Block-transitive 3-(v, k, 1) designs associated with alternating groups

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Abstract
Let \( D \) be a nontrivial 3-(v, k, 1) design admitting a block-transitive group \( G \) of automorphisms. A recent work of Gan and the second author asserts that \( G \) is either affine or almost simple. In this paper, it is proved that if \( G \) is almost simple with socle an alternating group, then \( D \) is the unique 3-(10, 4, 1) design, and \( G = \text{PGL}(2, 9) \), \( M_{10} \) or \( \text{Aut}(A_6) = S_6 : Z_2 \), and \( G \) is flag-transitive.

Keywords Block-transitive designs · t-(v, k, 1) designs · Steiner t-designs · Alternating groups · Primitive groups

Mathematics Subject Classification 05B05 · 20B25

1 Introduction
A t-(v, k, \( \lambda \)) design \( D = (\mathcal{P}, \mathcal{B}) \) is an incidence structure consisting of a set \( \mathcal{P} \) of \( v \) points and a set \( \mathcal{B} \) of \( k \)-subsets of \( \mathcal{P} \) called blocks such that, each block in \( \mathcal{B} \) has size \( k \), and each \( t \)-subset of \( \mathcal{P} \) lies in exactly \( \lambda \) blocks from \( \mathcal{B} \). Design \( D \) is said to be trivial if \( \mathcal{B} \) consists of all the \( k \)-subsets of \( \mathcal{P} \). A flag of \( D \) is a pair \((\alpha, B)\) where \( \alpha \) is a point and \( B \) is a block.
containing \( \alpha \). An automorphism of \( D \) is a permutation on \( P \) which permutes the blocks among themselves. For a subgroup \( G \) of the automorphism group \( \text{Aut}(D) \) of \( D \), the design \( D \) is said to be \( G \)-\textit{block-transitive} if \( G \) acts transitively on the set of blocks, and is said to be \( G \)-\textit{block-transitive} if it is \( \text{Aut}(D) \)-block-transitive. The point-transitivity and flag-transitivity are defined similarly. Clearly, flag-transitivity infers block-transitivity, and by a theorem of Block (see [1]) block-transitivity implies point-transitivity.

For a nontrivial \( G \)-block-transitive \( t-(v, k, \lambda) \) design, it is proved by Cameron and Praeger [6] that \( G \) is \( \lceil t/2 \rceil \)-homogeneous on the point set and \( t \leq 7 \), and while if \( G \) is flag-transitive, then \( G \) is \( \lceil (t + 1)/2 \rceil \)-homogeneous and \( t \leq 6 \). Furthermore, Cameron and Praeger [6] conjectured that there exists no nontrivial block-transitive \( 6-(v, k, \lambda) \) designs. Huber [20] confirmed the nonexistence of nontrivial block-transitive \( 7-(v, k, 1) \) designs, and then in [19] he proved that apart from \( G = P\Gamma L_2(p^e) \) with \( p \in \{2, 3\} \) and \( e \) an odd prime power, there exists no nontrivial \( G \)-block-transitive \( 6-(v, k, 1) \) designs. The exceptional case was studied by Tan, Liu and Chen [36] for the case \( k \leq 10000 \).

A \( t-(v, k, \lambda) \) design with \( \lambda = 1 \) is called a \textit{Steiner} \( t \)-\textit{design}, and a \( 2-(v, k, 1) \) design is also called a \textit{linear space}. For a \( t-(v, k, 1) \) design, we need \( t \geq 2 \) and \( t < k < v \) to avoid trivial examples, and so a \( t-(v, k, 1) \) design is said to be \textit{nontrivial} if \( t \geq 2 \) and \( t < k < v \). There have been a great deal of efforts to classify block-transitive \( 2-(v, k, 1) \) designs in the past fifty years. For example, point \( 2 \)-transitive \( 2-(v, k, 1) \) designs are classified by Buekenhout et al [3]. In 2001, Camina and Praeger [7] proved that if a \( 2-(v, k, 1) \) design is \( G \)-block-transitive and \( G \)-point-quasiprimitive, then \( G \) is either an affine group or an almost simple group. This result has inspired the study of \( G \)-block-transitive \( 2-(v, k, 1) \) designs with \( G \) an almost simple group, such as alternating groups [9], sporadic simple groups [8], simple groups of Lie type of small ranks [14, 25–30, 32, 38–40], and large dimensional classical groups [10].

Compared with \( 2-(v, k, 1) \) designs, the results for block-transitive \( t-(v, k, 1) \) designs with \( t \in \{3, 4, 5\} \) are rare. The flag-transitive \( t-(v, k, 1) \) designs for \( t \in \{3, 4, 5\} \) has been classified by Huber [15–17]. Note that 1-homogenous groups are simply transitive groups, while 2-homogenous groups are almost 2-transitive (with only one affine group as exception, see [21]), and in particular, 2-homogenous groups are either affine or almost simple. Consequently, for a \( G \)-block-transitive \( t-(v, k, 1) \) design with \( t \in \{4, 5\} \), \( G \) is either affine or almost simple. Huber [18] proved that if \( G \) is an affine group, then there exists no \( G \)-block-transitive nontrivial \( t-(v, k, 1) \) design for \( t \in \{4, 5\} \) except the one-dimensional affine case. Thus the study of \( G \)-block-transitive \( t-(v, k, 1) \) design with \( t \in \{4, 5\} \) has been essentially reduced to the case that \( G \) is an almost simple 2-transitive group. Based on the study of Cameron and Praeger [5] on block-transitive and point-imprimitive \( t-(v, k, \lambda) \) designs, it is proved by Mann and Tuan [33, Corollary 2.3(a)] that there exists no block-transitive and point-imprimitive \( 3-(v, k, 1) \) design. Very recently, it is shown by Gan and the second author [13] that for a nontrivial \( G \)-block-transitive \( 3-(v, k, 1) \) design, the group \( G \) is either affine or almost simple. This suggests the following problem.

\textbf{Problem 1.1} Classify nontrivial \( G \)-block-transitive \( 3-(v, k, 1) \) designs, where \( G \) is an almost simple group.

Those \( \text{PSL}_n(q) \)-block-transitive \( 3-(v, k, 1) \) designs with \( v = (q^n - 1)/(q - 1) \) are determined by Tang, Liu and Wang [35]. This paper is devoted to solve Problem 1.1 in the case where the socle of \( G \) is an alternating group. Note that \( \text{Aut}(A_6) = S_6 \rtimes Z_2 \), and following
Atlas [11], Aut(A6) has three subgroups of index 2, namely A6,2i with i ∈ {1, 2, 3}. Moreover, A6,21 ∼= PΓL2(9) ∼= S6, A6,22 ∼= PGL(2, 9) and A6,23 ∼= M10 (M10 is the stabilizer of the Mathieu group M11 on its natural action of degree 11).

**Theorem 1.2** Let G be an almost simple group with alternating socle A6(n ≥ 5). Suppose that D is a nontrivial G-block-transitive 3-(v, k, 1) design. Then G = PGL2(9), M10 or S6 : Z2, and D is G-flag-transitive with parameters v = 10 and k = 4.

The 3-(10, 4, 1) design appearing in Theorem 1.2 is constructed in Example 1 in Sect. 3, and it is a special case of a family of flag-transitive 3-(v, k, 1) designs on PSL2(q) (noticing that PSL2(9) ∼= A6), which is introduced in [15, Theorem (2)] and [22, Theorem 3(b)].

Let D be a nontrivial G-block-transitive t-(v, k, 1) design, where G is an almost simple group with alternating socle. According to Kantor [22, Theorem 3], there exists no such design D if t ≥ 4. It is proved by Camina, Neumann and Praeger [9] that if t = 2 then G = A7 or A8 and D is the 1-skeleton of the 3-dimensional projective geometry over F2, which is G-flag-transitive and has parameter v = 15 and k = 3. Therefore, Theorem 1.2 together with [22, Theorem 3] and [9] give the complete classification of such designs.

