On the $e$-positivity of $(\text{claw}, 2K_2)$-free graphs

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Abstract

Motivated by Stanley and Stembridge’s conjecture about the $e$-positivity of claw-free incomparability graphs, Hamel and her collaborators studied the $e$-positivity of $(\text{claw}, H)$-free graphs, where $H$ is a four-vertex graph. In this paper we establish the $e$-positivity of generalized pyramid graphs and $2K_2$-free unit interval graphs, which are two important families of $(\text{claw}, 2K_2)$-free graphs. Hence we affirmatively solve one problem proposed by Hamel, Hoàng and Tuero, and another problem considered by Foley, Hoàng and Merkel.

Mathematics Subject Classifications: 05E05, 05C15

1 Introduction

Given a finite simple graph $G$ with vertex set $V$ and edge set $E$, a proper coloring of $G$ is a function $\kappa$ from $V$ to $\mathbb{P} = \{1, 2, \ldots\}$ such that $\kappa(u) \neq \kappa(v)$ whenever $uv \in E$. Stanley [13] defined the chromatic symmetric function $X_G$ as

$$X_G = \sum_{\kappa} \prod_{v \in V} x_{\kappa(v)},$$

where $\kappa$ ranges over all proper colorings of $G$. It is clear that $X_G$ is a homogeneous symmetric function of degree $n$, where $n$ is the cardinality of $V$. There have been many works focusing on the expansion of $X_G$ in terms of various bases of symmetric functions. A well known basis is composed of elementary symmetric functions which are indexed by integer partitions. Recall that an integer partition of $n$ is a weakly decreasing sequence $\lambda = \ldots, \lambda_2, \lambda_1$ supported in part by the Fundamental Research Funds for the Central Universities and the National Science Foundation of China (Nos. 11522110, 11971249).
$(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\sum_{i=1}^{k} \lambda_i = n$, denoted by $\lambda \vdash n$. Sometimes we consider $\lambda$ as an infinite sequence by appending infinite 0’s. The elementary symmetric function $e_{\lambda}$ is defined as

$$e_{\lambda} = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k},$$

where

$$e_0 = 1 \quad \text{and} \quad e_i = \sum_{1 \leq j_1 < j_2 < \cdots < j_i} x_{j_1} x_{j_2} \cdots x_{j_i} \quad \text{for} \quad i \geq 1.$$ 

It is well known that the set $\{e_{\lambda} \mid \lambda \vdash n\}$ forms a basis of homogeneous symmetric functions of degree $n$. A celebrated conjecture of Stanley and Stembridge states that the chromatic symmetric function $X_G$ of a claw-free incomparability graph $G$ is $e$-positive, namely, $X_G$ can be written as a nonnegative linear combination of $e_{\lambda}$’s, see [13] and [15]. If $X_G$ is $e$-positive, we also say that $G$ is $e$-positive for convenience. Stanley and Stembridge’s conjecture has been extensively studied, see for instance [1, 3, 4, 9, 12]. The main objective of this paper is to prove the $e$-positivity of two families of $(\text{claw}, 2K_2)$-free graphs.

Let us first recall some related concepts and give an overview of some background. Let $H$ be a set of graphs. A graph $G$ is said to be $H$-free if it does not contain any graph of $H$ as an induced subgraph. Hamel, Hoàng and Tuero [8] studied the $e$-positivity of $H$-free graphs, where $H$ is composed of one claw and another four-vertex graph. There are eleven graphs on four vertices, see Figure 1. Concerning the $e$-positivity of $(\text{claw}, F)$-free graphs with $F$ being a four-vertex graph other than claw, some progress has been made. Tsujie [16] proved the $e$-positivity for the case $F = P_4$. Hamel, Hoàng and Tuero proved the $e$-positivity for $F = \text{paw}$ and $F = \text{co-paw}$. They also showed that a $(\text{claw}, F)$-free graphs.

![Figure 1: List of four-vertex graphs.](image-url)
graph is not necessarily e-positive if \( F \) is a diamond, co-claw, \( K_4, 4K_1, 2K_2 \) or \( C_4 \). It remains to study the case that \( F \) is a co-diamond, and Hamel, Hoàng and Tuero proposed the following open problem.

**Problem 1** ([8, Open problem 6.2]). Are (claw, co-diamond)-free graphs e-positive?

By considering the structure of (claw, co-diamond)-free graphs, they reduced the above problem to determine the e-positivity of certain peculiar graphs, as illustrated in [8, Figure 3].

