Nonlinear Dynamics of a Simply Supported FGM Rectangular Plate under Combined Parametrical and External Excitations

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Abstract. The present investigation deals with nonlinear oscillation behavior of a simply supported functionally graded rectangular plate in thermal environment with in-plane parametric and transverse external excitations. Material properties are assumed to be temperature-dependent. Based on the Reddy’s third-order plate theory and the non-linear strain-displacement relations, the governing equations of motion for the FGM plate are derived by using the Hamilton’s principle. The method of multiple scales is utilized to obtain four-dimensional nonlinear averaged equations. Using a numerical method, the averaged equations are analyzed. These results show that under certain conditions the periodic, multi-periodic solutions and chaotic motions of the FGM plates are found.

1. Introduction
The use of functionally graded materials (FGM) has attracted much interest in many engineering applications as heat-shielding materials such as nuclear reactor, aircraft, aerospace, defense industries and chemical plants [1]. With the use of the FGM plates in more and more fields, the research of the nonlinear dynamics, bifurcations, and chaos of the FGM plates is necessary.

Among those research available, Praveen and Reddy [2] examined the thermoelastostatic and thermoelastodynamic response of plates subjected to pressure loading and thickness varying temperature fields. Yang and Shen [3-5], studied the dynamic response of initially stressed FGM rectangular thin plates. Huang and Shen [6] dealt with the nonlinear vibration and dynamic response of FGM plates in environment. Chen [7] investigated the nonlinear vibration of FGM plates with arbitrary initial stresses. Qian et al. [8] analyzed the static deformations, and free and forced vibrations of a thick rectangular functionally graded elastic plate by using a higher-order shear and normal deformable plate theory and a meshless local Petrov–Galerkin method. Senthil and Batra [9] gave a three-dimensional exact solution for free and forced vibrations of simply supported FGM rectangular plates. Then Cheng and Batra [10] used Reddy’s third order plate theory to study buckling and steady state vibrations of a simply supported functionally gradient isotropic polygonal plate resting on a Winkler Pasternak elastic foundation and subjected to uniform in-plane hydrostatic loads. Ng et al. [11] presented the parametric resonance of functionally graded rectangular plates under harmonic in-plane loading is presented. Yang et al [12] studied the large amplitude vibration of pre-stressed FGM
laminated plates that were composed of a shear deformable functionally graded layer and two surface-mounted piezoelectric actuator layers. Chen and Zhang [15] analyzed the low-velocity impact response of laminated composite plate with multiple delaminations.

To the authors’ knowledge, the studies of the bifurcation and chaos for the FGM plates which are the base of nonlinear dynamics analysis have been given quite a few investigations. This paper aims to consider a simply supported at the fore-edge, rectangular FGM plate subjected to in-plane and transversal excitation simultaneously in the uniform thermal environment. The studies are focused on the case of 1:3 internal resonance and primary parametric resonance, 1/2 subharmonic resonance. Materials properties of the constituents are graded in the thickness direction according to a power-law distribution. In the framework of Reddy’s third-order shear deformation plate theory and the non-linear strain-displacement relations, the governing equations of motion for the FGM plate are derived by the Hamilton’s principle. Then the equations of motion with two-degree-of-freedom under combined parametrical and external excitations can be obtained by using Galerkin’s method. The perturbation method of multiple scales is used to reduce the second-order nonautonomous nonlinear differential equations to the first-order nonlinear averaged equations. Using numerical method, the averaged equations are analyzed.

2. Formulation
A simply supported at the four-edges FGM rectangular plate subjected to in-plane and transversal excitations simultaneously is considered, shown in Figure 1. We assume that the plate is made from a mixture of ceramics and metals with continuously varying such that the top surface of the plate is ceramic rich, whereas the bottom surface is metal rich. The material properties such as Young’s modulus $E$, coefficient of thermal expansion $\alpha$, can be expressed as a function of temperature, see [13]. The temperature variation occurs in the thickness direction only and one-dimensional temperature field is assumed to be constant in the in-plane of the plate. According to Reddy’s third-order shear deformation plate theory, non-linearity strains-displacement relation and the Hamilton’s principle, the nonlinear equations of motion for the FGM rectangular plate in terms of generalized displacements are given as follows

