First-Order Rewritability and Complexity of Two-Dimensional Temporal Ontology-Mediated Queries

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Abstract

Aiming at ontology-based data access to temporal data, we design two-dimensional temporal ontology and query languages by combining logics from the (extended) DL-Lite family with linear temporal logic LTL over discrete time \((Z, <)\). Our main concern is first-order rewritability of ontology-mediated queries (OMQs) that consist of a 2D ontology and a positive temporal instance query. Our target languages for FO-rewritings are two-sorted FO\((<)\)—first-order logic with sorts for time instants ordered by the built-in precedence relation \(<\) and for the domain of individuals—its extension FO\((<, \equiv)\) with the standard congruence predicates \(t \equiv 0 \pmod{n}\), for any fixed \(n > 1\), and FO(RPR) that admits relational primitive recursion. In terms of circuit complexity, FO\((<, \equiv)\)- and FO(RPR)-rewritability guarantee answering OMQs in uniform \(AC^0\) and \(NC^1\), respectively.

We proceed in three steps. First, we define a hierarchy of 2D DL-Lite/LTL ontology languages and investigate the FO-rewritability of OMQs with atomic queries by constructing projections onto 1D LTL OMQs and employing recent results on the FO-rewritability of propositional LTL OMQs. As the projections involve deciding consistency of ontologies and data, we also consider the consistency problem for our languages. While the undecidability of consistency for 2D ontology languages with expressive Boolean role inclusions might be expected, we also show that, rather surprisingly, the restriction to Krom and Horn role inclusions leads to decidability (and EXPSPACE-completeness), even if one admits full Booleans on concepts. As a final step, we lift some of the rewritability results for atomic OMQs to OMQs with expressible positive temporal instance queries. The lifting results are based on an in-depth study of the canonical models and only concern Horn ontologies.

1. Introduction

Ontology-based data access (Calvanese, De Giacomo, Lembo, Lenzerini, & Rosati, 2007b), also known as virtual knowledge graphs (Xiao, Ding, Cogrel, & Calvanese, 2019), and more
general ontology-mediated query answering (Bienvenu & Ortiz, 2015) have recently become one of the most successful applications of ontologies. The main aim of ontology-based data access (OBDA, for short) is to facilitate access to possibly heterogeneous, distributed and incomplete data for non-IT-users. To this end, an ontology is employed to provide both a user-friendly and uniform vocabulary for formulating queries and a conceptual model of the domain for capturing background knowledge and obtaining more complete answers. Thus, instead of querying data directly by means of often convoluted and ad hoc database queries, one can use ontology-mediated queries (OMQs) of the form $q = (O, \varphi)$ with an ontology $O$ and a query $\varphi$ over the familiar and natural vocabulary provided by $O$. The answers to $q$ over a data instance $A$ (which can often be obtained via mappings from the original data) are then those tuples of individual names from $A$ that satisfy $\varphi$ in every model of $O$ and $A$. After almost 20 years of research, OMQ answering is now well understood both in theory and real-world applications; consult (Xiao, Calvanese, Kontchakov, Lembo, Poggi, Rosati, & Zakharyaschev, 2018; Xiao et al., 2019) for surveys.

One of the main challenges in OBDA has been to identify ontology languages that strike a good balance between the expressive power required for conceptual modelling and querying on the one hand and the computational complexity of answering OMQs on the other. The ontology languages employed nowadays for OMQ answering are either based on description logics (DLs) (Baader, Calvanese, McGuinness, Nardi, & Patel-Schneider, 2007; Baader, Horrocks, Lutz, & Sattler, 2017), and in particular the DL-Lite family (Poggi, Lembo, Calvanese, De Giacomo, Lenzerini, & Rosati, 2008; Artale, Calvanese, Kontchakov, & Zakharyaschev, 2009), or extensions of datalog and various forms of tuple-generating dependences (Abiteboul, Hull, & Vianu, 1995) such as linear and sticky sets of tgdts (Calì, Gottlob, & Lukasiewicz, 2012b; Calì, Gottlob, & Pieris, 2012a) and existential rules (Baget, Leclère, Mugnier, & Salvat, 2011). The data complexity of answering OMQs, according to which only the data instance is regarded as an input, while the OMQ is deemed to be fixed (or negligibly small compared to the data), and the rewritability of OMQs into conventional database queries that can be directly evaluated over the data without ontology reasoning have emerged as the most important measures of the efficiency of OMQ answering. Thus, the DL-Lite based ontology language OWL 2 QL\(^1\), which has been standardised by W3C specifically for OBDA and supported by systems such as Mastro\(^2\) and Ontop\(^3\), ensures rewritability of all OMQs with conjunctive queries into first-order (FO) queries, i.e., essentially SQL queries (Abiteboul et al., 1995). Complexity-wise, this means that such OMQs can be answered in LogTime uniform AC\(^0\), one of the smallest complexity classes (Immerman, 1999).

In many applications, data comes with timestamps indicating at which moment of time a fact holds. For instance, suppose we have a database on the submission and publication of papers in the area of computer science collected from various sources on the Web and elsewhere. The database may contain, among others, the facts

\[
\begin{align*}
\text{underSubmissionTo}(a, \text{JACM}, \text{Feb2017}), & \quad \text{UnderSubmission}(b, \text{Jan2021}), \\
\text{underSubmissionTo}(a, \text{JACM}, \text{Sep2020}), & \quad \text{Published}(b, \text{Oct2021}),
\end{align*}
\]

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1. https://www.w3.org/TR/owl2-profiles
2. https://www.obdasystems.com
3. https://ontopic.ai
stating that paper \(a\) was under submission to JACM in February 2017 and September 2020; paper \(b\), authored by Bob in May 2014, was under submission in January 2021 (to an unknown venue) and was published (again in an unknown venue) in October 2021; JACM was a journal in January 1954. Observe that the predicates in the snippet above have a timestamp as their last argument (e.g., Feb2017) and either one or two object domain arguments (e.g., \(a\), Bob, JACM).

While existing standard ontology languages designed for OBDA could in principle be used to support answering queries over such timestamped data, they do not have the expressive power to model even very basic temporal aspects of the domain and support the formulation of queries using temporal vocabulary. For example, in the domain of publishing papers introduced above, both ontology engineers and users need means for referring to a discrete linearly ordered temporal precedence relation so as to formulate background knowledge and queries such as ‘all published papers have previously been accepted’ and ‘find all journals \(x\) and months \(t\) such that all papers under submission at \(x\) in \(t\) were eventually published’. Ontology languages such as OWL 2 QL are not able to capture those.

In this article, we combine DLs from the DL-Lite family with the well-established linear temporal logic LTL (Demri, Goranko, & Lange, 2016) for time to obtain a hierarchy of ontology languages that support temporal conceptual modelling and OMQ answering over temporal data. Our main aim is to explore the trade-off between the expressive power and computational complexity/FO-rewritability of OMQs formulated in these combined languages that are interpreted over the two-dimensional Cartesian products of an object domain and a discrete linear order representing a flow of time.

Combinations of DLs with temporal formalisms have been widely investigated since the pioneering work of Schmiedel (1990) and Schild (1993) in the early 1990s; we refer the reader to (Gabbay, Kurucz, Wolter, & Zakharyaschev, 2003; Baader, Küsters, & Wolter, 2003; Artale & Franconi, 2005; Lutz, Wolter, & Zakharyaschev, 2008) for surveys and (Pagliarecci, Spalazzi, & Taccari, 2013; Artale, Kontchakov, Ryzhikov, & Zakharyaschev, 2014; Gutiérrez-Basulto, Jung, & Schneider, 2014, 2015; Gutiérrez-Basulto, Jung, & Ozaki, 2016b; Baader, Borgwardt, Koopmann, Ozaki, & Thost, 2020a) for more recent developments. However, the main reasoning task targeted in this line of research was concept satisfiability rather than OMQ answering, and the general aim was to identify and tailor combinations of temporal and DL constructs that ensure decidability of concept satisfiability with acceptable combined complexity.\(^4\) In contrast, our main concern in this article is OMQ answering, and thus we focus instead on the data complexity and FO-rewritability of ontology-mediated queries.

We use the standard discrete time model with the integers \(\mathbb{Z}\) and the order \(<\) as precedence. The temporal operators supplied by LTL are \(\bigcirc_p\) (at the next moment of time), \(\bigodot_p\) (eventually), \(\square_p\) (always in the future), \(\mathcal{U}\) (until), and their past-time counterparts \(\bigcirc_p\) (at the previous moment), \(\bigodot_p\) (some time in the past), \(\square_p\) (always in the past) and \(\mathcal{S}\) (since);

\(^4\) In a nutshell, the rule of thumb is that to be decidable a temporalised DL should be embeddable into the monadic fragment of first-order temporal logic, where no temporal operator is applied to a formula with two free variables (Hodkinson, Wolter, & Zakharyaschev, 2000). Even tiny additions to one-variable temporal FO such as the ‘elsewhere’ quantifier lead to undecidability (Hampson & Kurucz, 2015).
see (Manna & Pnueli, 1992; Gabbay, Hodkinson, & Reynolds, 1994; Demri et al., 2016) and further references therein.

Following the DL terminology, we refer to data instances as ABoxes; in the ABox snippet \( A \) above, we call the binary relations \textit{UnderSubmission}, \textit{Published} and \textit{Journal} concept names and the ternary relations \textit{underSubmissionTo} and \textit{authorOf} role names. In general, an ABox is a finite set of atoms of the form \( A(a, \ell) \) and \( P(a, b, \ell) \), where \( a, b \) are individual names, \( \ell \in \mathbb{Z} \) is a timestamp, \( A \) a concept name, and \( P \) a role name.

The combined DL-Lite/LTL ontologies we consider are finite sets of inclusions between (compound) concepts and between (compound) roles in the style of DL-Lite, in which temporal operators can be applied to both concepts and roles. However, in contrast to standard DL-Lite (and generally DLs), here we treat roles in the same way as concepts and thus also allow Boolean operators to be applied to roles. Concept and role inclusions are assumed to be true at all moments of time. We illustrate the expressive power of our languages using the domain of papers. By applying temporal operators to roles we can state, for instance, that \textit{underSubmissionTo} is convex using

\[
\Diamond_P \textit{underSubmissionTo} \cap \Diamond_P \textit{underSubmissionTo} \sqsubseteq \textit{underSubmissionTo},
\]

and that \textit{authorOf} is rigid (does not change in time):

\[
\Diamond_P \textit{authorOf} \sqsubseteq \textit{authorOf}, \quad \Diamond_P \textit{authorOf} \sqsubseteq \textit{authorOf}.
\]

Similarly, we can postulate that concept \textit{Journal} is rigid.\(^5\) We could state inclusion similar to (1) also for the concept name \textit{UnderSubmission}, but observe that, while convexity of \textit{UnderSubmissionTo} corresponds to the widespread policy that a paper can be submitted to the same venue only once, convexity of \textit{UnderSubmission} implies that rejected papers cannot be submitted to another venue. So far, the role names have not been linked to the corresponding concept names. From DL-Lite we inherit the capability of doing so via domain and range restrictions. Thus, we can extend our ontology with the equivalences

\[
\exists \textit{underSubmissionTo} \equiv \textit{UnderSubmission}, \quad \exists \textit{publishedIn} \equiv \textit{Published},
\]

the first of which, for example, says that every article that is under submission to some venue is submitted (and the other way round). Observe that a concept inclusion stating that \textit{UnderSubmission} and \textit{Published} are disjoint will imply that the roles \textit{underSubmissionTo} and \textit{PublishedIn} are also disjoint. The converse does not hold as there can be submitted papers that have already been published elsewhere. Additional vocabulary items can be introduced by stating, for example, that one can only submit to conferences or journals:

\[
\exists \textit{underSubmissionTo}^{-1} \sqsubseteq \text{Conference} \sqcup \text{Journal},
\]

where the role \textit{underSubmissionTo}^{-1} is the inverse of \textit{underSubmissionTo}. We denote the resulting ontology snippet by \( \mathcal{O} \). To illustrate OMQ answering, consider the atomic query \( \varphi_1 = \textit{UnderSubmission} \) that asks to find all pairs \((x, t)\) such that paper \( x \) is under submission

\(^5\) Subtler modelling would be to state that each instance of \textit{Journal} remains a journal within its lifespan, which could be done using an additional concept ‘exists’ that, for any moment of time, comprises those objects that exist at that moment; see, e.g., (Gabbay et al., 2003).
at time point \( t \). By inclusions (1) and (3) of \( \mathcal{O} \), \((b, \text{Jan2021})\) and all pairs in the interval \((a, \text{Feb2017}), \ldots, (a, \text{Sep2020})\) are answers to the OMQ \( q_1 = (\mathcal{O}, \varphi_1) \) over \( A \). Next, consider the OMQ \( q_2 = (\mathcal{O}, \varphi_2) \) with \( \varphi_2 = \exists \text{authorOf}(\text{UnderSubmission} \sqcap \diamond_f \text{Published}) \) asking for pairs \((x,t)\) such that \( x \) is an author of a paper under submission at time point \( t \) and eventually published. Then, by inclusion (2), \((\text{Bob}, \text{Jan2021})\) is the single answer to this query over \( A \). Observe that both OMQs are rewritable into \( \text{FO}(\langle \rangle) \), two-sorted \( \text{FO} \) with quantification over the convex closure of the time points in the ABox with the temporal precedence relation \( \langle \rangle \) supplemented by quantification over the ABox individuals. In fact, an \( \text{FO}(\langle \rangle) \)-rewriting of \( q_1 \) is

\[
q_1(x,t) = \text{UnderSubmission}(x,t) \lor \exists y, t', t'' ((t' \leq t \leq t'') \land \\
\text{underSubmissionTo}(x,y,t') \land \text{underSubmissionTo}(x,y,t'')) ,
\]

and

\[
q_2(x,t) = \exists y, t' (\text{authorOf}(x,y,t') \land q_1(y,t) \land \exists t'' ((t'' > t) \land \text{Published}(y,t''))
\]

is an \( \text{FO}(\langle \rangle) \)-rewriting of \( q_2 \). It follows that answering such OMQs is in \( \text{AC}^0 \) for data complexity and can be implemented using conventional relational database management systems (RDBMSs).

In this article, we determine classes of OMQs all of which are \( \text{FO}(\langle \rangle) \)-rewritable. To illustrate, call a concept basic if it is of the form \( A \lor \exists S \), where \( A \) is a concept name, and \( S \) a role name or the inverse thereof. The set of \( \mathcal{C}^{\mathcal{O}}_{\mathcal{P}rior} \)-concepts is obtained from basic concepts using arbitrary Boolean connectives and Prior’s temporal operators \( \square_p, \diamond_p, \square_f \) and \( \diamond_f \) (Prior, 1956; Vardi, 2008). \( \mathcal{C}^{\mathcal{O}}_{\mathcal{P}rior} \)-concept inclusions (CIs) are inclusions between \( \mathcal{C}^{\mathcal{O}}_{\mathcal{P}rior} \)-concepts. Also, let \( \mathcal{R}^\diamond_{\text{horn}^+} \) denote the class of role inclusions (RIs) \( R_1 \sqcap \cdots \sqcap R_n \subseteq R \) with roles \( R_i \) possibly prefixed by \( \diamond_p \) or \( \diamond_f \) and \( R \) either \( \square \) or a role possibly prefixed by \( \square_p \) or \( \square_f \). The ontology \( \mathcal{O} \) above is a union of \( \mathcal{C}^{\mathcal{O}}_{\mathcal{P}rior} \)-CIs and \( \mathcal{R}^\diamond_{\text{horn}^+} \)-RIs. An atomic OMQ (OMAQ, for short) takes the form \((\mathcal{O}, A)\), where \( A \) is a concept name. We obtain the following rewritability result:

**Theorem A.** All OMAQs \((\mathcal{O}, A)\), where \( \mathcal{O} \) is a union of \( \mathcal{C}^{\mathcal{O}}_{\mathcal{P}rior} \)-CIs and \( \mathcal{R}^\diamond_{\text{horn}^+} \)-RIs, are \( \text{FO}(\langle \rangle) \)-rewritable.

Our next aim is to go beyond OMAQs, as in Theorem A, and admit 2D instance queries that support both DL and LTL constructs. In fact, our main expressive instance query language is given by positive temporal concepts, \( \preceq \), that are constructed from concept names using \( \sqcap, \sqcup \), any temporal operators of LTL, and the DL construct \( \exists S.\preceq \), where \( S \) is a role name or its inverse. An OMQ \((\mathcal{O}, \preceq)\) with such a \( \preceq \) is called an ontology-mediated positive instance query (or OMPIQ). We cannot expect Theorem A to generalise to OMPIQs as already for the atemporal ontology consisting of the CI \( \forall A \sqcup B \) answering OMPIQs (even without temporal operators in the query) is \( \text{coNP} \)-hard (Schaerf, 1993) and thus of prohibitively poor data complexity. This \( \text{coNP} \)-hardness result also holds for OMPIQs with a single temporal CI \( A \sqsubseteq \diamond_f B \) in the ontology. Nevertheless, we can define an expressive ontology language with OMPIQs rewritable into \( \text{FO}(\langle \rangle) \). Let \( \mathcal{C}^\diamond_{\text{horn}} \) denote the class of CIs \( C_1 \sqcap \cdots \sqcap C_n \subseteq C \), where the \( C_i \) are basic concepts possibly prefixed by operators of the form \( \diamond_p, \square_p, \diamond_f, \square_f \), and \( C \) is either \( \square \) or a basic concept possibly prefixed by \( \square_p \) or \( \square_f \).
Theorem B. All OMPIQs \((\mathcal{O}, \preceq)\), where \(\mathcal{O}\) is the union of \(C_{\text{horn}}^\circ\)-CIs and \(R_{\text{horn}+}^\circ\)-RIs, are FO(<)-rewritable.

Notice that the ontology \(\mathcal{O}\) above without CI (4) is covered by Theorem B. The combined ontology languages considered up to now do not use the temporal operators \(\langle\cdot\rangle\) and \(\langle\cdot\rangle\). In fact, as observed by Artale, Kontchakov, Kovtunova, Ryzhikov, Wolter, and Zakharyaschev (2021), already the OMAQ \((\{A \sqaleq_p \lnot B, B \sqaleq_p \lnot A\}, A)\) is not FO(<)-rewritable. Following this work, we extend FO(<) to obtain a suitable target language for rewriting OMQs using \(\circ_p\) and \(\circ_p\) with the data complexity still in \(AC^0\). Let FO(<,≡) be the extension of FO(<) with the unary congruence predicates \(t \equiv 0 \mod n\), for any fixed \(n > 1\). To illustrate our rewritability results with the target language FO(<,≡), let a DL-Lite\(_{\text{core}}\) ontology contain any inclusion \(\vartheta_1 \sqaleq \vartheta_2\) or \(\vartheta_1 \sqaleq \vartheta_2 \sqaleq \bot\), in which \(\vartheta_1, \vartheta_2\) are either basic concepts or roles possibly prefixed by the operators \(\langle\cdot\rangle\) and/or \(\langle\cdot\rangle\). In other words, DL-Lite\(_{\text{core}}\) is the extension of the DL-Lite dialect underpinning OWL 2 QL with the operators \(\circ_p\) and \(\circ_p\).

Theorem C. All OMPIQs \((\mathcal{O}, \preceq)\) with a DL-Lite\(_{\text{core}}^\circ\) ontology \(\mathcal{O}\) are FO(<,≡)-rewritable.

Observe that rigidity of concepts and roles as in (2) can also be expressed using \(\langle\cdot\rangle\) and \(\langle\cdot\rangle\). To cover ontologies that are able to capture (1) and more general Horn-shaped inclusions as well as \(\circ_p\) and \(\circ_p\), we have to go beyond FO(<,≡) as the target language for rewritings and admit some form of recursion. Again following (Artale et al., 2021), we consider the extension FO(RPR) of FO(<) with relational primitive recursion. Rewritability into FO(RPR) implies that OMQ answering is in NC\(^1\) \(\subseteq L\) for data complexity. Note that FO(RPR)-queries can be expressed in SQL with recursion or procedural extensions, which are, in general, less efficient and not always supported by RDBMSs. A DL-Lite\(_{\text{horn}}^\circ\) ontology contains CIs and RIs of the form \(\vartheta_1 \sqaleq \cdots \sqaleq \vartheta_n \sqaleq \vartheta\), where each \(\vartheta_i\) is a basic concept/role possibly prefixed by operators of the from \(\circ_p, \circ_p, \circ_p, \circ_p\), \(\circ_p, \circ_p\), and \(\vartheta\) is either \(\bot\) or a basic concept/role possibly prefixed by operators of the form \(\circ_p, \circ_p, \circ_p, \circ_p\).

Theorem D. All OMPIQs \((\mathcal{O}, \preceq)\) with a DL-Lite\(_{\text{horn}}^\circ\) ontology \(\mathcal{O}\) are FO(RPR)-rewritable.

Theorems A–D only show a few ‘impressions’ of the results obtained in this article. To give a natural hierarchy of the combined DL-Lite/LTL OMQs and facilitate a systematic study of their rewritability and data complexity, we present the ontology languages in a rather different way than in Theorems A–D, using a clausal normal form to be introduced in Section 2. An overview of our rewritability and complexity results for ontologies in the normal form will be provided in Table 3 (Section 5) for OMAQs and in Table 4 (Section 6) for OMPIQs. The theorems above are then obtained by straightforward polynomial-time normalisations.

We establish our results in three steps:

(consistency) first, we reduce FO-rewritability of OMAQs to FO-rewritability of ‘consistent’ \(\bot\)-free OMAQs (whose ontology is consistent with all ABoxes);

(projection) then, where possible, we project consistent 2D DL-Lite/LTL OMAQs onto 1D LTL OMQs, which have been classified according to their rewritability properties by Artale et al. (2021);

(lifting) and, finally, we lift, where possible, FO-rewritability of Horn OMAQs to OMPIQs.
Here, by 1D LTL we mean propositional LTL speaking, intuitively, about how a single individual develops in time; input data is given by 1D ABoxes containing assertions of the form $A(\ell)$ with $\ell \in \mathbb{Z}$, and answers to 1D OMQs are sets of timestamps from the 1D ABox. Artale et al. (2021) obtain 1D rewritability results for the target languages FO($<$), FO($<$, $\equiv$) and FO(RPR) using automata-theoretic machinery with a few model-theoretic insights. We apply those results here in a black-box manner. Thus, despite the fact that ultimately the results obtained in this article heavily rely on automata theory, it is only present implicitly as the vehicle used by Artale et al. (2021) to obtain 1D rewritings.

By design, the interaction between concepts and roles in classical atemporal DL-Lite ontologies is rather weak: it can be captured by the one-variable fragment of FO if all role inclusions are Horn and by the two-variable fragment of FO otherwise (Kontchakov, Ryzhikov, Wolter, & Zakharyaschev, 2020). Temporalised concepts and roles interpreted over 2D Cartesian products of DL-Lite and LTL structures make the combined logic more expressive than both of its components. Answering OMAQs with Booleans on roles is similar to reasoning with the two-variable first-order LTL, which is known to be undecidable (Hodkinson et al., 2000). More unexpectedly, as we show in Theorem 27, answering OMAQs with Horn-shaped CIs and RIs having $\Box$-operators on roles only is NC$^1$-complete (and so needs recursion), while the OMAQs in the respective DL-Lite and LTL component fragments are FO($<$)-rewritable (in fact, the combined logic is capable of expressing the $\Box_P$- and $\Diamond_P$-operators). We construct a projection of 2D OMAQs with Boolean CIs and Horn RIs onto 1D LTL OMAQs by showing that the interaction between the component logics can be captured using (exponentially many, in general) ‘connecting axioms’ in the form of an implication between propositional variables possibly prefixed by a temporal operator that, in many cases, has to be $\Box_P$ or $\Diamond_P$, which explains the unexpected increase of expressivity mentioned above. This projection is sound and complete only if the input OMAQs are evaluated over ABoxes that are consistent with the ontology.

Therefore, as indicated in the (consistency) step, the problem of deciding whether an ABox is consistent with an ontology has to be analysed separately. As this problem is of independent interest for temporal conceptual modelling, we consider not only the data complexity of deciding consistency but also its combined complexity. Our results are given in Table 2 (Section 4) and provide a systematic investigation into the way how expressive role inclusions affect the complexity of consistency. While we confirm that full Boolean expressive power on roles leads to undecidability, we show that, rather surprisingly, the restriction to Krom and Horn RIs leads to decidability and ExpSpace-completeness, even if one admits full Booleans on concepts. The latter result is one of the very few instances breaking the monodicity barrier in temporal first-order logic, according to which in almost all instances reasoning about the temporal evolution of binary relations enables the encoding of undecidable tiling problems (Gabbay et al., 2003). The upper bounds are proved by relating the temporal DLs considered here to the one-variable fragment of first-order temporal logic.

The final and technically most difficult step of our construction shows how to lift a few rewritability and data complexity results from OMAQs to OMPIQs. This is not always possible. In particular, as already mentioned above, if the ontology language admits disjunction, answering OMPIQs becomes coNP-hard. We thus focus on OMPIQs whose ontology is given in the combined Horn DL-Lite/LTL language. We construct ourrewrit-
nings inductively from the known rewritings of the constituent OMAQs by describing possible embeddings of the query into the canonical model. The classical atemporal FO-rewritings of DL-Lite OMQs from, say (Poggi et al., 2008; Bienvenu, Kikot, Kontchakov, Podolskii, & Zakharyaschev, 2018), rely upon the key property of DL-Lite that can be characterised as ‘concept locality’ in the sense that the concepts a given individual belongs to in the canonical model only depend on the ABox concepts containing that element, if any, and the roles adjacent to it but not on other individuals. The rewratability results in the temporal case are only possible because concept locality comes together with the (exponential) periodicity of the temporal evolution of each pair of domain elements in the canonical models, which can also be captured in FO for any given Horn OMPIQ.

We have chosen to focus in this investigation on OMQs with positive instance queries because, under the open-world semantics, answers to queries involving negation, implication, or universal quantification are typically rather uninformative. To illustrate, there are no answers, under the open-world semantics, to the query ‘find all journals \( x \) and months \( t \) such that all papers under submission to \( x \) at \( t \) are eventually published in \( x' \),

\[ \varphi(x,t) = \forall y \left( \text{underSubmissionTo}(y,x,t) \rightarrow \Diamond \text{publishedIn}(y,x,t) \right) \],

over any ABox simply because, for any journal and month, we can add a paper under submission that was never published. The epistemic semantics for OMQs with FO-queries proposed by Calvanese, De Giacomo, Lembo, Lenzerini, and Rosati (2007a) and partially realised in SPARQL (Glimm & Ogbuji, 2013) yields more useful answers. In the query above, the answers \((x,t)\) are then such that every paper \( y \) known to be under submission to \( x \) at \( t \) is eventually published in \( x \). We observe that, as expected from the atemporal and 1D temporal cases, for OMQs with FO-queries that use positive temporal concepts as atoms interpreted under the epistemic semantics, the data complexity remains the same as for OMPIQs.

The article is structured as follows. In the remainder of this section, we briefly discuss related work. Section 2 introduces our classification of temporal DL-Lite logics, and Section 3 defines and illustrates OMAQs and OMPIQs with their semantics. Section 4 investigates the combined complexity of checking consistency of knowledge bases in our formalisms. Section 5 identifies classes of FO-rewritable OMAQs by projection to LTL, while Section 6 lifts the obtained results from Horn OMAQs to OMPIQs. We conclude in Section 8 by discussing open problems and directions of further research.

1.1 Related Work

In the previous section, we have already discussed the main related work on atemporal OBDA and on combining DLs and temporal logic. Here, we focus on two related topics: (1) work on OMQ answering over temporal data with discrete linear time and (2) work on temporal deductive databases. For detailed recent surveys of temporal OMQ answering in general we refer the reader to (Artale, Kontchakov, Kovtunova, Ryzhikov, Wolter, & Zakharyaschev, 2017; Ryzhikov, Walega, & Zakharyaschev, 2020).

In temporal OMQ answering, there is a basic distinction between formalisms where temporal constructs are added to both ontology and query languages (as in this article) and those in which only the query language is temporalised while ontologies are given in
some standard atemporal language. The main advantage of keeping the ontology language atemporal is that the increase in the complexity of query answering compared to the atemporal case (observed also in this article) is less severe, if any. Temporal OMQ answering with atemporal ontologies has been introduced and investigated in the context of semantic technologies for situation awareness; a detailed introduction is given by Baader, Borgwardt, Koopmann, Thost, and Turhan (2020b). A basic query language, $\text{LTL-CQ}$, that has been proposed in this framework is obtained from $\text{LTL}$ formulas by replacing occurrences of propositional variables by arbitrary conjunctive queries (CQs). Baader, Borgwardt, and Lippmann (2013, 2015b) analyse the data and combined complexity of answering $\text{LTL-CQ}$s with respect to $\mathcal{ALC}$ and $\mathcal{SHQ}$ ontologies. In this analysis, a fundamental distinction is between $\text{LTL-CQ}$ answering without any temporal constraints on the interpretation of concept and role names and $\text{LTL-CQ}$ answering when some concept and/or role names are declared rigid so that their interpretation does not change over time. For example, $\text{LTL-CQ}$ answering under $\mathcal{ALC}$ ontologies is ExpTime-complete in combined complexity without rigid names but $2\text{ExpTime}$-complete with rigid names.

Borgwardt and Thost (2015a, 2015b) and Baader, Borgwardt, and Lippmann (2015a) investigate the complexity of answering $\text{LTL-CQ}$s with respect to weaker ontology languages such as $\mathcal{EL}$ (see below) and members of the $\mathcal{DL-Lite}$ family, and Borgwardt, Lippmann, and Thost (2013, 2015) study the rewritability properties of $\text{LTL-CQ}$s. Again, a basic distinction is between query answering over interpretations with and without rigid names. Bourgaux, Koopmann, and Turhan (2019) investigate the problem of querying inconsistent data, and Koopmann (2019) proposes an extension to probabilistic data. As the monitoring of systems based on data streams is a major application area of these logics, it is of interest to be able to answer queries without storing the whole ABox. For results on the query language $\text{LTL-CQ}$s, we refer the reader to (Borgwardt et al., 2015). In the monitoring context, the temporal query language STARQL inspired by SPARQL has also been proposed and investigated. Its expressive power is quite different from $\text{LTL-CQ}$s (Özcep & Möller, 2014) as it extends SPARQL with operators to compute answers over time windows and comes with an epistemic semantics that is similar to our epistemic semantics. Another trend is stream reasoning, where the central research problem is efficient query answering over (temporal or atemporal) data instances under the assumption that only restricted portions of data (called windows in the temporal case) are available for querying; see, e.g., (Dell’Aglio, Eiter, Heintz, & Phuoc, 2019; Beck, Dao-Tran, & Eiter, 2018; Ajileye, Motik, & Horrocks, 2021).

We next consider work on OMQ answering with temporal ontologies that combines $\text{LTL}$ with $\mathcal{EL}$ rather than $\mathcal{DL-Lite}$. In contrast to $\mathcal{DL-Lite}$, $\mathcal{EL}$ admits qualified existential restrictions, $\exists R.C$, but does not admit inverse roles. $\mathcal{EL}$ underpins the $\text{OWL2 EL}$ profile of $\text{OWL} 2$ (Baader et al., 2017). In this case, since OMQ answering with atemporal $\mathcal{EL}$ is already P-complete for data complexity, a more expressive target language than $\text{FO(<)}$ is required even for the weakest combinations (unless atemporal $\mathcal{EL}$ is restricted, see below). Gutiérrez-Basulto, Jung, and Kontchakov (2016a) consider combinations of fragments of $\text{LTL}$ and $\mathcal{EL}$ and investigate the complexity and rewritability of atomic queries. A basic difference to the combined languages considered in this article is that no role inclusions are admitted and thus the only temporal expressive power available in the ontology language for roles is to declare them rigid or not (concept inclusions can contain temporal operators). In fact, rigid roles have a significant influence on the complexity of query answering. The
target language for rewritings is the extension datalog\textsubscript{1s} of datalog introduced by Chomicki and Imieliński (1988), and it is shown that answering atomic OMQs is P-complete for data and PSPACE-complete for combined complexity in an $\mathcal{EL}/\mathcal{LTL}$ combination without rigid roles, and PSPACE-complete for data and in ExpTime for combined complexity if rigid roles can only occur on the left-hand side of concept inclusions. It is also shown that, for acyclic ontologies, one obtains rewritability into the extension of FO($<$) with the standard function $+$. Recent work of Borgwardt, Forkel, and Kovtunova (2021) investigates temporal OMQ answering over sparse temporal data in an extension of $\mathcal{ELH}_{\bot}$ with metric temporal operators and different types of rigid concepts: rooted CQs with guarded negation under the minimal-world semantics are shown to be P-complete for data and ExpSPACE-complete for combined complexity.

The extension of standard relational databases to temporal deductive databases has been an active area of research for almost 40 years. It is beyond the scope of this discussion to provide a survey of this field, and we refer the reader instead to the Encyclopaedia of Database Systems for brief introductions into the main notions and developments and pointers to the literature (Liu & Özsu, 2018). Of particular relevance to this article is work on the extension of datalog by terms for the natural numbers or integers representing time points (and possibly constraints over them in the sense of constraint databases (Kuper, Libkin, & Paredaens, 2000)) or by temporal operators. Two basic such languages are datalog\textsubscript{1s}, which extends datalog by allowing the successor function to be applied to 0 and to variables, and Templog, which extends fragments of logic programming languages with the operators $\Box_F$, $\Diamond_F$, and $\circ_F$ (suitably restricted to admit Herbrand models) interpreted over the natural numbers (Baudinet, Chomicki, & Wolper, 1993); for further relevant languages see (Revesz, 2000). We share with this research direction the issue of having to deal with infinite intended models over the integers and thus answering queries over infinite domains. Another main concern of temporal deductive databases has been the problem of finding finite representations of infinite answers (Baudinet et al., 1993). This problem is beyond the scope of this article. As discussed above, answers to OMQs in this article are tuples from the finite domain of the input ABox, as in standard relational databases.

