Iterative Learning Control for Generalized Projective Synchronization of Fractional-order System with Unknown Local Lipschitz Function

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Abstract. This paper is devoted to the generalized projective synchronization problem of fractional-order systems subject to the unknown local Lipschitz conditions. Based on the definition of the fractional integral, a suitable variable is defined, thereby the generalized projective synchronization problem between leader and follower is transformed into the stabilization problem of the system consisting of the newly defined variable. An adaptive iterative learning controller is proposed to control the fractional-order systems. Based on the composite energy function method, the convergence of learning process is analysed. Consequently, the sufficient conditions are derived to guarantee that the leader-following FOSs can achieve the generalized projective synchronization in the finite time interval as the iteration step goes to infinity. Finally, a numerical simulation example is presented to demonstrate the effectiveness of the proposed method.

Keywords. Iterative learning control, local Lipschitz condition, fractional-order system, generalized projective synchronization, composite energy function.

1. Introduction
With the research of the fractional calculus [1], it is found that many phenomena are suitable to be described by the fractional-order differential equations [2], and numerous researchers turn their attention to the control problem of fractional-order systems (FOSs). Among various researches about the FOSs, the synchronization control problem is very important due to its application in the secure communication [3]. Recently, many synchronization results of FOSs have been obtained from various perspectives, such as phase synchronization [4], lag synchronization [5], and projective synchronization [6, 7]. The time delays [8] and disturbances [9] have also been addressed for the synchronization of FOSs, respectively.

The above literatures have investigated the asymptotic or finite-time synchronization of FOSs, while some practical control tasks require that the synchronization is precisely achieved during the overall process of executing the given tasks. The iterative learning control (ILC) method is suitable for this kind of precise control problems, because the ILC method can generate the appropriate control inputs for the controlled systems with any target trajectory in a finite time interval. There have existed some literatures about the ILC synchronization of FOSs. [10] has designed the ILC controller for the stabilization of fractional-order linear continuous-time switched systems. The varying trial length problems in the process of iteration learning have been addressed for non-instantaneous impulsive
fractional-order systems [11]. The iterative learning sliding mode control method have been proposed for the synchronization of uncertain fractional-order systems by combining the ILC method and the sliding mode control (SMC) method [12].

In the above ILC literatures of FOSs, the global Lipschitz conditions are required and the contraction mapping principle is used to analyse the convergence of learning process. To the best of author’s knowledge, the local Lipschitz conditions have not been addressed for the synchronization problems of FOSs until now, yet there have existed some literatures about the ILC synchronization of integer-order systems (IOSs) subject to the local Lipschitz conditions. [13] has designed the ILC controller for the synchronization of IOSs. It is shown in [14] that the synchronization performance of uncertain IOSs can be improved by combining the ILC method and SMC method. In [15], the synchronization problem of a class of one-sided Lipschitz nonlinear systems is investigated. However, due to the invalidation of the Leibniz rule in the fractional derivatives and the particularities of the local Lipschitz conditions, the existing ILC synchronization results for IOSs subject to the local Lipschitz conditions is hard to be directly extended to the synchronization for FOSs subject to the local Lipschitz conditions. Thus, it is still a challenging to design an ILC controller for the synchronization of the FOSs subject to the unknown local Lipschitz conditions.

Inspired by the limitations of the existing ILC synchronization results of FOSs, we investigate the generalized projective synchronization problem of leader-following FOSs subject to the unknown local Lipschitz conditions. The main contributions of this paper can be summarized as follows:

1) Addressing the generalized projective synchronization problem of FOSs subject to the unknown local Lipschitz conditions. The existing ILC literatures have investigated the synchronization of FOSs subject to the global Lipschitz conditions, while the local Lipschitz conditions haven’t been addressed in the FOSs framework.

2) Designing the adaptive ILC controller. By the composite energy function method, the convergence of learning process is analysed and the GPS condition is derived.

3) The derived sufficient condition guarantees that the leader-following FOSs can achieve the generalized projective synchronization in the finite time interval as the iteration step goes to infinity.

2. Problem formulation
Consider the leader-following fractional-order systems, which work in a repeatable control environment. The leader is described by

\[ D^\alpha x_d(t) = f_d(t, x_d(t)), \]  

where \( D^\alpha \) denotes the \( \alpha \)-order Caputo derivative of \( (\cdot) \) [16], \( \alpha \in (0,1) \), and \( t \in [0,T] \), \( T \) is the terminal time. \( x_d(t) \) and \( f_d(t, x_d(t)) \) are the state and nonlinear function of leader, respectively.

The follower is described by

\[ D^\alpha x_i(t) = \theta(t)x_i(t)^2 + u_i(t), \]

where \( i \) denotes the iteration step. \( x_i(t) \) and \( u_i(t) \) are the state and control input of follower, respectively. \( \theta(t) \) is a unknown vector function satisfying the following assumption.

