A VISCOSITY APPROACH TO STOCHASTIC DIFFERENTIAL GAMES
OF CONTROL AND STOPPING INVOLVING IMPULSIVE CONTROL

DAVID MGUNI∗

Abstract.
This paper analyses a stochastic differential game of control and stopping in which one of the players modifies a diffusion process using impulse controls, an adversary then chooses a stopping time to end the game. The paper firstly establishes the regularity and boundedness of the upper and lower value functions from which an appropriate variant of the dynamic programming principle (DPP) is derived. It is then proven that the upper and lower value functions coincide so that the game admits a value and that the value of the game is a unique viscosity solution to a HJBI equation described by a double obstacle quasi-integro-variational inequality.

Key words. Impulse control, Stochastic Differential Games, Optimal Stopping, Diffusion process, Dynkin Games, Viscosity solution

1. Introduction. This paper considers a stochastic differential game in which a controller modifies a diffusion process and an adversary stops the game. The objective of the controller is to exercise an impulse control that minimises an objective criterion whilst the adversary seeks to stop the process at a time which maximises the same function. A motivation for this investigation is a formal mathematical treatment of the probability of lifetime ruin problem which also includes financial transaction costs a problem in which an investor seeks to maximise their lifetime wealth by modifying their investment position whilst minimising the risk that their wealth process falls below some prefixed value.

Problems that combine discretionary stopping and stochastic optimal control have attracted much attention over recent years; in particular there is a significant amount of literature on models of this kind in which a single controller uses absolutely continuous controls to modify the system dynamics (see for example [4, 2, 5]). Game-theoretic formulations of the problem in which the task of controlling the system dynamics and exit time is divided between two players who act according to separated interests have also been studied [1].

Impulse control problems are stochastic control models in which the cost of control is bounded below by some fixed positive constant which prohibits continuous control, therefore augmenting the problem to one of finding both an optimal sequence of times to apply the control policy, in addition to determining optimal control magnitudes. We refer the reader to [6] as a general reference to impulse control theory and to [26, 21] for articles on applications. Impulse control frameworks therefore underpin the description of financial environments with transaction costs and liquidity risks and more generally, applications of optimal control theory in which the system dynamics are modified by a sequence of discrete actions - see [19] for an extensive survey of applications of impulse control models within finance.

Stochastic differential games with impulse control have also recently appeared in the stochastic impulse control literature. In [28] stochastic differential games in which one player uses impulse control and the other uses continuous controls were studied. Using

∗ (davidmguni@hotmail.com). Quantitative and Applied Spatial Economic Research Laboratory, University College London, Gower Street, London, WC1E 6BT, UK.
Centre for Doctoral Training in Financial Computing & Analytics, University College London, Gower Street, London, WC1E 6BT, UK.
a verification argument, the conditions under which the value of game is a solution to a HJBI equation is also shown in [28]. In [10], Cosso was the first to study a stochastic differential game in which both players use impulse control using viscosity theory. Thus, in [10] it is shown that the game admits a value which is a unique viscosity solution to a double obstacle quasi-variational inequality.

Despite the small but notable literature on impulse controls involving two players, the task of analysing stochastic differential games with a discretionary stopper and a controller who uses impulse controls remains unaddressed.

Stochastic differential games of absolutely continuous control and stopping have a number of applications within theoretical finance. A notable example is the investor lifetime ruin problem in which an investor, who operates in a Black-Scholes market seeks to maximise some utility criterion which is a function of his wealth process whilst seeking some optimal time to exit the market which minimises the risk of ruin (see for example [2] and [5]).

The restriction to absolutely continuous controls prohibits the inclusion of transaction costs for which the investor must take actions in a timed and discretised fashion therefore necessitating that the class of controls used to modify the system dynamics are impulse controls. The current paper therefore establishes the necessary results to construct optimal investment policies for the lifetime ruin problem with minimally bounded or transaction costs.

In control theory and more generally, differential game theory there are two main approaches to obtaining a solution to the problem. The first approach is a verification method which involves characterising the value function in terms of a set of (in general, non-linear, second order) PDEs or HJB equations (in the case of differential games, HJBI equations). The verification approach is a direct method that is initialised at the dynamic programming principle from which a classical limit procedure is used to then derive the HJB equations.

Though the verification approach offers a direct means of establishing a PDE characterisation of the value function, verification theorems impose smoothness conditions\footnote{The assumption that the value function is differentiable everywhere is sometimes referred to as the high contact or smooth fit principle.} on the value function that are not likely to hold in classes of control problems beyond those in which absolutely continuous controls are applied to diffusion processes without jumps. Indeed, verification theorems require that the value function must be everywhere differentiable and have smooth derivatives almost everywhere.\footnote{Given a Lipschitz continuous value function, we can however, by Rademacher’s lemma, deduce that any Lipschitz continuous function is almost everywhere differentiable.} In many control problems this condition is violated, in particular, it is unlikely that in problems in which the controller makes discrete modifications to a diffusion process that the smoothness assumptions are upheld. Indeed, in general it is not possible to invoke the mean value theorem to derive the HJBI equations via a classical limiting procedure. A second issue is that verification theorems do not address the question of existence of the value of stochastic differential games nor is the question of uniqueness considered: in general, there exists an infinite number of Lipschitz continuous functions that satisfy the HJBI equations of the verification theorems (see for example exercise 3.2 in [8]).

Fortunately, using viscosity theory it can be shown that the value functions of a wide class of stochastic control problems (and consequently stochastic differential games) do
in fact satisfy verification theorems when the HJBI equations are interpreted in a weaker, viscosity sense. Indeed, viscosity solutions generalise the notion of a solution to a PDE to a non-classical definition\(^3\). The main advantage of the viscosity solution approach is that it does not require that the (everywhere) smoothness of the value function be established.

2. Contribution. This paper analyses a stochastic differential game of control and stopping in which the controller uses impulse controls; the results cover a general setting in which the underlying state process is a diffusion process. We examine the problem using viscosity theory from which we show that the value of the game exists and is a unique viscosity solution to a HJBI equation. A similar game to the one considered in this paper is that contained within [10] in which a stochastic differential game in which both players modify the state process using impulse controls is studied. In our script however, in addition to studying a stochastic differential game of control and stopping we study a stochastic differential game in which the underlying state process is a diffusion process in contrast to [10]. Other related papers to the current are [1, 4, 3] in which conditions for a HJBI equation are proved for stochastic differential games of control and stopping in which the controller uses continuous controls.

The Dynamics: Canonical Description. We suppose then that the uncontrolled passive state \(X \in \mathcal{S} \subseteq \mathbb{R}^p (p \in \mathbb{N})\), evolves according to a diffusion on \((\mathcal{C}([t, \tau_S]; \mathbb{R}^p), (\mathcal{F}_{(t,s)} \sigma_{[t, \tau_S]}), \mathcal{F}, \mathcal{P})\) that is to say for \(s \in [t, \tau_S]\), \(X \in \mathcal{S} \subseteq \mathbb{R}^p\) the state process obeys the following SDE:

\[
    dX^t_{s,x_0} = \mu(s, X^t_{s,x_0})ds + \sigma(s, X^t_{s,x_0})dB(s), \quad X^t_{t,x_0} := x_0,
\]

where \(B(s) \in \mathbb{R}^p\) be a \(p\)-dimensional standard Brownian motion with state space \(\mathcal{S}\), \(\tilde{N}(ds, dz) = N(ds, dz) - \nu(dz)dt\) is a \(\mathcal{F}\)-Poisson random measure with \(\nu(\cdot) := \mathbb{E}[N(1, \cdot)]\) is a Lévy measure. Both \(\tilde{N}(ds, dz)\) and \(B(s)\) are supported by the filtered probability space and \(\mathcal{F}\) is the filtration of the probability space \((\Omega, \mathcal{F} = \{\mathcal{F}_s\}_{s \in [t, \tau_S]}\). We assume that \(N\) and \(B\) are independent.

We assume that the functions \(\mu: [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R}^p, \sigma: [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R}^{p \times m}\) and \(\gamma: \mathbb{R}^p \times \mathbb{R}^I \to \mathbb{R}^{p \times I}\) are deterministic, measurable functions that are Lipschitz continuous and satisfy a (polynomial) growth condition so as to ensure the existence of (2.1) [17] (see assumptions A.1.1, & A.2.).

As in [9], we note that the above specification of the filtration ensures stochastic integration and hence, the controlled diffusion is well defined (this is proven in [24]).

The generator of \(X\) (the uncontrolled process) is:

\[
    \mathcal{L}\phi(x) = \sum_{i=1}^{p} \mu_i(x) \frac{\partial \phi}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{p} (\sigma \sigma^T)_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + I\phi(x).
\]

The state process is influenced by impulse controls \(u \in \mathcal{U}\) exercised by player I where \(u(s) = \sum_{j \geq 1} \xi_j \cdot 1_{\tau_j \leq \tau_S}(s)\) for all \(s \in [t, \tau_S]\). The impulses \(\{\xi_j\} \in \mathcal{Z} \subseteq \mathcal{S}\) are exercised by player I who intervenes at \(\mathcal{F}\)-measurable stopping times \(\{\tau_i\}\) where \(t \leq \tau_1 < \tau_2 < \ldots \leq \tau_S\) and where \(\mathcal{S} \subseteq \mathbb{R}^p\) is a given set.

\(^3\)Viscosity solutions were introduced by Michael Crandall and Pierre-Louis Lions in 1983 [12] and was developed to handle first order HJB equations. The theory was subsequently developed to handle second order equations in part due to a comparison principle result introduced by Robert Jensen in 1988 [18].
We assume $U \subseteq \mathbb{R}^p$ is a convex cone which is the set of admissible control actions for player I and $Z$ is the set of admissible impulse values. Indeed, if we suppose that an impulse $\zeta \in Z$ determined by some admissible policy $w$ is applied at some $\mathcal{F}$–measurable stopping time $\tau$ when the state is $x' = X^{t,x_0,:}(\tau-) \equiv X^{t,x_0} \cdot \Gamma(x',\zeta)$, then the state immediately jumps from $x' = X^{t,x_0}(\tau-) \equiv X^{t,x_0,w}(\tau)$ to $X^{t,x_0,w}(\tau)$ where $\Gamma : \mathbb{R}^p \times Z \to \mathbb{R}^p$ is called the impulse response function and $(t,x_0) \in [t,\tau_S] \times \mathbb{R}^p$. We assume that the impulses $\xi_j \in Z$ are $U$–valued and are $\mathcal{F}$–measurable for all $j \in \mathbb{N}$.

For notational convenience, as in [9], we will use $u = [\tau_j, \xi_j]_{j \geq 1}$ to denote the control policy $u = \sum_{j \geq 1} \xi_j \cdot 1_{\tau_j \leq \tau_S}(s) \in \mathcal{U}$ which consists of $\mathcal{F}$–measurable stopping times $\{\tau_j\}_{j \in \mathbb{N}}$ and $\mathcal{F}$–measurable impulse interventions $\{\xi_j\}_{j \in \mathbb{N}}$.

The evolution of the state process with interventions is described by the equation:

\[
X_r = x + \int_t^r \mu(s, X_s^{t,x,u}) ds + \int_t^r \sigma(s, X_s^{t,x,u}) dB_s + \sum_{j \geq 1} \xi_j \cdot 1_{\{\tau_j \leq \tau_s\}}(r).
\]

The game is s.th. player II can choose some $\mathcal{F}$–measurable stopping time $\rho \in \mathcal{T}$ at which point the process is stopped and both players receive a terminal cost (reward) $G(X^{t,S \wedge \rho})$. Player I has a cost function which is also the player II gain (or profit) function. The corresponding payoff functions are given by the following expression which player I (resp., player II) minimises (resp., maximises):

\[
J^{u}(x) \equiv J(t, x_0; u, \rho) = \mathbb{E} \left[ \int_{t_0}^{\tau_{S \wedge \rho}} f(s, X_s^{t,x_0,u}) ds + \sum_{m \geq 1} c(\tau_m, \xi_m) \cdot 1_{\{\tau_m \leq \tau_S \wedge \rho\}}(\tau_{S \wedge \rho}) \right] + G(X_{t,S \wedge \rho}),
\]

where $x := (t, x_0) \in [t, \tau_S] \times \mathbb{R}^p, \tau_m \in [t, \tau_S], u \in \mathcal{U}, \rho \in [t, \tau_S]$ and $\xi_m \in Z$ and where the functions $f : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R}, G : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R}$ are the running cost function and the bequest function respectively.

The results contained in this paper are built exclusively under the following set of assumptions unless otherwise stated:

**Standing Assumptions.** A.1.1. Lipschitz Continuity

We assume there exist real-valued constants $c_{\mu}, c_{\sigma} > 0$ s.th. $\forall s \in [t, \tau_S], \forall x, y \in \mathbb{R}^p$ we have:

\[
|\mu(s, x) - \mu(s, y)| \leq c_{\mu}|x - y| \\
|\sigma(s, x) - \sigma(s, y)| \leq c_{\sigma}|x - y|.
\]

A.1.2. Lipschitz Continuity

We also assume the Lipschitzianity of the running functions $h, g, \psi$ and $\phi$ and the so that we assume the existence of real-valued constants $c_h, c_g, c_\psi, c_\phi > 0$ s.th. $\forall s \in [t, \tau_S], \forall (x, y) \in \mathbb{R}^p$ we have for $R \in \{h, g, k, l, \psi, \phi\}$:

\[
|R(s, x) + R(s, y)| \leq c_R|x - y|.
\]
A.2. Growth Conditions

We assume the existence of a real-valued constants \( d_\mu, d_\sigma > 0 \) s.th. \( \forall (s, x) \in [t, \tau_S] \times \mathbb{R}^p \) we have:

\[
\begin{align*}
|\mu(s, x)| &\leq d_\mu (|1 + |x|^p|), \\
|\sigma(s, x)| &\leq d_\sigma (|1 + |x|^p|).
\end{align*}
\]

We also make the following assumptions on the cost function \( c : [t, \tau_S] \times \mathbb{R}^p \rightarrow \mathbb{R} \):

A.3.

Let \( \tau, \tau' \in [t, \tau_S] \) be \( F \)-measurable stopping times s.th. \( t \leq \tau < \tau' \leq \tau_S \) and let \( \xi, \xi' \in \mathcal{Z} \) be measurable impulse interventions. Then we assume that the following statements hold:

\[
\begin{align*}
(2.4) &\quad c(\tau, \xi + \xi') \leq c(\tau, \xi) + c(\tau, \xi'), \\
(2.5) &\quad c(\tau, \xi) \geq c(\tau', \xi).
\end{align*}
\]

A.4.

We also assume that the there exists a constant \( \lambda \) > 0 s.th. \( \inf_{\xi \in \mathcal{Z}} c(s, \xi) \geq \lambda_c \forall s \in [t, \tau_S] \) where \( \xi \in \mathcal{Z} \) is a measurable impulse intervention.

