Quantum contextuality and joint measurement of three observables of a qubit

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Whereas complementarity manifests itself via two incompatible observables, quantum contextuality can only be revealed via the joint measurements among at least three observables. By incorporating unsharp measurements and joint measurements into a realistic model, we reestablish an inequality due to Liang, Spekkens, and Wiseman rigorously based on the assumption of noncontextuality alone. Its violation therefore unambiguously pinpoints the quantum contextuality of a two-level system. The maximal violation is attained by three triplewise jointly measurable observables that are pairwise jointly measured in an incompatible way. We also present the necessary and sufficient condition of triplewise joint measurability of three unbiased observables of a qubit.

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Complementarity and quantum contextuality are two fundamental essential elements of quantum theory, both originate in the fact that there are incompatible observables that cannot be measured jointly in a single measurement apparatus. As stated by Bohr’s complementarity principle \[^1\], there are mutually exclusive but equally real aspects of quantum systems. Two exclusive properties or two incompatible observables are enough to demonstrate complementarity, e.g., wave versus particle and momentum versus position. Quantitatively, the complementarity can be demonstrated by various kinds of uncertainty relationships for preparation \[^2\] as well as for the joint measurement of two noncommuting observables \[^3\].

Unlike complementarity, quantum contextuality can only be revealed by the joint measurements among at least three observables, e.g., via the pairwise joint measurements. A joint measurement of two or more observables defines a measurement context and at least two different contexts must be present to demonstrate the contextuality. By non-contextuality we mean that the outcomes of a faithful measurement of a given observable are predetermined regardless of what other compatible observables might be measured along. For systems with three or more distinguishable states, Kochen and Specken (KS) \[^9\], as well as by Bell \[^10\] proved quantum contextuality by showing that non-contextual realistic models cannot reproduce all the predictions of quantum mechanics via mere logical contradictions. With the help of non-contextuality inequalities \[^11\] \[^12\], or KS inequalities, quantum contextuality can be put to experimental tests, just like Bell inequalities. There are also state-independent proofs \[^13\] \[^14\] that do not arise from KS-type logical contradictions.

In the case of two-level systems, or qubits, proofs of quantum contextuality inevitably involve some additional assumptions other than non-contextuality. For examples, Cabello and Nakamura \[^17\] tried to extract a logical contradiction of KS type by assuming the outcome determinism for unsharp measurements. However, a non-contextual model \[^18\] \[^19\] was found to explain the contradictions away. Accardi \[^15\] and Fujikawa \[^16\] used the conditional probabilities of sequential measurements, assuming the existence of their counterparts in quantum theory, in their arguments. Busch \[^20\], as well as Caves et al, \[^21\] proved Gleason’s theorem for qubit by assuming the additivity of unsharp measurements.

Notably, Spekkens \[^22\] formulated a kind of generalized notion of measurement contextuality, assuming certain linearity for unsharp measurement. Later on, based on this notion, Liang, Spekkens, and Wiseman (LSW) \[^23\] derived an inequality on the average anti-correlations in the Specker’s scenario \[^24\] in which three pairwise joint measurements of three observables define three measurement contexts. Recently Runjwal and Ghosh \[^25\] found out a violation to LSW’s inequality in a qubit. However, because of some additional assumptions used in the original derivation of LSW’s inequality, the violation to LSW’s inequality does not clearly pinpoint the quantum contextuality of a qubit.

The purpose of this Letter is twofold. One is classical: we shall at first model unsharp measurements and joint measurements in a non-contextual realistic model and then re-derive LSW’s inequality based rigorously on the assumption of non-contextuality alone. One is quantum: we shall derive the maximal violation to LSW’s inequality by a qubit and the necessary and sufficient condition for three unbiased observables of a qubit to be triplewise joint measurable. Contrary to the customary expectations, it is the compatibility of pairwise joint measurements instead of triplewise joint measurability that is relevant to the quantum contextuality of a qubit.

Quantum theory can be regarded a special kind of operational probabilistic theory, making statistical predictions on possible measurements. The issue of contextuality arises when one tries to attribute properties to the system independent of the measurements, i.e., to understand the statistical predictions from a non-contextual and realistic point of view. In a non-contextual realistic model all observables have realistic values predetermined by some hidden variables, denoted collectively as \(\lambda\), distributed...
according to some probability distribution \( g_\lambda \) normalized to 1. Measurements are physical processes capable of revealing these predetermined values. Some measurements are faithful, called here as \textit{sharp} measurements, and some might not be so faithful, called here as \textit{unsharp} measurements. How a measuring apparatus responds to the predetermined values is recorded by the so-called \textit{response function}, which was first introduced by Spekkens [22]. A sharp measurement of an observable \( A \) yields outcomes that are identical to the predetermined values \( A(\lambda) \) and therefore it has an ideal response function \( \chi_\mu[A(\lambda)] \), which equals to 1 if \( A(\lambda) = \mu \) and 0 otherwise. The probability of obtaining outcome \( k \) by a sharp measurement of \( A \) reads

\[
P(\mu|A) = \int d\lambda g_\lambda \chi_\mu[A] := \langle \chi_\mu[A] \rangle
\]

(1)

where \( \chi_\mu[A] = \sum_\mu' P_{\mu|\mu'}(\lambda) \chi_{\mu'}[A] \) is defined to be the response function of an unsharp measurement of \( A \). In general, the response function \( \tilde{\chi}_\mu[A] \) of a measurement of \( A \) with outcome \( \mu \) is defined to be a function of the hidden variable \( \lambda \) satisfying

\[
\text{RF1. } \tilde{\chi}_\mu[A] \geq 0, \quad \text{RF2. } \sum_\mu \tilde{\chi}_\mu[A] = 1
\]

(2)

such that RF3. (c.f. Eq.(1)) \textit{the probability of obtaining outcome } \( \mu \text{ is given by the average of the response function.} \) Condition RF3 defines the response functions whereas conditions RF1 and RF2 are justified by the fact that the probability is nonnegative and normalized, respectively, for any distribution \( g_\lambda \) of hidden variables. For unsharp measurements the response functions may differ from the predetermined non-contextual values and can even be contextual in a non-contextual model as will be shown below. In comparison, Spekkens [22] assumed that the response functions are predetermined and non-contextual albeit non-deterministic values of observables.

