A filtered version of the Bipolar Theorem of Brannath and Schachermayer

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Abstract

We extend the Bipolar Theorem of Brannath and Schachermayer (1999) to the space of nonnegative càdlàg supermartingales on a filtered probability space. We formulate the notion of fork-convexity as an analogue to convexity in this setting. As an intermediate step in the proof of our main result we establish a conditional version of the Bipolar theorem. In an application to mathematical finance we describe the structure of the set of dual processes of the utility maximization problem of Kramkov and Schachermayer (1999) and give a budget-constraint characterization of admissible consumption processes in an incomplete semimartingale market.

Key words: bipolar theorem, stochastic processes, positive supermartingales, duality, mathematical finance

1 INTRODUCTION

The classical Bipolar Theorem of functional analysis states that the bipolar $D^{oo}$ of a subset $D$ of a locally convex vector space is the smallest closed, balanced and convex set containing $D$. The locally convex structure of the underlying space is of great importance since the proof relies heavily on the Hahn-Banach Theorem. In their recent article, [BS99] exploit the order structure of $L^0_0(\Omega, \mathcal{F}, \mathbb{P})$ - (the space of all nonnegative measurable functions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the topology of convergence in measure) - to obtain an extension of the Bipolar Theorem to this (generally not locally convex) space. Indeed, if $\mathbb{P}$ is a diffuse measure, the topological dual of $L_0$ reduces to $\{0\}$ (see e.g. [KPR84], Theorem 2.2). Brannath and Schachermayer consider a dual pair of convex cones $<L^0_+, L^0_+>$ with the scalar product

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< f, g > → \mathbb{E}[fg] taking values in [0, \infty] and successfully identify the bipolar of a subset \( \mathcal{D} \) of \( L_0^+ \) as the smallest convex, closed in probability and solid set containing \( \mathcal{D} \). The motivation for this extension comes from mathematical finance, where it is customary to consider the natural duality between the set of attainable contingent claims and a variant of the set of all equivalent local martingale measures. For the problem of maximizing the utility of the terminal wealth in general incomplete semimartingale securities market model, the set of all (Radon-Nikodym densities of) equivalent local martingale measures turns out to be too small - in terms of closedness and compactness properties. The appropriate enlargement, as described in [KS99], is obtained by passing to the bipolar. This is where an operative description - provided by the Bipolar Theorem for Subsets of \( L_0^+ \) - is a sine qua non.

Inspired by, and heavily relying on, the result of Brannath and Schachermayer, we decided to go one step further and derive an analogue of the Bipolar Theorem for sets of stochastic processes. Additional motivation came from mathematical finance - from an attempt to characterize the optimal intratemporal consumption policy for an investor in an incomplete semimartingale market. Here it is not enough to study the relationship between equivalent local martingale measures and attainable contingent claims. The time-dependent nature of the problem forces us to consider the whole wealth process and the corresponding dual "density processes" of equivalent local martingale measures. Also, the enlargement necessary to rectify the lack of closedness and compactness properties of the set of all density processes (see [KS99]) must take place in a considerably more 'hostile' environment - the set of nonnegative adapted stochastic processes. Specifically, for a set of nonnegative càdlàg processes \( \mathcal{X} \) defined in terms of stochastic integrals with respect to a fixed semimartingale, the set \( \mathcal{Y}^c \) of density processes corresponds to all strictly positive càdlàg martingales \( Y \) with \( Y_0 = 1 \) such that \((Y_t X_t)_{t \in [0,T]}\) is a local martingale for all \( X \in \mathcal{X} \). The enlargement (as proposed in [KS99]) \( \mathcal{Y} \) of \( \mathcal{Y}^c \) consists of all nonnegative càdlàg supermartingales \( Y \) with \( Y_0 \leq 1 \) such that \((Y_t X_t)_{t \in [0,T]}\) is a supermartingale for each \( X \in \mathcal{X} \).

In this paper we abstract the important properties of such an enlargement and phrase it in terms of a suitably defined notion of the polar. In the manner of [BS99] we put the set of all nonnegative adapted càdlàg processes in duality with itself. However, this time the scalar product is no longer a numerical function anymore and it takes values in a suitably chosen quotient space of the space of nonnegative stochastic processes.

For our analysis we focus on sets of nonnegative supermartingales endowed with mild additional properties. These properties are analogous to those of the set of all density processes of equivalent local martingale measures. In this context the new notion of fork-convexity turns out to be the right analogue for the concept of convexity in the classical case. We identify the bipolar of a set of supermartingales as its fork-convex, solid and closed hull, with notions of solidity and closedness suitably defined. As a by-product, we also obtain a conditional version of the Bipolar Theorem which is, at least to the author, an interesting result in its own sake. We then apply the obtained results to describe the structure of the enlarged set \( \mathcal{Y} \) of dual density processes for the problem of optimal consumption. For this case we give a simple budget-constraint characterization of all admissible consumption densities. The results of this paper can also be successfully
applied to the problem of optimal consumption in an incomplete semimartingale market (see [ˇZit99]).

The paper is divided into 4 sections. Section 1 is the introduction. In Section 2 we present the setting and state main theorems. Section 3 contains the proofs, and in Section 4 we discuss applications to mathematical finance.

2 PRELIMINARIES AND THE MAIN RESULT

In [BS99], the following environment is introduced. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $L^0(\mathcal{F})$ denote the set of all (equivalence classes) of real valued $\mathcal{F}$-measurable functions defined on $\Omega$. $L^0(\mathcal{F})$ becomes a topological vector space if we endow it with the topology of convergence in measure. With $L^0_+(\mathcal{F}) = \{ f \geq 0 : f \in L^0(\mathcal{F}) \}$ being the positive orthant of $L^0(\mathcal{F})$, it is possible to define a ‘scalar product’ on $L^0_+(\mathcal{F})$ by setting

$$< f, g > \mapsto \mathbb{E}[fg] \in [0, \infty].$$

In this way $L^0_+(\mathcal{F})$ is placed in duality with itself. In this setting, Brannath and Schachermayer give the following:

**Definition 1.** Let $D$ be a subset of $L^0_+$. The set

$$D^\circ = \{ g \in L^0_+ : < f, g > \leq 1 \text{ for all } f \in D \}$$

is called the polar of $D$. A subset $D$ of $L^0_+$ is called

a) **solid** if for $f \in D$ and $g \in L^0_+(\mathcal{F})$, $g \leq f$ a.s. implies $g \in D$

b) **closed** if it is closed with respect to the topology of convergence in probability.

The Bipolar Theorem for Subsets of $L^0_+$ is given in the following:

**Theorem 1 (Brannath and Schachermayer (1999)).** Let $D$ be a subset of $L^0_+(\mathcal{F})$. The bipolar $(D^\circ)^\circ$ of $D$ is the smallest closed, solid and convex subset of $L^0_+(\mathcal{F})$ containing $D$.

In our setting we would like to derive a similar theorem in the context of stochastic processes. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$, where $T > 0$ is the time horizon, and assume that $(\mathcal{F}_t)_{t \in [0,T]}$ satisfies the usual conditions with $\mathcal{F}_0$ being the completed trivial $\sigma$-algebra. We also introduce the following notation and terminology for certain classes of stochastic processes:

1. A nonnegative adapted càdlàg stochastic process we will call a **positive process** and we denote the set of all positive processes by $\mathcal{P}$.

