RINGS OF INVARIANTS FOR MODULAR REPRESENTATIONS OF ELEMENTARY ABELIAN $p$-GROUPS

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Abstract. We initiate a study of the rings of invariants of modular representations of elementary abelian $p$-groups. With a few notable exceptions, the modular representation theory of an elementary abelian $p$-group is wild. However, for a given dimension, it is possible to parameterise the representations. We describe parameterisations for modular representations of dimension two and of dimension three. We compute the ring of invariants for all two-dimensional representations; these rings are generated by two algebraically independent elements. We compute the ring of invariants of the symmetric square of a two-dimensional representation; these rings are hypersurfaces. We compute the ring of invariants for all three-dimensional representations of rank at most three; these rings are complete intersections with embedding dimension at most five. We conjecture that the ring of invariants for any three-dimensional representation of an elementary abelian $p$-group is a complete intersection.

Introduction

We initiate a study of the rings of invariants of modular representations of elementary abelian $p$-groups. With a few notable exceptions, the modular representation theory of an elementary abelian $p$-group is wild; see, for example, [B1, §4.4]. However, for a given dimension, it is possible to parameterise the representations. We describe parameterisations for modular representations of dimension two and
of dimension three. We compute the ring of invariants for all two-dimensional representations; these rings are generated by two algebraically independent elements. We compute the ring of invariants of the symmetric square of a two-dimensional representation; these rings are hypersurfaces. We compute the ring of invariants for all three-dimensional representations of rank at most three; these rings are complete intersections with embedding dimension at most five. We conjecture that the ring of invariants for a three-dimensional representation of rank \( r \) is a complete intersection with embedding dimension at most \( \lceil r/2 \rceil + 3 \).

Let \( V \) denote an \( n \)-dimensional representation of a group \( G \), over a field \( \mathbf{F} \) of characteristic \( p \), for a prime number \( p \). We will usually assume that \( G \) is finite and that \( p \) divides the order of \( G \); in other words, that \( V \) is a modular representation of \( G \). We view \( V \) as a left module over the group ring \( \mathbf{F}G \) and the dual, \( V^* \), as a right \( \mathbf{F}G \)-module. Let \( \mathbf{F}[V] \) denote the symmetric algebra on \( V^* \). The action of \( G \) on \( V^* \) extends to an action by degree-preserving algebra automorphisms on \( \mathbf{F}[V] \). Choosing a basis \( \{x_1, x_2, \ldots, x_n\} \) allows us to identify \( \mathbf{F}[V] \) with the algebra of polynomials \( \mathbf{F}[x_1, x_2, \ldots, x_n] \). Our convention that \( \mathbf{F}[V] \) is a right \( \mathbf{F}G \)-module is consistent with the convention used by the invariant theory package in the computer algebra software Magma [BCP]. The ring of invariants, \( \mathbf{F}[V]^G \), is the subring of \( \mathbf{F}[V] \) consisting of those polynomials fixed by the action of \( G \). Note that elements of \( \mathbf{F}[V] \) represent polynomial functions on \( V \) and that elements of \( \mathbf{F}[V]^G \) represent polynomial functions on the set of orbits \( V/G \). For \( G \) finite and \( \mathbf{F} \) algebraically closed, \( \mathbf{F}[V]^G \) is the ring of regular functions on the categorical quotient \( V//G \). For background material on the invariant theory of finite groups, see [B2], [CW], [DK], or [NS].

Choosing a basis for \( V \) (or \( V^* \)) determines a group homomorphism \( \rho : G \to \text{GL}_n(\mathbf{F}) \). Conversely, a group homomorphism \( \rho \) can be used to define a right \( \mathbf{F}G \)-module structure on \( \mathbf{F}^n \). Our philosophy is to use the set of group homomorphisms \( \text{hom}(G, \text{GL}_n(\mathbf{F})) \) to parameterise representations of \( G \) over \( \mathbf{F} \) of dimension \( n \). From this point of view, every group homomorphism determines a subring \( \mathbf{F}[x_1, \ldots, x_n]^{\rho(G)} \subseteq \mathbf{F}[x_1, \ldots, x_n] \). Conjugation can be used to define a left action of \( \text{GL}_n(\mathbf{F}) \) on \( \text{hom}(G, \text{GL}_n(\mathbf{F})) \) and the left action of the automorphism group \( \text{Aut}(G) \) on \( G \) induces a right action of \( \text{Aut}(G) \) on \( \text{hom}(G, \text{GL}_n(\mathbf{F})) \). Equivalent representations give conjugate group homomorphisms and the corresponding subrings are isomorphic but not necessarily equal. The action of \( \text{Aut}(G) \) preserves the image of the group homomorphism and, therefore, automorphisms in the same \( \text{Aut}(G) \)-orbit determine the same subring. Our goal is to compute the subrings of \( \mathbf{F}[x_1, \ldots, x_n] \) corresponding to the \( \text{GL}_n(\mathbf{F}) \)-orbits of \( \text{hom}(G, \text{GL}_n(\mathbf{F}))/\text{Aut}(G) \). In practice, we consider a variety \( V \subset \text{hom}(G, \text{GL}_n(\mathbf{F})) \) and a subgroup \( H < \text{GL}_n(\mathbf{F}) \), which acts on \( V \) by conjugation, and compute the subrings corresponding to the \( H \)-orbits of \( V//\text{Aut}(G) \). If \( G = (\mathbf{Z}/p)^r = \langle e_1, \ldots, e_r \rangle \) is an elementary abelian \( p \)-group, then \( \text{Aut}(G) \cong \text{GL}_r(\mathbf{F}_p) \).

We make extensive use of the theory of SAGBI bases to compute rings of invariants. A SAGBI basis is the Subalgebra Analogue of a Gröbner Basis for Ideals. The concept was introduced independently by Robbiano–Sweedler [RS] and Kapur–Madlener [KM]; a useful reference is Chapter 11 of Sturmfels [S]. We adopt the convention that a monomial is a product of variables and a term is a mono-
mial with a coefficient. For a polynomial \( f \in \mathbb{F}[x_1, \ldots, x_n] \), we denote the lead monomial of \( f \) by \( \operatorname{LM}(f) \) and the lead term of \( f \) by \( \operatorname{LT}(f) \). For \( \mathcal{B} = \{f_1, \ldots, f_\ell\} \subseteq \mathbb{F}[x_1, \ldots, x_n] \) and \( I = (i_1, \ldots, i_\ell) \), a sequence of non-negative integers, denote \( \prod_{j=1}^\ell f_j^{i_j} \) by \( f^I \). A \textit{tête-à-tête} for \( \mathcal{B} \) is a pair \( (f^I, f^J) \) with \( \operatorname{LM}(f^I) = \operatorname{LM}(f^J) \); we say that a tête-à-tête is \textit{non-trivial} if the support of \( I \) is disjoint from the support of \( J \). The reduction of an \( S \)-polynomial is a fundamental calculation in the theory of Gröbner bases. The analogous calculation for \( \text{SAGBI} \) bases is the \textit{subduction} of a tête-à-tête. A subset \( \mathcal{B} \) of a subalgebra \( A \subseteq \mathbb{F}[x_1, \ldots, x_n] \) is a \( \text{SAGBI} \) basis for \( A \) if the lead monomials of the elements of \( \mathcal{B} \) generate the lead term algebra of \( A \) or, equivalently, every non-trivial tête-à-tête for \( \mathcal{B} \) subducts to zero. For background material on term orders and Gröbner bases, we recommend [AL].

We conclude the introduction with a summary. In Section 1, we introduce the \text{SAGBI}/Divide-by-\( x \) algorithm which, in principle, can be used to compute the ring of invariants for any modular representation of a \( p \)-group. We also prove a result (Theorem 1.1) which provides sufficient conditions to show that a set of invariants generates the full ring of invariants. In the later sections we repeatedly use this result to show that we have correctly constructed a set of generators for the ring of invariants. Section 2 is devoted to describing the two-dimensional modular representations of \( p \)-groups and computing the corresponding rings of invariants. We observe that any \( p \)-group \( G \) having a faithful two-dimensional modular representation must be elementary abelian and the corresponding ring of invariants is a polynomial algebra generated by two algebraically independent polynomials, one of degree 1 and the other of degree \( |G| \). An explicit description of the degree \( |G| \) invariant plays an important rôle in the calculation of the ring of invariants for the symmetric square of the dimension two representation. This calculation is given in Section 3; in all cases the ring of invariants is a hypersurface with 4 generators and 1 relation. In Section 4, we describe the three-dimensional representations of \( (\mathbb{Z}/p)^r \). Section 5 includes the construction of a particularly nice generating set for the field of fractions of the ring of invariants for a generic three-dimensional representation of \( (\mathbb{Z}/p)^r \). In Section 6, we compute the ring of invariants for all three-dimensional representations of \( (\mathbb{Z}/p)^2 \) and in Section 7 we compute the ring of invariants for all three-dimensional representations of \( (\mathbb{Z}/p)^3 \). We classify these representations first by using their fixed point sets and then by polynomial conditions. In all cases the ring of invariants is a complete intersection. The final section is devoted to conclusions and conjectures.

1. The SAGBI/Divide-by-\( x \) Algorithm

In this section \( G \) is a \( p \)-group, \( \mathbb{F} \) is an arbitrary field of characteristic \( p \) and \( V \) is an \( \mathbb{F} \)-module of dimension \( n \). Choose a basis \( \{x, y_1, \ldots, y_{n-1}\} \) for \( V^* \) so that \( x \in (V^*)^G \) and \( y_i G \in \operatorname{Span}_\mathbb{F}\{x, y_j \mid j \leq i\} \), i.e., the action of \( G \) is by upper-triangular matrices. We use the graded reverse lexicographic order with \( x < y_1 < y_2 < \cdots < y_{n-1} \).

\textbf{Theorem 1.1.} For homogeneous \( f_1, \ldots, f_\ell \in \mathbb{F}[V]^G \) with \( \operatorname{LM}(f_i) \in \mathbb{F}[y_1, \ldots, y_{n-1}] \), define \( \mathcal{B} := \{x, f_1, \ldots, f_\ell\} \) and let \( A \) denote the algebra generated by \( \mathcal{B} \). Suppose \( A[x^{-1}] = \mathbb{F}[V]^G[x^{-1}] \), \( \mathcal{B} \) is a SAGBI basis for \( A \) and \( \mathbb{F}[V]^G \) is an integral extension
of $A$. Then $A = F[V]^G$ and $B$ is a SAGBI basis for $F[V]^G$.

Proof. Since $A$ and $F[V]^G$ have the same field of fractions and $F[V]^G$ is an integral extension of $A$, to prove $A = F[V]^G$ it is sufficient to show that $A$ is normal, i.e., integrally closed in its field of fractions. Since $A[x^{-1}] = F[V]^G[x^{-1}]$ is a normal domain, to show that $A$ is normal, it is sufficient to show that $xA$ is a prime ideal. (See, for example, [Ke, Exercise 8.5].)