By a \textit{joint measurement} of two observables \( A_1 \) and \( A_2 \) we mean any unsharp measurement that outputs a joint probability distribution of measurement results of \( A_1 \) and \( A_2 \) in any distribution \( g_\lambda \) of the hidden variables. The response function \( \tilde{\chi}_{\mu\nu}[A_{12}] \) of the joint measurement has two response functions of \( A_1 \) and \( A_2 \) as marginals, i.e.,

\[
\tilde{\chi}_\mu[A_1] = \sum_\nu \tilde{\chi}_{\mu\nu}[A_{12}], \quad \tilde{\chi}_\nu[A_2] = \sum_\mu \tilde{\chi}_{\mu\nu}[A_{12}].
\]

(3)

The joint measurement of three or more observables can be defined similarly. Unlike quantum cases, in a non-contextual realistic model all observables are jointly measurable since the product of all the response functions defines a joint measurement.

A joint measurement defines a measurement context and different measurement contexts may be incompatible even classically. Three pairwise joint measurements \( \tilde{\chi}_{\mu\nu}[A_{jk}] \) of three observables \( A_k \) with \( j < k \) and \( j, k = 1, 2, 3 \) are \textit{compatible} if there exists a joint measurement \( \tilde{\chi}_{\mu\nu\tau}[A_{123}] \) that has those three response functions of pairwise joint measurements as marginals, e.g.,

\[
\tilde{\chi}_{\mu\nu}[A_{12}] = \sum_\tau \tilde{\chi}_{\mu\nu\tau}[A_{123}].
\]

(4)

Specifically, we consider in what follows binary observables taking values \( \pm 1 \). The response function of a sharp measurement of a binary observable is given by \( \chi_\mu[A] = (1+\mu A)/2 \). In an unsharp measurement of \( A \) there might be a probability \( P_+(\lambda) \) or \( P_-(\lambda) = 1 - P_+(\lambda) \) of obtaining an outcome \( \mu = \pm 1 \) if the predetermined value of \( A \) is actually \( \mu \) or \( \bar{\mu} = -\mu \), respectively. Denote by \( \eta(\lambda) = P_+(\lambda) - P_-(\lambda) \) the \textit{local unsharpness} and the corresponding response function reads

\[
\tilde{\chi}_\mu[A] = P_+(\lambda) \chi_\mu[A] + P_-(\lambda) \chi_{\bar{\mu}}[A] = \frac{1 + \mu \eta(\lambda) A}{2}.
\]

(5)

The \textit{(global) sharpness} \( \eta \) of an unsharp measurement can be understood in a theory independent fashion as

\[
\eta = \min_{A = \pm} \max_{\lambda} |P(+) - P(-)|
\]

(6)

with minimization taken over all possible states in which observable \( A \) has definite values. In a realistic model \( \eta = \min(|\eta(\lambda)|) \) over all distributions \( g_\lambda \) of \( \lambda \).

Given three binary observables \( A_1, A_2 \) and \( A_3 \), the most general pairwise joint measurement of observable \( A_j \) and \( A_k \) has a response function

\[
\tilde{\chi}_{\mu\nu}[A_{jk}] = \frac{1 + \mu \eta_j(\lambda) A_j + \nu \eta_k(\lambda) A_k + \mu \nu C_{jk}}{4}
\]

(7)

where \( C_{jk} \) is an observable whose predetermined values satisfy \( 1 \geq C_{jk} \geq |\eta_j(\lambda) A_j \pm \eta_k(\lambda) A_k| \) to ensure \( \tilde{\chi}_{\mu\nu}[A_{jk}] \geq 0 \). The anti-correlation, i.e., the probability of obtaining different outcomes in a joint measurement, is given by the average of \( (1 + C_{jk})/2 \).

\textit{Theorem 1.} Three pairwise joint measurements of three binary observables are compatible if and only if

\[
1 - |C_{13} - C_{23}| \geq C_{12} \geq |C_{13} + C_{23}| - 1.
\]

(8)

Proof is given in Supplemental Material [38] and we note an interesting similarity to Accardi’s inequality on conditional probabilities [15]. For three compatible pairwise joint measurements, because of the triplywise joint
measurement, all the outcomes of pairwise joint measurements can be accounted for in a non-contextual manner, i.e., they are determined by the hidden variables alone and independent of which observables might be measured along. In other words, the probability of giving a false response to an actual value is independent of what other observables that might be measured along. However three observables can also be pairwise jointly measured in an incompatible manner. For example, the pairwise joint measurements of three observables given by \( \eta_k = \eta < 1/2 \) for \( k = 1, 2, 3 \) and \( C_{jk} = \eta(1 + A_j A_k) - 1 \) violate the condition Eq. (8) and therefore are incompatible. In this case a non-contextual account for all the long-run statistics is impossible because of the absence of a joint probability distribution. Thus even a non-contextual realistic model may exhibit measurement contextuality.

