Off-shell superconformal higher spin multiplets in four dimensions

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Abstract

We formulate off-shell $\mathcal{N} = 1$ superconformal higher spin multiplets in four spacetime dimensions and briefly discuss their coupling to conformal supergravity. As an example, we explicitly work out the coupling of the superconformal gravitino multiplet to conformal supergravity. The corresponding action is super-Weyl invariant for arbitrary supergravity backgrounds. However, it is gauge invariant only if the supersymmetric Bach tensor vanishes. This is similar to linearised conformal supergravity in curved background.
1 Introduction

The role of conformal field theories as cornerstones for the exploration of more general quantum field theories, which are connected to them via renormalization group flows, has been appreciated since a long time ago. Higher spin gauge theories \[1, 2, 3, 4\] have an even longer history and have attracted considerable interest recently. It is quite natural to combine the two symmetry principles and to study conformal higher spin theories \[5\]. A further symmetry which is compatible with conformal and higher spin symmetry is supersymmetry. This leads to superconformal higher spin theories, first advocated in \[6\], which are the main focus of this note. More specifically, we introduce off-shell \(\mathcal{N} = 1\) superconformal higher spin multiplets in four dimensions and analyse in some detail the problem of lifting such supermultiplets to curved backgrounds. Our main technical tool, as far as the supersymmetry and supergravity aspects are concerned, is superspace and we refer to \[7\] for a thorough introduction to this formalism.

We first study superconformal higher spin theories in flat superspace. In Sect. 2 we review superconformal transformations and the important notion of superconformal primaries. In Sect. 3 we construct off-shell superconformal higher spin multiplets: starting from prepotentials and their transformation laws under higher spin gauge transformations and under superconformal transformations, we construct field strengths and invariant actions. The two cases of half-integer and integer superspin as well as the superconformal gravitino multiplet have to be treated separately. The component fields of these multiplets are totally symmetric traceless tensor and tensor-spinor fields. More general fields will be briefly discussed in the last part of Sect. 3. In Sect. 4 we couple the superconformal higher spin multiplets to conformal supergravity where the notion of superconformal transformations is replaced by that of super-Weyl transformations. While super-Weyl invariance is easy to achieve, gauge invariance requires non-minimal couplings. We explicitly discuss the gravitino multiplet, but defer the general case to the future. Sect. 5 contains concluding comments, including the explicit expressions for conserved higher spin current multiplets that correspond to the superconformal higher spin prepotentials.

The main body of the paper is accompanied by two technical appendices. Appendix A contains those results concerning the Grimm-Wess-Zumino superspace geometry \[8\], which are important for understanding the supergravity part of this paper. Appendix B contains the essential information about the super-Weyl transformations \[9\].

There are different ways to describe \(\mathcal{N} = 1\) conformal supergravity in superspace. \[1\]

\[1\]See \[5\] for a nice review of \(\mathcal{N} = 1\) conformal supergravity and the complete list of references.
The simplest option is to make use of the superspace geometry of \cite{8}, which underlies
the Wess-Zumino approach \cite{10} to the old minimal formulation for $\mathcal{N} = 1$ supergravity
developed independently in \cite{11}. Another option is to work with the $\text{U}(1)$ superspace
proposed by Howe \cite{12}. Finally, one can make use of the so-called conformal superspace
\cite{13}. The three superspace approaches to $\mathcal{N} = 1$ conformal supergravity are equivalent,
although each of them has certain advantages and disadvantages (see \cite{13} for a detailed
discussion of the relationship between these formulations). In this paper we make use of
the oldest and simplest superspace setting \cite{8}.

2 Superconformal transformations

In this section we briefly recall the structure of $\mathcal{N} = 1$ superconformal transformations
in Minkowski superspace $\mathbb{M}^{4|4}$, see \cite{7} for more details. We denote by $z^A = (x^a, \theta^\alpha, \bar{\theta}^\dot{\alpha})$
the Cartesian coordinates for $\mathbb{M}^{4|4}$, and use the notation $D^A = (\partial_a, D_\alpha, \bar{D}^\dot{\alpha})$ for the superspace
covariant derivatives.

Let $\xi = \xi^B D_B = \xi^b \partial_b + \xi^\beta D_\beta + \bar{\xi}^\dot{\alpha} \bar{D}^{\dot{\alpha}}$ be a real supervector field on $\mathbb{M}^{4|4}$. It is called
conformal Killing if it obeys the equation

$$\left[\xi + \frac{1}{2} K^{bc}[\xi] M_{bc}, D_A\right] + \delta_{\sigma[\xi]} D_A = 0 , \quad (2.1)$$

for some local Lorentz ($K^{bc}[\xi]$) and super-Weyl ($\sigma[\xi]$) parameters. The super-Weyl trans-
formation of the covariant derivatives is defined in (B.1). Choosing $A = \alpha$ and $A = \dot{\alpha}$ in
(2.1) implies that the spinor components of $\xi^A$ as well as the parameters $K^{bc}[\xi]$ and $\sigma[\xi]$ are expressed in terms of the vector components of $\xi^A$:

$$\xi^\alpha = -\frac{i}{8} D_\dot{\alpha} \xi^{\dot{\alpha} \alpha} , \quad D_\dot{\gamma} \xi^\alpha = 0 , \quad (2.2a)$$

$$K_{\alpha\beta}[\xi] = D_{(\alpha} \xi_{\beta)} , \quad \bar{D}_{\dot{\gamma}} K_{\alpha\beta}[\xi] = 0 , \quad (2.2b)$$

$$\sigma[\xi] = \frac{1}{3} (D_\alpha \xi^\alpha + 2 D^\dot{\alpha} \bar{\xi}^{\dot{\alpha}}) , \quad D_\dot{\gamma} \sigma[\xi] = 0 . \quad (2.2c)$$

The vector components of $\xi^A$ obey the equations

$$D_{(\alpha} \xi_{\beta)} = 0 \iff \bar{D}_{(\dot{\alpha} \beta)} = 0 , \quad (2.3)$$

which imply

$$D^2 \xi_{\beta\beta} = 0 \iff \bar{D}^2 \xi_{\dot{\beta}\dot{\beta}} = 0 , \quad (2.4)$$
as well as the ordinary conformal Killing equation

\[ \partial_a \xi_b + \partial_b \xi_a = \frac{1}{2} \eta_{ab} \partial_c \xi^c. \]  

(2.5)

A useful corollary of (2.1) with \( A = \alpha \) is

\[ D_\gamma K^{\alpha \beta}[\xi] = \delta^{(\alpha}_\gamma D^{\beta)[\sigma[\xi] \quad \implies \quad D^2 \sigma[\xi] = 0. \]  

(2.6)

Another consequence of (2.1) is

\[ \partial_a \sigma[\xi] = \partial_a \bar{\sigma}[\xi] \quad \implies \quad \partial_a D_\beta \sigma[\xi] = 0. \]  

(2.7)

The most general conformal Killing supervector field proves to be

\[ \xi^{\dot{\alpha} \alpha} = a^{\dot{\alpha} \alpha} + \frac{1}{2} (\sigma + \bar{\sigma}) x^{\dot{\alpha} \alpha} + \bar{K}^{\dot{\alpha} \beta} x^+_{\dot{\beta} \alpha} + x^+_{\dot{\alpha} \beta} K^{\beta \alpha} + x^+_{\dot{\alpha} \beta} x^+_{\dot{\beta} \alpha} + 4i \epsilon^{\dot{\alpha} \theta} - 4x^{\dot{\alpha} \beta} \eta^\beta \theta^\alpha, \]  

(2.8a)

\[ \xi^\alpha = \epsilon^\alpha + (\bar{\sigma} - \frac{1}{2} \sigma) \theta^\alpha + \theta^\beta K^\beta \alpha + + \theta^\beta b_{\beta \dot{\beta}} x^+_{\dot{\beta} \alpha} - i \bar{\eta} \beta x^+_{\dot{\beta} \alpha} + 2 \theta^2 \eta^\alpha, \]  

(2.8b)

where we have introduced the complex four-vector

\[ \xi^a = \xi^a + \frac{i}{8} \xi \sigma^\alpha \bar{\theta}, \quad \bar{\xi}^a = \xi^a, \]  

(2.9)

along with the complex bosonic coordinates \( x^a_+ = x^a + i \theta^a \sigma \bar{\theta} \) of the chiral subspace of \( \mathbb{M}^{4|4} \). The constant bosonic parameters in (2.8) correspond to the spacetime translation \((a^{\dot{\alpha} \alpha})\), Lorentz transformation \((K^{\beta \alpha}, \bar{K}^{\dot{\alpha} \dot{\beta}})\), special conformal transformation \((b_{\alpha \beta})\), and combined scale and \( R \)-symmetry transformations \((\sigma = \tau - \frac{2}{3} i \varphi)\). The constant fermionic parameters in (2.8) correspond to the \( Q \)-supersymmetry \((\epsilon^\alpha)\) and \( S \)-supersymmetry \((\eta^\alpha)\) transformations. The constant parameters \( K_{\alpha \beta} \) and \( \sigma \) are obtained from \( K_{\alpha \beta}[\xi] \) and \( \sigma[\xi] \), respectively, by setting \( z^A = 0 \).

