ENTANGLEMENT IN MULTI-QUBIT PURE FERMIONIC COHERENT STATES

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Received May 8, 2012

In this paper we investigate the entanglement of multi-qubit fermionic coherent states described by anticommutative Grassmann numbers. Choosing an appropriate weight function, we show that it is possible to construct some entangled pure states, consisting of GHZ, W, Bell and biseparable states, by tensor product of fermion coherent states. Moreover a comparison with maximal entangled bosonic coherent states is presented and it is shown that in some cases they have fermionic counterpart which are maximal entangled after integration with suitable weight functions.

Key words: Entanglement, coherent states, Grassmann number, concurrence.

PACS: 03.65.Ud.

1. INTRODUCTION

Understanding the entanglement properties of fermionic systems remains one of the main goals of quantum information theory [1, 2]. As fermionic optics research grows [3], fermionic coherent state (FCS) which is defined as eigenstate of the fermionic annihilation operator with Grassmann-valued eigenvalue, becomes important. Fermionic coherent states can be introduced by parametrizing with Grassmann numbers rather than complex numbers, which overcomes challenges due to anticommutativity relations [4–7].

Besides the idea of constructing bosonic or fermionic coherent states, there is a great deal of interest in studying entanglement of them. For bosonic case, some attempts have been made to quantify the entanglement of multipartite coherent states [8–18]. The required conditions for the maximally entangled nonorthogonal states have been explicitly investigated and some maximally entangled coherent states have been classified [11]. But, despite these attempts, not much work has been done for investigating the entangled fermionic coherent states [19]. The problem is that, strictly within the framework of fermion fields, Grassmann numbers which arise from Pauli’s exclusion principle, anticommute with each other.

The aim of our present work is to elucidate a connection between fermionic coherent states and multi-qubit pure entangled states, treated almost always separately.
We find throughout this work that it is quite possible to construct multi-qubit pure entangled states after integration over tensor product of FCS with suitable weight function. In particular, for example, we can construct a family of maximally entangled states like \( \text{GHZ}, \text{W} \), Bell and Bell-like states [20] for three qubit systems and then generalize to the multi-qubit cases. There exist also FCSs which yield biseparable states in multi-qubits systems except for these cases we have to consider FCS with different Grassmann numbers. Also, we make a comparison between maximal entangled bosonic and coherent states, with one complex or Grassmann number in entries respectively. Somewhat surprisingly, it is shown that in some particular cases the MESs for fermions, have maximal counterpart for bosonic coherent states obtained in Ref. [11] via concurrence measure [21].

The paper is organized as follows. In section 2, the FCS for two level system is introduced. In section 3, explicit examples of multi-qubit entangled states such as Bell, Bell-like, \( \text{W}, \text{GHZ} \) and biseparable states are constructed by tensor product of FCS with appropriate weight functions. Section 4 is devoted to compare some special MESs of bosonic and fermionic coherent states with just one complex or Grassmann number in each entries. The paper ends with a brief conclusion.

### 2. GRASSMANNIAN COHERENT STATES

For our purpose, it is necessary to study the mathematical structure of the anticommuting mathematical objects, so called Grassmann algebra which are needed in order to construct relevant coherent states [4–7]. To describe this algebra we consider \( n \) generators \( \{ \theta_1, \ldots, \theta_n \} \) satisfying the relations:

\[
\{ \theta_i, \theta_j \} = 0 \quad \forall \ i, j = 1, 2, \ldots, n, \tag{1}
\]

and clearly we have

\[
\theta_i^2 = 0 \quad \forall i = 1, 2, \ldots, n. \tag{2}
\]

Any linear combination of \( \theta_i \) with the complex number coefficients is called Grassmann number. In other words, we consider Taylor expansion of a Grassmann function as follows

\[
g(\theta_1, \theta_2, \ldots \theta_n) = c_0 + \sum_i c_i \theta_i + \sum_{i<j} c_{i,j} \theta_i \theta_j + \ldots,
\]

where \( c_0, c_i, c_{i,j}, \ldots \in \mathbb{C} \). For instance, \( \exp(\theta_1 \theta_2) = 1 + \theta_1 \theta_2 \). The complex conjugate of the Grassmann number \( \theta \) is also defined by \( (\theta)^* = \theta^* \), which is treated as another Grassmann number. A Grassmann function is called "of \( n \) degree" if it contains a term with \( n \) multiple of Grassmann numbers like \( \theta_1 \theta_2 \ldots \theta_n \). The Grassmann
integration and differentiation over the complex Grassmann variables are given by

\[ \int d\theta f(\theta) = \frac{\partial f(\theta)}{\partial \theta} \quad (3) \]

\[ \int d\theta = 0, \quad \int d\theta = 1, \quad \int d\theta^* = 0, \quad \int d\theta^* = \theta^* \quad (4) \]

\[ \frac{\partial}{\partial \theta} \theta = 1, \quad \frac{\partial}{\partial \theta^*} \theta = 0, \quad \frac{\partial}{\partial \theta^*} \theta = 1, \quad \frac{\partial^2}{\partial \theta^* \partial \theta^*} = \frac{\partial^2}{\partial \theta \partial \theta} = 0 \quad (5) \]