![Fig. 1 The model of an FGMs Rectangular plate](image-url)
\[
\begin{align*}
A_1 \frac{\partial^2 u_0}{\partial x^2} + A_{60} \frac{\partial^2 u_0}{\partial y^2} + (A_{12} + A_{60}) \frac{\partial^2 v_0}{\partial x \partial y} + (B_{11} + c_i E_{11}) \frac{\partial^2 \phi_0}{\partial x^2} + (B_{60} + c_i E_{60}) \frac{\partial^2 \phi_0}{\partial y^2} & \\
+ (B_{12} - c_i E_{12} + B_{60} - c_i E_{60}) \frac{\partial^2 \phi_y}{\partial x \partial y} + A_{11} \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} + A_{66} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial y^2} + (A_{12} + A_{60}) \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial y^2} & \\
c_i E_{11} \frac{\partial^2 w_0}{\partial x^2} - c_i (E_{12} + 2E_{60}) \frac{\partial^2 w_0}{\partial x \partial y} = I_0 \dot{\nu}_0 + (I_1 - c_i I_3) \ddot{\phi}_y - c_i I_3 \frac{\partial \ddot{w}_0}{\partial x},
\end{align*}
\]
in the form of
\[
\pi = \sin \frac{3\pi y}{a} \sin \frac{\pi y}{b} + \sin \frac{\pi y}{a} \sin \frac{3\pi y}{b}.
\]  
(2)
where \(\pi_1\) and \(\pi_2\) are the dimensionless amplitudes of two modes respectively.

The transverse excitation can be represented as
\[
\bar{F} = \bar{F}_1 \sin \frac{\pi y}{a} \sin \frac{3\pi y}{b} + \bar{F}_2 \sin \frac{\pi y}{a} \sin \frac{\pi y}{b}.
\]  
(3)
where \(\bar{F}_1\) and \(\bar{F}_2\) represents the dimensionless amplitude of the transverse forcing excitation corresponding to the two nonlinear modes. For simplicity, we drop the overbars in the following analysis. Following as [14], here neglecting all inertia terms in equation (1c) and the in-plane displacements inertia terms in equation (1c) and substituting equation (2) into equation (1), we can obtain the expressions of the displacements \(u, v, \phi_x, \phi_y\) in the form of \(w\). Substituting equation (2) and equation (3) into equation (1c) and applying the Galerkin procedure yield the governing
differential equations of transverse motion for the FGM rectangular plate for the dimensionless as follows

\[ \ddot{w}_1 + \omega_1^2 w_1 + a_1 \dot{w}_1 + a_2 w_1 \cos \Omega_2 t + a_3 w_1^2 + a_4 w_1 + a_5 w_1^2 + a_6 w_1^3 + a_7 w_1 w_2 = f_1 \cos \Omega_2 t, \quad (4a) \]

\[ \ddot{w}_2 + \omega_2^2 w_2 + b_1 \dot{w}_2 + b_2 w_2 \cos \Omega_2 t + b_3 w_1^2 + b_4 w_2^2 + b_5 w_2^2 + b_6 w_1 w_2 + b_7 w_2 = f_2 \cos \Omega_2 t. \quad (4b) \]

The perturbation method of multiple scales is utilized to obtain four-dimensional nonlinear averaged equations. Only the primary parametric resonance and 1:3 internal resonance are considered in this paper.

The above equations, which include the quadratic, cubic terms, parametric and transverse excitations, describe nonlinear vibrations of the FGM rectangular plate in the first two modes. To consider the influence of the quadratic terms on nonlinear dynamic characteristics of the FGM rectangular plate, we need to obtain the second-order approximate solution of equation (4). The method of multiple scales is used to get the averaged equations as follows

\[ \dot{x}_1 = (g_{01} - g_{02} - f_{12}) x_2 + g_{04} \left( x_1^2 + x_2^2 \right) + g_{05} x_1 \left( x_1^2 + x_2^2 \right) + \frac{29}{16} g_{03} - g_{06} x_1^2 x_2 + g_{07} (x_1^2 x_4 + x_2^2 x_4)
+ 2x_1 x_4 (x_3 + x_4) + g_{09} x_1 (x_1^2 + x_4^2) + (g_{09} - g_{10}) x_1 (x_1^2 + x_4^2) + (g_{11} - g_{12}) x_1 x_4 (x_1^2 + x_4^2) + (g_{11} + g_{12}) x_1 x_4 (x_1 + x_4) + g_{13} (x_1^2 x_2 x_3 + x_1^2 x_4 + x_2^2 x_1 x_4 + x_2^2 x_4 x_4)
+ 2g_{15} x_1^2 x_2^2 + g_{15} (x_1^2 x_2 + x_3^2), \quad (5a) \]

\[ \dot{x}_2 = \left[ (g_{01} + g_{02} + f_{1}) x_1 + \frac{19}{16} g_{03} + g_{06} \right] x_1^2 - \frac{29}{16} g_{03} - g_{06} - g_{05} \right] x_1 x_2 - g_{05} x_2^3 - g_{07} x_4 (x_1^2)
+ x_2^2 + 2x_1 x_4 (x_3 + x_4) + g_{09} x_2 (x_1^2 + x_4^2) + (g_{09} + g_{10}) x_2 (x_1^2 + x_4^2) + (g_{11} + g_{12}) x_2 (x_1 + x_4) + g_{13} (x_2^3 + x_3^2 + x_1 x_2 x_4 + x_1 x_2 x_4^2 + x_1 x_2 x_4^3) + g_{15} (2x_1^2 x_2^2 + x_3^2 + x_1 x_2^3), \quad (5b) \]