Assumptions A.1.1. and A.2. ensure the existence and uniqueness of a solution to (2.1) (c.f. [17]). Assumption A.3. (i) (subadditivity) is required in the proof of the uniqueness of the value function. Assumption A.3. (ii) (the player cost function is a decreasing function in time) and may be interpreted as a discounting effect on the cost of interventions. Assumption A.1.2. is required to prove the regularity of the value function (see for example [10] and for the single-player case, see for example [13]). Assumption A.3. (ii) was introduced (for the two-player case) in [27] though is common in the treatment of single-player case problems (e.g. [13, 9]). Assumption A.4. is integral to the definition of the impulse control problem.

Throughout the script we adopt the following standard notation (e.g. [8, 9, 24]):

**Notation.** Let \( \Omega \) be a bounded open set on \( \mathbb{R}^{p+1} \). Then we denote by: \( \bar{\Omega} \) - The closure of the set \( \Omega \).

\( Q(s, x; R) = (s', x') \in \mathbb{R}^{p+1} : \max |s' - s|, |x' - x| < R, s' < s \).

\( \partial \Omega \) - the parabolic boundary \( \Omega \) i.e. the set of points \((s, x) \in \bar{\mathcal{S}} \) s.th. \( R > 0, Q(s, x; R) \notin \bar{\Omega} \).

\( C^{[1,2]}([t, \tau_S], \Omega) = \{ h \in C^{[1,2]}(\Omega) : \partial_s h, \partial_{x_i} h, h \in C(\Omega) \} \), where \( \partial_s \) and \( \partial_{x_i} \) denote the temporal differential operator and second spatial differential operator respectively.

\( \nabla \phi = (\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_p}) \) - The gradient operator acting on some function \( \phi \in C^1([t, \tau_S] \times \mathbb{R}^p) \).

\( \cdot \) - The Euclidean norm to which \( \langle x, y \rangle \) is the associated scalar product acting between two vectors belonging to some finite dimensional space.

3. **Statement of Main Results.** In this paper, we prove two main results for the game that characterise the conditions for a HJBI equation in both zero-sum and non-zero-sum impulse controller-stopper stochastic differential games.

We prove that the stochastic differential game with a diffusion process and in which one of the players uses impulse controls and the other player chooses when to end the game admits a value.

We prove that the value of the game satisfied a double obstacle quasi-integro-variational equality and is a unique viscosity solution to a HJBI equation.
In particular, in section V we will show that equality (4.3) holds by firstly showing that $V^+(x)$ (resp.; $V^-(x)$) is a viscosity supersolution (resp.; subsolution) to the following non-linear obstacle problem:

$$\begin{align*}
\{\max\{\min&[-\frac{\partial V}{\partial s} - \mathcal{L}V - f, V - G], V - \mathcal{M}V\} = 0 \\
V(X^{t,x,u}(\tau_\rho)) &= G(X^{t,x,u}(\tau_\rho)).
\end{align*}$$

(3.1)

where $\mathcal{L}$ is the local stochastic generator operator associated to the process and $\mathcal{M}$ is the non-local intervention operator - we will use $\mathcal{L}$ to denote the local stochastic generator for the controlled process, where it will not cause confusion we will also employ the shorthand $r(s, X_s) \equiv r(X_s)$ where $r \in f, G$ - the constituent functions of the payoff function $J$.

To our knowledge, this is the first game that involves impulse controls in which the role of one of the players is to stop the game at a desirable point.

**Organisation.** In section III, we introduce some of the technical definitions in order to provide a description of the game. Here, we also introduce the underlying concepts required to study the game. In section IV, we prove some preliminary results that under-pin the main analysis which is performed in section V - in particular, we prove that the upper and lower value functions are regular and bounded. Where it is of no detriment to the main body of ideas we postpone some of the technical proofs of the section to the appendix. In section V, we introduce the viscosity theory framework - here we show by way of a dynamic programming principle and comparison principle, that the value of the game is a unique viscosity solution to a HJBI equation. Lastly, the paper is finalised by an appendix to which some of the technical proofs from sections IV - V are deferred.

**Definitions.** We now give some definitions which we shall need to describe the system dynamics modified by impulse controls:

**Definition 3.1.** Denote by $T_{[t,\tau]}$ the set of all $\mathcal{F}-$measurable stopping times in the interval $[t, \tau]$, where $\tau$ is some stopping time s.th. $\tau' \leq \tau_\rho$, if $\tau' = \tau_\rho$ then we will denote by $T \equiv T_{[t, \tau_\rho]}$. Let $u = [\tau_j, \xi_j]_{j \in \mathbb{N}}$ be a control policy where $\{\tau_j\}_{j \in \mathbb{N}}$ and $\{\xi_j\}_{j \in \mathbb{N}}$ are $\mathcal{F}_{\tau_\rho}-$measurable stopping times and interventions respectively, then we denote by $\mu_\tau u$ the number of impulses the agent executes within the interval $[t, \tau]$ under the control policy $u$ for some $\tau \in T$.

**Definition 3.2.** Let $u$ be an impulse control policy. We say that an impulse control is admissible on $[t, \tau_\rho]$ if the number of impulse interventions is finite $\mathbb{P}-$a.s, that is to say we have that $\mathbb{E}[\mu_{[t,\tau_\rho]}(u)] < \infty$.

We shall hereon use the symbol $\mathcal{U}$ (resp., $\mathcal{T}-$ which belongs to the set of all $\mathcal{F}-$measurable stopping times in $[t, \tau_\rho]$) to denote the set of admissible controls for player I (resp., player II).

Given two player I controls $u \in \mathcal{U}$ and $u' \in \mathcal{U}$; we interpret the notion $u \equiv u'$ on $[t, \tau_\rho]$ iff $P(u = u')$ a.e. on $[t, \tau_\rho] = 1$.

Similarly, given two player II stopping times $\rho \in \mathcal{T}$ and $\rho' \in \mathcal{T}$, we interpret the notion $\rho \equiv \rho'$ on $[t, \tau_\rho]$ analogously.

**Definition 3.3.** Let $u(s) = \sum_{j \geq 1} \xi_j \cdot 1_{[\tau_j, \tau_{j+1})}(s) \in \mathcal{U}$ be an impulse control defined over $[t, \tau_\rho]$, further suppose that $\tau$ and $\tau'$ are two $\mathcal{F}-$measurable stopping times contained within the interval $[t, \tau_\rho]$, then we define the restriction $u_{[\tau, \tau']}$ of the impulse control $u(s)$ to be $u_{[\tau, \tau']}(s) = \sum_{j \geq 1} \xi_{\mu_{[\tau, \tau']}} \cdot 1_{[\tau_{\mu_{[\tau, \tau']}}, \tau'_{\leq s}]}(s)$ where $\tau \leq s < \tau'$.  

We denote by \( \mathcal{U}_{[\tau, \tau']} \subset \mathcal{U}_{[t, \tau_s]} \equiv \mathcal{U} \) the (restricted) set of admissible controls over the interval \([\tau, \tau']\).

**Strategies.** A player strategy is a map from the other player’s set of controls to the player’s own set of controls. An important feature of the players’ strategies is that they are non-anticipative - neither player may guess in advance, the future behavior of other players given his current information.

We formalise this condition by constructing non-anticipative strategies which were used in the viscosity solution approach to differential games in [15]. Non-anticipative strategies were introduced by [23, 14, 25, 14]. Hence, in this game, one of the players chooses his control and the other player responds by selecting a control according to some strategy.

**Definition 3.4.** A non-anticipative strategy on \([t, \tau_s]\) for Player I is a measurable mapping which we shall denote by \( \alpha : \mathcal{U}_{[t, \tau_s]} \times \Omega \times \mathcal{T} \to \mathcal{U} \) and for any stopping time \( \tau : \Omega \to \mathcal{T} \) and any \( \mathcal{F} \)-measurable player II stopping times \( \rho_1, \rho_2 \in \mathcal{T} \) with \( \rho_1 \equiv \rho_2 \) on \([t, \tau]\) we have that \( \alpha(\rho_1) \equiv \alpha(\rho_2) \) on \([t, \tau]\).

We define the Player II non-anticipative strategy \( \beta : [t, \tau_s] \times \Omega \times \mathcal{U} \to \mathcal{T} \) analogously. Hence, \( \alpha \) and \( \beta \) are Elliott-Kalton strategies.

We denote the set of all non-anticipative strategies for Player I (resp., Player II) by \( \mathcal{A}_{(t, \tau_s)} \) (resp., \( \mathcal{B}_{(t, \tau_s)} \)).

**Remark 3.5.** The intuition behind definition 3.4 is as follows: suppose player I uses the control \( u_1 \in \mathcal{U} \) and the system follows a path \( \omega \) and that player II employs the strategy \( \beta \) against the control \( u_1 \) and some other player I control \( u_2 \in \mathcal{U} \) then controls \( u_1 \) and \( u_2 \) induce the same response from the player II strategy that is to say \( \beta(u_1) \equiv \beta(u_2) \).

Note that when \( \mathcal{U} \) is a singleton the game is degenerate and collapses into a classical optimal stopping problem for player II with a value function and solution as that in ch.3 in [20]. Similarly, when \( \mathcal{T} \) is a singleton the game collapses into a classical impulse control problem for player I with a value function and solution as that in ch.7 in [20].

**Definition 3.6.** Suppose we denote the space of measurable functions by \( \mathcal{H} \), suppose also that the function \( \phi : [t, \tau_s] \times \mathbb{R}^p \to \mathbb{R}^p \) s.th. \( \phi \in \mathcal{H} \). Let \( \tau \in [t, \tau_s] \) and \( \rho \in \mathcal{T} \) be \( \mathcal{F} \)-measurable stopping times; we define the [non-local] Player I-intervention operator \( \mathcal{M} : \mathcal{H} \to \mathcal{H} \) acting at \( \tau \) by the following expression:

\[
\mathcal{M}[(\phi)] := \inf_{z \in \mathbb{R}} [\phi(\Gamma(X(\tau^-), z)) + c(\tau, z) \cdot 1_{\tau \leq \rho \wedge \tau_s}]
\]

where \( \Gamma : \mathbb{R}^p \times \mathcal{Z} \to \mathbb{R}^p \) is the impulse response function defined earlier.

**Remark 3.7.** Suppose that the value of the game exists and that we denote the value by \( V \). If \( V \in \mathcal{H} \), then the term \( \mathcal{M}V(s, x) \) that is, the non-local intervention operator \( \mathcal{M} \) acting the value function associated to the game, represents the value of the player I strategy that consists of performing the best possible intervention at some given time \( s \in [t, \tau_s] \) when the state is at \( x \in \mathbb{R}^p \), then performing optimally thereafter.

Suppose \( \tau \in \mathcal{T} \) is some intervention time then the equality \( \mathcal{M}V(\tau, x) = V(\tau, x) \) holds at the points of intervention \( \forall \tau \in \mathbb{R}^p \), we note however, that an immediate intervention may not be optimal; that is we have the following lemma:

**Lemma 3.8.** Suppose that the value of the game \( V \) exists and that \( V \in \mathcal{H} \), then the non-local intervention operator \( \mathcal{M} \) satisfies the following inequality pointwise on
$[t, \tau_S] \times \mathbb{R}^p$:

\begin{equation}
\mathcal{M}V(s, x) \geq V(s, x).
\end{equation}

4. Preliminaries. Given the remarks of section III, we now define the value functions of the game. As in [15], we define the value functions in terms of Elliot-Kalton strategies introduced in [14]:

Value Functions. The two value functions associated to the game are given by the following expressions:

\begin{align}
V^-(x) &= \inf_{\alpha \in \tilde{A}_i(t, \tau_s)} \sup_{\rho' \in T \setminus \{t\}} J(t, x_0; \alpha(\rho'), \rho'), \\
V^+(x) &= \sup_{\beta \in \hat{B}_i(t, \tau_s)} \inf_{u' \in \bar{U}_i(t, \tau_s)} J(t, x_0; u', \beta(u')).
\end{align}

We refer to $V^-$ and $V^+$ as the upper and lower value functions respectively.

We say that the value of the game exists if $\forall x \in [t, \tau_s] \times \mathbb{R}^p$ we can commute the supremum and infimum operators in (4.2) and (4.1) where after we can deduce the existence of a function $V \in \mathcal{H}$ s.th. $V \equiv V^-(x) = V^+(x)$. We will use the notation $V^\pm$ to mean any element drawn from the set $\{V^+, V^-\}$.

Theorem 4.1. For any $x \in [t, \tau_s] \times \mathbb{R}^p$ the value of the game exists and is given by:

\begin{equation}
V(x) = V^-(x) = V^+(x).
\end{equation}

Remark 4.2. By definition of the value functions we automatically have

\begin{equation}
V^-(x) \geq V^+(x),
\end{equation}

for all $x \in [t, \tau_s] \times \mathbb{R}^p$.

To prove Theorem 4.1, it therefore remains to establish the reverse inequality of (4.4).

We now prove the regularity of the value functions associated to the game. Related results can be found in [10] where the regularity (Lipschitzianity in the spatial variable, Hölder continuity in time) for the corresponding two-player impulse control game with an uncontrolled process.

Lemma 4.3. The functions $V^-$ and $V^+$ may be equivalently written as:

\begin{align}
V^-(t, x) &= \inf_{\alpha \in \tilde{A}_i(t, \tau_s)} \sup_{\rho' \in T \setminus \{t\}} J(t, x, \alpha(\rho'), \rho'), \\
and \quad V^+(t, x) &= \sup_{\beta \in \hat{B}_i(t, \tau_s)} \inf_{u' \in \bar{U}_i(t, \tau_s)} J(t, x, u, \beta(u)),
\end{align}

where $\tilde{U}_i(t, \tau_s)$ is the set of player I admissible controls which have no impulses at time $s = t$ and correspondingly, $\hat{A}_i(t, \tau_s)$ (resp., $\hat{B}_i(t, \tau_s)$ is the set of all player I (resp., player II) non-anticipative strategies with controls drawn from the set $\hat{U}_i(t, \tau_s)$ (resp., $T \setminus \{t\}$).
Proof

The proof of the lemma is similar to that of Lemma 3.3 in [10] with some modifications:

The main idea is to prove that for all \((t, x_0) \in [t_0, \tau_S] \times \mathbb{R}^p\) there exists a control \(\bar{u} \in \mathcal{U} \setminus \mathcal{U}(t, \tau_S)\) and an \(\mathcal{F}\)-measurable stopping time \(\bar{\rho} \in \mathcal{T} \setminus t'\) where \(\mathcal{U}(t, \tau_S) \subset \mathcal{U}\) is the set of admissible impulse controls \(\mathcal{U}\) which excludes impulses at time \(t'\), for which the following inequality holds \(\forall u \in \mathcal{U}, \mathcal{F}\)-measurable stopping time \(\rho \in \mathcal{T}\) and for some \(\epsilon > 0\):

\[
(4.7) \quad |J(t', x, u, \beta(u)) - J(t', x, \bar{u}, \bar{\beta}(\bar{u}))| + |J(t', x, \alpha(\rho), \rho) - J(t', x, \bar{\alpha}(\bar{\rho}), \bar{\rho})| \leq \epsilon.
\]

We prove the result for the case in which player I exercises only one intervention at the point \(t\) since the extension to multiple interventions is straightforward.