However this measurement contextuality induced by unsharp measurements cannot account for the quantum contextuality. To show this we consider the average anti-correlation, i.e., the average probability of obtaining different outcomes, in three pairwise joint measurements

\[
R_3 = \frac{1}{3} \sum_{j<k} \sum_{\mu=\pm} P(\mu, -\mu | A_{jk})
\]

which is first introduced by LSW [23]. For compatible pairwise joint measurements, using Theorem 1, it holds \( R_3 \leq 2/3 \) [23]. A violation \( R_3 > 2/3 \) does not mean that those three observables are not triplewise jointly measurable. Instead, it means that these three observables are pairwise-jointly measured in an incompatible way.

**Theorem 2.** In a non-contextual realistic model the average anti-correlation of three unsharp measurements with sharpness \( \eta_1 \geq \eta_2 \geq \eta_3 \) satisfies

\[
R_3 \leq 1 - \frac{\eta_1}{3}.
\]

Proof is given in Supplemental Material [26]. In appearance this is just a trivial generalization of LSW’s inequality to the case of unequal sharpness. However there are two main differences in their derivations. First, in its original proof [23] the response function of unsharp measurement is obtained on the assumption of certain linearity of response functions, which may not hold in some non-contextual models [26], so that only global sharpness is considered. Second, in its original proof [23] only a subset of response functions Eq. (9), in which \( C_{ijk} \) is implicitly assumed to be a function of \( A_j \) and \( A_k \), was taken into account. Here we have taken into account all possible response functions conforming to the long-run statistics and, because of the local sharpness, the response function needs not to be linear. As a result we have established LSW’s inequality rigorously based on the assumption of non-contextuality alone, i.e., it is valid for any non-contextual model admitting unsharp measurements and joint measurements. All relevant quantities such as sharpness, joint measurements, anti-correlations are also well defined in quantum theory.

Quantum mechanically the most general measurement is a positive operator valued measure (POVM), a set of positive operators \( \{O_\mu \geq 0\} \) summed up to the identity, playing the role of response function. Two observables \( O_\mu^1 \) and \( O_\mu^2 \), are jointly measurable if there exists an observable \( \{M_{\mu \nu}^{12}\} \) having two given POVMs as marginals, i.e., \( O_\mu^1 = \sum_\nu M_{\mu \nu}^{12} \) and \( O_\nu^2 = \sum_\mu M_{\mu \nu}^{12} \). Three observables \( \{O_\mu^k\} \) with \( k = 1, 2, 3 \) are called triplewise jointly measurable if there is a joint observable \( \{M_{\mu \nu \tau}\} \) having the three given observables as marginals, e.g., \( O_\mu^1 = \sum_{\nu \tau} M_{\mu \nu \tau} \). Three pairwise joint measurements \( \{M_{\mu \nu}^{ij}\}_{i<j} \) are compatible if there exists a triplewise joint observable \( \{M_{\mu \nu \tau}\} \) such that these three pairwise measurements arise as marginals, e.g., \( M_{\mu \nu}^{12} = \sum_\tau M_{\mu \nu \tau} \). Obviously triplewise jointly measurable observables are pairwise jointly measurable and three observables having compatible pairwise joint measurements are triplewise jointly measurable. However, triplewise joint measurable observables may have incompatible pairwise joint measurements.

An unbiased observable of a qubit refers to a two-outcome POVM \( \{O_\pm(\vec{x}) = \frac{1}{2}(1 \pm \vec{x} \cdot \vec{\sigma})\} \) with \( \eta = |\vec{x}| \leq 1 \) being exactly the global sharpness. It is unbiased in the sense that the outcomes of the measurement are purely random if the system is in the maximally mixed state. The necessary and sufficient condition for the joint measurability of two most general unsharp observables of a qubit has been found [27][31]. Two unbiased observables \( \{O_\pm(\vec{x}_{i,j})\} \) are joint measurable if and only if

\[
H_{ij} := 1 - |\vec{x}_{i,j}|^2 - |\vec{x}_{i,j}|^2 + (\vec{x}_i \cdot \vec{x}_j)^2 \geq 0.
\]

**Theorem 3.** For three pairwise jointly measurable unbiased observables \( \{O_\pm(\vec{x}_i)\} \) with the same unsharpness
$|\vec{\lambda}_i| = \eta$ for $i = 1, 2, 3$, it holds
\begin{equation}
R_3 \leq \begin{cases} \frac{1}{2} + \frac{\eta^2}{2} + \frac{1}{2} \sqrt{1 - 2\eta^2 + \frac{\eta^4}{4}}, & \eta \leq \sqrt{0.3}, \\ \frac{1}{2} - \frac{3}{2}\eta^2, & \eta \geq \sqrt{0.3}. \end{cases} \end{equation}
(11)

The upper bound is attained by trine spin observables, i.e., $\vec{\lambda}_1 \cdot \vec{\lambda}_2 = -1/2$, in the case of $\eta \leq \sqrt{0.3}$ and three parallel observables, i.e., $\vec{\lambda}_1 = \vec{\lambda}_2 = -\vec{\lambda}_3$, in the case of $\eta \geq \sqrt{0.3}$. The optimal state is a pure state whose Bloch vector $\vec{\varrho} = \text{Tr}(\vec{\varrho} \vec{\sigma})$ is orthogonal to $\vec{\lambda}_{1,2,3}$. The optimal pairwise joint measurements are composed of four rank-1 effects
\begin{equation}
M_{ij}^{\mu\nu} = \frac{1}{4} (1 + \mu \nu \vec{\lambda}_i \cdot \vec{\lambda}_j)(I + \vec{m}_{ij}^{\mu\nu} \cdot \vec{\delta}) \end{equation}
where $\vec{m}_{ij}^{\mu\nu} \propto \mu \vec{\lambda}_i + \nu \vec{\lambda}_j - \mu \nu \vec{\varrho} \sqrt{H_{ij}}$ are unit vectors for $i < j$ and $i, j = 1, 2, 3$.

