INTERSECTION SPACE COHOMOLOGY OF THREE-STRATA PSEUDOMANIFOLDS

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ABSTRACT. The theory of intersection spaces assigns cell complexes to certain stratified topological pseudomanifolds depending on a perversity function in the sense of intersection homology. The main property of the intersection spaces is Poincaré duality over complementary perversities for the reduced singular (co)homology groups with rational coefficients. This (co)homology theory is not isomorphic to intersection homology, instead they are related by mirror symmetry. Using differential forms, Banagl extended the intersection space cohomology theory to 2-strata pseudomanifolds with a geometrically flat link bundle. In this paper we use differential forms on manifolds with corners to generalize the intersection space cohomology theory to a class of 3-strata spaces with flatness assumptions for the link bundles. We prove Poincaré duality over complementary perversities for the cohomology groups. To do so, we investigate fiber bundles on manifolds with boundary and use the Hodge-Morrey-Friedrichs-Decomposition to provide a geometric cotruncation for differential forms on manifolds with boundary. At the end, we give examples for the application of the theory.

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1. INTRODUCTION

In this paper, we give a de Rham description of the intersection space cohomology theory extending it to a class of Thom-Mather-stratified pseudomanifolds with three strata.

To prove Poincaré duality for the resulting cohomology theory, we introduce a proof technique called the method of iterated triangles. Roughly speaking, one proves Poincaré duality by induction on the stratification depth using intermediate complexes of forms, distinguished triangles in the derived category over the reals and a Five-Lemma argument in the induction step. This technique is also applicable for arbitrary large stratification depth and might be the guideline to generalize the intersection space cohomology theory to a class of Thom-Mather-stratified pseudomanifolds of arbitrary stratification depth.

The theory of intersection spaces, first introduced by M. Banagl in [Ban10], assigns CW complexes $I^pX$ to certain types of topological stratified pseudomanifolds. Those depend on a perversity function $\bar{p}$ in the sense of Goresky and MacPherson, see [GM80, GM83]. Their main property is Poincaré duality over complementary perversities for the reduced singular (co)homology groups with coefficients in a field. Additionally, using regular singular (co)homology, one gets perversity internal cup products for cohomology.
The construction of the intersection spaces is built upon a homotopy theoretic technique called Moore approximation or spatial homology truncation. The links of the singularities are replaced by a CW-complex (which is not always a subcomplex) with truncated homology. Note, that Moore approximation is an Eckmann-Hilton dual notion of Postnikov approximation. If the singularities are not isolated, one has to perform the Moore approximation equivariantly, see [BC16]. Having a perversity internal cup product, intersection space cohomology cannot be isomorphic to intersection cohomology. This is underlined by the behaviour of both theories on cones of smooth manifolds: Intersection (co)homology of a cone equals the truncated (co)homology of the manifold, while intersection space (co)homology of a cone is equal to the cotruncated (co)homology of the manifold.

In [Ban16], Banagl gives a de Rham description of intersection space cohomology, using differential forms on the top stratum of a Thom-Mather-stratified pseudomanifold of stratification depth one with geometrically flat link bundles. A bundle is called flat if the transition functions are locally constant and geometrically flat if, in addition, the structure group of the bundle is contained in the isometries of the fiber. Flat link bundles occur in reductive Borel-Serre compactifications of locally symmetric spaces and in foliated stratified spaces. The latter play a role in the work of Farrell and Jones on the topological rigidity of negatively curved manifolds, for instance, see [FJ88, FJ89]. For such bundles, the Leray-Serre spectral sequence with real coefficients collapses at the $E_2$ page, see [Ban13, Theorem 5.1]. Examples of flat sphere bundles with nonzero real Euler class, constructed by Milnor in [Mil58], show that one cannot always equip the link of a flat bundle with a Riemannian metric such that the bundle becomes geometrically flat. Banagl uses Riemannian Hodge theory to cotruncate the de Rham complex on the fiber of the link bundle. The geometrical flatness condition then allows to perform that cotruncation fiberwisely. The de Rham complex computing intersection space cohomology consists of all forms on the top stratum of the pseudomanifold with restriction to a collar neighbourhood of the boundary equaling the pullback of a fiberwisely cotruncated form on the boundary.

Banagl establishes a de Rham isomorphism for pseudomanifolds of depth one with only isolated singularities. Examples of applications of the intersection space cohomology theory contain K-theory ([Ban10, Chapter 2.8]), deformation of singular varieties in algebraic geometry ([BM12]), perverse sheaves ([BBM14]), geometrically flat bundles and equivariant cohomology ([Ban13]) and string theory in theoretical physics ([Ban10, Chapter 3] and [BBM14]).

The purpose of the present paper is a generalization of intersection space cohomology via the de Rham approach to certain pseudomanifolds of stratification depth two. The approach we pursue might be suitable to generalize the theory to pseudomanifolds of arbitrary stratification depth. However,
not all the technical difficulties do already arise in the current setting. In [Ban12], Banagl uses homotopy pushouts of 3-diagrams of spaces to define intersection spaces for first cases of depth two pseudomanifolds. By using the de Rham approach, we enlarge the class of depth two pseudomanifolds intersection space cohomology is applicable to. Let $X$ be a compact, oriented, Thom-Mather-stratified pseudomanifold with three strata of different dimension, closed stratum (the stratum of maximum codimension) a finite set of points, and a geometrically flat link bundle for the intermediate stratum. We then define a subcomplex $\Omega_{\bar{p}}^I$ of the complex of smooth differential forms on the blowup $M$ of $X$. We prove the following Poincaré duality theorem for the cohomology groups $HI_{\bar{p}}^I(X) = H^\bullet(\Omega_{\bar{p}}^I(M))$ if the pseudomanifold satisfies a Witt-type condition.

**Theorem 7.4.1:** (Poincaré duality for $HI$)

For all $r \in \mathbb{Z}$, integration induces nondegenerate bilinear forms

$$
\int : HI_{\bar{p}}^r(X) \times HI_{\bar{q}}^{n-r}(X) \to \mathbb{R}
$$

$$
([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta.
$$

We give a short overview about the construction of the complex $\Omega_{\bar{p}}^I$ in the depth two case and the idea of the proof of the Poincaré duality theorem. If $X$ has three strata of different dimension, the blowup of the pseudomanifold is a $⟨2⟩$--manifold $M$. The boundary decomposes into two smooth manifolds with boundary $E$ and $W$, glued along their common boundary. $E$ is the total space of a geometrically flat link bundle, while the connected components of $W$ are (trivially) fibered over points. $M$ comes equipped with a system of collars for $E$, $W$ and $\partial E = \partial W$ (which is induced by the Thom-Mather control data of $X$). The intermediate complex $\tilde{\Omega}_{\bar{p}}^I(M)$ is defined to contain the smooth forms on $M$ with restriction to the collar neighbourhood of $E$ equaling the pullback of a fiberwisely cotruncated multiplicatively structured form on $E$. $\Omega_{\bar{p}}^I(M)$ is then defined to contain the forms of $\tilde{\Omega}_{\bar{p}}^I(M)$ with restriction to the collar of $W$ equaling the pullback of a cotruncated form on $W$. So, forms in $\Omega_{\bar{p}}^I(M)$ satisfy two different pullback-cotruncation properties. Thus, the restriction of the forms to the intersection of the two collar neighbourhoods of the boundary parts $E$ and $W$ (which is the collar neighbourhood of $\partial E$) has to be both the pullback of an appropriate form on $E$ as well as the pullback of an appropriate form on $W$, hence the pullback of some form on $\partial E = \partial W$. This is the main difficulty in the proof of the Poincaré duality theorem for the cohomology of $\Omega_{\bar{p}}^I(M)$ is, To deal with this problem, we have to impose a Witt-type condition on the pseudomanifold, see Section 7.1. As mentioned before, we use the method of iterated triangles to prove Poincaré duality. That means that we use a chain of intermediate complexes and prove Poincaré-Lefschetz duality statements for them using distinguished triangles and 5-Lemma arguments. This results
in Poincaré duality for $HI$. In this paper we need only one intermediate complex, namely $\tilde{\Omega}^*_{\rho}(M)$. For a pseudomanifold of stratification depth $d$ (with flatness and additional Witt-type conditions) one would need $d - 1$ intermediate complexes.

**General Notation.** For a smooth $(n)$--manifold $M$ with boundary $\partial M = \partial M_1 \cup ... \cup \partial M_p$, a collar of the boundary part $\partial M_i$ is denoted by $c_i : \partial M_i \times [0, 1) \hookrightarrow M$ with $c_i|_{\partial M_i \times \{0\}} = \text{id}_{\partial M_i}$. We mainly work with $(2)$-manifolds, i.e. manifolds with corners and two boundary parts $\partial M_1 = E$ and $\partial M_2 = W$, $\partial M = E \cup_{\partial E = \partial W} W$. The inclusion of the boundary parts is denoted by $j_E : E \hookrightarrow M$ and $j_W : W \hookrightarrow M$ and the inclusion of the corner $\partial E = \partial W$ by $j_{\partial W} = j_{\partial E} : \partial W \hookrightarrow M$. The image of a collar, $\text{im} c_i \subset M$ is called a collar neighbourhood. For a real vector space $V$, we denote the linear dual $\text{Hom}(V, \mathbb{R})$ by $V^\dagger$.

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2. Collars on Bundles and Manifolds with Corners

2.1. Width of a collar. In order to prove Poincaré duality for the later defined complexes on manifolds with boundaries we need the following relations between finite open covers and collars on manifolds with boundary. Since we only consider compact manifolds in the remainder of the paper, open covers will always have finite subcovers.

**Definition 2.1.1** (Small collars). Let $B$ be a manifold with nonempty boundary $\partial B$, $c : \partial B \times [0, 1) \hookrightarrow B$ an open collar of the boundary and $U := \{U_\alpha\}_{\alpha \in I}$ an open cover of $B$. Let $I_\emptyset \subset I$ be the index set containing the indices of the $U_\alpha$ with nonempty intersection with the boundary $\partial B$. The collar $c$ is called small with respect to the cover $U$ if the following conditions are satisfied.

1. $U_\alpha \cap B_\emptyset \neq \emptyset$, for every $\alpha \in I$, where $B_\emptyset := B - c(\partial B \times [0, 1))$.
2. There exist $W_\alpha \subset \partial B$ open with $c(W_\alpha \times [0, 1)) \subset U_\alpha$ for each $\alpha \in I_\emptyset$ and such that $\{W_\alpha \times [0, 1)\}_{\alpha \in I_\emptyset}$ is an open cover of $\partial B \times [0, 1)$.

**Lemma 2.1.2.** Let $B$ be a manifold with non-empty boundary $\partial B$, let $c : \partial B \times [0, 1) \hookrightarrow B$ be an open collar of $\partial B$ in $B$ and let $U := \{U_\alpha\}_{\alpha \in I}$ be a finite open cover of $B$. Then there is an $\varepsilon \in (0, 1]$ such that the subcollar $c| : \partial B \times [0, \varepsilon) \hookrightarrow B$.
is small with respect to $U$.

Proof. Let $C = c(\partial B \times [0,1])$. If there are no $U_\alpha \in U$ such that $U_\alpha \subset C$ we take $\epsilon = 1$ and are done. So suppose $U_\alpha \subset C$. Since $U_\alpha \subset B$ is open, there must be an $N_\alpha \in \mathbb{N}$ such that $U_\alpha \not\subset c(B \times [0,1/n])$ for all $n \geq N_\alpha$. (Otherwise $U_\alpha$ would be contained in $\partial B = c(\partial B \times \{0\})$.) Choose such an $N_\alpha$ for each $\alpha \in I$ and set $\epsilon := (\max_{\alpha \in I} N_\alpha)^{-1} \in (0,1]$. This is well defined since the index set $I$ is finite and the first relation in the definition is satisfied for that $\epsilon$.

Assume without loss of generality that we could choose $\epsilon = 1$ in the above. Take an $\alpha \in I_\partial$. Let $x \in U_\alpha \cap \partial B$. Since $c^{-1}(U_\alpha \cap C) \subset \partial B \times [0,1]$ is open, there exist $W_x \subset \partial B$ open and an $\epsilon_x \in (0,1]$ such that $W_x \times [0,\epsilon_x) \subset c^{-1}(U_\alpha \cap C)$. If $x$ is contained in more than one of the $U_\alpha$’s, choose $W_x$ and $\epsilon_x$ so small that $c(W_x \times [0,\epsilon_x))$ is contained in all the $U_\alpha$’s. Since $\partial B$ is compact there are finitely many $x_1, \cdots, x_k \in \partial B$ such that the $W_{x_i}$ cover $\partial B$. Let $\epsilon := \min_{x_i} \epsilon_{x_i}$ and set $W_\alpha := \bigcup_{x_i \in U_\alpha} W_{x_i}$. By definition, $c(W_\alpha \times [0,\epsilon)) \subset U_\alpha$ and $\partial B \times [0,\epsilon) \subset \bigcup_{\alpha \in I_\partial} W_\alpha \times [0,\epsilon)$. Hence, the collar $c : \partial B \times [0,\epsilon) \hookrightarrow B$ is small with respect to $U$. □

2.2. $p$-related Collars on Fiber Bundles. We start with a proposition on $p$-related collars on a fiber bundle $p : E \to B$ over a base manifold $B$ with boundary $\partial B$.

**Definition 2.2.1. ($p$-related collars)**

Let $p : E \to B$ be a smooth fiber bundle with closed smooth fiber $F$ and $B$ a compact smooth manifold with boundary $\partial B$. Let

$$c_{\partial E} : \partial E \times [0,1) \to E$$

be a smooth collar on the manifold with boundary $E$ and

$$c_{\partial B} : \partial B \times [0,1) \to B$$

a smooth collar on $B$. Then $c_{\partial E}$ and $c_{\partial B}$ are called $p$-related if and only if the diagram

$$\begin{array}{ccc}
\partial E \times [0,1) & \xrightarrow{c_{\partial E}} & E \\
\downarrow{p \times \text{id}} & & \downarrow{p} \\
\partial B \times [0,1) & \xrightarrow{c_{\partial B}} & B
\end{array}$$

commutes.

**Example 2.2.2.** Let $E = L \times B$ be a trivial link bundle. We then can take any collar $c_{\partial B} : \partial B \times [0,1) \hookrightarrow \partial B$ in $B$ and take $c_{\partial E} := \text{id}_L \times c_{\partial B} : \partial E \times [0,1) \hookrightarrow E$. $c_{\partial E}$ is indeed a collar of $\partial E = \partial B \times L$, since we work
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with closed fibers $L$. Hence the diagram

$$\partial E \times [0,1) \xrightarrow{c_{\partial E}} E = L \times B$$

$$\downarrow \pi_2 \times \text{id} \quad \downarrow \pi_2$$

$$\partial B \times [0,1) \xrightarrow{c_{\partial B}} B$$

commutes and the collars are $p$-related.

**Proposition 2.2.3.** For any smooth fiber bundle $p : E \to B$ with base space a compact smooth manifold with boundary $(B, \partial B)$ and closed smooth fiber $L$ there is a pair of $p$-related collars

$$c_{\partial E} : \partial E \times [0,1) \hookrightarrow E,$$

$$c_{\partial B} : \partial B \times [0,1) \hookrightarrow B.$$  

Moreover, if a collar $c_{\partial B} : \partial B \times [0,1) \hookrightarrow B$ is given then a collar $c_{\partial E} : \partial E \times [0,1) \hookrightarrow E$ can be chosen such that $c_{\partial E}$ and $c_{\partial B}$ are $p$-related for some subcollar of $c_{\partial B}$. (In detail, we take a subcollar $c_{\partial B}|_{\partial B \times (0,\alpha)}$ for some $\alpha \in (0,1]$ and reparametrize it to get a map $\partial B \times [0,1) \hookrightarrow B$.)

This is also true the other way round: If you start with a collar $c_{\partial E} : \partial E \times [0,1) \hookrightarrow E$ on $E$, you can choose a collar $c_{\partial B} : \partial B \times [0,1) \hookrightarrow B$ such that $c_{\partial E}$ and $c_{\partial B}$ are $p$-related for some subcollar of $c_{\partial E}$.

**Proof.** We start with the first part and therefore proceed as follows:

1. First we construct a vector field $X$ on $B$ which is nowhere tangent to $\partial B$. The flow of this vector field then gives the collar $c_{\partial B}$ on $B$.
2. By locally lifting this vector field, we construct a vector field $Y$ on $E$ that is nowhere tangent to $\partial E$ and $p$-related to $X$, i.e. for each $e \in E$ we have

$$p_* Y_e = X_{p(e)}.$$

3. By [AMR88, Prop 4.2.4], we then have the relation

$$p \circ \eta^Y_t = \eta^X_t \circ p.$$  

for the flows $\eta^X$ of $X$ and $\eta^Y$ of $Y$. That relation implies the statement of the proposition.

The first step is quite simple and standard: Take a finite good open cover $\{U_\alpha\}_{\alpha \in I}$ of $B$ such that the bundle trivializes with respect to this cover. Then let $J \subseteq I$ denote the set of those $\alpha \in I$ with $U_\alpha \cap \partial B \neq \emptyset$. For each $\alpha \in J$ define a vector field $X_\alpha$ on $U_\alpha$ by taking the induced vector field of $\partial_b$ on $\mathbb{R}^d_b$ by the coordinate map $\phi_\alpha$. Then take a partition of unity $\{\rho_\alpha\}_{\alpha \in I}$ subordinate to the cover $\{U_\alpha\}$ and define

$$X := \sum_{\alpha \in I} \rho_\alpha X_\alpha.$$
To obtain the vector field $Y \in \mathfrak{X}(E) = \Gamma(TE)$ we proceed as follows: Since there is a natural isomorphism between vector bundles

$$T(U_\alpha) \times T(L) \xrightarrow{\cong} T(U_\alpha \times L)$$

for all $\alpha \in I$, we can lift the vector field $\rho_\alpha X_\alpha \in \mathfrak{X}(U_\alpha)$ to $(\rho_\alpha X_\alpha, 0)$, a section of $T(U_\alpha) \times T(L) \cong T(U_\alpha \times L)$, which still has compact support in $U_\alpha \times L$.

Since $p : E \to B$ is a fiber bundle with fiber $L$ and $\{U_\alpha\}_{\alpha \in I}$ a covering of the base $B$ with respect to which the fiber trivializes, we have a diffeomorphism $\psi_\alpha : U_\alpha \times L \xrightarrow{\cong} p^{-1}(U_\alpha)$, for all $\alpha \in I$, such that the diagram

$$
\begin{array}{ccc}
p^{-1}(U_\alpha) & \xleftarrow{\psi_\alpha} & U_\alpha \times L \\
p \downarrow & & \pi_1 \\
U_\alpha & \xleftarrow{\cong} & U_\alpha
\end{array}
$$

commutes. Note that since the $\psi_\alpha$ are diffeomorphisms, there exist push-forward vector fields $\psi_\alpha* (\rho_\alpha X_\alpha, 0) \in \mathfrak{X}(p^{-1}(U_\alpha))$, which still have compact support (in $p^{-1}(U_\alpha)$).

Since the family $\{p^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of $E$, such that the sets in $\{p^{-1}(U_\alpha)\}_{\alpha \in J}$ cover an open neighbourhood of the boundary $\partial E$ of $E$, we can set

$$Y := \sum_{\alpha \in J} \psi_\alpha* (\rho_\alpha X_\alpha, 0)$$

to get a vector field $Y \in \mathfrak{X}(E)$ that is nowhere tangent to $\partial E$. Let $x \in \partial E$, then $x \in p^{-1}(U_{\alpha_1} \cap \ldots \cap U_{\alpha_r})$ for some $\alpha_1, \ldots, \alpha_r \in J$. Then $Y_x$ is not tangent to $\partial E$ if and only if $((\psi_{\alpha_1}^{-1})_x Y)_{\psi_{\alpha_1}^{-1}(x)}$ is not tangent to $\partial B \times U_{\alpha_1} \times L$.

$$((\psi_{\alpha_1}^{-1})_x Y)_{\psi_{\alpha_1}^{-1}(x)} = \sum_{i=1}^{r} \rho_{\alpha_i}(p(x)) ([\text{id} \times (\pi_2 \circ \psi_{\alpha_1}^{-1} \circ \psi_{\alpha_i})]_x (X_{\alpha_i}, 0))_{\psi_{\alpha_1}^{-1}(x)}$$

$$= \sum_{i=1}^{r} \rho_{\alpha_i}(p(x))(X_{\alpha_i})_{p(x)}.$$

Now this is of course not tangent to the boundary since by definition of the $X_\alpha \in \mathfrak{X}(U_\alpha)$ we have (with again the $\phi_\alpha$ the coordinate maps of the base):

$$(\phi_{\alpha_1}^{-1})_x X_\alpha = (\phi_{\alpha_1}^{-1} \circ \phi_\alpha)_x \partial_b = \sum_{i=1}^{b} a_i \partial_i$$

with $a_b > 0$ since the transition maps are maps between manifolds with boundary.

