On the $L_{q,p}$-cohomology of Riemannian Manifolds with Negative Curvature

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Dedicated to the Memory of Sergei L'vovich Sobolev

Abstract

We prove a non-vanishing result for the $L_{q,p}$-cohomology of complete simply-connected Riemannian manifolds with pinched negative curvature.

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1 Introduction

In the paper [2], we have established a connection between Sobolev inequalities for differential forms on a Riemannian manifold $(M, g)$ and an invariant called the $L_{q,p}$-cohomology $(H_{q,p}^k(M))$ of that manifold. It is thus important to try and compute this cohomology, and in this paper we shall prove some non-vanishing results for the $L_{q,p}$-cohomology of simply connected complete manifolds with negative curvature.

1.1 $L_{q,p}$-cohomology and Sobolev inequalities

To define the $L_{q,p}$-cohomology of a Riemannian manifold $(M, g)$, we first need to remember the notion of weak exterior differential of a locally integrable differential form. Let us denote by $C_c^\infty(M, \Lambda^k)$ the space of smooth differential forms of degree $k$ with compact support on $M$.

Definition 1. One says that a form $\theta \in L^1_{loc}(M, \Lambda^k)$ is the weak exterior differential of a form $\phi \in L^1_{loc}(M, \Lambda^{k-1})$ and one writes $d\phi = \theta$ if for each $\omega \in C_c^\infty(M, \Lambda^{n-k})$, one has

$$\int_M \theta \wedge \omega = (-1)^k \int_M \phi \wedge d\omega.$$
The Sobolev space $W^{1,p}(M,A^k)$ of differential $k$-forms is then defined to be the space of $k$-forms $\phi$ in $L^p(M)$ such that $d\phi \in L^p(M)$ and $d(\ast \phi) \in L^p(M)$, where $\ast : A^k \to A^{n-k}$ is the Hodge star homomorphism. But we are interested in a different “Sobolev type” space of differential forms, that will be denoted by $\Omega^k_{q,p}(M)$. This is the space of all $k$-forms $\phi$ in $L^q(M)$ such that $d\phi \in L^p(M)$ (1 $\leq q, p \leq \infty$), and it is a Banach space for the graph norm

$$\|\omega\|_{\Omega^k_{q,p}} := \|\omega\|_{L^q} + \|d\omega\|_{L^p}. \quad (1.1)$$

When $k = 0$ and $q = p$, the space $\Omega^0_{p,p}(M)$ coincides with the classical Sobolev space $W^{1,p}(M)$ of functions in $L^p$ with gradient in $L^p$. Let us stress that the more general space $\Omega^0_{q,p}(M)$ has been considered in [10] in the context of embedding theorems and Sobolev inequalities.

To define the $L^q,p$–cohomology of $(M,g)$, we also introduce the space of weakly closed forms

$$Z^k_p(M) = \{ \omega \in L^p(M,A^k) \mid d\omega = 0 \},$$

and the space of differential forms in $L^p(M)$ having a primitive in $L^q(M)$

$$B^k_{q,p}(M) = d(\Omega^{k-1}_{q,p}).$$

Note that $Z^k_p(M) \subset L^p(M,A^k)$ is always a closed subspace but that is generally not the case of $B^k_{q,p}(M)$, and we will denote by $\overline{B}_{q,p}(M)$ its closure in the $L^p$-topology. Observe also that $\overline{B}_{q,p}(M) \subset Z^k_p(M)$ (by continuity and because $d \circ d = 0$), we thus have

$$B^k_{q,p}(M) \subset \overline{B}_{q,p}(M) \subset Z^k_p(M) = \overline{Z}_p^k(M) \subset L^p(M,A^k).$$

**Definition 2.** The $L^q,p$-cohomology of $(M,g)$ (where $1 \leq p, q \leq \infty$) is defined to be the quotient

$$H^k_{q,p}(M) := Z^k_p(M)/B^k_{q,p}(M),$$

and the reduced $L^q,p$-cohomology of $(M,g)$ is

$$\overline{H}^k_{q,p}(M) := Z^k_p(M)/\overline{B}^k_{q,p}(M).$$

The reduced cohomology is naturally a Banach space and the unreduced cohomology is a Banach space if and only if it coincides with the reduced one.

