Algorithms for creating ruled surfaces according to various initial conditions

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Abstract. The article considers the algorithms for creating the ruled surfaces according to various initial conditions. Ruled surfaces are widely used in the development of technical objects, building structures, surfaces of road dumps, dams, etc. This is due to the manufacturability of such surfaces’ production. They can consist of rectilinear elements from various materials widely produced by modern industry. A lot of research is devoted to the development of algorithms for creating various types of the ruled surfaces. But most of them consider certain types of surfaces. In this paper, an attempt to create the most general algorithm for creating various types of the ruled surfaces is made. This will significantly expand the range of the ruled surfaces used in practical problems. The algorithms are implemented using the Object ARX technology for the AutoCAD system, which can greatly simplify the development, while using the extensive capabilities of AutoCAD. The algorithms developed in the article can be applied in the design tasks of various objects using ruled surfaces, as well as for developing the new types of ruled surfaces and studying their properties.

Introduction
When designing the surfaces of technical objects, architectural and building structures, etc. the linear surfaces are widely used.

In the general case, a ruled surface is formed by continuous movement in space of a straight line (generatrix) according to the given law [1, 2]. Thus, the main point in creating a ruled surface is the law of the generatrix movement in space. The combination of the generatrix and its movement law, according to the terminology adopted in [2], is called the surface determinant. It is convenient to set the law of displacement in the form of certain relations of the moving generatrix with other geometric objects. As it is adopted in [2], we will distinguish the geometric part of the determinant, which includes the geometric objects (including the generatrix) involved in the determinant, and the algorithmic one — that includes the generatrix connections and the given geometric objects that determine the law of its movement in space. Thus, the determinant of the surface will be written as

\[ \Sigma : (G); \{A\}, \]  

where \((G)\) – is the geometric part of the determinant, \((A)\) – is the algorithmic part of the determinant.
In many works, the forming ruled surfaces’ methods are considered [3–5], but in most works some specific way of their formation is considered. This work is an attempt to create the most general software algorithm for creating the ruled surfaces.

The algorithms presented in the article are implemented using the ObjectARX technology [6] for AutoCAD [7] in the structural text C++ [8]. The use of this technology significantly reduces the time of the software development, as it allows to use wide graphical capabilities and a convenient system interface AutoCAD.

The Problem Statement
The objective of this study is to develop the software algorithms for creating the ruled surfaces according to various conditions, i.e. to implement various types of such surfaces’ determinants in program form.

The formation of ruled surfaces

Figure 1. Defining a straight line in space

As noted above, a ruled surface is formed by the continuous movement of a straight line generatrix according to a certain law, therefore, it can be considered as a one-parameter set of the straight lines in space.

A straight line in space is determined by any pair of mismatched points belonging to it. Figuratively, it is possible to imagine that these points can freely move along this straight line (Figure 1). Each point in three-dimensional space is defined by three independent parameters, but since these points can move along the straight line, two parameters are enough to set each of them. Thus, the set of direct space is a four parametric set. Let us define it as $\mathbb{A}^4$. By imposing the condition on the lines’ position, we exclude one of the parameters. Therefore, to highlight the ruled surface in space, it is necessary to impose three conditions on the lines’ position. In [9, 10], these conditions are considered in detail.

we introduce the generatrix $g$ and three geometric objects $O_1$, $O_2$, $O_3$ into the geometric part of the surface determinant and will call them the bearing objects. In the algorithmic part of the determinant, we introduce three conditions that must be satisfied at each position of the generatrix of the line, relative to the baring objects. We denote these conditions in a general form $C_i(g, O_i), C_2(g, O_2), C_3(g, O_3)$. Then the determinant of the surface (1) has the form

$$
\Sigma: (g, O_1, O_2, O_3);\{C_1(g, O_1), C_2(g, O_2), C_3(g, O_3)\}.
$$

we will take lines, planes or surfaces as the bearing objects as in [9, 10]. If a line is taken as bearing, then the required generator in the corresponding condition in each position intersects it, for example, $g_i \cap k \neq \emptyset$. This condition is always taken as the first condition. If the bearing line is a plane, then it is required that the generatrix forms a predetermined angle with it ($\angle g, P = \alpha$). The condition imposed on the generators, in the case when the bearing object is a surface, is the contact of this surface. This condition in the determinant is denoted as follows: $g_i \Omega \neq \emptyset$. Q
Figure 2. The lines’ complex

