UNCERTAINTY PRINCIPLES FOR THE FOURIER TRANSFORMS IN QUANTUM CALCULUS

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ABSTRACT. Some properties of the $q$-Fourier-sine transform are studied and $q$-analogues of the Heisenberg uncertainty principle is derived for the $q$-Fourier-cosine transform studied in [5] and for the $q$-Fourier-sine transform.

1. Introduction

One of the basic principles in classical Fourier analysis is the impossibility to find a function $f$ being arbitrarily well localized together with its Fourier transform $\hat{f}$. There are many ways to get this statement precise. The most famous of them is the so called Heisenberg uncertainty principle, a consequence of Cauchy-Schwarz’s inequality which states that for $f \in L^2(\mathbb{R})$,

$$\left( \int_{-\infty}^{\infty} x^2 \ | f(x) |^2 \ dx \right) \left( \int_{-\infty}^{\infty} \lambda^2 \ | \hat{f}(\lambda) |^2 \ d\lambda \right) \geq \frac{1}{4} \left( \int_{-\infty}^{\infty} | f(x) |^2 \ dx \right)^2$$

with equality only if $f(x)$ is almost everywhere equal to a constant multiple of $e^{-px^2}$ for some $p > 0$. Here

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\lambda}dx.$$

In this paper we shall prove that similarly to the classical theory, a nonzero function and its $q$-Fourier ($q$-Fourier-cosine and $q$-Fourier-sine) transform cannot both be sharply localized. For this purpose we will prove a $q$-analogue of the Heisenberg uncertainty principle. This paper is organized as follows: in Section 2, we present some preliminaries results and notations that will be useful in the sequel. In Section 3, we study some $q$-harmonic results and state $q$-analogues of the Heisenberg uncertainty principle.

2. Notations and Preliminaries

Throughout this paper, we will fix $q \in [0, 1)$ such that $\frac{\text{Log}(1 - q)}{\text{Log}(q)} \in \mathbb{Z}$. We recall some usual notions and notations used in the $q$-theory (see [8] and [12]). We refer to the book by G. Gasper and M. Rahman [8], for the definitions, notations and properties of the $q$-shifted factorials and the $q$-hypergeometric functions.
We note $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$ and $\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$.

We also denote
\[ [x]_q = \frac{1 - q^x}{1 - q}, \quad x \in \mathbb{C} \]
and
\[ [n]_q! = \frac{(q; q)_n}{(1 - q)^n}, \quad n \in \mathbb{N}. \]

The $q$-derivatives $D_q f$ and $D_q^+ f$ of a function $f$ are given by
\[ (D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad (D_q^+ f)(x) = \frac{f(q^{-1}x) - f(x)}{(1 - q)x}, \quad \text{if } x \neq 0, \]
\[ (D_q f)(0) = f'(0) \text{ and } (D_q^+ f)(0) = q^{-1}f'(0) \text{ provided } f'(0) \text{ exists.} \]

The $q$-Jackson integrals from 0 to $a$ and from 0 to $\infty$ are defined by (see [10])
\[ \int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} f(aq^n)q^n, \]
\[ \int_0^\infty f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n)q^n, \]
provided the sums converge absolutely.

The $q$-Jackson integral in a generic interval $[a, b]$ is given by (see [10])
\[ \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \]

The improper integral is defined in the following way (see [14])
\[ \int_0^\infty f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f\left(\frac{A}{q^n}\right) \frac{A^n}{q^n}. \]

We remark that for $n \in \mathbb{Z}$, we have
\[ \int_0^{\infty} f(x) d_q x = \int_0^{\infty} f(x) d_q x. \]

The $q$-integration by parts is given for suitable functions $f$ and $g$ by
\[ \int_a^b g(x)D_q f(x) d_q x = f(b)g(b) - f(a)g(a) - \int_a^b f(qx)D_q g(x) d_q x. \]

**Remark** A second $q$-analogue of the integration by parts theorem is given for a suitable function $f$ and $g$ by (see [13])
\[ \int_a^b g(x)D_q f(x) d_q x = f(b)g(q^{-1}b) - f(a)g(q^{-1}a) - \int_a^b f(x)D_q^+ g(x) d_q x. \]
Proposition 1. The $q$-analogue of the integration theorem by change of variable for $u(x) = \alpha x^\beta$, $\alpha \in \mathbb{C}$ and $\beta > 0$ is as follows

\begin{equation}
\int_{u(a)}^{u(b)} f(u) du = \int_{a}^{b} f(u(x)) D_{q^\beta} u(x) d_{q^\beta} x.
\end{equation}

