Optimal Warping Paths are unique for almost every pair of Time Series

Brijnesh J. Jain and David Schultz
Technische Universität Berlin, Germany
e-mail: brijnesh.jain@gmail.com

An optimal warping path between two time series is generally not unique. The size and form of the set of pairs of time series with non-unique optimal warping path is unknown. This article shows that optimal warping paths are unique for almost every pair of time series in a measure-theoretic sense. All pairs of time series with non-unique optimal warping path form a negligible set and are geometrically the union of zero sets of quadratic forms. The result is useful for analyzing and understanding adaptive learning methods in dynamic time warping spaces.

1 Introduction

In pattern recognition and data mining of temporal data, it is often necessary to compare two time series and measure the extent to which they differ. Comparison of time series is used for classification, clustering, and retrieval [3, 4, 19]. The challenge is that time series can differ in length and vary in speed from one segment to another. In these situations, there is no natural correspondence between the time points in one time series and those in the other. The general approach to compare two time series aligns both time series by expanding and compressing different segments such that the shape of the aligned time series is as similar as possible. Technically, aligning aims at finding an optimal correspondence between the time points of both time series by minimizing a cost function over all admissible correspondences. The cost of an optimal correspondence is then used to measure the dissimilarity between the time series under consideration.

One standard technique of time series comparison is dynamic time warping (DTW) [15]. Warping paths determine the set of admissible correspondences. The cost of an optimal warping path defines the DTW distance. Optimal warping paths between two time series are generally not unique, whereas the DTW distance is well-defined, because the costs of different optimal warping paths are identical.

Non-uniqueness of optimal warping paths is not critical for pairwise dissimilarity approaches such as hierarchical clustering algorithms or nearest-neighbor methods for classification and retrieval. Such approaches only depend on the DTW distance but not on the choice of an optimal warping path. However, non-uniqueness of optimal warping paths becomes an issue for adaptive learning methods in DTW spaces. Examples of adaptive methods in DTW spaces are time series averaging [10, 12, 16], k-means clustering [6, 13, 14, 17], self-organizing maps [9], learning vector quantization [18, 8], and linear classifiers [7]. All these examples have in common that their adaptive update rules are based on the choice of an optimal warping path. Non-uniqueness of an optimal warping path can cause several problems for adaptive learning in DTW spaces. For example, different optimal warping paths usually result in different update directions. This in turn can degrade stability of the learning process and impedes a theoretical analysis.

The extent of the difficulties caused by non-uniqueness depends on how common it is that two time series have more than one optimal warping path. The above highlighted problems can be handled easier when non-uniqueness occurs exceptionally rather than almost always. The size and form of the set of pairs of time series that can be aligned by a unique optimal warping path is an open question.
In this article, we prove that optimal warping paths are unique almost everywhere, that is for almost every pair of time series. We show that the set of time series for which an optimal warping set is not unique corresponds to the union of zero sets of quadratic forms. The results assume the Borel-Lebesgue measure space, the squared Euclidean norm as local cost function and hold for uni- as well as multivariate time series.

The colloquial term "almost everywhere" has a precise measure-theoretic meaning. A measure quantifies the size of a set. It generalizes the concepts of length, area, and volume of a solid body defined in one, two, and three dimensions. The term "almost everywhere" finds its roots in the notion of a "negligible set". Negligible sets are sets contained in a set of measure zero. When working in a measure space, a negligible set contains the exceptional cases we do not care about and often ignore. For example, the set of all points where a function is discontinuous on the negligible set contains the exceptional cases we do not care about and often ignore. For example, the function
\[ f(x) = \begin{cases} 1 : & x \neq 0 \\ 0 : & x = 0 \end{cases} \]
is discontinuous on the negligible set \{0\} with measure zero. We say, function \( f \) is continuous almost everywhere, because the set where \( f \) is not continuous is negligible. More generally, we say a property \( P \) is true "almost everywhere" if the set where \( P \) is false is negligible. When integrating \( f \) over \([-1, 1]\), for example, we do not care about the behavior of \( f \) on a negligible set. The situation is a bit different for adaptive learning in DTW spaces. In this case, we can not plainly ignore points from a negligible set, but we are still in the position to properly cope with these points.

Here, we show that the set of pairs of time series with non-unique optimal warping paths is a negligible set. Then we can conclude that the property of a unique optimal warping path holds almost everywhere.

