On the quenched functional CLT for stationary random fields under projective criteria

Lucas Reding* and Na Zhang

Université de Rouen Normandie. Saint-Étienne-du-Rouvray, 76801, France.
Email: lucas.reding@univ-rouen.fr
Department of Mathematics, Towson University, Towson, MD 21252-0001, USA.
Email: nzhang@towson.edu

Abstract

In this work we study and establish some quenched functional central limit theorems (CLT) for stationary random fields under a projective criteria. These results are functional generalizations of the theorems obtained by Zhang et al. [35] who themselves extended the quenched functional CLT for ortho-martingales established by Peligrad and Volný [28] to random fields satisfying a Hannan type projective condition. In the work of Zhang et al. [35], the authors have already proven a quenched functional CLT however the assumptions were not optimal (as they require the existence of a $2 + \delta$-moment). In this article, we establish the results under weaker assumptions, namely we only require a Orlicz space condition to hold.

1 Introduction

Developments within the Markovian theory led to the question of the conditions under which a central limit theorem could be derived for Markov chains; in particular what restrictions were sufficient on the initial distribution and the transition operator to have this kind of result. Seminal results were obtained by Gordin and Lifšic [14] (see also Borodin and Ibragimov [2] or Derriennic and Lin [11]) for Markov chains endowed with the stationary measure as their initial distribution as well as Kipnis and Varadhan [19] (see also Derriennic and Lin [11]) for additive functionals of reversible Markov chains. Additionally Derriennic and Lin [11] also obtained a CLT for Markov chains starting from a fixed point (in other words endowed with $\delta_x$ the Dirac measure in the state $x$ as their initial distribution). This kind of theorems are called quenched CLT. Another way of representing these results is to consider a fixed past and to study the convergence in distribution with respect to that past. The difficulties during the proof arise from the fact that this fixed past causes the process to not be stationary anymore. An extensive literature exists on the subject, one can cite the following works by Barrera et al. [1], Cuny and Peligrad [4], Cuny and Merlevède [5], Cuny and Volný [7], Dedecker et al. [10], Peligrad [26] and Volný and Woodroofe [34]. Note that some counter examples to quenched central limit theorems under specific conditions were found by Ouchti and Volný [25] and Volný and Woodroofe [33]. Functional versions of these quenched central limit theorems, also called quenched weak invariance principles, have also been the subject of numerous research articles such as the ones by Barrera et al. [1], Cuny and Merlevède [5], Cuny and Volný [7] and Peligrad [26].

Random fields naturally appear as a generalization of sequences of random variables, however extending the one-dimensional results to greater dimension is much harder than one would think. The first problem we are faced with is to correctly define the notion of past. The approach we have implemented in this paper is to use the notion of commuting filtrations. In particular, this property is satisfied by filtrations generated by fields of independent random variables or even by fields with independent columns (or equivalently independent rows). As a lot of process can be expressed as a functional of i.i.d. random variables, these type of filtrations are quite common and merit interest. A lot of work has been done under commuting filtrations (see Volný [30] and Cuny et al. [6]).

*Corresponding author
As usual we will require some kind of dependency condition on the field studied. Namely, in this paper, we will use Hannan’s projective condition [17]. The problem we are interested in has been studied by Cuny and Volný [7] for time series but it has yet to be investigated for higher dimensions which is the purpose of this article. Though the problem we focus on hasn’t been studied yet, one can note that fields satisfying Hannan’s condition have been quite extensively studied and numerous CLTs and functional CLTs (both annealed and quenched) have been obtained. One could refer to the following: Volný and Wang [32], Klicnarová et al. [23] and Zhang et al. [35].

The proof of the main theorem in this paper is based upon the use of a martingale-coboundary decomposition that can be found in [32] (some more recent and general results can be found in [12, 31, 13], see also [15]) as well as the central limit theorem and the weak invariance principle established by Peligrad and Volný [28] for ortho-martingales. Once the main theorems are established, we derive corollaries in the spirit of the results obtained by Zhang et al. [35]. As shown by the previous results in the literature, it will be required to address two situations separately: first when the summations are done over cubic regions of $Z^d$ and, after that, when the regions are only required to be rectangular.

This paper will be structured as follows: in Section 2, we present our results, the notations used throughout our article and the main results obtained in this work. In particular, we will split the results into two categories; those who deal with summations over cubic regions only and those who deal with more general rectangular regions. The proofs of these theorems will appear in Section 3 and in Section 4 we improve on the two applications studied by Zhang et al. [35]. These two examples concern the linear and Volterra random fields satisfying Hannan’s condition. They are a common occurrence in the field of financial mathematics and economics.

## 2 Framework and results

In all that follows, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and all the random variables considered thereafter will be real-valued and defined on that probability space. We start by introducing multiple notations that will be used throughout this article: $d$ will be an integer greater than 1, bold characters will designate multi-indexes, denote $\mathbf{n} := (n_1, \ldots, n_d)$ and $|\mathbf{n}| := \prod_{i=1}^d n_i$ for any $\mathbf{n} \in \mathbb{Z}^d$. The set of integers $\{1, \ldots, d\}$ will be denoted by $[1, d]$. In order to define the concept of past trajectory, it is necessary to define an order on $\mathbb{Z}^d$: if $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$ are multi-indexes such that for all $k \in [1, d], u_k \leq v_k$, then we will write $\mathbf{u} \leq \mathbf{v}$.

Convergence of fields indexed by $\mathbb{Z}^d$ will be interpreted in the following sense. If $\mathbf{n} = (n_1, \ldots, n_d)$ is a multi-index, then the notation $\mathbf{n} \to \infty$ is to be interpreted as the convergence of $\min\{n_1, \ldots, n_d\}$ to $\infty$.

Before introducing the field we are interested in, we define some transformations on $\Omega$. We let $T_i : \Omega \to \Omega$, $i \in \{1, \ldots, d\}$ be invertible measure-preserving commuting transforms on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we make use of the operators notation (i.e. if $U$ and $V$ are two transformations on $\Omega$, we denote $UV := U \circ V$).

We consider a sigma-field $\mathcal{F}_0 \subset \mathcal{F}$ such that $\mathcal{F}_0 \subset T^{-1}\mathcal{F}_0$ and a random variable $X_0 \in L^2_0$ where $L^2_0 = L^2_0(\Omega, \mathcal{F}_0, \mathbb{P})$ is the set of all $\mathcal{F}_0$-measurable square integrable centered random variables. For every $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, set

$$X_n = X_0 \circ T^n,$$

(1)

and

$$\mathcal{F}_n = T^{-n}\mathcal{F}_0,$$

(2)

where $T^n = T_1^{n_1} \cdots T_d^{n_d}$. As a result $X_n$ is $\mathcal{F}_n$-measurable.

Suppose that the family $(\mathcal{F}_k)_{k \in \mathbb{Z}^d}$ is a commuting filtration, that is, for every integrable random
variable $X$, it holds
\[ E_i [E_j [X]] = E_{i \land j} [X], \]
where $E_i [X] = E [X | \mathcal{F}_i]$ and $i \land j$ is the coordinate-wise minimum between $i$ and $j$.

We recall the notion of ortho-martingale which were introduced by Cairoli [3] (see also Khoshnevisan [20]). We say that a random field $(D_i)_{i \in \mathbb{Z}^d}$ is an ortho-martingale difference field if it is integrable and satisfies the equation $E_a [D_n] = 0$ as long as there exists an integer $k \in [1, d]$ such that $a_k < b_k$. Then, if $M_n := \sum_{u=0}^{d} D_u$ the random field $(M_n)_{n \in \mathbb{N}^d}$ will be called an ortho-martingale.

Suppose also that the random variable $X_0$ is regular with respect to the filtration $\mathcal{F}$, that is $E [X_0 | \mathcal{F}_{-\infty e_i}] = 0$ for every $i \in \{1, \ldots, d\}$, where $e_i$ is the multi-index whose $i$-th coordinate is equal to 1 and the others are equal to 0 and using the convention that $\infty \times 0 = 0$.

Following the work of Krasnosel’skii and Rutitskii [21], we define the Luxemburg norm associated to the Young function $\Phi : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+ \setminus \{0\}$ as
\[ \|f\|_\Phi = \inf \{t > 0 : E [\Phi (|f|/t)] \leq 1\}. \]

In everything that follows, we will consider the Young function $\Phi : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+ \setminus \{0\}$ defined for every $x \in \mathbb{R}^+ \setminus \{0\}$ by
\[ \Phi(x) = x^2 (\log(1 + |x|))^{d-1}. \]

We consider the projection operators defined, for any $n \in \mathbb{Z}^d$, by $P_n = \prod_{i=1}^{d} (E_n - E_{n-e_i})$. For every $\omega \in \Omega$, we denote by $P_\omega$ a regular version of the conditional probability given $\mathcal{F}_0$, that is $P_\omega = P (\cdot | \mathcal{F}_0) (\omega)$.

