THE POLAR CONE OF THE SET OF MONOTONE MAPS

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Abstract. We prove that every element of the polar cone to the closed convex cone of monotone transport maps can be represented as the divergence of a measure field taking values in the positive definite matrices.

1. Introduction

The one-dimensional pressureless gas dynamics equations
\[
\begin{cases}
\frac{\partial}{\partial t} \varrho + \nabla \cdot (\varrho u) = 0 \\
\frac{\partial}{\partial t} (\varrho u) + \nabla \cdot (\varrho u \otimes u) = 0
\end{cases}
\]
in \([0, \infty) \times \mathbb{R}\)

has recently been shown equivalent (in the regime of sticky particles) to a first-order differential inclusion on the space of monotone transport maps from the reference measure space \(([0, 1], L^1|_{[0,1]}) =: (\Omega, m)\) (where \(L^1\) is the one-dimensional Lebesgue measure) to \(\mathbb{R}\); see [7]. More precisely, to every density/velocity \((\varrho, u)\) solving (1.1) one can associate a unique map \(X \in L^2(\Omega, m)\) that is monotone, and where \(\partial I_K\) is the subdifferential of the indicator function of \(K\) (the closed convex cone of all transport maps \(X \in L^2(\Omega, m)\) that are monotone). If \(X\) satisfies (1.3) and is related to \(\varrho\) through (1.2), then the Eulerian velocity \(u\) can be recovered from the Lagrangian velocity \(V := \dot{X}\) through
\[
V(t, \cdot) = u(t, X(t, \cdot)) \quad \text{for all } t \in [0, \infty).
\]

Assuming finite kinetic energy, it is natural to require that
\[
V(t, \cdot) \in L^2(\Omega, m), \quad u(t, \cdot) \in L^2(\mathbb{R}, \varrho(t, \cdot)).
\]
The relation (1.4) in particular determines the initial Lagrangian velocity \(\dot{V}\) in (1.3) in terms of the initial data \((\varrho, u)(0, \cdot) =: (\bar{\varrho}, \bar{u})\) of the system (1.1).

It is shown in [7] that the solution of (1.3) can be written explicitly as
\[
X(t, \cdot) = P_K(\bar{X} + t\dot{V}) \quad \text{for all } t \in [0, \infty),
\]
with \(\bar{X} := X(0, \cdot) \in \mathcal{K}\) given by (1.2). Here \(P_K\) denotes the metric projection onto the cone \(\mathcal{K}\). The connection between (1.1) and (1.3) makes it possible to apply classical results from the theory of first-order differential inclusions in Hilbert spaces...
to study the pressureless gas dynamics equations, which form a system of hyperbolic conservation laws. We refer the reader to [3,7] for further information.

It is known that if \( X \) satisfies (1.5), then the difference \( (\bar{X} + t\bar{V}) - X(t, \cdot) \) must be an element of the polar cone \( N_{\mathcal{H}}(X(t, \cdot)) \) of \( \mathcal{H} \), which is defined as

\[
N_{\mathcal{H}}(X) := \left\{ Y \in L^2(\Omega, \mathbb{m}) : \int_{\Omega} Y(X' - X) \leq 0 \quad \text{for all} \quad X' \in \mathcal{H} \right\}
\]

for all \( X \in \mathcal{H} \). We observe that \( N_{\mathcal{H}}(X) \) coincides with the subdifferential \( \partial I_{\mathcal{H}}(X) \). Since \( \mathcal{H} \) is a cone, one can choose \( X' = 2X \) in (1.6) to obtain that

\[
Y \in N_{\mathcal{H}}(X) \iff \int_{\Omega} YX = 0, \int_{\Omega} YX' \leq 0 \quad \text{for all} \quad X' \in \mathcal{H}.
\]

One is therefore naturally led to the problem of characterizing the polar cone of the set of monotone transport maps, beyond the basic definition (1.6). It is shown in [7] that if \( Y \in L^2(\Omega, \mathbb{m}) \) is an element of the polar cone \( N_{\mathcal{H}}(X) \), then \( Y \) coincides with the derivative of a nonnegative function. We refer the reader to [7] for more details, and to [4,6,9] for similar results.

