TENSOR PRODUCTS OF MODULAR REPRESENTATIONS OF
$SL_2(F_p)$ AND A RANDOM WALK ON THEIR
INDECOMPOSABLE SUMMANDS

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Abstract

In this paper we give a novel, concise and elementary proof of the decomposition of tensor products of simple modular $SL_2(F_p)$-representations. This result is used to decompose tensor products involving their projective covers and to decompose symmetric squares. We define a Markov chain on the simple modular $SL_2(F_p)$-representations via tensoring with a fixed simple module and choosing an indecomposable summand according to a specified weighting; we show this chain is reversible and find its stationary distributions.

1. Introduction

Let $p$ be prime, let $k$ be an algebraically closed field of characteristic $p$, and let $F_p$ be the prime subfield of $k$. Let $G \leq SL_2$ be a subgroup of the group of invertible $2 \times 2$ matrices with entries in $k$. We use “module” to mean “finite-dimensional module”. In this paper we consider tensor products of the following representations of $G$, especially in the case when $G = SL_2(F_p)$.

Definition. For $n \geq 1$, let $V_n$ be the $n$-dimensional $kG$-module consisting of homogeneous polynomials over $k$ of degree $n - 1$ in two variables $X$ and $Y$, with $G$-action given by

$$(a \ b \ c \ d) f(X, Y) = f(aX + cY, bX + dY).$$

Let $P_n$ be the projective cover of $V_n$.

Note that $V_2$ is the natural $kG$-module, and that $V_n \cong \text{Sym}^{n-1} V_2$. More details about these representations when $G = SL_2(F_p)$ are given in Section 2; most importantly, $\{V_n \mid 1 \leq n \leq p\}$ is a complete set of simple $kSL_2(F_p)$-modules.

Rules giving the decompositions of the tensor products of simple modules are known as Clebsch–Gordan rules. The rule for $SU_2(\mathbb{C})$ (equivalently, for the Lie algebra $sl_2(\mathbb{C})$) in characteristic 0 is well-known [Hall, Appendix C]. This rule, as well as those for other Lie groups which appear as physical symmetry groups, is of importance in quantum physics, where simple modules of a symmetry group represent fundamental objects and tensor products represent compound systems which can be better understood by decomposing.

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This paper offers a novel proof of a Clebsch–Gordan rule for $G = \text{SL}_2(\mathbb{F}_p)$ in characteristic $p$. The rule can be found from [Glo78, (5.5) and (6.3)] or from [Kou90a, Corollary 1.2(a) and Proposition 1.3(c)], both of which prove the required decompositions via repeated tensoring by the natural module (the former initially working with the semigroup of $2 \times 2$ matrices over $\mathbb{F}_p$, before restricting to $\text{GL}_2(\mathbb{F}_p)$ and $\text{SL}_2(\mathbb{F}_p)$). In this paper, we define a new family of surjective $\text{GL}_2(k)$-homomorphisms (Lemmas 3.3 and 3.4) and find their kernels when restricted to $\text{SL}_2(k)$ (Proposition 3.5); we then obtain a concise proof of the rule for $\text{SL}_2(\mathbb{F}_p)$ by showing these homomorphisms split. The decompositions can also be found in terms of tilting modules in [EH02, Lemma 4], where the proofs rely on tilting theory.

This paper extends the Clebsch–Gordan rule to give decompositions of all tensor products of a simple module and a projective indecomposable module, and of two projective indecomposable modules. An iterative approach to finding these decompositions is described in [Kou90a, remark following Proposition 1.3], whereas here we give a convenient, intuitive method through pairing up subquotients. The rule and its extension are summarised in Theorem 1.1 below; Sections 3 and 4 comprise its proof.

In Section 5, we decompose symmetric and exterior squares. We first find an explicit $k\text{SL}_2(k)$-isomorphism between $\text{Sym}^2 V_n$ and $\bigwedge^2 V_{n+1}$. This, together with the Clebsch–Gordan rule, allows us to inductively decompose $\text{Sym}^2 V_n$ into indecomposable modules for $1 \leq n \leq p$. Various results on symmetric and exterior powers of representations of $\text{GL}_2(\mathbb{F}_p)$ and $\text{SL}_2(\mathbb{F}_p)$ are given in [Kou90b], but these typically ignore projective summands.

In Sections 6 and 7, we investigate a Markov chain on the simple $k\text{SL}_2(\mathbb{F}_p)$-modules, defined by tensoring by a fixed simple module and choosing a non-projective indecomposable summand of the result with probability depending on a weighting given to each simple module. This is motivated by [BDLT18], which considers a similar Markov chain but chooses from the composition factors of the tensor product rather than the indecomposable summands. Here we exclude projective summands as otherwise they form an absorbing set. The case of tensoring by $V_2$ (the natural module) and choosing uniformly from the non-projective indecomposable summands results in a familiar Markov chain: a symmetric random walk in one dimension with reflecting boundaries. Although [BDLT18] focuses on tensoring by the natural module, we consider tensoring by any non-projective simple module. We find the connected components of these chains, show that they are reversible and find their stationary distributions. Identifying properties of these chains also reveals facts about the representation theory of $G$ (Lemma 6.1 and Proposition 6.2).

Throughout this paper, we make use of Iverson bracket notation, and the notation $[r] = \{1, 2, \ldots, r\}$. We introduce the following notation for a family of sets that occur frequently.

**Definition.** For $n \geq m \geq 1$, let the $(n,m)$-string be the set

$$\langle n,m \rangle = \{n + m - 1, n + m - 3, \ldots, n - m + 3, n - m + 1\},$$

and let $\langle n,0 \rangle = \emptyset$. 

**Theorem 1.1.** Suppose $G = SL_2(\mathbb{F}_p)$, and suppose $1 \leq n, m \leq p$. If $n \geq m$, we have

$$V_n \otimes V_m \cong \bigoplus_{i \in \langle n, m \rangle} V_i \oplus \bigoplus_{i \in \langle n, m \rangle} P_i [n = m = p] V_p.$$  

If $m \neq 1$ and $n \notin \{1, p\}$, we have

$$P_n \otimes V_m \cong \begin{cases} \bigoplus_{i \in \langle n, m \rangle} P_i \oplus \bigoplus_{i \in \langle n, m \rangle} P_{2p-i} [n = m] V_p & \text{if } n \geq m, \\ \bigoplus_{i \in \langle m, n \rangle} P_i \oplus \bigoplus_{i \in \langle m, n \rangle} P_{2p-i} \oplus P_{p-(m-n)} \oplus P_{i}^{\otimes 2} & \text{if } n < m < p, \\ \bigoplus_{i \in \langle p, n \rangle} P_i^{\otimes 2} \oplus P_{i}^{\otimes 2} \oplus P_n & \text{if } m = p. \end{cases}$$

In terms of tensor products already found, for all $1 \leq n \leq p$ we have

$$P_1 \otimes V_n \cong P_n \oplus [n > 2] (V_p \otimes V_{n-2}) \oplus [n = p] V_p$$

and

$$P_1 \otimes P_n \cong P_n^{\otimes 2} \oplus [p > 2] (V_{p-2} \otimes P_n),$$

and lastly, for $2 \leq n, m \leq p-1$, we have

$$P_n \otimes P_m \cong (P_n \otimes V_m) \oplus (P_m \otimes V_n)$$

$$\oplus \begin{cases} (P_{p-1} \otimes V_{2p-(n+m)}) & \text{if } n + m \geq p, \\ (P_{p-1} \otimes V_{p+1-(n+m)}) \oplus (P_{n+m} \otimes V_{p-1}) & \text{if } n + m < p. \end{cases}$$

We illustrate how to use our Clebsch–Gordan rule to decompose the tensor product of two simple modules with the following example.

**Example 1.2.** Let $G = SL_2(\mathbb{F}_p)$ and $p = 17$, and we consider $V_{14} \otimes V_9$. We draw the $(14, 9)$-string below, and indicate those elements $i$ for which $2p - i \in \langle 14, 9 \rangle$ by joining $i$ and $2p - i$ with a dotted line. The summand of $V_{14} \otimes V_9$ that arises out of each element of $\langle 14, 9 \rangle \cap [17]$ is written below it.

![Diagram of tensor product decomposition]

$$V_{14} \otimes V_9 \cong V_6 \oplus V_8 \oplus V_{10} \oplus P_{12} \oplus P_{14} \oplus P_{16}$$

The pairing-up of $i$ and $2p - i$ in fact corresponds to an isomorphism

$$V_{2p-i} \cong \frac{P_i}{\langle V_i \oplus [i = 1] V_p \rangle}$$

proved in Corollary 4.2.
We make several immediate observations about the tensor product of simple modules $V_n$ and $V_m$ (where $1 \leq m \leq n \leq p$):

(i) all non-projective summands of $V_n \otimes V_m$ are simple;
(ii) $V_n \otimes V_m$ is semisimple if and only if $n + m \leq p + 1$, in which case $V_n \otimes V_m \cong \bigoplus_{i \in [n,m]} V_i$, which is exactly the rule for analogously defined representations of SU$_2(\mathbb{C})$ over $\mathbb{C}$;
(iii) $V_n \otimes V_m$ is projective if and only if $n = p$, in which case $V_p \otimes V_m \cong \bigoplus_{i \in [p,m] \cap [p]} P_i \oplus [m = p] V_p$;
(iv) in the sense of indecomposable summands, $V_n \otimes V_m$ is multiplicity-free unless $n = m = p$ (when $V_p$ occurs with multiplicity 2, and all other indecomposable summands occur only once).

2. Background on representation theory of SL$_2(\mathbb{F}_p)$

In this section, we take $G = \text{SL}_2(\mathbb{F}_p)$, and give some useful facts about the representations of $G$ discussed in this paper.

The $kG$-modules $V_1, \ldots, V_p$ are simple [Alp86, pp. 14–16] (in fact, this proof holds for any SL$_2(\mathbb{F}_p) \leq G \leq \text{GL}_2(k)$). Furthermore, the set $\{ V_n \mid 1 \leq n \leq p \}$ is a complete set of simple $kG$-modules up to isomorphism, since the number of $p$-regular conjugacy classes in $G$ is $p$. In particular, there is a unique simple $kG$-module of each dimension less than or equal to $p$, and so the simple modules are self-dual. Also, it follows that the set $\{ P_n \mid 1 \leq n \leq p \}$ is a complete set of projective indecomposable $kG$-modules. This means that Theorem 1.1 gives decompositions of tensor products of all possible pairs of simple and projective indecomposable modules.

