Quantum quench in long-range field theories

M. A. Rajabpour\textsuperscript{1} and S. Sotiriadis\textsuperscript{2}

\textsuperscript{1} Instituto de Física, Universidade Federal Fluminense, Av. Gal. Milton Tavares de Souza s/n, Gragoatá, 24210-346, Niterói, RJ, Brazil
\textsuperscript{2} Dipartimento di Fisica dell’Università di Pisa and INFN, Sezione Pisa 56127 Pisa, Italy

(Dated: September 24, 2014)

We study equilibration properties of observables in long-range field theories after the mass quench in $d = 1, 2$ and $3$ dimensions. We classify the regimes where we expect equilibration or its absence in different dimensions. In addition we study the effect of the initial state in the equilibration properties of our system. In the case of free field theories we show that whenever we have equilibration the long-time limit of correlations can be described by the Generalized Gibbs Ensemble. We prove that all integrals of motions in our system are non-local.

PACS numbers: 05.30.-d

I. INTRODUCTION

In recent years due to interesting experiments, long-range interacting systems moved into the focus of research. Using ultracold atoms and trapped ions techniques many different quantum long-range spin models were engineered \cite{1} and their time dependent properties were investigated \cite{2}. There has been also considerable progress in the theoretical understanding of quantum long-range systems \cite{3}. Different properties such as entanglement entropy \cite{4, 5} dynamics of entanglement \cite{6} and equilibration properties \cite{7, 8} were studied extensively. One of the interesting features of long-range systems is the absence of Lieb-Robinson velocity \cite{9}. Based on the Lieb-Robinson theorem for short-range systems, the effect of a perturbation in part of the system at a given time becomes exponentially small with the distance outside a region called the causal region \cite{11}. This effect which is usually called quasi-locality helps to define a horizon-like region inside which the effect of the perturbation is non-zero while outside it is exponentially small. This picture is consistent with the presence of a finite Lieb-Robinson velocity that plays the same role as the speed of light in Lorentz invariant field theories. In the presence of long-range interactions the above picture is no-longer true. Although in particular regimes one can still find a generalized Lieb-Robinson bound \cite{10} for these systems the general form of the causal region is no longer cone-like, as when there exists a maximum velocity. The causal region in these cases grows logarithmically for large distances \cite{10}.

One of the important features of long-range systems is the presence of a power-law like dispersion relation, e.g. $\omega(k) = |k|^{\alpha/2}$ with $0 < \alpha < 2$. For example starting from a system of hard-core bosons with hopping $|i - j|^{-(d+\alpha)}$ one arrives at the dispersion relation $\omega(k) = |k|^{\alpha/2}$ in momentum space \cite{12}. As we will see soon, the same is true also for coupled harmonic oscillators and it is widely believed that this should be the case also for long-range spin systems \cite{13}. Although, as we mentioned, the long-range systems have been already studied with many experimental and numerical methods, a direct study of a bosonic system with the simple dispersion relation $\omega^2(k) = |k|^\alpha + m^\alpha$ has initiated just recently in the context of entanglement entropy \cite{5}. The interest on this model is two fold: firstly this is among the few systems where one can do analytical calculations and secondly despite its simplicity it captures most of the interesting features of the more complicated long-range interacting systems.

Having these two motivations in mind we are interested in studying the evolution of the two-point correlation functions after a sudden change of some parameter in a long-range system of harmonic oscillators. Such out-of-equilibrium problems belong to the general category of quantum quenches \cite{14, 15}. We first define our quench protocol and then we demonstrate different properties emerging from the dynamics. In particular we focus on two important features: the presence of horizon effect and the occurrence of equilibration. We show in which circumstances we expect a casual region or horizon and we study in detail the regimes where we expect equilibration of correlations. We also discuss the physical interpretation of this equilibration in the so-called semiclassical approximation \cite{16, 17, 18, 19}. Next we verify that, in the regimes where equilibration occurs, the long-time properties can be described by a statistical ensemble known as Generalized Gibbs Ensemble (GGE) \cite{20, 21, 22, 23}. The latter is deduced by an entropy maximization principle, subject to all infinite constraints imposed by the free dynamics. We also point out that there is no way to express the GGE in terms of local integrals of motion, in contrast to the usual case \cite{24, 25, 26}. Finally we also discuss how turning on the interactions can modify our results.

The outline of the paper is as follows: In the next section we first introduce the model we will study and summarize its properties. In the third section we define our quantum quench protocol and then study the time evolution of correlations. We explicitly detect the parameter regimes where a causal region exists and also the regimes where correlation functions exhibit equilibration. In the fourth section we show that the long-time properties of those cases that lead to equilibration can be described.
by the GGE and comment on its non-local form. In the fifth section we study the equilibration properties of the momentum operator. The section six is dedicated to the extension of our results to interacting field theories. In the last section we summarize our findings and conclusions.

II. DEFINITION OF THE MODEL

The system that we are interested in is a system of coupled harmonic oscillators that are described in momentum space by the following quadratic Hamiltonian

\[ H = \sum_k \left( \frac{1}{2} p_k^2 + \frac{1}{2} \omega^2(k) \phi_k \phi_{-k} \right) \]

We will consider two different types of dispersion relations (Fig. 2)

- Type-1: \[ \omega_1(k) = \sqrt{|k|^\alpha + m^2} \]
- Type-2: \[ \omega_2(k) = (k^2 + m^2)^\alpha \]

where \( m \) will be called the “mass” parameter and \( \alpha \) is a real parameter that controls the long-range form of harmonic coupling. When \( m = 0 \) both types are identical and give the massless dispersion relation \( \omega(k) = |k|^{\alpha} \). Also they both reduce to the simple local relativistic dispersion relation for \( \alpha = 2 \).

For \( 0 < \alpha < 2 \), type-1 dispersion relation corresponds to long-range couplings in real space. Indeed the Hamiltonian in real space has, in this case, the form

\[ H = \frac{1}{2} \sum_r \pi^2(r) + \frac{1}{2} \sum_{r,r'} \phi(r) \left( \frac{1}{|r-r'|^d+\alpha} + m^\alpha \delta_{r,r'} \right) \phi(r') \]

For type-1, the asymptotic behavior of the integral for large \( r \) gives

\[ G(r) \sim \frac{1}{r^{\alpha+d}}, \quad m \neq 0, 0 < \alpha < 2. \] (6)

The situation is different for type-2 dispersion relation: the integral can be done explicitly and the result is

\[ G(r) \propto r^{\frac{\alpha}{2} - \frac{d+1}{2}} e^{-mr}, \quad m \neq 0, 0 < \alpha \leq 2. \] (7)

