Abstract: We consider the nonlinear fractional Langevin equation involving two fractional orders with initial conditions. Using some basic properties of Prabhakar integral operator, we find an equivalent Volterra integral equation with two parameter Mittag–Leffler function in the kernel to the mentioned equation. We used the contraction mapping theorem and Weissinger’s fixed point theorem to obtain existence and uniqueness of global solution in the spaces of Lebesgue integrable functions. The new representation formula of the general solution helps us to find the fixed point problem associated with the fractional Langevin equation which its contractivity constant is independent of the friction coefficient. Two examples are discussed to illustrate the feasibility of the main theorems.

Keywords: fractional Langevin equation; Mittag–Leffler function; Prabhakar integral operator; existence; uniqueness

1. Introduction

Dynamical behavior of physical processes are usually represented by differential equations. If the model of physical system in some ways possesses a memory and hereditary properties, for instance, viscoelastic deformation [1], anomalous diffusion [2], stock market [3], bacterial chemotaxis [4] and complex networks [5], relaxation in filled polymer networks [6], relaxation and reaction kinetics of polymers [7], description of mechanical systems subject to damping [8], Behavior of Biomedical Materials [9]; the corresponding models can be described by the fractional differential equations.

Langevin equation is a fundamental theory of the Brownian motion to describe the evolution of physical phenomena in fluctuating environments [10,11]. Fractional Langevin equation as a generalization of classical one gives a fractional Gaussian process parametrized by two indices, which is more flexible for modeling fractal processes [12–16].

The virtually simultaneous development of fractional derivatives, various generalizations of the Langevin equation were proposed and studied by various researchers during recent years. Despite the widespread use of many of the applications [17–22], the fractional Langevin equation is extensively studied in literature both from theoretical and numerical points of view. Authors in [23] studied nonlinear fractional Langevin equation involving two fractional orders in different intervals as a generalized form of three point third order nonlocal boundary value problem of nonlinear ordinary differential equations. In [24], the authors have studied fractional Langevin equations with nonlocal integral boundary conditions. Recently, anti-periodic boundary value problem for Langevin equation involving two fractional orders has been studied in [25]. Existence and uniqueness results for coupled and uncoupled systems of fractional Langevin equations of Riemann-Liouville and Hadamard types has been discussed in [26]. Guo et al. [27] gave an efficient numerical method for solving the fractional
Langevin equation with or without an external force. Some more recent work on Langevin equation can be found in [28–37].

In the current paper, we mainly focus on the existence and uniqueness result for the fractional Langevin equation involving two fractional orders:

\[
\begin{aligned}
D^{\delta}(D^{\alpha} + \lambda) x(t) &= f(t, x(t)), \quad 0 < t \leq 1, \\
x^{(i)}(0) &= \mu_i, \quad 0 \leq i < l, \\
x^{(i+\alpha)}(0) &= \nu_i, \quad 0 \leq i < n,
\end{aligned}
\]  

(1)

where \( m - 1 < \alpha \leq m, n - 1 < \beta \leq n, l = \max\{m, n\}, m, n \in \mathbb{N} \), \( D^{\alpha} \) is the Caputo fractional derivative, \( x(t) \) is the particle displacement, \( x^{(i+\alpha)}(0) \) equals \( D^{\alpha} D^{\beta} x(0) \), in the sequential sense, \( \lambda \in \mathbb{R} \) is the friction coefficient and \( f : [0,1] \times \mathbb{R} \to \mathbb{R} \) is a given function which represents a noise term.

Based on the criteria specified in [38], the problem (1) is a general form of anomalous systems governed by a generalized Langevin equation with long-range memory. In contrast to the classical Langevin equation, we use \( D^{\beta} x(t) \) and \( D^{\alpha} D^{\beta} x(t) \) instead of the ordinary definition of the velocity and acceleration as the first and second derivatives of the displacement to derive a generalized Langevin equation involving friction memory kernel. For example, if \( 0 < \beta \leq 1, \alpha = 1 \), then according to the standard definition of the Caputo fractional derivative operator, we have a special case of generalized Langevin equation involving friction memory kernel equal to \( \frac{\lambda}{\Gamma(1-\beta)} t^{\beta - 1} \). Based on the calculations in ([39], Section B), in this case, the resulting motion is in fact subdiffusive. Furthermore, it is worth noting that, if \( \alpha + \beta > 2 \), then we do not have any physical meaning for the main problem. For this case, it is only a valuable problem in the theory of fractional differential equations as a sequential fractional differential equation with initial conditions.

