VERTICES OF INTERSECTION POLYTOPES AND RAYS OF GENERALIZED KOSTKA CONES

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Abstract. Let \( \mathcal{K}(G) \) be the rational cone consisting of pairs \((\lambda, \mu)\) where \( \lambda \) and \( \mu \) are dominant integral weights and \( \mu \) is a nontrivial weight space in the representation \( V_\lambda \) of \( G \). We produce all extremal rays of \( \mathcal{K}(G) \) by considering the vertices of corresponding intersection polytopes \( IP_\mu \), the set of points in \( \mathcal{K}(G) \) with first coordinate \( \lambda \). We show that vertices of \( IP_\mu \) arise as lifts of vertices coming from cones \( \mathcal{K}(L) \) associated to simple Levi subgroups containing \( \mu \). As corollaries we obtain a complete description of all extremal rays, as well as polynomial formulas describing the numbers of extremal rays depending on type.

1. Introduction

Let \( G \) be a simple, simply-connected linear algebraic group over \( \mathbb{C} \). We choose a maximal torus and Borel subgroup \( T \subset B \subset G \). Then the irreducible finite-dimensional representations of \( G \) are indexed by the dominant integral weights of \( T \). For such a weight \( \lambda \), the associated representation, \( V_\lambda \), possesses a weight space decomposition with respect to \( T \): \( V_\lambda = \bigoplus V_\lambda(\mu) \), where \( V_\lambda(\mu) \) is the subspace upon which \( T \) acts through scalar multiplication by the character \( \mu \).

As is well-known, \( V_\lambda(\mu) \neq (0) \) if and only if

- (a) \( \lambda - \mu \) lies in the root lattice for \( G \) and
- (b) \( \mu \) is contained inside the Weyl polytope \( \text{conv}(W \cdot \lambda) \),

where \( W \) denotes the Weyl group and \( W \cdot \lambda \) its orbit through \( \lambda \). However, there is a simpler criterion if \( \mu \) is already known to be a dominant weight (cf. [Ste98]): for \( \lambda, \mu \) both dominant, \( V_\lambda(\mu) \neq (0) \) if and only if

- (a') \( \lambda - \mu \) is a linear combination of simple roots with nonnegative integral coefficients.

Since \( \dim V_\lambda(\mu) = \dim V_\lambda(w\mu) \) for any \( w \in W \), restricting our attention to \( \mu \) dominant does not, in fact, lose any information about the representation \( V_\lambda \). It is customary to write \( \lambda \geq \mu \) for the statement \( V_\lambda(\mu) \neq (0) \) and to call \( \geq \) the dominance order.

1.1. Rays of the Kostka cone. In type \( A \), the multiplicity \( m_{\lambda, \mu} = \dim V_\lambda(\mu) \) has classically been called a Kostka number and has meaning in a variety of contexts, such as symmetric functions and representations of the symmetric group, cf. [Ful97]. We set \( \mathcal{K}(G) = \{ (\lambda, \mu) | \lambda, \mu \text{ dominant, } V_\lambda(\mu) \neq (0) \} \) and call \( \mathcal{K}(G) \) the Kostka cone for \( G \). In [GKOY], the third author and S. Gao, G. Orelowitz, and A. Yong completely describe the extremal rays of \( \mathcal{K}(GL_n) \) using the language of partitions and Young tableaux and make further investigations into the Hilbert basis of the Kostka cone. Inspired by that work, we now seek to extend one of their results:

Question: What are the extremal rays of \( \mathcal{K}(G) \), for \( G \) simple and simply-connected?

Let us ensure that the above question makes sense. If \( X^{*}(T) \) denotes the space of all weights of \( T \), criterion (a') tells us that \( \mathcal{K}(G) \subset (X^{*}(T))^2 \) is exactly the solution space to a system of linear inequalities which we list in Proposition 2.2. In the ambient vector space \( (X^{*}(T) \otimes \mathbb{Q})^2 \), the set \( \mathcal{K}(G)_\mathbb{Q} \) of rational solutions to those inequalities thus forms a pointed, rational, polyhedral cone, and it makes sense to ask what its extremal rays are.

Our first observation is that, given a dominant weight \( \lambda \), the affine slice through \( \mathcal{K}(G)_\mathbb{Q} \) defined by fixing the first coordinate to be \( \lambda \) is a convex polytope \( IP_\lambda \); it can be viewed as the intersection of
the Weyl polytope $\text{conv}(W \cdot \lambda)$ and the dominant chamber of $X^*(T)_Q$. Examining these polytopes, we find that the vertices of $IP_\lambda$ depend linearly on $\lambda$, which allows us to conclude:

**Theorem 1.1.** The extremal rays of $\mathcal{K}(G)_Q$ are all of the form $(\lambda, \mu)$ where $\lambda = \varpi_j$ is a fundamental weight and $\mu$ is a vertex of $IP_{\varpi_j}$. Conversely, every such pair $(\lambda, \mu)$ produces an extremal ray of $\mathcal{K}(G)_Q$.

Thus in order to completely describe the set of extremal rays of $\mathcal{K}(G)_Q$, it is sufficient for us to enumerate the vertices of all of the intersection polytopes $IP_{\varpi_j}$ for all fundamental weights $\varpi_j$ of $T$. We complete this work in Section 5, with the following result.

**Theorem 1.2.** The vertices of $IP_{\varpi_j}$ consist of the following:

(a) $\varpi_j$, and

(b) $\varpi_j - \sum_i c_i \alpha_i$, where $I$ stands for a connected subdiagram of the Dynkin diagram for $G$ containing node $j$ and the coefficients $c_i$ are the entries of the $j$th column of the inverse transpose of the Cartan matrix associated to $I$.

**Remark 1.3.** It is natural to consider (a) as a special case of (b) where $I$ is the empty subdiagram.

**Remark 1.4.** In type $A$, the vertices of the polytopes $IP_\lambda$ were enumerated by Hoffman [Hof53] while reproducing a theorem of Hardy, Littlewood, and Pólya which characterizes dominance order using doubly-stochastic matrices.

1.2. **Two examples.** First take $G = SL_2$. For nonnegative integers $\ell, m$, the weight space $V_\ell(m)$ is nonempty and only if $\ell \geq m$ and $\ell - m$ is even. So the cone $\mathcal{K}(SL_2)$ looks like this:

Notice that, intersecting the cone with the dashed line through $\ell = 1$, we obtain the polytope isomorphic to the interval $[0, 1]$, whose two vertices give extremal rays as depicted.

