Exterior products of operators and superoptimal analytic approximation

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ABSTRACT

We give a new algorithm for the construction of the unique superoptimal analytic approximant of a given continuous matrix-valued function on the unit circle, using exterior powers of operators in preference to spectral or Wiener–Masani factorizations.

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1. Introduction

In this paper we put forward a new algorithm for the computation of the superoptimal analytic approximation of a continuous matrix-valued function on the circle, a notion that arises naturally in the context of the classical ‘Nehari problem’, and also in the ‘robust stabilization problem’ in control engineering.

To explain the term ‘superoptimal’, let us start from the elementary observation that a measure of the ‘size’ of a compact operator \( T \) between Hilbert spaces is provided by the operator norm \( \| T \| \) of \( T \). However, a single number can only ever provide a coarse measure of the size of a multi-dimensional object, and there is a well-developed classical theory \([12]\) of s-numbers or singular values of an operator or matrix, which provides much more refined information about an operator than the operator norm. Consider Hilbert spaces \( \mathcal{H}, \mathcal{K} \) and an
operator $T : H \to K$, and let $j \geq 0$. The quantity $s_j(T)$ is defined to be the distance, with respect to the operator norm, of $T$ from the set of operators of rank at most $j$:

$$s_j(T) \overset{\text{def}}{=} \inf \{ \| T - R \| : R \in \mathcal{L}(H,K), \text{rank } R \leq j \}.\] Here, for Hilbert spaces $H, K$, we define by $\mathcal{L}(H,K)$ the Banach space of bounded linear operators from $H$ to $K$ with the operator norm. We denote by $\mathcal{K}(H,K)$ the Banach space of compact linear operators from $H$ to $K$ with the operator norm. In the setting of matrices $T$ (that is, in the case that $H$ and $K$ are finite-dimensional), $s_j(T)$ is often called the $j$th singular value of $T$. In this setting one can show that the singular values of $T$ are precisely the eigenvalues of $\sqrt{T^*T}$. The largest singular value of $T$ is the spectral radius of $\sqrt{T^*T}$, that is, $\| T \|$, and so clearly the set of all singular values of $T$ contains much more information than the norm $\| T \|$ alone. The use of $s$-numbers immediately gives rise to a measure of the error in an approximation of an operator- or matrix-valued function. Consider, for example, an $m \times n$ matrix valued function $G$ on the unit circle $\mathbb{T}$, and suppose we wish to approximate $G$ by a matrix-valued function $Q$ of a specified form (such as a rational function of a prescribed McMillan degree). It is natural to regard the difference $G - Q$ as the ‘error’ in the approximation, and to regard the quantities

$$s_j^\infty(G - Q) \overset{\text{def}}{=} \text{ess sup}_{z \in \mathbb{T}} s_j(G(z) - Q(z))$$

for $j \geq 0$ as measures of how good an approximation $Q$ is to $G$. We set

$$s^\infty(G - Q) \overset{\text{def}}{=} (s_0^\infty(G - Q), s_1^\infty(G - Q), \ldots, s_j^\infty(G - Q), \ldots),$$

and say that $AG$ is a superoptimal approximation of $G$ in a given class $\mathcal{F}$ of functions if $s^\infty(G - Q)$ attains its minimum with respect to the lexicographic ordering of the set of sequences of non-negative real numbers over $\mathcal{F}$ when $Q = AG$.

The notion of superoptimality pertains to matricial or operator-valued functions, and is therefore particularly relevant to control engineering and electrical networks more generally, since in these fields one must analyze engineering constructs whose mathematical representations are typically matrix-valued functions on the circle or the real line. In particular, a primary application is to the problem of designing automatic controllers for linear time-invariant plants with multiple inputs and outputs. Such design problems are often formulated in the frequency domain, that is, in terms of the Laplace or $z$-transform of signals. By this means the problem becomes to construct an analytic matrix-valued function in a disc or half-plane, subject to various constraints. An important requirement is usually to minimize, or at least to bound, some cost or penalty function. In practical engineering problems a wide variety of constraints and cost functions arise, and the engineer must take account of many complications, such as the physical limitations of devices and the imprecision of models. Engineers have developed numerous ways to cope with these complications [9, 11]. One of them, developed in the 1980s, is $H^\infty$ control theory [10]. It is a wide-ranging theory, that makes pleasing contact with some problems and results of classical analysis; a seminal role is played by Nehari’s theorem on the best approximation of a bounded function on the circle by an analytic function in the disc. Also important in the development of the theory was a series of deep papers by Adamyan, Arov and Krein [1, 2] which greatly extend Nehari’s theorem and which apply to matrix-valued functions.

In this context the notion of a superoptimal analytic approximation arose very naturally. Simple diagonal examples of a $2 \times 2$-matrix-valued function $G$ on $\mathbb{T}$ show that the set of best analytic approximants to $G$ in the $L^\infty$ norm typically comprises an entire infinite-dimensional ball of functions, and so one is driven to ask for a stronger optimality criterion, and preferably one which will provide a unique optimum. The very term ‘superoptimal’ was coined by engineers even before its existence had been proved in generality. The paper [25] proved that the superoptimal approximant does indeed exist, and moreover is unique, as long
as the approximand $G$ is the sum of a continuous function and an $H^\infty$ function on the circle. In engineering examples $G$ is usually rational and so continuous on the circle.

Let us first provide some preliminary definitions and then formulate the problem. Throughout the paper, $\mathbb{C}^{m \times n}$ denotes the space of $m \times n$ complex matrices with the operator norm and $\mathbb{D}, \mathbb{T}$ denote the unit disc and the unit circle, respectively.

**Definition 1.1.** Let $E$ be a Banach space.

$H^\infty(\mathbb{D}, E)$ denotes the space of bounded analytic $E$-valued functions on the unit disk with supremum norm:

$$\|Q\|_{H^\infty} \overset{\text{def}}{=} \|Q\|_\infty \overset{\text{def}}{=} \sup_{z \in \mathbb{D}} \|Q(z)\|_E.$$  

$L^\infty(\mathbb{T}, \mathbb{C}^{m \times n})$ is the space of essentially bounded weakly measurable $E$-valued functions on the unit circle with essential supremum norm

$$\|f\|_{L^\infty} = \text{ess sup}_{|z|=1} \|f(z)\|_E,$$

and with functions equal almost everywhere identified.

Also, $C(\mathbb{T}, E)$ is the space of continuous $E$-valued functions from $\mathbb{T}$ to $E$.

Naturally engineers need to be able to compute the superoptimal approximant of $G$.

**Problem 1.2** (The superoptimal analytic approximation problem). Given a function $G \in L^\infty(\mathbb{T}, \mathbb{C}^{m \times n})$, find a function $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ such that the sequence $s^\infty(G - Q)$ is minimized with respect to the lexicographic ordering.

In general, the superoptimal analytic approximant may not be unique. However, it has been proved that if the given function $G$ belongs to $H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$, then Problem 1.2 has a unique solution. The following theorem, which was proved by Peller and Young in [25], asserts what we have just stated.

**Theorem 1.3** [25, p. 303]. Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$. Then the minimum with respect to the lexicographic ordering of $s^\infty(G - Q)$ over all $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ is attained at a unique function $AG$. Moreover, the singular values $s_j(G(z) - AG(z))$ are constant almost everywhere on $\mathbb{T}$ for $j \geq 0$.

The topic of this paper is not the existence and uniqueness of the function $AG$ described in Theorem 1.3, but rather the construction of $AG$. In the proof of the validity of our construction, we have no compunction in making any use of results proved in [25], such as the existence of some special matrix functions. For example, to justify our algorithm we shall prove, using results of [25], that certain operators that we introduce are unitarily equivalent to block Hankel operators, which fact enables us to use general properties of Schmidt vectors of Hankel operators, without the need to calculate the symbols of those Hankel operators.

The existence proof in [25] can in principle be turned into an algorithm, but into a very computationally intensive one. The construction is recursive, and at each step of the recursion one must augment a column-matrix function to a unitary matrix-valued function on the circle with some special properties. Computationally this step requires a spectral factorization of a positive semi-definite matrix-valued function on the circle. There are indeed algorithms for this step, but they involve an iteration which may be slow to converge and badly conditioned, especially if some function values have eigenvalues on or close to the unit circle.
It is certainly desirable to avoid the matricial spectral factorization step if it is possible to do so. Our aim in this project was to devise an algorithm in which the iterative procedures are as few and as well conditioned as possible. Iteration cannot be completely avoided; even in the scalar case, optimal error is the norm of a certain operator, and the best approximant is given by a simple formula involving the corresponding Schmidt vectors. Thus, one has to perform a singular value decomposition. In the case that the approximand \( G \) is of type \( m \times n \) one must expect to solve \( \min(m,n) \) successive singular value problems. However, from the point of view of numerical linear algebra, singular value decomposition is regarded as a fast, accurate and well-behaved operation. In this paper we describe an algorithm that is, in a sense, parallel to the construction of \([26]\) and that in addition to the spectral factorization of \( \text{scalar} \) functions, requires only rational arithmetic and singular-value decompositions. Several engineers have developed alternative approaches \([16, 29]\) based on state-space methods. These too are computationally intensive.

For practical purposes, before even looking for an algorithm for the construction of \( \mathcal{A}G \), we need to know that the problem of superoptimal analytic approximation is well posed, in sense that arbitrarily small perturbations of \( G \) do not result in large fluctuations in \( \mathcal{A}G \). This issue arises even for scalar \( G \), and in fact it is known \([22]\) that, for general continuous functions \( G \), \( \mathcal{A}G \) does not depend continuously on \( G \). However, Peller and Khruschev have shown in \([22]\) that, for \( G \) in suitable subspaces \( X \) of the continuous functions on \( T \), the best analytic approximation operator is continuous for \( \| \cdot \|_X \), and so it makes sense to compute it. A similar assertion holds for matrix-valued functions \( G \), as was shown by Peller and Young in \([28]\).

We believe that the present method, which uses exterior powers of Hilbert spaces and operators, provides a conceptual approach to the construction of superoptimal approximants which is a promising basis for computation. The theoretical justification of the algorithm we present in this paper is lengthy and elaborate. However, the implementation of the algorithm should be straightforward. It will be very interesting to see whether it leads to an efficient numerical method in the future.

For vector-valued \( L^p \) spaces we use the terminology of \([33]\).

**Definition 1.4.** Let \( E \) be a separable Hilbert space and let \( 1 \leq p < \infty \). Define

(i) \( L^p(T, E) \) to be the normed space of measurable (weakly or strongly, which amounts to the same thing, in view of the separability of \( E \)) \( E \)-valued maps \( f : T \to E \) such that

\[
\|f\|_p = \left( \frac{1}{2\pi} \int_0^{2\pi} \|f(e^{i\theta})\|_E^p d\theta \right)^{1/p} < \infty;
\]

(ii) \( H^p(D, E) \) to be the normed space of analytic \( E \)-valued maps \( f : D \to E \) such that

\[
\|f\|_p = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_E^p d\theta \right)^{1/p} < \infty,
\]

the left-hand side of this inequality defining a norm on \( H^p(D, E) \).

Our algorithm provides a solution \( \mathcal{A}G \) to Problem 1.2. By computing the value of each \( t_k \) at every step, we obtain each term \( s_k^\infty(G - \mathcal{A}G) \) of the sequence \( s^\infty(G - \mathcal{A}G) \). First, we need the notion of a \emph{Hankel operator} and the definitions of some long-established standard function spaces; for a more detailed account of these spaces see \([33, \text{Chapter V}]\).

If \( E \) is a separable Hilbert space, then every function \( f \in H^2(D, E) \) has a radial limit at almost every point of \( T \), by a theorem of Fatou \([33, \text{Chapter V}]\), and the map that takes a function \( f \in H^2(D, E) \) to its radial limit function embeds \( H^2(D, E) \) isometrically in \( L^2(T, E) \). In this paper we only envisage the case that \( E \) is separable, and so we can always regard \( H^2(D, E) \) as a closed subspace of \( L^2(T, E) \). The operators \( P_+, P_- \) on \( L^2(T, E) \) are the operators
of orthogonal projection onto the closed subspaces \( H^2(\mathbb{D}, E) \) and \( H^2(\mathbb{D}, E)^\perp \) respectively.

**Definition 1.5.** Let \( E \) be a separable Hilbert space, and let \( \varphi \) be an essentially bounded measurable \( \mathcal{L}(E) \)-valued function on \( \mathbb{T} \); then the Hankel operator \( H_\varphi \) is the operator from \( H^2(\mathbb{D}, E) \) to \( H^2(\mathbb{D}, E)^\perp \) given by

\[
H_\varphi x = P_-(\varphi x) \quad \text{for } x \in H^2(\mathbb{D}, E), \quad \text{where } (\varphi x)(z) = \varphi(z)x(z) \text{ for } z \in \mathbb{T}.
\]

**Remark 1.6.** In this paper we call an operator \( U : H \to K \) between Hilbert spaces \( H, K \) a unitary operator if \( U \) is both isometric and surjective. Some authors restrict the name “unitary operator” to the case that \( H = K \). Such authors would use a terminology like “isometric isomorphism” for our “unitary operator” in the case that \( H \neq K \).

For any vector \( x \) in a Hilbert space \( E \), we denote by \( x^* \) the linear functional \( \langle \cdot, x \rangle_E \) on \( E \). For an \( E \)-valued function \( x \) on a set \( S \subset \mathbb{C} \), we define an \( E^* \)-valued function \( x^* \) on \( S \) by \( x^*(z) = x(z)^* \) for all \( z \in S \). We observe that if \( x \in L^p(\mathbb{T}, E) \), where \( 1 \leq p \leq \infty \), then \( x^* \in L^p(\mathbb{T}, E) \) and \( \|x^*\|_p = \|x\|_p \).

If \( x, y \in E \), then \( xy^* \) denotes the operator of rank one on \( E \) defined by \( xy^*(u) = \langle u, y \rangle_E x \) for all \( u \in E \). This operator is sometimes denoted by \( x \otimes y \) (see, for example, [3, equation (1.17)]). If \( x, y \) are \( E \)-valued functions on a set \( S \subset \mathbb{C} \), then \( xy^* \) is the function from \( S \) to \( \mathcal{L}(E) \) given by \( xy^*(z) = x(z)y(z)^* \) for all \( z \in S \).

**Definition 1.7** [38, p. 206]. Let \( H, K \) be Hilbert spaces and let \( T : H \to K \) be a compact operator. Suppose that \( s \) is a singular value of \( T \). A Schmidt pair for \( T \) corresponding to \( s \) is a pair \((x, y)\) of non-zero vectors, with \( x \in H \), \( y \in K \), such that

\[
Tx = sy, \quad T^*y = sx.
\]

The following lemma is elementary.

**Lemma 1.8.** Let \( T \in \mathcal{L}(H, K) \) be a compact operator and let \( x \in H \), \( y \in K \) be such that \((x, y)\) is a Schmidt pair for \( T \) corresponding to \( s = \|T\| \). Then \( x \) is a maximizing vector for \( T \), \( y \) is a maximizing vector for \( T^* \), and \( \|x\|_H = \|y\|_K \).

**Definition 1.9** [33, p. 190]. (i) The matrix-valued bounded analytic function \( \Theta \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) is called inner if \( \Theta(e^{it}) \) is an isometry from \( \mathbb{C}^n \) to \( \mathbb{C}^m \) for almost every \( e^{it} \) on \( \mathbb{T} \).

(ii) An analytic \((m \times n)\)-matrix-valued function \( \Phi \) on \( \mathbb{D} \) is said to be outer if

\[
\Phi H^2(\mathbb{D}, \mathbb{C}^n) = \{ \Phi f : f \in H^2(\mathbb{D}, \mathbb{C}^n) \}
\]

is a norm-dense subspace of \( H^2(\mathbb{D}, \mathbb{C}^m) \), and co-outer if

\[
\Phi^T H^2(\mathbb{D}, \mathbb{C}^m) = \{ \Phi^T g : g \in H^2(\mathbb{D}, \mathbb{C}^m) \}
\]

is dense in \( H^2(\mathbb{D}, \mathbb{C}^n) \).

The following is a brief summary of our algorithm. A full account of all the steps, with definitions and justifications will be given in Section 4. Our method uses exterior powers \( \wedge^p E \) of a finite-dimensional Hilbert space \( E \) and of ‘pointwise wedge products’ of \( E \)-valued functions \( f, g \) on \( \mathbb{D} \) or \( \mathbb{T} \), defined by

\[
(f \wedge g)(z) = f(z) \wedge g(z) \quad \text{for all } z \in \mathbb{D} \text{ or for all } z \in \mathbb{T}.
\]

These notions are explained more fully in Subsection 3.2.
Algorithm: For a given $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$, the superoptimal analytic approximant $AG \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ can be constructed as follows.

(i) Step 0. Let $T_0 = H_G$ be the Hankel operator with symbol $G$. Let $t_0 = \|H_G\|$. If $t_0 = 0$, then $H_G = 0$, which implies $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$. In this case, the algorithm terminates, we define $r$ to be zero and the superoptimal approximant $AG$ is given by $AG = G$.

Let $t_0 \neq 0$. The Hankel operator $H_G$ is a compact operator and so there exists a Schmidt pair $(x_0, y_0)$ corresponding to the singular value $t_0 = \|H_G\|$ of $H_G$. By the definition of a Schmidt pair $(x_0, y_0)$,

$$x_0 \in H^2(\mathbb{D}, \mathbb{C}^n), \quad y_0 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

are non-zero vector-valued functions such that

$$H_G x_0 = t_0 y_0, \quad H_G^* y_0 = t_0 x_0.$$

The functions $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $\bar{y}_0 \in H^2(\mathbb{D}, \mathbb{C}^m)$ admit the inner–outer factorizations

$$x_0 = \xi_0 h_0, \quad \bar{y}_0 = \eta_0 h_0 \quad (1.1)$$

for some scalar outer factor $h_0 \in H^2(\mathbb{D}, \mathbb{C})$ and column-matrix inner functions $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, $\eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)$. Then,

$$\|x_0(z)\|_{\mathbb{C}^n} = |h_0(z)| = \|y_0(z)\|_{\mathbb{C}^m} \quad \text{almost everywhere on } \mathbb{T}. \quad (1.2)$$

We write equations (1.1) as

$$\xi_0 = \frac{x_0}{h_0}, \quad \eta_0 = \frac{\bar{y}_0}{h_0}. \quad (1.3)$$

Then

$$\|\xi_0(z)\|_{\mathbb{C}^n} = 1 = \|\eta_0(z)\|_{\mathbb{C}^m} \quad \text{almost everywhere on } \mathbb{T}. \quad (1.4)$$

There exists a function $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ which is at minimal distance from $G$; any such function satisfies

$$(G - Q_1) x_0 = t_0 y_0, \quad y_0^* (G - Q_1) = t_0 x_0^*. \quad (1.5)$$

Choose any function $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ which satisfies the equations (1.5).

(ii) Step 1. Let

$$X_1 = \xi_0 \wedge H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \wedge^2 \mathbb{C}^n), \quad (1.6)$$

and let

$$Y_1 = \eta_0 \wedge H^2(\mathbb{D}, \mathbb{C}^m) \subset H^2(\mathbb{D}, \wedge^2 \mathbb{C}^m)^\perp. \quad (1.7)$$

$X_1$ is a closed linear subspace of $H^2(\mathbb{D}, \wedge^2 \mathbb{C}^n)$. $Y_1$ is a closed linear subspace of $H^2(\mathbb{D}, \wedge^2 \mathbb{C}^m)^\perp$.

Define the operator

$$T_1 : X_1 \to Y_1$$

by

$$T_1(\xi_0 \wedge x) = P_{Y_1}(\eta_0 \wedge (G - Q_1) x) \quad \text{for all } x \in H^2(\mathbb{D}, \mathbb{C}^n), \quad (1.8)$$

where $P_{Y_1}$ is the projection from $L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)$ on $Y_1$. We show that $T_1$ is well defined.

If $T_1 = 0$, then the algorithm terminates, we define $r$ to be 1 and the superoptimal approximant $AG$ is given by the formula

$$G - AG = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2} = \frac{t_0 y_0 x_0^*}{|h_0|^2},$$

where $t_i$ are the singular values of $H_G$ and $y_i x_i^*$ are the Schmidt pair with $t_i = \|y_i x_i^*\|$. 

The remaining case, $T_1 \neq 0$, defines $r$ as the smallest integer for which $T_1 = 0$. In this case, $AG$ is not in $H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$, it is a bounded analytic operator (in other words, a superoptimal analytic approximant) $AG : H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \to H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$, and $AG = G$.
and the solution is
\[ AG = G - \frac{t_0 y_0 x_0^*}{|h_0|^2}. \]

If \( T_1 \neq 0 \), let \( t_1 = \|T_1\| > 0 \). \( T_1 \) is a compact operator and so there exist \( v_1 \in H^2(\mathbb{D}, \mathbb{C}^n) \), \( w_1 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp \) such that \((\xi_0 \wedge v_1, \bar{\eta}_0 \wedge w_1)\) is a Schmidt pair for \( T_1 \) corresponding to \( t_1 \). Let \( h_1 \) be the scalar outer factor of \( \xi_0 \wedge v_1 \) and let
\[ x_1 = (I_{\mathbb{C}^n} - \xi_0 \xi_0^*)v_1, \quad y_1 = (I_{\mathbb{C}^m} - \bar{\eta}_0 \eta_0^T)w_1, \tag{1.9} \]
where \( I_{\mathbb{C}^n} \) and \( I_{\mathbb{C}^m} \) are the identity operators in \( \mathbb{C}^n \) and \( \mathbb{C}^m \), respectively. Then
\[ \|x_1(z)\|_{\mathbb{C}^n} = |h_1(z)| = \|y_1(z)\|_{\mathbb{C}^m} \text{ almost everywhere on } T. \tag{1.10} \]

There exists a function \( Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) such that both \( s_0^\infty(G - Q_2) \) and \( s_1^\infty(G - Q_2) \) are minimized and
\[ s_1^\infty(G - Q_2) = t_1. \]

Any such \( Q_2 \) satisfies
\[ (G - Q_2)x_0 = t_0 y_0, \quad y_0^*(G - Q_2) = t_0 x_0^*, \tag{1.11} \]
\[ (G - Q_2)x_1 = t_1 y_1, \quad y_1^*(G - Q_2) = t_1 x_1^*. \]

Choose any function \( Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) which satisfies the equations (1.11). Define
\[ \xi_1 = \frac{x_1}{h_1}, \quad \eta_1 = \frac{y_1}{h_1}. \tag{1.12} \]

Then \( \|\xi_1(z)\|_{\mathbb{C}^n} = 1 = \|\eta_1(z)\|_{\mathbb{C}^m} \) almost everywhere on \( T \).

**Definition 1.10.** Let \( E \) be a Hilbert space. We say that a collection \( \{\gamma_j\} \) of elements of \( L^2(\mathbb{T}, E) \) is pointwise orthonormal on \( \mathbb{T} \) if, for almost all \( z \in \mathbb{T} \) with respect to Lebesgue measure, the collection of vectors \( \{\gamma_j(z)\} \) is orthonormal in \( E \).

(iii) **Inductive step.** Suppose that we have constructed
\[
\begin{align*}
t_0 &\geq t_1 \geq \cdots \geq t_j > 0 \\
x_0, x_1, \ldots, x_j &\in L^2(\mathbb{T}, \mathbb{C}^n) \\
y_0, y_1, \ldots, y_j &\in L^2(\mathbb{T}, \mathbb{C}^m) \\
h_0, h_1, \ldots, h_j &\in H^2(\mathbb{D}, \mathbb{C}) \text{ outer} \\
\xi_0, \xi_1, \ldots, \xi_j &\in L^\infty(\mathbb{T}, \mathbb{C}^n) \text{ pointwise orthonormal on } \mathbb{T} \\
\eta_0, \eta_1, \ldots, \eta_j &\in L^\infty(\mathbb{T}, \mathbb{C}^m) \text{ pointwise orthonormal on } \mathbb{T} \\
X_0 &\in H^2(\mathbb{D}, \mathbb{C}^n), X_1, \ldots, X_j \\
Y_0 &\in H^2(\mathbb{D}, \mathbb{C}^m)^\perp, Y_1, \ldots, Y_j \\
T_0, T_1, \ldots, T_j &\text{ compact operators.}
\end{align*}
\]

There exists a function \( Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) such that
\[
(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \ldots, s_j^\infty(G - Q_{j+1}))
\]
is lexicographically minimized. Any such function \( Q_{j+1} \) satisfies
\[
(G - Q_{j+1})x_i = t_i y_i, \quad y_i^*(G - Q_{j+1}) = t_i x_i^*, \quad i = 0, 1, \ldots, j. \tag{1.13}
\]

Choose any function \( Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) which satisfies the equations (1.13).
Define
\[ X_{j+1} = \xi_0 \hat{\lambda} \hat{\xi}_1 \cdots \hat{\lambda} \hat{\xi}_j \hat{H}^2(D, \mathbb{C}^n), \]  
and let
\[ Y_{j+1} = \eta_0 \hat{\lambda} \hat{\eta}_1 \cdots \hat{\lambda} \hat{\eta}_j \hat{H}^2(D, \mathbb{C}^m)^\perp. \]

\( X_{j+1} \) is a closed subset of \( H^2(D, \Lambda^{j+2} \mathbb{C}^n) \), and \( Y_{j+1} \) is a closed subspace of \( H^2(D, \Lambda^{j+2} \mathbb{C}^m)^\perp \).

Consider the operator
\[ T_{j+1} : X_{j+1} \to Y_{j+1} \]
given, for all \( x \in H^2(D, \mathbb{C}^n) \), by
\[ T_{j+1}(\xi_0 \hat{\lambda} \hat{\xi}_1 \cdots \hat{\lambda} \hat{\xi}_j \hat{A}x) = P_{Y_{j+1}}(\eta_0 \hat{\lambda} \hat{\eta}_1 \cdots \hat{\lambda} \hat{\eta}_j \hat{A}(G - Q_{j+1})x). \]  
(1.16)

\( T_{j+1} \) is well defined.

If \( T_{j+1} = 0 \) then the algorithm terminates, we define \( r \) to be \( j + 1 \), and the superoptimal approximant \( AG \) is given by the formula
\[ G - AG = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2} = \sum_{i=0}^{j} \frac{t_i y_i x_i^*}{|h_i|^2}. \]

Otherwise, we define \( t_{j+1} = ||T_{j+1}|| > 0 \). Then \( T_{j+1} \) is a compact operator and hence there exist \( v_{j+1} \in H^2(D, \mathbb{C}^n) \), \( w_{j+1} \in H^2(D, \mathbb{C}^m)^\perp \) such that
\[ (\xi_0 \hat{\lambda} \hat{\xi}_1 \cdots \hat{\lambda} \hat{\xi}_j \hat{A}v_{j+1}, \eta_0 \hat{\lambda} \hat{\eta}_1 \cdots \hat{\lambda} \hat{\eta}_j \hat{A}w_{j+1}) \]  
(1.17)
is a Schmidt pair for \( T_{j+1} \) corresponding to the singular value \( t_{j+1} \).

Let \( h_{j+1} \) be the scalar outer factor of \( \xi_0 \hat{\lambda} \hat{\xi}_1 \cdots \hat{\lambda} \hat{\xi}_j \hat{A}v_{j+1} \), and let
\[ x_{j+1} = (I_{C^n} - \xi_0^* \xi_0 - \cdots - \xi_j^* \xi_j) v_{j+1}, \quad y_{j+1} = (I_{C^m} - \eta_0^* \eta_0 - \cdots - \eta_j^* \eta_j) w_{j+1}, \]
and define
\[ \xi_{j+1} = \frac{x_{j+1}}{h_{j+1}}, \quad \eta_{j+1} = \frac{y_{j+1}}{h_{j+1}}. \]  
(1.19)

One can show that \( ||\xi_{j+1}(z)||_{C^n} = 1 \) and \( ||\eta_{j+1}(z)||_{C^m} = 1 \) almost everywhere on \( T \).

This completes the recursive step. The algorithm terminates after at most \( \min(m, n) \) steps, so that, \( r \leq \min(m, n) \) and the superoptimal approximant \( AG \) is given by the formula
\[ G - AG = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2}. \]

**Remark 1.11.** Observe that, in step \( j \) of the algorithm, we define an operator \( T_j \) in terms of any function \( Q_j \in H^\infty(D, \mathbb{C}^{m \times n}) \) that satisfies the equations
\[ (G - Q_j)x_i = t_i y_i, \quad y_i^*(G - Q_j) = t_i x_i^*, \quad i = 0, 1, \ldots, j - 1. \]  
(1.20)

This constitutes a system of linear equations for \( Q_j \) in terms of the computed quantities \( x_i, t_i \) and \( y_i \) for \( i = 0, \ldots, j - 1 \), and we know, from Proposition 10.5, that the system has a solution for \( Q_j \) in \( H^\infty(D, \mathbb{C}^{m \times n}) \). By Proposition 8.1 \( T_j \) is independent of the choice of \( Q_j \) that satisfies equations (1.20).

**Remark 1.12.** At each step we need to find \( ||T_j|| \) and a Schmidt pair
\[ (\xi_0 \hat{\lambda} \hat{\xi}_1 \cdots \hat{\lambda} \hat{\xi}_{j-1} \hat{A}v_j, \eta_0 \hat{\lambda} \hat{\eta}_1 \cdots \hat{\lambda} \hat{\eta}_{j-1} \hat{A}w_j) \]  
(1.21)
for \( T_j \) corresponding to the singular value \( t_j \). Then we compute the scalar outer factor \( h_j \) of \( \xi_0 \hat{\lambda} \hat{\xi}_1 \cdots \hat{\lambda} \hat{\xi}_{j-1} \hat{A}v_j \in H^2(D, \Lambda^{j+1} \mathbb{C}^n) \). These are the only spectral factorizations needed.
in the algorithm. Note that if \( f \in H^2(\mathbb{D}, \mathbb{C}^n) \) has the inner–outer factorization \( f = hg \), with \( h \in H^2(\mathbb{D}, \mathbb{C}) \) a scalar outer function and \( g \in H^\infty(\mathbb{D}, \mathbb{C}^n) \) inner, then \( (f^*f)(z) = |h(z)|^2 \) almost everywhere on \( \mathbb{T} \), and so the calculation of \( h \) requires us to find a spectral factorization of the positive scalar-valued function \( f^*f \) on the circle.

**Remark 1.13.** In a numerical implementation of the algorithm one would need to find a way to compute the norms and Schmidt vectors of the compact operators \( T_j \). For this purpose it would be natural to choose convenient orthonormal bases of the cokernel \( X_j \ominus \ker T_j \) and the range \( \text{ran } T_j \). It is safe to assume that in most applications \( G \) will be a rational function, in which case the cokernel and range will be finite-dimensional. At step 0, \( T_0 \) is a Hankel operator, and the calculation of the matrix of \( T_0 \) with respect to suitable orthonormal bases is a known task [36]; we believe that similar methods will work for step \( j \).

In Theorem 10.15 we arrive at the following conclusion about the superoptimal approximant \( AG \).

**Theorem 1.14.** Let \( G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n}) \). Let \( T_i, x_i, y_i, h_i \), for \( i \geq 0 \), be defined by the algorithm above. Let \( r \) be the least index \( j \geq 0 \) such that \( T_j = 0 \). Then \( r \leq \min(m, n) \) and the superoptimal approximant \( AG \) is given by the formula

\[
G - AG = \sum_{i=0}^{r-1} t_i y_i x_i^* |h_i|^2.
\]

Wedge products, and in particular pointwise wedge products, along with their properties are studied in detail in Section 3.

### 2. History and recent work

The Nehari problem of approximating an essentially bounded Lebesgue measurable function on the unit circle \( \mathbb{T} \) by a bounded analytic function on the unit disk \( \mathbb{D} \), has been attracting the interest of both pure mathematicians and engineers since the middle of the 20th century. The problem was first formulated and studied from the viewpoint of scalar-valued functions, and, in the years that followed, from the operator-valued perspective also, which motivated research into the superoptimal approximation problem.

The Nehari problem in the scalar case first appeared in the paper of Nehari [17]. Given an essentially bounded complex valued function \( g \) on \( \mathbb{T} \), one seeks its distance from \( H^\infty \) with respect to the essential supremum norm, and wishes to determine for which elements of \( H^\infty \) this distance is attained. It is also of interest to know whether the distance is attained at a uniquely determined function. Such problems have been studied in detail by Nehari [17], Sarason [30] and Adamjan, Arov and Krein in [1, 2]. These authors proved that the distance of \( g \) from \( H^\infty \) is equal to the norm of the Hankel operator \( H_g \) with symbol \( g \). Moreover, if \( H_g \) has a maximizing vector in \( H^2 \), then the bounded analytic complex-valued function \( q \) that minimizes the essential supremum norm \( \|g - q\|_L^\infty \) is uniquely determined and can be explicitly calculated (see, for example, [38, p. 196]). Furthermore, if the essential norm \( \|H_g\|_e \) is less than \( \|H_g\|_e \), then \( g \) has a unique best approximant.

Pure mathematicians and engineers started seeking analogues of those results for matrix- and operator-valued functions. These generalizations are not only mathematically interesting but are essential for applications in engineering, and especially in control theory. There has accordingly been an explosion of research in this field since 1980, on the part of both pure mathematicians and engineers.
Page [18] and Treil [34] gave various extensions of the results of Adamjan, Arov and Krein to operator-valued functions. Page proved that for operator-valued mappings $T \in L^\infty(\mathbb{T}, \mathcal{L}(E_1, E_2))$, $\inf \{\|T - \Phi\| : \Phi \in H^\infty(\mathbb{D}, \mathcal{L}(E_1, E_2))\}$ is equal to $\|H_T\|$. Here $E_1, E_2$ are Hilbert spaces and $\mathcal{L}(E_1, E_2)$ denotes the Banach space of bounded linear operators from $E_1$ to $E_2$. Treil extended the Adamjan, Arov and Krein theorem in [2] to an operator-valued analogue.

However, in the matrix-valued setting there are typically infinitely many functions that minimize the $L^\infty$ norm of the error function. This fact is simply illustrated by the following example. Let $G(z) = \text{diag}\{\bar{z}, 0\}$, for $z \in \mathbb{T}$. The norm of $H_G$ in this case is easily seen to be 1, and hence all matrix-valued functions $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2})$ of the form $Q(z) = \text{diag}\{0, q(z)\}$, where $q \in H^\infty$ and $\|q\|_{H^\infty} \leq 1$, minimize the norm $\|G - Q\|_{L^\infty}$, yielding the error 1. However, if one goes on to minimize in turn the essential suprema of both singular values of $G(z) - Q(z)$ over $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2})$, one finds that such a minimum occurs uniquely when $q(z)$ is equal to 0. This type of example suggests that the enhanced approximation criterion based on successive singular values generates the ‘very best’ amongst the best approximants to $G$ by an element of $H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2})$.

Such reflections led to the formulation of a strengthened approximation problem, the superoptimal approximation problem 1.2 as explained above. In [37] Young introduced this strengthened notion of optimal analytic approximation, subsequently called superoptimal approximation. Given a $G$ as above, find a $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that the sequence $s^\infty(G - Q)$ is lexicographically minimized. This criterion obviously constitutes a considerable strengthening of the notion of optimality, as one needs to determine a $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ that not only minimizes $\|G - Q\|_{L^\infty}$ but minimizes the $L^\infty$ norm of all the subsequent singular values $s_j(G(z) - Q(z))$ for $j \geq 0$.

A good starting point for the superoptimal approximation problem of matrix functions is [25]. As we have said, the problem is to find, for a given

$$G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n}),$$

a function $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that the sequence $s^\infty(G - Q)$ is lexicographically minimized. Peller and Young proved some requisite preparatory results on ‘thematic factorizations’, on the analyticity of the minors of unitary completions of inner matrix columns and on the compactness of some Hankel-type operators with matrix symbols. These results provided the foundation for their main theorem, namely that if $G$ belongs to $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$, then there exists a unique $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that the sequence $s^\infty(G - Q)$ is lexicographically minimized as $Q$ varies over $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$; moreover for this $Q$, the singular values $s_j(G(z) - Q(z))$ are constant almost everywhere for $z \in \mathbb{T}$, for $j = 0, 1, 2, \ldots$.

Later, in [26] Peller and Young presented a conceptual algorithm for the computation of the superoptimal approximant. Their algorithm is based on the theory developed in [25]. Also in [26], the algorithm was applied to a concrete example of a rational $2 \times 2$ matrix-valued function $G$ in $H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2}) + C(\mathbb{T}, \mathbb{C}^{2\times 2})$ and the superoptimal approximant $AG$ was calculated by hand.

Additionally, Peller and Young in [27] studied superoptimal approximation by meromorphic matrix-valued functions, that is, matrix-valued functions in $H^\infty$ that have at most $k$ poles for some prescribed integer $k$. They modified the results of [25] and established a uniqueness criterion in the case that the given matrix-valued function $G$ is in $H^\infty + C$ and has at most $k$ poles. In addition, they provided an algorithm for the calculation of the superoptimal approximant.

One can extend the above results to operator-valued functions on the circle; the operator-valued superoptimal approximation problem was studied by Peller in [20]. He generalized the notions of [25] and proved that there exists a unique superoptimal approximant in $H^\infty(B)$ for functions that belong to $H^\infty(B) + C(C)$, where $B$ denotes the space of bounded linear operators and $C(C)$ denotes the space of continuous functions on the circle taking values in the space of compact operators.
Very badly approximable functions, that is, functions that have the zero function as a superoptimal approximant, were studied in the years that followed and a considerable amount of work was published. Peller and Young’s paper [25] provided the motivation for the study of this problem, where they were able to algebraically characterize the very badly approximable matrix functions of class \( H^\infty(D, \mathbb{C}^{m \times n}) + C(T, \mathbb{C}^{m \times n}) \). Their results were extended in [23] to the case of matrix functions \( G \) for which \( \|H_G\|_e \) is less than the smallest non-zero superoptimal singular value of \( G \). Very badly approximable matrix functions with entries in \( H^\infty + C \) were completely characterized in [24].

Recent work in [4] by Baratchart, Nazarov and Peller explores the analytic approximation of matrix-valued functions in \( L^p \) of the unit circle by matrix-valued functions from \( H^p \) of the unit disk in the \( L^p \) norm for \( p \leq 2 \). They proved that if a given matrix-valued function \( \Psi \in L^p(T, \mathbb{C}^{m \times n}) \) is a ‘respectable’ matrix function, then its distance from \( H^p(D, \mathbb{C}^{m \times n}) \) is equal to \( \|H_\Psi\|_e \), and they obtained a characterization of that distance also in the case \( \Psi \) is a ‘weird’ matrix-valued function. Furthermore, they established the notion of \( p \)-superoptimal approximation and illustrated the fact that every \( n \times n \) rational matrix function has a unique \( p \)-superoptimal approximant for \( 2 \leq p < \infty \). For the case \( p = \infty \) they provided a counterexample.

In a more recent paper of Condori [6], the author considered the relation between the sum of the superoptimal singular values of admissible functions in \( L^\infty(T, \mathbb{C}^{m \times n}) \) and the superoptimal analytic approximation problem in the space \( L^\infty(T, S^m_p) \), where \( S^m_p \) denotes the space of \( m \times n \) matrices endowed with the Schatten-von Neumann norm \( \|\cdot\|_{S^m_p} \). He illustrated the fact that if \( \Phi \in L^\infty(T, \mathbb{C}^{n \times n}) \) is an admissible matrix function of order \( k \), then \( Q \in H^\infty(D, \mathbb{C}^{n \times n}) \) is a best approximant function under the \( L^\infty(T, S^m_p) \)-norm and the singular values \( s_j((\varphi - Q)(z)) \) are constant almost everywhere on \( T \) for \( j = 0, 1, \ldots, k - 1 \) if and only if \( Q \) is a superoptimal approximant to \( \Phi \),

\[
\text{ess sup}_{z \in T} s_j((\Phi - Q)(z)) = 0
\]

for \( j \geq k \), and the sum of the superoptimal singular values of \( \Phi \) is equal to

\[
\sup \left| \int_T \text{trace}(\Phi(\zeta)\Psi(\zeta)) \, dm(\zeta) \right|,
\]

where \( m, n > 1 \), \( 1 \leq k \leq \min(m, n) \) and the supremum is taken over all \( \Psi \in H^1(\mathbb{D}, \mathbb{C}^{n \times m}) \) for which \( \|\Psi\|_{L^1(T, \mathbb{C}^{n \times m})} \leq 1 \) and \( \text{rank} \Psi(\zeta) \leq k \) almost everywhere on \( T \).

### 3. Exterior powers of Hilbert spaces

In this section we recall the well-established notion of the wedge product of Hilbert spaces. One can find definitions and properties of wedge products in [7, 13, 19, 31, 32, 35]. Here we present a concise version of this theory which we need for the new superoptimal algorithm.

#### 3.1. Exterior powers

In this subsection, we first present some results concerning the action of permutation operators on tensors, then we recall the definition of antisymmetric tensors and we define an inner product on the space of all antisymmetric tensors. In the following \( E \) denotes a Hilbert space. We shall assume known the notion of the algebraic tensor product of vector spaces, which is precisely explained in [14]. For the Hilbert space tensor product of Hilbert spaces, see [8]. One can find proofs of many statements given below, in our paper [5].

**Definition 3.1.** \( \otimes^p E \) is the \( p \)-fold algebraic tensor product of \( E \) and is spanned by tensors of the form \( x_1 \otimes x_2 \otimes \cdots \otimes x_p \), where \( x_j \in E \) for \( j = 1, \ldots, p \).
**Definition 3.2.** An inner product on $\otimes^p E$ is defined on elementary tensors by

$$
\langle x_1 \otimes x_2 \otimes \cdots \otimes x_p, y_1 \otimes y_2 \otimes \cdots \otimes y_p \rangle_{\otimes^p E} = p! \langle x_1, y_1 \rangle_E \cdots \langle x_p, y_p \rangle_E,
$$

for any $x_1, \ldots, x_p, y_1, \ldots, y_p \in E$, and is extended to $\otimes^p E$ by sesqui-linearity.

**Definition 3.3.** $\otimes^p_H E$ is the completion of $\otimes^p E$ with respect to the norm $\|u\| = \langle u, u \rangle_{\otimes^p E}^{1/2}$, for $u \in \otimes^p E$.

**Definition 3.4.** Let $S_p$ denote the symmetric group on $\{1, \ldots, p\}$, with the operation of composition. For $\sigma \in S_p$, we define $S_\sigma : \otimes^p E \to \otimes^p E$ on elementary tensors by

$$
S_\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_p) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(p)},
$$

and we extend $S_\sigma$ to $\otimes^p E$ by linearity, that is, for $u = \sum_{i=1}^n \lambda_i x_i^1 \otimes \cdots \otimes x_i^p$, we define

$$
S_\sigma(u) = \sum_{i=1}^n \lambda_i S_\sigma(x_i^1 \otimes \cdots \otimes x_i^p),
$$

for any $x_i^j \in E$ and $\lambda_i \in \mathbb{C}$.

**Remark 3.5.** Clearly if $\sigma$ is a bijective self-map of $\{1, \ldots, p\}$, then so is its inverse map $\sigma^{-1}$, and $(S_p, \circ)$ is a group under composition. Moreover,

$$
\sigma \circ \sigma^{-1} = \text{id} = \sigma^{-1} \circ \sigma,
$$

where id $\in S_p$ is the identity map on $\{1, \ldots, p\}$. Then, if $\epsilon_\sigma$ denotes the signature of the permutation $\sigma$,

$$
\epsilon_{\sigma \circ \sigma^{-1}} = \epsilon_\sigma \epsilon_{\sigma^{-1}} = 1,
$$

hence $\epsilon_\sigma = \epsilon_{\sigma^{-1}}$.

**Proposition 3.6.** Let $E$ be a Hilbert space, and let $p$ be any positive integer. Then, for any $\sigma \in S_p$, $S_\sigma$ is a linear operator on the normed space $(\otimes^p E, \|\cdot\|)$, which extends to an isometry $S_\sigma$ on $(\otimes^p_H E, \|\cdot\|)$. Furthermore, $S_\sigma$ is a unitary operator on $\otimes^p_H E$, $S_\sigma^* = S_{\sigma^{-1}}$, and therefore

$$
S_\sigma^* S_\sigma = S_{\sigma^{-1}} S_\sigma = I
$$

is the identity operator on $\otimes^p_H E$.

Henceforth we shall denote the extended operator $S_\sigma$ by $S_\sigma$.

**Definition 3.7.** A tensor $u \in \otimes^p_H E$ is said to be symmetric if $S_\sigma(u) = u$ for all $\sigma \in S_p$. A tensor $u \in \otimes^p_H E$ is said to be antisymmetric if $u = \epsilon_\sigma S_\sigma u$ for all $\sigma \in S_p$ where $\epsilon_\sigma$ is the signature of $\sigma$.

**Definition 3.8.** The space of all antisymmetric tensors in $\otimes^p_H E$ will be denoted by $\wedge^p E$.

**Theorem 3.9.** Let $E$ be a Hilbert space. Then $\wedge^p E$ is a closed linear subspace of the Hilbert space $\otimes^p_H E$ for any $p \geq 2$. 

Proof. For $\sigma \in \mathfrak{S}_p$ define the operator
\[
 f_\sigma \overset{\text{def}}{=} S_\sigma - \epsilon_\sigma I : \mathcal{O}_H^p E \to \mathcal{O}_H^p E,
\]
where $I$ denotes the identity operator on $\mathcal{O}_H^p E$. Since $S_\sigma$ is a continuous linear operator on $\mathcal{O}_H^p E$, $f_\sigma$ is a continuous linear operator. The kernel of the operator $f_\sigma$ is
\[
 \ker f_\sigma = \{ u \in \mathcal{O}_H^p E : (S_\sigma - \epsilon_\sigma I)(u) = 0 \} = \{ u \in \mathcal{O}_H^p E : \epsilon_\sigma S_\sigma(u) = u \}.
\]
Since $f_\sigma$ is a continuous linear operator on $\mathcal{O}_H^p E$, $\ker f_\sigma$ is a closed linear subspace of $\mathcal{O}_H^p E$. Thus, $\wedge^p E$ is a closed linear subspace of $\mathcal{O}_H^p E$, since
\[
 \wedge^p E = \{ u \in \mathcal{O}_H^p E : \epsilon_\sigma S_\sigma(u) = u \text{ for all } \sigma \in \mathfrak{S}_p \} = \bigcap_{\sigma \in \mathfrak{S}_p} \ker f_\sigma.
\]
\[\square\]

Theorem 3.9 implies that the orthogonal projection from $\mathcal{O}_H^p E$ onto $\wedge^p E$ is well defined on $\mathcal{O}_H^p E$.

**Definition 3.10.** Let $E$ be a Hilbert space. For $x_1, \ldots, x_p \in E$, define $x_1 \wedge x_2 \wedge \cdots \wedge x_p$ to be the orthogonal projection of the elementary tensor $x_1 \otimes x_2 \otimes \cdots \otimes x_p$ onto $\wedge^p E$, that is,

\[
x_1 \wedge x_2 \wedge \cdots \wedge x_p = P_{\wedge^p E}(x_1 \otimes \cdots \otimes x_p).
\]

One can find a proof of the following statement in [5].

**Theorem 3.11.** Let $E$ be a Hilbert space. For all $u \in \mathcal{O}_H^p E$,

\[
P_{\wedge^p E}(u) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \epsilon_\sigma S_\sigma(u).
\]

**Proposition 3.12.** Let $E$ be a Hilbert space. The inner product in $\wedge^p_H E$ is given by

\[
\langle x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_p \rangle_{\wedge^p H E} = \det \begin{pmatrix} \langle x_1, y_1 \rangle_E & \cdots & \langle x_1, y_p \rangle_E \\ \vdots & \ddots & \vdots \\ \langle x_p, y_1 \rangle_E & \cdots & \langle x_p, y_p \rangle_E \end{pmatrix},
\]

for all $x_1, \ldots, x_p, y_1, \ldots, y_p \in E$.

**Proof.** By Theorem 3.11, we have

\[
\langle x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_p \rangle_{\wedge^p H E}
\]

\[= \left( \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \epsilon_\sigma S_\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_p), \frac{1}{p!} \sum_{\tau \in \mathfrak{S}_p} \epsilon_\tau S_\tau(y_1 \otimes y_2 \otimes \cdots \otimes y_p) \right)_{\mathcal{O}_H^p E}
\]

\[= \frac{1}{p!} \sum_{\sigma' \in \mathfrak{S}_p} \epsilon_{\sigma'} \langle x_1 \otimes x_2 \otimes \cdots \otimes x_p, S_{\sigma'}(y_1 \otimes y_2 \otimes \cdots \otimes y_p) \rangle_{\mathcal{O}_H^p E}
\]

\[= \sum_{\sigma' \in \mathfrak{S}_p} \epsilon_{\sigma'} \prod_{i=1}^p \langle x_i, y_{\sigma'(i)} \rangle_E
\]

\[= \det \begin{pmatrix} \langle x_1, y_1 \rangle_E & \cdots & \langle x_1, y_p \rangle_E \\ \vdots & \ddots & \vdots \\ \langle x_p, y_1 \rangle_E & \cdots & \langle x_p, y_p \rangle_E \end{pmatrix}, \text{ by Leibniz’ formula.}
\]
See [5, Proposition 2.14] for slightly more detail.

**Corollary 3.13.** Let $E$ be a Hilbert space and let $x_1, \ldots, x_p \in E$. Then $x_1 \wedge \cdots \wedge x_p = 0$ if and only if $x_1, \ldots, x_p$ are linearly dependent.

**Proof.** Note that $x_1 \wedge \cdots \wedge x_p = 0$ if and only if $\|x_1 \wedge \cdots \wedge x_p\|_{\wedge^p E}^2 = 0$, which, by Proposition 3.12, holds if and only if $\det([\langle x_i, x_j \rangle]_{i,j=1}^p) = 0$.

Thus, $x_1 \wedge \cdots \wedge x_p = 0$ if and only if there exist complex numbers $\lambda_1, \ldots, \lambda_p$, which are not all zero, such that

$$
\begin{pmatrix}
\langle x_1, y_1 \rangle_E & \cdots & \langle x_1, y_p \rangle_E \\
\vdots & \ddots & \vdots \\
\langle x_p, y_1 \rangle_E & \cdots & \langle x_p, y_p \rangle_E
\end{pmatrix}
\begin{pmatrix}
\bar{\lambda}_1 \\
\vdots \\
\bar{\lambda}_p
\end{pmatrix} = 0.
$$

This holds if and only if there exist complex numbers $\lambda_1, \ldots, \lambda_p$, which are not all zero, such that

$$
\left\langle x_i, \sum_{j=1}^p \lambda_j x_j \right\rangle_E = 0 \quad \text{for } i = 1, \ldots, p.
$$

The latter statement is equivalent to the assertion that there exist complex numbers $\lambda_1, \ldots, \lambda_p$, which are not all zero, such that

$$
\left\langle \sum_{i=1}^p \lambda_i x_i, \sum_{j=1}^p \lambda_j x_j \right\rangle_E = 0,
$$

which in turn is equivalent to the condition that there exist complex numbers $\lambda_1, \ldots, \lambda_p$, not all zero, such that

$$
\sum_{j=1}^p \lambda_j x_j = 0.
$$

The latter statement is equivalent to the linear dependence of $x_1, \ldots, x_p$ as required. □

**Lemma 3.14.** Suppose that $\{u_1, \ldots, u_n\}$ is an orthonormal set in $E$. Then, for $j = 1, \ldots, n - 1$ and for every $x \in E$,

$$
\|u_1 \wedge \cdots \wedge u_j \wedge x\|_{\wedge^{j+1} E} = \|x - \sum_{i=1}^j \langle x, u_i \rangle u_i\|_E.
$$

See [5, Lemma 2.15].

**Definition 3.15.** Let $(E, \| \cdot \|_E)$ be a Hilbert space. The $p$-fold Cartesian product of $E$ is defined to be the set

$$
E \times \cdots \times E \text{ \(p\)-times} = \{(x_1, \ldots, x_p) : x_i \in E\}.
$$
Moreover, we define a norm on $E \times \cdots \times E$ by
\[
\| (x_1, \ldots, x_p) \| = \left\{ \sum_{i=1}^{p} \| x_i \|^2_E \right\}^{1/2}.
\]

**Definition 3.16.** Let $E$ be a Hilbert space. We define the multi-linear operator
\[
\Lambda: E \times \cdots \times E \to \wedge^p E
\]
by
\[
\Lambda(x_1, \ldots, x_p) = x_1 \wedge \cdots \wedge x_p \quad \text{for all } x_1, \ldots, x_p \in E.
\]

**Proposition 3.17** (Hadamard’s inequality [15, p. 477]). For any matrix
\[
A = (a_{ij}) \in \mathbb{C}^{n \times n},
\]
\[
| \det(A) | \leq \prod_{j=1}^{n} \left( \sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2}
\]
and
\[
| \det(A) | \leq \prod_{i=1}^{n} \left( \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.
\]

**Proposition 3.18.** Let $E$ be a Hilbert space. Then the multi-linear mapping
\[
\Lambda: E \times \cdots \times E \to \wedge^p E
\]
is bounded and
\[
\| \Lambda(x_1, \ldots, x_p) \|_{\wedge^p E}^2 \leq \prod_{j=1}^{p} \| x_j \|_E \left( \sum_{i=1}^{p} \| x_i \|^2_E \right)^{1/2}.
\]
(3.1)

See [5, Proposition 2.19] for more detail.

3.2. **Pointwise wedge products and pointwise creation operators**

For the purposes of this paper we need to consider the wedge product of mappings defined on the unit circle or in the unit disk that take values in Hilbert spaces. To this end we introduce a notion of pointwise wedge product and we study its properties.

**Definition 3.19.** Let $E$ be a Hilbert space and let $f, g: \mathbb{D} \to E$ ($f, g: \mathbb{T} \to E$) be $E$-valued maps. We define the **pointwise wedge product of $f$ and $g$**, $f \wedge g: \mathbb{D} \to \wedge^2 E$ ($f \wedge g: \mathbb{T} \to \wedge^2 E$) by
\[
(f \wedge g)(z) = f(z) \wedge g(z) \quad \text{for all } z \in \mathbb{D} \quad \text{(for almost all } z \in \mathbb{T}).
\]

**Definition 3.20.** Let $E$ be a Hilbert space and let $\chi_1, \ldots, \chi_n: \mathbb{D} \to E(\chi_1, \ldots, \chi_n: \mathbb{T} \to E)$ be $E$-valued maps. We call $\chi_1, \ldots, \chi_n$ **pointwise linearly dependent** on $\mathbb{D}$ (or on $\mathbb{T}$) if for all $z \in \mathbb{D}$ (for almost all $z \in \mathbb{T}$, respectively) the vectors $\chi_1(z), \ldots, \chi_n(z)$ are linearly dependent in $E$.

**Remark 3.21.** If $x_1, \ldots, x_n$ are pointwise linearly dependent on $\mathbb{T}$, then
\[
(x_1 \wedge \cdots \wedge x_n)(z) = 0
\]
for almost all $z \in \mathbb{T}$.
PROPOSITION 3.22. Let $E$ be a Hilbert space and $x_1, x_2, \ldots, x_n : \mathbb{D} \to E$ be analytic $E$-valued maps on $\mathbb{D}$. Then,

$$x_1 \hat{x}_2 \hat{x}_3 \cdots \hat{x}_n : \mathbb{D} \to \wedge^n E$$

is also analytic on $\mathbb{D}$ and

$$(x_1 \hat{x}_2 \hat{x}_3 \cdots \hat{x}_n)'(z) = x_1'(z) \wedge x_2(z) \wedge \cdots \wedge x_n(z) + \cdots + x_1(z) \wedge x_2(z) \wedge \cdots \wedge x_n(z)$$

for all $z \in \mathbb{D}$.

The proof is straightforward. It follows from Proposition 3.12, continuity of $\Lambda$ (Proposition 3.18) and Hadamard’s inequalities 3.1.

DEFINITION 3.23. Let $E$ be a separable Hilbert space. If $x \in L^2(\mathbb{T}, E)$ and $y \in L^\infty(\mathbb{T}, E)$, then $y^* x \in L^2(\mathbb{T}, E)$ is given by $(y^* x)(z) = \langle x(z), y(z) \rangle_E$ almost everywhere on $\mathbb{T}$.

PROPOSITION 3.24. Let $E$ be a separable Hilbert space, let $x \in H^2(\mathbb{D}, E)$ and let $y \in H^\infty(\mathbb{D}, E)$. Then

$$x \hat{y} \in H^2(\mathbb{D}, \wedge^2 E).$$

See [5, Proposition 3.8].

DEFINITION 3.25. Let $E$ be a Hilbert space. We say that a family $\{f_\lambda\}_{\lambda \in \Lambda}$ of maps from $\mathbb{T}$ to $E$ is pointwise orthonormal on $\mathbb{T}$, if for all $z$ in a set of full measure in $\mathbb{T}$, the family of vectors $\{f_\lambda(z)\}_{\lambda \in \Lambda}$ is orthonormal in $E$.

