CRITICAL BRANCHING AS A PURE DEATH PROCESS COMING DOWN
FROM INFINITY

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Abstract

We consider the critical Galton–Watson process with overlapping generations stemming from a single founder. Assuming that both the variance of the offspring number and the average generation length are finite, we establish the convergence of the finite-dimensional distributions, conditioned on non-extinction at a remote time of observation. The limiting process is identified as a pure death process coming down from infinity. This result brings a new perspective on Vatutin’s dichotomy, claiming that in the critical regime of age-dependent reproduction, an extant population either contains a large number of short-living individuals or consists of few long-living individuals.

Keywords: Galton–Watson process with overlapping generations; Bellman–Harris process; Sevastyanov process; Crump–Mode–Jagers process; convergence of finite-dimensional distributions; Vatutin’s dichotomy

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1. Introduction

Consider a self-replicating system evolving in the discrete-time setting according to the following rules:

Rule 1: The system is founded by a single individual, the founder, born at time 0.

Rule 2: The founder dies at a random age $L$ and gives a random number $N$ of births at random ages $\tau_j$ satisfying $1 \leq \tau_1 \leq \ldots \leq \tau_N \leq L$.

Rule 3: Each new individual lives independently from others according to the same life law as the founder.

An individual that was born at time $t_1$ and dies at time $t_2$ is considered to be alive during the time interval $[t_1, t_2 - 1]$. Letting $Z(t)$ stand for the number of individuals alive at time $t$, we study the random dynamics of the sequence

$$Z(0) = 1, Z(1), Z(2), \ldots,$$

which is a natural extension of the well-known Galton–Watson process, or $GW$ process for short; see [13]. The process $Z(\cdot)$ is the discrete-time version of what is usually called the
Crump–Mode–Jagers process or the general branching process; see [5]. To emphasise the discrete-time setting, we call it a GW process with overlapping generations, or GWO process for short.

Put $b := \frac{1}{2} \text{var}(N)$. This paper deals with the GWO processes satisfying
\[ E(N) = 1, \quad 0 < b < \infty. \]  
(1)
The condition $E(N) = 1$ says that the reproduction regime is critical, implying $E(Z(t)) \equiv 1$ and making extinction inevitable, provided $b > 0$. According to [1, Chapter I.9], given (1), the survival probability
\[ Q(t) := \text{P}(Z(t) > 0) \]
of a GW process satisfies the asymptotic formula $tQ(t) \to b^{-1}$ as $t \to \infty$ (this was first proven in [6] under a third moment assumption). A direct extension of this classical result for the GWO processes,
\[ tQ(ta) \to b^{-1}, \quad t \to \infty, \quad a := E(\tau_1 + \ldots + \tau_N), \]
was obtained in [3, 4] under the conditions (1), $a < \infty$,
\[ t^2 \text{P}(L > t) \to 0, \quad t \to \infty, \]  
(2)
plus an additional condition. (Notice that by our definition, $a \geq 1$, and $a = 1$ if and only if $L \equiv 1$, that is, when the GWO process in question is a GW process.) Treating $a$ as the mean generation length (see [5, 8]), we may conclude that the asymptotic behaviour of the critical GWO process with short-living individuals (see the condition (2)) is similar to that of the critical GW process, provided time is counted generation-wise.

New asymptotic patterns for the critical GWO processes are found under the assumption
\[ t^2 \text{P}(L > t) \to d, \quad 0 \leq d < \infty, \quad t \to \infty, \]  
(3)
which, compared to (2), allows the existence of long-living individuals given $d > 0$. The condition (3) was first introduced in the pioneering paper [12] dealing with the Bellman–Harris processes. In the current discrete-time setting, the Bellman–Harris process is a GWO process subject to two restrictions: (a) $\text{P}(\tau_1 = \ldots = \tau_N = L) = 1$, so that all births occur at the moment of an individual’s death, and (b) the random variables $L$ and $N$ are independent. For the Bellman–Harris process, the conditions (1) and (3) imply $a = E(L), a < \infty$, and according to [12, Theorem 3], we get
\[ tQ(t) \to h, \quad t \to \infty, \quad h := a + \sqrt{a^2 + 4bd^2} \]  
(4)
As was shown in [11, Corollary B] (see also [7, Lemma 3.2] for an adaptation to the discrete-time setting), the relation (4) holds even for the GWO processes satisfying the conditions (1), (3), and $a < \infty$.

The main result of this paper, Theorem 1 of Section 2, considers a critical GWO process under the above-mentioned set of assumptions (1), (3), $a < \infty$, and establishes the convergence of the finite-dimensional distributions conditioned on survival at a remote time of observation. A remarkable feature of this result is that its limit process is fully described by a single parameter $c := 4bda^{-2}$, regardless of complicated mutual dependencies between the random variables $\tau_j, N, L$. 
Our proof of Theorem 1, requiring an intricate asymptotic analysis of multi-dimensional probability generating functions, is split into two sections for the sake of readability. Section 3 presents a new proof of (4) inspired by the proof of [12]. The crucial aspect of this approach, compared to the proof of [7, Lemma 3.2], is that certain essential steps do not rely on the monotonicity of the function $Q(t)$. In Section 4, the technique of Section 3 is further developed to finish the proof of Theorem 1.

We conclude this section by mentioning the illuminating family of GWO processes called the Sevastyanov processes [9]. The Sevastyanov process is a generalised version of the Bellman–Harris process, with possibly dependent $L$ and $N$. In the critical case, the mean generation length of the Sevastyanov process, $a = E(LN)$, can be represented as

$$a = \text{cov}(L, N) + E(L).$$

Thus, if $L$ and $N$ are positively correlated, the average generation length $a$ exceeds the average life length $E(L)$.

Turning to a specific example of the Sevastyanov process, take

$$P(L = t) = p_1 t^{-3} (\ln t)^{-1}, \quad P(N = 0|L = t) = 1 - p_2, \quad P(N = n_t|L = t) = p_2, \quad t \geq 2,$$

where $n_t := \lfloor t(\ln t)^{-1} \rfloor$ and $(p_1, p_2)$ are such that

$$\sum_{t=2}^{\infty} P(L = t) = p_1 \sum_{t=2}^{\infty} t^{-3} (\ln t)^{-1} = 1, \quad E(N) = p_1 p_2 \sum_{t=2}^{\infty} n_t t^{-3} (\ln t)^{-1} = 1.$$

In this case, for some positive constant $c_1$,

$$E(N^2) = p_1 p_2 \sum_{t=1}^{\infty} n_t^2 t^{-3} (\ln t)^{-1} < c_1 \int_{2}^{\infty} \frac{d(\ln t)}{(\ln t)^2 \ln \ln t} < \infty,$$

implying that the condition (1) is satisfied. Clearly, the condition (3) holds with $d = 0$. At the same time,

$$a = E(NL) = p_1 p_2 \sum_{t=1}^{\infty} n_t t^{-2} (\ln t)^{-1} > c_2 \int_{2}^{\infty} \frac{d(\ln t)}{(\ln t)(\ln \ln t)} = \infty,$$

where $c_2$ is a positive constant. This example demonstrates that for the GWO process, unlike for the Bellman–Harris process, the conditions (1) and (3) do not automatically imply the condition $a < \infty$.

2. The main result

**Theorem 1.** For a GWO process satisfying (1), (3) and $a < \infty$, there holds a weak convergence of the finite-dimensional distributions

$$(Z(ty), 0 < y < \infty|Z(t) > 0) \overset{\text{fdd}}{\rightarrow} (\eta(y), 0 < y < \infty), \quad t \rightarrow \infty.$$

The limiting process is a continuous-time pure death process $(\eta(y), 0 \leq y < \infty)$, whose evolution law is determined by a single compound parameter $c = 4bda^{-2}$, as specified next.
The finite-dimensional distributions of the limiting process $\eta(\cdot)$ are given below in terms of the $k$-dimensional probability generating functions $E(z_1^{\eta(y_1)} \cdots z_k^{\eta(y_k)})$, $k \geq 1$, assuming

\[
0 = y_0 < y_1 < \ldots < y_j < 1 \leq y_{j+1} < \ldots < y_k < y_{k+1} = \infty, \quad 0 \leq j \leq k, \quad 0 \leq z_1, \ldots, z_k < 1.
\]

Here the index $j$ highlights the pivotal value 1 corresponding to the time of observation $t$ of the underlying GWO process.