## 2 Preliminaries

For a finite group G, the socle of G, denoted by Soc(G), is the product of all its minimal normal subgroups, and the group G is said to be **almost simple** if Soc(G) is a nonabelian simple group. For an element g of G, let C(G)(g) denote the centralizer of g in G. If G is a permutation group on a finite set Ω, then we use FixΩ(g) to denote the set of points in Ω fixed by g. For a point α ∈ Ω, denote by Gα the stabilizer of α in G, and by αG the orbit of G containing α. The alternating group and the symmetric group on a finite set Ω is denoted by Alt(Ω) and Sym(Ω), respectively. For notations of groups, we follow Atlas [11]. In particular, for two finite groups H and K, we use H : K to denote the semidirect product of H by K. For a real number x, let ⌊x⌋ be the largest integer less than or equal to x.

### 2.1 3-(v, k, 1) designs

In this subsection, we introduce some results about 3-(v, k, 1) designs. The following fact is well known, and we refer the readers to [12, p. 180].

**Lemma 2.1** Let D = (P, B) be a 3-(v, k, 1) design.

(a) |B| = \frac{v(v - 1)(v - 2)}{k(k - 1)(k - 2)}.

(b) For every point α in P, the number of blocks containing α is

\[ \lambda_1 := \frac{(v - 1)(v - 2)}{(k - 1)(k - 2)} \]

(c) For two distinct points α and β in P, the number of blocks containing α and β is

\[ \lambda_2 := \frac{v - 2}{k - 2} \]

The parameters v and k have an important relation.
Lemma 2.2 ([4]) Let \( D \) be a 3-(v, k, 1) design. Then
\[
v \geq k^2 - 3k + 4.
\]
In particular, \( k \leq \lfloor \sqrt{v} \rfloor + 2 \).

The idea of the following lemma is original from [9, Proposition 2.7].

Lemma 2.3 Let \( D = (P, B) \) be a G-block-transitive 3-(v, k, 1) design. Suppose that G has an element \( g \neq 1 \) such that \( \langle g \rangle \) has an orbit on \( P \) with length at least 3. Then
\[
\frac{(v - 1)(v - 2)}{k(k - 1)(k - 2)} \leq \frac{|G|}{|C_G(g)|}.
\]

Proof Let \( \Omega \) be an orbit of \( \langle g \rangle \) with length at least 3. Take three distinct points \( \alpha, \beta, \gamma \in \Omega \) such that \( \beta = \alpha^g \) and \( \gamma = \alpha^{g^2} \). Let \( X = G_{\alpha} \) and let \( U = G_{\alpha} \cap G_{\beta} \cap G_{\gamma} \). Note that
\[
\alpha^{g^i}C_{G_{\alpha}}(g) = \alpha^{C_{G_{\alpha}}(g)g^i} = \alpha^{g^i} \text{ for all } i \in \{1, 2\}.
\]
Therefore, \( C_{G_{\alpha}}(g) \) fixes \( \alpha^g = \beta \) and \( \alpha^{g^2} = \gamma \), which implies that \( U \geq C_{G_{\alpha}}(g) = X \cap C_G(g) \). Then we have
\[
\frac{|X|}{|U|} \leq \frac{|X|}{|X \cap C_G(g)|} = \frac{|XC_G(g)|}{|C_G(g)|} \leq \frac{|G|}{|C_G(g)|}.
\]
Let \( B \) be the unique block containing the 3-subset \( \{\alpha, \beta, \gamma\} \). Then \( U \leq GB \). Hence
\[
|B| = \frac{|G|}{|GB|} \leq \frac{|G|}{|U|} = \frac{|G|}{|X|} \leq \frac{|G|}{|C_G(g)|}.
\]
This together with Lemma 2.1(a) prove the lemma. \( \square \)

Next we collect several important properties of block-transitive 3-(v, k, 1) designs from [13].

Lemma 2.4 ([13, Lemma 2.5]) Let \( D = (P, B) \) be a G-block-transitive 3-(v, k, 1) design. If there exists an element \( g \) of G with order 3 and \( g \) fixes no points, then \( k \) divides \( v \).

Lemma 2.5 ([13, Lemma 2.7]) Let \( D = (P, B) \) be a G-block-transitive 3-(v, k, 1) design. If \( v \) is divisible by \( k \) and 4, then \( D \) is G-flag-transitive.

Let \( G \) be a transitive permutation group on a set \( \Omega \) and let \( \alpha \in \Omega \). A \( G_{\alpha} \)-orbit on \( \Omega \) is called a suborbit of \( G \) relative to \( \alpha \), and the length of a \( G_{\alpha} \)-orbit on \( \Omega \) is called a subdegree of \( G \). Clearly, \( G_{\alpha} \) has a trivial orbit \( \{\alpha\} \). A subdegree is said to be nontrivial if the corresponding suborbit is not \( \{\alpha\} \).

Lemma 2.6 ([13, Lemma 2.8]) Let \( D = (P, B) \) be a G-block-transitive 3-(v, k, 1) design. Then
(a) \( (v - 1)(v - 2) \) divides \( k(k - 1)(k - 2)|G_{\alpha}| \) for every \( \alpha \in P \).
(b) \( (v - 1)(v - 2) \) divides \( k(k - 1)(k - 2)d(d - 1) \) for every nontrivial subdegree \( d \) of \( G \) on \( P \).

Lemma 2.7 ([13, Lemma 3.2]) Let \( D = (P, B) \) be a G-block-transitive 3-(v, k, 1) design. Suppose that \( 1 \neq H \leq G \) has an orbit on \( P \) with length at least 3. Then
\[
|\text{Fix}_P(H)| \geq \frac{2(v - k)}{k - 2} + k - 2.
\]
We end this subsection with an observation which is useful in computation.

**Lemma 2.8** Let \( v, k, c \) be positive integers such that \( 3 < k < v \) and 

\[
v \geq k^2 - 3k + 4, \text{ and } (v-1)(v-2) \leq ck(k-1)(k-2).
\]

Then \( v < (c+2)^2 \).

**Proof** Since \( v \geq k^2 - 3k + 4 \), we have \( v \geq (k-2)^2 \), and \( v-2 \geq k^2 - 3k + 2 = (k-1)(k-2) \). It follows that

\[
\sqrt{v-2} < \frac{(\sqrt{v}-2)(v-1)}{v-4} = \frac{v-1}{\sqrt{v}+2} \leq \frac{v-1}{k} \leq \frac{v-2}{k(k-1)(k-2)} \leq c.
\]

Hence, \( v < (c+2)^2 \). \( \square \)

### 2.2 Subdegrees of two classes of permutation groups

In this subsection, we study the subdegrees of two classes of permutation groups associated with the alternating groups and symmetric groups. The first one is well-known.

**Lemma 2.9** Let \( m \) and \( n \) be positive integers with \( n \geq 5 \) and \( n \geq 2m+1 \), and let \( \Omega \) be the set of \( m \)-subsets of \( \{1, 2, \ldots, n\} \). Let \( G = S_n \) and \( L = A_n \) acting naturally on \( \Omega \). Then both \( G \) and \( L \) have \( m+1 \) suborbits on \( \Omega \), and have the same nontrivial subdegrees:

\[
d_i = \binom{m}{i} \binom{n-m}{m-i} = \frac{m!(n-m)!}{i!(m-i)!(n+i-2m)!} \text{ for all } i \in \{0, 1, \ldots, m-1\}.
\]

**Proof** Fix \( \alpha \in \Omega \). For every \( i \in \{0, 1, \ldots, m\} \), let

\[
\Delta_i = \{ \beta \in \Omega \mid |\alpha \cap \beta| = i \}.
\]

Clearly, \( \Delta_m = \{\alpha\} \), and it is the trivial suborbit of \( G \) with respect to \( \alpha \). Note that

\[
G_\alpha = \text{Sym}(\alpha) \times \text{Sym}(\overline{\alpha}) \text{ with } \overline{\alpha} = \{1, 2, \ldots, n\} \setminus \alpha.
\]

Let \( i \in \{0, 1, \ldots, m-1\} \). Take \( \beta, \gamma \) from \( \Delta_i \) and let \( \beta_1 = \beta \cap \alpha, \beta_2 = \beta \cap \overline{\alpha}, \gamma_1 = \gamma \cap \alpha \) and \( \gamma_2 = \gamma \cap \overline{\alpha} \). Since \( |\beta_1| = |\gamma_1| = i \) and \( |\beta_2| = |\gamma_2| = m-i \), one can find permutations \( g_1 \in \text{Sym}(\alpha) \) and \( g_2 \in \text{Sym}(\overline{\alpha}) \) such that \( g_1 \text{ swaps } \beta_1 \text{ and } \gamma_1 \), and \( g_2 \text{ swaps } \beta_2 \text{ and } \gamma_2 \). Consequently, \( g_1g_2 \text{ swaps } \beta \text{ and } \gamma \). This implies that \( \beta \) and \( \gamma \) are in the same orbit of \( G_\alpha \). The arbitrariness of the choices of \( \beta \) and \( \gamma \) implies that \( \Delta_i \) is contained in a \( G_\alpha \)-orbit.

