ENDOMORPHISMS OF NILPOTENT GROUPS OF FINITE RANK

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ABSTRACT. We obtain sufficient criteria for endomorphisms of torsion-free nilpotent groups of finite rank to be automorphisms, by considering the induced maps on the torsion-free abelianisation and the centre. Whilst these results are known in the finitely generated case removing this assumption introduces several difficulties.

1. Introduction

Suppose an endomorphism $\sigma$ of a torsion-free finitely generated nilpotent group $N$ induces an isomorphism on the centre. It is known by (independent) work of Farkas [2] and Wehrfritz [4] that $\sigma$ is necessarily an automorphism. Meanwhile Wehrfritz [ibid.] demonstrates that for $N$ of finite (Prüfer) rank, still torsion-free and nilpotent, the result is false in general. For $\pi$ a set of rational primes, we obtain sufficient criteria in the $\pi$-divisible case by considering a generalisation of integer-like endomorphisms to so-called $\pi$-like endomorphisms. Invertible integer-like endomorphisms (the case $\pi = \emptyset$) are discussed in the context of nilpotent groups in [1]. The integer-like endomorphisms are those that preserve a maximal rank torsion-free finitely generated abelian group in the Lie algebra of the Mal’cev completion. As a special case of theorem (3.6), we then have the following.

**Theorem.** Let $N$ be torsion-free nilpotent of finite rank, and $\sigma$ an integer-like endomorphism with $\det \sigma|_{Z(N)} = \pm 1$. Then $\sigma$ is an automorphism of $N$.

We give counterexamples (see 3.7) to show that one must have a condition on the determinant and furthermore that some version of integer-like must be considered. Meanwhile Farkas and Wehrfritz, again independently, show that an endomorphism of a polycyclic-by-finite group which induces an isomorphism on the Zaleskii subgroup is an automorphism - see also the later paper of Wehrfritz [5] in this context. We give an example (see 3.8) to show that this cannot be generalised to finitely generated minimax groups.

On the other hand, we show (theorem (2.4)) that the torsion-free abelianisation always detects the surjectivity of an endomorphism, a more straightforward result. This is the content of the first section of the paper, since the tools are required later.

2. Detecting Surjectivity on the Torsion-Free Abelianisation

The following proposition is standard, but we give here a precise formulation for our specific application later. For any group $G$ and $i \geq 1$, denote by $\gamma_i(G)$ the $i$-th term of its lower central series, and $\Gamma_i(G)$ its isolator, that is, the preimage of the torsion subgroup of

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\( G / \gamma_i(G). \) As usual we often write \( G' = \gamma_2(G). \) We say that an endomorphism \( \sigma \) preserves \( H \) if \( \sigma(H) \leq H \) and stabilises \( H \) if \( \sigma(H) = H. \)

**Lemma 2.1 ([3, 1.2.11]).** Let \( N \) be a torsion-free nilpotent group and \( \sigma \) an endomorphism of \( N. \) For each \( i \geq 1, \) commutation in \( N \) induces a \( \mathbb{Z}[\sigma] \)-module homomorphism

\[
\alpha_i : \left( N / \Gamma_2(N) \right)^{\otimes i} \longrightarrow \Gamma_i(N) / \Gamma_{i+1}(N),
\]

where we equip the tensor power with the diagonal action. Moreover, the image of \( \alpha_i \) is precisely

\[
\gamma_i(N) \Gamma_{i+1}(N) / \Gamma_{i+1}(N).
\]

**Proof.** We remark only that the formulation here is a standard generalisation of the cited result. \( \square \)

We now prove here a certain rigidity result for torsion-free abelian groups of finite rank. It is presumably well-known, but we are unaware of a reference.

We begin with the following elementary lemma. Notation: for an abelian group \( A, \) and natural number \( l \geq 0, \) set \( A[r] \) to be the subgroup of \( A \) consisting of elements of order dividing \( l. \)

**Lemma 2.2.** Let \( p \) be a prime, and \( T \) a \( p \)-torsion abelian group with \( T[r] \) finite. Then every injective endomorphism of \( T \) is surjective.

