TOWARDS A CHARACTERIZATION OF THE STAR-FREE SETS OF INTEGERS

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Abstract. Let $U$ be a numeration system, a set $X \subseteq \mathbb{N}$ is $U$-star-free if the set made up of the $U$-representations of the elements in $X$ is a star-free regular language. Answering a question of A. de Luca and A. Restivo [10], we obtain a complete logical characterization of the $U$-star-free sets of integers for suitable numeration systems related to a Pisot number and in particular for integer base systems. For these latter systems, we study as well the problem of the base dependence. Finally, the case of $k$-adic systems is also investigated.

1. Introduction

In the study of numeration systems, a natural question is to determine if a set of non-negative integers has a “simple” representation within the considered number system. Otherwise stated, is it possible for a given set $X \subseteq \mathbb{N}$, to find a “simple” algorithm (a finite automaton) testing membership in $X$? This question has given rise to a lot of papers dealing with the so-called recognizable sets. A subset $X$ of $\mathbb{N}$ is said to be $k$-recognizable if the language made up of the $k$-ary expansions of all the elements in $X$ is regular (i.e., recognizable by a finite automaton).

Since the work of A. Cobham [3], it is well-known that the recognizability of a set depends on the base of the numeration system. If $k$ and $l$ are two multiplicatively independent integers then the only subsets of $\mathbb{N}$ which are simultaneously $k$-recognizable and $l$-recognizable are exactly the ultimately periodic sets.

Among the recognizable sets, it could be interesting to describe the sets whose corresponding languages of representations belong to a specific subset of regular languages. Among the regular languages, the “simplest” are certainly the star-free languages because the automata accepting those languages are counter-free. Having in mind this idea of “simpler” representation of a set, A. de Luca and A. Restivo have considered in [10] the problem of determining the existence of a suitable base $k$ such that the $k$-ary representations of the elements belonging to a set $X \subseteq \mathbb{N}$ made up a regular language of (unrestricted) star-height 0 (such a set is then said to be $k$-star-free). One of the main results of [10] is that if a $l$-recognizable set $X$ is such that its density function is bounded by $c(\log n)^d$, for some constants $c$ and $d$, then there exists a base $k$ such that $X$ is $k$-star-free.

The star-free languages have been extensively studied in the literature [12, 14, 15, 16]. In particular M.P. Schützenberger has shown that the star-free languages — i.e., the languages expressed in terms of extended regular
expressions without the star operation — are exactly the aperiodic languages \[14\]. We recall that a language \(L \subset \Sigma^*\) is aperiodic if there exists a positive integer \(n\) such that for all words \(u, v, w \in \Sigma^*\),
\[ uv^n w \in L \iff uv^{n+1} w \in L. \]

In the present paper, we answer some of the remaining open questions addressed in \[10\] about sets of integers having a representation of star height 0. Especially, we give a complete characterization of the \(k\)-recognizable sets such that the language of \(k\)-ary expansions is aperiodic. To obtain this result, we use the first-order logical characterization of the star-free languages given by R. McNaughton and S. Papert \[12\].

In the first two sections, for the sake of simplicity we consider the case of the binary system. Next, we show how our results can be extended not only to the \(k\)-ary systems but also to numeration systems defined by a linear recurrent sequence whose characteristic polynomial is the minimal polynomial of a Pisot number (a Pisot number is an algebraic integer \(\theta > 1\) such that the other roots of the minimal polynomial of \(\theta\) have modulus less than one). In this wider framework, we have to consider the additional assumption that the set of all the representations computed by the greedy algorithm is star-free. For instance, this assumption is satisfied for the Fibonacci system. In Section 5, we consider the problem of the base dependence of the aperiodicity of the representations for integer base systems. The obtained result can be related to the celebrated Cobham’s theorem: only ultimately periodic sets can be \(k\)- and \(l\)-star-free for two multiplicatively integers \(k\) and \(l\) but if the period \(p\) of an ultimately periodic set is greater than 1 then this set is \(k\)-star-free for some \(k\) depending on \(p\) but not for all \(k \in \mathbb{N}\). In particular, we show that a set is \(k\)-star-free if and only if it is \(k^n\)-star-free for any \(n \geq 1\).

In the last section, we consider the case of the unambiguous \(k\)-adic numeration system. It is worth noticing that the unique \(k\)-adic representation of an integer is not computed through the greedy algorithm and therefore this system differs from the other systems encountered in this paper. It appears unsurprisingly that the star-free sets with respect to this latter system are exactly the \(k\)-star-free sets.

In the following, we assume the reader familiar with basic formal languages theory (see for instance, \[3\]). Finite automata will be denoted as 5-tuples \(\mathcal{M} = (Q, q_0, F, \Sigma, \delta)\) where \(Q\) is the set of states, \(q_0\) is the initial states, \(F \subseteq Q\) is the set of final states, \(\Sigma\) is the input alphabet and \(\delta : Q \times \Sigma \rightarrow Q\) is the transition function.

### 2. Logical characterization of star-free languages

Let us consider the alphabet \(\Sigma_2 = \{0, 1\}\). A word \(w\) in \(\Sigma_2^+\) can be identified as a finite model \(\mathfrak{M}_w = (M, <, P_1)\) where \(M = \{1, \ldots, |w|\}\) (\(|w|\) is the length of \(w\)), < is the usual binary relation on \(M\) and \(P_1\) is a unary predicate for the set of positions in \(w\) carrying the letter 1. For our convenience, positions are counted from right to left. As an example, the word \(w = 1101001\) corresponds to the model \((M, <, P_1)\) where \(M = \{1, \ldots, 7\}\) and \(P_1 = \{1, 4, 6, 7\}\). For further purposes, as in \[16\] we expand this model with its maximal element \(\text{max}\) (in the latter example, \(\text{max} = 7\) — notice
that \( \max \) is definable in terms of \(<\). So to each nonempty word in \( \Sigma_2^+ \) is associated a model \((M, <, P_1, \max)\).

