Detailed study of transient anomalous electric field vector focused by parabolic mirror

Kazunori Shibata¹, Mitsuharu Uemoto², Mayuko Takai³ and Shinichi Watanabe³

¹ Institute of Laser Engineering, Osaka University, 2-6 Yamada-Oka, Suita, Osaka 565-0871 Japan
² Center for Computational Sciences, University of Tsukuba, Tennodai 1-1-1, Tsukuba, Ibaraki 305–8577, Japan
³ Department of Physics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama, Kanagawa 223–8522, Japan

E-mail: shibata-ka@ile.osaka-u.ac.jp

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Abstract

This paper provides a detailed theoretical analysis of the unexpected transient divergent and rotational distributions of the focused electric field vector reported in Shibata et al (2015 Phys. Rev. A 92 053806). We reveal the physical origin of these distributions. More quantitatively, we derive the semi-analytic expressions and clarify how these distributions depend on the mirror size, offset angle, and the intensity distribution of the incident parallel light. We compare the formulas with numerical calculations and evaluate the area where linearity holds. If the wavelength and the mirror size are sufficiently shorter than the focal length, the radius of the linear area becomes longer than the wavelength. These formulas and evaluations are useful for studies, which require high spatio-temporal resolution.

Keywords: parabolic mirror, electric field vector, spatio-temporal distribution

(Some figures may appear in colour only in the online journal)

Introduction

The collection of light, which is typically done using a lens or a mirror, is one of the most fundamental processes in physics. A parabolic mirror (PM) is particularly useful for collecting broadband parallel light, because it has the advantage of having no chromatic aberration. This feature is leveraged for radio telescope [1], detection of far infrared radiations, and ultrafast spectroscopy [2–4]. In addition, it is used for tighter focusing [5] and imaging [6]. There have been some reports on the focusing properties using parabolic mirrors [7–9]. These papers treated an axisymmetrical PM and time-averaged electric field, or, intensity. On the other hand, we treat on- and off-axis parabolic mirrors without time-averaging. These differences lead us to find the transient divergent and rotational distributions of the focused electromagnetic waves. In addition, aberrations of reflection with a mirror segment [10, 11] have been studied previously.

Recently, starting from time-domain spectroscopy (TDS) in the terahertz frequency range, a technology has been developed to directly observe the temporal waveform of a focused electromagnetic wave from visible light [12] to far infrared light [13] within the temporal scale of a single cycle. Moreover, the spatial distribution of the focused electromagnetic field has been observed with a resolution of sub-wavelength scale [14]. In order to pursue light–matter interaction within these scales, a precise understanding of the time evolution of the electromagnetic field vector at the vicinity of the focus is necessary. Such studies are essential for designing an optical system that utilizes the information of polarization, such as a terahertz ellipsometry [15] or an ultrafast...
polarization imaging system [16]. Furthermore, it will be useful for developing novel applications such as ultrafast high-resolution field vector imaging [17].

In the previous paper [18], we reported peculiar distributions of the electromagnetic field vector focused by a PM. Such distributions appear momentarily within a period. For linearly polarized incident light, the electromagnetic field around the focus is almost uniform at the peak time, i.e. the time when the main component of the electric field at the focus achieves the maximum. On the other hand, the electromagnetic field in the focal plane shows divergent or rotational distributions corresponding to the incident polarization at the zero-crossing time, i.e. the time 1/4 period after the peak time. The electromagnetic field is proportional to the displacement from the focus. These distributions were indeed observed in an experiment using a terahertz TDS [18]. However, their physical origin has not been clarified. Moreover, there were limitations for the theoretical treatment, i.e. the mirror size was supposed to be much shorter than the focal length and the incident lights were supposed to be uniform. In experimental systems, especially in a terahertz TDS system, the size of the PM is comparable to the focal length and the spatial intensity distribution of the incident terahertz wave is not always uniform. These situations can significantly affect the divergent and rotational distributions at the zero-crossing time. Therefore, it is necessary to extend the theoretical analysis to realistic experimental conditions. In addition, there was no discussion on the area where linearity holds.