The projective indecomposable $kG$-modules are constructed in [Alp86, pp. 48–52] (using the special case $m = 2$ of our Proposition 3.1), from which follows the Brauer trees for $G$ in [Alp86, p. 123]. We here describe the projective indecomposable modules. Firstly, $P_p \cong V_p$ is projective and simple. When $p = 2$, there is only one other projective indecomposable module: $P_1$, which is of composition length 2 (and hence has composition factors only $V_1$). For $p > 2$, all other projective indecomposable modules have composition length 3, and so the only structural information which is missing is their heart. The heart of $P_1$ is $V_{p-2}$, the heart of $P_{p-1}$ is $V_2$, and for $2 \leq n \leq p - 2$ the heart of $P_n$ is $V_{p-n-1} \oplus V_{p-n+1}$; these structures are illustrated below.

\begin{center}
\begin{tikzpicture}[scale=1]

\node (P1) at (0,0) {$P_1$};
\node (P2) at (1,0) {$P_2$};
\node (Pp-2) at (2,0) {$P_{p-2}$};
\node (Pp-1) at (3,0) {$P_{p-1}$};
\node (V1) at (0,-1) {$V_1$};
\node (V2) at (1,-1) {$V_2$};
\node (Vp-2) at (2,-1) {$V_{p-2}$};
\node (Vp-1) at (3,-1) {$V_{p-1}$};
\node (Vp-3) at (1,-2) {$V_{p-3}$};
\node (Vp-4) at (2,-2) {$V_{p-4}$};
\node (Vp-5) at (3,-2) {$V_{p-5}$};

\draw (P1) -- (V1);
\draw (P2) -- (V2);
\draw (Pp-2) -- (Vp-2);
\draw (Pp-1) -- (Vp-1);
\draw (P1) -- (Vp-3);
\draw (P2) -- (Vp-4);
\draw (Pp-2) -- (Vp-5);
\draw (P1) -- (V1);
\draw (P2) -- (V2);
\draw (Pp-2) -- (Vp-2);
\draw (Pp-1) -- (Vp-1);
\end{tikzpicture}
\end{center}
Note that $P_1$ and $P_p$ are both $p$-dimensional, while all other projective indecomposable $kG$-modules are $2p$-dimensional.

We can now write down the Cartan matrix. It is most convenient to give the simple modules and their covers the ordering

$$V_1, V_{p-2}, V_3, \ldots, V_{2p}, V_{p-1}, V_2, V_{p-3}, \ldots, V_{p+1}, V_p$$

where $\varepsilon \in \{\pm 1\}$ and $\varepsilon \equiv p \pmod{4}$. For $p = 2$, the Cartan matrix is simply $\left( \begin{array}{c} 2 & 0 \\ 0 & 1 \end{array} \right)$. For $p > 2$, let $C$ be the $\frac{p-1}{2} \times \frac{p-1}{2}$ matrix

$$\begin{pmatrix}
2 & 1 & 1 & \cdots & 1 \\
1 & 2 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
1 & 2 & 1 & \cdots & 1 \\
1 & 3 & 1 & \cdots & 1
\end{pmatrix}$$

where $C = (3)$ when $p = 3$. Then the Cartan matrix, in block diagonal form, is

$$\begin{pmatrix}
C & 0 \\
0 & 1
\end{pmatrix}.$$

### 3. Short exact sequences

**Definition.** Let $\mu: V_n \otimes V_m \to V_{n+m-1}$ be the multiplication map, defined by $k$-linear extension of $\mu(f \otimes g) = fg$. The dependence of $\mu$ on $n$ and $m$ is suppressed, since it is always clear from context.

It is easily seen that $\mu$ is surjective and $GL_2(k)$-equivariant. The following result identifying the kernel of $\mu$ is well-known (see \cite{Glo78, (5.1)}, or \cite{Kou90a, Proposition 1.2(a)} for the case $m = 2$).

**Proposition 3.1.** Suppose $G \leq SL_2(k)$ and suppose $n, m \geq 2$. Then the kernel of $\mu$ is isomorphic to $V_{n-1} \otimes V_{m-1}$, and hence there is a short exact sequence

$$0 \to V_{n-1} \otimes V_{m-1} \to V_n \otimes V_m \xrightarrow{\mu} V_{n+m-1} \to 0.$$

**Proof.** Consider the map $\theta: V_{n-1} \otimes V_{m-1} \to V_n \otimes V_m$ defined by $k$-linear extension of $\theta(f \otimes g) = Xf \otimes Yg - Yf \otimes Xg$. Observe that $\theta$ is $SL_2(k)$-equivariant: for $t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(k)$, we have

$$t\theta(f \otimes g) = t(Xf \otimes Yg - Yf \otimes Xg)$$

$$= (aX + cY)sf \otimes (bX + dY)sg - (bX + dY)sf \otimes (aX + cY)sg$$

$$= (ad - bc)Xsf \otimes Ysg - (ad - bc)Ysf \otimes Xsg$$

$$= \det(t)(Xsf \otimes Ysg - Ysf \otimes Xsg)$$

$$= Xsf \otimes Ysg - Ysf \otimes Xsg$$

$$= \theta(t(f \otimes g)).$$

It is easy to see that $\text{im} \, \theta \leq \ker \mu$. Because $\mu$ is surjective, we have that $\dim(\ker \mu) = \dim(V_n \otimes V_m) - \dim(V_{n+m-1}) = \dim(V_{n-1} \otimes V_{m-1})$, and so it remains only to show that $\theta$ is injective.
Let \( e_{i,j} = X^i Y^{n-2-i} \otimes X^j Y^{m-2-j} \in V_{n-1} \otimes V_{m-1} \), so that \( \{ e_{i,j} \mid 0 \leq i \leq n-2, 0 \leq j \leq m-2 \} \) is a linear basis for \( V_{n-1} \otimes V_{m-1} \). For \( 0 \leq r \leq n+m-4 \), let \( U_r = \langle e_{i,j} \mid i+j = r \rangle \subseteq V_{n-1} \otimes V_{m-1} \), and note that as vector spaces \( V_{n-1} \otimes V_{m-1} = \bigoplus_{r=0}^{n+m-4} U_r \).

Similarly, let \( f_{i,j} = X^i Y^{n-1-i} \otimes X^j Y^{m-1-j} \in V_n \otimes V_m \), so that \( \{ f_{i,j} \mid 0 \leq i \leq n-1, 0 \leq j \leq m-1 \} \) is a linear basis for \( V_n \otimes V_m \). For \( 0 \leq r \leq n+m-2 \), let \( W_r = \langle f_{i,j} \mid i+j = r \rangle \subseteq V_n \otimes V_m \), and note that as vector spaces \( V_n \otimes V_m = \bigoplus_{r=0}^{n+m-2} W_r \).

Observe that \( \theta(e_{i,j}) = f_{i+1,j} - f_{i,j+1} \). Then \( \theta(U_r) \subseteq_k W_{r+1} \), and thus it suffices to show that \( \theta|_{U_r} \) is injective for each \( 0 \leq r \leq n+m-4 \). Fix \( r \) in this range, and let \( i_0 = \max\{0, r-(m-2)\} \) and \( j_0 = \max\{0, r-(n-2)\} \) so that \( U_r = \langle e_{i,r-i} \mid i_0 \leq i \leq r-j_0 \rangle \). Then the images under \( \theta \) of these basis vectors for \( U_r \) are as follows.

\[
\begin{align*}
\theta(e_{i_0,r-i_0}) &= f_{i_0+1,r+1-(i_0+1)} - f_{i_0,r+1-i_0} \\
\theta(e_{i_0+1,r-(i_0+1)}) &= f_{i_0+2,r+1-(i_0+2)} - f_{i_0+1,r+1-(i_0+1)} \\
\theta(e_{i_0+2,r-(i_0+2)}) &= f_{i_0+3,r+1-(i_0+3)} - f_{i_0+2,r+1-(i_0+2)} \\
&\quad \vdots \\
\theta(e_{r-(j_0+1),j_0+1}) &= f_{r+1-(j_0+1),j_0+1} - f_{r+1-(j_0+2),j_0+2} \\
\theta(e_{r-j_0,j_0}) &= f_{r+1-j_0,j_0} - f_{r+1-(j_0+1),j_0+1}
\end{align*}
\]

Thus the \( (r-i_0-j_0+1) \times (r-i_0-j_0) \) matrix representing \( \theta \) with respect to these bases is

\[
\begin{pmatrix}
1 & 1 & 1 & \cdots \\
-1 & -1 & -1 & \cdots \\
& & 1 & \cdots \\
& & & -1
\end{pmatrix},
\]

which is of full (column) rank. Thus \( \theta|_{U_r} \) is injective as required. \( \square \)

Remark. Unlike \( \mu \), the map \( \theta \) is not \( \text{GL}_2(k) \)-equivariant: \( t(\theta(f \otimes g)) = \det(t)\theta(t(f \otimes g)) \) for \( t \in \text{GL}_2(k) \), so \( \theta \) is not \( G \)-equivariant for any subgroup \( G \) which contains a matrix with determinant not equal to 1. For an extension of this proposition to such subgroups, see [Glo78 (5.1)].

Definition. For an algebra \( A \), the Grothendieck group \( G_0(A) \) is the abelian group with:

- a generator \( [V] \) for every \( A \)-module \( V \), and
- a relation \( [W] = [U] + [V] \) for every short exact sequence \( 0 \to U \to W \to V \to 0 \).

The important property of the Grothendieck group for our purposes is that \( [U] = [V] \) if and only if \( U \) and \( V \) have the same multiset of composition factors.

Corollary 3.2. Suppose \( G \leq \text{SL}_2(k) \) and suppose \( 1 \leq m \leq n \). Then \( V_n \otimes V_m \) has a filtration

\[
0 = U_m \subseteq U_{m-1} \subseteq \cdots \subseteq U_1 \subseteq U_0 = V_n \otimes V_m
\]

where \( U_i \cong V_{n-i} \otimes V_{m-i} \) and \( U_{i+1}/U_{i+1} \cong V_{n+m-1-2i} \).
In particular, in the Grothendieck group,
\[ [V_n \otimes V_m] = \sum_{i \in \langle n, m \rangle} [V_i]. \]

Proof. By induction on \( m \). The case \( m = 1 \) is immediate. For \( m \geq 2 \), the short exact sequence involving \( \mu \) gives that there is \( U_1 \subseteq V_n \otimes V_m \) such that
\[ U_1 \cong V_{n-1} \otimes V_{m-1} \]
and
\[ V_n \otimes V_m \setminus U_1 \cong V_{n+m-1}. \]
Applying the inductive hypothesis to \( U_1 \) gives the rest of the filtration. The equality in the Grothendieck group follows because \( \langle n, m \rangle = \{ n + m - 1 - 2i \mid 0 \leq i \leq m - 1 \} \).

Remark. The proof of Proposition 3.1 holds equally well if \( k \) is of characteristic 0. In this case the simple modules are also projective and so the short exact sequences split, and we obtain \( V_n \otimes V_m \cong \bigoplus_{i \in \langle n, m \rangle} V_i \) (recovering the well-known Clebsch–Gordan rule for \( SU_2(\mathbb{C}) \)). The same decomposition is obtained when \( G \subseteq \text{SL}_2(k) \) is finite with \( p \nmid |G| \).

We next introduce a novel family of maps, which generalise the map \( \delta \) defined in [Glo78, (5.2)] (corresponding to \( n = 1 \) below). These maps allow us to see the inclusion of the bottom layer of the above filtration into \( V_n \otimes V_m \), and they split in more cases than \( \mu \) does.

Definition. For \( n \geq 1 \) and \( m \geq 2 \), let \( \lambda : V_n \otimes V_m \to V_{n+1} \otimes V_{m-1} \) be the map defined by \( k \)-linear extension of
\[ \lambda(f \otimes g) = Xf \otimes \frac{\partial g}{\partial X} + Yf \otimes \frac{\partial g}{\partial Y}. \]
The dependence of \( \lambda \) on \( n \) and \( m \) is suppressed, since it is always clear from context.