For the next uses, we require the following quantization relations between Fock states and Grassmann numbers

\[ \theta |0\rangle = |0\rangle \theta, \quad \theta |1\rangle = -|1\rangle \theta, \quad \theta |0\rangle = \langle 0|\theta, \quad \theta |1\rangle = -\langle 1|\theta \quad (7) \]

which implies that Grassmann numbers commute with \(|0\rangle\langle 0|\), and \(|1\rangle\langle 1|\), while anti-commute with \(|1\rangle\langle 0|\), and \(|0\rangle\langle 1|\). Now let \(a\) and \(a^\dagger\) be annihilation and creation operators for a fermionic system respectively. These operators satisfy the anti-commutation relations

\[ \{a, a^\dagger\} = 1, \quad \{a, a\} = \{a^\dagger, a^\dagger\} = 0 \quad (8) \]

Clearly, \(a\) and \(a^\dagger\) are nilpotent. We shall also assume that Grassmann variables anti-commute with fermionic operators

\[ \{a, \theta\} = \{a^\dagger, \theta\} = 0 \quad (9) \]

A fermionic coherent state, like the bosonic case, is defined as eigen-state of the annihilation operator

\[ a|\theta\rangle = \theta|\theta\rangle \quad (10) \]

which is satisfied by the following state

\[ |\theta\rangle = \exp \left( -\frac{\theta^* \theta}{2} \right) (|0\rangle - \theta |1\rangle) = D(\theta)|0\rangle \quad (11) \]

where

\[ D(\theta) := \exp (a^\dagger \theta - \theta^* a) \quad (12) \]

Note that the displacement operator \(D(\theta)\) is a unitary operator \(i.e. \; D(\theta)D(\theta)^\dagger = I\).

3. ENTANGLEMENT AND FCS

In this section, we show that one can get the well known maximally entangled pure states such as \(\text{GHZ, W, Bell and Bell-like states}\) [20], through integrating over tensor product of FCSs with suitable choice of weight function.
3.1. Bell and Bell-like States

Let us consider the simple cases that yield the following Bell states

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle),$$

(13)

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle).$$

(14)

Regarding the FCS of Eq. (11), we product two states $|\theta\rangle$ and $|\pm \theta\rangle$ as follows

$$|\theta\rangle|\pm \theta\rangle = \exp(-\theta^*\theta)|00\rangle \pm |11\rangle,$$

(15)

and by subtracting the above states we get

$$|\theta\rangle|\pm \theta\rangle - |\mp \theta\rangle|\mp \theta\rangle = \mp 2\theta(|01\rangle \pm |10\rangle).$$

(17)

Our task is to find the weight function such that when we integrate over Grassmann numbers, $\theta$ and $\theta^*$ yield the $|\Psi^{(\pm)}\rangle$. To this aim let

$$w(\theta, \theta^*) = c_0 + c_1 \theta + c_2 \theta^* + c_3 \theta^* \theta,$$

(18)

be such weight function. Then

$$\int \frac{d\theta^* d\theta}{(\theta, \theta^*)} w(\theta, \theta^*) ([|\theta\rangle |\pm \theta\rangle - |\mp \theta\rangle|\mp \theta\rangle]) = |\Psi^{\pm}\rangle,$$

where it is satisfied with $c_2 = \pm \frac{1}{2\sqrt{2}}$, and the other coefficients are arbitrary which may be taken as $c_0 = c_1 = c_3 = 0$ i.e., $w(\theta, \theta^*) = \pm \frac{1}{2\sqrt{2}} \theta^*$. Note that instead of subtracting the states $|\theta\rangle |\mp \theta\rangle$ and $|\mp \theta\rangle|\mp \theta\rangle$, we can add them and integrate on weight function $w(\theta, \theta^*) = \pm \frac{1}{2\sqrt{2}}$, which in turn yields the separable state $|00\rangle$. Another tensor product of FCS which yields the Bell state $|\Psi^-\rangle$ are

$$\int \frac{d\theta^* d\theta}{(\theta, \theta^*)} w(\theta, \theta^*) |\pm \theta\rangle |\pm \theta\rangle = |\Psi^-\rangle,$$

with $w(\theta, \theta^*) = \pm \frac{1}{\sqrt{2}} \theta^*.$

This is not the only way to construct $|\Psi^{(\pm)}\rangle$ states by FCS. We can get the same result if we define the states

$$|\theta\rangle_{\pm} = |\theta\rangle \pm |\mp \theta\rangle,$$

(19)

and by integration as follows we have

$$\int d\theta^* \frac{d\theta}{(\theta, \theta^*)} w(\theta, \theta^*) [|\theta\rangle_{+} |\theta\rangle_{-} \pm |\theta\rangle_{-} |\theta\rangle_{+}] = |\Psi^{\pm}\rangle,$$

