A PRIORI BOUNDS AND EXISTENCE OF NON-REAL EIGENVALUES OF INDEFINITE STURM-LIOUVILLE PROBLEMS

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Abstract. The present paper gives a priori bounds on the possible non-real eigenvalues of regular indefinite Sturm-Liouville problems and obtains sufficient conditions for such problems to admit non-real eigenvalues.

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1. Introduction

The present paper is concerned with the indefinite spectral problem
\begin{equation}
- y'' + qy = \lambda wy, \quad y(-1) = y(1) = 0, \quad \text{in } L^2_{|w|}[−1, 1]
\end{equation}
under the standing hypothesis that $q$ and $w$ are real-valued functions satisfying
\begin{equation}
w(x) \neq 0 \text{ a.e. on } [-1, 1], \quad q, w \in L^1[-1, 1],
\end{equation}
and $w(x)$ changes sign on $[-1, 1]$. The indefinite problem (1.1) has discrete, real eigenvalues, unbounded from both below and above, and may also admit non-real eigenvalues. Such problems occur in certain physical models, particularly in transport theory and statistical physics. The indefinite nature of the problem was noticed by Haupt [9] and Richardson [12] at the beginning of the last century. For a review of the early work in this direction, see [11].

As a simple example of (1.1), the Richardson problem
\begin{equation}
- y'' - \mu y = \lambda \text{sgn}(x)y, \quad x \in [-1, 1], \quad y(-1) = 0 = y(1)
\end{equation}
(1.3)
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was studied by many authors, such as Turyn [14], Atkinson and Jabon [1], Fleckinger and Mingarelli [8], and P. Binding and H. Volkmer [6].

For the indefinite problem (1.1), non-real eigenvalues might appear only if the corresponding right-definite problem

\begin{equation}
-y'' + qy = \lambda|w|y, \quad y(-1) = y(1) = 0 \quad \text{in} \quad L^2_{|w|}[-1, 1]
\end{equation}

has negative eigenvalues, namely, here holds the following result.

**Proposition 1.1.** (cf. [10, Theorem 2, p. 523] and [7, Corollary 1.7])

If the problem (1.4) has \( n \) negative eigenvalues, then the problem (1.1) has at most \( 2n \) non-real eigenvalues.

Although the upper bound given in Proposition 1.1 is sharp [12, 2], determining a priori bounds and the exact number of non-real eigenvalues are still difficult and interesting open problems in Sturm-Liouville theory (see [11] and [15, p. 126]). Recently, by means of the operator theory in Krein spaces, Behrndt, Katatbeh and Trunk [2, Theorem 2.3, Corollary 2.4] have given sufficient conditions for the existence of non-real eigenvalues of the singular indefinite Sturm-Liouville operator

\begin{equation}
(Af)(x) := \text{sgn}(x)(-f''(x) + V(x)f(x)) = \lambda f(x), \quad x \in \mathbb{R},
\end{equation}

and if \( V \in L^\infty(\mathbb{R}) \), Behrndt, Philipp and Trunk [3, Theorem 4,2] have obtained explicit bounds on the non-real eigenvalues of (1.5) in terms of \( V \).

In the present paper, we will first obtain a priori bounds for possible non-real eigenvalues and then find sufficient conditions for the existence of non-real eigenvalues of (1.1). These results will answer or partially answer several open problems posed in [11]. We state these results in this section and prove them in Sections 2 and 3.

Denote by \( \| \cdot \|_p \) the norm of the space \( L^p[-1, 1] \) and by \( \| \cdot \|_C \) the maximum norm of \( C[-1, 1] \). If \( xw(x) > 0 \) a.e. on \([-1, 1]\), we set

\begin{equation}
S_1(\varepsilon) = \{ x \in [-1, 1] : \ xw(x) < \varepsilon \}, \quad m_1(\varepsilon) = \text{mes} S_1(\varepsilon).
\end{equation}

If \( w \in AC_{\text{loc}}[-1, 1], \ w' \in L^2[-1, 1], \) we set

\begin{equation}
S_2(\varepsilon) = \{ x \in [-1, 1] : \ w^2(x) < \varepsilon \}, \quad m_2(\varepsilon) = \text{mes} S_2(\varepsilon).
\end{equation}