Lemma 3.3. The map \( \lambda \) is \( \text{GL}_2(k) \)-equivariant.

Proof. Let \( t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) \( \in \text{GL}_2(k) \), and let \( f \in V_n \) and \( g \in V_m \). Then
\[ t\lambda(f \otimes g) = t \left( Xf \otimes \frac{\partial g}{\partial X} + Yf \otimes \frac{\partial g}{\partial Y} \right) \]
\[ = (aX + cY)tf \otimes t\frac{\partial g}{\partial X} + (bX + dY)tf \otimes t\frac{\partial g}{\partial Y} \]
\[ = Xtf \otimes \left( at\frac{\partial g}{\partial X} + bt\frac{\partial g}{\partial Y} \right) + Ytf \otimes \left( ct\frac{\partial g}{\partial X} + dt\frac{\partial g}{\partial Y} \right) \]
and
\[ \lambda(t(f \otimes g)) = Xtf \otimes \frac{\partial(tg)}{\partial X} + Ytf \otimes \frac{\partial(tg)}{\partial Y}. \]
So it suffices to show that \( \frac{\partial(tg)}{\partial X} = at\frac{\partial g}{\partial X} + bt\frac{\partial g}{\partial Y} \) and that \( \frac{\partial(tg)}{\partial Y} = ct\frac{\partial g}{\partial X} + dt\frac{\partial g}{\partial Y} \).
Without loss of generality, suppose $g$ is a monomial; write $g = X^i Y^j$ (where $i + j = m - 1$). Then $tg = (aX + cY)^i(bX + dY)^j$, and

$$
\frac{\partial (tg)}{\partial X} = \frac{\partial (aX + cY)^i}{\partial X} (bX + dY)^j + (aX + cY)^i \frac{\partial (bX + dY)^j}{\partial X}
$$

$$= ia(aX + cY)^i-1(bX + dY)^j + jb(aX + cY)^i(bX + dY)^j-1
$$

$$= atg + b \frac{\partial g}{\partial X}
$$

and similarly $\frac{\partial (tg)}{\partial Y} = ctg + d \frac{\partial g}{\partial Y}$.

\[\square\]

**Lemma 3.4.** Suppose $n \geq m$ and $2 \leq m \leq p$. Then the map $\lambda$ is surjective.

**Proof.** Let $f = X^i Y^j \in V_{n+1}$, $g = X^i Y^j \in V_{m-1}$ be monomials. We have $i + j' = n + m - 2 \geq 2(m - 1)$, and hence either $i + j \geq m - 1$ or $i' + j' \geq m - 1$. We show that $f \otimes g \in \text{im } \lambda$ by downward induction on $j$ whenever $i + j \geq m - 1$; then by analogy the same holds whenever $i' + j' \geq m - 1$.

Note first that if $i + j \geq m - 1$, then $i \geq 1$ (since $0 \leq j \leq m - 2$) and so $\frac{1}{X}f$ is a polynomial (in $V_n$).

If $j = m - 2$, then $g = X^{m-2}$ so $\frac{\partial (Xg)}{\partial X} = (m-1)X^{m-2}$ and $\frac{\partial g}{\partial Y} = 0$. Then

$$\lambda \left( \frac{1}{X}f \otimes Xg \right) = (m-1)f \otimes g
$$

and $m - 1$ is invertible (since $2 \leq m \leq p$), so $f \otimes g \in \text{im } \lambda$.

Now suppose $0 \leq j < m - 2$. Then

$$\lambda \left( \frac{1}{X}f \otimes Xg \right) = (j + 1)f \otimes g + \frac{Y}{X}f \otimes X \frac{\partial g}{\partial Y}.
$$

But by the inductive hypothesis $\frac{Y}{X}f \otimes X \frac{\partial g}{\partial Y} \in \text{im } \lambda$ (since $X \frac{\partial g}{\partial Y}$ has a higher power of $X$ than $g$, and the sum of the powers of $X$ in $\frac{Y}{X}f$ and $X \frac{\partial g}{\partial Y}$ is $i + j \geq m - 1$). Then since $j + 1$ is invertible, we have $f \otimes g \in \text{im } \lambda$. \[\square\]

**Proposition 3.5.** Suppose $G \leq \text{SL}_2(k)$ and suppose $n \geq m$ and $2 \leq m \leq p$. Then the kernel of $\lambda$ is isomorphic to $V_{n-m+1}$, and hence there is a short exact sequence

$$0 \longrightarrow V_{n-m+1} \longrightarrow V_n \otimes V_m \xrightarrow{\lambda} V_{n+1} \otimes V_{m-1} \longrightarrow 0.
$$

**Proof.** Define $\text{GL}_2(k)$-equivariant variations on the multiplication map by

$$\mu^{(r)} : V_{n_1} \otimes V_{m_1} \otimes \cdots \otimes V_{n_r} \otimes V_{m_r} \rightarrow V_{N-(r-1)} \otimes V_{M-(r-1)}
$$

$$f_1 \otimes g_1 \otimes \cdots \otimes f_r \otimes g_r \mapsto f_1 \cdots f_r \otimes g_1 \cdots g_r
$$

extended $k$-linearly, where $N = \sum_{i=1}^r n_i$ and $M = \sum_{i=1}^r m_i$. Let $g_m \in V_m \otimes V_m$ be the element

$$g_m = \mu^{(m-1)}((X \otimes Y - Y \otimes X) \otimes \cdots \otimes (X \otimes Y - Y \otimes X))
$$

$$= \sum_{i=0}^{m-1} (-1)^{m-1-i} \binom{m-1}{i} X^i Y^{m-1-i} \otimes X^{m-1-i} Y^i.
$$

By the first expression, it is clear that $tg_m = (\det t)^{m-1} g_m$ for any $t \in \text{GL}_2(k)$.\[\square\]
Now define a $k$-linear map $\eta: V_{n-m+1} \to V_n \otimes V_m$ by

$$\eta(f) = \mu(2)(f \otimes 1 \otimes g_m) = \sum_{i=0}^{m-1} (-1)^{m-1-i} \binom{m-1}{i} fX^iY^{m-1-i} \otimes X^{m-1-i}Y^i.$$

Then for any $t \in \text{GL}_2(k)$, we have $t\eta(f) = (\det t)^{m-1}\eta(tf)$, and so $\eta$ is $G$-equivariant. Clearly the expression above is zero if and only if $f = 0$, so $\eta$ is injective. Furthermore,

$$\lambda \eta(f) = \sum_{i=0}^{m-2} (-1)^{m-1-i} \binom{m-1}{i} (m-1-i)fX^{i+1}Y^{m-1-i} \otimes X^{m-2-i}Y^i$$

$$+ \sum_{i=1}^{m-1} (-1)^{m-1-i} \binom{m-1}{i} ifX^iY^{m-i} \otimes X^{m-1-i}Y^{i-1}$$

$$= 0,$$

where the final equality can be seen by replacing $i$ with $i-1$ in the first sum, and noting that $\binom{m-1}{i} = (m-1)\binom{m-2}{i-1} = (m-1)(m-i)$. Thus $V_{n-m+1} \cong \text{im} \eta \cong \ker \lambda$.

Since $n \geq m$ and $2 \leq m \leq p$, by Corollary 3.2 we have that $\lambda$ is surjective, and then by counting dimensions we have $V_{n-m+1} \cong \ker \lambda$. \hfill \Box

**Remark.** Using Corollary 3.2 and comparing the filtrations of $V_n \otimes V_m$ and $V_{n+1} \otimes V_{m-1}$, we see immediately that $[\ker \lambda] = [V_{n-m+1}]$ (when $\lambda$ is surjective). In the case $n-m+1 \leq p$ and $G \geq \text{SL}_2(\mathbb{F}_p)$, we have that $V_{n-m+1}$ is simple, and we could then deduce this proposition immediately without considering $\eta$.

We prove one more isomorphism before we use the short exact sequences to decompose tensor products. This isomorphism, for representations of the semigroup of $2 \times 2$ matrices over $\mathbb{F}_p$, is established in [Glo78, (5.3)].

**Lemma 3.6.** Suppose $\mathbb{F}_q \leq k$ is a finite subfield of order $q$ (where $q$ is a power of $p$) and $G \leq \text{GL}_2(\mathbb{F}_q)$. Then there is an isomorphism $V_n \otimes V_q \cong V_{nq}$.

**Proof.** Let $\psi: V_n \to V_{nq}$ be the map defined by $\psi(f(X,Y)) = f(X^q,Y^q)$. It is $k$-linear (indeed, it is the $k$-linear extension of $X^qY^j \mapsto X^{qj}Y^{qj}$). Then let $\varphi: V_n \otimes V_q \to V_{nq}$ be the map defined by $k$-linear extension of $\varphi(f \otimes g) = \psi(f)g$.

We immediately see that $\varphi$ is surjective: given $X^rY^{nq-1-r} \in V_{nq}$, write $r = iq+j$ with $0 \leq j \leq q-1$, and then $\varphi(X^iY^{nq-1-i} \otimes X^jY^{q-1-j}) = X^rY^{nq-1-r}$. Then $\varphi$ is also injective, since $\dim(V_n \otimes V_q) = nq = \dim(V_{nq})$. To obtain an isomorphism $V_n \otimes V_q \cong V_{nq}$, it remains only to show that $\varphi$ is $G$-equivariant. For this it suffices to show that $\psi$ is $G$-equivariant.
Let $t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Recall that $x^q = x$ for any $x \in \mathbb{F}_q$, and that $(y + z)^q = y^q + z^q$ for any $y, z$ in any ring of characteristic $p$. Then
\[
t\psi(f(X, Y)) = tf(X^q, Y^q) \\
= f((aX + cY)^q, (bX + dY)^q) \\
= f(aX^q + cY^q, bX^q + dY^q) \\
= \psi(f(aX + cY, bX + dY)) \\
= \psi(tf(X, Y))
\]
as required. \hfill \Box

4. Decompositions of tensor products

Let $G = \text{SL}_2(\mathbb{F}_p)$ throughout this section.

