Symmetry breaking in noncommutative finite temperature $\lambda\phi^4$ theory with a nonuniform ground state

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Abstract. We consider the CJT effective action at finite temperature for a noncommutative real scalar field theory, with noncommutativity among space and time variables. We study the solutions of a stripe type nonuniform background, which depends on space and time. The analysis in the first approximation shows that such solutions appear in the planar limit, but also under normal anisotropic noncommutativity. Further we show that the transition from the uniform ordered phase to the non uniform one is first order and that the critical temperature depends on the nonuniformity of the ground state.

1. Introduction
Noncommutativity in field theory has received much attention in recent years, as a means to explore high energy effects [1]. An interesting question is the finite temperature behavior and phase transitions. In particular there is a breaking of translational invariance [2], which leads to the appearance of a stripe phase, related to the UV/IR mixing [2, 3, 4]. It is known that the noncommutativity of time with space variables leads to causality problems [5]. However, at finite temperature, in the imaginary time formalism, we could consider full noncommutativity [6], which could be related to nonequilibrium processes. This type of processes has been studied in [7, 8] by means of the effective action of Cornwall, Jackiw and Tomboulis (CJT) [9]. This effective action allows to make selective summations to higher loop graphs [10], which on the other side can be seen as consistent truncations which describe dissipative processes [7]. In [11] we considered the conditions for the existence of a stripe phase, in a scalar field theory with $\lambda\phi^4$ interaction at finite temperature, with noncommutativity among all variables. We included a dependence of the stripe background on the imaginary time. The study non uniform solutions was made by means of a Rayleigh-Ritz variation of the CJT effective action. We obtained that there are consistent solutions in the planar limit, as already found in [12, 2], but also with anisotropic noncommutativity, with a temperature dependent background. Moreover there is a discontinuity from the uniform phase to the non uniform one, which points to a first order transition between these phases, thus confirming the phase transition in four dimensions conjectured in [2], corresponding to a Lifshitz point.
2. CJT effective action
Let us consider an action of scalar fields $I(\Phi)$. The CJT [9] effective action $\Gamma(\phi, G)$ is obtained including a quadratic source term in the generating functional of the Green functions,

$$Z(J, K) = \int D\Phi \exp \left\{ i \left[ I(\Phi) + \int d^4x \Phi(x) J(x) + \frac{1}{2} \int d^4x d^4y \Phi(x) K(x, y) \Phi(y) \right] \right\},$$

and then doing a Legendre transformation of the generating functional of the connected diagrams

$$\Gamma(\phi, G) = W(J, K) - \int d^4x \phi(x) J(x) - \frac{1}{2} \int d^4x d^4y \phi(x) K(x, y) \phi(y) - \frac{1}{2} \int d^4x d^4y G(x, y) K(y, x).$$

Defining

$$\frac{\delta W(J, K)}{\delta J(x)} = \phi(x), \quad \text{and} \quad \frac{\delta W(J, K)}{\delta K(x, y)} = \frac{1}{2} [\phi(x) \phi(y) + G(x, y)],$$

it turns out that $\phi(x)$ is the vacuum expectation value of $\Phi(x)$, and $G(x, y)$ is the connected two point function. The physical solutions are obtained from the stationarity conditions,

$$\frac{\delta}{\delta \phi(x)} \Gamma(\phi, G) = 0, \quad \text{and} \quad \frac{\delta}{\delta G(x, y)} \Gamma(\phi, G) = 0.$$

The resulting effective action is [9]

$$\Gamma(\phi, G) = I(\phi) + i \frac{1}{2} Tr \ln G^{-1} + \frac{i}{2} Tr \left[ \Delta^{-1}(\phi) G \right] + \Gamma_2(\phi, G) - \frac{i}{2} Tr (1),$$

where $\Gamma_2(\phi, G)$ is the 2PI diagram expansion, $iD^{-1}(x - y) = -\left( \partial^\mu \partial_\mu + m^2 \right) \delta^4(x - y)$ and

$$i\Delta^{-1}(x - y) = \frac{\delta^2 I(\phi)}{\delta \phi(x) \delta \phi(y)} = -\left( \partial^\mu \partial_\mu + m^2 \right) \delta^4(x - y) + \frac{\delta^2 I_{\text{int}}(\phi)}{\delta \phi(x) \delta \phi(y)}.$$