As will be shown in Section 4.2, if $j = 0$, then

\[
E(z_1^{\eta(y_1)} \cdots z_k^{\eta(y_k)}) = 1 - \frac{1 + \sqrt{1 + \sum_{i=1}^{k} z_1 \cdots z_{i-1}(1 - z_i)\Gamma_i}}{(1 + \sqrt{1 + c})y_1}, \quad \Gamma_i := c(y_1/y_i)^2,
\]

and if $j \geq 1$,

\[
E(z_1^{\eta(y_1)} \cdots z_k^{\eta(y_k)}) = \frac{\sqrt{1 + \sum_{i=1}^{j} z_1 \cdots z_{i-1}(1 - z_i)\Gamma_i} + cz_1 \cdots z_j}{(1 + \sqrt{1 + c})y_1} - \frac{1 + \sum_{i=1}^{k} z_1 \cdots z_{i-1}(1 - z_i)\Gamma_i}{(1 + \sqrt{1 + c})y_1}.
\]

In particular, for $k = 1$, we have

\[
E(z^{\eta(y)}) = \frac{\sqrt{1 + c(1 - z) + cz^2} - \sqrt{1 + c(1 - z)}}{(1 + \sqrt{1 + c})y}, \quad 0 < y < 1,
\]

\[
E(z^{\eta(y)}) = 1 - \frac{1 + \sqrt{1 + c(1 - z)}}{(1 + \sqrt{1 + c})y}, \quad y \geq 1.
\]

It follows that $P(\eta(y) \geq 0) = 1$ for $y > 0$, and moreover, putting here first $z = 1$ and then $z = 0$ yields

\[
P(\eta(y) < \infty) = \frac{\sqrt{1 + cy^2} - 1}{(1 + \sqrt{1 + c})y} \cdot 1_{0 < y < 1} + \left(1 - \frac{2}{(1 + \sqrt{1 + c})y}\right) \cdot 1_{(y \geq 1)},
\]

\[
P(\eta(y) = 0) = \frac{y - 1}{y} \cdot 1_{(y \geq 1)},
\]

implying that $P(\eta(y) = \infty) > 0$ for all $y > 0$. In fact, letting $y \to 0$, we may set $P(\eta(0) = \infty) = 1$.

To demonstrate that the process $\eta(\cdot)$ is indeed a pure death process, consider the function

\[
E\left(z_1^{\eta(y_1) - \eta(y_2)} \cdots z_k^{\eta(y_{k-1}) - \eta(y_k)} z_k^{\eta(y_k)}\right)
\]

determined by

\[
E\left(z_1^{\eta(y_1) - \eta(y_2)} \cdots z_k^{\eta(y_{k-1}) - \eta(y_k)} z_k^{\eta(y_k)}\right) = E\left(z_1^{\eta(y_1)}(z_2/z_1)^{\eta(y_2)} \cdots (z_k/z_{k-1})^{\eta(y_k)}\right).
\]
This function is given by two expressions:

\[
\frac{(1 + \sqrt{1 + c})y_1 - 1 - \sqrt{1 + \sum_{j=1}^{k} (1 - z_j)\gamma_i}}{(1 + \sqrt{1 + c})y_1}, \quad \text{for } j = 0,
\]

\[
\frac{\sqrt{1 + \sum_{j=1}^{k} (1 - z_j)\gamma_i + (1 - z_j)\Gamma_j + cz_j^2} - \sqrt{1 + \sum_{j=1}^{k} (1 - z_j)\gamma_i}}{(1 + \sqrt{1 + c})y_1}, \quad \text{for } j \geq 1,
\]

where \(\gamma_i := \Gamma_i - \Gamma_{i+1}\) and \(\Gamma_{k+1} = 0\). Setting \(k = 2\), \(z_1 = z\), and \(z_2 = 1\), we deduce that the function

\[
E(z^{\eta(y_1) - \eta(y_2)}; \eta(y_1) < \infty), \quad 0 < y_1 < y_2, \quad 0 \leq z \leq 1,
\]

is given by one of the following three expressions, depending on whether \(j = 2\), \(j = 1\), or \(j = 0\):

\[
\frac{\sqrt{1 + cy_1^2 + c(1 - z)(1 - (y_1/y_2)^2) - \sqrt{1 + c(1 - z)(1 - (y_1/y_2)^2)}}}{(1 + \sqrt{1 + c})y_1}, \quad y_2 < 1,
\]

\[
\frac{\sqrt{1 + cy_1^2 + c(1 - z)(1 - y_1^2) - \sqrt{1 + c(1 - z)(1 - y_1^2)}}}{(1 + \sqrt{1 + c})y_1} - 1 - \frac{1 + \sqrt{1 + c(1 - z)(1 - (y_1/y_2)^2)}}{(1 + \sqrt{1 + c})y_1}, \quad 1 \leq y_1.
\]

Since the generating function (6) is finite at \(z = 0\), we conclude that

\[
P(\eta(y_1) < \eta(y_2); \eta(y_1) < \infty) = 0, \quad 0 < y_1 < y_2.
\]

This implies

\[
P(\eta(y_2) \leq \eta(y_1)) = 1, \quad 0 < y_1 < y_2,
\]

meaning that unless the process \(\eta(\cdot)\) is sitting at the infinity state, it evolves by negative integer-valued jumps until it gets absorbed at zero.

Consider now the conditional probability generating function

\[
E(z^{\eta(y_1) - \eta(y_2)}|\eta(y_1) < \infty), \quad 0 < y_1 < y_2, \quad 0 \leq z \leq 1.
\]

In accordance with the three expressions given above for (6), the generating function (7) is specified by the following three expressions:

\[
\frac{\sqrt{1 + cy_1^2 + c(1 - z)(1 - (y_1/y_2)^2) - \sqrt{1 + c(1 - z)(1 - (y_1/y_2)^2)}}}{\sqrt{1 + cy_1^2 - 1}}, \quad y_2 < 1,
\]

\[
\frac{\sqrt{1 + cy_1^2 + c(1 - z)(1 - y_1^2) - \sqrt{1 + c(1 - z)(1 - y_1^2)}}}{\sqrt{1 + cy_1^2 - 1}} - 1 - \frac{1 + \sqrt{1 + c(1 - z)(1 - (y_1/y_2)^2)}}{(1 + \sqrt{1 + c})y_1 - 2}, \quad 1 \leq y_1.
\]
In particular, setting \( z = 0 \) here, we obtain

\[
P(\eta(y_1) - \eta(y_2) = 0|\eta(y_1) < \infty) = \begin{cases} 
\frac{\sqrt{1+c(1-y_1^2-y_2^2)} - \sqrt{1+c(1-y_1/y_2)^2}}{\sqrt{1+c^2}} - 1 & \text{for } 0 < y_1 < y_2 < 1, \\
\sqrt{1+c(1-y_1/y_2)^2} & \text{for } 0 < y_1 < 1 \leq y_2, \\
1 - \frac{\sqrt{1+c(1-y_1/y_2)^2} - 1}{(1+\sqrt{1+c})y_1} & \text{for } 1 \leq y_1 < y_2.
\end{cases}
\]

Notice that given \( 0 < y_1 \leq 1 \),

\[
P(\eta(y_1) - \eta(y_2) = 0|\eta(y_1) < \infty) \to 0, \quad y_2 \to \infty,
\]

which is expected because of \( \eta(y_1) \geq \eta(1) \geq 1 \) and \( \eta(y_2) \to 0 \) as \( y_2 \to \infty \).