In this paper, we address the above-mentioned problems. First, we explain the physical origin of the transient divergent and rotational distributions. These distributions are understood by the spatial symmetry of the PM and the intensity distribution of the incident light. Second, we give semi-analytic (SA) expressions of the distributions for a practical size of the PM and for a general intensity distribution of the incident light. We compare the SA expressions with numerical calculations to evaluate the area where linearity holds. Within the used parameters, the radius of the area is longer than 1/4 of the wavelength.

System and problem
We consider the collection of parallel incident light by a PM. We calculate the electric field vector at the vicinity of the focus. Figure 1(a) shows the coordinate system. The PM is a part of a paraboloid \( z = (x^2 + y^2)/(4f) \) with focal length \( f \). At least for the present, its shape is arbitrary as far as it is symmetric about the \( XZ \) plane. Its projection over the \( XY \) plane is symmetric about the \( X \) axis and the center of mass is on the \( X \) axis. We denote the \( X \) coordinate by \( x_i \) and the center of the PM is given by \( (x_i, 0, 0^2/(4f)) \). We introduce a new coordinate system \( X'Y'Z' \) as follows. The origin is the focus. The positive \( Z' \) direction is along the vector from the center of the mirror to the focus. The \( Y' \) axis is parallel to the \( Y \) axis. The unit vector of the \( X' \) axis is given by the outer product of the unit vectors of the \( Y' \) and \( Z' \) axes. The offset angle \( \theta \) is defined by the angle between the \( Z \) and \( Z' \) axes. It can be seen that \( x_i = 2f \tan(\theta/2) \). If a position is expressed in the \( X'Y'Z' \) system, we attach a prime to the parentheses, e.g. the focus is given by \( (0, 0, f) = (0, 0, 0') \). The position vector \( x' \) expresses the position at the vicinity of the focus in the \( X'Y'Z' \) system. Thus, \( x' = (x', y', z') = (x' \cos \theta - z' \sin \theta, y', x' \sin \theta + z' \cos \theta + f) \).

The incident light propagates toward the negative \( Z \) direction. Let \( M(x, y) \in [0, 1] \) be the square root of the intensity distribution. It is convenient to introduce the ‘\( Y \) even’ function as \( M_E(x, y) = [M(x, y) + M(x, -y)]/2 \) and the ‘\( Y \) odd’ function as \( M_O(x, y) = [M(x, y) - M(x, -y)]/2 \). The incident light is supposed to be a linearly polarized monochromatic wave with wavelength \( \lambda = 2\pi/k = 2\pi c/\omega \), with \( c \) being the speed of light. Throughout the paper, a steady state is considered. We refer to the incident light as \( X(Y) \) polarization if the incident electric field contains only the \( X(Y) \) component. Figure 2 represents the time evolution of the focused electric field in

Figure 1. Configuration of the PM. (a) The definitions of the \( XYZ \) and \( X'Y'Z' \) systems, the paraboloid, and the incident polarization are shown. In the former discussion and the supplementary material, the shape of the PM is arbitrary as far as it is symmetric about the \( XZ \) plane. The large arrows indicate the propagation direction of the incident light. (b) The circle represents the projection of the PM over the \( XY \) plane for the calculations of figure 2 and the latter part.
the focal plane. Figure 2(a) indicates the main component at the focus. The peak time is defined as the time when the main component achieves the maximum. The zero-crossing time is defined as the time 1/4 period after the peak time. Figures 2(b) and (c) show the electric field vector distribution for the X polarization. At the peak time, almost uniform distribution appears around the focus. However, at the zero-crossing time the divergent distribution appears as is theoretically and experimentally reported in [18]. Figures 2(d) and (e) show the results for the Y polarization. The rotational distribution appears at the zero-crossing time. We study in detail these divergent and rotational distributions. Note that...
each vector in figures 2(b) and (d) is shortened to 1/10. Thus, the intensity distribution around the focus almost corresponds to the square of the electric field at the peak time.