(20)

where

$$w(\theta, \theta^*) = \frac{1}{4\sqrt{2}} \theta^*.$$
It is interesting that we can construct all Bell states if we assume the FCS to be constructed by two Grassmann numbers $\theta_1$ and $\theta_2$ and their complex conjugations. Such a possible state may be $|\theta\rangle|\theta^*\rangle$, whose integration with weight function $w(\theta, \theta^*) = \exp(\pm \theta \theta^*)$, gives

$$\int d\theta^* d\theta \left(\pm \frac{1}{\sqrt{2}} e^{\pm \theta \theta^*}\right) |\theta^*\rangle |\theta\rangle = |\Phi^\pm\rangle,$$

(21)

And also

$$\int d\theta^* d\theta \frac{1}{\sqrt{2}} (\theta^* \pm \theta) |\theta^*\rangle |\theta\rangle = |\Psi^\pm\rangle.$$  

(22)

To generalize more we take some other states, the first of which, goes as follows

$$|\theta_1\rangle|\theta_2\rangle = \exp \left[ -\frac{1}{2} (\theta_1^* \theta_1 + \theta_2^* \theta_2) \right] (|00\rangle - \theta_2|01\rangle - \theta_1|10\rangle - \theta_1 \theta_2|11\rangle)$$

(23)

The above state leads to Bell states via suitable weight functions. For example

$$\int d\theta^*_1 d\theta_1 d\theta^*_2 d\theta_2 \frac{1}{\sqrt{2}} (\theta_1^* \theta_1^* \theta_2^* \theta_2^* + \theta_1^* \theta_2^* \theta_1^* \theta_2^*) |\theta_1\rangle |\theta_2\rangle = |\Phi^\pm\rangle,$$

(24)

$$\int d\theta^*_1 d\theta_1 d\theta^*_2 d\theta_2 \frac{1}{\sqrt{2}} (\theta_1^* \theta_1^* \theta_2^* \theta_2^* + \theta_1^* \theta_2^* \theta_1^* \theta_2^*) |\theta_1\rangle |\theta_2\rangle = |\Psi^\pm\rangle.$$  

(25)

We note that in the case $|\theta\rangle|\theta^*\rangle$, it is impossible to choose a weight function of degree three or more, while in the case $|\theta_1\rangle|\theta_2\rangle$, it is possible. Now consider the symmetric and anti-symmetric FCSs $|\Lambda_\pm(\theta_1, \theta_2)\rangle = |\theta_1\rangle|\theta_2\rangle \pm |\theta_2\rangle|\theta_1\rangle = \pm |\Lambda_\pm(\theta_2, \theta_1)\rangle$, for which the following maximal entangled and separable states are deduced

$$\int d\theta^*_1 d\theta_1 d\theta^*_2 d\theta_2 \left(\frac{2}{\sqrt{2}} \theta_1^*\right) |\Lambda_\pm(\theta_1, \theta_2)\rangle = |\Psi^{(\pm)}\rangle,$$

(26)

$$\int d\theta^*_1 d\theta_1 d\theta^*_2 d\theta_2 w(\theta_1, \theta_1^*, \theta_2, \theta_2^*) |\Lambda_\pm(\theta_1, \theta_2)\rangle = \begin{cases} |00\rangle & \text{with } w = \frac{1}{2} \theta_1^* \theta_2^* \\ |11\rangle & \text{with } w = \frac{1}{2} \theta_1^* \theta_2^* \theta_2^* \theta_2. \end{cases}$$

(27)

The anti-symmetric state $|\Lambda_-(\theta_1, \theta_2)\rangle$ only gives Bell state $|\Psi^-\rangle$ which is anti-symmetric, and the symmetric state $|\Lambda_+(\theta_1, \theta_2)\rangle$ only gives Bell state $|\Psi^+\rangle$ which is symmetric.

Another MESs which can be manipulated by FCSs are Bell-like states

$$|\Psi^+\rangle_{BL} = \frac{1}{\sqrt{2}} \left(e^{i\frac{\pi}{2}} |01\rangle \pm e^{-i\frac{\pi}{2}} |10\rangle \right).$$

(28)

where

$$\int d\theta^* d\theta \frac{1}{\sqrt{2}} \left(e^{i\frac{\pi}{2}} \theta^* \pm e^{-i\frac{\pi}{2}} \theta \right) |\theta^*\rangle |\theta\rangle = |\Psi^+\rangle_{BL},$$

(29)

$$\int d\theta^*_1 d\theta_1 d\theta^*_2 d\theta_2 \frac{1}{\sqrt{2}} \left(e^{i\frac{\pi}{2}} \theta_1 \theta_1^* \theta_2^* \pm e^{-i\frac{\pi}{2}} \theta_1^* \theta_2^* \theta_2 \right) |\theta_1\rangle |\theta_2\rangle = |\Psi^\pm\rangle_{BL}.$$  

