TRANSVERSE BIFURCATION OF VISCOUS SLOW MHD SHOCKS

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Abstract. We study by a combination of analytical and numerical Evans function techniques multi-D viscous and inviscid stability and associated transverse bifurcation of planar slow Lax MHD shocks in a channel with periodic boundary conditions. Notably, this includes the first multi-D numerical Evans function study for viscous MHD. Our results suggest that, rather than a planar shock, a nonplanar traveling wave with the same normal velocity is the typical mode of propagation in the slow Lax mode. Moreover, viscous and inviscid stability transitions appear to agree, answering (for this particular model and setting) an open question of Zumbrun and Serre.

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1. INTRODUCTION

In this paper, continuing and extending investigations of [46, 47, 24, 6, 2, and 42], we study by a combination of analytical and numerical Evans function techniques multi-D viscous and inviscid stability and associated transverse bifurcation of planar viscous slow Lax magnetohydrodynamic (MHD) shocks in a channel with periodic boundary conditions. Notably, this includes the first multi-D numerical Evans function study for viscous MHD, a computationally intensive problem representing the current state of the art, and, together with the treatment of gas-dynamical shocks in [29], the first such study for viscous shock profiles of any physical system in multi-D.

We obtain also new detail on the inviscid stability problem, while at the same time unifying and somewhat simplifying previously obtained results. In particular, we give a general framework for the treatment of constraints, or involutions [16], such as arise in multi-D MHD or elasticity, recovering and extending to the viscous case the fundamental results obtained by Blokhin et al (see [13, 24] and references therein) for inviscid MHD in a way apparently special to that case. The latter result answers in the affirmative the fundamental open problem posed in [40] whether Evans function stability in the sense defined there is necessary as well as sufficient for nonlinear multi-dimensional stability of viscous shocks in the presence of a constraint.

Our main physical conclusions are two: first, we make a mathematical connection between the spectral instability observed for slow inviscid MHD shocks [13, 24], and “corrugation instabilities” observed in the astrophysics community [34, 45], via a viscous bifurcation analysis as in [46, 47, 42]. Namely, our results suggest that, rather than a planar shock, a nonplanar “wrinkled” or “corrugated” traveling wave with nearby normal velocity is the typical mode of propagation in the slow Lax mode.

In particular, we demonstrate transitions from stability to instability satisfying the bifurcation hypotheses proposed in [42] (parallel magnetic field case) and [47] (nonparallel case), implying bifurcation in a mode transverse to the direction of shock propagation, i.e., lying in the direction parallel to the front, the first examples for which these scenarios have been shown to occur.

Second, continuing 1-D investigations of [2], we show numerically that the transition to instability for viscous multi-D slow Lax MHD shocks coincides with the transition to instability observed in the inviscid case. As shown in [55, 50, 51], for a rather general class of physical systems generalizing the “Kawashima class” of [37], viscous stability implies inviscid stability: that is, viscous effects may destabilize, but never stabilize a planar shock wave. The question posed in [55, 50, 51] whether and under what circumstances these two conditions coincide is a fundamental open problem in the theory of shock waves. On the one hand, it is much simpler to determine inviscid as opposed to viscous stability, so that pre-knowledge of coincidence would be a great help in applications; on the other, destabilization due to viscous effects would be physically extremely interesting. Our results here give the first information in this direction for multi-D, and (along with [29]) a first set of data for multi-D viscous systems. They are obtained by the introduction of an algorithm for numerical determination of the “refined stability condition” of [55, 51, 52, 10], detecting concavity at transition of the associated “critical” spectral curve through the origin, a new tool of general use.
1.1. Problem and background. The Navier–Stokes, or viscous, equations for isentropic 2-D MHD are given, in vectorial notation, by

\[ \rho_t + \text{div}(\rho u) = 0 \]  
\[ (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla q = \mu \Delta u + (\eta + \mu) \text{div} u \]  
\[ h_t - \nabla \times (u \times h) = \nu \Delta h, \]  

where \( u = (u_1, u_2, 0) \) is the velocity field, \( h = (h_1, h_2, 0) \) is the magnetic field, and \( \sum := \eta \text{div}(u) I + \mu (\nabla u + (\nabla u)^T) \), \( q = p + \frac{|h|^2}{2} \), where \( p = p(\rho) \) is gas-dynamical pressure \[\text{[7, 35, 15, 37, 17]}\]. Here, \((x_1, \ldots, x_3)\) is spatial location and \( t \) is time, with the solution independent of \( x_3 \). We take \( p(\rho) = a \rho^\gamma \) corresponding to a \( \gamma \)-law, or polytropic equation of state. The corresponding Euler, or inviscid, equations are given by Eqs. \[\text{[1.1]}\] with righthand sides set to zero. In either (viscous or inviscid) case, the magnetic field must satisfy in addition the constraint

\[ \text{div}(h) \big|_{t=0} = 0, \]  

which if satisfied at initial time \( t = 0 \), may be seen to persist for all \( t > 0 \).

Our aim is to study the spectral stability of both viscous and inviscid planar (without loss of generality standing) shock waves \( u(x, t) \equiv \overline{u}(x_1) \) in dimension 2, either as solutions on the whole space, or- which amounts to restricting discrete Fourier modes to a finite set- as solutions on a two-dimensional channel, \( x_1 \in \mathbb{R}, x_2 \in [0, 1] \) with periodic boundary conditions in \( x_2 \). In the parallel case \( u = (u_1, 0, 0), h = (h_1, 0, 0) \), Eqs. \[\text{[1.1]}\] decouple into the equations of nonmagnetic isentropic gas dynamics in \((\rho, u)\) and a heat equation for \( h \), from which we may readily deduce that the set of parallel planar MHD shocks consists precisely of the set of nonmagnetic gas-dynamical shocks in \((\rho, u)\), adjoined with \( h_1 \equiv \text{constant} \) (for details, see for instance \[\text{[6, 41, 24]}\]). As existence/transversality of traveling wave profiles for viscous polytropic gas dynamics is well known both in the nonsentropic \[\text{[19, 27]}\] and isentropic \[\text{[53]}\] case, one obtains thereby immediately existence/transversality of parallel MHD profiles, and, by perturbation, of near-parallel profiles as well; see \[\text{[6]}\] for details.

The “type” of an MHD shock is defined by the number of characteristics at plus and minus infinity moving inward toward the shock.\footnote{A standard detail suppressed here is that the equations must first be recast in noncharacteristic form; see below.} For parallel shocks, this is determined by the strength of the normal magnetic field \( |h| \), being “fast Lax” type for \( 0 \leq |h| < H_\ast \), “intermediate” type for \( H_\ast < |h| < H^\ast \), and “slow Lax” type for \( H^\ast < |h| \) \[\text{[11, 24]}\], where

\[ H_\ast = u_1^+ \sqrt{\rho^+}, \quad H^\ast = u_1^- \sqrt{\rho^-}, \]  

(for \( u_1^+ < u_1^- \); see Lemma \[\text{3.1(iii)}\]). Fast shocks are somewhat analogous to gas-dynamical shocks, and indeed reduce to this case in the zero-magnetic field limit \( |h| \equiv 0 \). Intermediate shocks are of nonclassical “overcompressive” type not appearing in gas dynamics \[\text{[22, 50, 51, 6]}\]. Slow shocks are of classical Lax type, but separated in parameter space from the fast type and exhibiting somewhat different properties.

Inviscid numerical studies \[\text{[19, 48]}\] indicate that fast parallel shocks are typically stable, while slow parallel shocks are typically unstable. Intermediate shocks, since overcompressive, are always inviscid unstable, and will not be discussed here. (Nonetheless, they appear to play an important role in viscous behavior \[\text{[22, 55, 52]}\] where they have been seen numerically to be at least 1-D stable \[\text{[6]}\].) Indeed, it has been shown analytically \[\text{[12, 25]}\] that fast parallel MHD shocks are stable under the gas-dynamical stability condition of Majda \[\text{[39]}\], notably for a polytropic equation of state; likewise, it has been shown analytically \[\text{[11, 13, 24]}\] that slow MHD shocks are unstable in the infinite-magnetic field limit \( |h| \to \infty \). In particular, in the brief but suggestive paper \[\text{[24]}\] Freistühler and Trakhinin, among other things, showed analytically the inviscid instability of slow...
Lax shocks for parallel MHD for sufficiently large magnetic field, extending to the parallel case (degenerate in this context [24]) the fundamental results of Blokhin et al [11, 13].

The result of Freistühler and Trakhinin [24] corroborates and puts on more solid mathematical ground an earlier study on instability of slow planar shocks in MHD performed by Stone and Edelman in astrophysics [45], where it is thought to play a role for example in the dynamics of accretion disks of binary dwarf stars. According to [45], the loss of stability through oscillations in the slow magnetosonic shock front is known as corrugation instability. Even though most of Stone and Edelman’s results rely on formal linear stability analysis, they also study the phenomena numerically through a time evolution code. Furthermore, they observe an oscillatory nature for the instability which results in fingers that end up destroying the planar structure of the shock front; their numerical results in [45, Section 3.1 and 3.3] indicate that the onset of instability is associated with a loss of the planar structure of the viscous profile, i.e., appearance of the above-mentioned corrugations.

This latter phenomenon has been verified rigorously in the form of a steady transverse bifurcation in a general \( O(2) \)-symmetric strictly parabolic system of conservation laws [42] relevant to the parallel MHD case, under appropriate spectral bifurcation conditions, namely, that transition to instability occurs through a pair of real eigenvalues corresponding to nonzero transverse Fourier modes passing through the origin. In the non-\( O(2) \) symmetric case, corresponding to nonparallel magnetic field, a similar Hopf bifurcation result was shown in [47], under the assumption that loss of stability occurs through the passage of a complex conjugate pair of eigenvalues associated with nonzero transverse modes. Our ultimate goal is to verify these spectral scenarios by a detailed numerical study of the eigenmodes of the linearized operator about the shock.

1.2. Main results and outline of the paper. The first logical step in this work consists of combining the analytical conclusions of [24] of inviscid instability in the infinite-magnetic field limit with numerical Evans function results showing that slow Lax shocks can be stable for smaller magnetic fields. Putting these observations together, one may conclude the existence of a stability transition, associated with which one might hope to observe bifurcation in wave structure. This is far from obvious at the inviscid level, where such transitions are associated with infinitely many Fourier modes simultaneously entering the right half of the complex plane (see [8]); nor is it clear a priori that there is a corresponding stability transition at the viscous level, since viscous and inviscid spectra can vary greatly at mid- and high frequencies. Nevertheless, by the Zumbrun-Serre Lemma [55] connecting viscous and inviscid spectra in the low frequency regime, one may conjecture the associated appearance of more standard bifurcations in the better-behaved viscous case, involving finitely many low-frequency modes; see Section 1.3 or [54] for further discussion. In the simplest situation that the single (necessarily real) double eigenvalue pair (double by \( O(2) \) symmetry) for large magnetic field moves into the stable half plane as magnetic field is decreased, without meeting any other eigenvalues along the way, this would necessarily be a “steady” spectral bifurcation, passing through \( \lambda = 0 \). This simple scenario is likewise not a priori guaranteed, but our numerical investigations confirm that it is indeed what occurs.

To carry out these numerical investigations requires some interesting extensions of the standard Evans function framework [26, 55, 50, 51, 52] to handle the presence of constraints such as (1.2), similar to what was done for inviscid MHD by Blokhin et al [13, 24], and (partially) for viscous MHD by Métivier et al [41]. In the process, we unify and simplify these previous analyses, at the same time obtaining a new formulation of the MHD equations- the “\( \beta \)-model”- that is particularly convenient for numerics, combining in one model the desirable properties of noncharacteristicity, hyperbolicity, and conservation form. All of this discussion is developed in Section 2.

To describe the main issues, a starting point is the observation that, in the presence of constraint (1.2), the equations of MHD are not prescribed uniquely, but only up to the addition of multiples
of the constraint. Indeed, in the study of spectral stability, one could adjoin the constraint as an additional equation if desired. Thus, one must take care to choose a form of the equations possessing properties under which the standard Evans function and Lopatinski determinants used to study viscous and inviscid stability are well defined, namely noncharacteristicity, hyperbolicity, and conservative form. There is a standard reformulation of the equations in which they become symmetric hyperbolic-parabolic and noncharacteristic [15, 13, 11] and another, different, formulation in which they become conservative; the standard approach has been to use ad hoc combinations of these in the analysis, depending on the need at hand. Here, we introduce for our single formulation a different analytical framework encompassing both viscous and inviscid cases. This gives for the first time an Evans condition necessary and sufficient for viscous MHD stability in the presence of constraint (1.2), answering a problem posed in [41] (where necessity was established); in passing, we rederive and further illuminate the inviscid results of [13, 24]. These issues are discussed in Section 2, where the $\beta$-model and basic analytic framework are introduced.

In section 3, we provide a description of the Rankine-Hugoniot conditions and the Lopatinski determinant condition, giving the foundations for a careful study of inviscid instabilities. In particular, we (i) recapitulate in the more convenient $\beta$-model framework the large-magnetic field asymptotics of [24] showing instability, at the same time correcting certain computation errors in [24]; and (ii) carry out a numerical Lopatinski study both verifying our asymptotics and extending the analysis to the small-magnetic field regime (see Figs. 3 and 2(b), respectively).

Two notable conclusions are that:

(i) the large-magnetic field asymptotics are quite accurate, extending even down to rather small magnetic field strengths (see Fig. 1);

(ii) there do exist inviscid stable slow parallel MHD shocks for sufficiently small magnetic field and sufficiently small amplitude of the shock.

The latter conclusion has the important implication that a stability transition, with potential for bifurcation, occurs. Finally, in Section 3.1.3 we compare our analytical and numerical multi-D results with those in [24] and [15]. Notably, we find that our large-magnetic field asymptotics improve by 20% on the accuracy of previous analyses (see again Fig. 1). In fact, when compared with results in [24 Eq. (61)], the calculation (3.4) of $\lambda_2$ show a better agreement with numerical results (see Figure 1 and also Section 3); one of the sources for the inaccuracy in their result is explained in the appendix C, where we show that [24 Eq. (44)] - a dynamic Rankine-Hugoniot
Figure 2. Bifurcation diagram plotting $h_1 - H^*$ against $u_1^+ - u_1^+$ indicating the boundary between stable and unstable waves in the (a) viscous model and (b) the inviscid model when $\gamma = 5/3$. A red dot corresponds to instability while a black plus sign indicates stability. In the viscous case, to examine stability, we considered $\xi \in [0.001, 0.004, 0.007, 0.01, 0.04, 0.07, 0.1, 0.14, 0.17, 0.2]$. The dashed line in both figures indicates the critical transition parameter for the Lopatinski determinant.

Figure 3. Log-log plot of the relative error between the numerical and asymptotic descriptions of the zero of the Lopatinski determinant against $\varepsilon = \frac{1}{h_1}$ when $\gamma = 5/3$ and $u_1^+ = 0.6$.

condition - was inaccurately computed. Overall, we highlight that our analysis is valid for all $\gamma \geq 1$ for isentropic $\gamma$-law, considerably improving the analysis in [21 §3.4] (constrained to $\gamma \in [1,2]$).

In Section 4, we connect the inviscid analysis with the viscous theory of stability of planar shocks based on the work of [55], through a study of the low-frequency limit (see Table 2 and Fig. 5); this gives an additional check on correctness of our Lopatinski computations through its asymptotic agreement with the Evans function, an object computed in a completely different way.

We go on to carry out a complete numerical Evans function study of viscous stability over all parameters and frequencies (see Fig. 2(a)), verifying that there occurs the same stability transition, at the same parameter values, that were seen in the inviscid case. We present, further, numerical results verifying the bifurcation conditions assumed in the abstract results of Texier-Zumbrun [47] and Monteiro [42] (see also the related [43]), consisting of the absence of other neutrally stable
eigenvalues (see Fig. 6) and nonzero speed of crossing of the imaginary axis as the magnetic field is varied (see Figure 8). We compute also approximate zero-eigenfunctions at the bifurcation point, yielding the approximate shape of the bifurcating nonplanar wave (see Figure 4).

