Distributed Cooperative Online Estimation
With Random Observation Matrices,
Communication Graphs and Time-Delays

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Abstract

We analyze convergence of distributed cooperative online estimation algorithms by a network of multiple nodes via information exchanging in an uncertain environment. Each node has a linear observation of an unknown parameter with randomly time-varying observation matrices. The underlying communication network is modeled by a sequence of random digraphs and is subjected to nonuniform random time-varying delays in channels. Each node runs an online estimation algorithm consisting of a consensus term taking a weighted sum of its own estimate and neighbours’ delayed estimates, and an innovation term processing its own new measurement at each time step. By stochastic time-varying system, martingale convergence theories and the binomial expansion of random matrix products, we transform the convergence analysis of the algorithm into that of the mathematical expectation of random matrix products. Firstly, for the delay-free case, we show that the algorithm gains can be designed properly such that all nodes’ estimates converge to the real parameter in mean square and almost surely if the observation matrices and communication graphs satisfy the stochastic spatial-temporal persistence of excitation condition. Especially, this condition holds for Markovian switching communication graphs and observation matrices, if the stationary graph is balanced with a spanning tree and the measurement model is spatially-temporally jointly observable. Secondly, for the case with time-delays, we introduce delay matrices to model the random time-varying communication delays between nodes, and propose a mean square convergence condition, which quantitatively shows the intensity of spatial-temporal persistence of excitation to overcome time-delays.

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Distributed online estimation, cooperative estimation, random graph, random time-delay, mean square convergence, almost sure convergence.

I. INTRODUCTION

Estimation algorithms have important applications in many fields, e.g. navigation systems, space exploration, machine learning and power systems ([1]-[4]), etc. In a power system, measurement devices such as remote terminal units and phasor measurement units, send the measured active and reactive power flows, bus injection powers and voltage amplitudes to the Supervisory Control and Data Acquisition (SCDA) system, then the voltage amplitudes and phase angles at all buses are estimated for secure and stable operation of the system ([5]-[6]). Generally speaking, there are mainly two categories of estimation algorithms in term of information structure, i.e. centralized and distributed algorithms. In a centralized algorithm, a fusion center is used to collect all nodes’s measurements and gives the global estimate. The centralized information structure heavily relies on the fusion center and lacks robustness and security. In a distributed algorithm, a network of multiple nodes is employed to cooperatively estimate the unknown parameter via information exchanging, where each node is an entity with integrated capacity of sensing, computing and communication, and occasional node/link failures may not destroy the entire estimation task. Hence, distributed cooperative estimation algorithms are more robust than centralized ones ([7]-[8]).

There exist various kinds of uncertainties in real networks. For example, sensors are usually powered by chemical or solar cells, and the unpredictability of cell power leads to random node/link failures, which can be modeled by a sequence of random communication graphs. Besides, node sensing failures or measurement losses ([9]) can be modeled by a sequence of random observation matrices (regression matrices). There are lots of literature on distributed online estimation problems with random graphs. Ugrinovskii [10] studied distributed estimation with Markovian switching graphs. Kar & Moura [11] and Sahu et al [12] considered distributed estimation with i.i.d. graph sequences, where Kar & Moura [11] showed that the algorithm achieves weak consensus under a weak distributed detectability condition and Sahu et al [12] proved that the algorithm converges almost surely if the mean graph is balanced and strongly connected. Simões & Xavier [13] proposed a distributed estimation algorithm with i.i.d. undirected graphs and proved that the convergence rate of mean square estimation error is asymptotically equal to that of the centralized algorithm. Distributed cooperative online estimation based on diffusion strategies was addressed in [14]-[18] with spatially-temporally...
independent observation matrices, i.e. the sequence of observation matrices of each node is an independent random process and those of different nodes are mutually independent. Piggott & Solo [19]-[20] studied distributed estimation with temporally correlated observation matrices and a fixed communication graph. Ishihara & Alghunaim [21] studied distributed estimation with spatially independent observation matrices. Kar et al [22] and Kar & Moura [23] proposed consensus+innovation distributed estimation algorithms with random graphs and observation matrices, where the sequences of communication graphs and observation matrices are both i.i.d. and the mathematical expectation of observation matrices needs to be known. They proved that the algorithm converges almost surely if the mean graph is balanced and strongly connected. Zhang & Zhang [24] considered distributed estimation with finite Markovian switching graphs and i.i.d. observation matrices, and proved that the algorithm converges in mean square and almost surely if all graphs are balanced and jointly contain a spanning tree. Zhang et al [25] proposed a robust distributed estimation algorithm with the communication graphs and observation matrices being mutually independent with each other and both uncorrelated sequences. In summary, most existing literature on distributed cooperative estimation algorithms required balanced mean graphs and special statistical properties of communication graphs and observation matrices, such as i.i.d. or Markovian switching graph sequences, spatially or temporally independent observation matrices with the fixed mathematical expectation, which are also independent of communication graphs.

Besides random communication graphs and observation matrices, random communication delays are also common in real systems ([26]-[28]). Due to congestions of communication links and external interferences, time-delays are usually random and time-varying, whose probability distribution can be approximately estimated by statistical methods. However, to our best knowledge, there has been no literature on distributed online estimation with general random time-varying communication delays. Zhang et al [29] and Millán et al [30] considered distributed estimation with uniform deterministic time-invariant and time-varying communication delays, respectively, where Millán et al [30] established a LMI type convergence condition by the Lyapunov-Krasovskii functional method.

In this paper, we analyze convergence of distributed cooperative online parameter estimation algorithms with random observation matrices, communication graphs and time-delays. Each node’s algorithm consists of a consensus term taking a weighted sum of its own estimate and delayed estimates of its neighbouring nodes, and an innovation term processing its own new measurement at each time step. The sequences of observation matrices, communication graphs and time-delays are not required to satisfy special statistical properties, such as
mutual independence and spatial-temporal independence. Furthermore, neither the sample paths of the random graphs nor the mean graphs are necessarily balanced. These relaxations together with the existence of random time-varying time-delays bring essential difficulties to the convergence analysis, and most existing methods are not applicable. For examples, the frequency domain approach ([29],[31]) is only suitable for deterministic uniform time-invariant time-delays, and the Lyapunov-Krasovskii functional method leads to a non-explicit LMI type convergence condition ([30]). Liu et al [32] and Liu et al [33] addressed distributed consensus with deterministic time-varying communication delays and i.i.d. communication graphs, whose analysis method relying on the condition of balanced mean graphs, is not applicable to unbalanced mean graphs.

We introduce delay matrices to model the random time-varying communication delays between each pair of nodes. By stochastic time-varying system, martingale convergence theories and the binomial expansion of random matrix products, we transform the convergence analysis of the algorithm into that of the mathematical expectation of random matrix products. Firstly, for the delay-free case, we show that the algorithm gains can be designed properly such that all nodes’ estimates converge to the real parameter in mean square and almost surely if the observation matrices and communication graphs satisfy the stochastic spatial-temporal persistence of excitation condition. Especially, we show that for Markovian switching communication graphs and observation matrices, this condition holds if the stationary graph is balanced with a spanning tree and the measurement model is spatially-temporally jointly observable. Secondly, for the case with time-delays, we propose a mean square convergence condition, which explicitly relies on the conditional expectations of delay matrices, observation matrices and weighted adjacency matrices of communication graphs over a sequence of fixed-length time intervals. This condition quantitatively shows the intensity of spatial-temporal persistence of excitation to overcome additional effects of time-delays. Compared with the existing literature, our contributions are summarized as below.

- The delay-free case
  - We show that it is not necessary that the sequences of observation matrices and communication graphs are mutually independent or spatially-temporally independent. Also, the mean graphs are not necessarily time-variant and balanced. We establish the stochastic spatial-temporal persistence of excitation condition under which the distributed cooperative online estimation algorithm with random graphs and observation matrices converges in mean square and almost surely. For a network consisting of completely isolated nodes, the stochastic spatial-temporal persistence
of excitation condition degenerates to several independent stochastic persistence of excitation conditions for centralized algorithms.

– Especially, for the case with Markovian switching communication graphs and observation matrices, we prove that the stochastic spatial-temporal persistence of excitation condition holds if the stationary graph is balanced with a spanning tree and the measurement model is spatially-temporally jointly observable, implying that neither local observability of each node nor instantaneous global observability of the entire measurement model is necessary.

• The case with time-delays
  – We introduce delay matrices to model the random time-varying time-delays between each pair of nodes. By the method of binomial expansion of random matrix products, we obtain a mean square convergence condition, which explicitly relies on the conditional expectations of the delay matrices, observation matrices and weighted adjacency matrices of communication graphs over a sequence of fixed-length time intervals, and shows that the communication graphs and observation matrices need to be persistently excited with enough intensity to attenuate the random time-delays.
  – The nonuniform random time-varying communication delays considered in this paper are more general, and we allow correlated communication delays, graphs and observation matrices.

The rest of the paper is arranged as follows. In Section II, we formulate the problem. In Section III, we describe the distributed cooperative online parameter estimation algorithm with random observation matrices, communication graphs and time-delays. The convergence analysis for the delay-free case and the case with random time-varying time-delays are given in Sections IV and V, respectively. Finally, we conclude the paper and give some future topics in Section VI.

Notation and symbols:
- $\odot$: Hadamard product;
- $\otimes$: Kronecker product;
- $\text{Tr}(A)$: trace of matrix $A$;
- $\|A\|$ : 2-norm of matrix $A$;
- $A^T$: transpose of matrix $A$;
- $\mathbb{P}\{A\}$: probability of event $A$;
- $I_n$: $n$ dimensional identity matrix;
- $\rho(A)$: spectral radius of matrix $A$;
\( |a| \): absolute value of real number \( a \);
\( \mathbb{R}^n \): \( n \) dimensional real vector space;
\( A \geq B \): \( A - B \) is positive semidefinite;
\( \lfloor x \rfloor \): the largest integer less than or equal to \( x \);
\( \lceil x \rceil \): the smallest integer greater than or equal to \( x \);
\( \mathbb{E}[\xi] \): mathematical expectation of random variable \( \xi \);
\( \lambda_{\text{min}}(A) \): minimum eigenvalue of real symmetric matrix \( A \);
\( \mathbf{1}_n \): \( n \) dimensional column vector with all entries being one;
\( \mathbf{0}_{n \times m} \): \( n \times m \) dimensional matrix with all entries being zero;
\( b_n = O(r_n) \): \( \limsup_{n \to \infty} \frac{|b_n|}{r_n} < \infty \), where \( \{b_n, n \geq 0\} \) is a sequence of real numbers, \( \{r_n, n \geq 0\} \) is a sequence of real positive numbers; \( b_n = o(r_n) \): \( \lim_{n \to \infty} \frac{b_n}{r_n} = 0 \);

II. PROBLEM FORMULATION

A. Measurement model

Consider a network of \( N \) nodes. Each node is an estimator with integrated capacity of sensing, computing, storage and communication. The estimators/nodes cooperatively estimate an unknown parameter vector \( x_0 \in \mathbb{R}^n \) via information exchanging. The relation between the measurement vector \( z_i(k) \in \mathbb{R}^{n_i} \) of estimator \( i \) and the unknown parameter \( x_0 \) is represented by

\[
 z_i(k) = H_i(k)x_0 + v_i(k), \quad i = 1, \ldots, N, \quad k \geq 0. \tag{1}
\]

Here, \( H_i(k) \in \mathbb{R}^{n_i \times n} \) is the random observation (regression) matrix at time instant \( k \) with \( n_i \leq n \), and \( v_i(k) \in \mathbb{R}^{n_i} \) is the additive measurement noise. Denote \( z(k) = [z_1^T(k), \ldots, z_N^T(k)]^T, \quad H(k) = [H_1^T(k), \ldots, H_N^T(k)]^T \) and \( v(k) = [v_1^T(k), \ldots, v_N^T(k)]^T \). Rewrite (1) by the compact form

\[
 z(k) = H(k)x_0 + v(k), \quad k \geq 0. \tag{2}
\]
Remark 1. In many real applications, the relations between the unknown parameter and the measurements can be represented by (1). For examples, in the distributed multi-area state estimation in power systems, the grid is partitioned into multiple geographically non-overlapping areas, and each area is regarded as a node. The grid state $x_0$ to be estimated consists of voltage amplitudes and phase angles at all buses. The measurement $z_i(k)$ of each area/node consists of the active and reactive power flow, bus injection powers and voltage amplitude information measured by remote terminal units and phasor measurement units in the $i$-th area. By the DC power flow approximation ([34]), the grid state degenerates to the voltage phase angles at all buses and the relation between the measurement of each area and the grid state can be represented by (1). In distributed parameter identification, each node’s measurement equation is given by

$$z_i(k) = \sum_{j=1}^{n} c_j z_i(k-j) + v_i(k) = [z_i(k-1), \ldots, z_i(k-n)][c_1, \ldots, c_n]^T + v_i(k).$$

For this case, the unknown parameter $x_0 = [c_1, \ldots, c_n]^T$ and the observation matrix (generally called regressor) $H_i(k) = [z_i(k-1), \ldots, z_i(k-n)]$ is an $n$ dimensional row vector. In addition, sensing failures in real networks can be modeled by a Markov chain or an i.i.d. sequence of Bernoulli variables $\{\delta_i(k), k \geq 0\}$. Then $H_i(k) = \delta_i(k)H_i'(k)$, where $\{H_i'(k), k \geq 0\}$ is the sequence of observation matrices without sensing failures.