Finally we introduce the sum that we will be studying, for every $n \in (\mathbb{N}^*)^d$,
\[ S_n = \sum_{i=1}^{n} X_i, \]
and we also set
\[ S_n = S_n - R_n \quad \text{where} \quad R_n = \sum_{i=1}^{d} (-1)^{i-1} \sum_{1 \leq j_1 < \cdots < j_i \leq d} E_{n^{(j_1, \ldots, j_i)}} [S_n], \]
where $n^{(j_1, \ldots, j_i)}$ is the multi-index obtained by replacing with 0 all the $j_1, \ldots, j_i$-th coordinates of the multi-index $n$ and leaving the rest unchanged.

In dimension $d = 1$, this reduces to the following expression:
\[ S_n = S_n - E [S_n | \mathcal{F}_0], \quad \text{for} \ n \in \mathbb{N}^*. \]
This case was investigated by Cuny and Volný in [7] and therefore, we will always consider $d > 1$ in the rest of the paper. In dimension $d = 2$, the definition of $S_n$ reduces down to
\[ S_{n,m} = S_{n,m} - E [S_{n,m} | \mathcal{F}_{n,0}] - E [S_{n,m} | \mathcal{F}_{0,m}] + E [S_{n,m} | \mathcal{F}_{0,0}], \quad \text{for} \ (n, m) \in (\mathbb{N}^*)^2. \]

2.1 Functional CLT over squares
Here we present the quenched functional CLT over cubic regions of $\mathbb{Z}^d$. These results expand to the functional framework Theorem 4.1, the second part of Corollary 4.3 and Theorem 4.4 (a) obtained in [35]. It is also possible to view these results as an extension to higher dimensions of Theorem 1 established by Cuny and Volný in [7]. As noted in [35], the proof of these theorems essentially reduce down to a particular case of the proof of the functional central limit theorems over rectangular regions of $\mathbb{Z}^d$. The differences in the proofs between the two frameworks will be specified in greater details in Section 3.
Theorem 2.1 Assume that \((X_n)_{n \in \mathbb{Z}^d}\) is defined by (1) and the filtrations are commuting. Also assume that one of the transformations \(T_i, 1 \leq i \leq d\) is ergodic and in addition
\[
\sum_{u \geq 0} \|P_0(X_u)\|_2 < \infty,
\]
(4)

Then, for \(\mathbb{P}\)-almost all \(\omega \in \Omega\),
\[
\left( \frac{1}{n^{d/2}} \bar{S}_{[nt]} \right)_{t \in [0,1]^d} \xrightarrow{\mathcal{L}} (\sigma W_t)_{t \in [0,1]^d} \quad \text{under} \quad \mathbb{P}^\omega,
\]
where \(\sigma^2 := \mathbb{E} \left[ D_0^2 \right]\) with \(D_0 = \sum_{i \in \mathbb{Z}^d} P_0(X_i)\), \((W_t)_{t \in [0,1]^d}\) is a standard Brownian sheet, \(kt := (kt_1, \ldots, kt_d)\) for \(k \in \mathbb{Z}\) and the convergence happens in the Skhorohod space \(D([0,1]^d)\) endowed with the uniform topology.

In Theorem 2.1 the random centering \(R_{n,n}\) cannot be avoided without additional hypothesis. As a matter of fact, for \(d = 1\), Volný and Woodroofe [33] constructed an example showing that the CLT for partial sums needs not be quenched. It should also be noticed that, for a stationary ortho-martingale, the existence of a finite second moment is not enough for the validity of a quenched CLT when the summation is taken over rectangles (see Peligrad and Volný [28]). That being said, the following corollary gives a sufficient condition to get rid of the stochastic centering \(R_n\) in the previous theorem.

Corollary 2.2 Assume the hypothesis of Theorem 2.1 holds and assume that in addition, for every \(i \in \{1, \ldots, d\}\), it holds
\[
\frac{1}{n^d} \mathbb{E}_0 \left[ \max_{1 \leq m \leq n} \mathbb{E}_{m^{(i)}} [S_m] \right] \xrightarrow{\text{a.s.}} 0
\]
where we recall that \(m^{(i)}\) is the multi-index obtained by replacing with 0 the \(i\)-th coordinate of the multi-index \(m\) and leaving the rest unchanged. Then, for almost all \(\omega \in \Omega\),
\[
\left( \frac{1}{n^{d/2}} S_{[nt]} \right)_{t \in [0,1]^d} \xrightarrow{\mathcal{L}} (\sigma W_t)_{t \in [0,1]^d} \quad \text{under} \quad \mathbb{P}^\omega,
\]
(5)

where \((W_t)_{t \in [0,1]^d}\) is a standard Brownian sheet and the convergence happens in the Skhorohod space \(D([0,1]^d)\) endowed with the uniform topology.

To end this section, we give a condition that is easier to verify but still guarantees that the convergence (5) holds.

Theorem 2.3 Assume that
\[
\sum_{u \geq 1} \frac{\|\mathbb{E}_1(X_u)\|_2}{|u|^2} < \infty.
\]
(6)

Then, for almost all \(\omega \in \Omega\), the conclusion of Corollary 2.2 holds.

Once again we note that this result is an extension of Corollary 2 in [7] to random fields and an extension to the functional framework of Theorem 2.6 (a) found in [35].

2.2 Functional CLT over rectangles

In order to obtain a functional CLT when we sum over rectangles, a stronger projective condition than (4) is necessary. Indeed Peligrad and Volný [28] have given a counter example to a quenched CLT over rectangles for some stationary ortho-martingale. This leads us to consider a stronger projective condition.

Theorem 2.4 Assume that \((X_n)_{n \in \mathbb{Z}^d}\) is defined by (1) and the filtrations are commuting. Also assume that one of the transformations \(T_i, 1 \leq i \leq d\) is ergodic and in addition
\[
\sum_{u \geq 0} \|P_0(X_u)\|_0 < \infty,
\]
where we recall that \( \Phi(x) = x^2(\log(1 + |x|))^{d-1} \). Then, for \( \mathbb{P} \)-almost all \( \omega \in \Omega \),

\[
\left( \frac{1}{\sqrt{|n|}} S_{\lfloor tn \rfloor} \right)_{t \in [0,1]^d} \xrightarrow{D} (W_t)_{t \in [0,1]^d} \quad \text{under} \quad \mathbb{P}^\omega,
\]

where \( \lfloor tn \rfloor \) is the integer part of the vector \( tn := (t_1n_1, \ldots, t_dn_d) \), \( \sigma^2 = \lim_{n \to \infty} \frac{E[S_n^2]}{|n|} \), \( (W_t)_{t \in [0,1]^d} \) is a Brownian sheet and the convergence happens in the Skhorohod space \( D([0,1]^d) \).

We remark that this result and the ones following expand on Theorem 4.2, the first part of Corollary 4.3 and Theorem 2.6 (b) in [35] by obtaining a functional version of these theorems.

**Corollary 2.5** Assume the hypothesis of the previous theorem hold and assume that in addition, for every \( 1 \leq i \leq d \),

\[
\frac{1}{n} \mathbb{E}_0 \left[ \max_{1 \leq m \leq n} \mathbb{E}_2 m(i) [S_m] \right] \xrightarrow{n \to \infty} 0.
\]

Then, for \( \mathbb{P} \)-almost all \( \omega \in \Omega \),

\[
\left( \frac{1}{\sqrt{|n|}} S_{\lfloor tn \rfloor} \right)_{t \in [0,1]^d} \xrightarrow{D} (W_t)_{t \in [0,1]^d} \quad \text{under} \quad \mathbb{P}^\omega,
\]

where \( (W_t)_{t \in [0,1]^d} \) is a Brownian sheet and the convergence happens in the Skhorohod space \( D([0,1]^d) \).

**Theorem 2.6** Assume that the hypothesis of Theorem 2.4 and (6) hold. Then for almost all \( \omega \in \Omega \), (7) holds.

This last Theorem not only extend Theorem 4.4 (b) in [35] to the functional case but also reduces the required condition even in the classical CLT case.

**Theorem 2.7** Assume that

\[
\sum_{u \geq 1} \frac{\|E_1 [X_u]\|_{\Phi}}{\Phi^{-1}(|u|)} < \infty.
\]

Then, for almost all \( \omega \in \Omega \), the conclusion of Corollary 2.5 holds.

