Reflection subgroups of Euclidean reflection groups.

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1 Introduction

Let $X$ be an $n$-dimensional Euclidean, spherical, or hyperbolic space. A convex polytope in $X$ is called a Coxeter polytope if all its dihedral angles are integer submultiples of $\pi$. A group generated by reflections with respect to the facets of a Coxeter polytope is discrete; a fundamental domain of this group is the initial Coxeter polytope.

From now on by polytope we mean a finite volume polytope in the $n$-dimensional Euclidean space $E^n$. A group generated by reflections with respect to the facets of Euclidean Coxeter polytope we call a Euclidean reflection group. Finite groups generated by reflections we call spherical groups.

A classification of reflection subgroups of spherical reflection groups may be deduced from [1]. A classification of reflection subgroups of Euclidean and hyperbolic reflection groups is still incomplete. Papers [2], [3], [4] and [5] are devoted to reflection subgroups of hyperbolic reflection groups with simplicial fundamental domains.

In this paper, we classify reflection subgroups of discrete Euclidean reflection groups.

In section 3 we make use results of [1] to classify reflection subgroups of indecomposable spherical reflection groups. In fact, any reflection subgroup in any decomposable reflection group $G$ is a direct product of reflection subgroups of indecomposable components of $G$ (Lemma 1). Hence, it is sufficient to describe subgroups of indecomposable spherical and Euclidean reflection groups. All the indecomposable Euclidean compact Coxeter polytopes are simplices (see [6]). Thus, to obtain a classification of reflection subgroups of Euclidean reflection groups we only need to classify reflection subgroups of groups generated by reflections in the facets of Euclidean simplices.

In Section 4 we classify all indecomposable finite index reflection subgroups of indecomposable Euclidean reflection groups. We also prove that any indecomposable reflection subgroup is determined by its index up to an automorphism of the whole group. Furthermore, in Section 5 we give a general description of reflection subgroups in terms of affine root systems. In Section 6 we classify decomposable maximal reflection subgroups. In Section 7 we consider infinite index reflection subgroups.

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2 Definitions and notation

A hyperplane $\mu$ is called a mirror of a reflection group if the group contains a reflection in $\mu$. Mirrors of a reflection group decompose the space into connected
components (fundamental chambers); any fundamental chamber is a fundamental domain for the group action. Reflections in facets of any fundamental chamber of a reflection group generate the whole group.

Let $H$ be a finite index reflection subgroup of a reflection group $G$. Then a fundamental chamber of $H$ consists of several copies of a fundamental chamber of $G$. If two such copies have a facet in common then they are symmetric to each other with respect to this facet.

To describe Coxeter polytopes we use Coxeter diagrams: nodes $v_1, \ldots, v_k$ of the diagram correspond to the facets $f_1, \ldots, f_k$ of the polytope; the nodes $v_i$ and $v_j$ are joined by an $(m_{ij} - 2)$-fold edge if the dihedral angle formed up by $f_i$ and $f_j$ is equal to $\frac{\pi}{m_{ij}}$ (if $f_i$ is orthogonal to $f_j$ then the nodes $v_i$ and $v_j$ are disjoint); the nodes $v_i$ and $v_j$ are joined by a bold edge if $f_i$ is parallel to $f_j$.

Let $\Sigma$ be a Coxeter diagram with $k$ nodes $v_1, \ldots, v_k$. Denote by $Gr_\Sigma = (g_{ij})$ a symmetric $k \times k$ matrix with $g_{ii} = 1$ ($i = 1, \ldots, k$), $g_{ij} = -\cos(\frac{\pi}{m_{ij}})$ when $i \neq j$ and $v_i$ is connected with $v_j$ by an $(m_{ij} - 2)$-fold edge, $g_{ij} = -1$ when $i \neq j$ and $v_i$ is connected with $v_j$ by a bold edge.

A connected Coxeter diagram $\Sigma$ is called elliptic if $Gr_\Sigma$ is positively defined; a connected diagram $\Sigma$ is called parabolic if $Gr_\Sigma$ is degenerate and any subdiagram of $\Sigma$ is elliptic. A Coxeter polytope is called indecomposable if its Gram matrix is indecomposable (or, similarly, its Coxeter diagram is connected). Connected elliptic Coxeter diagrams coincide with Coxeter diagrams of indecomposable spherical Coxeter simplices, and connected parabolic diagrams coincide with Coxeter diagrams of Euclidean Coxeter simplices. A Coxeter diagram of any compact Euclidean Coxeter polytope is a disjoint union of connected parabolic diagrams. Table 1 contains the list of Coxeter diagrams of indecomposable spherical and Euclidean Coxeter simplices. We also use the notation $B_1 = C_1 = A_1$, $D_2 = 2A_1$ and $D_3 = A_3$ (see [1]).

Denote by $\Sigma(P)$ the Coxeter diagram of a Coxeter polytope $P$ and by $G_P$ the group generated by reflections in the facets of $P$. The matrix $Gr_{\Sigma(P)}$ coincides with the Gram matrix of $P$.

A survey about reflection groups and their fundamental polytopes may be found in [2].

Let $P$ be a Euclidean Coxeter polytope. There exists at least one vertex $V$ of $P$ such that the stabilizer of $V$ contains linear parts of all elements of $G_P$ (see [3] Ch. 6]). Vertices of that type are called special vertices of polytope $P$.

Now let $P$ be an indecomposable Euclidean Coxeter polytope, i.e. a Coxeter simplex; let $V$ be any special vertex of $P$ and $\mu$ be a facet opposite to $V$. Let $v$ be a node of $\Sigma(P)$ corresponding to $\mu$. We call $v$ a special node of $\Sigma(P)$. Notice that $\Sigma(P)$ may contain several special vertices. However, the diagram $\Sigma(P) \setminus v$ does not depend on the choice of special vertex (in other words, any two diagrams of this type are equivalent under an automorphism of the diagram $\Sigma(P)$). Denote $\Sigma(P) \setminus v$ by $\Sigma'(P)$. Denote by $G'_P(V)$ the group generated by reflections in the facets of $P$ different from $\mu$. This group does not depend on the choice of special vertex $V$ (i.e. for any pair of special vertices $V_1$ and $V_2$ of $P$ there exists an automorphism of $G_P$ taking $V_1$ to $V_2$. When the choice of $V$ is not important we write $G'_P$ instead of $G'_P(V)$. Note that $\Sigma'(P)$ is a Coxeter diagram of a fundamental chamber of $G'_P$.

A subgroup $G_P \subset G_F$ is called maximal if there is no simplex $T$ such that $G_P \subset$
Table 1. Coxeter diagrams. Connected elliptic and parabolic Coxeter diagrams are listed in left and right columns respectively. Special nodes are colored in white.

| $A_n$ ($n \geq 1$) | $\tilde{A}_1$ |
|---------------------|--------------|
| $\tilde{A}_n$ ($n \geq 2$) | $\tilde{A}_n$ ($n \geq 2$) |
| $B_n = C_n$ ($n \geq 2$) | $\tilde{B}_n$ ($n \geq 3$) |
| $\tilde{C}_n$ ($n \geq 2$) | $\tilde{C}_n$ ($n \geq 2$) |
| $D_n$ ($n \geq 4$) | $\tilde{D}_n$ ($n \geq 4$) |
| $G_2$ | $\tilde{G}_2$ |
| $F_4$ | $\tilde{F}_4$ |
| $E_6$ | $\tilde{E}_6$ |
| $E_7$ | $\tilde{E}_7$ |
| $E_8$ | $\tilde{E}_8$ |
| $H_3$ | $\tilde{H}_3$ |
| $H_4$ | $\tilde{H}_4$ |
$G_T \subset G_F$.

A reflection group is called \textit{indecomposable} if its fundamental Coxeter polytope is indecomposable. Any reflection group is a direct product of several indecomposable reflection groups. We call these factors \textit{components} of a reflection group.

\textbf{Lemma 1.} Any reflection subgroup $H$ of decomposable reflection group $G$ is a direct product of reflection subgroups of indecomposable components of $G$.

\textit{Proof.} Let $G = G_1 \times G_2 \times \ldots \times G_k$, and let $\{r_1^1, \ldots, r_k^1\}$ be reflections generating the group $G_i$. Denote by $[G, G]$ the commutator subgroup of $G$. Furthermore, notice that

$$G/[G, G] \cong \mathbb{Z}_2^m,$$

where $M$ is the number of conjugacy classes of reflections in $G$. In other words, any set of reflections generating $G$ must contain representatives of all conjugacy classes of reflections in $G$. Hence, any reflection $r \in H$ is conjugated in $G$ to some $r_i^j \in G_i$. Since $G_i \triangleleft G$, we obtain that $r \in G_i$, and the lemma is proved.

\section{Spherical reflection subgroups}

In paper [1] Dynkin classified regular semisimple subalgebras of semisimple Lie algebras. For this he listed all root subsystems in finite root systems. This problem is very close to the classification of reflection subgroups of finite reflection groups. However, it is not the same.

In this section we make use the results of [1] to classify reflection subgroups of finite reflection groups.

\subsection{Root subsystems}

Let $G$ be a finite reflection group different from $H_3$, $H_4$ and $G_2^{(m)}$ (if $m \neq 2, 3, 4, 6$). Then $G$ may be thought as a Weyl group of some finite root system $\Delta$ (see [1]). Mirrors of $G$ are hyperplanes on which the roots of $\Delta$ vanish. A type of $G$ coincide with the type of the root system $\Delta$.

