ZIGZAG NANORIBBONS IN EXTERNAL ELECTRIC AND MAGNETIC FIELDS

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ABSTRACT. We consider the Schrödinger operators on zigzag nanoribbons (quasi-1D tight-binding models) in external magnetic fields and an electric potential $V$. The magnetic field is perpendicular to the plane of the ribbon and the electric field is perpendicular to the axis of the nanoribbon and the magnetic field. If the magnetic and electric fields are absent, then the spectrum of the Schrödinger (Laplace) operator consists of two non-flat bands and one flat band (an eigenvalue with infinite multiplicity) between them. If we switch on the magnetic field, then the spectrum of the magnetic Schrödinger operator consists of some non-flat bands and one flat band between them. Thus the magnetic field changes the continuous spectrum but does not flatten the flat band. If we switch on a weak electric potential $V \to 0$, then there are two cases: (1) the flat band splits into the small spectral band. We determine the asymptotics of the spectral bands for small fields. (2) the unperturbed flat band remains the flat band. We describe all potentials when the unperturbed flat band remains the flat band and when one splits into the small band of the continuous spectrum. Moreover, we solve inverse spectral problems for small potentials.

1. Introduction

There are a lot of papers about the electronic structure of carbon materials such as carbon nanoribbons, nanotubes and fullerenes during the past two decades because of fundamental scientific interest in nanomaterials and because of their versatile electronic properties that are expected to be important for future nanoelectronics [DDE], [Ha], [SDD]. Among the carbon nanostructures, a simple variation of graphene, ribbons has been studied extensively. The graphene nanoribbons with varying widths can be realized either by cutting [Hi] mechanically exfoliated graphenes [No1], or by patterning epitaxially grown graphenes [ZTSK].

In our paper we concentrate on carbon nanoribbons, the recently discovered two-dimensional carbon crystal [No1], [No2]. The graphene forms a hexagonal graph (the honeycomb lattice) embedded in $\mathbb{R}^2$ with all edges of constant length, see Fig.1-3. In physics one considers only two type of ribbons: zigzag and armchair. When a single graphite layer is terminated by zigzag edges on both sides, then this layer is a zigzag graphene nanoribbon (ZGNR), see Fig.1. When a single graphite layer is terminated by armchair edges on both sides, then this layer is an armchair graphene nanoribbon, see Fig.1. Following conventional notation, a ZGNR is specified by the number $N$ of zigzag chains along the ribbon forming the width and is referred as a N-ZGNR. For example, in Fig.1 we have the 4-ZGNR.

The main interest in physics is to study electronic structure of ribbons, see [DDE], [Ha], [SDD] and references therein. Moreover, there are numerous physical papers devoted to
graphene nanotubes and nanoribbons in an external field, see [KLYH], [N], [No3], [SCL], [SDD] and references therein.

There are some mathematical papers about graphene-media, see [KL], [KL1], [K], [KuP] and [Pa], devoted to continuous models, see also [KS], [RR] about other models. But in the physical literature the most commonly used model is the tight-binding model, corresponding to discrete Hamiltonians. The zigzag and armchair nanotubes in external electric and magnetic fields (discrete models) were considered in [KKu4]. The zigzag nanoribbons (discrete models) in external electric field were considered in [KKu3].

We consider now the $N$-ZGNR, $N \geq 1$ in external fields, see Fig. 2. In fact we have a 2D problem. Firstly, there is a 2D nanoribbon $\subset \mathbb{R}^2$. The ZGNRs are assumed to be infinite along the x direction (horizontal). In general, the boundary of the ZGNR can be very complicated. We consider only the ZGNRs with the simple boundary, see Fig. 2. Secondly, an external 2-dim transverse electric field $E_{ext} \in \mathbb{R}^2$ is applied across the ZGNR and along the y direction (vertical), see Fig. 2 with the 3-ZGNR in an external electric potential. Moreover, the homogeneous magnetic field is perpendicular to the ribbon and along the z direction.

We will show that the spectrum of the Laplacian (an unperturbed operator) on the ZNRB consists of two spectral bands separated by a gap and exactly one eigenvalue with infinite multiplicity (a flat band) in this gap. If we switch on an external magnetic field, then we will show that the unperturbed spectral bands change, but the flat band (unperturbed eigenvalue) will be still a flat band. If additionally we switch on an small external electric field, then we will show that the spectral bands slightly change and the flat band will splits into either a new spectral band or will be still a flat band. Our main goal is to describe this transformation. For applications of our model see ref. in [Ha], [SDD]. The second part of our paper is devoted to the inverse problems.

Denote a $N$-ZGNR by $\Gamma = \Gamma_N$. We consider the ribbon $\Gamma \subset \mathbb{R}^2$ as graph. This graph is just a discrete set of vertices $V = \{v_\kappa : \kappa = (n,k) \in \mathbb{Z} \times N_p\}$, where $p = 2N + 1, N = \{1, \ldots, N\}$ and a set of undirected edges $E$ such that $[v, v'] \in E$ if the vertices $v$ and $v'$ are connected by an edge. Each vertex inside the ribbon is connected with some other 3 vertices, the vertex on the boundary is connected with 1 or 2 other vertices. If the vertices $v$ and $v'$ are connected by an edge, we denote this by $v \sim v'$. Introduce the discrete Hilbert space $\ell^2(\Gamma)$ consisting of functions $f = (f_\kappa)_{\kappa \in \mathbb{Z} \times N_p}$ on the set of vertices $V$ equipped with the norm $\|f\|_{\ell^2(\Gamma)}^2 = \sum |f_\kappa|^2$. We define the magnetic Hamiltonian $\Delta_b$ on the nanoribbon $\Gamma$ in an external magnetic fields $B = B(0,0,1) \in \mathbb{R}^3$ by

$$
(\Delta_b f)_\kappa = \sum_{v_\kappa \sim v_{\kappa'}} e^{ia_{\kappa \kappa'}} f_{\kappa'}, \quad \kappa = (n,k) \in \mathbb{Z} \times N_p, \quad p = 2N + 1, \quad b = \frac{B\sqrt{3}}{2},
$$
where \( f = (f_x)_{x \in \mathbb{Z} \times N_p} \in \ell^2(\Gamma) \) satisfies the Dirichlet boundary conditions
\[
  f_{n,0} = f_{n,p} = 0, \quad n \in \mathbb{Z}. \tag{1.1}
\]
The factor \( e^{ia_x.x'} \) is associated with the magnetic field \( \mathcal{B} \) and if \( \mathcal{B} = 0 \), then \( e^{ia_x.x'} = 1 \).

We define the discrete Hamiltonian \( H_b = \Delta_b + V \) on the nanoribbon \( \Gamma \) (a tight-binding model of single-wall nanoribbons, see [SDD], [N]) in an external electric potential \( V \) and the uniform magnetic field \( \mathcal{B} = B(0,0,1) \in \mathbb{R}^3 \). The magnetic field is perpendicular to the plane of the ribbon and the electric field is perpendicular to the axis of the nanoribbon and the magnetic field.

Our model nanoribbon \( \Gamma \) is a graph, which is a set of vertices \( v_{n,k} \) and edges \( E_{n,k,j} \) given by
\[
\begin{align*}
  v_{n,2k+1} &= (\sqrt{3}(2n+k), 3k), \quad k \in \mathbb{N}_N^0, \\
  v_{n,2k} &= (\sqrt{3}(2n+k), 3k-2), \quad k \in \mathbb{N}, \\
  E_{n,k,1} &= [v_{n,2k}, v_{n,2k+1}], \\
  E_{n,k,2} &= [v_{n,2k}, v_{n,2k-1}], \\
  E_{n,k,3} &= [v_{n,2k-1}, v_{n+1,2k-1}],
\end{align*}
\]
where \( \mathbb{N}_k = \{1, \ldots, k\} \subset \mathbb{N} \) and \( \mathbb{N}_k^0 = \mathbb{N}_k \cup \{0\} \); see Fig. 2 and 3 for the case \( N = 3 \).

Our Schrödinger operator has the form \( H_b = \Delta_b + V, b = \frac{Bx}{2} \in T = \mathbb{R}/(2\pi\mathbb{Z}), \) where the magnetic operator \( \Delta_b \) (see more in Section 4) and the electric potential \( V \) are given by
\[
\begin{align*}
  (\Delta_b f)_{n,2k+1} &= e^{-ib(2n+k)} f_{n,2k} + e^{ib(n+2k)} f_{n-1,2k+2} + e^{ib(n-k)} f_{n,2k+2}, \quad k \in \mathbb{N}_N^0, \\
  f_{n,0} &= f_{n,p+1} = 0, \quad p = 2N + 1, \\
  (\Delta_b f)_{n,2k} &= e^{-ib(n-k+1)} f_{n,2k-1} + e^{-ib(n+2k-1)} f_{n+1,2k-1} + e^{ib(2n+k)} f_{n,2k+1}, \quad k \in \mathbb{N}_N, \\
  (V f)_x &= v_k f_x, \quad x = (n,k) \in \mathbb{Z} \times \mathbb{N}_p, \tag{1.3}
\end{align*}
\]
where \( n \in \mathbb{Z} \) and \( f = (f_x)_{x \in \mathbb{Z} \times \mathbb{N}_p} \in \ell^2(\Gamma) \) and \( v = (v_k^p) \in \mathbb{R}^p \). In fact we consider the Schrödinger operator \( H_b \) on the ribbon \( \Gamma \) (on the set \( \mathbb{Z} \times \mathbb{N}_p \)) with the Dirichlet boundary conditions \( f_{n,0} = f_{n,p+1} = 0 \). Note that (1.3) gives \( H_{b+2\pi} = H_b \) for all \( b \in \mathbb{R} \). Our electric
potential $V$ is given by $v \in \mathbb{R}^p$, since the electric field is perpendicular to the axis of the nanoribbon. We formulate our preliminary result.