Furthermore, for every \( g \in G_\alpha \) and for every \( \beta \in \Delta_i \), since

\[
|\alpha \cap \beta^g| = |\alpha \cap \beta| = |\alpha \cap \beta| = i,
\]

we conclude that \( \Delta_i \) is exactly a \( G_\alpha \)-orbit. Therefore, \( G \) has \( m \) nontrivial suborbits \( \Delta_i \) for all \( i \in \{0, 1, \ldots, m-1\} \) on \( \Omega \), and the nontrivial subdegrees are

\[
d_i = |\Delta_i| = \binom{m}{i} \binom{n-m}{m-i} = \frac{m!(n-m)!}{i!(m-i)!(n+i-2m)!}.
\]

Now we consider the action of \( L \) on \( \Omega \). Note that for every \( \beta \in \Delta_i \) with \( i \in \{0, 1, \ldots, m-1\} \), writing \( \overline{\beta} = \{1, 2, \ldots, n\} \setminus \beta \), there holds that

\[
G_{\alpha \beta} := G_\alpha \cap G_\beta = \text{Sym}(\alpha \cap \beta) \times \text{Sym}(\alpha \cap \overline{\beta}) \times \text{Sym}(\overline{\alpha} \cap \beta) \times \text{Sym}(\overline{\alpha} \cap \overline{\beta}).
\]
Since \( n \geq 5 \), we conclude that both \( G_\alpha \) and \( G_{\alpha\beta} \) contain an odd permutation. It follows that \( |G_\alpha|/|L_\alpha| = 2 \) and \( |G_{\alpha\beta}|/|L_{\alpha\beta}| = 2 \), and hence \( |L_\alpha|/|L_{\alpha\beta}| = |G_\alpha|/|G_{\alpha\beta}| \). This implies that both \( G \) and \( L \) have \( m + 1 \) suborbits on \( \Omega \), and have the same nontrivial subdegrees.

\[ \Box \]

**Lemma 2.10** Let \( m \geq 3 \) be a positive integer and let \( \Omega \) be the set of partitions of \( \{1, 2, \ldots, 2m\} \) with 2 blocks of size \( m \). Let \( G = S_{2m} \) and \( L = A_{2m} \) acting naturally on \( \Omega \). Then both \( G \) and \( L \) have \( \lfloor m/2 \rfloor + 1 \) suborbits on \( \Omega \), and have the same nontrivial subdegrees:

\[
d_i := 2^{-\lfloor \frac{m}{2} \rfloor} \left( \frac{m!}{(m-i)!i!} \right)^2 \quad \text{for all } i \in \{1, 2, \ldots, \lfloor m/2 \rfloor \}.
\]

**Proof** Fix \( \alpha = \{V_1, V_2\} \in \Omega \). By \([37, \text{Lemma 2.9}]\), we conclude that \( \beta = \{W_1, W_2\} \) and \( \gamma = \{U_1, U_2\} \) are in the same \( G_\alpha \)-orbit if and only if there exist permutation matrices \( P \) and \( Q \) such that

\[
\begin{align*}
\left( \frac{|V_1 \cap U_1|}{|V_2 \cap U_1|} \right) \left( \frac{|V_1 \cap U_2|}{|V_2 \cap U_2|} \right) &= P \left( \frac{|V_1 \cap W_1|}{|V_2 \cap W_1|} \right) \left( \frac{|V_1 \cap W_2|}{|V_2 \cap W_2|} \right) Q.
\end{align*}
\]

Note that in matrices \( (|V_i \cap U_i|)_{2 \times 2} \) and \( (|V_i \cap W_i|)_{2 \times 2} \), the sum of every row equals to \( m \) and the sum of every column still equals to \( m \). Therefore, the nontrivial suborbits of \( G \) relative to \( \alpha \) are the following sets \( \Delta_i \) for all \( i \in \{1, 2, \ldots, \lfloor m/2 \rfloor \} \):

\[
\Delta_i := \left\{ \{W_1, W_2\} \mid \left( \frac{|V_1 \cap W_1|}{|V_2 \cap W_1|} \right) \left( \frac{|V_1 \cap W_2|}{|V_2 \cap W_2|} \right) = \left( \begin{array}{cc} i & m-i \\ m-i & i \end{array} \right) \right\}.
\]

Consequently, \( G \) has \( \lfloor m/2 \rfloor + 1 \) suborbits on \( \Omega \).

Now we compute the length of \( \Delta_i \) for every \( i \in \{1, 2, \ldots, \lfloor m/2 \rfloor \} \). Let \( \beta = \{W_1, W_2\} \in \Delta_i \). Then \( G_\alpha = \text{Sym}(V_1) \times \text{Sym}(V_2) : \langle g \rangle \) with \( g \) an involution swapping \( V_1 \) and \( V_2 \), and \( G_\beta = \text{Sym}(W_1) \times \text{Sym}(W_2) : \langle h \rangle \) with \( h \) an involution swapping \( W_1 \) and \( W_2 \). Let \( N = \text{Sym}(V_1) \times \text{Sym}(V_2) \) be the kernel of \( G_\alpha \) acting on the partition \( \alpha = \{V_1, V_2\} \), and let \( G_{\alpha\beta} = G_\alpha \cap G_\beta \). It is easy to find one involution \( x \) which swaps \( V_2 \setminus V_1 \) and \( V_1 \setminus V_2 \), and swaps \( V_1 \cap W_1 \) and \( V_2 \cap W_2 \). This involution \( x \) stabilizes the partitions \( \alpha \) and \( \beta \), and so \( x \in G_{\alpha\beta} \). Since \( x \) swaps \( V_1 \) and \( V_2 \), we see that \( x \notin N \) and hence

\[
G_{\alpha\beta} / (N \cap G_{\alpha\beta}) \cong G_{\alpha\beta}/N \cong S_2,
\]

which implies \( |G_{\alpha\beta}| = 2|N \cap G_{\alpha\beta}| \). Now for every \( z \in N \cap G_{\alpha\beta} \), we have \( V_1^z = V_1 \) and \( V_2^z = V_2 \), and either \( (W_1, W_2)^z = (W_1, W_2) \) or \( (W_1, W_2)^z = (W_2, W_1) \). Note that if \( (W_1, W_2)^z = (W_1, W_1) \), then

\[
i = |V_1 \cap W_1| = |(V_1 \cap W_1)^z| = |V_1 \cap W_1| = |V_1 \cap W_2| = m - i.
\]

Suppose that \( 2i \neq m \). Then \( i \neq m - i \), which implies that \( (W_1, W_2)^z = (W_1, W_2) \) for all \( z \in N \cap G_{\alpha\beta} \). Thus \( N \cap G_{\alpha\beta} = \text{Sym}(V_1 \cap W_1) \times \text{Sym}(V_1 \cap W_2) \times \text{Sym}(V_2 \cap W_1) \times \text{Sym}(V_2 \cap W_2) \), and so

\[
G_{\alpha\beta} = (\text{Sym}(V_1 \cap W_1) \times \text{Sym}(V_1 \cap W_2) \times \text{Sym}(V_2 \cap W_1) \times \text{Sym}(V_2 \cap W_2)) : \langle x \rangle.
\]

