Very Well-Covered Graphs of Girth at least Four and Local Maximum Stable Set Greedoids*

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Abstract

A maximum stable set in a graph $G$ is a stable set of maximum cardinality. $S$ is a local maximum stable set of $G$, and we write $S \in \Psi(G)$, if $S$ is a maximum stable set of the subgraph induced by $S \cup N(S)$, where $N(S)$ is the neighborhood of $S$.

Nemhauser and Trotter Jr. [20], proved that any $S \in \Psi(G)$ is a subset of a maximum stable set of $G$. In [12] we have shown that the family $\Psi(T)$ of a forest $T$ forms a greedoid on its vertex set. The cases where $G$ is bipartite, triangle-free, well-covered, while $\Psi(G)$ is a greedoid, were analyzed in [14], [15], [17], respectively.

In this paper we demonstrate that if $G$ is a very well-covered graph of girth $\geq 4$, then the family $\Psi(G)$ is a greedoid if and only if $G$ has a unique perfect matching.

Keywords: very well-covered graph, local maximum stable set, greedoid, triangle-free graph, König-Egerváry graph.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. $K_n, C_n$ denote respectively, the complete graph on $n \geq 1$ vertices and the chordless cycle on $n \geq 3$ vertices. If $A, B \subset V$ and $A \cap B = \emptyset$, then $(A, B)$ stands for the set \{e = ab : a \in A, b \in B, e \in E\}.

The neighborhood of a vertex $v \in V$ is the set $N(v) = \{u : u \in V \text{ and } vu \in E\}$. For $A \subset V$, we denote $N(A) = \{v \in V - A : N(v) \cap A \neq \emptyset\}$ and $N[A] = A \cup N(A)$.

A stable set in $G$ is a set of pairwise non-adjacent vertices. A stable set of maximum size will be referred to as a maximum stable set of $G$, and the stability number of $G$, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in $G$. Let $\Omega(G)$ stand for the set of all maximum stable sets of $G$.

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A matching in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ such that no two edges of $M$ share a common vertex. A maximum matching is a matching of maximum cardinality. By $\mu(G)$ is denoted the cardinality of a maximum matching. A matching is perfect if it saturates all the vertices of the graph.

Let us recall that $G$ is a König-Egerváry graph provided $\alpha(G) + \mu(G) = |V(G)|$ [4], [23]. As a well-known example, any bipartite graph is a König-Egerváry graph [5], [10].

**Theorem 1.1** [13] If $G$ is a König-Egerváry graph, then every maximum matching is contained in $(S, V(G) - S)$, for each $S \in \Omega(G)$.

A matching $M = \{a_i b_i : a_i, b_i \in V(G), 1 \leq i \leq k\}$ of graph $G$ is called a uniquely restricted matching if $M$ is the unique perfect matching of $G[\{a_i, b_i : 1 \leq i \leq k\}]$ [8]. For instance, all the maximum matchings of the graph $G$ in Figure 1 are uniquely restricted, while the graph $H$ from the same figure has both uniquely restricted maximum matchings (e.g., $\{uv, xw\}$) and non-uniquely restricted maximum matchings (e.g., $\{xy, tv\}$).

![Figure 1: The unique cycle of $H$ is alternating with respect to the matching $\{yv, tx\}$](image)

Recall that $G$ is well-covered if all its maximal stable sets have the same cardinality [21], and $G$ is very well-covered if, in addition, it has no isolated vertices and $|V(G)| = 2\alpha(G)$ [6].

![Figure 2: Only $C_4$ and $G_1$ are very well-covered graphs.](image)

It is easy to prove that every graph having a perfect matching consisting of pendant edges is very well-covered. The converse is not generally true; e.g., the graphs $C_4$ and $G_1$ depicted in Figure 2. Moreover, there are well-covered graphs without perfect matchings; e.g., $K_3$. Nevertheless, having a perfect matching is a necessary condition for very well-coveredness.

**Theorem 1.2** [6] For a graph $G$ without isolated vertices the following are equivalent:

(i) $G$ is very well-covered;
(ii) there exists a perfect matching in $G$ that satisfies property $P$;
(iii) there exists at least one perfect matching in $G$ and every perfect matching in $G$ satisfies property $P$.