Further, for a $\lambda\Phi^4$ action

$$I(\Phi) = \int_{-\infty}^{\infty} d^4x \left( \frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{\lambda}{4!} \Phi^4 \right),$$

$\Gamma_2$ is given to first order in $\lambda$ by [9],

$$\Gamma_2(\phi, G) = -\frac{\lambda}{8} \int_{-\infty}^{\infty} d^4x G^2(x, x).$$

3. Noncommutative finite temperature action
Let us consider the noncommutative action for a real scalar field with a $\lambda\Phi^4$ potential,

$$I(\Phi) = \int_{-\infty}^{\infty} d^4x \left( \frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{\lambda}{4!} \Phi^4 \Phi \Phi \Phi \right).$$

The noncommutative Weyl-Moyal product is given by,

$$\Phi(x) * \Psi(y) = e^{i \frac{\theta_{\mu\nu} \partial^\mu \partial^\nu}{\hbar} \Phi(x) \Psi(y)} \bigg|_{y=x}. $$
We are interested on symmetry breaking of noncommutative finite temperature field theory. Action (9), as well as (7), is symmetric under $\Phi \rightarrow - \Phi$. The imaginary time formulation is obtained from the relativistic field theory by setting the fields periodic in the imaginary time variable $\tau = i t$, $\Phi(\tau + \beta) = \Phi(\tau)$. The imaginary time integration interval is $[0, \beta]$, and $i \int d^3 x \rightarrow \int_0^\beta d\tau \int d^3 x$. The Fourier transform of the fields is then given by

$$
\phi(x) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int d^3 p \frac{e^{i (\omega_n \tau + \vec{p} \cdot \vec{x})}}{(2\pi)^3} \phi_n(p),
$$

where $\omega_n = \frac{2\pi n}{\beta}$ is the Matsubara frequency, see e.g. [14]. Thus the free action entering into the CJT effective action will be,

$$
I_0(\phi) = -\frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int d^3 p \frac{\left(p_n^2 + m^2\right) \phi_n(p) \phi_{-n}(-p)}{(2\pi)^3},
$$

where $p_n \equiv (\omega_n, \vec{p})$, i.e. $p_n^2 = \omega_n^2 + \vec{p}^2$. Further, if $\Theta_{\tau l} = i \theta^{0l}$ and $\Theta_{ij} = \theta^{ij}$, we write

$$
\phi(x) * \phi(x) = e^{i \int \left[ \Theta_{\tau l} \phi(x) \phi(x) \right]} = e^{i \int \left[ \Theta_{\tau l} \phi(x) \phi(x) \right]} \left. \right|_{y=x} = \left. \phi(x) \phi(x) \right|_{y=x}.
$$

Therefore, if $(\Theta_{\tau l})_i = \Theta_{\tau l}, \Theta_k = \epsilon_{ijk} \Theta_{jk}$, for the interaction action we get

$$
I_{int}(\phi) = \frac{\lambda}{4!} (2\pi)^2 \frac{3}{4} \sum_{k=1}^4 \left[ \int d^3 p \frac{\phi_n(p_k) \phi_{-n}(-p_k)}{(2\pi)^3} \right] e^{i \int \left[ \Theta_{\tau l} \phi(x) \phi(x) \right]} \left. \right|_{y=x} = \left. \phi(x) \phi(x) \right|_{y=x},
$$

where $p_n \wedge p_m = \Theta_{\tau l} (\omega_n p_l - \omega_m p_l) + \Theta_{ij} p_l \times p_m$. The noncommutative CJT action can be obtained from (9) and (5), as follows

$$
i \Gamma(\phi, \dot{G}) = i I(\phi) - \frac{1}{2} Tr \ln G^{-1} + \frac{1}{2} Tr \left[ \frac{\delta^2}{\delta \phi \delta \dot{\phi}} (i I(\phi)) \right] + i \Gamma_2(\phi, \dot{G}) + \frac{1}{2} Tr \left(1\right),
$$

which becomes real by the continuation to imaginary time. Thus we get, including the nonplanar two loop terms,

$$
i \Gamma(\phi, \dot{G}) = -\frac{1}{2\beta} \sum_{l} \int d^3 p \frac{\left(p_l^2 + m^2\right) \phi_l(p) \phi_{-l}(-p)}{(2\pi)^3} + \frac{\lambda}{4!} \left(\frac{2\pi}{\beta}\right)^3 \frac{3}{4} \sum_{k=1}^4 \left[ \int d^3 p \frac{\phi_n(p_k) \phi_{-n}(-p_k)}{(2\pi)^3} \right] e^{i \int \left[ \Theta_{\tau l} \phi(x) \phi(x) \right]} \left. \right|_{y=x} = \left. \phi(x) \phi(x) \right|_{y=x}.
$$