(30)
3.2. GHZ AND W STATES

Here, we proceed the same way as above to construct three qubit MESs, known as W and GHZ states which are used widely in quantum information theory. Then we generalize them for n-qubit cases. For W case, consider tensor product of three FCSs of the form

\[ |\theta\rangle |\theta\rangle |\theta\rangle = \exp \left( -\frac{3}{2} \theta^* \theta \right) \left[ |000\rangle - \theta (|001\rangle + |010\rangle + |100\rangle) \right], \]

thus with a convenient weight function we get

\[ \int d\theta^* d\theta \frac{\theta^*}{\sqrt{3}} \left| \theta\rangle |\theta\rangle |\theta\rangle = \frac{1}{\sqrt{3}} \left( |000\rangle + |001\rangle + |010\rangle + |100\rangle \right). \] (31)

One can easily generalize this to n-qubit W state as follows

\[ \int d\theta^* d\theta \frac{\theta^*}{\sqrt{n}} \left| \theta\rangle |\theta\rangle \ldots |\theta\rangle = |W^{(n)}\rangle, \] (32)

where

\[ |W^{(n)}\rangle = \frac{1}{\sqrt{n}} (|00\ldots0\rangle + |01\ldots0\rangle + \ldots + |0\ldots00\rangle). \] (33)

It is convenient to write the n-qubit W states with respect to FCSs of Eq.(19) as

\[ \int d\theta^* d\theta \frac{1}{2^n \sqrt{n}} \theta^* |\psi\rangle = |W^{(n)}\rangle, \] (34)

where

\[ |\psi\rangle = |\theta\rangle + |\theta\rangle + \ldots |\theta\rangle - |\theta\rangle + |\theta\rangle + \ldots |\theta\rangle + \ldots + |\theta\rangle + |\theta\rangle + \ldots |\theta\rangle + |\theta\rangle. \] (35)

To construct the three qubit GHZ state we have to use tensor product of three FCSs with different Grassmann numbers \(|\theta_1\rangle |\theta_2\rangle |\theta_3\rangle\). Then, the integration goes as

\[ \int d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 d\theta_3^* d\theta_3 \left[ \frac{1}{\sqrt{2}} (\theta_1^* \theta_1 \theta_2^* \theta_2 \theta_3^* \theta_3 + \theta_1^* \theta_2^* \theta_3^* \theta_3) \right] |\theta_1\rangle |\theta_2\rangle |\theta_3\rangle \]

\[ = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle). \] (36)

In a similar way as n-qubit W states, we can create the general n-qubit GHZ states by using FCSs. To this aim we take \(|\theta_1\rangle |\theta_2\rangle \ldots |\theta_n\rangle\) together with weight function as

\[ w = \frac{1}{\sqrt{2}} (\theta_1^* \theta_1 \theta_2^* \theta_2 \ldots \theta_n^* \theta_n + \theta_1^* \theta_2^* \ldots \theta_n^*), \] (37)
we get
\[
\int d\theta_1^* d\theta_2^* d\theta_3^* d\theta_3 \left[ \frac{1}{\sqrt{2}} (\theta_1^* \theta_2^* \theta_3^* \theta_3^* + \theta_1^* \theta_1^* \theta_2^* \theta_3^* \theta_3^* + \theta_1^* \theta_1^* \theta_2^* \theta_3^* \theta_3^* ) \right] |\theta_1 \rangle |\theta_2 \rangle |\theta_3 \rangle
= \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) = |\text{GHZ}^{(n)}\rangle.
\]

