In this work, we irradiate a superconducting artificial molecule composed of two magnetic-flux-tunable transmons with microwave light while monitoring its state via joint dispersive readout. At certain fluxes, the molecule demonstrates a complex spectrum deviating qualitatively from the solution of the Schrödinger equation without driving. We reproduce the observed extra spectral lines accurately by numerical simulations, and find them to be a consequence of an Autler-Townes-like effect when a single tone is simultaneously dressing the system and probing the transitions between new eigenstates. We present self-consistent analytical models accounting both these processes at the same time that agree well with both experiment and numerical simulation. This study is an important step towards understanding the behaviour of complex systems of many atoms interacting coherently with strong radiation.

I. INTRODUCTION

Over the past twenty years, superconducting artificial atoms (SAAs) were used in numerous experiments in a compelling demonstration of the validity of fundamental quantum mechanical laws [1,2]. Their Hamiltonians can be pre-designed and engineered which makes them a particularly versatile tool for studies in quantum optics, and high controllability of their parameters allows direct observation of novel physical effects previously inaccessible for natural systems.

One of the most prominent milestones that superconducting quantum circuits have reached so far is the strong coupling with light in circuit QED [3,4] when the relaxation and decoherence rates appear smaller than the Rabi frequency. Currently, they are surpassing all other implementations of strong coupling in terms of coherence [5]. However, in sharp contrast with natural atoms and molecules, SAAs do not even require confined radiation to implement strong coupling with light: they may be coupled unprecedently strongly to free-propagating electromagnetic waves in 1D on-chip waveguides [6] without using cavities at all. The Rabi frequency in this case may reach 50% of the transition frequency [7] which is even beyond strong coupling regime [5]. To correctly describe the behaviour of an atom in these conditions, the so-called dressed atom approach [8] is employed when the radiation has to be directly included in the Hamiltonian and affects its level structure.

So far, experiments with intense driving fields resulting in the dressing effects in on-chip quantum optics were focused on single artificial atoms [6,9–16]. In all these works, light dressing of the atom manifests itself through Mollow triplets or Autler-Townes (A-T) splittings of different kinds. However, despite the recent successes in control of large arrays of interacting SAAs [17,19], there were, up to our best knowledge, no experiments concerning interaction of similar composite structures with intense driving field. While there were studies on dressing of multi-atomic systems in a cavity [20–23], their interaction with external dressing field is not less attractive since its frequency may be easily tuned to observe various absorption processes for arbitrary configurations of the system parameters.

In this work, we study a pair of strongly coupled artificial atoms: a superconducting artificial molecule [24] (SAM). We use two Xmon-type [25] transmons [26] interacting with each other through a cavity bus [27]. Microwave radiation is applied to this system through an on-chip coplanar antenna while its state can be monitored by joint dispersive readout using the same coupling cavity [28]. We find that strong interaction with microwaves not only results in multiphoton transitions of various orders between SAM states, but also significantly modifies its level structure. Even in a simple diatomic molecule, this leads to complex Autler-Townes-like effects involving single- and multiphoton transitions that can only be explained in the dressed picture. We note that although the A-T splittings have been investigated before in a wide range of quantum systems (including natural molecules [29,30]), we find qualitatively new spectral manifestations of light dressing when SAAs are irradiated unequally. We note that there were no such experiments with natural molecules, which could in principle demonstrate same effects, because they lack controllability and coherence of superconducting quantum devices, and previous studies of SAAs under strong driving field either involved just a single atom or have demonstrated only the standard spectral signatures known from the quantum optics
Besides its fundamental importance, this experiment may be useful for future quantum computers: a quantum engineer should account for the observed behaviours carefully controlling the driving power (for instance, as we will show, the bSWAP gate \[31\] is directly affected by light dressing).

The manuscript consists of four main parts and an Appendix. Section I is this introduction; Section II is devoted to the approaches that were used in our study; Section III contains the results of our experimental and theoretical research, including numerical simulations and analytical analysis; finally, in Section IV we make a conclusion of our work and discuss future prospects. The Appendix contains important details of the theoretical framework that we use.

II. METHODS

A. Device design and control

We have designed the SAM as a pair of transmon (Xmon) SAAs with asymmetric SQUIDs \[32\] (maximal Josephson energies \(E^1_J/h = 24.3\) GHz, \(E^2_J/h = 18.3\) GHz, charging energies \(E^1_C/h = 220\) MHz, asymmetries \(d^1/2 = 0.75\) ) coupled to a single notch-type \[33\] \(\lambda/4\) resonator \((f_r = 7\) GHz, \(\kappa = 1/100\) ns\(^{-1}\)) which both couples \[27\] the transmons and allows joint readout \[28\] of their states. In Fig. 1(a), the optical photograph of the device is shown, where the layout of the components is presented. The resonator in its upper part is connected to a coplanar waveguide through which the readout is performed. In its lower part, it is coupled to the transmons by a dual "claw" \[25\] coupler. Flux lines that allow independent control of the transmon frequencies are coming from the sides. Finally, the excitation waveguide coming from below directs microwave signal towards the SAM. In Fig. 1(b), the equivalent electrical circuit of the device is shown. The resonator fundamental mode is approximated as an LC-circuit (in the middle), the transmons are shown in color on the sides; their SQUIDs are represented as tunable Josephson junctions. A capacitively coupled excitation line is also shown in green. Finally, in Fig. 1(c) we show schematically the frequencies of the transmons depending on electric current \(I\) which we apply to an external coil wound around the sample holder. Since the effective junction of the first transmon is larger, its spectrum (orange) lies higher in frequency than the spectrum of the second transmon (blue). Using individual flux-control lines, it is possible to align the SAAs so that the lower sweet-spot of the orange transmon is just below the upper sweet-spot of the blue transmon (see the dashed rectangle, Fig. 1(c)). This configuration is, as we will show in the following, convenient to track the energies of highly excited levels via multi-photon transitions in a single spectroscopic scan. Additionally, the transmons are better protected from the flux noise in this area being close to their sweet spots.

![Fig. 1. (a) Optical image of the device (false coloured). Two transmons (orange, 1 and blue, 2) are coupled capacitively to a \(\lambda/4\) coplanar resonator. Frequency control lines come from both sides, and from below a waveguide for microwave excitation is connected (green). (b) The equivalent electrical circuit. Tunable Josephson junctions are SQUIDs with magnetic flux control. (c) Frequencies of the fundamental transitions of the transmons \(\omega_1(I)\) and \(\omega_2(I)\) depending on the current \(I\) in an external coil when correctly aligned by the individual flux control lines. In this work, we focus on the area inside the dashed rectangle.](image)

| Parameter       | Transmon 1 | Transmon 2 |
|-----------------|------------|------------|
| \(\omega/2\pi\) [GHz] | 5.12 - 6.3 | 4 - 5.45    |
| \(\alpha/2\pi\) [MHz] | -220       | -220       |
| \(T_1\) [\(\mu\)s] | 6.82       | 4.41       |
| \(T_2\) [\(\mu\)s] | 5.14       | 3.33       |
| \(J/2\pi\) [MHz] | 3          |

TABLE I. SAM parameters. Transmons are only different in the frequency tuning range and coherence times measured in the lower sweet spot for the 1\(^{st}\) one and in the higher sweet spot for the 2\(^{nd}\) one. The coupling strength \(J\) depends on the transmon frequencies and is specified here for \(\omega_1/2\pi = \omega_2/2\pi = 5.32\) GHz.