Now take $G = Sp_2$ (type $C_2$). The intersection polytope $IP_\lambda$ for $\lambda = \varpi_1 + \varpi_2$ has $4 = 2^2$ vertices as predicted by Remark 3.8.

In contrast, the polytope $IP_{\varpi_1}$ has the three vertices $\varpi_1$, $\frac{1}{2} \varpi_2 = \varpi_1 - \frac{1}{2} \alpha_1$, and $0 = \varpi_1 - \frac{1}{2}(2\alpha_1 + \alpha_2)$; note that $\frac{1}{2}(1)$ and $\frac{1}{2} \left( \begin{array}{c} 2 \\ 1 \end{array} \right)$ are indeed the corresponding columns of the inverse transposes of the Levi Cartan matrices.
1.3. **Levi induction.** To each vertex \( v \) of an intersection polytope \( IP_\lambda \), we assign a Levi subgroup \( L \subseteq G \) and show that \( v \) can be lifted from a corresponding vertex in an intersection polytope for \( L \). This lifting procedure therefore induces a map \( \text{Ind} : \mathcal{K}(L) \to \mathcal{K}(G) \) that takes extremal rays to extremal rays. We describe this procedure in Section 4. It is tempting to compare this to the induction of extremal rays for the eigencone from Levi subgroup described in [BK19].

1.4. **Formulas for the numbers of extremal rays.** We also count the number of extremal rays of \( \mathcal{K}(G) \) and produce, for each Lie type, a polynomial formula as a function of the rank. As a consequence of Theorem 1.2, these formulas do not depend on the lacing of the associated Dynkin diagrams, so there are only three cases to consider: types \( A_r, D_r \), and \( E_r \). We also point out that these counting polynomials each begin with leading term \( r^3/6 \). See Remark 7.1.

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## 2. Notation and Background

We fix \( G \) a simple, simply-connected linear algebraic group over \( \mathbb{C} \). We choose a maximal torus and Borel subgroup \( T \subset B \subset G \). We denote by \( X^*(T) \) the lattice of weights of \( T \), and by \( X_*(T) \) the lattice of coweights. Their natural pairing is denoted by \( \langle \cdot , \cdot \rangle \). We let \( \Phi \) denote the set of roots of \( G \) with respect to \( T \), and denote by \( \Phi^+ \) the set of positive roots of \( G \) with respect to \( B \). We let \( \Phi^\vee \) denote the set of coroots, so \( (\Phi, X^*(T), \Phi^\vee, X_*(T)) \) is a root datum for \( G \). For a subset \( I \) of \( \{1, \ldots, r\} \), denote by \( L_I \) the semisimple part of the corresponding Levi subgroup, where \( \alpha_i \in \Phi(L_I) \) for all \( i \in I \). We write \( \Lambda^+ \) for the set of dominant weights. We denote by \( W \) the Weyl group of \( G \). We denote by \( \Delta = \{\alpha_1, \ldots, \alpha_r\} \) (resp., \( \{\alpha_1^\vee, \ldots, \alpha_r^\vee\} \)) the set of simple roots in \( \Phi \) (resp., simple coroots in \( \Phi^\vee \)). We will write \( x_i \) for the fundamental coweights, so \( \langle \alpha_i, x_j \rangle = \delta_{ij} \). We write \( \mathfrak{C}(e) \) for the dominant chamber, \( \mathfrak{C}(e) = \{\lambda \in h^*_\mathbb{Q}^+ | \langle \lambda, \alpha^\vee \rangle \geq 0, \forall \alpha \in \Phi^+ \} \).

If \( \lambda \in \Lambda^+ \), we write \( V_\lambda \) for the irreducible representation of \( G \) of highest weight \( \lambda \). If \( \mu \in X^*(T) \) we write \( V_\lambda(\mu) \) for the subspace of weight \( \mu \). Given a dominant weight \( \lambda \in h^*_\mathbb{Q} \), we can associate to it the Weyl polytope \( \text{conv}(W \cdot \lambda) \), which we denote by \( W_\lambda \). For an \( n \)-dimensional polytope \( P \), we call a face of dimension \( n-1 \) a facet.

**Definition 2.1.** Denote by \( \mathcal{K}(G) \) the set of pairs \( (\lambda, \mu) \in \Lambda^+ \times \Lambda^+ \) such that \( V_\lambda(\mu) \neq 0 \). Let \( \mathcal{K}(G)_{\mathbb{Q}^+} = \mathcal{K}(G) \otimes_{\mathbb{Z}_{\geq 0}} \mathbb{Q}^{\geq 0} \). Since we work over \( \mathbb{Q} \) for this whole paper, we will abuse notation and write \( \mathcal{K}(G) \) for \( \mathcal{K}(G)_{\mathbb{Q}^+} \).

The following proposition is well-known (see [Ste98]):

**Proposition 2.2.** \( \mathcal{K}(G) \) is a rational cone. Moreover, \( (\lambda, \mu) \in \mathcal{K}(G) \) if and only if the inequalities

\[
\langle \lambda - \mu, x_i \rangle \geq 0
\]

\[
\langle \lambda, \alpha_i^\vee \rangle \geq 0
\]

\[
\langle \mu, \alpha_i^\vee \rangle \geq 0
\]

for each \( i \in \{1, \ldots, r\} \) are satisfied.

## 3. Vertices of the Intersection Polytope

In order to study the extremal rays of the Kostka cone associated to \( G \), we study the following associated polytope.

**Definition 3.1.** We define the intersection polytope \( IP_\lambda := \mathfrak{C}(e) \cap W_\lambda \).
This is clearly a convex polytope, as it is the intersection of the convex polytope $W_\lambda$ with a finite collection of half-spaces. The facets of $IP_\lambda$ fall into two classes.

The first class of facets are sub-polytopes of the hyperplanes defining $\mathcal{C}(e)$. These facets $F_i^\lambda$, associated to simple coroots, are defined as the intersection

$$F_i^\lambda = IP_\lambda \cap \{ \mu \in h^*_L | \langle \mu, \alpha_i^\vee \rangle = 0 \}.$$  

Similarly, we have a second class of facets $E_j^\lambda$, associated to fundamental coweights, which are sub-polytopes of some of the facets of $W_\lambda$, defined by

$$E_j^\lambda = IP_\lambda \cap \{ \mu \in h_Q^* | \langle \lambda - \mu, x_j \rangle = 0 \}.$$  

The candidates for vertices of $IP_\lambda$ are suitable intersections $pX F_i^\lambda q X pX E_j^\lambda q$.

First we explore why it is at all reasonable to expect these intersections to be well-behaved.