Proof is given in Supplemental Material [30]. As shown in Fig.1a LSW’s inequality is violated as long as $\eta \neq 0, 1$. The maximal violation $\delta = R_3 - 1 + \eta/3$ to LSW’s inequality is found numerically to be attained at $\eta_{\text{c}} \approx 0.456619$ with $R_3 = 0.937439$ by trine spin observables. To attain this optimal value, three observables are triplywise jointly measurable with incompatible pairwise joint measurements. The optimal pairwise joint measurements in this case are illustrated in Fig.2b. In order to further investigate the relevance of triplywise joint measurability to the violation to LSW’s inequality, we shall derive the condition for the triplywise joint measurability of three unbiased observables.

The joint measurement of three unbiased orthogonal observables was first considered by Busch [32] with a sufficient condition that is proved by Barnet [34] to be also necessary. Liang, Spekkens, and Wiseman [25] provided the necessary and sufficient condition for trine spin observables. Pal and Ghosh [35] proved a necessary condition for the triplywise joint measurability of three general unsharp observables in terms of the Fermat-Toricelli (FT) vector. By definition, a FT vector of a set of three or more vectors $\{\vec{v}_k\}$ in Euclidean space is the vector $\vec{v}$ that minimizes the total distances $\sum_k |\vec{v} - \vec{v}_k|$. The FT vector always exists and is unique [35]. Pal and Ghosh’s necessary condition [35] can be proved (Supplemental Material [30]) to be sufficient for three unbiased observables:

**Theorem 4.** Three unbiased observables $\{O_{\pm}(\vec{\lambda}_k)\}^3_{k=1}$ are triplywise jointly measurable if and only if
\begin{equation}
\sum_{a=0}^{3} \left| \vec{\lambda}_{a} - \vec{\lambda}_{\text{FT}} \right| \leq 4 \end{equation}
(13)
where $\vec{\lambda}_{\text{FT}}$ denotes the FT vector of four vectors
\begin{equation}
\vec{\lambda}_0 = \vec{\lambda}_1 + \vec{\lambda}_2 + \vec{\lambda}_3, \quad \vec{\lambda}_k = 2\vec{\lambda}_k - \vec{\lambda}_0 \quad (k = 1, 2, 3). \end{equation}

The FT vector of four general vectors does not have an analytical expression. In some special cases such as coplanar vectors and one vector being orthogonal to other two vectors the FT vector can be found explicitly.

**Example 1: coplanar observables.** Without loss of generality we can suppose that three coplanar directions $\{\vec{\lambda}_i\}$ lie in the same half plane and $\vec{\lambda}_1$ lies between directions $\vec{\lambda}_1$ and $\vec{\lambda}_2$ as shown in Fig.1(a) and Fig.1(b), in which dark gray shaded region $\Delta_{12}$ denotes the triangle formed by vectors $\vec{\lambda}_{1,2}$ and the zero vector. If $\vec{\lambda}_3 \in \Delta_{12}$ then four vectors $\{\vec{\lambda}_a\}^3_{a=0}$ form a convex quadrilateral as shown in Fig.1(a) and the FT vector can be easily found to be the intersection of two diagonals, shown as red dots in Fig.1.

The condition of triplywise joint measurability Eq.(13) turns out to be exactly the condition of the joint measurability of two observables $\{O_{\pm}(\vec{\lambda}_{1,2})\}$. That means the observable that is a convex combination of two jointly measurable observables can also be measured jointly for free. If $\vec{\lambda}_3 \notin \Delta_{12}$ then vector $\vec{\lambda}_3$ falls in the triangle formed by three other vectors and coincides with the FT vector. As a result the condition Eq.(13) becomes
\begin{equation}
|\vec{\lambda}_1 + \vec{\lambda}_2| + |\vec{\lambda}_1 - \vec{\lambda}_3| + |\vec{\lambda}_2 - \vec{\lambda}_3| \leq 2. \end{equation}
(15)

In particular if all three coplanar vectors $\vec{\lambda}_k$ have the same length $\eta = |\vec{\lambda}_k|$ then we have $\vec{\lambda}_3 \notin \Delta_{12}$. Denoting by $\phi_k$ the angles spanned by $\vec{\lambda}_3$ and $\vec{\lambda}_k$ for $k = 1, 2$, the triplywise joint measurability condition Eq.(13) becomes
\begin{equation}
\eta \leq \left( \cos \frac{\phi_1 + \phi_2}{2} + \sin \frac{\phi_1}{2} \right)^{-1}. \end{equation}
(16)

In the case of trine spin observables where $\phi_1 = \phi_2 = \pi/3$ we reproduce the known condition $\eta \leq \frac{5}{2}$ [23].

