POLYNOMIALLY CONVEX EMBEDDINGS OF EVEN-DIMENSIONAL COMPACT MANIFOLDS

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Abstract. We show that, for $k > 1$, any $2k$-dimensional compact submanifold of $\mathbb{C}^{3k-1}$ can be perturbed to be polynomially convex and totally real except at a finite number of points. This lowers the known bound on the number of smooth functions required on every $2k$-manifold $M$ to generate a dense subalgebra of $\mathcal{C}(M)$. We also show that the obstruction to isotropic embeddability of all $2k$-dimensional manifolds in $\mathbb{C}^{3k-1}$ does not persist if we allow for Kähler forms with isolated degeneracies.

1. Introduction and main results

Polynomial convexity is an important notion largely owing to the Oka-Weil theorem which states that holomorphic functions in a neighbourhood of a polynomially convex set $M$ (see Section 2 for relevant definitions) can be approximated uniformly on $M$ by holomorphic polynomials. Although polynomial convexity imposes topological restrictions on $M$, it is known that if $M$ is a nonmaximally totally real submanifold of $\mathbb{C}^n$, it can be deformed via a small perturbation into a polynomially convex one, as proved by Forstnerič-Rosay [11], Forstnerič [9], and Løw-Wold [18]. The condition that any abstract $m$-dimensional compact real manifold admits a totally real embedding into $\mathbb{C}^n$ is well understood: one must have $\lfloor \frac{3m}{2} \rfloor \leq n$. Thus, any $m$-dimensional compact manifold can be embedded as a totally real polynomially convex submanifold of $\mathbb{C}^n$ provided that $n \geq \lfloor \frac{3m}{2} \rfloor$ and $(m, n) \neq (1, 1)$.

The bound discussed above is sharp for manifolds without boundary, see [15]. That is, if $n < \lfloor \frac{3m}{2} \rfloor$, then certain $m$-dimensional compact manifolds necessarily acquire complex tangent directions when embedded into $\mathbb{C}^n$. The points where the tangent space of $M \subset \mathbb{C}^n$ contains complex directions are called the CR-singularities of $M$. The size of these singularities is governed by Thom’s transversality theorem (see, e.g., [13]): for a generic $m$-dimensional submanifold $M \subset \mathbb{C}^n$, CR-singular points (if exist) form a submanifold in $M$ of dimension $3m - 2n - 2$, $m \leq n$. CR-singularities encode topological information about $M$, such as its Euler characteristic and Pontryagin numbers; see Lai [17], Webster [26], and Domrin [7].

The simplest nontrivial case of CR-singularities is that of complex points of a real surface in $\mathbb{C}^2$. This has been an actively explored area starting from the seminal work of Bishop [4]. Stable complex points are classified into two types: elliptic and hyperbolic. A surface is locally polynomially convex near its hyperbolic points (Forstnerič-Stout [12]) but never near its elliptic points ([4]). Regardless of the type of complex points a compact real closed surface $M$ may have in $\mathbb{C}^2$, it can never be (globally) polynomially convex; see Stout [25].

In this paper we consider the only other case when CR-singularities are generically discrete and $m \leq n$, namely when $m = 2k$ and $n = 3k - 1$, $k > 1$, (if $m > n$, problems of polynomial convexity become rather trivial). As discussed above, any $2k$-dimensional real compact manifold $M$ admits an embedding into $\mathbb{C}^{3k-1}$ that is totally real except for a finite number of isolated CR-singularities. Beloshapka [3] for $k = 2,$
and Coffman [6] for \( k > 2 \), constructed the normal form (2.1) for generic CR-singularities of this kind. Our principal result is to show that, unlike the case of complex points of real surfaces, \( M \) is locally polynomially convex near any such CR-singularity, and as a result, there exists a polynomially convex embedding of \( M \) in \( \mathbb{C}^{3k-1} \). More precisely the following holds.

**Theorem 1.1.** Suppose \( M \) is a 2k-dimensional \( (k > 1) \) smooth compact submanifold (closed or with boundary) of \( \mathbb{C}^{3k-1} \). Then, given any \( s \geq 2 \), there exists a \( C^s \)-small perturbation \( M' \) of \( M \) that is polynomially convex. The submanifold \( M' \) is totally real with finitely many generic CR-singularities.

The proof is based on the idea of perturbation of \( M \) away from the set of CR-singularities where \( M \) is already locally polynomially convex; a general result of this type is contained in Arosio-Wold [1]. When \( M \) has nonempty boundary, \( M' \) can be further perturbed to be totally real and polynomially convex. This can be done by ‘pushing’ any CR-singularity of \( M' \) to one of its boundary components and then removing a thin collar neighbourhood of the boundary, leaving the manifold with no CR-singularities. A small perturbation can now be used to further make it polynomially convex.

Our main application of Theorem 1.1 deals with the structure of the space of continuous complex-valued functions on real manifolds. To illustrate this, consider first an elementary example. Any continuous function on the circle \( S^1 \subset \mathbb{C} \) can be uniformly approximated on \( S^1 \) by a sequence of polynomial combinations of \( z \) and \( 1/z \). This follows from the Stone-Weierstrass approximation theorem, but can alternatively be deduced from the fact that the map \( z \mapsto (z, 1/z) \) embeds \( S^1 \) in \( \mathbb{C}^2 \) as a polynomially convex and totally real submanifold, thus allowing the combined use of the Oka-Weil theorem and an approximation result by Nirenberg-Wells [20, Theorem 1]. Generally, given a real manifold \( M \) we say that the space \( \mathcal{C}(M) \) has \( n \)-polynomial density if there is a tuple \( F = (f_1, \ldots, f_n) \) of \( n \) functions in \( \mathcal{C}^\infty(M) \) such that the set

\[
\{ P \circ F : P \text{ is a holomorphic polynomial on } \mathbb{C}^n \}
\]

is dense in \( \mathcal{C}(M) \). We refer to our paper [14] for a detailed discussion of polynomial and rational density of the spaces \( \mathcal{C}^\ell(M) \), \( \ell \geq 0 \), and its connection with polynomially convex and isotropic embeddings. In complete analogy with \( S^1 \) above, a totally real and polynomially convex embedding \( F : M \hookrightarrow \mathbb{C}^n \) guarantees \( n \)-polynomial density of \( \mathcal{C}(M) \). Thus, any compact real \( m \)-manifold has \( n = \lfloor \frac{3m}{2} \rfloor \)-polynomial density. This value of \( n \) does not appear to be sharp, with the optimal \( n \) being somewhere in the range \( m < n < \lfloor \frac{3m}{2} \rfloor \). While it is an open problem to find the best \( n \) for all \( m \)-dimensional manifolds, Theorem 1.1 gives the following improvement for even-dimensional manifolds.

**Corollary 1.2.** Let \( M \) be a 2k-dimensional \( (k > 1) \) compact manifold. Then, \( \mathcal{C}(M) \) has \( (3k-1) \)-polynomial density. Furthermore, if \( M \) has nonempty boundary, then \( \mathcal{C}^\ell(M) \) has \( (3k-1) \)-polynomial density for all \( \ell \geq 0 \).

Recall that a real submanifold \( \iota : M \hookrightarrow \mathbb{C}^n \) is called Lagrangian (isotropic) with respect to a Kähler form \( \omega \) on \( \mathbb{C}^n \) if \( \dim M = n \) (\( \dim M < n \)) and \( \iota^* \omega = 0 \). Our second application can be viewed as a variation of the Gromov-Lees theorem [2], which in turn is an application of Gromov’s h-principle. The Gromov-Lees theorem says that a compact \( n \)-dimensional manifold \( M \) admits a Lagrangian immersion into \( (\mathbb{C}^n, \omega_M) \) if and only if its complexified tangent bundle is trivializable. This is the same topological condition that completely characterizes the totally real immersability of a manifold \( M \) in \( \mathbb{C}^n \) (see [10, Prop. 9.1.4]). Subcritical versions of these results imply that any compact \( m \)-dimensional manifold admits an
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isotropic embedding into \((\mathbb{C}^n, \omega_{\text{K}})\) for \(n \geq \left\lfloor \frac{3m}{2} \right\rfloor\). Furthermore, there exist \(m\)-dimensional manifolds that do not admit such embeddings when \(n < \left\lfloor \frac{3m}{2} \right\rfloor\); see [14] for details. Despite this fact, our next result says that if \(m\) is even, any \(m\)-dimensional \(M\) can be embedded as an isotropic submanifold in \(\mathbb{C}^{\left\lfloor \frac{3m}{2} \right\rfloor - 1}\) with respect to some degenerate Kähler form.

