Existence and uniqueness of solutions for coupled system of fractional differential equations involving proportional delay by means of topological degree theory

Anwar Ali, Muhammad Sarwar, Mian Bahadur Zada and Thabet Abdeljawad

Abstract

In this manuscript, we obtain sufficient conditions required for the existence of solution to the following coupled system of nonlinear fractional order differential equations:

\[ D^\gamma \omega(\ell) = \mathcal{F}(\ell, \omega(\lambda \ell), \upsilon(\lambda \ell)), \]
\[ D^\delta \upsilon(\ell) = \mathcal{F}(\ell, \omega(\lambda \ell), \upsilon(\lambda \ell)), \]

with fractional integral boundary conditions

\[ a_1 \omega(0) - b_1 \omega(\eta) - c_1 \omega(1) = \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma - 1} \phi(\rho, \omega(\rho)) \, d\rho \quad \text{and} \]
\[ a_2 \upsilon(0) - b_2 \upsilon(\xi) - c_2 \upsilon(1) = \frac{1}{\Gamma(\delta)} \int_0^1 (1 - \rho)^{\delta - 1} \psi(\rho, \upsilon(\rho)) \, d\rho, \]

where \( \ell \in \mathbb{Z} = [0, 1], \gamma, \delta \in (0, 1], 0 < \lambda < 1, D \) denotes the Caputo fractional derivative (in short CFD), \( \mathcal{F}, \mathcal{F} : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( \phi, \psi : \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \) are continuous functions. The parameters \( \eta, \xi \) are such that \( 0 < \eta, \xi < 1, \) and \( a_i, b_i, c_i \) (\( i = 1, 2 \)) are real numbers with \( a_i \neq b_i + c_i \) (\( i = 1, 2 \)). Using topological degree theory, sufficient results are constructed for the existence of at least one and unique solution to the concerned problem. For the validity of our result, an appropriate example is presented in the end.

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1 Introduction

It has been proved that fractional differential equations (in short FDEs) are a powerful tool for modeling various phenomena of physical and chemical as well as biological sciences.
Besides, it has also been proved that FDEs have numerous applications in various scientific and engineering disciplines such as chemistry, physics, biology, and optimization theory [1–5].

Many mathematicians give much attention to the existence theory of FDEs with multipoint boundary conditions, and there is rapidly growing area of research due to its wide range of applications in real world problems [6–10]. For the existence and uniqueness of solutions of FDEs, different methods are used like topological degree theory and fixed point theory. Here we use topological degree theory. After studying the present literature, we noticed that FDEs having fractional integral type boundary conditions are not well examined through topological degree theory. Very few articles used topological degree theory for simple initial and boundary value problems (BVPs) having CFD [11–15]. If viewed carefully, the existence of solutions to FDEs having integral boundary conditions has a wide range of applications in optimization theory, viscoelasticity, fluid mechanics, and quantitative theory which have been studied by many researchers [16–21]. Keeping in mind the applications of topological degree theory, Ali et al. [22] studied the existence of solutions to the following FDE:

\[
\begin{align*}
\frac{cD^\gamma}{2} \omega(\ell) &= F(\ell, \omega(\ell)), \\
\quad &\quad 1 < \gamma \leq 2, \ell \in \mathbb{Z}, \\
a_1 \omega(0) + b_1 \omega(1) &= F_1(\omega), \\
a_2 \omega'(0) + b_2 \omega'(1) &= F_2(\omega),
\end{align*}
\]

where \( F_1, F_2 : C(\mathbb{Z}, \mathbb{R}) \to \mathbb{R} \) and \( F : \mathbb{Z} \to \mathbb{R} \) are continuous functions and \( a_i, b_i \) are real numbers with \( a_i + b_i \neq 0, i = 1, 2 \). Using fixed point theory, Cabada et al. [23] discussed the following problem:

\[
\begin{align*}
\frac{cD^\gamma}{2} \omega(\ell) + F(\ell, \omega(\ell)) &= 0, \quad \ell \in (0, 1), \\
\omega(0) + \omega''(0) &= 0, \\
\quad &\quad \omega(1) = a \int_0^1 \omega(\rho) d\rho,
\end{align*}
\]

where \( 2 < \gamma < 3, 0 < a < 2, D \) is the CFD and \( F : \mathbb{Z} \times [0, \infty) \to [0, \infty) \).

Motivated by [22] and [23], we examine the results for the existence of solution to the following nonlinear coupled system of FDEs through topological degree theory:

\[
\begin{align*}
D^\gamma \omega(\ell) &= F(\ell, \omega(\ell), \nu(\ell)), \\
D^\delta \nu(\ell) &= F(\ell, \omega(\ell), \nu(\ell)), \\
a_1 \omega(0) - b_1 \omega(\eta) - c_1 \omega(1) &= \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma-1} \phi(\rho, \omega(\rho)) d\rho, \\
a_2 \nu(0) - b_2 \nu(\xi) - c_2 \nu(1) &= \frac{1}{\Gamma(\delta)} \int_0^1 (1 - \rho)^{\delta-1} \psi(\rho, \nu(\rho)) d\rho,
\end{align*}
\]

where \( \ell \in \mathbb{Z}, \gamma, \delta \in (0, 1], 0 < \lambda < 1, D \) denotes the CFD. Further \( F, F : \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \) and \( \phi, \psi : \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \) are continuous functions. The parameters \( \eta, \xi \) are such that \( 0 < \eta, \xi < 1 \) and \( a_i, b_i, c_i (i = 1, 2) \) are real numbers with \( a_i \neq b_i + c_i \).

