PREVALENCE OF STABLE PERIODIC SOLUTIONS IN THE FORCED RELATIVISTIC PENDULUM EQUATION

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Abstract. We study the prevalence of stable periodic solutions of the forced relativistic pendulum equation for external force which guarantees the existence of periodic solutions. We prove the results for a general planar system.

1. Introduction. During the last decade, there are some important progress on the study of the forced relativistic pendulum equation

\[
\left( \frac{x'}{\sqrt{1-x'^2}} \right)' + \mu \sin x = f(t),
\]

in the literature. Most results are related to the existence of its periodic solutions. The common tools include Leray-Schauder degree, the method of upper and lower solutions and variational methods. For example, Brezis and Mawhin proved in [4] that (1) has at least one periodic solution for any \( \mu \neq 0 \) and any forcing term \( f \in \mathcal{F} \) by finding a minimum for the corresponding functional. Throughout this paper, we
shall use $X = C(\mathbb{R}/T\mathbb{Z})$ to denote the set of continuous $T$-periodic functions and use
\[
\mathcal{F} = \left\{ f \in X : \bar{f} = \frac{1}{T} \int_0^T f(t) dt = 0 \right\}.
\]
Obviously, $\mathcal{F}$ becomes a Banach space with the norm $\|f\|_\infty = \max_{t \in \mathbb{R}} |f(t)|$. Later, the second periodic solution of (1) was found by Bereanu and Torres in [3] by using Szulkin’s critical point theory [21]. Equation (1) can be seen as a special case of the more general periodic $\phi$-problem
\[
(\phi(x'))' + g(t, x) = f(t),
\]
where $\phi$ is a suitable increasing homeomorphism with $\phi(0) = 0$. It was proved by Cid and Torres in [8] that (2) has at least two periodic solutions not differing by a multiple of $\omega$ for any forcing $f \in \mathcal{F}$ and the continuous function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ with its primitive $G(t, x)$ satisfying $G(t, x) = G(t, x + \omega)$ for all $(t, x) \in [0, T] \times \mathbb{R}$. We refer the reader to [1, 2, 4, 8, 10, 14, 22] for more results on the existence of periodic solutions for equation (1) and the general problem (2). We also refer to [14] for a relatively complete introduction to the global results for the non-relativistic forced pendulum equations.

Compared to the existence of periodic solutions, the study of the dynamical behavior of periodic solutions is more difficult but there are also some interesting results. The Lyapunov stability of the equilibrium of (1) without the external force was proved in [5]. The existence and stability of periodic solutions of relativistic singular differential equations were established in [6] based on a known connection between the index of a periodic solution and its stability. In [13], by applying a version of the Poincaré-Birkhoff theorem due to Franks, Marò proved the existence of at least two geometrically different periodic solution with winding number $N$ for (1). The instability of a solution and the existence of twist dynamics are also given in [13]. The stability of periodic solutions for the more general problem (2) was studied in [8].

In this paper, we continue to study the dynamical behavior of periodic solutions for equation (1). To be precise, we shall focus on the prevalence of its stable periodic solutions. A prevalent set can be seen as the analogue of a set of full measure in infinite dimension. For more information on prevalence, we refer to the paper by Ott and York [20]. We are motivated by several recent works [7, 16, 17, 18]. In [16, 17], Ortega proved that the forced non-relativistic pendulum equations
\[
x'' + \beta \sin x = f(t)
\]
has at least one stable $T$-periodic solution for almost every forcing $f \in \mathcal{F}$ when $0 < \beta \leq (\pi/T)^2$, which is equivalent to say that the set
\[
\{ f \in \mathcal{F} : (3) \text{ has a stable } T\text{-periodic solution} \}
\]
is prevalent in $\mathcal{F}$. Two examples were constructed in [18] to explain that the upper bound $(\pi/T)^2$ for $\beta$ is optimal. The prevalence of stable periodic solutions of the dissipative case and the conservative Duffing equations has been studied by the first two authors in [7].

Note that equation (1) is equivalent to the planar system
\[
\begin{align*}
x' &= \phi^{-1}(y), \\
y' + \mu \sin x &= f(t),
\end{align*}
\]
for all $(t, f(t)) \in [0, T] \times \mathbb{R}$.
where 

\[ \phi^{-1}(y) = \frac{y}{\sqrt{1 + y^2}}. \]

In order to make our results more applicable, we prove the prevalence of stable periodic solutions for the following general planar system

\[
\begin{align*}
    x' &= g_1(y), \\
    y' + g_2(x) &= f(t),
\end{align*}
\]

where \( f \in \mathcal{F} \) and \( g_1, g_2 \) satisfy some additional conditions (see Section 3). Under reasonable conditions, we shall prove that \( S \) is prevalent in \( E \), where

\[ E = \{ f \in \mathcal{F} : (5) \text{ has at least one } T\text{-periodic solution} \}, \]

and

\[ S = \{ f \in \mathcal{F} : (5) \text{ has at least one stable } T\text{-periodic solution} \}. \]

As an application, we are able to show that if \( \mu \) is less than some constant, (1) has at least one stable \( T \)-periodic solution for almost every forcing \( f \in \mathcal{F} \).

The paper is organized as follows. In Section 2, we present some preliminary results and some basic facts on planar linear Hamiltonian system. In Sections 3, we state and prove the main results of this paper.

2. Preliminaries.

2.1. Two prevalent results. Let \( E \) be a separable Banach space of infinite dimension. A subset \( N \) of \( E \) is Haar-null if there exists a Borel set \( B \) and a Borel measure \( \mu \) on \( E \) such that

\- \( N \subset B \),
\- \( 0 < \mu(C) < \infty \) for some compact subset \( C \) of \( E \),
\- \( \mu(e + B) = 0 \) for each \( e \in E \).

The above definition of Haar-null set is the same as in [16]. We refer to [20] for more information on the notion of Haar-null sets. A subset of \( E \) is prevalent if its complement is Haar-null. We remark that Lemma 2.1 and Lemma 2.2 below are proved in [16, 17] under a very slightly different definition of Haar-null set but the results still hold.