A. Generalized Lieb-Robinson bound

In this subsection we would like to classify different regimes where a Lieb-Robinson bound or its generalized form exists. We first present the Lieb-Robinson theorem [11]: consider the Hamiltonian \( H_\Lambda = \sum_{X \subseteq \Lambda} h_X \) defined on the lattice \( \Lambda \) with finite range interactions supported in \( X \). Then for arbitrary observables \( O_A \) and \( O_B \) supported in the disjoint sets \( A \) and \( B \) we have

\[ \||O_A(t), O_B(t)|| \leq e^{-a(d(A,B) - v|t|)}, \]

where \( d(A,B) \) is the distance between the sets \( A \) and \( B \) and \( ||O|| \) is the operator norm of the observable. In the above equation \( v \) is called Lieb-Robinson velocity and \( a \) is just a constant. As it is clear from the right hand side of the equation, for \( d(A,B) > v|t| \) we expect an exponentially small left hand side which is a direct manifestation of cone-like horizon. For our models the Lieb-Robinson bound can be applied only for type-2 dispersion relation because this dispersion relation corresponds to exponentially decaying interactions. For the other cases we have a generalized Lieb-Robinson bound [10] that we now define: consider the Hamiltonian \( H_\Lambda \) with interactions satisfying \( \sum ||h_X|| \leq \lambda(1+d(r,r'))^{-(d+\alpha)} \) (for the power-law
decaying interactions \( \frac{1}{|x_1|^{\alpha+x_{\infty}}} \) one can show that this condition is satisfied as far as \( \alpha > 0 \) \([6]\) then we have

\[
\|[O_A(t), O_B(t)]\| \leq c \frac{e^{\gamma|t|} - 1}{(1 + d(A, B))^{d+\gamma}}. \tag{10}
\]

As it is clear from the right hand side of the above equation one can not expect cone-like casual region in this case. However, some other kinds of casual regions have been anticipated in the recent paper \([8]\) for systems with long-range interactions. As far as we take \( \alpha > 0 \) for the massive and massless type-1 dispersion relation our model satisfies the assumptions of the theorem and so one can try to investigate the consequences of the generalized Lieb-Robinson bound. However, this is not the path that we are going to choose, instead we will show that most of the results can be understood with a simple semiclassical approximation which is just an intuitive way of understanding most of the results concerning the Lieb-Robinson bound.

### III. CORRELATIONS AFTER A MASS QUENCH

The two point correlation function after a quench of (any parameter of) the dispersion relation, \( \omega_0 \to \omega \), in a system described by a quadratic Hamiltonian, is given by the following expression \([15]\)

\[
G(r, t) \equiv \langle \phi(0,t)\phi(0,t) \rangle = \int \frac{dk}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}(\omega_0^2(k) - \omega^2(k))(1 - \cos(2\omega(k)t))}{\omega^2(k)\omega_0(k)}. \tag{11}
\]

Initial correlations have been subtracted in the definition of \( G(r, t) \), so that we can focus on the correlations that develop due to the quench. Details of the derivation of this formula can be found in \([15]\). Eq. (11) can be split into a time independent and a time dependent part

\[
\tilde{G}(r) = \int \frac{dk}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}(\omega_0^2(k) - \omega^2(k))}{\omega^2(k)\omega_0(k)}, \tag{12}
\]

\[
\tilde{G}(r, t) = -\int \frac{dk}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}(\omega_0^2(k) - \omega^2(k))}{\omega^2(k)\omega_0(k)} \cos(2\omega(k)t). \tag{13}
\]

As long as the integral in \( \tilde{G}(r) \) is convergent, the integral in \( \tilde{G}(r, t) \) vanishes for large \( t \) by virtue of the lemma Riemann-Lebesgue, and therefore \( G(r, t) \) becomes stationary. If instead \( \tilde{G}(r) \) is not convergent, then \( G(r, t) \) may not become stationary for large times, i.e. it may exhibit persistent oscillations or increase indefinitely with time.

We are interested in studying the consequences of a mass quench in the separate cases of the two dispersion relations introduced above. Before we proceed to the derivation of our results, we can anticipate some of them based on a semiclassical approach introduced next.

#### A. Semiclassical approximation

The semiclassical approximation provides a general physical interpretation of several features of quantum quenches, not only qualitatively but also quantitatively \([15]\). This has been already applied to the case of coupled harmonic oscillators, conformal field theories, spin chains and some interacting integrable models \([14, 31-34]\). A quantum quench in a system where both the pre-quench and the post-quench Hamiltonians are essentially non-interacting (free or interacting but mappable to free) and invariant under spatial translations and under parity of the fields, has as a result the instantaneous production of pairs of quasiparticles with opposite momenta (this is because under the above conditions the relation between the pre-quench and post-quench creation and annihilation operators is a simple Bogoliubov transformation). If the initial state has short-range correlations then both quasiparticles in each of these pairs are produced within a distance of the order of the initial correlation length \( \xi \). If instead the initial state has long-range correlations then quasiparticle pairs can be emitted from points at any distance. After the quench, the two correlated quasiparticles of each pair spread the initial correlations by traveling through the system. According to the semiclassical approximation, the quasiparticles move ballistically with velocity given by the group velocity \( v(k) = d\omega(k)/dk \) corresponding to their wavelength \( k \). This physical mechanism is pictorially shown in Fig. 1.

**FIG. 1: Physical interpretation of the evolution of correlations after a quantum quench in the semiclassical approximation.**

The evolution after the quench can be explained by the collective effect of the quasiparticle motion. For example, let us assume that the initial correlations are short-range with a vanishingly small correlation length \( \xi \to 0 \) (which means that there are zero initial correlations between any two different points) and that after the quench the dispersion relation \( \omega(k) \) has a maximum group velocity \( v_{\text{max}} \), as in the relativistic case. Then the equal time two-point correlation function is expected to exhibit the horizon effect: correlations between two points at distance \( r \) are frozen to their initial zero value until the time \( t^*(r) = |r|/(2v_{\text{max}}) \) when a pair of the fastest quasiparticles emitted from the middle between these two points, reach them and transmit the first signal of the quench.
If $\xi$ is not small, the first quasiparticle pairs to reach the two points, would be emitted not from the same but from points at distances of the order of $\xi$ and the correlations out of the horizon, i.e. in the region $t < t^*(r)$, will not be zero but will decay exponentially with the distance, as in the Lieb-Robinson theorem. If the initial correlations are long-range, i.e. $\xi \to \infty$ and they decay as a power law, one would expect that correlations in the region $t < t^*(r)$ decay also as a power law with the distance. In this case where the correlations behave differently inside and outside of the horizon, but do not decay exponentially outside, the Lieb-Robinson bound is not present and the horizon is “fake”. Finally, if the post-quench dispersion relation does not have a maximum group velocity, then correlations at distant points develop immediately after the quench and there is no horizon at all. This is a physical explanation of the horizon effect.