B. Communication models

Assume that there exist nonuniform random time-varying communication delays for the communication links between each pair of nodes. We use a sequence of random variables $\{\lambda_{ji}(k) \in \{0, \ldots, d\}, k \geq 0\}$ to represent the time-delays associated with the link from node $j$ to node $i$, where the positive integer $d$ represents the maximum time-delay. This sequence is subjected to the discrete probability distribution

$$\mathbb{P}\{\lambda_{ji}(k) = q\} = p_{ji,q}(k) \text{ with } \sum_{q=0}^{d} p_{ji,q}(k) = 1. \quad (3)$$

We stipulate that $\mathbb{P}\{\lambda_{ii}(k) = 0\} = 1, i = 1, \ldots, N, k \geq 0$. Denote the $N$ dimensional matrices $\mathcal{I}(k,q) = [\mathcal{I}_{\lambda_{ji}(k),q}]_{1 \leq j,i \leq N}, 0 \leq q \leq d, k \geq 0$, called delay matrices. By the definition of Kronecker function, we know that for each $q = 0, 1, \ldots, d$, $\{\mathcal{I}(k,q), k \geq 0\}$ is a sequence of random matrices and its sample paths are sequences of $0-1$ matrices. By (3), we know that $\mathbb{E}[\mathcal{I}_{\lambda_{ji}(k),q}] = p_{ji,q}(k)$ and

$$\sum_{q=0}^{d} \mathcal{I}(k,q) = 1_N 1_N^T \text{ a.s.} \quad (4)$$
We use a sequence of random communication graphs \( \{G(k) = (\mathcal{V}, \mathcal{A}_{G(k)})\}, k \geq 0 \) to describe the possible link failures among nodes, where \( \mathcal{V} = \{1, \cdots, N\} \) is the node set and \( \mathcal{A}_{G(k)} = [a_{ij}(k)]_{1 \leq i,j \leq N} \) is the weighted adjacency matrix of the communication graph in which \( a_{ii}(k) = 0 \) a.s. for all \( i \in \mathcal{V} \) and \( k \geq 0 \) and \( a_{ij}(k) \neq 0 \) if and only if the link from node \( j \) to node \( i \) exists at time instant \( k \) for all \( i \neq j \). The neighborhood of node \( i \) is \( \mathcal{N}_i(k) = \{j | a_{ij}(k) \neq 0\} \). The degree matrix of the graph is \( D_{G(k)} = diag(\sum_{j=1}^{N} a_{1j}(k), \cdots, \sum_{j=1}^{N} a_{Nj}(k)) \) and the Laplacian matrix of the graph is \( L_{G(k)} = D_{G(k)} - \mathcal{A}_{G(k)} \) ([35]). Let

\[
\overline{A}(k, q) = (\mathcal{A}_{G(k)} \circ \mathcal{I}(k, q)) \otimes I_n. \tag{5}
\]

Then, by (4) and the above, we have

\[
\sum_{q=0}^{d} \overline{A}(k, q) = \mathcal{A}_{G(k)} \otimes I_n. \tag{6}
\]

III. DISTRIBUTED COOPERATIVE ONLINE ESTIMATION ALGORITHM

Let \( x_i(k) \in \mathbb{R}^n \) be the estimate by node \( i \) for the unknown parameter \( x_0 \) at time instant \( k \). Starting at the initial estimate \( x_i(0) \), at any time instant \( k \geq 0 \), node \( i \) takes a weighted sum of its own estimate and delayed estimates received from its neighbours, and then adds a correction term based on the local measurement information (innovation) to update the estimate \( x_i(k+1) \). Specifically, the distributed cooperative online parameter estimation algorithm with random observation matrices, communication graphs and time-delays is given by

\[
x_i(k+1) = x_i(k) + a(k)H_i^T(k)(z_i(k) - H_i(k)x_i(k)) + b(k) \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k)(x_j(k) - \lambda_{ji}(k)) - x_i(k), \quad i \in \mathcal{V}, \; k \geq 0, \tag{7}
\]

where \( a(k) \) and \( b(k) \) are called the innovation gain and the consensus gain, respectively.

Denote the \( \sigma \)-fields \( \mathcal{F}(k) = \sigma(\mathcal{A}_{G(s)}, v(s), H_i(s), \lambda_{ji}(s), \; j, i \in \mathcal{V}, \; 0 \leq s \leq k), \; k \geq 0 \), with \( \mathcal{F}(-1) = \{\Omega, \emptyset\} \). For the algorithm (7), we have the following assumptions.

**A1.a** The sequence \( \{v(k), k \geq 0\} \) is independent of \( \{H(k), k \geq 0\}, \{\mathcal{A}_{G(k)}, k \geq 0\} \) and \( \{\lambda_{ji}(k), j, i \in \mathcal{V}, k \geq 0\} \).

**A1.b** The sequence \( \{v(k), \mathcal{F}(k), k \geq 0\} \) is a martingale difference sequence and there exists a constant \( \beta_v > 0 \) such that \( \sup_{k \geq 0} \mathbb{E}[\|v(k)\|^2 | \mathcal{F}(k-1)] \leq \beta_v \) a.s.

**A2.a** \( \sup_{k \geq 0} \|H(k)\| < \infty \) a.s. and \( \sup_{k \geq 0} \|\mathcal{A}_{G(k)}\| < \infty \) a.s.

**A2.b** There exist positive constants \( \beta_a \) and \( \beta_H \) such that \( \max_{i,j \in \mathcal{V}} \sup_{k \geq 0} |a_{ij}(k)| \leq \beta_a \) a.s. and \( \max_{i \in \mathcal{V}} \sup_{k \geq 0} \|H_i(k)\| \leq \beta_H \) a.s.
A3.a \{a(k), k \geq 0\} and \{b(k), k \geq 0\} are positive real sequences monotonically decreasing to zero, satisfying \(a(k) = O(b(k)), b^2(k) = o(a(k)), k \to \infty\) and \(\sum_{k=0}^{\infty} a(k) = \infty\).

A3.b \(\sum_{k=0}^{\infty} b^2(k) < \infty\).

A3.c \[\sup_{k \geq 0} b(k) < \sup_{0 < \kappa < 1} \min \left\{ \frac{\kappa}{2[N\beta_a + N\sqrt{N}\beta_a + C_a\beta^2_H]}, \frac{(1 - (1 - \kappa)^{-1})\kappa}{2N\sqrt{N}\beta_a(1 - (1 - \kappa)^{-(d+1)})} \right\},\]

where the constant \(C_a\) satisfies \(a(k) \leq C_a b(k), \forall k \geq 0\).

Remark 2. Note that, in Assumption A1.a, neither mutual independence nor spatial-temporal independence is assumed on the observation matrices, communication graphs and time-delays.

Remark 3. It is easy to find \(a(k)\) and \(b(k)\) satisfying Assumptions A3.a and A3.b. For example, if \(a(k) = b(k) = \frac{1}{k^\tau}, 0.5 < \tau \leq 1\), or \(a(k) = \frac{1}{k^\tau}, b(k) = \frac{1}{k^\tau_2}, 0.5 < \tau_2 < \tau_1 \leq 1\), then Assumptions A3.a and A3.b hold.

By the definition of \(\mathcal{L}_{\lambda_{ji}(k),q}\), we know that \(x_j(k - \lambda_{ji}(k)) = \sum_{q=0}^{d} x_j(k-q)\mathcal{L}_{\lambda_{ji}(k),q}\). Then by (7), we have

\[
x_i(k+1) = x_i(k) + a(k)H^T_i(k)[z_i(k) - H_i(k)x_i(k)] + b(k)\sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) \left[ \sum_{q=0}^{d} x_j(k-q)\mathcal{L}_{\lambda_{ji}(k),q} - x_i(k) \right], i \in \mathcal{V}.
\] (8)

Denote \(\mathcal{H}(k) = diag\{H_1(k), \cdots, H_N(k)\}\) and \(x(k) = [x_1^T(k), \cdots, x_N^T(k)]^T\). By (5), rewrite (8) as

\[
x(k+1) = [I_N - b(k)\mathcal{D}_{G(k)} \otimes I_n - a(k)\mathcal{H}^T(k)\mathcal{H}(k)]x(k) + b(k)\sum_{q=0}^{d} \bar{A}(k,q)x(k-q) + a(k)\mathcal{H}^T(k)z(k).
\] (9)

Denote the overall estimation error vector \(e(k) = x(k) - 1_N \otimes x_0\). Note that \((\mathcal{L}_{G(k)} \otimes I_n)(1_N \otimes x_0) = 0\). By (2) and (6), subtracting \(1_N \otimes x_0\) on both sides of (9) leads to

\[
e(k+1)
\]

\[
= [I_N - b(k)\mathcal{D}_{G(k)} \otimes I_n - a(k)\mathcal{H}^T(k)\mathcal{H}(k)]x(k) + b(k)\sum_{q=0}^{d} \bar{A}(k,q)x(k-q)
\]

\[
+ a(k)\mathcal{H}^T(k)z(k) - 1_N \otimes x_0
\]

\[
= [I_N - b(k)\mathcal{D}_{G(k)} \otimes I_n - a(k)\mathcal{H}^T(k)\mathcal{H}(k)](x(k) - 1_N \otimes x_0 + 1_N \otimes x_0)
\]

\[
+ b(k)\sum_{q=0}^{d} \bar{A}(k,q)(x(k-q) - 1_N \otimes x_0 + 1_N \otimes x_0) + a(k)\mathcal{H}^T(k)z(k) - 1_N \otimes x_0
\]
\[ [I_{Nn} - b(k) \mathcal{D}_{\mathcal{G}(k)} \otimes I_n - a(k) \mathcal{H}^T(k) \mathcal{H}(k)](e(k) + 1_N \otimes x_0) \\
+ b(k) \sum_{q=0}^{d} \bar{A}(k, q)(e(k - q) + 1_N \otimes x_0) + a(k) \mathcal{H}^T(k)z(k) - 1_N \otimes x_0 \]

Noting that \( \mathcal{H}(k)(1_N \otimes x_0) = H(k)x_0 \), by the above, we obtain the overall estimation error equation

\[ e(k + 1) = [I_{Nn} - b(k) \mathcal{D}_{\mathcal{G}(k)} \otimes I_n - a(k) \mathcal{H}^T(k) \mathcal{H}(k)]e(k) \\
+ b(k) \sum_{q=0}^{d} \bar{A}(k, q)e(k - q) + a(k) \mathcal{H}^T(k)v(k), \quad k \geq 0. \tag{10} \]

**IV. THE DELAY-FREE CASE**

In this section, we give the convergence conditions of the algorithm (7) for the delay-free case, i.e., \( \lambda_{ji}(k) = 0 \), a.s. \( \forall \, j, i \in \mathcal{V} \), \( \forall \, k \geq 0 \). All proofs of this section are put in Appendix B.

Denote \( \widehat{\mathcal{L}}_{\mathcal{G}(k)} = \frac{\mathcal{L}_{\mathcal{G}(k)}^0 + \mathcal{L}_{\mathcal{G}(k)}^{T}}{2} \). Specifically, if \( \mathcal{G}(k) \) is balanced, then \( \widehat{\mathcal{L}}_{\mathcal{G}(k)} \) is the Laplacian matrix of the symmetrized graph of \( \mathcal{G}(k), k \geq 0 \) ([36]). For any given positive integers \( h \) and \( m \), denote

\[ \lambda_m^h = \lambda_{\min} \left[ \sum_{k=mh}^{(m+1)h-1} \left( \frac{b(k)}{a(k)} \mathbb{E} [\mathcal{L}_{\mathcal{G}(k)} | \mathcal{F}(mh - 1) \otimes I_n + \mathbb{E} [\mathcal{H}^T(k) \mathcal{H}(k) | \mathcal{F}(mh - 1)]] \right) \right]. \]

**Theorem IV.1.** If Assumptions A1.a, A1.b and A3.a hold, and there exist a positive integer \( h \) and positive constants \( \theta \) and \( \rho_0 \) such that

\[ \begin{align*}
(b.1) \quad & \inf_{m \geq 0} \lambda_m^h \geq \theta > 0 \text{ a.s.}; \\
(b.2) \quad & \sup_{k \geq 0} \mathbb{E} [|\| \mathcal{L}_{\mathcal{G}(k)} \otimes I_n + \mathcal{H}^T(k) \mathcal{H}(k) \|^2 | \mathcal{F}(k - 1)]] \leq \rho_0 \text{ a.s.,}
\end{align*} \]

then the algorithm (7) converges in mean square, i.e., \( \lim_{k \to \infty} \mathbb{E} \| x_i(k) - x_0 \|^2 = 0, \quad i \in \mathcal{V} \).

**Theorem IV.2.** If the conditions in Theorem IV.1 hold and Assumptions A2.a and A3.b hold, then the algorithm (7) converges almost surely, i.e., \( \lim_{k \to \infty} x_i(k) = x_0, \quad i \in \mathcal{V} \) a.s.
Remark 4. The condition (b.1) in Theorems IV.1 and IV.2 is the key convergence condition. We call it the stochastic spatial-temporal persistence of excitation condition, where “spatial-temporal” represents the reliance of the condition on all nodes’ observation matrices and communication graphs (spatial dimension) over a sequence of fixed-length time intervals (temporal dimension) and “persistence of excitation” represents that the minimum eigenvalues of matrices consisting of spatial-temporal observation matrices and Laplacian matrices are uniformly bounded away from zero. Guo [37] considered centralized estimation algorithms with random observation matrices and proposed the “stochastic persistence of excitation” condition to ensure convergence. The condition (b.1) can be regarded as the generalization of “stochastic persistence of excitation” condition in [37] to that for distributed algorithms. For a network with $N$ isolated nodes, $L_G(k) = 0_{N \times N}$ a.s., and the condition (b.1) degenerates to $N$ independent “stochastic persistence of excitation” conditions.

Remark 5. Most existing literature on distributed estimation required balanced mean graphs ([22],[24]). Here, the condition (b.1) may still holds even if the mean graphs are unbalanced. For example, consider a simple fixed graph $G = \langle V = \{1, 2\}, A_G = [a_{ij}]_{2 \times 2} \rangle$ with $a_{12} = 1, a_{21} = 0.3$ and let $H_1 = 0, H_2 = 1$ and $a(k) = b(k)$. Obviously, $G$ is unbalanced. By some direct calculations, we have $\lambda_m = \lambda_{\min}(\hat{L}_G + H^T H) = 0.4829 > 0$, which implies the condition (b.1).

In the most existing literature, it was also required that the sequence of observation matrices is i.i.d. and independent of the sequence of communication graphs, neither of which is necessary in Theorems IV.1 and IV.2. Subsequently, we further give more intuitive convergence conditions for Markovian switching communication graphs and observation matrices, as stated in the following assumption.

$A4 \{\langle H_l, A_{G(l)} \rangle, k \geq 0 \} \subseteq S$ is a homogeneous and uniform ergodic Markov chain with a unique stationary distribution $\pi$.

Here, $S = \{\langle H_l, A_l \rangle, l = 1, 2, ... \}$ with $H_l = \text{diag}(H_{1,l}, \cdots, H_{N,l})$, where $\{H_{i,l} \in \mathbb{R}^{n_i \times n}, l = 1, 2, \ldots \}$ is the state space of observation matrices of node $i$ and $\{A_l, l = 1, 2, \ldots \}$ being the state space of the weighted adjacency matrices, $\pi = [\pi_1, \pi_2, \ldots]^T, \pi_l \geq 0, l = 1, 2, \ldots$, and $\sum_{l=1}^{\infty} \pi_l = 1$ with $\pi_l$ representing $\pi(\langle H_l, A_l \rangle)$.