A language is said to be star-free if it is obtained from finite sets by a finite number of Boolean operations (union, intersection and complementation) and concatenation products. McNaughton and Papert have shown that these languages are exactly those defined by first-order sentences when words are considered as finite ordered models \([12]\) (a sentence is a formula whose all variables are bound). As an example, the language \( 1^+0^* \) is star-free because if we denote by \( \overline{X} \) the complement \( \Sigma_2^* \setminus X \) of \( X \) then

\[
1^+0^* = \{1\overline{00}\overline{01}\overline{10}\} \text{ where } \emptyset = \{0\} \cap \{00\}
\]

and this language is also defined by the formula

\[
(1) \quad (\exists x)[P_1(x) \land (\forall y)(x < y \rightarrow P_1(y)) \land (\forall y)(y < x \rightarrow \neg P_1(y))].
\]

The language of all the formulas defining star-free languages will be denoted by \( L_{SF} \) (if necessary, to recall the alphabet \( \Sigma_2 \) we can write \( L_{SF,2} \)). Notice that with these finite models, we are not considering the empty word.

To be precise, if \( \varphi(x_0, \ldots, x_n) \) is a formula having at most \( x_0, \ldots, x_n \) as free variables, the interpretation of \( \varphi \) in a word-model \( M_w \) having \( M \) as domain and \( r_0, \ldots, r_n \) as \( M \)-elements is defined in a natural manner and we write \( M_w \models \varphi[r_0, \ldots, r_n] \) if \( \varphi \) is satisfied in \( M_w \) when interpreting \( x_i \) by \( r_i \).

The language defined by a formula \( \varphi \) is

\[
\{w \in \Sigma_2^+ | M_w \models \varphi\}.
\]

2.1. Syntax of logical formulas in \( L_{SF} \). The first-order language of the finite ordered models representing words is defined as follows. The variables are denoted \( x, y, z, \ldots \) and are ranging over \( M \)-elements. The terms are obtained from the variables and the constant \( \max \). The atomic formulas are obtained by the following rules:

1. if \( \tau_1 \) and \( \tau_2 \) are terms then \( \tau_1 < \tau_2 \) and \( \tau_1 = \tau_2 \) are atomic formulas
2. if \( \tau \) is a term then \( P_1(\tau) \) is an atomic formula.

Finally, we obtain the set \( L_{SF} \) of all the formulas by using the Boolean connectives \( \neg, \land, \lor, \rightarrow, \leftrightarrow \) and the first-order quantifiers \( (\exists x) \ldots \) and \( (\forall x) \ldots \) where \( x \) is a variable.

Notice that for convenience we are somehow redundant in our definitions, \( \varphi \lor \psi \) stands for \( \neg(\neg \varphi \land \neg \psi) \), \( x = y \) stands for \( \neg((x < y) \lor (y < x)) \), \( \varphi \rightarrow \psi \) stands for \( \neg \varphi \lor \psi \) and \( \varphi \leftrightarrow \psi \) stands for \( (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi) \). We also write \( x \leq y \) for \( (x < y) \lor (x = y) \).

It is worth noting that in this formalism we can define \( y = x + 1 \), where \( x \) and \( y \) are variables,

\[
y = x + 1 \equiv (x < y) \land (\forall z)(x < z \rightarrow y \leq z)
\]

but the form \( z = x + y \) is not allowed, if \( x, y \) and \( z \) are variables \([12]\).

3. Logical characterization of recognizable sets of integers

In the present section, we consider the binary numeration system. If \( x \) is an non-negative integer, the binary expansion of \( x \) computed through the greedy algorithm (the normalized 2-representation of \( x \)) is denoted \( \rho_2(x) \) (for a presentation of the greedy algorithm, see \([4]\)). Notice that \( \rho_2(0) \) is the
empty word $\varepsilon$ and we allow leading zeroes in normalized 2-representations. Thus, the set of all the normalized representations is

$$\mathcal{N}_2 = 0^\ast \{ \rho_2(n) \mid n \in \mathbb{N} \}.$$ 

A set $X$ of integers is said to be 2-recognizable if the set $\rho_2(X)$ of the normalized 2-representations of all the elements in $X$ is a regular language.

Remark 1. Allowing leading zeroes does not change the star-free behavior of a language made up of representations. Indeed, let $\Sigma = \{0, \ldots, k-1\}$, $k \geq 2$ and $L \subset \Sigma^* \setminus 0\Sigma_k^*$ be a regular language consisting of words which do not begin with 0. Then $L$ is star-free if and only if $0^\ast L$ is star-free. Indeed,

$$0^\ast L = \overline{0}\{1, \ldots, k-1\}\overline{0}L \text{ and } L = 0^\ast L \setminus \{0\overline{0}\}.$$ 

Definition 2. A set $X \subset \mathbb{N}$ is said to be 2-star-free if $\rho_2(X)$ (or equivalently $0^\ast \rho_2(X)$) is a regular aperiodic language.

It is well-known that the 2-recognizable sets are exactly those definable in the first-order structure $\langle \mathbb{N}, +, V_2 \rangle$ (see [3, Theorem 6.1] or [8, 17]) where $V_2(x)$ is the greatest power of 2 dividing $x$ (and $V_2(0)$ is 1). Thus $X \subset \mathbb{N}$ is said to be 2-definable if there exists a formula $\varphi$ of $\langle \mathbb{N}, +, V_2 \rangle$ such that

$$X = \{ n \in \mathbb{N} \mid \langle \mathbb{N}, +, V_2 \rangle \models \varphi(n) \}.$$ 

Instead of $V_2(x)$, we shall use the binary relation $\epsilon_2(x, y)$ defined by “$y$ is a power of 2 occurring in the normalized 2-representation of $x$”. As an example $(74, 8)$ belongs to $\epsilon_2$ because $\rho_2(74) = 1001010$ but $(74, 16)$ and $(74, 31)$ do not. Thus we can write

$$x = \sum_{\epsilon_2(x, y)} y.$$ 

Observe also that $(x, x)$ belongs to $\epsilon_2$ if and only if $x$ is a power of 2.