**Example 2: $\vec{\lambda}_3 \perp \vec{\lambda}_{1,2}$.** In this case the the FT vector of four vectors $\{\vec{\lambda}_a\}$ can be found explicitly $\vec{\lambda}_{\text{FT}} \propto \vec{\lambda}_3$.
The triplewise joint measurability condition Eq. 13 now becomes
\[ |\bar{\lambda}_1 + \bar{\lambda}_2| + |\bar{\lambda}_1 - \bar{\lambda}_2| \leq 2\sqrt{1 - |\bar{\lambda}_3|^2}. \] (17)
In the case of \( \bar{\lambda}_{1,2} \) also being orthogonal we reproduce the known condition \( \sum_i |\bar{\lambda}_i|^2 \leq 1 \) [34] for the joint measurement of three unbiased orthogonal observables.

Interestingly, there are three observables that are not triplewise jointly measurable but cannot violate LSW’s inequality no matter how each two observables are jointly measured. For example, we consider three co-planar observables \( \{O_\mu(\hat{\lambda}_k)\} \) with identical sharpness \( \eta = 1/\sqrt{2} \) and suppose that the angles spanned by \( \hat{\lambda}_3 \) and \( \hat{\lambda}_{1,2} \) are \( \phi_1 = \phi_2 = 3\pi/4 \), respectively. These three observables are obviously pairwise jointly measurable but not triplewise jointly measurable according to condition Eq. 13. In this case the maximal average anti-correlation reads
\[ R_3 = (3 + \sqrt{2})/6 < \sqrt{2}/6 = 1 - \eta/3. \]

In conclusion, by modeling unsharp measurement and joint measurements in realistic models, we have established LSW’s inequality based rigorously and solely on the assumption of non-contextuality. Thus LSW’s inequality can be regarded as a genuine KS inequality involving three observables and can be put to experimental tests. The introduction of unsharp and joint measurements in a realistic model allows the room for some kind of measurement contextuality even classically. However this kind of measurement contextuality is not enough to explain all the quantum mechanical predictions on two-level systems as LSW’s inequality can be violated, showing that even for a two-level system the attribution of predetermined values to observables is at odds with quantum mechanics. The maximal violation to LSW’s inequality is found to be attained by three pairwise jointly measurable observables that are pairwise measured in an incompatible way. Also we derive the necessary and sufficient condition for the joint measurement of three unbiased observables and show that there are three observables that are not triplewise joint measurable cannot give rise to a violation of LSW’s inequality.

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[36] See Supplemental Material for the proofs of Theorems 1 to 4.
Supplemental Material

To recapitulate, in a non-contextual realistic model, all observables possess predetermined values that can be revealed by either a sharp measurement or an unsharp measurement. While a sharp measurement always gives the correct response, unsharp measurement may give wrong response to the predetermined values. A general unsharp measurement is characterized by a response function satisfying three conditions

RF1. $\tilde{x}_\mu[A] \geq 0$,  
RF2. $\sum_\mu \tilde{x}_\mu[A] = 1$ (s.1)

such that the probability of obtaining outcome $k$ is given by the average of the response function, i.e.,

RF3. $P(\mu|\tilde{A}) = \langle \tilde{x}_\mu[A] \rangle$. (s.2)

A joint measurement outputs a joint probability distribution of measurement results of $A_1$ and $A_2$ in any distribution of the hidden variables $g_\lambda$. Thus it has a response function $\tilde{x}_{\mu\nu}[A_{12}]$ with two response functions of the given measurements of $A_1$ and $A_2$ as marginals

$$\tilde{x}_\mu[A_1] = \sum_\nu \tilde{x}_{\mu\nu}[A_{12}], \quad \tilde{x}_\nu[A_2] = \sum_\mu \tilde{x}_{\mu\nu}[A_{12}].$$ (s.3)

Three pairwise joint measurements $\tilde{x}_{\mu\nu}[A_{jk}]$ of three observables $A_k$ with $j < k$ and $j, k = 1, 2, 3$ are compatible if there exists a joint measurement $\tilde{x}_{\mu\nu\tau}[A_{123}]$ having those three response functions of pairwise joint measurements as marginals, e.g.,

$$\tilde{x}_{\mu\nu}[A_{12}] = \sum_\tau \tilde{x}_{\mu\nu\tau}[A_{123}].$$ (s.4)

Specifically, a general measurement of a binary observable $A_k$ taking values $\pm$ has the following response function

$$\tilde{x}_\mu[A_k] = \frac{1 + \mu \eta_k(\lambda)A_k}{2} := \frac{1 + \mu \tilde{A}_k}{2}.$$ (s.5)

The most general joint measurement of two observables $A_j$ and $A_k$ has the following response function

$$\tilde{x}_{\mu\nu}[A_{jk}] = \frac{1 + \mu \eta_j(\lambda)A_j + \nu \eta_k(\lambda)A_k + \mu \nu C_{jk}}{4},$$ (s.6)

where $C_{jk}$ is an arbitrary observable whose predetermined values satisfy $1 \pm C_{jk} \geq |\eta_j(\lambda)A_j \pm \eta_k(\lambda)A_k|$. The (global) sharpness $\eta$ of an unsharp measurement can be understood in a theory independent fashion as

$$\eta = \min_{A = \pm} \frac{1}{2} \left| P(+|\tilde{A}) - P(-|\tilde{A}) \right|$$ (s.7)

with minimization taken over all possible states in which observable $A$ has definite values. In a realistic model $\eta = \min(\langle \eta(\lambda) \lambda \rangle)$ over all distributions $g_\lambda$.