**Proposition 3.2.** Suppose $I \cup J = \{1, \ldots, r\}$ (disjoint union). Then the collection

$$\{\alpha_i^\vee\}_{i \in I} \cup \{x_j\}_{j \in J}$$

form a basis of $h$.

**Proof.** If we can show they are linearly independent, that will be sufficient. Assume there is a relation

$$\sum a_i \alpha_i^\vee + \sum b_j x_j = 0.$$  

Then for every $i_0 \in I$,

$$\alpha_{i_0} \left( \sum a_i \alpha_i^\vee \right) = 0.$$  

Let $L_I$ be the Levi associated to $I$. Then the element $\sum a_i \alpha_i^\vee \in h_{L_I}$ must be identically 0 (since the $\alpha_i, i \in I$ form a basis of $h^*_{L_I}$). So each $a_i = 0$ in our relation. But

$$\sum b_j x_j = 0$$

forces the $b_j = 0$ since the $x_j$ are linearly independent. \hfill \Box

We use the following lemma from [LT92]:

**Lemma 3.3.** Let $G$ be be semisimple. Let $\lambda$ be a dominant weight. Then for any $j$, $\langle \lambda, x_j \rangle \geq 0$. Furthermore, if $G$ is simple, this is strict.

**Definition 3.4.** Suppose $I, J \subseteq \{1, \ldots, r\}$, and take $x \in h_Q^*$ and $\lambda$ to be a dominant weight. If the system of equations

$$\langle x, \alpha_i^\vee \rangle = 0, \ i \in I$$

$$\langle \lambda - x, x_j \rangle = 0, \ j \in J$$

has a unique solution, we will denote it by $v_{I,J}$.

**Lemma 3.5.** The solutions $v_{I,J}$ where $I \cup J = \{1, \ldots, r\}$ are vertices of $IP_\lambda$.

**Proof.** By Proposition 3.2, there exists a unique solution in $h_Q^*$ to the system of equations

$$\langle x, \alpha_i^\vee \rangle = 0, \ i \in I$$

$$\langle \lambda - x, x_j \rangle = 0, \ j \in J.$$  

To ensure that $v_{I,J}$ is indeed inside $IP_\lambda$, it must satisfy the following two additional systems of inequalities: (a) $i \notin I \implies \langle v_{I,J}, \alpha_i^\vee \rangle \geq 0$ and (b) $j \notin J \implies \langle \lambda - v_{I,J}, x_j \rangle \geq 0$. The $J$ equations tell us that

$$\lambda = v_{I,J} + \sum_{i \in I} a_i \alpha_i$$
for suitable rational numbers $a_i$. To establish (b), we must show that each $a_i \geq 0$. Indeed, restricted as weights for $L = L_I$, we have agreement

$$\lambda|_{h_L} = \sum_{i \in I} a_i \alpha_i|_{h_L}. $$

Furthermore, the $\alpha_i|_{h_L}$ are the simple roots of the $L$ root system and $\lambda|_{h_L}$ is still dominant. Therefore, by Lemma 3.3 applied to $L$, each $a_i \geq 0$. Given that (b) holds, we can now verify (a): if $i \notin I$ then

$$\langle v_{I,J}, \alpha_i^\vee \rangle = \langle \lambda, \alpha_i^\vee \rangle + \sum_{i' \in I} a_i \langle -\alpha_{i'}, \alpha_i^\vee \rangle \geq 0$$

since $\langle \alpha_{i'}, \alpha_i^\vee \rangle \leq 0$ when $i' \neq i$.

Next, we are to show that any vertex of $IP_{\lambda}$ is of the form $v_{I,J}$ for suitable $I, J$ satisfying $I \cup J = \{1, \ldots, r\}$ and $I \cap J = \emptyset$. Certainly any vertex of $IP_{\lambda}$ is the intersection of the facets on which it lies, so every vertex is the unique solution to a system of the following form:

\begin{align*}
  x & \in IP_{\lambda} \\
  x & \in F_i^\lambda \forall i \in I \\
  x & \in E_j^\lambda \forall j \in J
\end{align*}

for some $I, J \subseteq \{1, \ldots, r\}$, not necessarily disjoint. Note that a vertex $v$ satisfying the above system may satisfy yet more equalities of the form $\langle \lambda - v, x_i \rangle$ for $l \notin J$. For any such further equality satisfied by $v$, we add $l$ to $J$ until $J$ is maximal, so that $J = \{j|x \in E_j^\lambda\}$. We now show that we can always assume $I \cup J = \{1, \ldots, r\}$. We begin with the following lemma.

**Lemma 3.6.** Suppose $I, J \subseteq \{1, \ldots, r\}$ and that $v$ is the unique solution to (3), with $J$ maximal. If $k \in I \cap J$, then $\langle \lambda, \alpha_k^\vee \rangle = 0$. Furthermore, $v$ is the unique solution to the weaker system

\begin{align*}
  x & \in IP_{\lambda} \\
  x & \in F_i^\lambda \forall i \in I - \{k\} \\
  x & \in E_j^\lambda \forall j \in J.
\end{align*}

**Proof.** Take $k \in I \cap J$. By $v \in IP_{\lambda}$ and $v \in E_j^\lambda$ for all $j \in J$, we may write

$$v = \lambda - \sum_{\ell \notin J} k_\ell \alpha_\ell$$

for some rational numbers $k_\ell \geq 0$. Furthermore, since $J$ is assumed to be maximal, we know each $k_\ell > 0$. Because $k \in J$, $\langle \alpha_\ell, \alpha_k^\vee \rangle \leq 0$ for all $\ell \notin J$. Therefore, given that $\langle v, \alpha_k^\vee \rangle = 0$,

$$0 \leq \langle \lambda, \alpha_k^\vee \rangle = \langle v, \alpha_k^\vee \rangle + \sum_{\ell \notin J} k_\ell \langle \alpha_\ell, \alpha_k^\vee \rangle \leq 0,$$

so all expressions appearing are 0. In particular, $\langle \lambda, \alpha_k^\vee \rangle = 0$. Furthermore, each $\langle \alpha_\ell, \alpha_k^\vee \rangle = 0$ since each $k_\ell > 0$.