**Proof.** Let \( \sigma \) be an injective endomorphism of \( T. \) Since \( T = \bigcup_{i \geq 0} T[p^i] \) and these are fully invariant subgroups, it suffices to show that the restriction of \( \sigma \) to \( T[p^i] \) is surjective for each \( i. \) Since \( \sigma \) is injective, it will suffice to show that these subgroups are all finite. The case \( i = 1 \) is our hypothesis, so assume that \( T[p^i] \) is finite and note that \( T[p^{i+1}] \) is an extension of \( T[p^i] \) by \( pT[p^{i+1}] \leq T[p^i]. \) Conclude. \( \square \)

We may now show

**Proposition 2.3.** Let \( A \) be a torsion-free abelian group of finite rank and \( B \) a subgroup of the same rank. Suppose \( \sigma \) is an endomorphism of \( A \) which stabilises \( B. \) Then \( \sigma \in \text{Aut}(A). \)

**Proof.** By the five lemma, it will suffice to show that the induced endomorphism on \( A/B \) is surjective. One verifies immediately from the necessary injectivity of \( \sigma \) on \( A \) that this is also injective on this quotient. Moreover, decomposing the torsion group \( A/B \) into its primary components, we may assume that \( A/B \) is \( p \)-torsion for some prime \( p. \) Lemma (2.2) now applies due to our finiteness of rank assumption. \( \square \)

We are now in a position to prove

**Theorem 2.4.** Let \( N \) be torsion-free nilpotent of finite Prüfer rank, and \( \sigma \) an endomorphism of \( N \) with \( \sigma(N) \Gamma_2(N) = N. \) Then \( \sigma \in \text{Aut}(N). \)
Proof. Note firstly that $\sigma$ is necessarily injective by a Hirsch length argument. We induct on the class $c$ of $N$, the case $c = 1$ being clear. For $N$ of class $c > 1$, a further Hirsch length argument shows that the induced map $\bar{\sigma} : N/\Gamma_c(N) \to N/\Gamma_c(N)$ is injective. By induction we deduce that $N = \sigma(N)\Gamma_c(N)$.

Consider now the $\mathbb{Z}[\sigma]$-module homomorphism $\alpha^c$ as in lemma (2.1). Since by hypothesis $\sigma$ acts as an automorphism on $N/\Gamma_2(N)$, the image $\gamma_c(N)$ of $\alpha^c$ is stabilised by $\sigma$. Proposition (2.3) applies and we deduce that $\sigma(\Gamma_c(N)) = \Gamma_c(N)$. Finally $N = \sigma(N)\Gamma_c(N) = \sigma(N)$, as desired. □

3. Detecting Surjectivity via the Centre

We introduce first the main tool required to obtain our second result. For a group $G$ and $i \geq 0$, denote the $i$-th term of the upper central series of $G$ by $Z^i(G)$. We often simply write $Z^1(G) = Z(G)$.

Lemma 3.1 ([3, 1.2.19]). Let $N$ be a nilpotent group and $\omega$ an automorphism of $N$. Then for each $i > 0$ there is an $\omega$-equivariant split monomorphism of abelian groups

$$
\beta_i : Z^{i+1}(N)/Z^i(N) \longrightarrow \text{Hom}(N/N', Z^i(N)/Z^{i-1}(N))
$$

$$
\bar{w} \longmapsto (\bar{x} \mapsto [w, x]),
$$

where $\omega$ acts on the right hand side by sending some $\theta$ to the map $\theta^\omega$, which for $\bar{x}$ in $N/N'$ is defined by $\theta^\omega(\bar{x}) = \omega\theta(\omega^{-1}(x))$.