4. Nonuniform solutions

In [2], following [15], the ground state field is chosen in a stripe phase, $\phi(x) = A \cos(\vec{Q} \cdot \vec{x})$. Considering the possibility of non closed systems [16], we take a dependence on the imaginary time

$$
\phi(\tau, \vec{x}) = A \cos(\omega_n \tau + \vec{Q} \cdot \vec{x}).
$$
These fields are commutative, i.e. \( \phi(\tau, \vec{x}) \ast \phi(\tau, \vec{x}) = \phi^2(\tau, \vec{x}) \). For simplicity, following [13], we keep \( G(x, y) = G(x - y) \) translational invariant. The Fourier expansion coefficients of (17) are

\[
\phi_l(\vec{p}) = (2\pi)^3 \frac{\beta}{2} A \left[ \delta_{ln} \delta^3(\vec{p} - \vec{Q}) + \delta_{l,-n} \delta^3(\vec{p} + \vec{Q}) \right].
\]

Thus

\[
i \Gamma(\phi, G) = \left\{ -\frac{1}{4} \beta A^2 \left( Q_n^2 + m^2 + \frac{\lambda}{16} A^2 \right) \right\}
+ \frac{1}{2} \sum_l \int \frac{d^3q}{(2\pi)^3} \left\{ (q_l^2 + m^2)G(q_l) + \log[G(q_l)] \right\}
+ \frac{\lambda}{8} \sum_l \int \frac{d^3q}{(2\pi)^3} G(q_l) \left[ \frac{1}{2} \cos(Q_n \wedge q_l) \right]
- \frac{\lambda}{8\beta} \sum_{k,l} \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} G(p_k)G(q_l) \left[ 1 + \frac{1}{2} \cos(p_k \wedge q_l) \right] \right\} V_3
+ \frac{1}{4} \beta A^2 \left[ -\frac{5\lambda}{48} A^2 - m^2 + \frac{3\lambda}{4\beta} \sum_l \int \frac{d^3q}{(2\pi)^3} G(q_l) \right] \delta_{n,0} \delta_{Q^3} V_3 + \text{const.}
\]

(19)

where \( \delta_Q \) vanishes if \( \vec{Q} \neq \vec{0} \), and is 1 when \( \vec{Q} = \vec{0} \). Further \( V_3 = \int d^3x \) is the three dimensional volume. This expression is discontinuous at \( \vec{Q} \neq \vec{0} \). Due to the dependence of (17) on parameters, a Rayleigh-Ritz variation with respect to \( A, Q \) and \( G(p_k) \) of (19) is in order [9]. Thus

\[
A \left\{ \frac{\lambda}{8} A^2 + Q_n^2 + m^2 - \frac{\lambda}{2\beta} \sum_l \int \frac{d^3q}{(2\pi)^3} G(q_m) \left[ 1 + \frac{1}{2} \cos(Q_n \wedge q_l) \right] \right\}
+ \left[ \frac{5\lambda}{24} A^2 + m^2 - \frac{3\lambda}{4\beta} \sum_l \int \frac{d^3q}{(2\pi)^3} G(q_l) \right] \delta_{n,0} \delta_{Q^3} = 0,
\]

(20)

\[
-A^2 \left[ -Q + \frac{\lambda}{2\beta} \sum_l \int \frac{d^3q}{(2\pi)^3} G(q_l) (2\pi \epsilon_l \Theta_I T + \epsilon_{ijk} \Theta_J q_k) \sin(Q_n \wedge q_l) \right] = 0,
\]

(21)

\[
G^{-1}(p_k) + p_k^2 + m^2 + \frac{\lambda}{4} A^2 \left[ 1 + \frac{1}{2} \cos(p_k \wedge q_l) \right] + \frac{3\lambda}{8} A^2 \delta_{n,0} \delta_{Q^3} = 0.
\]