The random times

\[ T = \sup\{u : \eta(u) = \infty\}, \quad T_0 = \inf\{u : \eta(u) = 0\} \]

are major characteristics of a trajectory of the limit pure death process. Since

\[
P(T \leq y) = E(z^{\eta(y)}) \bigg|_{z=1}, \quad P(T_0 \leq y) = E(z^{\eta(y)}) \bigg|_{z=0},
\]

in accordance with the above-mentioned formulas for \( E(z^{\eta(y)}) \), we get the following marginal distributions:

\[
P(T \leq y) = \frac{\sqrt{1+cy^2} - 1}{(1+\sqrt{1+c})y} \cdot 1_{\{0 \leq y < 1\}} + \left(1 - \frac{2}{(1+\sqrt{1+c})y}\right) \cdot 1_{\{y \geq 1\}},
\]

\[
P(T_0 \leq y) = \frac{y-1}{y} \cdot 1_{\{y \geq 1\}}.
\]

The distribution of \( T_0 \) is free from the parameter \( c \) and has the Pareto probability density function

\[ f_0(y) = y^{-2}1_{\{y > 1\}}. \]

In the special case (2), that is, when (3) holds with \( d = 0 \), we have \( c = 0 \) and \( P(T = T_0) = 1 \). If \( d > 0 \), then \( T \leq T_0 \), and the distribution of \( T \) has the following probability density function:

\[
f(y) = \begin{cases} 
\frac{1}{(1+\sqrt{1+c})y^2} \left(1 - \frac{1}{\sqrt{1+c^2}}\right) & \text{for } 0 \leq y < 1, \\
\frac{2}{(1+\sqrt{1+c})y^2} & \text{for } y \geq 1,
\end{cases}
\]

which has a positive jump at \( y = 1 \) of size \( f(1) - f(0) = (1+c)^{-1/2} \); see Figure 1. Observe that \( \frac{f(1)}{f(0)} \to \frac{1}{2} \) as \( c \to \infty \).

Intuitively, the limiting pure death process counts the long-living individuals in the GWO process, that is, those individuals whose life length is of order \( t \). These long-living individuals may have descendants, however none of them would live long enough to be detected by the
Finite-dimensional distributions at the relevant time scale, see Lemma 2 below. Theorem 1 suggests a new perspective on Vatutin’s dichotomy (see [12]), claiming that the long-term survival of a critical age-dependent branching process is due to either a large number of short-living individuals or a small number of long-living individuals. In terms of the random times $T \leq T_0$, Vatutin’s dichotomy discriminates between two possibilities: if $T > 1$, then $\eta(1) = \infty$, meaning that the GWO process has survived thanks to a large number of individuals, while if $T \leq 1 < T_0$, then $1 \leq \eta(1) < \infty$, meaning that the GWO process has survived thanks to a small number of individuals.

3. Proof that $tQ(t) \to h$

This section deals with the survival probability of the critical GWO process

$$Q(t) = 1 - P(t), \quad P(t) := P(Z(t) = 0).$$

By its definition, the GWO process can be represented as the sum

$$Z(t) = 1_{\{L > t\}} + \sum_{j=1}^{N} Z_j\left(t - \tau_j\right), \quad t = 0, 1, \ldots,$$

involving $N$ independent daughter processes $Z_j(\cdot)$ generated by the founder individual at the birth times $\tau_j, j = 1, \ldots, N$ (here it is assumed that $Z_j(t) = 0$ for all negative $t$). The branching property (8) implies the relation

$$1\{Z(t) = 0\} = 1_{\{L \leq t\}} \prod_{j=1}^{N} 1\{Z_j(t - \tau_j) = 0\},$$

which says that the GWO process goes extinct by the time $t$ if, on one hand, the founder is dead at time $t$ and, on the other hand, all daughter processes are extinct by the time $t$. After taking expectations of both sides, we can write

$$P(t) = E\left(\prod_{j=1}^{N} P\left(t - \tau_j\right) ; L \leq t\right).$$

As shown next, this nonlinear equation for $P(\cdot)$ implies the asymptotic formula (4) under the conditions (1), (3), and $a < \infty$. 

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**Figure 1.** The dashed line is the probability density function of $T$; the solid line is the probability density function of $T_0$. The left panel illustrates the case $c = 5$, and the right panel illustrates the case $c = 15$. 

3.1. Outline of the proof of (4)

We start by stating four lemmas and two propositions. Let

$$\Phi(z) := E((1 - z)^N - 1 + Nz),$$

(10)

$$W(t) := (1 - ht^{-1})^N + Nht^{-1} - \sum_{j=1}^{N} Q(t - \tau_j) - \prod_{j=1}^{N} P(t - \tau_j),$$

(11)

$$D(u, t) := E\left(1 - \prod_{j=1}^{N} P(t - \tau_j) ; u < L \leq t\right) + E\left((1 - ht^{-1})^N - 1 + Nht^{-1}; L > u\right),$$

(12)

$$E_u(X) := E(X; L \leq u),$$

(13)

where $0 \leq z \leq 1$, $u > 0$, $t \geq h$, and $X$ is an arbitrary random variable.

**Lemma 1.** Given (10), (11), (12), and (13), assume that $0 < u \leq t$ and $t \geq h$. Then

$$\Phi(ht^{-1}) = P(L > t) + E_u\left(\sum_{j=1}^{N} Q(t - \tau_j)\right) - Q(t) + E_u(W(t)) + D(u, t).$$

**Lemma 2.** If (1) and (3) hold, then $E(N; L > ty) = o(t^{-1})$ as $t \to \infty$ for any fixed $y > 0$.

**Lemma 3.** If (1), (3), and $a < \infty$ hold, then for any fixed $0 < y < 1$,

$$E_{ty}\left(\sum_{j=1}^{N} \left(\frac{1}{t - \tau_j} - \frac{1}{t}\right)\right) \sim at^{-2}, \quad t \to \infty.$$

**Lemma 4.** Let $k \geq 1$. If $0 \leq f_j, g_j \leq 1$ for $j = 1, \ldots, k$, then

$$\prod_{j=1}^{k} (1 - g_j) - \prod_{j=1}^{k} (1 - f_j) = \sum_{j=1}^{k} (f_j - g_j)r_j,$$

where $0 \leq r_j \leq 1$ and

$$1 - r_j = \sum_{i=1}^{j-1} g_i + \sum_{i=j+1}^{k} f_i - R_j,$$

for some $R_j \geq 0$. If moreover $f_j \leq q$ and $g_j \leq q$ for some $q > 0$, then

$$1 - r_j \leq (k - 1)q, \quad R_j \leq kq, \quad R_j \leq k^2 q^2.$$

**Proposition 1.** If (1), (3), and $a < \infty$ hold, then $\lim \sup_{t \to \infty} tQ(t) < \infty$.

**Proposition 2.** If (1), (3), and $a < \infty$ hold, then $\lim \inf_{t \to \infty} tQ(t) > 0$.

According to these two propositions, there exists a triplet of positive numbers $(q_1, q_2, t_0)$ such that

$$q_1 \leq tQ(t) \leq q_2, \quad t \geq t_0, \quad 0 < q_1 < h < q_2 < \infty. \quad (14)$$

The claim $tQ(t) \to h$ is derived using (14) by accurately removing asymptotically negligible terms from the relation for $Q(\cdot)$ stated in Lemma 1, after setting $u = ty$ with a fixed $0 < y < 1$, and then choosing a sufficiently small $y$. In particular, as an intermediate step, we will show that

$$Q(t) = E_{ty}\left(\sum_{j=1}^{N} Q(t - \tau_j)\right) + E_{ty}(W(t)) - ah^{-2} + o(t^{-2}), \quad t \to \infty. \quad (15)$$
Then, restating our goal as \( \phi(t) \to 0 \) in terms of the function \( \phi(t) \), defined by

\[
Q(t) = \frac{h + \phi(t)}{t}, \quad t \geq 1,
\]

we rewrite (15) as

\[
\frac{h + \phi(t)}{t} = E_{\gamma}\left( \sum_{j=1}^{N} \frac{h + \phi(t - \tau_j)}{t - \tau_j} \right) + E_{\gamma}(W(t)) - aht^{-2} + o(t^{-2}), \quad t \to \infty.
\] (17)

