Galois Relations on Knot Invariants

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Abstract

We discuss the existence of Galois relations obeyed by certain link invariants. Some of these relations have recently been identified and exploited within the context of conformal field theory and Lie/Kac-Moody representation theory. These relations should aid in computing knot invariants. They probably have an interpretation in terms of quasitriangular (quasi) Hopf algebras. They could also have a topological interpretation, and may serve as a concrete model for related ideas of Degiovanni, Drinfeld, Grothendieck, Ihara, and others.

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1. Introduction

The classification and study of knots is an old problem \[1,2\]. One of the most fruitful ideas has been to assign to each knot a set of objects (e.g. numbers or polynomials): two knots will be equivalent only if these objects are identical. The first knot polynomial is due to Alexander in 1928, but many more invariants have been discovered since (see e.g. \[3\]). In fact there is now an embarrassment of riches, but it is still an open question whether all the known invariants will completely distinguish all knots.

Three important goals in the subject are: to find convenient aids for computing these invariants; to obtain a topological understanding of these invariants; and to bring some order to the multitude of invariants by finding systematic relationships among them. The traditional approach to computation has been to use some sort of skein relation, but these are practical only for the simplest invariants and knots. An adequate topological understanding is still missing \[1\] for most invariants. Most invariants are redundant; the most significant relationship known among them is the theorem of Birman-Lin \[4\] that the Vassiliev invariants contain the Lie algebra ones.

In the program introduced in this paper, we hope to provide new means to approach the first two goals, by using the Galois group \(\text{Gal}(\mathbb{Q}_{ab}/\mathbb{Q})\) (by \(\mathbb{Q}_{ab}\) we mean a maximal abelian extension of \(\mathbb{Q}\)) to directly address the latter. In \[5\] Witten establishes connections between 3-dimensional topology and 2-dimensional rational conformal field theory (RCFT); we will exploit that connection as we generalize relations recently obtained in RCFT. For concreteness we will focus here on the Witten link invariants \[5\], which equal the quantum group link invariants \[6\] for the quantum enveloping algebras \(U_q(g)\), where \(g\) is a Lie algebra and \(q\) a root of unity. The special property possessed by these invariants is the presence of the Lie algebra – it makes the appearance of Galois relations more explicit. Later in this paper we will discuss the possibility that there is an action of the group \(\text{Gal}(\mathbb{Q}_{ab}/\mathbb{Q})\) on knots themselves, and its possible relation to certain ideas of Grothendieck and others.

Each (directed) link \(L\) in each closed, connected and orientable 3-manifold\(^1\) \(\mathcal{M}\) (i.e. an embedding \((S^1)^t \rightarrow \mathcal{M}\)) possesses several Witten invariants. Namely, choose any non-twisted affine Kac-Moody algebra \(\hat{g} = X^{(1)}_r\), any integer \(k > 0\), and any level \(k\) highest weights \(\lambda^1, \ldots, \lambda^t \in P_k^+(\hat{g})\) (we will review affine algebra representation theory in the following section; see also \[7\]). Then there is a Witten invariant \[5\]

\[
\mathcal{Z}(L, \hat{g}, \{\lambda^i\}) \in \mathbb{C}. \tag{1}
\]

We will discuss eq.(1) in section 2; these numbers \(\mathcal{Z}\) (topological invariants for \(L\)) are the starting point for this paper.

We will find a Galois relation mixing some of these \(\mathcal{Z}\), which in some cases will be non-linear, and can involve different \(L\) (and even \(\mathcal{M}\)). This project is motivated in part by \[8\], in which was found a simple Galois relation\(^2\) for the Kac-Peterson matrix \(S\) and

\(^1\) To be well-defined, the Witten invariant requires the 3-manifold to be framed \[5\], i.e. a trivialization of its tangent bundle must be chosen. A different choice however merely multiplies the invariant by a certain root of unity. In this paper we can and will safely ignore this complication.

