Quantum Entanglement of Fermions-Antifermions Pair Creation modes in Non-commutative Bianchi I Space-time

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The non-commutative Bianchi I curved space-time vierbeins and spin connections are derived. Moreover, the corresponding non-commutative Dirac equation as well as its solutions are presented. As an application within the quantum field theory approach using Bogoliubov transformations, the von Neumann fermion-antifermion pair creation quantum entanglement entropy is studied. It is shown that its behaviour is strongly dependent on the value of the non-commutativity \( \theta \) parameter, \( k_{\perp} \)-modes frequencies and the structure of the curved space-time. Various discussions of the obtained features are presented.

Keywords: Bianchi I universe, Non-commutative geometry, Pair creation, Fermion-antifermion quantum entanglement entropy, Quantum information.

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1. Introduction

During the last few years, non-commutative (N.C.) Seiberg-Witten (S.W.) space-time geometry has played an important role in understanding various phenomena for example in particle physics and cosmology [1,2]. Furthermore, quantum entanglement (Q.E.) has been extensively studied in non-relativistic flat-space setups and expanding universes [4-15]. Increasing interest to the emerging field of relativistic quantum information and entanglement has attracted many people [16-22]. Recently, in Refs. [17] and [18] Q.E. of fermionic and bosonic particles in a certain type
of Freedman-Robertson-Walker (F.R.W.) universe has been shown to have special $k$-modes frequencies and mass dependence. In fact, as it was pointed out in Ref. [17], the response of Q.E. to the dynamics of the expansion of the universe is affected by the particular choice of quantum field theory employed and the geometric structure of space-time. Information about the rate and volume of the expansion is codified in the frequency and amount of the entangled modes. In our case, as we will see later, some of the thermodynamical quantities are better evaluated employing estimation techniques that use Q.E. generated between fermions and antifermions in cosmology scenarios which is very sensitive to anisotropies of the universe expansion and deformation of the space-time. What we have to know is how much information about anisotropies, deformation and thermodynamics is encoded into this entanglement created by the dynamics of the Bianchi I space-time. It is worth mentioning that there is no physical principle or mathematical formalism which can constrain the order of magnitude of the non-commutativity parameter nor the scale for which the non-commutative model in consideration is relevant. The related energy scale could be as low as a few TeV [23–34], the same order of magnitude of energies employed in collider experiments (LHC, ILC, etc.), or Planck scale, as it is the case of string or quantum gravity [35–40]. The N.C. gravity, which we present in this paper, can be considered as an effective theory. Thus, the non-commutativity of space-time can be reinterpreted as an extra interaction term on a commutative space-time, and therefore in this way the theory is equivalent to a higher derivative or curvature extension of ordinary gauge gravity [2]. Moreover, even if the non-commutativity of space-time is relevant at the Planck scale, the physical phenomena which are induced, like pair creation and entanglement, can appear at larger scales. The main goal of this paper is not the derivation of the Dirac equation from the first principles and Lagrangian formalism in itself but to study the von Neumann Q.E. of fermionic-antifermionic modes created by the dynamics of the N.C. Bianchi I universe and understand the new features generated by both, the space-time expansion and the non-commutativity $\theta$ parameter. In section 2 we present the N.C. mathematical formalism. In section 3 we derive the expression of the bipartite fermion-antifermion Q.E. codification of the information as a function of non-commutativity $\theta$ parameter and $k_\perp$-modes frequencies as well as the pair creation density. We will see also the role of Q.E. in determining some of the thermodynamical properties of this space-time. Finally in section 4 we show the main results and draw our conclusions.

2. Mathematical Formalism

The N.C. space-time is characterised by the coordinates operators $\hat{x}^\mu$ ($\mu = 0, 3$) satisfying the following commutation relation:

$$[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu\nu} \tag{1}$$

where $\theta^{\mu\nu}$ are antisymmetric matrix elements that control the non-commutativity of the space-time. The Dirac equation for a massless S.W. spinor particle $\psi$ in a
curved N.C. space-time is shown to take the following form (see Appendix A) [41]:

$$\gamma^f \left( i \partial_f + A_f \right) + \gamma^f \gamma^5 \tilde{B}_f \right) \ast \psi = 0$$ (2)

where

$$\tilde{A}^f = \Im \left( \hat{e}_f^\mu \sum_{a=1}^4 \hat{\omega}_a^\mu \right) + \Re \left[ \hat{e}^\mu_d \left( \hat{\omega}_d^f - \hat{\omega}_d^f \right) \right]$$ (3)

$$\tilde{B}^f = \Im \left[ \left( \hat{e}^{\mu d} \hat{\omega}_d^{ab} \right) + \frac{1}{4} \theta^{\alpha \beta} \left( \partial_\mu \partial_\alpha \hat{e}^{\mu d} \right) \left( \partial_\sigma \partial_\beta \hat{\omega}_d^{ab} \right) \right] \tilde{e}_{f, ab}$$ (4)

($\Im$ and $\Re$ stand for imaginary and real parts respectively). Here $\hat{e}_f^\mu$ and $\hat{\omega}_d^f$ are the N.C. S.W. vierbeins and spin connections respectively (see Ref. [42]). Their general expressions are given in terms of the corresponding commutative quantities in Appendix B. In what follows, the N.C. Dirac matrices $\gamma^\mu$ and derivative $\partial_\mu$ in a curved space-time are related to the ones of the Minkowski flat-space through the relations:

$$\gamma^\mu = \hat{e}_f^\mu \gamma^f$$ (5)

and

$$\partial_\mu = \hat{e}_f^\mu \partial_f$$ (6)

(Greek and Latin indices are for curved and flat spaces respectively), where the following vierbeins orthogonality relation holds

$$\frac{1}{2} \left( \hat{e}_a^\mu \hat{e}_b^{+\mu} + \hat{e}_b^\mu \hat{e}_a^{+\mu} \right) = \delta_b^a$$ (7)

In what follows, we consider a Bianchi I universe where the metric has the form:

$$ds^2 = -dt^2 + t^2 (dx^2 + dy^2) + dz^2$$ (8)

with dimensionless space-time coordinates, (here the time $t$ is related to the cosmological parameters of the model). We take the signature convention of the space-time to be $(-,+,+,+)$ and for simplicity, make the choice:

$$\theta_{\mu \nu} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\theta & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$ (9)

