The Dirichlet problem for the $\alpha$-translating soliton equation on a strip

Rafael López*
Departamento de Geometría y Topología
Instituto de Matemáticas (IEMath-GR)
Universidad de Granada
18071 Granada, Spain
rcamino@ugr.es

Abstract

We prove the existence of classical solutions to the Dirichlet problem for the $\alpha$-translating soliton equation defined in a strip of $\mathbb{R}^2$. We use the Perron method where a family of grim reapers are employed as barriers for solving the Dirichlet problem when the boundary data is formed by two copies of a convex function.

*Partially supported by MEC-FEDER grant no. MTM2017-89677-P
1 Introduction and statement of results

Let $\Omega \subset \mathbb{R}^2$ be a smooth domain and a given constant $\alpha > 0$. We consider the Dirichlet problem

$$\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = \left( \frac{1}{\sqrt{1 + |Du|^2}} \right)^\alpha$$ in $\Omega$, $u \in C^2(\overline{\Omega}) \quad (1)$

$$u = \varphi \text{ on } \partial \Omega, \quad (2)$$

where $D$ and div are the gradient and divergence operators and $\varphi$ is a continuous function in $\partial \Omega$. Equation (1) is called the $\alpha$-translating soliton equation and the graph $\Sigma_u = \{(x, u(x)) : x \in \Omega\}$ is an $\alpha$-translating soliton whose boundary is the graph of $\varphi$. In the limit case $\alpha = 0$, Equation (1) is the known constant mean curvature equation. The motivation for the study of Equation (1) comes from the case $\alpha = 1$, where 1-translating solitons, or simply, translating solitons, appear in the singularity theory of the mean curvature flow in $\mathbb{R}^3$ as the limit flow by a proper blow-up procedure near type II singular points: [2, 9, 10, 24]. When $\alpha \neq 1$, Equation (1) extends to the case of the flow of surfaces by powers of the mean curvature: [17, 18, 19].

However, and besides of this interest in the mean curvature flow, Equation (1) already appeared in the classical article of Serrin [16] on the Dirichlet problem for quasilinear equations of divergence type. Possibly due to the extension of this article and its focus on the constant mean curvature equation, Equation (1) seemed be forgotten. Indeed, in [16, p.477], Serrin considered Equation (1) written as

$$(1 + |Du|^2)^{\frac{3}{2}} \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = c(1 + |Du|^2)^{\frac{3\alpha}{2}}, \quad (3)$$

where $c > 0$ is a constant and he studied the solvability of the Dirichlet problem for mean domains. Recall that $\Omega$ is said to be mean convex if the mean curvature $H_{\partial \Omega}$ with respect to the inward normal is non-negative, $H_{\partial \Omega} \geq 0$. For arbitrary dimension, Serrin proved the following result in [16, p.478].

**Theorem 1.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Then there exists a unique solution of (1)-(2) for any continuous function $\varphi$ if and only if:
1. \( \Omega \) is mean convex when \( \alpha \geq 1 \).

2. \( H_{\partial \Omega} > 0 \) when \( 0 < \alpha < 1 \).

This result has been recently revisited in the literature: see \([3, 12, 13, 15]\). In the context of translating solitons \((\alpha = 1)\), the existence of solutions of (1) has been studied in \([1, 22, 23]\), including also Neumann boundary conditions. In this article we study the solvability of (1)-(2) when \( \Omega \) is a strip of \( \mathbb{R}^2 \). The interest to the case that \( \Omega \) is a strip comes by the result in \([23]\) where Wang found convex translating solitons that are graphs on a strip, although our methods and results in the present article differ on them.

A first example of a translating soliton that is a graph on a strip is the grim reaper \( u(x, y) = -\log(\cos(y)) \) where \( u \) is defined in the strip \( \Omega = \{(x, y) \in \mathbb{R}^2 : -\pi/2 < y < \pi/2\} \). Let us observe that \( u \to \infty \) as \( |y| \to \pm\pi/2 \). If we narrow the strip to other strip included in \( \Omega \), then the restriction of \( u \) to the new domain has constant boundary values. For the case \( \alpha > 0 \), it is possible to have the generalization of the grim reapers when we consider solutions of (1) that depend only on one variable and these solutions will be candidates to be supersolutions of (1).

In this paper we solve (1)-(2) when the boundary data is formed by two copies of a convex function of \( \mathbb{R} \) extending what occurs for the grim reapers. Firstly, we need to introduce the next notation. Without loss of generality, suppose that the strip is \( \Omega_m = \{(x, y) \in \mathbb{R}^2 : -m < y < m\}, m > 0 \). Let \( f \) be a smooth convex function defined in \( \mathbb{R} \) and we extend \( f \) to \( \partial \Omega_m \) by defining \( \varphi_f(x, \pm m) = f(x) \).

