Canonical Nonlinear Connections in the Multi-Time Hamilton Geometry

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Abstract

In this paper we study some geometrical objects (d-tensors, multi-time semisprays of polymomenta and nonlinear connections) on the dual 1-jet vector bundle $J^1(T, M) \to T \times M$. Some geometrical formulas, which connect the last two geometrical objects, are also derived. Finally, a canonical nonlinear connection produced by a Kronecker $h$-regular multi-time Hamiltonian is given.

Mathematics Subject Classification (2000): 53B40, 53C60, 53C07.

Key words and phrases: dual 1-jet vector bundle, d-tensors, multi-time semisprays of polymomenta, Kronecker $h$-regular multi-time Hamiltonians, canonical nonlinear connections.

1 Introduction

From a geometrical point of view, we point out that the 1-jet spaces are fundamental ambient mathematical spaces used in the study of classical and quantum field theories (in their contravariant Lagrangian approach). For this reason, the differential geometry of these spaces was intensively studied by many authors (please see, for example, Saunders [15] or Asanov [1] and references therein). In this direction, it is important to note that, following the geometrical ideas initially stated by Asanov in [1], a multi-time Lagrange contravariant geometry on 1-jet spaces (in the sense of distinguished connection, torsions and curvatures) was recently constructed by Neagu and Udriște [12], [14] and published by Neagu in the book [13]. This geometrical theory is a natural multi-parameter extension on 1-jet spaces of the already classical Lagrange geometrical theory on the tangent bundle elaborated by Miron and Anastasiei [10].

From the point of view of physicists, the differential geometry of the dual 1-jet spaces was also studied because the dual 1-jet spaces represent the polymomentum phase spaces for the covariant Hamiltonian formulation of the field theory (this is a natural multi-parameter, or multi-time, extension of the classical Hamiltonian formalism from Mechanics). Thus, in order to quantize the covariant Hamiltonian field theory (this is the final purpose in the framework of quantum field theory), the covariant Hamiltonian differential geometry was developed in three distinct ways:
the multisymplectic covariant geometry elaborated by Gotay, Isenberg, Marsden, Montgomery and their co-workers [6], [7];

• the polysymplectic covariant geometry investigated by Giachetta, Mangiarotti and Sardanashvily [5];

• the De Donder-Weyl covariant Hamiltonian geometry intensively studied by Kanatchikov (please see [8] and references therein).

These three distinct geometrical-physics variants differ by the multi-time phase space and the geometrical techniques used in study.

Inspired by the Cartan covariant Hamiltonian approach of classical Mechanics, the studies of Miron [9], Atanasiu [2], [3] and their co-workers led to the development of the Hamilton geometry on the cotangent bundle exposed in the book [11]. We underline that, via the Legendre duality of the Hamilton spaces with the Lagrange spaces, it was shown in [11] that the theory of Hamilton spaces has the same symmetry like the Lagrange geometry, giving in this way a geometrical framework for the Hamiltonian theory of Analytical Mechanics.

In such a physical and geometrical context, suggested by the multi-time framework of the De Donder-Weyl covariant Hamiltonian formulation of Physical Fields, the aim of this paper is to present some basic geometrical concepts on dual 1-jet spaces (we refer to distinguished tensors, multi-time semisprays of polymomenta and nonlinear connections), necessary to the development of a subsequent multi-time covariant Hamilton geometry (in the sense of d-connections, d-torsions and d-curvatures [4]), which to be a natural multi-parameter, or poly-momentum, generalization of the Hamilton geometry on the cotangent bundle [11].

Finally, we would like to point out that the multi-time Legendre jet duality between our subsequent multi-time covariant Hamilton geometry and the already constructed multi-time contravariant Lagrange geometry [13] is a part of our work in progress and represents a general direction of our future studies.

2 The dual 1-jet vector bundle $J^{1*}(T, M)$

We start our geometrical study considering the smooth real manifolds $T^m$ and $M^n$ of dimension $m$, respectively $n$, whose local coordinates are $(t^a)_{a=1,m}$, respectively $(x^i)_{i=1,n}$.

Remark 2.1 i) In this work all geometrical objects and all mappings are considered of class $C^\infty$. This thing is expressed by the words differentiable or smooth.

ii) We point out that, throughout this paper, the indices $a, b, c, d, f, g$ run over the set $\{1, 2, \ldots, m\}$ and the indices $i, j, k, l, r, s$ run over the set $\{1, 2, \ldots, n\}$.

Let $(t_0, x_0)$ be an arbitrary point of the product manifold $T \times M$ and let $C^\infty(T, M)$ be the set all smooth maps between the manifolds $T$ and $M$. We
define on the space \( C^\infty (T, M) \) the relation of equivalence

\[
\rho \sim_{(t_0, x_0)} \sigma \iff \begin{cases} 
\rho(t_0) = \sigma(t_0) = x_0 \\
\partial_0 \rho = \partial_0 \sigma.
\end{cases}
\]

Obviously, having two arbitrary smooth maps \( \rho, \sigma \in C^\infty (T, M) \), the relation of equivalence \( \rho \sim_{(t_0, x_0)} \sigma \) takes the local form

\[
\begin{cases} 
x^i(t^0_0) = y^i(t^0_0) = x^i_0 \\
\frac{\partial x^i}{\partial t^a}(t^0_0) = \frac{\partial y^i}{\partial t^a}(t^0_0),
\end{cases}
\]

where \( t^b(t_0) = t^b_0 \), \( x^i(x_0) = x^i_0 \), \( x^i = x^i \circ \rho \) and \( y^i = x^i \circ \sigma \).