A tensor superfield \( \mathcal{T} \) (with its indices suppressed) is said to be superconformal primary of weight \((p,q)\) if its superconformal transformation law is

\[ \delta \xi \mathcal{T} = \left( \xi + \frac{1}{2} K^{bc}[\xi] M_{bc} \right) \mathcal{T} + \left( p\sigma[\xi] + q\bar{\sigma}[\xi] \right) \mathcal{T}, \]  

(2.10)

for some parameters \( p \) and \( q \). The dimension of \( \mathcal{T} \) is \((p+q)\), while \((p-q)\) is proportional to its \( R \)-symmetry charge. If \( \mathcal{T} \) is superconformal primary and chiral, \( \bar{D}_\alpha \mathcal{T} = 0 \), then \( \mathcal{T} \) cannot possess dotted indices, i.e. \( \bar{M}_{\dot{\alpha} \dot{\beta}} \mathcal{T} = 0 \), and it must hold that \( q = 0 \). In the chiral case, it suffices to say that \( \mathcal{T} \) is superconformal primary of dimension \( p \).
Given a real scalar $L$, which is superconformal primary of weight $(1,1)$,
\[
\delta \xi L = \xi L + (\sigma [\xi] + \bar{\sigma} [\bar{\xi}]) L = \partial_\alpha (\xi^\alpha L) - D_\alpha (\xi^\alpha L) - \bar{D}^{\dot{\alpha}} (\bar{\xi}^\dot{\alpha} L),
\]  
the functional
\[
S = \int d^4x d^2\theta d^2\bar{\theta} L
\]  
is invariant under superconformal transformations. Given a chiral scalar $L_c$, which is superconformal primary of dimension $+3$,
\[
\bar{D}_\dot{\alpha} L_c = 0, \quad \delta \xi L_c = \xi L_c + 3\sigma [\xi] L_c = \partial_\alpha (\xi^\alpha L_c) - D_\alpha (\xi^\alpha L_c),
\]  
the functional
\[
S_c = \int d^4x d^2\theta L_c
\]  
is invariant under superconformal transformations.

3 Off-shell superconformal multiplets in flat space

In this section we introduce off-shell superconformal higher spin multiplets. We first consider the half-integer and integer superspin cases, and then give some generalisations of the constructions proposed. Strictly speaking, the notion of superspin is defined only for super-Poincaré multiplets. The rationale for our use of this name in the superconformal framework is that our superconformal multiplets will be described solely in terms of the gauge prepotentials corresponding to the off-shell massless higher spin multiplets constructed in [14,15]. Each of these massless multiplets also involves certain compensator superfields, in addition to the gauge prepotential.

3.1 Half-integer superspin

Let $s$ be a positive integer. In the superspin-$(s + \frac{1}{2})$ case, the conformal prepotential $H_{\alpha_1...\alpha_s \dot{\alpha}_1...\dot{\alpha}_s} \equiv H_{(s+\frac{1}{2})}$ is a real superfield, which is symmetric in its undotted indices and, independently, in its dotted indices. The gauge transformation law of $H_{\alpha_1...\alpha_s \dot{\alpha}_1...\dot{\alpha}_s}$ is
\[
\delta H_{\alpha_1...\alpha_s \dot{\alpha}_1...\dot{\alpha}_s} = \bar{D}_{(\dot{\alpha}_1 \Lambda_{\alpha_1...\alpha_s \dot{\alpha}_2...\dot{\alpha}_s})} - D_{(\alpha_1 \bar{\Lambda}_{\alpha_2...\alpha_s \dot{\alpha}_1...\dot{\alpha}_s})},
\]  
with unconstrained gauge parameter $\Lambda_{\alpha_1...\alpha_s \dot{\alpha}_1...\dot{\alpha}_s}$. 

In the $s = 1$ case, the transformation law (3.1) corresponds to linearised conformal supergravity [16]. The same transformation of $H_{a\bar{a}}$ occurs in all off-shell models for linearised $\mathcal{N} = 1$ supergravity, see [7] for a review. Such actions involve not only the gravitational superfield $\mathcal{H}_{16}$, but also certain compensators. For $s > 1$, the gauge transformation law (3.1) was introduced in [14] in the framework of the (two dually equivalent) off-shell formulations for the massless superspin-$(s + 1/2)$ multiplet. The massless actions of [14] involve not only the gauge prepotential $H_{a(s)\bar{a}(s)}$ but also certain compensators (see [7] for a pedagogical review).

The superconformal transformation law of $H_{a(s)\bar{a}(s)}$ is

$$\delta_{\xi} H_{a(s)\bar{a}(s)} = \left( \xi + \frac{1}{2} K^{bc}[\xi] M_{bc} \right) H_{a(s)\bar{a}(s)} - \frac{s}{2} \left( \sigma[\xi] + \bar{\sigma}[\xi] \right) H_{a(s)\bar{a}(s)}. \quad (3.2)$$

This transformation law is uniquely determined if one requires both the gauge superfield $H_{a(s)\bar{a}(s)}$ and the gauge parameter $\Lambda_{a(s)\bar{a}(s)}$ in (3.1) to be superconformal primary (see also [19 20]). It follows from (3.1) that the chiral symmetric spinor

$$W_{a_1...a_{2s+1}} = -\frac{1}{4} D^2 \partial_{(a_1} \dot{\beta}_1 \ldots \partial_{a_s} \dot{s} \ldots D_{a_{s+1}} H_{a_{s+2}...a_{2s+1}} \dot{\beta}_1 \ldots \dot{\beta}_s \quad (3.3)$$

is gauge invariant. Our crucial observation is that $W_{a_{(2s+1)}}$ is superconformal primary of dimension 3/2. We conclude that the gauge-invariant action

$$S_{s+\frac{1}{2}} = \int d^4x d^2\theta W^{a_1...a_{2s+1}} W_{a_1...a_{2s+1}} + \int d^4x d^2\bar{\theta} \bar{W}_{\bar{a}_1...\bar{a}_{2s+1}} \bar{W}^{\bar{a}_1...\bar{a}_{2s+1}} \quad (3.4)$$

is superconformal. In the $s = 1$ case, it coincides with the action for linearised conformal supergravity [16]. One may check that

$$\int d^4x d^2\theta W^{a_1...a_{2s+1}} W_{a_1...a_{2s+1}} = \int d^4x d^2\bar{\theta} \bar{W}_{\bar{a}_1...\bar{a}_{2s+1}} \bar{W}^{\bar{a}_1...\bar{a}_{2s+1}}. \quad (3.5)$$

We briefly comment on the component structure of the superconformal theory (3.4). The gauge parameter $\bar{D}(\dot{a}_1 \Lambda_{a_1...a_s \bar{a}_2...\bar{a}_s})$ in (3.1) may be represented as

$$\bar{D}(\dot{a}_1 \Lambda_{a_1...a_s \bar{a}_2...\bar{a}_s})(\theta, \bar{\theta}) = e^{i\theta_0} \left\{ g_{a(s)\bar{a}_1...\bar{a}_s} + i\bar{\theta}(\dot{a}_1 \rho_{a(s)\bar{a}_2...\bar{a}_s}) + i\theta^2 \chi_{a(s)\bar{a}_1...\bar{a}_s} \right. \right.$$

$$\left. + \theta^2 v_{a(s)\bar{a}_1...\bar{a}_s} + \theta^2 \bar{\theta}(\dot{a}_1 \omega_{a(s)\bar{a}_2...\bar{a}_s}) + \bar{\theta}^2(\dot{a}_1 \omega_{a(s)\bar{a}_2...\bar{a}_s}) \right\}. \quad (3.6)$$

---

\(^2\text{The chiral superfield (3.3) is the only gauge-invariant field strength which remains non-vanishing on-shell in the supersymmetric higher spin theories introduced in [14]. In a model independent framework of superfield representations, field strengths of the form (3.3) appeared in [21].}\)
where $\mathcal{H}_0 = \theta \sigma^a \bar{\theta} \partial_a$ and all component gauge parameters are complex. The parameters $\chi_{\beta, \alpha(s)} \bar{\alpha(s)}$ and $f_{\beta, \alpha(s)} \bar{\alpha(s-1)}$ transform in the representation $2 \otimes (2s + 1)$ of $\text{SL}(2, \mathbb{C})$ with respect to their undotted indices. It follows from (3.1) and (3.6) that a Wess-Zumino

$$H_{a_1...a_s \bar{a}_1...\bar{a}_s}(\theta, \bar{\theta}) = \theta^3 \bar{\theta}^3 \bar{h}(\bar{a}_{a_1...a_s}) \bar{\phi}(\bar{a}_{a_1...a_s}) + \bar{\theta}^2 \theta^2 \psi(\bar{a}_{a_1...a_s}) \bar{a}_1...\bar{a}_s$$

$$- \bar{\theta}^2 \bar{\theta}^2 \bar{\psi}(\bar{a}_{a_1...a_s}) \bar{a}_1...\bar{a}_s,$$

(3.7)

where the bosonic fields $h_{a(s+1)\bar{a}(s+1)}$ and $h_{a(s)\bar{a}(s)}$ are real. The residual gauge freedom is generated by

$$\bar{D}(\bar{a}_1 \Lambda_{a(s)\bar{a}_2...\bar{a}_s})(\theta, \bar{\theta}) = e^{\mathcal{H}_0} \left\{ - \frac{i}{2} \zeta_{a(s)\bar{a}_2...\bar{a}_s} + i \bar{\theta}(\bar{a}_1 \bar{\rho}_{a(s)\bar{a}_2...\bar{a}_s}) - i \theta(a_1 \rho_{a_2...a_s}) \bar{a}_1...\bar{a}_s$$

$$+ \frac{s}{s + 1} \theta^3 \bar{\theta}^3 \tilde{\zeta}(a_{1} \zeta_{a_2...a_s}) \bar{a}_1...\bar{a}_s - \frac{1}{2} \frac{s^2}{(s + 1)^2} \theta(a_1 \bar{\rho}(\bar{a}_1 \zeta_{a_2...a_s}) \bar{a}_1...\bar{a}_s)$$

$$- \frac{s}{s + 1} \theta^2 \bar{\theta}^2 \tilde{\zeta}(a_1 \zeta_{a_2...a_s}) \bar{a}_1...\bar{a}_s \right\},$$

(3.8)

where the bosonic parameters $\zeta_{a(s)\bar{a}(s)}$ and $\zeta_{a(s-1)\bar{a}(s-1)}$ are real. The residual gauge transformations are:

$$\delta h_{a_1...a_{s+1} \bar{a}_1...\bar{a}_{s+1}} = \partial(a_1 \zeta_{a_2...a_{s+1}} \bar{a}_2...\bar{a}_{s+1})$$

(3.9a)

$$\delta h_{a_1...a_s \bar{a}_1...\bar{a}_s} = \partial(a_1 \zeta_{a_2...a_s} \bar{a}_2...\bar{a}_s)$$

(3.9b)

$$\delta \bar{\psi}_{a_1...a_{s+1} \bar{a}_1...\bar{a}_{s+1}} = \partial(a_1 \bar{\rho}_{a_2...a_{s+1}} \bar{a}_2...\bar{a}_{s+1}) \bar{a}_1...\bar{a}_s.$$

(3.9c)

These transformation laws correspond to conformal higher spin fields [3].