Remark 3. The structures $\langle \mathbb{N}, +, V_2 \rangle$ and $\langle \mathbb{N}, +, \epsilon_2 \rangle$ are equivalent (i.e., for any formula $\varphi(n)$ of $\langle \mathbb{N}, +, V_2 \rangle$ there exists a formula $\varphi'(n)$ of $\langle \mathbb{N}, +, \epsilon_2 \rangle$ such that $\{ n \in \mathbb{N} \mid \langle \mathbb{N}, +, V_2 \rangle \models \varphi(n) \} = \{ n \in \mathbb{N} \mid \langle \mathbb{N}, +, \epsilon_2 \rangle \models \varphi'(n) \}$ and conversely). Indeed, $\epsilon_2(x, y)$ is defined in $\langle \mathbb{N}, +, V_2 \rangle$ by

$$(V_2(y) = y) \land (\exists z)(\exists t)(x = t + y + z \land z < y \land (y < V_2(t) \lor t = 0))$$
and $V_2(x) = y$ is defined in $\langle \mathbb{N}, +, \epsilon_2 \rangle$ by

$$\epsilon_2(x, y) \land (\forall z)(\epsilon_2(x, z) \rightarrow y \leq z).$$

To be complete, notice that the binary relation $<$ is definable in the Presburger arithmetic $\langle \mathbb{N}, + \rangle$ by

$$x < y \equiv (\exists z)(\neg(z = 0) \land y = x + z).$$

For our purposes, we introduce a subset $\mathcal{L}_{2,n}$ of formulas $\varphi(n)$ in $\langle \mathbb{N}, +, \epsilon_2 \rangle$ defined as follows.
3.1. Syntax of logical formulas in \( \mathcal{L}_{2,n} \). The variables are ranging over \( \mathbb{N} \) and denoted \( b, n, x, y, z, \ldots \) (when specified, \( b \) and \( n \) have some special role). Roughly speaking \( n \) is dedicated to be the only free variable and \( b \) plays the role of an upper limit for all the bound variables occurring in a formula. The only terms are the variables. The atomic formulas are obtained with the following rules:

1. If \( x \) and \( y \) are variables (\( \neq b, n \)) then \( x < y \) and \( x = y \) are atomic formulas.
2. If \( x \) is a variable (\( \neq b, n \)) then \( e_2(n, x) \) is an atomic formula.

If \( \varphi \) is a formula whose \( x \) is a free variable (\( x \neq b, n \)) then

\[
(\exists x)^{<b}_2 \varphi \equiv (\exists x)(e_2(x, x) \land x < b \land \varphi)
\]

and

\[
(\forall x)^{<b}_2 \varphi \equiv (\forall x)(e_2(x, x) \land x < b \land \varphi)
\]

are formulas. To obtain formulas, we can also use the usual Boolean connectives \( \neg, \land, \lor, \to, \leftrightarrow \) either for formulas or atomic formulas. We are now able to define \( \mathcal{L}_{2,n} \). If \( \varphi \) is a formula in which the only free variables are (possibly) \( n \) and \( b \) then

\[
(\exists b)(e_2(b, b) \land \varphi)
\]

is a formula of \( \mathcal{L}_{2,n} \) having (possibly) a single free variable \( n \).

Example 4. The formula \( \varphi(n) \) given by

\[
\varphi(n) \equiv (\exists b)\{e_2(b, b) \land (\exists x)^{<b}_2[e_2(n, x) \land (\forall y)^{<b}_2(x < y \to e_2(n, y)) \land (\forall y)^{<b}_2(y < x \to \neg e_2(n, y))]\}
\]

belongs to \( \mathcal{L}_{2,n} \). We shall see that the set \( X = \{ n \mid \langle \mathbb{N},+,e_2 \rangle \models \varphi(n) \} \) is such that \( \rho_2(X) = 1^*0^* \). Thus \( \varphi(n) \) actually defines a 2-star-free set of integers. As another example, the set \( Y \) of the powers of 2 is 2-star-free because \( \rho_2(Y) = 1^* \) and it is also definable in \( \mathcal{L}_{2,n} \) by the formula

\[
\psi(n) \equiv (\exists b)[e_2(b, b) \land (\exists x)^{<b}_2(e_2(n, x) \land (\forall y)^{<b}_2(e_2(n, y) \to x = y))].
\]

With this definition of \( \mathcal{L}_{2,n} \), we obtain quite easily the following result.

Theorem 5. A set \( X \subseteq \mathbb{N} \) is 2-star-free (i.e., \( \rho_2(X) \) is regular aperiodic) if and only if \( X \) is definable by a first-order formula of \( \mathcal{L}_{2,n} \).

Proof. Let us first show that the condition is sufficient. Let \( X \subseteq \mathbb{N} \) be a set defined by a formula \( \psi \) of \( \mathcal{L}_{2,n} \). This formula is of the form

\[
\psi \equiv (\exists b)(e_2(b, b) \land \varphi)
\]

