Vanishing preons in the fifth dimension

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Received 2 October 2006, in final form 22 November 2006
Published 12 December 2006
Online at stacks.iop.org/CQG/24/417

Abstract
We examine supersymmetric solutions of $N = 2, D = 5$ gauged supergravity coupled to an arbitrary number of Abelian vector multiplets using the spinorial geometry method. By making use of methods developed in Gran et al (2006 Preprint hep-th/0606049) to analyse preons in type IIB supergravity, we show that there are no solutions preserving exactly $3/4$ of the supersymmetry.

PACS numbers: 04.50.+h, 04.65.+e, 11.30.Pb

1. Introduction

Considerable research activity has been devoted recently to the analysis and the study of black holes and other gravitational configurations in $N = 2, D = 5$ gauged supergravity coupled to Abelian vector multiplets [2]. It can be said that this, to a large extent, has been motivated by the AdS/CFT conjectured equivalence [3]. For example, string solutions preserving $1/4$ of supersymmetry have been found in [4]. Examples of $1/2$ supersymmetric solutions are the domain wall solutions in [4], as well as the solutions given in [5–7] and [8] which correspond to black holes without regular horizons, i.e., the solutions either have naked singularities or closed timelike curves.

More recently, motivated by the method of [9], a systematic approach has been employed in order to classify $1/4$ supersymmetric solutions of the minimal gauged five-dimensional supergravity [10]. The basic idea is to assume the existence of a Killing spinor (i.e., to assume that the solution preserves at least one supersymmetry) and construct differential forms as bilinears in the Killing spinor. The algebraic and differential conditions satisfied by these forms are sufficient to determine the local form of the spacetime metric and the rest of the bosonic fields of the theory. This general framework provides a more powerful method for obtaining many new interesting black holes than the method of guessing an Ansatz. The first examples of explicit $1/4$ supersymmetric regular asymptotically $AdS_5$ supersymmetric solutions were given in [11]. The classification of $1/4$ supersymmetric solutions and more
explicit regular solutions of the gauged supergravity with vector multiplets were later given in [12, 13]. Further solutions were considered in [14] and [15].

The results obtained in the literature so far seem to have focused mainly on the classification of supersymmetric solutions of $N = 2, D = 5$ gauged supergravity which preserve 2 of the 8 supersymmetries. In $N = 2, D = 5$ gauged supergravity, it is known that the only solution which preserves all 8 of the supersymmetries is $AdS_5$ with vanishing gauge field strengths and constant scalars. Moreover, the Killing spinor equations are linear over $\mathbb{C}$ when written in terms of Dirac spinors. Hence it follows that supersymmetric solutions of this theory preserve either 2, 4, 6 or 8 of the supersymmetries.

In particular, this immediately excludes the possibility of solutions preserving exactly $7/8$ of the supersymmetry. Such solutions would be lower-dimensional analogues of hypothetical preon solutions in $D = 11$ supergravity [16], which, if possible, preserve $31/32$ of the supersymmetry. Properties of preons in ten and eleven dimensions have also been investigated in [1, 17–20]. It has also been shown that there are no exactly $3/4$ supersymmetric solutions of minimal $N = 2, D = 4$ gauged supergravity for which one of the Killing spinors generates a null Killing vector [21].

Having eliminated the possibility of preonic solutions of $N = 2, D = 5$ gauged supergravity, it is natural to investigate whether solutions preserving the next highest proportion of supersymmetry, i.e. exactly $3/4$ supersymmetric solutions, can exist. In this paper, we present a proof that such solutions also do not exist. In order to construct the non-existence proof, it will be particularly useful to consider the spinors as differential forms [22–24]. This method of writing spinors as forms has been used to classify solutions of supergravity theories in ten and eleven dimensions (see, for example, [1, 25–27].)

The plan of the paper is as follows. In section 2, we review some of the properties of five-dimensional gauged supergravity coupled to Abelian vector multiplets. In section 3, we show how spinors of the theory can be written as differential forms, and how the $Spin(4, 1)$ gauge freedom present in the theory can be used to reduce a spinor to one of three ‘canonical’ forms. We also define a $Spin(4, 1)$-invariant non-degenerate bilinear form $B$ on the space of spinors. In section 4, we show how solutions preserving $3/4$ of the supersymmetry can be placed into three classes according to the canonical form of the spinor which is orthogonal (with respect to $B$) to the Killing spinors. This method of characterizing supersymmetric solutions by the spinors which are orthogonal to the Killing spinors was originally developed in [1] where it was used to show that there are no preons in type IIB supergravity. For each class of solutions, we prove that the algebraic Killing spinor equations constrain the solution in such a manner that the solution reduces to a solution of the minimal gauged five-dimensional supergravity. Finally, in section 5, we show that for all three possible types of solutions, the integrability conditions of the Killing spinor equations in the minimal five-dimensional gauged supergravity fix the gauge field strengths to vanish, and constrain the spacetime geometry to be $AdS_5$. However, it is known that $AdS_5$ is the unique maximally supersymmetric solution of this theory. It therefore follows that there can be no exactly $3/4$ supersymmetric solutions of $N = 2, D = 5$ gauged supergravity coupled to arbitrary many vector multiplets.

