REGULARIZATION OF CURRENT ALGEBRA

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ABSTRACT In this talk I want to explain the operator substractions needed to regularize gauge currents in a second quantized theory. The case of space-time dimensions $3 + 1$ is considered in detail. In presence of chiral fermions the regularization effects a modification of the local commutation relations of the currents by local Schwinger terms. In $1 + 1$ dimensions one gets the usual central extension (Schwinger term does not depend on background gauge field) whereas in $3 + 1$ dimensions one gets an anomaly linear in the background potential.

1. INTRODUCTION

Chiral fermions in a nonabelian external gauge field are quantized as follows. Let $G$ be a compact gauge group, $g$ its Lie algebra, $M$ the physical space, and $\mathcal{A}$ the space of smooth $g$ valued vector potentials in $M$. For each $A \in \mathcal{A}$ one constructs a fermionic Fock space $\mathcal{F}_A$ containing a Dirac vacuum $\psi_A$. The Hilbert space $\mathcal{F}_A$ carries an irreducible representation of the canonical anticommutation relations.

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\[ a^*(u)a(v) + a(v)a^*(u) = (u,v) \text{ all other anticommutators} = 0. \]

The representation is characterized by the property

\[ a^*(u)\psi_A = 0 = a(v)\psi_A \text{ for } u \in H_-(A) \text{ and } v \in H_+(A) \]

where \( H_+(A) \) is the subspace of the one-particle fermionic Hilbert space \( H \) spanned by the eigenvectors of the Dirac-Weyl Hamiltonian

\[ D_A = i\gamma_k(\nabla_k + A_k) \]

belonging to nonnegative eigenvalues and \( H_-(A) \) is the orthogonal complement of \( H_+(A) \). Here \( \nabla_k \)'s are covariant derivatives in directions given by a (local) orthonormal basis, with respect to a fixed Riemannian metric on \( M \). In the following we shall concentrate to the physically most interesting case \( \dim M = 3 \) and the \( \gamma \)-matrices can be chosen as the Pauli matrices \( \sigma_1, \sigma_2, \sigma_3 \) with \( \sigma_1\sigma_2 = i\sigma_3 \) (and similarly for cyclic permutations of the indices) and \( \sigma_2^2 = 1 \).

The group \( \mathcal{G} = Map(M,G) \) of smooth gauge transformations acts on \( A \) as \( g \cdot A = gAg^{-1} + dg \cdot g^{-1} \). The Fock spaces \( \mathcal{F}_A \) form a vector bundle over \( A \). A natural question is then: How does \( \mathcal{G} \) act in the total space \( \mathcal{F} \) of the vector bundle? Since the base base \( A \) is flat there obviously is a lift of the action on the base to the total space. However, we have the additional physical requirement that

\[ \hat{g}\hat{D}_A\hat{g}^{-1} = \hat{D}_{g \cdot A} \]

where \( \hat{D}_A \) is the second quantized Hamiltonian and \( \hat{g} \) is the lift of \( g \) to \( \mathcal{F} \). This condition has as a consequence that \( \hat{g}\psi_A \) should be equal, up to a phase, to the

\[ \text{vacuum } \psi_{g \cdot A} \]
A complication in all space-time dimensions higher than \(1 + 1\) is that the representations of CAR in the different fibers of \(\mathcal{F}\) are inequivalent, \([A]\). The effect of this is that a proper mathematical definition of the infinitesimal generators of \(\mathcal{G}\) (current algebra) involves further renormalizations in addition to the normal ordering prescription. In one space dimensions the situation is simple. The current algebra is contained in a Lie algebra \(\mathfrak{gl}_1\) which by definition consists of all bounded operators \(X\) in \(H\) satisfying \([\epsilon, X] \in L_2\), where \(\epsilon\) is the sign operator \(\frac{D_0}{|D_0|}\) associated to the free Dirac operator and \(L_2\) is the space of Hilbert-Schmidt operators. In general, we denote by \(L_p\) the Schatten ideal of operators \(T\) with \(|T|^p\) a trace-class operator. Let \(a^*_n = a^*(u_n)\), where \(D_0u_n = \lambda_nu_n\) and the eigenvalues are indexed such that \(\lambda_n \geq 0\) for \(n \geq 0\) and \(\lambda_n < 0\) for \(n < 0\). Denoting the matrix elements of a one-particle operator \(X\) by \((X_{nm})\), the second quantized operator \(\hat{X}\) is

\[
\hat{X} = \sum X_{nm} : a^*_n a_m :
\]

where the normal ordering is defined by

\[
:a^*_n a_m := \begin{cases} 
  a_m a^*_n & \text{if } n = m < 0 \\
  a^*_n a_m & \text{otherwise}.
\end{cases}
\]