Then

\[
\frac{|\Delta_i|}{|G_{\alpha\beta}|} = \frac{2(m!)^2}{2((m-i)!i!)^2} = \left( \frac{m!}{(m-i)!i!} \right)^2 = 2^{-\lfloor \frac{m}{2} \rfloor} \left( \frac{m!}{(m-i)!i!} \right)^2.
\]
Suppose that $2i = m$. Now $|\Omega| = 2m = 4i$ and $|V_s \cap W_r| = i$ for all $s, r \in \{1, 2\}$. Write $\Omega = \{t_1, t_2, \ldots, t_{4i}\}$ and

\[
V_1 \cap W_1 = \{t_1, t_2, \ldots, t_i\}, \quad V_1 \cap W_2 = \{t_{i+1}, t_{i+2}, \ldots, t_{2i}\}, \\
V_2 \cap W_1 = \{t_{2i+1}, t_{2i+2}, \ldots, t_{3i}\}, \quad V_2 \cap W_2 = \{t_{3i+1}, t_{3i+2}, \ldots, t_{4i}\}.
\]

Let $z = (t_1, t_{i+1})(t_2, t_{i+2}) \cdots (t_{i+1}, t_{3i+1})(t_{i+2}, t_{3i+2}) \cdots (t_{3i}, t_{4i})$. Then $z$ swaps $V_1 \cap W_1$ and $V_1 \cap W_2$, and swaps $V_2 \cap W_1$ and $V_2 \cap W_2$. Since $z$ fixes both $V_1$ and $V_2$, and swaps $W_1$ and $W_2$, it follows that $z \in N = \text{Sym}(V_1) \times \text{Sym}(V_2)$, and $z$ fixes both partitions $\alpha = \{V_1, V_2\}$ and $\beta = \{W_1, W_2\}$, that is, $z \in G_{ab}$. Therefore, $z \in N \cap G_{ab}$. Let $M = \text{Sym}(V_1) \times \text{Sym}(V_2)$. Since $z$ swaps $W_1$ and $W_2$, we deduce that $M \cap N \cap G_{ab}$ has index 2 in $N \cap G_{ab}$. Note that $M \cap N \cap G_{ab} = M \cap N = \text{Sym}(V_1 \cap W_1) \times \text{Sym}(V_1 \cap W_2) \times \text{Sym}(V_2 \cap W_1) \times \text{Sym}(V_2 \cap W_2)$. Thus

\[
G_{ab} = (\text{Sym}(V_1 \cap W_1) \times \text{Sym}(V_1 \cap W_2) \times \text{Sym}(V_2 \cap W_1) \times \text{Sym}(V_2 \cap W_2)) : \langle z, x \rangle.
\]

For every $s \in \{1, 2\}$ and every $r \in \{1, 2\}$, since

\[
(V_s \cap W_r)^x = (V_{3-s} \cap W_{3-r})^z = V_{3-s} \cap W_r, \\
(V_s \cap W_r)^z = (V_s \cap W_{3-r})^x = V_{3-s} \cap W_r,
\]

we conclude that $zx = xz$ and so $\langle x, z \rangle \cong S^2$. Then

\[
|\Delta| = |\Delta_{m/2}| = \frac{|G_a|}{|G_{ab}|} = \frac{2(2^m)^2}{4((m-1)!/2!)^2} = 2^{-\frac{m}{2}} \left(\frac{m!}{(m-i)!i!}\right)^2.
\]

Now we consider the action of $L$ on $\Omega$. Since $m \geq 3$ and $i \leq m/2$, we see that $m-i \geq 2$, which implies both $G_a$ and $G_{ab}$ contain an odd permutation. It follows that $|G_a|/|L_a| = 2$ and $|G_{ab}|/|L_{ab}| = 2$, and hence $|L_a|/|L_{ab}| = |G_a|/|G_{ab}|$. This implies that both $G$ and $L$ have $|m/2| + 1$ suborbits on $\Omega$, and have the same nontrivial subdegrees. \qed

### 3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We make the following hypothesis throughout.

**Hypothesis 3.1** Let $G$ be an almost simple group such that $\text{Soc}(G) = A_n$ with $n \geq 5$. Suppose that $D = (P, B)$ is a nontrivial $G$-block-transitive $3-(v, k, 1)$ design. Let $\alpha$ be a point in $P$ and let $B$ be a block in $B$ such that $\alpha \in B$.

Since there exists no $G$-block-transitive and $G$-point-imprimitive $3-(v, k, 1)$ design by [33, Corollary 2.3(a)], we conclude that $G$ acts primitively on $P$, and hence the group $G_\alpha$ is maximal in $G$. Note that for every $n \neq 6$, the automorphism group of $A_n$ is $S_n$, while $\text{Aut}(A_6) \cong \text{PGL}_2(9) = A_6.S_2$. For convenience, we shall first deal with $A_5$ and $A_6$. For the case $n \geq 7$, from the classification of maximal subgroups of $A_n$ and $S_n$ given by Liebeck, Praeger and Saxl [24], we conclude that one of the following holds:

(a) $G_\alpha = (S_m \times S_{m-m}) \cap G$, with $2m < n$ and $m \geq 1$ (intransitive case);
(b) $G_\alpha = (S_m : S_\ell) \cap G$, with $n = m\ell$, and $m \geq 2$ and $\ell \geq 2$ (imprimitive case);
(c) $G_\alpha = AGL_d(p) \cap G$, with $n = p^d$ (affine case);
(d) $G_\alpha = (T^{\ell}:((\text{Out}(T) \times S_\ell))) \cap G$, with $T$ a nonabelian simple group, $\ell \geq 2$ and $n = |T|^{\ell-1}$ (diagonal case);
(e) $G_\alpha = (S_m : S_\ell) \cap G$, with $n = m\ell$, $m \geq 5$ and $\ell \geq 2$ (wreath case);
Suppose for a contradiction that the socle of $G$ is $A_5$. Then $G = A_5$ or $S_5$. Note that $k > 3$ as $D$ is nontrivial. By Lemma 2.2, that is, $v \geq k^2 - 3k + 4$, we conclude that $v \geq 8$. By Atlas [11], we see that $(G, G_\alpha) = (A_5, O, S_3, O)$ with $O \leq 2$, and $v = 10$. Again, from $v \geq k^2 - 3k + 4$ we conclude that $k = 4$. By Lemma 2.1(a), $|B| = 30$. Suppose first that $G = A_5$. Then $G_\alpha \cong S_3$ and $G_B \cong C_2$. Note that $A_5$ has only one non-conjugate subgroup isomorphic to $S_3$, and has only one non-conjugate subgroup isomorphic to $C_2$. Since $G$ acts transitively on $\mathcal{P}$, we may let

$$G = \{(2, 3, 5)(4, 7, 10)(6, 9, 8), (1, 2, 4)(3, 6, 7)(5, 8, 10)\},$$

$$G_\alpha = \{(2, 3, 5)(4, 7, 10)(6, 9, 8), (2, 4)(3, 10)(5, 7)(6, 8)\}.$$

(Computation in MAGMA[2] shows that, up to permutation equivalence, the above permutation group $G$ is the unique permutation representation of $A_5$ of degree 10.) Further, since $G$ acts transitively on $\mathcal{B}$, the group $H := \langle (2, 4)(3, 10)(5, 7)(6, 8) \rangle$ is the stabilizer of some block $C$ in $G$, that is, $G_C = H$. Then $G_C$ has six orbits on $\mathcal{P}$, namely,

$$O_1 := \{1\}, O_2 := \{9\}, O_3 := \{2, 4\}, O_4 := \{3, 10\}, O_5 := \{5, 7\}, O_6 := \{6, 8\}.$$

Since $|C| = k = 4$, there are 10 choices for $C$, namely $C_i$ for all $i \in \{1, 2, \ldots, 10\}$ as follows:

$$C_1 := \{1, 9, 2, 4\}, C_2 := \{1, 9, 3, 10\}, C_3 := \{1, 9, 5, 7\}, C_4 := \{1, 9, 6, 8\},$$

$$C_5 := \{2, 4, 3, 10\}, C_6 := \{2, 4, 5, 7\}, C_7 := \{2, 4, 6, 8\}, C_8 := \{3, 10, 5, 7\},$$

$$C_9 := \{5, 7, 6, 8\}, C_{10} := \{3, 10, 6, 8\}.$$ 

Computation in MAGMA[2] shows that

- $|\langle C_i \rangle^G| < 30$ for $i \in \{1, 2, 3, 4\}$;
- $|\langle C_5 \rangle^G| = 30$, while $\langle C_5 \rangle^G$ contains $\{1, 3, 6, 10\}$ and $\{1, 6, 8, 10\}$;
- $|\langle C_6 \rangle^G| = 30$, while $\langle C_6 \rangle^G$ contains $\{1, 2, 6, 10\}$ and $\{1, 6, 9, 10\}$;
- $|\langle C_7 \rangle^G| = 30$, while $\langle C_7 \rangle^G$ contains $\{1, 1, 6, 10\}$ is not contained in any block in $\langle C_7 \rangle^G$;
- $|\langle C_8 \rangle^G| = 30$, while $\langle C_8 \rangle^G$ contains $\{1, 1, 6, 10\}$ is not contained in any block in $\langle C_8 \rangle^G$;
- $|\langle C_9 \rangle^G| = 30$, while $\langle C_9 \rangle^G$ contains $\{1, 2, 6, 10\}$ and $\{1, 6, 9, 10\}$;
- $|\langle C_{10} \rangle^G| = 30$, while $\langle C_{10} \rangle^G$ contains $\{1, 3, 6, 10\}$ and $\{1, 6, 8, 10\}$.

Therefore, the above computation results imply that the block $C \neq C_i$ for any $i \in \{1, 2, \ldots, 10\}$, a contradiction.

The case $G = S_5$ is treated similarly, and computation in MAGMA[2] shows that the case $G = S_5$ is also impossible. 

For the case $A_6$, it turns out that there exists an example for $\mathcal{D}$.

Note that $A_6 \cong \text{PSL}_2(9)$ and $\text{Aut}(A_6) \cong S_6 : Z_2$.

**Example 1** Let $G$ be a subgroup of $S_{10}$ generated by permutations

$$(3, 6, 8, 5, 7, 10, 9, 4), (1, 8, 2)(3, 4, 5)(6, 10, 7), (3, 7)(4, 6)(5, 10).$$
Computation in MAGMA [2] shows that $G \cong S_6 : Z_2$. Let $D_{3,10,4} = (P, B)$ where

$P = \{1, 2, \ldots, 10\}$,

$B = \{\{1, 5, 7, 6\}, \{1, 7, 8, 10\}, \{3, 6, 8, 10\}, \{1, 3, 4, 10\}, \{1, 5, 9, 10\}, \{2, 6, 7, 8\}, \{1, 3, 5, 8\}, \{5, 6, 8, 9\}, \{2, 4, 7, 10\}, \{4, 5, 7, 8\}, \{1, 3, 6, 9\}, \{1, 4, 7, 9\}, \{3, 7, 8, 9\}, \{2, 5, 8, 10\}, \{1, 2, 6, 10\}, \{2, 3, 4, 8\}, \{3, 2, 10, 9\}, \{4, 8, 9, 10\}, \{2, 3, 5, 6\}, \{3, 5, 7, 10\}, \{1, 4, 6, 8\}, \{4, 5, 6, 10\}, \{2, 4, 9, 6\}, \{2, 5, 7, 9\}, \{1, 2, 3, 7\}, \{1, 2, 8, 9\}, \{1, 2, 4, 5\}, \{3, 4, 5, 9\}, \{6, 7, 9, 10\}, \{3, 4, 6, 7\}\}.$

It is a straightforward verification that $D$ is a 3-(10, 4, 1)-design. Computation in MAGMA [2] shows that $G$ acts transitively on $B$, and $G_{(1,5,6,7)}$ is generated by

$$\{(1, 5)(2, 4)(3, 8), (1, 6, 5)(2, 9, 4, 3, 10, 8), (2, 8)(3, 4)(6, 7)\},$$

which implies that $G_{(1,5,6,7)}$ is transitive on $\{1, 5, 6, 7\}$. Therefore, $D$ is $G$-flag-transitive. Moreover, computation in MAGMA [2] shows that both $A_6, S_2 \cong \text{PGL}(2, 9)$ and $A_6, S_3 \cong \text{M}_{10}$ acts flag-transitively on $D$, and $G$ is maximal in $S_{10}$, which implies that $\text{Aut}(D_{3,10,4}) = G = S_6 : Z_2$.

**Lemma 3.3** Suppose that $\text{soc}(G) = A_6$. Then $G = \text{PGL}_2(9)$, $M_{10}$ or $S_6 : Z_2$, and $D \cong \text{Aut}(D_{3,10,4})$.

**Proof** By Atlas [11], the possibilities for $G$ are $A_6$, $A_6, S_2 \cong \text{PGL}(2, 9)$, $A_6, S_3 \cong \text{M}_{10}$ and $\text{Aut}(A_6) \cong S_6 : Z_2$.

Since $k > 3$ and $v \geq k^2 - 3k + 4$, we have $v \geq 8$.

$G = A_6$. Since $v \geq 8$, by Atlas [11] we have $(G, v) = (3^2 : 4, 10)$ or $(S_4, 15)$.

Suppose $(G, v) = (3^2 : 4, 10)$. Then $k = 4$ as $v \geq k^2 - 3k + 4$. By Lemma 2.1(a) we have $|B| = 30$, and so $|G_B| = 12$. Computation in MAGMA [2] shows that $G$ has two non-conjugate subgroups of order 12, and both of them have two orbits on $P$ with lengths 4 and 6. This implies that $G_B$ is transitive on $B$ and hence $D$ is $G$-flag-transitive. From [15, p. 208], we see that $G$ should be 3-homogeneous on $P$. However, computation in MAGMA [2] shows that $A_6$ is not 3-homogeneous on a set of 10 points, a contradiction. Actually, computation in MAGMA [2] shows that for those two subgroups of order 12 in $A_6$, their orbits of length 4 admits setwise stabilizer of order 24 in $A_6$.

Suppose $(G, v) = (S_4, 15)$. Then $k = 4$ or 5. However, by Lemma 2.1(c), $v - 2$ is divisible by $k - 2$. Thus, both $k = 4$ and $k = 5$ are impossible.

$G = S_6$. Since $v \geq 8$, by Atlas [11] we have $(G, v) = (3^2 : D_8, 10)$ or $(S_4 \times 2, 15)$. This case is ruled out with similar arguments as in the case $G = A_6$. Note that computation in MAGMA [2] shows that the action of $A_6, S_2$ on 10 points is also not 3-homogeneous.

$G = \text{PGL}_2(9)$. Since $v \geq 8$, by Atlas [11] we have $(G, v) = (D_{20}, 36)$, $(3^2 : 8, 10)$ or $(D_{16}, 45)$.

Suppose $(G, v) = (D_{20}, 36)$. Then $k \in \{4, 5, 6, 7\}$ as $k > 3$ and $v \geq k^2 - 3k + 4$. Since $v - 2$ is divisible by $k - 2$ by Lemma 2.1(c), we deduce that $k = 4$. Then $|B| = 1785$ by Lemma 2.1(a). This is a contradiction because $|G| = 720$ is divisible by $|B|$.