We finalize the paper with an appendix: in A we briefly explain how numerical winding number computations were carried out; in B following [13, 24], we present an alternative proof by direct computation of Proposition 2.11 in the special case of inviscid MHD, showing that the constraint $\text{div}(h)\big|_{t=0} = 0$ persists throughout the dynamics, namely, $\text{div}(h(t)) \equiv 0$, for all $t \geq 0$; last, in C we give another perspective on the dynamical Rankine-Hugoniot condition of [13, 24], at the same time correcting details of some related calculations in [24].

1.3. Discussion and open problems. Our general results on Evans functions for systems with constraints pave the way for a unified treatment of multi-dimensional viscous shock stability in MHD, elasticity, and related equations arising in continuum mechanics. Our introduction of the $\beta$-model, though more special to MHD, by putting the equations into a standard symmetrizable conservative form, has the tremendous advantage that it allows computations using existing “off-the-shelf” code in the numerical stability package STABLAB [5]. As discussed in [29, 8, 4], success or failure of multi-dimensional computations is highly dependent on the specific algorithm used,
Figure 6. Viscous stability bifurcation diagram when $\gamma = 7/5$, with red open circles corresponding to instability and black plus signs indicating stability.

with a number of catastrophic possible pitfalls that must be avoided. Thus, the ability to use existing, already-tested algorithms is of significant practical advantage.

Figure 7. Diagram indicating the ways in which a subcritical and critical Hopf bifurcation might manifest itself in the bifurcation diagrams when taking into account the relationship between the Evans function and the Loptinski determinant.

As regards our numerical results, we point out some further background and implications from a more general perspective contrasting viscous and inviscid stability. As shown in [55], viscous stability is closely related at low frequencies to inviscid stability, hence uniform viscous stability implies uniform inviscid stability. This means that, as shock or magnetic field strength is increased from a stable regime, a transition to inviscid instability implies a corresponding transition to viscous instability, occurring in low-frequency modes. The reverse is not true, as it is possible that a transition to viscous instability could occur in advance of the transition to inviscid instability due to destabilization of an intermediate- or high-frequency mode unrelated to the inviscid problem.

Indeed, let $\xi$ be the Fourier frequency parameter in the direction transversal to the shock front. Then, as depicted in Figure 7 there are essentially 3 different scenarios for viscous vs. inviscid stability transitions in a finite-cross section channel, depending mainly on concavity vs. convexity of the neutral spectral curve $\lambda = \lambda(\xi)$ for the viscous case, bifurcating from $\lambda(0) = 0$, given by the sign of $\lambda''(\cdot)$ near $\xi = 0$. Recall [55], that this curve is tangent at $(\xi, \lambda) = (0, 0)$ to the corresponding inviscid stability curve, given by homogeneity by a ray through the origin. The viscous spectral curve $\lambda(\cdot)$ is depicted in various cases in Figure 7 along with its tangent inviscid ray. In the first case, $\lambda(\cdot)$ is concave ($\lambda''(\cdot) < 0$) and we see that the transition to instability occurs simultaneously in the whole space (for which $\xi \in \mathbb{R}$) for viscous and inviscid problems, and slightly later for a duct.
of finite cross-section (for which $\xi \in \mathbb{Z}$, hence viscous spectra lags behind inviscid by a fixed finite amount). In the second, $\lambda(\cdot)$ is convex ($\lambda''(\cdot) > 0$) and viscous instability occurs slightly before inviscid instability as the bifurcation parameter is varied, for either the whole space or finite cross-section. In the third, $\lambda(\cdot)$ is concave, but high-frequency instabilities cause the viscous problem to destabilize first. This is consistent with the results of [55] in the whole space, where it is shown that the viscous transition occurs not later than the inviscid one. For the whole space problem ($\xi \in \mathbb{R}$), our discussion above shows that simultaneously precisely in case 1, and strictly sooner in cases 2 and 3.

Our numerical results (see Section 4.5) for the typical parameters $\gamma = 5/3$, $u_1^+ = 0.86$. indicate that the viscous transition occurs at approximately $H^* = 2$, while the inviscid transition occurs at approximately $H^* = 1.995$. Likewise, one can see from Figure 8 that the second derivative of the spectral parameter $\lambda = \lambda(\xi)$ with respect to $\xi$ is negative. Indeed, we find this to be the case for all parameter values; see Section 4.6 and particularly Figure 11(a). That is, we appear to be in the first case depicted in Figure 7. This has the important consequence that, considered in the whole space, viscous and inviscid stability transitions coincide for the 2-D transverse instabilities considered here, similarly as was seen in [2] for the 1-D longitudinal instabilities considered there. Thus, though viscosity can in principle according to the results of [55] hasten the onset of instability, for the two studies carried out so far (in [2] and here) for gas dynamics and MHD, this possibility was not in practice observed. It remains a very interesting open problem whether such “viscosity-enhanced instability” can occur for physically relevant models of gas dynamics or MHD, both philosophically, and (since inviscid transitions may often be computed explicitly, whereas viscous transitions require substantial numerical computation) from a practical point of view. Indeed, we note that it has been shown in [21] that the neutral inviscid stability curve, where the Lopatinski condition precisely vanishes for $\lambda = 0$, $\xi \neq 0$, may be determined explicitly, making this a practical condition indeed.

Our numerics confirm not only stability transition but also spectral bifurcation scenarios like those studied in [46, 47, 42]. For fully parabolic “artificial viscosity” versions of MHD, the actual nonlinear bifurcation would follow from the results presented in [46, 47, 42] restricted to the space of divergence free magnetic field functions, provided that the corresponding spectral scenario were verified. Extending this result to the physical, “real” viscosity case considered here is an important open problem. Presumably, one could expect a similar spectral bifurcation for the artificial viscosity
case (giving the full nonlinear result), but we do not investigate this here. The extension of our investigations to a complete “all-parameters” study of MHD shocks analogous to that done for gas dynamics in [29] is another important direction for further study: likewise, spectral stability of small amplitude nonextreme shocks as considered here, both inviscid and viscous.

It is worthwhile to emphasize the loss of planar structure observed here (see Section 4), in the numerical experiments of Edelman and Stone (cf. [45, Section 3.1 and 3.3]) and in the context of steady bifurcations in a $O(2)$-symmetric strictly parabolic system of conservation laws (see [42]). This indicates that the resolution of the 2-D-Riemann problem for slow shocks in MHD is not realized through planar shocks, but generically involve nonplanar “corrugated” fronts as component slow waves; see also the recent numerical results of [18]. Spectral and nonlinear stability of these nonplanar waves is another very interesting open problem for further investigation.

Finally, we mention an interesting related analysis of Freistühler, Kleber, and Schropp [21] for the inviscid isothermal ($\gamma = 1$) case, in which they find by explicit computation the inviscid stability boundary $\Delta(0, \pm 1) = 0$ for slow parallel shocks; a similar computation should be possible for general $\gamma$, sharpening our description of the inviscid boundary in Figure 2(b). Freistühler et al investigate numerically also the fast shock case, showing that parallel isothermal fast shocks are uniformly stable, but nonparallel ones experience a transition to instability across a particular parameter surface. Similarly, for the general isentropic case ($\gamma > 1$), Trakhinin [48] has shown that fast nonparallel shocks can be unstable in some regimes. A very interesting further study would be to carry out the corresponding analysis of viscous stability transition in these cases for fast shocks as we have done here for slow shocks.

1.4. Notation. In this paper we write $\text{Re}(z)$ to denote the real part of $z \in \mathbb{C}$. We let $u \in \mathbb{R}^n$ be a vector of states assuming values $u^\pm$ across a shock. Given any function $u \mapsto f(u)$ we write $[f(u)] = f(u^+) - f(u^-)$. We denote the real part (resp., imaginary part) of a number $z \in \mathbb{C}$ by $\text{Re}(z)$ (resp., $\text{Im}(z)$). Given a matrix $A \in \mathbb{C}^{n \times n}$, we write $A^{-1}$ to denote its inverse, and $\sigma(A)$ to represent its eigenvalues.

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2. The $\beta$-model: inviscid case

We now introduce the $\beta$-model, obtained by adding a multiple of $\text{div}(h)$ to the equation (1.1c) in (1.1). Equations (1.1) are rewritten as

$$
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u - h \otimes h) + \nabla q = 0, \\
h_t - \nabla \times (u \times h) + \beta \text{div}(h)e_1 = 0,
$$

where $\beta$ is a real valued parameter and $e_1 = (1, 0, 0)^t$. Noting the lack of dependence on the third coordinate, we shall most of the time write these vectors as $u = (u_1, u_2)$ and $h = (h_1, h_2)$. Likewise, assuming the solutions independent of $x_3$ we consider these as equations in $(x_1, x_2) \in \mathbb{R} \times [0, 2\pi]$, with periodic boundary conditions in $x_2$.

Recalling the constraint $\text{div}(h) = 0$ we see that (1.1) and (2.1) are equivalent. However, (2.1) has certain practical advantages, as we now explain. An important reason to modify (1.1) is that,

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2 For extreme shocks, i.e., 1- or 5-shocks in the artificial viscosity case, see [23] [20].
considered without the constraint $\text{div}(h) = 0$, (1.1) is not hyperbolic. A standard resolution of this problem is to make the substitution
\[
\nabla \times (h \times u) = \text{div}(u)h + (u \cdot \nabla)h - (\text{div}h)u - (h \cdot \nabla)u,
\]
which corresponds to adding the nonconstant multiple $u\text{div}(h)$ of the constraints to (1.1c). This has
the advantage of both recovering hyperbolicity, and providing a symmetrizable system of equations
(cf. discussion in [40, §7]). On the other hand, this handy device results in loss of conservative
form of the equations, as a result of which jump conditions across shocks are not computable for
this version of the equations.

The $\beta$-model is a modification in a similar spirit. It is clear that the $\beta$-model preserves the
conservative structure of the equations; however, it is not symmetrizable. Nonetheless, as we will
see next, it maintains the important property of hyperbolicity. Moreover, it has the property of
consistent splitting (see Def. 2.9), which will be needed for construction of a Lopatinski determinant
and numerical stability investigation.

The rest of this section will be devoted to the proof of these properties. System (2.1) may be
written in the form
\[
(f_0(W))_t + \sum_{i=1}^{2} [f_i(W)]_{x_i} = 0,
\] (2.2)
where $W = (\rho, u_1, u_2, h_1, h_2)^T$. Consider a stationary planar shock profile $W$. Existence of planar
traveling wave profiles and conditions for a parallel shock are encoded in the following conditions,
assumed throughout the paper:

(A1) We assume that the equations (2.1) support a planar shock profile
\[
W(x_1, x_2, t) = W(x_1) = (\bar{\rho}, \bar{u}_1, 0, \bar{h}_1, 0)^T(x_1).
\]

(A2) (Parallel shocks) In this paper we will study parallel shocks, i.e., cases in which the normal
to the shock front is parallel to the magnetic field. Henceforth assume that the state $W^+$
is defined as $W^+ = (\bar{\rho}^+, \vec{u}_1^+, \vec{h}_1^+, 0)$.

Then the linearization of (2.2) about $W$ is given by
\[
A^\pm_i W_t + A^\pm_i W_{x_1} + A^\pm_2 W_{x_2} = 0, \tag{2.3}
\]
and $A^\pm_i = A_i(W^\pm) = Df_i(W)$, for $i \in \{0, 1, 2\}$. With regards to the shock profile $W(x, t)$ we have the
following Rankine-Hugoniot conditions:
\[
[f_1(W)] = \begin{bmatrix}
\bar{\rho}\bar{u}_1 \\
\bar{\rho}u_2 - \bar{\rho}\bar{h}_1 \\
\bar{\rho}\bar{u}_1 \bar{u}_2 - \bar{\rho}\bar{h}_1 \bar{h}_2 \\
\beta\bar{h}_1 \\
\bar{\rho}_1 \bar{h}_1 - \bar{\rho}_2 \bar{h}_1
\end{bmatrix} = 0, \tag{2.4}
\]
where $[f_1(W)] := f_1(W^+) - f_1(W^-)$.

2.1. The $\beta$-model: hyperbolicity and further properties. To explore the properties of this
model we find it convenient to work in abstract settings, following the discussions in [17, §5.4],
afterwards specializing to our model. To begin with, we write $V = f_0(W)$, where $f_0(\cdot)$ is assumed
to be a (local) diffeomorphism from $\mathbb{R}^n$ to itself. We can write (2.2) in the standard form
\[
V_t + F(V) = 0, \quad F(V) := \sum_j f_j(V)_{x_j}, \tag{2.5}
\]
where $\mathcal{V}$ is a zero-speed planar background shock in $x_1$ direction:

$$V(x,t) = \mathcal{V}(x_1) = V^\pm,$$

for $x_1 \geq 0$, (2.6)

satisfying Rankine-Hugoniot conditions:

$$[f_1] = 0.$$

Furthermore, we have a constant coefficient constraint:

$$\Gamma \mathcal{V} := \sum_j \Gamma_j \partial_{x_j} \mathcal{V} \equiv 0,$$

satisfying Dafermos’ compatibility condition (involution) (cf. [16]):

$$\Gamma \mathcal{F}(\mathcal{V}) = \mathcal{M} \Gamma \mathcal{V},$$

(2.8)

$$\mathcal{M} := \sum_j \mathcal{M}_j \partial_{x_j},$$

constant-coefficient, noncharacteristic:

$$\text{det } \mathcal{M}_1 \neq 0,$$

(2.9)

and hyperbolic:

$$\sigma \left( \sum_j ik \mathcal{M}_j \right) \text{ real and semisimple.}$$

(2.10)

Note that (2.8) implies also the linearized version:

$$\Gamma \mathcal{L} = \mathcal{M} \Gamma,$$

(2.11)

where $\mathcal{L} := \sum_j A_j^\pm \partial_{x_j}$, $x \geq 0$, is the linearized operator about the background wave, with

$$A_j^\pm := f_j(V^\pm).$$

(2.12)

Remark 2.1. In our case, $\mathcal{V} = (\rho, \rho u_1, \rho u_2, h_1, h_2)$, $\Gamma \mathcal{V} = \text{div}(h)$, $\mathcal{M} = \beta \partial_{x_1}$, for $\beta \neq 0$.

2.1.1. Property I: hyperbolicity.

Proposition 2.2. Under the above assumptions, system (2.5) is weakly hyperbolic without constraint (2.7), in the sense that its characteristic modes are real, if and only if it is weakly hyperbolic with the constraint. In particular, $\mu$, $r$ are an eigenvalue, eigenvector pair of $\mathcal{L}(k)$ if and only if either (i) $\Gamma(k)r = 0$, or (ii) $\gamma := \Gamma(k)r$ is an eigenvector of $\mathcal{M}(k)$ with eigenvalue $\mu$, $0 \neq k \in \mathbb{R}^d$.

Likewise a shock is Lax type and noncharacteristic without the constraint if and only if it is so with the constraint taken into account.

Proof. Applying (2.8) to $(*)(\mathcal{L}(k) - \mu)r = 0$, we obtain $(\mathcal{M}(k) - \mu)\gamma = 0$, with $\gamma = \Gamma(k)r$, whence the second assertion follows. The first assertion then follows by hyperbolicity of $\mathcal{M}$. Noncharacteristicity holds always for the eigenvalues corresponding to $\sigma(\mathcal{M})$, by reality plus nonvanishing due to the assumption of hyperbolicity, and thus depends only on the eigenvalues for which the constraint is satisfied. Finally, we note that the type of the shock is not affected by eigenvalues of $\mathcal{M}$, as their signs are independent of $z$, so are the same at $z \to \pm \infty$, hence they correspond to transverse modes.