3.3. BISEPARABILITY

Here, we use FCSs to obtain biseparable states which, depending on how one considers partition for given state, there exists an entanglement in their subsystems partially. For example if a pure state $|\psi\rangle_{ABC}$ involves the three subsystems $A, B$ and $C$, the partition $\{A\}$ may be separable while $\{B, C\}$ are entangled. As an illustration, let us consider three and four partite cases as some examples. Hence, the entanglement of bipartite states can be made by FCS $|\theta_1 \rangle |\theta_2 \rangle |\theta_3 \rangle$ as follows
\[
\int d\theta_1^* d\theta_2^* d\theta_3^* d\theta_3 \left[ \frac{1}{\sqrt{2}} (\theta_1^* \theta_2^* \theta_3^* \theta_3^* + \theta_1^* \theta_1^* \theta_2^* \theta_3^* \theta_3^* + \theta_1^* \theta_1^* \theta_2^* \theta_3^* \theta_3^* ) \right] |\theta_1 \rangle |\theta_2 \rangle |\theta_3 \rangle
= |0\rangle_1 |\Psi^{\pm}\rangle_{2,3},
\]
where it implies that the entanglement is just between the second and third qubits. Other biseparable states are
\[
\int d\theta_1^* d\theta_2^* d\theta_2^* d\theta_3^* d\theta_3 \left[ \frac{1}{\sqrt{2}} (\theta_1^* \theta_2^* \theta_3^* \theta_3^* + \theta_1^* \theta_1^* \theta_2^* \theta_3^* \theta_3^* + \theta_1^* \theta_1^* \theta_2^* \theta_3^* \theta_3^* ) \right] |\theta_1 \rangle |\theta_2 \rangle |\theta_3 \rangle
= |\Psi^{\pm}\rangle_{1,2} |0\rangle_3,
\]
\[
\int d\theta_1^* d\theta_2^* d\theta_2^* d\theta_3^* d\theta_3 \left[ \frac{1}{\sqrt{2}} (\theta_1^* \theta_2^* \theta_3^* \theta_3^* + \theta_1^* \theta_1^* \theta_2^* \theta_3^* \theta_3^* + \theta_1^* \theta_1^* \theta_2^* \theta_3^* \theta_3^* ) \right] |\theta_1 \rangle |\theta_2 \rangle |\theta_3 \rangle
= |0\rangle_2 |\Psi^{\pm}\rangle_{1,3}
\]
where
\[
|0\rangle_2 |\Psi^{\pm}\rangle_{1,3} = \frac{1}{\sqrt{2}} (|001\rangle \pm |100\rangle)
\]
Furthermore one can easily see that
\[
\int d\theta_1^* d\theta_2^* d\theta_2^* d\theta_3^* d\theta_3 \left[ \frac{1}{\sqrt{2}} (\theta_1^* \theta_2^* \theta_2^* \theta_3^* \theta_3^* + \theta_1^* \theta_1^* \theta_2^* \theta_3^* \theta_3^* + \theta_1^* \theta_1^* \theta_2^* \theta_3^* \theta_3^* ) \right] |\theta_1 \rangle |\theta_2 \rangle |\theta_3 \rangle
= |0\rangle_1 |\Psi^{\pm}\rangle_{2,3}.
\]
The biseparable states $|0\rangle_2 |\Phi^{\pm}\rangle_{1,3}$ and $|0\rangle_3 |\Phi^{\pm}\rangle_{1,2}$ can be obtained in a same manner as above with different weight functions. We can also construct four qubit biseparable states like the three qubit case. To do this we take $|\theta_1 \rangle |\theta_2 \rangle |\theta_3 \rangle |\theta_4 \rangle$, and choose a
weight function as

\[ w = \frac{1}{\sqrt{3}} (\theta_1 \theta_1^* \theta_2 \theta_3 \theta_4^* \theta_4^* + \theta_1 \theta_1^* \theta_2 \theta_3 \theta_4^* \theta_4^* + \theta_1 \theta_1^* \theta_2 \theta_3 \theta_4^* \theta_4^*), \]  

(43)

then we have

\[ \int d\theta_4^* d\theta_1 d\theta_2^* d\theta_3 d\theta_4 w |\theta_1\rangle |\theta_2\rangle |\theta_3\rangle |\theta_4\rangle = |0\rangle_1 |W^{(3)}\rangle_{2,3,4}, \]  

(44)

where it means that the first qubit is not entangled with the other three qubit related to partition \{2, 3, 4\}. One can obtain biseparable states \(|s\rangle_{j,k,l}, (s = 0, 1)\) which may be any partition as Eq.(44). From both the partition and type of entanglement point of view, there are some other possibilities such as

\[ \int d\theta_1^* d\theta_2^* d\theta_3 d\theta_4^* d\theta_4 w |\theta_1\rangle |\theta_2\rangle |\theta_3\rangle |\theta_4\rangle = |0\rangle_1 |\text{GHZ}^{(3)}\rangle_{2,3,4}, \]  

(45)

where

\[ w = \frac{1}{\sqrt{2}} (\theta_1 \theta_1^* \theta_2 \theta_3 \theta_4^* \theta_4^* + \theta_1 \theta_1^* \theta_2 \theta_3 \theta_4^* \theta_4^*), \]  

and

\[ \int d\theta_1^* d\theta_2^* d\theta_3 d\theta_4^* d\theta_4 w |\theta_1\rangle |\theta_2\rangle |\theta_3\rangle |\theta_4\rangle = |00\rangle_{1,2} |\Phi^\pm_3\rangle_{3,4}, \]  

(46)

with

\[ w = \frac{1}{\sqrt{2}} (\theta_1 \theta_1^* \theta_2 \theta_3 \theta_4^* \theta_4^* \pm \theta_1 \theta_1^* \theta_2 \theta_3 \theta_4^* \theta_4^*), \]  

and also

\[ \int d\theta_1^* d\theta_2^* d\theta_3 d\theta_4^* d\theta_4 w |\theta_1\rangle |\theta_2\rangle |\theta_3\rangle |\theta_4\rangle = |\Psi^+\rangle_{1,3} |\Phi^+\rangle_{3,4}, \]  

(47)

with

\[ w = \frac{1}{2} (\theta_1 \theta_2 \theta_3 \theta_4^* \theta_4^* \pm \theta_1 \theta_1^* \theta_2 \theta_3 \theta_4^* \theta_4^* + \theta_1 \theta_1^* \theta_2 \theta_3 \theta_4^* \theta_4^* + \theta_1 \theta_1^* \theta_2 \theta_3 \theta_4^* \theta_4^*). \]  

It is easy to develop this discussion to more general forms.