PROPOSITION 3.26. Let $E$ be a separable Hilbert space, and let $\xi_0, \xi_1, \ldots, \xi_j \in L^\infty(\mathbb{T}, E)$ be a pointwise orthonormal set on $\mathbb{T}$, and let $x \in L^2(\mathbb{T}, E)$. Then

$$\xi_0 \hat{\xi}_1 \hat{\xi}_2 \cdots \hat{\xi}_j \hat{x} \in L^2(\mathbb{T}, \wedge^{j+2} E)$$

and

$$\|\xi_0 \hat{\xi}_1 \hat{\xi}_2 \cdots \hat{\xi}_j \hat{x}\|_{L^2(\mathbb{T}, \wedge^{j+2} E)} \leq \|x\|_{L^2(\mathbb{T}, E)}.$$

It follows from Lemma 3.14. See [5, Proposition 3.11] for a proof of this proposition.

DEFINITION 3.27. Let $E$ be a separable Hilbert space. Let $\xi \in H^\infty(\mathbb{D}, E)$. We define the pointwise creation operator

$$C_\xi : H^2(\mathbb{D}, E) \to H^2(\mathbb{D}, \wedge^2 E)$$

by

$$C_\xi f = \xi \hat{f} \text{ for } f \in H^2(\mathbb{D}, E).$$

REMARK 3.28. Let $E$ be a separable Hilbert space. Let $\xi \in H^\infty(\mathbb{D}, E)$ and let $f \in H^2(\mathbb{D}, E)$. By the generalized Fatou’s Theorem [33, Chapter V], the radial limits

$$\lim_{r \to 1} \xi(re^{i\theta}) = \tilde{\xi}(e^{i\theta}), \quad \lim_{r \to 1} f(re^{i\theta}) = \tilde{f}(e^{i\theta}) \quad (0 < r < 1)$$

exist almost everywhere on $\mathbb{T}$ and define functions $\tilde{\xi} \in L^\infty(\mathbb{T}, E)$ and $\tilde{f} \in L^2(\mathbb{T}, E)$, respectively, which satisfy the relations

$$\lim_{r \to 1} \|\xi(re^{i\theta}) - \tilde{\xi}(e^{i\theta})\|_E = 0, \quad \lim_{r \to 1} \|f(re^{i\theta}) - \tilde{f}(e^{i\theta})\|_E = 0 \quad (0 < r < 1)$$

for almost all $e^{i\theta} \in \mathbb{T}$. 
LEMMA 3.29. Let $E$ be a separable Hilbert space. Let $\xi \in H^\infty(\mathbb{D}, E)$ and let $f \in H^2(\mathbb{D}, E)$. Then the radial limits $\lim_{r \to 1} (\xi(re^{i\theta}) \wedge f(re^{i\theta}))$ exist for almost all $e^{i\theta} \in \mathbb{T}$ and define functions in $L^2(\mathbb{T}, \Lambda^2 E)$.

One can find a proof of this statement in [5, Lemma 4.3].

REMARK 3.30. Let $E$ be a separable Hilbert space. By [33, Chapter 5, Section 1], for any separable Hilbert space $E$, the map $f \mapsto \tilde{f}$ is an isometric embedding of $H^2(\mathbb{D}, E)$ in $L^2(\mathbb{T}, E)$, where $\tilde{f}(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$. Since $H^2(\mathbb{D}, E)$ is complete and the embedding is isometric, the image of the embedding is complete, and therefore is closed in $L^2(\mathbb{T}, E)$. Therefore, the space $H^2(\mathbb{D}, E)$ is identified isometrically with a closed linear subspace of $L^2(\mathbb{T}, E)$. In future we shall use the same notation for $f$ and $\tilde{f}$.

DEFINITION 3.31. Let $E$ be a separable Hilbert space. Let $F$ be a subspace of $L^2(\mathbb{T}, E)$ and let $X$ be a subset of $L^2(\mathbb{T}, E)$.

We define the pointwise orthogonal complement of $X$ in $F$ to be the set

$$\text{POC}(X, F) = \{ f \in F : f(z) \perp \{ x(z) : x \in X \} \text{ for almost all } z \in \mathbb{T} \}.$$ 

PROPOSITION 3.32. Let $E$ be a separable Hilbert space. Let $\eta \in L^2(\mathbb{D}, E)$. Then

(i) the space $V = \{ f \in H^2(\mathbb{D}, E) : \langle f(z), \eta(z) \rangle_E = 0 \text{ for almost all } z \in \mathbb{T} \}$ is a closed subspace of $H^2(\mathbb{D}, E)$;
(ii) the space $V = \{ f \in L^2(\mathbb{T}, E) : \langle f(z), \eta(z) \rangle_E = 0 \text{ for almost all } z \in \mathbb{T} \}$ is a closed subspace of $L^2(\mathbb{T}, E)$.

One can find a proof of this statement in [5, Proposition 4.6].

4. Superoptimal analytic approximation

In this section we present our main result, which is an algorithm for the superoptimal analytic approximation of a matrix-valued function on the circle. In Subsection 4.1 we recall certain known results and Peller and Young’s algorithm (Theorem 4.19). In Subsection 4.2 we present an alternative algorithm for the superoptimal approximant, based on exterior powers of Hilbert spaces. The proof of the validity of the new algorithm relies on the cited work given in Subsection 4.1.

4.1. Known results

THEOREM 4.1 (Hartman’s theorem [21, p. 74]). Let $E, F$ be separable Hilbert spaces and let $\Phi \in L^\infty(\mathbb{T}, \mathcal{L}(E, F))$. The following statements are equivalent:

(i) the Hankel operator $H_\Phi$ is compact on $H^2(\mathbb{D}, E)$;
(ii) $\Phi \in H^\infty(\mathbb{D}, \mathcal{L}(E, F)) + C(\mathbb{T}, \mathcal{K}(E, F))$;
(iii) there exists a function $\Psi \in C(\mathbb{T}, \mathcal{K}(E, F))$ such that $\hat{\Phi}(n) = \hat{\Psi}(n)$ for $n < 0$.

THEOREM 4.2 [18]. For any matrix-valued $\varphi \in L^\infty(\mathbb{T}, \mathbb{C}^{m \times n})$,

$$\inf_{Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})} \| \varphi - Q \|_\infty = \| H_\varphi \|$$

and the infimum is attained.
Definition 4.3 [25, p. 306]. The class of quasi-continuous functions is defined by
\[ QC = (H^\infty(D, C^{m \times n}) + C(T, C^{m \times n})) \cap (H^\infty(D, C^{m \times n}) + C(T, C^{m \times n})). \]
In other words this class consists of functions on the circle which belong to \( H^\infty + C \) and have the property that their complex conjugates belong to \( H^\infty + C \) as well.

We shall also need the class of functions of vanishing mean oscillation, as described, for example, in [21, Appendix 2, Section 5].

Definition 4.4. For any function \( f \in L^1(T) \) and any arc \( I \) in \( T \) let
\[ f_I = \frac{1}{m(I)} \int_I f \, dm, \]
where \( m \) is Lebesgue measure on \( T \). Thus, \( f_I \) is the mean of \( f \) over \( I \). The function \( f \) is said to have vanishing mean oscillation if
\[ \lim_{m(I) \to 0} \frac{1}{m(I)} \int_I |f - f_I| \, dm = 0. \]
The space of functions of vanishing mean oscillation on \( T \) is denoted by \( \text{VMO} \).

VMO is also related to the compactness of Hankel operators. The following is [21, Theorem 5.8].

Theorem 4.5. Let \( \varphi \in L^2 \). Then \( H_\varphi \) is compact if and only if \( P_\varphi \in \text{VMO} \).

It is therefore not surprising that the spaces \( QC \) and \( \text{VMO} \) are closely related. In fact

Theorem 4.6 [21, page 729].
\[ QC = \text{VMO} \cap L^\infty. \]

It follows from another characterization of \( \text{VMO} \), to wit
\[ \text{VMO} = \{ f + \tilde{g} : f, g \in C(T) \}, \]
where \( \tilde{g} \) denotes the harmonic conjugate of \( g \) [21, Theorem A2.8].

Remark 4.7. For \( G \in H^\infty(D, C^{m \times n}) + C(T, C^{m \times n}) \) we will say that a function \( Q \in H^\infty(D, C^{m \times n}) \) which minimizes the norm \( \|G - Q\|_{L^\infty} \) is a function at minimal distance from \( G \). By Nehari's Theorem, all such functions \( Q \) satisfy \( \|G - Q\|_{L^\infty} = \|H_G\| \).

Definition 4.8 [33, p. 190]. For a separable Hilbert space \( E \), a function \( \xi \in H^\infty(D, E) \) will be called inner if for almost every \( z \in T \),
\[ \|\xi(z)\|_E = 1. \]

Theorem 4.9 [25, Theorem 1.1]. Let \( \varphi \) be an \( n \times 1 \) inner matrix function. There exists a co-outer function \( \varphi_c \in H^\infty(D, C^{n \times (n-1)}) \) such that
\[ \Phi = (\varphi \quad \varphi_c) \]
is unitary-valued on \( T \) and all minors of \( \Phi \) on the first column are in \( H^\infty \).

For a function \( G : T \to C^{m \times n} \) and a space \( X \) of scalar functions on \( T \), we write \( G \in X \) to mean that each entry of \( G \) belongs to \( X \).
Next we describe some properties that a space $X$ of equivalence classes of scalar functions on the circle may possess [25, p. 330]. Define the non-linear operator $A = A^{(m,n)}$ on the space of $m \times n$ functions $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$ by saying that $A^{(m,n)}G$ is the unique superoptimal approximation in $H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ to $G$.

We say that $X$ is hereditary for $A$ if, for every scalar function $g \in X$, the best analytic approximation $Ag$ of $g$ belongs to $X$.

Consider the following conditions on $X$ from [22]:

(α1) $X$ contains all polynomial functions and $X \subset \text{VMO}$;
(α2) $X$ is hereditary for $A$;
(α3) if $f \in X$ then $\bar{z}f \in X$ and $P_+f \in X$;
(α4) if $f, g \in X \cap L^\infty$ then $fg \in X \cap L^\infty$;
(α5) if $f \in X \cap H^2$ and $h \in H^\infty$ then $T_h f \in X \cap H^2$.

The relevance of these properties is contained in the following statement, which is [25, Lemma 5.3]. Recall that a function $f \in L^\infty$ is said to be badly approximable if the best analytic approximant to $f$ is the zero function. In view of Nehari’s Theorem, $f$ is badly approximable if and only if $\|f\|_\infty = \|H_f\|$.

**Lemma 4.10.** Let $X$ satisfy (α1) to (α5) and let $\varphi \in X$ be an $n \times 1$ inner function. Let $\varphi_c$ be an $n \times (n - 1)$ function in $H^\infty$ such that $[\varphi \varphi_c]$ is unitary-valued almost everywhere on $\mathbb{T}$ and has all its minors on the first column in $H^\infty$. Then each entry of $\varphi_c$ belongs to $X$.

Below we shall use a modified version of [25, Theorem 0.2].

**Theorem 4.11** [25, Theorem 0.2]. Let $\varphi \in L^\infty(\mathbb{T}, \mathbb{C}^{m \times n})$ be such that $H_\varphi$ has a Schmidt pair $(v, w)$ corresponding to the singular value $t = \|H_\varphi\|$. Let $Q$ be a function in $H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ at minimal $L^\infty$-distance from $\varphi$. Then

\[(\varphi - Q)v = tw\]

and

\[(\varphi - Q)^*w = tv.\]

Moreover

\[\|w(z)\|_{\mathbb{C}^m} = \|v(z)\|_{\mathbb{C}^n} \text{ almost everywhere on } \mathbb{T} \quad (4.1)\]

and

\[\|\varphi(z) - Q(z)\| = t \text{ almost everywhere on } \mathbb{T}.\]

**Proof.** By Nehari’s theorem, $\|\varphi - Q\|_{L^\infty} = t$ and, by hypothesis,

$H_\varphi v = tw, \quad H_\varphi^*w = tv.$

If $t = 0$ then $\varphi \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$, so that $\varphi = Q$ and the statement of the theorem is trivially true. We may therefore assume $t > 0$. Thus $H_\varphi^*H_\varphi v = t^2 v$, and so $v$ is a maximizing vector for $H_\varphi$. We can assume that $v$ is a unit vector in $H^2(\mathbb{D}, \mathbb{C}^n)$, and then $w$ is a unit vector in $H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ and is a maximizing vector for $H_\varphi^*$. We have

\[t = \|H_\varphi v\| = \|H_\varphi - Qv\| = \|P_-(\varphi - Q)v\| \leq \|\varphi - Q\|_{L^\infty} = t.\]

The inequalities must hold with equality throughout, and therefore

\[\|P_-(\varphi - Q)v\| = \|\varphi - Q\|_{L^\infty},\]

which implies that $(\varphi - Q)v \perp H^2$ and so

$H_\varphi v = P_-(\varphi - Q)v = (\varphi - Q)v$. 

Furthermore, $\| \varphi - Q \| v = \| \varphi - Q \|_{L^\infty} \| v \|$ and since $v(z)$ is therefore a maximizing vector for $\varphi(z) - Q(z)$ for almost all $z$, we have $\| \varphi(z) - Q(z) \| = \| H_\varphi \|$. Likewise,

$$t = \| H_\varphi^* \| = \| H_{\varphi - Q}^* \| = \| H_{\varphi - Q}^* w \| = \| P_+(\varphi - Q)^* w \|_{L^2} \leq \| (\varphi - Q)^* w \|_{L^2}$$

$$\leq \| (\varphi - Q)^* \|_{L^\infty} \| w \|_{L^2} = \| (\varphi - Q)^* \|_{L^\infty} = t.$$  

Again, the inequalities hold with equality throughout, and in particular

$$\| P_+(\varphi - Q)^* w \|_{L^2} = \| (\varphi - Q)^* w \|_{L^2},$$  

so that $(\varphi - Q)^* w \in H^2$ and

$$(\varphi - Q)^* w = H_\varphi^* w = tv. \quad \Box$$

**Lemma 4.12** [25, p. 315–316]. Let $m, n > 1$, let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$ and $t_0 = \| H_G \| \neq 0$. Suppose that $v$ is a maximizing vector of $H_G$ and let

$$H_G v = t_0 w.$$  

Then $v, \bar{z}w \in H^2(\mathbb{D}, \mathbb{C}^n)$ have the factorizations

$$v = v_0 h, \quad \bar{z}w = \varphi w_0 h$$  

for some scalar outer function $h$, some scalar inner $\varphi$, and column-matrix inner functions $v_0, w_0$. Moreover there exist unitary-valued functions $V, W$ of types $n \times n, m \times m$, respectively, of the form

$$V = [v_0 \quad \bar{a}], \quad W^T = [w_0 \quad \bar{\beta}],$$  

where $\alpha, \beta$ are inner, co-outer functions, quasi-continuous functions of types $n \times (n - 1), m \times (m - 1)$, respectively, and all minors on the first columns of $V, W^T$ are in $H^\infty$. Furthermore, every $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ which is at minimal distance from $G$ satisfies

$$W(G - Q)V = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F \end{pmatrix}$$  

for some $F \in H^\infty(\mathbb{D}, \mathbb{C}^{(m - 1) \times (n - 1)}) + C(\mathbb{T}, \mathbb{C}^{(m - 1) \times (n - 1)})$ and some quasi-continuous function $u_0$ given by

$$u_0 = \frac{\bar{z}\varphi h}{h}$$  

with $|u_0(z)| = 1$ almost everywhere on $\mathbb{T}$.

In the statement of the lemma, in saying that an $m \times n$ matrix-valued function $\alpha$ is co-outer we mean that each column of $\alpha$ is in $H^\infty_m$ and $\alpha^T H^2_m$ is dense in $H^2_n$. (In [33, p. 190], such a function $\alpha$ is said to be $*$-outer.)

**Proof.** First we construct $V$ and $W$ with the properties (4.2)–(4.5). By equation (4.1), $\| v(z) \| = \| w(z) \|$ almost everywhere, and so the column-vector functions $v, \bar{z}w$ in $H^2$ have the same (scalar) outer factor $h$. This property yields the inner–outer factorizations (4.3) for some column inner functions $v_0, w_0$. By Theorem 4.9, there exists an inner co-outer function $\alpha$ of type $n \times (n - 1)$ such that $V \overset{\text{def}}{=} [v_0 \bar{\alpha}]$ is unitary-valued almost everywhere on $\mathbb{T}$ and all minors on the first column of $V$ are in $H^\infty$. Likewise there exists an inner co-outer function $\beta$ of type $m \times (m - 1)$ such that $W \overset{\text{def}}{=} [w_0 \bar{\beta}]^T$ is unitary-valued almost everywhere on $\mathbb{T}$ and all minors on the first column of $W^T$ are in $H^\infty$. 


Next we show that \( u_0 \) given by equation (4.6) is quasi-continuous. Let \( Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) be at minimal distance from \( G \). Then
\[
\|G - Q\|_\infty = \|H_G\| = t_0.
\]
By Theorem 4.11,
\[
(G - Q)v = t_0w
\]
and by the factorizations (4.3) we have
\[
(G - Q)v_0h = t_0\bar{z}\bar{\varphi}h\bar{w}_0
\]
and by equations (4.4) and (4.6)
\[
(G - Q)V (1 \ 0 \ \cdots \ 0)^T = W^* (t_0u_0 \ 0 \ \cdots \ 0)^T.
\]
Thus,
\[
W(G - Q)V = \begin{pmatrix} t_0u_0 & F \\ 0 & F \end{pmatrix}
\]
for some \( f \in L^\infty(\mathbb{T}, \mathbb{C}^{1 \times (n - 1)}) \), \( F \in L^\infty(\mathbb{T}, \mathbb{C}^{(m - 1) \times (n - 1)}) \).

Because \( t_0 = \|H_G\| \), it follows that \( |u_0| = 1 \) almost everywhere, and from Nehari’s Theorem
\[
\|W(G - Q)V\|_\infty = \|G - Q\| = \|H_G\| = t_0,
\]
and we have \( f = 0 \). So, \( W(G - Q)V \) has the form (4.5). Now, \( \|H_{u_0}\| \leq \|u_0\|_\infty = 1 \) and \( \|H_{u_0} h\| = \|\bar{z}\bar{\varphi}h\| = \|h\| \). Hence
\[
\|H_{u_0}\| = 1 = \|u_0\|_\infty,
\]
which implies that \( u_0 \) is badly approximable. The (1,1) entries of equation (4.5) are
\[
w_0^T(G - Q)v_0 = t_0u_0.
\]

Since \( v_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n) \), \( w_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m) \) and \( H^\infty(\mathbb{D}, \mathbb{C}) + C(\mathbb{T}, \mathbb{C}) \) is an algebra, \( u_0 \in H^\infty + C \). By a result in [22, Section 3.1], if \( u_0 \in H^\infty + C \) and \( u_0 \) is badly approximable then \( \bar{u}_0 \in H^\infty + C \). Thus, \( u_0 \) is quasi-continuous.

Now we show that \( v_0, w_0 \in QC \). It follows from Nehari’s theorem that
\[
(G - Q)^*w = t_0v,
\]
much as in the proof of [25, Theorem 0.2]. Indeed, since \( H_G^*w = t_0v \) and \( H_G^* = P_+ M_{(G - Q)^*}H^2_+ \), we have (assuming, as we may, that \( v \) and \( w \) are unit vectors),
\[
t_0 = \|H_G^*w\| = \|P_+ (G - Q)^*w\|
\]
\[
\leq \|(G - Q)^*w\| \leq \|G - Q\|_\infty \|w\| = t_0.
\]
It follows that the inequalities hold with equality, and so
\[
\|P_+ (G - Q)^*w\| = \|(G - Q)^*w\|
\]
when
\[
P_+ (G - Q)^*w = (G - Q)^*w,
\]
and so
\[
(G - Q)^*w = H_G^*w = t_0v, \tag{4.7}
\]
as claimed.
Taking complex conjugates in the last equation we have 
$$(G - Q)^T \tilde{w} = t_0 \tilde{v}.$$ 
Thus, by equation (4.3),
$$(G - Q)^T z \varphi w_0 h = t_0 \tilde{v} \tilde{v}_0$$
for some outer function $h$ and scalar inner $\varphi$. Therefore,
$$\tilde{v}_0 = \frac{(G - Q)^T z \varphi w_0 h}{t_0 h}.$$ 
Recall that $u_0 = \tilde{\varphi} \tilde{h}/h$, and so
$$\tilde{v}_0 = \frac{1}{t_0} (G - Q)^T u_0 w_0.$$ 
Since $u_0 \in QC$, $G - Q \in H^\infty + C$ and $w_0 \in H^\infty$, it follows that $\tilde{v}_0 \in H^\infty + C$. Since also $v_0 \in H^\infty$, we have $v_0 \in QC$.

To complete the proof of Lemma 4.12, all that remains is to show that $\alpha, \beta$ are quasi-continuous and $F \in H^\infty + C$. This will follow from Lemma 4.10.

The space VMO satisfies conditions (a1) to (a5), as stated on [25, p. 335], and we have $v_0 \in QC \subset \text{VMO}$. Hence, we may apply Lemma 4.10 with $\varphi = v_0$ to deduce that $\alpha \in \text{VMO}$. Since also $\alpha \in L^\infty$, it follows from Theorem 4.6 that $\alpha \in QC$. Likewise, $\beta \in QC$.

To show that $F \in H^\infty + C$, for $1 < i \leq m, 1 < j \leq n$ consider the $2 \times 2$ minor of equation (4.5) with indices $1i, 1j$:
$$\sum_{r<s, k<l} W_{i,r,s} (G - Q)_{r,s,k,l} V_{k,l,j} = t_0 u_0 F_{i-1,j-1}. \quad (4.8)$$
By the analytic minors property of $W, V$,
$$V_{k,l,j}, W_{i,r,s} \in H^\infty.$$ 
Since $G - Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$, the left-hand side of equation (4.8) is in $H^\infty(\mathbb{D}, \mathbb{C}^{2 \times 2}) + C(\mathbb{T}, \mathbb{C}^{2 \times 2})$ and hence $u_0 F \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1) \times (n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1) \times (n-1)})$. Thus,
$$F = \tilde{u}_0 (u_0 F) \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1) \times (n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1) \times (n-1)}). \quad \square$$

**Definition 4.13.** We say that a unitary-matrix-valued function $V$ is a thematic completion of a column-matrix inner function $v_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ if $V = (v_0 \alpha)$, for some co-outer function $\alpha \in H^\infty(\mathbb{D}, \mathbb{C}^{n \times (n-1)})$ such that $V(z)$ is a unitary matrix for almost all $z \in \mathbb{T}$ and such that all minors on the first column of $V$ are analytic.

**Remark 4.14.** By [25, Theorem 1.1], every column-matrix inner function has a thematic completion. Thematic completions are not unique, for if $V = (v_0 \alpha)$ is a thematic completion of $v_0$, then so is $(v_0 \alpha U)$ for any constant $(n - 1)$-square unitary matrix $U$. However, by [25, Corollary 1.6], the thematic completion of $v_0$ is unique up to multiplication on the right by a constant unitary matrix of the form $\text{diag} \{1, U\}$ for some constant $(n-1)$-square matrix $U$, and so it is permissible to speak of ‘the thematic completion of $v_0$’.

Furthermore, by [25, Theorem 1.2], thematic completions have constant determinants almost everywhere on $\mathbb{T}$, and hence $\alpha, \beta$ are inner matrix functions. Observe that, as we showed above, if the column $v_0$ belongs to VMO, then the thematic completion of $v_0$ is quasi-continuous. Similarly, if the column $w_0$ belongs to VMO, then the thematic completion of $w_0$ is quasi-continuous. Thus $\alpha, \beta$ are inner, co-outer, quasi-continuous functions of types $n \times (n - 1)$ and $m \times (m - 1)$, respectively.
Lemma 4.15 [25, p. 316]. Let $m, n > 1$, let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$, let $\|H_G\| = t_0$ and let $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ be at minimal distance from $G$, so that in the notation of Lemma 4.12,

$$W(G - Q_1)V = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F \end{pmatrix}$$

for some $F \in H^\infty(\mathbb{D}, \mathbb{C}^{m-1 \times n-1}) + C(\mathbb{T}, \mathbb{C}^{m-1 \times n-1})$. Let

$$\mathcal{E} = \{G - Q : Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}), \|G - Q\|_\infty = t_0\}.$$

Then

$$W\mathcal{E}V = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F + H^\infty(\mathbb{D}, \mathbb{C}^{m-1 \times n-1}) \end{pmatrix} \cap B(t_0),$$

(4.10)

where $B(t_0)$ is the closed ball of radius $t_0$ in $L^\infty(\mathbb{T}, \mathbb{C}^{m \times n})$.

Proof. Let $E_1 = G - Q_1 \in \mathcal{E}$ and for $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ consider

$$E = E_1 - Q.$$

By Lemma 4.12, there exists a function $g \in L^\infty(\mathbb{D}, \mathbb{C}^{m-1 \times n-1})$ such that

$$WEV = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & g \end{pmatrix}.$$

The latter equation combined with equation (4.9), yields

$$WQV = W(G - Q_1)V - WEV = \begin{pmatrix} 0 & 0 \\ 0 & F - g \end{pmatrix} \in \begin{pmatrix} 0 & 0 \\ 0 & L^\infty(\mathbb{T}, \mathbb{C}^{m-1 \times n-1}) \end{pmatrix} \cap WH^\infty(\mathbb{D}, \mathbb{C}^{m \times n})V.$$

By [25, Lemma 1.5], $WQV \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$, say $F - g = q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$. Then

$$WEV = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F - q \end{pmatrix},$$

which proves the inclusion (\subseteq) in equation (4.10).

Conversely, suppose $q \in H^\infty(\mathbb{D}, \mathbb{C}^{m-1 \times n-1})$ and

$$\|F - q\|_\infty \leq t_0.$$

By [25, Lemma 1.5], there exists a function $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ such that

$$WQV = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$$

and thus

$$W(E_1 - Q)V = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F - q \end{pmatrix}.$$

Then

$$E_1 - Q = G - (Q_1 - Q) \in \mathcal{E},$$

and so

$$\begin{pmatrix} t_0 u_0 & 0 \\ 0 & F - q \end{pmatrix} \in W\mathcal{E}V.$$

Hence, equality holds in equation (4.10). \qed
Lemma 4.16 [26, p. 16]. Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$ and let $(x_0, y_0)$ be a Schmidt pair for the Hankel operator $H_G$ corresponding to the singular value $t_0 = \|H_G\|$. Let $x_0 = \xi_0 h_0$ be the inner–outer factorization of $x_0$, where $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ is the inner and $h_0 \in H^2(\mathbb{D}, \mathbb{C})$ is the scalar outer factor of $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$, and let

$$V_0 = (\xi_0 \overline{\alpha_0})$$

be a unitary-valued function on $\mathbb{T}$, where $\alpha_0 \in H^\infty(\mathbb{D}, \mathbb{C}^{n \times (n-1)})$ is co-outer. Then

$$V_0 \begin{pmatrix} 0 & H^2(\mathbb{D}, \mathbb{C}^{n-1}) \end{pmatrix}^T$$

is the orthogonal projection of $H^2(\mathbb{D}, \mathbb{C}^n)$ onto the pointwise orthogonal complement of $x_0$ in $L^2(\mathbb{T}, \mathbb{C}^n)$. Similarly

$$V_0^* \begin{pmatrix} 0 & H^2(\mathbb{D}, \mathbb{C}^{n-1})^\perp \end{pmatrix}$$

is the orthogonal projection of $H^2(\mathbb{D}, \mathbb{C}^n)^\perp$ onto the pointwise orthogonal complement of $x_0$ in $L^2(\mathbb{T}, \mathbb{C}^n)$.

Lemma 4.17 [26, p. 16]. Let $G, x_0, y_0$ be defined as in Lemma 4.16 and let $\mathcal{K}, \mathcal{L}$ be the projections of $H^2(\mathbb{D}, \mathbb{C}^n), H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ onto the pointwise orthogonal complements of $x_0, y_0$ in $L^2(\mathbb{T}, \mathbb{C}^n), L^2(\mathbb{T}, \mathbb{C}^m)$, respectively. Let $Q_0 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ be at minimal distance from $G$, let $F$ be the (2,2) block of $W_0(G - Q_0)\nu_0$, as in Lemma 4.12, that is,

$$V_0 = (\xi_0 \overline{\alpha_0}), \quad W_0 = (\eta_0 \overline{\beta_0})^T \quad (4.11)$$

are unitary-valued functions on $\mathbb{T}$, $\alpha_0, \beta_0$ are co-outer functions of size $n \times (n-1), m \times (m-1)$, respectively, and all minors on the first columns of $V_0, V_0^T$ are in $H^\infty$. Let $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ satisfy

$$(G - Q)x_0 = \|H_G\|y_0, \quad y_0^*(G - Q) = \|H_G\|x_0^*.$$  

Then $H_F$ is a unitary multiple of the operator

$$\Gamma : = P_{\mathcal{K}} M_{G - Q}|\mathcal{K}, \quad (4.12)$$

where $M_{G - Q} : L^2(\mathbb{T}, \mathbb{C}^n) \to L^2(\mathbb{T}, \mathbb{C}^m)$ is the operator of multiplication by $G - Q$. More explicitly, if $U_1 : H^2(\mathbb{D}, \mathbb{C}^{n-1}) \to \mathcal{K}, U_2 : H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp \to \mathcal{L}$ are defined by

$$U_1 \chi = V_0 \begin{pmatrix} 0 & \chi \end{pmatrix}, \quad U_2 \psi = W_0^* \begin{pmatrix} 0 & \psi \end{pmatrix} \quad \text{for all} \quad \chi \in H^2(\mathbb{D}, \mathbb{C}^{n-1}), \psi \in H^2(\mathbb{D}, \mathbb{C}^{m-1}), \quad (4.13)$$

then $U_1, U_2$ are unitaries and

$$H_F = U_2^* \Gamma U_1.$$

Lemma 4.18 [25, p. 337]. Let $\alpha \in \text{QC}$ of type $m \times n$, where $m \geq n$, be inner and co-outer. There exists $A \in H^\infty(\mathbb{D}, \mathbb{C}^{n \times m})$ such that $A\alpha = I_n$. Here $I_n$ denotes the $n \times n$ identity matrix.

Theorem 4.19 gives the algorithm for the superoptimal analytic approximant constructed in [26].

Theorem 4.19 [26, p. 17]. Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$. The superoptimal approximant $\mathcal{A}G$ to $G$ is given by the following formula.

If $H_G = 0$, then $\mathcal{A}G = G$. Otherwise define spaces $K_j \subset L^2(\mathbb{T}, \mathbb{C}^n), N_j \subset L^2(\mathbb{T}, \mathbb{C}^m)$, vectors $\chi_j \in K_j, \psi_j \in N_j, H^\infty$ functions $Q_j$, operators $\Gamma_j$ and positive $\lambda_j$ as follows.

Let

$$K_0 = H^2(\mathbb{D}, \mathbb{C}^n), \quad N_0 = H^2(\mathbb{D}, \mathbb{C}^m)^\perp, \quad Q_0 = 0.$$
Let

$$\Gamma_j = P_{N_j}M_{G-Q_j}|K_j : K_j \to N_j, \quad \lambda_j = \|\Gamma_j\|,$$

where $P_{N_j}$ is the orthogonal projection onto $N_j$. If $\lambda_j = 0$ set $r = j$ and terminate the construction. Otherwise let $\chi_j, \psi_j$ be a Schmidt pair for $\Gamma_j$ corresponding to the singular value $\lambda_j$. Let $K_{j+1}$ be the range of the orthogonal projection of $K_j$ onto the pointwise orthogonal complement of $\chi_0, \ldots, \chi_j$ in $L^2(\mathbb{T}, \mathbb{C}^n)$. Let $N_{j+1}$ be the projection of $N_j$ onto the pointwise orthogonal complement of $\psi_0, \ldots, \psi_j$ in $L^2(\mathbb{T}, \mathbb{C}^n)$. Let $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ be chosen to be the projection of $N_j$ onto $H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$. Let $G_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ be chosen to satisfy, for $0 \leq k \leq j$,

$$Q_{j+1}\chi_k = G\chi_k - t_k\psi_k, \quad \psi_k^*Q_{j+1} = \psi_k^*G - t_k\chi_k^*.$$

(4.14)

Then each $\Gamma_j$ is a compact operator, $Q_j$ with the above properties does exist, the construction terminates with $r \leq \min(m, n)$ and

$$G - AG = \sum_{j=0}^{r-1} \frac{\lambda_j\psi_j\chi_j^*}{|h_j|^2}.$$

(4.15)

We shall derive a similar formula for the superoptimal analytic approximant $AG$, using exterior products of Hilbert spaces.

### 4.2. Algorithm for superoptimal analytic approximation

In this section we consider the superoptimal analytic approximation problem for a function $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$. We first state the algorithm for the solution of Problem 1.2; later we shall prove the claims that are made in this description of the algorithm. We will assume here the result of Peller and Young [25] that Problem 1.2 has a unique solution (see Theorem 1.3). For convenience, we give citations of the steps in this paper where the corresponding claims are proved.

**Algorithm:** Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$. In this subsection we shall give a fuller and more precise statement of the algorithm for $AG$ outlined in the Introduction, Section 1, in preparation for a subsequent formal proof of Theorem 10.15, which asserts that if entities $r, t_i, x_i, y_i, h_i$ for $i = 0, \ldots, r - 1$, are generated by the algorithm, then the superoptimal approximant is given by equation

$$AG = G - \sum_{i=0}^{r-1} \frac{t_iy_ix_i^*}{|h_i|^2}.$$

The proof will be by induction on $r$, which is the least index $j \geq 0$ such that $T_j = 0$, where $T_0 = H_G, T_1, T_2, \ldots$ is a sequence of operators recursively generated by the algorithm.

**Step 0.** Let $t_0 = \|H_G\|$. If $t_0 = 0$, then $H_G = 0$, which implies $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$. In this case, the algorithm terminates, we define $r$ to be zero and the superoptimal approximant $AG$ is given by $AG = G$, in agreement with the formula

$$G - AG = \sum_{j=0}^{r-1} \frac{t_jy_jx_j^*}{x_j^*x_j}.$$

(4.16)

contained in the statement of Theorem 10.15 (since the sum on the right-hand side of equation (4.16) is empty, and therefore by convention is interpreted as being zero).

Otherwise, $t_0 > 0$. By Theorem 4.1 and Lemma 4.12, $H_G$ is a compact operator and so there exists a Schmidt pair $(x_0, y_0)$ corresponding to the singular value $t_0$ of $H_G$. By the definition
of the Schmidt pair \((x_0, y_0)\) corresponding to \(t_0\) for the Hankel operator \(H_G : H^2(\mathbb{D}, \mathbb{C}^n) \to H^2(\mathbb{D}, \mathbb{C}^m)^\perp\),

\[
x_0 \in H^2(\mathbb{D}, \mathbb{C}^n), \quad y_0 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp
\]

are non-zero vector-valued functions such that

\[
H_G x_0 = t_0 y_0 \quad \text{and} \quad H_G^* y_0 = t_0 x_0.
\]

By Lemma 4.12, \(x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)\) and \(\bar{z}y_0 \in H^2(\mathbb{D}, \mathbb{C}^m)\) admit inner–outer factorizations

\[
x_0 = \xi_0 h_0, \quad \bar{z}y_0 = \eta_0 h_0
\]

for some scalar outer factor \(h_0 \in H^2(\mathbb{D}, \mathbb{C})\) and column-matrix inner functions \(\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)\), \(\eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)\). Then

\[
\|x_0(z)\|_{\mathbb{C}^n} = |h_0(z)| = \|y_0(z)\|_{\mathbb{C}^m} \quad \text{almost everywhere on } \mathbb{T}.
\]

We write equations (4.17) as

\[
\xi_0 = \frac{x_0}{h_0}, \quad \eta_0 = \frac{\bar{z}y_0}{h_0}.
\]

By equations (4.18) and (4.19),

\[
\|\xi_0(z)\|_{\mathbb{C}^n} = 1 = \|\eta_0(z)\|_{\mathbb{C}^m} \quad \text{almost everywhere on } \mathbb{T}.
\]

Let \(t_0 \neq 0\). By Lemma 4.12, every function \(Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})\) which is at minimal distance from \(G\) satisfies

\[
(G - Q_1)x_0 = t_0 y_0, \quad y_0^* (G - Q_1) = t_0 x_0^*.
\]

**Step 1.** Let

\[
X_1 \overset{\text{def}}{=} \xi_0 \hat{\wedge} H^2(\mathbb{D}, \mathbb{C}^n).
\]

By Proposition 6.1, \(X_1\) is a closed subspace of \(H^2(\mathbb{D}, \wedge^2 \mathbb{C}^n)\).

Similarly,

\[
\eta_0 \hat{\wedge} zH^2(\mathbb{D}, \mathbb{C}^m) \subset zH^2(\mathbb{D}, \wedge^2 \mathbb{C}^m)
\]

and therefore

\[
\eta_0 \hat{\wedge} zH^2(\mathbb{D}, \mathbb{C}^m) \subset \bar{z}H^2(\mathbb{D}, \wedge^2 \mathbb{C}^m),
\]

that is, if

\[
Y_1 \overset{\text{def}}{=} \eta_0 \hat{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp,
\]

then

\[
Y_1 \subset H^2(\mathbb{D}, \wedge^2 \mathbb{C}^m)^\perp.
\]

By Proposition 7.1, \(Y_1\) is a closed subspace of \(H^2(\mathbb{D}, \wedge^2 \mathbb{C}^m)^\perp\).

Choose any function \(Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})\) which satisfies the equations (4.21). Consider the operator \(T_1 : X_1 \to Y_1\) defined by

\[
T_1(\xi_0 \hat{\wedge} x) = P_{Y_1}(\eta_0 \hat{\wedge} (G - Q_1)x) \quad \text{for all } x \in H^2(\mathbb{D}, \mathbb{C}^n),
\]

where \(P_{Y_1}\) is the projection from \(L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)\) on \(Y_1\).

By Corollary 7.2 and Proposition 8.1, \(T_1\) is well defined.

If \(T_1 = 0\), then the algorithm terminates, we define \(r\) to be 1 and, in agreement with Theorem 10.15, the superoptimal approximant \(AG\) is given by the formula

\[
G - AG = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2} = \frac{t_0 y_0 x_0^*}{|h_0|^2},
\]
and the solution is
\[ AG = G - \frac{t_0 y_0 x_0^*}{|h_0|^2}. \]

If \( T_1 \neq 0 \), let \( t_1 = \| T_1 \| > 0 \). By Theorem 9.1, \( T_1 \) is a compact operator and so there exist \( v_1 \in H^2(\mathbb{D}, \mathbb{C}^n) \), \( w_1 \in H^2(\mathbb{D}, \mathbb{C}^m) \) such that \( (\xi_0 \hat{\lambda} v_1, \eta_0 \hat{\lambda} w_1) \) is a Schmidt pair for \( T_1 \) corresponding to \( t_1 \). Let \( h_1 \) be the scalar outer factor of \( \xi_0 \hat{\lambda} v_1 \) and let
\[ x_1 = (I_{\mathbb{C}^m} - \xi_0 \xi_0^*) v_1, \quad y_1 = (I_{\mathbb{C}^m} - \eta_0 \eta_0^*) w_1, \quad (4.25) \]
where \( I_{\mathbb{C}^n} \) and \( I_{\mathbb{C}^m} \) are the identity operators in \( \mathbb{C}^n \) and \( \mathbb{C}^m \), respectively. Then, by Proposition 5.1,
\[ \|x_1(z)\|_{\mathbb{C}^n} = |h_1(z)| = \|y_1(z)\|_{\mathbb{C}^m} \text{ almost everywhere on } T. \quad (4.26) \]

By Theorem 1.3, there exists a function \( Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) such that both \( s_0^\infty(G - Q_2) \) and \( s_1^\infty(G - Q_2) \) are minimized and
\[ s_1^\infty(G - Q_2) = t_1. \]

By Proposition 9.16, any such \( Q_2 \) satisfies
\[(G - Q_2)x_0 = t_0 y_0, \quad y_0^*(G - Q_2) = t_0 x_0^* \]
\[(G - Q_2)x_1 = t_1 y_1, \quad y_1^*(G - Q_2) = t_1 x_1^*. \quad (4.27) \]
Choose any function \( Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) which satisfies the equations (4.27).
Define
\[ \xi_1 = \frac{x_1}{h_1}, \quad \eta_1 = \frac{y_1}{h_1}. \quad (4.28) \]
By equations (4.26) and (4.28), \( \|\xi_1(z)\|_{\mathbb{C}^n} = 1 = \|\eta_1(z)\|_{\mathbb{C}^m} \text{ almost everywhere on } T. \]

Step 2. Define
\[ X_2 \overset{\text{def}}{=} \xi_0 \hat{\lambda} \xi_1 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n) \]
\[ Y_2 \overset{\text{def}}{=} \eta_0 \hat{\lambda} \eta_1 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^m) \perp. \]
Note that, by Proposition 6.2, \( X_2 \) is a closed linear subspace of \( H^2(\mathbb{D}, \Lambda^3 \mathbb{C}^n) \), and, by Proposition 7.3, \( Y_2 \) is a closed linear subspace of \( H^2(\mathbb{D}, \Lambda^3 \mathbb{C}^m) \perp. \)

Now consider the operator \( T_2 : X_2 \to Y_2 \) given by
\[ T_2(\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} x) = P_{Y_2}(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} (G - Q_2)x), \quad (4.29) \]
where \( P_{Y_2} \) is the projection from \( L^2(T, \mathbb{C}^m) \) on \( Y_2 \).

By Corollary 7.4 and Proposition 8.1, \( T_2 \) is well defined, that is, it does not depend on the choice of \( Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) satisfying equations (4.27). If \( T_2 = 0 \), then the algorithm terminates, we define \( r \) to be 2 and, according to Theorem 10.15, the superoptimal approximant \( AG \) is given by the formula
\[ G - AG = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2} - \frac{t_0 y_0 x_0^*}{|h_0|^2} + \frac{t_1 y_1 x_1^*}{|h_1|^2}. \]
If \( T_2 \neq 0 \), then let \( t_2 = \| T_2 \| \). By Theorem 9.26, \( T_2 \) is a compact operator and hence there exist \( v_2 \in H^2(\mathbb{D}, \mathbb{C}^n) \), \( w_2 \in H^2(\mathbb{D}, \mathbb{C}^m) \) such that
\[ (\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} v_2, \eta_0 \hat{\lambda} \eta_1 \hat{\lambda} w_2) \]
is a Schmidt pair for \( T_2 \) corresponding to \( \| T_2 \| = t_2 \).
Let \( h_2 \) be the scalar outer factor of \( \xi_0 \wedge_1 \wedge_2 v_2 \). Note that, by Proposition 6.2, \( \xi_0 \wedge_1 \wedge_2 v_2 \in H^2(\mathbb{D}, \wedge^3 \mathbb{C}^n) \). Let

\[
x_2 = (I_{\mathbb{C}^m} - \xi_0 \xi_0^\top - \xi_1 \xi_1^\top) v_2, \quad y_2 = (I_{\mathbb{C}^m} - \eta_0 \eta_0^\top - \eta_1 \eta_1^\top) w_2.
\]

Then, by Proposition 5.1,

\[
\|x_2(z)\|_{\mathbb{C}^n} = |h_2(z)| = \|y_2(z)\|_{\mathbb{C}^m} \quad \text{almost everywhere on } \mathbb{T}.
\]

Define

\[
\xi_2 = \frac{x_2}{h_2}, \quad \eta_2 = \frac{\bar{z} y_2}{h_2}.
\]

Clearly \( \|\xi_2(z)\|_{\mathbb{C}^n} = 1 \) and \( \|\eta_2(z)\|_{\mathbb{C}^m} = 1 \) almost everywhere on \( \mathbb{T} \).

**Recursive step.** Suppose that, for \( j \leq \min(m, n) - 2 \), we have constructed

\[
t_0 \geq t_1 \geq \cdots \geq t_j > 0
\]

\[
x_0, x_1, \ldots, x_j \in L^2(\mathbb{T}, \mathbb{C}^n)
\]

\[
y_0, y_1, \ldots, y_j \in L^2(\mathbb{T}, \mathbb{C}^m)
\]

\[
h_0, h_1, \ldots, h_j \in H^2(\mathbb{D}, \mathbb{C}) \text{ outer}
\]

\[
\xi_0, \xi_1, \ldots, \xi_j \in L^\infty(\mathbb{T}, \mathbb{C}^n) \text{ pointwise orthonormal on } \mathbb{T}
\]

\[
\eta_0, \eta_1, \ldots, \eta_j \in L^\infty(\mathbb{T}, \mathbb{C}^m) \text{ pointwise orthonormal on } \mathbb{T}
\]

\[
X_0 = H^2(\mathbb{D}, \mathbb{C}^m), X_1, \ldots, X_j
\]

\[
Y_0 = H^2(\mathbb{D}, \mathbb{C}^m)^\perp, Y_1, \ldots, Y_j
\]

\[
T_0, T_1, \ldots, T_j \text{ compact operators.}
\]

By Theorem 1.3, there exists a function \( Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) such that

\[
(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \ldots, s_{j+1}^\infty(G - Q_{j+1}))
\]

is lexicographically minimized. By Proposition 10.5, any such function \( Q_{j+1} \) satisfies

\[
(G - Q_{j+1}) x_i = t_i y_i, \quad \bar{y}_i (G - Q_{j+1}) = t_i x_i^\ast, \quad i = 0, 1, \ldots, j.
\]

Choose any function \( Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) which satisfies the equations (4.34). Define

\[
X_{j+1} \overset{\text{def}}{=} \xi_0 \wedge_1 \wedge_j \wedge H^2(\mathbb{D}, \mathbb{C}^n)
\]

\[
Y_{j+1} \overset{\text{def}}{=} \eta_0 \wedge_1 \wedge_j \wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp
\]

Note that, by Proposition 6.1, \( X_{j+1} \) is a subset of \( H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^n) \), and, by Proposition 10.8, \( Y_{j+1} \) is a closed subspace of \( H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^m)^\perp \). Consider the operator

\[
T_{j+1} : X_{j+1} \to Y_{j+1}
\]

given, for any \( x \in H^2(\mathbb{D}, \mathbb{C}^n) \), by

\[
T_{j+1}(\xi_0 \wedge_1 \wedge_j \wedge \xi_j \wedge x) = P_{Y_{j+1}}(\bar{\eta}_0 \wedge_1 \wedge_j \wedge \bar{\eta}_j \wedge (G - Q_{j+1}) x).
\]

By Corollary 7.4 and Proposition 8.1, \( T_{j+1} \) is well defined and does not depend on the choice of \( Q_{j+1} \) subject to equations (4.34).
If $T_{j+1} = 0$, then the algorithm terminates, we define $r$ to be $j + 1$, and, according to Theorem 10.15, the superoptimal approximant $AG$ is given by the formula

$$G - AG = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2}. $$

Otherwise, we define $t_{j+1} = |T_{j+1}| > 0$. By Theorem 9.1, $T_{j+1}$ is a compact operator and hence there exist $v_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^r)$, $w_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^m)\perp$ such that

$$(\xi_0 \wedge \xi_1 \wedge \cdots \wedge \xi_j \wedge v_{j+1}, \bar{\eta}_0 \wedge \bar{\eta}_1 \wedge \cdots \wedge \bar{\eta}_j \wedge w_{j+1})$$

is a Schmidt pair for $T_{j+1}$ corresponding to the singular value $t_{j+1}$.

Let $h_{j+1}$ be the scalar outer factor of $\xi_0 \wedge \xi_1 \wedge \cdots \wedge \xi_j \wedge v_{j+1}$, and let

$$x_{j+1} = (I_{\mathbb{C}^n} - \xi_0 \xi_0^* - \cdots - \xi_j \xi_j^*) v_{j+1}, \quad y_{j+1} = (I_{\mathbb{C}^m} - \bar{\eta}_0 \eta_0^T - \cdots - \bar{\eta}_j \eta_j^T) w_{j+1}.$$ (4.39)

Then, by Proposition 5.1,

$$\|x_{j+1}(z)\|_{\mathbb{C}^n} = |h_{j+1}(z)| = \|y_{j+1}(z)\|_{\mathbb{C}^m} \quad \text{almost everywhere on } \mathbb{T}. $$ (4.40)

We define

$$\xi_{j+1} = \frac{x_{j+1}}{h_{j+1}}, \quad \eta_{j+1} = \frac{\bar{\eta} y_{j+1}}{h_{j+1}}.$$ (4.41)

Clearly $\|\xi_{j+1}(z)\|_{\mathbb{C}^n} = 1$ and $\|\eta_{j+1}(z)\|_{\mathbb{C}^m} = 1$ almost everywhere on $\mathbb{T}$.

This completes the recursive step. The algorithm terminates after at most $\min(m, n)$ steps, so that $r \leq \min(m, n)$ and, in accordance with Theorem 10.15 the superoptimal approximant $AG$ is given by the formula

$$G - AG = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2}. $$

5. **Pointwise orthonormality of $\{\xi_i\}_{i=1}^j$ and $\{\bar{\eta}_i\}_{i=1}^j$ almost everywhere on $\mathbb{T}$**

These orthonormality properties will be needed for the justification of the main algorithm.

**Proposition 5.1.** Let $m, n$ be positive integers with $\min(m, n) \geq 2$, let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$ and let $0 \leq j \leq \min(m, n) - 2$. Suppose that we have applied steps $0, \ldots, j$ of the superoptimal analytic approximation algorithm from Subsection 4.2 to $G$ and we have obtained $x_i, y_i$ as in equations (4.39), and $\xi_i, \eta_i$ as in equations (4.41) for $i = 0, \ldots, j$. Then

(i) $\xi_0 \wedge v_1 = \xi_0 \wedge x_1, \quad \xi_0 \wedge \cdots \wedge \xi_{j-1} \wedge v_j = \xi_0 \wedge \cdots \wedge \xi_{j-1} \wedge x_j, \bar{\eta}_0 \wedge w_1 = \bar{\eta}_0 \wedge y_1, \quad \text{and} \quad \bar{\eta}_0 \wedge \cdots \wedge \bar{\eta}_{j-1} \wedge w_j = \bar{\eta}_0 \wedge \cdots \wedge \bar{\eta}_{j-1} \wedge y_j$;

(ii) $\|x_j(z)\|_{\mathbb{C}^n} = \|y_j(z)\|_{\mathbb{C}^m} = |h_j(z)|$ almost everywhere on $\mathbb{T}$;

(iii) The sets $\{\xi_i(z)\}_{i=0}^j$ and $\{\bar{\eta}_i(z)\}_{i=0}^j$ are orthonormal in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, for almost every $z \in \mathbb{T}$.

**Proof.** We will prove statement (ii) in Propositions 9.13 and 9.29. Statement (i) is proven below in equations (5.5), (5.9), (5.14). Let us prove assertion (iii).

Since the function $G$ belongs to $H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$, by Hartman’s theorem, the Hankel operator with symbol $G$, denoted by $H_G$, is a compact operator, and so there exist functions

$$x_0 \in H^2(\mathbb{D}, \mathbb{C}^n), \quad y_0 \in H^2(\mathbb{D}, \mathbb{C}^m)\perp$$
such that \((x_0, y_0)\) is a Schmidt pair corresponding to the singular value \(t_0 = \|H_G\| \neq 0\). By Lemma 4.12, \(x_0, \tilde{y}_0\) admit the inner–outer factorizations

\[
x_0 = \xi_0 h_0, \quad \tilde{y}_0 = \eta_0 h_0
\]

for column-matrix inner functions \(\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n), \eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)\) and some scalar outer factor \(h_0 \in H^2(\mathbb{D}, \mathbb{C}).\) By Theorem 4.11,

\[
\|x_0(z)\|_{\mathbb{C}^n} = |h_0(z)| = \|y_0(z)\|_{\mathbb{C}^m} \quad \text{almost everywhere on } \mathbb{T}.
\]

Hence, Proposition 5.1(iii) holds for \(\{\xi_j(z)\}_{j=0}^\infty\) in the case that \(j = 0.\)

Let \(T_1\) be given by equation (4.24). By the hypothesis (4.33), \(T_1\) is a compact operator, and if \(T_1 \neq 0\), then there exist \(v_1 \in H^2(\mathbb{D}, \mathbb{C}^n)\) and \(w_1 \in H^2(\mathbb{D}, \mathbb{C}^m)^*\) such that \((\xi_0 \wedge v_1, \eta_0 \wedge w_1)\) is a Schmidt pair corresponding to \(\|T_1\| = t_1\). By Proposition 3.24, \(\xi_0 \wedge v_1 \in H^2(\mathbb{D}, \Lambda^2 \mathbb{C}^n)\). Let \(h_1\) be the scalar outer factor of \(\xi_0 \wedge v_1\). We define

\[
x_1 = (I_n - \xi_0 \xi_0^*) v_1
\]

and

\[
\xi_1 = \frac{x_1}{h_1}.
\]

Then, for \(z \in \mathbb{D},\)

\[
\xi_1(z) = \frac{1}{h_1(z)} v_1(z) - \frac{1}{h_1(z)} \xi_0(z) \xi_0(z)^* v_1(z).
\]

Note that by equation (5.2),

\[
\xi_0(z) \xi_0(z) = \langle \xi_0(z), \xi_0(z) \rangle_{\mathbb{C}^n} = 1 \quad \text{almost everywhere on } \mathbb{T},
\]

hence

\[
\langle \xi_1(z), \xi_0(z) \rangle_{\mathbb{C}^n} = \xi_0^*(z) \xi_1(z) = \frac{1}{h_1(z)} \xi_0(z)^* v_1(z) - \frac{1}{h_1(z)} \xi_0(z)^* \xi_0(z) \xi_0(z)^* v_1(z) = 0
\]

almost everywhere on \(\mathbb{T}\). Note that, by equation (5.3), for almost every \(z \in \mathbb{T},\)

\[
\xi_0(z) \wedge v_1(z) = \xi_0(z) \wedge (x_1(z) + \xi_0(z) \xi_0(z)^* v_1(z))
\]

\[
= \xi_0(z) \wedge x_1(z) + \xi_0(z) \wedge \xi_0(z) \xi_0(z)^* v_1(z)
\]

\[
= \xi_0(z) \wedge x_1(z),
\]

the last equality following from the pointwise linear dependence of the vectors \(\xi_0\) and \(z \mapsto \xi_0(z) \wedge v_1(z), \xi_0(z)\) almost everywhere on \(\mathbb{T}\).

Moreover, since \(h_1\) is the scalar outer factor of \(\xi_0 \wedge v_1\), for almost every \(z \in \mathbb{T},\) we have

\[
|h_1(z)| = \|\xi_0(z) \wedge v_1(z)\|_{\Lambda^2 \mathbb{C}^n} = \|\xi_0(z) \wedge x_1(z)\|_{\Lambda^2 \mathbb{C}^n}.
\]

By Lemma 3.14,

\[
\|\xi_0(z) \wedge x_1(z)\|_{\Lambda^2 \mathbb{C}^n} = \|x_1(z) - \langle x_1(z), \xi_0(z) \rangle_{\mathbb{C}^n} \xi_0(z)\|_{\mathbb{C}^n} = \|x_1(z)\|_{\mathbb{C}^n}
\]

almost everywhere on \(\mathbb{T}\). Hence, for almost every \(z \in \mathbb{T},\)

\[
|h_1(z)| = \|x_1(z)\|_{\mathbb{C}^n}
\]

and thus

\[
\|\xi_1(z)\|_{\mathbb{C}^n} = \frac{\|x_1(z)\|_{\mathbb{C}^n}}{|h_1(z)|} = 1 \quad \text{almost everywhere on } \mathbb{T}.
\]
Consequently, \( \{\xi_0(z), \xi_1(z)\} \) is an orthonormal set in \( \mathbb{C}^n \) for almost every \( z \in T \). Hence, Proposition 5.1(iii) holds for \( \{\xi_i(z)\}_{i=0}^j \) in the case that \( j = 1 \).