**Corollary 1.3.** Suppose \(M\) is a \(2k\)-dimensional \((k > 1)\) smooth compact closed submanifold of \(\mathbb{C}^{3k-1}\). Then, given any \(s \geq 2\), there exists a \(C^s\)-small perturbation \(M'\) of \(M\) such that \(M'\) is isotropic with respect to \(dd^c\varphi\), for some plurisubharmonic function \(\varphi\) on \(\mathbb{C}^{3k-1}\) that is strictly plurisubharmonic on \(\mathbb{C}^{3k-1}\) except at the isolated CR-singularities of \(M'\).

One should note that the form \(dd^c\phi\) in the above corollary is not arbitrary, and is specific to the perturbation of \(M\). The proof no longer relies on the h-principle. Instead, we use the connection between rational convexity and isotropy with respect to Kähler forms established by Duval-Sibony [8].

2. Background material

The reader can refer to this section for the notation, terminology and definitions used in this paper. We begin with some notation.

- \(\mathbb{D}_\varepsilon\) and \(\overline{\mathbb{D}}_\varepsilon\) denote the open and closed discs, respectively, of radius \(\varepsilon\) centred at the origin in \(\mathbb{C}\).
- \(B_p(r)\) and \(\overline{B}_p(r)\) denote the open and closed Euclidean balls, respectively, of radius \(r\) centred at \(p\) in \(\mathbb{C}^n\), \(n > 1\).
- \(O\) denotes the origin in \(\mathbb{C}^n\) (the ‘\(n\)’ will be clear from the context).
- \(Z = (z, w_1, ..., w_{2k-2}, \zeta_1, ..., \zeta_k)\) denotes the complex coordinates in \(\mathbb{C}^{3k-1}\), where

\[
\begin{align*}
  z &= x + iy, \\
  w_\tau &= u_\tau + iv_\tau, \quad 1 \leq \tau \leq 2k - 2, \\
  \zeta_\sigma &= \xi_\sigma + iv_\sigma, \quad 1 \leq \sigma \leq k,
\end{align*}
\]

is the decomposition of the coordinates into their real and imaginary parts.
- \(Z' = (z, w_1, ..., w_{2k-2}, w)\) denotes the complex coordinates in \(\mathbb{C}^{2k}\).
- \(\xi^*\) is the conjugate transpose of the vector \(\xi \in \mathbb{C}^n\) (viewed as a matrix).
- \(J_C f(Z)\) denotes the complex Jacobian at \(Z\) of the map \(f : \mathbb{C}^{3k-1} \to \mathbb{C}^m\).
- \(\text{Hess}_C f(Z)\) denotes the complex Hessian of \(f : \mathbb{C}^{3k-1} \to \mathbb{R}\) at \(Z\).
- For any compact set \(X \subset \mathbb{C}^n\), \(C(X)\) is the algebra of complex-valued continuous functions on \(X\), and \(\mathcal{P}(X)\) is the closure in \(C(X)\) of the subalgebra generated by all the holomorphic polynomials restricted to \(X\).

Using the notation introduced above, it is of interest to determine when \(C(X) = \mathcal{P}(X)\). A necessary condition is that \(X \subset \mathbb{C}^n\) must coincide with its polynomially convex hull

\[
\hat{X} := \left\{ x \in \mathbb{C}^n : |P(x)| \leq \sup_{z \in X} |P(z)|, \text{ for all polynomials } P \in \mathbb{C}^n \right\}.
\]

If \(X = \hat{X}\), we say that \(X\) is polynomially convex. If \(X = M\) is an embedded submanifold in \(\mathbb{C}^n\), it is known that \(M\) cannot be polynomially convex unless \(m = \dim M \leq n\) (see [25, Section 2.3]). We will henceforth assume that \(m \leq n\).
A sufficient condition for a polynomially convex submanifold $M \subset \mathbb{C}^n$ to satisfy $C(M) = P(M)$ is that $M$ be totally real, i.e., $T_p M \cap i T_p M = \{0\}$ for all $p \in M$, where $T_p M$ denotes the real tangent space of $M$ at $p$. Thus, $C(M) = P(M)$ if $M$ is a totally real and polynomially convex submanifold of $\mathbb{C}^n$. In situations where CR-singularities necessarily occur, the following result due to O’Farrel-Preskenis-Walsch (see [21]; also see [25]) is useful. Let $X$ be a compact holomorphically convex set in $\mathbb{C}^n$, and let $X_0$ be a closed subset of $X$ for which $X \setminus X_0$ is a totally real subset of the manifold $\mathbb{C}^n \setminus X_0$. A function $f \in C(X)$ can be approximated uniformly on $X$ by functions holomorphic on a neighbourhood of $X$ if and only if $f|_{X_0}$ can be approximated uniformly on $X_0$ by functions holomorphic on $X$. One can also achieve approximation results in higher norms. This leads to the following general definition.

Let $M$ be a smooth compact abstract manifold. We say that $C^\ell(M)$, the space of $\ell$-times continuously differentiable $\mathbb{C}$-valued functions on $M$, has $n$-polynomial density if there is a tuple $F = (f_1, \ldots, f_n)$ of $n$ functions in $C^\infty(M)$ such that the set

$$\{P \circ F : P \text{ is a holomorphic polynomial on } \mathbb{C}^n\}$$

is $C^\ell$-dense in $C^\ell(M)$. If $F$ exists, we call $\{f_1, \ldots, f_n\}$ a PD-basis of $C^\ell(M)$. The notions of rational density and an RD-basis can be defined analogously. The existence of 2-RD bases for surfaces is discussed in Shafikov-Sukhov [23]. The optimal value of $n$ for which $C^\ell(M)$ has $n$-polynomial density for all $m$-dimensional manifolds without boundary is known for $\ell \geq 1$ (see [14]) but not for $\ell = 0$.

As discussed earlier, it is not always possible to arrange $M \subset \mathbb{C}^n$ to be totally real everywhere. Given a point $p \in M$, let $H_p M$ denote the maximal complex-linear subspace of $T_p M$. A point $p \in M$ is called a CR-singularity of order $\mu$ if $\dim_{\mathbb{C}}(H_p M) = \mu$. As a consequence of Thom’s transversality theorem, the set $S$ of CR-singularities of a generically embedded $M \subset \mathbb{C}^n$ is either empty or is a smooth submanifold of codimension $2(n - m) + 2$ in $M$. Moreover, $S$ is stratified by smooth submanifolds $S_\mu$ of codimension $2\mu^2 + 2\mu(m - n)$ (if nonempty) in $M$, each $S_\mu$ being the set of CR-singularities of order $\mu$ in $M$, see Domrin [7] for more details. For global polynomial convexity, it is desirable to find an embedding of $M$ that is, at the very least, locally polynomially convex near $S$. The situation is nontrivial even when $S$ is a discrete set, i.e., when $m = 2k$ and $n = 3k - 1$.

When $k = 1$ (or $m = n = 2$), the only possible CR-singularities are complex points. These were classified by Bishop as follows. Given an isolated nondegenerate complex point $p$ of a surface $M$, one can find local holomorphic coordinates in which $M$ can be written as

$$w = \begin{cases} \frac{1}{\alpha} z \overline{z} + \frac{1}{\alpha^2} (z^2 + \overline{z}^2) + o(|z|^2), & \text{if } 0 \leq \alpha < \infty, \\ z \overline{z} + o(|z|^2), & \text{if } \alpha = \infty. \end{cases}$$

Depending on whether $\alpha \in [0, 1)$, $\alpha = 1$ or $\alpha \in [1, \infty]$, $p$ is said to be a hyperbolic, parabolic or elliptic complex point, respectively. Parabolic points are not generic, and have varying local convexity properties (see [27] and [16]). Although, elliptic and hyperbolic points are both stable under small $C^2$-deformations, a surface is locally polynomially convex only near its hyperbolic complex points. In [24], Slapar proves a (possibly stronger) result for flat hyperbolic points ([12]), i.e., when local holomorphic coordinates can be chosen so that $\text{Im} o(|z|^2) \equiv 0$, i.e., $M$ is locally contained in $\mathbb{C} \times \mathbb{R}$. It is shown that, near a flat hyperbolic $p$, $M$ is the zero set of a nonnegative function that is strictly plurisubharmonic in its domain except at $p$. To achieve global polynomial convexity in our setting, we will rely on Slapar’s technique to obtain a similar result.
The case $k > 1$ is qualitatively different because of higher codimension ($m < n$). We are interested in understanding the local polynomial hull of a $2k$-submanifold $M$ in $\mathbb{C}^{3k-1}$ near an isolated CR-singularity. Unlike the $k = 1$ case, stable CR-singularities do not show diverse behaviour in this regard. In fact, we show that it suffices to understand one special case of $M$ to answer this question.