### 4. COMPARISON WITH BOSONIC COHERENT STATES

It is tempting to compare the fermion and boson coherent states. A bosonic coherent state can be defined as eigen-state of the annihilation operator

\[ b(\alpha) = \alpha |\alpha\rangle, \]  

(48)

where \(\alpha\) is a complex number, and \(b\) is annihilation operator for the bosonic coherent state

\[ |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = D(\alpha)|0\rangle. \]  

(49)
where the displacement operator $D(\alpha)$ is

$$D(\alpha) := \exp(b^\dagger \alpha - \alpha^* b)$$

(50)

There are different measures to quantify the entanglement of a quantum system (for a good review see [22]). One of them is entanglement of formation which gives the exact formula based on the often used two-qubit concurrence defined as [21]

$$C = |\langle \zeta | \sigma_y \otimes \sigma_y | \zeta^* \rangle|,$$

(51)

where $\sigma_y$ is $y$ component of the usual Pauli spin matrices. The concurrence of the following state

$$|\zeta\rangle = \mu |\alpha\rangle |\beta\rangle + \nu |\gamma\rangle |\delta\rangle,$$

(52)

in the subspace spanned by $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$ and $|\delta\rangle$ is

$$C = \frac{\mu \nu \sqrt{(1 - |\langle \alpha | \gamma \rangle|^2)(1 - |\langle \beta | \delta \rangle|^2)}}{\mu^2 + |\nu|^2 + \mu \nu^* \langle \gamma | \delta \rangle + \mu^* \nu \langle \alpha | \gamma \rangle \langle \beta | \delta \rangle},$$

(53)

where $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$ and $|\delta\rangle$ are bosonic coherent states. The denominator of the above concurrence come from the normalization of $|\zeta\rangle$. When $C = 1$ then $|\zeta\rangle$ is MES that is the conditions for maximality of entanglement for nonorthogonal four bosonic coherent states in Eq. (52) are [11]

$$\mu = \nu e^{i\varphi} \quad \text{and} \quad \langle \alpha | \gamma \rangle = -\langle \delta | \beta \rangle e^{i\varphi}.$$

(54)

For the particular cases we will discuss two following examples

$$|k_1 \alpha\rangle |k_2 \alpha\rangle \pm |k_3 \alpha\rangle |k_4 \alpha\rangle, \quad k_i \in \mathbb{C},$$

(55)

where due to their different behaviours, under imposing the maximality conditions, we will treat them separately.

### 4.1. EXAMPLE 1

Consider the following bosonic coherent state

$$|k_1 \alpha\rangle |k_2 \alpha\rangle - |k_3 \alpha\rangle |k_4 \alpha\rangle, \quad k_i \in \mathbb{C},$$

(56)

where its concurrence goes as

$$C = \frac{2 \left[ \left(1 - e^{-\frac{1}{2} |\alpha|^2 (f_{13} + f_{24})} \right) \left(1 - e^{-\frac{1}{2} |\alpha|^2 (f_{24} + f_{24})} \right) \right]^{1/2}}{2 - e^{-\frac{1}{2} |\alpha|^2 (f_{13} + f_{24})} - e^{-\frac{1}{2} |\alpha|^2 (f_{13} + f_{24})}},$$

(57)

where

$$f_{ij} = |k_i|^2 + |k_j|^2 - 2 k_i^* k_j.$$ 

Regarding the conditions (54) we have $f_{13} = f_{24}$, which implies that they have the same real and imaginary parts i.e.,

$$|k_1 - k_3| = |k_2 - k_4|, \quad \text{or} \quad (k_1 - k_3) = (k_2 - k_4) e^{i\phi}.$$ 

(58)
\[ \text{Im}(k_1^*k_3) = \text{Im}(k_4^*k_2). \]  
(59)

Some special cases of MES for the state (56), up to a normalization factor, are deduced \[11\]
\[
|\psi\rangle_{\text{boson}} = \begin{cases} 
|\alpha\rangle - |\alpha\rangle - |\alpha\rangle - 3|\alpha\rangle, \\
|\alpha\rangle - |\alpha\rangle - |\alpha\rangle - |\alpha\rangle, \\
|\alpha\rangle - |\alpha\rangle - i|\alpha\rangle - i|\alpha\rangle, \\
|\alpha\rangle - |\alpha\rangle - i|\alpha\rangle - |\alpha\rangle.
\end{cases}
\]  
(60)

Now we return to tensor product of FCSs and consider the same form of equation (56), where the complex parameter \(\alpha\) is replaced by Grassmann number \(\theta\) as follows
\[
|k_1\theta\rangle|k_2\theta\rangle - |k_3\theta\rangle|k_4\theta\rangle.
\]  
(61)

We call a FCS maximal, if there is a Grassmann weight function whose integration over that FCS gives a MES. If we take the weight function as
\[
w(\theta, \theta^*) = \frac{1}{m\sqrt{2}} \theta^*,
\]  
(62)

then
\[
|\Psi\rangle_{\text{max}} = \int d\theta^* d\theta w(\theta, \theta^*)[|k_1\theta\rangle|k_2\theta\rangle - |k_3\theta\rangle|k_4\theta\rangle]
\]  
(63)