(22)

These equations have solutions, at least for low temperatures, only if \( m^2 < 0 \). Thus, in the following we suppose that this is the case and we set \( \mu^2 = -m^2 \). From the third equation, we see that the integrals \( \int d^3q G(q_l) \) have UV divergences and in order to regularize them, a high-momentum \( \Lambda \) cutoff is made, and we define the (cut-off regularized) integral

\[
I(p_k) = -\sum_l \int_{\Lambda} \frac{d^3q}{(2\pi)^3} G(q_l) \left[ 1 + \frac{1}{2} \cos(p_k \wedge q_l) \right].
\]

(23)

For the ordered phase, at \( A = 0 \), (22) has the consistent, \( Q_n \) independent, solution

\[
G^{-1}(p_k) = -p_k^2 + \mu^2 - \frac{\lambda}{2(2\pi)^3 \beta} I(p_k)|_{A=0}.
\]

(24)
Further, equations (20) and (22) are discontinuous at $\bar{Q} = 0$, which points to a phase transition from the uniform, constant $\phi$ field, to the stripe phase (17). Indeed, defining $A_0 = A|_{\phi = \text{constant}}$, $G_0(p_k) \equiv G(p_k)|_{\phi = \text{constant}}$, $G(q_l, 0) = G(q_l, Q_n)|_{\bar{Q}, n = 0}$, where we made explicit the dependence on $Q_n$, and $\Delta G(p_k) = G(p_k, 0) - G_0(p_k)$, then the nonvanishing solution of (20) leads to the following equations for the discontinuity

$$A^2(0) - A^2_0 = 18 \frac{\lambda}{\beta} \sum_l \int_\Lambda \frac{d^3q}{(2\pi)^3} \Delta G(q_l),$$  

(25)

$$G^{-1}(p_k, 0) - G^{-1}_0(p_k) = \frac{3}{2} \mu^2 - \frac{\lambda}{4\beta} \sum_l \int_\Lambda \frac{d^3q}{(2\pi)^3} \left\{ -\frac{9}{2} G_0(q_l) + \Delta G(q_l) \left( 7 - \cos(p_k \wedge q_l) \right) \right\}.$$  

(26)

Further, as (21) is a very complicated equation, we will look for solutions which, by consistency, are necessarily non vanishing. For $A \neq 0$, if we define the adimensional quantities, $\bar{\theta} = \bar{\Theta} \Lambda^2$, $\bar{\Theta} = \Theta \bar{\Lambda}^2$, $\bar{k} = \frac{Q}{\Lambda}$, $\bar{\mu} = \frac{\mu}{\Lambda}$, $\bar{\theta} = \frac{\theta}{\Lambda}$ and $\bar{r} = \frac{r}{\Lambda}$ for the integration variables, (21) turns into

$$\kappa_i = -\frac{\lambda \bar{\theta}}{2} \sum_l \int_\Lambda \frac{d^3r}{(2\pi)^3} G(r_l) (2\pi l \hat{\theta}_r + \epsilon_{ijk} \hat{\theta}_j r_k) \sin \left[ 2\pi n_\perp (\bar{l} \bar{r} - n \bar{r}) + \bar{\theta} (\bar{r} \times \bar{r}) \right],$$  

(27)

where $G(r_l) = \bar{r}^2 - \mu^2 + \frac{2(\mu^2 - \kappa^2) - \lambda \kappa}{\Lambda} I(\kappa_2) \left\{ 1 + \frac{1}{2} \cos \left[ 2\pi \bar{\theta}_r (l \bar{r} - n \bar{r}) + \bar{\theta} (\bar{r} \times \bar{r}) \right] \right\}$

$$- \frac{\lambda \bar{\theta}}{2\Lambda^3} I(r_l).$$  

As expected, $\kappa_i$ vanish in the commutative limit. As we are interested on non vanishing solutions, we will look at the limit of the r.h.s., when $\kappa_i \to 0$. If the integral tends to zero, then $\kappa_i = 0$ is a consistent solution, otherwise, if the integral remains finite, then $\kappa_i \neq 0$. For finite values of $\bar{\Theta}$ and $\bar{\theta}$, when $\kappa_i$ tend to zero we get

$$\kappa_i \approx \frac{\lambda \bar{\theta}}{2} \sum_l \int_\Lambda \frac{d^3r}{(2\pi)^3} \frac{\epsilon_{ijk} \hat{\theta}_j r_k}{r^2 + \bar{\mu}^2 \left[ 1 + \cos \left( \frac{2\pi n_\perp \bar{\theta}_r \bar{r} \right) \right]} + O((\lambda \bar{\theta})^2),$$  