### 3 Proofs of the results

Before we prove the previous results, we start by defining some additional notations:

- if \( h : \Omega \to \mathbb{R} \) is a measurable function, we will denote by \( h_u, u \in \mathbb{Z}^d \), the function \( h \circ T^u \);
- for any \( n \in (\mathbb{N}^*)^d \) and for any measurable function \( h : \Omega \to \mathbb{R} \), we denote

\[
S_n(h) = \sum_{1 \leq i \leq n} h_i,
\]

and

\[
\bar{S}_n(h) = S_n(h) - R_n(h) \quad \text{where} \quad R_n(h) = \sum_{i=1}^d (-1)^{i-1} \sum_{1 \leq j_1 < \cdots < j_i \leq d} \mathbb{E}_n(j_1, \cdots, j_i) [S_n(h)],
\]

and \( n^{(j_1, \cdots, j_i)} \) is the multi-index whose \( j_1, \cdots, j_i \)-th coordinates are 0 and the others are equal to the corresponding coordinates of \( n \);
- for any \( i \in [1, d] \) and for any \( \ell \in \mathbb{N} \), we denote

\[
F_{\ell}^{(i)} = \bigvee_{k \in \mathbb{Z}^d, k_i \leq \ell} F_{k}.
\]
Lemma 3.2

For any function \( h \) such that \( \mathbb{E} [h^2 \max(0, \ln(|h|))^{d-1}] < \infty \); if \( \mathcal{G} = \mathcal{F} \), we simply write \( L^2 \ln^{d-1} L(\mathcal{G}) = L^2 \ln^{d-1} L \);

- if \( h \in L^2 \ln^{d-1} L \), then we define the maximal operator \( h^* = \sup_{|m| > 0} \frac{1}{|m|} \sum_{1 \leq i \leq m} |h| \circ T^i \).

Let us start with the proof of Theorem 2.4 as it is the most general result. Moreover the computations used in the proof of Theorem 2.1 are a particular case of the computations for the proof of Theorem 2.4 and will be largely skipped.

The proof of Theorem 2.4 relies on the following important lemma which we will refer to as the main lemma in the rest of the paper.

Lemma 3.1

For any function \( h \in L^2 \ln^{d-1} L \), \( \mathcal{F}_0 \)-measurable and satisfying the following condition:

\[
\sum_{u \geq 0} \| \mathcal{P}_0(h_u) \|_\phi < \infty, \tag{9}
\]

there exists an integrable function \( g \) such that for all \( N \in (\mathbb{N}^*)^d \),

\[
\sqrt{\mathbb{E}_0 \left[ \max_{1 \leq n \leq N} \frac{1}{n} |\mathcal{S}_n(h)|^2 \right]} \leq g \quad \mathbb{P} \text{-a.s.}
\]

In order to establish this lemma, we shall first obtain this intermediary lemma.

Lemma 3.2

For any function \( h \in L^2 \ln^{d-1} L \), there exists a constant \( C > 0 \) such that for all \( h \in \mathbb{Z}^d \), we have

\[
\left\| \sqrt{\left( |\mathcal{P}_0(h_u)|^2 \right)^*} \right\|_1 \leq C \| \mathcal{P}_0(h_u) \|_\phi.
\]

Proof of Lemma 3.2.

Let \( h \in L^2 \ln^{d-1} L \), \( u \in \mathbb{Z}^d \) and \( t > \| \mathcal{P}_0(h_u) \|_\phi \). We let

\[
\Omega_t = \left\{ \omega \in \Omega : 4 (\mathcal{P}_0(h_u))^2 (\omega) > t^2 \right\}.
\]

According to Corollary 1.7 of Chapter 6 in Krengel [22], there exists a constant \( C_d > 0 \) such that

\[
\mathbb{P} \left( \sup_{n \in (\mathbb{N}^*)^d} \frac{1}{|n|} \sum_{1 \leq i \leq n} \left( \mathcal{P}_0(h_u) \circ T^i \right)^2 > t^2 \right) \leq C_d \int_{\Omega_t} \frac{4 (\mathcal{P}_0(h_u))^2}{t^2} \left( \ln \left( \frac{4 (\mathcal{P}_0(h_u))^2}{t^2} \right) \right)^{d-1} d\mathbb{P}
\]

\[
\leq 2^{d-1} C_d \int_{\Omega_t} \frac{(2 \mathcal{P}_0(h_u))^2}{t^2} \left( \ln \left( 1 + \frac{2 |\mathcal{P}_0(h_u)|}{t} \right) \right)^{d-1} d\mathbb{P}
\]

\[
\leq 2^{d+1} C_d t^{-2} |\mathcal{P}_0(h_u)|^2_\phi.
\]

The last inequality results from the fact that

\[
|\mathcal{P}_0(h_u)|_\phi = \inf \left\{ t > 0 : \mathbb{E} \left[ \Phi \left( \frac{|\mathcal{P}_0(h_u)|}{t} \right) \right] \leq 1 \right\}.
\]

Indeed, by letting \( t_0 = |\mathcal{P}_0(h_u)|_\phi \), we have

\[
\mathbb{E} \left[ (\mathcal{P}_0(h_u))^2 \left( \ln \left( 1 + \frac{|\mathcal{P}_0(h_u)|}{t_0} \right) \right)^{d-1} \right] \leq t_0^2.
\]

Hence since \( t > t_0 \),

\[
\int_{\Omega} \frac{(\mathcal{P}_0(h_u))^2}{t^2} \left( \ln \left( 1 + \frac{|\mathcal{P}_0(h_u)|}{t} \right) \right)^{d-1} d\mathbb{P} \leq t_0^2 t^{-2};
\]
Therefore, applying this inequality to \( h' = 2h \in L^2 \log^{d-1} L \), we get

\[
\left\| \sqrt{\left( \mathcal{P}_0(\mathcal{h}_u)^2 \right)} \right\|_1 = \int_0^\infty \mathbb{P} \left( \sup_{n \in (\mathbb{N}^*)^d} \frac{1}{n} \sum_{1 \leq i \leq n} \left( \mathcal{P}_0(\mathcal{h}_u) \circ T^i \right)^2 > t^2 \right) dt
\leq \int_0^{t_0} 1 dt + \int_{t_0}^\infty \mathbb{P} \left( \sup_{n \in (\mathbb{N}^*)^d} \frac{1}{n} \sum_{1 \leq i \leq n} \left( \mathcal{P}_0(\mathcal{h}_u) \circ T^i \right)^2 > t^2 \right) dt
\leq \left( 2^{d+1} C_d + 1 \right) \left\| \mathcal{P}_0(\mathcal{h}_u) \right\|_{\mathcal{F}}.
\]

**Proof of the main lemma.** We consider a measurable function \( h \) satisfying the hypothesis of the lemma and we let \( n, N \in (\mathbb{N}^*)^d \) such that \( n \leq N \). Then, we start by studying the quantity \( S_n(h) \) using the following projective decomposition (see [27]):

\[ S_n(h) - R_n(h) = \sum_{1 \leq i \leq n} \mathcal{P}_i \left( \sum_{i \leq u \leq n} h_u \right) = \sum_{1 \leq i \leq n} \mathcal{P}_0 \left( \sum_{0 \leq u \leq n-i} h_u \right) \circ T^i. \]

By exchanging the sums, we get

\[ \tilde{S}_n(h) = \sum_{0 \leq u \leq n-1} \sum_{1 \leq i \leq n-u} \mathcal{P}_0(h_u) \circ T^i. \]

Then, recalling that \( n \leq N \), we obtain

\[ |\tilde{S}_n(h)| \leq \sum_{0 \leq u \leq N-1} \max_{1 \leq k \leq N} \left| \sum_{1 \leq i \leq k} \mathcal{P}_0(h_u) \circ T^i \right|. \]

Note that for all \( u \geq 0 \), the partial sum \( \sum_{1 \leq i \leq k} \mathcal{P}_0(h_u) \circ T^i \) is an ortho-martingale. Using Cairoli’s inequality for ortho-martingales (see [3]), we find that

\[ \sum_{0 \leq u \leq N-1} \mathbb{E}_0 \left[ \max_{1 \leq k \leq N} \left| \sum_{1 \leq i \leq k} \mathcal{P}_0(h_u) \circ T^i \right|^2 \right] \leq 2^{2d} \sum_{u \geq 0} \mathbb{E}_0 \left[ \left( \sum_{1 \leq i \leq N} \mathcal{P}_0(h_u) \circ T^i \right)^2 \right]. \]