Now take any $v'$ satisfying (4). Again we can write $v' = \lambda - \sum_{i \notin J} b_i \alpha_i$. Note immediately that by the above conditions on $\lambda$ and $\alpha_i$ for $i \notin J$, $\langle v', \alpha_k^\vee \rangle = 0$. Thus $v'$ satisfies (3), and so $v' = v$. □
Now let $v$ be an arbitrary vertex of $IP_{\lambda}$. Then for some $I, J \subseteq \{1, \ldots, r\}$, $v$ is defined by the following properties:

$$v \in \left(\bigcap_{j \in J} E^\lambda_j\right) \cap \left(\bigcap_{i \in I} F^\lambda_i\right)$$

(5)

$$\langle v, \alpha^\vee_i \rangle \geq 0, i \notin I$$

$$\langle \lambda - v, x_j \rangle \geq 0, j \notin J$$

By Lemma 3.6, we assume $I \cap J = \emptyset$ and $J$ is maximal. If $I \cup J = \{1, \ldots, r\}$, then we are done. Otherwise, set $K = \{1, \ldots, r\} - (I \cup J)$. From Lemma 3.5, the point $v_{I \cup K, J}$ is a vertex of $IP_{\lambda}$; furthermore $v_{I \cup K, J}$ satisfies the system (5), so $v = v_{I \cup K, J}$. This completes the proof of the following

**Theorem 3.7.** Let $\lambda \in \Lambda^+$. The vertices of $IP_{\lambda}$ are exactly the points $v_{I, J}$ for $I, J$ satisfying $I \cup J = \{1, \ldots, r\}$.

**Remark 3.8.** Note that for $\lambda$ regular dominant, the $v_{I, J}$ are all distinct for different pairs $(I, J)$ satisfying $I \cup J = \{1, \ldots, r\}$; in particular there are $2^r$ vertices. This follows from Lemma 3.6. However if $\lambda$ is not regular then it is possible to have $v_{I, J} = v_{I, J'}$. We explore this in further detail in Sections 4 and 5 below.

### 4. Lifting Extremal Rays from Levi Subgroups

From Theorem 3.7, it is clear that, since

$$\mathcal{H}(G) = \bigcup_{\lambda \in \Lambda^+_0} \{\lambda\} \times IP_{\lambda},$$

any subset of $\mathcal{H}(G)$ which generates all pairs $(\lambda, v_{I, J}(\lambda))$ also generates $\mathcal{H}(G)$. Our aim is now to find such a generating set which is finite and minimal—this will be the set of extremal rays of $\mathcal{H}(G)$. First we describe a method of producing elements of $\mathcal{H}(G)$ via a “lifting” from $\mathcal{H}(L)$ where $L$ is a Levi subgroup. It will turn out that extremal rays of the latter give extremal rays of the former.

So let $L$ be a choice of semisimple Levi subgroup of $G$, and let $\Delta(L)$ be the corresponding collection of simple root indices. For such a simple Levi subgroup we can identify the simple roots in $\Phi(L)$ with a subset of the simple roots in $\Phi(G)$. We write $\bar{\alpha}_i$ for the fundamental coweights of $L$. Thus for $i, j \in \Delta(L)$, $\langle \alpha_j, x_i \rangle_G = \langle \bar{\alpha}_j, \bar{\alpha}_i \rangle_L$.

**Definition 4.1.** Given a Levi subgroup $L \hookrightarrow G$ and a dominant integral weight $\lambda_L \in \Lambda^+_L$, we associate to $\lambda_L$ a dominant integral weight $\lambda \in \Lambda^+_G$, as follows: if $i \in \Delta(L)$ then $\langle \lambda, \alpha^\vee_i \rangle = \langle \lambda_L, \alpha^\vee_i \rangle$ and if $j \notin \Delta(L)$ then $\langle \lambda, \alpha^\vee_j \rangle = 0$. We call this new weight $\lambda$ the extension by 0 of $\lambda_L$.

Let $(\lambda_L, \mu_L)$ be an element of $\mathcal{H}(L)$, and write

$$\lambda_L - \mu_L = \sum_{k \in \Delta(L)} c_k \alpha_k.$$

Then define the weight $\lambda$ for $G$, extending $\lambda_L$ by zero on each $\alpha^\vee_k, k \notin \Delta(L)$, and set

$$\mu := \lambda - \sum_{k \in \Delta(L)} c_k \alpha_k,$$

as a weight for $G$.

**Definition 4.2.** Given a point $(\lambda_L, \mu_L) \in (\Lambda^+_L)^2$ for $\lambda_L = \sum_i a_i \alpha_i$ and $\mu_L = \lambda_L - \sum_i c_i \alpha_i |_L$, we write $\text{Ind}^G_L(\lambda_L, \mu_L) = (\lambda, \mu) \in (\Lambda^+_G)^2$ where $\lambda = \sum \alpha_i, \bar{\alpha}_i$ is the extension of $\lambda_L$ by 0 and $\mu = \lambda - \sum c_i \alpha_i$.

**Lemma 4.3.** The map $\text{Ind}^G_L : (\Lambda^+_L)^2 \rightarrow (\Lambda^+_G)^2$ induces a map $\mathcal{H}(L) \rightarrow \mathcal{H}(G)$. By abuse of notation we will name this induced map $\text{Ind}^G_L$ as well.
Proof. First, note that for \( s \in \Delta(L) \) we have \(<\mu, \alpha_s^\vee> = <\mu_L, \alpha_s^\vee> \geq 0\), since \( \mu_L \) is a dominant weight of \( L \). Else, if \( \alpha_s \notin \Delta(L) \), we have
\[
<\mu, \alpha_s^\vee> = <\lambda, \alpha_s^\vee> - \sum_{k \in \Delta(L)} c_k <\alpha_k, \alpha_s^\vee>
\]
\[
= 0 - \sum_{k \in \Delta(L)} c_k <\alpha_k, \alpha_s^\vee>
\]
\[
\geq 0
\]
as \(<\lambda, \alpha_s^\vee> = 0\) and for each \( k \) we have \( c_k \geq 0 \) and \(<\alpha_k, \alpha_s^\vee> \leq 0\).

Since \( \mu \) is a dominant weight with \( \mu = \lambda - \sum_{k \in \Delta(L)} c_k \alpha_k \), we see that \( V_\lambda(\mu) \neq 0 \) by Proposition 2.2. \( \square \)

Not only do elements of \( \mathcal{K}(L) \) lift to elements of \( \mathcal{K}(G) \), but vertices of \( IP_\lambda \) lift to vertices of \( IP_\lambda \).

If \( L \hookrightarrow G \) is a simple Levi, with the associated map \( \Delta(L) \rightarrow \Delta(G) \), we enumerate the nodes of the Dynkin diagram of \( L \) such that they are compatible with the enumeration on the Dynkin diagram of \( G \). We can conveniently identify the vertices of the polytope \( IP_\lambda \) as \( v_{I',J'}(\lambda_L) \) where \( I' \sqcup J' = \Delta(L) \).