2. $\mathcal{S}$ will denote the set of all supermartingales in $\mathcal{P}$, and $\mathcal{S}_1$ the set of all supermartingales $Y$ in $\mathcal{P}$ such that $Y_0 \leq 1$. 


3. A subset $D$ of $S_1$ is called **far-reaching** if there is an element $Y \in D$ such that $Y_T > 0$ a.s.

4. $V$ denotes the set of all nonincreasing processes $B$ in $P$ such that $B_0 \leq 1$.

**Definition 2.** Let $D$ be a subset of $P$. The **(process)-polar** of $D$ is the set of all positive processes $Y$, such that $XY = (X_t Y_t)_{t \in [0,T]}$ is a supermartingale with $(XY)_0 \leq 1$ for all $X \in D$.

**Remark 1.** There is a formal analogy between our definition of the polar and that for random variables in [BS99]. To show how $P$ is placed in duality with itself we first have to define a suitable range space for the scalar product. Let $\preceq_p$ be a binary relation on $P$ defined by

$$X_0 \leq Y_0 \text{ and } X - Y \text{ is a supermartingale}.$$ 

Defined as it is, $\preceq_p$ is not a partial order. However, if we set $R$ to be the quotient space obtained from $P$ by identifying processes whose difference is a local martingale null at 0, the natural projection $\preceq_p$ of $\preceq_p$ to $R$ will define a partial order on $R$. If we denote by $F$ the natural projection from $P$ onto $R$, we can see that a polar of a subset $D$ of $P$ is given by

$$D^\times = \{ Y \in P : F[XY] \preceq 1 \text{ for all } X \in D \}.$$ 

Our next task is to define analogues of solidity, closedness and convexity. It turns out that the right substitute for solidity is the following concept, multiplicative in nature. We recall that $V$ stands for the set of all nonincreasing processes $B$ in $P$ such that $B_0 \leq 1$.

**Definition 3.** Let $D$ be a subset of $P$. $D$ is called *(process) solid* if for each $Y \in D$ and each $B \in V$ we have $Y B \in D$.

To define the appropriate notion of closedness, we recall the concept of Fatou-convergence. It is an analogue of convergence a.s. in the context of càdlàg processes and was used for example in [Kra96], [FK97] and [DS99].

**Definition 4.** Let $(Y^{(n)})_{n \in \mathbb{N}}$ be a sequence of positive processes. We say that $(Y^{(n)})_{n \in \mathbb{N}}$ **Fatou converges** to a positive process $Y$ if there is a countable dense subset $T$ of $[0,T]$ such that

$$Y_t = \lim \inf_{s \uparrow t, s \in T} \inf_n Y_s^{(n)}$$

for all $t$. We interpret (2.1) to mean $Y_t = \lim_n Y_t^{(n)}$ for $t = T$.

Fatou convergence has a number of desirable properties, especially when applied to sequences in $S$. The following proposition is an easy consequence of the Fatou Lemma:

**Proposition 1.** Let $(Y^{(n)})_{n \in \mathbb{N}}$ be a sequence in $S$, Fatou converging to a positive process $Y$. Then $Y$ is in $S$ as well. If additionally, $Y^{(n)} \in S_1$ for all $n$, then so is $Y$. 


Definition 5. Let $\mathcal{D}$ be a subset of $\mathcal{P}$. $\mathcal{D}$ is called closed if it is closed with respect to Fatou convergence.

Finally, we define the concept of fork-convexity for subsets of $\mathcal{S}$. We want to look at processes in $\mathcal{S}$ as built up of multiplicative increments. In order to be able to do this we have to make sure that these increments are well-defined. We refer the reader to ([RY91], Prop. II.3.4, page 66) for the proof of the following proposition.

**Proposition 2.** If $X$ is a nonnegative right-continuous supermartingale and

$$T = \inf \{ t : X_t = 0 \} \wedge \inf \{ t > 0 : X_{t-} = 0 \},$$

then, for almost every $\omega \in \Omega$, $X(\omega)$ vanishes on $[T(\omega), \infty)$.

This result, together with the convention that $\frac{0}{0} = 0$ (which we freely use throughout the paper) allows us to define random variables of the form $Y_t Y_s$ for $Y \in \mathcal{S}$ and $t \geq s$.

**Definition 6.** A subset $\mathcal{D}$ of $\mathcal{S}$ is called fork-convex if for any $s \in [0, T]$, any $h \in L^0_+ (\mathcal{F}_s)$ with $h \leq 1$ a.s. and any $Y^{(1)}, Y^{(2)}, Y^{(3)} \in \mathcal{D}$, the process $Y$, defined by

$$Y_t = \begin{cases} Y_t^{(1)} & t < s \\ Y_s^{(1)} (h Y_s^{(2)} + (1 - h) Y_s^{(3)}) & t \geq s \end{cases}$$

belongs to $\mathcal{D}$.

**Remark 2.** The motivation for the introduction of fork-convexity comes from mathematical finance. It can easily be shown that the set of density processes of equivalent local martingale measures for a semimartingale $S$ is fork-convex. By density process of a probability measure $Q$ equivalent to $P$ we intend the càdlàg version of the martingale $Y^Q_t = E[\frac{dQ}{dP} | \mathcal{F}_t]$. We refer the reader to [FK97] for the related concept of predictable convexity.

Now we can state the main result of this paper.

**Theorem 2. [Filter Bipolar Theorem]** Let $\mathcal{D}$ be a far-reaching subset of $\mathcal{S}_1$. The process bipolar $\mathcal{D}^{\times \times} = (\mathcal{D}^\times)^\times$ of $\mathcal{D}$ is the smallest closed, fork-convex and solid subset of $\mathcal{S}_1$ containing $\mathcal{D}$.

An important ingredient in the proof of our main result is the Conditional Bipolar Theorem. This conditional version may be interpreted as the Filtered Bipolar Theorem in the setting of a discrete two-element time set. Before stating the theorem, we give the necessary definitions.

**Definition 7.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. A subset $\mathcal{D}$ of $L^0_+ (\mathcal{F})$ is called $\mathcal{G}$-convex if for all $f, g \in \mathcal{D}$ and every $h \in L^0_+ (\mathcal{G})$ with $h \leq 1$ a.s., we have $hf + (1 - h)g \in \mathcal{D}$.
Definition 8. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(\mathcal{G}\) be a sub-\(\sigma\)-algebra of \(\mathcal{F}\). For a subset \(D\) of \(L^0_+ (\mathcal{F})\) the set
\[
[D|\mathcal{G}^\circ = \{ g \in L^0_+ (\mathcal{F}) : \mathbb{E}[fg|\mathcal{G}] \leq 1, \text{ a.s. for all } f \in D \}.
\]
is called the **conditional polar of \(D\) with respect to \(\mathcal{G}\)**.

We also recall the definition of boundedness in probability.

Definition 9. A subset \(D \subseteq L^0_+ (\mathcal{F})\) is said to be **bounded in probability** if for each \(\varepsilon > 0\) there is an \(M > 0\) such that
\[
\mathbb{P}[f > M] < \varepsilon \text{ for all } f \in D.
\]

Remark 3. We will use the following easy consequence of boundedness in probability: Let \((f_n)_{n \in \mathbb{N}}\) be a sequence in \(L^0_+ (\mathcal{F})\) converging a.s. to a random variable \(f\) with values in \([0, \infty]\). If \((f_n)_{n \in \mathbb{N}}\) is bounded in probability, then so is \(\{f\} \cup \{f_n : n \in \mathbb{N}\}\) and \(f < \infty\) a.s.

