Controlling Human Utilization of Failure-Prone Systems via Taxes
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Abstract—We consider a game-theoretic model where individuals compete over a shared failure-prone system or resource. We investigate the effectiveness of a taxation mechanism in controlling the utilization of the resource at the Nash equilibrium when the decision-makers have behavioral risk preferences, captured by prospect theory. We first observe that heterogeneous prospect-theoretic risk preferences can lead to counter-intuitive outcomes. In particular, for resources that exhibit network effects, utilization can increase under taxation and there may not exist a tax rate that achieves the socially optimal level of utilization. We identify conditions under which utilization is monotone and continuous, and then characterize the range of utilizations that can be achieved by a suitable choice of tax rate. We further show that resource utilization is higher when players are charged differentiated tax rates compared to the case when all players are charged an identical tax rate.

I. INTRODUCTION

Large-scale cyber-physical systems form the basis of much of society’s critical infrastructure [2], and thus must be designed to be resilient to failures and attacks in order to avoid catastrophic social and economic consequences. While there are a variety of angles to designing such systems to be more resilient (including the design of secure control schemes [3], [4], interconnection topologies [5], [6], and resilient communication mechanisms [7]), there is an increasing realization that the resilience of these systems also depends crucially on the humans that use them [8], [9]. Therefore, in order to design more resilient socio-cyber-physical systems, it is critical to understand (in a rigorous mathematical framework) the decisions made by humans in decentralized and uncertain environments, and to influence those decisions to obtain better outcomes for the entire system [9]–[11].

In this paper, we investigate the impacts of human decision-making on the resiliency of a shared system in a game-theoretic framework. Game theory has emerged as a natural framework to investigate the impacts of decentralized decision-making on the efficiency, security and robustness of large-scale systems [9], [12]. When the utilities of the decision-makers or players are uncertain (e.g., due to risk of system failure or cyber-attack), their risk preferences play a significant role in shaping their behavior. With the exception of a few recent papers, most of the existing theoretical literature involving uncertainty in game-theoretic settings models decision-makers as risk neutral (expectation maximizers) or risk averse (expected utility maximizers with respect to a concave utility function). However, empirical evidence has shown that the preferences of human decision-makers systematically deviate from the preferences of a risk neutral or risk averse decision-maker [13], [14]. Specifically, humans compare different outcomes with a reference utility level, and exhibit different attitudes towards gains and losses. In their Nobel-prize winning work, Kahneman and Tversky proposed prospect theory [13] in order to capture these attitudes with appropriately defined utility and probability weighting functions. Prospect theory has been one of the most widely accepted models of human decision-making, and has shown its relevance in a broad range of disciplines [14]–[16], including recent applications in engineering [17]–[21].

Motivated by the strong empirical and behavioral foundations of prospect theory, we study how to control the behavior of human decision-makers with prospect-theoretic utilities in a game-theoretic setting. We consider a broad class of games where users compete over a shared failure-prone system. We use the term “resource” to refer to this shared system to maintain consistency with related game-theoretic models. Specifically, in our setting, a set of players split their budget between a safe resource with a constant return and a shared “common pool” resource (CPR). As total investment or utilization by all players in the CPR increases, it becomes more likely for the CPR to fail, in which case the players do not receive any return from it. If the CPR does not fail, then the players receive a return per unit investment according to a rate of return function. Shared resources with increasing rates of return exhibit so-called network effects [22]: examples include peer to peer file sharing systems CPRs with decreasing rates of return model congestion effects and describe engineered systems such as transportation and communication networks [25], [26] and natural resources such as fisheries [27]. We consider CPRs with both network and congestion effects in this work. In Section III we further discuss how this general model captures the externalities present in several applications.

A. Contributions

We study a tax mechanism where each player is charged a tax amount proportional to her investment in the CPR. A
central authority chooses the tax rate to control the utilization of the shared resource. Analysis of this taxation scheme is quite challenging under prospect-theoretic preferences. Building upon the analysis in [28] (where we analyzed users’ equilibrium strategies in the absence of taxation), we first show that the game admits a unique pure Nash equilibrium (PNE). We refer to the total investment in the CPR as its utilization, and the failure probability as its fragility.

Our primary goal is to identify conditions under which:

1) there exists a tax rate that achieves a desired (e.g., socially optimal) level of CPR utilization, and
2) there exists an optimal tax rate that maximizes a continuous function of the tax rate and utilization (such as the revenue).

In order to answer these questions, we provide conditions under which utilization is monotone and continuous in the tax rate. It is perhaps natural to expect that a higher tax rate will reduce the utilization in a continuous manner. However, for CPRs that exhibit network effects, we find that behavioral risk preferences can sometimes cause utilization (and fragility) to increase with a higher tax rate. Furthermore, we illustrate that utilization can be discontinuous as the tax rate increases, both as a consequence of the shape of the utility function, and under heterogeneous prospect-theoretic preferences. We (separately) identify (i) conditions on the CPR characteristics and prospect-theoretic parameters under which utilization decreases monotonically with tax rate, and (ii) the range of tax rates under which the utilization varies continuously. In contrast to CPRs that exhibit network effects, we show that for CPRs that exhibit congestion effects, utilization is continuous and monotonically decreasing in the tax rate under heterogeneous prospect-theoretic preferences of the players. Building upon these insights, we then identify the range of utilization that can be achieved via our taxation scheme.

Finally we explore the implications of imposing different (player-specific) tax rates on the players. We show that imposing different tax rates on a set of homogeneous loss averse players leads to a higher utilization than imposing a uniform tax rate (equal to the mean of the heterogeneous tax rates).

B. Related work

Within the game-theoretic framework, controlling the resource utilization levels through economic incentives such as taxes and rewards has been studied extensively [29]–[31]. In [29], the authors study how a taxation scheme known as Pigovian tax improves social welfare at a PNE in a CPR game. The effect of player-specific tax sensitivities on the price of anarchy were studied in [30], [32] in the context of nonatomic congestion games. In contrast, our game formulation is an instance of atomic splittable congestion games [33]. To the best of our knowledge, there has been no investigation of the impact of behavioral risk preferences on users’ strategies under taxation in congestion or CPR games.

II. PROSPECT THEORY

As discussed in the previous section, our focus is on behavioral preferences captured by the utility function of prospect theory [13]. Specifically, consider a gamble that has an outcome with value $z \in \mathbb{R}$. A prospect-theoretic individual perceives its utility in a skewed manner, via the function

$$v(z, z_0) = \begin{cases} (z - z_0)^\alpha, & \text{when } z \geq z_0 \\ -k(z_0 - z)^\alpha, & \text{otherwise} \end{cases}$$

(1)

where $z_0$ is the reference point, $\alpha \in (0, 1]$ is the sensitivity parameter and $k \in [0, 1]$ is referred to as the loss aversion index. Increase in utility with respect to the reference point ($z \geq z_0$) is referred to as a gain and decrease in utility is referred to as a loss ($z < z_0$).

The parameter $\alpha$ shapes the utility function according to observed behavior, i.e., the utility function is concave in the domain of gains and convex in the domain of losses. Accordingly, the decision maker is said to be “risk averse” in gains and “risk seeking” in losses. As its name indicates, the parameter $k$ captures loss aversion behavior. Specifically, when $\alpha = 1$, a loss of $1$ feels like a loss of $k$ to the player. A value of $k > 1$ implies that the individual is loss averse, while $k < 1$ implies that the individual is gain seeking. When the reference point is an exogenous constant, the values $k = 1$ and $\alpha = 1$ capture risk neutral behavior. A smaller $\alpha$ implies greater deviation from risk neutral behavior. The shape of the value function is shown for different values of $k$ in Figure 1 with $\alpha = 0.5$ and $z_0 = 0$.

III. FRAGILE COMMON POOL RESOURCE GAME

We start by introducing the Fragile Common Pool Resource game [28]. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ be the set of players. Each player has an endowment or wealth equal to 1 which she must split between a safe resource and a shared common pool resource (CPR). We define the strategy of a player $i \in \mathcal{N}$ as her investment in the CPR, denoted by $x_i \in [0, 1]$. The total investment by all players in the CPR is denoted by $x_T = \sum_{i \in \mathcal{N}} x_i$. Following conventional notation, we denote the profile of investments by all players other than $i$ as $x_{-i} \in [0, 1]^{n-1}$. Furthermore, let $\bar{x}_{-i} = \sum_{j=1, j \neq i}^{n} x_j$, be the total investment of all players other than $i$.

Players receive returns on their investments from both resources. The return per unit investment from the safe resource is normalized to 1, i.e., player $i$ investing $1 - x_i$ in the safe resource receives a return of $1 - x_i$. The return from the CPR is subject to risk, captured by a probability of failure $p(x_T)$, which is a function of the aggregate investment in the CPR. If

Fig. 1: Prospect-theoretic utility function (1) with $\alpha = 0.5$ and reference point $z_0 = 0$. 
the CPR fails, players do not receive any return from it. If the CPR does not fail, it has a per unit return that is a function of the total investment \(x_T\), denoted by \(\bar{r}(x_T)\). In other words, player \(i\) gets \(x_i\bar{r}(x_T)\) from the CPR when it does not fail.

The formulation described above has been studied in many different contexts.

1) The above formulation was studied as common pool resource games to model competition over failure-prone shared resources such as fisheries [27], [34].

2) CPR games, without resource failure, are equivalent to an instance of atomic splittable congestion games on a network with two nodes and two parallel links joining them. One link corresponds to the CPR described above and the second has a constant delay of 1. This class of games has been extensively studied in the context of transportation and communication networks [26], [33].

3) Fragile CPR games are related to the setting in [18], where players are microgrid operators who decide the fraction of energy to store for potentially selling at a higher price in the event of an emergency [3]. Both settings are related if we define the investment of a player as the fraction of stored energy, and \(p(x_T)\) as the probability that the energy requirement during emergency is smaller than the total stored energy (i.e., energy price does not increase and the players incur losses).

4) In resource dilemma games [35], players bid for utilizing a fraction of a shared resource with unknown size. If the total demand exceeds the size of the resource, no player receives any benefit. This model is potentially relevant when a set of users compete over a shared energy storage system [36]. This class of games is closely related to Fragile CPR games where \(x_i\) is the bid of player \(i\), and \(p(x_T)\) is the distribution of resource size.

Given the breadth of applications where this formulation arises, the goal of this paper is to understand to what extent we can control the utilization \((x_T)\) of the resource at the Nash equilibrium by imposing taxes on players’ investments.

IV. PROSPECT-THEORETIC UTILITY UNDER TAXATION

We first consider the case where a central authority imposes a uniform tax rate \(t \geq 0\) per unit investment in the CPR on the players. Figure 2 represents the schematic of our setting.

Under this taxation scheme, a player \(i\) with investment \(x_i\) in the CPR is charged \(tx_i\) as tax. We will consider the implications of player-specific tax rates in Section LX. Each player is prospect-theoretic, with a player-specific loss aversion index \(k_i \in \mathbb{R}_{>0}\) and sensitivity parameter \(\alpha_i \in (0, 1]\).