Jackson [10] defined the $q$-analogue of the Gamma function by

\begin{equation}
\Gamma_q(x) = \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}} (1 - q)^{1-x}, \quad x \neq 0, -1, -2, \ldots.
\end{equation}

It is well known that it satisfies

\begin{equation}
\Gamma_q(x + 1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1 \quad \text{and} \quad \lim_{q \to 1^{-}} \Gamma_q(x) = \Gamma(x), \quad \Re(x) > 0.
\end{equation}

The third $q$-Bessel function (see [9, 16]) is given and denoted by M. E. H. Ismail as

\begin{equation}
J_{\alpha}(z; q^2) = \frac{z^\alpha}{(1 - q^2)^{\alpha}} \frac{\varphi_1(0; q^{2\alpha+2}; q^2, q^2 z^2)}{\Gamma_q(\alpha + 1)}
\end{equation}

It verifies for $\alpha, \beta$ reals (see [16])

\begin{equation}
J_{\alpha}(q^\beta; q^2) = J_{\beta}(q^\alpha; q^2),
\end{equation}

and we have the following orthogonality relation (see [16])

\begin{equation}
\sum_{k=-\infty}^{\infty} q^{2k} q^{n+m} J_{n+k}(x; q^2) J_{m+k}(x; q^2) = \delta_{n,m}, \quad |x| < q^{-1}, \quad n, m \in \mathbb{Z}.
\end{equation}

Moreover, if $\alpha > -1$, we have (see [4, 16]),

\begin{equation}
\forall x \in \mathbb{R}_{q,+}, \quad |J_{\alpha}(x; q^2)| \leq \frac{(-q; q^2)_{\infty} (-q^{2\alpha+1}; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left\{ \begin{array}{ll}
\frac{1}{q^{(\log q)^2}} & \text{if } x \leq 1 \\
q^{(\log q)^2} & \text{if } x > 1.
\end{array} \right.
\end{equation}

and

\begin{equation}
\forall \nu \in \mathbb{R}, \quad J_{\alpha}(x; q^2) = o(x^{-\nu}) \quad \text{as } x \to +\infty \quad \text{in } \mathbb{R}_{q,+}.
\end{equation}

The $q$-trigonometric functions $q$-cosine and $q$-sine are defined by (see [5]):

\begin{equation}
\cos(x; q^2) = \varphi_1\left(0, q; q^2, (1 - q)^2 x^2 \right) = \sum_{n=0}^{\infty} (-1)^n q^n (n-1) \frac{x^{2n}}{[2n]_q!}
\end{equation}

and

\begin{equation}
\sin(x; q^2) = x \varphi_1\left(0, q^3; q^2, (1 - q)^2 x^2 \right) = \sum_{n=0}^{\infty} (-1)^n q^n (n-1) \frac{x^{2n+1}}{[2n+1]_q!}.
\end{equation}

Note that we have the relations

\begin{equation}
\cos(x; q^2) = \frac{\Gamma_q(\frac{1}{2})}{q (1 + q^{-1})^{\frac{1}{2}}} x^{\frac{1}{2}} J_{\frac{1}{2}}\left(1 - \frac{q}{x}; q^2 \right),
\end{equation}
\begin{equation}
\sin(x; q^2) = \frac{\Gamma_{q^2}(\frac{1}{2})}{(1 + q^{-1})^{\frac{1}{2}}} x^{\frac{1}{2}} J_{\frac{1}{2}}\left(\frac{1 - q}{q} x; q^2\right)
\end{equation}

and they verify

\begin{equation}
D_q \cos(x; q^2) = -\frac{1}{q} \sin(qx; q^2)
\end{equation}

and

\begin{equation}
D_q \sin(x; q^2) = \cos(x; q^2).
\end{equation}

3. q-UNCERTAINLY PRINCIPLE

We define the $q$-Fourier-cosine and the $q$-Fourier-sine as (see [5] and [16])

\begin{equation}
\mathcal{F}_q(f)(x) = c_q \int_0^{\infty} f(t) \cos(xt; q^2) dq t
\end{equation}

and

\begin{equation}
q\mathcal{F}(f)(x) = c_q \int_0^{\infty} f(t) \sin(xt; q^2) dq t,
\end{equation}

where

\begin{equation}
c_q = \frac{(1 + q^{-1})^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})}.
\end{equation}