Suppose $f, g \in A$ with $fg \in xA$. Since $A$ is graded we may assume $f$ and $g$ are homogeneous. Since $xF[V]$ is prime, without loss of generality we may assume $f \in xF[V]$. Hence the lead monomial $LM(f)$ is divisible by $x$. $B$ is a SAGBI basis for $A$ and $f \in A$. Thus $f$ subducts to 0. Using the grevlex order with $x$ small, every monomial of degree $\deg(f)$, less than $LM(f)$, is divisible by $x$. At each stage of the subduction, the lead monomial of the remaining polynomial is less than $LM(f)$ and is therefore divisible by $x$. Since $x$ is the only element of $B$ whose lead monomial is divisible by $x$, there must be a factor of $x$ at each step of the subduction. Hence $f \in xA$ and $xA$ is a prime ideal. \hfill \Box

The usual SAGBI bases algorithms applied to $B$ proceed by subducting tête-à-têtes and adjoining non-zero subductions to produce a sequence $B = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ with each $B_i$ a generating set for $A$ (see, for example, [S, Chap. 11]). Here we introduce an new algorithm, SAGBI/Divide-by-$x$, as an extension of the SAGBI algorithm: if a non-zero subduction $f$ has lead monomial $x^m y^E$, then $fx^{-m}$ is adjoined rather than $f$. The SAGBI/Divide-by-$x$ algorithm produces a sequence of sets $B = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ and a sequence of ring extensions $A \subseteq A_1 \subseteq A_2 \subseteq \cdots$, with $B_i$ a generating set of $A_i$.

Theorem 1.2. Suppose that $x \in B$, the elements of $B$ are homogeneous, $B$ generates $A$, and $A[x^{-1}] = F[V]^G[x^{-1}]$. Further suppose there exists $h_1, \ldots, h_{n-1}$ in $B$ such that $LM(h_i) = y_i^{a_i}$ for positive integers $a_i$. Then the SAGBI/Divide-by-$x$ algorithm applied to $B$ terminates with a SAGBI basis for $F[V]^G$.

Proof. Let $H$ denote the lead term algebra of $F[x, h_1, \ldots, h_{n-1}]$, in other words, $H = F[x, y_1^{a_1}, \ldots, y_{n-1}^{a_{n-1}}]$. Then $F[V]$ is a finite module over the Noetherian ring $H$. Let $B_i$ denote the algebra generated by the lead terms of the elements of $B_i$. Thus $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ is an ascending chain of $H$-modules in the Noetherian $H$-module $F[V]$. Hence this sequence terminates, say with $B_j$. Thus we have a SAGBI basis $B_j$ for the algebra $A_j$. Since $\{x, h_1, \ldots, h_{n-1}\} \subseteq A \subseteq A_j$ is a homogeneous system of parameters for $F[V]^G$ and $A_j[x^{-1}] = A[x^{-1}] = F[V]^G[x^{-1}]$, the hypotheses of Theorem 1.1 are met and $B_j$ is a SAGBI basis for $F[V]^G$. \hfill \Box

Remark 1.3. In practice, $B_0$ should be chosen with care to minimise the number of iterations required by the SAGBI/Divide-by-$x$ algorithm. As a first step in the construction of $B_0$, choose the homogeneous system of parameters $\{x, h_1, \ldots, h_{n-1}\} \subseteq F[V]^G$; if necessary, take $h_i$ to the orbit product of $y_i$. For many examples, $x$ is in the radical of the image of the transfer map. In this case, choose $a \in F[V]$ with $x^\ell = \text{tr}(a) := \sum_{g \in G} a \cdot g$ and define $\rho : F[V][x^{-1}] \to F[V]^G[x^{-1}]$ by $\rho(f) = x^{-\ell} \text{tr}(a f)$. Then $\rho(f) = f$ for $f \in F[V]^G$, and $\rho$ is a surjective map of $F[x, h_1, \ldots, h_{n-1}]$-modules. If $C$ is a set of homogeneous module generators for
\( \mathbb{F}[V] \), then \( \mathcal{B}_0 := \{ x, h_1, \ldots, h_{n-1} \} \cup \{ \text{tr}(\beta) \mid \beta \in \mathcal{C} \} \) satisfies the hypotheses of Theorem 1.2.

2. Two dimensional representations

Let \( \mathbb{F} \) denote a field of characteristic \( p \) and consider a finite subgroup \( W \) of the additive group \((\mathbb{F}, +)\). The order of \( W \) is \( p^r \) for some non-negative integer \( r \) and \((W, +)\) is isomorphic to the elementary abelian \( p \)-group \(((\mathbb{Z}/p)^r, +)\). Choosing an isomorphism from \((\mathbb{Z}/p)^r\) to \( W \) is equivalent to choosing a basis for \( W \) as a vector space over the finite field \( \mathbb{F}_p \). Define a group homomorphism \( \rho : \mathbb{F} \to \text{GL}_2(\mathbb{F}) \) by

\[
\rho(c) := \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.
\]

The restriction of \( \rho \) to \( W \) gives a representation of \( W \).

**Proposition 2.1.** Two subgroups of \( \text{GL}_2(\mathbb{F}) \) of the form \( \rho(W) \) and \( \rho(W') \) are conjugate if and only if \( W' = \alpha W \) for some \( \alpha \in \mathbb{F}^* \).

**Proof.** Since \( \rho(W) \) and \( \rho(W') \) have the same socle, only an upper-triangular matrix can conjugate \( \rho(W) \) to \( \rho(W') \). For an invertible matrix

\[
M = \begin{bmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{bmatrix},
\]

define \( \alpha := \frac{m_{11}}{m_{22}} \in \mathbb{F}^* \). Then \( M \rho(c) M^{-1} = \rho(\alpha c) \) and \( W' = \alpha W \). \( \square \)

Every faithful two-dimensional \( \mathbb{F} \)-representation of \((\mathbb{Z}/p)^r\) is equivalent to \( \rho \circ \varphi \) for some group monomorphism \( \varphi : (\mathbb{Z}/p)^r \to \mathbb{F} \) and therefore has image \( \rho(W) \) for some finite subgroup \( W \). Two such representations, say \( \rho \circ \varphi \) and \( \rho \circ \varphi' \), are equivalent if and only if \( \varphi' = \alpha \varphi \) for some \( \alpha \in \mathbb{F}^* \).

Let \( V_2 \) denote the natural left \( \text{GL}_2(\mathbb{F}) \)-module, i.e., the space of two-dimensional column vectors. Identify the right \( \text{GL}_2(\mathbb{F}) \)-module \( V_2^* \) with the space of two-dimensional row vectors. Taking \( x = [0 \ 1] \) and \( y = [1 \ 0] \), we have \( x \rho(c) = x \) and \( y \rho(c) = y + cx \). This action extends to a right action of \((\mathbb{F}, +)\) on the polynomial algebra \( \mathbb{F}[x, y] = \mathbb{F}[V_2] \) by degree-preserving algebra automorphisms. For any finite subgroup \((W, +)\) of \((\mathbb{F}, +)\) we denote the ring of invariants of the restriction of \( \rho \) to \( W \) by \( \mathbb{F}[V_2]^W \). Clearly \( x \in \mathbb{F}[V_2]^W \). A second invariant is given by the orbit product

\[
N_W(y) := \prod_{c \in W} y \rho(c) = \prod_{c \in W} (y + cx).
\]

Using the graded reverse lexicographic order with \( x < y \), the leading term of \( N_W(y) \) is \( y^r \). Thus \( \{ x, N_W(y) \} \) is a homogeneous system of parameters with \( \text{deg}(x) \text{deg}(N_W(y)) = |W| \). Hence applying [DK, 3.7.5] gives the following proposition.

**Proposition 2.2.** \( \mathbb{F}[V_2]^W = \mathbb{F}[x, N_W(y)] \).

Define

\[
F_W(t) := \prod_{c \in W} (t - c) \in \mathbb{F}[t].
\]
Thus $F_W$ is the monic polynomial whose roots are the elements of $W$. Note that for $\alpha \in \mathbf{F}^*$ we have $F_{\alpha W}(t) = \alpha^{|W|} F_W(t/\alpha)$. Furthermore, $N_W(y) = x^{|W|} F_W(y/x)$ and $N_{\alpha W}(y) = (\alpha x)^{|W|} F_W(y/(\alpha x))$. Therefore the same monomials appear with non-zero coefficients in both $N_W(y)$ and $N_{\alpha W}(y)$.

Choose an $\mathbf{F}_p$-vector space basis $\{c_1, \ldots, c_r\}$ for $W \subseteq \mathbf{F}$. Since $\mathbf{F}$ is an $\mathbf{F}_p$-algebra, taking $\psi(x_i) = c_i$ defines an $\mathbf{F}_p$-algebra map $\psi : \mathbf{F}_p[x_1, \ldots, x_r] \rightarrow \mathbf{F}$. A change of basis for $W$ corresponds to the action of an element of $\text{GL}_r(\mathbf{F}_p)$ on $\mathbf{F}_p[x_1, \ldots, x_r]$. Thus the restriction of $\psi$ to $\mathbf{F}_p[x_1, \ldots, x_r]^{\text{GL}_r(\mathbf{F}_p)}$ is determined by $W$ and is independent of the choice of basis of $W$. We denote the restriction by $\psi_W$.

Let $U$ denote the $\mathbf{F}_p$-span of $\{x_1, \ldots, x_r\}$. Recall that the Dickson invariants, $d_1, \ldots, d_r$, may be defined as the coefficients of the polynomial

$$D(t) := \prod_{v \in U} (t - v) = t^{p^r} + \sum_{i=0}^{r-1} d_{r-i} t^{p^i} \in \mathbf{F}_p[x_1, \ldots, x_r][t]$$

and that $\mathbf{F}_p[x_1, \ldots, x_r]^{\text{GL}_r(\mathbf{F}_p)} = \mathbf{F}_p[d_1, \ldots, d_r]$ (see [D1] or [W]). Define $d_i(W) := \psi_W(d_i) \in \mathbf{F}$.

**Theorem 2.3.** $N_W(y) = y^{p^r} + \sum_{i=0}^{r-1} d_{r-i}(W) y^{p^i} x^{p^r-p^i}$.

**Proof.** Applying $\psi$ to the polynomial $D(t)$ gives $F_W(t)$. Hence

$$N_W(y) = x^{p^r} \psi(D(y/x)) = x^{p^r} \left((y/x)^{p^r} + \sum_{i=0}^{r-1} d_{r-i}(W)(y/x)^{p^i}\right)$$

$$= y^{p^r} + \sum_{i=0}^{r-1} d_{r-i}(W) y^{p^i} x^{p^r-p^i}$$

as required. \qed

**Remark 2.4.** For $\mathbf{F} = \mathbf{F}_p(x_1, \ldots, x_r)$ and $W = \text{Span}_{\mathbf{F}_p} \{x_1, \ldots, x_r\}$, the map $\psi$ is injective and $d_i(W) \neq 0$ for $i = 1, \ldots, r$. However, if $W = \mathbf{F}_p^{r^*} \subseteq \mathbf{F}$, then $F_W(t) = t^{p^r} - t$, giving $d_r(W) = -1$ and $d_i(W) = 0$ for $1 \leq i < r$.