B. Theoretical background

A single transmon SAA can be regarded as an oscillator with a quartic perturbation describing the leading-order anharmonicity. Therefore, in the main text we do not use the charge and phase operators and write down its Hamiltonian using only the annihilation operator \(\hat{b}\):

\[
\hat{H}_r = \hbar \omega \hat{b}^\dagger \hat{b} + \frac{\hbar}{2} \alpha \hat{b}^\dagger \hat{b} (\hat{b}^\dagger \hat{b} - 1),
\]

where \(\omega\) is its \(\ket{0} \rightarrow \ket{1}\), or fundamental, transition frequency and \(\alpha\) is the anharmonicity of the transmon. By applying magnetic flux to the transmon’s SQUID (either via an individual on-chip line or via an external coil) it is possible \[26\] to directly control \(\omega\). In our modelling, we take into account the three lowest states of the transmon \((\ket{0}, \ket{1}\) and \(\ket{2}\)).
describes the SAA without interaction with light. To model a monochromatic microwave signal of frequency $\omega_d$ applied through a capacitively coupled transmission line, the following driving term should be included in the Hamiltonian:

$$\hat{H}_d = \hbar \Omega (b + b^\dagger) \cos \omega_d t,$$

where $\Omega$ is the driving amplitude coinciding with the frequency of the Rabi oscillations between $|0\rangle$ and $|1\rangle$.

Next, we assemble the model for two coupled transmons with the corresponding annihilation operators $\hat{b}$ and $\hat{c}$, the fundamental frequencies $\omega_{1,2}$ and anharmonicities $\omega_{1,2}$. The corresponding Hamiltonian of the SAM contains two terms representing each transmon, two terms representing the interaction of each transmon with the driving field at the same frequency $\omega_d$, and the transmon-transmon interaction term:

$$\hat{H} = \hat{H}_{i1} + \hat{H}_{i2} + \hat{H}_{d1} + \hat{H}_{d2} + \hat{H}_{int},$$

where $\hat{H}_{int} = \hbar J (b + b^\dagger)(c + c^\dagger)$ is the transverse interaction between the atoms. Strictly speaking, $J = J(\omega_n, \omega_d)$ depends on the transmon frequencies [26], but we take $J$ to be a constant as in Table I due to its negligible variation for our range of frequencies.

For brevity, the SAM Hamiltonian without driving terms and the corresponding eigenenergies will be referred below as “unperturbed”. Since we take 3 levels for each transmon, there will be total 9 basis states of the SAM $|i\rangle \otimes |j\rangle = |ij\rangle$, where $i$ and $j$ show the number of excitations in the first transmon, and in the second one, respectively.

In the following, we will also transform Eq. (3) into the frame rotating with both drives by an operator

$$\hat{R} = \exp[-i(t(\omega_1^2b^2 + \omega_2^2c^2)t)],$$

arriving at

$$\hat{H}_R = \hat{R}^\dagger \hat{H} \hat{R} - i \hat{R}^\dagger \partial_t \hat{R}.$$  

Here we use subscripts 1 and 2 to denote first and second SAAs. After the transformation and application of the RWA

$$\omega_{1(2)} \rightarrow \Delta_{1(2)} = \omega_{1(2)} - \omega_d^{(1,2)},$$

$$\hat{H}_{int} \rightarrow \hbar J \left[\hat{b}^\dagger e^{i(\omega_1^1 - \omega_2^1)t} \hat{b} + \hat{c}^\dagger e^{-i(\omega_1^2 - \omega_2^2)t} \hat{c}\right],$$

$$\hat{H}_{d1} \rightarrow \frac{\hbar \Omega}{2} (b + b^\dagger), \hat{H}_{d2} \rightarrow \frac{\hbar \Omega}{2} (c + c^\dagger).$$

Besides the unitary evolution, we take into account the incoherent processes of relaxation and dephasing for each transmon. They are modelled using the Lindblad equation with the following collapse operators [34]:

$$\hat{O}_{\gamma_11} = \sqrt{\gamma_1} \hat{b}, \hat{O}_{\phi_11} = \sqrt{\gamma_0} \hat{b}^\dagger \hat{b},$$

$$\hat{O}_{\gamma_12} = \sqrt{\gamma_2} \hat{c}, \hat{O}_{\phi_22} = \sqrt{\gamma_0} \hat{c}^\dagger \hat{c},$$

where $\gamma_{1,2}$ are the individual relaxation rates, and $\gamma_{0(1,2)}$ are the pure dephasing rates. As one can see, the collapse operators are in a separable form, i.e. acting only upon a single transmon each. This is a valid approach until the coupling strength $J$ is not too large [35]. Therefore, the complete evolution equation for the system density matrix $\hat{\rho}$ is

$$\partial_t \hat{\rho} = \frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \sum_{\alpha, \beta} \mathcal{D}[O_{\alpha \beta}] \hat{\rho} = \mathcal{L} \hat{\rho},$$

where $\mathcal{D}[O] = \hat{O} \hat{\rho} \hat{O}^\dagger - \frac{1}{2} [\hat{O}^\dagger \hat{O}, \hat{\rho}]$ and $\mathcal{L}$ is the Liouvillean superoperator, or the Liouvillian; $(R)$ denotes if the Hamiltonian and the corresponding solution density matrix are in the rotating frame with RWA. In this work, we do not alter the dissipator terms when changing the reference frame despite that it may not be correct in general [36].

The readout resonator is not included in the model since in the dispersive regime it does not affect the dynamics of the SAM. Hence, the readout is modelled just by some measurement operator $\hat{M}(f_p)$ that can be obtained by finding the transmission $S_{21}(f_p)$ ($f_p$ is the probe frequency near the resonance) through the sample after preparing various states of the SAM. Another way to find $\hat{M}(f_p)$ is to calculate $S_{21}(f_p)$ via offsetting the experimental resonance curve measured when the SAM is in the ground state by the dispersive shifts corresponding to the SAM basis states [28][37]. Since we have not directly measured these dispersive shifts, in our modelling they are treated as fitting parameters. Finally, the observable value for any state $\hat{\rho}$ is calculated as $S_{21}(f_p) = \text{Tr}[\hat{M}(f_p)\hat{\rho}]$.

C. Numerical solution in qutip

Numerical simulations are necessary for studying Eq. 8 since is does not have an analytical solution. We have been using the qutip [38] (quantum optics toolbox in Python) to simulate the dynamics and to find the steady state of the system for various parameter combinations of the Liouvillian. The source code for the simulations can be found on GitHub [39].

Two distinct modes of simulation were used in the modelling. The first one is for the Liouvillians that do not explicitly depend on time. In this case, the steady state $\rho_s$ of the system should be calculated from the set of linear equations obtained from Eq. 8:

$$\partial_t \hat{\rho} = 0 \Rightarrow \mathcal{L} \hat{\rho} = 0.$$  

This equation is solved with the qutip’s steadystate function [40]. This method is applicable when the driving frequency for both transmons is the same because from Eq. 6 it follows that $\hat{H}_{int}$ is time independent when $\omega_d = \omega_2^d$.