We define the reference utility of a player \(i\) as her utility when she invests entirely in the safe resource, i.e., chooses \(x_i = 0\). Accordingly, the reference utility is 1 for every player. Now consider a strategy profile \(\{x_T\}_{i \in \mathcal{N}}\) with total investment \(x_T\). In the event of CPR failure, each player \(i\) with a nonzero \(x_i\) experiences a loss \(-(1 + t)x_i\), which comprises of the lost income from not investing \(x_i\) in the safe resource, and the tax payment. If the CPR succeeds, the reference-dependent return is \(x_i(\bar{r}(x_T) - 1 - t)\), which could be positive (representing a gain) or negative (representing a loss) depending on the values of \(t\) and \(x_T\). For ease of exposition, we define \(r(x_T) := \bar{r}(x_T) - 1\), and henceforth refer to \(r(x_T)\) as the rate of return function.

Using the prospect-theoretic utility function (1), player \(i\)’s perception of gains and losses is

\[
u_i(x_i, x_{-i}) := \begin{cases} x_i^\alpha \left(\max(r(x_T) - t, 0)\right)^{\alpha - 1} & \\
-k_i(1 - x_i)\left(1 + (1 + t)x_i\right)^{\alpha - 1}, \text{w.p. } 1 - p(x_T), \\
-k_i(1 + t)x_i^{\alpha - 1}, \text{w.p. } p(x_T). \\
\end{cases}
\]

Player \(i\) maximizes the expected utility with respect to the above utility function given by

\[
\mathbb{E}(u_i(x_i, x_{-i})) = x_i^\alpha f_i(x_T, t),
\]

where

\[
f_i(x_T, t) := \begin{cases} (r(x_T) - t)^\alpha (1 - p(x_T)) - k_i(1 + t)^\alpha p(x_T), & \text{when } r(x_T) - t \geq 0, \\
-k_i((t - r(x_T))^\alpha (1 - p(x_T)) + (1 + t)^\alpha p(x_T)), & \text{otherwise.}
\end{cases}
\]

We refer to \(f_i(x_T, t)\) as the effective rate of return of player \(i\). Figure 3 shows the shapes of \(f_i(x_T, t)\) for different parameters. We denote this class of Fragile CPR games as \(\Gamma(N, \{u_i\}_{i \in \mathcal{N}})\). In this paper, we consider Fragile CPR games that satisfy the following assumptions.

**Assumption 1:** The class of Fragile CPR games \(\Gamma(N, \{u_i\}_{i \in \mathcal{N}})\) has the following properties.

1) The failure probability \(p(x_T)\) is convex, strictly increasing and continuously differentiable for \(x_T \in [0, 1]\) and \(p(x_T) = 1\) for \(x_T \geq 1\).

2) The rate of return \(r(x_T)\) is concave, positive, strictly monotonic and continuously differentiable.

3) Define

\[
\bar{t} := \sup\{t \geq 0 : \max_{x_T \in [0, 1]} \max_{i \in \mathcal{N}} f_i(x_T, t) > 0\}.
\]

We assume that \(\bar{t} > 0\), and the tax rate \(t \in [0, \bar{t})\). These assumptions capture a fairly broad class of resources, while retaining analytical tractability.

To explain the third point in Assumption 1 note from the definition of \(\bar{t}\) that for any tax rate \(t \geq \bar{t}\), the effective rate of return is nonpositive for every player and every \(x_T \in [0, 1]\). Accordingly, all players have expected utility equal to 0 at any PNE. On the other hand, for \(t < \bar{t}\), there exist player(s) who make a nonzero investment at the PNE. Furthermore, at a PNE with nonzero CPR investments, the total investment must
be such that $r(x_T) - t \geq 0$ (from (4), we have $f_i(x_T, t) \geq 0 \implies r(x_T) - t \geq 0$). Accordingly, most of our analysis will focus on the range of total investments that lie within a subset $S_i \subseteq [0, 1]$ such that $r(x_T) - t \geq 0$ for $x_T \in S_i$. When $r(x_T)$ is strictly decreasing, we have $S_i := [0, b_t)$, where

$$b_t := \begin{cases} 1, & \text{if } r(1) \geq t, \\ r^{-1}(t), & \text{if } r(1) < t, \end{cases}$$

(6)

where $r^{-1}(t) = \{ y \in [0, 1] | r(y) = t \}$. On the other hand, when $r(x_T)$ is strictly increasing, we have $S_i := [a_t, 1)$, where

$$a_t := \begin{cases} 0, & \text{if } r(0) \geq t, \\ r^{-1}(t), & \text{if } r(0) < t. \end{cases}$$

(7)

Note that for $t \in [0, \bar{t})$, $S_i$ is well defined and is nonempty.

Remark 1: The taxation scheme introduced here can be viewed as a subsidy on the safe resource (which increases the rate of return of the safe resource to $1 + t$). The reference-dependent utility under this subsidy is identical to (2). Such a subsidy scheme was studied in [29].

In the following section, we investigate the existence and uniqueness of PNE in Fragile CPR games under taxes.

V. CHARACTERISTICS OF PURE NASH EQUILIBRIUM

Consider a Fragile CPR game with a fixed tax rate $t \in [0, \bar{t})$. We define the best response correspondence of a player $i$ as $B_i(x_{-i}) := \arg\max_{x_i \in [0, 1]} E_{i,t}(x_i, x_{-i})$, where $E_{i,t}(\cdot)$ is defined in (3). While $E_{i,t}(x_i, x_{-i})$ is not concave in $x_i \in [0, 1]$, we will show that $B_i(\cdot)$ is single-valued and continuous in $x_i$. We start with the following lemma. While the proof largely follows from identical arguments as the proof of Lemma 1 in [28] (where we considered Fragile CPR games without taxation), we present it here as the proof formally defines several important quantities that are useful in the analysis throughout the paper. Recall that $x_{-i}$ denotes the total investment by all players other than $i$.

Lemma 1: Consider a Fragile CPR game with a fixed tax rate $t \in [0, \bar{t})$. Then, for any player $i$, the following are true.

1) There exists a unique $y_i^* \in [0, 1]$, such that 0 is a unique best response for player $i$ if and only if $\bar{x}_{-i} \geq y_i^*$.

2) When $y_i^* > 0$, $f_i(y_i^*, t) = 0$, and there exists an interval $I_i^* \subseteq [0, y_i^*) \subset S_i$ such that if $\bar{x}_{-i} < y_i^*$, then each best response $b_t \in B_i(x_{-i})$ (i) is positive, and (ii) satisfies $b_t + \bar{x}_{-i} \in I_i^*$.

3) For $x_T \in I_i^*$, we have $f_i(x_T, t) > 0$ and $f_i(x_T, t) := \frac{\partial g_i(x_T, t)}{\partial x_T} < 0$.

Proof: We consider the following two cases.

Case 1: $r(x_T)$ is decreasing. From the definition of $b_t$ (equation (5)), we obtain $f_i(b_t, t) < 0$ in (4). Straightforward calculation shows that $f_i(x_T, t)$ is strictly decreasing in $x_T$ when $x_T \in [0, b_t]$. If $f_i(0, t) \leq 0$, we define $y_i^* = 0$.

On the other hand, if $f_i(0, t) > 0$, we define $y_i^* \in S_i$ as the unique investment where $f_i(y_i^*, t) = 0$. If $\bar{x}_{-i} \geq y_i^*$, any nonzero investment $\epsilon$ by player $i$ will lead to $f_i(\epsilon + \bar{x}_{-i}, t) < 0$, and consequently a strictly negative utility. Therefore, $B_i(x_{-i}) = \{ 0 \}$ in this case. On the other hand, if $\bar{x}_{-i} < y_i^*$, there exists $\delta > 0$ such that $\delta + \bar{x}_{-i} < y_i^*$, and therefore, $f_i(\delta + \bar{x}_{-i}, t) > 0$. Therefore, the interval $I_i^* := [0, y_i^*)$ has the desired properties.

Case 2: $r(x_T)$ is increasing. If $f_i(x_T, t) \leq 0$ for $x_T \in [a_t, 1]$, we define $y_i^* = 0$, and $B_i(x_{-i}) := \{ 0 \}$ for every $x_{-i}$.

Now suppose there exists $x_T \in [a_t, 1]$ where $f_i(x_T, t) > 0$. Straightforward calculation shows that $f_i(x_T, t)$ is strictly concave in $x_T$ when $x_T \in [a_t, 1]$. Therefore, there exists a unique maximizer of $f_i(x_T, t)$ given by $z_i^* := \arg\max_{x_T \in [a_t, 1]} f_i(x_T, t)$. Note that we must have $z_i^* < 1$ since $f_i(1, t) < 0$. From the strict concavity of $f_i$, we have $f_i(x_T, t) := \frac{\partial g_i(x_T, t)}{\partial x_T} < 0$ for $x_T > z_i^*$. Thus, there exists a unique investment $y_i^* \in (z_i^*, 1)$ such that $f_i(y_i^*, t) = 0$. We argue that $y_i^*$ and $I_i^* := (z_i^*, y_i^*)$ have the desired properties; $I_i^*$ satisfies the third property by definition.

Now suppose the total investment by players other than $i$ satisfies $\bar{x}_{-i} \geq y_i^*$. Then any $x_i > 0$ would imply $f_i(x_i + \bar{x}_{-i}, t) < 0$, and 0 is the unique best response. On the other hand, if $\bar{x}_{-i} < y_i^*$, there exists $\delta > 0$ such that $f_i(\delta + \bar{x}_{-i}, t) > 0$, and thus, all best responses must be positive. Note that we must necessarily have $\delta + \bar{x}_{-i} > a_t$. Now suppose $x_T \in B_i(x_{-i})$. Then it must necessarily satisfy the first order condition of optimality $\frac{\partial g_i(x_T, t)}{\partial x_T} = 0$ for the utility in (3), leading to

$$x_T f_i(x_T, t + \bar{x}_{-i}, t) + \alpha_t f_i(x_T + \bar{x}_{-i}, t) = 0. \quad (8)$$

Since $f_i(x_T + \bar{x}_{-i}, t) > 0$, we must have $f_i(x_T, t + \bar{x}_{-i}, t) < 0$, and therefore, $x_T + \bar{x}_{-i} \in I_i^*$.

Remark 2: Figure 3 illustrates the quantities introduced in the above lemma; the subscript $i$ is dropped for convenience. Figure [33] shows that $y_i^* = 0.8359$ and $y_i^* = 0.6166$ for a CPR with decreasing rate of return. Note from the figure that $f(y_i^*, t) = 0$ in both cases. Figure 33 and 32 show the values of $y_i^* \bar{t}^*$ and $z_i^* \bar{t}^*$ for a CPR with $r(x_T) = 3x_T + 1$ and $p(x_T) = 0.2 + 0.8x_T$ for different tax rates and risk preferences. Note from the figures that $z_i^*$ is the maximizer of $f_i(x_T, t)$, and $f_i(y_i^*, t) = 0$. The kinks in the last two figures occur at the respective $a_t$ values.

We now build upon the above discussion, and introduce a few other important quantities. For a player $i$, we define $g_i(x_T, t) := \frac{\alpha_t f_i(x_T, t)}{1 - f_i(x_T, t)} x_T \in S_i$.

It follows from the first order optimality condition in (5) that a nonzero best response $x_T^* \in B_i(x_{-i})$ satisfies $x_T^* = g_i(x_T^* + \bar{x}_{-i}, t)$. Note that $g_i(x_T, t)$ is a natural extension of the function $g(x_T)$ defined in (28). Accordingly, at a fixed tax rate $t$, we have the following result on the monotonicity of the function $g_i(x_T, t)$ with respect to $x_T$.