It turns out that the three terms involving \( h \), outside \( W(t) \), effectively cancel each other, yielding

\[
\frac{\phi(t)}{t} = E_{\gamma}\left( \sum_{j=1}^{N} \frac{\phi(t - \tau_j)}{t - \tau_j} + W(t) \right) + o(t^{-2}), \quad t \to \infty.
\] (18)

Treating \( W(t) \) in terms of Lemma 4 yields

\[
\frac{\phi(t)}{t} = E_{\gamma}\left( \sum_{j=1}^{N} \frac{\phi(t - \tau_j)}{t - \tau_j} r_j(t) \frac{t}{t - \tau_j} \right) + o(t^{-1}),
\] (19)

where \( r_j(t) \) is a counterpart of \( r_j \) in Lemma 4. To derive from here the desired convergence \( \phi(t) \to 0 \), we will adapt a clever trick from Chapter 9.1 of [10], which was further developed in [12] for the Bellman–Harris process, with possibly infinite var(\( N \)). Define a non-negative function \( m(t) \) by

\[
m(t) := |\phi(t)| \ln t, \quad t \geq 2.
\] (20)

Multiplying (19) by \( \ln t \) and using the triangle inequality, we obtain

\[
m(t) \leq E_{\gamma}\left( \sum_{j=1}^{N} m(t - \tau_j) r_j(t) \frac{t \ln t}{(t - \tau_j) \ln (t - \tau_j)} \right) + v(t),
\] (21)

where \( v(t) \geq 0 \) and \( v(t) = o(t^{-1} \ln t) \) as \( t \to \infty \). It will be shown that this leads to \( m(t) = o(\ln t) \), thereby concluding the proof of (4).

### 3.2. Proof of lemmas and propositions

**Proof of Lemma 1.** For \( 0 < u \leq t \), the relations (9) and (13) give

\[
P(t) = E_u\left( \prod_{j=1}^{N} P(t - \tau_j) \right) + E\left( \prod_{j=1}^{N} P(t - \tau_j); u < L \leq t \right).
\] (22)

On the other hand, for \( t \geq h \),

\[
\Phi(h^{-1}) = E_u\left( (1 - h t^{-1})^N - 1 + Nh t^{-1} \right) + E\left( (1 - h t^{-1})^N - 1 + Nh t^{-1}; L > u \right).
\]

Adding the latter relation to

\[
1 = P(L \leq u) + P(L > t) + P(u < L \leq t)
\]
and subtracting (22) from the sum, we get
\[
\Phi(h^{-1}) + Q(t) = E_u\left((1 - ht^{-1})^N + Nht^{-1} - \prod_{j=1}^N P(t - \tau_j)\right) + P(L > t) + D(u, t),
\]
with \(D(u, t)\) defined by (12). After a rearrangement, we obtain the statement of the lemma. \(\Box\)

**Proof of Lemma 2.** For any fixed \(\epsilon > 0\),
\[
E(N; L > t) = E(N; N \leq \epsilon, L > t) + E(N; 1 < N(\epsilon)^{-1}, L > t)
\]
\[
\leq \epsilon P(L > t) + (\epsilon)^{-1}E(N^2; L > t).
\]
Thus, by (1) and (3),
\[
\limsup_{t \to \infty} \epsilon E(N; L > t) \leq d\epsilon,
\]
and the assertion follows as \(\epsilon \to 0\). \(\Box\)

**Proof of Lemma 3.** For \(t = 1, 2, \ldots\) and \(y > 0\), put
\[
B_t(y) := t^2 E_y\left(\sum_{j=1}^N \left(\frac{1}{t - \tau_j} - \frac{1}{t}\right)\right) - a.
\]

For any \(0 < u < ty\), using
\[
a = E_u(\tau_1 + \ldots + \tau_N) + A_u, \quad A_u := E(\tau_1 + \ldots + \tau_N; L > u),
\]
we get
\[
B_t(y) = E_u\left(\sum_{j=1}^N \frac{t}{t - \tau_j}\right) + E\left(\sum_{j=1}^N \frac{t}{t - \tau_j}; u < L \leq ty\right)
\]
\[
- E_u(\tau_1 + \ldots + \tau_N - A_u)
\]
\[
= E\left(\sum_{j=1}^N \frac{\tau_j}{1 - \tau_j/t}; u < L \leq ty\right) + E_u\left(\sum_{j=1}^N \frac{\tau_j^2}{t - \tau_j}\right) - A_u.
\]

For the first term on the right-hand side, we have \(\tau_j \leq L \leq ty\), so that
\[
E\left(\sum_{j=1}^N \frac{\tau_j}{1 - \tau_j/t}; u < L \leq ty\right) \leq (1 - y)^{-1}A_u.
\]

For the second term, \(\tau_j \leq L \leq u\) and therefore
\[
E_u\left(\sum_{j=1}^N \frac{\tau_j^2}{t - \tau_j}\right) \leq \frac{u^2}{t - u}E_u(N) \leq \frac{u^2}{t - u}.
\]

This yields
\[
-A_u \leq B_t(y) \leq (1 - y)^{-1}A_u + \frac{u^2}{t - u}, \quad 0 < u < ty < t,
\]
implying
\[-A_u \leq \liminf_{i \to \infty} B_i(y) \leq \limsup_{i \to \infty} B_i(y) \leq (1 - y)^{-1} A_u.\]

Since \(A_u \to 0\) as \(u \to \infty\), we conclude that \(B_i(y) \to 0\) as \(t \to \infty\).

\[\square\]

**Proof of Lemma 4.** Let
\[r_j := (1 - g_1) \ldots (1 - g_{j-1}) (1 - f_{j+1}) \ldots (1 - f_k), \quad 1 \leq j \leq k.\]

Then \(0 \leq r_j \leq 1\), and the first stated equality is obtained by telescopic summation of
\[
(1 - g_1) \prod_{j=2}^{k} (1 - f_j) - \prod_{j=1}^{k} (1 - f_j) = (f_1 - g_1)r_1,
\]
\[
(1 - g_1)(1 - g_2) \prod_{j=3}^{k} (1 - f_j) - (1 - g_1) \prod_{j=2}^{k} (1 - f_j) = (f_2 - g_2)r_2, \ldots.\]
\[
\prod_{j=1}^{k} (1 - g_j) - \prod_{j=1}^{k} (1 - g_j) = (f_k - g_k)r_k.
\]

The second stated equality is obtained with
\[R_j := \sum_{i=j+1}^{k} f_i(1 - (1 - f_{i+1}) \ldots (1 - f_{i-1})) \]
\[+ \sum_{i=1}^{j-1} g_i(1 - (1 - g_1) \ldots (1 - g_{i-1}) (1 - f_{i+1}) \ldots (1 - f_k)),\]

by performing telescopic summation of
\[
1 - (1 - f_{j+1}) = f_{j+1},
\]
\[
(1 - f_{j+1}) - (1 - f_{j+1}) (1 - f_{j+2}) = f_{j+2} (1 - f_{j+1}) \ldots.\]
\[
\prod_{i=j+1}^{k} (1 - f_i) - \prod_{i=j+1}^{k} (1 - f_i) = f_k \prod_{i=j+1}^{k} (1 - f_i),
\]
\[
\prod_{i=j+1}^{k} (1 - f_i) - (1 - g_1) \prod_{i=j+1}^{k} (1 - f_i) = g_1 \prod_{i=j+1}^{k} (1 - f_i), \ldots.\]
\[
\prod_{i=1}^{j-2} (1 - g_i) \prod_{i=j+1}^{k} (1 - f_i) - \prod_{i=1}^{j-1} (1 - g_i) \prod_{i=j+1}^{k} (1 - f_i) = g_{j-1} \prod_{i=1}^{j-2} (1 - g_i) \prod_{i=j+1}^{k} (1 - f_i).
\]

By the above definition of \(R_j\), we have \(R_j \geq 0\). Furthermore, given \(f_j \leq q\) and \(g_j \leq q\), we get
\[R_j \leq \sum_{i=1}^{j-1} g_i + \sum_{i=j+1}^{k} f_i \leq (k - 1)q.
\]