2. $N = 2, D = 5$ supergravity

In this section, we review briefly some aspects of the $N = 2, D = 5$ gauged supergravity with field content consisting of the graviton, the gravitino, $n$ vector potentials, $n − 1$ gauginos and $n − 1$ scalars. The bosonic action of this theory is [2]
\[
S = \frac{1}{16\pi G} \int \left( -5 R + 2\chi^2 \mathcal{V} - Q_{IJ} F^I \wedge \star F^J + Q_{IJ} dX^I \wedge \star dX^J - \frac{1}{6} C_{IJK} F^I \wedge F^J \wedge A^K \right) 
\]

where \( I, J, K \) take values 1, \ldots, \( n \) and \( F^I = dA^I \). \( \chi \) is a nonzero constant, and \( C_{IJK} \) are constants that are symmetric on \( IJK \); we will assume that \( Q_{IJ} \) is invertible, with inverse \( Q^{IJ} \).

The metric has signature \((-+, -,-,-,-)\).

The \( X^I \) are scalars which are constrained via
\[
\frac{1}{6} C_{IJK} X^I X^J X^K = 1. 
\]

We may regard the \( X^I \) as being functions of \( n-1 \) unconstrained scalars \( \phi^a \). In addition, the coupling \( Q_{IJ} \) depends on the scalars via
\[
Q_{IJ} = \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K 
\]

so in particular
\[
Q_{IJ} X^I = \frac{3}{2} X_I, \quad Q_{IJ} \partial_a X^I = -\frac{3}{2} \partial_a X_I. 
\]

The scalar potential can be written as
\[
\mathcal{V} = 9 V_I V_J \left( X^I X^J - \frac{1}{2} Q^{IJ} \right) 
\]

where \( V_I \) are constants.

For a bosonic background to be supersymmetric there must be a spinor \( \epsilon \) for which the supersymmetry variations of the gravitino and the superpartners of the scalars vanish. We shall investigate the properties of these spinors in greater detail in the next section. The gravitino Killing spinor equation is
\[
\left[ \nabla_\mu + \frac{1}{8} \gamma_\mu X_I F^I_{\mu \nu} \gamma^{\nu} - \frac{3}{4} X_I F^I_{\mu \nu} \gamma^{\nu} \right] \epsilon + \frac{i\chi}{2} V_I (X^I \gamma_\mu - 3 A^I_{\mu}) \epsilon = 0 
\]

and the algebraic Killing spinor equations associated with the variation of the scalar superpartners are
\[
F^I_{\mu \nu} \gamma^{\mu \nu} \epsilon = [X^I X_J F^I_{\mu \nu} \gamma^{\mu \nu} + 2 \gamma^{\mu} \nabla_\mu X^I] \epsilon + 4i\chi (X^I V_J X^J - \frac{3}{2} Q^{IJ} V_J) \epsilon. 
\]

We shall refer to (8) as the dilatino Killing spinor equation. We also require that the bosonic background should satisfy the Einstein, gauge field and scalar field equations obtained from the action (1); however we will not make use of these equations in our analysis, as it will be sufficient to work with the Killing spinor equations alone.

3. Spinors in five dimensions

Dirac spinors in five dimensions can be written as complexified forms on \( \mathbb{R}^2 \) (this construction is also given in an appendix of [28]). The space of these spinors will be denoted by \( \Delta = \Lambda^*(\mathbb{R}^2) \otimes \mathbb{C} \). A generic spinor \( \eta \) can therefore be written as
\[
\eta = \lambda_1 + \mu^i e^i + \sigma e^{12} 
\]

where \( e^1, e^2 \) are 1-forms on \( \mathbb{R}^2 \), and \( i = 1, 2; e^{12} = e^1 \wedge e^2 \). \( \lambda, \mu^i \) and \( \sigma \) are complex functions.
The action of $\gamma$-matrices on these forms is given by
\begin{align}
\gamma_i &= i(e^i \wedge + i e_i) \\
\gamma_{i+2} &= -e^i \wedge + i e_i
\end{align}
for $i = 1, 2$. $\gamma_0$ is defined by
\begin{align}
\gamma_0 = \gamma_{1234}
\end{align}
and satisfies
\begin{align}
\gamma_0 1 &= 1, \\
\gamma_0 e^{12} &= e^{12}, \\
\gamma_0 e^i &= -e^i, \quad i = 1, 2.
\end{align}
The charge conjugation operator $C$ is defined by
\begin{align}
C 1 &= -e^{12}, \\
C e^{12} &= 1, \\
C e^i &= -\epsilon_{ij} e^j, \quad i = 1, 2
\end{align}
where $\epsilon_{ij} = \epsilon_{ij}$ is antisymmetric with $\epsilon_{12} = 1$.

We note the useful identity
\begin{align}
(\gamma_M)^* = -\gamma_0 C \gamma_M \gamma_0 C.
\end{align}
It will be particularly useful to complexify the gamma operators via
\begin{align}
\gamma_p &= \frac{1}{\sqrt{2}} (\gamma_p - i \gamma_{p+2}) = \sqrt{2} i e^p \wedge \\
\gamma_p &= \frac{1}{\sqrt{2}} (\gamma_p + i \gamma_{p+2}) = \sqrt{2} i e^p.
\end{align}

### 3.1. Gauge transformations and canonical spinors

There are two types of gauge transformations which can be used to simplify the Killing spinors of this theory. First, there are local $U(1)$ gauge transformations of the type
\begin{align}
\epsilon \to e^{i\theta} \epsilon
\end{align}
for real functions $\theta$, and there are also local $\text{Spin}(4, 1)$ gauge transformations of the form
\begin{align}
\epsilon \to e^{i f_{MN} \gamma_{MN}} \epsilon
\end{align}
for real functions $f_{MN}$.