W.l.o.g., we can employ the following short-hands \(\beta(u) \equiv \rho \in \mathcal{T}, \bar{\beta}(\bar{u}) \equiv \bar{\rho} \in \mathcal{T'}, \alpha(\rho) \equiv u \in \mathcal{U}\) and \(\bar{\alpha}(\bar{\rho}) \equiv \bar{u} \in \mathcal{U}(t, \tau_S)\). The result is proven by constructing the following control and stopping times:

\[
u_n = \xi \cdot 1_{[\tau_n, \tau_S]} + u'
\]

where \(\tau_n = (\tau + \frac{1}{n}) \cdot 1_{\tau=\tau'} + \tau \cdot 1_{\tau > \tau'}\) and,

\[
\rho_n = (\rho + \frac{1}{n}) \cdot 1_{\rho=\rho'} + \rho \cdot 1_{\rho > \rho'}.
\]

where \(u' = \sum_{j \geq 1} \xi_j \cdot 1_{[\tau_j, \tau_S]}\). Using this we find that:

\[
J(t', x_0; u, \rho) - J(t', x_0; u_n, \rho_n)
\]

\[
= \mathbb{E} \left[ \int_{t'}^{\rho \land \tau_S} (f(s, X_{s}^{t', x_0, u}) - f(s, X_{s}^{t', x_0, u_n}))ds - \int_{\rho \land \tau_S}^{\rho_n \land \tau_S} f(s, X_{s}^{t', x_0, u_n})ds
\]

\[
- \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\rho \leq \tau_j < \rho_n} + G(X_{\rho_n \land \tau_S}^{t', x_0, u_n}) - G(X_{\rho \land \tau_S}^{t', x_0, u}) \right],
\]

We now readily observe that \(X_{s}^{t', x_0, u_n} \to X_{s}^{t', x_0, u}\) and by construction, \(\rho_n \to \rho\) as \(n \to \infty\), \(\mathbb{P}\)-a.s. hence, after invoking the dominated convergence theorem we can deduce the existence of an integer \(N \geq 1\) s.t. \(\forall \epsilon > 0\) and \(\forall n \geq N\) we deduce that:

\[
J(t', x_0; u, \rho) - J(t', x_0; u_n, \rho_n) \leq \epsilon.
\]

The proof can be extended to multiple impulse case (at time \(t')\) straightforwardly after employing conditions A.3. (iii) and A.3. (iv) where after the proof easily reduces to the single impulse case.

Given Lemma 4.3, we can now deduce that the upper and lower value functions associated to the game are Lipschitz continuous in the spatial variable and Hölder continuous in time, that is, we observe the following proposition:

**Proposition 4.4.** We can deduce the existence of constants \(c_1, c_2 > 0\) and some \(\beta \in (0, 1]\) s.t. the following result holds for all \((s, x'), (s, x), (t', x), (t, x) \in [t, \tau_S] \times \mathbb{R}^p\):
i. \[ |V^{-}(s, x') - V^{-}(s, x)| + |V^{+}(s, x') - V^{+}(s, x)| \leq c_{1}|x' - x| \]
ii. \[ |V^{-}(t', x) - V^{-}(t, x)| + |V^{+}(t', x) - V^{+}(t, x)| \leq c_{2}|t' - t|^{\frac{1}{2}}. \]

**Proof of Proposition 4.4.**

We separate the proof into two parts, proving the spatial Lipschitzianity (i) first, then the temporal \( \frac{1}{2} \)-Hölder-continuity (ii) last.

To show that the value functions are Lipschitz continuous in the spatial variable, it suffices to show that the property is satisfied for the function \( J \).

The proof is straightforward since the result follows as an immediate consequence of the Lipschitzianity of the constituent functions. In particular, we have:

\[
|J(t, x, \cdot, \cdot) - J(t, x', \cdot, \cdot)| \leq c|x - x'|.
\]

(4.9)

We note also that since the constituent functions of \( J \) are bounded, \( J \) is also bounded; hence by (4.9) and by Lemma 3.6 in [8], we therefore conclude that:

\[
|J(t, x, \cdot, \cdot)| \leq c|x - x'|,
\]

(4.10)

for some constant \( c > 0 \) as required.

To show that the value functions are Lipschitz continuous in the temporal variable, we show that (ii) is satisfied by the function \( V^{+} \) with the proof for the function \( V^{-} \) being analogous.

We firstly note that:

\[
V^{+}(t', x) - V^{+}(t, x) = \sup_{\mu \in B_{i,(t,\tau_{2})}} \inf_{u \in U} J[t', x_{0}; u, \mu(u)] - \sup_{\mu \in B_{i,(t,\tau_{2})}} \inf_{u \in U} J[t, x_{0}; u, \mu(u)]
\]

(4.11)

By (i), we can deduce the existence of some \( \epsilon \)-optimal strategy \( \hat{\mu} \in B_{i,(t,\tau_{2})} \) against \( V^{+}(t, x) \) s.t. \( V^{+}(t, x) - \epsilon \leq \inf_{u \in U} J[t, x, u, \hat{\mu}(u)] \) \( \forall (t, x) \in [t, \tau_{2}] \times \mathbb{R}^{p} \) where \( \epsilon > 0 \) is arbitrary. Hence, by (4.11) we have that:

\[
V^{+}(t', x) - V^{+}(t, x) - 2\epsilon \leq \inf_{u \in U} J[t', x, u, \hat{\mu}(u)] - \inf_{u \in U} J[t, x, u, \hat{\mu}(u)].
\]

(4.12)

Let us now construct the control \( u_{c} = \sum_{i \geq 1} \xi_{i} \cdot 1_{[\tau_{i}, \tau_{2}]} \) which is associated with the strategy \( \alpha^{c} \in \mathcal{A}_{i,(t,\tau_{2})} \). Let us also construct the control \( u'_{c} \in U_{[t', \tau_{2}]} \) using the following expression: \( u'_{c} = \sum_{i \leq \ell} \xi_{i} \cdot 1_{[\tau_{i}, \tau_{2}]} + \sum_{i > \ell} \xi_{i} \cdot 1_{[\tau_{i}, \tau_{2}]} \) which is associated to the strategy \( \hat{\alpha}^{c} \) so that the control \( u'_{c} \) is simply the control \( u_{c} \in U \) except that the impulse interventions within the interval \( [t, t') \) are now pushed to \( t' \).

Now thanks to Lemma 4.3, we have that \( |J(t, x; u, \cdot) - J(t, x; \hat{u}, \cdot)| < \epsilon \) where \( \epsilon > 0 \) is arbitrary and where \( \hat{u} \in \hat{U} \) is the set of player I admissible controls that have no impulses
at time $t$. Hence, by Lemma 4.3 and using the $\epsilon-$optimality of the strategy $\hat{\mu} \in B(t, \tau_s)$, we can therefore deduce the following inequality:

$$V^+(t', x) - V^+(t, x) - 3\epsilon \leq J(t', x; u'_e, \mu(u'_e)) - J(t, x; u_e, \hat{\mu}(u_e)).$$

Let us denote by $\hat{\rho} := \mu(u'_e) \equiv \hat{\mu}(u_e)$ and define $\rho \in [t, \tau_S]$ by:

$$\hat{\rho} := \begin{cases} t', & \{\hat{\rho} < t'\} \\ \rho, & \{\hat{\rho} \geq t'\} \end{cases}$$

Hence, we have that:

$$J[t', x; u'_e, \hat{\mu}(u'_e)] - J[t, x; u_e, \hat{\mu}(u_e)] = J[t', x; u'_e, \hat{\rho}] - J[t, x; u_e, \hat{\rho}]$$

From which we can now calculate that:

$$J[t', x; u'_e, \hat{\rho}] - J[t, x; u_e, \hat{\rho}]$$

$$= -E \left[ \int_{t'}^{\hat{\rho} \wedge \tau_S} f(s, X_{t,s}^t, x, u_e)ds + \sum_{j \geq 1} c(\tau_j^t, \xi_j^t) \cdot 1\{t \leq \tau_j^t < \hat{\rho} \wedge \tau_S\} \right. - \left. \left( G(X_{t,s}^{t', x, u'_e}) - G(X_{t,s}^{t, x, u_e}) \right) \cdot 1\{\hat{\rho} \leq t'\} \right]$$

$$+ E \left[ \int_{t'}^{\hat{\rho} \wedge \tau_S} f(s, X_{t,s}^t, x, u_e)ds - \int_{t'}^{\hat{\rho} \wedge \tau_S} f(s, X_{t,s}^{t', x, u'_e})ds + \sum_{j \geq 1} c(\tau_j^t, \xi_j^t) \cdot 1\{t \leq \tau_j^t < k\} \right.$$  

$$- \left. \left( G(X_{t,s}^{t', x, u'_e}) - G(X_{t,s}^{t, x, u_e}) \right) \cdot 1\{\hat{\rho} > t'\} \right]$$

$$= E \left[ \int_{t'}^{\hat{\rho} \wedge \tau_S} f(s, X_{t,s}^t, x, u_e)ds - \int_{t'}^{\hat{\rho} \wedge \tau_S} f(s, X_{t,s}^{t', x, u'_e})ds \right] + E \left[ c(t', \sum_{\tau_j^t \leq t'} \xi_j^t) - \sum_{\tau_j^t \leq t'} c(\tau_j^t, \xi_j^t) \right]$$

$$+ E \left[ \left( G(X_{t,s}^{t', x, u'_e}) - G(X_{t,s}^{t, x, u_e}) \right) \cdot 1\{\hat{\rho} > t'\} + \left( G(X_{t,s}^{t, x, u_e}) - G(X_{t,s}^{t', x, u'_e}) \right) \cdot 1\{\hat{\rho} \leq t'\} \right].$$

Now by assumption A.3, we have that:

$$\sum_{\tau_j^t \leq t'} c(\tau_j^t, \xi_j^t) \geq c(t', \sum_{\tau_j^t \leq t'} \xi_j^t).$$

Hence, we find that:

$$J(t', x_0; u'_e, \hat{\mu}(u'_e)) - J(t, x_0; u_e, \hat{\mu}(u_e))$$

$$\leq E \left[ \int_{t'}^{\hat{\rho} \wedge \tau_S} f(X_{t,s}^t, x_0, u_e)ds - \int_{t'}^{\hat{\rho} \wedge \tau_S} f(X_{t,s}^{t', x_0, u'_e})ds \right]$$

$$+ \sup_{\rho' \in [t', \hat{\rho}]} E \left[ G(X_{t,s}^{t', x_0, u'_e}) - G(X_{t,s}^{t', x_0, u'_e}) \right].$$

(4.16)
where we have used (4.15) to remove the cost terms. Now by the Lipschitz property of the function \(G\) (c.f. A.1.2.) and Lemma A.3., we can deduce the existence of a constant \(c > 0\) s.th.:

\[
\mathbb{E} \left[ |G(X_{s}^{t,\cdot, x_0}) - G(X_{s}^{t,\cdot, x_0'})| \right] \leq c \sup_{s \in [t, \tau_S]} \mathbb{E} \left[ |X_{s}^{t,\cdot, x_0'} - X_{s}^{t,\cdot, x_0}| \right] \leq c |t - t'|^{\frac{1}{2}},
\]

where \(c\) denotes some arbitrary constant (which may differ in each step).

Moreover, we can arrive at the result after observing a boundedness property of \(f\) and invoking the Lipschitz property then appealing to the statements of Lemma (4.18).

Now, since \(\sup_{s \in [t, \tau_S]} \mathbb{E} \left[ |X_{s}^{t,\cdot, x_0'} - X_{s}^{t,\cdot, x_0}| \right] \leq c |t - t'|^{\frac{1}{2}}\),

we have that:

\[
\mathbb{E} \left[ \int_{t}^{t'} f(s, X_{s}^{t,\cdot, x_0'}) ds \right] \leq c \mathbb{E} \left[ \int_{t}^{t'} \sup_{s \in [t, t']} (1 + |X_{s}^{t,\cdot, x_0'}|) ds \right] \leq c \int_{t}^{t'} \sup_{s \in [t, t']} (1 + |X_{s}^{t,\cdot, x_0'}|) ds \leq c (t' - t)(1 + |x|) \in L,
\]

for some \(c > 0\) and where we have used the continuity of \(f\) in the last step. Therefore, for all \(\mathbb{E}[\int_{t}^{\tau_S} f(\cdot) ds]\) is bounded from above by \(c(t' - t)\) for some \(c > 0\). Now, we firstly observe using Fubini’s Theorem, we can deduce the existence of some constant \(c > 0\) s.th.:

\[
\mathbb{E} \left[ \int_{t}^{\tau_S} f(s, X_{s}^{t,\cdot, x_0'}) ds \right] \leq c \int_{t}^{\tau_S} \mathbb{E} \left[ X_{s}^{t,\cdot, x_0'} - X_{s}^{t,\cdot, x_0} \right] ds + c \int_{t}^{t'} \sup_{s \in [t, t']} (1 + \mathbb{E} \left[ |X_{s}^{t,\cdot, x_0'}| \right]) ds \leq c |t - t'|^{\frac{1}{2}}.
\]

(4.19)

using the Lipschitzianity of \(f\), (4.18) and (ii) and (iii) of Lemma A.3.

Hence, after plugging (4.19) and (4.17) into (4.16) and (4.13); and then applying (ii) and (iii) of Lemma A.3., we can deduce that there exists a constant \(c > 0\) s.th. for \(\epsilon > 0\) the following estimate holds for all \((t, x_0), (t', x_0) \in [t, \tau_S] \times \mathbb{R}^p\):

\[
|V^+(t', x_0) - V^+(t, x_0)| \\
\leq |J(t', x_0; u_\epsilon, \bar{\mu}(u_\epsilon)) - J(t, x_0; u_\epsilon, \bar{\mu}(u_\epsilon))| + 3\epsilon \\
\leq c |t - t'|^{\frac{1}{2}} + 3\epsilon.
\]

(4.20)

Now, since \(\epsilon\) in (4.20) is arbitrary, we can deduce the existence of a constant \(c > 0\) s.th. for all \((t, x_0), (t', x_0) \in [t, \tau_S] \times \mathbb{R}^p\) and for some \(\beta \in (0, 1)\):

\[
|V^+(t, x_0) - V^+(t', x_0)| \leq c |t - t'|^{\frac{1}{2}}.
\]

(4.21)

after which we deduce (ii) holds for the function \(V^+\).