(28)

where the term proportional to $2\pi l \bar{\theta}$ in the numerator of (27) vanishes because the integrand is an odd function. Decomposing $\theta_j$ in the numerator of (27) parallel and perpendicular to $\bar{\theta}_r$, $\bar{\theta} = \bar{\Theta} \bar{\perp} + \bar{\Theta} \bar{\parallel}$, the part of the integral with $\bar{\Theta} \bar{\perp}$ vanishes as well because the integrand is odd, hence

$$\kappa_i \approx \frac{\lambda \bar{\theta}}{2} \sum_l \int_\Lambda \frac{d^3r}{(2\pi)^3} \frac{\epsilon_{ijk} (\hat{\theta}_j r_k)}{r^2 + \bar{\mu}^2 \left[ 1 + \cos \left( \frac{2\pi n_\parallel \bar{\theta}_r \bar{r} \right) \right]} + O((\lambda \bar{\theta})^2),$$  

(29)

which does not vanish if $n \neq 0$, and there is a nonvanishing solution, $\kappa_i \neq 0$. The components $\bar{\theta}_r$ correspond to the maximal rank parametrization taken in [2] for $\Theta_{\mu\nu}$, and $\bar{\Theta} \bar{\perp}$ corresponds to anisotropic noncommutativity, $\bar{\theta} \times \bar{\perp} \sim (\Theta^2)_{0i}$. Therefore for maximal rank noncommutativity there is no stripe phase, but for anisotropic noncommutativity $\kappa_i \neq 0$. In the planar limit, $\bar{\Theta} \bar{\perp}$ and $\bar{\theta}$ tend to infinity, letting at the same time $\kappa_i$ tend to zero, such that $\alpha_r = \bar{\theta} \bar{r} \bar{\perp}$ and $\bar{r} = \bar{\perp} \times \bar{r} \bar{\parallel}$ remain constant. We multiply (27) by $\kappa_i$ and proceed to these limits, as follows.

If $n = 0$

$$\kappa^2 \approx -\frac{\lambda \bar{\theta}}{2} \sum_l \int_\Lambda \frac{d^3r}{(2\pi)^3} \frac{2\pi \bar{\theta} \bar{\perp} \alpha_r + \bar{\Theta} \bar{\perp} r^2 + \bar{\mu}^2 \left[ 1 + \cos \left( 2\pi \bar{\theta} \bar{\perp} \alpha_r + \bar{\Theta} \bar{\perp} r \right) \right]} + O((\lambda \bar{\theta})^2).$$  

(30)
If \( n \neq 0 \)

\[
\kappa^2 \simeq -\frac{\lambda \partial}{2} \sum_i \int \frac{d^3 r}{(2\pi)^3} \left( \frac{2\pi l \partial \alpha + \partial r}{r^2 + \hat{\mu}^2} \sin \left( \frac{2\pi n \partial \tau}{r^2 + \hat{\mu}^2} \right) \right) + O[(\lambda \partial)^2].
\]

(31)

Both integrals differ by the argument of the trigonometric functions, which is finite when \( n = 0 \), and tends to infinity when \( n \neq 0 \). In both cases the denominator is positive. For \( n = 0 \), in general the integral does not vanish. For \( n \neq 0 \), the highly oscillating trigonometric functions in the numerator and in the denominator have same frequency and phase, and the integral does not vanish. Therefore in the limit \( \hat{\theta} \to \infty \), \( \kappa \) does not vanish, consistently with [2, 12].