The commutation relations are

\[
[\hat{X}, \hat{Y}] = [\hat{X,Y}] + c(X,Y)
\]

where \(c\) is the Lundberg’s cocycle, \([L]\),

\[
c(X,Y) = \frac{1}{4} \text{tr}[\epsilon, X][\epsilon, Y].
\]

When \(X, Y\) are infinitesimal gauge transformations on a circle the right-hand-side is equal to the central term of an affine Kac-Moody algebra, \([PS]\),

\[
c(X,Y) = \frac{i}{2} \int \text{tr} X'Y.
\]
In this talk I want to explain the regularizations needed in \(3 + 1\) space-time dimensions and the generalization of (1.4) through (1.7).

2. ACTION OF THE GROUP OF GAUGE TRANSFORMATIONS IN THE FOCK BUNDLE

Let \(\epsilon(A) = \frac{D_A}{|D_A|}\); if \(D_A\) has zero modes define \(\epsilon(A)\) to be +1 in the zero mode subspace. For \(A \in \mathcal{A}\) denote by \(P_A\) the set of unitary operators \(h : H \to H\) such that

\[
\quad [\epsilon, h^{-1}\epsilon(A)h] \in L_2.
\]

If \(h \in P_A\) then also \(hs \in P_A\) for any \(s \in U_1\), where \(U_1\) is the group of unitary operators \(s\) with the property \([\epsilon, s] \in L_2\). The spaces \(P_A\) form a principal bundle over \(\mathcal{A}\) with the structure group \(U_1\).

Since \(\mathcal{A}\) is flat the bundle \(P\) is trivial and we may choose a section \(A \mapsto h_A \in P_A\). Define

\[
\omega(g; A) = h_{g,A}^{-1}T(g)h_A
\]

where \(T(g)\) is the one-particle representation of \(g \in \mathcal{G}\). By construction, \(\omega\) satisfies the 1-cocycle condition

\[
\omega(gg'; A) = \omega(g; g' \cdot A)\omega(g'; A).
\]

Using \(T(g)D_AT(g)^{-1} = D_{g \cdot A}\) we get \(T(g)\epsilon(A)T(g)^{-1} = \epsilon(g \cdot A)\) which implies

\[
\quad h_{g,A}[\epsilon, \omega(g; A)]h_A^{-1} = (h_{g,A}\epsilon h_{g,A}^{-1})T(g) - T(g)(h_A\epsilon h_A^{-1})
\]

\[
= \epsilon(g \cdot A)T(g) - T(g)\epsilon(A) \text{ mod } L_2 = 0.
\]
Since $L_2$ is an operator ideal this equation implies

\[(2.4) \quad [\epsilon, \omega(g; A)] \in L_2.\]

Thus the 1-cocycle $\omega$ takes values in the group $U_1$.

The group valued cocycle $\omega$ gives rise to a Lie algebra cocycle $\theta$ by

\[\theta(X; A) = \frac{d}{dt}\omega(e^{tX}; A)|_{t=0}
= h_A^{-1}dT(X)h_A - h_A^{-1}\mathcal{L}_X h_A.\]

It satisfies the Lie algebra cocycle condition

\[(2.5) \quad \theta([X,Y]; A) - [\theta(X; A), \theta(Y; A)] + \mathcal{L}_X \theta(Y; A) - \mathcal{L}_Y \theta(X; A) = 0,\]

where $\mathcal{L}_X$ is the Lie derivative in the direction of the infinitesimal gauge transformation $X$, $\mathcal{L}_X f(A) = \frac{d}{dt} f(e^{tX} \cdot A)|_{t=0}$. We denote by $dT$ the Lie algebra representation in $H$ corresponding to the representation $T$ of finite gauge transformations. For each $A \in \mathcal{A}$ and $X \in \text{Map}(M, g)$ the operator $\theta(X; A) \in \mathfrak{gl}_1$.