By orthogonality, we obtain

\[
\sqrt{\mathbb{E}_0 \left[ \max_{1 \leq n \leq N} \left| \tilde{S}_n(h) \right|^2 \right]} \leq 2^d \sum_{u \geq 0} \sqrt{\mathbb{E}_0 \left[ (\mathcal{P}_0(h_u))^2 \circ T^i \right]} \leq 2^d \sqrt{N} \sum_{u \geq 0} \sqrt{\left( \mathcal{P}_0(h_u)^2 \right)}.
\]

Hence for all \( N \in (\mathbb{N}^*)^d \)

\[ \sqrt{\mathbb{E}_0 \left[ \max_{1 \leq n \leq N} \frac{1}{n} \left( \tilde{S}_n(h) \right)^2 \right]} \leq 2^d \sum_{u \geq 0} \sqrt{\left( \mathcal{P}_0(h_u)^2 \right)} \tag{10} \]

However, according to Lemma 3.2 and hypothesis (9), there exists \( C > 0 \) such that

\[ \left\| \sum_{u \geq 0} \sqrt{\left( \mathcal{P}_0(h_u)^2 \right)} \right\|_1 \leq C \sum_{u \geq 0} \| \mathcal{P}_0(h_u) \|_{\mathcal{F}} < \infty. \]

This concludes the proof of the main lemma. ■

**Proof of Theorem 2.4.** For any \( n \in \mathbb{N}^* \), we let

\[ X^{(n)}_0 = \sum_{j \in [-n,0]^d} \mathcal{P}_j(X_0) \]
Given the regularity of $X_0$, the sequence of random variable $(X_0 - X_0^{(n)})_{n \in \mathbb{N}}$ converges almost surely to 0 and using (10), we get the convergence

$$
\limsup_{N \to \infty} \sqrt{E_0 \left[ \max_{1 \leq m \leq N} \frac{1}{|m|} \left| S_m (X_0 - X_0^{(n)}) \right|^2 \right]} \leq 2^d \sum_{u \geq 0} \sqrt{\left( \|P_0 \left( (X_0 - X_0^{(n)}) \circ T^u \right) \|_F \right)^*}
$$

for all $n \in \mathbb{N}^*$. Then, according to the proof of the main lemma, there exists a constant $C$ such that

$$
\left\| \sqrt{\limsup_{N \to \infty} E_0 \left[ \max_{1 \leq m \leq N} \frac{1}{|m|} \left| S_m (X_0 - X_0^{(n)}) \right|^2 \right]} \right\|_1 \leq C \sum_{u \geq 0} \|P_0 \left( (X_0 - X_0^{(n)}) \circ T^u \right) \|_F \quad \text{a.s.} \to 0.
$$

Therefore, there exists an increasing sequence of integers $(n_k)_{k \in \mathbb{N}}$ such that

$$
\lim \limsup_{k \to \infty} \limsup_{N \to \infty} E_0 \left[ \max_{1 \leq m \leq N} \frac{1}{|m|} \left| S_m (X_0 - X_0^{(n_k)}) \right|^2 \right] = 0 \quad \text{a.s.}
$$

(11)

Moreover, we also have, for all $n \in \mathbb{N}^*$

$$
\frac{1}{|N|} E_0 \left[ \max_{1 \leq i \leq N} \left| R_i (X_0^{(n)}) \right|^2 \right] \to 0 \quad \text{a.s.} \quad \text{as } N \to \infty
$$

(12)

Indeed, using the triangular inequality, it is enough to show that for all $i \in \mathbb{Z}^d$

$$
\frac{1}{|N|} E_0 \left[ \max_{1 \leq j \leq N} \left| R_j (P_i (X_0)) \right|^2 \right] \to 0.
$$

This holds true by applying following lemma.

**Lemma 3.3** For any square integrable function $F_0$-measurable $h$, the condition

$$
\sum_{u \geq 1} \frac{\|E_1 [h_u]\|_2}{|u|^{1/2}} < \infty
$$

implies

$$
\frac{1}{|N|} E_0 \left[ \max_{1 \leq n \leq N} \left| R_n (h) \right|^2 \right] \to 0 \quad \text{a.s.} \quad \text{as } N \to \infty
$$

We delay the proof of this lemma to latter in this section.

Remark that the proof of Proposition 4.1 in [32] can be easily adapted to the case of Orlicz spaces such that for some fixed $n \in \mathbb{N}^*$, we get the following martingale-codoundary decomposition

$$
X_0^{(n)} = \sum_{S \subseteq [1,d]} h^{(n)}_S \circ \prod_{j \in S^c} (I - T_j),
$$

where $h^{(n)}_S \in \bigcap_{i \in S} \left( L^2 \ln^{d-1} L \left( F_0^{(i)} \right) \right) \cap L^2 \ln^{d-1} L \left( F_{-1}^{(i)} \right)$ for all $S \subseteq [1,d]$ and using the convention $\prod_{j \in \emptyset} (I - T_j) = I$. Moreover

$$
h^{(n)}_{[1,d]} = \sum_{i \in \mathbb{Z}^d} P_0 \left( X_0^{(n)} \circ T^i \right).
$$

According to the proof of Remark 11 in [28] (see also the proof of Theorem 7 in the same article), the following almost-sure convergence

$$
\mathbb{P}^\omega \left( \max_{1 \leq m \leq N} \frac{1}{|m|} \left| S_m (X_0^{(n)} - d_n) \right|^2 \geq \epsilon \right) \to 0 \quad \text{a.s.} \quad \text{as } N \to \infty
$$

(13)
holds for all $\epsilon > 0$, where $d_n = h_{\lfloor [1,d] \rfloor}$. Moreover, letting $N \in (\mathbb{N}^*)^d$ and $D_0 = \sum_{i \in \mathbb{Z}^d} P_0(X_i)$, we get
\[
S_N(D_0 - d_n) = \sum_{1 \leq i \leq N} \left( D_i - D_i^{(n)} \right),
\]
where
\[
D_i = \sum_{j \in \mathbb{Z}^d} P_i(X_{i-j}) \quad \text{et} \quad D_i^{(n)} = \sum_{j \in [-n,0]^d} P_i(X_{i-j}).
\]
Hence, given that $\left( D_i - D_i^{(n)} \right)_{i \in \mathbb{Z}^d}$ is an ortho-martingale differences field and according to Cairoli’s inequality, we have
\[
P^\omega \left( \frac{1}{\sqrt{|N|}} \max_{1 \leq i \leq N} |S_i(D_0 - d_n)| > \epsilon \right) \leq \frac{2^d}{\epsilon^2 |N|} \sum_{1 \leq i \leq N} \mathbb{E}_0 \left[ \left( D_i - D_i^{(n)} \right)^2 \right].
\]
Let us note that
\[
\sqrt{\frac{1}{|N|}} \sum_{1 \leq i \leq N} \mathbb{E}_0 \left[ \left( D_i - D_i^{(n)} \right)^2 \right] \leq \sum_{j \notin [-n,0]^d} \sqrt{\frac{1}{|N|}} \sum_{1 \leq i \leq N} \mathbb{E}_0 \left[ (P_0(X_{-j}))^2 \circ T^i \right].
\]
According to the ergodic Theorem 1.1 of Chapter 6 in Krengel [22] for Dunford Schwarz operators and Lemma 7.1 in [10], we have the following convergence :
\[
\lim_{N \to \infty} \frac{1}{|N|} \sum_{1 \leq i \leq N} \mathbb{E}_0 \left[ (P_0(X_{-j}))^2 \circ T^i \right] = \mathbb{E} \left[ (P_j(X_0))^2 \right] \quad \text{a.s.}
\]
for all $j \notin [-n,0]^d$. Since $\sum_{j \in \mathbb{Z}^d} \|P_0(X_j)\|_2 < \infty$, we get
\[
\lim_{n \to \infty} \lim_{N \to \infty} \frac{2^d}{\epsilon^2 |N|} \sum_{1 \leq i \leq N} \mathbb{E}_0 \left[ \left( D_i - D_i^{(n)} \right)^2 \right] = 0 \quad \text{a.s.} \quad (14)
\]
Combining (11), (12), (13) and (14), we obtain that for all $\epsilon > 0$,
\[
\limsup_{N \to \infty} \mathbb{P}^\omega \left( \frac{1}{\sqrt{|N|}} \max_{1 \leq m \leq N} |\tilde{S}_m - s_m(D_0)| \geq \epsilon \right) = 0 \quad \text{a.s.}
\]
We conclude by noticing that the field $(D_0 \circ T^i)_{i \in \mathbb{Z}^d}$ satisfies a functional central limit theorem (according to Theorem 5 in [28]) and therefore the expected result is obtained by applying Theorem 3.1 in [24].