The class of equivalence of an element \( \rho \in C^\infty (T, M) \) is denoted by

\[
[\rho]_{(t_0, x_0)} = \{ \sigma \in C^\infty (T, M) \mid \sigma \sim_{(t_0, x_0)} \rho \}.
\]

If we denote by

\[
J^1_{t_0, x_0}(T, M) = C^\infty (T, M) / \sim_{(t_0, x_0)},
\]

the quotient space obtained by the factorization of the space \( C^\infty (T, M) \) with respect to the relation of equivalence ” \( \sim_{(t_0, x_0)} \)” , we can construct the total space of the jets of order one, putting

\[
J^1(T, M) = \bigcup_{(t_0, x_0) \in T \times M} J^1_{t_0, x_0}(T, M).
\]

Now, let us organize \( J^1(T, M) \) like a vector bundle over the base space \( T \times M \) endowed with the differentiable structure of a product manifold. For this, we start with an arbitrary smooth map \( \rho \in C^\infty (T, M) \), locally given by

\[
(t^1_0, \ldots, t^m_0) \rightarrow (x^1(t^1_0, \ldots, t^m_0), \ldots, x^n(t^1_0, \ldots, t^m_0)).
\]

Developing the maps \( x^i \) in Taylor series around the point \((t^1_0, \ldots, t^m_0) \in \mathbb{R}^m\), we obtain the local expressions

\[
x^i(t^1, \ldots, t^m) = x^i_0 + (t^a - t^a_0) \frac{\partial x^i}{\partial t^a}(t^1_0, \ldots, t^m_0) + O(2),
\]

where \( (t^1, \ldots, t^m) \in \mathbb{R}^m \) is an arbitrary point from a convenient neighbourhood of the point \((t^1_0, \ldots, t^m_0) \in \mathbb{R}^m\), that is \( \|(t^1 - t^1_0, \ldots, t^m - t^m_0)\| < \varepsilon \). Considering now the smooth map \( \overline{\rho} \in C^\infty (T, M) \), defined by the set of local functions

\[
\overline{\rho}(t^1, \ldots, t^m) = x^i_0 + (t^a - t^a_0) \frac{\partial x^i}{\partial t^a}(t^1_0, \ldots, t^m_0), \quad \|(t^1 - t^1_0, \ldots, t^m - t^m_0)\| < \varepsilon,
\]

we deduce that \( \overline{\rho} \sim_{(t_0, x_0)} \rho \). In other words, the affine linear approximation \( \overline{\rho} \) of the map \( \rho \) is a very good representative of the class of equivalence \([\rho]_{(t_0, x_0)}\).
Let $\pi^1 : J^1(T, M) \to T \times M$ be the canonical projection defined by

$$\pi^1([\rho]_{(t_0, x_0)}) = (t_0, \rho(t_0) = x_0).$$

It is obvious that the map $\pi^1$ is well defined and surjective. Using this projection, for each local chart $U \times V \subset T \times M$, we can define the bijection

$$\phi_{U \times V} : (\pi^1)^{-1}(U \times V) \to U \times V \times \mathbb{R}^{mn},$$

setting

$$\phi_{U \times V}([\rho]_{(t_0, x_0)}) = \left(t_0, x_0, \frac{\partial x^i}{\partial t^a}(t_0)\right), \quad x_0 = \rho(t_0).$$

In conclusion, the 1-jet space $J^1(T, M)$ can be endowed with a differentiable structure of dimension $m + n + mn$, such that the maps $\phi_{U \times V}$ to be diffeomorphisms. In this context, the local coordinates on $J^1(T, M)$ are $(t^a, x^i, x^i_a)$, where

$$t^a([\rho]_{(t_0, x_0)}) = t^a(t_0),$$

$$x^i([\rho]_{(t_0, x_0)}) = x^i(x_0),$$

$$x^i_a([\rho]_{(t_0, x_0)}) = \frac{\partial x^i}{\partial t^a}(t_0).$$

Using the above coordinates on the 1-jet space $J^1(T, M)$, we get that the projection $\pi^1 : J^1(T \times M) \to T \times M$ has the local expression

$$\pi^1(t^a, x^i, x^i_a) = (t^a, x^i).$$

Moreover, the differential map $\pi^1_*$ of the map $\pi^1$ is locally determined by the Jacobi matrix

$$\begin{pmatrix}
\delta_{ab} & 0 & 0 \\
0 & \delta_{ij} & 0 \\
\end{pmatrix} \in \mathcal{M}_{m+n,m+n+mn}(\mathbb{R}).$$

Of course, the map $\pi^1_*$ is a surjection (i. e., rank $\pi^1_* = m + n$) and therefore the map $\pi^1$ is a submersion. Consequently, we have

**Proposition 2.2** The 1-jet space $J^1(T, M)$ is a vector bundle over the base space $T \times M$, having the fibre type $\mathbb{R}^{mn}$.

**Remark 2.3** From a physical point of view, the manifold $T$ can be regarded as a temporal manifold or, better, a multi-time manifold, while the manifold $M$ can be regarded as a spatial one. Moreover, the 1-jet vector bundle $J^1(T, M) \to T \times M$ can be regarded as a bundle of configurations. This terminology is justified by the fact that in the particular case $T = \mathbb{R}$ (i. e., the temporal manifold $T$ coincides with the usual time axis represented by the set of real numbers $\mathbb{R}$), we recover the bundle of configurations which characterizes the classical non-autonomous, or rheonomic, Mechanics.
Taking into account the form of the changes of coordinates on the product manifold \( T \times M \), we easily deduce

**Proposition 2.4** The transformations of coordinates \((t^a, x^i, x^i_a) \mapsto (\tilde{t}^a, \tilde{x}^i, \tilde{x}^i_a)\) induced from \( T \times M \) on the 1-jet space \( J^1(T, M) \) are given by

\[
\begin{align*}
\tilde{t}^a &= \tilde{t}^a(t^b) \\
\tilde{x}^i &= \tilde{x}^i(x^j) \\
\tilde{x}^i_a &= \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial t^b}{\partial t^a} x^j_b,
\end{align*}
\]  

where \( \det(\partial \tilde{t}^a / \partial t^b) \neq 0 \) and \( \det(\partial \tilde{x}^i / \partial x^j) \neq 0 \).

Now, using the general theory of vector bundles (please see [10], for example), let us consider the dual 1-jet vector bundle \( J^1(T, M) \to T \times M \), whose total space is

\[
J^1(T, M) = \bigcup_{(t^a, x^i) \in T \times M} J^1(t^a_0, x^i_0)(T, M),
\]

where

\[
J^1(t^a_0, x^i_0)(T, M) = \{ \omega(t^a, x^i_0) : J^1(t^a_0, x^i_0)(T, M) \to \mathbb{R} \, | \, \omega(t^a_0, x^i_0) \text{ is } \mathbb{R}\text{-linear} \},
\]

and which has the fibre type \((\mathbb{R}^{mn})^* \equiv \mathbb{R}^{mn}\). The local coordinates on the dual 1-jet vector bundle \( J^1(T, M) \) are denoted by \((t^a, x^i, p^a_i)\).

