Abstract. Let $X$ be a nondegenerate Peano unicoherent continuum. The family $CB(X)$ of proper subcontinua of $X$ with connected boundaries is a $G_\delta$-subset of the hyperspace $C(X)$ of all subcontinua of $X$. If every nonempty open subset of $X$ contains an open subset homeomorphic to $\mathbb{R}^n$ (such space is called $\pi$-$n$-Euclidean) and $2 \leq n < \infty$, then $C(X) \setminus CB(X)$ is recognized as an $F_\sigma$-absorber in $C(X)$; if additionally, no one-dimensional subset separates $X$, then the family of all members of $CB(X)$ which separate $X$ is a $D_2(F_\sigma)$-absorber in $C(X)$, where $D_2(F_\sigma)$ denotes the small Borel class of differences of two $\sigma$-compacta.

All continua in the paper are metric. For a continuum $X$, we consider the hyperspaces
$$2^X = \{A \subset X : A \text{ is closed and nonempty}\}$$
and
$$C(X) = \{A \in 2^X : A \text{ is connected}\}$$
with the Hausdorff metric. Define
$$CB(X) = \{A \in C(X) \setminus \{X\} : Bd(A) \text{ is connected}\},$$
where $Bd(A)$ denotes the boundary of $A$ in $X$.

1. Evaluation of the Borel complexity of $CB(X)$

K. Kuratowski observed in [7, p. 156] that, for any compact nondegenerate $X$, the function $\alpha : 2^X \setminus \{X\} \rightarrow 2^X$, $\alpha(A) = \overline{X \setminus A}$ is lower semicontinuous, hence of the first Borel class, while the function $Bd : 2^X \setminus \{X\} \rightarrow 2^X$, $A \mapsto Bd(A) = A \cap \overline{X \setminus A}$, is of the second Borel class for any nondegenerate continuum $X$. It means that, for each nondegenerate continuum $X$, the preimage $\alpha^{-1}(D)$ of a closed
subset $D \subset 2^X$ is $G_\delta$ and the preimage $Bd^{-1}(D)$ is $F_{\sigma\delta}$ in $2^X \setminus \{X\}$. It follows that the preimages are also $G_\delta$ and $F_{\sigma\delta}$ subsets of $2^X$. Since $CB(X) = C(X) \cap Bd^{-1}(C(X))$, we have the following proposition.

**Proposition 1.1.** If $X$ is a nondegenerate continuum, then $CB(X)$ is an $F_{\sigma\delta}$-subset of $2^X$.

Actually, in many cases the Borel class of $CB(X)$ can be reduced. For example, one can easily see that

1. if $X$ is the circle $S^1$ then $CB(X)\cap \text{top}$ is compact;
2. $CB(I)$ is homeomorphic to the real line $\mathbb{R}$, so it is an absolute $G_\delta$-set;
3. if $X$ is an indecomposable nondegenerate continuum, then the family $CB(X)$ is also $G_\delta$ since it is equal to $C(X) \setminus \{X\}$.

**Lemma 1.2.** If $X$ is a nondegenerate unicoherent continuum then $CB(X) = C(X) \cap \alpha^{-1}(C(X))$.

**Proof.** If $C \in CB(X)$, then $Bd(C) = C \cap \overline{X \setminus C}$ is connected and since $X = C \cup \overline{X \setminus C}$ is connected, the set $X \setminus C$ is also connected by a well known fact [8, Corollary 5, p. 133]. Thus $C \in C(X) \cap \alpha^{-1}(C(X))$.

The converse implication follows directly from the unicoherence of $X$. $\square$

Since $\alpha$ is of the first Borel class, Lemma 1.2 implies the following proposition.

**Proposition 1.3.** If $X$ is a nondegenerate unicoherent continuum then $CB(X)$ is a $G_\delta$-subset of $C(X)$.

2. **Subcontinua with connected boundaries in $\pi$-Euclidean unicoherent Peano continua**

Recall that a family $\mathcal{U}$ of nonempty open subsets of a space $Y$ is a $\pi$-base of $Y$ if each nonempty open subset of $Y$ contains some $U \in \mathcal{U}$.

**Definition 2.1.** A space $Y$ is said to be $\pi$-Euclidean if there exist $n \in \mathbb{N}$ and a $\pi$-base of $Y$ whose elements are homeomorphic to $\mathbb{R}^n$; such $Y$ will be also called $\pi$-$n$-Euclidean.

In this section we assume that $X$ is a nondegenerate unicoherent Peano continuum without free arcs. Then the hyperspace $C(X)$ is a Hilbert cube [4].