More specifically, according to the above picture a semiclassical expression for the correlation function is of the form (Fig. 1)

$$G_{sc}(x_1, x_2; t) = \int ds ds' G_{qp}(x_1, s; t)G_0(s, s')G_{qp}(x_2, s'; t)$$

$$= \int dr G_{qp}(|(x_2 - x_1 - r)/2; t)|^2G_0(r)$$

(14)

where $G_{qp}(x, x'; t)$ is the semiclassical propagator of a quasiparticle moving from $x'$ to $x$ over time $t$ after the quench with velocity $v = |x' - x|/t$ and $G_0(x, x')$ are the initial correlations between points $x$ and $x'$. The integration finally runs over only one coordinate variable $r = |s' - s|$ since the quasiparticle pairs have opposite velocities $v$ and $-v$, constraining geometrically the coordinates $s$ and $s'$. Obviously, by translational invariance each of the functions $G_{sc}, G_{qp}$ and $G_0$ depend only on the coordinate distances, apart from the time. Moreover since the propagation of quasiparticles is ballistic, $G_{qp}(x, x'; t)$ depends only on the velocity $v = |x' - x|/t$ and, if the dispersion relation has a maximum group velocity $v_{max}$, $G_{qp}(x, x'; t)$ vanishes for $|x' - x| > v_{max}t$. In this case from (14) we see that, in the region $|x_2 - x_1| > 2v_{max}t$, $G_{sc}(x_1, x_2; t)$ decays at large distances in the same way that $G_0(x_1, x_2)$ decays at large distances. In particular, if the latter decays exponentially the same holds for $G_{sc}(x_1, x_2; t)$ and therefore it exhibits a horizon in the strict sense. If instead $G_0(x_1, x_2)$ decays as a power law, then $G_{sc}(x_1, x_2; t)$ decays also as a power law and the horizon is fake. In all other cases, no horizon effect should be expected.

In the present case it is simple to see that the group velocity for the above two types of dispersion relations has the following form (Fig. 2)

$$v_1(k) = \frac{\alpha |k|^{\alpha - 1}}{2\sqrt{|k|^2 + m^2}}$$

$$v_2(k) = \frac{\alpha |k|(k^2 + m^2)^{\frac{\alpha}{2} - 1}}{2}$$

(15) (16)

From these expressions we deduce that in the massless case there is no maximum group velocity for $0 < \alpha < 2$ but instead the group velocity diverges as $k \to 0$. In the massive case, for type-1 dispersion relation, there exists a maximum group velocity only for $\alpha \geq 1$ while for $\alpha < 1$ it also diverges as $k \to 0$. For type-2 there always exists a maximum group velocity. This means that, according to the discussion above, we should expect that in the case of a type-2 massive-to-massive quench, causality holds and
the horizon effect is present, since both conditions are satisfied: the initial correlations are exponentially decaying and the evolution is governed by an upper-bounded group velocity of excitations. In the case of a type-1 massive-to-massive quench the initial correlations decay as a power law and for $\alpha \geq 1$ there is also a maximum group velocity in this type of dispersion relation, so we expect that there is a fake horizon. In all other cases, no horizon effect should be expected.

Apart from the explanation of the horizon effect, the semiclassical approximation explains also the equilibration at large times. Indeed, if $\omega(k)$ has a local minimum or, more generally, a stationary point, then the destructive interference (dephasing) of all incoherent quasiparticles that finally arrive at any point from very distant initial points, has as a result the stationary behavior of the correlations at large times, according to a simple stationary phase argument. In the present problem, $\omega(k)$ has a stationary point (more specifically a minimum at $k = 0$) only in the massive case of type-2 dispersion relation, therefore equilibration is expected to occur in this case. In the massless dispersion relation or the massive type-1, the minimum at $k = 0$ is not smooth but non-analytic, however application of the stationary phase method is still possible using a change of integration variable, so equilibration may occur also in this case. In fact, as we will see below, equilibration occurs in all massive-to-massive quenches in both types of dispersion relations, as well as in the massless case for certain ranges of values of $\alpha$.

B. Exact results

We now proceed to the study of the behavior of $G(r, t)$ given by (11), using both analytical and numerical methods. There are two distinct cases to study: the massless case $m = 0$ and the massive one $m > 0$, where $m$ is the post-quench mass parameter. These should be studied separately as they exhibit different convergence properties. In addition, we should distinguish between the two different types of dispersion relations. Firstly, we will consider the case where the dispersion relation is of type-1, in which the initial correlations are long-range, i.e. decay as a power law at large distance. Later, in subsection III B 3 we will consider the case of type-2 dispersion relation instead, thus studying also the effect of the initial state on the dynamics.

1. Massless post-quench Hamiltonian

Let us focus on the massless case $m = 0$, assuming that before the quench the system is described by the type-1 dispersion relation. The integral in $G(r)$ is written as

$$G(r) = \int \frac{d^d k}{(2\pi)^d} e^{i k \cdot r} \frac{m_0^\alpha}{|k|^\alpha (m_0^\alpha + |k|^\alpha)^2}. \quad (17)$$

and it is convergent for all values $d - 1 \leq \alpha < d$. For $\alpha < d - 1$ it has an ultraviolet (UV) divergence, which however can be ignored since it is not relevant for the macroscopic behavior of the system. By introducing a suitable UV cutoff we can regularize the integral without spoiling its (physically interesting) macroscopic behavior. On the other hand, for $\alpha > d$ the integral exhibits an infrared (IR) divergence. In one spatial dimension ($d = 1$)

![FIG. 3: (Color online) Dependence of the evolution of correlations on the parameter $\alpha$. The plots are density plots of the correlations ($|G(r, t)|$) as given by (11) in space-time, for the quench from $m_0 = 1$ to zero mass in type-1 dispersion relation and $1d$. The parameter $\alpha$ takes values $\frac{3}{2}$ (top) and $\frac{1}{2}$ (bottom). In the upper plot contours are also drawn. The insets show plots of the correlation function as a function of $t$ at some fixed distances ($r = 25$ and $r = 2$ respectively). The correlations increase with time for $\alpha = \frac{3}{2}$, while for $\alpha = \frac{1}{2}$ they do not. For $\alpha = \frac{1}{2}$ the system equilibrates, even though as we see in the inset, the time decay of the oscillations is very slow. Note that since the results are symmetric with respect to the vertical axis we plotted just the positive side of the real line.](image)
FIG. 4: (Color online) The effect of dimensionality on the evolution of correlations. The plots are density and contour plots of the correlations in space-time for the quench to zero mass in type-1 dispersion relation with $\alpha = 3/2$ and $m_0 = 1$ in 2$d$ (top) and 3$d$ (bottom). The contours become asymptotically vertical straight lines for large times, meaning that the correlations tend to stationary values. Compare with Fig. 3.

this means that there is equilibration for $\alpha < 1$, while for $\alpha > 1$ we find that $G(r, t)$ does not become stationary but increases indefinitely with time instead, as it can be easily seen also numerically. In two spatial dimensions ($d = 2$) the above constraint means that there is always equilibration for $\alpha < 2$ while for the short-range case $\alpha = 2$ it is known that $G(r, t)$ increases logarithmically with time [25]. Lastly, in three spatial dimensions ($d = 3$), $G(r, t)$ equilibrates for any value of $\alpha$ (as already explained, the case $\alpha > 2$ is not physically interesting in any dimension).