**Proof of the Theorem 2.1.** The proof of this theorem is very similar to the previous one, with the exception of using Theorem 2.8 instead of Theorem 1.1 of Chapter 6 in Krengel [22] and Lemma 1.4 in the same Chapter (applied to the abstract maximal operator $Mf := \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{1 \leq i \leq n} |f| \circ T^i$, see definition 1.3 of Chapter 6 and Corollary 2.2 of Chapter 1 in Krengel[22]) instead of Corollary 1.7 in order to obtain the $L^2$ versions of intermediary lemma and the main lemma mentioned below.

**Lemma 3.4 (Version $L^2$)** For any function $h \in L^2(F_0)$ satisfying the following condition :
\[
\sum_{n \geq 0} \|P_0(h_{\lfloor u \rfloor})\|_2 < \infty, \quad (15)
\]
there exist an integrable function $g$ such that for all $N \in \mathbb{N}^*$,
\[
\sqrt{\mathbb{E}_0 \left[ \max_{1 \leq n \leq N} \frac{1}{n} \left| \tilde{S}_{n1}(h) \right|^2 \right]} \leq g \quad \mathbb{P} \text{- a.s.}
\]
Lemma 3.5 (Version $L^2$ of Lemma 3.2) For all functions $h \in L^2$, there exists a constant $C > 0$ such that for all $u \in \mathbb{Z}^d$, we have

$$\left\| \sqrt{\left| \mathcal{P}_n(hu) \right|} \right\|_1 \leq C \left\| \mathcal{P}_n(hu) \right\|_2,$$

where $h^* = \sup_{n \in \mathbb{N}} \frac{1}{n^d} \sum_{1 \leq i \leq n} |h| \circ T^i$.

The following proof of Corollary 2.5 can be easily adapted to obtain Corollary 2.2 by using the Theorem 2.1 instead of Theorem 2.4.

Proof of Corollary 2.5. According to Theorem 2.4 and Theorem 3.1 in [24], it is enough to show

$$\frac{1}{|n|} \mathbb{E}_0 \left[ \max_{1 \leq i \leq n} R_i^2 \right] \xrightarrow{\text{a.s.}} 0.$$

Let $n \in \mathbb{Z}^d$, and recall that

$$R_n = \sum_{i=1}^d (-1)^{i-1} \sum_{1 \leq j_1 < \cdots < j_i \leq d} \mathbb{E}_{n^{(j_1, \cdots, j_i)}} \left[ S_n \right],$$

where $n^{(j_1, \cdots, j_i)}$ is the multi-index such that the $j_k$-th, $1 \leq k \leq i$ coordinates are zero and the other are equal to the corresponding coordinates of $n$.

Using the triangular inequality, it is enough to establish that

$$\frac{1}{|n|} \mathbb{E}_0 \left[ \max_{1 \leq i \leq n} \mathbb{E}_{n^{(j_1, \cdots, j_i)}} \left[ S_n \right]^2 \right] \xrightarrow{\text{a.s.}} 0 \quad (16)$$

for any $j_1 < \cdots < j_i, 1 \leq i \leq d$.

The terms satisfying $i = 1$ have this property according to the hypothesis of the corollary and all the other terms are treated in a similar fashion. By induction, the corollary will be proven if we can show that if property (16) is verified for all $j_1 < \cdots < j_i$, it also holds true for all $j_1 < \cdots < j_{i+1}$. For the sake of simplicity and without loss of generality, we will only establish that

$$\forall j \in [1, d], P(j) \implies P(1, 2).$$

In other words, we use the hypothesis of the corollary to show

$$\frac{1}{|n|} \mathbb{E}_0 \left[ \max_{1 \leq i \leq n} \left( \mathbb{E}_{q(1, 2)} \left[ S_i \right] \right)^2 \right] \xrightarrow{\text{a.s.}} 0.$$

According to Jensen’s inequality we have

$$\mathbb{E}_0 \left[ \max_{1 \leq i \leq n} \left( \mathbb{E}_{q(1, 2)} \left[ S_i \right] \right)^2 \right] = \mathbb{E}_0 \left[ \max_{1 \leq i \leq n} \left( \mathbb{E}_{q(1, 2)} \left[ \mathbb{E}_{q(1)} \left[ S_i \right] \right] \right)^2 \right] \leq \mathbb{E}_0 \left[ \max_{1 \leq i \leq n} \left( \mathbb{E}_{q(1)} \left[ S_i \right] \right)^2 \right].$$

So

$$\frac{1}{|n|} \mathbb{E}_0 \left[ \max_{1 \leq i \leq n} \left( \mathbb{E}_{q(1, 2)} \left[ S_i \right] \right)^2 \right] \leq \frac{1}{|n|} \mathbb{E}_0 \left[ \max_{1 \leq i \leq n} \left( \mathbb{E}_{q(1)} \left[ S_i \right] \right)^2 \right] \xrightarrow{\text{a.s.}} 0.$$

Before continuing with the proof, we establish Lemma 3.3.

Proof of Lemma 3.3. We show that for any $i \in [1, d]$ and any $1 \leq j_1 < \cdots < j_i \leq d$, we have the convergence

$$\frac{1}{|N|} \mathbb{E}_0 \left[ \max_{1 \leq n \leq N} \mathbb{E}_{n^{(j_1, \cdots, j_i)}} \left[ S_n(h) \right]^2 \right] \xrightarrow{\text{a.s.}} 0.$$
In order to do so, we use an induction on $k = d - i$. For $k = 0$ and in the same way as in the proof of Lemma 3.2 in [35], we establish that
\[ \mathbb{E}_0 \left[ \frac{S_n(h)}{|n|} \right] \xrightarrow{\text{a.s.}} 0. \]
Hence
\[ \frac{1}{|N|} \max_{1 \leq n \leq N} \mathbb{E}_0 \left[ \frac{S_n(h)}{|n|} \right] \xrightarrow{\text{a.s.}} 0. \]
Now, we suppose that the desired property holds true for $k - 1 < d - 1$. Without loss of generality, we establish the property only for $(j_1, \ldots, j_n) = (1, \ldots, n)$; it is enough to show that
\[ \frac{1}{|N|} \mathbb{E}_0 \left[ \max_{1 \leq n \leq N} \left( \mathbb{E}_n^{(j_1, \ldots, j_k)} [S_n(h)] - \mathbb{E}_n^{(j_2, \ldots, j_k)} [S_n(h)] \right)^2 \right] \xrightarrow{\text{a.s.}} 0. \]
Let $n, N \in (\mathbb{N}^*)^d$ such that $n \leq N$ then the following decomposition holds (see the proof of Lemma 3.2 in [35])
\[ \mathbb{E}_0 \left[ \max_{1 \leq n \leq N} \left( \mathbb{E}_n^{(j_1, \ldots, j_k)} [S_n(h)] - \mathbb{E}_n^{(j_2, \ldots, j_k)} [S_n(h)] \right)^2 \right] = \mathbb{E}_0 \left[ \max_{1 \leq n \leq N} \left( \sum_{i=1}^{n_1} P^{(1)}_{n^{(j_1, \ldots, j_k)}, i} (S_n(h)) \right)^2 \right], \]
where $P^{(1)}_{n^{(j_1, \ldots, j_k)}, i}(S_n(h)) = \mathbb{E}_{e_1 + n^{(j_1, \ldots, j_k)}} [S_n(h)] - \mathbb{E}_{(i-1)e_1 + n^{(j_1, \ldots, j_k)}} [S_n(h)]$ and $e_1$ is the multi-index whose coordinates are all zero except for the first one which is 1.