The lines intersecting a given line \( k \) in space form a three parametric set \((\infty^3)\), called a complex (Figure - 2). Let the line \( k \) be defined by the equation

\[
\vec{r} = \vec{r}_k(u),
\]

then the equation of each line in this complex can be written as

\[
\hat{\vec{r}}(u, \phi, \varphi, t) = \vec{r}_k(u) + \vec{r}_l(\phi, \varphi)t,
\]

where the parameters \( u, \phi \), \( \varphi \) determine the line included in the complex. The parameter \( u \) sets the points on the line \( k \), through which the line passes, \( \phi \) and \( \varphi \) determine the coordinates of the guide vector \( \vec{r}_l \) straight; they are equal.

\[
\begin{align*}
    x_1 &= d \cos \phi \cos \varphi; \\
    y_1 &= d \sin \phi \cos \varphi; \\
    z_1 &= d \sin \varphi; \\
    0 &\leq \phi < 2\pi; \quad 0 \leq \varphi < \pi.
\end{align*}
\]

In equations (4) \( d \) – is the length of the line’s directing vector, it can be of any value; \( \phi \) – is the angle formed by the orthogonal projection of the line’s directing vector \( \vec{r}_l \) to the plane \( xOy \), with an axis \( x \); \( \varphi \) – defines the angle made by the directing vector of the line \( \vec{r}_l \) with the plane \( xOy \) (Figure - 3). The parameter \( t \) in (3) determines the position of a point on a straight complex.

We impose the second condition on the lines’ position. There are three possible options:

- intersection of the second line \( l \);
- position at a given angle to the plane \( P \);
- contacting a given surface \( \Omega \).

Let us consider the first option. Let the line \( l \) be set by the equation in space, besides the line \( k \):

\[
\vec{r} = \vec{r}_l(u).
\]

Figure 3. The directing vector of the direct complex

The set of lines intersecting both given lines is a two-parameter set \((\infty^2)\) and is called congruence. We will consider it as a one-parameter set of cones which vertices are located on the line \( k \), the guide line of each of them is a line \( l \) (Figure - 4a). The congruence equation has the form:

\[
\hat{\vec{r}}(u, v, t) = \vec{r}_k(u) + (\vec{r}_l(v) - \vec{r}_k(u))t,
\]

(5)
where the parameter \( v \) defines the points of the line \( l \). In equation (5), the parameters \( u \) and \( v \) are the direct congruences, \( t \) – is the line point position.

**Figure 4.** Congruences of lines

When choosing the lines inclination angles’ equality to a given guide plane \( P \) as the second condition, we get a congruence, which is a one-parameter set of circular cones. The tops of the cones are located on the guide line \( k \), the generators make up a predetermined angle with the guide plane \( \alpha \) (Figure – 4b). We take the circles, which are their sections of a given guide plane as the guide cones’ lines. We find the equations of the cones’ guiding circles:

\[
\mathbf{r} = \mathbf{r}_g(u, v).
\]

The parameter (6) includes \( u \), this means that each of the congruence cones has its own guide line, unlike the previous case.

We determine the form of equation (6). Let the plane be given by the points \( A(x_A, y_A, z_A) \), \( B(x_B, y_B, z_B) \), \( C(x_C, y_C, z_C) \) and the top of the cone \( S(x_S, y_S, z_S) \) (Figure – 5). The point coordinates \( S \) depend on the parameter \( u \). We find the center of the circle – \( O(x_O, y_O, z_O) \). It is the intersection point of the perpendicular through the point \( S \), with a given plane. To determine the coordinates of this point, it is necessary to solve the following system of linear equations:

\[
\begin{align*}
ax + by + cz + d &= 0; \\
x &= x_s + av; \\
y &= y_s + bv; \\
z &= z_s + cv;
\end{align*}
\]

where

\[
\begin{align*}
a &= \begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}; \\
b &= \begin{vmatrix} x_B - x_A & z_B - z_A \\ x_C - x_A & z_C - z_A \end{vmatrix}; \\
c &= \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}; \\
d &= -ax_s - by_s - cz_s.
\end{align*}
\]

**Figure 5.** Determination of the cone guide circle (option 4b).
The first equation of the system defines a plane, the next three equations define a straight line in a parametric form. Having solved the system (7), for the point parameter $O$ on the line, we get:

$$v_o = -\frac{ax_s + by_s + cz_s + d}{a^2 + b^2 + c^2}.$$ (7)

For the coordinates $O$, we have:

$$x_o = x_s + av_x; \quad y_o = y_s + bv_y; \quad z_o = z_s + cv_z.$$ 

The radius of the circle is

$$r = h \cot \alpha,$$

where $h = \sqrt{(x_y - x_o)^2 + (y_y - y_o)^2 + (z_y - z_o)^2}$.

