Numerical implementation of the Cubic Galileon model in PINOCCHIO

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ABSTRACT

We present a perturbative treatment of nonlinear galaxy clustering in the context of the cubic Galileon modified gravity model, in terms of 2nd order Lagrangian Perturbation theory and an extension of ellipsoidal collapse that includes Vainshtein screening. We numerically implement such prescriptions in the approximate PINOCCHIO code, and use it to generate realisations of the matter density field and halo catalogues with different prescriptions for ellipsoidal collapse. We investigate the impact of three different approximations in the computation of collapse times on the halo mass function, halo bias and matter power spectrum. In the halo mass function, both the modified gravity effect and the screening effect are significant in the high mass end, similar to what is found for other MG models. We perform a comparison with N-body simulations to assess the validity of our approach, and show that we can reproduce the same trend observed in simulations for all quantities considered. With a simple modification to the grouping algorithm of PINOCCHIO to take into account the gravity modification, and without the need to re-calibrate the algorithm, we show that we can reproduce the linear halo bias and the mildly-nonlinear matter power spectrum of simulations with good accuracy, especially for the implementation with Vainshtein screening. We stress that, while approximate, our method is orders of magnitude faster than a full N-body simulation, making it an optimal tool for the quick generation of large sets of halo catalogues for cosmological observables.

Key words: (cosmology:) large-scale structure of Universe, (cosmology:) dark energy, Galaxy: halo

1 INTRODUCTION

Within the standard ΛCDM cosmological model, the observed accelerated expansion of the Universe is ascribed to the presence of dark energy (DE) in the form of a cosmological constant Λ: a perfect fluid with constant energy and equation of state, whose negative pressure \( P = -\rho \) is responsible for the accelerated expansion. DE makes up most of the energy density of the Universe today together with cold dark matter (CDM), however, its nature remains elusive. In the standard framework, the cosmological constant \( \Lambda \) is thought to be related to vacuum energy: an interpretation that, despite the theoretical problems (Carroll 2001). Additionally, tensions in some of the cosmological parameters as measured by early and late time probes such as the Cosmic Microwave Background and redshift-space distortions of galaxies (Aghanim et al. 2020; Heymans et al. 2021; DES Collaboration et al. 2021) have prompted the study of structure formation in the context of beyond-ΛCDM models. Many alternatives have been proposed, from exotic DE models to modified gravity (MG, see Bull et al. (2016); Koyama (2018); Ishak (2019) for recent reviews). While the former are mainly characterized by a dynamically evolving equation of state, the latter focus on the possibility that General Relativity (GR) is not the correct theory to describe gravity on cosmological scales, and introduce an additional fifth force that drives cosmic acceleration.

On the other hand GR has successfully passed several stringent tests, therefore any viable MG model should be able to evade such constraints and reduce to GR on small scales. In order to be consistent with both small- and large-scale observations, the additional MG fifth force has to be shielded in high density regions. This is achieved by means of a so-called screening mechanism. Screening mechanisms can be classified in three general categories (Lombriser 2016), based on the condition that the Newtonian gravitational potential \( \Psi_N \) or its derivatives exceed a certain threshold \( \Lambda_T \):

(i) screening at large field values, such as the chameleon (Khoury & Weltman 2004) or symmetron (Hinterbichler & Khoury 2010) models. These are very similar to the large field inflation model: the screening effect turns on in regions where the Newtonian gravitational potential exceeds a certain threshold, \( |\nabla \Psi_N| > \Lambda_T \). A high density environment plays a key role in this mechanism;

(ii) screening with first derivatives, such as in k-mouflage models (Babichev et al. 2009). This screening effect operates when the local gravitational acceleration is above a certain threshold, \( |\nabla \Psi_N| > \Lambda^2_T \);
(iii) screening with second derivatives, such as Vainshtein screening (Vainshtein 1972). This screening mechanism activates when the local curvature is large, \( \nabla^2 \Psi_N > \Lambda^2 \). Unlike the chameleon mechanism, Vainshtein screening does not rely on the environment, the screening radius is jointly determined by the Schwarzschild radius of the object and the Hubble radius.

The investigation of alternatives to the standard cosmological model is indeed one of the key targets of modern cosmology. Upcoming Stage-IV galaxy surveys, such as Euclid \(^1\), LSST \(^2\), WFIRST \(^3\), DESI \(^4\), J-PAS \(^5\), CSST \(^6\), will probe the clustering of galaxies to high precision, allowing to test different MG models in the linear and mildly nonlinear regime of structure formation (Amendola et al. 2013; Alam et al. 2020).

It is therefore key that accurate theoretical modelling beyond GR is prepared in advance in order to compare to observations, with a particular focus on the nonlinear and mildly nonlinear regimes. The most reliable tool to trace the growth of structures deep into the nonlinear regime are N-body simulations. However, N-body simulations are computationally extremely expensive, in particular when they include MG.

This paper studies the modelling of nonlinearities in the framework of perturbation theory (PT) in the context of the cubic Galileon model (Nicolis et al. 2009). In particular, we focus on a numerical implementation in the PINOCCHIO code (PINpointing Orbit-Crossing Collapsed Hierarchical Objects, Monaco et al. (2002a,b); Taffoni et al. (2002); Monaco et al. (2013); Munari et al. (2017)), an algorithm to quickly generate simulated dark matter halo catalogues based on Lagrangian PT and ellipsoidal collapse.

We organize the papers as follows: in section 2 we describe the background, linear and nonlinear perturbation evolution of the cubic Galileon in detail. In section 3, we present the 1st and 2nd order Lagrangian PT for the cubic Galileon model. In section 4, we give the prescription of the “G3-PINOCCHIO” algorithm, including the extension of ellipsoidal collapse. In section 5, we show the nonlinear matter power spectrum and mass function obtained with the code, as well as a prediction for the linear bias. Finally, we present our conclusions in section 6.