**Theorem 4.1** (Clebsch–Gordan rule for $\text{SL}_2(\mathbb{F}_p)$ in characteristic $p$). Suppose $1 \leq m \leq n \leq p$. Then
\[
V_n \otimes V_m \cong \bigoplus_{i \in (n, m) \cap [p]} V_i \oplus \bigoplus_{i \in (n, m) \cap [p]} P_i \oplus [n = m = p] V_p.
\]

**Proof.** We show that if the theorem holds for $(n + 1, m - 1)$ then the short exact sequence involving $\lambda$ splits and hence the theorem holds for $(n, m)$ (where $2 \leq m \leq n \leq p - 1$). We also show, using the short exact sequence involving $\mu$, that if the theorem holds for $(p - 1, m - 1)$ then it holds for $(p, m)$ (where $2 \leq m \leq p$). It then suffices to show the theorem holds for $(n, 1)$ for $1 \leq n \leq p$ (as illustrated in the case $p = 7$ in Figure 1). But these cases are trivial, since $V_n \otimes V_1 \cong V_n$ (and $P_p \cong V_p$).

\[\begin{array}{c c c c c c c c c}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}\]

**Figure 1.** An illustration of how the implications we prove suffice to prove the entire theorem, in the case $p = 7$. The dot in position $(n, m)$ represents the theorem holding for that pair of values, the hollow dots being the trivial cases with $m = 1$; the arrows represent the implications we prove here.
Suppose the theorem holds for $(n+1, m-1)$ (where $2 \leq m \leq n \leq p-1$); that is,

\[ V_{n+1} \otimes V_{m-1} \cong \bigoplus_{i \in (n+1, m-1) \cap [p]} V_i \oplus \bigoplus_{i \in (n+1, m-1) \cap [p]} P_i. \]

Observe that the proposed decomposition of $V_n \otimes V_m$ differs from that of $V_{n+1} \otimes V_{m-1}$ only by an additional summand of $V_{n+1-m}$. Thus, to show the theorem holds for $(n, m)$, it suffices to show that the short exact sequence

\[ 0 \longrightarrow V_{n-m+1} \longrightarrow V_n \otimes V_m \xrightarrow{\lambda} V_{n+1} \otimes V_{m-1} \longrightarrow 0 \]

from Proposition 3.5 splits.

Let $Q \cong \bigoplus_{i \in (n+1, m-1) \cap [p]} P_i$ be the projective part of $V_{n+1} \otimes V_{m-1}$. Then the projection of $\lambda$ onto $Q$ splits, and so there is a module $W$ such that

\[ V_n \otimes V_m \cong W \oplus Q \]

and such that there is a short exact sequence

\[ 0 \longrightarrow V_{n-m+1} \longrightarrow W \longrightarrow \bigoplus_{i \in (n+1, m-1) \cap [p]} V_i \longrightarrow 0. \]

It now suffices to show that this sequence splits. Indeed, suppose, towards a contradiction, the sequence does not split. Then $W$, and hence $V_n \otimes V_m$, has as an indecomposable summand some non-split extension $T$ of $V_{n-m+1}$ by a module with composition factors a nonempty subset of $\{ V_i \mid i \in (n+1, m-1) \cap [p] \}$. This set of composition factors does not contain $V_{n-m+1}$ itself, so $T$ is not self-dual. Furthermore, the dual of $T$ is not a summand of $W$, since $V_{n-m+1}$ occurs only once as a composition factor of $W$, and nor is it a summand of $Q$, since $V_{n-m+1}$ does not occur as the head of any of the projective summands of $Q$. Thus the dual of $T$ is not a summand of $V_n \otimes V_m$, contradicting the self-duality of $V_n \otimes V_m$. So the sequence splits as required.

Now suppose the theorem holds for $(p-1, m-1)$ (where $2 \leq m \leq p$). Then, using that $(p-1, m-1) \cap [p] = ((p, m) \setminus \{ p + m - 1 \}) \cap [p] = (p, m) \cap [p]$, we have

\[ V_{p-1} \otimes V_{m-1} \cong V_{p-m+1} \bigoplus_{i \in (p, m) \cap [p]} P_i. \]

Then by Proposition 3.1 we have a short exact sequence

\[ 0 \longrightarrow V_{p-m+1} \bigoplus_{i \in (p, m) \cap [p]} P_i \longrightarrow V_p \otimes V_m \xrightarrow{\mu} V_{p+m-1} \longrightarrow 0. \]

Thus $\bigoplus_{i \in (p, m) \cap [p]} V_i$ is isomorphic to a submodule of $\text{soc}(V_p \otimes V_m)$. But since $V_p$ is projective, so is $V_p \otimes V_m$ (because the tensor product of a projective module with any other module is projective, as shown in [Alp86, Lemma 4, p. 47]). Thus $\bigoplus_{i \in (p, m) \cap [p]} P_i$ is isomorphic to a submodule of $V_p \otimes V_m$.

We proceed by counting dimensions, recalling that the projective indecomposable $k\text{SL}_2(\mathbb{F}_p)$-modules are $2p$-dimensional, except for $P_1$ and $P_p \cong V_p$ which are $p$-dimensional.
First suppose \( m \neq p \), so that 1 \( \notin \langle p, m \rangle \) and also \( p > 2 \). If \( m \) is even, then \( p \notin \langle p, m \rangle \) and \( \langle p, m \rangle \cap [p] = \emptyset \), so \( \dim(\bigoplus_{i \in \{p, m\} \cap [p]} P_i) = \frac{m}{2} \cdot 2p = mp = \dim(V_p \otimes V_m) \). If \( m \) is odd, then \( p \in \langle p, m \rangle \) and \( \langle p, m \rangle \cap [p] = \frac{m+1}{2} \), so \( \dim(\bigoplus_{i \in \{p, m\} \cap [p]} P_i) = p + \frac{m-1}{2} \cdot 2p = mp = \dim(V_p \otimes V_m) \). Thus, in either case, \( V_p \otimes V_m \cong \bigoplus_{i \in \{p, m\} \cap [p]} P_i \).

Finally suppose \( m = p \). Then 1 \( \in \langle p, p \rangle \), and so in the count above one of the \( 2p \)-dimensional modules is replaced with a \( p \)-dimensional module, which leaves us with \( \dim(\bigoplus_{i \in \{p, p\} \cap [p]} P_i) = \dim(V_p \otimes V_p) - p \) (and if \( p = 2 \) then \( \langle p, p \rangle = \{1, 3\} \) and \( \bigoplus_{i \in \{p, p\} \cap [p]} P_i = P_1 \) is of dimension \( p = 2 = p^2 - p \) as well). Since \( V_p \otimes V_p \) is projective, these \( p \) dimensions must be accounted for by an additional copy of either \( P_1 \) or \( V_p \).

Recall \( V_p \) is self-dual, so \( V_p \otimes V_p \cong \text{Hom}_k(V_p, V_p) \). Now, the direct sum of all trivial submodules of \( \text{Hom}_k(V_p, V_p) \) is \( \text{Hom}_k(V_p, V_p) \), which is isomorphic to \( V_1 \) by Schur’s Lemma. Thus \( V_1 \) occurs in the socle of \( V_p \otimes V_p \) with multiplicity 1, and so the missing summand is \( V_p \).

In the remainder of this section, we use Theorem 4.1 to decompose tensor products of combinations of simple and projective indecomposable \( kG \)-modules.

The remaining combinations all involve at least one projective indecomposable module, and hence the tensor product is projective. It follows from the invertibility of the Cartan matrix that a projective module is uniquely determined by its composition factors; this is useful, as it means to decompose a projective module, it suffices to write its image in the Grothendieck group as a sum of classes of projective modules.

Since the composition factors of the projective indecomposable modules are known (see Section 2), inverting the Cartan matrix gives us a simple method to do this: use our Clebsch–Gordan rule to find all the composition factors of the tensor product, then multiply by the inverse of the Cartan matrix to find the multiplicities of the the projective indecomposable summands.

Nevertheless, in this paper we use a different approach that avoids this computation, and (in most cases) avoids using the structure of the projective indecomposable modules. The trick is to use the result below to pair up classes of (not necessarily simple) modules into classes of projective modules. Such pairings are also made when applying our Clebsch–Gordan rule in the manner described in Example 1.2.

**Corollary 4.2.** Suppose \( 1 \leq n \leq p - 1 \). Then

\[
V_{2p-n} \cong \frac{P_n}{V_n} \oplus [n = 1] V_p.
\]

**Remark.** The structure of the projective indecomposable modules is known (see Section 2), so this corollary gives us the structure of \( V_i \) for \( p + 1 \leq i \leq 2p - 1 \).

**Proof.** Let \( 2 \leq m \leq p \). Via \( \mu \), we have an isomorphism

\[
V_{p+m-1} \cong V_p \otimes V_m \bigg/ V_{p-1} \otimes V_{m-1}.
\]
Then, applying Theorem 4.1 we have

\[
V_{p+m-1} \cong \bigoplus_{i \in (p,m) \cap [p]} P_i \oplus [m = p]V_p
\]

\[
\left( V_{p-m+1} \oplus \bigoplus_{i \in (p,m) \cap [p]} P_i \right)
\]

\[
\cong P_{p-m+1} \bigcap V_{p-m+1} \oplus [m = p]V_p.
\]

Taking \( n = p - m + 1 \) gives the result. \( \square \)

**Corollary 4.3.** Suppose \( 2 \leq n, m \leq p - 1 \) (and in particular \( p > 2 \)). Then:

\[
P_n \otimes V_m \cong \begin{cases} 
\bigoplus_{i \in (n,m)} P_i \oplus \bigoplus_{i \in (n,m)} P_{2p-i} \oplus [n = m]V_p & \text{if } m \leq n, \\
\bigoplus_{i \in (m,n)} P_i \oplus \bigoplus_{i \in (m,n)} P_{2p-i} \oplus P_{p-(m-n)} \oplus \bigoplus_{i \in (p,m-n-1)} P_i & \text{if } m > n.
\end{cases}
\]

**Proof.** We have that \( V_n \otimes V_m \) is isomorphic to a submodule of \( P_n \otimes V_m \). Using Corollary 4.2 for \( 2 \leq n \leq p - 1 \) we have

\[
P_n \otimes V_m / V_n \otimes V_m \cong P_n \otimes V_n \oplus V_{p-n} \otimes V_m.
\]

That is, in the Grothendieck group,

\[
[P_n \otimes V_m] = [V_n \otimes V_m] + [V_{2p-n} \otimes V_m].
\]

Suppose first that \( m \leq n \). Then by Corollary 3.2 and observing that \( \langle 2p - n, m \rangle = 2p - \langle n, m \rangle \), we have

\[
[P_n \otimes V_m] = \sum_{i \in (n,m)} [V_i] + \sum_{i \in (2p-n,m)} [V_i]
\]

\[
= \sum_{i \in (n,m)} [V_i] + [V_{2p-i}].
\]

But Corollary 4.2 tells us that \( [V_i] + [V_{2p-i}] = [P_{\min\{i,2p-i\}}] + [i \in \{1, 2p - 1\}] [V_p] \) for \( 1 \leq i \leq 2p - 1 \) and \( i \neq p \). Thus

\[
[P_n \otimes V_m] = \sum_{i \in (n,m)} [P_i] + \sum_{i \in (n,m)} [P_{2p-i}] + [1 \in (n, m)] [V_p]
\]

\[
= \left[ \bigoplus_{i \in (n,m)} P_i \oplus \bigoplus_{i \in (n,m)} P_{2p-i} \oplus [n = m]V_p \right],
\]

which completes the first case.