Corollary IV.1. If Assumptions A1.a, A1.b, A3.a, A3.b and A4 hold, $\sup_{l \geq 1} \|A_l\| < \infty$, $\sup_{l \geq 1} \|H_l\| < \infty$, and

(c.1) the stationary weighted adjacency matrix $\sum_{l=1}^{\infty} \pi_l A_l$ is nonnegative and its associated
graph is balanced with a spanning tree;

(c.2) the measurement model (1) is *spatially-temporally jointly observable*, i.e.,

\[
\lambda_{\min}\left(\sum_{i=1}^{N} \left(\sum_{l=1}^{\infty} \pi_l H_{i,l}^T H_{i,l}\right)\right) > 0,
\]

then the algorithm (7) converges in mean square and almost surely, i.e., \(\lim_{k \to \infty} \mathbb{E}\|x_i(k) - x_0\|^2 = 0, i \in \mathcal{V}\) and \(\lim_{k \to \infty} x_i(k) = x_0, i \in \mathcal{V}\) a.s.

**Remark 6.** Most of the existing distributed estimation algorithms used the mathematical expectation of observation matrices which is restricted to be time-invariant and difficult to be obtained ([22],[24]). They required instantaneous global observability in the statistical sense for the measurement model, i.e., \(\sum_{i=1}^{N} H_i \sum_{l=1}^{\infty} \pi_l H_{i,l}^T H_{i,l}\) is positive definite, where \(H_i\) is a fixed matrix with \(\mathbb{E}[H_i(k)] \equiv H_i\), for all \(k \geq 0, i = 1, 2, ..., N\). Differently, we only use the sample paths of observation matrices in the algorithm (7). The mathematical expectations of observation matrices are allowed to be time-varying. We prove that for homogeneous and uniform ergodic Markovian switching observation matrices and communication graphs, the *stochastic spatial-temporal persistence of excitation* condition holds if the stationary graph is balanced with a spanning tree and the measurement model is spatially-temporally jointly observable, that is, (11) holds, implying that neither local observability of each node, i.e. \(\lambda_{\min}(\sum_{l=1}^{\infty} \pi_l H_{i,l}^T H_{i,l}) > 0, i \in \mathcal{V}\), nor instantaneous global observability of the entire measurement model, i.e. \(\lambda_{\min}(\sum_{i=1}^{N} H_{i,l}^T H_{i,l}) > 0, l = 1, 2, ...,\) is needed.

V. THE CASE WITH RANDOM TIME-VARYING COMMUNICATION DELAYS

In this section, we further analyze the convergence of the algorithm (7) with random observation matrices, communication graphs and time-delays simultaneously. All proofs of this section are put in Appendix C.

The random time-varying communication delays bring about that the mean square convergence analysis of the algorithm becomes very difficult. To this end, we transform (10) into the following equivalent system ([32]-[33]).

\[
\begin{align*}
    r(k+1) &= F(k)r(k) + g(k), \\
    g(k) &= \sum_{q=1}^{d} C_q(k)g(k-q) + a(k)\mathcal{H}^T(k)v(k),
\end{align*}
\]

where \(F(k), C_q(k), 1 \leq q \leq d, k \geq 0\) satisfy

\[
F(k) + C_1(k) = I_{Nn} - b(k)\mathcal{D}_{\mathcal{G}(k)} \otimes \mathbb{I}_n - a(k)\mathcal{H}^T(k)\mathcal{H}(k) + b(k)\mathcal{A}(k,0),
\]
Lemma V.1. If Assumptions A2.b and A3.c hold, then there exists a constant \( \kappa \in (0, 1) \) such that \( F(k) \) is invertible a.s. and \( \|F^{-1}(k)\| \leq (1 - \kappa)^{-1} \) a.s., \( \forall k \geq 0 \).

If Assumptions A2.b and A3.c hold, then \( F(k) \) is invertible a.s. by Lemma V.1. Then by (13), we have

\[
F(k) = I_{Nn} - b(k)D_{G(k)} \otimes I_n - a(k)H^T(k)H(k) + b(k)A(k, 0) - C_1(k)
\]
\[
= I_{Nn} - b(k)D_{G(k)} \otimes I_n - a(k)H^T(k)H(k) + b(k)A(k, 0) - (C_2(k) - b(k)A(k, 1))F^{-1}(k - 1)
\]
\[
= I_{Nn} - \left[ b(k)D_{G(k)} \otimes I_n + a(k)H^T(k)H(k) \right.
\]
\[
- b(k) \sum_{q=0}^d A(k, q) \left[ \Phi_F(k - 1, k - q) \right]^{-1}
\]
\[
= I_{Nn} - G(k), \quad k \geq 0, \quad (14)
\]

where

\[
G(k) = b(k)D_{G(k)} \otimes I_n + a(k)H^T(k)H(k) - b(k) \sum_{q=0}^d A(k, q) \left[ \Phi_F(k - 1, k - q) \right]^{-1}. \quad (15)
\]

For any given positive integers \( h \) and \( m \), denote

\[
\lambda_{m}^{h'} = \lambda_{\min} \left[ \sum_{k=mh}^{(m+1)h-1} \left( 2 \frac{b(k)}{a(k)} \mathbb{E}[\hat{\mathcal{L}}_{G(k)} | \mathcal{F}(mh - 1)] \otimes I_n + 2\mathbb{E}[H^T(k)H(k)|\mathcal{F}(mh - 1)] \right.
\]
\[
- \frac{b(k)}{a(k)} \sum_{q=0}^d \mathbb{E}[A(k, q)|[\Phi_F(k - 1, k - q)]^{-1} - I_{Nn}]|\mathcal{F}(mh - 1)]
\]
\[
- \frac{b(k)}{a(k)} \sum_{q=0}^d \mathbb{E}[[\Phi_F(k - 1, k - q)]^{-1} - I_{Nn}]^T A^T(k, q)|\mathcal{F}(mh - 1)] \right]. \quad (16)
\]
Theorem V.1. If Assumptions A1.a, A1.b, A2.b, A3.a and A3.c hold, and there exist a positive integer $h$ and a constant $\theta$ such that $\inf_{m \geq 0} \lambda^h_{m} \geq \theta > 0$ a.s., then the algorithm (7) converges in mean square, i.e. $\lim_{k \to \infty} \mathbb{E} \|x_i(k) - x_0\|^2 = 0$, $i \in \mathcal{V}$.

For any given positive integers $h$ and $m$, denote

$$\Delta^h_m = \sum_{k=mh}^{(m+1)h-1} \frac{b(k)}{a(k)} \left[ \sum_{q=0}^{d} \|\mathbb{E}[A(k,q) \cdot \Phi_F(k-1,k,q)]^{-1} - I_{Nn}] |F(mh-1)|\right] .$$

Subsequently, we present a corollary which reflects the impact of communication delays more intuitively.

Corollary V.1. If Assumptions A1.a, A1.b, A2.b, A3.a and A3.c hold and there exist a positive integer $h$ and a constant $\theta$ such that $\inf_{m \geq 0} (\lambda^h_{m} - \Delta^h_m) \geq \theta > 0$ a.s., then the algorithm (7) converges in mean square, i.e. $\lim_{k \to \infty} \mathbb{E} \|x_i(k) - x_0\|^2 = 0$, $i \in \mathcal{V}$.

Remark 7. Theorem V.1 gives an explicit convergence condition under which all nodes’ estimates converge to the real parameter in mean square. Existing literature used the Lyapunov-Krasovskii functional method to deal with time-delays and obtained the non-explicit LMI type convergence condition ([30]). In this section, we transform the system with random time-varying communication delays into an equivalent delay-free system by introducing an auxiliary system and then adopt the method of binomial expansion of random matrix products to transform the mean square convergence analysis of the delay-free system into that of the mathematical expectation of random matrix products, and obtain the key convergence condition $\inf_{m \geq 0} \lambda^h_{m} \geq \theta > 0$ a.s. which explicitly relies on the conditional expectations of delay matrices, observation matrices and weighted adjacency matrices of communication graphs over a sequence of fixed-length time intervals. In Corollary V.1, we further obtain the more intuitive convergence condition $\inf_{m \geq 0} (\lambda^h_{m} - \Delta^h_m) \geq \theta > 0$ a.s. which shows that the communication graphs and observation matrices need to be persistently excited with enough intensity to attenuate additional effects of time-delays. When time-delays don’t exist, these conditions both degenerate to the stochastic spatial-temporal persistence of excitation condition in Theorem IV.1.

VI. CONCLUSION

In this paper, we analyzed the convergence of the distributed cooperative online parameter estimation algorithm in an uncertain environment. Each node has a partial linear observation of the unknown parameter with random time-varying observation matrices. The underlying
communication network is modeled by a sequence of random digraphs and is subjected to nonuniform random time-varying delays in channels. For the delay-free case, we proved that if the observation matrices and the graph sequence satisfy the *stochastic spatial-temporal persistence of excitation* condition, then the algorithm gains can be designed properly such that all nodes’ estimates converge to the real parameter in mean square and almost surely. Specially, for Markovian switching communication graphs and observation matrices, this condition holds if the stationary graph is balanced with a spanning tree and the measurement model is spatially-temporally jointly observable. For the case with communication delays, we introduced delay matrices to model the random time-varying communication delays, adopted the method of binomial expansion of random matrix products to transform the mean square convergence analysis of the algorithm into that of the mathematical expectation of random matrix products, and obtained mean square convergence conditions explicitly relying on the conditional expectations of delay matrices, observation matrices and weighted adjacency matrices of communication graphs over a sequence of fixed-length intervals and showing that the communication graphs and observation matrices need to be persistently excited with enough intensity to attenuate additional effects of time-delays. Furthermore, when time-delays don’t exist, these conditions degenerate to the *stochastic spatial-temporal persistence of excitation* condition obtained for the delay-free case.

Future topics may include generalizing this work to case with asynchronous measurements and communication, the case with input delays and communication noises. Meanwhile, the convergence rate analysis is also an interesting topic for future investigation.

**APPENDIX A**

**SEVERAL USEFUL LEMMAS**

**Definition A.1.** ([38]) A Markov chain on a countable state space \(S\) with a stationary distribution \(\pi\), and transition function \(P(x, \cdot)\) is called uniform ergodic, if there exist positive constants \(r > 1\) and \(R\) such that for all \(x \in S\),

\[
\|P^n(x, \cdot) - \pi\| \leq Rr^{-n}.
\]

Here, \(\|P^n(x, \cdot) - \pi\| = \sum_y |P^n(x, y) - \pi_y|\).

**Lemma A.1.** ([39]) For any given matrix \(P\), denote \(W = I - P\). If there exists a constant \(\kappa \in (0, 1)\) such that \(\|P\| \leq \kappa\), then \(W\) is invertible and \(\|W^{-1}\| \leq (1 - \|P\|)^{-1} \leq (1 - \kappa)^{-1}\).
Lemma A.2. \((\ref{40A})\) Assume that \(\{s_1(k), k \geq 0\}\) and \(\{s_2(k), k \geq 0\}\) are real sequences satisfying \(0 \leq s_2(k) < 1\) and \(\sum_{k=0}^{\infty} s_2(k) = \infty\). Then
\[
\lim_{k \to \infty} \sum_{i=1}^{k} s_1(i) \prod_{l=i+1}^{k} (1 - s_2(l)) = \lim_{k \to \infty} \frac{s_1(k)}{s_2(k)}.
\]

Lemma A.3. \((\ref{41A})\) Assume that \(\{x(k), F(k)\}, \{\alpha(k), F(k)\}, \{\beta(k), F(k)\}\) and \(\{\gamma(k), F(k)\}\) are all nonnegative adaptive sequences, satisfying
\[
E[x(k + 1)| F(k)] \leq (1 + \alpha(k))x(k) - \beta(k) + \gamma(k), k \geq 0 \text{ a.s.}
\]
If \(\sum_{k=0}^{\infty} (\alpha(k) + \gamma(k)) < \infty \text{ a.s.}\), then \(x(k)\) converges to a finite random variable a.s. and \(\sum_{k=0}^{\infty} \beta(k) < \infty \text{ a.s.}\).

For the subsequent Lemmas A.4 and A.5, the readers may be referred to Theorem 6.4 and its next paragraph in Ch. 6 of [42].

Lemma A.4. \((\text{Conditional Lyapunov inequality})\) Denote the probability space by \((\Omega, \mathcal{F}, P)\). Let \(\mathcal{F}_1\) be a sub \(\sigma\)-algebra of \(\mathcal{F}\) and \(\xi\) be a random variable on \((\Omega, \mathcal{F}, P)\). Then \((E[|\xi|^s|\mathcal{F}_1])^{\frac{1}{s}} \leq (E[|\xi|^t|\mathcal{F}_1])^{\frac{1}{t}} \text{ a.s.}, 0 < s < t\).

Lemma A.5. \((\text{Conditional Hölder inequality})\) Denote the probability space \((\Omega, \mathcal{F}, P)\). Let \(\mathcal{F}_1\) be a sub \(\sigma\)-algebra of \(\mathcal{F}\). Let \(\xi\) and \(\eta\) be two random variables on \((\Omega, \mathcal{F}, P)\). Let constants \(p \in (1, \infty), q \in (1, \infty)\) and \(1/p + 1/q = 1\). If \(E[|\xi|^p] < \infty\) and \(E[|\eta|^q] < \infty\), then \(E[|\xi\eta||\mathcal{F}_1] \leq (E[|\xi|^p|\mathcal{F}_1])^{\frac{1}{p}}(E[|\eta|^q|\mathcal{F}_1])^{\frac{1}{q}} \text{ a.s.}\).

Lemma A.6. For any random matrix \(A \in \mathbb{R}^{m \times n}, \|E[AA^T]\| \leq m\|E[A^TA]\|\).

Proof. By the properties of matrix trace, we have
\[
\|E[AA^T]\| = \lambda_{\text{max}}(E[AA^T]) \leq \text{Tr}(E[AA^T]) = \text{Tr}(E[A^TA]) \leq m\lambda_{\text{max}}(E[A^TA]) = m\|E[A^TA]\|.
\]

Lemma A.7. Let \(A = [a_{ij}]_{N \times N}\) be a weighted adjacency matrix of an undirected graph with \(N\) nodes and \(\mathcal{L}\) be the associated Laplacian matrix. Let \(x = [x_1^T, ..., x_N^T]^T \in \mathbb{R}^N\) be any given
nonzero \( N \times n \)-dimensional vector where \( x_i \in \mathbb{R}^n, \ i = 1, 2, ..., N \) and there exists \( i \neq j \), such that \( x_i \neq x_j \). If \( a_{ij} \geq 0, \ i, j = 1, 2, ..., N \) and the graph is connected, then \( x^T (\mathcal{L} \otimes I_n) x > 0 \).