and we can assume that \( \psi \) has \( n \) as only free variable. (If \( \psi \) is a sentence, then \( X \) is equal to \( \mathbb{N} \) or \( \emptyset \) and the result is obvious.) Let us now proceed to some syntactical transformations. In \( \psi \), we keep only \( \varphi \) in which we replace each occurrence of \( e_2(n, x) \), \( (\forall x)^{<}_2 \) and \( (\exists x)^{<}_2 \) with respectively \( P_1(x') \), \( (\forall x') \) and \( (\exists x') \). The remaining variables \( x \) are naturally replaced with \( x' \). It is clear that the obtained formula \( \varphi' \) has no free variable and belongs to \( \mathcal{L}_{SF} \). Indeed, \( n \) appears in \( \varphi \) only through terms of the form \( e_2(n, x) \). (As an example, the reader can consider the formulas (2) and (3).) Therefore, \( \varphi' \) defines a star-free language \( L \) over \( \{0,1\} \). To conclude this part of the proof, we have to show that \( \rho_2(X) = L \). Let \( n \) be such that \( \langle \mathbb{N},+,e_2 \rangle \models \psi(n) \).
Assume that $\epsilon_2(n, x)$ appears in $\varphi$ with $x = 2^l$ for some $l < \log_2 b$ because $x$ is within the scope of a quantifier $(\forall x)_2^\leq$ or $(\exists x)_2^\leq$. It means that $\rho_2(n)$ has a $1$ in the $(l + 1)$th position (counting positions from right to left and beginning with $1$). In $\varphi'$ corresponding to $\epsilon_2(n, x)$, we have $P_1(x')$ which means that the model of a word — i.e., the representation of an integer — satisfying $\varphi'$ has a $1$ in position $x'$. Thus, we obtain the result when $x'$ is identified as $1 + \log_2 x$. The upper limit in $\psi$ given by $b$ and appearing in the quantifiers $(\forall x)_2^\leq$ and $(\exists x)_2^\leq$ is clearly understood in $\varphi'$ since in $\mathcal{L}_{SF}$ we consider words as finite models. It is the reason for removing the first part of $\psi$ to obtain $\varphi'$ and the constant $\max$ can be identified as $\log_2 b$.

Let us now assume that $X$ is a 2-star-free set. By McNaughton and Papert’s theorem, $\rho_2(X)$ is defined by a sentence $\varphi$ in $\mathcal{L}_{SF}$ where the bound variables are denoted $x, y, z, \ldots$ ($\neq n, b$). In $\varphi$, we replace $(\forall x), (\exists x)$ and $P_1(x)$ with respectively $(\forall x)_2^\leq$, $(\exists x)_2^\leq$ and $\epsilon_2(n, x)$ to obtain a formula $\psi'$. It is clear that $\psi \equiv (\exists b)(\epsilon_2(b, b) \land \psi')$ is a formula of $\mathcal{L}_{2,n}$ and has possibly $n$ as single free variable. To conclude the proof, it is clear that

$$X = \{ n \in \mathbb{N} | \langle \mathbb{N}, +, \epsilon_2 \rangle \models \psi(n) \}.$$  

One can view $b$ as $2^{\max}$ if $\max$ is a constant of the finite model associated to a word. 

**Example 6.** The set $10^*$ can be defined by the following formula of $\mathcal{L}_{SF}$

$$(\exists x)(P_1(x) \land (\forall y)(P_1(y) \rightarrow x = y)).$$

The reader can check that this formula corresponds exactly to the formula (3) in $\mathcal{L}_{2,n}$ if one proceeds to the transformations indicated in the second part of the proof.

**Remark 7.** From the logical characterization of the 2-star-free sets given in the previous theorem, other equivalent models can be obtained. In [4], it is shown how a finite automaton $M$ can be effectively derived from a formula $\varphi$ of $\langle \mathbb{N}, +, V_2 \rangle$ defining a 2-recognizable set $X$. Using classical results [4], it is also clear that the characteristic sequence of this $X$ is 2-automatic and the morphisms generating it can be derived from $M$ and thus from $\varphi$.

4. Generalization to linear numeration systems

For the sake of simplicity, we have up to now considered the binary numeration system but Theorem 3 can be extended to more general numeration systems.

**Definition 8.** A linear numeration system $U$ is a strictly increasing sequence $(U_n)_{n \in \mathbb{N}}$ of integers such that $U_0 = 1$, $U_{n+1}/U_n$ is bounded and satisfying for all $n \in \mathbb{N}$ a linear recurrence relation

$$U_{n+k} = c_{k-1}U_{n+k-1} + \cdots + c_0U_n, \quad c_i \in \mathbb{Z}, \quad c_0 \neq 0.$$  

By analogy to the binary system, the normalized representation of $x$ is denoted by $\rho_U(x)$ (with leading zeroes allowed) and $V_U(x)$ is the greatest
$U_n$ appearing in the greedy decomposition of $x$ with a non-zero coefficient
($V_U(0) = U_0 = 1$). A set $X \subseteq \mathbb{N}$ is $U$-recognizable if $\rho_U(X)$ is regular.

In the following, we shall only consider the class $\mathcal{U}$ of linear numeration
systems $(U_n)_{n \in \mathbb{N}}$ whose characteristic polynomial is the minimal polynomial
of a Pisot number. For instance, the $k$-ary system and the Fibonacci systems
belong to $\mathcal{U}$. The choice of the class $\mathcal{U}$ relies mainly upon the following
result. If $U = (U_n)_{n \in \mathbb{N}}$ is a numeration system belonging to $\mathcal{U}$ then the
$U$-recognizable sets are exactly those definable in $\langle \mathbb{N}, +, V_U \rangle$ (see [1, Theorem
16]). In fact, $\mathcal{U}$ is up to now the largest set of numeration systems having
well-known and useful properties such as the recognizability of addition.

Let $U = (U_n)_{n \in \mathbb{N}} \in \mathcal{U}$. Since sup $\frac{U_{n+1}}{U_n}$ is bounded, the alphabet of the
normalized representations is finite and is denoted $A_U = \{0, \ldots, c\}$. Naturally
words over $A_U$ will be interpreted as finite models $(M, <, P_1, \ldots, P_c, \text{max})$
and the star-free languages are exactly those defined by first-order sentences
in this formalism (the extension of $\mathcal{L}_{SF}$ defined in Section 2.1 is left to the
reader). As an example, if $w = 1230112$ then $P_1 = \{2, 3, 7\}$, $P_2 = \{1, 6\}$ and
$P_3 = \{5\}$.