A matching $M$ in a graph $G$ satisfies Property $P$ if

"$N(x) \cap N(y) = \emptyset$, and each $v \in N(x) - \{y\}$ is adjacent to all vertices of $N(y) - \{x\}$" hold for every edge $xy \in M$. 

2
For example, the perfect matching \( M = \{ab, xy, uv\} \) of the graph \( G_2 \) from Figure 2 does not satisfy Property \( P \), since \( uv \in M, b \in N(u), y \in N(v) \), but \( b \notin E(G_2) \). Hence, \( G_2 \) is not a very well-covered graph. Moreover, \( G_2 \) is not well-covered, because no maximum stable set of \( G_2 \) includes the stable set \( \{b, v\} \). However, \( G_2 \) is a König-Egerváry graph. Notice that \( K_4 \) is well-covered, has perfect matchings, but is neither a König-Egerváry graph, nor a very well-covered graph.

**Theorem 1.3** [11] A graph is very well-covered if and only if it is a well-covered König-Egerváry graph.

A set \( A \subseteq V(G) \) is a local maximum stable set of \( G \) if \( A \in \Omega(G[N[A]]) \) [12]; by \( \Psi(G) \) we denote the family of all local maximum stable sets of the graph \( G \). For instance, \( \{a\}, \{a, e\} \in \Psi(G) \), while \( \{c\}, \{b, f\} \notin \Psi(G) \), where \( G \) is from Figure 1. Notice also that in the same graph, the stable sets \( \{a, e\}, \{b, f\} \) are contained in some maximum stable sets of \( G \), while for \( \{a, c\}, \{c, e\} \) this is not true.

**Theorem 1.4** [20] Every local maximum stable set of a graph is a subset of a maximum stable set.

**Definition 1.5** [1], [9] A greedoid is a pair \((V, F)\), where \( F \subseteq 2^V \) is a non-empty set system satisfying the following conditions:

- **Accessibility:** for every non-empty \( X \in F \) there is an \( x \in X \) such that \( X - \{x\} \in F \);
- **Exchange:** for \( X, Y \in F \), \( |X| = |Y| + 1 \), there is an \( x \in X - Y \) such that \( Y \cup \{x\} \in F \).

In the sequel we use \( F \) instead of \((V, F)\), as the ground set \( V \) will be, usually, the vertex set of some graph.

![Figure 3: Ψ(G) is not a greedoid, Ψ(H) is a greedoid.](image)

The graphs from Figure 3 are non-bipartite König-Egerváry graphs, and all their maximum matchings are uniquely restricted. Let us remark that both graphs are also triangle-free, but only \( \Psi(H) \) is a greedoid. It is clear that \( \{b, c\} \in \Psi(G) \), while \( \{b\}, \{c\} \notin \Psi(G) \). Notice also that \( G[N[\{b, c\}]] \) is not a König-Egerváry graph, and, as one can see from the following theorem, this is a good reason for \( \Psi(G) \) not to be a greedoid.

**Theorem 1.6** [15] If \( G \) is a triangle-free graph, then the following assertions are equivalent:

(i) \( \Psi(G) \) is a greedoid;

(ii) all maximum matchings of \( G \) are uniquely restricted and the closed neighborhood of every local maximum stable set of \( G \) induces a König-Egerváry graph.

The cases of trees, bipartite graphs, unicycle graphs, whose family of local maximum stable sets forms a greedoid, were analyzed in [12], [14], [18], respectively.

In this paper we characterize very well-covered graphs of girth at least four, whose families of local maximum stable sets are greedoids.
2 Results

Let us remark that the very well-covered graph $G_1$ in Figure 2 has a $C_4$ and one of the edges of this $C_4$ belongs to the unique perfect matching of $G_1$.

**Lemma 2.1** No edge of some $C_q$, for $q = 3$ or $q \geq 5$, belongs to a perfect matching in a very well-covered graph.

**Proof.** If the graph $G$ is very well-covered, then by Theorem 1.2, $G$ has a perfect matching, say $M$, and each perfect matching satisfies Property $P$.

Let $xy \in M$. Then, Property $P$ implies that $N(x) \cap N(y) = \emptyset$, i.e., $xy$ belongs to no $C_3$ in $G$. Further, if $v \in N(x) - \{y\}$ and $u \in N(y) - \{x\}$, Property $P$ assures that $vu \in E(G)$, i.e., $xy$ belongs to no $C_q$, for $q \geq 5$. 