**Theorem 1.1.**

i) The operator $H_b = \Delta_b + V$ is unitarily equivalent to the operator $\int_{[0, 2\pi]} J_t \frac{dt}{2\pi}$, where $J_t \equiv J_t(b, v)$ is a Jacobi operator, acting on $\mathbb{C}^p$ and given by

$$(J_t y)_n = a_{n-1} y_{n-1} + a_n y_{n+1} + v_n y_n, \quad a_n = \begin{cases} 1 & \text{even } n, \\ 2 | \cos(\frac{t}{2} - \frac{3n-2b}{2})| & \text{odd } n, \end{cases}$$  \hspace{1cm} (1.5)

where $y = (y_n)^p \in \mathbb{C}^p$, $y_0 = 0 = y_{p+1}$, $p = 2N + 1$.

ii) The spectrum of $H_b$ is given by

$$\sigma(H_b) = \bigcup_{k \in \mathbb{Z}_N} \sigma_k, \quad \sigma_k = \lambda_k([0, 2\pi]) = \begin{cases} [\lambda_k^-, \lambda_k^+], & k \geq 0, \\ [\lambda_k^+, \lambda_k^-], & k < 0, \end{cases} \quad k \in \mathbb{Z}_N = \{-N, ..., N\},$$  \hspace{1cm} (1.6)

where $\lambda_{-N}(t) \leq \lambda_{-N+1}(t) \leq ... \leq \lambda_N(t)$ are eigenvalues of $J_t$, $t \in [0, 2\pi)$. Moreover, $\lambda_n(\cdot)$ is real analytic for any $t$, where $a_{2k-1} \neq 0, k \in \mathbb{N}_N$.

**Remark.**

1) Below we will sometimes write $\lambda_k(t, b, v), \sigma_k(b, v), J_t(b, v), ...$ instead of $\lambda_k(t), \sigma_k, J_t, ...$, when several potentials $v$ or magnetic fields are being dealt with. The operator $J_t = J_t(b, v)$ is the symmetric $p \times p$ matrix given by

$$J_t(b, v) = \begin{pmatrix} v_1 & a_1 & 0 & 0 & ... & 0 \\ a_1 & v_2 & 1 & 0 & ... & 0 \\ 0 & 1 & v_3 & a_3 & ... & 0 \\ 0 & 0 & a_3 & v_4 & 1 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & 0 & 1 & v_p \end{pmatrix} = J_t(b, 0) + \text{diag}(v_n)^1_p, \quad v = (v_n)^p \in \mathbb{R}^p.$$  \hspace{1cm} (1.7)

2) We take the specific boundary of the ZGNR, see Fig. 2. If we change the boundary of the ZGNR, then the corresponding operator $J_t$ will be more complicated. In the last case even the exact calculation of the spectrum for the unperturbed operator $\Delta_0$ can be the problem.

3) If $\lambda_k(t) = \text{const}$ for all $t \in [0, 2\pi)$, then $\sigma_k$ is a flat band. Otherwise, $\sigma_k$ is a non-flat spectral band. In the periodic spectral theory the basic problem is to describe all flat bands. Examples of the dispersive curves for $\lambda_k(\cdot)$ are given in Fig. 5.

We recall the result about the spectrum of $\sigma(\Delta_0)$ of the Laplacian $\Delta_0$ with $v = 0$ and $b = 0$ from [Ku3] (for more details see [2.3]–[2.5])

$$\sigma(\Delta_0) = \sigma_{ac}(\Delta_0) \cup \sigma_{pp}(\Delta_0), \quad \sigma_{pp}(\Delta_0) = \{0\}, \quad \sigma_{ac}(\Delta_0) = [-\lambda^0, \lambda^0] \setminus [-s_1, s_1],$$  \hspace{1cm} where $\lambda^0 = (5 + 4c_1)^{\frac{1}{2}}, \quad c_\alpha = \cos \frac{\alpha \pi}{N + 1}, \quad s_\alpha = \sin \frac{\alpha \pi}{N + 1}, \quad \alpha \in \mathbb{R}.$  \hspace{1cm} (1.8)

Recall that $\mathbb{Z}_N = \{-N, ..., N\}$. Now we describe the spectrum of the magnetic operator $\Delta_b$.

**Theorem 1.2.**

i) Let $b \in \mathbb{T}$. Then the spectrum of $\Delta_b$ is given by

$$\sigma(\Delta_b) = \sigma_{ac}(\Delta_b) \cup \sigma_{pp}(\Delta_b), \quad \sigma_{ac}(\Delta_b) = \bigcup_{k \in \mathbb{Z}_N \setminus \{0\}} \lambda_k([0, 2\pi]), \quad \sigma_{pp}(\Delta_b) = \{0\},$$  \hspace{1cm} (1.9)

where $\sigma_0(\Delta_b) = \{0\}$ is a flat band and $\lambda_k(\cdot)$ is defined in Theorem [1.4].
ii) Let $b \in \mathbb{T}$ and $b \to 0$. Then $\sigma_{ac}(\Delta_b) = [\mu_1^-, \mu_2^+] \cup [\mu_2^+, \mu_1^+]$ and $\mu_n^\pm$ has asymptotics

$$\mu_1^\pm = \pm (5 + 4c_1)\sqrt{\frac{1}{2}} + O(b^2), \quad \mu_2^\pm = \pm s_1 \mp \frac{3c_1}{2(N+1)} b + O(b^2).$$

(1.10)

We describe the spectrum of $H_b$ for $b$ and $v = (v_n)_1^p \in \mathbb{R}^p$ for which $\sigma_{pp}(H_b) \neq \emptyset$.

**Theorem 1.3.** Let $(b, v) \in \mathbb{T} \times \mathbb{R}^p$. Then $\sigma_{pp}(H_b) \neq \emptyset$ iff $v_{2n+1} = v_1$ for any $n \in \mathbb{N}_N$. Moreover, if $\sigma_{pp}(H_b) \neq \emptyset$, then $\sigma_{pp}(H_b) = \sigma_0(H_b) = \{v_1\}$ and each $|\sigma_n(H_b)| > 0$ and $\sigma_n(H_b) \subset (v_1, +\infty), \sigma_{-n}(H_b) \subset (-\infty, v_1)$ for all $n \in \mathbb{N}_N$.

**Remark.** (1) Theorem 1.3 generalizes results from [KKu3], devoted to the operator $\Delta_0 + V$.

(2) If $\sigma_{pp}(H_b) = \{v_1\}$, then a flat band $\sigma_0(H_b) = \{v_1\}$ lies in the gap of the continuous spectrum, see Fig 3 and 4.

**Fig 3.** Spectral bands $\sigma_k$ for the case $v, b = 0$ and $N = 15$.

**Fig 4.** Spectral bands for the case $N = 3$.

We will use notations $\lambda_k(t, b, v) \equiv \lambda_k(t)$ for our spectral curves $\lambda_k(t)$. The perturbation theory gives the standard asymptotics

$$\lambda_k(t, b, v) = \lambda_k(t, b, 0) + O(\|v\|) \quad \text{as } v \to 0.$$  

(1.11)

In Theorem 1.4 we determine asymptotics of the eigenvalue $\lambda_0(t, b, v)$ as $v \to 0$ at fixed $b$, recall that $\lambda_0(t, b, 0) = 0$. Below we will use this result to solve inverse spectral problem for small potentials.

**Theorem 1.4.** Let $b \in \mathbb{T}$ and let $v \to 0$. Then

$$\lambda_0(t) = \sum_{n=0}^N v_{2k+1} \eta_k(t) + O(\|v\|^2),$$  

(1.12)
\[ \eta_k(t) = \frac{\beta_k^2(t)}{\sum_{s=0}^{N} \beta_s^2(t)}, \quad \beta_0 = 1, \quad \beta_k = \prod_{j=1}^{2k} a_j(t), \quad k \geq 1, \quad (1.13) \]

uniformly in \( t \in [0, 2\pi] \).

\[ \lambda_0(t) \equiv 0 \]

is a flat band

1) unperturbed potential \( b, v = 0 \)

\[ \lambda_0(t) \equiv 0 \]

Fig 5. Spectral curves \( \lambda_k(t) \) and bands \( \sigma_k \) for the case \( N = 3 \).

**Strong electric fields.** We consider now the nanoribbon in strong electric fields \( \tau V \) as the coupling constant \( \tau \to \infty \) and \( V \) is fixed. Our operator has the form \( H_b(\tau) = \Delta_b + \tau V \). We determine asymptotics of the spectral bands \( \sigma_k(\tau v) = [\lambda_k^+(\tau), \lambda_k^-(\tau)], k \in \mathbb{Z}_N \) of the operator \( H_b(\tau) \) as the coupling constant \( \tau \to \infty \).

\[ \frac{s}{\tau v_1} \quad \frac{s}{\tau v_2} \quad \frac{s}{\tau v_3} \quad \frac{s}{\tau v_4} \quad \frac{s}{\tau v_5} \quad \frac{s}{\tau v_6} \quad \lambda \]

Fig. 6. Spectral bands for the case of strong electric field

**Theorem 1.5.** Let \( H_b(\tau) = \Delta_b + \tau V \), where \( \tau \to \infty \) and let the vector \( v = (v_b)_1^p \in \mathbb{R}^p, p = 2N + 1 \) satisfy \( v_1 < \ldots < v_p \). Then the spectral bands \( \sigma_j(\tau v) = [\lambda_j^+(\tau), \lambda_j^{-}(\tau)], j \in \mathbb{Z}_N \) have asymptotics:

\[ \lambda_j^s(\tau) = \tau v_k - \frac{\xi_j^s}{\tau} + O(\tau^{-1}), \quad \xi_j^s = \frac{r_j^k}{s} = \frac{r_j^k}{\tau v_{k-1} - v_k} + \frac{r_j^k}{\tau v_{k+1} - v_k}, \quad s = (-1)^k, \quad k = j+N+1, \]

\[ v_0 = v_{p+1} = 0, \quad r_j^+ = r_j^+ = 0, \quad r_j^{-} = r_j^{-} = 0, \quad r_j^{4n} = 4, \quad n \in \mathbb{N}_N, \quad (1.14) \]

\[ |\sigma_j(\tau v)| = \frac{4 + O(\tau^{-1})}{\tau |v_{k-(-1)^k} - v_k|}, \quad j \neq N, \quad \text{and} \quad |\sigma_N(\tau v)| = O(\tau^{-2}). \quad (1.15) \]

Moreover, the operator \( H_b(\tau) \) has not a flat bands for \( \tau \) large enough.
Remark. 1) If $\tau \to \infty$, then roughly speaking the spectrum of operator $H_b(\tau)$ consists of $p$ bands $\sigma_k(\tau v), k \in \mathbb{N}_p$ with lengths $|\sigma_k(\tau v)| > 0$, separated by $2N$ large gaps. In this case all $2N$ gaps are realized. In same sense it is clear, since we have the operator $\frac{H_b}{\varepsilon} = V + \varepsilon \Delta_b$ with small coupling constant $\varepsilon = \frac{1}{b}$. The operator $V$ acting on $\ell^2(\Gamma)$ has only flat bands (eigenvalues) $\{\nu_k\}, k \in \mathbb{N}_p$. Under the small perturbation $\varepsilon \Delta_b$ these flat bands $\{\nu_k\}, k \neq N$ become the small spectral bands $\sigma_k(\tau v)$ of the continuous spectrum with lengths $> 0$. It is important that only Theorem 1.3 shows that $\sigma_N(\tau v)$ is not a flat band, since asymptotics (1.15) are not sharp for $j = N$.