The Beloshapka-Coffman normal form denotes the manifold

$$\mathcal{M}_k := \left\{ Z \in \mathbb{C}^{3k-1} : \begin{array}{l} v_\tau = 0, \quad 1 \leq \tau \leq 2k - 2, \\ \zeta_1 = |z|^2 + \overline{\tau}(u_1 + iu_2), \\ \zeta_\sigma = \overline{\tau}(u_{2\sigma - 1} + iu_{2\sigma}), \quad 2 \leq \sigma \leq k - 1, \\ \zeta_k = z^2 \end{array} \right\}. \quad (2.1)$$

Note that $\dim \mathcal{M}_k = 2k$ and it has an isolated CR-singularity of order 1 at the origin. In [3] and [6], Beloshapka ($k = 2$) and Coffman ($k > 2$) showed that a nondegenerate CR-singularity $p$ of a $2k$-dimensional submanifold $M$ of $\mathbb{C}^{3k-1}$ is locally formally equivalent to $\mathcal{M}_k$ at the origin. The nondegeneracy conditions appearing in their work are the full-rank conditions on matrices involving the second-order derivatives of the graphing functions of $M$ at $p$. Any isolated CR-singular point can, thus, be made nondegenerate with the help of a small $C^\ell$-perturbation, $\ell \geq 2$. In [5], Coffman further proved that if $M$ is also real analytic in a neighbourhood of $p$, then there is a local normalizing transformation that is given by a convergent power series. Since any smooth $M$ near a nondegenerate CR-singularity $p$ can be made real analytic after a small $C^\ell$-perturbation, we will only concern ourselves with real analytic nondegenerate CR-singularities. These will be referred to as generic CR-singularities in this paper. We will rely heavily on the fact that any $M$ at a generic CR-singularity $p$ is locally biholomorphic to $\mathcal{M}_k$ at $O$.

3. A defining function for the Beloshapka-Coffman normal form

In this section and the next, we collect the technical tools required to prove our main results. The primary result here is the existence of a special plurisubharmonic defining function for $\mathcal{M}_k$ at $O$, which allows us to control the hull of a generic $2k$-manifold near its CR-singularities under small perturbations.

**Proposition 3.1.** Let $k > 1$. There exists a smooth plurisubharmonic function $\psi$ defined in some neighbourhood $U$ of $O$ in $\mathbb{C}^{3k-1}$ such that

(a) $\{ \psi = 0 \} = \mathcal{M}_k \cap U$,
(b) $\psi > 0$ on $U \setminus \mathcal{M}_k$, and
(c) $\psi$ is strictly plurisubharmonic on $U \setminus \{ O \}$.

**Proof.** We work with an auxiliary family of $2k$-manifolds in $\mathbb{C}^{2k}$. Let $\alpha < 1$. Set

$$S_\alpha = \left\{ Z' \in \mathbb{C}^{2k} : \begin{array}{l} \text{Im}(w_1) = \cdots = \text{Im}(w_{2k-2}) = 0, \\ w = \alpha \frac{1}{2}|z|^2 + \frac{1}{4}(z^2 + \overline{z}^2) \end{array} \right\}. \quad (3.2)$$

Note that each slice $S_\alpha \cap \{ Z' \in \mathbb{C}^{2k} : (w_1, \ldots, w_{2k-2}) = (s_1, \ldots, s_{2k-2}) \}$, where $(s_1, \ldots, s_{2k-2}) \in \mathbb{R}^{2k-2}$, is a totally real surface with an isolated hyperbolic complex point at the origin in $\mathbb{C}_z^2$. In [24], Slapar has constructed plurisubharmonic defining functions with additional properties for such surfaces. A slight modification of Slapar’s construction yields the following key ingredient of our proof.
Lemma 3.2. For each $\alpha < 0.46$, there is a neighbourhood $V_\alpha$ of the origin in $\mathbb{C}^{2k}$ and a smooth plurisubharmonic function $\rho_\alpha : V_\alpha \to \mathbb{R}$ such that

* $\{\rho_\alpha = 0\} = S_\alpha \cap V_\alpha$,
* $\rho_\alpha > 0$ on $V_\alpha \setminus S_\alpha$, and
* $\rho_\alpha$ is strictly plurisubharmonic on $V_\alpha \setminus Y$, where

$$Y := \{z' \in \mathbb{C}^{2k} : z = \text{Im} w_1 = \cdots = \text{Im} w_{2k-2} = w = 0\}.$$  

(3.1)

We relegate the proof of this lemma to the appendix (see Section 6). To continue with the proof of Proposition 3.1, we produce holomorphic maps that send the Beloshapka-Coffman normal form $\mathcal{M}_k$ into $S_\alpha$. These allow us to pull back $\rho_\alpha$ to $\mathbb{C}^{3k-1}$ (locally near $O$) to give plurisubharmonic functions that vanish on $\mathcal{M}_k$. The required $\psi$ will be designed from these pulled-back functions. For this purpose, let $f_\alpha : \mathbb{C}^{3k-1} \to \mathbb{C}^{2k}$ be the map

$$Z \mapsto \left(z + \frac{\alpha w_1}{2}, w_1, \ldots, w_{2k-2}, \frac{1}{2} z + \frac{\alpha w_2}{2} + \frac{1}{2} w_1, w_1, \ldots, w_{2k-2}, \xi \right).$$

For $1 \leq \sigma \leq k - 1$, let $f_\sigma^\alpha : \mathbb{C}^{3k-1} \to \mathbb{C}^{2k}$ be the map given by

$$f_\sigma^\alpha = f_\alpha \circ F^\sigma,$$

where $F^\sigma : \mathbb{C}^{3k-1} \to \mathbb{C}^{3k-1}$ is the automorphism

$$\left(z, w_1, \ldots, w_{2k-2}, \frac{1}{2} z + \frac{1}{2} w_1, \ldots, w_{2k-2}, \frac{1}{2} w_1, \ldots, w_{2k-2}, \frac{1}{2} \xi_1, \ldots, \xi_k\right) \mapsto$$

$$\left(z, \frac{w_1 + w_{2\sigma-1}}{2}, \frac{w_2 + w_{2\sigma}}{2}, w_3, \ldots, w_{2k-2}, \frac{1}{2} \xi_1, \ldots, \xi_k\right).$$

Each $f_\sigma^\alpha$ is holomorphic on $\mathbb{C}^{3k-1}$ and has the following properties.

- $(f_\sigma^\alpha)^{-1}(S_\alpha) = M_\sigma^\alpha$, where

$$M_\sigma^\alpha = \left\{ Z \in \mathbb{C}^{3k-1} : \alpha \left[\frac{\xi_1 + \xi_\sigma}{2} + \frac{1}{2} \right] + \frac{i \alpha}{2} \left[ w_{2\sigma-1} + i \left( w_2 + w_{2\sigma} \right) \right] + \frac{\xi_\sigma - \overline{\xi}_2}{2} = 0, \right\}.$$

- $(f_\sigma^\alpha)^{-1}(Y) = X_\sigma^\alpha$, where

$$X_\sigma^\alpha = \left\{ Z \in \mathbb{C}^{3k-1} : \alpha \left[ \frac{\xi_1 + \xi_\sigma}{2} + \frac{1}{2} \right] + \frac{i \alpha}{2} \left[ w_2 + w_{2\sigma} \right] = 0, \right\}.$$

- ker $J_C(f_\sigma^\alpha)(Z) = \left\{ \left(0, \ldots, 0, \xi_1, \ldots, \xi_{k-1}, -\alpha \left[ \frac{\xi_1 + \xi_\sigma}{2} \right] \right) : \left( \xi_1, \ldots, \xi_{k-1} \right) \in \mathbb{C}^{k-1} \right\}$.

Next, let $\psi_\sigma^\alpha := \rho_\alpha \circ f_\sigma^\alpha$ on $U_\sigma^\alpha := (f_\sigma^\alpha)^{-1}(V_\alpha)$, where $\rho_\alpha$ and $V_\alpha$ are as in Lemma 3.2. Then, owing to the properties of $\rho_\alpha$ and $f_\sigma^\alpha$, we have that $\psi_\sigma^\alpha$ is a plurisubharmonic function on $U_\sigma^\alpha$, satisfying the following properties (compare with the required properties (a)-(c)).