In this paper, we will give a generalization of this result to the multi-dimensional case. We are interested in the following setting: We assume that a Borel probability measure \( \varrho \) on \( \mathbb{R}^d \) is given with finite second moments, so that \( \int_{\mathbb{R}^d} |x|^2 \varrho(dx) < \infty \). We consider the closed convex cone of monotone transport maps

\[
\mathcal{H}_\varrho := \left\{ f \in L^2(\mathbb{R}^d, \varrho) : f \text{ is monotone} \right\}.
\]

Here we call any Borel map \( f : \mathbb{R}^d \rightarrow \mathbb{R}^d \) monotone if the support of the induced transport plan \( \gamma_f := (\text{id}, f)\#\varrho \), which is a Borel probability measure on the product space \( \mathbb{R}^d \times \mathbb{R}^d \), is a monotone set. Recall that \( \Gamma \subset \mathbb{R}^d \times \mathbb{R}^d \) is monotone if

\[
(y_1 - y_2) \cdot (x_1 - x_2) \geq 0 \quad \text{for all} \quad (x_i, y_i) \in \Gamma \quad \text{with} \quad i = 1..2,
\]

where \( \cdot \) denotes the Euclidean inner product on \( \mathbb{R}^d \). Our goal is to find a representation of elements of the polar cone \( \mathcal{H}_\varrho^\perp \) (at the zero map), defined as

\[
\mathcal{H}_\varrho^\perp := \left\{ g \in L^2(\mathbb{R}^d, \varrho) : \int_{\mathbb{R}^d} g(x) \cdot f'(x) \varrho(dx) \leq 0 \quad \text{for all} \quad f' \in \mathcal{H}_\varrho \right\}.
\]

Notice that since \( \varrho \) has finite second moments, any smooth monotone function with at most linear growth at infinity (see details below) is an element of \( \mathcal{H}_\varrho \). Moreover, whenever \( g \in \mathcal{H}_\varrho^\perp \) is given, then the product \( g\varrho \) is in fact an \( \mathbb{R}^d \)-valued finite Borel measure, because of Cauchy-Schwarz inequality. We will show below in Theorem 2.31 that for any \( g \in \mathcal{H}_\varrho^\perp \) the measure \( g\varrho \) can be written as the divergence of a finite Borel measure taking values in the symmetric, positive semidefinite matrices. In the one-dimensional case, we therefore obtain the derivative of a nonnegative function (measure) as in [7]. Our proof relies on an application of the Hahn-Banach theorem and is inspired by a similar argument in [2] for the construction of Michell trusses. It is possible to prove a representation of the polar cone \( \mathcal{H}_\varrho^\perp \) similar to ours by using a characterization of polar cones from [10] and subharmonic functions; see [6,9] for instance. Compared to these presentations, our proof is shorter and simpler.
2. The main result

We will denote by \( x \cdot y \) the Euclidean inner product of \( x, y \in \mathbb{R}^k \), and by \(|x|\) the induced norm. We write \( \mathbb{R}^{l \times l} \) for the space of real matrices. For any \( A, B \in \mathbb{R}^{l \times l} \) with components \( A = (a_{ij}) \) and \( B = (b_{ij}) \) we define an inner product

\[
\langle A, B \rangle := \text{tr}(AB^T) = \sum_{i,j=1}^l a_{i,j}b_{i,j}
\]

(with \( B^T \) the transpose of \( B \)), which induces the Frobenius norm

\[
\|A\| := \sqrt{\text{tr}(A A^T)} = \sum_{i,j=1}^l a_{i,j}^2.
\]

We denote by \( S^l \) the space of symmetric real matrices and by \( S^l_+ \) the subset of positive semidefinite symmetric matrices. The space of all positive definite, but not necessarily symmetric matrices will be denoted by \( \mathbb{R}^+_{l \times l} \). Recall that

\[
A \in \mathbb{R}^+_{l \times l} \iff v \cdot (Av) \geq 0 \text{ for all } v \in \mathbb{R}^l.
\]

Equivalently, we have \( A \in \mathbb{R}^+_{l \times l} \) if and only if \( A^{\text{sym}} := (A + A^T)/2 \in S^l_+ \).