The second mode is required when it is not possible to avoid the time-dependence of the Liouvillian or if one wants to solve the master equation in the laboratory frame. For example, when the transmons have to be excited at different frequencies ($\omega_d^1 - \omega_d^2 = \delta \neq 0$), from Eq. 6 we find that even in the doubly-rotating frame, $\hat{H}_{int}$ is oscillating, and it is not possible to simply drop this term in RWA because it is inherent for the SAM. In this case, to find the steady state of the system one can employ the qutip’s propagator and propagator_steady-state functions. The propagator is a completely positive map $\mathcal{A}(t_1, t_0): \hat{\rho}(t_0) \rightarrow \hat{\rho}(t_1)$ describing the time evolution of the system density matrix; for Eq. 8 it is defined
where $T_n$ is the time-ordering superoperator. Since the Liouvillean is periodic with a period of $T = 2\pi/\delta$, it is possible to calculate the steady state as the eigenvector $\hat{\rho}_{ss}$ of the single-period propagator $\Lambda(T, 0)$ corresponding to its largest eigenvalue $\vert 41, 42 \rangle$. Upon infinitely many applications of $\Lambda$,

$$
\lim_{n \to \infty} [\Lambda(T, 0)]^n \hat{\rho} = \Lambda(nT, 0)\hat{\rho} \to \hat{\rho}_{ss}.
$$

### III. RESULTS

#### A. Spectroscopy: experiment and numerical simulation

Experiment was conducted as described in Appendix D. We use high-power two-tone spectroscopy to probe transitions between eigenstates of the SAM; the result is shown in Fig. 2 (a). As one can see, some new spectral lines are visible besides the fundamental ones that were shown in Fig. 1 (c). Their frequency also depends on the applied current, and at several points they become resonant with each other. At three pairs of such resonant points, we observe distinct features shown in insets marked with Roman numbers. Some secondary details of such resonant points, we observe distinct features shown in insets marked with Roman numbers. Some secondary details are shown with Arabic-numbered markers.

In areas I, II and III, we observe peculiar avoided crossings forming triplet transitions. In I and II, they are barely visible, however in III the splitting is very pronounced. To check whether our theoretical model summarized in Eq. 8 can reproduce the experimental spectrum, we have solved the master equation Eq. 8 finding the steady states of the system $\hat{\rho}_{ss}$ and the corresponding expected measurement outcomes $Re \text{Tr}[M\hat{\rho}_{ss}]$ depending on the excitation frequency and magnetic flux. The results are shown in Fig. 2 (b) where all the experimental details are immediately reproduced with just 9 SAM states (i.e., 3 levels for each transmon). We have solved Eq. 8 in the rotating frame with RWA using Eq. 9 and in the lab frame (using the propagator approach Eq. 10) and did not find any noticeable difference in the results; though, the runtime of the whole 401 by 401 points simulation was 9 hours without RWA vs. 3 minutes with RWA. As can be seen from three slices of Fig. 2 (a),(b) in Fig. 2 (c), the numerical results are in a good agreement with the experiment (the spurious resonance is softened). The parameters of model were chosen to match with the experimental data.

Remarkably, features I, II and III turn out to be impossible to explain using only the unperturbed Hamiltonian; we confirm this by numerical diagonalization taking the same 9 SAM states as in the previous simulation. All possible transitions between all resulting eigenlevels of the SAM depending on the coil current are shown in Fig. 2 (d) with solid and dashed lines. While correctly reproducing, for instance, the avoided crossing labelled as feature 3 in Fig. 2 (a), it completely misses the Roman-numbered effects.

So far, we have established the fact that the same model SAM yields different spectra depending on the presence of the driving, and that the full model [Eq. 8] agrees well with experimental data. To understand the nature of I, II and III, we describe Fig. 2 in more detail in a separate section below.

#### B. Analysis of the spectra

**a. Identification of the spectral lines.** Though it is not possible to describe all of the experimental features with the unperturbed model, we can still use it to identify most of the observed transitions as follows. By numerical diagonalization, we calculate the unperturbed frequencies of the undriven SAM as $\omega_{mn} = (E_n - E_m)/\hbar$ where $n > m$, $n, m = 0, 9$. In Fig. 2 (d), we place them over the heatmap data optionally dividing $\omega_{mn}$ by some integer $n$ for $n$-photon processes. Since the number of SAM states is finite, we can quickly find all matches between the heatmap and the calculated eigenlevels.

Next, at the sweet spot, we name the found lines as $i/j/n$ where $i, j$ denote the final basis state $|ij\rangle$ of the SAM after the transition from the ground state, and $n$ is optional for an $n$-photon process. We can do that because at the sweet spot the transmons are detuned from each other, and thus SAM eigenstates are nearly factorized (dispersive regime). Using this notation, we label lines in Fig. 2 (b),(c), as well. One should be careful because for Fig. 2 (d) this notation is valid only near the sweet spot before any two lines form an avoided crossing. When they do, their names should be swapped.

As one can see, various multiphoton transitions besides the main lines at $\omega_{12}(I)/2\pi$ (or 01 and 10 lines) are visible. The 02/2 and 02/2 are very commonly found for transmons and lie $|\alpha_{12}|/2$ lower than the main lines $\omega_{12}(I)$. Another two-photon process appears as 11/2 line when two SAAs are excited simultaneously. This process is used, for example, for doing the bSWAP gate [31]; in our case, it is taking part in feature I. A three-photon process 12/3 is also clearly visible just below the 02/2 line. As we will see, processes 12/3 and 21/3 are involved in forming features III and II, respectively. Notably, transitions 11/2, 21/3 and 12/3 are forbidden when there is no interaction between transmons ($J = 0$). Therefore, it is expected that all Roman-numbered features should only appear with non-vanishing $H_{int}$.

When two lines intersect, they may form an avoided crossing even for an unperturbed model if the corresponding matrix element of $H_{int}$ is non-zero. In such cases, the SAM becomes inseparable with some of the eigenstates becoming superpositions of the factorized states. For a two-transmon SAM, such points are, for instance, anticrossings between 01,10 or 11,02(20) visible clearly in Fig. 2 (d). Additionally, marked as feature 3 in Fig. 2 (d) we see an avoided crossing between 12/3, 21/3 at the same current where 01, 10 intersect.