Inverse spectral problem for odd potentials. Below we solve inverse problems for sufficiently small ”odd” potentials. Define the space of odd potentials

$$\mathcal{V}_{odd} = \{v = (v_k)_p \in \mathbb{R}^P: \ v_{2k} = 0, \ k \in \mathbb{N}_N\}$$

equipped with the standard norm $\| \cdot \|$ in $\mathbb{R}^P$. Let $(b, v) \in \mathbb{T} \times \mathcal{V}_{odd}$ and let $0 \leq t_0 < t_1 < \ldots < t_N \leq \pi$. We define the mapping $\Lambda: \mathcal{V}_{odd} \to \mathbb{R}^{N+1}$ by

$$v \mapsto \Lambda(v) = (\lambda_0(t_j, b, v))_{0}^{N},$$

where $\lambda_0(t_j, b, v) \in \sigma_0 = \lambda_0([0, 2\pi], b, v)$, and $\lambda_0$ is defined in Theorem 1.1. Let $B_r = \{v \in \mathcal{V}_{odd}: \ |v| < r\}, r > 0$ be the ball in $\mathcal{V}_{odd}$.

Theorem 1.6. i) Let $b \in \mathbb{T}$ be sufficiently small. Then the mapping $\Lambda: B_r \to \Lambda(B_r)$ is a real analytic bijection for some $r > 0$, where $\Lambda(B_r) \subset \mathbb{R}^{N+1}$ is an open domain and $0 \in \Lambda(B_r)$.

ii) In the case of i) for any $\varepsilon \in (0, r)$ there exists a potential $v \in B_\varepsilon$ and a potential $w \notin B_\varepsilon$ such that $\Lambda(w) = \Lambda(v)$, i.e. there is no a global injection.

iii) Let $b \in \mathbb{T}$ and let $t_k \in [0, 2\pi], k \in \mathbb{N}_p$ be a sequence of a distinct numbers. Then the vector $(\lambda_0(t_k, b, v))^p_1$ uniquely determines the potential $v \in B_r$ for sufficiently small $r > 0$.

![Fig 6](image)

Fig 6. The case $N = 3$, small perturbation $b, v$. Spectral curve $\lambda_0(t)$ covers the band $\sigma_0$; $2(N + 1)$-periodic eigenvalues $\Lambda_k = \lambda_0(\frac{k\pi}{N+1})$ belong to $\sigma_0$.

Remark. Let $t_k = \frac{\pi k}{N+1}, k \in \mathbb{N}_N^0$. Then $\lambda_0(t_k)$ are so-called $2(N+1)$-periodic eigenvalues for the Jacobi matrix $J_b$. Thus, if $b, r$ are small enough, then by Theorem 1.6.i), $2(N+1)$-periodic eigenvalues $\lambda_0(t_k) \in \sigma_0, k \in \mathbb{N}_N^0$ uniquely determine the odd potential $v$ (and operator $V$). Note that we take only some part of $2(N+1)$-periodic eigenvalues to determine the potential $V$, since $\lambda_j(t_k), j \neq 0$ are $2(N + 1)$-periodic eigenvalues too.
Inverse problems for monotonic electric potentials without the magnetic field.

We consider the case $b = 0$. Define the mapping in terms of antiperiodic eigenvalues
\[ \Psi : v \rightarrow \Psi(v) = (\lambda_k(\pi, 0, v))_{N}^{N}. \]

Here $\lambda_k(\pi, 0, v)$ is a antiperiodic eigenvalues for Jacobi operator $J_t$, since $t = \pi$. Define the bounded set of monotonic potentials by
\[ \hat{V}_\alpha = \{ v \in \mathbb{R}^p : 0 \leq v_1 < v_2 < v_3 < \ldots < v_{2N+1} \leq \alpha \}, \]
\[ \hat{V}_\alpha = \{ v \in \mathbb{R}^p : \alpha \geq v_1 > v_2 > v_3 > \ldots > v_{2N+1} \geq 0 \}. \]

**Theorem 1.7.** i) The mapping $\Psi : \hat{V}_\alpha \to \mathbb{R}^{N+1}$ is an injection iff $\alpha \in [0, 1]$.

ii) The mapping $\Psi : \hat{V}_\alpha \to \mathbb{R}^{N+1}$ is an injection iff $\alpha \in [0, 1]$.

iii) In these both cases there exist the algorithm of recovering of $v$ in terms of $\Psi(v) = (\lambda_k(\pi, 0, v))_{N}^{N}$.

iv) Spectral curves $\lambda_k$, $k \in \mathbb{Z}_N$ satisfy $\lambda_k(\pi + t, 0, v) = \lambda_k(\pi - t, 0, v)$ for any $(t, v) \in [0, \pi] \times \mathbb{R}^p$.

If $v \in \hat{V}_1$ or $v \in \hat{V}_1$ then all $\lambda_k(\pi, 0, v)$, $k \in \mathbb{Z}_N$ are distinct numbers.

**Remark.** 1) This Theorem shows that the set of all antiperiodic eigenvalues uniquely determine the increasing (or decreasing) bounded potential. Moreover, in the proof of Theorem 1.7 we show how to determine the monotonic potential $v$ by antiperiodic eigenvalues $\Psi(v) = (\lambda_k(\pi, 0, v))_{N}^{N}$.

2) Theorem 1.7(iv) shows us that $\lambda_k$ reach the maximum or minimum at the point $t = \pi$. Then for sufficiently small monotonic potentials $v$ the spectrum change multiplicity at any of anti-periodic eigenvalues $\lambda_k(\pi, 0, v)$.

3) Note that in the paper [KKu2] we proved that some finite set of $k$-periodic eigenvalues uniquely determines the spectrum (including multiplicity) of a periodic matrix-valued Jacobi operator. In the present paper Theorem 1.6 gives the stronger result and shows that some of $N + 1$-periodic eigenvalues uniquely determine the operator $H_b$ for potentials $v$ small enough.

In the proof of our theorems we vary various asymptotics for periodic Jacobi operators with specific coefficients given by [15]. Note that there exist a lot of papers devoted to asymptotics and estimates for periodic Jacobi operators, see e.g. [KKu1], [KKu], [La], [vMou].

We present the plan of our paper. In Section 4 we prove Theorem 1.1 with a technical proof, where we use arguments from [KKu3]. In Sect. 2 we describe the properties $\Delta_b$ and prove Theorem 1.2. In Sect. 3 we prove Theorems 1.3 - 1.7.

2. Properties of the magnetic operator $\Delta_b$

Theorem 1.1 is proved in Section 4. In this section we describe the spectral properties of $\Delta_b$. Recall that Theorem 1.1 gives that the operator $H_b = \Delta_b + V$ is unitarily equivalent to the operator $\int_{[0, 2\pi]} J_{\pi} dv$, where $J_t \equiv J_t(b, v)$ is a $p \times p$ Jacobi operator. Introduce a Jacobi operator $\tilde{J}_t \equiv \tilde{J}_t(b, v)$ acting in $\mathbb{C}^p$ and given by
\[ (\tilde{J}_t y)_n = \tilde{a}_{n-1} y_{n-1} + \tilde{a}_n y_{n+1} + v_n y_n, \quad \tilde{a}_n = \begin{cases} 1 & \text{even } n \\ 2 \cos(\frac{x}{2} - \frac{3n-2}{2} b) & \text{odd } n \end{cases} \]
and recall that $J_t \equiv J_t(b, v)$ is given by
\[ (J_t y)_n = a_{n-1} y_{n-1} + a_n y_{n+1} + v_n y_n, \quad a_n = |\tilde{a}_n|, \]
where $y = (y_n)_1^p \in \mathbb{C}^p$ and $y_0 = 0 = y_{p+1}, p = 2N + 1$. Note that the matrices $\tilde{J}_t = \tilde{J}_t(b, v)$ and $J_t = J_t(b, v)$ are unitarily equivalent. Then the eigenvalues $\lambda_k(t) = \lambda_k(t, b, v), k \in \mathbb{Z}_N$ of $J_t$ (see Theorem 1.3.ii) are eigenvalues of $\tilde{J}_t$. Sometimes below we will use the matrix $\tilde{J}_t$ instead of $J_t$ to study the functions $\lambda_k(t)$ and the bands $\sigma_k$, defined in Theorem 1.3.ii).

The Laplacian $\Delta_0$. We describe the spectrum $\sigma(\Delta_0)$ of the unperturbed operator $\Delta_0$ at $V = 0$ and $b = 0$. Let $\lambda_{0-N}^0(t) \leq \lambda_{0-N+1}^0(t) \leq \ldots \leq \lambda_{0}^0(t)$ be eigenvalues of the Jacobi matrix $J_0^0 = J_t(0, 0)$ corresponding to $\Delta_0$ (see Theorem 1.3.ii). We recall the results about the spectrum of $\sigma(\Delta_0)$ from [KKu3]:

$$
\lambda_k^0(t) = (a^2 - 2ck^2 + 1) \frac{k}{|k|}, \quad 1 \leq |k| \leq N, \quad \lambda_0^0(t) = 0, \quad t \in [0, 2\pi], \quad a = 2|\cos \frac{t}{2}|,
$$

$$
\sigma = \sigma_{ac} \cup \sigma_{pp} = \bigcup_{-N}^N \sigma_k^0, \quad \sigma_{pp}^0 = \sigma_0^0 = \{0\}, \quad \sigma_{ac}^0 = [-\lambda_{N}^0(0), \lambda_{N}^0(0)] \setminus [-s_1, s_1],
$$

$$
\sigma_{-k}^0 = -\sigma_k^0, \quad \sigma_k^0 = [\lambda_k^{0-}, \lambda_k^{0+}] = \begin{cases} 
[\lambda_k^0(\tau), \lambda_k^0(0)], & \text{if } c_k < 0, \\
[\lambda_k^0(\pi), \lambda_k^0(0)], & \text{if } c_k > 0, 
\end{cases} \quad \text{all } k = 1, \ldots, N,
$$

for all $k = 1, \ldots, N$, where $\lambda_k^0(\tau) = s_k, 2\cos \frac{\tau}{2} = c_k$, and $\lambda_k^0(\pi) = 1, \lambda_k^0(0) = (5 + 4c_k)^{\frac{1}{2}}$.