A value of \( x \) about which \( w(x) \) changes its sign will be called a turning point [10]. If \( w(x) \) has only one turning point, we will obtain the following a priori bounds for possible non-real eigenvalues.
Theorem 1.2. Suppose that $\lambda$ is, if it exists, a non-real eigenvalue of (1.1). If $xw(x) > 0$ a.e. on $[-1, 1]$, then
\begin{equation}
|\text{Re} \lambda| \leq \frac{4}{\varepsilon_1} (\|q_{-}\|_1 + 4\|q_{-}\|^2), \quad |\text{Im} \lambda| \leq \frac{4}{\varepsilon_1} \|q_{-}\|_1.
\end{equation}
where $\varepsilon_1 > 0$ satisfies $8\|q_{-}\|^2 m_1(\varepsilon_1) < 1$ and $q_{-}(x) = -\min\{0, q(x)\}$.

In the case where $w(x)$ is allowed to have more turning points, we will obtain

Theorem 1.3. Suppose that $\lambda$ is, if it exists, a non-real eigenvalue of (1.1). If $w \in AC[-1, 1]$ and $w' \in L^2[-1, 1]$, then
\begin{equation}
|\text{Re} \lambda| \leq \frac{8}{\varepsilon_2} \|q_{-}\|_1^2 (3\|w\|_C + \|w'\|_2), \quad |\text{Im} \lambda| \leq \frac{8}{\varepsilon_2} \|w'\|_2 \|q_{-}\|^2.
\end{equation}
where $\varepsilon_2 > 0$ is chosen such that $8\|q_{-}\|^2 m_2(\varepsilon_2) < 1$.

In the particular case where $q \geq 0$, we see by Theorems 1.2 and 1.3 that (1.1) has no any non-real eigenvalues, which is in accordance with the conclusion in Proposition 1.1 since now (1.4) does not have any negative eigenvalues.

In what follows, we impose the symmetry conditions on $q$ and $w$, namely,
\begin{equation}
q(x) = q(-x), \quad w(-x) = -w(x).
\end{equation}
In this case, more accurate a priori bounds on imaginary eigenvalues can be found if $q$ is bounded below and $w$ keeps away from zero.

Theorem 1.4. Suppose that (1.10) holds and $xw(x) > 0$ a.e. on $[-1, 1]$. If, for some $q_0 < 0$ and $w_0 > 0$,
\begin{equation}
q(x) \geq q_0, \quad |w(x)| \geq w_0 \quad \text{a.e.} \quad x \in [-1, 1],
\end{equation}
then for any possible pure imaginary eigenvalue $\lambda$ of (1.1), there holds
\begin{equation}
|\text{Im} \lambda| \leq \frac{4(-q_0)^{3/2}}{w_0}.
\end{equation}

In view of (1.10), using the spectral theory of operators in Krein spaces, we obtain an existence result for non-real eigenvalues of the indefinite problem (1.1).

Theorem 1.5. Let (1.10) be fulfilled. If the eigenvalue problem
\begin{equation}
-y'' + q(x)y = \lambda y, \quad y(-1) = y(1) = 0
\end{equation}
has one negative eigenvalue and the rest eigenvalues are all positive, then (1.1) has exactly two purely imaginary eigenvalues.
Immediate consequences of Proposition 1.1, Theorems 1.4 and 1.5 are the existence and bounds for non-real eigenvalues of Richardson problem.

Corollary 1.6. For $\mu \in \left(\frac{\pi^2}{4}, \pi^2\right)$, the Richardson eigenvalue problem (1.3) has exactly two purely imaginary eigenvalues whose moduli are bounded by $4\mu^3/2$.

Remark 1. Theorems 1.2–1.5 can be generalized to the problem

\[
\begin{cases}
-(p(x)y')' + q(x)y = \lambda w(x)y, \\
\alpha_1y(-1) + \beta_1y'(-1) = 0, \\
\alpha_2y(1) + \beta_2y'(1) = 0,
\end{cases}
\]

where $p(x) > 0$ a.e. on $[-1, 1]$, $1/p \in L^1[-1, 1]$, $\alpha_j, \beta_j \in \mathbb{R}$ for $j = 1, 2$ and $\alpha_1\beta_2 + \alpha_2\beta_1 = 0$, but we do not pursue this here.