**Proposition 4.4.** Let \( \mu_L \) be a vertex of \( IP_\lambda \), so that \( \mu_L = v_{I',J'}(\lambda_L) \). Then \( \text{Ind}_{\mu_L}^\mathcal{G}(\lambda_L, \mu_L) = (\lambda, \mu) \), where \( \mu \) is vertex of \( IP_\lambda \). In particular, \( \mu \) coincides with \( v_{I,J}(\lambda) \) where \( I = I' \) and \( J = I'' \). \( \square \)

**Proof.** Since \( \mu_L = v_{I',J'}(\lambda_L) \), by definition it satisfies the following linear equations: \(<\mu_L, \alpha_i^\vee> = 0, i \in I'\) and \(<\lambda_L - \mu_L, \alpha_j^\vee> = 0, j \in J'\). Now we take \( I = I' \) and \( J = I'' \). For any finite-dimensional representation \( \lambda, \mu \), we may form \( \langle \lambda - \mu, x_j \rangle = \langle \sum c_i \alpha_i, x_j \rangle = 0 \) again by assumption.

We have the following immediate compatibility:

**Lemma 4.5.** Let \( L' \subset L \subset G \) be simple Levi subgroups. Then \( \text{Ind}_{\mu_L}^\mathcal{G}(\text{Ind}_{\mu_L}^\mathcal{L}(\lambda, \mu))) = \text{Ind}_{\mu_L}^\mathcal{L}(\lambda, \mu)) \).

Thus we observe that, if \( \lambda \) is the extension by \( 0 \) associated to \( \lambda_L \) for some simple Levi \( L \hookrightarrow G \), then a subset of the \( v_{I,J}(\lambda) \) vertices of \( IP_\lambda \) are in fact lifts associated to \( v_{I',J'}(\lambda_L) \), where \( I' \subset I \).

Since whenever we lift a vertex \( v_{I,L,J}(\lambda_L) \) from a Levi subgroup it always lifts to \( v_{I',L,J}(\lambda) \) and by the compatibility in 4.5, we can make the following notational simplification.

**Notation 1.** We write \( v_{\Delta(L)}(\lambda) \) for a vertex of \( IP_\lambda \) lifted from Levi \( L \), where \( L \) is the smallest possible such Levi.

**Definition 4.6.** If \( C_1, C_2 \) are rational semigroups inside ambient vector spaces \( W_1, W_2 \), we may form the direct sum \( C_1 \oplus C_2 = \{(c_1, c_2) \mid c_1 \in C_1, c_2 \in C_2 \} \subseteq W_1 \oplus W_2 \), which is again a semigroup under \((c_1, c_2) + (d_1, d_2) = (c_1 + d_1, c_2 + d_2)\) and admits scalar multiplication by nonnegative rational numbers \( q \cdot (c_1, c_2) = (qc_1, qc_2) \). Furthermore, if \( C_1 \) and \( C_2 \) are convex rational cones (i.e., defined by a finite system of rational linear inequalities), so is \( C_1 \oplus C_2 \) (defined by the union of the inequalities for \( C_1 \) and \( C_2 \)).

**Lemma 4.7.** Let \( L \) be a non-simple Lie group, \( L = \coprod L_i \) where each \( L_i \) is simple. Then \( \mathcal{K}(L) = \bigoplus \mathcal{K}(L_i) \).

**Proof.** If \( L \) is non-simple then the root system \( \Phi(L) = \bigcup \Phi(L_i) \). For any finite-dimensional representation \( V_{\lambda_L} \) we have \( V_{\lambda_L} = \bigotimes V_{\lambda_{L_i}} \); here each simple factor \( L_i \) acts only on \( V_{\lambda_{L_i}} \). The statement on Kostka cones follows. \( \square \)

**Remark 4.8.** This allows us to consider only simple \( G \), as was assumed in the introduction.
Proposition 5.1. Suppose \( \lambda \) and \( \mu \) are some subset \( J \) of elements in \( \mathbb{R}^n \). Then
\[
\sum \operatorname{Ind}_{L_i}(\lambda_{L_i}, \mu_{L_i}) = (\sum \lambda_i, v_{\Delta(L)}(\sum \lambda_i))
\]
where \( \lambda_i \) is the extension by 0 of \( \lambda_{L_i} \).

Proof. This follows by the same type of argument as in the proof of Proposition 4.4. \( \square \)

5. Enumeration of the Extremal Rays

We begin by naming some extremal rays of \( \mathcal{H}(G) \).

Proposition 5.1. Suppose \( I \sqcup J = \{1, \ldots, r\} \). Then \((\varpi_i, v_{I,J}(\varpi_i))\) gives an extremal ray of \( \mathcal{H}(G) \).

Proof. Suppose \((\varpi_i, v_{I,J}(\varpi_i)) = (\lambda_1, \mu_1) + (\lambda_2, \mu_2)\) where each \((\lambda_j, \mu_j) \in \mathcal{H}(G)\). We wish to show the \((\lambda_j, \mu_j)\) are parallel. Since the \( \lambda_j \) are both dominant and sum to \( \varpi_i \), we must have \( \lambda_1 = a_1 \varpi_i \) and \( \lambda_2 = a_2 \varpi_i \) where \( a_1, a_2 \geq 0 \) and \( a_1 + a_2 = 1 \). It follows that \( \mu_1 \) is inside the polytope \( IP_{a_1 \varpi_i} \) and \( \mu_2 \) inside \( IP_{a_2 \varpi_i} \). Since \( a_1 = 0 \) makes \((\lambda_1, \mu_1) = (0, 0)\), in which case the pair \((\lambda_j, \mu_j)\) are trivially parallel, we may assume \( a_1 > 0 \). Likewise, we assume \( a_2 > 0 \). By scaling, \( \frac{1}{a_j} \mu_j \) for \( j = 1, 2 \) belongs to the polytope \( IP_{\varpi_i} \). Furthermore, the vertex \( v_{I,J}(\varpi_i) \) is equal to the convex sum
\[
a_1 \left( \frac{1}{a_1} \mu_1 \right) + a_2 \left( \frac{1}{a_2} \mu_2 \right)
\]
of elements in \( IP_{\varpi_i} \). By the following lemma (a standard result in convex geometry), this forces \( \frac{1}{a_j} \mu_j = v_{I,J}(\varpi_i) \) for each \( j = 1, 2 \). \( \square \)

Lemma 5.2. Let \( P \) be a compact, convex polytope inside of \( \mathbb{R}^n \) defined as the solution space to a system of linear inequalities (given by linear functions \( f_j: \mathbb{R}^n \to \mathbb{R}, j = 1, \ldots, m \)):
\[
P = \{ \bar{x} \in \mathbb{R}^n | f_j(\bar{x}) \geq 0 \ \forall j \}
\]
Let \( v \in P \) be a vertex (that is, the unique point of intersection of \( P \) with the hyperplanes \( f_j = 0 \) for some subset \( J \subseteq \{1, \ldots, m\} \)). If \( v = t_1 x_1 + t_2 x_2 \) for points \( x_i \in P \) and positive real numbers \( t_i \) such that \( t_1 + t_2 = 1 \), then each \( x_i = v \).