The expressive power of the ontologies and OMQs considered in this article and the queries considered in temporal datalog extensions are incomparable. A major insight of research into OBDA was the need for existential restrictions on the right hand side of concept inclusions. The main datalog extensions considered in deductive temporal databases do not admit such existential quantifiers (except possibly over the time points). On the other hand, plain datalog without additional temporal features is already much more expressive than OMQs with ontologies in Horn DL-Lite dialects without time. As temporal extensions of datalog inherit this expressive power, query answering is much harder in data complexity than in the languages we are investigating. Finally, we mention the recent temporalisations of datalog using the operators of metric temporal logic $\text{MTL}$ rather than $\mathcal{LTL}$ (Brandt, Kalayci, Ryzhikov, Xiao, & Zakharyaschev, 2018; Walega, Cuenca Grau, Kaminski, & Kostylev, 2020; Walega, Cuenca Grau, Kaminski, & Kostylev, 2020a; Cucala, Walega, Cuenca Grau, & Kostylev, 2021).
2. Temporal DL-Lite Logics

We begin by defining the ontology languages we are going to investigate in the context of temporal OBDA via FO-rewriting. The W3C standardised ontology language OWL 2 QL for atemporal OBDA is based on the DL-Lite family (Calvanese et al., 2007b; Artale et al., 2009) of description logics that was designed as a compromise between two aims:

- the logics should be expressive enough to represent basic conceptual modelling constructs (thereby providing a link between relational databases and ontologies), and
- simple enough to guarantee uniform reducibility of OMQ answering to standard database query evaluation (that is, FO-rewritability, which also ensures that answering OMQs can be done in AC0 for data complexity).

Conceptual data models for temporal databases (Chomicki, Toman, & B¨ ohlen, 2001) were analysed and encoded in 2D temporalised DL-Lite logics by Artale et al. (2014), but the rewritability properties of those logics have remained open.

The temporal DL-Lite logics we are about to define are constructed similarly to those of Artale et al. (2014). However, following the standard atemporal OWL 2 QL, we opted not to include cardinality constraints in our languages as their interaction with role inclusions ruins FO-rewritability already in the atemporal case (Artale et al., 2009). On the other hand, we allow role inclusion axioms of the same shape as concept inclusion axioms.

The alphabet of our temporal DL-Lite logics contains countably infinite sets of individual names $a_0, a_1, ...$, concept names $A_0, A_1, ...$, and role names $P_0, P_1, ...$. We then construct roles $S$, temporalised roles $R$, basic concepts $B$, and temporalised concepts $C$ by means of the following grammar:

\[
S ::= P_i \mid P_i^-, \\
R ::= S \mid \Box_F R \mid \Box_P R \mid \bigcirc_F R \mid \bigcirc_P R, \\
B ::= A_i \mid \exists S, \\
C ::= B \mid \Box_F C \mid \Box_P C \mid \bigcirc_F C \mid \bigcirc_P C
\]

with the temporal operators $\Box_F$ (always in the future), $\Box_P$ (always in the past), $\bigcirc_F$ (at the next moment) and $\bigcirc_P$ (at the previous moment). A concept or role inclusion takes the form

\[
\vartheta_1 \sqcap \cdots \sqcap \vartheta_k \sqsubseteq \vartheta_{k+1} \sqcup \cdots \sqcup \vartheta_{k+m}, \tag{5}
\]

where the $\vartheta_i$ are all temporalised concepts of the form $C$ or, respectively, temporalised roles of the form $R$. As usual, we denote the empty $\sqcap$ by $\top$ and the empty $\sqcup$ by $\bot$. When it does not matter if we talk about concepts or roles, we refer to the $\vartheta_i$ as terms. A TBox $\mathcal{T}$ and an RBox $\mathcal{R}$ are finite sets of concept inclusions (CIs, for short) and, respectively, role inclusions (RIs); their union $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$ is called an ontology.

Following (Artale et al., 2021), we classify ontologies depending on the form of their concept and role inclusions and the temporal operators that are allowed to occur in them. Let $c, r \in \{g\text{-}bool, bool, horn, krom, core\}$ and $o \in \{\Box, \bigcirc, \Box\bigcirc\}$. We denote by DL-Lite$^o_{c/r}$ the temporal description logic whose ontologies contain inclusions of the form (5) with the (future and past) operators indicated by $o$ (for example, $o = \Box$ means that only $\Box_F$ and $\Box_P$ can be used); in addition, CIs in DL-Lite$^o_{c/r}$ satisfy the following restrictions on $k$ and $m$ in (5) indicated by $c$: 
(horn) \( m \leq 1 \) if \( c = \text{horn} \),

(krom) \( k + m \leq 2 \) if \( c = \text{krom} \),

(core) \( k + m \leq 2 \) and \( m \leq 1 \) if \( c = \text{core} \),

(g-bool) \( k \geq 1 \) and any \( m \) if \( c = \text{g-bool} \),

(boots) any \( k \) and \( m \) if \( c = \text{bool} \),

and RIs in DL-Lite\(_{c/r}^0\) satisfy analogous restrictions on \( k \) and \( m \) indicated by \( r \). Whenever \( c = r \), we use a single subscript: DL-Lite\(_{c}^0 = DL-Lite_{c/r}^0\). We shall also require the fragment DL-Lite\(_{c/horn}^{0+}\) of DL-Lite\(_{c/horn}^0\) that disallows temporal operators on the left-hand side of RIs. Note that all of our logics feature disjointness inclusions of the form \( \vartheta_1 \cap \vartheta_2 \subseteq \bot \).

However, unlike the standard atemporal DL-Lite logics (Calvanese et al., 2007b; Artale et al., 2009), which can have various types of CIs but allow only core RIs (of the form \( S_1 \subseteq S_2 \) and \( S_1 \cap S_2 \subseteq \bot \)), we treat CIs and RIs in a uniform way and impose restrictions on the clausal structure of CIs and RIs separately (the complexity of reasoning with such atemporal DLs is discussed in Section 2.1 below).

An ABox (or data instance), \( A \), is a finite set of atoms of the form \( A_i(a, \ell) \) and \( P_i(a, b, \ell) \), where \( a \) and \( b \) are individual names and \( \ell \in \mathbb{Z} \) is a timestamp. We write \( P^-(a, b, \ell) \in A \) whenever \( P(b, a, \ell) \in A \). We denote by \( \text{ind}(A) \) the set of individual names that occur in \( A \), by \( \min A \) and \( \max A \) the minimal and maximal integer numbers occurring in \( A \), and set \( \text{tem}(A) = \{ n \in \mathbb{Z} \mid \min A \leq n \leq \max A \} \). To simplify constructions and without much loss of generality, we assume that \( 0 = \min A \) and \( 1 = \max A \), implicitly adding ‘dummies’ such as \( D(a,0) \) and \( D(a,1) \) if necessary, where \( D \) is a fresh concept name (which will never be used in queries). A DL-Lite\(_{c/r}^0\) knowledge base (KB) is a pair \((\mathcal{O}, A)\), where \( \mathcal{O} \) is a DL-Lite\(_{c/r}^0\) ontology and \( A \) an ABox. The size \( |\mathcal{O}| \) of an ontology \( \mathcal{O} \) is the number of occurrences of symbols in \( \mathcal{O} \); the size of a TBox, RBox, ABox, and knowledge base is defined analogously assuming that the numbers (timestamps) in ABoxes are given in unary.

A (temporal) interpretation is a pair \( \mathcal{I} = (\Delta^\mathcal{I}, I^\mathcal{I}(n)) \), where \( \Delta^\mathcal{I} \neq \emptyset \) and, for each \( n \in \mathbb{Z} \),

\[
\mathcal{I}(n) = (\Delta^\mathcal{I}, a^\mathcal{I}_0, \ldots, a^\mathcal{I}_n, \ldots, P^\mathcal{I}_n, \ldots)
\]

is a standard (atemporal) description logic interpretation with \( a^\mathcal{I}_i \in \Delta^\mathcal{I} \), \( A^\mathcal{I}_n \subseteq \Delta^\mathcal{I} \) and \( P^\mathcal{I}_n \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I} \). Thus, we assume that the domain \( \Delta^\mathcal{I} \) and the interpretations \( a^\mathcal{I}_i \in \Delta^\mathcal{I} \) of the individual names are the same for all \( n \in \mathbb{Z} \). (However, we do not adopt the unique name assumption.) The description logic and temporal constructs are interpreted in \( \mathcal{I}(n) \) as follows:

\[
(P^{-})^\mathcal{I}(n) = \{ (u, v) \mid (v, u) \in P^\mathcal{I}_n \}, \quad (\exists S)^\mathcal{I}(n) = \{ u \mid (u, v) \in S^\mathcal{I}(n), \text{ for some } v \},
\]

\[
(\Box_p \vartheta)^\mathcal{I}(n) = \bigcap_{k>n} \vartheta^\mathcal{I}(k), \quad (\Box_p \vartheta)^\mathcal{I}(n) = \bigcap_{k<n} \vartheta^\mathcal{I}(k),
\]

\[
(\Diamond_p \vartheta)^\mathcal{I}(n) = \vartheta^\mathcal{I}(n+1), \quad (\Diamond_p \vartheta)^\mathcal{I}(n) = \vartheta^\mathcal{I}(n-1).
\]

As usual, \( \bot \) is interpreted by \( \emptyset \), and \( \top \) by \( \Delta^\mathcal{I} \) for concepts and by \( \Delta^\mathcal{I} \times \Delta^\mathcal{I} \) for roles. CIs and RIs are interpreted in \( \mathcal{I} \) globally in the sense that inclusion (5) is true in \( \mathcal{I} \) if

\[
\vartheta_1^\mathcal{I}(n) \cap \ldots \cap \vartheta_k^\mathcal{I}(n) \subseteq \vartheta_{k+1}^\mathcal{I}(n) \cup \ldots \cup \vartheta_{k+m}^\mathcal{I}(n), \quad \text{for all } n \in \mathbb{Z}.
\]

6. Here, \( g \) stands for ‘guarded’ (Andréka, Némethi, & van Benthem, 1998).
Given an inclusion $\alpha$, we write $I \models \alpha$ if $\alpha$ is true in $I$. We call $I$ a model of $(O, A)$ and write $I \models (O, A)$ if $I \models \alpha$ for all $\alpha \in O$, $a^I \in A^I(\ell)$ for all $A(a, \ell) \in A$, and $(a^I, b^I) \in P^I(\ell)$ for all $P(a, b, \ell) \in A$. We say that $O$ is consistent if there is an interpretation $I$, a model of $O$, such that $I \models \alpha$, for all $\alpha \in O$; we also say that $A$ is consistent with $O$ if there is a model of $(O, A)$. For an inclusion $\alpha$, we write $O \models \alpha$ if $I \models \alpha$ for every model $I$ of $O$. A concept $C$ is consistent with $O$ if there is a model $I$ of $O$ and $n \in \mathbb{Z}$ such that $C^I(n) \neq \emptyset$; consistency of roles with $O$ is defined analogously.

It is not hard to see that, for any $DL-Lite^O_{c/r}$ ontology $O$, one can construct a $DL-Lite^O_{c/r}$ ontology $O'$, possibly using some fresh concept and role names, such that $O'$ contains no nested temporal operators, the size of $O'$ is linear in the size of $O$, and $O'$ is a model-conservative extension\(^7\) of $O$, which implies that $O$ and $O'$ give the same certain answers to queries (to be defined in Section 3). For example, the inclusion $\Diamond P \Box P.A \subseteq B$ in $O$ can be replaced with two inclusions $\Box P.A \subseteq A'$ and $\Diamond P.A' \subseteq B$, where $A'$ is a fresh name. In what follows and where convenient, we assume without loss of generality that our ontologies do not contain nested temporal operators.

Although we do not include the standard LTL operators $\Diamond F$ (eventually), $\Diamond P$ (some time in the past), $U$ (until) and $S$ (since) in our ontology languages, all of them can be expressed in the $bool$ fragment. For example, $A \sqsubseteq \Diamond P.B$ can be simulated by two $krom$ inclusions with $\Box P$ and a fresh name $A'$: namely, $A \sqcap \Box P.A' \subseteq \bot$ and $\top \sqsubseteq A' \sqcup B$; more details and examples can be found in (Artale et al., 2021). Each of our ontology languages can say that a concept $A$ is expanding (by means of $A \sqsubseteq \Box P.A$ in the languages with $\Box$ and $A \sqsubseteq \Box P.A$ in the languages with $\square$) or rigid (using, in addition, $A \sqsubseteq \Box P.A$ and $A \sqsubseteq \square P.A$, respectively), and similarly for roles. Finally, note that $DL-Lite_{horn/horn^+}$ extends the temporal ontology language $TQL$ (Artale, Kontchakov, Wolter, & Zakharyaschev, 2013b) since $\Diamond P.A \subseteq B$ is equivalent to $A \subseteq \Box P.B$. Thus, convexity axioms such as (1) for both concepts and roles are expressible in $DL-Lite_{horn^+}$:

\[
underSubmissionTo \sqsubseteq \Box P.wasUST, \quad underSubmissionTo \sqsubseteq \Box P.willBeUST, \\
waseUST \sqcap willBeUST \sqsubseteq underSubmissionTo,
\]

where $wasUST$ and $willBeUST$ are two fresh roles names.

### 2.1 Remarks on the Underlying Description Logics

We conclude Section 2 with a brief summary of what is known about the computational complexity of reasoning with the DL fragments of our temporal $DL-Lite$ logics; for more details the reader is referred to (Kontchakov et al., 2020).

Table 1(a) shows the known results on the combined complexity of checking whether a given knowledge base is consistent (with both ontology and ABox regarded as input). Observe first that the most expressive $DL-Lite$ logic, which admits Boolean CIs and RIs, is as complex as the two-variable fragment $FO_2$ of first-order logic (FO). In fact, it has the same expressive power as the language $\mathcal{ALC}^{\top, \id, \sqcap, \neg}$ (Lutz, Sattler, & Wolter, 2001)\(^7\). An ontology $O'$ is a model conservative extension of an ontology $O$ if $O' \models O$, the signature of $O$ is contained in the signature of $O'$, and every model of $O$ can be expanded to a model of $O'$ by providing interpretations of the fresh symbols of $O'$ and leaving the domain and the interpretation of the symbols in $O$ unchanged.

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\(^7\) An ontology $O'$ is a model conservative extension of an ontology $O$ if $O' \models O$, the signature of $O$ is contained in the signature of $O'$, and every model of $O$ can be expanded to a model of $O'$ by providing interpretations of the fresh symbols of $O'$ and leaving the domain and the interpretation of the symbols in $O$ unchanged.
without the identity role, and it also almost captures the class of linear existential disjunctive rules with two variables (Bourhis, Manna, Morak, & Pieris, 2016). By disallowing RIs of the form $\top \sqsubseteq \vartheta_1 \sqcup \cdots \sqcup \vartheta_m$, we obtain a DL-Lite logic whose CIs and RIs are translated into guarded FO-sentences (Andréka et al., 1998): for example, the standard FO-translation of the CI $\top \sqsubseteq A \sqcup B$ is the guarded sentence $\forall x ((x = x) \rightarrow A(x) \lor B(x))$. This logic approximates the two-variable guarded fragment GF$_2$ of FO, inheriting its ExpTime-completeness. The DL-Lite logics with Horn/Krom RIs are polynomially reducible (preserving Horness/Kromness) to propositional logic. Finally, the DL-Lite logics with core RIs are well-documented in the literature: the one with core CIs goes under the monikers DL-Lite$_R$ (Calvanese et al., 2007b) and DL-Lite$_H^{core}$ (Artale et al., 2009); the one with Horn CIs is known as DL-Lite$_R^{\sqcup}$ (Calvanese, De Giacomo, Lembo, Lenzerini, & Rosati, 2006) and DL-Lite$_H^{horn}$ (Artale et al., 2009); the remaining two logics are called DL-Lite$_H^{krom}$ and DL-Lite$_H^{bool}$ (Artale et al., 2009).

In the context of answering ontology-mediated queries to be discussed in the sequel, we are interested in the data complexity (when only the ABox is regarded as input) of the instance checking problem for concepts constructed from concept names using $\sqcap$, $\sqcup$ and the existential restriction $\exists S.C$ (in other words, $\mathcal{ELUI}$-concepts). The coNP upper bound in Table 1(b) follows, e.g., from the results on the two-variable FO with counting quantifiers (Pratt-Hartmann, 2009), the coNP lower bound for the ontology $\{\top \sqsubseteq A \sqcup B\}$ with a single Krom CI was established by Schaefer (1993); see also Section 6. In fact, for DL-Lite ontologies with Boolean CIs and RIs, this lower bound already holds for atomic queries (concept names) as we can encode instance queries in the ontology. This Krom CI can obviously be modelled with the help of a Krom RI; using the guarded RI $P \sqsubseteq R \sqcup Q$, one can capture the CI $A \sqsubseteq B \sqcup C$, for which instance checking is also coNP-hard; see, e.g., (Gerasimova, Kikot, Kurucz, Podolskii, & Zakharyaschev, 2020). The $\text{AC}^0$ upper bound (FO-rewritability, to be more precise) for Horn CIs and core RIs is a classical result of Calvanese et al. (2006, 2007b), which can be readily extended to Horn RIs (Kontchakov et al., 2020).

| role concept inclusions | role concept inclusions |
|------------------------|------------------------|
| $\text{bool}$          | $\text{bool}$          |
| $\text{g-bool}$        | $\text{g-bool}$        |
| $\text{krom}$          | $\text{coNP}$          |
| $\text{horn}$          | $\text{coNP}$          |
| $\text{core}$          | $\text{in AC}^0$       |

Table 1: Underlying non-temporal description logics: (a) combined complexity of consistency checking; (b) data complexity of instance checking.
3. Ontology-Mediated Atomic and Positive Instance Queries

Using (variations) of LTL to query temporal data has a long tradition, going back more than 30 years (Chomicki, 1994). In fact, by Kamp’s famous theorem (Kamp, 1968), propositional temporal logic over discrete (and more general Dedekind complete) linear orders has the same expressive power as monadic first-order logic, and so LTL supplies a user-friendly query language without sacrificing expressivity. An analogous result can be proved for the positive fragment of LTL (Artale et al., 2021). Unfortunately, expressive completeness of LTL is lost in the two-dimensional case when compared with two-sorted first-order logic; see (Toman & Niwinski, 1996) in the continuous case and (Abiteboul, Herr, & Van den Bussche, 1996) in the discrete case. This non-expressive completeness result is less relevant under the open-world semantics of OBDA, however, since the use of negation, implication, or universal quantification hardly yields satisfactory answers to queries in OBDA. For those connectives, an epistemic semantics appears to be more appropriate.

We now introduce our basic language for querying temporal knowledge bases. It consists of positive temporal concepts, \( \kappa \), and positive temporal roles, \( \rho \), that are defined by the following grammar:

\[
\kappa ::= \top \mid A_k \mid \exists S \kappa \mid \kappa_1 \sqcap \kappa_2 \mid \kappa_1 \sqcup \kappa_2 \mid \text{op}_1 \kappa \mid \kappa_1 \text{op}_2 \kappa_2,
\]

\[
\rho ::= S \mid g_1 \sqcap g_2 \mid g_1 \sqcup g_2 \mid \text{op}_1 \rho \mid g_1 \text{op}_2 g_2,
\]

where \( \text{op}_1 \in \{ \odot_F, \odot_P, \Box_F, \Box_P, \Diamond_F, \Diamond_P \} \) and \( \text{op}_2 \in \{ U, S \} \). Let \( I = (\Delta^I, \mathcal{I}^I) \) be an interpretation. The extensions \( \mathcal{I}^I(n) \) of \( \kappa \) in \( I \), for \( n \in \mathbb{Z} \), are computed using (7)–(8) and the following:

\[
\begin{align*}
\top^{I(n)} &= \Delta^I, \\
(\kappa_1 \sqcap \kappa_2)^{I(n)} &= \mathcal{I}^{I(n)}(\kappa_1 \sqcap \kappa_2), \quad (\kappa_1 \sqcup \kappa_2)^{I(n)} = \mathcal{I}^{I(n)}(\kappa_1 \sqcup \kappa_2), \\
(\exists S \kappa)^{I(n)} &= \{ u \in \Delta^I : (u, v) \in S^{I(n)} \text{ for some } v \in \mathcal{I}^{I(n)} \}, \\
(\odot_F \kappa)^{I(n)} &= \bigcup_{k > n} \mathcal{I}^{I(n)}(\kappa), \quad (\odot_P \kappa)^{I(n)} = \bigcup_{k > n} \mathcal{I}^{I(n)}(\kappa), \\
(\kappa_1 U \kappa_2)^{I(n)} &= \bigcup_{k > n} \left( \mathcal{I}^{I(n)}(\kappa_1) \right) \cap \bigcap_{n < m < k} \mathcal{I}^{I(n)}(\kappa_2), \quad (\kappa_1 S \kappa_2)^{I(n)} = \bigcup_{k > n} \left( \mathcal{I}^{I(n)}(\kappa_1) \right) \cap \bigcap_{n < m < k} \mathcal{I}^{I(n)}(\kappa_2).
\end{align*}
\]

The definition of \( \rho^{I(n)} \) is analogous. Note that positive temporal concepts \( \kappa \) and roles \( \rho \) include all temporalised concepts \( A \) and roles \( R \), respectively (\( \exists S \) is a shortcut for \( \exists S. \top \)).

A DL-Lite\(_{c/r}^O\) ontology-mediated positive instance query (OMPIQ) is a pair of the form \( q = (O, \kappa) \) or \( q = (O, \rho) \), where \( O \) is a DL-Lite\(_{c/r}^O\) ontology, \( \kappa \) is a positive temporal concept and \( \rho \) a positive temporal role (which can use all temporal operators, not necessarily only those in \( o \)). If \( \kappa \) is a basic concept (i.e., \( A \) or \( \exists S \)) and \( \rho \) a role, then we refer to \( q \) as an ontology-mediated atomic query (OMAQ).

A certain answer to an OMQ (\( O, \kappa \)) over an ABox \( A \) is a pair \( (a, \ell) \in \text{ind}(A) \times \text{tem}(A) \) such that \( a^I \in \kappa^{I(\ell)} \) for every model \( I \) of \( (O, A) \). A certain answer to \( (O, \rho) \) over \( A \) is a triple \( (a, b, \ell) \in \text{ind}(A) \times \text{ind}(A) \times \text{tem}(A) \) such that \( (a^I, b^I) \in \rho^{I(\ell)} \) for every model \( I \) of \( (O, A) \). The set of all certain answers to \( q \) over \( A \) is denoted by \( \text{ans}(q, A) \). As a technical tool in our constructions, we also require ‘certain answers’ in which \( \ell \) ranges over the whole \( \mathbb{Z} \) rather than only the active temporal domain \( \text{tem}(A) \); we denote the set of such certain answers over \( A \) and \( \mathbb{Z} \) by \( \text{ans}^\mathbb{Z}(q, A) \).
Example 1. Suppose $\tau = \exists P . \bigcirc_{p} \exists Q^- . B$, $g = P \cap \bigcirc_{p} Q$ and

$$T = \{ A \subseteq \exists P \}, \quad R = \{ P \subseteq \bigcirc_{p} Q \}.$$  

For $A_1 = \{ P(a, b, 0), Q(c, b, 1), B(c, 1) \}$ and $q_1 = (\emptyset, \tau)$, we have $\text{ans}(q_1, A_1) = \{(a, 0)\}$ because $a^T \in \tau^{(0)}$ for any model $I$ of $A_1$; see Fig. 1a. For $A_2 = \{ P(a, b, 0), B(a, 1) \}$ and $q_2 = (R, \tau)$, we also have $\text{ans}(q_2, A_2) = \{(a, 0)\}$ because $(a^T, b^T) \in Q^{(1)}$, for every model $I$ of $A_2$ and $R$; see Fig. 1b. For $A_3 = \{(a, 0), B(a, 1)\}$ and $q_3 = (T \cup R, \tau)$, we again have $\text{ans}(q_3, A_3) = \{(a, 0)\}$ because, in every model $I$ of $(T \cup R, A_3)$, there exists $u \in \Delta^I$ with $(a^T, u) \in P^{(0)}$ and $(a^T, u) \in Q^{(1)}$; see Fig. 1c. Finally, for $A_4 = \{(a, 0)\}$ and $q_4 = (T \cup R, g)$, we obviously have $\text{ans}(q_4, A_4) = \{(a, b, 0)\}$, while $\text{ans}(q_5, A_4) = \emptyset$ for $q_5 = (T \cup T', g)$, where $T' = \{ \exists P^- \subseteq \bigcirc_{p} \exists Q^- \}$.

By the OMPIQ answering problem for DL-Lite$_{c/f}^\circ$ we understand the decision problem for the set $\text{ans}(q, A)$, where $q$ is a DL-Lite$_{c/f}^\circ$ OMPIQ and $A$ an ABox. In the context of OBDA, we are usually interested in the data complexity of this problem when $q$ is considered to be fixed or negligibly small compared to data $A$, which is regarded as the only input to the problem (Vardi, 1982). In the atemporal case, the data complexity of answering conjunctive queries mediated by ontologies from the DL-Lite family is well understood (Calvanese et al., 2007b; Artale et al., 2009): it ranges from $\text{AC}^0$—which guarantees FO-rewritability (see below)—to $P$ and further to $\text{coNP}$ (see also Table 1). The data complexity of answering LTL OMQs is either $\text{AC}^0$ or $\text{NC}^1$ (Artale et al., 2021); we remind the reader that

$$\text{AC}^0 \subsetneq \text{NC}^1 \subsetneq L \subseteq P \subseteq \text{coNP}.$$  

As our aim here is to identify families of OMPIQs answering which can be done in $\text{AC}^0$ or $\text{NC}^1$, we now look at these two complexity classes in more detail.

Let $A$ be an ABox with $\text{ind}(A) = \{ a_0, \ldots, a_m \}$. Without loss of generality, we always assume that $\max A \geq m$ (if this is not so, we simply add a ‘dummy’ $D(a_0, m)$ to $A$). We represent $A$ as a first-order structure $\mathfrak{A}$ with domain $\text{tem}(A)$ ordered by $<$ such that

$$\mathfrak{A} \models A(k, \ell) \iff A(a_k, \ell) \in A \quad \text{and} \quad \mathfrak{A} \models P(k, k', \ell) \iff P(a_k, a_{k'}, \ell) \in A,$$  

Figure 1: Fragments of models in Example 1.

8. The fine-grained combined complexity of answering conjunctive queries mediated by OWL 2 QL ontologies was investigated by Bienvenu et al. (2018).
for any concept and role names $A$, $P$ and any $k, k', \ell \in \text{tem}(A)$. To simplify notation, we often identify an individual name $a_k \in \text{ind}(A)$ with its number representation $k \in \text{tem}(A)$. As a technical tool in our constructions, we also use infinite first-order structures $\mathcal{S}^\infty_A$ with domain $\mathbb{Z}$ that are defined in the same way as $\mathcal{S}_A$ but over the whole $\mathbb{Z}$.

The structure $\mathcal{S}_A$ represents a temporal database over which we can evaluate first-order formulas (queries) with data atoms of the form $A(x,t)$ and $P(x,y,t)$ as well as atoms with the order $t_1 < t_2$ and congruence $t \equiv 0 \pmod{n}$, for $n > 1$. It is well-known (Immerman, 1999) that the evaluation problem for FO-formulas with numerical predicates (such as $+$ and $\times$) is in non-uniform $\text{AC}^0$ for data complexity, the class of languages computable by bounded-depth polynomial-size circuits with unary NOT-gates and unbounded fan-in AND- and OR-gates. In this paper, we require FO($<$)-formulas, which only use data and order atoms, and FO($<$, $\equiv$)-formulas, which in addition may contain congruence atoms. Evaluation of FO($<$, $\equiv$)-formulas is known to be in LogTime-uniform $\text{AC}^0$ for data complexity (Immerman, 1999). We also use FO(RPR)-formulas, that is, FO-formulas extended with relational primitive recursion (RPR), whose evaluation is in $\text{NC}^1$ for data complexity (Compton & Laflamme, 1990), the class computed by a family of polynomial-size logarithmic-depth circuits with gates of at most two inputs. We remind the reader that, using RPR, we can construct formulas $\Phi(z, z_1, \ldots, z_n)$ such as

$$\begin{align*}
Q_1(z_1, t) &\equiv \Theta_1(z_1, t, Q_1(z_1, t-1), \ldots, Q_n(z_n, t-1)) \\
\cdots \\
Q_n(z_n, t) &\equiv \Theta_n(z_n, t, Q_1(z_1, t-1), \ldots, Q_n(z_n, t-1))
\end{align*}$$

where the part of $\Phi$ within [...] defines recursively, via the FO(RPR)-formulas $\Theta_i$, the interpretations of the predicates $Q_i$ in the FO(RPR)-formula $\Psi$. The recursion starts at $t = 0$ assuming that $Q_i(z_i, -1)$ is false for all $Q_i$ and $z_i$, $1 \leq i \leq n$. Thus, the truth value of $Q_i(z_i, 0)$ is computed by substituting falsehood $\bot$ for all $Q_i(z_i, -1)$. We allow the relation variables $Q_i$ to occur in only one recursive definition [...], so it makes sense to write $\mathcal{S}_A \models Q_i(n_i, k)$, for any tuple $n_i$ in tem($A$) and $k \in \text{tem}(A)$, if the computed value is ‘true’. Using thus defined truth-values, we compute inductively the truth-relation $\mathcal{S}_A \models \Psi(n, n_1, \ldots, n_n)$, and so $\mathcal{S}_A \models \Phi(n, n_1, \ldots, n_n)$, as usual in first-order logic.

**Definition 2.** Let $\mathcal{L}$ be one of the three classes of FO-formulas introduced above. A constant-free $\mathcal{L}$-formula $Q(x,t)$ is said to be an $\mathcal{L}$-rewriting of an OMPIQ $q = (O, \prec)$ if ans($q, A) = \{ (a, \ell) \in \text{ind}(A) \times \text{tem}(A) \mid \mathcal{S}_A \models Q(a, \ell) \}$, for any ABox $A$. Similarly, a constant-free $\mathcal{L}$-formula $Q(x,y,t)$ is an $\mathcal{L}$-rewriting of an OMPIQ $q = (O, \prec)$ if we have ans($q, A) = \{ (a,b,\ell) \in \text{ind}(A) \times \text{ind}(A) \times \text{tem}(A) \mid \mathcal{S}_A \models Q(a,b, \ell) \}$, for any ABox $A$.

It follows from the definition that answering FO($<$)- and FO($<$, $\equiv$)-rewritable OMPIQs is in $\text{AC}^0$ for data complexity, and answering FO(RPR)-rewritable OMPIQs is in $\text{NC}^1$. We illustrate these three types of FO-rewriting by instructive examples.

**Example 3.** Consider the $\text{DL-Lite}_{\text{core}}$ OMPIQs $q = (T \cup R, Q)$ and $q' = (T \cup R, \prec)$, where $T$, $R$ and $\prec$ are as in Example 1. It is readily seen that $Q(x, y, t) = Q(x, y, t) \lor P(x, y, t-1)$ is an FO($<$)-rewriting of $q$, where $P(x, y, t-1)$ abbreviates $\exists t' (P(x, y, t') \land (t' < t) \land \neg \exists s (t' < s) \land (s < t))$;
see (Artale et al., 2021, Remark 3). The ABoxes $\mathcal{A}_1$--$\mathcal{A}_3$ from Example 1 show three sets of atoms at least one of which must be present in an ABox $\mathcal{A}$ for $(a,0)$ to be a certain answer to $q'$ over $\mathcal{A}$; see Fig. 1. This observation implies that an FO($<$)-rewriting of $q'$ can be defined as $Q'(x, t) = Q_1(x, t) \lor Q_2(x, t) \lor Q_3(x, t)$, where

$$Q_1(x, t) = \exists y, z \ (P(x, y, t) \land Q(z, y, t + 1) \land B(z, t + 1)),$$

$$Q_2(x, t) = \exists y \ (P(x, y, t) \land B(x, t + 1)),$$

$$Q_3(x, t) = A(x, t) \land B(x, t + 1),$$

and $B(x, t + 1)$ and $B(z, t + 1)$ are shortcuts similar to $P(x, y, t - 1)$ above.

**Example 4.** Consider the DL-Lite$^\Box_{\text{horn/core}}$ OMQ $q = (T \cup R, A \sqcap \exists S)$ with

$$\mathcal{T} = \{ \exists R \sqsubseteq \exists S \}, \quad \mathcal{R} = \{ S \sqsubseteq \diamond_p T, S \sqsubseteq \square_p \ominus_p \diamond_p P, \ \diamond_p \ominus_p \diamond_p T \cap \square_p P \sqsubseteq R \}.$$ 

We first observe that $\mathcal{R} \models S \sqsubseteq \ominus_p \diamond_p R$, and so $\mathcal{T} \cup \mathcal{R} \models \exists S \sqsubseteq \ominus_p \diamond_p \exists S$, although we do not have $\mathcal{T} \cup \mathcal{R} \models S \sqsubseteq \ominus_p \diamond_p S$; see Fig. 2. Now, for every ABox $\mathcal{A}_n = \{ S(a, b, 0), A(a, n) \}$, $n \geq 0$, we have $\text{ans}(q, \mathcal{A}_n) = \{(a, n)\}$ iff $n$ is even; otherwise $\text{ans}(q, \mathcal{A}_n) = \emptyset$. This suggests the following FO($<$, $\exists$)-rewriting of $q$:

$$Q(x, t) = A(x, t) \land \exists y, s \ ([R(x, y, s) \lor S(x, y, s)] \land (t - s \in 0 + 2\mathbb{N})).$$

where $p + q\mathbb{N}$ is the set $\{ p + qk \mid k \geq 0 \}$, for $p, q \geq 0$, and $t - s \in 0 + 2\mathbb{N}$ is an FO($<$, $\exists$)-formula $\varphi(t, s)$ such that $\mathcal{S}_A \models \varphi(n, m)$ iff $n - m \in p + q\mathbb{N}$; see (Artale et al., 2021, Remark 3). Note, however, that $q$ is not FO($<$)-rewritable since properties such as ‘the size of the domain is even’ are not definable by FO($<$)-formulas, which can be established using a standard Ehrenfeucht-Fraisse argument (Straubing, 1994; Libkin, 2004). On the other hand, all LTL$^\Box_{\text{horn}}$ OMQs are FO($<$)-rewritable (Artale et al., 2021).