A two-dimensional representation of $(\mathbb{Z}/p)^r$ is given by a point $(c_1, \ldots, c_r) \in \mathbf{F}^r$: $(\mathbb{Z}/p)^r = \langle e_1, \ldots, e_r \rangle$ and $e_i \mapsto c_i$. The representation is faithful if the dimension of $\text{Span}_{\mathbf{F}_p} \{c_1, \ldots, c_r\}$ is $r$. Two points $v, v' \in \mathbf{F}^r$ give equivalent representations if and only if $v' = \alpha v$ for some $\alpha \in \mathbf{F}^*$. Thus equivalence classes of non-trivial representations are parameterised by points in projective space $P(\mathbf{F}^r)$. The action of the automorphism group $\text{Aut}((\mathbb{Z}/p)^r) = \text{GL}_r(\mathbf{F}_p)$ corresponds to changing generators for $(\mathbb{Z}/p)^r$ and permutes representations while preserving the ring of invariants. Thus the rings of invariants for non-trivial representations are parameterised by the projective variety associated to $\mathbf{F}^r / \text{GL}_r(\mathbf{F}_p)$. The coordinate ring of this variety is $\mathbf{F}[x_1, \ldots, x_r]^{\text{GL}_r(\mathbf{F}_p)} = \mathbf{F}[d_1, \ldots, d_r]$ and thus it is just a weighted projective space of dimension $r - 1$ (with weights $p^r - p^i$ for $i = 0, 1, \ldots, r - 1$). The representation given by $v \in \mathbf{F}^r$ is faithful if and only if $d_r(v) \neq 0$. To see this, observe that $d_r$ is the product of all non-trivial $\mathbf{F}_p$-linear combinations of $\{x_1, \ldots, x_r\}$. 
3. Symmetric square representations

In this section we assume \( p > 2 \). Let \( V_3 \) denote the representation of \((\mathbf{F},+)\) dual to the symmetric square of \( V_2^* \). Since \( x^2 \rho(c) = x^2 \), \((xy)\rho(c) = xy + cx^2\) and \( y^2 \rho(c) = (y + cx)^2\), \( V_3 \) is the left \((\mathbf{F},+)\)-module given by

\[
\rho_2(c) := \begin{bmatrix}
1 & 2c & c^2 \\
0 & 1 & c \\
0 & 0 & 1
\end{bmatrix}.
\]

Taking \( x = [0 \ 0 \ 1] \), \( y = [0 \ 1 \ 0] \), and \( z = [1 \ 0 \ 0] \), we have \( x\rho_2(c) = x \), \( y\rho_2(c) = y + cx \), and \( z\rho_2(c) = z + 2cy + c^2x \). Note that “multiplication by \( x \)” embeds \( V_2^* \) as a submodule of \( V_3^* \), justifying our convention of using \( x \) and \( y \) to denote elements of both \( V_2^* \) and \( V_3^* \). Dual to this embedding, we have the \((\mathbf{F},+)\)-module surjection \( V_3 \to V_3/V_3^F \cong V_2 \). The action of \((\mathbf{F},+)\) on \( V_3^* \) extends to a right action on the polynomial algebra \( \mathbf{F}[x, y, z] = \mathbf{F}[V_3] \) by degree-preserving algebra automorphisms.

For any finite subgroup \((W,+1\) of \((\mathbf{F},+)\) we denote the ring of invariants of the restriction of \( \rho_2 \) to \( W \) by \( \mathbf{F}[V_3]^W \). It is clear that \( x \in \mathbf{F}[V_3]^W \). A simple calculation shows that

\[
\delta := y^2 - xz = \mathbf{F}[V_3]^W.
\]

Since

\[
\prod_{c \in W} y\rho_2(c) = \prod_{c \in W} (y + cx) = N_W(y),
\]

we also have \( N_W(y) \in \mathbf{F}[V_3]^W \). Define

\[
N_W(z) := \prod_{c \in W} z\rho_2(c) = \prod_{c \in W} (z + 2cy + c^2x).
\]

The rest of this section is devoted to the proof that \( \mathbf{F}[V_3]^W \) is generated by \( \mathcal{G} := \{x, \delta, N_W(y), N_W(z)\} \). We will use the graded reverse lexicographic order with \( z > y > x \) and show that \( \mathcal{G} \) is a SAGBI basis for \( \mathbf{F}[V_3]^W \) with respect to this order. With this order we have \( \text{LT}(\delta) = y^2 \), \( \text{LT}(N_W(y)) = yp^r \) and \( \text{LT}(N_W(z)) = zp^r \). Thus there is a single non-trivial tête-à-tête among the elements of \( \mathcal{G} \):

\[
\delta p^r - N_W(y)^2 = \left(y^{2p^r} - (xz)^{p^r}\right) - \left(y^{p^r} + \sum_{i=0}^{r-1} d_{r-i}(W)y^{p^i}x^{p^r-p^i-p^j}\right)^2.
\]

Note that

\[
N_W(y)^2 - y^{2p^r} = 2 \sum_{0 \leq i < j \leq r} d_{r-i}(W)d_{r-j}(W)y^{p^i+p^j}x^{2p^r-p^i-p^j}
\]

\[
+ \sum_{i=0}^{r-1} d_{r-i}(W)^2 y^{2p^i}x^{2(p^r-p^i)}
\]

is a polynomial in \( x \) and \( y^2 \). Define \( H_W(s, t) \in \mathbf{F}[s, t] \) so that \( H_W(x, y^2) = N_W(y)^2 - y^{2p^r} \).
Lemma 3.1. $\delta^{p^r} - NW(y)^2 + x^{p^r}NW(z) + HW(x, \delta) = 0$.

Proof. Consider $F := \delta^{p^r} - NW(y)^2 + HW(x, \delta)$. Note that the coefficient of $z^{p^r}$ in $F$ is $-x^{p^r}$. Also, working modulo the ideal in $F[V_3]$ generated by $z$ we have $F \equiv (z)^{y^{2p^r}} - NW(y)^2 + HW(x, y^2) = 0$. Thus $z$ divides $F$. However, since $F \in F[V_3]^W$, each element in the $W$-orbit of $z$ divides $F$. By definition, $NW(z)$ is the product of the elements in the $W$-orbit of $z$. Also, observe that if $\alpha$ and $\beta$ are elements in the $W$-orbit of $z$, then $\alpha$ divides $\beta$ if an only if $\alpha = \beta$. Therefore $NW(z)$ divides $F$. Since $-x^{p^r}$ is the coefficient of $z^{p^r}$ in $F$, we must have $F = -x^{p^r}NW(z)$, as required. □

Let $A$ denote the subalgebra of $F[V_3]^W$ generated by $G$. Using the relation given by the lemma, the sole non-trivial tête-à-tête subducts to 0, giving the following.

Theorem 3.2. $G$ is a SAGBI basis for $A$.

Theorem 3.3. $F[V_3]^W$ is the hypersurface generated by $\{x, \delta, NW(y), NW(z)\}$ subject to the relation $\delta^{p^r} - NW(y)^2 + x^{p^r}NW(z) + HW(x, \delta) = 0$. Furthermore, this generating set is a SAGBI with respect to the graded reverse lexicographic order with $z > y > x$.

Proof. Note that LT($\delta$) = $y^2$ and LT($NW(z)$) = $z^{p^r}$. Thus $(x, \delta, NW(z))F[V_3]$ is a zero-dimensional ideal and $\{x, \delta, NW(z)\}$ is a homogeneous system of parameters. Hence $A \subseteq F[V_3]^W$ is an integral extension. Since $F(x, y)^W = F(x, NW(y))$ and $\delta$ is degree 1 in $z$, applying [CC, Thm. 2.4] gives $F(V_3)^W = F(x, NW(y), \delta)$ (see also [Ka]). Furthermore, since the coefficient of $z$ in $\delta$ is $-x$, we have $F[V_3]^W[x^{-1}] = F[x, NW(y), \delta][x^{-1}] = A[x^{-1}]$. (Note that using the relation given in Lemma 3.1, we obtain an explicit expression for $NW(z) \in F[x, \delta, NW(y)][x^{-1}]$.) Applying Theorem 1.1 gives $A = F[V_3]^W$. □

Remark 3.4. A proof of Theorem 3.3 for the special case $W = F_{p^r}$ is given in Section 2 of [HS].

4. Classifying the three-dimensional representations

In this section we describe all three-dimensional representations of $G := (\mathbb{Z}/p)^r$. We first sort the non-trivial representations $V$ of $G$ by the dimensions of the factors in the socle series.

Type (2, 1): $\dim_F(V^G) = 2$ and $\dim_F((V/V^G)^G) = 1$. The image of the representation is of the form

$$\left\{ \begin{bmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix} \bigg| \ (c_1, c_2) \in U \right\}$$

for some finite subgroup $U \leq (F^2, +)$. By [CW, Thm. 3.9.2] (or [LS]), the rings of invariants for these representations are polynomial algebras. Note that this case includes Stong’s example (see [CW, §8.1]) so the image is not necessarily a Nakajima group. (A Nakajima group is a subgroup of the unipotent upper triangular matrices whose ring of invariants is generated by the orbit products of linear forms, see [CW, Chap. 8] for details.)
Type $(1, 2)$: \( \dim_\mathbf{F}(V^G) = 1 \) and \( \dim_\mathbf{F}((V/V^G)^G) = 2 \). The image of the representation is of the form
\[
\begin{bmatrix}
1 & c_1 & c_2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
for some finite subgroup \( U \leq (\mathbf{F}^2, +) \). Here \( \mathbf{F}[V]^G = \mathbf{F}[x, y, N_U(z)] \) where
\[
N_U(z) := \prod_{(c_1, c_2) \in U} (z + c_1 y + c_2 x)
\]
by [DK, 3.7.5].

Type $(1, 1, 1)$: \( \dim_\mathbf{F}(V^G) = 1 \) and \( \dim_\mathbf{F}((V/V^G)^G) = 1 \). In this case the image of the representation contains at least one element whose Jordan form consists of a single Jordan block; hence \( p > 2 \). This case includes the symmetric square representations of Section 3. Define a group homomorphism \( \sigma : \mathbf{F}^2 \to \text{GL}_3(\mathbf{F}) \) by
\[
\sigma(c_1, c_2) := \begin{bmatrix}
1 & 2c_1 & c_1^2 + c_2 \\
0 & 1 & c_1 \\
0 & 0 & 1
\end{bmatrix}.
\]

**Proposition 4.1.** For any representation of type $(1, 1, 1)$, there exists a choice of basis for which the image of the representations is given by \( \sigma(U) \) for some finite subgroup \( U \leq (\mathbf{F}^2, +) \).

**Proof.** Since the representation is of type $(1, 1, 1)$ there is at least one element whose Jordan form consists of a single Jordan block. Note that this element determines the socle series of the representation. Choose a basis for \( V \) so that the matrix representing this element is \( \sigma(1, 0) \). With respect to this basis, the other group elements are represented by upper triangular unipotent matrices. Furthermore, these matrices must commute with \( \sigma(1, 0) \). A straightforward computation shows that any upper triangular unipotent matrix which commutes with \( \sigma(1, 0) \) is in the image of \( \sigma \). \( \square \)

**Remark 4.2.** If \( V \) is a decomposable representation of type $(2, 1)$ then there exists a choice of basis for which the image of the representation is contained in \( \{ \sigma(0, c) \mid c \in \mathbf{F}\} \), the centre of the upper triangular unipotent group.