They further explored the e-positivity of (claw, co-diamond, \( F \))-free graphs where \( F \) is a four-vertex graph. The e-positivity of (claw, co-diamond, \( F \))-free graphs is unknown for the cases \( F = C_4, F = 2K_2 \) and \( F = K_4 \). Hamel, Hoàng and Tuero showed that if a peculiar graph is (claw, co-diamond, \( 2K_2 \))-free, then it can be characterized as a generalized pyramid \( GP(r, s, t) \), as illustrated in Figure 2, where \( a, b, c \) are three pairwise nonadjacent vertices, the vertices of \( S_{a,b} (S_{a,c} \) or \( S_{b,c} \)) form a clique of size \( r \) (resp. \( s \) or \( t \)), and each vertex of \( S_{a,b} (S_{a,c} \) or \( S_{b,c} \)) is adjacent to every vertex of \( GP(r, s, t) \) other than \( c \) (resp. \( b \) or \( a \)). In particular, they came up with the following problem.

![Figure 2: The generalized pyramid graph GP(\( r, s, t \)).](image)

**Problem 2** ([8, Open problem 6.1]). Are generalized pyramids e-positive?

In this paper we give an affirmative answer to this problem.

The second part of this paper is devoted to the study of the e-positivity of \( 2K_2 \)-free unit interval graphs. Guay-Paquet [7] proved that if unit interval graphs are e-positive, then any claw-free incomparability graph \( G \) is e-positive, as conjectured by Stanley and Stembridge. Based on Guay-Paquet’s work, Foley, Hoàng and Merkel [5] considered the e-positivity of \( F \)-free unit interval graphs, where \( F \) is a four-vertex graph. It was shown that for any four-vertex graph \( F \) other than co-diamond, \( K_4, 4K_1 \) and \( 2K_2 \), each \( F \)-free unit interval graph is e-positive. Foley, Hoàng and Merkel proved some special cases of \( 2K_2 \)-free unit interval graphs are e-positive. In this paper we show that any \( 2K_2 \)-free unit interval graph is e-positive, which provides further evidence in favor of Stanley and Stembridge’s conjecture.
The paper is organized as follows. In Section 2 we prove the $e$-positivity of generalized pyramid graphs based on the monomial expansion of the corresponding chromatic symmetric functions. In Section 3 we prove the $e$-positivity of $2K_2$-free unit interval graphs by showing that such graphs must be co-triangle free graphs or generalized bull graphs.

## 2 Generalized pyramid graphs

This section is devoted to proving the $e$-positivity of the generalized pyramid graphs $GP(r,s,t)$. By using Stanley’s result on the monomial expansion of the chromatic symmetric function of a graph, we first obtain the monomial expression of $X_{GP(r,s,t)}$. Then based on the transition matrix between the monomial basis and the elementary basis, we explicitly determine the coefficients in the expansion of $X_{GP(r,s,t)}$ in terms of elementary symmetric functions. Finally, we prove that all these coefficients are nonnegative.

Now let us recall some related definitions and results. Given an integer partition $\lambda$, the monomial symmetric function $m_\lambda$ is defined as

$$m_\lambda = \sum_\alpha x^\alpha,$$

where $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2} \cdots$ and $\alpha = (\alpha_1, \alpha_2, \ldots)$ arranges over all distinct permutations of $\lambda = (\lambda_1, \lambda_2, \ldots)$. If $\lambda$ has $r_i$ parts equal to $i$, we also use $\langle 1^{r_1}2^{r_2} \cdots \rangle$ to represent $\lambda$. The augmented monomial symmetric function $\tilde{m}_\lambda$ is defined as

$$\tilde{m}_\lambda = r_1!r_2! \cdots m_\lambda.$$

It is clear that $\{m_\lambda \mid \lambda \vdash n\}$ forms a basis of homogeneous symmetric functions of degree $n$, and hence so does $\{\tilde{m}_\lambda \mid \lambda \vdash n\}$. Let $G$ be a graph with vertex set $V$ and edge set $E$. By using the notion of stable partitions of $G$, Stanley [13] gave a combinatorial interpretation of the coefficients in the expansion of $X_G$ in terms of $\{\tilde{m}_\lambda\}$. Recall that a stable set of $G$ is a subset $S$ of $V$ such that no two vertices of $S$ are adjacent, and a stable partition $\pi$ of $G$ is a set partition of $V$ such that each block of $\pi$ is a stable set. The type of $\pi$ is defined to be the integer partition obtained by rearranging the block sizes of $\pi$ in decreasing order. Stanley’s result can be stated as follows.

**Lemma 3.** [13, Proposition 2.4] Let $G$ be a graph with $n$ vertices and $a_\lambda$ be the number of stable partitions of $G$ of type $\lambda$. Then

$$X_G = \sum_{\lambda \vdash n} a_\lambda \tilde{m}_\lambda.$$  

We now consider the monomial expansion of the chromatic symmetric function of a generalized pyramid graph $GP(r,s,t)$ in Figure 2.