Reducing the actions (3.4) from superspace to components, for $s = 1, 2, \ldots$, reproduces the conformal higher spin actions introduced by Fradkin and Tseytlin [3],

### 3.2 Integer superspin

In the superspin-$s$ case, the superconformal multiplet is described in terms of an unconstrained prepotential $\Psi_{a(s)\bar{a}(s-1)} \equiv \Psi_{a_1...a_s \bar{a}_1...\bar{a}_{s-1}}$ and its conjugate $\bar{\Psi}_{a(s-1)\bar{a}(s)}$. The prepotential is symmetric in its undotted indices and, independently, in its dotted indices. For $s > 1$ the gauge freedom is

$$\delta \Psi_{a_1...a_s \bar{a}_1...\bar{a}_{s-1}} = D(a_1 \zeta_{a_2...a_s} \bar{a}_1...\bar{a}_{s-1}) + \bar{D}(\bar{a}_1 \zeta_{a_1...a_s} \bar{a}_2...\bar{a}_{s-1}).$$

(3.10)
with unconstrained gauge parameters \( \tilde{\Lambda}_{\alpha(s-1)}\tilde{\alpha}(s-1) \) and \( \zeta_{\alpha(s)\dot{\alpha}(s-2)} \). The choice \( s = 1 \) will be considered in subsection 3.3.

As was shown in [15], the prepotential \( \Psi_{\alpha(s)\dot{\alpha}(s-1)} \) naturally originates within the so-called \textit{longitudinal} formulation for the massless superspin-\( s \) multiplet, which also makes use of a real unconstrained compensator \( H_{\alpha(s-1)\dot{\alpha}(s-1)} \). The prepotential \( \Psi_{\alpha(s)\dot{\alpha}(s-1)} \) enters the action functional of [15] only via the \textit{longitudinal linear} field strength \( G_{\alpha(s)\dot{\alpha}(s)} := \bar{D}(\alpha_{1}\Psi_{\alpha(s)\dot{\alpha}_{2}...\dot{\alpha}_{s}}) \), which is manifestly invariant under the \( \zeta \)-transformation (3.10). On the other hand, in the non-superconformal case the gauge parameter \( \Lambda \) is not arbitrary but instead has the form \( \tilde{\Lambda}_{\alpha(s-1)}\dot{\alpha}(s-1) = D_{\beta_{1}}\bar{L}_{\dot{\beta}_{1}}...D_{\beta_{s-1}}\bar{L}_{\dot{\beta}_{s-1}}\tilde{\Lambda}_{\alpha(s-1)}\dot{\alpha}(s-1) \), with \( L_{\alpha(s)\dot{\alpha}(s-1)} \) unconstrained. This is not critical since one may always make \( \tilde{\Lambda}_{\alpha(s-1)}\dot{\alpha}(s-1) \) unconstrained at the cost of introducing an additional compensator (in complete analogy with the massless gravitino case considered in [22] and reviewed in [7]). For the massless superspin-\( s \) multiplet, there exists another off-shell formulation which was constructed in [15] and called \textit{transverse}. It is dual to the longitudinal one. It does not appear to be suitable to describe a superconformal multiplet.

The superconformal transformation law of \( \Psi_{\alpha(s)\dot{\alpha}(s-1)} \) is postulated to be
\[
\delta_{\xi}\Psi_{\alpha(s)\dot{\alpha}(s-1)} = \left( \xi + \frac{1}{2}K^{bc}[\xi]M_{bc} \right)\Psi_{\alpha(s)\dot{\alpha}(s-1)} - \frac{1}{2}(s\sigma[\xi] + (s-1)\bar{\sigma}[\xi])\Psi_{\alpha(s)\dot{\alpha}(s-1)} .
\] (3.11)

It follows from (3.10) that the following chiral descendants of the prepotentials
\[
\mathcal{W}_{\alpha_{1}...\alpha_{2s}} = -\frac{1}{4}\bar{D}^{2}\partial_{(\alpha_{1}\dot{\beta}_{1})...\partial_{\alpha_{s-1}\dot{\beta}_{s-1}}D_{\alpha_{s+1}...\alpha_{2s}}\Psi_{\alpha_{s+2}...\alpha_{2s}}\dot{\beta}_{1}...\dot{\beta}_{s} ,
\] (3.12a)
\[
\mathcal{Z}_{\alpha_{1}...\alpha_{2s}} = -\frac{1}{4}\bar{D}^{2}\partial_{(\alpha_{1}\dot{\beta}_{1})...\partial_{\alpha_{s-1}\dot{\beta}_{s-1}}D_{\alpha_{s+1}...\alpha_{2s}}\bar{\Psi}_{\alpha_{s+2}...\alpha_{2s}}\dot{\beta}_{1}...\dot{\beta}_{s} ,
\] (3.12b)
are gauge invariant. As before, the crucial observation is that the field strengths \( \mathcal{W}_{\alpha(2s)} \) and \( \mathcal{Z}_{\alpha(2s)} \) are superconformal primaries of dimension 1 and 2, respectively. This allows us to construct a superconformal and gauge-invariant action
\[
S_{s} = i\int d^{4}xd^{2}\theta\mathcal{W}_{\alpha_{1}...\alpha_{2s}}\mathcal{Z}_{\alpha_{1}...\alpha_{2s}} - i\int d^{4}xd^{2}\bar{\theta}\bar{\mathcal{W}}_{\dot{\alpha}_{1}...\dot{\alpha}_{2s}}\bar{\mathcal{Z}}_{\dot{\alpha}_{1}...\dot{\alpha}_{2s}} .
\] (3.13)

One checks that
\[
\int d^{4}xd^{2}\theta\mathcal{W}_{\alpha_{1}...\alpha_{2s}}\mathcal{Z}_{\alpha_{1}...\alpha_{2s}} + \int d^{4}xd^{2}\bar{\theta}\bar{\mathcal{W}}_{\dot{\alpha}_{1}...\dot{\alpha}_{2s}}\bar{\mathcal{Z}}_{\dot{\alpha}_{1}...\dot{\alpha}_{2s}} = 0 .
\] (3.14)

We now comment on the component structure of (3.13). One may choose a Wess-Zumino gauge of the form
\[
\Psi_{\alpha_{1}...\alpha_{s}\dot{\alpha}_{1}...\dot{\alpha}_{s-1}}(\theta, \bar{\theta}) = \theta^{\beta}\bar{\theta}^{\dot{\beta}}\psi_{(\beta\alpha_{1}...\alpha_{s})(\dot{\beta}\dot{\alpha}_{1}...\dot{\alpha}_{s-1})} + \bar{\theta}^{2}\theta^{2}B_{(\beta\alpha_{1}...\alpha_{s})\dot{\alpha}_{1}...\dot{\alpha}_{s-1}}
\]
\footnote{The field strength (3.12a) was introduced in [15].}
\[-\theta^2 \hat{\phi}^2 \hat{h}_{\alpha_1...\alpha_s} (\beta \hat{a}_1...\hat{a}_{s-1}) + \theta^2 \hat{\phi}^2 \hat{\psi}_{\alpha_1...\alpha_s} (\beta \hat{a}_1...\hat{a}_{s-1}) , \]  

(3.15)

where the bosonic fields \( h_{\alpha(s)} \hat{a}(s) \) and \( B_{\alpha(s+1)} \hat{a}(s-1) \) are complex. In the Wess-Zumino gauge chosen, the bosonic fields \( h_{\alpha(s)} \hat{a}(s) \) and \( B_{\alpha(s+1)} \hat{a}(s-1) \) and the fermionic fields \( \psi_{\alpha(s+1)} \hat{a}(s) \) and \( \psi_{\alpha(s)} \hat{a}(s-1) \) are defined modulo gauge freedom of the type \([3.9]\).