Any reflection subgroup $H \subset G$ is a Weyl group of some root system $\Delta_1 \subset \Delta$. The root system $\Delta_1$ consists of those roots of $\Delta$ which vanish on mirrors of $H$. Conversely, a Weyl group of any root system $\Delta_1 \subset \Delta$ is a reflection subgroup of $G$.

In this way we obtain a one-to-one correspondence between reflection subgroups $H \subset G$ and root systems $\Delta_1 \subset \Delta$.

A root system $\Delta_1 \subset \Delta$ is called a \textit{root subsystem} if the following holds:

$$\text{if } \alpha, \beta \in \Delta_1 \text{ and } \alpha + \beta \in \Delta, \text{ then } \alpha + \beta \in \Delta_1 \quad (*)$$

Root subsystems of finite root systems are classified in [1]. So, for each finite reflection group $G$ we only need to list root systems $\Delta_1 \subset \Delta$ which are not root subsystems.

Condition (*) may be reformulated in the following way. Let $\Delta_1 \subset \Delta$ be root systems, and let $\{\alpha_1, \ldots, \alpha_n\}$ be simple roots of $\Delta_1$. The root system $\Delta_1$ is a root
subsystem of $\Delta$ if and only if the following holds:

$$\alpha_i - \alpha_j \notin \Delta \text{ if } i \neq j.$$  (***)

What does condition (*** mean from geometric point of view? Let $\Delta_1 \subset \Delta$ be root systems, let $\alpha_i, \alpha_j \in \Delta_1$ be simple roots of $\Delta_1$, and let $\alpha_i - \alpha_j \in \Delta$. Consider the root system generated by $\alpha_i$ and $\alpha_j$. It should coincide with one of $2A_1, A_2, C_2$ or $G_2$. For each of these systems the angle formed up by roots $\alpha_i$ and $\alpha_j$ is cut by the line orthogonal to the root $\alpha_i - \alpha_j$ (simple roots are the outward normal vectors to the sides of the angle). The angle formed up by simple roots can not be acute, so we have

$$(\alpha_i - \alpha_j)^2 = \alpha_i^2 + \alpha_j^2 - 2(\alpha_i, \alpha_j) \geq \alpha_i^2 + \alpha_j^2.$$ 

Since any finite root system contains roots of at most two different lengths, the equality holds if and only if $\alpha_i$ and $\alpha_j$ are short roots of $\Delta$, $(\alpha_i, \alpha_j) = 0$ when $\Delta \neq G_2$, and $(\alpha_i, \alpha_j) = -\frac{1}{2}$ when $\Delta = G_2$.

In particular, we obtain the following

**Lemma 2.** Let $\Delta_1 \subset \Delta$ be root systems, let $\Pi_1$ and $\Pi$ be simple root systems of $\Delta_1$ and $\Delta$ respectively. Suppose that one of the following holds:

1) all the real roots of $\Delta$ have the same length;
2) $\Pi_1 \setminus \Pi$ contains no short root.

Then $\Delta_1$ is a root subsystem of $\Delta$.

**Corollary 1.** All reflection subgroups of the reflection groups $A_n, D_n, E_6, E_7$ and $E_8$ are listed in [1].

Hence, we only need to list root systems $\Delta_1 \subset \Delta$ that are not root subsystems, when $\Delta = B_n$ (or $C_n$), $F_4$ and $G_2$, as well as reflection subgroups of reflection groups $H_3, H_4$ and $G_2^{(m)}$ (when $m \geq 5$, $m \neq 6$).

### 3.2 Subgroups of $B_n$

Consider the group $G = B_n$ as the Weyl group of the root system $\Delta = B_n$. By [1, Table 9] (see also [10]), the root system $B_n$ contains subsystems of the type

$$A_{k_1} + \ldots + A_{k_s} + D_{m_1} + \ldots + D_{m_r} + B_m,$$

$$\sum_{i=1}^s (k_i + 1) + \sum_{i=1}^r m_i + m \leq n, \quad k_1 \geq \ldots \geq k_s \geq 0, \quad m_1 \geq \ldots \geq m_r > 1, \quad m \geq 0.$$ 

From the other hand, the group $B_n$ may be considered as the Weyl group of the root system $C_n$. Changing lengths of roots in root subsystems of $C_n$ (see [1, Table 9]), we see that $B_n$ contains root subsystems of the type

$$A_{k_1} + \ldots + A_{k_s} + B_{l_1} + \ldots + B_{l_p},$$

where $\sum_{i=1}^s (k_i + 1) + \sum_{i=1}^p l_i \leq n, \quad k_1 \geq \ldots \geq k_s \geq 0, \quad l_1 \geq \ldots \geq l_p > 0$.

An elementary check shows that all root systems $\Delta_1 \subset \Delta$ consist of components of the systems described above. More precisely, the following holds:
Lemma 3. Let $\Delta = B_n$ and $\Delta_1 \subset \Delta$. Then $\Delta_1$ is of the type

$$A_{k_1} + \ldots + A_{k_s} + B_{l_1} + \ldots + B_{l_p} + D_{m_1} + \ldots + D_{m_r},$$

where $\sum_{i=1}^{s}(k_i+1) + \sum_{i=1}^{p}l_i + \sum_{i=1}^{r}m_i \leq n$, $k_1 \geq \ldots \geq k_s \geq 0$, $l_1 \geq \ldots \geq l_p > 0$, $m_1 \geq \ldots \geq m_r > 1$.

For any two root systems of the same type there exists an automorphism of $\Delta$ taking one system to another.

Notice that components of the type $A_1$ and $D_2$ consist of long roots, and components of the type $B_1$ consist of short roots. Furthermore, two different root systems $\Delta_1, \Delta_2 \subset \Delta$ may have Weyl groups $H'$ and $H''$ of the same type. For example, Weyl groups of root systems $\Delta_1 = A_3 + D_2$ and $\Delta_2 = D_3 + 2A_1$ in $\Delta = B_7$ are reflection groups with Coxeter diagram

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At the same time, there is no automorphism of $G$ taking one of these groups to another.

It is easy to see that the following is true (cf. [10]):

Lemma 4. Given reflection subgroups $H'$ and $H''$ of $G$ of the same type, there exists an automorphism of $G$ taking $H'$ to $H''$ if and only if the corresponding root systems $\Delta_1, \Delta_2 \subset \Delta$ contain the same number of components of the same type (i.e. we differ components of the type $A_3$ and $D_3$, $A_1$ and $B_1$, as well as $D_2$, $2A_1$ and $2B_1$).

Corollary 2. Maximal rank reflection subgroups $H'$ and $H''$ of the same type cannot be taken one to another by any automorphism of $G$ if and only if the corresponding root systems $\Delta_1, \Delta_2 \subset \Delta$ are of the type

$$\Delta_1 = B_{l_1} + \ldots + B_{l_p} + lB_1 + D_{m_1} + \ldots + D_{m_r} + mD_2,$$

$$\Delta_2 = B_{l_1} + \ldots + B_{l_p} + (l+2k)B_1 + D_{m_1} + \ldots + D_{m_r} + (m-k)D_2,$$

where $\sum_{i=1}^{p}l_i + \sum_{i=1}^{r}m_i + l + 2m = n$, $l_1 \geq \ldots \geq l_p > 1$, $m_1 \geq \ldots \geq m_r > 2$, $k \neq 0$.

Now we list maximal reflection subgroups of $B_n$.

Lemma 5. Maximal reflection subgroups of the reflection group $B_n$ are either $D_n$ or $B_k + B_{n-k}$, where $k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Maximality of the groups under consideration follows immediately from Lemmas 1 and 3.

Consider a reflection subgroup $H \subset G = B_n$. The corresponding root system $\Delta_1 \subset \Delta$ is of the type described in Lemma 3, i.e.

$$\Delta_1 = A_{k_1} + \ldots + A_{k_s} + B_{l_1} + \ldots + B_{l_p} + D_{m_1} + \ldots + D_{m_r},$$

$$\sum_{i=1}^{s}(k_i+1) + \sum_{i=1}^{p}l_i + \sum_{i=1}^{r}m_i \leq n.$$
The structure of the root system $B_n$ (see [8] or [9]) shows that for any component $A_{k_i}$, there exists a root system of the type $D_{k_i+1}$ which contains $A_{k_i}$ and which is orthogonal to all the rest components of $\Delta_1$. Thus,

$$ \Delta_1 \subset \Delta_2 = B_{l_1} + \ldots + B_{l_p} + D_{k_1+1} + \ldots + D_{k_s+1} + D_{m_1} + \ldots + D_{m_r}, $$

$$ \sum_{i=1}^s (k_i + 1) + \sum_{i=1}^p l_i + \sum_{i=1}^r m_i \leq n. $$

Furthermore, there exists a root system of the type $D_m$, $m = \sum_{i=1}^s (k_i + 1) + \sum_{i=1}^r m_i$, which contains the system $D_{k_1+1} + \ldots + D_{k_s+1} + D_{m_1} + \ldots + D_{m_r}$ and which is orthogonal to all the components of $\Delta_2$ of the type $B_{l_i}$. Hence,

$$ \Delta_1 \subset \Delta_2 \subset \Delta_3 = B_{l_1} + \ldots + B_{l_p} + D_m, \quad \sum_{i=1}^p l_i + m \leq n. $$

Notice also that any component $D_m$ is contained in a root system $B_m$ which is orthogonal to all components of $\Delta_3$ of the type $B_{l_i}$. Collecting, in the similar way, all the components of the type $B_{l_i}$ into root system $B_l$ for $l = \sum_{i=1}^p l_i$, we obtain the claim of the lemma. 

\[ \square \]

### 3.3 Subgroups of $F_4$

Consider the group $G = F_4$ as the Weyl group of the root system $\Delta = F_4$. Table 10 of [1] contains the list of all root subsystems of the root system $F_4$. Now we ”forget” lengths of roots to consider these root subsystems as Coxeter diagrams of reflection subgroups of the reflection group $F_4$.