Suppose $(G, v) = (3^2 : 8, 10)$. From $k > 3$ and $v \geq k^2 - 3k + 4$ we obtain $k = 4$. Then $|B| = 30$ and $|G_B| = 24$. Computation in MAGMA [2] shows that $G$ has only one conjugate classes of subgroup of order 24, and moreover, $G_B$ has two orbits of lengths 4 and 6 on $P$. This implies that $D$ is flag-transitive, and a block is the $G_B$-orbit of length 4. We may identify $G = \text{PGL}_2(9)$ with a subgroup of the group $S_6 : Z_2$ constructed in Example 1, that is,

$$G = ((1, 4)(2, 10)(3, 5)(6, 7)(8, 9), (2, 6, 10)(3, 8, 5)(4, 9, 7)).$$
Then we may take $G_B$ as the subgroup generated by permutations

$$(1, 6, 7)(2, 8, 10)(3, 4, 9), (1, 5, 7)(2, 9, 4)(3, 10, 8), (2, 10)(3, 9)(4, 8)(5, 7).$$

Now the $G_B$-orbit of length 4 is $\{1, 5, 6, 7\}$. Computation shows that $D$ is exactly the design $D_{3,10,4}$ in Example 1.

Suppose $(G_\alpha, v) = (D_{16}, 45)$. Then from $k > 3$ and $v \geq k^2 - 3k + 4$ we conclude that $k \in \{4, 5, 6, 7, 8\}$. However, $v - 2$ is not divisible by $k - 2$ for any $k \in \{4, 5, 6, 7, 8\}$, contradicting Lemma 2.1(c).

$G = M_{10}$. Since $v \geq 8$, by Atlas [11] we have $(G_\alpha, v) = (5: 4, 36), (3^2 : 8, 10)$ or $(8: 2, 45)$. The arguments for this case are similar to that for the case $G = PGL_2(9)$.

$G = S_6 : Z_2$. From Atlas [11] we see that $(G_\alpha, v) = (10 : 4, 36), (3^2 : [2^4], 10)$ or $([2^5], 45)$. The arguments are also similar.

### 3.2 The intransitive case

In this subsection, we assume that $n \geq 7$ and deal with the case (a), where $G_\alpha$ acts intransitively on $\{1, 2, \ldots, n\}$. In this case, we may identify $\mathcal{P}$ with the set of $m$-subsets of $\{1, 2, \ldots, n\}$. Recall that the nontrivial subdegrees of $G$ are given in Lemma 2.9.

**Lemma 3.4** Suppose that $\text{Soc}(G) = A_n$ with $n \geq 7$. Then $G_\alpha$ is not of intransitive case.

**Proof** Suppose that $G_\alpha = (S_m \times S_{n-m}) \cap G$, with $m \geq 1$ and $n \geq 2m + 1$. If $m = 1$, then $v = n, G_\alpha = A_{n-1}$ or $S_{n-1}$, and $G$ acts 2-transitively on $\mathcal{P}$, and the case is proved to be impossible by [22]. Therefore, $m \geq 2$.

Since $m \geq 2$, the subgroup $S_m$ of $S_m \times S_{n-m}$ contains an odd permutation on $\{1, 2, \ldots, n\}$, which implies that $(S_m \times S_{n-m}) \cap A_n$ is of index 2 in $A_n$. Hence

$$v = \frac{|S_n|}{|S_m| \cdot |S_{n-m}|} = \frac{n!}{m!(n-m)!} = \frac{n(n-1) \cdots (n-m+1)}{m!} = \binom{n}{m}. \quad (3.1)$$

Let $g = (1, 2, 3) \in G$. Since $G$ acts faithfully on $\mathcal{P}$, it follows that there exists at least one point not fixed by $g$, and hence $\langle g \rangle$ has an orbit of length 3 on $\mathcal{P}$. Note that now we identify $\mathcal{P}$ with the set of $m$-subsets of $\{1, 2, \ldots, n\}$. Let $\beta \in \mathcal{P}$ be a $m$-subset fixed by $g$. If $m = 2$, then $\{1, 2, 3\} \subseteq \{1, 2, \ldots, n\} \setminus \beta$ and hence there are $\binom{n-3}{m}$ choices for $\beta$. If $m \geq 3$, then either $\{1, 2, 3\} \subseteq \{1, 2, \ldots, n\} \setminus \beta$ or $\{1, 2, 3\} \subseteq \beta$, and hence there are $\binom{n-3}{m-3} + \binom{n-3}{m}$ choices for $\beta$. It follows that

$$|\text{Fix}_\mathcal{P}((g))| = \binom{n-3}{m} \text{ if } m = 2, \text{ and } |\text{Fix}_\mathcal{P}((g))| = \binom{n-3}{m-3} + \binom{n-3}{m} \text{ if } m \geq 3.$$ 

In particular,

$$|\text{Fix}_\mathcal{P}((g))| \geq \binom{n-3}{m}. \quad (3.2)$$

Since $v \geq k^2 - 3k + 4$ by Lemma 2.2, we have $v - k \geq k^2 - 4k + 4 = (k-2)^2$. Then from Lemma 2.7 we conclude that

$$|\text{Fix}_\mathcal{P}((g))| \leq \frac{2(v-k)}{k-2} + k - 2 = \frac{2(v-k) + (k-2)^2}{k-2} \leq \frac{3(v-k)}{k-2} < \frac{3v}{k-2}.$$
This implies that
\[
    k < \frac{3v}{|\text{Fix}_\mathcal{P}(\{g\})|} + 2 \leq 3\left(\frac{n}{m}\right)\left(\frac{n-3}{m}\right) + 2 = \frac{3n(n-1)(n-2)}{(n-m)(n-m-1)(n-m-2)} + 2.
\]
By Lemma 2.9, \(G\) has a nontrivial subdegree \(d := m(n-m)\). Then from Lemma 2.6(b), we conclude that
\[
    (v-2)^2 < (v-1)(v-2) \leq k(k-1)(k-2)\ell(d-1) = k(k-1)(k-2)m^2(n-m)^2.
\]
\[
(3.4)
\]
Suppose that the pair \((m, n)\) satisfying one of the following:
- \(m = 2\) and \(7 \leq n \leq 31\);
- \(m = 3\) and \(7 \leq n \leq 16\);
- \(m = 4\) and \(9 \leq n \leq 15\);
- \((5, 11), (5, 12), (5, 13), (5, 14), (6, 13), (6, 14), (7, 15)\).

Recall that the nontrivial subdegrees of \(G\) are given in Lemma 2.9. For every pair \((n, m)\) above and every \(k\) satisfying (3.3) and \(k > 3\), computation shows that Lemma 2.6 holds only if \(n = 8, m = 3\) and \(k = 11\). In this case, we have \(v = 56\) and \(|B| = 168\). Suppose that \(G = S_8\). We may let \(G_\alpha\) be the stabilizer of subset \(\{1, 2, 3\}\) in \(G\). Computation in Magma[2] shows that there are two non-conjugate subgroups of index 168 in \(G\), say \(H_1\) and \(H_2\), and the orbits of \(H_i\) on \(\mathcal{P}\) are \(O_i\) for \(1 \leq i \leq 6\) with \(|O_1| = 1, |O_2| = 5, |O_3| = |O_4| = |O_5| = 10\) and \(|O_6| = 20\), and the orbits of \(H_2\) on \(\mathcal{P}\) are \(Q_i\) for \(1 \leq j \leq 3\) with \(|Q_1| = 6, |Q_2| = 20\) and \(|Q_3| = 30\). Since \(k = 11\), that is, a block consists of 11 points, we derive that \(G_B\) is conjugate to \(H_1\), and \(B = (O_1 \cup \cup O_3)^G\), \((O_1 \cup O_4)^G\) or \((O_1 \cup O_5)^G\). However, for every candidate for \(B\), computation in Magma[2] shows that there exists some 3-subset of \(\{1, 2, \ldots, v\}\) which is contained in at least two blocks, a contradiction. The computation for the case \(G = A_8\) is similar, and it turns out that the case \(G = A_8\) is still impossible.

Suppose \(m = 2\) and \(n \geq 32\). By (3.3) we see that
\[
    k < \frac{3n(n-1)(n-2)}{(n-2)(n-3)(n-4)} + 2 < 3\left(\frac{n-1}{n-4}\right)^2 + 2 \leq 3\left(\frac{32-1}{32-4}\right)^2 + 2 < 6.
\]
Thus \(k \leq 5\). Then (3.4) is reduced to
\[
    \left(\frac{n(n-1)}{2} - 2\right)^2 < 5 \cdot 4 \cdot 3 \cdot 2^2 \cdot (n-2)^2.
\]
Computation shows that the above inequality does not hold for any \(n \geq 32\), a contradiction.