We use notation $\tilde{J}_t(b, v)$, $J_t(b, v)$ and $\lambda_k(t, b, v)$ for matrices $\tilde{J}_t$, $J_t$ (defined in (2.1), (1.5)) and its eigenvalues.

For $k \neq 0$ introduce

$$
\Phi_k^0(t) = (\varphi_k(t)\varphi_k(t))_{n=1}^N, \quad \varphi_k(t) = \begin{cases} 
(1)^{n+1}s_{nk}, & \text{if } c_k < 0, \\
(1)^n(2s_{nk}\cos \frac{t}{2} - s_{(n-1)k}), & \text{if } c_k > 0
\end{cases} \quad \text{for } t \in (0, \pi),
$$

$$
\varphi_{k,2n-1}(t) = \frac{(1)^{n+1}(2s_{nk}\cos \frac{t}{2} - s_{(n-1)k})}{\sqrt{N + 1} \lambda_k^0(t)}, \quad \varphi_{k,2n-1}(t + \pi) = \varphi_{k,2n}(t) = \varphi_{N+1-k,2n}(t), \quad t \in (0, \pi).
$$

Below we use notation $\langle \cdot, \cdot \rangle$ for the standard scalar product in $\mathbb{C}^n$. We consider the case $V = 0$ and $b \to 0$.

**Lemma 2.1.** i) The following identities hold true

$$
J_0^0\Phi_k^0(t) = \lambda_k^0(t)\Phi_k^0(t), \quad \|\Phi_k^0(t)\| = 1, \quad k \neq 0, \quad t \in [0, 2\pi],
$$

i.e. $\Phi_k^0(t)$, $k \neq 0$ are orthonormal eigenvectors for $J_0^0$.

ii) For any $t \in [0, 2\pi] \setminus \{\pi\}$ the following identity holds true

$$
\partial_t(\lambda_k(t), 0, 0) = \frac{\sin \frac{t}{2}}{(N + 1)\lambda_k^0(t)} \left( (N + 1)(3N + 1) \left( 2\cos \frac{t}{2} + c_k \text{sign}(t - \pi) \right) - 6 \cos \frac{t}{2} \right),
$$

**Proof.** Direct calculations gives i).

ii) Using the perturbation theory, we obtain

$$
(\lambda_k)_b(t, 0, 0) = \langle (\tilde{J}_t)_b(0, 0)\Phi_k(t), \Phi_k(t) \rangle = 2\sin \frac{t}{2} \sum_{n=1}^N (6n - 5)\varphi_{k,2n-1}^0(t)\varphi_{k,2n}^0(t).
$$
Define a Jacobi matrix

$$\frac{2 \sin \frac{t}{2}}{(N + 1) \lambda_k^2(t)} \sum_{n=1}^{N} (6n - 5)(2s_{nk}^2 \cos \frac{t}{2} - s_{nk}s_{(n-1)k})$$

$$= \frac{2 \sin \frac{t}{2}}{(N + 1) \lambda_k^2(t)} \sum_{n=1}^{N} (12n^2_{nk} \cos \frac{t}{2} - 10s_{nk}^2 \cos \frac{t}{2} - 6n s_{nk}s_{(n-1)k} + 5s_{nk}s_{(n-1)k}) =$$

$$= \frac{2 \sin \frac{t}{2}}{(N + 1) \lambda_k^2(t)} \left( (N + 1)(3N + 2) \cos \frac{t}{2} - (N + 1)(3N + 1) \frac{c_k}{2} \right),$$

where we use identities

$$\sum_{n=0}^{N} c_{2nk+\alpha} = \sum_{n=0}^{N} s_{2nk+\alpha} = 0, \quad 1 \leq |k| \leq N, \quad \alpha \in \mathbb{R},$$

$$\sum_{n=1}^{N} s_{nk} = \sum_{n=1}^{N} \frac{1 - c_{2nk}}{2} = \frac{N + 1}{2},$$

$$\sum_{n=1}^{N} s_{nk}s_{(n-1)k} = \frac{1}{2} \sum_{n=1}^{N} (c_k - c_{(2n-1)k}) = \frac{1}{2} (Nc_k + c_{-k}) = \frac{N + 1}{2} c_k,$$

$$\sum_{n=1}^{N} nc_{2kn} = -\frac{N + 1}{2}, \quad \sum_{n=1}^{N} ns_{2kn} = -\frac{N + 1}{2} c_k, \quad 1 \leq |k| \leq N,$$

$$\sum_{n=1}^{N} ns_{nk} = \sum_{n=1}^{N} \frac{1 - c_{2nk}}{2} = \frac{(N + 1)^2}{4},$$

$$\sum_{n=1}^{N} ns_{nk}s_{(n-1)k} = \frac{1}{2} \sum_{n=1}^{N} n(c_k - c_{(2n-1)k}) = \frac{1}{2} \sum_{n=1}^{N} n(c_k - c_{2nk}c_k - s_{2nk}s_k) =$$

$$= \frac{(N + 1)(N + 2)}{4} c_k. \blacksquare$$

**Lemma 2.2.** Let $b, t \in [0, 2\pi]$ and $v_{2n+1} = 0$ for all $n \in \mathbb{N}_{N}^\mathbb{N}$. Then $\lambda = 0$ is a simple eigenvalue of $J_t$.

**Proof.** Define a Jacobi matrix $A$ by

$$A = \begin{pmatrix}
0 & a_1 & 0 & \ldots & 0 \\
a_1 & v_2 & a_2 & \ldots & 0 \\
0 & a_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & a_{2n} & 0 \\
\end{pmatrix}, \quad a_{2k} \neq 0, \text{ for all } k \in \mathbb{N}_n,$$

for some $n \in \mathbb{N}$. Then $\lambda = 0$ is a simple eigenvalue of $A$ and $x = (x_k)_{2n+1}^{2n+1}$ is a corresponding eigenvector, where $x_1 = 1, x_{2k+1} = (-1)^{k \alpha_{1} \cdots \alpha_{2k-1}} a_{2k}, x_{2k} = 0, k \in \mathbb{N}_n$. \blacksquare

**Proof of Theorem 1.1.2** i) Lemma 2.2 gives that $\{0\}$ is a flat band. Then Theorem 1.3 i) and the Theorem 1.2 give the proof of i).

ii) Relations (2.3)-(2.5) imply

$$\mu_2^0(0) = \min_{t \in [0, 2\pi]} \lambda_1^0(t) = \lambda_1^0(z_1^0) = \lambda_1^0(z_2^0) = s_1, \quad s_1 < \lambda_1^0(t), \quad t \in [0, 2\pi] \setminus \{z_1^0, z_2^0\}, \quad (2.10)$$
where \( z^0_1, z^0_2 \in (0, 2\pi) \), \( z^0_1 \neq z^0_2 \) such that \( 2|\cos \frac{z^0_j}{2}| = c_1 \), \( j = 1, 2 \). We denote \( \lambda^0_j(t, b) \equiv \lambda_1(t, b, 0) \). Perturbation Theory gives us that \( \lambda^0_j(t, b) \) is an analytic function in some neighbourhood of points \( t = z^0_1, z^0_2 \) and \( b = 0 \), because \( \lambda^0_1(z^0_1, 0) \) and \( \lambda_1(z^0_2, 0) \) are simple eigenvalues of analytic matrix function \( J(t, b) \). Using

\[
\partial_2 \lambda^0_1(z^0_j, 0) = 0, \quad \partial_2^2 \lambda^0_1(z^0_j, 0) > 0, \quad j = 1, 2, \tag{2.11}
\]

and the Implicit Function Theorem we deduce that there exist only two functions \( z_1(b), z_2(b) \) (analytic in \( |b| \leq \delta \) for some \( \delta > 0 \)) such that

\[
z_j(0) = z^0_j, \quad \partial_2 \lambda^0_1(z_j(b), b) = 0, \quad \partial_2^2 \lambda^0_1(z_j(b), b) > 0, \quad j = 1, 2 \tag{2.12}
\]

for all \( |b| \leq \delta \). Identities (2.10) and (2.12) yield that \( z_j(b) \) are points of local minimum of function \( \lambda^0_1(t, b) \) and one of these is a point of global minimum, that is

\[
\mu_2^+(b) = \min_{t \in [0, 2\pi]} \lambda^0_1(t, b) = \min_{j \in \mathbb{N}_2} \{ \lambda^0_1(z_j(b), b) \}, \tag{2.13}
\]

\[
\lambda^0_1(z_j(b), b) = \lambda_1^0(z^0_j) + b(\partial_b \lambda^0_1(z_j(b), b)|_{b=0}) + O(b^2) = s_1 + b(\partial_b \lambda^0_1(z_j(b), b)|_{b=0}) + O(b^2), \tag{2.14}
\]

\[
\partial_b \lambda^0_1(z_j(b), b)|_{b=0} = \partial_2 \lambda^0_1(z^0_j, 0) \partial_2 z_j(b) + \partial_b \lambda^0_1(z^0_j, 0) = \partial_b \lambda^0_1(z^0_j, 0), \quad j = 1, 2 \tag{2.15}
\]

Then, substituting \( \partial_b \lambda^0_1(z^0_j, 0) \) and \( \partial_2 \lambda^0_1(z^0_j, 0) \) from (2.10), (2.12) to (2.13), substituting (2.15) to (2.14) and using (2.13) we obtain asymptotics for \( \mu^+_2 \). Other cases are proved similar. \( \blacksquare \)

3. PROOF OF THE MAIN RESULTS

In order to prove Theorem 1.3 we need two Lemmas.