Proof. Let \( j \in J \). Then \( t_1 f_j(x_1) \geq 0 \) and \( t_2 f_j(x_2) \geq 0 \). Since
\[
0 \leq t_1 f_j(x_1) + t_2 f_j(x_2) = f_j(v) = 0,
\]
it must furthermore be true that \( t_i f_j(x_i) = 0 \) for each \( i = 1, 2 \), which implies \( f_j(x_i) = 0 \) for each \( i = 1, 2 \).

Since the choice of \( j \in J \) was arbitrary, both \( x_1 \) and \( x_2 \) are in the intersection of \( P \) with the hyperplanes \( f_j = 0, j \in J \). So each \( x_i = v \). \( \square \)

To complete the enumeration of all extremal rays of \( \mathcal{H}(G) \), we now demonstrate that every extremal ray has the special form above.

Proposition 5.3. The vertices \( v_{I,J}(\lambda) \) of \( IP_{\lambda} \) depend linearly on \( \lambda \).

Proof. The \( v_{I,J}(\lambda) \) are solutions to a linear system depending linearly on \( \lambda \), as we now explain. Suppose \( I \sqcup J = \{1, \ldots, r\} \). The vertex \( v_{I,J}(\lambda) \) of \( IP_{\lambda} \) is the unique solution to the system
\[
\langle x, \alpha_i \rangle = 0, \ i \in I
\]
and likewise the vertex \( v_{I,J}(\mu) \) of \( IP_{\mu} \) is the unique solution to
\[
\langle x, \alpha_i \rangle = 0, \ i \in I
\]
Let \( \alpha \) be a nontrivial weight on \( \Lambda \). Consider, therefore, the sum \( v_{I,J}(\lambda) + v_{I,J}(\mu) \), which satisfies
\[
\langle v_{I,J}(\lambda) + v_{I,J}(\mu), \alpha_i \rangle = 0, \quad i \in I
\]
\[
\langle \lambda + \mu - (v_{I,J}(\lambda) + v_{I,J}(\mu)), x_j \rangle = 0, \quad j \in J;
\]
this system has only one solution, namely \( v_{I,J}(\lambda + \mu) \).

This allows us to relate the vertices of distinct polytopes and establishes that \( IP_{\lambda+\mu} \) is the Minkowski sum of \( IP_\lambda \) and \( IP_\mu \).

**Proposition 5.4.** If \((\lambda, \mu)\) is an extremal ray of \( \mathcal{X}(G) \), then, up to scaling, \( \lambda = \varpi_i \) for some \( i \).

**Proof.** Let \((\lambda, \mu) \in \mathcal{X}(G)\). Since \( \mu \in IP_\lambda \), we can write \( \mu = \sum a_{I,J}v_{I,J}(\lambda) \) where \( \sum a_{I,J} = 1 \). Moreover we see that \( \lambda = \sum a_{I,J}\lambda \). Thus we can rewrite \( (\lambda, \mu) = (\sum a_{I,J}\lambda, \sum a_{I,J}v_{I,J}(\lambda)) = \sum a_{I,J}(\lambda, v_{I,J}(\lambda)) \).

Now we write \( \lambda = \sum b_k \varpi_k \), and using that the \( v_{I,J}(\lambda) \) are linear in \( \lambda \) we have
\[
(\lambda, \mu) = \sum a_{I,J}(\lambda, v_{I,J}(\lambda))
= \sum a_{I,J}(\sum b_k \varpi_k, v_{I,J}(\sum b_k \varpi_k))
= \sum a_{I,J}(\sum b_k \varpi_k, \sum b_k v_{I,J}(\varpi_k))
= \sum a_{I,J} \sum b_k (\varpi_k, v_{I,J}(\varpi_k)).
\]

Thus the generators of extremal rays of \( \mathcal{X}(G) \) coincide with the collection of \((\varpi_i, v_{I,J}(\varpi_i))\). There could be redundancy among the \( v_{I,J}(\varpi_i) \)s (indeed there is), so we now wish to enumerate these extremal rays without repetition.

In Section 4 we see that for \( \lambda \) an extension by 0 of some \( \lambda_L \), we can lift extremal rays of the form \((\lambda_L, \mu_L)\) from \( \mathcal{X}(L) \) to \( \text{Ind}(\lambda_L, \mu_L) \) in \( \mathcal{X}(G) \). Of course, each \( \varpi_i \in \Lambda^+ \) is precisely the extension by 0 of \((\varpi_i)_L\) for any Levi \( L \) containing \( \alpha_i \).

We first show that the extremal ray \( \text{Ind}(\varpi_i, v_{\Delta(L)}(\varpi_i)) \) depends only on the simple summand of \( L \) where \( \varpi_i \) is a nontrivial weight.

**Lemma 5.5.** Let \( L = \prod L_i \) be a direct product of simple Levis, and let \( \varpi_k \) be a nontrivial weight on some simple \( L_i \). Let \( I \subset \Delta(L) \) and let \( I' = I \cap \Delta(L_i) \) with \( k \in I' \). Then
\[
\text{Ind}(\varpi_k, v_{I,i}(\varpi_k)) = \text{Ind}(\varpi_k, v_{I,i'(L')}(\varpi_k)).
\]

**Proof.** This follows from Proposition 4.9 and Lemma 4.5, since the only way to choose dominant weights \( \lambda_{L_j} \) for each \( L_j \) such that \( \sum \lambda_j = \varpi_k \) (where here the sum denotes summing the extensions by 0 in \( \Lambda_G \)) is for \( \lambda_{L_j} = 0 \) unless \( L_j = L_i \), in which case \( \lambda_{L_i} \) is taken to be \( \varpi_k \).