Physical origin of transient divergence and rotation

First, we state the physical origin of the divergence (for the X polarization) and the rotation (for the Y polarization) appear at the zero-crossing time. This prospect is irrelevant to the intensity distribution of the incident light and the shape of the PM. In this section, we consider each electric field component separately.

Let \( A = (x_A, y_A, z_A) \) be a point on the PM. Because the PM is symmetric about the XZ plane, the symmetric point \((-y_A, x_A, z_A)\) is also on the PM. We denote it by \( A' \). Let \( O \) be the focus and \(|AO|\) be the distance between \( A \) and \( O \). It is possible to separately discuss for each combination of the incident polarization, the \( Y \) even or \( Y \) odd function of the intensity distribution of the incident light, and the considered component. We consider the electric field generated by the point currents at \( A \) and \( A' \). At the focus, their amplitudes are the same, because of the geometrical symmetry. As for the phases, they are limited to either in-phase or anti-phase. Whether the two phases are in-phase or anti-phase depends on the combination. However, it is important that there is no dependence on the choice of the specific point \( A \) on the PM. Therefore, for a qualitative discussion on the coordinate dependence of the electric field around the focus, we do not have to consider all the points of the PM. Instead, it is sufficient to consider the contributions from the coordinates \( A \) and \( A' \). Thus, we suppose that \( y_A > 0 \) without loss of generality.

Let \( O' \) be a point that is shifted from \( O \) by \( \Delta x', \) and we discard its second or higher order. If the displacement is along the \( X' \) axis, \( \Delta x' = (x', 0, 0) \) and

\[
|AO'| \approx |AO| - \frac{x_A \cos \theta + (z_A - f) \sin \theta}{|AO|} x'.
\]  
(1)

Similarly, if \( \Delta x' = (0, y', 0) \),

\[
|AO'| \approx |AO| - \frac{y_A}{|AO|} y',
\]  
(2)

and if \( \Delta x' = (0, 0, z') \),

\[
|AO'| \approx |AO| + \frac{x_A \sin \theta - (z_A - f) \cos \theta}{|AO|} z'.
\]  
(3)

As we have mentioned, the electric fields generated by the point currents at \( A \) and \( A' \) are in-phase or anti-phase at the focus. For each case, we calculate the electric field at \( O' \). Taking a suitable origin of the time, the electric field component from \( A, A' \) at the focus and time \( t \) can be expressed by \( E_A \sin \omega t \) and \( E_{A'} \sin \omega (t + \phi) \), respectively (\( E_{A'} = 0 \)).

In the in-phase case, \( \phi = 0 \) and the electric field component at the focus is \( 2E_A \sin \omega t \). It is not identically zero. We consider the electric field component at \( O' \). If the displacement is along the \( X' \) axis, \( |AO'| = |AO| \approx |AO| - \varepsilon \), where \( \varepsilon = [x_A \cos \theta + (z_A - f) \sin \theta] x'/|AO| \). Since \( x' \) is sufficiently small, \( \varepsilon \) is also small. For the contribution of the point current at \( A \), we can assume that \( A, O, \) and \( O' \) are arranged nearly in a straight line. Let \( \Delta t = \varepsilon/c \). The electric field at \( O' \) and time \( t \) is approximated by those at \( O \) and \( t + \Delta t \), and given by \( E_A \sin \omega (t + \Delta t) \). Similarly, the contribution of \( A' \) is given by \( E_{A'} \sin \omega (t + \Delta t) \). Discarding the second or higher order of \( x' \), the electric field at \( O' \) is given by

\[
2E_A \left[ \sin \omega t + (\cos \omega t) \frac{x_A \cos \theta + (z_A - f) \sin \theta}{|AO|} x' \right].
\]  
(4)