This state is MES if
\[
(k_1 - k_3) = e^{i\phi}(k_2 - k_4) = m.
\]  
(64)

which is equivalent to (58). Now we are interested in finding some especial cases of \(|\Psi\rangle_{\text{max}}\) which lead to MESs similar to the four bosonic coherent states of the form \(|\psi\rangle_{\text{boson}}\). We distinguish the following cases:

**Cases 1,2:** Let \(|\Psi\rangle_{\text{max}} = |\Psi^\pm\rangle\), \(\phi = 0\) and \(m = \pm 2\). Imposing \(k_1 = -k_2 = -k_3\), the Eq. (61) reduces to
\[
|\theta\rangle - |\theta\rangle - |\theta\rangle - 3|\theta\rangle, \\
|\theta\rangle - |\theta\rangle - |\theta\rangle - |\theta\rangle,
\]  
(65)

where first FCS refers to plus sign and the second one refers to minus sign.

**Cases 3,4:** Let \(|\Psi\rangle_{\text{max}} = |\Psi^{\pm}_{\text{BL}}\rangle\), \(\phi = 0\) and \(m = \pm \sqrt{2}\). If we take \(k_1 = 1, k_2 = \pm 1, k_3 = k_4 = \mp i\), then FCS (61) is reduced to the following states
\[
|\theta\rangle - |\pm \theta\rangle - |\theta\rangle - i|\theta\rangle,
\]  
(66)

The above FCSs obtained in cases 1-4 could be compared with the maximal bosonic coherent states \(|\psi\rangle_{\text{boson}}\). Of course, we deliberately call these FCSs maximally entangled as done for bosonic coherent states mentioned in reference \[11\]. Furthermore,
the following bosonic and fermionic coherent states

\[ |\psi'\rangle_{\text{boson}} = \frac{1}{\sqrt{2}}|\alpha\rangle_+ + |\alpha\rangle_- + |\alpha\rangle_- + |\alpha\rangle_+ , \quad (68) \]

\[ |\psi'\rangle_{\text{fermion}} = \frac{1}{\sqrt{2}}|\theta\rangle_+ + |\theta\rangle_- + |\theta\rangle_- + |\theta\rangle_+ , \quad (69) \]

have the same form and both are MES, in the sense that the \( |\psi'\rangle_{\text{boson}} \) is MES by itself [13] and

\[ \int d\theta^* d\theta \frac{\theta^*}{4} |\psi'\rangle_{\text{fermion}} = |\Psi^+\rangle , \quad (70) \]

which is clearly maximal entangled state. There are some other FCSs that lead to MESs for fermionic coherent systems in integration method which also have the maximally entangled bosonic counterpart obtained by concurrence. For example, in the case 1, 2, we take the plus case and \( k_1 = 3, k_2 = -1, k_3 = 1, k_4 = -3 \), then the state (61) reduces to

\[ |3\theta\rangle|\theta\rangle - |\theta\rangle|3\theta\rangle , \]

while its bosonic counterpart

\[ |3\alpha\rangle|\alpha\rangle - |\alpha\rangle|3\alpha\rangle \]

is also MES [11, 13]. Thus, according to (64), allocating an arbitrary value to \( m \) and accounting for proper conditions among \( k_1, k_2, k_3, k_4 \), one can obtain other MESs for FCSs. As another example, let \( m = 3, k_1 = k_2 = 1, k_3 = k_4 = -2 \), then the state (61) reduces to

\[ |\theta\rangle|\theta\rangle - |2\theta\rangle|2\theta\rangle , \]

which just like the above cases has the maximally entangled bosonic counterpart. It is perhaps worth pointing out that, although it is possible to find maximal FCSs (which have the same form in the bosonic maximal coherent state of the form (56)), it can be shown that the inverse does not hold. For instance, we have

\[ |\Psi\rangle_{\text{max}} = \int d\theta^* d\theta \left( \frac{\theta^*}{\sqrt{2(i-1)}} \right) ([i\theta^*]|i\theta\rangle - |\theta\rangle|\theta\rangle] \]

\[ = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) = |\Psi^+\rangle , \quad (71) \]

which is clearly maximal while its bosonic counterpart

\[ |i\alpha\rangle|\alpha\rangle - |\alpha\rangle|i\alpha\rangle \]

is not. This is due to the fact that although \( \text{Re}(f_{13}) = \text{Re}(f_{24}^*) \), we have \( \text{Im}(f_{13}) \neq \text{Im}(f_{24}^*) \), which implies that \( f_{13} \neq f_{24}^* \), so the condition (59) is not satisfied. In fact, the \( k_i \)s that satisfy (58) and (59) for bosonic coherent states (56), must also satisfy the relaxed conditions (64) for FCSs (61).
4.2. EXAMPLE 2