Lemma 2: For $x_T \in I_i^*$, we have $\frac{\partial g_i(x_T, t)}{\partial x_T} < 0$.

The proof is analogous to the proof of Lemma 4 in [28], and thus we omit it. However, $g_i(x_T, t)$ is not always decreasing in $t$ as we will explore later. As a consequence of the above two lemmas, we have the following result.

Proposition 1: Consider a Fragile CPR game with a fixed tax rate $t$ satisfying Assumption 1. Then there exists a unique PNE of the game.

Proof: In Lemma 1 we showed that when a player $i$ has a nonzero best response, the total investment in the CPR lies in the interval $I_i^*$. When $x_T \in S_i$, the rate of return function is monotone, concave and positive. Therefore, the results on the
have z and a player Supp in functions from the proof of Lemma 1. Support p the utilization x of the proof of Theorem 1 in [28].

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x

1

3

≤ 0.05. Here y0 = 0.9961, z0 = 0.6083, y1.5 = 0.6052, z1.5 = 0.0,

and z1.21 = 0.85.

Fig. 3: Shapes of the effective rate of return function under different CPR characteristics and risk preferences.

uniqueness and continuity of best responses from Lemma 2 and 3 in [28] carry over to the present setting under taxation. As a consequence of Brouwer’s fixed point theorem [37], a PNE exists. The uniqueness of PNE follows the monotonicity of nonzero best responses shown in Lemma 2. Its proof follows identical arguments as the proof of Theorem 1 in [28].

At a given tax rate t, we denote the total investment in the CPR at corresponding PNE as x^t_{NE}, and refer to it as the utilization (of the CPR). Furthermore, we refer to the corresponding failure probability p(x^t_{NE}) as its fragility. We now define the support of a PNE.

Definition 1: The support of the PNE of the game Γ, denoted Supp(Γ), is the set of players who have a nonzero investment in the CPR. In particular, by Lemma 1, Supp(Γ) := \{i ∈ N | x^t_{NE} < y^i\}.

Accordingly, the total investment at the PNE satisfies

\[ x^t_{NE} = \sum_{i \in Supp(Γ)} g_i(x^t_{NE}, t). \]  

(10)

From our earlier discussion, we have x^t \in I^t_i for i \in Supp(Γ). Therefore, for increasing rate of return functions, x^t_{NE} = z^t_i for every i \in Supp(Γ), since I^t_i = (z^t_i, y^i) for such functions from the proof of Lemma 1.

We conclude this section with two lemmas that will be useful in several proofs later in the paper, and a brief discussion on the social welfare. The first shows the monotonicity of z^t_i in t for certain risk preferences.

Lemma 3: Consider a CPR game with increasing r(x,T), and a player i with α_i = 1, and let t_2 < t_1 < t. If k_i < 1, then z^t_2 ≤ z^t_1, and vice versa.

Proof: When x^t \in S^t, the effective rate of return function in (11) for player i is given by

\[ f_i(x, t, x, t) = r(x, t)(1 - p(x, t)) - k(x, t) - t(k - 1)p(x, t). \]  

(11)

Let z^t_2 > a_i > 0; otherwise the result follows directly. According to the first order optimality condition for z^t_i, we have

\[ f_i(z^t_2, t_2) = \frac{\partial f_i}{\partial z_i}(z^t_2, t_2) = 0. \]  

Since k_i < 1, and p(x, t) is strictly increasing, it is easy to see that f_i(x, z^t_2, t_1) > 0 implying z^t_1 ≤ z^t_2. The same reasoning applies to the converse.

Observe that in Figure 3, z^t_i is indeed increasing in t in accordance with the above lemma. Now, let Γ_1 and Γ_2 be two instances of Fragile CPR games with identical resource characteristics and tax rates t_1 and t_2, respectively. Let the respective total PNE investments be x^t_{NE,1} and x^t_{NE,2}. We prove the following result which holds for CPRs with both increasing and decreasing r(x, t).

Lemma 4: If t_1 > t_2, we have y^i_{t_1} ≤ y^i_{t_2} for every player i with α_i \in (0, 1) and k_i > 0. In addition, if t_1 > t_2 and x^t_{NE,1} > x^t_{NE,2}, we have Supp(Γ_1) ⊆ Supp(Γ_2).

Proof: Let max_{x \in S^t_i} f_i(x, t_1) > 0; otherwise y^i_{t_1} = 0, and the first statement trivially holds. When y^i_{t_1} > 0, it follows from Lemma 1 that f_i(y^i_{t_1}; t_1) = 0. When t_2 < t_1, it is easy to see (from (6) and (7)) that S^t_1 ⊆ S^t_2. Furthermore, note from (4) that f_i is decreasing in the second argument t for both increasing and decreasing rate of return functions. Accordingly, f_i(y^i_{t_1}; t_2) > 0, and therefore, y^i_{t_1} ≤ y^i_{t_2}. From Definition 1, we have

\[ x^t_{NE,1} < y^i_{t_1} \implies x^t_{NE,2} < x^t_{NE,1} < y^i_{t_1} \leq y^i_{t_2}. \]

As a result, j \in Supp(Γ_1). This concludes the proof.

As discussed in the introduction, one of the key motivations behind this work is to understand conditions under which a socially desirable level of utilization can be achieved via taxation. In the context of congestion games and CPR games, the metric that is often used to capture a socially desired level of utilization is the resource utilization that maximizes the sum of utilities of all players in the absence of taxation (also referred to as the social welfare) [30, 31, 38]. The following result shows that the CPR utilization and fragility are higher at the PNE compared to their counterparts at a social welfare maximizing strategy profile.

Proposition 2: Let x^{opt} = (x_{1,0}, x_{2,0}, \ldots, x_{n,0}) \in [0, 1]^N be a set of investments that maximizes Ψ(x) := \sum_{x \in N} \sum_{i \in N} x_{i, x_i, x_i} at t = 0. Then, the resulting total CPR investment at the social optimum is at most the total CPR investment at the PNE, i.e., x^{opt} ≤ x^0_{NE}.

The result holds under heterogeneous prospect-theoretic preferences of the users. We refer to Appendix A for the proof.

In order to answer questions such as the existence of a tax rate that achieves x^0_{opt}, we now investigate whether utilization, as
VI. MONOTONICITY OF UTILIZATION UNDER TAXATION

We now investigate whether the total investment $x^t_{BE}$ at the PNE (i.e., the utilization of the CPR) is monotone in $t$. Note from (3) that the utility of a player in a game with a tax rate $t$ is equivalent to that in a game without taxes, but with a smaller rate of return function $(r(x_T) - t)$, and a larger index of loss aversion $(k_i(1 + t)^\alpha)$. Therefore, intuition suggests that an increase in tax rate would lead to smaller utilization. We start this section by showing that this is not always the case for CPRs with increasing $r(x_T)$ and under prospect theory.

A. Monotonicity of utilization under network effects (or lack thereof)

Recall that online platforms such as peer to peer file sharing systems are instances of CPRs that exhibit network effects. In the following example, we show that imposing a higher tax rate can lead to higher utilization and fragility.

Example 1: Consider a Fragile CPR game with $n = 3$ players. Let $r(x_T) = 8x_T + 5$ and $p(x_T) = x_T$. Let $\alpha = 0.15$ and $k = 1.2$ for all players, i.e., all players are loss averse, and the deviation from risk neutral behavior ($\alpha = 1$ and $k = 1$) is significant. As shown in Figure 4 when $t$ increases from 0 to 4.9, the fragility is not monotonically decreasing.

Recall from Figure 1 that $\alpha < 1$ gives rise to risk seeking behavior in losses and risk averse behavior in gains. When the value of $\alpha$ approaches 0, the higher tax rate $t$ increases their investment into the CPR. Such behavior is not limited to the case when $\alpha$ is very small. In the conference version of this paper, we provided additional examples where a higher tax rate leads to higher utilization when the index of loss aversion is smaller than 1. In both instances, players increase their investments to receive a higher return and compensate for the tax payment (at the cost of increased risk of resource failure).

We now obtain a sufficient condition under which a higher tax rate does indeed lead to smaller utilization; a central authority can check this condition, and avoid the counter productive outcome observed in the previous subsection.

Recall that $a_i \in [0, 1]$ is the unique investment such that $r(a_i) = t$. For $t \in [0, \bar{t})$, $x_T \in (a_t, 1]$, let

$$q_i(x_T, t) := \frac{r'(x_T)(1 - p(x_T))^2}{(r(x_T) + 1)p'(x_T) - \alpha_i r'(x_T)(1 - p(x_T))p(x_T) \times \left( \frac{1 + t}{r(x_T) - t} \right)^{1 - \alpha_i}}.$$  \hspace{1cm} (12)

The following lemma proves a few useful properties of this function.

Lemma 5: The function $q_i(\cdot, \cdot)$ defined above has the following properties.

1) Let $x_T \in [0, 1]$ and $t \in \{t \geq 0 | x_T \in (a_t, 1]\}$. If $k_i > q_i(x_T, t)$, then $\frac{\partial q_i(x_T, t)}{\partial t} < 0$.

2) Let $x_T \in \mathcal{I}_i^1$. Then, the denominator of $q_i(\cdot, \cdot)$ is positive. Specifically, $(r(x_T) + 1)p'(x_T) > (r(x_T) - t)p'(x_T) > \alpha_i r'(x_T)(1 - p(x_T))p(x_T)$.

3) For $x_T \in (a_t, 1]$, $q_i(x_T, t)$ is strictly decreasing in $x_T$.

The proof is presented in Appendix 3. We are now ready to prove the main result of this subsection.

Proposition 3: Consider a Fragile CPR game with increasing $r(x_T)$. Let $t_2 < t_1 < t$ with $x^t_{BE} > a_{t_1}$. Suppose $k_i > q_i(x^t_{BE}, t_1)$ for every player $i$, where $q_i(\cdot, \cdot)$ is defined in (12). Then, $x^t_{BE} \leq x^{t_2}_{BE}$.

Proof: Assume on the contrary that $x^t_{BE} > x^{t_2}_{BE} > 0$. According to Lemma 4 we have $\text{Supp}(\Gamma_1) \subseteq \text{Supp}(\Gamma_2)$. From the characterization of PNE in equation (10), we obtain

$$\Rightarrow \sum_{j \in \text{Supp}(\Gamma_1)} g_j(x^t_{BE}, t_1) > \sum_{j \in \text{Supp}(\Gamma_2)} g_j(x^{t_2}_{BE}, t_2) \Rightarrow \sum_{j \in \text{Supp}(\Gamma_1)} g_j(x^t_{BE}, t_1) > \sum_{j \in \text{Supp}(\Gamma_1)} g_j(x^{t_2}_{BE}, t_2).$$  \hspace{1cm} (13)

In the remainder of the proof, our goal is to contradict the inequality in equation (13). In particular, for each player $j \in \text{Supp}(\Gamma_1)$, we show that $g_j(x^t_{BE}, t_1) < g_j(x^{t_2}_{BE}, t_2)$.