2. A priori bounds of non-real eigenvalues

In this section we will prove Theorems 1.2, 1.3, and 1.4.

The proof of Theorem 1.2. Let $\lambda$ be a non-real eigenvalue of (1.1) and $\phi(x)$ the corresponding eigenfunction with $\|\phi\|_2 = 1$. Multiplying both sides of $-\phi'' + q\phi = \lambda w\phi$ by $\overline{\phi}$ and integrating over the interval $[x, 1]$ we have

\[
(\phi'\overline{\phi})(x) + \int_x^1 |\phi'|^2 + \int_x^1 q|\phi|^2 = \lambda \int_x^1 w|\phi|^2.
\]

Separating the real and imaginary parts of both sides of (2.1) yields

\[
\begin{align*}
\text{Re} \lambda \int_x^1 w|\phi|^2 &= \text{Re}(\phi'\overline{\phi})(x) + \int_x^1 |\phi'|^2 + \int_x^1 q|\phi|^2, \\
\text{Im} \lambda \int_x^1 w|\phi|^2 &= \text{Im}(\phi'\overline{\phi})(x).
\end{align*}
\]

We will use (2.2) and (2.3) to estimate $\text{Re} \lambda$ and $\text{Im} \lambda$. To do this, let $x = -1$ in (2.3). From $\text{Im} \lambda \neq 0$ and $\phi(-1) = 0$, we have $\int_{-1}^1 w|\phi|^2 = 0$, and hence, by (2.2),

\[
\int_{-1}^1 (|\phi'|^2 + q|\phi|^2) = 0.
\]
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Set \( Q(x) = \int_{-1}^{x} q_-(t) dt \). Then \( \max |Q(x)| \leq \| q_- \|_1 \) and

\[
\int_{-1}^{1} q_- (x) |\phi(x)|^2 dx = \int_{-1}^{1} Q'(x) |\phi(x)|^2 dx
\]

\[- 2 \Re \left( \int_{-1}^{1} Q(x) \phi'(x) \overline{\phi(x)} dx \right),
\]

which, together with \( \| \phi \|_2 = 1 \), yields that

\[(2.5) \quad \int_{-1}^{1} q_- |\phi|^2 \leq 2 \| q_- \|_1 \int_{-1}^{1} |\phi'| \| \phi \| \leq 2 \| q_- \|_1 \| \phi' \|_2 \leq 2 \| q_- \|_1^2 + \frac{1}{2} \| \phi' \|_2^2.
\]

Then, from (2.4), we get

\[(2.6) \quad \| \phi' \|_2^2 \leq 4 \| q_- \|_1^2, \quad \int_{-1}^{1} q_- |\phi|^2 \leq 4 \| q_- \|_1^2.
\]

From \( \phi(x) = \int_{-1}^{x} \phi'(t) dt \), by Cauchy-Schwarz inequality, we have

\[
|\phi(x)|^2 = \left| \int_{-1}^{x} \phi'(t) dt \right|^2 \leq (x + 1) \int_{-1}^{x} \phi'(t)^2 dt \leq \int_{-1}^{0} \phi'^2 \leq \| \phi' \|_2^2
\]

for \(-1 \leq x \leq 0\). From \( \phi(x) = -\int_{x}^{1} \phi'(t) dt \), one similarly proves

\[
|\phi(x)|^2 \leq \| \phi' \|_2^2 \text{ for } x \in [0, 1], \text{ and so,}
\]

\[(2.7) \quad |\phi(x)|^2 \leq \| \phi' \|_2^2, \quad x \in [-1, 1].
\]

Since \( xw(x) > 0 \), a.e. on \([-1, 1]\), one can find \( \varepsilon_1 > 0 \) such that

\[
8 \| q_- \|_1^2 m_1(\varepsilon_1) < 1,
\]

where \( m_1(\varepsilon) \) is defined in (1.3). Using \( \int_{-1}^{1} w |\phi|^2 = 0 \), from (2.6) and (2.7), we have

\[(2.8) \quad \int_{-1}^{1} \int_{-1}^{x} w(t) |\phi(t)|^2 dt dx = \int_{-1}^{1} xw(x) |\phi(x)|^2 dx
\]

\[
\geq \varepsilon_1 \left( \int_{-1}^{1} |\phi(x)|^2 dx - \int_{S_1(\varepsilon_1)} |\phi(x)|^2 dx \right)
\]