Given a vector \( e \in E \) with norm \( \|e\| \), the open ball of radius \( r \) centered at \( e \) is denoted by

\[ B(e, r) = \{ f \in E : \|f - e\| < r \}. \]

The norm of a vector \( \xi \) in the space of finite dimension \( \mathbb{R}^d \) will be denoted by \( |\xi| \).

We consider a map

\[ h : \mathbb{R}^d \times E \to \mathbb{R}^d, \quad (\xi, e) \mapsto h(\xi, e), \]

and define

\[ Z = \{ (\xi, e) \in \mathbb{R}^d \times E : h(\xi, e) = 0 \}. \]

**Lemma 2.1.** [16, Theorem 2] Assume that the following conditions hold:

\( (C_1) \) \( h \in C^1(\mathbb{R}^d \times E, \mathbb{R}^d) \) satisfies the periodicity condition

\[ h(T(\xi), e) = h(\xi, e) \text{ with } T(\xi_1, \xi_2, \ldots, \xi_d) = (\xi_1 + 2\pi, \xi_2, \ldots, \xi_d); \]

\( (C_2) \) there exists a compact set \( K \subset E \) such that the linear operator \( \partial_2 h(\xi, e) : E \to \mathbb{R}^d \) is onto if \( (\xi, e) \in Z \) and \( e \notin K \);

\( (C_3) \) given \( b > 0 \), there exists \( B > 0 \) such that if \( (\xi, e) \in Z \) and \( \|e\| \leq b \), then \( |\xi| \leq B \), where \( \xi = (\xi_2, \ldots, \xi_d) \).
Then the set
\[ \mathcal{R} = \{ e \in \mathbb{E} : 0 \text{ is a regular value of } h(\cdot, e) \} \]
is open and prevalent, where 0 is a regular value is equivalent to
\[ \det[\partial_1 h(\xi, e)] \neq 0 \]
for each \( \xi \in \mathbb{R}^d \) such that \( h(\xi, e) = 0 \).

**Lemma 2.2.** [17, Proposition 4] Let \( G \) be an open and prevalent subset of \( \mathbb{E} \). Assume that there exist a family \( \{ U_\alpha \}_{\alpha \in A} \) of open subsets of \( \mathbb{E} \) and functionals \( d_\alpha \in C^1(U_\alpha, \mathbb{R}) \) such that
\[ G \subset \bigcup_{\alpha \in A} U_\alpha, \]
\[ d_\alpha'(e) \neq 0 \]
for each \( e \in U_\alpha, \alpha \in A \).

Let \( C \) be a Borel subset of \( \mathbb{R} \) with zero measure. Then the set
\[ \tilde{G} = \bigcup_{\alpha \in A} d_\alpha^{-1}(\mathbb{R} \setminus C) \]
is prevalent in \( \mathbb{E} \).

### 2.2. Discriminant of planar linear system.

Consider the planar linear Hamiltonian system
\[ y' = JB(t)y, \tag{6} \]
where
\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}, \]
with \( \alpha(t), \beta(t), \gamma(t) \) being smooth \( T \)-periodic functions. It was proved in [11] that there exists a smooth function \( t \mapsto \psi(t) \) such that the change of variables
\[ y = R_{\psi(t)}x \]
will transform (6) into a simpler linear Hamiltonian system
\[ x_1' = b(t)x_2, \quad x_2' = -a(t)x_1, \]
where
\[ R_{\psi(t)} = \begin{pmatrix} \cos \psi(t) & -\sin \psi(t) \\ \sin \psi(t) & \cos \psi(t) \end{pmatrix}, \]
and the function \( \psi(t) \) was constructed explicitly in [11]. If \( B(t) \) is \( T \)-periodic, \( R_{\psi(t)} \) is also \( T \)-periodic. After these spatial changes, without loss of generality, we only consider the following linear systems
\[ \begin{cases} x' = b(t)y, \\ y' = -a(t)x, \end{cases} \tag{7} \]
where \( a, b \in C(\mathbb{R}/T\mathbb{Z}) \) and \( b(t) > 0 \) for all \( t \in \mathbb{R} \). We remark that the condition on the sign of \( b \) will be used in the proof of Lemma 2.3 below. Moreover, such a condition holds in many applied models, for example, the forced relativistic pendulum equation (1). The Poincaré matrix of (7) is
\[ M_T = \begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \psi_1(T) & \psi_2(T) \end{pmatrix}, \]
where \( (\phi_1(t), \psi_1(t)) \) and \( (\phi_2(t), \psi_2(t)) \) are real-valued solutions of (7) satisfying \( \phi_1(0) = 1, \psi_1(0) = 0 \) and \( \phi_2(0) = 0, \psi_2(0) = 1 \), respectively. The eigenvalues
\( \lambda_{1,2} \) of \( M_T \) are called the Floquet multipliers of (7). Obviously \( \lambda_1 \cdot \lambda_2 = 1 \). We can distinguish (7) in the following three cases:

- elliptic: \( \lambda_1 = \lambda_2, |\lambda_1| = 1, \lambda_1 \neq \pm 1 \);
- hyperbolic: \( 0 < |\lambda_1| < 1 < |\lambda_2| \);
- parabolic: \( \lambda_1 = \lambda_2 = \pm 1 \).

**Lemma 2.3.** The three functions \( \phi_1^2, \phi_1 \phi_2 \) and \( \phi_2^2 \) are linearly independent on any open interval \( I \subset \mathbb{R} \).