It was shown in [2] that $\mathcal{F}_q$ is an isomorphism of $L^2_q(\mathbb{R}_q,+)$ and we have $\mathcal{F}_q^{-1} = \mathcal{F}_q$ and the following Plancherel formula:

\[
\| \mathcal{F}_q(f) \|_{q,2} = \| f \|_{q,2}, \quad f \in L^2_q(\mathbb{R}_q,+),
\]

where $L^2_q(\mathbb{R}_q,+)$ is the set of functions defined on $\mathbb{R}_q$ such that \( \int_0^{\infty} | f(t) |^2 dq t < \infty \), equipped with the norm $\| f \|_{q,n} = \left( \int_0^{\infty} | f(t) |^n dq t \right)^{\frac{1}{n}}$.

The $q$-Fourier-sine verifies the following properties.

**Proposition 2.** For $f \in L^1_q(\mathbb{R}_q,+)$, we have

1) $\forall \lambda \in \mathbb{R}_q, | q\mathcal{F}(f)(\lambda) | \leq \frac{(1 + q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)(q; q)^{\frac{1}{2}}_{\infty}} \cdot \| f \|_{q,1};$

2) $\lim_{\lambda \to \infty} q\mathcal{F}(f)(\lambda) = 0.$

**Proof.** Using the inequality (see [5])

\[
| \sin(x; q^2) | \leq \frac{1}{(q; q)^{\frac{1}{2}}_{\infty}}, \quad x \in \mathbb{R}_q,
\]
we obtain
\[ | f(t) \sin(\lambda t; q^2) | \leq \frac{1}{(q; q^2)^2} | f(t) |, \; \lambda, t \in \mathbb{R}_q. \]

Which gives, after integration, the first inequality and together with the Lebesgue theorem it gives the limit. \(\blacksquare\)

In the following proposition, we shall try to prove a Plancherel formula for the \(q\)-Fourier-sine transform. We begin by the following useful result:

**Lemma 1.** For all \(x, y \in \mathbb{R}_{q,+}\), we have
\[ \sqrt{xy} \int_0^\infty \sin(xt; q^2) \sin(yt; q^2) dq t = \frac{q^2 \Gamma^2_q\left(\frac{1}{2}\right)}{(1 + q^{-1})(1 - q)} \delta_{x,y}. \]

**Proof.** Let \(x = q^n\) and \(y = q^m\), \(m, n \in \mathbb{Z}\) be two elements of \(\mathbb{R}_{q,+}\). The orthogonality relation \((17)\) leads to
\[ \sum_{k=-\infty}^{\infty} q^{2k} q^{n+m} J_{n+k}(q^{1/2}; q^2) J_{m+k}(q^{1/2}; q^2) = \delta_{n,m}, \]
which is equivalent to
\[ \sum_{k=-\infty}^{\infty} q^{2k} q^{n+m} J_{\frac{1}{2}}(q^{n+k}; q^2) J_{\frac{1}{2}}(q^{m+k}; q^2) = \delta_{n,m}. \]

Using the relation \((23)\), we obtain
\[ \sum_{k=-\infty}^{\infty} \frac{(1 - q)(1 + q^{-1})}{q q^2(\frac{1}{2})^2} q^{n+m} q^k \sin \left( \frac{q}{1 - q} q^{n+k}; q^2 \right) \sin \left( \frac{q}{1 - q} q^{m+k}; q^2 \right) = \delta_{n,m}. \]

Then
\[ \frac{1 + q^{-1}}{q q^2(\frac{1}{2})^2} \int_0^\infty \sin \left( \frac{q}{1 - q} q^n t; q^2 \right) \sin \left( \frac{q}{1 - q} q^m t; q^2 \right) dq t = \delta_{n,m}. \]

The change of variable \(u = \frac{q}{1-q} t\) gives
\[ \frac{(1 + q^{-1})(1 - q)}{q^2 q^2(\frac{1}{2})^2} \int_0^\infty \sin \left( q^n t; q^2 \right) \sin \left( q^m t; q^2 \right) dq t = \delta_{n,m}. \]

