Product of Polycyclic-by-Finite Groups (PPFG)

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Abstract: In this paper we show that if the soluble-by-finite group \( G=AB \) is the product of two polycyclic-by-finite subgroups \( A \) and \( B \), then \( G \) is polycyclic-by-finite.

Keywords: polycyclic-by-finite, soluble group, maximal condition, finite group AMS classification: 20F32.

INTRODUCTION

In 1955 N. Itô (see [7]) found an impressive and very satisfying theorem for arbitrary factorized groups. He proved that every product of two abelian groups is metabelian. Besides that, there were only a few isolated papers dealing with infinite factorized groups. P.M. Cohn (1956) (see[21]) and L.Reedi (1950)(see [22]) considered products of cyclic groups, and around 1965 O.H.Kegel (See [30, 31]) looked at linear and locally finite factorized groups.

In 1968 N.F. Sesekin (see [19]) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition. He and Amberg independently obtained a similar result for the maximal condition around 1972 (See [20, 1]). Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his Habilitationsschrift (1973) he started a more systematic investigation of the following general question. Given a (soluble) product \( G \) of two subgroups \( A \) and \( B \) satisfying a certain finiteness condition \( \mathcal{F} \), when does \( G \) have the same finiteness condition \( \mathcal{F} \)? (See [20])

For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is ‘yes’ for soluble (and even for soluble-by-finite) groups. This combines theorems of B. Amberg (see [1-4] and [6]), N.S. Chernikov (see [5]), S. Franciosi, F. de Giovani (see [3, 6, 32-36]), O.H.Kegel (see [8]), J.C.Lennox (see [12]), D.J.S. Robinson(see [9] and [15]), J.E. Roseblade (see [13]), Y.P.Sysak(see [37-40]), J.S.Wilson (see [41]), and D.I.Zaitsev(see [11] and [18]).

Now, In this paper we show that if the soluble-by-finite group \( G=AB \) is the product of two polycyclic-by-finite subgroups \( A \) and \( B \), then \( G \) is polycyclic-by-finite.

Preliminaries: (elementary properties and theorems.)

Definition: Recall that the FC-centre of a group \( G \) is the subgroup of all elements of \( G \) with a finite number of conjugates. A group is an FC-group if it coincides with its FC-centre.

Lemma: Let the group \( G=AB \) be the product of two abelian subgroups \( A \) and \( B \), and let \( S \) be a factorized subgroup of \( G \). Then the centralizer \( C_G(S) \) is factorized. Moreover, every term of the upper central series of \( G \) is factorized.

Proof: Since \( S \) is factorized, we have that \( S=(A \cap S)(B \cap S) \). Let \( x=ab \) be an element of \( S \), where \( a \) is in \( A \cap S \) and \( b \) is in \( B \cap S \). If \( c=ab-1b-1 \) is an element of \( C_G(S) \), with \( a-1 \) in \( A \) and \( b-1 \) in \( B \), it follows that.

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\[ [a_1, x] = [a_1, ab] = [a_1, b] = [cb_1, b] = [c, b]^{b_1} = 1. \]

Therefore \( a_1 \) belongs to \( C_0(S) \), and \( C_0(S) \) is factorized by Lemma 1.1.1 of [4]. In particular, the center of \( G \) is factorized. It follows from Lemma 1.1.2 of [4] that also every term of the upper central series of \( G \) is factorized.

**Lemma:** Let the group \( G=AB \) be the product of two subgroups \( A \) and \( B \). If \( A_1, B_1, \) and \( F \) are the FC-centers of \( A, B, \) and \( C \), respectively, then \( F=A\cap B \cap B:F \). In particular, if \( A \) and \( B \) are FC-groups, the FC-centre of \( G \) is factorized subgroup.

**Proof:** Let \( x \) be an element of \( A \cap B \).\( F \), and write \( x=au \) where \( a \) is in \( A_1 \) and \( u \) is in \( F \). Since the centralizers \( C_0(a) \) and \( C_0(u) \) have finite index in \( A_1 \), the index \( [A_1, C_0(x)] \) is also finite. Similarly, \( C_0(x) \) has finite index in \( B \). Therefore \( |G:C_0(x)\times C_0(x)| \) is finite by Lemma 1.2.5 of [4]. It follows that \( C_0(x) \) has finite index in \( G \) and hence \( x \) belongs to \( F \). Thus \( F=A\cap B \cap B:F \).

**Lemma:** (See [7]) Let the finite non-trivial group \( G=AB \) be the product of two abelian subgroups \( A \) and \( B \). Then there exists a non-trivial normal subgroup of \( G \) contained in \( A \) or \( B \).