**Theorem 1.2.** Let \( \Omega_m \subset \mathbb{R}^2 \) be a strip. For each convex function \( f \) defined in \( \mathbb{R} \), there exists a solution of (1) for boundary values \( u = \varphi_f \) on \( \partial \Omega_m \) in the next cases:

1. For any \( \alpha > 1 \) and \( m > 0 \).

2. If \( 0 < \alpha \leq 1 \), provided the width of \( \Omega_m \) satisfies \( m < d(\alpha) \), where \( d(\alpha) > 0 \) depends only on \( \alpha \).

This result was proved for the constant mean curvature equation \((\alpha = 0 \text{ in } (1))\) by Collin in \([3]\).
This paper is organized as follow. In Section 2 we find the solutions of the one-dimensional case of (1) and then in Section 3 we recall the maximum principle for Equation (1) and some properties of the solutions when $\Omega$ is a bounded domain. Finally in Section 4 we prove Theorem 1.2 by means of the Perron process of sub and supersolutions.

2 The family of $\alpha$-grim reapers

In this section we generalize the grim reapers for every $\alpha > 0$. Consider $\mathbb{R}^3$ the Euclidean space with canonical coordinates $(x, y, z)$. It is immediate that any translation of the space and a rotation about an axis parallel to $e_3 = (0, 0, 1)$ preserves the condition to be an $\alpha$-translating soliton. In this section we study those $\alpha$-translating solitons that depend only on one variable, or in other words, the surface is a cylindrical surface where all the rulings are parallel to a fixed direction. For our convenience, we consider firstly solutions that are invariant in one direction orthogonal to $e_3$, namely, to the $x$-axis. The surface is then generated by a curve $\gamma(s) = (y(s), z(s))$ contained in the $yz$-plane, which we suppose parametrized by the arc-length. Let $y'(s) = \cos \phi(s)$, $z'(s) = \sin \phi(s)$ for some angle function $\phi$. The surface parametrizes by $X(x, s) = x(1, 0, 0) + (0, \gamma(s)) = (x, y(s), z(s))$ and its mean curvature is $H(x, s) = \phi'(s)/2$, where $\phi'(s)$ is the curvature of $\gamma(s)$. Definitively the surface $X(x, s)$ is an $\alpha$-translating soliton if and only if $\gamma$ satisfies

\begin{align}
y'(s) &= \cos \phi(s) \\
z'(s) &= \sin \phi(s) \\
\phi'(s) &= (\cos \phi(s))^\alpha. \tag{4}
\end{align}

It is immediate that a vertical line satisfies (4), indeed, $\gamma$ can be expressed as $\gamma(s) = (y_0, \pm s + z_0), (y_0, z_0) \in \mathbb{R}^2$, with $\phi(s) = \pm \pi/2$. After a translation in $\mathbb{R}^2$, we can suppose that $\gamma$ goes through the origin, hence $\gamma(0) = (0, 0)$.

**Proposition 2.1.** Let $\alpha > 0$. There exists a family of solutions $\gamma = \gamma(s)$ of (4) satisfying

\begin{align}
y(0) = z(0) = 0, \phi(0) = 0, \tag{5}
\end{align}

and with the following properties:
1. \( \gamma \) is a symmetric convex graph about the \( y \)-axis with one global minimum.

2. Let \( \gamma(y) = (y, z(y)) \), \( z : (-d, d) \to \mathbb{R} \) with \( d = d(\alpha) \leq \infty \) and \( (-d, d) \) is the maximal domain of \( z(y) \). Then we have:

   (a) \( d(\alpha) = \infty \) if \( \alpha > 1 \).

   (b) \( d(\alpha) < \infty \) if \( 0 < \alpha \leq 1 \) and \( \gamma \) is asymptotic to the vertical lines \( y = \pm d(\alpha) \).

Proof. Since the derivatives in (4) are bounded, then the maximal domain of the solution \( \gamma \) of (4) is \( \mathbb{R} \). We prove that \( \phi \) never attains the values \( \pm \pi/2 \). If at some point \( s = s_0 \), it holds \( \phi(s_0) = \pi/2 \) (similar in case \( -\pi/2 \)) and it is immediate that

\[
\bar{y}(s) = y(s_0), \quad \bar{z}(s) = s - s_0 + z(s_0), \quad \bar{\phi}(s) = \pi/2
\]

is a solution of (4) with the same initial condition at \( s = s_0 \) as the initial solution \( \{y(s), z(s), \phi(s)\} \). By uniqueness, \( \gamma(s) = (y(s_0), s - s_0 + z(s_0)) \), \( \gamma \) is a straight line and \( \phi(s) = \pi/2 \) for all \( s \in \mathbb{R} \). This is a contradiction because the initial condition at \( s = 0 \) is \( \phi(0) = 0 \).