Thus
\[ \sqrt{xy} \int_0^\infty \sin \left( xt; q^2 \right) \sin \left( yt; q^2 \right) dq t = \frac{q^2 \Gamma^2_q\left(\frac{1}{2}\right)}{(1 + q^{-1})(1 - q)} \delta_{x,y}. \] \(\blacksquare\)
Proposition 3. 1) For $f \in L^2_q(\mathbb{R}_q,+)$, we have $q\mathcal{F}(f) \in L^2_q(\mathbb{R}_q,+)$ and
$$
\|q\mathcal{F}(f)\|_{q,2} = q \|f\|_{q,2}.
$$
2) $q\mathcal{F}$ is an isomorphism of $L^2_q(\mathbb{R}_q,+)$ and $(q\mathcal{F})^{-1} = \frac{1}{q^2} q\mathcal{F}$.

Proof. 1) For $x \in \mathbb{R}_q,+$, we have
$$
q\mathcal{F}(f)(x) = c_q \int_0^\infty f(t) \sin(qt; q^2) dq t
= c_q (1 - q) \sum_{n=-\infty}^\infty q^n f(q^n) \sin(q^n; q^2).
$$
So, for $x \in \mathbb{R}_q,+$,
$$
(q\mathcal{F}(f)(x))^2 = c_q^2 (1 - q)^2 \sum_{n=-\infty}^\infty q^{2n} f^2(q^n) \sin^2(q^n; q^2)
+ c_q^2 (1 - q)^2 \sum_{n,m=-\infty, n \neq m}^\infty q^{m+n} f(q^m) f(q^n) \sin(q^m; q^2) \sin(q^n; q^2).
$$
By integration, we obtain
$$
\int_0^\infty (q\mathcal{F}(f)(x))^2 dq x = c_q^2 (1 - q)^2 \int_0^\infty \sum_{n=-\infty}^\infty q^{2n} f^2(q^n) \sin^2(q^n; q^2) dq x
+ c_q^2 (1 - q)^2 \int_0^\infty \sum_{n,m=-\infty, n \neq m}^\infty q^{m+n} f(q^m) f(q^n) \sin(q^m; q^2) \sin(q^n; q^2) dq x.
$$
The previous lemma, the relation [18] and Fubini’s theorem imply that we can exchange the integral and the sum signs and we have:
$$
\int_0^\infty (q\mathcal{F}(f)(x))^2 dq x = c_q^2 (1 - q)^2 \sum_{n=-\infty}^\infty q^n f^2(q^n) q^n \int_0^\infty \sin^2(q^n; q^2) dq x
= c_q^2 (1 - q)^2 \frac{q^2 \Gamma_q^2(\frac{1}{2})}{(1 + q^{-1})(1 - q)} \sum_{n=-\infty}^\infty q^n f^2(q^n)
= q^2 \int_0^\infty f^2(t) dt t.
$$
2) Using the same arguments, we can see that for $y \in \mathbb{R}_q,+$, we have
$$
\int_0^\infty q\mathcal{F}(f)(x) \sin(xy; q^2) dq x = c_q (1 - q) \sum_{n=-\infty}^\infty q^n f(q^n) \int_0^\infty \sin(q^n; q^2) \sin(xy; q^2) dq x
= \frac{q^2}{c_q} f(y).
$$
The following result gives a relation between the $q$-Fourier-cosine and the $q$-Fourier-
sine.

**Lemma 2.** For $f \in L^1_q(\mathbb{R}_q, +)$ such that $D_qf \in L^1_q(\mathbb{R}_q, +)$, we have:

1) 

\[(30) \quad q\mathcal{F}(D_qf)(\lambda) = -\frac{\lambda}{q} q\mathcal{F}(f) \left( \frac{\lambda}{q} \right), \quad \lambda \in \mathbb{R}_q, +.\]

2) Additionally, if $f(0) = 0$ then

\[(31) \quad \mathcal{F}_q(D_qf)(\lambda) = \frac{\lambda}{q^2} q\mathcal{F}(f)(\lambda), \quad \lambda \in \mathbb{R}_q, +.\]