### 2 Preliminaries

In this section we recollect some facts, definitions, and results. Throughout this work \( U = C(\mathbb{Z}, \mathbb{R}), V = C(\mathbb{Z}, \mathbb{R}) \) represent the Banach spaces for all continuous function defined
on $\mathbb{R}$ into $\mathbb{R}$ with norm $\|\omega\| = \sup\{|\omega(\ell)| : 0 \leq \ell \leq 1\}$. The product space $\mathcal{U} \times \mathcal{V}$ is a Banach space with norm $\|(\omega, \upsilon)\| = \|\omega\| + \|\upsilon\|$.

**Definition 2.1** ([24]) Let $\mathcal{H} : V \to \mathcal{U}$ be a continuous bounded map, where $V \subseteq \mathcal{U}$. Then $\mathcal{H}$ is

1. $\sigma$-Lipschitz if there exists $h \geq 0$ such that $\sigma(\mathcal{H}(S)) \leq h\sigma(S)$ for all bounded subsets $S \subseteq V$;
2. strict $\sigma$-contraction if there exists $0 \leq h < 1$ with $\sigma(\mathcal{H}(S)) \leq h\sigma(S)$ for all bounded subsets $S \subseteq V$;
3. $\sigma$-condensing if $\sigma(\mathcal{H}(S)) < \sigma(S)$ for all bounded subsets $S \subseteq V$ having $\sigma(S) > 0$. In other words, $\sigma(\mathcal{H}(S)) \geq \sigma(S)$ implies $\sigma(S) = 0$.

Moreover, $\mathcal{H} : V \to \mathcal{U}$ is Lipschitz whenever there is $h > 0$ such that

$$\|\mathcal{H}(\omega) - \mathcal{H}(\upsilon)\| \leq h|\omega - \upsilon| \quad \text{for all } \omega, \upsilon \in V.$$

Further $\mathcal{H}$ will be a strict contraction if $h < 1$.

**Proposition 2.1** ([25]) If $\mathcal{H}, G : V \to \mathcal{U}$ are $\sigma$-Lipschitz with constants $h_1$ and $h_2$ respectively, then $\mathcal{H} + G$ is $\sigma$-Lipschitz with constant $h_1 + h_2$.

**Proposition 2.2** ([25]) If $\mathcal{H} : V \to \mathcal{U}$ is Lipschitz with constant $h$, then $\mathcal{H}$ is $\sigma$-Lipschitz with the same constant $h$.

**Proposition 2.3** ([25]) If $\mathcal{H} : V \to \mathcal{U}$ is compact, then $\mathcal{H}$ is $\sigma$-Lipschitz with constant $h = 0$.

**Theorem 2.1** ([25]) Let $\mathcal{H} : \mathcal{U} \to \mathcal{U}$ be $\sigma$-condensing such that

$$\Lambda = \{\omega \in \mathcal{U} : \text{there exists } 0 \leq \vartheta \leq 1 \text{ such that } \omega = \vartheta \mathcal{H}(\omega)\}.$$

If $\Lambda$ is bounded in $\mathcal{U}$, so there exists $r > 0$ such that $\Lambda \subseteq S_r(0)$, then the degree

$$\mathcal{D}(I - \vartheta \mathcal{H}, S_r(0), 0) = 1 \quad \text{for all } \vartheta \in [0, 1].$$

Consequently, $\mathcal{H}$ has at least one fixed point which lies in $S_r(0)$.

**Definition 2.2** ([26]) The fractional order integral of a function $F : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$I^\gamma F(\ell) = \frac{1}{\Gamma(\gamma)} \int_0^\ell (\ell - \rho)^{\gamma-1} F(\rho) \, d\rho. \quad (2.1)$$

**Definition 2.3** ([26]) The CFD of order $\gamma > 0$ of a function $F : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$D^\gamma F(\ell) = \frac{1}{\Gamma(n-\gamma)} \int_0^\ell (\ell - \rho)^{n-\gamma-1} F^{(n)}(\rho) \, d\rho. \quad (2.2)$$
Lemma 3.1

If \( h \) is a solution with integral type boundary conditions

\[
\int^\ell 0 \alpha_1 \Omega(0) - b_1 \Omega(\eta) - c_1 \Omega(1) = \frac{1}{\Gamma(\gamma)} \int^1_0 (1 - \rho)^{\gamma - 1} \mu(\rho, \omega(\rho)) \, d\rho,
\]

has a solution

\[
\omega(\ell) = \frac{1}{\alpha_1 + b_1} \int^\ell 0 \alpha_1 \Omega(0) - b_1 \Omega(\eta) - c_1 \Omega(1) + \frac{1}{\Gamma(\gamma)} \int^\ell (\ell - \rho)^{\gamma - 1} \mu(\rho, \omega(\rho)) \, d\rho
\]
Applying fractional integrable operator $I^\gamma$ to $D^\gamma \omega(\ell) = h(\ell)$ and using Lemma 2.1, we get

$$\omega(\ell) = c_0 + I^\gamma h(\ell). \tag{3.1}$$

On applying boundary conditions to (3.1), we have

$$c_0(\alpha_1 - b_1 - c_1) - b_1 I^\gamma h(\eta) - c_1 I^\gamma h(1) = \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma-1} \phi(\rho, \omega(\rho)) \, d\rho.$$