As we have seen in the papers cited above about analysis of fractional Langevin equation, using various classical fixed point theorems is a common and useful technique for obtaining the existence and uniqueness results for fractional Langevin equation involving different initial or boundary conditions. In the mentioned papers, the contractivity constant of the fixed point problem associated with the fractional Langevin equation depended on the friction coefficient \( \lambda \). For example, in the obtained existence and unique results in [33,34], the contractivity constants \( R_1, R_2 \) satisfy the following conditions

\[
R_1 = \sup_{0 \leq t \leq 1} \left( \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} a(s) \, ds + \frac{|\lambda|}{\Gamma(\alpha+1)} \right) < 1,
\]

(2)

and

\[
R_2 = k \left( \frac{\|a\|}{\Gamma(\alpha+\beta)} \right) + \frac{|\lambda|}{\Gamma(\alpha)} < 1,
\]

(3)

where \( k = \left( \frac{1}{1-q(1-\alpha)} \right)^{\frac{1}{q}} \) and \( p^{-1} + q^{-1} = 1 \), respectively. As stated in relations (2) and (3), the contractivity constants \( R_1, R_2 \) depend on the friction constant \( \lambda \). Therefore, from (2), we can not discuss the problems involving the friction constant \( |\lambda| \geq \Gamma(\alpha+1) \). Similarly, from (3), we can not study the problems involving the friction constant \( |\lambda| \geq \Gamma(\alpha) \). Note that \( 0 < 1 - q(1-\alpha) < 1 \). Therefore, we cannot discuss the existence and uniqueness of solutions for the problems involving large friction coefficient \( \lambda \). In this paper, we strive to overcome this major limitation. First we propose a new construction of the general solution for the Equation (1) using two parameter Mittag–Leffler functions and some of the basic properties of Prabhakar operator. This is done in Section 2. Then we obtain a new existence and uniqueness results under some weak conditions by using contractive mapping theorem and Weissinger’s fixed point theorem. This is content of Section 3. Two examples are given in Section 4 to illustrate our results.
2. Preliminaries and Auxiliary Results

In the following section, we apply some technical calculations related to fractional calculus to build a new general solution corresponding to initial value problem (1) which provides an extremely powerful tool for the proof of the main result. Furthermore, we present some preliminaries and notations regarding fractional calculus for the reader’s convenience. For details, see [40–46].

**Definition 1.** The Riemann-Liouville fractional integral of order $\alpha > 0$ for the function $x : [0, 1] \to \mathbb{R}$, $x \in L^1[0, 1]$ is defined as

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}x(s)ds.$$  

**Definition 2.** The Caputo fractional derivative of order $\alpha > 0$ of a function $x : [0, 1] \to \mathbb{R}$ is defined as

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1}x^n(s)ds,$$

where $n - 1 < \alpha \leq n$ and $n \in \mathbb{N}$, provided that the right-hand-side integral exists and is finite.

**Definition 3 ([46]).** Let $\alpha, \beta > 0$, $\lambda \in \mathbb{R}$ and $x \in L^1[0, 1]$. The Prabhakar integral can be written as

$$E[\alpha, \beta, \lambda]x(t) = \int_0^t (t-s)^{\beta-1}E_{\alpha, \beta}(\lambda(t-s)\alpha)x(s)ds,$$

where $E_{\alpha, \beta}(\cdot)$ is the so-called two parameter Mittag-Leffler function, defined by

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}$$

and $E_{\alpha}(\cdot) = E_{\alpha,1}(\cdot)$. Like the Mittag–Leffler function $E_{\alpha}(z)$, $E_{\alpha, \beta}(z)$ is an entire function of order $\frac{1}{\alpha}$.

**Lemma 1 ([46]).** Let $\alpha, \beta, \gamma \geq 0$ and $x \in L^1[0, 1]$. Then

$$I^\gamma E[\alpha, \beta, \lambda]x(t) = E[\alpha, \beta, \lambda]I^\gamma x(t) = E[\alpha, \beta + \gamma, \lambda]x(t),$$

holds almost everywhere on $[0, 1]$. Furthermore, $E[\alpha, \beta, \lambda]t^\gamma = \Gamma(\gamma + 1)t^{\gamma+\beta}E_{\alpha, \beta}(\lambda t^\alpha)$.

**Lemma 2.** The general solution of (1) is given by

$$x(t) = \sum_{j=0}^{m-1} \mu_j t^j E_{\alpha+j}(-\lambda t^\alpha) + \sum_{i=0}^{n-1} v_i t^{\alpha+i} E_{\alpha, \alpha}(-\lambda t^\alpha) + \sum_{i=0}^{n-1} \mu_i t^i \left( \frac{1}{\Gamma(i+1)} - E_{\alpha}(-\lambda t^\alpha) \right)$$

$$+ \int_0^t (t-s)^{\alpha+\beta-1} E_{\alpha, \alpha+\beta}(-\lambda(t-s)^\alpha) f(s, x(s))ds.$$  

**Proof.** Let $x(t)$ be a solution of the problem (1), we have

$$(D^\alpha + \lambda) x(t) = \sum_{i=0}^{n-1} a_i t^i + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s, x(s))ds.$$
By using the initial conditions for the initial problem (1), we find that \( a_i = \frac{\nu_i + \lambda \mu_i}{\Gamma(i+1)} \), \( i = 0, 1, \ldots, n-1 \). Therefore, we have

\[
(D^a + \lambda) x(t) = \sum_{i=0}^{n-1} v_i + \lambda \mu_i t^i + \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds.
\]