Let \( \mathcal{C}_*(\mathbb{R}^d; \mathbb{R}^{l \times l}) \) be the space of all continuous functions \( w : \mathbb{R}^d \rightarrow \mathbb{R}^{l \times l} \) with the property that \( \lim_{|x| \rightarrow \infty} w(x) \in \mathbb{R}^{l \times l} \) exists. Note that we can write

\[
\mathcal{C}_*(\mathbb{R}^d; \mathbb{R}^{l \times l}) = \mathbb{R}^{l \times l} + \mathcal{C}_0(\mathbb{R}^d; \mathbb{R}^{l \times l}),
\]

where \( \mathcal{C}_0(\mathbb{R}^d; \mathbb{R}^{l \times l}) \) is the closure of the space of all compactly supported continuous \( \mathbb{R}^{l \times l} \)-valued maps, w.r.t. the sup-norm. In an analogous way, we define \( \mathcal{C}_*(\mathbb{R}^d; S^l) \) and \( \mathcal{C}_*(\mathbb{R}^d; S^l_+) \). For any map \( u \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}^d) \) we denote by

\[
e(u(x)) := Du(x)^{\text{sym}} \text{ for all } x \in \mathbb{R}^d
\]

its deformation tensor, which is an element of \( \mathcal{C}(\mathbb{R}^d; S^d) \). Let

\[
\mathcal{C}^1_*(\mathbb{R}^d; \mathbb{R}^d) := \{u \in \mathcal{C}^1_*(\mathbb{R}^d; \mathbb{R}^d) : Du \in \mathcal{C}_*(\mathbb{R}^d; \mathbb{R}^{d \times d})\},
\]

\[
\text{MON}(\mathbb{R}^d) := \{u \in \mathcal{C}^1_*(\mathbb{R}^d; \mathbb{R}^d) : u \text{ is monotone}\},
\]

so that \( e(u) \in \mathcal{C}_*(\mathbb{R}^d; S^d_+) \) if \( u \in \text{MON}(\mathbb{R}^d) \). The cone \( \text{MON}(\mathbb{R}^d) \) contains all linear maps \( u(x) := Ax \) for \( x \in \mathbb{R}^d \), where \( A \in \mathbb{R}^d_{d \times d} \). See \[\text{MON}\] for more details.

We will denote by \( \mathcal{M}(\mathbb{R}^d; \mathbb{R}^k) \) the space of finite \( \mathbb{R}^k \)-valued Borel measures. In an analogous way, we define \( \mathcal{M}(\mathbb{R}^d; S^l) \) and \( \mathcal{M}(\mathbb{R}^d; S^l_+) \). If \( f_i, i = 1 \ldots k \), are the components of \( F \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^k) \) and \( u \in \mathcal{C}_0(\mathbb{R}^d; \mathbb{R}^k) \) we write

\[
\int_{\mathbb{R}^d} u(x) \cdot F(dx) = \sum_{i=1}^k u_i(x) f_i(dx).
\]

We will say that \( F \) has finite first moment if \( \sum_{i=1}^k \int_{\mathbb{R}^d} |f_i|(dx) < \infty \). If \( \mu_{i,j} = \mu_{j,i}, i, j = 1 \ldots l \), are the components of \( M \in \mathcal{M}(\mathbb{R}^d; S^l) \) and \( v \in \mathcal{C}_0(\mathbb{R}^d; S^l) \), then

\[
\int_{\mathbb{R}^d} \langle v(x), M(dx) \rangle = \sum_{i,j=1}^l v_{i,j}(x) \mu_{i,j}(dx).
\]
For any $\mathbf{M} = (\mu_{i,j}) \in \mathcal{M}(\mathbb{R}^d; \mathcal{S}^1)$ we have $\mathbf{M} \in \mathcal{M}(\mathbb{R}^d; \mathcal{S}_+^d)$ if and only if
\[\sum_{i,j=1}^{l} \mu_{i,j} v_i v_j \] is a positive measure for all $v \in \mathbb{R}^l$.

We can now state our representation result.

**Theorem 2.1** (Stress tensor). Assume that there exist a measure $\mathbf{F} \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$ with finite first moment and a matrix-valued field $\mathbf{H} \in \mathcal{M}(\mathbb{R}^d; \mathcal{S}_+^d)$ with
\[G(\mathbf{u}) := -\int_{\mathbb{R}^d} \mathbf{u}(x) \cdot \mathbf{F}(dx) - \int_{\mathbb{R}^d} \langle e(\mathbf{u}(x)), \mathbf{H}(dx) \rangle \geq 0\]
for all $\mathbf{u} \in \text{MON}(\mathbb{R}^d)$. Then there exists $\mathbf{M} \in \mathcal{M}(\mathbb{R}^d; \mathcal{S}_+^d)$ such that
\[\int_{\mathbb{R}^d} \text{tr}(\mathbf{M}(dx)) \leq -\int_{\mathbb{R}^d} \mathbf{x} \cdot \mathbf{F}(dx) - \int_{\mathbb{R}^d} \text{tr}(\mathbf{H}(dx)).\]

Notice that the integrals in (2.1) are finite for any choice of $\mathbf{u} \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}^d)$, by our assumptions on $\mathbf{F}$ and $\mathbf{H}$. Recall that the trace of a symmetric matrix is equal to the sum of its eigenvalues, which in the case of a positive semidefinite matrix are all nonnegative. Therefore (2.3) controls the size of the measure $\mathbf{M}$.