- (a') $\{\psi_\sigma^\alpha = 0\} = M_\sigma^\alpha \cap U_\sigma^\alpha$,
- (b') $\psi_\sigma^\alpha > 0$ on $U_\sigma^\alpha \setminus M_\sigma^\alpha$, and
- (c') $\xi^* \cdot \text{Hess}_C \psi_\sigma^\alpha(Z) \cdot \xi > 0$, when $Z \in U_\sigma^\alpha \setminus X_\sigma^\alpha$ and $\xi \in \mathbb{C}^{3k-1} \setminus \text{ker} \ J_C(f_\sigma^\alpha)(Z)$. 

As $\mathcal{M}_k \subseteq M_\sigma^*$, we need to ‘correct’ $\psi_\sigma$. For this, let

$$g(Z) = |\zeta_k - Z|^2 + \sum_{\sigma=2}^{k-1} |\zeta_\sigma - \overline{z(w_{2\sigma-1} + i w_{2\sigma})}|^2.$$ 

Since $M_\sigma^* \cap g^{-1}(0) = \mathcal{M}_k$, and $\xi \cdot \text{Hess}_g g(Z) \cdot \xi > 0$ for any $Z \in \mathbb{C}^{3k-1}$ and any nonzero $\xi \in \ker J_\xi(f_\sigma^*)(Z)$, we have that each $g + \psi_\sigma^*$ is a plurisubharmonic function on $U_\sigma^*$ satisfying properties (a), (b) and

$$(c') \quad \psi_\sigma^* + g \text{ is strictly plurisubharmonic on } U_\sigma^* \setminus X_\sigma^*.$$ 

Finally, to obtain property (c), we observe that

$$\bigcap_{\sigma=1}^{k-1} (X_\sigma^* \cap X_\sigma^*) = \{0\}$$

when $\alpha \neq \beta$. Thus, choosing $\alpha = 1/4$ and $\beta = 1/3$, we have that

$$(3.2) \quad \psi := g + \sum_{\sigma=1}^{k-1} \left( \psi_{1/4}^\sigma + \psi_{1/3}^\sigma \right)$$

is a plurisubharmonic function on $U := \bigcap_{\sigma=1}^{k-1} (U_{1/4}^\sigma \cap U_{1/3}^\sigma)$ with the required properties (a)-(c). This completes the proof of Proposition 3.1. \hfill $\Box$

We now have the means to construct a defining function for a generic $2k$-dimensional real submanifold $M$ in $\mathbb{C}^{3k-1}$, which is strictly plurisubharmonic except at finitely many points in $M$. This gives a degenerate Kähler form — defined only in a neighbourhood of $M$ — with respect to which $M$ is isotropic. We will use this construction to prove Corollary 1.3.

**Lemma 3.3.** Suppose $M \subset \mathbb{C}^{3k-1}$ is a real $2k$-dimensional $(k > 1)$ smooth compact submanifold that is totally real except for finitely many generic CR-singularities $p_1, \ldots, p_n \in M$. Then there exists a smooth function $\Psi$ defined in a neighbourhood $\mathcal{U}$ of $M$ such that

1. $\{ \Psi = 0 \} = M$,
2. $\Psi \geq 0$, and
3. $\Psi$ is strictly plurisubharmonic on $\mathcal{U} \setminus \{ p_1, \ldots, p_n \}$.

In particular, $M$ admits a Stein neighbourhood basis.

**Proof.** As each $p_j$, $j = 1, \ldots, n$, is a generic CR-singularity of $M$, we can use the biholomorphic equivalence of $(M, p_j)$ and $(M(k), O)$, together with Proposition 3.1, to conclude that there exist pairwise disjoint open sets $U_j \ni p_j$ and smooth plurisubharmonic functions $\psi_j : U_j \to \mathbb{R}$, $j = 1, \ldots, n$, such that

(a) $\{ \psi_j = 0 \} = M \cap U_j$,
(b) $\psi_j > 0$ on $U_j \setminus M$, and
(c) $\psi_j$ is strictly plurisubharmonic on $U_j \setminus \{ p_j \}$.

Let $\widetilde{M} := M \setminus \bigcup_{1 \leq j \leq n} U_j$. Then, as $\widetilde{M}$ is totally real in $\mathbb{C}^{3k-1}$, $\psi_0(z) := \text{dist}^2(z, M)$ is strictly plurisubharmonic on some neighbourhood $U_0$ of $M$ in $\mathbb{C}^{3k-1}$. Now, let $\mathcal{U} := \bigcup_{0 \leq j \leq n} U_j$. The neighbourhoods $U_j$’s in the above construction should be chosen small enough so that $\pi : \mathcal{U} \to M$ given by $z \mapsto p$, where
\[ \text{dist}(p, M) = \text{dist}(p, z), \] is well-defined and smooth. Let \( \{\chi_j\}_{0 \leq j \leq n} \) be a partition of unity subordinate to \( \{U_j \cap M\}_{0 \leq j \leq n} \). Define

\[ \Psi(z) := \sum_{j=0}^{n} \chi_j(\pi(z))\psi_j(z). \]

Since \( M \cap U_j \subseteq \{\nabla \psi_j = 0\} \), we have that

\[ \frac{\partial}{\partial c} \Psi(p) = \sum_{0}^{n} \chi_j(p)\frac{\partial}{\partial c} \psi_j(p), \]

when \( p \in M \). Thus, \( \frac{\partial}{\partial c} \Psi(p) \) is strictly positive on any compact subset of \( M \setminus \{p_1, \ldots, p_n\} \). Moreover, since \( \chi_j \equiv 1 \) near \( p_j \), \( \frac{\partial}{\partial c} \Psi = \frac{\partial}{\partial c} \psi_j \) near \( p_j \). Thus, shrinking \( U \) if necessary, we have that \( \Psi \) is plurisubharmonic on \( U \) and strictly plurisubharmonic on \( U \setminus \{p_1, \ldots, p_n\} \).

To obtain a Stein neighbourhood basis of \( M \), one can take \( \{\Psi < \varepsilon\} \) where \( \Psi \) as above and \( \varepsilon > 0 \) is chosen small enough.

\section{4. Local polynomial convexity at CR-singularities}

For the purpose of our main theorem, we establish the local polynomial convexity of \( M \) at a generic CR-singularity by constructing a global version of \( \psi \) from Proposition 3.1.

\textbf{Proposition 4.1.} Let \( M \) be a \( 2k \)-dimensional submanifold of \( \mathbb{C}^{3k-1} \) with a generic CR-singularity at \( p \in M \). Then there exists a neighbourhood \( U \) of \( p \in \mathbb{C}^{3k-1} \) such that for any ball \( B \subset U \), there exists a smooth plurisubharmonic function \( \rho : \mathbb{C}^{3k-1} \to \mathbb{R} \) such that

- \( \{\rho = 0\} = M \cap \bar{B} \),
- \( \rho \geq 0 \), and
- \( \rho \) is strictly plurisubharmonic on \( \mathbb{C}^{3k-1} \setminus \{p\} \).

In particular, \( M \) is locally polynomially convex at \( p \).

\textbf{Proof.} Owing to the biholomorphic equivalence of \((M, p)\) and \((M_k, O)\) (see Section 2) and Proposition 3.1, there is a neighbourhood \( U \subset \mathbb{C}^{3k-1} \) of \( p \in M \) and a smooth plurisubharmonic function \( \psi : U \to \mathbb{R} \) such that

(a) \( \{\psi = 0\} = M \cap \bar{B} \),
(b) \( \psi > 0 \) on \( U \setminus M \), and
(c) \( \psi \) is strictly plurisubharmonic on \( U \setminus \{p\} \).

Now, let \( B \subset U \) be a ball. As \( \bar{B} \) is polynomially convex, there is a smooth plurisubharmonic function \( \sigma \) on \( \mathbb{C}^{3k-1} \setminus \{p\} \) such that

(i) \( \{\sigma = 0\} = \bar{B} \),
(ii) \( \sigma > 0 \) on \( \mathbb{C}^{3k-1} \setminus \bar{B} \), and
(iii) \( \sigma \) is strictly plurisubharmonic on \( \mathbb{C}^{3k-1} \setminus \bar{B} \).