The first term in the brackets is the same as the electric field at the focus. The second term linearly depends on \( x' \), i.e., the displacement along the \( X' \) axis. If the displacement is along the \( Y' \) axis, a similar discussion holds. In equation (4), \( [x_A \cos \theta + (z_A - f) \sin \theta] x' \) is replaced by \(-[x_A \sin \theta - (z_A - f) \cos \theta] z' \). Thus, the electric field linearly depends on the displacement along the \( Y' \) axis. On the other hand, if the displacement is along the \( Y' \) axis, let \( \varepsilon' = y_A y'/|AO| \) and \( \Delta \varepsilon' = \varepsilon'/c \). If \( \varepsilon' > 0 \), \( |AO'| < |AO| = |A'O| < |A'O'| \). Therefore, the electric field component at \( O' \) and \( t \) is given by \( E_A \sin \omega (t + \Delta t') + E_{A'} \sin \omega (t - \Delta t') \approx 2E_A \sin \omega t \). The same result is obtained for \( \varepsilon' < 0 \). This means that the Electric field component is invariant for small displacement along the \( Y' \) axis. As a result, the electric field component at the vicinity of the focus can be expressed in a form of

\[
\beta_0 + \beta_1 x' + \beta_2 z',
\]  
(5)

where \( \beta_{0,1,2} \) are independent of the coordinates. In other words, the electric field component is not identically zero at the focus. It linearly depends on \( x' \) and \( z' \). The dependence on \( y' \) is at least of the second order.

In the anti-phase case, \( \phi = \pi \) and the electric field component at the focus is always zero. Let us consider the electric field component at \( O' \). If the displacement is along the \( X' \) axis, the electric field from \( A \) is \( E_A \sin \omega (t + \Delta t) \) and that from \( A' \) is \(-E_A \sin \omega (t + \Delta t) \). Thus, the sum is zero. This indicates the electric field component remains zero with displacement along the \( X' \) axis. The same result is obtained for the \( Y' \) axis. If the displacement is along the \( Y' \) axis, the electric field component at \( O' \) and \( t \) is

\[
E_A \left[ \sin \omega (t + \Delta t') - \sin \omega (t - \Delta t') \right]
\]  
\[
\approx 2E_A \cos \omega t \frac{y_A}{|AO|} y',
\]  
(6)

As a result, the electric field component can be expressed in the form of

\[
\beta_3 y',
\]  
(7)

where \( \beta_3 \) is independent of the coordinates. In other words, if the waves from \( A \) and \( A' \) are anti-phase, the value at the focus is always zero. The value linearly depends on \( y' \) and its dependence on \( x', z' \) is at least of the second order.

All we have to do is to determine whether the two phases are in-phase or anti-phase for each combination of the incident polarization (X pol. or Y pol.), the \( Y \) even or \( Y \) odd function of the incident distribution \( M_E \) or \( M_O \), and the
component of the electric field \((X', Y', \text{or} \ Z')\). Let their combination be expressed as \([X \text{pol.}, M_E, X']\). The combinations that lead to the in-phase case are \([X \text{pol.}, M_E, X'], [X \text{pol.}, M_E, Z'], [X \text{pol.}, M_O, Y'], [Y \text{pol.}, M_E, Y'], [Y \text{pol.}, M_O, X'], \text{and} \ [Y \text{pol.}, M_O, Z']\). The remaining combinations lead to the anti-phase case. This result is in agreement with the equations derived in the supplementary material (equations (S.6), (S.14), (S.20), and (S.26)). With this classification and equations (5) and (7), the distribution at the zero-crossing time and in the focal plane \((z' = 0)\) is directly understood. If \(M_O\) is identically zero, for the \(X\) polarization, the \(X'\) component is proportional to \(x'\) and the \(Y'\) component is proportional to \(y'\). It is a divergent-like distribution (‘like’ means the coefficients can be zero or of opposite signs). Similarly, for the \(Y\) polarization, the \(X'\) component is proportional to \(y'\) and the \(Y'\) component is proportional to \(x'\). It is a rotation-like distribution.

**Properties of transient divergence and rotation**

Next, we present a quantitative discussion on the coefficients in equations (5) and (7). In particular, we derive semi-analytical formulas for a general PM and the intensity distribution of the incident light. Thus, this discussion is an extension of the previous paper. In the steady state, all the physical quantities are separated as \(E(x)e^{-i\omega t}\) and the real part is the physical quantity. \(F(x)\) is the spatial part.