\(^2\) More generally, this holds for any RCFT \[8\], which suggests that the comments in this paper should also
corresponding fusion coefficients $N_{\lambda,\mu}^\nu$ (see eqs.(3)). $S_{\lambda,\mu}$ and $N_{\lambda,\mu}^\nu$ turn out to be examples of link invariants, so it is natural to ask if other link invariants will also obey reasonably simple Galois relations. This is made much more plausible by the observation that the Witten invariants can be obtained from surgery (which involves $S$) and skein relations (which involve e.g. roots of unity). This is reviewed and generalized in section 3, where we also make some remarks about what the general Galois relations will look like and how they can be found.

The Galois symmetry on $S$ has been very effective in the classification of RCFTs (see e.g. [10]). The action on $N$ has been used [11] to construct new exceptional partition functions of RCFTs. They were used in [12] to obtain relations among weight multiplicities for representations of finite-dimensional Lie algebras. Further applications$^3$ are discussed in [14]. Thus it is certainly possible that the generalization of these relations to other link invariants will yield other valuable formulae, with applications lying outside the context of RCFT. This should be particularly true once a more topological understanding of these invariants is found. In the final section we address some of these points.

A well-known collection of relationships among the Witten invariants involves the transformation $\lambda \mapsto C\lambda$, taking a Lie algebra highest weight to the one contragredient to it. Witten’s invariants are preserved if $C$ is applied simultaneously to all weights $\lambda^i$:

$$Z(\mathcal{L}, \hat{g}, \{C\lambda^i\}) = Z(\mathcal{L}, \hat{g}, \{\lambda^i\}) . \quad (2a)$$

This has a topological meaning: the Witten invariants of a link $\mathcal{L}$ are equal to those of its reverse $\mathcal{L}^r$:

$$Z(\mathcal{L}, \hat{g}, \{\lambda^i\}) = Z(\mathcal{L}^r, \hat{g}, \{\lambda^i\}) . \quad (2b)$$

$\mathcal{L}^r$ is obtained from $\mathcal{L}$ by switching the orientations of all of its components. Since some knots are inequivalent to their reverse, eq.(2b) implies that the Witten invariants cannot distinguish all inequivalent knots.

$C$ also has a privileged role in RCFT, but in [8] it is found that $C$ is only one among a whole family of Galois “symmetries” $\sigma \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ which obey similar equations ($C$ corresponds to complex conjugation); whenever $C$ possesses some property in RCFT, an analogue is usually possessed by all other $\sigma$. In this paper we propose that a similar situation should hold in knot theory, i.e. that analogues of eq.(2a) and possibly eq.(2b) hold for all $\sigma$.

In this short note we can do little more than briefly state the general ideas, give some simple examples, and speculate about possible future developments. Much more work is required before their scope and usefulness can be uncovered. However, we believe the possibilities are intriguing enough to justify publication of this work at this early stage.

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$^3$ Galois has made other appearances in RCFT – see e.g. [13].

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hold in non-Lie theoretic contexts – for example the quasi-Hopf link invariants [9] – though perhaps at the expense of ease of computation.
2. Witten link invariants

We begin by briefly describing some objects in affine algebra representation theory (see [7] for more details), then proceed to give a few simple facts about links (see also [1,2]), and end by reviewing the basic theory of Witten invariants (see e.g. [5]).

There is a one-to-one correspondence between finite-dimensional semi-simple Lie algebras \( X_r \), and nontwisted affine Kac-Moody algebras \( \hat{g} = X_r^{(1)} \). The integrable highest weight representations of \( \hat{g} \) are characterized by their highest weights \( \mu = \sum_{i=0}^r \mu_i w^i \), where each \( \mu_i \in \mathbb{Z}_{\geq} \) (the \( w^i \) are the fundamental weights). The set of all highest weights for \( \hat{g} \) are partitioned into finite subsets \( P^+_k(\hat{g}) \), indexed by the level \( k \) (e.g. for \( \hat{g} = A_r^{(1)} \), \( k = \sum_{i=0}^r \mu_i \)). The characters \( \chi_\mu \) of \( \hat{g} \), for \( \mu \in P^+_k(\hat{g}) \), define a representation of the modular group \( SL_2(\mathbb{Z}) \):