Now suppose \( m > n \). As before, we use Corollary 4.2 and Corollary 3.2 and this time we find

\[
[P_n \otimes V_m] = \sum_{i \in (m,n)} [V_i] + \sum_{i \in (2p-n,m)} [V_i]
\]
and we cannot pair up the summands as we did in the case \( m \leq n \). However, we do find that

\[
(2p - n, m) = \{2p - n - m + 1, 2p - n - m + 3, \ldots, 2p - n - m + (2n - 1), 2p - n - m + (2n + 1), \ldots, 2p - n + m - 3, 2p - n + m - 1\}
\]

\[
= \{2p - (m + n - 1), 2p - (m + n - 3), \ldots, 2p - (m - n + 1), 2p - (m - n) + 1, \ldots, 2p + (m - n) - 3, 2p + (m - n) - 1\}
\]

\[
= (2p - (m, n)) \sqcup (2p, m - n).
\]

Thus

\[
[P_n \otimes V_m] = \sum_{i \in (m, n)} (V_i + [V_{2p-i}]) + \sum_{i \in (2p, m-n)} V_i
\]

\[
= [P_m \otimes V_n] + [V_{2p} \otimes V_{m-n}]
\]

\[
= [P_m \otimes V_n] + [P_{p-1} \otimes V_{m-n}],
\]

where the final equality holds because \( V_{2p} \cong V_2 \otimes V_p \) by \( \text{Lemma 3.6} \) and \( V_2 \otimes V_p \cong P_{p-1} \) for \( p > 2 \) by \( \text{Theorem 4.1} \).

We can now use the first case to decompose each of the products in this sum (or, if \( m - n = 1 \), simply using \( P_{p-1} \otimes V_1 \cong P_{p-1} \)). The second product becomes

\[
[P_{p-1} \otimes V_{m-n}] = \sum_{i \in (p-1, m-n)} [P_i] + \sum_{i \in (p-1, m-n)} [P_{2p-i}]
\]

\[
= [P_{p-(m-n)}] + \sum_{i \in (p, m-n-1) \cap [p]} 2[P_i],
\]

as required. \( \square \)

**Corollary 4.4.** Suppose \( 2 \leq m \leq p - 1 \) (and in particular \( p > 2 \)). Then

\[
V_p \otimes P_m \cong \bigoplus_{i \leq p} P_i^{\otimes 2} \oplus \bigoplus_{i \not\leq p} P_i^{\otimes 2} \oplus P_m.
\]

**Proof.** We have

\[
V_p \otimes P_m \big / V_p \otimes V_m \cong V_p \otimes V_{2p-m}.
\]

Now

\[
(2p - m, p) = \{p - m + 1, p - m + 3, \ldots, 3p - m - 1\}
\]

\[
= \langle p, m \rangle \sqcup \{p + m + 1, \ldots, 3p - m - 1\}
\]

\[
= \langle p, m \rangle \sqcup (2p, p - m)
\]

and so \([V_{2p-m} \otimes V_p] = [V_p \otimes V_m] + [V_{2p} \otimes V_{p-m}]\). But \( V_{2p} \cong P_{p-1} \), so we have

\[
V_p \otimes P_m \cong (V_p \otimes V_m)^{\otimes 2} \oplus (P_{p-1} \otimes V_{p-m}).
\]

Using the modular Clebsch–Gordan rule and \( \text{Corollary 4.3} \) gives the decomposition into indecomposable modules. \( \square \)
Corollary 4.5. Suppose $2 \leq m \leq n \leq p - 1$ (and in particular $p > 2$). Then

$$P_n \otimes P_m \cong (P_n \otimes V_m) \oplus (P_m \otimes V_n)$$

$$+ \left\{ \begin{array}{ll}
(P_{p-1} \otimes V_{2p-(n+m)}) & \text{if } n + m \geq p, \\
(P_{p-1} \otimes V_{p+1-(n+m)}) \oplus (P_{n+m} \otimes V_{p-1}) & \text{if } n + m < p.
\end{array} \right.$$  

Proof. We have

$$P_n \otimes P_m \cong P_n \otimes V_m$$

and

$$P_n \otimes V_{2p-m} \cong V_{2p-n} \otimes V_{2p-m},$$

and so

$$[P_n \otimes P_m] = [P_n \otimes V_m] + [V_n \otimes V_{2p-m}] + [V_{2p-n} \otimes V_{2p-m}].$$

Now,

$$\langle 2p - m, 2p - n \rangle = \{ n - m + 1, m - n + 3, \ldots, 4p - n - m - 1 \} = (n, m) \cup \{ n + m + 1, \ldots, 4p - n - m - 1 \} = (n, m) \cup \langle 2p, 2p - (n + m) \rangle.$$  

Thus $[V_{2p-m} \otimes V_{2p-n}] = [V_n \otimes V_m] + [V_{2p} \otimes V_{2p-(n+m)}]$. But $[V_n \otimes V_{2p-m}] + [V_n \otimes V_m] = [V_n \otimes P_m]$ and $V_{2p} \cong P_{p-1}$ (for $p > 2$), so

$$[P_n \otimes P_m] = [P_n \otimes V_m] + [P_m \otimes V_n] + [P_{p-1} \otimes V_{2p-(n+m)}].$$

If $n + m \geq p$, we are done.

If $n + m < p$, and since also $n + m > 1$, we have $V_{2p-(n+m)} \cong P_{n+m} \setminus V_{n+m}$.

Then $[P_{p-1} \otimes V_{2p-(n+m)}] = [P_{p-1} \otimes P_{n+m}] - [P_{p-1} \otimes V_{n+m}]$. We use the first case to decompose

$$P_{p-1} \otimes P_{n+m} \cong (P_{p-1} \otimes V_{n+m}) \oplus (P_{n+m} \otimes V_{p-1}) \oplus (P_{p-1} \otimes V_{p+1-(n+m)}),$$

and so $[P_{p-1} \otimes V_{2p-(n+m)}] = [P_{p-1} \otimes V_{p+1-(n+m)}] + [P_{n+m} \otimes V_{p-1}]$ giving the result. \qed

We have so far avoided using the structure of the projective indecomposable modules, but for the case of tensoring with $P_1$ it is most convenient to make use of our knowledge of their composition factors. As described in Section 2 for $p = 2$ we have $[P_1] = 2[V_1]$ whilst for $p > 2$ we have:

$$[P_1] = 2[V_1] + [V_{p-2}],$$

$$[P_{p-1}] = 2[V_{p-1}] + [V_2],$$

$$[P_n] = 2[V_n] + [V_{p-n-1}] + [V_{p-n+1}] \text{ for } 2 \leq n \leq p - 2.$$  

Proposition 4.6. Suppose $1 \leq n \leq p$. Then

$$P_1 \otimes P_n \cong P_n \otimes P_1 \cong [p > 2] (V_{p-2} \otimes P_n).$$

Proof. Immediate from the structure of $P_1$. \qed

Proposition 4.7. Suppose $1 \leq n \leq p - 1$. Then

$$P_1 \otimes V_n \cong P_n \otimes V_n \cong [n > 2] (V_p \otimes V_{n-2}).$$
Proof. The case \( n = 1 \) is trivial. For the remaining cases, we have \( p > 2 \). Observe that

\[
[P_1 \otimes V_n] = 2[V_n] + [V_{p-2} \otimes V_n].
\]

For \( n = 2 \), we have \( V_{p-2} \otimes V_2 \cong V_{p-3} \oplus V_{p-1} \), and so

\[
[P_1 \otimes V_2] = 2[V_2] + [V_{p-3}] + [V_{p-1}] = [P_2].
\]

Next suppose \( 3 \leq n \leq p-2 \). Then

\[
2[V_n] + [V_{p-2} \otimes V_n] = 2[V_n] + \sum_{i \in (p-2,n)} [V_i] = 2[V_n] + [V_{p-n-1}] + [V_{p-n+1}] = [P_n] + [V_p \otimes V_{n-2}].
\]

Finally, for \( n = p-1 \), we have

\[
2[V_{p-1}] + [V_{p-1} \otimes V_{p-2}] = 2[V_{p-1}] + \sum_{i \in (p-1,p-2)} [V_i] = 2[V_{p-1}] + [V_2] + \sum_{i \in (p,p-3)} [V_i] = [P_{p-1}] + [V_p \otimes V_{p-3}]
\]

as required. \( \square \)

This completes the proof of Theorem 1.1 describing the decomposition of a tensor product of any combination of simple or projective indecomposable \( kG \)-modules.

5. Symmetric and exterior squares

Suppose \( G \leq \text{SL}_2(k) \) and \( p \neq 2 \) throughout this section.

Lemma 5.1. Suppose \( n \geq 1 \). Then \( \text{Sym}^2 V_n \cong \Lambda^2 V_{n+1} \).

Proof. Define a map \( \zeta : \text{Sym}^2 V_n \to \Lambda^2 V_{n+1} \) by \( \zeta(fg) = Xf \wedge Yg - Yf \wedge Xg \) (extended \( k \)-linearly). This is well-defined because \( \zeta(fg) = Xf \wedge Yg - Yf \wedge Xg = Xg \wedge Yf - Yg \wedge Xf = \zeta(gf) \). Furthermore, \( \zeta \) is \( \text{SL}_2(k) \)-equivariant, exactly as the map \( \theta \) was in Proposition 3.1 (with tensors replaced by wedges).

We aim to show \( \zeta \) is surjective; since \( \dim(\text{Sym}^2 V_n) = \binom{n+1}{2} = \dim(\Lambda^2 V_{n+1}) \), this suffices to show \( \zeta \) is an isomorphism.

Let \( e_{i,j} = X^i Y^{n-1-j} \), \( e_{i,j} \in \text{Sym}^2 V_n \), so that \( \{ e_{i,j} \mid 0 \leq i \leq j \leq n-1 \} \) is a linear basis for \( \text{Sym}^2 V_n \). Similarly, let \( f_{i,j} = X^i Y^{n-1-j} \), \( f_{i,j} \in \Lambda^2 V_{n+1} \), so that \( \{ f_{i,j} \mid 0 \leq i < j \leq n \} \) is a linear basis for \( \Lambda^2 V_{n+1} \). Observe that \( \zeta(e_{i,j}) = f_{i+1,j} - f_{i,j+1} \).
Fix $0 \leq i < j \leq n$, and we aim to show $f_{i,j} \in \text{im} \zeta$. Let $l = \min(i, j, n)$; note $l \geq 1$. Then:

\[
\begin{align*}
\zeta(-e_{i-1,i+j-l}) &= -f_{l,i+j-l} + f_{i-1,i+j-l+1} \\
\zeta(-e_{i-2,i+j-l+1}) &= -f_{l-1,i+j-l+1} + f_{i-2,i+j-l+2} \\
\zeta(-e_{i-3,i+j-l+2}) &= -f_{l-2,i+j-l+2} + f_{i-3,i+j-l+3} \\
& \quad \vdots \\
\zeta(-e_{i+1,j-2}) &= -f_{i+2,j-2} + f_{i+1,j-1} \\
\zeta(-e_{i,j-1}) &= -f_{i+1,j-1} + f_{i,j} \\
\zeta(e_{i-1,j}) &= f_{i,j} - f_{i-1,j+1} \\
\zeta(e_{i-2,j+1}) &= f_{i-1,j+1} - f_{i-2,j+2} \\
& \quad \vdots \\
\zeta(e_{i+j-l+2,l-3}) &= f_{i+j-l+3,l-3} - f_{i+j-l+2,l-2} \\
\zeta(e_{i+j-l+1,l-2}) &= f_{i+j-l+2,l-2} - f_{i+j-l+1,l-1} \\
\zeta(e_{i+j-l,l-1}) &= f_{i+j-l+1,l-1} - f_{i+j-l,l} 
\end{align*}
\]