**Remark 2.5** i) In order to simplify the notations, we will use the notations \( E = J^1(T, M) \) and \( E^* = J^1(T, M) \).

ii) According to the Kanatchikov’s physical terminology [8], which generalizes the Hamiltonian terminology from Analytical Mechanics, the coordinates \( p^a_i \) are called **polymomenta** and the dual 1-jet space \( E^* \) is called the **polymomentum phase space**.

It is easy to see that a transformation of coordinates on the product manifold \( T \times M \) produces the following results:

**Proposition 2.6** The transformations of coordinates \((t^a, x^i, x^i_a) \mapsto (\tilde{t}^a, \tilde{x}^i, \tilde{x}^i_a)\) induced from \( T \times M \) on the dual 1-jet space \( E^* \) have the expressions

\[
\begin{align*}
\tilde{t}^a &= \tilde{t}^a(t^b) \\
\tilde{x}^i &= \tilde{x}^i(x^j) \\
\tilde{x}^i_a &= \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial t^b}{\partial t^a} x^j_b,
\end{align*}
\]  

where \( \det(\partial \tilde{t}^a / \partial t^b) \neq 0 \) and \( \det(\partial \tilde{x}^i / \partial x^j) \neq 0 \).
Corollary 2.7 The dual 1-jet space $E^*$ is an orientable manifold having the dimension $m + n + mn$.

Now, doing a transformation of coordinates (2.2) on $E^*$, we obtain

Proposition 2.8 The elements of the local natural basis

$$\left\{ \frac{\partial}{\partial t^a}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial p^a_i} \right\}$$

of the Lie algebra of vector fields $X(E^*)$ transform by the rules

$$\frac{\partial}{\partial t^b} = \frac{\partial t^b}{\partial t^a} \frac{\partial}{\partial t^a},$$

$$\frac{\partial}{\partial x^i} = \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial}{\partial \tilde{x}^j},$$

$$\frac{\partial}{\partial p^a_i} = \frac{\partial p^a_i}{\partial \tilde{t}^b} \frac{\partial}{\partial \tilde{t}^b} + \frac{\partial p^a_i}{\partial \tilde{x}^j} \frac{\partial}{\partial \tilde{x}^j}. \quad (2.3)$$

Proposition 2.9 The elements of the local natural cobasis $\{ dt^a, dx^i, dp^a_i \}$ of the Lie algebra of covector fields $X^*(E^*)$ transform by the rules

$$dt^a = \frac{\partial t^a}{\partial \tilde{t}^b} d\tilde{t}^b,$$

$$dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j,$$

$$dp^a_i = \frac{\partial p^a_i}{\partial \tilde{t}^b} d\tilde{t}^b + \frac{\partial p^a_i}{\partial \tilde{x}^j} d\tilde{x}^j + \frac{\partial p^a_i}{\partial x^l} \frac{\partial}{\partial \tilde{t}^b} \frac{\partial}{\partial \tilde{t}^b} dt^a. \quad (2.4)$$

Remark 2.10 Let us remark that, in the particular case $T = \mathbb{R}$, we find the momentum phase space $J^1(\mathbb{R}, M) \equiv \mathbb{R} \times T^* M$ (we have a punctual identification), where $T^* M$ is the cotangent bundle. This particular momentum phase space $\mathbb{R} \times T^* M$ is regarded as a vector bundle over the product of manifolds $\mathbb{R} \times M$. Its coordinates are denoted by $(t, x^i, p_i)$ and the corresponding transformations group (2.2) becomes

$$\left\{ \tilde{t} = \tilde{t}(t) \right.,
\left. \tilde{x}^i = \tilde{x}^i(x^j) \right.,
\left. \tilde{p}_i = \frac{\partial x^l}{\partial \tilde{x}^j} \frac{\partial}{\partial \tilde{t}^b} \frac{\partial}{\partial \tilde{t}^b} dt^a. \quad (2.5) \right.$$
group of transformations

\[
\begin{align*}
\tilde{t} &= t \\
\tilde{x}^i &= \tilde{x}^i (x^j), \\
\tilde{p}_i &= \frac{\partial x^i}{\partial \tilde{x}^j} p_j
\end{align*}
\] (2.6)

of the trivial bundle \( \mathbb{R} \times T^* M \to T^* M \) used in the non-autonomous Hamilton geometry for the study of the metrical structure

\[
g^{ij} (t, x, p) = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j},
\]

where \( H : \mathbb{R} \times T^* M \to \mathbb{R} \) is a Hamiltonian function.

3 d-Tensors, multi-time semisprays of polymomenta and nonlinear connections

It is well known the importance of tensors in the development of a fertile geometry on a vector bundle. Following the geometrical ideas developed in the books [10] and [11], in our study upon the geometry of the dual 1-jet bundle \( E^* \), a central role is played by the distinguished tensors or, briefly, d-tensors.

**Definition 3.1** A geometrical object \( T = \left( T_{ai(k)(d) \ldots}^{bj(c)(l) \ldots} \right) \) on the dual 1-jet vector bundle \( E^* \), whose local components, with respect to a transformation of coordinates (2.2) on \( E^* \), transform by the rules

\[
T_{ai(k)(d) \ldots}^{bj(c)(l) \ldots} = T_{exp(\tau)(h) \ldots}^{exp(\rho)(s) \ldots} \frac{\partial x^i}{\partial \tilde{x}^c} \frac{\partial \tilde{x}^d}{\partial \tilde{t}^p} \left( \frac{\partial x^k}{\partial \tilde{x}^c} \frac{\partial \tilde{x}^d}{\partial \tilde{t}^p} \right) \left( \frac{\partial \tilde{t}^p}{\partial \tilde{x}^c} \frac{\partial \tilde{t}^p}{\partial \tilde{x}^d} \right) \ldots,
\]

is called a d-tensor or distinguished tensor field on the dual 1-jet space \( E^* \).