Based on the warping embedding formalism introduced in \([16]\), the proof casts the problem of measuring the set of time series with non-unique optimal warping paths to the problem of measuring the zero sets of quadratic forms in the corresponding Lebesgue-Borel measure space.

## 2 Almost Everywhere Uniqueness of Optimal Warping Paths

This section first introduces warping paths and then defines the notions of 'negligible' and 'almost everywhere'. Finally, we state the almost everywhere result for uniqueness and describe the form of the non-unique set.

### 2.1 Time Series and Warping Paths

We first define time series. Let \( \mathcal{F} = \mathbb{R}^d \) denote the \( d \)-dimensional feature space. A \( d \)-variate time series of length \( m \) is a sequence \( x = (x_1, \ldots, x_m) \) consisting of features \( x_i \in \mathcal{F} \). By \( \mathcal{F}^m \) we denote the set of all time series of length \( m \) with features from \( \mathcal{F} \).

Next, we describe warping paths. Let \([n] = \{1, \ldots, n\} \), where \( n \in \mathbb{N} \). A \((m \times n)\)-lattice is a set of the form \( \mathcal{L}_{m,n} = [m] \times [n] \). A warping path in lattice \( \mathcal{L}_{m,n} \) is a sequence \( p = (p_1, \ldots, p_L) \) of \( L \) points \( p_l = (i_l, j_l) \in \mathcal{L}_{m,n} \) such that

1. \( p_1 = (1,1) \) and \( p_L = (m,n) \) (boundary conditions)
2. \( p_{l+1} - p_l \in \{(1,0), (0,1), (1,1)\} \) for all \( l \in [L-1] \) (step condition)

By \( \mathcal{P}_{m,n} \) we denote the set of all warping paths in \( \mathcal{L}_{m,n} \). A warping path departs at the upper left corner \((1,1)\) and ends at the lower right corner \((m,n)\) of the lattice. Only cast \((0,1)\), south \((1,0)\), and southeast \((1,1)\) steps are allowed to move from a given point \( p_l \) to the next point \( p_{l+1} \) for all \( 1 \leq l < L \).

Finally, we introduce optimal warping paths. A warping path \( p \in \mathcal{P}_{m,n} \) defines an alignment (warping) between time series \( x \in \mathcal{F}^m \) and \( y \in \mathcal{F}^n \) by relating features \( x_i \) and \( y_j \) if \((i,j) \in p\). The cost of aligning time series \( x \) and \( y \) along warping path \( p \) is defined by
\[
C_p(x,y) = \sum_{(i,j) \in p} \| x_i - y_j \|^2.
\]
where \( \|z\| \) denotes the Euclidean norm on \( \mathcal{F} \). A warping path \( p_* \in \mathcal{P}_{m,n} \) between \( x \) and \( y \) is \textit{optimal} if

\[
p_* = \arg\min_{p \in \mathcal{P}_{m,n}} C_p(x,y).
\]

By \( \mathcal{P}_*(x,y) \) we denote the set of all optimal warping paths between time series \( x \) and \( y \).

2.2 Almost Everywhere Uniqueness

Our goal is to measure the size of the subset \( N \subseteq \mathcal{F}^m \times \mathcal{F}^n \) of pairs of time series \( x \in \mathcal{F}^m \) and \( y \in \mathcal{F}^n \) that have multiple optimal warping paths. Measure theory provides a systematic way to measure the content of suitable sets satisfying certain calculation rules. For example, in one, two, and three dimensions, the length, area, and volume are examples of a measure of a suitable subset. We first introduce the necessary measure-theoretic concepts and then state our main results on almost everywhere uniqueness.

2.2.1 Basic Definitions from Measure Theory

We briefly introduce basic concepts from measure theory. For details, we refer to [5].

One issue in measure theory is that not every subset of a given set \( \mathcal{X} \) is measurable. A family \( \mathcal{A} \) of measurable subsets of a set \( \mathcal{X} \) is called \( \sigma \)-algebra in \( \mathcal{X} \). A measure is a function \( \mu : \mathcal{A} \to \mathbb{R}_+ \) that assigns a non-negative value to every measurable subset of \( \mathcal{X} \) such that certain conditions are satisfied.