**Proof.** By the definition of Laplacian matrix, we have
\[
x^T (\mathcal{L} \otimes I_n) x = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \| x_i - x_j \|^2.
\]
Noting that there exists \( i \neq j \), such that \( x_i \neq x_j \) and the graph is connected, by \( a_{ij} \geq 0, \ i, j = 1, 2, ..., N \), we get
\[
x^T (\mathcal{L} \otimes I_n) x > 0.
\]

**APPENDIX B**

**PROOFS IN SECTION IV**

Let
\[
P(k) = I_{Nn} - D(k), \tag{17}
\]
where
\[
D(k) = b(k) \mathcal{L}_{G(k)} \otimes I_n + a(k) \mathcal{H}^T(k) \mathcal{H}(k). \tag{18}
\]

The proof of Theorem IV.1 needs the following lemma.

**Lemma B.1.** If Assumption A3.a holds and there exist a positive integer \( h \) and positive constants \( \theta \) and \( \rho_0 \) such that
\[
(b.1) \quad \inf_{m \geq 0} \lambda_{m}^h \geq \theta > 0 \text{ a.s.;}
\]
\[
(b.2) \quad \sup_{k \geq 0} \mathbb{E}[\| \mathcal{L}_{G(k)} \otimes I_n + \mathcal{H}^T(k) \mathcal{H}(k) \|^{2 \max\{h, 2\}} | \mathcal{F}(k - 1) \] \( \frac{1}{2^{\max\{h, 2\}}} \) \( \| \mathcal{F}(k - 1) \| \) \( \leq \rho_0 \) a.s.,
\]
then
\[
\lim_{k \to \infty} \mathbb{E}[\Phi_P(k, 0) \Phi_P^T(k, 0)] = 0. \tag{19}
\]

**Proof.** By (17), we have
\[
\Phi_P((m + 1)h - 1, mh) \Phi_P^T((m + 1)h - 1, mh)
\]
\[
= (I_{Nn} - D((m + 1)h - 1)) \cdots (I_{Nn} - D(mh))
\]
\[
\times (I_{Nn} - D^T(mh)) \cdots (I_{Nn} - D^T((m + 1)h - 1)). \tag{20}
\]
Taking conditional expectation w.r.t. \( \mathcal{F}(mh - 1) \) on both sides of the above, by the binomial expansion, we have
\[
\mathbb{E}[\Phi_P((m + 1)h - 1, mh) \Phi_P^T((m + 1)h - 1, mh) | \mathcal{F}(mh - 1)]
\]
\[
= \mathbb{E}[(I_{Nn} - D((m + 1)h - 1)) \cdots (I_{Nn} - D(mh))
\]
\[
\times (I_{Nn} - D^T(mh)) \cdots (I_{Nn} - D^T((m + 1)h - 1)) | \mathcal{F}(mh - 1)]
\]
Here, $M_i(m), i = 2, \cdots , 2h$ represent the $i$-th order terms in the binomial expansion of $\Phi_P((m+1)h - 1, mh)\Phi_T((m+1)h - 1, mh)$.

Since the 2-norm of a symmetric matrix is equal to its spectral radius, by the definition of spectral radius, we have

$$
\| I_{Nn} - \sum_{k=mh}^{(m+1)h-1} \mathbb{E}[D(k) + D^T(k)|\mathcal{F}(mh - 1)] \| \\
= \rho \left( I_{Nn} - \sum_{k=mh}^{(m+1)h-1} \mathbb{E}[D(k) + D^T(k)|\mathcal{F}(mh - 1)] \right),
$$

$$
= \max_{1 \leq i \leq Nn} \left| \lambda_i \left( I_{Nn} - \sum_{k=mh}^{(m+1)h-1} \mathbb{E}[D(k) + D^T(k)|\mathcal{F}(mh - 1)] \right) \right|
$$

$$
= \max_{1 \leq i \leq Nn} \left| 1 - \lambda_i \left( \sum_{k=mh}^{(m+1)h-1} \mathbb{E}[D(k) + D^T(k)|\mathcal{F}(mh - 1)] \right) \right|. \tag{22}
$$

By the condition (b.2), Assumption A3.a and (18), we know that there exists a positive integer $m_1$, which is independent of the sample paths, such that

$$
\lambda_i \left( \sum_{k=mh}^{(m+1)h-1} \mathbb{E}[D(k) + D^T(k)|\mathcal{F}(mh - 1)] \right) \leq 1, \ i = 1, \cdots , Nn, \ \forall \ m \geq m_1 \text{ a.s.}
$$

This together with (21) and (22) leads to

$$
\| \mathbb{E}[\Phi_P((m+1)h - 1, mh)\Phi_T((m+1)h - 1, mh)|\mathcal{F}(mh - 1)] \| \\
\leq 1 - \lambda_{\min} \left( \sum_{k=mh}^{(m+1)h-1} \mathbb{E}[D(k) + D^T(k)|\mathcal{F}(mh - 1)] \right) + \| \mathbb{E}[M_2(m) + \cdots + M_{2h}(m)|\mathcal{F}(mh - 1)] \| , \ \forall \ m \geq m_1 \text{ a.s.} \tag{23}
$$

For the first term on the right side of the above, by definitions of $D(k)$ and $\lambda_m$, Assumption A3.a and the condition (b.1), we have

$$
1 - \lambda_{\min} \left( \sum_{k=mh}^{(m+1)h-1} \mathbb{E}[D(k) + D^T(k)|\mathcal{F}(mh - 1)] \right)
$$

$$
= 1 - \lambda_{\min} \left( \sum_{k=mh}^{(m+1)h-1} \mathbb{E}[2b(k)\hat{\mathcal{G}}(k) \otimes I_n + 2a(k)\mathcal{H}^T(k)\mathcal{H}(k)|\mathcal{F}(mh - 1)] \right)
$$
By Lemma A.4 and the condition (b.2), it follows that

\[
\sup_{k \geq 0} \mathbb{E}[\|\tilde{D}(k)\|^i | \mathcal{F}(k - 1)] \leq \sup_{k \geq 0} [\mathbb{E}[\|\tilde{D}(k)\|^{2h} | \mathcal{F}(k - 1)]]^{\frac{i}{2h}} \leq \rho_0^i \text{ a.s., } 2 \leq i \leq 2^h,
\]

where \(\tilde{D}(k) = \mathcal{L}_{\mathcal{G}(k)} \otimes I_n + \mathcal{H}^T(k) \mathcal{H}(k)\). Note that

\[
\mathbb{E}[\|\tilde{D}(k)\|^i | \mathcal{F}(mh - 1)] = \mathbb{E}[\|\tilde{D}(k)\|^i | \mathcal{F}(k - 1)], \ 2 \leq l \leq 2^h, \ k \geq mh.
\]

From definitions of \(M_i(m), i = 2, \cdots, 2h\), Assumption A3.a, and the above, by termwise multiplication and using Lemma A.5 repeatedly, then, for the second term on the right side of (23), we have

\[
\|\mathbb{E}[M_2(m) + \cdots + M_{2h}(m) | \mathcal{F}(mh - 1)]\| \leq b^2 (mh) \left( \sum_{i=2}^{2h} C_{2h}^i \rho_0^i \right) = b^2 (mh) \alpha,
\]

where \(\alpha = (1 + \rho_0)^{2h} - 1 - 2h \rho_0\) and \(C_m^p\) denotes the combinatorial number of choosing \(p\) elements from \(m\) elements. By (23)-(25), we have

\[
\|\mathbb{E}[\Phi_P((m + 1)h - 1, mh) \Phi_P^T((m + 1)h - 1, mh) | \mathcal{F}(mh - 1)]\|
\leq 1 - a((m + 1)h) \theta + b^2 (mh) \alpha, \ m \geq m_1 \text{ a.s.} \quad (26)
\]

Denote \(m_k = \lfloor \frac{k}{h} \rfloor\). By the properties of the conditional expectation, (26) and Lemma A.6, we have

\[
\|\mathbb{E}[\Phi_P(k, 0) \Phi_P^T(k, 0)]\|
\leq Nn \|\mathbb{E}[\Phi_P^T(k, 0) \Phi_P(k, 0)]\|
= Nn \|\mathbb{E}[\Phi_P^T(m_k h - 1, 0) \Phi_P^T(k, m_k h) \Phi_P(k, m_k h) \Phi_P(m_k h - 1, 0)]\|
\leq Nn \|\mathbb{E}[\Phi_P^T(m_k h - 1, 0) \|\Phi_P(k, m_k h)\|^2 \Phi_P(m_k h - 1, 0)]\|
= Nn \|\mathbb{E}[\Phi_P^T(m_k h - 1, 0) \|\Phi_P(k, m_k h)\|^2 \Phi_P(m_k h - 1, 0) | \mathcal{F}(m_k h - 1)]\|
= Nn \|\mathbb{E}[\Phi_P^T(m_k h - 1, 0) \Phi_P(k, m_k h) \|^2 \mathcal{F}(m_k h - 1)] \Phi_P(m_k h - 1, 0)]\|. \quad (27)
\]

By the condition (b.2), it follows that there exists a positive constant \(\rho_1\) such that

\[
\sup_{k \geq 0} \mathbb{E}[\|\Phi_P(k, m_k h)\|^2 | \mathcal{F}(m_k h - 1)] \leq \rho_1 \text{ a.s.,} \quad (28)
\]

which together with (27) implies

\[
\|\mathbb{E}[\Phi_P(k, 0) \Phi_P^T(k, 0)]\|
\]
\begin{align*} 
&\leq \rho_1 N n \| \mathbb{E}[\Phi_P^T(m_k h - 1, 0) \Phi_P(m_k h - 1, 0)] \| \\
&= \rho_1 N n \| \mathbb{E}[\Phi_P^T(m_1 h - 1, 0) \Phi_P^T(m_k h - 1, m_1 h) \Phi_P(m_k h - 1, m_1 h) \Phi_P(m_1 h - 1, 0)] \| \\
&= \rho_1 N n \| \mathbb{E}[\Phi_P^T(m_1 h - 1, 0) \Phi_P^T(m_k h - 1, m_1 h) \\
&\times \Phi_P(m_k h - 1, m_1 h) \Phi_P(m_1 h - 1, 0) | \mathcal{F}(m_1 h - 1)] \| \\
&\leq \rho_1 N n \| \Phi_P^T(m_k h - 1, 0) \\
&\times \| \Phi_P^T(m_k h - 1, m_1 h) \Phi_P(m_k h - 1, m_1 h) | \mathcal{F}(m_1 h - 1)] \| \Phi_P(m_1 h - 1, 0)] . \tag{29} 
\end{align*}

Note that for any given random variable \( \xi \) and \( \sigma \)-algebra \( \mathcal{F}_1 \subseteq \mathcal{F}_2 \), we have

\[ \mathbb{E}[\xi | \mathcal{F}_1] = \mathbb{E}[\mathbb{E}[\xi | \mathcal{F}_2] | \mathcal{F}_1]. \tag{30} \]

Then by (26), we have

\begin{align*}
&\| \mathbb{E}[\Phi_P^T(m_k h - 1, m_1 h) \Phi_P(m_k h - 1, m_1 h) | \mathcal{F}(m_1 h - 1)] \| \\
&= \| \mathbb{E}[\Phi_P^T((m_k - 1) h - 1, m_1 h) \Phi_P^T(m_k h - 1, (m_k - 1) h) \Phi_P(m_k h - 1, (m_k - 1) h) \\
&\quad \times \Phi_P((m_k - 1) h - 1, m_1 h) | \mathcal{F}(m_1 h - 1)] \| \\
&= \| \mathbb{E}[\Phi_P^T((m_k - 1) h - 1, m_1 h) \\
&\quad \times \Phi_P^T(m_k h - 1, (m_k - 1) h) \Phi_P^T(m_k h - 1, (m_k - 1) h) \Phi_P(m_k h - 1, (m_k - 1) h) \\
&\quad \times \Phi_P((m_k - 1) h - 1, m_1 h) | \mathcal{F}(m_1 h - 1)] \| \\
&\leq \| \mathbb{E}[\Phi_P^T((m_k - 1) h - 1, m_1 h) \\
&\quad \times \| \mathbb{E}[\Phi_P^T(m_k h - 1, (m_k - 1) h) \Phi_P(m_k h - 1, (m_k - 1) h) | \mathcal{F}(m_1 h - 1)] \| \\
&\quad \times \Phi_P((m_k - 1) h - 1, m_1 h) | \mathcal{F}(m_1 h - 1)] \| \\
&\leq [1 - a((m_k h) h + b^2((m_k - 1) h) \alpha] \\
&\quad \times \| \mathbb{E}[\Phi_P^T((m_k - 1) h - 1, m_1 h) \Phi_P((m_k - 1) h - 1, m_1 h) | \mathcal{F}(m_1 h - 1)] \| \\
&\leq \prod_{s=m_1}^{m_k - 1} [1 - a((s + 1) h) \theta + b^2(s h) \alpha] \text{ a.s., } \tag{31} 
\end{align*}

which together with (29) leads to

\begin{align*}
&\| \mathbb{E}[\Phi_P(k, 0) \Phi_P^T(k, 0)] \| \\
&\leq \rho_1 N n \| \mathbb{E}[\Phi_P^T(m_1 h - 1, 0) \Phi_P(m_1 h - 1, 0)] \| \prod_{s=m_1}^{m_k - 1} [1 - a((s + 1) h) \theta + b^2(s h) \alpha]. \tag{32} 
\end{align*}