Therefore \( \gamma \) is a graph on the \( y \)-axis and we write \( z = z(y) \). Taking into account that \( \phi(s) \in (-\pi/2, \pi/2) \), we find that \( \phi'(s) > 0 \) and this implies that the (signed) curvature of \( \gamma \) is positive proving that \( z = z(y) \) is a convex function. On the other hand, Equations (4)-(5) write now as

\[
\frac{z''}{1 + z'^2} = \frac{1}{(1 + z'^2)^{\alpha/2}}, \quad z(0) = 0, \quad z'(0) = 0.
\]

Hence it is immediate that \( z = z(y) \) is symmetric about \( y = 0 \) with a global minimum at \( y = 0 \). The integration of (6) is given in terms of first hypergeometric function \( 2F_1(a, b; c; x) \), where it is is known that when \( \alpha \in (0, 1] \), the domain of \( z = z(y) \) is a bounded (symmetric) interval \( (-d, d) \), \( d = d(\alpha) \) depending on \( \alpha \) with \( \lim_{y \to \pm d} z(y) = \pm \infty \) and if \( \alpha > 1 \), then \( z = z(y) \) is defined on the entire \( y \)-axis.

Remark 2.2. Some explicit solutions of (4) can be obtained by simple quadratures:

1. Case \( \alpha = 1 \). Then \( z(y) = -\log(\cos(y)) \), \( \gamma \) is the grim reaper and \( d = \pi/2 \).
2. Case $\alpha = 2$. Then $z(y) = \cosh(y)$, $\gamma$ is the catenary and $d = \infty$.

3. Case $\alpha = 3$. Then $z(y) = y^2/2$, $\gamma$ is the parabola and $d = \infty$.

In the limit case $\alpha = 0$, we have $z(y) = -\sqrt{1 - y^2}$, $\gamma$ is a halfcircle and $d = 1$.

Recall that each solution $\gamma$ given in Proposition 2.1 corresponds with an $\alpha$-translating soliton where the rulings are orthogonal to the vector $e_3$. In order to find supersolutions in the Perron process, we need to use $\alpha$-translating solitons whose rulings are not necessary orthogonal to $e_3$. Having this in mind, we consider the $\alpha$-translating solitons of Proposition 2.1 up to scaling and rotating about the $y$-axis.

**Definition 2.3.** Let $\alpha > 0$. If $w = w(y)$ is a solution of (6), we define the uniparametric family of $\alpha$-grim reapers $w_\theta = w_\theta(x, y)$ as

$$w_\theta(x, y) = \frac{1}{(\cos \theta)^{\alpha+1}} w((\cos \theta)^\alpha y) + (\tan \theta)x + a,$$

where $\theta \in (-\pi/2, \pi/2)$ and $a \in \mathbb{R}$.

As consequence of Proposition 2.1 if $\alpha > 1$ $w_\theta$ is defined in $\mathbb{R}^2$ and if $0 < \alpha \leq 1$, then the domain of $w_\theta$ is the strip

$$\Omega_{d,\theta} = \{(x, y) \in \mathbb{R}^2 : -\frac{d}{(\cos \theta)^\alpha} < y < \frac{d}{(\cos \theta)^\alpha}\}.$$

In particular, if $0 \leq \theta_1 < \theta_2$, it follows that $\Omega_{\theta_1} \subset \Omega_{\theta_2}$ and thus the domain

$$\Omega_d := \Omega_{d,0} = \{(x, y) \in \mathbb{R}^2 : -d < y < d\}$$

is contained in $\Omega_{d,\theta}$ for all $\theta \in (-\pi/2, \pi/2)$.

### 3 Some properties of the solutions of Equation (1)

In this section we collect some properties of the solutions of (1) with a special interest in the control of $|u|$ and $|Du|$ when $\Omega$ is a bounded domain. Here we make use of
explicit examples of \( \alpha \)-translating solitons to get a priori \( C^0 \) estimates. Firstly we need to recall that Equation (1) satisfies a maximum principle which is a consequence of the comparison principle [7, Th. 10.1]).