Note in particular that $\frac{1}{2} (\gamma_{12} + \gamma_{34})$, $\frac{1}{2} (\gamma_{13} - \gamma_{24})$ and $\frac{1}{2} (\gamma_{14} + \gamma_{23})$ generate a $SU(2)$ which leaves $1$ and $e^{12}$ invariant and acts on $e^1$, $e^2$; whereas $\frac{1}{2} (\gamma_{12} - \gamma_{34})$, $\frac{1}{2} (\gamma_{13} + \gamma_{24})$ and $\frac{1}{2} (\gamma_{14} - \gamma_{23})$ generate another $SU(2)$ which leaves the $e^i$ invariant but acts on $1$ and $e^{12}$. In addition, $\gamma_{03}$ generates a $SO(1, 1)$ which acts (simultaneously) on $1$, $e^1$ and $e^{12}$, whereas $\gamma_{04}$ generates another $SO(1, 1)$ which acts (simultaneously) on $1$, $e^2$ and $e^1$, $e^{12}$.

So, one can always use $\text{Spin}(4, 1)$ gauge transformations to write a single spinor as
\begin{align}
\epsilon = f 1
\end{align}
or
\begin{align}
\epsilon = f e^1
\end{align}
or
\begin{align}
\epsilon = f (1 + e^1)
\end{align}
for some real function $f$. 
3.2. A \textit{Spin}(4, 1) invariant bilinear form on spinors

In order to analyse the $3/4$ supersymmetric solutions it is necessary to construct a non-degenerate $\text{Spin}(4, 1)$ invariant bilinear form on the space of spinors. We first define a Hermitian inner product on the space of spinors via

$$\langle z^0 + z^1 e^1 + z^2 e^2 + z^3 e^{12}, w^0 + w^1 e^1 + w^2 e^2 + w^3 e^{12} \rangle = \bar{z}^a w^a$$

(summing over $a = 0, 1, 2, 3$). However, this is not $\text{Spin}(4, 1)$ gauge-invariant. We define a bilinear form $B$ by

$$B(\eta, \epsilon) = \langle C\eta^*, \epsilon \rangle.$$ 

(22)

$B$ satisfies the identities

$$B(\eta, \epsilon) + B(\epsilon, \eta) = 0$$

$$B(\gamma_M \eta, \epsilon) - B(\eta, \gamma M \epsilon) = 0$$

$$B(\gamma_{MN} \eta, \epsilon) + B(\eta, \gamma_{MN} \epsilon) = 0$$

for all spinors $\eta, \epsilon$.

The last of the above constraints implies that $B$ is $\text{Spin}(4, 1)$ invariant. Note that $B$ is linear over $\mathbb{C}$ in both arguments.

$B$ is also non-degenerate: if $B(\epsilon, \eta) = 0$ for all $\eta$ then $\epsilon = 0$.

4. $3/4$ supersymmetric solutions

We now proceed to examine solutions preserving six out of the eight allowed supersymmetries. This implies the existence of three Killing spinors, which we shall denote by $\epsilon_0, \epsilon_1, \epsilon_2$, which are linearly independent over $\mathbb{C}$.

Suppose we denote the span (over $\mathbb{C}$) of $\epsilon_0, \epsilon_1, \epsilon_2$ by $W$. Any complex three-dimensional subspace of $\mathbb{C}^4$ can be uniquely specified by its one (complex) dimensional orthogonal complement with respect to the standard inner product on $\mathbb{C}^4$. It follows that one can specify $W$ via its orthogonal complement with respect to $B$. If the one-dimensional $B$-orthogonal subspace to $W$ is spanned by $\hat{\epsilon}$, one has

$$W = W_\epsilon = \{ \psi \in \Delta : B(\psi, \hat{\epsilon}) = 0 \}$$

(24)

for some fixed non-vanishing $\hat{\epsilon} \in \Delta$. As $B$ is $\text{Spin}(4, 1)$ invariant, it will be most convenient to use a $\text{Spin}(4, 1)$ gauge transformation in order to write the spinor $\hat{\epsilon}$ in one of the three canonical forms; i.e. either $\hat{\epsilon} = 1$, or $\hat{\epsilon} = e^i$ or $\hat{\epsilon} = 1 + e^i$ (up to an overall scaling which plays no role in our analysis and can be removed).

If $\hat{\epsilon} = 1$ then $W$ is spanned by $\eta_0 = 1, \eta_1 = e^1, \eta_2 = e^2$. If $\hat{\epsilon} = e^i$ then $W$ is spanned by $\eta_0 = 1, \eta_1 = e^i, \eta_2 = e^{12}$. If $\hat{\epsilon} = 1 + e^i$ then $W$ is spanned by $\eta_0 = 1, \eta_1 = e^i, \eta_2 = -e^2 + e^{12}$.

In all cases the Killing spinors $\epsilon_0, \epsilon_1, \epsilon_2$ are related to the spinors $\eta_A$ for $A = 0, 1, 2$ via

$$\epsilon_A = z^A_B \eta_B$$

(25)

where $z$ is a complex $3 \times 3$ matrix such that $\det z \neq 0$.