In the cases where equilibration occurs, the approach of the correlation function $G(r, t)$ to its stationary large-time value $\bar{G}(r)$, is given by the decay of the time dependent part $\tilde{G}(r, t)$ for large times $t$, with the distance $r$ kept fixed. The asymptotic behavior of this integral is

$$\tilde{G}(r, t \to \infty) \sim \frac{1}{t^{2(d/\alpha) - 2}}$$ (18)

i.e. a power law with an exponent dependent both on $d$ and $\alpha$ (the asymptotic behavior of such integrals, as well as of the ones below, can be derived by suitable choice of the integration variable and application of the stationary phase method or the steepest descent method which in most cases reduces to a simple Wick rotation of the integration variable in the complex plane). On the other hand, the large-distance behavior of the stationary correlations $\bar{G}(r)$ turns out to be

$$\bar{G}(r \to \infty) \sim \frac{1}{r^{d-\alpha}}, \quad \alpha < d.$$

We have checked the behavior of the system regarding equilibration and we will now check whether it exhibits a horizon. As mentioned in the introduction the Lieb-Robinson causal region is characterized by the exponential way in which the correlations decay outside the causal region. The presence of horizon can be checked by studying the behavior of $G(r, t)$ keeping $t$ fixed and taking the limit $r \to \infty$. We find that the decay at large distance is given by

$$G(r \to \infty, t) \sim \frac{1}{r^{d+\alpha}}.$$ (20)

Since this decay is not exponential, there is no horizon. Fig. 3 and 4 show typical plots of the correlations in 1, 2 and 3 spatial dimensions.

The above results are in agreement with the semiclassical approximation. Indeed, the absence of causality and the associated horizon effect can be attributed to the fact that the group velocity is unbounded for all values of $\alpha$ in the massless case. On the other hand, since the dispersion relation has no stationary point but a non-analytic minimum at $k = 0$, naively we would not expect equilibration at large times. However a simple change of variables allows us to apply the stationary phase method from which we find that equilibration does occur.

2. Massive post-quench Hamiltonian

Let us switch to the massive case $m > 0$. In this case there are no IR divergences in $\bar{G}(r)$ and so $\tilde{G}(r, t)$ decays always to zero for large $t$ and the system equilibrates. The small $k$ behavior of the type-1 dispersion relations is

$$\omega_1(k) \approx m^{\alpha/2} + \frac{1}{2} \frac{k^\alpha}{m^{\alpha/2}} + ...$$ (21)

and the stationary phase method gives

$$\tilde{G}(r, t \to \infty) \sim t^{-d/\alpha} \cos(2m^{\alpha/2}t),$$ (22)
The parameter $\alpha$ is $\frac{3}{2}$ (top) and $\frac{1}{2}$ (bottom) with $m_0 = 2$ and $m = 1$. Top: The evolution seems to exhibit a horizon, however this is not true horizon since the decay of correlations with the distance turns out to be a power law. The dashed green line indicates the slope that corresponds to the maximum group velocity $v_{\text{max}} = \sqrt{3}/2^{2/3} \approx 0.5456$ for this particular value of $\alpha$, as derived from Eq. (15). Bottom: There is no sign of horizon in this case. Even if this is not evident from the density plots due to the limited temporal range, the system equilibrates in both cases.

The large distance decay of the stationary correlations $\bar{G}(r)$ is given by a power law

$$\bar{G}(r \to \infty) \sim \frac{1}{r^{d+\alpha}}.$$  \hfill (23)

due to the branch cut of the integrand in the complex $k$-plane, which starts from the origin $k = 0$. The same asymptotic behavior is valid for large distances at any fixed time $\bar{G}(r \to \infty, t)$.

As far as the horizon effect is concerned, we find that for $\alpha < 1$ correlations start developing even at arbitrarily long distances soon after the quench, therefore no horizon exists, while for $\alpha > 1$ they seem to form a horizon, however their decay outside of it is power law therefore even in this case no true Lieb-Robinson horizon occurs. This is demonstrated in Fig. 5 where we show plots for the values $\alpha = \frac{3}{2}$ and $\alpha = \frac{1}{2}$. As before, the semiclassical approximation can explain this behavior. According to the discussion of section IIIA since the initial correlations for the type-1 dispersion relation are long-range, i.e. decay as a power-law with the distance, the same is true for the evolution of correlations after the quench and a true horizon effect is not possible. However, since the group velocity has a maximum for $\alpha \geq 1$, a generalized horizon with power-law decaying correlations outside of it is present in this case. In contrast, for $\alpha < 1$ the group velocity is unbounded and there is no horizon at all.

3. The effect of the initial state

The same analysis can be repeated in the case where the dispersion relation is of type-2. According to (8) the initial correlations are then short-range, i.e. they decay exponentially with the distance.

In the massive-to-massless case, as in the type-1 case of subsection III B 1, equilibration occurs only for $\alpha < d$ since the integral in $\bar{G}(r)$ is then finite, while otherwise it is IR divergent. Furthermore, there is no horizon effect in this case either, as we would also expect following the semiclassical analysis. Fig. 6 shows a typical plot of such
branch cuts that start from \( k \) the integrand in \( G \) state correlations under the type-2 dispersion relation) decaying correlations outside of it. The reason for this, this is a true horizon effect i.e. with exponentially \( \text{exponentially decaying with distance, which according to the}\) horizon based on contour lines. From the slope of this line \( \text{with linear horizon. The dashed red line is an estimate of the}\) horizon based on contour lines. From the slope of this line \( \text{with linear horizon. The dashed red line is an estimate of the}\) horizon based on contour lines. From the slope of this line \( \text{with linear horizon. The dashed red line is an estimate of the}\) horizon based on contour lines. From the slope of this line \( \approx 0.14 \) which is in perfect agreement with the exact \( \text{value} \, v_{\text{max}} = 3^{3/8} / 7^{1/8} / 2 \approx 0.1375 \) as derived from Eq. (16).

In addition the stationary correlations at large times are
\[
\tilde{G}(r \to \infty) \sim r^{-d/2} \cos(2m^0/2t). \tag{24}
\]
In addition the stationary correlations at large times are
\[
\tilde{G}(r \to \infty) \sim r^{-d/2} \cos(2m^0/2t). \tag{25}
\]
where in the last case we assumed (as is typically done) that \( m_0 > m \).

Lastly, in type-2 massive-to-massive quench the horizon effect is prominent (Fig. 7, as in the type-1 case. However it is worth to mention that, in contrast to type-1, this is a true horizon effect i.e. with exponentially decaying correlations outside of it. The reason for this exponential decay is that (as in the case of the ground state correlations under the type-2 dispersion relation) the integrand in \( G(r,t) \) exhibits, in the complex \( k \)-plane, branch cuts that start from \( k = \pm im \) and \( k = \pm im_0 \) and since \( m_0 > m \) the closest to the origin are the former. Once again, this behavior can be explained by the semiclassical approximation. Indeed, in massive type-2 dispersion relation there always exists a maximum group velocity and, in addition, the initial correlations decay exponentially with the distance, which according to the semiclassical approximation are the conditions for a true Lieb-Robinson horizon. This is demonstrated in Fig. 7, where the slope of the horizon is accurately predicted from the maximum group velocity of the post-quench dispersion relation.