Since $h$ is $\mathcal{F}_0$-measurable, we have
\[ \left| \sum_{i=1}^{n_1} P^{(1)}_{n^{(j_1, \ldots, j_k)}, i} (S_n(h)) \right| \leq \sum_{1 \leq u \leq N} \max_{1 \leq k \leq N_1} \sum_{i=1}^{k} P^{(1)}_{u^{(j_1, \ldots, j_k)}, i} (h_u) \]
Therefore, according to Doob’s inequality for martingales, it follows
\[ \sqrt{\mathbb{E}_0 \left[ \max_{1 \leq n \leq N} \left( \sum_{i=1}^{n_1} P^{(1)}_{n^{(j_1, \ldots, j_k)}, i} (S_n(h)) \right)^2 \right]} \leq 2 \sum_{1 \leq u \leq N} \sqrt{\mathbb{E}_0 \left[ \left( \sum_{i=1}^{N_1} P^{(1)}_{u^{(j_1, \ldots, j_k)}, i} (h_u) \right)^2 \right]} \]
Let $c > 0$, we use the following decomposition
\[ \frac{1}{\sqrt{|N|}} \sum_{1 \leq u \leq N} \sqrt{\mathbb{E}_0 \left[ \left( \sum_{i=1}^{N_1} P^{(1)}_{u^{(j_1, \ldots, j_k)}, i} (h_u) \right)^2 \right]} = I_{N,c} + II_{N,c} \]
where
\[ I_{N,c} = \frac{1}{\sqrt{|N|}} \sum_{1 \leq u_1 \leq N_1} \sum_{1 \leq u_2, \ldots, u_d \leq c} \sqrt{\mathbb{E}_0 \left[ \left( \sum_{i=1}^{N_1} P^{(1)}_{u^{(j_1, \ldots, j_k)}, i} (h_u) \right)^2 \right]} \]
and $II_{N,c}$ is the remainder of the initial sum.

Let us show that
\[ \limsup_{N \to \infty} I_{N,c} = 0 \quad \text{a.s.} \]
Indeed, by an orthogonality argument
\[ I_{N,c} \leq \frac{e^{d-1}}{\sqrt{|N|}} \sup_{1 \leq u_2, \ldots, u_d \leq c} \sum_{a_1 \geq 0} \sqrt{\mathbb{E}_0 \left[ \left( P^{(1)}_{u^{(j_1, \ldots, j_k)}, i} (h_u) \right)^2 \circ T_1 \right]} . \]
However, according to Theorem 1.1 of Chapter 6 in [22] and Lemma 7.1 in [10], we get the convergence
\[
\lim_{N_1 \to \infty} \frac{1}{N_1} \sum_{1 \leq i \leq N_1} \mathbb{E}_0 \left[ \left( P_{u(i_1, \ldots, i_k), (h_u)}^{(1)} \right)^2 \right] = \mathbb{E} \left[ \left( P_{u(i_1, \ldots, i_k), 0}^{(1)} \right)^2 \right] \quad \text{a.s.}
\]
where \( \mathcal{I}_1 \) is the invariant \( \sigma \)-algebra associated to \( T_1 \). Hence
\[
\limsup_{N \to \infty} I_{N,c} = 0 \quad \text{a.s.}
\]
The different sums appearing in II_{N,c}, admit at least one direction (different from the first one) for which the indices in the sums for this direction are at least equal to \( c + 1 \). Without loss of generality, we suppose that it is the second one. Then
\[
\sqrt{\frac{1}{N}} \sum_{1 \leq u_1 \leq N_1} \sum_{c+1 \leq u_2 \leq N_2} \sum_{1 \leq u_3 \leq N_3} \cdots \sum_{1 \leq u_d \leq N_d} \sqrt{\frac{1}{N_1} \sum_{i=1}^{N_1} \mathbb{E}_0 \left[ \left( P_{u(i_1, \ldots, i_k), (h_u)}^{(1)} \right)^2 \right]}.
\]
Applying, once again Theorem 1.1 of Chapter 6 in [22] and Lemma 7.1 in [10], we obtain
\[
\limsup_{N \to \infty} II_{N,c} \leq \sum_{u_2 \geq c+1} \sum_{u_1 \geq 0} \sum_{u_2 \geq 1} \cdots \sum_{u_d \geq 1} \sqrt{\mathbb{E} \left[ \left( P_{u(i_1, \ldots, i_k), 0}^{(1)} \right)^2 \right] \mathbb{P} \left( \mathcal{I}_1 \right)}.
\]
Using (45) in [35], we get
\[
\lim \limsup_{N \to \infty} II_{N,c} = 0 \quad \text{a.s.}
\]
This concludes the proof. ■

Theorems 2.6 and 2.3 are direct consequences of this lemma and the previous results.

**Proof of Theorem 2.6.** This theorem is a consequence of Lemma 3.3 and of Theorem 2.5. ■

**Proof of Theorem 2.3.** The theorem is a consequence of Lemma 3.3, (47) in [35] (for \( q = 2 \) and the proof of Theorem 4.2 and Lemma 3.3 in [35]) and Theorem 2.1. ■

The rest of this section will be dedicated to proving Theorem 2.7. We start by making a few remarks concerning the Luxemburg norms. Let us note that for \( x \geq 0 \) and \( 0 < \lambda \leq e - 1 \),
\[
\ln \left( 1 + \frac{x}{\lambda} \right) \ln(1 + \lambda) \leq \ln(1 + x).
\]
Recall that the function \( \Phi : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+ \setminus \{0\} \) is defined by
\[
\Phi(x) = x^2(\ln(1 + |x|))^{d-1}
\]
for all \( x \in \mathbb{R}^+ \setminus \{0\} \). Then, we deduce the following remarkable property of the function \( \Phi \).

For \( x > 0 \) and \( 0 < \lambda \leq e - 1 \),
\[
\Phi \left( \frac{x}{\lambda} \right) = \left( \frac{x}{\lambda} \right)^2 \left( \ln \left( 1 + \frac{x}{\lambda} \right) \right)^{d-1} \leq \frac{x^2(\ln(1 + x))^{d-1}}{\lambda^2(\ln(1 + \lambda))^{d-1}} = \frac{\Phi(x)}{\Phi(\lambda)}
\]
(17)

Besides, since \( \Phi \) is a convex function, we also have
\[
\Phi \left( \frac{x}{\lambda} \right) = \Phi \left( \frac{x}{\lambda} + \left( 1 - \frac{1}{\lambda} \right) \cdot 0 \right) \leq \frac{\Phi(x)}{\lambda} + \left( 1 - \frac{1}{\lambda} \right) \Phi(0) = \frac{\Phi(x)}{\lambda}, \]
for \( x \geq 0 \) and \( \lambda > 1 \).

Obviously the function \( \Phi \) defined by (3) is bijective and we denote by \( \Phi^{-1} \) its inverse function. The following lemma might be well-known but we could not find it in the literature.

**Lemma 3.6** Let \( X \in L^2 \ln^{d-1} L \). If \( \Phi^{-1}(\mathbb{E}[\Phi(X)]) \leq e - 1 \), then

\[
\|X\|_\Phi \leq \Phi^{-1}(\mathbb{E}[\Phi(X)]),
\]
and if \( \mathbb{E}[\Phi(X)] > 1 \), then

\[
\|X\|_\Phi \leq \mathbb{E}[\Phi(X)].
\]

**Proof of Lemma 3.6.** If \( X = 0 \) almost surely, then the property is evident. Else, suppose that \( \mathbb{P}(X = 0) \neq 1 \), then by the definition of Luxembourg norm

\[
\|X\|_\Phi = \inf \left\{ \lambda > 0 : \mathbb{E} \left[ \Phi \left( \frac{X}{\lambda} \right) \right] \leq 1 \right\}.
\]

Note that by the properties of \( \Phi \) for any \( 0 < \lambda \leq e - 1 \)

\[
\mathbb{E} \left[ \Phi \left( \frac{X}{\lambda} \right) \right] \leq \frac{\mathbb{E}[\Phi(X)]}{\Phi(\lambda)}.
\]

From this inequality it follows that if \( \lambda \) is the solution to the equation \( \mathbb{E}[\Phi(X)] = \Phi(\lambda) \) we have necessarily that \( \mathbb{E} \left[ \Phi \left( \frac{X}{\lambda} \right) \right] \leq 1 \), and then \( \|X\|_\Phi \leq \lambda = \Phi^{-1}(\mathbb{E}[\Phi(X)]) \).

For the case \( \mathbb{E}[\Phi(X)] > 1 \), the proof is similar using property (18) of \( \Phi \).