Theorem 3. [Conditional Bipolar Theorem] Let \(D\) be a subset of \(L^0_+ (\mathcal{F})\) which is bounded in probability. Then \([D|\mathcal{G}]^\circ = ([D|\mathcal{G}]^\circ |\mathcal{G})^\circ\) is the smallest \(\mathcal{G}\)-convex, solid and closed subset of \(L^0_+ (\mathcal{F})\) containing \(D\).

3 PROOFS OF THE THEOREMS

We start this section by proving the Conditional Bipolar Theorem via a number of auxiliary lemmata. \(\mathcal{B}_+ (\mathcal{G})\) denotes the set of all nonnegative \(\mathcal{G}\)-measurable functions on \(\Omega\) with expectation less than or equal to 1. In other words, \(\mathcal{B}_+ (\mathcal{G})\) is the intersection of \(L^0_+ (\mathcal{F})\) with the unit ball of \(L^1 (\mathcal{G})\).

Lemma 1. For \(D \subseteq L^0_+ (\mathcal{F})\), \([D|\mathcal{G}]^\circ\) is solid, \(\mathcal{G}\)-convex and closed.

Proof. Let \((g_n)_{n \in \mathbb{N}}\) be a sequence in \([D|\mathcal{G}]^\circ\) converging in probability to \(g \in L^0_+ (\mathcal{F})\). By passing to a subsequence we can assume \(g_n \to g\) a.s. By the Conditional Fatou Lemma, for every \(f \in D\),
\[
\mathbb{E}[fg|\mathcal{G}] = \mathbb{E}[\lim_n f g_n|\mathcal{G}] \leq \liminf_n \mathbb{E}[f g_n|\mathcal{G}] \leq 1 \text{ a.s.},
\]
so \(g \in [D|\mathcal{G}]^\circ\), i.e. \([D|\mathcal{G}]^\circ\) is closed. \(\mathcal{G}\)-convexity and solidity follow easily from the definition. \(\square\)

For \(f \in L^0_+ (\mathcal{F})\) and \(D \subseteq L^0_+ (\mathcal{F})\) we put \(fD = \{fg : g \in D\}\) and \(\frac{1}{f}D = \{h \in L^0_+ (\mathcal{F}) : hf \in D\}\).

Lemma 2. Let \(D \subseteq L^0_+ (\mathcal{F})\). Then
\[
[D|\mathcal{G}]^\circ = \bigcap_{l \in \mathcal{B}_+ (\mathcal{G})} \frac{1}{l}D^\circ,
\]
and
\[
[D|\mathcal{G}]^{\circ\circ} = \bigcap_{l \in \mathcal{B}_+ (\mathcal{G})} \frac{1}{l} \left( \bigcup_{k \in \mathcal{B}_+ (\mathcal{G})} kD \right)
\]
where \(D^\circ = \{g \in L^0_+ (\mathcal{F}) : \mathbb{E}[fg] \leq 1 \forall f \in D\}\) is the (unconditional) polar of \(D\) and \(\bigcup\) denotes the convex, solid and closed hull.
Proof. We observe that for $Z \in L_+^0(\mathcal{F})$, $\mathbb{E}[Z|\mathcal{G}] \leq 1$ if and only if $\mathbb{E}[Zl] \leq 1$ for all $l \in B_+(\mathcal{G})$ (the easy proof is left to the reader). Then

$$[D|\mathcal{G}]^\circ = \{g \in L_+^0(\mathcal{F}) : \mathbb{E}[fg|\mathcal{G}] \leq 1, \forall f \in D\}$$

$$= \{g \in L_+^0(\mathcal{F}) : \mathbb{E}[fg] \leq 1, \forall f \in D, \forall l \in B_+(\mathcal{G})\}$$

$$= \bigcap_{l \in B_+(\mathcal{G})} \{g \in L_+^0(\mathcal{F}) : \mathbb{E}[fg] \leq 1, \forall f \in D\}$$

$$= \bigcap_{l \in B_+(\mathcal{G})} \frac{1}{l}D^\circ.$$

If we reiterate the same procedure and use the following simple relations

$$(kD)^\circ = \frac{1}{k}D^\circ$$

for any $k \in L_+^0(\mathcal{F})$, and any $D \subseteq L_+^0(\mathcal{F})$ and

$$\bigcup_{\alpha \in I} D_\alpha = \bigcap_{\alpha \in I} D_\alpha$$

for any family $(D_\alpha)_{\alpha \in I}$ of subsets of $L_+^0(\mathcal{F})$, we obtain

$$[D|\mathcal{G}]^{\circ\circ} = \bigcap_{l \in B_+(\mathcal{G})} \frac{1}{l}(D|\mathcal{G})^\circ = \bigcap_{l \in B_+(\mathcal{G})} \frac{1}{l}\left(\bigcap_{k \in B_+(\mathcal{G})} \frac{1}{k}D^\circ\right)^\circ$$

$$= \bigcap_{l \in B_+(\mathcal{G})} \left(\frac{1}{l}\bigcup_{k \in B_+(\mathcal{G})} kD\right)^\circ = \bigcap_{l \in B_+(\mathcal{G})} \frac{1}{l}\left(\bigcup_{k \in B_+(\mathcal{G})} kD\right)^\circ,$$

by the (unconditional) Bipolar Theorem 1.

Lemma 3. Let $D \subseteq L_+^0(\mathcal{F})$ be bounded in probability. Then

$$\left(\bigcup_{k \in B_+(\mathcal{G})} kD\right) = \bigcup_{k \in B_+(\mathcal{G})} k\overline{D}^G,$$

and

$$[D|\mathcal{G}]^{\circ\circ} = \bigcap_{l \in B_+(\mathcal{G})} \left(\frac{1}{l}\bigcup_{k \in B_+(\mathcal{G})} k\overline{D}^G\right)^\circ,$$

where $(\overline{ })^G$ denotes the $\mathcal{G}$-convex, solid and closed hull.

Remark 4. Lemma 3 can be restated as follows: For each $f \in [D|\mathcal{G}]^{\circ\circ}$ and each $l \in B_+(\mathcal{G})$ there are $h \in \overline{D}^G$ and $k \in B_+(\mathcal{G})$ such that $fl = hk$.

Proof. From Lemma 1 and that $\overline{D|\mathcal{G}} = [D|\mathcal{G}]^{\circ\circ}$, we can assume without loss of generality that $D$ is already $\mathcal{G}$-convex, solid and closed, because taking a $\mathcal{G}$-convex, solid and closed hull preserves boundedness in probability. Let $(\overline{ })$ denote the closure with respect to convergence in probability. We only need to prove that

$$\left(\bigcup_{k \in B_+(\mathcal{G})} kD\right) \subseteq \bigcup_{k \in B_+(\mathcal{G})} k\overline{D},$$

and

$$[D|\mathcal{G}]^{\circ\circ} = \bigcap_{l \in B_+(\mathcal{G})} \left(\frac{1}{l}\bigcup_{k \in B_+(\mathcal{G})} k\overline{D}\right)^\circ.$$
since \( \cup_{k \in B_+(G)} kD \) is a convex and solid subset of \( \bigcup_{k \in B_+(G)} kD \). Let \( f \in \bigcup_{k \in B_+(G)} kD \). Then, there is a sequence \( f_n \) converging to \( f \) in probability, and each \( f_n \) is of the form \( l_n h_n \) for some \( l_n \in B_+(G) \) and \( h_n \in D \). By passing to a subsequence, we can assume that \( f_n \to f \) a.s. The sequence \( l_n \) is bounded in \( L^1 \), so by Komlos’ Theorem (see [Sch86] and references therein for a good exposition and generalizations) there is a sequence of convex combinations

\[
k_n \in \text{conv} \left( l_n, l_{n+1}, \ldots \right) \subseteq B_+(G)
\]

converging to a random variable \( l' \) a.s. By Fatou Lemma, \( E[l'] \leq 1 \) so \( l' \in B_+(G) \). If \( k_n \) is of the form \( \sum_{j=n}^{m_n} \alpha_j^n l_j \), we define (recalling that \( \frac{0}{0} = 0 \))

\[
k_j^n = \frac{\alpha_j^n l_j}{k_n} \quad \text{and} \quad \hat{h}_n = \sum_{j=n}^{m_n} k_j^n h_j.
\]

