,dx\,dy\right)^{\frac{1}{p}}.
\]

The set of all holomorphic functions $\phi$ on $D$ with $\|\phi\|_{B_p} < \infty$ is denoted by $B_p(D)$ and called the analytic $p$-Besov space on $D$. By ignoring the difference
of complex constants, $B_p(D)$ is a complex Banach space with this norm. For details, see [35 Sect.5.3].

Let $H(z) = -i(z + 1)/(z - 1)$ be the Cayley transformation, which maps $D$ onto $U$. For a conformal map $f$ of $D$, we take the conjugate $\hat{f} = H \circ f \circ H^{-1}$. Moreover, for a holomorphic function $\phi$ on $D$, we take the push-forward $\phi_\ast = \phi \circ H^{-1}$. Then,

$$\|\phi\|_{B_p} = \left( \int_U |\phi_\ast(\zeta)|^p (2 \Im \zeta)^{p-2} d\zeta d\eta \right)^{1/p} =: \|\phi_\ast\|_{B_p},$$

and the set of all such holomorphic functions $\phi_\ast$ on $U$ with $\|\phi_\ast\|_{B_p} < \infty$ is defined to be the analytic $p$-Besov space $B_p(U)$ on $U$. Similarly, for the Schwarzian derivative $S \phi$, let $\varphi_\ast = S_j = S \circ f_\circ H^{-1}$ on $U$ and let $\|\varphi_\ast\|_{A_p} := \|\varphi_\ast\|_{A_p}$. The Banach space $A_p(U) = \{\varphi_\ast | \varphi \in A_p(D)\}$ is defined by this norm.

**Proposition 5.1.** Let $p > 1$. The following conditions are equivalent for a conformal homeomorphism $f$ of $D$ extending to a quasiconformal homeomorphism of $\mathbb{C}$ with $f(1) = 1$ and $f(\infty) = \infty$:

(i) $S_j \in A_p(D)$; (ii) $\log f' \in B_p(D)$; (iii) $S_j \in A_p(U)$; (iv) $\log f' \in B_p(U)$.

**Proof.** The equivalence (i) $\Leftrightarrow$ (ii) is proved in [12 Th.1], (iii) $\Leftrightarrow$ (iv) is proved in [24 Th.4.4] and [44 Th.1.3]. See [32 Th.7.1]. (i) $\Leftrightarrow$ (iii) is due to the Möbius invariance.

Any function $\phi \in B_p(D)$ has a non-tangential limit at almost every point of $S$ (see [35 Lem.10.13]). This is also true for $\phi_\ast \in B_p(U)$. These define the boundary functions $\phi|_{S}$ on $S$ and $\phi_\ast|_{U}$ on $U$.

**Definition 5.2.** A locally integrable complex-valued function $u$ on $S$ belongs to the $p$-Besov space $B_p(S)$ on $S$ for $p > 1$ if

$$\|u\|_{B_p(S)} = \left( \int_S \int_S \frac{|u(x_1) - u(x_2)|^p}{|x_1 - x_2|^2} (dx_1)(dx_2) \right)^{1/p} < \infty.$$ 

The $p$-Besov space $B_p(R)$ on $R$ is the space of all functions $u_\ast = u \circ H^{-1}$ for $u \in B_p(S)$ with the norm $\|u_\ast\|_{B_p} := \|u\|_{B_p}$. We regard them as complex Banach spaces by ignoring additive constants.

The next claim says that the boundary function $\phi|_{S}$ of $\phi \in B_p(D)$ belongs to $B_p(S)$ for $p > 1$. This statement is in [35 p.131]. An explicit proof can be found in [19 Th.2.1, 5.1] as mentioned in [21 p.505].

**Lemma 5.3.** There is a constant $C_p \geq 1$ depending only on $p > 1$ such that for every $\phi \in B_p(D)$,

$$C_p^{-1} \|\phi\|_{B_p} \leq \|\phi|_{S}\|_{B_p} \leq C_p \|\phi|_{S}\|_{B_p}.$$ 

**Remark 5.4.** For $p = 2$, $B_2(D)$ is nothing but the analytic Dirichlet space on $D$ and the corresponding $B_2(S)$ coincides with the Sobolev space $H^2(S)$. In this case, the Douglas formula gives the equality $\|\phi|_{S}\|_{B_p} = 2\sqrt{\pi} \|\phi|_{S}\|_{B_p}$ (see [2 Th.2.5]).

The above lemma is also valid for $B_p(U)$ and $B_p(L)$ defined similarly on the lower half-plane $L$ both of which have the Besov space $B_p(R)$ on $R$ as the space of boundary extensions.

In general, any quasisymmetric homeomorphism $f^\mu|R$ for $\mu \in M(U)$ can be represented as conformal welding $f^\mu|R = g_{\mu^{-1}} \circ f_\mu|R$, where $f_\mu$ is conformal on $L$ and has the complex dilatation $\mu$ on $U$, and $g_{\mu^{-1}}$ is conformal on $U$ and has the complex dilatation $\mu^{-1}$ of $(f^\mu)^{-1}$ on $L$. Here, $f^\mu : U \to U$. All the mappings are normalized so that they fix $0$, $1$, and $\infty$.