**Lemma 6.** A group $2B_2$ is a unique maximal rank reflection subgroup of $F_4$ not appearing in Table 10 of [1].

For any two maximal rank reflection subgroups of the same type there exists some automorphism of $F_4$ taking one of them to another.

**Proof.** Let $H \subseteq G$ be a reflection subgroup of rank 4, such that a Coxeter diagram of $H$ does not appear in Table 10 of [1], and let $\Delta_1 \subseteq \Delta$ be the corresponding root system. Denote by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ simple roots of $\Delta_1$.

Since $\Delta_1$ is not a root subsystem of $\Delta$, there exist short roots $\alpha_i$ and $\alpha_j$, $i = 1, \ldots, 4$, such that $\alpha_i - \alpha_j \in \Delta$. We may assume that $i = 1, j = 2$. In particular, $\alpha_1$ is orthogonal to $\alpha_2$.

Note that a change of lengths of all roots of $\Delta$ induces an automorphism of the Weyl group $G$. This automorphism takes $H$ to another maximal rank reflection subgroup $H'$ corresponding to some root system $\Delta'_1 \subseteq \Delta$; a root system $\Delta'_1$ differs from $\Delta_1$ by lengths of all roots only. The images $\alpha'_i \in \Delta'_1$ of the roots $\alpha_i \in \Delta_1$ are simple roots of $\Delta'_1$.

A Coxeter diagram of $H'$ is obviously the same as one of $H$, hence $\Delta'_1$ is not a root subsystem of $\Delta$, either. Since the roots $\alpha'_1$ and $\alpha'_2$ are long, we see that $\alpha'_3 - \alpha'_4 \in \Delta$. In particular, $\alpha'_3$ and $\alpha'_4$ are mutually orthogonal short roots.
Therefore, the set of simple roots of $\Delta'$ consists of two short roots and two long roots, and any two roots of the same length are mutually orthogonal. Since two roots of different lengths cannot form an angle $\frac{\pi}{3}$, we obtain that $H'$ is either of the type $4A_1$, or $2A_1 + B_2$, or $2B_2$, so the first claim of the lemma is proved.

Further, it is shown in [1] that any two maximal rank root subsystems of $\Delta = F_4$ of the same type are conjugated by some element of the Weyl group $G$. It follows that any two maximal rank reflection subgroups corresponding to root systems of the same type are conjugated in $G$. Extending the inner automorphism group of $G$ by the outer automorphism described above, we see that for any two maximal rank reflection subgroups of the same type (different from $2B_2$) there exists an automorphism of $G$ taking one to another.

Now consider subgroups of the type $2B_2$. Any of them is contained in some subgroup of the type $B_4$, and for any two subgroups of this type there exists an automorphism of $G$ taking one of them to another. Furthermore, any two subgroups of the type $2B_2$ are conjugated in $B_4$, and any automorphism of $B_4$ may be extended to an automorphism of $G$. This proves the second statement of the lemma.

Lemma 7. A maximal reflection subgroup of $F_4$ is either of the type $B_4$ or $2A_2$.

Proof. As we have proved before, subgroups of the type $2B_2$ are not maximal. Thus, any maximal subgroup should either correspond to maximal root subsystem of $\Delta$ or be a subgroup of smaller rank. Arguments of the proof of Lemma 8 applied to a subgroup of smaller rank show that any such a subgroup is equivalent modulo the automorphism group of $G$ to a subgroup corresponding to a root subsystem of $\Delta$. Hence, we only need to consider subgroups corresponding to maximal root subsystems.

All the maximal root subsystems of $\Delta = F_4$ are listed in Table 12 of [1] (a subsystem $A_3 + A_1$ is not maximal, see [10]). Any subgroup of the type $B_3 + B_1$ is contained in some subgroup of the type $B_4$, so it is not maximal, either. Subgroups of the type $B_4$ are obviously maximal ($[F_4 : B_4] = 3$). The same for subgroups of the type $2A_2$ follows from Lemmas 1 and 8 and from the description of reflection subgroups of $B_4$ (Lemma 3) and $D_4$ (Cor. 1 and [1, Table 9]).

3.4 Subgroups of $G_2$

Rank 2 subgroups of the reflection group $G_2$ are either of the type $A_2$ or $2A_1$. All the subgroups of the type $A_2$ are conjugated in $G_2$; any two subgroups of the type $2A_1$ are equivalent modulo the automorphism group of $G_2$.

We present another elementary corollary of results provided in sections 3.2–3.4 and paper [1].

Lemma 8. Let $G$ be a finite reflection group different from $H_3$, $H_4$ and $G_2^{(m)}$ (when $m \neq 2, 3, 4, 6$). Then any two maximal rank indecomposable reflection subgroups of $G$ of the same type are equivalent modulo the automorphism group of $G$. 

3.5 Subgroups of $G_{2}^{(m)}$, $H_3$ and $H_4$

Rank 2 reflection subgroups of $G_{2}^{(m)}$ are of the type $G_{2}^{(d)}$, for some $d \mid m$. Maximal rank subgroups of the groups $H_3$ and $H_4$ are classified in [11]. In particular, Theorem 1 of [11] implies that rank 2 subgroups of $H_3$ are either of the type $G_{2}^{(3)}$ or $2A_1$. Moreover, it follows from the same theorem that all subgroups of $H_4$ of rank less than 4 are not maximal. So, the classification of reflection subgroups of $H_3$ and $H_4$ immediately follows from the classification of reflection subgroups of the rest finite reflection groups.

4 Indecomposable subgroups

In this section, we study indecomposable reflection subgroups of Euclidean reflection groups.

Let $P$ and $F$ be Euclidean Coxeter simplices and $G_P$ be a subgroup of $G_F$. The group $G_F$ contains an infinite number of mutually parallel mirrors. Hence, $P$ may be similar to $F$, i.e. $\Sigma(P)$ may coincide with $\Sigma(F)$.

4.1 Similar simplices

In this section we assume that $\Sigma(F) = \Sigma(P)$.

Let $V$ be a special vertex of $P$. Without loss of generality we may assume that $F$ is a fundamental simplex of $G_F$, such that $F$ is contained in $P$ and contains the vertex $V$. By the definition of a special vertex, $V$ is a special vertex of $F$.

Lemma 9. 1) Let $F$ be a Coxeter simplex in $\mathbb{E}^n$. For any $k \in \mathbb{N}$ there exists a Coxeter simplex $T$ such that $\Sigma(T) = \Sigma(F)$, $G_T \subset G_F$ and $[G_F : G_T] = k^n$. 2) If simplex $F$ is homothetic to $P$ and $G_P \subset G_F$, then the dilation factor $k$ is positive integer number and $[G_F : G_P] = k^n$.

Proof. 1) Let $V$ be a special vertex of $F$ and $\mu$ be a facet of $F$ opposite to $V$. Since $V$ is a special vertex, there exists a mirror $\nu$ containing $V$ and parallel to $\mu$.

Let $K$ be a simplicial cone with apex $V$ and facets containing the facets of $F$. Consider a group $G$ generated by reflections in $\mu$ and $\nu$. Mirrors of $G$ cut simplices $T_k$ out of $K$. It is clear that $\Sigma(T_k) = \Sigma(F)$, $G_{T_k} \subset G_F$ and $[G_F : G_{T_k}] = k^n$.

2) Since $F$ is a fundamental domain of $G_F$, no mirror of $G_F$ parallel to $\mu$ separates $\mu$ and $\nu$. Thus, simplices $T_k$ are only simplices contained in $K$ which are homothetic to $F$ and bounded by mirrors of $G_P$. In other words, $P = T_k$ for some $k$, and $[G_F : G_P] = k^n$.

Lemma 10. Let $F \neq P$ be Coxeter simplices in $\mathbb{E}^n$ such that $\Sigma(F) = \Sigma(P)$ and $G_P \subset G_F$. Let $V$ be a common special vertex of $F$ and $P$, and $\mu$ be a facet of $P$ opposite to $V$. The following three conditions are equivalent:

1) $1 < [G_F : G_P] < 2^n$.
2) No mirror of $G_F$ parallel to $\mu$ intersects the interior of $P$.
3) One of the following three opportunities holds:

- $\Sigma(F) = \Sigma(P) = \tilde{C}_2$, $[G_F : G_P] = 2$;
\[ \Sigma(F) = \Sigma(P) = \tilde{G}_2, \ [G_F : G_P] = 3; \]
\[ \Sigma(F) = \Sigma(P) = \tilde{F}_4, \ [G_F : G_P] = 4. \]

For any of these three cases the subgroup \( G_P \) is determined uniquely up to an automorphism of \( G_F \).