**Proof.** It is sufficient to prove that the Wronskian \( W = W(t) \) of functions \( \phi_1^2, \phi_1 \phi_2 \) and \( \phi_2^2 \) never vanishes. A simple computation shows that

\[
W = \det \begin{pmatrix}
\phi_1^2 & \phi_1 \phi_2 & \phi_2^2 \\
2b\phi_1 \psi_1 & b(\phi_1 \psi_2 + \phi_2 \psi_1) & 2b\phi_2 \psi_2 \\
w_{31} & w_{32} & w_{33}
\end{pmatrix},
\]

where

\[
w_{31} = -2ab\phi_1^2 + 2b'\phi_1 \psi_1 + 2b^2 \psi_1^2, \\
w_{32} = -2ab\phi_1 \phi_2 + b'\phi_1 \psi_2 + b'\phi_2 \psi_1 + 2b^2 \psi_1 \psi_2, \\
w_{33} = -2ab\phi_2^2 + 2b'\phi_2 \psi_2 + 2b^2 \psi_2^2.
\]

In particular,

\[
W(0) = \det \begin{pmatrix}
1 & 0 & 0 \\
0 & b(0) & 0 \\
-2a(0)b(0) & b'(0) & 2b^2(0)
\end{pmatrix} = 2b^3(0).
\]

First we assume that the functions \( a(t) \) and \( b(t) \) are of class \( C^1 \). The three functions \( z_{ij} = \phi_i \phi_j, 1 \leq i \leq j \leq 2 \), are the solutions of the third order linear equation

\[ b^2 z''' - 3bb' z'' + (3b^2 + 4ab^3 - bb')z' + 2b^2(a'b - ab')z = 0. \]

Liouville’s formula can be applied to this equation and it follows that the Wronskian \( W(t) = 2b^3(t) \) never vanishes because \( b(t) > 0 \) for all \( t \in \mathbb{R} \). If the functions \( a(t) \) and \( b(t) \) are only continuous, then they can be approximated in the uniform sense by \( C^1 \) functions \( a_n(t) \) and \( b_n(t) \). By continuous dependence, we are able to show that the corresponding Wronskians \( W_n \) of the system

\[
\begin{aligned}
x' &= b_n(t)y, \\
y' &= -a_n(t)x,
\end{aligned}
\]

converge to \( W \). \( \Box \)

The discriminant of (7) is defined as

\[
\Delta := \phi_1(T) + \psi_2(T) = \text{trace}(M_T) = \lambda_1 + \lambda_2.
\]

It is easy to verify that the equation (7) is elliptic if \(|\Delta| < 2\), hyperbolic if \(|\Delta| > 2\) and parabolic if \(|\Delta| = 2\). See [12] for some properties of the discriminant. The existence of non-trivial \( T \)-periodic solutions for (7) is equivalent to \( \lambda_1 = \lambda_2 = 1 \) or \( \Delta = 2 \).

In the elliptic case the monodromy matrix \( M_T \) is similar to a rotation. More precisely, there exist a number \( \theta \in (0, 2\pi), \theta \neq \pi \), and \( 2 \times 2 \) matrix \( P \) with \( \det P = 1 \) such that \( M_T = PR[\theta]P^{-1} \) with

\[
R[\theta] = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]
In this case the discriminant is \( \Delta = 2 \cos \theta \).

The discriminant of the linear system (7) can be interpreted as a functional depending on the coefficients \( a(t) \) and \( b(t) \). More precisely, we consider the functional

\[
\Delta : C(\mathbb{R}/TZ) \times C(\mathbb{R}/TZ) \to \mathbb{R}, \quad (a, b) \mapsto \Delta[a, b],
\]

where \( \Delta[a, b] \) is the discriminant of (7).

**Lemma 2.4.** The discriminant \( \Delta[a, b] \) of the linear system (7) has continuous partial derivatives and has exactly two critical values \( \Delta = \pm 2 \).

**Proof.** Given \( \xi, \eta \in C(\mathbb{R}/TZ) \). The partial derivatives along \( \xi, \eta \) are given by

\[
\Delta'_a[a, b]\xi := \frac{d}{ds}\Delta[a + s\xi, b]|_{s=0} = x_1(T) + y_2(T),
\]

\[
\Delta'_b[a, b]\eta := \frac{d}{ds}\Delta[a, b + s\eta]|_{s=0} = x_3(T) + y_4(T),
\]

where \( (x_i, y_i)(i = 1, 2) \) are the solutions of

\[
\begin{aligned}
x_i' &= b(t)y_i, \\
y_i' &= -a(t)x_i - \phi_1(t)\xi(t), \\
x_i(0) = y_i(0) = 0,
\end{aligned}
\]

and \( (x_j, y_j)(j = 3, 4) \) are the solutions of

\[
\begin{aligned}
x_j' &= b(t)y_j + \psi_1(t)\eta(t), \\
y_j' &= -a(t)x_j, \\
x_j(0) = y_j(0) = 0.
\end{aligned}
\]

Using the method of variation of constants, we deduce that

\[
\begin{aligned}
x_1(t) &= \int_0^t G_1(t, s)\xi(s)ds, \\
x_2(t) &= \int_0^t G_2(t, s)\xi(s)ds, \\
x_3(t) &= \int_0^t G_3(t, s)\eta(s)ds, \\
x_4(t) &= \int_0^t G_4(t, s)\eta(s)ds,
\end{aligned}
\]

where

\[
\begin{aligned}
G_1(t, s) &= \phi_1(t)\phi_1(s)\phi_2(s) - \phi_2(t)\phi_1^2(s), \\
G_2(t, s) &= \psi_1(t)\phi_2^2(s) - \psi_2(t)\phi_1(s)\phi_2(s), \\
G_3(t, s) &= \phi_1(t)\psi_1(s)\psi_2(s) - \phi_2(t)\psi_1^2(s), \\
G_4(t, s) &= \psi_1(t)\psi_2^2(s) - \psi_2(t)\psi_1(s)\psi_2(s).
\end{aligned}
\]

Then

\[
\begin{aligned}
\Delta'_a[a, b]\xi &= \int_0^T \chi_a(s)\xi(s)ds, \\
\Delta'_b[a, b]\eta &= \int_0^T \chi_b(s)\eta(s)ds,
\end{aligned}
\]

where

\[
\begin{aligned}
\chi_a(s) &= -\phi_2(T)\phi_1^2(s) + (\phi_1(T) - \psi_2(T))\phi_1(s)\phi_2(s) + \psi_1(T)\phi_2^2(s), \\
\chi_b(s) &= -\phi_2(T)\psi_1^2(s) + (\phi_1(T) - \psi_2(T))\psi_1(s)\psi_2(s) + \psi_1(T)\psi_2^2(s).
\end{aligned}
\]

From the above facts, using standard analysis, we can readily show that the discriminant \( \Delta[a, b] \) has continuous partial derivatives.