For $\mathbf{H} \equiv 0$ we obtain the representation announced in the introduction:
\[\int_{\mathbb{R}^d} \mathbf{u}(x) \cdot \mathbf{F}(dx) = -\int_{\mathbb{R}^d} \langle \mathbf{D}\mathbf{u}(x), \mathbf{M}(dx) \rangle\]
for all test functions $\mathbf{u}$. Recall that $\mathbf{M}$ takes values in the symmetric matrices. The more general form of (2.1) is motivated by a variational time discretization for the compressible Euler equations, for which a minimization problem of the form
\[\inf_{\mathbf{f} \in \mathcal{K}_0} \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\mathbf{h}(x) - \mathbf{f}(x)|^2 \varrho(dx) + \int_{\mathbb{R}^d} e(x) \det(\mathbf{Df}(x)^{\text{sym}})^{1-\gamma} dx \right\}\]
for suitable $\mathbf{h} \in \mathcal{L}^2(\mathbb{R}^d, \varrho)$ and nonnegative $e \in \mathcal{L}^1(\mathbb{R}^d)$ must be solved, with $\gamma > 1$ some constant. Denoting by $\mathbf{f} \in \mathcal{K}_0$ the minimizer of (2.4) and letting $\mathbf{g} := \mathbf{h} - \mathbf{f}$, we can write the corresponding first-order optimality condition (formally) as
\[-\int_{\mathbb{R}^d} \mathbf{g}(x) \cdot \mathbf{f}'(x) \varrho(dx) - (\gamma - 1) \int_{\mathbb{R}^d} e(x) \det(\mathbf{Df}(x)^{\text{sym}})^{-\gamma} \text{tr} \left( \text{cof}(\mathbf{Df}(x)^{\text{sym}})^{T} \mathbf{Df}'(x) \right) dx \geq 0\]
for all $\mathbf{f}' \in \mathcal{K}_0$. From this, assumption (2.1) follows if we define
$$\mathbf{F} := \mathbf{g}\varrho \quad \text{and} \quad \mathbf{H} := (\gamma - 1) e \det(\mathbf{Df}^{\text{sym}})^{-\gamma} \text{cof}(\mathbf{Df}^{\text{sym}})^{T}.$$ One can then check that $\mathbf{F}$ has finite first moments and that $\mathbf{H} \in \mathcal{M}(\mathbb{R}^d; \mathcal{S}_+^d)$. This application will be discussed in more detail in an upcoming publication.
2.1. Positive functionals. In this section, we will discuss a general result about extensions of positive functionals, which is due to Riedl [8]. Let us start with some notation: In the following, we denote by $E$ a normed vector space. We call positive cone any subset $C \subset E$ with $C \neq E$ with the following properties:

$$C + C \subset C, \quad \lambda C \subset C, \quad \text{for all } \lambda > 0, \quad C \cap (-C) = \{0\}.$$  

The positive cone $C$ induces a partial ordering $\geq$ on the space $E$ by

$$y \geq x \iff y - x \in C.$$  

A linear map $F: L \to \mathbb{R}$ defined on a subspace $L \subset E$ is called positive if

$$F(x) \geq 0 \quad \text{for all } x \in L \cap C.$$  

A linear map $F: E \to \mathbb{R}$ is called functional if it is continuous.

**Proposition 2.2.** Let $E$ be a Banach space, partially ordered by a positive cone $C$. If some subspace $L \subset E$ contains an interior point of $C$, then every positive linear map $F_0: L \to \mathbb{R}$ can be extended to a positive functional $F: E \to \mathbb{R}$.