**b. Analysing features I, II and III.** First, we have reproduced the avoided crossing of feature III in an additional numerical simulation taking only 2 levels for the transmon 1 and three levels for the transmon 2. Upon this, it has become clear that features II and III are actually of the same nature and differ only by the ordering of the transmons: for II, they appear whenever the transmon 01 intersects with the two-photon one 20/2, and for III when 10 intersects 02/2; one can see this
From Fig. 2 (d) we conclude that avoided crossing in III is between two transitions: 12/3 (three-photon transition |00⟩ → |12⟩) and 10 − 02 (|10⟩ → |02⟩) which are of the same frequency when \(\omega_3 = \omega_1 + \alpha_1/2\). The latter process is depopulating 10 and it is better discernible in Fig. 2 (d) than in Fig. 2 (a), (b). For II, the opposite is true: 21/3 and 01 − 20 are crossing when \(\omega_2 = \omega_1 + \alpha_1/2\). From additional measurements and simulations, we find that the splitting depends on the driving power; the experimental and simulated results for II are shown in colour in Fig. 3 (a), (b), respectively. As one can see, the growth of the splitting with increasing power is linear: it is roughly equal to the FWHM of the 01 spectral line. To fully quantify the shape of this splitting, in Section III C we derive analytical expressions for the dashed curves fitting the spectral lines in Fig. 3. In Fig. 3 (c), blue points, we present splitting sizes extracted by fitting of the model to the data as in Fig. 3 (a) for various power values of the microwave generator connected to the excitation waveguide. As one can see from the linear approximation of the points, the splitting indeed is simply proportional to the RMS voltage of the signal. From the model, we expect that the minimal distance between curves in the anticrossing is equal to \(\sqrt{\frac{3}{2}} \Omega_2\); using this relation, we extract the proportionality coefficient
between $\Omega_2$ and $V_{rms}$ is around 0.23 MHz/mV from the linear fit.

In Fig. 4 we demonstrate a similar power dependence of feature I: increasing drive amplitude on one of the transmons while keeping the other one small and constant again elicits avoided crossings with the proportional size. We show only the simulation results; an experiment is not possible with our sample because, since we have just a single excitation line, there is no way for us to control the driving amplitudes $\Omega_1, 2$ individually. We note that two qualitatively different patterns arise depending on which of the transmons is driven stronger than the other. From this and from the shape of the splitting in feature I of Fig. 2 (a), we can infer that $\Omega_1/\Omega_2 \approx 2$ there (also consistent with the 01, 10 linewidths in Fig. 2 (c)). If in contrast both transmons are driven with equal amplitudes, the avoided crossing vanishes. As can be seen from black dashed lines in Fig. 4 all these cases are explained well by our analytical model described in detail in Sections III C and III D.

From all presented observations, we conclude that effects I – III are caused by light dressing. In the case III, the first transmon is dressed by a strong resonant field; in case II, the second one; finally, in case I, both transmons may be dressed at the same time. We will discuss these effects in greater detail in Sections III C and III D.
FIG. 4. Simulated population of the ground state when the transmons are driven at the same frequency but at different amplitudes $\Omega_{1,2}$. In the top row (middle row), $\Omega_{1(2)}$ is increased while $\Omega_{2(1)} = \text{const}$. $\Omega_{2(1)} \ll \Omega_{1(2)}$; two topologically different types of anticrossings arise depending on which transmon is driven stronger. In the bottom row, we show how the splittings vanish when the weaker driving is increased to match with the stronger one. Gray dashed lines show the solution without driving, in black are the model curves based on Sections III C and III D.

**c. Secondary features.** Using Fig. 2 (d), we can get an insight into the features 1 – 5 as well.

Feature 1 is a small avoided crossing between 02/2 and 21/3. It is missing in the unperturbed solution and thus is caused by the light dressing just as I – III. Feature 2 is its twin: 12/3 and 20/2 intersect there but the anticrossing is smaller due to the asymmetry of the driving strengths and is not resolved.

Feature 3 is a large avoided crossing between three-photon processes 12/3 and 21/3. It is predicted by the unperturbed model, and direct diagonalization yields the splitting of $4J$. A remarkable detail here is that the dim lower branch implies the presence of a dark state with respect to the driving operator in the third order.

Feature 4 is also explained by the unperturbed model and is caused by several spectral lines and a pair of avoided crossings near a single point (dotted lines in Fig. 2 (d)). It appears at the point where 02/2 intersects 20/2 and is just barely visible in the experimental data because of the noise.

Feature 5 is located at the intersection between 20/2 and 10 – 02, and can be found in the experimental data, too.

In conclusion to this section, we note that when the coupling is turned off ($J = 0$) in the simulation, the system does not demonstrate any of the described details. This means that all these effects can only be attributed to the SAM as a whole.

**C. Explaining extra avoided crossings**

Since we had already connected the additional avoided crossings with the light dressing, it was natural to expect an Autler-Townes-like effect to be at the root of the additional spectral lines. For a three-level system, the standard A-T effect is revised in Appendix A. However, in our case the level structure and the effect itself are more complicated.

First of all, since during the spectroscopy we apply only a single microwave tone ($\omega_{1} = \omega_{2}$), it has to be simultaneously the coupler and the probe in terms of the standard A-T effect; moreover, the probe must be much weaker than the coupler. Fortunately, all these conditions are satisfied near feature II(III) when $\omega_{1(2)} = \omega_{1(2)}$ and $\omega_{1(2)/2}$ where we can simultaneously excite transitions 01(10) and 20(20)/2. Since the two-photon Rabi frequency is much smaller than the single-photon $\Omega_{2}$, it is natural to view this two-photon excitation as the probing process which does not affect the level structure. In contrast, single-photon excitation is strong enough to dress the system. In other words, in our case the A-T coupling operator is $H_{A}$ (strong), and the probing operator is $H_{3}$ (weak in the two-photon regime). However, their separation in our case is not in frequency, but in the Hilbert subspaces they act upon and in the number of participating photons.

For feature I, the simultaneously excited transitions are...
Features II and III. (a) Schematic of the transitions near III (not to scale). Black dashed line shows the current (0.36 mA) where the first transmon (orange) is below the second (blue), and the first transmon driving is not shown here. Action of \( \hat{H}_{\text{int}} \) in RWA is depicted as orange-blue circles. (b) System level structure at the resonant point and the first transmon driving at its own frequency. (c) In the frame rotating with \( \hat{R}_1^d \), states \(|j\rangle, |\bar{j}\rangle \) become nearly degenerate \((\omega_1 \rightarrow \Delta_1)\). Dressing increases this splitting to \( \Omega^R_1 \). (d) Transitions in the dressed system induced by \( \hat{R}_2^d \) (coupled level subspaces are shown with blue ellipses). In the left part of the panel, all possible two-photon transitions near \( \omega_0 \) are depicted: the blue transitions are not shifted in frequency, the light blue ones are shifted by \( \pm \Omega^R_2/2 \). In the right part, the most contributing trajectories are depicted: gray crosses show transitions that are forbidden without the coupling \( J \).

By \( \hat{H}^d_2 \) is shown with orange ellipses while much weaker two-photon driving by \( \hat{H}^d_2 \) is not shown because it does not alter the structure of energy levels. Since from numerical simulations we know that the third level of the first transmon is not necessary to observe the splitting, \(|20\rangle \) is shown transparent, and states \(|21\rangle, |22\rangle \) are not shown.

The next step is to view the system in the frame rotating with the first transmon (Fig. 5(c)) and then move to the dressed picture similarly to Appendix A. Now, the first transmon splitting equals \( \Omega^R_1 = \sqrt{\Omega^2_1 + \Delta^2_1} \), and its new eigenstates (dressed states) are denoted \(|a\rangle \) and \(|b\rangle \). Meanwhile, the second transmon energies stay the same.