Further, we have to show that $X$ and $Y$ are $p$-related, i.e. it holds that $p_* Y_e = X_{p(e)}$ for every $e \in E$. This is equivalent to the statement that for all smooth functions on an open subset of $B$ it holds that

$$Y(f \circ p) = (X f) \circ p.$$
(see e.g. [Lee13, Lemma 3.17]). For let \( f : U \to \mathbb{R} \) be a smooth function on an open subset \( U \subset B \) and let \( x \in p^{-1}(U) \). Then

\[
Y(f \circ p)(x) = Y_x(f \circ p)
\]

\[
= \sum_{\alpha \in \tilde{J}} \psi_{\alpha*}(\rho_\alpha X_\alpha, 0)_{\psi_{\alpha}^{-1}(x)} (f \circ p) \quad \text{with} \quad \tilde{J} = \{ \alpha \in J | x \in p^{-1}(U_\alpha) \}
\]

\[
= \sum_{\alpha \in \tilde{J}} (\rho_\alpha X_\alpha, 0)_{\psi_{\alpha}^{-1}(x)} (f \circ p \circ \psi_\alpha)
\]

\[
= \sum_{\alpha \in \tilde{J}} \rho_\alpha (p(x)) ((X_\alpha)_{p(x) = \pi_1 \circ \psi_{\alpha}^{-1}(x)}, 0_{\pi_2 \circ \psi_{\alpha}^{-1}(x)}) (f \circ \pi_1)
\]

\[
= \sum_{\alpha \in \tilde{J}} \rho_\alpha (p(x))(X_\alpha)_{p(x)}(f) = X_{p(x)}(f) = (Xf)(p(x)).
\]

As mentioned, for every \( t \), this implies the relation

\[
p \circ \eta^X_t = \eta^X_t \circ p
\]

for the flows \( \eta^X \) of the vector field \( X \in \mathfrak{X}(B) \) and \( \eta^Y \) of \( Y \in \mathfrak{X}(E) \). This relation implies the claim since there are open neighbourhoods \( W_B \subset \partial B \times [0, \infty) \) of \( \partial B \) and \( W_E \subset \partial E \times [0, \infty) \) of \( \partial E \) respectively, such that the flows \( \eta^X \) and \( \eta^Y \) are defined on these open subsets. But then there are constants \( \epsilon_B, \epsilon_E > 0 \) such that \( \partial B \times [0, \epsilon_B) \subset W_B \) and \( \partial E \times [0, \epsilon_E) \subset W_E \). Let \( \epsilon := \min(\epsilon_B, \epsilon_E) \) and let \( f : [0, \infty) \to [0, \epsilon) \) be a diffeomorphism. Then we have collar embeddings

\[
c_{\partial B} : \partial B \times [0, \infty) \xrightarrow{id \times f} \partial B \times [0, \epsilon) \xrightarrow{\eta^X} B
\]

and

\[
c_{\partial E} : \partial E \times [0, \infty) \xrightarrow{id \times f} \partial E \times [0, \epsilon) \xrightarrow{\eta^Y} E
\]

such that

\[
p \circ c_{\partial E} (x, t) = (p \circ \eta^Y_{f(t)})(x) = \eta^X_{f(t)}(p(x)) \quad \text{(by eq. (1))}
\]

\[
= c_{\partial B} \circ (p \times id)(x, t)
\]

For the second part of the proof we proceed likewise, but take a special vector field in step 1: The collar allows us to define a vector field \( \tilde{X} \in \mathfrak{X}(C_{\partial B}) \) (with \( C_{\partial B} = \text{im} c_{\partial B} \)) by taking the pushforward of \( \partial_t : \tilde{X} = c_\alpha \partial_t \). Then for any \( q \in \partial B \) and any \( f \in C^\infty(C_{\partial B}) \) it holds that

\[
(\tilde{X}f)(c(\tau, q)) = (\partial_t(f \circ c))(\tau, q) = \frac{d}{dt}|_{t=\tau} (f \circ c)(t, q).
\]

This means that the flow of the vector field restricted to the boundary \( \partial B \) is the given collar \( c_{\partial B} \). We then "lift" this vector field as before, not to a vector field on the whole total space \( E \) but rather to a vector field \( Y \in \)
\[ X(p^{-1}(C_{\partial B})), \text{ where } p^{-1}(C_{\partial B}) \text{ is an open neighbourhood of the boundary,} \]

by setting

\[ Y = \sum_{\alpha \in J} \psi_{\alpha \ast}(\rho_{\alpha} \tilde{X}|, 0). \]

As before this defines a nowhere vanishing vector field which is nowhere tangent to the boundary \( \partial E \). The rest is a complete analogy to the first step. Note that it suffices to have the vector fields on open neighbourhoods of the boundary since we later only need the flow of the vector fields restricted to the boundary.

The arguments to prove the third part of the statement are slightly easier versions of the previous ones. The collar \( c_{\partial E} \) on \( E \) defines a vector field \( \tilde{Y} \in X(C_{\partial E}) \) by taking the pushforward of \( \partial_t \). The flow of this vector field again is the collar we started with. We now project this vector field down, i.e. we set \( X := p_{\ast} \tilde{Y} \). Trivially, both vector fields are \( p \)-related and nowhere tangent to the boundaries. The rest is done as before. \( \square \)

2.3. Collars on Manifolds with Corners. We are going to work with differential forms on a smooth manifold with corners \( M^n \), the boundary of which can be subdivided as \( \partial M = E \cup_{\partial E = \partial W} W \), satisfying certain conditions near the boundary parts \( E \) and \( W \). In order to define "near \( E, W \)" precisely we have to investigate how the concept of a collar on a manifold with boundary generalizes to manifolds with corners of that type. In [Ver84], the author proves a theorem, see [Ver84, Theorem 6.5], that can be interpreted as a transition between Thom-Mather-stratified pseudomanifolds and manifolds with faces: Any such pseudomanifold can be obtained from a manifold with faces by making certain identifications on the faces.

Definition 2.3.1. (Manifolds with Faces)

Let \( M^n \) be an \( n \)-dimensional manifold with corners and for each \( x \in M \) let \( c(x) \) denote the number of zeroes of \( \phi(x) \in \mathbb{R}^n_+ = [0, \infty)^n \) for any coordinate chart \( \phi : U \to \mathbb{R}^n_+ \) with \( x \in U \). A face is the closure of a connected component of the set \( \{ p \in M | c(p) = 1 \} \). Then \( M \) is called a manifold with faces if each \( x \in M \) is contained in \( c(x) \) different faces.

Example 2.3.2. 2-dimensional disc with one corner is a manifold with corners but not with faces, since the corner point does not lie in 2 faces but only in one.

As mentioned, manifolds with faces are considered by Verona in [Ver84, Chapter 4] to examine triangulability of stratified mappings, but also by Alpert in [Alp16, Section 3] to estimate simplicial volume, by Jänich in [Jän68] to classify \( O(n) \)-Manifolds and by Laures in [Lau00] to investigate cobordisms on manifolds with corners.

The latter two authors also define \( \langle n \rangle \)-manifolds, which are manifolds with faces together with a decomposition of the boundary into \( n \) faces that satisfy the following relations.
Definition 2.3.3. (⟨n⟩-manifolds) [See [Jän68, Def. 1]]
A manifold with faces \( M \) together with a n-tuple of faces \((\partial_0 M, \ldots, \partial_{n-1} M)\) is called an \langle n \rangle-manifold if

1. \( \partial M = \bigcup_{i=0}^{n-1} \partial_i M \),
2. \( \partial_i M \cap \partial_j M \) is a face of both \( \partial_i M \) and \( \partial_j M \) if \( i \neq j \).

Note that a \langle 0 \rangle-manifold is just a usual manifold (without boundary) and a \langle 1 \rangle-manifold is a manifold with boundary. Simple examples of \langle n \rangle-manifolds for arbitrary \( n \in \mathbb{N} \) are \( \mathbb{R}^n \) or the standard \( n \)-simplex. We focus on \( n = 2 \).

So let \( M^n \) be an \( n \)-dimensional \langle 2 \rangle-manifold with faces \( E, W \) (hence \( \partial E = \partial W \)). By [Lau00, Lemma 2.1.6] there are collars

\[
\begin{align*}
c_{\partial E} &: \partial E \times [0,1) \hookrightarrow E, \\
c_{\partial W} &: \partial W \times [0,1) \hookrightarrow W, \\
c_E &: E \times [0,1) \hookrightarrow M \quad \text{and} \\
c_W &: W \times [0,1) \hookrightarrow M,
\end{align*}
\]

with \( C_X := c_X(X \times [0,1)) \) for \( X = \partial E, \partial W, E, W \) such that

\[
c_E|_{\partial E \times [0,1)} = c_{\partial W}
\]

and

\[
c_W|_{\partial W \times [0,1)} = c_{\partial E}.
\]

Proposition 2.3.4. Let \( M^n \) be a \langle 2 \rangle-manifold with boundary \( \partial M = E \cup W \) as before. Then any two collars \( c_{\partial E} : \partial E \times [0,1) \hookrightarrow E \) and \( c_{\partial W} : \partial W \times [0,1) \hookrightarrow W \) extend to collars

\[
\begin{align*}
c_E &: E \times [0,1) \hookrightarrow M, \\
c_W &: W \times [0,1) \hookrightarrow M,
\end{align*}
\]

i.e. \( c_E|_{\partial W \times [0,1)} = c_{\partial W} \) and \( c_W|_{\partial E \times [0,1)} = c_{\partial E} \).

Proof. The proof is simple: Interpret the collars as flows of vector fields on \( E, W \) which do not vanish on the boundaries and point inwards and extend them to vector fields on \( M \) (for example using an arbitrary collar on \( M \)) which do not vanish anywhere on \( W, E \), respectively, and point into \( M \). The flows of these vector fields are collars \( c_W \) and \( c_E \) with the desired properties. \( \square \)

Corollary 2.3.5. As before, let \( M \) be a \langle 2 \rangle-manifold with boundary \( \partial M = E \cup_{\partial E} W \). Assume furthermore that \( E \) is the total space of a geometrically flat fiber bundle \( p : E \to B \) with closed fiber \( L \) and a compact base manifold with boundary \( B \). Then there are collars \( c_E, c_W \) of \( E, W \) in \( M \) and \( c_{\partial B} \) of \( \partial B \) in \( B \) such that \( c_W|_{\partial E \times [0,1)} = c_{\partial E} \) and \( c_{\partial B} \) are \( p \)-related.

Proof. One can either take a pair of \( p \)-related collars of \( \partial E \) in \( E \) and of \( \partial B \) in \( B \) and any collar of \( \partial W \) in \( W \) and then use the previous Proposition to
extend them to collars of $E$, $W$ in $M$ or, alternatively, one can take any pair of collars $c_E, c_W$ of $E$ and $W$ in $M$ and construct a collar of $\partial B$ in $B$ by pushing forward the vector field on $E$ induced by $c_W|_{\partial W \times [0,1]}$ with the bundle map $p$, as was done in the proof of Proposition 2.2.3. \qed

3. Thom-Mather-stratified Pseudomanifolds with depth 2

3.1. Thom-Mather-stratified Spaces. If one wants to work with differential forms there has to be some smooth structure. Hence we do not work with topological stratified pseudomanifold as defined for example in [Ban07, Definition 4.1.1] but use Thom-Mather smooth stratified spaces. We use the definition of B. Hughes and S. Weinberger, cf. [HW01, sect. 1.2]. (Another older reference is [Mat12].) In this paper we work with $C^\infty$-Thom-Mather stratified pseudomanifolds. In [Mat12] and [Mat73], Mather proved that every Whitney stratified space has a $C^\infty$-Thom-Mather stratification. Since Whitney showed in [Whi65] that any complex or real analytic set admits a Whitney stratification, those are examples for the type of spaces we consider.

Note further, that Mather also proved, using Thom’s isotopy lemmas, that any stratum $X_i$ in a Thom-Mather stratified space has a neighbourhood $N$ such that the pair $(N, X_i)$ is homeomorphic to the pair $(\text{cyl}(f), X_i)$, with $\text{cyl}(f)$ the mapping cylinder of some fiber bundle $p : E \to X_i$, which is called the link bundle of the stratum. We will later assume these bundles to satisfy flatness conditions.

As we already mentioned in Section 2.3, Verona observes in [Ver84] that Thom-Mather-stratified pseudomanifolds are closely related to manifolds with faces. The Thom-Mather control data corresponds to a system of collars on these manifolds.

As a side remark, we also allude that, by a theorem of Goresky (see [Gor78]), each $C^\infty$-Thom-Mather stratified pseudomanifold can be (smoothly) triangulated by a triangulation compatible with the filtration and hence is a PL-pseudomanifold.

3.2. Flat Fiber Bundles. We recall the definition of geometrically flat fiber bundles.

**Definition 3.2.1.** \((Geometrically) \text{ Flat fiber bundles})\n
A fiber bundle $p : E \to B$ of smooth manifolds with fiber $L$ is called flat if there is an atlas $\mathcal{U} := \{U_\alpha\}_{\alpha \in I}$ of the bundle such that the corresponding transition functions are locally constant. That means that for the local trivialization maps $\phi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times L$, $\pi_1 \circ \phi_\alpha = p$, it holds that

$$\phi_\beta \circ \phi_\alpha^{-1} = \text{id} \times g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times L \to (U_\alpha \cap U_\beta) \times L$$

with $g_{\alpha\beta} \in \text{Diffeo} (L)$ if $U_\alpha \cap U_\beta$ is connected.

A fiber bundle is called geometrically flat if it is flat and if there is a Riemannian metric on the fiber such that the structure group of the bundle is the isometry group of the link with respect to that metric, i.e. the $g_{\alpha\beta}$ in the above definition are isometries of $L$. 

Note that if the base $B$ is a smooth manifold with boundary, then the same holds for the total space $E$ and the restriction of the bundle to the boundary $p| : \partial E \to \partial B$ is also a fiber bundle with the same flatness properties as the original bundle $p$.

### 3.3. The Depth One Setting

In [Ban16], Banagl investigates oriented, compact smooth Thom-Mather stratified pseudomanifolds with filtration

$$X = X_n \supset X_b = \Sigma$$

with $\Sigma^b$ a $b$-dimensional connected closed manifold with geometrically flat link bundle. That means there is an open neighbourhood $N$ of $\Sigma$ in $X$, such that the boundary of the compact manifold $M = X - N$ is the total space of a geometrically flat link bundle $p : \partial M \to \Sigma$ with fiber an oriented, closed smooth Riemannian manifold $L^m$ of dimension $m = n - 1 - b$. There are two strata in this setting: $X_b = \Sigma$ and $X_n - X_b$.

Banagl defines a complex of differential forms $\Omega^I_p$ on the nonsingular part $M$ of $X$ using cotruncation in the fiber direction for multiplicatively structured forms on the boundary $\partial M$.

The flat link bundle condition allows us to define a complex of multiplicatively structured differential forms on the boundary. Let therefore $U := \{U_\alpha\}_{\alpha \in I}$ be a good open cover of $\Sigma$ such that the bundle trivializes with respect to this cover, i.e. for each $\alpha \in I$ there are diffeomorphisms $\psi_\alpha : U_\alpha \times L \to p^{-1}(U_\alpha)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
p^{-1}(U_\alpha) & \xleftarrow{\psi_\alpha} & U_\alpha \times L \\
p| & \downarrow & \pi_1 \\
U_\alpha & \xleftarrow{\pi_2} & L
\end{array}
$$

We are then able to define the following subcomplex of the complex $\Omega^I_p(\partial M)$ of differential forms on $\partial M$, using the projections $\pi_1 : U_\alpha \times L \to U_\alpha$ and $\pi_2 : U_\alpha \times L \to L$:

$$
\Omega_{MS}^I(\Sigma) := \left\{ \omega \in \Omega^I(\partial M) \mid \omega|_{U_\alpha} = \phi_\alpha^* \sum_j \pi_1^* \eta_j \wedge \pi_2^* \gamma_j \right\},
$$

with $\eta_j \in \Omega^I(U_\alpha), \gamma_j \in \Omega^I(L)$.

These forms can be truncated or cotruncated in the link direction (see [Ban16, section5]) and the mentioned complex $\Omega^I_p$ is defined as containing the forms that look like the pullback of a fiberwise cotruncated multiplicative structured form near $\partial M$ in a collar neighbourhood of the boundary.

The cohomology of that complex then satisfies generalized Poincaré-duality over complementary perversities and is isomorphic to the cohomology of the associated intersection space if the link bundle is trivial. For arbitrary flat link bundle we do not yet know how to construct the intersection space.
3.4. Spaces of Stratification Depth Two. The aim of the thesis is to generalize the above construction to certain classes of pseudomanifolds with stratification depth two. Strictly speaking, we consider smooth Thom-Mather stratified pseudomanifolds \( X \) of dimension \( n \) with filtration \( X = X_n \supset X_b \supset X_s \) with \( n - 2 \geq b > s \) and additional conditions on the regular neighbourhoods of the singular strata. The strata here are \( X_s \) and \( X_b - X_s \), which are the singular strata, and \( X_n - X_b \). We mainly consider zero dimensional bottom strata, i.e. \( s = 0 \) and \( X_s = \{ x_0, \ldots, x_d \} \).

We consider Thom-Mather-stratified pseudomanifolds \( X \) with filtration

\[
X = X_n \supset X_b \supset X_0 = \{ x_0, \ldots, x_d \},
\]

where the bottom stratum is zero dimensional and the middle stratum satisfies a geometrical flatness condition. To define intersection space cohomology on these stratified pseudomanifolds we first remove a regular neighbourhood \( R_0 \) of \( X_0 \) homeomorphic to \( \text{cone}(L_0) \), with \( L_0 \) a stratified pseudomanifold of dimension \( n - 1 \). The result is a stratified pseudomanifold \( X' = X - R_0 \) with boundary and one singular stratum

\[
B := X'_b = X_b - R_0 \cap X_b,
\]

a \( b \)-dimensional compact smooth manifold with boundary \( \partial B \). We assume that this singular stratum has a geometrically flat link bundle in \( X' \), i.e. there is an open tubular neighbourhood \( T_b \) of \( B \) in \( X' \) such that

\[
M := X' - R_b
\]

is a smooth \( (2) \)-manifold with boundary decomposed as

\[
\partial M = E \cup_{\partial E = \partial W} W
\]

such that \( E \) is the total space of a geometrically flat link bundle

\[
\begin{array}{ccc}
L & \longrightarrow & E \\
\downarrow & & \downarrow p \\
& & B
\end{array}
\]

over the compact base manifold with boundary \( B \) with link a closed smooth Riemannian manifold \( L^0 \). The manifold with boundary \( W \) is

\[
W = L_0 - T_b \cap L_0,
\]

with boundary \( \partial W = \partial E \). It is the regular part of the pseudomanifold \( L_0 = \partial X' \), the link of \( X_0 \). Note that in order to prove Poincaré duality for \( H_I^p \) over complementary perversities, we have to impose an additional Witt-type condition on \( W \).