By rearranging, we get

$$c_0 = \frac{1}{\alpha_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma-1} \phi(\rho, \omega(\rho)) \, d\rho$$

$$+ \frac{c_1}{\alpha_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma-1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho$$

$$+ \frac{b_1}{\alpha_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^\eta (\eta - \rho)^{\gamma-1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho.$$

By Lemma 3.1, the solution of system (1.1) is a solution of the following system of integral equations:

$$\left\{ \begin{array}{l}
\omega(\ell) = \frac{1}{\alpha_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma-1} \phi(\rho, \omega(\rho)) \, d\rho \\
+ \frac{1}{\Gamma(\gamma)} \int_0^1 (\ell - \rho)^{\gamma-1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho \\
+ \frac{c_1}{\alpha_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma-1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho \\
+ \frac{b_1}{\alpha_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^\eta (\eta - \rho)^{\gamma-1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho,
\end{array} \right. \tag{3.2}$$

$$\nu(\ell) = \frac{1}{\alpha_2 - (c_2 + b_2)} \frac{1}{\Gamma(\delta)} \int_0^1 (1 - \rho)^{\delta-1} \psi(\rho, \nu(\rho)) \, d\rho$$

$$+ \frac{1}{\Gamma(\delta)} \int_0^1 (\ell - \rho)^{\delta-1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho$$

$$+ \frac{1}{\alpha_2 - (c_2 + b_2)} \frac{1}{\Gamma(\delta)} \int_0^\xi (\xi - \rho)^{\delta-1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho$$

Define the operator $\mathcal{J} : U \times \mathcal{V} \rightarrow U \times \mathcal{V}$ by

$$\mathcal{J}(\omega, \nu)(\ell) = (\mathcal{J}_1 \omega(\ell), \mathcal{J}_2 \nu(\ell)),$$

where

$$\mathcal{J}_1 \omega(\ell) = \frac{1}{\alpha_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma-1} \phi(\rho, \omega(\rho)) \, d\rho$$

and

$$\mathcal{J}_2 \nu(\ell) = \frac{1}{\alpha_2 - (c_2 + b_2)} \frac{1}{\Gamma(\delta)} \int_0^1 (1 - \rho)^{\delta-1} \psi(\rho, \nu(\rho)) \, d\rho.$$
Also define the operator \( G : \mathcal{U} \times \mathcal{V} \rightarrow \mathcal{U} \times \mathcal{V} \) by
\[
G(\omega, \nu)(\ell) = (G_1(\omega, \nu)(\ell), G_2(\omega, \nu)(\ell)),
\]
where
\[
G_1(\omega, \nu)(\ell) = \frac{c_1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma - 1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho
+ \frac{b_1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^\eta (\eta - \rho)^{\gamma - 1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho
+ \frac{1}{\Gamma(\gamma)} \int_0^\ell (\ell - \rho)^{\gamma - 1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho
\]
and
\[
G_2(\omega, \nu)(\ell) = \frac{c_2}{a_2 - (c_2 + b_2)} \frac{1}{\Gamma(\delta)} \int_0^1 (1 - \rho)^{\delta - 1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho
+ \frac{b_2}{a_2 - (c_2 + b_2)} \frac{1}{\Gamma(\delta)} \int_0^\xi (\xi - \rho)^{\delta - 1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho
+ \frac{1}{\Gamma(\delta)} \int_0^\ell (\ell - \rho)^{\delta - 1} F(\rho, \omega(\lambda \rho), \nu(\lambda \rho)) \, d\rho.
\]

Further, we define \( T = J + G \). Then the system of integral equations (3.2) can be written as an operator form
\[
(\omega, \nu) = T(\omega, \nu) = J(\omega, \nu) + G(\omega, \nu),
\]
which is the solution of system (1.1) in the operator form.

**Lemma 3.2** The operator \( J \) satisfies the Lipschitz condition
\[
\| J(\omega, \nu) - J(\overline{\omega}, \overline{\nu}) \| \leq k \| (\omega, \nu) - (\overline{\omega}, \overline{\nu}) \|. \tag{3.3}
\]

**Proof** For arbitrary \((\omega, \nu), (\overline{\omega}, \overline{\nu}) \in \mathcal{U} \times \mathcal{V}\), we have
\[
| J_1 \omega - J_1 \overline{\omega} | = \left| \frac{1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma - 1} \phi(\rho, \omega(\rho)) \, d\rho 
- \frac{1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma - 1} \phi(\rho, \overline{\omega}(\rho)) \, d\rho \right|
= \left| \frac{1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma - 1} \left[ \phi(\rho, \omega(\rho)) - \phi(\rho, \overline{\omega}(\rho)) \right] \, d\rho \right|
\]
which implies that
\[
\| J_1 \omega - J_1 \overline{\omega} \| \leq \frac{k_\phi}{|a_1 - (c_1 + b_1)|} \| \omega - \overline{\omega} \|. \tag{3.4}
\]
Similarly,

\[ \| J_2 u - J_2 v \| \leq \frac{k_\phi}{|a_2 - (c_2 + b_2)|} \| u - v \|. \tag{3.5} \]