**Proof of Theorem 1.** — Consider three observables $A_k$ each of which is measured unsharply with some sharpness $\eta_k(\lambda)$ and response functions $\tilde{x}_{\mu}[A]$. Let $\tilde{x}_{\mu\nu}[A_{jk}]$ be the response function of three pairwise joint measurements for $j < k$. If these three pairwise joint measurements are compatible then there exists a triplewise joint measurement with response function

$$6 \tilde{x}_{\mu\nu\tau}[A_{123}] = 1 + \mu \tilde{A}_1 + \nu \tilde{A}_2 + \tau \tilde{A}_3 + \mu \nu C_{12} + \mu \tau C_{13} + \nu \tau C_{23} + \mu \nu \tau C,$$ (s.8)

where $C$ is an arbitrary observable whose predetermined values must ensure $\tilde{x}_{\mu\nu\tau}[A_{123}] \geq 0$, which are equivalent to $1 + \tau C \geq \Gamma_{\mu\nu}$ for all $\mu, \nu, \tau = \pm$ where

$$\Gamma_{\mu\nu} = \mu (\tilde{A}_1 + \tau \tilde{A}_3) + \nu (\tilde{A}_2 + \tau \tilde{A}_3) - \mu \nu (C_{12} + \tau \tilde{A}_3).$$ (s.9)

As a result we obtain

$$2 \geq \Gamma_{\mu\nu} + \Gamma_{\nu',\mu'} \quad (\mu, \nu, \mu', \nu' = \pm).$$ (s.10)

In the case of $\mu' = \nu' = -$, we obtain

$$1 + \mu C_{12} + \nu C_{23} + \mu \nu C_{13} \geq 0, \quad (\mu, \nu = \pm)$$

from which it follows Eq.(s.8). On the other hand, by noticing that Eq.(s.10) is ensured by $\tilde{x}_{\mu\nu\tau}[A_{123}] \geq 0$ for $j < k$ in the case of $(\mu', \nu') \neq (-, -)$ and by the condition Eq.(s.8) in the case of $(\mu', \nu') = (-, -)$, the choice

$$C = \max_{\mu, \nu} \Gamma_{\mu\nu} - 1$$

makes Eq.(s.8) a response function of a triplewise joint measurement with three given pairwise joint measurements as marginals, i.e., they are compatible. □

**Proof of Theorem 2.** — By substituting the response functions Eq.(7) of the most general pairwise measurements into $R_3$ we obtain

$$R_3 = \frac{1}{3} \sum_{j<k} \left( P(+, -|A_{jk}) + P(-, +|A_{jk}) \right)$$

$$= \frac{1}{3} \sum_{j<k} \langle \tilde{x}_{+-}|A_{jk} \rangle + \langle \tilde{x}_{-+}|A_{jk} \rangle$$ (s.11)

$$= \frac{1}{3} \sum_{j<k} \left[ 1 - \frac{1}{2} (C_{jk}) \right] = 1 - \frac{1}{3} \sum_{j<k} \frac{1}{2} (C_{jk})$$

$$\leq 1 - \frac{1}{6} \sum_{j<k} |\langle \eta_j(\lambda)A_j \pm \eta_k(\lambda)A_k \rangle|$$ (s.12)

$$\leq 1 - \frac{1}{3} \max_k \langle \eta_k(\lambda) \rangle$$ (s.13)

$$\leq 1 - \frac{1}{3} \max_k |\langle \eta_k(\lambda) \rangle| \leq 1 - \frac{1}{3} \max_k \eta_k.$$ (s.14)

Here Eq.(s.11) is due to the defining property RF3 of response function and the first inequality Eq.(s.12) is due
to $\chi_\mu[A_{jk}] \geq 0$ while the second inequality Eq. \((s.13)\) is due to the triangle inequality
\[
\sum_{j<k} |\eta_j(\lambda)A_j + \eta_k(\lambda)A_k| \geq 2|\eta_\lambda(\lambda)|,
\]
considering $|A_j| = 1$, for any $i = 1, 2, 3$. The last inequality Eq. \((s.14)\) is due to the definition of the global sharpness $\eta_\lambda = \min(|\eta_\lambda(\lambda)|) \leq |\langle \eta_\lambda(\lambda)\rangle|$. □