4.1. Reduction to minimal solutions

The first stage in the analysis is to show that the dilatino Killing spinor equations (8) imply that the $3/4$ supersymmetric solutions correspond to solutions of the minimal theory. In particular, we shall show that the scalars $X_I$ must be constant, that there exists a nonzero real constant $\xi$ such that

$$X_I = \xi V_I$$

(26)
and that the 2-form field strengths $F^I$ satisfy
\[ F^I = X^I H \] (27)
where $H$ is a closed 2-form.

To show this we first note that the algebraic constraints (8) are linear over $\mathbb{C}$. Hence (8) is equivalent to
\[ F^I_{\mu\nu}\gamma^{\mu\nu}\eta_A = (X^I X^I) F^I_{\mu\nu} \gamma^{\mu\nu} + 2 \gamma^\mu \nabla_\mu X^I) \eta_A + 4i \chi (X^I V_J X^J - \frac{3}{2} Q^{IJ} V_J) \eta_A \] (28)
for $A = 0, 1, 2$. In order to compute (28), it is first useful to evaluate (8) acting on the spinor $\lambda_{\gamma}\epsilon^p \epsilon^e_{12}$. We obtain
\[ -\sqrt{2} i F^I_{0m}\epsilon^m_\eta + \square \lambda \epsilon_0 F^I_{0q} = F^I_{\mu\eta} \mu q + 2 F^I_{\alpha\beta}\mu_\alpha \] (29)
and
\[ \sqrt{2} i F^I_{0m}\epsilon^m_\eta - \lambda F^I_{mn} \eta \mu m = X^I (\sqrt{2} i H_{mn}\epsilon^m_\eta + \sqrt{2} i \lambda H_{0q} - H_{mn}\mu q + 2 H_{mq}\mu m) - \partial_0 X^I \mu q - \sqrt{2} i \lambda \partial_0 X^I \eta_0 \] (30)
where $\mu q \equiv \delta_0 \epsilon_{12}$, and we have defined $H = X^I X^I$. (29), (30), and (31) correspond to the $1, e q$ and the $e_1 2$ components of (8) respectively.

4.1.1. Solutions with $B$-orthogonal spinors to 1. For solutions with spinors $\epsilon_A$ such that $B(\epsilon_A, 1) = 0$, we compute the constraints obtained from (28), taking $\eta_0 = 1, \eta_1 = e^1, \eta_2 = e^2$, using (29)–(31) to read off the components of the constraints.

Evaluating (29) on $\eta_0 = 1$ we find the constraint
\[ -F^I_{0m} = -X^I H_{0m} + \partial_0 X^I + 2i \chi (X^I X^J - \frac{3}{2} Q^{IJ} V_J) \] (32)
Splitting this expression into its real and imaginary parts we find
\[ \partial_0 X^I = 0 \] (33)
and
\[ F^I_{mn} = X^I H_{mn} - 2i \chi (X^I X^J - \frac{3}{2} Q^{IJ} V_J) V_J. \] (34)
Evaluating (29) on $\eta_m = \epsilon^m$ we find
\[ F^l_{0m} = X^I H_{0m} + \partial_m X^I. \] (35)
Next, we evaluate (31) acting on $\eta_0 = 1$, to find the constraint
\[ F^I_{mn} = X^I H_{mn} \] (36)
and the constraint from (31) acting on $\eta_m = \epsilon^m$ is equivalent to (35). Finally, we evaluate (30) acting on $\eta_0 = 1$ to find
\[ F^I_{0m} = X^I H_{0m} - \partial_m X^I \] (37)
and evaluating (30) acting on \( \eta_q = e^q \) we find

\[
-F^I_{m} m \delta_{pq} + 2 F^I_{pq} = X^I \left(-H^m m \delta_{pq} + 2 H_{pq}\right)
\]

\[
- \partial_0 X^I \delta_{pq} + 2 i \chi \left(X^I X^J - \frac{3}{2} Q^{IJ}\right) V_J \delta_{pq}.
\]

(38)

First compare (35) with (37) to find

\[
\partial_0 X^I = 0.
\]

(39)

This together with (33) implies that the \( X_I \) are constant. Substituting back into (35) we find

\[
F^I_{0m} = X^I H_{0m}.
\]

(40)

Next take the trace of (38) to obtain the constraint

\[
\left(X^I X^J - \frac{3}{2} Q^{IJ}\right) V_J = 0.
\]

(41)

This is equivalent to

\[
X_I V_J X^J - V_J = 0.
\]

(42)

Hence, if \( V_J X^J = 0 \) at any point, then \( V_I = 0 \) for all \( I \). As we are interested in solutions of the gauged theory, we discard this case. Hence there is a non-zero constant \( \xi \) such that

\[
X_I = \xi V_I.
\]

(43)

Substituting this back into (34) we obtain

\[
F^I_{m} m = X^I H^m m.
\]

(44)

Finally, substituting this back into (38) we find

\[
F^I_{pq} = X^I H_{pq}.
\]

(45)

Hence we have the identity

\[
F^I = X^I H
\]

(46)

which completes the reduction of these solutions to solutions of the minimal theory.

4.1.2. Solutions with \( B \)-orthogonal spinors to \( 1 + e^1 \). For solutions with spinors \( \epsilon_A \) such that \( B(\epsilon_A, 1 + e^1) = 0 \), we compute the constraints obtained from (28), taking \( \eta_0 = 1, \eta_1 = e^1, \eta_2 = e^{12} - e^2 \), using (29)–(31) to read off the components of the constraints.