To deduce that (ii) holds for the function \(V^-\), we note that analogous to (4.11), we have that:

\[
V^-(t', x) - V^-(t, x) = \inf_{\alpha \in \mathcal{A}(t, \tau_S)} \sup_{\rho \in \mathcal{T}} J(t', x_0; \alpha(\rho), \rho) - \inf_{\alpha \in \mathcal{A}(t, \tau_S)} \sup_{\rho \in \mathcal{T}} J(t, x_0; \alpha(\rho), \rho).
\]

(4.22)
In a similar way to the proof of (ii) for $V^+$ we can deduce the existence of a constant $c > 0$ s.th. for all $(t, x_0), (t', x_0) \in [t, \tau_S] \times \mathbb{R}^p$:

\begin{equation}
|V^-(t, x_0) - V^-(t', x_0)| \leq c|t - t'|^\beta.
\end{equation}

from which we deduce the thesis.

**Proposition 4.5.** The value functions $V^\pm$ are both bounded.

**Proof**

We do the proof for the function $V^-$ with the proof for $V^+$ being analogous.

Recall that:

\begin{equation}
V^-(x) = \inf_{\alpha \in A(t, \tau_S)} \sup_{\rho \in \mathcal{T}} \mathbb{E} \left[ \int_t^{\rho \wedge \tau_S} f(s, X_s^{t, x_0, \alpha(p)}) ds + \sum_{j \geq 1} c(\tau_j(\rho), \xi_j(\rho)) \cdot 1_{\{\tau_j(\rho) \leq \rho \wedge \tau_S\}} \right] + G(X_{\rho \wedge \tau_S})
\end{equation}

Now let $u_0 \in \mathcal{U}$ be the player I control with which no impulses exercised. Then

\begin{equation}
X_{\tau_S}^{t_0, x_0, u} = X_{\tau_S}^{t_0, x_0, u_0} + \sum_{j \geq 1} \xi_j.
\end{equation}

Now, since $u \in \mathcal{U}$ (hence $\mathbb{E}[\mu_{(t_0, \tau_S)}(u)] < \infty$) we can find some $\lambda > 0$ s.th.\n\[\sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < \tau_S \wedge \rho\}} \leq \lambda\] hence, if we denote by $Y_{\tau_S}^{t_0, x_0, u} := X_{\tau_S}^{t_0, x_0, u_0} + \sum_{j \geq 1} \xi_j \cdot 1_{\{\tau_j < \tau_S \wedge \rho\}}$ then we have that:

\begin{align*}
\mathbb{E} \left[ \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < \tau_S \wedge \rho\}} + G(X_{\rho \wedge \tau_S}^{t, x_0, \alpha(p)}) \right] \\
= \mathbb{E} \left[ \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < \tau_S \wedge \rho\}} + ((G(Y_{\tau_S}^{t_0, x_0, \alpha(p)}) - G(X_{\tau_S}^{t_0, x_0, u_0})) + G(X_{\tau_S}^{t_0, x_0, u_0})) \right] \\
\leq \mathbb{E} \left[ (\lambda + c_1|Y_{\tau_S}^{t_0, x_0, u} - X_{\tau_S}^{t_0, x_0, u_0}|) + G(X_{\tau_S}^{t_0, x_0, u_0}) \right] \\
\leq \mathbb{E} \left[ (G(X_{\tau_S}^{t_0, x_0, u_0}) + (\lambda + c_1((\rho \wedge \tau_S) - t_0)^{\beta})) \right],
\end{align*}

for some $c_1 > 0$ and where the last line follows by Lemma A.3. followed by Gronwall’s lemma.

Hence, since by similar reasoning we can deduce that $\mathbb{E}[\int_t^{\rho \wedge \tau_S} f(s, X_s^{t, x_0, u}) ds] \leq \mathbb{E}[\int_t^{\rho \wedge \tau_S} f(s, X_s^{t, x_0, u_0}) ds + c((\rho \wedge \tau_S) - t_0)^{\beta} \cdot 1_{\{\mu_{(t_0, \tau_S)}(u)\}}]$ for some $c > 0$ and $\beta \in (0, 1)$:
using the growth condition on the functions $f$ and $G$ we find that:

$$V^-(x)$$

$$\leq \sup_{\rho \in T} \mathbb{E} \left[ \int_t^{\rho \wedge \tau_S} f(s, X_{s}^{t,x_0,u_0}) ds + G(X_{\tau_S}^{t_0,x_0,u_0}) $$

$$+ (\lambda + c_1(\rho \wedge \tau_S) - t_0^{\frac{1}{4}}) \cdot 1_{\{\mu(t_0,\tau_S)=u\}} \right]$$

$$\leq \sup_{\rho \in T} \mathbb{E} \left[ \int_t^{\rho \wedge \tau_S} c_2(1 + \mathbb{E}[X_{s}^{t,x_0,u_0}]) ds + c_3(1 + \mathbb{E}[X_{\rho \wedge \tau_S}^{t_0,x_0,u_0}]) $$

$$+ (\lambda + c_1(\rho \wedge \tau_S) - t_0^{\frac{1}{4}}) \cdot 1_{\{\mu(t_0,\tau_S)=u\}} \right]$$

$$\leq \sup_{\rho \in T} \mathbb{E} \left[ \alpha + (\lambda + c_1(\rho \wedge \tau_S) - t_0^{\frac{1}{4}}) \cdot 1_{\{\mu(t_0,\tau_S)=u\}} \right] ,$$

using Lemma A.3. and where $\alpha := (\rho \wedge \tau_S) \cdot [c_2 + c_3(1 + |x|)]$ and $c_1 > 0$ and $c_2 > 0$ are constants. We then deduce the thesis since each of the terms inside the square bracket is bounded.

**Lemma 4.6.** Let $V \in H$ be a bounded function and $(\tau, x) \in [t, \tau_S] \times \mathbb{R}^p$ where $\tau$ is some $\mathcal{F}$-measurable stopping time, then the set $\Xi(\tau, x)$ defined by:

$$(4.26) \quad \Xi(\tau, x) := \{ \xi \in \mathcal{Z} : MV(\tau-, x) = V(\tau, x + \xi) + c(\tau, \xi) \cdot 1_{\{\tau \leq \tau_S\}} \}$$

is non-empty.

The proof of the lemma is straightforward since we need only prove that the infimum is in fact a minimum. This follows directly from the fact that the cost function is minimally bounded (c.f. A.4.) and that the value functions are also bounded by Proposition 4.5.

A proof of the following lemma is reported in [28], Lemma 3.6 and similar result may be found in (Lemma 3.10 in [9]):

We give the following result without demonstration:

**Lemma 4.7.** The non-local intervention operator $\mathcal{M}$ is continuous wherein we can deduce the existence of a constants $c_1, c_2 > 0$ s.th. $\forall x, y \in \mathbb{R}^p$ and $s < s'$ with $s, s' \in [t, \tau_S]$:  

i) $|\mathcal{M}V^\pm(s, x) - \mathcal{M}V^\pm(s, y)| \leq c_1|x - y|.$ 

ii) $|\mathcal{M}V^\pm(t, x) - \mathcal{M}V^\pm(t, s)| \leq c_2|t - s|^{\frac{1}{4}}.$

5. **A Viscosity Theoretic Approach.** We now approach problem (3.1) using a viscosity theory approach. We shall firstly start by proving the existence of a value of the game and that the value is a unique viscosity solution to the HJBI equation, some ideas for the proofs in the section come from [10], [8, 16].

The outline of the scheme is as follows:

i. Using the established regularity results for the value functions of the game, prove a dynamic programming principle for each of the value functions.

ii. Prove that the upper (resp., lower) value function is a viscosity subsolution (resp., supersolution) to the HJBI equation (3.1).
iii. Prove a comparison theorem and prove the reverse inequality to (4.4) therefore proving equality of the value functions.

iv. Using Isaacs’ condition, from (iii) deduce the existence of a value of the game and that the value is a unique solution to the HJBI equation.

Let us therefore now introduce some key definitions relating to viscosity solutions:

**Definition 5.1.** A locally bounded lower (resp., upper) semicontinuous function $\psi : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R}$ is a viscosity supersolution (resp., subsolution) to the HJBI equation (3.1) if:

For all $(s, x) \in [t, \tau_S] \times \mathbb{R}^p$ and $\psi \in C^{1,2}([t, \tau_S]; \mathbb{R}^p)$ s.th. $(s, x)$ is a local minimum (resp., maximum) of $V - \psi$, we have that:

\begin{align}
\max \{ \min [ -\partial_s \psi(s, x) - (\mathcal{L}\psi(s, x) + f(s, x)), \psi(s, x) - G(x)] & \\
\quad , \psi(s, x) - \mathcal{M}\psi(s, x) \} & \geq 0 \quad (\text{resp., } \leq 0)
\end{align}

A locally bounded lower (resp., upper) semicontinuous function $\psi : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R}$ is a viscosity solution to the HJBI equation (3.1) if it is both a subsolution and supersolution of (5.2).

**Remark 5.2.** If the value functions are known a priori to be continuous (or deterministic or in the discrete case) derivation of the DPP is straightforward. Otherwise, in general, we must use one of two arguments: a measurable selection argument or establish the regularity of the value functions then construct a measurable selection i.e. partition the state space then construct a measurable selection (this uses the Lindelöf property of the canonical space).

We now state the dynamic programming principle for the game:

**Theorem 5.3. Dynamic Programming Principle for stochastic differential games of control and stopping with Impulse Controls**

Let $u \in \mathcal{U}$ be an admissible player I control and suppose $\rho \in \mathcal{T}$ is an $\mathcal{F}$–measurable stopping time, then for all $(t, x_0) \in [t, \tau_S] \times \mathbb{R}^p$ and for a sufficiently small $h \in \mathbb{R}^+$ (i.e. $h \leq \tau_S - t$) the following variants of the dynamic programming principle hold for the functions $V^+$ and $V^-$:

$$
V^-(x) = \inf_{\alpha \in \mathcal{A}(t, \tau_S)} \sup_{\rho \in \mathcal{T}} \mathbb{E} \left[ \int_t^{\tau + h} f(s, X_s^{t, x_0, \alpha}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq (t+h) \land \rho\}} + G(X_{\rho \land t+h}^{t, x_0, \alpha}) \cdot 1_{\{\rho \land \tau_S < t+h\}} + V^-((t+h) \land \rho, X_{t+h}^{t, x_0, \alpha}) \cdot 1_{\{\rho \land \tau_S \geq t+h\}} \right]
$$

and

$$
V^+(x) = \sup_{\beta \in \mathcal{B}(t, \tau_S)} \inf_{u \in \mathcal{U}} \mathbb{E} \left[ \int_t^{\tau + h} f(s, X_s^{t, x_0, u}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq (t+h) \land \beta\}} + G(X_{\beta \land t+h}^{t, x_0, u}) \cdot 1_{\{\beta(u) \land t+h\}} + V^-((t+h) \land \beta(u), X_{t+h}^{t, x_0, u}) \cdot 1_{\{\beta(u) \geq t+h \land \beta(u)\}} \right].
$$
Proof of Theorem 5.3.

We begin by proving:

\[ V^+(t_0, x_0) \geq \sup_{\beta \in \mathcal{B}(t, \tau_S)} \inf_{u \in \mathcal{U}} \mathbb{E} \left[ \int_{t_0}^{t_0 + h \wedge \beta(u)} f(s, X_s^{t_0, x_0, u}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1\{\tau_j < t_0 + h\} \right] + G(X_{t_0}^{t_0, x_0, u}) \cdot 1\{\rho \leq t + h\} + V^+((t_0 + h) \wedge \beta(u), X_{t_0 + h}^{t_0, x_0, u}) \cdot 1\{\rho \geq t_0 + h\}. \]

Having established the uniform continuity of the functions \( V^- \) and \( V^+ \), a countable selection argument is sufficient in order to derive a dynamic programming principle thus avoiding measurable selection arguments directly. Indeed, using Proposition 4.4 we can find a set of controls that produce values of \( J \) that are arbitrarily close to the values of \( V^- \) and \( V^+ \) at some given point.

Hence, let \((A^i)_{i \in \mathbb{N}}\) be a partition of \( \mathbb{R}^p \). Let \( \hat{\mu} \in \mathcal{B}(t, \tau_S) \) be some \( \epsilon \)-optimal strategy against \( \sup_{\beta(u) \in \mathcal{B}(t_0, \tau_S)} \inf_{u \in \mathcal{U}} V^+(t_0 + h \wedge \beta(u), X_{t_0 + h}^{t_0, x_0, u}) \). Note by Lemma A.1 we can deduce that since \( \hat{\mu} \) is an \( \epsilon \)-optimal strategy against \( \sup_{\beta(u) \in \mathcal{B}(t_0, \tau_S)} \inf_{u \in \mathcal{U}} V^+(t_0 + h \wedge \beta(u), X_{t_0 + h}^{t_0, x_0, u}) \) then there exists some \( 2\epsilon \)-optimal strategy \( \hat{\mu}^x \in \mathcal{B}(t_0, \tau_S) \) against \( \sup_{\beta(u) \in \mathcal{B}(t_0, \tau_S)} \inf_{u \in \mathcal{U}} V^+(t_0 + h \wedge \beta(u), x, y) \) s.th.

\[ \sup_{\beta(u) \in \mathcal{B}(t_0, \tau_S)} \inf_{u \in \mathcal{U}} V^+(t_0 + h \wedge \beta(u), x, y) \leq \inf_{u \in \mathcal{U}} J(t_0 + h, x, u, X_{t_0 + h, \tau_S}^{t_0, x_0, u}, \hat{\mu}^x(u, t_0 + h, \tau_S)) \] where \( \epsilon > \delta > 0 \). Hence, we deduce that the strategy \( \hat{\mu}^x \) is a \( 2\epsilon \)-optimal strategy against \( \sup_{\beta(u) \in \mathcal{B}(t_0, \tau_S)} \inf_{u \in \mathcal{U}} V^+(t + h \wedge \beta(u), y) \) for all \( y \in B_{\delta}(x) \) within some radius \( 0 < \delta < \epsilon \).