Thus when finding extremal rays of the form \((\varpi_i, v_{I,J}(\varpi_i))\), we need only consider the following cases: in the first case, \( i \notin I \), which we consider below. The second case corresponds to \( i \in I \), and then Lemma 5.5 allows us to restrict our attention to the simple factor \( L' \) of \( L \) containing \( \alpha_i \) as a root. If \( I \not\subset \Delta(L') \), then Lemma 4.5 allows us to restrict to the simple sub-Levi \( L' \subset L \) such that \( I = \Delta(L') \).

**Lemma 5.6.** Let \( L_1 \) and \( L_2 \) be distinct simple Levi subgroups such that \( \alpha_i \in \Phi(L_1) \) and \( \alpha_i \in \Phi(L_2) \). Then
\[
\text{Ind}_{L_1}^G(\varpi_i, v_{\Delta(L_1)}(\varpi_i)) \neq \text{Ind}_{L_2}^G(\varpi_i, v_{\Delta(L_2)}(\varpi_i)).
\]
Proof. Since $L_1$ and $L_2$ are distinct, there must be some root $\alpha_j$ which is not in both root systems. Assume that $\alpha_j \in \Phi(L_1)$ but $\alpha_j \notin \Phi(L_2)$. We can write

$$\varpi_i - v_{\Delta(L_1)}(\varpi_i) = \sum_{m \in \Delta(L_1)} k_n \alpha_n$$

with each $k_n > 0$ (since otherwise we could reduce to a Levi $L' \subset L_1$), and similarly

$$\varpi_i - v_{\Delta(L_2)}(\varpi_i) = \sum_{m \in \Delta(L_2)} k_m \alpha_m$$

with $k_m > 0$.

Now pairing with $x_j$, we see $\langle \varpi_i - v_{\Delta(L_1)}(\varpi_i), x_j \rangle = k_j$, whereas $\langle \varpi_i - v_{\Delta(L_2)}(\varpi_i), x_j \rangle = 0$. □

**Theorem 5.7.** The extremal rays of $\mathcal{H}(G)$ with first coordinate $\varpi_i$ coincide precisely with the set \{Ind$^G_L(\varpi_i, v_{I,J}(\varpi_i))\}$, where either

(a) $L$ is a simple Levi such that $\alpha_i \in \Phi(L)$ (note $L = G$ is one such simple Levi),

(b) $L = e$, which corresponds to the extremal ray $(\varpi_i, \varpi_i)$.

**Proof.** By Lemma 5.5, we know we need only consider two cases; the first being $(\varpi_i, v_{I,J}(\varpi_i))$ where $i \notin I$, and the second being Ind$^G_L(\varpi_i, v_{\Delta(L)}(\varpi_i))$ for simple Levis $L$ such that $\alpha_i \in \Phi(L)$. By Lemma 5.6 we know that all those extremal rays coming from the second case (corresponding to distinct Levis) are distinct. All that remains to show is that the first case, $(\varpi_k, v_{I,J}(\varpi_k))$ where $k \notin I$, corresponds to the extremal ray $(\varpi_k, \varpi_k)$.

By assumption, $v_{I,J}(\varpi_k)$ satisfies $\langle v_{I,J}(\varpi_k), \alpha_i \rangle = 0$ for all $i \in I$ as well as $\langle \varpi_k - v_{I,J}(\varpi_k), x_j \rangle = 0$ for all $j \in I^c$. Moreover we have

$$v_{I,J}(\varpi_k) = \varpi_k - \sum_{l \in I} a_l \alpha_l$$

for non-negative $a_l$. We restrict to the Levi $L$ such that $\Delta(L) = I$. Then $\varpi_k|_L = 0$, and $v_{I,J}(\varpi_k)|_L = -\sum_{l \in I} a_l \alpha_l$. This latter term must be a dominant weight, which forces $a_l = 0$ for all $l$, or in other words that $v_{I,J}(\varpi_i) = \varpi_i$, by Lemma 3.3.

Proposition 5.4 and Theorem 5.7 taken together give a complete description of the extremal rays of $\mathcal{H}(G)$.

**Corollary 5.8.** Let $L$ be a simple Levi with $i \in \Delta(L)$. The extremal ray $(\varpi_i, v_{\Delta(L)}(\varpi_i)) = \text{Ind}^G_L(\varpi_i, 0)$.

**Proof.** Simply note that $\langle 0, \alpha_k \rangle = 0$ for all $k \in I$. □

6. DETAILED EXAMPLE IN C4

In this section, we present an explicit computation of extremal rays of the form $(\varpi_3, -)$ in type $G = C_4$ using Corollary 5.8 and Theorem 5.7. We consider extremal rays of the form $(k\varpi_3, v_{\Delta(L)}(k\varpi_3))$ for all simple Levi subgroups such that $3 \in \Delta(L)$ (of course the $(\varpi_3, v_{\Delta(L)}(\varpi_3))$ are also rays). By Corollary 5.8, in order to get integral points of $\mathcal{H}(G)$ instead of just rational extremal rays, all that is required is to find $k$ such that $k\varpi_i$ is in the root lattice; then $(k\varpi, 0)$ is an integral point of $\mathcal{H}(L)$ and its lift will be integral as well. The expression of $\varpi_i$ as a rational sum of roots is encoded by the transpose inverse Cartan matrix associated to $L$; if $(C^{-1}_L)^t$ is the transpose inverse Cartan matrix associated to $L$ then taking $k = \text{det}(C_L)$ will always yield $k\varpi_i$ on the root lattice. Note however that $(\text{det}(C_L)\varpi_i, 0)$ may not be the first integral point on the ray $(\varpi_i, 0)$, as seen in case 4 below corresponding to Levi $\Delta(L) = \{2, 3, 4\}$.

In the examples below, $\Delta(L)$ are denoted by solid nodes, and a circle is placed around node 3. The transpose inverse Cartan matrix of $L$ is given below the diagrams, with columns aligned with the
corresponding simple roots of \( L \). By Corollary 5.8, all extremal rays are given as lifts of \( \text{Ind}_L^G(k\varpi_3, 0) \), where \( k \) is the determinant of the Cartan matrix of \( L \) (to preserve integrality).