Next we prove that all values in \( \mathbb{R} \setminus \{-2, 2\} \) are regular. Assume that \( a, b \in C(\mathbb{R}/TZ) \) are such that \( \Delta[a, b] \neq \pm 2 \). Going back to the formula (8), we notice that we must prove that both \( \chi_a \) and \( \chi_b \) are not identically zero. Assume by contradiction that \( \chi_a(s) = 0 \) or \( \chi_b(s) = 0 \) for any \( s \in \mathbb{R} \). Here we state only when \( \chi_a(s) = 0 \) because similar contradiction is obtained when \( \chi_b(s) = 0 \). In particular, \( \chi_a(0) = 0 \) and \( \chi'_a(0) = 0 \). This implies that \( \phi_2(T) = 0 \) and \( \phi_1(T) - \psi_2(T) = 0 \).
and so the function $\chi_{\alpha}$ takes the simplified form $\chi_{\alpha}(s) = \psi_1(T)\phi_2^3(s)$. Now we conclude that $\psi_1(T) = 0$. Then the monodromy matrix $M_T$ takes the form $\alpha I_2$ with $\alpha = \phi_1(T) = \psi_2(T)$. Since $M_T$ has determinant one, we deduce that $M_T = \pm I_2$. This is impossible if $\Delta[a, b] \neq \pm 2$.

Finally, we choose $a_1(t) = (\frac{2}{T})^2$, $b_1(t) \equiv 1$ and $a_2(t) = (\frac{2}{T})^2$, $b_2(t) \equiv 1$. It is easy to verify that

$$\Delta[a_1, b_1] = 2, \quad \Delta[a_2, b_2] = -2$$

and

$$\Delta'_2[a_1, b_1] = 0, \quad \Delta'_2[a_2, b_2] = 0, \quad i = 1, 2,$$

which implies that $\pm 2$ are critical values. \hfill $\Box$

2.3. Ellipticity and index. We say that a $T$-periodic solution $(x, y)$ of the system (5) is non-degenerate if its variational system

$$\begin{cases}
\dot{u} = g_1'(y(t))v, \\
\dot{v} + g_2'(x(t))u = 0,
\end{cases}$$

(9)

has only the trivial $T$-periodic solution. Let us consider the following two sets

$$E_1 = \{f \in \mathcal{F} : (5) \text{ has at least one non-degenerate } T\text{-periodic solution}\},$$

and

$$E_2 = \{f \in \mathcal{F} : (5) \text{ has at least one elliptic } T\text{-periodic solution}\}.$$

Let $f \in E_1$ and $(x, y)$ be a non-degenerate $T$-periodic solution. It has an associated index which can be computed as

$$\gamma(x, y) = \text{sign}\{(1 - \lambda_1)(1 - \lambda_2)\},$$

where $\lambda_1, \lambda_2$ are the eigenvalues of the corresponding Poincaré matrix $M_T$ of the variational system associated to $(x, y)$.

Lemma 2.5. [19, Lemma 5.2] Assume that

$$\int_0^T a(t)dt \cdot \int_0^T b(t)dt \leq 4.$$

Then (7) does not admit negative Floquet multipliers.

Theorem 2.6. Let $(x, y)$ be a non-degenerate $T$-periodic solution of the system (5). In addition assume that

$$\int_0^T g_1'(y(t))dt \cdot \int_0^T g_2'(x(t))dt < 4 \quad (10)$$

Then $\gamma(x, y) = 1$ (resp. $\gamma(x, y) = -1$) if and only if $(x, y)$ is elliptic (resp. hyperbolic).

Proof. Denote by $\lambda_1, \lambda_2$ ($|\lambda_1| \geq |\lambda_2|$) the Floquet multipliers of (9). By Lemma 2.5 the multipliers are either conjugate complex numbers or positive real numbers. The elliptic case $\lambda_1 = \bar{\lambda}_2$ holds if and only if

$$\gamma(x, y) = \text{sign}\{\det(I_2 - M_T)\} = \text{sign}\{|1 - \lambda_1|^2\} = 1.$$

The hyperbolic case $0 < \lambda_1 < 1 < \lambda_2$ holds if and only if

$$\gamma(x, y) = \text{sign}\{\det(I_2 - M_T)\} = \text{sign}\{(1 - \lambda_1)(1 - \lambda_2)\} = -1.$$

The parabolic case is excluded because 1 cannot be a Floquet multiplier since $(x, y)$ is non-degenerate. \hfill $\Box$
3. Main results.

3.1. Prevalence of non-degenerate periodic solutions. Let us assume that

\( (\sigma_1) \) \( g_1, g_2 \in C^1 \) are bounded, \( g_1(0) = 0, g_1'(y) > 0 \) for all \( y \in \mathbb{R} \); \( g_2 \) is not locally trivial and \( g_2(s + 2\pi) = g_2(s) \) for each \( s \in \mathbb{R} \).

Here \( g_2 \) is not locally trivial means that for every open and non-empty interval \( I \subset \mathbb{R} \) there exists some \( x \in I \) such that \( g_2(x) \neq 0 \).

**Theorem 3.1.** Assume that \((\sigma_1)\) holds. Then the set \( E_1 \) is prevalent in \( E \).

**Proof.** We will apply Lemma 2.1 with \( \mathbb{E} = E \) and \( d = 2 \). We work with column vectors \( \xi = (\xi_1, \xi_2)^* \in \mathbb{R}^2 \) and the norm \( |\xi| = |\xi_1| + |\xi_2| \). In the space of \( 2 \times 2 \) matrices \( \mathbb{R}^{2 \times 2} \), we use the norm

\[
|A| = \max_{|\xi| \leq 1} |A\xi| = \max\{|a_{11}| + |a_{21}|, |a_{12}| + |a_{22}|\},
\]

where \( A = (a_{ij}) \in \mathbb{R}^{2 \times 2} \). The space of bounded linear operator from \( \mathcal{F} \) to \( \mathbb{R}^2 \) will denoted by \( \mathcal{L}(\mathcal{F}, \mathbb{R}^2) \) with norm

\[
\|L\| = \sup\{|Lf| : f \in \mathcal{F}, \|f\|_{\infty} \leq 1\}
\]

if \( L \in \mathcal{L}(\mathcal{F}, \mathbb{R}^2) \).