**Proof.** Since $f$ is in $L^1_q(\mathbb{R}_q, +)$ then for all $\lambda \in \mathbb{R}_q, +$, $f(t) \sin(\lambda t; q^2)$ and $f(t) \sin(\lambda t; q^2)$ tend to 0 as $t$ tends to $\infty$. So by $q$-integrations by parts, we obtain

\[
q\mathcal{F}(D_qf)(\lambda) = c_q \int_0^\infty D_qf(t) \sin(\lambda t; q^2) d_qt
= -c_q \int_0^\infty \lambda ft(\lambda; q^2) \cos(\lambda t; q^2) d_qt
= -\frac{\lambda}{q} q\mathcal{F}(f) \left( \frac{\lambda}{q} \right)
\]

and

\[
\mathcal{F}_q(D_qf)(\lambda) = c_q \int_0^\infty D_qf(t) \cos(\lambda t; q^2) d_qt
= \frac{c_q}{q} \int_0^\infty \lambda ft(\lambda; q^2) \sin(\lambda t; q^2) d_qt
= \frac{\lambda}{q^2} q\mathcal{F}(f)(\lambda)
\]

Now, we are in a situation to state a $q$-analogues of the Heisnberg uncertainty principle.

**Theorem 1.** Let $f$ be in $L^2_q(\mathbb{R}_q, +)$ such that $D_qf$ is in $L^2_q(\mathbb{R}_q, +)$. Then

\[(32) \quad \left( \int_0^\infty t^2 \mid f(t) \mid^2 d_qt \right)^{1/2} \left( \int_0^\infty x^2 \mid \mathcal{F}_q(f)(x) \mid^2 d_qx \right)^{1/2} \geq \frac{q}{q^2 + 1} \| f \|_{q, 2}^2 \]
Proof. First, using the previous lemma and Proposition \[3\] we have
\[
\int_0^\infty \lambda^2 |F_q(f)(\lambda)|^2 \, d\lambda = \frac{1}{q} \int_0^\infty \frac{\lambda}{q} F_q(f)(\frac{\lambda}{q})^2 \, d\lambda
\]
\[= \frac{1}{q} \int_0^\infty |qF(D_qf)(\lambda)|^2 \, d\lambda
\]
\[= q \int_0^\infty |D_qf(t)|^2 \, dt.
\]
The relation
\[D_q(f\overline{f})(t) = D_qf(t)\overline{f} + f(qt)D_q\overline{f}(t)
\]
leads to
\[
\left| \int_0^\infty tD_q(f\overline{f})(t) \, dt \right| \leq \int_0^\infty \left| tD_qf(t)\overline{f}(t) \right| \, dt + \int_0^\infty \left| tf(qt)D_q\overline{f}(t) \right| \, dt
\]
\[\leq \left( \int_0^\infty |\overline{f}(t)|^2 \, dt \right)^{1/2} \left( \int_0^\infty |D_qf(t)|^2 \, dt \right)^{1/2}
\]
\[+ \left( \int_0^\infty |f(qt)|^2 \, dt \right)^{1/2} \left( \int_0^\infty |D_q\overline{f}|^2 \, dt \right)^{1/2}
\]
\[= \left( \frac{q^{3/2} + 1}{q^2} \right) \left( \int_0^\infty t^2 |f(t)|^2 \, dt \right)^{1/2} \left( \int_0^\infty x^2 |F_q(f)(x)|^2 \, dx \right)^{1/2}.
\]
On the other hand, since \(f\) is in \(L_q^2(\mathbb{R}_{q,+})\) then \(|f(t)|^2\) tends to 0 as \(t\) tends to \(\infty\) in \(\mathbb{R}_{q,+}\). So by \(q\)-integration by parts, we obtain
\[
\int_0^\infty tD_q(f\overline{f})(t) \, dt = - \int_0^\infty |f(qt)|^2 \, dt = -1/q \int_0^\infty |f(t)|^2 \, dt.
\]
Finally
\[
\left( \int_0^\infty t^2 |f(t)|^2 \, dt \right)^{1/2} \left( \int_0^\infty x^2 |F_q(f)(x)|^2 \, dx \right)^{1/2} \geq \frac{q}{q^2 + 1} \|f\|^2_{q,2}.
\]
Similarly, we have an uncertainty principle for the \(q\)-Fourier-sine transform.