Remark 2.3. To establish full hyperbolicity, it seems one must either show that characteristics of $\mathcal{M}$ are distinct from those of $\mathcal{L}$ with the constraint taken into account, or else show that there are no Jordan blocks in the eigenstructure of $\mathcal{L}$. This is true for the MHD $\beta$-model by direct computation, as one sees that the “extra” eigenvalue coming from $\mathcal{M}$ must be the single eigenvector depending on $\beta$ (see the proof of Lemma 3.1). Only weak hyperbolicity is needed for our purposes here.
2.1.2. Property II: compatibility with constraints. Recall from, e.g., [55], the normal modes equations after shifting to a frame \((z, x_2, \ldots, x_d)\) with discontinuity at \(z = 0\):

\[
\lambda \hat{V} + A_1^\pm \hat{V}_z + \sum_{j \neq 1} A_j^\pm \hat{V}_{x_j} = 0, \quad z \gtrless 0,
\]

(2.13) (interior equation) and

\[
Y \left( \lambda [V] + \sum_{j \neq 1} ik_j [f_j(V)] \right) - [A \hat{V}] = 0,
\]

(2.14) (linearized jump conditions), where \([\cdot]\) denotes jump across \(z = 0\). Here, \(\hat{V}\) is the Laplace–Fourier transform of the normal mode, \(Y \in \mathbb{C}\) the Laplace–Fourier transform of the front location, and \(\lambda\) a spectral parameter indicating growth \(\sim e^{\lambda t}\) of the normal mode \(e^{\lambda t + \sum_{j \neq 1} ik_j x_j \hat{V}(z)}\), with \(\text{Re}(\lambda) > 0\) corresponding to instability.

**Lemma 2.4.** Under the assumptions above, defining \(\gamma := \Gamma \hat{V}\) with \(\hat{V}\) piecewise smooth, satisfying \(2.13) - (2.14)\), we have

\[(\lambda + M)\gamma = 0, \quad \text{whenever } x \gtrless 0; \quad [M_1 \gamma] = 0, \quad \text{whenever } x = 0.\]

(2.15)

**Corollary 2.5.** Under the assumptions above, for \(v\) piecewise smooth satisfying \(2.13) - (2.14)\) with \(\text{Re}(\lambda) > 0\), we have \(\gamma := \Gamma \hat{V} \equiv 0\). That is, unstable normal modes automatically satisfy the constraint.

**Proof.** It follows immediately that \(\gamma\) is a weak solution of \((\lambda + M)\gamma = 0\), observing that the jump condition for piecewise smooth weak solutions of \((2.15) (i)\) with a single jump at \(z = 0\) is \((2.15) (ii)\). By Hersh’s Lemma (cf. [28], or [9, Chap. 4, Lemma 4.1]), \(\hat{V}\) and thus \(\gamma\) decays exponentially as \(z \to \pm \infty\). Mollifying \(\hat{V}\) by convolution with a standard smoothing kernel \(\eta_\varepsilon\), we obtain a family of \(C^\infty\) solutions \(\gamma_\varepsilon := \gamma \ast \eta_\varepsilon\) of \((\lambda - M)f = 0\), each decaying as \(z \to \pm \infty\), and converging as \(\varepsilon \to 0\) in \(L^1_{\text{loc}}\) to \(\gamma\). These solutions, if nontrivial, would represent eigenfunctions of \(M\) with eigenvalue \(\lambda\). However, \(M\), being constant coefficient, has only continuous spectrum, and so each \(\gamma_\varepsilon\) must vanish identically, as therefore does \(\gamma\) in the limit as \(\varepsilon \to 0\).

**Remark 2.6.** Note that for Corollary 2.5 it is not necessary that \(M\) be hyperbolic, but only constant-coefficient. Hyperbolicity of \(M\) is used rather to achieve hyperbolicity of \((2.5)\) and set up the framework to define the Lopatinski condition in the first place.

We now recall the *uniform stability*, or “uniform Lopatinski” condition of Majda [39], that there exist no normal mode solutions for frequencies \(|\xi, \lambda| = 1\) with \(\text{Re}(\lambda) \geq 0\), under which there holds short-time nonlinear stability of inviscid shock waves. We recall also the weak Lopatinski condition that there exist no normal modes for \(\text{Re}(\lambda) > 0\), a necessary condition for linearized stability, failure of which implies exponential instability.

**Corollary 2.7.** Under the above assumptions, uniform stability in the sense of Majda holds if the uniform Lopatinski condition is satisfied for system \((2.5)\), ignoring the constraint; strong instability holds if the weak Lopatinski condition fails for system \((2.5)\), ignoring the constraint.

**Proof.** Stability ignoring the constraint implies stability with the constraint, as the constraint is preserved under the flow of the time-evolution equations. Failure of the weak Lopatinski condition, on the other hand, implies existence of a normal mode solution \(\hat{V}\) of \((2.13) - (2.14)\) with \(\text{Re}(\lambda) > 0\), which by Corollary 2.5 must satisfy the constraint.

It remains to establish the key Lemma 2.4.
Proof of Lemma 2.4. Taking Laplace-Fourier transform, we obtain symbols
\[ \mathcal{L}(k) = A_1^+ \partial_z + \sum_{j \neq 1} ik_j A_j^+, \quad \Gamma(k) = \Gamma_1 \partial_z + \tilde{\Gamma}(k), \quad \text{and} \quad \mathcal{M}(k) := \mathcal{M}_1 \partial_z + \tilde{\mathcal{M}}(k), \]
where \( \tilde{\Gamma}(k) = \sum_{j \neq 1} ik_j \Gamma_j \) and \( \tilde{\mathcal{M}}(k) = \sum_{j \neq 1} ik_j \mathcal{M}_j \). Comparing derivatives of different orders, we find from the key relation (2.11) that
\[ \Gamma_1 A_1 = \mathcal{M}_1 \Gamma_1, \quad \tilde{\Gamma} A = \tilde{\mathcal{M}} \tilde{\Gamma}, \quad \tilde{\Gamma} A_1 + \Gamma_1 \tilde{A} = \tilde{\mathcal{M}} \Gamma_1 + \tilde{\mathcal{M}} \Gamma. \] (2.16)
Likewise, from (2.8), we have
\[ \Gamma_1 \sum_{j \neq 1} ik f_j(\mathcal{V}) = (\tilde{\mathcal{M}} \Gamma_1 + \mathcal{M}_1 \tilde{\Gamma}) f_1(\mathcal{V}), \quad \tilde{\Gamma} \sum_{j \neq 1} ik f_j(\mathcal{V}) = \sum_{j \neq 1} ik \mathcal{M}_j \tilde{\Gamma} \mathcal{V}. \] (2.17)
From the fact that \( \partial_z \mathcal{V} = 0 \) and \( \Gamma \mathcal{V} = 0 \), we find, further, that
\[ \tilde{\Gamma} \mathcal{V} = 0, \quad [\Gamma_1 \mathcal{V}] = 0, \] (2.18)
and from the Rankine-Hugoniot relation that \([f_1(\mathcal{V})] = 0\), so that in fact
\[ \Gamma_1 \sum_{j \neq 1} ik [f_j(\mathcal{V})] = 0, \quad \tilde{\Gamma} \sum_{j \neq 1} ik f_j(\mathcal{V}) = 0. \] (2.19)
Applying \( \tilde{\Gamma} \) now to (2.14), and using (2.19), we obtain \([\tilde{\Gamma} A_1 \hat{\mathcal{V}}] = 0\). Using (2.16)(iii), we may rewrite this as \([\mathcal{M}_1 \Gamma_1] + [\tilde{\mathcal{M}} \Gamma_1] = [\Gamma_1 \mathcal{A} \mathcal{V}]\), and, using (2.13) to express \( \Gamma_1 \mathcal{A} \mathcal{V} \) as \(-\Gamma_1(\lambda + A_1 \partial_z) \hat{\mathcal{V}}\), then by (2.16)(i) as \(-\mathcal{A}_1(\lambda + \mathcal{M}_1 \partial_z) \hat{\mathcal{V}}\), finally as
\[ [\mathcal{M}_1 \gamma] = -([\tilde{\mathcal{M}} + \lambda] \Gamma_1 \hat{\mathcal{V}}]. \] (2.20)
On the other hand, applying \( \Gamma_1 \) to (2.14) and using (2.18), (2.16), we obtain
\[ 0 = [\Gamma_1 A_1 \hat{\mathcal{V}}] = [\mathcal{M}_1 \Gamma_1 \hat{\mathcal{V}}], \]
hence, by the assumed invertibility of \( \mathcal{M}_1, [\Gamma_1 \hat{\mathcal{V}}] = 0 \). Substituting into the righthand side of (2.20), we have (2.15)(ii). Eq. (2.15)(i) follows immediately upon applying \( \Gamma \) to (2.13) and using (2.11). \( \square \)

Remark 2.8. In the case of the MHD \( \beta \)-model, the identity \([\Gamma_1 \hat{\mathcal{V}}] = 0\) may be recognized as the relation \([\beta h_1] = 0\) that we computed earlier by hand/force. It may be recognized as the analog in our setting of identity \([u_1 h_1] = 0\) derived ad hoc in [24] by a series of clever and special computations.

2.1.3. Property III: consistent splitting.

Definition 2.9. (Following [1]) Taking Laplace-Fourier transform of (2.13) in the variables \( x_j \), for \( j \neq 1 \) we obtain a pair of ODEs
\[ \partial_z \hat{\mathcal{V}} = -(A_1^+) \partial_z + \sum_{j \neq 1} N \xi_j A_j^+ = 0 \]
\[ \hat{\mathcal{V}} = : A^+ \left( \lambda, \xi, z \right) \hat{\mathcal{V}}, \quad z \geq 0, \] (2.21)
where \( \hat{\mathcal{V}} \) denotes the Laplace-Fourier transform of \( \mathcal{V}(\cdot) \in H^s(\mathbb{R}^N) \) for \( s > 0 \) sufficiently large. We define the domain of consistent splitting for problem (2.21) as the open subset of parameters \( \lambda \in \mathbb{C} \) such that the limiting matrices \( \Lambda^+ = \lim_{z \to \infty} \Lambda(\lambda, z) \) and \( \Lambda^- = \lim_{z \to -\infty} \Lambda(\lambda, z) \) are hyperbolic (no center subspace), and the dimensions of their stable subspace \( S^+ \) and \( S^- \), respectively, are the same.

Unfortunately, even though the \( \beta \)-model is hyperbolic, it is not symmetrizable. We are able to prove that consistent splitting holds through the following -Hersh Lemma type of idea: hyperbolicity “prevents” eigenvalues from switching the sign of their real part.
Proposition 2.10. Consistent splitting holds for the \( \beta \)-model, \( \forall \beta \in \mathbb{R}^* \).

**Proof.** We provide two different proofs of this result: initially, we recall that hyperbolicity consists in showing that the eigenvalues of the matrix \( A_0^{-1}(A_1 + \xi A_2) \) are real valued. One can show that this matrix is similar (therefore, same spectrum) to

\[
\begin{pmatrix}
  u_1 & \rho & \rho \xi & 0 & 0 \\
  p_u & u_1 & 0 & 0 & 0 \\
  p_u \xi & 0 & h_1 \xi & -h_1 \xi^2 & -h_1 \\
  0 & 0 & 0 & \beta & 0 \\
  0 & 0 & -h_1 & 0 & u_1 \\
\end{pmatrix}
\]  

(2.22)

By inspection, one can readily see that \( \beta \) is an eigenvalue, real valued, and in the stable half plane for the full viscous problem, namely, constant coefficient limiting problem. Furthermore, all the other eigenvalues are independent of \( \beta \), thanks to the Block-structure of the matrix in (2.22). On the other hand, we also know that for the particular choice \( \beta \), the other eigenvalues are independent of constraint \( \beta \), are already real and nonvanishing, while the \( \beta \)-dependent eigenvalue is real and nonvanishing by inspection. By homotopy the number of positive eigenvalues depends only on sign of \( \beta \). Observing that \( u_1^+ \) and \( u_1^- \) have the same sign and recalling that consistent splitting is satisfied for the standard nonlinear model with \( \beta \) replaced by \( u \), we find that consistent splitting is satisfied for the \( \beta \)-model as well.

A second proof goes as follows: we know also that under the divergent-free constraint all eigenvalues are the same for the \( \beta \)-model as for the usual MHD equations. We also know for any \( \text{Re}(\lambda) \geq 0 \) except \( \lambda = 0 \) that the constraint is satisfied for any spectra of \( \mathcal{L}_\beta \) by looking at the spectra of \( M_\beta(\Gamma \mathcal{L} = \mathcal{L} \Gamma) \) and noticing that it is stable. So that means that there are no unstable spectra for the limiting operators at \( \pm \infty \), else: a) the constraint would be satisfied, so b) the unstable spectra would agree with that of the usual MHD, which is stable, a contradiction. This proves that the dispersion curves are in the stable complex half-plane, except at \( \lambda = 0 \), which then implies consistent splitting by the standard Hersh argument (cf. [28]).  

\[ \square \]

2.1.4. Property IV: persistence of constraint condition. Persistence under evolution of the divergence-free constraint in the magnetic field was shown by Freistühler and Trakhinin via “by-hand” computation [24, Remark 3.2]. We give in this section a more general argument that contains and generalizes their result in a more abstract fashion. We offer a second proof of this statement in appendix B through an argument closer to that in [24]. We remark that our result for the \( \beta \)-model in fact implies the result of [24, Remark 3.2] for their version of the MHD equations, since solutions of both of these agree for piecewise smooth solutions with entropic shocks.

**Proposition 2.11.** Let \( \mathcal{V} \) be a weak solution to system (2.1) and \( h \) be the magnetic field component of this solution. If \( h \) satisfies the constraint (1.2) at \( t = 0 \) then \( \text{div}(h) = 0 \), \( \forall t > 0 \).

**Proof.** First, a weak solution of \( \mathcal{V}_t = \mathcal{F}(\mathcal{V}) \) is equivalent to \( \langle \phi, \mathcal{V}_t - \mathcal{F}(\mathcal{V}) \rangle = 0 \) for all test functions \( \phi \), which implies \( \langle \Gamma^* \phi, \mathcal{V}_t - \mathcal{F}(\mathcal{V}) \rangle = 0 \) (since \( \Gamma \) is constant coefficient, so \( \Gamma^* \phi \) is another test function.), and thus \( \langle \phi, \Gamma(\mathcal{V}_t - \mathcal{F}(\mathcal{V})) \rangle = 0 \), or \( \langle \phi, (\partial_t - \mathcal{M}) \mathcal{V} \rangle = 0 \), i.e., \( \gamma = \Gamma \mathcal{V} \) is a weak solution of \( \gamma_t - \mathcal{M} \gamma = 0 \), with \( \gamma \equiv 0 \) at \( t = 0 \). Since \( \mathcal{M} \) is constant coefficient hyperbolic, we have uniqueness of weak solutions, giving \( \gamma \equiv 0 \) for \( t \geq 0 \).

\[ \square \]

**Remark 2.12** (Persistence of constraints with respect to the dynamics). The persistence of constraints is also studied in other contexts and might not be preserved through the dynamics in a variable coefficient case: for instance, vorticity can be created in the context of wave breaking, which makes the study of these phenomena both numerically and analytically challenging [38].
Now that we have verified all the good qualities of the $\beta$-model we can study the viscous and inviscid stability of parallel shocks.

3. INVISCID STABILITY ANALYSIS: THE LOPATINSKI DETERMINANT

The main goal in this section is to introduce the Lopatinski determinant, which will be the main tool used in the study of inviscid stability. Consider the equations (2.3) and assumptions (A1), (A2). Initially we derive some properties of the shock type. Recall that shocks are categorized as Lax, undercompressive and compressive depending on the number of characteristics entering the shock. Lax shocks are further categorized by their characteristic field: the unique family entering on both sides. To begin with, we obtain a useful parametrization of the shocks we study.