2 CUBIC GALILEON

Galileon gravity (Nicolis et al. 2009) is proposed in the inspiration of flat space quantum field theories. The latter are invariant under the transformation \( \phi \rightarrow \phi + b_\mu x^\mu + c \), where \( b_\mu \) and \( c \) are a constant vector and scalar in flat space respectively. The existence of \( b_\mu \) and \( c \) indicates respectively the Galilean and shift symmetry. To avoid the Ostrogradski instability, the field equation includes up to second order time derivatives. When extending to curved spacetime, the non-minimal coupling between the Galileon field and the metric breaks the Galilean symmetry, but keeps the shift symmetry preserved. Such model, called covariant Galileon model, has a Lagrangian of the form

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{M_\pi^2}{2} R - \frac{1}{2} c_2 X + \frac{c_3}{M^3} X \Box \phi + \frac{c_4}{4M^6} X^2 R \right. \\
- \frac{c_4}{M^6} X [(\Box \phi)^2 - \phi \mu \nu \phi_{\mu \nu}] + \frac{3c_5}{4M^9} X^2 G_{\mu \nu} \phi^{\mu \nu} \\
+ \frac{c_5}{2M^9} X [(\Box \phi)^3 - 3 \Box \phi \phi_{\mu \nu} \phi^{\mu \nu} + 2 \phi^{\mu \nu} \phi_{\mu \sigma} \phi^{\sigma \nu}] \right\},
\]

where \( M_\pi \) is the Planck mass \( M_\pi^2 = 8\pi G, g, R G_{\mu \nu} \) are respectively the determinant of the metric, the Ricci scalar and the Einstein tensor, \( c_2, c_3, c_4, c_5 \) are dimensionless constants and \( M^2 = M_\pi H_0^2 \), with \( H_0 \) the present Hubble parameter. \( \Box = \phi_{\mu \nu} \phi^{\mu \nu} \) is the kinetic term of the scalar field. The semicolon in Eq. (1) represents the covariant derivative. There are three branches of covariant Galileon: the G3 branch, known as cubic Galileon, with \( c_3 \neq 0, c_4 = c_5 = 0 \), the G4 branch, also called quartic Galileon, with \( c_3, c_4 \neq 0, c_5 = 0 \) and the G5 branch, also known as quintic Galileon, with \( c_3, c_4, c_5 \neq 0 \). Some recent studies that focus on constraining the covariant Galileon model by means of various cosmological observations include Barreira et al. (2012, 2013b, 2014b); Neveu et al. (2013); Peirone et al. (2018); Renk et al. (2017); Frusciante et al. (2020).

The breakthrough discovery of gravitational waves (GW) from the merger of a neutron star binary (Abbott et al. 2017a) by the LIGO and Virgo collaborations puts a tight constraint on the GW speed (Abbott et al. 2017b), effectively ruling out several MG models. In particular, the G4 and G5 branches are ruled out (Creminelli & Vernizzi 2017; Ezquiaga & Zumalacárregui 2017; Baker et al. 2017) with a high confidence level. In the G3 branch however, the scalar field is minimally coupled to gravity, and the GW speed is unaltered. Hence, G3 is still a viable MG model even according to GW constraints.

An additional reason to investigate the cubic Galileon model is that it is related to the screening mechanism it features. Since the G3 field drives cosmic acceleration via the non-canonical kinetic energy, the extra gravitational force is screened via the Vainshtein mechanism. While chameleon screening has been extensively discussed during the past decades, in particular in the context of \( f(R) \) gravity (for example in Li et al. (2012); Puchwein et al. (2013); Llinares et al. (2014)), Vainshtein screening is usually studied as feature of nDGP modified gravity (e.g. Winther et al. (2015); Barreira et al. (2015); Winther et al. (2017); Hernández-Aguayo et al. (2021)). Studies of this type of screening mechanism that focus on cubic Galileon are instead more rare; some examples include the work of Schmidt (2009); Barreira et al. (2013a); Li et al. (2013).

A first step to include MG in the PINOCCHIO code focused indeed on \( f(R) \) gravity, which features scale-dependent growth and chameleon screening (Moretti et al. 2020). In this work, we present an implementation of scale-independent MG models with Vainshtein screening, focusing in particular on the G3 model, and study its performance with PINOCCHIO.

2.1 Background evolution under cubic Galileon

The Einstein equation and the scalar field equation can be obtained by varying the action with respect to the metric and scalar field respectively. In the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric, one can get the 1st- and 2nd- Friedmann

\(^1\) http://sci.esa.int/euclid
\(^2\) http://www.lsst.org
\(^3\) https://wfirst.gsfc.nasa.gov
\(^4\) https://www.desi.lbl.gov
\(^5\) http://www.j-pas.org/wiki/index.php/Main_Page
\(^6\) http://www.bao.ac.cn/csst/
The matter component satisfies the continuity equation \( \dot{\bar{\rho}} = \frac{\dot{\rho}}{a^3} + \frac{2c_3}{M^3} \dot{\phi}^2 \). \( \dot{\rho}_r \) follows we show that the number of parameters in G3 is actually the same as those in \( \Lambda \)CDM. By substituting the tracker solution Eq. (6) into the 1st Friedmann equation, one can get
\[ E^2 = (\Omega_{m,0}a^{-3} + \Omega_{\phi,0}a^{-4})E^2 + \frac{1}{6}c_2 \dot{\xi}^2 + 2c_3 \dot{\xi}^3, \] (7)
where \( E \) is the dimensionless Hubble parameter \( E = H/H_0 \), \( \Omega_{m,0} = \rho_{m,0}/(3M_{*}^2H_0^2) \) and \( \Omega_{\phi,0} = \rho_{\phi,0}/(3M_{*}^2H_0^2) \) are the present matter density and radiation density respectively and \( a \) is the scale factor.

Defining
\[ \Omega_{\Lambda,0} = 1 - \Omega_{m,0} - \Omega_{\phi,0} = \frac{1}{6}c_2 \dot{\xi}^2 + 2c_3 \dot{\xi}^3, \] (8)
we can get a parameterised Hubble parameter
\[ H = H_0 \sqrt{\frac{\Omega_{m,0}a^{-3} + \Omega_{\phi,0}a^{-4} + \sqrt{(\Omega_{m,0}a^{-3} + \Omega_{\phi,0}a^{-4})^2 + 4\Omega_{\Lambda,0}}}{2}} \] (9)

To avoid the scaling degeneracy and without loss of generality (Barreira et al. 2014b), one can assume \( c_2 = -1 \). Combining the scalar field equation and the tracker solution, we can get a constraining equation on the G3 parameters:
\[ c_2 \dot{\xi} + 6c_3 \dot{\xi}^2 = 0 \] (10)
From Eq. (8) and (10), we have
\[ c_3 = \frac{1}{6 \sqrt{6\Omega_{\Lambda,0}}} \quad \xi = \sqrt{6\Omega_{\Lambda,0}}. \] (11)
One can see that, once \( \Omega_{\Lambda,0} \) is given, all the G3 model parameters are fixed and there are no additional parameters with respect to \( \Lambda \)CDM. Through the 2nd Friedmann equation, we obtain the acceleration parameter
\[ \ddot{H} = \frac{H^2}{H^2 - \frac{8}{3}H_0^2\Omega_{m,0}a^{-3} + 2\Omega_{r,0}a^{-4} - 4\Omega_{\Lambda,0}}. \] (12)
The linear perturbation regime of MG models has been extensively studied in the past few years. In particular, it has been shown both theoretically and numerically that single field models can be re-expressed in the language of the Effective Field Theory of Dark Energy (EFTofDE, Gubitosi et al. (2013); Bloomfield et al. (2013); Piazza & Vernizzi (2013); Hu et al. (2014); Zumalacárregui et al. (2017); Frusciante & Perenon (2020)). The action in the EFTofDE reads:

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{M^2_{\text{pl}}}{2} \left[ (\partial \Phi)^2 / a^2 + \Lambda(t) - c(t) \delta g^{00} \right] + \frac{M^2_{\text{pl}}}{2} (\delta g^{ij})^2 + \frac{M^2_{\text{pl}}}{2} \delta K_{ij} \delta K^{ij} - \frac{M^2_{\text{pl}}}{2} \delta R^{00} + \frac{M^2_{\text{pl}}}{2} \delta g^{ij} \delta R^{ij} \right\} + m^2_{\text{pl}} \bar{g}_{ij} \bar{\chi}^{ij},
\]

where \( \delta g^{00}, \delta K_{ij}, \delta R^{00}, \delta R^{ij} \) are perturbations of the time-time component of the metric, the external curvature and its trace, and the 3-dimensional Ricci scalar in the constant time hypersurface, \( S_m \) is the minimally coupled term for all matter fields \( \chi_i \) with metric \( \bar{g}_{ij} \), and \( \bar{M}_i, \bar{M}_j, \) and \( \bar{M}_l \) are the EFT functions. The perturbation of the scalar field can be expressed by the infinitesimal time diffeomorphism, \( t \rightarrow t + \pi(x^i) \), where \( \pi \) is the perturbation of the scalar field.

Following Pogosian & Silvestri (2016), we combine the Einstein equation and the scalar field equation under the quasi-static approximation, which applies only to scales below the sound horizon of the scalar field. Under this approximation, we ignore the time derivatives of the gravitational potential and the scalar field. One can then derive the following expression for \( \mu^L(a,k) \) in the EFTofDE framework:

\[
\mu^L = \frac{f_1 + f_3 a^2 / k^2}{f_3 + f_4 a^2 / k^2},
\]

with

\[
\begin{align*}
f_1 &= c - \frac{1}{2} (H + \dot{\chi}) \bar{M}_i^2, \\
f_2 &= -3c \bar{H} + \frac{3}{2} (3H^2 + \bar{H} \dot{\chi} + \ddot{\chi}) \bar{M}_i^2, \\
f_3 &= 2M^2_{\text{pl}} \Omega_f f_1 - \frac{1}{2} \bar{M}_i^2, \\
f_4 &= 2M^2_{\text{pl}} f_2, \tag{16}
\end{align*}
\]

In the upper panel of Figure 2, we show the linear part of the \( \mu \) function as a function of redshift \( z \) and wavenumber \( k \). The values are encoded in the color bar. On top of that, we show contours with constant \( \mu^L \) values. One can see that the contours are vertically distributed, in line with the fact that the modification of the gravitational force is scale independent. Furthermore, the \( \mu \) values are significantly larger at low redshifts: the gravity enhancement reaches a value of around 2 at present time, while it approaches unity for redshifts \( z > 1 \). In the lower panel of Figure 2 we show the nonlinear \( \mu^{NL} \) function that includes Vainshtein screening, discussed in the next section, as a function of both redshift \( z \) (horizontal axis) and the local density contrast \( \delta \) (vertical axis). One can see that with higher local density, the extra gravitational force is shielded for all redshifts considered.

### 2.3 Nonlinear clustering in cubic Galileon

We now consider the 2nd order perturbation contribution. Under the quasi-static approximation, after ignoring the terms suppressed by the time derivatives and Hubble expansion, the 00 component of the Einstein equation reads (Frusciante & Pace 2020)

\[
\frac{\partial^2 \Phi}{a^2} = 4\pi G \rho_m \delta_m - 8\pi G \frac{c_3}{M^3} \frac{\partial^2}{a^2} \left( \delta \Phi \right),
\]

where \( \partial \) denotes derivatives with respect to comoving spacial coordinates. The equation for the scalar field is

\[
-\frac{c_3}{M^2} \frac{\partial^2}{a^2} \left( \frac{\partial^2 \Phi}{a^2} \right) = \left[ \frac{1}{c_2} + \frac{c_2}{M^2} \frac{\partial^2}{a^2} \left( \Phi + 2H \dot{\Phi} \right) \right] \frac{\partial^2 \Phi}{a^2} + \frac{c_1}{M^2} \left( \frac{\partial^2 \Phi}{a^2} \right)^2 - \left( \frac{\partial \delta \Phi}{a^2} \right)^2.
\]

Figure 2. Two dimensional contour plot of the \( \mu \) function. The upper panel shows the linear part \( \mu^L \) as a function of redshift \( z \) (horizontal axis) and wavenumber \( k \) (vertical axis), with lines showing the isocontours. The lower panel shows the same quantity but including Vainshtein screening \( \mu^{NL} \) (Eq. 25), again as a function of redshift \( z \) (horizontal axis), but with the local matter density contrast \( \delta \) on the vertical axis. Values different from 1 mark a deviation from GR.
When the matter perturbations are large such that $\delta$ with a top-hat density profile $\delta$ same as the linear $\delta$, one gets

$$
\frac{\partial^2 \langle \delta \phi \rangle}{a^2} + \chi^2(a) \left( \frac{\partial^2 \langle \delta \phi \rangle}{a^2} \right)^2 - \left( \frac{\partial \delta (\frac{\partial \phi}{a^2})}{a^2} \right)^2 = -4\pi G\zeta(a) \rho_m \delta_m,
$$

(19)

where

$$
\chi^2(a) = \frac{c_3/M^3}{\pi^2} 2 \left( 2H \phi + 8\pi G\frac{c_3}{M^3} \delta^2 \phi^4 \right), \quad \zeta(a) = \chi^2 \phi^2.
$$

Assuming a spherically symmetric profile for $\delta$, Eq. (19) becomes

$$
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d \delta \phi}{dr} \right) - 2r \frac{d \delta \phi}{dr} = -4\pi G\zeta \rho_m \delta_m,
$$

(20)

a second order ordinary differential equation. After integrating once, one

$$
r^2 \frac{d \delta \phi}{dr} - 2 \chi \frac{d \delta \phi}{dr} = -G \zeta m(r),
$$

(21)

where $m(r) = 4\pi \int_0^r r'^2 \rho_m (r') \delta_m (r') dr'$ is the mass enclosed in a sphere of radius $r$. Considering a top-hat density profile with radius $R$, when $r \leq R$ the physical solution of Eq. (21) reads (Frusciante et al. 2020)

$$
\frac{d \delta \phi}{dr} = \frac{r}{4\chi^2} \left( 1 - \sqrt{1 + \frac{r^2}{R^2}} \right),
$$

(22)

where $r_v$ is the Vainshtein radius of the enclosed mass, given by

$$
r_v^3 = 8\pi G m(r) \chi \phi.
$$

(23)