**Theorem 4.** For any nonnegative integers $r, s, t$, we have

$$X_{GP(r,s,t)} = \tilde{m}_{(3,1^{r+s+t})} + (rst)\tilde{m}_{(2,2,2,1^{r+s+t-3})} + (rt + rs + st + r + s + t) \cdot$$

$$\tilde{m}_{(2,1^{r+s+t-1})} + (r + s + t + 3)\tilde{m}_{(2,1^{r+s+t+1})} + \tilde{m}_{(1^{r+s+t+3})}.$$  

(2)
Proof. From Figure 2 we see that there exists no stable set of size greater than or equal to 4. Moreover, there exists a unique stable set of size 3, namely \( \{a, b, c\} \). A stable set of size 2 can only be of the form \( \{a, u\} \) with \( u \in S_{a,c} \cup \{b, c\} \), or \( \{b, v\} \) with \( v \in S_{a,c} \cup \{a, c\} \), or \( \{c, w\} \) with \( w \in S_{a,b} \cup \{a, b\} \). Thus, any admissible stable partition of \( GP(r, s, t) \) is of type \((3, 1^{r+s+t+1})\), \((2, 1^{r+s+t+1})\), \((2, 2, 1^{r+s+t-1})\), \((2, 2, 2, 1^{r+s+t-3})\) or \((1^{r+s+t+3})\). Moreover, we will show that

\[
\begin{align*}
    a_{(3, 1^{r+s+t+1})} &= 1, \\
    a_{(2, 1^{r+s+t+1})} &= r + s + t + 3, \\
    a_{(2, 2, 1^{r+s+t-1})} &= rt + rs + st + r + s + t, \\
    a_{(2, 2, 2, 1^{r+s+t-3})} &= rst, \\
    a_{(1^{r+s+t+3})} &= 1.
\end{align*}
\]

The above formulas can be proven in the same manner. As an example we prove the fourth formula. Note that a stable partition of type \((2, 2, 2, 1^{r+s+t-3})\) is uniquely determined by the set of three stable sets of size 2, which can only be of the form \( \{\{a, u\}, \{b, v\}, \{c, w\}\} \) with \( u \in S_{b,c} \), \( v \in S_{a,c} \), \( w \in S_{a,b} \). It is clear that \( u \) has \( t \) choices, \( v \) has \( s \) choices and \( w \) has \( r \) choices. Hence the fourth formula holds. This completes the proof.

Next we shall give the expansion of \( X_{GP(r, s, t)} \) in terms of elementary symmetric functions. To this end, we need to use some results concerning the transition matrix between the bases \( \{m_\lambda : \lambda \vdash n\} \) and \( \{e_\lambda : \lambda \vdash n\} \). Let \( Par(n) \) denote the set of all partitions of \( n \). Given two partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( \mu = (\mu_1, \mu_2, \ldots) \) of \( Par(n) \), we say that \( \mu \leq \lambda \) if

\[
\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i \quad \text{for all } i \geq 1.
\]

The conjugate of \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is defined as the partition \( \lambda' = (\lambda'_1, \lambda'_2, \ldots) \) where \( \lambda'_i = |\{j : \lambda_j \geq i\}| \). We have the following result.

**Lemma 5.** [14, Chapter 7] Let \( \lambda \vdash n \). If

\[
e_\lambda = \sum_{\mu \vdash n} M_{\lambda \mu} e_\mu,
\]

then \( M_{\lambda \mu} \) is equal to the number of \((0, 1)\)-matrices \( A = (a_{ij})_{i,j \geq 1} \) satisfying \( \text{row}(A) = \lambda \) and \( \text{col}(A) = \mu \), where \( \text{row}(A) \) (resp., \( \text{col}(A) \)) is the vector of row sums (resp., column sums) of \( A \). Moreover, \( M_{\lambda \mu} = 0 \) unless \( \lambda \leq \mu' \), and \( M_{\lambda \lambda'} = 1 \).

Combining Theorem 4 and Lemma 5, we obtain the following result.

**Theorem 6.** For any nonnegative integers \( r, s, t \), we have

\[
X_{GP(r, s, t)} = A \cdot e_{(r+s+t+1, 1, 1, 1)} + B \cdot e_{(r+s+t+1, 3)} + C \cdot e_{(r+s+t+1, 2, 1)} + D \cdot e_{(r+s+t+2, 1)} + E \cdot e_{(r+s+t+3)},
\]

(3)
where

\[ A = (r + s + t)! , \]
\[ B = (r + s + t - 3)! \cdot 6rst , \]
\[ C = (r + s + t - 3)! \cdot 2(r + s + t - 1) \cdot [(r^2s + rs^2 - 2rs) + (rt^2 + r^2t - 2rt) + (s^2t + st^2 - 2st)] , \]
\[ D = (r + s + t - 2)! \cdot [(r^4 + r^3 - 2r^2) + (3r^2s - 2rs) + (3rs^2 - 2s^2) + (3r^2t - 2rt) + (9rst - 2st) + (3rt^2 - 2t^2) + 3s^2t + 5rs^2t + 5r^2st + 2s^2t + \]
\[ + t^3 + 2rt^2 + 2st^2 + t^4 + 2r^3s + 2r^2s^2 + s^3 + 2rs^3 + s^4] , \]
\[ E = (r + s + t - 1)! \cdot (3 + r + s + t)(r + s)(r + t)(s + t) . \]