Now consider the following state

\[ |k_1 \alpha \rangle |k_2 \alpha \rangle + |k_3 \alpha \rangle |k_4 \alpha \rangle, \quad k_i \in \mathbb{C}, \]

which has concurrence

\[ C = \frac{2 \left[ \left( 1 - e^{-\frac{1}{2} |\alpha|^2 (f_{13} + f_{14}^*)} \right) \left( 1 - e^{-\frac{1}{2} |\alpha|^2 (f_{24} + f_{24}^*)} \right) \right]^{1/2}}{2 + e^{-\frac{1}{2} |\alpha|^2 (f_{13} + f_{24}^*)} + e^{-\frac{1}{2} |\alpha|^2 (f_{13} + f_{24})}}, \]

thus, the state (72) is maximally entangled when

\[ f_{13} = f_{24}^* + \frac{2\pi i}{|\alpha|^2}, \]

in other word

\[ |k_1 - k_3| = |k_2 - k_4|, \quad \text{or} \quad (k_1 - k_3) = (k_2 - k_4)e^{i\phi} \]

and

\[ \text{Im}(k_1^* k_2) - \text{Im}(k_3^* k_4) = \frac{\pi}{|\alpha|^2}. \]

Now let us consider the same state as (72) but complex \( \alpha \) is replaced by Grassmann number \( \theta \), i.e.,

\[ |k_1 \theta \rangle |k_2 \theta \rangle + |k_3 \theta \rangle |k_4 \theta \rangle. \]

Again we take \( w(\theta, \theta^*) = \frac{1}{m \sqrt{2}} \theta^* \), then

\[ |\Psi\rangle_{max} = \int d\theta^* d\theta \left( \frac{\theta^*}{m \sqrt{2}} \right) \left[ |k_1 \theta \rangle |k_2 \theta \rangle + |k_3 \theta \rangle |k_4 \theta \rangle \right] \]

\[ = \frac{1}{m \sqrt{2}} [k_2 + k_4] |01\rangle + \frac{1}{m \sqrt{2}} [k_1 + k_3] |10\rangle. \]

This state is MES if

\[ (k_1 + k_3) = e^{i\phi} (k_2 + k_4) = m. \]

Here we can treat three cases separately.

**Case 1:** In the first case, let \( k_i \)s satisfy all the conditions (74), (75) and (78). Hence there is a fermionic counterpart for any bosonic MES and vice versa. For example if \( k_1 = k_2 = \frac{m}{2 |\alpha|^2}, \quad k_3 = k_4 = 1 \) and \( \phi = 0 \), then the Eq.(77) gives MES for following FCS

\[ |\frac{i\pi}{2 |\alpha|^2} \rangle |\frac{i\pi}{2 |\alpha|^2} \rangle + |\theta \rangle |\theta \rangle, \]

and the state (72) reduces to the following MES for bosonic coherent state

\[ |\frac{i\pi}{2 |\alpha|^2} \rangle |\frac{i\pi}{2 |\alpha|^2} \rangle + |\alpha \rangle |\alpha \rangle, \]

These states are counterparts of each other.

**Case 2:** In the second case, let \( k_i \)s satisfy the conditions (74), (75) but the condition (78) does no hold. Therefore we have a set of bosonic MESs which have no similar
fermionic maximally entangled counterparts. For example

\[ |(\frac{\pi}{2|\alpha|^2} + i\alpha)\rangle |(\frac{\pi}{|\alpha|^2} + i\alpha)\rangle + |(\frac{\pi}{2|\alpha|^2} - i\alpha)\rangle |(\frac{\pi}{|\alpha|^2} - i\alpha)\rangle. \tag{81} \]

Clearly the fermionic counterpart of this state does not lead to a MES with any choice of weight function.

**Case 3:** In the third case, let \( k_i \)s satisfy the conditions (78), but the condition (74) or (75) does no hold. Hence we have a set of fermionic coherent states that, according to Eq.(77), give MESs while the bosonic counterpart of them are not MESs. To give an example we can take FCS

\[ |k\theta\rangle |l\theta\rangle + |l\theta\rangle |k\theta\rangle, \quad k, l \in \mathbb{C}, \tag{82} \]

which have no maximally entangled bosonic counterpart.

**5. CONCLUSION**

In summary, we have shown the some well-known entangled pure states like GHZ, W, Bell, Bell-like and biseparable states can be constructed by tensor product of fermion coherent states with integration over proper Grassmann weight functions. For three qubit GHZ and W states, the construction can be easily generalized to multi-qubit cases, however there is an important difference between GHZ and W constructions: in the former case, we must use tensor product of FCSs with different Grassmann numbers, while in the latter case the tensor product of \( n \) FCSs \(|\theta\rangle\) is sufficient to this aim. We called a FCS *maximal*, if there is a Grassmann weight function whose integration over that FCS gives a MES. As we saw in the last section, some maximally entangled BCSs have FCSs counterparts, but it is of course perfectly possible to find simple examples of maximal FCSs, using the integration method, which have no maximal BCSs counterparts and vice versa.

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