From Eq. (22) one can see that $d\delta \phi/dr \propto r$ within $r \leq R$, so $\partial^2 (\phi) \partial r^2$ is constant. Taking the derivative of Eq. (22) and substituting it in Eq. (17), we get the modified Poisson equation which is constant within the Top Hat

$$
\frac{\partial^2 \mu}{a^2} = 4\pi G\mu^{NL}(a,R) \rho_m \delta_m,
$$

(24)

where $\mu^{NL}(a,R)$ includes the nonlinear contribution

$$
\mu^{NL}(a,R) = 1 + 2(\mu^L - 1) \left( \frac{R}{R_V} \right)^3 \left( \sqrt{1 + \frac{R^3}{R_V^3}} - 1 \right).
$$

(25)

In the above equation, $\mu^L = 1 + 8\pi G\frac{c_3}{M^3} \hat{\phi}^2 \zeta(a)$ is the linear part of the $\mu$ function. $R_V$ is the Vainshtein radius of the enclosed mass with a top-hat density profile $R_V^3 = 8\pi G m(R) \chi \phi \propto R^3$. Then, one can write

$$
\left( \frac{R}{R_V} \right)^3 \left( 1 - \frac{1}{4\Omega m^2 H^2 \chi^2 \phi \delta_m} \right).
$$

(26)

When the matter perturbations are large such that $R \ll R_V$, $\mu^{NL} \rightarrow 1$ recovering GR, while for small perturbations when $R \gg R_V$, $\mu^{NL} \rightarrow \mu^L$, recovering the linear result. In the lower panel of Figure 2, we plot $\mu^{NL}$ as a function of redshift $z$ and the top-hat object density $\delta$. One can see that for low redshifts $z < 1$, it deviates from GR, same as the linear $\mu^L$ function. When the density contrast is about 0.01–0.1, $\mu^{NL}$ is similar to $\mu^L$, roughly depending on time only. For large values of $\delta > 1000$, $\mu^{NL} \rightarrow 1$, recovering GR. At fixed redshift, the value of $\mu^{NL}$ for a high density contrast is smaller than that of a low density contrast.

3 LAGRANGIAN PERTURBATION THEORY FOR CUBIC GALILEON

In this section we briefly review the linear and 2nd order Lagrangian perturbation theory (LPT) for the G3 model Song & Hu (2021). The equation of motion for the displacement field ($\delta$) in Lagrangian coordinates are

$$
\nabla_i \cdot T \delta = -\mathcal{A}(a) \delta - B(a) \left( \frac{\partial^2 \langle \delta \phi \rangle}{a^2} \right)^2 - \left( \frac{\partial \delta (\frac{\partial \phi}{a^2})}{a^2} \right)^2,
$$

(27)

where

$$
\mathcal{T} = \frac{d^2}{dt^2} + 2H \frac{d}{dt},
$$

(28)

$$
\mathcal{A}(a) = 4\pi G\zeta_m \left( 1 + 8\pi G\frac{c_3}{M^3} \delta^2 \zeta(a) \right),
$$

(29)

$$
\mathcal{B}(a) = 8\pi G\frac{c_3}{M^3} \delta^2 \chi(a).
$$

(30)

Notice that the differential operation $\nabla_i$ is in Eulerian coordinates. Transforming to Lagrangian coordinates one can write

$$
\nabla_i \cdot \mathcal{T} \delta = (J^{-1})_{ij} J S_{ij} = (\delta_{ij} - S_{ij}) \mathcal{T} S_{ij} = \mathcal{T} S_{ij} - S_{ij} \mathcal{T} S_{ij} ,
$$

(31)

where $J$ is the Jacobian of the coordinate transformation, and $S_{ij}, S_{ij}$ are now spatial derivatives with respect to Lagrangian coordinates. Substituting Eq. (31) in Eq. (27), we can write the displacement field equation in Lagrangian coordinates

$$
\mathcal{T} S_{ij} - S_{ij} \mathcal{T} S_{ij} = -\mathcal{A}(a) \delta - \mathcal{B}(a) \left( \frac{\partial^2 \langle \delta \phi \rangle}{a^2} \right)^2 - \left( \frac{\partial \delta (\frac{\partial \phi}{a^2})}{a^2} \right)^2.
$$

(32)

Expanding $\delta, \delta \phi, \delta \phi$ with respect to a small parameter $\varepsilon$

$$
\delta = \varepsilon \delta^{(1)} + \varepsilon^2 \delta^{(2)} + ..., \quad \delta \phi = \varepsilon \delta \phi^{(1)} + \varepsilon^2 \delta \phi^{(2)} + ..., \quad \delta \phi = \varepsilon \delta \phi^{(1)} + \varepsilon^2 \delta \phi^{(2)} + ..., \quad \delta \phi = \varepsilon \delta \phi^{(1)} + \varepsilon^2 \delta \phi^{(2)} + ...,
$$

(33)

one can split the above equations into serial differential equations according to their orders. At 1st order, the displacement field equation reads

$$
(\mathcal{T} - \mathcal{A}(a)) S^{(1)}_{ij} (t, \vec{q}) = 0.
$$

(36)

Since $\mathcal{T} - \mathcal{A}(a)$ only depends on time, $S^{(1)}_{ij} (t, \vec{q})$ can be separated into time- and space-dependent components. Transforming the above equation to Fourier space, we obtain

$$
S^{(1)}_{ij} (\vec{k}, t) = -D_1(t) \delta^{(1)}(\vec{k}, t_0),
$$

(37)

where the 1st order growth factor $D_1(t)$ satisfies

$$
(\mathcal{T} - \mathcal{A}(a)) D_1(t) = 0.
$$

(38)

Like in standard GR, the linear growth in G3 only has temporal dependence. For 2nd order in PT, the displacement field equation reads

$$
\mathcal{T} S^{(2)}_{ij} (\vec{k}, t) - S^{(1)}_{ij} \mathcal{T} S^{(1)}_{ij} (\vec{k}, t) = -\mathcal{A}(a) \delta^{(2)}(\vec{k}, t) - B(a) \left( \frac{\partial^2 \langle \delta \phi \rangle^{(1)}}{a^2} \right)^2 - \left( \frac{\partial \delta (\frac{\partial \phi}{a^2})^{(1)}}{a^2} \right)^2 (\vec{k}, t).
$$

(39)

Through Eq. (19) we obtain

$$
\frac{\vec{k}^2}{a^2} \delta^{(1)}(\vec{k}, t) = 4\pi G\zeta(a) \rho_m D_1(t) \delta^{(1)}(\vec{k}, t_0).
$$