**Assumption 1.** The continuous function \( \theta(t) \) on \( t \in [0,T] \) can’t be known in advance but satisfies \( \theta(t) \leq \theta_m \), where \( \theta_m \) is a known positive constant and is call as the supreme of \( \theta(t) \).

**Control objective:** This paper will design the iterative learning controller such that the generalize projective synchronization between leader and follower can be achieved in the finite time interval as the iteration step goes to infinity.

3. Main results
Let the generalize projective synchronization error \( e_i(t) = x_i(t) - \lambda x_d(t) - \eta \), where \( \lambda, \eta > 0 \) are the coefficients of generalize projective synchronization. Then the generalize projective synchronization error system can be described by
\[
D_\alpha^\tau e_i(t) = -\lambda f_i(t, x_i(t)) - \eta + \Theta(t)x_i^2(t) + u_i(t).
\]
(3)

To simplify the analysis, the following assumption is needed.

**Assumption 2:** [10] The initial states of follower are reset to satisfy the condition (4) after each execution.
\[
e_i(0) = x_i(0) - \lambda x_d(0) - \eta = 0.
\]
(4)

**Remark 1:** To better focus on the main problem, here the initial GPS errors are assumed to be zero. In fact, some researches have addressed the varying initial state/output error problems in the ILC-based synchronizations [17].

Define a variable \( s_i(t) \) as \( D_\alpha^\tau e_i(t) = s_i(t) \). Based on Definitions 1 and 2, we have
\[
\int_0^\tau D_\alpha^\tau e_i(\tau)d\tau = s_i(t).
\]
Then, (3) can be rewritten as
\[
s_i(t) = -\lambda f_i(t, x_i(t)) - \eta + \Theta(t)x_i^2(t) + u_i(t).
\]
(5)

**Remark 2:** It is easy to understand from (3), (5) and Assumption 1 that \( \lim_{i \to \infty} s_i(t) = 0 \) is equivalent to \( \lim_{i \to \infty} e_i(t) = 0 \). Thus, based on (3) and (5), the synchronization control problem between (1) and (2) is transformed into the stabilization problem of (5).

To stabilize (5), the ILC controller is designed as
\[
u_i(t) = \hat{\alpha} f_i(t, x_i(t)) + \eta - k s_i(t) - \hat{\theta}_i(t)x_i^2(t),
\]
(6)
where \( k > 0 \) is learning gain that will be designed later. \( \hat{\theta}_i(t) \) is an estimate of unknown parameter \( \theta_i(t) \), and the adaptive updating law of \( \hat{\theta}_i(t) \) is designed as
\[
\dot{\hat{\theta}}_i(t) = \hat{\theta}_{i-1}(t) + x_i^2(t) s_i(t),
\]
(7)
where \( \hat{\theta}_{i-1}(t) = 0 \).

Based on the previous results, the following theorem can be obtained.

**Theorem 1:** Consider the leader-following FOSs (1) and (2) satisfying Assumptions 1 and 2, and let the ILC controller (6) with the adaptive updating law (7) is applied. For a given constant \( c \in (0, 0.5) \), if there exist learning gain \( k \) satisfying
\[
k s_0^2(t) + (0.5 - c) \phi_0^2(t) \leq \Theta_m^2 / 4c,
\]
then
\[
\lim_{k \to \infty} x_k(t) = \lambda x_d(t) + \eta.
\]
(8)

**Proof.**

Construct the composite energy function as
\[
E_i(t) = 0.5s_i^2(t) + 0.5\int_0^t \phi_i^2(\tau)d\tau,
\]
(10)
where \( \phi_i(t) = \hat{\theta}_i(t) - \theta(t) \).

Based on (10) and Assumption 1, we have
\[
E_i(0) = 0.5s_0^2(0) + 0.5\int_0^t \phi_0^2(\tau)d\tau = 0.
\]
(11)
Thus, noticing $\dot{\theta}_0(t)=\chi^2_0(t)s_0(t)$, the first-order derivative of $E_0(t)$ is
\[ \frac{dE_0(t)}{dt}=s_0(t)\ddot{s}_0(t)+0.5\dot{\phi}_0^2(t)=-k\chi_0^2(t)-\dot{\phi}_0(t)\dot{\theta}(t)-0.5\dot{\phi}_0^2(t). \] (12)
Substituting $c\dot{\phi}_0^2(t)+\theta_0^2(t)/4c\geq-\dot{\phi}_0(t)\dot{\theta}(t)$ into (14), and noticing $c\in(0,0.5)$ and (8), we obtain $\frac{dE_0(t)}{dt}\geq0$, which indicates that $E_0(t)$ is bound in the finite time $t\in[0,T]$.