To combine the properties of \( \psi \) and \( \sigma \), we choose balls \( B' \) and \( B'' \) such that \( B \Subset B' \Subset B'' \subset U \). Let \( \chi \in C^\infty(\mathbb{C}^{3k-1}) \) be such that \( 0 \leq \chi \leq 1 \), and

\[
\chi(Z) = \begin{cases} 
1, & \text{if } Z \in B'; \\
0, & \text{if } Z \in \mathbb{C}^{3k-1} \setminus B''. 
\end{cases}
\]
Then, for any $C > 0$, the function $\chi \psi + C\sigma$ is well-defined on $\mathbb{C}^{3k-1}$ as $\chi$ vanishes outside $U$. Moreover, because of property (c) of $\psi$ and (iii) of $\sigma$, there is a large enough $C > 0$ such that $\rho = \chi \psi + C\sigma$ is strictly plurisubharmonic on $\mathbb{C}^{3k-1} \setminus \{0\}$. It also follows that $\rho$ vanishes precisely on $M \cap \overline{B}$ and is positive everywhere else. Since plurisubharmonic hulls coincide with polynomial hulls, $M \cap \overline{B}$ is polynomially convex. As $B \subseteq U$ is an arbitrarily chosen ball, $M$ is locally polynomially convex at $p$. \hfill \Box

**Remark.** We note that there is a more direct way of proving the local polynomial convexity of $M_k$ at $O$. For this, we recall the following criterion (an iterated version of Theorem 1.2.16 from [25]). If $X \subseteq \mathbb{C}^n$ is a compact subset and if $G : X \to \mathbb{R}^m$ is a map whose components are in $\mathcal{P}(X)$, then $X$ is polynomially convex if and only if $G^{-1}(t)$ is polynomially convex for each $t \in \mathbb{R}^m$. Now, choosing the restriction to $M_k$ of $G : \mathbb{C}^{3k-1} \to \mathbb{C}^{2k-2}$ that maps $Z \mapsto (w_1, \ldots, w_{2k-2})$, and noting that the subalgebra generated by $z$ and $\overline{z}$ in $\mathcal{C}(\overline{D}_z)$ coincides with $\mathcal{C}(\overline{D}_z)$ (see [19]), we have that every fibre of $G$ is polynomially convex. Hence, by the criterion stated above, $M_k$ is locally polynomially convex at $O$.

We now use Propositions 3.1 and 4.1 to prove a lemma that allows us to simultaneously handle multiple CR-singularities.

**Lemma 4.2.** Suppose $M \subset \mathbb{C}^{3k-1}$ is a real $2k$-dimensional ($k > 1$) smooth compact submanifold that is totally real except for finitely many generic CR-singularities $p_1, \ldots, p_n \in M$. For $r > 0$, define

$$B_p(r) := \bigcup_{j \leq n} \overline{B}_{p_j}(r), \quad p = \{p_1, \ldots, p_n\}.$$ 

Then, there exists an $r > 0$ such that

1. $B_p(r')$ is polynomially convex for all $r' \leq 3r$,
2. $B_p(r') \cap M$ is polynomially convex for all $r' \leq 3r$, and
3. For small enough $\varepsilon > 0$ and any diffeomorphism $\phi$ of $\mathbb{C}^{3k-1}$ satisfying $\phi = \text{Id}$ on $B_p(r)$ and outside a tubular neighbourhood of $M$, and $\|\phi - \text{Id}\|_{C^3(\mathbb{C}^{3k-1})} < \varepsilon$,

the set $\phi(M) \cap B_p(2r)$ is polynomially convex.

**Proof.** To prove (1), we use induction on the number of CR-singularities of $M$ to show that $B_p(3r)$ is polynomially convex for some $r > 0$. Let us assume that if $M$ has $m$ CR-singularities, then there is an $r > 0$ such that $B_p(3r)$ is polynomially convex, where $p = \{p_1, \ldots, p_m\}$ is the set of CR-singularities of $M$. Since the union of two disjoint balls is polynomially convex, this assumption holds if $m \leq 2$. Now suppose the set of CR-singularities of $M$ is $p' = \{p_1, \ldots, p_m, p_{m+1}\}$. We claim that for some $r > 0$, $B_{p'}(3r) = B_p(3r) \cup \overline{B}_{p_{m+1}}(3r)$ is polynomially convex. To see this, first we choose $r > 0$ so that $B_p(3r)$ is polynomially convex (the induction hypothesis grants this), and $B_p(3r)$ and $\overline{B}_{p_{m+1}}(3r)$ are disjoint. As $B_p(3r)$ is rationally convex and $p_{m+1} \notin B_p(3r)$, there is a polynomial $H$ on $\mathbb{C}^{3k-1}$ such that $H(p_{m+1}) = 0$ but $H$ does not vanish on $B_p(3r)$. Thus, $1/H$ is holomorphic in a neighbourhood of $B_p(3r)$. By the Oka-Weil theorem for polynomially convex sets, there exists a sequence $\{Q_j\}_{j \in \mathbb{N}}$ of polynomials on $\mathbb{C}^{3k-1}$ that converges to $1/H$ uniformly on $B_p(3r)$. Thus, the sequence of polynomials $\{HQ_j\}_{j \in \mathbb{N}}$ converges to the constant function 1 on $B_p(3r)$ and vanishes identically at $p_{m+1}$. Hence, for some sufficiently large $j \in \mathbb{N}$ and a sufficiently small $r > 0$, the polynomial $P := HQ_j$ maps $B_p(3r)$ to the point 1 and $\overline{B}_{p_{m+1}}(3r)$ into $\overline{D}_{1/2}$ in $\mathbb{C}$. By Kalinin’s lemma, $B_p(3r) \cup \overline{B}_{p_{m+1}}(3r) = B_{p'}(3r)$ is polynomially convex. Thus, we have (1) for $r' = 3r$ by induction. The proof of (1) for $r' < 3r$ now follows easily.
For (2), we will repeat the technique used in the proof of Proposition 4.1. First, we assume that \( r > 0 \) is such that (1) holds, and there exists a smooth plurisubharmonic function \( \psi \) defined in a neighbourhood \( U \) of \( \bar{B}_p(3r) \) such that

(a) \( \{ \psi = 0 \} = M \cap U \),
(b) \( \psi > 0 \) on \( U \setminus M \), and
(c) \( \psi \) is strictly plurisubharmonic on \( U \setminus \{ p \} \).

The latter can be arranged due to Proposition 3.1. Now, let \( r' \leq 3r \). As \( B_p(r') \) is polynomially convex, there is a smooth plurisubharmonic function \( \sigma \) on \( \mathbb{C}^{3k-1} \) such that

(i) \( \{ \sigma = 0 \} = B_p(r') \),
(ii) \( \sigma > 0 \) on \( \mathbb{C}^{3k-1} \setminus B_p(r') \), and
(iii) \( \sigma \) is strictly plurisubharmonic on \( \mathbb{C}^{3k-1} \setminus B_p(r') \).

Next, we choose \( r_1, r_2 > 0 \) such that \( r' < r_1 < r_2 \) and \( B_p(r_2) \Subset U \), and define \( \chi \in C^\infty(\mathbb{C}^{3k-1}) \) such that \( 0 \leq \chi \leq 1 \), and

\[
\chi(Z) = \begin{cases} 
1, & \text{if } Z \in B_p(r_1); \\
0, & \text{if } Z \in \mathbb{C}^{3k-1} \setminus B_p(r_2).
\end{cases}
\]

As in the proof of Proposition 4.1, there is a large enough \( C > 0 \) such that \( \rho = \chi \psi + C \sigma \) is strictly plurisubharmonic on \( \mathbb{C}^{3k-1} \setminus p \) and vanishes precisely on \( M \cap B_p(r') \). Thus, (2) is proved.