(40)
Plugging Eq. (40), (37) into Eq. (39), we can write

\[
\langle \hat{T} - A(a) \rangle S_{ij}^{(2)}(\vec{k}, t) = \left[ \frac{1}{2} \Lambda(a) + B(a)(4\pi G\rho_0 \zeta(a))^2 \right] \times \\
\int_{k_1 = \vec{k}} \left[ 1 - \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_1^2 k_2^2} \right] D_1^2(t) \delta_i^{(1)}(t_0) \delta_j^{(1)}(t_0) \\
= -C(a) \int_{k_1 = \vec{k}} \left[ 1 - \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_1^2 k_2^2} \right] D_1^2(t) \delta_i^{(1)}(t_0) \delta_j^{(1)}(t_0),
\]

(41)

where \( \int_{k_1 = \vec{k}} \) is short-hand notation for \( \int d^2k_1 d^2k_2 \delta(\vec{k} - \vec{k}_1 - \vec{k}_2), \) and

\[ C(a) = \frac{1}{2} A(a) + B(a)(4\pi G\rho_0 \zeta(a))^2. \]

The 2nd order displacement field then reads

\[
S_{ij}^{(2)}(\vec{k}, t) = \int_{k_1 = \vec{k}} \tilde{D}_2(\vec{k}_1, \vec{k}_2, t) D_1^2(t) \delta_i^{(1)}(t_0) \delta_j^{(1)}(t_0),
\]

(42)

where \( \tilde{D}_2(\vec{k}_1, \vec{k}_2, t) \) is a normalized second order growth factor that satisfies

\[
\langle \hat{T} - A(a) \rangle \tilde{D}_2(\vec{k}_1, \vec{k}_2, t) = -C(a) \left[ 1 - \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_1^2 k_2^2} \right] .
\]

(43)

Since \( \langle \hat{T} - A(a) \rangle \) depends only on time, \( \tilde{D}_2(\vec{k}_1, \vec{k}_2, t) \) can also be separated into the product of time component \( D_2(t) \) and space component \( \left[ 1 - \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_1^2 k_2^2} \right]. \) Thus Eq. (42) can be rewritten as

\[
S_{ij}^{(2)}(\vec{k}, t) = D_1^2(t) \tilde{D}_2(t) \int_{k_1 = \vec{k}} \left[ 1 - \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_1^2 k_2^2} \right] \delta_i^{(1)}(t_0) \delta_j^{(1)}(t_0).
\]

(44)

Redefining \(-2D_1^2(t)\tilde{D}_2(t)\) as \( D_2(t) \), then the second order growth factor \( D_2(t) \) satisfies

\[
\langle \hat{T} - A(a) \rangle D_2(t) = 2C(a)D_1^2(t).
\]

(45)

Again, \( D_2 \) is independent of \( k \), as it is in the standard GR case.

To get the second order displacement field \( S_{ij}^{(2)}(\vec{k}, t) \), one just needs to convolve the initial linear density fields with a kernel as in Eq. (44). The difference between G3 and \( \Lambda \)CDM lies in the temporal evolution of the linear and 2nd order growth factors. When \( A(a) = 4\pi G\rho_0, B(a) = 0, C(a) = 2\pi G\rho_0 \), the equations for the 1st and 2nd order displacements recover \( \Lambda \)CDM (Song & Hu 2021).

We choose the initial conditions for the \( D_1 \) differential equation to match an Einstein-de Sitter universe, i.e. \( D_1(a) = a, \frac{dD_1}{da} = 1 \). In the top panel of Fig. 3 we show \( D_1(a) \) in G3 (red solid line) and \( \Lambda \)CDM (blue dashed line), with their ratio in the second panel, as a function of the scale factor \( a \). One can see that \( D_1 \) in G3 is the same as in \( \Lambda \)CDM at early times. Starting from \( a \approx 0.2 \), \( D_1 \) in G3 increases faster than in \( \Lambda \)CDM, with their ratio reaching \( \sim 15\% \) at present times. The initial conditions for \( D_2 \) are assumed to be \( D_2(a) = \frac{a^2}{2}, \frac{dD_2}{da} = \frac{a}{2}, \) corresponding to matter domination. In the third panel of Figure 3 we show \( D_2(a) \) in G3 (red solid line) and \( \Lambda \)CDM (blue dashed line), with their ratio in the bottom panel, as a function again of the scale factor \( a \). Similarly to \( D_1 \), at early times \( D_2 \) for G3 and \( \Lambda \)CDM are the same. Moreover, at \( a \approx 0.2 \), \( D_2 \) in G3 increases much faster than in \( \Lambda \)CDM. The difference is more sizable than in \( D_1 \), about 60% for \( a = 1 \). Since the growth functions are enhanced, cubic Galileon predicts stronger galaxy clustering than \( \Lambda \)CDM. In Figure 4, we show the ratio of \( D_2/D_1^2 \). This ratio is exactly 3/7 in the Einstein-de Sitter universe. The extra clustering power in G3 comes from a larger effective gravitational constant \( G_{	ext{eff}} \) compared to the standard Newton constant \( G_N \).

\[
\delta_i^{(1)}(\vec{k}, a_0) = -\delta_i^{(1)}(\vec{k}, a_0),
\]

(46)

\[
S_{ij}^{(2)}(\vec{k}, a_0) = \frac{1}{2} \int_{k_1 = \vec{k}} \left[ 1 - \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_1^2 k_2^2} \right] \delta_i^{(1)}(a_0) \delta_j^{(1)}(a_0).
\]

(47)

The initial time \( a_0 \) is arbitrary and related to the normalisation choice for the linear power spectrum. PINOCCHIO chooses to normalise at \( a_0 = 1 \) so that \( D_1(a_0) = 1 \). For convenience, one can define nor-
malised growth functions

\[ D'_1(a) = \frac{D_1(a)}{D_1(a = 1)} \quad D'_2(a) = \frac{D_2(a)}{D_2(a = 1)}. \] (48)

Then the positions of dark matter particles then are given by

\[ \vec{x}(a) = \vec{q} + D'_1(a) \vec{S}^{(1)}(\vec{q}, a_0) + D'_2(a) \vec{S}^{(2)}(\vec{q}, a_0), \] (49)

where \( \vec{S}^{(i)}(\vec{q}, a) \) are the Fourier transforms of the displacements of Eq. 46.