Consider a player $j \in \text{Supp}(\Gamma_1)$. From Lemma 1 $x^t_{BE} < y^t_j$. Furthermore, from Lemma 4 we have $y^{t_1}_j \leq y^{t_2}_j$. Together with our assumption, we obtain

$$x^t_{BE} < x^t_{BE} \leq y_j^{t_1} \leq y_j^{t_2} \Rightarrow [x^t_{BE}, x^t_{BE}] \subset I^1_j.$$  \hspace{1cm} (14)

From the monotonicity of $g_j(x_T, t)$ in $x_T$ in Lemma 2 we obtain $g_j(x^t_{BE}, t_1) < g_j(x^{t_2}_{BE}, t_2)$.

It is now sufficient to show that $g_j(x^t_{BE}, t_1) < g_j(x^{t_4}_{BE}, t_2)$. Since $x^t_{BE} > x^{t_2}_{BE} > a_{t_1}$, part three of Lemma 5 yields

$$g_j(x^t_{BE}, t_1) > q_j(x^{t_1}_{BE}, x^{t_1}_{BE}).$$

Furthermore, from its definition, $q_j(x, t)$ is strictly increasing in $t$. Thus, for $t \in [t_2, t_1]$,

$$q_j(x^t_{BE}, t_1) > q_j(x^{t_1}_{BE}, t_1);$$

note that $x^t_{BE} > a_{t_1}$ for $t \in [t_2, t_1]$. Combining these observations, we obtain

$$k_j > q_j(x^t_{BE}, t_1) > q_j(x^{t_1}_{BE}, t_1) > q_j(x^{t_1}_{BE}, t) \not\Rightarrow \frac{\partial q_j(x^t_{BE}, t)}{\partial t} < 0,$$

for $t \in [t_2, t_1]$ (following the first part of Lemma 5). Therefore, $g_j(x^t_{BE}, t_1) < g_j(x^{t_1}_{BE}, t_2)$, which contradicts (14).
The above result establishes a condition that we can check to ensure that a higher tax rate will lead to smaller utilization. All else being equal, if players become more loss averse (i.e., as \( k_i \) increases), it becomes more likely that the condition \( k_i > q_i(x_T, t) \) is satisfied. Furthermore, when all players have identical \( \alpha \), we only need to check the condition for the player with the smallest loss aversion index.\(^4\)

When \( \alpha = 1 \), \( q(x_T, t) \) is independent of \( t \) from (12). We now obtain the following result.

**Proposition 4**: Consider a Fragile CPR game with increasing \( r(x_T) \). Let \( \alpha = 1 \) and \( k \geq 1 \) for every player. Let \( t_2 < t_1 < t \). Then, \( x_{BE}^{t_2} \leq x_{BE}^{t_1} \).

**Proof**: Assume on the contrary that \( x_{BE}^{t_1} > x_{BE}^{t_2} \). Following analogous arguments as the first part of the proof of Proposition 3 it suffices to show that for a player \( j \in \text{Supp}(\Gamma_1) \), \( g_j(x_{BE}^{t_1}, t_1) < g_j(x_{BE}^{t_2}, t_2) \).

Note that \( x_{BE}^{t_2} > x_{BE}^{t_1} \). Therefore, \( f_j(x_{BE}^{t_1}, t) \) is defined for \( t \in [t_2, t_1] \). Since \( \alpha_j = 1 \), we have
\[
f_j(x_T, t) = (r(x_T) - t)(1 - p(x_T)) - k_j(1 + t)p(x_T),
\]
and
\[
f_j, x(x_T, t) = \frac{\partial f_j}{\partial x_T}(x_T, t) = r'(x_T)(1 - p(x_T)) - r(x_T)p'(x_T) - k_j p'(x_T) - tp'(x_T)(k_j - 1).
\]
It is easy to see that \( \frac{\partial f_j}{\partial x_T}(x_{BE}^{t_1}, t) < 0 \). Furthermore, when \( k_j \geq 1 \), \( \frac{\partial^2 f_j}{\partial x_T^2}(x_{BE}^{t_1}, t) \leq 0 \) for \( t \in [t_2, t_1] \). Therefore, \( g_j(x_{BE}^{t_1}, t) \) is decreasing in \( t \) for \( t \in [t_2, t_1] \). The result now follows from identical arguments as the proof of Proposition 3.

The above result states that when users have \( \alpha_j = 1 \), and are risk neutral or loss averse (i.e., have \( k_j \geq 1 \)), a higher tax rate leads to smaller CPR utilization. Therefore, the lack of monotonicity observed in the earlier examples is a consequence of prospect-theoretic risk preferences.

**B. Monotonicity of utilization under congestion effects**

In contrast with the observations in the above subsection, for resources with a decreasing \( r(x_T) \), we show here that an increase of tax rate always leads to smaller utilization of the CPR. The result holds when \( k_i \in \mathbb{R}_{>0} \) and \( \alpha_j \in (0, 1] \) are player-specific, and potentially heterogeneous.

We start with the following useful lemma, which holds for the general form of the utility function (4) with \( \alpha_j \in (0, 1] \).

**Lemma 6**: Let \( x_T \in [0, 1] \). For a player \( j \), let \( T_j^{x_T} := \{ t \in [0, \bar{t}] | f_j(x_T, t) > 0 \} \). Let \( g_j(x_T, t) \) be the function defined in (2). Then, \( \frac{\partial g_j(x_T, t)}{\partial t} < 0 \) for \( t \in T_j^{x_T} \).

In other words, at a given utilization level \( x_T \), the function \( g_j(x_T, t) \) is strictly decreasing in the tax rate \( t \) over the range of tax rates at which the effective rate of return remains positive. The proof is presented in Appendix B. We are now ready the state the main result of this subsection.

**Proposition 5**: Consider a Fragile CPR game with decreasing \( r(x_T) \). Let \( 0 \leq t_2 < t_1 < \bar{t} \). Then, \( x_{BE}^{t_2} \leq x_{BE}^{t_1} \).

**Proof**: Assume on the contrary that \( x_{BE}^{t_2} > x_{BE}^{t_1} \). From Lemma 4 we have \( \text{Supp}(\Gamma_1) \subseteq \text{Supp}(\Gamma_2) \), where \( \text{Supp}(\Gamma_k), k \in \{1, 2\} \) is the support of the PNE at tax rate \( t_k \). Following identical arguments as the proof of Proposition 3 we obtain
\[
\sum_{j \in \text{Supp}(\Gamma_1)} g_j(x_{BE}^{t_1}, t_1) > \sum_{j \in \text{Supp}(\Gamma_1)} g_j(x_{BE}^{t_2}, t_2). \tag{14}
\]
We obtain a contradiction to the above equation as follows.

Consider a player \( j \in \text{Supp}(\Gamma_1) \). Then \( x_{BE}^{t_1} \in T_j^{t_1} \), where \( T_j^{t_1} \) is the interval defined in Lemma 1 in particular, when \( x_{BE}^{t_1} \in T_j^{t_1} \), we have \( f_j(x, t_1) > 0 \) and \( f_j(x, t_1) < 0 \). When \( r(x) \) is decreasing in \( x \), and \( x_{BE}^{t_2} \geq x_{BE}^{t_1} \), then \( x_{BE}^{t_2} \in T_j^{t_1} \). Therefore, from Lemma 2 we have \( g_j(x_{BE}^{t_2}, t_1) < g_j(x_{BE}^{t_2}, t_1) \).

Since \( x_{BE}^{t_2} \in T_j^{t_1} \), we have \( f_j(x_{BE}^{t_2}, t_1) > 0 \). Since \( f_j(x, t) \) is also decreasing in \( t \), \( f_j(x_{BE}^{t_2}, t) > 0 \) for \( t \in [t_2, t_1] \). From Lemma 6 we obtain \( g_j(x_{BE}^{t_2}, t_1) < g_j(x_{BE}^{t_2}, t_2) \). Thus, for every player \( j \in \text{Supp}(\Gamma_1) \), \( g_j(x_{BE}^{t_1}, t_1) < g_j(x_{BE}^{t_2}, t_2) \), which contradicts (14).

We recall that we want to characterize the set of total investments that can be achieved at a PNE via taxation. In order to answer this question, we now investigate whether utilization varies as a continuous function of the tax rate.

**VII. CONTINUITY OF UTILIZATION UNDER TAXATION**

**A. Continuity of utilization under network effects (or lack thereof)**

We start with a few numerical examples to illustrate how utilization can be discontinuous in the tax rate. We discuss the mathematical intuition behind such discontinuities, followed by identifying conditions (on the prospect-theoretic preferences) under which utilization remains continuous.

First consider a player \( i \) who is risk neutral (i.e., \( \alpha_i = 1 \) and \( k_i = 1 \)). For such a player \( i \), the effective rate of return (4) is given by
\[
f_i(x_T, t) = r(x_T)(1 - p(x_T)) - p(x_T) - t.
\]
Therefore, for such a player, \( z_i^t := \max_{x_T \in [0, 1]} f_i(x_T, t) \) is independent of \( t \). Now suppose \( z_i^t = 0 > 0 \). If at a PNE, player \( i \) has a nonzero investment, then the utilization \( x_{BE}^{z_i^t} > z_i^t \). In the following example, we show that as \( t \) exceeds \( \bar{t} \), \( f_i(x_{BE}^{z_i^t}, t) \) becomes negative, and \( x_{BE}^{z_i^t} \) drops abruptly to 0.

**Example 2**: Consider a Fragile CPR game with two risk neutral players. Let \( p(x_T) = 0.2 + 0.8x_T^2 \) and \( r(x_T) = 3x_T + 1 \). Note that \( \bar{t} = 1.655 \), i.e., \( \max_{x_T \in [0, 1]} f(x, t) = 0 \). In Figure 35, we show how utilization \( x_{BE}^{z_i^t} \) decreases as \( t \) increases (in accordance with Proposition 3); there is a discontinuity at \( \bar{t} \) where \( x_{BE}^{z_i^t} \) drops from 0.5665 to 0. The socially optimal level of utilization in the absence of taxes is \( x_{OPT}^{z_i^t} = 0.6829 \). For \( t \in [0, \bar{t}] \), \( x_{BE}^{z_i^t} \) is monotone, continuous and as \( t \rightarrow \bar{t} \) approaches \( t \) from below, \( x_{BE}^{z_i^t} \rightarrow z_i^t = 0.5665 \). Therefore, there exists a tax rate \( t^* \) such that \( x_{BE}^{z_i^t} = x_{OPT}^{z_i^t} \).

In the following example, we show that when players exhibit gain seeking behavior (i.e., \( k < 1 \)), there are instances where the socially optimal utilization cannot be achieved.

**Example 3**: Consider a Fragile CPR game with \( p(x_T) = 0.2 + 0.8x_T^2 \), and \( r(x_T) = 3x_T + 1 \) as before. We consider homogeneous players with \( \alpha = 1, k = 0.05 \). Recall that the shape of \( f(x_T, t) \) for these parameters were shown in Figure 36 for different values of tax rates. Note from Figure 35 that...
\[\alpha\] and \(\beta\) these parameters at \(t\)ous in \(t < t\)on Berge’s maximum theorem \([37]\), also stated formally in Appendix \(C\). The result states that utilization remains continuous over a subset of tax rates for which utilization is nonzero. Recall that in Example \(H\) \(\hat{t}_1 = 1.583\), while \(\hat{t}_2 = 2.19\). As shown in Figure \(5c\) utilization is continuous for \(t \in [0, 1.583]\) in accordance with the above result, and has a discontinuous jump at \(t = 1.583\). Several important consequences of this result will be discussed in the next section.