In order to prove (3), we rely on the function \( \rho \) constructed above. Let \( r > 0 \) be such that (1) and (2) hold. Let \( \varepsilon > 0 \) be such that for any diffeomorphism \( \phi \) of \( \mathbb{C}^{3k-1} \) satisfying

- \( \phi = \text{Id on } B_p(r) \) and outside a tubular neighbourhood of \( M \), and
- \( ||\phi - \text{Id}||_{C^2(\mathbb{C}^{3k-1})} < \varepsilon \),

the set \( K := \phi(M \cap B_p(3r)) \) satisfies \( \mathcal{P}(K) = \mathcal{C}(K) \). This is because, for \( \varepsilon > 0 \) small enough, the function \( \tilde{\rho} := \rho \circ \phi^{-1} \) is strictly plurisubharmonic on \( \mathbb{C}^{3k-1} \setminus p \). Since \( \tilde{\rho} \) vanishes precisely on \( K \), \( K \) is polynomially convex (and holomorphically convex). Moreover, \( K \setminus p \) is a totally real submanifold of \( \mathbb{C}^{3k-1} \setminus p \), and any continuous function on \( p \) can be approximated uniformly on \( p \) by holomorphic polynomials (\( p \) is polynomially convex). Thus, by a result due to O’Farrel-Preskenis-Walsch (stated in Section 2) and the Oka-Weil theorem for polynomially convex sets, \( \mathcal{P}(L) = \mathcal{C}(L) \) for all compact subsets \( L \) of \( K \). Shrinking \( \varepsilon > 0 \) further, if necessary, we ensure that for \( \phi \) as above, \( \phi(M) \cap B_p(2r) \) is a compact subset of \( \phi(M \cap B_p(3r)) \). Thus, \( \mathcal{P}(\phi(M) \cap B_p(2r)) = \mathcal{C}(\phi(M) \cap B_p(2r)) \). This implies that \( \phi(M) \cap B_p(2r) \) is polynomially convex. The proof of the lemma is now complete. \( \square \)

5. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. Let \( \iota : M \hookrightarrow \mathbb{C}^{3k-1} \) be the inclusion map of a smooth \( 2k \)-dimensional submanifold \( M \subset \mathbb{C}^{3k-1} \). Fix \( s \geq 2 \). By Thom’s Transversality Theorem, there exists a \( C^s \)-small perturbation \( j \) of \( \iota \) such that \( j(M) \) is smooth and totally real except at a finite number of CR-singular points (see [7, Section 1] for details). Without loss of generality, we may further assume that \( j(M) \) has generic CR-singular points (see end of Section 2). Let \( p \) denote the set of CR-singularities of \( j(M) \). From Lemma 4.2, we know that there exists an open neighbourhood \( U \) of \( p \) contained in \( j(M) \) such that \( K := \overline{U} \) is polynomially convex and \( j(M) \setminus U \) is a compact submanifold of \( \mathbb{C}^{3k-1} \) with boundary. Since, \( j(M) \setminus U \) is totally real, we can now apply the following result due to Arosio-Wold (see [1, Theorem 1.4]). Let \( N \) be a compact smooth
This is possible because of O’Farrel-Preskenis-Walsch \cite{18}. Let \( N \) be a compact smooth manifold (possibly with boundary) of dimension \( d < n \) and let \( f : N \to \mathbb{C}^n \) be a totally real \( C^\infty \)-embedding. Let \( K \subset \mathbb{C}^n \) be a compact polynomially convex set. Then for all \( s \geq 1 \) and for all \( \varepsilon > 0 \), there exists a totally real \( C^\infty \)-embedding \( f_\varepsilon : N \to \mathbb{C}^n \) such that

1. \( ||f - f_\varepsilon||_{C^s(N)} < \varepsilon \)
2. \( f_\varepsilon = f \) near \( f^{-1}(K) \), and
3. \( K \cup f_\varepsilon(N) \subset U \cup f_\varepsilon(N) \).

In our situation, \( N = M \setminus j^{-1}(U) \), \( f = j|_N \) and \( K = \overline{U} \). Let \( \varepsilon > 0 \) be arbitrary. We set \( M' := f_\varepsilon(M \setminus j^{-1}(U)) \cup U \) to obtain a polynomially convex perturbation of \( M \) that is totally real outside of the finite set \( p \).

**Remark.** The proof of the Arosio-Wold result used above depends crucially on the following lemma (an earlier version is due to Loe-Wold in \cite{18}). Let \( N \) be a compact smooth manifold (possibly with boundary) of dimension \( d < n \) and let \( f : N \to \mathbb{C}^n \) be a totally real \( C^\infty \)-embedding. Let \( K \subset \mathbb{C}^n \) be a compact polynomially convex set, and \( U \) be a neighbourhood of \( K \). Then for all \( s \geq 1 \) and for all \( \varepsilon > 0 \), there exists a totally real \( C^\infty \)-embedding \( f_\varepsilon : N \to \mathbb{C}^n \) such that

1. \( ||f - f_\varepsilon||_{C^s(N)} < \varepsilon \)
2. \( f_\varepsilon = f \) near \( f^{-1}(K) \), and
3. \( K \cup f_\varepsilon(N) \subset U \cup f_\varepsilon(N) \).

In our setting, our result can be obtained directly from this lemma using the following argument. Choose \( r > 0 \) granted by Lemma 4.2. Let \( K = B_p(r) \), \( U = \text{int} B_p(2r) \) and \( N = M \setminus j^{-1}(U) \). Note that \( r \) is chosen so that \( K \) is polynomially convex. Then, letting \( \varepsilon > 0 \) be small enough, we obtain an \( f_\varepsilon : N \to \mathbb{C}^{3k-1} \) such that

\[
(5.1) \quad K \cup f_\varepsilon(N) \subset \text{int} B_p(2r) \cup f_\varepsilon(N).
\]

We extend \( f_\varepsilon \circ f^{-1} \) to a diffeomorphism \( \phi \) of \( \mathbb{C}^{3k-1} \) satisfying

* \( \phi = \text{Id} \) on \( B_p(r) \) and outside a tubular neighbourhood of \( M \), and
* \( ||\phi - \text{Id}||_{C^2(\mathbb{C}^{3k-1})} < \varepsilon \).

This is possible because \( f_\varepsilon \circ f^{-1} \) is identity near \( K \). Now, (5.1) can be rewritten as

\[
\overline{\phi(M)} \subset \text{int} B_p(2r) \cup \phi(M).
\]

But, \( \phi(M) \cap B_p(2r) \) is polynomially convex (by Lemma 4.2). By Rossi’s maximum principle, we have that

\[
\overline{\phi(M)} \subset \phi(M) \cup (\phi(M) \cap B_p(2r)).
\]

Thus, \( M' := \phi(M) \) is polynomially convex (and totally real away from \( p \)).

**Proof of Corollary 1.2.** Let \( M \) be a compact \( 2k \)-dimensional abstract manifold without boundary. By Theorem 1.1, there exists a \( C^\infty \)-smooth embedding \( F = (f_1, \ldots, f_{3k-1}) : M \to \mathbb{C}^{3k-1} \) such that \( F(M) \) is polynomially convex and totally real outside a finite set \( p \subset F(M) \). For any compact set \( X \subset \mathbb{C}^n \), we let

\[
\mathcal{O}(X) = \{ f|_X : f \text{ is holomorphic in some open neighbourhood of } X \}.
\]

Note that \( X := F(M) \) and \( X_0 := p \) satisfy the hypothesis of the result due to O’Farrel-Preskenis-Walsch \cite{21} stated in Section 2. Hence, \( \mathcal{O}(F(M)) = C(F(M)) \). Further, by the Oka-Weil theorem for polynomially
convex sets, we have that \( \mathcal{P}(X) = \overline{\mathcal{O}(F(M))} \). Thus, \( \{ P \circ F : P \text{ is a holomorphic polynomial on } \mathbb{C}^{3k-1}\} \) is dense in \( \mathcal{C}(M) \). In other words, \( \{ f_1, \ldots, f_{3k-1}\} \) is a PD-basis of \( \mathcal{C}(M) \).