**Lemma 3.7** Condition (8) implies \( \sum_{u \geq 0} \|\mathcal{P}_0(X_u)\|_\Phi < \infty \).

**Proof.** Let \( a, b \in \mathbb{Z}^d \) such that \( a \leq b \). Denote by \( \psi \) the conjugate function associated with \( \Phi \) defined in the following way

\[
\psi(x) = \sup_{y \geq 0} (xy - \Phi(y))
\]
for \( x \in \mathbb{R}^+ \setminus \{0\} \). By the generalized Holder inequality for Orlicz spaces (see [29], p.58) we have

\[
\sum_{u \geq 1} \|\mathcal{P}_0(X_u)\|_\Phi = \sum_{n \geq 0} \sum_{v=2^n}^{2^{n+1}-1} \|\mathcal{P}_0(X_v)\|_\Phi
\leq C \sum_{n \geq 0} \inf \left\{ \eta > 0 : \sum_{v=2^n}^{2^{n+1}-1} \psi \left( \frac{1}{\eta} \right) \leq 1 \right\} \cdot \inf \left\{ \eta > 0 : \sum_{v=2^n}^{2^{n+1}-1} \Phi \left( \frac{\|\mathcal{P}_0(X_v)\|_\Phi}{\eta} \right) \leq 1 \right\}.
\]

We have that

\[
\inf \left\{ \eta > 0 : \sum_{v=2^n}^{2^{n+1}-1} \psi \left( \frac{1}{\eta} \right) \leq 1 \right\} = \frac{1}{\psi^{-1}(2^{-n})},
\]
where \( 2^{-n} = (2^{n_1} \cdots 2^{n_d})^{-1} \). It is enough to establish the result in the case \( \mathbb{E} \left[ \Phi \left( \frac{\mathcal{P}_0(X_0)}{\lambda} \right) \right] \leq \Phi(e^{-1}) \) for any \( v \geq 0 \), as the other case can be proved using similar arguments and is left to the reader.

By Lemma 3.6 above, if \( \mathbb{E} \left[ \Phi \left( \frac{\mathcal{P}_v(X_0)}{\lambda} \right) \right] \leq \Phi(e^{-1}) \) for a \( v \geq 0 \), we also have

\[
\Phi \left( \frac{\|\mathcal{P}_v(X_0)\|_\Phi}{\lambda} \right) \leq \mathbb{E} \left[ \Phi \left( \frac{\mathcal{P}_v(X_0)}{\lambda} \right) \right].
\]
Since
\[ \sum_{v=2^n}^{2^{n+1}-1} \Phi \left( \frac{\| P_{-v}(X_0) \|_\phi}{\lambda} \right) \leq \sum_{u=2^n}^{2^{n+1}-1} E \left[ \Phi \left( \frac{P_{-v}(X_0)}{\lambda} \right) \right]. \]

Also by convexity and the Jensen inequality as well as the definition of \( \Phi \) it follows that
\[ \sum_{v=2^n}^{2^{n+1}-1} E \left[ \Phi \left( \frac{P_{-v}(X_0)}{\lambda} \right) \right] \leq E \left[ \Phi \left( \sum_{v=2^n}^{2^{n+1}-1} \frac{P_{-v}(X_0)}{\lambda} \right) \right] = E \left[ \Phi \left( \frac{1}{\lambda} \sum_{i=0}^{d} (-1)^i \sum_{1 \leq j_1 < \cdots < j_i \leq d} E_{-2^{n+1}(j_1, \ldots, j_i)} [X_0] \right) \right] \leq \frac{1}{2^d} \sum_{i=0}^{d} (-1)^i \sum_{1 \leq j_1 < \cdots < j_i \leq d} E \left[ \Phi \left( \frac{2^d}{\lambda} E_{-2n} [X_0] \right) \right] \leq E \left[ \Phi \left( \frac{2^d}{\lambda} E_{-2n} [X_0] \right) \right]. \]

So
\[
\inf \left\{ \eta > 0 : \sum_{v=2^n}^{2^{n+1}-1} \Phi \left( \frac{\| P_0(X_v) \|_\phi}{\eta} \right) \leq 1 \right\} \leq \inf \left\{ \eta > 0 : E \left[ \Phi \left( \frac{2^d}{\eta} E_{-2n} [X_0] \right) \right] \leq 1 \right\} = 2^d \| E_{-2n} [X_0] \|_\phi.
\]

Since \( \| E_{-n} [X_0] \|_\phi \) is nonincreasing in all directions of \( n \), we obtain that for any \( n \) such that \( n_k > 0 \) for all \( k \in \llbracket 1, d \rrbracket \), we have
\[ \frac{2^n}{\Phi^{-1}(2^n)} \| E_{-2n} [X_0] \|_\phi \leq \sum_{u=2^n}^{2^n-1} \frac{\| E_0 [X_u] \|_\phi}{\Phi^{-1}(|u|)}. \]

So, for some positive constant \( K \)
\[ \sum_{v \geq 1} \| P_0(X_v) \|_\phi \leq K \sum_{n \geq 1} \frac{\Phi^{-1}(2^n)}{2^n \psi^{-1}(2^{-n})} \sum_{u=2^n}^{2^n-1} \frac{\| E_{-u} (X_0) \|_\phi}{\Phi^{-1}(|u|)}. \]

However, there exits a constant \( K' > 0 \) such that \( \Phi^{-1}(2^n) \sim 2^n \psi^{-1}(2^{-n}) K' \). Hence, according to the previous inequalities, we have shown that (8) implies
\[ \sum_{u \geq 1} \| P_0(X_u) \|_\phi < \infty. \]

In the same way, we have for every \( i \in \llbracket 1, d \rrbracket \) and for every \( (j_1, \ldots, j_i) \in \llbracket 1, d \rrbracket^i \) such that \( j_1 < \cdots < j_i \),
\[ \sum_{u_{(j_1, \ldots, j_i)} \geq 1} (j_1, \ldots, j_i) \| P_0(X_{(j_1, \ldots, j_i)}) \|_\phi < \infty. \]

Hence (2.4) is fullfilled. ■

**Proof of Theorem 2.7.** This theorem is a consequence of Theorem 2.4 and Lemmas 3.3 and 3.7. ■
4 Examples

In this section, we present two examples of application of the previous results we obtained. We will focus on both a linear and a non-linear example. We improve on the results by Zhang et al. [35] by requiring weaker assumptions on both the moment of the innovations and the coefficients which appear in each example. More precisely, we obtain a functional CLT despite only requiring that the i.i.d. innovations belongs to the Orlicz space $L^2 \ln d^{-1} L$ instead of the Lebesgue space $L^q$ with $q > 2$ as is required by Zhang et al. [35]. Throughout this section, as before, we denote by $C > 0$ a generic constant, which may be different from line to line. Recall that the function $\Phi : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+ \setminus \{0\}$ is bijective and defined by (3).

Example 4.1 (Linear field) Let $(\xi_n)_{n \in \mathbb{Z}^d}$ be a random field of independent, identically distributed random variables, which are centered and satisfy $E[|\xi_0|^2 (\log(1 + |\xi_0|))^{d-1}] < \infty$. For $k \geq 0$ define

$$X_k = \sum_{j \geq 0} a_j \xi_{k-j}. $$

Assume that the following condition hold.

$$\sum_{k \geq 1} \frac{1}{\Phi^{-1}(|k|)} \left( \sum_{j \geq k-1} a_j^2 \right)^{1/2} < \infty. \quad (19)$$

Then the quenched functional CLT in Corollary 2.5 holds.

Remark 4.2 Condition (19) is fulfilled whenever the stronger but more practical condition

$$\sum_{k \geq 1} \frac{(\log(|k|))^{1/2}}{|k|^{1/2}} \left( \sum_{j \geq k-1} a_j^2 \right)^{1/2} < \infty$$

is satisfied.

The results obtained by Zhang et al. [35] (Remark 6.2 (c)) required the existence of $q$-th moment, with $q > 2$, of the innovation $\xi_0$ to obtain the quenched functional CLT; meanwhile, we only require that $\xi_0$ satisfy a weaker Orlicz condition to obtain that result. Additionally we require weaker assumptions on the coefficients $a_u, u \in \mathbb{Z}^d$.

Proof of Example 4.1. According to the independence of $\xi_n$, we have

$$E_1[X_u] = \sum_{j \geq u-1} a_j \xi_{k-j}. $$

Then, applying the Burkholder inequality for Orlicz spaces (see Meyer and Dellacherie [8], p.304, VII - 92), we obtain

$$\|E_1[X_u]\|_\Phi = \left\| \sum_{j \geq u-1} a_j \xi_{k-j} \right\|_\Phi \leq C \left( \sum_{j \geq u-1} a_j^2 \|\xi_{k-j}\|^2_\Phi \right)^{1/2}.$$ 

By stationarity and since $\|\xi_0\|_\Phi < \infty$, we obtain, by using assumption (19), that condition (8) is satisfied. Hence the result.
proof. Ito. and Dellacherie [8], p.304, VII - 92), we get

\[
\sum_{k \geq 1} \frac{1}{\Phi^{-1}(|k|)} \left( \sum_{u,v \geq (k-1,k-1)} a_{u,v}^2 \right)^{1/2} < \infty,
\]

(20)

Then the quenched functional CLT in Theorem 2.5 holds.