(5)

Now, using the approach of Kilbas et al. ([40], Section 3.1), the solution of the Equation (5) is given by the following expression

\[
x(t) = \sum_{i=0}^{m-1} \mu_i t^i E_{a,i+1}(-\lambda t^a) + \int_0^t (t-s)^{\alpha-1} E_{a,a}(-\lambda(t-s)^{a}) \left( \sum_{i=0}^{n-1} v_i + \lambda \mu_i s^i + I^\beta f(\cdot, x(\cdot))(s) \right) ds.
\]

(6)

Note \( E_{a,a}(z) = a E'(\alpha, z) \) and so \((t-s)^{\alpha-1} E_{a,a}(-\lambda(t-s)^{a}) = \frac{d}{ds} \left( \frac{1}{\lambda} E_a(-\lambda(t-s)^{a}) \right) \). This yields that

\[
\int_0^t (t-s)^{\alpha-1} E_{a,a}(-\lambda(t-s)^{a}) s^i ds = \frac{1}{\lambda} \left( s^i E_a(-\lambda(t-s)^{a}) \right)^{t}_{0} - \int_0^t E_a(-\lambda(t-s)^{a}) s^{i-1} ds,
\]

(7)

for each \( i \in \mathbb{N} \). Applying Lemma 1 to the second term in the right-hand side of (7), we conclude

\[
\int_0^t (t-s)^{\alpha-1} E_{a,a}(-\lambda(t-s)^{a}) s^i ds = \frac{1}{\lambda} \left( t^i \Gamma(i+1) E_a(-\lambda t^{a}) \right),
\]

for each \( i \in \mathbb{N} \). Therefore

\[
x(t) \bigg| \bigg. = \sum_{i=0}^{m-1} \mu_i t^i E_{a,i+1}(-\lambda t^a) + \sum_{i=0}^{n-1} v_i + \sum_{i=0}^{n-1} \lambda \mu_i t^i E_{a,a}(-\lambda(t-s)^{a}) s^i ds
\]

\[
+ \sum_{i=0}^{n-1} \lambda \mu_i t^i \int_0^t (t-s)^{\alpha-1} E_{a,a}(-\lambda(t-s)^{a}) s^i ds + \int_0^t (t-s)^{\alpha-1} E_{a,a}(-\lambda(t-s)^{a}) \left( t^\beta f(\cdot, x(\cdot))(s) \right) ds
\]

\[
= \sum_{i=0}^{m-1} \mu_i t^i E_{a,i+1}(-\lambda t^a) + \sum_{i=0}^{n-1} v_i t^{i+1} E_{a,a}(-\lambda t^a) + \sum_{i=0}^{n-1} \lambda \mu_i t^i \int_0^t (t^i \Gamma(i+1) E_a(-\lambda t^{a}) - E_a(-\lambda t^{a}))
\]

\[
+ \int_0^t (t-s)^{\alpha-1} E_{a,a}(-\lambda(t-s)^{a}) f(s, x(s)) ds,
\]

which is the desired result.

Now, we state Weissinger’s fixed point theorem ([41], Theorem D.7) as a generalization of the so-called contraction mapping theorem which is needed to prove Theorem 3.

Theorem 1. Let \( X \) to be a Banach space and let \( \theta_n \geq 0 \) for every \( n \in \mathbb{N} \cup \{0\} \) such that \( \sum_{n=0}^{\infty} \theta_n \) converges. Furthermore, assume \( T : X \to X \) is a nonlinear mapping which satisfies the inequality \( \| T^n x - T^n y \| \leq \theta_n \| x - y \| \) for every \( n \in \mathbb{N} \) and every \( x, y \in X \). Then, \( T \) has a unique fixed point \( x^* \). Moreover, the sequence \( \{ T^n x_0 \}_{n=0}^{\infty} \) converges to this fixed point \( x^* \), for any \( x_0 \in X \).

3. Existence and Uniqueness

Our aim in the following section is to deeply investigate the existence and uniqueness results for the main problem (1) in the Lebesgue space.

Theorem 2. Let \( \max\{1, \frac{1}{\alpha+q}\} \leq p \leq \infty, \ p^{-1} + q^{-1} = 1 \) and the following hypotheses 1–3 hold:
Hypothesis 1. \( f(t, 0) \in L^q[0, 1] \).

Hypothesis 2. There exists nonnegative \( a \in L^p[0, 1] \) such that \( |f(t, x_2) - f(t, x_1)| \leq a(t)|x_2 - x_1| \), for each \( t \in [0, 1] \) and \( x_1, x_2 \in \mathbb{R} \).

Hypothesis 3. \( R := \frac{M_1 \| \xi \|_p}{(1 - q + q(\alpha + \beta))^q} < 1 \) where \( M_1 = \sup_{t \in [0,1]} |E_{\alpha, \alpha + \beta}(-\lambda t^\alpha)| \).

Then the integral Equation (4) has a unique solution in \( L^q[0,1] \).