Now let $N$ be a torsion-free nilpotent group of finite rank and $\sigma$ an injective endomorphism of $N$. Furthermore denote by $R$ the Mal’cev completion of $N$ and $\sigma_\times$ the induced automorphism of $R$. Note that the upper central series of $N$ is indeed preserved by $\sigma$, a consequence for example of the elementary fact that for each $i \geq 0$, we have $Z^i(N) = Z^i(R) \cap N$, so that

$$
\sigma(Z^i(N)) = \sigma_\times(Z^i(R)) \cap \sigma(N) = Z^i(R) \cap \sigma(N) \leq Z^i(N).
$$

Together with the map $\beta_1$ described in lemma (3.1), the natural maps $Z(N) \to Z(R)$ and $N/N' \to R/R'$ induce maps which fit together as below.

$$
\begin{array}{ccc}
Z^2(N)/Z^1(N) & \xrightarrow{\beta_1} & \text{Hom}(N/N', Z(N)) \\
\downarrow & & \downarrow \gamma \\
\text{Hom}(N/N', Z(R)) & \xrightarrow{s} & \text{Hom}(R/R', Z(R))
\end{array}
$$

(3.1)

The required properties of this sequence are detailed in the following proposition.

Proposition 3.2. Let $N, R, \sigma, \sigma_\times$ be as above and consider diagram (3.1). We claim the following.
3.1 Proof. The first claim is an immediate consequence of the injectivity of $Z(N) \to Z(R)$.

That $\delta$ is an isomorphism follows from the fact that $Z(R)$ is a $\mathbb{Q}$-vector space and the map $N/N' \to R/R'$ is naturally isomorphic to tensoring with $\mathbb{Q}$.

For the third part, note that $\sigma_x$ has a well defined action (as specified in lemma (3.1)) on Hom($R/R'$, $Z(R)$) since it is an automorphism of $R$. In order to show it is equivariant, let $wZ$ be an element of $Z^2(N)/Z^1(N)$. It is required to show that we have an equality of maps

$$(\delta^{-1} \circ \gamma \circ \beta_1)(\sigma(w)Z) = (\delta^{-1} \circ \gamma \circ \beta_1)(wZ)^{\sigma_x} : R/R' \to Z(R).$$

Since $\delta$ is an isomorphism it suffices to check that these are equal after precomposing with the natural map $N/N' \to R/R'$. It thus suffices to show that if $x \in N$ that

$$(\delta^{-1} \circ \gamma \circ \beta_1)(\sigma(w)Z)(xR') = \sigma_x\left(\left(\delta^{-1} \circ \gamma \circ \beta_1\right)(wZ)\right)(\sigma_x^{-1}(x)R').$$

The left hand side now immediately reduces to $[\sigma(w), x]$, so that this equality holds if and only if

$$(3.2) \quad \sigma_x^{-1}[\sigma(w), x] = \left(\delta^{-1} \circ \gamma \circ \beta_1\right)(wZ)(\sigma_x^{-1}(x)R').$$

Note that $\sigma_x^{-1}(x)$ may not lie in $N$, but there certainly exists some $l \geq 1$ for which $\sigma_x^{-1}(x)^l \in N$. Taking the $l$-th multiple in $Z(R)$ of the right hand side of equation (3.2) we thus see that

$$l \cdot \left(\delta^{-1} \circ \gamma \circ \beta_1\right)(wZ)(\sigma_x^{-1}(x)R') = \left(\delta^{-1} \circ \gamma \circ \beta_1\right)(wZ)(\sigma_x^{-1}(x)^lR')$$

$$= [w, \sigma_x^{-1}(x)^l]$$

$$= l \cdot [w, \sigma_x^{-1}(x)].$$

Appealing to the unique divisibility of $Z(R)$ we deduce that the right hand side of equation (3.2) is precisely $[w, \sigma_x^{-1}(x)] = \sigma_x^{-1}[\sigma(w), x]$, as desired.

For the final part, note that the fact that the abelian group Hom($N/N'$, $Z(N)$) is a $\mathbb{Z}[\sigma_x^{-1}]$-module is where the additional hypothesis on $\sigma$ plays its role. The proof of equivariance is both similar to and simpler than the one above, so we omit it.

In all that follows, $\pi$ will denote a (possibly empty) set of prime numbers. We begin by recalling the following standard notions. A $\pi$-number is a rational integer with all prime divisors contained in $\pi$, and we note $\mathbb{Z}[1/\pi] := \{m/n : m \in \mathbb{Z}, n \text{ a non-zero } \pi\text{-number}\}$. Finally a $\pi$-unit is a unit in this ring. If $\pi = \{p\}$, we note $\mathbb{Z}[1/p] = \mathbb{Z}[1/p]$ as usual. We now introduce the notion of $\pi$-like morphisms.
Definition 3.3. Let $V$ be a rational vector space of dimension $n < \infty$, and let $\nu$ be an automorphism of $V$. We say that $\nu$ is $\pi$-like if one of the following equivalent conditions hold.