**Example 5.** As shown in (Artale et al., 2021, Example 5), the OMAQ $q = (O, B_0)$ with

$$O = \{ \ominus_p B_k \cap A_0 \sqsubseteq B_k, \ \ominus_p B_{1-k} \cap A_1 \sqsubseteq B_k \mid k = 0, 1 \}$$

encodes PARITY in the following sense. For any binary word $e = (e_1, \ldots, e_n) \in \{0,1\}^n$, let $\mathcal{A}_e = \{ B_0(a, 0) \} \cup \{ A_{e_i}(a, i) \mid 0 < i \leq n \}$. Then $(a, n)$ is a certain answer to $q$ over $\mathcal{A}_e$ iff
the number of 1s in \( e \) is even. As Parity is not in \( \text{AC}^0 \) (Furst, Saxe, & Sipser, 1984), \( q \) is not rewritable to any FO-formula with numeric predicates. An FO(RPR)-rewriting of \( q \) inspired by (Artale et al., 2021, Example 5) is shown below:

\[
Q(x,t) = \left[ \begin{array}{c}
Q_0(x,t) \equiv \Theta_0 \\
Q_1(x,t) \equiv \Theta_1
\end{array} \right] Q_0(x,t),
\]

where

\[
\Theta_k(x,t,Q_0(x,t-1),Q_1(x,t-1)) = \\
B_k(x,t) \lor (Q_k(x,t-1) \land A_0(x,t)) \lor (Q_{1-k}(x,t-1) \land A_1(x,t)), \text{ for } k = 0, 1.
\]

We obtain our FO-rewritability results for temporal \( \text{DL-Lite} \) OMPIQs in three steps. First, we reduce FO-rewritability of OMAQs to FO-rewritability of \( \perp \)-free OMAQs (whose ontology is consistent with all ABoxes). Then, we further reduce, where possible, FO-rewritability of \( \perp \)-free OMAQs to FO-rewritability of \( \text{LTL} \) OMAQs, which has been thoroughly investigated by Artale et al. (2021). This ‘projection’ of a 2D formalism onto one of its ‘axes’ relies on the fact that atemporal \( \text{DL-Lite} \) ontologies with Horn or Krom RIs can be encoded in the one-variable fragment of first-order logic (Artale et al., 2009). The third step reduces FO-rewritability of some \( \text{DL-Lite}^\perp_{\text{horn}} \) OMPIQs to FO-rewritability of OMAQs using model-theoretic considerations.

In the remainder of Section 3, we first briefly remind the reader of \( \text{LTL} \) OMAQs and OMPIQs (Artale et al., 2021) and make two basic observations on the reducibility of answering \( \text{DL-Lite}^\perp_{\text{horn}} \) OMAQs that do not contain interacting concepts and roles to answering \( \text{LTL}^\perp_\circ \) and \( \text{LTL}^\perp_\circ \) OMAQs.

### 3.1 LTL OMAQs and OMPIQs

\( \text{LTL}^\perp_\circ \) OMAQs and OMPIQs can be defined as \( \text{DL-Lite}^\perp_\circ \) OMAQs and OMPIQs of the form \( q = (\mathcal{O}, A) \) and, respectively, \( q = (\mathcal{O}, \tau) \) that do not contain occurrences of roles. In this case, \( O \) is referred to as an \( \text{LTL}^\perp_\circ \) ontology. We assume (without loss of generality) that an \( \text{LTL} \) ABox \( \mathcal{A} \) has one fixed individual name \( a \) and consists of assertions of the form \( C(a, \ell) \) with a role-free temporalised concept \( C \). In a temporal interpretation \( \mathcal{I} \) for \( \text{LTL} \), we have \( \Delta^\mathcal{I} = \{ a \} \) and \( P^\mathcal{I}_i(n) = \emptyset \) for all role names \( P_i \) and \( n \in \mathbb{Z} \). To simplify notation, we write \( \mathcal{I} \models C(n) \) instead of \( a^\mathcal{I} \in C^{\mathcal{I}(n)} \) and \( C(\ell) \in \mathcal{A} \) instead of \( C(a, \ell) \in \mathcal{A} \).

The key property of \( \text{LTL}^\perp_{\text{horn}} \) we need is that every consistent \( \text{LTL}^\perp_{\text{horn}} \) KB \( (\mathcal{O}, \mathcal{A}) \) has a canonical model \( C_{\mathcal{O},\mathcal{A}} \), which can be defined as the intersection of all of its models:

\[
C_{\mathcal{O},\mathcal{A}} \models A_i(n) \iff \mathcal{I} \models A_i(n), \text{ for all models } \mathcal{I} \text{ with } \mathcal{I} \models (\mathcal{O}, \mathcal{A}), \quad (9)
\]

for every concept name \( A_i \) (Artale, Kontchakov, Ryzhikov, & Zakharyaschev, 2013a). (An alternative, syntactic, definition of the canonical models is given in Section 6.1 below.)

Denote by \( \text{sub}_\mathcal{O} \) the set of all (sub)concepts occurring in \( \mathcal{O} \) and their negations. By \( \tau_{\mathcal{O},\mathcal{A}}(n) \) we denote the set of all \( C \in \text{sub}_\mathcal{O} \) with \( C_{\mathcal{O},\mathcal{A}} \models C(n) \). It is known that every satisfiable \( \text{LTL} \)-formula is satisfied in an ultimately periodic model (Sistla & Clarke, 1985). Since \( C_{\mathcal{O},\mathcal{A}} \) is the minimal model in the sense of (9), it is also ultimately periodic, which is formalised in the next lemma that follows immediately from (Artale et al., 2021, Lemmas 19 and 21). This periodic structure will be used for defining our FO-rewritings in the sequel.
Lemma 6. (i) For any consistent LTL\textsuperscript{horn}_\text{\textcircled{}} KB (O, A) with \{C | C(\ell) \in A\} \subseteq \text{sub}_O, there are positive integers s\textsubscript{O,A} \leq 2^{\|O\|} and p\textsubscript{O,A} \leq 2^{2\|O\|} such that

\[
\tau_{O,A}(n) = \tau_{O,A}(n - p_{O,A}), \text{ for } n \leq \min A - s_{O,A}, \\
\tau_{O,A}(n) = \tau_{O,A}(n + p_{O,A}), \text{ for } n \geq \max A + s_{O,A}.
\]

(10)

If O is an LTL\textsuperscript{horn} ontology, one can take s\textsubscript{O,A} \leq \|O\| and p\textsubscript{O,A} = 1.

(ii) For any LTL\textsuperscript{horn} ontology O, there are positive integers s\textsubscript{O} \leq 2^{\|O\|} and p\textsubscript{O} \leq 2^{2\|O\|} such that, for any ABox A consistent with O and with \{C | C(\ell) \in A\} \subseteq \text{sub}_O, we have

\[
\tau_{O,A}(n) = \tau_{O,A}(n - p_{O}), \text{ for } n \leq \min A - s_{O}, \\
\tau_{O,A}(n) = \tau_{O,A}(n + p_{O}), \text{ for } n \geq \max A + s_{O}.
\]

(11)

If O is an LTL\textsuperscript{horn} ontology, one can take s\textsubscript{O} \leq \|O\| and p\textsubscript{O} = 1.

3.2 Projecting Temporal DL-Lite OMAQs to LTL: Initial Observations

Consider a DL-Lite\textsubscript{\tau} OMAQ q = (T, B) such that the ontology does not have role inclusions and the TBox T does not contain \bot. Define an LTL\textsuperscript{\tau} OMAQ \textsc{q}\textsuperscript{\tau} = (\textsc{T}\textsuperscript{\tau}, B\textsuperscript{\tau}) as follows. For basic concepts A and \exists S, set

\[ A\textsuperscript{\tau} = A \quad \text{and} \quad (\exists S)\textsuperscript{\tau} = E_S, \]

where E_S is a fresh concept name, the surrogate of \exists S. The TBox \textsc{T}\textsuperscript{\tau} is obtained from T by replacing every basic concept B with B\textsuperscript{\tau}. The LTL-translation C\textsuperscript{\tau} of a temporalised concept C is defined analogously. Given an ABox A and a \in \text{ind}(A), denote by A\textsubscript{\textsuperscript{\tau}} an LTL ABox that consists of all ground atoms A(\ell) for A(a, \ell) \in A and E_S(a, \ell) for S(a, b, \ell) \in A (remembering that S(\text{\textsuperscript{\tau}}(a, b, \ell) \in A). We claim that

\[ \text{ans}^Z(q, A) = \{ (a, n) | a \in \text{ind}(A) \text{ and } n \in \text{ans}^Z(q\textsuperscript{\tau}, A\textsubscript{\textsuperscript{\tau}}) \}. \]

(12)

Indeed, it suffices to show that, for any a \in \text{ind}(A) and n \in \mathbb{Z}, there is a model \mathcal{I} of (T, A) with a \notin B\textsuperscript{\tau}(n) iff there is a model \mathcal{I}_a of (\textsc{T}\textsuperscript{\tau}, A\textsubscript{\textsuperscript{\tau}}) with \mathcal{I}_a \models B\textsuperscript{\tau}(n). Direction (⇒) is easy: every model \mathcal{I} of (T, A) gives rise to the LTL-interpretations \mathcal{I}_a, for a \in \text{ind}(A), defined by taking \mathcal{I}_a \models A(n) iff a \in A\textsuperscript{\tau}(n), and \mathcal{I}_a \models E_S(n) iff a \in \exists S\textsuperscript{\tau}(n). It is readily seen by induction that, for any temporalised concept C and any n \in \mathbb{Z}, we have a \in C\textsuperscript{\tau}(n) iff \mathcal{I}_a \models C\textsuperscript{\tau}(n), and so the \mathcal{I}_a are models of (\textsc{T}\textsuperscript{\tau}, A\textsubscript{\textsuperscript{\tau}}).

For (⇐), a bit more work is needed. For each b \in \text{ind}(A) different from the given a, we take any model \mathcal{I}_b of (\textsc{T}\textsuperscript{\tau}, A\textsubscript{\textsuperscript{\tau}}). Further, for any role S and any n \in \mathbb{Z}, we take a model \mathcal{I}_{S,n} of (T, \{\exists S\textsuperscript{\tau}^{-}(w_S, n)\}), for some fresh individual name w_S. These models exist because T is \bot-free. Finally, define an interpretation \mathcal{I} by taking, for all b, c \in \text{ind}(A) and n \in \mathbb{Z}: b \in A\textsuperscript{\tau}(n) iff \mathcal{I}_b \models A(n), (b, c) \in S\textsuperscript{\tau}(n) iff S(b, c, n) \in A, and (b, w_S) \in S\textsuperscript{\tau}(n) iff \mathcal{I}_b \models E_S(n). Note that b \in \exists S\textsuperscript{\tau}(n) iff \mathcal{I}_n \models E_S(n). Using this fact, it is not hard to show by induction that, for any temporalised concept C and any n \in \mathbb{Z}, we have b \in C\textsuperscript{\tau}(n) iff \mathcal{I}_b \models C\textsuperscript{\tau}(n). It follows that \mathcal{I} is the model of (T, A) as required.

Equality (12) gives us the following:
Proposition 7. Let $\mathcal{L}$ be one of FO($<$), FO($<, \forall$) or FO(RPR). A $\bot$-free DL-Lite$_{c/r}^g$ OMAQ $(T, B)$ is $\mathcal{L}$-rewritable whenever the LTL$^g_\mathcal{L}$ OMAQ $(T^\dagger, B^\dagger)$ is $\mathcal{L}$-rewritable.

Proof. We obtain an $\mathcal{L}$-rewriting of $(T, B)$ from an $\mathcal{L}$-rewriting $Q^\dagger(t)$ of $(T^\dagger, B^\dagger)$ by replacing every variable $s$ with $A(s, x, s)$, every $E_P(s)$ with $\exists y P(x, y, s)$ and every $E_{P^-}(s)$ with $\exists y P(y, x, s)$. In the case of FO(RPR), we additionally replace every $Q(t_1, \ldots, t_k)$ for a relation variable $Q$, with $Q(x, t_1, \ldots, t_k)$. $\blacksquare$

Answering $\bot$-free DL-Lite$_{c/r}^g$ OMPIQs 
$q = (T \cup R, g)$ with a positive temporal role $g$ can also be reduced to answering LTL$^g_{\mathcal{L}}$ OMPIQs. We assume that $R$ is closed under taking the inverses of roles in RIs in the sense that, together with every RI (say, $\diamond_P P_1 \cap \Box_P P_2 \subseteq P_3$), $R$ contains the corresponding RI for the inverse roles ($\diamond_P P_1 \cap \Box_P P_2 \subseteq P_3$ in our example). For every role name $S$, we introduce two concepts names $A_P$ and $A_{P^-}$. Let $\mathcal{R}^\dagger$ and $g^\dagger$ be the results of replacing each role $S$ in them with $A_S$, and let $q^\dagger = (\mathcal{R}^\dagger, g^\dagger)$. Given any temporal DL-Lite ABox $A$ and $a, b \in \text{ind}(A)$, denote by $A_{a,b}^\dagger$ the LTL ABox with the atoms $A_S(\ell)$ such that $S(a, b, \ell) \in A$. We claim that

$$\text{ans}^Z(q, A) = \{(a, b, n) \mid a, b \in \text{ind}(A) \text{ and } n \in \text{ans}^Z(q^\dagger, A_{a,b}^\dagger)\}.$$  

Indeed, it is easy to ‘reformulate’ models of $(\mathcal{R}, A)$ in terms of models of $(\mathcal{R}^\dagger, A_{a,b}^\dagger)$ and (since $R$ is $\bot$-free) the other way round. As $T$ is $\bot$-free, it cannot add more tuples to the left-hand side of (13). Thus, we obtain the following transfer result for OMPIQs:

Proposition 8. Let $\mathcal{L}$ be one of FO($<$), FO($<, \forall$) or FO(RPR). A $\bot$-free DL-Lite$_{c/r}^g$ OMPIQ $(T \cup R, g)$ is $\mathcal{L}$-rewritable whenever the LTL$^g_{\mathcal{L}}$ OMPIQ $(\mathcal{R}^\dagger, g^\dagger)$ is $\mathcal{L}$-rewritable.

Proof. Suppose $Q^\dagger(t)$ is an $\mathcal{L}$-rewriting of $(\mathcal{R}^\dagger, g^\dagger)$. We can assume that $Q^\dagger(t)$ is constructed from atoms of the form $A_P(s)$ and $A_{P^-}(s)$, with $P$ occurring in $(T \cup R, g)$, and the built-in predicates and constructs of $\mathcal{L}$. We obtain the required $\mathcal{L}$-rewriting $Q(x, y, t)$ by replacing every $A_P(s)$ in $Q^\dagger(t)$ with $P(x, y, s)$, every $A_{P^-}(s)$ with $P(y, x, s)$ in the case of FO(RPR), by additionally replacing every occurrence of $Q(t_1, \ldots, t_k)$, for a relation variable $Q$, with $Q(x, y, t_1, \ldots, t_k)$. $\blacksquare$

In Section 5, we show that answering OMAQs in temporal DL-Lite can be reduced to answering OMAQs with $\bot$-free ontologies. This reduction relies on our ability to decide consistency of temporal DL-Lite KBs, which will be studied in the next section.

4. Consistency of Temporal DL-Lite Knowledge Bases

Observe first that the translation $\dagger$ defined in Section 3.2 reduces the consistency problem for temporal DL-Lite KBs that do not contain CIs to LTL satisfiability (which is known to be PSPACE-complete):

Proposition 9. A DL-Lite$_{c/r}$ KB $(\mathcal{R}, A)$ is inconsistent iff there are $a, b \in \text{ind}(A)$ such that the LTL$^g_{\mathcal{L}}$ KB $(\mathcal{R}^\dagger, A_{a,b}^\dagger)$ is inconsistent.

On the other hand, both consistency and OMAQ answering with DL-Lite$_{g}$-boot, and so DL-Lite$_{g}$-boot ontologies turn out to be undecidable (even for data complexity):
It is readily checked that

\[ q \text{ is consistent with } \mathcal{A} \iff \{I(a,0)\} \text{ is consistent with } \mathcal{O}_{\mathcal{T}} \text{ iff } \mathcal{T} \text{ can tile the } \mathbb{N} \times \mathbb{N} \text{ grid.} \]

(ii) Using the representation of the universal Turing machine by means of tiles (Börger, Grädel, & Gurevich, 1997), we obtain a set \( \mathcal{U} \) of tile types for which the following problem is undecidable: given a finite sequence of tile types \( i_0, \ldots, i_n \), decide whether \( \mathcal{U} \) can tile the \( \mathbb{N} \times \mathbb{N} \) grid so that tiles of types \( i_0, \ldots, i_n \) are placed on \( (0,0), \ldots, (n,0) \), respectively. Given such \( i_0, \ldots, i_n \), we take the ABox \( \mathcal{A} = \{I(a,0), R_{i_0}(a,b,0), \ldots, R_{i_n}(a,b,n)\} \). Then \( \mathcal{U} \) can tile \( \mathbb{N} \times \mathbb{N} \) with \( i_0, \ldots, i_n \) on the first row iff \( \mathcal{A} \) is consistent with \( \mathcal{O}_{\mathcal{U}} \) iff \( \mathcal{A}(a,0) \) is not a certain answer to OMAQ \( (\mathcal{O}_{\mathcal{U}}, \mathcal{A}) \) over \( \mathcal{A} \), where \( A \) is a fresh concept name. Similar considerations apply to the case of a fresh role \( S \).

Table 2: Combined complexity of the consistency problem for \( DL-Lite^\phi_{c/r} \).

| fragment \( DL-Lite^\phi_{c/r} \) | complexity |
|----------------------------------|------------|
| bool and guarded bool (\( \bigcirc \) only) | undecidable [Th. 10] |
| bool/krom (\( \bigcirc \) only) | \( \text{EXPSPACE} \) [Th. 12] |
| bool/horn and horn | \( \text{EXPSPACE} \) [Th. 16] |
| bool/core and horn/core | \( \text{PSPACE} \) [Th. 18] |

**Theorem 10.** (i) Checking consistency of \( DL-Lite^\phi_{g\text{-bool}} \) and \( DL-Lite^\phi_{g\text{-bool}} \) KBs is undecidable.

(ii) There are \( DL-Lite^\phi_{g\text{-bool}} \) OMAQs \( q_1 = (\mathcal{O}, \mathcal{A}) \) and \( q_2 = (\mathcal{O}, S) \) such that the problems whether, given an ABox \( \mathcal{A} \), the pair \( (a,0) \) is a certain answer to \( q_1 \) over \( \mathcal{A} \) and \( (a,b,0) \) is a certain answer to \( q_2 \) over \( \mathcal{A} \) are undecidable.

Proof. (i) The proof is by reduction of the \( \mathbb{N} \times \mathbb{N} \)-tiling problem, which is known to be undecidable (Berger, 1966): given a finite set \( \mathcal{T} \) of tile types \( \{1, \ldots, m\} \), decide whether \( \mathcal{T} \) can tile the \( \mathbb{N} \times \mathbb{N} \) grid. For each \( \mathcal{T} \), we denote by \( up(i) \), \( down(i) \), \( left(i) \) and \( right(i) \) the colours on the four edges of any tile type \( i \in \mathcal{T} \). Define a \( DL-Lite^\phi_{g\text{-bool}} \) ontology \( \mathcal{O}_{\mathcal{T}} \), where the \( R_i \) are role names associated with tile types \( i \in \mathcal{T} \), by taking

\[
I \subseteq \bigcup_{i \in \mathcal{T}} \exists R_i, \quad \exists R_i \sqcap \exists R_j \subseteq \bot, \quad \text{for } i, j \in \mathcal{T} \text{ with } i \neq j,
\]

\[
\exists R_i^\top \subseteq \bigcup_{j \in \mathcal{T}} \exists R_j \quad \text{and} \quad R_i \subseteq \bigcup_{j \in \mathcal{T}} \top_i \circ j R_j, \quad \text{for } i \in \mathcal{T}.
\]

It is readily checked that \( \{I(a,0)\} \) is consistent with \( \mathcal{O}_{\mathcal{T}} \) iff \( \mathcal{T} \) can tile the \( \mathbb{N} \times \mathbb{N} \) grid.

(ii) Using the representation of the universal Turing machine by means of tiles (Börger, Grädel, & Gurevich, 1997), we obtain a set \( \mathcal{U} \) of tile types for which the following problem is undecidable: given a finite sequence of tile types \( i_0, \ldots, i_n \), decide whether \( \mathcal{U} \) can tile the \( \mathbb{N} \times \mathbb{N} \) grid so that tiles of types \( i_0, \ldots, i_n \) are placed on \( (0,0), \ldots, (n,0) \), respectively. Given such \( i_0, \ldots, i_n \), we take the ABox \( \mathcal{A} = \{I(a,0), R_{i_0}(a,b,0), \ldots, R_{i_n}(a,b,n)\} \). Then \( \mathcal{U} \) can tile \( \mathbb{N} \times \mathbb{N} \) with \( i_0, \ldots, i_n \) on the first row iff \( \mathcal{A} \) is consistent with \( \mathcal{O}_{\mathcal{U}} \) iff \( \mathcal{A}(a,0) \) is not a certain answer to OMAQ \( (\mathcal{O}_{\mathcal{U}}, \mathcal{A}) \) over \( \mathcal{A} \), where \( A \) is a fresh concept name. Similar considerations apply to the case of a fresh role \( S \). \( \square \)
the domain of $M$ as usual, while the LTL operators are interpreted over $(Z, <)$, also as usual, given an assignment of $x$ to some domain element of $\Delta^n$. For a formula with free variables $x_1, \ldots, x_k$ and a tuple of individual constants $a_1, \ldots, a_k$, we write $M, n \models \varphi(a_1, \ldots, a_k)$ to say that, under the assignment $x_i \mapsto a_i \in \Delta^n$, $1 \leq i \leq k$, the formula $\varphi$ is true at moment $n$ in $M$. We also abbreviate $\Box_p \Box_p$ by $\Box$ and a sequence of $\ell$-many $\Box_p$ by $\Box^\ell_p$. Also, $\Box^k p$ stands for $\Box^{k-1} p$ if $k > 0$, $\varnothing$ if $k = 0$, and $\Box^{-k} p$ if $k < 0$.

4.1 Consistency of DL-Lite$^\Diamond_{bool/krom}$ KBs

Let $K = (T \cup R, A)$ be a DL-Lite$^\Diamond_{bool/krom}$ KB. We assume that $K$ has no nested temporal operators, $R$ is closed under taking the inverses of roles in RIs, and that, in RIs of the form $T \subseteq R_1 \sqcup R_2$ and $R_1 \sqcap R_2 \subseteq \bot$ from $R$, both $R_i$ are plain (atemporal) roles. We construct a FO-LTL$_1$ sentence $\Phi_K$ with one variable $x$. First, we set $\Phi_K = \bot$ if $(R, A)$ is inconsistent, which can be checked in polynomial time by Proposition 9 and (Artale et al., 2014, Lemma 5.3). If, however, $(R, A)$ is consistent, then we treat basic concepts in $K$ as unary predicates (e.g., $A(x)$ for a concept name $A$ and $\exists P^-(x)$ for $\exists P^-$ with a role name $P$) and define $\Phi_K$ as a conjunction of the following FO-LTL$_1$ sentences:

\[
\square \forall x [C_1(x) \land \cdots \land C_k(x) \rightarrow C_{k+1}(x) \lor \cdots \lor C_{k+m}(x)], \quad (14)
\]

for all $C_1 \sqcap \cdots \sqcap C_k \subseteq C_{k+1} \sqcup \cdots \sqcup C_{k+m}$ in $T$,

\[
\square \forall x [\exists S_1(x) \lor \exists S_2(x)] \land \square \forall x \exists S_1(x) \lor \forall x \exists S_2(x), \quad (15)
\]

for all $T \subseteq S_1 \sqcup S_2$ in $R$,

\[
\Box^\ell_p A(a), \quad (16)
\]

for all $A(a, \ell)$ in $A$,

\[
\Box^\ell_p \exists P(a) \land \Box^\ell_p \exists P^-(b), \quad (17)
\]

for all $P(a, b, \ell)$ in $A$,

\[
\Box \exists P(x) \leftrightarrow \exists x \exists P^-(x), \quad (18)
\]

for all role names $P$ in $T$,

\[
\square \forall x [\Box^1 \exists S_1(x) \rightarrow \exists S_2(x)], \quad (19)
\]

for all $\Box^1 \exists S_1 \subseteq \exists S_2$, with $R \models \Box^1 \exists S_1 \subseteq \exists S_2$.

where each $\Box_i$ is $\Box_p$, $\Box_p$ or blank, and $\Box^1 \exists S_1$ can be $\top$ and $\Box^1 \exists S_2$ can be $\bot$. Note that the problem of checking whether $R \models \Box^1 \exists S_1 \subseteq \exists S_2$ is in P (Artale et al., 2014, Lemma 5.3), and so $\Phi_K$ can be constructed in polynomial time.

**Lemma 11.** A DL-Lite$^\Diamond_{bool/krom}$ KB $K$ is consistent iff $\Phi_K$ is satisfiable.

**Proof.** ($\Rightarrow$) Suppose $I \models K$. Treating $I$ as a temporal FO-interpretation, we show that $I \models \Phi_K$. The only non-standard sentences are (15). Suppose $I \models T \subseteq S_1 \sqcup S_2$ and $I, n \not\models \exists S_1(d)$, for some $d \in I^T$ and $n \in Z$. Then, for every $e \in I^T$, we have $I, n \models S_2(d, e)$, and so $I, n \models \exists S_2(e)$.

($\Leftarrow$) Suppose $M \models \Phi_K$. We require the following property of $M$, which follows, for any RI $T \subseteq S_1 \sqcup S_2$ in $R$, from (15): for any $n \in Z$ and $d, e \in M^n$, either

\[
M, n \models \exists S_1(d) \text{ and } M, n \not\models \exists S_2^-(e) \quad \text{or} \quad M, n \not\models \exists S_1(d) \text{ and } M, n \models \exists S_2^-(e). \quad (20)
\]

We construct a model $\mathcal{I}$ of $K$ in a step-by-step manner, regarding $\mathcal{I}$ as a set of ground atoms. To begin with, we put in $\mathcal{I}$ all $P(a, b, n) \in A$ and then proceed in three steps.

**Step 1:** If $T \subseteq S_1 \sqcup S_2$ is in $R$ with $M, n \models \exists S_1(a)$ and either $M, n \not\models \exists S_2(a)$ or $M, n \not\models \exists S_2^-(b)$, for $n \in Z$ and $a, b \in \text{ind}(A)$, then, by (20), $M, n \models \exists S_2^-(b)$, and we add

\[
23
\]
Step 2 and repeat the process. Suppose otherwise, that is, there are some $P(a,b,n)$ and $R(a,b,n)$ in $\mathcal{I}$ with $P \cap R \subseteq \bot$ in $\mathcal{R}$. Two cases need consideration. (i) If both atoms were added at Step 1 because of some RI $\top \subseteq P \cup Q$ and $\top \subseteq R \cup S$, then $\mathcal{R} \models P \subseteq S$ and so, by (19), $\mathcal{M}, n \models \exists S(a)$ and $\mathcal{M}, n \models \exists P(b)$, contrary to the definition of Step 1. (ii) Otherwise, as $A$ is consistent with $\mathcal{R}$, the only other possibility is that $R(a,b,n) \in A$ and $P(a,b,n)$ was added at Step 1 because of some RI $\top \subseteq P \cup Q$. In this case, $\mathcal{R} \models R \subseteq Q$, whence, by (17), $\mathcal{M}, n \models \exists Q(a)$ and $\mathcal{M}, n \models \exists Q(b)$, contrary to $P(a,b,n)$ being added at Step 1.

**Step 2:** For all roles $P$ and $R$ and all $n, k \in \mathbb{Z}$, if $P(a,b,n) \in \mathcal{I}$ and $\mathcal{R} \models P \subseteq \bigcirc^k R$, then we add $R(a,b,n+k)$ to $\mathcal{I}$. We show that the resulting $\mathcal{I}$ remains consistent with $\mathcal{R}$. Suppose otherwise, that is, there are $R_1(a,b,n), R_2(a,b,n) \in \mathcal{I}$ with $R_1 \cap R_2 \subseteq \bot$ in $\mathcal{R}$. Suppose that $R_i(a,b,n)$, for $i = 1,2$, was added to $\mathcal{I}$ for $\mathcal{R} \models P_i \subseteq \bigcirc^k R_i$ and a $P_i$-atom constructed at Step 1. As $\mathcal{R} \models \neg R_i \subseteq \neg \bigcirc^{-k} P_i$, we arrive to a contradiction with the consistency of $\mathcal{I}$ at Step 1.

**Step 3:** For each RI $\top \subseteq P \cup Q$ in $\mathcal{R}$, each $a, b \in \text{ind}(A)$ and each $n \in \mathbb{Z}$ such that $\mathcal{M}, n \models \exists P(a)$, $\mathcal{M}, n \models \exists P^-(b)$, $\mathcal{M}, n \models \exists Q(a)$, $\mathcal{M}, n \models \exists Q^-(b)$, but neither $P(a,b,n)$ nor $Q(a,b,n)$ are in $\mathcal{I}$, we add one of them, say $P(a,b,n)$, to $\mathcal{I}$. The result remains consistent with $\mathcal{R}$: indeed, if we had $S(a,b,n) \in \mathcal{I}$ with $P \cap S \subseteq \bot$ in $\mathcal{R}$, then $Q(a,b,n)$ would have been added to $\mathcal{I}$ at Step 2 because $\mathcal{R} \models S \subseteq Q$. We take the closure of $P(a,b,n)$ as at Step 2 and repeat the process.

We conclude the first stage of constructing $\mathcal{I}$ by extending it with all $B(a,n)$ such that $\mathcal{M}, n \models B(a)$, for $n \in \mathbb{Z}$ and $a \in \text{ind}(A)$. By construction, $P(a,b,n) \in \mathcal{I}$ implies $\exists P(a,n) \cup \exists P^-(b,n) \in \mathcal{I}$, but not necessarily the other way round. So, suppose $\exists P(a,n) \in \mathcal{I}$ but there is no $P(a,b,n)$ in $\mathcal{I}$. Take a fresh individual $w_P$, add it to the domain of $\mathcal{I}$ and add $P(a,w_P,n)$ to $\mathcal{I}$. By (19) and (18), the result is consistent with $\mathcal{R}$. By (18), there is $d \in \Delta^{\mathcal{M}}$ with $\mathcal{M}, n \models \exists P^-(d)$. So, we add $B(w_P,m)$ to $\mathcal{I}$ for each basic concept $B$ and $m \in \mathbb{Z}$ with $\mathcal{M}, m \models B(d)$. We then apply to $\mathcal{I}$ the three-step procedure described above and repeat this ad infinitum (a similar unravelling construction is given in detail in the proof of Lemma 13 below).

It is readily seen that the obtained interpretation $\mathcal{I}$ is a model of $\mathcal{K}$.\qed

It follows from the proof of Lemma 11 that it is always possible to construct a model $\mathcal{I}$ of $\mathcal{R}$ from a model $\mathfrak{M}$ of $\Phi_{\mathcal{K}}$ if $\mathfrak{M}$ satisfies the domain/range restrictions for the roles in $\mathcal{R}$ encoded by (15) and (19). One reason why this encoding is enough is that, e.g., the CI $\Box_P \exists P \subseteq \exists Q$ is sufficient to capture the effect of the RI $\Box_P P \subseteq Q$ on the domains/ranges. However, $\Box_P P \subseteq Q$ does not entail $\Box \exists P \subseteq \exists Q$, and it is not clear what domain/range axioms can be used to capture the impact of the RI on domains/ranges in the presence of $\Box$-operators. The complexity of the consistence problem for $\text{DL-Lite}^\Box_{\text{bool/krom}}$ KBs remains open. We show in Section 4.2 how Horn RIs with $\Box$-operators can be encoded in $\text{FOLTL}_1$.