Recall that \( v = (v_k)_1^p \), \( p = 2N + 1 \) and \( a_{2k} = 1, k \in \mathbb{N}_N \) and \( \tilde{a}_{2k+1} = \tilde{a}_{2k+1}(t, b) = 2 \cos \left( \frac{t}{2} - \frac{b - k - 1}{2} \right) \), \( k \in \mathbb{N}_N \). These functions are analytic for \( t \in \mathbb{C} \), below we will consider some cases when \( t \in \mathbb{C} \) not only \( t \in [0, 2\pi] \) as we did above. Introduce the matrices

\[
T_k(t, b, v) = \begin{pmatrix}
-1 & -v_{2k-1} \\
 v_{2k} & v_{2k}v_{2k-1} - \tilde{a}_{2k-1}
\end{pmatrix}, \quad k \in \mathbb{N}_{N+1}, \quad v_{2N+2} = 0. \tag{3.1}
\]

Introduce the functions \( u_k \equiv u_k(t, b, v), k \in \mathbb{N}_{N+1}^0 \) by

\[
\begin{pmatrix}
u_0 \\
u_1
\end{pmatrix} = \begin{pmatrix} 0 \ 1 \end{pmatrix}, \quad \begin{pmatrix} u_2k \\
u_{2k+1}
\end{pmatrix} = T_k \begin{pmatrix} u_{2k-2} \\
u_{2k-1}
\end{pmatrix}, \quad k \in \mathbb{N}_{N+1}. \tag{3.2}
\]

Note that \( u_k \) is an analytic functions for \( t \in \mathbb{C} \). Below we need a simple fact.

**Lemma 3.1.** Let \( t, b \in [0, 2\pi] \) and let \( v \in \mathbb{R}^p \). Suppose \( \tilde{a}_{2k-1}(t, b) \neq 0 \) for all \( k = 1, \ldots, N \). Suppose \( \lambda = 0 \) is eigenvalue for \( J(t, b) \). Then \( u_{2N+2}(t, b, v) = 0 \).

**Proof.** Let \( J(t, b)F = \lambda F \), where \( F = (f_n)_1^p \neq 0 \) is the corresponding eigenvector. Then the definition (2.2) implies

\[
v_1 f_1 + a_{2k} f_2 = 0, \quad \tilde{a}_{2k-1} f_{2k-1} + v_{2k} f_{2k} + \tilde{a}_{2k} f_{2k+1} = 0, \quad k \in \mathbb{N}_N, \quad \tilde{a}_{2N} f_{2N} + v_{2N+1} f_{2N+1} = 0,
\]

which yields

\[
f_1 = f_1 u_1, \quad \begin{pmatrix} f_{2k} \\
u_{2k+1}
\end{pmatrix} = \frac{1}{a_{2k-1}} T_k \begin{pmatrix} f_{2k-2} \\
u_{2k-1}
\end{pmatrix} = f_1 \xi_k \begin{pmatrix} u_{2k} \\
u_{2k+1}
\end{pmatrix}, \quad k \in \mathbb{N}_{N+1},
\]
\[ \xi_k = \prod_{n=1}^{k} \tilde{a}_{2n-1} \] which yields \( f_1 \neq 0 \), since \( F \neq 0 \). Then

\[ 0 = \tilde{a}_{2N} f_{2N} + v_{2N+1} f_{2N+1} = \frac{\tilde{a}_{2N} u_{2N} + v_{2N+1} u_{2N+1}}{f_1 \xi_N} = -\frac{u_{2N+2}}{f_1 \xi_N}, \]

which yields \( u_{2N+2} = 0 \). \( \blacksquare \)

**Lemma 3.2.** Let \( b \in \mathbb{T}, v \in \mathbb{R}^\nu \). The following asymptotics hold true

\[ |u_{2k}(it)| = |v_{2k-1}|e^{2(k-1)t} + O(e^{2(k-2)t}), \quad t \to +\infty, \quad i = \sqrt{-1}, \quad k \in \mathbb{N}_{N+1}, \]

\[ |u_{2k+1}(it)| = e^{2kt} + O(e^{2(k-1)t}), \quad t \to +\infty, \quad i = \sqrt{-1}, \quad k \in \mathbb{N}. \quad (3.3) \]

**Proof.** Note that

\[ \tilde{a}_k(it) = e^t + O(1), \quad t \to +\infty, \quad k \in \mathbb{N}_p. \quad (3.4) \]

For \( k = 1 \) identities (3.4) and (3.1)-(3.2) yield

\[ \begin{pmatrix} u_{2k}(it) \\ u_{2k+1}(it) \end{pmatrix} = (-1)^k \begin{pmatrix} v_{2k-1}e^{(k-1)t} \\ e^{kt} \end{pmatrix} + \begin{pmatrix} O(e^{(k-2)t}) \\ O(e^{(k-1)t}) \end{pmatrix}, \quad t \to +\infty. \quad (3.5) \]

Substituting (3.5) for \( k = 1 \) into (3.2) and using (3.1), (3.4) we deduce that (3.3) is true for \( k = 2 \). Repeating this procedure we deduce that (3.3) is true for any \( k \in \mathbb{N}_p \), which yield asymptotic (3.3). \( \blacksquare \)

**Proof of Theorem 1.3** If \( v_1 = 0 \) then **Sufficiency** follows from Lemma 2.2. If \( v_1 \neq 0 \) then the proof is similar, since we may consider new operator \( H_b = \Delta_b + (V - v_1 I) \), where \( I \) is an identity operator.

**Necessity.** We have \( \sigma_{pp} \neq \emptyset \). Without lost of generality we may assume \( 0 \in \sigma_{pp} \) (in other case we shift spectrum by adding some constant \( c \) to any of diagonal components \( v_k \)). Then \( \lambda = 0 \) is an eigenvalue for matrix \( \tilde{J}_t(b, v) \) for \( t \in I_1 \), for some set \( I_1 \subset [0, 2\pi] \) and \( |I_1| > 0 \). There exists infinite set \( I_2 \subset I_1 \), satisfying \( \prod_{n=1}^{\infty} \tilde{a}_{2n-1}(t, b) \neq 0 \) for any \( t \in I_2 \). Then, by Lemma 3.1 we have \( u_{2N+2}(t, b, v) = 0, t \in I_2 \). Then \( u_{2N+2}(t, b, v) = 0 \) for any \( t \in \mathbb{C} \), since \( u_{2N+2} \) is an analytic function by \( t \). Using (3.3) we deduce \( |u_{2N+2}(it)| \sim |v_{2N+1}|e^{2nt}, t \to +\infty \), which yields \( v_{2N+1} = 0 \). Then, using (3.1), (3.2) and \( v_{2N+1} = 0 \), we obtain \( u_{2N}(t, b, v) = -u_{2N+2}(t, b, v) = 0, t \in \mathbb{C} \). Similarly to above, we deduce \( v_{2N-1} = 0 \) and so on.

By (2.3) we know that \( \lambda_{-k}(t, 0, 0) < \lambda_0(t, 0, 0) \equiv 0 < \lambda_k(t, 0, 0) \) for any \( (t, k) \in [0, 2\pi] \). Let some \( b \in \mathbb{T} \) and \( v \in \mathbb{R}^\nu \) be such that \( v_1 = v_3 = \ldots = v_{2N-1} = 0 \). We fix \( t \in [0, 2\pi] \) and let variable \( \tau \in [0, 1] \), perturbation theory gives us that \( \lambda_k(t, \tau b, \tau v), k \in \mathbb{Z}_N \) are continuous functions depending on \( \tau \in [0, 1] \), since these are eigenvalues of analytic matrix-function \( \tilde{J}_t(\tau b, \tau v) \). Note that \( \lambda_0(\tau b, \tau v) = 0 \) for any \( \tau \in [0, 1] \). Then inequalities \( \lambda_{-k}(t, \tau b, \tau v) < \lambda_0(\tau b, \tau v) \equiv 0 < \lambda_k(t, \tau b, \tau v), k \in \mathbb{Z}_N \) remain for any \( \tau \in [0, 1] \), since by Lemma 2.2 \( \lambda_0 = 0 \) is always simple eigenvalue of \( \tilde{J}_t(\tau b, \tau v) \). Then by (1.10) \( \sigma_{-k}(b, v) \subset (-\infty, 0) \) and \( \sigma_k(b, v) \subset (0, +\infty) \) for any \( k \in \mathbb{N}_N \). If \( v_1 = v_3 = \ldots = v_{2N+1} \neq 0 \) then we just shift spectrum by adding \( -v_1 \) to any of diagonal components \( v_k \) of the matrix \( \tilde{J}_t \) and repeat the reasoning. \( \blacksquare \)

**Proof of Theorem 1.4** If \( V = 0 \), then Theorem 1.2 gives that \( \sigma_0 = \{0\} \) is a flat band for \( H^b \). Also \( \lambda_0 = 0 \) is a simple eigenvalue of matrix \( J_1(b, 0) \). Using perturbation theory, for
\( v \to 0 \) we obtain

\[
\lambda_0(t, b, v) = \lambda_0(t, b, 0) + \sum_{n=1}^{2N+1} (\partial_{v_n} \lambda_0)(t, b, 0)v_n + O(\|v\|^2), \quad \text{uniformly for } t \in [0, 2\pi],
\]

where \( \Phi^b(t) = (\varphi^b(t))_1^p \) is a corresponding eigenvector for an eigenvalue \( \lambda_0(t, b, 0) = 0 \), i.e. \( J_1(b, 0)\Phi^b(t) = 0 \) and \( \|\Phi^b(t)\| = 1 \). Using results from the proof of Lemma 2.2, we obtain \( \varphi^b_0(t) = 0, \varphi^b_0(t) = (-1)^n\beta_n(t) \) (recall that \( \beta_n \) are defined in (1.13)), since \( J_1(b, v) \) has the form (4.11).

We start to prove our results about the inverse problems. It is convenient to prove Theorem 1.6 after the proof of Theorem 1.4, since we use the same notations.