**Remark 3.2** The utilization between parentheses of certain indices of the local components \( T_{ai(k)(d) \ldots}^{bj(c)(l) \ldots} \) is necessary for clearer future contractions. For the moment, we point out only that \((k)\) or \((d)\) behaves like a single double index.

**Example 3.3** i) If \( H : E^* \to \mathbb{R} \) is a Hamiltonian function depending on the polymomenta \( p_i^a \), then the local components

\[
G_{(a)(b)}^{(i)(j)} = \frac{1}{2} \frac{\partial^2 H}{\partial p_i^a \partial p_j^b}
\]

represent a d-tensor field \( G = \left( G_{(a)(b)}^{(i)(j)} \right) \) on the dual 1-jet space \( E^* \), which is called the fundamental vertical metrical d-tensor associated to the Hamiltonian function of polymomenta \( H \). This is because if \( T = \mathbb{R} \) and \( H \) is a regular Hamiltonian function, then the d-tensor field \( G \) can be regarded
as the fundamental metrical d-tensor \( g^{ij}(t, x, p) \) from the theory of rheonomic Hamilton spaces.

ii) Let us consider the d-tensor \( C^* = \left( C^{(a)}_{(i)} \right) \), where \( C^{(a)}_{(i)} = p^a_i \). The distinguished tensor \( C^* \) is called the \textbf{Liouville-Hamilton d-tensor field of polymomenta} on the dual 1-jet space \( E^* \). Remark that, for the particular case \( T = \mathbb{R} \), we recover the classical Liouville-Hamilton vector field

\[
C^* = p_i \frac{\partial}{\partial p_i}
\]
on the cotangent bundle \( T^* M \), which is used in the Hamilton geometry [11].

iii) Let \( h_{ab}(t) \) be a semi-Riemannian metric on the temporal manifold \( T \). The geometrical object \( L = \left( L^{(c)}_{(j)ab} \right) \), where

\[
L^{(c)}_{(j)ab} = h_{ab} \delta^i_j
\]

is a d-tensor field on \( E^* \), which is called the \textbf{polymomentum Liouville-Hamilton d-tensor field associated to the metric} \( h_{ab}(t) \).

iv) Using the preceding metric \( h_{ab}(t) \), we can construct the d-tensor field \( J = \left( J^{(i)}_{(a)bj} \right) \), where

\[
J^{(i)}_{(a)bj} = h_{ab} \delta^i_j.
\]

The distinguished tensor \( J \) is called the \textbf{d-tensor of h-normalization} on the dual 1-jet vector bundle \( E^* \).

It is obvious that any d-tensor field on \( E^* \) is a tensor field on \( E^* \). Conversely, this statement is not true. As examples, we construct two tensors on \( E^* \), which are not distinguished tensors on \( E^* \).

**Definition 3.4** A global tensor \( G \) on \( E^* \), locally expressed by

\[
G = p^a_i dx^i \otimes \frac{\partial}{\partial t^a} - 2G^{(b)}_{1(j)i} dx^i \otimes \frac{\partial}{\partial p^b_j},
\]
is called a \textbf{temporal semispray} on the dual 1-jet vector bundle \( E^* \).

Taking into account that the temporal semispray \( G \) is a global tensor on \( E^* \), by a direct calculation, we obtain

**Proposition 3.5** i) With respect to a transformation of coordinates \( 2.2 \), the components \( G^{(b)}_{1(j)i} \) of the global tensor \( G \) transform by the rules

\[
2G^{(c)}_{1(k)c} = 2L^{(b)}_{1(j)i} \frac{\partial x^i}{\partial x^k} \frac{\partial x^k}{\partial x^c} - \frac{\partial x^i}{\partial x^k} \frac{\partial x^j}{\partial x^c} \frac{\partial p^c_k}{\partial p^a_j} p^a_i.
\]

In other words, the temporal semispray \( G \) is not a d-tensor.

ii) Conversely, to give a temporal semispray on \( E^* \) is equivalent to give a set of local functions \( G = \left( G^{(b)}_{1(j)i} \right) \) which transform by the rules \( 3.1 \).
Moreover, if we start with
\[
G_1 = G_1^a dx^i \otimes \frac{\partial}{\partial t^a} - 2G_1^{(b)} dx^i \otimes \frac{\partial}{\partial \tilde{p}_j^b},
\]
a global tensor on \(E^*\) and \(h_{ab}(t)\) an arbitrary semi-Riemannian metric on the temporal manifold \(T\), then we easily find the following result:

**Proposition 3.6** The tensor \(G_1\) is a temporal semispray on \(E^*\) if and only if
\[
J_{(a)bj} G_1^{cr} = L_{(j)ab}^{(c)},
\]
where \(J\) is the d-tensor of \(h\)-normalization and \(L\) is the multi-time Liouville-Hamilton d-tensor field associated to the metric \(h_{ab}(t)\).

**Example 3.7** If \(\kappa_{abc}(t)\) are the Christoffel symbols of a semi-Riemannian metric \(h_{ab}(t)\) of the temporal manifold \(T\), then the local components
\[
0_{(a)} G_1^{(b)} = \frac{1}{2} \kappa_{abc} \tilde{p}_j^b \partial_{p^c} \partial_{x^j}
\] (3.2)
represent a temporal semispray \(\tilde{G}_1\) on the dual 1-jet vector bundle \(E^*\).

**Definition 3.8** The temporal semispray \(\tilde{G}_1\) given by (3.2) is called the canonical temporal semispray associated to metric \(h_{ab}(t)\).

A second example of tensor on the dual 1-jet space \(E^*\), which is not a distinguished tensor, is offered by

**Definition 3.9** A global tensor \(G_2\) on \(E^*\), locally expressed by
\[
G_2 = \delta_1^i dx^i \otimes \frac{\partial}{\partial x^j} - 2G_2^{(b)} dx^i \otimes \frac{\partial}{\partial \tilde{p}_j^b},
\]
is called a spatial semispray on the dual 1-jet vector bundle \(E^*\).