**Proposition 3.1** (Touching principle). Let \( \Sigma_i \) be two \( \alpha \)-translating solitons, \( i = 1, 2 \). If \( \Sigma_1 \) and \( \Sigma_2 \) have a common tangent interior point and \( \Sigma_1 \) lies above \( \Sigma_2 \) around \( p \), then \( \Sigma_1 \) and \( \Sigma_2 \) coincide at an open set around \( p \). The same holds if \( p \) is a common boundary point and the tangent boundaries coincide at \( p \).

As a direct application of the touching principle, there do not exist compact \( \alpha \)-translating solitons because if \( \Sigma \) were a such surface, we can place a vertical plane \( \Pi \) tangent to \( \Sigma \) and leaving \( \Sigma \) in one side of \( \Pi \): this is impossible by the touching principle because both \( \Pi \) and \( \Sigma \) are \( \alpha \)-translating solitons.

Besides the \( \alpha \)-grim reapers defined in Section 2, other family of useful \( \alpha \)-translating solitons is formed by the surfaces of revolution about a vertical axis, or equivalently, the radial solutions of Equation (1). There are two types of rotational \( \alpha \)-translating solitons. The first type are convex entire graphs on \( \mathbb{R}^2 \) and the second ones are of winglike-shape: we refer to the reader to [4] when \( \alpha = 1 \) and to [20, Th. 1.1] for the general \( \alpha > 0 \). We are interested in the first ones to find \( C^0 \) estimates of the solutions of (1). After a horizontal translation, we suppose that the rotation axis is the \( z \)-axis.

**Definition 3.2.** For each \( \alpha > 0 \), and up to a constant, there exists an entire radially symmetric strictly convex solution of (1), namely, \( b = b(r), r^2 = x^2 + y^2 \), with a global minimum in the \( z \)-axis. We call the corresponding graph \( B \) as the \( \alpha \)-bowl soliton.

Up to a constant, suppose that the minimum of \( b = b(r) \) is the value 0, that is, \( b(0) = 0 \), so the origin of \( \mathbb{R}^3 \) is the minimum of \( B \). For \( t > 0 \), we intersect \( B \) with the horizontal plane of equation \( z = t \) obtaining a compact cap \( B_R \) whose boundary is a circle of radius \( R > 0 \) included in the plane \( z = t \). Moreover, \( R \to \infty \) as \( t \to \infty \).

As a conclusion, we have proved that for any \( R > 0 \), there exists a radial solution of (1)-(2) with \( \phi = 0 \) on \( \partial \Omega \) and \( \Omega \) is a disk of radius \( R > 0 \).

**Proposition 3.3.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain. If \( u \) is a solution of (1)-(2), we have
1. The solution is unique.

2. There exists \( C = C(\Omega, \varphi) \) a constant depending only on \( \varphi \) and \( \Omega \) such that
\[
C \leq u \leq \max_{\partial \Omega} \varphi \quad \text{in} \ \Omega. \tag{8}
\]

3. The maximum of the gradient is attained at some boundary point:
\[
\max_{\Omega} |Du| = \max_{\partial \Omega} |Du|.
\]

Proof. 1. The uniqueness of solutions holds because the right hand side of (1) is non-decreasing on \( u \) \( (\text{Th. 10.1}) \).

2. Since the right hand side of (1) is non-negative, then \( \sup_{\Omega} u = \max_{\partial \Omega} u = \max_{\partial \Omega} \varphi \). The lower estimate for \( u \) is obtained by using \( \alpha \)-bowl solitons as comparison surfaces. Indeed, let us take a round disc \( D_R \subset \mathbb{R}^2 \) of radius \( R > 0 \) sufficiently big so \( \Omega \subset D_R \). Consider the \( \alpha \)-bowl soliton \( B_R \) given by \( b(r) \) defined in \( D_R \) with \( b|_{\partial D_R} = 0 \) and denote by \( b_m \) the minimum value of \( b \). We move down \( B_R \) sufficiently so \( \Sigma_u \) lies above \( B_R \), that is, if \( (x, y, z) \in \Sigma_u \), \( (x, y, z') \in B_R \), then \( z > z' \). Then we move up \( B_R \) until the first touching point with \( \Sigma_u \). If the first contact occurs at some interior point, then the touching principle implies \( \Sigma_u \subset B_R \). In other case, if \( \Sigma_u \not\subset B_R \), the first contact point occurs when \( B_R \) touches a boundary point of \( \Sigma_u \). In both cases, we conclude \( b_m \leq u - \min_{\partial \Omega} \varphi \) and consequently \( C := b_m + \min_{\partial \Omega} \varphi \leq u \).