**Proof of Theorem 1.6.** We use notation from Theorem 1.4.

i) Let \( b = 0 \). For any fix \( t \in \mathbb{R} \) the function \( \lambda_0(t, 0, \cdot) \) is real analytic for small arguments, since \( \lambda_0(t, 0, 0) \) is a single eigenvalue of analytic matrix-function \( \tilde{J}_1(0, 0) \). Then for small \( v \in \mathcal{V}_{odd} \) there exist Frechet derivative \( \partial_v \Lambda(v) : \mathcal{V}_{odd} \to \mathbb{R}^{N+1} \) is a real analytic function. Using (1.12), (1.13) and (1.5) we deduce

\[
f(t_n) = 0, \quad n \in \mathbb{N}_N^0, \quad \text{where } f(t) = \sum_{m=0}^{N} a^m v_{2m+1}, \quad a = \left(\frac{2\cos \frac{t}{2}}{t}\right)^2.
\]

Then \( f \equiv 0 \) and \( v_{2m+1} = 0, m \in \mathbb{N}_N^0, \) since this is a polynomial of degree \( N \) has \( N + 1 \) zeroes. We have \( D_0 \) is an isomorphism and \( \Lambda(\cdot) \) is a bijection for small arguments. If \( b \neq 0 \) is a sufficiently small value then \( D_0 \) is an isomorphism too, since \( D_0 \) analytically depends on \( b \), then \( \Lambda \) is a bijection for small arguments.

ii) Define the potential \( v^0 = (v^0_j)_1^p \in \mathcal{V}_{odd} \), where \( v^0_1 = -1, v^0_{2n+1} = n, v^0_{2n} = 0, n \in \mathbb{N}_N \). Let \( s > 0 \) be sufficiently small. Then using (1.5) we obtain

\[
J_1(b, s^{-1}v^0) = s^{-1}A(s), \quad A(s) = \text{diag}(v^0) + sJ_1(b, 0).
\]

Let \( \tilde{\lambda}_{-N}(s) \leq \ldots \leq \tilde{\lambda}_N(s) \) be eigenvalues of the matrix \( A(s) \). We have \( A(0) = \text{diag}(v^0) \) and then \( \tilde{\lambda}_{-N}(0) = -1, \tilde{\lambda}_{-N+n}(0) = 0 \), if \( n \in \mathbb{N}_N \) and \( \tilde{\lambda}_n(0) = n, n \in \mathbb{N}_N \).

Consider the asymptotic for \( \tilde{\lambda}_{-N+n}(s), n \in \mathbb{N}_N \) for \( s \to 0 \). We have \( \tilde{\varepsilon}_n = (\delta_{2n,j})_1^p \) (where \( \delta_{i,j} \) is a Kronecker symbol) are eigenvectors of \( A(0) \), corresponding to eigenvalues \( \lambda_{-N+n}(0) \). Using famous results of the perturbation theory, we obtain that the numbers \( \tilde{\lambda}_{-N+n}(0), n \in \mathbb{N}_N \) are eigenvalues of matrix \( \tilde{A} = ((A'(0)\tilde{e}_j, \tilde{e}_k))_1^N_{j,k} \). Easy calculations gives \( A = 0 \), then \( \tilde{\lambda}_{-N+n}(0) = 0, n \in \mathbb{N}_N \), and then

\[
\tilde{\lambda}_{-N+n}(s) = \tilde{\lambda}_{-N+n}(0) + s\tilde{\lambda}'_{-N+n}(0) + O(s^2) = O(s^2) \quad \text{as } s \to 0, \quad n \in \mathbb{N}_N.
\]

Using the fact, that \( \lambda_n(t, b, s^{-1}v^0), n \in \mathbb{Z}_N \) are eigenvalues of \( J_1(b, sv^0) \) and using (3.7), (3.8) we obtain

\[
\lambda_0(t, b, s^{-1}v^0) = s^{-1}\tilde{\lambda}_0(s) = O(s), \quad s \to 0.
\]
Then \( \Lambda(s^{-1}v^0) = O(s) \), \( s \to 0 \). Using i) we obtain, that for any sufficiently small \( s > 0 \) there exists unique sufficiently small potential \( w^s \in \mathcal{V}_{odd} \), which satisfy \( \Lambda(s^{-1}v^0) = \Lambda(w^s) \), and \( w^s \neq s^{-1}v^0 \), since \( s^{-1}v^0 \) has large norm.

iii) Define the mapping \( \hat{\Lambda}(v) = (\lambda_0(t_k, b, v))^0_0 \). We use similar arguments as in i). For any fix \((t, b) \in \mathbb{R}^2 \) the function \( \hat{\lambda}_0(\cdot) \equiv \lambda_0(t, b, \cdot) \) is real analytic for small arguments, since for small \( v \) the number \( \lambda_0(t, b, v) \) is a single eigenvalue of analytic matrix-function \( J_t(b, v) \). Then there exists Frechet derivative \( \partial_v \hat{\Lambda}(v) : \mathcal{V}_{odd} \to \mathbb{R}^{2N+1} \), which is real analytic function for sufficiently small \( v \in \mathcal{V}_{odd} \). Using (1.12), (1.13) we obtain

\[
D_0 \equiv \partial_v \hat{\Lambda}(0) = (\eta_{2m+1}(t_n))_{n=0,m=0}^{2N,N} \in \mathbb{R}^{(2N+1) \times (N+1)}.
\]

We need to show that the matrix (operator) \( D_0 \) is an injection. Suppose \( D_0 v^0 = 0 \) for some \( v^0 = (v_j)_1^{2N+1} \in \mathcal{V}_{odd} \). Then, using (1.12), (1.13), we deduce

\[
f(t_n) = 0, \quad n \in \mathbb{N}_0^{2N}, \quad \text{where} \quad f(t) = \sum_{m=0}^{N} \beta_m^2(t)v_{2m+1}.
\]  

(3.10)

Using (1.3) and (1.5) we obtain

\[
\beta_m^2(t) = \sum_{j=0}^{m} (c_{j,m} \cos jt + s_{j,m} \sin jt), \quad c_{m,m} \neq 0, \quad m \in \mathbb{N}_0^N
\]

(3.11)

for some constants \( c_{j,m}, s_{j,m} \). Then using (3.10), (3.11) we deduce that \( f(t) \equiv 0 \), since \( f \) is a trigonometrical polynomial of degree \( N \) and it has \( 2N + 1 \) zeroes in the interval \([0, 2\pi]\). Also we deduce that \( v_{2m+1} = 0, \quad m \in \mathbb{N}_0^N \), since by (3.11) we have that \( \beta_m^2(t), \quad m \in \mathbb{N}_0^N \) are linearly independent functions. Then \( v^0 = 0 \) and \( D_0 = \partial_v \hat{\Lambda}(0) \) is an injection. Also \( \hat{\Lambda}(\cdot) \) is an injection for small arguments, since \( \partial_v \hat{\Lambda}(\cdot) \) is a real analytic function for small arguments. 

We consider the case of the strong electric fields.

**Proof of Theorem 1.5.** In the case \( \Delta + \tau V \) the corresponding Jacobi operator depends on \( \tau \) and is given by

\[
(\tilde{J}_t(b, \tau v)y)_n = \tilde{a}_{-n-1}y_{n+1} + \tilde{a}_n y_{n+1} + \tau v_n y_n, \quad y = (y_n)_{n \in \mathbb{Z}} \in \mathbb{C}^p.
\]  

(3.12)

Using (2.11) we rewrite \( \tilde{J}_t(b, \tau v) \), \( \tau \to \infty \) in the form

\[
\tilde{J}_t(b, tv) = \tau(\tilde{V} + \tau^{-1}\tilde{J}_t(b, 0)) = \tau(\tilde{V} + \varepsilon \tilde{J}_t(b, 0)), \quad \tilde{V} = \text{diag}(v_j)_1^p, \quad \text{as} \quad \varepsilon = \frac{1}{\tau} \to 0.
\]

Then the perturbation theory (see Sect. XII, 1, [RS]) for \( \tilde{V} + \varepsilon \tilde{J}_t(b, 0) \) gives

\[
\lambda_{k-N-1}(t, b, \tau v) = \tau(v_k - \alpha_k \varepsilon^2 + O(\varepsilon^3)),
\]

\[
\alpha_k = \sum_{j \in \mathbb{N}_p \setminus \{k\}} \frac{u_{k,j}^2}{v_j - v_k}, \quad u_{k,j} = (\varepsilon_j^0, \tilde{J}_t(b, 0)v_k^0),
\]  

(3.13)
Proof of ii) is similar to the Proof of i).

Below we use the notation $\{0 \leq k, j, n \in \mathbb{N}, k < j \}$. The matrix $J_i(b, 0) = \{v_{k,j} \}$ is given by \((2.1)\), where

$$
\begin{align*}
    v_{k,j} = v_{k,j}(t, b) = \begin{cases} 
    0, & |k - j| \neq 1, \\
    2 \cos\left(\frac{\pi}{2} - \frac{6k - 5b}{2}\right), & j = k + 1, j \in 2\mathbb{N}, \\
    2 \cos\left(\frac{\pi}{2} - \frac{6j - 5b}{2}\right), & j = k - 1, k \in 2\mathbb{N}, \\
    1, & \text{other cases}
    \end{cases}.
\end{align*}
$$

(3.14)

Then (3.13), (3.14) yield (1.14), (1.15). □

Proof of Theorem 1.7

i) Sufficiency. Sufficiently to show that $\Psi$ is an injection for $\alpha = 1$, since $\hat{\psi}_\beta \subset \hat{\psi}_\alpha$ for $\beta \leq \alpha$. Using (1.5) we get $a_{2n-1}(\pi, 0) = 0$, $a_{2n} = 1$, $n \in \mathbb{N}$, so direct calculations give

$$
\bigcup_{k=-N}^{N} \{\lambda_k(\pi, 0, v)\} = \sigma(J_\pi(0, v)) = \{v_1\} \cup \bigcup_{k=1}^{N} \{\lambda : (\lambda - v_{2k})(\lambda - v_{2k+1}) - 1 = 0\}.
$$

(3.15)

Below we use the notation $\lambda_k^{(0)} \equiv \lambda_k(\pi, 0, v)$. Firstly, let $p_j(\lambda) = (\lambda - x_j)(\lambda - y_j) - 1$, $j = 1, 2$ be two quadratic polynomials for some $0 \leq x_1 < y_1 < x_2 < y_2 \leq 1$. Let $\mu_1 < \mu_2$ be roots of $p_j$, then it is not difficult to show that

$$
\mu_1 < \mu_2 < 0 < 1 < \nu_1 < \nu_2.
$$

(3.16)

Using this fact, monotonicity of $v$ and (3.15) we deduce that

$$
\lambda_{0,N}^{(0)} < \ldots < \lambda_{0,-1}^{(0)} < 0 \leq \lambda_0^{(0)} \leq 1 < \lambda_1^{(0)} < \ldots < \lambda_N^{(0)},
$$

(3.17)

where $\lambda_0^{(0)} = v_1$ and $\lambda_{-1}^{(0)} = v_{-1}$, $\lambda_k^{(0)}$ are roots of the polynomial $(\lambda - v_{2k})(\lambda - v_{2k+1}) - 1$ for $k \in \mathbb{N}$. Then $\Psi(v)$ uniquely determine monotonic potential $v$ and then $\Psi : \mathcal{V}_\alpha \to \mathbb{R}^p$ is an injection.