Remark 3.4.1 (Thom-Mather control data and Collars). The control data of the Thom-Mather stratification of a pseudomanifold induces a system of collars on the \( (2) \)-manifold \( M \). In particular, also a collar of \( \partial B \) on \( B \) is induced such that we get compatible collars on the fiber bundle, as was explained in Corollary 2.3.5. We always work with this system of collars.
3.4.1. **Cotruncation Values.** If we have a stratified pseudomanifold $X$ with stratification $X = X_n \supset X_b \supset X_s$ and complementary perversities $\bar{p}$, $\bar{q}$, then, unless otherwise stated, we set $\dim(L = \text{Link } X_b) := m := n - 1 - b$ and $\dim(F = \text{Link } X_s) := f := n - 1 - s$ and define the cutoff values $K := m - \bar{p}(m + 1)$, $K' := m - \bar{q}(m + 1)$ and $L := f - \bar{p}(f + 1)$, $L' := f - \bar{q}(f + 1)$.

These cutoff values are the cotruncation degrees for the complexes of multiplicatively structured differential forms near the respective strata.

4. **Cotruncation on Manifolds with Boundary**

In this section we establish the cotruncation of the cochain complex of smooth differential forms on manifolds with boundary. Recall that on a closed Riemannian manifold $M$ the Hodge decomposition provides orthogonal splittings

\[
\Omega^r(M) = \text{im } d \oplus \ker d^* = \text{im } d \oplus \text{im } d^* \oplus \mathcal{H}^r(M)
\]

with $\mathcal{H}^r(M) := \{ \omega \mid \Delta \omega = (d d^* + d^* d) \omega = 0 \}$ the harmonic $r$-forms on $M$. This allows us to define the cotruncated subcomplex of smooth differential forms:

\[
\tau \geq L \Omega^\bullet(M) := \cdots \rightarrow \ker d^* \rightarrow \Omega^{L+1}(M) \rightarrow \Omega^{L+2}(M) \rightarrow \cdots \subset \Omega^\bullet(M).
\]

By the Hodge decomposition this complex has the following properties:

\[
H^r(\tau \geq L \Omega^\bullet(M)) = \begin{cases} 0 & \text{for } r < L, \\ H^r(M) & \text{for } r \geq L \end{cases}
\]

and

\[
\tau \geq L \Omega^\bullet(M) \oplus \tau < L \Omega^\bullet = \Omega^\bullet(M),
\]

where $\tau < L \Omega^\bullet(M) := \cdots \rightarrow \Omega^{L-1}(M) \rightarrow \text{im } d \rightarrow 0 \rightarrow \cdots \subset \Omega^\bullet(M)$.

For manifolds with boundary the Hodge decomposition (2) is not true in general, so we cannot define the subcomplex of cotruncated differential forms $\tau \geq L \Omega^\bullet(M)$ in the same way as before. But there is a natural substitute for the Hodge decomposition: The so called Hodge–Morrey–Friedrichs decomposition. In principle, the difference to the Hodge decomposition for closed manifolds is that one has to impose boundary conditions for the differential forms. In particular, if the boundary is the empty set ($M$ closed) these conditions vanish and the decomposition reduces to the well known Hodge decomposition on closed manifolds.

4.1. **The Hodge–Morrey–Friedrichs Decomposition.** Let $(M^n, g)$ be an oriented and compact smooth Riemannian manifold with boundary $\partial M$ and Riemannian metric $g$. Let $\omega \in \Omega^r(M)$ be a smooth $r$-form on $M$ and let $t \omega$ denote the tangential and $n \omega$ the normal component of $\omega$. 

Definition 4.1.1. For $r \in \mathbb{Z}$ we define the spaces of Dirichlet forms $\Omega^r_D(M)$, Neumann forms $\Omega^r_N(M)$, coclosed Neumann forms $cC^r_N(M)$, exact Dirichlet forms $E^r_D(M)$, coexact Neumann forms $cE^r_N(M)$, exact harmonic forms $\mathcal{H}^r_{\text{ex}}(M)$ and Neumann harmonic forms $\mathcal{H}^r_N(M)$ as follows:

$$
\begin{align*}
\Omega^r_D(M) &:= \{ \omega \in \Omega^r(M) \mid \text{tr} \, \omega = 0 \} \\
\Omega^r_N(M) &:= \{ \omega \in \Omega^r(M) \mid \text{tr} \, \omega = 0 \} \\
cC^r_N(M) &:= \{ \omega \in \Omega^r_N(M) \mid d^*\omega = 0 \} \\
E^r_D(M) &:= \{ \text{d}\alpha \mid \alpha \in \Omega^{-1}_{D}^{r}(M) \} \\
cE^r_N(M) &:= \{ d^*\xi \mid \xi \in \Omega^{r+1}_{N}(M) \} \\
\mathcal{H}^r_{\text{ex}}(M) &:= \{ d\eta \mid \eta \in \Omega^{-1}_{D}^{r}(M), d^*d\eta = 0 \} \\
\mathcal{H}^r_N(M) &:= \{ \eta \in \Omega^r_N(M) \mid d^*\eta = 0, \, d\eta = 0 \}
\end{align*}
$$

Note that $E^r_D(M) \subset \Omega^r_D(M)$ and $cE^r_N(M) \subset \Omega^r_N(M)$ for all $r \in \mathbb{Z}$. We then have the following decomposition of $\Omega^r(M)$ into orthogonal direct summands established by C. B. Morrey and K. O. Friedrichs:

**Theorem 4.1.2.** *(The Hodge–Morrey–Friedrichs Decomposition)*

On a compact oriented smooth Riemannian manifold $(M^n, g)$ with boundary $\partial M$ we have, for each $r \in \mathbb{Z}$, the orthogonal direct sum decomposition

$$
\Omega^r(M) = E^r_D(M) \oplus \mathcal{H}^r_{\text{ex}}(M) \oplus cE^r_N(M) \oplus \mathcal{H}^r_N(M).
$$

**Proof:** By [Sch95, Corollary 2.4.9] the above orthogonal decomposition holds for $L^2$-forms and forms of arbitrary Sobolev class. But then a standard argument involving the Sobolev lemma and some regularity results gives the desired decomposition for smooth forms. For more details, see page 85 of [Sch95] and [Sch95, Section 2.2]).

\[ \square \]

**Corollary 4.1.3.** Let $r \in \mathbb{Z}$. Then for $E^r(M) := \{ d\omega \mid \omega \in \Omega^{-1}(M) \}$ and $cC^r_N(M)$ there are orthogonal direct splittings

$$
E^r(M) = E^r_D(M) \oplus \mathcal{H}^r_{\text{ex}}(M)
$$

and

$$
cC^r_N(M) = E^r_N(M) \oplus \mathcal{H}^r_N(M).
$$

**Proof:** The main tool for proving this corollary is **Green’s formula** ([Sch95, Prop. 2.1.2]). It implies that for two smooth forms $\omega \in \Omega^{-1}(M), \eta \in \Omega^{r}(M)$ we have

$$
\ll d\omega, \eta \gg = \ll \omega, d^*\eta \gg + \int_{\partial M} \text{tr} \, \omega \wedge *\text{n} \, \eta,
$$

where $\ll \alpha, \beta \gg = \int_{M} \alpha \wedge *\beta$ denotes the $L^2$-metric on $\Omega^r(M)$. We first show that

$$
E^r(M) = E^r_D(M) \oplus \mathcal{H}^r_{\text{ex}}(M): \tag{3}
$$
For let $d\omega \in E^r(M)$, $d^* \alpha \in cE^r_N(M)$ and $\beta \in H^r_N(M)$. Then by Green’s formula
\[
\ll d\omega, d^* \alpha \gg = \ll d\omega, (d^*)^2 \alpha \gg + \int_{\partial M} t \ \omega \wedge *n \ \alpha = 0,
\]
since $(d^*)^2 = 0$ and $n \alpha = 0$. On the other hand
\[
\ll d\omega, \beta \gg = \ll d\omega, d^* \beta \gg + \int_{\partial M} t \ \omega \wedge *n \ \beta = 0,
\]
since $\beta \in H^r_N(M)$. Therefore by the above Theorem 4.1.2 we have $\omega \in E^r_D(M) \oplus H^r_N(M)$, i.e. $E^r(M) \subset E^r_D(M) \oplus H^r_N(M)$. This implies (3), since the converse is trivially true.
The second step is to show that
\[
(4) \quad cC^r_N(M) = cE^r_N(M) \oplus H^r_N(M).
\]
For let $\omega \in cC^r_N(M)$, $d\alpha \in E^r_D(M)$ and $d\beta \in H^r_N(M)$. Again by Green’s formula we obtain
\[
\ll \omega, d\alpha \gg = \ll d\omega, \alpha \gg + \int_{\partial M} t \ \alpha \wedge *n \ \omega = 0
\]
and
\[
\ll \omega, d\beta \gg = \ll d\omega, \beta \gg + \int_{\partial M} t \ \beta \wedge *n \ \omega = 0
\]
(by definition of $cC^r_N(M)$). Therefore by Theorem 4.1.2 $\omega \in E^r_N(L) \oplus H^r_N(M)$, i.e. $cC^r_N(M) \subset E^r_N(L) \oplus H^r_N(M)$, and since the converse inclusion is trivially true, the corollary is established.

4.2. Cotruncation on Manifolds with Boundary. Using the results of the previous subsection, in particular the Hodge–Morrey–Friedrichs decomposition we now can establish the cotruncated subcomplex of the complex of differential forms on a Riemannian manifold with boundary $M$.

**Definition 4.2.1.** Let $k \in \mathbb{N}$. Then we define
\[
\tau_{\geq k} \Omega^\bullet(M) := \ldots \to 0 \to cC^k_N(M) \to \Omega^{k+1}(M) \to \Omega^{k+2}(M) \to \ldots
\]

**Lemma 4.2.2.** The subcomplex inclusion $i : \tau_{\geq k} \Omega^\bullet(M) \hookrightarrow \Omega^\bullet(M)$ induces an isomorphism
\[
i^* : H^r(\tau_{\geq k} \Omega^\bullet(M)) \xrightarrow{\cong} H^r(M) \quad \text{for } r \geq k.
\]
On the other hand
\[
H^r(\tau_{\geq k} \Omega^\bullet(M)) = 0 \quad \text{for } r < k.
\]
Proof: For \( r \geq k + 2 \) the statement is obvious since then \( \tau_{\geq k} \Omega^r(M) = \Omega^r(M) \) and \( \tau_{\geq k} \Omega^{r-1} = \Omega^{r-1}(M) \).

Let \( r = k + 1 \).

\[
H^{k+1}(\tau_{\geq k} \Omega^\bullet(M)) = \frac{\ker d^{k+1}}{d^k(cC^k_N(M))}
\]

But Corollary 4.1.3 implies that

\[
d^k(cC^k_N(M)) = d^k(cC^k_N(M) \oplus E^k(M)) = d^k(\Omega^k(M)) = \text{im} d^k
\]

and hence

\[
H^k(\tau_{\geq k} \Omega^\bullet(M)) = \frac{\ker d^{k+1}}{\text{im} d^k}.
\]

Now let \( r = k \).

\[
H^k(\tau_{\geq k} \Omega^\bullet(M)) = \frac{\ker d^k \cap cC^k_N(M)}{d^{k-1}(0)} = \ker d^k \cap cC^k_N(M) = H^k_N(M)
\]

Let \( \omega \in \Omega^k(M) \). By Theorem 4.1.2 and Corollary 4.1.3 there are forms \( d\alpha \in E^k(M), d^*\beta \in cE^k_N(M) \) and \( \sigma \in \mathcal{H}_N^k(M) \) such that \( \omega = d\alpha + d^*\beta + \sigma \).

Now if \( \omega \) is closed, \( d\omega = 0 \), then \( dd^*\beta = d\omega - d^2\alpha - d\sigma = 0 \) and by Green’s formula we therefore have

\[
\ll d^*\beta, d^*\beta \gg = \ll dd^*\beta, \beta \gg - \int_{\partial M} t \ d^*\beta \wedge n \beta = 0
\]

by the definition of \( \Omega^{k+1}_N(M) \). Therefore \( d^*\beta = 0 \) and we have

\[
\omega \in \ker d^k \iff \omega = d\alpha + \sigma.
\]

Hence, by the orthogonality of the Hodge–Morrey–Friedrichs decomposition we have

\[
H^k(\tau_{\geq k} \Omega^\bullet(M)) = \ker d^k \cap cC^k_N(M) = \mathcal{H}_N^k(M) \cong \frac{\ker d^k}{\text{im} d^{k-1}} = H^k(M).
\]

The second statement is obvious since, by definition, \( \tau_{\geq k} \Omega^r(M) = 0 \) for \( r < k \).

\( \square \)

4.3. Cotruncation of other Complexes of Differential Forms. To be able to prove Poincaré duality for \( HI \) on 3-strata pseudomanifolds we will need cotruncation of subcomplexes of the complex of differential forms. Even more we need this cotruncation to be consistent with the cotruncation \( \tau_{\geq k} \Omega^\bullet(M) \). This means, if we have a subcomplex \( S^\bullet \subset \Omega^\bullet(M) \), where \( M \) is a smooth compact manifold (with or without boundary), we want a cotruncated subcomplex \( \tau_{\geq k} S^\bullet \) that satisfies

\[
(1) \ H^r(\tau_{\geq k} S^\bullet) = \begin{cases} 0 & \text{if } r < k, \\ H^r(S^\bullet) & \text{if } r \geq k, \end{cases}
\]

\( (2) \ \tau_{\geq k} S^\bullet \subset \tau_{\geq k} \Omega^\bullet(M) \) is a subcomplex.
Definition 4.3.1. (Geometrically cotruncatable subcomplexes of $\Omega^\bullet(M)$)
A subcomplex $S^\bullet \subset \Omega^\bullet(M)$ is called geometrically cotruncatable in degree $k \in \mathbb{N}$ if
$$\text{im} d^k_{M} \cap S^k = d^k_{S}(S^{k-1}).$$
Note that the inclusion "⊃" in the relation is satisfied for any subcomplex while the inclusion "⊂" does usually not hold.

Lemma 4.3.2. (An equivalent condition for cotruncatability)
Let $S^\bullet \subset \Omega^\bullet(M)$ be a subcomplex. Then $S^\bullet$ is geometrically cotruncatable in degree $k \in \mathbb{N}$ if and only if subcomplex inclusion induces and injection $H^k(S^\bullet) \hookrightarrow H^k(M)$.

Proof: This is just the definition of the geometric cotruncatability and cohomology. $\square$

The next lemma shows why we call a complex satisfying the condition of Definition 4.3.1 geometrically cotruncatable: The intersection of $S^\bullet$ with the cotruncated complex $\tau_{\geq k}\Omega^\bullet(M)$ is a cotruncation of $S^\bullet$ in degree $k$:

Lemma 4.3.3. If $S^\bullet \subset \Omega^\bullet(M)$ is geometrically cotruncatable in degree $k \in \mathbb{N}$, then there is an orthogonal direct sum decomposition
$$S^k = d^{k-1}(S^{k-1}) \oplus S^k \cap cC_N^k(M)$$
and the complex $\tau_{\geq k}S^\bullet$ defined by
$$\tau_{\geq k}S^\bullet := \ldots \rightarrow 0 \rightarrow S^k \cap cC_N^k(M) \rightarrow S^{k+1} \rightarrow \ldots$$
is a suitable cotruncation in the above sense, i.e. $\tau_{\geq k}S^\bullet \subset \tau_{\geq k}\Omega^\bullet(M)$ is a subcomplex and
$$H^r(\tau_{\geq k}S^\bullet) = \begin{cases} 0 & \text{if } r < k, \\ H^r(S^\bullet) & \text{if } r \geq k. \end{cases}$$
(Note that if $M$ is closed then $cC_N^r(M) = cC^r(M) = \{ \omega \in \Omega^r(M) | d^r\omega = 0 \}$ for all $r$.)

Proof: The first equation follows from the orthogonal direct sum composition
$$\Omega^k(M) = \text{im} d^{k-1} \oplus cC_N^k(M)$$
for $\Omega^k(M)$ given by Theorem 4.1.2 and Corollary 4.1.3. This gives the direct sum decomposition
$$S^k = S^k \cap \Omega^k(M) = \underbrace{S^k \cap \text{im} d^{k-1} \oplus S^k \cap cC_N^k(M)}_{=d^{k-1}(S^{k-1})}.$$
\[ r > k + 1 \text{ then } \tau_{\geq k}^r S^r = S^r \text{ and } \tau_{\geq k}^r S^{r+1} = S^{r+1} \text{ and hence } H^r(\tau_{\geq k}^r S^\bullet) = H^r(S^\bullet). \] The only nontrivial degrees are \( r = k \) and \( r = k + 1 \): We have
\[
S^k = S^k \cap \Omega^k(M) = S^k \cap (cC^k_N(M) \oplus \text{im } d^{k-1}) \\
= (S^k \cap cC^k_N(M)) \oplus (S^k \cap \text{im } d^{k-1})
\]
Hence
\[
H^k(S^\bullet) = \frac{\ker d^k \cap S^k}{d^{k-1}(S^{k-1})} \cong S^k \cap cC^k_N(M) \cap \ker d^k = H^k(\tau_{\geq k} S^\bullet)
\]
and
\[
H^{k+1}(S^\bullet) = \frac{\ker d^{k+1} \cap S^{k+1}}{d^k(S^k)} \cong \frac{\ker d^{k+1} \cap S^{k+1}}{d^k(cC^k_N(M) \cap S^k)} = H^{k+1}(\tau_{\geq k} S^\bullet).
\]

Remark: Note that the subscript "\( N \)" in \( cC^\bullet_N(M) \) stands for Neumann boundary conditions and can be dropped if the manifold \( M \) has empty boundary.

Example 4.3.4. Let \( S^\bullet \subset \Omega^\bullet(M) \) be a subcomplex with \( H^k(S^\bullet) = 0 \), for some \( k \in \mathbb{Z} \). Then \( \text{im } d^{k-1} \cap S^k \subset \ker d^k \cap S^k = d^{k-1} S^{k-1} \subset \text{im } d^{k-1} \cap S^k \) and hence \( S^\bullet \) is geometrically cotruncatable in degree \( k \).

Remark 4.3.5. (Truncation of arbitrary subcomplexes \( S^\bullet \subset \Omega^\bullet(M) \))
Note that it is always possible to define truncation for arbitrary subcomplexes \( S^\bullet \subset \Omega^\bullet(M) \) (without additional assumptions on \( S^\bullet \)) as a subcomplex of \( \tau_{<k} \Omega^\bullet(M) \):
\[
\tau_{<k} S^\bullet := \ldots \rightarrow S^{k-1} \rightarrow d^{k-1}(S^{k-1}) \rightarrow 0 \rightarrow \ldots
\]
Indeed, \( \tau_{<k} S^\bullet \) satisfies
\[
H^r(\tau_{<k} S^\bullet) = \begin{cases} H^r(S^\bullet) & \text{if } r < k, \\ 0 & \text{else.} \end{cases}
\]
If in addition \( S^\bullet \) is geometrically cotruncatable in degree \( k \), then there is an orthogonal direct sum decomposition
\[
S^\bullet = \tau_{<k} S^\bullet \oplus \tau_{\geq k} S^\bullet
\]
(induced by the direct sum decomposition \( \Omega^\bullet(M) = \tau_{<k} \Omega^\bullet(M) \oplus \tau_{\geq k} \Omega^\bullet(M) \)) and hence the composition
\[
\tau_{<k} S^\bullet \hookrightarrow S^\bullet \xrightarrow{\text{proj}} \frac{S^\bullet}{\tau_{\geq k} S^\bullet}
\]
is an isomorphism of differential complexes.
5. The method of iterated triangles

To prove the main theorem 7.4.1, we iterate 5-Lemma arguments involving diagrams containing long exact sequences that are induced by distinguished triangles. In the setting of this paper we need the intermediate complex $\widetilde{\Omega}_I^\bullet$, see Section 6. The two distinguished triangles 6.2.2, 6.2.6 relate the absolute and relative $\widetilde{\Omega}_I^\bullet$--complexes to the absolute and relative complexes of differential forms on the blowup $M$ of $X$ and complexes of fiberwisely (co)truncated multiplicatively structured forms on the boundary part $E$. A 5-Lemma argument proves Poincaré-Lefschetz duality for the cohomology of $\widetilde{\Omega}_I^\bullet$, using the Poincaré-Lefschetz statements for forms on $M$ and the fiberwisely (co)truncated multiplicatively structured forms on $E$.