\[
\geq \varepsilon_1 \left[ 1 - 4 \| q_- \|_1^2 m_1(\varepsilon_1) \right] \geq \frac{\varepsilon_1}{2}.
\]
Set \( q_+(x) = \max\{0, q(x)\} \). Then \( q = q_+ - q_- \) and \( |q| = q_+ + q_- = q + 2q_- \). Repeatedly using (2.4), we have
\[
\left| \int_{-1}^{1} \int_{x}^{1} (|\phi'|^2 + q|\phi|^2) dtdx \right| = \left| \int_{-1}^{1} x(|\phi'|^2 + q|\phi|^2) dx \right|
\leq \int_{-1}^{1} (|\phi'|^2 + q|\phi|^2 + 2q_-|\phi|^2) dx = 2 \int_{-1}^{1} q_-|\phi|^2 dx.
\]
Now, by (2.6), the integration of (2.2) gives
\[
|\text{Re} \lambda| \int_{-1}^{1} \int_{x}^{1} w|\phi|^2 = \left| \int_{-1}^{1} \text{Re}(\phi\overline{\phi}) dx + \int_{-1}^{1} \int_{x}^{1} (|\phi'|^2 + q|\phi|^2) dtdx \right|
\leq \|\phi\|_2^2 + 2 \int_{-1}^{1} q_-|\phi|^2 dx \leq 2\|q_\|_1 + 8\|q_\|_1^2.
\]
Therefore, in view of (2.8), we conclude that
\[
(2.9) \quad |\text{Re} \lambda| \leq \frac{4}{\varepsilon_1} (\|q_-\|_1 + 4\|q_-\|_1^2).
\]
Moreover, integrating (2.3) and using (2.8) and (2.6), we have
\[
(2.10) \quad \frac{\varepsilon_1}{2} |\text{Im} \lambda| \leq |\text{Im} \lambda| \int_{-1}^{1} \int_{x}^{1} w|\phi|^2 = \left| \int_{-1}^{1} \text{Im}(\phi\overline{\phi}) \right| \leq \|\phi\|_2 \leq 2\|q_\|_1,
\]
and (1.8) follows immediately. This completes the proof of Theorem 1.2.

The Proof of Theorem 1.3. Let \( \lambda \) be a non-real eigenvalue of (1.1) and \( \phi \) the corresponding eigenfunction with \( \|\phi\|_2 = 1 \). In this case we still can make use of (2.1), (2.2) and (2.3). From (2.3), since \( \text{Im} \lambda \neq 0 \), one sees that \( \int_{-1}^{1} w|\phi(x)|^2 dx = 0 \). Thus, (2.4), (2.6) and (2.7) hold, and particularly,
\[
(2.11) \quad |\phi(x)|^2 \leq \|\phi\|_2^2, \ x \in [-1, 1], \ \|\phi\|_2^2 \leq \int_{-1}^{1} q_-|\phi|^2 \leq 4\|q_-\|_1^2.
\]
Multiplying \(-\phi'' + q\phi = \lambda w\phi \) by \( w\overline{\phi} \) and integrating by parts, we get
\[
(2.12) \quad \int_{-1}^{1} w|\phi'|^2 + \int_{-1}^{1} w'\phi\overline{\phi} + \int_{-1}^{1} wq|\phi|^2 = \lambda \int_{-1}^{1} w^2|\phi|^2.
\]
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Separating the real and imaginary parts of the both sides of (2.12) yields

\begin{align}
(2.13) \quad \text{Re} \lambda \int_{-1}^{1} w^2|\phi|^2 &= \text{Re} \left( \int_{-1}^{1} w'\phi \overline{\phi} \right) + \int_{-1}^{1} w(|\phi'|^2 + q|\phi|^2), \\
(2.14) \quad \text{Im} \lambda \int_{-1}^{1} w^2|\phi|^2 &= \text{Im} \left( \int_{-1}^{1} w'\phi \overline{\phi} \right).
\end{align}