Summing all these expressions together, we have that $2f_{i,j} - f_{i,i+j-l} - f_{i+j-l,i} \in \text{im} \zeta$. But $f_{r,s} = -f_{s,r}$ for any $r, s$, and since $2$ is invertible when $p \neq 2$, we have $f_{i,j} \in \text{im} \zeta$. \qed

Recall that $\text{Sym}^2 U \oplus \bigwedge^2 U \cong U \otimes U$ for any module $U$. Then, by Corollary 3.2

\[
[\text{Sym}^2 V_n \oplus \bigwedge^2 V_n] = [V_n \otimes V_n] = \sum_{i \in (n, n)} [V_i] = \sum_{i \equiv 1 (\text{mod } 2)} [V_i].
\]

**Proposition 5.2.** Suppose $G = \text{SL}_2(\mathbb{F}_p)$ and $1 \leq n \leq p$. Then:

\[
\text{Sym}^2 V_n \cong \bigoplus_{i \in (n,n) \cap [p]} V_i \oplus \bigoplus_{i \equiv 2n-1 (\text{mod } 4)} \bigoplus_{i \equiv 2n-1 (\text{mod } 4)} P_i.
\]

**Proof.** We use induction on $n$. The case $n = 1$ is immediate: $\text{Sym}^2 V_1 \cong V_1$.

Suppose the proposition holds for $1 \leq n \leq p - 1$. Then using Lemma 5.1 we have

\[
\bigwedge^2 V_{n+1} \cong \text{Sym}^2 V_n \cong \bigoplus_{i \in (n,n) \cap [p]} V_i \oplus \bigoplus_{i \equiv 2n-1 (\text{mod } 4)} P_i.
\]

Observe that if $i \in \{2n+1, 2p - (2n + 1)\}$ then $i \not\equiv 2n - 1 (\text{mod } 4)$, and so neither $V_i$ nor $P_i$ appear in the above sum. Thus:

\[
\bigwedge^2 V_{n+1} \cong \bigoplus_{i \equiv (n+1,n+1) \cap [p]} V_i \oplus \bigoplus_{i \equiv 2n-1 (\text{mod } 4)} P_i.
\]
Then

\[
\text{Sym}^2 V_{n+1} \cong V_{n+1} \otimes V_{n+1} \bigg/ \bigoplus_{i \in (n+1,n+1) \cap [p]} V_i \bigoplus_{i \in (n+1,n+1) \cap [p]} P_i \bigoplus_{i \equiv 2n+1 \ (\text{mod} \ 4)} P_i \bigoplus_{i \equiv 2n-1 \ (\text{mod} \ 4)} [n+1 = p] V_p
\]

as required. \(\square\)

**Remark.** For \(n > p\), and for \(\text{SL}_2(\mathbb{F}_p) < G \leq \text{SL}_2(k)\), there may no longer be an isomorphism here, but by a similar inductive proof there is equality in the Grothendieck group:

\[
\text{[Sym}^2 V_n] = \sum_{i \equiv 2n-1 \ (\text{mod} \ 4)} [V_i].
\]

### 6. Tables of Multiplicities

Let \(G = \text{SL}_2(\mathbb{F}_p)\) and \(p \neq 2\) throughout this section.

We examine the table of multiplicities of simple modules as indecomposable summands of tensor products of simple modules, as well as the graph which has this table as its adjacency matrix. This table has symmetries that reveal properties of the tensor products of representations of \(G\). Furthermore, the Markov chain defined in the following section is shown to be a walk on this graph, so our observations here aid our understanding of that Markov chain. We use \([:\] to denote multiplicity as an indecomposable summand.

**Definition.** For \(n \in [p-1]\), let \(A^{(n)}\) be the matrix with entries \(A^{(n)}_{i,j} = [V_i \otimes V_n : V_j]\). Let \(G^{(n)}\) be the (directed) graph (with loops) whose adjacency matrix is \(A^{(n)}\). The parameter \(n\) is suppressed unless there is need to emphasise it.

The matrix \(A\) is depicted in [Figure 2]. It is visually apparent that \(A\) is symmetric; this motivates our next result.

**Lemma 6.1.** Suppose \(1 \leq i, j, l \leq p-1\). The following are equivalent:

1. \(V_i\) is a summand of \(V_i \otimes V_j\);
2. \(V_i\) is a summand of \(V_j \otimes V_i\);
3. \(V_j\) is a summand of \(V_i \otimes V_i\);
4. \(i + j + l \equiv 1 \ (\text{mod} \ 2)\), \(i + j + l < 2p\), and \(l < i + j\), \(i < j + l\) and \(j < l + i\).

In particular, \(A\) is a symmetric matrix.

**Proof.** Observe that \([\text{iv}]\) is symmetric in \(i, j\) and \(l\), and so it suffices to show that \([\text{i}]\) and \([\text{iv}]\) are equivalent. Indeed, Theorem 4.1 tells us that \([\text{i}]\) holds if and only if \(l \equiv i + j - 1 \ (\text{mod} \ 2)\) and \(\max\{i-j,j-i\} < l < \min\{i+j,2p-(i+j)\}\), which easily rearranges to \([\text{iv}]\) \(\square\)
Thus $\mathcal{G}$ can be viewed as an undirected graph (with loops); we do so from now on. Some small examples of $\mathcal{G}$ are depicted in Figure 3.

Figure 3. The graphs $\mathcal{G}^{(n)}$, for $p = 7$ and $2 \leq n \leq p - 2$. 
There is another visually apparent symmetry of the adjacency matrix $A$: it is invariant under rotation by 180 degrees. We give various interpretations of this fact in Proposition 6.2. To give these interpretations, we make the following definitions.

**Definition.** Let $T$ be the $(p - 1) \times (p - 1)$ matrix defined by $T_{i,j} = [i + j = p]$.

That is, $T$ is the matrix with 1s on the antidiagonal:

$$T = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$ 

It is the basis-change matrix for reversing the order of the basis, and is self-inverse. Also:

- left-multiplying by $T$ reflects a matrix in the horizontal midline;
- right-multiplying by $T$ reflects a matrix in the vertical midline;
- conjugating by $T$ rotates a matrix by 180 degrees.

**Definition.** Let $\Omega^0 (-)$ denote the projective-free part of a module.

**Definition.** Let $\mathfrak{p}$ the subgroup of the Grothendieck group $G_0(kG)$ generated by classes of projective modules.

Note that $G_0(kG)$ can be made into a (commutative) ring via tensoring, and that $\mathfrak{p}$ is an ideal of this ring. Recall that a quotient ring is naturally a (left) module for the original ring by (left) multiplication.

**Proposition 6.2.** The following statements hold:

(a) $V_l$ is a summand of $V_i \otimes V_j$ if and only if $V_l$ is a summand of $V_{p-i} \otimes V_{p-j}$, for all $1 \leq i, j, l \leq p - 1$;

(b) $A^{(n)} = TA(p-n) = A(p-n)T$;

(c) $TAT = A$;

(d) the map $i \mapsto p - i$ is a graph automorphism of $G$;

(e) $\Omega^0 (V_i \otimes V_j) \cong \Omega^0 (V_{p-i} \otimes V_{p-j})$ for all $1 \leq i, j \leq p - 1$;

(f) $[V_i \otimes V_j] + \mathfrak{p} = [V_{p-i} \otimes V_{p-j}] + \mathfrak{p}$ for all $1 \leq i, j \leq p - 1$;

(g) the $k$-linear automorphism $\rho$ of $G_0(kG) / \mathfrak{p}$ defined by $\rho: [V_i] + \mathfrak{p} \mapsto [V_{p-i}] + \mathfrak{p}$ is $G_0(kG)$-equivariant.

**Proof.** Statement (a) and the first equality in (b) are equivalent, and the second equality in (b) follows from the first since $A$ and $T$ are symmetric. The statements (c) and (d) are equivalent, and are implied by (b). Given that the projective-free parts of the tensor products of simple modules are multiplicity-free sums of simple modules, the statements (a), (e) and (f) are equivalent.

To see that (g) follows from (b) let $A \subseteq [p - 1]$ be such that $\Omega^0 (V_j \otimes V_i) \cong \bigoplus_{l \in A} V_l$. Then, by the second equality in (b) we have $\Omega^0 (V_j \otimes V_{p-i}) \cong \bigoplus_{l \in A} V_{p-l}$; thus

$$\rho([V_j \otimes V_i] + \mathfrak{p}) = \rho \left( \sum_{l \in A} [V_l] + \mathfrak{p} \right)$$

$$= \sum_{l \in A} [V_{p-l}] + \mathfrak{p}$$

$$= [V_j \otimes V_{p-i}] + \mathfrak{p}.$$
Thus it suffices to show (a) holds. Indeed, condition (iv) of Lemma 6.1 is invariant under taking both $i \mapsto p - i$ and $j \mapsto p - j$. □

Remark. Because the automorphism in (d) swaps the parity of each vertex, the induced subgraph on even vertices is isomorphic to the induced subgraph on odd vertices (via the isomorphism $i \mapsto p - i$).

We next observe that a certain submatrix of $A$ contains all the information of $A$, and use the resulting simplification of the structure of $A$ to identify the connected components of $G$.

Definition. Let $\bar{A}^{(n)}$ be the $\frac{p-1}{2} \times \frac{p-1}{2}$ submatrix of (a conjugate of) $A$ defined by

$$
\bar{A}^{(n)}_{i,j} = \begin{cases} 
A_{2i-1,2j-1}^{(n)} & \text{if } n \text{ is odd;} \\
A_{2i-1,p+1-2j}^{(n)} & \text{if } n \text{ is even.}
\end{cases}
$$

That is, if the vertices are reordered to $1, 3, \ldots, p - 2, p - 1, p - 3, \ldots, 4, 2$ (the odd integers followed by the even integers, with the former in ascending order and the latter in descending order), then $\bar{A}$ is the upper-left block of $A$ when $n$ is odd and is the upper-right block of $A$ when $n$ is even.

Lemma 6.3. The matrix $\bar{A}$ has the following properties:

(a) under the ordering $1, 3 \ldots p - 2, p - 1, \ldots, 4, 2$, the matrix $A$ is of the form

$$
A = \begin{cases} 
(\bar{A} & *) \\
(\ast & \bar{A}) \\
(\bar{A} & \ast) & \text{if } n \text{ is odd,} \\
(*) & (*) & \text{if } n \text{ is even,}
\end{cases}
$$

where $*$ denotes an unspecified matrix;

(b) $\bar{A}^{(n)} = 1$ if and only if $2|i - j| < r < 2(i + j - 1) < 2p - r$, where $r = n$ if $n$ is odd and $r = p - n$ if $n$ is even.

(c) $\bar{A}^{(p-n)} = \bar{A}^{(n)}$;

(d) $\bar{A}$ is symmetric;

(e) for $1 < n < p - 1$, the graph with adjacency matrix $\bar{A}$ is connected.

Proof. By Proposition 6.2(c) we have $A_{2i-1,2j-1} = A_{p+1-2i,p+1-2j}$, and so (under the new ordering) the upper-left and lower-right blocks of $A$ are the same. Similarly the upper-right and lower-left blocks are the same, and (a) follows.