The distinguished triangles 7.2.3, 7.2.4 then relate $\Omega_I^\bullet$ to the absolute and relative $\widetilde{\Omega}_I^\bullet$--complexes and the complexes of (co)truncated $\Omega_I^\bullet$--forms on the boundary part $W$, which can be seen as the blowup of the link of $X_0$. The Poincaré duality for $HI$ can then be deduced with a 5-Lemma argument, using the Poincaré-Lefschetz duality of $\widetilde{\Omega}_I^\bullet$ and the Poincaré duality of $HI^\bullet(W)$, which follows from [Ban16, Theorem 8.2].

Note that we used the triangle and 5-Lemma argument twice, where two is also the stratification depth of $X$ and hence the number of boundary parts of the blowup. It might be possible, that an analogous construction might help to prove Poincaré duality for $HI$ for pseudomanifolds of greater stratification depth with one pair of distinguished triangles for each singular stratum. In analogy to our setting, these triangles would relate a chain of intermediate complexes and certain complexes on the boundary parts of the respective singular strata.

6. The Partial de Rham Intersection Complex

We define the intermediate complexes $\widetilde{\Omega}_I^\bullet(M)$ and $\widetilde{\Omega}_I^\bullet(M,C_W)$. They consist of forms whose restriction to $C_E$ is the pullback of a fiberwisely cotruncated form on $E$ and whose restriction to $C_W$ is either the pullback of some form on $W$ or zero for the relative group. We show that the corresponding cohomology groups $H^r(\widetilde{\Omega}_I^\bullet(M))$ and $H^{n-r}(\widetilde{\Omega}_I^\bullet(M,C_W))$ are Poincaré-Lefschetz dual to each other, see Theorem 6.5.4.

Not till then we define the actual complex of intersection space forms on $M$, $\Omega_I^\bullet$, and show Poincaré duality for it.

Before we give the definitions of $\widetilde{\Omega}_I^\bullet(M)$ and $\widetilde{\Omega}_I^\bullet(M,C_W)$ we recall the definitions of the complex of multiplicatively structured forms as well as the complex of fiberwisely truncated and cotruncated multiplicatively structured forms from [Ban16, Sections 3 and 6]:

**Definition 6.0.1 (Multiplicatively structured forms).** Let $p : E \to B$ be a flat bundle with base $B$ a compact manifold with boundary $\partial B$ and fiber a Riemannian manifold $L$ and let $U = \{U_\alpha\}_{\alpha \in I}$ be a good open cover of $B$
such that the bundle trivializes with respect to that cover. Let further \( U \subset B \) be open. We then define

\[
\Omega_{\text{MS}}^\bullet(U) := \{ \omega \in \Omega^\bullet(p^{-1}(U)) \mid \forall \alpha \in I : \omega|_{p^{-1}(U_\alpha)} = \phi_\alpha^* \sum_{j_a} \pi_1^* \eta_{j_a} \wedge \pi_2^* \gamma_{j_a} \\
\text{with } \eta_{j_a} \in \Omega^\bullet(U \cap U_\alpha), \ \gamma_{j_a} \in \Omega^\bullet(L) \} \]

Here, the \( \phi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times L \) denote the local trivializations of the bundle.

To define the complexes of fiberwisely truncated and cotruncated multiplicatively structured forms we need the complexes of truncated and cotruncated forms of the closed (Riemannian) manifold \( L \) from [Ban16, Section 4].

**Definition 6.0.2** (Fiberwisely (co)truncated forms). Let \( p : E \to B \) be a flat bundle with base \( B \) a compact manifold with boundary \( \partial B \) and fiber a closed manifold \( L \) and let \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in I} \) be a good open cover of \( B \) such that the bundle trivializes with respect to that cover as in the previous definition. Let further \( U \subset B \) be open. We then define, for any integer \( K \), the complex of (in degree \( K \)) fiberwisely truncated multiplicatively structured forms by

\[
ft_{<K} \Omega_{\text{MS}}^\bullet(U) := \{ \omega \in \Omega_{\text{MS}}^\bullet(U) \mid \forall \alpha \in I : \omega|_{p^{-1}(U_\alpha)} = \phi_\alpha^* \sum_{j_a} \pi_1^* \eta_{j_a} \wedge \pi_2^* \gamma_{j_a} \\
\text{with } \gamma_{j_a} \in \tau_{<K} \Omega^\bullet(L) \} \]

If the fiber is a Riemannian manifold and the bundle is geometrically flat, moreover we define the complex of fiberwisely cotruncated multiplicatively structured forms by

\[
ft_{\geq K} \Omega_{\text{MS}}^\bullet(U) := \{ \omega \in \Omega_{\text{MS}}^\bullet(U) \mid \forall \alpha \in I : \omega|_{p^{-1}(U_\alpha)} = \text{as in (5)} \\
\text{with } \gamma_{j_a} \in \tau_{\geq K} \Omega^\bullet(L) \} .
\]

All of these complexes \( \Omega_{\text{MS}}^\bullet(U), ft_{<K} \Omega_{\text{MS}}^\bullet(U), \) and \( ft_{\geq K} \Omega_{\text{MS}}^\bullet(U) \), for \( U \subset B \) open, are subcomplexes of the complex of forms \( \Omega^\bullet(p^{-1}(U)) \).

**Definition 6.0.3** (The Partial de Rham Intersection Complex).

\[
\tilde{\Omega}I_p^\bullet(M) := \{ \omega \in \Omega^\bullet(M) \mid c_E^* \omega = \pi_E^* \eta \text{ for some } \eta \in ft_{\geq K} \Omega_{\text{MS}}^\bullet(B) \\
c_W^* \omega = \pi_W^* \rho \text{ for some } \rho \in \Omega^\bullet(W) \}
\]

\( \pi_E, \ \pi_W \) denote the projections \( \pi_E : E \times [0, 1) \to E, \ \pi_W : W \times [0, 1) \to W \).

This is a subcomplex of the complex \( \Omega^\bullet(M) \) of forms on \( M \).

**Definition 6.0.4** (The relative Partial de Rham Intersection Complex).

\[
\tilde{\Omega}I_p^\bullet(M, C_W) := \{ \omega \in \tilde{\Omega}I_p^\bullet(M) \mid c_W^* \omega = 0 \} \subset \Omega^\bullet(M, C_W).
\]
In the rest of this section we prove Poincaré duality between $\widehat{\Omega}^\bullet_p(M)$ and $\widehat{\Omega}^\bullet_q(M,W)$.

**Theorem 6.5.4 (Poincaré Duality for Partial Intersection Forms)**

For complementary perversities $\bar{p}$ and $\bar{q}$, integration induces a nondegenerate bilinear form

$$\widehat{H}^\bar{p}_\bar{p}(M) \times \widehat{H}^{n-\bar{r}}_\bar{q}(M,C_W) \to \mathbb{R}$$

$$(\omega, \eta) \mapsto \int_M \omega \wedge \eta,$$

where $\widehat{H}^\bar{p}_\bar{p}(M) := H^\bar{p}(\widehat{\Omega}^\bullet_p(M))$ and $\widehat{H}^{n-\bar{r}}_\bar{q}(M,C_W) := H^{n-\bar{r}}(\widehat{\Omega}^\bullet_q(M,C_W))$.

6.1. **Multiplicative Forms that are constant near the end.** The proof of Theorem 6.5.4 is geared on the proof of Poincaré duality for intersection forms in the two strata case, see [Ban16, Sect. 8]. However, the additional stratum produces additional technical difficulties, as one might have expected. We first deal with the fact that in $\widehat{\Omega}^\bullet_p$ we do not just demand that the forms restricted to a collar neighbourhood of $E$ come from a form in $ft_{\geq K}\Omega^\bullet_{MS}(B)$ but also that they are constant in the collar direction in a collar neighbourhood of $W$.

**Definition 6.1.1** (Fiberwise cotruncated forms that are in $\Omega^\bullet_{2k}(E)$). We recall that the collar $c_{\partial E} : \partial E \times [0,1) \hookrightarrow E$ of $E$ in $E$ is the restriction of the collar of $W$ in $M$, $c_{\partial E} = c_W|_{\partial W \times [0,1)}$, and define

$$P^\bullet(B) := \{ \omega \in \Omega^\bullet_{MS}(B) \mid \exists \eta \in \Omega^\bullet_{MS}(\partial B) : c^\bullet_{\partial E}\omega = \pi^\bullet_{\partial E}\eta \}\).$$

Analogously, we define

$$P_{\geq K}(B) := \{ \omega \in ft_{\geq K}\Omega^\bullet_{MS}(B) \mid \exists \eta \in ft_{\geq K}\Omega^\bullet_{MS}(\partial B) : c^\bullet_{\partial E}\omega = \pi^\bullet_{\partial E}\eta \}\)$$

and

$$P_{< K}(B) := \{ \omega \in ft_{< K}\Omega^\bullet_{MS}(B) \mid \exists \eta \in ft_{< K}\Omega^\bullet_{MS}(\partial B) : c^\bullet_{\partial E}\omega = \pi^\bullet_{\partial E}\eta \}\).$$

We want to show that these complexes are quasi-isomorphic to the analogous complexes without the condition at the end of the manifold. The argument of the proof of [Ban16, Prop. 2.4], which uses integration of the forms on the collar, is applicable to multiplicatively structured forms as well.

**Lemma 6.1.2.** The subcomplex inclusions $i : P^\bullet(B) \hookrightarrow \Omega^\bullet_{MS}(B)$, $i_{\geq K} : P_{\geq K}^\bullet(B) \hookrightarrow ft_{\geq K}\Omega^\bullet_{MS}(B)$, $i_{< K} : P_{< K}^\bullet(B) \hookrightarrow ft_{< K}\Omega^\bullet_{MS}(B)$ are quasi-isomorphisms.

**Proof.** We give a proof for the non-truncated case that transfers literally to the truncated and cotruncated one. Take a slight extension of the $p$–related collars $c_{\partial E}$ and $c_{\partial B}$ to a pair $\tilde{c}_{\partial E} : \partial E \times [0,\tfrac{3}{2}) \hookrightarrow E, \tilde{c}_{\partial B} : \partial B \times [0,\tfrac{3}{2}) \hookrightarrow B$ of collars which are still $p$–related. Define a smooth cutoff function $\xi : [0,\tfrac{3}{2}) \to \mathbb{R}$ with compact support and $\xi|[0,1] = 1$. This induces a cutoff function $p^*(\tilde{c}_{\partial B}^{-1})\pi^*\xi$ on $E$, which we also denote by $\xi$. Trivially, this is a
multiplicative function. Let \( \omega \in \Omega^\bullet(E) \) be any form. Then \( c_{\partial E}^\bullet \omega \) decomposes as \( \omega_0 + dt \land \omega_1 \), with \( \omega_0(t), \omega_1(t) \in \Omega^\bullet(\partial E) \) for each \( t \in [0, \frac{1}{2}] \). Let \( \alpha \in (0, 1) \) and define a map \( \rho : \Omega^\bullet(E) \to \Omega^\bullet_{\partial C}(E) \), where the latter is the subcomplex of forms \( \omega \) with \( c_{\partial E}^\bullet \omega \) constant in the collar direction, by

\[
\rho(\omega) = (1 - \xi)\omega + \xi c_{\partial E}^\bullet \pi_1^* \omega_0(a) - d\xi \land \int_a^t \omega_1 dt.
\]

By the argument of Banagl, this map is a chain homotopy equivalence with homotopy inverse the subcomplex inclusion \( \Omega^\bullet_{\partial C}(E) \hookrightarrow \Omega^\bullet(E) \). The homotopy is given by \( K(\omega) = \xi \int_a^t \omega_1 dt \). To be able to apply the arguments, we must show that \( \rho \) and \( K \) restrict to complexes of multiplicatively structured forms. By our choice of \( \xi \), this can be achieved by proving that for a multiplicative \( \omega_1 \) in the above decomposition, integration yields a multiplicative form \( \int_a^t \omega_1 dt \). So let \( \omega_1 \) be a multiplicatively structured form. Recall, that we work with a collar that is small with respect to the (finite) open cover \( U = \{ U_\alpha \}_{\alpha \in I} \) with respect to which the bundle trivializes, see Definition 2.1.1 and Lemma 2.1.2. Let \( I_\partial \) and the \( W_\alpha \) be as in the aforesaid definition. Let \( \{ \rho_\alpha \}_{\alpha \in I_\partial} \) be a partition of unity on \( \partial B \) with respect to the cover \( \{ W_\alpha \}_{\alpha \in I_\partial} \). This gives a partition of unity \( \{ \tilde{\eta}_\alpha := c^\bullet \pi_1^* \rho_\alpha \}_{\alpha \in I_\partial} \) of the collar of \( \partial B \) in \( B \). Hence \( \omega_1 = \sum_\alpha p^* \tilde{\eta}_\alpha \omega_1 \). Since \( \omega_1 \) is multiplicatively structured, we can write it as follows.

\[
\omega_1 = \sum_{\alpha \in I_\partial} \phi_\alpha^* \sum_{j_\alpha} \pi_1^* (\tilde{\eta}_\alpha \eta_{j_\alpha}) \land \pi_2^* \gamma_{j_\alpha}.
\]

Note that \( \tilde{\eta}_\alpha \) is always independent of the collar coordinate. The \( p \)-related collars \( c_{\partial E} \) and \( c_{\partial B} \) then allow us to write the integration of \( \omega_1 \) in the collar direction as a multiplicative form.

\[
\int_a^t \omega_1(\tau) \, d\tau = \sum_{\alpha \in I_\partial} \phi_\alpha^* \sum_{j_\alpha} \pi_1^* (\tilde{\eta}_\alpha \int_a^t \eta_{j_\alpha}(\tau) \, d\tau) \land \pi_2^* \gamma_{j_\alpha}.
\]

We want to show that this form is multiplicatively structured. Let \( \alpha \in I \) and let \( \alpha_1, \ldots, \alpha_k \in I_\partial \) be all the indices such that \( U_\alpha \cap c_{\partial B}(W_{\alpha_i} \times [0, 1)) \neq \emptyset \). Restricting \( \int_a^t \omega_1(\tau) \, d\tau \) to \( p^{-1}(U_\alpha) \), we then get

\[
\phi_\alpha^* \sum_{i=1}^k \sum_{j_{\alpha_i}} \pi_1^* (\tilde{\eta}_{\alpha_i} \int_a^t \eta_{j_{\alpha_i}}(\tau) \, d\tau) \bigg|_{U_{\alpha_i}} \land \pi_2^* (g_{\alpha_2 \gamma_{j_{\alpha_2}}}),
\]

with \( g_{\alpha_2 \gamma_{j_{\alpha_2}}} \) the transition functions of the bundle. In summary, we have shown that \( \int_a^t \omega_1(\tau) \, d\tau \) is multiplicatively structured and hence Banagl’s proof is applicable.

6.2. Two Distinguished Triangles for \( \tilde{\Omega}^\bullet_{\partial'} \). To prove Theorem 6.5.4, we use a five lemma argument and therefore need two distinguished triangles in \( D(\mathbb{R}) \), the derived category over the reals.
Definition 6.2.1 (Forms that are multiplicative near $E$).

$$\Omega^r_{EMS}(M) := \{ \omega \in \Omega^r(M) \mid \exists \eta \in \Omega^r_{MS}(B) : e_E^* \omega = \pi_E^* \eta \}.$$ 

Lemma 6.2.2. In $\mathcal{D}(\mathbb{R})$, the derived category of complexes of real vector spaces, there is a distinguished triangle

\begin{equation}
\widetilde{\Omega}^\bullet(M) \to \Omega^\bullet_{EMS}(M) \to ft_{<K} \Omega^\bullet_{MS}(B) \to \widetilde{\Omega}^\bullet(M)[+1]
\end{equation}

Proof. There is a short exact sequence

\begin{equation}
0 \to \widetilde{\Omega}^\bullet(M) \to \Omega^\bullet_{EMS}(M) \to Q^\bullet(M) := \frac{\Omega^\bullet_{EMS}(M)}{\Omega_{I}^\bullet(M)} \to 0
\end{equation}

We have to show that there is a quasi-isomorphism $Q^\bullet(M) \to ft_{<K} \Omega^\bullet_{MS}(B)$. Let $\sigma_E : E \hookrightarrow E \times [0, 1)$ be the inclusion at 0. Then the map $J_E := c_E \circ \sigma_E$ induces maps

$$J^*_E : \Omega^\bullet_{EMS}(M) \xrightarrow{c_E^*} \Omega^\bullet_{EMS}(E \times [0, 1)) \xrightarrow{\sigma_E^*} \Omega^\bullet_{MS}(B)$$

$$\widetilde{J}^*_E : \widetilde{\Omega}^\bullet(M) \xrightarrow{c_E^*} \widetilde{\Omega}^\bullet(E \times [0, 1)) \xrightarrow{\sigma_E^*} P_{\geq K}(B)$$

$$\overline{J}^*_E : Q^\bullet(M) \xrightarrow{c_E^*} Q^\bullet(E \times [0, 1)) \xrightarrow{\sigma_E^*} Q^\bullet(B) := \frac{\Omega^\bullet_{MS}(B)}{P_{\geq K}(B)}.$$

The induced maps $J^*_E$ and $\widetilde{J}^*_E$ are surjective by the standard argument of enlarging the collar and using a bump function. By standard homological algebra, the map $\overline{J}^*_E : Q^\bullet(M) \to Q^\bullet(B)$ is an isomorphism. By Lemma 6.1.2, subcomplex inclusion induces a quasi-isomorphism

$$\overline{\gamma} : Q^\bullet(B) \xrightarrow{qis} \frac{\Omega^\bullet_{MS}(B)}{ft_{<K} \Omega^\bullet_{MS}(B)}$$

and since we work with a flat bundle $E$ over $B$, there is a quasi-isomorphism

\begin{equation}
\gamma_B : ft_{<K} \Omega^\bullet_{MS}(B) \to \frac{\Omega^\bullet_{MS}(B)}{ft_{<K} \Omega^\bullet_{MS}(B)}
\end{equation}

by [Ban16, Lemma 6.7]. All in all we get a fraction of quasi-isomorphisms

\begin{equation}
\xymatrix{ Q_E(M) \ar[r]^{qis} & \frac{\Omega^\bullet_{MS}(B)}{ft_{<K} \Omega^\bullet_{MS}(B)} \ar[l]_{\overline{\gamma}_B} \\
\frac{\Omega^\bullet_{MS}(B)}{ft_{<K} \Omega^\bullet_{MS}(B)} \ar[u]_{\overline{J}_E} \ar[r]^{qis} & \frac{\Omega^\bullet_{MS}(B)}{ft_{<K} \Omega^\bullet_{MS}(B)} \ar[u]_{\gamma_B}}
\end{equation}

in the derived category $\mathcal{D}(\mathbb{R})$ which allows us to replace $Q^\bullet(M)$ in (7) to get the desired distinguished triangle in $\mathcal{D}(\mathbb{R})$. \qed
Definition 6.2.3 (Relative de Rham complexes).
\[
\Omega^\bullet_{rel}(M) := \{ \omega \in \Omega^\bullet(M) | c^*_E \omega = 0, c^*_W \omega = 0 \}
\]
\[
\tilde{\Omega}^\bullet_p(M,C_W) := \{ \omega \in \tilde{\Omega}^\bullet_p(M) | c^*_W \omega = 0 \}
\]
\[
f_{t \geq K} \Omega^\bullet_{MS}(B,C_{\partial B}) := \{ \omega \in f_{t \geq K} \Omega^\bullet_{MS}(B) | c^*_E \omega = 0 \}.
\]

Remark 6.2.4. Note that since we work with \(p\)-related collars \(c^*_E, c^*_W\) on \(E\) and \(B\), we can rewrite \(f_{t \geq K} \Omega^\bullet_{MS}(B,C_{\partial B})\): For each coordinate chart \(U \subset B\) with respect to which the bundle trivializes we have
\[
\phi_U(C_{\partial E} \cap U) = (C_{\partial B} \cap U) \times L.
\]
Hence we have for \(\omega \in f_{t \geq K} \Omega^\bullet_{MS}(B,C_{\partial B})\) and each coordinate chart \(U \subset B\):
\[
0 = (c^*_E \omega)|_{p^{-1}(U)} = \phi_U^* \sum \pi^*_i c^*_\gamma_{\partial B} \eta_j \wedge \pi^*_2 \gamma_j,
\]
implying \(c^*_\gamma_{\partial B} \eta_j = 0\) for all \(j\). To see this, we use that \(U\) is a coordinate chart. Therefore, we can write
\[
\sum_j \pi^*_i \eta_j \wedge \pi^*_2 \gamma_j = \sum_{I} \sum_{j=1}^{k_I} f^I_j dx^I \wedge \gamma^I_j,
\]
where we sum over all multi-indices \(I\). We then can treat each multi-index \(I\) separately. Assume that there is an \(j_0 \in \{1, \ldots, k_I\}\) and an \(x \in C_{\partial B}\) such that \(f^I_{j_0}(x) \neq 0\). Contracting with \(\partial^I_{x_j}\) and evaluating at \(x\), this gives:
\[
\gamma^I_{j_0} = -\sum_{j \neq j_0} \frac{f^I_{j_0}(x)}{f^I_{j_0}(x)} \gamma^I_j.
\]
Therefore we can write
\[
\sum_{j=1}^{k_I} f^I_j dx^I \wedge \gamma^I_j = \sum_{j \neq j_0} (f^I_j - \frac{f^I_{j_0}(x)}{f^I_{j_0}(x)} f^I_{j_0}) dx^I \wedge \gamma^I_j.
\]
If these new coefficient functions \(f^I_j - \frac{f^I_{j_0}(x)}{f^I_{j_0}(x)} f^I_{j_0}\) vanish on \(C_{\partial B}\) we are done. Otherwise, repeat this process inductively to reduce the above sum to just one summand \(f^I dx^I \wedge \gamma^I\), for some \(\gamma^I\), which still must equal the sum we started with and is thereby zero on \(C_{\partial B} \times L\). Then either \(\gamma^I = 0\) or \(f^I|_{C_{\partial B}} = 0\).