**Theorem 12.** Checking consistency of $\text{DL-Lite}^\Box_{\text{bool/krom}}$ KBs is EXPSPACE-complete.

**Proof.** The upper bound follows from Lemma 11. We prove hardness by reduction of the $\mathbb{N} \times (2^n - 1)$ corridor tiling problem, which is known to be EXPSPACE-complete (Van Emde Boas, 1997): given a finite set $\mathcal{S}$ of tile types $\{1, \ldots, m\}$ with four colours $\text{up}(i)$, 

\begin{align*}
S_1(a,b,n) \rightarrow I. 
\end{align*}
Figure 3: Structure of models in the proof of Theorem 12. The tile at \((c, d)\) is denoted by \(\tau(c, d)\), and the horizontal and diagonal dashed arrows show the left-right and the top-down neighbouring tiles, respectively. Roles \(S_i\) are not shown, and roles \(Q_i\) are only partially depicted: if \(Q_i\) contains some \((a^T, v_i)\) at \(t\), then it contains all \((u, v)\) at \(t\).

down(i), left(i) and right(i) and a distinguished colour \(W\), decide whether \(\mathcal{X}\) can tile the grid \(\{(t, s) \mid t \in \mathbb{N}, 1 \leq s < 2^n\}\) so that

- (b_1) tile 0 is placed at \((0, 1)\),
- (b_2) any tile \(i\) placed at any \((c, 1)\) has \(\text{down}(i) = W\), and
- (b_3) any tile \(i\) placed at any \((c, 2^n - 1)\) has \(\text{up}(i) = W\).

Let \(A = \{A(a, 0)\}\) and let \(O\) contain the following axioms:

\[
A \subseteq D, \quad D \subseteq \bigcap_{j=0}^{2^n} P_j, \quad \exists P^{-} \subseteq \bigcup_{i \in \mathcal{I}} T_i, \\
T_i \subseteq \bigcap_{j \in \mathcal{I}} T_j, \quad T_i \cap \exists S_i^{-} \subseteq \bot, \quad \top \subseteq S_i \sqcup Q_i, \quad \text{for } i \in \mathcal{I}, \\
\exists Q_i \cap \bigcap_{j \in \mathcal{I}} \exists Q_j \subseteq \bot, \quad \text{for } i, j \in \mathcal{I} \text{ with } \text{up}(i) \neq \text{down}(j).
\]

Observe that \((O, A)\) is consistent if there is a placement of tiles on the \(\mathbb{N} \times (2^n - 1)\) grid: each of the \((2^n - 1)\) \(P\)-successors of \(a\) created at moments \(1, \ldots, 2^n - 1\) represents a column of the corridor; see Fig. 3. Note, however, that the size of the CIs is exponential in \(n\). We now describe how they can be replaced by polynomial-size CIs.

Consider, for example, a CI \(D \subseteq \bigcap_{j=0}^{2^n} D\). We express it using the following CIs:

\[
D \subseteq \bigcap_{j=0}^{2^n} (\neg B_{n-1} \cap \cdots \cap \neg B_0), \\
B_{n-1} \cap \cdots \cap B_0 \subseteq D, \\
\neg B_k \cap B_{k-1} \cap \cdots \cap B_0 \subseteq \bigcap_{j=0}^{2^n} (B_k \cap \neg B_{k-1} \cap \cdots \cap \neg B_0), \quad \text{for } 0 \leq k < n,
\]
\[ \neg B_j \cap \neg B_k \subseteq \neg \varphi B_j \quad \text{and} \quad B_j \cap \neg B_k \subseteq \varphi B_j, \quad \text{for } 0 \leq k < j < n, \]

which have to be converted into normal form (5). Intuitively, they encode a binary counter from 0 to \(2^n - 1\), where \(\neg B_i\) and \(B_j\) stand for `the ith bit of the counter is 0' and, respectively, `the ith bit of the counter is 1'. Other CIs of the form \(C_1 \subseteq \varphi C_2\) are handled similarly. For \(A \subseteq \bigcap_{1 \leq s < 2^n} \varphi \exists P\), we use the \(B_k \subseteq \exists P\), for \(0 \leq k < n\), instead of \(B_{n-1} \cap \cdots \cap B_0 \subseteq \exists P\).

To ensure that \((b_1)\)–\((b_3)\) are satisfied, we add to \(O\) the following CIs:

\[
\begin{align*}
A \cap \varphi \exists Q_i & \subseteq \bot, & \text{for } i \in \mathcal{S} \setminus \{0\}, \\
D \cap \varphi \exists Q_i & \subseteq \bot, & \text{for } \text{down}(i) \neq W, \\
\varphi D \cap \exists Q_i & \subseteq \bot, & \text{for } \text{up}(i) \neq W.
\end{align*}
\]

It is readily seen that \((O, A)\) is as required.

4.2 Consistency of \(DL-Lite_{\varnothing/horn}^{\bigcirc}\) KBs

Let \(\mathcal{K} = (O, A)\) be a \(DL-Lite_{\varnothing/horn}^{\bigcirc}\) KB with \(O = \mathcal{T} \cup \mathcal{R}\). Denote by \(\mathrm{sub}_{\mathcal{T}}\) the set of (sub)concepts in \(\mathcal{T}\) and their negations. For \(\tau \subseteq \mathrm{sub}_{\mathcal{T}}\), let \(\tau^\uparrow\) be the result of replacing each \(B\) in \(\tau\) with \(B^\uparrow\). A concept type \(\tau\) for \(\mathcal{T}^\uparrow\) is a maximal subset \(\tau\) of \(\mathrm{sub}_{\mathcal{T}}\) such that \(\tau^\uparrow\) is consistent with \(\mathcal{T}^\uparrow\) (note, however, that \(\tau\) is not necessarily consistent with \(\mathcal{T}\); for example, \(\tau^\uparrow\) can be consistent with \(\mathcal{T}^\uparrow\) even if \(\exists P \in \tau\) and \(\mathcal{T}\) contains \(\exists P^\bot \subseteq \bot\)). A beam \(b\) for \(\mathcal{T}\) is a function from \(\mathcal{Z}\) to the set of all concept types for \(\mathcal{T}^\uparrow\) such that, for all \(n \in \mathcal{Z}\), we have the following:

\[
\begin{align*}
\varphi C & \in b(n) & \text{iff } & C \in b(n+1), \\
\neg \varphi C & \in b(n) & \text{iff } & C \in b(n-1), \quad (21) \\
\varphi C & \in b(n) & \text{iff } & C \in b(k), \text{ for all } k > n, \quad \neg \varphi C & \in b(n) & \text{iff } & C \in b(k), \text{ for all } k < n. \quad (22)
\end{align*}
\]

Given an interpretation \(\mathcal{I}\) and \(u \in \Delta^\mathcal{I}\), the function \(b^\mathcal{I}_{u}: n \mapsto \{ C \in \mathrm{sub}_{\mathcal{T}} \mid u \in C^{\mathcal{I}(n)} \}\) is a beam; we refer to it as the beam of \(u\) in \(\mathcal{I}\).

As before, we assume that \(\mathcal{R}\) is closed under taking the inverses of roles in \(\mathcal{RI}\) and contains all roles that occur in \(\mathcal{T}\). Denote by \(\mathrm{sub}_{\mathcal{R}}\) the set of (sub)roles in \(\mathcal{R}\) and their negations. A role type \(\rho\) for \(\mathcal{R}\) is a maximal subset of \(\mathrm{sub}_{\mathcal{R}}\) consistent with \(\mathcal{R}\) (equivalently, \(\rho^\uparrow\) is consistent with the \(LTL_{horn}^{\bigcirc}\) ontology \(\mathcal{R}^\uparrow\), and a rod \(r\) for \(\mathcal{R}\) is a function from \(\mathcal{Z}\) to the set of all role types for \(\mathcal{R}\) such that (21) and (22) hold for all \(n \in \mathcal{Z}\) with \(b\) replaced by \(r\) and \(C\) by temporalised roles \(R\). Given an interpretation \(\mathcal{I}\) and \(u, v \in \Delta^\mathcal{I}\), the function \(r^\mathcal{I}_{u,v}: n \mapsto \{ R \in \mathrm{sub}_{\mathcal{R}} \mid (u, v) \in R^{\mathcal{I}(n)} \}\) is a rod for \(\mathcal{R}\); we call it the rod of \((u, v)\) in \(\mathcal{I}\).

In this section we consider Horn \(\mathcal{RI}\), and the key property of \(LTL_{horn}^{\bigcirc}\) is that every consistent \(LTL_{horn}^{\bigcirc}\) KB \((\mathcal{R}^\uparrow, \mathcal{A}^\uparrow)\) has a canonical model (see Section 3.1) \(\mathcal{C}_{\mathcal{R}^\uparrow, \mathcal{A}^\uparrow}\). Since the \(\mathcal{RI}\) in \(\mathcal{R}\) are Horn, given any \(LTL\) ABox \(\mathcal{A}^\uparrow\), which is a set of atoms of the form \(S^\uparrow(\ell)\), we define the \(\mathcal{R}\)-canonical rod \(r_{\mathcal{A}^\uparrow}\) for \(\mathcal{A}^\uparrow\) (provided that \(\mathcal{A}^\uparrow\) is consistent with \(\mathcal{R}^\uparrow\)) by taking \(r_{\mathcal{A}^\uparrow}: n \mapsto \{ R \in \mathrm{sub}_{\mathcal{R}} \mid R(n) \in \mathcal{C}_{\mathcal{R}^\uparrow, \mathcal{A}^\uparrow}\}. \) In other words, \(\mathcal{R}\)-canonical rods are the minimal rods for \(\mathcal{R}\) `containing' all atoms of \(\mathcal{A}^\uparrow\): for any \(R\) and \(n \in \mathcal{Z}\),

\[
R \in r_{\mathcal{A}^\uparrow}(n) \quad \text{iff} \quad R \in r(n), \text{ for all rods } r \text{ for } \mathcal{R} \text{ with } S \in r(\ell) \text{ for } S^\uparrow(\ell) \in \mathcal{A}^\uparrow. \quad (23)
\]

Finally, given a beam \(b\), we say a rod \(r\) is \(b\)-compatible if \(\exists S \in b(n)\) whenever \(S \in r(n)\), for all \(n \in \mathcal{Z}\) and basic concepts \(\exists S\). We are now fully equipped to prove the following characterisation of \(DL-Lite_{\varnothing/horn}^{\bigcirc}\) KBs consistency.
Figure 4: The beams and rods in Example 14 (left) and the temporal interpretation $\mathcal{I}$ obtained by unravelling them (right).

**Lemma 13.** Let $\mathcal{K} = (\mathcal{O}, \mathcal{A})$ be a DL-Lite$^{\bigodot}$ KB with $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$. Let

$$\Xi = \text{ind}(\mathcal{A}) \cup \{ w_P, w_P^- \mid P \text{ a role name in } \mathcal{O} \}.$$ 

Then $\mathcal{K}$ is consistent iff there are beams $b_w, w \in \Xi$, for $\mathcal{T}$ satisfying the following conditions:

1. $A \in b_a(\ell)$, for all $A(a, \ell) \in \mathcal{A}$, if $\exists S \in b_w(n)$, then $\exists S^- \in b_w S^-(k)$, for some $k \in \mathbb{Z}$, for every $a, b \in \text{ind}(\mathcal{A})$, there is a $b_a$-compatible rod $r$ for $\mathcal{R}$ such that $S \in r(\ell)$, for all $S(a, b, \ell) \in \mathcal{A}$, if $\exists S \in b_w(n)$, then there is a $b_w$-compatible rod $r$ for $\mathcal{R}$ such that $S \in r(n)$.

Moreover, for any beams $b_w, w \in \Xi$, for $\mathcal{T}$ satisfying (24)–(27), there is a model $\mathcal{I}$ of $\mathcal{K}$ such that

- for each $a \in \text{ind}(\mathcal{A})$, the beam of $a^\mathcal{T}$ in $\mathcal{I}$ coincides with $b_a$,

- for each $u \in \Delta^\mathcal{T} \setminus \{ a^\mathcal{T} \mid a \in \text{ind}(\mathcal{A}) \}$, there are $S$ and $n \in \mathbb{Z}$ with $b_a^u(k) = b_w a S^u(k + n)$, for all $k \in \mathbb{Z}$, and

- for each $a, b \in \text{ind}(\mathcal{A})$, the rod of $(a^\mathcal{T}, b^\mathcal{T})$ in $\mathcal{I}$ is the $\mathcal{R}$-canonical rod for $\mathcal{A}^\mathcal{T}_{a,b}$.

Before proving the lemma, we illustrate the construction by an example.

**Example 14.** Consider the KB $\mathcal{K} = (\mathcal{O}, \{Q(a, b, 0)\})$, where $\mathcal{O}$ consists of

$$\exists Q \land \Box_P A \subseteq \bot, \quad \top \subseteq A \cup \exists P \quad \text{and} \quad P^- \subseteq \Box_P Q,$$

which is the result of converting $\exists Q \subseteq \Diamond_P \exists P$ and $P^- \subseteq \Box_P Q$ into normal form (5). Beams $b_a$, $b_b$, $b_w P^-$ and $b_w P$ are depicted in Fig. 4 (left) by horizontal lines: the type contains $\exists P$ or $\exists Q$ whenever the large node is grey; similarly, the type contains $\exists P^-$ or $\exists Q^-$ whenever
the large node is white (the label of the arrow specifies the role); we omit $A$ to avoid clutter. The rods are the arrows between the pairs of horizontal lines. For example, the rod required by (26) for $a$ and $b$ is labelled by $r_{a,b}$; it contains only $Q$ at 0 (we specify only the positive components of the types); the rod required by (26) for $b$ and $a$ is labelled by $r_{b,a}$, and in this case, it is the mirror image of $r_{a,b}$. In fact, if we choose $\mathcal{R}$-canonical rods in (26), then the rod for any $b, a$ will be the mirror image of the rod for $a, b$. The rod $r_{P,a}$ required by (27) for $\exists P$ on $b_a$ at moment 2 is depicted between $b_a$ and $u_{P,2}$: it contains $P$ at 2 and $Q$ at 3. In fact, it should be clear that, if in (27) we choose $\mathcal{R}$-canonical rods, then they will all be isomorphic copies of at most $|\mathcal{O}|$ rods: more precisely, the rods will be of the form $r_{\{S^m(n)\}}$, for a role $S$ from $\mathcal{O}$. In the proof of Lemma 13, we show how this collection of beams and $\mathcal{R}$-canonical rods can be used to obtain a model $\mathcal{I}$ of $\mathcal{K}$ depicted on the right-hand side of Fig. 4 (again, with $A$ omitted). We now proceed with the proof of the lemma.

**Proof.** ($\Leftarrow$) Suppose that we have the required collection of beams $b_w$ for $T$. We construct by induction on $m < \omega$ a sequence of temporal interpretations $I_m = (\Delta I_m, I_{m}(n))$ and maps $f_m : \Delta I_m \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$. The meaning of $f_m$ is as follows: if $f_m(\lambda, k) = (w, l)$, then the type of the element $\lambda$ in $I_m$ at moment $k$ is ‘copied’ from the beam $w$ at moment $l$. For the basis of induction, set $\Delta I_0 = \text{ind}(A)$, $A^{I_0(k)} = \{ a \mid A \in b_a(k) \}$ and $P^{I_0(k)} = \{ (a, b) \mid P \in r_{a,b}(k) \}$, for all concept and roles names $A$ and $P$ and all $k \in \mathbb{Z}$, and $f_0(a, k) = (a, k)$, for all $a \in \text{ind}(A)$ and $k \in \mathbb{Z}$, where $r_{a,b}$ is the $\mathcal{R}$-canonical rod for $A^{I_0}_{a,b}$, which exists by (26) and which, by (23), is compatible with $b_a$, and its inverse is compatible with $b_b$.

Suppose next that $I_m$ and $f_m$, for $m \geq 0$, have already been defined, that the elements of $\Delta I_m$ are words of the form $\lambda = aS^m_1 \ldots S^m_i$, for $a \in \text{ind}(A)$, $n_i \in \mathbb{Z}$ and $l \geq 0$. We call a pair $(\lambda, n)$ an $S$-defect in $I_m$ if (i) $f_m(\lambda, n) = (w, n')$, (ii) $\exists S \in b_w(n')$ and (iii) $(\lambda, \lambda') \notin S^{I_m(n)}$ for any $\lambda' \in \Delta I_m$. For any role $S$ and any $S$-defect $(\lambda, n)$ in $I_m$, we add the word $\lambda S^n$ to $\Delta I_m$ and denote the result by $\Delta I_{m+1}$. By (25), we have $\exists S^- \in b_w(\cdot)(l)$, for some $l \in \mathbb{Z}$. We fix one such $l$ and extend $f_m$ to $f_{m+1}$ by setting $f_{m+1}(\lambda S^n, k) = (w_{S^-}, k - n + l)$, for any $k \in \mathbb{Z}$. We also define $A^{I_{m+1}(k)}$ by extending $A^{I_m(k)}$ with those $\lambda S^n$ for which $A \in b_{w_{S^-}}(k - n + l)$, and we define $P^{I_{m+1}(k)}$ by extending $P^{I_m(k)}$ with $(\lambda, \lambda S^n)$ for which $P \in r_{S, S^n}(k)$ and with $(\lambda S^n, \lambda)$ for which $P^- \in r_{S, S^n}(k)$, where $r_{S, S^n}$ is the $\mathcal{R}$-canonical rod for $\{S^n\}$, which exists by (27) and which, by (23), is compatible with the respective beams.

Finally, let $I$ and $f$ be the unions of all $I_m$ and $f_m$, for $m < \omega$, respectively. We show that $I$ is a model of $\mathcal{K}$. It follows immediately from the construction that $I$ is a model of $\mathcal{R}$ and $\mathcal{A}$. To show that $I$ is also a model of $\mathcal{T}$, it suffices to prove that, for any $\lambda \in \Delta I$ and any role $Q$, we have $\lambda \in (\exists Q)I^{(k)} \iff \exists Q \in b_w(l)$, where $f(\lambda, k) = (w, l)$. The implication ($\Leftarrow$) follows directly from the procedure of ‘curing defects’. Let $\lambda \in (\exists Q)I^{(k)}$, and so $(\lambda, \lambda') \in Q^{I(k)}$, for some $\lambda' \in \Delta I$. Two cases are possible now.

- If $\lambda, \lambda' \in \text{ind}(A)$, then $Q \in r_{\lambda, \lambda'}(k)$. Then, by (26), $\exists Q \in b_{\lambda}(k)$. It remains to recall that $f(\lambda, k) = f_0(\lambda, k) = (\lambda, k)$.

- If $\lambda' \notin \text{ind}(A)$, then $\lambda' = \lambda S^n$, for some $S$ and $n$, and $Q \in r_{S, n}(k)$. We also have $\exists S \in b_w(n')$, where $f(\lambda, n) = (w, n')$. By (27), there is a rod $r$ for $\mathcal{R}$ such that $S \in r(n')$, and so, we must have $Q \in r(l)$. Since $r$ is compatible with $b_w$, we obtain $\exists Q \in b_w(l)$, as required.
(⇒) Given a model $\mathcal{I}$ of $\mathcal{K}$, we construct beams $b_w$ for $\mathcal{T}$ as follows. Set $b_a = b_a^w$, for all $a \in \text{ind}(A)$. For each $S$, if $S^{T(n)} \neq \emptyset$, for some $n \in \mathbb{Z}$, then set $b_{ws} = b_{a}^w$, for $u \in (\exists S)^{T(n)}$; otherwise, set $b_{ws} = b_a^w$, for an arbitrary $a \in \text{ind}(A)$. It is straightforward to check that these beams are as required.

We now reduce the existence of the required collection of beams to the EXPSPACE-complete satisfiability problem for $\text{FO-LTL}_1$ (Halpern & Vardi, 1989; Gabbay et al., 2003), thereby establishing the upper complexity bound for $\text{DL-Lite}^{\bigcirc\bigcirc}_{\text{bool/horn}}$. Let $\mathcal{K} = (\mathcal{O}, \mathcal{A})$ be a $\text{DL-Lite}^{\bigcirc\bigcirc}_{\text{bool/horn}}$ KB with $\mathcal{O} = T \cup \mathcal{R}$. We assume that $\mathcal{R}$ is closed under taking the inverses of roles in RIs. Let $\Xi$ be as in Lemma 13. We treat its elements as constants in the first-order language and define a translation $\Psi_\mathcal{K}$ of $\mathcal{K}$ into $\text{FO-LTL}_1$ with a single individual variable $x$ as a conjunction of the following sentences, for all constants $w \in \Xi$:

\[
\begin{align*}
\Box (C_1(w) \land \cdots \land C_k(w) & \rightarrow C_{k+1}(w) \lor \cdots \lor C_{k+m}(w)), \\
& \text{for all } C_1 \cap \cdots \cap C_k \subseteq C_{k+1} \cup \cdots \cup C_{k+m} \text{ in } T, \\
\Box \forall x (R_1(w,x) \land \cdots \land R_k(w,x) & \rightarrow R(w,x)), \\
& \text{for all } R_1 \cap \cdots \cap R_k \subseteq R \text{ in } \mathcal{R}, \\
\Box \forall x (R_1(w,x) \land \cdots \land R_k(w,x) & \rightarrow \bot), \\
& \text{for all } R_1 \cap \cdots \cap R_k \subseteq \bot \text{ in } \mathcal{R}, \\
\Box_p \forall P(a,b), & \text{ for all } P(a,b) \in \mathcal{A}, \\
\Box_p \exists S(w) & \rightarrow \exists \forall_x \exists S^-(wS^+), \\
& \text{for all roles } S \in \mathcal{O}, \\
\Box (\exists S(w) & \leftrightarrow \exists x S(w,x)), \\
& \text{for all roles } S \in \mathcal{O}.
\end{align*}
\]

Thus, in $\Psi_\mathcal{K}$, we regard concept names $A$ and basic concepts $\exists S$ as unary predicates and roles $S$ as binary predicates, assuming that $S^-(w,x) = S(x,w)$. The ‘interesting’ conjuncts in $\Psi_\mathcal{K}$ are (33) and (34), which reflect the interaction between $T$ and $\mathcal{R}$.

**Lemma 15.** A $\text{DL-Lite}^{\bigcirc\bigcirc}_{\text{bool/horn}}$ KB $\mathcal{K}$ is consistent iff $\Psi_\mathcal{K}$ is satisfiable.

**Proof.** Each collection of beams $b_w$, $w \in \Xi$, for $\mathcal{T}$ gives rise to a model $\mathcal{M}$ of $\Psi_\mathcal{K}$: the domain of $\mathcal{M}$ comprises $\Xi$ and elements $u_{S,m}$, for a role $S$ and $m \in \mathbb{Z}$. We fix $\mathcal{R}$-canonical rods $r_{a,b}$ for $A_{a,b}^I$, which are guaranteed to exist by (26), and $\mathcal{R}$-canonical rods $r_{S,m}$ for $\{S^+(m)\}$ for each $S$ and $m \in \mathbb{Z}$ with $\exists S \in b_w(m)$, for some $w \in \Xi$, which are guaranteed to exist by (27), and set

\[
\begin{align*}
\mathcal{M}, n & \models B(w) \text{ iff } B \in b_w(n), \text{ for all } w \in \Xi, n \in \mathbb{Z} \text{ and basic concepts } B, \\
\mathcal{M}, n & \models P(a,b) \text{ iff } P \in r_{a,b}(n), \text{ for all } a, b \in \text{ind}(A), n \in \mathbb{Z}, \text{ and role names } P, \\
\mathcal{M}, n & \models S'(w,u_{S,m}) \text{ iff } S' \in r_{S,m}(n), \\
& \text{for all } w \in \Xi, n, m \in \mathbb{Z} \text{ and roles } S, S' \text{ with } \exists S \in b_w(m).
\end{align*}
\]

It is readily checked that $\mathcal{M}$ is as required; see also Fig. 4, where, in the context of Example 14, the $u_{S,m}$ are represented explicitly by grey horizontal lines. Conversely, it can be verified that every model $\mathcal{M}$ of $\Psi_\mathcal{K}$ gives rise to the required collection of beams for $\mathcal{T}$.

**Theorem 16.** Checking consistency of $\text{DL-Lite}^{\bigcirc\bigcirc}_{\text{bool/horn}}$ and $\text{DL-Lite}^{\bigcirc\bigcirc}_{\text{horn}}$ KBs is EXPSPACE-complete.
Proof. The upper bound follows from Lemma 15 and ExpSPACE-completeness of $FOLTL_1$. The hardness is proved by reduction of the non-halting problem for deterministic Turing machines with exponential tape. We assume that the head of a given machine $M$ never runs beyond the first $2^n$ cells of its tape on an input word $a$ of length $m$, where $n = p(m)$ for some polynomial $p$; we also assume that it never attempts to access cells before the start of the tape. We construct a $DL-Lite_{horn}^{\square}$ ontology that encodes the computation of $M$ on $a$ using a single individual $o$. The initial configuration is spread over the time instants $1, \ldots, 2^n$, from which the first $m$ instants represent $a$ and the remaining ones encode the blank symbol #. The second configuration uses the next $2^n$ instants, $2^n+1, \ldots, 2^n+2^n$, etc. The configurations are encoded with the following concept names:

- $H_{q,a}$ contains $o$ at the moment $i2^n+j$ whenever the machine head scans the $j$th cell of the $i$th configuration and sees symbol $a$, with $q$ being the current state of the machine;
- $S_a$ contains $o$ at $i2^n+j$ whenever the $j$th cell of the $i$th configuration contains $a$ but is not scanned by the head.

We can encode computations of the Turing machine $M$ with tape alphabet $\Gamma$ and transition function $\delta$: $Q \times \Gamma \to Q \times \Gamma \times \{R,L\}$ by means of the following $DL-Lite_{horn}^{\square}$ CIs (here and below, we use $\vartheta \sqsubseteq C_0(\vartheta_1 \cap \cdots \cap \vartheta_k)$ as an abbreviation for the $\vartheta \sqsubseteq C_0^{\infty}(\vartheta_1 \cap \cdots \cap \vartheta_k)$:

\[
\begin{align*}
&\forall \varphi S_{a'} \sqcap H_{q,a} \sqcap \forall \varphi S_{a''} \sqsubseteq \forall \varphi^n (\forall \varphi S_{a'} \sqcap S_b \sqcap \forall \varphi H_{q',a''}), & \text{for } \delta(q,a) = (q',b,R) \text{ and } a',a'' \in \Gamma, \\
&\forall \varphi S_{a'} \sqcap H_{q,a} \sqcap \forall \varphi S_{a''} \sqsubseteq \forall \varphi^n (\forall \varphi H_{q',a''} \sqcap S_b \sqcap \forall \varphi S_{a''}), & \text{for } \delta(q,a) = (q',b,L) \text{ and } a',a'' \in \Gamma, \\
&\forall \varphi S_{a'} \sqcap S_a \sqcap \forall \varphi S_{a''} \sqsubseteq \forall \varphi^n S_{a''}, & \text{for } a,a',a'' \in \Gamma, \\
&I \sqsubseteq \forall \varphi^{n} \sqcap \forall \varphi F, & \\
&E \sqsubseteq \forall \varphi^{n} E, & \\
&\forall \varphi E \sqcap F \sqsubseteq S_{\#}, & \\
&H_{q,a} \sqsubseteq \bot, & \text{for each accepting and rejecting state } q \text{ and } a \in \Gamma,
\end{align*}
\]

where $I$ and $E$ mark the start and the end of the input $a$, respectively, and $F$ marks the cells of the first configuration (to be filled with blanks #); we also assume that $S_{\#}$ also holds at moment 0. These CIs are of exponential size, and our next task is to show how to convert them into a $DL-Lite_{horn}^{\square}$ ontology of polynomial size.

Consider a CI of the form $A \sqsubseteq \forall \varphi^n B$. First, we replace the temporalised concept $\forall \varphi^n B$ by $\exists P$ and add the CI $\exists Q \sqsubseteq B$ to the TBox, where $P$ and $Q$ are fresh role names. Then, we add the following RIs to the RBox, $R$, for fresh role names $P_0, \ldots, P_{n-1}, \bar{P}_0, \ldots, \bar{P}_{n-1}$:

\[
\begin{align*}
P &\equiv \forall \varphi (P_{n-1} \cap \cdots \cap P_0), \\
P_{n-1} \sqcap \cdots \sqcap P_0 &\equiv Q, \\
P_k \sqcap P_{k-1} \sqcap \cdots \sqcap P_0 &\equiv \forall \varphi (P_k \sqcap \bar{P}_{k-1} \sqcap \cdots \sqcap \bar{P}_0), & 0 \leq k < n, \\
P_j \sqcap \bar{P}_k &\equiv \forall \varphi \bar{P}_j & & 0 \leq k < j < n, \quad \text{and } \quad P_j \sqcap \bar{P}_k \equiv \forall \varphi P_j,
\end{align*}
\]

Intuitively, these RIs encode a binary counter from 0 to $2^n - 1$, where roles $P_i$ and $\bar{P}_i$ stand for ‘the $i$th bit of the counter is 0’ and, respectively, ‘is 1’, and ensure that $R \models P \sqsubseteq \forall \varphi^n Q$ but $R \not\models P \sqsubseteq \forall \varphi^n Q$ for any $i \neq 2^n$ (remember that $\exists P$ on $o$ generates different $P$-successors
at different time points). Further details are left to the reader. Note that the encoding of $\mathcal{O}_c^{p_0}$ on concepts using an RBox of size polynomial in $n$ is based on the same idea as the encoding of $\mathcal{O}_c$ on concepts using an RBox with $\Box$ operators only in Example 5 and Theorem 27.

### 4.3 Consistency of DL-Lite$^{\Box \mathcal{O}}_{\text{bool/core}}$ KBs

We now adapt the technique of Section 4.2 to reduce consistency of DL-Lite$^{\Box \mathcal{O}}_{\text{bool/core}}$ KBs to that of LTL KBs. The modification is based on the observation that the consequences of core RIs, and so the auxiliary $\mathcal{R}$-canonical rods as well, have a simpler structure than those of Horn RIs: intuitively, while core RIs can entail $\mathcal{R} \models P \sqsubseteq \mathcal{O}^n Q$ for $n$ exponentially large in the size of $\mathcal{R}$, it must also be the case that $\mathcal{R} \models P \sqsubseteq \mathcal{O}^i Q$ for either all $i < n$ or all $i > n$. Using this property and a trick with binary counters (see below), we reduce satisfiability of $\mathcal{K}$ to satisfiability of a polynomial-size LTL KB.

Let $\mathcal{R}$ be a DL-Lite$^{\Box \mathcal{O}}_{\text{bool/core}}$ RBox and $r$ the $\mathcal{R}$-canonical rod for some $A_R^1 = \{R(0)\}$. Then $S \in r(n)$ iff one of the following conditions holds:

- $(\mathcal{R}')^1, A_R^1 \models S^1(n)$, where $\mathcal{R}'$ is obtained from $\mathcal{R}$ by removing the RIs with $\Box$,
- there is $m > n$ with $|m| \leq 2^{|\mathcal{R}|}$ and $\Box_p S \in r(m)$,
- there is $m < n$ with $|m| \leq 2^{|\mathcal{R}|}$ and $\Box_p S \in r(m)$.

Let $\min_{R,S}$ be the minimal integer with $\Box_p S \in r(m)$; if it exists, then $|\min_{R,S}| \leq 2^{|\mathcal{R}|}$. The maximal integer $\max_{R,S}$ with $\Box_p S \in r(m)$ has the same bound (if exists). The following example shows that these integers can indeed be exponential in $|\mathcal{R}|$.

**Example 17.** Let $\mathcal{R}$ be the DL-Lite$^{\Box \mathcal{O}}_{\text{bool/core}}$ RBox with the following RIs:

\[
P \sqsubseteq R_0, \quad R_i \sqsubseteq \Box_p R_{i+1} \ (\text{mod } 2), \quad \text{for } 0 \leq i < 2, \quad R_1 \sqsubseteq Q,
\]

\[
P \sqsubseteq Q_0, \quad Q_i \sqsubseteq \Box_p Q_{i+1} \ (\text{mod } 3), \quad \text{for } 0 \leq i < 3, \quad Q_1 \sqsubseteq Q, \quad Q_2 \sqsubseteq Q,
\]

\[
P \sqsubseteq Q, \quad P \sqsubseteq \Box_p Q.
\]

Clearly, $\mathcal{R} \models P \sqsubseteq \mathcal{O}_c^6 \Box_p Q$. If instead of the 2- and 3-cycles we use $p_i$-cycles, where $p_i$ is the $i$th prime number and $1 \leq i \leq n$, then $\mathcal{R} \models P \sqsubseteq \mathcal{O}_c^{p_1 \times \cdots \times p_n} \Box_p Q$.

In any case, the existence and binary representation of $\min_{R,S}$ and $\max_{R,S}$ can be computed in PSPACE.

**Theorem 18.** Checking consistency of DL-Lite$^{\Box \mathcal{O}}_{\text{bool/core}}$ and DL-Lite$^{\Box \mathcal{O}}_{\text{horn/core}}$ KBs is PSPACE-complete.