**Proof.** Let \( i = r + s + t \) and \( P = \{(2^3, 1^{i-3}), (3, 1^i), (2^2, 1^{i-1}), (1^{i+1}), (1^{i+3})\} \). In order to give the elementary expansion of \( X_{GP(r,s,t)} \), by Theorem 4 and Lemma 5 it suffices to consider the monomial expansion of those \( e_\lambda \)'s such that \( \lambda \leq \mu \) for some \( \mu \in P \). It is straightforward to verify that the set of such partitions \( \lambda \) is composed of \( \{(i, 3), (i + 1, 1, 1), (i + 1, 2), (i + 2, 1), (i + 3)\} \). Using Lemma 5, we will show

\[ e_{(i,3)} = m_{(2,2,2,1^{i-3})} + (i - 1)m_{(2,2,1^{i-1})} + \left(\frac{i + 1}{2}\right) m_{(2,1^{i+1})} + \left(\frac{i + 3}{3}\right) m_{(1^{i+3})} , \]
\[ e_{(i+1,1,1)} = m_{(3,1^i)} + (2i + 3)m_{(2,1^{i+1})} + 2m_{(2,2,1^{i-1})} + 2\left(\frac{i + 3}{2}\right) m_{(1^{i+3})} , \]
\[ e_{(i+1,2)} = m_{(2,2,1^{i-1})} + (i + 1)m_{(2,1^{i+1})} + \left(\frac{i + 3}{2}\right) m_{(1^{i+3})} , \]
\[ e_{(i+2,1)} = m_{(2,1^{i+1})} + (i + 3)m_{(1^{i+3})} , \]
\[ e_{i+3} = m_{(1^{i+3})} . \]

The above formulas are easy to prove. As an example we prove that the coefficient of \( m_{(2,1^{i+1})} \) in \( e_{(i+1,2)} \) is \( i + 1 \). By Lemma 5, we only need to count the number of \((0, 1)\)-matrices \( A = (a_{pq})_{p,q \geq 1} \) with row \( (A) = (i + 1, 2) \) and col \( (A) = (2, 1^{i+1}) \). Since row \( (A) = (i + 1, 2) \), there are \( i + 1 \) entries equal to 1 in the first row of matrix \( A \) and two entries equal to 1 in the second row. Since col \( (A) = (2, 1^{i+1}) \), we must have \( a_{11} = a_{21} = 1 \) and \( a_{pq} = 0 \) for \( p \geq 3 \) or \( q \geq i + 3 \). Moreover, the submatrix

\[
\begin{pmatrix}
  a_{12} & a_{13} & \cdots & a_{1,i+2} \\
  a_{22} & a_{23} & \cdots & a_{2,i+2}
\end{pmatrix}
\]

can be any \( 2 \times (i + 1) \) matrix composed of \( i \) column vectors \( \binom{1}{0} \)'s and one column vector \( \binom{0}{1} \). Hence we have \( M_{(i+1,2),(2,1^{i+1})} = i + 1 \).
By using the above $m$-expansion formulas we can get the $e$-expansion of those monomial symmetric functions appearing in (2). Substituting the resulted $e$-expansion formulas into (2), we complete the proof.

We proceed to prove the main result of this section.

**Theorem 7.** For any nonnegative integers $r, s, t \geq 0$ the generalized pyramid graph $GP(r, s, t)$ is $e$-positive.

**Proof.** Note that if $r = s = t = 0$, then $X_{GP(r,s,t)} = e_1^3$, which is obviously $e$-positive. If only two of $r, s, t$ are zero, then $GP(r, s, t)$ belongs to one class of $e$-positive graphs studied by Hamel, Hoàng and Tuero, see [8, Lemma 9]. If exactly one of $r, s, t$ is zero, then $GP(r, s, t)$ is a generalized bull graph in Figure 3, whose positivity is already known, see Foley, Hoàng and Merkel [5, Theorem 11] and Cho, Huh [2, Theorem 3.7].

From now on we assume that $r, s, t$ are positive integers. In order to show the $e$-positivity of $X_{GP(r,s,t)}$, it suffices to show that the coefficients $A, B, C, D, E$ in (3) are nonnegative. Clearly, $A, B$ and $E$ are always nonnegative.

We continue to prove $C \geq 0$. Since $r, s \geq 1$, we have

$$r^2 s + rs^2 - 2rs \geq r^2 + s^2 - 2rs \geq 0,$$

Similarly, we have

$$r^2 t + rt^2 - 2rt \geq 0,$$

and

$$st^2 + s^2 t - 2st \geq 0.$$

Therefore, $C \geq 0$.