From (3.4) and (3.5), we have

\[ \| J(\omega, v) - J(\bar{\omega}, \bar{v}) \| = \| J_1 \omega(\ell) - J_1 \bar{\omega}(\ell) + J_2 v(\ell) - J_2 \bar{v}(\ell) \| \]

\[ \leq \frac{k_\phi}{|a_1 - (c_1 + b_1)|} \| \omega - \bar{\omega} \| + \frac{k_\psi}{|a_2 - (c_2 + b_2)|} \| v - \bar{v} \| \]

\[ \leq k \| (\omega, v) - (\bar{\omega}, \bar{v}) \| , \]

where \( k = \max\left(\frac{k_\phi}{|a_1 - (c_1 + b_1)|}, \frac{k_\psi}{|a_2 - (c_2 + b_2)|}\right) \). Thus \( J \) is Lipschitz with constant \( k \), and therefore by Proposition 2.2, \( J \) is \( \sigma \)-Lipschitz with constant \( k \). \( \square \)

**Lemma 3.3** The operator \( G : U \times V \to U \times V \) is continuous.

**Proof** Consider a sequence \( \{(\omega_n, v_n)\}_{n \in \mathbb{N}} \) in a bounded set

\[ B_r = \{ (\omega, v) : \| (\omega, v) \| \leq r, (\omega, v) \in U \times V \} \]

such that \( (\omega_n, v_n)_{n \in \mathbb{N}} \to (\omega, v) \) as \( n \to +\infty \) in \( B_r \). To check that \( G \) is continuous, we have to prove that

\[ \| G(\omega_n, v_n)(\ell) - G(\omega, v)(\ell) \| \to 0 \quad \text{as} \quad n \to +\infty. \]

For this, we have

\[ |G_1(\omega_n, v_n)(\ell) - G_1(\omega, v)(\ell)| \]

\[ = \left| \frac{c_1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^\ell (1 - \rho)^{\gamma-1} F(\rho, \omega_n(\lambda, \rho), v_n(\lambda, \rho)) \, d\rho \right| \]

\[ + \frac{1}{\Gamma(\gamma)} \int_0^\ell (\ell - \rho)^{\gamma-1} F(\rho, \omega_n(\lambda, \rho), v_n(\lambda, \rho)) \, d\rho \]

\[ + \frac{b_1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^\eta (\eta - \rho)^{\gamma-1} F(\rho, \omega_n(\lambda, \rho), v_n(\lambda, \rho)) \, d\rho \]

\[ - \frac{c_1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^\ell (1 - \rho)^{\gamma-1} F(\rho, \omega(\lambda, \rho), v(\lambda, \rho)) \, d\rho \]

\[ - \frac{b_1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^\eta (\eta - \rho)^{\gamma-1} F(\rho, \omega(\lambda, \rho), v(\lambda, \rho)) \, d\rho \]

\[ \leq \frac{|c_1|}{|a_1 - (c_1 + b_1)|} \frac{1}{\Gamma(\gamma)} \int_0^\ell (1 - \rho)^{\gamma-1} \left| F(\rho, \omega_n(\lambda, \rho), v_n(\lambda, \rho)) - F(\rho, \omega(\lambda, \rho), v(\lambda, \rho)) \right| \, d\rho \]
Similarly other terms approach 0 as \( n \to +\infty \).

From the continuity of \( F \), it follows that
\[
F(\rho, \omega_n(\lambda \rho), v_n(\lambda \rho)) \to F(\rho, \omega(\lambda \rho), v(\lambda \rho)) \quad \text{as} \quad n \to +\infty.
\]

For every \( \ell \in 3 \) and by using (C3), we get
\[
\left\| \mathcal{G}_1(\omega_n, v_n)(\ell) - \mathcal{G}_1(\omega, v)(\ell) \right\| \to 0 \quad \text{as} \quad n \to +\infty.
\]

That is, \( \mathcal{G}_1 \) is continuous. Proceeding the same way as above, we can show that
\[
\left\| \mathcal{G}_2(\omega_n, v_n)(\ell) - \mathcal{G}_2(\omega, v)(\ell) \right\| \to 0 \quad \text{as} \quad n \to +\infty.
\]

That is, \( \mathcal{G}_2 \) is continuous and hence \( \mathcal{G} \) is continuous.

\[\text{Lemma 3.4} \quad \text{The operators} \quad \mathcal{J} \quad \text{and} \quad \mathcal{G} \quad \text{satisfy the following growth conditions:}\]
\[
\| \mathcal{J}(\omega, v) \| \leq C \| (\omega, v) \|^{q_1} + M \quad \text{for each} \quad (\omega, v) \in \mathcal{U} \times \mathcal{V}
\]
\[
\text{and}
\]
\[
\| \mathcal{G}(\omega, v) \| \leq \Delta \left( \| (\omega, v) \|^{q_2} + M^* \right) \quad \text{for each} \quad (\omega, v) \in \mathcal{U} \times \mathcal{V},
\]

respectively, where \( c = \max(c_1, c_2), \quad d = \max(d_1, d_2), \quad C = \max\left( \frac{c_\phi}{|a_1 - (c_1 + b_1)|}, \frac{c_\psi}{|a_2 - (c_2 + b_2)|} \right), \quad \Delta = \max\left( \frac{c_\phi}{|a_1 - (c_1 + b_1)|}, \frac{c_\psi}{|a_2 - (c_2 + b_2)|} \right). \]