**B. Continuity of utilization under congestion effects**

We now show that for CPRs with decreasing rate of return functions, the total investment at the PNE is continuous in \(t\) for \(t \in [0, \hat{t}]\), i.e., the entire range of tax rates with nonzero utilization. Our analysis relies on Berge’s maximum theorem. In order to apply Berge’s maximum theorem, we need to express the total PNE investment \(x_{RE}\) as the unique maximizer of a function that is jointly continuous in the total investment and the tax rate.

First we define the following function. For a player \(i\), \(x_{T} \in [0, 1]\) and \(t \in [0, \hat{t}]\), let

\[\hat{g}_i(x_{T}, t) := \begin{cases} \alpha_i f_i(x_{T}, t) - \beta_i f_i(x_{T}, t), & x_{T} \in [0, y_i^t) \\ 0, & \text{otherwise.} \end{cases}\]  

(15)

Note that \(\hat{g}_i(x_{T}, t)\) is bounded, and therefore well-defined. In the following lemma, we prove the (joint) continuity of \(\hat{g}_i(\cdot, \cdot)\).

**Lemma 7:** The function \(\hat{g}_i(x_{T}, t), x_{T} \in [0, 1], t \in [0, \hat{t}]\) defined in (15) is jointly continuous in \(x_{T}\) and \(t\).

**Proof:** First observe that at a given \(t\), \(\hat{g}_i(x_{T}, t)\) is continuous and monotone in \(x_{T}\) for \(x_{T} \in [0, 1]\); since \(f_i(y_i^t, t) = 0\), we have \(\hat{g}_i(y_i^t, t) = 0\), and the monotonicity follows from Lemma 2. Following (19), it now suffices to show that \(\hat{g}_i(x_{T}, t)\) is continuous in \(t\) at a given \(x_{T}\).

Since \(f_i(x_{T}, t)\) is strictly decreasing in \(t\), the condition \(x_{T} \in [0, y_i^t)\) is equivalent to \(t \in [0, \hat{t}_i^{x_{T}})\), where \(\hat{t}_i^{x_{T}} := \min\{t \in [0, \hat{t}_i) : f_i(x_{T}, t) \leq 0\}\). For \(t \in [0, \hat{t}_i^{x_{T}})\), \(\hat{g}_i(x_{T}, t)\) is continuous in \(t\) as both numerator and denominator are continuous in \(t\).

\(\Box\)

**References:**

1. **Appendix C:** The function \(x_{RE} : [0, \hat{t}] \rightarrow [0, 1]\) is continuous in \(t\) for \(t \in [0, \min_{i \in N} \hat{t}_i]\).

The proof is presented in Appendix \(C\). We largely rely on Berge’s maximum theorem \([37]\), also stated formally in Appendix \(C\).
For $t \geq \tbar_i$, $\hat{g}_i(x_t, t) = 0$. Furthermore, when $\tbar_i^2 > 0$, $f_i(x_t, \tbar_i^2) = 0$. Thus, $\hat{g}_i(x_t, t)$ is continuous in $t$ at a given $x_t \in [0, 1]$.

We now show that the total PNE investment can be stated as a maximizer of a function that is continuous in both the total investment and the tax rate.

Lemma 8: Define

$$h^C(x_t, t) := -[x_t - \sum_{i \in \mathbb{N}} \hat{g}_i(x_t, t)]^2, x_t \in [0, 1], t \in [0, \tbar].$$

Then, at a given $t$, argmax$_{x_t \in [0, 1]} h^C(x_t, t)$ is single-valued, and is equal to $x^*_R$.

The proof follows from identical arguments as the proof of Lemma 11 in Appendix C and is omitted.

As before, let $x_{\mathbb{R}} : [0, t] \rightarrow [0, 1]$ such that $x_{\mathbb{R}}(t) := x^*_R$.

We are now ready to prove the following result.

Proposition 7: The function $x_{\mathbb{R}} : [0, t] \rightarrow [0, 1]$ is continuous in $t$ for $t \in [0, \tbar]$.

Proof: Consider a set-valued map or correspondence $C : [0, \tbar] \rightarrow [0, 1]$ such that $C(t) = [0, 1]$ for every $t \in [0, \tbar]$. From its definition, $C$ is compact-valued, and is both upper and lower hemi-continuous at every $t \in [0, \tbar]$. From Lemma 7, $h^C(x_t, t)$ is jointly continuous in $x_t$ and $t$. Following Berge’s maximum theorem (see Appendix C), the set-valued map argmax$_{x_t \in C(t)} h^C(x_t, t)$ is upper hemi-continuous. From Lemma 8 we have argmax$_{x_t \in C(t)} h^C(x_t, t) = \{x_{\mathbb{R}}(t)\}$, i.e., the set-valued map is in fact single-valued. Therefore, $x_{\mathbb{R}}(t)$ is continuous in $t$ for $t \in [0, \tbar]$.

We now show that $x_{\mathbb{R}}(t)$ is continuous at $t = \tbar$. From the strict monotonicity and continuity of $f(\cdot)$ and the definition of $\tbar$, we have max$_{i \in \mathbb{N}} f_i(0, 0) = 0$. Therefore, $x_{\mathbb{R}}(0) = 0$.

Now, recall from Lemma 1 and Definition 1 that $x_{\mathbb{R}}(t) <$ max$_{i \in \mathbb{N}} y_i^t$ at any tax rate $t$. Furthermore, as $t \uparrow \tbar$, we have

$$\max_{i \in \mathbb{N}} f_i(0, t) \rightarrow 0 \implies \max_{i \in \mathbb{N}} y_i^t \rightarrow 0 \implies x_{\mathbb{R}}(t) \rightarrow 0.$$

This concludes the proof.

Our results on the monotonicity and continuity of utilization have several implications which we discuss below.

VIII. MAIN RESULTS

Recall from the introduction that our goal is to characterize the range of utilizations (including the socially optimal level of utilization $x^0_{\mathbb{R}}$) that can be achieved at the PNE by a suitable choice of tax rate. We answer this question in this section.

A. CPRs with network effects

Our analysis on the monotonicity and continuity of utilization leads to the following theorem. Recall that $\tbar_i := \sup \{ t \geq 0 \mid \max_{x_t \in S_i} f_i(x_t, t) > 0 \}$.

Theorem 1: Consider a Fragile CPR game with increasing $r(x_t)$, and player-specific prospect-theoretic preferences.

- For all continuous functions $w(x_{\mathbb{R}}(t), t)$ and $\delta > 0$, there exists a $t^* \in [0, \min_{i \in \mathbb{N}} \tbar_i - \delta]$ that maximizes $w(x_{\mathbb{R}}(t), t)$.
- Let $j := \operatorname{argmin}_{i \in \mathbb{N}} \tbar_i$, and let $\bar{x}_j := \lim_{t \uparrow \tbar_j} z^j$. If $x^0_{\mathbb{R}} > \bar{x}_j$ (respectively, $x^0_{\mathbb{R}} < \bar{x}_j$), then for any given level of utilization $x^* \in (\bar{x}_j, x^0_{\mathbb{R}})$ (respectively, $x^* \in (x^0_{\mathbb{R}}, \bar{x}_j))$ there exists a tax rate $t$ such that $x^* = x^*_R$.

- If $\bar{x}_j < x^0_{\mathbb{R}}$, then there exists a tax rate $t^*$ such that $x^*_R = x^0_{\mathbb{R}}$.

Proof: The first statement follows from Proposition 6 and the extreme value theorem. For the second part, note from the definition of $\tbar_i$ that it is the smallest tax rate at which the maximum value of $f_i(x_t, \tbar_i) = 0$, $x_t \in S_i$. From the definitions of $z^j$ and $\bar{x}_j$, we have $\lim_{t \uparrow \tbar_j} z^j = \lim_{t \uparrow \tbar_j} y_i^t$.

The second statement now follows from Proposition 6 and intermediate value theorem. The third statement now follows from Proposition 2 which states that $x^0_{\mathbb{R}} \leq x^0_{\mathbb{R}}$.

Remark 3: Recall that in Example 2, $x_j = 0.5665 < 0.6829 = x^0_{\mathbb{R}}$. As shown in Figure 5a, there exists a tax rate at which PNE utilization equals $x^0_{\mathbb{R}}$ in accordance with Theorem 1. On the other hand, in Example 3, $x_j = 0.8541 > 0.7351 = x^0_{\mathbb{R}}$. While $x^0_{\mathbb{R}}$ is continuous in $t \in [0, \tbar]$ following Proposition 6, there does not exist a tax rate that achieves $x^0_{\mathbb{R}}$ as shown in Figure 5b.

We now describe several important implications of the above result for special cases of prospect-theoretic preferences.

When all players have identical $\alpha$, the player with the largest loss aversion index has the smallest $\tbar_i$. When all players have identical $\alpha$ and $t$, $\tbar_i$ is identical for every player, and thus $t = \min_{i \in \mathbb{N}} \tbar_i$. Therefore, the conclusions of the above result holds over the entire range of tax rates under which PNE utilization is non-zero. Finally, let all players have homogeneous preferences with $\alpha = 1$ and $k \geq 1$. Recall that in this case, their utilities are linear or piecewise concave, and reflects risk neutral or risk averse preferences. We have the following corollary.

Corollary 1: Let all players have $\alpha = 1$ and $k \geq 1$. Then there exists a tax rate $t^*$ such that $x^*_R = x^0_{\mathbb{R}}$.

Proof: From Lemma 3, we know that as the tax rate increases, $z^j$ decreases for every player. Furthermore, from Proposition 4 in our prior work [28], $x^0_{\mathbb{R}}$ is equal to the investment by a single player when she invests in isolation. Therefore, $\bar{x} = \lim_{t \uparrow \tbar_j} z^j < z^j < x^0_{\mathbb{R}}$. The result now follows from Theorem 1.

B. CPRs with congestion effects

The counterpart of the above result is much stronger for CPRs with congestion effects, and is stated below.

Theorem 2: Consider a Fragile CPR game with decreasing $r(x_t)$, and player-specific prospect-theoretic preferences.

- For all continuous functions $w(x_{\mathbb{R}}(t), t)$, there exists a tax rate $t^* \in [0, \tbar]$ that maximizes $w(x_{\mathbb{R}}(t), t)$.
- For any given level of utilization $x^* \in [0, x^0_{\mathbb{R}}]$, there exists a tax rate $t \in [0, \tbar]$ such that $x^* = x^*_R$. Specifically, there exists a tax rate $t^*$ such that $x^0_{\mathbb{R}} = x^*_R$. In addition, for any $x^* > x^0_{\mathbb{R}}$, there does not exist a positive tax rate that achieves $x^*_R$.

Proof: The first statement follows from Proposition 7 and the extreme value theorem. Furthermore, Proposition 3 states that $x^0_{\mathbb{R}} \leq x^0_{\mathbb{R}}$. Therefore, the second part follows from the monotonicity and continuity of utilization in the tax rate stated in Proposition 5 and Proposition 7 respectively.