**Recursive step.** Suppose that the entities in equations (4.33) have been constructed and have the stated properties. Since by the inductive hypothesis \( T_j \) is a compact operator, there exist

\[
v_j \in H^2(D, \mathbb{C}^n), \quad w_j \in H^2(D, \mathbb{C}^m)^\perp
\]

such that

\[
(\xi_0 \hat{\xi}_1 \ldots \hat{\xi}_{j-1} \hat{v}_j, \tilde{\eta}_0 \hat{\eta}_1 \ldots \hat{\eta}_{j-1} \hat{w}_j)
\]

is a Schmidt pair for \( T_j \) corresponding to \( \|T_j\| = t_j \). Let us first prove that \( \xi_0 \hat{\xi}_1 \ldots \hat{\xi}_{j-1} \hat{v}_j \) is an element of \( H^2(D, \wedge^{j+1} \mathbb{C}^n) \). By hypothesis,

\[
x_i = (I_n - \xi_0 \xi_1^* - \cdots - \xi_{i-1} \xi_{i-1}^*) v_i \quad \text{and} \quad \xi_i = \frac{x_i}{h_i}
\]

for \( i = 0, \ldots, j-1 \). Then, for all \( z \in D \),

\[
(\xi_0 \hat{\xi}_1 \ldots \hat{\xi}_{j-1} \hat{v}_j)(z) = \left( \frac{x_1}{h_1} \ldots \frac{x_{j-1}}{h_{j-1}} \right) (z) = \left( \frac{1}{h_1} \ldots \frac{1}{h_{j-1}} \xi_0 \hat{v}_1 \ldots \hat{v}_{j-1} \hat{v}_j \right) (z).
\]

We obtain

\[
(\xi_0 \hat{\xi}_1 \ldots \hat{\xi}_{j-1} \hat{v}_j)(z) = \left( \frac{1}{h_1} \ldots \frac{1}{h_{j-1}} \xi_0 \hat{v}_1 \ldots \hat{v}_{j-1} \hat{v}_j \right) (z), \text{ for all } z \in D,
\]

due to pointwise linear dependence of \( \xi_k \) and \( z \mapsto \xi_k(z) \langle v_j(z), \xi_k(z) \rangle_{\mathbb{C}^n} \) on \( D \), for all \( k = 0, \ldots, j-1 \). By Proposition 3.22,

\[
\frac{1}{h_1} \ldots \frac{1}{h_{j-1}} \xi_0 \hat{v}_1 \ldots \hat{v}_{j-1} \hat{v}_j
\]

is analytic on \( D \). Moreover, by Proposition 3.26, since \( \xi_0, \xi_1, \ldots, \xi_{j-1} \) are pointwise orthogonal on \( T \),

\[
\|\xi_0 \hat{\xi}_1 \ldots \hat{\xi}_{j-1} \hat{v}_j\|_{L^2(T, \wedge^{j+1} \mathbb{C}^n)} < \infty.
\]

Therefore,

\[
\xi_0 \hat{\xi}_1 \ldots \hat{\xi}_{j-1} \hat{v}_j \in H^2(D, \wedge^{j+1} \mathbb{C}^n).
\]

Let \( h_j \) be the scalar outer factor of \( \xi_1 \hat{\xi}_2 \ldots \hat{\xi}_{j-1} \hat{v}_j \). We define

\[
x_j = (I_n - \xi_0 \xi_1^* - \cdots - \xi_{j-1} \xi_{j-1}^*) v_j \quad (5.7)
\]

and

\[
\xi_j = \frac{x_j}{h_j} \quad (5.8)
\]

Let us show that \( \{\xi_0(z), \ldots, \xi_{j-1}(z), \xi_j(z)\} \) is an orthonormal set in \( \mathbb{C}^n \) almost everywhere on \( T \). We have

\[
\xi_j = \frac{1}{h_j} v_j - \frac{1}{h_j} \xi_0 \xi_0^* v_j - \cdots - \frac{1}{h_j} \xi_{j-1} \xi_{j-1}^* v_j,
\]
and so for \( i = 0, \ldots, j - 1, \)
\[
\langle \xi_j(z), \xi_i(z) \rangle_{C^n} = \frac{1}{h_j(z)} \xi_i^*(z)v_j(z) - \frac{1}{h_j(z)}\xi_0^*(z)\xi_0^*(z)v_1(z) - \cdots \\
- \frac{1}{h_j(z)}\xi_i^*(z)\xi_{j-1}(z)\xi_{j-1}(z)v_1(z)
\]
almost everywhere on \( T. \) Note that by the inductive hypothesis, for \( i, k = 0, 1, \ldots, j - 1 \) and for almost all \( z \in T, \)
\[
\xi_i^*(z)\xi_k(z) = \begin{cases} 
0, & \text{for } i \neq k \\
1, & \text{for } i = k
\end{cases}
\]
Thus, for \( i = 0, \ldots, j - 1, \)
\[
\langle \xi_j(z), \xi_i(z) \rangle_{C^n} = \frac{1}{h_j(z)} \xi_i^*(z)v_j(z) - \frac{1}{h_j(z)}\xi_0^*(z)\xi_0^*(z)v_1(z) - \cdots \\
- \frac{1}{h_j(z)}\xi_i^*(z)\xi_{j-1}(z)\xi_{j-1}(z)v_1(z) = 0
\]
almost everywhere on \( T, \) and hence, by induction on \( j \) and for all integers \( j = 0, \ldots, r - 1,\{\xi_0(z), \ldots, \xi_j(z)\} \) is an orthogonal set in \( C^n \) for almost all \( z \in T. \)
Let us show that
\[
\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge v_j(z) = \xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge x_j(z)
\]
almost everywhere on \( T. \) Equation (5.7) yields
\[
\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge v_j(z) \\
= \xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge (x_j(z) + \xi_0(z)\xi_0^*(z)v_j(z)) \\
+ \cdots + \xi_{j-1}(z)\xi_{j-1}(z)v_j(z) \\
= \xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge (x_j(z) + \xi_0(z)v_j(z), \xi_0(z))_{C^n} + \cdots \\
+ \cdots + \xi_{j-1}(z)v_j(z), \xi_{j-1}(z))_{C^n}
\]
almost everywhere on \( T. \)
Note that, for \( i = 0, \ldots, j - 1, \) the vectors \( \xi_i \) and \( z \mapsto \xi_i(z)(v_j(z), \xi_i(z))_{C^n} \) are pointwise linearly dependent almost everywhere on \( T. \) Thus, for all \( i = 0, \ldots, j - 1, \)
\[
\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge \xi_i(z)(v_{i+1}(z), \xi_i(z))_{C^n} = 0
\]
almost everywhere on \( T. \)
Hence,
\[
\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge v_j(z) = \xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge x_j(z) \text{ almost everywhere on } T. \tag{5.9}
\]
Next, we shall show that \( \|\xi_j(z)\|_{C^n} = 1 \) for almost all \( z \in T. \) Recall that \( h_j \) is the scalar outer factor of \( \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{j-1} \wedge v_j, \) and therefore
\[
|h_j(z)| = \|\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge v_j(z)\|_{\wedge i+1C^n} = \|\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge x_j(z)\|_{\wedge i+1C^n}
\]
almost everywhere on \( T. \)
By the inductive hypothesis, \( \{\xi_0(z), \ldots, \xi_{j-1}(z)\} \) is an orthonormal set in \( C^n \) for almost all \( z \in T, \) hence, by Lemma 3.14,
\[
|h_j(z)| = \|\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge x_j(z)\|_{\wedge i+1C^n} \\
= \|x_j(z) - \sum_{i=0}^{j-1} \langle x_j(z), \xi_i(z) \rangle \xi_i(z)\|_{C^n} \\
= \|x_j(z)\|_{C^n} \text{ almost everywhere on } T. \tag{5.10}
\]
Thus,

$$\|\xi_j(z)\|_{C^n} = \frac{\|x_j(z)\|_{C^n}}{|h_j(z)|} = 1$$

almost everywhere on $\mathbb{T}$, and hence, by induction on $j$, $\{\xi_0(z), \ldots, \xi_{j-1}(z), \xi_j(z)\}$ is an orthonormal set in $\mathbb{C}^n$ for almost all $z \in \mathbb{T}$, and for all integers $j = 0, \ldots, r - 1$.

Next, we will prove inductively that the set $\{\bar{\eta}_h(z)\}_{i=0}^r$, defined in equations (4.41), is orthonormal. For $i = 0$, by equation (5.1), we have

$$\|\bar{\eta}_0(z)\|_{C^n} = 1 \text{ almost everywhere on } \mathbb{T}. \quad \text{(5.11)}$$

Let $T_1$ be given by equation (4.24). $T_1$ is assumed to be a compact operator, and if $T_1 \neq 0$, there exist $v_1 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $w_1 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ such that $(\xi_0, \bar{\eta}_0)^\perp$ is a Schmidt pair corresponding to $\|T_1\| = t_1$. Suppose that $h_1$ is the scalar outer factor of $\xi_0, v_1$. Let

$$y_1 = (I_m - \bar{\eta}_0 h_0^T)w_1 = w_1 - \bar{\eta}_0 h_0^T w_1 \quad \text{(5.12)}$$

and let

$$\eta_1(z) = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T}. \quad \text{By equation (5.11), } \|\eta_1(z)\|_{C^n} = \frac{z y_1(z)}{h_1(z)} \text{ almost everywhere on } \mathbb{T. \end{equation}}$$

Recall that $h_1$ is the scalar outer factor of $\xi_0, v_1$. By equation (5.6) and Proposition 9.13,

$$\|x_1(z)\|_{C^n} = \|y_1(z)\|_{C^n} = |h_1(z)| \quad \text{almost everywhere on } \mathbb{T},$$

thus

$$\|\bar{\eta}_1(z)\|_{C^n} = \frac{\|z y_1(z)\|_{C^n}}{|h_1(z)|} = 1 \text{ almost everywhere on } \mathbb{T}. \quad \text{(5.11)}$$

Consequently, $\{\bar{\eta}_0(z), \eta_1(z)\}$ is an orthonormal set in $\mathbb{C}^m$ for almost every $z \in \mathbb{T}$. Hence, Proposition 5.1(iii) holds for $\{\bar{\eta}_j\}_{j=0}^r$ in the case $j = 1$.

**Recursive step:** Suppose that the entities in equations (4.33) have been constructed and have the stated properties. Since by the inductive hypothesis $T_j$ is a compact operator, there exist

$$v_j \in H^2(\mathbb{D}, \mathbb{C}^n), \quad w_j \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

such that

$$(\xi_1, \xi_2, \ldots, \xi_j, v_j, \bar{\eta}_0, \bar{\eta}_1, \ldots, \bar{\eta}_{j-1}, w_j)$$
is a Schmidt pair for $T_j$ corresponding to $\|T_j\| = t_j$. We have proved above that
\[ \xi_0 \hat{\land} \cdots \hat{\land} \xi_{j-1} \hat{\land} v_j \in H^2(D, \mathbb{C}^{j+1}). \]

Let $h_j$ be the scalar outer factor of $\xi_0 \hat{\land} \xi_1 \hat{\land} \cdots \hat{\land} \xi_{j-1} \hat{\land} v_j$. We define
\[ y_j = (I_m - \bar{\eta}_0 \eta_0^T - \cdots - \bar{\eta}_{j-1} \eta_{j-1}^T) w_j \]
and
\[ \bar{\eta}_j = \frac{zy_j}{h_j} \quad (5.13) \]

Let us show that $\{\bar{\eta}_0(z), \ldots, \bar{\eta}_j(z)\}$ is an orthonormal set in $\mathbb{C}^m$ almost everywhere on $T$. We have
\[ \bar{\eta}_j = \frac{zw_j}{h_j} - \cdots - \frac{z \bar{\eta}_{j-1} \eta_{j-1}^T w_j}{h_j} \]
and so, for $i = 0, \ldots, j - 1$,
\[ \langle \bar{\eta}_j(z), \bar{\eta}_i(z) \rangle_{\mathbb{C}^m} = \eta_i^T(z) \bar{\eta}_j(z) = \frac{z \eta_i^T(z) w_j(z)}{h_j(z)} - \cdots - \frac{z \eta_i^T(z) \eta_{j-1}(z) \eta_{j-1}^T(z) w_j(z)}{h_j(z)} \]
almost everywhere on $T$.

Note that, by the inductive hypothesis, for $i, k = 0, \ldots, j - 1$ and for almost all $z \in T$,
\[ \bar{\eta}_i(z) \bar{\eta}_k(z) = \begin{cases} 0, & \text{for } i \neq k \\ 1, & \text{for } i = k \end{cases} \]
Hence, for $i = 0, \ldots, j - 1$,
\[ \langle \bar{\eta}_j(z), \bar{\eta}_i(z) \rangle_{\mathbb{C}^m} = \frac{z \eta_i^T(z) w_j(z)}{h_j(z)} - \cdots - \frac{z \eta_i^T(z) \eta_{j-1}(z) \eta_{j-1}^T(z) w_j(z)}{h_j(z)} = 0 \]
almost everywhere on $T$. Thus by induction on $j$, for all integers $j = 0, \ldots, r - 1$, $\{\bar{\eta}_0(z), \ldots, \bar{\eta}_j(z)\}$ is an orthogonal set in $\mathbb{C}^m$ almost everywhere on $T$.

To complete the proof, we have to prove that $\|\bar{\eta}_j(z)\|_{\mathbb{C}^m} = 1$ for almost all $z \in T$. Recall that $h_j$ is the scalar outer factor of $\xi_0 \hat{\land} \xi_1 \hat{\land} \cdots \hat{\land} \xi_{j-1} \hat{\land} v_j$. By Proposition 10.13,
\[ |h_j(z)| = \|x_j(z)\|_{\mathbb{C}^m} = \|y_j(z)\|_{\mathbb{C}^m} \]
almost everywhere on $T$, thus
\[ \|\bar{\eta}_j(z)\|_{\mathbb{C}^m} = \frac{\|zy_j(z)\|_{\mathbb{C}^m}}{\|h_j(z)\|_{\mathbb{C}^m}} = 1 \]
almost everywhere on $T$, and hence, $\{\bar{\eta}_0(z), \ldots, \bar{\eta}_j(z)\}$ is an orthonormal set in $\mathbb{C}^m$ almost everywhere on $T$.

Note that, for $j = 1, \ldots, r - 1$,
\[ \bar{\eta}_0 \hat{\land} \cdots \hat{\land} \bar{\eta}_{j-1} \hat{\land} y_j = \bar{\eta}_0 \hat{\land} \cdots \hat{\land} \bar{\eta}_{j-1} \hat{\land} (I_m - \bar{\eta}_0 \eta_0^T - \cdots - \bar{\eta}_{j-1} \eta_{j-1}^T) w_j \]
\[ = \bar{\eta}_0 \hat{\land} \cdots \hat{\land} \bar{\eta}_{j-1} \hat{\land} w_j - \sum_{k=0}^{j-1} \bar{\eta}_0 \hat{\land} \cdots \hat{\land} \bar{\eta}_{j-1} \hat{\land} \bar{\eta}_{k} \eta_{k}^T w_j \]
\[ = \bar{\eta}_0 \hat{\land} \cdots \hat{\land} \bar{\eta}_{j-1} \hat{\land} w_j \quad (5.14) \]
on account of the pointwise linear dependence of $\bar{\eta}_k$ and $z \mapsto \bar{\eta}_k(z) \langle w_j(z), \bar{\eta}_k(z) \rangle_{\mathbb{C}^m}$ almost everywhere on $T$. \(\square\)
6. The closed subspace $X_{j+1}$ of $H^2(\mathbb{D}, \Lambda^{j+2} \mathbb{C}^n)$

Note that, although $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $\xi_0$ is inner, $x_i$ and $\xi_i$ might not be in $H^2(\mathbb{D}, \mathbb{C}^n)$ in general for $i = 1, \ldots, \min(m,n) - 1$. However, for every $x \in H^2(\mathbb{D}, \mathbb{C}^n)$, the pointwise wedge product

$$\xi_0 \wedge \cdots \wedge \xi_j \wedge x$$

is an element of $H^2(\mathbb{D}, \Lambda^{j+2} \mathbb{C}^n)$ as the following proposition asserts.

**Proposition 6.1.** Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$, and let $j \leq n - 1$. Let the vector-valued functions $\xi_0, \xi_1, \ldots, \xi_j$ be constructed after applying steps $0, \ldots, j$ of the algorithm above and be given by equations (4.41). Then

$$\xi_0 \wedge \cdots \wedge \xi_j \wedge H^2(\mathbb{D}, \mathbb{C}^n)$$

is a subset of $H^2(\mathbb{D}, \Lambda^{j+2} \mathbb{C}^n)$.

**Proof.** For $j = 0$, since $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$, the Hankel operator $H_G$ is compact. There exist $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n), y_0 \in H^2(\mathbb{D}, \mathbb{C}^n)^\perp$ such that $(x_0, y_0)$ is a Schmidt pair for the Hankel operator $H_G$ corresponding to the singular value $\|H_G\|$. By Lemma 4.12, $x_0, y_0$ admit the inner–outer factorizations

$$x_0 = \xi_0 h_0, \quad \bar{y} y_0 = \eta_0 h_0$$

for some inner $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, $\eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)$ and some scalar outer $h_0 \in H^2(\mathbb{D}, \mathbb{C})$.

Then, by Proposition 3.24, $\xi_0 \wedge H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \Lambda^{j+2} \mathbb{C}^n)$.

Let us now consider the case where $j = 1$. By definition,

$$X_1 = \xi_0 \wedge H^2(\mathbb{D}, \mathbb{C}^n), \quad Y_1 = \bar{\eta}_0 \wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

and, by the inductive hypothesis, $T_1 : X_1 \to Y_1$ given by equation (4.24) is a compact operator. Suppose $\|T_1\| \neq 0$ and let $(\xi_0 \wedge v_1, \bar{\eta}_0 \wedge w_1)$ be a Schmidt pair corresponding to $\|T_1\|$, where $v_1 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $w_1 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. We define

$$x_1 = (I_n - \xi_0 \bar{\xi}_0) v_1.$$

Note that, by Proposition 3.24, $\xi_0 \wedge v_1 \in H^2(\mathbb{D}, \Lambda^2 \mathbb{C}^n)$. Let $h_1 \in H^2(\mathbb{D}, \mathbb{C})$ be the scalar outer factor of $\xi_0 \wedge v_1 \in H^2(\mathbb{D}, \Lambda^2 \mathbb{C}^n)$. Then we define

$$\xi_1 = \frac{x_1}{h_1}.$$

Note that $\xi_0$ and $z \mapsto \xi_0(z) \langle v_1(z), \xi_0(z) \rangle_{\mathbb{C}^n}$ are pointwise linearly dependent on $\mathbb{D}$, since $\xi_0 v_1$ is a mapping from $\mathbb{D}$ to $\mathbb{C}$. Thus, for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $z \in \mathbb{D}$, we have

$$(\xi_0 \wedge \xi_1 \wedge x)(z) = \xi_0(z) \wedge \xi_1(z) \wedge x(z) = \xi_0(z) \wedge \frac{x_1(z)}{h_1(z)} \wedge x(z),$$
and by substituting the value of $x_1$, we find

\[
\xi_0(z) \wedge \frac{x_1(z)}{h_1(z)} \wedge x(z)
\]

\[
= \frac{1}{h_1(z)} \xi_0(z) \wedge (v_1(z) - \xi_0(z)\xi_0(z)^* v_1(z)) \wedge x(z)
\]

\[
= \frac{1}{h_1(z)} \xi_0(z) \wedge v_1(z) \wedge x(z) - \frac{1}{h_1(z)} \xi_0(z) \wedge \xi_0(z)^* v_1(z) \wedge x(z)
\]

\[
= \left( \frac{1}{h_1(z)} \xi_0 \wedge v_1 \wedge x \right)(z).
\]

Note that $v_1 \in H^2(\mathbb{D}, \mathbb{C}^n)$, $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ and $h_1 \in H^2(\mathbb{D}, \mathbb{C})$ is the scalar outer factor of $\xi_0 \wedge v_1$. By Proposition 3.22, for every $x \in H^2(\mathbb{D}, \mathbb{C}^n)$,

\[
\frac{1}{h_1(z)} \xi_0 \wedge v_1 \wedge x
\]

is analytic on $\mathbb{D}$. By Proposition 3.26, since $\xi_0$ and $\xi_1$ are pointwise orthogonal almost everywhere on $\mathbb{T}$,

\[
\|\xi_0 \wedge \xi_1 \wedge x\|_{L^2(\mathbb{T}, \mathbb{C}^n)} < \infty.
\]

Hence,

\[
\xi_0 \wedge \xi_1 \wedge x \in \frac{1}{h_1} \xi_0 \wedge v_1 \wedge H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \wedge^3 \mathbb{C}^n).
\]

**Recursive step:** Suppose that we have constructed vector-valued functions $\xi_0, \ldots, \xi_{j-1}$, $\eta_0, \ldots, \eta_{j-1}$, spaces $X_j, Y_j$ and a compact operator $T_j : X_j \rightarrow Y_j$ after applying steps $0, \ldots, j$ of the algorithm from Subsection 4.2 satisfying

\[
\xi_0 \wedge \cdots \wedge \xi_{j-1} \wedge H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \wedge^{j+1} \mathbb{C}^n).
\]

(6.1)

Since $T_j$ is a compact operator, there exist vector-valued functions $v_j \in H^2(\mathbb{D}, \mathbb{C}^n), w_j \in H^2(\mathbb{D}, \mathbb{C}^n)^\perp$ such that

\[
(\xi_0 \wedge \cdots \wedge \xi_{j-1} \wedge \eta_0 \wedge \cdots \wedge \eta_{j-1} \wedge w_j)
\]

is a Schmidt pair for $T_j$ corresponding to $\|T_j\|$. Define

\[
x_j = (I_n - \xi_0 \xi_0^* - \cdots - \xi_{j-1} \xi_{j-1}^*) v_j.
\]

(6.2)

By assumption, $\xi_0 \wedge \cdots \wedge \xi_{j-1} \wedge v_j$ lies in $H^2(\mathbb{D}, \wedge^{j+1} \mathbb{C}^n)$. Let $h_j \in H^2(\mathbb{D}, \mathbb{C})$ be the scalar outer factor of $\xi_0 \wedge \cdots \wedge \xi_{j-1} \wedge v_j$. Define $\xi_j = \frac{x_j}{h_j}$. Note that $\xi_i(z)$ and $z \mapsto \xi_i(z) (v_j(z), \xi_j(z))_{\mathbb{C}^n}$ are pointwise linearly dependent on $\mathbb{D}$ for $i = 0, \ldots, j - 1$. Thus, for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$ and all $z \in \mathbb{D}$,

\[
(\xi_0 \wedge \cdots \wedge \xi_{j-1} \wedge \xi_j \wedge x)(z) = (\xi_0 \wedge \cdots \wedge \xi_{j-1} \wedge \frac{x_j}{h_j} \wedge x)(z)
\]

\[
= \xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge \frac{1}{h_j(z)} (v_j(z) - \xi_0(z)\xi_0(z)^* v_j(z)) - \cdots
\]

\[
- \xi_{j-1}(z)\xi_{j-1}^*(z) v_j(z)) \wedge x(z)
\]

\[
= \xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge \frac{1}{h_j(z)} v_j(z) \wedge x(z)
\]
Thus, for every $i$, by equation (6.3), for all $z \in \mathbb{D}$,

$$\left(\frac{1}{h_j} \xi_0 \hat{\Lambda} \cdots \hat{\Lambda} \xi_{j-1} \hat{\Lambda} x\right)(z) = \left(\frac{1}{h_j} \xi_0 \hat{\Lambda} \cdots \hat{\Lambda} \xi_{j-1} \hat{\Lambda} v_j \hat{\Lambda} x\right)(z).$$

Substituting $\xi_i$ for $\xi_i$ in the latter equation, where $x_i$ are given by equation (6.2) for $i = 1, \ldots, j - 1$, we obtain

$$\left(\frac{1}{h_1 h_2} \cdots \frac{1}{h_j} \xi_0 \hat{\Lambda} \cdots \hat{\Lambda} v_1 \hat{\Lambda} \cdots \hat{\Lambda} v_{j-1} \hat{\Lambda} v_j \hat{\Lambda} x\right)(z), \ z \in \mathbb{D}$$

on account of the pointwise linear dependence of $\xi_k$ and $z \mapsto \langle v_k(z), \xi(z) \rangle_{\mathbb{C}^n} \xi_k(z)$ on $\mathbb{D}$, for $k = 0, \ldots, j$. By Proposition 3.22, for every $x \in H^2(\mathbb{D}, \mathbb{C}^n)$,

$$\left(\frac{1}{h_1} \frac{1}{h_2} \cdots \frac{1}{h_j} \xi_0 \hat{\Lambda} \cdots \hat{\Lambda} v_1 \hat{\Lambda} \cdots \hat{\Lambda} v_j \hat{\Lambda} x\right)$$

is analytic on $\mathbb{D}$. By Proposition 3.26, since $\xi_0, \xi_1, \ldots, \xi_j$ are pointwise orthogonal on $\mathbb{T}$,

$$\langle \xi_0 \hat{\Lambda} \cdots \hat{\Lambda} \xi_j \hat{\Lambda} x, \xi_j \hat{\Lambda} x \rangle_{L^2(\mathbb{T}, \wedge^{j+2} \mathbb{C}^n)} < \infty.$$

Thus, for every $x \in H^2(\mathbb{D}, \mathbb{C}^n)$,

$$\xi_0 \hat{\Lambda} \cdots \hat{\Lambda} v_1 \hat{\Lambda} \cdots \hat{\Lambda} v_j \hat{\Lambda} x \in H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^n)$$

and the claim has been proved. \(\square\)

**Proposition 6.2.** In the notation of Proposition 6.1,

$$\xi_0 \hat{\Lambda} \cdots \hat{\Lambda} v_1 \hat{\Lambda} \cdots \hat{\Lambda} v_j \hat{\Lambda} x$$

is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^n)$.

**Proof.** Let us first show that $\xi_0 \hat{\Lambda} H^2(\mathbb{D}, \mathbb{C}^n)$ is a closed subspace of $H^2(\mathbb{D}, \wedge^2 \mathbb{C}^n)$. Observe that, by Proposition 3.24, $\xi_0 \hat{\Lambda} H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \wedge^2 \mathbb{C}^n)$. Let

$$\Xi_0 = \{ f \in H^2(\mathbb{D}, \mathbb{C}^n) : \langle f(z), \xi_0(z) \rangle_{\mathbb{C}^n} = 0 \ \text{almost everywhere on} \ \mathbb{T}\}.$$  

Consider a vector-valued function $w \in H^2(\mathbb{D}, \mathbb{C}^n)$. For all $z \in \mathbb{D}$, we may write $w$ as

$$w(z) = w(z) - \langle w(z), \xi_0(z) \rangle_{\mathbb{C}^n} \xi_0(z) + \langle w(z), \xi_0(z) \rangle_{\mathbb{C}^n} \xi_0(z).$$

Then, for all $w \in H^2(\mathbb{D}, \mathbb{C}^n)$ and for all $z \in \mathbb{D}$,

$$(\xi_0 \hat{\Lambda} w)(z) = \xi_0(z) \wedge (w(z) - \langle w(z), \xi_0(z) \rangle_{\mathbb{C}^n} \xi_0(z) + \langle w(z), \xi_0(z) \rangle_{\mathbb{C}^n} \xi_0(z))$$

$$= \xi_0(z) \wedge (w(z) - \langle w(z), \xi_0(z) \rangle_{\mathbb{C}^n} \xi_0(z))$$
on account of the pointwise linear dependence of $\xi_0$ and $z \mapsto \langle w(z), \xi_0(z) \rangle_{C^n} \xi_0(z)$ on $\mathbb{D}$.

Note that

$$w(z) - \langle w(z), \xi_0(z) \rangle_{C^n} \xi_0(z) \in \Xi_0,$$

thus

$$\xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n) \subset \xi_0 \hat{\lambda} \Xi_0.$$

By Corollary 3.32, $\Xi_0$ is a closed subspace of $H^2(\mathbb{D}, \mathbb{C}^n)$, hence

$$\xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n) \supset \xi_0 \hat{\lambda} \Xi_0,$$

and so,

$$\xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n) = \xi_0 \hat{\lambda} \Xi_0.$$

Consider the mapping

$$C_{\xi_0} : \Xi_0 \to \xi_0 \hat{\lambda} \Xi_0$$

given by

$$C_{\xi_0} w = \xi_0 \hat{\lambda} w$$

for all $w \in \Xi_0$. Note that, by Proposition 5.1, $\|\xi_0(e^{i\theta})\|_{C^n}^2 = 1$ for almost every $e^{i\theta} \in \mathbb{T}$. Therefore, for any $w \in \Xi_0$, we have

$$\|\xi_0 \hat{\lambda} w\|_{L^2(\mathbb{T}, \wedge^2 \mathbb{C}^n)}^2 = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi_0 \hat{\lambda} w, \xi_0 \hat{\lambda} w \rangle(e^{i\theta}) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\|\xi_0(e^{i\theta})\|_{C^n}^2 \|w(e^{i\theta})\|_{C^n}^2 - |\langle w(e^{i\theta}), \xi_0(e^{i\theta}) \rangle|^2) d\theta$$

$$= \|w\|_{L^2(\mathbb{T}, \mathbb{C}^n)}^2,$$

since $w$ is pointwise orthogonal to $\xi_0$ almost everywhere on $\mathbb{T}$. Thus, the mapping

$$C_{\xi_0} : \Xi_0 \to \xi_0 \hat{\lambda} \Xi_0$$

is an isometry. Furthermore, $C_{\xi_0} : \Xi_0 \to \xi_0 \hat{\lambda} \Xi_0$ is a surjective mapping, thus $\Xi_0$ and $\xi_0 \hat{\lambda} \Xi_0$ are isometrically isomorphic. Therefore, since $\Xi_0$ is a closed subspace of $H^2(\mathbb{D}, \mathbb{C}^n)$, the space $\xi_0 \hat{\lambda} \Xi_0$ is complete, hence is a closed subspace of $H^2(\mathbb{D}, \wedge^2 \mathbb{C}^n)$. Hence, $\xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n)$ is a closed subspace of $H^2(\mathbb{D}, \wedge^2 \mathbb{C}^n)$.

To prove that $\xi_0 \hat{\lambda} \cdots \hat{\lambda} \xi_2 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n)$ is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^n)$, let us consider

$$\Xi_j = \{ f \in H^2(\mathbb{D}, \mathbb{C}^n) : \langle f(z), \xi_i(z) \rangle_{C^n} = 0 \text{ for } i = 0, \ldots, j \}$$

which is the pointwise orthogonal complement of $\xi_0, \ldots, \xi_j$ in $H^2(\mathbb{D}, \mathbb{C}^n)$. Let $\psi \in H^2(\mathbb{D}, \mathbb{C}^n)$. We may write $\psi$ as

$$\psi(z) = \psi(z) - \sum_{i=0}^j \langle \psi(z), \xi_i(z) \rangle_{C^n} \xi_i(z) + \sum_{i=0}^j \langle \psi(z), \xi_i(z) \rangle_{C^n} \xi_i(z).$$

Then, for all $\psi \in H^2(\mathbb{D}, \mathbb{C}^n)$ and for almost all $z \in \mathbb{T}$,

$$(\xi_0 \hat{\lambda} \cdots \hat{\lambda} \xi_j \hat{\lambda} \psi)(z) = \xi_0(z) \wedge \cdots \wedge \left( \psi(z) - \sum_{i=0}^j \langle \psi(z), \xi_i(z) \rangle_{C^n} \xi_i(z) \right).$$
due to the pointwise linear dependence of \( \xi_k \) and \( z \mapsto \xi_k(z) \langle \psi(z), \xi_k(z) \rangle_{C^2} \) almost everywhere on \( T \).

Note that \( \langle \psi(z), \xi_i(z) \rangle_{C^2} \) is in \( \Xi_j \), thus

\[
\xi_0 \wedge \ldots \wedge \hat{\xi}_j \wedge H^2(\mathbb{D}, \mathbb{C}^n) \subset \xi_0 \wedge \ldots \wedge \hat{\xi}_j \wedge \Xi_j.
\]

The reverse inclusion holds by the definition of \( \Xi_j \), hence

\[
\xi_0 \wedge \ldots \wedge \hat{\xi}_j \wedge H^2(\mathbb{D}, \mathbb{C}^n) = \xi_0 \wedge \ldots \wedge \hat{\xi}_j \wedge \Xi_j.
\]

Consequently, in order to prove the proposition it suffices to show that \( \xi_0 \wedge \ldots \wedge \hat{\xi}_j \wedge \Xi_j \) is a closed subspace of \( H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^n) \). By Corollary \( 3.32 \), \( \Xi_j \) is a closed subspace of \( H^2(\mathbb{D}, \mathbb{C}^n) \), being an intersection of closed subspaces. For any \( f \in \Xi_j \),

\[
\|\xi_0 \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge f\|^2_{L^2(T, \wedge^{j+2} \mathbb{C}^n)} = \frac{1}{2\pi} \int_0^{2\pi} \det\left(\begin{array}{cccc}
\|\xi_0(e^{i\theta})\|^2_{\mathbb{C}^n} & \cdots & \cdots & \|\xi_0(e^{i\theta}), f(e^{i\theta})\|_{\mathbb{C}^n} \\
\langle \xi_1(e^{i\theta}), \xi_0(e^{i\theta}) \rangle_{\mathbb{C}^n} & \|\xi_1(e^{i\theta})\|^2_{\mathbb{C}^n} & \cdots & \langle \xi_1(e^{i\theta}), f(e^{i\theta}) \rangle_{\mathbb{C}^n} \\
\vdots & \vdots & \ddots & \vdots \\
\langle f(e^{i\theta}), \xi_0(e^{i\theta}) \rangle_{\mathbb{C}^n} & \cdots & \cdots & \|f(e^{i\theta})\|^2_{\mathbb{C}^n}
\end{array}\right) d\theta.
\]

Note that \( f \) and \( \xi_i \) are pointwise orthogonal almost everywhere on \( T \), and, by Proposition \( 5.1 \), \( \{\xi_0(z), \ldots, \xi_j(z)\} \) is an orthonormal set for almost every \( z \in T \). Hence,

\[
\|\xi_0 \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge f\|^2_{L^2(T, \wedge^{j+2} \mathbb{C}^n)} = \frac{1}{2\pi} \int_0^{2\pi} \det\left(\begin{array}{cccc}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \|f(e^{i\theta})\|^2_{\mathbb{C}^n}
\end{array}\right) d\theta = \|f\|^2_{L^2(T, \mathbb{C}^n)}.
\]

Thus,

\[
\xi_0 \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge : \Xi_j \to \xi_0 \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge \Xi_j
\]

is an isometry. Furthermore,

\[
(\xi_0 \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge): \Xi_j \to \xi_0 \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge \Xi_j
\]

is a surjective mapping, thus \( \Xi_j \) and \( \xi_0 \wedge \ldots \wedge \hat{\xi}_j \wedge \Xi_j \) are isometrically isomorphic. Therefore, since \( \Xi_j \) is a closed subspace of \( H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^n) \), the space \( \xi_0 \wedge \ldots \wedge \hat{\xi}_j \wedge \Xi_j \) is a closed subspace of \( H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^n) \). Hence,

\[
\xi_0 \wedge \ldots \wedge \hat{\xi}_j \wedge H^2(\mathbb{D}, \mathbb{C}^n)
\]

is a closed subspace of \( H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^n) \).

\[\square\]

7. The closed subspace \( Y_{j+1} \) of \( H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^m) \)

**Proposition 7.1.** Given \( \tilde{\eta}_0 = \frac{z \eta_0}{\|z \eta_0\|} \) as constructed in the algorithm in Subsection 4.2 the space \( \tilde{\eta}_0 \wedge H^2(\mathbb{D}, \mathbb{C}^m) \) is a closed subspace of \( H^2(\mathbb{D}, \wedge^2 \mathbb{C}^m) \).
Proof. As in Proposition 6.1, one can show that
\[ \eta_0 \hat{\omega} H^2(\mathbb{D}, \mathbb{C}^m) \subset z H^2(\mathbb{D}, \Lambda^2 \mathbb{C}^m) \]
and therefore
\[ \tilde{\eta}_0 \hat{\omega} z H^2(\mathbb{D}, \mathbb{C}^m) \subset \tilde{z} H^2(\mathbb{D}, \Lambda^2 \mathbb{C}^m). \]
Hence,
\[ \tilde{\eta}_0 \hat{\omega} H^2(\mathbb{D}, \mathbb{C}^m) \perp \subset H^2(\mathbb{D}, \Lambda^2 \mathbb{C}^m) \perp. \]
By virtue of the fact that complex conjugation is a unitary operator on \( L^2(\mathbb{T}, \mathbb{C}^m) \), an equivalent statement to Proposition 7.1 is that \( \eta_0 \hat{\omega} H^2(\mathbb{D}, \mathbb{C}^m) \) is a closed subspace of \( z H^2(\mathbb{D}, \Lambda^2 \mathbb{C}^m) \).

Let
\[ V = \{ f \in z H^2(\mathbb{D}, \mathbb{C}^m) : \langle f(z), \eta_0(z) \rangle_{\mathbb{C}^m} = 0 \text{ for almost all } z \in \mathbb{T} \} \]
be the pointwise orthogonal complement of \( \eta_0 \) in \( z H^2(\mathbb{D}, \mathbb{C}^m) \).

Consider \( g \in z H^2(\mathbb{D}, \mathbb{C}^m) \). We may write \( g \) as
\[ g(z) = g(z) - \langle g(z), \eta_0(z) \rangle_{\mathbb{C}^m} \cdot \eta_0(z) + \langle g(z), \eta_0(z) \rangle_{\mathbb{C}^m} \cdot \nu_0(z) \]
for every \( z \in \mathbb{D} \). Then, for all \( g \in z H^2(\mathbb{D}, \mathbb{C}^m) \) and for all \( z \in \mathbb{D} \),
\[ \langle \eta_0 \hat{\omega} g(z), \eta_0(z) \rangle = \eta_0(z) \cdot [g(z) - \langle g(z), \nu_0(z) \rangle_{\mathbb{C}^m} \cdot \nu_0(z)] \]
on account of the pointwise linear dependence of \( \eta_0 \) and \( z \mapsto \langle g(z), \eta_0(z) \rangle_{\mathbb{C}^m} \cdot \nu_0(z) \) on \( \mathbb{D} \).

Note that \( g(z) - \langle g(z), \eta_0(z) \rangle_{\mathbb{C}^m} \cdot \nu_0(z) \in V \), thus
\[ \eta_0 \hat{\omega} z H^2(\mathbb{D}, \mathbb{C}^m) \subset \eta_0 \hat{\omega} V. \]
The reverse inclusion is obvious, hence
\[ \eta_0 \hat{\omega} z H^2(\mathbb{D}, \mathbb{C}^m) = \eta_0 \hat{\omega} V. \]
To prove the proposition, it suffices to show that \( \eta_0 \hat{\omega} V \) is a closed subspace of \( z H^2(\mathbb{D}, \Lambda^2 \mathbb{C}^m) \).

Consider the mapping
\[ C_{\eta_0} : V \rightarrow \eta_0 \hat{\omega} V \]
defined by
\[ C_{\eta_0} \nu = \eta_0 \hat{\omega} \nu \]
for all \( \nu \in V \). Note that, by Proposition 5.1, \( \| \eta_0(e^{i\theta}) \|_{\mathbb{C}^m}^2 = 1 \) for almost every \( e^{i\theta} \in \mathbb{T} \). Then, for any \( \nu \in V \), we have
\[ \| \eta_0 \hat{\omega} \nu \|_{L^2(\mathbb{T}, \Lambda^2 \mathbb{C}^m)}^2 = \frac{1}{2\pi} \int_0^{2\pi} \langle \eta_0 \hat{\omega} \nu, \eta_0 \hat{\omega} \nu \rangle(e^{i\theta}) d\theta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \left( \| \eta_0(e^{i\theta}) \|_{\mathbb{C}^m}^2 \| \nu(e^{i\theta}) \|_{\mathbb{C}^m}^2 - |\langle \nu(e^{i\theta}), \eta_0(e^{i\theta}) \rangle|^2 \right) d\theta \]
\[ = \| \nu \|_{L^2(\mathbb{T}, \mathbb{C}^m)}^2, \]
since \( \nu \) is pointwise orthogonal to \( \eta_0 \) almost everywhere on \( \mathbb{T} \). Thus, the mapping \( C_{\eta_0} : V \rightarrow \eta_0 \hat{\omega} V \) is an isometry.

Note that by Corollary 3.32, \( V \) is a closed subspace of \( z H^2(\mathbb{D}, \mathbb{C}^m) \). Furthermore,
\[ C_{\eta_0} : V \rightarrow \eta_0 \hat{\omega} V \]
is a surjective mapping, thus $V$ and $\eta_j \wedge V$ are isometrically isomorphic. Therefore, since $V$ is a closed subspace of $zH^2(\mathbb{D}, \mathbb{C}^m)$, the space $\eta_j \wedge V$ is complete and therefore a closed subspace of $zH^2(\mathbb{D}, \wedge^2 \mathbb{C}^m)$. Hence, $\bar{\eta}_0 \wedge \overline{H^2(\mathbb{D}, \mathbb{C}^m)}$ is complete and therefore a closed subspace of $H^2(\mathbb{D}, \wedge^2 \mathbb{C}^m)$.

\[ \square \]

**Corollary 7.2.** The orthogonal projection $P_{Y_1}$ from $L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)$ onto $\bar{\eta}_0 \wedge H^2(\mathbb{D}, \mathbb{C}^m)$ is well defined.

**Proof.** By Proposition 3.30, $H^2(\mathbb{D}, \wedge^2 \mathbb{C}^m)$ can be identified with a closed subspace of $L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)$, thus we have

\[ H^2(\mathbb{D}, \wedge^2 \mathbb{C}^m) \perp L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m) \cap H^2(\mathbb{D}, \wedge^2 \mathbb{C}^m). \]

Now the assertion follows immediately from Proposition 7.1. \[ \square \]

**Proposition 7.3.** Let $0 \leq j \leq m - 2$. Let the functions $\bar{\eta}_j$ be given by equations (4.41) in the algorithm from Subsection 4.2, that is, $\bar{\eta}_j = \frac{\bar{v}_j}{h_i}$ for $i = 0, \ldots, j$. Then, the space

\[ \bar{\eta}_0 \wedge \bar{\eta}_1 \wedge \cdots \wedge \bar{\eta}_j \wedge \overline{H^2(\mathbb{D}, \mathbb{C}^m)} \]

is a closed linear subspace of $H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^m)$. \[ \square \]

**Proof.** First let us show that, for every $x \in H^2(\mathbb{D}, \mathbb{C}^m)$,

\[ \eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge z x \in zH^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^m). \]

Recall that

\[ \eta_i \wedge \cdots \wedge \eta_{j-1} \wedge z \bar{y}_j = \eta_0 \wedge \cdots \wedge \eta_{j-1} \wedge (z \bar{w}_j - \sum_{i=0}^{j-1} \eta_i \eta_{j-i} \bar{z} \bar{w}_j) = \eta_0 \wedge \cdots \wedge \eta_{j-1} \wedge \bar{z} \bar{w}_j \quad (7.1) \]

because of the pointwise linear dependence of $\eta_i$ and $z \mapsto \langle \bar{z} \bar{w}_{j+1}(z), \eta_i(z) \rangle_{\mathbb{C}^m} \eta_i(z)$ on $\mathbb{D}$.

By Proposition 10.13,

\[ |h_i(z)| = \|y_i(z)\|_{\mathbb{C}^m} \]

almost everywhere on $\mathbb{T}$.

Substituting $\eta_i = \frac{\eta_i}{h_i}$ for all $i = 0, \ldots, j - 1$ in equation (7.1), we obtain

\[ \eta_0 \wedge \cdots \wedge \eta_{j-1} \wedge \bar{z} \bar{y}_j = \frac{1}{h_0} \frac{1}{h_1} \cdots \frac{1}{h_j} \eta_0 \wedge \bar{z} \bar{w}_1 \wedge \cdots \wedge \bar{z} \bar{w}_j. \]

Observe that, by Proposition 3.22, for every $x \in H^2(\mathbb{D}, \mathbb{C}^m)$,

\[ \frac{1}{h_0} \frac{1}{h_1} \cdots \frac{1}{h_j} \chi(\eta_0 \wedge \bar{z} \bar{w}_1 \wedge \cdots \wedge \bar{z} \bar{w}_j) \]

is analytic on $\mathbb{D}$. By Proposition 3.26, for all $x \in H^2(\mathbb{D}, \mathbb{C}^m)$, since $\eta_0, \ldots, \eta_j$ are pointwise orthogonal on $\mathbb{T}$,

\[ \|\eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge z x\|_{L^2(\mathbb{T}, \wedge^{j+2} \mathbb{C}^m)} < \infty. \]

Hence, for every $x \in H^2(\mathbb{D}, \mathbb{C}^m)$,

\[ \eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge z x = \frac{1}{h_0} \frac{1}{h_1} \cdots \frac{1}{h_j} \eta_0 \wedge \bar{z} \bar{w}_1 \wedge \cdots \wedge \bar{z} \bar{w}_j \wedge z x \]

is in $zH^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^m)$. \[ \square \]
Taking complex conjugates, we infer that
\[ Y_{j+1} \triangleq \eta_0 \wedge \cdots \wedge \eta_{j-1} \wedge \eta_j \wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp \subset H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^m)^\perp. \]

Let us prove that \( Y_{j+1} \) is a closed linear subspace of \( H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^m)^\perp \). Since complex conjugation is a unitary operator on \( L^2(\mathbb{T}, \mathbb{C}^m) \), an equivalent statement to the above is that
\[ \eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge H^2(\mathbb{D}, \mathbb{C}^m) \]
is a closed linear subspace of \( z H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^m) \).

Let
\[ V_j = \{ \varphi \in z H^2(\mathbb{D}, \mathbb{C}^m) : \langle \varphi(z), \eta_i(z) \rangle_{\mathbb{C}^m} = 0, \text{ for } i = 0, \ldots, j \} \]
be the pointwise orthogonal complement of \( \eta_0, \ldots, \eta_j \) in \( z H^2(\mathbb{D}, \mathbb{C}^m) \). Consider \( f \in z H^2(\mathbb{D}, \mathbb{C}^m) \). We may write \( f \) as
\[ f(z) = f(z) - \sum_{i=0}^j \langle f(z), \eta_i(z) \rangle \eta_i(z) + \sum_{i=0}^j \langle f(z), \eta_i(z) \rangle \eta_i(z). \]
Then, for all \( f \in z H^2(\mathbb{D}, \mathbb{C}^m) \) and for almost all \( z \in \mathbb{T} \),
\[ \langle \eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge f(z) \rangle = \eta_0(z) \wedge \eta_1(z) \wedge \cdots \wedge \eta_j(z) \wedge \left( f(z) - \sum_{i=0}^j \langle f(z), \eta_i(z) \rangle \eta_i(z) \right). \]
Note that \( (f(z) - \sum_{i=0}^j \langle f(z), \eta_i(z) \rangle \eta_i(z)) \in V_j \), thus
\[ \eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge H^2(\mathbb{D}, \mathbb{C}^m) \subset \eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge V_j. \]
The reverse inclusion holds by the definition of \( V_j \), hence
\[ \eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge H^2(\mathbb{D}, \mathbb{C}^m) = \eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge V_j. \]
Consequently, in order to prove the proposition it suffices to show that \( \eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge V_j \) is a closed subspace of \( z H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^m) \). By Corollary 3.32, \( V_j \) is a closed subspace of \( z H^2(\mathbb{D}, \mathbb{C}^m) \), being an intersection of closed subspaces. For any \( f \in V_j \), we have
\[ \| \eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge f \|^2_{L^2(\mathbb{T}, \wedge^{j+2} \mathbb{C}^m)} = \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} \| \eta_0(e^{i\theta}) \|^2_{\mathbb{L}^2_{\mathbb{C}^m}} & \cdots & \langle \eta_0(e^{i\theta}), f(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle \eta_1(e^{i\theta}), \eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \cdots & \| \eta_1(e^{i\theta}) \|^2_{\mathbb{L}^2_{\mathbb{C}^m}} \\ \vdots & \ddots & \vdots \\ \langle f(e^{i\theta}), \eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \cdots & \| f(e^{i\theta}) \|^2_{\mathbb{L}^2_{\mathbb{C}^m}} \end{pmatrix} d\theta. \]
Note that \( f \) and \( \eta_i \) are pointwise orthogonal almost everywhere on \( \mathbb{T} \) and, by Proposition 5.1, \( \{ \eta_0(z), \ldots, \eta_j(z) \} \) is an orthonormal set for almost every \( z \in \mathbb{T} \). Hence,
\[ \| \eta_0 \wedge \eta_1 \wedge \cdots \wedge \eta_j \wedge f \|^2_{L^2(\mathbb{T}, \wedge^{j+2} \mathbb{C}^m)} = \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \| f(e^{i\theta}) \|^2_{\mathbb{L}^2_{\mathbb{C}^m}} \end{pmatrix} d\theta \]
\[ = \| f \|^2_{L^2(\mathbb{T}, \mathbb{C}^m)}. \]
Thus,
\[ \eta_0 \hat{\lambda}_{\eta_1} \cdots \hat{\lambda}_{\eta_j} \cdot : V_j \to \eta_0 \hat{\lambda}_{\eta_1} \cdots \hat{\lambda}_{\eta_j} \hat{\lambda} V_j \]
is an isometry. Furthermore,
\[ (\eta_0 \hat{\lambda}_{\eta_1} \cdots \hat{\lambda}_{\eta_j} \hat{\lambda} : V_j \to \eta_0 \hat{\lambda}_{\eta_1} \cdots \hat{\lambda}_{\eta_j} \hat{\lambda} V_j \]
is a surjective mapping, thus \( V_j \) and \( \eta_0 \hat{\lambda}_{\eta_1} \cdots \hat{\lambda}_{\eta_j} \hat{\lambda} V_j \) are isometrically isomorphic. Therefore, since \( V_j \) is a closed subspace of \( zH^2(\mathbb{D}, \mathbb{C}^m) \), the space \( \eta_0 \hat{\lambda}_{\eta_1} \cdots \hat{\lambda}_{\eta_j} \hat{\lambda} V_j \) is a closed subspace of \( zH^2(\mathbb{D}, \Lambda^{j+2}\mathbb{C}^m) \). Hence,

\[ \eta_0 \hat{\lambda}_{\eta_1} \cdots \hat{\lambda}_{\eta_j} \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^m) \]
is a closed subspace of \( H^2(\mathbb{D}, \Lambda^{j+2}\mathbb{C}^m) \). \( \square \)

**Corollary 7.4.** Let \( 0 \leq j \leq m - 2 \). The orthogonal projection
\[ P_{Y_j} : L^2(\mathbb{T}, \Lambda^{j+2}\mathbb{C}^m) \to Y_j \]
is well defined.

**Proof.** By Proposition 3.30, \( H^2(\mathbb{D}, \Lambda^{j+2}\mathbb{C}^m) \) can be identified with a closed subspace of \( L^2(\mathbb{T}, \Lambda^{j+2}\mathbb{C}^m) \), thus we have
\[ H^2(\mathbb{D}, \Lambda^{j+2}\mathbb{C}^m) \|^\perp = L^2(\mathbb{T}, \Lambda^{j+2}\mathbb{C}^m) \cap H^2(\mathbb{D}, \Lambda^{j+2}\mathbb{C}^m). \]
Now the assertion follows immediately from Proposition 7.3. \( \square \)

8. \( T_j \) is a well-defined operator

**Proposition 8.1.** Let \( G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n}) \) and let \( 0 \leq j \leq \min(m, n) - 2 \). Let the functions \( \xi_i, \eta_i \) be defined by equations (4.41), that is,

\[ \xi_i = \frac{x_i}{h_i}, \quad \eta_i = \frac{\bar{\eta}_i}{h_i} \quad (8.1) \]

for \( i = 0, \ldots, j \) and let
\[ X_i = \xi_0 \hat{\lambda}_{\xi_1} \cdots \hat{\lambda}_{\xi_{i-1}} \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \Lambda^{i+1}\mathbb{C}^n), \quad i = 0, \ldots, j, \]
\[ Y_i = \eta_0 \hat{\lambda}_{\eta_1} \cdots \hat{\lambda}_{\eta_{i-1}} \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^m) \perp \subset H^2(\mathbb{D}, \Lambda^{i+1}\mathbb{C}^m) \perp, \quad i = 0, \ldots, j. \]
Let \( Q_i \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) \) satisfy
\[ (G - Q_i) x_k = t_k y_k, \quad (G - Q_i)^* y_k = t_k x_k \quad (8.2) \]
for \( k = 0, \ldots, i - 1 \).

Then, the operators \( T_i : X_i \to Y_i, \quad i = 0, \ldots, j \), given, for \( x \in H^2(\mathbb{D}, \mathbb{C}^n) \), by
\[ T_i(\xi_0 \hat{\lambda}_{\xi_1} \cdots \hat{\lambda}_{\xi_{i-1}} \hat{\lambda} x) = P_{Y_i}(\eta_0 \hat{\lambda}_{\eta_1} \cdots \hat{\lambda}_{\eta_{i-1}} \hat{\lambda}(G - Q_i)x) \quad (8.3) \]
are well defined and are independent of the choice of \( Q_i \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) \) satisfying equations (8.2).

**Proof.** By Corollary 7.4, the projections \( P_{Y_i} \) are well defined for all \( i = 0, \ldots, j \). Hence, it suffices to show that, for all \( i = 0, 1, \ldots, j \), \( T_i \) maps a zero from its domain to a zero in its range and that \( T_i \) does not depend on the choice of \( Q_i \), which satisfies equations (8.2).

For \( i = 0 \), the operator \( T_0 \) is the Hankel operator \( H_G \). If \( f_0 \equiv 0 \), then \( H_G f_0 = 0 \) and, moreover, \( H_G \) is independent of the choice of any \( Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) \) as \( H_{G - Q} = H_G \). Thus, \( T_0 \) is well defined.
For $i = 1$, let $(x_0, y_0)$ be a Schmidt pair for the compact operator $H_G$ corresponding to $t_0 = ||H_G||$, where $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $y_0 \in H^2(\mathbb{D}, \mathbb{C}^m)$. By Lemma 4.12, $x_0, \overline{y_0}$ admit the inner–outer factorizations $x_0 = \xi_0 h_0$, $\overline{y_0} = \eta_0 h_0$, where $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, $\eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)$ are inner vector-valued functions and $h_0 \in H^2(\mathbb{D}, \mathbb{C})$ is scalar outer. The spaces $X_1$ and $Y_1$ are given by the formulas

$$X_1 = \xi_0 \wedge H^2(\mathbb{D}, \mathbb{C}^n), \quad Y_1 = \eta_0 \wedge H^2(\mathbb{D}, \mathbb{C}^m).$$

The operator $T_1 : X_1 \rightarrow Y_1$ is given by

$$T_1(\xi_0 \wedge x) = P_Y(\eta_0 \wedge (G - Q_1)x)$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$, where $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ satisfies equations (8.2).