**Proof.** We encode the given KB $\mathcal{K}$ in LTL following the proof of Lemma 15 and representing $\Psi_\mathcal{K}$ as an LTL-formula with propositional variables of the form $C_u$ and $R_{u,v}$, for $u, v \in \Xi$, assuming that $R_{u,v} = R_{v,u}$; in particular, $\exists S(w)$ denotes the propositional variable for the unary atom $\exists S(w)$ of $\Psi_\mathcal{K}$. Sentences (28)–(33) can be translated into LTL by simply instantiating all universal quantifiers by constants in $\Xi$. Sentences (34), however, require a special treatment. First, we take

\[
\Box (\Box_1 \exists S_1 w \rightarrow \Box_2 \exists S_2 w), \quad \text{for every } \Box_1 S_1 \sqsubseteq \Box_2 S_2 \text{ in } \mathcal{R}, \quad (35)
\]
Table 3: Rewritability and data complexity of $DL$-Lite$^\circ_{c/r}$ OMAQs.

|             | $DL$-Lite$^\circ_{c/r}$ | $DL$-Lite$^\circ_{c/r}$ and $DL$-Lite$^\circ_{c/r}$ |
|-------------|--------------------------|-----------------------------------------------|
| $\text{bool}$ | $\text{coNP}$-hard [Sec. 2.1] | undecidable [Thm. 10] |
| $\text{bool/horn}$ | $\text{FO(RPR)}$ [Cor. 26] | $\text{FO(RPR)}$, $\text{NC}^1$-hard$^*$ |
| $\text{horn}$ | $\text{FO(<)}$ [Cor. 30] | $\text{FO(RPR)}$, $\text{NC}^1$-hard (Artale et al., 2021) |
| $\text{core}$ | $\text{FO(<)}$ [Cor. 30] | $\text{FO(<, \equiv)}$ [Cor. 26] |

$^*$The lower bound follows from (Artale et al., 2021, Theorem 8).

where each $\bigcirc_i$ is $\bigcirc_F$, $\bigcirc_P$ or blank. We also require the consequences of $R$ of the form $\exists R \subseteq \bigcirc_{\text{max}, R,S} \bigcirc_P \exists S$ and $\exists R \subseteq \bigcirc_{\text{min}, R,S} \bigcirc_P \exists S$, for all $R$ and $S$ with defined $\text{max}_{R,S}$ and $\text{min}_{R,S}$ that are not entailed by (35). These integers can be represented in binary using $n$ bits, where $n$ is polynomial in $|\mathcal{R}|$. So, for example, if $\text{max}_{R,S} \geq 0$, then we can encode $\exists R \subseteq \bigcirc_{\text{max}, R,S} \bigcirc_P \exists S$ by

$$\Box (\Box_F \Box_P (\exists R)_w \rightarrow \Box_P (\exists S)_w),$$

$$\Box ((\exists R)_w \land \neg \Box_P (\exists R)_w \rightarrow \bigcirc_{\text{max}, R,S} \bigcirc_P (\exists S)_w),$$

where $\bigcirc_{\text{max}, R,S}$ can be expressed by $O(n^2)$ formulas using the binary counter (similar to those in the proof of Theorem 12). To explain the meaning of (36)–(37), consider any $w \in \Delta^I$ in a model $I$ of $\mathcal{K}$. If $w \in (\exists R)^I(n)$ for infinitely many $n > 0$, then $w \in (\exists S)^I(n)$ for all $n$, which is captured by (36). Otherwise, there is $n$ such that $w \in (\exists R)^I(n)$ and $w \notin (\exists R)^I(m)$, for $m > n$, whence $w \in (\exists S)^I(k)$, for any $k < n + \text{max}_{R,S}$, which is captured by (37). The LTL translation $\Psi_\mathcal{K}'$ of $\mathcal{K}$ is a conjunction of (28)–(33), (35) as well as (36)–(37) for all $R$ and $S$ with defined $\text{max}_{R,S}$ with their counterparts for $\exists R \subseteq \bigcirc_{\text{min}, R,S} \bigcirc_P \exists S$. One can show that $\mathcal{K}$ is satisfiable iff $\Psi_\mathcal{K}'$ is satisfiable.

The PSPACE lower bound follows from the fact that $LTL_{\text{horn}}^\bigcirc$ is PSPACE-complete and every $LTL_{\text{horn}}^\bigcirc$-formula is equisatisfiable with some $DL$-Lite$^\circ_{\text{horn/core}}$ KB.

5. Rewriting $DL$-Lite$^\bigcirc_{\text{bool/horn}}$ OMAQs by Projection to LTL

Our aim in this section is to identify classes of FO-rewritable $DL$-Lite$^\bigcirc_{\text{bool/horn}}$ OMAQs, which will be done by projecting them to LTL OMAQs and using the classification of Artale et al. (2021). Initial steps in this direction have been made in Section 3.2 for OMAQs without $\bot$ and interacting concepts and roles. The results of this section are summarised in Table 3.

We begin by showing how to get rid of $\bot$ from $DL$-Lite$^\bigcirc_{\text{bool/horn}}$ OMAQs.
Proof. Suppose \( O = T \cup R \). We construct \( O' = T' \cup R' \) as follows. The RBox \( R' \) is obtained by first replacing every occurrence of \( \bot \) in \( R \) with a fresh role name \( P_1 \) and then adding to the result the RIs \( P \subseteq P_1 \) for all role names \( P \) in \( O \) that are inconsistent with \( O \).

(1) We show how to construct a model \( I\) of \( O' \), and so \( \text{ans}(O', A_1, A) = \emptyset \). Moreover, \( \text{ans}(O, B, A) \supseteq \text{ans}^Z(O', B, A) \) and \( \text{ans}^Z(O, \emptyset, A) \supseteq \text{ans}^Z(O', \emptyset, A) \).

To show the converse direction for (i), suppose \((O', A_1)\) has no certain answers over \( A \). We show how to construct a model \( I\) of \((O, A)\). Obviously, \( O' \) and \( A \) are consistent. For each \( a \in \text{ind}(A) \), there is a model \( I_a \) of \((O', A)\) such that \( a^{I_a} \notin A_1^{(n)} \) for all \( n \in \mathbb{Z} \) (recall that \( A_1 \) is global). Also, for each role \( S \) consistent with \( O \), there is a model \( I_S \) of \((O', \{ S(w, u, 0) \}) \) such that \( w^{I_S} \notin A_1^{(n)} \) for all \( n \in \mathbb{Z} \) (again, \( A_1 \) is global). We take, for each \( a \in \text{ind}(A) \), the beam \( b_a \) for \( a^{I_a} \) in \( I_a \), and for each role \( S \) consistent with \( O \), the beam \( b_{os} \) of \( w^{I_S} \) in \( I_S \), and apply Lemma 13 to obtain a model \( I \) of \((O', A)\) such that \( A_1^{(n)} = \emptyset \) and \( P_1^{(n)} = \emptyset \). By construction, \( I \) is also a model of \((O, A)\).

(2) To complete the proof of (ii), it remains to show that, for consistent \( O \) and \( A \), we have \( \text{ans}(O, B, A) \subseteq \text{ans}(O' \cup \{ A_1 \subseteq B \}, B, A) \). Suppose \((a, \ell) \notin \text{ans}(O' \cup \{ A_1 \subseteq B \}, B, A) \). Then there is a model \( I_a \) of \((O', A)\) such that \( a^{I_a} \notin B^{\ell} \), and so \( a^{I_a} \notin A_1^{(n)} \) for all \( n \in \mathbb{Z} \); we take the beam \( b_a \) of \( a^{I_a} \) in \( I_a \). As \( \text{ans}(O', A_1, A) = \emptyset \), for every \( b \in \text{ind}(A) \setminus \{ a \} \), there is a model \( I_b \) of \((O', A)\) such that \( b^{I_b} \notin A_1^{(n)} \) for all \( n \in \mathbb{Z} \); we take the beam \( b_b \) of \( b^{I_b} \) in \( I_b \). We now construct a model \( I \) of \((O, A)\) with \( a^{I_a} \notin B^{\ell} \) from the chosen beams using the same unravelling procedure as in the proof of (i) above. The case of role OMPQs is similar.

As a consequence of Theorem 19, we immediately obtain the following:

**Corollary 20.** Let \( L \) be one of \( \text{FO}(\leq) \), \( \text{FO}(\leq, \subseteq) \) or \( \text{FO}(\text{RPR}) \), \( r \in \{ \text{core, horn} \} \) and \( O \) a DL-Lite\(_{c/r}\) ontology. Suppose \( O' \) is the \( \bot \)-free ontology provided by Theorem 19 for \( O \) and let \( \chi_1 = \exists x, t Q_1(x, t) \), where \( Q_1(x, t) \) is an \( L \)-rewriting of the OMAQ \((O', A_1)\).
(i) If \( Q'(x,t) \) is an \( \mathcal{L} \)-rewriting of the OMAQ \((\mathcal{O}' \cup \{A_\bot \subseteq B\}, B)\), for a basic concept \( B \), then \( Q'(x,t) \lor \chi_L \) is an \( \mathcal{L} \)-rewriting of \((\mathcal{O}, B)\).

(ii) If \( Q'(x,y,t) \) is an \( \mathcal{L} \)-rewriting of the OMAPIQ \((\mathcal{O}', q)\), for a positive temporal role \( q \), then \( Q'(x,y,t) \lor \chi_L \) is an \( \mathcal{L} \)-rewriting of \((\mathcal{O}, q)\).

Now we focus on projecting \( \bot \)-free \( DL-Lite^{\Box \bowtie} \) OMAQs with interacting concepts and roles to the \( LTL \) axis.

5.1 \( \mathcal{L} \)-rewritability of \( DL-Lite^{\Box \bowtie} \) OMAQs

We begin with two examples illustrating the interaction between the DL and temporal dimensions in \( DL-Lite^{\Box \bowtie} \), we need to take into account when constructing rewritings.

Example 21. Let \( \mathcal{T} = \{ B \subseteq \exists P, \exists Q \subseteq A \} \) and \( \mathcal{R} = \{ P \subseteq \Box_P Q \} \). An obvious idea of constructing a rewriting for the OMAQ \( q = (\mathcal{T} \cup \mathcal{R}, A) \) would be to find first a rewriting of the \( LTL \) OMAPIQ \((\mathcal{T}', A')\) obtained from \((\mathcal{T}, A)\) by replacing the basic concepts \( \exists P \) and \( \exists Q \) with surrogates \((\exists P)^\dagger = E_P\) and \((\exists Q)^\dagger = E_Q\), respectively. This would give us the first-order query \( A(t) \lor EQ(t) \). By restoring the intended meaning of \( A \) and \( EQ \) (see the proof of Proposition 7), we would then obtain \( A(x,t) \lor \exists y Q(x,y,t) \). The second step would be to rewrite, using the RBox \( \mathcal{R} \), the atom \( Q(x,y,t) \) into \( Q(x,y,t) \lor P(x,y,t-1) \). Alas, the resulting formula
\[
A(x,t) \lor \exists y (Q(x,y,t) \lor P(x,y,t-1))
\]
falls short of being an FO(\(<\))-rewriting of \( q \) as it does not return the certain answer \((a,1)\) over \( A = \{ B(a,0) \} \). The reason is that, in our construction, we did not take into account the CI \( \exists P \subseteq \Box_P \exists Q \), which is a consequence of \( \mathcal{R} \). If we now add the ‘connecting axiom’ \((\exists P)^\dagger \subseteq \Box_P (\exists Q)^\dagger\) to \( \mathcal{T}' \), then in the first step we obtain \( A(t) \lor EQ(t) \lor E_P(t-1) \lor B(t-1) \), which gives us the correct FO(\(<\))-rewriting
\[
A(x,t) \lor \exists y (Q(x,y,t) \lor P(x,y,t-1)) \lor \exists y P(x,y,t-1) \lor B(x,t-1)
\]
of \( q \), where the third disjunct is obviously redundant and can be omitted.

Example 22. Consider now the OMAQ \( q = (\mathcal{T} \cup \mathcal{R}, A) \) with
\[
\mathcal{T} = \{ \exists Q \subseteq \Box_P A \}, \quad \mathcal{R} = \{ P \subseteq \Box_P P_1, T \subseteq \Box_P T_1, T_1 \subseteq \Box_P T_2, P_1 \cap T_2 \subseteq Q \}.
\]
The two-step construction outlined in Example 21 would give us first the formula
\[
\Phi(x,t) = A(x,t) \lor \exists t' ((t < t') \lor \exists y Q(x,y,t'))
\]
as a rewriting of \((\mathcal{T}, A)\). It readily seen that the following formula is a rewriting of \((\mathcal{R}, Q)\):
\[
\Psi(x,y,t') = Q(x,y,t') \lor [(P_1(x,y,t') \lor \exists t'' ((t'' < t') \land P(x,y,t''))) \lor (T_2(x,y,t') \lor \exists t'' ((t'' < t') \land (T_1(x,y,t'') \lor \exists t'''' ((t'''' < t'') \land T(x,y,t'''))))]).
\]
However, the result of replacing \( Q(x,y,t') \) in \( \Phi(x,t) \) with \( \Psi(x,y,t') \) is not an FO-rewriting of \((\mathcal{O}, A)\): when evaluated over \( A = \{ T(a,b,0), P(a,b,1) \} \), it does not return the certain
together with symmetrical CIs for $\cdash$.

In Example 21, for the role type $\rho$ the set of all such CIs for all possible role types $\rho$ for $T$ so

$\exists$ the $R_Z$ evaluated the obtained 'rewriting' over $\rho$ such that $\rho$

that a role type is non-empty if it contains some role $R$

use the positive part of role types (ignoring all of the negated roles); in particular, we say

answers $(a, 0)$ and $(a, 1)$; see Fig. 5. (Note that these answers would be found had we

evaluated the obtained 'rewriting' over $Z$ rather than $\{0, 1\}$.)

This time, in the two-step construction of the rewriting, we are missing the 'consequence'$\exists(\Box_\rho P_1 \cap \Box_\rho T_2) \subseteq \Box_\rho \exists Q$ of $\mathcal{R}$ and $\mathcal{T}$. To fix the problem, we can take a fresh role name $G_\rho$, for $\rho = \{ \Box_\rho P_1, \Box_\rho T_2 \}$, and add the 'connecting axiom'$\exists G_\rho \subseteq \Box_\rho \exists Q$ to $\mathcal{T}$. Then, in the first step, we rewrite the extended TBox and $A$ into the formula

$\Phi'(x, t) = A(x, t) \lor \exists t' ((t < t') \land \exists y Q(x, y, t')) \lor \exists t' \exists y G_\rho(x, y, t'),$

where we replace $Q(x, y, t')$ with $\Psi(x, y, t')$ as before, and restore the intended meaning of $G_\rho(x, y, t')$ by rewriting $(\mathcal{R}, \Box_\rho P_1 \cap \Box_\rho T_2)$ into

$P(x, y, t') \land (T_1(x, y, t') \lor \exists t'' ((t'' < t') \land T(x, y, t'')))$

and substituting it for $G_\rho(x, y, t')$ in $\Phi'(x, t)$.

We now formally define the connecting axioms for a given $\bot$-free $DL$-Lite$|^\Box_\rho$horn ontology $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$. We assume that $\mathcal{R}$ contains all of the role names from $\mathcal{T}$. Recall that a role type $\rho$ for $\mathcal{R}$ is a maximal subset of $sub_\mathcal{R}$ consistent with $\mathcal{R}$. In this section, we only use the positive part of role types (ignoring all of the negated roles); in particular, we say that a role type is non-empty if it contains some role $R$. Given a role type $\rho$, we consider the $\mathcal{R}$-canonical rod $r_\rho$ (see Section 4.2) for $\{R^i(0) \mid R \in \rho \}$. Note that, by definition, we have $r_\rho(0) = \rho$. By Lemma 6 (i), we can find positive integers $s^\rho \leq |\mathcal{R}|$ and $p^\rho \leq 2^{2|\mathcal{R}|}$ such that

$r_\rho(n) = r_\rho(n - p^\rho),$ for $n \leq -s^\rho$, and $r_\rho(n) = r_\rho(n + p^\rho),$ for $n \geq s^\rho$.

For a role type $\rho$ for $\mathcal{R}$, we take a fresh role name $G_\rho$ and fresh concept names $D_\rho^n$, for $-s^\rho - p^\rho < n < s^\rho + p^\rho$, and define the following CIs:

$\exists G_\rho \subseteq D_\rho^0,$ $D_\rho^n \subseteq \Box_\rho D_\rho^{n+1},$ for $0 \leq n < s^\rho + p^\rho - 1,$ $D_\rho^{s^\rho+p^\rho-1} \subseteq \Box_\rho D_\rho^{s^\rho},$

and $D_\rho^n \subseteq \exists S,$ for roles $S \in r_\rho(n)$ and $0 \leq n < s^\rho + p^\rho,$

together with symmetrical CIs for $-s^\rho - p^\rho \leq n \leq 0$ for the past-time 'loop'. Let (con) be the set of all such CIs for all possible role types $\rho$ for $\mathcal{R}$, and let $\mathcal{T}_R = \mathcal{T} \cup (\text{con})$

Example 23. In Example 21, for the role type $\rho = \{ P, \Box_\rho Q \}$, we have $s^\rho = 2$, $p^\rho = 1$, and so $\mathcal{T}_R$ contains the following:

$\exists P \subseteq D_\rho^0,$ $D_\rho^0 \subseteq \Box_\rho D_\rho^1,$ $D_\rho^1 \subseteq \Box_\rho D_\rho^2,$ $D_\rho^2 \subseteq \Box_\rho D_\rho^3,$ and $D_\rho^0 \subseteq \exists P,$ $D_\rho^1 \subseteq \exists Q,$
which imply $\exists P \subseteq \circ \exists Q$. In the context of Example 22, for the role type $\rho = \{ \boxdot T_1, \boxdot T_2 \}$, we have $s^{\rho} = 1$, $p^{\rho} = 1$, and so $T_R$ contains the following CIs:

$$\exists G_\rho \subseteq D_\rho^0, D_\rho^0 \subseteq \circ \rho D_\rho^1, D_\rho^1 \subseteq \circ \rho D_\rho^1, \text{ and } D_\rho^1 \subseteq \exists P, D_\rho^1 \subseteq \exists T_2, D_\rho^1 \subseteq \exists Q.$$ 

Note that, in this case, instead of two CIs $D_\rho^0 \subseteq \circ \rho D_\rho^1$ and $D_\rho^1 \subseteq \circ \rho D_\rho^1$, we could use a single $D_\rho^0 \subseteq \circ \rho D_\rho^1.$

Denote by $T_R^1$ the $\text{LTL}^{\circ \circ}$ TBox obtained from $T_R$ by replacing every basic concept $B$ in it with $B^!$. Consider an ABox $A$. For any $a, b \in \text{ind}(A)$, let $r_{a,b}$ be the $R$-canonical rod for $A_{a,b}^1$. We split $A$ into the concept and role components, $U$ and $B$, as follows:

$$U = \{ (a, \ell) \mid (a, \ell) \in A \},$$

$$B = \{ \exists G_\rho(a, \ell) \mid a \in \text{ind}(A), \ell \in \text{tem}(A) \text{ and } \rho = r_{a,b}(\ell) \neq \emptyset, \text{ for some } b \in \text{ind}(A) \}.$$

We denote by $U^1_\rho$ and $B^1_\rho$ the sets of all atoms $A^{\dagger}(\ell)$, for $(a, \ell) \in U$, and $(\exists G_\rho)^{\dagger}(\ell)$, for $\exists G_\rho(a, \ell) \in B$, respectively. Observe that the connecting axioms are such that $(\text{con})^{\dagger}$ is an $\text{LTL}^{\circ \circ}$ core ontology, and the ABox $B$ is defined so that, for any $a \in \text{ind}(A)$ and $n \in \mathbb{Z}$,

$$S \in r_{a,b}(n) \text{ for some } b \in \text{ind}(A) \quad \text{iff} \quad (\exists S)^{\dagger}(n) \in C_{(\text{con})^{\dagger}, B^1_\rho} \text{ for any role } S \text{ in } R. \quad (38)$$

Indeed, let $S \in r_{a,b}(n)$. If $n \in \text{tem}(A)$, then $S \in \rho$ and $\exists G_\rho(a, n) \in B$, for $\rho = r_{a,b}(n)$, whence $(\exists G_\rho)^{\dagger}(n) \in B^1_\rho$, and so $(\exists S)^{\dagger}(n) \in C_{(\text{con})^{\dagger}, B^1_\rho}$. If $n > \text{max}(A)$, then we consider $\rho = r_{a,b}(\text{max}(A))$. It should be clear that the canonical model of $(R^{\dagger}, \{ R^{\dagger}(\text{max}(A) \mid R \in \rho) \})$ contains $S(n)$. So $\rho \neq \emptyset$ and $\exists G_\rho(a, \text{max}(A)) \in B$, whence $(\exists G_\rho)^{\dagger}(\text{max}(A)) \in B^1_\rho$, and so $(\exists S)^{\dagger}(n) \in C_{(\text{con})^{\dagger}, B^1_\rho}$. The case $n < \text{min}(A)$ is symmetric. The converse implication follows directly from the definition of $(\text{con})$. We use (38) to establish the following key result:

**Lemma 24.** Let $O = T \cup R$ be a DL-Lite$^{\circ \circ}$ horn ontology and $B$ be a basic concept in the alphabet of $O$. Then, for any ABox $A$, we have

$$\text{ans}^{Z}(O, B, A) = \{ (a, n) \mid a \in \text{ind}(A) \text{ and } n \in \text{ans}^{Z}(T_R^1, B^1, U^1_\rho \cup B^1_\rho) \}.$$ 

**Proof.** ($\subseteq$) Suppose that $n \notin \text{ans}^{Z}(T_R^1, B^1, U^1_\rho \cup B^1_\rho)$. Then there is an $\text{LTL}$ model $I_a$ of $(T_R^1, U^1_\rho \cup B^1_\rho)$ with $I_a \models B^1(n)$. We define a model $I$ of $(O, A)$ with $a^{\dagger} \notin B^2(n)$ using unravelling (Lemma 13) similarly to the proof of Theorem 19. To begin with, we take the beam $b_a: n \rightharpoonup \{ C \in \text{sub}_{T} \mid I_a \models C(n) \}$; note that $b_a$ is a beam for $T$ because $T_R^1$ extends $T$. By (38), the $R$-canonical rod $r_{a,b}$ for $A_{a,b}^1$ is $b_a$-compatible, for each $b \in \text{ind}(A)$. Next, we fix a model $J$ of $(O, A)$ and, for every $b \in \text{ind}(A) \setminus \{ a \}$, take the beam $b_b$ of $b_J^J$ in $J$. By Lemma 13, we obtain a model $I$ of $(O, A)$ with $a^{\dagger} \notin B^2(n)$.

The inclusion ($\supseteq$) is straightforward. $\square$

We now use this technical result to construct rewritings for OMAQs $(O, B)$ from rewritings of suitable $\text{LTL}$ OMPIQs, where we identify a role type $\rho$ with the intersection of all $R \in \rho$, i.e., $\rho = \bigcap_{R \in \rho} R$, and $p^{\dagger}$ with the intersection of all $R^{\dagger}$, for $R \in \rho$;

**Theorem 25.** Let $L$ be one of $\text{FO}(\langle \rangle), \text{FO}(\langle \rangle, \exists)$ or $\text{FO}(\text{RPR})$. A $\bot$-free DL-Lite$^{\circ \circ}$ horn OMAQ $q = (O, B)$ with $O = T \cup R$ is $L$-rewritable whenever
the \( \text{LTL}^{\Box \Box} \) OMAQs \( q^\dag = (\mathcal{T}_R^\dag, B^\dag) \) are \( \mathcal{L} \)-rewritable, for role types \( \rho \) for \( R \).

Proof. By Proposition 8, we have an \( \mathcal{L} \)-rewriting \( Q^\rho(x,y,t) \) of every \( q^\dag = (R, \rho) \). Similarly to the proof of Proposition 7, we claim that the \( \mathcal{L} \)-formula \( Q(x,t) \) obtained from an \( \mathcal{L} \)-rewriting \( \mathcal{Q}^\dag(t) \) of \( q^\dag \) by replacing every \( A(s) \) in it with \( A(x,s) \), every \( (\exists P)^\dag \) with \( \exists y P(x,y,s) \), every \( (\exists P^-)^\dag \) with \( \exists y P(y,x,s) \), every \( (\exists G)^\dag \) with \( \exists y Q^\rho_p(x,y,s) \) and, in the case of FO(RPR), by replacing every \( Q(t_1,\ldots,t_k) \), for a relation variable \( Q \), with \( R(x,t_1,\ldots,t_k) \) is an \( \mathcal{L} \)-rewriting of \( q^\dag \).

Indeed, we show that \( \mathcal{O}, A \models B(a,\ell) \) iff \( \mathcal{S}_A \models Q^\rho(a,\ell) \), for any ABox \( A \), any \( \ell \in \text{tem}(A) \) and any \( a \in \text{ind}(A) \). If \( B \) uses a concept or role name not in \( \mathcal{O} \), then the claim is trivial. Otherwise, by Lemma 24, \( \mathcal{O}, A \models B(a,\ell) \) iff \( \mathcal{T}^\dag \cup A^\dag \models B^\dag(\ell) \). As \( Q^\dag(\ell) \) is an \( \mathcal{L} \)-rewriting of \( (\mathcal{T}^\dag, B^\dag) \), the latter is equivalent to \( \mathcal{S}_{A^\dag} \models Q^\dag(\ell) \). Now, since \( Q^\rho(x,y,t) \) is an \( \mathcal{L} \)-rewriting of \( q^\dag \), for all \( b \in \text{ind}(A) \) and \( n \in \text{tem}(A) \), we have \( (\exists G)^\dag(n) \in B^\dag \) iff \( \rho = r_{b,c}(n) \) for the \( \mathcal{R} \)-canonical rod for \( A^\dag_{b,c} \), for some \( c \in \text{ind}(A) \), iff \( \mathcal{S}_A \models \exists y Q^\rho_b(b,y,n) \). Then, \( \mathcal{S}_{A^\dag} \models Q^\dag(\ell) \) iff \( \mathcal{S}_A \models Q(a,\ell) \), as required.

As a consequence of Corollary 20 (i), Theorem 25 and FO(\(<,\forall/\exists \))\( /\text{FO(RPR)} \)-rewritability of \( \text{LTL}^{\Box \Box} /\text{LTL}^{\Box \Box} \) OMAQs and \( \text{LTL}^{\Box \Box} /\text{LTL}^{\Box \Box} \) OMAQs (Artale et al., 2021, Theorems 16, 8, 24 and 27), we obtain:

Corollary 26. (i) All \( \text{DL-Lite}^{\Box \Box}_{\text{core}/\text{krom}} \) and \( \text{DL-Lite}^{\Box \Box}_{\text{core}/\text{krom}} \) OMAQs are FO(\(<,\forall/\exists \))-rewritable.

(ii) All \( \text{DL-Lite}^{\Box \Box}_{\text{boof}/\text{horn}} \) OMAQs are FO(RPR)-rewritable.

Despite FO(\(<\))-rewritability of \( \text{LTL}^{\Box \Box} \) OMAQs, we cannot obtain FO(\(<\))-rewritability of \( \text{DL-Lite}^{\Box \Box}_{\text{horn}} \) OMAQs because the connecting axioms (\text{con}) use the next-time operator \( \Box \). We now show that actually there are \( \text{DL-Lite}^{\Box \Box}_{\text{horn}} \) OMAQs that are not FO(\(<\))- and not even FO(\(<,\forall/\exists \))-rewritable:

Theorem 27. There is a \( \text{DL-Lite}^{\Box \Box}_{\text{horn}} \) OMAQ, answering which is NC\(^1\)-hard for data complexity.

Proof. Consider the \( \text{DL-Lite}^{\Box \Box}_{\text{horn}} \) OMAQ \( q = (\mathcal{O}, \exists S_0) \) with \( \mathcal{O} = \mathcal{T} \cup \mathcal{R} \) and

\[
\mathcal{T} = \{ \exists R_k \sqcap A_0 \sqsubseteq \exists S_k, \quad \exists R_k \sqcap A_1 \sqsubseteq \exists S_{1-k} \mid k = 0, 1 \} \cup \{ B \sqsubseteq \exists S_0 \},
\mathcal{R} = \{ S_k \sqsubseteq F_k, \quad S_k \sqsubseteq \Box F_k, \quad S_k \sqsubseteq \Box P_k, \quad \Box P_k \sqcap P_k \sqsubseteq R_k \mid k = 0, 1 \};
\]

cf. Example 5. For \( e = (e_0, \ldots, e_{n-1}) \in \{0, 1\}^n \), let \( \mathcal{A}_e = \{ B(a,n) \} \cup \{ A_e(i,a) \mid i < n \} \). Then \( (a,0) \) is a certain answer to \( q \) over \( \mathcal{A}_e \) iff the number of 1s in \( e \) is even—see Fig. 6 and note that \( \exists S_k(a,n) \) always generates a fresh witness \( aS^n_k \). The same idea can be used to simulate arbitrary NFAs as in the proof of (Artale et al., 2021, Theorem 10), which shows NC\(^1\)-hardness. We leave details to the reader. \( \square \)
is to give a sufficient condition under which the connecting axioms to encode the LTL in DL-Lite
As follows from Theorem 27, DL-Lite turns out to be fundamentally different from LTL\textsuperscript{horn}; in the proof, we used basic concepts of the form \( \exists Q \) and Horn RIIs with \( \Box \) and \( \square \) to encode the \( \bigcirc \) operator on concepts in the sense that \( \mathcal{O} \models \exists S_k \subseteq \circ \exists R_k \). Our aim now is to give a sufficient condition under which the connecting axioms (con) can be expressed in DL-Lite\textsubscript{core}, which would guarantee FO(\( < \))-rewritability of \( q^\dagger \) in Theorem 25.

We say that a DL-Lite\textsubscript{bool/horn} RBox \( \mathcal{R} \) (and an ontology with such \( \mathcal{R} \)) is role-monotone if, for any role type \( \rho \) for \( \mathcal{R} \) and any role \( S \),

\[
S \in r_\rho(n) \text{ and } n \neq 0 \quad \text{implies} \quad S \in r_\rho(k) \text{ for all } k \geq n \text{ or for all } k \leq n. \tag{39}
\]

In other words, \( \{ n \in \mathbb{Z} \mid S \in r_\rho(n) \} = I_0^S \cup I_0^S \cup I_0^S \), where each of \( I_0^S, I_0^S \) and \( I_0^S \) is either empty or an interval of the form \((\infty, m'], [0)\) or \([m, \infty)\), respectively; see Fig. 7 for an illustration of the possible cases. The proof of Theorem 27 gives an example of an RBox that is not role-monotone: for the role type \( \rho \) containing \( S_k, F_k, \Box F_k \) and \( \Box F_k \), we have \( R_k \in r_\rho(-1) \), but neither \( R_k \in r_\rho(-2) \) nor \( R_k \in r_\rho(0) \).

We now show how to replace (con) for a role-monotone \( \mathcal{R} \) with a set of DL-Lite\textsubscript{core} CIs. Let \( \rho \) be a role type for \( \mathcal{R} \). We consider the following four groups of cases for each role \( S \).

(i) If \( I_0^S = I_0^S = \emptyset \), then we take the CI

\[
\exists G_\rho \subseteq \exists S, \quad \text{if } I_0^S \neq \emptyset; \tag{i1}
\]

otherwise—that is, in case \( S \notin r_\rho(n) \), for all \( n \in \mathbb{Z} \)—we do not need any CIs.

(ii) If \( I_0^S = \emptyset \) and \( I_0^S = [m, \infty) \), then we take the following CIs, depending on \( m \) and \( I_0^S \) (note that \( I_0^S \subseteq I_0^S \) if \( m \leq 0 \)):

\[
\exists G_\rho \subseteq \Box^m \exists S, \quad \text{if } m > 0 \text{ and } I_0^S = \emptyset; \tag{ii0}
\]

\[
\exists G_\rho \subseteq \exists S \quad \text{and} \quad \exists G_\rho \subseteq \Box^m \exists S, \quad \text{if } m > 0 \text{ and } I_0^S \neq \emptyset; \tag{ii1}
\]

\[
\exists G_\rho \subseteq \Box^m D_{\rho} S \quad \text{and} \quad \Box^m+1 D_{\rho} S \subseteq \exists S, \quad \text{if } m \leq 0, \text{ for a fresh } D_{\rho} S. \tag{ii2}
\]

(iii) If \( I_0^S = (\infty, m'] \) and \( I_0^S = \emptyset \), then we take the mirror image of (ii), with \( \Box \), \( < \), \( \geq \) and \( m' \) in place of \( \Box \), \( > \), \( \leq \) and \( m \), respectively.
Denote by $\rho$ the set of CIs for all role types $(\text{con})$.

For any role-monotone $\mathcal{R}$, Lemma 24 and Theorem 25 hold for $\mathcal{T}_{\text{mon-}R}$; thus, we obtain:

**Theorem 28.** For any role-monotone $\mathcal{O}$, Lemma 24 and Theorem 25 hold for $\mathcal{T}_{\text{mon-}R}$ in place of $\mathcal{T}_R$.

We show now that all $DL\text{-Lit}_{\text{core}}$ RBoxes as well as $DL\text{-Lite}_{\text{horn}}$ RBoxes without $\Box$-operators on the left-hand side of RIs are role-monotone. Recall that the fragment of $DL\text{-Lite}_{\text{bool/horn}}$ with RIs of the latter type is denoted by $DL\text{-Lite}_{\text{bool/horn}+}$. Note that instead of RIs with $\Box$-operators on the right-hand side only we can take RIs with $\Diamond$-operators on the left-hand side only (see the discussion on TQL in Section 2).

**Theorem 29.** All $DL\text{-Lite}_{\text{bool/core}}$ and $DL\text{-Lite}_{\text{bool/horn}+}$ ontologies are role-monotone.

**Proof.** Suppose first that $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$ is a $DL\text{-Lite}_{\text{bool/core}}$ ontology and $\rho$ a role type for $\mathcal{R}$. Denote by $\tau^\rho_R(n)$ the set of temporalised roles $R$ such that $R^3(n) \in \mathcal{R}_R^2(\{R^3(0) \mid R \in \rho\});
Table 4: Rewritability and data complexity of $DL$-Lite$^\Box_{c/r}$ OMPIQs.

|                     | $DL$-Lite$^\Box_{c/r}$ | $DL$-Lite$^\Box_{c/r}$ and $DL$-Lite$^\Box_{c/r}$ |
|---------------------|------------------------|--------------------------------------------------|
| $bool$              | coNP-hard              | undecidable                                      |
| $bool/horn$         | coNP-hard              |                                                  |
| $bool/horn^+$       | coNP-hard              |                                                  |
| $bool/core$         | FO(RPR)               |                                                  |
| $krom$              | FO(RPR)               | FO(RPR) [Cor. 40]                                |
| $krom/core$         | FO($<$)                | FO(\leq, \equiv) [Cor. 40]                      |

As a consequence of the above and FO($<$)-rewritability of $LTL^\Box_{bool}$ OMAQs and $LTL^\Box_{horn}$ OMPIQs (Artale et al., 2021, Theorems 11 and 24), we obtain:

**Corollary 30.** All $DL$-Lite$^\Box_{bool/core}$ and $DL$-Lite$^\Box_{bool/horn^+}$ OMAQs are FO($<$)-rewritable.

6. Rewriting $DL$-Lite$^\Box_{horn}$ OMPIQs

Our next aim is to lift, where possible, the rewritability results obtained in Section 5 for temporal $DL$-Lite OMAQs to OMPIQs. Our results are summarised in Table 4.