\section{IV. LONG TIME LIMIT AND GGE}

\subsection{A. Validity of the GGE}

Having shown that, under specified conditions, the long-range systems we consider equilibrate for large times, we will now check whether the asymptotic stationary behavior can be described by the GGE. As mentioned in the introduction, the GGE takes into account all integrals of motion, which in practice means that the GGE density matrix can be written as
\[
\rho_{\text{GGE}} = Z^{-1} e^{-\sum \lambda_k I_k}, \quad Z = \text{Tr} \, e^{-\sum \lambda_k I_k}, \tag{26}
\]
where the \( I_k \)'s are a set of commuting and independent integrals of motions and the \( \lambda_k \)'s are the associated Lagrange multipliers, fixed by the self-consistency requirement that the values of \( I_k \) are the same in the initial state and in the ensemble
\[
\langle I_k \rangle_{t=0} = \langle I_k \rangle_{\text{GGE}}. \tag{27}
\]
A natural choice of integrals of motions that one can consider in the present problem is the number of particles with a certain momentum, i.e. \( n_k = a_k^\dagger a_k \), where
\[
a_k^\dagger = \frac{1}{\sqrt{2\omega(k)}} (\omega(k) \phi_+ - i\pi_-), \quad a_k = \frac{1}{\sqrt{2\omega(k)}} (\omega(k) \phi_- + i\pi_+). \tag{28}
\]
are the usual creation and annihilation operators. With this choice of integrals of motion, it is easy to see that the large-\( t \) stationary part of the correlations \( \tilde{G}(r,t \to \infty) = \tilde{G}(r) \) is automatically equal to the GGE prediction [12]. Indeed \( \tilde{G}(r) \) turns out to be directly related to the initial momentum density \( \langle n_k \rangle_{t=0} \): it is essentially its Fourier transform. On the other hand, the GGE prediction for the two-point correlator is the Fourier transform of the momentum density in the GGE \( \langle n_k \rangle_{\text{GGE}} \). Therefore by the very definition of the GGE that requires \( \langle n_k \rangle_{\text{GGE}} = \langle n_k \rangle_0 \), we see that, as long as the correlation function equilibrates (i.e. \( \tilde{G}(r,t \to \infty) \to 0 \)) its stationary value \( \tilde{G}(r) \) is automatically equal to the GGE prediction.

The explicit values of the Lagrange multipliers \( \lambda(k) \) are given by [15]
\[
\lambda(k) = 2 \log \left| \frac{\omega_0(k) + \omega(k)}{\omega_0(k) - \omega(k)} \right|. \tag{29}
\]
B. Non-locality of the GGE

We have just shown that the large time behavior of the 2-point correlation function is described by the GGE \[^{[20]}\] where the Lagrange multipliers are given by \(^{[29]}\).

The general belief is that the integrals of motion used in the GGE should be local conserved charges, meaning integrals over the whole coordinate space of local density operators (see \[^{[31]}\] and references therein). In fact all quantum systems, even non-integrable ones, possess infinite integrals of motion: the projections onto their eigenstates, which, ignoring the case of degeneracies, contain all information about the initial state. An ensemble based on such projections onto all post-quench eigenstates is called the diagonal ensemble and therefore describes correctly (but in a trivial way) the time average of any local observable (or its large time value, provided it equilibrates). However, a statistical ensemble is useful only if it is economic, in the sense that it does not require all information about the initial state. Integrable models are special in that they possess a set of local integrals of motion which are much less in number than the dimension of the Hilbert space (their number grows only linearly with the system size) and are sufficient for their exact solvability. Therefore it is sensible to assume that they are also sufficient for the description of their large time behavior. Of course the above argument works equally well for any system with a minimal set of integrals of motion that is smaller in number than the dimension of the Hilbert space but sufficient for its exact solvability, even if they are not local. The locality requirement is usually introduced so that the GGE reduced density matrix corresponding to any subsystem can be constructed out of extensive quantities that therefore depend only on the size of that subsystem. This is because in standard thermodynamics we would expect the “generalized free energy” \(F_{\text{GGE}} = -\log Z[\rho_{\text{GGE}}]\) to be extensive on the subsystem’s size. (In fact this is a stronger requirement than the condition that \(F_{\text{GGE}}\) can be written as a sum over all local integrals of motion, since such an infinite sum of local quantities is not necessarily local or extensive itself.)

However, if the Hamiltonian of the system is itself non-local, then it may not be possible to find a set of local integrals of motion to construct the GGE. In such a case, according to the above discussion, if a GGE based on a minimal set of non-local integrals of motion (less in number than the dimension of the Hilbert space) describes correctly the system’s stationary behavior, it is also an economic ensemble, because it depends only on partial information about the initial state. For example, it was recently pointed out that this is the case in the harmonic Calogero model \[^{[31]}\].

In the form \(^{[20]}\) the GGE does not seem to satisfy the locality requirement, since the conserved quantities from which it is constructed are the momentum occupation number operators \(n(k)\) which are not local but global operators. However it may still be possible to rewrite them in a local form. One can easily see that in the earlier studied short-range case \(\alpha = 2\), the generalized free energy \(F_{\text{GGE}} = \int dk \lambda(k)n(k)\) can be cast as an infinite sum of local quantities. To see this, it is sufficient to write \(n(k)\) in terms of local fields using its definition

\[
n(k) = a_k^\dagger a_k = \frac{1}{2} (\omega(k)\phi_{-k}\phi_k + i\phi_{-k}\pi_k - i\pi_{-k}\phi_k + \pi_{-k}\pi_k/\omega(k)) = \frac{1}{2} \int dxdy e^{i\lambda(x-y)}[\omega(k)\phi(x)\phi(y) + i(\phi(x)\pi(y) - \pi(x)\phi(y)) + \pi(x)\pi(y)/\omega(k)]
\]

and substitute in the generalized free energy

\[
F_{\text{GGE}} = \frac{1}{2} \int dxdy [D_1(x-y)\phi(x)\phi(y) + iD_2(x-y) (\phi(x)\pi(y) - \pi(x)\phi(y)) + D_3(x-y)\pi(x)\pi(y)],
\]

where

\[
D_1(s) = \int dk e^{iks}\lambda(k)\omega(k),
D_2(s) = \int dk e^{iks}\lambda(k),
D_3(s) = \int dk e^{iks}\lambda(k)/\omega(k).
\]

Using the commutation relation \([\phi(x),\pi(y)]=i\delta(x-y)\) and the fact that \(D_2(s)\) is an even function (since \(\lambda(k)\) is such too) we can further write \(^{[31]}\) as

\[
F_{\text{GGE}} = \frac{1}{2} \int dxdy [D_1(x-y)\phi(x)\phi(y) + D_3(x-y)\pi(x)\pi(y)] - \pi\lambda(0)
\]