The spatial part of the electric field at the position \(x\) is given by the Stratton–Chu equation [19] as

\[
E(x) = i\omega\mu_0 \int d\mathbf{x} g^{(3)}(k; x, \hat{x}) \mathbf{f}(\hat{x}),
\]

where \(\mu_0\) is the magnetic permeability in vacuum. The integral is over the mirror surface, \(g^{(3)}\) denotes the dyadic Green function. The surface current \(\mathbf{j}(\hat{x})\) can be approximated by the physical optics method [20].

\[
\mathbf{j}(\hat{x}) = \frac{2}{\mu_0 \omega} \mathbf{n} \times [\mathbf{k} \times E^{\text{inc}}(\hat{x})] \delta[\mathbf{\hat{x}}^2 - (\mathbf{\hat{x}} \cdot \mathbf{\hat{x}})^2/(4\omega)],
\]

where \(E^{\text{inc}}\) is the incident electric field and \(\mathbf{n}\) is the outward normal vector at the surface point \(\hat{x}\). We expand the integral to the first order of \(\mathbf{x}'\).

We first calculate the \(X\) polarization. The incident electric field is given by \(E^{\text{inc}}(\mathbf{x}) = E_0 M(\mathbf{x}, \mathbf{\hat{x}})\exp(-ik\mathbf{x})e_y\), where \(E_0\) is the amplitude and \(e_y\) is the unit vector of the \(X\) axis. Substituting it into equation (9), the SA electric field is approximated by

\[
E_{x, y}^{\text{SA}}(x') = P_1 + iP_3 x' + iQ_1 y' + iP_5 z',
E_y^{\text{SA}}(x') = Q_2 + iQ_3 x' + iP_6 y' + iQ_4 z',
E_z^{\text{SA}}(x') = P_5 + iP_3 x' + iQ_3 y' + iP_5 z'.
\]

The electric field is normalized by \(\exp(ikf/)E_0(f/\lambda)\). The coefficients \(P_n, Q_n\) \((n\text{ is an integer})\) are independent of \(x'\) and \(\lambda\). They are derived in the supplementary material. The contributions of \(M_E\) and \(M_O\) are included in \(P_n\) and \(Q_n\), respectively. For uniform incident light, \(M_E(x, y) = 1\) and \(M_O(x, y) = 0\), and therefore, \(Q_n = 0\). In this case, equation (10) shows that the electric field at the focus \((x' = 0)\) contains \(X'\) and \(Z'\) components. The maximum values of these components are given by \(E_{x, y, z}^{\text{SA}} = |P_3|, E_{x, y, z}^{\text{SA}} = |P_5|\). In the following, we suppose that the projection is a circle with radius \(L\) (see figure (1(b)) and \(P_n\) are given in closed forms (see equation (S.9) in the supplementary material). Furthermore, in order to treat a practical PM, we restrict the parameters as \(0 < L/f < 2\) and \(0<\theta<90^\circ\). Within these ranges, \(P_1 > 0, P_5 < 0\), and we obtain \(E_{x, y, z}^{\text{SA}} / E_{x, y, z}^{\text{max}} > 1.9\). Note that \(E_{x, y, z}^{\text{SA}}\) can be larger than \(E_{x, y, z}^{\text{max}}\) without the ranges. For comparison with the previous paper, we regard the \(X'\) component as the main component and define the zero-crossing time \(t_0\) by

\[
E_{x, y, z}^{\text{SA}}(0, t_0) = 0 \quad \text{and} \quad \frac{d}{dt} E_{x, y, z}^{\text{SA}}(0, t_0) < 0. \tag{11}
\]

At the zero-crossing time, the approximated electric field is expressed by

\[
E_{x, y, z}^{\text{SA}}(x', t_0) / E_{x, y, z}^{\text{SA}} = K_{XX} x' / \lambda + K_{ZZ} z' / \lambda,
E_{x, y, z}^{\text{SA}}(x', t_0) / E_{x, y, z}^{\text{SA}} = K_{XX} y' / \lambda,
E_{x, y, z}^{\text{SA}}(x', t_0) / E_{x, y, z}^{\text{SA}} = K_{XX} x' / \lambda + K_{ZZ} z' / \lambda. \tag{12}
\]