**Proposition 4.3.** Two subgroups of \( \text{GL}_3(\mathbf{F}) \) of the form \( \sigma(U) \) and \( \sigma(U') \) are conjugate if and only if there exist \( \alpha \in \mathbf{F}^* \) and \( \gamma \in \mathbf{F} \) such that
\[
U' = \{ \alpha(c_1, c_1 + \alpha c_2) \mid (c_1, c_2) \in U \}.
\]

**Proof.** Since the subgroups \( \sigma(U) \) and \( \sigma(U') \) have the same socle series, only an upper triangular matrix can conjugate \( \sigma(U) \) to \( \sigma(U') \). For an invertible matrix
\[
M := \begin{bmatrix}
m_{11} & m_{12} & m_{13} \\
0 & m_{22} & m_{23} \\
0 & 0 & m_{33}
\end{bmatrix},
\]
define \( \alpha := m_{11}/m_{22} \in \mathbf{F}^* \) and \( \gamma := (m_{12}/m_{22}) - 2(m_{23}/m_{33}) \in \mathbf{F} \). The condition \( M \sigma(c_1, c_2) M^{-1} \in \sigma(\mathbf{F}) \) forces \( m_{22}/m_{33} = \alpha \) and gives \( M \sigma(c_1, c_2) M^{-1} = \sigma(\alpha c_1, \alpha^2 c_2 + \alpha \gamma c_1) \). \( \square \)
We encode this conjugation as a left action of $\mathbf{F} \times \mathbf{F}^*$ on $\mathbf{F}^2$:

$$(\gamma, \alpha) \cdot \left[ \begin{array}{c} c_1 \\ c_2 \end{array} \right] = \left( \begin{array}{cc} \alpha & 0 \\ \alpha \gamma & \alpha^2 \end{array} \right) \left[ \begin{array}{c} c_1 \\ c_2 \end{array} \right],$$

where $\mathbf{F}^*$ acts on $\mathbf{F}$ by multiplication. A matrix

$$M = \left( \begin{array}{cccc} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \end{array} \right) \in \mathbf{F}^{2 \times r}$$

determines a three-dimensional representation $V_M$ of $G = (\mathbb{Z}/p)^r = \langle e_1, \ldots, e_r \rangle$ by $e_i \mapsto \sigma(c_{1i}, c_{2i})$. Every representation of type $(1, 1, 1)$ is equivalent to $V_M$ for some choice of $M$. Furthermore, $V_M$ is of type $(1, 1, 1)$ if $c_{1i} \neq 0$ for some $i$. If $V_M$ is of type $(1, 1, 1)$ and $c_{2i} = 0$ for all $i$, then $V_M$ is a symmetric square representation.

We choose a basis $\{x, y, z\}$ for $V_M^*$ so that $x \sigma(c_1, c_2) = x, y \sigma(c_1, c_2) = y + c_1 x$ and $\sigma(c_1, c_2) = z + 2c_1 y + (c_1^2 + c_2) x$. Let $U$ denote the $\mathbf{F}_p$-span of the columns of $M$. Then $\mathbf{F}[V_M]^G = \mathbf{F}[x, y, z]^{\sigma(U)}$. For any $f \in \mathbf{F}[x, y, z]$, let $N_G(f)$ denote the product of the elements of $G$-orbit of $f$.

**Theorem 4.4.** If $\deg(N_G(y)) \deg(N_G(\delta)) = 2|G|$, then

$$\mathbf{F}[V_M]^G[x^{-1}] = \mathbf{F}[x, N_G(y), N_G(\delta)][x^{-1}].$$

**Proof.** Since $\delta = y^2 - xz$, it is clear that $\mathbf{F}[x, y, z][x^{-1}] = \mathbf{F}[x, y, \delta/x][x^{-1}]$. Observe that $\delta \sigma(c_1, c_2) = \delta - c_2 x^2$. Thus $G$ acts on $\mathbf{F}[x, y, \delta/x]$ by degree-preserving algebra automorphisms. By hypothesis, $\deg(N_G(y)) \deg(N_G(\delta/x)) = |G|$. Therefore, by [DK, 3.7.5], $\mathbf{F}[x, y, \delta/x]^G = \mathbf{F}[x, N_G(y), N_G(\delta/x)]$. Thus $\mathbf{F}[V_M]^G[x^{-1}] = \mathbf{F}[x, N_G(y), N_G(\delta)[x^{-1}] = \mathbf{F}[x, N_G(y), N_G(\delta)][x^{-1}]$. □

**Remark 4.5.** The right action of $\text{GL}_r(\mathbf{F}_p)$ on $\mathbf{F}^{2 \times r}$ corresponds to changing generators for $(\mathbb{Z}/p)^r$ and permutes representations while preserving the ring of invariants. The left action of $\mathbf{F} \times \mathbf{F}^*$ on $\mathbf{F}^{2 \times r}$ corresponds to a change of basis for $V_M$.

The rings of invariants for representations of the form $V_M$ are parameterised by $(\mathbf{F} \times \mathbf{F}^*)$-orbits in the variety $\mathbf{F}^{2 \times r} / \text{GL}_r(\mathbf{F}_p)$ (assuming $\mathbf{F}$ is algebraically closed). The coordinate ring of this variety is $\mathbf{F}[\mathbf{F}^{2 \times r}]^{\text{GL}_r(\mathbf{F}_p)}$. In Sections 6 and 7 we use elements of $\mathbf{F}[\mathbf{F}^{2 \times r}]^{\text{GL}_r(\mathbf{F}_p)}$ to partition the $(\mathbf{F} \times \mathbf{F}^*)$-orbits of $\mathbf{F}^{2 \times r} / \text{GL}_r(\mathbf{F}_p)$; we then compute the corresponding rings of invariants.

5. The field of fractions for the generic case

Consider the ring $\mathbf{F}_p[x_{ij}] := \mathbf{F}_p[x_{1j}, x_{2j} \mid j = 1, 2, \ldots, r]$ and its quotient field $k := \mathbf{F}_p(x_{1j}, x_{2j} \mid j = 1, 2, \ldots, r)$. In this section, we work over $k$. The action of $G = (\mathbb{Z}/p)^r = \langle e_1, \ldots, e_r \rangle$ is given by $e_j \mapsto \sigma(x_{1j}, x_{2j})$. Therefore, we assume $p > 2$. Define a $2(r + 1) \times r$ matrix by

$$\Gamma := \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1r} \\ x_{21} & x_{22} & \cdots & x_{2r} \\ x_{1p} & x_{1p} & \cdots & x_{1p} \\ x_{2p} & x_{2p} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1p'} & x_{1p'} & \cdots & x_{1p'} \\ x_{2p'} & x_{2p'} & \cdots & x_{2p'} \end{bmatrix}.$$
For a subsequence $I = (i_1, \ldots, i_r)$ of $(1, 2, \ldots, 2r + 2)$, let $\gamma_I \in k$ denote the associated $r \times r$ minor of $\Gamma$. Note that $\gamma_I$ is invariant under the natural right action of $\text{SL}_r(F_p)$. Form a $(2r + 1) \times (r + 1)$ matrix $\tilde{\Gamma}$ by augmenting $\Gamma$ with the column
\[
v = [y/x, -\delta/x^2, (y/x)^p, (-\delta/x^2)^p, \ldots, (y/x)^p, (-\delta/x^2)^p]^T.
\]
For a subsequence $J = (j_1, \ldots, j_{r+1})$ of $(1, 2, \ldots, 2r + 2)$, let $\tilde{f}_J \in k[x, y, z][x^{-1}]$ denote the associated $(r + 1) \times (r + 1)$ minor of $\tilde{\Gamma}$. Let $f_J$ denote the element of $k[x, y, z]$ constructed by minimally clearing the denominator of $\tilde{f}_J$. Observe that $y \Delta_j = x x_{1j}$ and $\delta \Delta_j = -x^2 x_{2j}$, where $\Delta_j := e_j - 1 \in kG$. Therefore
\[
v \Delta_j = [\Gamma_{1j}, \Gamma_{2j}, \ldots, \Gamma_{(2r+1)j}]^T,
\]
the $j$th column of $\Gamma$. Thus $\tilde{f}_J \in k[x, y, z]^G[x^{-1}]$ and $f_J \in k[x, y, z]^G$.

**Lemma 5.1.** $k[x, y, z]^G[x^{-1}] = k[x, f_{(1, 3, 5, \ldots, 2r+1)}, f_{(1, 2, 3, 5, \ldots, 2r-1)}][x^{-1}]$.

**Proof.** Since $f_{(1, 3, 5, \ldots, 2r+1)}$ is a scalar multiple of $N_G(y)$ and $f_{(1, 2, 3, 5, \ldots, 2r-1)}$ is an invariant which is degree 1 in $z$, applying [CC, Thm. 2.4] shows that $k(x, y, z)^G$ is generated by $\{x, f_{(1, 3, \ldots, 2r+1)}, f_{(1, 2, 3, 5, \ldots, 2r-1)}\}$. Furthermore, the coefficient of $z$ in $f_{(1, 2, 3, 5, \ldots, 2r-1)}$ is a scalar times a power of $x$. Hence, we have a generating set once $x$ has been inverted. \(\square\)

Observe that $\text{LM}(f_{(1, 3, \ldots, 2r+1)}) = y^p$ and $\text{LM}(f_{(1, 2, 3, 5, \ldots, 2r-1)}) = y^{p-1}$. Define $f_1 := f_{(1, 2, 3, \ldots, r+1)}$ and $s := [r/2]$. For $r$ odd, define $f_2 := f_{(1, 2, \ldots, r, r+2)}$ and observe that $\text{LM}(f_1) = y^{2p^s-1}$ and $\text{LM}(f_2) = y^{p^s}$. For $r$ even, define
\[
f_2 := \frac{\gamma_{(1, 2, \ldots, r)} f_{(1, 2, \ldots, r, r+2)} + f_1^2}{2 x^{p^s-2p^{s-1}}}.\]

In this case $\text{LM}(f_1) = y^{p^s}$ and a straight-forward calculation gives
\[
\text{LT}(f_2) = \gamma_{(1, 2, \ldots, r)} \gamma_{(1, 2, \ldots, r-1, r+1)} y^{p^{s+2p^{s-1}}}.\]

The rest of this section is devoted to the proof of the following.

**Theorem 5.2.** $k[x, y, z]^G[x^{-1}] = k[x, f_1, f_2][x^{-1}]$.

For a subsequence $K = (k_1, k_2, \ldots, k_{r+2})$ of $(1, 2, \ldots, 2r + 2)$, let $K_\ell$ denote the subsequence of $K$ formed by omitting $\ell$ and let $K_{\ell, m}$ denote the subsequence formed by omitting $\ell$ and $m$.

**Lemma 5.3.** For any subsequence $(\ell_1, \ell_2, \ell_3)$ of $K$,
\[
(-1)^{\epsilon_1} \gamma_{K_{\ell_1, \ell_2}} \tilde{f}_{K_{\ell_3}} + (-1)^{\epsilon_2} \gamma_{K_{\ell_2, \ell_3}} \tilde{f}_{K_{\ell_1}} + (-1)^{\epsilon_3} \gamma_{K_{\ell_1, \ell_3}} \tilde{f}_{K_{\ell_2}} = 0
\]
for some choice of $\epsilon_i \in \{0, 1\}$. 
Proof. Form a matrix $\Lambda$ by adding the row $[0, 0, 0, \ldots, 0, 1]$ to the bottom of $\tilde{\Gamma}$. Thus the minors of $\Lambda$ which include the last row are, up to sign, minors of $\Gamma$, and the minors of $\Lambda$ which do not include the last row are minors of $\tilde{\Gamma}$. Consider the Plücker relation for $\Lambda$ determined by the sequence $K$, of length $r + 2$, and the sequence of length $r$ formed by omitting $\ell_1, \ell_2, \ell_3$ from $K$ and adding $2r + 3$. All but three of the terms in this Plücker relation are zero and the non-zero terms give the required relation.