**Proof.** Let \( \mu_1, ..., \mu_n \) be the facets of \( P \) containing \( V \). Consider a diagram \( \Sigma'(F) = \Sigma(F) \setminus v \), where \( v \) is a special node of \( \Sigma(F) \) corresponding to \( \mu \). Suppose that \( \Sigma(F) \neq \tilde{C}_2, \tilde{G}_2 \) and \( \tilde{F}_4 \). Then any automorphism of the diagram \( \Sigma'(F) \) is a restriction of some automorphism of the diagram \( \Sigma(F) \). Thus, the dihedral angles formed up by \( \mu \) and \( \mu_1, ..., \mu_n \) are uniquely defined. Hence, \( P \) is homothetic to \( F \), and condition (2) implies that \( \Sigma(F) = \tilde{C}_2, \tilde{G}_2 \) or \( \tilde{F}_4 \). For any of these cases there exists a unique automorphism of \( \Sigma'(F) \) that is not a restriction of some automorphism of \( \Sigma(F) \). These automorphisms lead to the subgroups shown in Table 2.

Therefore, condition (2) implies (3). Now suppose that condition (1) holds. In this case \( P \) is not homothetic to \( F \) by the second part of Lemma 9. Thus, \( \Sigma(F) = \tilde{C}_2, \tilde{G}_2 \) or \( \tilde{F}_4 \), and condition (3) holds.

Evidently, condition (3) implies (1) and (2).

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**Table 2. Three exclusions.** The table contains all subgroups \( G_P \subset G_F \) such that \( \Sigma(F) = \Sigma(P) \) and \( [G_F : G_P] < 2^n \). Vectors \( \xi_1, ..., \xi_{n+1} \) are the outward normals to the facets \( f_i \) of \( F \). The facets are indexed as it is shown in the second column.

By \( r_i \) we denote the reflection with respect to \( f_i \). The normals to the facets of \( P \) are expressed in terms of \( \xi_1, ..., \xi_{n+1} \).

| Type | \( \Sigma(F) \) | \( \Sigma(P) \) | additional vector | index |
|------|----------------|----------------|------------------|------|
| \( \tilde{C}_2 \) | ![Diagram](Diagram1) | ![Diagram](Diagram2) | \( \xi_4 = r_1(\xi_2) \) | 2 |
| \( \tilde{G}_2 \) | ![Diagram](Diagram3) | ![Diagram](Diagram4) | \( \xi_4 = r_1r_2(\xi_3) \) | 3 |
| \( \tilde{F}_4 \) | ![Diagram](Diagram5) | ![Diagram](Diagram6) | \( \xi_6 = r_5r_4r_3(\xi_2) \) | 4 |

**Lemma 11.** Let \( P \) and \( F \) be similar Euclidean Coxeter simplices such that \( G_P \subset G_F \). Then

1) The index \( [G_F : G_P] \) determines the subgroup \( G_P \subset G_F \) by a unique way up to an automorphism of \( G_F \).

2) If the subgroup \( G_P \subset G_F \) is maximal then either \( F \) is homothetic to \( P \), or \( G_P \subset G_F \) is one of the subgroups presented in Table 2.

**Proof.** Let \( V \) be a common special vertex of \( P \) and \( F \), and assume that \( P \) contains \( F \).

Suppose that \( \Sigma(F) \neq \tilde{C}_2, \tilde{G}_2 \) and \( \tilde{F}_4 \). Then we can assume that \( P = h(F) \), where \( h \) is a homothety centered at \( V \) (see Lemma 10), and both statements of the lemma are evident.
Now suppose that $\Sigma(F) = \tilde{C}_2, \tilde{G}_2$ or $\tilde{F}_4$. Let $\mu_P$ be the facet of $P$ opposite to $V$. Let $\mu$ be the closest to $V$ mirror of $G_F$ parallel to $\mu_P$ such that $V \notin \mu$ and $\mu$ intersects $P$ ($\mu$ may coincide with $\mu_P$). The mirror $\mu$ cuts some simplex $T$ out of $P$, $T = h(P)$, where $h$ is a homothety centered at $V$ with dilation factor $\frac{k}{h}$, $k \in \mathbb{N}$ ($k$ may equal one). Consider the subgroup $G_T \subset G_F$. Notice that $G_P \subset G_T$, since $V$ is a special vertex, and $\mu$ is the closest mirror to $V$ parallel to $\mu_P$ and not containing $V$. By Lemma 11 either $T = F$ or $[G_F : G_T] = 2, 3, 4$ respectively for $\Sigma(F) = \tilde{C}_2, \tilde{G}_2$ and $\tilde{F}_4$. Thus, $[G_F : G_P] = mk^n$, where $k \in \mathbb{N}$, and either $m = 1$ or $m = 2, 3$ and $4$ respectively for $\Sigma(F) = \tilde{C}_2, \tilde{G}_2$ and $\tilde{F}_4$. Clearly, such a subgroup $G_P \subset G_F$ is completely determined by $m$ and $k$, and the numbers $m$ and $k$ are determined uniquely by the index $[G_F : G_P] = mk^n$. This proves the first part of the lemma. If the subgroup $G_P \subset G_F$ is maximal then either $m$ or $k$ is equal to one. This implies the second statement.

\[\square\]

We provide another one fact concerning similar simplices in $\mathbb{E}^n$.

**Lemma 12.** Let $F$ and $P$ be simplices in $\mathbb{E}^n$, where $\Sigma(F) = \Sigma(P)$ and $G_P \subset G_F$. Then any automorphism of $G_P$ is a restriction of some automorphism of $G_F$.

**Proof.** Let $\varphi$ be an automorphism of $G_P$. It is well-known (see, e.g., [12]) that $\varphi$ takes reflections to reflections. This means that we may consider $\varphi$ as an isometry of $\mathbb{E}^n$ preserving the set of mirrors of $G_P$.

Let $V$ be a special vertex of $P$, and $F_1$ be a fundamental simplex of $G_F$ that contains the vertex $V$ and lies inside of $P$. Such a simplex $F_1$ is unique since $\Sigma(F) = \Sigma(P)$. Then $\varphi(V)$ is a special vertex of a fundamental simplex $\varphi(P)$ of $G_P$, and $\varphi(F_1)$ is a simplex congruent to $F$ contained in $\varphi(P)$ and containing the vertex $\varphi(V)$. Denote by $F_2$ the fundamental simplex of $G_F$ contained in $\varphi(P)$ and containing $\varphi(V)$.

If $\Sigma(F) \neq \tilde{C}_2, \tilde{G}_2$ and $\tilde{F}_4$, then there exists a unique simplex of given volume with Coxeter diagram $\Sigma(F)$ containing $\varphi(V)$ as a special vertex and contained in $\varphi(P)$. Hence, $F_2 = \varphi(F_1)$, so $\varphi$ is an automorphism of $G_F$.

Suppose now that $\Sigma(F) = \tilde{C}_2, \tilde{G}_2$ or $\tilde{F}_4$. If $\varphi(F_1) = F_2$ we have nothing to prove. Suppose that $\varphi(F_1) \neq F_2$. Then either $F_1$ is homothetic to $P$, $F_2$ is not homothetic to $\varphi(P)$, or $F_1$ is not homothetic to $P$, $F_2$ is homothetic to $\varphi(P)$. Thus, one of the indices $[G_{F_1} : G_P]$ and $[G_{F_2} : G_{\varphi(P)}]$ equals $k^n$, and another one equals $2k^n$, $3k^n$ and $4k^n$ respectively for the cases $\Sigma(F) = \tilde{C}_2, \tilde{G}_2$ and $\tilde{F}_4$. This contradicts to the fact that $G_{F_1} = G_{F_2}$ and $G_P = G_{\varphi(P)}$.

\[\square\]

### 4.2 Simplices with distinct Coxeter diagrams

Now suppose that $P$ is not similar to $F$.

Let $\mu(v)$ denote the facet of Coxeter simplex $T$ corresponding to a node $v$ of $\Sigma(T)$, and let $V$ be the vertex of $T$ opposite to $\mu(v)$. Denote by $G_T\setminus\mu(v)$ the subgroup of $G_T$ generated by reflections in the facets of $T$ different from $\mu(v)$. In other words, $G_T\setminus\mu(v)$ is the stabilizer of $V$ in $G_T$. 
Lemma 13. Let $F$ and $P$ be Coxeter simplices in $\mathbb{E}^n$, where $G_P \subset G_F$. Then for any node $v$ of $\Sigma(P)$ there exists a node $w$ of $\Sigma(F)$ such that $G_{P \setminus \mu(v)} \subset G_{F \setminus \mu(w)}$ for some fundamental simplex $F_w$ of $G_F$.

Proof. Let $V$ be the vertex of $P$ opposite to $\mu(v)$. Let $F_w$ be a fundamental simplex of $G_F$ such that $F_w$ is contained in $P$ and contains $V$. Let $w$ be the node of the diagram $\Sigma(F)$ corresponding to the facet of $F_w$ opposite to $V$. The stabilizer of $V$ in $G_P$ is a subgroup of the stabilizer of $V$ in $G_F$. At the same time, the stabilizers of $V$ in $G_P$ and $G_F$ coincide with $G_{P \setminus \mu(v)}$ and $G_{F_w \setminus \mu(w)}$ respectively, so the lemma is proved.