Since \( \theta > 0 \), by Assumption A3.a, we know that there exists a positive integer \( m_2 \) such that

\[ b^2(m h) \alpha \leq \frac{1}{2} a((m + 1) h) \theta, \forall m \geq m_2, \tag{33} \]

and

\[ \sum_{s=0}^{\infty} a((s + 1) h) \geq \frac{1}{h} \sum_{s=0}^{\infty} \sum_{i=(s+1)h}^{(s+2)h-1} a(i) = \frac{1}{h} \sum_{k=h}^{\infty} a(k) = \infty. \tag{34} \]
Denote \( m_3 = \max\{m_2, m_1\} \) and \( r_1 = \prod_{s=m_3}^{m_2-1} [1 - a((s+1)h)\theta + b^2(sh)\alpha] \). By (33)-(34), we have

\[
\lim_{k \to \infty} \prod_{s=m_2}^{m_1-1} [1 - a((s+1)h)\theta + b^2(sh)\alpha] \\
\leq \lim_{k \to \infty} r_1 \prod_{s=m_3}^{m_1-1} [1 - \frac{1}{2} a((s+1)h)\theta] \\
\leq \lim_{k \to \infty} r_1 \exp \left( - \frac{1}{2} \theta \sum_{s=m_3}^{m_1-1} a((s+1)h) \right) \\
= r_1 \exp \left( - \frac{1}{2} \theta \sum_{s=m_3}^{\infty} a((s+1)h) \right) = 0, \quad k \geq (m_3 + 1)h. \tag{35}
\]

Note that \(|\mathbb{E}[\Phi_P^T(m_1h - 1, 0)\Phi_P(m_1h - 1, 0)]| < \infty\) by the condition (b.2). Hence, by (32) and (35), we have (19). The lemma is proved. \(\square\)

**Proof of Theorem IV.1.** If \( \lambda_{ji}(k) = 0 \) a.s., \( \forall \ j, i \in \mathcal{V}, \forall \ k \geq 0 \), then the error equation (10) becomes

\[
e(k + 1) = [I_{Nn} - b(k)\mathcal{L}_{g(k)}] \otimes I_n - a(k)\mathcal{H}^T(k)\mathcal{H}(k))e(k) + a(k)\mathcal{H}^T(k)v(k) \\
= P(k)e(k) + a(k)\mathcal{H}^T(k)v(k) \\
= \Phi_P(k, 0)e(0) + \sum_{i=0}^{k} a(i)\Phi_P(k, i + 1)\mathcal{H}^T(i)v(i), \quad k \geq 0, \tag{36}
\]

which further leads to

\[
\mathbb{E}[e(k + 1)e^T(k + 1)] = \mathbb{E}[\Phi_P(k, 0)e(0)e^T(0)\Phi_P^T(k, 0)] \\
+ \mathbb{E} \left[ \phi_P(k, 0)e(0) \sum_{i=0}^{k} a(i)\phi_P(k + 1)\mathcal{H}^T(i)v(i)]^T \right] \\
+ \mathbb{E} \left[ \sum_{i=0}^{k} a(i)\phi_P(k + 1)\mathcal{H}^T(i)v(i)\phi_P(k, 0)e(0)^T \right] \\
+ \mathbb{E} \left[ \sum_{i=0}^{k} a(i)\phi_P(k + 1)\mathcal{H}^T(i)v(i) \right] \\
\times \left[ \sum_{i=0}^{k} a(i)\phi_P(k + 1)\mathcal{H}^T(i)v(i) \right]^T. \tag{37}
\]

By Assumptions A1.a and A1.b, we know that the second and third terms on the right side of (37) are both equal to zero. Moreover, from

\[
\mathbb{E}[v(i)v^T(j)] = \mathbb{E}[\mathbb{E}[v(i)v^T(j)|\mathcal{F}(i - 1)]] = \mathbb{E}[\mathbb{E}[v(i)|\mathcal{F}(i - 1)]v^T(j)] = 0, \quad \forall \ i > j, \tag{38}
\]

we have

\[
\mathbb{E} \left[ \left( \sum_{i=0}^{k} a(i)\phi_P(k + 1)\mathcal{H}^T(i)v(i) \right) \left( \sum_{i=0}^{k} a(i)\phi_P(k + 1)\mathcal{H}^T(i)v(i) \right)^T \right]
\]
where \( \rho \) Substituting the above into (37) and taking the 2-norm leads to
\[
\| \mathbb{E}(e(k + 1)e^T(k + 1)) \| \\
\leq \| \mathbb{E}[\Phi_P(k, 0)\Phi_P(k, 0)] \| \| e(0) \| ^2 \\
+ \sum_{i=0}^{\infty} a^2(i) \| \mathbb{E}[\Phi_P(k, i + 1)\mathcal{H}^T(i)v(i)v^T(i)\mathcal{H}(i)\Phi_P(k, i + 1)] \|.
\] (39)

For the two terms on the right side of (39), by Lemma B.1, we know that the first term converges to zero. Next we prove that the second term converges to zero. By Lemma A.6, (28) and (30), we have
\[
\| \mathbb{E}[\Phi_P(k, i + 1)\Phi_P^T(k, i + 1)|\mathcal{F}(i)] \|
\leq Nn\| \mathbb{E}[\Phi_P^T(k, i + 1)\Phi_P(k, i + 1)|\mathcal{F}(i)] \|
= Nn\| \mathbb{E}[\Phi_P^T(\tilde{m}_i h - 1, i + 1)\Phi_P^T(m_k h - 1, \tilde{m}_i h)\Phi_P(k, m_k h) \\
\times \Phi_P(m_k h - 1, \tilde{m}_i h)\Phi_P(\tilde{m}_i h - 1, i + 1)|\mathcal{F}(i)] \|
= Nn\| \mathbb{E}[\Phi_P^T(\tilde{m}_i h - 1, i + 1)\Phi_P^T(m_k h - 1, \tilde{m}_i h)\Phi_P(k, m_k h)|\mathcal{F}(m_k h - 1)] \\
\times \Phi_P(m_k h - 1, \tilde{m}_i h)\Phi_P(\tilde{m}_i h - 1, i + 1)|\mathcal{F}(i)] ||
\leq Nn\rho_1\| \mathbb{E}[\Phi_P^T(\tilde{m}_i h - 1, i + 1)\Phi_P^T(m_k h - 1, \tilde{m}_i h)|\mathcal{F}(i)] || \\
\times \Phi_P(m_k h - 1, \tilde{m}_i h)\Phi_P(\tilde{m}_i h - 1, i + 1)|\mathcal{F}(i)] || a.s.,
\] (40)

where \( \tilde{m}_i = \lceil \frac{i}{\theta} \rceil \). Similarly to (31) in the proof of Lemma B.1, we have
\[
\| \mathbb{E}[\Phi_P^T(m_k h - 1, \tilde{m}_i h)\Phi_P(m_k h - 1, \tilde{m}_i h)|\mathcal{F}(\tilde{m}_i h - 1)] || \\
\leq \prod_{s=\tilde{m}_i}^{m_k - 1} [1 - a((s + 1)h)\theta + b^2(sh)\alpha] \\
\leq \rho_2 \prod_{s=i+1}^{k} [1 - a((s + 1)h)\theta + b^2(sh)\alpha] a.s., 0 \leq i \leq k,
\]

where \( \rho_2 \) is a positive constant. Thus, from the above and (40), we have
\[
\| \mathbb{E}[\Phi_P(k, i + 1)\Phi_P^T(k, i + 1)|\mathcal{F}(i)] || \\
\leq Nn\rho_1\| \mathbb{E}[\Phi_P^T(\tilde{m}_i h - 1, i + 1)\Phi_P^T(m_k h - 1, \tilde{m}_i h) \\
\times \Phi_P(m_k h - 1, \tilde{m}_i h)\Phi_P(\tilde{m}_i h - 1, i + 1)|\mathcal{F}(i)] ||
= Nn\rho_1\| \mathbb{E}[\Phi_P^T(\tilde{m}_i h - 1, i + 1)\Phi_P^T(m_k h - 1, \tilde{m}_i h) \\
\times \Phi_P(m_k h - 1, \tilde{m}_i h)\Phi_P(\tilde{m}_i h - 1, i + 1)|\mathcal{F}(\tilde{m}_i h - 1)] || \mathcal{F}(i)] ||
= Nn\rho_1\| \mathbb{E}[\Phi_P^T(\tilde{m}_i h - 1, i + 1)\Phi_P^T(m_k h - 1, \tilde{m}_i h)\Phi_P(m_k h - 1, \tilde{m}_i h)|\mathcal{F}(\tilde{m}_i h - 1)] ||
\]
\[ \times \Phi_P(\tilde{\theta}_i - 1, i + 1)\mathcal{F}(i) \right| \\
\leq Nn\rho_1 \left| \mathbb{E}[\Phi_P^T(\tilde{\theta}_i - 1, i + 1)\mathcal{F}(\tilde{\theta}_i - 1, i + 1)] \right| \\
\leq Nn\rho_1 \rho_2 \prod_{s=i+1}^k [1 - a((s + 1)h)\theta + b^2(sh)\alpha] \\
\times \left| \mathbb{E}[\Phi_P^T(\tilde{\theta}_i - 1, i + 1)\mathcal{F}(\tilde{\theta}_i - 1, i + 1)] \right| \\
\leq Nn\rho_1 \rho_2 \rho_3 \prod_{s=i+1}^k [1 - a((s + 1)h)\theta + b^2(sh)\alpha] \text{ a.s.,} \tag{41} \]

where \( \rho_3 \) is a constant satisfying \( \sup_{i \geq 0} \mathbb{E}[|\Phi_P(\tilde{\theta}_i - 1, i + 1)|^2 | \mathcal{F}(i)] \leq \rho_3 \) a.s. By (41), the condition (b.2), Assumptions A1.a and A1.b, it follows that

\[ \left| \mathbb{E}[\Phi_P(k, i + 1)\mathcal{H}^T(i)v(i)v^T(i)\mathcal{H}(i)\Phi_P^T(k, i + 1)] \right| \\
\leq Nn\beta_\nu \rho_0 \rho_1 \rho_2 \rho_3 \prod_{s=i+1}^k [1 - a((s + 1)h)\theta + b^2(sh)\alpha] , 0 \leq i \leq k, \]

which implies

\[ \sum_{i=0}^k a^2(i) \left| \mathbb{E}[\Phi_P(k, i + 1)\mathcal{H}^T(i)v(i)v^T(i)\mathcal{H}(i)\Phi_P^T(k, i + 1)] \right| \\
\leq Nn\beta_\nu \rho_0 \rho_1 \rho_2 \rho_3 \sum_{i=0}^k a^2(i) \prod_{s=i+1}^k [1 - a((s + 1)h)\theta + b^2(sh)\alpha] \text{ a.s.} \tag{42} \]

Furthermore, by some direct calculations, we have

\[ \sum_{i=0}^k a^2(i) \prod_{s=i+1}^k [1 - a((s + 1)h)\theta + b^2(sh)\alpha] \\
= \sum_{i=0}^k a^2(i) \prod_{s=i+1}^k [1 - a((s + 1)h)\theta + b^2(sh)\alpha] \\
+ \sum_{i=m_3}^k a^2(i) \prod_{s=i+1}^k [1 - a((s + 1)h)\theta + b^2(sh)\alpha] \\
\leq \sum_{i=0}^k a^2(i) \prod_{s=i+1}^k [1 - a((s + 1)h)\theta + b^2(sh)\alpha] \\
+ \sum_{i=m_3}^k a^2(i) \prod_{s=i+1}^k [1 - \frac{1}{2}a((s + 1)h)\theta]. \tag{43} \]

By Assumption A3.a and the finiteness of \( m_3 \), similarly to (35), we have

\[ \lim_{k \to \infty} \sum_{i=0}^{m_3-1} a^2(i) \prod_{s=i+1}^k [1 - a((s + 1)h)\theta + b^2(sh)\alpha] = 0. \tag{44} \]
From Assumption A3.a and Lemma A.2, it follows that
\[
\lim_{k \to \infty} \sum_{i=m_3}^{k} a^2(i) \prod_{s=i}^{k} \left(1 - \frac{1}{2} \alpha((s + 1)h)\theta\right) = \lim_{k \to \infty} \frac{2a^2(k)}{\theta \alpha((k + 1)h)} = 0. \tag{45}
\]
Then, by (42)–(45), we have
\[
\lim_{k \to \infty} \sum_{i=0}^{k} a^2(i) \| \mathbb{E}[\Phi_P(k, i + 1) \mathcal{H}^T(i)v(i)v^T(i)\mathcal{H}(i)\Phi^T_P(k, i + 1)] \| = 0,
\]
which together with Lemma B.1 and (39) gives \( \lim_{k \to \infty} \| \mathbb{E}[e(k)e^T(k)] \| = 0. \) Since \( \mathbb{E}\|e(k)\|^2 \leq Nn\|\mathbb{E}[e(k)e^T(k)]\|, \) it follows that \( \lim_{k \to \infty} \mathbb{E}\|e(k)\|^2 = 0. \) The proof is completed. \( \square \)

**Proof of Theorem IV.2.** By (36), it follows that
\[
e((m + 1)h) = \Phi_P((m + 1)h - 1, mh)e(mh)
\]
\[
+ \sum_{k=mh} a(i)\Phi_P((m + 1)h - 1, k + 1)\mathcal{H}^T(k)v(k), \ m \geq 0,
\]
which gives
\[
\|e((m + 1)h)\|^2
\]
\[
e^T(mh)\Phi^T_P((m + 1)h - 1, mh)\Phi_P((m + 1)h - 1, mh)e(mh)
\]
\[
+ \left[ \sum_{k=mh} a(k)\Phi_P((m + 1)h - 1, k + 1)\mathcal{H}^T(k)v(k) \right]^T
\]
\[
\times \left[ \sum_{k=mh} a(k)\Phi_P((m + 1)h - 1, k + 1)\mathcal{H}^T(k)v(k) \right]
\]
\[
+ 2e^T(mh)\Phi^T_P((m + 1)h - 1, mh) \left[ \sum_{k=mh} a(k)\Phi_P((m + 1)h - 1, k + 1)\mathcal{H}^T(k)v(k) \right].
\]
Taking conditional expectation w.r.t. \( \mathcal{F}(mh - 1) \) on both sides of the above, by Lemma A.1 in [35], Assumptions A1.a and A1.b, we have
\[
\mathbb{E}[\|e((m + 1)h)\|^2|\mathcal{F}(mh - 1)]
\]
\[
e^T(mh)\mathbb{E}[\Phi^T_P((m + 1)h - 1, mh)\Phi_P((m + 1)h - 1, mh)|\mathcal{F}(mh - 1)]e(mh)
\]
\[
+ \sum_{k=mh} a^2(k)\mathbb{E}[\|\Phi_P((m + 1)h - 1, k + 1)\mathcal{H}^T(k)v(k)\|^2|\mathcal{F}(mh - 1)]. \tag{46}
\]
In the light of the condition (b.2), Assumptions A1.a and A1.b, we know that there exists a constant \( \rho_4 \) such that
\[
\sum_{k=mh}^{(m+1)h-1} \mathbb{E}[\|\Phi_P((m + 1)h - 1, k + 1)\mathcal{H}^T(k)v(k)\|^2|\mathcal{F}(mh - 1)] \leq \rho_4 \ \text{a.s.,} \ \forall \ m \geq 0,
\]
which together with (26) and (46) gives
\[
\mathbb{E}[\|e((m+1)h)\|^2|\mathcal{F}(mh-1)] \\
\leq \|\mathcal{E}[\Phi^T_P((m+1)h-1,mh)\Phi_P((m+1)h-1,mh)|\mathcal{F}(mh-1)]\|\|e(mh)\|^2 \\
+ a^2(mh) \sum_{k=mh}^{\infty} \mathbb{E}[\|\Phi_P((m+1)h-1,k+1)\mathcal{H}^T(k)v(k)\|^2|\mathcal{F}(mh-1)] \\
\leq (1 + b^2(mh)\alpha)\|e(mh)\|^2 + a^2(mh)\rho_4 \text{ a.s.}
\]