To begin with, recall that all $LTL$ OMPIQs are FO(RPR)-rewritable, and so answering them can always be done in NC$^1$ for data complexity. On the other hand, as follows from (Schaerf, 1993), answering the atemporal $DL$-Lite$^\Box_{krom}$ OMPIQ

$$\big( \{ T \sqsubseteq A \sqcup B \}, \exists R. (\exists P_1.A \sqcap \exists P_2.A \sqcap \exists N_1.B \sqcap \exists N_2.B)\big)$$

is coNP-complete for data complexity. In a similar manner, one can show that answering OMPIQs with the $DL$-Lite$^\Box_{krom}$ ontology $\{ A \sqsubseteq \Diamond p.B \}$ is coNP-complete because its normal form is $\{ A \sqcap \Box p.A' \sqsubseteq \bot, T \sqsubseteq A' \sqcup B \}$. There are also $DL$-Lite$^\Box_{krom}$ OMPIQs that are complete for P, NL and L (Gerasimova et al., 2020), with the inclusion NC$^1$ $\subseteq$ L believed to be strict. (Note that these results require ABoxes with an unbounded number of individual names.) Thus, in the remainder of Section 6, we focus on $DL$-Lite$^\Box_{horn}$ OMPIQs.

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6.1 Canonical Models

The proofs of the rewritability results for OMPIQs require canonical models for fragments of DL-Lite$_{ow}$, which generalise the canonical models of Artale et al. (2021).

Suppose we are given a DL-Lite$_{ow}$ ontology $\mathcal{O}$ and an ABox $\mathcal{A}$. Let $\Lambda$ be a countable set of atoms of the form $\bot$, $C(w, n)$ and $R(w_1, w_2, n)$, where $C$ is a temporalised concept, $R$ a temporalised role and $n \in \mathbb{Z}$. To simplify notation, we refer to the concept and role atoms $C(w, n)$ and $R(w_1, w_2, n)$ as $\vartheta(w, n)$, calling $w$—that is, $w$ or $(w_1, w_2)$—a tuple. Denote by $\text{cl}_\mathcal{O}(\Lambda)$ the result of applying non-recursively to $\Lambda$ the following rules, where $S$ is a role:

(mp) if $\mathcal{O}$ contains $\vartheta_1 \cap \cdots \cap \vartheta_m \models \vartheta$ and $\vartheta_i(w, n) \in \Lambda$ for all $i$, $1 \leq i \leq m$, then we add $\vartheta(w, n)$ to $\Lambda$;

(cls) if $\mathcal{O}$ contains $\vartheta_1 \cap \cdots \cap \vartheta_m \sqsubseteq \bot$ and $\vartheta_i(w, n) \in \Lambda$ for all $i$, $1 \leq i \leq m$, then we add $\bot$;

($\Box^{-}_p$) if $\Box_p \vartheta(w, n) \in \Lambda$, then we add all $\vartheta(w, k)$ with $k > n$;

($\Box^{-}_p$) if $\vartheta(w, k) \in \Lambda$ for all $k > n$, then we add $\Box_p \vartheta(w, n)$;

($\bigcirc^{-}_p$) if $\bigcirc_p \vartheta(w, n) \in \Lambda$, then we add $\vartheta(w, n+1)$;

($\bigcirc^{-}_p$) if $\vartheta(w, n+1) \in \Lambda$, then we add $\bigcirc_p \vartheta(w, n)$;

(inv) if $S(w_1, w_2, n) \in \Lambda$, then we add $S^-(w_2, w_1, n)$ to $\Lambda$ (assuming that $(S^-)^- = S$);

($\exists^{-}$) if $S(w_1, w_2, n) \in \Lambda$, then we add $\exists S(w_1, n)$;

($\exists^{-}$) if $\exists S(w, n) \in \Lambda$, then we add $S(w, wS^m, n)$, where $wS^m$ is a fresh individual name called the witness for $\exists S(w, n)$.

We set $\text{cl}^0_{\mathcal{O}}(\Lambda) = \Lambda$ and, for any successor ordinal $\xi + 1$ and limit ordinal $\zeta$,

$$\text{cl}^{\xi+1}_{\mathcal{O}}(\Lambda) = \text{cl}_{\mathcal{O}}(\text{cl}^{\xi}_{\mathcal{O}}(\Lambda)), \quad \text{and} \quad \text{cl}^\zeta_{\mathcal{O}}(\Lambda) = \bigcup_{\xi < \zeta} \text{cl}^{\xi}_{\mathcal{O}}(\Lambda).$$  

(40)

Let $\mathcal{C}_{\mathcal{O}, \mathcal{A}} = \text{cl}^{\omega}_{\mathcal{O}}(\mathcal{A})$, where $\omega_1$ is the first uncountable ordinal (as $\text{cl}^{\omega}_{\mathcal{O}}(\Lambda)$ is countable, there is an ordinal $\alpha < \omega_1$ such that $\text{cl}^{\alpha}_{\mathcal{O}}(\Lambda) = \text{cl}^{\omega}_{\mathcal{O}}(\Lambda)$, for all $\beta \geq \alpha$). We regard $\mathcal{C}_{\mathcal{O}, \mathcal{A}}$ as both a set of atoms of the form $\bot$ and $\vartheta(w, n)$ and as an interpretation whose domain, $\Delta^{\mathcal{C}_{\mathcal{O}, \mathcal{A}}}$, comprises $\text{ind}(\mathcal{A})$ and the witnesses $wS^m$ that are used in the construction of $\mathcal{C}_{\mathcal{O}, \mathcal{A}}$, and the interpretation function is defined by taking $w \in \eta^{\mathcal{C}_{\mathcal{O}, \mathcal{A}}(n)}$ iff $\eta(w, n) \in \mathcal{C}_{\mathcal{O}, \mathcal{A}}$, for any concept or role name $\eta$.

**Example 31.** The interpretation $\mathcal{C}_{\mathcal{O}, \mathcal{A}}$ for $\mathcal{A} = \{ A(a, 0) \}$ and

$$\mathcal{O} = \{ A \sqsubseteq \Box_p \exists P, \quad P \sqsubseteq \Box_p R, \quad R \sqsubseteq \Box_p R, \quad \Box_p R \sqsubseteq S, \quad \exists S^+ \sqsubseteq \exists T, \quad \exists T^- \sqsubseteq A \}$$

is shown in Fig. 8. Note that its construction requires $\omega^2$ applications of $\text{cl}$: $\omega^2 + 2$ applications are needed to derive $S(a, aP^1, 1)$ (in fact, all $S(a, aP^k, n)$, for $n \geq k > 0$), then a $T$-successor with $A$ is created, and the process is repeated again and again. The resulting interpretation has infinite branching and infinite depth.
Theorem 32. Let $\mathcal{O}$ be a DL-Lite$_{horn}^0$ ontology and $\mathcal{A}$ an ABox. Then the following hold:

(i) for any temporalised concept or role $\vartheta$, any tuple $w$ in $\Delta_{\mathcal{C},\mathcal{A}}$ and any $n \in \mathbb{Z}$, we have $w \in \eta_{\mathcal{C},\mathcal{A}}(w,n)$ iff $\vartheta(w,n) \in \mathcal{C}_{\mathcal{O},\mathcal{A}}$;

(ii) for any model $\mathcal{I}$ of $\mathcal{O}$ and $\mathcal{A}$, there exists a homomorphism $h$ from $\mathcal{C}_{\mathcal{O},\mathcal{A}}$ to $\mathcal{I}$ such that $h(w) \in \mathcal{I}$ implies $h(w) \in \vartheta(\mathcal{I},n)$, for any temporalised concept or role $\vartheta$, any tuple $w$ in $\Delta_{\mathcal{C},\mathcal{A}}$ and any $n \in \mathbb{Z}$;

(iii) if $\top \in \mathcal{C}_{\mathcal{O},\mathcal{A}}$, then $\mathcal{O}$ and $\mathcal{A}$ are inconsistent; otherwise, $\mathcal{C}_{\mathcal{O},\mathcal{A}}$ is a model of $\mathcal{O}$ and $\mathcal{A}$;

(iv) if $\mathcal{O}$ and $\mathcal{A}$ are consistent, then, for any OMPIQ $q = (\mathcal{O}, \varrho)$ and any $n \in \mathbb{Z}$, we have $(a,n) \in \text{ans}_{\varrho}(q,n)$ iff $a \in \mathcal{C}_{\mathcal{C},\mathcal{A}}(n)$; similarly, for any OMPIQ $q = (\mathcal{O}, \vartheta)$, we have $(a,b,n) \in \text{ans}_{\vartheta}(q,n)$ iff $(a,b) \in \vartheta_{\mathcal{C},\mathcal{A}}(n)$.

Proof. Claim (i) is proved by induction on the construction of $\vartheta$. The basis of induction (for a concept or role name $\vartheta$) follows from the definition of $\mathcal{C}_{\mathcal{O},\mathcal{A}}$. Suppose $\vartheta = \exists S$ and $w \in \exists S_{\mathcal{C},\mathcal{A}}(n)$. Then $(w,w') \in S_{\mathcal{C},\mathcal{A}}(n)$, for some $w' \in \Delta_{\mathcal{C},\mathcal{A}}$, and so, by the induction hypothesis, $S(w,w',n) \in \mathcal{C}_{\mathcal{O},\mathcal{A}}$, which gives $\exists S(w,n) \in \mathcal{C}_{\mathcal{O},\mathcal{A}}$ by ($\exists^{-}$). Conversely, if $\exists S(w,n) \in \mathcal{C}_{\mathcal{O},\mathcal{A}}$ then, by ($\exists^{-}$), we have $S(w,w,S^n,n) \in \mathcal{C}_{\mathcal{O},\mathcal{A}}$, whence, by the induction hypothesis, $(w,w,S^n) \in S_{\mathcal{C},\mathcal{A}}(n)$, and so $w \in \exists S_{\mathcal{C},\mathcal{A}}(n)$. The case of $\vartheta = \top$ is straightforward by (inv). For $\vartheta = \square_{\top} \vartheta_1$, suppose first that $w \in \vartheta_1_{\mathcal{C},\mathcal{A}}(n)$. Then $w \in \vartheta_1_{\mathcal{C},\mathcal{A}}(k)$ for all $k > n$, whence, by the induction hypothesis, $\vartheta_1(w,k) \in \mathcal{C}_{\mathcal{O},\mathcal{A}}$, and so, by ($\square_{\top}^{-}$),
In this section we prove our main technical result reducing rewritability of rewritability of DL-Litehorn. Suppose \(Cw\) and \(\varphi\) and role names \(\varphi\) and \(\varphi\), then, by \((\Box^\omega)\), we have \(\varphi Cw\) for all \(k > n\). By the induction hypothesis, \(w \in \varphi Cw(k)\) for all \(k > n\), and so \(w \in (\Box^\omega)\varphi Cw(n)\). The other temporal operators, \(\Box^\omega\), \(\varnothing^\omega\) and \(\varnothing^\omega\), are treated similarly.

(ii) Suppose \(I\) is a model of \(O\) and \(A\). By induction on \(\alpha < \omega_1\), we construct a map \(h^\alpha\) from \(\Delta^{Cw}(\varphi)\) to \(\Delta^I\) such that, for any \(\varphi\), any tuple \(w\) from \(\Delta^{Cw}(\varphi)\) and any \(n \in \mathbb{Z}\), if \(\varphi(w, n) \in \varphi Cw(\varphi)\) then \(h^\alpha(w) \in \varphi Cw(\varphi)\). For the basis of induction, we set \(h^0(a) = a^I\). Next, for a successor ordinal \(\xi + 1\), we define \(h^{\xi + 1}\) by extending \(h^\xi\) to the new witnesses \(wS^n\) introduced by the rule \((\exists^\omega)\) at step \(\xi + 1\). If \(\exists S(w, n) \in \varphi Cw(\varphi)\), then \(h^{\xi + 1}(w) \in (\exists S)^2(\varphi)\) by the induction hypothesis, and so there is \(w' \in \Delta^I\) such that \((h^\xi(w), w') \in S^2(\varphi)\). We set \(h^{\xi + 1}(wS^n) = w'\) for all such \(wS^n\), and keep \(h^\xi\) the same as \(h^\xi\) on the domain elements of \(\varphi Cw(\varphi)\). That \(\varphi(w, n) \in \varphi Cw(\varphi)\) implies \(h^{\xi + 1}(w) \in \varphi Cw(\varphi)\) follows from the induction hypothesis and the fact that \(I\) is ‘closed’ under all of our rules save \((\exists^\omega)\). For a limit ordinal \(\zeta\), we take \(h^\zeta\) to be the union of all \(h^\xi\), for \(\xi < \zeta\).

(iii) Suppose first that \(\perp \in \varphi Cw(\varphi)\). Then there is an axiom \(\varphi_1 \land \cdots \land \varphi_m \Rightarrow \perp\) in \(O\) and \(n \in \mathbb{Z}\) such that \(\varphi_i(w, n) \in \varphi Cw(\varphi)\) for all \(i (1 \leq i \leq m)\). By (ii), \(w \in \varphi Cw(\varphi)\), for all models \(I\) of \(O\) and \(A\) and all \(i (1 \leq i \leq m)\). But then \(A\) is inconsistent with \(O\). On the other hand, if \(A\) is consistent with \(O\), then \(\varphi Cw(\varphi)\) is a model of \(O\) by (i) and closure under the rules (mp) and (cls) since \(\perp \notin \varphi Cw(\varphi)\); \(\varphi Cw(\varphi)\) is a model of \(A\) by definition.

To show (iv), observe that, for any interpretations \(I\) and \(J\), if \(\eta^I \subseteq \eta^J\) for all concepts and role names \(\eta\), then \(\varphi Cw(\varphi)\) is a positive temporal concepts and role \(\varphi\). Now, that \(\perp \in \varphi Cw(\varphi)\) implies \(w \in \varphi Cw(\varphi)\) follows from (iii) from the fact that \(\varphi Cw(\varphi)\) is a model of \(O\) and \(A\), while the converse direction follows from (ii) and the above observation.

We also require the following simple properties of the canonical models:

**Lemma 33.** Let \(O = T \cup R\) be a DL-Litehorn ontology with a TBox \(T\) and an RBox \(R\), and let \(A\) be an ABox. If \(O\) and \(A\) are consistent, then, for any role \(S\) and \(n \in \mathbb{Z}\), we have

(i) \(wS^n \in \Delta^{Cw}(\varphi)\) if \(\exists S(w, n) \in \varphi Cw(\varphi)\), for any \(w \in \Delta^{Cw}(\varphi)\);

(ii) \(R(w, wS^n, m) \in \varphi Cw(\varphi)\) if \(R \models S \sqsubseteq \Diamond^n \neg R\), for any temporalised role \(R\), \(wS^n \in \Delta^{Cw}(\varphi)\) and \(m \in \mathbb{Z}\);

(iii) \(C(wS^n, m) \in \varphi Cw(\varphi)\) if \(O \models S \sqsubseteq \Diamond^n \neg C\), for any temporalised concept \(C\), any \(wS^n \in \Delta^{Cw}(\varphi)\) and any \(m \in \mathbb{Z}\).

**Proof.** Claim (i) follows immediately from rule \((\exists^\omega)\) in the definition of \(\varphi Cw(\varphi)\).

(ii) Suppose \(R \models S \sqsubseteq \Diamond^n \neg R\) and \(wS^n \in \Delta^{Cw}(\varphi)\). Then \(S(w, wS^n, n) \in \varphi Cw(\varphi)\) by the definition of \(\varphi Cw(\varphi)\). By Theorem 32 (iii), \(\varphi Cw(\varphi) \models R\), and so \(R(w, wS^n, m) \in \varphi Cw(\varphi)\). Conversely, suppose \(R(w, wS^n, m) \in \varphi Cw(\varphi)\). By the construction of \(\varphi Cw(\varphi)\), we have \(S(w, wS^n, n) \in \varphi Cw(\varphi)\) and \(R(w, wS^n, m) \in \varphi Cw(\varphi)\). It follows, by (9), that \(\models S \sqsubseteq \Diamond^n \neg R\).

The proof of (iii) is similar.

### 6.2 Rewriting DL-Litehorn OMPIQs

In this section we prove our main technical result reducing rewritability of DL-Litehorn OMPIQs to rewritability of DL-Litehorn OMAQs.
First, for OMPIQs of the form \((\mathcal{O}, \varnothing)\) with a positive temporal role \(\varnothing\), Corollary 20 (ii), Proposition 8, Theorem 25 (for the consistency OMAQ in Corollary 20 (ii)) and (Artale et al., 2021, Theorems 24 and 8) give us the following:

**Corollary 34.** (i) All DL-Lite\(\sqcup_{\text{horn}}\) OMPIQs of the form \((\mathcal{O}, \varnothing)\) are FO\((<)\)-rewritable.

(ii) All DL-Lite\(\sqcup_{\text{core}}\) OMPIQs of the form \((\mathcal{O}, \varnothing)\) are FO\((<,=)\)-rewritable.

(iii) All DL-Lite\(\bigcirc_{\text{boot}}\) OMPIQs of the form \((\mathcal{O}, \varnothing)\) are FO\((\text{RPR})\)-rewritable.

So, from now on we only consider DL-Lite\(\square_{\text{horn}}\) OMPIQs of the form \(q = (\mathcal{O}, \varnothing)\) with a positive temporal concept \(\varnothing\). Without loss of generality, we assume that all concept and roles names in \(\varnothing\) occur in \(\mathcal{O} = T \cup R\) and that a role name occurs in \(T \) iff it occurs in \(R\).

In order to encode the structure of the infinite canonical model in a finite way, we use phantoms (Artale et al., 2021), which are formulas that encode certain answers to OMPIQs beyond the active temporal domain \(\text{tem}(\mathcal{A})\). In the context of DL-Lite, we require the following modification of the original definition.

**Definition 35.** Let \(L\) be one of FO\((<)\), FO\((<,=)\) or FO\((\text{RPR})\). An \(L\)-phantom of the given OMPIQ \(q = (\mathcal{O}, \varnothing)\) for \(k \neq 0\) is an \(L\)-formula \(\Phi^k_q(x)\) such that, for any ABox \(\mathcal{A}\),

\[
\mathcal{A} \models \Phi^k_q(a) \text{ iff } (a, \sigma_\mathcal{A}(k)) \in \text{ans}^\mathcal{A}(q, \mathcal{A}), \quad \text{for any } a \in \text{ind}(\mathcal{A}),
\]

where

\[
\sigma_\mathcal{A}(k) = \begin{cases} 
\max \mathcal{A} + k, & \text{if } k > 0, \\
\min \mathcal{A} + k, & \text{if } k < 0.
\end{cases}
\]

Similarly, an \(L\)-phantom of an OMPIQ \(q = (\mathcal{O}, \varnothing)\) for \(k \neq 0\) is an \(L\)-formula \(\Phi^k_q(x, y)\) such that, for any ABox \(\mathcal{A}\),

\[
\mathcal{A} \models \Phi^k_q(a, b) \text{ iff } (a, b, \sigma_\mathcal{A}(k)) \in \text{ans}^\mathcal{A}(q, \mathcal{A}), \quad \text{for any } a, b \in \text{ind}(\mathcal{A}).
\]

If \(\mathcal{O}\) is \(\bot\)-free, then, for any OMPIQ \(q = (\mathcal{O}, \varnothing)\), we can, by (13), construct \(L\)-phantoms \(\Phi^k_q(x, y)\) from \(L\)-phantoms \(\Phi^k_{q,\dagger}\) for the LTL OMPIQ \(q^{\uparrow} = (\mathcal{R}, \varnothing^{\uparrow})\), provided that they exist. For OMAQs of the form \(q = (\mathcal{O}, B)\), we have the following analogue of Theorem 25:

**Theorem 36.** Let \(L\) be one of FO\((<)\), FO\((<,=)\) or FO\((\text{RPR})\). A \(\bot\)-free DL-Lite\(\bigcirc_{\text{horn}}\) OMAQ \((\mathcal{O}, B)\) with \(\mathcal{O} = T \cup R\) has an \(L\)-phantom for \(k \neq 0\) whenever

- the LTL\(\bigcirc_{\text{horn}}\) OMAQ \(q^{\uparrow} = (\mathcal{R}, B^{\uparrow})\) has an \(L\)-phantom for \(k \neq 0\) and

- the LTL\(\bigcirc_{\text{horn}}\) OMPIQs \(q_{\rho}^{\uparrow} = (\mathcal{R}, \rho^{\uparrow})\) are \(L\)-rewritable, for role types \(\rho\) for \(R\).

The proof relies on the fact that Lemma 24 holds for all \(n \in \mathbb{Z}\) and proceeds by taking an \(L\)-phantom for \(q^{\uparrow}\) and replacing every \(A(s)\) with \(A(x, s)\), every \((\exists P)^{\uparrow}(s)\) with \(\exists y \, P(x, y, s)\), every \((\exists P)^{\uparrow}(s)\) with \(\exists y \, P(y, x, s)\) and every \((\exists G_{\varnothing})^{\uparrow}(s)\) with \(\exists y \, Q_{\rho}(x, y, s)\), where \(Q_{\rho}(x, y, s)\) is an \(L\)-rewriting for \((\mathcal{R}, \rho)\) provided by Proposition 8 for each \(q_{\rho}^{\uparrow}\).

Since the LTL canonical models have ultimately periodic structure, there are only finitely many non-equivalent phantoms for any OMPIQ \(q = (\mathcal{O}, \varnothing)\), and we have the following:
Lemma 37. Let $\mathcal{O}$ be a DL-Lite$^\Diamond$ horn ontology and $\mathcal{A}$ an ABox consistent with $\mathcal{O}$. For $w = aS^m_1 \ldots S^m_l \in \Delta^{\mathcal{C}}_{\mathcal{O},\mathcal{A}}$, with $l \geq 0$, denote

$$M^w = \max\{ \max \mathcal{A}, m_1, \ldots, m_l \}$$ and $$\bar{M}^w = \min\{ \min \mathcal{A}, m_1, \ldots, m_l \}.$$

For any positive temporal concept $\varphi$ (all of whose concept and role names occur in $\mathcal{O}$) and any $w \in \Delta^{\mathcal{C}}_{\mathcal{O},\mathcal{A}}$, there are positive integers $s_\mathcal{O}$ and $p_\mathcal{O}$ such that

$$\mathcal{C}_{\mathcal{O},\mathcal{A},k} \models \varphi(w) \iff \mathcal{C}_{\mathcal{O},\mathcal{A},k + p_\mathcal{O}} \models \varphi(w), \text{ for any } k \geq M^w + s_\mathcal{O} + |\varphi| p_\mathcal{O},$$

$$\mathcal{C}_{\mathcal{O},\mathcal{A},k} \models \varphi(w) \iff \mathcal{C}_{\mathcal{O},\mathcal{A},k - p_\mathcal{O}} \models \varphi(w), \text{ for any } k \leq \bar{M}^w - s_\mathcal{O} - |\varphi| p_\mathcal{O}.$$

The bounds on $s_\mathcal{O}$ and $p_\mathcal{O}$ are double- and triple-exponential in the size of $\mathcal{O}$, and the proof of this lemma, which generalises Lemma 6 in Section 3.1, can be found in Appendix A.2; cf. (Artale et al., 2021, Lemma 22) for the LTL case.

Now we have all the required ingredients to establish Theorem 39, our main technical result in Section 6. First, we illustrate the rather involved construction in its proof by an example.

Example 38. Consider the OMPIQ $q = (\mathcal{O}, \varphi)$ with $\varphi = \exists Q. \Diamond F B$ and $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$, where

$$\mathcal{T} = \{ A \subseteq \varphi \exists P, \exists P^- \subseteq \varphi^2 B, \exists Q \subseteq \exists Q \}$$ and $$\mathcal{R} = \{ P \subseteq \varphi Q \}$$

(recall that we require that the TBox contains the same role names as the RBox). We construct an FO($<, \equiv$)-rewriting $Q_{\mathcal{O},\varphi}(x,t)$ of $q$ by induction. To illustrate, consider an ABox $\mathcal{A}$ consistent with $\mathcal{O}$, $a \in \text{ind}(\mathcal{A})$ and $\ell \in \text{tem}(\mathcal{A})$ such that $a \in (\exists Q. \Diamond F B)^{\mathcal{C}}_{\mathcal{O},\mathcal{A}}(\ell)$. Our rewriting has to cover two main cases; see Fig. 9.

Case 1: If there is $b \in \text{ind}(\mathcal{A})$ with $(a,b) \in Q_{\mathcal{O},\varphi}(x,t)$ and $b \in (\Diamond F B)^{\mathcal{C}}_{\mathcal{O},\mathcal{A}}(\ell)$, then we can describe this configuration by the formula

$$\exists y \left[ Q_{\mathcal{R},Q}(x,y,t) \land Q_{\mathcal{O},\Diamond F B}(y,t) \right],$$ (41)
where $Q_{\mathcal{R},Q}(x,y,t) = \exists s \left( (s > t) \land Q(x,s) \right)$ and $Q_{\mathcal{O},\diamond_P B}(x,t)$ is a rewriting of the OMAQ $(\mathcal{R},Q)$ provided by Proposition 8, and $Q_{\mathcal{O},\diamond_P B}(x,t)$ is a rewriting of a simpler OMPIQ $(\mathcal{O},\diamond_P B)$, which can be defined as follows:

$$Q_{\mathcal{O},\diamond_P B}(x,t) = \exists s \left( (s > t) \land Q_{\mathcal{O},B}(x,s) \right) \lor \bigvee_{k \in (0,N]} \Phi^k_{\mathcal{O},B}(x),$$

where $Q_{\mathcal{O},B}(x,t)$ is the rewriting and the $\Phi^k_{\mathcal{O},B}(x)$ are the phantoms of the OMAQ $(\mathcal{O},B)$ provided by Theorems 25 and 36, respectively, and $N = s_{\mathcal{O}} + |s|_{P_{\mathcal{O}}}$; see Lemma 37. That is, $N$ is a suitable integer that depends on $q$ and reflects the periodicity of the canonical model of $\mathcal{O}$: a ‘witness’ $B(b,m)$ for $(\diamond_P B)(b,\ell)$ is either in the active temporal domain $\text{tem}(\mathcal{A})$ (see Case 1.1 in Fig. 9) or at a distance $k \leq N$ from $\max \mathcal{A}$ (see Case 1.2 in Fig. 9). The latter requires phantoms $\Phi^k_{\mathcal{O},B}(x)$; note that due to $\diamond_P$ we look only at integers larger than $\ell$ (in particular, only positive $k$).

Case 2: If there exists $n \in \mathbb{Z}$ such that

- the canonical model for $(\mathcal{O},\mathcal{A})$ contains $P(a,aP^n,n)$,
- the canonical model for $(\mathcal{R},\{P(a,aP^n,n)\})$ contains $Q(a,aP^n,\ell)$,
- the canonical model for $(\mathcal{O},\{P(a,aP^n,n)\})$ contains $(\diamond_P B)(aP^n,\ell)$,

then we consider two further options.

Case 2.1: If $n \in \text{tem}(\mathcal{A})$, then, for the first item, we use a rewriting $Q_{\mathcal{O},\exists_P}(x,s)$ of the OMAQ $(\mathcal{O},\exists_P)$. For the second, we need a formula $\Theta_{P,\exists_Q}(s,t)$ such that $\mathcal{A} \models \Theta_{P,\exists_Q}(n,\ell)$ iff the canonical model for $(\mathcal{R},\{P(a,b,n)\})$ contains $Q(a,b,\ell)$, for $n,\ell \in \text{tem}(\mathcal{A})$. To capture the third condition, we need a formula $\bar{\Psi}_{P,\diamond_P B}(x,s,t)$ such that $\mathcal{A} \models \bar{\Psi}_{P,\diamond_P B}(a,n,\ell)$ iff the canonical model for $(\mathcal{O},\{P(a,b,n)\})$ contains $(\diamond_P B)(b,\ell)$, for $n,\ell \in \text{tem}(\mathcal{A})$. Assuming that these formulas are available, we can express Case 2.1 by the formula

$$\exists s \left[ Q_{\mathcal{O},\exists_P}(x,s) \land \Theta_{P,\exists_Q}(s,t) \land \bar{\Psi}_{P,\diamond_P B}(x,s,t) \right]. \quad (42)$$

Case 2.2: If $n \notin \text{tem}(\mathcal{A})$, then we need phantom versions $\Phi_{\mathcal{O},\exists_P}(x), \Theta_{P,\exists_Q}(t)$ and $\bar{\Psi}_{P,\diamond_P B}(x,t)$ of the subformulas in (42). For example, $\bar{\Psi}_{P,\diamond_P B}(x,t)$ should be such that $\mathcal{A} \models \bar{\Psi}_{P,\diamond_P B}(a,n,\ell)$ iff the canonical model for $(\mathcal{O},\{P(a,b,\sigma_\mathcal{A}(\mu))\})$ contains $(\diamond_P B)(b,\ell)$, for $\ell \in \text{tem}(\mathcal{A})$. Provided that such phantoms are available, Case 2.2 can be represented by the formula

$$\bigvee_{\mu \in [-N,0) \cup (0,N]} \left[ \Phi^\mu_{\mathcal{O},\exists_P}(x) \land \Theta^\mu_{P,\exists_Q}(t) \land \bar{\Psi}_{P,\diamond_P B}(x,t) \right]; \quad (43)$$

notice that the quantifier $\exists s$ (corresponding to the choice of $n$) is replaced by the (finite) disjunction over $\mu$, while the argument $s$ of the subformulas is now shifted to the superscript $\mu$ of the phantoms.

The required $\text{FO}(\prec,\exists)$-rewriting $Q_{\mathcal{O},\prec}(x,t)$ of $q$ is a disjunction of (41), (42) and (43). All auxiliary formulas mentioned above are constructed using the same type of analysis that will be explained in full detail below. For example, in formula (42), that is, Case 2.1 with $n \in \text{tem}(\mathcal{A})$, one can take

$$\bar{\Psi}_{P,\diamond_P B}(x,s,t) = \exists s' \left( (s' > t) \land \bar{\Psi}_{P,B}(x,s,s') \right) \lor \bigvee_{k \in (0,N]} \bar{\Psi}^k_{P,B}(x,s),$$

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Table 5: Semantic characterisations of the auxiliary formulas in rewritings: $a \in \text{ind}(A)$, $n, n_1, \ldots, n_t \in \text{tem}(A)$, $k \in \mathbb{Z} \setminus \{0\}$ and $\mu_1, \ldots, \mu_t \in \mathbb{Z}$.

| Expression                                                                 | \(\mathcal{S}_A \models \Theta_{S \sim S_1}(n, n_1) \iff \mathcal{R} \models S \subseteq \bigcirc^{n_1-n} S_1\) | \(\mathcal{S}_A \models \Theta_{S \sim S_1}(n_1) \iff \mathcal{R} \models S \subseteq \bigcirc^{n_1-\sigma_{A(k)} S_1}\) | \(\mathcal{S}_A \models \Theta_{S \sim S_1}(n) \iff \mathcal{R} \models S \subseteq \bigcirc^{\sigma_{A(k)}-n} S_1\) | \(\mathcal{S}_A \models \Theta_{S \sim S_1}(k_1) \iff \mathcal{R} \models S \subseteq \bigcirc^{\sigma_{A(k_1)}-\sigma_{A(k)} S_1}\) |
|---------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------|
| \(\mathcal{S}_A \models \Xi_{B \sim B_1}(n, n_1) \iff \mathcal{O} \models B \subseteq \bigcirc^{n_1-n} B_1\)                  | \(\mathcal{S}_A \models \Xi_{B \sim B_1}(n_1) \iff \mathcal{O} \models B \subseteq \bigcirc^{n_1-\sigma_{A(k)} B_1}\)          | \(\mathcal{S}_A \models \Xi_{B \sim B_1}(n) \iff \mathcal{O} \models B \subseteq \bigcirc^{\sigma_{A(k_1)}-n} B_1\)          | \(\mathcal{S}_A \models \Xi_{B \sim B_1}(k_1) \iff \mathcal{O} \models B \subseteq \bigcirc^{\sigma_{A(k_1)}-\sigma_{A(k)} B_1}\) |
| \(\mathcal{S}_A \models \Psi_{S_{1, \ldots, S_{1, \ldots, S_{1, \ldots, S_{1, \ldots, S_1}}}}(a, n_1, \ldots, n_t, n) \iff \mathcal{S}_1(w_0, w_1, m_1), \ldots, S_l(w_l-1, w_l, m_l) \in \mathcal{C} \text{ and } w_l \in \mathcal{X}_{\mathcal{C}(n)}, \) \(\mathcal{S}_A \models \Psi_{S_{1, \ldots, S_{1, \ldots, S_{1, \ldots, S_{1, \ldots, S_1}}}}(a, n_1, \ldots, n_t) \iff \mathcal{S}_1(w_0, w_1, m_1), \ldots, S_l(w_l-1, w_l, m_l) \in \mathcal{C} \text{ and } w_l \in \mathcal{X}_{\mathcal{C}(\sigma_{A(k)})}, \) where \(\mathcal{C} = \mathcal{C}_{\mathcal{O}, \exists S_1(a, m_1)}\), \(w_0 = a\) and \(w_l = w_l-1 S_l^{m_l}, \text{ for all } i, 1 \leq i \leq l, \) and \(m_i = \begin{cases} n_i, & \text{if } \mu_i = 0, \\ \sigma_{A}(\mu_i), & \text{otherwise, } \end{cases} \text{ for all } i, 1 \leq i \leq l. \) |}

where \(\Psi_{P, B}(x, s)\) is such that \(\mathcal{S}_A \models \Psi_{P, B}(a, n)\) iff the canonical model for \((\mathcal{O}, \{P(a, b, n)\})\) contains \(B(b, \sigma_{A(k)})\). Note the similarity to formula \(Q_{\mathcal{O}, \exists P, B}(x, t)\) in Case 1 above: the first disjunct covers the case when the canonical model contains \(B(b, m)\) with \(n\) in the active temporal domain \(\text{tem}(A)\), while the second disjunct covers the case when \(B(b, m)\) is found beyond \(\text{tem}(A)\), and so we have to resort to the phantoms.