Lemma 13 gives a necessary condition for $G_P$ to be a subgroup of $G_F$ in terms of Coxeter diagrams $\Sigma(F)$ and $\Sigma(P)$. This condition is easy to check: the groups $G_{P \setminus \mu(v)}$ and $G_{g(F) \setminus \mu(u)}$ act in the spherical space $S^{n-1}$, and we can use results of Section 3. Straightforward check of the condition described in Lemma 13 gives rise to the following claim:

Lemma 14. Let $F$ and $P$ be Coxeter simplices in $\mathbb{E}^n$ such that $G_P \subset G_F$ and $\Sigma(P) \neq \Sigma(F)$. Then the pair of diagrams $(\Sigma(P), \Sigma(F))$ coincides with one of the pairs listed in Table 3.

In particular, Lemma 13 implies that if $G_P \subset G_F$ and $\Sigma(P) = \tilde{C}_2$, $\tilde{G}_2$ or $\tilde{F}_4$, then $\Sigma(F) = \Sigma(P)$.

Further, for any pair $(F, P)$ with $\Sigma(P) \neq \Sigma(F)$ satisfying Lemma 13 we find simplices $F_1$ and $P_1$ such that $\Sigma(F_1) = \Sigma(F)$, $\Sigma(P_1) = \Sigma(P)$ and $G_{P_1} \subset G_{F_1}$. Examples of such simplices are presented in the right column of Table 3.

Lemma 15. Let $F$ and $P$ be Coxeter simplices in $\mathbb{E}^n$ such that the subgroup $G_P \subset G_F$ is maximal and $\Sigma(P) \neq \Sigma(F)$. Then there exist a special vertex $V$ of $P$ and a fundamental simplex $F_1$ of $G_F$ such that $V$ is a special vertex of $F_1$.

Proof. Let $L$ be the set of all mirrors of $G_F$ parallel to mirrors of $G_P$. Denote by $G_L$ the group generated by reflections in all mirrors contained in $L$. Obviously, $G_P \subseteq G_L \subseteq G_F$ and the subgroup $G_P \subset G_F$ is maximal. Thus, either $G_L = G_P$ or $G_L = G_F$.

Let $F_1$ be a fundamental simplex of $G_F$ contained in $P$, and let $V$ be a special vertex of $F_1$. For any mirror $\mu$ of $G_F$ there exists a mirror containing $V$ and parallel to $\mu$. If $G_L = G_P$ then any of these mirrors is contained in $G_P$, so $V$ is a special vertex of $P$.

Now, suppose that $G_L = G_F$. Then the set of linear parts of elements of $G_P$ coincides with the set of linear parts of elements of $G_F$. It follows that either $\Sigma(F) = \tilde{C}'_n$, $\Sigma(P) = \tilde{B}'_n$ or $\Sigma(F) = \tilde{B}_n$, $\Sigma(P) = \tilde{C}_n$. In both cases each special vertex of $P$ is a special vertex of some fundamental simplex of $G_F$.

Lemma 16. Let $G_P \subset G_F$ be a maximal subgroup, where $F$ and $P$ are Euclidean Coxeter simplices with $\Sigma(P) \neq \Sigma(F)$. Then $G_P$ is determined by the pair $(\Sigma(P), \Sigma(F))$ uniquely up to an automorphism of $G_F$.
Theorem 1. Let $P_1$ and $P_2$ be simplices similar to $P$, and suppose the subgroups $G_{P_1} \subset G_F$ to be maximal. It is sufficient to show that there exists an automorphism of $G_F$ taking $P_1$ to $P_2$.

By Lemma 14 there exist a special vertex $V_1$ of $P_1$ and a fundamental simplex $F_1$ of $G_F$ such that $V_1$ is a special vertex of $F_1$. Let $V_2$ be a special vertex of $P_2$, and $F_2$ be a fundamental simplex of $G_F$ such that $V_2$ is a special vertex of $F_2$.

Consider an automorphism $\varphi$ of $G_F$ taking $F_1$ to $F_2$. Notice that for any Euclidean Coxeter simplex $T$ the following holds: for any two special vertices $V$ and $U$ of $T$ there exists a symmetry of $T$ exchanging $U$ and $V$. Hence, there exists an automorphism $\psi$ of $G_F$ taking $F_2$ to itself and satisfying $\psi \circ \varphi(V_1) = V_2$. Denote $\psi \circ \varphi(P_1)$ by $P_3$.

Let $K_2$ and $K_3$ be the minimal cones with an apex $V_2$ containing $P_2$ and $P_3$ respectively.

Consider the stabilizers $G'_F(V_2)$ and $G'_P(V_2)$ of $V_2$ in $G_F$ and $G_P$ respectively. As it is shown in Section 3 (Lemma 8), an indecomposable maximal rank finite subgroup $G'_P(V_2) \subset G'_F(V_2)$ is determined by $G'_F(V_2)$ by $\Sigma'(P)$ and $\Sigma'(F)$ uniquely up to an automorphism of $G'_F(V_2)$. Hence, there exists an automorphism $\rho'$ of $G'_F(V_2)$ sending $K_3$ to $K_2$. Since $\Sigma(P) \neq \Sigma(F)$, we may assume that $\Sigma(P) \neq \tilde{C}_2, \tilde{G}_2$ and $\tilde{F}_4$. Therefore, $\rho'$ is a restriction of some automorphism $\rho$ of $G_F$. Let $P_4 = \rho(P_3)$.

Since $\Sigma(P) \neq \tilde{C}_2, \tilde{G}_2$ and $\tilde{F}_4$, any two mirrors cutting a simplex similar to $P$ out of $K_2$ are mutually parallel. The subgroup $G_{P_1} \subset G_F$ is maximal, consequently $P_2$ is cut off by a closest to $V$ mirror described above. The same is true for $P_4$. Thus, $P_4 = P_2$, so $\rho \circ \psi \circ \varphi(P_1) = P_2$.

\[ \Box \]

Lemma 17. Let $G_P \subset G_{P_1} \subset G_F$, where $F$ and $P$ are Euclidean Coxeter simplices with a common vertex $V$. Suppose that $F$ is contained in $P$, and let $P$ be the image of $P_1$ under the homothety centered at $V$ with dilation factor $k \in \mathbb{N}$. Then there exists a homothety with dilation factor $k$ taking $F$ to $F_1$, such that $G_P \subset G_{F_1} \subset G_F$.

The proof is evident: take $V$ as the center of homothety.

Summing up the above, we obtain the following

Theorem 1. Let $F$ and $P$ be Coxeter simplices in $\mathbb{E}^n$, and $G_P \subset G_F$. Then there exists a sequence of subgroups $G_P = G_{F_1} \subset G_{F_{i-1}} \subset \ldots \subset G_{F_1} \subset G_{F_0} = G_F$, where $G_{F_{i+1}} \subset G_{F_i}$ is a subgroup described either in Table 4 or in Table 5 or in Lemma 6.

The subgroup $G_P \subset G_F$ is determined by the index $[G_F : G_P]$ uniquely up to an automorphism of $G_F$.

Proof. Since $[G_F : G_P] < \infty$, there exists a sequence $G_P = G_{F_1} \subset G_{F_{i-1}} \subset \ldots \subset G_{F_i} \subset G_{F_0} = G_F$ such that any subgroup $G_{F_{i+1}} \subset G_{F_i}$ is maximal. Consider those parts of the sequence, for which $\Sigma(F_{i+1}) = \Sigma(F_i)$. By Lemma 14 a subgroup $G_{F_{i+1}} \subset G_{F_i}$ is one described either in Table 4 or in Lemma 6. Now consider those parts for which $\Sigma(F_{i+1}) \neq \Sigma(F_i)$. By Lemmas 14 and 16 these parts are described in Table 5 and the first statement of the theorem is proved.

We only left to show that the index $[G_F : G_P]$ determines the subgroup $G_P \subset G_F$ uniquely up to an automorphism of $G_F$. 

We say that $\Sigma_q \subset \Sigma_{q-1} \subset \ldots \subset \Sigma_1$ is an admissible sequence of diagrams for the subgroup $G_P \subset G_F$ if there exist simplices $T_q = P, T_{q-1}, \ldots, T_1, T_0 = F$ satisfying the following conditions:

1. $\Sigma(T_i) = \Sigma_i$, $1 \leq i \leq q$,
2. $\Sigma_1 = \Sigma(F)$,
3. $G_{T_{i+1}} \subset G_{T_i}$ is a maximal subgroup if $\Sigma(T_i) \neq \Sigma(T_{i+1})$,
4. $\Sigma(T_i) = \Sigma(T_j)$, $i < j$, if and only if $i = 0$, $j = 1$.

The sequence of subgroups $G_P = G_{T_q} \subset G_{T_{q-1}} \subset \ldots \subset G_{T_1} \subset G_{T_0} = G_F$ satisfying conditions (1)–(4) we also call admissible.

Now we will show that for any subgroup $G_P \subset G_F$ there exists an admissible sequence of subgroups.

Let $G_P = G_{F_1} \subset G_{F_{i-1}} \subset \ldots \subset G_{F_i} \subset G_{F_0} = G_F$ be a sequence of subgroups described in the theorem. Suppose that $\Sigma(F_j) = \tilde{C}_2, \tilde{G}_2, \tilde{F}_4$, and $j > 0$. Then $\Sigma(F) = \Sigma(F_1) = \ldots = \Sigma(F_j)$, so the subgroup $G_{F_j} \subset G_{F_0}$ is determined by the index $[G_{F_0} : G_{F_j}]$ by Lemma 11. Hence, we may assume that $\Sigma(F_j) \neq \tilde{C}_2, \tilde{G}_2, \tilde{F}_4$ if $j > 1$.