3. Equation (1) can be expressed as
\[
(1 + |Du|^2)\Delta u - u_i u_j u_{ij} - (1 + |Du|^2) \frac{3 - \alpha}{2} u_i (1 + |Du|^2)^{\frac{1-\alpha}{2}} = 0. \tag{9}
\]
where \( u_i = \partial u / \partial x_i, \ i = 1, 2 \), and we assume the summation convention of repeated indices. Define the function \( v^k = u_k, \ k = 1, 2 \) and we differentiate (9) with respect to \( x_k \), obtaining for each \( k = 1, 2 \),
\[
((1 + |Du|^2)\delta_{ij} - u_i u_j) v^k_{ij} + 2 \left( u_i \Delta u - u_j u_{ij} - \frac{3 - \alpha}{2} u_i (1 + |Du|^2)^{\frac{1-\alpha}{2}} \right) v^k_i = 0, \tag{10}
\]
where \( \delta_{ij} \) is the Kronecker delta. We deduce that \( |v^k| \) and then \( |Du| \) has not an interior maximum. In particular, if \( u \) is a solution of (1), the maximum of
$|Du|$ on the compact set $\overline{\Omega}$ is attained at some boundary point, proving the result.

\[ \square \]

4 Proof of Theorem 1.2

In this section we prove Theorem 1.2 in a successive number of steps. First, we define the operator

\[ Q[u] = \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) - \left( \frac{1}{\sqrt{1 + |Du|^2}} \right)^{\alpha}. \]

For the Perron process we need to have a subsolution of (1)-(2). In the next result, $f$ is not necessarily a convex function.

**Proposition 4.1.** Let $\Omega \subset \mathbb{R}^2$ be a strip. If $f$ is a continuous function defined in $\mathbb{R}$, then there exists a solution $v^0$ of the Dirichlet problem

\[ \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0 \quad \text{in} \ \Omega \]

\[ u = \varphi_f \quad \text{on} \ \partial \Omega \]

with the property $f(x) < v^0(x, y)$ for all $(x, y) \in \Omega$.

This result was proved in [5] following ideas of [11]: see Remark 2 in [5] where $v^0$ is called the maximal solution of (11).

We begin with the classical Perron method of sub and supersolutions for the Dirichlet problem (11)-(12): see [6, pp. 306-312], [7, Sec. 6.3]). Let $u \in C^0(\overline{\Omega})$ be a continuous function and let $D$ be a closed round disk in $\Omega$. We denote by $\bar{u} \in C^2(D)$ the unique solution of the Dirichlet problem

\[ \begin{cases} Q[\bar{u}] = 0 & \text{in} \ D \\ \bar{u} = u & \text{on} \ \partial D \end{cases} \]
whose existence is assured by Theorem 1.1 and its uniqueness by Proposition 3.3. We extend \( \bar{u} \) by continuity to \( \Omega \) by defining \( M_D[\bar{u}] \) in \( \Omega \) as

\[
M_D[\bar{u}] = \begin{cases} 
\bar{u}, & \text{in } D \\
u, & \text{in } \Omega \setminus D.
\end{cases}
\]

The function \( u \) is said to be a supersolution in \( \Omega \) if satisfies

\[
M_D[u] \leq u
\]

for every closed round disk \( D \) in \( \Omega \).

**Example.** For any domain \( \Omega \subset \mathbb{R}^2 \), the function \( u = 0 \) in \( \Omega \) is a supersolution in \( \Omega \). This is because if \( D \subset \Omega \) is a closed round disk, then \( \bar{u} < 0 \) since \( Q[0] < Q[\bar{u}] = 0 \) and the maximum principle applies. Thus \( M_D[u] \leq 0 \).

Moreover, for each \( p \in \Omega \), there exists a supersolution \( u \) with \( u(p) < 0 \). Indeed, let \( D \subset \Omega \) be a closed round disk centered at \( p \), which suppose to be the origin of \( \mathbb{R}^2 \). Let \( b = b(r) \) be that \( \alpha \)-bowl soliton with \( b|_{\partial D} = 0 \). Then the function \( u \) defined as \( u = b \) in \( D \) and \( u = 0 \) in \( \Omega \setminus D \) is a supersolution.