**Proof.** See Theorem 10.10 of [8]. We include a proof for the reader’s convenience.

**Step 1.** We first observe that $E = L - C$. Indeed if $x_0 \in L$ is an inner point of $C$, then there exists a $\delta > 0$ with $B_\delta(x_0) \subset C$. Moreover, for all $x \in E$ there exists $\lambda > 0$ (choose $\lambda := 2\|x\|/\delta$, for example) with the property that

$$x/\lambda \subset B_\delta(0) = x_0 - B_\delta(x_0) \subset x_0 - C.$$

Since $L$ is a subspace we obtain, using $\lambda C \subset C$ for all $\lambda > 0$, that

$$E \subset \bigcup_{\lambda > 0} \lambda(x_0 - C) \subset \mathbb{R}x_0 - C \subset L - C.$$

**Step 2.** Since $E = L - C$, for every $x \in E$ there exist $y_+ \in L$ and $z_\pm \in C$ such that

$$\pm x = y_\pm - z_\pm,$$

which implies that $y_+ \geq x \geq y_-$. We now define

$$p(x) := \inf \left\{ F_0(y) : y \in L, y \geq x \right\} \quad \text{for all } x \in E.$$  

Then $p(x) \leq F_0(y_+) < \infty$. On the other hand, for every $y \in L$ with $y \geq x$ we have $y \geq -y_-$. Since $y + y_- \in L \cap C$, we have $F_0(y + y_-) \geq 0$, by positivity of $F_0$. This implies that $F_0(y) \geq -F_0(y_-) > -\infty$. We conclude that $p(x)$ is finite for all $x \in E$.

It is easy to check that for all $x_1, x_2 \in E$ and for all $\lambda > 0$ we have

$$p(x_1 + x_2) \leq p(x_1) + p(x_2), \quad p(\lambda x_1) = \lambda p(x_1).$$

For every $x \in L$ and $z \in E$ with $z \geq x$, we have $F_0(x) \leq p(z)$ (in particular, we may choose $z = x$). Indeed for every $y \in L$ with $y \geq z$, we have $y \geq x$, thus $y - x \in L \cap C$. Hence $F_0(y - x) \geq 0$, by positivity, which yields $F_0(y) = F_0(x) + F_0(y - x) \geq F_0(x)$. Taking the inf over all $y \in L$ with $y \geq z$, we obtain the estimate.

**Step 3.** We can now apply the Hahn-Banach theorem and obtain a linear map $F: E \to \mathbb{R}$ with $F(x) \leq p(x)$ for all $x \in E$. In order to show that $F$ is positive, let $x \in C$. Then $0 \geq -x$ and $0 \in L$, so we may choose $y = 0$ in the definition of $p(-x)$ (see (2.7)) to obtain $p(-x) \leq 0$. Therefore $F(-x) \leq p(-x) \leq 0$, and so $F(x) \geq 0$ for all $x \in C$. To prove that $F$ is an extension of $F_0$, let $x \in L$. Then we may choose $y = -x$ in (2.7) to obtain $p(-x) \leq F_0(-x)$ for all $x \in L$. Then

$$-F(x) = F(-x) \leq p(-x) \leq F_0(-x) = -F_0(x),$$
We now observe that both $\bar{F}_0(x) \leq F(x)$. Applying the same argument to $-x \in L$, we get $F_0(x) \geq F(x)$. It follows that $F_0(x) = F(x)$ for all $x \in L$. Therefore $F$ is an extension of $F_0$.

**Step 4.** It remains to prove that $F$ is continuous. Let $x_0$ be the interior point of $C$ from Step 1, for which $B_{\delta}(x_0) \subset C$. Then $B_{\delta}(0) \subset \pm(x_0 - C)$. Let $\lambda := F(x_0) > 0$ (recall that $F(x) > 0$ for all $x \in C$). Then for all $x \in B_{\delta}(0)$ we have $x_0 - x \in C$, thus $F(x_0 - x) > 0$. It follows that $F(x_0) \geq F(x)$. Similarly, we obtain $F(x) \geq -F(x_0)$. Then either $F$ vanishes (if $\lambda = 0$), or the preimage of the nonempty interval $(-\lambda, \lambda)$ contains a neighborhood of 0, and so $F$ (being linear) is continuous. $\square$

### 2.2. Proof of Theorem 2.1

We apply Proposition 2.2 with

$$
E := C^*_s(\mathbb{R}^d; S^d), \quad C := C^*_s(\mathbb{R}^d, S^d_+), \quad L := \{e(u): u \in C^1_s(\mathbb{R}^d; \mathbb{R}^d)\}.
$$

Clearly $C$ satisfies conditions (2.5). The identity map $\text{id}$ is an element of $\text{MON}(\mathbb{R}^d)$, with constant deformation tensor $e(\text{id})$ equal to the identity matrix $1 \in S^d_+$. Since the eigenvalues of a symmetric matrix depend continuously on the matrix entries, we have that $e(\text{id}) = 1$ is an interior point of $C$: For all $\|v - \text{id}\|_E$ sufficiently small, the eigenvalues of $v(x)$ are larger than 1/2 for all $x \in \mathbb{R}^d$ and $v \in E$.