As in the case of a temporal semispray, we can prove without difficulties the following statements:

**Proposition 3.10** i) To give a spatial semispray on \(E^*\) is equivalent to give a set of local functions \(G_2^{(b)}\) which transform by the rules
\[
2\tilde{G}_2^{(d)} = 2\tilde{G}_2^{(b)} \frac{\partial t^d}{\partial x^j} \frac{\partial x^i}{\partial x^j} \frac{\partial x^j}{\partial \tilde{x}^k} \frac{\partial \tilde{x}^k}{\partial x^i} - \frac{\partial x^i}{\partial \tilde{x}^s} \frac{\partial \tilde{p}_s^d}{\partial \tilde{x}^k} \frac{\partial \tilde{x}^k}{\partial x^j}
\] (3.3)
ii) A global tensor on $E^*$, defined by

$$ G = G_1^i dx^i \otimes \frac{\partial}{\partial x^i} - 2G_2^{(b)}_{(i)j} dx^i \otimes \frac{\partial}{\partial p_j^b}, $$

is a spatial semispray on $E^*$ if and only if

$$ J^{(i)}_{(a)b} G_2^k = J^{(k)}_{(a)b}, $$

where $J$ is the d-tensor of $h$-normalization for an arbitrary semi-Riemannian temporal metric $h_{ab}(t)$.

**Example 3.11** If $\gamma_{jk}^i(x)$ are the Christoffel symbols of a semi-Riemannian metric $\varphi_{ij}(x)$ of the spatial manifold $M$, then the local components

$$ 0 G_2^{(b)_{(j)k}} = -\frac{1}{2} \gamma_{jk}^i p_i^b $$

define a spatial semispray $0 G_2$ on the dual 1-jet space $E^*$.

**Definition 3.12** The spatial semispray $0 G_2$ given by (3.4) is called the canonical spatial semispray associated to the metric $\varphi_{ij}(x)$.

**Remark 3.13** It is obvious that the difference between two temporal (spatial, respectively) semisprays is a d-tensor.

Using the Remark 3.13 and the preceding notations, we easily deduce

**Theorem 3.14** Let $(T, h)$ and $(M, \varphi)$ be two semi-Riemannian manifolds and let $G = (G_1^{(b)}_{(i)j}, G_2^{(b)}_{(i)j})$, respectively, be an arbitrary temporal (spatial, respectively) semispray on the dual 1-jet space $E^*$. In this context, the following equalities are true:

$$ G_1^{(b)_{(i)j}} = \frac{1}{2} \gamma_{cd}^b p_i^d + T_1^{(b)_{(i)j}}, \quad G_2^{(b)_{(i)j}} = -\frac{1}{2} \gamma_{ij}^k p^b_k + T_2^{(b)_{(i)j}}, $$

where $T_1^{(b)_{(i)j}}, T_2^{(b)_{(i)j}}$ are unique d-tensors with the preceding properties.

**Definition 3.15** A pair $G = (G_1, G_2)$ consisting of a temporal semispray $G_1$ and a spatial one $G_2$ is called a multi-time semispray of polymomenta on the dual 1-jet space $E^*$.

**Remark 3.16** The Theorem 3.14 emphasizes the central role played by the canonical semispray of polymomenta $0 G_1$, associated to a pair of semi-Riemannian metrics $(h_{ab}(t), \varphi_{ij}(x))$, in the description of an arbitrary multi-time semispray of polymomenta $G = (G_1, G_2)$ on the dual 1-jet space $E^*$. 

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Definition 3.17 A pair of local functions \( N = \left( N^{(c)}_1, N^{(c)}_2 \right) \) on \( E^* \), which transform by the rules

\[
\begin{align*}
\tilde{N}^{(b)}_1 &= N^{(c)}_1 \frac{\partial \tilde{b}}{\partial \hat{e}} \frac{\partial \hat{x}^b}{\partial \hat{x}^a} \frac{\partial a}{\partial \hat{p}^b} - \frac{\partial \tilde{b}}{\partial \hat{e}} \frac{\partial \hat{x}^b}{\partial \hat{x}^a}, \\
\tilde{N}^{(b)}_2 &= N^{(c)}_2 \frac{\partial \tilde{b}}{\partial \hat{e}} \frac{\partial \hat{x}^b}{\partial \hat{x}^a} \frac{\partial x^i}{\partial \hat{p}^b} - \frac{\partial \tilde{b}}{\partial \hat{e}} \frac{\partial \hat{x}^b}{\partial \hat{x}^a} \frac{\partial x^i}{\partial \hat{p}^b},
\end{align*}
\]

is called a nonlinear connection on the dual 1-jet bundle \( E^* \).

Remark 3.18 The nonlinear connections are very important in the study of the differential geometry of the dual 1-jet space \( E^* \) because they produce the adapted distinguished 1-forms

\[
\delta p_i^a = dp_i^a + N^{(a)}_1 \partial \tilde{b}^{(b)} \frac{\partial a}{\partial \tilde{b}^b} d\hat{x}^b + N^{(a)}_2 \partial \tilde{b}^{(b)} \frac{\partial x^i}{\partial \tilde{b}^b} dx^i,
\]

which are necessary for the adapted local description of the geometrical objects involved in study, such as the \( \mathbf{d}\)-connections, the \( \mathbf{d}\)-torsions or the \( \mathbf{d}\)-curvatures. For more details, please see the paper [4].

Now, let us expose the connection between the notions of multi-time semispray of polymomenta and nonlinear connection on the dual 1-jet space \( E^* \). Thus, in our context, using the transformation rules \( \widetilde{N}^{(b)}_1 \) and \( \widetilde{N}^{(b)}_2 \) of the geometrical objects taken in study, we can easily prove the following statements:

Proposition 3.19 i) If \( g^{(a)}_{(j)k} \) are the components of a temporal semispray \( G^1 \) on \( E^* \) and \( \varphi_{ij}(x) \) is a semi-Riemannian metric on the spatial manifold \( M \), then the local components

\[
N^{(a)}_1 = \varphi_{ij} \frac{\partial G^{(a)}_{(j)k}}{\partial \hat{p}_i^b} \varphi_{ir}
\]

represent the temporal components of a nonlinear connection \( N_G \) on \( E^* \).

ii) Conversely, if \( N^{(a)}_1 \) are the temporal components of a nonlinear connection \( N \) on \( E^* \), then the local components

\[
g^{(a)}_{(j)i} = \frac{1}{2} N^{(a)}_1 \partial_{\hat{p}_i} \partial_{\hat{p}_j}
\]

represent a temporal semispray \( G_N \) on \( E^* \).