Ellipsoidal collapse is used to predict the collapsing time of each dark matter particle. The dynamics of ellipsoidal collapse can be completely described by following the evolution of the three principal axes of the ellipsoid (Bond & Myers 1996). The physical length of ellipsoid semi axes is \( r_i = a_i q \), where \( q \) is the comoving radius of the “Lagrangian sphere” (a sphere concentric with and sharing the same mass as the ellipsoid, but with density equal to the background density) and \( a_i \) denotes the time evolution of each axis. By enforcing mass conservation we can write \( a_i^3 q_i \rho_{m,e} = a_1 a_2 a_3 q^3 \rho_{m,e} \), where \( \rho_{m,e} \) is the density within the ellipsoid, so that the nonlinear overdensity \( \delta \) is

\[ \delta = \frac{a_3^3}{a_1 a_2 a_3} - 1. \] (50)

Following the approach of Bond & Myers (1996), we can derive the evolution for the \( a_i \) for the G3 model:

\[ \frac{d^2 a_i}{dt^2} = -\frac{a_i}{a} - 4\pi G\mu \rho_m \left[ \delta + \frac{5}{3} \frac{b'_i(t)}{\delta} \right] a_i, \] (51)

where

\[ b'_i(t) = -\frac{2}{3} a_1(t) a_2(t) a_3(t) \int_{a_i(t)}^{a_i(\infty)} \frac{d\tau}{[a_i^2(t) + \tau]^{1/2} [a_j^2(t) + \tau]^{1/2}}, \] (52)

and the deviation from \( \Lambda \)CDM is encoded in the \( \mu \) function discussed in the previous sections. Eq. (51) is an integro-differential equation, which should in principle be solved for each particle, resulting in a computationally expensive prescription. In its standard version, PINOCCHIO relies on an approximation based on LPT to compute collapse times. Such an approximation however does not hold in the case of models where growth is substantially different than \( \Lambda \)CDM. For our purposes, we adopt the description of the ellipsoid’s evolution presented in Nadkarni-Ghosh & Singhal (2016). This is equivalent to the Bond & Myers (1996) approach, but avoids the integral of Eq. 52 by resorting to a set of nine dimensionless parameters:

\[ \lambda_{a,i} = 1 - \frac{a_i}{a}, \] (53)

\[ \lambda_{v,i} = \frac{1}{H a_i} - 1, \] (54)

\[ \lambda_{d,i} = \delta + \frac{5}{3} \frac{b'_i(t)}{\lambda_{d,i}}. \] (55)

Here \( \lambda_{a,i} \) correspond to the eigenvalues of the deformation tensor and characterize the shape of the ellipsoid, \( \lambda_{v,i} \) describe the deviation of the velocity of the \( i \)th axis from the background Hubble flow and \( \lambda_{d,i} \) correspond to the eigenvalues of the tensor of second derivatives of the gravitational potential. Taking the time derivative of \( \lambda_{a,i}, \lambda_{v,i}, \lambda_{d,i} \), we obtain the second order ordinary differential equations

\[ \frac{d\lambda_{a,i}}{d\ln a} = -\lambda_{v,i}(1 - \lambda_{a,i}), \] (56)

\[ \frac{d\ln \lambda_{a,i}}{d\ln a} = -\frac{3}{2} \mu \Omega_m \lambda_{d,i} - \lambda_{v,i}(2 + \frac{H}{H_T}) - \lambda_{a,i}^2, \] (57)

\[ \frac{d\lambda_{d,i}}{d\ln a} = -(1 + \delta)(\delta + \frac{5}{2})^{-1}(\lambda_{d,i} + \frac{5}{6}) \sum_{j=1}^3 \lambda_{a,j} + \] (58)

\[ \sum_{j \neq i} \left[ \lambda_{d,j} - \lambda_{d,i} \right] \left[ \lambda_{a,j}^2 - \lambda_{a,i}^2 \right] - \lambda_{d,i}^2 \delta, \] where \( \delta = \lambda_{d,1} + \lambda_{d,2} + \lambda_{d,3} \). The initial conditions for this set of equations are

\[ \lambda_{a,i} = \frac{D'_{1,im} \lambda_{d,i}}{D'_{1,im} \lambda_{d,i} - 1}, \] (59)

\[ \lambda_{v,i} = \frac{D'_{1,im} \lambda_{d,i}}{D'_{1,im} \lambda_{d,i} - \lambda_{a,i}}, \] (60)

\[ \lambda_{d,i} = \frac{D'_{1,im} \lambda_{d,i}}{D'_{1,im} \lambda_{d,i} - \lambda_{a,i}}. \] (61)

where \( \lambda_{d,i} \) is the eigenvalue of \( -S^{(1)}_{ij}(\vec{k}, a_0) \), and \( D'_{1,im} \) is the normalized growth factor \( D'_1 \) at an initial time \( a \sim 10^{-5} \). At this initial time the Zel’dovich approximation is accurate enough, hence we can compute the 1st order initial displacement field \( S^{(1)}_{ij}(\vec{k}, a_0) \) from the realisation of the density contrast. Then we diagonalise \( S^{(1)}_{ij}(\vec{k}, a_0) \) to get its eigenvalues. Finally, by solving Eqs. (56) we can get the collapse time for each particle. The collapse time is defined as the moment of collapse of the first axis of the ellipsoid collapsed, \( \lambda_{a,i} \rightarrow 1 \).

Collapsed particles may become part of dark matter halos or of the filament network that connects them. Halos are constructed with an algorithm that mimics their hierarchical formation: (i) for each collapsing particle the code checks their 6 neighbours in Lagrangian space; (ii) a particle that collapses before its neighbours becomes a 1-particle halo; (iii) a collapsing particle may accrete on a halo if it "touches" it in Lagrangian space. The accretion condition is satisfied if, after the halo and the particle are moved using Eq. (49), their distance is less than a threshold distance (discussed below); (v) collapsed particles that don’t accrete on halos become filaments, and may accrete later if a neighbour is accreted to a halo; (vi) halo mergers are checked each time a collapsing particle "touches" two halos.
the merger takes place if the halos, moved again with Eq. (49), get nearer than a threshold distance.

In the PINOCCHIO code, the threshold distance for the collapsed particles being accreted into halos is defined as

\[ d_{\text{hl}}^2 = \begin{cases} (f_s R_e)^2 + (f_{200} R_e)^2, & D_1' \sigma \leq \sigma' \ , \\ (f_s R_e)^2 [1 + f_{\text{ra}}(D_1' \sigma - \sigma')]]^2 + (f_{200} R_e)^2, & D_1' \sigma > \sigma' \ , \end{cases} \tag{58} \]

where \( R = (M_h)^{1/3} \), and \( M_h \) is the halo mass. The threshold distance for merging halos is defined as

\[ d_{\text{hl}}^2 = \begin{cases} (f_m R_{\text{lar}} e)^2 + (f_{200} R_{\text{lar}} e)^2, & D_1 \sigma \leq \sigma' \ , \\ (f_m R_{\text{lar}} e)^2 [1 + f_{\text{rma}}(D_1' \sigma - \sigma')]]^2 + (f_{200} R_{\text{lar}} e)^2, & D_1' \sigma > \sigma' \ , \end{cases} \tag{59} \]

where \( R_{\text{lar}} = (M_{h,\text{lar}})^{1/3} \), with \( M_{h,\text{lar}} \) is the mass of the larger halo, \( \sigma \) is the variance of the linear density contrast on the grid, and is a function of time. \( \sigma' \) is a free parameter that controls the change of the "virial radius", and \{e.g., \( f_s, f_m, f_{\text{ra}}, f_{\text{rma}}, f_{200} \)\} are additional free parameters that have been calibrated with N-body simulation in \( \Lambda \)CDM. Due to the enhancement of gravitational force in G3, the accretion and merging conditions are easier to be satisfied in G3, even for the same halo mass. Via the Poisson equation, one can absorb the gravitational enhancement into the redefined density and eventually the mass:

\[ \frac{\dd^2 \Psi}{a^2} = 4\pi G \rho_0 = 4\pi G \delta \rho_{\text{m eff}} \ , \tag{60} \]

where \( \delta \rho_{\text{m eff}} = \mu \delta \rho_0 \). The true halo mass is \( M_h = \rho_0 \int_0^{r_{200}} 4\pi r^2 \delta \rho_0 dr \). After the redefinition, the effective halo mass reads

\[ M_{h,\text{eff}} = \rho_0 \int_0^{r_{200}} 4\pi r^2 \delta \rho_{\text{m eff}} dr = \rho_0 \int_0^{r_{200}} 4\pi r^2 \mu \delta \rho_0 dr \ . \tag{61} \]