To introduce these concepts formally, we assume that \( 2^\mathcal{X} \) denotes the power set of a set \( \mathcal{X} \), that is the set of all subsets of \( \mathcal{X} \). A system \( \mathcal{A} \subseteq 2^\mathcal{X} \) is called a \( \sigma \)-algebra in \( \mathcal{X} \) if it has the following properties:

1. \( \mathcal{X} \in \mathcal{A} \)
2. \( U \in \mathcal{A} \) implies \( \mathcal{X} \setminus U \in \mathcal{A} \)
3. \( (U_i)_{i \in \mathbb{N}} \in \mathcal{A} \) implies \( \bigcup_{i \in \mathbb{N}} U_i \in \mathcal{A} \).

A measure on \( \mathcal{A} \) is a function \( \mu : \mathcal{A} \to [0, +\infty] \) that satisfies the following properties:

1. \( \mu(U) \geq 0 \) for all \( U \in \mathcal{A} \)
2. \( \mu(\emptyset) = 0 \)
3. For a countable collection of disjoint sets \( (U_i)_{i \in \mathbb{N}} \in \mathcal{A} \), we have

\[
\mu\left(\bigcup_{i \in \mathbb{N}} U_i\right) = \sum_{i \in \mathbb{N}} \mu(U_i).
\]

A triple \( (\mathcal{X}, \mathcal{A}, \mu) \) consisting of a set \( \mathcal{X} \), a \( \sigma \)-algebra \( \mathcal{A} \) in \( \mathcal{X} \), and a measure \( \mu \) defined on \( \mathcal{A} \) is called a \textit{measure space}.

The Borel-algebra \( \mathcal{B} \) in \( \mathbb{R}^d \) is the \( \sigma \)-algebra generated by the open sets of \( \mathbb{R}^d \). The Lebesgue-measure \( \mu \) on \( \mathcal{B} \) generalizes the concept of \( d \)-volume of a box in \( \mathbb{R}^d \). The triple \( (\mathbb{R}^d, \mathcal{B}, \mu) \) is called Borel-Lebesgue measure space.

Let \( (\mathcal{X}, \mathcal{A}, \mu) \) be a measure space, where \( \mathcal{X} \) is a set, \( \mathcal{A} \) is a \( \sigma \)-algebra in \( \mathcal{X} \), and \( \mu \) is a measure defined on \( \mathcal{A} \). A set \( N \subseteq \mathcal{X} \) is \( \mu \)-\textit{negligible} if there is a set \( N' \in \mathcal{A} \) such that \( \mu(N') = 0 \) and \( N \subseteq N' \). A property of \( \mathcal{X} \) is said to hold \( \mu \)-\textit{almost everywhere} if the set of points in \( \mathcal{X} \) where this property fails is \( \mu \)-negligible.

2.2.2 Almost Everywhere Uniqueness

To state Theorem 2.1 we use the following terminology: Let \( m, n \in \mathbb{N} \) and \( k = m + n \). We write \( \mathcal{X} = \mathcal{F}^m \times \mathcal{F}^n \) and \( \mathcal{X}^2 = \mathcal{F}^{k \times k} \). The notation \( (x, y) \in \mathcal{X} \) and \( z \in \mathcal{X} \) means that \( x \in \mathcal{F}^m \), \( y \in \mathcal{F}^n \), and \( z \in \mathcal{F}^k \). The \textit{multipath} set of \( \mathcal{X} \) is the subset defined by

\[
N_\mathcal{X} = \{(x, y) \in \mathcal{X} : |\mathcal{P}_*(x, y)| > 1\}.
\]

The next result shows that the property "\( (x, y) \in \mathcal{X} \) has a unique optimal warping path" holds \( \mu \)-almost everywhere.
Theorem 2.1. Let \((\mathcal{X}, \mathcal{B}, \mu)\) be the Lebesgue-Borel measure space. Then the multipath set \(\mathcal{N}_X\) of \(X\) is \(\mu\)-negligible.

Next, we describe the geometric form of the multipath set \(\mathcal{N}_X\). By \(Z(f) = \{z \in X : f(z) = 0\}\) we denote the zero set of a function \(f : X \to \mathbb{R}\).