In [2, Theorem 6.1], we have established the following connection between Sobolev inequalities for differential forms on a Riemannian manifold $(M,g)$ and its $L^q,p$-cohomology of $(M,g)$:
Theorem 1. $H^k_{q,p}(M, g) = 0$ if and only if there exists a constant $C < \infty$ such that for any closed $p$-integrable differential form $\omega$ of degree $k$ there exists a differential form $\theta$ of degree $k - 1$ such that $d\theta = \omega$ and

$$\|\theta\|_{L^q} \leq C \|\omega\|_{L^p}.$$ 

Suppose $k = 1$. If $M$ is simply connected (or more generally $H^1_{\text{deRham}}(M) = 0$), then any $\omega \in Z^1_p(M)$ has a primitive locally integrable function $f$, $df = \omega$. It means that for simply connected manifolds the space $Z^1_p(M)$ coincides with the seminormed Sobolev space $L^1_p(M)$, $\|f\|_{L^1_p(M)} := \|df\|_{L^p(M)}$. The previous Theorem then says that

Corollary 2. Suppose $(M, g)$ is a simply connected Riemannian manifold, then $H^1_{q,p}(M, g) = 0$ if and only if there exist a constants $C < \infty$ depending only on $M$, $(q, p)$ and a constant $a_f < \infty$ depending also on $f \in L^1_p(M, g)$ such that

$$\|f - a_f\|_{L^q} \leq C \|df\|_{L^p}.$$ 

for any $f \in L^1_p(M, g)$. 

In the present paper, we prove nonvanishing results on the $L^q_p$-cohomology of simply connected complete manifolds with negative curvature i.e. results about non existence of Sobolev inequality for such pairs $(q, p)$.

1.2 Statement of the main result

The main goal of the present paper is to prove the following nonvanishing result on the $L^q_p$-cohomology of simply connected complete manifolds with negative curvature.

**Theorem 3.** Let $(M, g)$ be an $n$-dimensional Cartan-Hadamard manifold\(^4\) with sectional curvature $K \leq -1$ and Ricci curvature $\text{Ric} \geq -(1+\epsilon)^2(n-1)$.

(A) Assume that

$$\frac{1+\epsilon}{p} < \frac{k}{n-1} \quad \text{and} \quad \frac{k-1}{n-1} + \epsilon < \frac{1+\epsilon}{q},$$

then $H^k_{q,p}(M) \neq 0$.

(B) If furthermore

$$\frac{1+\epsilon}{p} < \frac{k}{n-1} \quad \text{and} \quad \frac{k-1}{n-1} + \epsilon < \min \left\{ \frac{1+\epsilon}{q}, \frac{1+\epsilon}{p} \right\},$$

then $\overline{H}^k_{q,p}(M) \neq 0$.

\(^4\) recall that a Cartan-Hadamard manifold is a complete simply-connected Riemannian manifold of non positive sectional curvature.
Theorem 3 together with Theorem 1 has the following (negative) consequence about Sobolev inequalities for differential forms:

**Corollary 4.** Let \((M, g)\) be a Cartan-Hadamard manifold as above. If \(q\) and \(p\) satisfy the condition \((A)\) of Theorem 3, then there is no finite constant \(C\) such that any smooth closed \(k\)-form \(\omega\) on \(M\) admits a primitive \(\theta\) such that \(d\theta = \omega\) and

\[
\|\theta\|_{L^q(M)} \leq C \|\omega\|_{L^p(M)}.
\]

The proof of Theorem 3 will be based on a duality principle proved in [2] and a comparison argument inspired from the chapter 8 of the book of M. Gromov [8]. This will be explained below, but let us first discuss some particular cases.

- If \(M\) is the hyperbolic plane \(\mathbb{H}^2\) \((n = 2, \epsilon = 0)\), Theorem 3 says that \(\overline{H}^k_{q,p}(\mathbb{H}^2) \neq 0\) for any \(q, p \in (1, \infty)\); and another proof can be found in [2, Theorem 10.1].

- For \(q = p\), the Theorem says that \(\overline{H}^k_{p,p}(M) \neq 0\) provided

\[
\frac{k - 1}{n - 1} + \epsilon < \frac{1 + \epsilon}{p} < \frac{k}{n - 1},
\]

this result was already known by Gromov (see [8, page 244]). The inequalities (1.2) can also be written in terms of \(k\) as follows:

\[
\frac{n - 1}{p} < k < \frac{n - 1}{p} + \tau
\]

with \(\tau = 1 - \epsilon(n - 1)\).

- By contrast, Pierre Pansu has proved that \(H^k_{p,p}(M) = 0\) if the sectional curvature satisfies \(-(1 + \epsilon)^2 \leq K \leq -1\) and

\[
(1 + \epsilon) p \leq \frac{n - 1}{k} + \epsilon,
\]

see [12, Théorème A].