Evaluating (29) on \( \eta_0 = 1 \) we find the constraint

\[
-F^I_{m} m = -X^I H^m m + \partial_0 X^I + 2i \chi \left(X^I X^J - \frac{3}{2} Q^{IJ}\right) V_J.
\]

(47)

Splitting this expression into its real and imaginary parts we find

\[
\partial_0 X^I = 0
\]

(48)

and

\[
F^I_{m} m = X^I H^m m - 2i \chi \left(X^I X^J - \frac{3}{2} Q^{IJ}\right) V_J.
\]

(49)

Next, evaluate (29) on \( \eta_1 = e^1 \) to find

\[
F^I_{01} = X^I H_{01} + \partial_1 X^I
\]

(50)

and evaluating (29) on \( \eta_2 = e^{12} - e^2 \) gives

\[
\sqrt{2} F^I_{02} + F^I_{mn} \epsilon^{mn} = \sqrt{2} i X^I H_{02} + X^I H_{mn} \epsilon^{mn} + \sqrt{2} i \partial_2 X^I.
\]

(51)
Evaluating (30) on \( \eta_0 = 1 \) we find
\[
F^I_{0p} = X^I H_{0p} - \partial_p X^I
\] (52)
and on \( \eta_1 = e^1 \) we obtain (simplifying using (48))
\[
-F^I_{m} m \delta_{1q} + 2 F^I_{1q} = X^I \left( -H^m_m \delta_{1q} + 2 H_{1q} \right) + 2i\chi \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) V_J \delta_{1q}
\] (53)
and on \( \eta_2 = e^{12} - e^2 \) we obtain
\[
\sqrt{2} i F^I_{0m} \epsilon^m_q + F^I_{m} m \delta_{2q} - 2 F^I_{2q} = X^I \left( \sqrt{2} i H^m_m \epsilon^m_q + H^m_m \delta_{2q} - 2 H_{2q} \right)
- 2i\chi \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) V_J \delta_{2q} - \sqrt{2} i \partial_m X^I \epsilon^m_q.
\] (54)
This expression can be further simplified using (52) to give
\[
F^I_{m} m \delta_{2q} - 2 F^I_{2q} = X^I \left( H^m_m \delta_{2q} - 2 H_{2q} \right) - 2i\chi \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) V_J \delta_{2q}.
\] (55)
Combining this expression with (53) we obtain
\[
F^I_{m} m \delta_{pq} - 2 F^I_{pq} = X^I \left( H^m_m \delta_{pq} - 2 H_{pq} \right) - 2i\chi \left( X^I X^J - \frac{3}{2} Q^{IJ} \right) V_J \delta_{pq}.
\] (56)
Taking the trace we find that
\[
\left( X^I X^J - \frac{3}{2} Q^{IJ} \right) V_J = 0.
\] (57)
Next consider (31) acting on \( \eta_0 = 1 \); we obtain
\[
F^I_{mn} = X^I H_{mn}
\] (58)
and (31) acting on \( \eta_1 = e^1 \) implies
\[
F^I_{02} = X^I H_{02} + \partial_2 X^I.
\] (59)
Combining this with (50) we obtain
\[
F^I_{0p} = X^I H_{0p} + \partial_p X^I.
\] (60)
However, comparing this expression with (52) it is clear that
\[
\partial_p X^I = 0.
\] (61)
This together with (48) implies that the \( X^I \) are again constant, and then (57) implies, by the reasoning used in the previous subsection, that there exists a non-zero constant \( \xi \) such that
\[
X^I = \xi V_I.
\] (62)
Then (52) implies
\[
F^I_{0p} = X^I H_{0p}.
\] (63)
Next consider (56). This may be simplified using (57) to give
\[
F^I_{m} m \delta_{pq} - 2 F^I_{pq} = X^I \left( H^m_m \delta_{pq} - 2 H_{pq} \right)
\] (64)
and then further simplified using (49) to obtain
\[
F^I_{pq} = X^I H_{pq}.
\] (65)
Hence it follows that
\[
F^I = X^I H
\] (66)
which completes the reduction of these solutions to those of the minimal theory.
4.1.3. Solutions with $B$-orthogonal spinors to $e^1$. For solutions with spinors $\epsilon_A$ such that $B(\epsilon_A, e^1) = 0$, we compute the constraints obtained from (28), taking $\eta_0 = 1, \eta_1 = e^1, \eta_2 = e^{12}$, using (29)–(31) to read off the components of the constraints.

Evaluating (29) on $\eta_0 = 1$ we find the constraint

$$-F^I_m = -X^I H_m + \partial_0 X^I + 2i \chi (X^I X^J - \frac{3}{2} Q^{IJ}) V_J.$$  \hspace{1cm} (67)

Splitting this expression into its real and imaginary parts we find

$$\partial_0 X^I = 0$$ \hspace{1cm} (68)

and

$$F^I_m = X^I H_m - 2i \chi (X^I X^J - \frac{3}{2} Q^{IJ}) V_J.$$ \hspace{1cm} (69)

Evaluating (29) on $\eta_1 = e^1$ and $\eta_2 = e^{12}$ we find

$$F^I_{01} = X^I H_{01} + \partial_1 X^I$$ \hspace{1cm} (70)

and

$$F^I_{mn} = X^I H_{mn}$$ \hspace{1cm} (71)

respectively.