Corollary 2.2. Let \(\mathcal{N}_X\) be the multipath set in \(X\). Then there are \(D \in \mathbb{N}\) symmetric matrices \(A_1, \ldots, A_D \in \mathbb{R}^2\) such that

\[
\mathcal{N}_X = \bigcup_{i=1}^D Z(z^T A_i z).
\]

From Corollary 2.2 follows that the multipath set \(\mathcal{N}_X\) is the union of zero sets defined by quadratic forms. The number \(D = D_{m,n}\) is the Delannoy number

\[
D_{m,n} = \min\{m,n\} \sum_{i=0}^\infty 2^i \binom{m}{i} \binom{n}{i}.
\]

The Delannoy number \(D_{m,n}\) corresponds the number of warping paths in lattice \(L_{m,n}\) and grows exponentially in \(m\) and \(n\).

3 Proofs

This section presents the proof of Theorem 2.1 and Corollary 2.2. We first consider the univariate case \((d = 1)\) in Sections 3.1 and 3.2. Finally, Section 3.3 generalizes the results to the multivariate case.

3.1 Time Warping Embeddings

Suppose that \(d = 1\). For the univariate case, we write \(\mathbb{R}^m\) instead of \(\mathcal{F}^m\) to denote a time series of length \(m\). By \(e^k \in \mathbb{R}^m\) we denote the \(k\)-th standard basis vector of \(\mathbb{R}^m\) with elements

\[
e^k_i = \begin{cases} 1 & : i = k \\ 0 & : i \neq k \end{cases}.
\]

Definition 3.1. Let \(p = (p_1, \ldots, p_L) \in \mathcal{P}_{m,n}\) be a warping path with points \(p_i = (i, j_i)\). Then

\[
\Phi = (e^{i_1}, \ldots, e^{i_L})^T \in \mathbb{R}^{L \times m}, \quad \Psi = (e^{j_1}, \ldots, e^{j_L})^T \in \mathbb{R}^{L \times n}
\]

is the pair of embedding matrices induced by warping path \(p\).

The embedding matrices have full column rank \(n\) due to the boundary and step condition of the warping path. Thus, we can regard the embedding matrices of warping path \(p\) as injective linear maps \(\Phi : \mathbb{R}^m \to \mathbb{R}^L\) and \(\Psi : \mathbb{R}^n \to \mathbb{R}^L\) that embed time series \(x \in \mathbb{R}^m\) and \(y \in \mathbb{R}^n\) into \(\mathbb{R}^L\) by matrix multiplication \(\Phi x\) and \(\Psi y\). We can express the cost \(C_p(x, y)\) of aligning time series \(x\) and \(y\) along warping path \(p\) by the Euclidean distance between their induced embeddings.

Proposition 3.2. Let \(\Phi\) and \(\Psi\) be the embeddings induced by warping path \(p \in \mathcal{P}_{m,n}\). Then

\[
C_p(x, y) = \|\Phi x - \Psi y\|^2.
\]

for all \(x \in \mathbb{R}^m\) and all \(y \in \mathbb{R}^n\).

Proof. [10], Proposition A.2. \(\square\)

Next, we define the warping and valence matrix of a warping path.
Definition 3.3. The valence matrix $V \in \mathbb{R}^{m \times m}$ and warping matrix $W \in \mathbb{R}^{m \times n}$ of warping path $p \in P_{m,n}$ are defined by

$$V = \Phi^T \Phi$$
$$W = \Phi^T \Psi,$$

where $\Phi$ and $\Psi$ are the embedding matrices induced by $p$.

The definition of valence and warping matrix are oriented in the following sense: The warping matrix $W \in \mathbb{R}^{m \times n}$ aligns a time series $y \in \mathbb{R}^n$ to the time axis of time series $x \in \mathbb{R}^m$. The diagonal elements $v_{ii}$ of the valence matrix $V \in \mathbb{R}^{m \times m}$ count the number of elements of $y$ warped onto the same element $x_i$ of $x$. Alternatively, we can define the complementary valence and warping matrix of $w$ by

$$V = \Psi^T \Psi$$
$$W = \Psi^T \Phi = W^T.$$

The complementary warping matrix $W \in \mathbb{R}^{n \times m}$ warps time series $x \in \mathbb{R}^m$ to the time axis of time series $y \in \mathbb{R}^n$. The diagonal elements $v_{ii}$ of the complementary valence matrix $V \in \mathbb{R}^{n \times n}$ counts the number of elements of $x$ warped onto the same element $y_i$ of $y$.