Finally, in Fig. 5(d) we demonstrate possible two-photon transitions between the dressed states induced by the second transmon driving \( \hat{R}_2^d \). In the left part of the panel, one can find the unmodified two-photon transition \( |02\rangle \rightarrow |11\rangle \) at \( \omega_1 \) and two sidebands at \( \omega_0 \pm \Omega^R_2/2 \). This picture finally explains the observed triplet transition of feature III. The right part describes the mechanism of these two-photon processes through virtual excitations of the intermediate states. From here it becomes obvious that without the transmon-transmon interaction the sideband transitions are forbidden due the selection rule: \( \langle a|j \rangle \otimes \hat{n}_2 |b, j + 1\rangle = 0 \) since \( \langle a|b\rangle = 0 \). However, we will show that they become allowed in the second order in \( J \) when the coupling is turned on.

Now, we will repeat this reasoning using a quantitative mathematical model. We start from the initial Hamiltonian \( \hat{H}_1 \). To move to the rotating frame with Eq. 5 and apply the RWA, we use the following operator:

\[
R = \exp \left[ -i t \omega^1_0 \left( b^\dagger b + c^\dagger c \right) \right].
\] (11)

Note that now we rotate both transmon subspaces simultaneously in contrast to what is shown in Fig. 5 because it will be convenient to have time-independent \( \hat{H}_{\text{int}} \). Now,

\[
\omega^{(1,2)} = \omega^{(1)} - \omega^{(2)};
\]

\[
\hat{H}_{\text{int}} \rightarrow \hbar J \left[ \hat{\sigma}_z \hat{c} + \hat{\sigma}_z \hat{c}^\dagger \right],
\]

\[
\hat{R}^d_1 \rightarrow \frac{\hbar \Omega^1_1}{2} \hat{\sigma}_x, \quad \hat{R}^d_2 \rightarrow \frac{\hbar \Omega^2_2}{2} \left( \hat{c} \hat{e}^{i \delta t} + \hat{c}^\dagger \hat{e}^{-i \delta t} \right),
\] (12)

where \( \delta = \omega^{(2)} - \omega^{(1)} \).

FIG. 5. Explaining feature III. (a) Schematic of the transition frequencies near III (not to scale). Black dashed line shows the current (0.36 mA) where the first transmon (orange) is below the second (blue) and several transitions become resonant. Here we assume that each transmon is driven at its own frequency. Action of \( \hat{H}_{\text{int}} \) in RWA is depicted as orange-blue circles. In the left part of the panel, all possible two-photon transitions near \( \omega_0 \) are depicted: the blue transitions are not shifted in frequency, the light blue ones are shifted by \( \pm \Omega^R_2/2 \). In the right part, the most contributing trajectories are depicted: gray crosses show transitions that are forbidden without the coupling \( J \).

FIG. 6. Explaining feature I. (a) Schematic of the transitions near I. (b) Transitions in the frame rotating with the second transmon with dressing. As will be shown in Section [IH5] in our experiment there can only be one visible transition to the left or to the right from the resonance.
Since \( \hat{H}_d \) is now time-independent, we can move to the dressed basis by applying a transformation \( S \) which diagonalizes the first transmon. After that, the Hamiltonian may be split into three parts

\[
\hat{H}^D / \hbar = \left[ \frac{\Delta_1}{2} \hat{\sigma}_z \right] + \Delta_2 \hat{b}^\dagger \hat{b} + \frac{1}{2} \alpha \hat{b}^\dagger \hat{b} (\hat{b}^\dagger \hat{b} - 1),
\]

\[
\tilde{V}_J = S^\dagger \hat{H}_d S,
\]

\[
\tilde{V}_I = S^\dagger \hat{H}_d S = \hat{H}_d^2.
\]

The first part is diagonal, and the remaining two will be treated as perturbations. To simplify further calculations, we consider the point where \( \Delta_1 = 0 \) and \( \Delta_2 = -\alpha/2 \). In these conditions, \( \hat{H}^D \) becomes degenerate as can be seen in Eq. B2 of Appendix B. Using the degenerate perturbation theory for \( \hat{V}_J \) summarized therein, we find the first-order corrected wave functions of \( \hat{H}^D + \hat{V}_J \), labelled \(|k\rangle \), \( k = 1..6 \). We find the mean relative element-wise error of around 3% between numerically obtained eigenvectors and perturbative ones for our experimental parameters.

The time-dependent perturbation theory that we use to calculate the transition rates of the two-photon processes stimulated by \( \tilde{V}_I \) is reviewed in Appendix C. Using the corrected eigenstates \(|k\rangle \) and neglecting small terms, we obtain the following expressions for the transition rates per unit time (13):

\[
R^{(2)}_{1 \to 5} \approx \pi \Omega_1^2 \alpha^4 \sqrt{3\alpha^2 + 7\Omega_1^2},
\]

\[
R^{(2)}_{1 \to 6} \approx \pi \Omega_2^2 \alpha^4 \sqrt{(5\alpha - 3\Omega_2)^2},
\]

when \( \alpha > \Omega_1, \Omega_2, \Omega \) which is true for our setup. Tailor expansion here leads to errors of less than 0.5%. From Eq. 14 follows that the second transmon is prohibited without the interaction in the SAM (\( J = 0 \)), and the extra avoided crossings will not be observed.

b. Feature I. Finally, we discuss the last Roman-numbered effect. It is illustrated in Fig. 6 for the case when \( \Omega_2 \gg \Omega_1 \); the second transmon is dressed, and the splitting has the shape shown in Fig. 6 (a). For the opposite case (\( \Omega_1 \gg \Omega_2 \)), the logic is the same.

As one can see from Fig. 4 near the 01, 10 intersection only the two-photon transition 11/2 is affected and deviates from unperturbed spectrum while the spectral lines 01, 10 do not shift. Similarly to II and III, in the frame rotating with the second transmon it turns into two different single-photon transitions for \( \hat{H}_d^2 \) located at \( \omega = \pm \Omega_2^R \), see Fig. 6 (b). We will show below that in the experiment with a single excitation frequency, it is not possible to observe both transitions simultaneously which is also clear from Fig. 4 where we observe one line to the left and the other to the right from the 01, 10 intersection.

D. Self-consistent equations for the avoided crossings

Before, we have assumed that the transmons are driven independently at two different frequencies. In this section, we will employ the same model from Fig. 5 to explain the experimentally observed splittings when only a single frequency \( \omega_d \) is sent at the SAM.

a. Features II, III. We start again with feature III. As shown in Fig. 5 (d), the sideband transitions are formed by two-photon processes with \( \hat{H}_d^2 \). On the other hand, we know from the unperturbed solution that in the laboratory frame the two transitions forming the avoided crossing are 10 – 02 (one-photon) and 12/3 (three-photon). This means that there should be a smooth transformation between these 1-, 2- and 3-photon processes when the system approaches the resonant point where \( \Omega_2 + 2\omega_1 = 2\omega_0 \). Let us consider hypothetical two-photon transitions (10 – 02)/2 and 12/2. In the frame rotating with the first transmon like in Fig. 5 (c) when \( \Omega_1 = 0 \), their frequencies are

\[
\omega_{10-02}/2 = (\omega_2 + 2\omega_1 - \Delta_1)/2,
\]

\[
\omega_{12}/2 = (\omega_2 + 2\omega_1 + \Delta_1)/2.
\]

When the first transmon becomes dressed by \( \Omega_1 \neq 0 \), its splitting \( \Delta_1 \) changes to \( \Omega_1^R = \sqrt{\Omega_1^2 + (\omega_1 - \omega_d)^2} \). Substituting this new splitting into the above equations we find that \( \omega_{10-02}/2 \) and \( \omega_{12}/2 \) are exactly equal to the two-photon sideband frequencies \( \Omega_1 \pm \Omega_1^R/2 \) established in the previous section. Therefore, it is logical to use them to model the splitting behaviour beyond the resonant point.