Next consider (30) acting on $\eta_0 = 1$. This implies

$$F^I_{0p} = X^I H_{0p} - \partial_p X^I.$$ \hspace{1cm} (72)

(30) acting on $\eta_1 = e^1$ together with (68) imply that

$$-F^I_m \delta_{1q} + 2F^I_{1q} = X^I (-H_m \delta_{1q} + 2H_{1q}) + 2i \chi (X^I X^J - \frac{3}{2} Q^{IJ}) V_J \delta_{1q}$$ \hspace{1cm} (73)

and (30) acting on $\eta_2 = e^{12}$ is equivalent to (72).

Next note that (31) acting on $\eta_0 = 1$ is equivalent to (71), and (31) acting on $\eta_2 = e^1$ implies

$$F^I_{02} = X^I H_{02} + \partial_2 X^I.$$ \hspace{1cm} (74)

Combining this with (70) we obtain

$$F^I_{0p} = X^I H_{0p} + \partial_p X^I.$$ \hspace{1cm} (75)

Comparing this expression with (72) yields

$$\partial_p X^I = 0$$ \hspace{1cm} (76)

which together with (68) implies that the $X^I$ are constant. Substituting this back into (72) implies that

$$F^I_{0p} = X^I H_{0p}.$$ \hspace{1cm} (77)

Finally, consider (31) acting on $\eta_2 = e^{12}$. This implies

$$F^I_m = X^I H_m + 2i \chi (X^I X^J - \frac{3}{2} Q^{IJ}) V_J.$$ \hspace{1cm} (78)

Comparing (69) with (78) implies that

$$(X^I X^J - \frac{3}{2} Q^{IJ}) V_J = 0.$$ \hspace{1cm} (79)

As the $X_I$ are constant, this constraint implies, using the reasoning in the previous sections, that there exists a non-zero constant $\xi$ such that

$$X_I = \xi V_I.$$ \hspace{1cm} (80)
and hence
\[ F^I_m^m = X^I H_m^m. \]  
(81)

Then (73) implies that
\[ F^I_{1\bar q} = X^I H_{1\bar q}. \]  
(82)

This constraint, together with (81) implies that
\[ F^I_{p\bar q} = X^I H_{p\bar q}. \]  
(83)

Hence we have shown that
\[ F^I = X^I H \]  
(84)

which completes the reduction of these solutions to solutions of the minimal theory.

5. 3/4-supersymmetric solutions of the minimal theory

Having shown that all 3/4 supersymmetric solutions correspond to solutions of the minimal theory, it remains to consider the gravitino Killing spinor equations of the minimal theory obtained from (7). We substitute
\[ \xi V_I \]  
(85)

into (7) and define A by
\[ A = \xi V_I A^I \]  
(86)

so that \( H = dA \), with \( F^I = X^I H \). Lastly, it is convenient to define \( \tilde{\chi} = \chi \xi^{-1} \), and then drop the hat.

In order to analyse these solutions, we shall consider the integrability conditions associated with (7). These can be written as
\[ \tilde{R}_{MN} \eta_A = \frac{1}{2} \left( S^2_{MN} \right)_{N_1N_2} Y_{N_1N_2} + \left( S^1_{MN} \right)_L Y^L + \frac{i}{2} \left( T^2_{MN} \right)_{N_1N_2} Y_{N_1N_2} \]
\[ + i \left( T^1_{MN} \right)_L Y^L + \frac{3i\chi}{2} H_{MN} \eta_A = 0 \]  
(87)

for \( A = 0, 1, 2 \), where
\[ \left( S^2_{MN} \right)_{N_1N_2} = -\frac{1}{2} R_{MN,N_1N_2} - \frac{1}{4} \epsilon_{L_1L_2N_1N_2}[M \nabla_N] H^{L_1L_2} - \frac{1}{4} H_{LM} H^L_{[N_1N_2]N} \]
\[ + \frac{1}{2} H_{LN} H^L_{[N_1N_2]L} + \frac{1}{2} H_M[N_1N_2]_N + \left( \frac{1}{2} H_{L_1L_2} H^{L_1L_2} + \chi^2 \right) g_{MN[N_1N_2]N} \]  
(88)
\[ \left( S^1_{MN} \right)_L = -\frac{1}{2} \nabla_L H_{MN} + \frac{1}{4} H^{L_1L_2} H^L_{[M}\epsilon_{N_1L_1L_2L]N} \]  
(89)
\[ \left( T^2_{MN} \right)_{N_1N_2} = \chi \left( H_{M[N_1N_2]N} - H_{N[N_1N_2]M} \right) \]  
(90)

and
\[ \left( T^1_{MN} \right)_L = -\frac{\chi}{4} \epsilon_{NLL,L_1L_2} H^{L_1L_2}. \]  
(91)

In all cases, we shall show that the integrability condition \( \tilde{R}_{MN} \eta_A = 0 \) for \( A = 0, 1, 2 \) can be used to obtain constraints involving only \( T^2, T^1 \) and \( H \). These constraints are sufficient to fix \( H = 0 \), and so \( T^1 = T^2 = S^1 = 0 \). Furthermore, in all cases, the integrability conditions
then imply that $S^2 = 0$, or equivalently
\[ R_{MNN_{1}N_{2}} = 2 \chi^2 g_{M(N_{1}N_{2})N}. \]  

(92)

This implies that the spacetime geometry is $AdS_5$. However, it is known that $AdS_5$ is the unique maximally supersymmetric solution of this theory. Hence there can be no solutions preserving exactly $3/4$ of the supersymmetry.