\section*{Lemma 5.4.}
For $J$ a length $r + 1$ subsequence of $(1, 2, \ldots, 2r + 2)$,

$$\tilde{f}_J \in \text{Span}_k \{ \tilde{f}^p_{(1,2,3,\ldots,r+1)}, \tilde{f}^p_{(2,3,\ldots,r+2)} \mid i = 0, 1, 2, \ldots \}.$$ 

Proof. Since the $p^{th}$ power map is $F_p$-linear, $\tilde{f}^p_J = \tilde{f}^p_{(j_1+2,j_2+2,\ldots,j_{(r+1)}+2)}$.

The proof is by induction on $t = j_{r+1} - j_1$, starting with $t = r$. If $t = r$ and $j_1$ is even, say $j_1 = 2i + 2$, then $\tilde{f}_J = \tilde{f}^p_{(2,3,\ldots,r+2)}$. If $t = r$ and $j_1$ is odd, say $j_1 = 2i + 1$, then $\tilde{f}_J = \tilde{f}^p_{(1,2,3,\ldots,r+1)}$.

Suppose $t > r$. Choose an integer $m \not\in \{j_1, \ldots, j_{r+1}\}$ with $j_1 < m < j_{r+1}$. Insert $m$ into the sequence $J$ to produce a sequence $K$ of length $r + 2$. Apply Lemma 5.3 to the subsequence $(j_1, m, j_{r+1})$ of $K$. This allows us to write $\tilde{f}_J$ as a $k$-linear combination of $\tilde{f}^p_{K_{j_1}}$ and $\tilde{f}^p_{K_{(r+1)}}$, both of which lie in

$$\text{Span}_k \{ \tilde{f}^p_{(1,2,3,\ldots,r+1)}, \tilde{f}^p_{(2,3,\ldots,r+2)} \mid i = 1, 2, \ldots \}$$

by induction.

Using Lemma 5.4 and Lemma 5.1, we see that

$$k[x, y, z]^G[x^{-1}] = k[x, \tilde{f}^p_{(1,2,\ldots,r+1)}, \tilde{f}^p_{(2,3,\ldots,r+2)}][x^{-1}].$$

Applying Lemma 5.3 and clearing denominators completes the proof of Theorem 5.2.

\section*{Remark 5.5.}
It follows from Theorem 1.2 that applying the SAGBI/Divide-by-$x$ algorithm to $\{ x, f_1, f_2, N_G(z) \}$ will produce a SAGBI basis for $k[V]^G$.

\section*{6. The invariants for rank two dimension three}

In this section $G = (\mathbb{Z}/p)^2 = \langle e_1, e_2 \rangle$. We start with the generic representation over $k := F_p(x_{11}, x_{12}, x_{21}, x_{22})$ given by $e_i \mapsto \sigma(x_{1i}, x_{2i})$; hence, we assume $p > 2$. Consider $B := \{ x, f_1, f_2, N_G(z) \}$ with $f_1$ and $f_2$ defined as in Section 5. Thus $\text{LT}(f_1) = \gamma_{12}y^p$ and $\text{LT}(f_2) = \gamma_{12}\gamma_{13}y^{p+2}$. There is a single non-trivial tête-à-tête: $(f_1^{p+2}, f_2^{p})$.

Define

$$\tilde{N} := \gamma_{13}f_1^{p+2} - \gamma_{12}f_2^p + c_1x^{p-2}f_1^pf_2 + c_2x^pf_1^{p+1} + c_3x^{2p-2}f_2f_1^{p-1} + c_4x^{2p-1}f_2^{(p+1)/2}f_1^{(p-3)/2}$$

where, for $p > 3$, we define $c_1 := -2\gamma_{13}^p$, $c_2 := \gamma_{12}\gamma_{23}^p$, $c_3 := \gamma_{12}\gamma_{14}\gamma_{13}^{p-1} - \gamma_{23}^p$, $c_4 := \gamma_{12}\gamma_{13}^{(p-3)/2} - (\gamma_{13}\gamma_{24} - \gamma_{23}\gamma_{14})$, and for $p = 3$ we take $c_1 := \gamma_{13}^3$, $c_2 := \gamma_{12}\gamma_{23}^3 + (\gamma_{13}\gamma_{12})^2$, $c_3 := \gamma_{12}\gamma_{13}^2\gamma_{14} - \gamma_{12}\gamma_{23}^3 - (\gamma_{13}\gamma_{12})^2$ and $c_4 := \gamma_{12}\gamma_{34} - \gamma_{12}\gamma_{13}\gamma_{14} + \gamma_{12}\gamma_{23}^3 + (\gamma_{13}\gamma_{12})^2$. 
Lemma 6.1. \( \text{LT}(\tilde{N}) = -\frac{1}{2}\gamma_{12}^2 p^2 x^2 p z^p p^2 \).

Proof. For \( p = 3 \), verifying the result is a Magma calculation. Suppose \( p > 3 \). We work modulo the ideal in \( k[x, y, z] \) generated by \( x^{2p+1} \) and \( y^{2p} \), which we denote by \( n \). By definition, \( f_1 = \gamma_{12} y^p + \gamma_{13} y^p x^{p-2} + \gamma_{23} x y^p \). For \( p \geq 5 \), we have \( p^2 - 2p > 2p + 1 \) and \( 2p + p - 4 > 2p \). Thus \( f_1^n = n \gamma_{12} y^p \) and \( x^{2p-2} f_1 = n \gamma_{12} x^{2p-2} y^p \). By definition \( 2x^{p-2} f_2 = \gamma_{12} f_1 + 12 = f_1 - \gamma_{12} x^{p-2} + \gamma_{12} y^p + \gamma_{12} y^p x^{2p-2} + \gamma_{12} y^p x^{2p-1} \),

\[
f_2 = \gamma_{12} \gamma_{13} y^p + \gamma_{12} \gamma_{23} x y^p + \gamma_{12} \gamma_{14} \gamma_{23} y^p + \gamma_{12} y^p x^{p-2} + \gamma_{12} y^p x^{2p-2} + \gamma_{12} y^p x^{2p-1} + \gamma_{12} y^p x^{2p-2} + \gamma_{12} y^p x^{2p-1}.
\]

Substituting and simplifying gives

\[
\tilde{N} = -\gamma_{12} x^{2p} y^p z^p,
\]

as required. \( \square \)

Theorem 6.2. For the generic rank two representation, \( B = \{ x, f_1, f_2, N_G(z) \} \) is a SAGBI basis for \( k[V_3]^G \). Furthermore, there is a single relation among the generators constructed by subducting \( \gamma_{13} f_1^{p+2} f_2 = \gamma_{12} f_2^p \).

Proof. Define \( N := \tilde{N}/x^{2p} \). It follows from Lemma 6.1 that \( B' := \{ x, f_1, f_2, N \} \) is a SAGBI basis for the algebra it generates. Applying Theorem 5.2 and Theorem 1.1 proves that \( B' \) is a SAGBI basis for \( k[V_3]^G \). Thus the lead term algebra of \( k[V_3]^G \) is generated by \( \{ x, y^p, y^{p+2}, z^p \} \). Since \( LM(B) = \{ x, y^p, y^{p+2}, z^p \} \), \( B \) is also a SAGBI basis for \( k[V_3]^G \). The single non-trivial tête-à-tête subducts to produce the relation. \( \square \)

For \( F \) an arbitrary field of characteristic \( p \), consider the representation \( V_M \) of \( (\mathbb{Z}/p)^2 \) determined by

\[
M := \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}
\]

with \( c_{ij} \in F \), i.e., \( e_i \rightarrow \sigma(c_{ii}) \) for \( i = 1, 2 \). Let \( \psi_M : F_p[x_{ij}] \rightarrow F \) denote the evaluation map: \( \psi_M(x_{ij}) = c_{ij} \). For \( f \in k(x, y, z) \), let \( \overline{f} \) denote the element of \( F[x, y, z] \) constructed by minimally clearing denominators to obtain an element of \( F_p[x_{ij}] \) and then applying \( \psi_M \) to the coefficients of that element. Note that we can also interpret elements of \( F_p[x_{ij}] \) as regular functions on \( F^{2 \times 2} \), the space of representations of the given type.

Theorem 6.3. If \( \gamma_{12} \neq 0 \) and \( \gamma_{13} \neq 0 \) then \( \{ x, \overline{f_1}, \overline{f_2}, N_G(z) \} \) is a SAGBI basis for \( F[V_M]^G \). Furthermore, there is a single relation among the generators constructed by subducting \( \overline{\gamma_{13}} \overline{f_1}^{p+2} \overline{f_2}^p = \overline{\gamma_{12}} \overline{f_2}^p \).

Proof. We show that, in essence, the proof of Theorem 6.2 survives evaluation. Let \( A \) denote the algebra generated by \( B' := \{ x, \overline{f_1}, \overline{f_2}, \overline{N} \} \). Using Lemma 6.1, \( B' \) is a SAGBI basis for \( A \) and the lead term algebra of \( A \) is generated by \( \{ x, y^p, y^{p+2}, z^p \} \). Thus it is sufficient to show that \( A[x^{-1}] = F[V_M]^G[x^{-1}] \). Note that \( \overline{f_1} \) is degree
1 in $z$ with coefficient $-\gamma_{13} z^{p-1}$. Hence $F[x,f_1,N_G(y)][x^{-1}] = F[V_M]^G[x^{-1}]$. Therefore, to complete the proof, we need only show that $N_G(y) \in A[x^{-1}]$.

Using the notation of Section 5, $\bar{f}_{135} = \gamma_{13} N_G(y)$. Recall that $f_1 = f_{123}$ and that $\gamma_{12} f_{124} = 2x^{p-2} f_2 - f_1^p$. Applying Lemma 5.3 to the subsequence $(1, 4, 5)$ of $(1, 3, 4, 5)$ shows that $\gamma_{34} f_{135}$ is an $F_p$-linear combination of $\gamma_{35} f_{134}$ and $\gamma_{13} f_{345}$. However, $\gamma_{34} = \gamma_{12}^p$, $\gamma_{35} = \gamma_{13}^p$, and $\bar{f}_{345} = \bar{f}_{123}^p$. Applying Lemma 5.3 to the subsequence $(2, 3, 4)$ of $(1, 2, 3, 4)$ shows that $\gamma_{12} f_{134}$ is an $F_p$-linear combination of $\gamma_{13} f_{124}$ and $\gamma_{14} f_{123}$. Thus $\gamma_{12}^2 f_{135}$ can be written as a polynomial in $x$, $f_1$ and $f_2$ with coefficients in $F_p[x]$. Thus $N_G(y) \in A$, as required. □

For a representation with $\gamma_{13} = 0$ and $\gamma_{12} \neq 0$, define $h := \bar{f}_2 / (\gamma_{12} x)$. Working from the expression for $f_2$ given in the proof of Lemma 6.1 gives

$$h = \gamma_{23} y^{p+1} + \frac{1}{2} \left( \gamma_{12} x z^p + x^{p-1} \left( \gamma_{14} \delta + \gamma_{23} y^2 \right) + \gamma_{24} y x^p \right).$$

**Theorem 6.4.** If $\gamma_{12} \neq 0$ and $\gamma_{13} = 0$ then $\{x, N_G(y), h, N_G(z)\}$ is a SAGBI basis for $F[V_M]^G$. Furthermore, there is a single relation among the generators constructed by subducting $\gamma_{23} N_G(y)^{p+1} - h^p$.