**Lemma 3.1** (Parametrization of MHD planar shocks with zero speed). Let

$$W(x_1, x_2, t) = W^\pm(x_1) = (\rho^\pm, u_1^\pm, u_2^\pm, h_1^\pm, h_2^\pm)(x_1),$$

$x_1 \geq 0$, be a planar shock solution satisfying the Rankine-Hugoniot conditions in (2.4). Assume that the shock is parallel, i.e., $u_2^+ = 0$ and $h_2^+ = 0$.

(i) One can parametrize the slow shocks connecting to the state $(\rho^+, u_1^+, 0, h_1^+, 0)$ to the right (i.e., $x_1 > 0$) using variables $R$ and $M$, defined as

$$R = \frac{\rho^+}{\rho^-} = \frac{u^-}{u^+},$$

$$M^2 = \frac{R^\gamma - 1}{\gamma R^\gamma (R - 1)},$$

where $M^2 = \frac{(u_1^+)^2}{\rho_0^+ p_0^+}$ ($M$ is also known as the downstream Mach number), and $u_2^- = 0$, $h_2^- = h_1^-; (ii)$ For large magnetic field, slow shocks, i.e., 2-shocks, are characterized by $R > 1$ and $M < 1$; (iii) There exist two numbers $H^*_\ast = u_1^+ \sqrt{\rho^+}$, $H^* = u_1^- \sqrt{\rho^-}$, according to which three scenarios are possible: fast Lax shocks (i.e., extreme, or gas-dynamical type) for $0 \leq |h| \leq H^*_\ast$, intermediate shocks for $H^*_\ast \leq |h| \leq H^*$ and slow shocks for $H^* \leq |h|$ (in particular $|h| \to \infty$).

We remark that in the context of 3D MHD, the 2-D slow shocks seen here are 2-shocks (as in, for instance, [24, Section 2-3]).

**Proof.** Considering the jump conditions in (2.4) we readily observe that the ratios $\rho^+ / \rho^-$ and $u^- / u^+$ are in fact equal, so that one can define the variable $R$ as in (3.1). Both properties $u_2^+ = 0$, $h_2^- = 0$ follow from solving the third and fifth rows of the jump conditions in (2.4); the fourth equation in (2.4) implies that $h_1^+ = h_1^-$. Setting $M := \frac{u_1^+}{\sqrt{\rho_0^+ p_0^+}}$, we obtain the rightmost condition in (3.1) from the second relation in the jump condition (2.4). This establishes (i).

With regards to (ii), classical compressibility conditions in gas dynamics, namely $p_\rho(\cdot) > 0$, gives that $R > 1$; by inspection of (3.1), one can see that this condition implies that $0 \leq M < 1$, for $\gamma \geq 1$.

Last, we prove (iii): according to Hersh’s Lemma (cf. [28], or [9, Chap. 4, Lemma 4.1]), it suffices to study the symbol of the operator (2.3) in the 1-D scenario, i.e., when $x_1$ is the only spatial variable taken into account, which is equivalent to studying the spectrum of the matrix.
\[ A_0^{-1}A_1 \] on both sides of the shock:

\[
A_0^{-1}A_1 = \begin{pmatrix}
    u_1 & \rho & 0 & 0 & 0 \\
    \frac{\rho_0}{\rho} & u_1 & 0 & 0 & 0 \\
    0 & 0 & u_1 & 0 & -\frac{h_1}{\rho} \\
    0 & 0 & 0 & \beta & 0 \\
    0 & 0 & -h_1 & 0 & u_1
\end{pmatrix}, \quad \sigma((A_0^{-1}A_1)^\pm) = \left\{ u_1^\pm \pm \sqrt{\frac{p_0^\pm}{\rho^\pm}}, \beta, u_1^\pm \pm \frac{h_1}{\sqrt{\rho^\pm}} \right\}.
\]

In the large magnetic field scenario \( h_1 \to \infty, \beta > 0 \) and a zero speed shock we count 2 negative/3 positive (resp. 1 negative/4 positive) eigenvalues for \( x_1^\pm > 0 \) (resp., \( x_1^\pm < 0 \)) whenever \( u_1^\pm \pm \sqrt{\frac{p_0^\pm}{\rho^\pm}} \leq 0 \); using the definitions in (3.1) we can see that this is equivalent to \( M < 1 \) (using the constraints for \( x_1^\pm > 0 \)) and \( R > 1 \) (using the constraints for \( x_1^\pm < 0 \)). Therefore, the discussion above says that we just need to analyze the signs of \( 1 - \frac{h_1}{u_1^\pm \sqrt{\rho^\pm}} \) and \( 1 - \frac{h_1}{u_1^- \sqrt{\rho^-}} = 1 - \frac{h_1}{u_1^+ \sqrt{R\rho^+}} \). As \( R > 1 \) we have

\[ 1 - \frac{h_1}{u_1^+ \sqrt{\rho^+}} < 1 - \frac{h_1}{u_1^- \sqrt{\rho^-}} = 1 - \frac{h_1}{u_1^+ \sqrt{R\rho^+}}. \]

Therefore, recalling \( R = \frac{u_1^-}{u_1^+} \), we obtain the following table corresponding with (1.3):

| Parameter range | Num of positive e-vals (Left,Right) | Shock type                  |
|-----------------|-----------------------------------|-----------------------------|
| \( h_1 > H^* := u_1^- \sqrt{\rho^-} \) | (4,3)                            | Lax 2-shock (Slow)          |
| \( h_1 < H_s := u_1^+ \sqrt{\rho^+} \) | (5,4)                            | Lax 1-shock (Fast)          |
| \( H_s < h_1 < H^* \)     | (5,3)                            | Doubly overcompressive      |

Notice that whenever \( \beta > 0 \) the \( \beta \)-model preserves slow shocks, but this property can also be verified whenever a positive multiple of \( \text{div}(h) \) is added to any upstream/downstream side of the propagating shock. In particular, one may add different multiples to upstream and downstream sides, as for example in [21, 24], allowing one to symmetrize the equations while preserving 2-shock structure. Last, we remark that the case \( \beta = 0 \) is degenerate, for the matrix \( A_1 \) is not invertible in this case; the invertibility of \( A_1 \) is necessary in the construction of the Lopatinski determinant, which we discuss next.

### 3.1. Lopatinski determinant: construction and asymptotic analysis.

In this section we study the Lopatinski determinant associated to inviscid parallel shocks and the onset of instability. Initially, we study the behavior of the system (2.3) by taking its Laplace-Fourier transform (in \( t \) and \( x_2 \), respectively). We obtain

\[
\lambda A_0^\pm v + A_1^\pm v_{x_1} + i\xi A_2^\pm v = 0 \implies v_{x_1} = -A_0^\pm (\lambda A_0^\pm + i\xi A_2^\pm) v,
\]

Nonconstant \( \beta \) is inconvenient however for the viscous case, destroying conservative structure.
where \( v \) is the Laplace-Fourier transform of \( u \) and

\[
-(A_1^\pm)^{-1} (\lambda A_0^\pm + i\xi A_2^\pm) = \begin{pmatrix}
-\frac{\lambda_{u_1}}{u_1^2-p_0} & \frac{\lambda_{\rho}}{u_1^2-p_0} & -\frac{i\rho_1\xi}{i\rho_1} & 0 & 0 \\
\frac{\rho u_1^2}{u_1^2-p_0} & -\frac{\lambda_{\rho}}{u_1^2-p_0} & \frac{i\rho_1\xi}{i\rho_1} & 0 & 0 \\
0 & \frac{\rho u_1^2}{u_1^2-p_0} & -\frac{\lambda_{\rho}}{u_1^2-p_0} & \frac{i\rho_1\xi}{i\rho_1} & 0 \\
0 & 0 & \frac{\rho u_1^2}{u_1^2-p_0} & -\frac{\lambda_{\rho}}{u_1^2-p_0} & \frac{i\rho_1\xi}{i\rho_1} \\
-\frac{i\rho_1\xi}{i\rho_1} & 0 & -\frac{\lambda_{\rho}}{u_1^2-p_0} & \frac{i\rho_1\xi}{i\rho_1} & -\frac{\lambda_{\rho}}{u_1^2-p_0} \\
\end{pmatrix},
\]

where the variables \((u_1, \rho, p_0)\) should be read \((u_1^+, \rho^+, p_0^+)\) (resp., \((u_1^-, \rho^-, p_0^-)\)) whenever \( x_1 > 0 \) (resp. \( x_1 < 0 \)). Equation (3.2) consists of two systems of ODEs in the interior \((x_1 \geq 0)\), coupled through Rankine-Hugoniot conditions at \( x_1 = 0 \) (Appendix C; see also [55] for further discussion on the technique). The Lopatinski determinant is defined as

\[
\Delta(\lambda, \xi) = \det(A_1^+ \mathcal{E}^+, A_1^- \mathcal{E}^-, \lambda[f_0] + i\xi[f_2])\bigg|_{x_1=0}.
\]

The parameter \( \lambda \) is a spectral parameter indicating solutions to the system (2.3) with growth \( \sim e^{\lambda t} \) (thus, \( \text{Re}(\lambda) > 0 \) corresponding to instability); \( \mathcal{E}^\pm(\lambda, \xi) \) denote manifolds of (spatial) decaying solutions in \( x_1 \geq 0 \). In the rest of the paper we omit the dependence of these spaces on \( \lambda \) and \( \xi \), simply writing \( \mathcal{E}^\pm \).

3.1.1. Large-\( h_1 \) asymptotics. Following [24] [13] we now study the large magnetic field \( \mathcal{F}_1 \) asymptotics. It is convenient to define the quantity \( \varepsilon := \frac{1}{h_1} \), which parametrizes the underlying viscous profile \( \mathcal{V} = \mathcal{V}(\varepsilon) \). Our analysis consists of Taylor expanding the roots of the Lopatinski determinant defined in (3.3) considered as a function in \( \varepsilon \) with \( \xi \) held fix at 1, i.e., \( \varepsilon \mapsto \lambda(\varepsilon) \). It is shown that

\[
\lambda(\varepsilon) = \lambda_2 \varepsilon^2 + \mathcal{O}(\varepsilon^3)
\]

where \( \lambda_2 > 0 \). One can conclude that an unstable regime occurs in the large magnetic field scenario, as verified in [24].

The study of the spaces \( \mathcal{E}^\pm \) is equivalent to analyzing the eigenvalues of \( -(A_1^\pm)^{-1}(\lambda A_0^\pm + i\xi A_2^\pm) \) and their associated eigenspaces. The approach we adopt relies on careful analytical estimates allied with the use of symbolic computations (carried out in SAGE; see [44]). The main idea is the following: assume that the spectral parameter \( \lambda \) can be expanded as

\[
\lambda = \lambda_0 + \lambda_1 \varepsilon + \lambda_2 \varepsilon^2 + \ldots
\]

Thus each element \( \mu \in \sigma(-(A_1^\pm)^{-1}(\lambda A_0^\pm + i\xi A_2^\pm)) \) can be expanded as

\[
\mu = \mu(\lambda) = \mu_0(\lambda_0) + \varepsilon \mu_1(\lambda_0, \lambda_1) + \varepsilon^2 \mu_2(\lambda_0, \lambda_1, \lambda_2) + \mathcal{O}(\varepsilon^3),
\]

Afterwards we find a set of eigenvectors of \( -(A_1^\pm)^{-1}(\lambda A_0^\pm + i\xi A_2^\pm) \) spanning the spaces \( \mathcal{E}^\pm \); namely, a mapping

\[
\mu \mapsto X^\pm(\mu) : \sigma\left(-\left(A_1^\pm\right)^{-1}\left(\lambda A_0^\pm + i\xi A_2^\pm\right)\right) \to \mathcal{E}^\pm.
\]

For the sake of convenience we drop the indexes “\( \pm \)” for now, since these formulas work for both cases if one use the notation in (3.1). If we also expand the eigenvector \( X(\mu) \) in (3.7) in \( \varepsilon \) terms we have

\[
X(\mu) = X^{(0)}(\mu_0) + \varepsilon X^{(1)}(\mu_0, \mu_1) + \varepsilon^2 X^{(2)}(\mu_0, \mu_1, \mu_2) + \ldots
\]

Observation 3.2. With regards to the symbolic computations, a few remarks are in hand:
(i) In order to find the expansion in (3.6) we find the characteristic polynomial $p(\cdot)$ of the matrix $-(A_1^+)^{-1}(\lambda A_0^+ + i\xi A_2^+)$ when (3.5) holds; we conclude by matching coefficients. The characteristic polynomial $p(\mu)$ can be shown to expand as $p(\mu) = p(\mu_0, \mu_1, \mu_2, \ldots) = p_0(\mu_0) + \varepsilon p_1(\mu_0, \mu_1) + \varepsilon^2 p_2(\mu_0, \mu_1, \mu_2) + \ldots$, with $p_j$ computed explicitly using SAGE;

(ii) Higher order terms in the expansion (3.6) in terms of $\varepsilon$ can be easily obtained using SAGE, because once we have $\mu_0 \ldots \mu_\ell$ the problem of finding $\mu_{\ell+1}$ is linear; these terms won’t be written explicitly here though;

(iii) It is not hard to show that $-\frac{\lambda}{\beta}$ is an eigenvalue of $-(A_1^+)^{-1}(\lambda A_0^+ + i\xi A_2^+)$: This also holds true in the nonparallel case, as we will show later;

(iv) Notice that, upon scaling, we can take $\xi = 1$ (redefine $\lambda \to \lambda \xi$);

(v) Let $X_i$ denote the $i$th column of a square matrix $A = [X_1, X_2, \ldots, X_n]$ and $X_i(\varepsilon) = X_i(0) + \varepsilon X_i(1) + \varepsilon^2 (X_i(2)) + O(\varepsilon^3)$. Then

$$det(A) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + O(\varepsilon^3),$$

where $A_k = \sum_{a_1+\ldots+a_n=k} det(X_1^{(a_1)}, \ldots, X_n^{(a_n)})$.

Taking into account these observations, we plug (3.5), (3.6) and (3.7) into (3.3), to obtain

$$\Delta(\varepsilon)(\lambda, 1) = \Delta_0 + \varepsilon \Delta_1 + \varepsilon^2 \Delta_2 + O(\varepsilon^3).$$

(3.8)

In what follows we shall exploit the multi-linearity of the determinant function in order to find the terms in this expansion; each term $\Delta_i$ is a function of $(\lambda_i)_{i \in \mathbb{N}}$, but this dependence will be most of the time omitted and made explicit as we carry out our computations. We make use of (3.8) to verify the condition $\Delta(\varepsilon)(\lambda, 1) \equiv 0$: we look for $(\lambda_i)_{i \in \mathbb{N}}$ that gives $\Delta_i(\lambda) = 0$ for all $i \in \mathbb{N}$. An important step in the analysis consists of an explicit representation of the manifolds $\mathcal{E}^\pm$ referred to in (3.3), an investigation that we reformulate as the study of the eigenvalues of the matrices in (3.2). Indeed, one can observe by inspection that $-\frac{\lambda}{\beta}$ is an eigenvalue of $-(A_1^+)^{-1}(\lambda A_0^+ + i\xi A_2^+)$, while an expansion of the eigenvalues of the form (3.6) readily shows that their zeroth order terms $\mu_0$ for $x_1 \geq 0$ are

$$\pm\frac{1}{u_1 + \sqrt{p_\rho}}(\lambda_0 + \varepsilon \lambda_1) + O(\varepsilon^2), \quad \text{if } \frac{1}{u_1 + \sqrt{p_\rho}} > 0,$$

$$\pm\frac{1}{u_1 - \sqrt{p_\rho}}(\lambda_0 + \varepsilon \lambda_1) + O(\varepsilon^2), \quad \text{if } \frac{1}{u_1 - \sqrt{p_\rho}} < 0,$$

where all the variables $(u_1, \rho, p_\rho)$ should be read $(u_{1,0}^+, p_\rho^+, p_\rho^-)$ (resp., $(u_{1,0}^-, p_\rho_-, p_\rho^0)$) whenever $x_1 > 0$ (resp, $x_1 < 0$). Therefore, since we are looking for “decaying” manifolds we must have, for $x_1 > 0$, eigenspaces associated to the following eigenvalues:

$$-1 + O(\varepsilon^2), \quad -\frac{1}{u_1 + \sqrt{p_{\rho^+}}}(\lambda_0 + \varepsilon \lambda_1) + O(\varepsilon^2), \quad \text{and } \frac{\lambda}{\beta}.$$ 

Analogously, “decaying” manifolds in $x_1 < 0$ must be eigenspaces associated to the eigenvalue $1 + O(\varepsilon^2)$. Notice that the number of eigenvalues in each interior $x_1 \geq 0$ is consistent with the analysis derived from Hersh’s Lemma, hence it suffices to analyze the number of positive and negative eigenvalues of $-A_1^{-1}A_0$.