Proof. We define the operator \( T \) as follows:

\[
Tx(t) = \int_0^t (t-s)^{\alpha + \beta - 1} E_{\alpha, \alpha + \beta}(-\lambda (t-s)^\alpha) f(s, x(s)) \, ds + \phi(t),
\]

(8)

where

\[
\phi(t) = \sum_{j=0}^{m-1} \mu_j t^j E_{\alpha, \alpha + \beta}(-\lambda t^\alpha) + \sum_{i=0}^{n-1} \nu_i t^i E_{\alpha, \alpha + \beta}(-\lambda t^\alpha) + \sum_{i=0}^{n-1} \mu_i t^i \left( \frac{1}{\Gamma(i+1)} - E_{\alpha}(-\lambda t^\alpha) \right).
\]

(9)

Let \( M(t) = t^{\alpha + \beta - 1} E_{\alpha, \alpha + \beta}(-\lambda t^\alpha) \), \( M_1 = \sup_{t \in [0,1]} |E_{\alpha, \alpha + \beta}(-\lambda t^\alpha)| \) and \( M_2 = \sup_{t \in [0,1]} |\phi(t)| \). Note that the generalized Mittag–Leffler functions are entire functions \([43,44]\). For each \( x \in L^q[0,1] \), we have

\[
|Tx(t)| \leq \left| \int_0^t M(t-s) f(s, x(s)) \, ds \right| + M_2
\]

\[
\leq \int_0^t |M(t-s)||f(s,0)| + |M(t-s)||f(s, x(s)) - f(s,0)| \, ds + M_2
\]

\[
\leq \int_0^t |M(t-s)|^{\frac{1}{p}} |f(s,0)| + |M(t-s)|^{\frac{1}{p}} ds + \int_0^t |M(t-s)||x(s)||a(s)| \, ds + M_2
\]

\[
\leq \left( \int_0^t |M(t-s)| \, ds \right)^{\frac{1}{p}} \left( \int_0^t |f(s,0)| \, ds \right)^{\frac{1}{q}} + M_1 \left( \int_0^t |x(s)|^q \, ds \right)^{\frac{1}{q}} \left( \int_0^t |a(s)|^p \, ds \right)^{\frac{1}{q}} + M_2
\]

\[
\leq M_1 \left( \int_0^t \frac{|f(s,0)|^q}{(t-s)^{1-\alpha + \beta}} ds \right)^{\frac{1}{q}} + M_1 \left( \int_0^t \frac{|x(s)|^q}{(t-s)^{1-\alpha + \beta}} ds \right)^{\frac{1}{q}} + M_2
\]

\[
= M_1 \left\| f(s,0) \right\|_q + M_1 \left\| x \right\|_q + M_2.
\]

Therefore, we have

\[
\|Tx\|_q \leq \frac{M_1}{(\alpha + \beta)^{\frac{\alpha}{p}}} \left( \int_0^t \frac{|f(s,0)|^q}{(t-s)^{1-\alpha + \beta}} ds \right)^{\frac{1}{q}} + M_1 \left\| a \right\|_p \left( \int_0^t \frac{|x(s)|^q}{(t-s)^{1-\alpha + \beta}} ds \right)^{\frac{1}{q}} + M_2
\]

\[
= \frac{M_1}{(\alpha + \beta)^{\frac{\alpha}{p}}} \left( \int_0^1 \frac{|f(s,0)|^q}{(t-s)^{1-\alpha + \beta}} ds \right)^{\frac{1}{q}} + M_1 \left\| a \right\|_p \left( \int_0^1 \frac{|x(s)|^q}{(t-s)^{1-\alpha + \beta}} ds \right)^{\frac{1}{q}} + M_2
\]

\[
= \frac{M_1}{(\alpha + \beta)^{\frac{\alpha}{p}}} \left\| f(s,0) \right\|_q + \frac{M_1}{(1 - q + q(\alpha + \beta))^q} \left\| a \right\|_p \left\| x \right\|_q + M_2.
\]
Theorem 3. Let \( E \) and \( x \) for all \( z > 0 \), that is, \( E \) Remark 1. initial problem (1). By the Banach contraction principle, \( T \) Mathemat

\[
|T x(t) - T y(t)| \leq \int_0^t |M(t-s)||f(s,x(s)) - f(s,y(s))| \, ds \\
\leq \int_0^t |M(t-s)||x(s) - y(s)||a(s)| \, ds \\
\leq M_1 \left( \int_0^t \frac{|x(s) - y(s)|^q}{(t-s)^{q(a + \beta)}} \, ds \right)^{\frac{1}{q}} \left( \int_0^t |a(s)|^p \, ds \right)^{\frac{1}{p}} \\
= M_1 ||a||_p \left( \int_0^t \frac{|x(s) - y(s)|^q}{(t-s)^{q(a + \beta)}} \, ds \right)^{\frac{1}{4}} ,
\]
which implies that