The last expression is a double instead of a single spatial integral. However, if both \(\omega(k)\) and \(\lambda(k)\) can be Taylor expanded in terms of \(k = 0\), then each of \(D_1(s), D_2(s)\) and \(D_3(s)\) can be expressed by means of differential operators so that the whole expression is a single spatial integral. In fact it is sufficient that \(\omega(k)\) and \(\omega_0(k)\) are Taylor expandable because \(\lambda(k)\), being a function of these two, is Taylor expandable whenever each of them is such. For example, in this case \(D_1(s)\) reads

\[
D_1(s) = \sum_{n=0}^{\infty} c_n \int dk k^n e^{iks} = \left[ \sum_{n=0}^{\infty} c_n \left(-i\frac{d}{ds}\right)^n \right] \int dk e^{iks} = 2\pi \hat{D}_s^{(1)} \delta(s),
\]

where

\[
\hat{D}_s^{(1)} = \sum_{n=0}^{\infty} c_n \left(-i\frac{d}{dx}\right)^n
\]
where in the last line we used the equations of motion $\dot{\phi}_k = -\omega^2(k)\phi_k$. Therefore

$$\pi_k\pi_{-k} = \frac{1}{2} \frac{d^2}{dt^2} (\phi_k\phi_{-k}) + \omega^2(k)\phi_k\phi_{-k}$$  \hspace{1cm} (39)$$

From the above relation, analogously to the expression for the $\phi$ correlation function Eq. (11), we can find that the two-point correlation function of $\pi$ is given by

$$F(r,t) \equiv \langle \pi(r,t)\pi(0,t) \rangle - \langle \pi(r,0)\pi(0,0) \rangle = \int \frac{d^d k}{(2\pi)^d} e^{i k r} \frac{\omega^2(k) - \omega^2(0)}{\omega_0(k)} (\cos(2\omega(k)t) - 1).$$  \hspace{1cm} (40)$$

The large time behavior of the last expression can be studied in the same way as that of (11): it always becomes stationary, since for $m \neq 0$ its time independent part never exhibits IR divergences (only physically irrelevant UV divergences). Therefore we conclude that the $\pi$ correlations equilibrate for any values of the parameter $\alpha$ and of the masses.

VI. THE EFFECT OF INTERACTIONS

An interesting question is how our conclusions would be modified by the presence of interactions, i.e. if the Hamiltonian that describes the system after the quench was not quadratic but included anharmonic terms as

$$H_\alpha = \frac{1}{2} \sum_r \pi^2(r) + \frac{1}{2} \sum_{r,r'} \phi(r) \left( \frac{1}{|r-r'|^{d+\alpha}} + m^\alpha \delta_{r,r'} \right) \phi(r')$$

$$+ \frac{\lambda}{4!} \sum_r \phi^4(r).$$  \hspace{1cm} (41)$$

As far as equilibration is concerned, an answer to the question posed at the beginning of the section can be induced by earlier studies based on self-consistent perturbation theory. In [25] it was shown that in the short-range case, the leading order effect of the interactions as given by a self-consistent approximation (Hartree-Fock or ‘loop’ correction), is to shift the value of the mass to a higher effective value. Due to this shifting effect, equilibration is expected to remain unaffected by the inclusion of interactions, but also to extend to the massless case, since the zero mass would now be shifted to a positive effective value, allowing the system to equilibrate. This conclusion is expected to be correct in short-range systems at least in 2d and 3d where perturbation theory works fine also at equilibrium. Following this reasoning we could argue that the same is true also in the long-range case. Based on the self-consistent approach of [25] one finds that the effective mass at large times satisfies the equation

$$m^{\ast \alpha} = m^\alpha + \frac{\lambda}{2} \int \frac{d^d k}{(2\pi)^d} \left( \frac{(\omega_0(k) - \omega^*(k))^2}{4\omega_0(k)\omega^2(k)} + \frac{\omega(k) - \omega^*(k)}{2\omega(k)\omega^*(k)} \right).$$  \hspace{1cm} (42)$$
where $\omega^*(k) = \sqrt{|k|^\alpha + m^{*\alpha}}$. In principle, since in $d = 2$ and $3$ the system equilibrates even when we do not have interactions, based on the above argument one can simply conclude that this will be the case also when we do have interactions. The interesting case is the massless case in $d = 1$ with $1 \leq \alpha < 2$ where, as we saw, in the absence of interactions there is no equilibration. After making the integrals in the equation (42) dimensionless we have

$$m^*\alpha = m^\alpha + \frac{\lambda}{2\pi} \left[ m^0_0 (1 - \alpha/2) f_\alpha \left( \frac{m^*}{m_0} \right) + m^1_{-\alpha/2} h_\alpha \left( \frac{m}{m^*} \right) \right],$$

where

$$f_\alpha(s) \equiv \int_0^\infty dk \frac{(\sqrt{k^\alpha + 1} - \sqrt{k^\alpha + s^\alpha})^2}{4\sqrt{k^\alpha + 1}(k^\alpha + s^\alpha)}$$

(44)

$$h_\alpha(s) \equiv \int_0^\infty dk \frac{\sqrt{k^\alpha + s^\alpha} - \sqrt{k^\alpha + 1}}{2\sqrt{k^\alpha + s^\alpha}\sqrt{k^\alpha + 1}}\left(1 - s^{1-\alpha/2}\right) \Gamma(1/\alpha) \Gamma(1/2 - 1/\alpha)/(2\sqrt{\pi}\alpha).$$

Solving numerically the equation (43) for $m = 0$ and $1 \leq \alpha < 2$ we find that the solution $m^*$ is real and positive (non-zero). The results are shown in Fig. 8 where the solutions $m^*$ are plotted as functions of $\alpha$ for various values of $\lambda$. This is a remarkable result because it actually tells us that in $1d$, if we quench the system to zero mass and $1 \leq \alpha < 2$, according to the above approximation the interactions will produce a non-zero effective mass $m^*$, which will lead the system to equilibration. This is in contrast to what happens in the absence of interactions, where the system does not equilibrate for these parameter values.

![FIG. 8: Plots of the effective mass $m^*$ for $m = 0$ as a function of $\alpha$ in the range from 1 to 2, for various values of $\lambda$ (0.001 to 100) in units of $m_0 = 1$. The results are obtained by numerical solution of the self-consistency equation (43). The solutions are real positive. Note also that the curves tend to zero in the short-range limit $\alpha = 2$ with the same slope for all $\lambda$.](image)

In the above calculation, we did not include explicitly in the Hamiltonian (41) a short-range kinetic term, i.e. $g(\nabla^2 \phi(r))^2$, which appears anyway when one studies long-range spin systems [52]. In the last forty years different versions of renormalization group have been applied to the field theory $H_g = H_\alpha + g(\nabla^2 \phi(r))^2$ with different contradicting conclusions [52,54]. The main difference between these papers is that the short-range regime of the Ising model starts from $\alpha = 2$ or $\alpha = 2 - \eta$, where $\eta$ is the anomalous dimension of spin operator (in $d = 1$ it is equal to $\frac{1}{2}$). What all the studies agree on is that for $\alpha < 2 - \eta$ one can ignore the short-range kinetic part i.e. just set $g = 0$ and do the renormalization group analysis [55]. In principle it is not obvious that this line of reasoning can be applied also in the case of the quench problem. In non-equilibrium systems one would have to use an argument from boundary renormalization group in a slab geometry [25] which in the case of long-range systems is completely unexplored in the literature. This is mainly because in the presence of long-range systems it is very difficult to control the boundary conditions. For this reason we are unable to make any concrete prediction regarding the equilibration properties of the Hamiltonian $H_g$. It is worth mentioning that the quench problem in the case of long-range interacting Ising model is even more complicated because in that case one can not even rely on the Hamiltonian $H_g$.