Further, let $\Sigma(F_i) = \Sigma(F_j)$, $i < j$, $1 < j$. Using Lemma 17 we can subtract the subgroups $G_{F_{i+1}}, \ldots, G_{F_j}$ from the sequence in the following way: if $\Sigma(F) = \Sigma(F_1)$ then apply a homothety with dilation factor $[G_{F_i} : G_{F_j}]$ to the simplices $F_1, \ldots, F_i$; if $\Sigma(F) \neq \Sigma(F_1)$ then apply a homothety with factor $[G_{F_i} : G_{F_j}]$ to the simplices $F_0, \ldots, F_i$ and insert obtained subgroups between $G_F$ and $G_{F_{j+1}}$. Note that after any of these procedures simplex $T_1$ is similar to $F$.

Thus, we need at most $l - 1$ steps to transform the initial sequence to the required one (if no simplex in the initial sequence is similar to $F$ we simply insert $T_1 = F$ between $F$ and $F_1$). Now, using Table 3 and Lemma 16 it is easy to find all admissible sequences of diagrams $\Sigma_q \subset \Sigma_{q-1} \subset \ldots \subset \Sigma_1$. The sequences with $q \geq 3$ are listed below (for the case $q = 2$ see Table 3).

| $\Sigma_q \subset \ldots \subset \Sigma_1$ | $[G_{T_1} : G_P]$ |
|------------------------------------------|------------------|
| $\tilde{D}_4 \subset \tilde{B}_4 \subset \tilde{F}_4$ | $2 \cdot 3$ |
| $\tilde{C}_4 \subset \tilde{B}_4 \subset \tilde{F}_4$ | $2^3 \cdot 3$ |
| $\tilde{D}_n \subset \tilde{B}_n \subset \tilde{C}_n$ | $4$ |

For each subgroup we found some admissible sequence of diagrams. Notice that for any pair $(\Sigma_q, \Sigma_1)$ there exists at most one admissible sequence $\Sigma_q \subset \Sigma_{q-1} \subset \ldots \subset \Sigma_1$. Hence, it is sufficient to show that each admissible sequence of diagrams corresponds to at most one subgroup of given index.

We are rest with two cases: $q = 2$ and $q = 3$ (see Lemma 11 for the case $q = 1$).

Suppose that $q = 2$. By Lemma 16 the subgroup $G_P \subset G_{T_1}$ is determined uniquely up to an automorphism of $G_{T_1}$. In particular, $[G_{T_1} : G_P]$ is uniquely determined. Hence, the index $[G_F : G_{T_1}]$ is determined, too. By Lemma 11 the subgroup $G_{T_1} \subset G_F$ is determined up to an automorphism of $G_F$. Since $\Sigma(F) =$
Given a Euclidean reflection group $G$, the theorem.

5 General description of subgroups

Given a Euclidean reflection group $G$, we present an algorithm to find all reflection subgroups of $G$.

Let $F$ be a Coxeter simplex, and $P$ be a compact Coxeter polytope in $\mathbb{E}^n$, where $G_P \subset G_F$. Let $P_1, ..., P_s$ be indecomposable factors of $P$: $P = P_1 \times ... \times P_s$, where $P_i$ are Euclidean simplices. For each diagram $\Sigma(P_i)$, $i = 1, ..., s$, take an arbitrary node $u_i$. Then $G_{P_1 \setminus \mu(u_1)} \times ... \times G_{P_s \setminus \mu(u_s)}$ is a stabilizer of some vertex $U$ of $P$ in $G_P$. Hence, $\text{Fix}(U, G_F)$ contains $G_{P_1 \setminus \mu(u_1)} \times ... \times G_{P_s \setminus \mu(u_s)}$. Without loss of generality we may assume that $U$ is a vertex of $F$.

Therefore, any finite index reflection subgroup of $G_F$ can be obtained as a result of the following procedure:

1) Choose a vertex $U$ of $F$ and a maximal rank reflection subgroup $H$ of $\text{Fix}(U, G_F)$ (i.e. $U$ is the only point of $\mathbb{E}^n$ fixed by $H$). Denote by $K$ a fundamental cone of $H$. We have $K = K_1 \times ... \times K_s$, where $K_i$ are indecomposable cones.

2) For each cone $K_i$ take a mirror $\mu_i$ of $G_F$ such that $\mu_i \cap K_j = \emptyset$, $j \neq i$, and $\mu_i$ cuts an acute-angled simplex $P_i$ out of $K_i$. Let $P = P_1 \times ... \times P_s$. Then $G_P$ is a finite index subgroup of $G_F$.

To give an explicit description of mirrors $\mu_i$ we use affine root systems.

Let $\Delta_F$ be an affine root system such that $G_F$ is the Weyl group of $\Delta_F$. More precisely, for each multiple edge of $\Sigma(F)$ put an arrow to arrange a Dynkin diagram $S(F)$. We choose direction of arrows in order to obtain a diagram contained in Table “Aff1” (see [S] Ch. 4). The root system $\Delta_F$ consists of two disjoint parts: the set of real roots $\Delta_F^\text{re}$ and the set of imaginary roots $\Delta_F^\text{im} = \{\pm \delta, \pm 2\delta, ...\}$ (see [S] Prop. 5.10]). Here $\delta = \sum a_i \alpha_i$, $\alpha_i$ are simple roots of $\Delta_F$, and $a_i$ are the coefficients of the linear dependency between the columns of generalized Cartan matrix. Further, let $V$ be a special vertex of $F$, and $M$ be a set of mirrors of $G_F$ containing $V$. Let $\Delta_F' \subset \Delta_F$ be a finite root system that consists of all roots vanishing on mirrors contained in $M$. By Prop. 6.3. of [S], $\Delta_F^\text{re} = \{\alpha + n\delta | \alpha \in \Delta_F', n \in \mathbb{Z}\}$.

Let $P$ be a compact Coxeter polytope in $\mathbb{E}^n$, and $G_P \subset G_F$. Consider roots of $\Delta_F$ vanishing on the facets of $P$. These roots compose a set of simple roots for some root system $\Delta \subset \Delta_F$. The Weyl group of $\Delta$ coincides with $G_P$.

Following the procedure described above, consider an arbitrary maximal rank finite root system $\Delta' \subset \Delta_F$. Let $\Delta' = \Delta_1' + ... + \Delta_s'$, where $\Delta_i'$, $i = 1, ..., s$, are indecomposable components, and let $\Pi_i$ be a set of simple roots of $\Delta_i'$. For each of $\Pi_i$ we should add a root $\beta_i$ vanishing on $\mu_i$ and satisfying the following two conditions: $\beta_i$ is orthogonal to each root of $\Delta_j'$ if $i \neq j$; for any $\gamma \in \Pi_i$ the angle formed up by $\beta_i$ and $\gamma$ is not acute.
Table 3. Indecomposable maximal subgroups. Finite index indecomposable maximal reflection subgroups of Euclidean reflection groups are listed in the table. Notation is the same as in Table 2.

| $\Sigma(F)$       | $\Sigma(P)$       | index |
|-------------------|-------------------|-------|
| $\tilde{B}_n$ $n \geq 3$ | $\tilde{D}_n$ | 2     |
| $\begin{array}{cccccc}
1 & 2 & 3 & 4 & \ldots & n+1 \\
\end{array}$ | $\begin{array}{cccccc}
1 & 2 & 3 & 4 & \ldots & n+2 \\
\end{array}$ | $\xi_{n+2} = r_{n+1}(\xi_n)$ |

| $\tilde{C}_n$ $n \geq 3$ | $\tilde{B}_n$ | 2     |
|-----------------------------|----------------|-------|
| $\begin{array}{cccccc}
1 & 2 & 3 & 4 & \ldots & n+1 \\
\end{array}$ | $\begin{array}{cccccc}
2 & 3 & 4 & \ldots & n+2 \\
\end{array}$ | $\xi_{n+2} = r_1(\xi_2)$ |

| $\tilde{E}_8$ | $\tilde{D}_8$ | $2 \cdot 3^3 \cdot 5$ |
|----------------|----------------|------------------------|
| $\begin{array}{cccccccccccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}$ | $\begin{array}{cccccccccccccccccccccccccccccccc}
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}$ | $\xi_{10} = r_1 r_2 r_3 r_4 r_5 r_6 r_7 r_8 r_9 r_{10}(\xi_3)$ |

| $\tilde{A}_8$ | $\tilde{A}_7$ | $2^7 \cdot 3^2 \cdot 5$ |
|----------------|----------------|------------------------|
| $\begin{array}{cccccccccccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}$ | $\begin{array}{cccccccccccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 10 \\
\end{array}$ | $\xi_{10} = r_9 r_2 r_3 r_4 r_5 r_6 r_7 r_8 r_9 r_{10}(\xi_3)$ |

| $\tilde{G}_2$ | $\tilde{A}_2$ | 2 |
|----------------|----------------|-------|
| $\begin{array}{cccccccccccccccccccccccccccccccc}
1 & 2 & 3 \\
\end{array}$ | $\begin{array}{cccccccccccccccccccccccccccccccc}
2 & 3 & 4 & 5 & 6 \\
\end{array}$ | $\xi_6 = r_1 r_2(\xi_3)$ |

| $\tilde{F}_4$ | $\tilde{B}_4$ | 3 |
|----------------|----------------|-------|
| $\begin{array}{cccccccccccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array}$ | $\begin{array}{cccccccccccccccccccccccccccccccc}
2 & 3 & 4 & 5 & 6 \\
\end{array}$ | $\xi_{6} = r_1 r_2(\xi_3)$ |