\[\text{Proof} \quad \text{For the growth condition on} \quad \mathcal{J}, \quad \text{consider}\]
\[
\| \mathcal{J}(\omega, v) \| = \| (\mathcal{J}_1 \omega(\ell), \mathcal{J}_2 v(\ell)) \|
\]
\[
= \left\| \mathcal{J}_1 \omega(\ell) \right\| + \left\| \mathcal{J}_2 v(\ell) \right\|
\]
\[
= \left\| \frac{1}{|a_1 - (c_1 + b_1)|} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma - 1} \phi(\rho, \omega(\rho)) \, d\rho \right\|
\]
\[
+ \left\| \frac{1}{|a_2 - (c_2 + b_2)|} \frac{1}{\Gamma(\delta)} \int_0^1 (1 - \rho)^{\delta - 1} \psi(\rho, v(\rho)) \, d\rho \right\|
\]
\[
\leq \frac{c_\phi \| \omega \|^{q_1}}{|a_1 - (c_1 + b_1)|} + M_\phi + \frac{c_\psi \| v \|^{q_1}}{|a_2 - (c_2 + b_2)|} + M_\psi
\]
\[
\leq C \| (\omega, v) \|^{q_1} + M,
\]
where $M = \max(M\phi, M\psi)$, which is the growth condition for $\mathcal{J}$. Now, for the growth condition on $\mathcal{G}$, we have

\[
\|G_1(\omega, v)(\ell)\| = \left| \frac{c_1}{a_1 - (c_1 + b_1)} \Gamma(\gamma) \int_0^1 (1 - \rho)^{\gamma - 1} F(\rho, \omega(\lambda, \rho), v(\lambda, \rho)) \, d\rho \right|
\]

\[
+ \frac{1}{\Gamma(\gamma)} \int_0^\ell (\ell - \rho)^{\gamma - 1} F(\rho, \omega(\lambda, \rho), v(\lambda, \rho)) \, d\rho
\]

\[
+ \frac{b_1}{a_1 - (c_1 + b_1)} \Gamma(\gamma) \int_0^\eta (\eta - \rho)^{\gamma - 1} F(\rho, \omega(\lambda, \rho), v(\lambda, \rho)) \, d\rho
\]

\[
\leq \frac{|c_1|}{|a_1 - (c_1 + b_1)|} \left( c_1 \|\omega\|^q_2 + c_2 \|v\|_q^2 + M_F \right)
\]

\[
+ \frac{|b_1| \eta^\gamma}{|a_1 - (c_1 + b_1)|} \left( c_1 \|\omega\|^q_2 + c_2 \|v\|_q^2 + M_F \right)
\]

\[
+ \ell^\gamma \left( c_1 \|\omega\|^q_2 + c_2 \|v\|_q^2 + M_F \right),
\]

which implies that

\[
\|G_1(\omega, v)(\ell)\| \leq \left( \frac{2|c_1| + 2|b_1| + |a_1|}{|a_1 - (c_1 + b_1)|} \right) c_1 \|\omega\|^q_2 + c_2 \|v\|_q^2 + M_F. \tag{3.8}
\]

Similarly,

\[
\|G_2(\omega, v)(\ell)\| \leq \left( \frac{2|c_2| + 2|b_2| + |a_2|}{|a_2 - (c_2 + b_2)|} \right) d_1 \|\omega\|^q_2 + d_2 \|v\|_q^2 + M_\mathcal{F}. \tag{3.9}
\]

Now, from (3.8) and (3.9), we have

\[
\|G(\omega, v)(\ell)\| = \|G_1(\omega, v)(\ell)\| + \|G_2(\omega, v)(\ell)\|
\]

\[
\leq \left( \frac{2|c_1| + 2|b_1| + |a_1|}{|a_1 - (c_1 + b_1)|} \right) c_1 \|\omega\|^q_2 + c_2 \|v\|_q^2 + M_F
\]

\[
+ \left( \frac{2|c_2| + 2|b_2| + |a_2|}{|a_2 - (c_2 + b_2)|} \right) d_1 \|\omega\|^q_2 + d_2 \|v\|_q^2 + M_\mathcal{F}
\]

\[
\leq \Delta \left( \|\omega, v\|^{q_2} + M^* \right),
\]

where $M^* = \max(M_\mathcal{F}, M_\mathcal{F})$. Hence $G$ satisfies the growth condition. \hfill \square

**Lemma 3.5** The operator $\mathcal{G} : U \times V \to U \times V$ is compact.

**Proof** Let $B$ be a bounded subset of $B_\mathcal{F} \subseteq U \times V$ and $\{(\omega_n, v_n)\}_{n \in \mathbb{N}}$ be a sequence in $B$, then by using the growth condition of $\mathcal{G}$, it is clear that $\mathcal{G}(B)$ is bounded in $U \times V$. Now, we need to show that $\mathcal{G}$ is equicontinuous. Let $0 \leq \ell \leq \tau \leq 1$, then we have

\[
\|G_1(\omega_n, v_n)(\ell) - G_1(\omega_n, v_n)(\tau)\|
\]

\[
= \left| \frac{1}{\Gamma(\gamma)} \int_0^{\ell} (\ell - \rho)^{\gamma - 1} F(\rho, \omega_n(\lambda, \rho), v_n(\lambda, \rho)) \, d\rho \right|
\]