**Proof.** For $\gamma_{12} \neq 0$, we can choose generators for $G$ using the left $\SL_2(F_p)$ action and a basis for $V_M$ using the right $F \times F^*$ action, so that

$$M := \begin{bmatrix} 1 & c_{12} \\ 0 & c_{22} \end{bmatrix}$$

and $\gamma_{12} = c_{22}$. Since $\gamma_{13} = 0$, we have $c_{12}^p = c_{12}$. Thus $c_{12} \in F_p$. Again using the $\SL_2(F_p)$ action to change generators, we can take

$$M := \begin{bmatrix} 1 & 0 \\ 0 & c_{22} \end{bmatrix}.$$ 

Hence $\deg(N_G(y)) = p$ and $\deg(N_G(\delta)) = 2p$. Therefore, using Theorem 4.4, we have $F[V_M]^G[x^{-1}] = F[x, N_G(y), N_G(\delta)][x^{-1}]$. An explicit calculation gives $N_G(\delta) = \delta^p - \delta x^{2p-2} c_{22}^{p-1}$. Evaluating gives $\bar{f}_{124} = -c_{22} N_G(\delta)$. Similarly, $\bar{f}_1 = c_{22} N_G(y)$. Using the definition of $f_2$, we have $\gamma_{12} f_{124} = 2f_2 x^{p-2} - f_1^p$. Thus $N_G(\delta)$ is in the algebra generated by $\{x, h, N_G(y)\}$. Hence $F[x, N_G(y), h][x^{-1}] = F[V_M]^G[x^{-1}]$.

Referring to the definition of $\widetilde{N}$ given above and using Lemma 6.1, $N' := 2N' / (c_{22} x)^p$ has lead term $x^{p^2}$. Define $B' := \{x, N_G(y), h, N'\}$. Applying Theorem 1.1 shows that $B'$ is a SAGBI basis for $F[V_M]^G$. Thus the lead term algebra of $F[V_M]^G$ is generated by $\{x, y^p, y^{p+1}, z^{p^2}\}$. Therefore $\{x, N_G(y), h, N_G(z)\}$ is a SAGBI basis for $F[V_M]^G$. Note that while the right $F \times F^*$ action results in a change of variables, the lead term algebra is preserved. □

For a representation with $\gamma_{12} = 0$ and $\gamma_{13} \neq 0$, define $d := \frac{\bar{f}_1}{x^{p-2}} = \gamma_{13} \delta + \gamma_{23} y x$. 

\[ \text{(14)} \]
Theorem 6.5. If $\gamma_{13} \neq 0$ and $\gamma_{12} = 0$ then \{x, d, NG_G(y), NG_G(z)\} is a SAGBI basis for $F[V_M]^G$. Furthermore, there is a single relation among the generators constructed by subducting $d^p - \gamma_{13}^2 NG_G(y)^2$.

Proof. For $\gamma_{12} = 0$ and $\gamma_{13} \neq 0$, we can choose generators for $G$ using the left $\text{SL}_2(F_p)$ action and a basis for $V_M$ using the right $F \times F^*$ action, so that

$$M := \begin{bmatrix} 1 & c_{12} \\ 0 & 0 \end{bmatrix}.$$ 

With this choice, $\gamma_{13} = \gamma_{12}^p - c_{12}$, $\gamma_{23} = 0$ and $d = \gamma_{12} \delta$. Since $V_M$ is a symmetric square representation, the result follows from Theorem 3.3. \qed

Remark 6.6. A representation with $\gamma_{13} = 0$ and $\gamma_{12} = 0$ is not faithful. If the image of the representation is not equal to the identity, then the representation is a rank one symmetric square representation and, using Theorem 3.3, the ring of invariants is a hypersurface with generators in degrees $(1, 2, p, p)$ and a relation in degree $2p$. This is a familiar example, first computed by Dickson [D2, Lecture III §7].

7. The invariants for rank three dimension three

In this section $G = (\mathbb{Z}/p)^3 = \langle e_1, e_2, e_3 \rangle$. We start with the generic representation over $k := F_p(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})$ given by $e_i \mapsto \sigma(x_{ij}, x_{ji})$. Thus, we assume $p > 2$. Consider $\mathcal{B} := \{x, f_1, f_2, NG_G(z)\}$ with $f_1$ and $f_2$ defined as in Section 5:

$$f_1 = -\gamma_{123} \delta^p - \gamma_{124} x^p y^p - \gamma_{134} \delta x^{2p^2} - \gamma_{234} y x^{2p^2 - 1},$$

$$f_2 = \gamma_{123} y^{p^2} - \gamma_{125} y^{p^2 - p} - \gamma_{135} \delta x^{p^2 - 2} - \gamma_{235} y x^{p^2 - 1}.$$ 

Thus $\text{LT}(f_1) = -\gamma_{123} y^{p^2}$ and $\text{LT}(f_2) = \gamma_{123} y^{p^2}$. There is a single non-trivial tête-à-tête: $(f_1^p, f_2^p)$.

Lemma 7.1.

$$\text{LT}(f_1^p + \gamma_{123}^p f_2^2 + 2(-\gamma_{123}(p-3)/2) \gamma_{125} x^{p^2 - p} f_1^{p+1}/2) = -2(\gamma_{123}^{p-1}) \gamma_{135} x^{p^2 - 2} y^{p^2 + 2}.$$ 

Proof. We work modulo the ideal in $k[x, y, z]$ generated by $x^{p^2 - 1}$, which we denote by $m$. Using the formulae given above, $f_1^p \equiv_m -\gamma_{123} x^{2p^2}, x^{p^2-p} f_1 \equiv_m -\gamma_{123} x^{p^2-p} y^{p^2}$ and

$$f_2^2 \equiv_m \gamma_{123} x^{2p^2} - 2\gamma_{123} \gamma_{125} x^{p^2-p} y^{p^2+p} - 2\gamma_{123} \gamma_{135} x^{p^2-2} \delta y^{p^2}.$$ 

Substituting and simplifying gives

$$f_1^p + \gamma_{123}^{p-2} f_2^2 + 2(-\gamma_{123}(p-3)/2) \gamma_{125} x^{p^2-p} f_1^{p+1}/2 \equiv_m -2(\gamma_{123}^{p-1}) \gamma_{135} x^{p^2-2} y^{p^2} \delta$$ 

and the result follows. \qed
Define
\[
f_3 := \frac{f_1^p + \gamma_{123}^{-2} f_2^2 + 2 (-\gamma_{123})^{(p-3)/2} \gamma_{125} x p^{2-p} f_1^{(p+1)/2}}{-2 x p^{2-2}}
\]
and \( B_1 := B \cup \{ f_3 \} \). There is a single new non-trivial tête-à-tête among the elements of \( B_1 \): \((f_3^p, f_1 f_2^p)\).

**Lemma 7.2.** For \( h := (x^3, x^2 y) k[x, y, z] \),
\[
f_3 \equiv h \gamma_{123}^{-1} (\gamma_{135} \delta y^3 + \gamma'_{235} x y p^{2+1} - \frac{1}{2} \gamma_{123} x^2 z y^3).
\]

**Proof.** We consider \( x p^{2-2} f_3 \) modulo \( h' := (x p^{2+1}, y x p^2) k[x, y, z] \). Substituting for \( f_1 \) and \( f_2 \) and simplifying modulo \( h' \) gives
\[
-2 x p^{2-2} f_3 \equiv h' \gamma_{123} x^2 p^2 z y^3 - 2 \gamma_{123}^{-1} x p^{2-p} (\gamma_{135} x p^{2-3} \delta y^3 + \gamma'_{235} x^{p-1} \delta y^3 + 1),
\]
and the required description of \( f_3 \) modulo \( h \) follows. \( \square \)

**Lemma 7.3.** There exist \( c_1, c_2, c_3 \in k \) such that
\[
\tilde{N} := f_3^p + \gamma_{135} \gamma_{123}^{p^2-2-p-1} f_1 f_2^p - c_1 x p f_1^{(p+1)/2} - c_2 x p^{2-p} f_2^{-1} f_3 - c_3 x p^{2-p-1} f_3^{(p+1)/2} f_2^{(p-3)/2} f_1^{(p-1)/2}
\]
has lead term \( -\frac{1}{2} \gamma_{123}^p x^2 p z^3 y^3 \).

**Proof.** We work modulo the ideal in \( k[x, y, z] \) generated by \( x^{2p+1} \) and \( y x^p \), which we denote by \( n \). By definition, \( f_2^p \equiv n \gamma_{123}^p y^3 p^3 \). Thus
\[
f_3^p + \gamma_{135} \gamma_{123}^{p^2-2-p-1} f_1 f_2^p \equiv n f_3^p - \gamma_{135} \gamma_{123}^{p^2-2-p-1} f_1 f_2^p.
\]

Using Lemma 7.2 and the definition of \( f_1 \) gives
\[
f_3^p + \gamma_{135} \gamma_{123}^{p^2-2-p-1} f_1 f_2^p \equiv n \tilde{c}_1 x^p y^3 p^3 + \tilde{c}_2 x^2 p^{2-2} \delta y^3 p^3 + \tilde{c}_3 x^{2-p-1} y^3 p^3 + 1 - \frac{1}{2} \gamma_{123}^p x^2 p z^3 y^3
\]
for some \( \tilde{c}_i \in k \). Using the definitions and Lemma 7.2, we have
\[
x^p f_1^{(p+1)/2} \equiv n (-\gamma_{123})^{(p^2+1)/2} x^p y^3 p^3 + p,
\]
\[
x^{2p-2} f_2^{-1} f_3 \equiv n \gamma_{123}^{2p-2} (x^{2p-2} \gamma_{135} \delta y^3 p^3 + \gamma_{235} x^{2p-2} y), \quad \text{and}
\]
\[
x^{2p-1} f_3^{(p+1)/2} f_2^{(p-3)/2} f_1^{(p-1)/2} \equiv n (-1)^{(p-1)/2} \beta x^{2p-2} y p^3 + 1,
\]
with \( \beta \) a monomial in \( \gamma_{123} \) and \( \gamma_{135} \). We then solve for the required \( c_i \). \( \square \)

**Theorem 7.4.** For the generic rank three representation, \( B = \{ x, f_1, f_2, f_3, N_G(z) \} \) is a SAGBI basis for \( k[V_3]^G \). Furthermore, there are two relations among the generators, one constructed by subducting \( f_3^p - \gamma_{135} \gamma_{123}^{p^2-2p-1} f_1 f_2^p \) and the other given by
\[
f_1^p + \gamma_{123}^{p^2} f_2^2 + 2 (-\gamma_{123})^{(p-3)/2} \gamma_{125} x p^{2-p} f_1^{(p+1)/2} + 2 x p^{2-2} f_3 = 0.
\]
Proof. Define $N := \tilde{N}/x^{2p}$. It follows from Lemma 7.3 and the definition of $f_3$ that $B' := \{x, f_1, f_2, f_3, N\}$ is a SAGBI basis for the algebra it generates. Applying Theorem 5.2 and Theorem 1.1 proves that $B'$ is a SAGBI basis for $k[V_3]_G$. Thus the lead term algebra of $k[V_3]_G$ is generated by $\{x, y^{2p}, y^{p^2}, y^{p^2+2}, x^{p^2}\}$. Since $\text{LM}(B) = \{x, y^{2p}, y^{p^2}, y^{p^2+2}, z^{p^3}\}$, $B$ is also a SAGBI basis for $k[V_3]_G$. The two non-trivial tête-à-têtes subduct to produce the relations. □

For $F$ an arbitrary field of characteristic $p$, consider the representation $V_M$ for

$$M := \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

with $c_{ij} \in F$, i.e., $e_i \mapsto \sigma(c_{1i}, c_{2i})$. Let $\psi_M : F_p[x_{ij}] \to F$ denote the evaluation map: $\psi_M(x_{ij}) = c_{ij}$. For $f \in k[x, y, z]$, let $\overline{f}$ denote the element of $F[x, y, z]$ constructed by minimally clearing denominators and applying $\psi_M$ to the coefficients. Note that we can also interpret elements of $F_p[x_{ij}]$ as regular functions on $F^{2 \times 3}$, the space of representations of the given type.