We can always take $\theta_i + k_i \delta$ as $\beta_i$, where $\theta_i$ is the lowest root of $\Delta_i$, and $k_i \in \mathbb{N}$. However, sometimes there exist additional roots satisfying the conditions above. In more details, let $S_i$ be a Dynkin diagram of $\Pi_i$. If $S_i = B_l, C_l$ ($l \geq 3$), $F_4, G_2$ or $C_2$, there exists a family of mutually parallel mirrors such that any of these mirrors cuts
Table 4. An additional root $\theta'$.

|   | $S_i$ | $\theta'$       |
|---|-------|----------------|
| $B_l$ | $1\ 2\ \ldots\  l-1\ l$ | $\frac{1}{2}(-\gamma_1 + \theta)$ |
| $C_l$ | $1\ 2\ \ldots\  l-1\ l$ | $\gamma_1 + \theta$ |
| $G_2$ | $1\ 2$ | $\gamma_1 + \gamma_2 + \theta$ |
| $C_2$ | $1\ 2$ | $\gamma_1 + \theta$ |
| $F_4$ | $1\ 2\ 3\ 4$ | $\gamma_1 + \gamma_2 + \gamma_3 + \theta$ |

an acute-angled polytope out of $K_i$. Namely, the roots vanishing on these family equal $\theta' + k\delta$, where $k \in \mathbb{N}$, and $\theta'$ are listed in Table 4.

It is easy to check that there are no other possibilities for $\beta_i$.

Given a pair of Coxeter polytopes $F$ and $P$ one can ask if it is possible that $G_P \subset G_F$. The following theorem gives a criterion in terms of Coxeter diagrams $\Sigma(F)$ and $\Sigma(P)$.

**Theorem 2.** Let $F$ and $P$ be Euclidean Coxeter polytopes. A polytope $T$ satisfying $\Sigma(P) = \Sigma(T)$ and $G_T \subset G_F$ exists if and only if there exists an embedding of $G_P$ into $G_F$.

**Proof.** To prove that the condition is necessary, assume that $T = P$. Since $G_P \subset G_F$, we may also assume that $P$ contains $F$, and a special vertex $V$ of $P$ is also a vertex of $F$. Then $G'_P(V)$ is a subgroup of the stabilizer $\text{Fix}(V,G_F)$ of $V$ in $G_F$. Clearly, there exists an embedding of $\text{Fix}(V,G_F)$ into $G'_F$ (that takes any mirror $\mu \in \text{Fix}(V,G_F)$ to a mirror parallel to $\mu$ and containing some fixed special vertex of $F$). Hence, we found an embedding $G'_P \hookrightarrow G'_F$.

To prove that the condition is sufficient, consider an image $G$ of $G'_P$ under the embedding $G'_P \hookrightarrow G'_F$. Let $K$ be a fundamental chamber of $G$, and $\Delta' \subset \Delta_F$ be a finite root system whose simple roots vanish on the facets of $K$. Let $\Delta' = \Delta'_1 + \ldots + \Delta'_s$, where $\Delta'_i$, $i = 1, \ldots, s$, are indecomposable components, and let $\Pi_i$ be a set of simple roots of $\Delta'_i$. For each of $\Pi_i$ we add a root $\theta_i + \delta$, where $\theta_i$ is the lowest root of $\Delta'_i$. Denote by $T_1$ a fundamental chamber of the Weyl group of the direct sum of resulting root systems. Clearly, $G_{T_1} \subset G_F$.

By the construction of $T_1$, the diagram $\Sigma(T_1)$ is very similar to $\Sigma(P)$. The only possible difference is that some indecomposable components $\tilde{B}_l$ may be substituted by $\tilde{C}_l$ (and some $\tilde{C}_l$ may be substituted by $\tilde{B}_l$). For each of these components we take an index 2 (index $2^{l-1}$ respectively) reflection subgroup described in Table 2. In this way we obtain a subgroup with fundamental chamber whose Coxeter diagram coincides with $\Sigma(P)$, and the theorem is proved.
Remark on the uniqueness

In general, a decomposable subgroup \( G_P \subset G_F \) is not determined by \((\Sigma(P), \Sigma(F))\) and \([G_F : G_P]\). For example, see Fig. 1 for fundamental polygones of three different subgroups \( G_P \subset G_F \) with \( \Sigma(F) = \tilde{C}_2, \Sigma(P) = 2\tilde{A}_1, \) \([G_F : G_P] = 2 \cdot 4.\) Clearly, none of these subgroups is equivalent to another modulo the automorphism group of \( G_F \).

However, some additional conditions imply the uniqueness (up to automorphism of \( G_F \)).

Let \( G_P \subset G_F \) be a subgroup, and let \( M \) be a set of linear parts of all elements of \( G_P \). Suppose that \( G_P \) contains reflections in all those mirrors of \( G_F \) whose linear part is contained in \( M \). Then we say that \( G_P \) is a block-maximal subgroup of \( G_F \).

Lemma 18. A block-maximal subgroup \( G_P \subset G_F \) is determined by \((\Sigma(P), \Sigma(F))\) and \([G_F : G_P]\) uniquely up to an automorphism of \( G_F \).

Proof. Without loss of generality we may assume that \( F \) is contained in \( P \). Let \( V \) be a special vertex of \( F \). For any mirror \( \mu \) of \( G_P \) there exists a mirror containing \( V \) and parallel to \( \mu \). Since the subgroup \( G_P \subset G_F \) is block-maximal, any of these mirrors is contained in \( G_P \), so \( V \) is a special vertex of \( P \).

By the definition of a block-maximal subgroup, \( G_P \) is completely determined by the stabilizer \( G_P' \) of the special vertex \( V \). As it is shown in Section \( \Box \) an embedding of the maximal rank finite subgroup \( G_P' \) into \( G_F' \) is usually unique (up to an automorphism of \( G_F' \)). The only exclusions are the subgroups described in Cor. \( \Box \).

However, for these subgroups the indices \([G_F : G_P]\) differ by factor \( 2^k \), where \( k \neq 0 \) is the number defined in Cor. \( \Box \).

Suppose that \( \Sigma(F) \neq \tilde{C}_2, \tilde{G}_2 \) and \( \tilde{F}_4 \). Then any automorphism of \( G_F' \) can be extended to an automorphism of \( G_F \). Hence, the subgroup \( G_P \subset G_F \) is determined by \((\Sigma(P), \Sigma(F))\) uniquely up to an automorphism of \( G_F \).

Now, suppose that \( \Sigma(F) = \tilde{C}_2, \tilde{G}_2 \) or \( \tilde{F}_4 \). For each of these cases there exists the only symmetry of \( \Sigma'(F) \) that can not be extended to a symmetry of \( \Sigma(F) \). In other words, there are only two different embeddings of \( G_P' \) into \( G_F' \) (up to an automorphism of \( G_F \)). It is easy to check that for these embeddings the indices \([G_F : G_P]\) differ by factor 2, 3 and 4 respectively for \( \Sigma(F) = \tilde{C}_2, \tilde{G}_2 \) and \( \tilde{F}_4 \).
6 Decomposable maximal subgroups

In this section we describe decomposable maximal reflection subgroups.

Let $G$ be a reflection group, $H \subset G$ be a finite reflection subgroup of $G$, and $M$ be a set of all mirrors of $G$ that are parallel to mirrors of $H$. Let $G_M$ be a group generated by all reflections with respect to the mirrors contained in $M$. Then we say that $G_M$ is a $G$-extension of $H$. Obviously, $G_M$ is a block-maximal subgroup of $G$.

Lemma 19. Let $F$ be a Coxeter simplex in $E^n$, $P$ be a decomposable polytope in $E^n$, and $G_P \subset G_F$ be a maximal reflection subgroup. Then $G_P$ is a $G_F$-extension of some maximal reflection subgroup of $G'_F$.

Conversely, a $G_F$-extension of decomposable maximal reflection subgroup of $G'_F$ is a maximal reflection subgroup of $G_F$.

Proof. Let $L$ be the set of all mirrors of $G_F$ that are parallel to mirrors of $G_P$. Denote by $G_L$ the group generated by reflections in all mirrors contained in $L$. Since $G_P \subset G_L \subset G_F$ and the subgroup $G_P \subset G_F$ is maximal, either $G_L = G_P$ or $G_L = G_F$. The case $G_L = G_F$ is impossible, since $G_P$ is decomposable and $G_F$ is not. Thus, $G_L = G_P$ and $G_P$ is a $G_F$-extension of $G'_P$.

Furthermore, a spherical subgroup $G'_P$ is a maximal finite subgroup of $G'_F$ (indeed, if there exists a reflection subgroup $H$ satisfying $G'_P \subset H \subset G'_F$, then the $G_F$-extension of $H$ is a subgroup of $G_F$ containing $G_P$). Therefore, any decomposable maximal subgroup is a $G_F$-extension of some maximal finite (decomposable) subgroup of $G_F$.

From the other hand, it is clear that a $G_F$-extension of any maximal finite subgroup is a maximal subgroup of $G_F$.