- A Poincaré duality for reduced \(L^p\)-cohomology has been proved in [5], it says that for a complete Riemannian manifold, we have \(\overline{H}_k^{p,p}(M) = \overline{H}_{k,p'}^{q}(M)\) with \(p' = p/(p - 1)\), this duality, together with the result of Pansu and some algebraic computations, implies that for a manifold \(M\) as in Theorem 3 we also have \(\overline{H}_k^{p,p}(M) = 0\) if

\[
p \geq \frac{(n - 1) + \epsilon(n - k)}{k - 1}.
\]

- Consider for instance the case of the hyperbolic space \(\mathbb{H}^n\), this is a Cartan-Hadamard manifold with constant sectional curvature \(K \equiv -1\) and the reduced cohomology is known. Indeed, we have \(\epsilon = 0\) and the three inequalities above say in this case that \(\overline{H}_k^{p,p}(\mathbb{H}^n) \neq 0\) if and only if \(p \in \left(\frac{n - 1}{k}, \frac{n - 1}{k - 1}\right)\) (or, equivalently, for \(\frac{n - 1}{p} < k < \frac{n - 1}{p} + 1\)). This result also follows from the computation of the \(L^p\)-cohomology of warped cylinders given in [6, 7].
When \( \epsilon > 0 \), there remains a gap between the vanishing and the non vanishing result for \( L^{p,p} \)-cohomology. When \( \epsilon \geq \frac{1}{n-1} \), the estimate (1.2) no longer gives any information on \( L^{p,p} \)-cohomology. Note by contrast that Theorem 3 always produces some non vanishing \( L^{q,p} \)-cohomology.

2 Manifolds with a contraction onto the closed unit ball

As an application of a concept of almost duality from [2], we have the following Theorem which is inspired from [8] and will be used in the proof of Theorem 3. Recall that by the Rademacher theorem a Lipschitz map \( f : M \to N \) is differentiable for almost any \( x \in M \) and its differential \( df_x \) defines a homomorphism

\[
\Lambda^k f_x : \Lambda^k(T_f x N) \to \Lambda^k(T_x M).
\]

We shall denote by \( |\Lambda^k f_x| \) the norm of this homomorphism.

**Theorem 5.** Let \((M, g)\) be a complete Riemannian manifold, and let \( f : M \to \mathbb{B}^n \) be a Lipschitz map such that

\[
|\Lambda^k f| \in L^p(M) \quad \text{and} \quad |\Lambda^{n-k} f| \in L^{q'}(M),
\]

where \( \mathbb{B}^n \) is the closed unit ball in \( \mathbb{R}^n \) and \( q' = q/(q - 1) \), assume also that

\[
f^* \omega \in L^1(M) \quad \text{and} \quad \int_M f^* \omega \neq 0,
\]

where \( \omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \) is the standard volume form on \( \mathbb{B}^n \). Then \( H^k_{q,p}(M) \neq 0 \).

Furthermore, if \( |\Lambda^{n-k} f| \in L^{q'}(M) \) for \( p' = \frac{p}{p-1} \), then \( \overline{H}^k_{q,p}(M) \neq 0 \).

The proof will use the following “almost duality” result:

**Proposition 6.** Assume that \((M, g)\) is a complete Riemannian manifold. Let \( \alpha \in Z^k_p(M) \), and assume that there exists a closed \((n-k)\)-form \( \gamma \in Z^{n-k}_q(M) \) for \( q' = \frac{q}{q-1} \), such that \( \gamma \wedge \alpha \in L^1(M) \) and

\[
\int_M \gamma \wedge \alpha \neq 0,
\]

then \( H^k_{q,p}(M) \neq 0 \). Furthermore, if \( \gamma \in Z^{n-k}_p(M) \cap Z^{n-k}_q(M) \) for \( p' = \frac{p}{p-1} \) and \( q' = \frac{q}{q-1} \), then \( \overline{H}^k_{q,p}(M) \neq 0 \).
This result is contained in [2, Proposition 8.4 and 8.5].

We will also need some fact on locally Lipschitz differential forms:

**Lemma 7.** For any locally Lipschitz functions \( g, h_1, ..., h_k : M \to \mathbb{R} \), we have

\[
\frac{\partial}{\partial x^i} (g \, dh_1 \wedge \ldots \wedge dh_k) = \frac{\partial g}{\partial x^i} \wedge dh_1 \wedge \ldots \wedge dh_k
\]

in the weak sense.