The difference of $E_i(t)$ is
\[ \Delta E_i(t)=E_i(t)-E_{i-1}(t)=0.5\chi_i^2(t)+0.5\int_0^t(\dot{\phi}_i^2(\tau)-\dot{\phi}_{i-1}^2(\tau))d\tau-0.5\chi_{i-1}^2(\tau). \] (13)
The first term on the right side of (13) can be rewritten as
\[ 0.5\chi_i^2(t)=\int_0^t\left(\ddot{s}_i(\tau)s_i(\tau)\right)d\tau=\int_0^t[-\dot{\phi}_i(\tau)\chi_i^2(\tau)s_i(\tau)-k\chi_i^2(\tau)]d\tau. \] (14)
The second term on the right side of (13) can be rewritten as
\[ 0.5\int_0^t(\dot{\phi}_i^2(\tau)-\dot{\phi}_{i-1}^2(\tau))d\tau=\int_0^t(\dot{\phi}_i(\tau)\chi_i^2(\tau)s_i(\tau)-0.5\chi_i^4(\tau)s_i^2(\tau))d\tau. \] (15)
Substituting (14) and (15) into (13), we have
\[ \Delta E_i(t)=-\int_0^t k\chi_i^2(\tau)d\tau-0.5\int_0^t\chi_i^4(\tau)s_i^2(\tau)d\tau-0.5\chi_{i-1}^2(\tau)\leq0. \] (16)
Thus, we have
\[ \lim_{i\to\infty}\sum_{j=1}^{i-1}\int_0^t k\chi_j^2(\tau)d\tau=0 \quad \text{and} \quad \lim_{i\to\infty}\sum_{j=1}^{i-1}\chi_j^2(t)=0. \] (17)
Note that both $E_i(t)$ and $E_0(t)$ are positive definite, and $E_0(t)$ is bounded on $t\in[0,T]$, we have
\[ \lim_{i\to\infty}\sum_{j=1}^{i-1}\int_0^t k\chi_j^2(\tau)d\tau=0 \quad \text{and} \quad \lim_{i\to\infty}\sum_{j=1}^{i-1}\chi_j^2(t)=0. \] (18)
which is equivalent to $\lim_{i\to\infty}s_i(t)=0$. According to Remark 2, we have $\lim_{i\to\infty}e_i(t)=0$, which indicates that the generalized projective synchronization between (1) and (2) can be achieved in the finite time interval $t\in[0,T]$ as the iteration step $l$ goes to infinity.

This ends the proof of Theorem 1.

Theorem 1 provides a sufficient condition guaranteeing the generalized projective synchronization between (1) and (2) though the dynamical equation of follower is partly unknown and doesn’t satisfy the global Lipschitz condition.

**Remark 3.** The literatures [10-12] have investigated the ILC synchronization of FOSs subject to the global Lipschitz conditions, and the convergence of learning process is analyzed by the contraction mapping principle. In this paper, the GPS of FOSs subject to the unknown local Lipschitz conditions is investigated. Due to the existence of local Lipschitz conditions, the contraction mapping method isn’t suitable, thus the composite energy function method is used to analyze the convergence of learning process.

**Remark 4.** The local Lipschitz conditions have been addressed in the literatures of ILC synchronization of IOSs [13-15]. In contrast, this paper addresses the local Lipschitz conditions for the GPS of the FOSs.

4. **Simulation results**

Consider the leader-following FOSs. The dynamical equation of leader is described by
\[ D^2_0\chi_0(t)=0.98\cos(\pi t). \] The dynamical equation of follower is described by
\[ D^2_0\chi(t)=(1.2+0.8\sin(\pi t))\chi_0(t)^2+u(t). \]
It is easy to see from (23) that $\theta_M = 2$. In the simulation, the initial states of leader and follower are set as $x_L(0) = 0.5$ and $x_F(0) = 0.6$, respectively. The coefficients of generalize projective synchronization: $\lambda = 0.6$ and $\eta = 0.3$. Obviously, Assumption 2 is satisfied. The terminal time is set as $T = 1$, and the constant $c = 0.25$. The learning gain $k$ is set as $k = 3$. It is easy to verify the condition (8) is satisfied. According to Theorem 1, the GPS between leader and follower can be achieved.

**Figure 1.** $\lambda = 0.6$, $\eta = 0.3$, under the ILC controller (6) and the adaptive updating law (7) with $k = 3$, the trajectories of $\lambda x_d(t) + \eta$ and $x(t)$ for different iteration numbers: (a) $i = 10$; (b) $i = 30$; (c) $i = 180$.

**Figure 2.** $\lambda = 0.6$, $\eta = 0.3$, supremum norms of GPS errors versus iteration numbers under the controller (6) and the adaptive updating law (7) with $k = 3$. (Color online)

5. Conclusions

In this paper, the generalized projective synchronization (GPS) of FOSs subject to the unknown local Lipschitz conditions is investigated. Based on the relations between the integer-order derivative and the Caputo fractional-order derivative, a variable is defined, thereby the control problem of GPS between leader and follower is transformed into the stabilization problem of the system consisting of the newly defined variable. Based on the composite energy function method, the adaptive ILC controller is proposed and the convergence condition is derived to ensure the GPS of leader-following FOSs. Both theoretical study and simulation results show that the proposed method can effectively deal with the local Lipschitz conditions and achieve the generalized projective synchronization of FOSs.

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