In other words, any desired utilization $x^* \in [0, x^0_{\mathbb{R}}]$ can be achieved by an appropriate choice of tax rate.
Our discussion thus far assumes that the central authority imposes an identical tax rate \( t \) on every player. In the following section, we compare the utilization when the central authority imposes different tax rates on different players with the utilization under a uniform tax rate for all players.

IX. Uniform versus Differentiated Tax Rates

In order to isolate the effect of differentiated tax rates, we assume that all players have identical loss aversion indices \( k > 1 \) and \( \alpha = 1 \). Specifically, we show that the total investment at the PNE is smaller when all players are charged an identical tax rate, compared to the PNE of a game where tax rates are heterogeneous.

Without loss of generality, let the tax rate imposed on player \( i \) be \( \gamma_i t \) where \( \gamma_i \in [0, 1] \) is referred to as her tax sensitivity. Let \( S_{i,t} \subseteq [0, 1] \) be the interval such that \( r(x_T) - \gamma_i t \geq 0 \) for \( x_T \in S_{i,t} \). We also define \( \bar{f}(x_T) := r(x_T)(1 - p(x_T)) - kp(x_T) \), and \( v(x_T) := 1 + (k - 1)p(x_T) \) for \( x_T \in [0, 1] \). Following equation (4), the expected utility of player \( i \) at a strategy profile with \( x_T \in S_{i,t} \) is given by

\[
\mathbb{E}(u_i(x_i, x_{-i})) = x_i(r(x_T) - \gamma_i t)(1 - p(x_T)) - k(1 + \gamma_i t)x_i p(x_T) = x_i f_i(x_T, t) = [\bar{f}(x_T) - \gamma_i t v(x_T)].
\]  

Remark 4: Our results on PNE existence and uniqueness rely on the uniqueness, continuity and monotonicity properties of the best response. These properties remain unchanged with a linear scaling of the tax rate, and accordingly a PNE exists and is unique when the utilities are defined as in (15).

Remark 5: Note that \( \gamma_i \) can be viewed as the sensitivity of player \( i \) to the tax rate \( t \). Impacts of tax sensitivities on price of anarchy in congestion games was studied recently in [30], outside of the context of behavioral risk attitudes. Player-specific tax sensitivities can arise when players have different reference utilities. In particular, the utility in (15) arises if the reference utility of player \( i \) is \(- (1 - \gamma_i) x_i t \). In this case, player \( i \) perceives her tax payment as part of her reference utility as opposed to considering it entirely as a loss. If \( \gamma_i = 0 \), the results are the same as the case without taxation. If \( \gamma_i = 1 \), then we have the identical setting as the previous sections.

We now consider the family \( \Gamma_m \) of fragile CPR games with \( n \) players each with \( \alpha = 1 \) and \( k > 1 \), \( r(x_T) \) and \( p(x_T) \) satisfying Assumption 1 and the mean of the tax sensitivities is \( \gamma_m \in [0, 1] \). With a slight abuse of notation, we sometimes refer a player with sensitivity \( \gamma_m \) as player \( m \). Let \( t_m := \sup \{ t \geq 0 \mid \max_{x_T \in S_{m,t}} f_m(x_T, t) > 0 \} \). The following result holds for both increasing and decreasing rate of return functions. The main ideas behind the proof is analogous to the ideas used in the proof of Theorem 5 in [28].

Proposition 8: Let \( t \in [0, t_m] \). Let \( \Gamma_M \subseteq \Gamma_m \) be the game where the sensitivity parameter is \( \gamma_m \) for every player. Then, among all games in \( \Gamma_M \), CPR utilization is smallest in \( \Gamma_M \).

Proof: Let \( \Gamma_H \in \Gamma_m \) be a Fragile CPR game where the sensitivity parameters \( \{ \gamma_i, i \in \{1, 2, \ldots, n\} \} \) are player-specific. Without loss of generality, let \( 0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n \leq 1 \), with \( \sum_{i=1}^{n} \gamma_i = n \gamma_m \). Furthermore, let the utilizations at the respective PNEs of \( \Gamma_H \) and \( \Gamma_M \) be \( x_H \) and \( x_M \).

Suppose \( x_H = 0 \). Then, we have \( \bar{f}(x_T) - \gamma_i t v(x_T) \leq 0 \) for \( x_T \in S_{i,t} \). Since \( \gamma_m \geq \gamma_i \), we also have \( \bar{f}(x_T) - \gamma_m t v(x_T) \leq 0 \) for \( x_T \in S_{m,t} \), which implies \( x_M = 0 \). Since \( t < t_m \), the case \( x_H = 0 \) does not arise.

Therefore, \( x_H > 0 \). For \( j \notin \text{Supp}(\Gamma_H) \), we have

\[
\bar{f}(x_T) - \gamma_j t v(x_T) \leq 0 \implies \bar{f}(x_T) - \gamma_j t v(x_T) \leq 0,
\]

for every \( \gamma \geq \gamma_j \). Therefore, \( \text{Supp}(\Gamma_H) \) consists of a set of players with smallest sensitivity parameters. Since \( x_H > 0 \), player \( 1 \in \text{Supp}(\Gamma_H) \). From equation (10) for \( \Gamma_H \), we have

\[
x_H = \sum_{i \in \text{Supp}(\Gamma_H)} g_i(x_H, t) = \sum_{i=1}^{n} \max(g_i(x_H, t), 0) = \sum_{i=1}^{n} \max(h_{x_H,t}(\gamma_i), 0),
\]

where \( h_{x_H,t}(\cdot) \) is a function of the tax sensitivity \( \gamma \) at a given total investment \( x_H \) and tax rate \( t \). Note that, since the players are loss averse, we have \( v'(x_H) = (k - 1)p'(x_H) > 0 \). As a result, for \( \gamma \geq \gamma_j \), the numerator of \( h_{x_H,t}(\gamma) \) is strictly decreasing in \( \gamma \), while the denominator is strictly increasing in \( \gamma \). We now define an interval \( J \subseteq [\gamma_1, 1] \) as follows.

If \( x_{H,t}(\gamma) = 0 \), then \( J = [\gamma_1, 1] \). Otherwise, \( J = [\gamma_1, \gamma_j] \). Then, \( \bar{f}(x_H) - \gamma_j t v(x_H) > 0 \) and \( -\bar{f}(x_H) + \gamma_j t v(x_H) > 0 \), which implies

\[
\bar{f}(x_H)v'(x_H) > \gamma_j t v(x_H)v'(x_H) > \bar{f}(x_H)v(x_H).
\]

For \( \gamma \in J \), straightforward calculations yield

\[
h_{x_H,t}^H(\gamma) = \frac{\bar{f}(x_H)v'(x_H) - \bar{f}(x_H)v'(x_H)}{\gamma_j t v(x_H)} < 0,
\]

\[
h_{x_H,t}^H(\gamma) = \frac{-2v'(x_H)v'(x_H) - \bar{f}(x_H)v'(x_H) - \bar{f}(x_H)v'(x_H)}{\gamma_j t v(x_H)}.
\]

Following (17), we have \( h_{x_H,t}^H(\gamma) > 0 \) for \( \gamma \in J \). Therefore, \( \max(h_{x_H,t}(\gamma), 0) \) is continuous and convex for \( \gamma \in [\gamma_1, 1] \), applying Jensen’s inequality, we obtain

\[
x_H = \sum_{i=1}^{n} \max(h_{x_H,t}(\gamma_i), 0) \geq n \max(h_{x_H,t}(\gamma_m), 0)
\]

We now consider two cases. First, suppose \( h_{x_H,t}(\gamma_m) \leq 0 \). Note that \( -\bar{f}(x_H) + \gamma_j t v'(x_H) > 0 \) (since \( \gamma_m \geq \gamma_i \) and \( v'(x_H) > 0 \)). Thus, we have \( \bar{f}(x_H) - \gamma_m t v(x_H) \leq 0 \). When \( r(x_T) \) is decreasing, it is easy to see that \( \bar{f}(x_T) - \gamma_j t v(x_T) < 0 \) for \( x_T \in (x_H, 1] \). For an increasing and concave \( r(x_T) \), \( \bar{f}(x_T) - \gamma_j t v(x_T) \) is strictly concave in \( x_T \). Since \( \bar{f}(x_H) - \gamma_m t v'(x_H) < 0 \) and \( \bar{f}(x_H) - \gamma_m t v'(x_H) \leq 0 \), we have \( \bar{f}(x_T) - \gamma_m t v(x_T) \leq 0 \) for \( x_T \in (x_H, 1] \). Thus, \( x_M \leq x_H \).

Now suppose \( h_{x_H,t}(\gamma_m) > 0 \), i.e., \( \bar{f}(x_H) - \gamma_j t v(x_H) > 0 \) and \( \bar{f}(x_H) - \gamma_m t v'(x_H) > 0 \). Assume on the contrary that \( x_M > x_H \). Thus, we have \( \{x_H, x_M\} \subseteq \mathbb{I}_m \), where \( \mathbb{I}_m \) is the interval defined in Lemma 1 for a player \( m \) with tax sensitivity \( \gamma_m \) at tax rate \( t \). Following Lemma 2, we obtain

\[
x_H \geq n h_{x_H,t}(\gamma_m) = n g_m(x_H, t) > n g_m(x_M, t) = x_M,
\]
which is a contradiction. Therefore, $x_{H} \geq x_{M}$.

The above result shows that when players are risk averse with $\alpha = 1$, charging different tax rates to different players leads to higher utilization and fragility of the CPR. Since players with identical loss aversion indices make identical investments in the CPR, our result suggests that a central authority should impose similar tax rates on players who make similar investments at the equilibria without taxation.

The analysis is significantly more involved when $\alpha < 1$. On the other hand, since the utilities are continuous in $\alpha$, we should expect the above result to hold when $\alpha$ is close to 1. We conclude with the following remark.

Remark 6: In addition to utilization, another important quantity of interest to the central authority is the revenue, defined as $\sum_{i=1}^{n} \gamma_{i} x_{i, H}$. Straightforward calculations show that the function $\gamma x_{H, i, \gamma}(\gamma)$ defined in the above proof is concave in $\gamma$. Therefore, if the support of both games $\Gamma_{H}$ and $\Gamma_{M}$ contain all $n$ players, then it can be shown that the revenue is higher in $\Gamma_{M}$, even though utilization is lower.

X. CONCLUSION

We investigated the effectiveness of a taxation mechanism in controlling the utilization of a failure-prone shared resource under prospect-theoretic risk preferences of users. We first showed the existence and uniqueness of PNE in Fragile CPR games under taxation. We then showed that for resources that exhibit network effects, heterogeneous prospect-theoretic utilities of the players can lead to increase in utilization and fragility with higher tax rates, and the utilization at the Nash equilibrium can be discontinuous in the tax rate. In contrast, for resources with a decreasing rate of return or congestion effects, utilization is always decreasing and continuous in the tax rate. Building upon these insights, we identified the range of utilization that can be achieved by a suitable choice of tax rate for both classes of resources. Finally, we showed that for loss averse players, imposing differentiated tax rates results in higher utilization compared to the case where all players are charged an identical tax rate.

Our results highlight the nuances of controlling human behavior under uncertainty, and provide compelling insights on how to identify and control their utilization of shared systems via economic incentives. Future work will focus on characterizing revenue maximizing taxation schemes, and designing mechanisms for human users in dynamic environments.