The trivial Levi \( \{ e \} \)

\((\varpi_3, \varpi_3)\)

\(\frac{1}{2} (1)\)

\((2\varpi_3, v_{3,124}(2\varpi_3)) = (2\varpi_3, 2\varpi_2 - \alpha_3) = (2\varpi_3, \varpi_2 + \varpi_4)\)

\(\frac{1}{2} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\)

\((2\varpi_3, v_{34,12}(2\varpi_3)) = (2\varpi_3, 2\varpi_3 - 2\alpha_3 - \alpha_4) = (2\varpi_3, 2\varpi_2)\)

\(\frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\)

\((3\varpi_3, v_{23,14}(3\varpi_3)) = (3\varpi_3, 3\varpi_3 - \alpha_2 - 2\alpha_3) = (3\varpi_3, \varpi_1 + 2\varpi_4)\)

\(\frac{1}{2} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 4 \\ 4 \\ 3 \end{pmatrix}\)

\((2\varpi_3, v_{234,1}(2\varpi_3)) = (2\varpi_3, 2\varpi_3 - 2\alpha_2 - 4\alpha_4 - 2\alpha_2) = (2\varpi_3, 2\varpi_1)\)

\(\frac{1}{4} \begin{pmatrix} 3 \\ 2 \\ 2 \\ 1 \\ 2 \\ 4 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}\)

\((4\varpi_3, v_{123,4}(4\varpi_3)) = (4\varpi_3, 4\varpi_3 - \alpha_1 - 2\alpha_2 - 3\alpha_3) = (4\varpi_3, 3\varpi_4)\)
7. Polynomial Formulas for the Number of Extremal Rays

We now give formulas for the number of extremal rays in $\mathcal{H}(G)$ for each type of simple Lie group $G$ as polynomials in the rank. In the classical setting of Kostka numbers (i.e., for $G = A_r$), these are due to the third author, Gao, Orelowitz, and Yong [GKOY]. Our computations are based on the correspondence

\[(2\omega_3, v_{1234,\delta}(2\omega_3)) = (2\omega_3, 2\omega_3 - 2\alpha_1 - 4\alpha_2 - 6\alpha_3 - 3\alpha_4) = (2\omega_3, 0)\]

Remark 7.1. As we are looking at choices of simple Levi subgroups, the only influential factor is the underlying graph of the Dynkin diagram, and not the lacing of the diagram. Thus, we have that $G = A_r$, $B_r$, and $C_r$ will have the same number of extremal rays for each $r$, as will the pairs $A_2$ and $G_2$, and $A_4$ and $F_4$. We therefore reduce to computations for $A_r$, $D_r$, and $E_r$. We find it interesting that each polynomial begins with leading term $r^3/6$.

We take as convention the numbering scheme of Bourbaki [Bou02] for the simple Dynkin diagrams.

Proposition 7.2.  
(a) For $G$ of type $A_r$, the number of extremal rays in $\mathcal{H}(G)$ is $\binom{r+1}{3} + \binom{r+1}{2} + \binom{r+1}{1} - 1$.
(b) For $G$ of type $D_r$, the number of extremal rays in $\mathcal{H}(G)$ is $\binom{r+3}{3} + 3\binom{r+3}{2} + 2\binom{r+1}{1} - 3$.
(c) For $G$ of type $E_r$, the number of extremal rays in $\mathcal{H}(G)$ is $\binom{r+3}{3} + 4\binom{r+2}{2} + \binom{r+1}{1} - 8$.

Proof.  
(a) We proceed by induction on $r$. Set $R_{i,r}$ to be the number of extremal rays of the form $(k\omega_i, v_{I,J}(k\omega_i))$ for $A_r$. By the correspondence (7), we have as the base case that $R_{1,1} = 2$, which agrees with the formula (with the convention that $\binom{0}{2} = 0$). Now suppose that the formula holds for $r$. Again using the correspondence (7), we know that $R_{i,r+1} - R_{i,r}$ $(1 \leq i \leq r)$ is precisely the number of new Levi subgroups in $A_{r+1}$ containing node $i$, under the usual embedding $A_r \hookrightarrow A_{r+1}$, and that $R_{r+1,r+1} = r + 2$. As can easily be seen from the Dynkin diagram, we have for $1 \leq i \leq r$

$$R_{i,r+1} - R_{i,r} = i.$$ 

Therefore, we have

$$\sum_{i=1}^{r+1} R_{i,r+1} = \left(\sum_{i=1}^{r} R_{i,r+1}\right) + R_{r+1,r+1}$$

$$= \sum_{i=1}^{r} (R_{i,r} + i) + r + 2$$

$$= \binom{r+1}{3} + \binom{r+1}{2} + \binom{r+1}{1} - 1 + \frac{r(r+1)}{2} + r + 2$$

$$= \binom{r+2}{3} + \binom{r+2}{2} + \binom{r+2}{1} - 1.$$
as desired.

(b) We again proceed by induction on \( r \). Denote similarly in this case \( R_{i,r} \). Then we have as a base case

\[
R_{1,4} = R_{3,4} = R_{4,4} = 6, \quad R_{2,4} = 9,
\]

which can be obtained using the correspondence (7), and agrees with the formula. Now, choose the standard Levi embedding \( D_r \rightarrow D_{r+1} \), noting that now node \( i \) is labeled \( i + 1 \). An investigation of the Dynkin diagram in this case gives \( R_{1,r+1} = r + 3 \) and

\[
R_{i,r+1} = \begin{cases} 
R_{i-1,r} + (r + 1 - i) + 2, & 2 \leq i \leq r - 1 \\
R_{r-1,r} + 2, & i = r \\
R_{r,r} + 2, & i = r + 1
\end{cases}
\]

Therefore, we have

\[
\sum_{i=1}^{r+1} R_{i,r+1} = r + 3 + \left( \sum_{i=2}^{r-1} R_{i,r+1} \right) + R_{r,r+1} + R_{r+1,r+1} \\
= r + 3 + \left( \sum_{i=2}^{r-1} R_{i-1,r} + r + 3 - i \right) + R_{r-1,r} + 2 + R_{r,r} + 2 \\
= r + 7 + \sum_{i=1}^{r} R_{i,r} + \sum_{i=2}^{r-1} (r + 3 - i) \\
= r + 7 + \binom{r}{3} + 3 \binom{r}{2} + 2 \binom{r}{1} - 3 + \frac{1}{2}(r^2 + 3r - 10) \\
= \frac{r^3}{6} + \frac{3r^2}{2} + \frac{10r}{3} - 1 \\
= \binom{r+1}{3} + 3 \binom{r+1}{2} + 2 \binom{r+1}{1} - 3
\]

as desired. Note that this formula is still valid for \( D_3 = A_3 \) and \( D_2 = A_1 \times A_1 \).

(c) This is done by direct computation. In particular, the number of extremal rays in types \( E_6, E_7, \) and \( E_8 \) are 78, 118, and 168, respectively, which fits the formula. It should be noted that this also holds when making the associations \( E_5 = D_5, E_4 = A_4, \) and \( E_3 = A_2 \times A_1 \).

\[ \square \]

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