\[
||T x - T y||_q = M_1 ||a||_p \left( \int_0^t \int_0^1 \frac{|x(s) - y(s)|^q}{(t-s)^{q(a + \beta)}} \, ds \, dt \right)^{\frac{1}{4}} \\
= M_1 ||a||_p \left( \int_0^t \int_s^1 \frac{|x(s) - y(s)|^q}{(t-s)^{q(a + \beta)}} \, dt \, ds \right)^{\frac{1}{4}} \\
= M_1 ||a||_p \left( \int_0^t \frac{1 - q + q(a + \beta)}{1 - q + q(a + \beta)} |x(s) - y(s)|^q \, ds \right)^{\frac{1}{4}} \\
\leq M_1 ||a||_p \frac{1}{(1 - q + q(a + \beta))^{\frac{1}{4}}} ||x - y||_q, \\
= R ||x - y||_q.
\]

Note that \( 1 - q + q(a + \beta) \geq 0 \) because of \( p \geq \frac{1}{a+\beta} \). Therefore, \( T \) is a contraction since \( R < 1 \).

By the Banach contraction principle, \( T \) has a unique fixed point, which is the unique solution of the initial problem (1). \( \square \)

Remark 1. We recall from \([43,44]\) that \( E_{\alpha\beta}(z) \) is completely monotonic function for \( 0 < \alpha \leq 1 \) and \( \beta \geq \alpha \), that is, \( E_{\alpha\beta}(z) \) possesses derivatives \( d^n \frac{d^n}{dz^n} \left( E_{\alpha\beta}(z) \right) \) for all \( n = 0, 1, 2, \cdots \) and \( (-1)^n \frac{d^n}{dz^n} \left( E_{\alpha\beta}(z) \right) \geq 0 \) for all \( z > 0 \). Therefore, \( E_{\alpha\alpha+\beta}(\lambda t\alpha) \leq E_{\alpha\alpha+\beta}(0) = \frac{1}{(a+\beta)} \) for \( \lambda \geq 0 \) and \( 0 \leq t \leq 1 \).

Theorem 3. Let \( 1 \leq q \leq \infty \) and the following hypotheses 4 and 5 hold:

Hypothesis 4. \( f(t,0) \in L^q[0,1] \).

Hypothesis 5. There exists \( L > 0 \) such that \( |f(t,x_2) - f(t,x_1)| \leq L|x_2 - x_1| \), for almost every \( t \in [0,1] \) and \( x_1, x_2 \in \mathbb{R} \).

Then the integral Equation (4) has a unique solution in \( L^q[0,1] \).

Proof. With notations as in the proof of Theroen 2, and using the same arguments, we obtain

\[
|T x(t)| \leq \frac{M_1}{(a+\beta)^\frac{1}{q}} \left( \int_0^t \frac{|f(s,0)|^q}{(t-s)^{1-a-\beta}} \, ds \right)^{\frac{1}{4}} + \frac{M_1 L}{(a+\beta)^\frac{1}{q}} \left( \int_0^t \frac{|x(s)|^q}{(t-s)^{1-a-\beta}} \, ds \right)^{\frac{1}{4}} + M_2,
\]
and therefore,

\[
\|Tx\|_q \leq \frac{M_1}{(a + \beta)^\frac{1}{q}} \left( \int_0^1 \int_0^t |f(s,0)|^q (t-s)^{1-a-\beta} ds dt \right)^\frac{1}{q} + \frac{M_1 L}{(a + \beta)^\frac{1}{q}} \left( \int_0^1 \int_0^t |x(s)|^q (t-s)^{1-a-\beta} ds dt \right)^\frac{1}{q} + M_2
\]

\[
= \frac{M_1}{(a + \beta)^\frac{1}{q}} \left( \int_0^1 \int_s^1 |f(s,0)|^q |t-s|^{1-a-\beta} ds dt \right)^\frac{1}{q} + \frac{M_1 L}{(a + \beta)^\frac{1}{q}} \left( \int_0^1 \int_s^1 |x(s)|^q |t-s|^{1-a-\beta} ds dt \right)^\frac{1}{q} + M_2
\]

\[
\leq \frac{M_1}{(a + \beta)^\frac{1}{q}} (\|f(0,\cdot)\|_q + L\|x\|_q) + M_2,
\]

which yields \( T : L^q[0,1] \rightarrow L^q[0,1] \). On the other hand, for every \( n \in \mathbb{N} \) and for each \( t \in [0,1] \), we have

\[
|T^n x(t) - T^n y(t)| \leq \int_0^1 \left| \int_0^t |f(s, T^{n-1} x(s)) - f(s, T^{n-1} y(s))| \, ds \right| \, dt
\]

\[
\leq M_1 L \int_0^t \int_0^t \left| f(s, T^{n-1} x(s)) - f(s, T^{n-1} y(s)) \right| ds \, dt
\]

\[
\leq (M_1 L)^2 \int_0^t \int_0^t \left| f(s, T^{n-1} x(s)) - f(s, T^{n-1} y(s)) \right| ds \, dt
\]