![Figure 4: Both $H_1 = G[N[\{x,y\}]]$ and $H_2 = G[N[\{y,z\}]]$ are König-Egerváry graphs.](image)

Let us mention that if $G$ is very well-covered, $S$ is a stable set such that $G[N[S]]$ is a König-Egerváry graph, then $S$ does not necessarily belong to $\Psi(G)$; e.g., the set $S_1 = \{x,y\}$ is stable in the graph $G$ depicted in Figure 4, and $S_1 \notin \Psi(G)$, while $H_1 = G[N[S_1]]$ is a König-Egerváry graph. Notice that $S_2 = \{y,z\} \in \Psi(G)$ and $H_2 = G[N[S_2]]$ is a König-Egerváry graph. The following finding, firstly presented in [15], shows that this phenomenon is true for very well-covered graphs in general. We repeat the proof for the sake of self-containment.

**Theorem 2.2** If $G$ is a very well-covered graph, then $G[N[S]]$ is a König-Egerváry graph, for every $S \in \Psi(G)$.

**Proof.** By Theorem 1.3, $G$ is a König-Egerváry graph. According to Theorem 1.2, $G$ has a perfect matching, say $M$, and each perfect matching satisfies Property $P$.

Suppose by way of contradiction that there is $S = \{v_i : 1 \leq i \leq k\} \in \Psi(G)$, such that $G[N[S]]$ is not a König-Egerváry graph.

Since $G$ is well-covered, there exists some $W \in \Omega(G)$, with $S \subseteq W$. By Theorem 1.1, $M \subseteq (W, V(G) - W)$, and because $M$ is a perfect matching and $S \subseteq W$, we infer that $S$ is matched by $M$ into $N(S)$, and this implies $|S| \leq |N(S)|$. The assumption that $G[N[S]]$ is not a König-Egerváry graph leads to $|N(S)| > |S|$. It means that there exists a vertex $x \in N(S) - M(S)$, where $M(S) = \{w_i : v_i w_i \in M, 1 \leq i \leq k\}$.

In the following, we will prove that the set $\{x\} \cup M(S)$ is stable.

Firstly, $x$ must be adjacent to some vertex, say $v_1$, from $S$, otherwise $S \cup \{x\}$ is a stable set larger than $S$ in $G[N[S]]$, in contradiction with $S \in \Psi(G)$. By Lemma 2.1, $x$ is not adjacent to $w_1$, since $v_1 w_1 \in M$. Thus, $\{x, w_1\}$ is a stable set.

One of $x, w_1$ must be adjacent to one vertex, say $v_2$, from $S$, because, otherwise, the set $\{x, w_1\} \cup \{v_i : 2 \leq i \leq k\}$ would be stable in $G[N[S]]$, larger than $S$. If $w_1 v_2 \in E(G)$, then Property $P$, applied to the edge $v_1 w_1 \in M$, ensures that $xw_2 \in E(G)$.

In other words, $x$ must be adjacent to $v_1$. Moreover, the set $\{x, w_1, w_2\}$ is stable, because $xw_2 \notin E(G)$ according to Lemma 2.1 while for $w_1 w_2 \in E(G)$ we get, by Property $P$, that $xw_1 \in E(G)$, in contradiction with the fact that $\{x, w_1\}$ is a stable set.
Assume that for some \( j < k \), the set

\[
A_j = \{x\} \cup \{w_i : 1 \leq i \leq j\}
\]

is stable, and \( x \) is adjacent to each \( v_i, 1 \leq i \leq j \). Then, there is an edge joining a vertex, say \( a \), belonging to \( A \), and a vertex, say \( v_{j+1} \), from the set \( \{v_i : j+1 \leq i \leq k\} \). Otherwise,

\[
A_j \cup \{v_i : j+1 \leq i \leq k\}
\]

is a stable set in \( G[N[S]] \), larger than \( S \). If \( a = w_t \), then by Property \( P \), when the edge \( v_tw_t \) is concerned, the vertex \( x \) must be adjacent to \( v_{j+1} \). Thus, no matter where \( a \) is located, the vertex \( x \) is adjacent to the vertex \( v_{j+1} \) (see Figure 5(a)).