\[
\chi_\mu(\tau + 1, z, u) = \sum_{\nu \in P^+_k} T_{\mu,\nu} \chi_\nu(\tau, z, u) \tag{3a}
\]

\[
\chi_\mu\left(\frac{-1}{\tau}, \frac{z}{\tau}, u - \frac{(z | z)}{\tau}\right) = \sum_{\nu \in P^+_k} S_{\mu,\nu} \chi_\nu(\tau, z, u). \tag{3b}
\]

\( S \) and \( T \) are called the Kac-Peterson matrices. They are both unitary. \( T \) is diagonal, while the elements of \( S \) are related to certain values of the characters of \( X_r \). The map \( C \) appearing in eq.(2a) corresponds to the permutation matrix \( S^2 \).

We are also interested in the fusion coefficients \( N^\nu_{\lambda,\mu} \), which are defined by Verlinde’s formula [15]:

\[
N^\nu_{\lambda,\mu} = \sum_{\gamma \in P^+_k} S_{\lambda,\gamma} \frac{S_{\mu,\gamma}}{S_{k\omega_0,\gamma}} S^*_{\nu,\gamma}. \tag{3c}
\]

The \( N^\nu_{\lambda,\mu} \) will always be non-negative integers, and are closely related to the tensor product coefficients of \( X_r \). They have appeared in several different contexts in recent years, including of course RCFT.

A link is an embedding of \( S^1 \times \cdots \times S^1 \) into a 3-manifold \( M \). To avoid certain wild knots (e.g. infinite connected sums), additional conditions (e.g. smoothness or polygonality) are required [1,2], but will not be given here. All links here are directed. A knot is a link with precisely one connected component.

A link invariant is a function which assigns to each link \( L \) an object \( f(L) \) in such a way that equivalent links are assigned equivalent (usually equal, or equal up to some factor) objects. A simple example is the number of connected components in the link.

We call two links \( L, L' \) equivalent when they are ambient isotopic [2], i.e. if there exists an orientation-preserving homeomorphism \( h : M \to M \) with \( h(L) = L' \). In \( S^3 \) this is the same as saying their link diagrams (i.e. projections) can be related by some sequence of the familiar Reidemeister moves I, II, III [1]. Two links are called regular isotopic when their diagrams are invariant under Reidemeister II, III. In the literature both ambient and regular isotopic link invariants are discussed: the Jones polynomial and its generalization HOMFLY [3], as well as Conway’s normalization of the Alexander polynomial, are ambient; while ribbon invariants, Kauffman’s polynomial, and Witten’s invariants are regular.
Reidemeister I says that a localized loop in a string can be straightened; the special thing about this is that it changes the “self-winding number” of the knot. In the case of the Kauffman polynomial and the Witten invariants, ambient isotopy can always be recovered by introducing a phase factor depending on this self-winding number. The problem is that the resulting ambient isotopic invariant will depend on an arbitrary choice of overall phase factor, reflecting the fact that different ribbons can have the same knot as their “spine”.

In the language of quantum field theory, the Witten invariant in eq.(1) is the un-normalized expectation value of Wilson lines $L$ in the Chern-Simons theory whose gauge group is the compact exponentiation of $X_r$. The Lagrangian is proportional to $k$, and the components of $L$ carry the representations of highest weights $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^r)$. Mathematically, $Z$ is definable using the Reshetikhin-Turaev functor $F$ [6], which relates a category of directed ribbon graphs coloured by weights in $P_k(\hat{g})$ to a category of finite-dimensional representations of $U_q(X_r)$. An alternative is the approach given in [16] for Lie algebra invariants, using “Chinese character diagrams”.