On the subspace $L \subset E$, we define the functional $F_0$ as

$$
F_0(v) := -\int_{\mathbb{R}^d} u(x) \cdot F(dx) - \int_{\mathbb{R}^d} \langle v(x), H(dx) \rangle \quad \text{where } v = e(u).
$$

Note that $F_0$ is well defined: If there exists another map $\tilde{u} \in C^*_s(\mathbb{R}^d; \mathbb{R}^d)$ such that $e(\tilde{u}(x)) = v(x)$ for all $x \in \mathbb{R}^d$, then we have $e(u - \tilde{u}) \equiv 0$, by linearity. Consequently, there exist an antisymmetric matrix $B \in \mathbb{R}^{d \times d}$ and $c \in \mathbb{R}^d$ such that

$$
\tilde{u}(x) := u(x) - \tilde{u}(x) = Bx + c \quad \text{for all } x \in \mathbb{R}^d.
$$

Indeed assume that $e(\tilde{u}(x)) = 0$ and define

$$
W\tilde{u}(x) := \frac{D\tilde{u}(x) - D\tilde{u}(x)^T}{2} \quad \text{for all } x \in \mathbb{R}^d.
$$

Then $\partial_k(W\tilde{u}), i, j, k \equiv 0$ for all indices $i, j, k$. Since $D\tilde{u} = e(\tilde{u}) + W\tilde{u}$ it follows that $D\tilde{u}$ is a constant matrix with vanishing symmetric part, so $\tilde{u}$ is a rigid deformation. We now observe that both $\pm \tilde{u} \in \text{MON}(\mathbb{R}^d)$, which implies $F_0(e(\tilde{u})) = 0$ because of (2.1). As $F_0$ is linear, we conclude that $F_0$ is well defined. Similarly, one can check that $F_0(v) \geq 0$ for all $v \in L \cap C$, so the linear map $F_0: L \rightarrow \mathbb{R}$ is positive.

Applying Proposition 2.2, we obtain that $F_0$ can be extended to a continuous linear map $F: E \rightarrow \mathbb{R}$. Notice that $C^*_s(\mathbb{R}^d; \mathbb{R})$ is a separable and closed subalgebra of the space $C^*_b(\mathbb{R}^d; \mathbb{R})$ of bounded continuous $S^d$-functions. As is well known, to any closed subalgebra of a space of bounded continuous functions, there corresponds a compactification of the domain. In our case, we obtain the one-point (also called Alexandroff) compactification of $\mathbb{R}^d$, which we will denote by $\beta\mathbb{R}^d$. Then $C^*_s(\mathbb{R}^d; S^d)$ is isomorphic to $C(\beta\mathbb{R}^d; S^d)$. We refer the reader to [5] Section 4.8 for more details.

By the Riesz representation theorem, there therefore exists a finite Radon measure $\mathbf{M} \in \mathcal{M}(\beta\mathbb{R}^d; S^d)$ that represents the functional $F$ in the sense that

$$
F(v) = \int_{\beta\mathbb{R}^d} \langle v(x), \mathbf{M}(dx) \rangle \quad \text{for all } v \in C^*_s(\mathbb{R}^d; S^d).
$$
Since $F(v) \geq 0$ for all $v \in C_\ast(R^d; \mathbb{S}^d_+)$ we obtain that $M$ takes in fact values in $\mathbb{S}^d_+$. Moreover, as $F$ is an extension of $F_0$, the following identity holds:

$$F_0(v) = -\int_{R^d} u(x) \cdot F(dx) - \int_{\beta R^d} \langle v(x), H(dx) \rangle = \int_{\beta R^d} \langle v(x), M(dx) \rangle$$

for any $v = e(u)$ and $u \in C_\ast^1(R^d; \mathbb{R}^d)$; see (2.2). In particular, we may choose $u = \text{id}$ (with $e(\text{id}) = 1$) to obtain the control (recall that $M$ is $\mathbb{S}^d_+$-valued)

$$\int_{\beta R^d} \text{tr}(M(dx)) = -\int_{R^d} x \cdot F(dx) - \int_{R^d} \text{tr}(H(dx)).$$

Restricting the representation from $\beta R^d$ to $R^d$, we obtain the result.

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