Returning to the phase transition between the non uniform and the uniform phases, i.e. at \( Q_t \to 0 \), define \( \Delta G(p_k) = G(p_k, 0) - G_0(p_k) \). Thus, from (25) and (26), it can be shown that

\[
\Delta G(p_k) = -\frac{3}{2} \left( \frac{\mu^2}{\mu^2 + 2\mu^2} - \frac{\mu^2}{\mu^2 + 2\mu^2} \right) + O(\lambda),
\]

(32)

\[
\frac{\lambda}{8} [A^2(0) - A^2_0] = \frac{1}{4} \mu^2 - \frac{3\lambda \mu \theta}{16} \sum_i \left\{ 1 - \frac{4\sqrt{4l^2 \pi^2 \beta^2 + 2\mu^2}}{\sqrt{4l^2 \pi^2 \beta^2 + 2\mu^2}} \arctan \left( \frac{1}{\sqrt{4l^2 \pi^2 \beta^2 + 2\mu^2}} \right) + \frac{3\sqrt{4l^2 \pi^2 \beta^2 + 7\mu^2}}{2} \arctan \left( \frac{1}{\sqrt{4l^2 \pi^2 \beta^2 + 7\mu^2}} \right) \right\} + O(\lambda^2).
\]

(33)

which can be integrated to give

\[
\frac{\lambda}{8}\left( A^2 - A^2_0 \right) = \frac{1}{4} \mu^2 - \frac{3\lambda \mu \theta}{16} \sum_i \left\{ 1 - \frac{4\sqrt{4l^2 \pi^2 \beta^2 + 2\mu^2}}{\sqrt{4l^2 \pi^2 \beta^2 + 2\mu^2}} \arctan \left( \frac{1}{\sqrt{4l^2 \pi^2 \beta^2 + 2\mu^2}} \right) + \frac{3\sqrt{4l^2 \pi^2 \beta^2 + 7\mu^2}}{2} \arctan \left( \frac{1}{\sqrt{4l^2 \pi^2 \beta^2 + 7\mu^2}} \right) \right\} + O(\lambda^2).
\]

(34)

The r.h.s. of this expression is positive and does not vanish, for all temperatures below the cutoff. Therefore, there is a first order phase transition from the uniform to the non-uniform phase. This equation can be also applied to the transition from the non uniform to the disordered phase. The r.h.s. decreases as the temperature increases, but it reaches the maximum temperature (given by the cut-off) before reaching the zero value, and this phase transition is also first order.

Another question we are interested to explore is the critical temperature, at which the symmetry \( \phi \to -\phi \) is restored. We have from (20) for nonvanishing \( A \) and \( \hat{Q} \)

\[
\frac{\lambda}{8} A^2 = -Q_n^2 - \mu^2 - \frac{\lambda}{2(2\pi)^3} I(Q_n),
\]

(35)

which substituted into (22) gives

\[
G^{-1}(p_k) = -p_k^2 + \mu^2 + 2 (-\mu^2 + Q_n^2) \left[ 1 + \frac{1}{2} \cos(Q_n \wedge p_k) \right] + \frac{\lambda}{2(2\pi)^3} \left\{ \left[ 1 + \frac{1}{2} \cos(Q_n \wedge p_k) \right] I(Q_n) - \frac{1}{2} I(p_k) \right\}.
\]

(36)

Further, in order to see how are these solutions, we must consider the integral (23)

\[
I(Q_n) = \sum_i \int d^3 q \frac{1 + \frac{1}{2} \cos(Q_n \wedge q)}{2(\mu^2 - Q_n^2) \left[ 1 + \frac{1}{2} \cos(Q_n \wedge q) \right] + q_i^2 - \mu^2 - \frac{\lambda l}{(2\pi)^3} I(Q_n)},
\]

(37)
The denominator of the r.h.s. of (37), at zeroth order in $\lambda$

$$G^{-1(0)}(q_l, T) = \frac{(2\pi)^2}{(2\pi)^2} \left\{ -2n^2 \left( 1 + \frac{1}{2} \cos(Q_n \wedge q_l) \right) + \frac{3}{2} \lambda \right\} T^2$$

which is positive for sufficiently high values of $q_l^2 = (2\pi T)^2 + q^2$. Otherwise it can be negative, as happens with $G^{-1(0)}(0, T) = 2(\mu^2 - \frac{3}{2} Q^2_\lambda)$ for high temperature. Eq. (39) vanishes in general at various values of the temperature, whose minimum we call $T_0$. Further, we consider the nonuniformity and the noncommutativity as small perturbations. We suppose also that at the temperatures we are considering, $|Q_n \wedge q_l| < \frac{\pi}{2}$, as can be verified later. Thus, if $\mu^2 > Q^2_\lambda$, the second row in (39) is positive, and if $l^2 < 2n^2$, the term containing $T^2$ will be negative. Hence under these assumptions, the temperature at which (39) vanishes