The condition for $A_{i,j}$ to be nonzero is obtained from condition (iv) of Lemma 6.1 with the appropriate values of $i$ and $j$ substituted. Properties (c) and (d) are easily verified using this condition.

It follows from (b) that $\bar{A}$ has nonzero entries precisely in a rectangle bounded by the straight lines determined by these inequalities; we draw matrix $\bar{A}$ in Figure 4.

The connectedness of its graph is then clear provided $r \neq 1$. □

Lemma 6.4.

(a) If $n$ is odd, then $G$ is disconnected, with each connected component a subset of either the odd integers or the even integers.

(b) If $n$ is even, then $G$ is bipartite, with classes the odd integers and the even integers.
Figure 4. The matrix $\tilde{A}$ (here with $r < p - r$), where $r = n$ if $n$ is odd and $r = p - n$ if $n$ is even.

(c) When the vertices are ordered as $1, 3, \ldots, p - 2, p - 1, p - 3, \ldots, 4, 2$, we have

$$A = \tilde{A} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{n+1}.$$  

Proof. Let $1 \leq i \leq p - 1$. Observe that the neighbours of $i$ are all elements of $\langle i, n \rangle$ or $\langle n, i \rangle$ (according to whether $i \geq n$ or $i \leq n$). Furthermore, elements of these strings are all of the same parity, which is the parity of $i + n - 1$. Thus if $n$ is odd, the neighbours of $i$ are of the same parity as $i$, whilst if $n$ is even, the neighbours of $i$ are of different parity to $i$. The statements (a) and (b) are then immediate.

That is, under the new ordering, when $n$ is even the diagonal blocks of $A$ are zero, and when $n$ is odd the off-diagonal blocks are zero. The expression as a Kronecker product then follows from Lemma 6.3(a).

Proposition 6.5.

(a) If $n$ is odd and $n > 1$, then $\mathcal{G}$ has precisely two connected components, the odd integers and the even integers, and they are isomorphic.

(b) If $n$ is even and $n < p - 1$, then $\mathcal{G}$ is connected.

Proof. For $n$ odd, $\tilde{A}$ is the adjacency matrix for the subgraphs of $\mathcal{G}$ on odd vertices and on even vertices, so (a) follows immediately from Lemma 6.3(e).
For $n$ even, $\tilde{A}$ is the adjacency matrix for the quotient graph of $G$ with $i$ and $p - i$ identified. Again using [Lemma 6.3(e)]\(^\dagger\), since $G$ is bipartite (with each of $i$ and $p - i$ in a distinct class), to show (b) it suffices to show that $i$ is reachable from $p - i$ for some $i$. Indeed, $\tilde{A}$ has a nonzero diagonal entry (at $\frac{p - 1}{2}$), and so the two vertices identified to form the corresponding vertex of the quotient are adjacent. \hfill \Box

We conclude this section by finding the degrees of the vertices in $G$. The degree of $i$ in $G$ is also the number of nonzero entries in the $i$th row of $A$, and is the number of non-projective indecomposable summands of $V_i \otimes V_n$.

**Definition.** For $1 \leq i \leq p - 1$, let $d(i)$ be the degree of $i$ in $G$ (where a loop is considered to contribute 1 to the degree). The dependence of $d$ on $n$ is suppressed, since it is always clear from context.

**Lemma 6.6.** For $1 \leq i \leq p - 1$, we have

$$d(i) = \min\{i, p - i, n, p - n\}.$$  

Furthermore,

$$\sum_{i=1}^{p-1} d(i) = n(p - n).$$

**Proof.** Clearly $d(i)$ is symmetric in $i$ and $n$, so for the first equality it suffices to show that $d(i) = \min\{i, p - n\}$ when $i \leq n$. By [Theorem 4.1], the number of simple non-projective summands of $V_n \otimes V_i$ is the number of elements $j$ of $\langle n, i \rangle$ for which $2p - j / \notin \langle n, i \rangle$.

If $i + n - 1 < p$ (equivalently, $i \leq p - n$) then this is all the elements of $\langle n, i \rangle$, of which there are $i$.

If $i + n - 1 \geq p$ (equivalently, $i > p - n$), then the number of $j \in \langle n, i \rangle$ such that $2p - j \notin \langle n, i \rangle$ is

$$2 \left\lfloor \frac{i + n - 1 - p}{2} \right\rfloor + [i + n - 1 \text{ is odd}] = i + n - p,$$

and so $d(i) = i - (i + n - p) = p - n$.

We now find the sum of the $d(i)$. Let $m = \min\{n, p - n\}$. We have:

$$\sum_{i=1}^{p-1} d(i) = \sum_{i=1}^{p-1} \min\{i, p - i, n, p - n\}$$

$$= 2 \sum_{i=1}^{p-1} \min\{i, m\}$$

$$= 2 \left( \frac{p - 1}{2} - m \right) m + 2 \sum_{i=1}^{m} i$$

$$= m(p - 1 - 2m) + m(m + 1)$$

$$= m(p - m)$$

$$= n(p - n)$$ \hfill \Box
7. Random walks on indecomposable modules

Let $G = \text{SL}_2(\mathbb{F}_p)$ and $p \neq 2$ throughout this section.

We investigate the long-run behaviour of tensoring by a fixed simple $kG$-module by considering the Markov chain defined below. In particular, we assess the properties of reversibility, diagonalisability, irreducibility and periodicity, as well as calculating stationary distributions.

**Definition** (Non-projective summand random walk). Fix $n \in [p-1]$, $w$ a function that assigns a positive weight to each non-projective indecomposable $kG$-module, and $\nu$ a distribution on the non-projective simple $kG$-modules. Let the non-projective summand random walk be the (discrete time) Markov chain on the set of non-projective indecomposable $kG$-modules with initial distribution $\nu$ in which the probability of a step from $U$ to $V$ is

$$Q_{UV}^{(n)} = \frac{w(V)\left[U \otimes V_{n} : V\right]}{\sum_W w(W)\left[U \otimes V_{n} : W\right]},$$

where the sum is over all non-projective indecomposable modules $W$ (and $[ : ]$ denotes multiplicity as an indecomposable summand, as in Section 6). The parameter $n$ is suppressed unless there is need to emphasise it.

**Remarks.**

(i) If $U$ is a simple non-projective $kG$-module, Theorem 4.1 implies that $U \otimes V_{n}$ indeed has non-projective indecomposable summands, and that these summands are simple. Thus the chain is well-defined and remains on simple non-projective $kG$-modules throughout. The states of the chain can therefore be labelled with the dimensions of the modules, taking values in the finite set $[p-1]$.

(ii) Theorem 4.1 also implies that the non-projective part of a tensor product of simple modules is multiplicity-free, so $[U \otimes V_{n} : W] \in \{0, 1\}$ for all $W$.

(iii) If we were to allow steps to projective indecomposable modules, these modules would form an absorbing set (in the sense that once the chain hit a projective module it would stay on projective modules for all time). This definition allows us to consider a recurrent chain on the (non-projective) simple modules.

(iv) There are two trivial cases to be excluded: if $n = 1$, we never step away from the initial state; if $n = p - 1$, then $V_{p-i}$ is the unique non-projective indecomposable summand of $V_{i} \otimes V_{p-1}$, so at each step we switch between the initial state $i$ and $p - i$. From now on we assume $2 \leq n \leq p - 2$.

An illustrative example of our chain is given below. Note that when $w \equiv 1$, the summands are chosen uniformly at random; this case, and the case where $w(i) = i$ (in which modules are weighted by their dimension), are described for general $n$ at the end of this section.

**Example 7.1.** Suppose $w \equiv 1$ and $n = 2$. We have that

$$V_{i} \otimes V_{2} \cong \begin{cases} V_{2} & \text{if } i = 1, \\ V_{i-1} \oplus V_{i+1} & \text{if } 2 \leq i \leq p - 2, \\ V_{p-2} \oplus P_{p} & \text{if } i = p - 1. \end{cases}$$
Thus the non-projective summand random walk is a symmetric random walk in one dimension with reflecting boundaries. The transition matrix is

$$
\begin{pmatrix}
\frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
$$

and the stationary distribution is $\frac{1}{2(\rho-2)}(1, 2, 2, \ldots, 2, 1)$.

Our key observation while studying the non-projective summand random walk is that it is the random walk on the graph $\mathcal{G}$ (defined in Section 6) in which the probability of moving from a vertex $i$ to a neighbour $j$ is proportional to $w(j)$. Indeed, the transition matrix $Q$ has nonzero entries precisely where $A$ (the adjacency matrix for $\mathcal{G}$) does, and in both cases the transition probabilities are proportional to the weight of the destination. That is,

$$Q_{i,j} = \frac{w(j)}{\sum_{l \in E(\mathcal{G})} w(l)} A_{i,j}.$$

We use the properties of $\mathcal{G}$ given in Section 6 to shed light on the non-projective summand random walk. The first relevant property of $\mathcal{G}$ is that it is undirected, which implies that the communicating classes of our Markov chain are all closed (that is, they are irreducible chains themselves) and they are precisely the connected components of $\mathcal{G}$. Moreover, by the following lemma, it implies the chain is reversible and diagonalisable, and we are able to find a stationary distribution.

**Lemma 7.2.** Let $\mathcal{H}$ be any finite graph (with loops) and $u$ a function assigning a positive weight to each vertex of $\mathcal{H}$. Let $R$ be the transition matrix for the random walk on $\mathcal{H}$ defined by

$$R_{i,j} = \frac{u(j)}{\sum_{l \in E(\mathcal{H})} u(l)} \cdot [ij \in E(\mathcal{H})].$$

Let $\pi$ be the distribution defined by

$$\pi_i = \frac{u(i) \sum_{l \in E(\mathcal{H})} u(l)}{C},$$

where $C = \sum_{x \in V(\mathcal{H})} \sum_{xy \in E(\mathcal{H})} u(y)$.

Then $\pi$ is a stationary distribution in detailed balance with $R$, and the random walk is reversible and diagonalisable.

**Proof.** It suffices to verify the detailed balance equations for $\pi$ (noting that diagonalisability follows from reversibility [PR13, Section 2.4]). Observe:

$$\pi_i R_{i,j} = \frac{u(j)}{\sum_{l \in E(\mathcal{H})} u(l)} \cdot \frac{u(i) \sum_{l \in E(\mathcal{H})} u(l)}{C} \cdot [ij \in E(\mathcal{H})]
= \frac{u(i) u(j)}{C} \cdot [ij \in E(\mathcal{H})]
= \pi_j R_{j,i}.$$

$\square$
Next, we make use of our results about the connectedness and periodicity of $G$.

**Proposition 7.3.**

(a) If $n$ is odd, then the non-projective summand random walk is reducible into two chains, one on the even states and one on the odd states, which are each irreducible and aperiodic.

(b) If $n$ is even, then the non-projective summand random walk is irreducible and periodic with period 2.

**Proof.** The description of the irreducible components follows immediately from the description of the connected components of $G$ in [Proposition 6.5](#).