3.1.2. Asymptotic instability. Choosing an appropriate parametrization of the decaying manifolds $\mathcal{E}^\pm$ one can show that $\Delta_0 = \Delta_1 = 0$ in equation (3.8). A careful computation shows then that
\[\Delta_2 = \Delta_2(\lambda_0, \mu_0) = \det(X^{(1)}_1, X^{(0)}_2, X^{(0)}_3, X^{(1)}_4, X^{(0)}_5).\] We have
\[(X^{(1)}_1, X^{(0)}_2, X^{(0)}_3, X^{(1)}_4, X^{(0)}_5) =
\begin{pmatrix}
0 & E_\ast & 0 & 0 & \frac{2\lambda_0 u^+_1 (\rho^- - \rho^+)}{(u^+_1)^2 - \rho^+}
0 & \frac{i\lambda_0}{u^+_1 + \sqrt{p_{\rho^+}}} & 0 & 0 & \frac{\lambda_0 (\rho^- - \rho^+)}{\rho^+}
-(\lambda_0 - u^+_1)^2 & 0 & 0 & -\frac{u^-_1 (\lambda_0 + u^-_1)}{u^-_1} & 0
i u^+_1 (\lambda_0 - u^+_1) & 0 & 0 & -\lambda_0 & 0
\lambda_0 (\lambda_0 - u^+_1) & 0 & i u^+_1 & \frac{\lambda_0 u^+_1}{u^-_1} (\lambda_0 + u^-_1) & 0
\end{pmatrix},
\]
where \(E_\ast = \frac{i\lambda_0 (u^+_1)^2}{(u^+_1 + \sqrt{p_{\rho^+}}) p_{\rho^+} + \left(\frac{u^+_1}{u_1} + \sqrt{p_{\rho^+}}\right)^2 p_{\rho^+}}\) and \(\frac{1}{\rho^+} \frac{\lambda_0 (\rho^- - \rho^+)}{\rho^+}\).

The block structure of this matrix allows us to see right away that \(\Delta_2 = O(\lambda_0^3)\) and \(\Delta_2 = o(\lambda_0)\). There exists a \(\lambda_0 = O(1)\) satisfying \(\Delta_2 = 0\) at which, however, the parametrization of the manifolds \(\xi^\pm\) through their eigenvectors is lost because two of those eigenvectors coincide. These points are called \(\text{glancing modes}\); this issue was also pointed out in [24 §3.4]. Resorting to generalized eigenvectors one can show that the Lopatinski determinant does not vanish for this value of \(\lambda_0\), hence the only solution to \(\Delta_2 = 0\) is \(\lambda_0 = 0\).

In the search for roots of \(\Delta_3 = 0\) the now look for \(\lambda_1\). Thanks to the multilinearity of the determinant function, we readily obtain that \(\Delta_3 = 0\). We go to the next term in (3.8): thanks to Obs. 3.2 it is not hard to see that \(\Delta_4 = \det(X^{(1)}_1, X^{(0)}_2, X^{(0)}_3, X^{(1)}_4, X^{(0)}_5)\). A computation shows that the latter determinant has order \(O(\lambda_0^2)\), but not order \(o(\lambda_0^2)\), which implies that \(\lambda_1 = 0\). A similar analysis leads to the expression of the next term, that is,
\[\Delta_5 = \det(X^{(1)}_1, X^{(0)}_2, X^{(0)}_3, X^{(1)}_4, X^{(0)}_5).\]

Since the first two rows are linearly dependent, we can see that the determinant of the later matrix is zero. A more involved analysis is necessary in dealing with the \(\varepsilon^6\)-order term in (3.8): \(\Delta_6\) can be written as
\[
\Delta_6 = \det(X^{(2)}_1, X^{(2)}_2, X^{(0)}_3, X^{(1)}_4, X^{(1)}_5) + \det(X^{(1)}_1, X^{(2)}_1, X^{(0)}_2, X^{(0)}_3, X^{(2)}_4, X^{(1)}_5) + \\
\det(X^{(1)}_1, X^{(0)}_2, X^{(0)}_3, X^{(1)}_4, X^{(2)}_5) + \det(X^{(1)}_1, X^{(0)}_2, X^{(0)}_3, X^{(1)}_4, X^{(1)}_5).
\]

A simple analysis shows that the last determinant is zero (due to the structure of columns 1, 3, 4 and 5). We end up with
\[
\Delta_6 = \frac{i (R - 1)^2 M^3 R^3}{M^2 - 1} L_3 \lambda_0 (u^+_1)^7 - \frac{2 (i R - i) M^2 R^3}{M - 1} (u^+_1)^4.
\]

Setting \(\Delta_6 = 0\) and solving for \(\lambda_2\), we obtain
\[
\lambda_2 = \frac{M R^3 (u^+_1)^3 - M R^2 (u^+_1)^3}{2 (M + 1)} = \frac{(R - 1) M R^2 (u^+_1)^3}{2 (M + 1)}.
\]

This function is clearly positive for all \(R > 1\), from where we readily conclude inviscid instability for all values of \(\gamma \geq 1\), a result that extends and improves the results of [24 Page 3036], which were limited to the case \(\gamma \in [1, 2]\). Furthermore, our result shows a higher level of accuracy when compared to the results of [24] (see also the appendix C) and those of [35]; this improvement is clear once we compare the predicted analytically determined value for the instability of the Lopatinski determinant with those values observed numerically, as discussed in Section 4.
3.1.3. Comparison with previous results: parallel case. We investigate the Lopatinski determinant and its roots numerically for various values of $\gamma$, corresponding to monoatomic gas ($\gamma = 5/3$) and diatomic gas ($\gamma = 7/5$; for instance, $O^2$), $\gamma = 3$ (artificial gas) and compare those values to the analytically predicted result in equation (3.9); the comparison is shown in table 1. Another representative description of the good agreement between the analytical result and the numerical study can be also seen in Figures 3 and 1. In particular, Figure 1 points out the accuracy of our study can be also seen in Figures 3 and 1. In particular, Figure 1 points out the accuracy of our

| $\gamma$ \(\backslash\) $R$ | 1.5   | 2     | 2.5   | 3     | 3.5   | 4     | 4.5   | 5     | 5.5   | 6     |
|-----------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\gamma = 5/4$, exact       | 7.40e-2 | 1.01e-1 | 1.13e-1 | 1.18e-1 | 1.20e-1 | 1.20e-1 | 1.19e-1 | 1.18e-1 | 1.16e-1 | 1.10e-1 |
| $\gamma = 5/4$, est         | 6.65e-2 | 9.76e-2 | 1.10e-1 | 1.18e-1 | 1.16e-1 | 1.16e-1 | 1.16e-1 | 1.18e-1 | 1.13e-1 | 1.10e-1 |
| $\gamma = 5/4$, err         | 1.01e-1 | 3.64e-2 | 2.98e-2 | 2.54e-2 | 2.64e-2 | 2.72e-2 | 2.16e-2 | 2.26e-3 | 2.29e-2 | 3.52e-2 |
| $\gamma = 7/5$, exact       | 7.33e-2 | 0.100  | 1.11e-1 | 1.15e-1 | 1.17e-1 | 1.17e-1 | 1.16e-1 | 1.14e-1 | 1.13e-1 | 1.11e-1 |
| $\gamma = 7/5$, est         | 6.65e-2 | 9.97e-2 | 1.10e-1 | 1.15e-1 | 1.16e-1 | 1.16e-1 | 1.16e-1 | 1.15e-1 | 1.10e-1 | 1.10e-1 |
| $\gamma = 7/5$, err         | 9.44e-2 | 2.32e-3 | 8.06e-3 | 6.05e-3 | 4.70e-1 | 3.36e-3 | 1.51e-2 | 4.11e-3 | 2.02e-2 | 6.98e-3 |
| $\gamma = 5/3$ exact        | 7.24e-2 | 9.77e-2 | 1.08e-1 | 1.11e-1 | 1.12e-1 | 1.12e-1 | 1.11e-1 | 1.09e-1 | 1.07e-1 | 1.05e-1 |
| $\gamma = 5/3$ est          | 6.92e-2 | 9.84e-2 | 1.09e-1 | 1.12e-1 | 1.11e-1 | 1.12e-1 | 1.12e-1 | 1.11e-1 | 1.10e-1 | 1.05e-1 |
| $\gamma = 5/3$ err          | 4.37e-2 | 7.23e-3 | 7.70e-3 | 1.73e-3 | 3.43e-3 | 3.02e-3 | 1.01e-2 | 2.54e-2 | 4.65e-3 | 1.88e-3 |
| $\gamma = 3$ exact          | 6.78e-2 | 8.77e-2 | 9.40e-2 | 9.53e-2 | 9.47e-2 | 9.32e-2 | 9.13e-2 | 8.93e-2 | 8.73e-2 | 8.54e-2 |
| $\gamma = 3$ est            | 7.11e-2 | 9.41e-2 | 9.98e-2 | 1.02e-1 | 1.02e-1 | 1.05e-1 | 1.01e-1 | 9.60e-2 | 9.84e-2 | 9.29e-2 |
| $\gamma = 3$ err            | 4.89e-2 | 7.30e-2 | 6.17e-2 | 7.26e-2 | 7.98e-2 | 1.23e-1 | 1.10e-1 | 7.50e-2 | 1.26e-1 | 8.86e-2 |

Table 1. Comparison of exact coefficient with numerical coefficient. Parameters $u_1 = 1$, $\rho^\circ = 1$ are fixed; recall that $u_1^\circ = u_1/R$. For each value of $\gamma$ and $R$ we record the exact answer, the numerical estimate (est), and the relative error (err) between the two. Numerical estimates of the coefficient were determined by computing the roots $\lambda(h_1)$ of the Lopatinski determinant for several values of $h_1 \in [2,16]$ and then using curve fitting of $\log(\lambda)$ and $\log(1/h_1)$. The maximum relative error is 4.70e-1 and the average relative error is 4.83e-2.

representative description of the good agreement between the analytical result and the numerical study can be also seen in Figures 3 and 1. In particular, Figure 1 points out the accuracy of our results when compared to those presented in [24]. However, it is worthwhile to stress that the analysis in the latter paper gives the correct order for the root of the Lopatinski determinant, i.e., $\lambda = O(e^2)$; apparently, this result was already known in the astrophysics community since the late 80’s, as one can see in the formal linear analysis performed in [45 §2.1].

3.2. Full inviscid stability diagram. In this section, following the scaling of [6], we fix $u_1^\circ = 1$, $\rho^\circ = 1$ so that $\rho^-u_1^\circ = \rho^+u_1^\circ = 1$ (thanks to the Rankine-Hugoniot conditions (2.4)). With regards to the parametrization of Lemma (3.1), it consists with $R = 1/\bar{u}_1$. Note for this choice of parameters that the slow shock classification of (1.3) simplifies to $h_1 > H^* = 1$.

We complete our study of parallel inviscid shock stability by a numerical stability analysis over all parameters, complementing the asymptotic study of the previous subsections. Recall [24] that 1-D stability, or nonvanishing of $\Delta(\lambda, 1)$, has previously been verified. Thus, without loss of generality, we may fix $\xi = 1$ by homogeneity (see Observation 3.2(iv)), reducing the question of stability to nonvanishing of $\Delta(\lambda, 1)$ on $\Re(\lambda) \geq 0$. Noting that $\Delta(\lambda, 1)$ is analytic in $\lambda$, this can be done by a winding number computation. Indeed, if we evaluate $\Delta(\lambda, 1)$ along a contour in the complex plane that encloses any possible unstable roots of $\Delta(\lambda, 1)$, and if the resulting image contour has winding number 0, then the associated shock is stable, and if the winding number is positive, then the shock is unstable. We numerically evaluate $\Delta(\lambda, 1)$ along the contour $\partial\{z \in \mathbb{C} : \Re(z) \geq 0\} \cap \{z \in \mathbb{C} : |z| = r\}$ where $R > 0$ is sufficiently large. In practice, we took $r = 10$, which appears to be...
ample large. When \( \Re(\lambda) = 0 \), the real part of the eigenvalues of (3.1) collapse to zero making it difficult numerically to detect the correct bases for evaluating \( \Delta(\lambda, 1) \). To get around this technical difficulty, we simply in practice shift to the right of the contour on which we compute \( \Delta(\lambda, 1) \) by \( 1e - 4 \). As displayed in Figure 2 (b), we examine inviscid stability for \( u_1^+ \in \{0.05, 0.1, \ldots, 0.9, 0.95\} \) and \( h_1 \in \{1.1, 1.2, \ldots, 3.9, 4\} \).

The results displayed in Figure 2 (b) indicate a single stability transition for each fixed \( u_1^+ \) sufficiently close to \( u_{-1} = 1 \) as \( h_1 \) is increased from \( H^* = 1 \) (stability) to \( \infty \) (instability). For smaller \( u_1^+ \), corresponding to larger-amplitude waves, all slow shocks appear to be multi-d unstable independent of the strength of the magnetic field \( h_1 \), hence there is no stability transition. Moreover, we observe, similarly as in the large-\( h_1 \) case, that in the unstable case the instability corresponds to a double real root (winding number two), so that the stability transition as described in the introduction corresponds to passage of a double root through the origin \( \lambda = 0 \).

3.2.1. The critical destabilization parameter. Based on the above observations, to pinpoint the location of the stability transition \( h_1 \) for a given fixed \( u_1^+ \), we have only to numerically solve \( \Delta(0, 1) = 0 \), considered as an equation in \( h_1 \). (Here, we are using the fact, a consequence of reflection symmetry, that \( \Delta(\lambda, \xi) \) may be normalized to be real for \( \lambda \) real.) Again, to avoid technical difficulties to do with pure imaginary \( \lambda \), we solve the approximate equation \( \Delta(\lambda_0, 1) = 0 \) where \( \lambda_0 = 1e - 5 \) via the bisection method, where \( \Delta(\lambda, 1) \) is normalized by \( \Delta(0.1, 1) = 1 \). The result is displayed in Figure 2(a)(b) together with the results of our more complete coarse-mesh computations. The thick dashed line in Figure 2 marks the critical destabilization parameter. Apparently the stability region is exclusively determined by this critical destabilization parameter curve.

3.3. A remark on the nonparallel case. Our approach to the nonparallel case is similar to the parallel case: we compute an eigenbasis associated to decaying manifolds on both \( x_1 \geq 0 \) sides, expand their entries in \( \varepsilon \) in order to expand the Lopatinski determinant \( \Delta \) in \( \varepsilon \) (as in (3.8)). One obtains an expansion of the type

\[
\Delta = \frac{1}{\varepsilon^6} \Delta_{-6} + \frac{1}{\varepsilon^5} \Delta_{-5} + \frac{1}{\varepsilon^4} \Delta_{-4} + \ldots
\]

(3.10)

It is not hard to see that \( \Delta_{-6} = \Delta_{-5} = 0 \), since both corresponding matrices have rows of zeros. In order to find instability we need to show that there exists a \( \Re(\lambda_0) > 0 \) such that \( \Delta_{-4} = \Delta_{-4}(\lambda_0) = 0 \). The analytical study of this determinant is very complex, even if we use symbolic computations, so we approach this part numerically: we could verify that there exists an order 1 root with positive real part, so we do have instability; this result confirms and elucidates the assertions in [24, Remark 3.5] regarding the nonparallel case. However, the root we found does not agree with the explicit formula given in [24, Remark 3.5]; as the proof of this formula is not given in [24] we are unable to determine the reason for this discrepancy. Further, one can see in Figure 9 that in the nonparallel case the roots are not real valued. This appearance of complex roots corresponds to a break of \( O(2) \) symmetry upon linearization (see also [22, 43, 24]).