\[
- \frac{1}{\Gamma(\gamma)} \int_0^{\tau} (\tau - \rho)^{\gamma - 1} F(\rho, \omega_n(\lambda, \rho), v_n(\lambda, \rho)) \, d\rho \right|
\]
\[
\begin{align*}
&= \left| \frac{1}{\Gamma(y)} \int_0^\tau \left[ (\ell - \rho)^{y-1} - (\tau - \rho)^{y-1} \right] F(\rho, \omega_n(\lambda \rho), \upsilon_n(\lambda \rho)) d\rho \\
&\quad \quad - \frac{1}{\Gamma(y)} \int_\ell^\tau (\tau - \rho)^{y-1} F(\rho, \omega_n(\lambda \rho), \upsilon_n(\lambda \rho)) d\rho \right| \\
&\leq \frac{1}{\Gamma(y)} \int_0^\tau \left[ (\ell - \rho)^{y-1} - (\tau - \rho)^{y-1} \right] |F(\rho, \omega_n(\lambda \rho), \upsilon_n(\lambda \rho))| d\rho \\
&\quad \quad + \frac{1}{\Gamma(y)} \int_\ell^\tau (\tau - \rho)^{y-1} |F(\rho, \omega_n(\lambda \rho), \upsilon_n(\lambda \rho))| d\rho \\
&\leq \frac{1}{\Gamma(y + 1)} \left[ (\ell' - \ell)^{y'} - 2(\ell' - \ell)^y \right] c_1 \|\omega\|^{q_2} + c_2 \|\upsilon\|^{q_2} + M_{\mathcal{F}}.
\end{align*}
\]

Taking limit as \( \ell \to \tau \), we get

\[
\|G_1(\omega_n, \upsilon_n)(\ell) - G_1(\omega_n, \upsilon_n)(\tau)\| \to 0.
\]

That is, there exists \( \epsilon > 0 \) such that

\[
\|G_1(\omega_n, \upsilon_n)(\ell) - G_1(\omega_n, \upsilon_n)(\tau)\| < \frac{\epsilon}{2}, \quad (3.10)
\]

Similarly,

\[
\|G_2(\omega_n, \upsilon_n)(\ell) - G_2(\omega_n, \upsilon_n)(\tau)\| < \frac{\epsilon}{2}, \quad (3.11)
\]

From (3.10) and (3.11), it follows that

\[
\|G(\omega_n, \upsilon_n)(\ell) - G(\omega_n, \upsilon_n)(\tau)\| < \epsilon. \quad (3.12)
\]

Hence \( G \) is equicontinuous. Therefore \( G(\mathcal{B}) \) is compact in \( \mathcal{U} \times \mathcal{V} \) and hence by Proposition 2.1, \( G \) is \( \sigma \)-Lipschitz with constant zero. \( \square \)

**Theorem 3.1** Under assumptions \((C_1)-(C_3)\), BVP (1.1) has at least one solution \((\omega, \upsilon) \in \mathcal{U} \times \mathcal{V}\). Moreover, the solution set of (1.1) is bounded in \( \mathcal{U} \times \mathcal{V} \).

**Proof** From Lemma 3.2, \( \mathcal{J} \) is Lipschitz with constant \( k \in [0, 1) \), and from Lemma 3.5, \( \mathcal{G} \) is Lipschitz with constant 0. It follows by Proposition 2.1 that \( \mathcal{T} \) is a \( \sigma \)-contraction with constant \( k \). Define

\[
\mathcal{B} = \{ (\omega, \upsilon) \in \mathcal{U} \times \mathcal{V} : \text{there exists } \varphi \in \mathcal{J}(\omega, \upsilon) = \varphi \mathcal{T}(\omega, \upsilon) \}.
\]

We have to show that \( \mathcal{B} \) is bounded in \( \mathcal{U} \times \mathcal{V} \). Choose \((\omega, \upsilon) \in \mathcal{B}\), then by using (3.6) and (3.7) we have

\[
\| (\omega, \upsilon) \| = \| \varphi \mathcal{T}(\omega, \upsilon) \|
\]

\[
= \varphi(\| \mathcal{J}(\omega, \upsilon) + G(\omega, \upsilon) \|)
\]

\[
\leq \varphi(\| \mathcal{J}(\omega, \upsilon) \| + \| G(\omega, \upsilon) \|)
\]

\[
\leq \varphi(C \| (\omega, \upsilon) \|^{q_1} + \Delta (\| (\omega, \upsilon) \|^{q_2} + M^*))
\]

\[
= \varphi(C \| (\omega, \upsilon) \|^{q_1} + \Delta (\| (\omega, \upsilon) \|^{q_2} + \varphi(M + \Delta M^*).\]
Thus \( \mathcal{B} \) is bounded in \( \mathcal{U} \times \mathcal{V} \). Therefore Theorem 2.1 guarantees that \( \mathcal{T} \) has at least one fixed point; consequently, BVP (1.1) has at least one solution. \( \square \)

**Theorem 3.2** Under assumptions \((C_1)\)–\((C_4)\), assume that \( G^* < 1 \), then BVP (1.1) has a unique solution, where

\[
G^* = k + \frac{L_F[2|c_1| + 2|b_1| + |a_1|]}{|a_1 - (c_1 + b_1)|} + \frac{L_F[2|c_2| + 2|b_2| + |a_2|]}{|a_2 - (c_2 + b_2)|}.
\]