Let us therefore construct the strategy \( \mu \) by:

\[ \hat{\mu}(u)(s) = \begin{cases} \mu(u)(s), & s \in [t, t + h) \\ \hat{\mu}^x(u)(t + h, \tau_S)(s), & s \in [t + h, \tau_S], X_{t + h, \tau_S}^{t, x_0, u} \in B_{\delta}(x) \end{cases} \]

Now for any \((t, x_0) \in [t, \tau_S] \times \mathbb{R}^p, u \in \mathcal{U}, \mu \in \mathcal{B}(t, \tau_S) \) and \( \forall u \in [t + h, \tau_S] \in \mathcal{U}(t + h, \tau_S) \) using Lemma 4.3, we have:

\[ \mathbb{E} \left[ \int_{t}^{\hat{\mu}(u) \wedge \tau_S} f(s, X_s^{t, x_0, u}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1\{\tau_j \leq \hat{\mu}(u)\} + G(X_{\hat{\mu}(u) \wedge \tau_S}^{t, x_0, u}) \right] \]

\[ \geq \mathbb{E} \left[ \int_{t}^{(t + h) \wedge \mu(u)} f(s, X_s^{t, x_0, u}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1\{\tau_j < t + h \wedge \mu(u)\} \right] + G(X_{\mu(u) \wedge \tau_S}^{t, x_0, u}) \cdot 1\{\mu(u) < t + h\} \]

\[ + \mathbb{E} \left[ \int_{t + h}^{\hat{\mu}(u) \wedge \tau_S} f(s, X_s^{t, x_0, u}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1\{\hat{\mu}(u) \wedge \tau_S \geq \tau_j \geq t + h\} \right] + G(X_{\hat{\mu}(u) \wedge \tau_S}^{t, x_0, u}) \cdot 1\{\hat{\mu}(u) \geq t + h\} - \epsilon \]

(5.7)
for some arbitrary \( \epsilon > 0 \). Using the properties of \( X \), we can further rewrite (5.7) as:

\[
\begin{align*}
E \left[ \int_t^{(t+h)\wedge \mu(u)} f(s, X_{s}^{t,x_{0},u}) \, ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < t+h \wedge \mu(u)\}} + G(X_{\mu(u)}^{t,x_{0},u}) \cdot 1_{\{\mu(u) < t+h\}} \right] \\
+ E \left[ \int_{t+h}^{\hat{\mu}(u) \wedge \tau_S} f(s, X_{s}^{t+h,x_{0},u}) \, ds + G(X_{\hat{\mu}(u) \wedge \tau_S}^{t+h,x_{0},u}) \right] - \epsilon.
\end{align*}
\]

We now exploit the regularity of \( V \) (Proposition 4.4) and the \( \epsilon \)-optimality of \( \mu \) by which we deduce that:

\[
\begin{align*}
E \left[ \int_{t+1}^{\hat{\mu}(u) \wedge \tau_S} f(s, X_{s}^{t+h,x_{0},u}) \, ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \geq t+h\}} + G(X_{\hat{\mu}(u) \wedge \tau_S}^{t+h,x_{0},u}) \right] - \epsilon \\
\geq \sum_{j \in \mathbb{N}} \left( E \left[ \int_{t+1}^{\hat{\mu}(u) \wedge \tau_S} f(s, X_{s}^{t+h,x_{0},u}) \, ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \geq t+h\}} + G(X_{\hat{\mu}(u) \wedge \tau_S}^{t+h,x_{0},u}) \right] - c\delta - \epsilon \\
\geq V^+(t+h \wedge \hat{\mu}(u), X_{t+h}^{x_{0},u}) - 2\epsilon - c\delta,
\end{align*}
\]

(5.9)

using the \( \epsilon \)-optimality of the strategy \( \hat{\mu} \) against \( \sup_{\beta \in B(t, \tau_S)} \inf_{u \in U} V^+(t+h \wedge \beta(u), X_{t+h}^{x_{0},u}) \).
Hence, putting (5.9) together with (5.8) we find that:
\[
E \left[ \int_t^{(t+h)\wedge \mu(u)} f(s, X_s^{t,x_0,u}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < t+h \wedge \mu(u)\}} \right] \\
+ G(X_{\mu(u)}^{t,x_0,u}) \cdot 1_{\{\mu(u) < t+h\}}
\]
\[
+ E \left[ \int_{t+h}^{(t+h)\wedge \mu(u)} f(s, X_s^{t,x_0,u}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\mu(u) \wedge \tau_S \geq t+h\}} \right] \\
+ G(X_{\mu(u) \wedge \tau_S}^{t,x_0,u}) \cdot 1_{\{\mu(u) \geq t+h\}}
\]
\[
\geq E \left[ \int_{t}^{(t+h)\wedge \mu(u)} f(s, X_s^{t,x_0,u}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < t+h \wedge \mu(u)\}} \\
+ G(X_{\mu(u)}^{t,x_0,u}) \cdot 1_{\{\mu(u) < t+h\}} + V^+(t + h \wedge \hat{\mu}(u), X_{t+h}^{t,x_0,u}) \cdot 1_{\{\mu(u) \geq t+h\}} \right] - c\delta - 2\epsilon,
\]
from which after successively applying the inf and sup operators we deduce the first result since \(\delta\) and \(\epsilon\) can be chosen arbitrarily. We prove the reverse inequality in an analogous manner.

We now prove inequality for the function \(V^-\). Indeed, by Proposition 4.4 we can deduce the existence of a strategy \(\alpha^{(1,\epsilon)} \in A_{(t_0,\tau_S)}\) against \(V^-(t, x)\) s.th.:

\[
\inf_{\alpha \in A_{(t_0,\tau_S)}} \sup_{\rho \in T} E \left[ \int_{t}^{(t+h)\wedge \rho} f(s, X_s^{t,x_0,\alpha(\rho)}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < t+h \wedge \rho\}} \right] \\
+ G(X_{\rho}^{t,x_0,\alpha(\rho)}) \cdot 1_{\{\rho < t+h\}} + V^-(t + h, X_{t+h}^{t,x_0,\alpha(\rho)}) \cdot 1_{\{\rho \geq t+h\}}
\]
\[
\geq E \left[ \int_{t}^{(t+h)\wedge \rho} f(s, X_s^{t,x_0,\alpha^{(1,\epsilon)}(\rho)}) ds + \sum_{j \geq 1} c(\tau_j^{1,\epsilon}, \xi_j^{1,\epsilon}) \cdot 1_{\{\tau_j^{1,\epsilon} < t+h \wedge \rho\}} \right] \\
+ G(X_{\rho}^{t,x_0,\alpha^{(1,\epsilon)}(\rho)}) \cdot 1_{\{\rho < t+h\}} + V^-(t + h, X_{t+h}^{t,x_0,\alpha^{(1,\epsilon)}(\rho)}) \cdot 1_{\{\rho \geq t+h\}} \right] - \epsilon,
\]
(5.10)

where we use the shorthand \(\tau_j(\rho) \equiv \tau_j\) and \(\xi_j(\rho) \equiv \xi_j \forall j \in \mathbb{N}\). We now build the strategy \(\alpha\) by:

\[
\alpha(\rho)(s) = \begin{cases} \\
\alpha^{(1,\epsilon)}(\rho)(s), & s \in [t, t+h) \\
\alpha^A(\rho')(s), & s \in [t+h, \tau_S], X_{t+h}^{t,x_0,\alpha^{(1,\epsilon)}} \in A_i
\end{cases}
\]
where we have used \(\rho'\) to denote the player II stopping time s.th. \(\rho' \in T_{(t+h,\tau_S)} \subseteq [t+h, \tau_S]\).
Let \( \alpha^{(2,\epsilon)} \in \mathcal{A}_{(t_0, \tau_S)} \) be an \( \epsilon \)-optimal strategy against \( \sup_{\rho \in \mathcal{T}} V^-((t+h) \wedge \rho, x) \). Using Lemma 4.3 and by similar reasoning as in part (i), we can also deduce the existence of a strategy \( \alpha^{A^i} \in \mathcal{A}_{(t+h, \tau_S)} \) s.t. \( \forall y \in A_i, \rho' \in \mathcal{T}_{(t+h, \tau_S)} \) and some \( \epsilon > 0 \) the following inequality holds:

\[
V^-((t+h) \wedge \rho, y) \geq J((t+h) \wedge \rho, y, \alpha^{A^i}(\rho'), \rho') - \epsilon.
\]

(5.12)

We therefore observe that:

\[
\begin{align*}
E \left[ V^- (t_0 + h, X_{t_0+h}^{t,x_0,\alpha^{(1,\epsilon)}(\rho')} \right] & = E \left[ \sum_{i \geq 1} V^- (t_0 + h, X_{t_0+h}^{t,x_0,\alpha^{(1,\epsilon)}(\rho')} \cdot 1_{\{X_{t_0+h}^{t,x_0,\alpha^{(1,\epsilon)}(\rho')} \in A_i\}} \right] \\
& \geq E \left[ \sum_{i \geq 1} J(t_0 + h, X_{t_0+h}^{t,x_0,\alpha^{(1,\epsilon)}(\rho')}, \alpha^{A^i}(\rho')) \cdot 1_{\{X_{t_0+h}^{t,x_0,\alpha^{(1,\epsilon)}(\rho')} \in A_i\}} \right] - \epsilon \\
& = J(t_0 + h, X_{t_0+h}^{t,x_0,\alpha^{(1,\epsilon)}(\rho')}, \sum_{i \geq 1} \alpha^{A^i}(\rho') \cdot 1_{\{X_{t_0+h}^{t,x_0,\alpha^{(1,\epsilon)}(\rho')} \in A_i\}}) - \epsilon.
\end{align*}
\]

(5.13)

Let us now construct the strategy \( \bar{\alpha}^{(2,\epsilon)}(\rho) \in \mathcal{A}(t_0 + h) \) defined by: \( \bar{\alpha}^{(2,\epsilon)}(\rho) := \sum_{i \geq 1} \alpha^{A^i}(\rho') \cdot 1_{\{X_{t_0+h}^{t,x_0,\alpha^{(1,\epsilon)}(\rho')} \in A_i\}} \). Now, after introducing the strategy \( \bar{\alpha}^{(2,\epsilon)} \) to (5.13) we deduce that:

\[
\begin{align*}
E \left[ \int_{t}^{(t+h)\wedge \rho} f(s, X_{s}^{t,x_0,\alpha^{(1,\epsilon)}(\rho)})ds + \sum_{j \geq 1} c(\tau_j^{1,\epsilon}, \xi_j^{1,\epsilon}) \cdot 1_{\{\tau_j^{1,\epsilon} \leq t+h \wedge \rho\}} \\
+ G(X_{t}^{t,x_0,\alpha^{(1,\epsilon)}(\rho)}) \cdot 1_{\{\rho < t+h\}} + V^-((t+h) \wedge \rho, x, \alpha^{A^i}(\rho')) \cdot 1_{\{\rho \geq t+h\})} \right] - \epsilon \\
\geq E \left[ \int_{t}^{(t+h)\wedge \rho} f(s, X_{s}^{t,x_0,\alpha^{(1,\epsilon)}(\rho)})ds + \sum_{j \geq 1} c(\tau_j^{1,\epsilon}, \xi_j^{1,\epsilon}) \cdot 1_{\{\tau_j^{1,\epsilon} < t+h \wedge \rho\wedge \tau_S\}} \\
+ G(X_{t}^{t,x_0,\alpha^{(1,\epsilon)}(\rho)}) \cdot 1_{\{\rho < t+h\}} + J(t_0 + h, X_{t_0+h}^{t,x_0,\alpha^{(1,\epsilon)}(\rho')}, \bar{\alpha}^{(2,\epsilon)}(\rho')) \cdot 1_{\{\rho \geq t+h\})} \right] - 2\epsilon.
\end{align*}
\]

(5.14)

We lastly construct the strategy \( \alpha^{\epsilon} \in \mathcal{A}(t_0) \) which consists of the strategy \( \alpha^{(1,\epsilon)} \) which is played up to time \( t_0 + h \) at which point the strategy \( \bar{\alpha}^{(2,\epsilon)} \) is then played.
Hence, after putting (5.14) and (5.10) together we observe that:

\[ V^-(t, x) \geq \mathbb{E} \left[ \int_t^{(t+h) \land \rho} f(s, X_s^{t,x_0,\alpha'}(\rho))ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < (t+h) \land \rho\}} \right. \]

\[ + G(X_p^{t,x_0,\alpha'(\rho)}) \cdot 1_{\{\rho < t + h\}} + V^-(t + h, X_{t+h}^{t,x_0,\alpha'(\rho)}) \cdot 1_{\{\rho \geq t + h\}} \left. \right] - \epsilon \]

Moreover, since \( \epsilon \) is arbitrary, we readily deduce that:

\[ V^-(t, x_0) \geq \inf_{\alpha \in A_{(t_0, \tau_S)}} \sup_{\rho \in \mathcal{T}} \mathbb{E} \left[ \int_t^{(t+h) \land \rho} f(s, X_s^{t,x_0,\alpha(\rho)})ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < (t+h) \land \rho\}} \right. \]

\[ + G(X_p^{t,x_0,\alpha(\rho)}) \cdot 1_{\{\rho < t + h\}} + V^-(t + h, X_{t+h}^{t,x_0,\alpha(\rho)}) \cdot 1_{\{\rho \geq t + h\}} \left. \right], \]

from which we readily deduce the required result. We can prove the reverse inequality in an analogous manner for which, in conjunction with (5.4) proves the thesis.

A classical consequence of the dynamic programming principles (5.3) and (5.4) is that we find that the function \( V^- \) (resp., \( V^+ \)) is the subsolution (resp., supersolution) to an associated HJBI equation, namely (3.1). Moreover, if the game admits a value \( V \), s.th. \( V \in C^{1,2}(t, \tau_S, \mathbb{R}^p) \), then the \( V \) is a classical solution to an associated HJBI equation.

The following lemma characterises the conditions in which the value of the game satisfies a HJBI equation:

**Lemma 5.4.** The function \( V^-(t, x_0) \) is a viscosity supersolution of (3.1) and the \( V^+(t, x_0) \) is a viscosity subsolution of (3.1).

**Proof of Lemma 5.4.**

The lemma is proven by contradiction.

We begin by proving that \( V^+(t, x_0) \) is a viscosity subsolution of (3.1). Suppose \( \psi : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R} \) is a test function with \( \psi \in C^{1,2}([t, \tau_S], \mathbb{R}^p) \) and let \( x \equiv (t, x) \in [t, \tau_S] \times \mathbb{R}^p \) are s.th. \( V^+ - \psi \) attains a local minimum at \( (t, x) \) with \( V^+(x) - \psi(x) = 0 \). We note that it remains only to show that \( \forall(s, x) \in [t, \tau_S] \times \mathbb{R}^p, \frac{\partial \psi}{\partial x}(s, x) + \mathcal{L} \psi(s, x) + f(s, x) \geq 0 \) whenever \( \psi(s, x) - G(x) > 0 \) which follows as a classical consequence of the dynamic programming principle.