**Theorem 39.** Let \(\mathcal{L}\) be one of \(\text{FO}(<)\), \(\text{FO}(<, \equiv)\) or \(\text{FO}(\text{RPR})\). A DL-Lite\(^\Box\) OMPIQ \(q = (\mathcal{O}, \mathcal{X})\) with a \(\bot\)-free ontology \(\mathcal{O} = \mathcal{T} \cup \mathcal{R}\) is \(\mathcal{L}\)-rewritable whenever

- for any OMAQ \(q = (\mathcal{O}, B)\) with \(B\) in the alphabet of \(\mathcal{O}\), there exist an \(\mathcal{L}\)-rewriting \(Q(x, t)\) of \(q\) and \(\mathcal{L}\)-phantoms \(\Phi_{q, k}(x)\) of \(q\) for all \(k \neq 0\);

- for any OMAQ \(q = (\mathcal{O}, P)\) with \(P\) from \(\mathcal{O}\), there exist an \(\mathcal{L}\)-rewriting \(Q(x, y, t)\) of \(q\) and \(\mathcal{L}\)-phantoms \(\Phi_{q, (x, y)}\) of \(q\) for all \(k \neq 0\).

**Proof.** We require a number of auxiliary FO-formulas (similar to \(\Theta_{P, \sim Q}(t)\) and \(\Psi_{P, \exists P, B}(x, t)\) in Example 38), which are characterised semantically in Table 5 and defined syntactically in Appendix A. For the last two items in Table 5, the structure of the canonical model \(\mathcal{C}_{\mathcal{O}, \exists S_1(a, m_1)}\) is illustrated in Fig. 10 for \(l = 3\), where, in general, the order of the time instants \(m_1, \ldots, m_t, n\) can be arbitrary. Note that if \(\mu_i \neq 0\), then the formula \(\Psi_{\exists S_1 \ldots \exists S_{1, \ldots, S_{1, \ldots, S_{1, \ldots, S_1}}}}(x, t_1, \ldots, t_l, t)\) does not depend on \(t_i\). In the context of Example 38, we have \(\Psi_{P, \exists}(x, s, s') = \Psi_{P, \exists}(x, s, s')\), \(\Psi_{P, \exists}(x, t) = \Psi_{P, \exists}(x, 0, t)\) and \(\Psi_{k, \exists}(x, s) = \Psi_{k, \exists}(x, s)\), where \(0\) is used in place of the dummy argument because \(\Psi_{k, \exists}(x, t_1, t)\) does not depend on \(t_1\).
As we shall see in Appendix A, the formulas in Table 5 are all in FO(<,≡) if \( \mathcal{O} \) is a DL-Lite\(^{\text{horn}}\) ontology, and in FO(<) if \( \mathcal{O} \) is in DL-Lite\(^{\text{horn/core}}\) and DL-Lite\(^{\text{horn/horn+}}\). Using these formulas, we now construct an \( \mathcal{L} \)-rewriting \( Q_{\mathcal{O},x'}(x,t) \) of \( q \) and \( \mathcal{L} \)-phantoms \( \Phi^k_{\mathcal{O},x'}(x) \) for \( k \neq 0 \). For the given DL-Lite\(^{\text{horn}}\) ontology \( \mathcal{O} = \mathcal{T} \cup \mathcal{R} \), we take \( s_\mathcal{O} \) and \( p_\mathcal{O} \) as defined in Lemma 37, in which, without loss of generality, we assume that \( p_\mathcal{O} \geq s_\mathcal{O} \).

**Case** \( x = A \). An \( \mathcal{L} \)-rewriting \( Q_{\mathcal{O},A}(x,t) \) and \( \mathcal{L} \)-phantoms \( \Phi^k_{\mathcal{O},A}(x) \) are given by the formulation of the theorem.

**Cases** \( x = x_1 \sqcap x_2 \) and \( x = x_1 \sqcup x_2 \) are trivial.

**Case** \( x = \Box_P x' \). Using Lemma 37, we take \( N = s_\mathcal{O} + |x|p_\mathcal{O} \) and set
\[
Q_{\mathcal{O},\Box_P x'}(x,t) = \forall s \left( (s > t) \rightarrow Q_{\mathcal{O},x'}(x,s) \right) \land \bigwedge_{k \in (0,N]} \Phi^k_{\mathcal{O},x'}(x),
\]
\[
\Phi^k_{\mathcal{O},\Box_P x'}(x) = \begin{cases} 
\bigwedge_{i \in (k,k+N]} \Phi^i_{\mathcal{O},x'}(x), & \text{if } k > 0, \\
\bigwedge_{i \in (k,0]} \Phi^i_{\mathcal{O},x'}(x) \land Q_{\mathcal{O},\Box_P x'}(x,0), & \text{if } k < 0,
\end{cases}
\]

The cases of the other temporal operators are similar and left to the reader.

**Case** \( x = \exists S.x' \). In the rewriting, we reflect Cases 2.1 and 2.2 from Figure 9. For the former, in order to find when a suitable witness was created, we can try each time instant in the (finite) active domain. However, in Case 2.2, there are infinitely many time instant outside the active domain. Lemma 43 from Appendix A shows that actually it is enough to consider only a bounded number of time instants before \( \min A \) and after \( \max A \). Using this observation, we take \( N = s_\mathcal{O} + |x|p_\mathcal{O} \) and set
\[
Q_{\mathcal{O},\exists S,x'}(x,t) = \exists y \left( Q_{\mathcal{R},S}(x,y,t) \land Q_{\mathcal{O},x'}(y,t) \right) \lor \\
\bigvee_{\text{role } S_1 \text{ in } \mathcal{O}} \left[ \exists t_1 \left( Q_{\mathcal{O},\exists S_1}(x,t_1) \land Q_{\mathcal{O},x'}(t_1,t) \land \Psi_{S_1,x'}^0(x,t_1,t) \right) \lor \\
\bigvee_{\mu \in [-N,0] \cup [0,N]} \left( Q_{\mathcal{O},\exists S_1}(x) \land \Theta_{S_1\rightarrow S}^\mu(t) \land \Psi_{S_1,x'}^\mu(x,0,t) \right) \right].
\]

The three groups of disjuncts correspond to Cases 1, 2.1 and 2.2 in Example 38, respectively. We define the phantoms for \( k > 0 \) as follows:
and $L_k$ (formula built from atoms of the form $\kappa^x$) constructs in $\mathcal{O}$ keep the open-world interpretation of positive temporal concepts and roles, the individual $L_k$ be one of FO($a$) where

$$q\text{temporal are straightforward, so this section will be brief. A (2012; Gutierrez, Hurtado, & Vaisman, 2007). The definition and main rewritability result (2007a) and the SPARQL 1.1 entailment regimes (Glimm & Ogbuiji, 2013); cf. also (Motik, query language. It is inspired by the epistemic queries introduced by Calvanese et al.}

We use positive temporal concepts and roles as building blocks for our most expressive

7. First-Order Temporal OMQs under the Epistemic Semantics

We use positive temporal concepts and roles as building blocks for our most expressive query language. It is inspired by the epistemic queries introduced by Calvanese et al. (2007a) and the SPARQL 1.1 entailment regimes (Glimm & Ogbuiji, 2013); cf. also (Motik, 2012; Gutierrez, Hurtado, & Vaisman, 2007). The definition and main rewritability result are straightforward, so this section will be brief. A (temporal) ontology-mediated query (OMQ) is a pair $q = (\mathcal{O}, \psi(x, t))$, in which $\mathcal{O}$ is an ontology and $\psi(x, t)$ a first-order formula built from atoms of the form $\kappa(x, t)$, $\varrho(x, y, t)$ and $t < t'$, where $\kappa$ and $\varrho$ are a positive temporal concept and role, respectively, $x$ and $y$ are individual variables, and $t$ and $t'$ temporal variables; the free variables $x$ and $t$ of $\psi$ are called the answer variables of $q$. Given an ABox $\mathcal{A}$, the OMQ $q$ is evaluated over a two-sorted structure $\mathfrak{S}_{\mathcal{O}, \mathcal{A}}$ that extends the structure $\mathfrak{S}_\mathcal{A}$ associated with $\mathcal{A}$ by the set of all positive temporal concepts and roles entailed by $\mathcal{O}$ and $\mathcal{A}$. More precisely, let $a$ be an assignment that maps individual and temporal variables to elements of $\text{ind}(\mathcal{A})$ and $\text{tem}(\mathcal{A})$, respectively. The relation $\mathfrak{S}_{\mathcal{O}, \mathcal{A}} \models^a \psi$ (‘$\psi$ is true in $\mathfrak{S}_{\mathcal{O}, \mathcal{A}}$ under $a$’) is defined as follows:

$$\mathfrak{S}_{\mathcal{O}, \mathcal{A}} \models^a \kappa(x, t) \quad \text{iff} \quad (a(x), a(t)) \in \text{ans}(\mathcal{O}, \kappa, \mathcal{A}),$$

$$\mathfrak{S}_{\mathcal{O}, \mathcal{A}} \models^a \varrho(x, y, t) \quad \text{iff} \quad (a(x), a(y), a(t)) \in \text{ans}(\mathcal{O}, \varrho, \mathcal{A}),$$

$$\mathfrak{S}_{\mathcal{O}, \mathcal{A}} \models^a t < t' \quad \text{iff} \quad a(t) < a(t')$$

plus the standard clauses for the Boolean connectives and first-order quantifiers over both $\text{ind}(\mathcal{A})$ and $\text{tem}(\mathcal{A})$. Let $x = x_1, \ldots, x_k$ and $t = t_1, \ldots, t_m$ be the free variables of $\psi$. We say that $(a_1, \ldots, a_k, \ell_1, \ldots, \ell_m)$ is an answer to the OMQ $q = (\mathcal{O}, \psi(x, t))$ over $\mathcal{A}$ if $\mathfrak{S}_{\mathcal{O}, \mathcal{A}} \models^a \psi$, where $a(x_i) = a_i$, for all $i$, $1 \le i \le k$, and $a(t_j) = \ell_j$, for all $j$, $1 \le j \le m$. Thus, we keep the open-world interpretation of positive temporal concepts and roles, the individual and temporal variables of OMQs range over the active domains only, and the first-order constructs in $\psi$ are interpreted under the epistemic semantics (Calvanese et al., 2007a). Let $L$ be one of FO($<$), FO($<$, $\equiv$) or FO($\text{RPR}$). We call an OMQ $q = (\mathcal{O}, \psi(x, t))$ $L$-rewritable if there is an $L$-formula $Q(x, t)$ such that, for any ABox $\mathcal{A}$ and any tuples $a$ and $\ell$ in $\text{ind}(\mathcal{A})$ and $\text{tem}(\mathcal{A})$, respectively, the pair $(a, \ell)$ is an answer to $q$ over $\mathcal{A}$ iff $\mathfrak{S}_\mathcal{A} \models Q(a, \ell)$.
is straightforward to construct an $\mathcal{L}$-rewriting of $q$ by replacing all occurrences of positive temporal concepts and roles in $\psi$ with their $\mathcal{L}$-rewritings. Thus, we obtain the following result:

**Theorem 41.** Let $\mathcal{L}$ be one of $\text{FO}(\prec)$, $\text{FO}(\prec, \equiv)$ or $\text{FO}(\text{RPR})$ and $q = (\mathcal{O}, \psi)$ an OMQ. If all OMPIQs $(\mathcal{O}, \kappa)$ and $(\mathcal{O}, \varrho)$ with positive temporal concepts $\kappa$ and roles $\varrho$ in $\psi$ are $\mathcal{L}$-rewritable, then $q$ is also $\mathcal{L}$-rewritable.

8. Conclusions

In this article, aiming to extend the well-developed theory of ontology-based data access (OBDA) to temporal data, we designed a family of 2D ontology languages that combine logics from the $\text{DL-Lite}$ family for representing knowledge about object domains and clausal fragments of linear temporal logic $\text{LTL}$ over $(\mathbb{Z}, \prec)$ for capturing knowledge about the evolutions of objects in time. We also suggested a 2D query language that integrates first-order logic for querying the object domains with positive temporal concepts and roles as FO-atoms. The FO-constructs in these queries are interpreted under the epistemic semantics, while the temporal concepts and roles under the open world semantics. The resulting ontology-mediated queries (OMQs) can be regarded as temporal extensions of SPARQL queries under the (generalised) $\text{OWL 2 QL}$ direct semantics entailment regime (Kontchakov, Rezk, Rodriguez-Muro, Xiao, & Zakharyaschev, 2014). Our main result is the identification of classes of OMQs that are $\text{FO}(\prec)$-, $\text{FO}(\prec, \equiv)$- or $\text{FO}(\text{RPR})$-rewritable, with the first two types of rewriting guaranteeing OMQ answering in $\text{AC}^0$ and the third one in $\text{NC}^1$ for data complexity. In particular, we proved that all $\text{DL-Lite}^\text{core}$ OMQs are $\text{FO}(\prec)$-rewritable and all $\text{DL-Lite}^\text{core/bool/horn}$ OMQs are $\text{FO}(\prec, \equiv)$-rewritable, which means that classical atemporal OBDA with the W3C standard ontology language $\text{OWL 2 QL}$ and SPARQL queries can be extended to temporal ontologies, queries and data without sacrificing the data complexity of OMQ answering.

Having designed suitable languages for temporal OBDA and established their efficiency in terms FO-rewritability and data complexity, we are facing a number of further open questions:

**(succinctness)** What is the size of (various types of) minimal rewritings of temporal OMQs compared to that of $\text{OWL 2 QL}$ OMQs investigated in (Bienvenu et al., 2018; Bienvenu, Kikot, Kontchakov, Podolskii, Ryzhikov, & Zakharyaschev, 2017; Bienvenu, Kikot, Kontchakov, Ryzhikov, & Zakharyaschev, 2017)? The FO-rewritings constructed in this article seem to be far from optimal.

**(parameterised complexity)** What is the combined complexity of answering temporal OMQs, in which, for example, the temporal depth of positive temporal concepts and roles is regarded as the parameter? Or what is the data complexity of OMQ answering over data instances where the number of individual objects and/or the number of timestamps are treated as parameters.

**(non-uniform approach)** How complex is it to decide, given an arbitrary $\text{DL-Lite}^\text{core/bool/horn}$ OMAQ or OMPIQ, whether it is $\text{FO}(\prec)$-, $\text{FO}(\prec, \equiv)$- or $\text{FO}(\text{RPR})$-rewritable? For atemporal OMQs with DL-ontologies, this non-uniform approach to OBDA has been
actively developed since the mid 2010s (Bienvenu, ten Cate, Lutz, & Wolter, 2014; Bienvenu, Hansen, Lutz, & Wolter, 2016; Barceló, Berger, Lutz, & Pieris, 2018). In the temporal case, first steps have recently been made for LTL OMQs in (Ryzhikov, Savateev, & Zakharyaschev, 2021).

(two-sorted conjunctive queries) Is it possible to generalise the results of this article from OMPIQs to two-sorted first-order conjunctive queries under the open world semantics? Which of these two languages or their fragments would be more suitable for industrial users?

(Krom RIs) Are all DL-Lite\textsubscript{krom} OMAQs FO(<)-rewritable? Are all DL-Lite\textsubscript{krom} OMAQs FO(<, ≡)-rewritable? What is the combined complexity of the consistency problem for DL-Lite\textsubscript{bool}/krom KBs? To answer these questions, one may need a type-based technique similar to the approach in the proof of (Artale et al., 2021, Theorem 16) as Theorem 25 is not applicable to Krom role inclusions.

We conclude by emphasising another aspect of the research project we are proposing in this paper. It has recently been observed (Xiao et al., 2019) that OBDA should be regarded as a principled way to integrate and access data via virtual knowledge graphs (VKGs). In VKGs, instead of structuring the integration layer as a collection of relational tables, the rigid structure of tables is replaced by the flexibility of graphs that are kept virtual and embed domain knowledge. In this setting, we propose to integrate temporal data via virtual temporal knowledge graphs.

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References

Abiteboul, S., Hull, R., & Vianu, V. (1995). Foundations of Databases. Addison-Wesley.

Abiteboul, S., Herr, L., & Van den Bussche, J. (1996). Temporal versus first-order logic to query temporal databases. In Hull, R. (Ed.), Proceedings of the Fifteenth ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems, June 3-5, 1996, Montreal, Canada, pp. 49–57. ACM Press.

Ajileye, T., Motik, B., & Horrocks, I. (2021). Streaming partitioning of RDF graphs for datalog reasoning. In Verborgh, R., Hose, K., Paulheim, H., Champin, P., Maleshkova, M., Corcho, Ò., Ristoski, P., & Alam, M. (Eds.), The Semantic Web - 18th International Conference, ESWC 2021, Virtual Event, June 6-10, 2021, Proceedings, Vol. 12731 of Lecture Notes in Computer Science, pp. 3–22. Springer.

Andréka, H., Németi, I., & van Benthem, J. (1998). Modal languages and bounded fragments of predicate logic. J. Philos. Log., 27(3), 217–274.

Artale, A., Calvanese, D., Kontchakov, R., & Zakharyaschev, M. (2009). The DL-Lite family and relations. J. Artif. Intell. Res. (JAIR), 36, 1–69.
Artale, A., Kontchakov, R., Ryzhikov, V., & Zakharyaschev, M. (2013a). The complexity of clausal fragments of LTL. In Proc. of the 19th Int. Conf. on Logic for Programming, Artificial Intelligence and Reasoning, LPAR 2013, Vol. 8312 of Lecture Notes in Computer Science, pp. 35–52. Springer.

Artale, A., Kontchakov, R., Wolter, F., & Zakharyaschev, M. (2013b). Temporal description logic for ontology-based data access. In Proc. of the 23rd Int. Joint Conf. on Artificial Intelligence (IJCAI). IJCAI/AAAI.

Artale, A., & Franconi, E. (2005). Temporal description logics. In Handbook of Temporal Reasoning in Artificial Intelligence, Vol. 1 of Foundations of Artificial Intelligence, pp. 375–388. Elsevier.

Artale, A., Kontchakov, R., Kovtunova, A., Ryzhikov, V., Wolter, F., & Zakharyaschev, M. (2017). Temporal ontology-mediated querying: A survey. In Proc. of the 24th International Symposium on Temporal Representation and Reasoning (TIME17), Mons, Belgium. Dagstuhl Publishing.

Artale, A., Kontchakov, R., Kovtunova, A., Ryzhikov, V., Wolter, F., & Zakharyaschev, M. (2021). First-order rewritability of ontology-mediated queries in linear temporal logic. Artificial Intelligence, 299.

Artale, A., Kontchakov, R., Ryzhikov, V., & Zakharyaschev, M. (2014). A cookbook for temporal conceptual data modelling with description logics. ACM Trans. Comput. Log., 15(3), 25:1–25:50.

Baader, F., Borgwardt, S., Koopmann, P., Ozaki, A., & Thost, V. (2020a). Metric temporal description logics with interval-rigid names. ACM Trans. Comput. Log., 21(4), 30:1–30:46.

Baader, F., Borgwardt, S., Koopmann, P., Thost, V., & Turhan, A. (2020b). Semantic technologies for situation awareness. Künstliche Intell., 34(4), 543–550.

Baader, F., Borgwardt, S., & Lippmann, M. (2013). Temporalizing ontology-based data access. In Proc. of the 24th Int. Conf. on Automated Deduction, CADE-24, Vol. 7898 of Lecture Notes in Computer Science, pp. 330–344. Springer.

Baader, F., Borgwardt, S., & Lippmann, M. (2015a). Temporal conjunctive queries in expressive description logics with transitive roles. In Proc. of the 28th Australasian Joint Conf. on Advances in Artificial Intelligence, AI’15, Vol. 9457 of Lecture Notes in Computer Science, pp. 21–33. Springer.

Baader, F., Borgwardt, S., & Lippmann, M. (2015b). Temporal query entailment in the description logic SHQ. J. Web Semant., 33, 71–93.

Baader, F., Calvanese, D., McGuinness, D. L., Nardi, D., & Patel-Schneider, P. F. (Eds.). (2007). The Description Logic Handbook (2 edition). Cambridge University Press.

Baader, F., Horrocks, I., Lutz, C., & Sattler, U. (2017). An Introduction to Description Logic. Cambridge University Press.

Baader, F., Küsters, R., & Wolter, F. (2003). Extensions to description logics. In The Description Logic Handbook, pp. 219–261. Cambridge University Press.
Baget, J., Leclère, M., Mugnier, M., & Salvat, E. (2011). On rules with existential variables: Walking the decidability line. *Artif. Intell.*, 175(9-10), 1620–1654.

Barceló, P., Berger, G., Lutz, C., & Pieris, A. (2018). First-order rewritability of frontier-guarded ontology-mediated queries. In Lang, J. (Ed.), *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI 2018, July 13-19, 2018, Stockholm, Sweden*, pp. 1707–1713. ijcai.org.

Baudinet, M., Chomicki, J., & Wolper, P. (1993). Temporal deductive databases. In Tansel, A. U., Clifford, J., Gadia, S. K., Segev, A., & Snodgrass, R. T. (Eds.), *Temporal Databases: Theory, Design, and Implementation*, pp. 294–320. Benjamin/Cummings.

Beck, H., Dao-Tran, M., & Eiter, T. (2018). LARS: A logic-based framework for analytic reasoning over streams. *Artif. Intell.*, 261, 16–70.

Berger, R. (1966). *The Undecidability of the Domino Problem*. American Mathematical Society.

Bienvenu, M., Hansen, P., Lutz, C., & Wolter, F. (2016). First order-rewritability and containment of conjunctive queries in horn description logics. In Lenzerini, M., & Penaloza, R. (Eds.), *Proceedings of the 29th International Workshop on Description Logics, Cape Town, South Africa, April 22-25, 2016*, Vol. 1577 of *CEUR Workshop Proceedings*. CEUR-WS.org.

Bienvenu, M., Kikot, S., Kontchakov, R., Podolskii, V. V., Ryzhikov, V., & Zacharyaschev, M. (2017). The complexity of ontology-based data access with OWL 2 QL and bounded treewidth queries. In *Proc. of the 36th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS 2017*, pp. 201–216. ACM.

Bienvenu, M., Kikot, S., Kontchakov, R., Podolskii, V. V., & Zacharyaschev, M. (2018). Ontology-mediated queries: Combined complexity and succinctness of rewritings via circuit complexity. *J. ACM*, 65(5), 28:1–28:51.

Bienvenu, M., Kikot, S., Kontchakov, R., Ryzhikov, V., & Zacharyaschev, M. (2017). On the parametrised complexity of tree-shaped ontology-mediated queries in OWL 2 QL. In *Proc. of the 30th Int. Workshop on Description Logics, DL 2017*, Vol. 1879. CEUR-WS.

Bienvenu, M., & Ortiz, M. (2015). Ontology-mediated query answering with data-tractable description logics. In *Web Logic Rules: Tutorial Lectures at the 11th Int. Summer School on Reasoning Web, RW 2015*, Vol. 9203 of *Lecture Notes in Computer Science*, pp. 218–307. Springer.

Bienvenu, M., ten Cate, B., Lutz, C., & Wolter, F. (2014). Ontology-based data access: A study through disjunctive datalog, csp, and MMSNP. *ACM Trans. on Database Systems*, 39(4), 33:1–33:44.

Börger, E., Grädel, E., & Gurevich, Y. (1997). *The Classical Decision Problem*. Perspectives in Mathematical Logic. Springer.

Borgwardt, S., Forkel, W., & Kottunova, A. (2021). Temporal minimal-world semantics for sparse ABoxes. *Theory and Practice of Logic Programming*, TO BE UPDATED, 1–36.
Borgwardt, S., Lippmann, M., & Thost, V. (2013). Temporal query answering in the description logic DL-Lite. In *Proc. of the 9th Int. Symposium on Frontiers of Combining Systems, FroCoS’13*, Vol. 8152 of Lecture Notes in Computer Science, pp. 165–180. Springer.

Borgwardt, S., Lippmann, M., & Thost, V. (2015). Temporalizing rewritable query languages over knowledge bases. *J. Web Semant.*, 33, 50–70.

Borgwardt, S., & Thost, V. (2015a). Temporal query answering in DL-Lite with negation. In *Proc. of the Global Conf. on Artificial Intelligence, GCAI15*, Vol. 36 of EPiC Series in Computing, pp. 51–65.

Borgwardt, S., & Thost, V. (2015b). Temporal query answering in the description logic $\mathcal{EL}$. In *Proc. of the 24h Int. Joint Conf. on Artificial Intelligence, IJCAI’15*, pp. 2819–2825. AAAI Press.

Bourgaux, C., Koopmann, P., & Turhan, A. (2019). Ontology-mediated query answering over temporal and inconsistent data. *Semantic Web*, 10(3), 475–521.

Bourhis, P., Manna, M., Morak, M., & Pieris, A. (2016). Guarded-based disjunctive tuple-generating dependencies. *ACM Trans. Database Syst.*, 41(4), 27:1–27:45.

Brandt, S., Kalayci, E. G., Ryzhikov, V., Xiao, G., & Zakharyaschev, M. (2018). Querying log data with metric temporal logic. *J. Artif. Intell. Res.*, 62, 829–877.

Cali, A., Gottlob, G., & Pieris, A. (2012a). Towards more expressive ontology languages: The query answering problem. *Artificial Intelligence*, 193, 87–128.

Cali, A., Gottlob, G., & Lukasiewicz, T. (2012b). A general Datalog-based framework for tractable query answering over ontologies. *J. Web Semant.*, 14, 57–83.

Calvanese, D., De Giacomo, G., Lembo, D., Lenzerini, M., & Rosati, R. (2007a). EQL-Lite: Effective first-order query processing in description logics. In *Proc. of the 20th Int. Joint Conf. on Artificial Intelligence (IJCAI 2007)*, pp. 274–279.

Calvanese, D., De Giacomo, G., Lembo, D., Lenzerini, M., & Rosati, R. (2007b). Tractable reasoning and efficient query answering in description logics: The DL-Lite family. *J. Autom. Reasoning*, 39(3), 385–429.

Calvanese, D., De Giacomo, G., Lembo, D., Lenzerini, M., & Rosati, R. (2006). Data complexity of query answering in description logics. In *Proc. of the 10th Int. Conf. on Principles of Knowledge Representation and Reasoning, Lake District of the United Kingdom, June 2-5, 2006*, pp. 260–270. AAAI Press.

Chomicki, J. (1994). Temporal query languages: A survey. In Gabbay, D. M., & Ohlbach, H. J. (Eds.), *Temporal Logic, First International Conference, ICTL ’94, Bonn, Germany, July 11-14, 1994, Proceedings*, Vol. 827 of Lecture Notes in Computer Science, pp. 506–534. Springer.

Chomicki, J., & Imieliński, T. (1988). Temporal deductive databases and infinite objects. In *Proceedings of the Seventh ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems, PODS ’88*, pp. 61–73, New York, NY, USA. ACM.

Chomicki, J., Toman, D., & Böhlen, M. H. (2001). Querying ATSQL databases with temporal logic. *ACM Trans. Database Syst.*, 26(2), 145–178.
Compton, K. J., & Laflamme, C. (1990). An algebra and a logic for NC$^1$. *Inf. Comput.*, 87(1/2), 240–262.

Cucala, D. J. T., Walega, P. A., Cuenca Grau, B., & Kostylev, E. V. (2021). Stratified negation in datalog with metric temporal operators. In *Thirty-Fifth AAAI Conference on Artificial Intelligence, AAAI 2021, Thirty-Third Conference on Innovative Applications of Artificial Intelligence, IAAI 2021, The Eleventh Symposium on Educational Advances in Artificial Intelligence, EAAI 2021, Virtual Event, February 2-9, 2021*, pp. 6488–6495. AAAI Press.

Dell’Aglio, D., Eiter, T., Heintz, F., & Phuoc, D. L. (2019). Special issue on stream reasoning. *Semantic Web*, 10(3), 453–455.

Demri, S., Goranko, V., & Lange, M. (2016). *Temporal Logics in Computer Science*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press.

Furst, M. L., Saxe, J. B., & Sipser, M. (1984). Parity, circuits, and the polynomial-time hierarchy. *Mathematical Systems Theory*, 17(1), 13–27.

Gabbay, D., Hodkinson, I., & Reynolds, M. (1994). *Temporal Logic: Mathematical Foundations and Computational Aspects*, Vol. 1. Oxford University Press.

Gabbay, D., Kurucz, A., Wolter, F., & Zakharyaschev, M. (2003). *Many-Dimensional Modal Logics: Theory and Applications*, Vol. 148 of *Studies in Logic*. Elsevier.

Gerasimova, O., Kikot, S., Kurucz, A., Podolskii, V. V., & Zakharyaschev, M. (2020). A data complexity and rewritability tetrachotomy of ontology-mediated queries with a covering axiom. In Calvanese, D., Erdem, E., & Thielscher, M. (Eds.), *Proceedings of the 17th International Conference on Principles of Knowledge Representation and Reasoning, KR 2020, Rhodes, Greece, September 12-18, 2020*, pp. 403–413.

Glimm, B., & Ogbuji, C. (2013). SPARQL 1.1 entailment regimes. W3C Recommendation.

Gutierrez, C., Hurtado, C. A., & Vaisman, A. A. (2007). Introducing time into RDF. *IEEE Trans. Knowl. Data Eng.*, 19(2), 207–218.

Gutiérrez-Basulto, V., Jung, J. C., & Kontchakov, R. (2016a). Temporalized $\mathcal{EL}$ ontologies for accessing temporal data: Complexity of atomic queries. In *Proc. of the 25th Int. Joint Conf. on Artificial Intelligence, IJCAI’16*, pp. 1102–1108. IJCAI/AAAI.

Gutiérrez-Basulto, V., Jung, J. C., & Ozaki, A. (2016b). On metric temporal description logics. In *Proc. of the 22nd European Conf. on Artificial Intelligence, ECAI 2016*, Vol. 285 of *FAIA*, pp. 837–845. IOS Press.

Gutiérrez-Basulto, V., Jung, J. C., & Schneider, T. (2014). Lightweight description logics and branching time: A troublesome marriage. In *Proc. of the 14th Int. Conf. on Principles of Knowledge Representation and Reasoning, KR’14*, pp. 278–287. AAAI Press.

Gutiérrez-Basulto, V., Jung, J. C., & Schneider, T. (2015). Lightweight temporal description logics with rigid roles and restricted TBoxes. In *Proc. of the 24th Int. Joint Conf. on Artificial Intelligence, IJCAI 2015*, pp. 3015–3021. IJCAI/AAAI.

Halpern, J. Y., & Vardi, M. Y. (1989). The complexity of reasoning about knowledge and time. i. lower bounds. *J. Comput. Syst. Sci.*, 38(1), 195–237.
Hampson, C., & Kurucz, A. (2015). Undecidable propositional bimodal logics and one-variable first-order linear temporal logics with counting. *ACM Trans. Comput. Log.*, 16(3), 27:1–27:36.

Hodkinson, I. M., Wolter, F., & Zakharyaschev, M. (2000). Decidable fragment of first-order temporal logics. *Ann. Pure Appl. Log.*, 106(1-3), 85–134.

Immerman, N. (1999). *Descriptive Complexity*. Springer.

Kamp, H. W. (1968). *Tense Logic and the Theory of Linear Order*. PhD thesis, Computer Science Department, University of California at Los Angeles, USA.

Kontchakov, R., Rezk, M., Rodriguez-Muro, M., Xiao, G., & Zakharyaschev, M. (2014). Answering SPARQL queries over databases under OWL 2 QL entailment regime. In *Proc. of the 13th Int. Semantic Web Conf. (ISWC 2014)*, Part I, Vol. 8796 of LNCS, pp. 552–567. Springer.

Kontchakov, R., Ryzhikov, V., Wolter, F., & Zakharyaschev, M. (2020). Boolean role inclusions in dl-lite with and without time. In Calvanese, D., Erdem, E., & Thielscher, M. (Eds.), *Proceedings of the 17th International Conference on Principles of Knowledge Representation and Reasoning, KR 2020, Rhodes, Greece, September 12-18, 2020*, pp. 582–591.

Koopmann, P. (2019). Ontology-based query answering for probabilistic temporal data. In *Proc. of the 33rd AAAI Conference on Artificial Intelligence, AAAI 2019*, pp. 2903–2910.

Kuper, G. M., Libkin, L., & Paredaens, J. (Eds.). (2000). *Constraint Databases*. Springer.

Libkin, L. (2004). *Elements Of Finite Model Theory*. Springer.

Liu, L., & Özsu, M. T. (Eds.). (2018). *Encyclopedia of Database Systems, Second Edition*. Springer.