**Theorem 7.5.** If $\gamma_{123} \neq 0$ and $\gamma_{135} \neq 0$, then $\{x, y, z\} \cup \{\gamma_{123}, \gamma_{135}\}$ is a SAGBI basis for $F[V_M]_G$. Furthermore, there are two relations among the generators, one constructed by subducting $\overline{f^p} - \overline{\gamma_{135} \gamma_{123} \gamma_{135}^{p-2p-1} \gamma_{123}^{p-1}}$ and the other generated by

$$\overline{f_1^p} + \overline{\gamma_{123} \gamma_{135} \gamma_{123}^{-2} \gamma_{135}^{-2}} + 2(\overline{\gamma_{123})}^{(p-3)/2} \overline{\gamma_{123}} x^{p^2} - p \overline{f_1^{(p+1)/2}} + 2x^{p^2-2} \overline{\gamma_{123}} = 0.$$ 

Proof. We show that, in essence, the proof of Theorem 7.4 survives evaluation. Referring to the proof of Lemma 7.3, we see that $\tilde{c}_i \in F_p[x_{jk}]$ and that we can solve for the $\tilde{c}_i$ after inverting $\gamma_{123}$ and $\gamma_{135}$. Therefore, with $N := \tilde{N}/x^{2p}$, we have $\text{LM}(\overline{N}) = z^{p^3}$. Let $A$ denote the algebra generated by $B' := \{x, y, z\} \cup \{\gamma_{123}, \gamma_{135}\}$. Since $\gamma_{123} \neq 0$, we have $\text{LM}(\overline{f_1}) = y^{2p}$ and $\text{LM}(\overline{f_2}) = y^{p^2}$. Using Lemma 7.2, $\text{LM}(\overline{f_3}) = y^{p^2+2}$. The tête-à-tête $\overline{f_1} \overline{f_2}$ subducts to zero using the definition of $\overline{f_3}$ and the tête-à-tête $\overline{f_1} \overline{f_2}$ subducts to zero using the definition of $\overline{N}$. Thus $B'$ is a SAGBI basis for $A$.

Referring to the definition of $f_2$, we see that $\overline{f_3}$ has degree 1 as a polynomial in $z$ with coefficient $\overline{\gamma_{135}} x^{p^2-2}$. Thus $F[x, y, z, N_G(y)] [x^{-1}] = F[V_M]_G [x^{-1}]$. Using the notation of section 5, observe that $N_G(y) = \overline{\gamma_{135} \gamma_{135}}$. By definition $f_1 = f_{1234}$ and $f_2 = f_{1235}$. Applying Lemma 5.3 to the subsequence $(1, 4, 7)$ of $(1, 3, 4, 5, 7)$ shows that $\gamma_{345} \overline{f_1} \overline{f_3}$ can be written as an $F_p$-linear combination of $\gamma_{345} \overline{f_3} \overline{f_3}$ and $\gamma_{345} \overline{f_1} \overline{f_3}$. Observe that $\gamma_{345} = \gamma_{123}^p$ and $\overline{f_3} \overline{f_3} = \overline{f_3} \overline{f_3}$. Iteratively applying Lemma 5.3 as in the proof of Lemma 5.4 allows us to write $\gamma_{123} \overline{f_3} \overline{f_3}$ as an element of the $F_p[x_{jk}]$-span of $\{\overline{f_3} \overline{f_3}, \overline{f_3} \overline{f_3} \mid i = 0, 1\}$. Thus $N_G(y)$ is in the algebra generated by $\{x, y, z\}$. Therefore $A[x^{-1}] = F[V_M]_G [x^{-1}]$ and the result follows from Theorem 1.1. □

For a representation with $\gamma_{123} \neq 0$ and $\gamma_{135} = 0$, arguing as in the proof of Lemma 7.2, $\overline{f_3}/z$ is equivalent to $\overline{\gamma_{123}^{-1} (\overline{\gamma_{135}}^{p^2} - \gamma_{135} x z^{p^2})}$, modulo the ideal $(x^2, xy)F[x, y, z]$. The Plücker relation for $\Gamma$ determined by the sequences $(1, 3)$ and $(2, 3, 4, 5)$ gives $\gamma_{123} \gamma_{135} - \gamma_{134} \gamma_{235} + \gamma_{135} \gamma_{234} = 0$. Since $\gamma_{123} = \gamma_{345}$ and $\overline{\gamma_{123}} = 0$, we have $\overline{\gamma_{134}} \overline{\gamma_{235}} = \overline{\gamma_{123}}^p \neq 0$. Hence $\text{LM}(\overline{f_3}/x) = y^{p^2+1}$.
Theorem 7.6. If $\gamma_{123} \neq 0$ and $\gamma_{135} = 0$, then \( \{x, \overline{T}_1, N_G(y), \overline{T}_3/x, N_G(z)\} \) is a SAGBI basis for \( F[V_M]^G \). Furthermore, there are two relations among the generators constructed by subducting the tête-à-têtes \( (\overline{T}_3/x)^{p}, \overline{T}_1^{(p^{2}+1)/2}) \) and \( (\overline{T}_1^{p}, N_G(y)^2) \).

Proof. Define \( \overline{N} := (\overline{T}_3/(x\overline{T}_{123}))^p - (\overline{T}_{235}/\overline{T}_{123})^p (\overline{T}_{123}/\overline{T}_{123})^{(p^{2}+1)/2} \). Working modulo the ideal \( (x^{p+1}, x^{p}y)F[x, y, z] \), we see that \( LT(\overline{N}) = (-x z^{p}/2)^p \). Define \( N := \overline{N}/x^p \) and \( B := \{x, \overline{T}_1, \overline{T}_2, \overline{T}_3/x, N\} \). Let \( A \) denote the algebra generated by \( B \). There are two non-trivial tête-à-têtes among the elements of \( B \): \( (\overline{T}_1^p, \overline{T}_2^p) \) which subducts to zero using the definition of \( f_3 \), and \( (\overline{T}_3/x)^{p}, \overline{T}_1^{(p^{2}+1)/2}) \) which subducts to zero using the definition of \( N \). Therefore \( B \) is a SAGBI basis for \( A \).

For $\gamma_{123} \neq 0$ and $\gamma_{135} = 0$, we can choose generators for \( G \) and a basis for \( V_M \) so that

\[
M = \begin{bmatrix} 1 & c_{12} & 0 \\ 0 & c_{22} & c_{23} \end{bmatrix}.
\]

Using this description of \( V_M \), the orbit of \( y \) is \( \{y + (\alpha + c_{12}\beta)x \mid \alpha, \beta \in F_p\} \). Calculating the orbit product gives \( N_G(y) = \overline{T}_2/\gamma_{123} \in A \).

We now show that \( A[x^{-1}] = F[V_M]^G[x^{-1}] \). As in the proof of Theorem 4.4, observe that \( F[x, y, z][x^{-1}] = F[x, y, \delta/x][x^{-1}] \). Since \( f_1 = -\gamma_{123}\delta^{p} - \gamma_{124}\delta^{p}y^{p} - \gamma_{134}\delta^{p}x^{2p-2} - \gamma_{234}\delta^{p}x^{2p-1} \), we see that \( x^{-p}\overline{f}_1 \in F[x, y, \delta/x]^G \). Furthermore, \( N_G(y) \in F[x, y, \delta/x]^G \) and \( \deg(N_G(y)) \) \( \deg(x^{-p}\overline{f}_1) = 3 = |G| \). Therefore, using [DK, 3.7.5], we have \( F[x, y, \delta/x]^G = F[x, N_G(y), x^{-p}\overline{f}_1] \). Hence

\[
F[x, y, z]^G[x^{-1}] = F[x, y, \delta/x]^G[x^{-1}] = F[x, \overline{T}_1, \overline{T}_2][x^{-1}] = A[x^{-1}].
\]

Since \( A[x^{-1}] = F[V_M]^G[x^{-1}] \) and \( F[V_M]^G \) is integral over \( A \), it follows from Theorem 1.1 that \( B \) is a SAGBI basis for \( F[V_M]^G \). Hence the lead term algebra of \( F[V_M]^G \) is generated by \( \{x, y^{2p}, y^{p}, y^{p+1}, z^{3p} \} \) and \( \{x, \overline{T}_1, N_G(y), \overline{T}_3/x, N_G(z)\} \) is a SAGBI basis for \( F[V_M]^G \). \( \square \)

For the rest of this section, we consider representations for which \( \gamma_{123} = 0 \). If \( c_{1j} = 0 \) for all \( j \in \{1, 2, 3\} \), then \( F[V_M]^G = F[x, y, N_G(z)] \). Thus we will also assume that at least one \( c_{1j} \) is non-zero. Therefore, we can choose generators for \( G \) and a basis for \( V_M \) so that

\[
M = \begin{bmatrix} 1 & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \end{bmatrix}
\]

and \( c_{23}(c_{12} - c_{12}) = c_{22}(c_{13} - c_{13}) \). If \( \gamma_{124} = 0 \), we can take

\[
M = \begin{bmatrix} 1 & c_{12} & 0 \\ 0 & c_{22} & c_{23} \end{bmatrix}
\]

and if \( \gamma_{135} = 0 \), we can take

\[
M = \begin{bmatrix} 1 & c_{12} & 0 \\ 0 & c_{22} & c_{23} \end{bmatrix}.
\]
**Theorem 7.7.** If $V_M$ is a representation of type $(1, 1, 1)$ and $\gamma_{123} = \gamma_{124} = \gamma_{135} = 0$, then $V_M$ is not faithful.

**Proof.** Since $V_M$ has type $(1, 1, 1)$, at least one of the $c_1$ is non-zero. Therefore, since $\gamma_{124} = \gamma_{135} = 0$, we can choose generators for $G$ and a basis for $V_M$ so that

$$M = \begin{bmatrix} 1 & c_{12} & 0 \\ 0 & 0 & c_{23} \end{bmatrix}.$$ 

Thus $\gamma_{123} = c_{23}(c_{12} - c_{12}^p) = 0$. If $c_{23} = 0$ then $e_3$ acts trivially. If $c_{12}^p - c_{12} = 0$ then $c_{12} \in \mathbb{F}_p$ and $c_{12} e_1 - e_2$ acts trivially. \(\square\)

**Theorem 7.8.** If $\gamma_{123} = 0$, $\gamma_{124} = 0$ and $\gamma_{135} \neq 0$, then $V_M$ is a symmetric square representation.