4. Viscous stability analysis: Evans function

We also study the viscous linear stability analysis using Evans function techniques. Roughly speaking, the study of the Evans function proceeds as follows: consider the system,

\[
f_0(u)_t + \sum_{j=1}^{d} A_j(u)\partial_{x_j} u = \sum_{j,k=1}^{d} (B_{jk}(u)u_{x_k})_{x_j},
\]
and make the change of coordinates $x_1 \rightarrow x_1 - st$ to obtain,

$$f_0(u)_t - sf_0(u)_{x_1} + \sum_{j=1}^{d} A_j(u) \partial_{x_j} u = \sum_{j,k=1}^{d} (B_{jk}(u) u_{x_k})_{x_j}.$$  

We linearize about a planar traveling wave solution $\bar{u}$, traveling in the direction $x_1$ to obtain,

$$(A^0u)_t - s(A^0u)_{x_1} + \bar{C}u + \sum_{j=1}^{d} \bar{A}_j \partial_{x_j} u = \sum_{j,k=1}^{d} \bar{B}_{jk} u_{x_k x_j} + \sum_{j=1}^{d} (dB_{j1}(\bar{u})(u, \partial_{x_1} u)_{x_j},$$

where $\bar{A}_j := A_j(\bar{u})$ and the linearization of $A_1(u) \partial_{x_1} u$ is $\bar{C}u + \bar{A}_1 \partial_{x_1} u$. Next we take the Fourier transform in the variables $\xi_2, \ldots, \xi_d$, and the Laplace transform in $t$ to obtain the eigenvalue problem,

$$\lambda \bar{A}^0u - (\bar{A}_1)'u + \bar{C}u + \sum_{j=2}^{d} i \xi_j \bar{A}_j u + \sum_{j,k=2}^{d} \xi_j \xi_k \bar{B}_{jk} u = \left( \bar{B}_{11} u' + \sum_{j=2}^{d} i \xi_j \bar{B}_j u - \bar{A}_1 u \right)' \tag{4.1}$$

where $\bar{A}_j := \bar{A}_j + (\bar{B}_{j1})'$, $\bar{B}_j := \bar{B}_{j1} + B_{jj}$, $\bar{A}_1 := \bar{A}_1 u - sA^0u - dB_{11}(\bar{u})(u, \partial_{x_1} u)$, and $\bar{A}_j := \bar{A}_j u - dB_{j1}(\bar{u})(u, \partial_{x_1} u)$ for $j \geq 2$. We obtain our Evans function coefficient matrix by using the flux form, only wherever we see $\bar{A}^\xi := \sum_{j=2}^{d} \xi_j \bar{A}_j$, we instead use $i(\bar{A}_1)' - i\bar{C} + \sum_{j=2}^{d} \xi_j \bar{A}_j$ (for further discussion, see [3]).

The Evans function $(\lambda, \xi) \rightarrow D(\lambda, \xi)$ consists of a measurement at $x_1 = 0$ of the transversality between the decaying manifolds of the ODE (4.1) below when restricted to the spaces $x_1 \geq 0$; it is an analytic function of both its parameters whenever $\lambda$ is in the domain of consistent splitting (see [2,9]). For our study of stability in a channel the relevant values of $\xi$ are $\xi \in \frac{2\pi}{L} \mathbb{Z}$, $L$ being the width of the channel we are studying.

According to the results in [55], the Lopatinski determinant is a first order approximation of the Evans function in the low frequency regime; consequently, inviscid instability implies viscous instability. The latter implication is our main motivation in the search for zeros of the Evans function, i.e., values of the spectral parameter $\lambda$ such that $D(\lambda, \xi) = 0$ for some $\xi \in \frac{2\pi}{L} \mathbb{Z}$.

Apart from constraint issues, substantial new difficulties in going from 1-D to multi-D Evans function computations arise:

(i) Number of equations/parameters: we deal with $5 \times 5$ system of equations with downstream/upstream shock conditions. The complexity, both mathematical and numerical, is enormous. Symbolic computations are necessary to create the code without risk of human error. Fortunately, the $\beta$-model allows use of previously tested code (STABLAB; see more in [3]);
(ii) Unexpected issues with Evans function asymptotics related to Eulerian vs Lagrangian coordinates make computation of the multi-D numerical Evans function practically impossible; issues were only recently resolved in this project and, to knowledge of the authors, in only one other project (29).

4.1. Computing the profile. To solve for the viscous profile numerically, we cut the domain in half and use a coordinate change to reflect the interval \((-\infty, 0]\) to \([0, \infty)\). We use matching conditions at \(x = 0\) and projective boundary conditions at \(x = \infty\) that select the decaying solution. To solve the resulting three point boundary value problem, we use MATLAB’s bvp5c solver with relative and absolute error tolerances set respectively to \(10^{-6}\) and \(10^{-8}\).

4.2. Computation of decaying manifolds and eigenfunctions. We recall that the Evans function takes the form

\[
D(\lambda; \xi) = \det\left( [W_1^{-\infty}(0; \lambda, \xi), \ldots, W_k^{-\infty}(0; \lambda, \xi), W_{k+1}^{+\infty}(0; \lambda, \xi), \ldots, W_n^{+\infty}(0; \lambda, \xi)] \right),
\]

where

\[
\frac{d}{dx} W_j^{\pm\infty}(x; \lambda, \xi) = A(x; \lambda, \xi) W_j^{\pm\infty}(x; \lambda, \xi)
\]

(4.2)

and \(W_1^{-\infty}, \ldots, W_k^{-\infty}\) and \(W_{k+1}^{+\infty}, \ldots, W_n^{+\infty}\) form a basis for the solution space of (4.2) that decays as \(x \to -\infty\) and as \(x \to +\infty\), respectively. If \((\lambda_0, \xi_0, v_0)\) is an eigenvalue, Fourier mode, eigenfunction triple, then \(D(\lambda_0, \xi_0) = 0\) and \(v_0\) can be expressed as a linear combination of \(W_1^{-\infty}, \ldots, W_k^{-\infty}\) when \(x \in (-\infty, 0]\) and as a linear combination of \(W_{k+1}^{+\infty}, \ldots, W_n^{+\infty}\) when \(x \in [0, +\infty)\). Hence, to solve for an eigenvalue \(\lambda_0\) and eigenfunction \(v_0\) corresponding to a fixed \(\xi_0\), we may do as in [30], and solve (4.2) as a boundary value problem with \(\lambda\) as a free parameter. At \(x = \pm\infty\), we use projective boundary conditions, \(P^{\pm\infty}v_0(\pm\infty) = 0\), which force the projection of \(v_0\) onto the unstable and stable manifolds at \(x = \pm\infty\), respectively, to be zero. The projective boundary conditions at \(x = \pm\infty\) provide \(k\) and \(n - k\) boundary conditions, which leaves one additional boundary condition corresponding to the free parameter. We provide a phase condition, such as \(\|v_0(0)\| = 1\) or a component of \(v_0(0)\) is unity, which selects an eigenfunction from the family \(\{cv_0 : c \in \mathbb{C}, c \neq 0\}\). In practice, to numerically approximate \(v_0\) we divide the domain into two parts, \((-\infty, 0]\) and \([0, +\infty)\), and then perform the change of coordinates \(x \to -x\) on \((-\infty, 0]\), thus doubling the dimension of the system (4.2) now posed on \([0, +\infty)\). We then pose the boundary value problem on the finite interval \([0, L]\) where \(L\) is the truncation value approximating infinity as determined in solving the traveling wave profile. In the end, there is one phase condition given at \(x = 0\) and \(n\) projective boundary conditions given at \(x = L\).

To obtain an initial guess for the boundary value problem, we apply STABLAB’s built in root finding capabilities, such as the method of moments or a two-dimensional bisection method using squares in the complex plane, to the Evans function to find a \(\tilde{\lambda}_0\) which approximates the eigenvalue \(\lambda_0\) of interest. To approximate the eigenfunction \(v_0\), we set

\[
W_L(x) := [W_1^{-\infty}(x; \tilde{\lambda}_0, \xi_0), \ldots, W_k^{-\infty}(x; \tilde{\lambda}_0, \xi_0)]
\]

and \(W_R(x) := [W_{k+1}^{+\infty}(x; \tilde{\lambda}_0, \xi_0), \ldots, W_n^{+\infty}(x; \tilde{\lambda}_0, \xi_0)]\) and then find \(C = (c_L, c_R)^T\) that minimizes \(\| [W_L(0), -W_R(0)] C \|\) in the least squares sense subject to \(\|C\| = 1\). In practice, solving for \(W_L(x)\) and \(W_R(x)\) is difficult because of competing modes of \(A(x; \lambda, \xi)\) as \(x \to \pm\infty\). Thus, we compute \(W_L(x)\) and \(W_R(x)\) using the method of continuous orthogonalization of [32]. In this method, we set \(W_L = \Omega_L a_L\) and \(W_R = \Omega_R a_R\) where \(\Omega_L\) and \(\Omega_R\) are orthonormal basis of \(W_L\) and \(W_R\) respectively. In particular, as detailed in [32], \(\Omega_L\) and \(\alpha_L\) satisfy the well-conditioned ODEs

\[
\Omega' = (I - \Omega\Omega^*) \Omega
\]

\[
\alpha' = (\Omega^* A \Omega) \alpha.
\]
Thus, we can solve the ODE for $\Omega_L$ and $\Omega_R$ and then minimize $\|\Omega_L(0) - \Omega_R(0)\|C\|$ subject to $\|C\| = 1$, then afterward solve for $\alpha_L$ and $\alpha_R$ by initializing the associated ODE at $x = 0$. We note that solving for $\alpha_L$ from $x = 0$ to $x = -L$, or for $\alpha_R$ from $x = 0$ to $x = L$, is numerically well posed as error decays in this direction of integration. We then recover $W^L$ and $W^R$, which provides an initial guess for the boundary value problem described previously.

Figure 4 exemplifies the applicability of the numerical construction here described: it shows a graph of the real part of the variable $u_1$ of the eigenfunction associated to the bifurcating eigenvalue. Notice the loss in the planar structure, which is also pointed out in [42] in the strictly parabolic case for $O(2)$ steady bifurcations.

4.3. Viscous stability diagram. To determine stability of the viscous shock waves, we compute the Evans function, similar to the Lopatinski determinant, on a contour $\Omega_r := \partial(\{z \in \mathbb{C} : \Re(z) \geq 0\} \cap \{z \in \mathbb{C} : |z| = r\}$, where $r$ is now chosen by curve fitting the Evans function to within 0.2 relative tolerance of its asymptotic behavior $D(\lambda) \sim C_1 e^{C_2 \sqrt{\lambda}}$, indicating that any zeros of the Evans function that may exist lie within $\Omega_r$; see Appendix A for more details. However, we do limit $R \leq 128$ for practical reasons since the time to compute $D(\lambda)$ becomes unreasonable for $|\lambda|$ too large. To compute the Evans function, we use the method of continuous orthogonalization (33) described in Section 4.2, computed in pseudo-Lagrangian coordinates for better conditioning as described in [4]. We initialize the Evans ODE with a basis that varies analytically in $\lambda$, as described in [31]. All of these methods are built into the STABLAB platform with which we perform our computations [5].

When $\gamma = 5/3$, we sample the winding number for the Fourier coefficient $\xi \in [0.001, 0.004, 0.007, 0.01, 0.04, 0.07, 0.1, 0.14, 0.17, 0.2]$ for various values of $u_1^+$ and $h_1$ and plot the resulting stability diagram in Figure 2(a). When $\gamma = 7/5$, we obtain the corresponding stability diagram plotted in Figure 6. We note that the z-axis in Figure 6 indicates the value of $\xi$ to give a sense of which modes are unstable. In Table 2 we indicate for various parameters the radius needed to enclose any potentially unstable eigenvalues and we indicate the unstable root when it exists.

4.4. On the symmetry of eigenfunctions and equivariance of the Evans function. It is not possible to conclude from our analytical results that the bifurcating eigenvalues have associated dimension 2 rather than $2n$, $n \in \{2, 3, \ldots\}$. On the other hand, as we discuss next, an interesting conclusion can be derived with regards to the symmetry of the decaying manifolds discussed in this section.

By definition, in $O(2)$ symmetric systems, if $v(x, y)$ is a solution then $Rv(x, -y)$ is also a solution, for $R$ an orthogonal matrix. Fourier transforming in the y-direction, we observe that any real eigenvalue $\lambda$ has an eigenfunction $e^{iky}w(x)$ and also an eigenfunction $e^{-iky}\bar{R}w(x)$. However, by complex symmetry so is $e^{iky}\bar{w}(x)$. Likewise, $e^{iky}R\bar{w}(x)$. This suggests that an eigenvalue $\lambda = 0$ associated with $k_s \neq 0$ should have total multiplicity 4, 2 for each of $\pm k_s$, unless the apparently non-generic situation occurs that $Rw$ and $\bar{w}$ are constant multiples of one another.

The analogy to complex conjugation is apparent: given any eigenfunction $w$, form $(w + R\bar{w})$, the real part, and $(1/i)(w - R\bar{w})$, the imaginary part of the eigenfunction $w$. It is easily seen that both of these are invariant under

\[ T: f \rightarrow R\bar{f}, \]

and span the space $\text{span}\{w, R\bar{w}\}$ contained in the eigenspace of $L_{k_s}$. The same reasoning gives a symmetric basis of the subspaces of decaying solutions of $(\lambda - \bar{L})w = 0$ at $\pm \infty$. So, we can construct an Evans function from these eigenfunctions, and whenever there is a zero, we can find an eigenfunction given by a real linear combination of them, which is thus itself symmetric under the mapping $T$, that is, invariance under $T$ is in fact generic. For there to be non-symmetric eigenfunctions, there would have to be a higher multiplicity of linear dependence. Moreover, MHD
| \( \gamma \) | \( u_1^+ \) | \( h_1 \) | \( \xi \) | WND | Root | Radius | Num Pnts | Run time |
|-------|-------|-------|-------|-----|------|--------|---------|--------|
| 5/3   | 0.0001| 1.1   | 0.001 | 0   | NA   | 128    | 539     | 197    |
| 5/3   | 0.0001| 1.1   | 0.1   | 1   | 1.2280e-04 | 128 | 491     | 162    |
| 5/3   | 0.0001| 1.5   | 0.05  | 1   | 7.0364e-05 | 128 | 517     | 171    |
| 5/3   | 0.0001| 8     | 0.2   | 0   | NA   | 128    | 543     | 381    |
| 7/5   | 0.0001| 1.5   | 0.005 | 0   | NA   | 128    | 577     | 182    |
| 7/5   | 0.0001| 4     | 0.05  | 1   | 1.1829e-05 | 128* | 599     | 282    |
| 7/5   | 0.0001| 16    | 0.1   | 0   | NA   | 128    | 527     | 747    |
| 5/3   | 0.01  | 1.1   | 0.001 | 1   | 2.8902e-05 | 128* | 341     | 161    |
| 5/3   | 0.01  | 1.1   | 0.1   | 1   | 0.0024 | 128  | 297     | 128    |
| 5/3   | 0.01  | 1.5   | 0.05  | 1   | 7.5206e-04 | 64*  | 281     | 128    |
| 5/3   | 0.01  | 8     | 0.2   | 0   | NA   | 128    | 361     | 295    |
| 7/5   | 0.01  | 1.1   | 0.8   | 1   | 0.0050 | 128* | 287     | 139    |
| 7/5   | 0.01  | 2     | 1.6   | 0   | NA   | 64*    | 237     | 132    |
| 7/5   | 0.01  | 8     | 1.6   | 0   | NA   | 8*     | 133     | 163    |
| 5/3   | 0.2   | 1.1   | 0.005 | 1   | 4.7035e-04 | 16*  | 137     | 93.2   |
| 5/3   | 0.2   | 1.1   | 0.2   | 1   | 0.0137 | 4*   | 141     | 84.2   |
| 5/3   | 0.2   | 1.5   | 0.1   | 1   | 0.0044 | 32*  | 159     | 112    |
| 5/3   | 0.2   | 8     | 0.8   | 0   | NA   | 8*     | 171     | 203    |
| 7/5   | 0.2   | 1.5   | 0.05  | 1   | 0.0024 | 32*  | 165     | 113    |
| 7/5   | 0.2   | 4     | 0.1   | 1   | 3.7694e-04 | 4*  | 221     | 138    |
| 7/5   | 0.2   | 16    | 0.1   | 0   | NA   | 8*     | 251     | 521    |

Table 2. Table providing computational details of the viscous stability study. The fifth through ninth columns respectively show the winding number of the computation, the location of the root (if applicable) computed with absolute tolerance of \( 5 \times 10^{-7} \), the outer radius of the contour on which the Evans function was computed, the number of points on the contour, and the time in seconds the computation took to run. The Evans function was computed on a semi-annulus with inner radius \( 10^{-5} \) and outer radius as stated. A * indicates that the radius was taken large enough that curve fitting the Evans function with its asymptotic behavior yields a relative error no greater than 0.2.

gives an explicit example where the multiplicity is in fact 2. Indeed, if there are only 2 eigenvalues, then, choosing the representatives of eigenfunctions having symmetry, we see that this uses up all the dimensions and there cannot be more. Consequently, eigenfunctions may always be chosen with \( O(2) \) symmetry.