**Lemma 8.2.** Let $\xi_0 \wedge u = \xi_0 \wedge v$ for some $u, v \in H^2(\mathbb{D}, \mathbb{C}^n)$. Then

$$\eta_0 \wedge (G - Q_1)u = \eta_0 \wedge (G - Q_1)v.$$

**Proof.** Suppose that $\xi_0 \wedge u = \xi_0 \wedge v$ for some $u, v \in H^2(\mathbb{D}, \mathbb{C}^n)$. Let $x = u - v$, then $\xi_0 \wedge x = 0$, and so $x$ and $\xi_0$ are pointwise linearly dependent in $\mathbb{C}^n$ on $\mathbb{D}$. Therefore, there exist maps $\beta, \lambda : \mathbb{D} \rightarrow \mathbb{C}$, having no common zero in $\mathbb{D}$, such that

$$\beta(z)\xi_0(z) = \lambda(z)x(z) \quad \text{in} \quad \mathbb{C}^n, \quad (8.4)$$

for all $z \in \mathbb{D}$. By assumption, $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ satisfies equations (8.2). Thus, for all $z \in \mathbb{D},$

$$t_0 y_0(z) = (G - Q_1)(z)x_0(z). \quad (8.5)$$

By equations (8.1) and (8.4),

$$\beta(z)x_0(z) = \beta(z)h_0(z)\xi_0(z) = h_0(z)\lambda(z)x(z) \quad (8.6)$$

for all $z \in \mathbb{D}$. By equations (8.5) and (8.6), for all $z \in \mathbb{D},$

$$t_0 y_0(z) = (G - Q_1)(z)x_0(z), \quad \beta(z)t_0 z h_0(z) h_0(z) = (G - Q_1)(z)h_0(z)\lambda(z)x(z) \frac{z}{h_0(z)}.$$

Therefore, by equations (8.1), for all $z \in \mathbb{D},$

$$t_0 \beta(z) \overline{y_0}(z) = (G - Q_1)(z)x(z)\mu(z) \quad \text{in} \quad \mathbb{C}^m,$$

where

$$\mu(z) = \frac{zh_0(z)\lambda(z)}{h_0(z)}, \quad \text{for all} \quad z \in \mathbb{D}.$$ 

Hence, by Definition 3.20, $\overline{y_0}$ and $(G - Q_1)x$ are pointwise linearly dependent in $\mathbb{C}^m$ on $\mathbb{D}$, and so

$$\eta_0 \wedge (G - Q_1)x = 0.$$

Consequently,

$$\eta_0 \wedge (G - Q_1)u = \eta_0 \wedge (G - Q_1)v.$$

Therefore, the formula (8.3) (with $i = 1$) does uniquely define $T_1 u \in Y_1$. Next, we show that the operator $T_1$ is independent of the choice of $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ satisfying equations (8.2). Suppose $Q_1, Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ satisfy

$$(G - Q_1)x_0 = t_0 y_0, \quad y_0^*(G - Q_1) = t_0 x_0^* \quad (8.7)$$
and
\[(G - Q_2)x_0 = t_0y_0, \quad y_0^*(G - Q_2) = t_0x_0^*.\] (8.8)
Then, we claim that, for all \(x \in H^2(\mathbb{D}, \mathbb{C}^n)\),
\[P_{Y_1}(\bar{\eta}_0 \hat{\lambda}(G - Q_1)x) = P_{Y_1}(\bar{\eta}_0 \hat{\lambda}(G - Q_2)x),\]
that is,
\[P_{Y_1}(\bar{\eta}_0 \hat{\lambda}(Q_1 - Q_2)x) = 0.\]
The latter equation is equivalent to the statement that, for all \(x \in H^2(\mathbb{D}, \mathbb{C}^n)\), \(\bar{\eta}_0 \hat{\lambda}(Q_2 - Q_1)x\) is orthogonal to \(Y_1\), that is, to \(\bar{\eta}_0 \hat{\lambda} \varrho\) for all \(\varrho \in H^2(\mathbb{D}, \mathbb{C}^n)\). As a matter of convenience, set
\[Ax = (Q_2 - Q_1)x, \quad x \in H^2(\mathbb{D}, \mathbb{C}^n).\]
We have to prove that
\[\langle \bar{\eta}_0 \hat{\lambda}Ax, \bar{\eta}_0 \hat{\lambda} \varrho \rangle_{L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)} = 0\]
for all \(x \in H^2(\mathbb{D}, \mathbb{C}^n)\) and all \(\varrho \in H^2(\mathbb{C}^m)^\perp\). Note that
\[\langle \bar{\eta}_0 \hat{\lambda}Ax, \bar{\eta}_0 \hat{\lambda} \varrho \rangle_{L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)} = \frac{1}{2\pi} \int_0^{2\pi} \langle \bar{\eta}_0(e^{i\theta}) \hat{\lambda} A(e^{i\theta})x(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \hat{\lambda} \varrho(e^{i\theta}) \rangle_{\wedge^2 \mathbb{C}^m} d\theta,\]
which by Proposition 3.12 yields
\[\frac{1}{2\pi} \int_0^{2\pi} \det \left( \begin{array}{cc} \langle \bar{\eta}_0(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \bar{\eta}_0(e^{i\theta}), \varrho(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle A(e^{i\theta})x(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle A(e^{i\theta})x(e^{i\theta}), \varrho(e^{i\theta}) \rangle_{\mathbb{C}^m} \end{array} \right) d\theta \]
\[= \frac{1}{2\pi} \int_0^{2\pi} \| \bar{\eta}_0(e^{i\theta}) \|^2_{\mathbb{C}^m} \langle A(e^{i\theta})x(e^{i\theta}), \varrho(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta - \frac{1}{2\pi} \int_0^{2\pi} \langle A(e^{i\theta})x(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} \langle \bar{\eta}_0(e^{i\theta}), \varrho(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta.\]
By Proposition 5.1, \(\| \bar{\eta}_0(e^{i\theta}) \|^2_{\mathbb{C}^m} = 1\) for almost every \(e^{i\theta} \in \mathbb{T}\). Since \(Ax \in H^2(\mathbb{D}, \mathbb{C}^m)\) and \(\varrho \in H^2(\mathbb{C}^m)^\perp\),
\[\frac{1}{2\pi} \int_0^{2\pi} \langle A(e^{i\theta})x(e^{i\theta}), \varrho(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta = \langle Ax, \varrho \rangle_{L^2(\mathbb{T}, \mathbb{C}^m)} = 0.\]
Thus,
\[\langle \bar{\eta}_0 \hat{\lambda}Ax, \bar{\eta}_0 \hat{\lambda} \varrho \rangle_{L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)} = \frac{1}{2\pi} \int_0^{2\pi} \langle A(e^{i\theta})x(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} \langle \bar{\eta}_0(e^{i\theta}), \varrho(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta \]
\[= \frac{1}{2\pi} \int_0^{2\pi} \bar{\eta}_0^*(e^{i\theta}) A(e^{i\theta})x(e^{i\theta}) \langle \bar{\eta}_0(e^{i\theta}), \varrho(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta.\]
Recall that by equation (4.17), \( \bar{\eta}_0(z) = \frac{z y_0(z)}{h_0(z)} \), \( z \in \mathbb{T} \), so that

\[
\bar{\eta}_0(e^{i\theta}) = \left( \frac{e^{i\theta} y_0(e^{i\theta})}{h_0(e^{i\theta})} \right)^* = \frac{e^{-i\theta} y_0^*(e^{i\theta})}{h_0(e^{i\theta})}.
\]

Therefore,

\[
\langle \bar{\eta}_0 Ax, \bar{\eta}_0 \rangle_{L^2(\mathbb{T}, \mathbb{R}^2)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-i\theta} y_0^*(e^{i\theta})}{h_0(e^{i\theta})} A(e^{i\theta}) x(e^{i\theta}) \langle \bar{\eta}_0(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta.
\]

Recall our initial assumption was that \( Q_1, Q_2 \) satisfy equations (8.7) and (8.8), consequently,

\[
y_0^*(G - Q_i) = t_0 x_0^*, \text{ for } i = 1, 2.
\]

Hence, for \( z \in \mathbb{T} \),

\[
y_0^*(z) A(z) x(z) = y_0^*(z) (G - Q_1)(z) x(z) - y_0^*(z) (G - Q_2)(z) x(z)
\]

\[
= (t_0 x_0^* - t_0 x_0^*) x(z) = 0.
\]

We deduce that

\[
\frac{1}{2\pi} \int_0^{2\pi} \bar{\eta}_0^*(e^{i\theta}) A(e^{i\theta}) x(e^{i\theta}) \langle \bar{\eta}_0(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta = 0.
\]

To conclude, we have proved that, if \( Q_1, Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) satisfy equations (8.7) and (8.8), then

\[
P_{Y_1}(\bar{\eta}_0 \hat{\lambda}(G - Q_1)x) = P_{Y_1}(\bar{\eta}_0 \hat{\lambda}(G - Q_2)x),
\]

that is, \( T_1 \) is independent of the choice of \( Q_1 \) subject to equations (8.7), (8.8).

Recursive step: suppose that functions \( x_{i-1} \in L^2(\mathbb{T}, \mathbb{C}^n) \), \( y_{i-1} \in L^2(\mathbb{T}, \mathbb{C}^m) \), outer functions \( h_{i-1} \in H^2(\mathbb{D}, \mathbb{C}) \), positive numbers \( t_i \), matrix-valued functions \( Q_i \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \), spaces \( X_i, Y_i \) and compact operators \( T_i : X_i \to Y_i \) are constructed inductively by the algorithm for \( i = 0, \ldots, j \).

Let us prove that \( T_j : X_j \to Y_j \), given by equation (4.37), is well defined for \( 0 \leq j \leq r \). Note, by Corollary 7.4, the projection \( P_{Y_j} \) is well defined.

We must show that if an element of \( X_j \) has two different expressions as an element of \( \hat{\xi}_0 \hat{\lambda} \ldots \hat{\xi}_{j-1} \hat{\lambda} \hat{\xi}_j \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n) \), say

\[
u = \xi_0 \hat{\lambda} \ldots \hat{\xi}_{j-1} \hat{\lambda} x = \xi_0 \hat{\lambda} \ldots \hat{\xi}_{j-1} \hat{\lambda} \tilde{x}
\]

for some \( x, \tilde{x} \in H^2(\mathbb{D}, \mathbb{C}^n) \), then the two corresponding formulae for \( T_j \nu \) given by the defining equation (4.37) agree, that is,

\[
P_{Y_j}(\bar{\eta}_0 \hat{\lambda} \bar{\eta}_1 \hat{\lambda} \ldots \hat{\eta}_{j-1} \hat{\lambda}(G - Q_j)x) = P_{Y_j}(\bar{\eta}_0 \hat{\lambda} \bar{\eta}_1 \hat{\lambda} \ldots \hat{\eta}_{j-1} \hat{\lambda}(G - Q_j)x),
\]

or equivalently,

\[
P_{Y_j}(\bar{\eta}_0 \hat{\lambda} \bar{\eta}_1 \hat{\lambda} \ldots \hat{\eta}_{j-1} \hat{\lambda}(G - Q_j)(x - \tilde{x})) = 0,
\]

which is to say that we need to show that

\[
\bar{\eta}_0 \hat{\lambda} \bar{\eta}_1 \hat{\lambda} \ldots \hat{\eta}_{j-1} \hat{\lambda}(G - Q_j)(x - \tilde{x}) \in Y_j^\perp.
\]

If \( x, \tilde{x} \) satisfy equation (8.9), then

\[
\xi_0 \hat{\lambda} \ldots \hat{\xi}_{j-1} \hat{\lambda}(x - \tilde{x}) = 0,
\]
and so, by Corollary 3.13, \(\xi_0, \xi_1, \ldots, \xi_{j-1}, x - \bar{x}\) are pointwise linearly dependent almost everywhere on \(\mathbb{T}\).

It follows immediately that, for almost all \(z \in \mathbb{T}\), the vectors
\[
(G - Q_j)\xi_0(z), \ldots, (G - Q_j)\xi_{j-1}(z), (G - Q_j)\xi_j(z), (G - Q_j)(x - \bar{x})(z)
\]
are linearly dependent in \(\mathbb{C}^m\).

Since \(y_j = \bar{z}\bar{h}_j\bar{n}_j\), by equations (1.19), equations (8.2) imply, for \(i = 0, \ldots, j - 1\) and almost all \(z \in \mathbb{T}\),
\[
(G - Q_j)\xi_i(z) = (G - Q_j)\frac{x_i}{\bar{h}_i}(z) = \frac{t_i}{\bar{h}_i} y_i(z) = \frac{t_i}{\bar{h}_i} \bar{z}\bar{h}_i\bar{n}_i(z).
\]
Thus, for almost all \(z \in \mathbb{T}\), the vectors
\[
\frac{t_0}{\bar{h}_0} \bar{z}\bar{h}_0\bar{n}_0(z), \ldots, \frac{t_{j-1}}{\bar{h}_{j-1}} \bar{z}\bar{h}_{j-1}\bar{n}_{j-1}(z), (G - Q_j)(x - \bar{x})(z)
\]
are linearly dependent in \(\mathbb{C}^m\). Since \(t_0 \geq t_1 \geq \cdots \geq t_j > 0\), it follows that
\[
\bar{n}_0(z), \ldots, \bar{n}_{j-1}(z), (G - Q_j)(x - \bar{x})(z)
\]
are linearly dependent for almost all \(z \in \mathbb{T}\) and so, by Corollary 3.13,
\[
\bar{n}_0 \wedge \cdots \wedge \bar{n}_{j-1} \wedge (G - Q_j)(x - \bar{x}) = 0,
\]
which certainly implies the desired relation (8.10).

Thus, \(T_j : X_j \to Y_j\) is well defined.

For the operator \(T_j\) to be uniquely defined in the algorithm, it remains to prove \(T_j\) is independent of the choice of \(Q_j \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})\) subject to equations (8.2). Let \(Q_1, Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})\) satisfy
\[
(G - Q_1)x_i = t_i y_i, \quad (G - Q_2)x_i = t_i y_i, \quad y_i^*(G - Q_1) = t_i x_i^*, \quad y_i^*(G - Q_2) = t_i x_i^* \quad (8.11)
\]
for \(i = 0, \ldots, j - 1\). We shall prove that, for all \(x \in H^2(\mathbb{D}, \mathbb{C}^n)\),
\[
P_{Y_j} \langle \bar{n}_0 \wedge \cdots \wedge \bar{n}_{j-1} \wedge (G - Q_1)x \rangle = P_{Y_j} \langle \bar{n}_0 \wedge \cdots \wedge \bar{n}_{j-1} \wedge (G - Q_2)x \rangle.
\]
The latter equality holds if and only if, for all \(x \in H^2(\mathbb{D}, \mathbb{C}^n)\),
\[
P_{Y_j} \langle \bar{n}_0 \wedge \cdots \wedge \bar{n}_{j-1} \wedge (Q_2 - Q_1)x \rangle = 0
\]
which is equivalent to the assertion that \(\bar{n}_0 \wedge \cdots \wedge \bar{n}_{j-1} \wedge (Q_2 - Q_1)x\) is orthogonal to \(\bar{n}_0 \wedge \cdots \wedge \bar{n}_{j-1} \wedge q\) for all \(x \in H^2(\mathbb{D}, \mathbb{C}^n)\) and for all \(q \in H^2(\mathbb{D}, \mathbb{C}^n)\)\(\perp\).

Equivalently
\[
\langle \bar{n}_0 \wedge \cdots \wedge \bar{n}_{j-1} \wedge (Q_2 - Q_1)x, \bar{n}_0 \wedge \cdots \wedge \bar{n}_{j-1} \wedge q \rangle_{L^2(\mathbb{T}, \wedge^j \mathbb{C}^m)} = 0
\]
for all \(x \in H^2(\mathbb{D}, \mathbb{C}^n)\) and for all \(q \in H^2(\mathbb{C}^m)\)\(\perp\). Set \(Ax = (Q_2 - Q_1)x, \; x \in H^2(\mathbb{D}, \mathbb{C}^n)\).

By Proposition 3.12,
\[
\hat{n}_0 \wedge \cdots \wedge \hat{n}_{j-1} \wedge (Q_2 - Q_1)x, \hat{n}_0 \wedge \cdots \wedge \hat{n}_{j-1} \wedge q \rangle_{L^2(\mathbb{T}, \wedge^j \mathbb{C}^m)}
\]
is equal to
\[
\frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix}
\langle \hat{n}_0(e^{i\theta}), \hat{n}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \cdots & \langle \hat{n}_0(e^{i\theta}), \hat{n}_{j-1}(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \hat{n}_0(e^{i\theta}), q(e^{i\theta}) \rangle_{\mathbb{C}^m} \\
\cdots & \cdots & \cdots & \cdots \\
\langle \hat{n}_{j-1}(e^{i\theta}), \hat{n}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \cdots & \langle \hat{n}_{j-1}(e^{i\theta}), \hat{n}_{j-1}(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \hat{n}_{j-1}(e^{i\theta}), q(e^{i\theta}) \rangle_{\mathbb{C}^m} \\
\langle A(e^{i\theta})x(e^{i\theta}), \hat{n}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \cdots & \langle A(e^{i\theta})x(e^{i\theta}), \hat{n}_{j-1}(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle A(e^{i\theta})x(e^{i\theta}), q(e^{i\theta}) \rangle_{\mathbb{C}^m}
\end{pmatrix} d\theta.
\]
Note that $Ax$ and $q$ are orthogonal in $L^2(\mathbb{T}, \mathbb{C}^m)$ and, by Proposition 5.1, \( \{\tilde{\eta}_i(z)\}_{i=0}^{j-1} \) is an orthonormal sequence in $\mathbb{C}^m$ almost everywhere on $\mathbb{T}$. Also, for $i = 0, \ldots, j - 1$, by equations (8.11),
\[
\langle A(e^{i\theta})x(e^{i\theta}), \tilde{\eta}_i(e^{i\theta}) \rangle_{\mathbb{C}^m} = \eta_i^T(e^{i\theta})A(e^{i\theta})x(e^{i\theta}) = \frac{e^{-i\theta} y_i^*(e^{i\theta})}{h_i(e^{i\theta})} A(e^{i\theta})x(e^{i\theta})
\]
\[
= \frac{e^{-i\theta}}{h_i(e^{i\theta})} (y_i^*(e^{i\theta})(G - Q_1)(z)x(z) - y_i^*(e^{i\theta})(G - Q_2)(z)x(z))
\]
\[
= \frac{e^{-i\theta}}{h_i(e^{i\theta})} (t_ix^*_i x - t_ix_i x) = 0.
\]

Thus,
\[
\langle \tilde{\eta}_0 \hat{\lambda} \cdots \hat{\lambda} \tilde{\eta}_j \hat{\lambda}(Q_2 - Q_1)x, \tilde{\eta}_0 \hat{\lambda} \cdots \hat{\lambda} \tilde{\eta}_j \hat{\lambda} q \rangle_{L^2(\mathbb{T}, \mathbb{A}^{j+1}\mathbb{C}^m)} = 1 \begin{vmatrix} 1 & 0 & \cdots & \langle \tilde{\eta}_0(e^{i\theta}), q(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ 0 & 1 & \cdots & \langle \tilde{\eta}_2(e^{i\theta}), q(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle A(e^{i\theta})x(e^{i\theta}), q(e^{i\theta}) \rangle_{\mathbb{C}^m} \end{vmatrix} d\theta
\]
\[
= \langle Ax, q \rangle_{L^2(\mathbb{T}, \mathbb{C}^m)} = 0.
\]

Consequently
\[
P_{Y_j}(\hat{\lambda} \tilde{\eta}_0 \hat{\lambda} \cdots \hat{\lambda} \tilde{\eta}_j \hat{\lambda}(G - Q_1)x) = P_{Y_j}(\hat{\lambda} \tilde{\eta}_0 \hat{\lambda} \cdots \hat{\lambda} \tilde{\eta}_j - 1 \hat{\lambda}(G - Q_2)x),
\]
and so $T_j$ is independent of the choice of $Q_j$ subject to equations (8.2). \(\Box\)

9. Compactness of the operators $T_1$ and $T_2$

Here we use notations from the algorithm of Subsection 4.2 to prove the compactness of the operator $T_j$ given by equation (4.37) for $j = 1, 2$. The proof requires several steps. Let us first prove that the operator $T_1$ is compact.

Recall that since $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$, by Hartman’s theorem, the operator $T_0 = H_G$ is compact and hence there exist $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $y_0 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ such that $(x_0, y_0)$ is a Schmidt pair for $H_G$ corresponding to the singular value $\|H_G\| = t_0$.

By Lemma 4.12, $x_0, \bar{z}y_0$ admit the inner–outer factorizations
\[
x_0 = \xi_0 h_0, \quad \bar{z}y_0 = \eta_0 \overline{h_0},
\]
where $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, $\eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)$ are vector-valued inner functions and $h_0 \in H^2(\mathbb{D}, \mathbb{C})$ is a scalar outer function. Moreover there exist unitary-valued functions of types $n \times n, m \times m$, respectively, of the form
\[
V_0 = (\xi_0 \ \alpha_0), \quad W_0 = (\eta_0 \ \overline{\beta_0})^T,
\]
where $\alpha_0, \beta_0$ are inner, co-outer, quasi-continuous functions of types $n \times (n - 1), m \times (m - 1)$, respectively, and all minors on the first columns of $V_0, W_0^T$ are in $H^\infty$. Furthermore, every $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ which is at minimal distance from $G$ satisfies
\[
W_0(G - Q_1)V_0 = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F_1 \end{pmatrix}
\]
for some
\[
F_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1) \times (n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1) \times (n-1)})
\]
and some quasi-continuous function $u_0$ with $|u_0(z)| = 1$ almost everywhere on $T$.

Recall that

$$X_1 = \xi_0 \dot{\lambda} H^2(D, \mathbb{C}^n), \quad Y_1 = \eta_0 \dot{\lambda} H^2(D, \mathbb{C}^m)^\perp$$

and $T_1: X_1 \to Y_1$ is given by

$$T_1(\xi_0 \dot{\lambda} x) = P_{Y_1}[\eta_0 \dot{\lambda} (G - Q_1)x] \quad \text{for all } x \in H^2(D, \mathbb{C}^n).$$

Our first endeavor in this section is to prove the following theorem.

**Theorem 9.1.** Let

$$K_1 \overset{\text{def}}{=} V_0 \left( H^2(D, \mathbb{C}^{n-1}) \right), \quad L_1 \overset{\text{def}}{=} W_0^* \left( H^2(D, \mathbb{C}^{m-1}) \right),$$

(9.3)

and let the maps

$$U_1: H^2(D, \mathbb{C}^{n-1}) \to K_1,$$

$$U_2: H^2(D, \mathbb{C}^{m-1})^\perp \to L_1$$

be given by

$$U_1x = V_0 \left( \begin{pmatrix} 0 \\ x \end{pmatrix} \right), \quad U_2y = W_0^* \left( \begin{pmatrix} 0 \\ y \end{pmatrix} \right)$$

for all $x \in H^2(D, \mathbb{C}^{n-1})$, $y \in H^2(D, \mathbb{C}^{m-1})^\perp$. Consider the operator $\Gamma_1 = P_{c_i, M_{G - Q_1}}|K_1$.

Then

(i) the maps $U_1, U_2$ are unitaries;

(ii) the maps $(\xi_0 \dot{\lambda} \cdot): K_1 \to H^2(D, \dot{\lambda}^2 \mathbb{C}^n)$ and $(\eta_0 \dot{\lambda} \cdot): L_1 \to H^2(D, \mathbb{C}^m)^\perp$ are unitaries;

(iii) the following diagram is commutative:

$$H^2(D, \mathbb{C}^{n-1}) \xrightarrow{U_1} K_1 \xrightarrow{\Gamma_1} \xi_0 \dot{\lambda} H^2(D, \mathbb{C}^n) = X_1$$

$$H^2(D, \mathbb{C}^{m-1})^\perp \xrightarrow{U_2} L_1 \xrightarrow{T_1} \eta_0 \dot{\lambda} H^2(D, \mathbb{C}^m)^\perp = Y_1$$

(9.4)

(iv) $T_1$ is a compact operator;

(v) $\|T_1\| = \|\Gamma_1\| = t_1$.

**Proof.** Statement (i) follows from Lemma 4.17. Statement (ii) follows from Propositions 9.6 and 9.10, which are consequences of the following lemmas.

**Lemma 9.2.** In the notation of Theorem 9.1, the Hankel operator $H_G$ has a maximizing vector $x_0$ of unit norm such that $\xi_0$, which is defined by $\xi_0 = \frac{x_0}{\|x_0\|}$, is a co-outer function.

**Proof.** Choose any maximizing vector $x_0$. By Lemma 4.12, $x_0$ has the inner–outer factorization $x_0 = \xi_0 h_0$, where $h_0$ is a scalar outer factor. Then, the closure of $\xi_0^T H^2(D, \mathbb{C}^n)$, denoted by $\text{cis} (\xi_0^T H^2(D, \mathbb{C}^n))$, is a closed shift-invariant subspace of $H^2(D, \mathbb{C})$, so, by Beurling’s theorem,

$$\text{cis} (\xi_0^T H^2(D, \mathbb{C}^n)) = \varphi H^2(D, \mathbb{C})$$

for some scalar inner function $\varphi$. Hence,

$$\varphi \xi_0^T H^2(D, \mathbb{C}^n) \subset H^2(D, \mathbb{C}).$$

Thus, if $\xi_0^T = (\xi_{01}, \ldots, \xi_{0n})$, we have $\varphi \xi_{0j} \in H^\infty(D, \mathbb{C})$ for $j = 1, \ldots, n$, and so

$$\varphi \xi_0 \in H^\infty(D, \mathbb{C}^n).$$
Hence,
\[ \phi x_0 = \phi \xi_0 h_0 \in H^2(\mathbb{D}, \mathbb{C}^n). \]

Let \( Q \) be a best \( H^\infty \) approximation to \( G \). Since \( x_0 \) is a maximizing vector for \( H_G \), by Theorem 4.11,
\[ (G - Q)x_0 \in H^2(\mathbb{D}, \mathbb{C}^m) \]
and
\[ \| (G - Q)(z)x_0(z) \|_{\mathbb{C}^m} = \| H_G \| \| x_0(z) \|_{\mathbb{C}^n} \]
for almost all \( z \in \mathbb{T} \). Thus,
\[ (G - Q)\overline{\phi x_0} \in H^2(\mathbb{D}, \mathbb{C}^m) \]
and
\[ \| (G - Q)(\overline{\phi x_0}(z)) \|_{\mathbb{C}^m} = \| H_G \| \| \overline{\phi x_0}(z) \|_{\mathbb{C}^n} \]
for almost all \( z \in \mathbb{T} \).

Hence, \( \overline{\phi x_0} \in H^2(\mathbb{D}, \mathbb{C}^n) \) is a maximizing vector for \( H_G \), and \( \overline{\phi x_0} \) is co-outer. Then \( \overline{\phi x_0} \) is a co-outer maximizing vector of unit norm for \( H_G \). \( \square \)

**Lemma 9.3.** Let \( x_0 \) be a co-outer maximizing vector of unit norm for \( H_G \), and let \( x_0 = \xi_0 h_0 \) be the inner–outer factorization of \( x_0 \). Then
(i) \( \xi_0 \) is a quasi-continuous function and
(ii) there exists a function \( A \in H^\infty(\mathbb{D}, \mathbb{C}^n) \) such that
\[ A^T \xi_0 = 1. \]

**Proof.** Let us first show that
\[ \xi_0 \in \left( H^\infty(\mathbb{D}, \mathbb{C}^n) + C(\mathbb{D}, \mathbb{C}^n) \right) \cap \overline{H^\infty(\mathbb{D}, \mathbb{C}^n) + C(\mathbb{T}, \mathbb{C}^n)}. \]

Let \( Q \) be a best \( H^\infty \) approximation to \( G \). Then, by Theorem 4.11, the function \( Q \) satisfies the equation
\[ (G - Q)^* y_0 = t_0 x_0. \]
Taking complex conjugates in equations (9.1), we have
\[ (G - Q)^T \overline{y}_0 = t_0 \overline{x}_0. \]
Hence, for \( z \in \mathbb{T} \),
\[ (G - Q)^T z h_0 \eta_0 = t_0 \overline{h}_0 \overline{\xi}_0, \]
and therefore
\[ \frac{(G - Q)^T z h_0 \eta_0}{t_0 \overline{h}_0} = \overline{\xi}_0. \]
Recall, by equation (4.6) (with \( \varphi = 1 \)), \( u_0 = \frac{\overline{z} h_0}{h_0} \). By Lemma 4.12, \( u_0 \in QC \), hence \( \overline{u}_0 \in H^\infty + C \). Note \( \overline{u}_0 = \frac{\overline{z} h_0}{h_0} \), and hence
\[ \overline{\xi}_0 = \frac{(G - Q)^T \overline{w}_0 \eta_0}{t_0}. \]
Since \( H^\infty + C \) is an algebra and \( (G - Q)^T \), \( \eta_0 \in H^\infty + C \), it follows that \( \overline{\xi}_0 \in H^\infty + C \), thus
\[ \xi_0 \in \left( H^\infty(\mathbb{D}, \mathbb{C}^n) + C(\mathbb{T}, \mathbb{C}^n) \right) \cap \overline{H^\infty(\mathbb{D}, \mathbb{C}^n) + C(\mathbb{T}, \mathbb{C}^n)}. \]
The conclusion that there exists a function $A \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ such that $A^T \xi_0 = 1$ now follows directly from Lemma 4.18.

**Lemma 9.4.** In the notation of Theorem 9.1, let $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ be a vector-valued inner, co-outer, quasi-continuous function and let

$$V_0 = (\xi_0 \quad \bar{\alpha}_0)$$

be a thematic completion of $\xi_0$ as described in Lemma 4.12, where $\alpha_0$ is an inner, co-outer, quasi-continuous function of order $n \times (n - 1)$ and all minors on the first column of $V_0$ are analytic. Then,

$$\alpha^T_0 H^2(\mathbb{D}, \mathbb{C}^n) = H^2(\mathbb{D}, \mathbb{C}^{n-1}).$$

**Proof.** By Lemma 4.18, for the given $\alpha_0$, there exists $A_0 \in H^\infty(\mathbb{D}, \mathbb{C}^{(n-1) \times n})$ such that $A_0 \alpha_0 = I_{n-1}$. Equivalently, $\alpha^T_0 A^T_0 = I_{n-1}$.

Let $g \in H^2(\mathbb{D}, \mathbb{C}^{n-1})$. Then $g = (\alpha^T_0 A^T_0) g \in \alpha^T_0 A^T_0 H^2(\mathbb{D}, \mathbb{C}^{n-1})$, which implies that $g \in \alpha^T_0 H^2(\mathbb{D}, \mathbb{C}^n)$. Hence, $H^2(\mathbb{D}, \mathbb{C}^{n-1}) \subseteq \alpha^T_0 H^2(\mathbb{D}, \mathbb{C}^n)$.

For the reverse inclusion, note that since $\alpha_0$ is in $H^\infty(\mathbb{D}, \mathbb{C}^{n \times (n-1)})$, we have $\alpha^T_0 H^2(\mathbb{D}, \mathbb{C}^n) \subseteq H^2(\mathbb{D}, \mathbb{C}^{n-1})$. Thus,

$$\alpha^T_0 H^2(\mathbb{D}, \mathbb{C}^n) = H^2(\mathbb{D}, \mathbb{C}^{n-1}).$$

**Proposition 9.5.** Let $\xi_0, \alpha_0$ and $V_0$ be as in Lemma 9.4. Then

$$V_0^* \text{POC}((\xi_0), L^2(\mathbb{T}, \mathbb{C}^n)) = \begin{pmatrix} 0 \\ L^2(\mathbb{T}, \mathbb{C}^{n-1}) \end{pmatrix}.$$

**Proof.** Let $g \in V_0^* \text{POC}((\xi_0), L^2(\mathbb{T}, \mathbb{C}^n))$. Equivalently, $g$ can be written as $g = V_0^* f$ for some $f \in L^2(\mathbb{T}, \mathbb{C}^n)$ such that $f(z) \perp \xi_0(z)$ for almost all $z \in \mathbb{T}$. This in turn is equivalent to the assertion that $g = V_0^* f$ for some $f \in L^2(\mathbb{T}, \mathbb{C}^n)$ such that $(V_0^* f)(z) \perp (V_0^* \xi_0)(z)$ for almost all $z \in \mathbb{T}$, since $V_0(z)$ is unitary for almost all $z \in \mathbb{T}$.

Note that, by the fact that $V_0$ is unitary-valued almost everywhere on $\mathbb{T}$, we have

$$I_n = V_0^* (z) V_0(z)$$

$$= \begin{pmatrix} \xi_0(z) \\ \alpha^T_0(z) \end{pmatrix} \begin{pmatrix} \xi_0(z) & \bar{\alpha}_0(z) \end{pmatrix} = \begin{pmatrix} \xi_0(z) \xi_0^*(z) & \xi_0(z) \bar{\alpha}_0(z) \\ \alpha^T_0(z) \xi_0(z) & \alpha^T_0(z) \bar{\alpha}_0(z) \end{pmatrix}$$

almost everywhere on $\mathbb{T}$, \hspace{1cm} (9.5)

and so

$$V_0^* \xi_0 = \begin{pmatrix} \xi_0^* \\ \alpha^T_0 \end{pmatrix} \xi_0 = \begin{pmatrix} 1 \\ 0_{(n-1) \times 1} \end{pmatrix},$$

where $0_{(n-1) \times 1}$ denotes the zero vector in $\mathbb{C}^{n-1}$.

Hence, $g = V_0^* f$ with $(V_0^* f)(z)$ orthogonal to $(V_0^* \xi_0)(z)$ for almost every $z \in \mathbb{T}$, is equivalent to the statement $g \in L^2(\mathbb{T}, \mathbb{C}^n)$ and

$$g(z) \perp \begin{pmatrix} 1 \\ 0_{(n-1) \times 1} \end{pmatrix}$$

for almost all $z \in \mathbb{T}$, or equivalently, $g \in \left( L^2(\mathbb{T}, \mathbb{C}^{n-1}) \right)$. \hspace{1cm} $\square$.
Proposition 9.6. Under the assumptions of Theorem 9.1, where \( x_0 \) is a co-outer maximizing vector of unit norm for \( H_G, \xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n) \) is a vector-valued inner function given by \( \xi_0 = \frac{w}{\overline{w}^0} \), \( V_0 = (\xi_0, a_0) \) is a thematic completion of \( \xi_0 \) and \( K_1 \) is defined by

\[
K_1 = V_0 \left( \begin{array}{c} 0 \\ H^2(\mathbb{D}, \mathbb{C}^{n-1}) \end{array} \right) \subseteq L^2(\mathbb{D}, \mathbb{C}^n),
\]

we have

\[
\xi_0 \hat{\lambda} K_1 = \xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n)
\]

and the operator

\[
(\xi_0 \hat{\lambda} \cdot): K_1 \to \xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n)
\]

is unitary.

Proof. Let us first prove \( \xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n) \subseteq \xi_0 \hat{\lambda} K_1 \). Let \( \varphi \in H^2(\mathbb{D}, \mathbb{C}^n) \). Since \( V_0 \) is unitary-valued,

\[
\xi_0 \hat{\lambda} + \bar{\alpha}_0 \alpha_0^T = I_n.
\]

Thus,

\[
\xi_0 \hat{\lambda} \varphi = \xi_0 \hat{\lambda} (\xi_0 \hat{\lambda} \varphi + \bar{\alpha}_0 \alpha_0^T \varphi)
= \xi_0 \hat{\lambda} \xi_0 \hat{\lambda} \alpha_0 (\alpha_0^T \varphi)
= 0 + \xi_0 \hat{\lambda} \bar{\alpha}_0 (\alpha_0^T \varphi)
\]

on account of the pointwise linear dependence of \( \xi_0 \) and \( \xi_0 \hat{\lambda} \varphi \) on \( \mathbb{D} \). Recall that, by Lemma 9.4, \( \alpha_0^T \varphi \in H^2(\mathbb{D}, \mathbb{C}^{n-1}) \) and, by the definition of \( K_1 \),

\[
K_1 = \bar{\alpha}_0 H^2(\mathbb{D}, \mathbb{C}^{n-1}).
\]

Hence, for \( \varphi \in H^2(\mathbb{D}, \mathbb{C}^n) \),

\[
\xi_0 \hat{\lambda} \varphi = \xi_0 \hat{\lambda} \bar{\alpha}_0 \alpha_0^T \varphi \in \xi_0 \hat{\lambda} K_1,
\]

and thus

\[
\xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n) \subseteq \xi_0 \hat{\lambda} K_1.
\]

Let us now show that \( \xi_0 \hat{\lambda} K_1 \subseteq \xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n) \). Since \( K_1 = \bar{\alpha}_0 H^2(\mathbb{D}, \mathbb{C}^{n-1}) \), an arbitrary element \( u \in \xi_0 \hat{\lambda} K_1 \) is of the form

\[
u = \xi_0 \hat{\lambda} \bar{\alpha}_0 g,
\]

for some \( g \in H^2(\mathbb{D}, \mathbb{C}^{n-1}) \). Note that, by Lemma 9.4, there exists a function \( f \in H^2(\mathbb{D}, \mathbb{C}^n) \) such that \( g = \alpha_0^T f \). Hence, \( u = \xi_0 \hat{\lambda} \alpha_0 \alpha_0^T f \). By equation (9.5), \( \alpha_0^\dagger \bar{\alpha}_0 + \bar{\alpha}_0 \alpha_0^T = I_n \). Thus,

\[
u = \xi_0 \hat{\lambda} (I_n - \xi_0 \hat{\lambda} \alpha_0^T f).
\]

and so, \( \xi_0 \hat{\lambda} K_1 \subseteq \xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n) \). Combining the latter inclusion with the relation (9.6), we have

\[
\xi_0 \hat{\lambda} K_1 = \xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n).
\]

Now, let us show that the operator \( (\xi_0 \hat{\lambda} \cdot): K_1 \to \xi_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n) \) is unitary. As we have shown above, the operator is surjective. We will show it is also an isometry. Let \( f \in K_1 \). Then,

\[
\| \xi_0 \hat{\lambda} f \|_{L^2(\mathbb{T}, \mathbb{L}^2 \mathbb{C}^n)}^2 = (\xi_0 \hat{\lambda} f, \xi_0 \hat{\lambda} f)_{L^2(\mathbb{T}, \mathbb{L}^2 \mathbb{C}^n)}
= \frac{1}{2\pi} \int_0^{2\pi} \langle \xi_0 (e^{i\theta}) \hat{\lambda} f (e^{i\theta}), \xi_0 (e^{i\theta}) \hat{\lambda} f (e^{i\theta}) \rangle_{\mathbb{L}^2 \mathbb{C}^n} d\theta.
\]
By Proposition 3.12, the latter integral is equal to
\[
\frac{1}{2\pi} \int_{0}^{2\pi} \det \left( (\xi_{0}(e^{i\theta}), \xi_{0}(e^{i\theta})), \langle \xi_{0}(e^{i\theta}), f(e^{i\theta}) \rangle_{C^{n}} \right) \, d\theta
\]
and
\[
\frac{1}{2\pi} \int_{0}^{2\pi} \|\xi_{0}(e^{i\theta})\|_{C^{n}}^{2}, \langle f(e^{i\theta}), f(e^{i\theta}) \rangle_{C^{n}} - |\langle \xi_{0}(e^{i\theta}), f(e^{i\theta}) \rangle_{C^{n}}|^{2} \, d\theta.
\]
Note that, by Proposition 5.1, \(\|\xi_{0}(e^{i\theta})\|_{C^{n}} = 1\) for almost all \(e^{i\theta}\) on \(T\). Moreover, since
\[
K_{1} = \tilde{\alpha}_{0}H^{2}(\mathbb{D}, C^{n-1}),
\]
f = \(\tilde{\alpha}_{0}g\) for some \(g \in H^{2}(\mathbb{D}, C^{n-1})\). Hence,
\[
\langle \xi_{0}(e^{i\theta}), f(e^{i\theta}) \rangle_{C^{n}} = \langle \xi_{0}(e^{i\theta}), \tilde{\alpha}_{0}(e^{i\theta})g(e^{i\theta}) \rangle_{C^{n}} = \langle \alpha_{0}^{T}(e^{i\theta})\xi_{0}(e^{i\theta}), g(e^{i\theta}) \rangle_{C^{n-1}} = 0
\]
almost everywhere on \(T\), since \(V_{0} = \langle \xi_{0}, \tilde{\alpha}_{0} \rangle\) is unitary-valued. Thus,
\[
\|\xi_{0}\langle f \rangle_{L^{2}(\mathbb{T}, \Lambda^{2}C^{n})} = \|f\|_{L^{2}(\mathbb{T}, C^{n})}^{2},
\]
that is, the operator \((\xi_{0}\langle \cdot \rangle): K_{1} \to \xi_{0}\langle \cdot \rangle H^{2}(\mathbb{D}, C^{n})\) is an isometry. Therefore, the surjective operator \((\xi_{0}\langle \cdot \rangle)\) is unitary. \(\square\)

**Lemma 9.7.** Let \(u \in L^{2}(\mathbb{T}, C^{m})\) and let \(\eta_{0} \in H^{\infty}(\mathbb{D}, C^{m})\) be a vector-valued inner function. Then
\[
\langle \tilde{\eta}_{0}\langle u \rangle, \tilde{\eta}_{0}\langle z \rangle \rangle_{L^{2}(\mathbb{T}, \Lambda^{2}C^{m})} = 0 \quad \text{for all} \quad f \in H^{2}(\mathbb{D}, C^{m})
\]
if and only if the function
\[
z \mapsto u(z) - \langle u(z), \tilde{\eta}_{0}(z) \rangle_{C^{m}} \tilde{\eta}_{0}(z)
\]
belongs to \(H^{2}(\mathbb{D}, C^{m})\).

**Proof.** The statement that \(\tilde{\eta}_{0}\langle \cdot \rangle\) is orthogonal to \(\tilde{\eta}_{0}\langle \cdot \rangle f\) in \(L^{2}(\mathbb{T}, \Lambda^{2}C^{m})\) is equivalent to the equation \(I = 0\), where
\[
I = \frac{1}{2\pi} \int_{0}^{2\pi} \langle \tilde{\eta}_{0}(e^{i\theta}), \langle u(e^{i\theta}), \tilde{\eta}_{0}(e^{i\theta}) \rangle_{C^{m}} \rangle_{\Lambda^{2}C^{m}} \, d\theta.
\]
By Proposition 3.12,
\[
I = \frac{1}{2\pi} \int_{0}^{2\pi} \det \left( \langle \tilde{\eta}_{0}(e^{i\theta}), \langle u(e^{i\theta}), \tilde{\eta}_{0}(e^{i\theta}) \rangle_{C^{m}} \rangle_{\Lambda^{2}C^{m}} \right) \, d\theta.
\]
Note that, since \(\eta_{0}\) is an inner function, \(\|\tilde{\eta}_{0}(e^{i\theta})\|_{C^{m}} = 1\) almost everywhere on \(T\), and hence
\[
I = \frac{1}{2\pi} \int_{0}^{2\pi} \det \left( \langle u(e^{i\theta}), \tilde{\eta}_{0}(e^{i\theta}) \rangle_{C^{m}} \right) \, d\theta
\]
and
\[
I = \frac{1}{2\pi} \int_{0}^{2\pi} \langle u(e^{i\theta}), e^{-i\theta} \tilde{f}(e^{i\theta}) \rangle_{C^{m}} \, d\theta
\]
and
\[
I = \frac{1}{2\pi} \int_{0}^{2\pi} \langle u(e^{i\theta}), e^{-i\theta} \tilde{f}(e^{i\theta}) \rangle_{C^{m}} \, d\theta
\]
and
\[
I = \frac{1}{2\pi} \int_{0}^{2\pi} \langle u(e^{i\theta}), e^{-i\theta} \tilde{f}(e^{i\theta}) \rangle_{C^{m}} \, d\theta
\]
Thus, the condition (9.7) holds if and only if

\[
\frac{1}{2\pi} \int_0^{2\pi} \langle \bar{\eta}_0(e^{i\theta}) \lambda u(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \lambda e^{-i\theta} \bar{f}(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta = 0 \quad \text{for all} \ f \in H^2(\mathbb{D}, \mathbb{C}^m)
\]

if and only if

\[
\frac{1}{2\pi} \int_0^{2\pi} \langle u(e^{i\theta}) - \langle u(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} \bar{\eta}_0(e^{i\theta}) e^{-i\theta} \bar{f}(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta = 0
\]

for all \( f \in H^2(\mathbb{D}, \mathbb{C}^m) \), and the latter equation holds if and only if

\[
u(e^{i\theta}) - \langle u(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} \bar{\eta}_0(e^{i\theta})
\]

belongs to \( H^2(\mathbb{D}, \mathbb{C}^m) \).

**Lemma 9.8.** In the notation of Theorem 9.1,

\[
\mathcal{L}_1^\perp = \{ f \in L^2(\mathbb{T}, \mathbb{C}^m) : \beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-1}) \}.
\]

**Proof.** It is easy to see that \( \mathcal{L}_1 = \beta_0 H^2(\mathbb{D}, \mathbb{C}^{m-1}) \perp \). The general element of \( \beta_0 H^2(\mathbb{D}, \mathbb{C}^{m-1}) \perp \) is \( \beta_0 \bar{g} \) with \( g \in H^2(\mathbb{D}, \mathbb{C}^{m-1}) \). For \( f \in L^2(\mathbb{T}, \mathbb{C}^m) \), \( f \in \mathcal{L}_1^\perp \) if and only if

\[
\langle f, \beta_0 \bar{g} \rangle_{L^2(\mathbb{T}, \mathbb{C}^m)} = 0 \quad \text{for all} \ g \in H^2(\mathbb{D}, \mathbb{C}^{m-1}).
\]

Equivalently, \( f \in \mathcal{L}_1^\perp \) if and only if

\[
\frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{i\theta}), \beta_0(e^{i\theta}) e^{-i\theta} \bar{g}(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta = 0 \quad \text{for all} \ g \in H^2(\mathbb{D}, \mathbb{C}^{m-1})
\]

if and only if

\[
\frac{1}{2\pi} \int_0^{2\pi} \langle \beta_0(e^{i\theta})^* f(e^{i\theta}), e^{-i\theta} \bar{g}(e^{i\theta}) \rangle_{\mathbb{C}^{m-1}} d\theta = 0 \quad \text{for all} \ g \in H^2(\mathbb{D}, \mathbb{C}^{m-1}).
\]

The latter statement is equivalent to the assertion that \( \beta_0^* f \) is orthogonal to \( H^2(\mathbb{D}, \mathbb{C}^{m-1}) \perp \) in \( L^2(\mathbb{T}, \mathbb{C}^{m-1}) \), which holds if and only if \( \beta_0^* f \) belongs to \( H^2(\mathbb{D}, \mathbb{C}^{m-1}) \).

Hence,

\[
\mathcal{L}_1^\perp = \{ f \in L^2(\mathbb{T}, \mathbb{C}^m) : \beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-1}) \}
\]

as required.

**Proposition 9.9.** Under the assumptions of Theorem 9.1, let \( \eta_0 \) be defined by equation (9.1) and let \( W_0^T = (\eta_0, \beta_0) \) be a thematic completion of \( \eta_0 \), where \( \beta_0 \) is an inner, co-outer, quasi-continuous function of type \( m \times (m-1) \). Then,

\[
\beta_0^* H^2(\mathbb{D}, \mathbb{C}^m) \perp = H^2(\mathbb{D}, \mathbb{C}^{m-1}) \perp.
\]

**Proof.** By virtue of the fact that complex conjugation is a unitary operator on \( L^2(\mathbb{T}, \mathbb{C}^m) \), an equivalent statement is that \( \beta_0^* z H^2(\mathbb{D}, \mathbb{C}^m) = z H^2(\mathbb{D}, \mathbb{C}^{m-1}) \). By Lemma 4.18, since \( \delta_0 \) is an inner, co-outer and quasi-continuous function, there exists a matrix-valued function \( B_0 \in \mathcal{H}^\infty(\mathbb{D}, \mathbb{C}^{(m-1) \times m}) \) such that

\[
B_0 \delta_0 = I_{m-1}
\]

or, equivalently,

\[
\beta_0^T B_0^T = I_{m-1}.
\]
Let \( g \in zH^2(D, \mathbb{C}^{m-1}) \). Then,
\[
g = (\beta_0^T B_0^T) g \in \beta_0^T B_0^T zH^2(D, \mathbb{C}^{m-1}) \subseteq \beta_0^T zH^2(D, \mathbb{C}^m).
\]
Hence,
\[
zH^2(D, \mathbb{C}^{m-1}) \subseteq \beta_0^T zH^2(D, \mathbb{C}^m).
\]
Note that, \( \beta_0 \in H^\infty(D, \mathbb{C}^{m \times (m-1)}) \), and so,
\[
zH^2(D, \mathbb{C}^{m-1}) \subseteq \beta_0^T zH^2(D, \mathbb{C}^m) \subseteq zH^2(D, \mathbb{C}^{m-1}).
\]
Thus,
\[
\beta_0^T zH^2(D, \mathbb{C}^m) = zH^2(D, \mathbb{C}^{m-1}). \quad \Box
\]

**Proposition 9.10.** In the notation of Theorem 9.1, let \( \eta_0 \in H^\infty(D, \mathbb{C}^m) \) be a vector-valued inner function given by equation (9.1), let \( W_0^T = (\eta_0 \beta_0) \) be a thematic completion of \( \eta_0 \) given by equation (9.2), and let
\[
\mathcal{L}_1 = W_0^T \begin{pmatrix} 0 \\ H^2(D, \mathbb{C}^{m-1}) \end{pmatrix}. 
\]
Then
\[
\bar{\eta}_0 \bar{\lambda} \mathcal{L}_1 = \bar{\eta}_0 \bar{\lambda} H^2(D, \mathbb{C}^m)^\perp
\]
and the operator
\[
(\bar{\eta}_0 \bar{\lambda} \cdot) : \mathcal{L}_1 \to \bar{\eta}_0 \bar{\lambda} H^2(D, \mathbb{C}^m)^\perp
\]
is unitary.

**Proof.** Let us first prove that \( \bar{\eta}_0 \bar{\lambda} H^2(D, \mathbb{C}^m)^\perp \subseteq \bar{\eta}_0 \bar{\lambda} \mathcal{L}_1 \). Consider an element \( f \in H^2(D, \mathbb{C}^m)^\perp \). Note that, since \( W_0^T \) is unitary-valued, we have
\[
\bar{\eta}_0 \eta_0^T + \beta_0 \beta_0^* = I_m. \quad \text{(9.8)}
\]
Thus,
\[
\bar{\eta}_0 \bar{\lambda} f = \bar{\eta}_0 \bar{\lambda} (\bar{\eta}_0 \eta_0^T + \beta_0 \beta_0^*) f \\
= \bar{\eta}_0 \bar{\lambda} \bar{\eta}_0 \eta_0^T f + \bar{\eta}_0 \bar{\lambda} \beta_0 \beta_0^* f \\
= 0 + \bar{\eta}_0 \bar{\lambda} \beta_0 \beta_0^* f,
\]
the last equality following by the pointwise linear dependence of \( \bar{\eta}_0 \) and \( \bar{\eta}_0 (\eta_0^T f) \) on \( D \). By Proposition 9.9,
\[
\beta_0^* H^2(D, \mathbb{C}^m)^\perp = H^2(D, \mathbb{C}^{m-1})^\perp,
\]
and, by the definition of \( \mathcal{L}_1 \), we have
\[
\mathcal{L}_1 = \beta_0 H^2(D, \mathbb{C}^{m-1})^\perp.
\]
Hence, for \( f \in H^2(D, \mathbb{C}^m)^\perp \),
\[
\bar{\eta}_0 \bar{\lambda} f = \bar{\eta}_0 \bar{\lambda} \beta_0 \beta_0^* f \in \bar{\eta}_0 \bar{\lambda} H^2(D, \mathbb{C}^{m-1})^\perp,
\]
and thus
\[
\bar{\eta}_0 \bar{\lambda} H^2(D, \mathbb{C}^m)^\perp \subseteq \bar{\eta}_0 \bar{\lambda} \mathcal{L}_1.
\]
Let us show
\[
\bar{\eta}_0 \bar{\lambda} \mathcal{L}_1 \subseteq \bar{\eta}_0 \bar{\lambda} H^2(D, \mathbb{C}^m)^\perp.
\]
A typical element of $\eta_0 \wedge L_1$ is of the form $\eta_0 \wedge \beta_0 g$, for some $g \in H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp$. By Proposition 9.9, there exists a $\varphi \in H^2(\mathbb{D}, \mathbb{C}^{m})^\perp$ such that $\beta_0^T \varphi = g$. Then

$$\eta_0 \wedge \beta_0 g = \eta_0 \wedge \beta_0 \beta_0^T \varphi.$$  

By equation (9.8), we have

$$\bar{\eta}_0 \wedge \beta_0 g = \bar{\eta}_0 \wedge (I_m - \bar{\eta}_0 \eta_0^T) \varphi = \bar{\eta}_0 \wedge \varphi,$$

the last equality following by pointwise linear dependence of $\bar{\eta}_0$ and $\bar{\eta}_0 (\eta_0^T \varphi)$ on $\mathbb{D}$. Thus,

$$\bar{\eta}_0 \wedge \beta_0 g \in \bar{\eta}_0 \wedge H^2(\mathbb{D}, \mathbb{C}^{m})^\perp,$$

and so $\bar{\eta}_0 \wedge L_1 \subseteq \bar{\eta}_0 \wedge H^2(\mathbb{D}, \mathbb{C}^{m})^\perp$. Consequently

$$\bar{\eta}_0 \wedge L_1 = \bar{\eta}_0 \wedge H^2(\mathbb{D}, \mathbb{C}^{m})^\perp.$$

To prove that the operator

$$(\bar{\eta}_0 \wedge \cdot) : L_1 \to \bar{\eta}_0 \wedge H^2(\mathbb{D}, \mathbb{C}^{m})^\perp$$

is unitary, it suffices to show that it is an isometry, since the preceding discussion asserts that it is surjective. To this end, let $s \in L_1$. Then,

$$\|\bar{\eta}_0 \wedge s\|^2_{L^2(\mathbb{D}^2, \lambda^2 \wedge \mathbb{C}^{m})} = \langle \bar{\eta}_0 \wedge s, \bar{\eta}_0 \wedge s \rangle_{L^2(\mathbb{D}^2, \lambda^2 \wedge \mathbb{C}^{m})}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \langle \bar{\eta}_0(e^{i\theta}) \wedge s(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \wedge s(e^{i\theta}) \rangle_{\lambda^2 \wedge \mathbb{C}^{m}} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \det \left( \langle \bar{\eta}_0(e^{i\theta}) \wedge s(e^{i\theta}) \rangle_{\mathbb{C}^{m}}, \langle \bar{\eta}_0(e^{i\theta}) \wedge s(e^{i\theta}) \rangle_{\mathbb{C}^{m}} \right) d\theta.$$

By Proposition 5.1, $\|\bar{\eta}_0(z)\|_{\mathbb{C}^{m}} = 1$ almost everywhere on $\mathbb{T}$. Moreover, since $s \in L_1$, there exists a function $\psi \in H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp$ such that $s = \beta_0 \psi$. Then

$$\langle \bar{\eta}_0(e^{i\theta}), s(e^{i\theta}) \rangle_{\mathbb{C}^{m}} = \langle \bar{\eta}_0(e^{i\theta}), \beta_0(e^{i\theta}) \psi(e^{i\theta}) \rangle_{\mathbb{C}^{m}} = \langle \beta_0^T(e^{i\theta}) \bar{\eta}_0(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^{m}} = 0$$

almost everywhere on $\mathbb{T}$, which follows by the fact that $W_0$ is unitary-valued, and so

$$(W_0 W_0^*)(z) = \begin{pmatrix} \eta_0^T(z) \\ \beta_0^T(z) \end{pmatrix} \begin{pmatrix} \bar{\eta}_0(z) \\ \beta_0(z) \end{pmatrix} = \begin{pmatrix} \eta_0^T(z) \bar{\eta}_0(z) \\ \beta_0^T(z) \bar{\eta}_0(z) \end{pmatrix} = I_m$$

almost everywhere on $\mathbb{T}$.

Thus, for all $s \in L_1$,

$$\|\bar{\eta}_0 \wedge s\|^2_{L^2(\mathbb{D}^2, \lambda^2 \wedge \mathbb{C}^{m})} = \|s\|^2_{L^2(\mathbb{D}^2, \mathbb{C}^{m})},$$

which shows that the operator

$$(\bar{\eta}_0 \wedge \cdot) : L_1 \to \bar{\eta}_0 \wedge H^2(\mathbb{D}, \mathbb{C}^{m})^\perp$$

is an isometry. We have proved it is also surjective, hence the operator $(\bar{\eta}_0 \wedge \cdot)$ is unitary. \(\square\)

(iii) Continuation of the proof of Theorem 9.1. We have to prove that diagram (9.4) commutes. Recall, by Lemma 4.17, the left-hand square commutes, so it suffices to show that that the right-hand square, namely

$$\begin{array}{c}
K_1 \xrightarrow{\xi_0 \wedge \cdot} \xi_0 \wedge H^2(\mathbb{D}, \mathbb{C}^{n}) = X_1 \\
\downarrow \Gamma_1 \quad \downarrow \Gamma_1 \\
L_1 \xrightarrow{\bar{\eta}_0 \wedge \cdot} \bar{\eta}_0 \wedge H^2(\mathbb{D}, \mathbb{C}^{m})^\perp = Y_1
\end{array}$$

(9.9)
also commutes. That is, we wish to prove that, for all \( x \in \mathcal{K}_1 \),
\[
T_1(\xi_0 \wedge x) = \bar{\eta}_0 \hat{\lambda} \Gamma_1(x),
\]
where \( \Gamma_1(x) = P_{L_1}((G - Q_1)x) \) for any function \( Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) that satisfies the following equations:
\[
(G - Q_1)x_0 = t_0 y_0, \quad y_0^*(G - Q_1) = t_0 x_0^*.
\]
By Proposition 9.6,
\[
\xi_0 \wedge \mathcal{K}_1 = \xi_0 \wedge H^2(\mathbb{D}, \mathbb{C}^n),
\]
and so, for every \( x \in \mathcal{K}_1 \), there exists \( \hat{x} \in H^2(\mathbb{D}, \mathbb{C}^n) \) such that
\[
\xi_0 \wedge x = \xi_0 \wedge \hat{x}.
\]
Thus, for \( x \in \mathcal{K}_1 \),
\[
T_1(\xi_0 \wedge x) = T_1(\xi_0 \wedge \hat{x}) = P_{Y_1}((\bar{\eta}_0 \hat{\lambda} (G - Q_1) \hat{x})
\]
and
\[
\bar{\eta}_0 \hat{\lambda} \Gamma_1(x) = \bar{\eta}_0 \hat{\lambda} P_{L_1}((G - Q_1)x).
\]
Hence, to prove the commutativity of diagram (9.9), it suffices to show that, for all \( x \in \mathcal{K}_1 \),
\[
P_{Y_1}([\bar{\eta}_0 \hat{\lambda} (G - Q_1) \hat{x}]) = \bar{\eta}_0 \hat{\lambda} P_{L_1}((G - Q_1)x)
\]
in \( Y_1 \), where \( \xi_0 \wedge (x - \hat{x}) = 0 \). By Proposition 9.10,
\[
\bar{\eta}_0 \hat{\lambda} L_1 = \bar{\eta}_0 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^m)^\perp = Y_1,
\]
and so, for all \( x \in \mathcal{K}_1 \), \( \bar{\eta}_0 \hat{\lambda} P_{L_1}((G - Q_1)x) \in Y_1 \).