The result of this discussion enables us to write
\[
f_{t \geq K} \Omega^\bullet_{MS}(B,C_{\partial B}) = \{ \omega \in f_{t \geq K} \Omega^\bullet_{MS}(B,C_{\partial B})|_{p^{-1}(U)} = \sum_j \pi^*_i \eta_j \wedge \pi^*_2 \gamma_j
\]
with \(\eta_j \in \Omega^\bullet(U,U \cap C_{\partial B})\), \(\gamma_j \in \tau_{t \geq K} \Omega^\bullet(L)\).

Remark 6.2.5. The cohomology groups of the above defined complexes do not depend on the choice of a pair of \(p\)-related collars. This can be deduced by a spectral sequence argument, see [Ess16, Section 11].
Lemma 6.2.6. There is a second distinguished triangle

\[(9) \Omega^•_{rel}(M) \to \tilde{\Omega}^•_p(M, C_W) \to ft_{\geq K}\Omega^•_{MS}(B, C_{\partial B}) \to \Omega^•_{rel}(M)[+1]\]

Proof. The map \(\tilde{J}_E^* : \tilde{\Omega}^•_p(M, C_W) \to ft_{\geq K}\Omega^•_{MS}(B, C_{\partial B})\) is surjective by the same arguments we gave in previous proofs. The kernel of \(\tilde{J}_E^*\) are those forms \(\omega \in \tilde{\Omega}^•_p(M, C_W)\) with \(c^*E\omega = 0\) and hence \(\ker \tilde{J}_E^* = \Omega^•_{rel}(M)\) and we therefore have a commutative diagram

\[
0 \to \Omega^•_{rel}(M) \to \tilde{\Omega}^•_p(M, C_W) \xrightarrow{\tilde{J}_E^*} ft_{\geq K}\Omega^•_{MS}(B, C_{\partial B}) \to 0
\]

and in particular the Distinguished Triangle (9).

6.3. Poincaré Duality for Fiberwisely (Co)truncated Forms.

Proposition 6.3.1. For any \(r \in \mathbb{Z}\), integration induces a nondegenerate bilinear form

\[
\int : H^{r-1}(ft_{< K}\Omega^•_{MS}(B)) \times H^{n-r}(ft_{\geq K}\Omega^•_{MS}(B, C_{\partial B})) \to \mathbb{R},
\]

\[
([\omega], [\eta]) \mapsto \int_E \omega \wedge \eta.
\]

For being able to prove the above Proposition 6.3.1, we need two Poincaré Lemmata and a Bootstrap Principle:

Lemma 6.3.2. (Poincaré Lemma for fiberwisely truncated forms)

Let \(U \subset B\) be a chart intersecting, that means the bundle \(p : E \to B\) trivializes over \(U\). In detail, there is a diffeomorphism \(\phi_U : p^{-1}(U) \xrightarrow{\cong} U \times L\) with \(p = \pi_1 \circ \phi_U\). Let further denote \(\pi_2 : U \times L \to L\) the second factor projection and \(S_x : L \xrightarrow{at x} U \times L\) the inclusion at \(x \in U - (\partial B \cap U)\). Then the induced maps

\[
ft_{< K}\Omega^•_{MS}(U) \xrightarrow{S_x^*} \tau_{< K}\Omega^•(L)
\]

are chain homotopy inverses of each other. In particular both are homotopy equivalences.

Proof. The proof is an analogy to the proof of [Ban16, Lemma 5.1]. The only difference is that for charts intersecting the boundary, \(\mathbb{R}^0 \leftarrow \mathbb{R}_+\) is embedded at \(1 \in [0, \infty)\), which does not change the argument. \(\square\)
Definition 6.3.3. For any open subset $U \subset B$ we define
\[
\Omega_{\text{MS}}^\bullet(U,U \cap C_{\partial B}) := \{ \omega \in \Omega_{\text{MS}}^\bullet(U) \mid \omega|_{p^{-1}(U \cap C_{\partial B})} = 0 \},
\]
\[
\Omega_{\text{MS},c}^\bullet(U,U \cap C_{\partial B}) := \{ \omega \in \Omega_{\text{MS},c}^\bullet(U) \mid \omega|_{p^{-1}(U \cap C_{\partial B})} = 0 \}
\]
Analogously, we define the fiberwisely truncated and cotruncated subcomplexes.

In the following lemma we give the induction start for the Mayer–Vietoris argument, which makes use of the fact that the collar we work with is small with respect to the chosen good open cover $U$ (compare to 2.1.2).

Lemma 6.3.4. (Poincaré Lemma for relative forms with compact supports)

Let $U \in \mathcal{U}$ be an open chart (with respect to which the bundle trivializes, i.e. there is a diffeomorphism $\phi_U : p^{-1}(U) \to U \times L$ with $p|_{p^{-1}(U)} = \pi_1 \circ \phi_U$).

Then in particular there is a diffeomorphism $\psi : U \cong V$ with $V = \mathbb{R}^n_+$ or $V = \mathbb{R}^n$ and, by Lemma 2.1.2, $U$ is not completely contained in the collar neighbourhood $C_{\partial B} \supset \partial B$ of the boundary of $B$. Then there is a form $e \in \Omega^*_c(U,U \cap C_{\partial B}) = \{ \omega \in \Omega^*_c(U) \mid \omega|_{U \cap C_{\partial B}} = 0 \}$ such that the maps
\[
f_{t \geq K} \Omega_{\text{MS},c}^\bullet(U,U \cap C_{\partial B}) \xrightarrow{(\pi_2)_* \circ (\phi_U^{-1})^*} \tau_{t \geq K} \Omega^{\star,-n}(L),
\]
where
\[
\pi_2^*(\pi_1^* \eta \wedge \pi_2^* \gamma) = \begin{cases} (\int_U \eta) \gamma & \text{if } \eta \in \Omega^*_c(U,U \cap C_{\partial B}), \\ 0 & \text{else}, \end{cases}
\]
and
\[
e_*(\gamma) = \phi_U^* (e \wedge \pi_2^* \gamma),
\]
are chain homotopy inverses of each other and in particular are both chain homotopy equivalences.

Proof. First step: (Definition of the form $e$)

Independent of $U$ being diffeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$ we can assume that $\psi(U) = V \subset \mathbb{R}^n$ is arranged in such a way that for, say the $x^0$ component of elements $x \in V$ large enough, $x^0 > s$, one has $x \notin \psi(C_{\partial B} \cap U)$ (for $V = \mathbb{R}^n_+$, $x^0$ is also a component such that $\partial \mathbb{R}^n_+ = \{x^0 = 0\}$). We then take bump functions $\epsilon_i \in C^0_c(\mathbb{R})$ with $\int_{\mathbb{R}} \epsilon_i = 1$ for $i \in \{0,...,n-1\}$, such that in addition $\text{supp}(\epsilon_0) \subset (s,\infty)$. But then
\[
e := \psi^* \left( \prod_{i=0}^{n-1} \epsilon_i \right) dx^0 \wedge ... \wedge dx^{n-1} \in \Omega^*_c(U,U \cap C_{\partial B}).
\]

The map $e_* : \tau_{t \geq K} \Omega^\star(L) \to f_{t \geq K} \Omega^\star_{\text{MS},c}(U,U \cap C_{\partial B})$ is defined by relation (10) and by the definition of the form $e$ it holds that $(\pi_2)_* \circ \phi_U^* \circ e_* = \text{id}$. 
Second step: (Construction of the homotopy operator) As in the proof of [Ban16, Lemma 5.5] and in the proof of the previous Lemma 6.3.2, we prove by induction on $n$ that $e_* \circ (\pi_2)_* \circ \phi_U^* \simeq \text{id}$. In detail, we proceed as follows: First we show that the maps

$$e_{0*} : ft_{\geq K} \Omega_{MS,c}^n(\mathbb{R}^{n-1}) \rightarrow ft_{\geq K} \Omega_{MS,c}^n(U, U \cap C_{\partial B})$$

$$e_{0*}(\pi^1_1 \eta \wedge \pi^*_2 \gamma) := \phi_U^*(\pi^1_1 \psi^*(e_0 \wedge \pi^* \eta) \wedge \pi^*_2 \gamma)$$

with $\pi : V \rightarrow \mathbb{R}^{n-1}$ the projection, and

$$\pi_* : ft_{\geq K} \Omega_{MS}^n(U, U \cap C_{\partial B}) \rightarrow ft_{\geq K} \Omega_{MS,c}^n(\mathbb{R}^{n-1})$$

(integration along the first fiber coordinate) defined by

$$\pi_*(\phi_U^*(\pi^1_1 \psi^*(f(x,t)du^j) \wedge \pi^*_2 \gamma)) = 0$$

$$\pi_*(\phi_U^*(\pi^1_1 \psi^*(g(x,t)dt \wedge du^j) \wedge \pi^*_2 \gamma)) = \pi^1_1 \int_{\mathbb{R}} g(x,t)dt \ du^j \wedge \pi^*_2 \gamma$$

satisfy the relation $e_{0*} \circ \pi^* \simeq \text{id}$ and hence are mutually inverse homotopy equivalences. The homotopy operator

$$K : ft_{\geq K} \Omega_{MS,c}^n(U, U \cap C_{\partial B}) \rightarrow ft_{\geq K} \Omega_{MS,c}^n(U, U \cap C_{\partial B})$$

satisfying $dK + Kd = e_{0*} \circ \pi_*$ is defined by

$$K(\phi_U^*(\pi^1_1 \psi^*(f(x,t)du^j) \wedge \pi^*_2 \gamma)) = 0$$

$$K(\phi_U^*(\pi^1_1 \psi^*(g(t,x)dt \wedge du^j) \wedge \pi^*_2 \gamma))$$

$$= \phi_U^*(\pi^1_1 \psi^*(\int_{-\infty}^{t} g(\tau,x)d\tau - \int_{-\infty}^{t} e_0) \int_{\mathbb{R}} g(\tau,x) \ d\tau \ du^j \wedge \pi^*_2 \gamma),$$

as usual. Note that by our definition of $e_0$, $K$ respects the vanishing condition. A standard calculation shows that $Kd + dK = e_{0*} \circ \pi_* - \text{id}$.

The second step is to put together the first step with the result of [Ban16, Lemma 5.5]: The following diagram commutes

$$\begin{array}{ccc}
ft_{\geq K} \Omega_{MS,c}^n(U, U \cap C_{\partial B}) & \xrightarrow{\pi_*} & ft_{\geq K} \Omega_{MS,c}^n(\mathbb{R}^{n-1}) \\
\xrightarrow{e_{0*}} & \xrightarrow{\pi_*} & \tau_{\geq K} \Omega_{MS,c}^{n}(L) \\
& \xrightarrow{\tau_{\geq K} \Omega_{MS,c}^{n-1}} &
\end{array}$$

Note that $\bar{e}_*$ and $\bar{\pi}_*$ denote the mutually inverse homotopy equivalences of [Ban16, Lemma 5.5]. The commutativity of this diagram then implies the statement of the lemma: Since $e_* = \bar{e}_* \circ e_{0*}$ and $\pi_2 = \pi_* \circ \bar{\pi}_*$ are the composition of mutually inverse homotopy equivalences, they are also mutually inverse homotopy equivalences.
To use a Mayer–Vietoris type argument we need a bootstrap principle. The following lemma will provide one in our case:

**Lemma 6.3.5. (Bootstrap principle)**

Let \( U, V \subset B \) be open sets and let \( b := \dim B, m = \dim L \). Then if

\[
\int : H^r(\mathfrak{f}_\leq K \mathcal{OM}_S(Y)) \times H^{b+m-r}(\mathfrak{f}_\geq K \mathcal{OM}_S, c(Y, Y \cap C_{dB})) \to \mathbb{R}
\]

\[
([\omega], [\eta]) \mapsto \int_{p^{-1}(Y)} \omega \wedge \eta
\]

is nondegenerate for \( Y = U, V, U \cap V \), so it is for \( Y = U \cup V \).

**Proof.** The same arguments as in the proof of [Ban16, Lemma 5.10] apply, if the following claim is true.

**Claim:** For \( \omega \in \mathfrak{f}_\geq K \mathcal{OM}_S, c(U, U \cap C_{\partial B}) \) and \( f \in C^\infty(U) \) it holds that

\[
p^*(f) \omega \in \mathfrak{f}_\geq K \mathcal{OM}_S, c(U, U \cap C_{\partial B}).
\]

**Proof of the Claim:** Since, by definition of a fiber bundle, for a coordinate chart \( U_\alpha \) it holds that \( \pi_1 \circ \phi_\alpha = p\vert_{U_\alpha} \), \( p^*(f) \omega \in \mathfrak{f}_\geq K \mathcal{OM}_S, c(U) \). Further we have \((p^*(f) \omega) \vert_{C_{\partial E}} = 0\) since \( \omega \vert_{C_{\partial E}} = 0 \). Hence the claim is established and the argumentation of [Ban16, Lemma 5.10] is applicable. \( \square \)

**Remark 6.3.6.** Note, that the compactness of \( B \) implies that

\[
\mathfrak{f}_\geq K \mathcal{OM}_S, c(B, C_{\partial B}) = \mathfrak{f}_\geq K \mathcal{OM}_S(B, C_{\partial B}).
\]

The proof is literally the same as the proof of [Ban16, Lemma 5.11].

Together with the bootstrap principle of the above Lemma 6.3.5, we need an induction basis for being able to use the inductive Mayer–Vietoris argument.

**Lemma 6.3.7. (Local Poincaré Duality)**

For \( U \in \mathcal{U} \) a coordinate chart, the bilinear form

\[
\int : H^r(\mathfrak{f}_\leq K \mathcal{OM}_S(U)) \times H^{b+m-r}(\mathfrak{f}_\geq K \mathcal{OM}_S, c(U, U \cap C_{\partial B})) \to \mathbb{R},
\]

where again \( b = \dim B, m = \dim L \), is nondegenerate.

**Proof.** The map

\[
\int : H^r(\tau_\leq K \mathcal{OM}_S(L)) \to H^{m-r}(\tau_\geq K \mathcal{OM}_S(L))^\dagger
\]

\[
[\omega] \mapsto \int_L - \wedge \omega
\]

is an isomorphism. Since the isomorphisms of the two Lemmata 6.3.2 and 6.3.4 commute with integration, the statement of the lemma is established. \( \square \)
Now we have all the tools to establish Proposition 6.3.1

**Proof of Proposition 6.3.1:** By Remark 6.3.6, the statement of the proposition is equivalent to the statement that integration induces a map

\[ \int: H^r(ft < K \Omega^•_{MS}(B)) \times H^{b+m-r}(ft \geq K^\ast \Omega^•_{MS,c}(B, C_\partial B)) \rightarrow \mathbb{R} \]

that is nondegenerate for all \( r \).

In fact, we prove that the bilinear map

\[ \int: H^r(ft < K \Omega^•_{MS}(U)) \times H^{b+m-r}(ft \geq K^\ast \Omega^•_{MS,c}(U, U \cap C_\partial B)) \rightarrow \mathbb{R} \]

is nondegenerate for all \( r \) and all open subsets \( U \subset B \) of the form \( U = \bigcup_{i=1}^s U_{\alpha^i_0, \ldots, \alpha^i_{\beta^i}} \) with \( s \leq |I| \) by an induction on \( s \).

For \( s = 1 \) the statement was already proven in Lemma 6.3.7. The induction step follows from the bootstrap principle of Lemma 6.3.5, compare to [Ban16, Prop. 5.12]. This finishes the proof, since \( B \) is the finite union \( B = \bigcup_{\alpha \in I} U_\alpha \).

\[ \square \]

### 6.4. Integration on \( \widetilde{\Omega}_p^\bullet(M) \).

**Lemma 6.4.1.** For any \( r \in \mathbb{Z} \), integration defines a bilinear form

\[ \int: \Omega^r_{E,MS}(M) \times \Omega^{n-r}_{E,MS}(M) \rightarrow \mathbb{R}. \]

**Proof.** Bilinearity is obvious and the finiteness of the integral is ensured by the compactness of \( M \). \[ \square \]

**Corollary 6.4.2.** For any \( r \in \mathbb{Z} \), integration defines bilinear forms

\[ \int: \widetilde{\Omega}_p^r(M) \times \widetilde{\Omega}_p^{n-r}(M, C_W) \rightarrow \mathbb{R}. \]

To be able to prove Poincaré duality for \( \widetilde{\Omega}_p^\bullet(M) \) we need two technical lemmas:

**Lemma 6.4.3.** For \( \nu_0 \in ft_K \Omega^{r-1}_{MS}(B) \) and \( \eta_0 \in ft_K \Omega^{n-r}_{MS}(B, C_\partial B) \) we have

\[ \int_E \nu_0 \wedge \eta_0 = 0. \]

**Proof.** The proof is literally the same as the proof of [Ban16, Lemma 7.3]. \[ \square \]

**Lemma 6.4.4.** For \( \nu \in \widetilde{\Omega}_p^{r-1}(M) \), \( \eta \in \widetilde{\Omega}_q^{n-r}(M, C_W) \) we have

\[ \int_M d(\nu \wedge \eta) = 0. \]

**Proof.** The boundary of \( M \) is

\[ \partial M = E \cup \partial E \cup W. \]
To prove the lemma we compute
\[
\int_M d(\nu \wedge \eta) = \int_{\partial M} (\nu \wedge \mu) |_{\partial M} \quad \text{by Stokes' Theorem}
\]
\[
= \int_E \nu_0 \wedge \eta_0 + \int_W c_W^*(\nu \wedge \eta)
\]
for some \(\nu_0 \in ft_{\geq K}\Omega_{\tilde{\mathcal{M}}_S}^{r-1}(B), \eta_0 \in ft_{\geq K}\Omega_{\tilde{\mathcal{M}}_S}^{n-r}(B, C\partial B)\)
\[
= 0 + \int_W c_W^*(\nu) \wedge c_W^*(\eta) \quad \text{by Lemma 6.4.4}
\]
\[
= 0 \quad \text{since} \quad \eta \in \widetilde{\Omega}_{\tilde{q}}^{n-r}(M, C_W).
\]
\[
\square
\]

6.5. Poincaré Duality for \(\widetilde{\Omega}_{\tilde{q}}^*(M)\).

**Proposition 6.5.1.** For any \(r \in \mathbb{Z}\), integration on \(\widetilde{\Omega}_{\tilde{q}}^r(M)\) induces a bilinear form
\[
\int : \widetilde{H}_{\tilde{q}}^r(M) \times \widetilde{H}_{\tilde{q}}^{n-r}(M, C_W) \to \mathbb{R}
\]
\[
([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta.
\]

**Proof.** Let \(\omega \in \widetilde{\Omega}_{\tilde{q}}^r(M)\) closed, \(\tilde{\omega} \in \widetilde{\Omega}_{\tilde{q}}^{r-1}(M)\), \(\eta \in \widetilde{\Omega}_{\tilde{q}}(M, C_W)\) closed and \(\tilde{\eta} \in \widetilde{\Omega}_{\tilde{q}}^{n-r-1}(M, C_W)\).
\[
\int_M (\omega + d\tilde{\omega}) \wedge \eta = \int_M \omega \wedge \eta + \int_M d(\tilde{\omega} \wedge \eta) = \int_M \omega \wedge \eta,
\]
where the last step holds by the previous Lemma 6.4.4. By an analogous argument
\[
\int_M \omega \wedge (\eta + d\tilde{\eta}) = \int_M \omega \wedge \eta.
\]
\[
\square
\]

**Lemma 6.5.2.** The subcomplex inclusion
\[
\Omega_{\tilde{q}}^*(M) \hookrightarrow \Omega^*(M)
\]
is a quasi-isomorphism.