Necessity. Suppose that $\alpha > 1$. Define two potentials $v, w \in \hat{\psi}_\alpha$ by

$$
0 < v_1 = 2\varepsilon < \frac{1}{4} < v_2 < \ldots < v_{p-2} < \frac{3}{4} <
$$

$$
v_{p-1} = 1 + 5\varepsilon - \sqrt{2\varepsilon + \varepsilon^2} < v_p = 1 + 5\varepsilon + \sqrt{2\varepsilon + \varepsilon^2} < \min\left\{\alpha, \frac{5}{4}\right\},
$$

(3.18)

$$
0 < w_1 = 4\varepsilon < \frac{1}{4} < w_2 = v_2 < \ldots < w_{p-2} = v_{p-2} < \frac{3}{4} <
$$

$$
w_{p-1} = 1 + 4\varepsilon - 2\sqrt{\varepsilon + \varepsilon^2} < w_p = 1 + 4\varepsilon + 2\sqrt{\varepsilon + \varepsilon^2} < \min\left\{\alpha, \frac{5}{4}\right\},
$$

(3.19)

for some $\varepsilon > 0$ small enough. Direct calculations give us that $\Psi(v) = \Psi(w) = (\lambda_k^{(0)})_N^{-N}$, where $\lambda_{-N-1+k}^{(0)}$, $\lambda_k^{(0)}$ are roots of the polynomial $(\lambda - v_{2k})(\lambda - v_{2k+1}) - 1$ for $k \in \mathbb{N}_{N-1}$ and $\lambda_{-1}^{(0)} = 2\varepsilon$, $\lambda_0^{(0)} = 4\varepsilon$, $\lambda_{-N}^{(0)} = 2 + 6\varepsilon$. Then $\Psi : \mathcal{V}_\alpha \to \mathbb{R}^{2N+1}$ is not injection, since $\Psi(v) = \Psi(w)$. The Proof of ii) is similar to the Proof of i).

iii) Firstly, $\lambda_0^{(0)} = v_1$ determine the first component of potential. By the remark after (3.17) the components $v_{2k}, v_{2k+1}$ are defined uniquely as a roots of polynomial $(\lambda - \lambda_{-N-1+k}^{(0)})(\lambda - \lambda_k^{(0)}) + 1$. 
iv) By (1.5) we have that $J_{\pi+t}(0,v) = J_{\pi-t}(0,v)$, which yields symmetry $\lambda_k(\pi-t,0,v) = \lambda_k(\pi+t,0,v)$ and then $t = \pi$ is a point of local extremum for functions $\lambda_k$. If $v \in \mathcal{V}_1$ (or $\mathcal{V}_1$) then by (3.17) all components $\lambda_k^{(0)}$ are distinct numbers.

4. Appendix, proof of Theorem 1.1

For the magnetic field $\mathcal{B} = B(0,0,1) \in \mathbb{R}^3$ the corresponding magnetic vector potential is given by

$$\mathcal{A}(x) = \frac{1}{2}[\mathcal{B},x] = \frac{B}{2}(-x_2,x_1,0), \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3.$$  

Define the coefficients $a_{\nu,\rho}(t) = (\mathcal{A}(x_\nu + t\mathbf{e}_{\nu,\rho}), \mathbf{e}_{\nu,\rho})$, where $\nu, \rho \in \mathbb{Z} \times \mathbb{N}_p$ and $t \in [0,1]$. In Lemma 4.1 we will show that $a_{\nu,\rho}(t)$ does not depend on $t \in [0,1]$. Recall that the magnetic operator $\Delta_b$ is given by

$$(\Delta_b f)_\omega = e^{ia_{\omega,(n,2k)}}f_{n,2k} + e^{ia_{\omega,(n,2k+2)}}f_{n-1,2k+2} + e^{ia_{\omega,(n,2k+2)}}f_{n,2k+2}, \quad k \in \mathbb{N}_N^0,$$

$$f_{n,0} = f_{n,2N+2} = 0, \quad b = \frac{B\sqrt{3}}{2},$$

$$(\Delta_b f)_\sigma = e^{ia_{\sigma,(n,2k-1)}}f_{n,2k-1} + e^{ia_{\sigma,(n,2k-1)}}f_{n+1,2k-1} + e^{ia_{\sigma,(n,2k+1)}}f_{n,2k+1}, \quad k \in \mathbb{N}_N,$$  

where $\omega = (n,2k+1)$, $\sigma = (n,2k) \in \mathbb{Z} \times \mathbb{N}_p$ and $f = (f_{n,k})_{(n,k)\in \mathbb{Z} \times \mathbb{N}_p} \in \ell^2(\Gamma)$.

Lemma 4.1. Let a function $a_{\nu,\rho}(t) = (\mathcal{A}(x_\nu + t\mathbf{e}_{\nu,\rho}), \mathbf{e}_{\nu,\rho})$, where $\nu, \rho \in \mathbb{Z} \times \mathbb{N}_p$ and $t \in [0,1]$. Denote $\omega = (n,2k+1)$, $\sigma = (n,2k)$, then

$$a_{\omega,(n,2k+2)} = b(n-k), \quad a_{\omega,(n,2k)} = b(n+2k), \quad a_{\omega,(n,2k+2)} = -b(2n+k),$$

$$a_{\sigma,(n,2k-1)} = -b(n-k+1), \quad a_{\sigma,(n,2k)} = -b(n+2k-1), \quad a_{\sigma,(n,2k+1)} = b(n+k),$$

where $b = B\sqrt{3}$ and all $(t,n,k) \in [0,1] \times \mathbb{Z} \times \mathbb{N}_p$.

Proof. Identity $\mathcal{A}(r) = \frac{B}{2}[\mathbf{e}_0,r], \mathbf{e}_0 = (0,0,1), r \in \mathbb{R}^3$ yields for any $t \in [0,1]$

$$a_{\omega,\sigma}(t) = \frac{B}{2}([\mathbf{e}_0, x_\omega + t\mathbf{e}_{\omega,\sigma}], e_{\omega,\sigma}) = \frac{B}{2}([\mathbf{e}_0, x_\omega], e_{\omega,\sigma}) = a_{\omega,\sigma}(0) = a_{\omega,\sigma},$$

where $\sigma = (n,2k+2), (n-1,2k+2), (n,2k)$.

Recall $x_\omega = (\sqrt{3}(2n+k), 3k, 0)$. If $\sigma = (n,2k+2)$, then $e_{\omega,\sigma} = (\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$ and (4.4) yields

$$a_{\omega,(n,2k+2)} = \frac{B}{2}([\mathbf{e}_0, x_\omega], e_{\omega,\sigma}) = \frac{B}{2} \det \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ \sqrt{3}(2n+k) & 3k & 0 \end{pmatrix} = \frac{\sqrt{3}B}{2}(n-k).$$

If $\sigma = (n-1,2k+2)$, then $e_{\omega,\sigma} = (-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$ and (4.4) yields

$$a_{\omega,(n-1,2k+2)} = \frac{B}{2}([\mathbf{e}_0, x_\omega], e_{\omega,\sigma}) = \frac{B}{2} \det \begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ \sqrt{3}(2n+k) & 3k & 0 \end{pmatrix} = \frac{\sqrt{3}B}{2}(n+2k).$$

If $\sigma = (n,2k)$, then $e_{\omega,\sigma} = (0,-1,0)$ and (4.4) yields

$$a_{\omega,(n-1,2k+2)} = \frac{B}{2}([\mathbf{e}_0, x_\omega], e_{\omega,\sigma}) = \frac{B}{2} \det \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ \sqrt{3}(2n+k) & 3k & 0 \end{pmatrix} = -\frac{\sqrt{3}B}{2}(2n+k).$$
The proof of other cases is similar. □

Substituting identities from Lemma 4.1 into the magnetic operator \( \Delta_b \) is given by (4.1) we obtain (4.3).

**Proof of Theorem 1.1.** i) Recall that

\[
(\Delta_b f)_{n,2k+1} = e^{-ib(2n+k)}f_{n,2k} + e^{ib(n+2k)}f_{n-1,2k+2} + e^{ib(n-k)}f_{n,2k+2} + v_{2k+1}f_{n,2k+1}, \quad k \in \mathbb{N}_0^0,
\]

\[
f_{n,0} = f_{n,2N+2} = 0,
\]

\[
(\Delta_b f)_{n,2k} = e^{-ib(n-k+1)}f_{n,2k-1} + e^{-ib(n+2k-1)}f_{n+1,2k+1} + e^{ib(2n+k)}f_{n,2k+1} + v_{2k}f_{n,2k}, \quad k \in \mathbb{N}_N,
\]

(4.5)

Define the unitary operators \( U, S \) acting in \( \ell^2(\mathbb{Z}) \) by

\[
U(h_n)_{n \in \mathbb{Z}} = (\tau^n h_n)_{n \in \mathbb{Z}}, \quad \tau = e^{ib}, \quad Sh = (h_{n+1})_{n \in \mathbb{Z}}, \quad h = (h_n)_{n \in \mathbb{Z}} \quad (4.6)
\]

For each \( (f_{n,k})_{(n,k) \in \mathbb{N}_p \times \mathbb{N}_p} \in \ell^2(\Gamma) \) we introduce the function \( \psi_k = (f_{n,k})_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}), \quad k \in \mathbb{N}_p, p = 2N + 1 \) and \( \psi = (\psi_k)_{k \in \mathbb{N}_p} \in (\ell^2(\mathbb{Z})^p). \) Using (4.3) and \( (Vf)_{n,k} = v_k f_{n,k} \) for any \( (n, k) \in \mathbb{Z} \times \mathbb{N}_p, \) we obtain that the operator \( H : \ell^2(\Gamma) \rightarrow \ell^2(\Gamma) \) is unitarily equivalent to the operator \( K : \ell^2(\mathbb{Z})^p \rightarrow \ell^2(\mathbb{Z})^p, \) given by

\[
(K\psi)_{2k+1} = (\tau^k(U^*)^2\psi_{2k} + (1 + \tau S^*) \psi_{2k+2} + v_{2k+1}\psi_{2k+1}, \quad \psi_0 = \psi_{p+1} = 0, \quad k \in \mathbb{N}_N,
\]

\[
(K\psi)_{2k} = U^*(\tau^{k-1} + S \tau^{-k})\psi_{2k-1} + U^2\tau^{k}\psi_{2k+1}, \quad k \in \mathbb{N}_N.
\]