Let us show that, for \( x \in \mathcal{K}_1 \), \( \bar{\eta}_0 \hat{\lambda} (G - Q_1) \hat{x} - \bar{\eta}_0 \hat{\lambda} P_{L_1}((G - Q_1)x)
\]
is orthogonal to \( Y_1 \) in \( L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m) \), or equivalently, that for every \( f \in H^2(\mathbb{D}, \mathbb{C}^m) \),
\[
\langle \bar{\eta}_0 \hat{\lambda}((G - Q_1)x - P_{L_1}((G - Q_1)x), \bar{\eta}_0 \hat{\lambda} \hat{z} \hat{f} \rangle_{L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)} = 0 \tag{9.10}
\]
for \( x \in \mathcal{K}_1 \) and for any \( \hat{x} \in H^2(\mathbb{D}, \mathbb{C}^n) \) such that \( \xi_0 \wedge \hat{x} = \xi_0 \wedge x \). By Lemma 8.2,
\[
\bar{\eta}_0 \hat{\lambda} (G - Q_1)x = \bar{\eta}_0 \hat{\lambda} (G - Q_1) \hat{x}.
\]
Then equation (9.10) is equivalent to the equation
\[
\langle \bar{\eta}_0 \hat{\lambda} P_{L_1}((G - Q_1)x, \bar{\eta}_0 \hat{\lambda} \hat{z} \hat{f} \rangle_{L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)} = 0 \tag{9.11}
\]
for any \( x \in \mathcal{K}_1 \). By Lemma 9.7, equation (9.11) holds if and only if the function
\[
z \mapsto [P_{L_1}((G - Q_1)x)](z) - \langle [P_{L_1}((G - Q_1)x)](z), \bar{\eta}_0(z) \rangle_{\mathbb{C}^m} \bar{\eta}_0(z) \tag{9.12}
\]
belongs to \( H^2(\mathbb{D}, \mathbb{C}^m) \). By Lemma 9.8, there exists a function \( \psi \in L^2(\mathbb{T}, \mathbb{C}^m) \) such that
\[
P_{L_1}((G - Q_1)x) = \psi \tag{9.13}
\]
and
\[
\beta_0^* \psi \in H^2(\mathbb{D}, \mathbb{C}^{m-1}).
\]
Equation (9.13) implies
\[
(G - Q_1)x - \psi \in \mathcal{L}_1 = \beta_0 H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp.
\]
Hence, to prove that the function defined by equation (9.12) belongs to $H^2(\mathbb{D}, \mathbb{C}^m)$, we have to show that

$$\psi - (\eta_0^T \psi)\bar{\eta}_0 \in H^2(\mathbb{D}, \mathbb{C}^m).$$

Since $W_0 = (\eta_0 \bar{\eta}_0)^T$ is a unitary-valued function,

$$\bar{\eta}_0(z)\eta_0^T(z) + \beta_0(z)\beta_0^*(z) = I_m$$

almost everywhere on $\mathbb{T}$.

Since $\eta_0^T \psi$ is a scalar-valued function,

$$\psi - (\eta_0^T \psi)\bar{\eta}_0 = (I_m - \eta_0^T \bar{\eta}_0)\psi = \beta_0\beta_0^* \psi \in H^2(\mathbb{D}, \mathbb{C}^m).$$

Thus diagram (9.9) commutes.

(iv) By Lemma 4.12,

$$F_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times (n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1)\times (n-1)}).$$

Then, by Hartman’s Theorem 4.1, the Hankel operator $H_{F_1}$ is compact, and by (iii),

$$(\eta_0 \hat{\cdot}) \circ (U_2 H_{F_1} U_1^*) \circ (\xi_0 \hat{\cdot})^* = T_1.$$

By (i) and (ii), the operators $U_1, U_2, (\xi_0 \hat{\cdot})$ and $(\eta_0 \hat{\cdot})$ are unitary. Hence, $T_1$ is a compact operator.

(v) Since diagram (9.4) is commutative and $U_1, U_2, (\xi_0 \hat{\cdot})$ and $(\eta_0 \hat{\cdot})$ are unitaries,

$$\|T_1\| = \|\Gamma_1\| = \|H_{F_1}\|. \quad \Box$$

In what follows, we will prove an analogous statement to Theorem 9.1 for $T_2$. To this end, we need the following results.

**Lemma 9.11.** In the notation of Theorem 9.1, $v_1 \in H^2(\mathbb{D}, \mathbb{C}^m)$ and $w_1 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ are such that $(\xi_0 \hat{\cdot} v_1, \eta_0 \hat{\cdot} w_1)$ is a Schmidt pair for the operator $T_1$ corresponding to $\|T_1\|$. Then

(i) there exist $x_1 \in K_1$ and $y_1 \in L_1$ such that $(x_1, y_1)$ is a Schmidt pair for the operator $\Gamma_1$;

(ii) for any $x_1 \in K_1$ and $y_1 \in L_1$ such that

$$\xi_0 \hat{\cdot} x_1 = \xi_0 \hat{\cdot} v_1, \quad \bar{\eta}_0 \hat{\cdot} y_1 = \bar{\eta}_0 \hat{\cdot} w_1,$$

the pair $(x_1, y_1)$ is a Schmidt pair for $\Gamma_1$ corresponding to $\|\Gamma_1\|$.

**Proof.** (i) By Theorem 9.1, the diagram (9.4) commutes, $(\xi_0 \hat{\cdot})$ is unitary from $K_1$ to $X_1$, and $(\bar{\eta}_0 \hat{\cdot})$ is unitary from $L_1$ to $Y_1$. Thus, $\|\Gamma_1\| = \|T_1\| = t_1$. Moreover, by Lemma 4.17, the operator $\Gamma_1 : K_1 \to L_1$ is compact, hence there exist $x_1 \in K_1, y_1 \in L_1$ such that $(x_1, y_1)$ is a Schmidt pair for $\Gamma_1$ corresponding to $\|\Gamma_1\| = t_1$.

(ii) Suppose that $x_1 \in K_1, y_1 \in L_1$ satisfy

$$\xi_0 \hat{\cdot} x_1 = \xi_0 \hat{\cdot} v_1, \quad (9.14)$$

$$\bar{\eta}_0 \hat{\cdot} y_1 = \bar{\eta}_0 \hat{\cdot} w_1. \quad (9.15)$$

Let us show that $(x_1, y_1)$ is a Schmidt pair for $\Gamma_1$ corresponding to $t_1$, that is,

$$\Gamma_1 x_1 = t_1 y_1, \quad \Gamma_1 y_1 = t_1 x_1.$$

Since diagram (9.9) commutes,

$$T_1 \circ (\xi_0 \hat{\cdot}) = (\bar{\eta}_0 \hat{\cdot}) \circ \Gamma_1, \quad (\xi_0 \hat{\cdot})^* \circ T_1^* = \Gamma_1^* \circ (\bar{\eta}_0 \hat{\cdot})^*. \quad (9.16)$$

By hypothesis,

$$T_1(\xi_0 \hat{\cdot} v_1) = t_1(\bar{\eta}_0 \hat{\cdot} w_1), \quad T_1^*(\bar{\eta}_0 \hat{\cdot} w_1) = t_1(\xi_0 \hat{\cdot} v_1). \quad (9.17)$$
Thus, by equations (9.15), (9.16) and (9.17),
\[ \Gamma_1 x_1 = (\bar{\eta}_0 \hat{\cdot})^* T_1 (\xi_0 \hat{\cdot} v_1) \]
\[ = (\bar{\eta}_0 \hat{\cdot})^* t_1 (\bar{\eta}_0 \hat{\cdot} w_1) \]
\[ = t_1 (\bar{\eta}_0 \hat{\cdot})^* (\bar{\eta}_0 \hat{\cdot} y_1). \]
Hence,
\[ \Gamma_1 x_1 = t_1 (\bar{\eta}_0 \hat{\cdot})^* (\bar{\eta}_0 \hat{\cdot} y_1) = t_1 y_1. \]

By equation (9.14),
\[ x_1 = (\xi_0 \hat{\cdot})^* (\xi_0 \hat{\cdot} v_1), \]
and, by equation (9.15),
\[ (\bar{\eta}_0 \hat{\cdot})^* (\bar{\eta}_0 \hat{\cdot} w_1) = y_1. \]
Thus,
\[ \Gamma_1^* y_1 = \Gamma_1^* (\bar{\eta}_0 \hat{\cdot})^* (\bar{\eta}_0 \hat{\cdot} w_1) \]
\[ = (\xi_0 \hat{\cdot})^* T_1^* (\bar{\eta}_0 \hat{\cdot} w_1), \]
the last equality following by the second equation of (9.16). By equations (9.14) and (9.17), we have
\[ T_1^* (\bar{\eta}_0 \hat{\cdot} w_1) = t_1 (\xi_0 \hat{\cdot} v_1) = t_1 (\xi_0 \hat{\cdot} x_1), \]
and so,
\[ \Gamma_1^* y_1 = t_1 x_1. \]
Therefore, \((x_1, y_1)\) is a Schmidt pair for \(\Gamma_1\) corresponding to \(\|\Gamma_1\| = \|T_1\| = t_1. \)

**Lemma 9.12.** Suppose that \((\xi_0 \hat{\cdot} v_1, \bar{\eta}_0 \hat{\cdot} w_1)\) is a Schmidt pair for \(T_1\) corresponding to \(t_1\). Let
\[ x_1 = (I_n - \xi_0 \xi_0^*) v_1, \quad y_1 = (I_m - \bar{\eta}_0 \bar{\eta}_0^*) w_1, \]
and let
\[ \hat{x}_1 = \alpha_0^T x_1, \quad \hat{y}_1 = \beta_0^* y_1. \]
Then
\[ (i) \]
\[ x_1 = \alpha_0 \alpha_0^T x_1, \quad y_1 = \beta_0 \beta_0^* y_1; \]
\[ (ii) \] the pair \((\hat{x}_1, \hat{y}_1)\) is a Schmidt pair for \(H_{F_1}\) corresponding to \(\|H_{F_1}\| = t_1. \)

**Proof.** (i) Since \(V_0 = (\xi_0 \hat{\cdot} \alpha_0)\) is unitary-valued, \(I_n - \xi_0 \xi_0^* = \bar{\alpha}_0 \alpha_0^T\), and so
\[ \bar{\alpha}_0 \alpha_0^T x_1 = (I_n - \xi_0 \xi_0^*) (I_n - \xi_0 \xi_0^*) v_1 \]
\[ = (I_n - 2\xi_0 \xi_0^* + \xi_0 \xi_0^* \xi_0 \xi_0^*) v_1 \]
\[ = (I_n - \xi_0 \xi_0^*) v_1 = x_1. \]
(9.18)

Similarly, since \(W_0^T = (\eta_0 \hat{\cdot} \beta_0)\) is unitary valued, \(I_m - \bar{\eta}_0 \bar{\eta}_0^* = \beta_0 \beta_0^*\), and so
\[ \beta_0 \beta_0^* y_1 = (I_m - \bar{\eta}_0 \bar{\eta}_0^*) (I_m - \bar{\eta}_0 \bar{\eta}_0^*) w_1 \]
(9.19)
By equations (9.21) and (9.18), we have
\[ \chi \text{ for all } \eta \text{ and let } \]
\[ (I_m - 2\bar{\eta}_0\eta_0^T)w_1 \]
\[ = (I_m - \bar{\eta}_0\eta_0^T)w_1 = y_1. \quad (9.20) \]

(ii) Recall that, by Lemma 4.17, the maps
\[ U_1: H^2(\mathbb{D}, \mathbb{C}^{m-1}) \to \mathcal{K}_1, \quad U_2: H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp \to \mathcal{L}_1, \]
defined by
\[ U_1\chi = V_0 \left( \begin{array}{c} 0 \\ \chi \end{array} \right) = \bar{\alpha}_0\chi, \quad U_2\psi = W_0 \left( \begin{array}{c} 0 \\ \psi \end{array} \right) = \beta_0\psi \]
for all \( \chi \in H^2(\mathbb{D}, \mathbb{C}^{m-1}) \) and all \( \psi \in H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp \), are unitaries. By the commutativity of the diagram (9.4),
\[ H_{F_1} = U_2^*\Gamma_1 U_1. \quad (9.21) \]

By part (i), \( x_1 \in \mathcal{K}_1 \) and \( y_1 \in \mathcal{L}_1 \) and by Proposition 5.1,
\[ \xi_0\check{\lambda}x_1 = \xi_0\check{\lambda}v_1, \quad \bar{\eta}_0\check{\lambda}y_1 = \bar{\eta}_0\check{\lambda}w_1. \]
Thus, by Lemma 9.11, \((x_1, y_1)\) is a Schmidt pair for the operator \( \Gamma_1 \) corresponding to \( t_1 = \|\Gamma_1\| \), that is,
\[ \Gamma_1 x_1 = t_1 y_1, \quad \Gamma_1^* y_1 = t_1 x_1. \quad (9.22) \]

To prove that the pair \((\hat{x}_1, \hat{y}_1)\) is a Schmidt pair for \( H_{F_1} \) corresponding to \( \|H_{F_1}\| = t_1 \), we need to show that
\[ H_{F_1} \hat{x}_1 = t_1 \hat{y}_1, \text{ and } H_{F_1}^* \hat{y}_1 = t_1 \hat{x}_1. \]

By equations (9.21) and (9.18), we have
\[ H_{F_1} \hat{x}_1 = H_{F_1} \alpha_0^T \hat{x}_1 \]
\[ = U_2^* \Gamma_1 U_1 \alpha_0^T x_1 = U_2^* \Gamma_1 \bar{\alpha}_0 \alpha_0^T x_1 \]
\[ = U_2^* \Gamma_1 x_1 = t_1 \beta_0^* y_1 = t_1 \hat{y}_1. \quad (9.23) \]

Let us show that \( H_{F_1}^* \hat{y}_1 = t_1 \hat{x}_1 \). By equations (9.21) and (9.18), we have
\[ H_{F_1}^* \hat{y}_1 = H_{F_1}^* \beta_0^* y_1 \]
\[ = U_1^* \Gamma_1^* U_2 \beta_0^* y_1 = U_1^* \Gamma_1^* \beta_0 \beta_0^* y_1 \]
\[ = U_1^* \Gamma_1^* y_1 = t_1 U_1^* x_1 = t_1 \alpha_0^T x_1 = t_1 \hat{x}_1. \quad (9.24) \]

Therefore, \((\hat{x}_1, \hat{y}_1)\) is a Schmidt pair for \( H_{F_1} \) corresponding to \( \|H_{F_1}\| = t_1 \). \( \square \)

**Proposition 9.13.** Let \((\xi_0\check{\lambda}v_1, \bar{\eta}_0\check{\lambda}w_1)\) be a Schmidt pair for \( T_1 \) corresponding to \( t_1 \) for some \( v_1 \in H^2(\mathbb{D}, \mathbb{C}^n) \), \( w_1 \in H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp \), let \( h_1 \in H^2(\mathbb{D}, \mathbb{C}) \) be the scalar outer factor of \( \xi_0\check{\lambda}v_1 \), let
\[ x_1 = (I_n - \xi_0^* \xi_0) v_1, \quad y_1 = (I_m - \bar{\eta}_0^T \eta_0^*) w_1, \]
and let
\[ \hat{x}_1 = \alpha_0^T x_1, \quad \hat{y}_1 = \beta_0^* y_1. \]
Then

\[ \|\hat{x}_1(z)\|_{C^n} = \|\hat{y}_1(z)\|_{C^m} = \|h_1(z)\|, \]

\[ \|x_1(z)\|_{C^n} = \|y_1(z)\|_{C^m} = \|h_1(z)\| \]

and

\[ \|\xi_0(z) \wedge v_1(z)\|_{\Lambda^2C^n} = \|\tilde{\eta}_0(z) \wedge w_1(z)\|_{\Lambda^2C^m} = \|h_1(z)\| \]

almost everywhere on \( T \).

Proof. By Lemma 9.12, \((\hat{x}_1, \hat{y}_1)\) is a Schmidt pair for \( H_{F_1} \), corresponding to \( \|H_{F_1}\| = t_1 \).

Hence,

\[ H_{F_1}\hat{x}_1 = t_1\hat{y}_1 \quad \text{and} \quad H_{F_1}^*\hat{y}_1 = t_1\hat{x}_1. \]

By Theorem 4.11, for the Hankel operator \( H_{F_1} \) and the Schmidt pair \((\hat{x}_1, \hat{y}_1)\), we have

\[ \|\hat{y}_1(z)\|_{C^m} = \|\hat{x}_1(z)\|_{C^n} \]  \hspace{1cm} (9.25)

almost everywhere on \( T \).

By equations (9.18),

\[ x_1 = \bar{\alpha}_0\alpha_0^T x_1 = \bar{\alpha}_0\hat{x}_1, \quad y_1 = \beta_0\beta_0^* y_1 = \beta_0\hat{y}_1. \]

Since \( \bar{\alpha}_0(z) \) and \( \beta_0(z) \) are isometric for almost every \( z \in T \),

\[ \|x_1(z)\|_{C^n} = \|\hat{x}_1(z)\|_{C^n} \quad \text{and} \quad \|y_1(z)\|_{C^m} = \|\hat{y}_1(z)\|_{C^m} \]

almost everywhere on \( T \). By equations (9.25), we deduce

\[ \|x_1(z)\|_{C^n} = \|y_1(z)\|_{C^m} \]  \hspace{1cm} (9.26)

almost everywhere on \( T \).

By Theorem 9.1, \((\xi_0,\lambda)\) is an isometry from \( K_1 \) to \( X_1 \), and \((\tilde{\eta}_0,\lambda)\) is an isometry from \( L_1 \) to \( Y_1 \). By Proposition 5.1,

\[ \xi_0\lambda x_1 = \xi_0\lambda v_1, \quad \tilde{\eta}_0\lambda y_1 = \tilde{\eta}_0\lambda w_1. \]

Hence,

\[ \|\xi_0(z) \wedge v_1(z)\|_{\Lambda^2C^n} = \|\xi_0(z) \wedge x_1(z)\|_{\Lambda^2C^n} = \|x_1(z)\|_{C^n} \]

almost everywhere on \( T \). Also

\[ \|\tilde{\eta}_0(z) \wedge w_1(z)\|_{\Lambda^2C^m} = \|\tilde{\eta}_0(z) \wedge y_1(z)\|_{\Lambda^2C^m} = \|y_1(z)\|_{C^m} \]

almost everywhere on \( T \). Thus, by equation (9.26),

\[ \|\xi_0(z) \wedge v_1(z)\|_{\Lambda^2C^n} = \|\tilde{\eta}_0(z) \wedge w_1(z)\|_{\Lambda^2C^m} \]

almost everywhere on \( T \). Recall that \( h_1 \) is the scalar outer factor of \( \xi_0\lambda v_1 \).

Hence,

\[ \|\xi_0(z) \wedge v_1(z)\|_{\Lambda^2C^n} = \|\tilde{\eta}_0(z) \wedge w_1(z)\|_{\Lambda^2C^m} = \|h_1(z)\|, \]

\[ \|x_1(z)\|_{C^n} = \|y_1(z)\|_{C^m} = \|h_1(z)\| \]

and

\[ \|\hat{x}_1(z)\|_{C^n} = \|\hat{y}_1(z)\|_{C^m} = \|h_1(z)\| \]

almost everywhere on \( T \).
Definition 9.14. Given \( G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(T, \mathbb{C}^{m \times n}) \) and \( 0 \leq j \leq \min(m, n) \), define \( \Omega_j \) to be the set of level \( j \) superoptimal analytic approximants to \( G \), that is, the set of \( Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) which minimize the tuple
\[
\left( s_0^\infty(G - Q), s_1^\infty(G - Q), \ldots, s_j^\infty(G - Q) \right)
\]
with respect to the lexicographic ordering over \( Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \). For \( Q \in \Omega_j \) we call \( G - Q \) a level \( j \) superoptimal error function, and we denote by \( \mathcal{E}_j \) the set of all level \( j \) superoptimal error functions, that is,
\[
\mathcal{E}_j = \{ G - Q : Q \in \Omega_j \}.
\]

Proposition 9.15. Let \( m, n \) be positive integers such that \( \min(m, n) \geq 2 \). Let \( G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(T, \mathbb{C}^{m \times n}) \). In line with the algorithm from Subsection 4.2, let \( Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) satisfy
\[
(G - Q_1)x_0 = t_0 y_0, \quad (G - Q_1)^* y_0 = t_0 x_0.
\]
Let the spaces \( X_1, Y_1 \) be given by
\[
X_1 = \xi_0 \hat{\Lambda} H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \Lambda^2 \mathbb{C}^n), \quad Y_1 = \eta_0 \hat{\Lambda} H^2(\mathbb{D}, \mathbb{C}^m)^\perp \subset H^2(\mathbb{D}, \Lambda^2 \mathbb{C}^m)^\perp,
\]
and consider the compact operator \( T_1 : X_1 \to Y_1 \) given by
\[
T_1(\xi_0 \hat{\Lambda} x) = P_{Y_1}(\eta_0 \hat{\Lambda}(G - Q_1)x)
\]
for all \( x \in H^2(\mathbb{D}, \mathbb{C}^n) \). Let \( (\xi_0 \hat{\Lambda} v_1, \eta_0 \hat{\Lambda} w_1) \) be a Schmidt pair for the operator \( T_1 \) corresponding to \( t_1 = \|T_1\| \), let \( h_1 \in H^2(\mathbb{D}, \mathbb{C}) \) be the scalar outer factor of \( \xi_0 \hat{\Lambda} v_1 \), let
\[
x_1 = (I_m - \xi_0 \xi_0^*) v_1, \quad y_1 = (I_m - \eta_0 \eta_0^*) w_1
\]
and let
\[
\xi_1 = \frac{x_1}{h_1}, \quad \eta_1 = \frac{y_1}{h_1}.
\]
Then, there exist unitary-valued functions \( \tilde{V}_1, \tilde{W}_1 \) of types \( (n - 1) \times (n - 1), (m - 1) \times (m - 1) \), respectively, of the form
\[
\tilde{V}_1 \overset{\text{def}}{=} \left( \alpha_0^T \xi_1 \bar{\eta}_1 \right)
\]
and
\[
\tilde{W}_1 \overset{\text{def}}{=} \left( \beta_0^T \eta_1 \bar{\xi}_1 \right),
\]
where \( \alpha_1, \beta_1 \) are inner, co-outer, quasi-continuous functions of types \( (n - 1) \times (n - 2), (m - 1) \times (m - 2) \), respectively, and all minors on the first columns of \( \tilde{V}_1, \tilde{W}_1 \) are in \( H^\infty \).

Furthermore, the set of all level 1 superoptimal functions \( \mathcal{E}_1 \) satisfies
\[
\mathcal{E}_1 = W_0^* \begin{pmatrix} 1 & 0 & 0 \\ 0 & W_1 \end{pmatrix} \begin{pmatrix} t_0 u_0 & 0 \\ t_1 u_1 & 0 \\ 0 & \end{pmatrix} \begin{pmatrix} F_2 + H^\infty(\mathbb{D}, \mathbb{C}^{(m-2) \times (n-2)}) \cap B(t_1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V_1^* \end{pmatrix} V_0^*,
\]
where \( F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2) \times (n-2)}) + C(T, \mathbb{C}^{(m-2) \times (n-2)}) \), \( u_1 = \bar{z} h_1 / h_1 \) is a quasi-continuous unimodular function and \( V_0, W_0^T \) are as in Theorem 9.1, and \( B(t_1) \) is the closed ball of radius \( t_1 \) in \( L^\infty(T, \mathbb{C}^{(m-2) \times (n-2)}) \).
Proof. By Theorem 9.1, the following diagram commutes:

\[
\begin{align*}
H^2(\mathbb{D}, \mathbb{C}^{n-1}) & \xrightarrow{\mathcal{U}_1} \mathcal{K}_1 \xrightarrow{\xi_{\theta} \Lambda} \xi_{\theta} \Lambda H^2(\mathbb{D}, \mathbb{C}^n) = X_1 \\
H^2(\mathbb{D}, \mathbb{C}^{m-1}) & \xrightarrow{\mathcal{V}_2} \mathcal{L}_1 \xrightarrow{\eta_{\theta} \Lambda} \eta_{\theta} \Lambda H^2(\mathbb{D}, \mathbb{C}^m) = Y_1.
\end{align*}
\]

(9.30)

Let \( \hat{x}_1 = \alpha_0^T x_1, \) \( \hat{y}_1 = \beta_0^* y_1. \) By Lemma 9.12, \((\hat{x}_1, \hat{y}_1)\) is a Schmidt pair for \( F_{\mathcal{I}} \) corresponding to \( t_1. \) By equations (9.18),

\[
x_1 = \bar{x}_1 = \alpha_0 \alpha_0^T x_1 = \bar{x}_1 = \beta_0 \beta_0^* y_1 = \beta_0 \beta_0^* y_1.
\]

We want to apply Lemma 4.12 to \( F_{\mathcal{I}} \) and the Schmidt pair \((\hat{x}_1, \hat{y}_1)\) to find unitary-valued functions \( \hat{V}_1, \hat{W}_1 \) such that, for any function \( \hat{Q}_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}) \) which is at minimal distance from \( F_{\mathcal{I}} \), the following equation holds:

\[
F_{\mathcal{I}} - \hat{Q}_1 = \hat{W}_1^* \begin{pmatrix} t_1 u_1 & 0 \\ 0 & F_2 \end{pmatrix} \hat{V}_1^*,
\]

for some \( F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)}) \). For this purpose we find the inner–outer factorizations \( \hat{x}_1 \) and \( \hat{y}_1. \) By Proposition 9.13,

\[
\|\hat{x}_1(z)\|_{\mathbb{C}^{n-1}} = \|x_1(z)\|_{\mathbb{C}^n} = \|\xi_0(z) \hat{v}_1(z)\|_{\Lambda^2 \mathbb{C}^n} = |h_1(z)|
\]

and

\[
\|\hat{y}_1(z)\|_{\mathbb{C}^{m-1}} = \|y_1(z)\|_{\mathbb{C}^m} = \|\eta_0(z) \hat{w}_1(z)\|_{\Lambda^2 \mathbb{C}^m} = |h_1(z)|
\]

(9.31)

almost everywhere on \( \mathbb{T}. \) Equations (9.31) imply that \( h_1 \in H^2(\mathbb{D}, \mathbb{C}) \) is the scalar outer factor of both \( \hat{x}_1 \) and \( \hat{y}_1. \) By Lemma 4.12, \( \hat{x}_1, \hat{y}_1 \) admit the inner–outer factorizations

\[
\hat{x}_1 = \xi_1 h_1, \quad \hat{y}_1 = \eta_1 h_1,
\]

for some inner vector-valued \( \hat{\xi}_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{n-1}) \) and \( \hat{\eta}_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m-1}) \). Recall that

\[
\hat{x}_1 = \alpha_0^T x_1 = \alpha_0^T \xi_1 h_1, \quad \hat{y}_1 = \beta_0^T y_1 = \beta_0^* \eta_1 h_1,
\]

which imply

\[
\hat{\xi}_1 = \alpha_0^T \xi_1 \quad \text{and} \quad \hat{\eta}_1 = \beta_0^T \eta_1.
\]

Let us show that \( \alpha_0^T \xi_1, \beta_0^T \eta_1 \) are inner in order to apply Lemma 4.12. Recall that, since \( V_0, W_0^T \) are unitary-valued, we have

\[
I_n - \xi_0 \xi_0^* = \bar{x}_1, \quad I_m - \eta_0 \eta_0^* = \beta_0 \beta_0^*.
\]

Therefore,

\[
x_1 = (I_n - \xi_0 \xi_0^*) v_1 = \bar{x}_1 \alpha_0^T v_1, \quad y_1 = (I_m - \eta_0 \eta_0^*) w_1 = \beta_0 \beta_0^* w_1.
\]

Then,

\[
\alpha_0^T x_1 = \alpha_0^T v_1, \quad \beta_0^T y_1 = \beta_0^T w_1,
\]

(9.32)

and since

\[
\xi_1 = \frac{x_1}{h_1}, \quad \eta_1 = \frac{y_1}{h_1},
\]

the functions

\[
\alpha_0^T \xi_1 = \frac{x_1}{h_1}, \quad \beta_0^T \eta_1 = \frac{y_1}{h_1},
\]

(9.32)
are analytic. Furthermore, by Proposition 9.13,
\[ \|x_1(z)\|_{C^m} = \|y_1(z)\|_{C^m} = |h_1(z)| = \|\hat{x}_1(z)\|_{C_\kappa} = \|\hat{y}_1(z)\|_{C_{\kappa}} \]
almost everywhere on \( T \). Thus,
\[ \|\alpha_0^T(z)x_1(z)\|_{C_{\kappa}} = \|\alpha_0^T(z)v_1(z)\|_{C_{\kappa}} = |h_1(z)| \]
and
\[ \|\beta_0^T(z)\bar{z}\hat{y}_1(z)\|_{C_{\kappa}} = \|\beta_0^T(z)\bar{z}\hat{w}_1(z)\|_{C_{\kappa}} = |h_1(z)| \]
almost everywhere on \( T \). Hence,
\[ \|\alpha_0^T(z)\xi_1(z)\|_{C_{\kappa}} = 1, \quad \|\beta_0^T(z)\eta_1(z)\|_{C_{\kappa}} = 1 \]
almost everywhere on \( T \). Therefore, \( \alpha_0^T \xi_1, \beta_0^T \eta_1 \) are inner functions. By Lemma 4.12, there exist inner, co-outer, quasi-continuous functions \( \alpha_1, \beta_1 \) of types \((n-1) \times (n-2)\) and \((m-1) \times (m-2)\), respectively, such that
\[ \hat{V}_1 = (\alpha_0^T \xi_1 \bar{\eta}_1), \quad \hat{W}_1^T = (\beta_0^T \eta_1 \bar{\beta}_1) \]
are unitary-valued and all minors on the first columns are in \( H^\infty \). Furthermore, by Lemma 4.12, every \( Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1) \times (n-1)}) \) which is at minimal distance from \( F_1 \) satisfies
\[ F_1 - Q_1 = \hat{W}_1^T \left( \begin{array}{cc} t_1 u_1 & 0 \\ 0 & F_2 \end{array} \right) \hat{V}_1^*, \]
where \( F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2) \times (n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2) \times (n-2)}) \) and \( u_1 \) is a quasi-continuous unimodular function given by \( u_1 = \frac{\eta_1}{\eta_1} \).

By Lemma 4.15, the set
\[ \mathcal{E}_0 = \{ F_1 - \hat{Q} : \hat{Q} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1) \times (n-1)}), \|F_1 - \hat{Q}\|_{L^\infty} = t_1 \} \]
satisfies
\[ \mathcal{E}_0 = \hat{W}_1^* \left( \begin{array}{cc} t_1 u_1 & 0 \\ 0 & (F_2 + H^\infty(\mathbb{D}, \mathbb{C}^{(m-2) \times (n-2)}) \cap B(t_1)) \end{array} \right) \hat{V}_1^*, \]
for some \( F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2) \times (n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2) \times (n-2)}) \) and for the closed ball \( B(t_1) \) of radius \( t_1 \) in \( L^\infty(\mathbb{T}, \mathbb{C}^{(m-2) \times (n-2)}) \). Thus, by Lemma 4.15, \( \mathcal{E}_1 \) admits the factorization (9.29) as claimed. \( \square \)

**Proposition 9.16.** Suppose that the function \( Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) minimizes
\[ (s_0^\infty(G - Q), s_1^\infty(G - Q)). \]
Then \( Q_2 \) satisfies
\[ (G - Q_2)x_0 = t_0 y_0, \quad (G - Q_2)^*y_0 = t_0 x_0 \]
and
\[ (G - Q_2)x_1 = t_1 y_1, \quad (G - Q_2)^*y_1 = t_1 x_1, \]
where \( x_0, x_1, y_0, y_1, t_0, t_1 \) are as in Theorem 9.1.

**Proof.** Let \((x_0, y_0)\) be a Schmidt pair for the Hankel operator \( H_G \) corresponding to \( \|H_G\| = t_0 \). Then, by Theorem 4.11, every \( Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) which is at minimal distance from \( G \) satisfies
\[ (G - Q_2)x_0 = t_0 y_0, \quad (G - Q_2)^*y_0 = t_0 x_0, \]
and, by Lemma 4.12,
\[ W_0(G - Q_2) V_0 = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F_1 \end{pmatrix}, \]
where \( F_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1)\times(n-1)}) \).

Moreover, by Lemma 4.15, the set \( \mathcal{E}_0 = \{ G - Q : Q \in \Omega_0 \} \) of all level 0 superoptimal error functions satisfies
\[ W_0 \mathcal{E}_0 V_0 = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F_1 + H^\infty(\mathbb{D}, \mathbb{C}^{m-1\times n-1}) \end{pmatrix} \cap B(t_0). \tag{9.33} \]

Suppose \( Q_2 \in \Omega_0 \). Then
\[ W_0(G - Q_2) V_0 = \begin{pmatrix} \eta_0 \tilde{\beta}_0^T \\ \beta_0^T \end{pmatrix} (G - Q_2) (\xi_0 \alpha_0) = \begin{pmatrix} \eta_0^T (G - Q_2) \xi_0 & \eta_0^T (G - Q_2) \tilde{\alpha}_0 \\ \beta_0^T (G - Q_2) \tilde{\alpha}_0 & \beta_0^T (G - Q_2) \alpha_0 \end{pmatrix}. \]

By equation (9.33), for \( \tilde{Q}_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}) \) at minimal distance from \( F_1 \),
\[ \begin{pmatrix} \eta_0^T (G - Q_2) \xi_0 & \eta_0^T (G - Q_2) \tilde{\alpha}_0 \\ \beta_0^T (G - Q_2) \tilde{\alpha}_0 & \beta_0^T (G - Q_2) \alpha_0 \end{pmatrix} = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F_1 - \tilde{Q}_1 \end{pmatrix}. \tag{9.34} \]

Note that, by Theorem 4.2,
\[ \| F_1 - \tilde{Q}_1 \|_\infty = \| H_{F_1} \|, \]
and, by Theorem 9.1 (part (v)), \( \| H_{F_1} \| = t_1 \).

Consideration of the (2,2) entries of equation (9.34) yields
\[ F_1 - \tilde{Q}_1 = \beta_0^* (G - Q_2) \tilde{\alpha}_0. \tag{9.35} \]

Note that, if \((\tilde{x}_1, \tilde{y}_1)\) is a Schmidt pair for \( H_{F_1} \) corresponding to \( t_1 = \| H_{F_1} \| \), then, by Theorem 4.11,
\[ (F_1 - \tilde{Q}_1) \tilde{x}_1 = t_1 \tilde{y}_1, \quad (F - \tilde{Q}_1)^* \tilde{y}_1 = t_1 \tilde{x}_1. \]

In view of equation (9.35), the latter equations imply
\[ \beta_0^*(G - Q_2) \tilde{\alpha}_0 \tilde{x}_1 = t_1 \tilde{y}_1 \tag{9.36} \]
and
\[ \alpha_0^T(G - Q_2)^* \beta_0 \tilde{y}_1 = t_1 \tilde{x}_1. \tag{9.37} \]

By Lemma 9.12, we may choose the Schmidt pair for \( H_{F_1} \) corresponding to \( \| H_{F_1} \| \) to be
\[ \tilde{x}_1 = \alpha_0^T x_1, \quad \tilde{y}_1 = \beta_0^* y_1. \tag{9.38} \]

Recall that, by equations (9.18),
\[ x_1 = \alpha_0 \alpha_0^T x_1 \tag{9.39} \]
and
\[ y_1 = \beta_0 \beta_0^* y_1. \tag{9.40} \]

In view of equations (9.36) and (9.38), we obtain
\[ \beta_0^*(G - Q_2) \alpha_0 \alpha_0^T x_1 = t_1 \beta_0 \beta_0^* y_1. \]

Multiplying both sides of the latter equation by \( \beta_0 \), we have
\[ \beta_0 \beta_0^*(G - Q_2) \alpha_0 \alpha_0^T x_1 = t_1 \beta_0 \beta_0^* y_1, \]
which, by equation (9.39), implies
\[ \beta_0 \beta_0^* (G - Q_2) x_1 = t_1 \beta_0^* y_1, \]
or equivalently,
\[ \beta_0 \beta_0^* ((G - Q_2) x_1 - t_1 y_1) = 0. \]

Since, by Theorem 9.1, \( U_2^* = M_{\beta_0 \beta_0^*} \) is unitary, the latter equation yields
\[ (G - Q_2) x_1 = t_1 y_1. \]

Moreover, by equations (9.37) and (9.38), we obtain
\[ \alpha_0^T (G - Q_2)^* \beta_0 \beta_0^* y_1 = t_1 \alpha_0^T x_1. \]
Multiplying both sides of the latter equation by \( \bar{\alpha}_0 \), we have
\[ \bar{\alpha}_0 \alpha_0^T (G - Q_2)^* \beta_0 \beta_0^* y_1 = t_1 \bar{\alpha}_0 \alpha_0^T x_1. \]

In view of equation (9.40), the latter expression is equivalent to the equation
\[ \bar{\alpha}_0 \alpha_0^T (G - Q_2)^* y_1 = t_1 \bar{\alpha}_0 \alpha_0^T x_1, \]
or equivalently,
\[ \bar{\alpha}_0 \alpha_0^T ((G - Q_2)^* y_1 - t_1 x_1) = 0. \]

Since, by Theorem 9.1, \( U_1^* = M_{\bar{\alpha}_0 \alpha_0^*} \) is unitary, the latter equation yields
\[ (G - Q_2)^* y_1 = t_1 x_1. \]

Therefore, \( Q_2 \) satisfies the required equations. \( \square \)

The next few propositions are in preparation for Theorem 9.26 on the compactness of \( T_2 \).

**Proposition 9.17.** For a thematic completion of the inner matrix-valued function \( \beta_0^T \eta_1 \) of the form \( \bar{W}_1^T = (\beta_0^T \eta_1 \beta_1) \), where \( \beta_1 \) is an inner, co-outer, quasi-continuous function of type \( (m - 1) \times (m - 2) \), the following equation holds
\[ \beta_1^T H^2(\mathbb{D}, \mathbb{C}^{m-1}) = H^2(\mathbb{D}, \mathbb{C}^{m-2}) \]

**Proof.** By virtue of the fact that complex conjugation is a unitary operator on \( L^2(\mathbb{T}, \mathbb{C}^m) \), an equivalent statement is that \( \beta_1^T z H^2(\mathbb{D}, \mathbb{C}^{m-1}) = z H^2(\mathbb{D}, \mathbb{C}^{m-2}) \). By Lemma 4.18, there exists a matrix-valued function \( B_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2) \times (m-1)}) \) such that
\[ B_1 \beta_1 = I_{m-2} \]
or, equivalently,
\[ \beta_1^T B_1^T = I_{m-2}. \]

Let \( f \in z H^2(\mathbb{D}, \mathbb{C}^{m-2}) \). Then,
\[ f = (\beta_1^T B_1^T) f \in \beta_1^T B_1^T z H^2(\mathbb{D}, \mathbb{C}^{m-2}) \subseteq \beta_1^T z H^2(\mathbb{D}, \mathbb{C}^{m-1}). \]

Hence,
\[ z H^2(\mathbb{D}, \mathbb{C}^{m-2}) \subseteq \beta_1^T z H^2(\mathbb{D}, \mathbb{C}^{m-1}). \]
Note that, since \( \beta_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1) \times (m-2)}) \), we have
\[ \beta_1^T z H^2(\mathbb{D}, \mathbb{C}^{m-1}) \subseteq z H^2(\mathbb{D}, \mathbb{C}^{m-2}). \]
Thus,

\[ \beta_1^T z H^2(\mathbb{D}, \mathbb{C}^{m-1}) = z H^2(\mathbb{D}, \mathbb{C}^{m-2}). \]

**Lemma 9.18.** For a thematic completion of the inner matrix-valued function \( \alpha_0^T \xi_1 \) of the form \( \tilde{V}_1 = (\alpha_0^T \xi_1 \tilde{\alpha}_1) \), where \( \alpha_1 \) is an inner, co-outer, quasi-continuous function of type \((n-1) \times (n-2)\), the following equation holds:

\[ \alpha_1^T H^2(\mathbb{D}, \mathbb{C}^{n-1}) = H^2(\mathbb{D}, \mathbb{C}^{n-2}). \]

**Proof.** By Lemma 4.18, for the given \( \alpha_1 \), there exists \( A_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(n-2) \times (n-1)}) \) such that \( A_1 \alpha_1 = I_{n-2} \). Equivalently, \( \alpha_1^T A_1^T = I_{n-2} \).

Thus, by Proposition 5.1, the set \( \{ \xi_0(\xi_1), \xi_1(\xi_1) \} \) is orthonormal in \( \mathbb{C}^m \) for almost every \( z \in \mathbb{T} \).

Let us state certain identities that are useful for the next statements.

**Remark 9.19.** Let \( V_0 \) and \( \tilde{V}_1 \) be given by equations (4.11) and (9.27), respectively, and let \( V_1 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{V}_1 \end{pmatrix} \). Since \( V_0 \), \( \tilde{V}_1 \) and \( V_1 \) are unitary-valued, we have

\[ I_n = V_0 V_0^* = \xi_0 \xi_0^* + \alpha_0 \alpha_0^T, \quad \text{(9.41)} \]

\[ I_{n-1} = \tilde{V}_1 \tilde{V}_1^* = \alpha_0^T \xi_1 \xi_1^* \bar{\alpha}_0 + \bar{\alpha}_1 \alpha_1^T. \quad \text{(9.42)} \]

**Lemma 9.20.** Let \( V_0 \) and \( \tilde{V}_1 \) be given by equations (4.11) and (9.27), respectively. Let \( A_1 = \alpha_0 \alpha_1 \). Then

\[ I_n - \xi_0 \xi_0^* - \xi_1 \xi_1^* = \bar{\alpha}_0 \alpha_1 \alpha_1^T \alpha_0^T = \bar{A}_1 A_1^T. \quad \text{(9.43)} \]

Thus, almost everywhere on \( \mathbb{T} \).

**Proof.** By equation (9.42)

\[ \bar{\alpha}_1 \alpha_1^T = I_{n-1} - \alpha_0^T \xi_1 \xi_1^* \bar{\alpha}_0, \]

thus

\[ \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T = \bar{\alpha}_0 (I_{n-1} - \alpha_0^T \xi_1 \xi_1^* \bar{\alpha}_0) \alpha_0^T. \]

By equation (9.41),

\[ \bar{\alpha}_0 \alpha_0^T = I_n - \xi_0 \xi_0^*. \]

Hence,

\[ \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T = (I_n - \xi_0 \xi_0^*) - (I_n - \xi_0 \xi_0^*) \xi_1 \xi_1^* (I_n - \xi_0 \xi_0^*). \]

Since, by Proposition 5.1, the set \( \{ \xi_0(z), \xi_1(z) \} \) is orthonormal in \( \mathbb{C}^m \) for almost every \( z \in \mathbb{T} \),

\[ \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T = I_n - \xi_0 \xi_0^* - \xi_1 \xi_1^* \]

almost everywhere on \( \mathbb{T} \).

Let us state certain identities that are useful for the next statements.
Remark 9.21. Let $W_0$ and $\tilde{W}_1$ be given by equations (4.11) and (9.28), respectively, and let $W_1 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1 \end{pmatrix}$. Then

\[
I_m = W_0^*W_0 = \tilde{\eta}_0\eta_0^T + \beta_0\beta_0^*, \quad (9.44)
\]

\[
I_{m-1} = \tilde{W}_1^*\tilde{W}_1 = \beta_0^*\tilde{\eta}_1\eta_1^T\beta_0 + \beta_1\beta_1^*. \quad (9.45)
\]

\[
W_0^* \begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1^* \end{pmatrix} = (\tilde{\eta}_0 \beta_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \beta_0^* \tilde{\eta}_1 \\ \beta_1 \end{pmatrix} = (\tilde{\eta}_0 \beta_0\beta_0^*\tilde{\eta}_1 \beta_0\beta_1). \quad (9.46)
\]

\[
W_0^* \begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1^* \end{pmatrix} W_0 = (\tilde{\eta}_0 \beta_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \beta_0^* \tilde{\eta}_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \eta_0^T \\ \beta_1^* \eta_0^T \beta_0^* \beta_1 \end{pmatrix}
\]

\[
= \tilde{\eta}_0\eta_0^T + \beta_0\beta_0^*\tilde{\eta}_1\eta_1^T\beta_0 + \beta_0\beta_0^*\beta_1^*\beta_0^*.
\]

Furthermore,

\[
\tilde{\eta}_0\eta_0^T + \beta_0\beta_0^*\tilde{\eta}_1\eta_1^T\beta_0 + \beta_0\beta_0^*\beta_1^*\beta_0^* = \tilde{\eta}_0\eta_0^T + \beta_0(I_{m-1} - \beta_1\beta_1^* + \beta_1\beta_1^*)\beta_0^* = I_m. \quad (9.47)
\]

Equations (9.44) and (9.45) follow from the facts that $W_0, \tilde{W}_1$ and $W_1$ are unitary-valued on $\mathbb{T}$. Equations (9.47) follow from equations (9.44) and (9.45).

Lemma 9.22. Let $W_0$ and $\tilde{W}_1$ be given by equations (4.11) and (9.28), respectively. Let $B_1 = \beta_0\beta_1$. Then

\[
I_m - \tilde{\eta}_0\eta_0^T - \tilde{\eta}_1\eta_1^T = \beta_0\beta_1\beta_1^*\beta_0^* = B_1B_1^*. \quad (9.48)
\]

almost everywhere on $\mathbb{T}$.

Proof. By equation (9.45)

\[
\beta_1\beta_1^* = I_{m-1} - \beta_0\tilde{\eta}_1\eta_1^T\beta_0,
\]

thus

\[
\beta_0\beta_1\beta_1^*\beta_0^* = \beta_0(I_{m-1} - \beta_0^*\tilde{\eta}_1\eta_1^T\beta_0)\beta_0^*.
\]

By equation (9.44),

\[
\beta_0\beta_0^* = I_m - \tilde{\eta}_0\eta_0^T.
\]

Hence,

\[
\beta_0\beta_1\beta_1^*\beta_0^* = (I_m - \tilde{\eta}_0\eta_0^T) - (I_m - \tilde{\eta}_0\eta_0^T)\tilde{\eta}_1\eta_1^T(I_m - \tilde{\eta}_0\eta_0^T).
\]

Since, by Proposition 5.1, the set $\{\tilde{\eta}_0(z), \tilde{\eta}_1(z)\}$ is orthonormal in $\mathbb{C}^m$ for almost every $z \in \mathbb{T}$,

\[
\beta_0\beta_1\beta_1^*\beta_0^* = I_m - \tilde{\eta}_0\eta_0^T - \tilde{\eta}_1\eta_1^T
\]

almost everywhere on $\mathbb{T}$. 

□
PROPOSITION 9.23. With the notation of Proposition 9.15, let unitary completions of $\xi_0$ and $\alpha_0^T \xi_1$ be given by

$$V_0 = (\xi_0 \alpha_0), \quad \hat{V}_1 = (\alpha_0^T \xi_1 \alpha_1),$$

where $\alpha_0, \alpha_1$ are inner, co-outer, quasi-continuous matrix-valued functions of types $n \times (n - 1)$ and $(n - 1) \times (n - 2)$, respectively. Let

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & \hat{V}_1 \end{pmatrix}$$

and let

$$K_2 = V_0 V_1 \begin{pmatrix} 0_{2 \times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{n-2}) \end{pmatrix}.$$  \hspace{1cm} (9.49)

Then

$$\xi_0 \hat{\xi}_1 \hat{H}^2(\mathbb{D}, \mathbb{C}^n) = \xi_0 \hat{\xi}_1 \hat{K}_2$$

and the operator $(\xi_0 \hat{\xi}_1 \hat{\cdot}) : K_2 \to \xi_0 \hat{\xi}_1 \hat{H}^2(\mathbb{D}, \mathbb{C}^n)$ is unitary.

Proof. First let us prove that

$$\xi_0 \hat{\xi}_1 \hat{H}^2(\mathbb{D}, \mathbb{C}^n) = \xi_0 \hat{\xi}_1 \hat{K}_2.$$  

Recall that $A_1 = \alpha_0 \alpha_1$. Observe that, by definition,

$$K_2 = \alpha_0 \alpha_1 H^2(\mathbb{D}, \mathbb{C}^{n-2}) = \hat{A}_1 H^2(\mathbb{D}, \mathbb{C}^{n-2}).$$  \hspace{1cm} (9.50)

By Lemmas 9.4 and 9.18,

$$H^2(\mathbb{D}, \mathbb{C}^{n-2}) = \alpha_1^T H^2(\mathbb{D}, \mathbb{C}^{n-1})$$

$$= \alpha_1^T \alpha_2^T H^2(\mathbb{D}, \mathbb{C}^{n})$$

$$= A_1^T H^2(\mathbb{D}, \mathbb{C}^{n}).$$  \hspace{1cm} (9.51)

By equations (9.50) and (9.51),

$$K_2 = \alpha_0 \alpha_1 H^2(\mathbb{D}, \mathbb{C}^{n-2}) = \hat{A}_1 A_1^T H^2(\mathbb{D}, \mathbb{C}^{n}).$$  \hspace{1cm} (9.52)

By Lemma 9.20,

$$\hat{A}_1 A_1^T = I_n - \sum_{k=0}^1 \xi_k \xi_k^*.$$  \hspace{1cm} (9.53)

By Proposition 5.1, $\{\xi_i(z)\}_{i=0}^1$ is an orthonormal set in $\mathbb{C}^n$ for almost every $z \in \mathbb{T}$. Therefore, by equations (9.52) and (9.53),

$$\xi_0 \hat{\xi}_1 \hat{K}_2 = \xi_0 \hat{\xi}_1 \hat{\hat{A}_1 A_1^T H^2(\mathbb{D}, \mathbb{C}^{n})}$$

$$= \xi_0 \hat{\xi}_1 \hat{(I_n - \sum_{k=0}^1 \xi_k \xi_k^*) H^2(\mathbb{D}, \mathbb{C}^{n})}$$

$$= \xi_0 \hat{\xi}_1 \hat{H^2(\mathbb{D}, \mathbb{C}^{n})}.$$  \hspace{1cm} (9.54)

Let us show that the operator $(\xi_0 \hat{\xi}_1 \hat{\cdot}) : K_2 \to \xi_0 \hat{\xi}_1 \hat{H^2(\mathbb{D}, \mathbb{C}^{n})}$ is unitary. The foregoing paragraph asserts that the operator is surjective. It remains to be shown that it is an isometry. To this end, let $f \in K_2$. Then

$$\| \xi_0 \hat{\xi}_1 \hat{f} \|_{L^2(\mathbb{T}, \lambda^3 \mathbb{C}^n)}^2 = \langle \xi_0 \hat{\xi}_1 \hat{f}, \xi_0 \hat{\xi}_1 \hat{f} \rangle _{L^2(\mathbb{T}, \lambda^3 \mathbb{C}^n)}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \langle \xi_0(e^{i\theta}) \hat{\xi}_1(e^{i\theta}) \hat{f}(e^{i\theta}), \xi_0(e^{i\theta}) \hat{\xi}_1(e^{i\theta}) \hat{f}(e^{i\theta}) \rangle _{\lambda^3 \mathbb{C}^n} \ d\theta.$$
By Proposition 3.12, the latter integral is equal to

$$\frac{1}{2\pi} \int_0^{2\pi} \text{det} \begin{pmatrix}
\langle \xi_0(e^{i\theta}), \xi_0(e^{i\theta}) \rangle_{C^n} & \langle \xi_0(e^{i\theta}), \xi_1(e^{i\theta}) \rangle_{C^n} & \langle \xi_0(e^{i\theta}), f(e^{i\theta}) \rangle_{C^n} \\
\langle \xi_1(e^{i\theta}), \xi_0(e^{i\theta}) \rangle_{C^n} & \langle \xi_1(e^{i\theta}), \xi_1(e^{i\theta}) \rangle_{C^n} & \langle \xi_1(e^{i\theta}), f(e^{i\theta}) \rangle_{C^n} \\
\langle f(e^{i\theta}), \xi_0(e^{i\theta}) \rangle_{C^n} & \langle f(e^{i\theta}), \xi_1(e^{i\theta}) \rangle_{C^n} & \langle f(e^{i\theta}), f(e^{i\theta}) \rangle_{C^n}
\end{pmatrix} \, d\theta.$$ 

Note that, by Proposition 5.1, \{\xi_0(e^{i\theta}), \xi_1(e^{i\theta})\} is an orthonormal set for almost all \(e^{i\theta}\) on \(\mathbb{T}\). Moreover, since \(\mathcal{K}_2 = \hat{\alpha}_0\hat{\alpha}_1 H^2(\mathbb{D}, \mathbb{C}^{n-2})\), then \(f = \hat{\alpha}_0\hat{\alpha}_1 \varphi\) for some \(\varphi \in H^2(\mathbb{D}, \mathbb{C}^{n-2})\). Hence,

$$\langle \xi_0(e^{i\theta}), f(e^{i\theta}) \rangle_{C^n} = \langle \xi_0(e^{i\theta}), \hat{\alpha}_0(e^{i\theta})\hat{\alpha}_1(e^{i\theta})\varphi(e^{i\theta}) \rangle_{C^n}$$

$$= \langle \alpha_0^T(e^{i\theta})\xi_0(e^{i\theta}), \hat{\alpha}_1(e^{i\theta})\varphi(e^{i\theta}) \rangle_{C^{n-1}} = 0$$

almost everywhere on \(\mathbb{T}\), since \(V_0\) is unitary-valued. Similarly, since \(\hat{V}_1\) is unitary valued, we deduce that

$$\langle \xi_1(e^{i\theta}), f(e^{i\theta}) \rangle_{C^n} = \langle \alpha_1^T(e^{i\theta})\alpha_0^T(e^{i\theta})\xi_1(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{C^{n-2}} = 0$$

almost everywhere on \(\mathbb{T}\). Therefore,

$$\|\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} f\|_{L^2(\mathbb{T}, \hat{\lambda}^{3} C^n)}^2 = \frac{1}{2\pi} \int_0^{2\pi} \text{det} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \|f(e^{i\theta})\|_{C^n}^2
\end{pmatrix} \, d\theta = \|f\|_{L^2(\mathbb{T}, C^n)}^2,$$

that is, \((\xi_0 \hat{\lambda} \xi_1 \hat{\lambda}, \cdot) : \mathcal{K}_2 \to \xi_0 \hat{\lambda} \xi_1 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n)\) is an isometric operator. Thus, the operator \((\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} : \mathcal{K}_2 \to \xi_0 \hat{\lambda} \xi_1 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n)\) is unitary. \(\square\)

**Proposition 9.24.** Let \(\eta_0, \eta_1\) be defined by equations (4.17) and (4.28), respectively, and let \(\beta_0, \beta_1\) be inner, co-outer, quasi-continuous functions of types \(m \times (m - 1)\) and \((m - 1) \times (m - 2)\), respectively, such that the functions

\[
W_0^T = (\eta_0, \beta_0), \quad \hat{W}_1^T = (\beta_0^T \eta_1, \beta_1)
\]

are unitary-valued. Let

\[
W_1^T = \begin{pmatrix} 1 & 0 \\ 0 & \hat{W}_1^T \end{pmatrix}
\]

and let

\[
\mathcal{L}_2 = W_0^* W_1^* \left( H^2(\mathbb{D}, \mathbb{C}^{m-2}) \right)'.
\]

Then

\[
\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} \mathcal{L}_2 = \eta_0 \hat{\lambda} \eta_1 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^m)',
\]

and the operator \((\eta_0 \hat{\lambda} \eta_1 \hat{\lambda}, \cdot) : \mathcal{L}_2 \to \eta_0 \hat{\lambda} \eta_1 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^m)'\) is unitary.