Instead of $V_U(x)$, we shall use $c$ binary relations $\epsilon_{j,U}(x,y)$, $j = 1, \ldots, c$, meaning
that $y$ is an element of the sequence $(U_n)_{n \in \mathbb{N}}$ appearing in the
normalized decomposition of $x$ with a coefficient $j$. Thus

$$x = \sum_{j=1}^c \sum_{y \in U_n} j y.$$ 

**Remark 9.** The structures $\langle \mathbb{N}, +, V_U \rangle$ and $\langle \mathbb{N}, +, \epsilon_{1,U}, \ldots, \epsilon_{c,U} \rangle$ are equivalent, $\epsilon_{j,U}(x,y)$ is defined by

$$(V_U(y) = y) \land (\exists t)(\exists z)(x = t + jy + z \land z < y \land (y < V_U(t) \lor t = 0))$$

and $V_U(x) = y$ by

$$(\epsilon_{1,U}(x,y) \lor \cdots \lor \epsilon_{c,U}(x,y)) \land (\forall z)((\epsilon_{1,U}(x,z) \lor \cdots \lor \epsilon_{c,U}(x,z)) \rightarrow y \leq z).$$

By analogy to $\mathcal{L}_{2,n}$ introduced in the frame of the binary system, we can define a language $\mathcal{L}_{U,n}$ of formulas in $\langle \mathbb{N}, +, \epsilon_{1,U}, \ldots, \epsilon_{c,U} \rangle$ having possibly
a single free variable $n$. For instance,

$$(\exists x)(\exists \varphi \equiv (\exists x)(\epsilon_{1,U}(x,x) \land x < b \land \varphi)).$$

The reader could easily make up the complete definition of $\mathcal{L}_{U,n}$.

Let us just introduce two notations, if $\varphi$ is any formula of $\mathcal{L}_{U,n}$, we shall
denote by $\mathfrak{P}(\varphi)$ the main part of the formula (i.e., the largest sub-formula
in which $b$ is still free), namely the formula is necessarily of the form

$$\varphi \equiv (\exists b)(\epsilon_{1,U}(b,b) \land \mathfrak{P}(\varphi)).$$

If $\rho_U(\mathbb{N})$ is aperiodic then it is definable by a sentence $\chi$ of $\mathcal{L}_{SF}$. In $\chi$, we
replace $P_3(x)$, $(\forall x)$ and $(\exists x)$ with $\epsilon_{j,U}(n,x)$, $(\forall x)^P_U$ and
$(\exists x)^P_U$ respectively to obtain a formula $\chi_N$ being the main part of a formula in $\mathcal{L}_{U,n}$.

**Theorem 10.** Let $U$ be a numeration system in $\mathcal{U}$. If $N_U = 0^*\rho_U(\mathbb{N})$ is
aperiodic then a set $X \subseteq \mathbb{N}$ is $U$-star-free (i.e., $\rho_U(X)$ is regular aperiodic)
if and only if $X$ is definable by a first-order formula of $\mathcal{L}_{U,n}$ of the form

$$(\exists b)(\epsilon_{1,U}(b,b) \land \mathfrak{P}(\varphi) \land \chi_N).$$
where $\varphi$ is a first-order formula of $L_{U,n}$.

**Proof.** The only differences with the proof of Theorem 5 appear when we show that the condition is sufficient. Roughly speaking, we should have to be careful for the choice of a formula $\varphi$ in $L_{U,n}$ because we want to obtain a corresponding formula in $L_{SF}$ valid only for normalized representations interpreted as finite models. To avoid this problem we use the formula $X_N$.

Let $\varphi$ be any formula of $L_{U,n}$. It is necessarily of the form

$$(\exists b)(\epsilon_1U(b,b) \land P(\varphi)).$$

Assuming that $X_N$ and $P(\varphi)$ have different variables except for $n$ and $b$ then

$$\psi \equiv (\exists b)(\epsilon_1U(b,b) \land P(\varphi) \land X_N)$$

is again a formula of $L_{U,n}$. Adding the part $X_N$ in such a formula $\psi$ ensures that if we transform $\psi$ into a sentence $\psi'$ of $L_{SF}$ (following the scheme given in the proof of Theorem 5) then the words satisfying $\psi'$ are all normalized $U$-representations and the corresponding language is aperiodic.

**Remark 11.** Notice that for the $k$-ary system, the set of all the normalized $k$-representations (allowing leading zeroes) is aperiodic

$$0^*\rho_k(N) = \{0, \ldots, k-1\}^* = \emptyset$$

and any word of $\Sigma_k$ is a valid normalized $k$-representation. So in this special case, we do not need a formula $X_N$. To be precise, $X_N$ is a tautology. In particular, this explains the simpler form of Theorem 5 which holds for any numeration system with an integer base $k$.

**Example 12.** Let us consider the Fibonacci system given by $U_0 = 1$, $U_1 = 2$ and $U_{n+2} = U_{n+1} + U_n$. As a consequence of the greedy algorithm,

$$N_U = \overline{\{011\}}$$

is aperiodic and defined by the following sentence of $L_{SF}$

$$\mathcal{X} \equiv (\forall x)(\forall y)[(\exists z)(x < z < y) \lor \neg(P1(x) \land P1(y))$$

corresponding to

$$X_N \equiv (\forall x)(\forall y)[(\exists z)(x < z < y) \lor \neg(\epsilon_1U(x) \land \epsilon_1U(y))].$$

So any formula $\varphi$ of $L_{U,n}$ gives a new formula

$$(\exists b)(\epsilon_1U(b,b) \land P(\varphi) \land X_N)$$

defining a $U$-star-free subset of $\mathbb{N}$ (which could be finite or empty depending on the compatibility of the conditions given by $P(\varphi)$ and $X_N$).