By rotating the horizontal track, the algorithm of which is described in [16], we bring the plane to a position that coincides with the horizontal plane. Let us move the origin of the coordinate system to a point $O$. The point coordinates in the new system can be determined as:

$$(x', y', z') = (x, y, z) + (-x_o, -y_o, -z_o).$$ (8)

Let us rotate the coordinate system around the horizontal trace of the plane, so that it coincides with the plane $xOy$. In this system, the points have the following coordinates:

$$(x'', y'', z'') = (x', y', z') \begin{pmatrix} k^2 + \cos \varphi & kl(1 - \cos \varphi) & -l \sin \varphi \\ kl(1 - \cos \varphi) & l^2 + \cos \varphi(1 - l^2) & k \sin \varphi \\ -l \sin \varphi & k \sin \varphi & \cos \varphi \end{pmatrix}.$$ (9)

Where

$$k = \frac{b}{\sqrt{a^2 + b^2}}; \quad l = -\frac{a}{\sqrt{a^2 + b^2}}; \quad m = 0$$

the bearing cosines of the horizontal trace of the plane,

$$\varphi = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

The angle formed by the plane with the axis $z$. In this coordinate system, the equation of the bearing circle has the form:

$$x'' = r \cos \nu; \quad y'' = r \sin \nu; \quad z'' = 0; \quad 0 \leq \nu < 2\pi.$$ (10)

The derivative by parameter $\nu$ is equal to:
To obtain the coordinates of the bearing circle points and their derivatives in the original coordinate system, it is necessary to perform the inverse transformations (9) and (8).

In the third case, we take the spheres as bearing surfaces $\Omega$. Then the congruence is a one-parameter set of circular cones whose vertices are on the line $k$ generators and are in contact with the bearing sphere (Figure 4c). We will take the circles along which the generators are in contact with the sphere as the bearing lines of the cones. In general terms, their vector equations will be similar (6).

![Figure 6. Determination of the cone’s guide circle (option 4c)](image)

Let us find the center $O$ and radius $r$ bearing circle. Let the vertex of the cone be at the point $S$, center of the guide sphere in $O_\Omega$, its radius $- R$ (Figure – 6). Let us denote the coordinates of the line $SO_\Omega$ directing vector, which is also the normal to the plane of the bearing circle, through $a$, $b$, $c$, they are equal $a = x_{O_\Omega} - x_S; b = y_{O_\Omega} - y_S; c = z_{O_\Omega} - z_S$. From the right triangles $SMO_\Omega$ and $SOM$ follows that:

$$r = \frac{d \sqrt{a^2 + b^2 + c^2}}{d}; h = \frac{d^2 - R^2}{d}.$$  

where $d = SO_\Omega = \sqrt{a^2 + b^2 + c^2}$, $h = SO$ – is the cone height. For the coordinates of the bearing circle’s center, we have:

$$x_O = x_S + \frac{a}{\sqrt{a^2 + b^2 + c^2}} h; y_O = y_S + \frac{b}{\sqrt{a^2 + b^2 + c^2}} h; z_O = z_S + \frac{c}{\sqrt{a^2 + b^2 + c^2}} h.$$  

Further, according to the algorithm described above, we find the coordinates of the circle generatrix points, for a given parameter $v$.

We impose the third condition on the lines’ position. As in the previous case, this can be any of the three options considered. As a result, we obtain a one-parameter set of lines $(x^j)$, etc. as the ruled surface.

If we consider the lines that satisfy the first and third conditions, then we get one of the congruences shown in Figure 5. To determine the lines that satisfy the three given conditions, which are the generators of the ruled surface, we will find the intersection line of the pair of cones which vertices are at one point on the lines $k$ (figure – 7). In Figure 7, the bearing lines of the cones are denoted by $n_1$ and $n_2$. Their equations in vector form $\vec{r} = \vec{r}_{n_1}(u, v_1)$ and $\vec{r} = \vec{r}_{n_2}(u, v_2)$. Thus, the equations forming a ruled surface are determined by the following system of equations:

$$\begin{align*}
\frac{dx^j}{dv} &= -r \sin v; \frac{dy^j}{dv} = r \cos v; \frac{dz^j}{dv} = 0; 0 \leq v < 2\pi. (11) \\
\end{align*}$$