**Definition 4.2.** Let \( u \in C^0(\overline{\Omega}) \). We say \( u \) is a superfunction relative to \( f \) if \( u \) is a supersolution in \( \Omega \) and \( f \leq u \) on \( \partial \Omega \). Denote by \( S_f \) the class of all superfunctions relative to \( f \), that is,

\[
S_f = \{ u \in C^0(\overline{\Omega}) : M_D[u] \leq u \text{ for every closed round disk } D \subset \Omega, f \leq u \text{ on } \partial \Omega \}.
\]

**Lemma 4.3.** The set \( S_f \) is not empty.

**Proof.** We prove that \( v^0 \in S_f \), where \( v^0 \) is the minimal solution given in Proposition 4.1. Let \( D \subset \Omega \) be a closed round disk. Since \( v^0 \) is a minimal surface, then \( Q[v^0] < 0 \) and because \( \bar{v}^0 = v^0 \) in \( \partial D \), the maximum principle implies \( M_D[v^0] = \bar{v}^0 \leq v^0 \) in \( D \). On the other hand, \( v^0 = f \) on \( \partial \Omega \), proving definitively that \( v^0 \in S_f \). \( \Box \)

We now give some properties about superfunctions whose proofs are straightforward: in the case of the constant mean curvature equation, we refer [14]; in the context of \( \alpha \)-translating solitons, see [12, Lems. 4.2–4.4].

**Lemma 4.4.**

1. If \( \{u_1, \ldots, u_n\} \subset S_f \), then \( \min\{u_1, \ldots, u_n\} \in S_f \).

2. The operator \( M_D \) is increasing in \( S_f \).
3. If \( u \in S_f \) and \( D \) is a closed round disk in \( \Omega \), then \( M_D[u] \in S_f \).

We begin with the proof of Theorem 1.2. We take \( d(\alpha) \) the number given in Proposition 2.1 and let \( f \) be a convex function on \( \mathbb{R} \).

Consider the family of \( \alpha \)-grim reaper \( w_\theta \) (Definition 2.3) and recall that \( w_\theta \) is defined in \( \mathbb{R}^2 \) if \( \alpha > 1 \) or in the strip \( \Omega_{d,\theta} \). In particular, by the definition of \( \Omega_d \) in (7), we find \( \Omega_m \subset \Omega_d \subset \Omega_\theta \) for any \( \theta \). Thus it makes sense to restrict \( w_\theta \) to the strip \( \Omega_m \) and we keep the same notation for its restriction in \( \Omega_m \). Consequently \( w_\theta \) is a linear function on \( \partial \Omega_m \) and the boundary of \( \Sigma_{w_\theta} \) consists of two parallel lines.

Consider the subfamily of \( \alpha \)-grim reapers \( G = \{w_\theta : w_\theta \leq f \text{ on } \partial \Omega_m, \theta \in (-\pi/2, \pi/2)\} \).

Two observations are needed to state:

1. The set \( G \) is not empty because \( f \) is convex.

2. Let \( v^0 \) be the minimal surface given in Proposition 4.1 with \( v^0 = f \) on \( \partial \Omega_m \).

   Then \( Q[v^0] < 0 \). Since \( Q[w_\theta] = 0 \) for all \( w_\theta \in G \), the comparison principle asserts that \( w_\theta < v^0 \) in \( \Omega_m \) for all \( \theta \in (-\pi/2, \pi/2) \). This implies that \( v^0 \) will play the role of a subsolution for (1)-(2).

We are going to construct a solution of Equation (1) between the \( \alpha \)-grim reapers of \( G \) and the minimal surface \( v^0 \). Let

\[
S_f^* = \{ u \in S_f : w_\theta \leq u \leq v^0, \forall w_\theta \in G \}.
\]

We point out that \( S_f^* \) is not empty because \( v^0 \in S_f^* \).

**Lemma 4.5.** The set \( S_f^* \) is stable for the operator \( M_D \), that is, if \( u \in S_f^* \), then \( M_D[u] \in S_f^* \).

**Proof.** Let \( u \in S_f^* \). We know by Lemma 4.4 that \( M_D[u] \in S_f \). On the other hand, since \( w_\theta \leq u \leq v^0 \) and because \( M_D \) is increasing (Lemma 4.4 again), then for every closed round disk \( D \) in \( \Omega_m \) it follows that \( M_D[w_\theta] \leq M_D[u] \leq M_D[v^0] \). Finally \( M_D[w_\theta] = w_\theta \), so \( w_\theta \leq M_D[u] \), and since \( v^0 \) is a supersolution, we conclude \( M_D[v^0] \leq v^0 \), hence \( M_D[u] \leq v^0 \). This proves the result.  \( \square \)
We are going to complete the proof of Theorem 1.2.

**Proposition 4.6** (Perron process). The function \( u : \Omega_m \to \mathbb{R} \) given by

\[
v(x, y) = \inf \{ u(x, y) : u \in S_f^* \}
\]

is a solution of (1) with \( v = \varphi_f \) on \( \partial \Omega_m \).