Proposition 3.20 i) If \( g^{(b)}_{(j)i} \) are the components of a spatial semispray \( G^2 \) on \( E^* \), then the local components

\[
N^{(b)}_2 = 2g^{(b)}_{(j)i}
\]

represent the spatial components of a nonlinear connection \( N_G \) on \( E^* \).
ii) Conversely, if \( N^{(b)}_{2(j)i} \) are the spatial components of a nonlinear connection \( N \) on \( E^* \), then the local functions
\[
G^{(b)}_{2(j)i} = \frac{1}{2} N^{(b)}_{2(j)i}
\]
represent a spatial semispray \( G^N \) on \( E^* \).

**Remark 3.21** The Propositions 3.19 and 3.20 emphasize that a multi-time semispray of polymomenta \( G = \left( G_1, G_2 \right) \) on the dual 1-jet space \( E^* \) naturally induces a nonlinear connection \( N_G \) on \( E^* \) and vice-versa, \( N \) induces \( G_N \).

**Definition 3.22** The nonlinear connection \( N_G \) on the dual 1-jet space \( E^* \) is called the canonical nonlinear connection associated to the multi-time semispray of polymomenta \( G = \left( G_1, G_2 \right) \) and vice-versa.

**Corollary 3.23** The canonical nonlinear connection \( N = \left( \frac{\partial^2 H}{\partial p^a_i \partial p^b_j} \right) \) produced by the canonical multi-time semispray of polymomenta \( G = \left( G_1, G_2 \right) \) associated to the pair of semi-Riemannian metrics \((h_{ab}(t), \varphi_{ij}(x))\) has the local components
\[
N_{1(i)b} = \kappa_{cb} p^c_i \quad \text{and} \quad N_{2(i)j} = -\gamma_{ik} p^k_i.
\]

### 4 Kronecker h-regularity. Canonical nonlinear connections

Let us consider a smooth multi-time Hamiltonian function \( H : E^* \to \mathbb{R} \), locally expressed by
\[
E^* \ni (t^a, x^i, p^a_i) \to H(t^a, x^i, p^a_i) \in \mathbb{R},
\]
whose fundamental vertical metrical d-tensor is defined by
\[
G^{(i)(j)}_{(a)(b)} = \frac{1}{2} \frac{\partial^2 H}{\partial p^a_i \partial p^b_j}.
\]

In the sequel, let us fix \( h = (h_{ab}(t^c)) \), a semi-Riemannian metric on the temporal manifold \( T \), together with a d-tensor \( g^{ij}(t^c, x^k, p^c_k) \) on the dual 1-jet space \( E^* \), which is symmetric, has the rank \( n = \dim M \) and a constant signature.

**Definition 4.1** A multi-time Hamiltonian function \( H : E^* \to \mathbb{R} \), having the fundamental vertical metrical d-tensor of the form
\[
G^{(i)(j)}_{(a)(b)}(t^c, x^k, p^c_k) = \frac{1}{2} \frac{\partial^2 H}{\partial p^a_i \partial p^b_j} = h_{ab}(t^c) g^{ij}(t^c, x^k, p^c_k),
\]
is called a Kronecker h-regular multi-time Hamiltonian function.
In this context, we can introduce the following geometrical concept:

**Definition 4.2** A pair \( MH^n_m = (E^* = J^1(T, M), H) \), where \( m = \text{dim} \, T \) and \( n = \text{dim} \, M \), consisting of the dual 1-jet space and a Kronecker \( h \)-regular multi-time Hamiltonian function \( H : E^* \to \mathbb{R} \), is called a **multi-time Hamilton space**.

**Remark 4.3** i) In the particular case \((T, h) = (\mathbb{R}, \delta)\), a multi-time Hamilton space will be called a **relativistic rheonomic Hamilton space** and it will be denoted by \( RH^n = (J^1(\mathbb{R}, M), H) \).

ii) If the temporal manifold \((T, h)\) is 1-dimensional, then, via a temporal reparametrization, we have \( J^1(T, M) \equiv J^1(\mathbb{R}, M) \). In other words, a multi-time Hamilton space having \( \text{dim} \, T = 1 \) is a reparametrized relativistic rheonomic Hamilton space.

**Example 4.4** Let us consider the following Kronecker \( h \)-regular multi-time Hamiltonian function \( H_1 : E^* \to \mathbb{R} \), defined by

\[
H_1 = \frac{1}{mc} h_{ab}(t) \varphi^{ij}(x) p^a_i p^b_j,
\]

where \( h_{ab}(t) \) (\( \varphi_{ij}(x) \), respectively) is a semi-Riemannian metric on the temporal (spatial, respectively) manifold \( T \) (\( M \), respectively) having the physical meaning of **gravitational potentials**, and \( m \) and \( c \) are the known constants from Physics representing the **mass of the test body** and the **speed of light**. Then, the multi-time Hamilton space

\[
GMH^n_m = (E^*, H_1)
\]

defined by the multi-time Hamiltonian function (4.1) is called the **multi-time Hamilton space of gravitational field**. This is because, for \((T, h) = (\mathbb{R}, \delta)\), we recover the classical Hamilton space of gravitational field from the book [11].

**Example 4.5** Using preceding notations, let us consider the Kronecker \( h \)-regular multi-time Hamiltonian function \( H_2 : E^* \to \mathbb{R} \), defined by

\[
H_2 = \frac{1}{mc} h_{ab}(t) \varphi^{ij}(x) p_i^a p_j^b - \frac{2e}{mc^2} A_{(a)}^{(i)}(x) p_i^a + \frac{e^2}{mc^3} F(t, x),
\]

where \( A_{(a)}^{(i)}(x) \) is a \( d \)-tensor on \( E^* \) having the physical meaning of **potential \( d \)-tensor of an electromagnetic field**, \( e \) is the **charge of the test body** and the function \( F(t, x) \) is given by

\[
F(t, x) = h^{ab}(t) \varphi_{ij}(x) A_{(a)}^{(i)}(x) A_{(b)}^{(j)}(x).
\]

Then, the multi-time Hamilton space

\[
EDMH^n_m = (E^*, H_2)
\]
defined by the multi-time Hamiltonian function \((4.2)\) is called the autonomous multi-time Hamilton space of electrodynamics. This is because, in the particular case \((T, h) = (R, \delta)\), we recover the classical Hamilton space of electrodynamics studied in the book \([11]\). The non-dynamical character (the independence of the temporal coordinates \(t^c\)) of the spatial gravitational potentials \(\varphi_{ij}(x)\) motivated us to use the term "autonomous".