Remark 4.4 In the same manner as before, we notice that (20) is satisfied whenever

\[
\sum_{k \geq 1} \frac{(\log(|k|))^{1/2}}{|k|^{1/2}} \left( \sum_{u,v \geq (k-1,k-1)} a_{u,v}^2 \right)^{1/2} < \infty.
\]

This is a generalization of the quenched functional CLT obtained in [35] (Example 6.3). Here, we only require an Orlicz space type condition on the innovation \( \xi_0 \) and we weaken the condition (54) in Zhang et al. [35] to condition (20).

Proof. Note that

\[
\mathbb{E}_1 [X_k] = \sum_{u,v \geq k-1} a_{u,v} \xi_{k-u} \xi_{k-v}
\]

Let \( (\xi'_n)_{n \in \mathbb{Z}^d} \) and \( (\xi''_n)_{n \in \mathbb{Z}^d} \) be two independent copies of \( (\xi_n)_{n \in \mathbb{Z}^d} \). By applying the decoupling inequality (see Theorem 3.1.1 in De la Peña and Giné [9], p.99) and Jensen’s inequality, we get for any \( t > 0 \),

\[
\mathbb{E} [\Phi (\mathbb{E}_1 [X_k] / t)] = \mathbb{E} \left[ \Phi \left( \frac{1}{t} \sum_{u,v \geq k-1} a_{u,v} \xi_{k-u} \xi_{k-v} \right) \right] \\
\leq \mathbb{E} \left[ \Phi \left( \frac{C}{t} \sum_{u,v \geq k-1} a_{u,v} \xi'_{k-u} \xi''_{k-v} \right) \right].
\]

Hence

\[
\|\mathbb{E}_1 [X_k]\|_\Phi \leq C \left\| \sum_{u,v \geq k-1} a_{u,v} \xi'_{k-u} \xi''_{k-v} \right\|_\Phi.
\]

Therefore, applying the Burkholder inequality for Orlicz spaces (see Meyer and Dellacherie [8], p.304, VII - 92), we get

\[
\|\mathbb{E}_1 [X_k]\|_\Phi \leq C \left( \sum_{(u,v) \geq (k-1,k-1)} a_{u,v}^2 \|\xi_{k-u}\|_\Phi \|\xi_{k-v}\|_\Phi \right)^{1/2}.
\]

By stationarity and since \( \|\xi_0\|_\Phi < \infty \), we obtain, by using assumption (20), that condition (8) holds. Thus the CLT in Corollary 2.5 holds. ■
Acknowledgment
The authors would like to thank in particular Dalibor Volný and Christophe Cuny for their helpful remarks during the writing of this article.

References

[1] Barrera, D., Peligrad, C. and Peligrad, M. (2016). On the functional CLT for stationary Markov chains started at a point. Stochastic Process. Appl. 126, 1885-1900.
[2] Borodin A. and Ibragimov I. (1995). Limit theorems for functionals of random walks. Mathematical Soc., T. 195.
[3] Cairoli, R. (1969). Un théorème de convergence pour martingales a indices multiples. C. R. Acad. Sci. Paris Sér. A-B 269, A587-A589.
[4] Cuny, C. and Peligrad, M. (2012). Central limit theorem started at a point for stationary processes and additive functional of reversible J. Theoret. Probab. 25, 171-188.
[5] Cuny, C. and Merlevède, F. (2014). On martingale approximations and the quenched weak invariance principle. Ann. Probab. 42, 760-793.
[6] Cuny, C., Dedecker J. and Volný, D. (2015). A functional central limit theorem for fields of commuting transformations via martingale approximation. J. Math. Sci. 219, 765-781.
[7] Cuny, C. and Volný, D. (2013). A quenched invariance principle for stationary processes. ALEA Lat. Am. J. Probab. Math. Stat. 10, 107-115.
[8] Dellacherie, C. and Meyer, P.A. (1980). Probabilités et potentiel: Chapitres V à VIII: Théorie des martingales. Actualités scientifiques et industrielles. Hermann.
[9] De la Peña, V. and Giné, E. (1999). Decoupling : from dependence to independence. Randomly stopped processes. U-statistics and processes. Martingales and beyond. Springer series in statistics. Probability and its applications. Springer, New York.
[10] Dedecker, J., Merlevède, F. and Peligrad, M. (2014). A quenched weak invariance principle. Annales de l'I.H.P. Probabilités et statistiques 50, 872-898.
[11] Derriennic, Y. and Lin, M. (2001). The central limit theorem for Markov chains with normal transition operators started at a point. Probab. Theory Relat. Fields 119, 508-528.
[12] El Machkouri, M. and Giraudo, D. (2016). Orthomartingale-coboundary decomposition for stationary random fields. Stochastics and Dynamics, 16(05), p.1650017.
[13] Giraudo, D. (2017). Invariance principle via ortho-martingale approximation. Stochastics and Dynamics. Online Ready.
[14] Gordin, M. I. and Liššic, B. A. (1978). The central limit theorem for stationary Markov processes. Doklady Akademii Nauk, 239(4), 766-767.
[15] Gordin, M. I. (2009). Martingale-coboundary representation for a class of stationary random fields. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 364, Veroyatnostn i Statistika. 14.2, 88-108, 236; and J. Math. Sci. (N. Y.) 163, 363-374.
[16] Hall, P. and Heyde, C. C. (1980). Martingale Limit Theory and Its Application. Probability and Mathematical Statistics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London.
[17] Hannan, E. J. (1973). Central limit theorems for time series regression. Z. Wahrscheinlichkeits-theorie und Verw. Gebiete 26, 157-170.
[18] Kipnis, C. and Landim, C. (1998). Scaling limits of interacting particle systems, volume 320. Springer Science & Business Media.

[19] Kipnis, C. and Varadhan, S.R.S. (1986). Central limit theorem for additive functionals of reversible markov processes and applications to simple exclusions. Communications in Mathematical Physics, 104(1) :1–19.

[20] Khoshnevisan, D. (2002). Multiparameter processes. An introduction to random fields. Springer Monographs in Mathematics. Springer-Verlag, New York.

[21] Krasnosel’skii, M.A. and Rutitskii, Ya.B. (1961). Convex functions and Orlicz spaces (Translated from the first Russian edition). P. Noordhoff Ltd., Groningen.

[22] Krengel, U. (1985). Ergodic Theorems. De Gruyter Studies in Mathematics, Berlin.

[23] Klicnarová, J., Volný, D. and Wang, Y. (2016). Limit theorems for weighted Bernoulli random fields under Hannan’s condition. Stoch. Proc. Appl. 126, 1819-1838.

[24] Neuhaus, G. (1971). On weak convergence of stochastic processes with multidimensional time parameter. The Annals of Mathematical Statistics, 42(4), 1285-1295.

[25] Ouchti, L. and Volný D. (2008). A conditional CLT which fails for ergodic components. Journal of Theoretical Probability, 21(3), 687-703.

[26] Peligrad, M. (2015). Quenched Invariance Principles via Martingale Approximation; in Asymptotic laws and methods in stochastics. The volume in honour of Miklós Csörgő work. Springer in the Fields Institute Communications Series. Springer-Verlag New York 76, 121-137.

[27] Peligrad, M. and Zhang, Na (2018). Martingale approximation for random fields. Electron. Commun. Probab. 23, Page NO. 28, 9.

[28] Peligrad, M. and Volný, D. (2020). Quenched invariance principles for ortho-martingale-like sequences. Journal of Theoretical Probability. 33(3), 1238-1265.

[29] Rao, M. and Ren, Z. (1991). Theory of Orlicz spaces. M. Dekker New York

[30] Volný, D. (2015). A central limit theorem for fields of martingale differences. C. R. Math. Acad. Sci. Paris 353, 1159-1163.

[31] Volný, D. (2017). Martingale-coboundary decomposition for stationary random fields. arXiv preprint arXiv:1706.07978.

[32] Volný, D. and Wang, Y. (2014). An invariance principle for stationary random fields under Hannan’s condition. Stochastic Proc. Appl. 124, 4012-4029.

[33] Volný, D. and Woodroofe, M. (2010). An example of non-quenched convergence in the conditional central limit theorem for partial sums of a linear process. Dependence in analysis, probability and number theory (The Phillipp memorial volume), Kendrick Press. 317-323.

[34] Volný, D. and Woodroofe, M. (2014). Quenched central limit theorems for sums of stationary processes. Statist. Probab. Lett. 85, 161-167.

[35] Zhang, N., Reding, L. and Peligrad, M. (2020). On the quenched CLT for stationary random fields under projective criteria. Journal of Theoretical Probability. 33, 2351-2379.