In the following sections, we present the integrability constraints used to prove this for all three possible types of $3/4$ supersymmetric solutions, according to whether the Killing spinors $\epsilon_A$ are orthogonal to 1, $1 + e^1$ or $e^1$. In what follows it will be convenient to suppress the $MN$ indices in the tensors $S_1, S_2, T_1, T_2$ and $H$, though these will be re-introduced explicitly in several places.

5.1. Minimal solutions with B-orthogonal spinors to 1

The integrability constraints obtained by requiring that $\tilde{R}_{MN} 1 = 0$ are
\[
-(S^2)^m_m + (S^1)_0 - i(T^2)^m_m + i(T^1)_0 + \frac{3i\chi}{2} H = 0 \\
i(S^2)_0 - i(S^1)_h - (T^2)_0 + (T^1)_h = 0 \\
(S^2)_{\alpha\bar{\alpha}}\epsilon^{\alpha\bar{\alpha}} + i(T^2)_{\alpha\bar{\alpha}}\epsilon^{\alpha\bar{\alpha}} = 0
\]  

(93)

and the integrability constraints obtained by requiring that $\tilde{R}_{MN} e^p = 0$ are
\[
-i(S^2)p - i(S^1)p + (T^2)_0 + (T^1)_0 = 0 \\
-(S^2)m^m \delta_{ph} + 2(S^2)_{ph} - (S^1)_0 \delta_{ph} \\
-i(T^2)m^m \delta_{ph} + 2i(T^2)_{ph} - i(T^1)_0 \delta_{ph} + \frac{3i\chi}{2} H \delta_{ph} = 0 \\
-i(S^2)0q - i(S^1)q + (T^2)_0q + (T^1)_q = 0
\]  

(94)

From these constraints it is straightforward to show that
\[
(T^2)_m^m = 0 \\
(S^2)_m^m = 3i\chi H \\
(S^1)_0 = 0 \\
(T^1)_0 = \frac{3\chi}{2} H
\]  

(95-98)

and
\[ (T^2)_{pq} = 0 \]  

(99)

\[ (S^2)_{pq} = \frac{3i\chi}{2} H \delta_{pq} \]  

(100)

and
\[ (S^2)_0^p = -(S^1)_p = i(T^2)_0^p = -i(T^1)_p. \]  

(101)

To proceed, note that imposing the constraint $(T^1_{MN})_0 = \frac{3\chi}{4} H_{MN}$ for all possible $M, N$ forces all components of $H$ to vanish. Hence $H = S^1 = T^1 = T^2 = 0$, and by the above constraints it follows that $S^2 = 0$ also. This implies that the spacetime geometry is $AdS_5$. 

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5.2. Minimal solutions with B-orthogonal spinors to $1 + e^1$

The integrability constraints obtained by requiring that $\bar{R}_{MN}1 = 0$ are

\[-(S^2)_m^m + (S^1)_0 - i(T^2)_m^m + i(T^1)_0 + \frac{3i\chi}{2}H = 0\]

\[i(S^2)_00 - i(S^1)_a - (T^2)_00 + (T^1)_a = 0\]  \hspace{1cm} (102)

\[(S^2)_{am} e^{am} + i(T^2)_{am} e^{am} = 0.\]

The integrability constraints obtained by requiring that $\bar{R}_{MN}(e^p - \delta^p e^{12}) = 0$ are

\[-\sqrt{2} i(S^2)_0p - \delta_{p2}(S^2)_{mn} e^{mn} - \sqrt{2} i(S^1)_p + \sqrt{2}(T^2)_0p - i\delta_{p2}(T^2)_{mn} e^{mn} + \sqrt{2}(T^1)_p = 0\]

\[\frac{3i\chi}{2}H + 3i = 0\]

The integrability constraints obtained by requiring that $\bar{R}_{MN}1 = 0$ are

\[-\sqrt{2} i(S^2)_0p - \delta_{p2}(S^2)_{mn} e^{mn} - \sqrt{2} i(S^1)_p + \sqrt{2}(T^2)_0p - i\delta_{p2}(T^2)_{mn} e^{mn} + \sqrt{2}(T^1)_p = 0\]

\[\frac{3i\chi}{2}H + 3i = 0\]

\[\frac{3i\chi}{2}H = 0.\]

From these constraints we obtain

\[(S^2)_m^m = i(T^2)_m^m\]  \hspace{1cm} (104)

\[(S^1)_0 = i(T^2)_m^m\]  \hspace{1cm} (105)

\[(S^2)_{0p} = -i(T^1)_0 - \frac{1}{\sqrt{2}}\delta p e^{mn}(T^2)_m^m\]  \hspace{1cm} (106)

\[(S^1)_p = -i(T^2)_0p - \frac{1}{\sqrt{2}}\delta p e^{mn}(T^2)_m^m\]  \hspace{1cm} (107)

\[(S^2)_{pq} = i(T^2)_m^m \delta p q - \sqrt{2}\delta p e^{q}(T^2)_{0} e^{q}(T^1)_{0} e^{q}(T^1)_{0} - i(T^1)_0 e^{q}(T^1)_{0}\]  \hspace{1cm} (109)

and from the last constraint in (103) we find

\[\delta_{p2}(T^2)_m^m - \sqrt{2} i e^{p} (T^2)_0 e^{q} e^{p2} (T^2)_m^m + \delta_{p2}(T^1)_0 - \sqrt{2} i e^{p} (T^1)_{0} + \frac{3\chi}{2} = 0.\]  \hspace{1cm} (110)

Choosing $p = 2$ in the above constraint allows us to express $H$ in terms of components of $T$:

\[\frac{3\chi}{2} H_{MN} = -(T^2)_{MN} + \sqrt{2} i (T^2)_{MN} + i(T^1)_{0} - \sqrt{2} i (T^1)_{0}.\]  \hspace{1cm} (111)

Evaluating this constraint for all possible choices of $M, N$ forces all components of $H$ to vanish. Hence $H = S^1 = T^1 = T^2 = 0$, and by the above constraints it follows that $S^2 = 0$ also. This implies that the spacetime geometry is once more $Ad S_5$.