**Proof.** First let us prove that

\[
\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} \mathcal{L}_2 = \eta_0 \hat{\lambda} \eta_1 \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^m)'.
\]

Let \(B_1 = \beta_0 \beta_1\). By equations (9.55) and (9.46),

\[
\mathcal{L}_2 = B_1 H^2(\mathbb{D}, \mathbb{C}^{m-2})'.
\]
By Lemmas 9.9 and 9.17,
\[
H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp = \beta_1^*(H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp = \beta_1^* \beta_0^*(H^2(\mathbb{D}, \mathbb{C}^m)^\perp = B_1^*H^2(\mathbb{D}, \mathbb{C}^m)^\perp.
\]
(9.58)

By equations (9.57) and (9.58),
\[
\mathcal{L}_2 = B_1H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp = B_1B_1^*H^2(\mathbb{D}, \mathbb{C}^m)^\perp.
\]
(9.59)

By Lemma 9.22,
\[
B_1B_1^* = I_m - \sum_{i=0}^{1} \tilde{\eta}_i \eta_i^T.
\]
(9.60)

Thus,
\[
\mathcal{L}_2 = (I_m - \sum_{i=0}^{1} \tilde{\eta}_i \eta_i^T)H^2(\mathbb{D}, \mathbb{C}^m)^\perp.
\]
(9.61)

By Proposition 5.1, \( \{\tilde{\eta}_i(z)\}_{i=0}^{1} \) is an orthonormal set in \( \mathbb{C}^m \) for almost every \( z \in \mathbb{T} \). Therefore, by equations (9.59) and (9.61),
\[
\tilde{\eta}_0 \check{\wedge} \tilde{\eta}_1 \check{\wedge} \mathcal{L}_2 = \tilde{\eta}_0 \check{\wedge} \tilde{\eta}_1 \check{\wedge} (I_m - \sum_{i=0}^{1} \tilde{\eta}_i \eta_i^T)H^2(\mathbb{D}, \mathbb{C}^m)^\perp = \tilde{\eta}_0 \check{\wedge} \tilde{\eta}_1 \check{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp.
\]
(9.62)

To complete the proof, let us show that the operator
\[
(\tilde{\eta}_0 \check{\wedge} \tilde{\eta}_1 \check{\wedge} \cdot) : \mathcal{L}_2 \to \tilde{\eta}_0 \check{\wedge} \tilde{\eta}_1 \check{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp
\]
is unitary. Observe that the foregoing paragraph asserts the operator is surjective. Hence, it suffices to prove that it is an isometry. To this end, let \( v \in \mathcal{L}_2 \). Then
\[
\|\tilde{\eta}_0 \check{\wedge} \tilde{\eta}_1 \check{\wedge} v\|^2_{L^2(\mathbb{T}, \Lambda^3 \mathbb{C}^m)} = \langle \tilde{\eta}_0 \check{\wedge} \tilde{\eta}_1 \check{\wedge} v, \tilde{\eta}_0 \check{\wedge} \tilde{\eta}_1 \check{\wedge} v \rangle_{L^2(\mathbb{T}, \Lambda^3 \mathbb{C}^m)},
\]
and, by Proposition 3.12,
\[
\langle \tilde{\eta}_0 \check{\wedge} \tilde{\eta}_1 \check{\wedge} v, \tilde{\eta}_0 \check{\wedge} \tilde{\eta}_1 \check{\wedge} v \rangle_{L^2(\mathbb{T}, \Lambda^3 \mathbb{C}^m)}
= \frac{1}{2\pi} \int_0^{2\pi} \det \left( \begin{array}{ccc} \langle \tilde{\eta}_0(e^{i\theta}), \tilde{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \tilde{\eta}_0(e^{i\theta}), \tilde{\eta}_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \tilde{\eta}_0(e^{i\theta}), v(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle \tilde{\eta}_1(e^{i\theta}), \tilde{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \tilde{\eta}_1(e^{i\theta}), \tilde{\eta}_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \tilde{\eta}_1(e^{i\theta}), v(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle v(e^{i\theta}), \tilde{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle v(e^{i\theta}), \tilde{\eta}_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle v(e^{i\theta}), v(e^{i\theta}) \rangle_{\mathbb{C}^m} \end{array} \right) d\theta.
\]
Note that, by Proposition 5.1, \( \{\tilde{\eta}_0(e^{i\theta}), \tilde{\eta}_1(e^{i\theta})\} \) is an orthonormal set almost everywhere on \( \mathbb{T} \). Further, since \( \mathcal{L}_2 = \beta_0 \beta_1 H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp \), \( v = \beta_0 \beta_1 \varphi \) for some \( \varphi \in H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp \). Hence,
\[
\langle \tilde{\eta}_0(e^{i\theta}), v(e^{i\theta}) \rangle_{\mathbb{C}^m} = \langle \tilde{\eta}_0(e^{i\theta}), \beta_0(e^{i\theta}) \beta_1(e^{i\theta}) \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m} = \langle \beta_0^*(e^{i\theta}) \tilde{\eta}_0(e^{i\theta}), \beta_1(e^{i\theta}) \varphi(e^{i\theta}) \rangle_{\mathbb{C}^{m-1}} = 0,
\]
since \( W_0^T \) is unitary-valued almost everywhere on \( \mathbb{T} \). Similarly, since, by Proposition 9.15, \( \hat{\mathcal{W}}_1^T \) is unitary-valued almost everywhere on \( \mathbb{T} \), we obtain
\[
\langle \tilde{\eta}_1(e^{i\theta}), v(e^{i\theta}) \rangle_{\mathbb{C}^m} = \langle \beta_1^*(e^{i\theta}) \beta_0^*(e^{i\theta}) \tilde{\eta}_1(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^{m-2}} = 0.
\]
Therefore,
\[
\|\tilde{\eta}_0 \check{\wedge} \tilde{\eta}_1 \check{\wedge} v\|^2_{L^2(\mathbb{T}, \Lambda^3 \mathbb{C}^m)} = \frac{1}{2\pi} \int_0^{2\pi} \det \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \|v(e^{i\theta})\|^2_{\mathbb{C}^m} \end{array} \right) d\theta = \|v\|^2_{L^2(\mathbb{T}, \mathbb{C}^m)},
\]

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that is, the operator \((\eta_0 \wedge \eta_1 \wedge \cdot) : L_2 \to \eta_0 \wedge \eta_1 \wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp\) is an isometry. Thus, the operator is unitary.

**Proposition 9.25.** Let \(\eta_0, \eta_1\) be defined by equations (4.17) and (4.28), respectively, and let \(\beta_0, \beta_1\) be inner, co-outer, quasi-continuous functions of types \(m \times (m - 1)\) and \((m - 1) \times (m - 2)\), respectively, such that the functions

\[
W_0^T = (\eta_0 \quad \beta_0), \quad \tilde{W}_1^T = (\beta_0^T \eta_1 \quad \beta_1)
\]

are unitary-valued. Let

\[
L_2 = W_0^* \begin{pmatrix} 1 & 0 \\ 0 & W_1^* \end{pmatrix} \begin{pmatrix} 0_{2 \times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp \end{pmatrix}.
\]

Then

\[
L_2^\perp = \{ f \in L^2(T, \mathbb{C}^m) : \beta_1^* \beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-2}) \}.
\]

**Proof.** Clearly \(L_2 = \beta_0 \beta_1 H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp\). The general element of \(\beta_0 \beta_1 H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp\) is \(\beta_0 \beta_1 \tilde{g}\) with \(g \in H^2(\mathbb{D}, \mathbb{C}^{m-2})\). A function \(f \in L^2(T, \mathbb{C}^m)\) belongs to \(L_2^\perp\) if and only if

\[
\langle f, \beta_0 \beta_1 \tilde{g} \rangle_{L^2(T, \mathbb{C}^m)} = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D}, \mathbb{C}^{m-2})
\]

if and only if

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \langle f(e^{i\theta}), \beta_0(e^{i\theta}) \beta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D}, \mathbb{C}^{m-2})
\]

if and only if

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \langle \beta_1^*(e^{i\theta}) \beta_0^*(e^{i\theta}) f(e^{i\theta}), e^{-i\theta} \tilde{g}(e^{i\theta}) \rangle_{\mathbb{C}^{m-2}} d\theta = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D}, \mathbb{C}^{m-2}),
\]

which in turn is equivalent to the assertion that \(\beta_1^* \beta_0^* f\) is orthogonal to \(H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp\) in \(L^2(T, \mathbb{C}^{m-2})\), which holds if and only if \(\beta_1 \beta_0 f\) belongs to \(H^2(\mathbb{D}, \mathbb{C}^{m-2})\). Thus,

\[
L_2^\perp = \{ f \in L^2(T, \mathbb{C}^m) : \beta_1^* \beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-2}) \}
\]

as required.

**Theorem 9.26.** Let \(m, n\) be positive integers such that \(\min(m, n) \geq 2\). Let \(G\) be in \(H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(T, \mathbb{C}^{m \times n})\). Let \((\xi_0 \wedge v_1, \tilde{\eta}_0 \wedge w_1)\) be a Schmidt pair for the operator \(T_1\), as given in equation (4.24), corresponding to \(t_1 = \|T_1\| \neq 0\), let \(h_1 \in H^2(\mathbb{D}, \mathbb{C})\) be the scalar outer factor of \(\xi_0 \wedge v_1\), let

\[
x_1 = (I_n - \xi_0 \xi_0^*) v_1, \quad y_1 = (I_m - \tilde{\eta}_0 \eta_0^T) w_1,
\]

and let

\[
\xi_1 = \frac{x_1}{h_1}, \quad \tilde{\eta}_1 = \frac{y_1}{h_1}.
\]

Let

\[
V_0 = (\xi_0 \quad \tilde{\eta}_0), \quad W_0^T = (\eta_0 \quad \beta_0)
\]

be given by equations (4.11), and let

\[
\tilde{V}_1 = (\alpha_0^T \xi_1 \quad \tilde{\alpha}_1), \quad \tilde{W}_1^T = (\beta_0^T \eta_1 \quad \beta_1)
\]

be given by equations (9.27) and (9.28), respectively. Let

\[
X_2 = \xi_0 \wedge \xi_1 \wedge H^2(\mathbb{D}, \mathbb{C}), \quad Y_2 = \tilde{\eta}_0 \wedge \tilde{\eta}_1 \wedge H^2(\mathbb{D}, \mathbb{C})^\perp,
\]
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\[ K_2 = V_0 \begin{pmatrix} 1 & 0 \\ 0 & \tilde{V}_1 \end{pmatrix} H^2(\mathbb{D}, \mathbb{C}^{n-2}) \]  
\[ L_2 = W_0^* \begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1 \end{pmatrix} H^2(\mathbb{D}, \mathbb{C}^{n-2}). \]  
(9.63)

Consider the operator \( T_2: X_2 \to Y_2 \) given by

\[ T_2(\xi_0 \wedge \xi_1 \lambda x) = P_{Y_2}(\tilde{\eta}_0 \wedge \tilde{\eta}_1 \lambda (G - Q_2)x), \]  
(9.64)

where \( Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) satisfies

\[ (G - Q_2)x_i = t_i y_i, \quad (G - Q_2)y_i = t_i x_i \quad \text{for} i = 0, 1. \]  
(9.65)

Let the operator \( \Gamma_2: K_2 \to L_2 \) be given by \( \Gamma_2 = P_{L_2}M_{G - Q_2}|_{K_2}. \) Then

(i) the maps

\[ M_{\bar{\alpha}_0 \bar{\alpha}_1}: H^2(\mathbb{D}, \mathbb{C}^{n-2}) \to K_2: x \mapsto \bar{\alpha}_0 \bar{\alpha}_1 x \]

and

\[ M_{\bar{\beta}_0 \bar{\beta}_1}: H^2(\mathbb{D}, \mathbb{C}^{n-2}) \to L_2: y \mapsto \bar{\beta}_0 \bar{\beta}_1 y \]

are unitaries;

(ii) the maps \( (\xi_0 \wedge \xi_1 \lambda): K_2 \to X_2, \quad (\tilde{\eta}_0 \wedge \tilde{\eta}_1 \lambda): L_2 \to Y_2 \) are unitaries;

(iii) the following diagram commutes:

\[ \begin{array}{ccc}
H^2(\mathbb{D}, \mathbb{C}^{n-2}) & \xrightarrow{M_{\bar{\alpha}_0 \bar{\alpha}_1}} & K_2 \\
\downarrow H_{F_2} & & \downarrow \Gamma_2 \\
H^2(\mathbb{D}, \mathbb{C}^{n-2}) & \xrightarrow{M_{\bar{\beta}_0 \bar{\beta}_1}} & L_2 \\
\downarrow & & \downarrow \\
H^2(\mathbb{D}, \mathbb{C}^{m-2}) & \xrightarrow{\xi_0 \wedge \xi_1 \lambda} & H^2(\mathbb{D}, \mathbb{C}^{n}) = X_2 \\
\end{array} \]

(9.66)

where \( F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2) \times (n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2) \times (n-2)}) \) is the function defined in Proposition 9.15;

(iv) \( T_2 \) is a compact operator;

(v) \( \|T_2\| = \|\Gamma_2\| = \|H_{F_2}\| = t_2. \)

**Proof.** (i) Follows from Lemma 4.16.

(ii) Follows from Propositions 9.23 and 9.24.

(iii) By Proposition 8.1, \( T_2 \) is well defined and is independent of the choice of \( Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) satisfying equations (9.65). We can choose \( Q_2 \) which minimizes

\[ (s_0^\infty(G - Q), s_1^\infty(G - Q)), \]

and therefore satisfies equations (9.65). By Lemma 4.17 and Theorem 4.11, the left-hand side of diagram (9.66) commutes. Let us show the right-hand side also commutes. A typical element of \( K_2 \) is of the form \( \bar{\alpha}_0 \bar{\alpha}_1 x \) where \( x \in H^2(\mathbb{D}, \mathbb{C}^{n-2}). \) Then, by equation (9.64),

\[ T_2(\xi_0 \wedge \xi_1 \lambda \bar{\alpha}_0 \bar{\alpha}_1 x) = P_{Y_2}(\tilde{\eta}_0 \wedge \tilde{\eta}_1 \lambda (G - Q_2)\bar{\alpha}_0 \bar{\alpha}_1 x). \]

By Proposition 9.15, every \( Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) which minimizes \( (s_0^\infty(G - Q), s_1^\infty(G - Q)) \) satisfies the following equation (see equation (9.29)):

\[ (G - Q_2)V_0 \begin{pmatrix} 1 & 0 \\ 0 & \tilde{V}_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = W_0^* \begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1 \end{pmatrix} \begin{pmatrix} 0 \\ F_2 x \end{pmatrix} \]

(9.67)

for some \( F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2) \times (n-2)}) + C(\mathbb{D}, \mathbb{T}^{(m-2) \times (n-2)}). \) This implies that

\[ (G - Q_2)\bar{\alpha}_0 \bar{\alpha}_1 x = \bar{\beta}_0 \bar{\beta}_1 F_2 x, \]

(9.68)
for \( x \in H^2(\mathbb{D}, \mathbb{C}^{n-2}) \). Hence,
\[
T_2(\xi_0 \hat{\eta} \xi_1 \hat{\eta} \alpha_0 \alpha_1 x) = P_{Y_2}(\eta_0 \hat{\eta} \eta_1 \beta_0 \beta_1 F_2 x).
\] (9.69)

Furthermore,
\[
(\eta_0 \hat{\eta} \eta_1 \eta_\cdot) \Gamma_2(\alpha_0 \alpha_1 x) = \eta_0 \hat{\eta} \eta_1 \eta_\cdot P_{L_2}[(G - Q_2) \alpha_0 \alpha_1 x].
\]

Hence, by equation (9.68),
\[
(\eta_0 \hat{\eta} \eta_1 \eta_\cdot) \Gamma_2(\alpha_0 \alpha_1 x) = \eta_0 \hat{\eta} \eta_1 \eta_\cdot P_{L_2}(\beta_0 \beta_1 F_2 x).
\] (9.70)

To show commutativity of the right-hand square in the diagram (9.66), we need to prove that, for every \( x \in H^2(\mathbb{D}, \mathbb{C}^{n-2}) \),
\[
T_2(\xi_0 \hat{\eta} \xi_1 \hat{\eta} \alpha_0 \alpha_1 x) = (\eta_0 \hat{\eta} \eta_1 \eta_\cdot) \Gamma_2(\alpha_0 \alpha_1 x).
\] (9.71)

By equations (9.69) and (9.70), it is equivalent to show that
\[
P_{Y_2}(\eta_0 \hat{\eta} \eta_1 \beta_0 \beta_1 F_2 x) = \eta_0 \hat{\eta} \eta_1 \eta_\cdot P_{L_2}(\beta_0 \beta_1 F_2 x).
\] (9.72)

Therefore, we need to show that
\[
\eta_0 \hat{\eta} \eta_1 \eta_\cdot P_{L_2}(\beta_0 \beta_1 F_2 x) \in Y_2
\]
and that
\[
\eta_0 \hat{\eta} \eta_1 \beta_0 \beta_1 F_2 x - \eta_0 \hat{\eta} \eta_1 \eta_\cdot P_{L_2}(\beta_0 \beta_1 F_2 x)
\]
is orthogonal to \( Y_2 \).

By Proposition 9.24, \( \eta_0 \hat{\eta} \eta_1 \eta_\cdot P_{L_2}(\beta_0 \beta_1 F_2 x) \) is indeed an element of \( Y_2 \). Furthermore,
\[
\eta_0 \hat{\eta} \eta_1 \beta_0 \beta_1 F_2 x - \eta_0 \hat{\eta} \eta_1 \eta_\cdot P_{L_2}(\beta_0 \beta_1 F_2 x) = \eta_0 \hat{\eta} \eta_1 [\beta_0 \beta_1 F_2 x - P_{L_2}(\beta_0 \beta_1 F_2 x)]
\]
\[
= \eta_0 \hat{\eta} \eta_1 \eta_\cdot P_{L_2}^\perp(\beta_0 \beta_1 F_2 x).
\]

Let us show that \( \eta_0 \hat{\eta} \eta_1 \eta_\cdot P_{L_2}^\perp(\beta_0 \beta_1 F_2 x) \) is orthogonal to \( Y_2 \).

It is so if and only if
\[
\left< \eta_0 \hat{\eta} \eta_1 \eta_\cdot P_{L_2}^\perp(\beta_0 \beta_1 F_2 x), \eta_0 \hat{\eta} \eta_1 \eta_\cdot g \right>_{L^2(T, \alpha, \lambda, \mathbb{C}^m)} = 0 \quad \text{for every } g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp. \quad (9.73)
\]

Let \( \Phi = P_{L_2}^\perp(\beta_0 \beta_1 F_2 x) \in L^2(T, \mathbb{C}^m) \). By Proposition 9.25,
\[
\beta_0^* \beta_0^* \Phi \in H^2(\mathbb{D}, \mathbb{C}^{m-2}). \quad (9.74)
\]

Then, by Proposition 3.12, assertion (9.73) is equivalent to the following assertion:
\[
\frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix}
\langle \eta_0(e^{i\theta}), \eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \eta_0(e^{i\theta}), \eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} \\
\langle \eta_1(e^{i\theta}), \eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \eta_1(e^{i\theta}), \eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m}
\end{pmatrix}
\begin{pmatrix}
\langle \Phi(e^{i\theta}), \eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), \eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m} \\
\langle \Phi(e^{i\theta}), \eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), \eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m}
\end{pmatrix} d\theta = 0
\]

for every \( g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp \), which in turn, by Proposition 5.1, is equivalent to the assertion
\[
\frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\langle \Phi(e^{i\theta}), \eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), \eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m} \\
\langle \Phi(e^{i\theta}), \eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), \eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m}
\end{pmatrix} d\theta = 0
\]

for every \( g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp \). The latter statement is equivalent to the assertion
\[
\frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix}
\langle \Phi(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m} \\
-\langle \Phi(e^{i\theta}), \eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), \eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m}
\end{pmatrix}
d\theta = 0
\]
for every $g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$, which in turn is equivalent to the statement that
\[
\frac{1}{2\pi} \int_0^{2\pi} g^*(e^{i\theta})\Phi(e^{i\theta}) - g^*(e^{i\theta})\tilde{\eta}_0\tilde{\eta}_0^T(e^{i\theta})\Phi(e^{i\theta}) - g^*(e^{i\theta})\tilde{\eta}_1(e^{i\theta})\tilde{\eta}_1^T(e^{i\theta})\Phi(e^{i\theta}) \, d\theta = 0
\]
for every $g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. Equivalently
\[
\frac{1}{2\pi} \int_0^{2\pi} g^*(e^{i\theta})(I_m - \tilde{\eta}_0(e^{i\theta})\tilde{\eta}_0^T(e^{i\theta}) - \tilde{\eta}_1(e^{i\theta})\tilde{\eta}_1^T(e^{i\theta}))\Phi(e^{i\theta}) \, d\theta = 0
\]
for every $g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ if and only if
\[
(I_m - \tilde{\eta}_0\tilde{\eta}_0^T - \tilde{\eta}_1\tilde{\eta}_1^T)\Phi = \beta_0\beta_1\beta_0^\ast\Phi.
\]
By Lemma 9.22,
\[
(I_m - \tilde{\eta}_0\tilde{\eta}_0^T - \tilde{\eta}_1\tilde{\eta}_1^T)\Phi = \beta_0\beta_1\beta_0^\ast\Phi.
\]
Recall that, by assertions (7.44), $\beta_0^\ast\beta_0^\ast\Phi \in H^2(\mathbb{D}, \mathbb{C}^{m-2})$, and so
\[
\beta_0\beta_1\beta_0^\ast\beta_0^\ast\Phi \in H^2(\mathbb{D}, \mathbb{C}^m).
\]
Thus, the right-hand square in the diagram (9.66) commutes, and so the diagram (9.66) commutes.

(iv) By Proposition 9.15,
\[
F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(T, \mathbb{C}^{(m-2)\times(n-2)}).
\]
Thus, by Hartman’s theorem, the Hankel operator $H_{F_2}$ is compact. By (iii),
\[
(\tilde{\eta}_0\wedge\tilde{\eta}_1\wedge^\ast) \circ (M_{\beta_0\beta_1}H_{F_2}M_{\alpha_0\alpha_1}^T) \circ (\xi_0\wedge\xi_1\wedge^\ast)^* = T_2.
\]
By (i) and (ii), the operators $M_{\alpha_0\alpha_1}, M_{\beta_0\beta_1}, (\xi_0\wedge\xi_1\wedge^\ast)$ and $(\tilde{\eta}_0\wedge\tilde{\eta}_1\wedge^\ast)$ are unitaries. Hence, $T_2$ is a compact operator.

(v) Since diagram (9.66) commutes and the operators $M_{\alpha_0\alpha_1}, M_{\beta_0\beta_1}, (\xi_0\wedge\xi_1\wedge^\ast)$ and $(\tilde{\eta}_0\wedge\tilde{\eta}_1\wedge^\ast)$ are unitaries,
\[
\|T_2\| = \|\Gamma_2\| = \|H_{F_2}\| = t_2.
\]

**Lemma 9.27.** In the notation of Theorem 9.26, let $v_2 \in H^2(\mathbb{D}, \mathbb{C}^m)$ and $w_2 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ be such that $(\xi_0\wedge\xi_1\wedge v_2, \tilde{\eta}_0\wedge\tilde{\eta}_1\wedge w_2)$ is a Schmidt pair for the operator $T_2$ corresponding to $\|T_2\|$. Then

(i) there exist $x_2 \in \mathcal{K}_2$ and $y_2 \in \mathcal{L}_2$ such that $(x_2, y_2)$ is a Schmidt pair for the operator $\Gamma_2$;

(ii) for any $x_2 \in \mathcal{K}_2$ and $y_2 \in \mathcal{L}_2$ such that
\[
\xi_0\wedge\xi_1\wedge v_2 = \xi_0\wedge\xi_1\wedge y_2 = \tilde{\eta}_0\wedge\tilde{\eta}_1\wedge w_2,
\]
the pair $(x_2, y_2)$ is a Schmidt pair for $\Gamma_2$ corresponding to $\|\Gamma_2\|$.

**Proof.** (i) By Theorem 9.26, the diagram (9.66) commutes, $(\xi_0\wedge\xi_1\wedge^\ast)$ is unitary from $\mathcal{K}_2$ to $X_2$, and $(\tilde{\eta}_0\wedge\tilde{\eta}_1\wedge^\ast)$ is unitary from $\mathcal{L}_2$ to $Y_2$ and $\|\Gamma_2\| = \|T_2\| = t_2$. Moreover, by the commutativity of diagram (9.66), the operator $\Gamma_2 : \mathcal{K}_2 \to \mathcal{L}_2$ is compact, hence there exist $x_2 \in \mathcal{K}_2$, $y_2 \in \mathcal{L}_2$ such that $(x_2, y_2)$ is a Schmidt pair for $\Gamma_2$ corresponding to $\|\Gamma_2\| = t_2$. 

(ii) Suppose that \( x_2 \in \mathcal{K}_2, y_2 \in \mathcal{L}_2 \) satisfy
\[
\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} x_2 = \xi_0 \hat{\lambda} \xi_1 \hat{\lambda} v_2 \quad \text{and} \quad \eta_0 \hat{\lambda} \eta_1 \hat{\lambda} y_2 = \eta_0 \hat{\lambda} \eta_1 \hat{\lambda} w_2.
\]
(9.75)
Let us show that \((x_2, y_2)\) is a Schmidt pair for \( \Gamma_2 \), that is,
\[
\Gamma_2 x_2 = t_2 y_2, \quad \Gamma_2^* y_2 = t_2 x_2.
\]
Since diagram (9.66) commutes,
\[
T_2 \circ (\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} \cdot) = (\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} \cdot) \circ \Gamma_2, \quad (\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} \cdot)^* \circ T_2^* = \Gamma_2^* \circ (\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} \cdot)^*.
\]
By hypothesis,
\[
T_2(\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} v_2) = t_2 (\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} w_2), \quad T_2^*(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} w_2) = t_2 (\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} v_2).
\]
(9.78)
Thus, by equations (9.75), (9.76) and (9.78),
\[
\Gamma_2 x_2 = (\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} \cdot)^* T_2(\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} v_2)
= (\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} \cdot)^* t_2(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} w_2)
= t_2(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} \cdot)^*(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} y_2).
\]
Hence,
\[
\Gamma_2 x_2 = t_2(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} \cdot)^*(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} y_2) = t_2 y_2.
\]
By equation (9.75),
\[
x_2 = (\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} \cdot)^*(\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} v_2),
\]
and, by equation (9.76),
\[
(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} \cdot)^*(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} w_2) = y_2.
\]
Thus,
\[
\Gamma_2^* y_2 = (\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} \cdot)^* T_2^*(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} w_2)
= (\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} \cdot)^* T_2^*(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} w_2),
\]
the last equality following by the second equation of (9.77). By equations (9.75) and (9.78), we have
\[
T_2^*(\eta_0 \hat{\lambda} \eta_1 \hat{\lambda} w_2) = t_2(\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} v_2) = t_2(\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} x_2),
\]
and so,
\[
\Gamma_2^* y_2 = t_2 x_2.
\]
Therefore, \((x_2, y_2)\) is a Schmidt pair for \( \Gamma_2 \) corresponding to \( t_2 \).

**Lemma 9.28.** Suppose that \((\xi_0 \hat{\lambda} \xi_1 \hat{\lambda} v_2, \eta_0 \hat{\lambda} \eta_1 \hat{\lambda} w_2)\) is a Schmidt pair for \( T_2 \) corresponding to \( t_2 \). Let
\[
x_2 = (I_n - \xi_0 \xi_0^\dagger - \xi_1 \xi_1^\dagger) v_2, \quad y_2 = (I_m - \eta_0 \eta_0^T - \eta_1 \eta_1^T) w_2,
\]
and let
\[
\hat{x}_2 = \alpha_1^T \alpha_0^T x_2, \quad \hat{y}_2 = \beta_1^s \beta_0^s y_2.
\]
Then the pair \((\hat{x}_2, \hat{y}_2)\) is a Schmidt pair for \( H_{F_2} \) corresponding to \( \|H_{F_2}\| = t_2 \).
**Proof.** Let us first show that \( \hat{x}_2 \in H^2(\mathbb{D}, \mathbb{C}^{n-2}) \) and \( x_2 \in \mathcal{K}_2 \). Recall that \( V_0 = (\xi_0 \, \alpha_0) \) and \( V_1 = (\alpha_0^T \xi_1 \, \tilde{\alpha}_1) \) are unitary-valued, that is, \( \alpha_0^T \xi_0 = 0 \), \( \alpha_1^T \alpha_0^T \xi_1 = 0 \),

\[
I_n - \xi_0 \xi_0^* = \alpha_0 \alpha_0^T \tag{9.79}
\]

and

\[
I_{n-1} - \alpha_0^T \xi_1 \xi_1^* \alpha_0 = \tilde{\alpha}_1 \alpha_1^T. \tag{9.80}
\]

Then

\[
\hat{x}_2 = \alpha_1^T \alpha_0^T x_2 \\
= \alpha_1^T \alpha_0^T (I_n - \xi_0 \xi_0^* - \xi_1 \xi_1^*) v_2 \\
= \alpha_1^T \alpha_0^T v_2 - \alpha_1^T \alpha_0^T \xi_0 \xi_0^* v_2 - \alpha_1^T \alpha_0^T \xi_1 \xi_1^* v_2 \\
= \alpha_1^T \alpha_0^T v_2, \tag{9.81}
\]

which, by Lemmas 9.4 and 9.18, implies that \( \hat{x}_2 \in H^2(\mathbb{D}, \mathbb{C}^{n-2}) \). Moreover, by Lemma 9.20, we obtain

\[
\bar{\alpha}_0 \bar{\alpha}_1 \hat{x}_2 = \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T v_2 \\
= (I_n - \xi_0 \xi_0^* - \xi_1 \xi_1^*) v_2 = x_2.
\]

Hence,

\[
x_2 = \bar{\alpha}_0 \bar{\alpha}_1 \hat{x}_2, \tag{9.82}
\]

and thus \( x_2 \in \mathcal{K}_2 \).

Next, we shall show that \( \hat{y}_2 \in H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp \) and \( y_2 \in \mathcal{L}_2 \). Note that since \( \bar{W}_1^T = (\beta_0^T \eta_1 \, \beta_1) \) and \( W_0^T = (\eta_0 \, \beta_0) \) are unitary-valued, \( \beta_0^T \eta_0 = 0 \), \( \beta_1^T \beta_0^T \eta_1 = 0 \),

\[
(I_{m-1} - \beta_0^T \eta_1 \eta_1^T \beta_0) = \beta_1 \beta_1^T, \tag{9.83}
\]

and

\[
(I_m - \bar{\eta}_0 \eta_0^T) = \beta_0 \beta_0^*. \tag{9.84}
\]

We have

\[
\hat{y}_2 = \beta_1^* \beta_0^* y_2 \\
= \beta_1^* \beta_0^* (I_m - \bar{\eta}_0 \eta_0^T - \bar{\eta}_1 \eta_1^T) w_2 \\
= \beta_1^* \beta_0^* w_2 - \beta_1^* \beta_0^* \bar{\eta}_0 \eta_0^T w_2 - \beta_1^* \beta_0^* \bar{\eta}_1 \eta_1^T w_2 \\
= \beta_1^* \beta_0^* w_2, \tag{9.85}
\]

which, by Propositions 9.9 and 9.17, implies that \( \hat{y}_2 \in H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp \). By Lemma 9.22, we have

\[
\beta_0 \beta_1 \hat{y}_2 = \beta_0 \beta_1 \beta_1^* \beta_0^* w_2 \\
= (I_m - \bar{\eta}_0 \eta_0^T - \bar{\eta}_1 \eta_1^T) w_2 = y_2.
\]

Hence,

\[
y_2 = \beta_0 \beta_1 \beta_1^* \beta_0^* w_2 = \beta_0 \beta_1 \hat{y}_2, \tag{9.86}
\]

and therefore \( y_2 \in \mathcal{L}_2 \).
By Theorem 9.26, the maps
\[ M_{\alpha_0 \alpha_1} : H^2(\mathbb{D}, \mathbb{C}^{n-2}) \to K_2, \quad M_{\beta_0 \beta_1} : H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp \to L_2 \]
are unitaries and
\[ H_{F_2} = M_{\beta_0 \beta_1}^* \Gamma_2 M_{\alpha_0 \alpha_1}. \]  
(9.87)

We need to show that
\[ H_{F_2} \hat{x}_2 = t_2 \hat{y}_2, \quad H_{F_2}^* \hat{y}_2 = t_2 \hat{x}_2. \]
By equations (9.81) and (9.82),
\[ x_2 = \bar{\alpha}_0 \alpha_1^T \alpha_0^T x_2. \]
(9.88)
Hence, equation (9.87) yields
\[ H_{F_2} \hat{x}_2 = \beta_1^* \beta_0^* \bar{\alpha}_0 \alpha_1 \hat{x}_2 = \beta_1^* \beta_0^* \Gamma_2 x_2. \]
(9.89)

By Proposition 5.1(ii),
\[ \xi_0 \lambda \xi_1 \lambda x_2 = \xi_0 \lambda \xi_1 \lambda v_2, \quad \eta_0 \lambda \eta_1 \lambda w_2 = \bar{\eta}_0 \lambda \bar{\eta}_1 \lambda w_2. \]
Thus, by Lemma 9.27, \((x_2, y_2)\) is a Schmidt pair for the operator \(\Gamma_2\) corresponding to \(t_2 = \|\Gamma_2\|\), that is,
\[ \Gamma_2 x_2 = t_2 y_2, \quad \Gamma_2^* y_2 = t_2 x_2. \]
(9.90)
Thus, equation (9.89) yields
\[ H_{F_2} \hat{x}_2 = \beta_1^* \beta_0^* \Gamma_2 x_2 = \beta_1^* \beta_0^* t_2 y_2 = t_2 \hat{y}_2 \]
as required. Let us show that \(H_{F_2}^* \hat{y}_2 = t_2 \hat{x}_2\). By equations (9.85) and (9.86),
\[ y_2 = \beta_0 \beta_1^* \beta_0^* y_2. \]
(9.91)
By equation (9.87),
\[ H_{F_2}^* = M_{\alpha_0 \alpha_1}^* \circ \Gamma_2^* \circ M_{\beta_0 \beta_1}. \]
Hence,
\[ H_{F_2}^* \hat{y}_2 = \alpha_1^T \alpha_0^T \Gamma_2^* \Gamma_2 \hat{y}_2 = \alpha_1^T \alpha_0^T \Gamma_2 \hat{y}_2, \]
(9.92)
and, by equations (9.90) and (9.92),
\[ H_{F_2}^* \hat{y}_2 = \alpha_1^T \alpha_0^T \Gamma_2 \hat{y}_2 = \alpha_1^T \alpha_0^T t_2 x_2 = t_2 \hat{x}_2. \]
Therefore, \((\hat{x}_2, \hat{y}_2)\) is a Schmidt pair for \(H_{F_2}\) corresponding to \(\|H_{F_2}\| = t_2\). \(\square\)

**Proposition 9.29.** Let \((\xi_0 \lambda \xi_1 \lambda v_2, \bar{\eta}_0 \lambda \bar{\eta}_1 \lambda w_2)\) be a Schmidt pair for \(T_2\) corresponding to \(t_2\) for some \(v_2 \in H^2(\mathbb{D}, \mathbb{C}^n), w_2 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp\), let \(h_2 \in H^2(\mathbb{D}, \mathbb{C})\) be the scalar outer factor of \(\xi_0 \lambda \xi_1 \lambda v_2\), let
\[ x_2 = (I_n - \xi_0^T \xi_0^*) v_2, \quad y_2 = (I_m - \bar{\eta}_0 \bar{\eta}_0^T - \bar{\eta}_1 \bar{\eta}_1^T) w_2, \]
and let
\[ \hat{x}_2 = \alpha_1^T \alpha_0^T x_2 \quad \text{and} \quad \hat{y}_2 = \beta_1^* \beta_0^* y_2. \]
(9.93)
Then
\[ \|\hat{x}_2(z)\|_{\mathbb{C}^{n-2}} = \|y_2(z)\|_{\mathbb{C}^{m-2}} = |h_2(z)|, \]
\[ \|x_2(z)\|_{\mathbb{C}^n} = \|y_2(z)\|_{\mathbb{C}^m} = |h_2(z)|. \]
and 
\[ \|\xi_0(z) \land \xi_1(z) \land v_2(z)\|_{\mathcal{A}_m} = \|\tilde{\eta}_0(z) \land \tilde{\eta}_1(z) \land w_2(z)\|_{\mathcal{A}_m} = |h_2(z)| \]
almost everywhere on \( T \).

**Proof.** By Lemma 9.28, \((\hat{x}_2, \hat{y}_2)\) is a Schmidt pair for \( H_{F_2} \) corresponding to \( \|H_{F_2}\| = t_2 \) (see Theorem 9.26 (v)). Hence, 
\[ H_{F_2} \hat{x}_2 = t_2 \hat{y}_2 \quad \text{and} \quad H_{F_2}^* \hat{y}_2 = t_2 \hat{x}_2. \]
Then, by Theorem 4.11, 
\[ \|\hat{y}_2(z)\|_{C^{m-2}} = \|\hat{x}_2(z)\|_{C^{m-2}} \]
almost everywhere on \( T \). Note that, by equations (9.93), 
\[ x_2 = \bar{a}_0 \tilde{a}_1 \hat{x}_2, \]
and since \( \bar{a}_0(z), \tilde{a}_1(z) \) are isometric for almost every \( z \in T \), we obtain 
\[ \|x_2(z)\|_{C^m} = \|\hat{x}_2(z)\|_{C^{m-2}}. \]
Furthermore, by equations (9.93), 
\[ y_2 = \beta_0 \beta_1 \hat{y}_2, \]
and since \( \beta_0(z), \beta_1(z) \) are isometric almost everywhere on \( T \), we have 
\[ \|y_2(z)\|_{C^m} = \|\hat{y}_2(z)\|_{C^{m-2}} \]
almost everywhere on \( T \). By equations (9.25), we deduce 
\[ \|x_2(z)\|_{C^m} = \|\hat{x}_2(z)\|_{C^{m-2}} = \|\hat{y}_2(z)\|_{C^{m-2}} = \|y_2(z)\|_{C^m} \quad (9.94) \]
almost everywhere on \( T \). By Proposition 5.1, 
\[ \xi_0 \land \xi_1 \wedge v_2 = \xi_0 \land \xi_1 \wedge x_2, \quad \tilde{\eta}_0 \land \tilde{\eta}_1 \land w_2 = \tilde{\eta}_0 \land \tilde{\eta}_1 \land y_2. \]
Hence, by Proposition 3.14, 
\[ \|\xi_0(z) \land \xi_1(z) \wedge v_2(z)\|_{\mathcal{A}_m} = \|\xi_0(z) \land \xi_1(z) \land x_2(z)\|_{\mathcal{A}_m} \]
\[ = \|x_2(z) - \sum_{i=0}^{1} \langle x_2(z), \xi_i(z) \rangle \xi_i(z)\|_{C^m} = \|x_2(z)\|_{C^m} \]
almost everywhere on \( T \). Furthermore, 
\[ \|\tilde{\eta}_0(z) \land \tilde{\eta}_1(z) \land w_2(z)\|_{\mathcal{A}_m} = \|\tilde{\eta}_0(z) \land \tilde{\eta}_1(z) \land y_2(z)\|_{\mathcal{A}_m} \]
\[ = \|y_2(z) - \sum_{i=0}^{1} \langle y_2(z), \tilde{\eta}_i(z) \rangle \tilde{\eta}_i(z)\|_{C^m} = \|y_2(z)\|_{C^m} \]
almost everywhere on \( T \). Thus, by equation (9.94), 
\[ \|\xi_0(z) \land \xi_1(z) \land v_2(z)\|_{\mathcal{A}_m} = \|\tilde{\eta}_0(z) \land \tilde{\eta}_1(z) \land w_2(z)\|_{\mathcal{A}_m} \]
almost everywhere on \( T \).
Recall that \( h_2 \) is the scalar outer factor of \( \xi_0 \land \xi_1 \wedge v_2 \). Hence, 
\[ \|\hat{x}_2(z)\|_{C^{m-2}} = \|\hat{y}_2(z)\|_{C^{m-2}} = |h_2(z)|, \]
\[ \|x_2(z)\|_{C^m} = \|y_2(z)\|_{C^m} = |h_2(z)| \]
and 
\[ \|\xi_0(z) \land \xi_1(z) \land v_2(z)\|_{\mathcal{A}_m} = \|\tilde{\eta}_0(z) \land \tilde{\eta}_1(z) \land w_2(z)\|_{\mathcal{A}_m} = |h_2(z)| \]
almost everywhere on \( T \). \( \square \)
Proposition 9.30. Let $m, n$ be positive integers such that $\min(m, n) \geq 2$. Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n})$. In line with the algorithm from Subsection 4.2, let $Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ satisfy

\begin{equation}
(G - Q_2)x_0 = t_0y_0, \quad (G - Q_2)^*y_0 = t_0x_0,
(G - Q_2)x_1 = t_1y_1, \quad (G - Q_2)^*y_1 = t_1x_1.
\end{equation}

(9.95)

Let the spaces $X_2, Y_2$ be given by

$$X_2 = \xi_0\hat{\xi}_1\hat{\lambda}^2(\mathbb{D}, \mathbb{C}^n), \quad Y_2 = \bar{\eta}_0\hat{\eta}_1\hat{\lambda}^2(\mathbb{D}, \mathbb{C}^m),$$
and consider the compact operator $T_2 : X_2 \to Y_2$ given by

$$T_2(\xi_0\hat{\xi}_1\hat{\lambda}x) = P_{Y_2}(\bar{\eta}_0\hat{\eta}_1\hat{\lambda}(G - Q_2)x)$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$. Let $(\xi_0\hat{\xi}_1\hat{\lambda}\nu_2, \bar{\eta}_0\hat{\eta}_1\hat{\lambda}\nu_2)$ be a Schmidt pair for the operator $T_2$ corresponding to $t_2 = ||T_2||$, let $h_2 \in H^2(\mathbb{D}, \mathbb{C})$ be the scalar outer factor of $\xi_0\hat{\xi}_1\hat{\lambda}\psi_2$, let

$$x_2 = (I_n - \xi_0\xi_0^* - \xi_1\xi_1^*)\nu_2, \quad y_2 = (I_m - \bar{\eta}_0\eta_0^T - \bar{\eta}_1\eta_1^T)\nu_2$$

and let

$$\xi_2 = \frac{x_2}{h_2}, \quad \eta_2 = \frac{\bar{y}_2}{h_2}.$$

Then there exist unitary-valued functions $\tilde{V}_2, \tilde{W}_2$ of types $(n - 2) \times (n - 2)$ and $(m - 2) \times (m - 2)$, respectively, of the form

$$\tilde{V}_2 = (\alpha_1^T \alpha_0^T \xi_2 \quad \hat{\alpha}_2), \quad \tilde{W}_2^T = (\beta_1^T \beta_0^T \eta_2 \quad \hat{\beta}_2),$$

where $\alpha_2, \beta_2$ are inner, co-outer, quasi-continuous and all minors on the first columns of $\tilde{V}_2, \tilde{W}_2^T$ are in $H^\infty$. Furthermore, the set $E_2$ of all level 2 superoptimal error functions for $G$ satisfies

$$E_2 = W_0^T W_1^T \left[ \begin{array}{ccc} I_2 & 0 & 0 \\ 0 & W_2 \end{array} \right] \left[ \begin{array}{ccc} 0 & 0 & 0 \\ t_0u_0 & 0 & 0 \\ 0 & t_1u_1 & 0 \\ 0 & t_2u_2 & 0 \\ 0 & 0 & (F_3 + H^\infty) \cap B(t_2) \end{array} \right] \left[ \begin{array}{c} I_2 \\ 0 \end{array} \right] V_1^* V_0^*,$$

for some $F_3 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-3)\times(n-3)}) + C(\mathbb{T}, \mathbb{C}^{(m-3)\times(n-3)})$, where $u_3 = (2\bar{h}_3/h_3)$ is a quasi-continuous unimodular function and $B(t_2)$ is the closed ball of radius $t_2$ in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-3)\times(n-3)})$.

Proof. By Theorem 9.26, the following diagram commutes:

\begin{equation}
\begin{array}{ccc}
H^2(\mathbb{D}, \mathbb{C}^{n-2}) & \xrightarrow{M_{\alpha_0}} & K_2 \\
H^2_\perp(\mathbb{D}, \mathbb{C}^{m-2}) & \xrightarrow{M_{\beta_0}} & L_2 \\
\end{array}
\end{equation}

(9.96)

Recall that the operators $M_{\alpha_0}, M_{\beta_0}$, $(\xi_0\hat{\xi}_1\hat{\lambda}\cdot)$ and $(\bar{\eta}_0\hat{\eta}_1\hat{\lambda}\cdot)$ are unitaries.

By Proposition 8.1, $T_2$ is well defined and is independent of the choice of $Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ satisfying conditions (9.95). Hence, we may choose $Q_2$ to minimize $(s_0^\infty (G - Q), s_1^\infty (G - Q))$, and then, by Proposition 9.16, the conditions (9.95) hold.

By Lemma 9.27, $(x_2, y_2)$ defined above is a Schmidt pair for $\Gamma_2$ corresponding to $t_2$. By Lemma 9.28, $(\hat{x}_2, \hat{y}_2)$ is a Schmidt pair for $H_{F_2}$ corresponding to $t_2$, where

$$\hat{x}_2 = \alpha_1^T \alpha_0^T x_2, \quad \hat{y}_2 = \beta_1^T \beta_0^T y_2.$$
We intend to apply Lemma 4.12 to $H_{F_2}$ and the Schmidt pair $(\hat{x}_2, \bar{y}_2)$ to find unitary-valued functions $\tilde{V}_2, \tilde{W}_2$ such that, for every $Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times (n-2)})$ which is at minimal distance from $F_2$, a factorization of the form

$$F_2 - \tilde{Q}_2 = W_2^* \begin{pmatrix} t_2 u_2 & 0 \\ 0 & F_3 \end{pmatrix} V_2^*$$

is obtained, for some $F_3 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times (n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2)\times (n-2)})$. For this purpose we find the inner–outer factorizations of $\hat{x}_2$ and $\bar{y}_2$.

By Lemma 9.29

$$\|\hat{x}_2(z)\|_{C^{m-2}} = |h_2(z)| \text{ and } \|\bar{y}_2(z)\|_{C^{m-2}} = |h_2(z)|$$

(9.97)

almost everywhere on $\mathbb{T}$. Equations (9.97) imply that $h_2 \in H^2(\mathbb{D}, \mathbb{C})$ is the scalar outer factor of both $\hat{x}_2$ and $\bar{y}_2$.

By Lemma 4.12, $\hat{x}_2, \bar{y}_2$ admit the inner–outer factorizations

$$\hat{x}_2 = \xi_2 h_2, \quad \bar{y}_2 = \eta_2 h_2,$$

for some inner $\xi_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{n-2}), \eta_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m-2})$. Then

$$\hat{x}_2 = \xi_2 h_2 = \alpha_1^T \alpha_0^T x_2, \quad \bar{y}_2 = \eta_2 h_2 = \bar{\beta}_1^T \beta_0^T \bar{y}_2,$$

from which we obtain

$$\xi_2 = \alpha_1^T \alpha_0^T \xi_2, \quad \eta_2 = \beta_1^T \beta_0^T \eta_2.$$

We show that $\alpha_1^T \alpha_0^T \xi_2, \beta_1^T \beta_0^T \eta_2$ are inner in order to apply Lemma 4.12 and obtain $\tilde{V}_2$ and $\tilde{W}_2$. Recall that, by Lemma 9.28,

$$x_2 = (I_n - \xi_0^s \xi_1^s) v_2 = \bar{\alpha}_0 \bar{\alpha}_1 \alpha_0^T v_2, \quad y_2 = (I_m - \eta_0 \eta_1^T) w_2 = \beta_0 \bar{\beta}_1 \beta_0^T w_2.$$

Thus,

$$\alpha_1^T \alpha_0^T x_2 = \alpha_1^T \alpha_0^T v_2, \quad \beta_1^T \beta_0^T y_2 = \beta_1^T \beta_0^T \bar{w}_2,$$

and since

$$\xi_2 = \frac{x_2}{h_2}, \quad \eta_2 = \frac{\bar{y}_2}{h_2},$$

we deduce that the functions

$$\alpha_1^T \alpha_0^T \xi_2 = \frac{\alpha_1^T \alpha_0^T v_2}{h_2}, \quad \beta_1^T \beta_0^T \eta_2 = \frac{\beta_1^T \beta_0^T \bar{w}_2}{h_2}$$

are analytic. Furthermore, $\|\xi_2(z)\|_{C^n} = 1$ and $\|\eta_2(z)\|_{C^m} = 1$ almost everywhere on $\mathbb{T}$, and, by equations (9.97),

$$\|\alpha_1^T(z) \alpha_0^T(z) x_2(z)\|_{C^{m-2}} = \|\alpha_1^T(z) \alpha_0^T(z) v_2(z)\|_{C^{m-2}} = |h_2(z)|$$

and

$$\|\beta_1^T(z) \beta_0^T(z) y_2(z)\|_{C^{n-2}} = \|\beta_1^T(z) \beta_0^T(z) \bar{w}_2(z)\|_{C^{n-2}} = |h_2(z)|$$

almost everywhere on $\mathbb{T}$. Hence,

$$\|\alpha_1^T(z) \alpha_0^T(z) \xi_2(z)\|_{C^{n-2}} = 1, \quad \|\beta_1^T(z) \beta_0^T(z) \eta_2(z)\|_{C^{m-2}} = 1$$

almost everywhere on $\mathbb{T}$. Thus, $\alpha_1^T \alpha_0^T \xi_2, \beta_1^T \beta_0^T \eta_2$ are inner functions.

By Lemma 4.12, there exist inner, co-outer, quasi-continuous functions $\alpha_2, \beta_2$ of types $(n-2) \times (n-3), (m-2) \times (m-3)$, respectively, such that the functions

$$\tilde{V}_2 = (\alpha_1^T \alpha_0^T \xi_2, \bar{\alpha}_2), \quad \tilde{W}_2^T = (\beta_1^T \beta_0^T \eta_2, \bar{\beta}_2)$$

are unitary-valued with all minors on the first columns in $H^\infty$. 


Furthermore, by Lemma 4.12, every \( \hat{Q}_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) \) which is at minimal distance from \( F_2 \) satisfies
\[
F_2 - \hat{Q}_2 = \hat{W}_2^* \begin{pmatrix} t_2 u_2 & 0 \\ 0 & F_3 \end{pmatrix} \chi_2^*,
\]
for some \( F_3 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-3)\times(n-3)}) + C(\mathbb{T}, \mathbb{C}^{(m-3)\times(n-3)}) \) and for the quasi-continuous unimodular function given by \( u_2 = \frac{\tilde{u}_2}{n^2} \).

By Lemma 4.15, the set
\[
\mathcal{E}_2 = \{ F_2 - \hat{Q} : \hat{Q} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}), \| F_2 - \hat{Q} \|_{L^\infty} = t_2 \}
\]
satisfies
\[
\mathcal{E}_2 = \hat{W}_2^* \begin{pmatrix} t_2 u_2 & 0 \\ 0 & (F_3 + H^\infty) \cap B(t_2) \end{pmatrix} \chi_2^*,
\]
where \( B(t_2) \) is the closed ball of radius \( t_2 \) in \( L^\infty(\mathbb{T}, \mathbb{C}^{(m-3)\times(n-3)}) \). Thus, by Proposition 9.15, \( \mathcal{E}_2 \) admits the factorization claimed.

**Proposition 9.31.** Every \( Q_3 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) \) which minimizes
\[
(s_0^\infty(G - Q), s_1^\infty(G - Q), s_2^\infty(G - Q))
\]
satisfies
\[
(G - Q_3)x_i = t_3 y_i, \quad (G - Q_3)^* y_i = t_3 x_i \quad \text{for} \quad i = 0, 1, 2.
\]

**Proof.** By Proposition 9.16, every \( Q_3 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) \) that minimizes
\[
(s_0^\infty(G - Q), s_1^\infty(G - Q))
\]
satisfies
\[
(G - Q_3)x_i = t_3 y_i, \quad (G - Q_3)^* y_i = t_3 x_i \quad \text{for} \quad i = 0, 1.
\]

Hence, it suffices to show that \( Q_3 \) satisfies
\[
(G - Q_3)x_2 = t_2 y_2, \quad (G - Q_3)^* y_2 = t_2 x_2.
\]

By Theorem 9.26, the following diagram commutes:
\[
\begin{array}{ccc}
H^2(\mathbb{D}, \mathbb{C}^{n-2}) & \xrightarrow{M_{\alpha_0 \gamma_1}} & K_2 \\
H^2(\mathbb{D}, \mathbb{C}^{m-2}) & \xrightarrow{M_{\alpha_0 \beta_1}} & \mathcal{L}_2 \\
\end{array}
\]
where the operator \( \Gamma_2 : K_2 \rightarrow \mathcal{L}_2 \) is given by \( \Gamma_2 = P_{\mathcal{L}_2} M_{G - Q_2} \big|_{K_2} \) and \( F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)}) \) is constructed as follows.

By Lemma 4.12 and Proposition 9.15, there exist unitary-valued functions
\[
\hat{V}_1 = (\alpha_0^T, \beta_0), \quad \hat{W}_1^T = (\beta_0^T, \eta_0),
\]
where \( \alpha_0, \beta_0 \) are inner, co-inner, quasi-continuous functions of types \( (n - 1) \times (n - 2) \) and \( (m - 1) \times (m - 2) \), respectively, and all minors on the first columns of \( \hat{V}_1, \hat{W}_1^T \) are in \( H^\infty \).

Furthermore, the set of all level 1 superoptimal functions \( \mathcal{E}_1 = \{ G - Q : Q \in \Omega_1 \} \) satisfies
\[
\mathcal{E}_1 = W_0^* \begin{pmatrix} 1 & 0 \\ 0 & \hat{W}_1 \end{pmatrix} \begin{pmatrix} t_0 u_0 & 0 \\ 0 & t_1 u_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ (F_2 + H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) \cap B(t_1)) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{V}_1^* \end{pmatrix} V_0^*,
\]
(9.98)
for some $F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(T, \mathbb{C}^{(m-2)\times(n-2)})$, for a quasi-continuous unimodular function $u_1 = (\bar{z}h_1/h_1)$, and for the closed ball $B(t_1)$ of radius $t_1$ in $L^\infty(T, \mathbb{C}^{(m-2)\times(n-2)})$.

Consider some $Q_3 \in \Omega_1$, so that, according to equation (9.98),

\[
\begin{pmatrix}
1 & 0 \\
0 & W_1
\end{pmatrix} W_0(G - Q_3) V_0 \begin{pmatrix}
1 & 0 \\
0 & \bar{v}_1
\end{pmatrix} = \begin{pmatrix}
t_0 u_0 & 0 & 0 \\
t_1 u_1 & 0 & 0 \\
0 & F_2 - \tilde{Q}_2
\end{pmatrix},
\]

for some $\tilde{Q}_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)})$, that is,

\[
\begin{pmatrix}
1 & \eta_1^T \beta_0 \\
0 & \beta_1^*
\end{pmatrix} \begin{pmatrix}
\eta_0^T \\
\beta_0^*
\end{pmatrix} (G - Q_3)(\xi_0, \bar{\alpha}_0) \begin{pmatrix}
1 & 0 \\
0 & \alpha_1^T\bar{\xi}_1
\end{pmatrix} = \begin{pmatrix}
t_0 u_0 & 0 & 0 \\
t_1 u_1 & 0 & 0 \\
0 & F_2 - \tilde{Q}_2
\end{pmatrix}.
\]

Observe

\[
\begin{pmatrix}
\eta_0^T \\
\beta_0^*
\end{pmatrix} (G - Q_3)(\xi_0, \bar{\alpha}_0) = \begin{pmatrix}
t_0 u_0 & 0 \\
0 & \beta_0^*(G - Q_3) \bar{\alpha}_0
\end{pmatrix},
\]

hence

\[
\begin{pmatrix}
1 & 0 \\
0 & \eta_1^T \beta_0
\end{pmatrix} \begin{pmatrix}
\eta_0^T \\
\beta_0^*
\end{pmatrix} (G - Q_3)(\xi_0, \bar{\alpha}_0) \begin{pmatrix}
1 & 0 \\
0 & \alpha_1^T\bar{\xi}_1
\end{pmatrix} = \begin{pmatrix}
t_0 u_0 & 0 & 0 \\
t_1 u_1 & 0 & 0 \\
0 & F_2 - \tilde{Q}_2
\end{pmatrix}.
\]

is equal to

\[
\begin{pmatrix}
t_0 u_0 & 0 \\
0 & \eta_1^T \beta_0 \beta_0^*(G - Q_3) \bar{\alpha}_0 \alpha_0^T \xi_1 \\
0 & \beta_1^* \beta_0^*(G - Q_3) \bar{\alpha}_0 \alpha_0^T \xi_1
\end{pmatrix} = \begin{pmatrix}
t_0 u_0 & 0 \\
t_1 u_1 & 0 \\
0 & F_2 - \tilde{Q}_2
\end{pmatrix},
\]

and so equation (9.99) yields

\[
\begin{pmatrix}
t_0 u_0 & 0 \\
0 & \eta_1^T \beta_0 \beta_0^*(G - Q_3) \bar{\alpha}_0 \alpha_0^T \xi_1 \\
0 & \beta_1^* \beta_0^*(G - Q_3) \bar{\alpha}_0 \alpha_0^T \xi_1
\end{pmatrix} = \begin{pmatrix}
t_0 u_0 & 0 \\
t_1 u_1 & 0 \\
0 & F_2 - \tilde{Q}_2
\end{pmatrix},
\]

which is equivalent to the following equations

\[
\eta_1^T \beta_0 \beta_0^*(G - Q_3) \bar{\alpha}_0 \alpha_0^T \xi_1 = t_1 u_1,
\]

\[
\eta_1^T \beta_0 \beta_0^*(G - Q_3) \bar{\alpha}_0 \alpha_1 = 0,
\]

\[
\beta_1^* \beta_0^*(G - Q_3) \bar{\alpha}_0 \alpha_0^T \xi_1 = 0,
\]

and

\[
\beta_1^* \beta_0^*(G - Q_3) \bar{\alpha}_0 \alpha_1 = F_2 - \tilde{Q}_2.
\]

By Theorem 4.11 applied to $H_{F_2}$, if $(\hat{x}_2, \hat{y}_2)$ is a Schmidt pair for $H_{F_2}$ corresponding to $t_2 = \|H_{F_2}\|$, then, for any $Q_2$ which is at minimal distance from $F_2$, we have

\[
(F_2 - \tilde{Q}_2) \hat{x}_2 = t_2 \hat{y}_2, \quad (F_2 - \tilde{Q}_2)^* \hat{y}_2 = t_2 \hat{x}_2.
\]

By equations (9.100) and (9.101),

\[
\beta_1^* \beta_0^*(G - Q_3) \bar{\alpha}_0 \alpha_1 \hat{x}_2 = t_2 \hat{y}_2
\]

and

\[
\alpha_1^T \alpha_0^T (G - Q_3)^* \beta_0 \beta_1 \hat{y}_2 = t_2 \hat{x}_2.
\]
Recall that, by equations (9.82) and (9.86),

$$\alpha_0\alpha_1\hat{x}_2 = x_2 \quad \text{and} \quad \hat{y}_2 = \beta_1^*\beta_0^*y_2.$$  \(\text{(9.104)}\)

Hence, by equation (9.102), we obtain

$$\beta_1^*\beta_0^*(G - Q_3)x_2 = t_2\beta_1^*\beta_0^*y_2,$$

or equivalently,

$$\beta_1^*\beta_0^*(G - Q_3)x_2 - t_2y_2 = 0.$$  

Since, by Theorem 9.26, \(M_{\beta_0,\beta_1}\) is unitary, the latter equation yields

$$(G - Q_3)x_2 = t_2y_2.$$  

Moreover, in view of equations (9.100), (9.101) and (9.104), equation (9.103) implies

$$\alpha_1^T\alpha_0^T(G - Q_3)^*y_2 = t_2\alpha_1^T\alpha_0^Tx_2,$$

which in turn is equivalent to the equation

$$\alpha_1^T\alpha_0^T((G - Q_3)^*y_2 - t_2x_2) = 0.$$  

By Theorem 9.26, \(M_{\alpha_0,\alpha_1}\) is unitary, hence the latter equation yields

$$(G - Q_3)^*y_2 = t_2x_2,$$

and therefore the assertion has been proved. \(\square\)

10. **Compactness of the operator** \(T_{j+1}\)

At this point, the reader has seen the proof of the compactness of the operators \(T_1\) and \(T_2\). Suppose that we have applied steps 0, ..., \(j\) of the superoptimal analytic approximation algorithm from Subsection 4.2 to \(G\), we have constructed

\[
\begin{align*}
t_0 & \geq t_1 \geq \cdots \geq t_j > 0, \\
x_0, x_1, \ldots, x_j & \in L^2(T, \mathbb{C}^n) \\
y_0, y_1, \ldots, y_j & \in L^2(T, \mathbb{C}^m) \\
h_0, h_1, \ldots, h_j & \in H^2(\mathbb{D}, \mathbb{C}) \text{ outer} \\
\xi_0, \xi_1, \ldots, \xi_j & \in L^\infty(T, \mathbb{C}^n) \text{ pointwise orthonormal on } T \\
\eta_0, \eta_1, \ldots, \eta_j & \in L^\infty(T, \mathbb{C}^m) \text{ pointwise orthonormal on } T \\
X_0 & = H^2(\mathbb{D}, \mathbb{C}^n), X_1, \ldots, X_j \\
Y_0 & = H^2(\mathbb{D}, \mathbb{C}^m)^\perp, Y_1, \ldots, Y_j \\
T_0, T_1, \ldots, T_j & \text{ compact operators,}
\end{align*}
\]

and all the claimed properties hold. We shall apply a similar method to show that the operator \(T_{j+1}\) as given in equation (4.37) is compact.