One can immediately notice that every subcontinuum of $X$ is approximated (in the Hausdorff metric) by arcs and the arcs are nowhere dense in $X$, so they belong to $CB(X)$. Thus, we have
Proposition 2.2. $CB(X)$ is a dense $G_δ$-subset of $C(X)$.

Imposing yet an additional structure on $X$ we can fully characterize the family $CB(X)$ in the next theorem which is our main result of this section.

Notice that if $n \geq 2$ then a $\pi$-Euclidean space contains no free arcs.

Theorem 2.3. If $X$ is $\pi$-Euclidean, $n \geq 2$, then there is a homeomorphism $h : I^∞ \to C(X)$ such that $h((0,1)^∞) = CB(X)$.

In proving the theorem we will rather concentrate on the complement $C(X) \setminus CB(X)$ and show that it is an $F_σ$-absorber in $C(X)$. For a basic terminology and facts on such absorbers the reader is referred to [9] and [3]. Recall here that, given a class $\mathcal{M}$ of spaces which is topological (i.e., if $M \in \mathcal{M}$ then each homeomorphic image of $M$ is in $\mathcal{M}$) and closed hereditary (i.e., each closed subset of $M \in \mathcal{M}$ is in $\mathcal{M}$), a subset $A$ of a Hilbert cube $Q$ is called an $\mathcal{M}$-absorber in $Q$ if

1. $A \in Q$,
2. $A$ is strongly $\mathcal{M}$-universal, i.e. for each $M \subset I^ω$ from the class $\mathcal{M}$ and each compact set $K \subset I^ω$, any embedding $f : I^ω \to Q$ such that $f(K)$ is a $Z$-set in $Q$ can be approximated arbitrarily closely by an embedding $g : I^ω \to Q$ such that $g(I^ω)$ is a $Z$-set in $Q$, $g|K = f|K$ and $g^{-1}(A) \setminus K = M \setminus K$.
3. $A$ is contained in a $σZ$-set in $Q$.

If $A \subset Q$ is an $\mathcal{M}$-absorber in $Q$ and $B \subset I^∞$ is an $\mathcal{M}$-absorber in $I^∞$, then there is a homeomorphism $h : I^∞ \to Q$ such that $h(B) = A$ [9, Theorem 5.5.2].

We will be interested in two classes $\mathcal{M}$: the Borel class of $F_σ$-subsets of the Hilbert cube and the small Borel class $D_2(F_σ)$ of all subsets of the Hilbert cube that are differences of two $F_σ$-sets. In the first case, property (3) above is redundant in presence of (1) and (2) (see [1, Theorem 5.3]). The pseudo-boundary $\partial(I^∞) = \{(t_i) \in I^∞ : t_i \in \{0,1\} \text{ for some } i\}$ is a standard $F_σ$-absorber in $I^∞$, while $\partial(I^∞) \times (0,1)^∞$ is a $D_2(F_σ)$-absorber in $I^∞ \times I^∞$.

Proposition 2.4. If $X$ contains an open subset homeomorphic to $\mathbb{R}^n$, $2 \leq n < ∞$, then $C(X) \setminus CB(X)$ is $F_σ$-universal, i.e., for each $F_σ$-subset $M$ of the Hilbert cube $I^∞$, there is an embedding $φ : I^∞ \to C(X)$ such that

\[
φ^{-1}(C(X) \setminus CB(X)) = M.
\]
Proof. For convenience, assume that $X$ contains $(-3, 3)^n$ as an open subset. Let us construct an embedding $\Theta : I^\infty \to C(X)$ such that

$$ (2.2) \quad \Theta((t_k)) \in C(X) \setminus CB(X) \quad \text{if and only if} \quad (t_k) \in \partial(I^\infty). $$

Denote $D := I^n \setminus \left(\frac{1}{3}, \frac{2}{3}\right)^n$ and, for $t \in [0, 1]$,

$$ D(t) := D \setminus \left(\frac{1}{3}, \frac{1}{3}(1 + t)\right) \times \left[0, \frac{1}{3}\right] \times \left(\frac{1}{3}, \frac{2}{3}\right)^{n-2} $$

(let us agree that $D(t) = D \setminus \left(\frac{1}{3}, \frac{1}{3}(1 + t)\right) \times \left[0, \frac{1}{3}\right]$ if $n = 2$ and $D(0) = D$),

$$ D_k(t) := \left(\frac{1}{2k}, 0, \ldots, 0\right) + \frac{1}{2k} D(t(1 - t)), \quad k \in \mathbb{N}, $$

and consider segments $L_k(t)$ in $(-3, 3)^n$ from

$$ \left(\frac{1}{2k}, -\frac{t}{2k}, 0, \ldots, 0\right) \quad \text{to} \quad \left(\frac{1}{2k}, \frac{1 + t}{2k}, 0, \ldots, 0\right). $$