5.3. Minimal solutions with B-orthogonal spinors to $e^1$

The integrability constraints obtained by requiring that $\bar{R}_{MN}1 = 0$ are

\[-(S^2)_m^m + (S^1)_0 - i(T^2)_m^m + i(T^1)_0 + \frac{3i\chi}{2}H = 0\]

\[i(S^2)_00 - i(S^1)_a - (T^2)_00 + (T^1)_a = 0\]  \hspace{1cm} (112)

\[(S^2)_{am} e^{am} + i(T^2)_{am} e^{am} = 0.\]
The integrability constraints obtained by requiring that $\tilde{R}_{MNE} = 0$ are
\begin{align*}
-i(S^2)_{01} - i(S^1)_1 + (T^2)_{01} + (T^1)_1 &= 0 \\
-(S^2)_m^m \delta_{1\bar{n}} + 2(S^2)_{1\bar{n}} - (S^1)_{0\bar{n}} &= 0 \\
-i(T^2)_m^m \delta_{1\bar{n}} + 2i(T^2)_{1\bar{n}} - i(T^1)_0 \delta_{1\bar{n}} + \frac{3i\chi}{2} H \delta_{1\bar{n}} &= 0 \\
i(S^2)_{0\bar{n}} + i(S^1)_{1\bar{n}} - (T^2)_{0\bar{n}} - (T^1)_1 &= 0.
\end{align*}
(113)

The integrability constraints obtained by requiring that $\tilde{R}_{MNE}^{12} = 0$ are
\begin{align*}
(S^2)_m^m \epsilon_{mn} + i(T^2)_m^m \epsilon_{mn} &= 0 \\
-i(S^2)_{0q} + i(S^1)_q + (T^2)_{0q} - (T^1)_q &= 0 \\
-(S^2)_m^m - (S^1)_0 - i(T^2)_m^m - i(T^1)_0 - \frac{3i\chi}{2} H &= 0.
\end{align*}
(114)

From these constraints we find
\begin{align*}
(S^1)_0 &= (T^2)_m^m = (S^2)_m^m = 0
\end{align*}
(115)
and
\begin{align*}
(T^2)_m^m &= (S^2)_m^m = 0
\end{align*}
(116)
and
\begin{align*}
i(T^1)_0 &= \frac{3i\chi}{2} H
\end{align*}
(117)
and
\begin{align*}
(S^2)_{1\bar{n}} + i(T^2)_{1\bar{n}} + \frac{3i\chi}{2} H \delta_{1\bar{n}} &= 0
\end{align*}
(118)
which implies that
\begin{align*}
(T^2)_{11} &= 0.
\end{align*}
(119)

We also find the constraints
\begin{align*}
(S^2)_{01} &= (S^1)_1 = -i(T^3)_{01} = -i(T^1)_1
\end{align*}
(120)
and
\begin{align*}
(S^2)_{02} &= (S^1)_2 = i(T^2)_{02} = i(T^1)_2.
\end{align*}
(121)

Finally note that imposing the constraint $(T^1_MN)_0 = -\frac{\chi}{2} H_{MN}$ for all possible $M$, $N$ forces all components of $H$ to vanish. Hence $H = S^1 = T^1 = T^2 = 0$, and by the above constraints it follows that $S^2 = 0$ also. This implies that the spacetime geometry is again $AdS_5$.

6. Conclusion

In conclusion, we have studied configurations preserving $3/4$ of supersymmetry for the theory of $N = 2$ five-dimensional gauged supergravity coupled to Abelian vector multiplets. In our analysis we have employed the method of writing spinors of the theory as differential forms. By exploiting the $Spin(4, 1)$ gauge freedom, it was shown that solutions preserving $3/4$ of the supersymmetry can be placed into three classes. For each class of solutions, the algebraic Killing spinor equations, coming from the vanishing of the dilatino supersymmetric variations, reduce our solutions to that of the minimal gauged five-dimensional supergravity. Furthermore, using the integrability conditions of the Killing spinor equations coming from
the vanishing of the gravitino supersymmetric variations, it was shown that the gauge field strengths must vanish. This means that the spacetime geometry is the unique maximally supersymmetric AdS$^5$ and therefore there are no exactly $3/4$ supersymmetric solutions of five-dimensional supergravity coupled to arbitrary many vector multiplets.

Acknowledgments

Jan Gutowski thanks the organizers of the XVth Oporto Meeting on Geometry, Topology and Physics, during which part of this work was completed. Jai Grover thanks the Cambridge Commonwealth Trusts for financial support. The work of Wafic Sabra was supported in part by the National Science Foundation under grant number PHY-0313416.

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