Let us denote by \( \text{Lip}^*(M) \) the algebra generated by locally Lipschitz functions and the wedge product. By the previous lemma \( \text{Lip}^*(M) \) is a graded differential algebra, an element in this algebra is called a locally Lipschitz form.

**Proposition 8.** For any locally Lipschitz map \( f : M \to N \) between two Riemannian manifolds, the pullback \( f^*(\omega) \) of any locally Lipschitz form \( \omega \) is a locally Lipschitz form and \( d(f^*(\omega)) = f^*(d\omega) \).

A proof of the lemma and the proposition can be found in [1]; see also [3] for some related results.

**Proof of Theorem 5**

Let us set \( \omega' = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k \) and \( \omega'' = dx_{k+1} \wedge dx_2 \wedge \ldots \wedge dx_n \).

Using the fact that \( |(f^*\omega)| \leq |\Lambda^k f| \cdot |\omega(x)| \), we observe that

\[
\|f^*\omega\|_{L^p(M, \Lambda^k)} = \left( \int_M |(f^*\omega)|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \left( \int_M \left( |\Lambda^k f|^p \cdot |\omega'\wedge \omega''| \right)^{\frac{p}{p'}} \, dx \right)^{\frac{1}{p'}}
\]

\[
\leq \left| \Lambda^k f \right|_{L^p(M)} \|\omega'\|_{L^\infty(M, \Lambda^k)}
\]

\[< \infty.\]

Let us set \( \alpha = f^*\omega' \). Because \( f \) is a Lipschitz map \( \alpha \) is a lipschitz form we have by Proposition 8 that \( \alpha \in L^p(M, \Lambda^k) \) and we thus have \( \alpha \in Z^k_p(M) \). The same argument shows that \( \gamma = f^*\omega'' \).

By hypothesis, we have \( \alpha \wedge \gamma = f^*(\omega' \wedge \omega'') = f^*(\omega) \in L^1(M) \) and

\[
\int_M \gamma \wedge \alpha = \int_M f^*\omega \neq 0,
\]

and we conclude from Proposition 8 that \( H^k_{q,p}(M) \neq 0 \).

If we also assume that \( \Lambda^{n-k} f_x \in L^p'(M) \) for \( p' = \frac{p}{p-1} \), then \( \gamma \in Z^{n-k}_{p'}(M) \) and by the second part of Proposition 8 we conclude that \( \check{H}^k_{q,p}(M) \neq 0 \).

The paper [1] contains other results relating \( L_{q,p} \)-cohomology and classes of mappings.
3 Proof of the main Theorem

Let $(M, g)$ be a complete simply connected manifold of negative sectional curvature of dimension $n$. Fix a base point $o \in M$ and identify $T_o M$ with $\mathbb{R}^n$ by a linear isometry. The exponential map $\exp_o : \mathbb{R}^n = T_o M \to M$ is then a diffeomorphism and we define the map $f : M \to \overline{\mathbb{B}}^n$ where $\overline{\mathbb{B}}^n \subset \mathbb{R}^n$ is the closed Euclidean unit ball by

$$f(x) = \begin{cases} \exp_o^{-1}(x) & \text{if } |\exp_o^{-1}(x)| \leq 1, \\ \frac{\exp_o^{-1}(x)}{|\exp_o^{-1}(x)|} & \text{if } |\exp_o^{-1}(x)| \geq 1. \end{cases}$$

Using polar coordinates $(r, u)$ on $M$, i.e. writing a point $x \in M$ as $x = \exp_o(r \cdot u)$ with $u \in S^{n-1}$ and $r \in [0, \infty)$, we can also write this map as $f(r, u) = \min(r, 1) \cdot u$. Because the exponential map is expanding, the map $f : M \to \overline{\mathbb{B}}^n$ is contracting and in particular it is a Lipschitz map.

Recall that $\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ is the volume form on $\overline{\mathbb{B}}^n$. It can also be written as $r^n dr \wedge d\sigma_0$ where $d\sigma_0$ is the volume form of the standard sphere $S^{n-1}$. It follows that $f^* \omega = 0$ on the set $\{x \in M \mid d(o, x) > 1\}$ and $f^* \omega$ has thus compact support and is in particular integrable. Let us denote by $U_1 = \{x \in M \mid d(o, x) < 1\}$ the Riemannian open unit ball in $M$, the restriction of $f$ to $U_1$ is a diffeomorphism onto $\overline{\mathbb{B}}^n$ and therefore

$$\int_M f^* \omega = \int_{U_1} f^* \omega = \int_{\overline{\mathbb{B}}^n} \omega = \text{Vol}(\overline{\mathbb{B}}^n) > 0.$$ 