**Proof:** Assume that \( \{1\} \) is the only normal subgroup of \( G \) contained in \( A \) or \( B \). By Lemma 2.11 have \( Z(G)=A \cap Z(G) \cap B \cap Z(G) = 1 \). The centralizer \( C = C_G(A \cap C_G(G')) \) contains \( AG' \), and so is normal in \( G \). Since \( B \cap (AZ(C)) = Z(G) = 1 \), it follows that \( AZ(C) = A(B \cap AZ(C)) = A \). This \( Z(G) \) is a normal subgroup of \( G \) contained in \( A \), and so \( Z(G)=1 \). Since \( G' \) is abelian by Theorem 2.9, we have \( A \cap G' \leq A \cap C_G(G') \leq Z(C) = 1 \).

**Theorem:** (See [8, 10]): If the finite group \( G=AB \) is the product of two nilpotent subgroups \( A \) and \( B \), then \( G \) is soluble.

**Proof:** See [4], (Theorem 2.4.3).

**Lemma:** Let \( A \) and \( B \) be subgroups of a group \( G \), and let \( A_1 \) and \( B_1 \) be subgroups of \( A \) and \( B \), respectively, such that \( |A : A_1| \leq m \) and \( |B : B_1| \leq n \). Then \( |A \cap B : A_1 \cap B_1| \leq mn \).

**Proof:** To each left coset \( x(A_1 \cap B_1) \) of \( A_1 \cap B_1 \) in \( A \cap B \) assign the pair of left cosets \( (xA_1, xB_1) \). Clearly this defines an injective map from the set of left cosets of \( A_1 \cap B_1 \) in \( A \cap B \) into the cartesian product of the set of left cosets of \( A_1 \) in \( A \) and the set of left cosets of \( B_1 \) in \( B \). The lemma is proved.

**Lemma:** (See [11]): Let the finitely generated group \( G=AB=AKBK \) be the product of two abelian-by-finite subgroups \( A \) and \( B \) and an abelian normal subgroup \( K \) of \( G \). Then \( G \) is nilpotent-by-finite.
Proof: Let $A$ and $B$ be abelian subgroups of finite index of $A$ and $B$, respectively, and let $n$ be a positive integer such that $|A:A| \leq n$ and $|B:B| \leq n$. Since $G$ is finitely generated, it has only finitely many subgroups of each finite index, and hence the intersection $H$ of all subgroups of $G$ with index at most $n^2$ also has finite index in $G$. In particular $H$ is finitely generated.

Consider a finite homomorphic image $H/N$ of $H$. Then $N$ has finite index in $G$, and hence also its core $N_0$ has finite index in $G$. Let $p_1, \ldots, p_t$ be the prime divisors of the order of the finite abelian group $K/(K \cap N_G)$. For each $j \leq t$, let $K_j/(K \cap N_G)$ be the $p_j$-component of $K/(K \cap N_G)$. Clearly each $K_j$ is normal in $G$ and

$$\bigcap_{j=1}^t K_j = K \cap N_G.$$  

The factor group $\overline{G} = G/K$, has the triple factorization $\overline{G} = \overline{A}\overline{B} = \overline{A}\overline{K} = \overline{B}\overline{K}$, where $\overline{K}$ is a finite normal $p_j$-subgroup of $\overline{G}$. Clearly

$$\left| \overline{G} : \overline{A} \cap \overline{B} \right| = \left| \overline{A} : \overline{A} \cap \overline{B} \right| \cdot \left| \overline{B} : \overline{A} \cap \overline{B} \right| = p_j^k$$

for some non-negative integer $k$. On the other hand, $\left| \overline{A} \cap \overline{B} : \overline{A}_j \cap \overline{B}_j \right| \leq n^2$ by Lemma 2.16, so that $\left| \overline{A} : \overline{A}_j \cap \overline{B}_j \right| \leq p_j^k n^2$. As $\overline{A}_j$ and $\overline{B}_j$ are abelian, the intersection $\overline{A}_j \cap \overline{B}_j$ is contained in the centre of $\overline{A}_j \cap \overline{B}_j$, and the factor group $\overline{A}_j \cap \overline{B}_j$ has order at most $p_j^k n^2$. Let $\overline{P} = (\overline{A}_j \cap \overline{B}_j)$ be a Sylow $p_j$-subgroup of $\overline{A}_j \cap \overline{B}_j$. Then $\left| \overline{A}_j \cap \overline{B}_j : \overline{P} \right| \leq n^2$, and since $\left| \overline{G} : \overline{A}_j \cap \overline{B}_j \right| \leq n^2$ by Lemma 2.2, we obtain $\left| \overline{G} : \overline{P} \right| \leq n^d$. Therefore $HK/K_j$ is contained in $\overline{P}$. As an extension of the central subgroup $\overline{A}_j \cap \overline{B}_j$ by a finite $p_j$-group, $\overline{P}$ is nilpotent, so that $H/(H \cap K_j) \simeq HK_j/K_j$ is also nilpotent for each $j$. Hence,

$$\frac{H}{\bigcap_{j=1}^t (H \cap K_j)} = H/(K \cap N_G)$$

is nilpotent. We have shown that each finite homomorphic image of $H$ is nilpotent. As $K$ is abelian, $H$ is soluble, and hence even nilpotent (Robinson 1972, Part 2, Theorem 10.51). Therefore $G$ is nilpotent-by-finite.