So, $D$ is the cube $I^n$ with the smaller open cube $D' = \left(\frac{1}{3}, \frac{2}{3}\right)^n$ subtracted; its boundary in $(-3, 3)^n$ has two components. $D(t)$ is obtained from $D$ by removing a cube $D'(t) = \left(\frac{1}{3}, \frac{1}{3}(1 + t)\right) \times \left[0, \frac{1}{3}\right] \times \left(\frac{1}{3}, \frac{2}{3}\right)^{n-2}$ adjacent to the both components; the size of $D'(t)$ continuously depends on parameter $t$ so that the boundary of $D(t)$ is connected if and only if $t > 0$. The sets $D_k(t_k)$ are copies of $D(t_k(1 - t_k))$ scaled by factors $\frac{1}{2k}$ and shifted by vectors $\left(\frac{1}{2k}, 0, \ldots, 0\right)$, correspondingly. The union of all $D_k(t_k)$’s compactified by the point $(0, \ldots, 0)$ is a continuum whose boundary is connected if and only if each parameter $t_k$ is strictly between 0 and 1; moreover, the continuum continuously depends on sequence $(t_k)$ but not in a one-to-one fashion. In order to get the one-to-one correspondence, we attach segments $L_k(t_k)$ (Figure 1). Formally, we define an embedding

$$ (2.3) \quad \Theta((t_k)) = \{(0, \ldots, 0)\} \cup \bigcup_{k=1}^{\infty} (L_k(t_k) \cup D_k(t_k)) \subset (-3, 3)^n. $$

Since the pseudo-boundary $\partial(I^\infty)$ is strongly $F_\sigma$-universal, it is $F_{\sigma}$-universal, in particular. So, there exists, for each $F_{\sigma}$-set $M \subset I^\infty$, an embedding $\chi : I^\infty \to I^\infty$ such that $\chi^{-1}(\partial(I^\infty)) = M$. Hence, the composition $\varphi = \Theta \chi : I^\infty \to C(X)$ satisfies (2.1).

$\square$
Lemma 2.5. If $X$ is $\pi$-$n$-Euclidean, $2 \leq n < \infty$, then $C(X) \setminus CB(X)$ is strongly $F_\sigma$-universal.

Proof. The proof is based on a technique developed in [2, 3]. For our purpose, we closely follow its rough description given in [6, Section 3.2]. Without loss of generality, we can assume that $\varphi$ from Proposition 2.4 satisfies

$$\varphi(I^\infty) \subset C((0,1)^n) \subset C(I^n) \subset C(X).$$

Given an $F_\sigma$-subset $M$ of $I^\omega$, for each open non-empty subset $U$ of $X$ and an open copy of $(0,1)^n$ in $U$, let $\varphi_U : I^\omega \to C(U)$ be a composition of $\varphi$ with an embedding of $C(I^n)$ into the hyperspace of that copy. Then

$$\varphi^{-1}_U(C(X) \setminus CB(X)) = M$$

which means that

$$\varphi_U(q) \in C(X) \setminus CB(X) \text{ if and only if } \varphi(q) \in C(X) \setminus CB(X) \text{ if and only if } q \in M.$$

Notice that the equivalences remain valid if, given $q \notin K$, we add to the one-dimensional part $A(q)$, appearing in the construction of embedding $g(q)$, finitely many pairwise disjoint sets $\varphi_{U_i}(q)$, $i < m$ for some $m \in \mathbb{N}$, such that $A(q) \cap \varphi_{U_i}(q)$ is a singleton for each $i$. It follows that $g^{-1}(C(X) \setminus CB(X)) \setminus K = M \setminus K$. Now, the construction of embedding $g$, as presented in [6, Section 3.2], satisfies all the required conditions.

\[\square\]

Proof of Theorem 2.3. By Proposition 1.3 and Lemma 1.2, $C(X) \setminus CB(X)$ is an $F_\sigma$-absorber in $Q = C(X)$. Hence, there exists a homeomorphism $h : I^\infty \to C(X)$ such that $h(\partial(I^\infty)) = C(X) \setminus CB(X)$. \[\square\]
Separators with connected boundaries.

Now, let $S(X) := \{C \in 2^X : C \text{ separates } X\}$, $N(X) := \{C \in 2^X : \text{int } C = \emptyset\}$.

By [6 Proposition 5.1] and Proposition 1.3 we get

**Proposition 2.6.** $S(X) \cap CB(X) \in D_2(F_\sigma)$.

It is proved in [6 Theorem 5.8] that $S(X) \cap N(X) \cap C(X)$ is a $D_2(F_\sigma)$-absorber in $C(X)$ if $X$ satisfies hypotheses of Theorem 2.3 $n \geq 3$ and no subset of dimension $\leq 1$ separates $X$. In a similar way, we will show that $S(X) \cap CB(X)$ is also a $D_2(F_\sigma)$-absorber in $C(X)$ for such $X$.