Moreover, since we are dealing with massless spinors, it is better to use the chiral representation of the Dirac $\gamma^\mu$ matrices that is:

$$\gamma_0 = \begin{bmatrix}
0_{2 \times 2} & 1_{2 \times 2} \\
1_{2 \times 2} & 0_{2 \times 2}
\end{bmatrix}, \quad \gamma_i = \begin{bmatrix}
0_{2 \times 2} & \sigma_{i \times 2} \\
-\sigma_{i \times 2} & 0_{2 \times 2}
\end{bmatrix}, \quad \gamma_5 = \begin{bmatrix}
1_{2 \times 2} & 0_{2 \times 2} \\
0_{2 \times 2} & -1_{2 \times 2}
\end{bmatrix}$$ (10)
where \((i = 1, 3)\). Using Maple 16 tensor package, the non-vanishing components of the vierbeins and spin connections have up to \(O(\theta^2)\) the following expressions:

\[
\begin{align*}
\hat{e}_0^0 &= \hat{e}_3^3 = 1 \\
\hat{e}_1^1 &= \hat{e}_2^2 = \frac{1}{t} \left( 1 - \frac{25}{128} \theta^2 \right) \\
\hat{\omega}_{23}^{13} &= -\hat{\omega}_{32}^{12} = \frac{1}{2} \hat{\omega}_3^{31} = -\frac{1}{2} \hat{\omega}_3^{21} = -\frac{i}{4} \\
\hat{\omega}_{23}^{12} &= \hat{\omega}_3^{13} = 1 + \frac{5 \theta^2}{128} \\
\hat{\omega}_{23}^{21} &= \hat{\omega}_3^{31} = -1 + \frac{7 \theta^2}{64}
\end{align*}
\]

(the tilde is for a curved space index). Since the metric presents a space-like singularity at \(t = 0\), it is difficult to define the particle state within the adiabatic approach [5, 46]. To do so, we first follow a quasi classical approach of Ref. 47 to identify the positive and negative modes frequencies and look for the asymptotic behaviour of the solutions at \(t \to 0\) and \(t \to \infty\). Secondly, we solve the Dirac equation and compare the solutions with the above quasi classical limit. Now, in order to find the solutions to the Dirac Eq. (2) in the N.C. Bianchi I space-time, we set:

\[
\hat{\psi} = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix}
\]

where

\[
\begin{align*}
\chi_1 &= f_1(t) e^{i\vec{k} \cdot \vec{x}} \\
\chi_2 &= f_2(t) e^{i\vec{k} \cdot \vec{x}} \\
\chi_3 &= f_3(t) e^{-i\vec{k} \cdot \vec{x}} \\
\chi_4 &= f_4(t) e^{-i\vec{k} \cdot \vec{x}}
\end{align*}
\]

the \(f_j(t)'s\) \((j = 1, 4)\) verify the second order differential equation:

\[
\frac{\partial^2 f_j(t)}{\partial t^2} + \frac{A_j}{t} \frac{\partial f_j(t)}{\partial t} + \left( \frac{C_j}{t^2} + \frac{D_j}{t} - k_j^2 \right) f_j(t) = 0
\]

with

\[
\begin{align*}
A_j &= 1 - 2i \Omega_j^2 \\
C_j &= \Omega_1^2 k_1^2 + \Omega_3^2 k_3^2 - \Omega_2^2 \\
D_j &= -i k_2^2 - 2k_2^2 \Omega_3^2
\end{align*}
\]
here
\[
k^2_x = k^2_x + k_y^2 \\
k^2_z = (-1)^{j+1}k_z \\
\Omega_1' = \Omega_1 \\
\Omega_2' = \Omega_2 \\
\Omega_3' = \Omega_3(\delta^1 - \delta^2 - \delta^3 + \delta^4)
\]
and
\[
\Omega_1 = -1 + \frac{25 \theta^2}{128} \\
\Omega_2 = -2\left(2 - \frac{59 \theta^2}{128}\right) \\
\Omega_3 = \frac{\theta}{2}
\]

Now, setting:
\[
f_j(t) = t^{\alpha_j} e^{\beta_j t} h_j(t)
\]
where
\[
\alpha_j = \frac{1 - A_j \pm \sqrt{(A_j - 1)^2 - 4C_j}}{2} \quad \text{(19a)} \\
\beta_j = -\frac{1}{2} \quad \text{(19b)}
\]

Eq.(14) can be rewritten as a Kummer’s differential equation of the form [48]:
\[
t \frac{d^2 h_j(t)}{dt^2} + (b_j - t) \frac{dh_j(t)}{dt} - a_j h_j(t) = 0 \quad \text{(20)}
\]
with
\[
a_j = -\frac{A_j}{2} - \alpha_j - D_j \quad \text{(21a)} \\
b_j = -2 \alpha_j - A_j \quad \text{(21b)}
\]
and the constraint:
\[
|k_z| = \frac{1}{2} \quad \text{(22)}
\]

The differential Eq.(20) has two solutions denoted by \(M(a_j, b_j, z)\) and \(U(a_j, b_j, z)\) such that:
\[
M(a_j, b_j, z) = \sum_{n=0}^{\infty} \frac{(a_j)_n}{(b_j)_n} z^n
\]
\[
U(a_j, b_j, z) = \frac{\pi}{\sin(\pi b)} \left( \frac{M(a_j, b_j, z)}{\Gamma(1 + a_j - b_j) \Gamma(b_j)} - z^{1-b'j} \frac{M(1 + a_j - b_j, 2 - b_j, z)}{\Gamma(a_j) \Gamma(2 - b_j)} \right)
\]

\[23a\] \[23b\]
where \((a)_n\) is defined by:
\[
(a)_n = a(a+1)(a+2)...(a+n-1)
\] (24)

Note that \((a)_0 = 1\). Thus, the solution of Eq. (20) is the following linear combination:

\[
f_j(t) = C_1^j t^{\alpha_j} e^{\frac{t}{2}} M(a_j,b_j,t) + C_2^j t^{\alpha_j} e^{\frac{t}{2}} U(a_j,b_j,t)
\] (25)

\((C_1^j, C_2^j\) are constants). Now, to better understand of the asymptotic behaviour at \(t \rightarrow 0\) ("in" fields) and \(t \rightarrow \infty\) ("out" fields) of the solutions Eq. (25), it is preferable to express the \(M(a_j,b_j,t)\) and \(U(a_j,b_j,t)\) Kummer functions in terms of Whittaker ones, such that:

\[
M\left(\frac{1}{2}+\mu, 1+2\mu, z\right) = e^{\frac{z}{2}} z^{-1} e^{-i\pi \lambda} W_{\lambda,\mu}(z) + e^{\frac{z}{2}} z^{-1} W_{-\lambda,\mu}(z)
\] (26a)

\[
U\left(\frac{1}{2}+\mu, 1+2\mu, z\right) = e^{\frac{z}{2}} z^{-1} W_{\lambda,\mu}(z)
\] (26b)

Note that \(M_{\lambda,\mu}(z)\) can be expressed in terms of \(W_{\lambda,\mu}(z)\) as:

\[
(W_{\lambda,\mu}(z))^* = W_{-\lambda,\mu}(-z)
\] (28)

and

\[
(M_{\lambda,\mu}(z))^* = (-1)\mu^* + \frac{1}{2} M_{-\lambda,\mu}(z)
\] (29)