The section $h_A$ of $P$ can be used to trivialize the bundle of Fock spaces over $\mathcal{A}$. Each fiber $\mathcal{F}_A$ is identified as the free Fock space $\mathcal{F}_0$. The Hamiltonian $D_A$ is quantized as

\[(2.7) \quad \hat{D}_A = q(h_A^{-1}D_A h_A),\]

that is, we first conjugate the one-particle operator $D_A$ by $h_A$ and then canonically quantize $h_A^{-1}D_A h_A$. The conjugated operator has a Dirac vacuum $\psi_A$ contained in $\mathcal{F}_0$ (but differing from the free vacuum $\psi_0$). The CAR algebra in the background $A$ is represented in $\mathcal{F}_0$ through the automorphism $a^*(u) \mapsto a^*_A(u) = a^*(h_A^{-1}u)$, $a(u) \mapsto a_A(u) = a(h_A^{-1}u)$ and using the free CAR representation for the operators on the right. The Hamiltonian $\hat{D}_A$ is then

\[(2.8) \quad \hat{D}_A = \sum a^*_A(y_n) a_A(y_n),\]
where the $u_n$’s for nonnegative (negative) indices are the eigenvectors of $D_A$ belonging to nonnegative (negative) eigenvalues. The normal ordering is defined with respect to the free vacuum.

Sections of the Fock bundle are now ordinary $\mathcal{F}_0$ valued functions. The effect of an infinitesimal gauge transformation consists of two parts: The Lie derivative $\mathcal{L}_X$ acting on the argument $A$ of the function and an operator acting in $\mathcal{F}_0,$

\begin{equation}
\hat{X} = \mathcal{L}_X + \sum \theta(X; A)_{nm} : a_n^* a_m :,
\end{equation}

where the $\theta(X; A)_{nm}$’s are matrix elements of $\theta(X; A)$ in the eigenvector basis $(v_n)$ of $D_0$. The commutation relations of the second quantized operators are modified by the Lundberg’s cocycle, [M1],

\begin{equation}
[\hat{X}, \hat{Y}] = [\hat{X}, \hat{Y}] + c(\theta(X; A), \theta(Y; A)).
\end{equation}

In the next section we want to compute the right-hand side of (2.10) more explicitly. We shall denote by $c_n(X, Y; A)$ ($n=\dim M$) the second term on the right. It is a Lie algebra 2-cocycle in the following sense:

\begin{equation}
c_n([X, Y], Z; A) + \mathcal{L}_X c_n(Y, Z; A) + \text{cyclic perm.} = 0.
\end{equation}
3. A COMPUTATION OF THE COCYCLE

First we shall construct the section $h_A$ explicitly as a function of the vector potential when $\dim M = 3$. We shall define $h_A$ through its symbol, as a pseudo-differential operator (PSDO) in the spin bundle over $M$. I claim that an operator with the following asymptotic expansion satisfies the requirement (2.1):

(3.1) \[ h_A = 1 + \frac{i}{4} \frac{[\xi, A]}{|\xi|^2} + \text{terms of lower order in } |\xi|. \]

Here $\xi = \sum \xi_k \sigma_k$ is the three-momentum; its components represent partial derivatives $-i\partial_k$ in $M$, with respect to some local coordinates. In order to make the discussion as simple as possible we assume that $M$ is the one-point compactification of $\mathbb{R}^3$ and we use standard coordinates in $\mathbb{R}^3$. We also use the notation $A = \sum A_k \sigma_k$.

An example of an unitary operator with the asymptotic expansion (3.1) is the operator

(3.2) \[ h_A = \exp\left(\frac{i}{4}(D_0^2 + \lambda)^{-1/2}[D_0, A](D_0^2 + \lambda)^{-1/2}\right) \]

where we have added a small positive constant $\lambda$ to the denominator in order to cancel the infrared singularity at $\xi = 0$; this has an effect in the asymptotic expansion only on terms of order $-2$ and lower in the momentum $\xi$. It is clear that the lower order terms do not have any effect on the condition (2.1) since any operator of order $\leq -2$ is automatically Hilbert-Schmidt when the dimension of $M$ is 3. Thus we have

(3.3) \[ \theta(X; A) = h_A^{-1} dT(X)h_A - h_A^{-1} \mathcal{L}_X h_A = X + \frac{i}{4} \frac{[\xi, A]}{|\xi|^2} + O(-2) \]
where \( O(-p) \) denotes terms of order \( \leq -p \). The symbol of the PSDO \( \epsilon \) is \( \frac{\xi}{|\xi|} \) and it is a simple computation to check that indeed \([\epsilon, \theta(X; A)] \in L_2\) using the product rule of symbols,

\[
(p * q)(\xi, x) = \sum_n \frac{(-i)^{|n|}}{n!} \partial^n_x \xi \partial^n_x q
\]

(3.4)

where the sum is over multi-indices \( n = (n_1, n_2, n_3) \in \mathbb{N}^3 \), \(|n| = n_1 + n_2 + n_3\), \(n! = n_1!n_2!n_3!\) and \( \partial^n_x = (\frac{\partial}{\partial x_1})^{n_1}(\frac{\partial}{\partial x_2})^{n_2}(\frac{\partial}{\partial x_3})^{n_3} \).