**Example 4.6** More general, if we take on \(E^*\) a symmetric d-tensor field \(g_{ij}(t, x)\) having the rank \(n\) and a constant signature, we can define the Kronecker \(h\)-regular multi-time Hamiltonian function \(H_3 : E^* \rightarrow \mathbb{R}\), setting

\[
H_3 = h_{ab}(t)g^{ij}(t, x)p_i^a p_j^b + U^{(i)}(t, x)p_i^a + F(t, x),
\]

where \(U^{(i)}(t, x)\) is a d-tensor field on \(E^*\) and \(F(t, x)\) is a function on \(E^*\). Then, the multi-time Hamilton space

\[
\mathcal{NEDM}H^m_n = (E^*, H_3)
\]
defined by the multi-time Hamiltonian function \((4.3)\) is called the non-autonomous multi-time Hamilton space of electrodynamics. The dynamical character (the dependence of the temporal coordinates \(t^c\)) of the gravitational potentials \(g_{ij}(t, x)\) motivated us to use the word "non-autonomous".

An important role and, at the same time, an obstruction for the subsequent development of a geometrical theory of the multi-time Hamilton spaces, is represented by

**Theorem 4.7** (of characterization of multi-time Hamilton spaces) If we have \(m = \text{dim} T \geq 2\), then the following statements are equivalent:

(i) \(H\) is a Kronecker \(h\)-regular multi-time Hamiltonian function on \(E^*\).

(ii) The multi-time Hamiltonian function \(H\) reduces to a multi-time Hamiltonian function of non-autonomous electrodynamic kind, that is we have

\[
H = h_{ab}(t)g^{ij}(t, x)p_i^a p_j^b + U^{(i)}(t, x)p_i^a + F(t, x).
\]

Proof. (ii) \(\Rightarrow\) (i) It is obvious (even if we have \(m = 1\)).

(i) \(\Rightarrow\) (ii) Let us suppose that \(m = \text{dim} T \geq 2\) and let us consider that \(H\) is a Kronecker \(h\)-regular multi-time Hamiltonian function, that is we have

\[
\frac{1}{2} \frac{\partial^2 H}{\partial p_k^a \partial p_j^b} = h_{ab}(t^c)g^{ij}(t^c, x^k, p_k^c).
\]

(1°) Firstly, let us suppose that there exist two distinct indices \(a\) and \(b\), from the set \(\{1, \ldots, m\}\), such that \(h_{ab} \neq 0\). Let \(k\) (c, respectively) be an arbitrary element of the set \(\{1, \ldots, n\}\) (\(\{1, \ldots, m\}\), respectively). Deriving the above relation, with respect to the variable \(p_k^c\), and using the Schwartz theorem, we obtain the equalities

\[
\frac{\partial g_{ij}}{\partial p_k^c} h_{ab} = \frac{\partial g^{jk}}{\partial p_i^c} h_{bc} = \frac{\partial g^{jk}}{\partial p_j^c} h_{bc}, \quad \forall \ a, b, c \in \{1, \ldots, m\}, \quad \forall \ i, j, k \in \{1, \ldots, n\}.
\]
Contracting now with \(h^{cd}\), we deduce that
\[
\frac{\partial g^{ij}}{\partial p^c_k} h_{ab} h^{cd} = 0, \quad \forall \ d \in \{1, \ldots, m\}.
\]

In this context, the supposing \(h_{ab} \neq 0\), together with the fact that the metric \(h\) is non-degenerate, imply that
\[
\frac{\partial g_{ij}}{\partial p^c_k} = 0,
\]
for any two arbitrary indices \(k\) and \(c\). Consequently, we have \(g^{ij} = g^{ij}(t^d, x^r)\).

(2°) Let us suppose now that \(h_{ab} = 0, \forall \ a \neq b \in \{1, \ldots, m\}\). It follows that \(h_{ab} = h_a(t)\delta^a_b, \forall \ a, b \in \{1, \ldots, m\}\), where \(h_a(t) \neq 0, \forall \ a \in \{1, \ldots, m\}\). In these conditions, the relations
\[
\frac{\partial^2 L}{\partial p^i_a \partial p^j_b} = 0, \quad \forall \ a \neq b \in \{1, \ldots, m\}, \quad \forall \ i, j \in \{1, \ldots, n\},
\]
\[
\frac{1}{2h_a(t)} \frac{\partial^2 L}{\partial p^i_a \partial p^j_a} = g_{ij}(t^c, x^k, p^a_k), \quad \forall \ a \in \{1, \ldots, m\}, \quad \forall \ i, j \in \{1, \ldots, n\},
\]
are true. If we fix now an index \(a\) in the set \(\{1, \ldots, m\}\), we deduce from the first relations that the local functions \(\frac{\partial L}{\partial p^i_a}\) depend only by the coordinates \((t^c, x^k, p^a_k)\).

Considering \(b \neq a\) another index from the set \(\{1, \ldots, m\}\), the second relations imply
\[
\frac{1}{2h_a(t)} \frac{\partial^2 L}{\partial p^i_a \partial p^j_a} = \frac{1}{2h_b(t)} \frac{\partial^2 L}{\partial p^i_b \partial p^j_b} = g_{ij}(t^c, x^k, p^b_k), \quad \forall \ a, b \in \{1, \ldots, m\}, \quad \forall \ i, j \in \{1, \ldots, n\}.
\]

Because the first term of the above equality depends only by the coordinates \((t^c, x^k, p^a_k)\), while the second term depends only by the coordinates \((t^c, x^k, p^b_k)\), and because we have \(a \neq b\), we conclude that \(g^{ij} = g^{ij}(t^d, x^r)\).