More specifically, the field \( B_{\alpha(s+1)} \hat{a}(s-1) \) belongs to a more general family of conformal fields than those described by the gauge transformation laws \([3.9]\). The point is that one may consider conformal higher spin fields \( \phi_{\alpha(m)} \hat{a}(n) \), where \( m \) and \( n \) are integers such that \( m > n > 0 \). Since \( m \neq n \), the field \( \phi_{\alpha(m)} \hat{a}(n) \) is complex. Postulating the gauge transformation law

\[
\delta \phi_{\alpha_1...\alpha_m} = \partial (\lambda \alpha_{a_1} \lambda \alpha_{a_2}...\lambda \alpha_{a_n}) \]

(3.16)

and requiring both the field \( \phi_{\alpha(m)} \hat{a}(n) \) and the gauge parameter \( \lambda_{\alpha(m-1)} \hat{a}(n-1) \) to be primary, the dimension of \( \phi_{\alpha(m)} \hat{a}(n) \) is fixed to be equal to \( 2 - \frac{1}{2} (m + n) \). We can define two gauge-invariant field strengths

\[
\tilde{C}_{\alpha_1...\alpha_m+n} = \partial (\lambda \alpha_{a_1} \lambda \alpha_{a_2}...\lambda \alpha_{a_n}) \phi_{\alpha_{n+1}...\alpha_{m+n}} \beta_{a_1}...\beta_{a_n}, \quad (3.17a)
\]

\[
\tilde{C}_{\alpha_1...\alpha_m+n} = \partial (\lambda \alpha_{a_1} \lambda \alpha_{a_2}...\lambda \alpha_{a_n}) \phi_{\alpha_{m+1}...\alpha_{m+n}} \beta_{a_1}...\beta_{a_n}. \quad (3.17b)
\]

They are conformal primaries of dimension \( 2 - \frac{1}{2} (m - n) \) and \( 2 - \frac{1}{2} (m - n) \), respectively. In terms of those we can write a gauge-invariant conformal action

\[
S = i^{m+n} \int d^4x \tilde{C}^{\alpha_1...\alpha_{m+n}} \tilde{C}_{\alpha_1...\alpha_{m+n}} + c.c. \quad (3.18)
\]

### 3.3 Superconformal gravitino multiplet

In the \( s = 1 \) case, the gauge transformation law \([3.10]\) has to be replaced with

\[
\delta \Psi_{\alpha} = D_{\alpha} \tilde{A} + \zeta_{\alpha}, \quad \tilde{D}_{\beta} \zeta_{\alpha} = 0. \quad (3.19)
\]

This gauge transformation was given in Ref. [22], which proposed the off-shell formulation for the massless gravitino multiplet in terms of the gauge spinor prepotential \( \Psi_{\alpha} \) in conjunction with two compensators, an unconstrained real scalar and a chiral scalar.

The prepotential \( \Psi_{\alpha} \) is required to be superconformal primary of weight \((-1,0)\), which is a special case of \([3.11]\). The superconformal primary superfields \([3.12]\) for \( s = 1 \) are obviously invariant under the gauge transformations \([3.19]\).

---

4The case \( s = 1 \) is not considered here. Its special feature is that \( B_{\alpha \beta} \) is not a gauge field.
3.4 Generalisations

Given two integers \( m > n > 0 \), we introduce a gauge prepotential \( \Phi_{\alpha(m)\dot{\alpha}(n)} \equiv \Phi_{\alpha_1...\alpha_m\dot{\alpha}_1...\dot{\alpha}_n} \) and its conjugate \( \bar{\Phi}_{\alpha(n)\dot{\alpha}(m)} \). The gauge transformation of \( \Phi_{\alpha(m)\dot{\alpha}(n)} \) is postulated to be

\[
\delta \Phi_{\alpha_1...\alpha_m\dot{\alpha}_1...\dot{\alpha}_n} = D_{(\alpha_1 \bar{\alpha}_2...\alpha_m)} \Phi_{\alpha_1...\alpha_m\dot{\alpha}_1...\dot{\alpha}_n} + \bar{D}_{(\dot{\alpha}_1 \bar{\alpha}_2...\dot{\alpha}_n)} \zeta_{\alpha_1...\alpha_m\dot{\alpha}_1...\dot{\alpha}_n} , \tag{3.20}
\]

with unconstrained gauge parameters \( \bar{\Lambda}_{\alpha(m-1)\dot{\alpha}(n)} \) and \( \zeta_{\alpha(m)\dot{\alpha}(n-1)} \). The superconformal transformation law of \( \Phi_{\alpha(m)\dot{\alpha}(n)} \) is

\[
\delta_{\xi} \Phi_{\alpha(m)\dot{\alpha}(n)} = \left( \xi + \frac{1}{2} K^{bc}[\xi] M_{bc} \right) \Phi_{\alpha(m)\dot{\alpha}(n)} - \frac{1}{2} (m \sigma[\xi] + n \bar{\sigma}[\xi]) \Phi_{\alpha(m)\dot{\alpha}(n)} , \tag{3.21}
\]

Given \( \Phi \), we can define two gauge-invariant chiral field strengths

\[
\mathbb{W}_{\alpha_1...\alpha_{m+n+1}} = -\frac{1}{4} \bar{D}^2 \partial_{(\alpha_1} \beta_1 \ldots \partial_{\alpha_n} \beta_n D_{\alpha_{n+1}} \Phi_{\alpha_{n+2}...\alpha_{m+n+1})\beta_1...\beta_n} , \tag{3.22a}
\]

\[
\mathbb{Z}_{\dot{\alpha}_1...\dot{\alpha}_{m+n+1}} = -\frac{1}{4} \bar{D}^2 \partial_{(\dot{\alpha}_1} \beta_1 \ldots \partial_{\dot{\alpha}_m} \beta_m D_{\dot{\alpha}_{m+1}} \bar{\Phi}_{\dot{\alpha}_{m+2}...\dot{\alpha}_{m+n+1})\beta_1...\beta_m} . \tag{3.22b}
\]

which are superconformal primaries of dimension \( \frac{1}{2}(3 + n - m) \) and \( \frac{1}{2}(3 + m - n) \), respectively. Therefore, the following gauge-invariant action

\[
S = i^{m+n} \int d^4x \chi^2 \mathbb{W}_{\alpha_1...\alpha_{m+n+1}} \mathbb{Z}_{\dot{\alpha}_1...\dot{\alpha}_{m+n+1}} + \text{c.c.} \tag{3.23}
\]

is superconformal.

4 Off-shell superconformal multiplets in supergravity

We now turn to exploring whether the superconformal higher spin multiplets introduced in the previous section may be consistently lifted to curved superspace backgrounds.

4.1 General considerations

Just as in the non-supersymmetric setting, where conformal invariance in Minkowski space is replaced by Weyl invariance, in a curved background geometry, superconformal invariance is replaced by super-Weyl invariance. In other words, super-Weyl invariance in curved superspace implies superconformal invariance in Minkowski superspace.
A tensor superfield $\mathcal{T}$ (with its indices suppressed) is said to be super-Weyl primary of weight $(p, q)$ if its super-Weyl transformation law is

$$\delta_{\sigma} \mathcal{T} = (p \sigma + q \bar{\sigma}) \mathcal{T}, \quad (4.1)$$

for some parameters $p$ and $q$. Similar to the rigid supersymmetric case (2.10), we will refer to $(p+q)$ as the dimension of $\mathcal{T}$. Given a covariantly chiral tensor superfield $\mathcal{T}$ defined on a general supergravity background, $\bar{D}_{\dot{\alpha}} \mathcal{T} = 0$, it may carry only undotted indices, $\bar{M}_{\dot{\alpha} \dot{\beta}} \mathcal{T} = 0$, as a consequence of (A.3b). If $\mathcal{T}$ is covariantly chiral and super-Weyl primary, eq. (4.1), then $q = 0$. An example is provided by the super-Weyl tensor $W_{\alpha \beta \gamma}$ with the transformation law (B.2c). In Appendix B we also collect the transformation properties of various other geometric quantities under super-Weyl transformations.

As reviewed in Appendix A, the curved superspace geometry of [8] does not possess torsion tensors of dimensions 1/2. This means that the gauge transformation (3.1) is uniquely extended to curved superspace as

$$\delta H_{\alpha_1 \ldots \alpha_s \dot{\alpha}_1 \ldots \dot{\alpha}_s} = \bar{D}_{(\dot{\alpha}_1} \Lambda_{\alpha_1 \ldots \alpha_s \dot{\alpha}_2 \ldots \dot{\alpha}_s)} - D_{(\alpha_1} \bar{\Lambda}_{\alpha_2 \ldots \alpha_s)\dot{\alpha}_1 \ldots \dot{\alpha}_s} . \quad (4.2)$$

It is compatible with the following super-Weyl transformation of the prepotential:

$$\delta_{\sigma} H_{\alpha(s) \dot{\alpha}(s)} = -\frac{s}{2} (\sigma + \bar{\sigma}) H_{\alpha(s) \dot{\alpha}(s)} . \quad (4.3)$$

The chiral field strength (3.3) may uniquely be lifted to curved superspace as a co-variantly chiral superfield of the general form

$$W_{\alpha(2s+1)} = -\frac{1}{4} (\bar{D}^2 - 4R) \left\{ D_{(\alpha_1} \dot{\beta}_1 \ldots D_{\alpha_s} \dot{\beta}_s} D_{\alpha_{s+1}} H_{\alpha_{s+2} \ldots \alpha_{2s+1}) \dot{\beta}_1 \ldots \dot{\beta}_s} + \ldots \right\}, \quad (4.4a)$$

$$\bar{D}_{\dot{\beta}} W_{\alpha(2s+1)} = 0 , \quad (4.4b)$$

with the super-Weyl transformation law

$$\delta_{\sigma} W_{\alpha(2s+1)} = \frac{3}{2} \sigma W_{\alpha(2s+1)} . \quad (4.5)$$