**Proof** To find the unique solution of system (1.1), we use the Banach contraction theorem, that is, we have to show that \( \mathcal{T} \) is a contraction. For this, let \( (\omega, \upsilon), (\overline{\omega}, \overline{\upsilon}) \in \mathcal{U} \times \mathcal{V} \), then from (3.3) in Lemma 3.2, we showed that

\[
|\mathcal{T}(\omega, \upsilon) - \mathcal{T}(\overline{\omega}, \overline{\upsilon})| \leq k \| (\omega, \upsilon) - (\overline{\omega}, \overline{\upsilon}) \|.
\]

Next

\[
|G_1(\omega, \upsilon) - G_1(\overline{\omega}, \overline{\upsilon})| = \left| \frac{c_1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma - 1} F(\rho, \omega(\lambda, \rho), \upsilon(\lambda, \rho)) \, d\rho \right|

+ \frac{1}{\Gamma(\gamma)} \int_0^\ell (\ell - \rho)^{\gamma - 1} F(\rho, \omega(\lambda, \rho), \upsilon(\lambda, \rho)) \, d\rho

+ \frac{b_1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^\eta (\eta - \rho)^{\gamma - 1} F(\rho, \omega(\lambda, \rho), \upsilon(\lambda, \rho)) \, d\rho

- \frac{c_1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma - 1} F(\rho, \overline{\omega}(\lambda, \rho), \overline{\upsilon}(\lambda, \rho)) \, d\rho

- \frac{b_1}{a_1 - (c_1 + b_1)} \frac{1}{\Gamma(\gamma)} \int_0^\eta (\eta - \rho)^{\gamma - 1} F(\rho, \overline{\omega}(\lambda, \rho), \overline{\upsilon}(\lambda, \rho)) \, d\rho

\leq \frac{|c_1|}{|a_1 - (c_1 + b_1)|} \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \rho)^{\gamma - 1} |F(\rho, \omega(\lambda, \rho), \upsilon(\lambda, \rho)) - F(\rho, \overline{\omega}(\lambda, \rho), \overline{\upsilon}(\lambda, \rho))| \, d\rho

+ \frac{|b_1|}{|a_1 - (c_1 + b_1)|} \frac{1}{\Gamma(\gamma)} \int_0^\eta (\eta - \rho)^{\gamma - 1} |F(\rho, \omega(\lambda, \rho), \upsilon(\lambda, \rho)) - F(\rho, \overline{\omega}(\lambda, \rho), \overline{\upsilon}(\lambda, \rho))| \, d\rho

+ \frac{1}{\Gamma(\gamma)} \int_0^\ell (\ell - \rho)^{\gamma - 1} |F(\rho, \omega(\lambda, \rho), \upsilon(\lambda, \rho)) - F(\rho, \overline{\omega}(\lambda, \rho), \overline{\upsilon}(\lambda, \rho))| \, d\rho

\leq \frac{|c_1|}{|a_1 - (c_1 + b_1)|} L_F(|\omega - \overline{\omega}| + |\upsilon - \overline{\upsilon}|)

+ \frac{|b_1|}{|a_1 - (c_1 + b_1)|} L_F(|\omega - \overline{\omega}| + |\upsilon - \overline{\upsilon}|)

+ L_F(|\omega - \overline{\omega}| + |\upsilon - \overline{\upsilon}|),
which implies that

$$\|G_1(\omega, \nu) - G_1(\overline{\omega}, \overline{\nu})\| \leq \frac{2|c_1| + 2|b_1| + |a_1|}{|a_1 - (c_1 + b_1)|} L_F \|(\nu, \omega) + (\overline{\omega}, \overline{\nu})\|. \quad (3.14)$$

Similarly,

$$\|G_2(\omega, \nu) - G_2(\overline{\omega}, \overline{\nu})\| \leq \frac{2|c_2| + 2|b_2| + |a_2|}{|a_2 - (c_2 + b_2)|} L_F \|(\nu, \omega) + (\overline{\omega}, \overline{\nu})\|. \quad (3.15)$$

From (3.14) and (3.15), it follows that

$$\|G(\omega, \nu) - G(\overline{\omega}, \overline{\nu})\| = \|G_1(\omega, \nu) - G_1(\overline{\omega}, \overline{\nu})\| + \|G_2(\omega, \nu) - G_2(\overline{\omega}, \overline{\nu})\|$$

$$\leq \frac{2|c_1| + 2|b_1| + |a_1|}{|a_1 - (c_1 + b_1)|} L_F \|(\nu, \omega) + (\overline{\omega}, \overline{\nu})\| + \frac{2|c_2| + 2|b_2| + |a_2|}{|a_2 - (c_2 + b_2)|} L_F \|(\nu, \omega) + (\overline{\omega}, \overline{\nu})\|,$$

which implies that

$$\|G(\omega, \nu) - G(\overline{\omega}, \overline{\nu})\| \leq \left( L_F \left[ 2\frac{|c_1| + 2|b_1| + |a_1|}{|a_1 - (c_1 + b_1)|} + \frac{L_F[2|c_2| + 2|b_2| + |a_2|]}{|a_2 - (c_2 + b_2)|} \right] \right) \|(\omega, \nu) + (\overline{\omega}, \overline{\nu})\|. \quad (3.16)$$