Continuing this example, we show that the set of even integers is not $U$-star-free although it is easily definable in the Presburger arithmetic by

$$\varphi(n) \equiv (\exists x)(n = x + x).$$

Indeed, $U_n$ is even if and only if $n \equiv 1 \pmod{3}$ and therefore, for any $n$, two but not the three words $1(01)^n$, $1(01)^{n+1}$ and $1(01)^{n+2}$ are in the language $\rho_U(N)$. So the set of even integers is not definable in $L_{U,n}$. 

5. Base dependence

In this section, we consider once again integer base numeration systems and study the base dependence of the star-free property. We show that the sets of integers are classified into four categories.

The proof of the first result in this section does not use the previous logical characterization of the $p$-star-free sets but relies mainly on automata theory arguments.

**Proposition 13.** Let $p, k \geq 2$. A set $X \subseteq \mathbb{N}$ is $p^k$-star-free if and only if it is $p^k$-star-free.

**Proof.** Let us first show that if $X \subseteq \mathbb{N}$ is $p^k$-star-free then $X$ is $p$-star-free. Assume that $\rho_{p^k}(X)$ is obtained by an extended regular expression over the alphabet $\Sigma_{p^k} = \{0, \ldots, p^k - 1\}$ without star operation. In this expression, one can replace each occurrence of a letter $j \in \Sigma_{p^k}$ with the word $0^{k-l}\rho_p(j)$ ($l = |\rho_p(j)|$) of length $k$. Since we only use concatenation product, the resulting expression defining the language $L \subset \{0, \ldots, p-1\}^*$ is still star-free and it is clear that $\rho_p(X) = L$.

**Example 14.** The set $X = \{3.4^n \mid n \in \mathbb{N}\}$ is 4-star-free, $\rho_4(X) = 30^* = \overline{\{3\} \emptyset \{1, 2, 3\} \emptyset}$.

The set $X$ is also 2-star-free, we simply have to replace, 0, 1, 2 and 3 with respectively 00, 01, 10 and 11 and $\rho_2(X) = 11(00)^* = \overline{\{11\} \emptyset \{01, 10, 11\} \emptyset}$.

Before continuing the proof, we recall another characterization of the star-free languages given by McNaughton and Papert.

**Definition 15.** A deterministic finite automaton is permutation free if there is no word that makes a nontrivial permutation (i.e., not the identity permutation) of any subset of the set of states. In the same way, a language is said to be permutation free if its minimal automaton is permutation free.

**Example 16.** The automaton depicted in Figure 1 is not permutation free. Indeed, the word 01 makes a non trivial permutation of the set $\{p, r\}$.

![Figure 1. A non permutation free automaton.](image)

**Theorem 17.** [12, Theorem 5.1] A language is star-free if and only if it is permutation free.
Let us also recall a well-known result from automata theory.

**Proposition 18.** [5, Section III.5] Let \( L \subset \Sigma^* \) be a regular language having \( M_L = (Q_L, q_0, F_L, \Sigma, \delta_L) \) as minimal automaton. If \( M = (Q, q_0, F, \Sigma, \delta) \) is an accessible deterministic automaton recognizing \( L \) then there exists an application \( \Phi : Q \to Q_L \) such that \( \Phi \) is onto and for each \( q \in Q \) and \( w \in \Sigma^* \),

\[
\Phi(\delta(q, w)) = \delta_L(\Phi(q), w).
\]

Assume now that \( X \subseteq \mathbb{N} \) is \( p \)-star-free. Using Remark 1 and Theorem 17, \( 0^* \rho_p(X) \) is a permutation free language and we denote by \( M = (Q, q_0, F, \Sigma_p, \delta) \) its minimal automaton. From \( M \), we build a new automaton \( M' = (Q, q_0, F, \Sigma_p^k, \delta') \) having the same set of states. The transition function \( \delta' \) of \( M' \) is defined as follows. For each \( j \in \Sigma_p^k \), \( p, q \in Q \), let \( w = 0^{k-l} \rho_p(j) \) where \( l = |\rho_p(j)| \), then \( \delta'(p, j) = q \) if and only if \( \delta(p, w) = q \).

**Example 19.** Let \( \Sigma_2 = \{0, 1\} \) and consider the automaton \( M \) depicted in Figure 2. If we consider the 4-ary numeration system, we build a new automaton \( M' \) depicted in Figure 3 by considering in \( M \) the paths of label 00, 01, 10 and 11 replaced respectively by 0, 1, 2 and 3.

![Figure 2. An automaton \( M \) over \( \Sigma_2 \).](image)

![Figure 3. The corresponding automaton \( M' \) over \( \Sigma_4 \).](image)
Let $\mathcal{M}'' = (Q'', q_0'', F'', \Sigma_{p^k}, \delta'')$ be the minimal automaton of $0^*\rho_{p^k}(X)$. Thanks to Proposition 18 we have an onto application $\Phi$ between the automata $\mathcal{M}'$ and $\mathcal{M}''$ namely between the sets $Q$ and $Q''$. To conclude the proof by applying Theorem 17, we have to show that $\mathcal{M}''$ is permutation free. Assume that there exists a subset $T$ of $Q''$ and a word $w \in \Sigma_{p^k}$ such that $w$ makes a nontrivial permutation of $T$. So there exists a state $q \in T$ such that $\delta''(q, w) \in T$ and $\delta''(q, w) \neq q$. Therefore $\mathcal{M}'$ is not permutation free, $w$ makes a nontrivial permutation of $\Phi^{-1}(T) \subseteq Q$. Indeed, there exist $r \in \Phi^{-1}(q) \subseteq \Phi^{-1}(T)$ and $s \in \Phi^{-1}(\delta''(q, w)) \subseteq \Phi^{-1}(T)$ such that $r \neq s$ and $\delta'(r, w) = s$. This is a contradiction because we have shown previously that $\mathcal{M}'$ is permutation free.