- The coefficients of the characteristic polynomial of $\nu$ lie in $\mathbb{Z}[1/\pi]$.
- There exists some $W \leq V$ preserved by $\nu$ with $W \cong \mathbb{Z}[1/\pi]^n$.

Now let $\sigma$ an injective endomorphism of a torsion-free nilpotent group $N$ of finite rank. We say that $\sigma$ is $\pi$-like if the induced automorphism of the associated rational Lie algebra of $N$ is $\pi$-like.

The equivalence of these two conditions is standard linear algebra. In case $\pi = \emptyset$, this is the notion of integer-like automorphisms, as considered in [1]. This is a particularly well-behaved class of endomorphisms, as we see next.

Proposition 3.4. Let $N$ be torsion-free nilpotent of finite rank and $\sigma$ an injective endomorphism of $N$. Then the following are equivalent.

1. $\sigma$ is $\pi$-like.
2. For any central series of $N$ preserved by $\sigma$ with torsion-free sections, the action of $\sigma$ on each section is $\pi$-like.
3. The induced map on $N/\Gamma_2(N)$ is $\pi$-like.

Proof. 1 $\implies$ 2: Mal’cev complete at the central series to obtain a decomposition of the associated rational Lie algebra $V$ as $0 = V_0 \leq V_1 \leq \cdots \leq V_r = V$ with $\sigma_x(V_i) = V_i$ for each $i$, where $\sigma_x$ is the induced automorphism. Let the characteristic polynomial of the induced automorphism on the section $V_i/V_{i-1}$ be $f_i$ and the characteristic polynomial of $\sigma_x$ be $f$. We obtain a factorisation $f = f_1 \cdots f_n$. By hypothesis $f$ has coefficients in $\mathbb{Z}[1/\pi]$, and Gauss’ Lemma implies that each $f_i$ has coefficients in this ring too.

2 $\implies$ 3: Trivial.

3 $\implies$ 1: Select a subgroup isomorphic to a free $\mathbb{Z}[1/\pi]$-module of maximal rank in $N/\Gamma_2(N)$ preserved by $\sigma$. The image of this subgroup under the tensor power maps described in lemma (2.1) show that the action on each section of the isolated central series is $\pi$-like. Upon Mal’cev completing, this factorises the characteristic polynomial of the induced automorphism $\sigma_x$ into polynomials with coefficients in $\mathbb{Z}[1/\pi]$. Conclude. □

A trivial consequence of the above proposition is the following, which we state separately for later clarity.

Corollary 3.5. Let $V$ be a finite dimensional $\mathbb{Q}$-vector space and $\omega$ a $\pi$-like automorphism of $V$. Suppose that $U$ is a subspace of $V$ stabilised by $\omega$. Then $\omega|_U$ is $\pi$-like.

We may now show the following.

Theorem 3.6. Let $N$ be a $\pi$-divisible torsion-free nilpotent group of finite rank, and let $\sigma$ be a $\pi$-like endomorphism of $N$ such that $\det \sigma|_{\mathbb{Z}(N)}$ is a $\pi$-unit. Then $\sigma \in \text{Aut}(N)$.