Indeed, by Proposition 4.4. we can deduce the existence of a \( \epsilon \)-optimal strategy \( \alpha^* \in A_{(t, \tau_S)} \) to which the associated control is \( \alpha^*(v) \equiv u^* \in \mathcal{U} \) for all \( v \in V \) (against
We now exploit the regularity of show that given \( \epsilon > 0 \) we have that:

\[
\psi(x) = V^+(x) \geq \inf_{u \in \mathcal{U}} \mathbb{E} \left[ \int_{t_0}^{T_{\mathcal{S} \wedge \rho}} f(s, X_{s}^{t_0, x_0, u}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\tau_j < T_{\mathcal{S} \wedge \rho}} + G(X_{s}^{t_0, x_0, u}) \cdot 1_{T_{\mathcal{S} \wedge \rho} < t_0 + h} + V^+(t_0 + h, X_{t_0 + h}^{t_0, x_0, u}) \cdot 1_{t_0 + h = \rho} \right] - \epsilon h.
\]

(5.15)

Let us now define:

\[
\phi^{[h]}(t_0, x_0) := \mathbb{E} \left[ \int_{t_0}^{T_{\mathcal{S} \wedge \rho}} f(X_{s}^{t_0, x_0, u}) ds + G(X_{s}^{t_0, x_0, u}) \cdot 1_{T_{\mathcal{S} \wedge \rho} < t_0 + h} + \psi(X_{t_0 + h}^{t_0, x_0, u}) \cdot 1_{t_0 + h = \rho} \right]
\]

(5.16)

where \( u_0 \in \mathcal{U} \) is the player I control s.th. no impulses are exercised. We firstly wish to show that given \( \epsilon > 0 \) we have that:

\[
\psi(x) \geq \phi^{[h]}(t_0, x_0) - 2\epsilon h.
\]

Indeed, we firstly note that

\[
X_{t_0 + h}^{t_0, x_0, u} \equiv X_{t_0 + h}^{t_0, x_0, u_0(t_0, t_0 + h)} + \sum_{j = 1}^{\mu_{t_0, t_0 + h} \cdot (u_0)} \xi_j.
\]

(5.18)

We now exploit the regularity of \( V \) and boundedness of the sequence of intervention costs, indeed we have that:

\[
\mathbb{E} \left[ \sum_{j \geq 1} c(\tau_j^*, \xi_j^*) \cdot 1_{\{\tau_j^* < t_0 + h \wedge \rho\}} + V^+(t_0 + h, X_{t_0 + h}^{t_0, x_0, u}) \cdot 1_{\{t_0 + h = \rho\}} \right]
\]

\[
= \mathbb{E} \left[ \sum_{j = 1}^{\mu_{t_0, t_0 + h} \cdot (u_0)} c(\tau_j^*, \xi_j^*) + V^+(t_0 + h, X_{t_0 + h}^{t_0, x_0, u_0(t_0, t_0 + h)}) + \sum_{j = 1}^{\mu_{t_0, t_0 + h} \cdot (u_0)} \xi_j^* \cdot 1_{\{t_0 + h = \rho\}} \right]
\]

\[
\geq \mathbb{E} \left[ \sum_{j = 1}^{\mu_{t_0, t_0 + h} \cdot (u_0)} c(\tau_j^*, \xi_j^*) + (V^+(t_0 + h, X_{t_0 + h}^{t_0, x_0, u}) - V^+(t_0 + h, X_{t_0 + h}^{t_0, x_0, u})) \right.
\]

\[
+ V^+(t_0 + h, X_{t_0 + h}^{t_0, x_0, u}) \cdot 1_{\{t_0 + h = \rho\}} \right]
\]

\[
\geq \mathbb{E} \left[ (V^+(t_0 + h, X_{t_0 + h}^{t_0, x_0, u}) + (\lambda - c(\rho - t_0))^2) \cdot 1_{\mu_{t_0, t_0 + h} \cdot (u_0) \geq 1} \cdot 1_{\{t_0 + h = \rho\}} \right]
\]

\[
\geq \mathbb{E} [\psi(t_0 + h, X_{t_0 + h}^{t_0, x_0, u}) + (\lambda - c(\rho - t_0))^2] \cdot 1_{\mu_{t_0, t_0 + h} \cdot (u_0) \geq 1} \cdot 1_{\{t_0 + h = \rho\}},
\]

(5.19)
where we have used the fact that \( \sum_{j \geq 1} \inf_{z \in \mathbb{Z}} c(\tau_j^z, z) \geq \lambda \cdot 1_{\mu(t_0, t_0 + h)(u_s)} \) for some \( \lambda > 0 \) and that if \( X^t = X_{t_0 + h}^{t_0, x_0, u_s(t_0, t_0 + h)} + \sum_{j \geq 1} \mu_j(t_0, t_0 + h)(u_s) \xi_j \), we have that \( V^+(t_0 + h, X^t_{t_0 + h}) = V^+(t_0 + h, X^t) + (V^+(t_0 + h, X^t_{t_0 + h}) - V^+(t_0 + h, X^t)) \leq V^+(t_0 + h, X^t) + ch^\frac{1}{2} \) for some \( c > 0 \) using Lemma A.3 and Gronwall’s lemma.

Using the same arguments we can similarly deduce that there exists some constant \( c > 0 \) such that:

\[
G(X^t_{t_0 + h}) + f(t_0 + h, X^t_{t_0 + h}) \geq f(t_0 + h, X^t_{t_0 + h}) + G(X^t_{t_0 + h}) - ch^\frac{1}{2}.
\]

(5.20)

Now, since \( (\lambda - c(\rho - t_0)\frac{1}{2}) \cdot 1_{\mu(t_0, t_0 + h)(u_s) \geq 1} \cdot 1_{t_0 + h = \rho} = (\lambda - ch^\frac{1}{2}) \cdot 1_{\mu(t_0, t_0 + h)(u_s) \geq 1} \) and since there exists \( \bar{s} \in (t_0, T_0) \) s.th. for \( h \in [t_0, \bar{s}] \) for any \( \epsilon > 0 \) we have that:

\[
(\lambda - ch^\frac{1}{2}) \cdot 1_{\mu(t_0, t_0 + h)(u_s) \geq 1} \geq -\epsilon h,
\]

(5.21)

we observe that after inserting (5.21) and (5.20) into (5.19) and (5.15), we deduce that (5.17) does indeed hold.

Hence, combining (5.16) and (5.17) we find that:

\[
\psi(x) = V^+(x)
\]

\[
\geq \inf_{u \in U} \mathbb{E} \left[ \int_{t_0}^{\tau_{t_0, u} \wedge \rho} f(s, X^t_{s, t_0, x_0, u}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < \tau_{t_0, u} \wedge \rho\}} \cdot G(X^t_{\tau_j, t_0, x_0, u}) \cdot 1_{\tau_j < \rho} \right.
\]

\[
+ \left. V^+(t_0 + h, X^t_{t_0 + h}) \cdot 1_{\{t_0 + h = \rho\}} \right]
\]

\[
\geq \mathbb{E} \left[ \int_{t_0}^{\tau_{t_0, u} \wedge \rho} f(s, X^t_{s, t_0, x_0, u}) ds + G(X^t_{\tau_{t_0, u} \wedge \rho}, 1_{\{\tau_{t_0, u} \wedge \rho < t_0 + h\}}) \right.
\]

\[
+ \left. \psi(t_0 + h, X^t_{t_0 + h}) \cdot 1_{t_0 + h = \rho} \right] - 2\epsilon h
\]

(5.22)

Let us now define as \( \Lambda(s, x) := (\mathcal{G} + \mathcal{L})[\psi](s, x) \). By Itô’s formula for càdlàg semimartingale processes (see for example Theorem II.33 of [22]), we have that:

\[
\psi(x) = \psi(t_0 + h, X^t_{t_0 + h}) - \int_{t_0}^{t_0 + h + \tau_{t_0, u} \wedge \rho} (\nabla_x f(s, X^t_{s, t_0, x_0, u}), \mu(s, X^t_{s, t_0, x_0, u})) dB(s)
\]

\[
- \int_{t_0}^{t_0 + h + \tau_{t_0, u} \wedge \rho} \Lambda(s, X^t_{s, t_0, x_0, u}) ds.
\]

(5.23)

In order to generate a contradiction, we assume that \( G(x) - V^+(x) = G(x) - \psi(x) \geq 0 \) and suppose that the supposition of the lemma is false so that \( \Lambda(s, x) + f(s, x) > 0 \). We can therefore consider constants \( h, \delta > 0 \) s.th. \( \forall(s, x) \in [t_0, t_0 + h] \times B_h(x) \) s.th. \( G(x) - \psi(s, x) \geq \delta \) and \( \Lambda(s, x) + f(s, x) \geq \delta \).

Let us now define the set \( E := \{\inf_{s \in [t_0, t_0 + h]} |X^t_{s, t_0, x_0, u} - x| > a\} \) then using Lemma A.3, (i.e. the \( \frac{1}{2} \)-Hölder continuity of \( X \)) and by Tchebychev’s inequality, we can deduce
the existence of a constant \( c > 0 \) that depends only on the parameters of \( X^{t,x} \) s.th. \( \mathbb{P}[E] \leq \inf_{s \in [t_0, t_0 + h]} \frac{(s-t_0)^2}{a^2} \leq \frac{c^2}{a^4} \). Then since:

\[
(5.24) \quad \mathbb{E} \left[ \psi(t_0 + h, X^t_{t_0 + h}) \right] - \psi(x) = \mathbb{E} \left[ \int_{t_0}^{t_0 + h \wedge \tau_S} \Lambda(s, X^{t_0, x_0, u^*_t}) ds \right],
\]

using the boundedness (from beneath) of \( \psi \) and \( \Lambda \), we have that:

\[
-\psi(x) = \mathbb{E} \left[ 1_{E^c} \cdot \left( \int_{t_0}^{t_0 + h \wedge \tau_S} \Lambda(s, X^{t_0, x_0, u^*_t}) ds - \psi(t_0 + h, X^{t_0, x_0, u^*_t}) \right) \right]
\]

\[
+ \mathbb{E} \left[ 1_{E^c} \cdot \left( \int_{t_0}^{t_0 + h \wedge \tau_S} \Lambda(s, X^{t_0, x_0, u^*_t}) ds - \psi(t_0 + h, X^{t_0, x_0, u^*_t}) \right) \right]
\]

\[
\geq \mathbb{E} \left[ 1_{E^c} \left( \int_{t_0}^{t_0 + h \wedge \tau_S} \Lambda(s, X^{t_0, x_0, u^*_t}) ds - \psi(t_0 + h, X^{t_0, x_0, u^*_t}) \right) \right] - \frac{c h^2}{a^4}.
\]

Hence, by the given assumptions and reinvoking the above reasoning, we have that:

\[
-\psi(x) \geq \mathbb{E} \left[ 1_{E^c} \cdot \left( \int_{t_0}^{t_0 + h \wedge \tau_S} (\delta - f(s, X^{t_0, x_0, u^*_t}) ds + (\delta - G(X^{t_0, x_0, u^*_t})) \cdot 1_{\{\rho < t_0 + h\}} \right)
\]

\[
- \psi(t_0 + h, X^{t_0, x_0, u^*_t}) \cdot 1_{\{t_0 + h = \rho\}} \right) \right] - \frac{c h^2}{a^4}
\]

\[
\geq \mathbb{E} \left[ \delta(h + \mathbb{E}[1_{\{\rho < t_0 + h\}}]) \right] - \int_{t_0}^{t_0 + h \wedge \tau_S} f(s, X^{t_0, x_0, u^*_t}) ds
\]

\[
- G(X^{t_0, x_0, u^*_t}) \cdot 1_{\{\tau_S < \rho < t_0 + h\}} - \psi(t_0 + h, X^{t_0, x_0, u^*_t}) \cdot 1_{\{t_0 + h = \rho\}} \right] - \frac{2 c h^2}{a^4} - c h.
\]

(5.25)

Therefore combining (5.25) and (5.22) and after rearranging we find that:

\[
(5.26) \quad \frac{2 c h^2}{a^4} + 3 c h \geq \mathbb{E} \left[ \delta(h + \mathbb{E}[1_{\{\rho < t_0 + h\}}]) \right]
\]

Since \( \mathbb{E}[\delta(h + \mathbb{E}[1_{\{\rho < t_0 + h\}}])] \geq \mathbb{E}[\delta h] \), after dividing through by \( h \) we find that:

\[
(5.27) \quad \frac{1}{2} \delta - \left( \frac{c}{a^2} h + \frac{3}{2} \right) \leq 0.
\]

We then deduce the result since both \( h \) and \( \epsilon \) can be made arbitrarily small which yields a contradiction.

Next we prove that \( V(t, x_0) \) is a viscosity supersolution of (3.1). As in part (i), we prove the result by generating a contradiction, hence now suppose \( \psi : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R} \) is a test function with \( \psi \in C^{1,2}([t, \tau_S], \mathbb{R}^p) \) and let \( x \equiv (t, x) \in [t, \tau_S] \times \mathbb{R}^p \) are s.th. \( M V^{-} - \psi \) attains a local maximum at \( (t, x) \).
Now, to generate a contradiction, we assume that $\mathcal{M}V^-(s,x) - V^-(s,x)$
$= \mathcal{M}V^-(s,x) - \psi(s,x) \leq 0$ and suppose that the supposition of the lemma is false so that
$-\Lambda(s,x) - f(s,x) > 0$, and consider constants $h, \delta > 0$ s.th. $\forall (s,x) \in [t_0, t_0 + h] \times B_h(x)$ s.th. $\mathcal{M}V^-(s,x) - \psi(s,x) \leq -\delta$ and $\Lambda(s,x) + f(s,x) \leq -\delta$. By Proposition 4.4 we can
deduce the existence of a $\epsilon$-optimal strategy $\beta^\ast \in \mathcal{B}(t,\tau_S)$ to which the associated stopping time is $\beta^\ast(u) \equiv \rho^\ast \in [t, \tau_S]$ for all $u \in \mathcal{U}$ (against $V^-(x)$) s.th.:

$$
\psi(x) \leq \sup_{\rho \in \mathcal{T}} \mathbb{E}\left[ \int_{t_0}^{t_0 + h \wedge \rho \wedge \tau_S} f(s, X_t^{s_0, x_0, \alpha(\rho)}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < t \wedge \rho \wedge \tau_S\}} 
+ G(X_{t_0, x_0, \alpha(\rho)}^s) \cdot 1_{\{\rho \wedge \tau_S < t_0 + h\}} + V^-(t_0 + h, X_{t_0 + h}^s) \cdot 1_{\{\rho \wedge \tau_S \geq t_0 + h\}} \right] 
\leq \mathbb{E}\left[ \int_{t_0}^{t_0 + h \wedge \rho^\ast \wedge \tau_S} f(s, X_t^{s_0, x_0, \alpha(\rho^\ast)}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < t + h\}} 
+ G(X_{\rho \wedge \tau_S}^{s_0, x_0, \alpha(\rho^\ast)} \cdot 1_{\{\rho \wedge \tau_S < t_0 + h\}} + V^-(t_0 + h, X_{t_0 + h}^s) \cdot 1_{\{\rho \wedge \tau_S \geq t_0 + h\}} \right] + \epsilon h.
$$