\[ \dot{x}_3 = F_2 x_2 + h_{04} x_4 + 2(h_{05} - h_{14}) x_1 x_2 x_4 + (h_{14} + h_{05} + h_{10}) x_1^2 x_4 - (h_{01} + h_{06} - h_{03}) x_1 x_4^2 - h_{03} x_3 (x_1^2)
+ x_2^2) + h_{03} x_3 (3x_1^2 - x_2^2) + h_{01} x_3 x_1 (x_1^2 + x_2^2) + h_{10} x_4 (x_1^2 + x_2^2)
+ h_{12} x_4 (x_2^2 + 2x_1 x_2^3) + h_{13} x_4 (x_2^3 + x_2 x_4^2) + h_{14} x_4^3 + (h_{13} + 2h_{14}) x_4 x_4^2, \quad (5c) \]

\[ \dot{x}_4 = -F_2 x_1 - h_{01} x_3 + h_{03} (x_1^2 + x_2^2) + h_{04} x_1 x_2 x_4 + (h_{04} + h_{06} + h_{10}) x_1^2 x_4 - (h_{04} + h_{06} - h_{03}) x_1 x_4^2 - h_{03} x_4 (x_1^2)
+ h_{06} x_4 (x_1^3 - x_2^3) + h_{09} x_4 (x_1^3 - x_2^3) - h_{09} x_4 (x_1^3 - x_2^3) + h_{10} x_4 (x_1^2 + x_2^2) + 2(h_{04} - h_{03}) x_1 x_2 x_4
+ h_{12} x_4 (x_1^3 + x_2^3 + x_3^2 x_1^2 + x_3^2 x_2^2 + x_3^2 x_4^3) + h_{13} x_4 (x_1^3 + x_2^3 + 2x_1 x_2^3) x_4 + h_{14} x_4^3 + (h_{13} + 2h_{14}) x_4 x_4^2 + x_1 x_2 x_3 + x_1 x_2 x_3^2) + h_{14} x_4 (x_1^3 + x_2^3 + 2x_1 x_2^3)), \quad (5d) \]

3. Numerical Simulations of Periodic and Chaotic Motions

In the following numerical simulations, the chaotic motions of the FGM rectangular plate subjected to parametrical and external excitations have been analyzed. We consider the averaged equation (5) to do numerical simulation. We choose that the forcing excitation \( F_2 \) and \( F_1 \) as a controlling parameter when the periodic and chaotic responses of the FGM rectangular plate are investigated. In order to demonstrate the nonlinear dynamic behavior of the FGM rectangular plate, the three-dimensional phase portrait and Poincare map are plotted as well as the two- dimensional phase portrait and the waveform. It can be clearly found from numerical simulations that the periodic and chaotic motions occur for the FGM rectangular plate with 1:3 internal resonance.
From figures 2-4 it can be shown that the motion of the FGM rectangular plate be changed from the periodic motion to the chaotic motion then periodic motion, as the forcing excitation $F_1$ and $F_2$ changes. Figures 2 illustrate the existence of the periodic motion for the FGM rectangular plate. Figure 3 shows that the chaotic motion also occurs for the FGM rectangular plate when the forcing excitation changes. Continuously increasing the forcing excitation, it is found that periodic solution for the both two mode occur in the FGM rectangular plate as shown in figure 4.

4. Conclusions
The nonlinear oscillations and chaotic dynamics of the FGM rectangular plate under combined transverse and in plane excitations are investigated by numerical approaches. The resonant case considered here is 1:3 internal resonance, principal parametric resonance-1/2 subharmonic resonance. The perturbation method of multiple scales is utilized to obtain four-dimensional nonlinear averaged equations. Using numerical method, the averaged equations are analyzed. Under certain conditions the periodic, multi-periodic solutions and chaotic motions of the FGM plates are found. It is found that there exist different kinds of chaotic motions in the FGM rectangular plate. The 4-order Runge-Kutta algorithm is utilized to give the phase portrait, waveform and Poincare map of the system.

The influence of the forcing excitation $F_1$ and $F_2$ on the nonlinear dynamic behavior of the FGM rectangular plate is investigated. It is thought that the forcing excitation $F_1$ and $F_2$ can change the form of motion for the FGM rectangular plate.

![Figure 2](image)

**Figure 2** The period-2 motion of the rectangular FGM plate exists when $F_1 = 4.44, F_2 = 1.913$
The chaotic motion of the rectangular FGM plate exists when $F_1 = 5.44 \times F_2 = 2.345$

The period-10 motion of the rectangular FGM plate exists when $F_1 = 5.46 \times F_2 = 2.351$

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