Given \( \xi = (\xi_1, \xi_2)^* \in \mathbb{R}^2 \) and \( f \in \mathcal{F} \), the solution of the initial value problem

\[
\begin{cases}
x' = g_1(y), \\
y' + g_2(x) = f(t), \\
x(0) = \xi_1, \ y(0) = \xi_2
\end{cases}
\]

will be denoted by \( (x(t; \xi, f), y(t; \xi, f)) \). Since \( g_1 \) and \( g_2 \) are bounded, such a solution is globally defined. The notations \( \Phi(t) \) and \( \Phi(t; \xi, f) \) will be employed for the matrix solution of

\[
\dot{Y} = A(t)Y, \quad Y(0) = I_2,
\]

with

\[
A(t) = \begin{pmatrix}
0 & g_1'(y(t; \xi, f)) \\
-g_2'(x(t; \xi, f)) & 0
\end{pmatrix}.
\]

The theorem on continuous dependence can be applied to (11)-(12). It implies that the map

\[
(t; \xi, f) \in \mathbb{R} \times \mathbb{R}^2 \times \mathcal{F} \rightarrow \Phi(t; \xi, f) \in \mathbb{R}^{2 \times 2}
\]

is continuous. In particular it is uniformly continuous on compact sets. This implies that if \( \xi_n \rightarrow \xi \) and \( \|f_n - f\|_{\infty} \rightarrow 0 \), then

\[
\Phi(t; \xi_n, f_n) \rightarrow \Phi(t; \xi, f)
\]

uniformly in \( t \in [0, T] \). We also consider the map

\[
h : \mathbb{R}^2 \times \mathcal{F} \rightarrow \mathbb{R}^2, \quad h(\xi, f) = (x(T; \xi, f) - \xi_1, y(T; \xi, f) - \xi_2)^*
\]

and observe that the zeros of \( h(\cdot, f) \) are the initial conditions producing \( T \)-periodic solutions. This map is continuous and the theorem on differentiability with respect to initial conditions and parameters implies that it is Gateaux differentiable with partial derivatives

\[
\partial_1 h(\xi, f) \in \mathbb{R}^{2 \times 2}, \quad \partial_2 h(\xi, f) \in \mathcal{L}(\mathcal{F}, \mathbb{R}^2)
\]

given by

\[
\partial_1 h(\xi, f) = \Phi(T; \xi, f) - I_d, \quad \partial_2 h(\xi, f)p = (u(T), v(T))^*.
\]
where \((u, v)\) is the solution of
\[
\begin{cases}
\dot{u} = g_1'(y(t; \xi, f))v, \\
\dot{v} + g_2'(x(t; \xi, f))u = p(t), \\
u(0) = 0, \quad v(0) = 0.
\end{cases}
\]

The formula of variation of constants implies that
\[
u(t) = \int_0^T K_1(t, s; \xi, f)p(s)ds, \quad t \in [0, T],
\]
(13)
\[
v(t) = \int_0^T K_2(t, s; \xi, f)p(s)ds, \quad t \in [0, T],
\]
(14)
where
\[
K_1(t, s; \xi, p) = \phi_2(t)\phi_1(s) - \phi_1(t)\phi_2(s), \quad K_2(t, s; \xi, p) = \psi_2(t)\phi_1(s) - \psi_1(t)\phi_2(s).
\]

The continuity of \(\Phi\) and the formulas (13) and (14) can be employed to prove the continuity of the partial derivatives of \(h\). In particular the continuity of
\[(\xi, f) \in \mathbb{R}^2 \times \mathcal{F} \to \partial_2 h(\xi, f) \in L(\mathcal{F}, \mathbb{R}^2)\]
is a consequence of the estimate
\[
\|\partial_2 h(\xi, f) - \partial_2 h(\hat{\xi}, \hat{f})\| \leq \int_0^T \{ |K_1(T, s; \xi, f) - K_1(T, s; \hat{\xi}, \hat{f})| + |K_2(T, s; \xi, f) - K_2(T, s; \hat{\xi}, \hat{f})| \}\, ds.
\]
The previous discussions show that \(h\) is Fréchet differentiable and \([(C_1), \text{Lemma 2.1}]\) holds. The condition \((\sigma_1)\) implies that \(x(t; T(\xi), f) = x(t; \xi, f) + 2\pi\). Then we can deduce that \(h\) satisfies the periodicity condition. Moreover, given \((\xi, f) \in Z\) we know that \((x(t; \xi, f), y(t; \xi, f))\) is a \(T\)-periodic solution of (5). Hence
\[
\|\dot{x}(\cdot; \xi, f)\|_\infty \leq \|g_1\|_\infty, \quad \|\dot{y}(\cdot; \xi, f)\|_\infty \leq \|g_2\|_\infty + \|f\|_\infty.
\]
The periodicity of \((x(t; \xi, f), y(t; \xi, f))\) and the equation (5) imply that for some \(T\), we have \(y(\tau; \xi, f) = 0\). Then
\[
|\xi_2| = |y(0; \xi, f)| = \int_0^\tau |\dot{y}(t)|\, dt \leq (\|g_2\|_\infty + \|f\|_\infty)T.
\]
The condition \((C_3)\) holds with \(B = (\|g_2\|_\infty + \|f\|_\infty)T\).