Some (but not all) of the numbers in eq.(1) for fixed $L$ and $\hat{g}$ may be collected and reinterpreted as polynomials in a formal parameter $q$, evaluated at the root of unity $q = \exp[2\pi i/(k+h^\vee)]$, where $h^\vee$ is the dual Coxeter number of $X_r$. For example, Jones’ original polynomial $J(z)$, originally defined only for $L \subset S^3$, is related to $Z(L, A_1(1), w^1+(k-1)w^0)$, where every strand of $L$ is coloured by the (level $k$ “lift” of the) fundamental representation $w^1$ of $A_1$. In particular, this Witten invariant $Z$ will equal $\omega J(\exp[2\pi i/(k+2)])$, where $\omega$ is some appropriately chosen root of unity ($J$ is uniquely determined by its value at those infinitely many points). HOMFLY turns out to be the generalization of the Jones polynomial to $A_r(1)$, but again the only representation which appears there is the first fundamental one of $A_r$.

Witten invariants are a subset of the quantum group invariants [6], which turn out [4] to be special cases of the Vassiliev invariants [17]. It is not yet known however whether Vassiliev invariants are actually more powerful than Witten invariants; both can be graded by a positive integer called degree, and for degree up to 9 the dimensions of the spaces spanned by these invariants remain equal [16]. It is also unknown whether Vassiliev invariants distinguish all knots.

One of the main problems is how to compute the $Z$, even in principle. Usually two steps are followed. The first is to replace $M$ with a more convenient manifold, like $S^3$ or $S^1 \times R^h$ ($R^h$ is the genus $h$ Riemann surface). The main way to do this is surgery about an appropriately chosen knot. The result [5] is an expression for the original $Z$ as a linear combination of $Z$’s for a new link; the coefficients of the linear combination are computable using the Kac-Peterson matrices $S$ and $T$. In some special cases, connected and disconnected sums of links are also effective.

So because of surgery we can compute any $Z(L, \hat{g}, \{\lambda^i\})$ if we know all $Z(L' \subset S^3, \hat{g}, \{\mu^j\})$, say. The problem then reduces to working in $M = S^3$, and trying to simplify the link $L$ to some link we can handle. The main way to do this involves skein relations, along with an expression saying what phase is picked up when Reidemeister I is performed on the knot. These relations are given in [18,19], and in general require a generalization [18] of eq.(1) from embeddings in $M$ of $(S^1)^i$, to embeddings in $M$ of disjoint unions of trivalent graphs – i.e. graphs with three segments at each vertex. Each segment is assigned
a weight, and \( Z \) will vanish unless the fusion coefficient at each vertex is non-zero. Ref. [16] gives an alternate but related approach.

The skein relations permit one to reduce the computation of the \( Z(\mathcal{L}) \) to the evaluation of invariants of “tetrahedron graphs” (trivalent graphs with 4 vertices and 6 sides). The coefficients of these skein relations are awkward: they involve the square-roots \( \sqrt{S_{0,\lambda}/S_{0,0}} \), along with “braiding matrix elements” (no explicit expression for these are known in general) and \( T_{\lambda,\lambda} \). In [19] it is suggested that these tetrahedron invariants can be computed as the solutions of linear equations (consistency conditions from the skein relations). However there is no proof these equations can always be inverted, and in any event this approach is very impractical except in the simplest cases.

For use in the following section, we give here a few of the simpler Witten invariants (for readability we will write ‘0’ for the weight ‘\( k\omega_0 \)’).

(i) \( t \) parallel (i.e. unlinked) unknots, in \( S^3 \) (independent of orientation):

\[
Z =: D_{\lambda^1,\ldots,\lambda^t} = S_{0,0} \prod_{i=1}^{t} \frac{S_{0,\lambda^i}}{S_{0,0}} ; \quad (4a)
\]

(ii) \( t \) parallel unknots \( S^1 \times \{p_i\}, 1 \leq i \leq t, \) in \( S^1 \times R^h \) (oriented the same way): this is the Verlinde dimension [15]

\[
Z = V_{\lambda^1,\ldots,\lambda^t}^{h,t} = \sum_{\mu \in P^k} (S_{0,\mu})^{2(1-g)} \frac{S_{\lambda^1,\mu}}{S_{0,\mu}} \cdots \frac{S_{\lambda^t,\mu}}{S_{0,\mu}} ; \quad (4b)
\]