Finally, we prove that $D \geq 0$. Since $r, s, t \geq 1$, it is straightforward to verify that $r^4 + r^3 - 2r^2, 3r^2 s - 2rs, 3rs^2 - 2s^2, 3r^2 t - 2rt, 9rst - 2st, 3rt^2 - 2t^2$ are all nonnegative. Thus, $D \geq 0$. This completes the proof.

\[\square\]

3 **$2K_2$-free unit interval graphs**

The aim of this section is to prove that $2K_2$-free unit interval graphs are $e$-positive. Our proof is based on the characterization of $2K_2$-free unit interval graphs due to Hempel and Kratsch [10], who actually gave a characterization of a larger family of graphs. Using their result, we show that $2K_2$-free unit interval graphs can only be either co-triangle-free graphs or generalized bull graphs, which are already known to be $e$-positive.

Let us first recall some related definitions and results. A co-triangle means a stable set of size 3. Stanley and Stembridge [15] proved the $e$-positivity of the complement graphs of bipartite graphs, which are a special class of co-triangle-free graphs. Using their result, we can also apply to the following general case.

**Lemma 8.** [14, Exercise 7.47] If $G$ is a co-triangle-free graph, then $X_G$ is $e$-positive.
The generalized bull graphs were introduced by Foley, Hoàng and Merkel [5], but their e-positivity was first proved by Cho and Huh [2]. A generalized bull graph can be characterized as Figure 3, where $K_r$, $K_s$, $K_t$ form a clique of size $r + s + t$, $a$ is adjacent to each vertex of $K_r$, and $b$ is adjacent to each vertex of $K_s$. We denote such a graph by $GB(r,s,t)$.

![Figure 3: The generalized bull graph GB(r, s, t).](image)

Cho and Huh [2] obtained the following result.

**Lemma 9.** [2, Theorem 3.7] For any positive integers $r, s, t$, the generalized bull graph $GB(r,s,t)$ is e-positive.

Note that Cho and Huh proved the above result based on the Schur expansion of $X_{GB(r,s,t)}$. To be self-contained, we would like to give a new proof, which parallels that of Theorem 7.

**Proof of Lemma 9.** We first give the monomial expansion of $X_{GB(r,s,t)}$. Using the same method as in the proof of Theorem 4, we get that

$$X_{GB(r,s,t)} = t \cdot \tilde{m}(3,1^r+s+t-1) + (t(t-1) + tr + sr + st) \cdot \tilde{m}(2,2,1^r+s+t-2) + (1 + 2t + s + r) \cdot \tilde{m}(2,1^r+s+t+1) + \tilde{m}(1^r+s+t+2).$$

(9)

Setting $k = r + s + t$ and $i = k - 1$ in (5), (6), (7) and (8), and then substituting these four equations into (9), we obtain

$$X_{GB(r,s,t)} = (r + s + t - 2)! \cdot [(r + s + t - 1)t \cdot e_{(r+s+t,1,1,1)} + 2rs \cdot e_{(r+s+t,2,2)} + (r^3 + r^2s + rs^2 + s^3 + 2r^2t + 2rst + 2s^2t + rt^2 + st^2 - r - s) \cdot e_{(r+s+t+1,1,1)} + (r + s + t + 2)(r + s + t - 1)rs \cdot e_{(r+s+t+2,2,2)}].$$

Since $r, s, t \geq 1$, the e-positivity of $X_{GB(r,s,t)}$ is obvious. \qed
We proceed to recall Hempel and Kratsch's characterization of $2K_2$-free unit interval graphs. As will be shown below, $2K_2$-free unit interval graphs are a special class of (claw, $AT$)-free graphs. Recall that an interval graph is formed from a set of intervals on the real line, with a vertex for each interval and an edge between vertices whose intervals intersect. A unit interval graph is an interval graph for which each of its intervals has length. It is well known that unit interval graphs must be claw-free and $C_4$-free. The notion of $AT$-free graphs was introduced by Lekkerkerker and Boland [11]. A co-triangle in a graph $G$ is called an asteroidal triple, denoted by AT for short, if for any pair of its vertices there exists a path between them which does not intersect with the neighborhood of the third vertex. It has been shown in [11] that interval graphs are exactly the class of chordal $AT$-free graphs, where a chordal graph is a graph such that every induced cycle in the graph has exactly three vertices. Meanwhile, unit interval graphs have been shown to be exactly the class of claw-free interval graphs [6]. Hence, $2K_2$-free unit interval graphs are equivalent to $(2K_2, \text{claw, AT})$-free chordal graphs. Given a graph $G$ with vertex set $V$ and edge set $E$ and a pair of vertices $u$ and $v$, let $\alpha(G)$ denote the maximum size of stable sets and let $d(u,v)$ denote the number of edges of the shortest path between $u$ and $v$. For any vertex $w \in V$, let $N_i(w) = \{x \in V \mid d(x,w) = i\}$ and $[N_i(w)]$ denote the induced subgraph on $N_i(w)$. In particular, $N_1(w)$ is the neighborhood of $w$, also denoted by $N(w)$. With these notations, Hempel and Kratsch's characterization of (claw, AT)-free graphs can be stated as follows.