In order to see the distribution in the \(X'Y'\) plane, we study \(K_{XX}\) and \(K_{YY}\). They are given by

\[
K_{XX} = 2\pi P_3 / P_5,
K_{YY} = 2\pi P_5 / P_1. \tag{13}
\]

Figure 3 shows the \(\theta\) dependence of \(K_{XX}\) and \(K_{YY}\) for \(L/f = 1, 2\). The magnitudes of the coefficients \(K_{XX}\) and \(K_{YY}\) differ. This result is not expected from the discussion for a small mirror. The calculation shows that \(|K_{XX}| \leq |K_{YY}|\) for all \(L\) and \(\theta\). In particular, figure 3(b) indicates that there will be an offset angle such that \(K_{XX} = 0\) for a sufficiently large mirror. Figure 4 shows the \(L/f\) dependence of the angle at which \(K_{XX} = 0\).

We compare the coefficients with the expressions derived for a small mirror. From the previous paper [18], one can obtain

\[
K_{XX}^{\text{pre}} = K_{YY}^{\text{pre}} = \frac{\pi L^2 \sin \theta \cos^2(\theta/2)}{A^2 - L^2 \cos^4(\theta/2)} \left(1 + \frac{8}{k^2 L^2}\right). \tag{14}
\]

Let us search the ranges of \(L\) and \(\theta\) where equation (14) is a good approximation of \(K_{XX}\) and \(K_{YY}\). For a constant \(\Delta_{\text{error}}\), we consider the following condition:

\[
[1 - K_{XX}/K_{XX}^{\text{pre}}] [1 - K_{YY}/K_{YY}^{\text{pre}}] < \Delta_{\text{error}}. \tag{15}
\]

The denominator becomes zero if and only if \(\theta = 0^\circ\). If \(\theta = 0^\circ\), \(K_{XX} = K_{YY} = 0\) and the coefficients are equal to the previous result (14). Thus, we do not have to evaluate equation (15) for zero degree. We start from \(L/f = 0.01\) and increase \(\theta\) from \(1^\circ\) to \(90^\circ\), at intervals of \(1^\circ\). If the condition (15) holds for all \(\theta\), we increase \(L/f\) by 0.01. Again, we vary \(\theta\)
and check the condition. The maximum radii at which the condition holds for all \( \theta \) are calculated to be \( L/f = 0.08 \) for \( \Delta_{\text{error}} = 0.01 \) and \( L/f = 0.28 \) for \( \Delta_{\text{error}} = 0.1 \). In particular, if the offset angle is set to \( \theta = 90^\circ \), the maximum radii at which the condition holds are obtained to be \( L/f = 0.19 \) for \( \Delta_{\text{error}} = 0.01 \) and \( L/f = 0.63 \) for \( \Delta_{\text{error}} = 0.1 \). These results are in agreement with those for wavelengths of \( \lambda/f = 10^{-2}, 10^{-4} \).

Next, we search the area in the \( XY \) plane where the linear approximation (12) holds at the zero-crossing time. For the numerical calculation (NC), the corresponding maximum \( X \) component at the focus and zero-crossing time are denoted by \( E_{X,\text{max}}^{\text{NC}} \) and \( t_0^\prime \), respectively. The electric field at \( t_0^\prime \) is expressed by \( E^{\text{NC}}(x^\prime, t_0^\prime) \). Then, we evaluate

\[
\left| \frac{E^{\text{NC}}(x^\prime,y^\prime,\varphi)(x^\prime, t_0^\prime)}{E_{X,\text{max}}^{\text{NC}}} - \frac{E_{X,\text{max}}^{\text{SA}}(x^\prime, t_0^\prime)}{E_{X,\text{max}}^{\text{SA}}} \right| < \Delta_{\text{error}}, \tag{16}
\]