**Proposition 10.1.** Let \(m, n\) be positive integers such that \(\min(m, n) \geq 2\). Let \(G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(T, \mathbb{C}^{m\times n})\). In line with the algorithm from Subsection 4.2, let \(Q_j \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})\) satisfy

\[
(G - Q_j)x_i = t_iy_i, \quad (G - Q_j)^*y_i = t_i^*x_i \quad \text{for} \quad i = 0, 1, \ldots, j - 1. \tag{10.1}\]

Let the spaces \(X_j, Y_j\) be given by

\[
X_j = \xi_0^*\hat{\xi}_1^*\cdots\hat{\xi}_{j-1}^*\hat{\lambda}H^2(\mathbb{D}, \mathbb{C}^n), \quad Y_j = \eta_0^*\hat{\eta}_1^*\cdots\hat{\eta}_{j-1}^*\hat{\lambda}H^2(\mathbb{D}, \mathbb{C}^m)^\perp,
\]
and consider the compact operator \( T_j : X_j \to Y_j \) given by

\[
T_j(\xi_0 \lambda_1 \cdots \xi_{i-1} \lambda x) = P_{Y_j}(\eta_0 \lambda \cdots \lambda \eta_{i-1} \lambda (G - Q_j)x)
\]

for all \( x \in H^2(\mathbb{D}, \mathbb{C}^n) \). Let \((\xi_0 \lambda_1 \cdots \lambda \xi_{i-1} \lambda v_j, \eta_0 \lambda \cdots \lambda \eta_{i-1} \lambda v_j)\) be a Schmidt pair for the operator \( T_j \) corresponding to \( t_j = \|T_j\| \), let \( h_j \in H^2(\mathbb{D}, \mathbb{C}) \) be the scalar outer factor of \( \xi_0 \lambda_1 \cdots \lambda \xi_{i-1} \lambda \), let

\[
x_j = (I_n - \xi_0 \lambda_1 \cdots \xi_{i-1} \lambda) v_j, \quad y_j = (I_m - \eta_0 h_0 - \cdots - \eta_{i-1} \lambda h_{i-1}) w_j
\]

and let

\[
\xi_j = \frac{x_j}{h_j}, \quad \eta_j = \frac{y_j}{h_j}.
\]

(10.2)

Let, for \( i = 0, 1, \ldots, j - 1 \),

\[
\tilde{V}_i = (\alpha_i^T \cdots \alpha_0^T \xi_i \ \tilde{\alpha}_i), \quad \tilde{W}_i^T = (\beta_i^T \cdots \beta_0^T \eta_i \ \tilde{\beta}_i)
\]

be unitary-valued functions, as described in Lemma 4.12 (see also Proposition 9.30 for \( \tilde{V}_2 \) and \( \tilde{W}_2^T \)), \( u_i = \frac{\beta_i}{\alpha_i} \) are quasi-continuous unimodular functions, and

\[
V_i = \begin{pmatrix} I_i & 0 \\ 0 & \tilde{V}_i \end{pmatrix}, \quad W_i = \begin{pmatrix} I_i & 0 \\ 0 & \tilde{W}_i \end{pmatrix}.
\]

There exist unitary-valued functions \( \tilde{V}_j, \tilde{W}_j \) of the form

\[
\tilde{V}_j = (\alpha_j^T \cdots \alpha_0^T \xi_j \ \tilde{\alpha}_j), \quad \tilde{W}_j^T = (\beta_j^T \cdots \beta_0^T \eta_j \ \tilde{\beta}_j),
\]

(10.4)

where \( \alpha_0, \ldots, \alpha_{j-1} \) and \( \beta_0, \ldots, \beta_{j-1} \) are of types \( n \times (n - 1), \ldots, (n - j - 1) \times (n - j - 2) \) and \( m \times (m - 1), \ldots, (m - j - 1) \times (m - j - 2) \), respectively, and are inner, co-outer and quasi-continuous.

Furthermore, the set of all level \( j \) superoptimal error functions \( \mathcal{E}_j \) satisfies

\[
\mathcal{E}_j = W_0^* W_1^* \cdots W_j^* \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 & 0_{1 \times (n-j-1)} \\ 0 & t_1 u_1 & \cdots & 0 & 0_{1 \times (n-j-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_j u_j & 0 \\ 0_{(m-j-1) \times 1} & 0_{(m-j-1) \times 1} & \cdots & \cdots & (F_{j+1} + H^\infty) \cap B(t_j) \end{pmatrix} V_j^* \cdots V_0^*,
\]

(10.5)

for some \( F_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1) \times (n-j-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-1) \times (n-j-1)}) \), for the quasi-continuous unimodular functions \( u_i = (z \tilde{h}_i / h_i) \), for all \( i = 0, \ldots, j \), for the closed ball \( B(t_j) \) of radius \( t_j \) in \( L^\infty(\mathbb{T}, \mathbb{C}^{(m-j-1) \times (n-j-1)}) \), and for the unitary valued functions

\[
V_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{V}_j \end{pmatrix}, \quad W_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{W}_j \end{pmatrix}.
\]

Proof. Suppose that we have applied steps \( 0, \ldots, j \) of the algorithm from Subsection 4.2 and that the following diagram commutes:

\[
\begin{array}{ccc}
H^2(\mathbb{D}, \mathbb{C}^{n-j}) & \xrightarrow{M_{\alpha_{j+1}} \cdots \alpha_{j-1}} & K_j \xrightarrow{\xi_{(j-1)}^\lambda} \xi_{(j-1)}^\lambda H^2(\mathbb{D}, \mathbb{C}^n) = X_j \\
H^2(\mathbb{D}, \mathbb{C}^{m-j}) & \xrightarrow{M_{\beta_{j} \cdots \beta_{j-1}}} & L_j \xrightarrow{\eta_{(j-1)}^\lambda} \eta_{(j-1)}^\lambda H^2(\mathbb{D}, \mathbb{C}^m) = Y_j,
\end{array}
\]

(10.6)
where the maps
\[ M_{\bar{\alpha}_0\cdots\bar{\alpha}_{j-1}} : H^2(\mathbb{D}, \mathbb{C}^{n-j}) \to K_j : x \mapsto \bar{\alpha}_0 \cdots \bar{\alpha}_{j-1} x, \]
\[ M_{\bar{\beta}_0\cdots\bar{\beta}_{j-1}} : H^2(\mathbb{D}, \mathbb{C}^{m-j})^\perp \to L_j : y \mapsto \bar{\beta}_0 \cdots \bar{\beta}_{j-1} y, \]
\[(\xi_{(j-1)^{\lambda_j}}) : K_j \to X_j \text{ and } (\bar{\eta}_{(j-1)^{\lambda_j}}) : L_j \to Y_j \]
are unitaries.

Let \((\xi_{(j-1)^{\lambda_j}}) \wedge v_j, \bar{\eta}_{(j-1)^{\lambda_j}} \wedge w_j\) be a Schmidt pair for the compact operator \(T_j\). Then \(x_j \in K_j\), \(y_j \in L_j\) are such that \((x_j, y_j)\) is a Schmidt pair for \(\Gamma_j\) corresponding to \(t_j = \|\Gamma_j\|\), and \((\hat{x}_j, \hat{y}_j)\) is a Schmidt pair for \(H_{F_j}\) corresponding to \(t_j = \|H_{F_j}\|\), where

\[ \hat{x}_j = \alpha_{j-1}^T \cdots \alpha_0^T x_j, \quad \hat{y}_j = \beta_{j-1}^T \cdots \beta_0^T y_j. \]

We intend to apply Lemma 4.12 to \(H_{F_j}\) and the Schmidt pair \((\hat{x}_j, \hat{y}_j)\) to find unitary-valued functions \(\tilde{V}_j, \tilde{W}_j\) such that, for every \(\tilde{Q}_j \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j) \times (n-j)})\) which is at minimal distance from \(F_j\), a factorization of the form

\[ F_j - \tilde{Q}_j = \tilde{W}_j^* \begin{pmatrix} t_j u_j & 0 \\ 0 & F_{j+1} \end{pmatrix} \tilde{V}_j \]

is obtained, for some \(F_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1) \times (n-j-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-j) \times (n-j)})\). For this purpose we find the inner–outer factorizations of \(\hat{x}_j\) and \(\tilde{z}_j \tilde{y}_j\).

By the inductive hypothesis (see Lemma 9.29 for \(j = 2\)), we have

\[ |h_j(z)| = \|\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge v_j(z)\|_{\lambda_j+1 \mathbb{C}^n} = \|\bar{\eta}_0(z) \wedge \cdots \wedge \bar{\eta}_{j-1}(z) \wedge w_j(z)\|_{\lambda_j+1 \mathbb{C}^m}, \]
\[ \|\hat{x}_j(z)\|_{\lambda_{n-j}} = \|\hat{y}_j(z)\|_{\lambda_{m-j}} = |h_j(z)|, \]
\[ \|x_j(z)\|_{\lambda_n} = \|y_j(z)\|_{\lambda_m} = |h_j(z)|, \]

almost everywhere on \(\mathbb{T}\). Equations (10.8) imply that \(h_j \in H^2(\mathbb{D}, \mathbb{C})\) is the scalar outer factor of both \(\hat{x}_j\) and \(\tilde{z}_j \tilde{y}_j\).

By Lemma 4.12, \(\hat{x}_j, \tilde{z}_j \tilde{y}_j\) admit the inner–outer factorizations

\[ \hat{x}_j = \check{\xi}_j h_j, \quad \tilde{z}_j \tilde{y}_j = \check{\eta}_j h_j, \]

where \(\check{\xi}_j \in H^\infty(\mathbb{D}, \mathbb{C}^{n-j})\) and \(\check{\eta}_j \in H^\infty(\mathbb{D}, \mathbb{C}^{m-j})\) are vector-valued inner functions.

By equations (10.7) and (10.9), we deduce that

\[ \check{\xi}_j = \alpha_{j-1}^T \cdots \alpha_0^T \xi_j, \quad \check{\eta}_j = \beta_{j-1}^T \cdots \beta_0^T \eta_j. \]

We shall show that \(\alpha_{j-1}^T \cdots \alpha_0^T \xi_j, \beta_{j-1}^T \cdots \beta_0^T \eta_j\) are inner in order to apply Lemma 4.12 and obtain \(\tilde{V}_j\) and \(\tilde{W}_j\) as required. We have

\[ \hat{x}_j = \alpha_{j-1}^T \cdots \alpha_0^T x_j \]
\[ = \alpha_{j-1}^T \cdots \alpha_0^T (I_n - \xi_0^* \xi_0 - \cdots - \xi_{j-1}^* \xi_{j-1}) v_j \]
\[ = \alpha_{j-1}^T \cdots \alpha_0^T v_j - \alpha_{j-1}^T \cdots \alpha_0^T \xi_0^* \xi_0 v_j - \cdots - \alpha_{j-1}^T \xi_{j-1} \xi_{j-1}^* v_j. \]

Recall that, by the inductive hypothesis, for \(i = 0, \ldots, j - 1\), each

\[ \tilde{V}_i = (\alpha_{i-1}^T \cdots \alpha_0^T \xi_i, \alpha_i) \]

is unitary-valued, and so \(\alpha_i^T \alpha_{i+1}^T \cdots \alpha_0^T \xi_i = 0\). Hence, if \(0 \leq i \leq j - 1\), we have

\[ \alpha_i^T \cdots \alpha_{j-1}^T \alpha_i = 0. \]
Thus, 
\[ \hat{x}_j = \alpha_{j-1}^T \cdots \alpha_0^T x_j = \alpha_{j-1}^T \cdots \alpha_0^T v_j, \]
that is, \( \hat{x}_j \in H^2(\mathbb{D}, \mathbb{C}^{n-j}) \) and 
\[ \alpha_{j-1}^T \cdots \alpha_0^T \xi_j = \frac{1}{\hat{R}_j} \alpha_{j-1}^T \cdots \alpha_0^T x_j = \frac{1}{\hat{R}_j} \alpha_{j-1}^T \cdots \alpha_0^T v_j \]
is analytic. Moreover, by equations (10.8),
\[ \|\alpha_{j-1}^T(z) \cdots \alpha_0^T(z) x_j(z)\|_{c^{n-j}} = \|\alpha_{j-1}^T(z) \cdots \alpha_0^T(z) v_j(z)\|_{c^{n-j}} = |h_j(z)| \]
almost everywhere on \( \mathbb{T} \), and hence 
\[ \|\alpha_{j-1}^T(z) \cdots \alpha_0^T(z) \xi_j(z)\|_{c^{n-j}} = 1 \]
almost everywhere on \( \mathbb{T} \). Therefore, \( \alpha_{j-1}^T \cdots \alpha_0^T \xi_j \) is inner.

Furthermore, 
\[ \hat{y}_j = \beta_{j-1}^* \cdots \beta_0^* y_j \]
\[ = \beta_{j-1}^* \cdots \beta_0^* (I_m - \eta_0 \eta_0^T - \cdots - \eta_{j-1} \eta_{j-1}^T) w_j \]
\[ = \beta_{j-1}^* \cdots \beta_0^* w_j - \beta_{j-1}^* \cdots \beta_0^* \eta_0 \eta_0^T w_j - \cdots - \beta_{j-1}^* \cdots \beta_0^* \eta_{j-1} \eta_{j-1}^T w_j. \]
Note that, by the inductive hypothesis, for \( i = 0, \ldots, j-1 \), each 
\[ \tilde{W}_i^T = (\beta_{j-1}^T \cdots \beta_0^T \eta_i \ \eta_i) \]
is unitary-valued, and so \( \beta_{j-1}^* \cdots \beta_0^* \eta_i = 0 \). Hence, if \( 0 \leq i \leq j-1 \), we have 
\[ \beta_{j-1}^* \cdots \beta_{i+1}^* \beta_i^* \beta_0^* \eta_i = 0. \]

Thus, 
\[ \hat{y}_j = \beta_{j-1}^* \cdots \beta_0^* w_j, \]
that is, \( \hat{y}_j \in H^2(\mathbb{D}, \mathbb{C}^{m-j})^\perp \) and 
\[ \beta_{j-1}^* \cdots \beta_0^* \eta_j = \frac{1}{\hat{R}_j} \beta_{j-1}^T \cdots \beta_0^T \bar{y}_j = \frac{1}{\hat{R}_j} \beta_{j-1}^T \cdots \beta_0^T \bar{z} \bar{w}_j \]
is analytic. Further, by equations (10.8),
\[ \|\beta_{j-1}^T(z) \cdots \beta_0^T(z) \bar{y}_j(z)\|_{c^{m-j}} = \|\beta_{j-1}^T(z) \cdots \beta_0^T(z) \bar{z} \bar{w}_j(z)\|_{c^{m-j}} = |h_j(z)| \]
almost everywhere on \( \mathbb{T} \), and therefore 
\[ \|\beta_{j-1}^T(z) \cdots \beta_0^T(z) \eta_j(z)\|_{c^m} = 1 \]
almost everywhere on \( \mathbb{T} \), that is, \( \beta_{j-1}^T \cdots \beta_0^T \eta_j \) is inner.

We apply Lemma 4.12 to the Hankel operator \( H_{\tilde{F}_j} \) and the Schmidt pair \( (\hat{x}_j, \hat{y}_j) \) to deduce that there exist inner, co-outer, quasi-continuous functions \( \alpha_j, \beta_j \) of types \( (n-j) \times (n-j-1), \)
\( (m-j) \times (m-j-1) \), respectively, such that 
\[ \tilde{V}_j = \left( \alpha_{j-1}^T \cdots \alpha_0^T \xi_j \ \tilde{\alpha}_j \right), \quad \tilde{W}_j^T = \left( \beta_{j-1}^T \cdots \beta_0^T \eta_j \ \beta_j \right) \]
are unitary-valued and all minors on the first columns of \( \tilde{V}_j, \tilde{W}_j \) are in \( H^\infty \). Moreover, every function \( \tilde{Q}_j \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j)-(n-j)}) \), which is at minimal distance from \( F_j \), satisfies 
\[ F_j - \tilde{Q}_j = \tilde{W}_j^* \begin{pmatrix} t_j u_j & 0 \\ 0 & F_{j+1}^* \end{pmatrix} \tilde{V}_j^*, \]
for some $F_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1) \times (n-j-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-1) \times (n-j-1)})$ and for the quasi-continuous unimodular function $u_j = \left(\tilde{h}_j / h_j\right)$.

By Lemma 4.15, the set
\[
\tilde{\mathcal{E}}_j = \{F_j - \hat{Q} : \hat{Q} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j) \times (n-j)}), \|F_j - \hat{Q}\|_{L^\infty} = t_j\}
\]
satisfies
\[
\tilde{\mathcal{E}}_j = \hat{W}_j^* \left( t_j u_j \begin{bmatrix} 0 & (F_{j+1} + H^\infty) \cap B(t_j) \end{bmatrix} \right) \hat{V}_j^*,
\]
where $B(t_j)$ is the closed ball of radius $t_j$ in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-j-1) \times (n-j-1)})$.

By the inductive hypothesis, the set of all level $j$ superoptimal error functions $\mathcal{E}_j$ satisfies
\[
\mathcal{E}_{j-1} = W_0^* W_1^* \cdots W_j^*,
\]
for some $F_j \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j) \times (n-j)}) + C(\mathbb{T}, \mathbb{C}^{(m-j) \times (n-j)})$, $u_i = \frac{\tilde{h}_i}{h_i}$ are quasi-continuous unimodular functions for all $i = 0, \ldots, j - 1$, and for the closed ball $B(t_{j-1})$ of radius $t_{j-1}$ in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-j) \times (n-j)})$.

Thus, by equation (10.10), $\mathcal{E}_j$ admits the factorization (10.5) as claimed. \hfill \Box

**Remark 10.2.** Let, for $i = 0, 1, \ldots, j$,
\[
\hat{V}_i = \left(\alpha_{i-1}^T \cdots \alpha_0^T \xi_i \ \bar{\alpha}_i\right), \quad \hat{W}_i^T = \left(\beta_{i-1}^T \cdots \beta_0^T \eta_i \ \bar{\beta}_i\right)
\]
be unitary-valued functions, as described in Lemma 4.12. Let
\[
V_j = \begin{pmatrix} I_j & 0 \\ 0 & \hat{V}_j \end{pmatrix}, \quad W_j = \begin{pmatrix} I_j & 0 \\ 0 & \hat{W}_j \end{pmatrix}.
\]

Let $A_j = \alpha_0 \alpha_1 \cdots \alpha_j$, $A_{-1} = I_n$, $B_j = \bar{\beta}_0 \beta_1 \cdots \beta_j$ and $B_{-1} = I_m$.

Note
\[
W_1 W_0 = \begin{pmatrix} 1 & 0 \\ 0 & \bar{\beta}_1 \end{pmatrix} \begin{pmatrix} \eta_0^T \\ \eta_0 B_0 B_0^* \end{pmatrix} = \begin{pmatrix} \eta_0^T B_0 B_0^* \\ \eta_0 \bar{B}_0 \end{pmatrix}
\]
and
\[
W_2 W_1 W_0 = \begin{pmatrix} I_2 & 0 \\ 0 & \eta_2 B_1 \end{pmatrix} \begin{pmatrix} \eta_1^T B_0 B_0^* \\ \eta_1 \bar{B}_0 \end{pmatrix} = \begin{pmatrix} \eta_0^T B_0 B_0^* \\ \eta_0 \bar{B}_0 \eta_2 B_1 B_1^* \end{pmatrix}.
\]

Similarly, one obtains
\[
W_j W_{j-1} \cdots W_0 = \begin{pmatrix} \eta_0^T \\ \eta_1^T B_0 B_0^* \\ \vdots \\ \eta_{j-1}^T B_{j-1} B_{j-1}^* \\ \eta_j \bar{B}_j \end{pmatrix}.
\]
Thus, \[ W_0^*W_1^* \cdots W_j^* = (\bar{\eta}_0 \quad B_0B_0^*\bar{\eta}_1 \quad \cdots \quad B_{j-1}B_{j-1}^*\bar{\eta}_j \quad B_j). \] (10.13)

Therefore, \[ I_m = W_0^*W_1^* \cdots W_j^*W_k \cdots W_kW_0 = \sum_{i=0}^{j} B_{i-1}B_{i-1}^*\bar{\eta}_i\eta_i^TB_{i-1}B_{i-1}^* + B_jB_j^*. \] (10.14)

Furthermore, \[ V_0V_1 = (\xi_0 \quad \bar{\alpha}_0) \begin{pmatrix} 1 & 0 & 0 \\ \alpha_0^T \xi_1 & 0 & \bar{\alpha}_1 \end{pmatrix} = (\xi_0 \quad A_0^T\xi_1 \quad \bar{A}_1) \]
and \[ V_0V_1V_2 = (\xi_0 \quad \bar{A}_0 A_1^T\xi_1 \quad \bar{A}_1) \begin{pmatrix} I_2 & 0 & 0 \\ 0 & A_1^T\xi_2 & 0 \end{pmatrix} = (\xi_0 \quad \bar{A}_0 A_1^T\xi_1 \quad \bar{A}_1 A_1^T\xi_2 \quad \bar{A}_2). \]

One can easily show by induction that \[ V_0 \cdots V_j = (\xi_0 \quad \bar{A}_0 A_j^T\xi_1 \quad \bar{A}_1 A_j^T\xi_2 \cdots \quad \bar{A}_{j-1} A_{j-1}^T\xi_j \quad \bar{A}_j). \] (10.15)

Therefore, \[ I_n = V_0 \cdots V_j V_j^* \cdots V_0^* = \xi_0\xi_0^* + \bar{A}_0 A_0^T\xi_1\xi_1^*\bar{A}_0 A_0^T + \cdots \bar{A}_{j-1} A_{j-1}^T\xi_j\xi_j^*\bar{A}_{j-1} A_{j-1}^T + \bar{A}_j A_j^T. \] (10.16)

**Lemma 10.3.** Let \[ \bar{V}_i = (\alpha_{i-1}^T \cdots \alpha_0^T \xi_i \quad \bar{\alpha}_i) \] (10.17)
be unitary-valued functions, for \( i = 0, 1, \ldots, j \), as described in Lemma 4.12. For \( i = 0, 1, \ldots, j \), let \( A_i = \alpha_0 \alpha_1 \cdots \alpha_i \) and \( A_{-1} = I_n \). Then, for \( i = 0, 1, \ldots, j \),
\[ \bar{A}_i A_i^T = I_n - \sum_{k=0}^{i} \xi_k\xi_k^* \] (10.18)
after everywhere on \( \mathbb{T} \).

**Proof.** By equation (10.16), for \( k = 0, \ldots, j \),
\[ \bar{A}_k A_k^T = I_n - \sum_{i=0}^{k} \bar{A}_{i-1} A_{i-1}^T\xi_i\xi_i^*\bar{A}_{i-1} A_{i-1}^T. \] (10.19)

Thus, to prove condition (10.18) it suffices to show that, for \( k = 0, \ldots, j \),
\[ \bar{A}_{k-1} A_{k-1}^T\xi_k\xi_k^*\bar{A}_{k-1} A_{k-1}^T = \xi_k\xi_k^*. \]

For \( k = 0 \),
\[ \bar{A}_{-1} A_{-1}^T\xi_0\xi_0^*\bar{A}_{-1} A_{-1}^T = \xi_0\xi_0^*, \]
and so, equation (10.19) yields
\[ \bar{A}_0 A_0^T = I_n - \xi_0\xi_0^*. \]

For \( k = 1 \),
\[ \bar{A}_0 A_0^T\xi_1\xi_1^*\bar{A}_0 A_0^T = (I_n - \xi_0\xi_0^*)\xi_1\xi_1^*(I_n - \xi_0\xi_0^*). \]
By Proposition 5.1, $\xi_1$ and $\xi_0$ are pointwise orthogonal almost everywhere on $\mathbb{T}$, hence

$$\bar{A}_0 A_0^T \xi_1 \xi_1^* \bar{A}_0 A_0^T = \xi_1 \xi_1^*,$$

and in view of equation (10.19), we get

$$\bar{A}_1 A_1^T = I_n - \xi_0 \xi_0^* - \xi_1 \xi_1^*.$$

Suppose

$$\bar{A}_{\ell-1} A_{\ell-1}^T \xi_\ell \xi_\ell^* \bar{A}_{\ell-1} A_{\ell-1}^T = \xi_\ell \xi_\ell^* \quad (10.20)$$

holds for every $\ell \leq k$, where $0 \leq k \leq j$, almost everywhere on $\mathbb{T}$. By equations (10.19) and (10.20), this implies

$$\bar{A}_k A_k^T = I_n - \sum_{i=0}^{k} \xi_i \xi_i^*.$$

Let us show that

$$\bar{A}_k A_k^T \xi_{k+1} \xi_{k+1}^* \bar{A}_k A_k^T = \xi_{k+1} \xi_{k+1}^*.$$

Note that

$$\bar{A}_k A_k^T \xi_{k+1} \xi_{k+1}^* \bar{A}_k A_k^T = (I_n - \sum_{i=0}^{k} \xi_i \xi_i^*) \xi_{k+1} \xi_{k+1}^* (I_n - \sum_{i=0}^{k} \xi_i \xi_i^*).$$

By Proposition 5.1, the set $\{\xi_i(z)\}_{i=0}^{k+1}$ is pointwise orthogonal almost everywhere on $\mathbb{T}$, and therefore

$$\bar{A}_k A_k^T \xi_{k+1} \xi_{k+1}^* \bar{A}_k A_k^T = \xi_{k+1} \xi_{k+1}^*.$$

Thus, by equation (10.19),

$$\bar{A}_{k+1} A_{k+1}^T = I_n - \sum_{i=0}^{k+1} \xi_i \xi_i^*$$

almost everywhere on $\mathbb{T}$, and the assertion has been proved. \qed

**Lemma 10.4.** Let

$$\tilde{W}_i^T = (\beta_{i-1}^T \cdots \beta_0 \eta_i \ \tilde{\beta}_i) \quad (10.21)$$

be unitary-valued functions, for $i = 0, 1, \ldots, j$, as described in Lemma 4.12. For $i = 0, 1, \ldots, j$, let $B_i = \beta_0 \beta_1 \cdots \beta_i$ and $B_{-1} = I_m$. Then, for $k = 0, 1, \ldots, j$,

$$B_k B_k^* = I_m - \sum_{i=0}^{k} \tilde{\eta}_i \tilde{\eta}_i^T \quad (10.22)$$

almost everywhere on $\mathbb{T}$.

**Proof.** By equation (10.14), for $k = 0, \ldots, j$,

$$B_k B_k^* = I_m - \sum_{i=0}^{k} B_{i-1} B_{i-1}^* \tilde{\eta}_i \tilde{\eta}_i^T B_{i-1} B_{i-1}^*.$$

(10.23)

Thus, to prove condition (10.22) it suffices to show that, for $k = 0, \ldots, j$,

$$B_{k-1} B_{k-1}^* \tilde{\eta}_k \tilde{\eta}_k^T B_{k-1} B_{k-1}^* = \tilde{\eta}_k \tilde{\eta}_k^T.$$
For $k = 0$, 
\[ B_{-1}B_{-1}^* \bar{\eta}_0 \eta_0^T B_{-1}B_{-1}^* = I_m \bar{\eta}_0 \eta_0^T I_m = \bar{\eta}_0 \eta_0^T, \]
and so, equation (10.23) yields 
\[ B_0 B_0^* = I_m - \bar{\eta}_0 \eta_0^T. \]

For $k = 1$, 
\[ B_0 B_0^* \bar{\eta}_1 \eta_1^T B_0 B_0^* = (I_m - \bar{\eta}_0 \eta_0^T) \bar{\eta}_1 \eta_1^T (I_m - \bar{\eta}_0 \eta_0^T). \]
By Proposition 5.1, $\eta_1$ and $\eta_0$ are pointwise orthogonal almost everywhere on $T$, hence 
\[ B_0 B_0^* \bar{\eta}_1 \eta_1^T B_0 B_0^* = \bar{\eta}_1 \eta_1^T, \]
and in view of equation (10.23), we get 
\[ B_1 B_1^* = I_m - \bar{\eta}_0 \eta_0^T - \bar{\eta}_1 \eta_1^T. \]

Suppose 
\[ B_{\ell - 1} B_{\ell - 1}^* \bar{\eta}_\ell \eta_\ell^T B_{\ell - 1} B_{\ell - 1}^* = \bar{\eta}_\ell \eta_\ell^T \]
holds for every $\ell \leq k$, where $0 \leq k \leq j$, almost everywhere on $T$. By equations (10.23) and (10.24), this implies 
\[ B_k B_k^* = I_m - \sum_{i=0}^k \bar{\eta}_i \eta_i^T. \]
Let us show that 
\[ B_k B_k^* \bar{\eta}_{k+1} \eta_{k+1}^T B_k B_k^* = \bar{\eta}_{k+1} \eta_{k+1}^T. \]

Note that 
\[ B_k B_k^* \bar{\eta}_{k+1} \eta_{k+1}^T B_k B_k^* = (I_m - \sum_{i=0}^k \bar{\eta}_i \eta_i^T) \bar{\eta}_{k+1} \eta_{k+1}^T (I_m - \sum_{i=0}^k \bar{\eta}_i \eta_i^T). \]
By Proposition 5.1, the set \( \{\bar{\eta}_i(z)\}_{i=0}^{k+1} \) is pointwise orthogonal almost everywhere on $T$, and therefore 
\[ B_k B_k^* \bar{\eta}_{k+1} \eta_{k+1}^T B_k B_k^* = \bar{\eta}_{k+1} \eta_{k+1}^T. \]
Thus, by equation (10.23), 
\[ B_{k+1} B_{k+1}^* = I_m - \sum_{i=0}^{k+1} \bar{\eta}_i \eta_i^T \]
almost everywhere on $T$, and the assertion has been proved. \( \square \)

The following statement asserts that any function $Q_{j+1} \in \Omega_j$ necessarily satisfies equations (4.34).

**Proposition 10.5.** Every $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ which minimizes \( (s_0^\infty(G - Q), s_1^\infty(G - Q), \ldots, s_j^\infty(G - Q)) \) satisfies 
\[ (G - Q_{j+1}) x_i = t_i y_i, (G - Q_{j+1})^* y_i = t_i x_i, \quad \text{for } i = 0, 1, \ldots, j. \]

**Proof.** By the recursive step of the algorithm from Subsection 4.2, every $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ that minimizes 
\( (s_0^\infty(G - Q), \ldots, s_{j-1}^\infty(G - Q)) \)
satisfies

$$(G - Q_{j+1})x_i = t_i y_i, \quad (G - Q_{j+1})^* y_i = t_i x_i \quad \text{for} \quad i = 0, 1, \ldots, j-1.$$  

Hence, it suffices to show that $Q_{j+1}$ satisfies

$$(G - Q_{j+1})x_j = t_j y_j, (G - Q_{j+1})^* y_j = t_j x_j.$$  

Note that, by the inductive step, the following diagram commutes:

$$\begin{array}{ccc}
H^2(\mathbb{D}, \mathbb{C}^{n-j}) & \xrightarrow{M_{\theta_0 \cdots \theta_{j-1}}} & K_j \\
\downarrow H_{F_j} & & \downarrow \gamma_J \\
H^2(\mathbb{D}, \mathbb{C}^{m-j}) & \xrightarrow{M_{\theta_0 \cdots \theta_{j-1}}} & L_j
\end{array}$$

$$(\xi_{j-1})^{\wedge} \longrightarrow (\eta_{j-1})^{\wedge} \longrightarrow \xi_{j-1} \hat{H}^2(\mathbb{D}, \mathbb{C}^n) = X_j \\
(\eta_{j-1})^{\wedge} \longrightarrow \eta_{j-1} \hat{H}^2(\mathbb{D}, \mathbb{C}^n) = Y_j,
$$

where the maps $M_{\theta_0 \cdots \theta_{j-1}}$, $M_{\theta_0 \cdots \theta_{j-1}}$, $(\xi_{j-1})^{\wedge}$: $K_j \to X_j$ and $(\eta_{j-1})^{\wedge}$: $L_j \to Y_j$ are unitaries, and $F_j \in H^{\infty}(\mathbb{D}, \mathbb{C}^{m-j} \times (n-j)) + C(\mathbb{T}, \mathbb{C}^{m-j} \times (n-j)).$

By equation (10.10), the set of all level $j-1$ superoptimal error functions

$$E_{j-1} = \{G - Q : Q \in \Omega_{j-1}\}$$

satisfies

$$E_{j-1} = W_0^* W_1^* \cdots W_{j-1}^*$$

for some $F_j \in H^{\infty}(\mathbb{D}, \mathbb{C}^{m-j} \times (n-j)) + C(\mathbb{T}, \mathbb{C}^{m-j} \times (n-j))$, for quasi-continuous unimodular functions $u_i = \frac{\partial}{\partial z_i}$, $i = 0, \ldots, j-1$, and the closed ball $B(t_{j-1})$ of radius $t_{j-1}$ in $L^{\infty}(\mathbb{T}, \mathbb{C}^{m-j} \times (n-j))$. Consider some $Q_{j+1} \in \Omega_{j-1}$, so that, according to equation (10.26),

$$\left( I_{j-1} \quad 0 \right) \cdots W_0(G - Q_{j+1}) V_0 \cdots \left( I_{n-j-1} \quad 0 \right)$$

$$= \left( \begin{array}{cccc}
t_0 u_0 & 0 & \cdots & 0 \\
0 & t_1 u_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & t_{j-1} u_{j-1} \\
0 & 0 & \cdots & F_j - \tilde{Q}_j \end{array} \right),$$

where $\tilde{Q}_j \in H^{\infty}(\mathbb{D}, \mathbb{C}^{m-j} \times (n-j))$ is at minimal distance from $F_j$. Let $B_j = \beta_0 \cdots \beta_j$ and let $A_j = \alpha_0 \cdots \alpha_j$. By equations (10.3), we have

$$\left( I_{j-1} \quad 0 \right) \cdots W_0(G - Q_{j+1}) V_0 \cdots \left( I_{n-j-1} \quad 0 \right)$$

$$= \left( \begin{array}{cccc}
t_0 u_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & t_{j-1} u_{j-1} & 0 \\
0 & \cdots & F_j & - \tilde{Q}_j \end{array} \right).$$
which, combined with equation (10.27), yields
\[ B^*_{j-1}(G - Q_{j+1}) \hat{A}_{j-1} = F_j - \hat{Q}_j. \] (10.28)

Since \( \hat{Q}_j \) is at minimal distance from \( F_j \),
\[ \|F_j - \hat{Q}_j\| = \|H_{F_j}\| = t_j. \]

Note that, if \((\hat{x}_j, \hat{y}_j)\) is a Schmidt pair for \(H_{F_j}\), corresponding to \(t_j\), then, by Theorem 4.11,
\[ (F_j - \hat{Q}_j)\hat{x}_j = t_j\hat{y}_j, \quad (F_j - \hat{Q}_j)^*\hat{y}_j = t_j\hat{x}_j. \]

In view of equation (10.28), the latter equations imply
\[ B^*_{j-1}(G - Q_{j+1})\hat{A}_{j-1}\hat{x}_j = t_j\hat{y}_j, \quad A^T_{j-1}(G - Q_{j+1})^*B_{j-1}\hat{y}_j = t_j\hat{x}_j. \]

By equation (10.7),
\[ \hat{x}_j = A^T_{j-1}x_j, \quad \hat{y}_j = B^*_{j-1}y_j \]

Thus,
\[ B^*_{j-1}(G - Q_{j+1})\hat{A}_{j-1}\hat{x}_j = B^*_{j-1}(G - Q_{j+1})x_j = t_jB^*_{j-1}y_j, \]
or equivalently,
\[ B^*_{j-1}((G - Q_{j+1})x_j - t_jy_j) = 0, \]
and since, by the inductive hypothesis, \( M_{B_{j-1}} \) is a unitary map, we have
\[ (G - Q_{j+1})x_j = t_jy_j. \]

Furthermore,
\[ A^T_{j-1}(G - Q_{j+1})^*B_{j-1}\hat{y}_j = A^T_{j-1}(G - Q_{j+1})^*y_j = t_jA^T_{j-1}x_j, \]
or equivalently,
\[ A^T_{j-1}((G - Q_{j+1})^*y_j - t_jx_j) = 0. \]

By the inductive hypothesis, \( M_{\hat{A}_{j-1}} \) is a unitary map, hence
\[ (G - Q_{j+1})^*y_j = t_jx_j, \]
and therefore \( Q_{j+1} \) satisfies the required equations. \( \square \)

**Lemma 10.6.** Let
\[ \hat{V}_i = (\alpha^T_{i-1} \cdots \alpha^T_0 \xi_i \ \bar{\alpha}_i), \quad \hat{W}_i^T = (\beta^T_{i-1} \cdots \beta^T_0 \eta_i \ \bar{\beta}_i), \quad i = 0, 1, \ldots, j, \] (10.29)
be unitary-valued functions, as described in Lemma 4.12. Then
\[ \alpha^T_l H^2(\mathbb{D}, \mathbb{C}^{n-l}) = H^2(\mathbb{D}, \mathbb{C}^{n-l-1}) \]
and
\[ \beta^T_l (H^2(\mathbb{D}, \mathbb{C}^{n-l}))^\perp = H^2(\mathbb{D}, \mathbb{C}^{n-l-1})^\perp, \]
for all \( l = 0, \ldots, j. \)

**Proof.** Recall that, by Lemma 4.18, for all \( l = 0, \ldots, j \), the inner, co-outer, quasi-continuous functions \( \alpha_l, \beta_l \) of types \((n - l) \times (n - l - 1) \) and \((m - l) \times (m - l - 1) \), respectively, are left invertible. The rest of the proof is similar to Lemmas 9.4 and 9.9. \( \square \)

As a preparation for proof of the main inductive step we prove several propositions.
Proposition 10.7. Let
\[
\hat{V}_i = (\alpha_{i-1}^T \cdots \alpha_0^T \xi_i \ \hat{\alpha}_i)
\] (10.30)
be unitary-valued functions, for \( i = 0, 1, \ldots, j \), as described in Lemma 4.12. Let \( A_i = \alpha_0 \alpha_1 \cdots \alpha_i \), for \( i = 0, 1, \ldots, j \), and \( A_{-1} = I_n \). Let
\[
V_i = \begin{pmatrix} I_i & 0 \\ 0 & V_j \end{pmatrix}, \text{ for } i = 0, 1, \ldots, j,
\]
and let
\[
K_{j+1} = V_0 \cdots V_j \left( \begin{array}{c} 0_{(j+1) \times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) \end{array} \right).
\] (10.31)
Let \( \xi(j) = \xi_0 \hat{\lambda} \cdots \hat{\lambda} \xi_j \). Then,
\[
\xi(j) \hat{\lambda} K_{j+1} = \xi(j) \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n)
\]
and the operator \((\xi(j) \hat{\lambda} \cdot): K_{j+1} \rightarrow \xi(j) \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n)\) is unitary.

Proof. First let us prove that
\[
\xi_0 \hat{\lambda} \cdots \hat{\lambda} \xi_j \hat{\lambda} K_{j+1} = \xi_0 \hat{\lambda} \cdots \hat{\lambda} \xi_j \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n).
\]
By equations (10.31) and (10.15)
\[
K_{j+1} = \hat{A}_j H^2(\mathbb{D}, \mathbb{C}^{n-j-1}).
\] (10.32)
By Lemma 10.6,
\[
\alpha_j^T H^2(\mathbb{D}, \mathbb{C}^{n-l}) = H^2(\mathbb{D}, \mathbb{C}^{n-l-1})
\]
for all \( l = 0, \ldots, j \). Thus,
\[
H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) = \alpha_j^T H^2(\mathbb{D}, \mathbb{C}^{n-j})
\]
\[
= \alpha_j^T \alpha_{j-1}^T H^2(\mathbb{D}, \mathbb{C}^{n-j+1})
\]
\[
= \alpha_j^T \alpha_{j-1}^T \alpha_{j-2}^T H^2(\mathbb{D}, \mathbb{C}^{n-j+2})
\]
\[
= \cdots
\]
\[
= \alpha_j^T \alpha_{j-1}^T \cdots \alpha_1^T H^2(\mathbb{D}, \mathbb{C}^{n-1})
\]
\[
= \alpha_j^T \alpha_{j-1}^T \cdots \alpha_1^T \alpha_0^T H^2(\mathbb{D}, \mathbb{C}^n)
\]
\[
= \hat{A}_j^T H^2(\mathbb{D}, \mathbb{C}^n).
\] (10.33)
By equations (10.32) and (10.33),
\[
K_{j+1} = \hat{A}_j H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) = \hat{A}_j \hat{A}_j^T H^2(\mathbb{D}, \mathbb{C}^n).
\] (10.34)
By Lemma 10.3,
\[
\hat{A}_j \hat{A}_j^T = I_n - \sum_{k=0}^{j} \xi_k \xi_k^*.
\] (10.35)
By Proposition 5.1, \( \{\xi_i(z)\}_{i=0}^{j} \) is an orthonormal set in \( \mathbb{C}^n \) for almost every \( z \in T \). Therefore, by equations (10.34) and (10.35),
\[
\xi_0 \hat{\lambda} \cdots \hat{\lambda} \xi_j \hat{\lambda} K_{j+1} = \xi_0 \hat{\lambda} \cdots \hat{\lambda} \xi_j \hat{\lambda} \hat{A}_j \hat{A}_j^T H^2(\mathbb{D}, \mathbb{C}^n)
\]
\[
= \xi_0 \hat{\lambda} \cdots \hat{\lambda} \xi_j \hat{\lambda} (I_n - \sum_{k=0}^{j} \xi_k \xi_k^*) H^2(\mathbb{D}, \mathbb{C}^n)
\]
\[
= \xi_0 \hat{\lambda} \cdots \hat{\lambda} \xi_j \hat{\lambda} H^2(\mathbb{D}, \mathbb{C}^n).
\] (10.36)
To show that the operator \((\xi_{(j)} \cdot) : K_{j+1} \to \xi_{(j)} \wedge H^2(\mathbb{D}, \mathbb{C}^n)\) is unitary, it suffices to prove that, for every \(\vartheta \in K_{j+1},\)

\[
\|\xi_{(j)} \cdot \vartheta\|_{L^2(\mathbb{T}, \wedge^{j+2} \mathbb{C}^n)} = \|\vartheta\|_{L^2(\mathbb{T}, \mathbb{C}^n)}.
\]

Let \(\vartheta \in K_{j+1}.\) Then, by Proposition 3.12, we have

\[
\|\xi_{(j)} \cdot \vartheta\|^2_{L^2(\mathbb{T}, \wedge^{j+2} \mathbb{C}^n)} = \langle \xi_{(j)} \cdot \vartheta, \xi_{(j)} \cdot \vartheta \rangle_{L^2(\mathbb{T}, \wedge^{j+2} \mathbb{C}^n)}
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \langle \xi_{(j)}(e^{i\theta}), \vartheta(e^{i\theta}) \rangle_{\mathbb{C}^n} d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix}
1 & 0 & \cdots & \langle \xi_0(e^{i\theta}), \vartheta(e^{i\theta}) \rangle_{\mathbb{C}^n} \\
0 & 1 & \cdots & \langle \xi_1(e^{i\theta}), \vartheta(e^{i\theta}) \rangle_{\mathbb{C}^n} \\
\vdots & \vdots & \ddots & \vdots \\
\langle \vartheta(e^{i\theta}), \xi_0(e^{i\theta}) \rangle_{\mathbb{C}^n} & \cdots & \langle \vartheta(e^{i\theta}), \vartheta(e^{i\theta}) \rangle_{\mathbb{C}^n}
\end{pmatrix} d\theta.
\]

By Proposition 5.1, \(\{\xi_i(z)\}_{i=0}^j\) is an orthonormal set in \(\mathbb{C}^n\) for almost every \(z \in \mathbb{T}.\) Thus, the latter integral is equal to

\[
\frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \langle \vartheta(e^{i\theta}), \vartheta(e^{i\theta}) \rangle_{\mathbb{C}^n}
\end{pmatrix} d\theta.
\]

Note that since \(\vartheta \in K_{j+1},\)

\[
\vartheta = A_i A_j^T \psi = (I_n - \sum_{i=0}^j \xi_i \xi_i^*) \psi
\]

for some \(\psi \in H^2(\mathbb{D}, \mathbb{C}^n).\) Then, for almost every \(e^{i\theta} \in \mathbb{T},\)

\[
\langle \xi_k(e^{i\theta}), \vartheta(e^{i\theta}) \rangle_{\mathbb{C}^n} = \langle \xi_k(e^{i\theta}), (I_n - \sum_{i=0}^j \xi_i \xi_i^*) (e^{i\theta}) \psi(e^{i\theta}) \rangle_{\mathbb{C}^n}
\]

\[
= \langle \xi_k(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^n} - \langle \xi_k, \psi(e^{i\theta}) \rangle_{\mathbb{C}^n} = 0.
\]

Hence,

\[
\|\xi_{(j)} \cdot \vartheta\|^2_{L^2(\mathbb{T}, \wedge^{j+2} \mathbb{C}^n)} = \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \langle \vartheta(e^{i\theta}), \vartheta(e^{i\theta}) \rangle_{\mathbb{C}^n}
\end{pmatrix} d\theta,
\]

which yields

\[
\frac{1}{2\pi} \int_0^{2\pi} \|\vartheta(e^{i\theta})\|^2 d\theta = \|\vartheta\|^2_{L^2(\mathbb{T}, \mathbb{C}^n)}.
\]

Therefore, the operator \((\xi_{(j)} \cdot) : K_{j+1} \to \xi_{(j)} \wedge H^2(\mathbb{D}, \mathbb{C}^n)\) is unitary. \(\square\)

**Proposition 10.8.** Let

\[
\tilde{W}_i^T = (\beta_{i-1}^T \cdot \beta_0^T \eta_i \cdot \bar{\beta}_i)
\]

(10.37)
be unitary-valued functions, for \( i = 0, 1, \ldots, j \), as described in Lemma 4.12. Let \( B_i = \beta_0 \beta_1 \ldots \beta_i \), for \( i = 0, 1, \ldots, j \), and \( B_{-1} = I_m \). Let \( W_i^T = \begin{pmatrix} I_i & 0 \\ 0 & W_i^T \end{pmatrix} \), for \( i = 0, 1, \ldots, j \), and let

\[
\mathcal{L}_{j+1} = W_0^* \cdots W_j^* \left( 0_{(j+1) \times 1}^{0 \times 1} \right),
\]

(10.38)

Let \( \bar{\eta}(j) = \bar{\eta}_0 \hat{\ldots} \hat{\eta}_j \). Then,

\[
\bar{\eta}(j) \hat{\mathcal{L}}_{j+1} = \bar{\eta}(j) \hat{H}^2(\mathbb{D}, \mathbb{C}^m)\perp
\]

and the operator \( (\bar{\eta}(j) \hat{\cdot}) : \mathcal{L}_{j+1} \to \bar{\eta}(j) \hat{H}^2(\mathbb{D}, \mathbb{C}^m)\perp \) is unitary.

**Proof.** First let us prove that

\[
\bar{\eta}_0 \hat{\ldots} \hat{\eta}_j \hat{\mathcal{L}}_{j+1} = \bar{\eta}_0 \hat{\ldots} \hat{\eta}_j \hat{H}^2(\mathbb{D}, \mathbb{C}^m)\perp.
\]

By equations (10.38) and (10.13),

\[
\mathcal{L}_{j+1} = B_j H^2(\mathbb{D}, \mathbb{C}^m)\perp.
\]

(10.39)

By Lemma 10.6,

\[
\beta_j^* (H^2(\mathbb{D}, \mathbb{C}^{m-\ell})\perp = H^2(\mathbb{D}, \mathbb{C}^{m-\ell-1})\perp,
\]

for all \( \ell = 0, \ldots, j \). Thus,

\[
H^2(\mathbb{D}, \mathbb{C}^m)\perp = \beta_j^* (H^2(\mathbb{D}, \mathbb{C}^m)\perp
\]

\( = \beta_j^* \beta_j^* \匈[-H^2(\mathbb{D}, \mathbb{C}^m)\perp]
\]

\( = \beta_j^* \beta_j^* \beta_j^* \beta_j^* (H^2(\mathbb{D}, \mathbb{C}^m)\perp)
\]

\( = \ldots
\]

(10.40)

By equations (10.39) and (10.40),

\[
\mathcal{L}_{j+1} = B_j H^2(\mathbb{D}, \mathbb{C}^m)\perp = B_j B_j^* H^2(\mathbb{D}, \mathbb{C}^m)\perp.
\]

(10.41)

By Lemma 10.4,

\[
B_j B_j^* = I_m - \sum_{i=0}^j \bar{\eta}_i \eta_i^T.
\]

(10.42)

Thus,

\[
\mathcal{L}_{j+1} = (I_m - \sum_{i=0}^j \bar{\eta}_i \eta_i^T) H^2(\mathbb{D}, \mathbb{C}^m)\perp.
\]

(10.43)

By Proposition 5.1, \( \{ \bar{\eta}_i(z) \}_{i=0}^j \) is an orthonormal set in \( \mathbb{C}^m \) for almost every \( z \in \mathbb{T} \). Therefore, by equations (10.41) and (10.43),

\[
\bar{\eta}_0 \hat{\ldots} \hat{\eta}_j \hat{\mathcal{L}}_{j+1} = \bar{\eta}_0 \hat{\ldots} \hat{\eta}_j \hat{(I_m - \sum_{i=0}^j \bar{\eta}_i \eta_i^T) H^2(\mathbb{D}, \mathbb{C}^m)\perp}
\]

\( = \bar{\eta}_0 \hat{\ldots} \hat{\eta}_j \hat{H}^2(\mathbb{D}, \mathbb{C}^m)\perp.
\]

(10.44)
To show that the operator \( \hat{\eta}(j) \cdot : L_{j+1} \to \eta(j) \wedge H^2(\mathbb{D}, \mathbb{C}^{m}) \perp \) is unitary, it suffices to prove that, for every \( \varphi \in L_{j+1} \),

\[
\| \hat{\eta}(j) \wedge \varphi \|_{L^2(\mathbb{T}, \wedge j+2 \mathbb{C}^m)} = \| \varphi \|_{L^2(\mathbb{T}, \mathbb{C}^m)}.
\]

By Proposition 3.12, we have

\[
\| \hat{\eta}(j) \wedge \varphi \|_{L^2(\mathbb{T}, \wedge j+2 \mathbb{C}^m)}^2 = \langle \hat{\eta}(j) \wedge \varphi, \hat{\eta}(j) \wedge \varphi \rangle_{L^2(\mathbb{T}, \wedge j+2 \mathbb{C}^m)}
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix}
\langle \bar{\eta}_0(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \bar{\eta}_0(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m} & \cdots & \langle \bar{\eta}_0(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m} \\
\langle \bar{\eta}_1(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \ associations
\end{pmatrix} d\theta.
\]

By Proposition 5.1, the set \( \{ \bar{\eta}_i(z) \}_{i=0}^{j} \) is orthonormal in \( \mathbb{C}^n \) for almost every \( z \in \mathbb{T} \). Then the latter integral is equal to

\[
\frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix}
1 & 0 & \cdots & \langle \varphi(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m} \\
0 & 1 & \cdots & \langle \varphi(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m} \\
\cdots & \cdots & \cdots & \cdots \\
\langle \varphi(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m} & \cdots & \cdots & \langle \varphi(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m}
\end{pmatrix} d\theta.
\]

Note that since \( \varphi \in L_{j+1} \),

\[
\varphi = (I_m - \sum_{i=0}^{j} \bar{\eta}_i \eta^T_i) \psi,
\]

for some \( \psi \in H^2(\mathbb{D}, \mathbb{C}^m) \). Then, for almost every \( e^{i\theta} \in \mathbb{T} \),

\[
\langle \bar{\eta}_k(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m} = \langle \bar{\eta}_k(e^{i\theta}), (I_m - \sum_{i=0}^{j} \bar{\eta}_i \eta^T_i)(e^{i\theta}) \psi(e^{i\theta}) \rangle_{\mathbb{C}^m}
\]

\[
= \langle \bar{\eta}_k(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} - \langle \bar{\eta}_k(e^{i\theta}), \bar{\eta}_k(e^{i\theta}) \rangle_{\mathbb{C}^m} \langle \bar{\eta}_k(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} = 0.
\]

Thus,

\[
\| \hat{\eta}(j) \wedge \varphi \|_{L^2(\mathbb{T}, \wedge j+2 \mathbb{C}^m)}^2 = \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \langle \varphi(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m}
\end{pmatrix} d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \langle \varphi(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m} d\theta = \| \varphi \|_{L^2(\mathbb{T}, \mathbb{C}^m)}^2.
\]

Hence, the operator \( \hat{\eta}(j) \cdot : L_{j+1} \to \eta(j) \wedge H^2(\mathbb{D}, \mathbb{C}^m) \perp \) is unitary. \( \square \)

**Proposition 10.9.** With the notation of Proposition 10.8

\[ L_{j+1}^+ = \{ f \in L^2(\mathbb{T}, \mathbb{C}^m) : \beta_j^* \cdot \beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1}) \} \].

**Proof.** Clearly \( L_{j+1}^+ = \beta_0^* \cdots \beta_j^* H^2(\mathbb{D}, \mathbb{C}^{m-j-1}) \perp \). The general element of \( \beta_0^* \cdots \beta_j^* H^2(\mathbb{D}, \mathbb{C}^{m-j-1}) \perp \) is \( \beta_0^* \cdots \beta_j^* \tilde{g} \) with \( g \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1}) \). A function \( f \in L^2(\mathbb{T}, \mathbb{C}^m) \) belongs to \( L_{j+1}^+ \) if and only if

\[
\langle f, \beta_0^* \cdots \beta_j^* \tilde{g} \rangle_{L^2(\mathbb{T}, \mathbb{C}^m)} = 0 \text{ for all } g \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1})
\]
if and only if
\[ \frac{1}{2\pi} \int_{0}^{2\pi} \langle f(e^{i\theta}), \beta_0(e^{i\theta}) \cdot \beta_j(e^{i\theta})e^{-i\theta} g(e^{i\theta}) \rangle_{C^\infty} d\theta = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1}) \]
if and only if
\[ \frac{1}{2\pi} \int_{0}^{2\pi} \langle \beta_j^*(e^{i\theta}) \cdot \beta_0^*(e^{i\theta})f(e^{i\theta}), e^{-i\theta} g(e^{i\theta}) \rangle_{C^\infty} d\theta = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1}). \]

The latter statement is equivalent to the assertion that \( \beta_j^* \cdot \beta_0^* f \) is orthogonal to \( H^2(\mathbb{D}, \mathbb{C}^{m-j-1}) \) in \( L^2(\mathbb{T}, \mathbb{C}^{m-j-1}) \), which holds if and only if
\[ \beta_j^* \cdot \beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1}). \]

Thus,
\[ L^\perp_{j+1} = \{ f \in L^2(\mathbb{T}, \mathbb{C}^m) : \beta_j^* \cdot \beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1}) \} \]
as required. \( \square \)

Let us proceed to the main theorem of this section.