To check \([(C_2), \text{Lemma 2.1}]\) we define \(K = \{0\}\) and prove that \(\partial_2 h(\xi, f) : \mathcal{F} \to \mathbb{R}^2\) is onto if \((\xi, f) \in Z\) and \(f \neq 0\). After some computations with the formulas (13) and (14) we obtain
\[
\partial_2 h(\xi, f)p = \Phi(T; \xi, f)J \int_0^T p(t)\left(\dot{\phi}_1(t), \dot{\phi}_2(t)\right)^* \, dt.
\]
Since \(\Phi(T; \xi, f)\) and \(J\) are invertible matrices, it is enough to prove that
\[
L : p \in \mathcal{F} \to \int_0^T p(t)\left(\dot{\phi}_1(t), \dot{\phi}_2(t)\right)^* \, dt \in \mathbb{R}^2
\]
is onto. In view of \([16, \text{Lemma 3.1}]\), we need to prove that \(\dot{\phi}_1, \dot{\phi}_2\) are linearly independent. Since \(\dot{\phi}_i = g_i'(y)\psi_i (i = 1, 2)\), we only need to prove that \(\psi_1, \psi_2\) are linearly independent. Actually we will prove that the Wronskian of these functions
\[
W(\psi_1, \psi_2) = \begin{vmatrix}
\psi_1 & \psi_2 \\
\dot{\psi}_1 & \dot{\psi}_2
\end{vmatrix} = -g_2'(x(t; \xi, f))\psi_1 - g_1'(y(t; \xi, f))\psi_2 = g_2'(x(t; \xi, f))\psi_1
\]
is not identically zero. Assume by contradiction that \(W(\psi_1, \psi_2) \equiv 0\). Then \(g_2(x(t; \xi, f))\) vanishes identically and so \(g_2(x(t; \xi, f))\) is a constant \(k\). From the system (5),
\[
f(t) = \dot{y} + k, \quad \dot{x} = g_1(y(t; \xi, f)).
\]
The solution \((x(t; \xi, f)), (y(t; \xi, f))\) is \(T\)-periodic and \(f\) has zero average and hence \(k = 0\). In consequence also \(g_2(x(t; \xi, f))\) vanishes identically and the assumption \((\sigma_1)\) implies that \((x(t; \xi, f))\) must be constant. Thus \(g_1(y(t; \xi, f)) \equiv 0\), and \((y(t; \xi, f))\) must be constant. Then \(f(t) = k = 0\) but this forcing has been excluded by the definition of \(K\). Notice that 0 is a regular value of \(h(\cdot, f)\) if and only if any \(T\)-periodic solution \((x, y)\) is non-degenerate. The proof is completed using Lemma 2.1.

3.2. Prevalence of elliptic periodic solutions.

**Lemma 3.2.** Let \((x, y)\) be a \(T\)-periodic solution of (5). Then
\[
|y(0)| \leq T(M + \|f\|).
\]

**Proof.** If \((x, y)\) is a \(T\)-periodic solution, we integrate the first equation over a period and obtain
\[
\int_0^T g_1(y(t))dt = 0.
\]
Hence, there exists \(\xi \in [0, T]\) such that \(y(\xi) = 0\). The second equation leads to the estimate
\[
|y(0)| = |y(0) - y(\xi)| = |y'(\tau)||\xi| \leq T(M + \|f\|),
\]
where \(M = \max_{x \in \mathbb{R}} |g_2(x)|\). \(\square\)

**Lemma 3.3.** Assume that \(f \in E_2\). Then there exists only a finite number of \(T\)-periodic solutions of (5) satisfying \(x(0) \in [0, 2\pi]\).

**Proof.** The planar system (5) has an associated Poincaré map defined as
\[
P_T : \mathbb{R}^2 \to \mathbb{R}^2, \quad P_T(\xi) = (x(T; \xi), y(T; \xi)),
\]
where \((x(t; \xi), y(t; \xi))\) is the solution of (5) satisfying \(x(0) = \xi_1, y(0) = \xi_2\) and \(\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\). The fixed points of \(P_T\) are the initial conditions of the periodic solutions with periodic \(T\). Given a \(T\)-periodic solution \((x(t), y(t))\) and the fixed point \(\xi_* = (x(0), y(0))\), the theorem on differentiability with respect to initial conditions implies that the derivative of \(P_T\) at the fixed point is precisely the monodromy matrix of (9), that is, \(P_T'(\xi_*) = M_T\). In consequence, if \(f \in E_2\), all fixed points of \(P_T\) will satisfy \(\det(I - P_T'(\xi_*)) \neq 0\). The implicit function theorem can be applied to deduce that all these fixed points are isolated. If we combine this fact with bound in Lemma 3.2 we can conclude that the set of fixed points
\[
\{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : P_T(\xi) = \xi, \xi_1 \in [0, 2\pi]\}
\]
is finite. \(\square\)

**Lemma 3.4.** [15, Lemma 3] Let \(\mathcal{R} = (\alpha_-, \alpha_+) \times (\beta_-, \beta_+)\) be open rectangle and \(F : \mathcal{R} \to \mathbb{R}^2\) be continuous and such that
\[
F(\alpha_-, y) = F(\alpha_+, y) \neq 0, \quad \text{for all } y \in [\beta_-, \beta_+],
\]
\[
F_1(x, \beta_+) < 0 < F_1(x, \beta_-), \quad \text{for all } x \in [\alpha_-, \alpha_+].
\]
Then \(\deg[F, \mathcal{R}, 0] = 0\).
Lemma 3.5. There exists \( \rho > 0 \) such that, for any \( (x_0, y_0) \in \mathbb{R}^2 \) verifying \( |y_0| > \rho \), one has

\[
\text{sign}(y_0)[x(T; x_0, y_0) - x_0] > 0.
\]

Proof. The solution \( x(t; x_0, y_0) \) of the system (5) with the initial condition \( x(0) = x_0, y(0) = y_0 \) satisfies

\[
x(t) = x_0 + \int_0^t g_1(y(s))ds, \quad y(t) = y_0 + \int_0^t [f(s) - g_2(x(s))]ds.
\]

Hence

\[
x(T; x_0, y_0) - x_0 = \int_0^T g_1(y(s))ds = \int_0^T g_1 \left( y_0 + \int_0^\tau [f(\tau) - g_2(x(\tau))]d\tau \right)ds.
\]