By \( G \)-convexity of \( D \), \( \hat{h}_n \in D \) because \( \sum_{j=n}^{m_n} k_j^n = 1 \). If we redefine \( \hat{h}_n \) by putting \( \hat{h}_n = 0 \) on \( \{ f = 0 \} \) we still have \( \hat{h}_n \in D \), and the relation

\[
f = \lim_n \sum_{j=n}^{m_n} \alpha_j^n l_j h_j = \lim_n k_n \hat{h}_n
\]

allows us to conclude that on \( \{ f > 0 \} \cap \{ l' = 0 \} \), \( \hat{h}_n \) must converge to \( +\infty \) a.s. However, \( D \) is bounded in probability so we must have \( P(\{ f > 0 \} \cap \{ l' = 0 \}) = 0 \). It is now clear that there is a finite random variable \( \hat{h} \in \mathcal{D} = D \) such that \( \hat{h}_n \to \hat{h} \) a.s. Therefore

\[
f = l' \hat{h} \in \bigcup_{k \in B_+(G)} kD.
\]

\[
\square
\]

**Proof (Conditional Bipolar Theorem 3).** Without loss of generality we assume \( D \) is already closed, solid, \( G \)-convex and bounded in probability.

For \( f \in \mathcal{D}|G|^\infty \) we define

\[
H^f = \{ h \in D : \{ f = 0 \} \subseteq \{ h = 0 \} \quad \text{and} \quad fE[h|G] = hE[f|G] \}.
\]

By Remark 4 we can choose \( h' \in D \) and \( g \in B_+(G) \) such that \( f = gh' \). Then let \( h = h'1_{\{ f=0 \}} \) and obtain \( hE[f|G] = ghE[h|G] = fE[h|G] \), so \( h \in H^f \) implying that \( H^f \) is not empty.

In order to prove the theorem we need to show that \( f \) is dominated by an element of \( H^f \). We first show that \( h_{\text{max}} = \text{esssup}H^f \in H^f \). For \( h_1, h_2 \in H^f \) define \( C = \{ E[h_1|G] > E[h_2|G] \} \). From the defining property of \( H^f \) we have

\[
(h_1 \vee h_2)E[f|G] = E[h_1|G] \vee E[h_2|G]f = (1_C \cdot E[h_1|G] + 1_{C^c} \cdot E[h_2|G])f = (1_C h_1 + 1_{C^c} h_2)E[f|G].
\]

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As \( h_1 = h_2 = 0 \) on \( \{E[f|G] = 0\} \) we immediately conclude that \( h_1 \lor h_2 = 1_C h_1 + 1_C^c h_2 \) and \( h_1 \lor h_2 \in D \) by \( G \)-convexity. We proceed further and note that
\[
E[h_1 \lor h_2 | G] f = E[1_C h_1 + h_2 1_C^c | G] f = (h_1 1_C + h_2 1_C^c) E[f|G] = (h_1 \lor h_2) E[f|G],
\]
so \( h_1 \lor h_2 \in H^f \) which proves that \( H^f \) is closed under pairwise maximization. By Theorem A.3. in [KS98], \( h_{\max} \) can be written as \( h_{\max} = \lim_n h_n \), where \( h_n \) is a nondecreasing sequence in \( H^f \). Boundedness in probability of \( D \) and the monotone convergence theorem imply that \( h_{\max} < \infty \) a.s. and \( h_{\max} E[f|G] = f E[h_{\max}|G] \). Finally, \( h_{\max} \in H^f \) because \( D \) is closed.

To see that \( f \) is dominated by \( h_{\max} \), we define \( A = \{E[f|G] > E[h_{\max}|G]\} \) and assume \( P[A] > 0 \). With \( l = 1_A / P[A] \in B_+(G) \), by Remark 4, there are \( h \in D \) and \( k \in B_+(G) \) such that \( ll = kh \). Without loss of generality we may assume that \( h \in H^f \). As \( kh_{\max} \geq kh \), through conditioning upon \( G \) we have that \( k E[h_{\max}|G] \geq ll E[f|G] \) so
\[
l E[f|G] = 1_A l E[f|G] \leq 1_A k E[h_{\max}|G] \leq 1_A k E[f|G].
\]
The fact that \( E[f|G] \) is strictly positive on \( A \) leads us to conclude that \( k \geq l \) on \( A \). As \( k \in B_+(G) \) and \( E[l] = 1 \) we must have \( k = l \) and, consequently, \( f \leq h_{\max} \) on \( A \). Taking conditional expectation we get \( P[A] = 0 \).

We have shown that \( E[f|G] \leq E[h_{\max}|G] \) and the definition of \( H^f \) immediately yields \( f \leq h_{\max} \). In other words, \( f \) is dominated by an element of \( D \), implying that \( f \in D \) by solidity. Thus \( [D|G] \in C^\infty \subseteq D \). The converse inclusion is obvious. \( \square \)

We may now proceed gradually to the proof of our main result. For a process \( Y \in S \) and \( t > s \), we denote by \( \Delta_t \) the multiplicative increment and we define \( \Delta_t Y = 1 \).

**Proposition 3.** Let \( C \subseteq P \) with \( 1 \in C \), where \( 1 \) denotes the constant process equal to 1. Then \( C^\times \) is a closed, solid and fork-convex subset of \( S_1 \).

**Proof.** Since \( 1 \) is in \( C \), obviously \( C^\times \subseteq S_1 \). Let \( (Y^n)_{n \in N} \) be a sequence in \( C^\times \), Fatou-converging to some \( Y \in S_1 \) and let \( X \in C \). Since all \( XY^n \) are in \( S_1 \) and \( XY^n \) Fatou-converges to \( XY \), \( XY \) is in \( S_1 \) as well by Proposition 1. This proves the closedness of \( C^\times \). Let \( Y \in C^\times \) and \( B \in V \). Then , for all \( X \in C \), \( BXY \) is a supermartingale because \( XY \) is one. Finally, let \( Y^{(1)}, Y^{(2)}, Y^{(3)} \in C^\times \), \( t_0 \in [0,T] \) and \( h \in L^1_+ (F_{t_0}) \) with \( 0 \leq h \leq 1 \) and let a process \( Y \) be defined by
\[
Y_t = \begin{cases} Y_t^{(1)} & t < t_0 \\ Y_t^{(1)} \left( h \frac{Y_t^{(2)}}{Y_{t_0}} + (1-h) \frac{Y_t^{(3)}}{Y_{t_0}} \right) & t \geq t_0 \end{cases}.
\]
We want to prove that \( E[Y_t X_t | F_{t_0}] \leq X_s Y_s \), for all \( t > s \) and all \( X \in C \). To do so, we only consider the case \( s = t_0, t = T \). The other cases can be dealt with analogously. By definition of \( Y \) and the fact that \( XY^{(1)}, XY^{(2)} \) and \( XY^{(3)} \) are supermartingales,
\[
E[Y_t X_t | F_{t_0}] = E[Y_{t_0}^{(1)} h \Delta_{T-t_0} Y_t^{(2)} X_t | F_{t_0}] + E[Y_{t_0}^{(1)} (1-h) \Delta_{T-t_0} Y^{(3)} X_t | F_{t_0}]
\]
\[
\leq Y_{t_0}^{(1)} (h X_t + (1-h) X_{t_0}) = Y_{t_0}^{(1)} X_{t_0} = Y_{t_0} X_{t_0}.
\]
$XY$ is, therefore, a supermartingale so $Y \in C^\times$.