Then one can simply replace the effective masses computed with the above formula in the expression for \( R \) and \( R_{\text{lar}} \). On linear scales, \( \mu \) is independent of radius. From Eq. (61), one can read \( M_{h,\text{eff}} = \mu^{L}(a) M_h \). On the nonlinear scale, \( \mu^{NL}(a, \delta) \) also depends on the radius or equivalently on the local density. For simplification, we fix \( \delta = 200 \), then \( M_{h,\text{eff}} = \mu^{NL}(a, 200) M_h \). As a consequence, the 5th force is screened in virialised objects. The output halo mass is computed as \( N \times m_p \), where \( N \) is the number of particles within the halo and \( m_p = 4.62 \times 10^8 M_\odot / h \) is the particle mass.

### 5 RESULTS

We extend the standard PINOCCHIO algorithm according to what described in the previous sections. The code, dubbed as "G3-PINOCCHIO", will soon be available in the official repository. In order to compare our results with full N-body simulations, we choose the box size, number of particles and cosmological parameters as in Barreira et al. (2013a, 2014a). We run "G3-PINOCCHIO" in a box with size 200 Mpc/h with 512^3 particles. The cosmological parameters we adopt are \( \Omega_{m,0} = 0.279 \), \( \Omega_{\Lambda,0} = 0.721 \), \( h = 0.731 \), \( \sigma_8 = 0.997 \), \( n_s = 0.953 \), both for G3 and \( \Lambda \)CDM. In what follows, we describe three different prescriptions for the G3 model, namely “linearG3”, “vainG3” and “grG3”. For all three G3 models, we compute the displacement fields of dark matter particles with 2LPT as presented in Section 3. For “linearG3”, we compute the collapse times with the linear expression for the gravitational slip function, \( \mu^L(a) \), with no screening. For “vainG3”, we use the Vainshtein screened gravitational slip function, \( \mu^{NL}(a, \delta) \). In “grG3” we compute ellipsoidal collapse as in GR: we choose this case as benchmark to demonstrate the effectiveness of the screening mechanism; in fact, the “grG3” case can be seen as an extreme case of screening. A similar approach is taken in Barreira et al. (2013a, 2014a), where the authors define a “Linear model”, equivalent to our “linearG3”, and a “Full model”, same as our “vainG3”. However, their “QCDM model” differs from our “grG3”; the former only modifies the background evolution, while in the latter we modify the 2LPT displacements. As a consequence, our “grG3” run already includes the power spectrum enhancement predicted by linear theory, i.e. the amplitude of the power spectrum on large scales is the same in “grG3” and “linearG3”, “vainG3”.

In Fig. 5 we plot the distribution of collapse times for each particle in a smaller simulation with 200^3 particles and 500 Mpc/h box size. The upper panel shows the number of collapsed particles for each redshift bin, while the lower panel shows the cumulative numbers. The red, pink, cyan and purple shaded regions denote respectively \( \Lambda \)CDM, “grG3”, “linearG3” and “vainG3” models. From the zoom-in sub-panel, we see that at redshift \( z = 0 \) the cumulative collapsed
particle numbers in ΛCDM and “vainG3” are almost the same. This is because we normalize the linear power spectra of both ΛCDM and G3 models with the same linear perturbation amplitude at $z = 0$. From the top panel of Fig. 5, one can see that the ΛCDM collapsing rate at low redshift is lower than the one of all G3 models. On the contrary, the ΛCDM model has more collapsed particles above redshift $\sim 1.8$. This is because $D_{\text{lin}}'$ for ΛCDM is larger than that for G3 by 10–15% (as shown in Fig. 3), thus the ellipsoidal collapse for ΛCDM begins with larger density perturbations. This is again an artifact due to the chosen normalization setup. In the same figure, the “vainG3” collapsing rate is lower than the one of “linearG3” between redshifts 0.5 and 0. This is due to the $\mu$ going back to unity at late times, resulting in a weaker gravitational force with respect to the cases without screening. As an extreme screening limit, “grG3” further suppresses gravitational collapse.

The input linear power spectrum can be computed with an Einstein-Boltzmann solver such as CAMB/CLASS (Lewis et al. 2000; Lesgourgues 2011) or using the analytic fitting formula of Eisenstein & Hu (1998). To suppress sample variance in the power spectrum, the moduli of Fourier-space modes of the linear density field are fixed to their expectation value given by the power spectrum, while phases are randomly distributed between 0 and 2$\pi$. Since the linear growth rate in G3 is larger than the one of ΛCDM, to reach the same final amplitude the G3 model has to start from a more uniform density initial condition. For this reason, the high redshift matter power spectra in G3 are lower than those in ΛCDM cases.

In Fig. 6 we show the relative differences in the cumulative mass function of “linearG3” (green lines) and “vainG3” (blue lines) with respect to our benchmark, “grG3”. The role of the Vainshtein mechanism can clearly be seen in the high mass end, where the fifth force is screened and the “vainG3” case shows a smaller deviation from “grG3” than “linearG3”.

Additionally, we compute the linear halo bias. We use the PowerI4 package$^8$ to read the halo catalogues and compute the halo power spectra at different redshifts, then we compute the square root of the ratio between the halo power spectrum and the linear matter power spectrum for $k < 0.2$, namely the linear halo bias.

In Fig. 7 we compare our results for the halo bias with the N-body simulations of Barreira et al. (2014a). The cross symbols are our results, the solid lines are the best-fitting Sheth-Tormen formalism to the cumulative halo mass function from simulations, and the triangles with error bars are the measurement of halo bias directly from the simulations. Different colors mark the different approximations for collapse times, as detailed in the legend. It can be seen that our results, despite being computed with a completely different method, match remarkably well the general trend of the N-body simulations.