The ellipsis in (4.4a) stands for terms involving the super-Ricci tensor $G_{\alpha \dot{\alpha}}$ and its co-variant derivatives. Such terms can always be found. A systematic construction is to start with conformal superspace [13], where $G_{\alpha \dot{\alpha}}$ appears as a connection, and then to implement the so-called de-gauging procedure in order to arrive at the ordinary curved superspace geometry of [8]. Details of the construction will be given elsewhere, but examples of the complete superfields for $s = 1$ and $s = 2$ are given below in (4.7) and

\[ \text{In conformal superspace, the required primary chiral field strength } W_{\alpha(2s+1)} \text{ has a minimal form } \]

\[ W_{\alpha(2s+1)} = -\frac{1}{4} \bar{\nabla}^2 \nabla_{(\alpha_1} \dot{\beta}_1 \ldots \nabla_{\alpha_s} \dot{\beta}_s \nabla_{\alpha_{s+1}} H_{\alpha_{s+2} \ldots \alpha_{2s+1}) \dot{\beta}_1 \ldots \dot{\beta}_s} , \text{ where } \nabla_A = (\nabla_a, \nabla_\alpha, \bar{\nabla}^{\dot{\alpha}}) \text{ denotes the corresponding covariant derivatives [13].} \]
Two observations, which are crucial for this construction, are that the descendant \( A_\alpha^{(s+1)} \delta(s) := D_\alpha^{(s+1)} H_{\alpha_1 ... \alpha_s} \delta_1 ... \delta_s \) is super-Weyl primary and obeys the constraint \( D_{(\alpha_1} A_{\alpha_2 ... \alpha_{s+2})} \delta(s) = 0 \).

We may now consider a minimal extension of (3.3) to curved superspace given by

\[
\int d^4x d^2\theta \mathcal{E} W_{\alpha_1 ... \alpha_{s+1}} W_{\alpha_1 ... \alpha_{s+1}} + \text{c.c.,}
\]

where \( \mathcal{E} \) is the chiral integration measure. It follows from (4.5) that this functional is super-Weyl invariant. However, for non-vanishing background super-Weyl tensor, \( W_{\alpha \beta \gamma} \neq 0 \), the field strength \( W_{\alpha_1 (s+1)} \) and, therefore, the action (4.6) are not gauge invariant. In general, the gauge variation \( \delta_\Lambda W_{\alpha_1 (s+1)} \) is proportional to the background super-Weyl tensor \( W_{\alpha \beta \gamma} \), its conjugate \( \bar{W}_{\dot{\alpha} \dot{\beta} \dot{\gamma}} \) and their covariant derivatives. The action (4.6) needs to be completed to include non-minimal terms which contain \( W_{\alpha \beta \gamma} \), \( \bar{W}_{\dot{\alpha} \dot{\beta} \dot{\gamma}} \) and their covariant derivatives. An example will be given in section 4.3, where we discuss the gravitino supermultiplet.

Let us first discuss the simplest case of \( W_{\alpha_1 (2s+1)} \), \( s = 1 \), which is linearised conformal supergravity. The linearised super-Weyl tensor is

\[
W_{\alpha \beta \gamma} = -\frac{1}{4}(\bar{D}^2 - 4R)\left\{ (\bar{D}_{(\alpha} \dot{\gamma} + i G_{(\alpha} \dot{\gamma}) \bar{D}_{\beta}) H_{\gamma)} \right\},
\]

modulo normalisation. It varies homogeneously under the super-Weyl transformation, in accordance with (4.3). However, \( W_{\alpha \beta \gamma} \) is not invariant under the gauge transformation (4.2) with \( s = 1 \). One may check that

\[
\delta_\Lambda W_{\alpha \beta \gamma} = \frac{1}{2}(\bar{D}^2 - 4R) \left[ (\bar{D}^\delta W_{\delta(\alpha \beta)}) A_{\gamma)} - D_{(\alpha} (W_{\beta \gamma) \delta A_{\delta}) \right].
\]

The important point is that each term in \( \delta_\Lambda W_{\alpha \beta \gamma} \) involves either the background super-Weyl tensor or its covariant derivative. The variation vanishes if the background superspace is conformally flat, \( W_{\alpha \beta \gamma} = 0 \). In this case the functional (4.6) is the required superconformal gauge-invariant action. Here 'superconformal' means that the action is invariant under arbitrary superconformal isometries of the background superspace.

As another example of \( W_{\alpha_1 (2s+1)} \), we consider the case \( s = 2 \). The field strength \( W_{\alpha (5)} \) is uniquely determined to be

\[
W_{\alpha_1 ... \alpha_5} = -\frac{1}{4}(\bar{D}^2 - 4R)\left\{ D_{(\alpha_1} \dot{\beta_1} D_{\alpha_2} \dot{\beta_2} + 3i G_{(\alpha_1} \dot{\beta_1} D_{\alpha_2} \dot{\beta_2} - 2G_{(\alpha_1} \dot{\beta_1} G_{\alpha_2} \dot{\beta_2}}
\]

\footnote{For instance, in the \( s = 1 \) case the variation \( \delta_\Lambda W_{\alpha (3)} \) is given by (4.8).}
\[-\frac{1}{4}\left(\mathcal{D}_{(\alpha_1}\,\mathcal{D}^{\beta_1}]G_{\alpha_2}\,\beta_2) + \frac{3}{2}i(\mathcal{D}_{(\alpha_1}\,\delta_{\alpha_2}\,\beta_2)\right)\mathcal{D}_{\alpha_3}H_{A_{\alpha_3}\beta_1\beta_2} . \quad (4.9)\]

It is a tedious exercise to check that $W_{\alpha(5)}$ is super-Weyl primary.

In the case of anti-de Sitter superspace AdS$^4[23, 24, 25]$ specified by

\[W_{\alpha\beta\gamma} = 0 , \quad G_{\alpha\dot{\alpha}} = 0 , \quad R \neq 0 , \quad (4.10)\]

the gauge-invariant chiral field strength $W_{\alpha(2s+1)}$ was found in [26]. It is

\[W_{\alpha(2s+1)} = -\frac{1}{4}(\bar{D}^2 - 4R)\mathcal{D}(\alpha_1\,\delta_{\beta_1} \cdots \mathcal{D}_{\alpha_{s+1}}H_{\alpha_{s+2}..\alpha_{2s+1}})\dot{\beta}_1 \cdots \dot{\beta}_s . \quad (4.11)\]

The curved-superspace extension of the gauge transformation (3.10) is

\[\delta\psi_{\alpha_1..\alpha_s\dot{\alpha}_1..\dot{\alpha}_{s-1}} = \mathcal{D}_{(\alpha_1\,\bar{\Lambda}_{\alpha_2..\alpha_s})\dot{\alpha}_1..\dot{\alpha}_{s-1}} + \mathcal{D}_{(\alpha_1\,\zeta_{\alpha_1..\alpha_s\dot{\alpha}_2..\dot{\alpha}_{s-1})} . \quad (4.12)\]

It is compatible with the following super-Weyl transformation of the prepotential

\[\delta_\sigma\psi_{\alpha(s)\dot{\alpha}(s-1)} = -\frac{1}{2}(s\sigma + (s-1)\bar{\sigma})\psi_{\alpha(s)\dot{\alpha}(s-1)} . \quad (4.13)\]

### 4.2 Superconformal gravitino multiplet

Our next example is the superconformal gravitino multiplet. It is characterised by the gauge freedom

\[\delta\psi_{\alpha} = \mathcal{D}_{\alpha}\bar{\Lambda} + \zeta_{\alpha} , \quad \bar{D}_{\beta}\zeta_{\alpha} = 0 , \quad (4.14)\]

and the super-Weyl transformation

\[\delta_\sigma\psi_{\alpha} = -\frac{1}{2}\sigma\psi_{\alpha} . \quad (4.15)\]

The following covariantly chiral field strengths

\[W_{\alpha\beta} = -\frac{1}{4}(\bar{D}^2 - 4R)\mathcal{D}(\alpha\beta) , \quad (4.16a)\]

\[Z_{\alpha\beta} = -\frac{1}{4}(\bar{D}^2 - 4R)\left[(\mathcal{D}_{\alpha\dot{\alpha}} + iG_{\alpha\dot{\alpha}})\mathcal{D}_{\beta}\bar{\psi}_{\dot{\alpha}}\right] . \quad (4.16b)\]

are super-Weyl primary of dimension +1 and +2, respectively. These superfields are not invariant under the gauge transformations (4.14). One finds the following non-vanishing variations of $W_{\alpha\beta}$ and $Z_{\alpha\beta}$:

\[\delta_{\zeta}W_{\alpha\beta} = 2W_{\alpha\beta\gamma}\zeta_{\gamma} , \quad (4.17a)\]
\[ \delta \mathcal{Z}_{\alpha \beta} = \frac{i}{2} W_{\alpha \beta \gamma} (\bar{D}^2 - 4R) \mathcal{D}^\gamma \Lambda + \frac{i}{2} (\bar{D}^2 - 4R) \left\{ \Lambda \mathcal{D}^\gamma W_{\alpha \beta \gamma} \right\} \]  

(4.17b)

Consider the action

\[ S_{\text{GM}} = i \int d^4 x d^2 \theta \mathcal{E} W_{\alpha \beta} \mathcal{Z}_{\alpha \beta} - 2i \int d^4 x d^2 \bar{\theta} d^2 \bar{\theta} \mathcal{E} W_{\alpha \beta \gamma} \Psi_{\alpha} (\mathcal{D}_{\beta \bar{\gamma}} + iG_{\beta \bar{\gamma}}) \mathcal{D}_{\gamma} \bar{\Psi}^\bar{\beta} + \bar{\mathcal{E}} + \mathcal{E} \]  