Let \( M = \max_{(t,x) \in [0,T] \times \mathbb{R}} |f(t) - g_2(x)| \). Then we obtain

\[
y_0 \left[ x(T; x_0, y_0) - x_0 \right] \geq y_0 T \cdot g_1 \left( y_0 + \text{sign}(y_0) MT \right) > 0,
\]

which implies that the result holds since \( g_1(y) \) is increasing in \( y \) and \( g_1(0) = 0 \). \( \Box \)

Lemma 3.6. Assume that the set of \( T \)-periodic solutions of (5) is finite and given by

\[
(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).
\]

Then

\[
\sum_{i=1}^n \gamma(x_i, y_i) = 0.
\]

Proof. Let \( \alpha \neq x_i(0), \ i = 1, 2, \ldots, n \). Consider the rectangle

\[
\mathcal{R} = (\alpha, \alpha + 2\pi) \times (-\rho, \rho),
\]

where \( \rho > 0 \) is given by Lemma 3.5. It follows from Lemma 3.4 that

\[
\text{deg}[id - P_T, \mathcal{R}, 0] = 0.
\]

Since \( \mathcal{R} \) contains all the fixed points of \( P_T \) (after the identification \( x(t) \equiv x(t) + 2n\pi \)), now the proof is completed. \( \Box \)

Theorem 3.7. Let the condition \((\sigma_1)\) be satisfied. If (10) holds for any \( T \)-periodic solution \((x, y)\), then the set \( E_2 \) is prevalent in \( E \).

Proof. For each \( f \in E_1 \), (5) has at least one of non-generate solutions, say \((x_1, y_1)\), which must satisfy \( |\gamma(x_1, y_1)| = 1 \). If \( \gamma(x_1, y_1) = 1 \), then \((x_1, y_1)\) is elliptic by Theorem 2.6. If \( \gamma(x_1, y_1) = -1 \), by Lemma 3.6 there must exist \((x_2, y_2)\) such that \( \gamma(x_2, y_2) = 1 \), which implies that \((x_2, y_2)\) is elliptic by Theorem 2.6. Thus \( E_2 = E_1 \), and the result follows from Theorem 3.1. \( \Box \)

3.3. Prevalence of stable periodic solutions. Assume that \((\sigma_1)\) and (10) hold. Suppose further that

\[
(\sigma_2) \ g_1 \in C^\infty, g_2 \in C^\infty, g_2'(x) + (g_2'(x))^2 > 0 \text{ for any } x \in \mathbb{R}, \text{ the zeros of } g_2''(x) \text{ are isolated and } g_2''(x) = 0 \text{ if } g_2(x) = 0.
\]

Following the main ideas come from [9, 17], we establish the stable result which will be used later.
Lemma 3.8. Let $C = \varphi(L \cup \mathbb{Q})$, where $\varphi(x) = 2\cos(2\pi x)$ and $L$ denotes the set of Liouville numbers. Assume that $(x, y)$ is a $T$-periodic solution of (5) and the discriminant of the linearized system (9) satisfies $|\Delta| < 2$ and $\Delta \notin C$. Then $(x, y)$ is stable.

Proof. The Poincaré map $\Pi$ associated to the differential system (5) is $C^\infty$ area-preserving diffeomorphism. Note that $(x, y)$ is a $T$-periodic solution of (5) equivalent to say that the initial condition $(x(0), y(0))$ is a fixed point of $\Pi$. Recall that $\Delta = 2 \cos \theta$. Then the discriminant satisfies $|\Delta| < 2$ and is not in $C$ and so $(x, y)$ is elliptic and $\frac{\theta}{2\pi} \in \mathbb{R} \setminus \{L \cup \mathbb{Q}\}$, namely $\frac{\theta}{2\pi}$ is Diophantine. Theorem 3.3 in [9] can be applied and therefore $(x, y)$ is stable. $\Box$

Given $f \in \mathcal{F}$ and a $T$-periodic solution $(x, y)$ of (5), we define $D = D[x, y]$ as the discriminant of the linearized system (9). To be precise on the domain of the functional, we introduce the set

$$M = \left\{(x, y) \in X^2 : \int_0^T g_1(y(t))dt = \int_0^T g_2(x(t))dt = 0\right\}.$$  

The tangent space at $(x, y) \in M$ is

$$T_{(x,y)}(M) = \left\{(u, v) \in X^2 : \int_0^T g''_2(x(t))u(t)dt = \int_0^T g''_1(y(t))v(t)dt = 0\right\}.$$  

The rigorous definition of the functional is

$$D : M \to \mathbb{R}, \quad D[x, y] = \Delta[g''_2(x), g''_1(y)].$$

From the chain rule we deduce that for each $(u, v) \in T_{(x,y)}(M),$ 

$$D'_x[x, y]u = \Delta'[g''_2(x), g''_1(y)]g''_2(x)u = \int_0^T \chi_1(t)g''_2(x(t))u(t)dt,$$

$$D'_y[x, y]u = \Delta'[g''_2(x), g''_1(y)]g''_1(y)v = \int_0^T \chi_2(t)g''_1(y(t))v(t)dt,$$

where $\chi_1 = \chi_a, \chi_2 = \chi_b$ with $a(t) = g'_2(x(t))$ and $b(t) = g'_1(y(t)).$ Consider the new domain for the functional

$$M_* = M \setminus \{(x_n, \cdot) : n \in \mathbb{Z}\},$$

where $x_n$ satisfies $g_2(x_n) = 0$. Of course we also have that $g''_2(x_n) = 0$. This is an open subset of $M$ and the restriction of the functional will be denoted by $D_* : M_* \to \mathbb{R}$.