Since we are simultaneously dressing the states by \( \hat{H}_d^2 \) and probing two-photon transitions between them with \( \hat{H}_d^2 \), self-consistent equations have to be solved to find at which \( \omega_d \) the sideband spectral lines will appear

\[
\omega_{10-02}/2 = \omega_d, \quad \omega_{12}/2 = \omega_d.
\]

When \( \Omega_1 = 0 \), these equations yield identical pairs of solutions due to the properties of the modulus \( |\Omega_1 - \omega_d| \). Substituting Eq. 16 into Eq. 15, we obtain the frequencies and identify them as the three-photon and single-photon transition frequencies in the laboratory frame:

\[
\omega_d = \left\{ \begin{array}{l}
\omega_2 + 2\omega_1 + \omega_0, \\
\omega_2 + 2\omega_1 - \omega_0.
\end{array} \right.
\]

When \( \Omega_1 \neq 0 \), we obtain the following solutions:

\[
\omega_d = \frac{2\omega_2}{3} + \frac{4\omega_1}{3} - \omega_0 \pm \frac{\sqrt{3\Omega_1^2 + (\omega_2 + 2(\omega_1 - \omega_0))^2}}{3}.
\]

The minimal splitting between these lines is found at the resonant point and equals \( \frac{\sqrt{3}}{3} \Omega_2 \). All these calculations can be repeated exactly for the feature II by swapping the transmons; in that case, the splitting will be \( \frac{\sqrt{3}}{3} \Omega_2 \).

We find excellent agreement between this self-consistent analytical model Eq. 18 and both the experimental and simulated data as can be seen in Fig. 3.
spectroscopy. The only free parameter left was $\Omega_2$. We have repeated this approximation for a range of microwave powers to confirm the linear dependence of the splitting on excitation amplitude (see Fig. 3(c)).

The upper branch of the fitted anticrossing is expected to deviate from the model (see Fig. 3) because of the avoided crossing marked as the secondary feature 3 which was not incorporated in the simple linear model. The small discrepancy between the experimental and simulated data in the upper branch is caused by a slight elevation of the lower sweet spot of the first transmon moving the avoided crossing 3 closer to the feature II; however, it is not important for our reasoning.

**b. Feature I.** Feature I may be explained using the same dressing model. Looking at Fig. 6 for the case $\Omega_2 \gg \Omega_1$, we can write another pair of self-consistent equations:

$$\omega_1 \pm \Omega_2^R = \omega_d. \quad (19)$$

Due to the fact that $\Omega_2^R = \sqrt{\Omega_1^2 + (\omega_2 - \omega_d)^2}$, this system has a single solution:

$$\omega_d = \frac{\omega_1 + \omega_2}{2} - \frac{\Omega_2^2}{2(\omega_1 - \omega_2)}, \quad (20)$$

which gives back the frequency of 11/2 in the laboratory frame when $\Omega_2$ is zero. When $\Omega_2$ is non-zero, the solution near the point $\omega_1 \approx \omega_2$ is a hyperbolic curve, which agrees well with the numerical simulation. However, in reality we do not observe the asymptotically vertical parts of this model since it becomes invalid when $|\omega_1 - \omega_2| \ll J$ due to the finite coupling $J$ between the transmons. Since they never reach each other in frequency, the denominator in (20) is limited from below and there is no divergence.

When $\Omega_1 \gg \Omega_2$, besides the replacement $\Omega_2 \rightarrow \Omega_1$, Eq. 20 changes the sign:

$$\omega_d = \frac{\omega_1 + \omega_2}{2} + \frac{\Omega_2^2}{2(\omega_1 - \omega_2)}, \quad (21)$$

yielding the reversed shape of the splitting.

Continuing this logic, we can write down the resonance condition in the doubly-rotating frame when both transmons are dressed simultaneously:

$$\Omega_1^R + \Omega_2^R = 0. \quad (22)$$

Solving it for $\omega_d$, we obtain the generalization of Eq. 20 and Eq. 21:

$$\omega_d = \frac{\omega_1 + \omega_2}{2} + \frac{\Omega_1^2 - \Omega_2^2}{2(\omega_1 - \omega_2)} \quad (23)$$

Eq. 23 is used to plot the black dashed curves in Fig. 4 taking the $\Omega_{1,2}$ values from the simulation parameters shown therein. The frequencies $\omega_{1,2}$ are extracted from the fits to the visible spectral lines. We again find good agreement between the model and the simulated data.

**IV. DISCUSSION**

We have performed spectroscopy measurements of an isolated diatomic superconducting artificial molecule (SAM) in a regime of strong interaction with classical light. Using joint dispersive readout to directly access population of the SAM eigenstates, we have located several anomalies in the spectral data that were impossible to explain using the unperturbed model of the system. By using extensive numerical modellng explicitly including the interaction with light, we have reproduced the experimentally discovered effects and attributed them to an altered version of the well-known Autler-Townes effect.

In the standard A-T effect, the coupler and the probe lasers are separated both in frequency and power. In our case, there is only a single spectroscopic tone interacting with the system; therefore, it has to be both the coupler tone and the probe tone at the same time. However, since there are two components in the driving operator (one for each transmon), the separation between the coupler and the probe occurs in the Hilbert space and in the number of photons involved rather than in frequency and amplitude. The effectively weak driving limit for the probe part is achieved when it stimulates a two-photon transition while the coupler part is resonant with a single-photon one being strong enough to dress the system. We have built self-consistent models in rotating frame to model the experimental splittings induced by the same field that probes them, and found a good agreement between the model, the experiment, and the numerical simulation. Interestingly, no new spectral lines appear in this more complex case. Instead, for example, spectral lines 12/3 (three-photon) and 10 − 02 (single photon) morph to an additional avoided crossing of a non-standard size of $\frac{2\sqrt{3}}{3} \Omega_1$.

Another interesting effect occurs when both components of the driving operator are in the single-photon regime. Now, both transmons are dressed and probed simultaneously. In this case, the 11/2 transition is being split at the resonance between 01 and 10 into a hyperbolic curve, though never forming a doublet transition. The shape of the splitting and its visibility depends qualitatively on the relation between the driving amplitudes; for instant, if they are equal, the splitting does not appear at all. In our sample, we have a fixed ratio of approximately 2 between $\Omega_1$ and $\Omega_2$ which still allows us to distinguish the splitting experimentally. Notably, the frequency of 11/2 transition far from resonance is also changed noticeably with changing power, which means that a fast bSWAP gate should be performed at a different frequency than the unperturbed model predicts.