Now, to identify the positive and negative modes frequencies in the "in" and "out" fields, we concentrate only on the \(f_1 \equiv f\) function (results can be easily extended to \(f_j, (j = 2,4)\)). It is easy to show that at \(t \rightarrow 0\), one has:

\[
f_{\text{in}}^+ \sim M_{\lambda,\mu}(t) \sim e^{-\frac{t}{2}} t^{\mu+\frac{1}{2}}
\] (30a)

\[
f_{\text{in}}^- \sim (M_{\lambda,\mu}(t))^* \sim (-1)^{\mu+\frac{1}{2}} M_{-\lambda,\mu}(t)
\] (30b)

and respectively for \(t \rightarrow \infty\):

\[
f_{\text{out}}^+ \sim W_{\lambda,\mu}(t) \sim e^{-\frac{t}{2}} t^{\lambda}
\] (31a)

\[
f_{\text{out}}^- \sim (W_{\lambda,\mu}(t))^* \sim W_{-\lambda,\mu}(-t)
\] (31b)

where for the both cases we have:

\[
\mu = \frac{1}{2} (b_1 - 1)
\] (32a)

\[
\lambda = \frac{1}{2} (b_1 - a_1)
\] (32b)

A direct consequence of the linear transformation properties of such functions is that the Bogoliubov transformations associated with the transformation between "in" and "out" solutions take the simple form:

\[
f_{\text{in}}^{\pm}(k_{\perp}, \theta, t) = \alpha_{k_{\perp}, \theta}^{\pm} f_{\text{in}}^{\pm}(k_{\perp}, \theta, t) + \beta_{k_{\perp}, \theta}^{\pm} (f_{\text{out}}^{\pm}(k_{\perp}, \theta, t))^*
\] (33)
where $\alpha^\pm_{k_\perp,\theta}$ and $\beta^\pm_{k_\perp,\theta}$ are the Bogoliubov coefficients. Of course, the curved space spinor solutions of the Dirac equation are defined by:

\begin{align}
U_{\text{in, out}}(\vec{k}, \vec{x}, t) = f_{\text{in, out}}(k_\perp, \theta, t) e^{i\vec{k}\cdot\vec{x}} U(0, s) \\
V_{\text{in, out}}(\vec{k}, \vec{x}, t) = (f^+_{\text{in, out}}(k_\perp, \theta, t))^* e^{-i\vec{k}\cdot\vec{x}} V(0, s)
\end{align}

(34a, 34b)

where $U(0, s)$ and $V(0, s)$ are the ordinary flat space-time spinors, ("s" runs over the spin states). Using the Bogoliubov transformations between the asymptotic terms "$\text{in}$" and "$\text{out}$" operators, the field in "$\text{in}$" and "$\text{out}$" regions can then be expanded as:

\begin{equation}
\psi_{\text{in, out}} = \int \frac{d^3k}{(2\pi)^2} \sum_s [a_{\text{in, out}}(\vec{k}, s) U_{\text{in, out}}(\vec{k}, \vec{x}, t) + b^+_{\text{in, out}}(\vec{k}, s) V_{\text{in, out}}(\vec{k}, \vec{x}, t)]
\end{equation}

(35)

($a_{\text{in, out}}$ and $b^+_{\text{in, out}}$ are the annihilation and creation operators of the fermions and antifermions respectively). Due to the form of the Bogoliubov transformation, we can show easily that the "$\text{in}$" tensor product of the particle and antiparticle vacuum states can be expressed in terms of "$\text{out}$" as:

\begin{equation}
|0\rangle_{\text{in}} \otimes |0\rangle_{\text{in}} = \prod_k (|0\rangle_{\text{out}} \otimes |0\rangle_{\text{out}} + A_0 |1_{k_\perp}\rangle_{\text{out}} \otimes |1_{-k_\perp}\rangle_{\text{out}})
\end{equation}

(36)

where $|1_{-k_\perp}\rangle$ (respectively $|1_{k_\perp}\rangle$) represents an antiparticle (respectively particle) mode with momentum $(-k_\perp)$ (respectively $(k_\perp)$). Now, to find the relation between the coefficients $A_0$ and $A_1$ we impose the relation:

\begin{equation}
b_{\text{in}}(k_\perp)|0\rangle_{\text{in}} \otimes |0\rangle_{\text{in}} = 0
\end{equation}

(37)

yielding to:

\begin{equation}
\alpha^*_{k_\perp} A_1 |1_{-k_\perp}\rangle + \beta^*_{k_\perp} A_0 |1_{-k_\perp}\rangle = 0
\end{equation}

(38)

($\alpha_{k_\perp}$ and $\beta_{k_\perp}$ stand for $\alpha^\pm_{k_\perp,\theta}$ and $\beta^\pm_{k_\perp,\theta}$ respectively), and thus, one gets:

\begin{equation}
A_1 = -\Delta_{k_\perp}^* A_0
\end{equation}

(39)

where

\begin{equation}
\Delta_{k_\perp} = \frac{\beta_{k_\perp}}{\alpha_{k_\perp}}
\end{equation}

(40)

The normalised vacuum state takes the form:

\begin{equation}
|0\rangle_{\text{in}} \otimes |0\rangle_{\text{in}} = \prod \left( \frac{|0\rangle_{\text{out}} \otimes |0\rangle_{\text{out}} - \Delta_{k_\perp}^* |1_{k_\perp}\rangle_{\text{out}} \otimes |1_{-k_\perp}\rangle_{\text{out}}}{\sqrt{1 + |\Delta_{k_\perp}|^2}} \right)
\end{equation}

(41)

and it is a pure entangled state of fermion-antifermion modes frequencies with a reduced density matrix:

\begin{equation}
\rho_{k_\perp} = \frac{1}{1 + |\Delta_{k_\perp}|^2} \left(|0\rangle_{\text{out}} \langle 0| + |\Delta_{k_\perp}|^2 |1_{k_\perp}\rangle_{\text{out}} \langle 1_{k_\perp}|\right)
\end{equation}

(42)
In this case, the entanglement can be quantified for each mode by the von Neumann quantum entropy $S(\rho_{k\perp})$ such that:

$$S(\rho_{k\perp}) = \log_2 \left( \frac{1 + |\Delta_{k\perp}|^2}{|\Delta_{k\perp}|^{2\pi|\Delta_{k\perp}|^2}} \right)$$  \hfill (43)

Eq.(43) is equivalent to Eq.(21) in Ref. [18]. Note that the quantum entropy can be also expressed in terms of the pair creation density $\hat{n}_{k\perp}$ as:

$$S(\hat{n}_{k\perp}) = \log_2 \left( \frac{(\hat{n}_{k\perp})^{1-2\hat{n}_{k\perp}}}{(1-\hat{n}_{k\perp})^{1-\hat{n}_{k\perp}}} \right)$$  \hfill (44)

For the N.C. Bianchi I space-time, it is straightforward to show that:

$$|\Delta_{k\perp}|^2 = \left| \frac{\Gamma(\frac{1}{2} + \mu + \lambda)}{\Gamma(\frac{1}{2} + \mu - \lambda)} \right|^2 e^{-i\pi(\lambda - \mu - \frac{1}{2})}$$  \hfill (45)

It is worth mentioning that the expression of $|\Delta_{k\perp}|^2$ depends strongly on the values of $\mu$ and $\lambda$ (real, complex and pure imaginary) as it is shown in Appendix D. In the case of interest, we distinguish three situations:

1. $\theta = 0$

$$|\Delta_{k\perp}|^2 = \frac{x \sinh(\pi x)}{(1 + y)^{2}(y \sinh(\pi y)} e^{-8\pi k_{\perp}}$$  \hfill (46)

where

$$x = k_{\perp} - \frac{1}{2} \hfill (47a)$$
$$y = 3k_{\perp} + \frac{1}{2} \hfill (47b)$$

2. $\theta > 0$

$$|\Delta_{k\perp}|^2 = \left[ \frac{x_2 \Gamma(-x_2) \sin(\pi x_2)}{x_1 \Gamma(-x_1) \sin(\pi x_1)} \right] \prod_{n=0}^{\infty} \left( 1 + \frac{y_2^2}{\langle n+x_2 \rangle^2} \right) \exp \left( -8\pi(y_2 + \frac{1}{2}) \right)$$  \hfill (48)

3. $\theta < 0$

$$|\Delta_{k\perp}|^2 = \left[ \frac{\pi^2 \prod_{n=0}^{\infty} \left( 1 + \frac{y_2^2}{\langle n+x_2 \rangle^2} \right)}{x_1 \Gamma(-x_1) \sin(\pi x_1) \Gamma(x_2)} \right] \prod_{n=0}^{\infty} \left( 1 + \frac{y_1^2}{\langle n+x_1 \rangle^2} \right) \exp \left( -8\pi(y_2 + \frac{1}{2}) \right)$$  \hfill (49)
where

$$x_1 = \frac{\theta}{2} - 1$$  \hspace{1cm} (50a)

$$x_2 = -\frac{\theta}{2}$$  \hspace{1cm} (50b)

$$y_1 = 3\sqrt{\left(1 - \frac{25\theta^2}{64}\right)k_\perp^2 + \frac{\theta^2}{4} + \frac{1}{2}}$$  \hspace{1cm} (50c)

$$y_2 = \sqrt{\left(1 - \frac{25\theta^2}{64}\right)k_\perp^2 + \frac{\theta^2}{4} - \frac{1}{2}}$$  \hspace{1cm} (50d)

Notice the presence of linear terms in the non-commutativity $\theta$ parameter in the expressions of $x_1$ and $x_2$. This is due essentially to the fact that the S.W. vierbein and spin connection used from Ref. 42 are in general complex (e.g. terms pure imaginary of the spin connection are proportional to $\theta$) see Eqs.(11) in our paper together with a Dirac wave function $\hat{\psi}$ which is also complex. Thus, we expect in addition to second order terms in $\theta$, the first order ones as well coming from the imaginary parts of $\hat{\omega}_{ab}^\mu$ and $\hat{\psi}$.

3. Numerical Results and Discussions

By analysing our numerical results concerning the quantum entanglement entropy $S_{Q.E}$ in N.C. Bianchi I universe as a function of the $k_\perp$-modes frequencies and non-commutativity $\theta$ parameter, we have distinguished two main regions denoted by I and II corresponding to $k_\perp \leq \frac{1}{2}$ and $k_\perp > \frac{1}{2}$ respectively. In region I, $S_{Q.E}$ has a maximal value at $k_\perp = 0$. Notice that if $\theta$ is relatively small, $S_{Q.E}$ decreases as $k_\perp$ increases until reaching a minimum value at approximately $k_\perp \approx \frac{1}{2}$. Then, it increases until reaching a peak (with a very small value) near $k_\perp \approx 0.55$. Finally it decreases again and vanishes at infinity. However, for relatively big values of $\theta$, $S_{Q.E}$ becomes a monotonically decreasing function of $k_\perp$ (see FIG.1 and FIG.2). Now, for a fixed value of $k_\perp$, we notice that in the region I, $S_{Q.E}$ is a decreasing function of $\theta$. However, in the region II, it is an increasing function (see FIG.3 and FIG.4). To be more explicit, we have displayed, in the contour plots of FIG.3 and in the 3D curve of FIG.4, the behaviour of $S_{Q.E}$ as a function of $k_\perp$-modes frequencies and non-commutativity $\theta$ parameter. The theoretical explanation of such features is that in the region I, one can show that $\Delta k_\perp$ is approximately proportional to $e^{-4\pi \theta}$. Therefore, it is clear that if $\theta$ increases, $\Delta k_\perp$ decreases and since $\frac{\partial S}{\partial \Delta k_\perp} = -\frac{1}{(1+\Delta k_\perp)^2} \log_2 \Delta k_\perp$ is positive for $\Delta k_\perp < 1$ (case of our interest), then $S_{Q.E}$ decreases (see FIG.1). The new important numerical result is that the upper bound of the fermionic $S_{Q.E}$ (where one has a maximally entangled state) in the physical interval of $k_\perp$ is no more $\log_2 N$ where $N$ is the Hilbert space dimension (in our case $N = 2$) corresponding to $k_\perp \approx \frac{1}{8\pi} \log_2 \left(\frac{5}{8}\right) < 0$ but it depends strongly on the non-commutative $\theta$ parameter values. It is worth mentioning that in the region I the pair creation density has a non-thermal behaviour. In fact, because of the anisotropy of the Bianchi I space-time (which is more complicated than the
isotropic F.R.W. of Ref. 17 and non-commutativity, if a created gas of fermions is observed at a time scale much larger (less energy) than the expansion time then the particle number density deviates from the quasi equilibrium distribution without a well defined temperature nor a chemical potential. Furthermore, if we have an anisotropic space-time, the created particle-antiparticle pair (with the same energy) can not reach an equilibrium state in all space directions, except if their energies exceed a certain critical value \((k_\perp \approx \frac{1}{2})\) beyond which the anisotropic effects be-
come negligible. Thus, for \( k_\perp \leq \frac{1}{2} \), the particle-antiparticle pair creation velocity (energy) is less than the expansion velocity in the \( x \) and \( y \) directions and the density of the pair creation is in a non-thermal out-of-equilibrium state. Concerning the privileged value of \( k_\perp \) \((\theta = 0)\) for which \( S_{Q.E} \) is maximum, it is related to the characteristic wavelength correlated to the underlying space-time structure and non-commutativity (deformation). In fact, contrary to the argument (which seems general) given in Ref. 17 using the Pauli exclusion principle which states that it is logical that it is much cheaper to excite smaller \( k \)-modes frequencies in an expanding space-time. Our numerical analysis shows that the behaviour of \( S_{Q.E} \) as a function of the \( k_\perp \)-modes frequencies depends not only on the particles species (fermions or bosons) but on the space-time structure and deformation as well. It is very important to stress on the fact that in the region I, the non-commutativity plays the role of gravity slowing down the expansion and leading to a decrease in the information (quantum entanglement between the fermion-antifermion pair) encoded in \( S_{Q.E} \). In the region II, \((k_\perp > \frac{1}{2})\) and contrary to the region I \((k_\perp \leq \frac{1}{2})\), \( S_{Q.E} \) is an increasing function of \( \theta \) for a fixed value of \( k_\perp \). Theoretically, the non trivial behaviour of \( S_{Q.E} \) can be explained as follows: in fact, for relatively small values of \( \theta \) and \( k_\perp \approx 0.5 \) (resp. \( k_\perp \approx 0.55 \)) we have obtained numerically a minimum (resp. maximum) value of \( S_{Q.E} \). However for relatively large values of \( \theta \), we have checked numerically that \( S_{Q.E} \) is a monotonically decreasing function of the \( k_\perp \)-modes frequencies (see FIG.2). Regarding the thermal behaviour of the pair creation number density \( \hat{n} \) and \( S_{Q.E} \), it is important to notice that if \( \Delta_{k_\perp} \ll 1 \) or equivalently \( k_\perp \\gg 1 \), one can show that \( \hat{n} \) behaves as \( e^{-8\pi k_\perp} \) for \( \theta = 0 \) (thermal behaviour) leading to \( S_{Q.E} \approx -\Delta_{k_\perp} \log_2 \Delta_{k_\perp} \). For \( \theta \neq 0 \), one has \( \hat{n} \propto \theta^2 e^{-8\pi k_\perp} \). It is clear that for fixed values of the \( k_\perp \)-modes frequencies, \( \Delta_{k_\perp} \) and \( S_{Q.E} \) are increasing functions of \( \theta \) \((\Delta_{k_\perp} < 1)\). This, explains clearly the behaviour shown in FIG.2. We remark also that in this region, the non-commutativity does not play the role of gravity but rather as a repulsive force (e.g. quintessence, dark energy, etc...). In fact, to show the role of the space-time non-commutativity \( \theta \) parameter and its effect on the quantum entanglement, we have noticed that in the region I, \( \Delta_{k_\perp} \) is more sensitive to the factor \( J = e^{-\pi \sqrt{64k_\perp^2 - 25k_\perp^2 \theta^2 + 16\theta^2}} \) (see Eq.(48) and Eq. (49)). In this case, one has \(-25k_\perp^2 \theta^2 + 16\theta^2 > 0 \) and therefore, \( J, \Delta_{k_\perp} \) and \( S_{Q.E} \) become decreasing functions of \( \theta \). However, in the region II for large values of \( k_\perp \), \( \Delta_{k_\perp} \) is more affected by the factor \( I = \theta^2 e^{-\pi \sqrt{64k_\perp^2 - 25k_\perp^2 \theta^2 + 16\theta^2}} \). In this case, it is clear that, \( I, \Delta_{k_\perp} \) and \( S_{Q.E} \) are increasing functions of \( \theta \) (see FIG.2). We conclude that the non-commutativity induces two compelling terms with opposite signs: one is given by the term \(+16\theta^2 \) (with a positive sign) playing the role of gravity or contributing to the matter density through the stress energy-momentum tensor and the other is given by the term \(-25k_\perp^2 \theta^2 \) (with a negative sign). Thus, if the first term is bigger (case of \( k_\perp \leq \frac{1}{2} \)), the N.C. correction term \(-25k_\perp^2 \theta^2 + 16\theta^2 \) will slow down the expansion (like the gravity) and the information does not go faster as \( \theta \) increases (decrease in \( S_{Q.E} \)). However, for the relatively large values of \( k_\perp \), the term \(-25k_\perp^2 \theta^2 + 16\theta^2 \) changes the sign (with respect to the first case) and
we will have exactly the opposite effect as $\theta$ increases. That is $\theta$ will increase the expansion rate and the information (quantum entanglement) goes faster (increase in $S_{Q.E}$). It is very important mentioning that since $S_{Q.E}$ is an increasing function
of the pair creation number density $\hat{n}$ (see Eq. (44)) such that:

\[
\frac{\partial S_{Q,E}}{\partial \hat{n}} = -2 \log_2 \hat{n} - \log_2(1 - \hat{n}) + \frac{1 - \hat{n}}{\hat{n}}
\]  

(51)
it is always a positive quantity (since $\hat{n} \leq 1$). Similar behaviours are obtained for $\hat{n}$. The results are summarised in FIG.4, FIG.5, FIG.6, and FIG.7. Now, FIG.8 displays the $k_{\perp}$-dependence of the chemical potential $\mu$ for various values of $\theta$ (in the region II where there is a quasi equilibrium and thermalisation). Notice that $\mu$ is negative and $|\mu|$ is a decreasing function of $\theta$. The reason is in the thermodynamical equilibrium where $\mu \propto -\frac{\Delta S_{Q.E}}{\Delta \hat{n}}$ and since in the region II $\Delta S_{Q.E}$ is an increasing function of $\Delta \hat{n}$, thus the ratio $\frac{\Delta S_{Q.E}}{\Delta \hat{n}}$ is positive and therefore $\mu$ is negative. Moreover, since the increase or decrease of the chemical potential $\mu$ is related to that of $\hat{n}$, we can easily explain the $\theta$-dependence of $\mu$ for a fixed value of $k_{\perp}$. FIG.9 shows the chemical potential $\mu$ as a function of the second peak values of $S_{Q.E}$ (denoted by $S_{Q.E}^{max}$) for various values of $\theta$. FIG.10 is the same as FIG.9 but for the number particle density $\hat{n}$ and the first peak values of $S_{Q.E}$ (where $k_{\perp} \approx 0$). Notice that, if we know the position of the first and second peak, one can get easily information about certain thermodynamical quantities (like the chemical potential $\mu$) of the N.C. Bianchi I universe. FIG.11 displays the $\theta$-dependence of the first peak of $S_{Q.E}$. Finally, using the constraint $64k_{\perp}^2 - 25k_{\perp}^2 \theta^2 + 16\theta^2 \geq 0$ one can get an upper bound $|\theta| \leq \frac{8}{5}$ which justifies the small numerical values of the non-commutativity $\theta$ parameter considered in our paper.

4. Conclusion

Throughout this paper, we have studied the creation of entanglement between massless Dirac fermion-antifermion particle pairs in the framework of N.C. Bianchi I universe characterised by non-commutative components of the vierbeins and spin connections of Eq. (11). In section 2, we have derived the expressions of these N.C. vierbeins, spin connections and presented the N.C. Dirac equation and its solutions. Due to the complexity of the N.C. anisotropic Bianchi I space-time structure, the behaviours of $S_{Q.E}$ as a function of the $k_{\perp}$-modes frequencies were not trivial and different from those obtained in Ref. [17] for the case of an isotropic F.R.W. universe (special solvable case). In fact, at $k = 0$, the authors of Ref. [17] claim that the vanishing of $S_{Q.E}$ (without any rigorous justification) is due to the Pauli exclusion principle and it seems (according to the authors) that it is a general characteristic fermionic feature independent of the space-time structure. According to our numerical results, we notice that the structure and deformation of the space-time as well as the type of the involved particles (fermions or bosons) affect the behaviour of $S_{Q.E}$ and the position of its optimal values as a function of the $k_{\perp}$-modes frequencies. In Ref. [17] the authors have noticed that for massless fermions $S_{Q.E} = 0$ (no entanglement) and their results depend on $k = \sqrt{k_{x}^2 + k_{y}^2 + k_{z}^2}$ (because of the space-time isotropy). Our results show that even with massless particles one can have a non-vanishing entanglement and depend only on $k_{\perp} = \sqrt{k_{x}^2 + k_{y}^2}$ (because of Bianchi I space-time anisotropy and the choice of $\theta$). Thus, the argument of Ref. [17] does not hold in general. Lastly, the behaviour of some thermodynamical quantities (like the chemical potential $\mu$) as a function of the optimal $k_{\perp}$-modes frequencies
and deformation of $\theta$ parameter was also discussed. We conclude that a knowledge of the thermodynamical properties of Bianchi I space-time (like chemical potential $\mu$) may give useful information on the entanglement and vice-versa. Thus, to summarise our conclusions, we have shown that:

i. the behaviour of $S_{Q,E}$ depends not only on the kind of the involved particles (bosons or fermions) during the pair creation process but also on the structure and deformation of the space-time;

ii. because of the space-time deformation the $S_{Q,E}$ does not vanish for massless fermionic particles (contrary to Ref. [17]);

iii. because of the Bianchi I anisotropy and the choice of the N.C. $\theta$ parameter, our results depend on the $k_\perp$ (transverse) and not on the whole values of $k$;

iv. upper new bound of $S_{Q,E}$ depends strongly on the N.C. $\theta$ parameter and not equals to $\log_2 N$ as it is in Ref. [17];

v. for the consistency for our theoretical calculation, we have obtained an upper bound of the N.C. $\theta$ parameter. Of course, the considered small values of $\theta$ in our paper verify this constraint;

vi. the non-commutativity of space-time induces two compelling terms of opposite signs: one plays the role of gravity and contributes to the matter density and the other represents a sort of repulsive force (quintessence, dark energy, etc.). Thus, the information obtained from the quantum entanglement depends on N.C. $\theta$ parameter;

vii. if we know the position of the optimal values of the $S_{Q,E}$ we can get information about certain thermodynamical quantities (like the chemical potential $\mu$) of the N.C. Bianchi I universe and vice-versa.

(more studies will be presented in a future publication).

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Appendix A. The N.C. Dirac Equation

The Dirac equation for a massless fermions in 4-dimensional N.C.S.W. space-time has the following expression:

$$\gamma^f \left[ \hat{\epsilon}^\mu_f \partial_\mu - \frac{i}{8} \left( \hat{\epsilon}^\mu_f \hat{\omega}^{ab}_\mu + \hat{\omega}^{ab}_\mu \hat{\epsilon}^\mu_f \right) \Sigma_{ab} \right] \ast \hat{\psi} = 0$$  (A.1)
where $\hat{\psi}$ is the N.C. Dirac 4-components spinor and:
\[ \Sigma_{ab} = \Sigma_{[ab]} + \Sigma_{(ab)} \]  
(A.2)
with
\[ \Sigma_{[ab]} = \frac{i}{2} [\gamma_a, \gamma_b] \]  
(A.3)
\[ \Sigma_{(ab)} = \frac{1}{2} \{ \gamma_a, \gamma_b \} = \eta_{ab} \mathbb{I}_{4 \times 4} \]  
(A.4)
using the fact that:
\[ \frac{1}{2} [\gamma_d \Sigma_{[ab]} + \Sigma_{[ab]} \gamma_d] = \varepsilon_{f dab} \gamma^f \gamma^5 \]  
(A.5)
\[ \frac{i}{2} [\gamma_d \Sigma_{[ab]} - \Sigma_{[ab]} \gamma_d] = g_{db} \gamma_a - g_{da} \gamma_b \]  
(A.6)
where $\varepsilon_{f dab}$ is the 4-rank totally antisymmetric tensor, one can easily show that the N.C. Dirac equation takes the form:
\[ \left[ \gamma^f \left( i \partial_f + \hat{A}_f \right) + \gamma^f \gamma^5 \hat{B}_f \right] * \hat{\psi} = 0 \]  
(A.7)
where
\[ \hat{A}_f = 3 \left( \tilde{e}^f_{\mu} \sum_{a=1}^{4} \tilde{\omega}_{\mu a}^a + \mathbb{R} \left[ \tilde{e}^f_{\mu} \left( \tilde{\omega}_{\mu d}^{fd} - \tilde{\omega}_{\mu f}^{df} \right) \right] \right) \]  
(A.8)
\[ \hat{B}_f = 3 \left[ (\tilde{e}^{\mu \nu} \tilde{\omega}_{\mu \nu}^{ab} + \frac{1}{4} \theta^{\rho \sigma \beta \alpha \delta} \left( \partial_\rho \partial_\alpha \tilde{e}^{\mu \nu} \right) \left( \partial_\sigma \partial_\beta \tilde{\omega}_{\mu \nu}^{ab} \right) \right] \varepsilon_{f dab} \]  
(A.9)
up to $O(\theta^2)$. (The notation $[ \ldots ]$ stands for the antisymmetric part).
The expressions of the S.W. maps $\tilde{\omega}_{\mu}, \tilde{e}^{\mu}$ and the gauge parameter $\hat{A}$ (used in our paper are those of Ref. [42] based on the work of Refs. [43, 44]) are necessary for the invariance of the action of the pure N.C. gauge gravity (which is a part of our total action of Eq. (C.1)) under $*$-gauge transformations of Eqs. (C.15)-(C.17). The symmetrisation used in the matter field term of the action (C.1) is due to the $*$-product ordering ambiguity. As a result and in order to maintain the N.C. gauge invariance of the total action, we have introduced an N.C. torsion terms (see 2nd and 3rd terms in the action of Eq.(C.1)). Notice that, we can avoid the introduction of the torsion terms and preserving the invariance of the total action by modifying the N.C. gauge transformations of the matter field. Of course the gauge group is no more $SO(1,3)*$, and therefore the matter fields are not those of S.W. types but in the limit $\theta \to 0$ we recover all ordinary gauge transformations of the Lorentz gauge group $SO(1,3)$, (more effects of Moyal $*$-product symmetrisation and anti-symmetrisation on the infinitesimal gauge transformations and S.W. maps can be found in Ref. [45]). It is worth mentioning that even if we did not use a symmetrised $*$-product $\tilde{e}^{\mu} * \tilde{\omega}_{\mu}^{ab} + \tilde{\omega}_{\mu}^{ab} * \tilde{e}^{\mu}$ in Eq.(A.1), we can not obtain a compact form like for example $\gamma^f (\tilde{e}^{\mu}_{\psi} \partial_{\mu} - \frac{4}{3} \tilde{e}^{\mu}_{f} * \tilde{\omega}_{\mu}^{ab} \Sigma_{ab}) * \psi = 0$ with $\tilde{e}^{\mu}, \tilde{\omega}_{\mu}^{ab}$ and $\hat{\psi}$ are S.W. fields from first principles and appropriate Lagrangian density without adding an extra terms like torsion, cotorsion etc.
Appendix B. N.C. Mathematical Formalism