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for every player $j$ and let $j \neq 0$

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and a given tax rate $t$

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\[ \text{APPENDIX A}

\[ \text{SOCIAL WELFARE}

Following the convention in the game theory literature \[25\],

we define the social welfare at a strategy profile $x \in [0, 1]^n$

and a given tax rate $t \in [0, \ell]$ as

\[ \Psi(x) = \sum_{i \in N} E u_i(x_i, x_{-i}) = \sum_{i \in N} x_i^{\alpha_i} f_i(x_T, t), \]

(18)

where $u_i$ is defined in \[2\]. Due to the continuity of $\Psi$, there always exists a social welfare maximizing set of investments.

We now prove Proposition \[2\].

\[ \text{Proof of Proposition } 2 \]

Recall from Assumption \[1\] that $\ell > 0$. Therefore, at $t = 0$, there exists a player $k$ with $\max_{x_T \in [0, 1]} f_k(x_T, 0) > 0$. As a result, $\Psi(x_{OPT}) > 0$. In the rest of the proof, we omit the superscript 0 and the second argument from $f$ for better readability.

Now, assume on the contrary that $x_{OPT} > x_{HE}$. Then there exists a player $i$ with respective CPR investments satisfying $x_{1,OPT} > x_{i,HE} > 0$.

First we claim that $f_i(x_{OPT}) > 0$. Suppose otherwise, and let $j$ be a different player with $f_j(x_{OPT}) > 0$\[6\]. Let $\epsilon \in [0, x_{OPT})$, and consider a different strategy profile $\hat{x}_{OPT} = (x_{1,OPT} - \epsilon, \ldots, x_{j,OPT} + \epsilon, \ldots, x_{n,OPT})$ with total utilization $x_{OPT}$. Then

\[ (x_{OPT})^{\alpha_i} f_i(x_{OPT}) + (x_{OPT})^{\alpha_j} f_j(x_{OPT}) < (x_{OPT} - \epsilon)^{\alpha_i} f_i(x_{OPT}) + (x_{OPT} + \epsilon)^{\alpha_j} f_j(x_{OPT}) \]

$$
\Rightarrow \Psi(\hat{x}_{OPT}) < \Psi(x_{OPT}),$$

\[\text{Note that such a player always exists; otherwise we have } f_j(x_{OPT}) \leq 0 \]

for every player $j$, which implies $\Psi(x_{OPT}) \leq 0$.

since $f_i(x_{OPT}) < 0$ and $f_j(x_{OPT}) > 0$. This contradicts the optimality of $x_{OPT}$.

Since $x_{OPT} > x_{HE}$ and $f_i(x_{OPT}) > 0$, $x_{OPT} \in \mathcal{I}$, where $\mathcal{I}$ is the interval defined in Lemma \[1\]. From the first order optimality condition for player $i$ at the PNE \[5\], we obtain

\[ x_{i,HE} f_i(x_{HE}) + \alpha_i f_i(x_{OPT}) = 0 \]

$$
\implies x_{i,OPT} > x_{i,HE} = \frac{\alpha_i f_i(x_{HE})}{f_i(x_{OPT})} > 0.
$$

(20)

where $x_{i,OPT}(x_T) = \frac{\partial f_i}{\partial x_T}(x_T)$, and the second inequality in the second line follows from Lemma \[2\].

We now show that for every player $j$ other than $i$, $x_{j,OPT}^f(x_{OPT}) \leq 0$. For decreasing rate of return functions, this is true since $f(\cdot)$ is strictly decreasing in the total investment. On the other hand, for increasing rate of return functions, we have the following two cases:

\[ \text{Case 1: } \max_{x_T \in [0, 1]} f_j(x_T) > 0. \]

Following the discussion in Lemma \[1\] we have $x_{HE} > x_{j,HE}$ in this case. Therefore, $f_j(x_{HE}) < 0$. Furthermore, $f(\cdot)$ is concave (following Lemma \[1\]), and $x_{OPT} > x_{HE}$, which implies $f_j(x_{OPT}) < 0$.

\[ \text{Case 2: } \max_{x_T \in [0, 1]} f_j(x_T) \leq 0. \]

Following identical arguments as the second paragraph of the proof, we have $x_{j,OPT} = 0$ in this case.

We are now ready to complete the proof. From the first order optimality condition for the social optimum, we obtain

\[ 0 = \frac{\partial \Psi}{\partial x_i}|_{x=x_{OPT}} = \alpha_i x_{i,OPT}^{\alpha_i} f_i(x_{OPT}) + \sum_{j=1,j\neq i}^n x_{j,OPT}^{\alpha_j} f_j(x_{OPT}) < 0, \]

following the above discussion. This contradicts our initial claim, and we must have $x_{OPT} \leq x_{HE}$.

\[\text{\textbf{APPENDIX B}}

\[\text{PROOFS PERTAINING TO MONOTONICITY OF UTILIZATION}

\[\text{Proof of Lemma } 5 \]

When it is clear from the context, we omit the arguments $x_T$, $t$ and $i$ in the following analysis for better readability. We now state the effective rate of return function under taxation, and compute its derivatives with respect to $x_T$ and $t$. Let $t \in [0, \ell]$ and $x_T \in (a_i, 1]$. Recall that $r(\cdot)$ is strictly increasing and concave, and $p(\cdot)$ is strictly increasing and convex. From \[4\], we have

\[ f(x_T, t) = (r - t)^{a_i}(1 - p) - k(1 + t)^{b_i} \]

$$
\Rightarrow f_x(x_T, t) = \frac{\partial f}{\partial x_T}(x_T, t) = (r - t)^{a_i-1} r' (1-p) - (r - t)^{a_i} p' k (1 + t)^{b_i-1}.
$$

(20)

Differentiating $f(x_T, t)$ with respect to $t$ for $t \in \{ t \geq 0 \mid x_T \in (a_i, 1] \}$, we obtain

\[ f_t(x_T, t) = \frac{\partial f}{\partial t}(x_T, t) = - \alpha (r - t)^{a_i-1} (1-p) - \alpha k (1 + t)^{a_i-1} p', \quad \text{and} \]

(21)

\[ f_{x,t}(x_T, t) = \frac{\partial^2 f}{\partial x_T \partial t}(x_T, t) = - \alpha (a - 1)(r - t)^{a_i-2} r' (1-p) + \alpha (r - t)^{a_i-1} p - \alpha k (1 + t)^{a_i-1} p'. \]

(22)
Therefore, a sufficient condition for \( l = (x, t)^{\alpha-1} \), \( \tilde{l}_t := \alpha(1 + t)^{\alpha-1} \), and \( \tilde{k}_t := \alpha(1 + t)^{\alpha-1} \). From (23), we obtain
\[
\frac{f f_{x,t} - f_f_x}{f_x} = \alpha(r - t)^{2\alpha - 2} r'(1 - p)^2 - \alpha(r - t)^{\alpha-1} (1 - p)^{p'}
\]
Similarly,
\[
f_{x,t} - f_f_t = - \alpha^2 (r - t)^{2\alpha - 2} r'(1 - p)^2 + \alpha(r - t)^{\alpha-1} (1 - p)^{p'} - \alpha^2 k(r - t)^{\alpha-1} r'(1 + t)^{\alpha-1} p(1 - p) + \alpha k^2 (1 + t)^{2\alpha - 1} p'.
\]
From the above analysis, we obtain
\[
ff_{x,t} - f_{x,t} = \alpha(r - t)^{2\alpha - 2} r'(1 - p)^2 - \alpha k(r - t)^{\alpha-1} (1 - p)^{p'} + \alpha^2 k(r - t)^{\alpha-1} r'(1 + t)^{\alpha-1} p(1 - p) + \alpha k^2 (1 + t)^{2\alpha - 1} p'.
\]
Since \( \alpha < 1 \), and \( r' > 0 \), the first term in (24) is negative. Therefore, a sufficient condition for \( ff_{x,t} - f_{x,t} < 0 \) is
\[
k(1 + t)^{\alpha-1} p'(r + 1)
\]
\[
\implies k(1 + t)^{\alpha-1} p'(r + 1) > \alpha k(1 + t)^{\alpha-1} p(1 - p)
\]
\[
\implies k > \frac{r'(1 - p)^2}{(r + 1)^{p'} - \alpha r'(1 - p)^{p'}} \left( \frac{1 + t}{r - t} \right) = q(x, t, t).
\]
Now let \( x, t \in T \). Therefore, \( f_{x,t} < 0 \), and accordingly we have
\[
0 < f p' - f_{x,t} = (r - t)^{\alpha-1} p' - \alpha (r - t)^{\alpha-1} r'(1 - p)^2 p + (r - t)^{\alpha-1} p'(1 - p)
\]
\[
\implies (r + 1)p' > \alpha r'(1 - p)^2.
\]
It remains to show that at a given \( t \), \( q(x, t, t) \) is monotonically decreasing in \( x_T \) for \( x_T \in \{a_t, 1\} \). Specifically, let \( L_1 := r''(1 - p)^2 \) and \( L_2 := (r + 1)^{p'} - \alpha r'(1 - p)^2 p > 0 \). Then
\[
L_1 = r''(1 - p)^2 - 2 r'(1 - p)^{p'} < 0, \quad \text{and} \quad L_2 = (r + 1)^{p'} - \alpha r'(1 - p)^2 p - \alpha r'(1 - p)^2 p + \alpha r'(1 - p)^2 p > 0.
\]
Therefore, the numerator of \( q(x_T, t) \) is decreasing in \( x_T \), and the denominator is positive and increasing in \( x_T \).}

**Proof of Lemma 6** Let \( \tilde{k} := k(1 + t)^{\alpha} \), and \( \tilde{k}_t := k(1 + t)^{\alpha-1} \). From (23), we obtain
\[
\frac{f f_{x,t} - f_f_x}{f_x} = (r - t)^{2\alpha - 2} r'(1 - p)^2 - \alpha^2 k(r - t)^{\alpha-1} r'(1 + t)^{\alpha-1} p(1 - p) + \alpha k^2 (1 + t)^{2\alpha - 1} p'.
\]
\( \alpha(r - t)^{2\alpha - 2} r'(1 - p)^2 - \alpha(1 - \alpha)(r - t)^{\alpha-2} r' k_p(1 - p) + \alpha(r - t)^{\alpha-1} r' k_t(1 - p) - \alpha(r - t)^{\alpha-1} k_p + (r - t)k_t p - \alpha(r - t)^{\alpha-1} k_p - (r - t)^{\alpha-1} k_t p' \)
\( \alpha(1 - p)^{r'(1 - t)^{\alpha-2}(f + \alpha k_p + (r - t)k_t p') - \alpha(r - t)^{\alpha-1} k_p(1 - (r - t)^{\alpha-1} k_t p') < 0, \)
when \( r' > 0 \) and \( f > 0 \). This concludes the proof. \( \blacksquare \)

**APPENDIX C**

**PROOFS PERTAINING TO CONTINUITY OF UTILIZATION**

**A. Preliminaries**

As stated earlier, our proofs largely rely on Berge’s maximum theorem which we state below.