To summarize this section, we found that equilibration happens also in the case of interacting long-range evolution in all dimensions and for all the values of $m$ and $\alpha$. In fact we saw that it happens even in the cases where it does not happen in the non-interacting case. Our approximation on which we based these results, is to map the interacting non-integrable model to a free and therefore integrable system, with an interaction dependent effective mass that is determined self-consistently. In this approximation the equilibration is described by the GGE with the mass replaced by its effective value. However, since the original interacting field theory that we considered is non-integrable, one would expect that after the emergence of this quasi-stationary state at intermediate times (prethermalization), the interactions will slowly transfer energy from one momentum mode to the other, so that the different effective temperatures of each mode will eventually converge to the same value and true thermalization will take place. The validity of this scenario remains under investigation [57,58,59] with a few recent studies having observed prethermalization also in interacting long-range systems [61,65].

**VII. CONCLUSIONS**

In this paper we studied the time evolution of correlations after a quantum quench of the mass parameter in long-range free field theory for different values of the power of the couplings $\alpha$ and for two different types of dispersion relations, in one, two and three spatial dimensions. We classified the parameter regimes in which the system equilibrates and those in which it exhibits a hori-
the effect of the interactions and, following earlier work, quantum quench problems. Furthermore, we considered power-law, the horizon turns from fake to true. These effects remain unaffected by this change, however when the initial correlations decay exponentially, as is the case of the Lieb-Robinson bound.

We also studied the effect of the initial state in the dynamics of the correlations, more specifically of the type of the dispersion relation which affects the decay of the initial correlations with the distance. We showed that both the presence of equilibration and horizon effect remain unaffected by this change, however when the initial correlations decay exponentially instead of like a power-law, the horizon turns from fake to true. These results can be successfully explained by the semiclassical approach, which in general describes many aspects of quantum quench problems. Furthermore, we considered the effect of the interactions and, following earlier work, we concluded that the presence of interactions in either the pre-quench or the post-quench Hamiltonian does not spoil but rather enhances the equilibration properties of the system.

Finally we showed that despite the fact that the integrals of motions in our system are non-local, in the regimes where equilibration occurs, one can still describe the long-time stationary values by a GGE.

Acknowledgments

MAR’s work was partially supported by FAPESP. SS’s work was supported by the ERC under Starting Grant 279391 EDEQS. This work was initiated during MAR’s visit to the Physics Department of the University of Pisa using the European fellowship IRSES Grant QICFT. We thank also Pasquale Calabrese, Mario Collura and Thiago Rodrigues de Oliveira for fruitful comments.

[1] K. Kim, M.-S. Chang, R. Islam, S. Korenblit, L.-M. Duan, and C. Monroe Phys. Rev. Lett. 103, 120502 (2009); J. W. Britton, B. C. Sawyer, A. C. Keith, C.-C. Joseph Wang, J. K. Freericks, H. Uys, M. J. Biercuk and J. J. Bollinger Nature 484, 489 (2012); R. Islam, C. Senko, W. C. Campbell, S. Korenblit, J. Smith, A. Lee, E. E. Edwards, C.-C. J. Wang, J. K. Freericks, C. Monroe, Science 340, 583 (2013).

[2] P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempel, P. Zoller, R. Blatt, C. F. Roos, Nature 511, 202205 (2014); P. Richerme, Zhe-Xuan Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakis, A. V. Gorshkov, C. Monroe, Nature 511, 198201 (2014).

[3] M. Kastner, Phys. Rev. Lett. 104, 240403 (2010) Phys. Rev. Lett. 106, 130601 (2011); M. Maik, P. Hauke, O. Dutta, J. Zakrzewski, and M. Lewenstein, New J. Phys. 14, 113006 (2012), D. Peter, S. Muller, S. Wessel, H. P. Buchler, Phys. Rev. Lett. 109, 025303 (2012) and Zhe-Xuan Gong, M. Foss-Feig, S. Michalakis, A. V. Gorshkov, Phys. Rev. Lett. 113, 030602 (2014).

[4] T. Koffel, M. Lewenstein, and L. Tagliacozzo, Phys. Rev. Lett. 109, 267203 (2012).

[5] M. Ghasemi Nezhadhaghighi, M. A. Rajabpour, EPL, 100 (2012) 60011; M. Ghasemi Nezhadhaghighi, M. A. Rajabpour, Phys. Rev. B 88, 045426 (2013).

[6] P. Hauke and L. Tagliacozzo, Phys. Rev. Lett. 111, 207202; J. Schachenmayer, B. P. Lanyon, C. F. Roos, and A. J. Daley, Phys. Rev. X 3, 031015 (2013) and M. Ghasemi Nezhadhaghighi, M. A. Rajabpour, arXiv:1408.3714.

[7] M. Kastner, Phys. Rev. Lett. 106, 130601 (2011); Z.-X. Gong and L. M. Duan, New J. Phys. 15, 113051 (2013) and N. Nesi, A. Iucci, and M. A. Cazalilla, arXiv:1401.1986.

[8] J. Eisert, M. van den Worm, S. R. Manmana, M. Kastner, Phys. Rev. Lett. 111, 260401 (2013).

[9] M. Kastner, M. van den Worm arXiv:1403.4187 and K. R. A. Hazzard, M. van den Worm, M. Foss-Feig, S. R. Manmana, E. Dalla Torre, T. Pfau, M. Kastner, A. M. Rey. arXiv:1406.0937.

[10] M. B. Hastings and T. Koma, Commun. Math. Phys. 265, 781 (2006) and B. Nachtergaele, Y. Ogata, and R. Sims, J. Stat. Phys. 124, 1 (2006).

[11] E.H. Lieb and D.W. Robinson, Commun. Math. Phys. 28 (1972), 251-257.

[12] M. Tezuka, A. M. García-García, M. A. Cazalilla, arXiv:1403.1739.

[13] A. Dutta and J. K. Bhattacharjee, Phys. Rev. B 64, 184106 (2001).

[14] P. Calabrese and J. Cardy, Phys. Rev. Lett. 96, 136801 (2006).

[15] P. Calabrese and J. Cardy, J. Stat. Mech. (2007) P06008.

[16] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. 98 (2007) 050405.

[17] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, Phys. Rev. Lett. 100, 030602 (2008).

[18] M. Cramer and J. Eisert, New J. Phys. 12, 055020 (2010).

[19] T. Barthel and U. Schollwoeck, Phys. Rev. Lett. 100, 100601 (2008).