**Theorem 2.** Let \(f\) be in \(L_q^2(\mathbb{R}_{q,+})\) such that \(D_qf\) is in \(L_q^2(\mathbb{R}_{q,+})\) and \(f(0) = 0\). Then
\[
(33) \left( \int_0^\infty t^2 |f(t)|^2 \, dt \right)^{1/2} \left( \int_0^\infty \lambda^2 |F_q(f)(\lambda)|^2 \, d\lambda \right)^{1/2} \geq \frac{q}{q^2 + 1} \|f\|^2_{q,2}.
\]
**Proof.** Owing to Lemma \[2\] and the Plancherel formula, we have
\[
\int_0^\infty \lambda^2 |qF(f)(\lambda)|^2 \, d\lambda = q^4 \int_0^\infty |F_q(D_qf)(\lambda)|^2 \, d\lambda
\]
\[= q^4 \int_0^\infty |D_qf(t)|^2 \, dt.
\]
Using the same steps of the previous proof, we have
\[
\frac{1}{q} \int_0^\infty |f(t)|^2 dt \leq \left( \int_0^\infty |\hat{f}(t)|^2 dt \right)^{1/2} \left( \int_0^\infty |D_q f(t)|^2 dt \right)^{1/2} + \left( \int_0^\infty |tf(t)|^2 dt \right)^{1/2} \left( \int_0^\infty |D_q \hat{f}(t)|^2 dt \right)^{1/2}
\]
\[
= \frac{1 + q^{-3/2}}{q^2} \left( \int_0^\infty t^2 |f(t)|^2 dt \right)^{1/2} \left( \int_0^\infty \lambda^2 |\mathcal{F}(f)(\lambda)|^2 d\lambda \right)^{1/2}.
\]

Remark. Note that when \(q\) tends to 1, the inequalities (32) and (33) tend at least formally to the corresponding classical ones.

4. Uncertainty Principle in Hilbert Space

For \(A\) and \(B\) operators on a Hilbert space \(H\), with domains \(D(A)\) and \(D(B)\) respectively, we note \([A, B] = AB - BA\) and \([A, B]_q = qAB - BA\). The commutator \([A, B]\) and the \(q-\) commutator \([A, B]_q\) are both defined on \(D[A, B] = D(AB) \cap D(BA)\), where \(D(AB) = \{u \in D(B) : Bu \in D(A)\}\) and likewise for \(D(BA)\). Let us begin by the following well-known result:

Lemma 3. (Cauchy-Schwarz’s inequality) For \(x, y\) in the Hilbert space \(H\) the following inequality
\[
|\langle x, y \rangle| \leq \|x\| \|y\|
\]
holds.

Using this lemma, one can prove easily the following proposition, which gives the uncertainty principle for normal operators.

Proposition 4. For \(s \geq 0\), note \([A, B]_s = sAB - BA\). If \(A\) and \(B\) are operators on the Hilbert space \(H\), then for all \(u \in D[A, B]\), we have
\[
\|Au\| \|Bu\| + s \|Bu\| \|A^* u\| \geq \langle [A, B]_s u, u \rangle.
\]

In addition, if \(A\) and \(B\) are normal on \(H\), we obtain
\[
\|Au\| \|Bu\| \geq \frac{1}{1 + s} |\langle [A, B]_s u, u \rangle|.
\]

Proof. For \(s \geq 0\), using the lemma we have , and \(u \in D([A, B])\)
\[
\langle (sAB - BA)u, u \rangle = |s \langle ABu, u \rangle - \langle BAu, u \rangle|
\]
\[
\leq |s \langle Bu, A^* u \rangle - \langle Au, B^* u \rangle|
\]
\[
\leq s \|Bu\| \|A^* u\| + \|Au\| \|B^* u\|.
\]
Additionally, if the operators $A$ and $B$ are normal, we obtain
\[ | < (sAB - BA)u, u > | \leq (1 + s) \| Au \| \| Bu \| . \]
\[ \square \]
Now, take $H = \{ f \in L^2_q(\mathbb{R}_+^q) : f(0) = 0 \}$, $A : f(x) \mapsto xf(x)$, $B : f(x) \mapsto iD_q f(x)$ and $C : f(x) \mapsto iD_q^x f(x)$. $A$ is self-adjoint on $H$, $B^* = C$ (according to (11)) and
\[ D[A, B] = D[A, C] = \{ f \in H : xf, D_q f \in L^2_q(\mathbb{R}_+^q) \} . \]
For all $f \in D[A, B]$, we have
\[ [A, B] f(x) = -i f(x) \]
and
\[ [A, C] f(x) = -iq^{-1} f(q^{-1}x) . \]
So, for all $f \in D[A, B]$, we have
\[ \| f \|_{q,2}^2 \leq ( \| D_q f \|_{q,2} + \| D_q f \|_{q,2} ) \left( \int_0^\infty x^2 |f(x)|^2 d_q x \right)^{1/2} \]
\[ = \frac{1 + q^{3/2}}{q^{1/2}} \| D_q f \|_{q,2} \left( \int_0^\infty x^2 |f(x)|^2 d_q x \right)^{1/2} \]
and
\[ \left| \int_0^\infty f(x) \overline{f(qx)} d_q x \right| \leq ( \| D_q f \|_{q,2} + \| D_q f \|_{q,2} ) \left( \int_0^\infty x^2 |f(x)|^2 d_q x \right)^{1/2} \]
\[ = (1 + \frac{1}{q^{1/2}}) \| D_q f \|_{q,2} \left( \int_0^\infty x^2 |f(x)|^2 d_q x \right)^{1/2} . \]
\[ \text{Remark. Using the fact that } \| D_q f \|_{q,2} = \frac{1}{\sqrt{q}} \left( \int_0^\infty \lambda^2 |F_q(f)(\lambda)|^2 d_q \lambda \right)^{1/2} , \text{ one can see that } (37) \text{ is exactly (32).} \]