Corollary 3. A decomposable maximal reflection subgroup $G_P$ of an indecomposable reflection group $G_F$ is block-maximal. Such a subgroup $G_P \subset G_F$ is determined by diagrams $\Sigma(F)$, $\Sigma(P)$ and the index $[G_F : G_P]$ uniquely up to an automorphism of $G_F$.

Proof. By Lemma 19 the subgroup $G_P \subset G_F$ is a $G_F$-extension of some maximal subgroup. Thus, $G_P$ is block-maximal. The uniqueness follows from Lemma 18.

Using Table 12 of [1] (see also [10]) and results of Section 3, it is easy to find all maximal rank decomposable maximal reflection subgroups of indecomposable finite reflection groups. Table 5 contains the complete list of them. The same table contains the list of maximal decomposable Euclidean reflection subgroups of indecomposable Euclidean reflection groups. To find the indices $[G_F : G_P]$ of these subgroups we calculated volumes of $P$ and $F$: $[G_F : G_P] = \frac{\text{Vol}(P)}{\text{Vol}(F)}$, where $\text{Vol}(T)$ is the volume of $T$.

Remark. Table 5 shows that $[G_F : G_P]$ is a multiple of $[G'_F : G'_P]$. The reason of this is the following. Let $V$ be a special vertex of $F$ and $O_V$ be the orbit of $V$ under the action of $G_F$. Let $V_1, \ldots, V_k$ be the points of $O_V$ contained either in $P$ or at the
Table 5. Maximal rank decomposable maximal reflection subgroups of spherical and Euclidean indecomposable reflection groups.

| $\Sigma'(F)$ | $\Sigma'(P)$ | $[G'_F : G'_P]$ | $\Sigma(F)$ | $\Sigma(P)$ | $[G_F : G_P]$ |
|-------------|-------------|-----------------|-------------|-------------|----------------|
| $B_n = C_n$ | $B_k + B_{n-k}$ | $(\binom{n}{k})$ | $\tilde{B}_n$ | $\tilde{B}_k + \tilde{B}_{n-k}$ | $2^{(\binom{n}{k})}$ |
| $D_n$ | $D_k + D_{n-k}$ | $2^{(\binom{n}{k})}$ | $\tilde{D}_n$ | $\tilde{D}_k + \tilde{D}_{n-k}$ | $2^{(\binom{n}{k})}$ |
| $G_2$ | $2A_1$ | $3$ | $\tilde{G}_2$ | $2\tilde{A}_1$ | $2 \cdot 3$ |
| $F_4$ | $2A_2$ | $2^5$ | $\tilde{F}_4$ | $2\tilde{A}_2$ | $2^5 \cdot 3$ |
| $E_6$ | $A_5 + A_1$ | $2^2 \cdot 3^2$ | $\tilde{E}_6$ | $\tilde{A}_5 + \tilde{A}_1$ | $2^3 \cdot 3^2$ |
| $E_6$ | $3A_2$ | $2^4 \cdot 5$ | $\tilde{E}_6$ | $3\tilde{A}_2$ | $2^4 \cdot 3^2 \cdot 5$ |
| $E_7$ | $D_6 + A_1$ | $3^2 \cdot 7$ | $\tilde{E}_7$ | $\tilde{D}_6 + \tilde{A}_1$ | $2 \cdot 3^2 \cdot 7$ |
| $E_7$ | $A_5 + A_2$ | $2^5 \cdot 3 \cdot 7$ | $\tilde{E}_7$ | $\tilde{A}_5 + \tilde{A}_2$ | $2^5 \cdot 3^2 \cdot 7$ |
| $E_8$ | $E_7 + A_1$ | $2^3 \cdot 3 \cdot 5$ | $\tilde{E}_8$ | $\tilde{E}_7 + \tilde{A}_1$ | $2^4 \cdot 3 \cdot 5$ |
| $E_8$ | $E_6 + A_2$ | $2^6 \cdot 5 \cdot 7$ | $\tilde{E}_8$ | $\tilde{E}_6 + \tilde{A}_2$ | $2^6 \cdot 3 \cdot 5 \cdot 7$ |
| $E_8$ | $2A_4$ | $2^8 \cdot 3^3 \cdot 7$ | $\tilde{E}_8$ | $2\tilde{A}_4$ | $2^8 \cdot 3^3 \cdot 5 \cdot 7$ |

boundary of $P$. Let $G_{P(i)}$ be the group generated by reflections in those facets of $P$ that contain $V_i$. To find the index $[G_F : G_P]$ it is sufficient to calculate the number of images of $F$ under $G_F$ contained in the polytope $P$:

$$[G_F : G_P] = \sum_{i=1}^{k} \frac{|G'_F|}{|G'_{P(i)}|}$$

where $|G|$ is the order of $G$. Since $G_{P(i)}$ is a subgroup of $G'_P$, we have

$$|G_{P(i)}| = \frac{|G'_P|}{|G'_{P(i)}|}.$$

Hence,

$$\frac{[G_F : G_P]}{[G'_F : G'_P]} = \sum_{i=1}^{k} [G'_P : G_{P(i)}] \in \mathbb{Z}.$$

7 Infinite index subgroups

In previous sections we assumed that the polytope $P$ is compact, so $G_P \subset G_F$ is a finite index subgroup. However, sometimes a fundamental chamber of a discrete group generated by reflections is not compact. In this section, we discuss infinite index reflection subgroups of discrete Euclidean indecomposable reflection group $G_F$ (where $F$ is a Coxeter simplex).
Let $W$ be a discrete group generated by reflections in $\mathbb{E}^n$, and $P$ be a fundamental chamber of $W$. Then $P$ is a generalized Coxeter polytope, which is a convex domain bounded by finite number of hyperplanes $f_1, ..., f_k$, where either $f_i$ is parallel to $f_j$, or $f_i$ and $f_j$ form an angle $\frac{\pi}{m_{ij}}$, $m_{ij} \in \mathbb{N}$. As it is shown in [6], a generalized Euclidean Coxeter polytope is a direct product of several simplices and simplicial cones. A Coxeter diagram of this polytope is a union of several connected parabolic and elliptic diagrams.

Let $P = P_1 \times \ldots \times P_s \times P_{s+1} \times \ldots \times P_{s+t}$ be a decomposition of a Coxeter polytope $P$ into indecomposable components, where $P_1, \ldots, P_s$ are Euclidean simplices and $P_{s+1}, \ldots, P_{s+t}$ are indecomposable simplicial cones. Then $\Sigma(P)$ is a union of $s$ connected parabolic diagrams and $t$ connected elliptic diagrams. Let $v_1, \ldots, v_s$ be special vertices of the diagrams $\Sigma(P_1), \ldots, \Sigma(P_s)$ respectively.

A direct generalization of arguments of Section 5 leads to the following description of subgroups of given indecomposable Euclidean reflection group.

Consider a simplex $F$ in $\mathbb{E}^n$ and a root system $\Delta_F$ described in Section 5. Then any reflection subgroup (that may be of infinite index) of $G_F$ can be obtained as a result of the following procedure: choose a finite reflection subgroup $G \subset G_F$ ($G$ may not be of maximal rank); for each indecomposable component $G_i$ of $G$ consider the corresponding root system $\Delta_i \subset \Delta_F$ and take a positive integer $k_i$. Now enlarge some of $\Delta_i$ by the roots $\theta_i + k_i \delta$, where $\theta_i$ is the lowest root of $\Delta_i$ (one can take $\theta'$ instead of $\theta$ for some components, see Table 4), the rest systems $\Delta_i$ leave unchanged. The Weyl group $W$ of the resulting root system is a reflection subgroup of $G_F$ ($W$ may be of infinite index).

Now, suppose that the subgroup $W \subset G_F$ is maximal. Finite index maximal reflection subgroups are classified in Sections 4 and 6. In the following theorem we list all infinite index maximal reflection subgroups.

**Theorem 3.** Let $F$ be a Euclidean Coxeter simplex and $W \subset G_F$ be an infinite index maximal reflection subgroup. Let $\Sigma$ be the Coxeter diagram of fundamental chamber of $W$. Then $(\Sigma(F), \Sigma)$ coincides with one of the following pairs: $(\tilde{A}_n, \tilde{A}_k + \tilde{A}_{n-1-k})$, $k = 0, 1, 2, ..., n-1$, $(\tilde{D}_n, \tilde{D}_{n-1})$, $(\tilde{D}_n, \tilde{A}_{n-1})$, $(\tilde{E}_6, \tilde{D}_5)$ and $(\tilde{E}_7, \tilde{E}_6)$.

**Proof.** Let $W' \subset G'_{F}$ be a finite maximal reflection subgroup of $W$, and $\Sigma'$ be a Coxeter diagram of fundamental domain of $W'$. Since $W \subset G_F$ is a maximal subgroup, the subgroup $W' \subset G'_{F}$ is also maximal. Results of Section 6 imply that the groups $B_n$, $F_4$ and $G_2$ have no maximal subgroups of non-maximal rank. By Cor. 11 all maximal reflection subgroups of non-maximal rank are listed in Table 12 of [11]. Namely, in this case the pair $(\Sigma'(F), \Sigma')$ coincides with one of the following: $(A_n, A_k + A_{n-1-k})$, $k = 0, 1, 2, ..., n-1$, $(D_n, D_{n-1})$, $(D_n, A_{n-1})$, $(E_6, D_5)$ and $(E_7, E_6)$. This proves the theorem.

\[\square\]

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