Now, from (3.13) and (3.16), it follows that

$$|T(\omega, \nu) - T(\overline{\omega}, \overline{\nu})| \leq |J(\omega, \nu) - J(\overline{\omega}, \overline{\nu})| + |G(\omega, \nu) - G(\overline{\omega}, \overline{\nu})|$$

$$\leq k \|(\omega, \nu) + (\overline{\omega}, \overline{\nu})\|$$

$$\times \left( \left( L_F \left[ 2\frac{|c_1| + 2|b_1| + |a_1|}{|a_1 - (c_1 + b_1)|} + \frac{L_F[2|c_2| + 2|b_2| + |a_2|]}{|a_2 - (c_2 + b_2)|} \right] \right) \|(\omega, \nu) + (\overline{\omega}, \overline{\nu})\| \right)$$

$$\leq k \left[ \left( L_F \left[ 2\frac{|c_1| + 2|b_1| + |a_1|}{|a_1 - (c_1 + b_1)|} + \frac{L_F[2|c_2| + 2|b_2| + |a_2|]}{|a_2 - (c_2 + b_2)|} \right] \right) \right] \|(\omega, \nu) + (\overline{\omega}, \overline{\nu})\|,$$

which implies that

$$\|T(\omega, \nu) - T(\overline{\omega}, \overline{\nu})\| \leq \mathcal{G}^* \|(\omega, \nu) + (\overline{\omega}, \overline{\nu})\|. \quad (3.17)$$

Thus $T$ is a contraction and hence problem (1.1) has a unique solution. \qed

To illustrate our results, we provide the following example.
Example 3.1 Consider the following BVP:

\[
\begin{cases}
   D^{2/3} \omega(t) = \frac{e^{-\tau t}}{10} + \frac{\sin|\omega(t)| + \sin|u(t)|}{5 + t^2}, & t \in [0,1], \\
   D^{3/4} v(t) = \frac{e^{-\tau t}}{20} + \frac{\sin|\omega(t)| + \sin|u(t)|}{60 + (e^t + 1)^2}, & t \in [0,1], \\
   \left[ \frac{1}{2} \right] \omega(0) - \frac{1}{2} \omega(\frac{1}{2}) - 7\omega(1) = \frac{1}{\gamma(\frac{3}{4})} \int_0^1 (1 - \rho) \frac{1}{2} \cos\omega(\rho) \, d\rho, \\
   \left[ \frac{1}{6} \right] v(0) - \frac{1}{3} v(\frac{1}{2}) - 9v(1) = \frac{1}{\gamma(\frac{3}{4})} \int_0^1 (1 - \rho) \frac{1}{2} \cos\omega(\rho) \, d\rho. 
\end{cases}
\]  

(3.18)

Here, \( \mathcal{F} = \frac{e^{-\tau t}}{10} + \frac{\sin|\omega(t)| + \sin|u(t)|}{5 + t^2} \), \( \mathcal{G} = \frac{e^{-\tau t}}{20} + \frac{\sin|\omega(t)| + \sin|u(t)|}{60 + (e^t + 1)^2} \), \( \gamma = \frac{2}{3}, \delta = \frac{3}{4}, a_1 = \frac{1}{5}, b_1 = \frac{1}{2}, c_1 = 7, a_2 = \frac{1}{0}, b_2 = \frac{1}{3}, c_2 = 9, \eta = \xi = \frac{1}{2}. \) Let \( \varrho = \frac{1}{2} \), then by routine calculation we can easily find that \( k_\varphi = c_\varphi = \frac{1}{2}, k_\psi = c_\psi = \frac{1}{3}, M_\varphi = M_\psi = 0, c_1 = c_2 = L_{\mathcal{F}} = \frac{1}{51}, d_1 = d_2 = L_{\mathcal{G}} = \frac{1}{61}, M_{\mathcal{F}} = \frac{1}{10}, M_{\mathcal{G}} = \frac{1}{20} \), hence assumptions \((C_1)-(C_4)\) are satisfied. Further

\[
|\mathcal{J}(\omega, v)(t) - \mathcal{J}(\varphi, \psi)(t)| \leq \frac{1}{17.896} \int_0^1 (1 - \rho)^{3/4} \left| \cos(\omega) - \cos(\varphi) \right| \, d\rho \\
+ \frac{1}{36.387} \int_0^1 (1 - \rho)^{1/2} \left| e^v(\rho) - e^{\varphi}(\rho) \right| \, d\rho \\
\leq \frac{2}{17.896} \|\omega - \varphi\| + \frac{2}{36.387} \|v - \varphi\| \\
\leq 0.112 \|\omega - \varphi\|
\]

which means that \( \mathcal{J} \) is \( \sigma \)-Lipschitz with constant 0.112 and \( \mathcal{G} \) is \( \sigma \)-Lipschitz with constant zero, this implies that \( \mathcal{T} \) is strict \( \sigma \)-Lipschitz with constant 0.112. Since

\[
\mathcal{B} = \{(\omega, v) \in \mathcal{U} \times \mathcal{V} : \text{there exists } \varrho \in \mathcal{Z}, (\omega, v) = \varrho \mathcal{T}(\omega, v)\},
\]

then, by routine calculation, we get

\[
\|\omega, v\| \leq 0.0076 \leq 1,
\]

which implies that \( \mathcal{B} \) is bounded, and in the light of Theorem 3.1, BVP (3.18) has at least one solution. Moreover, \( \mathcal{G}^* \leq 0.3348 < 1 \). Hence the problem has a unique solution.

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Author details
1 Department of Mathematics, University of Malakand, Chakdara, Khyber Pakhtunkhwa, Pakistan. 2 Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia. 3 Department of Medical Research, China Medical University, Taichung, Taiwan. 4 Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan.

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