Proof. The proof is by induction on the class $c$ of $N$. If $c = 1$ and $N$ has rank $n$, select $W \cong \mathbb{Z}[1/\pi]^n$ preserved by $\sigma$ and $W \leq A$. Then the hypothesis on the determinant implies that $\sigma(W) = W$, and we may conclude with proposition (2.3).
Thus suppose $c > 1$. By considering $N/Z(N)$ of smaller class and centre $Z^2(N)/Z^1(N)$, it will suffice to show that the determinant of the induced map on $Z^2(N)/Z^1(N)$ is also a $\pi$-unit. In order to proceed, we will show that the action of $\sigma^{-1}_x$ on $\text{Hom}(R/R', Z(R))$ is $\pi$-like. This will follow from the final part of proposition (3.2) once we know that the action of $\sigma^{-1}_x$ on $\text{Hom}(N/N', Z(N))$ is $\pi$-like. Since $Z(N)$ is torsion-free, it is equivalent to show this for $\text{Hom}(N/\Gamma_2(N), Z(N))$. Our hypothesis implies, by proposition (3.4), that the action of $\sigma$ on $N/\Gamma_2(N)$ is $\pi$-like. Moreover by our determinant hypothesis we may deduce that the action of $\sigma^{-1}$ on $Z(N)$ is $\pi$-like. Thus there are subgroups $S, T$ of $N/\Gamma_2(N)$ and $Z(N)$ respectively, both of maximal Hirsch length and isomorphic to direct sums of copies of $Z[1/\pi]$, with $\sigma(S) \leq S$ and $\sigma^{-1}(T) \leq T$. Consider the subgroup $H_{S,T} \leq \text{Hom}(N/\Gamma_2(N), Z(N))$, consisting by definition of those $f$ for which $f(S) \leq T$. One may conclude by observing that $H_{S,T}$ is preserved by $\sigma^{-1}$, and that $H_{S,T}$ is of maximal Hirsch length and also isomorphic to a direct sum of copies of $Z[1/\pi]$.

Now let $U$ be the $\mathbb{Q}$-span of the image of $Z^2(N)/Z^1(N)$ in $\text{Hom}(R/R', Z(R))$ under $\delta^{-1} \circ \gamma \circ \beta$, in the notation of proposition (3.2). Then in particular $\sigma^{-1}_x(U) = U$ and corollary (3.5) applies with $\omega = \sigma^{-1}_x$: denoting the map $\sigma_x$ induces on $U$ by $\bar{\sigma}_x$, we deduce that $(\det \bar{\sigma}_x)^{-1}$ is a $\pi$-number. Moreover $\det \bar{\sigma}_x$ is a $\pi$-number by proposition (3.4). Thus it is a $\pi$-unit, as desired. \hfill $\square$

We now justify why we prove theorem (3.6) under the stated hypotheses.

**Example 3.7.** We consider the nilpotent group

$$N := \left( \begin{array}{ccc} 1 & Z & Z[1/2] \\ 1 & Z[1/2] & 1 \end{array} \right) \leq \text{GL}_3(\mathbb{Q}), \quad Z(N) = \left( \begin{array}{ccc} 1 & 0 & Z[1/2] \\ 1 & 0 & 1 \end{array} \right),$$

with endomorphisms

$$\varphi_1 \left( \begin{array}{c} a \\ b \\ 1 \end{array} \right) = \left( \begin{array}{c} 2a \\ b/2 \\ 1 \end{array} \right), \quad \varphi_2 \left( \begin{array}{c} a \\ b \\ 1 \end{array} \right) = \left( \begin{array}{c} 2a \\ 2b \\ 1 \end{array} \right).$$

Both $\varphi_1$ and $\varphi_2$ are injective endomorphisms of $N$ which are not surjective, but induce isomorphisms on the centre.

The first example demonstrates that we must assume that the induced map on the torsion-free abelianisation is $\pi$-like. (Note that $\pi = \emptyset$ here.) However proposition (3.4) shows that this already implies that the whole endomorphism is $\pi$-like, whence this hypothesis. Meanwhile, the second endomorphism is $\pi$-like but the determinant on the centre is not a $\pi$-unit, whence our second hypothesis.

It is shown independently in [2] and [4] that an endomorphism of a polycyclic-by-finite group which restricts to an isomorphism of the Zaleskii subgroup is an automorphism. We show finally that this cannot hold in the finitely generated minimax setting.
Example 3.8. Let $N$ be as above and set
\[ t := \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}, \quad x := \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}. \]

Then $G := \langle N, x \rangle$ is finitely generated minimax with Fitting subgroup $N$. Conjugating by $t$ gives a proper inclusion $G^t < G$ and moreover induces $\varphi_2$ on $N$ above. The Zaleskii subgroup here is precisely the centre of the Fitting subgroup, where our endomorphism induces an isomorphism.

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