$$T^2 = \frac{2(\mu^2 - Q^2_\lambda)}{(2\pi)^2} \left\{ -2n^2 \left( 1 + \frac{1}{2} \cos(Q_n \wedge q_l) \right) + \frac{3}{2} \lambda \right\}$$

will increase when $q^2$ and $l$ increase. Thus (40) will be minimum for $q^2 = 0$ and $l = 0$ at

$$T_0^2 = \frac{1}{(2\pi n)^2} \left( \frac{2}{3} \mu^2 - Q^2_{\lambda} \right).$$

Hence, at least to the lowest order in $\lambda$ and for temperatures below $T_0$, the integral (37) will be positive, and it will increase and diverge as $T$ approaches $T_0$. A numerical evaluation for $n = 1$ and $l = 0$, shows that (39) decreases monotonically as the temperature rises and becomes zero at $T_0$. Further, for $\lambda = 0$ the r.h.s. of (35) vanishes at $T^2 = \frac{1}{(2\pi n)^2} \left( \mu^2 - Q^2_{\lambda} \right) > T_0$. Therefore at first order in $\lambda$, the critical temperature will be reached when the integral in the r.h.s. of (35), evaluated at $\lambda = 0$, increases its value as $T \to T_0$. Thus in this approximation, the symmetry will be restored at a temperature $T_c \lesssim T_0$. We could go to higher orders in $\lambda$, by a selective summation in (19). We consider (37) with the denominator at first order in $\lambda$. Thus

$$G^{-1(1)}(q_l, T) = 2(\mu^2 - Q^2_\lambda) \left\{ 1 + \frac{1}{2} \cos(Q_n \wedge q_l) \right\} + q^2 - \mu^2$$

$$- \frac{\lambda T}{(2\pi)^3} \left\{ 1 + \frac{1}{2} \cos(Q_n \wedge q_l) \right\} I^{(0)}(Q_n) - \frac{1}{2} I^{(0)}(q_l) \right\} + O(\lambda^2),$$

where the integrals $I^{(0)}(Q_n)$ and $I^{(0)}(q_l)$ are at zeroth order. From the previous analysis, we know that the $l$ independent part in (42) becomes zero at $T_0$, when $q = 0$ and $l = 0$, whereas $I^{(0)}(Q_l)$ and $I^{(0)}(q_l)$ diverge at the same temperature. Thus we must consider

$$G^{-1(0)}(0, T) = 2(\mu^2 - 3Q^2_\lambda - 3(2\pi n)^2 T^2 - \frac{\lambda T}{2(2\pi)^3} \left\{ 3I^{(0)}(Q_n) - I^{(0)}(0) \right\} + O(\lambda^2),$$

at which, when $T \to T_0$, the difference $3I^{(0)}(Q_n) - I^{(0)}(0)$ gives a positive contribution. Thus the vanishing temperature of (43) will be eventually somewhat smaller than $T_0$, by a correction of order $\lambda$, and the critical temperature $T_c$ will be corrected in a similar way. This consideration could be continued iteratively to any order in the autoenergy.
5. Conclusions
We consider the noncommutative effective action at finite temperature, for a real scalar field theory with $\lambda \Phi^4$ interaction, in 4 dimensions, with noncommutative time. We solve to first order in $\lambda$ the stationarity equations of the CJT effective action, with a stripe type ansatz for the background, which depends on temperature. The CJT effective action shows a discontinuity between the results for uniform and non uniform background, which is present also in the stationarity equations. This discontinuity originates first order phase transitions between the corresponding phases, in particular, as conjectured in [2], between the ordered uniform and non uniform phases. Non uniform solutions arise not only in the planar limit, but also in the case of anisotropic non commutativity parameters. Further, the critical temperature depends on the physical mass of the scalar field and on the wave vector of the nonuniform ground state field, hence it depends on noncommutativity implicitly, although we would expect an explicit dependence in a more detailed computation.

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