**Proof.** As in Lemma 6.1.2, we can apply the arguments of [Ban16, Prop. 2.4], integrating forms in the collar direction. \(\square\)

**Proposition 6.5.3.** Let \(N := M - \partial M\) and
\[
\Omega_{\text{rel}}^*(N) = \Omega^*(N, C) = \{\omega \in \Omega^*(N) | \omega|_{C\cap N} = 0\}.
\]
Then the subcomplex inclusion
\[
\Omega_{\text{rel}}^*(N) \hookrightarrow \Omega^*_c(N)
\]
is a quasi-isomorphism.

Proof. We factor the subcomplex inclusion $\Omega^\bullet_{rel}(N) \hookrightarrow \Omega^\bullet_c(N)$ as

$$\Omega^\bullet_{rel}(N) \hookrightarrow \Omega^\bullet_c(N, C_E) \hookrightarrow \Omega^\bullet_c(N),$$

with $\Omega^\bullet_c(N, C_E) = \Omega^\bullet_c(N) \cap \Omega^\bullet(N, C_E)$. We use a standard argument to prove that both subcomplex inclusions are quasi-isomorphisms. Let $\epsilon > 0$ be a small number and let $C_X := c_X((0, 1 + \epsilon) \times X) \subset N$ denote slightly larger collar neighbourhoods of $X = E, W$ in $N$. Let $\xi_X$ be a smooth cutoff function in collar direction on $N$ with $\xi|_{N - C_X} = 0$ and $c^*_W \xi = 1$, where, again, $X = E, W$.

We first prove that $H^\bullet_{rel}(N) \hookrightarrow H^\bullet_c(N, C_E)$ is an isomorphism. Injectivity: Let $\omega \in \Omega^\bullet_{rel}(N)$ be a closed form such that $\omega = d\eta$ for some $\eta \in \Omega^\bullet_c(N, C_E)$. To prove injectivity, we have to show that the cohomology class $[\omega] \in H^\bullet_{rel}(N)$ is also zero. We first decompose $\eta|_{C_W}$ into its tangential and normal component: $\eta|_{C_W} = \eta_T(t) + dt \wedge \eta_N(t)$. We then define a new form

$$\eta := \eta - d(\xi_W \int_0^t \eta_N(\tau)d\tau) \in \Omega^\bullet_c(N).$$

This new form also satisfies $d\eta = d\eta = \omega$ and it holds that $c^*_W \eta = c^*_W \eta - dt \wedge \eta_N(t) = \eta_T(t)$. Since $\omega = d\eta$ and $c^*_W \omega = 0$, this gives

$$0 = c^*_W (d\eta) = d_W \eta_T(t) + dt \wedge \eta_T(t).$$

Hence, $\eta_T(t) = 0$, i.e. $\eta_T$ is independent of the collar coordinate (for $t < 1$). Since $\eta$ has compact support in $N$, there is a $\delta > 0$, such that $\eta_T(\delta) \equiv 0$. Hence $c^*_W \eta = \eta_T \equiv 0$ and $\eta \in \Omega^\bullet_{rel}(N)$.

Surjectivity: Let $\tilde{\omega} \in \Omega^\bullet_c(N, C_E)$ be a closed form. We want to show that there is a closed form $\omega \in \Omega^\bullet_{rel}(N)$ and a form $\tilde{\eta} \in \Omega^\bullet_c(N, C_E)$ such that $\tilde{\omega} = \omega + d\tilde{\eta}$. As in the previous step we decompose $\tilde{\omega}|_{C_W} = \tilde{\omega}_T(t) + dt \wedge \tilde{\omega}_N(t)$ and define the closed form $\omega \in \Omega^\bullet_c(N)$ by

$$\omega := \tilde{\omega} - d(\xi_W \int_0^t \tilde{\omega}_N(\tau)d\tau).$$

Hence, as before, the normal part of $c^*_W \omega$ is zero and since $\tilde{\omega}$ is closed, $\tilde{\omega}'_T(t) = 0$ for $t < 1$. The fact that $\tilde{\omega}$ has compact support implies that there is a $\delta > 0$ with $c^*_W \tilde{\omega}(x, t) = 0$ for arbitrary $x \in W$ and all $0 < t < \delta$.

We deduce that $\omega \in \Omega^\bullet_{rel}(N)$, since $c^*_W \omega = \tilde{\omega}_T(t) = \tilde{\omega}_T(\delta/2) = 0$. On the other hand, $\tilde{\omega}_N(t) = 0$ for every $t \in (0, \delta)$ and therefore the support of the form $\tilde{\eta} := \xi_W \int_0^t \tilde{\omega}_N(\tau)d\tau$ is contained in $c_W (W \times [\delta, 1 + \epsilon])$ and therefore compact.

The proof of the quasi-isomorphy of $\Omega^\bullet_c(N, C_E) \hookrightarrow \Omega^\bullet_c(N)$ uses the same argument. The only difference is that we work on the collar neighbourhood of $E$ instead of $W$, use $\xi_E$ instead of $\xi_W$ and so on.

Finally we are able to prove Poincaré Duality for $\tilde{\Omega}^\bullet_p(M)$:
Theorem 6.5.4. (Poincaré duality for $\tilde{H}_p(M)$)

For any $r \in \mathbb{Z}$, the bilinear form

$$\int : \tilde{H}_p^r(M) \times \tilde{H}_q^{n-r}(M, C_W) \to \mathbb{R}$$

of Proposition 6.5.1 is nondegenerate.

Proof. First Step For $\Omega^\bullet(M, C) := \{\omega \in \Omega^\bullet(M) \mid \omega|_C = 0\}$ and $H^r(M, \partial M) := H^r(\Omega^\bullet(M, C))$ (by an analogue to the de Rham Theorem, this is isomorphic to the relative singular cohomology complex) integration induces an isomorphism

$$\int : H^r(M) \to H^{n-r}(M, \partial M)^\dagger$$

for all $r \in \mathbb{Z}$.

By the previous Lemma 6.5.2, the subcomplex inclusion $\Omega^\bullet_{EMS}(M) \subset \Omega^\bullet(M)$ induces an isomorphism $H^r_{EMS}(M) := H^r(\Omega^\bullet_{EMS}(M)) \xrightarrow{\cong} H^r(M)$ for any $r \in \mathbb{Z}$. The inclusion $i : N \hookrightarrow M$ is a homotopy equivalence and hence induces an isomorphism $i^* : H^r(M) \cong H^r(N)$ for all $r \in \mathbb{Z}$, as well as the isomorphism $i^* : H^r(M, \partial M) \cong H^r_{rel}(N)$. Since integration gives an isomorphism

$$\int : H^r(N) \cong H^{n-r}_{c}(N)^\dagger$$

for all $r \in \mathbb{Z}$ and the diagram

$$\begin{array}{ccc}
H^r(M) & \xrightarrow{i^*} & H^r(N) \\
\downarrow f & & \downarrow f \\
H^{n-r}(M, \partial M)^\dagger & \cong & H^{n-r}_{rel}(N)^\dagger \\
& \xleftarrow{i^*} & \\
& \cong & \xrightarrow{\cong} \\
& \xrightarrow{incl} & H^{n-r}_{c}(N)^\dagger
\end{array}$$

commutes for any $r$, the first statement is established.

Second Step By Proposition 6.3.1, integration gives an isomorphism

$$\int : H^r(ft_{<K}\Omega^\bullet_{MS}(B)) \cong H^{n-r-1}(ft_{<K}\Omega^\bullet_{MS}(B, C_{\partial B}))^\dagger.$$

Third Step The distinguished triangles of the two Lemmata 6.2.2 and 6.2.6 give the long exact sequences on cohomology, which fit into the following
Together with the 5-Lemma, proving that this diagram commutes (up to sign) establishes the desired result. We first prove that the top square (TS) in the diagram commutes and therefore describe the connecting homomorphism \( \delta : H^{r-1}(\Omega_{\overline{\Omega}}^* \omega_{\Omega_{\overline{\Omega}}}(B)) \to \Omega_{\overline{\Omega}}^* \omega_{\Omega_{\overline{\Omega}}}(M) \). Let \( \omega \in f_{<K} \Omega_{\overline{\Omega}}^* \omega_{\Omega_{\overline{\Omega}}}(B) \) closed, i.e. \( d\omega = 0 \). Then \( d\gamma_B \omega = 0 \) holds as well, where \( \gamma_B : f_{<K} \Omega_{\overline{\Omega}}^* \omega_{\Omega_{\overline{\Omega}}}(B) \to Q^* \omega \) is the quasi-isomorphism defined in equation (8). Since the map \( J_E^* : Q^* \omega_{\Omega_{\overline{\Omega}}}(M) \to Q^* \omega \) defined in Lemma 6.2.2 is an isomorphism, there is a \( \overline{\omega} \in Q^* \omega_{\Omega_{\overline{\Omega}}}(M) \) : \( J_E^* \overline{\omega} = \gamma_B \omega \). Since \( J_E^* \) is an isomorphism and \( 0 = d\gamma_B \omega = dJ_E^* \overline{\omega} = J_E^* d\overline{\omega} \), we have \( d\overline{\omega} = 0 \) in \( Q^* \omega \). Let \( \xi \in \Omega^* \omega_{\Omega_{\overline{\Omega}}}(M) \) be a representative of \( \overline{\omega} \), i.e. \( q(\xi) = \overline{\omega} \). Then \( d\xi \in \Omega^* \omega_{\Omega_{\overline{\Omega}}}(M) \) since \( dq(\xi) = d\overline{\omega} = 0 \). Hence \( (d\xi, \xi) \in C^*(i) \), the mapping cone of the subcomplex inclusion \( i : \Omega^* \omega_{\Omega_{\overline{\Omega}}}(M) \to \Omega^* \omega_{\Omega_{\overline{\Omega}}}(M) \) with \( d(-d\xi, \xi) = 0 \). Therefore by the definition of distinguished triangles and the induced long exact cohomology sequences, \( \delta[\omega] = [-d\xi] \). The relation \( q(\text{incl } \omega) = \gamma_B \omega = J_E^* q(\xi) = q(J_E^* \xi) = q(\eta_E^* \xi) \in Q^*(B) \) implies that \( \alpha := \eta_E^* \xi - \omega \in f_{>K} \Omega_{\overline{\Omega}}^* \omega_{\Omega_{\overline{\Omega}}}(B) \). For closed forms \( \omega \in f_{<K} \Omega_{\overline{\Omega}}^* \omega_{\Omega_{\overline{\Omega}}}(B), \eta \in \Omega_{\overline{\Omega}}^* \omega_{\Omega_{\overline{\Omega}}}(M, C_W) \) we hence get:

\[
\int_M \delta(\omega) \wedge \eta = -\int_M d\xi \wedge \eta = -\int_M d(\xi \wedge \eta)
\]

\[
= -\int_{M-C_E} d(\xi \wedge \eta) - \int_{C_E} d(\xi \wedge \eta)
\]

\[
= -\int_{M-C_E} d(\xi \wedge \eta),
\]
since

\[ d(\xi \wedge \eta)|_{C_E} = \psi_E^* \pi_E^* d(\xi_0 \wedge \eta_0) \]

for some \( \xi_0 \in \Omega^r_{\text{MS}}(B) \), \( \eta_0 \in \text{ft}_{\geq K} \cdot \Omega^{n-r}_{\text{MS}}(B, C_{\partial B}) \). But

\[
\int_{C_E} d(\xi \wedge \eta) = \int_E d(\xi_0 \wedge \eta_0) = 0
\]
as an integral of an \( n \)-form over a \((n-1)\)-dimensional manifold. Let \( J_W : W - C_{\partial W} \hookrightarrow W \hookrightarrow M \). Then by Stokes’ Theorem for manifolds with corners

\[
- \int_{M - C_E} d(\xi \wedge \eta) = - \int_E \sigma_E^* \xi \wedge \tilde{J}_E^* \eta + \int_{W - C_{\partial W}} J_W^* (\xi \wedge \eta)
\]

\[
= - \int_E \omega \wedge \tilde{J}_E^* \eta - \int_E \alpha \wedge \tilde{J}_E^* \eta = - \int_E \omega \wedge \tilde{J}_E^* \eta,
\]

where

\[
\int_{W - C_{\partial W}} J_W^* (\xi \wedge \eta) = 0,
\]
since \( \eta|_{C_W} = 0 \), and

\[
\int_E \alpha \wedge \tilde{J}_E^* \eta = 0
\]
by Lemma 6.4.3. Thus, (TS) commutes up to sign.

The commutativity of the middle square (MS) is obviously fulfilled since both the vertical maps are induced by the subcomplex inclusions \( \Omega^*_B(M) \hookrightarrow \Omega^*_E (M) \) and \( \Omega^*_r(M) \hookrightarrow \tilde{\Omega}^*_q(M, C_W) \).

To prove the commutativity of the bottom square (BS), we first investigate the connecting homomorphism \( D : H^{n-r-1}(\text{ft}_{\geq K} \cdot \Omega^r_{\text{MS}}(B, C_{\partial B})) \rightarrow H^n_{\text{rel}}(M) \). We look at the distinguished triangle (9).

For \( \eta \in \text{ft}_{\geq K} \cdot \Omega^{n-r-1}_{\text{MS}}(B, C_{\partial B}) \) closed, the surjectivity of \( \tilde{J}_E^* \) implies that there is a form \( \overline{\eta} \in \tilde{\Omega}^q_{\text{rel}}(M, C_W) \) such that \( \tilde{J}_E^* \overline{\eta} = \eta \). Since \( \tilde{J}_E^* \) is a chain map, \( \overline{\eta} \in \ker \tilde{J}_E^* = \Omega^{n-r}_{\text{rel}}(M) \). Let \( \rho : \Omega^*_r(M) \hookrightarrow \tilde{\Omega}^*_q(M, C_W) \) denote the subcomplex inclusion and \( C^*(\rho) \) its algebraic mapping cone. Then the map \( f : C^*(\rho) \rightarrow \text{ft}_{\geq K} \cdot \Omega^r_{\text{MS}}(B, C_{\partial B}) \), \( (\tau, \sigma) \mapsto \tilde{J}_E^* (\sigma) \) is a quasi-isomorphism (by the standard argumentation). The cocycle \( c := (-d\overline{\eta}, \overline{\eta}) \in C^{n-r-1}(\rho) \) satisfies the equation \( f(c) = \tilde{J}_E^* \overline{\eta} = \eta \) and hence \( D[\eta] \) can be described as \( D[\eta] = [-d\overline{\eta}] \). We next describe the map

\[
Q : H^r(\Omega^*_E (M)) \rightarrow H^r(\text{ft}<K \Omega^r_{\text{MS}}(B)),
\]

induced by the corresponding map in the distinguished triangle (6).

Let \( \omega \in \Omega^*_E (M) \) be a closed form. Then \( J_E^* \omega \in \Omega^r_{\text{MS}}(B) \) represents the image of \( \omega \) under the composition

\[
J_E \circ q : \Omega^*_E (M) \rightarrow Q^*_E (M) \rightarrow Q^*(B).
\]
Since $\gamma_B : ft_{< K}\Omega^*_MS(B) \to Q^*(B)$ is a quasi-isomorphism, there are forms $\omega \in ft_{< K}\Omega^r_MS(B)$ such that $\gamma_B(\omega) = \mathcal{J}_E^*q(\omega) + dq_B(\xi)$. The above map $Q$ is then given by $Q[\omega] = [\omega]$. Note that the form $\alpha := \omega - \mathcal{J}_E^*\omega \in \Omega^*_I(M)$ is contained in $ft_{\geq K}\Omega^*_MS(B)$. We can now verify the commutativity of (BS) by proving

\begin{align}
\int_M \omega \wedge (-d\eta) = \pm \int_E \omega \wedge \eta,
\end{align}

with $[-d\eta] = D[\eta]$ and $[\omega] = Q[\omega]$.

\begin{align}
\int_M \omega \wedge (-d\eta) &= -\int_M \omega \wedge d\eta = \pm \int_{M-C_E} d(\omega \wedge \eta) - \int_{C_E} \omega \wedge d\eta \\
&= \pm \int_{M-C_E} d(\omega \wedge \eta) \quad \text{(since $d\eta \in \Omega^r_I(M)$)} \\
&= \pm \int_E \mathcal{J}_E^*\omega \wedge \mathcal{J}_E^*\eta \pm \int_{W-CW} \mathcal{J}_W^*(\omega \wedge \eta) \quad \text{(Stokes)} \\
&= \pm \int_E (\omega - \alpha - d\xi) \wedge \mathcal{J}_E^*\eta \quad \text{(above + $\eta|_{CW} = 0$)} \\
&= \pm \int_E \omega \wedge \eta,
\end{align}

since $\mathcal{J}_E^*\eta = \eta$, $\int_E \alpha \wedge \eta = 0$ by Lemma 6.4.3 and

\begin{align}
\int_E d\xi \wedge \eta = \int_E d(\xi \wedge \eta) = \int_{\partial E} \xi \wedge \eta = 0
\end{align}

by Stokes’ Theorem and since $c^*_W\eta = 0$ and hence $j^*_E\eta = 0$. Thus (BS) commutes and the theorem is proven. □

7. The de Rham Intersection Complex $\Omega^*_I(M)$

7.1. Truncation and Cotruncation of $\Omega^*_I(W)$. To be able to define the de Rham intersection complex $\Omega^*_I(M)$, we need to note some observations about the boundary part $W \subset \partial M$.

Remark 7.1.1. The $(n - 1)$-dimensional compact manifold with boundary $W$ is the top stratum of the singular stratified space $\partial X'$ mentioned in Section 3.4. The boundary $\partial W$ of $W$ is the total space of the flat link bundle $q : \partial W = \partial E \to \partial B$, with $B = \Sigma$ the bottom stratum of the stratified pseudomanifold-with-boundary $X'$. Hence, following [Ban16], we can construct the chain complex of intersection forms $\Omega^*_I(W)$ as a subcomplex of the complex of differential forms of the top stratum for the stratified pseudomanifold $\partial X'$ with two strata and regular part $W$.

To prove Poincaré Duality for the intersection space cohomology groups $HI^*_I(X)$, we must cotruncate $\Omega^*_I(W)$ in degree $L$. Hence we demand for $W$ that the subcomplex $\Omega^*_I(W) \subset \Omega^*_I(W)$ is geometrically cotruncatable in degree $L$, see Definition 4.3.1. This is an additional Witt-type condition,
that does not occur in the two-strata setting. It seems to be strongly related to the strong Witt condition that Banagl needs in [Ban12] to define intersection spaces for a much smaller class of pseudomanifolds with three strata.