$$\begin{align*}
[\vec{r}(u, v_1, t_1) = \vec{r}_{k_1}(u) + (\vec{r}_{n_1}(u, v_1) - \vec{r}_{k_1}(u))t_1; \\
\vec{r}(u, v_2, t_2) = \vec{r}_{k_2}(u) + (\vec{r}_{n_2}(u, v_2) - \vec{r}_{k_2}(u))t_2. (12)]
\end{align*}$$
Let us find the line of the two cones intersection with a common vertex. As known, the cones having a common vertex intersect in generators. Let us denote the point at which the vertices of the cones are located by $S$. Then the system (12) takes the form:

\[
\begin{align*}
\vec{r}(v_1,t_1) &= \vec{r}_s + (\vec{r}_n(v_1) - \vec{r}_s)t_1; \\
\vec{r}(v_2,t_2) &= \vec{r}_s + (\vec{r}_n(v_2) - \vec{r}_s)t_2. 
\end{align*}
\]

where $\vec{r}_s$ – is the point radius vector $S$. We solve the system (13) by the Newton method [12].

**Figure 7. The cones’ intersection**

First, we determine the approximate values of the parameters. We cut both cones in a horizontal plane $z=0$, define the points $A_i$ belonging to the intersection line with the first cone $a$, take fixed parameter values $v_1 - v'_1$. Then from the first equation of the system (13) for the coordinates of the points $A_i$ we have:

\[
x'_i = x_s + (x'_n - x_s)t'_i; \quad y'_i = y_s + (y'_n - y_s)t'_i; \quad z'_i = 0.
\]

where $t'_i = \frac{z'_s}{z'_s - z'_n}$, $x'_n$, $y'_n$, $z'_n$ – are the coordinates of the radius vector $\vec{r}_n(v'_1)$. For the point coordinates $B_i$, the lines of restraint $b c$ horizontal plane with the second cone we get similarly (14).

We approximate the lines $a$ and $b$ the chords $A_iA_{i+1}$ and $B_iB_{i+1}$. Let us find the intersection points of the chords. The coordinates of the lines intersection points $C_{ij}$, on which the chords lie are defined as follows:

\[
x''_c = \frac{b d_j - b d_i}{a b_j - a b_i}; \quad y''_c = \frac{a d_i - a d_j}{a b_j - a b_i}; \quad z''_c = 0,
\]

where
\[ a_i = y_i^{j+i} - y_i^j; b_i = x_i^{j+i} - x_i^j; d_i = -a_i x_i^j - b_i y_i^j; a_j = y_j^{j+i} - y_j^j; b_j = x_j^{j+i} - x_j^j; d_j = -a_j x_j^j - b_j y_j^j. \]

An intersection point exists if \( a_i b_j - a_j b_i \neq 0 \), besides this point should belong to both chords

\[ C_{ij} \in [A_iA_{j+1}]; C_{ij} \in [B_jB_{j+1}]. \quad (16) \]

To satisfy the conditions (16), the following inequality is necessary

\[ \frac{\xi - \xi_1}{\xi_2 - \xi} \geq 0, \]

where \( \xi_1 \) and \( \xi_2 \) – are the abscissas, ordinates, or applicates of the chord endpoints, \( \xi \) – defines the corresponding coordinate of the intersection point.

As the initial approximate values of the parameters \( v_1 \) and \( v_2 \), will be accepted as:

\[ v_1^n = \frac{v_1^j + v_1^{j+i}}{2}; \quad v_2^n = \frac{v_2^j + v_2^{j+i}}{2}. \]

To determine the initial value of the parameter \( t_1 \), we define the nearest point on the generatrix of the first cone corresponding to the parameter \( v_1^n \), to the point \( C_{ij} \). We construct a plane perpendicular to this generator and passing through the point \( C_{ij} \). The intersection point of the plane with the cone generatrix is the desired one. To determine the coordinates of this point, we solve a system similar to (7), then

\[ t_1^n = -\frac{ax_s + by_s + cz_s + d}{a^2 + b^2 + c^2}, \]

where

\[ a = x_n^n - x_s; b = y_n^n - y_s; c = z_n^n - z_s; d = -ax_s^n - by_s^n - cz_s^n, \]

\( x_n^n, y_n^n, z_n^n \) – are the coordinates of the radius vector \( \vec{r}_n(v_n^n) \). Similarly, for the initial value \( t_2 \)

\[ t_2^n = -\frac{a'x_s + b'y_s + c'z_s + d'}{a'^2 + b'^2 + c'^2}, \]

where

\[ a' = x_n^n - x_s; b' = y_n^n - y_s; c = z_n^n - z_s; d = -a'x_s^n - b'y_s^n - c'z_s^n, \]

\( x_n^n, y_n^n, z_n^n \) – are the coordinates of the radius vector \( \vec{r}_n(v_n^n) \).