**Lemma 10.** [10, Lemma 6] For any connected (claw, $AT$)-free graph $G$, there exists a vertex $w$ such that $\alpha([N(w)]) \leq 2$ and for any $i \geq 2$ each $[N_i(w)]$ is a clique (which might be empty).

It is well known that $X_{G \cup H} = X_G X_H$, where $G \cup H$ is a disjoint union of graphs $G$ and $H$. Given a $2K_2$-free unit interval graph $G$, it is clear that every connected component of $G$ is also a $2K_2$-free unit interval graph. Thus when studying the e-positivity of $X_G$, we may assume that $G$ is connected. Based on the above result, we could give a characterization of connected $2K_2$-free unit interval graphs.

**Corollary 11.** If $G$ is a connected $2K_2$-free unit interval graph, then there exists a vertex $w$ such that $\alpha([N(w)]) \leq 2$, $[N_2(w)]$ is a clique, $|N_3(w)| \leq 1$, and $N_i(w) = \emptyset$ for any $i \geq 4$. Moreover, if $[N(w)]$ is connected and $\alpha([N(w)]) = 2$, then $|N_2(w)| \leq 2$ and $[N(p) \cap N(w)]$ is a clique for any $p \in N_2(w)$.

**Proof of Corollary 11.** Since $G$ is a $2K_2$-free unit interval graph, thus it must be (claw, AT)-free, as mentioned before Lemma 10. Thus, there exists $w$ such that $\alpha([N(w)]) \leq 2$ and for any $i \geq 2$ each $[N_i(w)]$ is a clique.

We proceed to show that $|N_3(w)| \leq 1$ and $N_i(w) = \emptyset$ for any $i \geq 4$. We first show that $N_i(w) = \emptyset$ for any $i \geq 4$. Otherwise, if $N_i(w) \neq \emptyset$ for some $i \geq 4$, then $N_j(w) \neq \emptyset$ for any $1 \leq j \leq i - 1$. Thus there exist $x \in N(w), y \in N_{i-1}(w)$ and $z \in N_i(w)$ such that the set $\{w, x, y, z\}$ induces a $2K_2$, a contradiction. We next show that $|N_3(w)| \leq 1$. Otherwise if $|N_3(w)| > 1$, then there exist $u, v \in N_3(w)$ such that $uv \in E$, since $[N_3(w)]$
is a clique. Then for any \( x \) in \( N(w) \), the set \( \{ w, x, u, v \} \) induces a \( 2K_2 \), a contradiction. Hence \( |N_3(w)| \leq 1 \).

It remains to show that if \([N(w)]\) is connected, \( |N_3(w)| = 0 \) and \( \alpha([N(w)]) = 2 \), then \( |N_2(w)| \leq 2 \) and \( [N(p) \cap N(w)] \) is a clique for any \( p \in N_2(w) \). Note that by definition a unit interval graph must be \( C_4 \)-free. We first show that \([N(p) \cap N(w)]\) is a clique for any \( p \in N_2(w) \). Suppose to the contrary there exist \( p \in N_2(w) \) and non-adjacent \( a, b \in [N(p) \cap N(w)] \). Then \( \{ p, a, b, w \} \) induces a \( C_4 \), a contradiction. We next show that \( |N_2(w)| \leq 2 \). Suppose \( |N_2(w)| = s \). We claim that for any \( a \in N(w) \) there are at least \( s - 1 \) vertices in \( N_2(w) \) which are adjacent to \( a \), namely, \( |N(a) \cap N_2(w)| \geq s - 1 \). Suppose to the contrary there exist \( a \in N(w) \) and \( x, y \in N_2(w) \) such that neither \( x \) nor \( y \) is adjacent to \( a \), and thus \( \{ x, y, a, w \} \) induces a \( 2K_2 \) in \( G \) since \( [N_2(w)] \) is a clique, a contradiction. Since \( \alpha([N(w)]) = 2 \), there exist \( a, b \in N(w) \) which are not adjacent. Moreover, \( a, b \) can not be adjacent to the same vertex \( x \) in \( N_2(w) \) for otherwise the set \( \{ x, a, b, w \} \) induces a \( C_4 \), a contradiction. This means that

\[
(N(a) \cap N_2(w)) \cap (N(b) \cap N_2(w)) = \emptyset.
\]

Hence

\[
s = |N_2(w)| \geq |N(a) \cap N_2(w)| + |N(b) \cap N_2(w)| \geq (s - 1) + (s - 1),
\]

yielding \( s \leq 2 \). Hence \( |N_2(w)| \leq 2 \). This completes the proof.