The proof of the following lemma was inspired by techniques in Kramkov\textsuperscript{(1996)}.

**Lemma 4.** Suppose $D$ is a fork-convex far-reaching subset of $S$. Let $t_1 \leq t_2 \in [0, T]$ and let $g \in L_+^1(F_{t_2})$ be such that $\text{esssup}_{Y \in D} E[g\Delta_{t_2,t}Y | F_{t_1}] \leq 1$ a.s. If we define the process $X$ by

$$X_t = \begin{cases} \text{esssup}_{Y \in D} E[g\Delta_{t_2,t}Y | F_t] & t < t_1 \\ g & t \in [t_1, t_2) \\ 0 & t \in [t_2, T], \end{cases}$$

then $XY$ is a supermartingale for each $Y \in D$ and $X$ permits a càdlàg modification.

**Proof.** Without any loss of generality we assume $t_2 = T$. First we will prove that, for $t \geq t_1$, there is a sequence $(Y^n)_{n \in \mathbb{N}} \in D$ such that

$$E[g\Delta_{T,t}Y^n | F_t] / X_t,$$

as $n \to \infty$. By Theorem A.3. in [KS98] it is enough to prove that the set $\{E[g\Delta_{T,t}Y | F_t] : Y \in D\}$ is closed under pairwise maximization. Let $Y^{(1)}, Y^{(2)}$ be in $D$. Put

$$h = 1\{E[g\Delta_{T,t}Y^{(1)} | F_t] \geq E[g\Delta_{T,t}Y^{(2)} | F_t]\}.$$

Then for the process $Y^{\text{max}}$ defined by

$$Y^{\text{max}}_s = \begin{cases} Y^{(1)}_s & s < t \\ Y^{(1)}_s \left(h \frac{Y^{(1)}_s}{Y^{(1)}_t} + (1 - h) \frac{Y^{(2)}_s}{Y^{(2)}_t}\right) & s \geq t \end{cases},$$

fork-convexity implies that $Y^{\text{max}} \in D$ and

$$E[g\Delta_{T,t}Y^{\text{max}} | F_t] = E[g\Delta_{T,t}Y^{(1)} | F_t] \vee E[g\Delta_{T,t}Y^{(2)} | F_t].$$

Fix $t_1 \leq s \leq t \leq t_2$ and a sequence $(Y^n)_{n \in \mathbb{N}}$ such that (3.1) holds. By the Monotone Convergence Theorem, for each $Y \in D$, we have

$$E[Y_t X_t | F_s] = E[\lim_n Y_t E[g\Delta_{T,t}Y^n | F_s] | F_s] = \lim_n E[Y_t g\Delta_{T,t}Y^n | F_s].$$

By fork-convexity, $\Delta_{T,t}Y^n \Delta_{t,s}Y_s$ is equal to $\Delta_{T,s}\hat{Y}$ for some $\hat{Y} \in D$, so

$$\lim_n E[g\Delta_{T,t}Y^n \Delta_{t,s}Y_s | F_s] \leq \text{esssup}_{Y \in D} E[g\Delta_{T,s}Y | F_s] = X_s.$$

Therefore $E[Y_t X_t | F_s] \leq Y_s X_s$ and so $XY$ is a supermartingale on $[t_1, t_2)$ for all $Y \in D$. Because of the condition $\text{esssup}_{Y \in D} E[g\Delta_{t_2,t}Y | F_{t_1}] \leq 1$ a.s., $XY$ is a supermartingale on the whole interval $[0, T]$.

To prove that $X$ has a càdlàg version, we will first prove that $S = X\hat{Y}$ has one, where $\hat{Y}$ is an element of $D$ such that $\hat{Y}_T > 0$ a.s. The process $S$ is a supermartingale so it is enough to
prove that \( t \mapsto \mathbb{E}[S_t] \) is right-continuous (see [RY91], Theorem II.2.9, page 61). We fix \( p \in [0, T] \) and a sequence \( (p_n)_{n \in \mathbb{N}} \) such that \( p_n \searrow p \), and consider the only non-trivial case - namely, when \( p \in [t_1, t_2] \). Let \( Y^n \) be a sequence in \( D \) such that

\[
\hat{Y}_p \mathbb{E}[g \Delta_{T, p} Y^n | F_p] \not\in S_p.
\]

For \( \varepsilon > 0 \) there is an \( n \in \mathbb{N} \) such that \( \mathbb{E}[\hat{Y}_p g \Delta_{T, p} Y^n] > \mathbb{E}[S_p] - \varepsilon \). By right-continuity of processes in \( D \) and Fatou Lemma

\[
\mathbb{E}[\hat{Y}_p X \Delta_{T, p} Y^n] = \mathbb{E}[\lim_k (\hat{Y}_{p_k} g \Delta_{T, p_k} Y^n)] \\
\leq \liminf_k \mathbb{E}[\hat{Y}_{p_k} \Delta_{T, p_k} Y^n] \\
\leq \lim_k \mathbb{E}[S_{p_k}].
\]

Now, \( \lim_k \mathbb{E}[S_{p_k}] \geq \mathbb{E}[S_p] - \varepsilon \) for all \( \varepsilon > 0 \), so \( t \mapsto \mathbb{E}[S_t] \) is right continuous. Therefore \( \hat{Y}_t X_t \) has a càdlàg modification \( P_t \). Since \( \hat{Y}_T > 0 \), \( \{(t, \omega) : \hat{Y}_t(\omega) = 0 \text{ or } \hat{Y}_{t^-}(\omega) = 0\} \) is an evanescent set so we conclude that \( P_t \) is a càdlàg modification of \( X_t \).

**Remark 5.** For \( t_1 < t_2 \in [0, T] \) and \( C \subseteq \mathcal{P} \) we put \( C_{t_2, t_1} = \{ \Delta_{t_2, t_1} X : X \in C \} \) whenever it is well-defined. We note that if \( X \in \mathcal{P} \) and \( \hat{Y} \in \mathcal{S} \) with \( \hat{Y}_T > 0 \) a.s. such that \( X\hat{Y} \) is a supermartingale, then \( X \) has the following property (inherited from \( \hat{Y}X \)): if \( t_1 < t_2 \in [0, T] \) and \( X_{t_1} = 0 \) on \( A \in \mathcal{F}_{t_1} \), then \( X_{t_2} = 0 \) on \( A \) as well. Therefore, \( \Delta_{t_2, t_1} X \) is well-defined, if \( C \) is a polar of a far-reaching subset of \( S \).