We proceed to the system solution (13) by the Newton method. We subtract the first equation from the second and write in coordinate form

\[
\begin{cases}
(x_n(v_1) - x_s) t_1 - (x_n(v_2) - x_s) t_2 = 0; \\
(y_n(v_1) - y_s) t_1 - (y_n(v_2) - y_s) t_2 = 0; \\
(z_n(v_1) - z_s) t_1 - (z_n(v_2) - z_s) t_2 = 0.
\end{cases} \quad (17)
\]
Since, in the system (14), we have three equations and four unknowns \((t_1, t_2, v_1, v_2)\), then one of the parameters can be taken arbitrarily, for example \(t_1 = t_1'\). We denote
\[
 f_i(t_2, v_1, v_2) = (x_n(v_i) - x_0)t_1'' - (x_n(v_2) - x_0)t_2;
\]
\[
 f_i(t_2, v_1, v_2) = (y_n(v_i) - y_0)t_1'' - (y_n(v_2) - y_0)t_2;
\]
\[
 f_i(t_2, v_1, v_2) = (z_n(v_i) - z_0)t_1'' - (z_n(v_2) - z_0)t_2,
\]
then
\[
 \frac{\partial f_i(t_2, v_1, v_2)}{\partial t_2} = x_n - x_0; \quad \frac{\partial f_i(t_2, v_1, v_2)}{\partial v_1} = \frac{dx_n(v_i)}{dv_1}t_1''; \quad \frac{\partial f_i(t_2, v_1, v_2)}{\partial v_2} = \frac{dx_n(v_2)}{dv_2}t_2;
\]
\[
 \frac{\partial f_i(t_2, v_1, v_2)}{\partial t_2} = y_n - y_0; \quad \frac{\partial f_i(t_2, v_1, v_2)}{\partial v_1} = \frac{dy_n(v_i)}{dv_1}t_1''; \quad \frac{\partial f_i(t_2, v_1, v_2)}{\partial v_2} = \frac{dy_n(v_2)}{dv_2}t_2;
\]
\[
 \frac{\partial f_i(t_2, v_1, v_2)}{\partial t_2} = z_n - z_0; \quad \frac{\partial f_i(t_2, v_1, v_2)}{\partial v_1} = \frac{dz_n(v_i)}{dv_1}t_1''; \quad \frac{\partial f_i(t_2, v_1, v_2)}{\partial v_2} = \frac{dz_n(v_2)}{dv_2}t_2.
\]

The increment of the unknown parameters is determined by the solution of the linear equations’ following system
\[
 \begin{align*}
 \frac{\partial f_i}{\partial t_2} \Delta t_2 + \frac{\partial f_i}{\partial v_1} \Delta v_1 + \frac{\partial f_i}{\partial v_2} \Delta v_2 &= -f_i, \\
 \frac{\partial f_i}{\partial t_2} \Delta t_2 + \frac{\partial f_i}{\partial v_1} \Delta v_1 + \frac{\partial f_i}{\partial v_2} \Delta v_2 &= -f_i, \\
 \frac{\partial f_i}{\partial t_2} \Delta t_2 + \frac{\partial f_i}{\partial v_1} \Delta v_1 + \frac{\partial f_i}{\partial v_2} \Delta v_2 &= -f_i.
\end{align*}
\]

In the next step of the iteration, the parameters are equal to:
\[
 t_2 = t_2 + \Delta t_2; \quad v_1 = v_1 + \Delta v_1; \quad v_2 = v_2 + \Delta v_2.
\]

The iteration process continues until the maximum increment in absolute value becomes less than the specified value \(\varepsilon\).

Defining the parameter values \(v_1\) and \(v_2\), from (14) we find the generators along which the cones intersect. These are the generators of the ruled surface. In figure 7 they are indicated by \(g\) and \(\bar{g}\). Performing this algorithm for a number of line points \(k\), we get the right number of generators.

Summary

The developed software, based on the algorithms presented in the article, can be used to solve various problems of designing the objects using ruled surfaces. The software package can be used independently or included, as a module, in specialized CAD systems.

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