(5.28)

After re-employing the estimate (5.17) we find that:

$$
\psi(x) \leq \mathbb{E}\left[ \int_{t_0}^{t_0 + h \wedge \rho^\ast \wedge \tau_S} f(s, X_t^{s_0, x_0, u}) ds + G(X_{\rho \wedge \tau_S}^{s_0, x_0, u}) \cdot 1_{\{\rho \wedge \tau_S < t_0 + h\}} 
+ V^-(t_0 + h, X_{t_0 + h}^s) \cdot 1_{\{\rho \wedge \tau_S \geq t_0 + h\}} \right] + 2\epsilon h.
$$

(5.29)

Now using Lemma 3.8, we have that $-\delta \geq \mathcal{M}V^-(s,x) - \psi(s,x) \geq V^-(s,x) - \psi(s,x)$
that is $\psi(s,x) \geq V^-(s,x) + \delta$ $\forall (s,x) \in [t_0, t_0 + h] \times B_h(x)$. Using the definition of $\Lambda$ and
the set $E$, introduced earlier and again applying Itô’s formula and by similar reasoning
as part (i), we find that:

$$
\psi(x) \geq \mathbb{E}\left[ \left( \int_{t_0}^{t_0 + h \wedge \tau_S} -\Lambda(s, X_t^{s_0, x_0, \alpha(\rho^\ast)}) ds + \psi(t_0 + h, X_{t_0 + h}^s) \right) \cdot 1_{\{E^c\}} \right] - \frac{\epsilon h^2}{a^4}
\geq \mathbb{E}\left[ \left( \int_{t_0}^{t_0 + h \wedge \tau_S} (\delta + f(s, X_t^{s_0, x_0, \alpha(\rho^\ast)}) ds + (\delta + V^-(s, X_t^{s_0, x_0, \alpha(\rho^\ast)})) \cdot 1_{\{E^c\}} \right) \right]
- \frac{\epsilon h^2}{a^4}.
$$
Employing similar reasoning as in part (i), and the estimate \((5.17)\) we find that:

\[
\psi(x) \geq \mathbb{E} \left[ \delta(h + \mathbb{E}[1_{\{\rho < t_0 + h\}}]) + \int_{t_0}^{t_0 + h \land \tau_S} f(s, X_s^{t_0,x_0,u_0}) ds 
\right.
\]

\[
+ G(X_{\tau_S \land \rho}^{t_0,x_0,u_0}) \cdot 1_{\{\tau_S \land \rho < t_0 + h\}} - V^-(s, X_s^{t_0,x_0,u_0}) \cdot \mathbb{1}_{\{t_0 + h = \rho\}} \right] - \frac{2 \epsilon h^2}{a^4} - 2 \epsilon h.
\]

\[(5.30)\]

where we have used the fact that \(h > 0\) which implies that \(\delta(h + \mathbb{E}[1_{\{\rho < t_0 + h\}}]) > \delta \mathbb{E}[1_{\{\rho < t_0 + h\}}])\).

Hence, combining (5.30) with (5.28) and

\[(5.31)\]

\[4 \epsilon h \geq \delta h - \frac{2 \epsilon h^2}{a^4},\]

for \(h\) small enough \(h < 1\). Hence, we find that:

\[(5.32)\]

\[\frac{1}{2} \delta - \left(2 \epsilon + \frac{\epsilon h}{a^4}\right) \leq 0,
\]

which is a contradiction since both \(\epsilon\) and \(h\) can be made arbitrarily small; hence we deduce the thesis.

**Theorem 5.5.** If the value of the game \(V\) exists, then \(V\) is a viscosity solution to the HJBI equation \((3.1)\).

**Proof**

Let us firstly recall that by \((5.19)\) and selecting \(h\) s.th. \(h < \tau_S - t_0\) we have the following inequality:

\[
\psi(x) \geq \mathbb{E} \int_{t_0}^{t_0 + h \land \rho} f(s, X_s^{t_0,x_0,u_0}) ds + G(X_{\rho}^{t_0,x_0,u_0}) \cdot 1_{\{\rho \leq t_0 + h\}}
\]

\[
+ \psi(t_0 + h, X_{t_0+h}^{t_0,x_0}) \cdot 1_{\{t_0+h < \rho\}} - 2 \epsilon h.
\]

\[(5.33)\]

Moreover, since \(V^+ - \psi\) attains a local minimum at \((t, x)\) we can deduce the existence of a constant \(\delta > 0\) s.th. for \((t, x) \in [t_0, \tau_S] \times \mathbb{R}^p:\)

\[(5.34)\]

\[V^+(t, x) - \psi(t, x) \geq 0 \text{ for } |(t, x) - x| \leq \delta.
\]

Additionally, by Lemma A.3., we can deduce the existence of a constant \(c > 0\) s.th. \(\forall t \in [t_0, \tau_S)\) we have that:

\[(5.35)\]

\[\mathbb{E}|X_t^{t_0,x_0} - x_0| \leq c|t - t_0|^{1/2}.
\]
We can therefore deduce the existence of a sequence $t_n \downarrow t_0$ for which $X^{t_n,x_0} \to x_0$ as $n \to \infty$. Let us now define the closed balls $B_n$ by the following:

$$B_n := \{|X^{t_n,x_0} - x_0| \leq \delta \ \forall \ m \geq n\}.$$ 

Further, let us now introduce the sequence of stopping times:

$$(5.36) \quad \tau_m = \sum_{n=1}^{\infty} t_{n+m} \cdot 1_{\{B_n \setminus B_{n-1}\}} \wedge \rho.$$ 

Hence by (5.19) we have that:

$$(5.37) \quad \psi(t_0, x) \geq \mathbb{E} \left[ \int_{t_0}^{t_m} f(X^{t_n,x_0,u_t}) ds + \psi(t_m, X^{t_m,x_0}) \right] - 2\epsilon(t_m - t_0).$$

After applying Itô’s formula for càdlàg semi-martingale processes to (5.37) we find that:

$$(5.38) \quad 0 \geq \mathbb{E} \left[ \int_{t_0}^{t_m} \frac{\partial \psi}{\partial t}(s, X^{t_n,x_0,u_t}) + \mathcal{L}\psi(s, X^{t_n,x_0,u_t}) + f(s, X^{t_n,x_0,u_t}) ds \right] - 2\epsilon(t_m - t_0).$$

Then, after dividing both sides of (5.38) by $(t_m - t_0)$ and taking the limit $m \to \infty$, we deduce that:

$$(5.39) \quad 0 \geq \frac{\partial \psi}{\partial t}(x) + \mathcal{L}\psi(x) + f(x),$$

which proves the subsolution property. We can prove the supersolution property analogously by firstly using (5.29) and applying similar steps after which the thesis is proved.

The following result establishes the equality of the two value functions $V^-$ and $V^+$; we defer the proof of the following result to the appendix:

**Theorem 5.6. Comparison Principle** Let $V^- : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R}$ be a continuous bounded viscosity subsolution to (3.1) and let $V^+ : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R}$ be a continuous bounded viscosity supersolution to (3.1).

Also suppose that for all $t \in [t, \tau_S]$ we have that $V^-(X^{t,x}_S) \leq V^+(X^{t,x}_S)$ then $\forall x \in [t, \tau_S] \times \mathbb{R}^p$ we have that:

$$(5.40) \quad V^-(x) \leq V^+(x).$$

**Corollary 5.7. The Game Admits a Value** To prove Theorem 4.1 it remains only to reverse the inequality (4.4) and thus prove that $V^-(X^{t,x}_S) \leq V^+(X^{t,x}_S)$ - a result that follows directly from the comparison principle for the game. Indeed, Theorem 4.1 and Corollary 5.7 then follow as direct consequences to the viscosity solutions results of Lemma 5.4 in conjunction with the comparison principle.

6. Appendix. 

**Lemma A.1.** Let $\hat{\mu} \in [t, \tau_S]$ be some $\epsilon$–optimal strategy against $V(t, x)$ for any $(t, x) \in [t, \tau_S] \times \mathbb{R}^p$ then there exists some $\eta > 0$ s.th. the strategy $\hat{\mu}$ remains $2\epsilon$–optimal against $V(t, y^j)$ for any $y^j \in B(x, \eta)$. 
Proof of Lemma A.1.

We do the proof for the function $V^-$ with the proof for $V^+$ being analogous. Denote by $\rho' \equiv \hat{\mu}(u)$ where $\rho' \in \left[t, \tau_S\right]$. Since the strategy $\hat{\mu}$ is $\epsilon$-optimal against $V^-(t, x)$ we have that for some $\epsilon > 0$:

\begin{equation}
V^-(t, x) \leq \inf_{u \in \mathcal{U}} J^{(u, \rho')}(t, x) + \epsilon.
\end{equation}

Now by Proposition 4.4 we can deduce the existence of some constants $c_1, c_2 > 0$ s.t. for any $y_j \in B(x, \eta), u \in \mathcal{U}, p \in \mathcal{T}$:

\begin{equation}
\left| J^{(u, \rho')}(t, y_j) - J^{(u, \rho')}(t, x) \right| \leq c_1 |x - y_j|,
\end{equation}

hence,

\begin{align*}
\inf_{u \in \mathcal{U}} J^{(u, \rho')}(t, y_j) & \geq \inf_{u \in \mathcal{U}} J^{(u, \rho')}(t, x) - c_1 |x - y_j| \\
& \geq V^-(t, x) - c_1 |x - y_j| - \epsilon \\
& \geq V^-(t, y_j) - (c_1 + c_2) |x - y_j| - \epsilon \\
& \geq V^-(t, y_j) - 2\epsilon,
\end{align*}

where the last line follows provided that $|x - y_j| \leq \eta := \epsilon(c_1 + c_2)^{-1}$, from which we deduce the thesis after applying $\sup_{u \in \mathcal{U}}$ to both sides and since $\epsilon$ is arbitrary.

The following results relate the dependence of $X^{t, x_0, u}()$ to the initial point $x \equiv (t, x_0) \in \left[t, \tau_S\right] \times \mathbb{R}^p$ where $u \in \mathcal{U}$; the following results also establish the Lipschitz condition and Hölder continuity of the state process.

**Lemma A.2.** For all $(t, x'), (t, x) \in \left[t, \tau_S\right] \times \mathbb{R}^p, s \in \left[t, \tau_S\right]$, we can deduce the existence of a constant $c > 0$ s.t.:

\begin{equation}
\mathbb{E} \left| X^t_s, x, x' - X^t_s, x', x' \right| \leq c |x' - x|.
\end{equation}

**Proof of Lemma A.2.**

Let us denote $x := X^t_{s, x}$ and $x' := X^t_{s, x'}$ using Itô’s lemma we readily observe that for all $(t, x), (t, x') \in \left[t, \tau_S\right] \times \mathbb{R}^p$:

\begin{align*}
\left| X^t_{s, x} - X^t_{s, x'} \right|^2 &= |x - x'| + 2 \int_{t_0}^s \left< X^t_r, x - X^t_r, x', \mu(r, X^t_r, x) - \mu(r, X^t_r, x') \right> dr \\
&+ 2 \int_{t_0}^s \left< X^t_r, x - X^t_r, x', \sigma(r, X^t_r, x) - \sigma(r, X^t_r, x') \right> dW_r \\
&+ \int_{t_0}^s \left| \sigma(r, X^t_r, x) - \sigma(r, X^t_r, x') \right|^2 dr
\end{align*}

(6.4)

After taking expectations in (6.4) and noting that by the standard properties of Brownian Motion the third term disappears under the expectation by assumption A.1.1. (Lips-
chitizianity of \( \sigma \) and \( \mu \) we observe that:

\[
E \left| X^{t,x,x'}_s - X^{t,x'}_s \right|^2 
\]

\[
\leq E \left[ \int_{t_0}^s \left\langle X^{t,x,x'}_r - X^{t,x'}_r, \mu(r, X^{t_0,x,u}_r) - \mu(r, X^{t,x'}_r) \right\rangle dr \right] 
\]

\[
+ \left| \sigma(r, X^{t_0,x,u}_r) - \sigma(r, X^{t,x'}_r) \right|^2 dr 
\]

\[
= E \left[ \int_{t_0}^s \left\langle X^{t,x,x'}_r - X^{t,x'}_r, \mu(r, X^{t_0,x,u}_r) - \mu(r, X^{t,x'}_r) \right\rangle dr \right] 
\]

\[
+ \left| \sigma(r, X^{t_0,x,u}_r) - \sigma(r, X^{t,x'}_r) \right|^2 dr 
\]

\[
\leq (\text{Lip}(\sigma) + \text{Lip}(\mu)) \int_{t_0}^s E \left[ \left| X^{t,x,x'}_r - X^{t,x'}_r \right|^2 \right] dr \leq C' \left| x - x' \right|^2 ,
\]

where \( C' := (\text{Lip}(\sigma) + C||\mu||_\infty)(s - t_0) \) and where the last line follows from Gronwall’s lemma, after which we can readily deduce the result.

In a similar way to Lemma A.2., we can deduce the following estimates:

**Lemma A.3.**

\[
E \left[ \sup_{s \in [t,\tau_\omega]} \left| X^{t,x}_s \right|^p \right] \leq c(1 + |x|^p).
\]

\[
E \left[ \sup_{s \in [t,h]} \left| X^{t,x}_s - x \right|^p \right] \leq c h^{p/2}(1 + |x|^p)
\]

\[
E \left[ \sup_{s \in [t,x']} \left| X^{t,x'} - X^{t_0,x_0} \right|^p \right] \leq c |x' - x|^p + |t' - t|^{p/2}(1 + |x|^p),
\]

where \( x' \equiv X^{t,x_0}_{t'} \in \mathbb{R}^p \) and \( x \equiv X^{t,x}_t \in \mathbb{R}^p \).