Finally, we plot our result for the matter power spectrum in Fig. 8. For PINOCCHIO, this is obtained by displacing with 2LPT particles that do not belong to halos, while halo particles are distributed around the halo center of mass assuming an NFW profile (Navarro et al. 1997, 1996). These power spectra are computed with the Pylians package$^9$. For comparison, we also show the simulation results with solid curves. The relative differences are with respect to the “grG3” model in our case, and with respect to the “QCDM” model in the case of the simulations. One can see that, at large scales, the two results are discrepant: in particular, the Barreira et al. (2013a) ratios also show the amplitude difference in the power spectra that is due to the different linear evolution between G3 and ΛCDM, while ours do not. As mentioned previously, this is due to the fact that our “grG3” is not exactly the same as “QCDM”, since we modify both the background and the Lagrangian perturbations up to the 2nd order while the “QCDM” model in Barreira et al. (2013a) only modifies the background evolution. For this reason, we cannot do a quantitative comparison with Barreira et al. (2013a), although it can be seen that our results reproduce the general trend of the simulations. We report in Appendix A a more detailed analysis of our halo catalogues. We compare the halo properties of matched and unmatched catalogues, including the halo mass distributions, the halo mass functions and halo power spectra.

6 CONCLUSIONS

This work presents 1st and 2nd order Lagrangian perturbation theory for the cubic Galileon model, as well as its ellipsoidal collapse dynamics. As in the case of ΛCDM, both the 1st and 2nd order growth factors of the cubic Galileon model are scale independent, despite the extra scalar field which drives the accelerated expansion is effectively massless. Its Compton wavelength is indeed on the
Hubble horizon scale: below this scale, the modifications of gravity keep constant in space and vary slowly with time. Since both growth functions are larger than their CDM counterparts, as shown in Fig. 3, clustering in enhanced. In particular, the maximum enhancement of gravity happens at redshift zero, when the effective gravitational constant is about twice the Newton constant. Once we lapse to the G3 model, we create a new branch of the growth functions are larger that their QCDM simulations, which does not correspond to our "grG3" case exactly since it only includes modifications to the background evolution.

Using both the extension of standard 2LPT and ellipsoidal collapse to the G3 model, we create a new branch of the PINOCCHIO code, dubbed as G3-PINOCCHIO: our extension provides a fast tool to generate approximated dark matter halo catalogues with the G3 modified gravity model. We run the G3-PINOCCHIO code in a box with size 200 Mpc/h and 512^3 particles and study the properties of the obtained halo catalogues at different redshifts. To illustrate the effect of Vainshtein screening, we run three different implementations of ellipsoidal collapse, namely “linearG3”, “vainG3” and “grG3”. From these realisations we compute the cumulative mass function, the linear halo bias and the matter power spectrum. We find that, as for other types of modified gravity, the Vainshtein screening mechanism in cubic Galileon also suppresses the extra gravitational force in the high mass end of the mass function. We compute the linear halo bias and the matter power spectrum and perform a qualitative comparison to N-body simulations, showing that we can reproduce all trends remarkably well. Given the significantly reduced computational time required to generate halo catalogues with PINOCCHIO with respect to more computationally expensive N-body simulations, our implementation provides an optimal tool for the fast generation of large sets of realisations. Additionally, the code can be readily extended to include other MG models that feature Vainshtein screening.

Figure 7. Linear halo bias at a = 1 (top panel) and a = 0.8 (bottom panel), as a function of the halo mass. We mark our predictions with cross symbols, and compare to the results of Barreira et al. (2014a) showing their simulations results with triangles with errorbars, and their prediction based on the Sheth-Tormen formula with solid lines. We show the results for the “grG3” case, the “linearG3” case and the “vainG3” case with red, green and blue lines and symbols respectively.

Figure 8. Matter power spectrum relative differences at a = 1 (top panel) and a = 0.8 (bottom panel). The green lines mark the ratio of the “linearG3” simulations to the “grG3” case, while we show the “vainG3” case in blue. Our results, measured from PINOCCHIO snapshots, are plotted as dashed curves. The shadowed areas mark the 1σ errors on the measurements. We also show the results from Barreira et al. (2013a) with solid lines, remarking that these ratios are obtained to their QCDM simulations, which does not correspond to our “grG3” case exactly since it only includes modifications to the background evolution.
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DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding author.

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APPENDIX A: HALO CATALOGUES

We describe here a more detailed analysis of the halo catalogues. Since we use the same initial conditions for the different implementations of the modified gravity model, namely “linearG3” and “vainG3”, we can match the IDs of halos that are formed in both simulations. We call them “matched” halos, while halos that are only present in one of the two realisations are called “unmatched”. In Fig. A1 we show the halo number density for the matched and unmatched halos. One can see that most halos appear in both realisations, while only ~1% of the halos in “linearG3” are unmatched and ~10% halos in the “vainG3” are unmatched. In Fig. A2 we show the mass distribution of the matched halos for the “linearG3” and “vainG3” prescriptions. Because the modification to the gravitational slip is stronger in the linear case with respect to the nonlinear “vainG3” one, the halo masses shift to the massive direction in the “linearG3” case (i.e. they mostly sit below the black solid line). In Fig. A3 we show the power halo spectra computed for the whole catalog, and for the catalogue split in matched and unmatched halos. The halo power spectra are computed with Power14, considering all halos with at least 100 particles. It can clearly be seen that the halo power spectra of the matched halos are very similar for both “linearG3” and “vainG3”, while they are significantly different for the unmatched halos. One can conclude that the differences in the final

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Figure A1. Halo number density $dN/dM$ as a function of the halo mass at $a = 1$ (top panel) and $a = 0.8$ (bottom panel). We show results for the “linearG3” and “vainG3” cases, for the whole halo catalogue (marked as all, shown with green and blue lines) as well as for the catalogue split in matched (purple and cyan lines) and unmatched (red and orange lines) halos.

Figure A2. Mass distribution of the matched halos at $a = 1$ (top panel) and $a = 0.8$ (bottom panel) for the “linearG3” versus “vainG3” cases.

Halo power spectra actually come from the unmatched halos. Finally, in Fig. A4 we show the relative differences in the halo power spectra with respect to the “grG3” model. At $a = 1$ we find a difference of $\sim 20\%$ on all scales for “linearG3”. Such difference is strongly reduced for “vainG3”, highlighting once again the effectiveness of the Vainshtein mechanism in screening the MG fifth force. Similar results can also be seen for $a = 0.8$, with differences of $\sim 10\%$ and $\sim 2\%$ respectively for “linearG3” and “vainG3”.

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Figure A3. Halo power spectra at $a = 1$ (top panel) and $a = 0.8$. As above, we show results for the whole catalog, the matched and unmatched halos.

Figure A4. Relative differences in the halo power spectra at $a = 1$ and $a = 0.8$, shown for the “linearG3” prescription (green line) and the “vainG3” prescription (blue line). In both cases the ratio is computed with respect to the “grG3” case.