\[
= \frac{(M_1 L)^2}{(a + \beta)^\frac{1}{q}} \left( \int_0^t \left| f(s, T^{n-1} x(s)) - f(s, T^{n-1} y(s)) \right| ds \right)^\frac{1}{q}
\]

\[
\leq \frac{(1 + \beta)M_1 L^2}{(a + \beta)^\frac{1}{q}} \left( \int_0^t \left| f(s, T^{n-1} x(s)) - f(s, T^{n-1} y(s)) \right| ds \right)^\frac{1}{q}
\]

\[
\leq \frac{(1 + \beta)M_1 L^2}{(a + \beta)^\frac{1}{q}} \left( \int_0^t \left| f(s, T^{n-1} x(s)) - f(s, T^{n-1} y(s)) \right| ds \right)^\frac{1}{q}
\]

\[
\leq \frac{(1 + \beta)M_1 L^2}{(a + \beta)^\frac{1}{q}} \left( \int_0^t \left| f(s, T^{n-1} x(s)) - f(s, T^{n-1} y(s)) \right| ds \right)^\frac{1}{q}
\]

\[
\leq \frac{(1 + \beta)M_1 L^2}{(a + \beta)^\frac{1}{q}} \left( \int_0^t \left| f(s, T^{n-1} x(s)) - f(s, T^{n-1} y(s)) \right| ds \right)^\frac{1}{q}
\]

Therefore, we conclude

\[
\|T^n x - T^n y\|_q \leq \frac{(1 + \beta)M_1 L^2}{(a + \beta)^\frac{1}{q}} \|x - y\|_q,
\]

for every \( n \in \mathbb{N} \) and all \( x, y \in L^q[0,1] \). Now let \( \theta_n = \frac{(1 + \beta)M_1 L^2}{(a + \beta)^\frac{1}{q}} \). From the definition of the generalized Mittag–Leffler functions, we have \( \sum_{n=0}^{\infty} \theta_n = E_{a+\beta} (\Gamma(a + \beta) M_1 L) \) and hence the series \( \sum_{n=0}^{\infty} \theta_n \) converges. Therefore, the existence of the unique fixed point of \( T \) follows from Weissinger’s fixed point Theorem. \( \square \)

4. Illustrative Examples

In this section, some examples are provided to show the applicability of the analytical achievements of the paper.

**Example 1.** Consider the initial value problem

\[
\begin{cases}
D^{\frac{3}{2}} (D^{\frac{1}{2}} + \lambda) x(t) = 1 + t^2 + \frac{\sin t + \arctan x(t)}{2e^{\sqrt{t}}} & 0 < t \leq 1, \\
x(0) = 1, \quad D^{\frac{3}{2}} x(0) = 1.
\end{cases}
\]
Here \( f(t, x) = 1 + t^2 + \frac{\sin t + \arctan x}{2t^2} \), \( \alpha = \frac{1}{2}, \beta = \frac{4}{5} \) and the friction constant \( \lambda \geq 0 \).

Let \( p = q = 2 \). Clearly, \( f(t, 0) = 1 + t^2 + \frac{\sin t}{2t^2} \) and \( f(t, 0) \in L^2[0,1] \). In fact, it is easily seen that \( \|f(t, 0)\|_2 \leq 1 + \left( \frac{1}{2} \right)^2 + \frac{1}{2} \left( \frac{\Gamma(1/4)}{\sqrt{2}} \right)^2 \). On the other hand, \( |f(t, x) - f(t, y)| \leq \frac{1}{2} |x - y| \) with \( a(t) = \frac{1}{2t^2} \). Similarly, we see that \( a \in L^2[0,1] \) and \( \|a\|_2 \leq \frac{1}{2} \left( \frac{\Gamma(1/4)}{\sqrt{2}} \right)^2 \). Further, from Remark 1 it follows that \( M_1 = \sup_{t \in [0,1]} |E^{\alpha, \beta}_{1/2, m}(-\lambda t^\beta)| \leq \frac{1}{\Gamma(\frac{3}{10})} \). Therefore,

\[
R = \frac{M_1 \|a\|_p}{(1 - q + q(\alpha + \beta))^{\frac{1}{2}}} < \frac{1}{\sqrt{1.64226}} = 0.64226 < 1.
\]

Note that the contraction constant \( R \) is independent of friction constant \( \lambda \). Thus, by Theorem 2, the initial value problem (10) has a unique solution in \( L^2[0,1] \).

**Example 2.** Consider the initial value problem

\[
\begin{align*}
D^{\frac{3}{2}} \left( D^{\frac{3}{5}} + \lambda \right) x(t) = g(t) \frac{|x(t)|}{1 + |x(t)|} & \quad 0 < t \leq 1, \\
x(0) = 1, \quad x'(0) = -1, \quad D^{\frac{3}{2}} x(0) = 1,
\end{align*}
\]

\( g \in L^\infty[0,1] \) and the friction constant \( \lambda \in \mathbb{R} \).

Observe that \( f(t, 0) = 0 \) and \( |f(t, x) - f(t, y)| \leq L|x - y| \) for almost every \( t \in [0,1] \) with \( L = \|g\|_\infty \).