Suppose \(m = 3\) and \(n \geq 17\). Then by (3.3) we have
\[
    k < \frac{3n(n-1)(n-2)}{(n-3)(n-4)(n-5)} + 2 < 3\left(\frac{n-2}{n-5}\right)^3 + 2 \leq 3\left(\frac{17-2}{17-5}\right)^3 + 2 < 8.
\]
This implies \(k \leq 7\). Then (3.4) is reduced to
\[
    \left(\frac{n(n-1)(n-2)}{6} - 2\right)^2 < 7 \cdot 6 \cdot 5 \cdot 3^2 \cdot (n-3)^2.
\]
Computation shows that there is no \(n \geq 17\) satisfying the above inequality.

To avoid many repeating arguments, we omit the proof for the following cases:
- \(m = 4\) and \(n \geq 16\);
• $m = 5$ and $n \geq 15$;
• $m = 6$ and $n \geq 15$;
• $m = 7$ and $n \geq 16$.

Finally, we suppose that $m \geq 8$. Since $n \geq 2m + 1$, we have $n \geq 17$, and $m \leq n - m - 1$ and $m \leq (n - 1)/2$. Then by Lemma 2.6(b), we conclude that

$$(v - 2)^2 < (v - 1)(v - 2) \leq k(k - 1)(k - 2)d(d - 1) < (k - 1)^3d^2$$

$$< \left( \frac{3n(n - 1)(n - 2)}{(n - m)(n - m - 1)(n - m - 2)} + 1 \right)^3 m^2(n - m)^2$$

$$< \left( \frac{4n(n - 1)(n - 2)}{(n - m)(n - m - 1)(n - m - 2)} \right)^3 (n - m - 1)^2(n - m)^2$$

$$= \frac{26n^3(n - 1)^3(n - 2)^3}{(n - m)(n - m - 1)(n - m - 2)^3} \leq \frac{26n^3(n - 1)^3(n - 2)^3}{\frac{1}{2} \frac{1}{2} \left( \frac{1}{2} \right)^3}$$

$$= \frac{211n^3(n - 1)^2}{n - 3} \cdot \frac{(n - 2)^3}{(n + 1)(n - 3)^2}.$$ 

Since $n \geq 17$, one has

$$(n - 2)^3 - (n + 1)(n - 3)^2 = -n^2 + 9n - 17 < 0,$$

$$(48n^2 - 2)^2(n - 3) - 2^{11}n^3(n - 1)^2 = 256n^5 - 2816n^4 - 2240n^3 + 576n^2 + 4n - 12 > 0.$$ 

Then

$$(v - 2)^2 < \frac{211n^3(n - 1)^2}{n - 3} \cdot \frac{(n - 2)^3}{(n + 1)(n - 3)^2} < \frac{211n^3(n - 1)^2}{n - 3} < (48n^2 - 2)^2,$$

and hence $v < 48n^2$, that is,

$$\frac{n(n - 1) \cdots (n - m + 1)}{m!} < 48n^2. \tag{3.5}$$

We shall show that (3.5) does not hold for any $m \geq 8$ and $n \geq 2m + 1$. To do this, we let

$$f(n, m) = \frac{n(n - 1) \cdots (n - m + 1)}{m! \cdot n^2} = \frac{(n - 1) \cdots (n - m + 1)}{m!n}.$$ 

Since $n \geq 2m + 1$, we have $m \leq (n - 1)/2$ and so

$$n^2 - (n + 1)(n - m + 1) \geq n^2 - (n + 1) \left( n - \frac{n - 1}{2} + 1 \right) = \frac{n^2 - 4n - 3}{2} > 0.$$ 

Therefore,

$$f(n + 1, m) \geq \frac{f(n, m)}{f(n, m)} = \frac{n}{n - m + 1} \cdot \frac{n}{n + 1} = \frac{n^2}{(n + 1)(n - m + 1)} > 1,$$

which implies $f(n, m) \geq f(2m + 1, m)$ for all $n \geq 2m + 1$. Let

$$h(m) = f(2m + 1, m) = \frac{2m(2m - 1) \cdots (m + 2)}{m!(2m + 1)}.$$ 

Then

$$\frac{h(m + 1)}{h(m)} = \frac{(2m + 2)(2m + 1)}{m + 2} \cdot \frac{1}{m + 1} \cdot \frac{2m + 1}{2m + 3} = \frac{2(2m + 1)^2}{(m + 2)(2m + 3)} > 1.$$ 

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Consequently, \( h(m) \geq h(8) \) for all \( m \geq 8 \), and hence \( f(n, m) \geq h(m) \geq h(8) \) for all \( n \geq 2m + 1 \) and all \( m \geq 8 \). Computation shows that \( h(8) > 84 \). Therefore, \( f(n, m) > 48 \) for all \( n \geq 2m + 1 \) and all \( m \geq 8 \), contradicting (3.5). \( \square \)

### 3.3 The imprimitive case

In this subsection, we assume that \( n \geq 7 \) and deal with the case (b), where \( G_\alpha \) acts imprimitively on \( \{1, 2, \ldots, n\} \). In this case, we may identify \( \mathcal{P} \) with the set of partitions of \( \{1, 2, \ldots, m\ell\} \) with \( \ell \geq 2 \) blocks of size \( m \geq 2 \). Recall that in the case \( \ell = 2 \), the nontrivial subdegrees of \( G \) are given in Lemma 2.9.

**Lemma 3.5** Suppose that \( \text{Soc}(G) = A_n \) with \( n \geq 7 \). Then \( G_\alpha \) is not of imprimitive case.

**Proof** Suppose that \( G_\alpha = (S_m : S_\ell) \cap G \), with \( n = m\ell \), and \( m \geq 2 \) and \( \ell \geq 2 \). Then

\[
v = \frac{|S_m|}{|S_m|^{\ell}|S_\ell|} = \frac{(m\ell)!}{m!\ell!}.
\]

Let \( g = (1, 2, 3) \in G \). Since \( G \) acts faithfully on \( \mathcal{P} \), it follows that \( \langle g \rangle \) has an orbit of length 3 on \( \mathcal{P} \). Note that \( C_{S_m}(g) = \langle g \rangle \times \text{Sym}([4, 5, \ldots, n]) \). Since \( n \geq 7 \), the group \( \text{Sym}([4, 5, \ldots, n]) \) contains odd permutations, and hence \( |C_{S_m}(g)|/|C_{A_n}(g)| = 2 \). Therefore, \( |G|/|C_G(g)| = n(n-1)(n-2)/3 \). This together with Lemma 2.3 and Lemma 2.8 prove the following

\[
v \leq \left( \frac{n(n-1)(n-2)}{3} + 2 \right) < \frac{n^6}{9}.
\]

Consequently, we have

\[
\frac{(m\ell)!}{m!\ell!} < \frac{(m\ell)^6}{9}.
\]

Suppose first that \( m \leq 20 \) and \( \ell \leq 20 \). Then computation shows that the pairs \((m, \ell)\) satisfying (3.8) are

(i) \( \ell = 2 \) and \( 4 \leq m \leq 15 \) (noting that \( n = m\ell \geq 7 \));

(ii) \( (3, 3), (4, 3), (5, 3), (6, 3), (2, 4), (3, 4), (2, 5), (2, 6), (2, 7) \).

For those \((m, \ell)\) satisfying (i), the nontrivial subdegrees of \( G \) are determined by Lemma 2.10, and for those \((m, \ell)\) satisfying (ii), the nontrivial subdegrees of \( G \) can be obtained by computation in MAGMA[2]. However, computation shows that for those \((m, \ell)\) satisfying (i) or (ii), there exists no integer \( k \) with \( 3 < k \leq \sqrt{\ell} + 2 \) such that Lemma 2.6 holds.