It remains to observe that
\[1 - r_j \leq 1 - (1 - q)^{k-1} \leq (k - 1)q,
\]
and from the definition of \(R_j\),
\[R_j \leq q \sum_{i=1}^{k-j-1} (1 - (1 - q)^i) + q \sum_{i=1}^{j-1} (1 - (1 - q)^{k-j+i-1}) \leq q^2 \sum_{i=1}^{k-2} i \leq k^2 q^2.
\]

\[\square\]
Proof of Proposition 1. By the definition of $\Phi(\cdot)$, we have
\[
\Phi(Q(t)) = E_u \left( P(t)^N - \prod_{j=1}^{N} P(t - \tau_j) \right) + P(L > u)
- E(1 - P(t)^N; L > u)
- \left( \prod_{j=1}^{N} P(t - \tau_j; u < L \leq t) \right).
\] (23)
We therefore obtain the upper bound
\[
\Phi(Q(t)) \leq E_u \left( P(t)^N - \prod_{j=1}^{N} P(t - \tau_j) \right) + P(L > u),
\] which together with Lemma 4 and the monotonicity of $Q(\cdot)$ implies
\[
\Phi(Q(t)) \leq E_u \left( \sum_{j=1}^{N} Q(t - \tau_j) - Q(t(t)) \right) + P(L > u). \tag{24}
\]
Borrowing an idea from [11], suppose to the contrary that $t_n := \min\{t: Q(t) \geq n\}$ is finite for any natural $n$. It follows that
\[
Q(t_n) \geq \frac{n}{t_n}, \quad Q(t_n - u) < \frac{n}{t_n - u}, \quad 1 \leq u \leq t_n - 1.
\]
Putting $t = t_n$ into (24) and using the monotonicity of $\Phi(\cdot)$, we find
\[
\Phi(nt_n^{-1}) \leq \Phi(Q(t_n)) \leq E_u \left( \sum_{j=1}^{N} \left( \frac{n}{t_n - \tau_j} - \frac{n}{t_n} \right) \right) + P(L > u).
\]
Setting $u = t_n/2$ here and applying Lemma 3 together with (3), we arrive at the relation
\[
\Phi(nt_n^{-1}) = O(nt_n^{-2}), \quad n \to \infty.
\]
Observe that under the condition (1), the L’Hospital rule gives
\[
\Phi(z) \sim b z^2, \quad z \to 0. \tag{25}
\]
The resulting contradiction, $n^2 t_n^{-2} = O(nt_n^{-2})$ as $n \to \infty$, finishes the proof of the proposition.

Proof of Proposition 2. The relation (23) implies
\[
\Phi(Q(t)) \geq E_u \left( P(t)^N - \prod_{j=1}^{N} P(t - \tau_j) \right) - E(1 - P(t)^N; L > u).
\]
By Lemma 4,
\[
P(t)^N - \prod_{j=1}^{N} P(t - \tau_j) = \sum_{j=1}^{N} (Q(t - \tau_j) - Q(t)) r^*_j(t),
\]
where $0 \leq r_j^*(t) \leq 1$ is a counterpart of the term $r_j$ in Lemma 4. By the monotonicity of $P(\cdot)$, we have, again referring to Lemma 4,

$$1 - r_j^*(t) \leq (N - 1)Q(t - L).$$

Thus, for $0 < y < 1$,

$$\Phi(Q(t)) \geq E_{ty} \left( \sum_{j=1}^{N} \left( Q(t - \tau_j) - Q(t) r_j^*(t) \right) \right) - E(1 - P(N; L > ty)). \quad (26)$$

The assertion $\liminf_{t \to \infty} tQ(t) > 0$ is proven by contradiction. Assume that $\liminf_{t \to \infty} tQ(t) = 0$, so that $t_n := \min \left\{ t : tQ(t) \leq n^{-1} \right\}$ is finite for any natural $n$. Plugging $t = t_n$ into (26) and using

$$Q(t_n) \leq \frac{1}{nt_n}, \quad Q(t_n - u) - Q(t_n) \geq \frac{1}{n(t_n - u)} - \frac{1}{nt_n}, \quad 1 \leq u \leq t_n - 1,$$

we get

$$\Phi\left( \frac{1}{nt_n} \right) \geq n^{-1} E_{ty} \left( \sum_{j=1}^{N} \left( \frac{1}{t_n - \tau_j} - \frac{1}{t_n} \right) r_j^*(t_n) \right) - \frac{1}{nt_n} E(N; L > t_ny).$$

Given $L \leq ty$, we have

$$1 - r_j^*(t) \leq NQ(t(1 - y)) \leq N \frac{q_2}{t(1 - y)},$$

where the second inequality is based on the already proven part of (14). Therefore,

$$E_{ty} \left( \sum_{j=1}^{N} \left( \frac{1}{t_n - \tau_j} - \frac{1}{t_n} \right) \left( 1 - r_j^*(t_n) \right) \right) \leq \frac{q_2 y}{t_n^2 (1 - y)^2} E(N^2),$$

and we derive

$$nt_n^2 \Phi\left( \frac{1}{nt_n} \right) \geq t_n^2 E_{ty} \left( \sum_{j=1}^{N} \left( \frac{1}{t_n - \tau_j} - \frac{1}{t_n} \right) \left( 1 - r_j^*(t_n) \right) \right) - \frac{E(N^2) q_2 y}{(1 - y)^2} - t_n E(N; L > t_ny).$$

Sending $n \to \infty$ and applying (25), Lemma 2, and Lemma 3, we arrive at the inequality

$$0 \geq a - y q_2 E(N^2) (1 - y)^{-2}, \quad 0 < y < 1,$$

which is false for sufficiently small $y$. \hfill \Box

3.3. Proof of (18) and (19)

Fix an arbitrary $0 < y < 1$. Lemma 1 with $u = ty$ gives

$$\Phi(ht^{-1}) = P(L > t) + E_{ty} \left( \sum_{j=1}^{N} Q(t - \tau_j) \right) - Q(t) + E_{ty}(W(t)) + D(ty, t). \quad (27)$$
Let us show that
\[ D(ty, t) = o(t^{-2}), \quad t \to \infty. \] (28)

Using Lemma 2 and (14), we find that for an arbitrarily small \( \epsilon > 0 \),
\[ E(1 - \prod_{j=1}^{N} P(t - \tau_j) : ty < L \leq t(1 - \epsilon)) = o(t^{-2}), \quad t \to \infty. \]

On the other hand,
\[ E(1 - \prod_{j=1}^{N} P(t - \tau_j) : t(1 - \epsilon) < L \leq t) \leq P(t(1 - \epsilon) < L \leq t), \]
so that in view of (3),
\[ E(1 - \prod_{j=1}^{N} P(t - \tau_j) : ty < L \leq t) = o(t^{-2}), \quad t \to \infty. \]

This, (12), and Lemma 2 imply (28).