(iii) a chain of \( t \) linked unknots, in \( S^3 \) (oriented in the same way):

\[
Z =: C_{\lambda^1,\ldots,\lambda^t} = S_{0,\lambda^1} \prod_{i=1}^{t-1} \frac{S_{\lambda^i,\lambda^{i+1}}}{S_{0,\lambda^i}} ; \quad (4c)
\]

(iv) a central unknot in \( S^3 \) with weight \( \lambda^0 \), with \( t \) unknots linked around it (like keys around a key chain), all oriented the same:

\[
Z =: S_{\lambda^0;\lambda^1,\ldots,\lambda^t} = S_{0,\lambda^0} \prod_{i=1}^{t} \frac{S_{\lambda^0,\lambda^i}}{S_{\lambda^0,0}} . \quad (4d)
\]

Notice that \( S_{\lambda^1,\ldots,\lambda^t} = D_{\lambda^1,\ldots,\lambda^t} \). Also, \( S_{\lambda,\mu} = S_{\lambda;\mu} = C_{\lambda,\mu} \) and \( N_{\nu} = V_{\lambda,\mu,\nu}^{0,3} \).

3. Galois relations for the Witten invariants

Next we review the presence of Galois in RCFT, and conjecture how it will extend to the Witten link invariants. What we are after is some kind of generalization of eq.\((2a)\) to any Galois automorphism \( \sigma \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \) of the underlying RCFT.
The restriction to \( \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \) is forced by the invariants: they must lie in cyclotomic extensions of \( \mathbb{Q} \). For links in \( S^3 \), this can be seen directly from [4], and since it is true for the \( S \) and \( T \) matrices, by surgery it is true for links in all other 3-manifolds \( \mathcal{M} \).

It was discovered in [8] (based on work in [20], and generalizing and reinterpreting [21]) that there is a Galois “symmetry” present in any RCFT. In particular, let \( M \) be the field extension of \( \mathbb{Q} \) obtained by adjoining to it all the matrix elements \( S_{\lambda,\mu} \). It turns out that \( M \) will always lie in a cyclotomic field (for RCFT based on an affine algebra, this follows directly from Kac-Peterson). Choose any \( \sigma \in \text{Gal}(M/\mathbb{Q}) \). Then there is a permutation \( \lambda \mapsto \sigma \lambda \) of \( P^+_k(\hat{g}) \), and a choice of signs \( \epsilon_\sigma(\lambda) \in \{ \pm 1 \} \), such that

\[
\sigma(S_{\lambda,\mu}) = \epsilon_\sigma(\lambda) S_{\sigma(\lambda),\mu} = \epsilon_\sigma(\mu) S_{\lambda,\sigma(\mu)} . \tag{5a}
\]

Both \( \epsilon_\sigma \) and \( \lambda \mapsto \sigma \lambda \) have a geometric meaning in terms of the affine Weyl group of \( \hat{g} \) [8], and can be readily calculated in practice. Eq.(5a) is the basic equation of [8], and the source of the name “Galois” for the family of relations considered in this paper. From this and eq.(3c), it is possible to derive [8] the expression

\[
N^\nu_{\sigma,\lambda,\mu} = \epsilon_\sigma(0) \epsilon_\sigma(\lambda) \sum_{\gamma \in P^+_k} \epsilon_\sigma(\gamma) N^\gamma_{\lambda,\mu} N^\nu_{\gamma,\sigma,0} . \tag{5b}
\]