Therefore, \((m, \ell)\) is not a pair such that \( m \leq 20 \) and \( \ell \leq 20 \). Let

\[
f(m, \ell) = \frac{(m\ell)!}{m!\ell!} \cdot \frac{(m\ell)^6}{9} = \frac{9(m\ell)!}{m!\ell!(m\ell)^5}.
\]

Then

\[
\frac{f(m + 1, \ell)}{f(m, \ell)} = \frac{(m + 1)!}{(m)!} \cdot \frac{m!\ell!}{(m + 1)!} \cdot \frac{(m + 1)!}{(m)!} \cdot \frac{(m\ell)^6}{9} = \frac{(m + 1)!}{(m)!} \cdot \frac{1}{(m + 1)!} \cdot \frac{m^6}{(m + 1)^6} \cdot \frac{(m\ell)^6}{9}.
\]

\[
\frac{f(m, \ell + 1)}{f(m, \ell)} = \frac{(m\ell + 1)!}{(m\ell)!} \cdot \frac{m!\ell!}{(m\ell + 1)!} \cdot \frac{(m\ell)^6}{9} = \frac{(m\ell + 1)!}{(m\ell)!} \cdot \frac{1}{m!} \cdot \frac{\ell^6}{(\ell + 1)^7}.
\]
Since
\[
\frac{(m+1)!}{(m\ell)!} = \prod_{i=1}^\ell (m\ell + i) > (m\ell)^\ell,
\]
we have
\[
\frac{f(m+1, \ell)}{f(m, \ell)} > \frac{(m\ell)^\ell m^6}{(m+1)^{6+\ell}} = \ell^\ell \left( \frac{m}{m+1} \right)^{6+\ell} = \left( \frac{\ell}{m+1} \right)^{6+\ell}.
\]
Suppose that \(2 \leq \ell \leq 20\). Then \(m \geq 21\) and so
\[
\frac{f(m+1, \ell)}{f(m, \ell)} > \left( \frac{\ell}{m+1} \right)^{6+\ell} \geq \left( \frac{2}{21+1} \right)^{6+\ell} > 1^{6+\ell} = 1,
\]
which implies that if \(\ell\) is fixed then \(f(m, \ell)\) is increasing as a function of \(m\) where \(m \geq 21\) and \(2 \leq \ell \leq 20\), i.e. \(f(m, \ell) \geq f(21, \ell)\). Since
\[
f(21, \ell+1) = \left( \prod_{i=1}^{21} (21\ell+i) \right) \left( \frac{\ell^6}{21!(\ell+1)^7} \right) = \left( \prod_{i=1}^{20} \frac{21\ell+i}{i} \right) \left( \frac{21\ell+21}{21(\ell+1)^7} \right)
\]
we have \(f(21, \ell) \geq f(21, 2) > 1\) for all \(2 \leq \ell \leq 20\), and so \(f(m, \ell) > 1\) for all \(m \geq 21\) and \(2 \leq \ell \leq 20\), contradicting (3.8).

Suppose \(\ell \geq 21\). Then \(m \geq 2\) and
\[
\frac{f(m+1, \ell)}{f(m, \ell)} > \left( \frac{\ell}{m+1} \right)^{6+\ell} \geq \left( \frac{21}{21+1} \right)^{6+\ell} > 1^{6+\ell} = 1,
\]
which implies that \(f(m, \ell) \geq f(2, \ell)\) for all \(m \geq 2\) and \(\ell \geq 21\). Since
\[
f(2, \ell+1) = \frac{(2\ell+2)(2\ell+1)e^6}{2(\ell+1)^7} = (2\ell+1) \left( \frac{\ell}{\ell+1} \right)^6 \geq (2 \cdot 21 + 1) \left( \frac{21}{21+1} \right)^6 > 1,
\]
we have \(f(2, \ell) \geq f(2, 21) > 1\) for all \(\ell \geq 21\) and so \(f(m, \ell) > 1\) for all \(m \geq 2\) and \(\ell \geq 21\), contradicting (3.8).

\[
\square
\]

3.4 The primitive case

In this subsection, we assume that \(n \geq 7\) and deal with the cases (c)–(f) in the beginning of Sect. 3, where \(G_\alpha\) acts primitively on \(\{1, 2, \ldots, n\}\) and \(A_n \not\leq G_\alpha\).

**Lemma 3.6** Suppose that \(\text{Soc}(G) = A_n\) with \(n \geq 7\). Then \(G_\alpha\) is not of primitive affine, diagonal, wreath product and almost case.

**Proof** Suppose for a contradiction that \(G_\alpha\) is one of the cases (c)–(f). Then \(G_\alpha\) acts primitively on \(\{1, 2, \ldots, n\}\) and \(A_n \not\leq G_\alpha\). Let \(g = (1, 2, 3)\). If \(g \in G_\beta\) for some \(\beta \in \mathcal{P}\), then we conclude from [12, Theorem 3.3A] that \(G_\beta \geq A_n\), a contradiction. Therefore, \(g\) fixes no point in \(\mathcal{P}\). Then \(k\) divides \(v\) by Lemma 2.4.

Suppose that \(v\) is odd. From [23, Theorem C] we conclude that one of the following holds:
• $G = A_7$ and $G_\alpha = \text{PSL}_3(2)$;
• $G = A_8$, and $G_\alpha = 2^3\cdot \text{SL}_3(2) \cong \text{AGL}_3(2)$.

Both of these two cases lead to $v = 15$. Since $k \geq 4$ and $k$ divides $v$, we obtain $k = 5$, which contradicts that $v - 2$ is divisible by $k - 2$ (see Lemma 2.1(c)).

Suppose that $v$ is twice an odd integer. From Atlas [11] we conclude $n \neq 7$ or 8, and hence $n \geq 9$. Note that $\text{Sym}(\{1, 2, \ldots, 8\})$ has a Sylow 2-subgroup $P$ generating by the following permutations

$$(1, 2), (3, 4), (5, 6), (7, 8), (1, 3)(2, 4), (5, 6)(7, 8), (1, 5)(2, 6)(3, 7)(4, 8).$$

By the Sylow theorems, we can take a Sylow 2-subgroup $Q$ of $G$ such that $Q \geq P \cap G$. Again by the Sylow theorems, together with the transitivity of $G$ on $P$, we see that there exists some $\beta \in P$ such that a Sylow 2-subgroup $R$ of $G_\beta$ is contained in $Q$. Since $|G|/|G_\beta| = v$, which is twice an odd integer, it follows that $R$ is of index 2 in $Q$. Let $g_1 = (1, 2)(3, 4)$ and $g_2 = (1, 2)(5, 6)$. Clearly, both $g_1$ and $g_2$ are in $P \cap G$ and hence in $Q$. Note that now $G_\beta$ acts primitively on $\{1, 2, \ldots, n\}$. According to [12, Theorem 3.3B], if $G_\beta$ contains one of $g_1$ and $g_2$, then $G_\beta \geq A_n$, a contradiction. Therefore neither $g_1$ nor $g_2$ is in $G_\beta$. Now it follows from $g_1, g_2 \not\subset R$ and $|Q|/|R| = 2$ that $g_1g_2 = (3, 4)(5, 6) \in R$. However, applying [12, Theorem 3.3B] again we derive that $G_\alpha \geq A_n$, a contradiction.

Therefore, $v$ is divisible by 4. Then it follows from Lemma 2.5 that $D$ is $G$-flag-transitive. However, this contradicts the result of Huber [15].

Remark 3.7 There is an alternative way to prove Lemma 3.6. It is shown in [34, Theorem 1.1] that the order of a primitive group of degree $n$ is no more than $n^{1+\lfloor \log_2(n) \rfloor}$ apart from a few exceptions. Using this upper bound and (3.7) we conclude

$$\frac{n^6}{9} > v > \frac{|A_m|}{|G_\alpha|} \geq \frac{n!}{2 \cdot n^{1+\lfloor \log_2(n) \rfloor}} \geq \frac{n!}{2 \cdot n \log_2(n)}.$$ 

Computation shows that the above inequality holds only if $n \leq 13$, and hence we only need to investigate primitive groups of degree at most 13.

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Declarations

Conflict of interest The authors declare they have no financial interests.

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