Lemma 3.9. All real numbers different from $\pm 2$ are regular values of $D_*.$

Proof. Assume that $x \in M_*$ is a critical point of $D_*$. Then for each $(u, v) \in T_{(x,y)}(M),$ 

$$\int_0^T \chi_1(t)g''_2(x(t))u(t)dt = 0, \quad \int_0^T \chi_2(t)g''_1(y(t))v(t)dt = 0.$$

The integrals can be interpreted as an inner product in Hilbert space $L^2(\mathbb{R}/T\mathbb{Z}) \times L^2(\mathbb{R}/T\mathbb{Z}),$

$$< (\chi_1 g''_2(x), \chi_2 g''_1(y)), (u, v) >_{L^2 \times L^2} = 0.$$
This is equivalent to saying that \((\chi_1 g_2''(x), \chi_2 g_1''(y))\) is orthogonal to the tangent space, 
\[(\chi_1 g_2''(x), \chi_2 g_1''(y)) \in T_{(x,y)}(M)^{\perp}.
\] 
From the general theory of Hilbert spaces we know that 
\[T_{(x,y)}(M)^{\perp} = V^{\perp},\]
where \(V\) is the closure of \(T_{(x,y)}(M)\) in \(L^2(\mathbb{R}/TZ) \times L^2(\mathbb{R}/TZ)\). The space \(V\) can also be described as the hyperplane orthogonal to the line spanned by \((g_2'(x), g_1'(y))\), 
\[V = L^\perp, \quad L = \{(\lambda g_2'(x), \lambda g_1'(y)) : \lambda \in \mathbb{R}\}.
\] Hence \((x, y)\) is a critical point of \(D\) if and only if 
\[(\chi_1 g_2''(x), \chi_2 g_1''(y)) \in V^\perp = (L^\perp)^\perp = L.
\] This means that 
\[(\chi_1 g_2''(x), \chi_2 g_1''(y)) = (\lambda g_2'(x), \lambda g_1'(y))
\] for some \(\lambda \in \mathbb{R}\). Since \(x \in M\), there exists an instant \(\tau\) such that \(g_2(x(\tau)) = 0\). By condition \((\sigma_2)\), we know that \(g_2''(x(\tau)) = 0\) and therefore 
\[\lambda g_2'(x(\tau)) = 0.
\] Using condition \((\sigma_2)\) again, \(g_2'(x(\tau)) \neq 0\) and we obtain \(\lambda = 0\). Hence for every \(t\) 
\[\chi_1(t) g_2''(x(t)) = 0.
\] Since the zeros of \(g_2''\) are isolated, there exists an interval \(I\) such that for every \(t \in I\) 
\[g_2''(x(t)) \neq 0.
\] Thus \(\chi_1(t) = 0\) for each \(t \in I\). It follows that from Lemma 2.3 that 
\[\phi_2(T) = \phi_1(T) - \psi_2(T) = \psi_1(T) = 0,
\] which means that the monodromy matrix \(M_T = \pm I_2\) and therefore \(D[x, y] = \pm 2\).

Let us consider the operator 
\[A : M \to \mathcal{F} \times \mathcal{F}, \quad A[x, y] = (\dot{x} - g_1(y), \dot{y} + g_2(x)).
\] Hence the periodic problem for \((5)\) with \(f \in \mathcal{F}\) is equivalent to the equation 
\[A[x, y] = (0, f).
\] The map \(A\) is smooth with derivative 
\[A'[x, y] : T_{(x,y)}(M) \to \mathcal{F} \times \mathcal{F}, \quad A'[x, y](u, v) = (\dot{u} - g_1'(y)v, \dot{v} + g_2'(x)u).
\]

**Lemma 3.10.** Given \((x, y) \in M\), the derivative \(A'[x, y]\) is an isomorphism if and only if \((x, y)\) is a non-degenerate \(T\)-periodic solution of the system \((5)\).

**Proof.** Assume first that \(A'[x, y]\) is an isomorphism and let \((u(t), v(t))\) be a \(T\)-periodic solution of the linearized system \((9)\). Integrating over a period, we obtain 
\[\int_0^T g_1'(y(t))v(t)dt = 0, \quad \int_0^T g_2'(x(t))u(t)dt = 0,
\] and so \((u, v) \in T_{(x,y)}(M)\). From the linearized system, \(A'[x, y](u, v) = (0, 0)\), and this implies \((u, v) = (0, 0)\) because \(\text{Ker} A'[x, y] = \{(0, 0)\}\).
Fredholm’s alternative we know that the non-homogeneous system are direct consequences of Theorem 3.1, Theorem 3.7 and Theorem 3.11. For the relativistic pendulum equation.

3.4. Forced relativistic pendulum equation. Now we apply the main results to the forced relativistic pendulum equation (1), where \( \mu > 0 \) and \( f \in \mathcal{F} \). Note that its equivalent planar system (4) is a special case of the system (5) with

\[
\begin{align*}
g_1(y) &= \Phi^{-1}(y) = \frac{y}{\sqrt{1 + y^2}}, \\
g_2(x) &= \mu \sin x.
\end{align*}
\]

Then \( g'_1(y) = \frac{1}{(1+y^2)^{3/2}} > 0 \) for all \( y \in \mathbb{R} \). Moreover, it is easy to see that conditions (\( \sigma_1 \)), (\( \sigma_2 \)) are satisfied. In this case, \( E = \mathcal{F} \). Therefore, the following two results are direct consequences of Theorem 3.1, Theorem 3.7 and Theorem 3.11.
Theorem 3.12. The set
\[ \{ f \in \mathcal{F} : (1) \text{ has at least one non-degenerate } T\text{-periodic solution} \} \]
is prevalent in \( \mathcal{F} \).

Theorem 3.13. Assume that \( 0 < \mu \leq 4/T^2 \). Then the set
\[ \{ f \in \mathcal{F} : (1) \text{ has at least one stable } T\text{-periodic solution} \} \]
is prevalent in \( \mathcal{F} \).

Proof. Let \((x, y)\) be a periodic solution of equation (1). When \( 0 < \mu \leq 4/T^2 \), we obtain that
\[ \int_0^T g_1'(y(t))dt \cdot \int_0^T g_2'(x(t))dt < T^2 \mu \leq 4, \]
and the condition (10) is fulfilled. Now the result follows from Theorem 3.7 and Theorem 3.11.

Open problem. We believe that the constant \( 4/T^2 \) in the above Theorem can be improved. However, up to now, we do not know how to obtain the sharp bound for \( \mu \) and leave it as an open problem.

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