Let $p \in P_{m,n}$ be a warping path of length $L$ with induced embedding matrices $\Phi \in \mathbb{R}^{L \times m}$ and $\Psi \in \mathbb{R}^{L \times n}$. The aggregated embedding matrix $\Theta$ induced by warping path $p$ is defined by

$$\Theta = (\Phi, -\Psi) \in \mathbb{R}^{L \times k},$$

where $k = m + n$. Then the symmetric matrix $\Theta^T \Theta$ is of the form

$$\Theta^T \Theta = \begin{pmatrix} V & -W \\ -W & V \end{pmatrix}.$$

We use the following notations:

Notation 3.4. Let $m, n \in \mathbb{N}$ and $k = m + n$. We write $\mathcal{X} = \mathbb{R}^m \times \mathbb{R}^n$ and $\mathcal{X}^2 = \mathbb{R}^{k \times k}$. The notations $(x, y) \in \mathcal{X}$ and $z \in \mathcal{X}$ mean that $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $z \in \mathbb{R}^k$.

The next result expresses the cost $C_p(x, y)$ by the matrix $\Theta^T \Theta$.

Lemma 3.5. Let $\Theta$ be the aggregated embedding matrix induced by warping path $p \in P_{m,n}$. Then we have

$$C_p(z) = z^T \Theta^T \Theta z$$

for all $z \in \mathcal{X}$.

Proof. Suppose that $\Theta = (\Phi, -\Psi)$, where $\Phi$ and $\Psi$ are the embedding matrices induced by $p$. Let $z = (x, y) \in \mathcal{X}$. Then we have

$$C_p(z) = \|\Phi x - \Psi y\|^2$$
$$= x^T \Phi^T \Phi x - x^T \Phi^T \Psi y - y^T \Psi^T \Phi x + y^T \Psi^T \Psi y$$
$$= x^T V x - x^T W y - y^T W x + y^T V y$$
$$= (x^T, y^T) \begin{pmatrix} V & -W \\ -W & V \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= z^T \Theta^T \Theta z.$$
3.2 Proof of Theorem 2.1 and Corollary 2.2

The proofs assumes the univariate case \((d = 1)\).

**Proof of Theorem 2.1**

Suppose that \(\mathcal{P}_{m,n} = \{p_1, \ldots, p_D\}\). We use the following notations for all \(i \in [D]\):

1. \(\Theta_i\) denotes the aggregated embedding matrix induced by warping path \(p_i\).
2. \(\nabla_i\) and \(\nabla_i\) are the valence and warping matrices of \(p_i\).
3. \(\Xi_i\) and \(\Xi_i\) are the complementary valence and warping matrices of \(p_i\).
4. \(C_i(z)\) with \(z = (x, y)\) denotes the cost \(C_{p_i}(x, y)\) of aligning \(x\) and \(y\) along warping path \(p_i\).

For every \(i, j \in [D]\) with \(i \neq j\) and for every \(z = (x, y) \in \mathcal{X}\), we have

\[
C_i(z) - C_j(z) = z^T \Theta_i^T \Theta_i z - z^T \Theta_j^T \Theta_j z = z^T A^{(ij)} z,
\]

where \(A^{(ij)} \in \mathcal{X}^2\) is a symmetric square matrix of the form

\[
A^{(ij)} = \Theta_i^T \Theta_i - \Theta_j^T \Theta_j = \begin{pmatrix}
V_i - V_j & -W_i - W_j \\
-W_i - W_j & V_i - V_j
\end{pmatrix}.
\]

Since \(i \neq j\), the warping paths \(p_i\) and \(p_j\) are different. Hence, their respective warping matrices \(W_i\) and \(W_j\) are also different. This implies that \(A^{(ij)}\) is non-zero. Hence, the quadratic form \(f_{ij}(z) = z^T A^{(ij)} z\) is a non-zero polynomial in several variables. According to \([2, 11]\) the set \(\mathcal{U}_{ij} = \{z \in \mathcal{X} : f_{ij}(z) = 0\}\) is \(\mu\)-negligible. Consequently, the finite union

\[\mathcal{U} = \bigcup_{i < j} \mathcal{U}_{ij}\]

is also \(\mu\)-negligible.