We rewrite \( K \) in the matrix form by

\[
K(\psi_k)^T = \begin{pmatrix}
v_1 & (1 + \tau S^*)U & 0 & 0 & \cdots & 0 \\
v_2 & 0 & U^2 \tau & 0 & \cdots & 0 \\
(v_3 & (\tau + \tau^3 S^*)U & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & (U^*)^2 \tau^N & U^2 \tau^N & v_p
\end{pmatrix}
\]

\[
\psi_1
\]

\[
\psi_2
\]

\[
\psi_3
\]

\[
\psi_p
\]

(4.7)

where \( \psi_k \in \ell^2(\mathbb{Z}). \) Note that \( K^* = K, \) since \( S^* = S^{-1}. \) We rewrite \( K \) in the matrix form by

\[
K = \begin{pmatrix}
v_1 & A_1 & 0 & 0 & \cdots & 0 \\
A_1^* & v_2 & A_2 & 0 & \cdots & 0 \\
0 & A_2^* & v_3 & A_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\n0 & \cdots & \cdots & \cdots & \cdots & v_p
\end{pmatrix}
\]

(4.8)

where \( A_{2k} = \tau^k U^2, k \in \mathbb{N}_N \) and \( A_{2k+1} = (\tau^k + \tau^{2k+1} S^*)U, k \in \mathbb{N}_N^0. \) Define the unitary operator \( \mathcal{U} = \text{diag}(u^p), \) where \( u_{2k} = U^{3k-2}, k \in \mathbb{N}_N \) and \( u_{2k+1} = U^{3k}, k \in \mathbb{N}_N^0. \) Using \( US^* = \tau SU^* \) we obtain

\[
K_1 = \mathcal{U} K \mathcal{U}^* = \begin{pmatrix}
v_1 & r_1 & 0 & 0 & \cdots & 0 \\
r_1^* & v_2 & r_2 & 0 & \cdots & 0 \\
0 & r_2^* & v_3 & r_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & r_{p-1}^* \\
0 & \cdots & \cdots & \cdots & \cdots & v_p
\end{pmatrix}
\]

(4.9)

where \( r_{2k} = \tau^k, k \in \mathbb{N}_N \) and \( r_{2k+1} = \tau^{-k}(\tau^{6k+1} S^* + 1), k \in \mathbb{N}_N^0. \)
Introduce the unitary operator \( \Phi : \ell^2(\mathbb{Z})^p \to \int_{[0,2\pi]}^\oplus \mathcal{H}_0 \frac{dt}{2\pi} \), \( \mathcal{H}_0 = \mathbb{C}^p \), by \( \Phi(\psi_k)^p = (\phi \psi_k)^p \), where \( \phi : \ell^2(\mathbb{Z}) \to L^2(0,2\pi) \) is an isometric operator given by

\[
\phi h = \sum_{n \in \mathbb{Z}} h_n \frac{e^{int}}{\sqrt{2\pi}}, \quad h = (h_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}), \ t \in [0,2\pi].
\]

Then we deduce that

\[
\Phi_p K_1 \Phi_p^{-1} = \int_{[0,2\pi]}^\oplus \frac{dt}{2\pi},
\]

where the operator \( J_t : \mathcal{H}_0 \to \mathcal{H}_0 \) has the matrix given by

\[
\tilde{J}_t = \begin{pmatrix}
  v_1 & 1 + \tau e^{-it} & 0 & \cdots & 0 \\
  1 + \tau e^{it} & v_2 & \tau & \cdots & 0 \\
  0 & \tau & v_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & \tau^N & v_p \\
\end{pmatrix}.
\]

The matrix \( \tilde{J}_t \) is unitarily equivalent to the matrix \( J_t \), given by

\[
J_t = \begin{pmatrix}
  a_1(t) & 0 & \cdots & 0 \\
  v_1 & a_1(t) & 0 & \cdots \\
  v_2 & a_2(t) & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & a_{p-1}(t) \\
\end{pmatrix},
\]

where \( a_k(t) = 1 \), \( k \in \mathbb{N}_N \) and \( a_{2k-1}(t) = 2|\cos(k t - \frac{6k-5}{2}b)|, k \in \mathbb{N}_N \).

Thus we deduce that the operator \( H \) is unitarily equivalent to the operator \( \int_{[0,2\pi]}^\oplus \frac{dt}{2\pi} \).

ii) Let \( \lambda_k(t), k \in \mathbb{Z}_N \) be eigenvalues of \( J_t \) satisfy \( \lambda_{-N}(t) \leq \lambda_{-N+1}(t) \leq \ldots \leq \lambda_N(t) \). From the spectral theory of Jacobi operators \([vM]\) we have that if all \( a_k(t) \neq 0 \) for some \( t \), then \( \lambda_{-N}(t) < \lambda_{-N+1}(t) < \ldots < \lambda_N(t) \) and perturbation theory gives us that any of these functions is analytic in some neighborhood of \( t \).

**References**

[DDE] G. Dresselhaus, M. S. Dresselhaus, and P. C. Eklund, Science of Fullerenes and Carbon Nanotubes: Their Properties and Applications, Academic, New York, 1996.

[Ha] Harris P. Carbon Nanotubes and Related Structures, Cambridge Univ. Press., Cambridge, 1999.

[Hi] Hiura, H. Tailoring graphite layers by scanning tunneling microscopy, Appl. Surf. Sci. 222 (2004), 374–381.

[KLYH] Er-Jun Kan, Zhenyu Li, Jinlong Yang; J. G. Hou Will zigzag graphene nanoribbon turn to half metal under electric field?, Applied physics letters 91(2007), 243116,1-3.

[K] Korotyaev, E. Effective masses for zigzag nanotubes in magnetic fields, Letters in Math. Phys., 83 (2008), No 1, 83-95.

[KKr] Korotyaev, E.; Krasovsky, I. Spectral estimates for periodic Jacobi matrices, Commun. Math. Phys. 234(2003), 517-532.

[KKu1] Korotyaev, E.; Kutsenko, A. Inverse problem for the discrete 1D Schrödinger operator with small periodic potentials, Commun. Math. Phys. 261(2006), 673–692.

[KKu2] Korotyaev, E.; Kutsenko, A. Lyapunov functions for periodic matrix-valued Jacobi operators, AMS translations Series 2, 225 (2008), 117-131.

[KKu3] Korotyaev, E.; Kutsenko, A. Zigzag nanoribbons in external electric fields, to appear in ”Asymptotic Analysis”.
ZIGZAG NANORIBBONS IN EXTERNAL FIELDS

[KKu4] Korotyaev, E.; Kutsenko, A., Zigzag and armchair nanotubes in external fields. In the book: Advances in Mathematics Research, Nova Science Publishers, Inc. Volume 10, 2009, 271–299.

[KL] Korotyaev, E.; Lobanov, I. Schrödinger operators on zigzag periodic graphs. Ann. Henri Poincaré 8 (2007), 1151–1176.

[KL1] Korotyaev, E.; Lobanov, I. Zigzag periodic nanotube in magnetic field, preprint 2006.

[KS] Kostrykin, V.; Schrader, R. Quantum wires with magnetic fluxes. Dedicated to Rudolf Haag. Comm. Math. Phys. 237 (2003), 161–179.

[Ku] P. Kuchment, Graph models for waves in thin structures, Waves in Random Media, 12 (2002), R1–R24.

[KuP] Kuchment, P.; Post, O. On the spectra of carbon nano-structures, Commun. Math. Phys. 275(2007), 805–826.

[La] Last, Y. On the measure of gaps and spectra for discrete 1D Schrodinger operators. Comm. Math. Phys. 149 (1992), no. 2, 347–360.

[vM] van Moerbeke, P. The spectrum of Jacobi matrices. Invent. Math. 37 (1976), no. 1, 45–81.

[N] D.S. Novikov, Electron properties of carbon nanotubes in a periodic potential, Physical Rev. B 72(2005), 235428-1-22.

[No1] Novoselov, K.; Geim, A.; Morozov S.; Jiang, D. Two-Dimensional Gas of Massless Dirac Fermions in Graphene, Nature, 438(7065) (2005), 197-200.

[No2] Novoselov, K.; Jiang, D.; Schedin, F.; Booth, T.; Khotkevich, V.; Morozov, S.; Geim, A. Two-dimensional atomic crystals, Proc. of the Nat. ac. of science of USA, 102(2005), 10451-10453.

[No3] Novoselov, K.; Geim, A.; Morozov, S.; Jiang, D.; Zhang, Y.; Dubonos, S.; Grigorieva, I.; Firsov, A. Electric field effect in atomically thin carbon films, Science 22 October 2004: Vol. 306. no. 5696, 666 - 669.

[Pa] Pankrashkin, P. Spectra of Schrodinger operators on equilateral quantum graphs, Lett. Math. Phys. 77 (2006) 139-154.

[RR] V. Rabinovich; S. Roch, Essential spectra of difference operators on $\mathbb{Z}^n$-periodic graphs. J. Phys. A: Math. Theor. 40 (2007) 1010910128.

[RS] M. Reed ; B. Simon. Methods of modern mathematical physics. IV. Analysis of operators. Academic Press, New York-London, 1978.

[SCL] Young-Woo Son; Marvin L. Cohen; Steven G. Louie. Half-metallic graphene nanoribbons. Nature 444 (16 November 2006), 347-349.

[SDD] Saito, R.; Dresselhaus, G.; Dresselhaus, M. Physical properties of carbon nanotubes, Imperial College Press, 1998.

[vMou] van Mouche, P. Spectral asymptotics of periodic discrete Schrodinger operators. I. Asymptotic Anal. 11 (1995), no. 3, 263–287.

[ZTSK] Y. Zhang, Y.-W. Tan, H. L. Stormer and P. Kim, Experimental observation of the quantum Hall effect and Berry’s phase in graphene Nature 438(2005), 201-204.

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