It is well-known that any finite union of arithmetic progressions is $p$-star-free for some $p$ (see [10, Theorem 1.4]). So a natural question is to determine if an arithmetic progression $r + s \mathbb{N} = \{r + sn \mid n \in \mathbb{N}\}$, $s > 1$, is $p$-star-free for any $p \geq 2$ or only for some specific bases $p$.

**Example 20.** The set of even integers is 2-star-free and therefore $2^n$-star-free for each $n$. But this set is not 3-star-free, indeed $\rho_3(2\mathbb{N})$ is the set of words over $\{0, 1, 2\}$ having an even number of 1 (and therefore the minimal automaton of this language is not counter-free, which is another way to say that the language is not permutation free). Notice also that 2 and 10 are multiplicatively independent but 2$\mathbb{N}$ is 10-star-free. Actually, it is easy to see that the set of even integers is $(2p)$-star-free, for any $p$. So with this example, we see that we obtain a slightly different phenomenon that the one encountered in Cobham’s theorem.

A finite union of arithmetic progressions being ultimately periodic, we can always write it as $\bigcup_{j=1}^n (r_j + s \mathbb{N}) \cup F$ where $F$ is a finite set and $s$ is the l.c.m. of the periods of the different progressions. Since aperiodicity is preserved up to a finite modification of a language, we can forget the finite set $F$ and assume that $r_j < s$. Union of aperiodic sets being again aperiodic, we shall consider a single set $r + s \mathbb{N}$.

**Proposition 21.** The set $r + s \mathbb{N}$, (with $r < s$ and $s > 1$) is $(is)$-star-free for any integer $i > 0$.

**Proof.** The reader can easily check that the language made up of the $(is)$-ary expansions of the elements in $r + s \mathbb{N}$ is

$$\Sigma^*_i \{r, r + s, \ldots, r + (i - 1)s\}$$

which is a definite language.

We even have a better situation.

**Proposition 22.** Let $r + s \mathbb{N}$ be such that $r < s$, $s > 1$ and the factorization of $s$ as a product of primes is of the form

$$s = p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \quad \alpha_i > 0$$

If $P = \Pi_{j=1}^k p_j$ then $r + s \mathbb{N}$ is $(iP)$-star-free for any integer $i > 0$.

---

1 A language $L \subset \Sigma^*$ is said to be definite if there exist finite languages $M$ and $N$ such that $L = N \cup \Sigma^*M$. So to test the membership of a word in $L$, we only have to look at its last letters.
Proof. Let \( \alpha = \sup_{j=1, \ldots, k} \alpha_j \). By definition of \( P \), it is clear that \((iP)^{\alpha+n}\) is a multiple of \( s \) for all integers \( n \geq 0 \) and \( i > 0 \). So in the \((iP)\)-ary expansion of an integer the digits corresponding to those powers of \( iP \) provide the decomposition with multiples of \( s \). To obtain an element of \( r + sN \), we thus have to focus on the last \( \alpha \) digits corresponding to the powers 1, \( iP \), \ldots, \((iP)^{\alpha-1}\) of weakest weight. Consider the finite set

\[
Y = \{ r + ns \mid n \in \mathbb{N} \text{ and } r + ns < (iP)^\alpha \}.
\]

For each \( y_j \in Y \), \( j = 1, \ldots, t \), consider the word \( \rho_{iP}(y_j) \) preceded by some zeroes to obtain a word \( y'_j \in \Sigma_{iP}^* \) of length \( \alpha \). To conclude the proof, observe that the language made up of the \((iP)\)-ary expansions of the elements in \( r + sN \) is

\[
\Sigma_{iP}^* \{ y'_1, \ldots, y'_t \}
\]

and is a definite language. \( \square \)

Remark 23. The situation of Proposition 22 cannot be improved. Indeed with the previous notations, consider an integer \( Q \) which is a product of some but not all the prime factors appearing in \( s \). For the sake of simplicity, assume that

\[
Q = p_2^{\beta_2} \cdots p_k^{\beta_k}, \quad 1 \leq \beta_j \leq \alpha_j.
\]

For each \( n \), \( Q^n \not\equiv 0 \pmod{s} \). Indeed, if \( Q^n = is \) then

\[
p_2^{n\beta_2} \cdots p_k^{n\beta_k} = ip_1^{\alpha_1} \cdots p_k^{\alpha_k}
\]

which is a contradiction since \( p_1 \) does not appear in the left hand side factorization. Moreover, it is clear that the sequence \((Q_n \mod s)_{n \in \mathbb{N}}\) is ultimately periodic. Therefore \( \rho_Q(r + sN) \) is regular but not star-free because, due to this periodicity, the corresponding automaton is not counter-free. As an example, one can check that \( 6N \) is neither 2-star-free nor 3-star-free.

To summarize the situation, the sets of integers can be classified into four categories:

1. The finite and cofinite sets are \( p \)-star-free for any \( p > 1 \).
2. The ultimately periodic sets of period \( s = p_1^{\alpha_1} \cdots p_k^{\alpha_k} > 1 \) are \((iP)\)-star-free for \( P = \prod_{j=1}^k p_j \) and any \( i > 0 \). In particular, these sets are \( P^m \)-star-free for \( m \geq 1 \).
3. Thanks to Cobham’s theorem, if a \( p \)-recognizable set \( X \) is not a finite union of arithmetic progressions then \( X \) is only \( p^k \)-recognizable for \( k \geq 1 \) (\( p \) being simple). So if a \( p \)-star-free set \( X \) is not ultimately periodic then \( X \) is only \( p^k \)-star-free for \( k \geq 1 \) (\( p \) being simple).
4. Finally, there are sets which are not \( p \)-star-free for any \( p > 1 \).