We would like to point out that the first part of Corollary 11 is already known to Foley, Hoang and Merkel [5], and the second part tells more information of a \( 2K_2 \)-free unit interval graph \( G \). In fact, if more constraints are added, we could get a clearer characterization of \( G \). The following result will be used to check the \( e \)-positivity of some special \( 2K_2 \)-free unit interval graphs.

**Corollary 12.** Given a connected \( 2K_2 \)-free unit interval graph \( G \), let \( w \) be as in Corollary 11. Suppose that \([N(w)]\) is connected, \( |N_2(w)| = 1 \), \( |N_3(w)| = 0 \) and \( \alpha([N(w)]) = 2 \). Let \( N_2(w) = \{ p \} \), \( A = N(p) \cap N(w) \) and \( B = N(w) \setminus A \), then \( |N(a) \cap B| \geq |B| - 1 \) and \( |N(a) \cap B| \) is a clique for any \( a \in A \).

**Proof.** Let us first prove that \( |N(a) \cap B| \geq |B| - 1 \) for any \( a \in A \). Suppose the contrary. Then there exist \( a \in A \) and \( b_1, b_2 \in B \) such that \( b_1 \) and \( b_2 \) are not adjacent to \( a \). If \( b_1 \) and \( b_2 \) are not adjacent in \( G \), then \( \{ a, b_1, b_2 \} \) is a stable set, contradicting \( \alpha([N(w)]) = 2 \). If \( b_1 \) and \( b_2 \) are adjacent, then \( \{ a, p, b_1, b_2 \} \) induces a \( 2K_2 \), a contradiction. Thus \( a \) is adjacent to at least \( |B| - 1 \) vertices in \( B \). Next we show that \([N(a) \cap B]\) is a clique for any \( a \in A \). Suppose to the contrary there exist some \( a \in A \) and non-adjacent \( b, b' \in N(a) \cap B \). Note that the set \( \{ a, p, b, b' \} \) induces a claw, which leads to a contradiction. This completes the proof.

Finally we come to the main result of this section.

**Theorem 13.** If \( G \) is a \( 2K_2 \)-free unit interval graph, then \( X_G \) is \( e \)-positive.
Proof. Without loss of generality, we may assume that $G$ is connected. By Corollary 11, there are six cases to check:

1. $[N(w)]$ is not connected;
2. $[N(w)]$ is connected and $|N_3(w)| = 1$;
3. $[N(w)]$ is connected, $|N_3(w)| = 0$ and $\alpha([N(w)]) = 1$;
4. $[N(w)]$ is connected, $|N_3(w)| = 0$, $\alpha([N(w)]) = 2$ and $|N_2(w)| = 2$;
5. $[N(w)]$ is connected, $|N_3(w)| = 0$, $\alpha([N(w)]) = 2$ and $|N_2(w)| = 1$;
6. $[N(w)]$ is connected, $|N_3(w)| = 0$, $\alpha([N(w)]) = 2$ and $|N_2(w)| = 0$;

where $w$ is given as in Corollary 11.

Foley, Hoàng and Merkel [5] showed that the theorem is true for the first three cases. Indeed, they showed that $G$ must be a co-triangle-free graph or a generalized bull graph. Hence we only need to consider the remaining three cases.

Let us first deal with Case (6). In this case, it is clear that $G$ is co-triangle-free. Thus $X_G$ is $e$-positive by Lemma 8.

Next we consider Case (4). Set $N_2(w) = \{p, q\}$, $A = N(p) \cap N(w)$ and $B = N(w) \setminus A$. By Corollary 11, both $[A]$ and $[N_3(w)]$ are cliques. We claim that any vertex $b \in B$ is adjacent to $q$. Otherwise if there exists some $b \in B$ such that $q$ and $b$ are not adjacent, then $\{p, q, b, w\}$ induces a 2$K_2$, a contradiction. Hence all vertices of $B$ are adjacent to $q$. By Corollary 11 the induced subgraph $[N(q) \cap N(w)]$ is a clique and hence $[B]$ is a clique. Thus $G$ can be characterized as a co-triangle-free graph, as depicted in Figure 4, where the dashed lines represent that there may exist some edges between $A$ and $B$, as well as between $q$ and $A$. Again by Lemma 8, we obtain the $e$-positivity of $X_G$.

![Figure 4: The structure of $G$ in Case (4).](image-url)

Finally, we prove that the theorem holds for Case (5). Now set $N_2(w) = \{p\}$, $A = N(p) \cap N(w)$ and $B = N(w) \setminus A$. By Corollary 11, $[A]$ is a clique. If $[B]$ is a clique, then it is easy to see that $G$ is co-triangle-free, see Figure 5, where the dashed line represents that there may exist some edges between $A$ and $B$. Hence $X_G$ is $e$-positive by Lemma 8.