Definition: A group $G$ has finite Prüfer rank $r = r(G)$ if every finitely generated subgroup of $G$ can be generated by at most $r$ elements, and $r$ is the least positive integer with this property. Clearly subgroups and homomorphic images of groups with finite Prüfer rank also have finite Prüfer rank.

Lemma: (See [13]) If $N$ is a maximal abelian normal subgroup of a finite $p$-group $G$, then $r(G) \leq \frac{1}{2} r(N)(5 r(N) - 1)$.

Proof: Since $C_G(N) = N$, the factor group $G/N$ is isomorphic with a $p$-group of automorphism of $N$. Thus $G/N$ has perüfer rank at most $\frac{1}{2} r(N)(5 r(N) - 1)$ (See [15], part 2, lemma 7.44), and hence $r(G) \leq \frac{1}{2} r(N)(5 r(N) + 1)$.

Theorem: (See [9] and [11]) If the locally soluble group $G = AB$ with finite Prüfer rank is the product of two subgroups $A$ and $B$, then the Prüfer rank of $G$ is bounded by the function of the Prüfer ranks of $A$ and $B$.

Proof: First, let $G$ be a finite $p$-group for some prime $p$. If $N$ is a maximal abelian normal subgroup of $G$, by Lemma 2.18 we have $r(G) \leq \frac{1}{2} r(N)(5 r(N) + 1)$. Hence it is enough to prove that $r(N)$ is bounded by a function of the maximum $s$ of $r(A)$ and $r(B)$. The socle $S$ of $N$ is an elementary abelian group of order $p^s$. Clearly it is sufficient to prove the theorem for the factorizer $X(S)$ of $S$. Therefore we may suppose that the group $G$ has a triple factorization $G = AB = AK = BK$, where $K$ is an elementary abelian normal subgroup of $G$ of order $p^s$. 
Let $e$ be the least positive integer such that $A^{p^e}$ is contained in $B$. By Lemma 4.3.3 of [4], we have

$$|A : A \cap B| \leq |A : A^{p^e}| \leq p^{eg(s)-s^2}$$

where $g(s) = \frac{1}{2} s(3s + 1)$. Since $G$ is a finite p-group, we have

$$|G| = \frac{|A|}{|A \cap B|} = \frac{|B|}{|B \cap K|}.$$

It follows that

$$|K| = |A : A \cap B| \cdot |B \cap K| \leq p^{eg(s)-s^2} p^s = p^{eg(s)-s^2 + s}.$$

Hence $r \leq eg(s) - s^2 + s \leq eg(s)$. Therefore it is enough to show that $e \leq g(s) + 3$. Therefore it is enough to show that $e \leq g(s) + 3$.

Clearly we may suppose that $e > 1$. Let $a$ be an element of $A$ such that $a^{p^{e-1}}$ is not in $B$, and write $a^{p^{e-1}} = xb$, with $x$ in $K$ and $b$ in $B$. Then $[x, a^{p^{e-2}}] \neq 1$, because otherwise

$$b^p = (x^{-1} a^{p^{e-2}})^p = x^{-p} a^{p^{e-1}} = a^{p^{e-1}},$$

contrary to the choice of $a$. As $K$ has exponent $p$, it follows from the usual commutator laws that

$$[x, a^{p^{e-2}}] = \prod_{i=1}^{p^{e-2}} [x, a^{(p^i e - e)}] = [x, p^{e}.2a].$$

Thus $[K, G, ..., G] \neq 1$, and so $|K| > p^{p^{e-2}}$ since $G$ is a finite p-group. Therefore $p^{p^{e-2}} < r \leq eg(s)$. If $e \geq g(s) + 4$, then $p^{p^{e-2}} \geq 2^{p^{e-2}} > (e + 1)(e - 4) \geq (e + 1)g(s) > eg(s)$.

This contradiction shows that $e \leq g(s) + 3$.