**Proof.** Using the form for $M$ given in Equation (1), $c_{22} = 0$ and $c_{23}(c_{12}^p - c_{12}) = 0$. However, $\gamma_{135} \neq 0$ means that $c_{12}^p - c_{12} \neq 0$. Thus $c_{23} = 0$ and $V_M$ is a symmetric square representation. \(\square\)

Consider the case $\gamma_{123} = 0$, $\gamma_{124} \neq 0$, and $\gamma_{135} \neq 0$. The Plücker relation for $\Gamma$ determined by the sequences $(1, 2), (1, 3, 4, 5)$ gives

$$\gamma_{123} \gamma_{145} - \gamma_{124} \gamma_{135} + \gamma_{125} \gamma_{143} = 0.$$ 

If $\gamma_{123} = 0$, $\gamma_{124} \neq 0$, and $\gamma_{135} \neq 0$, then

$$f := \tilde{T}_1(-\gamma_{124} x^p)^{-1} = \tilde{T}_2(-\gamma_{125} x^{p^2 - p})^{-1} = y^p + \frac{\gamma_{134} \delta x^{p^2}}{\gamma_{124}} + \frac{\gamma_{124} \gamma_{134}}{\gamma_{124}} y x^{p - 1}$$

with $c := \frac{\gamma_{134}}{\gamma_{124}} \neq 0$. Since $\gamma_{135} \neq 0$, we have

$$N_G(y) = y^p + \frac{\gamma_{137}}{\gamma_{135}} y^p x^{p^2 - p^2} + \frac{\gamma_{157}}{\gamma_{135}} y^p x^{p^3} - p + \frac{\gamma_{357}}{\gamma_{135}} y x^{p^3 - 1}.$$ 

Working modulo the ideal $(x^{p^3 - p^2 - p^2}) \mathbb{F}[V_M]$, we see that the lead term of $\tilde{h} := N_G(y) - f^{p^2} + c x f^{2p} x^{p^3} - 2 c x^{p^2} f^{p+2} x^{p^3} - p^2 - 2p$ is $-4 c^{p^2} + 1 y^{p^2} + 2 x^{p^3} - p^2 - 2$. Thus $h := \tilde{h}(-4 c^{p^2} + p^{p+1} x^{p^3} - p^{p^2} - p^2)^{-1}$ has lead term $y^{p^2} + p^2$.

**Theorem 7.9.** If $\gamma_{123} = 0$, $\gamma_{124} \neq 0$ and $\gamma_{135} \neq 0$, then $\{x, f, h, N_G(z)\}$ is a SAGBI basis for $\mathbb{F}[V_M]^G$. Furthermore, there is a single relation among the generators constructed by subducting $f^{p^2 + p^2} - h^p$.

**Proof.** Subducting the tête-à-tête $f^{p^2 + p^2} - h^p$ with respect to $\{x, f, h\}$ gives a remainder $\tilde{N}$ with lead monomial $z^{p^3} x^{p^2 + 2p}$. Define $\tilde{N} := N / x^{p^2 + p}$ and $B' := \{x, f, h, N\}$. Let $A$ denote the subalgebra generated by $B'$. By construction, $B'$ is a SAGBI basis for $A$. It follows from the definition of $\tilde{h}$ that $N_G(y) \in A$. Observe that $f$ is degree 1 in $z$ with coefficient $-c x^{p-1}$. Therefore, by \cite[Thm. 2.4]{CC}, $A[x^{-1}] = \mathbb{F}[V_M]^G[x^{-1}]$. Applying Theorem 1.1, we see that $B'$ is a SAGBI basis for $\mathbb{F}[V_M]^G$. Hence the lead term algebra of $\mathbb{F}[V_M]^G$ is generated by $\{x, y^p, y^{p^2 + p^2}, z^{p^3}\}$. Therefore $\{x, f, h, N_G(z)\}$ is a SAGBI basis for $\mathbb{F}[V_M]^G$. \(\square\)
Consider the case $\gamma_{123} = \gamma_{135} = 0$ and $\gamma_{124} \neq 0$. Since $\gamma_{123} = 0$, using the form for $M$ given in Equation 2, we see that $c_{23}(c_{12}^2 - c_{12}) = 0$. If we assume $V_M$ is faithful, then $c_{23} \neq 0$ and $c_{12} \in F_p$. In this case $N_G(y) = y^p - yx^{p-1}$ and

$$N_G(\delta) = d_1(U)\delta^p - d_2(U)\delta x^{2p-2}$$

where $d_1(U) = \overline{\gamma}_{120}/\overline{\gamma}_{124}$ and $d_2(U) = \overline{\gamma}_{146}/\overline{\gamma}_{124}$ are the Dickson invariants for $U = \text{Span}_{F_p}\{c_{22}, c_{23}\}$. Working modulo the ideal $(x^{p^2})F[V_M]$ we see that the lead term of

$$\tilde{g} := N_G(\delta) - N_G(y)^{p-1}x^{p(p-1)}$$

is $-2y^{p^2+1}x^{p^2-1}$. Thus $g := \tilde{g}(-2x^{p^2-1})^{-1}$ has lead term $y^{p^2+1}$.

**Theorem 7.10.** If $V_M$ is a faithful representation with $\gamma_{123} = \gamma_{135} = 0$ and $\gamma_{124} \neq 0$, then $\{x, N_G(y), g, N_G(z)\}$ is a SAGBI basis for $F[V_M]^G$. Furthermore, there is a single relation among the generators constructed by subducting $N_G(y)^{p^2+1} - g^p$.

**Proof.** Define $\tilde{N} := N_G(y)^{p^2+1} - g^p + gN_G(y)^{(p-1)}x^{p-1}$. Working modulo the ideal $(x^{p^2+1}, yx^p)F[x, y, z]$, an explicit calculation gives $\text{LT}(\tilde{N}) = \frac{1}{2}z^{p^3}x^p$. Define $N := \tilde{N}/x^p$ and $B' := \{x, N_G(y), g, N\}$. Let $A$ denote the subalgebra generated by $B'$ and note that $B'$ is a SAGBI basis for $A$. It follows from the definition of $\tilde{g}$ that $N_G(\delta) \in A$. Since $\deg(N_G(y))\deg(N_G(\delta)) = 2p^3 = 2|G|$, applying Theorem 4.4, we see that $F[V_M]^G[x^{-1}] = F[x, N_G(y), N_G(\delta)]|x^{-1}] = A[x^{-1}]$. It then follows from Theorem 1.1 that $B'$ is a SAGBI basis for $F[V_M]^G$. Hence the lead term algebra of $F[V_M]^G$ is generated by $\{x, y^p, y^{p^2+1}, z^{p^3}\}$. Therefore $\{x, N_G(y), g, N_G(z)\}$ is a SAGBI basis for $F[V_M]^G$. \[\square\]

8. Conjectures and conclusions

Numerous computer calculations using Magma [BCP] support the following.

**Conjecture 8.1.** Let $V$ denote the generic three-dimensional representation of $G = (\mathbb{Z}/p)^r = \langle e_1, \ldots, e_r \rangle$ over $k = F_p(x_{1j}, x_{2j} | j = 1, \ldots, r)$ and define $s = [r/2]$. Then $k[V]^G$ is a complete intersection with embedding dimension $s + 3$. Furthermore, there exists a SAGBI basis $\{x, f_1, \ldots, f_{s+1}, N_G(z)\}$ such that:

(a) if $r = 2s$ then

$$\text{LM}(f_1) = y^{p^s}, \quad \text{LM}(f_i) = y^{p^{s+i-2}+2p^{s-i+1}}$$

for $i > 1$ and the relations come from subducting the tête-à-têtes $(f_2^p, f_1^{p+2})$ and $(f_i^p, f_{i-1}^p)_{(p^2-1)p^{i-3}}$ for $i > 2$;

(b) if $r = 2s - 1$ then

$$\text{LM}(f_1) = y^{2p^{s-1}}, \quad \text{LM}(f_2) = y^{p^s}, \quad \text{LM}(f_i) = y^{p^{s+i-3}+2p^{s-i+1}}$$

for $i > 2$ and the relations are constructed by subducting the tête-à-têtes $(f_3^p, f_2^p), (f_3^p, f_1 f_2^p)$ and $(f_i^p, f_{i-1} f_2^p)_{(p^2-1)p^{i-4}}$ for $i > 3$.

In Section 4, we showed that the only three-dimensional representations of $G = (\mathbb{Z}/p)^r$ for which the ring of invariants fails to be a polynomial algebra,
are of the form $V_M$ for some $M \in \mathbf{F}^{2 \times r}$. Rings of invariants for these representations are parameterised by $\mathbf{F}^{2 \times r} \sslash \text{GL}_r(\mathbf{F}_p)$. In Section 6, we identified $\gamma_{12}, \gamma_{13} \in \mathbf{F}[\mathbf{F}^{2 \times 2}]^{\text{SL}_2(\mathbf{F}_p)}$ such that representations satisfying $\gamma_{12} \neq 0$ and $\gamma_{13} \neq 0$ are essentially generic. In particular, these representations have rings of invariants which are generated in degrees $1, p, p+2, p^2$ with a single relation in degree $p(p+2)$. The remaining rank 2, dimension 3 representations fall into three cases: $\gamma_{12} \neq 0$ and $\gamma_{13} = 0$, giving generators in degrees $1, p, p+1, p^2$ with a relation in degree $p(p+1)$; $\gamma_{12} = 0$ and $\gamma_{13} \neq 0$, giving generators in degrees $1, 2, p, p^2$ with a relation in degree $2p$; $\gamma_{12} = 0$ and $\gamma_{13} = 0$, which fails to be faithful. In Section 7, we used $\gamma_{123}, \gamma_{135}, \gamma_{124} \in \mathbf{F}[\mathbf{F}^{2 \times 3}]^{\text{SL}_3(\mathbf{F}_p)}$ to stratify $\mathbf{F}^{2 \times 3} \sslash \text{GL}_3(\mathbf{F}_p)$. Representations satisfying $\gamma_{123} \neq 0$ and $\gamma_{135} \neq 0$ are essentially generic. Representations satisfying $\gamma_{123} \neq 0$ and $\gamma_{135} = 0$ form a second family and representations satisfying $\gamma_{123} = 0$ fall into four cases: $\gamma_{135} \neq 0$ and $\gamma_{124} \neq 0$; $\gamma_{135} = 0$ and $\gamma_{124} \neq 0$; $\gamma_{135} \neq 0$ and $\gamma_{124} = 0$; $\gamma_{135} = 0$ and $\gamma_{124} = 0$. The representations with $\gamma_{123} \neq 0$ have rings of invariants which are complete intersections with embedding dimension 5. The rings of invariants for representations with $\gamma_{123} = 0$ are hypersurfaces. We believe that for an arbitrary $r$ it is possible to stratify $\mathbf{F}^{2 \times r} \sslash \text{GL}_r(\mathbf{F}_p)$ using elements of $\mathbf{F}[\mathbf{F}^{2 \times r}]^{\text{SL}_r(\mathbf{F}_p)}$, and that in the essentially generic case, the ring of invariants is a complete intersection with embedding dimension $\lceil r/2 \rceil + 3$, and that in all other cases, the ring of invariants is a complete intersection with embedding dimension at most $\lceil r/2 \rceil + 3$.

**Conjecture 8.2.** For a faithful modular three-dimensional representation of the group $(\mathbb{Z}/p)^r$, the ring of invariants is a complete intersection with embedding dimension less than or equal to $\lceil r/2 \rceil + 3$.

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