Therefore one can build an Evans function with \( O(2) \) symmetry and this detects “nice” eigenfunctions having the desired symmetry; there may well be others, but this would be “extra”, and there is no reason they would need to be there; that is, they are not generic.

4.5. **Finding the critical destabilization parameter** \( h_1 \). In the following discussion we assume \( \gamma = 5/3 \) and \( u_1^+ = 0.86 \). Determining exactly where the stability transition occurs in \( h_1 \) for the Evans function is a little difficult because the contour on which we compute the Evans function comes close to a root near the stability transition. A closer approximation was tried, but we could only confirm that, for \( \xi = 0.005 \), the Evans function has a root when \( h_1 = 2 \), but it does not when \( h_1 = 1.999 \). Several values of \( \xi \) were tested when \( h_1 = 1.999 \), and no zeros of the Evans function were found; on the other hand, a root is found when \( h_1 = 2 \), being approximately \( 2.86 \times 10^{-7} \). The
Lopatinski determinant for \( h_1 = 2 \) and \( \xi = 1 \) has a root at \( \lambda_0 = 4.26 \times 10^{-4} \), which for \( \xi = 0.005 \), corresponds to \( \lambda_0 = 2.13 \times 10^{-6} \). It was verified that the Lopatinski determinant has no root to the right of the vertical line \( \lambda = 10^{-4} \) for \( h_1 = 1.995 \), but it does for \( h_1 = 1.996 \). The contour cannot be taken much closer to the imaginary axis than \( 10^{-4} \) because of the essential spectrum.

In summary, we can estimate that the stability transition for the Lopatinski determinant occurs at \( h_1 \approx 1.995 \) and for the Evans function at approximately \( h_1 = 2.0 \). As discussed in the introduction, this is due to discretization of Fourier modes. The whole-space transition values agree, as a consequence of concavity of the associated spectral curves, illustrated in Figure 5.

4.6. **Verifying concavity.** In the previous subsection, we have verified concavity of the critical spectral curve/agreement of (whole space) viscous and inviscid transition values for one (typical) choice of parameters, essentially by force, by computing the critical spectral curve \( \lambda_*(\xi) \) and approximating the second derivative. In this subsection, we check concavity more efficiently using the implicit function theorem.

To verify concavity of the spectral curves \( \lambda(\xi) = 0 \) at the critical transition, we approximate the quantity

\[
\sigma := -\frac{\tilde{D}_\rho}{D_{\lambda_0}}|_{(\rho,\lambda_0,\xi_0)=(\epsilon,0,1)}
\]

using finite difference quotients, where \( \tilde{D}(\rho,\lambda_0,\xi_0) = D(\rho\xi_0,\rho\lambda_0) \) is the Evans function in polar coordinates. This may be recognized as the negative of the “effective viscosity coefficient” of [55, 51, 52, 10], with \( \lambda(\xi) = \sigma \xi^2 + O(\xi^3) \). Negativity of \( \sigma \) corresponds to the “refined stability condition” of the references.

To approximate \( \sigma \), we obtain an initial basis for the Evans function ODE by finding a basis \( B_{\pm} \) at \((\lambda,\xi) = (0,\epsilon)\), and then multiplying \( B_{\pm} \) on the left by an analytic projection onto the desired subspace, thus creating a locally analytically varying basis determining \( D \). We then compute the difference quotient approximation

\[
\sigma \approx \frac{D(0,\epsilon) - D(0,2\epsilon)}{D(\epsilon^2,\epsilon) - D(0,\epsilon)}
\]  

for various values of \( \epsilon > 0 \). We perform a convergence study to verify the correctness of our approximation of \( \sigma \), using five evenly spaced values of \( \epsilon \) between 1e-3 and 1e-6. We plot \( \sigma \) against \( u_1^- - u_1^+ \) in Figure 11(a), demonstrating that \( \sigma \) is always negative. Recall that \( \sigma > 0 \) indicates that instability may occur in the viscous system before it does in the inviscid system as the bifurcation parameter \( h_1 \) is increased. We also did a spot check to verify that we get the same value for \( \sigma \) when we interpolate the curves \( \lambda(\xi) = 0 \) with quadratic polynomials for various values of \( h_1 \), and
then interpolate the second derivative of these quadratic polynomials in the variable \( h_1 \) with a quartic polynomial, which we evaluate at \( h_1 = H_\ast \). In Figure 11(b), we demonstrate the quadratic interpolation of a typical curve \( \lambda(\xi) = 0 \). We note that we compute \( \sigma \) only for \( u_1^+ \) as small as \( 0.7005 \approx U_\ast \), where \( U_\ast \) is the value of \( u_1^+ \) at which shocks become inviscid stable at the minimum \( h_1 = H_\ast \) value for which they are 3-shocks.

The clear conclusion from Figure 11(a) is that the spectral curve is indeed concave at transition to instability, for all relevant values of physical parameters, in the case of a monatomic gas \( \gamma = 5/3 \).

4.6.1. Finer points: analyticity and glancing. The implicit function computation \( \sigma = -\bar{D}_\rho/\bar{D}_\lambda \) of the second-order coefficient in \( \lambda(\xi) = \sigma \xi^2 + \ldots \), or, equivalently

\[
\lambda(\xi) = \sigma + \ldots,
\]

defined by \( \bar{D}(\rho, \lambda_0(\rho), 1) = 0 \), presupposes analyticity of \( \bar{D} \) in \( (\rho, \lambda_0) \). As pointed out in [55, 50], analyticity holds away from “glancing points”, defined as frequencies \( \lambda_0 \) for which \( A_1^{-1}(\lambda_0 A_0 + i\xi_0 A_2) \) has neutral (i.e., zero real part) eigenvalues possessing a nontrivial Jordan block.

However, for the parallel MHD equations, [40, Lemma 7.2(ii)] specialized to the 2-D case considered here yields for \( (\lambda_0, \xi_0) = (0, 1) \) that there is always a Jordan block of dimension 2, hence an associated square-root singularity in the initializing decaying eigenspaces at both \( x \to \pm\infty \), inherited by the manifolds of decaying solutions. This is readily verified by direct computation of the zero-eigenspace of \( iA_1^{-1}A_2 \), which may be seen to have geometric multiplicity 2 but algebraic multiplicity 3. Thus, we cannot simply appeal to nonglancing to conclude analyticity: there is always glancing! On the other hand, the fact that both decaying manifolds at \( \pm\infty \) have a square-root singularity at \( \lambda = 0 \) implies that the Evans determinant obtained as their exterior product, by a monodromy argument, or simply by composing the two square roots to obtain a linear factor, is analytic, despite the presence of glancing modes. This justifies our computations above.

5. ADDITIONAL DESCRIPTION OF NUMERICS

To create Figure 8(a), we used the method of moments described in [14] to determine the roots of the Evans function. The moments were computed on the contour \( \partial((\{ z \in \mathbb{C} : \Re(z) \geq 0 \} \cap \{ z \in \mathbb{C} : |z| \leq 0.01 \}) \cup \{ z \in \mathbb{C} : |z| \geq 10^{-4} \}) \). To create Figure 8(b), we used a forward finite difference scheme to approximate the derivative using the data given in Figure 8(a).
APPENDIX A. ON WINDING NUMBER COMPUTATIONS

The study of unstable modes reduces to studying an eigenvalue equation: at the inviscid level through Lopatinski determinant, \( \Delta(\lambda_{\text{lop}}, \xi) \); at the viscous level through an Evans function, \( D(\lambda_{\text{ev}}, \xi) \). Due to analyticity of these two objects in their parameters, the search for growing modes corresponds to verifying if, for a fixed \( \xi^* \) there exists a root \( \lambda_{\text{lop}} \) of the Lopatinski determinant and a root \( \lambda_{\text{ev}} \) of the Evans function in the half space \( \text{Re}(z) > 0 \) of the complex space \( C \). This computation relies then on winding number computations, based upon the argument principle. An example of these computations can be seen in Figures 12a and 12b.

![Evans Function](image1)

(A) Plot of the image of the Evans function computed on a semi-circle contour, a contour with a vertical line centered at the origin and a half circle on the right connecting the end points of the vertical line, with radius 0.1 when \( \gamma = 5/3, u_1^+ = 0.4, \) and \( h_1 = 2 \).

![Lopatinski determinant](image2)

(B) Plot of the image of the Lopatinski determinant computed on a semi-circle of radius 0.5 shifted right by 0.01 when \( \gamma = 5/3, u_1^+ = 0.4, \) and \( h_1 = 2 \). A red plus sign marks the origin.

**Figure 12.** Winding number computations for the viscous and inviscid system. A red plus sign marks the origin.

APPENDIX B. GUIDE TO [24] AND [13] #1: PERSISTENCE OF CONSTRAINT CONDITION, A SECOND PROOF

We give here a second proof of Proposition 2.11 now following [24, Remark 3.2]. To begin with, we linearize (1.1c) about a shock profile \( \mathbf{V} = (\bar{\rho}, \bar{u}_1, 0, \bar{h}_1, 0) \) (thus, \( \bar{u}_2 = 0, \bar{h}_2 = 0 \), obtaining

\[
\begin{align*}
(h_1)_t + (\bar{h}_1 u_2 - h_2 \bar{u}_1)x_2 + \beta(h_{1,x_1} + h_{2,x_2}) &= 0; \\
(h_2)_t - (\bar{h}_1 u_2 - h_2 \bar{u}_1)x_1 &= 0.
\end{align*}
\]

(B.1)

In this section we will only make use of the first of these equations.

\[\begin{align*}
\text{("+" case)} & \quad (h_1^+)_t + (\bar{h}_1 u_2^+ - h_2^+ \bar{u}_1^+)x_2 + \beta(h_{1,x_1}^+ + h_{2,x_2}^+) = 0; \\
\text{("-" case)} & \quad (h_1^-)_t + (\bar{h}_1 u_2^- - h_2^- \bar{u}_1^-)x_2 + \beta(h_{1,x_1}^- + h_{2,x_2}^-) = 0.
\end{align*}\]

(B.2, B.3)

The linearized Rankine-Hugoniot conditions across a shock at \( x_1 = 0 \) reduces to

\[
\begin{align*}
h_1^+ &= h_1^-; \\
h_1^- (u_2^- - u_2^+) &= \bar{u}_1^- \left( h_2^+ - \frac{\bar{p}^+}{\bar{p}} h_2^- \right) = \bar{u}_1^+ \left( h_2^+ - Rh_2^- \right).
\end{align*}
\]

(B.4)
We subtract equations (B.2) and (B.3),
\[(h_1^+ - h_1^-)_t + (\vec{t}_1(u_2^+ - u_2^-) - h_2^+ \vec{w}_1^+ + h_2^- \vec{w}_1^-)_{x_2} + \beta(h_1^+,x_1 + h_2^+,x_2) - \beta(h_1^-,x_1 + h_2^-,x_2) = 0,\]
and we differentiate the equation (B.4a) (respectively, (B.4b)) with respect to \(t\) (respectively, \(x_2\)), which is legitimate since we are not differentiating in any direction perpendicular to the shock front. We end up with
\[(\vec{w}_1^+(h_2^+ - Rh_2^-) - h_2^+ \vec{w}_1^+ + h_2^- \vec{w}_1^-)_{x_2} + \beta(h_1^+,x_1 + h_2^+,x_2) - \beta(h_1^-,x_1 + h_2^-,x_2) = 0,\]
where \((\vec{w}_1^+(h_2^+ - Rh_2^-) - h_2^+ \vec{w}_1^+ + h_2^- \vec{w}_1^-)_{x_2} = 0\); thus,
\[\text{div}(h) = \text{div}(h) \quad \text{at} \quad x_1 = 0.\]
Now if we differentiate the equation (B.1a) (respectively, (B.1b)) with respect to \(x_1\) (respectively, \(x_2\)) we obtain
\[\partial_t \text{div}(h) + \beta \partial_{x_1} \text{div}(h) = 0.\]
Using that \(\text{div}(h) \big|_{t=0} = 0\) and the jump condition at \(x_1 = 0\) provided by the Rankine-Hugoniot condition (B.7), we see by the characteristic method that the only possible solution is \(\text{div}(h) = 0\) for \(t > 0\).

**Appendix C. Guide to [24] and [13] #2: A different approach towards the derivation of Rankine-Hugoniot conditions**

In this appendix we consider a different way to derive the Rankine-Hugoniot conditions upon linearization about a planar shock wave. There are several ways to do that, and in the context of MHD equations we refer to [24] and [40]. As pointed out by Majda in [39], the study of stability of planar shocks reduces to a free boundary problem, in which a parameter measuring the deformation of the planar structure of the shock is introduced. The main point of the analysis is to “trade” the later unknown deformation by introducing a dynamic boundary condition at the linearized shock front. Now that we gave the rough idea we can put this heuristic on solid mathematical ground: assume initially that we are in a 2-D spatial domain. Throughout this section, the usual \(L^2(\mathbb{R}^n)\) inner product is written \(\langle \cdot , \cdot \rangle\); a subindex \((\cdot)_a\) denotes the partial derivative with respect to the variable \(a\), for \(a \in \{t,x_1,x_2\}\). Let \(u(\cdot)\) be a planar traveling wave with speed \(s\) solving the following system of equations
\[(f_0(u))_t + \sum_{i=1}^{2} [f_i(u)]_{x_i} = 0,\]
where \(u(x_1 - st) = u^\pm \ (u^\pm \ \text{are constants})\) whenever \(x_1 - st \gtrless 0\) and \(f_0, f_1, f_2 \in \mathcal{C}\infty(\mathbb{R}; \mathbb{R}^n)\). The Rankine-Hugoniot conditions are given by
\[-s[f_0(u)] + [f_1(u)] = 0,\]
where \([\cdot]\) denotes the jump across the shock. We consider a perturbation of this system given by a function \(u(\cdot) + v(\cdot)\), taking into account also perturbations in the shock front of the form \(x_1 - \phi(x_2,t) = 0\) for \(\phi\) sufficiently smooth. In this case the Rankine-Hugoniot conditions are:
\[-\phi_x[f_0(u + v)] - \phi_{x_2}[f_2(u + v)] + [f_1(u + v)] = 0.\]
As pointed out in Section 3 we can take \(s=0\). Linearizing the Rankine-Hugoniot conditions about the shock profile \(u(\cdot)\) and the shock \(x_1 - st = 0\) we have
\[-\phi_x[f_0(u)] - \phi_{x_2}[f_2(u)] + [Df_1(u)v] = 0, \quad (C.1)\]
where \(Df_1(\cdot)\) denotes the Jacobian of the mapping \(f_1\). We start with a trivial linear algebra result:
Claim C.1. We can choose \( n - 2 \) vectors \( \{v_1, \ldots, v_{n-2}\} \) in \( \mathbb{R}^n \) so that
\[
\langle v_i, (-\phi_t[f_0(u)] - \phi_{x_2}[f_2(u)]) \rangle = 0,
\]
for all \( i \in 1, \ldots, n - 2 \). Further, thanks to (C.1), we must have that \( \langle v_i, [Df_1(u)v] \rangle = 0 \).