As we saw at the end of the last section, both \( S_{\lambda,\mu} \) and \( N^\nu_{\lambda,\mu} \) can be interpreted as link invariants, corresponding respectively to the Hopf link (a chain of 2 linked unknots) in \( S^3 \), and 3 parallel unknots in \( S^1 \times S^2 \). Since they both satisfy a Galois relation, then perhaps many other invariants do too. It is easy to find these relations when explicit expressions exist for the invariants. For example, the same calculation which gave us eq.(5b) can be used to give us:

\[
\mathcal{V}_{h,t}^{\lambda_1,\ldots,\lambda_{t-1},\lambda_t} = \frac{\epsilon_\sigma(\lambda_1) \cdots \epsilon_\sigma(\lambda_{t-1})}{\epsilon_\sigma(0)} \sum_{\mu \in P^+_k} \frac{\epsilon_\sigma(\mu)}{\epsilon_\sigma(0)} \mathcal{V}_{\lambda_1,\ldots,\lambda_{t-1},C\mu}^{h,t} \mathcal{V}_{\sigma,0,\ldots,\sigma,0,\lambda_t}^{0,2h+t} . \tag{5c}
\]

From eqs.(4b) and (5a) we find that each \( \mathcal{V}_{\lambda_1,\ldots,\lambda_t}^{h,t} \) is fixed by \( \sigma \) — in fact they are non-negative integers. Galois relations for the other invariants in eqs.(4) can also be found. Simple generalizations of eq.(5a) are

\[
\sigma \left( S_{\lambda_0;\lambda_1,\ldots,\lambda_t} \right) = \epsilon_\sigma(\lambda_0) S_{\sigma(\lambda_0);\lambda_1,\ldots,\lambda_t} \tag{5d}
\]

\[
S_{\sigma,\lambda_0;\lambda_1,\ldots,\lambda_t} = \prod_{i=0}^{t} \frac{\epsilon_\sigma(\lambda_i)}{\epsilon_\sigma(0)} S_{\lambda_0;\sigma,0} \frac{S_{\lambda_0;\sigma,0,\ldots,\sigma,0}}{S_{\lambda_0;\sigma,0,\ldots,\sigma,0}} . \tag{5e}
\]

By analogy with eqs.(5), we are looking for polynomial relations over \( \mathbb{Z} \) among link invariants. The link invariants may not all correspond to the same link \( \mathcal{L} \), or even the same background manifold \( \mathcal{M} \), but the weights appearing in them are either the inputs \( \lambda_i \), or 0, or their images \( \sigma(\lambda_i) \), \( \sigma(0) \) under some \( \sigma \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \) (as in eq.(5b)), there can also appear dummy variables \( \gamma \) involved in a sum — a trace — over all of \( P^+_k \). In addition, in some cases applying \( \sigma \) directly to the invariant will yield a simple expression.
There is a way, using the machinery developed in [12], to linearize eq.(5b), using the dominant weight multiplicities $m^\mu_\lambda$ of the highest weight module $L(\lambda)$ of $g$. In particular, we can write eq.(5b) in the equivalent form

$$N^\sigma_\mu\lambda_{\lambda,\sigma}(\nu) := \frac{||W\pi||}{||W\pi_\sigma(\nu)||} \sum_{\gamma \in P^+} \ell^\gamma_{\pi_\sigma(\nu)} N^\sigma_\mu\lambda_{\lambda,\gamma} = \epsilon_\sigma(\lambda) \epsilon_\sigma(\mu) N^\mu_\lambda_{\lambda,\id(\nu)}, \quad (6a)$$

where $\pi_\sigma(\nu) := \sigma(\nu - \rho) + \rho$, and the $\ell$'s are the elements of the matrix $L = M^{-1}$ inverse to the matrix $M$ formed from the multiplicities $m^\mu_\lambda$ (both $L$ and $M$ are lower triangular, with 1's along the diagonal). $||W\pi||$ is the order of the (finite) Weyl orbit of $\pi = (\nu_1, \ldots, \nu_r)$. A similar trick works for other nonlinear Galois relations; for example,

$$C_{L_{\id}(\lambda^1),\lambda^2,\lambda^3,\id(\lambda^4)} = \epsilon_{\sigma^{-1}}(\lambda^2) \epsilon_{\sigma}(\lambda^3) C_{L_{\sigma}(\lambda^1),\sigma^{-1}\lambda^2,\sigma\lambda^3,\sigma^{-1}(\lambda^4)} \cdot \quad (6b)$$

By the subscript $L_{\sigma}(\lambda)$ in eqs.(6) we mean the linear combination over that position as defined on the r.h.s. of eq.(6a).