From now on we assume that $[B]$ is not a clique. Then there exist non-adjacent vertices $x, y \in B$. Now set $A_1 = N(x) \cap A$, $A_2 = N(y) \cap A$ and $A_3 = B \setminus \{x, y\}$. We claim that
either $A_1 = \emptyset$ or $A_2 = \emptyset$. Suppose to the contrary that $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. Now we have $A_1 \cap A_2 = \emptyset$, otherwise there exists $a \in A_1 \cap A_2$ and then $\{a, x, y, p\}$ induces a claw, a contradiction. Moreover, we have $A_1 \cup A_2 = A$, otherwise there exists $b \in A \setminus (A_1 \cup A_2)$ such that $\{b, x, y\}$ is a stable set, contradicting $\alpha([N(w)]) \leq 2$. By Corollary 12, both $[\{x\} \cup A_1 \cup A_3]$ and $[\{y\} \cup A_2 \cup A_3]$ are cliques. A little thought shows that $\{x, y, p\}$ is an asteroidal triple, as shown in Figure 6. This contradicts the fact that $G$ is AT-free. Thus at least one of $A_1$ and $A_2$ is empty.

Without loss of generality, we may assume that $A_1 = \emptyset$. We proceed to show that $N(w) \setminus \{x\}$ induces a clique. By observing that $N(w) = A \cup B$ and $A$ is a clique, it suffices to show that each $a \in A$ and each $z \in B \setminus \{x\}$ are adjacent and $[B \setminus \{x\}]$ is a clique. For the former assertion, assume to the contrary that there exist non-adjacent $a \in A$ and $z \in B \setminus \{x\}$. If $x, z$ are not adjacent, then $\{w, x, a, z\}$ induces a claw, a contradiction. If $x, z$ are adjacent, then $\{a, p, x, z\}$ induces a $2K_2$, again a contradiction.
Hence $a$ and $z$ are adjacent. For the latter assertion, assume to the contrary that there exist non-adjacent vertices $b_1, b_2 \in B \setminus \{x\}$, but then for any $a \in A$ the set $\{a, p, b_1, b_2\}$ induces a claw, a contradiction. Thus $[N(w) \setminus \{x\}]$ is a clique.

If we set $B_1 = N(x) \cap (B \setminus \{x\})$ and $B_2 = B \setminus \{\{x\} \cup B_1\}$, then $G$ can be considered as a generalized bull graph, see Figure 7. Thus in case (5) if $[B]$ is not a clique, the graph $G$ is also $e$-positive by Lemma 9.

Figure 7: The structure of $G$ in Case (5) when $B$ is not a clique.

Combining all the above cases, we complete the proof.

4 Future work

So far we have established the $e$-positivity of certain $(\text{claw}, 2K_2)$-free graphs. It is a natural problem to consider how to construct new $e$-positive graphs from old ones. This kind of problems have been considered by Foley, Ho` ang and Merkel [5]. Given a graph $G$ and a vertex $a$, let $G^{(a)}$ be the graph obtained from $G$ by replacing $a$ by two adjacent vertices $x, y$, and then placing edges connecting every vertex $b$ of $G$ to $x$ and $y$ if $ab$ is an edge of $G$. Foley, Ho`ang and Merkel proposed the following conjecture.

**Conjecture 14** ([5, Conjecture 23]). If $G$ is $e$-positive, so is $G^{(a)}$ for any vertex $a$.

We have proved the $e$-positivity of the generalized pyramid graphs $\text{GP}(r, s, t)$ and the generalized bull graphs $\text{GB}(r, s, t)$. Motivated by the above conjecture, we wish to consider the following problem. Given positive integers $i, j, k, r, s, t$, let $\text{GP}(i, j, k; r, s, t)$ denote the graph obtained from the generalized pyramid $\text{GP}(r, s, t)$ by replacing $a$ (or $c$) in Figure 2 by a clique $K_i$ (resp. $K_j$ or $K_k$), and placing edges connecting every vertex of $K_i$ (resp. $K_j$, or $K_k$) to $S_{a,b}$ and $S_{a,c}$ ($S_{a,b}$ and $S_{b,c}$, or $S_{a,c}$ and $S_{b,c}$). Similarly, let $\text{GB}(i, j; r, s, t)$ denote the graph obtained from the generalized bull graph $\text{GB}(r, s, t)$ by replacing $a$ (or $b$) in Figure 3 by $K_i$ (resp. $K_j$), and placing edges connecting every vertex of $K_i$ (resp. $K_j$) to $K_r$ (resp. $K_s$). Following our approach to Theorem 6 and Lemma 9, for small values of $i, j, k$ it is possible to get the monomial expansion of $X_{\text{GP}(i, j, k; r, s, t)}$, as well as that of $X_{\text{GB}(i, j; r, s, t)}$. However, the enumeration of stable partitions becomes complicated.
for general $i, j, k$. Thus it would be interesting to explore the $e$-positivity of $X_{GP(i,j,k;r,s,t)}$ and $X_{GB(i,j;r,s,t)}$.

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