The next lemma implies that if

$$\frac{1 + \epsilon}{p} < \frac{k}{n - 1},$$

then $|\Lambda^k f| \in L^p(M)$ and that if

$$\frac{1 + \epsilon}{q'} < \frac{n - k}{n - 1},$$

then $|\Lambda^{n-k} f| \in L^{q'}(M)$. Observe that the inequality

$$\frac{1 + \epsilon}{q'} < \frac{n - k}{n - 1}$$

is equivalent to

$$\frac{k - 1}{n - 1} + \epsilon < \frac{1 + \epsilon}{q}.$$
since \( q' = q/(q-1) \). Likewise, \( |\Lambda^{n-k} f| \in L^p(M) \) if

\[
\frac{k-1}{n-1} + \epsilon < \frac{1 + \epsilon}{p}.
\]

In conclusion, the map \( f \) satisfies all the hypothesis of Theorem 5 as soon as the conditions of Theorem 3 (A) or (B) are fulfilled. The proof of Theorem 3 is complete.

Lemma 9. The map \( f : M \to \mathbb{R}^n \) satisfies \( |\Lambda^m f| \in L^s(M) \) as soon as

\[
\frac{1 + \epsilon}{s} < \frac{m}{n-1}.
\]

Proof. Using the Gauss Lemma from Riemannian geometry, we know that in polar coordinates \( M \simeq [0, \infty) \times \mathbb{S}^{n-1}/(\{0\} \times \mathbb{S}^{n-1}) \), the Riemannian metric can be written as

\[
g = dr^2 + g_r,
\]

where \( g_r \) is a Riemannian metric on the sphere \( \mathbb{S}^{n-1} \). The Rauch comparison theorem tells us that if the sectional curvature of \( g \) satisfies \( K \leq -1 \), then

\[
g_r \leq \bigg( \frac{\sinh(r)}{r} \bigg)^2 g_0, \tag{3.1}
\]

where \( g_0 \) is the standard metric on the sphere \( \mathbb{S}^{n-1} \) (see any textbook on Riemannian geometry, e.g. Corollary 2.4 in [13, section 6.2] or [9, Corollary 4.6.1]). Using the fact that the euclidean metric on \( \mathbb{R}^n = T_0 M \) writes in polar coordinates as \( ds^2 = dr^2 + r^2 g_0 \) together with the first inequality in (3.1), we obtain that

\[
|f^*(\theta)| \leq \frac{r}{\sinh(r)}|\theta|
\]

for any covector \( \theta \in T^*_{r,u} M \) that is orthogonal to \( dr \). Because \( f^*(dr) \) has compact support, we conclude that

\[
|f^*(\phi)| \leq \text{const.} \left( \frac{r}{\sinh(r)} \right)^m \phi
\]

for any \( m \)-form \( \phi \in \Lambda^m(T^*_{r,u} M) \). In other words, we have obtained the pointwise estimate

\[
|\Lambda^m f|_{(r,u)} \leq \text{const.} \left( \frac{r}{\sinh(r)} \right)^m. \tag{3.2}
\]

The Ricci curvature comparison estimate says that if \( Ric \geq -(1+\epsilon)^2(n-1) \), then the volume form of \( (M, g) \) satisfies

\[
dvol \leq \left( \frac{\sinh((1+\epsilon)r)}{1 + \epsilon} \right)^{n-1} dr \wedge d\sigma_0 \tag{3.3}
\]
where $d\sigma_0$ is the volume form of the standard sphere $S^{n-1}$ (see e.g \[13\] section 9.1.1). The previous inequalities give us a control of the growth of $|\Lambda^m f|_{(r,u)}\,d\text{vol}$. To be precise, let us choose a number $t$ such that

\[
\frac{m(1+\epsilon)}{n-1} < t < s,
\]

then (3.2) and (3.3) imply

\[
|\Lambda^m f|_{s,(r,u)}\,d\text{vol} \leq \text{const. } e^{-ar}\,dr \wedge d\sigma_0,
\]

with $a = mt - (n-1)(1+\epsilon) > 0$. The latter inequality implies the integrability of $|\Lambda^m f|_{s,(r,u)}$: we have indeed

\[
\int_M |\Lambda^m f|_{s,(r,u)}\,d\text{vol} \leq \text{Vol}(S^{n-1}) \int_0^\infty e^{-ar}\,dr < \infty.
\]

\[\square\]

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