By Lemma A.3, Assumptions A3.a and A3.b, we know that \(\{e(mh), m \geq 0\}\) converges almost surely, which, along with \(\lim_{m \to 0} \mathbb{E}\|e(mh)\|^2 = 0\) by Theorem IV.1, gives
\[
\lim_{m \to 0} e(mh) = 0_{N \times 1} \text{ a.s.}
\] (47)

For arbitrarily small \(\epsilon > 0\), by Markov inequality, we have
\[
\mathbb{P}\{a(k)\|v(k)\| \geq \epsilon\} \leq \frac{a^2(k)\mathbb{E}\|v(k)\|^2}{\epsilon^2}, \ k \geq 0,
\]
which together with Assumptions A1.b, A3.a and A3.b gives
\[
\sum_{k=0}^\infty \mathbb{P}\{a(k)\|v(k)\| \geq \epsilon\} \leq \sum_{k=0}^\infty \frac{a^2(k)\mathbb{E}\|v(k)\|^2}{\epsilon^2} \leq \beta_\epsilon \sum_{k=0}^\infty a^2(k) < \infty.
\]

Then by the Borel-Cantelli lemma, we have \(\mathbb{P}\{a(k)\|v(k)\| \geq \epsilon \ i.o.\} = 0\), which means
\[
a(k)\|v(k)\| \to 0, \ k \to \infty \ a.s.
\] (48)

By (36), we have
\[
\|e(k)\| \leq \|\Phi_P(k-1,m_k)\|\|e(m_k h)\| + \sum_{i=m_k h}^{k-1} a(i)\|v(i)\|\|\Phi_P(k-1,i+1)\|\|\mathcal{H}^T(i)\|. \tag{49}
\]

By Assumption A2.a and noting \(0 \leq k - m_k h \leq h\), we know that \(\sup_{k \geq 0} \|\Phi_P(k-1,m_k)\| < \infty \ a.s.\) and \(\sup_{k \geq 0} \|\Phi_P(k-1,i+1)\|\|\mathcal{H}^T(i)\| < \infty \ a.s.\), \(k-1 \leq i \leq m_k h\). Then by (47)-(49), we have \(\lim_{k \to \infty} e(k) = 0_{N \times 1} \ a.s.\) The proof is completed. \[\square\]

**Proof of Corollary IV.1.** By Assumption A4 and the one-to-one correspondence among \(A_G(k)\) and \(\mathcal{L}_G(k)\), we know that \(\mathcal{L}_G(k)\) is a homogeneous and uniform ergodic Markov chain (Definition A.1) with the unique stationary distribution \(\pi\). Denote the associated Laplacian matrix of \(A_l\) by \(\mathcal{L}_l\) and \(\tilde{\mathcal{L}}_l = \frac{\mathcal{L}_l + \mathcal{E}^T_l}{2}, \ l = 1, 2, ...\)

By Assumption A3.a, we know that there exists a constant \(c_0 > 0\) such that \(\sup_{k \geq 0} \frac{a(k)}{b(k)} \geq c_0\). Without loss of generality, we assume \(a(k) = b(k), k \geq 0\). By the definition of \(\lambda_m^h\), we have
\[
\lambda_m^h = \lambda_{\min}\left[\sum_{k=mh}^{(m+1)h-1} \mathbb{E}[\tilde{\mathcal{L}}_G(k) \otimes I_n + \mathcal{H}^T(k)\mathcal{H}(k)|\mathcal{F}(mh-1)]\right]
\]

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we know that the convergence is uniform, we know that there exists an integer $\lambda \geq 0$, $h \geq 1$.

By the conditions (c.1) and (c.2), it follows that $\pi_1(\hat{L}_i \otimes I_n + \mathcal{H}_i^T \mathcal{H}_i) > 0$. To see this, for any given $x \in \mathbb{R}^n$, $x \neq 0_{N \times 1}$, let $x = [x_1^T, \cdots, x_N^T]$, $x_i \in \mathbb{R}^n$; (i) if $x = 1_N \otimes a$, $\exists \ a \in \mathbb{R}^n$ and $a \neq 0_{n \times 1}$, i.e. $x_1 = x_2 = \cdots = x_N = a$, then by the condition (c.2), we have $x^T(\sum_{t=1}^{\infty} \pi_t(\hat{L}_t \otimes I_n + \mathcal{H}_t^T \mathcal{H}_t))x = a^T[\sum_{t=1}^{N} \sum_{i=1}^{\infty} (\pi_t H_{i,i}^T H_{i,i})]a > 0$; (ii) otherwise, there must be $x_i \neq x_j$, $\exists \ i \neq j$. By the condition (c.1), we know that $\sum_{t=1}^{\infty} \pi_t \hat{L}_t$ is the Laplacian matrix of a connected graph. Then by Lemma A.7, we have $x^T(\sum_{t=1}^{\infty} \pi_t(\hat{L}_t \otimes I_n + \mathcal{H}_t^T \mathcal{H}_t))x \geq x^T(\sum_{t=1}^{\infty} \pi_t \hat{L}_t \otimes I_n)x > 0$. Combining (i) and (ii), we get $\lambda_{\min}(\sum_{t=1}^{\infty} \pi_t(\hat{L}_t \otimes I_n + \mathcal{H}_t^T \mathcal{H}_t)) > 0$.

Since the function $\lambda_{\min}()$, whose arguments are matrices, is continuous, we know that for the given $\frac{\delta}{2}$, there exists a constant $\delta > 0$ such that for any given matrix $L$, $|\lambda_{\min}(L) - \lambda_{\min}(\sum_{t=1}^{\infty} \pi_t(\hat{L}_t \otimes I_n + \mathcal{H}_t^T \mathcal{H}_t))| \leq \frac{\delta}{2}$ provided $\|L - \sum_{t=1}^{\infty} \pi_t(\hat{L}_t \otimes I_n + \mathcal{H}_t^T \mathcal{H}_t)\| \leq \delta$. Since the convergence is uniform, we know that there exists an integer $h_0 > 0$ such that

$$\sup_{m \geq 0} \left\| \frac{1}{h} \sum_{k=mh}^{(m+1)h-1} \mathbb{E}[\hat{L}_{G(k)} \otimes I_n + \mathcal{H}_t^T(k) \mathcal{H}(k)|F(mh - 1)] - \sum_{t=1}^{\infty} \pi_t(\hat{L}_t \otimes I_n + \mathcal{H}_t^T \mathcal{H}_t) \right\|$$
\[ \leq \delta, \ h \geq h_0 \ \text{a.s.,} \]

which gives
\[
\sup_{m \geq 0} \left| \frac{1}{h} \lambda_m^h - \lambda_{\min} \left( \sum_{i=1}^{\infty} \pi_i (\mathcal{L} \otimes I_n + \mathcal{H}_i) \right) \right| \leq \frac{\mu}{2}, \ h \geq h_0 \ \text{a.s.}
\]

Thus,
\[
\inf_{m \geq 0} \lambda_m^h \geq \frac{h \mu}{2} \geq \frac{h \mu}{2} > 0, \ h \geq h_0 \ \text{a.s.}
\]

Then by Theorems IV.1 and IV.2, the proof is completed. \(\square\)

APPENDIX C

PROOFS IN SECTION V

Proof of Lemma V.1. We adopt the mathematical induction method to prove the lemma.

By (6) and (14), noting that 
\[ F(k) = I_{Nn}, -d \leq k \leq -1, \] we have
\[
F(0) = I_{Nn} - [b(0)D_{G(0)} \otimes I_n + a(0)\mathcal{H}^T(0)\mathcal{H}(0) - b(0) \sum_{q=0}^{d} \mathcal{A}(0, q)]
\]
\[
= I_{Nn} - [b(0)D_{G(0)} \otimes I_n + a(0)\mathcal{H}^T(0)\mathcal{H}(0) - b(0)\mathcal{A}_{G(0)} \otimes I_n].
\]

By Assumption A3.c, we know that there exists a constant \(\kappa \in (0, 1)\) such that
\[
\sup_{k \geq 0} b(k) \leq \min \left\{ \frac{\kappa}{2[N\beta_a + N\sqrt{N\beta_a} + C_a\beta_H^2]}, \frac{(1 - (1 - \kappa)^{-1})\kappa}{2N\sqrt{N\beta_a}(1 - (1 - \kappa)^{-(d+1)})} \right\}, \tag{52}
\]
which together with Assumption A2.b leads to
\[
\| b(0)D_{G(0)} \otimes I_n + a(0)\mathcal{H}^T(0)\mathcal{H}(0) - b(0)\mathcal{A}_{G(0)} \otimes I_n \| \\
\leq b(0) \sup_{k \geq 0} \| D_{G(k)} \| + a(0) \sup_{k \geq 0} \| \mathcal{H}^T(k)\mathcal{H}(k) \| + b(0) \sup_{k \geq 0} \| \mathcal{A}_{G(k)} \|
\leq b(0)[N\beta_a + C_a\beta_H^2 + N\sqrt{N\beta_a}] \leq \frac{\kappa}{2} < \kappa \ \text{a.s.,}
\]
which together with (51) and Lemma A.1 gives that \( F(0) \) is invertible a.s. and \( \| F^{-1}(0) \| \leq (1 - \kappa)^{-1} \ a.s. \)

Assume that \( F(k) \) is invertible a.s. and \( \| F^{-1}(k) \| < (1 - \kappa)^{-1} \ a.s. \) for \( k = 0, 1, 2, \ldots \).

Then,
\[
F(k + 1) = I_{Nn} - [b(k + 1)D_{G(k+1)} \otimes I_n + a(k + 1)\mathcal{H}^T(k + 1)\mathcal{H}(k + 1)
\]
\[
- b(k + 1) \sum_{q=0}^{d} \mathcal{A}(k+1, q)[\Phi_F(k, k - q + 1)]^{-1}].
\]

By Assumption A2.b and (52), we have
\[
\| b(k + 1)D_{G(k+1)} \otimes I_n + a(k + 1)\mathcal{H}^T(k + 1)\mathcal{H}(k + 1)
\]
Lemma C.1. By Lemma A.1, we know that

\[-b(k+1) \sum_{q=0}^{d} \mathcal{A}(k+1, q)[\Phi_F(k, k-q+1)]^{-1}\|\]

\[\leq b(k+1)[N\beta + C_a\beta_H^2] + b(k+1) N\sqrt{N}\beta_a \sum_{q=0}^{d} (1-\kappa)^{-q}\]

\[= b(k+1)[N\beta + C_a\beta_H^2] + b(k+1) N\sqrt{N}\beta_a \frac{1-(1-\kappa)^{-(d+1)}}{1-(1-\kappa)^{-1}} \leq \kappa \frac{\kappa}{2} = \kappa \text{ a.s.}\]

By Lemma A.1, we know that \(F(k+1)\) is invertible a.s. and \(\|F^{-1}(k+1)\| \leq (1-\kappa)^{-1} \) a.s.

By the mathematical induction, the proof is completed. \qed

Before proving Theorem V.1, we need the following lemma.

Lemma C.1. If Assumptions A2.b, A3.a, A3.c hold and there exist a positive integer \(h\) and a constant \(\theta\) such that \(\inf_{m \geq 0} \lambda_m^h \geq \theta > 0\) a.s., then \(\lim_{k \to \infty} \|\mathbb{E}(\Phi_F(k, 0)\Phi_F^T(k, 0))\| = 0\).

Proof. Similarly to (20)–(23) in the proof of Lemma B.1, we know that there exists a positive integer \(m'1\) such that

\[
\|\mathbb{E}[\Phi_F((m+1)h-1, mh)\Phi_F^T((m+1)h-1, mh)|\mathcal{F}(mh-1)]\|
= 1 - \lambda_{\min}\left(\sum_{k=mh}^{(m+1)h-1} \mathbb{E}[G(k) + G^T(k)|\mathcal{F}(mh-1)]\right)
+ \|\mathbb{E}[\mathcal{M}_2(m) + \cdots + \mathcal{M}_2h(m)|\mathcal{F}(mh-1)]\|, \forall m \geq m'1 \text{ a.s.} \quad \text{(53)}
\]

Here, the definitions of \(\mathcal{M}_i(m), i = 2, \cdots, 2h\) are similar to (21).