Thus, by Theorem 3, the initial value problem (11) has a unique solution in \( L^\infty[0,1] \).

5. Conclusions

In this article, we have considered initial value problem of nonlinear fractional Langevin equation involving two fractional orders. As a first step, by applying the tools of fractional calculus and using some basic properties of Prabhakar integral operator, we build a general structure of solutions associated with our proposed model. Once the fixed point operator equation is available, the existence results are established by means of contraction mapping theorem and Weissinger’s fixed point theorem. Finally, two examples were presented to support the result.

**Author Contributions:** Supervision, H.S.; Writing—review and editing, H.F. and J.J.N. All authors have read and agreed to the published version of the manuscript.

**Funding:** The authors are thankful to the Editor(s) and reviewers of the manuscript for their helpful comments. The work of H. Fazli and H. Sun was supported by the National Key R&D Program of China (2017YFC0405203), the National Natural Science Foundation of China under Grant No. 11972148. The researcher of J. J. Nieto was partially supported by Xunta de Galicia, ED431C 2019/02, and by project MTM2016-75140-P of AEI/FEDER (Spain).

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**

1. Ingman, D.; Suzdalnitsky, J. Application of differential operator with servo-order function in model of viscoelastic deformation process. *J. Eng. Mech. 2005*, 131, 763–767. [CrossRef]
2. Sun, H.G.; Chen, W.; Chen, Y.Q. Variable-order fractional differential operators in anomalous diffusion modeling. *Phys. A 2009*, 388, 4586–4592. [CrossRef]
3. Lo, A.W. Long-term memory in stock market prices. *Econometrica 1991*, 59, 1279–1313. [CrossRef]
4. Rida, S.Z.; El-Sayed, A.M.A.; Arafa, A.M.A. Effect of bacterial memory dependent growth by using fractional derivatives reaction-diffusion chemotactic model. *J. Stat. Phys. 2010*, 140, 797–811. [CrossRef]
5. West, B.J.; Geneston, E.L.; Grigolini, P. Maximizing information exchange between complex networks. Phys. Rep. 2008, 468, 1–99. [CrossRef]

6. Metzler, R.; Schick, W.; Kilian, H.G.; Nonnenmacher, T.F. Relaxation in filled polymers: A fractional calculus approach. J. Chem. Phys. 1995, 103, 7180–7186. [CrossRef]

7. Glöckle, W.G.; Nonnenmacher, T.F. A fractional calculus approach to self-similar protein dynamics. Biophys. J. 1995, 68, 46–53. [CrossRef]

8. Gaul, L.; Klein, P.; Kempfle, S. Damping description involving fractional operators. Mech. Syst. Signal Process. 1991, 5, 46–53. [CrossRef]

9. Voyiadjis, G.Z.; Sumelka, W. Brain modelling in the framework of anisotropic hyperelasticity with time fractional damage evolution governed by the Caputo-Almeida fractional derivative. J. Mech. Behav. Biomed. 2019, 89, 209–216. [CrossRef]

10. Zwanzig, R. Nonequilibrium Statistical Mechanics; Oxford University Press: Oxford, UK, 2001.

11. Coffey, W.T.; Kalmykov, Y.P.; Waldron, J.T. The Langevin Equation, 2nd ed.; World Scientific: Singapore, 2004.

12. Mainardi, F.; Pironi, P. The fractional Langevin equation: Brownian motion revisited. Extr. Math. 1996, 10, 140–154.

13. Lutz, E. Fractional Langevin equation. Phys. Rev. E 2001, 64, 051106. [CrossRef] [PubMed]

14. Sandev, T.; Metzler, R.; Tomovski, Z. Correlation functions for the fractional generalized Langevin equation in the presence of internal and external noise. J. Math. Phys. 2014, 55, 023301. [CrossRef]

15. Kobelev, V.; Romanov, E. Fractional Langevin equation to describe anomalous diffusion. Prog. Theor. Phys. 2000, 139, 470. [CrossRef]

16. Bruce J.W. Fractal physiology and the fractional calculus: A perspective. Front. Physiol. 2010, 1, 12. [CrossRef]

17. Vojta, T.; Skinner, S.; Metzler, R. Probability density of the fractional Langevin equation with reflecting walls. Phys. Rev. E 2019, 100, 042142. [CrossRef]

18. Kosinski, R.A.; Grabowski, A. Langevin equations for modeling evacuation processes. Acta Phys. Polon. B Proc. 2010, 3, 365–377.