for \( \Delta_{\text{error}} = 0.01 \). We set \( \theta = 90^\circ \). In the \( X’Y’ \) plane, \( z’ = 0 \), and the position \((x^\prime, y^\prime, 0)\) is expressed by the polar coordinates \((r, \varphi)\), as \( x^\prime = r \cos \varphi \) and \( y^\prime = r \sin \varphi \). We start from \( r/\lambda = 0.01 \) and vary \( \varphi \) from 0° to 359°, at intervals of 1°. If the condition (16) holds for all \( \varphi \), \( r/\lambda \) is increased by 0.01. For each \( L/f \), we obtained the maximum \( r_{\text{max}}/\lambda \) at which the condition (16) holds for all \( \varphi \). The results for \( \lambda/f = 10^{-2}, 10^{-3} \) are shown in figure 5. It can be seen that \( r_{\text{max}}/\lambda \) becomes larger for a smaller mirror or shorter wavelength. Note that \( r_{\text{max}} \) can be longer than \( \lambda \).

With regard to the evaluation of the difference, one might question whether it is better to consider a ratio of the SA approximation and the numerical calculation as equation (15), instead of equation (16). However, such a ratio is not suitable for the present issue. The reason is that there are many points at which the electric field becomes zero or is very close to zero, for example, the \( X’ \) component along the \( Y’ \) axis. At these points, a small error is inevitable and the ratio can be an unexpected large or small value. The electric field itself is so small that it is difficult to get rid of the error. Moreover, for experimental comparison, equation (16) is sufficient for the present accuracy of the experiments.

Next, we calculate the \( Y \) polarization. The incident electric field is given by \( E^{\text{inc}}(\tilde{x}) = E_0 \tilde{m}(\tilde{x}, \tilde{y}) \exp(-i k \tilde{z}) \tilde{e}_y \), where \( \tilde{e}_y \) is the unit vector of the \( Y \) axis. Substituting it into equation (9), the spatial part of the SA electric field at the
vicinity of the focus is given by

\[ \begin{align*}
E_x^{SA}(x') &= V_1 + iV_2kx' + iU_1ky' + iV_3kz', \\
E_y^{SA}(x') &= U_2 + iU_3kx' + iV_4ky' + iU_4kz', \\
E_z^{SA}(x') &= V_5 + iV_6kx' + iU_5ky' + iV_7kz'.
\end{align*} \]

The electric field is normalized by \(i\exp(ikf)E_0(f/\lambda)\). The coefficients \(U_n\), \(V_n\) are given in the supplementary material. The contributions of \(M_E\) and \(M_D\) are included in \(U_n\) and \(V_n\), respectively. The following discussion is basically the same as that for the \(X\) polarization. For uniform incident light, \(V_n = 0\). In addition, for the circular projection of the PM, the integrals of \(U_n\) can be executed analytically (see equation (S.21) in the supplementary material). The electric field at the focus contains only the \(Y\) component and its maximum value is given by \(E_{Y\text{max}}^{SA} = U_2 > 0\). Therefore, we can define the zero-crossing time \(t_0\) by

\[ E_{y}^{SA}(\theta, t_0) = 0, \quad \text{and} \quad \frac{d}{dt} E_{y}^{SA}(\theta, t_0) < 0. \]

The electric field at the zero-crossing time is given by

\[ \begin{align*}
E_x^{SA}(x', t_0)/E_{Y\text{max}}^{SA} &= K_{XY}\frac{y'}{\lambda}, \\
E_y^{SA}(x', t_0)/E_{Y\text{max}}^{SA} &= K_{XY}\frac{x'}{\lambda} + K_{YX}\frac{y'}{\lambda}, \\
E_z^{SA}(x', t_0)/E_{Y\text{max}}^{SA} &= K_{YX}\frac{y'}{\lambda}.
\end{align*} \]

\(K_{XY}\) and \(K_{YX}\) are given by

\[ \begin{align*}
K_{XY} &= 2\pi U_1/U_2, \\
K_{YX} &= 2\pi U_2/U_1.
\end{align*} \]

Figure 6 shows the \(\theta\) dependence of \(-K_{XY}, K_{XY}\) for \(L/f = 1, 2, 3\). As in the case of the \(X\) polarization, the coefficients \(-K_{XY}\) and \(K_{XY}\) are not the same. \(|K_{XY}| < |K_{XY}|\) holds for all \(L\) and \(\theta\).