**Proof.** The proof consists of two parts.

**Claim 1.** The function \( v \) is a solution of Equation (1).

The proof is standard and here we follow [7]. Let \( p \in \Omega_m \) be an arbitrary fixed point of \( \Omega_m \). Consider a sequence \( \{ u_n \} \subset S_f^* \) such that \( u_n(p) \to v(p) \) when \( n \to \infty \).

Let \( D \) be a closed round disk centered at \( p \) and contained in \( \Omega_m \). For each \( n \), define the function \( v_n(q) = \min \{ u_1(q), \ldots, u_n(q) \} \), \( q \in \overline{\Omega_m} \).

Then \( v_n \in S_f^* \) by Lemma 4.4. By the definition of \( M_D \), we deduce \( M_D[v_n](p) \to v(p) \) as \( n \to \infty \) (Lemma 4.5). Set \( V_n = M_D[v_n] \). Then \( \{ V_n \} \) is a decreasing sequence bounded from below by \( w_\theta \) for all \( w_\theta \in G \) and satisfying (1) in the disk \( D \). Consequently the functions \( V_n \) are uniformly bounded on compact sets \( K \) of \( D \). In each compact set \( K \), the norms of the gradients \( |DV_n| \) are bounded by a constant depending only on \( K \) and using Hölder estimates of Ladyzhenskaya and Ural’tseva, there exist uniform \( C^{1,\beta} \) estimates for the sequence \( \{ V_n \} \) on \( K \): see [21] for interior estimates for the mean curvature type equations in two variables and in the context of Equation (1), the estimates were proved in [8]. By compactness, there exists a subsequence of \( V_n \), that we denote \( V_n \) again, such that \( \{ V_n \} \) converges on \( K \) to a \( C^2 \) function \( V \) in the \( C^2 \) topology and by continuity, \( V \) satisfies (1). Moreover, by construction, at the fixed point \( p \) we have \( V(p) = v(p) \).

It remains to prove that \( V = v \) in \( \text{int}(D) \), not only at the fixed point \( p \). For \( q \in \text{int}(D) \), we do a similar argument for \( q \), so let \( \{ \tilde{u}_n \} \subset S_f^* \), \( \tilde{u}_n(q) \to v(q) \). Let \( \tilde{v}_n = \min \{ V_n, \tilde{u}_n \} \) and \( \tilde{V}_n = M_D[\tilde{v}_n] \). Again \( \tilde{V}_n \) converges on \( D \) in the \( C^2 \) topology to a \( C^2 \) function \( \tilde{V} \) satisfying (1) and \( \tilde{V}(q) = v(q) \). By construction, \( \tilde{V}_n \leq \tilde{v}_n \leq V_n \), hence \( \tilde{V} \leq V \). Since \( v \leq \tilde{V} \), we infer \( \tilde{V}(p) = v(p) = V(p) \). Thus \( V \) and \( \tilde{V} \) coincide at an interior point of \( D \), namely, the point \( p \), and both functions \( V \) and \( \tilde{V} \) satisfy the \( \alpha \)-translating soliton equation (1). Because \( \tilde{V} \leq V \), the touching principle implies...
$V = \tilde{V}$ in $\text{int}(D)$. In particular, $V(q) = \tilde{V}(q) = v(q)$. This shows that $V = v$ in $\text{int}(D)$ and the claim is proved.

Claim 2. The function $v$ is continuous up to $\partial \Omega_m$ with $v = \varphi_f$ on $\partial \Omega_m$.

In order to finish the proof of Theorem 1.2 we prove that the function $v$ takes the value $f$ on $\partial \Omega_m$ and consequently $v$ is continuous up to $\partial \Omega_m$ proving that $v \in C^2(\Omega_m) \cap C^0(\overline{\Omega_m})$. Let us observe that the graph of $\varphi_f$ consists of two copies of the graph of $f$, namely,

$$\Gamma_{\varphi_f} = \Gamma_1 \cup \Gamma_2 = \{(x, m, f(x)) : x \in \mathbb{R}\} \cup \{(x, -m, f(x)) : x \in \mathbb{R}\}.$$  

Let $p = (x_0, m) \in \partial \Omega_m$ be a boundary point of $\Omega_m$ (similar argument if $p = (x_0, -m)$). Because of the convexity of $f$, in the plane of equation $y = m$ the tangent line $L_p$ to the planar curve $\Gamma_1$ leaves $\Gamma_1$ above $L_p$. Taking into account the symmetry of $\varphi_f$ and the convexity of $f$, there exists an $\alpha$-grim reaper $w^p_\theta$ such that $w^p_\theta(p) = f(x_0)$ and $w^p_\theta < f$ in $\Gamma_{\varphi_f} \setminus \{(x_0, m, f(x_0)), (x_0, -m, f(x_0))\}$, or in other words, $\partial \Sigma_{w^p_\theta}$ lies strictly below $\partial \Sigma_v$ except at the points $(x_0, m, f(x_0))$ and $(x_0, -m, f(x_0))$, where $\Sigma_{w^p_\theta}$ and $\Sigma_v$ coincide.