Lutz, C., Sattler, U., & Wolter, F. (2001). Modal logic and the two-variable fragment. In Fribourg, L. (Ed.), *Computer Science Logic, 15th International Workshop, CSL 2001. 10th Annual Conference of the EACSL, Paris, France, September 10-13, 2001, Proceedings*, Vol. 2142 of Lecture Notes in Computer Science, pp. 247–261. Springer.

Lutz, C., Wolter, F., & Zakharyaschev, M. (2008). Temporal description logics: A survey. In *Proc. of the 15th Int. Symposium on Temporal Representation and Reasoning, TIME’08*, pp. 3–14. IEEE Computer Society.

Manna, Z., & Pnueli, A. (1992). *The temporal logic of reactive and concurrent systems - specification*. Springer.

Motik, B. (2012). Representing and querying validity time in RDF and OWL: A logic-based approach. *J. Web Semant.*, 12, 3–21.

Özcep, O., & Möller, R. (2014). Ontology based data access on temporal and streaming data. In *Proc. of the 10th Int. Summer School on Reasoning Web (RW 2014)*, Vol. 8714 of LNCS, pp. 279–312. Springer.

Pagliarecci, F., Spalazzi, L., & Taccari, G. (2013). Reasoning with temporal ABoxes: Combining DL-Litecore with CTL. In *Proc. of the 26th Int. Workshop on Description Logics, DL’13*, pp. 885–897. CEUR-WS.
FO-Rewritability of Two-Dimensional Temporal Ontology-Mediated Queries

Poggi, A., Lembo, D., Calvanese, D., De Giacomo, G., Lenzerini, M., & Rosati, R. (2008). Linking data to ontologies. J. on Data Semantics, 10, 133–173.

Pratt-Hartmann, I. (2009). Data-complexity of the two-variable fragment with counting quantifiers. Inf. Comput., 207(8), 867–888.

Prior, A. (1956). Time and Modality. Oxford University Press.

Revesz, P. Z. (2000). Datalog and constraints. In Constraint Databases, pp. 155–170. Springer.

Ryzhikov, V., Walega, P. A., & Zakharyaschev, M. (2020). Temporal ontology-mediated queries and first-order rewritability: A short course. In Manna, M., & Pieris, A. (Eds.), Reasoning Web. Declarative Artificial Intelligence - 16th International Summer School 2020, Oslo, Norway, June 24-26, 2020, Tutorial Lectures, Vol. 12258 of Lecture Notes in Computer Science, pp. 109–148. Springer.

Schraerf, A. (1993). On the complexity of the instance checking problem in concept languages with existential quantification. J. Intelligent Information Systems, 2(3), 265–278.

Schild, K. (1993). Combining terminological logics with tense logic. In Proc. of the 6th Portuguese Conf. on Progress in Artificial Intelligence, EPIA’93, Vol. 727 of Lecture Notes in Computer Science, pp. 105–120. Springer.

Schmiedel, A. (1990). Temporal terminological logic. In Proc. of the 8th National Conf. on Artificial Intelligence, AAAI’90, pp. 640–645. AAAI Press / The MIT Press.

Sistla, A. P., & Clarke, E. M. (1985). The complexity of propositional linear temporal logics. J. ACM, 32(3), 733–749.

Straubing, H. (1994). Finite Automata, Formal Logic, and Circuit Complexity. Birkhauser Verlag.

Toman, D., & Niwinski, D. (1996). First-order queries over temporal databases inexpressible in temporal logic. In Apers, P. M. G., Bouzeghoub, M., & Gardarin, G. (Eds.), Advances in Database Technology - EDBT’96, 5th International Conference on Extending Database Technology, Avignon, France, March 25-29, 1996, Proceedings, Vol. 1057 of Lecture Notes in Computer Science, pp. 307–324. Springer.

Van Emde Boas, P. (1997). The convenience of tilings. In In Complexity, Logic, and Recursion Theory, pp. 331–363. Marcel Dekker Inc.

Vardi, M. (1982). The complexity of relational query languages (extended abstract). In Proc. of the 14th ACM SIGACT Symp. on Theory of Computing (STOC’82), pp. 137–146.

Vardi, M. Y. (2008). From Church and Prior to PSL. In 25 Years of Model Checking - History, Achievements, Perspectives, Vol. 5000 of Lecture Notes in Computer Science, pp. 150–171. Springer.

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Appendix A.

We provide syntactic definitions of the auxiliary formulas from Table 5 for a given $DL$-$L_{\bf horn}^{□□}$ ontology $\mathcal{O}$ with a TBox $\mathcal{T}$ and an RBox $\mathcal{R}$. Recall from Section 3.2 that $\mathcal{R}^\dagger$ is the translation of $\mathcal{R}$ to $LTL_{horn}^{□□}$, which uses the concept names $A_S$ for roles $S$, while $\mathcal{T}^\dagger$ is the translation of $\mathcal{T} = \mathcal{T} \cup (\mathtt{con})$ to $LTL_{horn}^{□□}$ defined in Section 5.1, which uses concept names from $\mathcal{T}$ along with concept names $(\exists S)^\dagger = E_S$ for roles $S$.

In the auxiliary formulas we use the $FO(<,\equiv)$-abbreviation $t-t' \in r+p\mathbb{N}$, for $r,p \geq 0$, from (Artale et al., 2021, Remark 3 (ii)) with the following meaning: for all $n,n' \in \mathtt{tem}(A)$,

$$\mathcal{A} \models n-n' \in r+p\mathbb{N} \iff n' + r \in \mathtt{tem}(A) \text{ and } n = n' + r + pk, \text{ for some } k \in \mathbb{N}.$$ 

Note that, if $n'+r > \max \mathcal{A}$, then the formula evaluates to $\bot$. We use two more shortcuts: $t-t' = r$ for $t-t' \in r+0\mathbb{N}$ and $t \in r+p\mathbb{N}$ for $t-0 \in r+p\mathbb{N}$. Also, we need symbols max and min that are used in place of temporal variables: max stands for a variable $t_{\max}$ additionally satisfying $-\exists t (t > t_{\max})$; similarly, min stands for a variable with $-\exists t (t < t_{\min})$. Finally, as usual, the empty disjunction is $\bot$.

Observe that, if $p = 1$, then $FO(<,\equiv)$-formula $t-t' \in r+p\mathbb{N}$ is equivalent to an $FO(<)$-formula that we abbreviate by $t-t' > r$ and define by taking $t > t'$ for $r = 0$ and

$$\exists t_1, \ldots, t_r ((t > t_r) \land (t_r > t_{r-1}) \land \cdots \land (t_2 > t_1) \land (t_1 > t')) \quad \text{for } r > 0.$$ 

A.1 $\Theta$- and $\Xi$-formulas.

To define the four types of $\Theta$-formulas, we consider the $\mathcal{R}$-canonical rod $r$ for $\{S^\dagger(0)\}$; see Section 4.2. Let $s = s_{\mathcal{R}^\dagger\{S^\dagger(0)\}}$ and $p = p_{\mathcal{R}^\dagger\{S^\dagger(0)\}}$ be the integers provided by Lemma 6 (i). Let $0 \leq s_1 < \cdots < s_t \leq s$ be all the numbers with $S_1 \in r(s_i)$ and let $1 \leq p_1 < \cdots < p_m \leq p$ be all the numbers with $S_1 \in r(s \pm p_i)$. Symmetrically, let $0 \leq s'_1 < \cdots < s'_\nu \leq s$ and $1 \leq p'_1 < \cdots < p'_\mu \leq p$ be all the numbers with $S_1 \in r(-s'_i)$ and $S_1 \in r(-s-p'_i)$.

Formulas $\Theta_{S \rightarrow \neg S_1}(t_1,t_1)$ simply list the cases for the distance between $t_1$ and $t$ such that an $S$ at $t$ implies an $S_1$ at $t_1$: by Lemma 6 (i), the distance can be one of the $s_i$, or one of
the \(-s'_i\), or belong to one of the arithmetic progressions \(s + p_i + p\mathbb{N}\) or \(-s - p'_i - p\mathbb{N}\). So, we define the formula by taking

\[
\Theta_{S \rightarrow S_1}(t, t_1) = \bigvee_{1 \leq i \leq l} (t - t_1 = s_i) \lor \bigvee_{1 \leq i \leq m} (t - t_1 \in s + p_i + p\mathbb{N}) \lor \bigvee_{1 \leq i \leq l'} (t - t_1 = s'_i) \lor \bigvee_{1 \leq i \leq m'} (t - t_1 \in s + p'_i + p\mathbb{N}) ;
\]

note that the \(s'_i\) and the \(p'_i\) are non-negative, and we flip the sign of the arithmetic expressions when \(t_1 < t\).

Formulas \(\bar{\Theta}_{S \rightarrow S_1}^{k_1}(t)\) follow the same principle. If \(k_1 > 0\), then we list the cases when the distance between \(t\) and \(\max + k_1\) is suitable. Note, however, that, as \(t\) ranges over the active temporal domain only, we have \(t < \max + k_1\), and so, we need only cases with the \(s_i\) and the \(p_i\), but not with the \(s'_i\) and the \(p'_i\). So, we set

\[
\bar{\Theta}_{S \rightarrow S_1}^{k_1}(t) = \bigvee_{1 \leq i \leq l} \left( \max - t = s_i - k_1 \right) \lor \bigvee_{1 \leq i \leq m} \left( \left( \max - t \in s + p_i + (p - k_1 \mod p) + p\mathbb{N} \right) \lor \bigvee_{0 \leq q \leq s + p_i - q + p\mathbb{N}} \left( \max - t = q \right) \right).
\]

Since the value of \(\max + k_1\) does not belong to the active temporal domain, we cannot use the \(((\max + k_1) - t \in s + p_i + p\mathbb{N})\) abbreviation directly and have to consider two cases, depending on whether \(\max - t\) is larger or smaller than \(s + p_i\) (see Fig. 11) — these two cases are encoded in the two disjuncts in the second line of the definition of \(\bar{\Theta}_{S \rightarrow S_1}^{k_1}(t)\). The case \(k_1 < 0\) is similar and left to the reader.

Formulas \(\Theta_k^{k_1}_{S \rightarrow S_1}(t_1)\) are constructed similarly to \(\bar{\Theta}_{S \rightarrow S_1}^{k_1}(t)\).

Formulas \(\Theta_{S \rightarrow S_1}^{k,k_1}\) have no free variables, but follow the same principle. We only consider the most interesting case of \(k < 0\) and \(k_1 > 0\) (leaving the three remaining cases to the reader): in this formula, we check that the difference between \(\max + k_1\) and \(\min + k\) either
is one of the $s_i$ (again, the $s'_i$ are irrelevant because the difference is positive) or belongs to
one of the the arithmetic progressions $s + p_i + p\mathbb{N}$ (the $p'_i$ are irrelevant), where we again
have two cases, with max $-$ min larger/smaller than $s + p_i$. We denote $\bar{k} = k_1 - k$ and set

$$\Theta_{S\sim S_1}^{k,k_1} = \bigvee_{1 \leq i \leq l} \left( \max - \min = s_i - \bar{k} \right) \lor \bigvee_{1 \leq i \leq m} \left( \max - \min \in s + p_i + (p - \bar{k} \mod p) + p\mathbb{N} \right) \lor \bigvee_{0 \leq q \leq s + p_i \text{ with } k \in s + p_i - q + p\mathbb{N}} (\max - \min = q) \right].$$

We note that if $\mathcal{R}$ is a $DL$-Lite$^{\Box}_{horn}$ RBox, then, by Lemma 6 (i), we can take $p = 1$, and
so the formulas $\Theta_{S\sim S_1}(t, t_1)$, $\Theta_{S\sim S_1}^{k}(t_1)$, $\Theta_{S\sim S_1}^{k_1}(t)$ and $\Theta_{S\sim S_1}^{k,k_1}$ are equivalent to $FO(<)$-formulas.

The $\Xi$-formulas are constructed similarly to the $\Theta$-formulas using the ontology $\mathcal{T}_{\mathcal{R}} \uparrow$ (see
Lemma 24) and the beam in $C_{\mathcal{T}_k, \{B(0)\}}$ (see Section 4.2), instead of $\mathcal{T}_{\mathcal{R}} \uparrow$ and the $\mathcal{R}$-canonical rod for $\{S(0)\}$. Observe that, for role-monotone $DL$-Lite$^{\Box}_{horn}$-ontologies $\mathcal{O}$, those formulas
are in $FO(<)$ because $\mathcal{T}_{\mathcal{R}} \uparrow$ is in $LTL^{\Box}_{horn}$.

To sum up, we have shown the following:

**Lemma 42.** Let $\mathcal{O} = (\mathcal{T}, \mathcal{R})$ be a $DL$-Lite$^{\Box}_{horn}$ ontology, $k, k_1 \in \mathbb{Z} \setminus \{0\}$, $B, B_1$ basic concepts, and $S, S_1$ roles. Then there are $FO(<)$-formulas $\Theta_{S\sim S_1}(t, t_1)$, $\Theta_{S\sim S_1}^{k}(t_1)$, $\Theta_{S\sim S_1}^{k_1}(t)$, $\Theta_{S\sim S_1}^{k,k_1}$, as well as $\Xi_{B\sim B_1}(t, t_1)$, $\Xi_{B\sim B_1}^{k_1}(t_1)$, $\Xi_{B\sim B_1}^{k_1}(t)$, $\Xi_{B\sim B_1}^{k,k_1}$ satisfying the characterisations in Table 5.

Moreover, if $\mathcal{O}$ is a $DL$-Lite$^{\Box}_{horn/core}$ or $DL$-Lite$^{\Box}_{horn/horn+}$ ontology, then those formulas are in $FO(<)$.

**A.2 $Ψ$-formulas**

The formulas $Ψ_{S_1,\ldots,S_1,\bar{s}}(x, t_1, \ldots, t_l, t)$ and $Ψ_{S_1,\ldots,S_1,\bar{s}}^{k_1}(x, t_1, \ldots, t_l)$ have, however, more involved definitions, which take into account the periodicity of the canonical model $C_{\mathcal{O}, A}$. The next lemma characterises the temporal periodicity of $C_{\mathcal{O}, A}$ on both ABox and non-ABox elements.

**Lemma 37.** Let $\mathcal{O}$ be a $DL$-Lite$^{\Box}_{horn}$ ontology and $A$ an ABox consistent with $\mathcal{O}$. For $w = aS_1^{m_1} \cdots S_l^{m_l} \in \Delta_{\mathcal{C}_{\mathcal{O}, A}}$, with $l \geq 0$, denote

$$M^w = \max \{ \max A, m_1, \ldots, m_l \} \quad \text{and} \quad \tilde{M}^w = \min \{ \min A, m_1, \ldots, m_l \}.$$

For any positive temporal concept $\bar{s}$ (all of whose concept and role names occur in $\mathcal{O}$) and
any $w \in \Delta_{\mathcal{C}_{\mathcal{O}, A}}$, there are positive integers $s_\mathcal{O}$ and $p_\mathcal{O}$ such that

$$C_{\mathcal{O}, A}, k \models \bar{s}(w) \iff C_{\mathcal{O}, A}, k + p_\mathcal{O} \models \bar{s}(w), \quad \text{for any } k \geq M^w + s_\mathcal{O} + |s|p_\mathcal{O},$$

$$C_{\mathcal{O}, A}, k \models \bar{s}(w) \iff C_{\mathcal{O}, A}, k - p_\mathcal{O} \models \bar{s}(w), \quad \text{for any } k \leq \tilde{M}^w - s_\mathcal{O} - |s|p_\mathcal{O}.$$

**Proof.** We set $s_\mathcal{O} = s_{\mathcal{T}_\mathcal{R}}$ and $p_\mathcal{O} = p_{\mathcal{T}_\mathcal{R}}$. Since the number of role types is bounded by $2^{|\mathcal{R}|}$, the size of (con) is bounded by $2^{|\mathcal{R}|}(3 + 2(|\mathcal{R}| + 2^{|\mathcal{R}|})(1 + |\mathcal{R}|)) \leq 2^{|\mathcal{R}|}$ (provided that
Case 1. For any $wT_1^{n_1} \ldots T_r^{n_r} \in \Delta^{C_{O,A}}$, with $r > 0$, we have

$$C_{O,A}, k \models \tau(wT_1^{n_1} \ldots T_r^{n_r}) \iff C_{O,A}, k + p_O \models \tau(wT_1^{n_1+p_O} \ldots T_r^{n_r+p_O}),$$

for any $k \geq M^w + s_O$ provided that $n_1 \geq M^w + s_O$,

$$C_{O,A}, k \models \tau(wT_1^{n_1} \ldots T_r^{n_r}) \iff C_{O,A}, k - p_O \models \tau(wT_1^{n_1-p_O} \ldots T_r^{n_r-p_O}),$$

for any $k \leq M^w - s_O - |p_O|$ provided that $n_1 \leq M^w - s_O$.

(By Lemma 6 (ii), if $n_1 \geq M^w + s_O$ then $wT_1^{n_1+p_O} \ldots T_r^{n_r+p_O} \in \Delta^{C_{O,A}}$, and similarly, if $n_1 \leq M^w - s_O$, then $wT_1^{n_1-p_O} \ldots T_r^{n_r-p_O} \in \Delta^{C_{O,A}}$.)

Cases $\tau = A$ and $\tau = \exists S$. For both the main claim and Claim 1 follow from Lemmas 24 and 6 (ii).

Case $\tau = \exists S. \tau'$ with $|\tau'| \geq 1$. We first prove the main claim of the lemma. Suppose first that $C_{O,A}, k \models \tau'(w')$ and $C_{O,A}, k \models S(w, w')$, for some $w' \in \Delta^{C_{O,A}}$. There are two possible locations for $w'$.

- Let $w' = aS_1^{m_1} \ldots S_{l-1}^{m_{l-1}}$ (if $l > 0$). Since $M^w \leq M^w$, and so $k \geq M^w + s_O + |\tau'|p_O$, by the induction hypothesis, we obtain $C_{O,A}, k + p_O \models \tau'(w')$. On the other hand, we have $S^{-}(w', w, k) \in C_{R_{2} \{S(w', w, m_1)\}}$, and so $S^{-} \in r_t(k)$, where $r_t$ is the $R$-canonical rod for $\{S(w, m_1)\}$. Since $k - m_1 > s_O$, by Lemma 6 (ii), we obtain $S^{-} \in r_t(k + p_O)$ and $S^{-}(w', w, k + p_O) \in C_{R_{2} \{S(w, w, m_1)\}}$, whence $C_{O,A}, k + p_O \models S(w, w')$.

- Let $w' = wS_1^{m_1+1}$. We have two cases to consider.

  If $m_{l+1} < M^w + s_O$, then $M^w < M^w + s_O$. As $s_O \leq p_O$, we have $k \geq M^w + s_O + |\tau'|p_O$, and so, by the induction hypothesis, we obtain $C_{O,A}, k + p_O \models \tau'(w')$; see Fig. 12a. On the other hand, since $k - m_{l+1} > p_O \geq s_O$, we can apply the argument with the $R$-canonical rods as above and obtain $C_{O,A}, k + p_O \models S(w', w')$.

  If $m_{l+1} \geq M^w + s_O$, then, by applying Claim 1 (as the induction hypothesis) to $w'' = wS_1^{m_1+p_O}$, we obtain $C_{O,A}, k + p_O \models \tau'(w'')$; see Fig. 12b. On the other hand, $S(w, w', k) \in C_{R_{2} \{S_{l+1}(w, w', m_{l+1})\}}$, whence, by shifting the time instant for $S_{l+1}$, we obtain $S(w, w', k + p_O) \in C_{R_{2} \{S_{l+1}(w, w', m_{l+1} + p_O)\}}$, and so, $C_{O,A}, k + p_O \models S(w, w'')$.

Similarly, we can show that $C_{O,A}, k + p_O \models \tau'(w')$ and $C_{O,A}, k + p_O \models S(w, w')$ imply $C_{O,A}, k \models \tau'(w')$ and $C_{O,A}, k \models S(w, w')$. We now establish Claim 1 for $\tau = \exists S. \tau'$ distinguishing between the following two cases.

- Suppose $C_{O,A}, k \models \tau'(wT_1^{n_1} \ldots T_{r-1}^{n_{r-1}})$ and $C_{O,A}, k \models S(wT_1^{n_1} \ldots T_{r-1}^{n_{r-1}}, wT_1^{n_1} \ldots T_{r-1}^{n_{r-1}})$.

  Observe that, by shifting the time instant for $T_1$ (and therefore for all $T_2, \ldots, T_r$),
we obtain $C_{O,A}, k + p_O \models S(wT_{1}^{n_1+p_O} \ldots T_{r}^{n_r+p_O}, wT_{1}^{n_1+p_O} \ldots T_{r-1}^{n_{r-1}+p_O})$. If $r = 1$, then, by the induction hypothesis of the lemma, $C_{O,A}, k + p_O \models \mathcal{C}(w)$. Otherwise, by Claim 1 as the induction hypothesis, we have $C_{O,A}, k + p_O \models \mathcal{C}(wT_{1}^{n_1+p_O} \ldots T_{r-1}^{n_{r-1}+p_O})$. In either case, $C_{O,A}, k + p_O \models \exists S. \mathcal{C}(wT_{1}^{n_1+p_O} \ldots T_{r}^{n_r+p_O})$. The converse is similar.

- Suppose $C_{O,A}, k \models \mathcal{C}(wT_{1}^{n_1} \ldots T_{r}^{n_r+1})$ and $C_{O,A}, k \models S(wT_{1}^{n_1} \ldots T_{r}^{n_r}, wT_{1}^{n_1} \ldots T_{r+1}^{n_{r+1}})$. Again, by shifting the time instant for $T_1$ (and therefore for all $T_2, \ldots, T_{r+1}$), we obtain $C_{O,A}, k + p_O \models S(wT_{1}^{n_1+p_O} \ldots T_{r}^{n_r+p_O}, wT_{1}^{n_1+p_O} \ldots T_{r+1}^{n_{r+1}+p_O})$. On the other hand, by the induction hypothesis (Claim 1), $C_{O,A}, k + p_O \models \mathcal{C}(wT_{1}^{n_1+p_O} \ldots T_{r+1}^{n_{r+1}+p_O})$. Thus, $C_{O,A}, k + p_O \models \exists S. \mathcal{C}(wT_{1}^{n_1+p_O} \ldots T_{r+1}^{n_{r+1}+p_O})$. The converse implication is similar.

The cases of temporal operators, $\cap$ and $\sqcup$ are easy and left to the reader; the proof straightforwardly follows from that of (Artale et al., 2021, Lemma 22).

Lemma 43. Suppose $O$ is a DL-Lite$_{\{O\}}$ ontology, $A$ is any ABox consistent with $O$, $\mathcal{C}$ a positive temporal concept, $w = aS_{1}^{m_1} \ldots S_{l}^{m_l}$, for $l \geq 0$, an element of $\Delta_{O,A}^{C}$ and $T$ a role from $O$. Let $M^w = \max\{ \max A, m_1, \ldots, m_l \}$ and $M^m = \min\{ \min A, m_1, \ldots, m_l \}$. Then, for the positive integers $s_O$ and $p_O$ in Lemma 37 and for every $n \in \text{tem}(A)$,

\begin{align*}
    C_{O,A}, n \models \mathcal{C}(wT^k) & \iff C_{O,A}, n \models \mathcal{C}(wT^{k+p_O}), \quad \text{for every } k \geq M^w + s_O + |\mathcal{C}|p_O \\
    C_{O,A}, n \models \mathcal{C}(wT^k) & \iff C_{O,A}, n \models \mathcal{C}(wT^{k-p_O}), \quad \text{for every } k \leq M^w - s_O - |\mathcal{C}|p_O.
\end{align*}

As in the proof of Lemma 37, $s_O = s_{T_R}$ and $p_O = p_{T_R}$. We again assume without loss of generality that $s_{T_R} \geq s_R$ and $p_{T_R}$ is divisible by $p_R$. To prove this lemma, it is more convenient to show a more general result for an arbitrarily long role chain and disregard the active domain boundaries for $n$. 

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Figure 13: Examples of variations in $k_i^+$ in Lemma 43. For an ontology $\{\exists S^- \subseteq A\}$, we have $\mathcal{C}_{O,A}, n \models \exists(wT_r)$ iff $\mathcal{C}_{O,A}, n \models \exists(wT_r^{k_r^+})$ where $\exists = \exists S^-$. However, since $A$ is unique for a witness created by $S^-$, statement $\mathcal{C}_{O,A}, n \models A(wT_r S^n)$ iff $\mathcal{C}_{O,A}, n \models A(wT_r^{k_r+} S^n)$ holds.

Claim 2. For roles $T_1, \ldots, T_r$ from $O$, $k_1, \ldots, k_r \in \mathbb{Z}$, $r \geq 0$, the following holds:

$$\mathcal{C}_{O,A}, n \models \exists(wT_1^{k_1^+} \ldots T_r^{k_r^+}) \iff \mathcal{C}_{O,A}, n \models \exists(wT_1^{k_1^+,+} T_1^{k_1^+} \ldots T_r^{k_r^+})$$

for every $k_1 \geq M^w + s_O$ and every $n \leq k_1 - r \cdot s_O - |\exists|p_O$,

$$\mathcal{C}_{O,A}, n \models \exists(wT_1^{k_1^-} \ldots T_r^{k_r^-}) \iff \mathcal{C}_{O,A}, n \models \exists(wT_1^{k_1^-} \ldots T_r^{k_r^-})$$

for every $k_1 \leq M^w - s_O$ and every $n \geq k_1 + r \cdot s_O + |\exists|p_O$,

where

$$k_i^+ = \begin{cases} k_i, & \text{if there is } 2 \leq j \leq i \text{ with } k_j \leq k_{j-1} - s_O, \\ k_1 + p_O, & \text{otherwise}, \end{cases}$$

$$k_i^- = \begin{cases} k_i, & \text{if there is } 2 \leq j \leq i \text{ with } k_j \geq k_{j-1} + s_O, \\ k_1 - p_O, & \text{otherwise}. \end{cases}$$

In order to provide an intuition behind the rather technical result, Fig. 13 depicts different witness behaviours considered in the claim.

Proof. We are going to show only the former statement. A proof for the latter will follow the same line of reasoning. For the reading convenience, we abbreviate the sequences $T_1^{k_1^+} \ldots T_r^{k_r^+}$, $T_1^{k_1^+} \ldots T_r^{k_r^+}$, and $T_1^{k_1^+,+} \ldots T_r^{k_r^+,+}$ as $T_r$, $T_r^{+}$, and $T_r^{k_r^+,+}$.

Similarly to Claim 1, by Lemma 6 (ii) and the construction of $k_i^+$, for all $i \leq r$, if $k_1 \geq M^w + s_O$, we observe that $wT_r \in \Delta^{C_{O,A}}$ iff $wT_r^{+} \in \Delta^{C_{O,A}}$.

Now, we proceed by induction on the construction of $\exists$ and consider the following cases.

Case $\exists = A$. We observe that witnesses $wT_r$ and $wT_r^{+}$ are defined by the same beam. If $k_r^+ = k_r$, then the claim is immediate. Otherwise, $\mathcal{C}_{O,A}, n \models A(wT_r)$ iff $\mathcal{C}_{O,A}, n +
We can now define the remaining two formulas in Table 5, $\Psi_{\mu_l^{0 \mu_0, \kappa}(x, t_0, \ldots, t_l, t)}$ and $\Psi_{\mu_l^{0 \mu_0, \kappa}(x, t_1, \ldots, t_l, t)}$. The definition is by induction on the structure of $\kappa$. For convenience, we denote the sequence $S_l, \ldots, S_l$ by $S_l$, the sequence $t_1, \ldots, t_l$ by $t_l$ and the sequence $\mu_1, \ldots, \mu_l$ by $\mu_l$ and adopt similar notation for their prefixes; for example, $S_0$ is the empty list of roles. We shall also assume that $\Psi_{S_0,\kappa}(x, t) = \Phi_{\kappa}(x, t)$ and $\Psi_{S_0,\kappa}(x, t) = \Phi_{\kappa}(x, t)$, for any $\kappa$.

**Case $\kappa = A$.** If $\mu_l = 0$, then $\Psi_{\mu_l^{0 \mu_0, \kappa}(x, t_1, t_l, t)} = \Xi_{\exists S_l \rightarrow A}(t_1, t)$ and $\Psi_{\mu_l^{0 \mu_0, \kappa}(x, t_1, t_l, t)} = \Xi_{\exists S_l \rightarrow A}(t_1, t)$.

Otherwise, we take $\Psi_{\mu_l^{0 \mu_0, \kappa}(x, t_1, t_l, t)} = \Xi_{\exists S_l \rightarrow A}(t_1, t)$ and $\Psi_{\mu_l^{0 \mu_0, \kappa}(x, t_1, t_l, t)} = \Xi_{\exists S_l \rightarrow A}(t_1, t)$.

**Case $\kappa = \exists S, \kappa'$.** Let $N = (l + 1)(s_0 + |\kappa|p_0)$; here we propagate the worst-case creation time boundaries from Lemma 43 for a witness of depth $l + 1$.

If $\mu_l = 0$, then $\Psi_{\mu_l^{0 \mu_0, \kappa}(x, t_1, t_l, t)}$ is the following:

\[
\bigvee_{\text{role } S_{l+1}, \text{in } O} \left( \Xi_{\exists S_{l+1} \rightarrow A}(t_{l+1}) \wedge \Theta_{S_{l+1} \rightarrow A}(t_l, t_{l+1}) \wedge \Psi_{S_{l+1}, \kappa'}(x, t_l, t_{l+1}) \right) \wedge \left( \Xi_{\exists S_{l+1} \rightarrow A}(t_{l+1}) \wedge \Theta_{S_{l+1} \rightarrow A}(t_l, t_{l+1}) \wedge \Psi_{S_{l+1}, \kappa'}(x, t_l, t_{l+1}) \right) \wedge \left[ \Theta_{S_{l} \rightarrow A}(t_l, t_{l-1}) \wedge \Psi_{S_{l-1}, \kappa'}(x, t_{l-1}, t) \right],
\]
The cases of other temporal operators are similar, and the cases are trivial. Case $\kappa$ and the $\Psi^\mu_{S,\kappa}(x, t)$ are obtained from the above by removing occurrences of variable $t$ and adding the $k$ decoration instead:

$$\bigvee_{\text{role } S_{i+1} \in O} \left( \exists t_{i+1} \left[ \exists s \exists t \exists s_{i+1} (t_i, t_{i+1} + 1) \wedge \Theta^\mu_{S, l_{i+1} \rightarrow s_{i+1}}(x, t_i, t_{i+1}) \right] \wedge \Psi^\mu_{S, l_i, \kappa}(x, t_i, t_{i+1}) \right)$$

$$\bigvee_{i \in [-N, 0) \cup (0, N]} \left[ \Theta^\mu_{S, l_i \rightarrow s_{i+1}} \wedge \Psi^\mu_{S, l_i, \kappa}(x, t_i, t_{i+1}) \right]$$

If $\mu_t \neq 0$, then $\Psi^\mu_{S, \kappa}(x, t_i, t)$ is the following:

$$\bigvee_{\text{role } S_{i+1} \in O} \left( \exists t_{i+1} \left[ \exists s \exists t \exists s_{i+1} (t_i, t_{i+1} + 1) \wedge \Theta^{\mu_t}_{S, l_{i+1} \rightarrow s_{i+1}}(x, t_i, t_{i+1}, t) \wedge \Psi^{\mu_t}_{S, l_i, \kappa}(x, t_i, t_{i+1}, t) \right] \wedge \Psi^{\mu_t}_{S, l_i, \kappa}(x, t_i, t_{i+1}, t) \right)$$

and the $\Psi^{\mu_t}_{S, \kappa}(x, t_i)$ are again obtained by replacing $t$ with the $k$ decorations:

$$\bigvee_{\text{role } S_{i+1} \in O} \left( \exists t_{i+1} \left[ \exists s \exists t \exists s_{i+1} (t_i, t_{i+1} + 1) \wedge \Theta^{\mu_t}_{S, l_{i+1} \rightarrow s_{i+1}}(x, t_i, t_{i+1}, t) \wedge \Psi^{\mu_t}_{S, l_i, \kappa}(x, t_i, t_{i+1}, t) \right] \wedge \Psi^{\mu_t}_{S, l_i, \kappa}(x, t_i, t_{i+1}, t) \right)$$

and

$$\bigvee_{i \in [-N, 0) \cup (0, N]} \left[ \Theta^{\mu_t}_{S, l_i \rightarrow s_{i+1}} \wedge \Psi^{\mu_t}_{S, l_i, \kappa}(x, t_i, t_{i+1}, t) \right]$$

Case $\kappa = \square p \psi$. By Lemma 37, we take again $N = (l + 1)(s_O + |\psi|p_O)$ and set

$$\Psi^{\mu_t}_{S, \kappa}(x, t_i, t) = \forall s [(s > t) \rightarrow \Psi^{\mu_t}_{S, l_i, \kappa}(x, t_i, s)] \wedge \bigwedge_{k \in (0, N]} \Psi^{\mu_t}_{S, l_i, \kappa}(x, t_i)$$

$$\Psi^{\mu_t}_{S, l_i, \kappa}(x, t_i) = \begin{cases} \bigwedge_{i \in (k, k+N]} \Psi^{\mu_t}_{S, l_i, \kappa}(x, t_i), & \text{if } k > 0; \\
\forall s \Psi^{\mu_t}_{S, l_i, \kappa}(x, t_i, s) \wedge \bigwedge_{i \in (k, 0) \cup (0, N]} \Psi^{\mu_t}_{S, l_i, \kappa}(x, t_i), & \text{if } k < 0. \end{cases}$$

The cases of other temporal operators are similar, and the cases $\kappa = \kappa_1 \cap \kappa_2$ and $\kappa = \kappa_1 \cup \kappa_2$ are trivial.