**Theorem 3 (from [37]):** Let \( \Theta \) and \( X \) be two metric spaces, and let \( C : \Theta \rightrightarrows X \) be a compact-valued correspondence. Let the function \( \Phi : X \times \Theta \rightrightarrows \mathbb{R} \) be jointly continuous in both \( X \) and \( \Theta \). Define
\[
\sigma(\theta) := \arg\max_{x \in C(\theta)} \Phi(x, \theta), \quad \phi \in C(\theta), \forall \theta \in \Theta.
\]
If \( C \) is continuous at \( \theta \in \Theta \), then
1) \( \sigma : \Theta \rightrightarrows \mathbb{R} \) is compact-valued, upper hemicontinuous and closed at \( \theta \).
2) \( \phi : \Theta \rightrightarrows \mathbb{R} \) is continuous at \( \theta \).

In many instances, the correspondence \( C \) takes the form of a parametrized constraint set, i.e., \( C(\theta) = \{x \in X | l_j(x, \theta) \leq 0, j \in \{1, 2, \ldots, m\} \} \). For this class of constraints, we have the following sufficient conditions for the upper and lower hemi-continuity of \( C \) [40, Theorem 10,12].

**Theorem 4:** Let \( C : \Theta \rightrightarrows X \subseteq \mathbb{R}^k \) be given by \( C(\theta) = \{x \in X | l_j(x, \theta) \leq 0, j \in \{1, 2, \ldots, m\} \} \).
1) Let \( X \) be closed, and all \( l_j \)'s be continuous on \( X \). Then, \( C \) is upper hemicontinuous on \( \Theta \).
2) Let \( l_j \)'s be continuous and convex in \( x \) for each \( \theta \). If there exists \( (x, \theta) \) such that \( l_j(x, \theta) < 0 \) for all \( j \), then \( C \) is lower hemicontinuous at \( \theta \), and in some neighborhood of \( \theta \).

**Remark 7:** Some authors use the term semi-continuity or open/closed correspondences instead of hemi-continuity [40]. The definitions coincide for closed and compact-valued correspondences, which is the case in this paper.

**B. Proofs pertaining to continuity under network effects**

We first introduce certain notation, and prove some preliminary lemmas. In appropriate places in this section, we treat \( y_{i_t}^t, y_{z_t}^t, x_{z_t}^t \) as functions of \( t \) (from \( 0, i \to 0, 1 \)), with a slight abuse of notation. Furthermore, we denote the utilization \( x_T \) as \( x \), and \( \frac{\partial}{\partial x} f_{i,x}(x, t) \).
Recall from \(6\) and \(7\) that \(S_t := [0, b_t]\) when \(r(x)\) is strictly decreasing, and \(S_t := [a_t, 1]\) when \(r(x)\) is strictly increasing. Furthermore, recall that \(t_i := \sup\{t \geq 0\} \max_{x \in S_t} f_i(x, t) > 0\). For \(t < t_i\), \(z_i^t := \arg \max_{x \in S_t} f_i(x, t)\), and \(y_i^t \in (z_i^t, 1)\) such that \(f_i(y_i^t, t) = 0\). In addition, \(f_i(x, t)\) is positive and decreasing for \(x \in (z_i^t, y_i^t)\). We now define

\[
\hat{z}_i^t := \arg \max_{x \in (z_i^t, y_i^t)} \left[ \alpha_i f_i(x, t) + f_{i,x}(x, t) \right]^2.
\]

(25)

Note that at a given \(t < t_i\), \(f_i(x, t)\) is concave, and therefore, \(\alpha_i f_i(x, t) + f_{i,x}(x, t)\) is strictly decreasing for \(x \in [z_i^t, y_i^t]\).

Thus, \(\hat{z}_i^t = z_i^t\) when \(z_i^t = 0\) and \(f_{i,x}(0, t) < 0\). Otherwise, \(\alpha_i f_i(\hat{z}_i^t, t) + f_{i,x}(\hat{z}_i^t, t) = 0\). With the above quantities, we are now ready to define the following function. For a player \(i\), \(x \in [0, 1]\) and \(t \in [0, t_i]\), let

\[
\hat{g}_i^N(x, t) := \begin{cases} 
1, & x \in (0, \hat{z}_i^t), \\
\frac{\alpha_i f_i(x, t)}{f_{i,x}(x, t)}, & x \in (\hat{z}_i^t, y_i^t), \\
0, & \text{otherwise}.
\end{cases}
\]

(26)

Note that \(\hat{g}_i^N(x, t)\) is well-defined. It follows from \(23\) that when \(\hat{z}_i^t > 0\), the maximum value of \(\frac{\alpha_i f_i(x, t)}{f_{i,x}(x, t)}\) occurs at \(x = \hat{z}_i^t\). As a result, \(\hat{g}_i^N(x, t)\) is bounded. In Figure 4 we illustrate the shape of the function \(\hat{g}_i^N(x, t)\), and how it compares with \(g_i(x, t)\) defined in \(9\) for the CPR with the same characteristics as Example 4. Note that the denominator of \(g_i(x, t)\) is 0 at \(x = \hat{z}_i^t\), while \(\hat{g}_i^N(x, t)\) is bounded for \(x \in [0, 1]\) as it is defined in terms of \(z_i^t\).

We first establish the continuity of \(z_i^t, \hat{z}_i^t\), and \(y_i^t\), and then prove the (joint) continuity of \(\hat{g}_i^N(\cdot, \cdot)\).

**Lemma 9:** When viewed as functions of \(t\), \(z_i^t, \hat{z}_i^t, \) and \(y_i^t\) are continuous in \(t\) for \(t \in [0, t_i]\).

**Proof:** We start with a slight abuse of notation, we view the set \(S_t\) as a correspondence \(S : [0, t_i] \rightarrow [0, 1]\) with \(S(t) := \{x \in [0, 1] | r(x) \leq t\}\). Recall that \(z_i^t := \arg \max_{x \in S(t)} f_i(x, t); \hat{z}_i^t\) is single-valued since \(f_i(x, t)\) is strictly concave in \(x \in S(t)\) at a given \(t\). Since \(r(x)\) is continuous, concave, and for every \(t \in t_i\), \(t - r(t) < 0\), it follows from Theorem 4 that \(S(t)\) is both upper and lower hemiconcave. Note that \(f_i(x, t)\) is defined for \(x \in [0, 1], t \in [0, t_i]\), and is jointly continuous in \(x\) and \(t\). Therefore, following Berge’s maximum theorem, \(z_i^t\) is continuous in \(t\).

Recall that \(y_i^t \in (z_i^t, 1]\) such that \(f_i(y_i^t, t) = 0\). Furthermore, \(f_i(x, t)\) is strictly decreasing in \(x\) for \(x \in (z_i^t, 1]\). Therefore, we can alternatively let \(y_i^t := \arg \max_{x \in (z_i^t, 1]} (-f_i(x, t))^2\). Since \(z_i^t\) is continuous in \(t\), the correspondence \(t \mapsto (z_i^t, 1]\) is continuous following Theorem 4 Berge’s maximum theorem now implies that \(y_i^t\) is continuous in \(t\).

Following the above discussion, we have \(t \mapsto (z_i^t, 1]\) as continuous. From its definition \(25\), \(z_i^t\) is the unique maximizer of a function that is jointly continuous in both \(x\) and \(t\). Once again, from Berge’s maximum theorem, we conclude that \(z_i^t\) is continuous in \(t\).

**Lemma 10:** The function \(\hat{g}_i(x, t), x \in [0, 1], t \in [0, t_i]\) defined in \(15\) is jointly continuous in \(x\) and \(t\).

**Proof:** First observe that at a given \(t\), \(\hat{g}_i(x, t)\) is continuous and monotone in \(x\) for \(x \in [0, 1]\). In particular, \(\hat{g}_i(z_i^t, t) = f_i(z_i^t, t) = 0\), and the monotonicity follows from Lemma 2. Following \(39\), it now suffices to show that \(\hat{g}_i(x, t)\) is continuous in \(t\) at a given \(x\). However, this is true because \(z_i^t\) and \(y_i^t\) are continuous in \(t\) following Lemma 9 together with \(\frac{\alpha_i f_i(x, t)}{f_{i,x}(x, t)} = 1\), and \(\frac{\alpha_i f_i(x, t)}{f_{i,x}(x, t)} = 0\).

We now show that the total PNE investment can be stated as a maximizer of a function that is continuous in both the total investment and the tax rate.

**Lemma 11:** For \(x \in [0, 1], t \in [0, \min_{i \in N} \tilde{t}_i]\), define

\[
h_N(x, t) := -[x - \sum_{i \in N} \hat{g}_i^N(x, t)]^2.
\]

Then, at a given \(t\), \(\arg \max_{x \in [0, 1]} h_N(x, t)\) is single-valued, and is equal to \(x_{\text{big}}\).

**Proof:** From its definition, \(h_N(x, t) \leq 0\). Suppose there exists \(x^t \in [0, 1]\) such that \(h_N(x^t, t) = 0\), or equivalently \(x^t = \sum_{i \in N} \hat{g}_i^N(x^t, t)\). First we claim that \(x^t > z_i^t\) for every player \(j\). If this is not the case, then for a player \(j\) with \(x^t \leq z_i^t\), we have \(\hat{g}_i^N(x^t, t) = 1\) which implies \(x^t < \sum_{i \in N} \hat{g}_i^N(x^t, t)\).

Now consider the strategy profile \(\{x^t_j\}_{j \in N}\) where \(x^t_j = \hat{g}_i^N(x^t, t)\) for each player \(j\). Consider a player \(j\) with \(x^t_j \geq y_i^t\). Then, \(x^t_j = \hat{g}_i^N(x^t, t) = 0\). Following Lemma 4 the strategy of player \(j\), \(x^t_j\), is her best response. Now suppose \(x^t < y_i^t\). From the definition of \(\hat{g}_i^N(\cdot, \cdot)\), we have \(x^t \cdot f_{i,x}(x^t, t) + \alpha_i f_i(x^t, t) = 0\). Following \(8\), the investment of player \(j\) satisfies the first order optimality condition for her utility. Furthermore, the proof of Lemma 2 in \(28\) showed that the utility of player \(j\) is strictly concave in the range of investments which contains the investment at which the first order optimality condition is satisfied. Therefore, \(x^t_j\) is the unique best response of player \(j\) for the given strategies of others. Thus, \(\{x^t_j\}_{j \in N}\) corresponds to a PNE strategy profile.

Recall that a PNE exists, and is unique. Following Theorem 1 in \(23\), the total investment at the PNE is unique as well. Therefore, there is a unique \(x = x_{\text{PNE}}^t\) with \(h_N(x, t) = 0\), which also maximizes \(h_N(x, t)\) at a given \(t\).

The proof of Proposition 6 now follows from Berge’s maximum theorem as shown below.

**Proof of Proposition 6**
hemicontinuous at every $t \in [0, \min_{i \in \mathcal{N}} \tilde{t}_i)$. From Lemma 10, $h^N(x, t)$ is jointly continuous in $x$ and $t$. Following Berge’s maximum theorem, $\arg\max_{x \in C(t)} h^N(x, t)$ is upper hemicontinuous. From Lemma 11, $h^N(x, t)$ is single-valued. Therefore, $x_{\mathcal{N}}(t)$ is continuous in $t$ for $t \in [0, \min_{i \in \mathcal{N}} \tilde{t}_i)$. ■