Figure 5: (a) The vertex \( x \) is adjacent to all vertices from \( \{v_i : 1 \leq i \leq j\} \). (b) The vertices \( x, v_{j+1}, w_{j+1}, w_t, v_t \) span a five vertex cycle.

Since \( xv_{j+1} \in E(G) \) and \( v_{j+1}w_{j+1} \in M \), Lemma 2.1 implies that the vertices \( x \) and \( w_{j+1} \) are not adjacent. Moreover, no vertex from the set \( \{w_i : 1 \leq i \leq j\} \) is adjacent to \( w_{j+1} \). Otherwise, if some \( w_t \) is adjacent to \( w_{j+1} \), then \( \{x, v_{j+1}, w_{j+1}, w_t, v_t\} \) spans a five vertex cycle in \( G[N[S]] \) (see Figure 5(b)). In accordance with Property \( P \), when the edge \( v_tw_t \) is concerned, the vertex \( x \) must be adjacent to \( w_{j+1} \). Hence, \( \{x, v_{j+1}, w_{j+1}\} \) spans a triangle, which is impossible, by Lemma 2.1.

Therefore, the set \( A_{j+1} \) is stable. In this way one can eventually reach the set \( \{x\} \cup M(S) \), which must be stable in \( G[N[S]] \) like all its predecessors. Now the inequality

\[
|\{x\} \cup M(S)| > |S|
\]

stays in contradiction with the following facts:

\[
\{x\} \cup M(S) \subseteq N[S] \quad \text{and} \quad S \in \Psi(G).
\]

Consequently, \( G[N[S]] \) is a König-Egerváry graph.

Theorem 2.3 Let \( G \) be a very well-covered graph of girth at least 4. Then the following assertions are equivalent:

(i) \( \Psi(G) \) is a greedoid;
(ii) \( G \) has a unique maximum matching.

Proof. Firstly, Theorem 1.2 implies that each maximum matching of \( G \) is perfect.

(i) \( \Rightarrow (ii) \) Since the girth of \( G \) is greater or equal to 4, the graph \( G \) is triangle-free. Hence, according to Theorem 1.6, a perfect matching of \( G \) is unique.

(ii) \( \Rightarrow (i) \) In fact, \( G \) has a unique perfect matching. Consequently, every maximum matching of \( G \) is uniquely restricted. Combining the fact that \( G \) is triangle-free with Theorems 2.2 and 1.0, we conclude that \( \Psi(G) \) is a greedoid.

The structure of very well-covered graphs of girth at least 5 is more specific.
Theorem 2.4 [3, 16] Let $G$ be a graph of girth at least 5. Then $G$ is very well-covered if and only if $G = H \circ K_1$, for some graph $H$ of girth $\geq 5$.

Consequently, a very well-covered graph of girth $\geq 5$ has a unique perfect matching. Therefore, by Theorem 2.3 we get the following.

Corollary 2.5 [17] Each very well-covered graph of girth at least 5 generates a local maximum stable set greedoid.

It is known that the recognition of well-covered graphs is a co-NP-complete problem [2, 22]. Nevertheless, very well-covered graphs can be recognized in polynomial time. Actually, it goes directly from Favaron’s characterization. Namely, to recognize a graph as being very well-covered, we just need to show that it has a perfect matching which satisfies property $P$. To find a maximum matching one needs $O(|V|^2 \cdot |E|)$ time [19]. To check property $P$ one has to handle $O\left(|V|^3\right)$ pairs of vertices in the worst case. All in all, it gives us an $O\left(|V|^3\right)$ algorithm.

If our goal is to recognize very well-covered graphs with unique perfect matchings, then we may do better. The reason for this is that one can test whether the graph has a unique perfect matching, and find it if it exists, in $O\left(|E| \cdot \log^4 |V|\right)$ time [7]. Finally, Theorem 2.3 and Corollary 2.5 justify that one can decide in $O\left(|E| \cdot \log^4 |V|\right)$ time whether $\Psi(G)$ is a greedoid, for a given very well-covered graph $G$ of girth $\geq 4$.

3 Conclusions

In this paper we have proved that $\Psi(G)$ is a greedoid for those very well-covered graphs $G$ of girth $\geq 4$ that have a unique perfect matching.

Problem 3.1 Characterize very well-covered graphs of girth three producing local maximum stable set greedoids.

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