[20] M. Rigol, Phys. Rev. Lett. 103, 100403 (2009).

[21] M. Rigol, Phys. Rev. A 80, 053607 (2009).

[22] M. A. Cazalilla, Phys. Rev. Lett. 97, 156403 (2006); A. Iucci, and M. A. Cazalilla, Phys. Rev. A 80, 063619 (2009); A. Iucci, and M. A. Cazalilla, New J. Phys. 12, 055019 (2010).

[23] M. A. Cazalilla, A. Iucci, and M.-C. Chung, Phys. Rev. E 85, 011133 (2012).

[24] P. Calabrese, F.H.L. Essler and M. Fagotti, Phys. Rev. Lett. 106, 227203 (2011); P. Calabrese, F.H.L. Essler and M. Fagotti, J. Stat. Mech. P07016 (2012); P. Calabrese, F.H.L. Essler and M. Fagotti, J. Stat. Mech. P07022 (2012).

[25] S. Sotiriadis, J. Cardy, Phys. Rev. B 81, 134305 (2010).

[26] B. Bertini, D. Schuricht, F. H. L. Essler, arXiv:1405.4813.

[27] D. Ficazzola and G. Mussardo, New J. Phys. 12, 055015 (2010); B. Pozsgay, J. Stat. Mech. P01011 (2011); G. Mussardo, Phys. Rev. Lett. 111, 100401 (2013).

[28] F. H. L. Essler, S. Evangelisti, and M. Fagotti, Phys. Rev. Lett. 109, 247206 (2012).

[29] D. Schuricht and F. H. L. Essler, J. Stat. Mech. P04017 (2012).
[30] M. Rigol and M. Srednicki, Phys. Rev. Lett. 108, 110601 (2012).
[31] M. Fagotti and P. Calabrese, Phys. Rev. A 78, 010306(R) (2008).
[32] P. Calabrese and J. Cardy, J. Stat. Mech. P04010 (2005).
[33] H. Rieger and F. Igloi, Phys. Rev. B 84, 165117 (2011); F. Igloi and H. Rieger, Phys. Rev. Lett. 85, 3233 (2000).
[34] S. Evangelisti, J. Stat. Mech. P04003 (2013).
[35] L.F. Santos, A. Polkovnikov, M. Rigol, Phys. Rev. Lett. 107, 040601 (2011).
[36] M. Rigol, Phys. Rev. Lett. 112, 110601 (2014).
[37] M. Collura and D. Karevski, Phys. Rev. B 89, 214308.
[38] L. Bucciantini, M. Kormos, P. Calabrese, J. Phys. A: Math. Theor. 47, 175002 (2014).
[39] M. Collura, S. Sotiriadis and P. Calabrese, Phys. Rev. Lett. 110, 245301 (2013).
[40] M. Kormos, M. Collura and P. Calabrese, Phys. Rev. A 89, 033609 (2014).
[41] M. A. Rajabpour and S. Sotiriadis, Phys. Rev. A 89, 033620 (2014).
[42] S. Sotiriadis and P. Calabrese, J. Stat. Mech. P04010 (2005).
[43] H. Rieger and F. Igloi, Phys. Rev. B 84, 165117 (2011); F. Igloi and H. Rieger, Phys. Rev. Lett. 85, 3233 (2000).
[44] S. Evangelisti, J. Stat. Mech. P04003 (2013).
[45] M. Kormos, A. Shashi, Y.-Z. Chou, J.-S. Caux, A. Imambeckov, Phys. Rev. B 88, 205131 (2013); J. De Nardis, B. Wouters, M. Brockmann, J. S. Caux, Phys. Rev. A 89, 033601 (2014).
[46] J.-S. Caux and F. H. L. Essler, Phys. Rev. Lett. 110, 257203 (2013).
[47] S. Sotiriadis, G. Takacs, and G. Mussardo, Phys. Lett. B 734 (2014).
[48] M. Fagotti, M. Collura, F. H. L. Essler, and P. Calabrese, Phys. Rev. B 89, 125101 (2014).
[49] B. Wouters, J. De Nardis, M. Brockmann, D. Fioretto, M. Rigol, J.-S. Caux, Phys. Rev. Lett. 113, 117202 (2014); B. Pozsgay, M. Mestyan, M. A. Werner, M. Kormos, G. Zarakand, and G. Tukaes, Phys. Rev. Lett. 113, 117203 (2014).
[50] M. Fagotti, J. Stat. Mech. P03016 (2014).
[51] M. Fagotti, F. H..L. Essler, Phys. Rev. B 87, 245107 (2013).
[52] M. E. Fisher, S. K. Ma and B. G. Nickel, Phys. Rev. Lett. 29, 917 (1972).
[53] J. Sak, Phys. Rev. B 8, 281 (1973); E. Luijten, H. W. J. Blote, Phys. Rev. Lett. 89, 025703 (2002), M. C. Angelini, G. Parisi, F. Ricci-Tersenghi, Phys. Rev. E 89, 062120 (2014).
[54] T. Blanchard, M. Picco, M. A. Rajabpour, EPL 101, 66003 (2013).
[55] The basic idea of [53] is that in the presence of interaction the $\phi$ operator has an anomalous dimension $\eta$ at $\alpha = 2$. If we now turn on the long-range kinetic term this term will be irrelevant up to $\alpha = 2 - \eta$. This reasoning is in contradiction with the numerical results in [56] and analytical results in [54].
[56] M. Picco, [arXiv:1207.1018].
[57] G. Biroli, C. Kollath, and A.M. Lauchli, Phys. Rev. Lett. 105, 250401 (2010).
[58] S. R. Manmana, S. Wessel, R. M. Noack, and A. Muramatsu, Phys. Rev. Lett. 98, 210405 (2007); Phys. Rev. B 79, 155104 (2009); Phys. Rev. B 80, 245107 (2013); Phys. Rev. B 82, 155104 (2010).
[59] M. Moeckel and S. Kehrein, Phys. Rev. Lett. 100, 175702 (2008); Phys. Rev. Lett. 104, 187201 (2010).
[60] M. Kollar, F. A. Wolf, and M. Eckstein, Phys. Rev. B 84, 054304 (2011).
[61] M. Rigol, Phys. Rev. Lett. 103, 100403 (2009); M. Rigol, Phys. Rev. A 80, 053607 (2009); L. F. Santos and M. Rigol, Phys. Rev. E 81, 036206 (2010); L. F. Santos and M. Rigol, Phys. Rev. E 82, 031130 (2010).
[62] G. Brandino, J.-S. Caux, and R. M. Konik, [arXiv:1301.0308].
[63] F.H.L. Essler, S. Kehrein, S.R. Manmana, and N.J. Robinson, Phys. Rev. B 89, 165104 (2014).
[64] M. Marcuzzi, J. Marino, A. Gambassi, and A. Silva, Phys. Rev. Lett. 111, 197203 (2013).
[65] M. van den Worm, B. C. Sawyer, J. J. Bollinger, and M. Kastner, New J. Phys. 15, 083007 (2013).