Interestingly, the self-consistent models for the observed effects are implying that multi-photon processes may smoothly change their order. For example, for features II and III, we observe a continuous transformation of a three-photon and a single-photon processes to the second order in Eq. 16 and of a two-photon transition to a zero-photon in Eq. 22.

In overall, irradiating an individual diatomic artificial molecule with intense light calls forth a plethora of effects which can not be directly observed in natural systems and extend the validity of the well-known light-dressing models.
Relative ease in attaining powers that cause multiphoton transitions promises even more complex dynamics in multi-atom systems. We are looking forward to investigating strong interactions promises even more complex dynamics in multi-atom systems.

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Appendix A: Standard Autler-Townes effect for a three-level Ξ atom

The underlying cause of the A-T effect is the dressing of the atomic levels by strong EM radiation. There are two equivalent mathematical models for it: the light may be classical or quantized.

a. Classical derivation. In the classical case, the mathematical description goes as follows. At first, there is a three-level system which is driven by a strong radiation with an amplitude Ω at a frequency detuned by Δ from the |0⟩→|1⟩ transition with a frequency ω01 (the coupler tone). Additionally, we have a weak probe radiation at a frequency ωp and with an amplitude Ωp between |1⟩ and |2⟩ (the probe tone).

\[ \hat{H}_0/\hbar = \begin{bmatrix} 0 & \Omega_c \cos (\omega_{01} - \Delta) t & 0 \\ \Omega_c \cos (\omega_{01} - \Delta) t & \omega_{01} & \Omega_p \cos \omega_p t \\ 0 & \Omega_p \cos \omega_p t & \omega_{02} \end{bmatrix}. \]

This step is illustrated in [Fig. 7](a). The Hamiltonian for such system is as follows:

Next, we move to the rotating frame by using an operator

\[ \hat{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-i(\omega_{01} - \Delta)} & 0 \\ 0 & 0 & e^{-i(\omega_{01} - \Delta)} \end{bmatrix}. \]

Note here that level |2⟩ is also rotated: this is convenient to preserve the frequency of the driving term with Ωp. The new Hamiltonian should be calculated as follows: \( \hat{H}_1 = \hat{R} \hat{H}_0 \hat{R}^\dagger \). The level structure without driving is shown in the left part of [Fig. 7](b). Applying the RWA, we obtain:

Next, moving to the basis where the upper left 2x2 corner is diagonal (the right part of [Fig. 7](b)), we obtain:

\[ \hat{H}_3/\hbar = \begin{bmatrix} \Delta + \Omega^2/2 & 0 & \Omega_p \cos \theta/2 \\ 0 & \Delta - \Omega^2/2 & \Omega_p \sin \theta/2 \\ \Omega_p \cos \theta/2 & \Omega_p \sin \theta/2 & \omega_{12} + \Delta \end{bmatrix}. \]

One may see that now resonant conditions for the probe drive are \( \omega_p = \omega_{12} - (\Delta + \Omega^2)/2 \), where \( \Omega^2 = \sqrt{\Omega_c^2 + \Delta^2} \), \( \omega_{12} = \omega_{02} - \omega_{01} \), and its amplitude is renormalized by an angle \( \theta, \tan2\theta = -\Omega_c/\Delta \).

b. Quantum derivation. For the fully quantum interpretation, we model the incident radiation at the frequency \( \omega_{01} - \Delta \) as a single-mode quantum oscillator which is coupled to the system. The Hamiltonian for this model will be as follows:

\[ \hat{H}_0/\hbar = \omega_{01} |1⟩⟨1| + \omega_{02} |2⟩⟨2| + (\omega_{01} - \Delta) \hat{a}^\dagger \hat{a} + g(\hat{a}^\dagger \hat{a}) \otimes (|1⟩⟨1| + |0⟩⟨0|), \]

where \( g \) is the coupling strength and \( a \) is the photon annihilation operator. After moving to the rotating frame with \( \hat{R} = \exp[i(\omega_{01} - \Delta) \hat{a}^\dagger \hat{a} + |1⟩⟨1| + |2⟩⟨2|] \) and applying the RWA, the Hamiltonian transforms into

\[ \hat{H}_1/\hbar = \Delta |1⟩⟨1| + (\omega_{12} + \Delta) |2⟩⟨2| + \frac{\Omega_p}{2} \hat{a}^\dagger \hat{a} \otimes |0⟩⟨1| + \hat{a} \otimes |1⟩⟨0|, \]

If we presume that the resonator is in a coherent state \( |\alpha e^{-i(\omega_{01} - \Delta)}⟩ \) with \( \langle N⟩ = \alpha^2 \) photons and average \( \hat{H}_1^\dagger \) over the resonator subspace, from the interaction term we will obtain again the classical driving term \( \Omega_c (|0⟩⟨1| + |1⟩⟨0|), \)

FIG. 7. Illustrating the A-T splitting by the classical driving in the rotating frame. (a) A three-level system is driven strongly by the coupler tone of amplitude \( \Omega_c \) at frequency \( \omega_c = \omega_{01} - \Delta \). (b) In the frame rotating with the drive changes, the |0⟩→|1⟩ transition frequency changes to \( \Delta \). However, when the RWA is applied and Hamiltonian is re-diagonalized, the splitting between two lowest levels (dressed states |a⟩ and |b⟩) becomes \( \hbar \Omega_R \). Now, a doublet transition from these levels to the state |2⟩ at frequencies \( \omega_{12} + (\Delta \pm \Omega_R)/2 \) may be observed.
where $\Omega_c = 2g\sqrt{\langle N \rangle}$ in correspondence with the previous case. Similarly, the energy levels $|1, N-1\rangle$ and $|0, N\rangle$ become mixed due to the coupling, and their splitting changes from $\hbar\Delta$ to $\hbar\Omega_c$. The following steps completely reproduce the classical case if we add the probe tone $\Omega_p$ in the last equation.

Appendix B: Degenerate perturbation theory

Since the Hamiltonian in the dressed basis is a useful illustration for Fig. 5, we provide it below in Eq. B2 for the case $\Delta_1 = 0$, $\delta = \omega_f^2 - \alpha_0^2 = \omega_0^2 - \alpha_0$. The degeneracy between states $|a, 0\rangle$, $|a, 2\rangle$ and $|b, 0\rangle$, $|b, 2\rangle$ occurs only in the resonant case $2\Delta_2 + \alpha_2 = 0$ meaning that the second transmon frame is rotated at its two-photon transition frequency. In this situation, its single-photon subspace is detuned by $-\alpha/2$.

Note that $\hat{V}_f = \hat{S}^T \hat{H}_{\text{rat}} \hat{S}$ that we treat as perturbation now has an sub- and super-diagonal block form and couples all the states $|a, j\rangle$, $|a, j+1\rangle$ and $|b, j\rangle$, $|b, j+1\rangle$.