(4.18)

Here \( \mathcal{E} \) and \( \mathcal{E} \) denote the chiral measure and the full superspace measure, respectively. The action \( S_{\text{GM}} \) is super-Weyl invariant,

\[ \delta_{\sigma} S_{\text{GM}} = 0 \]  

(4.19)

The second and third terms on the right of (4.18) are fixed by requiring \( S_{\text{GM}} \) to be invariant under the \( \zeta \)-transformation (4.14),

\[ \delta_{\zeta} S_{\text{GM}} = 0 \]  

(4.20)

Finally, a lengthy calculation gives

\[ \delta_{\Lambda} S_{\text{GM}} = 2 \int d^4 x d^2 \theta d^2 \bar{\theta} \mathcal{E} B^{a \dot{a}} (\Psi_{a} \mathcal{D}_{\dot{a}} \Lambda + \Lambda \mathcal{D}_{\dot{a}} \Psi_{a}) + \text{c.c.} \]  

(4.21)

Here \( B^{a \dot{a}} \) denotes the \( \mathcal{N} = 1 \) supersymmetric extension of the Bach tensor,

\[ B^{a \dot{a}} = iD_{\beta \dot{a}} \mathcal{D}_{\gamma} W^{a \beta \gamma} + (D_{\beta} G_{\gamma \dot{a}}) W^{a \beta \gamma} + G_{\beta \dot{a}} \mathcal{D}_{\gamma} W^{a \beta \gamma} \]
\[ = iD_{\alpha \beta} \bar{\mathcal{D}}_{\dot{a}} \bar{W}^{a \beta \dot{a}} - (\bar{D}_{\dot{a}} G_{\alpha \dot{a}}) \bar{W}^{a \beta \dot{a}} - G_{\alpha \beta} \bar{\mathcal{D}}_{\dot{a}} \bar{W}^{a \beta \dot{a}}, \]  

(4.22)

with the super-Weyl transformation

\[ \delta_{\sigma} B^{a \dot{a}} = \frac{3}{2} (\sigma + \bar{\sigma}) B^{a \dot{a}}. \]  

(4.23)

One can rewrite \( B^{a \dot{a}} \) as a manifestly real form [7, 27]

\[ B^{a \dot{a}} = -D^b D_b G_{a \dot{a}} - W^{a \beta \gamma} D_{\beta} G_{\gamma \dot{a}} + \bar{W}^{a \beta \dot{a}} \bar{D}_{\dot{a}} G_{\alpha \dot{a}} \]
\[ + \frac{1}{4} \left( (D^2 R) D_{\beta} + (\bar{D}_{\dot{a}} \bar{R}) \bar{D}^\dot{a} \right) G_{a \dot{a}} - (\bar{D}_{\dot{a}} G^b) D_{a} G_b - 3R \bar{R} G_{a \dot{a}} \]
\[ + \frac{1}{8} G_{a \dot{a}} (D^2 \bar{R} + D^2 \bar{R}) + \frac{i}{4} D_{a \dot{a}} (D^2 \bar{R} - D^2 \bar{R}). \]  

(4.24)

We recall that the super-Bach tensor may be introduced (see [7, 27] for the technical details) as a functional derivative of the conformal supergravity action [28, 29],

\[ I_{\text{CSG}} = \int d^4 x d^2 \theta \mathcal{E} W^{a \beta \gamma} W_{a \beta \gamma} + \text{c.c.} \]  

(4.26)
with respect to the gravitational superfield, specifically
\[ \delta \int d^4 x d^2 \theta d^2 \bar{\theta} E W^{\alpha \beta \gamma} W_{\alpha \beta \gamma} = \int d^4 x d^2 \theta d^2 \bar{\theta} E \Delta H^{\alpha \dot{\alpha}} B_{\alpha \dot{\alpha}}, \tag{4.27} \]
with \( \Delta H^{\alpha \dot{\alpha}} \) the covariantised variation of the gravitational superfield defined in \[30\]. The super-Bach tensor obeys the conservation equation
\[ \mathcal{D}^\alpha B_{\alpha \dot{\alpha}} = 0 \iff \bar{\mathcal{D}}^{\dot{\alpha}} B_{\alpha \dot{\alpha}} = 0, \tag{4.28} \]
which expresses the gauge invariance of the conformal supergravity action.

It follows from \(4.20\) and \(4.21\) that the action \(4.18\) is gauge invariant if the background super-Bach tensor is equal to zero,
\[ B_{\alpha \dot{\alpha}} = 0. \tag{4.29} \]
This holds, e.g., for all Einstein superspaces, which are characterised by
\[ G_{\alpha \dot{\alpha}} = 0 \implies R = \text{const}. \tag{4.30} \]

### 4.3 Linearised conformal supergravity

The condition \(4.29\) is also required to define an off-shell superconformal multiplet of superspin 3/2 in curved superspace. The point is that \(4.29\) is the equation of motion for conformal supergravity, since varying the action \(4.26\) with respect to the gravitational superfield gives\(\text{7}\)
\[ \delta S_{\text{CSG}} = 2 \int d^4 x d^2 \theta d^2 \bar{\theta} E \Delta H^{\alpha \dot{\alpha}} B_{\alpha \dot{\alpha}}. \tag{4.31} \]
The gauge-invariant action for the superconformal superspin-3/2 multiplet in curved background is obtained by linearising the conformal supergravity action \(4.26\) around its arbitrary stationary point, \( B_{\alpha \dot{\alpha}} = 0 \). In accordance with \[30\] (see also \[7\] for a review), the linearised gauge transformation of the prepotential is given by \(4.12\) with \(s = 1\). The linearised conformal supergravity action is automatically invariant under the gauge and super-Weyl transformations. Its explicit structure will be described elsewhere.

\(\text{7}\)The two terms in the right-hand side of \(4.26\) differ by a total derivative related to the Pontryagin invariant \[7, 27\].
5 Concluding comments

In this paper we constructed the off-shell $\mathcal{N} = 1$ superconformal higher spin multiplets in four dimensions\(^8\) and also sketched the general scheme of coupling such multiplets to conformal supergravity. Our work opens two new approaches to interacting conformal higher spin theories. Firstly, every conformal higher spin field may be embedded into an off-shell superconformal multiplet (the latter actually contains several bosonic and fermionic conformal fields). Instead of trying to couple the original conformal field to gravity, we can look for a consistent interaction of the superconformal multiplet with conformal supergravity. Since the gravitational field belongs to the conformal supergravity multiplet (also known as the Weyl multiplet), this will automatically lead to a consistent coupling of the component conformal fields to gravity.

The second avenue to explore is a superfield extension of the effective action approach to conformal higher spin fields advocated in [33, 34, 35]. One may start with a free massless chiral scalar superfield $\Phi$, $\bar{D}_{\dot{\alpha}} \Phi = 0$, and couple it to an infinite tower of background superconformal higher spin prepotentials $H_{\alpha(s)\dot{\alpha}(s)}$ (source superfields) by the rule

$$S[\Phi, \bar{\Phi}; H] = \int d^4x d^2\theta d^2\bar{\theta} \left\{ \Phi \bar{\Phi} + \sum_{s=1}^{\infty} H^{(s)\dot{\alpha}(s)} J_{\alpha(s)\dot{\alpha}(s)} \right\}, \quad J_{\alpha(s)\dot{\alpha}(s)} = J_{\alpha(s)\dot{\alpha}(s)} . \quad (5.1)$$

Here $J_{\alpha(s)\dot{\alpha}(s)}$ denotes a composite primary superfield, which describes a conserved current multiplet when $\Phi$ is on-shell. Then it is natural to consider the generating functional for correlation functions of these conserved higher spin supercurrents defined by

$$e^{i\Gamma[H]} = \int {\mathcal{D}}\Phi {\mathcal{D}}\bar{\Phi} e^{iS[\Phi, \bar{\Phi}; H]} . \quad (5.2)$$

Similar to the non-supersymmetric analysis of [36], one may show that the action (5.1), properly deformed by terms nonlinear in $H^{(s)\dot{\alpha}(s)}$, has an exact non-Abelian gauge symmetry (associated with the conformal higher-spin superalgebra $\text{shsc}^{\infty}(4|1)$ described in [6]) which reduces to (4.2) at lowest level in the superfields $H_{\alpha(s)\dot{\alpha}(s)}$, $\Phi$ and $\bar{\Phi}$. Here we restrict our discussion to giving the explicit expressions for $J_{\alpha(s)\dot{\alpha}(s)}$.