Observe that
\[ bh^2 = ah + d. \] (29)

Combining (27), (28), and
\[ P(L > t) = \Phi(ht^{-1}) = dt^{-2} - bh^2t^{-2} + o(t^{-2}) = -ah^{-2} + o(t^{-2}), \quad t \to \infty, \]
we derive (15), which in turn gives (17). The latter implies (18) since by Lemmas 2 and 4,
\[ E_t \left( \sum_{j=1}^{N} \frac{h}{t - \tau_j} \right) - \frac{h}{t} = E_t \left( \sum_{j=1}^{N} \left( \frac{h}{t - \tau_j} - \frac{h}{t} \right) \right) - ht^{-1} E(N; L > ty) = ah^{-2} + o(t^{-2}). \]

Turning to the proof of (19), observe that the random variable
\[ W(t) = (1 - ht^{-1})^N - \prod_{j=1}^{N} \left( 1 - \frac{h + \phi(t - \tau_j)}{t - \tau_j} \right) + \sum_{j=1}^{N} \left( \frac{h}{t} - \frac{h + \phi(t - \tau_j)}{t - \tau_j} \right) \]
can be represented in terms of Lemma 4 as
\[ W(t) = \prod_{j=1}^{N} (1 - f_j(t)) - \prod_{j=1}^{N} (1 - g_j(t)) + \sum_{j=1}^{N} (f_j(t) - g_j(t)) \]
by assigning
\[ f_j(t) := ht^{-1}, \quad g_j(t) := \frac{h + \phi(t - \tau_j)}{t - \tau_j}. \] (30)

Here \( 0 \leq r_j(t) \leq 1 \), and for sufficiently large \( t \),
\[ 1 - r_j(t) \leq Nq^2 t^{-1}. \] (31)
After plugging into (18) the expression

\[ W(t) = \sum_{j=1}^{N} \left( \frac{h}{t} - \frac{h}{t - \tau_j} \right) (1 - r_j(t)) - \sum_{j=1}^{N} \frac{\phi(t - \tau_j)}{t - \tau_j} (1 - r_j(t)), \]

we get

\[ \frac{\phi(t)}{t} = E_y \left( \sum_{j=1}^{N} \frac{\phi(t - \tau_j)}{t - \tau_j} r_j(t) \right) + E_y \left( \sum_{j=1}^{N} \left( \frac{h}{t - \tau_j} - \frac{h}{t} \right) (1 - r_j(t)) \right) + o(t^{-2}), \quad t \to \infty. \]

The latter expectation is non-negative, and for an arbitrary \( \epsilon > 0 \), it has the following upper bound:

\[ E_y \left( \sum_{j=1}^{N} \left( \frac{h}{t - \tau_j} - \frac{h}{t} \right) (1 - r_j(t)) \right) \leq q_2 \epsilon E_y \left( \sum_{j=1}^{N} \left( \frac{h}{t - \tau_j} - \frac{h}{t} \right) \right) + \frac{q_2 h}{(1 - y)^2} E(N^2; N > t\epsilon). \]

Thus, in view of Lemma 3,

\[ \frac{\phi(t)}{t} = E_y \left( \sum_{j=1}^{N} \frac{\phi(t - \tau_j)}{t - \tau_j} r_j(t) \right) + o(t^{-2}), \quad t \to \infty. \]

Multiplying this relation by \( t \), we arrive at (19).

### 3.4. Proof of \( \phi(t) \to 0 \)

Recall (20). If the non-decreasing function

\[ M(t) := \max_{1 \leq j \leq t} m(j) \]

is bounded from above, then \( \phi(t) = O \left( \frac{1}{\ln t} \right) \), proving that \( \phi(t) \to 0 \) as \( t \to \infty \). If \( M(t) \to \infty \) as \( t \to \infty \), then there is an integer-valued sequence \( 0 < t_1 < t_2 < \ldots \), such that the sequence \( M_n := M(t_n) \) is strictly increasing and converges to infinity. In this case,

\[ m(t) \leq M_{n-1} < M_n, \quad 1 \leq t < t_n, \quad m(t_n) = M_n, \quad n \geq 1. \tag{32} \]

Since \( |\phi(t)| \leq \frac{M_n}{\ln t_n} \) for \( t_n \leq t < t_{n+1} \), to finish the proof of \( \phi(t) \to 0 \), it remains to verify that

\[ M_n = o(\ln t_n), \quad n \to \infty. \tag{33} \]

Fix an arbitrary \( y \in (0, 1) \). Putting \( t = t_n \) in (21) and using (32), we find

\[ M_n \leq M_n E_{t_n^y} \left( \sum_{j=1}^{N} r_j(t_n) \frac{t_n \ln t_n}{(t_n - \tau_j) \ln (t_n - \tau_j)} \right) + (t_n^{-1} \ln t_n) o_n. \]

Here and elsewhere, \( o_n \) stands for a non-negative sequence such that \( o_n \to 0 \) as \( n \to \infty \). In different formulas, the sign \( o_n \) represents different such sequences. Since

\[ 0 \leq \frac{t \ln t}{(t-u) \ln (t-u)} - 1 \leq \frac{u(1 + \ln t)}{(t-u) \ln (t-u)}, \quad 0 \leq u < t - 1, \]

and \( r_j(t_n) \in [0, 1] \), it follows that

\[ M_n - M_n E_{t_n^y} \left( \sum_{j=1}^{N} r_j(t_n) \right) \leq M_n E_{t_n^y} \left( \sum_{j=1}^{N} \tau_j \frac{(1 + \ln t_n)}{t_n(1 - y) \ln (t_n(1 - y))} \right) + (t_n^{-1} \ln t_n) o_n. \]
Recalling that $a = E(\sum_{j=1}^{N} \tau_{j})$, observe that

$$E_{t_{n}y}\left(\sum_{j=1}^{N} \frac{\tau_{j}(1 + \ln t_{n})}{t_{n}(1 - y) \ln (t_{n}(1 - y))}\right) \leq \frac{a(1 + \ln t_{n})}{t_{n}(1 - y) \ln (t_{n}(1 - y))} = (a(1 - y)^{-1} + o_{n})t_{n}^{-1}.$$  

Combining the last two relations, we conclude

$$M_{n}E_{t_{n}y}\left(\sum_{j=1}^{N} (1 - r_{j}(t_{n}))\right) \leq a(1 - y)^{-1}t_{n}^{-1}M_{n} + t_{n}^{-1}(M_{n} + \ln t_{n})o_{n}. \quad (34)$$

Now it is time to unpack the term $r_{j}(t)$. By Lemma 4 with (30),

$$1 - r_{j}(t) = \sum_{i=1}^{j-1} \frac{h + \phi(t - \tau_{i})}{t - \tau_{i}} + (N - j) \frac{h}{t} - R_{j}(t),$$

where, provided $\tau_{j} \leq ty$,

$$0 \leq R_{j}(t) \leq Nq_{2}t^{-1}(1 - y)^{-1}, \quad R_{j}(t) \leq N^{2}q_{2}^{2}t^{-2}(1 - y)^{-2}, \quad t > t^{*},$$

for a sufficiently large $t^{*}$. This allows us to rewrite (34) in the form

$$M_{n}E_{t_{n}y}\left(\sum_{j=1}^{N} \left(\sum_{i=1}^{j-1} \frac{h + \phi(t_{n} - \tau_{i})}{t_{n} - \tau_{i}} + (N - j) \frac{h}{t_{n}}\right)\right)$$

$$\leq M_{n}E_{t_{n}y}\left(\sum_{j=1}^{N} R_{j}(t_{n})\right) + a(1 - y)^{-1}t_{n}^{-1}M_{n} + t_{n}^{-1}(M_{n} + \ln t_{n})o_{n}.$$ 

To estimate the last expectation, observe that if $\tau_{j} \leq ty$, then for any $\epsilon > 0$,

$$R_{j}(t) \leq Nq_{2}t^{-1}(1 - y)^{-1}1_{\{N > t_{n}\}} + N^{2}q_{2}^{2}t^{-2}(1 - y)^{-2}1_{\{N \leq t_{n}\}}, \quad t > t^{*},$$

implying that for sufficiently large $n$,

$$E_{t_{n}y}\left(\sum_{j=1}^{N} R_{j}(t_{n})\right) \leq q_{2}t_{n}^{-1}(1 - y)^{-1}E(N^{2}; N > t_{n}\epsilon) + q_{2}^{2}\epsilon t_{n}^{-1}(1 - y)^{-2}E(N^{2}),$$

so that

$$M_{n}E_{t_{n}y}\left(\sum_{j=1}^{N} \left(\sum_{i=1}^{j-1} \frac{h + \phi(t_{n} - \tau_{i})}{t_{n} - \tau_{i}} + (N - j) \frac{h}{t_{n}}\right)\right)$$

$$\leq a(1 - y)^{-1}t_{n}^{-1}M_{n} + t_{n}^{-1}(M_{n} + \ln t_{n})o_{n}.$$ 

Since

$$\sum_{j=1}^{N} \sum_{i=1}^{j-1} \left(\frac{h}{t_{n} - \tau_{i}} - \frac{h}{t_{n}}\right) \geq 0,$$

we obtain

$$M_{n}E_{t_{n}y}\left(\sum_{j=1}^{N} \left(\sum_{i=1}^{j-1} \frac{\phi(t_{n} - \tau_{i})}{t_{n} - \tau_{i}} + (N - 1) \frac{h}{t_{n}}\right)\right)$$

$$\leq a(1 - y)^{-1}t_{n}^{-1}M_{n} + t_{n}^{-1}(M_{n} + \ln t_{n})o_{n}.$$
By (16) and (14), we have \( \phi(t) \geq q_1 - h \) for \( t \geq t_0 \). Thus, for \( \tau_j \leq L \leq t_n y \) and sufficiently large \( n \),

\[
\frac{\phi(t_n - \tau_i)}{t_n - \tau_i} \geq \frac{q_1 - h}{t_n(1 - y)}.
\]