REFERENCES

[1] N. G. de Bruijn, *Uncertainty principle in Fourier analysis*, Inequalities (O. Shisha, ed.), Academic Press, New York, (1967), 55-71.
[2] L. Dhaoudi, J. El Kamel and A. Fitouhi, *Positivity of $q$-even translation and inequalities in $q$-Fourier analysis*, to appear.
[3] A. Fitouhi, N. Bettaibi and K. Brahimi, *The Mellin transform in Quantum Calculus*, to appear in Constructive Approximation.
[4] A. Fitouhi, K. Brahimi and N. Bettaibi, *Asymptotic approximations in Quantum Calculus*, Journal of Nonlinear Mathematical Physics, 12, Nr 4, (2005), 586-606.
[5] A. Fitouhi and F. Bouzeffour, *q-Cosine Fourier Transform and $q$-Heat Equation*, to appear in Ramanujan Journal.
[6] A. Fitouhi, M. M. Hamza and F. Bouzeffour, *The $q$-$J_\alpha$ Bessel function*, J. Approx. Theory, 115, (2002), 144-166.
[7] G. B. Folland and A. Sitaram, *The Uncertainty Principle: A Mathematical Survey*, The Journal of Fourier Analysis and Applications, V 3, Nr 3, (1997), 207-238.

[8] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its application, Vol 35 Cambridge Univ. Press, Cambridge, UK, 1990.

[9] M. E. H. Ismail, *The zeros of basic Bessel functions, the Function $J_{
u+ax}(x)$, and associated orthogonal polynomials*, J. Math. Anal. Appl. 86 (1982), 1-19.

[10] F. H. Jackson, *On a q-Definite Integrals*. Quarterly Journal of Pure and Applied Mathematics 41, 1910, 193-203.

[11] J. P. Kahane and P. G. Lemarié-Rieusset, *Séries de Fourier et ondelettes*, Cassini, Paris, 1998.

[12] V. G. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, New York, (2002).

[13] T. H. Koornwinder, *q-Special Functions, a Tutorial*, in Deformation theory and quantum groups with applications to mathematical physics, M. Gerstenhaber and J. Stasheff (eds), Contemp. Math. 134, Amer. Math. Soc., (1992).

[14] T. H. Koornwinder, *Special Functions and q-Commuting Variables*, in Special Functions, q-Series and related Topics, M. E. H. Ismail, D. R. Masson and M. Rahman (eds), Fields Institute Communications 14, American Mathematical Society, (1997), pp. 131–166; arXiv:q-alg/9608008.

[15] T. H. Koornwinder, *The continuous Wavelet Transform*, Series in Approximations and decompositions, Vol.1, Wavelets: An Elementary Treatment of Theory and Applications. Edited by T. H. Koornwinder, World Scientific, 1993, 27–48.

[16] T. H. Koornwinder and R. F. Swarttouw, *On q-analogues of the Fourier and Hankel transforms*, Trans. Amer. Math. Soc. 333, 1992, 445-461.

[17] M. Rosler and M. Voit, *An uncertainty principle for Hankel transforms*, Proc. of Amer. Math. Soc., V 127, Nr 1, (1999), 183-194.

[18] R. S. Strichartz, *Uncertainty Principle in Harmonic Analysis*, Journal of functional analysis 84, (1989), 97-114.

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