The N.C.Vierbeins \( \hat{e}_\mu^a \) up to \( O(\theta^2) \) are (see Ref. [2]):

\[
\hat{e}_\mu^a = e_\mu^a - i \theta^{\nu\rho} e^{a\nu\rho} + \theta^{\nu\rho} \theta^{\lambda\tau} e^{a\nu\rho\lambda\tau} + O(\theta^3) \tag{B.1}
\]

where

\[
e^{a}_{\nu\rho} = \frac{1}{4} \left[ \omega^{ac}_{\nu} \partial_{\rho} e^d_{\mu} + \left( \partial_{\rho} \omega^{ac}_{\mu} + R^{ac}_{\rho\mu} \right) e^d_{\nu} \right] \eta_{cd} \tag{B.2}
\]

\[
e^{a}_{\mu\nu\rho\lambda\tau} = \frac{1}{32} \left[ 2 \left( R_{\tau\nu\nu}, R_{\rho\mu\mu} \right) e^c_{\lambda} - \omega^{ab}_{\lambda} \left( D_{\rho} R^{cd}_{\tau\mu} + \partial_{\rho} R^{cd}_{\tau\mu} \right) e^m_{\nu} \eta_{dm} \right.
\]

\[
- \left\{ \omega_{\nu}, \left( D_{\rho} R_{\tau\mu} + \partial_{\rho} R_{\tau\mu} \right) \right\} e^c_{\lambda} - \partial_{\tau} \left\{ \omega_{\nu}, \left( \partial_{\rho} \omega_{\mu} + R_{\rho\mu} \right) \right\} e^c_{\lambda}
\]

\[- \omega^{ab}_{\lambda} \partial_{\tau} \left( \omega^{cd}_{\nu} \partial_{\rho} e^m_{\mu} + \left( \partial_{\rho} \omega^{cd}_{\mu} + R^{cd}_{\rho\mu} \right) e^m_{\nu} \right) \eta_{dm} + 2 \partial_{\nu} \omega^{ab}_{\lambda} \partial_{\rho} \partial_{\tau} e^c_{\mu}
\]

\[-2 \partial_{\rho} \left( \partial_{\tau} \omega^{ab}_{\mu} + R_{\tau\mu}^{ab} \right) \partial_{\nu} e^c_{\lambda} - \left\{ \omega_{\nu}, \left( \partial_{\rho} \omega_{\lambda} + R_{\rho\lambda} \right) \right\} e^c_{\lambda},
\]

\[- \left( \partial_{\tau} \omega^{ab}_{\mu} + R_{\tau\mu}^{ab} \right) \left( \omega^{cd}_{\nu} \partial_{\rho} e^m_{\mu} + \left( \partial_{\rho} \omega^{cd}_{\mu} + R^{cd}_{\rho\mu} \right) e^m_{\nu} \right) \eta_{dm} \right) \eta_{bc} \tag{B.3}
\]

here \( R^{ab}_{\mu\nu} \) is the strength field associated with the commutative spin connections \( \omega^{ab}_{\mu} \)
and is defined as:

\[
R^{ab}_{\mu\nu} = \partial_{\mu} \omega^{ab}_{\nu} - \partial_{\nu} \omega^{ab}_{\mu} + \left( \omega^{ac}_{\mu} \omega^{db}_{\nu} - \omega^{ac}_{\nu} \omega^{db}_{\mu} \right) \eta_{cd} \tag{B.4}
\]

(\( \eta_{ab} \) is the Minkowski metric). The N.C. spin connections \( \dot{\omega}^{AB}_{\mu} \) up to \( O(\theta^2) \) are:

\[
\dot{\omega}^{AB}_{\mu} = \omega^{AB}_{\mu} - i \theta^{\nu\rho} \omega^{AB}_{\mu\nu\rho} + \theta^{\nu\rho} \theta^{\lambda\tau} \omega^{AB}_{\mu\nu\rho\lambda\tau} + .... \tag{B.5}
\]

where

\[
\omega^{AB}_{\mu\nu\rho} = \frac{1}{4} \left\{ \omega_{\nu}, \partial_{\rho} \omega_{\nu} + R_{\rho\nu} \right\}^{AB} \tag{B.6}
\]

\[
\omega^{AB}_{\mu\nu\rho\lambda\tau} = \frac{1}{32} \left[ - \left\{ \omega_{\nu}, \partial_{\rho} \left\{ \omega_{\nu}, \partial_{\mu} \omega_{\mu} + R_{\rho\mu} \right\} \right\} \right] + 2 \left\{ \omega_{\nu}, \left\{ R_{\tau\nu\nu}, R_{\rho\mu\mu} \right\} \right\}
\]

\[- \left\{ \omega_{\nu}, \left\{ \omega_{\nu}, D_{\rho} R_{\tau\mu} + \partial_{\rho} R_{\tau\mu} \right\} \right\} - \left\{ \omega_{\nu}, \left( \partial_{\rho} \omega_{\lambda} + R_{\rho\lambda} \right), \left( \partial_{\tau} \omega_{\mu} + R_{\tau\mu} \right) \right\}
\]

\[+ 2 \left[ \partial_{\nu} \omega_{\lambda}, \partial_{\rho} \left( \partial_{\tau} \omega_{\mu} + R_{\tau\mu} \right) \right] \tag{B.7}
\]

here

\[
\left\{ \alpha, \beta \right\}^{AB} = \alpha^{AC} \beta^{CB} + \beta^{AC} \alpha^{CB} \tag{B.8}
\]

\[
\left[ \alpha, \beta \right]^{AB} = \alpha^{AC} \beta^{CB} - \beta^{AC} \alpha^{CB} \tag{B.9}
\]

and

\[
D_{\mu} R_{\rho\sigma}^{AB} = \partial_{\mu} R_{\rho\sigma}^{AB} + \left( \omega^{AC}_{\mu} + R^{DB}_{\rho\sigma} + \omega^{BC}_{\mu} R^{DA}_{\rho\sigma} \right) \eta_{CD} \tag{B.10}
\]
Appendix C. Derivation of N.C. Dirac Equation From a Least Action

To derive the N.C. Dirac equation given by Eq.(2) from the least action principle, we consider the following N.C. action:

\[ S = \int d^4x \sqrt{-\hat{g}} \left\{ \left[ \frac{i}{2} \left( \bar{\psi} \gamma^\mu \hat{D}_\mu \hat{\psi} + c.c. \right) + \frac{i}{4} \bar{\psi} \hat{K}_{\mu\nu\rho} \hat{X}^{\mu\nu\rho} \hat{\psi} \right] - T^{\mu\nu\rho} \hat{K}_{\mu\nu\rho} + L_g \right\} \]  

(C.1)

where \( \hat{K}_{\mu\nu\rho} \) and \( \hat{T}^{\mu\nu\rho} \) are the N.C. cotorsion and modified torsion tensors respectively, and \( \hat{g} \) is the determinant of the N.C. metric.