Eq.(6a) is linear in the fusion coefficients; the price is the introduction of the “inverse multiplicities” $\ell^\mu_\lambda$. However these are easy to compute (see e.g. eqs.(2.6),(5.2) of [12]), and so can be regarded essentially as known constants. This new relation (6a) should serve as a useful means of computing fusions – for example it directly provides an expression for the fusion coefficients $N^\mu_{0,\lambda}$ using the $\ell^\nu_\gamma$’s and the affine reflection $r_0$ built into $\pi_\sigma$ (this is where the dependence of $N$ on $k$ enters). Eq.(6a) together with the “normalization” $N^0_{0,0} = 1$ may suffice in fact to determine all other fusion coefficients.

Finding Galois relations for more complicated $Z$’s will be difficult, until we possess a more reliable way of computing these $Z$’s. If we use the skein relations, we presumably must know Galois relations for e.g. the tetrahedron graphs, perhaps by studying the consistency conditions of [19]. All this can probably be avoided in the quantum group approach of [6]: in the most optimistic case we would be looking in that language for a Galois action on the two categories (the representation category, and the category of coloured directed ribbon graphs) which commutes with the covariant functor $F$ between them. An interesting alternative within the Chern-Simons theory comes from e.g. [22].

4. Conclusion and speculations

Arguing by analogy with RCFT, we suggest that there could be a generalization of eq.(2a) to the other automorphisms in $\text{Gal}(\mathbb{Q}_{ab}/\mathbb{Q})$ ($C$ in eq.(2a) corresponds to complex conjugation). We provide some simple examples supporting this claim. This Galois “symmetry” has been useful in RCFT, so it may also be of use in analyzing knot invariants. For example it should help in computing some invariants. It would be interesting to find out if these Galois relations exist merely because of the presence of Lie algebras in these Witten invariants, or whether in some form they will persist for more general link invariants (e.g. Vassiliev). If they do persist, then perhaps they will obey relations obtained from $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the absolute Galois group of $\mathbb{Q}$.

Hopf algebras and Galois theory have deep connections [23]. It is very possible that the ultimate source of these Galois relations, both in RCFT and for link invariants, lie in
quasitriangular (quasi-)Hopf algebras. This is then a natural direction to pursue the ideas in this paper, both to clarify and to generalize them.

In eq.(2b) we find that complex conjugation has a topological interpretation. It would be desirable to find one for the other elements of Gal(\(\mathbb{Q}^{ab}/\mathbb{Q}\)). Unfortunately a more systematic treatment of these Galois relations, both algebraically and topologically, will be difficult without a more systematic algorithm for computing the Witten invariants, or without a concrete understanding of the Galois actions (if they exist) on the two Reshetikhin-Turaev categories.

There is a certain resemblance of these ideas to much more ambitious ones sketched by Grothendieck [24], Drinfeld [25], Degiovanni [26], and others (see in particular Ihara [27] and references therein). Grothendieck proposed to study Gal(\(\overline{\mathbb{Q}}/\mathbb{Q}\)) using the Teichmüller tower formed from the moduli spaces \(\mathcal{M}_{h,t}\) of Riemann surfaces of genus \(h\) with \(t\) punctures; the elements of Gal(\(\overline{\mathbb{Q}}/\mathbb{Q}\)) act as outer automorphisms of the tower. Drinfeld related this to a universal braid transformation group acting on structures of quasitriangular quasi-Hopf algebras. An RCFT interpretation has been suggested in [26]: any RCFT may provide a projective representation of Grothendieck’s tower, and there could be an action of Gal(\(\overline{\mathbb{Q}}/\mathbb{Q}\)) on the data of RCFT (e.g. the \(S\) and \(T\) matrices). Perhaps the program introduced in this paper can serve as a concrete realization of some aspects of these deep ideas.

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