Now, using (2.11), \(|q| = q + 2q_\epsilon\) and \(\int_{-1}^{1} q|\phi|^2 = -\int_{-1}^{1} |\phi'|^2 < 0\), we obtain

\begin{align}
|\int_{-1}^{1} w|\phi'|^2| &\leq \|w\|_C|\phi'|^2 \leq 4\|w\|_C\|q_-\|^2, \\
|\int_{-1}^{1} wq|\phi|^2| &\leq \|w\|_C \int_{-1}^{1} |q| |\phi|^2 \leq 8\|w\|_C\|q_-\|^2, \\
|\int_{-1}^{1} w'q\phi\overline{\phi}| &\leq \|\phi\|^2 \|w'\|_2 \|\phi\|^2 \leq 4\|w'\|_2\|q_-\|^2.
\end{align}

Recall that \(m_2(\varepsilon_2) = \text{mes} S_2(\varepsilon_2)\) defined in (1.7) and \(w^2(x) \geq \varepsilon_2\) on the set \(\Omega(\varepsilon_2) := [-1, 1] \setminus S_2(\varepsilon_2)\). Then \(8\|q_-\|^2 m(\varepsilon_2) < 1\) yields that

\begin{align}
\int_{-1}^{1} w^2(x)|\phi(x)|^2dx &\geq \varepsilon_2 \int_{\Omega(\varepsilon_2)} |\phi|^2 = \varepsilon_2 \left( 1 - \int_{S(\varepsilon_2)} |\phi|^2 \right) \\
&\geq \varepsilon_2 \left( 1 - 4\|q_-\|^2 m(\varepsilon_2) \right) \geq \frac{\varepsilon_2}{2},
\end{align}

which, together with (2.13), (2.14) and (2.15), gives (1.9) and completes the proof. \(\square\)

Under the conditions (1.2) and (1.10), it is easy to see that if \(\lambda \in \mathbb{C}\) is an eigenvalue of (1.1) with an eigenfunction \(\phi\), then \(-\lambda\) is an eigenvalue of (1.1) with the eigenfunction \(\overline{\phi(-\cdot)}\). Thus, if \(\lambda = i\alpha\) with \(\alpha \in \mathbb{R}\), then \(\overline{\phi(-x)} = C\phi(x)\) for some \(C \neq 0\) since the geometric multiplicity is one. Then it follows that \(|C| = 1\) from \(\overline{\phi(0)} = C\phi(0)\), \(\overline{\phi'(0)} = -C\phi'(0)\), and \(|\phi(0)| + |\phi'(0)| \neq 0\). To sum up, we have

**Lemma 2.1.** Let (1.2) and (1.10) hold. If \(\lambda \in \mathbb{C}\) is an eigenvalue of (1.1) with an eigenfunction \(\phi\), then \(-\lambda\) is an eigenvalue of (1.1) with the eigenfunction \(\overline{\phi(-\cdot)}\). Particularly, if \(\lambda = i\alpha\) with \(\alpha \in \mathbb{R}\) and \(\alpha \neq 0\), then \(\overline{\phi(-\cdot)} = C\phi\) for some \(C \in \mathbb{C}\) with \(|C| = 1\).
The Proof of Theorem 1.4. Let $\phi$ be an eigenfunction corresponding to $\lambda = i\alpha$ with $\|\phi\|_2 = 1$. It follows from Lemma 2.1 that there exists an $\omega \in [0, 2\pi)$ such that $\overline{\phi(-x)} = e^{i\omega}\phi(x)$ and $-\phi'(-x) = e^{i\omega}\phi'(x)$. So, $|\phi(x)|$ and $|\phi'(x)|$ are even functions. We see that (2.17) hold for this $\phi$. Similarly to (2.7), we have

$$|(\phi(x)|^2 \leq (x + 1) \int_{-1}^{x} |\phi'(t)|^2 dt \leq \int_{-1}^{0} |\phi'(t)|^2 dt = \frac{1}{2}\|\phi\|_2^2, \ x \in [-1, 0],$$

since $|\phi'(x)|$ is even. Actually, (2.17) is true for $x \in [-1, 1]$ since $|\phi(x)|$ is even.