Proof of Theorem 3. — For given two unbiased observables $(O_{\lambda}(\lambda), j)$ that are jointly measurable, i.e., $H_{ij} \geq 0$ the most general joint measurement is given by
\[
M_{ij}^\mu = \frac{I + \mu \nu Z_{ij} + (\mu \tilde{\lambda}_i + \nu \tilde{\lambda}_j - \mu \nu \tilde{z}_{ij}) \cdot \tilde{\sigma}}{4} \quad (s.15)
\]
with real number $Z_{ij}$ and vector $\tilde{z}_{ij}$ making $M_{ij}^\mu \geq 0$ for all $\mu, \nu = \pm$. This positivity requirement is equivalent to
\[
|\tilde{z}_{ij}|^2 \leq (1 + \mu Z_{ij})^2 - |\tilde{\lambda}_i + \mu \tilde{\lambda}_j|^2 := L_\mu(Z_{ij}) \quad (s.16)
\]
for $\mu = \pm 1$. Obviously it is necessary for $M_{ij}^\mu \geq 0$. To show its sufficiency we note that for each allowed value of $Z_{ij}$ determined by $L_\mu(Z_{ij}) \geq 0$, we choose $\tilde{z}_{ij}$ to be a vector orthogonal to both $\tilde{\lambda}_i, j$ with a length $\min_{\mu} L_{\mu}(Z_{ij})^{1/2}$, which define a joint measurement via Eq. \((s.15)\). In a given state $\varrho$ with a Bloch vector $\tilde{g} = \text{Tr} [\varrho \tilde{\sigma}]$ the anti-correlation for a given joint measurement Eq. \((s.15)\) reads
\[
R_{ij} = \text{Tr} (M_{ij}^{\mu \mu} + M_{ij}^{\mu -}) = \frac{1 - Z_{ij} + \tilde{g} \cdot \tilde{z}_{ij}}{2} \leq \frac{1 - Z_{ij} + |\tilde{z}_{ij}|}{2} \leq \frac{1 - Z_{ij} + \min_{\mu} \sqrt{L_{\mu}(Z_{ij})}}{2} \leq \frac{1 - \tilde{\lambda}_i \cdot \tilde{\lambda}_j + \sqrt{H_{ij}}}{2} \quad (s.17)
\]
The last inequality is due to the fact that $\min_{\mu} L_{\mu}(Z_{ij}) \leq H_{ij}$ in the case of $Z_{ij} \geq \tilde{\lambda}_i \cdot \tilde{\lambda}_j$ and $-Z_{ij} + \sqrt{L_{\mu}(Z_{ij})}$ is an increasing function of $Z_{ij}$ in the case of $Z_{ij} \leq \tilde{\lambda}_i \cdot \tilde{\lambda}_j$.

In the case of identical sharpness $|\tilde{\lambda}_i \cdot \tilde{\lambda}_j| = \eta$ we denote $\tilde{\lambda}_i \cdot \tilde{\lambda}_j = \eta^2 x_k$ for $(i, j, k)$ being three cyclic permutation of $(1, 2, 3)$. As a result the average anti-correlation has the following upper bound
\[
R_3 \leq \frac{r(x_1) + r(x_2) + r(x_3)}{6} \quad (s.18)
\]
with $r(x) = 1 - \eta^2 x + \sqrt{1 - 2\eta^2 + \eta^4 x^2}$. At least two out of three $x_k$ should be negative to achieve the largest upper bound. Without loss of generality we suppose $x_3 \leq 0$. Taking into account the positive semi-definiteness of the Gram matrix $[|\tilde{\lambda}_i \cdot \tilde{\lambda}_j|] \geq 0$ for three vectors $\tilde{\lambda}_k$, it holds $x_3 \geq x_1 x_2 - \tilde{x}_1 \tilde{x}_2$ where $\tilde{x}_k = \sqrt{1 - x_k^2}$. Since the function $r(x)$ is a decreasing function of $x$ in the case of $x \leq 0$, the upper bound achieves the largest value when $x_3 = x_1 x_2 - \tilde{x}_1 \tilde{x}_2$ and in this case we denote by $U(x_1, x_2)$ the r.h.s. of Eq. \((s.18)\). This upper bound is actually attained by the trine observables as shown in [25].

The maximal value of $U(x_1, x_2)$ is either achieved at the boundary $x_{1,2} = \pm 1$ or at the critical points determined by $\partial_k U(x_1, x_2) = 0$ for $k = 1, 2$, or equivalently
\[
\tilde{x}_k r'(x_k) + (x_1 \tilde{x}_2 + x_2 \tilde{x}_1) r'(x_3) = 0. \quad (s.19)
\]
It turns out that for any given constant $r$ the equation $r = \tilde{x} r'(x) := \tilde{r}(x)$ has two most solutions. This conclusion follows from the fact that the equation $\tilde{r}'(x) = 0$, which is equivalent to
\[
x (\eta^2 x - \sqrt{1 - 2\eta^2 + \eta^4 x^2}) = \frac{\eta^2 (1 - 2\eta^2) (1 - x^2)}{1 - 2\eta^2 + \eta^4 x^2} \quad (s.20)
\]
for $x \neq \pm 1$, has at most one solution. In fact, in the case of $1 - 2\eta^2 > 0$, the equation has only nonpositive solutions and l.h.s. is an increasing function of $-x$ while the r.h.s. is a decreasing function of $-x$. Thus there is at most one solution in this case and at most two solutions to the equation $\tilde{r}(x) = r$. In the case of $1 - 2\eta^2 < 0$ the l.h.s.$\geq 0$ while the r.h.s.$< 0$ so that $\tilde{r}'(x) = 0$ has no solution which means $\tilde{r}(x) = r$ has at most one solution for any $r$.