**Lemma 5.** Let \( D \subseteq S_1 \) be a fork-convex, solid and far-reaching set. Then, for all \( t_1 < t_2 \in [0, T] \), \( D_{t_2, t_1} \) is solid, convex and

\[
[D_{t_2, t_1} | \mathcal{F}_{t_1}]^\circ = (D^\times)_{t_2, t_1},
\]

where all random variables in the definitions of solidity and conditional polar are assumed to be \( \mathcal{F}_{t_2} \)-measurable.

**Proof.** The solidity and convexity of \( D_{t_2, t_1} \) follow from the solidity and fork-convexity of \( D \). By the previous remark, \( D^\times_{t_2, t_1} \) is well defined. Let \( g \in [D_{t_2, t_1} | \mathcal{F}_{t_1}]^\circ \subseteq L^+_0(\mathcal{F}_{t_2}) \). Then

\[
\text{esssup}_{Y \in D} \mathbb{E}[g \Delta_{t_2, t_1} Y | \mathcal{F}_{t_1}] 
\leq 1
\]

so, by Lemma 4 the càdlàg version of the process

\[
X_t = \begin{cases} 
1 & t < t_1 \\
\text{esssup}_{Y \in D} \mathbb{E}[g \Delta_{t_2, t_1} Y | \mathcal{F}_t] & t \in [t_1, t_2] \\
\text{esssup}_{Y \in D} \mathbb{E}[g \Delta_{t_2, t_1} Y | \mathcal{F}_{t_2}] & t \in [t_2, T],
\end{cases}
\]

is in \( D^\times \). Moreover, \( \frac{X_{t_2}}{X_{t_1}} \geq g \), so, by solidity, \( g \in (D^\times)_{t_2, t_1} \). Conversely, let \( h \in D^\times_{t_2, t_1} \) be of the form \( h = \frac{X_{t_2}}{X_{t_1}} \). By definition, \( \mathbb{E}[X_{t_2} Y_{t_2} | \mathcal{F}_{t_1}] \leq X_{t_1} Y_{t_1} \), so \( \mathbb{E}[h \Delta_{t_2, t_1} Y | \mathcal{F}_{t_1}] \leq 1 \) for all \( Y \in D \). Therefore \( h \in [D_{t_2, t_1} | \mathcal{F}_{t_1}]^\circ \).
Lemma 6. Let \( D \) be a far-reaching subset of \( S_1 \). Pick \( 0 = t_0 < t_1 < t_2 < \ldots < t_m \leq T \) and \( Y \in D^{x \times} \). Then there is a sequence \( Y^n \) of elements of the solid and fork-convex hull \( \bar{D} \) of \( D \) such that \( \lim_n Y^n_{t_k} = Y_{t_k} \) a.s., for \( k = 0, \ldots, m \).

Proof. Let \( 0 \leq t_1 < t_2 \leq T \). The set \( \left\{ Y_0 : Y \in \bar{D} \right\} \) is a subinterval of \([0, \infty)\) containing \( 0 \) and hence the bipolar \( \{ Y_0 : Y \in D^{x \times}\} \) is just the closure of this interval. Therefore there is a sequence \( Y^{(0,n)} \in D, n \in \mathbb{N} \) such that \( \lim_n Y^{(0,n)} = Y_0 \). By the Bipolar Theorem 1 and the previous lemma,

\[
(\bar{D}_{t_1,0}) = (\bar{D}_{t_1,0})^\circ = (D_{t_1,0})^\circ = [D_{t_1,0}|\mathcal{F}_0]^\circ = D_{t_1,0}^{x \times},
\]

where \( (\_\_)\) denotes closure with respect to the topology of convergence in probability and we take all random variables in the definitions of polars involved to be \( \mathcal{F}_{t_1} \)-measurable. We conclude there is a sequence \( (Y^{(1,n)})_{n \in \mathbb{N}} \in \bar{D} \) such that \( \Delta_{t_1,0} Y^{(1,n)} \to \Delta_{t_1,0} Y \) when \( n \to \infty \). Similarly,

\[
(\bar{D}_{t_2-t_1}) = (D_{t_2-t_1}^{x \times})
\]

by the Conditional Bipolar Theorem 3, so there is a sequence \( (Y^{(2,n)})_{n \in \mathbb{N}} \in \bar{D} \) such that \( \Delta_{t_2,t_1} Y^{(2,n)} \to \Delta_{t_2,t_1} Y \) as \( n \to \infty \). We continue this procedure to construct sequences \( (Y^{(k,n)})_{n \in \mathbb{N}} \in \bar{D} \) such that \( \Delta_{t_k,t_{k-1}} Y^{(k,n)} \to \Delta_{t_k,t_{k-1}} Y \) as \( n \to \infty \) for \( k = 3, \ldots, m \).

By fork-convexity and solidity of \( \bar{D} \), there is a sequence \( Y^{(n)} \in \bar{D} \) such that \( Y^{(0,n)} = Y_0 \) and \( \Delta_{t_k,t_{k-1}} Y^{(n)} = \Delta_{t_k,t_{k-1}} Y^{(k,n)} \) so \( Y^{(n)}_{t_k} \to Y_{t_k} \) as \( n \to \infty \) for \( k = 0, 1, \ldots, m \). \( \square \)

Lemma 7. Let \( D \) be a far-reaching subset of \( S_1 \). For each \( Y \in D^{x \times} \) there is a sequence \( Y^{(n)} \) in the solid and fork-convex hull \( \bar{D} \) of \( D \) such that \( Y^{(n)} \to Y_q \) a.s., for all \( q \in Dy \), where \( Dy = \{ q T : q \) is a dyadic rational in \([0,1]\)\}.

Proof. Let \( Y \in D^{x \times} \). Define \( t_k^{(m)} = \frac{k}{2^m} T \) for \( m \in \mathbb{N}, k \in \{0,1,\ldots,2^m\} \). By the previous lemma, for each \( m \) we can find a sequence \( (Y^{(m,n)})_{n \in \mathbb{N}} \) such that

\[
\lim_n Y^{(m,n)}_{t_k^{(m)}} = Y^{(m)}_{t_k^{(m)}} \text{ a.s.,}
\]

for all \( m, k \). The sequence \( (Y^{(m,n)})_{n \in \mathbb{N}} \) can be chosen in such a way that for \( m \in \mathbb{N}, k \in \{0,1,\ldots,2^m\} \), \( n \in \mathbb{N} \),

\[
\mathbb{P}[|Y^{(m,n)}_{t_k^{(m)}} - Y^{(m)}_{t_k^{(m)}}| > 2^{-n}] < 2^{-n}.
\]

This will ensure that for the diagonal sequence \( Y^{(n)} = Y^{(n,n)} \), \( Y^{(n)} \to Y_q \) a.s. for all \( q \in Dy \). \( \square \)

Proof (of the Filtered Bipolar Theorem 2). Let \( D' \) be the smallest solid, fork-convex and closed subset of \( S_1 \) containing \( D \). By the previous lemma, for each \( Y \in D^{x \times} \) there is a sequence \( (Y^{(n)})_{n \in \mathbb{N}} \) such that \( Y^{(n)} \to Y_q \) as \( n \to \infty \) for each \( q \in Dy \). \( Y \) is càdlàg so it follows from the definition that \( Y^{(n)} \) Fatou converges to \( Y \), and, consequently, \( Y \in D' \) so \( D^{x \times} \subseteq D' \). Conversely, since \( 1 \in D^{x} \), Proposition 3 implies \( D' \subseteq D^{x \times} \), thus proving the claim of the theorem. \( \square \)
4 AN APPLICATION TO MATHEMATICAL FINANCE

Let $S$ be a semimartingale taking values in $\mathbb{R}^d$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$. We will interpret $S$ as the price process of $d$ risky assets in a securities market with the time horizon $T$. By taking the constant process 1 as the numéraire we assume our prices are already discounted.