Being multiplicatively dependent is an equivalence relation over \( \mathbb{N} \), the smallest element in an equivalence class is said to be simple. For instance, 2, 3, 5, 6, 7, 10, 11 are simple.
6. P-ADIC NUMBER SYSTEMS

The p-ary numeration system is built upon the sequence \( U_n = p^n \) and the representation of an integer is a word over the alphabet \( \{0, \ldots, p-1\} \) computed through the greedy algorithm. On the other hand, the p-adic numeration system is built upon the same sequence but representations are written over the alphabet \( \{1, \ldots, p\} \). It can be shown that each integer has a unique p-adic representation (see [13] for an exposition on p-adic number systems). For instance, the use of p-adic system may be relevant to remove the ambiguity due to the presence of leading zeroes in a p-ary representation. Indeed, 0 is not a valid digit in a p-adic representation (see for instance [9, p. 303] for a relation to L systems).

In this small section, we show that the p-star-free sets are exactly the sets of integers whose p-adic representations made up a star-free language.

Capital Greek letters will represent finite alphabets.

If \( \Delta \subset \mathbb{Z} \) is a finite alphabet and \( w = w_n \cdots w_0 \) is a finite word over \( \Delta \), we denote by \( \pi_p(w) \) the numerical value of \( w \),

\[
\pi_p(w) = \sum_{i=0}^{n} w_i p^i.
\]

For instance, 1001 and 121 are respectively the 2-ary and 2-adic representations of 9,

\[
\pi_2(1001) = \pi_2(121) = 9.
\]

Let \( w \in \Delta^* \) be such that \( \pi_p(w) \in \mathbb{N} \). The partial function \( \nu_p : \Delta^* \to \{0, \ldots, p-1\}^* \) mapping \( w \) onto \( \rho_p(\pi_p(w)) \) is called the normalization function. Thanks to a result of C. Frougny, the graph of this function is regular whatever the alphabet \( \Delta \) is [7]. For the case we are interested in, the language

\[
\hat{\nu}_p^R = \{(u, v) \mid u \in \{1, \ldots, p\}^*0^*, v \in \Sigma^*, |u| = |v|, \pi_p(u^R) = \pi_p(v^R)\}
\]

is the reversal of the graph of the normalization function mapping the p-adic representation of an integer onto its p-ary representation. The trim minimal automaton (the sink has not been represented) of \( \hat{\nu}_p^R \) is given in Figure 4 and is clearly permutation free. So thanks to Theorem [7], \( \hat{\nu}_p^R \) is star-free.

![Figure 4. From p-adic to p-ary representation.](image)

**Lemma 24.** A language \( L \subseteq \Sigma^* \) is star-free if and only if \( L^R \) is star-free.
Proof. Let $u, v, w \in \Sigma^*$. Assume that $L$ is aperiodic, for $n$ large enough

\begin{align*}
  u v^n w \in L^R & \iff w^R(v^R)^n u^R \in L \\
  & \iff w^R(v^R)^{n+1} u^R \in L \\
  & \iff u v^{n+1} w \in L^R.
\end{align*}

Thus $\nu_p$ is a star-free language.

We denote by $p_1$ and $p_2$ the canonical homomorphisms of projection, if $A$ and $B$ are sets, $p_1 : A \times B \rightarrow A : (a, b) \mapsto a$ and $p_2 : A \times B \rightarrow B : (a, b) \mapsto b$.

Lemma 25. Let $L \subseteq \Sigma^*$ be a star-free language. Then the language $L \oplus \Gamma^* = \{(x, y) \mid x \in L, y \in \Gamma^*, |x| = |y|\}$ is also star-free.

Proof. Let $u, v, w \in (\Sigma \times \Gamma)^*$. Since $L$ is star-free and $p_1$ and $p_2$ are letter-to-letter (length preserving homomorphisms), we have

\begin{align*}
  u v^n w \in L \oplus \Gamma^* & \iff p_1(u v^n w) \in L \land p_2(u v^n w) \in \Gamma^* \\
  & \iff p_1(u) p_1(v)^n p_1(w) \in L \land p_2(u) p_2(v)^n p_2(w) \in \Gamma^* \\
  & \iff p_1(u) p_1(v)^{n+1} p_1(w) \in L \land p_2(u) p_2(v)^{n+1} p_2(w) \in \Gamma^* \\
  & \iff p_1(u v^{n+1} w) \in L \land p_2(u v^{n+1} w) \in \Gamma^* \\
  & \iff u v^{n+1} w \in L \oplus \Gamma^*
\end{align*}

Naturally, we can also define the language $\Gamma^* \oplus L$ in a similar way.

Generally, the homomorphic image of a star-free language is not star-free [12, p. 12] but the following weaker result holds.

Lemma 26. If a language $L \subseteq (\Sigma \times \Gamma)^*$ of couples of words of the same length is star-free then $p_1(L) \subseteq \Sigma^*$ and $p_2(L) \subseteq \Gamma^*$ are also star-free.

Proof. One can apply the same reasoning as the one given in the proof of the previous lemma.

We are now able to prove the main result of this section (notice that a small mention to $p$-adic systems appears also in [11]).

Proposition 27. A set $X \subseteq \mathbb{N}$ is $p$-star-free if and only if the language made up of the $p$-adic representations of the elements in $X$ is star-free.

Proof. If $\rho_p(X)$ is a star-free language then thanks to Lemma [25],

\[(0, \ldots, p)^* \oplus \rho_p(X)\]

is star-free. Thanks to Lemma [24] and since the family of aperiodic languages is closed under boolean operations, the language

\[L = [(0, \ldots, p)^* \oplus \rho_p(X)] \cap \nu_p\]

is again star-free. To conclude the first part of the proof, we apply Lemma [23], $p_1(L)$ is star-free and it is clear that this language is exactly made up of the $p$-adic representations of the elements in $X$. 
Conversely, let $M \subset \{1, \ldots, p\}^*$ be such that $\pi_p(M) = X$. If $M$ is star-free then thanks to Remark 1 and using the previous lemmas,

$$p_2[(0^*M \oplus \{0, \ldots, p-1\}^*) \cap \hat{\nu}_p]$$

is star-free.

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