By noting that $\tilde{x}_3 = |x_1 \tilde{x}_2 + x_2 \tilde{x}_1|$ and the fact that the critical points of $U(x_1, x_2)$ are determined by $\tilde{r}(x_1) = \tilde{r}(x_2) = \tilde{r}(x_3)$ with $\alpha = -\text{sgn}(x_1 \tilde{x}_2 + x_2 \tilde{x}_1)$, we can conclude that at least two out three $x_{1,2,3}$ must be equal and negative. Therefore we have the upper bound
\[
R_3 \leq \frac{\max_{-1 \leq x \leq 0} \frac{g(x)}{6}, \ g(x) = 2r(x) + r(2x^2 - 1).}
\]
Function $g(x)$ is well defined in the interval a) $[-1,0]$ if $\eta \leq 1/\sqrt{2}$; b) $I_1 \cup I_2$ if $\sqrt{3} - 1 \geq \eta > 1/\sqrt{2}$; c) $I_1$, if $\eta > \sqrt{3} - 1$ where
\[
I_1 := \left[-1 - \sqrt{\frac{1 + \beta}{2}}, \beta \right], \quad I_2 := \left[-\sqrt{\frac{1 - \beta}{2}}, -\beta \right], \quad \text{with} \ \beta = \sqrt{2\eta^2 - 1/\eta^2}. \quad \text{In case c) function} \ g(x) \ \text{is decreasing so that its maximum is attained at} \ x = -1. \ \text{In case b) function} \ g(x) \ \text{is monotonously increasing in the interval} \ I_1 \ \text{and concave in} \ I_2 \ \text{with} \ x = -1/2 \ \text{as the unique critical point. Thus its maximum is achieved at} \ x = -1 \ \text{or} \ x = -1/2. \ \text{In case a), if} \ \eta < 1/\sqrt{2} \ \text{function} \ g(x) \ \text{has a unique critical point} \ x = -1/2. \ \text{If} \ 1/2 \leq \eta \leq 1/\sqrt{2} \ \text{function} \ g(x) \ \text{has two critical points with} \ x = -1/2 \ \text{being local maximum and the other being local minimum. As a result the maximal value of} \ g(x) \ \text{is attained either by the trine spin observables, i.e.,} \ x = -1/2 \ \text{or collinear observables} \ x = -1. \ \text{All these properties can be checked for the function} \ g(x) \ \text{of single variable numerically and ultimate upper bound is given by Eq. \((s.11)\) which is depicted in Fig.1a. On the other hand it is straightforward}
to check that the optimal pairwise joint measurements given in Eq.(12) and state specified in Theorem 3 actually attain the upper bound.

**Proof of Theorem 4.** — The necessary part has already been proved by Pal and Ghosh [33] in the case of three general unsharp observables. Here we include its proof for unbiased observables for the sake of completeness. The most general form of triplewise joint measurement, if exists, takes the following form

\[ 8M_{\vec{\mu}} = I + \sum_{i>j} \mu_i \mu_j (Z_{ij} + \vec{z}_{ij} \cdot \vec{\sigma}) + \sum_{i=1}^{3} \mu_i \vec{\lambda}_i \cdot \vec{\sigma} - \mu_1 \mu_2 \mu_3 \vec{z} \cdot \vec{\sigma} \]  

(s.21)

with real constants \( Z_{ij} \) and vectors \( \vec{z}_{ij} \) and \( \vec{z} \) making \( M \geq 0 \), which is equivalent to

\[ 1 + \sum_{i>j} \mu_i \mu_j Z_{ij} \geq \left| \sum_i \mu_i \vec{\lambda}_i + \sum_{i>j} \mu_i \mu_j \vec{z}_{ij} - \mu_1 \mu_2 \mu_3 \vec{z} \right|, \]  

(s.22)

By summing over all \( \mu_k = \pm 1 \) and separating two different cases \( \mu_1 \mu_2 \mu_3 = \pm 1 \) we obtain

\[ 8 \geq \sum_{\mu} \left| \sum_i \mu_i \vec{\lambda}_i - \sum_{i>j} \mu_i \mu_j \vec{z}_{ij} - \mu_1 \mu_2 \mu_3 \vec{z} \right| \]

\[ = \sum_{\mu_1 \mu_2 \mu_3 = 1} \left| \sum_i \mu_i \vec{\lambda}_i - \sum_{i>j} \mu_i \mu_j \vec{z}_{ij} - \vec{z} \right| + \sum_{\mu_1 \mu_2 \mu_3 = -1} \left| -\sum_i \mu_i \vec{\lambda}_i - \sum_{i>j} \mu_i \mu_j \vec{z}_{ij} + \vec{z} \right| \]

\[ \geq 2 \sum_{\mu_1 \mu_2 \mu_3 = 1} \left| \sum_i \mu_i \vec{\lambda}_i - \vec{z} \right| \]

\[ \geq 2 \sum_{\mu_1 \mu_2 \mu_3 = 1} \left| \sum_i \mu_i \vec{\lambda}_i - \vec{A}_{FT} \right| \]  

(s.23)

with the last inequality due to the definition of the FT vector \( \vec{A}_{FT} \) of four vectors \( \{ \sum_i \mu_i \vec{\lambda}_i \mid \mu_1 \mu_2 \mu_3 = 1 \} \).

To prove its sufficiency we consider eight operators \( M_{\vec{\mu}} \) as given in Eq.(s.21) with \( \vec{z} = \vec{A}_{FT} \), \( \vec{z}_{ij} = 0 \), and

\[ Z_{ij} = 1 - \frac{|\vec{\lambda}_i - \vec{A}_{FT}| + |\vec{\lambda}_j - \vec{A}_{FT}|}{2} \]

with \( i > j \) and \( i,j = 1,2,3 \). It is obvious that three given unbiased observables arise as marginals of \( \{ M_{\vec{\mu}} \} \), e.g., \( O_{\mu_1} (\vec{\lambda}_1) = \sum_{\mu_2 \mu_3} M_{\vec{\mu}} \) and the conditions Eq.(s.22), which are equivalent to \( M_{\vec{\mu}} \geq 0 \), are ensured by Eq.(13) for \( \vec{\mu} = (\pm \pm \pm) \) and become equalities otherwise.

**FT vector in the case of \( \vec{\lambda}_3 \perp \vec{\lambda}_{1,2} \).** — With the help of Lindloef and Sturm condition, which in this case reads

\[ \sum_{a=0}^{4} \frac{\vec{\lambda}_a - \vec{A}_{FT}}{|\vec{\lambda}_a - \vec{A}_{FT}|} = 0 \]