An agent with initial endowment $x$ investing in this market chooses a predictable $S$-integrable process $H$ and an adapted nondecreasing càdlàg process $C$ with $C_0 = 0$. The triple $(x, H, C)$ is called an investment-consumption strategy. The process $H_t$ can be interpreted as the amount of each asset held, and $C_t$ is the cumulative amount spent on consumption prior to time $t$.

With an investment-consumption strategy we associate a process $X^{x,H,C}$ defined by

$$X^{x,H,C} = x + \int_0^t H_u dS_u - C_t. \quad (4.1)$$

$X^{x,H,C}$ represents the value of the agent’s current holdings and is called the wealth process associated with investment-consumption strategy $(x, H, C)$. An investment-consumption strategy $(x, H, S)$ is called admissible if its wealth process remains nonnegative, i.e. if $X^{x,H,C}_t \geq 0$ for all $t$. The set of all wealth processes of admissible investment-consumption strategies with an initial endowment of less than or equal to $x$ will be denoted by $\mathcal{X}(x)$. If for an investment-consumption strategy $(x, H, C)$, we have $C \equiv 0$, the pair $(x, H)$ is called a pure investment strategy and the set of all wealth processes of admissible pure investment strategies with initial endowment less than or equal to $x$ is denoted by $\mathcal{X}(x)$.

To have a realistic model of the market, we assume a variant of the non-arbitrage property by postulating the existence of a probability measure $Q$ on $\mathcal{F}$, equivalent to $\mathbb{P}$, such that each $X \in \mathcal{X}(1)$ is a local martingale under $Q$. Any such measure $Q$ is called an equivalent local martingale measure, and the set of all such measures is denoted by $\mathcal{M}$ (we refer the reader to [DS93] and [DS98] for an in-depth analysis of the relation between existence of equivalent local martingale measures and the non-arbitrage properties). If $Y^Q$ is a càdlàg process of the form

$$Y^Q_t = \mathbb{E}_t^Q[dQ/d\mathbb{P} | \mathcal{F}_t]$$

for some $Q \in \mathcal{M}$, then $Y^Q$ is called a local martingale density and $\mathcal{Y}^x$ denotes the set of all such processes. The Optional Decomposition Theorem (see [EQ95] for the original result, [Kra96], [FK98], [FK97], and [DS99] for more general versions) is the fundamental tool in our analysis. In particular, in our setting, Theorem 2.1. in [Kra96] states that a nonnegative càdlàg process $X$ with $X_0 \leq x$ is in $\mathcal{X}(x)$ if and only if $X$ is a supermartingale under each $Q \in \mathcal{M}$. Similarly, $X$ is in $\mathcal{X}(x)$ if and only if $X$ is a local martingale under each $Q \in \mathcal{M}$.

Remark 6. The Bayes rule for stochastic processes (see Lemma 3.5.3, page 193. in [KS91]) and the fact that $x\mathcal{X}(1) = \mathcal{X}(x)$ imply that $\mathcal{X}(1) = (\mathcal{Y}^x)^x$, so a nonnegative càdlàg process $X$ is in $\mathcal{X}(x)$ if and only if $X_0 \leq x$ and $XY^Q$ is a nonnegative càdlàg supermartingale for all $Y^Q \in \mathcal{Y}^x$. 

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Under certain duality considerations (see Kramkov and Schachermayer (1999)) it is necessary to enlarge $\mathcal{Y}^e$ to

$$\mathcal{Y} = \{Y \in \mathcal{P} : Y_0 \leq 1, YX \text{ is a supermartingale for all } X \in \mathcal{X}(1)\} = (\mathcal{X}(1))^\times$$

in order to rectify poor closedness properties of $\mathcal{Y}^e$. The main result of this section describes the structure of $\mathcal{Y}$, $\mathcal{X}(1)$ and $\mathcal{XC}(1)$:

**Theorem 4.**

(a) $\mathcal{Y} = (\mathcal{XC}(1))^\times$, so $Y(X - C)$ is a supermartingale for all $X - C \in \mathcal{XC}(1)$.

(b) $\mathcal{Y}$ is closed, fork-convex and solid, $\mathcal{Y} = (\mathcal{Y}^e)^\times$, and for each $Y \in \mathcal{Y}$ there is a sequence $Y^{(n)}$ of elements in the solid hull of $\mathcal{Y}^e$ such that $Y^{(n)} \to Y$ in the Fatou sense.

(c) $(\mathcal{X}(1))^\times \times = (\mathcal{XC}(1))$.

**Proof.** (a) By Remark 6, it is sufficient to prove that $\mathcal{Y} \subseteq (\mathcal{XC}(1))^\times$. Let $Y \in \mathcal{Y}$ and let $Z_t = X_t - C_t \geq 0$ be an element of $\mathcal{XC}(1)$ with $X \in \mathcal{X}(1)$. Also, let $C$ an nondecreasing, adapted, càdlàg process such that $C_0 = 0$. We assume $X_0 = 1$, take $s < t$ in $[0, T]$ and find $H$, a predictable $S$-integrable process such that $X = 1 + \int_0^t H_u dS_u$. We define a process $K$ by putting $K_u = H_u$, for all $u$, on the set $\{X_s = C_s\}$ and, for $u \leq s$, on $\{X_s > C_s\}$. For $u > s$ on $\{X_s > C_s\}$ we set

$$K_u = \frac{X_s}{X_s - C_s} H_u.$$  \hfill (4.2)

so that $K$ is an $S$-integrable predictable process. To show that $(1, K)$ is an admissible pure investment scheme, we that (with convention $\frac{0}{0} = 0$)

$$1 + \int_0^v K_u dS_u = \begin{cases} X_v & v \leq s \\ X_s + \frac{X_s}{X_s - C_s} (X_v - X_s) & v > s \end{cases}$$  \hfill (4.3)

because that $X_v = X_s$ for $v \geq s$ on $\{X_s = C_s\}$ by Proposition 2, since $X - C$ is a supermartingale under each $Q \in \mathcal{M}$. Furthermore,

$$X_s + \frac{X_s}{X_s - C_s} (X_v - X_s) \geq X_s + \frac{X_s}{X_s - C_s} (X_v - C_v - (X_s - C_s))$$

$$= X_s \left( \frac{X_v - C_v}{X_s - C_s} \right) \geq 0.$$  \hfill (4.4)

Therefore $(1, K)$ is indeed an admissible pure investment scheme, and, by the definition of
\( \mathcal{Y} \), the process \( Y_t(1 + \int_0^t K_u dS_u) \) is a supermartingale. Thus we may write
\[
Y_s X_s = Y_s \left( 1 + \int_0^s K_u dS_u \right) \geq E \left[ Y_t \left( 1 + \int_0^t K_u dS_u \right) \mid \mathcal{F}_s \right] = \frac{X_s}{X_s - C_s} E \left[ Y_t (X_t - C_t) \mid \mathcal{F}_s \right].
\]

(4.5)

If we multiply both sides by \( \frac{X_s - C_s}{X_s} \) we get the desired supermartingale property.