**Theorem 10.10.** Let \( m, n \) be positive integers such that \( \min(m, n) \geq 2 \). Let \( G \) be in \( H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n}) \). In the notation of the algorithm 4.2, let

\[ (\xi_0 \wedge \ldots \wedge \xi_{j-1} \wedge v_j, \bar{\eta}_0 \wedge \ldots \wedge \bar{\eta}_{j-1} \wedge w_j) \]
be a Schmidt pair for \( T_j \) corresponding to \( t_j = \|T_j\| \neq 0 \). Let \( h_j \in H^2(\mathbb{D}, \mathbb{C}) \) be the scalar outer factor of

\[ \xi_0 \wedge \ldots \wedge \xi_{j-1} \wedge v_j. \]

Let
\[ x_j = (I_n - \xi_0 \xi_0^* - \cdots - \xi_{j-1} \xi_{j-1}^*)v_j, \]
\[ y_j = (I_m - \bar{\eta}_0 \eta_0^T - \cdots - \bar{\eta}_{j-1} \eta_{j-1}^T)w_j \]
and
\[ \xi_j = \frac{x_j}{h_j}, \quad \bar{\eta}_j = \frac{y_j}{h_j}. \]

For \( i = 0, 1, \ldots, j \), let
\[ \tilde{V}_i = (\alpha_i^T \cdots \alpha_0^T \xi_i \quad \bar{\alpha}_i), \quad \tilde{W}_i^T = (\beta_i^T \cdots \beta_0^T \eta_i \quad \bar{\beta}_i) \]
be unitary-valued functions, as described in Lemma 4.12. Let
\[ V_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{V}_j \end{pmatrix}, \quad W_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{W}_j \end{pmatrix}. \]

Let \( A_j = \alpha_0 \alpha_1 \cdots \alpha_j, A_{-1} = I_n, B_j = \beta_0 \beta_1 \cdots \beta_j \) and \( B_{-1} = I_m \). Let
\[ X_{j+1} = \xi_0 \wedge \ldots \wedge \xi_j \wedge H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^n), \]
and let
\[ Y_{j+1} = \bar{\eta}_0 \wedge \ldots \wedge \bar{\eta}_j \wedge H^2(\mathbb{D}, \mathbb{C}^m) \subset H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m). \]

Let
\[ T_{j+1}(\xi_0 \wedge \ldots \wedge \xi_j \wedge x) = P_{Y_{j+1}}(\bar{\eta}_0 \wedge \ldots \wedge \bar{\eta}_j \wedge (G - Q_{j+1})x) \]
for all \( x \in H^2(\mathbb{D}, \mathbb{C}^n) \), where \( Q_{j+1} \) satisfies
\[
(G - Q_{j+1})x_i = t_i y_i, \quad \text{and} \quad (G - Q_{j+1})^* y_i = t_i x_i, \quad \text{for} \ i = 0, 1, \ldots, j. \quad (10.46)
\]

Let
\[
\mathcal{K}_{j+1} = V_0 \cdots V_j \left( H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) \right), \quad \mathcal{L}_{j+1} = W_0^* \cdots W_j^* \left( H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp \right). \quad (10.47)
\]

Let the operator \( \Gamma_{j+1} : \mathcal{K}_{j+1} \to \mathcal{L}_{j+1} \) be given by
\[
\Gamma_{j+1} = P_{\mathcal{L}_{j+1}} M_{G - Q_{j+1}} |_{\mathcal{K}_{j+1}}.
\]

Then

(i) the maps
\[
M_{\tilde{A}_j} : H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) \to \mathcal{K}_{j+1} : x \mapsto \tilde{A}_j x, \quad \text{and} \quad M_{B_j} : H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp \to \mathcal{L}_{j+1} : y \mapsto B_j y
\]
are unitaries;

(ii) the maps \((\xi_0 \wedge \ldots \wedge \xi_j \wedge \cdot) : \mathcal{K}_{j+1} \to X_{j+1}, (\bar{\eta}_0 \wedge \ldots \wedge \bar{\eta}_j \wedge \cdot) : \mathcal{L}_{j+1} \to Y_{j+1}\) are unitaries;

(iii) the following diagram commutes:
\[
\begin{array}{ccc}
H^2(\mathbb{D}, \mathbb{C}^{n-j}) & \xrightarrow{\mathcal{M}_{0^{n-j}}} & \mathcal{K}_{j+1} \\
| & \downarrow F_{j+1} & \downarrow \xi_{(j)} \wedge \cdot \\
H^2(\mathbb{D}, \mathbb{C}^{m-j})^\perp & \xrightarrow{\mathcal{M}_{0^{n-j}}} & \mathcal{L}_{j+1} \\
\end{array}
\]
where \( F_{j+1} \in \mathcal{H}^\infty(\mathbb{D}, \mathbb{C}^{m-j}(n-j-1)) + C(\mathbb{D}, \mathbb{C}^{m-j}(n-j-1)) \) is the function defined in Proposition 10.1;

(iv) \( \Gamma_{j+1} \) and \( T_{j+1} \) are compact operators;

(v) \( \| T_{j+1} \| = \| \Gamma_{j+1} \| = \| H_{F_{j+1}} \| = t_{j+1} \).

Proof. (i) It follows from Lemma 4.16. (ii) Follows from Propositions 10.7 and 10.8. (iii) By Theorem 1.3, there exists a function \( Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) such that the sequence
\[
(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \ldots, s_{j+1}^\infty(G - Q_{j+1}))
\]
is lexicographically minimized. By Proposition 10.5, any such \( Q_{j+1} \) satisfies
\[
(G - Q_{j+1})x_i = t_i y_i, (G - Q_{j+1})^* y_i = t_i x_i, \quad \text{for} \ i = 0, 1, \ldots, j. \quad (10.49)
\]
By Proposition 8.1, \( T_{j+1} \) is well defined and is independent of the choice of \( Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \) satisfying equations (10.49). We can choose \( Q_{j+1} \) which minimizes
\[
(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \ldots, s_{j+1}^\infty(G - Q_{j+1})),
\]
and therefore satisfies equations (10.49). Consider the following diagram:
\[
\begin{array}{ccc}
\mathcal{K}_{j+1} & \xrightarrow{\xi_0 \wedge \ldots \wedge \xi_j \wedge \cdot} & \xi_0 \wedge \ldots \wedge \xi_j \wedge H^2(\mathbb{D}, \mathbb{C}^n) \\
| & \downarrow \Gamma_{j+1} & \downarrow \Gamma_{j+1} \\
\mathcal{L}_{j+1} & \xrightarrow{\bar{\eta}_0 \wedge \ldots \wedge \bar{\eta}_j \wedge \cdot} & \bar{\eta}_0 \wedge \ldots \wedge \bar{\eta}_j \wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp \\
\end{array}
\]
Let us prove first that diagram (10.50) commutes. By Proposition 10.1, every \( Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \), which minimizes
\[
(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \ldots, s_{j+1}^\infty(G - Q_{j+1})),
\]
satisfies the following equation (see equation \((10.5)\)):

\[
G - Q_{j+1} = W_0^* W_1^* \cdots W_j^* \\
\times \left( \begin{array}{cccc}
t_0 u_0 & 0 & \cdots & 0 \\
0 & t_1 u_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & t_j u_j \\
\end{array} \right) \begin{bmatrix}
0_{1 \times (n-j-1)} \\
0_{1 \times (n-j-1)} \\
\vdots \\
0_{1 \times (n-j-1)} \\
\end{bmatrix} \\
= V_j^* \cdots V_0^*.
\]

Thus,

\[
(G - Q_{j+1}) V_0 \cdots V_j \left( \begin{array}{c} \frac{0_{(j+1) \times 1}}{H^2(\mathbb{D}, \mathbb{C}^{m-j-1})} \end{array} \right) \\
W_0^* W_1^* \cdots W_j^* \\
\times \left( \begin{array}{cccc}
t_0 u_0 & 0 & \cdots & 0 \\
0 & t_1 u_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & t_j u_j \\
\end{array} \right) \begin{bmatrix}
0_{1 \times (n-j-1)} \\
0_{1 \times (n-j-1)} \\
\vdots \\
0_{1 \times (n-j-1)} \\
\end{bmatrix} \\
= (G - Q_{j+1}) V_0 \cdots V_j \left( \begin{array}{c} \frac{0_{(j+1) \times 1}}{H^2(\mathbb{D}, \mathbb{C}^{m-j-1})} \end{array} \right).
\]

for some \(F_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1) \times (n-j-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-1) \times (n-j-1)})\), for the quasi-continuous unimodular functions \(u_i = \frac{\tilde{h}_i}{h_i}\), for all \(i = 0, \ldots, j\), for the closed ball \(B(t_j)\) of radius \(t_j\) in \(L^\infty(\mathbb{T}, \mathbb{C}^{(m-j-1) \times (n-j-1)})\). By equation \((10.13)\),

\[
W_0^* W_1^* \cdots W_j^* = (\bar{\eta}_0 \ B_0 B_0^* \eta_1 \ \cdots \ B_{j-1} B_{j-1}^* \bar{\eta}_j \ B_j).
\]

By equation \((10.15)\),

\[
V_0 \cdots V_j = (\xi_0 \ A_0^T \xi_1 \ A_1^T \xi_2 \ \cdots \ A_{j-1}^T A_{j-1}^T \xi_j \ A_j).
\]

Therefore, by equation \((10.52)\), for every \(\chi \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1})\),

\[
(G - Q_{j+1}) A_j \chi = B_j F_{j+1} \chi.
\]

A typical element \(x \in K_{j+1}\) is of the form \(x = A_j \chi\), for some \(\chi \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1})\). Then, by Proposition \(10.7\),

\[
(x_0 \hat{\lambda} \cdots \hat{\lambda} x_j \hat{\lambda} \chi) A_j \chi = x_0 \hat{\lambda} \cdots \hat{\lambda} x_j \hat{\lambda} A_j \chi \in X_{j+1}.
\]

Therefore, by the definition of \(T_{j+1}\) and by equation \((10.54)\),

\[
T_{j+1}(x_0 \hat{\lambda} \cdots \hat{\lambda} x_j \hat{\lambda} A_j \chi) = P_{Y_{j+1}}(\hat{\eta}_0 \hat{\lambda} \cdots \hat{\lambda} \hat{\eta}_j \hat{\lambda} (G - Q_{j+1}) A_j \chi)
\]

\[
= P_{Y_{j+1}}(\hat{\eta}_0 \hat{\lambda} \cdots \hat{\lambda} \hat{\eta}_j \hat{\lambda} B_j F_{j+1} \chi).
\]

Furthermore, by the definition of \(\Gamma_{j+1}\) and by equation \((10.54)\),

\[
(\hat{\eta}_0 \hat{\lambda} \cdots \hat{\lambda} \hat{\eta}_j \hat{\lambda} \Gamma_{j+1}(A_j \chi) = \hat{\eta}_0 \hat{\lambda} \cdots \hat{\lambda} \hat{\eta}_j \hat{\lambda} P_{\Gamma_{j+1}}(G - Q_{j+1})(A_j \chi)
\]

\[
= \hat{\eta}_0 \hat{\lambda} \cdots \hat{\lambda} \hat{\eta}_j \hat{\lambda} P_{\Gamma_{j+1}} B_j F_{j+1} \chi.
\]

In order to prove the commutativity of diagram \((10.50)\), we need to show that, for every \(\chi \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1})\),

\[
T_{j+1}(x_0 \hat{\lambda} \cdots \hat{\lambda} x_j \hat{\lambda} A_j \chi) = (\hat{\eta}_0 \hat{\lambda} \cdots \hat{\lambda} \hat{\eta}_j \hat{\lambda} \chi) \Gamma_{j+1}(A_j \chi).
\]
Hence, we must prove that, for every \( \chi \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) \),
\[
\tilde{\eta}_0 \hat{\chi} \cdots \hat{\eta}_j \hat{P}_{L^j+1} B_j F_{j+1} \chi \in Y_{j+1}
\]
and that
\[
\tilde{\eta}_0 \hat{\chi} \cdots \hat{\eta}_j \hat{P}_{L^j+1} (B_j F_{j+1} \chi - P_{L^j+1} B_j F_{j+1} \chi), \text{ which is equal to } \tilde{\eta}_0 \hat{\chi} \cdots \hat{\eta}_j \hat{P}_{L^j+1} B_j F_{j+1} \chi,
\]
is orthogonal to \( Y_{j+1} \). Observe that, by Proposition 10.8, for any \( \chi \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) \),
\[
\tilde{\eta}_0 \hat{\chi} \cdots \hat{\eta}_j \hat{P}_{L^j+1} B_j F_{j+1} \chi \text{ is indeed an element of } Y_{j+1}. \text{ To prove that }
\]
\[
\tilde{\eta}_0 \hat{\chi} \cdots \hat{\eta}_j \hat{P}_{L^j+1} B_j F_{j+1} \chi
\]
is orthogonal to \( Y_{j+1} \), it suffices to prove that
\[
\langle \tilde{\eta}_0 \hat{\chi} \cdots \hat{\eta}_j \hat{P}_{L^j+1} B_j F_{j+1} \chi, \forall \psi \rangle = 0
\]
for \( \Phi = P_{L^j+1} B_j F_{j+1} \chi \), for all \( \chi \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) \) and for all \( \psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp \). By Proposition 3.12,
\[
\langle \tilde{\eta}_0 \hat{\chi} \cdots \hat{\eta}_j \hat{P}_{L^j+1} B_j F_{j+1} \chi, \forall \psi \rangle = 0
\]
for all \( \psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp \). Recall, by Proposition 5.1, the set \( \{\eta_i\}_{i=0}^j \) is an orthonormal set in \( \mathbb{C}^m \) almost everywhere on \( \mathbb{T} \). Hence,
\[
\langle \tilde{\eta}_0 \hat{\chi} \cdots \hat{\eta}_j \hat{P}_{L^j+1} B_j F_{j+1} \chi, \forall \psi \rangle = 0
\]
for all \( \psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp \).
for all $\psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ if and only if
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( I_m - \sum_{i=0}^{j} \bar{\eta}_i e^{i\bar{\theta}} \bar{\eta}_i^T(e^{i\theta}) \right) \Phi(e^{i\theta}), \psi(e^{i\theta}) \right)_{\mathbb{C}^m} = 0
\]
for all $\psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$, which holds if and only if
\[
\left( I_m - \sum_{i=0}^{j} \bar{\eta}_i \eta_i^T \right) \Phi \in H^2(\mathbb{D}, \mathbb{C}^m).
\]
By Lemma 10.4,
\[
B_j B_j^* = I_m - \sum_{i=0}^{j} \bar{\eta}_i \eta_i^T. \quad (10.55)
\]
Thus,
\[
\langle \bar{\eta}(j) \hat{\Phi}, \bar{\eta}(j) \hat{\psi} \rangle_{L^2(T, \mathbb{C}^{m+2})} = 0
\]
for all $\psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ if and only if
\[
\frac{1}{2\pi} \int_0^{2\pi} (B_{j-1}(e^{i\theta}) B_{j-1}(e^{i\theta}) \Phi(e^{i\theta}), \psi(e^{i\theta}))_{\mathbb{C}^m} = 0
\]
which holds if and only if $B_j B_j^* \Phi \in H^2(\mathbb{D}, \mathbb{C}^m)$. By Proposition 10.9, $\Phi = P_{L_j^+} B_j F_{j+1} \chi$ satisfies the following property $B_j^* \Phi \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1})$. Hence, diagram (10.50) commutes.

Recall that, by Lemma 4.17, the following diagram also commutes:
\[
\begin{array}{ccc}
H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) & \xrightarrow{M_{\lambda_j}} & K_{j+1} \\
\downarrow H_{F_{j+1}} & & \downarrow \Gamma_{j+1} \\
H^2(\mathbb{D}, \mathbb{C}^{m-j-1}) & \xrightarrow{M_{B_j}} & L_{j+1}.
\end{array} \quad (10.56)
\]

(iv) Since $F_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(n-j-1)\times(m-j-1)}) + C(\mathbb{T}, \mathbb{C}^{(n-j-1)\times(m-j-1)})$, by Hartman’s theorem, the Hankel operator $H_{F_{j+1}}$ is compact. Since diagram (10.56) commutes and the operators $M_{\lambda_j}$ and $M_{B_j}$ are unitaries, $\Gamma_{j+1}$ is compact. By (iii),
\[
(\bar{\eta}_0 \ldots \bar{\lambda} \bar{\eta}_j \lambda)^* \circ (M_{B_j} \circ H_{F_{j+1}} \circ M_{\lambda_j}) \circ (\xi_0 \ldots \lambda \xi_j \lambda)^* = T_{j+1}.
\]
By (i) and (ii), the operators $M_{\lambda_j}$, $M_{B_j}$, $(\xi_0 \ldots \lambda \xi_j \lambda)$ and $(\bar{\eta}_0 \ldots \bar{\lambda} \bar{\eta}_j \lambda)$ are unitaries, hence, $T_{j+1}$ is a compact operator.

(v) Since diagram (10.48) commutes and the operators $M_{\lambda_j}$, $M_{B_j}$, $(\xi_0 \ldots \lambda \xi_j \lambda)$ and $(\bar{\eta}_0 \ldots \bar{\lambda} \bar{\eta}_j \lambda)$ are unitaries,
\[
\|T_{j+1}\| = \|\Gamma_{j+1}\| = \|H_{F_{j+1}}\| = t_{j+1}. \quad \square
\]

**Lemma 10.11.** Let $v_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $w_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ be such that
\[
(\xi_0 \ldots \lambda \xi_j \lambda v_{j+1}, \bar{\xi}_0 \ldots \bar{\lambda} \bar{\xi}_j \bar{\lambda} w_{j+1})
\]
is a Schmidt pair for the operator $T_{j+1}$ corresponding to $\|T_{j+1}\|$. Then
\begin{enumerate}
\item[(i)] there exist $x_{j+1} \in K_{j+1}$ and $y_{j+1} \in L_{j+1}$ such that $(x_{j+1}, y_{j+1})$ is a Schmidt pair for the operator $\Gamma_{j+1}$;
\item[(ii)] for any $x_{j+1} \in K_{j+1}$ and $y_{j+1} \in L_{j+1}$ such that
\[
\xi_0 \ldots \lambda \xi_j \lambda x_{j+1} = \bar{\xi}_0 \ldots \bar{\lambda} \bar{\xi}_j \bar{\lambda} y_{j+1} = \bar{\eta}_0 \ldots \bar{\lambda} \bar{\eta}_j \bar{\lambda} w_{j+1},
\]
the pair $(x_{j+1}, y_{j+1})$ is a Schmidt pair for $\Gamma_{j+1}$ corresponding to $\|\Gamma_{j+1}\|$.
\end{enumerate}
Proof. (i) By Theorem 10.10, the operator $\Gamma_{j+1} : \mathcal{K}_{j+1} \to \mathcal{L}_{j+1}$ is compact and
$$\|\Gamma_{j+1}\| = \|T_{j+1}\| = t_{j+1}.$$ Hence, there exist $x_{j+1} \in \mathcal{K}_{j+1}$, $y_{j+1} \in \mathcal{L}_{j+1}$ such that $(x_{j+1}, y_{j+1})$ is a Schmidt pair for $\Gamma_{j+1}$ corresponding to $\|\Gamma_{j+1}\| = t_{j+1}$.

(ii) Suppose that $x_{j+1} \in \mathcal{K}_{j+1}$, $y_{j+1} \in \mathcal{L}_{j+1}$ satisfy
\begin{align*}
\xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} x_{j+1} &= \xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} v_{j+1}, \\
\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} y_{j+1} &= \bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} w_{j+1}.
\end{align*}
(10.57)

(10.58)

Let us show that $(x_{j+1}, y_{j+1})$ is a Schmidt pair for $\Gamma_{j+1}$, that is,
$$\Gamma_{j+1} x_{j+1} = t_{j+1} y_{j+1}, \quad \Gamma^*_ {j+1} y_{j+1} = t_{j+1} x_{j+1}.$$ Since diagram (10.50) commutes,
$$T_{j+1} \circ (\xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} \cdot) = (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} \cdot) \circ \Gamma_{j+1}$$
and
$$\Gamma_{j+1} T_{j+1} = T_{j+1} (\xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} v_{j+1}) = t_{j+1} (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} w_{j+1}).$$
(10.59)

By hypothesis,
$$T_{j+1} (\xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} v_{j+1}) = t_{j+1} (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} w_{j+1}).$$
(10.60)

Thus, by equations (10.57), (10.58) and (10.60),
$$\Gamma_{j+1} x_{j+1} = (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} \cdot) T_{j+1} (\xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} v_{j+1})$$
$$= (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} \cdot) t_{j+1} (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} w_{j+1})$$
$$= t_{j+1} (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} \cdot) (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} y_{j+1}).$$
Hence,
$$\Gamma_{j+1} x_{j+1} = t_{j+1} (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} \cdot) (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} y_{j+1}) y_{j+1} = t_{j+1} y_{j+1}.$$ By equation (10.57),
$$x_{j+1} = (\xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} \cdot) (\xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} v_{j+1}),$$
and, by equation (10.58),
$$(\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} \cdot) (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} w_{j+1}) = y_{j+1}.$$ Thus,
$$\Gamma^*_ {j+1} y_{j+1} = \Gamma^*_ {j+1} (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} \cdot) (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} w_{j+1})$$
$$= (\xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} \cdot) T^*_ {j+1} (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} w_{j+1}),$$
the last equality following by the second equation of (10.59). By equations (10.57) and (10.60), we have
$$T^*_ {j+1} (\bar{\eta}_0 \hat{\wedge} \cdots \hat{\wedge} \bar{\eta}_j \hat{\wedge} w_{j+1}) = t_{j+1} (\xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} x_{j+1}) = t_{j+1} (\xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} x_{j+1}),$$
and so,
$$\Gamma^*_ {j+1} y_{j+1} = t_{j+1} x_{j+1}.$$
Therefore, \((x_{j+1}, y_{j+1})\) is a Schmidt pair for \(\Gamma_{j+1}\) corresponding to \(||\Gamma_{j+1}|| = t_{j+1}\).

**Lemma 10.12.** Suppose that
\[
(\xi_0 \wedge \cdots \wedge \xi_j \wedge v_{j+1}, \eta_0 \wedge \cdots \wedge \eta_j \wedge w_{j+1})
\]
is a Schmidt pair for the operator \(T_{j+1}\) corresponding to \(||T_{j+1}|| = t_{j+1}\). Let
\[
x_{j+1} = (I_m - \xi_0 \xi_0^\ast - \cdots - \xi_j \xi_j^\ast) v_{j+1},
\]
\[
y_{j+1} = (I_m - \eta_0 \eta_0^\ast - \cdots - \eta_j \eta_j^\ast) w_{j+1},
\]
and let
\[
\hat{x}_{j+1} = A^T_j x_{j+1}, \quad \hat{y}_{j+1} = B^\ast_j y_{j+1}.
\]
Then
(i) the pair \((x_{j+1}, y_{j+1})\) is a Schmidt pair for the operator \(\Gamma_{j+1}\) corresponding to \(t_{j+1}\);
(ii) the pair \((\hat{x}_{j+1}, \hat{y}_{j+1})\) is a Schmidt pair for \(H_{F_{j+1}}\) corresponding to \(||H_{F_{j+1}}|| = t_{j+1}||\).

**Proof.** By Lemmas 10.3 and 10.4,
\[
x_{j+1} = (I_m - \xi_0 \xi_0^\ast - \cdots - \xi_j \xi_j^\ast) v_{j+1} = \bar{A}_j A_j^T v_{j+1} \tag{10.61}
\]
and
\[
y_{j+1} = (I_m - \eta_0 \eta_0^\ast - \cdots - \eta_j \eta_j^\ast) w_{j+1} = B_j B_j^\ast w_{j+1}. \tag{10.62}
\]
Hence,
\[
\hat{x}_{j+1} = A^T_j x_{j+1} = A^T_j \bar{A}_j A_j^T v_{j+1} = A^T_j v_{j+1} \tag{10.63}
\]
and
\[
\hat{y}_{j+1} = B_j^\ast y_{j+1} = B_j^\ast B_j B_j^\ast w_{j+1} = B_j^\ast w_{j+1}. \tag{10.64}
\]
These imply that \(\hat{x}_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1}), x_{j+1} \in K_{j+1}, \hat{y}_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1})^\perp\) and \(y_{j+1} \in L_{j+1}\). By Proposition 5.1,
\[
\xi_0 \wedge \cdots \wedge \xi_j \wedge x_{j+1} = \xi_0 \wedge \cdots \wedge \xi_j \wedge v_{j+1} \quad \text{and} \quad \eta_0 \wedge \cdots \wedge \eta_j \wedge y_{j+1} = \eta_0 \wedge \cdots \wedge \eta_j \wedge w_{j+1}.
\]
Thus, by Proposition 10.11, the pair \((x_{j+1}, y_{j+1})\) is a Schmidt pair for \(\Gamma_{j+1}\) corresponding to \(||\Gamma_{j+1}||\). Therefore,
\[
\Gamma_{j+1} x_{j+1} = t_{j+1} y_{j+1}, \quad \Gamma_{j+1}^\ast y_{j+1} = t_{j+1} x_{j+1}. \tag{10.65}
\]
To show that the pair \((\hat{x}_{j+1}, \hat{y}_{j+1})\) is a Schmidt pair for \(H_{F_{j+1}}\) corresponding to \(t_{j+1}\), we need to prove that
\[
H_{F_{j+1}} \hat{x}_{j+1} = t_{j+1} \hat{y}_{j+1}, \quad H_{F_{j+1}}^\ast \hat{y}_{j+1} = t_{j+1} \hat{x}_{j+1}.
\]
By Theorem 10.10,
\[
H_{F_{j+1}} = (M_{B_j})^\ast \circ \Gamma_{j+1} \circ M_{A_j}, \tag{10.66}
\]
and
\[
H_{F_{j+1}}^\ast = M_{A_j}^\ast \circ \Gamma_{j+1}^\ast \circ M_{B_j}. \tag{10.67}
\]
By equation (10.66), we have
\[
H_{F_{j+1}} \hat{x}_{j+1} = H_{F_{j+1}} A_j^T x_{j+1}
\]
\[
= B_j^\ast t_{j+1} \bar{A}_j A_j^T x_{j+1}.
\]

Note that, by equations (10.61) and (10.63),
\[ x_{j+1} = \tilde{A}_j A_j^T x_{j+1}. \]  
(10.69)

Hence, by equations (10.65) and (10.68), we obtain
\[ H_{F_{j+1}} \tilde{x}_{j+1} = B_j^* \Gamma_{j+1} x_{j+1} = t_{j+1} B_j^* y_{j+1} = t_{j+1} \tilde{y}_{j+1}. \]

Let us show that \( H_{F_{j+1}} \tilde{y}_{j+1} = t_{j+1} \tilde{x}_{j+1} \). By equations (10.67) and (10.65), we have
\[ H_{F_{j+1}} \tilde{y}_{j+1} = H_{F_{j+1}}^* B_j^* y_{j+1} = A_j^T \Gamma_{j+1}^* B_j^* y_{j+1}. \]
(10.70)

Observe that, in view of equations (10.62) and (10.64),
\[ y_{j+1} = B_j B_j^* y_{j+1}. \]
(10.71)

Hence, by equations (10.65) and (10.70), we obtain
\[ H_{F_{j+1}}^* \tilde{y}_{j+1} = A_j^T \Gamma_{j+1}^* y_{j+1} = t_{j+1} A_j^T x_{j+1} = t_{j+1} \tilde{x}_{j+1}. \]

Therefore, \((\tilde{x}_{j+1}, \tilde{y}_{j+1})\) is a Schmidt pair for the Hankel operator \( H_{F_{j+1}} \) corresponding to \( \|H_{F_{j+1}}\| = t_{j+1}. \)

**Proposition 10.13.** Let
\[(\xi_0 \hat{\wedge} \cdots \hat{\wedge} \xi_j \hat{\wedge} v_{j+1}, \eta_0 \hat{\wedge} \cdots \hat{\wedge} \eta_j \hat{\wedge} w_{j+1})\]
be a Schmidt pair for \( T_{j+1} \) corresponding to \( t_{j+1} \) for some \( v_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^n) \), \( w_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^m) \). Let
\[ x_{j+1} = (I_n - \xi_0 \xi_0^* - \cdots - \xi_j \xi_j^*) v_{j+1}, \quad y_{j+1} = (I_m - \eta_0 \eta_0^* - \cdots - \eta_j \eta_j^*) w_{j+1}, \]
and let
\[ \tilde{x}_{j+1} = A_j^T x_{j+1} \quad \text{and} \quad \tilde{y}_{j+1} = B_j^* y_{j+1}. \]
(10.72)

Then
\[ \|\xi_0(z) \hat{\wedge} \cdots \hat{\wedge} \xi_j(z) \hat{\wedge} v_{j+1}(z)\|_{\wedge^{j+2} \mathbb{C}^n} = \|\eta_0(z) \hat{\wedge} \cdots \hat{\wedge} \eta_j(z) \hat{\wedge} w_{j+1}(z)\|_{\wedge^{j+2} \mathbb{C}^m} = |h_{j+1}(z)|, \]
\[ \|\tilde{x}_{j+1}(z)\|_{\mathbb{C}^n} = \|\tilde{y}_{j+1}(z)\|_{\mathbb{C}^m} = |h_{j+1}(z)|, \quad \text{and} \]
\[ \|x_{j+1}(z)\|_{\mathbb{C}^n} = \|y_{j+1}(z)\|_{\mathbb{C}^m} = |h_{j+1}(z)|, \]
(10.73)
amost everywhere on \( \mathbb{T} \).

**Proof.** By Lemma 10.12, \((\tilde{x}_{j+1}, \tilde{y}_{j+1})\) is a Schmidt pair for \( H_{F_{j+1}} \) corresponding to \( \|H_{F_{j+1}}\| = t_{j+1} \). Hence,
\[ H_{F_{j+1}} \tilde{x}_{j+1} = t_{j+1} \tilde{y}_{j+1} \quad \text{and} \quad H_{F_{j+1}}^* \tilde{y}_{j+1} = t_{j+1} \tilde{x}_{j+1}. \]

By Theorem 4.11,
\[ t_{j+1} \|\tilde{y}_{j+1}(z)\|_{\mathbb{C}^m} = \|H_{F_{j+1}}  \| \|\tilde{x}_{j+1}(z)\|_{\mathbb{C}^n} \]
amost everywhere on \( \mathbb{T} \). Thus,
\[ \|\tilde{y}_{j+1}(z)\|_{\mathbb{C}^m} = \|\tilde{x}_{j+1}(z)\|_{\mathbb{C}^n}, \]
(10.74)
amost everywhere on \( \mathbb{T} \).
Note that $\tilde{A}_j(z)$ is isometric for almost every $z \in \mathbb{T}$, and therefore, by equations (10.72), we obtain

$$\|x_{j+1}(z)\|_{C^n} = \|\hat{x}_{j+1}(z)\|_{C^{n-j-1}}.$$  

Moreover, since $B_j(z)$ are isometries almost everywhere on $\mathbb{T}$, by equations (10.72), we have

$$\|y_{j+1}(z)\|_{C^m} = \|\hat{y}_{j+1}(z)\|_{C^{m-j-1}}$$

almost everywhere on $\mathbb{T}$. By equations (10.74), we deduce

$$\|x_{j+1}(z)\|_{C^n} = \|y_{j+1}(z)\|_{C^m}$$  \hspace{1cm} (10.75)

almost everywhere on $\mathbb{T}$.

By Proposition 5.1,

$$\xi_0 \hat{\cdots} \hat{\xi}_j \hat{x}_{j+1} = \xi_0 \hat{\cdots} \hat{\xi}_j \hat{v}_{j+1}$$  \hspace{1cm} (10.76)

and

$$\tilde{\eta}_0 \hat{\cdots} \hat{\tilde{\eta}}_j \hat{y}_{j+1} = \tilde{\eta}_0 \hat{\cdots} \hat{\tilde{\eta}}_j \hat{w}_{j+1}.$$  \hspace{1cm} (10.77)

Hence, by Proposition 3.14,

$$\|\xi_0(z) \wedge \cdots \wedge \xi_j(z) \wedge v_{j+1}(z)\|_{\Lambda^{j+2}C^n} = \|\xi_0(z) \wedge \cdots \wedge \xi_j(z) \wedge x_{j+1}(z)\|_{\Lambda^{j+2}C^n},$$

$$= \|x_{j+1}(z) - \sum_{i=0}^j (x_{j+1}(z), \xi_i(z))\xi_i(z)\|_{C^n} = \|x_{j+1}(z)\|_{C^n},$$

almost everywhere on $\mathbb{T}$. Furthermore,

$$\|\tilde{\eta}_0(z) \wedge \cdots \wedge \tilde{\eta}_j(z) \wedge w_{j+1}(z)\|_{\Lambda^{j+2}C^m} = \|\tilde{\eta}_0(z) \wedge \cdots \wedge \tilde{\eta}_j(z) \wedge y_{j+1}(z)\|_{\Lambda^{j+2}C^m}$$

$$= \|y_{j+1}(z) - \sum_{i=0}^j (y_{j+1}(z), \tilde{\eta}_i(z))\tilde{\eta}_i(z)\|_{C^m} = \|y_{j+1}(z)\|_{C^m},$$

almost everywhere on $\mathbb{T}$.

Thus, by equation (10.75),

$$\|\tilde{\eta}_0(z) \wedge \cdots \wedge \tilde{\eta}_j(z) \wedge w_{j+1}(z)\|_{\Lambda^{j+2}C^m} = \|\xi_0(z) \wedge \cdots \wedge \xi_j(z) \wedge v_{j+1}(z)\|_{\Lambda^{j+2}C^n}$$

almost everywhere on $\mathbb{T}$.

Recall that $h_{j+1}$ is the scalar outer factor of $\xi_0 \wedge \cdots \wedge \xi_j \wedge v_{j+1}$. Hence,

$$\|\hat{x}_{j+1}(z)\|_{C^{n-j-1}} = \|\hat{y}_{j+1}(z)\|_{C^{m-j-1}} = |h_{j+1}(z)|,$$

$$\|x_{j+1}(z)\|_{C^n} = \|y_{j+1}(z)\|_{C^m} = |h_{j+1}(z)|$$

and

$$\|\xi_0(z) \wedge \cdots \wedge \xi_j(z) \wedge v_{j+1}(z)\|_{\Lambda^{j+2}C^n} = \|\tilde{\eta}_0(z) \wedge \cdots \wedge \tilde{\eta}_j(z) \wedge w_{j+1}(z)\|_{\Lambda^{j+2}C^m} = |h_{j+1}(z)|,$$

almost everywhere on $\mathbb{T}$. \hfill $\square$

**Proposition 10.14.** In the notation of Theorem 10.10, there exist unitary-valued functions $\tilde{V}_{j+1}, \tilde{W}_{j+1}$ of types $(n-j-1) \times (n-j-2)$ and $(m-j-1) \times (m-j-2)$, respectively, of the form

$$\tilde{V}_{j+1} = (A_j \xi_{j+1} \tilde{\alpha}_{j+1}), \quad \tilde{W}_{j+1}^T = (B_j^T \eta_{j+1} \tilde{\beta}_{j+1}),$$
where $\alpha_{j+1}, \beta_{j+1}$ are inner, co-outer, quasi-continuous and all minors on the first columns of $\tilde{V}_{j+1}, \tilde{W}_{j+1}$ are in $H^\infty$. Furthermore, the set $\mathcal{E}_{j+1}$ of all level $j + 1$ superoptimal error functions for $G$ is equal to the following set

$$W_0^* \cdots \begin{pmatrix} I_{j+1} & 0 \\ 0 & \tilde{W}_{j+1}^* \end{pmatrix} \begin{pmatrix} t_0 u_0 & 0 & 0 & 0 \\ 0 & t_1 u_1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & t_{j+1} u_{j+1} & 0 \\ 0 & 0 & 0 & (F_{j+2} + H^\infty) \cap B(t_{j+1}) \end{pmatrix} \begin{pmatrix} I_{j+1} & 0 \\ 0 & \tilde{V}_{j+1}^* \end{pmatrix} \cdots V_0^*,$$

where $F_{j+2} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-2) \times (n-j-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-2) \times (n-j-2)})$, $u_{j+1} = (\bar{z}\tilde{h}_{j+1}/h_{j+1})$ is a quasi-continuous unimodular function and $B(t_{j+1})$ is the closed ball of radius $t_{j+1}$ in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-j-2) \times (n-j-2)})$.

Proof. Recall that, in diagrams (10.50) and (10.56), the operators $M_{\lambda_j}$, $M_{B_j}$, $(\xi_0 \lambda \cdots \lambda \xi_j)$ and $(\bar{\eta}_0 \lambda \cdots \lambda \bar{\eta}_j \lambda)$ are unitaries. Since both diagrams commute and $(x_{j+1}, y_{j+1})$ defined above is a Schmidt pair for $\Gamma_{j+1}$ corresponding to $t_{j+1}$, by Lemma 10.12, $(\tilde{x}_{j+1}, \tilde{y}_{j+1})$ is a Schmidt pair for $H_{F_{j+1}}$ corresponding to $t_{j+1}$, where

$$\tilde{x}_{j+1} = A_j x_{j+1}, \quad \tilde{y}_{j+1} = B_j^* y_{j+1}.$$  

We intend to apply Lemma 4.12 to $H_{F_{j+1}}$ and the Schmidt pair $(\tilde{x}_{j+1}, \tilde{y}_{j+1})$ to find unitary-valued functions $\tilde{V}_{j+1}, \tilde{W}_{j+1}$ such that, for every $\tilde{Q}_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1) \times (n-j-1)})$ which is at minimal distance from $F_{j+1}$, we obtain a factorization of the form

$$F_{j+1} - \tilde{Q}_{j+1} = \tilde{W}_{j+1}^* \begin{pmatrix} t_{j+1} u_{j+1} & 0 \\ 0 & F_{j+2} \end{pmatrix} \tilde{V}_{j+1}^*,$$

for some $F_{j+2} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-2) \times (n-j-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-2) \times (n-j-2)})$. For this purpose we find the inner–outer factorizations of $\tilde{x}_{j+1}$ and $\tilde{y}_{j+1}$.

By Proposition 10.13,

$$\|\tilde{x}_{j+1}(z)\|_{C^{n-j-1}} = |h_{j+1}(z)|$$

and

$$\|\tilde{y}_{j+1}(z)\|_{C^{n-j-1}} = |h_{j+1}(z)|,$$

almost everywhere on $\mathbb{T}$. Equations (10.78) and (10.79) imply that $h_{j+1} \in H^2(\mathbb{D}, \mathbb{C})$ is the scalar outer factor of both $\tilde{x}_{j+1}$ and $\tilde{y}_{j+1}$. Hence, by Lemma 4.12, $\tilde{x}_{j+1}, \tilde{y}_{j+1}$ admit the inner–outer factorizations

$$\tilde{x}_{j+1} = \hat{\xi}_{j+1} h_{j+1}, \quad \tilde{y}_{j+1} = \hat{\eta}_{j+1} h_{j+1},$$

for some inner $\hat{\xi}_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{n-j-1}), \hat{\eta}_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m-j-1})$. Then

$$\tilde{x}_{j+1} = \hat{\xi}_{j+1} h_{j+1} = A_j^T x_{j+1}, \quad \tilde{y}_{j+1} = \hat{\eta}_{j+1} h_{j+1} = \bar{z} B_j^T \bar{y}_{j+1},$$

from which we obtain

$$\hat{\xi}_{j+1} = A_j^T \xi_{j+1}, \quad \hat{\eta}_{j+1} = B_j^T \eta_{j+1}.$$

We wish to show that $A_j^T \xi_{j+1}, B_j^T \eta_{j+1}$ are inner functions in order to apply Lemma 4.12. Observe that, by equations (10.61) and (10.62),

$$x_{j+1} = A_j A_j^T v_{j+1}, \quad y_{j+1} = B_j B_j^* w_{j+1}.$$
Then,
\[ A_j^T x_{j+1} = A_j^T v_{j+1}, \quad B_j^T y_{j+1} = B_j^T w_{j+1}, \]
and since
\[ \xi_{j+1} = \frac{x_{j+1}}{h_{j+1}}, \quad \eta_{j+1} = \frac{y_{j+1}}{h_{j+1}}, \]
the functions
\[ A_j^T \xi_{j+1} = \frac{A_j^T v_{j+1}}{h_{j+1}}, \quad B_j^T \eta_{j+1} = \frac{B_j^T w_{j+1}}{h_{j+1}}, \]
are analytic. Furthermore, by Proposition 5.1 \( \|\xi_{j+1}(z)\|_{C^n} = 1 \) and \( \|\eta_{j+1}(z)\|_{C^m} = 1 \) almost everywhere on \( T \), and by equations (10.73),
\[ \|A_j^T(z)\xi_{j+1}(z)\|_{C^{n-j-1}} = 1, \quad \|B_j^T(z)\eta_{j+1}(z)\|_{C^{m-j-1}} = 1 \]
almost everywhere on \( T \). Thus, \( A_j^T \xi_{j+1}, B_j^T \eta_{j+1} \) are inner functions.

By Lemma 4.12, there exist inner, co-outer, quasi-continuous functions \( \alpha_{j+1}, \beta_{j+1} \) of types \((n-j-1) \times (n-j-2), (m-j-1) \times (m-j-2)\), respectively, such that the functions
\[ \tilde{V}_{j+1} = (A_j^T \xi_{j+1} \alpha_{j+1}), \quad \tilde{W}_{j+1} = (B_j^T \eta_{j+1} \beta_{j+1}) \]
are unitary-valued with all minors on the first columns in \( H^\infty \). Furthermore, by Lemma 4.12, every \( \tilde{Q}_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1) \times (n-j-1)}) \) which is at minimal distance from \( F_{j+1} \) satisfies
\[ F_{j+1} - \tilde{Q}_{j+1} = \tilde{W}_{j+1}^* \begin{pmatrix} t_{j+1} u_{j+1} & 0 \\ 0 & F_{j+2} \end{pmatrix} \tilde{V}_{j+1}, \]
for some \( F_{j+2} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-2) \times (n-j-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-2) \times (n-j-2)}) \), where \( u_{j+1} \) is a quasi-continuous unimodular function given by \( u_{j+1} = \tilde{z} / h_{j+1} \).

By Lemma 4.15, the set
\[ \tilde{E}_{j+1} = \{ F_{j+1} - \tilde{Q} : \tilde{Q} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1) \times (n-j-1)}), \| F_{j+1} - \tilde{Q} \|_{L^\infty} = t_{j+1} \} \]
satisfies
\[ \tilde{E}_{j+1} = \tilde{W}_{j+1}^* \begin{pmatrix} t_{j+1} u_{j+1} & 0 \\ 0 & (F_{j+2} + H^\infty) \cap B(t_{j+1}) \end{pmatrix} \tilde{V}_{j+1}, \]
where \( B(t_{j+1}) \) is the closed ball of radius \( t_{j+1} \) in \( L^\infty(\mathbb{T}, \mathbb{C}^{(m-j-2) \times (n-j-2)}) \). Thus, by Proposition 10.1, \( \tilde{E}_{j+1} \) admits the factorization claimed. \( \square \)

**Theorem 10.15.** Let \( G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) + C(\mathbb{T}, \mathbb{C}^{m \times n}) \). Let \( T_i, x_i, y_i, h_i, \) for \( i \geq 0 \), be defined by the algorithm from Subsection 4.2. Let \( r \) be the least index \( j \geq 0 \) such that \( T_j = 0 \). Then \( r \leq \min(m, n) \) and the superoptimal approximant \( AG \) is given by the formula
\[
G - AG = \sum_{i=0}^{r-1} \frac{t_i y_i x_i}{|h_i|^2}.
\]

**Proof.** First observe that, if \( T_0 = H_G = 0 \), then this implies \( G \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n}) \), and so \( AG = G \).

Otherwise, let \( t_0 = \|H_G\| > 0 \). If \( T_1 = 0 \), by Theorem 9.1, \( H_{F_1} = 0 \), that is,
\[ F_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1) \times (n-1)}). \]
Then, by Lemma 4.15, we have
\[ W_0(G - \mathcal{A}G)V_0 = \begin{pmatrix} t_0u_0 & 0 \\ 0 & 0 \end{pmatrix}. \]
Equivalently
\[ G - \mathcal{A}G = W_0^* \begin{pmatrix} t_0u_0 & 0 \\ 0 & 0 \end{pmatrix} V_0^* \]
\[ = (\tilde{\eta}_0 \beta_0) \begin{pmatrix} t_0u_0 & 0 \\ 0 & 0 \end{pmatrix} \left( \begin{pmatrix} \xi_0^* \\ \alpha_0^* \end{pmatrix} \right) \]
\[ = (\tilde{\eta}_0 t_0u_0 \eta_0) \begin{pmatrix} \xi_0^* \\ \alpha_0^* \end{pmatrix} = \tilde{\eta}_0 t_0u_0 \xi_0^* \]
\[ = t_0 \frac{zy_0 \bar{z}h_0 x_0^*}{h_0} = \frac{t_0y_0x_0^*}{|h_0|^2}. \]

Let \( j \) be a non-negative integer such that \( T_j = 0 \) and \( T_i \neq 0 \) for \( 1 \leq i < j \). By the commutativity of the diagrams (10.50) and (10.56), \( H_{F_j} = 0 \), and therefore \( F_j \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j) \times (n-j)}) \). By Proposition 10.14, the superoptimal analytic approximant \( \mathcal{A}G \) satisfies equation (10.5), that is,
\[ G - \mathcal{A}G = W_0^*W_1^* \cdots W_{j-1}^* \begin{pmatrix} t_0u_0 & 0 & \ldots & 0 \\ 0 & t_1u_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & t_{j-1}u_{j-1} & 0 \\ 0 & \ldots & \ldots & 0 \end{pmatrix} V_{j-1}^* \cdots V_1^* V_0^*, \] (10.80)

where, for \( i = 0, 1, \ldots, j-1 \),
\[ \tilde{V}_i = (\alpha_i^T \cdots \alpha_0^T \xi_i \tilde{\alpha}_i), \quad \tilde{W}_i^T = (\beta_i^T \cdots \beta_0^T \eta_i \tilde{\beta}_i) \]
are unitary-valued functions, as described in Proposition 10.4, \( u_i = (\bar{z}h_i/h_i) \) are quasi-continuous unimodular functions, and
\[ V_i = \begin{pmatrix} I_i & 0 \\ 0 & \tilde{V}_i \end{pmatrix}, \quad W_i = \begin{pmatrix} I_i & 0 \\ 0 & \tilde{W}_i \end{pmatrix}. \]
Recall that, by equations (4.41), for \( i = 0, \ldots, j-1 \),
\[ \xi_i = \frac{x_i}{h_i}, \quad \eta_i = \frac{\bar{z}y_i}{h_i}. \] (10.81)

By Proposition 10.13, for \( i = 0, \ldots, j-1 \),
\[ |h_i(z)| = \|x_i(z)\|_{\mathbb{C}^n} = \|y_i(z)\|_{\mathbb{C}^m} \]
almost everywhere on \( \mathbb{T} \).

With the aid of the formulae (10.13) and (10.15), equation (10.80) simplifies to
\[ G - \mathcal{A}G = \frac{t_0y_0x_0^*}{|h_0|^2} + t_1 \frac{1}{|h_1|^2} B_0B_0^* y_1 x_1^* \tilde{A}_0A_0^* + \ldots \\
+ t_{j-1} \frac{1}{|h_{j-1}|^2} B_{j-1}B_{j-1}^* y_{j-1} x_{j-1}^* \tilde{A}_{j-1}A_{j-1}^T. \] (10.82)

By equations (10.69) and (10.71), for \( i = 0, \ldots, j-1 \),
\[ x_i^* = x_i^* \tilde{A}_{i-1}A_{i-1}^T \text{ and } y_i = B_{i-1}B_{i-1}^* y_i. \]
Thus, 
\[ G - \mathcal{A}G = \sum_{i=0}^{r-1} t_i y_i x_i^* \]
and the assertion has been proved. □

11. Application of the algorithm

Let us now apply the new algorithm to the example Peller and Young solved in [26].

**Problem 11.1.** Let 
\[ G = B^{-1}A \in L^\infty(\mathbb{C}^{2 \times 2}), \]
where 
\[ A(z) = \begin{pmatrix} \sqrt{3} + 2z & 0 \\ 0 & 1 \end{pmatrix}, \quad B(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} z^2 & z \\ z & 1 \end{pmatrix} \]
for all \( z \in \mathbb{T} \).

Find the superoptimal singular values of \( G \) and its superoptimal approximant \( \mathcal{A}G \in H^\infty \), that is, the unique \( \mathcal{A}G \in H^\infty(\mathbb{D}, \mathbb{C}^{2 \times 2}) \) such that the sequence
\[ s^\infty(G - \mathcal{A}G) = (s_0^\infty(G - \mathcal{A}G), s_1^\infty(G - \mathcal{A}G), \ldots) \]
is lexicographically minimized.

On \( \mathbb{T} \) \( G \) is
\[ G(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{3}z^2 + 2z & z \\ \sqrt{3}z + 2 & -1 \end{pmatrix}. \]

**Step 0:** The operator \( H_G^* H_G \) with the respect to the orthonormal basis
\[ B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} z \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \]
of \((z^2H^2)_\perp\), has matrix representation
\[ H_G^* H_G \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 10 & 2\sqrt{3} & 2 & 0 \\ 2\sqrt{3} & 3 & \sqrt{3} & 0 \\ 2 & \sqrt{3} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

Then \( \|H_G\| = \sqrt{6} \) and a non-zero vector \( x_0 \in H^2(\mathbb{D}, \mathbb{C}^2) \) such that
\[ \|H_Gx_0\|_{H^2(\mathbb{D}, \mathbb{C}^2)_\perp} = \|H_G\| \|x_0\|_{H^2(\mathbb{D}, \mathbb{C}^2)} \]
is
\[ x_0(z) = \begin{pmatrix} 4 + \sqrt{3}z \\ 1 \end{pmatrix}. \]

For \((x_0, y_0)\) to be a Schmidt pair for \( H_G \) corresponding to \( \|H_G\| \), the vector \( y_0 \in H^2(\mathbb{D}, \mathbb{C}^2)_\perp \)
can be calculated by
\[ y_0(z) = \frac{H_Gx_0(z)}{\|H_G\|} = 2\bar{z} \begin{pmatrix} \bar{z} + \sqrt{3} \\ 1 \end{pmatrix} \in H^2(\mathbb{D}, \mathbb{C}^2)_\perp. \]

Perform the inner–outer factorizations
\[ x_0 = \xi_0h_0, \quad \bar{z}y_0 = \eta_0h_0 \]
for some inner $\xi_0, \eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^2)$ and some scalar outer $h_0 \in H^2(\mathbb{D}, \mathbb{C})$.

In this example
\[
\xi_0(z) = \frac{x_0}{h_0} = \frac{a}{4\sqrt{3}(1-\gamma z)} \begin{pmatrix} 4 + \sqrt{3}z \\ 1 \end{pmatrix},
\]
\[
\bar{\eta}_0(z) = \frac{\bar{z}y_0}{h_0} = \frac{2a}{4\sqrt{3}(1-\gamma \bar{z})} \begin{pmatrix} \bar{z} + \sqrt{3} \\ 1 \end{pmatrix},
\]
where
\[
h_0(z) = \frac{4\sqrt{3}}{a}(1-\gamma z),
\]
a = \sqrt{10 - 2\sqrt{13}} and $\gamma = -\frac{a^2}{4\sqrt{3}}$.

A function $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times2})$ that satisfies
\[
(G - Q_1)x_0 = t_0y_0, \quad (G - Q_1)^*y_0 = t_0x_0
\]
is
\[
Q_1(z) = \begin{pmatrix} 0 & \sqrt{6} \\ 2\sqrt{2} & -\sqrt{6}(z + \sqrt{3}) \end{pmatrix}.
\]

**Step 1:** Let $X_1 = \xi_0 \wedge H^2(\mathbb{D}, \mathbb{C}^2)$ and $Y_1 = \bar{\eta}_0 \wedge H^2(\mathbb{D}, \mathbb{C}^2)\perp$.

Let $T_1 : X_1 \to Y_1$ be given by
\[
T_1(\xi_0 \wedge x) = P_{Y_1}(\bar{\eta}_0 \wedge (G - Q_1)x)
\]
for all $x \in H^2(\mathbb{D}, \mathbb{C}^2)$.

Note that
\[
X_1 = \left\{ \xi_0 \wedge \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : f_i \in H^2(\mathbb{D}, \mathbb{C}) \right\}
= \left\{ \frac{a}{4\sqrt{3}} \frac{(4 + \sqrt{3}z)f_2 - f_1}{1 - \gamma z} : f_i \in H^2(\mathbb{D}, \mathbb{C}) \right\}.
\]

If we choose
\[
f_1 = -\frac{4\sqrt{3}}{a}(1-\gamma z)g \quad \text{and} \quad f_2 = 0
\]
for some $g \in H^2(\mathbb{D}, \mathbb{C})$, we obtain $X_1 = H^2(\mathbb{D}, \mathbb{C})$.

Also
\[
Y_1 = \left\{ \bar{\eta}_0 \wedge \begin{pmatrix} \bar{z}\varphi_1 \\ \bar{z}\varphi_2 \end{pmatrix} : \varphi_i \in H^2(\mathbb{D}, \mathbb{C}) \right\}
= \left\{ \frac{a\bar{z}}{2\sqrt{3}} \frac{(\bar{z} + \sqrt{3})\varphi_2 - \varphi_1}{1 - \gamma \bar{z}} : \varphi_i \in H^2(\mathbb{D}, \mathbb{C}) \right\}.
\]

If we choose
\[
\varphi_1 = -\frac{2\sqrt{3}}{a}(1-\gamma z)\psi, \quad \text{and} \quad \varphi_2 = 0
\]
for some $\psi \in H^2(\mathbb{D}, \mathbb{C})$, we find that $Y_1 = H^2(\mathbb{D}, \mathbb{C})\perp$.

We have
\[
T_1\left( \xi_0 \wedge \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right) = T_1\left( \xi_0 \wedge \begin{pmatrix} -\frac{4\sqrt{3}}{a}(1-\gamma z)g \\ 0 \end{pmatrix} \right) = \frac{u(\gamma)}{z - \gamma},
\]
where
\[ u(\gamma) = \sqrt{2} (1 - \gamma^2)(2\sqrt{3}\gamma + 1)g(\gamma). \]
Then \( t_1 = \|T_1\| = \sqrt{2}(4 - \sqrt{13}) \).
Since \( T_1 \) is a compact operator, there exist \( v_1 \in H^2(\mathbb{D}, \mathbb{C}^2) \), \( w_1 \in H^2(\mathbb{D}, \mathbb{C}^2)^\perp \) such that
\[ T_1(\xi_0 \hat{v}_1) = t_1(\eta_0 \hat{w}_1), \quad T_1^* (\eta_0 \hat{w}_1) = t_1(\xi_0 \hat{v}_1). \]
Here we can choose
\[ v_1(z) = \frac{4\sqrt{3}}{a} \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad w_1(z) = \frac{2\sqrt{3}}{a} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]
Perform the inner–outer factorization of \( \xi_0 \hat{v}_1 \in H^2(\mathbb{D}, \mathbb{C}^2) \).
The function \( h_1(z) = (1/(1 - \gamma z)) \) is the scalar outer factor of \( \xi_0 \hat{v}_1 \).
Let
\[ x_1 = (I - \xi_0(z)\xi_0^*(z))v_1(z), \quad y_1(z) = (I - \eta_0(z)\eta_0^T(z))w_1(z). \]
Then
\[ x_1 = \frac{\gamma}{\alpha} \frac{1}{(1 - \gamma z)(1 - \gamma \bar{z})} \begin{pmatrix} -4\sqrt{3}(1 - \gamma z)(1 - \gamma \bar{z}) - 19 - 4\sqrt{3}(z + \bar{z}) \\ -4 - \sqrt{3} \end{pmatrix} \]
and
\[ y_1 = \frac{2\gamma \bar{z}}{\alpha} \frac{1}{(1 - \gamma z)(1 - \gamma \bar{z})} \begin{pmatrix} \sqrt{3}(1 - \gamma z)(1 - \gamma \bar{z}) + 4 + \sqrt{3}(z + \bar{z}) \\ z + \sqrt{3} \end{pmatrix}. \]
Calculations yield
\[ x_1 = \frac{\gamma}{\alpha} \frac{1}{(1 - \gamma z)(1 - \gamma \bar{z})} \begin{pmatrix} 1 \\ -4 - \sqrt{3} \end{pmatrix}, \quad y_1 = \frac{2\gamma \bar{z}}{\alpha} \frac{1}{(1 - \gamma z)(1 - \gamma \bar{z})} \begin{pmatrix} 1 \\ -4 + \sqrt{3} \end{pmatrix}. \]
The algorithm stops after at most \( \min(m, n) \) steps, hence in this case after two steps. Then,
by Theorem 10.10, the unique analytic superoptimal approximant \( AG \) is given by the formula
\[ AG = G - \frac{t_0 y_0 x_0^*}{|h_0|^2} - \frac{t_1 y_1 x_1^*}{|h_1|^2}. \]
All terms of \( AG \) can be calculated now, to give
\[ AG = \frac{\sqrt{2}}{1 - \gamma z} \begin{pmatrix} -\gamma \\ 2 + \gamma \sqrt{3} - \gamma z \\ -\sqrt{3} + 4\gamma \end{pmatrix}, \]
which is the unique superoptimal analytic approximant for the given \( G \) in Problem 11.1.

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