This gives

\[
\sum_{j=1}^{N} \left( \sum_{i=1}^{j-1} \frac{\phi(t_n - \tau_i)}{t_n - \tau_i} + (N - 1) \frac{h}{t_n} \right) \geq \left( h + \frac{q_1 - h}{2(1 - y)} \right) t_n^{-1} N(N - 1),
\]

which, after multiplying by \( t_n M_n \) and taking expectations, yields

\[
\left( h + \frac{q_1 - h}{2(1 - y)} \right) M_n E_{t_n y}(N(N - 1)) \leq a(1 - y)^{-1} M_n + (M_n + \ln t_n) o_n.
\]

Finally, since

\[
E_{t_n y}(N(N - 1)) \to 2b, \quad n \to \infty,
\]

we derive that for any \( 0 < \epsilon < y < 1 \), there is a finite \( n_\epsilon \) such that for all \( n > n_\epsilon \),

\[
M_n(2bh(1 - y) + bq_1 - bh - a - \epsilon) \leq \epsilon \ln t_n.
\]

By (29), we have \( bh \geq a \), and therefore

\[
2bh(1 - y) + bq_1 - bh - a - \epsilon \geq bq_1 - 2bh y - y.
\]

Thus, choosing \( y = y_0 \) such that \( bq_1 - 2bh y_0 - y_0 = \frac{bq_1}{2} \), we see that

\[
\limsup_{n \to \infty} \frac{M_n}{\ln t_n} \leq \frac{2\epsilon}{bq_1},
\]

which implies (33) as \( \epsilon \to 0 \), concluding the proof of \( \phi(t) \to 0 \).

### 4. Proof of Theorem 1

We will use the following notational conventions for the \( k \)-dimensional probability generating function

\[
E\left( Z(t_1) \ldots Z(t_k) \right) = \sum_{i_1=0}^{\infty} \ldots \sum_{i_k=0}^{\infty} P(Z(t_1) = i_1, \ldots, Z(t_k) = i_k) z_1^{i_1} \ldots z_k^{i_k},
\]

with \( 0 < t_1 \leq \ldots \leq t_k \) and \( z_1, \ldots, z_k \in [0, 1] \). We define

\[
P_k(\tilde{t}, \tilde{z}) := P_k(t_1, \ldots, t_n; \tilde{z}_1, \ldots, \tilde{z}_k) := E\left( z_1^{Z(t_1)} \ldots z_k^{Z(t_k)} \right)
\]

and write, for \( t \geq 0 \),

\[
P_k(t + \tilde{t}, \tilde{z}) := P_k(t + t_1, \ldots, t + t_k; \tilde{z}_1, \ldots, \tilde{z}_k).
\]

Moreover, for \( 0 < y_1 < \ldots < y_k \), we write

\[
P_k(t\tilde{y}, \tilde{z}) := P_k(t y_1, \ldots, t y_k; \tilde{z}_1, \ldots, \tilde{z}_k),
\]
and assuming $0 < y_1 < \ldots < y_k < 1$,
\[
P^*_k(t, \bar{y}, \bar{z}) := E\left(\frac{t}{1}^{Z(y_1)} \cdots \frac{t}{k}^{Z(y_k)} ; Z(t) = 0\right) = P_{k+1}(ty_1, \ldots, ty_k, t; z_1, \ldots, z_k, 0).
\]

These conventions will be similarly applied to the functions
\[
Q_k(\bar{t}, \bar{z}) := 1 - P_k(\bar{t}, \bar{z}), \quad Q^*_k(t, \bar{y}, \bar{z}) := 1 - P^*_k(t, \bar{y}, \bar{z}). \tag{35}
\]

Our special interest is in the function
\[
Q_k(t) := Q_k(t + \bar{t}, \bar{z}), \quad 0 = t_1 < \ldots < t_k, \quad z_1, \ldots, z_k \in [0, 1), \tag{36}
\]

to be viewed as a counterpart of the function $Q(t)$ treated by Theorem 2. Recalling the compound parameters
\[
h = \frac{a + \sqrt{a^2 + 4bd}}{2b}
\]
and $c = 4bd - 2$, put
\[
h_k := h \frac{1 + \sqrt{1 + cg_k}}{1 + \sqrt{1 + c}}, \quad g_k := g_k(\bar{y}, \bar{z}) := \sum_{i=1}^{k} z_1 \cdots z_{i-1} (1 - z_i)^{y_i - 2}. \tag{37}
\]

The key step of the proof of Theorem 1 is to show that for any given $1 = y_1 < y_2 < \cdots < y_k$,
\[
tQ_k(t) \rightarrow h_k, \quad t_i := t(y_i - 1), \quad i = 1, \ldots, k, \quad t \rightarrow \infty. \tag{38}
\]

This is done following the steps of our proof of $tQ(t) \rightarrow h$ given in Section 3.

Unlike $Q(t)$, the function $Q_k(t)$ is not monotone over $t$. However, monotonicity of $Q(t)$ was used in the proof of Theorem 2 only for the proof of (14). The corresponding statement
\[
0 < q_1 \leq tQ_k(t) \leq q_2 < \infty, \quad t \geq t_0,
\]
follows from the bounds $(1 - z_1)Q(t) \leq Q_k(t) \leq Q(t)$, which hold by the monotonicity of the underlying generating functions over $z_1, \ldots, z_n$. Indeed,
\[
Q_k(t) \leq Q_k(t, t + t_2, \ldots, t + t_k; 0, \ldots, 0) = Q(t),
\]
amd on the other hand,
\[
Q_k(t) = Q_k(t, t + t_2, \ldots, t + t_k; z_1, \ldots, z_k) = E\left(1 - \frac{Z(t)}{1}Z(t+t_2) \cdots \frac{Z(t+t_k)}{k} \right) \geq E\left(1 - \frac{Z(t)}{1} \right),
\]
where
\[
E\left(1 - \frac{Z(t)}{1} \right) \geq E\left(1 - \frac{Z(t)}{1}; Z(t) \geq 1 \right) \geq (1 - z_1)Q(t).
\]

4.1. Proof of $tQ_k(t) \rightarrow h_k$

The branching property (8) of the GWO process gives
\[
\prod_{i=1}^{k} \frac{Z(t_i)}{z_i} = \prod_{i=1}^{k} \frac{1}{1-z_i} \prod_{j=1}^{N} \frac{Z(t_i-t_j)}{z_i}. \tag{8}
\]
Given $0 < t_1 < \ldots < t_k < t_{k+1} = \infty$, we use
\[
\prod_{i=1}^{k} z_i^{1\{L>t_i\}} = 1_{\{L \leq t_1\}} + \sum_{i=1}^{k} z_1 \cdots z_i 1_{\{t_i < L \leq t_{i+1}\}}
\]
to deduce the following counterpart of (9):
\[
P_k(\bar{\varphi}, \bar{z}) = E_{t_1} \left( \prod_{j=1}^{N} P_k(\bar{\varphi} - \tau_j, \bar{z}) \right) + \sum_{i=1}^{k} z_1 \cdots z_i E \left( \prod_{j=1}^{N} P_k(\bar{\varphi} - \tau_j, \bar{z}); t_i < L \leq t_{i+1} \right).
\]
This implies
\[
P_k(\bar{\varphi}, \bar{z}) = E_{t_1} \left( \prod_{j=1}^{N} P_k(\bar{\varphi} - \tau_j, \bar{z}) \right) + \sum_{i=1}^{k} z_1 \cdots z_i P(t_i < L \leq t_{i+1})
- \sum_{i=1}^{k} z_1 \cdots z_i E \left( 1 - \prod_{j=1}^{N} P_k(\bar{\varphi} - \tau_j, \bar{z}); t_i < L \leq t_{i+1} \right). \tag{39}
\]
Using this relation we establish the following counterpart of Lemma 1.