We turn to describing the structure of the conserved current multiplets $J_{\alpha(s)\dot{\alpha}(s)}$. In order for the source term

$$S_{\text{source}}^{(s+\frac{1}{2})} = \int d^4x d^2\theta d^2\bar{\theta} H^{(s)\dot{\alpha}(s)} J_{\alpha(s)\dot{\alpha}(s)} . \quad (5.3)$$

\(^8\)In three dimensions, the off-shell superconformal higher spin multiplets have recently been described in [31] and [32] for the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases, respectively.
to be invariant under the superconformal transformations, the superfield \( J_{\alpha(s)} \dot{\alpha(s)} \) must be superconformal primary of weight \((1 + \frac{s}{2}, 1 + \frac{s}{2})\). In order for \( S_{\text{source}}^{(s+\frac{1}{2})} \) to be invariant under the gauge transformations (3.1), \( J_{\alpha(s)} \dot{\alpha(s)} \) must obey the conservation equations

\[
D^\beta J_{\beta\alpha_1...\alpha_{s-1} \dot{\alpha}_1...\dot{\alpha}_s} = 0, \quad \bar{D}^\dot{\beta} J_{\alpha_1...\alpha_s \dot{\alpha}_{s+1}...\dot{\alpha}_{s+1}} = 0. \tag{5.4}
\]

The \( s = 1 \) case corresponds to the superconformal version \[16\] of the Ferrara-Zumino supercurrent \[37\]. Extension to the other cases \( s > 1 \) was given in \[38\] (building on \[39\]). The authors of \[38\] also postulated the prepotential \( H_{\alpha(s)} \dot{\alpha(s)} \) as the source to generate the Noether coupling (5.3), as well as the gauge transformation (3.1) as the transformation of \( H_{\alpha(s)} \dot{\alpha(s)} \) which leaves (5.3) invariant. However, no higher spin extensions of linearised conformal supergravity were given.

Consider a free on-shell massless chiral scalar \( \Phi \),

\[
\bar{D}_a \Phi = 0, \quad D^2 \Phi = 0. \tag{5.5}
\]

which is superconformal primary of dimension +1. By analogy with the construction of \[40\], the conserved current multiplets \( J_{\alpha(s)} \dot{\alpha(s)} \), with \( s = 1, 2, \ldots \), can be obtained as unique composites of \( \Phi \) and \( \bar{\Phi} \) of the form

\[
J_{\alpha(s)} \dot{\alpha(s)} = (2i)^{s-1} \sum_{k=0}^{s} (-1)^k \binom{s}{k} \times \left\{ \binom{s}{k+1} \partial_{(\alpha_1 \dot{\alpha}_1} \ldots \partial_{\alpha_k \dot{\alpha}_k} D_{\alpha_{k+1}} \Phi \bar{D}_{\dot{\alpha}_{k+1}} \partial_{\alpha_{k+2} \dot{\alpha}_{k+2}} \ldots \partial_{\alpha_s} \dot{\alpha}_s) \bar{\Phi} \right. \\
- 2i \binom{s}{k} \partial_{(\alpha_1 \dot{\alpha}_1} \ldots \partial_{\alpha_k \dot{\alpha}_k} \Phi \partial_{\alpha_{k+1} \dot{\alpha}_{k+1}} \ldots \partial_{\alpha_s} \dot{\alpha}_s) \bar{\Phi} \right\}, \tag{5.6}
\]

where one should keep in mind that

\[
\binom{s}{s+1} = 0.
\]

It is an instructive exercise to check that the conservation equations (5.4) are satisfied. Choosing \( s = 1 \) in (5.6) gives the well-known supercurrent \[37\]

\[
J_{\alpha\dot{\alpha}} = D_\alpha \Phi \bar{D}_{\dot{\alpha}} \bar{\Phi} + 2i(\Phi \partial_{\alpha\dot{\alpha}} \bar{\Phi} - \partial_{\alpha\dot{\alpha}} \Phi \bar{\Phi}). \tag{5.7}
\]

The higher spin supercurrent (5.6) may be compared with the 3D \( \mathcal{N} = 2 \) result reported in \[41\].

16
Similar to the bosonic superfield prepotentials $H_{\alpha(s)\dot{\alpha}(s)}$, one may define conserved higher spin current supermultiplets associated with the fermionic superfield prepotentials $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$. Consider a source term of the form

$$S_{\text{source}}^{(s)} = \int d^4x d^2\theta d^2\bar{\theta} \Psi^{\alpha(s)\dot{\alpha}(s-1)} J_{\alpha(s)\dot{\alpha}(s-1)} + \text{c.c.} \quad (5.8)$$

In order for $S_{\text{source}}^{(s)}$ to be invariant under the superconformal transformations, $J_{\alpha(s)\dot{\alpha}(s-1)}$ must be superconformal primary of weight $(1+\frac{s}{2}, \frac{1}{2}+\frac{s}{2})$. In order for $S_{\text{source}}^{(s)}$ to be invariant under the gauge transformations (3.10), the superfield $J_{\alpha(s)\dot{\alpha}(s-1)}$ with $s > 1$ must obey the conservation equations

$$D^\beta J_{\beta\alpha_1...\alpha_{s-1}1...s-1} = 0, \quad \bar{D}^{\dot{\beta}} J_{\alpha_1...\alpha_s\dot{\beta}\dot{\alpha}_1...s-2} = 0. \quad (5.9)$$

In the $s = 1$ case, the conservation equations are\footnote{SMK is grateful to Jessica Hutomo for pointing out a sign error in eq. (5.12) in the published version of this work.}

$$D^\beta J_\beta = 0, \quad \bar{D}^{\dot{\beta}} J_\dot{\beta} = 0, \quad (5.10)$$

as a consequence of (3.19).

Conserved current multiplets $J_{\alpha(s)\dot{\alpha}(s-1)}$, with $s = 1, 2, \ldots$, may be constructed from two free massless chiral superfields $\Phi_+$ and $\Phi_-,

$$\bar{D}_\alpha \Phi_\pm = 0, \quad D^2 \Phi_\pm = 0. \quad (5.11)$$

One may check that the following composite

$$J_{\alpha(s)\dot{\alpha}(s-1)} = (2i)^{s-1} \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \left\{ \binom{s}{k+1} \partial_{\alpha_1(\dot{\alpha}_1} \cdots \partial_{\alpha_k\dot{\alpha}_k} D_{\alpha_{s-1}\dot{\alpha}_{s-1}} \Phi_+ \partial_{\alpha_{k+1}\dot{\alpha}_{k+1}} \cdots \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} \Phi_- \right. $$

$$\left. - \binom{s}{k} \partial_{\alpha_1(\dot{\alpha}_1} \cdots \partial_{\alpha_k\dot{\alpha}_k} \Phi_+ \partial_{\alpha_{k+1}\dot{\alpha}_{k+1}} \cdots \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} D_{\alpha_{s-1}\dot{\alpha}_{s-1}} \Phi_- \right\} \quad (5.12)$$

obeys the conservation equations (5.9) for $s > 1$. One may also check that $J_{\alpha(s)\dot{\alpha}(s-1)}$ is (anti)symmetric, $J_{\alpha(s)\dot{\alpha}(s-1)} \rightarrow (-1)^s J_{\alpha(s)\dot{\alpha}(s-1)}$, with respect to the interchange $\Phi_+ \leftrightarrow \Phi_-$. Choosing $s = 1$ in (5.12) gives the composite

$$J_\alpha = \Phi_- \leftrightarrow D_\alpha \Phi_+, \quad (5.13)$$
which obeys the conservation equations (5.10). The superfield $J_\alpha$ contains a conserved
fermionic current $j_{\alpha\dot{\beta}} = j_{\dot{\beta}\alpha}$ that corresponds to the second supersymmetry current.\(^\text{10}\)

The conserved current multiplet (5.13) is obtained by reducing the $\mathcal{N} = 2$ supercurrent
of a free massless hypermultiplet to $\mathcal{N} = 1$ superspace, see [42] for more details.

In the case of a free $\mathcal{N} = 2$ hypermultiplet described in terms of two $\mathcal{N} = 1$ chiral scalars $\Phi_+ \text{ and } \Phi_-$, the action (5.1) should be replaced with

\[
S[\Phi_\pm, \Phi_\pm; H, \Psi, \bar{\Psi}] = \int d^4x d^2\theta d^2\bar{\theta} \left\{ \Phi_+ \dot{\Phi}_+ + \Phi_- \dot{\Phi}_- + \sum_{s=1}^{\infty} H^{(s)}(\alpha(s)) J_{\alpha(s)\dot{\alpha}}(s) \right. \\
+ \left. \sum_{s=1}^{\infty} \left[ \Psi^{(s)}(\alpha(-1)) J_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \right] \right\}.
\]  

The current superfields $J_{\alpha(s)\dot{\alpha}(s)}$ and $J_{\alpha(s)\dot{\alpha}(s-1)}$ are $\mathcal{N} = 1$ components of a conserved
$\mathcal{N} = 2$ supermultiplet.

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**A The Grimm-Wess-Zumino superspace geometry**

In describing the Grimm-Wess-Zumino superspace geometry [8], we follow the notation and conventions of [7].\(^\text{11}\) In particular, the coordinates of $\mathcal{N} = 1$ curved superspace

\(^\text{10}\)This conserved current may be chosen as $j_{\alpha\dot{\beta}} = \left\{ [D_{\alpha}, \bar{D}_{\dot{\beta}}] J_\alpha + \frac{2}{3} i \bar{D}_{(\dot{\beta})} J_\alpha \right\}_{\theta = 0}$. Also associated
with $J_\alpha$ is the off-shell conserved current $v_{\alpha\dot{\beta}} = \partial_{\dot{\beta}} J_\alpha - 2\varepsilon_{\alpha\dot{\beta}} \partial_{\gamma} J^{\gamma}$ such that $\partial^{\dot{\gamma}} v_{\alpha\dot{\beta}} = 0$ for any $J_\alpha$.