The choice of zero-order state vectors $|N^0\rangle$ from a degenerate subspace is arbitrary because any linear combination of basis vectors $|n\rangle$ from this subspace will satisfy the unperturbed Schrödinger equation. However, if we demand the change of $|N^0\rangle$ to be small under the perturbation $\hat{V}$, they become determined and are given by diagonalization of $\hat{V}$ in the degenerate subspace. Unfortunately, all matrix elements of $\hat{V} = \hat{V}_f$ are zero in both our degenerate subspaces, so technically any choice of zero-order states will diagonalize it. In other words, the degeneracy is not lifted in the first-order, and thus we have to diagonalize the matrix [44]

$$M_{nm} = \sum_m V_{nm} V_{nm}^*.$$ 

(B1)

Here, $|n\rangle$ and $|n'\rangle$ are the basis states from the degenerate subspace with energy $E_n$, and $V_{nm} = \langle m | n \rangle \hat{V}_f | n \rangle$. The sum is over all other zero-order states $|m\rangle$ outside the degenerate subspace.

Appendix C: Transition rates of the two-photon process

We will employ the time-dependent perturbation theory that gives analytical expressions for the transition rates for single and multi-photon processes [43].

Let’s consider a time-dependent perturbation $\hat{V}(t) = \frac{\hbar}{2} \left( e^{i\omega_0 t} + e^{-i\omega_0 t} \right)$ to the unperturbed Hamiltonian $\hat{H}$. In the interaction picture, the Schrödinger equation looks as

$$i\hbar \partial_t \psi(t) = \hat{V}_f(t) \psi(t),$$

where

$$\hat{V}_f(t) = e^{\frac{i}{\hbar} t} \hat{V}(t) e^{-\frac{i}{\hbar} t}. $$

The eigenstate of $\hat{H}$ with eigenvalue $E_j$ will be denoted as $|j\rangle$, whence it follows that

$$\langle j | \hat{V}_f(t) | i \rangle = e^{i\omega_0 t} \langle j | \hat{V}(t) | i \rangle,$$

where $\hbar \omega_j = E_j - E_i$. Formally, we can write down the solution of the Schrödinger equation in the interaction picture corresponding to the initial state $|i\rangle$ as

$$|\psi_i(t)\rangle = |i\rangle - \frac{i}{\hbar} \int_{-\infty}^t d\tau_1 \hat{V}_f(\tau_1) |\psi_i(\tau_1)\rangle.$$

(C1)
Solving Eq. C1 by simple iterations gives the series solution for it:

\[ U(t, -\infty) = 1 + \sum_{n=1}^{\infty} U^{(n)}(t, -\infty), \]  

\[ U^{(n)}(t, -\infty) = \left(-\frac{i}{\hbar}\right)^{n} \int_{-\infty}^{t} d\tau_{1} \hat{V}_{I}(\tau_{1}) \cdots \int_{-\infty}^{\tau_{n-1}} d\tau_{n} \hat{V}_{I}(\tau_{n}), \]  

where \( U(t, -\infty) \) is the evolution operator. For a weak perturbation, this series may be truncated at a finite \( n \) and the \( n \)th order transition amplitude \( \langle f | U^{(n)}(t, -\infty) | i \rangle \) can be evaluated:

\[
\langle f | U^{(1)}(t, -\infty) | i \rangle = -\frac{i}{\hbar} \int_{-\infty}^{t} d\tau_{1} \hat{V}_{I}(\tau_{1}) = \int_{-\infty}^{t} d\tau_{1} e^{i(\omega_{f}-\omega_{i})\tau_{1}} \langle f | \hat{c}^{\dagger} | i \rangle + \int_{-\infty}^{t} d\tau_{1} e^{i(\omega_{f}+\omega_{i})\tau_{1}} \langle f | \hat{c} | i \rangle.
\]

To cancel out one of the delta functions in the resulting expression, we can use the identity

\[
\delta(\omega_{f} - \omega_{i} - 2\omega_{d}) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T/2}^{T/2} e^{i(\omega_{f} - 2\omega_{d})t} dt = \frac{T}{2\pi}.
\]

A two-photon emission:

\[
W_{i \to f}^{(2)} = \frac{\pi\Omega_{d}^{2}}{8} | \sum_{j} \frac{\langle f | \hat{c}^{\dagger} | j \rangle \langle j | \hat{c}^{\dagger} | i \rangle}{\omega_{j} + \omega_{d}} |^{2} \delta(\omega_{f} + 2\omega_{d}) = \frac{\pi\Omega_{d}^{2}}{8} | \sum_{j} \langle f | \hat{c}^{\dagger} | j \rangle \langle j | \hat{c}^{\dagger} | i \rangle |^{2} \delta(\omega_{f} + 2\omega_{d}).
\]

A two-photon absorption:

\[
W_{f \to i}^{(2)} = \frac{\pi\Omega_{d}^{2}}{8} | \sum_{j} \frac{\langle f | \hat{c}^{\dagger} | j \rangle \langle j | \hat{c}^{\dagger} | i \rangle}{\omega_{j} - \omega_{d}} |^{2} \delta(\omega_{f} - 2\omega_{d}) = \frac{\pi\Omega_{d}^{2}}{8} | \sum_{j} \langle f | \hat{c}^{\dagger} | j \rangle \langle j | \hat{c}^{\dagger} | i \rangle |^{2} \delta(\omega_{f} - 2\omega_{d}).
\]

For our case, \( \omega_{d} = \delta \) which leads to Eq. 14.

Appendix D: Measurement setup and methods

The sample was measured in a BlueFors LD250 dilution refrigerator at 16 mK. For the readout a Keysight PNA-L N5223A VNA was used. For the coherent excitation of the SAM, we used an Agilent MXG N5183B analog signal generator. The sample was flux biased using Yokogawa GS200 current sources (two for the flux lines and one for the external coil wrapped around the sample).

Input microwave lines were isolated from high-temperature noise with 60 dB of attenuation (10 @ 4K, 10 @ 1K, 20 @ 100 and 16 mK) and custom-made IR filters. The additional on-chip attenuation (due to the small capacitive coupling) between the drive line and the transmons was calculated in Sonnet to be around 70 dB @ 6 GHz. Coaxial flux-bias lines were attenuated by 20 dB @ 4K and IR filtered as well. Output path contained two 20 dB isolators and two amplifiers: the 4-14 GHz LNF amplifier at the 4 K stage and the room-temperature LNF amplifier.

As the main experimental method, we have employed the two-tone spectroscopy which consists of exciting the SAM with monochromatic light at certain frequency until steady state is reached while simultaneously measuring the signal transmission at the readout resonator frequency. This measurement yields the average value of the joint measurement operator \( \hat{M} \) in the steady-state.

Appendix E: Sample fabrication

The device was fabricated on a high-resistivity Si wafer (10 kOhm-cm). First, Si wafers were cleaned with Piranha, HF and then coated with bilayer MMA/PMMA resist stack. The nominal after-bake thickness of the MAA and PMMA are 800 and 100 nm, respectively. The bilayer resist stack was exposed using a 50kV Raith Voyager EBL system and then developed. Next, the sample was placed into a high-vacuum
electron-beam evaporation chamber (Plassey) and after a gentle ion-milling step, a double-angle evaporation technique at 10 is used to deposit Al/AlOx/Al layer. Finally, hot NMP followed by IPA was used to lift off the resist mask stack.

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