**Proposition 2.7.** If $X$ contains an open subset homeomorphic to $\mathbb{R}^n$, $2 \leq n < \infty$, then $S(X) \cap CB(X)$ is $D_2(F_\sigma)$-universal in $C(X)$, i.e., for each $D_2(F_\sigma)$-subset $M$ of $I^\infty$, there is an embedding $f : I^\infty \to C(X)$ such that

$$f^{-1}(S(X) \cap CB(X)) = M.$$  

**Proof.** Without loss of generality, we can again assume that $(-3, 3)^n$ is an open subset of $X$. In [6 Proposition 5.2], an embedding $\Psi : I^\infty \to C(X)$ is constructed such that

$$\Psi((q_k)) \in S(X) \text{ if and only if } (q_k) \in \partial(I^\infty).$$  

For each $(q_k)$, continuum $\Psi((q_k))$ can be located in $[-1, 0] \times [-1, 1]^{n-1}$ as a subset consisting of the segment $L' = [-1, 0] \times \{0, \ldots, 0\}$ and of a subset of the union of cubes $[\frac{1}{2k}, \frac{1}{2k+1}] \times [\frac{1}{2k}, \frac{1}{2k}]^{n-1}$, $k \in \mathbb{N}$.

We can now combine both embeddings $\Theta(2.3)$ and $\Psi(2.6)$ to define an embedding $\Phi : I^\infty \times I^\infty \to C(X)$ satisfying

$$\Phi((q_k), (t_k)) \in S(X) \cap CB(X) \text{ if and only if } ((q_k), (t_k)) \in \partial(I^\infty) \times (0, 1)^\infty,$$

by putting

$$\Phi((q_k), (t_k)) = \Psi((q_k)) \cup \Theta((t_k)).$$

Since $\partial(I^\infty) \times (0, 1)^\infty$ is strongly $D_2(F_\sigma)$-universal in $I^\infty \times I^\infty$, there exists, for each $D_2(F_\sigma)$-set $M \subset I^\infty$, an embedding $\tau : I^\infty \to I^\infty \times I^\infty$ such that $\tau^{-1}(\partial(I^\infty) \times (0, 1)^\infty) = M$. Hence, the composition $f = \Phi \tau : I^\infty \to C(X)$ satisfies $2.8$. Similarly as in $2.4$, we can additionally assume that $(0, 1)^n$ is an open subset of $X$ and

$$f(I^\infty) \subset C(I^n) \subset C(X).$$

\qed
Theorem 2.8. If $X$ is $\pi$-$n$-Euclidean and no subset of dimension $\leq 1$ separates $X$, then there is a homeomorphism $h : I^\infty \times I^\infty \to C(X)$ such that $h(\partial I^\infty \times (0,1)^\infty) = S(X) \cap \text{CB}(X)$.

Proof. Since subsets of dimension $\leq 1$ do not separate $X$, $n = \dim X \geq 3$.

1. By Proposition 2.6, family $S(X) \cap \text{CB}(X)$ is $D_2(F_\sigma)$.

2. To prove the strong $D_2(F_\sigma)$-universality, we proceed similarly as in the proof of Lemma 2.5. We just replace $\varphi$ with embedding $f$ from Proposition 2.7 satisfying (2.9) and family $C(X) \setminus \text{CB}(X)$ with $S(X) \cap \text{CB}(X)$. Then, for an arbitrary fixed $D_2(F_\sigma)$-set $M \subset I^\infty$, we have

$$\varphi_U(q) \in S(X) \cap \text{CB}(X) \text{ if and only if } f(q) \in S(X) \cap \text{CB}(X) \text{ if and only if } q \in M.$$ 

For $q \notin K$, attaching finitely many pairwise disjoint sets $\varphi_U(q)$, to the one-dimensional part $A(q)$ at single points does not change the above equivalences (no one-dimensional set separates $X$) which means that the required embedding $g$ satisfies

$$g^{-1}(C(X) \setminus \text{CB}(X)) \setminus K = M \setminus K.$$ 

3. $S(X) \cap \text{CB}(X)$ is contained in the $F_\sigma$-absorber $S(X) \cap C(X)$ in $C(X)$ (see [6, Theorem 5.3]), so in a $\sigma Z$-set in $C(X)$.

Thus we conclude that $S(X) \cap \text{CB}(X)$ is a $D_2(F_\sigma)$-absorber in $C(X)$ and the required homeomorphism $h$ exists.

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