19. Wodkiewicz, K.; Zubairy, M.S. Exact solution of a nonlinear Langevin equation with applications to photoelectron counting and noise-induced instability. J. Math. Phys. 1983, 24, 1401–1404. [CrossRef]

20. Bouchaud, J.P.; Cont, R. A Langevin approach to stock market fluctuations and crashes. Eur. Phys. J. B 1998, 6, 543–550. [CrossRef]

21. Hinch, E.J. Application of the Langevin equation to fluid suspensions. J. Fluid Mech. 1975, 72, 499–511. [CrossRef]

22. Schluttig, J.; Alamanova, D.; Helms, V.; Schwarz, U.S. Dynamics of protein-protein encounter: A Langevin equation approach with reaction patches. J. Chem. Phys. 2008, 129, 155106. [CrossRef]

23. Ahmad, B.; Nieto, J.J.; Alsaaedi, A.; El-Shahed, M. A study of nonlinear Langevin equation involving two fractional orders in different intervals. Nonlinear Anal. Real World Appl. 2012, 13, 599–606. [CrossRef]

24. Salem, A.; Alzahrani, F.; Almaghamsi, L. Fractional Langevin equation involving three fractional orders. Frac. Calc. Appl. Anal. 2015, 18, 986–995. [CrossRef]

25. Fazli, H.; Nieto, J.J. Fractional Langevin equation with anti-periodic boundary conditions. Chaos Solitons Frac. 2018, 114, 332–337. [CrossRef]

26. Sudsutad, W.; Ntouyas, S.K.; Tariboon, J. Systems of fractional Langevin equations of Riemann-Liouville and Hadamard types. Adv. Differ. Equ. 2015, 2015, 235. [CrossRef]

27. Guo, P.; Zeng, C.; Li, C.; Chen, Y. Numerics for the fractional Langevin equation driven by the fractional Brownian motion. Fract. Calc. Appl. Anal. 2013, 16, 123–141. [CrossRef]

28. Zhou, H.; Alzabut, J.; Yang, L. On fractional Langevin differential equations with anti-periodic boundary conditions. Eur. Phys. J. Spec. Topics. 2017, 226, 3577–3590. [CrossRef]

29. Lim, S.; Li, M.; Teo, L. Langevin equation with two fractional orders. Phys. Lett. A 2008, 372, 6309–6320. [CrossRef]

30. Fa, K.S. Fractional Langevin equation and Riemann-Liouville fractional derivative. Eur. Phys. J. E 2007, 24, 139–143.

31. Darzi, R.; Agheli, B.; Nieto, J.J. Langevin Equation Involving Three Fractional Orders. J. Stat. Phys. 2020, 178, 986–995. [CrossRef]

32. Chen, A.; Chen, Y. Existence of solutions to nonlinear Langevin equation involving two fractional orders with boundary value conditions. Bound. Value Probl. 2011, 2011, 516481. doi:10.1155/2011/516481. [CrossRef]
33. Yu, T.; Deng, K.; Luo, M. Existence and uniqueness of solutions of initial value problems for nonlinear Langevin equation involving two fractional orders. *Commun. Nonlinear Sci. Numer. Simul.* 2014, 19, 1661–1668. [CrossRef]

34. Baghani, O. On fractional Langevin equation involving two fractional orders. *Commun. Nonlinear Sci. Numer. Simul.* 2017, 42, 675–681. [CrossRef]

35. Fazli, H.; Sun, H.G.; Aghchi, S. Existence of extremal solutions of fractional Langevin equation involving nonlinear boundary conditions. *Int. J. Comput. Math.* 2020, 1–10. [CrossRef]

36. Salem, A.; Alghamdi, B. Multi-strip and multi-point Boundary conditions for fractional Langevin equation. *Fractal. Fract.* 2020, 4, 18. [CrossRef]

37. Lü, K.; Bao, J.D. Numerical simulation of generalized Langevin equation with arbitrary correlated noise. *Phys. Rev. E* 2005, 72. [CrossRef]

38. Morgado, R.; Oliveira, F.A.; Batrouni, G.G.; Hansen, A. Relation between anomalous and normal diffusion in systems with memory. *Phys. Rev. Lett.* 2002, 89, 100601. [CrossRef]

39. Jeon, J.H.; Metzler, R. Fractional Brownian motion and motion governed by the fractional Langevin equation in confined geometries. *Phys. Rev. E* 2010, 81, 021103. [CrossRef]

40. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and Applications of Fractional Differential Equations. In *North-Holland Mathematics Studies*; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; Volume 204.

41. Diethelm, K. *The Analysis of Fractional Differential Equations*; Springer: Berlin, Germany, 2010.

42. Podlubny, I. Fractional Differential Equations. In *Mathematics in Science and Engineering*, Academic Press: Cambridge, MA, USA, 1999; Volume 198.

43. Miller, K.S.; Samko, S.G. Completely monotonic functions. *Integral Transform. Spec. Funct.* 2001, 12, 389–402. [CrossRef]

44. Berberan-Santos, M.N. Properties of the Mittag-Leffler relaxation function. *J. Math. Chem.* 2005, 38, 629–635. [CrossRef]

45. Hu, D.L.; Chen, W.; Liang, Y.J. Inverse Mittag-Leffler stability of structural derivative nonlinear dynamical systems. *Chaos Solitons Frac.* 2019, 123, 304–308. [CrossRef]

46. Prabhakar, T.R. A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math. J.* 1971, 19, 7–15.

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).