By (15), (16), Assumption A3.a and \(\inf_{m \geq 0} \lambda_m^h \geq \theta > 0\) a.s., we have

\[
1 - \lambda_{\min}\left(\sum_{k=mh}^{(m+1)h-1} \mathbb{E}[G(k) + G^T(k)|\mathcal{F}(mh-1)]\right)
= 1 - \lambda_{\min}\left(\sum_{k=mh}^{(m+1)h-1} \mathbb{E}\left[2b(k)\mathcal{D}_G(k) \otimes I_n + 2a(k)\mathcal{H}_G(k) \otimes I_n \right.\right.
\left.\left. - b(j) \sum_{q=0}^{d} [\mathcal{A}(k, q)[\Phi_F(k-1, k-q)]^{-1} + (\mathcal{A}(k, q)[\Phi_F(k-1, k-q)]^{-1})^T]|\mathcal{F}(mh-1)\right]\right)
= 1 - \lambda_{\min}\left(\sum_{k=mh}^{(m+1)h-1} \mathbb{E}\left[2b(k)\hat{\mathcal{L}}_G(k) \otimes I_n + 2a(k)\mathcal{H}_G(k) \otimes I_n \right.\right.
\left.\left. - b(k) \sum_{q=0}^{d} \mathcal{A}(k, q)\Phi_F(k-1, k-q)^{-1} - I_{Nn}\right]\right.
\left.\left. + (\mathcal{A}(k, q)[\Phi_F(k-1, k-q)]^{-1} - I_{Nn})^T]|\mathcal{F}(mh-1)\right]\right)
\]
By the above and the definition of $\mathbf{A}(k, q)[\Phi_F(k - 1, k - q)]^{-1} - I_{Nn}$

\[ + [[\Phi_F(k - 1, k - q)]^{-T} - I_{Nn}]\mathbf{A}^T(k, q)] \mathcal{F}(mh - 1) \]

\[ \leq 1 - a((m + 1)h)\lambda_m^{h'} \leq 1 - a((m + 1)h) \inf_{m \geq 0} \lambda_m^{h'} \leq 1 - a((m + 1)h)\theta \text{ a.s.,} \]

which together with (53) gives

\[ \left\| I_{Nn} - \sum_{k=mh}^{(m+1)h-1} \mathbb{E}[G(k) + G^T(k)|\mathcal{F}(mh - 1)] \right\| \leq 1 - a((m + 1)h)\theta, \ \forall \ m \geq m'_1 \text{ a.s. (54)} \]

From (15), Assumption A2.b and Lemma V.1, we have

\[ \|G(k)\| \leq b(k)\|\mathcal{D}_G(k) \otimes I_n\| + a(k)\|\mathcal{H}^T(k)\mathcal{H}(k)\|b(k)\|\sum_{q=0}^{d} \mathbf{A}(k, q)[\Phi_F(k - 1, k - q)]^{-1}\| \]

\[ \leq b(k)\left(N\beta_a + C_a\beta_H^2 + N\sqrt{N}\beta_a \frac{1 - (1 - \kappa)^{-(d+1)}}{1 - (1 - \kappa)^{-1}} \right) \text{ a.s., } \forall \ k \geq 0. \]

By the above and the definition of $\overline{M}_i(m)$, for $i = 2, \cdots, 2h$, we have

\[ \|\overline{M}_i(m)\| \leq b^2(mh)C_{2h}^i \left(N\beta_a + C_a\beta_H^2 + N\sqrt{N}\beta_a \frac{1 - (1 - \kappa)^{-(d+1)}}{1 - (1 - \kappa)^{-1}} \right)^i \text{ a.s.,} \]

where $C_m^p$ represent the combinatorial number of choosing $p$ elements from $m$ elements. Hence,

\[ \|\mathbb{E}[\overline{M}_2(m) + \cdots + \overline{M}_{2h}(m)|\mathcal{F}(mh - 1)]\| \]

\[ \leq b^2(mh) \sum_{i=2}^{2h} C_{2h}^i \left(N\beta_a + C_a\beta_H^2 + N\sqrt{N}\beta_a \frac{1 - (1 - \kappa)^{-(d+1)}}{1 - (1 - \kappa)^{-1}} \right)^i \]

\[ = b^2(mh)\gamma \text{ a.s., (55)} \]

where

\[ \gamma = \left( \left(N\beta_a + C_a\beta_H^2 + N\sqrt{N}\beta_a \frac{1 - (1 - \kappa)^{-(d+1)}}{1 - (1 - \kappa)^{-1}} \right) + 1 \right)^{2h} \]

\[ - 1 - 2h \left(N\beta_a + C_a\beta_H^2 + N\sqrt{N}\beta_a \frac{1 - (1 - \kappa)^{-(d+1)}}{1 - (1 - \kappa)^{-1}} \right). \]

By (53)-(55), we have

\[ \|\mathbb{E}[\Phi_F((m + 1)h - 1, mh)\Phi_F^T((m + 1)h - 1, mh)|\mathcal{F}(mh - 1)]\| \]

\[ \leq 1 - a((m + 1)h)\theta + b^2(mh)\gamma \text{ a.s., } m \geq m'_1. \]  

(56)

By (14) and Assumption A2.b, we know that there exists a positive constant $\overline{\kappa}$ such that

\[ \|F(k)\| \leq \overline{\kappa} \text{ a.s., } k \geq 0. \]  

(57)
Denote \( m_k = \lfloor \frac{k}{N} \rfloor \). By (57) and Lemma A.6, we have

\[
\| \mathbb{E}[\Phi_F(k, 0) \Phi_F^T(k, 0)] \| \leq Nn \| \mathbb{E}[\Phi_F^T(k, 0) \Phi_F(k, 0)] \| \leq Nn \| \mathbb{E}[\Phi_F^T(m_k h - 1, 0) \Phi_F^T(k, m_k h) \Phi_F(k, m_k h) \Phi_F^T(m_k h - 1, 0) \| \leq Nn \| \mathbb{E}[\Phi_F^T(m_k h - 1, 0) \| \Phi_F(k, m_k h) \|^2 \Phi_F(m_k h - 1, 0) \| \]

\[
\leq \pi^2 h Nn \| \mathbb{E}[\Phi_F^T(m_k h - 1, 0) \| \Phi_F(k, m_k h) \|^2 \Phi_F^T(m_k h - 1, 0) \| \leq \pi^2 h Nn \| \mathbb{E}[\| \Phi_F(k, m_k h) \|^2 \Phi_F^T(m_k h - 1, 0) \| \Phi_F(m_k h - 1, 0) \| ] \leq \pi^2 h Nn \| \mathbb{E}[\Phi_F^T(m_k h - 1, 0) \| \Phi_F^T(m_k h - 1, m_k' h) \| \Phi_F(m_k h - 1, m_k' h) \| \]

\[
\leq \pi^2 (h + m_k' h) Nn \| \mathbb{E}[\Phi_F^T(m_k h - 1, 0) \| \Phi_F^T(m_k h - 1, m_k' h) \| \Phi_F^T(m_k h - 1, m_k' h) \| \]
\]

(58)

From the properties of the conditional expectation and (56), it follows that

\[
\| \mathbb{E}[\Phi_F^T(m_k h - 1, m_k' h) \Phi_F(m_k h - 1, m_k' h)] \| = \| \mathbb{E}[\Phi_F^T((m_k - 1) h - 1, m_k' h) \Phi_F^T(m_k h - 1, (m_k - 1) h) \times \Phi_F((m_k - 1) h - 1, m_k' h)] \|
\]
\[
= \| \mathbb{E}[\Phi_F^T((m_k - 1) h - 1, m_k' h) \Phi_F^T(m_k h - 1, (m_k - 1) h) \Phi_F(m_k h - 1, (m_k - 1) h) \times \Phi_F((m_k - 1) h - 1, m_k' h)] \| \mathcal{F}((m_k - 1) h - 1)) \| \]
\[
\leq \| \mathbb{E}[\Phi_F^T((m_k - 1) h - 1, m_k' h) \times \| \mathbb{E}[\Phi_F^T(m_k h - 1, (m_k - 1) h) \Phi_F^T(m_k h - 1, (m_k - 1) h)] \| \mathcal{F}((m_k - 1) h - 1)) \| \times \Phi_F((m_k - 1) h - 1, m_k' h)] \|
\]
\[
\leq [1 - a((m_k h) \theta + b^2 ((m_k - 1) h) \gamma)]
\]
\[
\times \| E[\Phi_F^T((m_k - 1) h - 1, m_k' h) \Phi_F((m_k - 1) h - 1, m_k' h)] \|
\]
\[
\leq \prod_{s=m_k'}^{m_k-1} [1 - a((s + 1) h) \theta + b^2 (sh) \gamma] \text{ a.s.} \quad (59)
\]

Combining (58) and (59) implies

\[
\| \mathbb{E}[\Phi_F(k, 0) \Phi_F^T(k, 0)] \| \leq Nn \pi^2 (h + m_k' h) \prod_{s=m_k'}^{m_k-1} [1 - a((s + 1) h) \theta + b^2 (sh) \gamma] \text{ a.s.}
\]

Similarly to (32)–(35) in the proof of Lemma B.1, by Assumption A3.a and the above, we have \( \lim_{k \to \infty} \| \mathbb{E}[\Phi_F(k, 0) \Phi_F^T(k, 0)] \| = 0 \). The proof is completed. \( \square \)

**Proof of Theorem V.1.** Denote \( \mathcal{T}(k) = [r^T(k), g^T(k), \cdots, g^T(k - d + 1)]^T \), \( \tilde{\mathcal{T}} = [0_{Nn \times Nn}, \tilde{I}]^T \) and \( \tilde{\mathcal{I}} = [I_{Nn}, 0_{Nn \times Nn}, \cdots, 0_{Nn \times Nn}] \), where \( \tilde{\mathcal{T}} \) and \( \tilde{\mathcal{I}} \) are the \( Nn(d + 1) \) dimensional column block matrix and \( Nn d \) dimensional row block matrix with each block being the \( Nn \) dimensional matrix, respectively. Denote

\[
T(k) = \begin{pmatrix} F(k) & \tilde{\mathcal{I}} \\ 0_{Nn \times Nn} & C(k) \end{pmatrix}
\]
which gives
\[
\Phi_T(k, 0) = \begin{pmatrix}
\Phi_F(k, 0) & \sum_{i=0}^{k} \Phi_F(k, i) \tilde{\Phi}_C(i, 0) \\
0_{Nnd \times Nn} & \Phi_C(k, 0)
\end{pmatrix}.
\]

Denote
\[
C(k) = \begin{pmatrix}
C_1(k+1) & C_2(k+1) & \cdots & C_d(k+1) \\
I_{Nn} & 0_{Nn \times Nn} & \cdots & I_{Nn} \\
0_{Nn \times Nn} & \cdots & 0_{Nn \times Nn}
\end{pmatrix}.
\]

By the state augmentation approach and (12), we have
\[
\tau(k+1) = \tau(k + 1) + \Phi_T(k, 0)\tau(0) + \sum_{i=1}^{k+1} a(i)\Phi_T(k, i)\tilde{\mathbf{H}}^T(i)v(i), \quad k \geq 0.
\]

Premultiplying the \(Nn(d+1)\) dimensional row block matrix \(\mathbf{T} \triangleq [I_{Nn}, 0_{Nn \times Nn}, \cdots, 0_{Nn \times Nn}]\) on both sides of the above gives
\[
r(k+1) = \mathbf{T}\Phi_T(k, 0)\tau(0) + \sum_{i=1}^{k+1} a(i)\mathbf{T}\Phi_T(k, i)\tilde{\mathbf{H}}^T(i)v(i),
\]
which further leads to
\[
\begin{align*}
\mathbb{E}[r(k+1)r^T(k+1)] \\
= \mathbb{E}[\mathbf{T}\Phi_T(k, 0)\tau(0)\tau^T(0)\Phi_T^T(k, 0)\mathbf{T}^T] \\
+ \mathbb{E}\left[\mathbf{T}\Phi_T(k, 0)\tau(0)\left(\sum_{i=1}^{k+1} a(i)v^T(i)\mathbf{H}(i)\tilde{\mathbf{H}}^T(i)\Phi_T^T(k, i)\mathbf{T}^T\right)\right] \\
+ \mathbb{E}\left[\left(\sum_{i=1}^{k+1} a(i)\mathbf{T}\Phi_T(k, i)\tilde{\mathbf{H}}^T(i)v(i)\right)\tau^T(0)\Phi_T^T(k, 0)\mathbf{T}^T\right] \\
+ \mathbb{E}\left[\left(\sum_{i=1}^{k+1} a(i)\mathbf{T}\Phi_T(k, i)\tilde{\mathbf{H}}^T(i)v(i)\right)\left(\sum_{i=1}^{k+1} a(i)\mathbf{T}\Phi_T(k, i)\tilde{\mathbf{H}}^T(i)v(i)\right)^T\right].
\end{align*}
\]

By Assumptions A1.a and A1.b, we know that the second and third terms on the right side of the above are both equal to zero.

By (38), we have
\[
\begin{align*}
\mathbb{E}\left[\left(\sum_{i=1}^{k+1} a(i)\mathbf{T}\Phi_T(k, i)\tilde{\mathbf{H}}^T(i)v(i)\right)\left(\sum_{i=1}^{k+1} a(i)\mathbf{T}\Phi_T(k, i)\tilde{\mathbf{H}}^T(i)v(i)\right)^T\right] \\
= \sum_{i=1}^{k+1} a^2(i)\mathbb{E}[\mathbf{T}\Phi_T(k, i)\tilde{\mathbf{H}}^T(i)v(i)v^T(i)\mathbf{H}(i)\tilde{\mathbf{H}}^T(i)\Phi_T^T(k, i)]
\end{align*}
\]
Substituting the above into (61) and taking the 2-norm on both sides of (61), from Assumptions A1.a, A1.b and A2.b, it follows that

\[
\| \mathbb{E}[r(k+1)r^T(k+1)] \| \\
\leq r_0 \| \mathbb{E}[\tilde{T}_n(k,0) \tilde{T}_n^T(k,0)] \| \\
+ \sum_{i=1}^{k+1} a^2(i) \mathbb{E}[\tilde{T}_n(k,i) \tilde{T}_n(i) v^T(i) \mathcal{H}(i) \tilde{T}_n^T(k,i) \tilde{T}_n^T] \\
= r_0 \| \mathbb{E}[\tilde{T}_n(k,0) \tilde{T}_n^T(k,0)] \| \\
+ \sum_{i=1}^{k+1} a^2(i) \mathbb{E}[\tilde{T}_n(k,i) \tilde{T}_n(i) v^T(i) \mathcal{H}(i) \tilde{T}_n^T(k,i) \tilde{T}_n^T] \\
\leq r_0 \| \mathbb{E}[\tilde{T}_n(k,0) \tilde{T}_n^T(k,0)] \| \\
+ \beta_H \sup_{k \geq 0} \| \mathbb{E}[v(k)v^T(k)] \| \sum_{i=1}^{k+1} a^2(i) \| \mathbb{E}[\tilde{T}_n(k,i) \tilde{T}_n^T(k,i) \tilde{T}_n^T] \| \\
\leq r_0 \| \mathbb{E}[\tilde{T}_n(k,0) \tilde{T}_n^T(k,0)] \| + \beta_H^2 \beta_v \sum_{i=1}^{k+1} a^2(i) \| \mathbb{E}[\tilde{T}_n(k,i) \tilde{T}_n^T(k,i) \tilde{T}_n^T] \|,
\]

(62)

where \( r_0 = \| \pi(0) \pi^T(0) \| \). By definitions of \( \Phi_n(k,0) \) and \( \tilde{T}_n \), we have

\[
\tilde{T}_n(k,0) = \left( \Phi_n(k,0) \sum_{i=0}^{k} \Phi_n(k,i+1) \tilde{T}_n(i-1,0) \right).
\]

Substituting the above into (62) gives

\[
\| \mathbb{E}[r(k+1)r^T(k+1)] \| \\
\leq r_0 \| \mathbb{E}[\Phi_n(k,0) \Phi_n^T(k,0)] \| + \beta_H^2 \beta_v \sum_{i=1}^{k+1} a^2(i) \| \mathbb{E}[\Phi_n(k,i) \Phi_n^T(k,i)] \| \\
+ r_0 \| \mathbb{E}\left[ \left\{ \sum_{i=0}^{k} \Phi_n(k,i+1) \tilde{T}_n(i-1,0) \right\} \left\{ \sum_{i=0}^{k} \Phi_n^T(i-1,0) \tilde{T}_n^T \Phi_n^T(k,i+1) \right\} \right] \| \\
+ \beta_H^2 \beta_v \sum_{i=1}^{k+1} a^2(i) \| \mathbb{E}\left[ \left\{ \sum_{i=1}^{k} \Phi_n(k,i+1) \tilde{T}_n(i-1,0) \right\} \right] \| \\
\times \left\{ \sum_{j=i}^{k} \Phi_n(k,j+1) \tilde{T}_n(j-1,i) \right\} \|,
\]

(63)

By Lemma C.1, we know that the first term on the right side of the above converges to zero.