Since $q(x) \geq q_0$ on $[-1, 1]$, it follows from (2.4) and $\|\phi\|_2 = 1$ that $\|\phi'\|_2^2 = -\int_{-1}^{1} q |\phi|^2 \leq -q_0$, and then the integration of (2.3) produces

$$\int_{-1}^{1} w|\phi|^2 = \int_{-1}^{1} \text{Im}(\phi\phi^\prime) \leq \|\phi\|_2 \leq (q_0)^{1/2}. \quad (2.18)$$

Let $\delta = 1/(-2q_0)$. By (2.17), we have $1 = \int_{-1}^{1} |\phi|^2 \leq \|\phi'\|_2^2 \leq -q_0$ and

$$\int_{-1}^{1} \int_{-1}^{1} w|\phi|^2 \geq \int_{-1}^{1} xw(x)|\phi|^2 dx \geq w_0 \int_{-1}^{1} |x||\phi|^2 dx \geq w_0 \delta \int_{|x| \geq \delta} |\phi|^2 = w_0 \delta \left(1 - \int_{-\delta}^{\delta} |\phi|^2 \right) \geq w_0 \delta (1 - \delta(-q_0)) = \frac{-w_0}{4q_0}. \quad (2.19)$$

Now, (1.12) follows from (2.18) and (2.19). The proof is complete. $\square$

3. Existence of non-real eigenvalues

In this section we prove Theorem 1.5 and in the proof we will use the following result which was proved, e.g., in [7] and [5].

Lemma 3.1. (cf. [7] Proposition 2.6) If $w_j \in L^1[-1, 1]$ and $w_j(x) > 0$ a.e. on $[-1, 1]$ for $j = 1, 2$, then the two eigenvalue problems

$$-y'' + q(x)y = \lambda w_j(x)y, \ y(-1) = y(1) = 0, \ j = 1, 2 \quad (3.1)$$

have the same number of negative eigenvalues.

Let $K$ be the Krein space $L^2_{|x|}[1, 1]$, equipped with the indefinite inner product

$$[f, g] = \int_{-1}^{1} f(x)\overline{g(x)}w(x)dx, \ f, g \in L^2_{|x|}[1, 1] \quad (3.2)$$
and $T$ a self-adjoint operator in $K$ with domain $\mathcal{D}(T)$ [4, 2, 7]. We say that the operator $T$ has $k$ negative squares, $k \in \mathbb{N}_0$, if there exists a $k$-dimensional subspace $X$ of $K$ in $\mathcal{D}(T)$ such that $[Tf, f] < 0$ if $f \in X$ and $f \neq 0$, but no $(k + 1)$-dimensional subspace with this property.

The Proof of Theorem 1.5. Let $A$ and $B$ be the operators associated with $-y'' + q(x)y = \lambda w(x)y$ and $-y'' + q(x)y = \lambda |w(x)|y$ with the Dirichlet boundary conditions, respectively. Then $B$ is self-adjoint with respect to the definite inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)|w(x)|dx, \ f, g \in L^2_{|w|}[-1, 1]$$

and $A$ is self-adjoint with respect to the indefinite inner product (3.2).

It follows from Lemma 3.1 and the assumption in Theorem 1.5 that $B$ has one negative eigenvalue and the rest are positive, and hence, $A$ has exactly one negative square since $[Af, f] = (Bf, f)$ and 0 is a resolvent point of $A$. It is well known (see, e.g., [7, Proposition 1.5] or [4, Theorem 3.1]) that this implies the existence of exactly one eigenvalue $\lambda$ of (1.1) in $\mathbb{R}$ or the upper half-plane $\mathbb{C}^+$ and that if $\lambda \in \mathbb{R}$ with eigenfunction $\phi$ then $[A\phi, \phi] = \lambda[\phi, \phi] \leq 0$. Let $\lambda$ be such an eigenvalue with eigenfunction $\phi$. If $\lambda$ is real, then $-\lambda = -\overline{\lambda}$ is also an eigenvalue with the eigenfunction $\phi(\cdot)$ by Lemma 2.1 and

$$-\lambda[\phi(\cdot), \phi(\cdot)] = \lambda[\phi, \phi] \leq 0$$

by the odd symmetry of $w$. Thus, we get that $\lambda$ and $-\lambda$ are two such eigenvalues, which is a contradiction. Since $\lambda \in \mathbb{C}^+$ implies $-\overline{\lambda} \in \mathbb{C}^+$, we see that $\lambda = -\overline{\lambda}$, i.e., $\lambda$ is purely imaginary. The proof of Theorem 1.5 is complete. □

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