**Remark 7.1.2.** Recall that, by Example 4.3.4, $\Omega I^*_p(W)$ is geometrically cotruncatable in degree $L$ if $H^k(\Omega I^*_p(W)) = 0$.

### 7.2. Definition of the de Rham Intersection Complex

**Definition 7.2.1.** (The de Rham intersection complex $\Omega I^*_p(M)$) 

$$\Omega I^*_p(M) := \{ \omega \in \widetilde{\Omega I}^*_p(M) | \exists \eta \in \tau_{\geq L}\Omega^*_p(W) : \pi^*_W \omega = \pi^*_W \eta \}.$$ 

**Remark 7.2.2.** Let $C := C_E \cap C_W, C \cong \partial E \times [0,1)^2$, a collar neighbourhood of $\partial E = \partial W$ in $M$. Then for $\omega \in \Omega I^*_p(M)$ with $\pi^*_W \omega = \pi^*_W \eta_W$ for some $\eta_W \in \tau_{\geq L}\Omega^*_p(W)$ and $c_E^* \omega = \pi^*_E |_{\partial E}$ for some $\eta_E \in f t_{K\Omega^*_MS}(B)$ we have that

$$\omega|_C = (c_E^* \omega)|_C = (\pi^*_E \eta_E)|_C = \bar{\pi}^*(\eta_E|_{\partial E}),$$

with $\bar{\pi} : C \cong C_{\partial E} \times [0,1) \to C_{\partial E}$ the projection, as well as

$$\omega|_C = (c_W^* \omega)|_C = (\pi^*_W \eta_W)|_C = \bar{\pi}^*(\eta_W|_{\partial E}),$$

with $\bar{\pi} : C \cong C_{\partial W} \times [0,1) \to C_{\partial W}$ also the projection. Let now $(x,t,s)$ denote the coordinates on $\partial E \times [0,1) \times [0,1)$ directed to $E$ directed to $W$ independent of $s$ and by (13) it is independent of $t$. Hence there is a form $\eta \in \Omega^p(\partial E = \partial W)$ such that $\omega|_C = \pi^* \eta$ with $\pi : C \to \partial E$ the projection. In particular we get $j^*_{\partial E} \eta = \pi^* \eta$ and hence $\eta \in f t_{K\Omega^*_MS}(\partial B)$. Since it also holds that $j^*_{\partial W} \eta = \pi^* \eta$, we deduce the following results.

$$\Omega I^*_p(M) = \{ \omega \in \widetilde{\Omega I}^*_p(M) | \pi^*_W \omega = \pi^*_W \eta \text{ for some } \eta \in \tau_{\geq L}\Omega^*_p(W) \}$$

and $\omega \in \widetilde{\Omega I}^*_p(M) \Rightarrow \sigma^*_W \circ c^*_W \omega \in \Omega I^*_p(W)$.

We start to give the preparational material for the proof of Poincaré Duality for $\Omega I^*_p(M)$:

**Lemma 7.2.3.** The following is a distinguished triangle in $\mathcal{D}(\mathbb{R})$.

$$
\begin{array}{ccc}
\tau_{\geq L}\Omega I^*_p(W) & \rightarrow & \Omega I^*_p(M) \\
  \downarrow & & \downarrow \\
\widetilde{\Omega I}^*_p(M,W) & \rightarrow & \Omega I^*_p(M) \\
\end{array}
$$

**Proof.** The kernel of the surjective map $\sigma^*_W \circ c^*_W : \Omega I^*_p(M) \to \tau_{\geq L}\Omega I^*_p(W)$ is

$$\{ \omega \in \Omega I^*_p(M) | c^*_W \omega = 0 \} = \widetilde{\Omega I}^*_p(M,W).$$
Hence there is a short exact sequence
\[ 0 \longrightarrow \widehat{\Omega}_p^\bullet(M,W) \longrightarrow \Omega^\bullet_p(M) \overset{\sigma_W \circ c_W}{\longrightarrow} \tau_{\geq L} \Omega^\bullet_p(W) \longrightarrow 0 \]
and in particular a distinguished triangle of the desired form in \( \mathcal{D}(\mathbb{R}) \).

**Lemma 7.2.4.** There is a short exact sequence
\[ 0 \longrightarrow \Omega^\bullet_p(M) \longrightarrow \widehat{\Omega}_p^\bullet(M) \longrightarrow \tau_{< L} \Omega^\bullet_p(W) \longrightarrow 0. \]
In particular, this induces the following other distinguished triangle in \( \mathcal{D}(\mathbb{R}) \).

\[ \begin{array}{c}
\Omega^\bullet_p(M) \\
\uparrow 1 \\
\tau_{< L} \Omega^\bullet_p(W)
\end{array} \xrightarrow{\sim} \begin{array}{c}
\widehat{\Omega}_p^\bullet(M) \\
\downarrow \\
\Omega^\bullet_p(W)
\end{array} \]

**Proof.** Since \( \Omega^\bullet_p(M) \hookrightarrow \widehat{\Omega}_p^\bullet(M) \) is a subcomplex, there is a short exact sequence
\[ (14) \hspace{1cm} 0 \longrightarrow \Omega^\bullet_p(M) \longrightarrow \widehat{\Omega}_p^\bullet(M) \longrightarrow \frac{\widehat{\Omega}_p^\bullet(M)}{\Omega^\bullet_p(M)} \longrightarrow 0 \]
By Remark 7.2.2, for any \( \omega \in \widehat{\Omega}_p^\bullet(M) \) one has that \( \sigma_W \circ c_W \omega \in \Omega^\bullet_p(W) \) and for \( \omega \in \Omega^\bullet_p(M) \) one has \( \sigma_W \circ c_W \omega \in \tau_{\geq L} \Omega^\bullet_p(W) \). By the standard arguments (enlarging the collar and using a cutoff function) the maps
\[ \widehat{\Omega}_p^\bullet(M) \overset{\sigma_W \circ c_W}{\longrightarrow} \Omega^\bullet_p(W) \]
and
\[ \Omega^\bullet_p(M) \overset{\sigma_W \circ c_W}{\longrightarrow} \tau_{\geq L} \Omega^\bullet_p(W) \]
are surjective and by the same argument as in [Ban16, sect.6,p.43] (using the \( 3 \times 3 \)-lemma) we get an isomorphism
\[ \frac{\widehat{\Omega}_p^\bullet(M)}{\Omega^\bullet_p(M)} \cong \frac{\Omega^\bullet_p(W)}{\tau_{\geq L} \Omega^\bullet_p(W)}. \]
By Remark 4.3.5, we have \( \Omega^\bullet_p(W) = \tau_{< L} \Omega^\bullet_p(W) \oplus \tau_{\geq L} \Omega^\bullet_p(W) \) and hence the map
\[ \tau_{< L} \Omega^\bullet_p(W) \overset{\text{proj inj cl}}{\longrightarrow} \Omega^\bullet_p(W) \]
By composition we get an isomorphism
\[ \frac{\widehat{\Omega}_p^\bullet(M)}{\Omega^\bullet_p(M)} \cong \tau_{< L} \Omega^\bullet_p(W). \]
Lemma 7.2.5. Integration induces a nondegenerate bilinear form
\[ \int : \Omega I^r_p(W) \times \Omega I^n_{q-1}(W) \to \mathbb{R}. \]

Proof. Notice that \( \Omega I^r_p(W) \cong \Omega I^n_q(W - \partial W) \) and consider [Ban16, Theorem 8.2].

Lemma 7.2.6. Also integration induces a nondegenerate bilinear form
\[ \int : \Omega^r(\tau_{<L}\Omega^r_p(W)) \times \Omega^{n-1-r}(\tau_{\geq L'}\Omega^n_q(W)) \to \mathbb{R}. \]

Proof. For \( r \geq L \) we have that \( n-r-1 < L' \) and both complexes are zero and therefore also the cohomology groups. For \( r < L \) we have that \( n-r-1 \geq L' \) and hence \( \Omega^r(\tau_{<L}\Omega^r_p(W)) = \Omega^r_p(W) \) as well as \( \Omega^{n-1-r}(\tau_{\geq L'}\Omega^n_q(W)) \cong \Omega^{n-1-r}(W) \). Therefore we traced back the statement of the lemma to the result of the previous lemma.

7.3. Integration on \( \Omega I^r_p(M) \). In analogy to [Ban16, Lemma 7.1, Cor. 7.2] we have

Lemma 7.3.1. Integration defines bilinear forms
\[ \int : \Omega^r_p(M) \times \Omega^{n-r}_q(M) \to \mathbb{R}. \]

The following lemma is the extension of [Ban16, Lemma 7.4] to the 3-strata case:

Lemma 7.3.2. Let \( \omega \in \Omega^{r-1}_p(M) \), \( \eta \in \Omega^{n-r}_q(M) \), Then
\[ \int_M d(\omega \wedge \eta) = 0. \]

Proof. By Stokes’ Theorem on manifolds with corners we get:
\[ \int_M d(\omega \wedge \eta) = \int_W j^*_W (\omega \wedge \eta) + \int_E j^*_E (\omega \wedge \eta) \]
By definition of \( \Omega I^*_p(M) \), we have
\[ j^*_W (\omega \wedge \eta) = \omega_W \wedge \eta_W, \quad \omega_W \in Q^{r-1}_p(W), \quad \eta_W \in Q^{n-r}_q(W) \]
and
\[ j^*_E (\omega \wedge \eta) = \omega_E \wedge \eta_E, \quad \omega_E \in ft_{\geq K'} \Omega^{r-1}_{p_{\mathcal{M}S}}(B), \quad \eta_E \in ft_{\geq K'} \Omega^{n-r}_{p_{\mathcal{M}S}}(B). \]
For \( r-1 \geq L = n-1 - \bar{p}(n-1) = 2 + \bar{q}(n-1) \) we get \( n-r \leq n-3 - \bar{q}(n-1) < n-1 - \bar{q}(n-1) = L' \) and hence
\[ Q^{n-r}_q(W) = \Omega^{n-r}_q(W) \cap \tau_{\geq L'} \Omega^{n-r}(W) = 0. \]
This implies that \( \int_W \omega_W \wedge \omega_E = 0 \) for \( r-1 \geq L \). For \( r-1 < L \) we have
\[ Q^{r-1}_p(W) = \Omega^{r-1}_p(W) \cap \tau_{\geq L} \Omega^{r-1}(W) = 0 \]
and therefore also \( \int_W \omega_W \wedge \eta_W = 0 \) for \( r-1 < L \).
\[ \int_E \omega_E \wedge \eta_E = 0 \] holds by Lemma 6.4.3. \( \square \)
With the help of the previous lemma we get:

**Proposition 7.3.3.** Integration induces bilinear forms

\[ \int : HI^r_p(M) \times HI^{n-r}_q(M) \to \mathbb{R}, ([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta. \]

**Proof.** Let \( \omega \in \Omega^r_p(M) \) and \( \eta \in \Omega^{n-r}_q(M) \) be closed forms and \( \omega' \in \Omega^{r-1}_p(M), \eta' \in \Omega^{n-r-1}_q(M) \) be any forms. We then apply Lemma 7.3.2 to deduce the following.

\[
\int_M (\omega + d\omega') \wedge \eta = \int_M \omega \wedge \eta + \int_M \frac{d\omega' \wedge \eta}{d(\omega \wedge \eta)} = \int_M \omega \wedge \eta \quad \text{as well as}
\]

\[
\int_M \omega \wedge (\eta + d\eta') = \int_M \omega \wedge \eta + \int_M \frac{\omega \wedge d\eta'}{d(\omega \wedge \eta')} = \int_M \omega \wedge \eta.
\]

\[\square\]

### 7.4. Poincaré Duality for the Intersection de Rham Complex

Finally we can state and prove the Poincaré duality theorem for \( \Omega^*_p(M) \):

**Theorem 7.4.1.** (Poincaré duality for \( HI \))

Integration induces nondegenerate bilinear forms

\[ \int : HI^r_p(M) \times HI^{n-r}_q(M) \to \mathbb{R}, ([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta. \]

**Proof.** The two distinguished triangles of the Lemmata 7.2.3 and 7.2.4 induce long exact sequences on cohomology that fit into a diagram of the following form.

\[
\begin{array}{ccccccccc}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
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& & & & & & & & & \\
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& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]
We claim that this diagram commutes. To show this, we prove step by step that the individual squares in the diagram commute. We start with the top square (TS) and describe the connecting homomorphism \( \delta : H^{r-1}(\tau_{<L}\Omega^{r}_{p}(W)) \to H^{r}_{p}(M) \) first. Let \( \omega \in \tau_{<L}\Omega^{r-1}_{p}(W) \) be a closed form. Then
\[
\int W \omega = d\gamma W \omega = \gamma W(d\omega) = 0,
\]
where \( \gamma W := \text{proj} \circ \text{incl} : \tau_{<L}\Omega^{r}_{p}(W) \to \Omega^{r}_{p}(W) / Q^{r}_{p}(W) \)

is a chain map. Let \( i : \Omega^{r}_{p}(M) \to \Omega^{r}_{p}(M) \) denote the subcomplex inclusion and \( C^{\bullet}(i) \) its algebraic mapping cone, defined by
\[
C^{r}(i) := \Omega^{r+1}_{p}(M) \oplus \widehat{\Omega}^{r}_{p}(M),
\]
\[
d(\alpha, \beta) = (-d\alpha, \alpha + d\beta).
\]
Since the map \( J := c_{W} \circ \sigma_{W} \) induces an isomorphism
\[
\tilde{J} : \Omega^{r}_{p}(M) / Q^{r}_{p}(W) \cong \Omega^{r}_{p}(W),
\]
there is exactly one \( \pi \in \Omega^{r}_{p}(M) / Q^{r}_{p}(W) \) with representative \( \pi \) such that \( J(\pi) = [J^{\ast}\pi] = [\omega] \). We further have \( -d\pi = \text{proj}(d\pi) = d(\text{proj} \pi) = d\pi = 0 \) since \( d[\omega] = 0 \) and \( \tilde{J} \) is an isomorphism. Hence \( d\pi \in \Omega^{r}_{p}(M) \) and \( (d\pi, \pi) \in C^{\bullet}(i) \) with \( d(-d\kappa, \kappa) = (d^{2}\kappa, -d\kappa + d\kappa) = (0, 0) \).

As a result, the cohomology class \( \delta([\omega]) \) is described as
\[
\delta([\omega]) = [-d\kappa] \in H^{r}_{p}(X).
\]

To show that (TS) commutes we must show that for \( \omega \in \tau_{<L}\Omega^{r-1}_{p}(W) \) closed and \( \eta \in \Omega^{n-r}_{q}(M) \) closed it holds that
\[
\int W \omega \wedge \sigma_{W}^{\ast} \circ c_{W}^{\ast}(\eta) = \pm \int M -d\kappa \wedge \eta.
\]
Since \( d\eta = 0, -d\kappa \wedge \eta = -d(\kappa \wedge \eta) \) and hence by Stokes’ Theorem for manifolds with corners
\[
\int M (-dw) \wedge \eta = -\int W \sigma_{W}^{\ast} \circ c_{W}^{\ast}(\kappa \wedge \eta) - \int E \sigma_{E}^{\ast} \circ c_{E}^{\ast}(\kappa \wedge \eta).
\]
Since \( \kappa \in \widehat{\Omega}^{r-1}_{p}(M) \) and \( \eta \in \Omega^{n-r}_{q}(M) \), it holds that
\[
\sigma_{E}^{\ast} \circ c_{E}^{\ast} \kappa \in ft_{\geq K} \Omega^{r}_{q}(B) \quad \text{and} \quad \sigma_{E}^{\ast} \circ c_{E}^{\ast} \eta \in ft_{\geq K} \Omega^{n-r}_{q}(B).
\]
and hence 6.4.3 implies that
\[
\int_E \sigma_E^* \circ c_E^* (\kappa \wedge \eta) = 0.
\]
What remains is to calculate the integral \(\int_W \sigma_W^* \circ c_W^*(\kappa \wedge \eta)\): By definition we have
\[
[\sigma_W^* c_W^*(\kappa)] = [J_W^*(\kappa)] = J_W^* \text{proj}(\kappa) = J_W^* \nu = [\omega]
\]
and hence there is a form \(\alpha \in Q_p^{-1}(W)\) such that \(\sigma_W^* c_W^*(\kappa) = \omega + \alpha\). That result gives
\[
\int_W \sigma_W^* c_W^*(\kappa \wedge \eta) = \int_W \omega \wedge \sigma_W^* c_W^* \eta + \int_W \alpha \wedge \sigma_W^* c_W^* \eta = \int_W \omega \wedge \sigma_W^* c_W^* \eta
\]
since \(\alpha \in Q_p^{-1}(W)\) and \(\sigma_W^* c_W^* \eta \in Q^n_{\bar{q}}(W)\) and hence
\[
\int_W \alpha \wedge \sigma_W^* c_W^* \eta = 0,
\]
by the same arguments as in the proof of Lemma 7.3.2. Summing up, we have shown that (TS) commutes.