**Lemma 5.** Consider the function (36) and put $P_k(t) := 1 - Q_k(t) = P_k(t + \bar{\varphi}, \bar{z})$. For $0 < u < t$, the relation
\[
\Phi(h_k t^{-1}) = P(L > t) - \sum_{i=1}^{k} z_1 \cdots z_i P(t + t_i < L \leq t + t_{i+1})
+ E_u \left( \sum_{j=1}^{N} Q_k(t - \tau_j) \right) - Q_k(t) + E_u(W_k(t)) + D_k(u, t) \tag{40}
\]
holds with $t_{k+1} = \infty$,
\[
W_k(t) := (1 - h_k t^{-1})^N + N h_k t^{-1} - \sum_{j=1}^{N} Q_k(t - \tau_j) - \prod_{j=1}^{N} P_k(t - \tau_j), \tag{41}
\]
and
\[
D_k(u, t) := E \left( 1 - \prod_{j=1}^{N} P_k(t - \tau_j); u < L \leq t \right) + E \left( (1 - h_k t^{-1})^N - 1 + N h_k t^{-1}; L > u \right)
+ \sum_{i=1}^{k} z_1 \cdots z_i E \left( 1 - \prod_{j=1}^{N} P_k(t - \tau_j); t + t_i < L \leq t + t_{i+1} \right). \tag{42}
\]

**Proof.** According to (39),
\[
P_k(t) = E_u \left( \prod_{j=1}^{N} P_k(t - \tau_j) \right) + E \left( \prod_{j=1}^{N} P_k(t - \tau_j); u < L \leq t \right)
+ \sum_{i=1}^{k} z_1 \cdots z_i P(t + t_i < L \leq t + t_{i+1})
- \sum_{i=1}^{k} z_1 \cdots z_i E \left( 1 - \prod_{j=1}^{N} P_k(t - \tau_j); t + t_i < L \leq t + t_{i+1} \right).
\]
By the definition of $\Phi(\cdot)$,
\[
\Phi(h_k t^{-1}) + 1 = E_u \left( \left( 1 - h_k t^{-1} \right)^N + N h_k t^{-1} \right) + P(L > t) + \mathbb{E} \left( \left( 1 - h_k t^{-1} \right)^N - 1 + N h_k t^{-1}; L > u \right) + P(u < L \leq t),
\]
and after subtracting the two last equations, we get
\[
\Phi(h_k t^{-1}) + Q_k(t) = E_u \left( \left( 1 - h_k t^{-1} \right)^N + N h_k t^{-1} - \prod_{j=1}^{N} P_k(t - \tau_j) \right) + P(L > t)
- \sum_{i=1}^{k} z_1 \cdots z_i P(t + t_i < L \leq t + t_{i+1}) + D_k(u, t),
\]
with $D_k(u, t)$ satisfying (42). After a rearrangement, the relation (40) follows together with (41).

With Lemma 5 in hand, the convergence (38) is proven by applying almost exactly the same argument as used in the proof of $tQ(t) \to h$. An important new feature emerges because of the additional term in the asymptotic relation defining the limit $h_k$. Let $1 = y_1 < y_2 < \ldots < y_k < y_{k+1} = \infty$. Since
\[
\sum_{i=1}^{k} z_1 \cdots z_i P(t y_i < L \leq t y_{i+1}) \sim dt^{-2} \sum_{i=1}^{k} z_1 \cdots z_i \left( y_i^2 - y_{i+1}^2 \right),
\]
we see that
\[
P(L > t) - \sum_{i=1}^{k} z_1 \cdots z_i P(t y_i < L \leq t y_{i+1}) \sim dg_k t^{-2},
\]
where $g_k$ is defined by (37). Assuming $0 \leq z_1, \ldots, z_k < 1$, we ensure that $g_k > 0$, and as a result, we arrive at a counterpart of the quadratic equation (29),
\[
 bh_k^2 = ah_k + dg_k,
\]
which gives
\[
h_k = \frac{a + \sqrt{a^2 + 4bdg_k}}{2b} = h \frac{1 + \sqrt{1 + cg_k}}{1 + \sqrt{1 + c}},
\]
justifying our definition (37). We conclude that for $k \geq 1$
\[
\frac{Q_k(t \tilde{y}, \tilde{z})}{Q(t)} \to 1 + \sqrt{1 + c} \sum_{i=1}^{k} z_1 \cdots z_{i-1} (1 - z_i) y_i^{-2} \quad \frac{1 + \sqrt{1 + c}}{1 + \sqrt{1 + c}},
\]
where $1 = y_1 < \ldots < y_k, \quad 0 \leq z_1, \ldots, z_k < 1$. (43)

### 4.2. Conditioned generating functions

To finish the proof of Theorem 1, consider the generating functions conditioned on the survival of the GWO process. Given (5) with $j \geq 1$, we have
\[
Q(t) E \left( z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)} | Z(t) > 0 \right) = E \left( z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)} ; Z(t) > 0 \right) = P_k(t \tilde{y}, \tilde{z}) - E \left( z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)} ; Z(t) = 0 \right) \overset{(35)}{=} Q_j^*(t, \tilde{y}, \tilde{z}) - Q_k(t \tilde{y}, \tilde{z}),
\]
and therefore,
\[ E\left( \frac{Z^{(ty_1)} \cdots Z^{(ty_k)}}{Z(t) > 0} \right) = \frac{Q_j(t, \bar{y}, \bar{z})}{Q(t)} - \frac{Q_k(t\bar{y}, \bar{z})}{Q(t)}. \]

Similarly, if (5) holds with \( j = 0 \), then
\[ E\left( \frac{Z^{(ty_1)} \cdots Z^{(ty_k)}}{Z(t) > 0} \right) = 1 - \frac{Q_k(t\bar{y}, \bar{z})}{Q(t)}. \]

Letting \( t' = ty_1 \), we get
\[ \frac{Q_k(t\bar{y}, \bar{z})}{Q(t)} = \frac{Q_k(t', t'y_2/y_1, \ldots, t'y_k/y_1)}{Q(t')} \frac{Q(t_1)}{Q(t)}, \]
and applying the relation (43), we have
\[ \frac{Q_k(t\bar{y}, \bar{z})}{Q(t)} \to 1 + \sqrt{1 + \sum_{i=1}^{k} z_i \cdots z_{i-1}(1 - z_i)\Gamma_i (1 + \sqrt{1 + c})y_1}, \]
where \( \Gamma_i = c(y_1/y_i)^2 \). On the other hand, since
\[ Q_j^\ast(t, \bar{y}, \bar{z}) = Q_{j+1}(ty_1, \ldots, ty_j, t; z_1, \ldots, z_j, 0), \quad j \geq 1, \]
we also get
\[ \frac{Q_j^\ast(t, \bar{y}, \bar{z})}{Q(t)} \to 1 + \sqrt{1 + \sum_{i=1}^{j} z_i \cdots z_{i-1}(1 - z_i)\Gamma_i + cz_1 \cdots z_j y_1^2} (1 + \sqrt{1 + c})y_1. \]

We conclude that as stated in Section 2,
\[ E\left( \frac{Z^{(ty_1)} \cdots Z^{(ty_k)}}{Z(t) > 0} \right) \to E\left( \frac{Z^{(\eta y_1)} \cdots Z^{(\eta y_k)}}{Z(t) > 0} \right). \]

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