We set:

\[ \frac{i}{4} \hat{K}_{\mu\nu\rho} \hat{X}^{\mu\nu\rho} = \hat{K} \]  

(C.2)

such that

\[ \hat{K} = \sqrt{-\hat{g}} \left\{ \frac{i}{2} \gamma^a \hat{Y}_a + \frac{i}{2} \hat{e}_a^\mu \hat{Y}_a \hat{e}_a^\mu \gamma^a \right\} - \frac{i}{2} \partial \mu \left( \sqrt{-\hat{g}} \hat{e}_a^\mu \right) \gamma^a \]  

(C.3)

and

\[ \hat{Y}_f = \frac{i}{8} \left\{ \hat{e}_f^\mu, \hat{\omega}_{ab}^\mu \right\} \Sigma_{ab} \]  

(C.4)

where

\[ \{a, b\}_* = a \ast b + b \ast a \]  

(C.5)

here the N.C. covariant derivative \( \hat{D}_\mu \) is defined such that

\[ \hat{e}_a^\mu \hat{D}_\mu \hat{\psi} = \left( \hat{e}_a^\mu \partial \mu \hat{Y}_a \right) \hat{\psi} \]  

(C.6)

one can show easily that the variational principle of the action with respect to \( \hat{\psi} \) gives our N.C. Dirac equation (Eq.2). For a comparative illustration, one can get from Eq.(C.3) in the commutative case the following expression:

\[ \hat{K} \rightarrow K = \frac{i}{4} \omega_{abc} \left\{ \gamma^{[a} \gamma^b \gamma^c] - \gamma^{a} \gamma^b \gamma^c \right\} + \frac{i}{2} \gamma^{b} \gamma^a \]  

(C.7)

where

\[ \gamma^{b} \gamma^a = C_{ab}^b - C_{ba}^b = 2C_{ab}^b \]  

(C.8)

and

\[ C_{ab}^b = \frac{1}{2} e^a_{\mu} \left( e^b_{\nu} \partial_\mu e^\nu_{b} - 1 \right) - \frac{1}{2} e^a_{\mu} e^b_{\nu} \partial_\mu e^\nu_{a} \]  

(C.9)

[ . ] stands for antisymmetrisation with respect to the indices. Moreover, in this case one can show also that:

\[ \partial_\nu \left( \sqrt{-g} e^\mu_{a} \right) = \sqrt{-g} \gamma^{b} \gamma^a \]  

(C.10)
the cotorsion and the modified torsion tensors are given by:

\[ K_{\alpha \beta \mu} = -Q_{\alpha \beta \mu} - Q_{\mu \alpha \beta} + Q_{\beta \mu \alpha} \] (C.11)

and

\[ T_{\mu \nu}^\alpha = Q_{\mu \nu}^\alpha + \delta_{\mu}^\alpha \, Q_{\nu}^\rho - \delta_{\nu}^\alpha \, Q_{\rho}^\mu \] (C.12)

respectively. Here \( Q_{\alpha \beta \mu} \) is the torsion tensor and:

\[ Q_{\alpha} = Q_{\alpha \nu} \nu \] (C.13)

the N.C. modified torsion tensor \( \hat{T}_{\mu \nu}^\alpha \) has a similar form as in the commutative case:

\[ \hat{T}_{\mu \nu}^\alpha = \hat{Q}_{\mu \nu}^\alpha + \delta_{\mu}^\alpha \, \hat{Q}_{\nu}^\rho - \delta_{\nu}^\alpha \, \hat{Q}_{\rho}^\mu \] (C.14)

Concerning the invariance of the N.C. Lagrangian density with respect to the N.C. local Lorentz transformations (see Eqs.(C.15)-(C.17)), one can choose: \( \delta_{\hat{\Lambda}} \hat{K}_{\mu \nu \rho} \) and, or \( \hat{T}_{\mu \nu \rho} = \hat{g}_{\rho a} \ast \hat{T}_{\mu \nu}^\alpha \) (not all the N.C. components of \( \hat{K}_{\mu \nu \rho} \) are contributing to \( \hat{K} \)) such that:

\[ \delta_{\hat{\Lambda}} \hat{\psi} = \hat{\Lambda} \ast \hat{\psi} \] (C.15)

\[ \delta_{\hat{\Lambda}} \hat{e}^\mu = \hat{\Lambda} \ast \hat{e}^\mu = \hat{e}^\mu \ast \hat{\Lambda} = \left[ \hat{\Lambda}, \hat{e}^\mu \right] \ast \] (C.16)

\[ \delta_{\hat{\Lambda}} \hat{\omega}_{\mu} = -\partial_{\mu} \hat{\Lambda} + \left[ \hat{\Lambda}, \hat{\omega} \right] \ast \] (C.17)

and

\[ \delta_{\hat{\Lambda}} \left[ \hat{\bar{\psi}} \hat{K} \hat{\psi} \right] + \delta_{\hat{\Lambda}} \left( \hat{T}_{\mu \nu \rho} \hat{K}_{\mu \nu \rho} \right) + \delta_{\hat{\Lambda}} \hat{L}_0 = 0 \] (C.18)

where \( [A, B] \ast \) stands for \( A \ast B - B \ast A \), and

\[ \hat{\psi} = \gamma^\mu = \hat{\psi} \ast \gamma^a \] (C.19)

\[ \hat{\omega}_{\mu} = \hat{\omega}^{ab} \mu \Sigma_{ab} \] (C.20)

\[ \hat{\Lambda} = \hat{\Lambda}^{ab} \Sigma_{ab} \] (C.21)

Here \( \delta_{\hat{\Lambda}} \hat{L}_0 \) is the non vanishing part of \( \delta_{\hat{\Lambda}} \left[ \frac{i}{2} \hat{\bar{\psi}} \ast \gamma^\nu \ast \hat{D}_\mu \ast \hat{\psi} + c.c \right] \) when we use the N.C. gauge transformations of Eqs.(C.15)-(C.17). Of course \( \delta_{\hat{\Lambda}} \hat{L}_g = 0 \) (see Ref. 43).

This is a simple justification for the derivation of our N.C. Dirac equation (Eq.2) with S.W. fields and proof of invariance of our N.C. Lagrangian density with respect to the N.C. local Lorentz transformations of Eqs.(C.15)-(C.17).
Appendix D. Determination of $|\Gamma(x + iy)|^2$

1. If $x = 0$

$$|\Gamma(iy)|^2 = \frac{\pi}{y \sinh(\pi y)}$$  \hspace{1cm} (D.1)

2. If $y = 0$

(a) $x > 0$

$$|\Gamma(x)|^2 = [(x - 1)!]^2$$  \hspace{1cm} (D.2)

(b) $x < 0$

$$|\Gamma(x)|^2 = \frac{\pi^2}{-x \Gamma(-x) \sin(\pi x)}$$  \hspace{1cm} (D.3)

3. $x \neq 0$, $y \neq 0$

$$|\Gamma(iy)|^2 = \prod_{n=0}^{\infty} \left[ 1 + \frac{y^2}{(n + x)^2} \right]^{-1}$$  \hspace{1cm} (D.4)

4. If $\frac{y^2}{x^2} < 1$ ($x \neq 0$), then $|\Gamma(x + iy)|$ can be approximated by:

$$|\Gamma(x + iy)| \approx |\Gamma(x)| \left[ 1 - \frac{y^2}{2} \Phi(1, 2, x) \right]$$  \hspace{1cm} (D.5)

where $\Phi(1, 2, x)$ is the Herwitz zeta function such that:

$$\Phi(1, 2, x) = \sum_{n=0}^{\infty} \frac{1}{(x + n)^2}$$  \hspace{1cm} (D.6)

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