Suppose now that $G = AB$ is an arbitrary finite soluble group. For each prime $p$, by Corollary 2.7 there exist Sylow $p$-subgroups $A_p$ of $A$ and $B_p$ of $B$ such that $G_p = A_p B_p$ is a Sylow $p$-subgroup of $G$. As was shown above, $r(G_p)$ is bounded by a function $f(s)$ of the maximum $s$ of $r(A)$ and $r(B)$, and this does not depend on $p$. Thus every subgroup of prime-power order of $G$ can be generated by a function $f(s)$ of the maximum $s$ of $r(A)$ and $r(B)$, and this does not depend on $p$. Thus every subgroup of prime-power order of $G$ can be generated by at most $f(s)$ elements. Application of Theorem 4.2.1 of [4] yields that every subgroup of $G$ can be generated by at most $f(s)+1$ elements, and hence the Prüfer rank of $G$ is bounded by $f(s)+1$. This proves the theorem is the finite case.

Let $G = AB$ be an arbitrary locally soluble group with finite Prüfer rank. If $N$ is a finite normal subgroup of $G$, and $X = X(N)$ is its factorizer, then the index $[X : A \cap B]$ is finite by Lemma 1.1.5. Let $Y$ be the core of $A \cap B$ in $X$.

Since the factorized group $X/Y$ is finite, it follows from the first part of the proof that the Prüfer rank of $X/Y$ is bounded by a function of the Prüfer ranks of $A$ and $B$. As $r(N) \leq r(X) \leq r(Y) + r(X/Y) \leq r(A) + r(X/Y)$ (e.g., see Robinson 1972, Part 1, Lemma 1.44) we obtain that there exists a function $h$ such that $r(N) \leq h(r(A), r(B)) = k$, for every finite normal subgroup $N$ of $G$. Clearly the same holds for every finite normal section of $G$.

Let $T$ be the maximum periodic normal subgroup of $G$. If $p$ is a prime, the group $T = T/O_p(T)$ is Chernikov by Lemma 3.2.5 of [4] (See also [16]). Let $J$ be the finite residual of $T$, and $S$ the socle of $J$. Since $S$ and $T/J$ are finite, it follows that $r(T) \leq r(J) + r(T/J) = r(S) + r(T/J) \leq 2k$. 

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As the Sylow \( p \)-subgroups of \( T \) can be embedded in \( \overline{T} \), they have Prüfer rank at most \( 2k \). Application of Theorem 4.2.1 of [4] (See also [14]). yields that every finite subgroup of \( T \) can be generated by at most \( 2k+1 \) elements. Hence \( r(T) \leq 2k + 1 \).

The group \( G/T \) is soluble (See[15]), Part 2, Lemma 10.39), and so the setoff primes \( \pi(G/T) \) is finite by Lemma 4.1.5 of [5] (See also [15]). It follows from Lemma 4.1.4 of [4] (See also [15]) that there exists in \( G \) a normal series of finite length \( T \leq G_1 \leq G_2 \leq G \), where \( G/T \) is torsion-free nilpotent, \( G_2/G_1 \) is torsion-free abelian, and \( G/G_2 \) is finite. Therefore

\[
\begin{align*}
  r(G) &\leq r(T) + r(G_1/T) + r(G_2/G_1) + r(G/G_2) \\
  &\leq r(T) + r_0(G) + r(G/G_2) \\
  &\leq r_0(G) + 3k + 1.
\end{align*}
\]

By theorem 4.1.8 of [4] (See also [3]) we have that \( r_0(G) \leq r_0(A) + r_0(B) \).

Moreover, \( r_0(A) \leq r(A) \) and \( r_0(B) \leq r(B) \) by Lemma 4.3.4 of [4] (See also [9]). Therefore \( r(G) \leq r(A) + r(B) + 3k + 1 \). The theorem is proved.

**Lemma**: (See [17]): Every finitely generated abelian-by-polycyclic Group is residually finite.

**Proof**: See ([4], Lemma 4.4.1).

**MAIN Theorem**: If the soluble-by-finite group \( G=AB \) is the product of two polycyclic-by-finite subgroups \( A \) and \( B \), then \( G \) is polycyclic-by-finite.

**Proof**: Assume that \( G \) it not polycyclic-by-finite. Then \( G \) contains an abelian normal section \( U/V \) which is either torsion-free or periodic and is not finitely generated. Clearly the factorizer of \( U/V \) in \( G/V \) is also a counterexample. Hence we may suppose that \( G \) has a triple factorization \( G=AB=AK=BK \), Where \( K \) is an abelian normal subgroup of \( G \) which is either torsion-free or periodic. By Lemma 1.2.6(i) of [4] (See also [17]) the group \( G \) satisfies the maximal condition on normal subgroups, so that it contains a normal subgroup \( M \) which is maximal with respect to the condition that \( G/M \) is not polycyclic-by-finite. Thus it can be assumed that every proper factor group of \( G \) is polycyclic-by-finite.

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