Now, if \( M \) is a manifold with boundary, Theorem 1.1 guarantees a smooth embedding \( F : M \to \mathbb{C}^{3k-1} \) such that \( F(M) \) is totally real and polynomially convex (see the comment following the statement of Theorem 1.1). We fix an \( \ell \geq 0 \), a \( g \in \mathcal{C}^\ell(M) \) and an arbitrary \( \varepsilon > 0 \). Let \( \bar{c} = C\varepsilon \), where \( C \) is a constant to be determined later. Since \( F(M) \) is totally real, a result due to Range-Siu (see Theorem 1 in [22]; although not explicitly stated, the result therein works for compact manifolds with or without boundary) grants the existence of a neighbourhood \( U \) of \( F(M) \) and a \( h \in \mathcal{O}(U) \) such that

\[
\|g - h\|_{C^\ell(F(M))} < \bar{c}.
\]

Due to the polynomial convexity of \( F(M) \), we can find a neighbourhood \( V \Subset U \) of \( F(M) \), such that \( \mathcal{V} \) is polynomially convex. By the Oka-Weil approximation theorem, there is a polynomial \( P \in \mathcal{C}^n \) such that

\[
\|h - P\|_{C(\mathcal{V})} < \bar{c}.
\]

As \( F(M) \) is compact, there is an \( r > 0 \) such that \( B_r(r) \Subset V \) for all \( x \in F(M) \). We fix an \( x \in F(M) \). As \( h - P \in \mathcal{O}(B_x(r)) \), we can combine Cauchy estimates and (5.3) to obtain

\[
\left| h^{(j)}(x) - P^{(j)}(x) \right| \leq \frac{j!}{r^j} \sup_{y \in B_x(r)} |h(y) - P(y)| \leq \frac{j!}{r^j} \bar{c},
\]

for any \( j \in \mathbb{N}_+ \). So, we obtain from (5.2) and (5.4) that

\[
\begin{align*}
\|g - P\|_{C^\ell(F(M))} &\leq \|g - h\|_{C^\ell(F(M))} + \|h - P\|_{C^\ell(F(M))} \\
&= \|g - h\|_{C^\ell(F(M))} + \sum_{j=0}^{k} \|h^{(j)} - P^{(j)}\|_{C(F(M))} \\
&< \bar{c} \left( 1 + \sum_{j=0}^{k} \frac{j!}{r^j} \right) = C\varepsilon \left( 1 + \sum_{j=0}^{k} \frac{j!}{r^j} \right).
\end{align*}
\]

Setting \( C = \left( 1 + \sum_{j=0}^{k} \frac{j!}{r^j} \right)^{-1} \), we obtain that \( \|g - P\|_{C^\ell(M)} < \varepsilon \). Since \( \varepsilon \) and \( g \) were chosen arbitrarily, and \( C \) is independent of \( \varepsilon \), we conclude that polynomials are dense in the space of \( \mathcal{C}^\ell \)-smooth functions on \( F(M) \) in the \( \ell \)-norm. Thus, the components of \( F \) form a PD-basis of \( \mathcal{C}^\ell(M) \), for every \( \ell \geq 0 \).

\[\square\]

**Remark.** The following statement is implicit in the above proof. If we view a PD-basis of \( M \) from \( M \) to \( \mathbb{C}^{3k-1} \), then the set of all PD-bases of \( \mathcal{C}(M) \) is dense in \( \mathcal{C}(M; \mathbb{C}^{3k-1}) \). This follows from the fact that smooth embeddings of \( M \) into \( \mathbb{C}^{3k-1} \) are dense in \( \mathcal{C}(M; \mathbb{C}^{3k-1}) \) for \( k > 1 \).

**Proof of Corollary 1.3.** Suppose \( M \) is a \( 2k \)-dimensional smooth compact closed submanifold of \( \mathbb{C}^{3k-1} \). By Theorem 1.1, there exists a \( \mathcal{C}^\infty \)-small perturbation \( M' \) of \( M \) such that \( M' \) is totally real except at finitely many generic CR-singularities, say \( p_1, \ldots, p_n \), and \( M' \) is polynomially convex. In particular, \( M' \) is rationally convex. Thus, as a consequence of a characterization of rationally convex hulls due to Duval-Sibony (see [8, Remark 2.2]), there is a smooth plurisubharmonic function \( \theta : \mathbb{C}^{3k-1} \to \mathbb{R} \) such that \( \omega = dd^c \theta \) vanishes on \( M' \) and is strictly positive outside \( M' \). Moreover, as \( M' \) satisfies the hypothesis of Lemma 3.3, there is a neighbourhood \( U \) of \( M' \) and a smooth plurisubharmonic function \( \Psi : U \to \mathbb{R} \) such that \( \{ z \in U : \Psi(z) = 0 \} = M \) and \( \Psi \) is strictly plurisubharmonic on \( U \setminus \{ p_1, \ldots, p_n \} \). Let \( \chi : \mathbb{C}^{3k-1} \to \mathbb{R} \) be a compactly supported nonnegative function that is identically 1 on some neighbourhood \( W \Subset U \). For
a large enough \( C \), the function \( \varphi := C\theta + \chi\Psi \) is well-defined and strictly plurisubharmonic except at \( p_1, \ldots, p_n \). Since the gradient of \( \psi \) vanishes along \( M' \), we also have that \( \iota^*dd^c\varphi = \iota^*dd^c\Psi = d(\iota^*d^c\Psi) = 0 \), where \( \iota : M' \to \mathbb{C}^{3k-1} \) is the inclusion map. Thus, \( M' \) is isotropic with respect to the degenerate Kähler form \( dd^c\varphi \).

\[ \square \]

### 6. Appendix: Proof of Lemma 3.2

The main technical ingredient of this paper relies on Lemma 3.2. It is a mild generalization of Lemma 4 in Slapar’s work [24], whose proof has been omitted there due to its close analogy with the proof of Lemma 3 therein. For the sake of completeness, we reproduce Slapar’s technique to provide a full proof of Lemma 3.2. We continue to use the notation established in Section 2.

**Lemma 3.2.** Let

\[
S_\alpha = \left\{ Z' = (z, w_1, \ldots, w_{2k-2}, w) \in \mathbb{C}^{2k} : \begin{array}{l}
\text{Im}(w_1) = \cdots = \text{Im}(w_{2k-2}) = 0, \\
w = \frac{\alpha}{2}|z|^2 + \frac{1}{4}(z^2 + \overline{z}^2) \end{array} \right\}.
\]

For each \( \alpha < 0.46 \), there is a neighbourhood \( V_\alpha \) of the origin in \( \mathbb{C}^{2k} \) and a smooth plurisubharmonic function \( \rho_\alpha : V_\alpha \mapsto \mathbb{R} \) such that

\[
\begin{align*}
\star & \ \{ \rho_\alpha = 0 \} = S_\alpha \cap V_\alpha, \\
\star & \ \rho_\alpha > 0 \text{ on } V_\alpha \setminus S_\alpha. \\
\star & \ \rho_\alpha \text{ is strictly plurisubharmonic on } V_\alpha \setminus Y, \text{ where}
\end{align*}
\]

\[
Y := \{ Z' \in \mathbb{C}^{2k} : z = \text{Im} w_1 = \cdots = \text{Im} w_{2k-2} = w = 0 \}.
\]

**Proof.** We consider new real (nonholomorphic) coordinates in \( \mathbb{C}^{2k} \), given by

\[
\begin{aligned}
x &= \text{Re} z, \ y = \text{Im} z, \\
u_j &= \text{Re} w_j, \ v_j = \text{Im} w_j, \quad 1 \leq j \leq 2k-2, \\
u &= \text{Re} w - \frac{\alpha}{2}|z|^2 - \frac{1}{4}(z^2 + \overline{z}^2), \ v = \text{Im} w.
\end{aligned}
\]

(6.1)
For this expression to be nonnegative, it suffices for the following equalities and inequalities to hold.

\[
4\partial_{x,x} = \Delta_{x,y} - 2((\alpha + 1)x\partial_x + (\alpha - 1)y\partial_y + \alpha)\partial_u
\]

\[
+ (\alpha + 1)^2x^2 + (\alpha - 1)^2y^2)\partial_u,
\]

\[
4\partial_{x,y} = \partial_{x,u} + \partial_{y,v} + i(\partial_{x,v} - \partial_{y,u}) - (\alpha + 1)x(\partial_{u,j} + i\partial_{v,j})\partial_u
\]

\[
+ (\alpha - 1)y(\partial_{v,j} - i\partial_{u,j})\partial_u, \quad 1 \leq j \leq 2k - 2,
\]

\[
4\partial_{u,j} = \partial_{x,u} + \partial_{y,v} + i(\partial_{x,v} - \partial_{y,u}) - (\alpha + 1)x - i(\alpha - 1)y\partial_{u,u}
\]

\[
- (\alpha - 1)y + i(\alpha + 1)x)\partial_{u,u},
\]

\[
4\partial_{w,j} = \partial_{w,j} + \partial_{v,j} + i(\partial_{v,j} - \partial_{w,j})\partial_{u,u}, \quad 1 \leq j, l \leq 2k - 2,
\]

\[
4\partial_{w,j} = \partial_{u,j,v} + \partial_{v,j} + i(\partial_{u,j,v} - \partial_{v,j})\partial_{u,u}, \quad 1 \leq j \leq 2k - 2,
\]

\[
4\partial_{w,j} = \Delta_{u,j,v}, \quad 1 \leq j \leq 2k - 2,
\]

\[
4\partial_{w,j} = \Delta_{u,v}.
\]