The term of order -2 in \( \theta \) is important in computing the actual value of \( \theta \). It is equal to

\[
\theta_{-2} = -\frac{1}{4} \frac{[\sigma_k, A]}{|\xi|^2} \partial_k X + \frac{1}{2} \frac{[\xi, A]}{|\xi|^4} \xi_k \partial_k X
\]

\[
+ \frac{1}{16} \frac{[\xi, A]}{|\xi|^4} [\xi, dX].
\]

(3.5)

Note that all terms are linear in the vector potential \( A \). The computation of \( c_3(X, Y; A) = c(\theta(X; A), \theta(Y; A)) \) is greatly simplified when we keep in mind that it is only the cohomology class of the cocycle \( c_3 \) we are interested in. Another simplification is the following: Formally,

\[
\frac{1}{4} \text{tr}[\epsilon, P][\epsilon, Q] = -\frac{1}{2} \text{tr}[\epsilon, P]Q
\]

(3.6)

when \( P, Q \) are in \( \mathfrak{gl}_1 \). However, the operator on the right is not quite trace-class; only its diagonal blocks are trace-class. For this reason the trace is only conditionally convergent. It is convergent when evaluated with respect to a basis compatible with the polarization \( H = H_+ \oplus H_- \), for example, one can choose a basis of eigenvectors of \( D_0 \). The trace of an operator \( P \) with symbol \( p(\xi, x) \) on a \( n \)-dimensional manifold is

\[
\text{tr}P = (\frac{1}{2\pi})^n \int \text{tr} p(\xi, x)d^n \xi d^n x
\]

(3.7)
Note that $P$ is a trace-class operator iff the order of its principal symbol is less or equal to $-1 \cdot \text{dim} M$.

As an exercise, let us compute (3.6) when $M = S^1$ and $P, Q$ are multiplication operators (infinitesimal gauge transformations). In that case the symbols are just smooth functions of the coordinate $x$ on the circle. Now $\epsilon = \frac{\xi}{|\xi|}$ is a step function on the real line, its derivative is twice the Dirac delta function located at $\xi = 0$. It follows that the symbol of the commutator $\frac{1}{2}[\epsilon, P]$ is

$$(-i)\delta_\xi p'(x) + \frac{(-i)^2}{2!} \delta_\xi p''(x) + \ldots.$$ 

Applying the formula (3.7) to (3.6) we get

$$\frac{1}{4} \text{tr} \epsilon [\epsilon, P][\epsilon, Q] = \frac{i}{2\pi} \int_{S^1} \text{tr} p'(x) q(x) dx,$$

where the trace under the integral sign is an ordinary matrix trace. If one feels uneasy with singular symbols, one can approximate $\epsilon$ by a differentiable function $\frac{\xi}{|\xi| + \lambda}$ and at the very end let $\lambda \to 0$.

In the 3-dimensional case we have to insert $P = \theta(X; A), Q = \theta(Y; A)$ in (3.6). Using the asymptotic expansions for $P$ and $Q, p = \sum p_{-k}(\xi, x)$ one has

$$c_3(X, Y; A) = \sum_{j, k} \text{tr} \left[ \frac{\xi}{|\xi|}, p_{-j} \right] * q_{-k}$$

In fact, one needs to take into account only finite number of terms. The sum of terms with $j + k \geq 4$ is a coboundary of the 1-cochain

$$\sum_{k \geq 4} \text{tr} (\epsilon * \theta(X; A)_{-k})$$

Thus we may restrict the sum in (3.8) to indices $j + k < 4$. To take care of the infrared singularity in the integration in (3.7) we replace all denominators $|\xi| = \lambda$ by
\((|\xi|+\lambda)^{-k}\). One can then check by a direct computation that, modulo coboundaries, the result of the computation in (3.8) does not depend on the value of \(\lambda\) (i.e., one may take the limit \(\lambda \rightarrow 0\) in cohomology). The final result is in accordance with the cohomological [M, F-S], [M2], and perturbative arguments, [JJ],

\[
(3.10) \quad c_3(X,Y;A) = \frac{1}{24\pi^2} \int_M \text{tr}A[dX,dY].
\]

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