Therefore the function $w^p_\theta$ and the minimal surface $v^0$ form a modulus of continuity in a neighbourhood of $p$, namely, $w^p_\theta \leq v \leq v^0$. Because $w^p_\theta(p) = v^0(p) = f(p)$, we infer that $v(p) = f(p)$ and this completes the proof of Theorem 1.2.

Acknowledgements

The author has been partially supported by MEC-FEDER grant no. MTM2017-89677-P

References

[1] S. J. Altschuler, L. F. Wu, Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle, Calc. of Var. 2 (1994) 101–111
[2] S. Angenent, On the formation of singularities in the curve shortening flow, J. Differential Geometry, 33 (1991) 601–633.

[3] M. Bergner, The Dirichlet problem for graphs of prescribed anisotropic mean curvature in $\mathbb{R}^{n+1}$, Analysis (Munich) 28 (2008) 149–166.

[4] J. Clutterbuck, O. Schnürer, F. Schulze, Stability of translating solutions to mean curvature flow, Calc. Var. 29 (2007) 281–293.

[5] P. Collin, Deux exemples de graphes de courbure moyenne constante sur une bande de $\mathbb{R}^2$, C. R. Acad. Sci. Paris. 311 (1990) 539–542.

[6] R. Courant, D. Hilbert, Methods of Mathematical Physics, Vol. II, Interscience, New York, 1962.

[7] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Second edition. Springer-Verlag, Berlin, 1983.

[8] C. Gui, H. Jian, H. J. Ju, Properties of translating solutions to mean curvature flow, Discrete Contin. Dyn. Syst. 28B (2010) 441-453.

[9] G. Huisken, C. Sinestrari, Mean curvature flow singularities for mean convex surfaces, Calc. Var. 8 (1999) 1–14.

[10] T. Ilmanen, Elliptic regularization and partial regularity for motion by mean curvature, Mem. Amer. Math. Soc. 108 (1994).

[11] H. Jenkins, J. Serrin, Variational problems of minimal surface type, II. Boundary value problems for the minimal surface equation, Arch. Rat. Mech. Anal. 12 (1963) 185–212.

[12] H-Y. Jian, H-J. Ju, Existence of translating solutions to the flow by powers of mean curvature on unbounded domains, J. Differential Equations 250 (2011) 3967–3987.

[13] H. Ju, Y. Liu, Dirichlet problem for anisotropic prescribed mean curvature equation on unbounded domains, J. Math. Anal. Appl. 439 (2016) 709–724.

[14] R. López, Constant mean curvature graphs on unbounded convex domains, J. Differential Equations 171 (2001) 54–62.
[15] T. Marquardt, Remark on the anisotropic prescribed mean curvature equation on arbitrary domain, Math. Z. 264 (2010) 507–511.

[16] J. B. Serrin, The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables, Phil. Trans. R. Soc. Lond. 264 (1969) 413–496.

[17] F. Schulze, Evolution of convex hypersurfaces by powers of the mean curvature, Math. Z. 251 (2005) 721–733.

[18] F. Schulze, Nonlinear evolution by mean curvature and isoperimetric inequalities, J. Differential Geom. 79 (2008) 197–241.

[19] W. Sheng, C. Wu, On asymptotic behavior for singularities of the powers of mean curvature flow, Chin. Ann. Math. Ser. B 30 (2009) 51–66.

[20] W. Sheng, C. Wu, Rotationally symmetric translating soliton of $H^k$-flow, Sci. China Math. 53 (2010) 1011–1016.

[21] L. Simon, Equations of mean curvature type in 2 independent variable, Pacific J. Math. 69 (1977) 245–268.

[22] J. Spruck, L. Xiao, Complete translating solitons to the mean curvature flow in $\mathbb{R}^3$ with nonnegative mean curvature. Preprint arXiv 1703.01003.

[23] X-J. Wang, Convex solutions to the mean curvature flow, Ann. of Math. 173 (2011) 1185–1239.

[24] B. White, The nature of singularities in mean curvature flow of mean convex surfaces, J. Amer. Math. Soc. 16 (2003) 123–138.