Consider the following homogenous polynomial in \(\mathbb{R}[x^2, y^2, u]\) of degree 4.

\[
P_\alpha(x^2, y^2, u) = u^4 + ((4\alpha + c)x^2 - cy^2) u^3 + (Ax^4 + Bx^2y^2 + A'y^4) u^2.
\]

Using (6.2), we have that

\[
4\frac{\partial^2 P_\alpha}{\partial x\partial y} =
\]

\[
= \left[6(\alpha + 1)^2 - 3(4\alpha + c)(3\alpha + 2) + 6A + B\right] x^2 + [6(\alpha - 1)^2 + (9\alpha - 6)c + B + 6A'] y^2 \right] 2u^2
\]

\[
+ \left[(6\alpha + 1)^2(4\alpha + c) - 4A(5\alpha + 4)x^4 + (4\alpha' - 5\alpha)(9\alpha - 6) - 6(\alpha - 1)^2 c) y^{4}\right] u
\]

\[
+ \left[24\alpha((\alpha - 1)^2 - c) - 20\alpha B)x^2y^2\right] u
\]

\[
+ (6\alpha + 1)^2 x^2 + (\alpha - 1)^2 y^2 \right) (Ax^4 + Bx^2y^2 + A'y^4).
\]

For this expression to be nonnegative, it suffices for the following equalities and inequalities to hold.

(1) \(A = \frac{3(\alpha + 1)^2(4\alpha + c)}{2(5\alpha + 4)}\);

(2) \(A' = \frac{3(\alpha - 1)^2}{2(4 - 5\alpha)}\)

(3) \(B = \frac{6}{5}((\alpha - 1)^2 - c)\);

(4) \(6A + B + 6(\alpha + 1)^2 > 3(4\alpha + c)(3\alpha + 2)\);

(5) \(6A' + B + 6(\alpha - 1)^2 > (6 - 9\alpha)c\).

Also, we want that \(P_\alpha\) is strictly positive for \(u \neq 0\) and \((x, y)\) small. We use the following lemma for this.

**Lemma 6.1** ([24, Lemma 2.]). Let \(p(x, y, u) = u^2 + b_1(x, y)u + b_0(x, y)\), where \(b_0, b_1\) are continuous functions in a neighbourhood of the origin in \(\mathbb{R}^3\), both vanishing at \((0, 0)\). Suppose \(b_1^2 < 4b_0\) for small \((x, y) \neq (0, 0)\). Then, there exists a small neighbourhood \(U\) of the origin in \(\mathbb{R}^3\) such that \(p\) is strictly positive on \(U \setminus \{u = 0\}\).
The above lemma yields the following constraints on $A$, $A'$ and $c$.

(6) $(4\alpha + c)^2 < 4A$;
(7) $c^2 < 4A'$.

To find constants $A$, $B$ and $A'$ that are positive and satisfy inequalities (4) – (7), it suffices to find a $c > 0$ such that

$$c < \min \left\{ (\alpha - 1)^2, \frac{16 + 8\alpha - 64\alpha^2 - 60\alpha^3}{11 + 30\alpha + 20\alpha^2}, \frac{4(\alpha - 1)^2(4 - 5\alpha)}{20\alpha^2 - 30\alpha + 11}, \frac{6 - 4\alpha - 14\alpha^2}{4 + 5\alpha} \right\}.$$  

The above condition follows from the positivity assumption on $B$ and by writing inequalities (4) – (7) purely in terms of $c$ and $\alpha$ with the means of (1) – (3). As long as $\alpha < 0.46$, the right-hand side of (6.3) is positive. Thus, there exists a homogeneous polynomial $P_{\alpha}$ of degree 4 in $\mathbb{R}[x^2, y^2, u]$ such that

- $P_{\alpha} > 0$ for $u \neq 0$ and $(x, y)$ small enough;
- $\frac{\partial^2 P_{\alpha}}{\partial z \partial z} = 0$ when $(x, y) = (0, 0)$, but is strictly positive otherwise; and
- $\frac{\partial^2 P_{\alpha}}{\partial z \partial w} = q_1u^2 + q_3$, where $q_1, q_3 \in \mathbb{R}[x^2, y^2]$ are polynomials of degree 1 and 3, respectively, with strictly positive coefficients.

Now consider

$$Q_{\alpha}(z, w_1, ..., w_{2k-2}, w) = P_{\alpha}(x^2, y^2, u) + (x^2 + y^2)u^4 + \frac{1}{2} \left( \sum_{j=1}^{2k-2} v_j^2 + v^2 \right),$$

where the coordinates $(z, w_1, ..., w_{2k-2}, w)$ and $(x, y, u_1, v_1, ..., u_{2k-2}, v_{2k-2}, u, v)$ are as in (6.1). Note that

$$\frac{\partial^2 Q_{\alpha}}{\partial z \partial w} = u^4 - (4\alpha)u^3 + (q_2 + \varepsilon q_1)u^2 + (1 - \varepsilon)q_1u^2 + q_3,$$

where $q_2 \in \mathbb{R}[x^2, y^2]$ is of degree 2. By Lemma 6.1, for any $\varepsilon > 0$, $u^4 - (4\alpha)u^3 + (q_2 + \varepsilon q_1)u^2$ is strictly positive for $u \neq 0$ and $(x, y)$ small enough. So, there is a neighbourhood $\mathcal{V}_{\alpha}$ of the origin such that

$$\frac{\partial^2 Q_{\alpha}}{\partial z \partial w} \geq R_3,$$

where $R_3 = \sum r_{j,k,l} (x^2)^j (y^2)^k u^l$ is a homogeneous polynomial in $\mathbb{R}[x^2, y^2, u]$ of degree 3, which is nondegenerate in the sense that all $r_{j,k,l} > 0$ whenever $l$ is even. Next, we have that

$$\frac{\partial^2 Q_{\alpha}}{\partial w \partial u} = \frac{\Delta_{w,u} P_{\alpha}}{4} + 3(x^2 + y^2)u^2 + \frac{1}{4} > \frac{1}{8},$$

Using (6.2), we also note that

$$\left| \frac{\partial^2 Q_{\alpha}}{\partial z \partial w} \right|^2 < R_3,$$

where $R_3(x^2, y^2, u)$ is some homogeneous polynomial of degree 5. Combining these estimates, we have that

$$\frac{\partial^2 Q_{\alpha}}{\partial z \partial u} \frac{\partial^2 Q_{\alpha}}{\partial w \partial u} - \left| \frac{\partial^2 Q_{\alpha}}{\partial z \partial w} \right|^2 \geq R_3 - R_5,$$
which — owing to the nondegeneracy of $R_3$ — is positive on $V_\alpha$ (shrinking if necessary) as long as $(x, y, u) \neq (0, 0, 0)$. As the characteristic polynomial of $\text{Hess} \ Q_\alpha$ (in the variable $\lambda$) is

$$\left( \lambda - \frac{1}{4} \right)^{2k-2} \left( \lambda^2 - \left( \frac{\partial^2 Q_\alpha}{\partial \bar{z} \partial z} + \frac{\partial^2 Q_\alpha}{\partial \bar{w} \partial w} \right) \lambda + \frac{\partial^2 Q_\alpha}{\partial \bar{z} \partial z} \frac{\partial^2 Q_\alpha}{\partial \bar{w} \partial w} - \left( \frac{\partial^2 Q_\alpha}{\partial \bar{z} \partial w} \right)^2 \right),$$

we obtain that $Q_\alpha$ is a plurisubharmonic function on $V_\alpha$ satisfying

- $Q_\alpha^{-1}(0) \cap V_\alpha = S_\alpha \cap V_\alpha$.
- $Q_\alpha > 0$ on $V_\alpha \setminus S_\alpha$.
- $Q_\alpha$ is strictly plurisubharmonic on $V_\alpha \setminus \{x = y = u = 0\}$.

To complete the construction of $\rho_\alpha$, let $\eta(z, w_1, \ldots, w_{2k-2}, w) = \left( \frac{1}{2} + x^2 + y^2 \right) \left( \sum_{j=1}^{2k-2} v_j^2 + v^2 \right)$. In a small enough neighbourhood $V$ of the origin, $\eta$ is plurisubharmonic, and strictly plurisubharmonic when $(v_1, \ldots, v_{2k-2}, v) \neq (0, \ldots, 0, 0)$. Finally, to obtain the desired neighbourhood and function, set $V_\alpha := V \cap V$ and $\rho_\alpha := Q_\alpha + \eta$. This completes the proof of Lemma 3.2. \qed

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