It remains to show that \(\mathcal{N}_X \subseteq \mathcal{U}\). Suppose that \(z = (x, y) \in \mathcal{N}_X\). Then the set \(\mathcal{P}_s(x, y)\) of optimal warping paths between \(x\) and \(y\) consists of at least two elements. Hence, there are indices \(i, j \in [D]\) with \(i < j\) such that \(p_i\) and \(p_j\) are both optimal warping paths between \(x\) and \(y\). Since \(p_i\) and \(p_j\) are optimal, we have \(C_i(z) = C_j(z)\). From

\[0 = C_i(z) - C_j(z) = z^T A^{(ij)} z\]

follows that \(z \in \mathcal{U}_{ij} \subseteq \mathcal{U}\). This proves that \(\mathcal{N}_X\) is \(\mu\)-negligible and implies the assertion.

**Proof of Corollary 2.2**

We use the same notations as in the proof of Theorem 2.1. It is sufficient to show that \(\mathcal{U} \subseteq \mathcal{N}_X\). Let \(z = (x, y) \in \mathcal{U}\). Then there are indices \(i, j \in [D]\) with \(i < j\) such that \(z \in \mathcal{U}_{ij}\). By construction \(p_i\) and \(p_j\) are both optimal warping paths between \(x\) and \(y\). This shows \(z \in \mathcal{N}_X\).

3.3 Generalization to the Multivariate Time Series

We briefly sketch how to generalize the results from the univariate to the multivariate case. The basic idea is to reduce the multivariate case to the univariate case. In the following, we assume that \(x \in \mathcal{F}^m\) and \(y \in \mathcal{F}^n\) are two \(d\)-variate time series and \(p = (p_1, \ldots, p_L) \in \mathcal{P}_{m,n}\) is a warping path between \(x\) and \(y\) with elements \(p_l = (i_l, j_l)\).

First observe that a \(d\)-variate time series \(x \in \mathcal{F}^m\) consists of \(d\) individual component time series \(x^{(1)}, \ldots, x^{(d)} \in \mathcal{R}^m\). Next, we construct the embeddings of a warping path. The \(d\)-variate time warping embeddings \(\Phi_d : \mathcal{F}^m \to \mathcal{F}^L\) and \(\Psi_d : \mathcal{F}^m \to \mathcal{F}^L\) induced by \(p\) are maps of the form

\[
\Phi_d(x) = \begin{bmatrix}
x_{i_1} \\
\vdots \\
x_{i_L}
\end{bmatrix}, \quad \Psi_d(y) = \begin{bmatrix}
y_{j_1} \\
\vdots \\
y_{j_L}
\end{bmatrix}.
\]
The maps $\Phi_d$ and $\Psi_d$ can be written as

$$
\Phi_d(x) = \left( \Phi x^{(1)}, \ldots, \Phi x^{(d)} \right)
$$

$$
\Psi_d(x) = \left( \Psi y^{(1)}, \ldots, \Psi y^{(d)} \right),
$$

where $\Phi$ and $\Psi$ are the embedding matrices induced by $p$. Since $\Phi$ and $\Psi$ are linear, the maps $\Phi_d$ and $\Psi_d$ are also linear maps. We show the multivariate formulation of Prop. 3.2.

**Proposition 3.6.** Let $\Phi_d$ and $\Psi_d$ be the $d$-variate embeddings induced by warping path $p \in \mathcal{P}_{m,n}$. Then

$$
C_p(x, y) = \|\Phi_d(x) - \Psi_d(y)\|^2.
$$

for all $x \in \mathcal{F}^m$ and all $y \in \mathcal{F}^n$.

**Proof.** The assertion follows from

$$
\|\Phi_d(x) - \Psi_d(y)\|^2 = \sum_{k=1}^d \left\| \Phi x^{(k)} - \Psi y^{(k)} \right\|^2
$$

$$
= \sum_{k=1}^d \sum_{(i,j) \in p} \left( x^{(k)}_i - y^{(k)}_j \right)^2
$$

$$
= \sum_{(i,j) \in p} \left\| x_i - y_j \right\|^2
$$

$$
= C_p(x, y).
$$

Due to the properties of product spaces and product measures, the proofs of all other results can be carried out componentwise.

4 Conclusion

This article shows that optimal warping paths with squared error local costs are unique almost everywhere. The set of pairs of time series with non-unique optimal warping path is the union of zero sets of quadratic forms. The implication of the proposed results is that non-uniqueness of optimal warping paths occurs exceptionally. Hence, in a practical setting, difficulties caused by non-uniqueness can be safely neglected for adaptive learning in DTW spaces.

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