We intrinsically characterize a quasisymmetric homeomorphism $f^\mu|R$ with $\mu \in M_p(U)$. For $p \geq 2$, this is in [22 Th.1.1], [24 Th.1.3], and [27 Th.1.12].

**Theorem 5.5.** A Beltrami coefficient $\mu \in M(U)$ represents the Teichmüller class $[\mu]$ in $T_p$ for $p > 1$ if and only if the quasisymmetric homeomorphism $f^\mu|R$ is locally absolutely continuous and $\log(f^\mu) \gamma$ is in $B_p(R)$. The corresponding claim is true for $M(D)$ and $B_p(S)$.

**Proof.** By Lemma 5.3 we may assume that $f^\mu$ is bi-Lipschitz. We consider conformal welding

$$f^\mu|R = g_{\mu^{-1}} \circ f_\mu|R.$$ 

Lemma 5.1 implies that $\mu^{-1} \in M_p(L)$. Then, by Lemma 5.2 and Proposition 5.1, we have $\log(f_\mu)' \in B_p(L)$ and $\log(g_{\mu^{-1}})' \in B_p(U)$, and by Lemma 5.3, $\log(f_\mu)'|_{R} \in B_p(R)$ and $\log(g_{\mu^{-1}})'|_{R} \in B_p(R)$.

Moreover, the non-tangential limits of $(f_\mu)'$ and $(g_{\mu^{-1}})'$ coincide with the angular derivatives of $f_\mu$ and $g_{\mu^{-1}}$, and in fact, the limits for the derivatives are allowed to be taken without restriction (see [20 Th.5.5]). This implies that $\log(f_\mu)'|_{R} = \log(f^\mu)'|_{R} = \log(g_{\mu^{-1}})'|_{R}$. By (4), we see that $f^\mu|R$ is locally absolutely continuous because so are $f_\mu|R$ and $g_{\mu^{-1}}|_{R}$ (see [32 Lem.3.2]) with $(g_{\mu^{-1}})'|_{R} > 0$ a.e. Hence,

$$\log(g_{\mu^{-1}})'|_{R} \circ f^\mu|R + \log(f^\mu)'|_{R} = \log(f_\mu)'|_{R}.$$ 

The first term of the left side of (5) also belongs to $B_p(R)$ (see [6 Th.12]). Thus, $\log(f^\mu)'|_{R} \in B_p(R)$.

Conversely, [30 Th.4.5] asserts that the variant of Beurling–Ahlfors extension by the heat kernel due to Fefferman, Kenig and Pipher [11] (see also [29]) extends such a quasisymmetric homeomorphism $f$.
of \( \mathbb{R} \) with \( \log f' \in B_p(\mathbb{R}) \) for \( p > 1 \) to a quasiconformal self-homeomorphism of \( \mathbb{U} \) with its complex dilatation in \( M_p(U) \).

In the unit disk case, the proof for the “only-if” part is the same. The “if” part is in \cite{MT5.2}.

The correspondence \( \mu \mapsto \log(f''|_R) \) in Theorem \( \ref{thm:BMOA} \) induces an injection \( T_p : M_p(U) \to B_p(\mathbb{R}) \), where \( B_p(\mathbb{R}) \) is the real Banach subspace of \( B_p(\mathbb{R}) \) consisting of real-valued functions. This is continuous, and in fact, real-analytic as shown in \cite{MT32}. Conversely, \cite[Th.4.4]{MT30} proves that the variant of Beurling–Ahlfors extension yields a real-analytic map \( \Lambda : B_p(\mathbb{R}) \to M_p(U) \) such that \( \pi \circ \Lambda \) is the inverse of \( L \). This shows that \( L \) is a bi-real-analytic homeomorphism.

**Proposition 5.6.** The Teichmüller space \( T_p \) is real-analytically equivalent to \( B_p(\mathbb{R}) \) for \( p > 1 \). In particular, \( T_p \) is contractible.

**Remark 5.7.** For \( p \geq 2 \), the barycentric extension yields a global real-analytic right inverse for the Teichmüller projection \( \pi : M_p(D) \to T_p \), from which we know that \( T_p \) is contractible.

**References**

[1] L. V. Ahlfors, Lectures on Quasiconformal Mappings, 2nd ed., Univ. Lect. Ser. 38, Amer. Math. Soc., 2006.

[2] L. V. Ahlfors, Conformal Invariants: Topics in Geometric Function Theory, AMS Chelsea, 2010.

[3] V. Alberge and M. Brakalova, On smoothness of the elements of some integrable Teichmüller spaces, Math. Rep. (Bucur.) 23 (2021), 95–105.