Denote \( \tilde{m}_i = \left[ \frac{i}{n} \right] \). By (57) and noting the definition of \( m_k \) defined in the proof of Lemma C.1, we have

\[
\sum_{i=1}^{k+1} a^2(i) \| \mathbb{E}[\Phi_n(k,i) \Phi_n^T(k,i)] \|
\]
where

\[ l \]

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By Assumptions A2.b and A3.a, then there exist \( \epsilon \in (0, \frac{1-n}{\sqrt{Nnd}}) \) and a positive integer \( k(\epsilon) \), such that for \( \forall \ k \geq k(\epsilon) \), \( \|C_i(k)\|_\infty \leq \frac{\epsilon_i(1-\epsilon)}{\epsilon_i(1-\epsilon)^d} \) a.s., \( 1 \leq i \leq d \), where \( \| \cdot \|_\infty \) represents the infinite norm of a matrix. If \( d > 1 \), denote \( Y = \text{diag}\{I_{Nn}, \epsilon I_{Nn}, \epsilon^2 I_{Nn}, \ldots, \epsilon^{d-1} I_{Nn}\} \); if \( d = 1 \), denote \( Y = I_{Nn} \), which together with (60) leads to

\[
YC(k)Y^{-1} = \begin{pmatrix}
    C_1(k+1) & \epsilon^{-1}C_2(k+1) & \cdots & \epsilon^{1-d}C_d(k+1) \\
    \epsilon I_N & 0_{N \times N} & \cdots & \epsilon I_N \\
    \cdots & \cdots & \cdots & \cdots \\
    \epsilon I_N & 0_{N \times N} 
\end{pmatrix}
\]
Then, it follows that
\[
\|YC(k)Y^{-1}\|_\infty \leq \max\left\{ \sum_{i=1}^{d} \epsilon^{1-i}\|C_i(k+1)\|_\infty, \epsilon \right\} \leq \max\left\{ \frac{\epsilon(\epsilon - 1) \epsilon - \epsilon^{1-d}}{\epsilon - \epsilon^{1-d} \epsilon - 1}, \epsilon \right\} = \epsilon \text{ a.s.}
\]

From the relation between infinite norm and 2-norm of a matrix, we have
\[
\|YC(k)Y^{-1}\| \leq \sqrt{Nd}\|YC(k)Y^{-1}\|_\infty \leq \epsilon\sqrt{Nd} < 1 - \kappa \text{ a.s.} \tag{65}
\]

Noting that \(F(k)\) is invertible a.s., we have
\[
\left\| \mathbb{E}\left[ \sum_{i=0}^{k} \Phi_F(k, i+1)\tilde{\Phi}_C(i, 0)\right] \left\{ \sum_{i=0}^{k} \Phi_F(k, i+1)\tilde{\Phi}_C(i, 0)\right\}^T \right\| \leq \sum_{0 \leq i, j \leq k} \left\| \mathbb{E}[\Phi_F(k, i+1)\tilde{\Phi}_C(i, 0)\Phi_F^T(j, 0)\tilde{\Phi}_C^T(k, i+1)] \right\| 
\]
\[
\leq \sum_{0 \leq i, j \leq k} \left\| \mathbb{E}[\Phi_F(k, 0)[\Phi_F(i, 0)]^{-1}\tilde{\Phi}_C(i, 0)\Phi_F^T(j, 0)\tilde{\Phi}_C^T(k, i+1)] \right\| 
\]
\[
\leq \sum_{0 \leq i, j \leq k} \left\| \mathbb{E}[\Phi_F(k, 0)][\Phi_F(i, 0)]^{-1}\|\tilde{\Phi}_C(i, 0)\|\Phi_F^T(j, 0)\|\tilde{\Phi}_C^T(k, i+1)\right\| 
\times \|\Phi_F(j, 0)\|^T\|\Phi_F^T(k, 0)\| \tag{66}
\]

By Lemma V.1, it follows that
\[
\|[\Phi_F(i, 0)]^{-1}\| \leq (1 - \kappa)^{-i+1} \text{ and } \|[\Phi_F(j, 0)]^{-T}\| \leq (1 - \kappa)^{-j+1} \text{ a.s.} \tag{67}
\]

From (65), we obtain
\[
\|\tilde{\Phi}_C(i-1, 0)\Phi_F^T(j-1, 0)\tilde{\Phi}_C^T(k, i+1)\| \leq \|\Phi_C(i-1, 0)\|\|\Phi_C(j-1, 0)\| 
\]
\[
= \|Y^{-1}\Phi_CY^{-1}(i-1, 0)Y\|\|Y^{-1}\Phi_CY^{-1}(j-1, 0)Y\| 
\]
\[
\leq (\epsilon\sqrt{Nd})^{i+j-2} \text{ a.s.,} \tag{68}
\]

which combining (66) and (67) gives
\[
\left\| \mathbb{E}\left[ \sum_{i=0}^{k} \Phi_F(k, i+1)\tilde{\Phi}_C(i, 0)\right] \left\{ \sum_{i=0}^{k} \Phi_F(k, i+1)\tilde{\Phi}_C(i, 0)\right\}^T \right\| \leq (1 - \kappa)^{-2}\|\mathbb{E}[\Phi_F(k, 0)\Phi_F^T(k, 0)]\| \sum_{0 \leq i, j \leq k} (1 - \kappa)^{-1}\epsilon\sqrt{Nd}^{i+j} \text{ a.s.}
\]

Noting that \((1 - \kappa)^{-1}\epsilon\sqrt{Nd} < 1\), we have \(\sum_{0 \leq i, j \leq k} (1 - \kappa)^{-1}\epsilon\sqrt{Nd}^{i+j} < \infty\). Hence, by Lemma C.1, it follows that
\[
\lim_{k \to \infty} \left\| \mathbb{E}\left[ \sum_{i=0}^{k} \Phi_F(k, i+1)\tilde{\Phi}_C(i, 0)\right] \left\{ \sum_{i=0}^{k} \Phi_F(k, i+1)\tilde{\Phi}_C(i, 0)\right\}^T \right\| = 0.
\]

Thus, the third term on the right side of (63) converges to zero.
By (67)-(68) and similarly to (66), it follows that
\[
\sum_{i=1}^{k+1} a^2(i) \| \{ \sum_{j=i}^{k} \Phi_F(k, j+1) \tilde{T}_C(j-1, i) \} \{ \sum_{j=i}^{k} \Phi_F(k, j+1) \tilde{T}_C(j-1, i) \}^T \| \\
= \sum_{i=1}^{k+1} a^2(i) \left\| \sum_{i \leq j_1, j_2 \leq k} \mathbb{E}[\Phi_F(k, j_1 + 1) \tilde{T}_C(j_1 - 1, i) \tilde{T}_C(j_2 - 1, i) \Phi_F^T(k, j_2 + 1)] \right\|
\]
\[
= \sum_{i=1}^{k+1} a^2(i) \left\| \sum_{i \leq j_1, j_2 \leq k} \mathbb{E}[\Phi_F(k, i) (\Phi_F(j_1, i))^{-1} \tilde{T}_C(j_1 - 1, i) \times \Phi_F^T(j_2 - 1, i) \tilde{T}_C(j_1 - 1, i) \times \Phi_F^T(j_2 - 1, i)^{-1} \Phi_F^T(k, i)] \right\|
\]
\[
\leq (1 - \kappa)^{-6} \sum_{i=1}^{k+1} a^2(i) \left\| \mathbb{E}[\Phi_F(k, i) \Phi_F^T(k, i)] \right\| \sum_{i \leq j_1, j_2 \leq k} ((1 - \kappa)^{-1} \epsilon \sqrt{Nnd}(j_1 + j_2 - 2i))
\]
\[
= (1 - \kappa)^{-6} \sum_{i=1}^{k+1} a^2(i) \left\| \mathbb{E}[\Phi_F(k, i) \Phi_F^T(k, i)] \right\| \frac{1 - ((1 - \kappa)^{-1} \epsilon)^{2k-2i+1}}{1 - (1 - \kappa)^{-1} \epsilon \sqrt{Nnd}}
\]
\[
\leq \frac{(1 - \kappa)^{-6}}{1 - (1 - \kappa)^{-1} \epsilon \sqrt{Nnd}} \sum_{i=1}^{k+1} a^2(i) \left\| \mathbb{E}[\Phi_F(k, i) \Phi_F^T(k, i)] \right\| \text{ a.s.}
\]
In the light of (64), the above converges to zero.

So far, we have proved that all the four terms on the right side of (63) converge to zero. Thus, we have \( \lim_{k \to \infty} \| \mathbb{E}(r(k + 1)r^T(k + 1)) \| = 0 \), which, along with the facts that \( \mathbb{E}[r(k)]^2 = \mathbb{E}[\text{Tr}(r(k)r^T(k))] = \text{Tr}(\mathbb{E}(r(k)r^T(k))) \) and \( r(k) \) is equivalent to \( e(k) \), gives \( \lim_{k \to \infty} \mathbb{E}[\| e(k) \|^2] = 0 \). The proof is completed. \( \square \)

**Proof of Corollary V.1.** For the \( n \) dimensional matrix \( B \), we have \( \| B \| = \| B^T \| \). To see this, noting the following matrix equality,
\[
\begin{pmatrix}
I_n & B \\
0_{n \times n} & I_n
\end{pmatrix}
\begin{pmatrix}
0_{n \times n} & 0_{n \times n} \\
B^T & B^T B
\end{pmatrix}
= \begin{pmatrix}
BB^T & BB^T B \\
B^T & B^T B
\end{pmatrix}
= \begin{pmatrix}
BB^T & 0_{n \times n} \\
B^T & 0_{n \times n}
\end{pmatrix}
\begin{pmatrix}
I_n & B \\
0_{n \times n} & I_n
\end{pmatrix},
\]
then the matrix
\[
B_1 \triangleq \begin{pmatrix}
0_{n \times n} & 0_{n \times n} \\
B^T & B^T B
\end{pmatrix}
\]
and the matrix \( B_2 \triangleq \begin{pmatrix}
BB^T & 0_{n \times n} \\
B^T & 0_{n \times n}
\end{pmatrix} \)
are similar. Thus, the spectra of \( B^T B \) and \( BB^T \) are equal, which gives \( \| B \| = \sqrt{\lambda_{\text{max}}(B^T B)} = \sqrt{\lambda_{\text{max}}(BB^T)} = \| B^T \| \). Also, noting that for any \( n \) dimensional symmetric matrix \( B, B \geq 0 \),
\[ \lambda_{\text{min}}(B)I_n, \quad B \leq \|B\|I_n \] and by definitions of \( \lambda^h_m \) and \( \Delta^h_m \), we have

\[
\sum_{k=mh}^{(m+1)h-1} \left( \frac{b(k)}{a(k)} \mathbb{E}[\tilde{L}_G(k) | \mathcal{F}(mh-1)] \otimes I_n + 2\mathbb{E}[\mathcal{H}^T(k) \mathcal{H}(k) | \mathcal{F}(mh-1)] \right) \\
- \frac{b(k)}{a(k)} \sum_{q=0}^{d} \mathbb{E}[\mathcal{A}(k, q) ([\Phi_F(k-1, k-q)]^{-1} - I_{Nn}) | \mathcal{F}(mh-1)] \\
- \frac{b(k)}{a(k)} \sum_{q=0}^{d} \mathbb{E}[[\Phi_F(k-1, k-q)]^{-1} - I_{Nn}]^T A^T(k, q) | \mathcal{F}(mh-1)] \\
\geq \lambda_{\text{min}} \left[ \sum_{k=mh}^{(m+1)h-1} \left( \frac{b(k)}{a(k)} \mathbb{E}[\tilde{L}_G(k) | \mathcal{F}(mh-1)] \otimes I_n + 2\mathbb{E}[\mathcal{H}^T(k) \mathcal{H}(k) | \mathcal{F}(mh-1)] \right) I_{Nn} \right] \\
- 2 \sum_{k=mh}^{(m+1)h-1} \frac{b(k)}{a(k)} \sum_{q=0}^{d} \mathbb{E}[\mathcal{A}(k, q) ([\Phi_F(k-1, k-q)]^{-1} - I_{Nn}) | \mathcal{F}(mh-1)] \right] I_{Nn} \\
= 2(\lambda^h_m - \Delta^h_m) I_{Nn}
\]

By the definition of \( \lambda^h_m' \), we know that \( 2(\lambda^h_m - \Delta^h_m) \leq \lambda^h_m' \), and further \( \inf_{m \geq 0}(\lambda^h_m - \Delta^h_m) \leq \frac{1}{2} \inf_{m \geq 0} \lambda^h_m' \), which together with \( \inf_{m \geq 0}(\lambda^h_m - \Delta^h_m) \geq \theta > 0 \) a.s. leads to \( \inf_{m \geq 0} \lambda^h_m' \geq 2 \inf_{m \geq 0}(\lambda^h_m - \Delta^h_m) \geq 2\theta > 0. \) Then, by Theorem V.1, we get the conclusion of the corollary. 

\[ \square \]

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