Each one of these vectors provide a “static” constraint. It turns out that we can actually find another vector \( \tilde{w} \) - independent of \( v_1, \ldots, v_{n-2} \) - that is also orthogonal to \( -\phi_t[f_0(u)] - \phi_{x_2}[f_2(u)] \). In this case, however, since \( \phi_t \) and \( \phi_{x_2} \) have a dynamic behavior (i.e., time dependence), we must expect the same for \( \tilde{w} \). The idea consists of looking for a vector of the form \( \tilde{w} = r \partial_t + s \partial_{x_2} \), where \( r, s \in \mathbb{R}^n \) are unknowns still to be found. Recall that \( [f_0(u)] \) and \( [f_1(u)] \) are constant vectors, since \( u \) is constant on both sides of the shock. Our aim is to satisfy \( \langle r \partial_t + s \partial_{x_2}, -\phi_t[f_0(u)] - \phi_{x_2}[f_2(u)] \rangle = 0 \); expanding the latter, we obtain the equivalent expression
\[
\phi_t \langle r, [f_0(u)] \rangle + \phi_{x_2} \langle s, [f_2(u)] \rangle + \phi_{x_2} \langle s, [f_0(u)] \rangle + \langle r, [f_2(u)] \rangle = 0.
\]
In order to verify this formula, it suffices to find \( r \) and \( s \) in \( \mathbb{R}^n \) such that
\[
\langle s, [f_0(u)] \rangle + \langle r, [f_2(u)] \rangle = 0;
\]
\[
\langle r, [f_0(u)] \rangle = 0;
\]
\[
\langle s, [f_2(u)] \rangle = 0.
\]
(C.2)

Our next step consists of defining a projection \( u \mapsto P_v(u) \) that projects the vector \( u \in \mathbb{R}^n \) in the space spanned by the space \( v \in \mathbb{R}^n \). The following properties of this operator, whose proofs we omit, are standard results in linear algebra:

Claim C.2. Given a vector \( v \in \mathbb{R}^n, v \neq 0 \) and defining the operator \( \mathbb{R}^n \ni u \mapsto P_v(u) \), then the following properties are satisfied:

i. \( P_v(0) = 0 \);

ii. \( P_v(v) = v \);

iii. \( P_v(w) = 0 \iff v \perp w \iff P_w(v) = 0 \).

Let \( [f_0]^\perp \in \mathbb{R}^n \) (respectively, \( [f_1]^\perp \in \mathbb{R}^n \)) be defined so that \( \{[f_0]^\perp, [f_0], v_1, \ldots, v_{n-2}\} \) (respectively, \( \{[f_1]^\perp, [f_1], v_1, \ldots, v_{n-2}\} \) spans \( \mathbb{R}^n \). Define
\[
r = P_{[f_0]^\perp}([f_2]) \quad \text{and} \quad s = \tilde{\mu} P_{[f_2]^\perp}([f_0]),
\]
(C.3)

where \( \tilde{\mu} \in \mathbb{R} \) will be defined later. Without loss of generality, we assume that \( P_{[f_0]^\perp}([f_2]) \neq 0 \) and \( P_{[f_2]^\perp}([f_0]) \neq 0 \); indeed, using Claim C.2 (iii) and the fact that \( \{v_1, \ldots, v_{n-2}\} \perp \text{span}\{[f_0], [f_1]\} \), if one of these projections is vanishes then the \([f_0]\) and \([f_1]\) are linearly dependent, thus the choice of a \((n - 1)^{th}\) orthogonal vector is reduced to a trivial problem. On the other hand, it is easy to see that the last two equations in (C.2) are satisfied. So we proceed as follows: we plug \( r \) and \( s \) as defined in (C.3) in the first equation of (C.2) to find that \( \tilde{\mu} \) should be
\[
\tilde{\mu} = -\frac{\langle P_{[f_0]^\perp}([f_2]), [f_2(u)] \rangle}{\langle P_{[f_2]^\perp}([f_0]), [f_0(u)] \rangle};
\]
Applying the result to (C.1) we see that \( \langle r \partial_t + s \partial_{x_2}, -\phi_t[f_0(u)] - \phi_{x_2}[f_2(u)] \rangle = 0 \), which implies that
\[
\langle r \partial_t + s \partial_{x_2}, [Df_1(u)v] \rangle = 0.
\]
(C.4)
The latter equation is called a dynamical Rankine-Hugoniot condition; notice that it is independent of the perturbation \( \phi \) of the shock front.
Remark C.3. A generalization of the results in this appendix to cases with a higher number of vectors is a bit tricky. Indeed, one needs to find more vectors $r_{j,k}, 1 \leq j \leq d-1, 1 \leq k \leq d-1$, corresponding to $d-1$ dynamic conditions (this in the generic case that $[f_0], [f_2], ..., [f_d]$ are independent, so that there are $n-d$ static conditions and in total there will be the needed $n-d$ total conditions for extreme shock). These must have similar properties to those in [C.2], that is: a) $r_k(\lambda, \eta) := \lambda r_{1,i} + i n r_{2,k} + ... i \eta d, 1 \leq k \leq d-1$ be independent for each $(\lambda, \eta) \neq 0$, and b) $\langle r_k(\lambda, \eta), \lambda [f_0] + \eta [f_2] + ... \rangle = 0$. Now, the case $d = 2$ is easy, because we only need find one of these vectors, and there are $d + d(d - 1)/2$ homogeneous constraints, so we can always find a nontrivial solution; to see that it is non-vanishing for all $(\lambda, \eta)$ one just looks and sees a contradiction if one entry but not the other is vanishing. However, when $d \geq 3$, this is not a simple task. Furthermore, there are degenerate cases where $[f_0], [f_2], ..., [f_d]$ are not independent, which need to be treated slightly differently also it seems. In conclusion: the method works for the current purposes. In spite of its limitations, it illustrates a case where one can find Rankine-Hugoniot conditions explicitly, even when the shock front is an unknown.

C.1. The dynamic jump condition of [24]. Recall that $u_{2,2} = 0, h_{2,2} = 0$. We make use of equations (1.1) in order to derive the vectors $f_0$, $f_1$ and $f_2$.

$$
 f_0 = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ h_1 \\ h_2 \end{pmatrix}, \quad f_1 = \begin{pmatrix} \rho u_1 \\ \rho u_1 u_2 - h_1 h_2 \\ \rho u_1 u_2 - h_1 h_2 + a \rho^\gamma \\ 0 \\ u_1 h_2 - h_1 u_2 \end{pmatrix}, \quad f_2 = \begin{pmatrix} \rho u_2 \\ \rho u_2 u_2 - h_1 h_2 \\ \rho u_2 u_2 + a \rho^\gamma + h_1 h_2 - h_1 h_2 \\ a \{ (\rho^+)^\gamma - (\rho^-)^\gamma \} \end{pmatrix}.
$$

The jumps across the shock are:

$$
[f_0] = \begin{pmatrix} \overline{\rho}^+ - \overline{\rho}^- \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad [f_2] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a \{ (\rho^+)^\gamma - (\rho^-)^\gamma \} \end{pmatrix}.
$$

It is easy to find the vectors mentioned in claim [C.1] $v_1 = e_2, v_2 = e_4$ and $v_3 = e_5$. To calculate the static Rankine-Hugoniot conditions we need the Jacobian of $f_1$, $D f_1((\rho, u_1, 0, h_1, 0))$:

$$
D f_1(\overline{\rho}, \overline{u}_1, 0, \overline{h}_1, 0) = \begin{pmatrix} \overline{u}_1 \\ \overline{u}_1^2 + a \gamma \rho^{\gamma-1} \\ 2 a \rho u_1 \\ 0 \\ 0 \end{pmatrix}.
$$

Using the notation in [24 §3.2], the result in C.1 provides the static Rankine-Hugoniot conditions. The persistence of the divergence free condition (Section 2 and appendix B) implies that

$$
[h_1] = 0.
$$

(C.5)

The condition $\langle v_1, [D f_1(\overline{\rho}, \overline{u}_1, 0, \overline{h}_1, 0) v] \rangle = 0$ gives

$$
\overline{u}_1^+ \left( 1 + \frac{1}{M^2} \right) \rho^+ - \overline{u}_1^- R^2 \left( 1 + \frac{1}{M^2 R^\gamma+1} \right) \rho^- + 2 a \rho^+(u_1^+ - u_1^-) - \frac{\overline{h}_1}{(u_1^+ - u_1^-)} (h_1^+ - h_1^-) = 0. \quad (C.6)
$$

The equation

$$
\langle v_2, [D f_1(\overline{\rho}, \overline{u}_1, 0, \overline{h}_1, 0) v] \rangle = 0
$$

(C.7)
Furthermore, unlike the case presented before, in equation C.7 we would obtain 
\[ h_1^+ = h_1^- \], which we already know for MHD due to persistence of the constraint (1.2) (see Section 2.1.4).

Remark C.4. In the \( \beta \)-model the derivation of the Rankine-Hugoniot conditions using the reasoning presented in this section follows similar lines; the main differences are that

\[
\begin{align*}
\rho_0 &= \left( \begin{array}{c} \rho \\ \rho u_1 \\ \rho u_2 \\ h_1 \\ h_2 \\ \end{array} \right), & f_1 &= \left( \begin{array}{c} \rho u_1 - \frac{\rho u_1^2}{h_1} + \frac{\rho u_2}{\beta} + \rho \gamma \\ \rho u_2 - \frac{\rho u_1}{h_1} + \rho \gamma \\ u_1 h_2 - h_1 u_2 \\ \end{array} \right), & f_2 &= \left( \begin{array}{c} \frac{\rho u_1}{h_1} - \frac{\rho u_2}{h_2} \\ \frac{\rho u_1^2}{h_1} - \frac{\rho u_2}{h_2} \\ u_1 h_2 - h_1 u_2 \\ \end{array} \right). 
\end{align*}
\]

The jump across the shock are:

\[
\begin{align*}
[f_0] &= \left( \begin{array}{c} \bar{\rho}^+ - \bar{\rho}^- \\ 0 \\ 0 \\ 0 \\ 0 \\ \end{array} \right), & [f_2] &= \left( \begin{array}{c} a \{(\bar{\rho}^+)^\gamma - (\bar{\rho}^-)^\gamma\} \\ 0 \\ 0 \\ \end{array} \right), 
\end{align*}
\]

and finally,

\[
Df_1((\bar{\rho}, \bar{u}_1, 0, \bar{h}_1, 0)) = \left( \begin{array}{cccc}
\bar{u}_1 & \bar{\rho} & 0 & 0 \\
\bar{\rho}^2 + a(\bar{\rho}^+)^{-1} & 2\bar{m}_1 & 0 & -\bar{h}_1 \\
0 & \bar{m}_1 & 0 & -\bar{h}_1 \\
0 & 0 & \beta & 0 \\
0 & 0 & -\bar{h}_1 & \bar{m}_1 \\
\end{array} \right). 
\]

Furthermore, unlike the case presented before, in equation C.7 we would obtain 
\[ h_1^+ = h_1^- \], which we already know for MHD due to persistence of the constraint (1.2) (see Section 2.1.4).

Remark C.5. If we use the preserved constraint and appropriate normalization (see [24, §3.1]), equation (C.6) reduces to the first Rankine-Hugoniot condition in [24, Eq. (44)] and equation C.8 corresponds to which corresponds to the last equation in [24, Eq. (44)].
Remark C.6. If we normalize as in [24 §3.2], $\tilde{t} \simeq t \pi_1^+$, we have
\[
\mathcal{A}_1 \left\{ (u_2^+ - u_2^-) - \frac{q}{M^2} \left( h_2^+ - h_2^- \right) \right\} - \left( \tilde{t} \left( \frac{\rho^+ - R \rho^-}{R^2} \right) + \left( u_1^+ - u_1^- \right) \right) = 0 \Leftrightarrow \frac{(R-1)}{R} \left\{ (u_2^+ - u_2^-) - \frac{q}{M^2} \left( h_2^+ - h_2^- \right) \right\} - \frac{(R-1)}{R} \left( \frac{\rho^+ - R \rho^-}{R^2} \right) = 0.
\]
Plugging the first static Rankine-Hugoniot condition in [24 Eq. (44)] derived in the tangential direction $x_2$ in order to remove the $\tilde{u}_1^+$ variable and the last Rankine-Hugoniot condition in [24 Eq. (44)] in order to get rid of the $(u_2^+ - u_2^-)_{x_2}$ term we obtain:
\[
\left\{ \frac{1}{q} (h_2^+ - R h_2^-) - \frac{q}{M^2} (h_2^+ - h_2^-) \right\} - \left\{ R(1-b_1)\rho^+ + R(b_2 - R)\rho^- + (R-1)u_1^- \right\}_{x_2} = 0.
\]
Multiplying by $\frac{M^2}{M^2 q^2}$,
\[
\left\{ \frac{1}{q} \left( h_2^+ + \frac{q^2 - RM^2}{q(M^2 - q^2)} \right) (h_2^-) \right\} - \left\{ \frac{R \left( \frac{M^2 - 1}{M^2 - q^2} \right)}{-b_3} \rho^+ + \frac{R M^2}{M^2 - q^2} \left( b_2 - R \right) \rho^- + \frac{M^2 (R-1)}{M^2 - q^2} \right\} u_1^- = 0,
\]
which we rewrite as
\[
\left\{ \frac{1}{q} \left( h_2^+ + \frac{q^2 - RM^2}{q(M^2 - q^2)} \right) (h_2^-) \right\} + \left\{ b_3 \rho^+ + b_4 \rho^- + b_5 u_1^- \right\} = 0. \tag{C.9}
\]
This equation does not agree with equations [24 Eq. (44)]: in fact, according to their calculations,
\[
\left\{ u_2^+ - u_2^- \right\} + \left\{ b_3 \rho^+ + b_4 \rho^- + b_5 u_1^- \right\}_{x_2} = 0 \quad \text{and} \quad h_2^+ - R h_2^- - qu_2^+ + qu_2^- = 0 \quad \text{at} \quad x_1 = 0
\]
which implies that
\[
\left\{ \frac{1}{q} (h_2^+ - Rh_2^-) \right\} + \left\{ b_3 \rho^+ + b_4 \rho^- + b_5 u_1^- \right\} = 0 \quad \text{at} \quad x_1 = 0. \tag{C.10}
\]
A more careful analysis can shows that Eq. (C.10) is not a linear combination of (C.9) and the other jump conditions in (C.5)-(C.8) and the interior equations evaluated on $x_1 = 0$. Indeed, if that was the case then $\partial_t \{ h_2^- \} = 0$. However there is no way of obtaining $h_2^+$ as a linear combination of the jump conditions (because it would introduce a linear term in $h_2^+$) or interior equations evaluated on $x_1 = 0$ (because it would introduce variables that are spatial derivatives in $x_1$). One can conclude that the the dynamic Rankine-Hugoniot condition in [24 Eq. (44)] is incorrect. The source of the error is small and did not do much harm to their calculation (for instance, they obtained the correct order for the root of the Lopatinski determinant; see [24 Eq. (61)]), since in the large magnetic field we have $q \to \infty$ and the coefficients of (C.9) and (C.10) only differ by an $\mathcal{O}(\varepsilon^3)$ term.

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