Before proving the commutativity of the bottom square (BS) in (15), we describe the connecting homomorphism and the map \(\eta\). We first describe \(\Delta I^W\Omega\), and let \(g\) and \(\bar{\omega}\) denote the subcomplex inclusion and \(C^\bullet(\rho)\) the corresponding mapping cone,
\[
C^r(\rho) = \Omega^{n-r+1}_p(M, W) \oplus \Omega^n_p(M),
\]
\[
d(\alpha, \beta) := (-d\alpha, \alpha + d\beta),
\]
and let \(g : C^\bullet(\rho) \to \tau_{\geq L} \Omega^\bullet_q(W)\) be the quasi-isomorphism defined by
\[
g(\alpha, \beta) := \sigma_W^* c_W^* \beta.
\]
Let then \(\eta \in \tau_{\geq L} \Omega^+_q(W)\) be a closed form. By Lemma 7.2.3, there is a form \(\omega \in \Omega^{n-r-1}_q(M)\) such that
\[
\xi := \eta - \sigma_W^* c_W^* \omega \in d(\tau_{\geq L} \Omega^{n-r-2}_q(W)).
\]
Further, \(\sigma_W^* c_W^* (d\omega) = d(\sigma_W^* c_W^* \omega) = d\eta + d\xi = 0\), and therefore \(d\omega \in \Omega^n_q(M, W)\) and \(c := (-d\omega, \omega) \in C^{n-r-1}(\rho)\) with \(dc = (d^2 \omega, -d\omega + d\omega) = (0, 0)\).
Since \([g(c)] = [\sigma_W^* c_W^* \omega] = [\eta]\), we get
\[
(17) \quad \Delta[\eta] = [-d\omega] \in \Omega^{n-r}_q(M, W).
\]
Second, we give a description of the map $\Lambda$. Let $\theta \in \tilde{\Omega}_{p}^{r}(M)$ be a closed form. Then $\text{proj}(\sigma_{W}^{*}c_{W}^{*}\theta) = [\sigma_{W}^{*}c_{W}^{*}\theta] \in \frac{\Omega_{p}^{r}(W)}{\sigma_{W}^{*}c_{W}^{*}\theta}$ is also closed. By the arguments in the proof of Lemma 7.2.4, subcomplex inclusion followed by projection is a quasi-isomorphism $\tau_{<L}\Omega_{p}^{*}(W) \rightarrow \frac{\Omega_{p}^{*}(W)}{\sigma_{W}^{*}c_{W}^{*}\theta}$ and thus there is an $\xi \in \tau_{<L}\Omega_{p}^{*}(W)$ closed such that

\begin{equation}
[\sigma_{W}^{*}c_{W}^{*}\theta - \xi] = d[\nu]
\end{equation}

for some $\nu \in \Omega_{p}^{r-1}(W)$. We can then describe $\Lambda$ by

\begin{equation}
\Lambda[\theta] = [\xi] \in H^{r}(\tau_{<L}\Omega_{p}^{*}(W)).
\end{equation}

To prove the commutativity of the (BS) we have to show that for $\eta \in \tau_{\geq L}\Omega_{q}^{n-r-1}(W)$ and $\theta \in \tilde{\Omega}_{p}^{r}(M)$ closed with $\Delta[\eta] = [-d\omega]$ and $\Lambda[\theta] = [\xi]$ as above it holds that

\[\int_{M} \theta \wedge (-d\omega) = \pm \int_{W} \xi \wedge \eta.\]

By Stokes’ Theorem on manifolds with corners, we get

\[\int_{M} \theta \wedge (-d\omega) = \int_{M} d(\theta \wedge \omega) = \int_{E} \sigma_{E}^{*}c_{E}^{*}(\theta \wedge \omega) + \int_{W} \sigma_{W}^{*}c_{W}^{*}(\theta \wedge \omega).\]

By definition, there are $\theta_{E} \in ft_{\geq K}\Omega_{MS}^{r}(B)$, $\omega_{E} \in ft_{\geq K}\Omega_{MS}^{n-r-1}(B)$ with $\sigma_{E}^{*}c_{E}^{*}\theta = \theta_{E}$, $\sigma_{E}^{*}c_{E}^{*}\omega = \omega_{E}$. Hence by Lemma 6.4.3,

\[\int_{E} \sigma_{E}^{*}c_{E}^{*}(\theta \wedge \omega) = \int_{E} \theta_{E} \wedge \omega_{E} = 0.\]

This implies that

\[\int_{M} \theta \wedge d\omega = \int_{W} \sigma_{W}^{*}c_{W}^{*}(\theta \wedge \omega) = \int_{W} \sigma_{W}^{*}c_{W}^{*}\theta \wedge (\eta + d\alpha),\]

for some $\tau \in \tau_{\geq L}\Omega_{q}^{n-r-2}(W)$. Since $j_{\partial W}\theta \in ft_{\geq K}\Omega_{MS}^{r}(\partial B)$, $j_{\partial W}\tau \in ft_{\geq K}\Omega_{MS}^{*}(\partial B)$, applying Stokes’ Theorem and [Ban16, Lemma 7.3] afterwards we get

\[\int_{W} \sigma_{W}^{*}c_{W}^{*}\theta \wedge d\tau = \int_{\partial W} j_{\partial W}^{*}(\theta \wedge \tau) = 0.\]

So we arrive at the equation

\[\int_{M} \theta \wedge d\omega = \int_{W} \sigma_{W}^{*}c_{W}^{*}\theta \wedge \eta.\]

On the other hand, by (18) we get

\[\int_{W} \xi \wedge \eta = \int_{W} \sigma_{W}^{*}c_{W}^{*}(\theta \wedge \eta) + \int_{W} \alpha \wedge \eta - \int_{W} d\nu \wedge \eta,
\]

where $\alpha := \xi + d\nu - \sigma_{W}^{*}c_{W}^{*}\theta \in Q_{p}^{r}(W)$. As before we have: If $r \geq L$, then $n - r - 1 < L^{*}$ and hence $\eta \in \tau_{\geq L}\Omega_{q}^{n-r-1}(W) = \{0\}$, implying $\eta = 0$. If
$r < L$, then $\alpha \in \tau_{\geq L}^r \Omega^r(W) = \{0\}$, and hence $\alpha = 0$. In both cases we have
\[
\int_W \alpha \wedge \eta = 0.
\]
Since $\nu \in \Omega_{I_{\bar{p}}^{r-1}}^r(W)$, there is a $\nu_0 \in ft_{\geq r} \Omega^\bullet_{\bar{p}}(\partial B)$ such that $j_{\partial W}^* \nu = \pi_{\partial W}^* \nu_0$ and $\eta \in \tau_{\geq L}^r \Omega_{I_{\bar{q}}^r}^r(W)$ implies that there exists a form $\eta_0 \in ft_{\geq L} \Omega^\bullet_{\bar{q}}(\partial B)$ with $j_{\partial W}^* \eta = \pi_{\partial W}^* \eta_0$. Therefore (and since $d\eta = 0$), we get by Stokes’ Theorem:
\[
\int_W d\nu \wedge \eta = \int_W d(\nu \wedge \eta) = \int_{\partial W} (\nu \wedge \eta)|_{\partial W} = \int_{\partial W} \nu_0 \wedge \eta_0 = 0
\]
by [Ban16, Lemma 7.3]. Hence
\[
\int_W \xi \wedge \eta = \int_W \sigma_{W}^* \xi_{W}(\theta) \wedge \eta = \int_N \theta \wedge (-d\omega),
\]
which means that (BS) commutes (up to sign).

The middle square in (15) commutes, since the vertical maps are just inclusions and the horizontal maps both integration of wedge products of two forms.

The commutativity of the diagram (15) together with the fact that the map
\[
\int : H^r(\tau_{< L} \Omega_{I_{\bar{p}}}^r(W)) \to H^{n-r-1}(\tau_{\geq L} \Omega_{I_{\bar{q}}}^r(W))
\]
is an isomorphism for all $r \in \mathbb{Z}$ by Lemma 7.2.6 as well as the map
\[
\int : \tilde{H}_{\bar{p}}^r(M) \to \tilde{H}_{\bar{q}}^{n-r}(M, W)
\]
by Proposition 6.5.4 then enables us to apply the 5-Lemma to conclude the statement of the theorem.

8. Examples

We want to give a class of examples of depth two pseudomanifolds, we can apply the intersection space cohomology theory to. We make use of the relation between Thom-Mather stratified pseudomanifolds and compact manifolds with corners and iterated fibration structures, which is described in [ALMP12, Section 2]. In our setting, this is the correspondence between the described depth two pseudomanifolds and (2)-manifolds with one boundary-component fibered by a geometrically flat fiber bundle.

We consider flat principal $G$-bundles, with $G$ a compact connected Lie group. This example is based on ideas of Laures, see [Lau00].
8.1. Fiberwise Truncation on Principal Bundles. Before we introduce the examples, we calculate the cohomology of fiberwisely (co)truncated multiplicatively structured forms on flat principal $G$-bundles $p : E \to B$, with $G$ a compact connected Lie group. There are horizontal sections $s : U \to p^{-1}(U)$ that induce a trivialization of the bundle with locally constant transition maps by defining local trivializations
\[
\phi_U^{-1} : U \times G \to p^{-1}(U),
\]
\[
(x, g) \mapsto s(x) \cdot g.
\]
Byun and Kim prove in [BK17, Section 6] that the Leray-Hirsch Theorem is applicable to flat principal bundles and hence $H^\bullet(E) \cong H^\bullet(B) \otimes H^\bullet(G)$ as algebras. They do so by constructing, for any flat connection $\alpha$ on $E$, an algebra homomorphism $E_A : H^\bullet_{dR}(G) \to H^\bullet_{dR}(E)$ satisfying $\iota_y \circ E_A = \text{id}$ for any inclusion $\iota_y : G \to p^{-1}(p(y))$, $g \mapsto yg$. The cohomology morphism $E_A$ is induced by a map
\[
E_A : \mathcal{H}^\bullet(G) \to \Omega^\bullet(E),
\]
with $\mathcal{H}^\bullet(G) = \{ \alpha \in \Omega^\bullet(G) : \mathcal{L}_g^\bullet \alpha = R_g^\bullet \alpha = \alpha \forall g \in G \}$ the complex of bi-invariant forms on $G$, which is isomorphic to $H^\bullet_{dR}(G)$ by [CE48, Theorem 12.1]. If the connection is flat, $E_A(\alpha)$ is a closed form for any $\alpha \in \mathcal{H}^\bullet(G)$. Locally, $E_A(\alpha)$ is given as $E_A(\alpha)|_{p^{-1}(U)} = \kappa_\alpha$, with $\kappa_\alpha : p^{-1}(U) \to G$ defined by $y = s(p(y)) \cdot \kappa_\alpha(y)$ for a horizontal section $s$ on $U$. This implies, that $E_A(\alpha)$ is actually a multiplicative form. For let $U$ be any coordinate chart. Then $\kappa_{\phi_U^{-1}}(x, g) = \kappa_\alpha(s(x) \cdot g) = g$, by the definition of $\kappa_\alpha$. This means that locally $E_A(\alpha)|_{p^{-1}(U)} = \phi_U^{-1} \pi_2^\alpha$ and hence, $E_A(\alpha)$ is multiplicative.

**Proposition 8.1.1.** Take the naive truncation and cotruncation of bi-invariant forms $\mathcal{H}^\bullet_{<K}(G) = \mathcal{H}^\bullet(G)$ for $r < K$ and zero otherwise and $\mathcal{H}^\bullet_{\geq K}(G) = \mathcal{H}^\bullet(G)$ for $r \geq K$ and zero otherwise. There are isomorphisms
\[
H^\bullet_{dR}(B) \otimes \mathcal{H}^\bullet_{<K}(G) \cong H^\bullet(f t_{<K} \Omega^\bullet_{\text{MS}}(E))
\]
and
\[
H^\bullet_{dR}(B) \otimes \mathcal{H}^\bullet_{\geq K}(G) \cong H^\bullet(f t_{\geq K} \Omega^\bullet_{\text{MS}}(E)),
\]
defined by
\[
([\omega], \alpha) \mapsto [p^* \omega \wedge E_A^*(\alpha)].
\]

**Proof.** The product $p^* \omega \wedge E_A^*(\alpha)$ is always multiplicative since $E_A(\alpha)$ is multiplicative and $p_U = \pi_1 \circ \phi_U$ for each coordinate chart $U$. It satisfies the following formula.
\[
(p^* \omega \wedge E_A^*(\alpha))|_{p^{-1}(U)} = \phi_U^* (\pi_1^* \omega \wedge \pi_2^\alpha).
\]
This relation implies that the assignment $([\omega], \alpha) \mapsto [p^* \omega \wedge E_A^*(\alpha)]$ defines maps as in the statement of the proposition. This is true because $\mathcal{H}^\bullet_{<K} \subset \tau_{<K} \Omega^\bullet(G)$ and $\mathcal{H}^\bullet_{\geq K} \subset \tau_{\geq K} \Omega^\bullet(G)$. The first inclusion is trivial. The second is again true, since all (non-trivial) bi-invariant forms are no boundaries and hence contained in $\ker d^*$, the kernel of the coboundary operator.
The same Mayer-Vietoris type argument which is used to prove the Leray-Hirsch Theorem (together with the Poincaré Lemmas for fiberwise (co)-truncated multiplicatively structured forms of [Ban16]) then implies the statement of the proposition. \[ \square \]

8.2. Principal bundles over \((2)\)-manifolds. Let \(G\) be a compact Lie group and \(B\) a \((2)\)-manifold with boundary \(\partial B = \partial B_1 \cup \partial \partial B_2\) such that there is a group representation \(\rho : \pi_1(B) \to G\) of the fundamental group of \(B\). Let \(\pi : \tilde{B} \to B\) denote the universal covering of \(B\). Then the map \(p : M_{\rho} := \tilde{B} \times_\rho G \to B, (x, g) \mapsto \pi(x)\), defines a flat principal \(G\)-bundle. \(M\) is a \((2)\)-manifold as well and can be interpreted as the blow-up of a pseudomanifold with three strata, which is constructed by first blowing down \(p^{-1}(\partial B_1)\) fiberwisely and then blowing down the boundary of the resulting pseudomanifold with boundary.

**Example 8.2.1.** We first construct a (high dimensional) orientable closed manifold \(P\) with fundamental group \(\pi_1(P) = \mathbb{Z}/2\mathbb{Z}\). Let \(n \geq 4, k \geq 2\) and set \(N := n + k\). Starting with the manifold \(S^1 \times S^{n-1}\), let \(c\) be a closed embedded curve such that its homotopy class represents two times the generator of the fundamental group \(\pi_1(S^1 \times S^{n-1})\). If we cut out a tubular neighborhood \(c \times D^{n-2}\) of \(c\) and glue a disk to \(c\), we get the space

\[
P = ((S^1 \times S^{n-1}) \setminus (c \times D^{n-1})) \cup_{\partial D = c} (D^2 \times S^{n-2}),
\]

which is a closed oriented \(n\)-dimensional manifold. By the Seifert-van Kampen theorem, the fundamental group of \(P\) is \(\pi_1(P) = \mathbb{Z}/2\mathbb{Z}\).

The manifold \(M\) is then obtained by cutting an Euclidean ball \(B\) out of \(P \times S^k\), \(M := P \times S^k \setminus B\). Depending on the embedding of \(B\) in \(P \times S^k\) and the associated collars, we can view \(M\) either as a manifold with boundary \(\partial B = S^{n-1}\) or as a \((2)\)-manifold with decomposed boundary \(\partial B = S^{N-1} = D_1^{N-1} \cup_{S^{N-2}} D_2^{N-1}\). The Seifert-van Kampen theorem implies that the fundamental group of \(M\) is still \(\pi_1(M) = \mathbb{Z}/2\mathbb{Z}\), since \(\pi_1(M) \cong \pi_1(P \times S^k) \cong \pi_1(P) \times \pi_1(S^k) = \pi_1(P) = \mathbb{Z}/2\mathbb{Z}\). Hence there is a group homomorphism \(\rho : \pi_1(M) \cong \mathbb{Z}/2\mathbb{Z} \to S^1 \to S^1 \times S^1 = T^2\). Let \(\pi : \tilde{M} \to M\) be the universal covering of \(M\). The homomorphism \(\rho\) defines a flat principal \(T^2\)-bundle

\[
p : M_{\rho} := \tilde{M} \times_\rho T^2 \to M, \ [\tilde{x}, t] \mapsto \pi(\tilde{x}).
\]

The total space of this bundle can be either seen as the blowup of a 2–strata pseudomanifold with singular set \(\partial M\) and link bundle \(p|_{p^{-1}(\partial M)} \to \partial M\), which is a manifold with boundary \(\partial M\). Alternatively, one can interpret \(M\) as the blowup of a 3–strata pseudomanifold with filtration by singular sets \(X \supset \partial M \cong S^{N-1} \supset \{x_0\}\), with \(x_0 \in S^{N-1}\) some point.

We first calculate the intersection space cohomology of the associated 2–strata pseudomanifold \(Y\). We use the long exact cohomology sequence
induced by the distinguished triangle

\[
\begin{array}{ccc}
\Omega^\bullet(M_\rho, \partial M_\rho) & \xrightarrow{\text{+1}} & \Omega I_\rho^\bullet(M_\rho) \\
\downarrow \scriptstyle{\text{f}_t \geq K \Omega_{MS}^\bullet(\partial M_\rho)} & & \downarrow \scriptstyle{\text{j}^*} \\
\end{array}
\]

where \(\partial M_\rho = p^{-1}(\partial M)\). Proposition 8.1.1 implies, that

\[
H^\bullet(\text{f}_t \geq K \Omega_{MS}^\bullet(\partial M_\rho)) \cong H^\bullet(\partial M) \otimes H^\bullet_{\geq K}(T^2) = H^\bullet_{\geq K}(T^2)[N - 1].
\]

By Poincaré-Lefschetz duality, we have \(H^r(M) \cong H_{N-r}(P \times S^k, B)\), which is isomorphic to \(\mathbb{R}\) for \(r = 0, k, n\) and zero otherwise. Hence, the cohomology of the principal flat \(T^2\)-bundle \(M_\rho\) satisfies the following relation.

\[
H^\bullet(M_\rho) \cong H^\bullet(T^2) \oplus H^\bullet(T^2)[k] \oplus H^\bullet(T^2)[n].
\]

The cohomology of the boundary satisfies the similar relation \(H^\bullet(\partial M_\rho) \cong H^\bullet(S^N) \oplus H^\bullet(T^2) \cong H^\bullet(T^2) \oplus H^\bullet(T^2)[N - 1]\). For dimensional reasons, the restriction to the boundary induces the trivial map on cohomology in all degrees but zero. There, it induces an isomorphism. Together with the long exact sequence of the pair \((M_\rho, \partial M_\rho)\), this gives rise to the following formula.

\[
H^\bullet(M_\rho, \partial M_\rho) \cong H^\bullet(T^2)[k] \oplus H^\bullet(T^2)[n] \oplus H^\bullet(T^2)[N].
\]

Proposition 8.1.1 implies, that there is an injection \(H^\bullet(\text{f}_t \geq K \Omega_{MS}^\bullet(\partial M_\rho)) \hookrightarrow H^\bullet(\partial M_\rho)\). Hence, the intersection space cohomology groups of \(Y\) are given as

\[
HI_\rho(Y) = H^\bullet(T^2)[k] \oplus H^\bullet(T^2)[n] \oplus H^\bullet_{< K}(T^2)[N] \oplus H^\bullet_{\geq K}(T^2).
\]

Note, that the Poincaré duality isomorphism between complementary perversities interchanges the first two factors and the latter two, respectively.

Let us also calculate the intersection space cohomology groups of the associated 3-strata pseudomanifold \(X\). The approach to calculate these is geared to our proof technique for the Poincaré duality theorem. The two truncation values that are needed on the respective boundary parts are \(K := 2 - \tilde{p}(3)\) and \(L := N - 1 - \tilde{p}(N)\). We calculate the cohomology groups \(\tilde{HI}_\rho(M)\) first, using the distinguished triangle (6). The long exact sequence of the pair \((M_\rho, p^{-1}(D_1))\), together with the previous calculations, implies that \(H^\bullet(M_\rho, p^{-1}(D_1)) \cong H^\bullet(T^2)[k] \otimes H^\bullet(T^2)[n]\). Since the disc \(D_1\) is contractible, Proposition 8.1.1 gives \(H^\bullet(\text{f}_t \geq K \Omega_{MS}^\bullet(p^{-1}(D_1))) \cong H^\bullet_{\geq K}(T^2)\). Therefore, the connecting homomorphism in the long exact cohomology sequence induced by (6) is trivial and hence the cohomology groups \(\tilde{HI}_\rho(M)\) are a direct product,

\[
\tilde{HI}_\rho(M) \cong H^\bullet(M_\rho, p^{-1}(D_1)) \oplus H^\bullet(\text{f}_t \geq K \Omega_{MS}^\bullet(p^{-1}(D_1)))
\]

\[
\cong H^\bullet_{\geq K}(T^2) \oplus H^\bullet(T^2)[k] \oplus H^\bullet(T^2)[n].
\]
To calculate the intersection space cohomology groups $HI^\bullet_p(X)$, we use the distinguished triangle of Lemma 7.2.4. To calculate the groups $HI^\bullet_p(p^{-1}(D_2))$, we use the same argumentation as in the first part of this example. The corner of $M$ is $\partial M = \partial D_2 = S^{N-2}$. The relative groups are

$$H^\bullet(p^{-1}(D_2), p^{-1}(S^{N-2})) \cong H^\bullet(T^2)[N-1] \quad \text{and} \quad H^\bullet(f^1_{\geq K}\Omega^\bullet_{MS}(p^{-1}(S^{N-2}))) \cong H^\bullet(S^{N-2}) \otimes H^\bullet_{\geq K}(T^2).$$

Together with the mentioned long exact cohomology sequence, we get

$$HI^\bullet_p(p^{-1}(D_2)) \cong H^\bullet_{\geq K}(T^2) \oplus H^\bullet_{< K}(T^2)[N-1].$$

Since $L = N - 1 - \bar{p}(N) \leq N - 1$, cotruncating below that degree gives

$$\tau_{\leq L} HI^\bullet_p(p^{-1}(D_2)) \cong H^\bullet_{\geq K}(T^2).$$

This is exactly the image of the pullback $j^*_{\partial D_2} \left( \tilde{HI}^\bullet_p(p^{-1}(D_2)) \right)$, where $j_{\partial D_2} : S^{N-2} = \partial D_2 \hookrightarrow D_2$. Hence, the long exact cohomology sequence induced by the triangle of Lemma 7.2.4 gives

$$HI^\bullet_p(X) \cong H^\bullet(T^2)[k] \oplus H^\bullet(T^2)[n],$$

which does not depend on the chosen perversity. In particular, the groups are different from the ones calculated in the first part of the example. The Poincaré duality isomorphism interchanges the two factors in the direct sum.

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