We compare \(-K_{XY}\) and \(K_{XY}\) with the previous result for a small mirror [18],

\[ -K_{XY}^{(pre)} = K_{XY}^{(pre)} = \frac{\pi L^2 \sin \theta \cos^2(\theta/2)}{4f^2 - L^2 \cos^2(\theta/2)}. \]

The contributions of \(K_{XY}\), \(K_{XY}\), and \(K_{YX}\) for \(L/f = 2\). Within the considered ranges of \(L\) and \(\theta\), it is obtained that \(|K_{XY}| < |K_{XY}|\).

Figure 7. For the \(Y\) polarization, the radius \(r_{\text{max}}/\lambda\) of the area where linearity holds for \(\Delta_{\text{error}} = 0.01\). The offset angle is set to 90°. For \(\lambda/f = 10^{-2}, 10^{-3}\) the results are similar to those for the \(X\) polarization.

We search the ranges of \(L\) and \(\theta\) where the following holds,

\[ |1 - K_{XY}/K_{XY}^{(pre)}|, |1 - K_{XY}/K_{XY}^{(pre)}| < \Delta_{\text{error}}. \]

The calculation is performed in the same way as that for the \(X\) polarization. The maximum radii at which the condition holds for all \(\theta\) are calculated to be \(L/f = 0.11\) for \(\Delta_{\text{error}} = 0.01\) and \(L/f = 0.37\) for \(\Delta_{\text{error}} = 0.1\). In particular, if the offset angle is set to \(\theta = 90°\), the maximum radii are \(L/f = 0.28\) for \(\Delta_{\text{error}} = 0.01\) and \(L/f = 0.88\) for \(\Delta_{\text{error}} = 0.1\). These results are independent of \(\lambda < f\), because equations (20) and (21) are independent of \(\lambda\).

Next, we search the area in the \(X'Y'\) plane where the linear approximation at the zero-crossing time holds. For \(\Delta_{\text{error}} = 0.01\) and \(\theta = 90°\), we consider a condition that is similar to equation (16). The results for \(\lambda/f = 10^{-2}, 10^{-3}\) are shown in figure 7. The result is almost the same as that for the \(X\) polarization.

**Conclusion**

In summary, we discussed the electric field vector focused by a PM at the vicinity of the focus. At the zero-crossing time, the divergent or rotational distributions appear in the focal plane. The magnitudes of these transient distributions are...
proportional to the displacement from the focus. We showed the physical origin of the structures and their linearity. These issues remained to be addressed in the previous paper [18]. Furthermore, we derived the SA equations that are applicable to various PMs and the intensity distributions of the incident light. In particular, for uniform incident light, the integrals were executed and the coefficients were given by closed forms These equations are direct extensions of the previous results. We compared the SA equations with the equations for a small PM. For \( q = 90^\circ \), the difference was less than or equal to 10% for \( L/f \leq 0.63 \) (for the X polarization) and \( L/f \leq 0.88 \) (for the Y polarization). The difference between the \( X' \) and \( Y' \) components increases with the mirror size. This feature will be important when constructing a high-resolution vector imaging system. In addition, we compared the SA equations with numerical calculations and determined the area in the \( X'Y' \) plane where linearity holds. In particular, if \( L/f \) and \( \lambda/f \) are sufficiently small, linearity holds for a circle with a radius that is longer than the wavelength. Therefore, these distributions should be taken into account for focusing by a PM. Our formalization is useful for experiments that utilize the focused-light–matter interaction within a period such as ultrafast phenomena. Finally, we would like to comment that the appearance of the divergent and rotational distributions severely depend on alignment. Our numerical calculations show that if the incident light is tilted \( 1^\circ \), these distributions do not appear.

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