[4] K. Astala and M. Zinsmeister, Teichmüller spaces and \( BMOA \), Math. Ann. 289 (1991), 613–625.

[5] N. Bourbaki, Variétés différentielles et analytiques, Éléments de mathématique XXXIII, Fascicule de résultats–Paragraphes 1 à 7, Hermann, 1967.

[6] G. Bourdaud, Changes of variable in Besov spaces II, Forum Math. 12 (2000), 545–563.

[7] G. Cui, Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces, Sci. China Ser. A 43 (2000), 267–279.

[8] A. Douady and C. J. Earle, Conformally natural extension of homeomorphisms of the circle, Acta Math. 157 (1986), 23–48.

[9] C. J. Earle and S. Nag, Conformally natural reflections in Jordan curves with applications to Teichmüller spaces, Holomorphic Functions and Moduli II, Math. Sci. Res. Inst. Publ. vol. 11, Springer, pp. 179–194, 1988.

[10] J. Fan and J. Hu, Holomorphic contractibility and other properties of the Weil-Petersson and \( VMOA \) Teichmüller spaces, Ann. Acad. Sci. Fenn. Math. 41 (2016), 587–600.

[11] R. A. Fefferman, C. E. Kenig and J. Pipher, The theory of weights and the Dirichlet problems for elliptic equations, Ann. of Math. 134 (1991), 65–124.

[12] H. Guo, Integrable Teichmüller spaces, Sci. China Ser. A 43 (2000), 47–58.

[13] O. Lehto, Univalent Functions and Teichmüller Spaces, Grad. Texts in Math. 109, Springer, 1987.

[14] Q. Li and Y. Shen, Some notes on integrable Teichmüller space on the real line, Filomat 37 (2023), 2633–2645.

[15] X. Liu and Y. Shen, Integrable Teichmüller space, Math. Z. 302 (2022), 2233–2251.

[16] K. Matsuzaki, Rigidity of groups of circle diffeomorphisms and Teichmüller spaces, J. Anal. Math. 40 (2020), 511–548.

[17] K. Matsuzaki, Teichmüller space of circle diffeomorphisms with Hölder continuous derivatives, Rev. Mat. Iberoam. 36 (2020), 1333–1374.

[18] S. Nag, The Complex Analytic Theory of Teichmüller Spaces, Wiley-Interscience, 1988.

[19] M. Pavlović, On the moduli of continuity of \( H_p \)-functions with \( 0 < p < 1 \), Proc. Edinburgh Math. Soc. 35 (1992), 89–100.

[20] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Springer, 1992.

[21] A. Reijonen, Besov spaces induced by doubling weights, Constr. Approx. 53 (2021), 503–528.

[22] Y. Shen, Weil–Petersson Teichmüller space, Amer. J. Math. 140 (2018), 1041–1074.

[23] Y. Shen, \( VMO \)-Teichmüller space on the real line, Ann. Acad. Sci. Fenn. Math. 42 (2017), 57–82.

[24] Y. Shen, S. Tang and L. Wu, Weil–Petersson and little Teichmüller spaces on the real line, Ann. Acad. Sci. Fenn. Math. 43 (2018), 935–943.

[25] Y. Shen and H. Wei, Universal Teichmüller space and \( BMO \), Adv. Math. 234 (2013) 129–148.

[26] S. Tang, Some characterizations of the integrable Teichmüller space, Sci. China Math. 56 (2013) 541–551.

[27] S. Tang and Y. Shen, Integrable Teichmüller space, J. Math. Anal. Appl. 465 (2018), 658–672.

[28] L. Takhtajan and L. P. Teo, Weil-Petersson metric on the universal Teichmüller space, Mem. Amer. Math. Soc. 183(861) (2006).

[29] H. Wei and K. Matsuzaki, Beurling–Ahlfors extension by heat kernel, \( A_\infty \)-weights for \( VMO \), and vanishing Carleson measures, Bull. London Math. Soc. 53 (2021), 723–739.

[30] H. Wei and K. Matsuzaki, The \( p \)-Weil–Petersson Teichmüller space and the quasiconformal extension of curves, J. Geom. Anal. 32, 213 (2022).
[31] H. Wei and K. Matsuzaki, The VMO-Teichmüller space and the variant of Beurling–Ahlfors extension by heat kernel, Math. Z. 302 (2022), 1739–1760.

[32] H. Wei and K. Matsuzaki, Parametrization of the $p$-Weil–Petersson curves: holomorphic dependence, J. Geom. Anal. 33, 292 (2023).

[33] M. Yanagishita, Introduction of a complex structure on the $p$-integrable Teichmüller space, Ann. Acad. Sci. Fenn. Math. 39 (2014), 947–971.

[34] M. Yanagishita, Smoothness and strongly pseudo-convexity of $p$-Weil–Petersson metric, Ann. Acad. Sci. Fenn. Math. 44 (2019), 15–28.

[35] K. Zhu, Operator Theory in Function Spaces, Math. Surveys 138, Amer. Math. Soc., 2007.