\(^\text{11}\)These conventions are similar to those of Wess and Bagger [43]. To convert the notation of [7] to\(^\text{11}\) that of [8], one replaces $R \rightarrow 2R$, $G_{\alpha\dot{\alpha}} \rightarrow 2G_{\alpha\dot{\alpha}}$, and $W_{\alpha\dot{\beta}\gamma} \rightarrow 2W_{\alpha\dot{\beta}\gamma}$. In addition, the vector derivative has to be changed by the rule $D_a \rightarrow D_a + \frac{1}{2} \varepsilon_{abcd} G^b M^{cd}$, where $G_a$ corresponds to [7]. Finally, the spinor Lorentz generators $(\sigma_{ab})_{\alpha\dot{\beta}}$ and $(\bar{\sigma}_{ab})^{\alpha\dot{\beta}}$ used in [7] have an extra minus sign as compared with [43], specifically $\sigma_{ab} = -\frac{1}{4}(\sigma_a \bar{\sigma}_b - \sigma_b \bar{\sigma}_a)$ and $\bar{\sigma}_{ab} = -\frac{1}{4}(\bar{\sigma}_a \sigma_b - \bar{\sigma}_b \sigma_a)$.
\( \mathcal{M}^{1|4} \) are denoted \( z^M = (x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \). The superspace geometry is described by covariant derivatives of the form

\[
\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^\dot{\alpha}) = E_A + \Omega_A .
\]

(A.1)

Here \( E_A \) denotes the inverse vielbein, \( E_A = E_A^M \partial_M \), and \( \Omega_A \) the Lorentz connection,

\[
\Omega_A = \frac{1}{2} \Omega_A^{bc} M_{bc} = \Omega_A^{\beta\gamma} M_{\beta\gamma} + \Omega_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} ,
\]

(A.2)

with \( M_{bc} = -M_{cb} \Leftrightarrow (M_{\beta\gamma} = M_{\gamma\beta}, \bar{M}_{\dot{\beta}\dot{\gamma}} = \bar{M}_{\dot{\gamma}\dot{\beta}}) \) the Lorentz generators. The covariant derivatives obey the following anti-commutation relations:

\[
\{ \mathcal{D}_\alpha, \bar{\mathcal{D}}_\dot{\alpha} \} = -2i \mathcal{D}_{\alpha\dot{\alpha}} ,
\]

(A.3a)

\[
\{ \mathcal{D}_\alpha, \mathcal{D}_\beta \} = -4R M_{\alpha\beta} , \quad \{ \bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}} \} = 4R \bar{M}_{\dot{\alpha}\dot{\beta}} , \quad \{ \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\beta}} \} = 4R \bar{M}_{\dot{\alpha}\dot{\beta}} ,
\]

(A.3b)

\[
[\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{D}_\beta] = -i \varepsilon_{\dot{\alpha}\beta} \left( R \mathcal{D}_\beta + G_{\beta}^{\dot{\gamma}} \bar{D}_\dot{\gamma} - (\bar{D}^{\dot{\gamma}} G_{\beta}^{\dot{\gamma}}) \bar{M}_{\dot{\gamma}\delta} + 2 W_{\beta}^{\gamma\delta} M_{\gamma\delta} \right) - i (\mathcal{D}_\beta R) \bar{M}_{\dot{\alpha}\delta} ,
\]

(A.3c)

\[
[\mathcal{D}_\alpha, \mathcal{D}_\beta] = i \varepsilon_{\alpha\beta} \left( \bar{R} \bar{D}_{\dot{\beta}} + G_{\beta}^{\gamma} \bar{D}_{\gamma} - (\bar{D}_{\gamma} G_{\beta}^{\gamma}) M_{\gamma\delta} + 2 W_{\beta}^{\gamma\delta} \bar{M}_{\dot{\gamma}\delta} \right) + i (\bar{D}_{\dot{\beta}} \bar{R}) M_{\alpha\beta} ,
\]

(A.3d)

\[
[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = \varepsilon_{\dot{\alpha}\dot{\beta}} \psi_{\alpha\beta} + \varepsilon_{\alpha\beta} \psi_{\dot{\alpha}\dot{\beta}} ,
\]

(A.3e)

where

\[
\psi_{\alpha\beta} := -i G_{(\alpha}^\dot{\gamma} \mathcal{D}_{\beta)\dot{\gamma}} + \frac{1}{2} (\mathcal{D}_{(\alpha} R) \mathcal{D}_{\beta)} + \frac{1}{2} (\mathcal{D}_{(\alpha} G_{\beta)}^{\dot{\gamma}}) \bar{D}_{\dot{\gamma}} + W_{\alpha\beta} \mathcal{D}_{\gamma}
\]

\[
+ \frac{1}{4} \left( (\bar{D}^2 - 8 R) \bar{D}_{\dot{\beta}} \right) M_{\alpha\beta} + (\mathcal{D}_{(\alpha} W_{\beta)}^{\gamma\delta}) M_{\gamma\delta} - \frac{1}{2} (\mathcal{D}_{(\alpha} \bar{D}^{\dot{\gamma}} G_{\beta)}^{\dot{\gamma}}) \bar{M}_{\dot{\gamma}\delta} ,
\]

(A.3f)

\[
\psi_{\dot{\alpha}\dot{\beta}} := -i G_{(\alpha}^\gamma \mathcal{D}_{\beta)\gamma} - \frac{1}{2} (\mathcal{D}_{(\alpha} \bar{D}_{\dot{\beta}}) - \frac{1}{2} (\mathcal{D}_{(\alpha} G_{\beta)}^{\gamma} \bar{D}_{\dot{\gamma}} - W_{\alpha\beta} \mathcal{D}_{\gamma})
\]

\[
+ \frac{1}{4} \left( (\bar{D}^2 - 8 R) \bar{D}_{\dot{\beta}} \right) \bar{M}_{\dot{\alpha}\dot{\beta}} - (\mathcal{D}_{(\alpha} W_{\beta)}^{\gamma\delta}) \bar{M}_{\dot{\gamma}\delta} + \frac{1}{2} (\mathcal{D}_{(\alpha} \mathcal{D}^{\dot{\gamma}} G_{\beta)}^{\dot{\gamma}}) M_{\gamma\delta} .
\]

(A.3g)

The torsion tensors \( R, G_a = \bar{G}_a \) and \( W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)} \) satisfy the Bianchi identities

\[
\bar{D}_{\dot{\alpha}} R = 0 , \quad \bar{D}_{\dot{\alpha}} W_{\alpha\beta\gamma} = 0 ,
\]

(A.4a)

\[
\bar{D}^{\dot{\gamma}} G_{\alpha\dot{\gamma}} = \mathcal{D}_a R ,
\]

(A.4b)

\[
\bar{D}^{\gamma} W_{\alpha\beta\gamma} = i \mathcal{D}_{(\alpha}^{\dot{\gamma}} G_{\beta)}^{\dot{\gamma}} ,
\]

(A.4c)

A supergravity gauge transformation is defined to act on the covariant derivatives and any tensor superfield \( U \) (with its indices suppressed) by the rule

\[
\delta_K \mathcal{D}_A = [K, \mathcal{D}_A] , \quad \delta_K U = KU ,
\]

(A.5a)
where the gauge parameter $\mathcal{K}$ has the explicit form

$$\mathcal{K} = \xi^B D_B + \frac{1}{2} K^{bc} M_{bc} = \xi^B D_B + K^{\gamma\delta} M_{\gamma\delta} + \bar{K}^{\gamma\delta} \bar{M}_{\gamma\delta} = \bar{\mathcal{K}}$$

(A.5b)

and describes a general coordinate transformation generated by the supervector field $\xi = \xi^B E_B$ as well as a local Lorentz transformation generated by the antisymmetric tensor $K^{bc}$.

\section*{B Super-Weyl transformations}

The algebra of covariant derivatives (A.3) preserves its functional form under super-Weyl transformations \cite{9}

$$\delta_\sigma D_\alpha = (\bar{\sigma} - \frac{1}{2} \sigma) D_\alpha + (\bar{D}^\beta \sigma) M_{\alpha\beta} ,$$

(B.1a)

$$\delta_\sigma \bar{D}_{\dot{\alpha}} = (\sigma - \frac{1}{2} \bar{\sigma}) \bar{D}_{\dot{\alpha}} + (\bar{D}^\dot{\beta} \bar{\sigma}) \bar{M}_{\dot{\alpha}\dot{\beta}} ,$$

(B.1b)

$$\delta_\sigma D_{\alpha\dot{\alpha}} = \frac{1}{2} (\sigma + \bar{\sigma}) D_{\alpha\dot{\alpha}} + \frac{i}{2} (\bar{D}_{\dot{\alpha} \bar{\sigma}}) D_{\alpha} + \frac{i}{2} (\bar{D}_{\alpha \sigma}) \bar{D}_{\dot{\alpha}}$$

$$+ (\bar{D}^\beta_{\dot{\alpha}} \sigma) M_{\alpha\beta} + (\bar{D}^{\dot{\beta}}_{\alpha} \bar{\sigma}) \bar{M}_{\dot{\alpha}\dot{\beta}} ,$$

(B.1c)

where $\sigma$ is an arbitrary covariantly chiral scalar superfield, $\bar{D}_{\dot{\alpha}} \sigma = 0$. The torsion tensors in (A.3) transform as follows:

$$\delta_\sigma R = 2\sigma R + \frac{1}{4} (\bar{D}^2 - 4R) \bar{\sigma} ,$$

(B.2a)

$$\delta_\sigma G_{\alpha\dot{\alpha}} = \frac{1}{2} (\sigma + \bar{\sigma}) G_{\alpha\dot{\alpha}} + i D_{\alpha\dot{\alpha}} (\sigma - \bar{\sigma}) ,$$

(B.2b)

$$\delta_\sigma W_{\alpha\beta\gamma} = \frac{3}{2} \sigma W_{\alpha\beta\gamma} .$$

(B.2c)

The local transformations (A.5) and (B.1) constitute the gauge freedom of conformal supergravity.

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