**Proof**

Let us denote \( x := X^{t,x_0}_t \) and \( x' := X^{t,x_0}_{t'} \) using Itô’s lemma we readily observe that for all \((t, x), (t, x') \in [t, \tau_S] \times \mathbb{R}^p \) there exists some constants \( c', c > 0 \) s.t.:

\[
E \left| X^{t,x}_s \right|^2 \leq |x|^2 + E \left( \int_t^s \left| \mu(r, X^{t_0,x,u}_r) \right|^2 dr \right) + E \left( \int_t^s \left| \sigma(r, X^{t,x}_r) \right|^2 dr \right)
\]

\[
\leq |x|^2 + cE \left[ \int_t^s \left| X^{t,x}_r \right|^2 dr \right]
\]

\[
\leq |x|^2 + cE \left[ \int_t^s (|X^{t,x}_r|^2 + E[|X^{t,x'}_r - X^{t,x}_r|^2]) dr \right]
\]

\[
\leq c(1 + |x|^2) + c'E \left[ \int_{t_0}^s (1 + \sup_{r \in [t,s]} E[|X^{t,x}_r|^2]) dr \right],
\]

by assumptions A.2., A.1.1. and Fubini’s theorem (and the smoothing theorem). Hence, after applying Gronwall’s lemma to (6.5), we immediately deduce the existence of some
real-valued constant \( c > 0 \) s.th:

\[
\mathbb{E}[|X_{s}^{t,x}-X_{s}^{t,x}|^2] \leq c(1 + |x|^2),
\]

which is the required result.

The proof of (ii) proceeds in a very much a similar way to the proof of Lemma A.2. and is omitted.

To prove statement (iii), given Lemma A.2.; it suffices only to prove that:

\[
\mathbb{E} \left[ \sup_{s \in [t,h]} |X_{t}^{t',x'} - X_{s}^{t,x}|^2 \right] \leq ch^{1/2}(1 + |x|).
\]

Indeed,

\[
|X_{t}^{t',x'} - X_{s}^{t,x}|^2 \leq 2 \int_{t}^{s} \left\langle X_{r}^{t,x} - X_{r}^{t',x'}, \mu(r, X_{r}^{t,x}) - \mu(r, X_{r}^{t',x'}) \right\rangle + \left| \sigma(r, X_{r}^{t',x'}) - \sigma(r, X_{r}^{t,x}) \right|^2 dr
\]

\[
+ 2 \int_{t}^{s} \left\langle X_{r}^{t,x} - X_{r}^{t',x'}, \sigma(r, X_{r}^{t,x}) - \sigma(r, X_{r}^{t',x'}) \right\rangle dW_{r}
\]

after taking expectations and using similar steps to the proof of Lemma A.2. and (i), we find that:

\[
\mathbb{E} \left[ X_{s}^{t',x'} - X_{s}^{t,x} \right]^2
\]

\[
\leq \mathbb{E} \left[ 2 \int_{t}^{s} \left\langle X_{r}^{t,x} - X_{r}^{t',x'}, \mu(r, X_{r}^{t,x}) - \mu(r, X_{r}^{t',x'}) \right\rangle + \left| \sigma(r, X_{r}^{t',x'}) - \sigma(r, X_{r}^{t,x}) \right|^2 \right] dr
\]

\[
\leq c \mathbb{E} \left[ \int_{t}^{s} \left( 1 + \left| X_{r}^{t',x'} \right|^2 + \left| X_{r}^{t,x} \right|^2 \right) \right.
\]

\[
+ \mathbb{E} \left[ \left| X_{r}^{t,x} - X_{r}^{t',x'} \right|^2 \right] dr
\]

\[
\leq c \mathbb{E} \left[ \int_{t}^{s} \left( 1 + \mathbb{E} \left[ \left| X_{r}^{t,x} \right|^2 \right] + \mathbb{E} \left[ \left| X_{r}^{t',x'} \right|^2 \right] \right)
\]

\[
+ \mathbb{E} \left[ \left| X_{r}^{t,x} - X_{r}^{t',x'} \right|^2 \right] \right) dr
\]

\[
\leq c \mathbb{E} \left[ \int_{t}^{s} \left( 1 + \mathbb{E}[|X_{r}^{t,x}|^2] \right) dr \right] = c(s - t_0)(1 + |x|),
\]

(6.8)
where we have used Gronwall’s lemma and result (i). After applying the sup operator to both sides of (6.8) we can readily deduce the result.

The proof of the comparison principle is an adaptation of the standard comparison theorem result, indeed, we prove Theorem 5.7 by making the necessary adjustments to existing comparison theorem results as given in for example, [10].

To prove Theorem 5.7, we first require the following definition and result:

**Definition A.4.** Let \( \psi \in C([t, \tau_S]; \mathbb{R}^p) \) be a lower semicontinuous function, then the parabolic superjet of \( \psi \) at the point \( (t, x) \in [t, \tau_S] \times \mathbb{R}^p \) which we denote by \( J^{(2,-)}(t, x) \) is the set of triples \( (M, r, q) \in S(p) \times \mathbb{R} \times \mathbb{R}^p \) s.t.:

\[
\psi(s, y) \geq \psi(s, x) + r(s-t) + \langle q, y - x \rangle + \frac{1}{2} (M(y - x), y - x) + O(|s-t| + |y - x|^2)
\]

as \( s \to t \) or \( s \downarrow t \) when \( t = 0 \) and \( y \to x \). We can analogously define the parabolic superjet of \( \psi \) at the point \( (t, x) \in [t, \tau_S] \times \mathbb{R}^p \) which we denote by \( J^{(2,+)}(t, x) \) by the following:

\[
J^{(2,+)}(t, x) = -J^{(2,-)}(t, x).
\]

Let us also introduce the following notation the convenience of which will be immediate: suppose \( \Lambda : S(p) \times \mathbb{R}^p \times C([t, \tau_S]; \mathbb{R}^p) \times [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R} \) then we define \( \Lambda \) by:

\[
\Lambda(M, r, \psi, m, q) := m - \sum_{i=1}^p \mu_i(m, q)v_i + \frac{1}{2} \sum_{i,j=1}^p (\sigma \cdot \sigma^T)_{ij}(q) M_{ij}
\]

\[
+ \sum_{j=1}^q \int_{\mathbb{R}^p} \psi(m, q + \gamma^{(j)}(m, q, z_j)) - \psi(m, q) - r \cdot \gamma^{(j)}(m, q, z_j) \nu_j(dz_j) + f(m, q).
\]

We note that using definition A.4 we can obtain the following result - the proof of which is standard and therefore omitted:

**Lemma A.5.** Let \( \psi \in C([t, \tau_S]; \mathbb{R}^p) \) be a lower (resp., upper) semicontinuous function, then \( \psi \) is a viscosity supersolution (resp., subsolution) to the HJBI equation (3.1) iff \( \forall (t, x) \in [t, \tau_S] \times \mathbb{R}^p \) and \( \forall (M, r) \in J^{(2,-)}(t, x) \) (resp., \( J^{(2,+)}(t, x) \)) we have that:

\[
\max\{\min[-\Lambda(M, r, \psi, m, x), V(t, x) - G(x)], V(t, x) - MV(t, x)\} \geq 0 \quad \text{(resp.,} \leq 0),
\]

\( \forall x \in \mathbb{R}^p \), we have that:

\[
\max\{V(\tau_S, x) - G(x), V(\tau_S, x) - MV(\tau_S, x)\} \geq 0 \quad \text{(resp.,} \leq 0).
\]

Having stated the above results, we can now prove Theorem 5.7:

**Proof of Theorem 5.7.**

We prove the comparison principle using the standard technique as introduced in [11] - namely we prove the result by contradiction. Suppose that the functions \( V \) and \( U \) are a viscosity subsolution and supersolution (respectively) to the HJBI equation (3.1), then to prove the theorem we must prove that under assumptions A.1.1. - A.4. we have that \( V \leq U \) on \( [t, \tau_S] \times \mathbb{R}^p \).

Hence, let us firstly assume that \( \forall x \in \mathbb{R}^p \):

\[
V(\tau_S, x) \leq U(\tau_S, x).
\]
Moreover, let us also assume that:

\[(6.14) \quad M := \sup_{[t, \tau_S] \times \mathbb{R}^p} (V - U) > 0.\]

Now by Proposition 4.5 we know that \(V\) and \(U\) are bounded hence for some \(\epsilon, \alpha, \eta > 0\) s.th. \(\forall (t, s, x, y) \in [t, \tau_S]^2 \times \mathbb{R}^2p:\)

\[(6.15) \quad M_{\epsilon, \alpha, \eta} := \max_{(t, s, x, y) \in [t, \tau_S]^2 \times \mathbb{R}^2p} V(t, x) - U(s, y) - \frac{\alpha}{2}(|x|^2 + |y|^2) + \eta t \]

is both a finitely bounded quantity and has some maximum which is achieved by a point (which depends on \((\epsilon, \alpha, \eta)\)) which we shall denote by \((\bar{t}, \bar{s}, \bar{x}, \bar{y}) \in [t, \tau_S]^2 \times \mathbb{R}^2p\). Now since there exists values \((s, y) \in [t, \tau_S] \times \mathbb{R}^p\) s.th. \(M = M_{\epsilon, \alpha, \eta}\), we have that:

\[(6.16) \quad 0 < M \leq M_{\epsilon, \alpha, \eta} = V(\bar{t}, \bar{x}) - U(\bar{s}, \bar{y}) - \frac{(\bar{t} - \bar{s})^2 + (\bar{x} - \bar{y})^2}{2\epsilon} - \frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2) + \eta \bar{t}.\]

Hence,

\[(6.17) \quad \lim_{\epsilon \downarrow 0} \frac{(\bar{t} - \bar{s})^2 + (\bar{x} - \bar{y})^2}{2\epsilon} < V(\bar{t}, \bar{x}) - U(\bar{s}, \bar{y}) - \frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2) + \eta \bar{t}.\]

Now since the RHS is composed of finitely bounded terms and the LHS is non-negative, we readily conclude that \(\lim_{\epsilon \downarrow 0} \frac{(\bar{t} - \bar{s})^2 + (\bar{x} - \bar{y})^2}{2\epsilon} = 0\) and hence we observe that \((\bar{t} - \bar{s})^2 + |\bar{x} - \bar{y}|^2 \to 0\) as \(\epsilon \downarrow 0\).

Moreover, if we denote by \((s_n, y_n), (t_n, x_n) \in [t, \tau_S] \times \mathbb{R}\) and \(\epsilon_n > 0\) a triple of bounded sequences s.th. \((s_n, y_n), (t_n, x_n) \to (t, x)\) as \(\epsilon_n \to 0\) we have that:

\[(6.18) \quad M \leq \lim_{\epsilon, \alpha, \eta \downarrow 0} M_{\epsilon, \alpha, \eta} \]

\[\leq \sup_{\epsilon, \alpha, \eta \downarrow 0} \lim_{(t_n, x_n) \to (t, x)} \sup_{[t, \tau_S] \times \mathbb{R}^p} [V(t_n, x_n) - U(s_n, y_n) - \frac{(t_n - s_n)^2 + |x_n - y_n|^2}{2\epsilon} - \frac{\alpha}{2}(|x_n|^2 + |y_n|^2) + \eta t_n] \]

\[= \sup_{(t_n, x_n) \to (t, x)} \sup_{[t, \tau_S] \times \mathbb{R}^p} \lim_{\epsilon, \alpha, \eta \downarrow 0} [V(t_n, x_n) - U(s_n, y_n) - \frac{(t_n - s_n)^2 + |x_n - y_n|^2}{2\epsilon} - \frac{\alpha}{2}(|x_n|^2 + |y_n|^2) + \eta t_n] \]

\[= \sup_{[t, \tau_S] \times \mathbb{R}^p} [V(t, x) - U(t, x)] = M.\]

We therefore readily deduce that:

\[(6.19) \quad p^n_t - p^n_0 = \partial_n \psi_n(t_n, x_n, y_n) = 2(t_n - t_0),\]

\[(6.20) \quad q^n_t = D_x \psi_n(t_n, x_n, y_n),\]

\[(6.21) \quad q^n_0 = D_y \psi_n(t_n, x_n, y_n).\]
and
\[
\begin{pmatrix}
M_n & 0 \\
0 & -N_n
\end{pmatrix} \leq A_n + \frac{1}{2} A_n^2
\]
where
\[
A_n := D_{x,y}^2 \psi_n(t_n, x_n, y_n)
\]
Now we note that by the viscosity subsolution property of \(V\) we have that:
\[
(6.22) \quad V(t_n, x_n) - p_n^V - \langle \mu(t_n, x_n), q_n^V \rangle - \frac{1}{2} tr(\sigma \cdot \sigma'(t_n, x_n) M_n) - f(t_n, x_n) \leq 0,
\]
And similarly, by the viscosity supersolution property of \(U\) we have that:
\[
(6.23) \quad U(t_n, y_n) - p_n^U - \langle \mu(t_n, y_n), q_n^U \rangle - \frac{1}{2} tr(\sigma \cdot \sigma'(t_n, y_n) N_n) - f(t_n, y_n) \geq 0.
\]
Now subtracting (6.23) from (6.22) yields the following:
\[
(6.24) \quad V(t_n, x_n) - U(t_n, y_n)
\]
\[
\leq p_n^V - p_n^U + \langle \mu(t_n, x_n), q_n^V \rangle - \langle \mu(t_n, y_n), q_n^U \rangle + \frac{1}{2} tr(\sigma \cdot \sigma'(t_n, x_n) M_n)
\]
\[
- \frac{1}{2} tr(\sigma \cdot \sigma'(t_n, y_n) N_n) + f(t_n, x_n) - f(t_n, y_n) \leq 0.
\]
We now use the fact that \((s_n, y_n), (t_n, x_n) \to (t, x)\) from which we now observe the following limits as \(n \to \infty\):
\[
(6.25) \quad \lim_{n \to \infty} [p_n^V - p_n^U] = \lim_{n \to \infty} [t_n - t_0] = 0,
\]
and for some \(c > 0:\)
\[
(6.26) \quad \lim_{n \to \infty} \langle \mu(t_n, x_n), q_n^V \rangle - \langle \mu(t_n, y_n), q_n^U \rangle \leq c \lim_{n \to \infty} |x_n - y_n| = 0,
\]
using the Lipschitzianity of \(\mu\). Lastly we observe, using that:
\[
(6.27) \quad \frac{1}{2} \lim_{n \to \infty} [tr(\sigma \cdot \sigma'(t_n, x_n) M_n) - tr(\sigma \cdot \sigma'(t_n, y_n) N_n)] = 0.
\]
Hence, putting (6.25) - (6.27) together with (6.24) yields a contradiction since we observe that:
\[
(6.28) \quad \lim_{n \to \infty} [V(t_n, x_n) - U(t_n, y_n)] \leq 0,
\]
which is a contradiction to (6.14).

7. Conclusion. Using standard assumptions, we proved that stochastic differential games involving an impulse controller and a stopper with diffusion system dynamics admit a value. We also proved that the value of the game is a viscosity solution to a HJBI equation which is represented by a double obstacle quasi-integro-variational inequality.

The game studied investigated in this paper is one in which the payoff structure is zero-sum; an interesting question which arises naturally is the existence and characterisation of stable equilibria for the non-zero-sum equivalent of the game.

Expectedly, arguments similar to that given in [7] (wherein the Nash equilibrium payoffs and controls for a stochastic differential game with absolutely continuous controls were characterised) can be invoked without laborious adaptation.
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