A walk on an undirected graph necessarily has period at most 2 (since any vertex can be revisited after two steps). The walk has period equal to 2 if and only if the graph contains no odd cycles and no loops, which is if and only if the graph is bipartite—and the walk is aperiodic otherwise. Thus the periodicity claims follow from [Lemma 6.4(b)](#) and the observation that when $n$ is odd, each component of $G$ has loops (at $\frac{p-1}{2}$ and $\frac{p+1}{2}$).

**Remark.** Thus for $n$ even, the chain has a unique stationary distribution but it does not necessarily converge to it. Meanwhile, for $n$ odd, each subchain has a unique stationary distribution which it converges to, and the stationary distributions of the entire chain are precisely the convex combinations of these distributions.

If $w$ satisfies $w(i) = w(p - i)$ for all $i$, then $Q$ has the same rotational symmetry as $A$, and several of the results from [Section 6](#) carry over. Some of these results are helpful for identifying the remaining eigenvalues of $Q$; the rate of convergence to equilibrium is determined by the second-largest (in absolute value) eigenvalue, so this in turn is helpful for finding the mixing time for the Markov chain.

Let $\bar{Q}$ be the submatrix of (a conjugate of) $Q$ defined analogously to $\bar{A}$.

**Proposition 7.4.** Suppose $w(i) = w(p - i)$ for all $i$. Then:

(a) $Q^{(n)} = TQ^{(p-n)} = Q^{(p-n)}T$;

(b) $TQT = Q$;

(c) the non-projective summand random walk is invariant under the relabelling $i \mapsto p - i$;

(d) if $n$ is odd, the two irreducible subchains are isomorphic;

(e) $\bar{Q}^{(p-n)} = \bar{Q}^{(n)}$;

(f) with the vertices ordered as $1, 3, \ldots, p - 2, p - 1, p - 3, \ldots, 4, 2$, we have

$$Q = Q \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{n+1};$$

(g) if $n$ is odd, every eigenvalue of $Q$ has even multiplicity; if $n$ is even, the eigenvalues of $Q$ come in signed pairs;

(h) the non-projective summand random walk has mixing time

$$t_{\text{mix}}(\varepsilon) = \frac{1}{1 - \lambda_*} \log \left( \frac{1}{\varepsilon \min_i (\pi_i)} \right)$$

where $\lambda_* = \max\{ |\lambda| \mid \lambda \neq 1 \text{ is an eigenvalue of } \bar{Q} \}$.

**Proof.** Statements (a)–(f) are entirely analogous to results in [Section 6](#) using $w(i) = w(p - i)$ to deduce that the entries in the desired places of $Q$ are not only nonzero but also equal.
Once we have the Kronecker product expression in (f), we see immediately that if $\bar{Q}$ has eigenvector-eigenvalue pairs \{$(v_1, \lambda_1), \ldots, (v_{p-1}, \lambda_{p-1})$\}, then $Q$ has eigenvector-eigenvalue pairs

\[
\{(v_i \otimes (\frac{1}{2}), \lambda_i) \mid 1 \leq i \leq \frac{p-1}{2}\} \cup \{(v_i \otimes (\frac{0}{1}), \lambda_i) \mid 1 \leq i \leq \frac{p-1}{2}\}
\] if $n$ is odd; 

\[
\{(v_i \otimes (\frac{1}{1}), \lambda_i) \mid 1 \leq i \leq \frac{p-1}{2}\} \cup \{(v_i \otimes (\frac{1}{-1}), -\lambda_i) \mid 1 \leq i \leq \frac{p-1}{2}\}
\] if $n$ is even.

Both parts of (g) then follow.

Note that $\bar{Q}$ is the transition matrix for an irreducible aperiodic chain, so all its eigenvalues lie in $(-1, 1]$ and the eigenvalue 1 has multiplicity 1; therefore $\lambda^\star < 1$ and $\lambda^\star$ is the second-largest (absolute value of an) eigenvalue of $\bar{Q}$.

If $n$ is odd, $\lambda^\star$ is therefore the second-largest (absolute value of an) eigenvalue for each irreducible component of the chain. If $n$ is even, in order to eliminate periodicity, we define the lazy chain with transition matrix $\frac{1}{2}(Q + I)$ (which converges at half the rate of the original chain); since the eigenvalues of $\bar{Q}$ come in signed pairs, the lazy chain has second-largest eigenvalue $\frac{\lambda^\star + 1}{2}$. Then the value for the mixing time follows from [LP17, Theorem 12.4, p. 163] (halving the mixing time of the lazy chain when $n$ is even).

In fact, for $n$ even, the eigenvalues still come in signed pairs, regardless of the weighting: it is always the case that $Q$ has nonzero entries only in the off-diagonal $\frac{p-1}{2} \times \frac{p-1}{2}$ blocks, and if $(u, v)$ is an eigenvector with eigenvalue $\lambda$ for such a matrix, then $(u, -v)$ is an eigenvector with eigenvalue $-\lambda$. However, in general, there is not a simple relation between these off-diagonal blocks, or to the blocks of the transition matrix with $p-n$ in place of $n$.

We conclude by exhibiting our results in the cases $w \equiv 1$ and $w(i) = i$. Recall from Section 6 that $d(i)$ is the degree of $i$ in $\mathcal{G}$.

**Example 7.5.** Let $w \equiv 1$. Then 

\[Q_{i,j} = \frac{A_{i,j}}{d(i)}.\]

This transition matrix is shown explicitly in Figure 5. Of course, $w(i) = w(p-i)$, and so $Q$ satisfies $TQT = Q$, and for $n$ odd the two irreducible subchains are isomorphic.

By Lemma 6.6 and Lemma 7.2, a stationary distribution is

\[\pi_i = \frac{\min\{i, p-i, n, p-n\}}{n(p-n)}.\]

Observe that $\pi T = \pi$. In particular, this stationary distribution assigns equal probability to being on an even or an odd state; that is,

\[\sum_{i \equiv 0 \pmod{2}} \pi_i = \sum_{i \equiv 1 \pmod{2}} \pi_i = \frac{1}{2}.\]

Thus, for $n$ even, the chain converges to the stationary distribution, provided that the initial distribution $\nu$ has equal weighting for even and odd states or that the chain is made lazy by taking the transition matrix to be $\frac{1}{2}(Q + I)$. Meanwhile, for $n$ odd, $\pi$ is the stationary distribution with equal weighting given to the even-state and odd-state walks.
If $n \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$, it can be shown that the eigenvalues of $\bar{Q}$ are
\[ \{1, -\frac{1}{2}, \frac{1}{2}, \ldots, (-1)^{\frac{n+1}{2}} \frac{2}{p-1}\}. \]
Then (by the proof of Proposition 7.4(g)) the eigenvalues of $Q$ are the eigenvalues in this set each with multiplicity 2 if $n$ is odd, and are $\{\pm 1, \pm \frac{1}{2}, \ldots, \pm \frac{2}{p-1}\}$ if $n$ is even. Then by Proposition 7.4(h) the mixing time of the walk is
\[ t_{\text{mix}}(\varepsilon) = 2 \log \left( \frac{p^2 - 1}{4\varepsilon} \right). \]

**Example 7.6.** Suppose $w(i) = i$ for each $i$; that is, each module has a chance of being chosen proportional to its dimension. Then for fixed $i$ we have
\[ \sum_{ij \in E(G)} j = (\text{number of neighbours of } i) \times (\text{average value of the neighbours of } i) \]
\[ = d(i) \times \text{mean}\{j \mid V_j \text{ is a summand of } V_i \otimes V_n\}. \]
If $i + n \leq p$, all of the composition factors of $V_i \otimes V_n$ are summands, and so their average dimension is $\max\{i, n\}$, the midpoint of the $(i, n)$-string or the $(n, i)$-string (as appropriate). If $i + n \geq p$, the midpoint of the relevant section of the string is instead
\[ \frac{(i - n + 1) + (2p - (i + n - 1))}{2} = p - \min\{i, n\}. \]
Also, by Lemma 6.6
\[ d(i) = \begin{cases} \min\{i, n\} & \text{if } i + n \leq p, \\ p - \max\{i, n\} & \text{if } i + n \geq p. \end{cases} \]
Thus
\[ \sum_{ij \in E(G)} j = \begin{cases} d(i) \max\{i, n\} & \text{if } i + n \leq p, \\ d(i)(p - \min\{i, n\}) & \text{if } i + n \geq p. \end{cases} \]
\[ = \begin{cases} in & \text{if } i + n \leq p, \\ (p - i)(p - n) & \text{if } i + n \geq p. \end{cases} \]
Then
\[ Q_{i,j} = \begin{cases} \frac{1}{n} & \text{if } i + n \leq p \text{ and } A_{i,j} \neq 0, \\ \frac{j}{(p-i)(p-n)} & \text{if } i + n \geq p \text{ and } A_{i,j} \neq 0, \\ 0 & \text{otherwise}. \end{cases} \]
It can be shown that $\sum_{i \in [p-1]} \sum_{ij \in E(G)} j = \frac{1}{6}np(p - n)(2p - n)$. Then by Lemma 7.2 a stationary distribution is
\[ \pi_i = \begin{cases} \frac{6i^2}{p(p - n)(2p - n)} & \text{if } i + n \leq p, \\ \frac{6i(p - i)}{np(2p - n)} & \text{if } i + n \geq p. \end{cases} \]
Now $w(i) \neq w(p - i)$ (for all $i$), and so we do not have that the walk is invariant under the map $i \mapsto p - i$. In particular, the two irreducible chains when $n$ is odd are not isomorphic.
Figure 5. The transition matrix $Q$ when $w \equiv 1$, in the cases $2n < p$, top, and $2n > p$, bottom.
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References

[Alp86] J. L. Alperin. *Local Representation Theory: Modular Representations as an Introduction to the Local Representation Theory of Finite Groups*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1986.

[BDLT18] Georgia Benkart, Persi Diaconis, Martin W Liebeck, and Pham Huu Tiep. “Tensor Product Markov Chains”. In: *arXiv preprint arXiv:1810.00409* (Sept. 2018).

[EH02] Karin Erdmann and Anne Henke. “On Ringel duality for Schur algebras”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 132.1 (2002), pp. 97–116.

[Glo78] D.J Glover. “A study of certain modular representations”. In: *Journal of Algebra* 51.2 (1978), pp. 425–475.

[Hal15] Brian Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Vol. 222. Graduate Texts in Mathematics. Springer International Publishing, 2015.

[Kou90a] Frank M. Kouwenhoven. “The \(\lambda\)-structure of the green ring of GL(2, \(F_p\)) in characteristic P. I”. In: *Communications in Algebra* 18.6 (1990), pp. 1645–1671.

[Kou90b] Frank M. Kouwenhoven. “The \(\lambda\)-structure of the green ring of GL(2, \(F_p\)) in characteristic P. II”. In: *Communications in Algebra* 18.6 (1990), pp. 1673–1700.

[LP17] D.A. Levin and Y. Peres. *Markov Chains and Mixing Times*. American Mathematical Society, 2017.

[PR13] Giovanni Pistone and Maria Piera Rogantin. “The algebra of reversible Markov chains”. In: *Annals of the Institute of Statistical Mathematics* 65.2 (2013), pp. 269–293.