(b) \( \mathcal{Y}^c \) is obviously a subset of \( S_1 \), and it is far-reaching since for each \( Y \in \mathcal{Y}^c \), there is a \( Q \in \mathcal{M} \) such that \( Y_T = \frac{dQ}{dP} > 0 \) a.s. Since \( \mathcal{Y} = (\mathcal{X}C(1))^{\times \times} = (\mathcal{Y}^c)^{\times \times} \) and because of the Filtered Bipolar Theorem 2, \( \mathcal{Y} \) is the smallest solid, fork-convex and closed subset of \( S_1 \) containing \( \mathcal{Y}^c \). It is easy to check that \( \mathcal{Y}^c \) is fork-convex so from the proof of Theorem 2 we infer that for each \( Y \in \mathcal{Y} \) we can find a sequence \( Y^{(n)} \) is the solid hull of \( \mathcal{Y}^c \) such that \( Y^{(n)} \) Fatou-converges to \( Y \).

(c) From (b) and Remark 6,
\[
\mathcal{X}C(1) \subseteq (\mathcal{X}C(1))^{\times \times} = (\mathcal{Y}^c)^{\times \times} = (\mathcal{Y})^{\times \times} \subseteq (\mathcal{Y}^c)^{\times \times} = \mathcal{X}C(1),
\]

so \( \mathcal{X}C(1) = (\mathcal{Y})^{\times \times} = (\mathcal{X}(1))^{\times \times} \).

As a corollary to this result we also give a simple duality characterization of admissible consumption processes. Let \( \mu \) be a probability measure on the Borel sets of \([0, T]\), diffuse on \([0, T)\), (i.e. the only atom we allow is on \( \{T\} \)). A process \( C \) will be called an \textit{x-admissible consumption process} if there is an admissible investment-consumption strategy of the form \((x, H, C)\). If there is a progressively measurable nonnegative process \( c \) such that \( C_t = \int_0^t c(u) \mu(du) \), then \( C \) will be called an \textit{absolutely continuous consumption process} and \( c \) its consumption density.

**Corollary 1.** Let \( x > 0 \). A nonnegative progressively measurable process \( c \) is a consumption density of an \( x \)-admissible absolutely continuous consumption process if and only if
\[
\sup_{Y \in \mathcal{Y}} E \left[ \int_0^T Y_u c(u) \mu(du) \right] \leq x.
\]

(4.6)

Before we give the proof, we need the following lemma:

**Lemma 8.** If \( Y^{(n)} \) is a sequence in \( S_1 \), Fatou converging to \( Y \in S_1 \), then there is a countable set \( K \subseteq [0, T) \) such that for \( t \in [0, T] \setminus K \), we have \( Y_t = \lim \inf_n Y_t(n) \) a.s.
Proof. Let \( T \) be the countable dense subset of \([0, T]\) as in the definition of Fatou convergence for \((Y^{(n)})_{n \in \mathbb{N}}\). Putting \( Y'_t = \liminf_n Y^{(n)}_t \), it is easy to see that \( Y'_t \) is a nonnegative supermartingale. Let \( K \) be the set of points of right-discontinuity of the function \( t \mapsto \mathbb{E}[Y'_t] \). Since \( Y' \) is a supermartingale, \( t \mapsto \mathbb{E}[Y'_t] \) is nonincreasing so \( K \) is a countable subset of \([0, T]\). For \( t \in [0, T] \setminus K \), and \( q_n \downarrow t \), \( q_k \in T \), \((Y'_{q_k})_{k \in \mathbb{N}}\) is a backward supermartingale bounded in \( L^1 \). By the Backward Supermartingale Convergence Theorem (see [RY91], Theorem II.2.3, page 58) and the definition of Fatou convergence, \( Y'_{q_k} \to Y_t \) \( \mathbb{P} \)-a.s. and in \( L^1 \). Therefore

\[
\mathbb{E}[Y_t] = \mathbb{E}[Y'_t].
\]  

(4.7)

On the other hand, for \( t \in [0, T] \setminus K \),

\[
Y_t = \mathbb{E}[Y_t | \mathcal{F}_t] = \mathbb{E}[\liminf_k Y'_{q_k} | \mathcal{F}_t] \leq \liminf_k \mathbb{E}[Y'_{q_k} | \mathcal{F}_t] \leq Y'_t,
\]

(4.8)

since \( Y' \) is a supermartingale. From (4.7) and (4.8) we get \( Y'_t = Y_t \) \( \mathbb{P} \)-a.s. for \( t \in [0, T] \setminus K \).

Proof (of Corollary 1). If (4.6) holds, the Bayes rule for stochastic processes and Fubini’s theorem give \( \sup_{Q \in \mathcal{M}} \mathbb{E}_Q[C_T] \leq x \), where \( C_t = \int_0^t c(u) \mu(du) \). Following [Kra96] we define \( F_t = \text{esssup}_{Q \in \mathcal{M}} \mathbb{E}_Q[C_T | \mathcal{F}_t] \). By Theorem 2.1.1 in [EQ95], a modification of \( F_t \) can be chosen in such a way to make \( F_t \) a càdlàg supermartingale under each \( Q \in \mathcal{M} \). Furthermore, \( F_0 = x' \leq x \).

The Optional Decomposition Theorem guarantees the existence of an \( X \in \mathcal{X}(x') \) and a càdlàg nonincreasing process \( D \) with \( D_0 = 0 \) such that \( F_t = X_t - D_t \). Since \( C_t = \mathbb{E}_Q[C_T | \mathcal{F}_t] \leq \mathbb{E}_Q[C_T | \mathcal{F}_t] \) we conclude that \( F_t \geq C_t \) and so \((x - x') + X_t - C_t \geq X_1 - D_t - C_t = F_t - C_t \geq 0 \). Thus, \( C_t \) is an \( x \)-admissible consumption process.

Conversely, suppose \( c \) is a consumption density of an absolutely continuous consumption process \( C_T \). It is easy to see that \( \sup_{Q \in \mathcal{M}} \mathbb{E}_Q[C_T] \leq x \) must hold, so

\[
\sup_{Q \in \mathcal{Y}^c} \int_0^T \mathbb{E}[Y^{Q}_u c(u)] \mu(du) = \sup_{Q \in \mathcal{Y}^c} \mathbb{E} \left[ \int_0^T Y^{Q}_u c(u) \mu(du) \right] \leq x.
\]

Let \( Y \in \mathcal{Y} \). By Theorem 4, b), we can find a sequence \((Y^{(n)})_{n \in \mathbb{N}}\) in the solid hull of \( \mathcal{Y}^c \) such that \((Y^{(n)})_{n \in \mathbb{N}}\) Fatou converges to \( Y \). By the previous lemma, \( \liminf_n Y^{(n)}_t = Y_t \) for all \( t \in [0, T] \), except for maybe \( t \) in a countable subset of \([0, T]\) and thus for \( t \) \( \mu \)-a.e. By Fatou lemma and monotonicity,

\[
\mathbb{E} \left[ \int_0^T Y_u c(u) \mu(du) \right] = \mathbb{E} \left[ \int_0^T \liminf_n Y^{(n)}_u c(u) \mu(du) \right] \leq \liminf_n \mathbb{E} \left[ \int_0^T Y^{(n)}_u c(u) \mu(du) \right] \leq x.
\]

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