Finally, the equalities
\[
\frac{1}{2} \frac{\partial^2 H}{\partial p^i_a \partial p^j_a} = h_{ab}(t^c)g^{ij}(t^c, x^k), \quad \forall \ a, b \in \{1, \ldots, m\}, \quad \forall \ i, j \in \{1, \ldots, n\},
\]
imply without difficulties that the multi-time Hamilton function \(H\) is one of non-autonomous electrodynamic kind (4.4). \(\blacksquare\)

**Corollary 4.8** The fundamental vertical metrical d-tensor of a Kronecker h-regular multi-time Hamiltonian function \(H\) has the form
\[
G^{i(j)}_{(a)(b)} = \frac{1}{2} \frac{\partial^2 H}{\partial p^i_a \partial p^j_a} = \begin{cases} 
  h_{11}(t)g^{ij}(t, x^k, p_k), & m = \dim T = 1 \\
  h_{ab}(t^c)g^{ij}(t^c, x^k), & m = \dim T \geq 2.
\end{cases}
\]

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Remark 4.9 i) It is obvious that the Theorem 4.7 is an obstruction in the development of a fertile geometrical theory of the multi-time Hamilton spaces. This obstruction will be surpassed in other paper by the introduction of the more general geometrical concept of generalized multi-time Hamilton space. The generalized multi-time Hamilton geometry on the dual 1-jet space $E^*$ will be constructed using only a Kronecker $h$-regular fundamental vertical metrical d-tensor (not necessarily provided by a Hamiltonian function)

$$G^{(i)(j)}_{(a)(b)} = h_{ab}(t^c)g^{ij}(t^c, x^k, p^c_k),$$

together with an a priori given nonlinear connection $N$ on $E^*$.

ii) In the case $m = \dim T \geq 2$, the Theorem 4.7 obliges us to continue our geometrical study of the multi-time Hamilton spaces channeling our attention upon the non-autonomous multi-time Hamilton spaces of electrodynamics.

In the sequel, following the geometrical ideas of Miron from [9], we will show that any Kronecker $h$-regular multi-time Hamiltonian function $H$ produces a natural nonlinear connection on the dual 1-jet bundle $E^*$, which depends only by $H$. In order to do that, let us take a Kronecker $h$-regular multi-time Hamiltonian function $H$, whose fundamental vertical metrical d-tensor is given by (4.5). Also, let us consider the generalized spatial Christoffel symbols of the d-tensor $g_{ij}$, given by

$$\Gamma_{ij}^k = \frac{g^{kl}}{2} \left( \frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

In this context, using preceding notations, we can give the following result:

Theorem 4.10 The pair of local functions $N = \left( N_{1}^{(a)}_{(i)b}, N_{2}^{(a)}_{(i)j} \right)$ on $E^*$, where

$$N_{1}^{(a)}_{(i)b} = \chi^a_{ch} p^c_i,$$

$$N_{2}^{(a)}_{(i)j} = \frac{h_{ab}}{4} \left[ \frac{\partial g_{ij}}{\partial x^k} \frac{\partial H}{\partial p^k_1} - \frac{\partial g_{ij}}{\partial x^k} \frac{\partial H}{\partial p^k_i} + g_{ik} \frac{\partial^2 H}{\partial x^j \partial p^k_i} + g_{jk} \frac{\partial^2 H}{\partial x^i \partial p^k_j} \right], \quad (4.6)$$

represents a nonlinear connection on $E^*$, which is called the canonical nonlinear connection of the multi-time Hamilton space $MH^m_n = (E^*, H)$.

Proof. Taking into account the classical transformation rules of the Christoffel symbols $\chi^a_{ch}$ of the temporal semi-Riemannian metric $h_{ab}$, by direct local computations, we deduce that the temporal components $N_{1}^{(a)}_{(i)b}$ from (4.6) verify the first transformation rules from (3.5) (please see also the Corollary 3.23).

In the particular case when $m = \dim T = 1$, the spatial components

$$N_{2}^{(1)}_{(i)j} = \frac{h_{11}}{4} \left[ \frac{\partial g_{ij}}{\partial x^k} \frac{\partial H}{\partial p^k_k} - \frac{\partial g_{ij}}{\partial x^k} \frac{\partial H}{\partial p^k_i} + g_{ik} \frac{\partial^2 H}{\partial x^j \partial p^k_i} + g_{jk} \frac{\partial^2 H}{\partial x^i \partial p^k_j} \right]$$

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become (except the multiplication factor $h_{11}$) exactly the canonical nonlinear connection from the classical Hamilton geometry (please see [9] or [11, pp. 127]).

For $m = \dim T \geq 2$, the Theorem 4.7 (more exactly, the formula (4.4)) leads us to the following expression for the spatial components $N^{(a)}_{(2)(ij)}$ from (4.6):

$$N^{(a)}_{(2)(ij)} = -\Gamma^{k}_{ij} p^{a}_{k} + T^{(a)}_{(i)j},$$

(4.7)

where

$$T^{(a)}_{(i)j} = \frac{h^{ab}}{4} \left[ \frac{\partial g_{ik} U^{(k)}_{(b)}}{\partial x^{j}} + g_{ik} \frac{\partial U^{(k)}_{(b)}}{\partial x^{j}} + g_{jk} \frac{\partial U^{(k)}_{(b)}}{\partial x^{i}} \right].$$

Because $T^{(a)}_{(i)j}$ is a d-tensor on $E^{*}$ (we prove this by local computations, studying the transformation laws of $T^{(a)}_{(i)j}$), it immediately follows that the spatial components $N^{(a)}_{(2)(ij)}$ given by (4.7) transform as in the second laws of (4.4).

Finally, using the expression (4.7), by computations, we find

**Corollary 4.11** For $m = \dim T \geq 2$, the canonical nonlinear connection $N$ of a multi-time Hamilton space $MH^{m}_{n} = (E^{*}, H)$ (given by (4.3)) has the components

$$N^{(a)}_{(1)(ib)} = \gamma^{a}_{cb} p^{c}_{i},$$

$$N^{(a)}_{(2)(ij)} = -\Gamma^{k}_{ij} p^{a}_{k} + \frac{h^{ab}}{4} \left( U_{ib\bullet} + U_{jbs} \right),$$

where $U_{ib} = g_{ik} U^{(k)}_{(b)}$ and

$$U_{bk\bullet} = \frac{\partial U^{(k)}_{ib}}{\partial x^{r}} - U_{sb} \Gamma^{r}_{kr}.$$

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