Channels with Synchronization/Substitution Errors and Computation of Error Control Codes

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Abstract—We introduce the concept of an $f$-maximal error-detecting block code, for some parameter $f$ between 0 and 1, in order to formalize the situation where a block code is close to maximal with respect to being error-detecting. Our motivation for this is that constructing a maximal error-detecting code is a computationally hard problem. We present a randomized algorithm that takes as input two positive integers $N,\ell$, a probability value $f$, and a specification of the errors permitted in some application, and generates an error-detecting, or error-correcting, block code having up to $N$ codewords of length $\ell$. If the algorithm finds less than $N$ codewords, then those codewords constitute a code that is $f$-maximal with high probability. The error specification (also called channel) is modelled as a transducer, which allows one to model any rational combination of substitution and synchronization errors. We also present some elements of our implementation of various error-detecting properties and their associated methods. Then, we show several tests of the implemented randomized algorithm on various channels. A methodological contribution is the presentation of how various desirable error combinations can be expressed formally and processed algorithmically.

I. INTRODUCTION

We consider block codes $C$, that is, sets of words of the same length $\ell$, for some integer $\ell > 0$. The elements of $C$ are called codewords or $C$-words. We use $A$ to denote the alphabet used for making words and $A^\ell = \text{the set of all words of length } \ell$.

Our typical alphabet will be the binary one $\{0,1\}$. We shall use the variables $u,v,w,x,y,z$ to denote words over $A$ (not necessarily in $C$). The empty word is denoted by $\varepsilon$.

We also consider error specifications $\varepsilon r$, which we call combinatorial channels, or simply channels. A channel $\varepsilon r$ specifies, for each allowed input word $x$, the set $\varepsilon r(x)$ of all possible output words. We assume that error-free communication is always possible, so $x \in \varepsilon r(x)$. On the other hand, if $y \in \varepsilon r(x)$ and $y \neq x$ then the channel introduces errors into $x$. Informally, a block code $C$ is $\varepsilon r$-detecting if the channel cannot turn a given $C$-word into a different $C$-word. It is $\varepsilon r$-correcting if the channel cannot turn two different $C$-words into the same word.

In Section II, we make the above concepts mathematically precise, and show how known examples of combinatorial channels can be defined formally so that they can be used as input to algorithms. In Section III, we present two randomized algorithms: the first one decides (up to a certain degree of confidence) whether a given block code $C$ is maximal $\varepsilon r$-detecting for a given channel $\varepsilon r$. The second algorithm is given a channel $\varepsilon r$, an $\varepsilon r$-detecting block code $C \subseteq A^\ell$ (which could be empty), and integer $N > 0$, and attempts to add to $C$ $N$ new words of length $\ell$ resulting into a new $\varepsilon r$-detecting code. If less than $N$ words get added then either the new code is $95\%$-maximal or the chance that a randomly chosen word can be added is less than 5%. Our motivation for considering a randomized algorithm is that embedding a given $\varepsilon r$-detecting block code $C$ into a maximal $\varepsilon r$-detecting block code is a computationally hard problem—this is shown in Section IV.

In Section V, we discuss briefly some capabilities of the new module codes.py in the open source software package FAAdo [1], [4] and we discuss some tests of the randomized algorithms on various channels. In Section VI, we discuss a few more points on channel modelling and conclude with directions for future research.

We note that, while there are various algorithms for computing error-control codes, to our knowledge these work for specific channels and implementations are generally not open source.

II. CHANNELS AND ERROR CONTROL CODES

We need a mathematical model for channels that is useful for answering algorithmic questions pertaining to error control codes. While many models of channels and codes for substitution-type errors use a rich set of mathematical structures, this is not the case for channels involving synchronization errors [13]. We believe the appropriate model for our purposes is that of a transducer. We note that transducers have been defined as early as in [18], and are a powerful computational tool for processing sets of words—see [2] and pg 41–110 of [17].

Definition 1. A transducer is a 5-tuple $t = (S,A,I,T,F)$ such that $A$ is the alphabet, $S$ is the finite set of states, $I \subseteq S$ is the set of initial states, $F \subseteq S$ is the set of final states, and $T$ is the finite set of transitions. Each transition is a 4-tuple $(s_i,x_i/y_i,t_i)$, where $s_i,t_i \in S$ and $x_i,y_i$ are words over $A$.

The general definition of transducer allows two alphabets: the input and the output alphabet. Here, however, we assume that both alphabets are the same.
The word $x_i$ is the input label and the word $y_i$ is the output label of the transition. For two words $x$, $y$ we write $y \in t(x)$ to mean that $y$ is a possible output of $t$ when $x$ is used as input. More precisely, there is a sequence

$$
(s_0, x_1/y_1, s_1), (s_1, x_2/y_2, s_2), \ldots, (s_{n-1}, x_n/y_n, s_n)
$$

of transitions such that $s_0 \in I$, $s_n \in F$, $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$. The relation $R(t)$ realized by $t$ is the set of word pairs $(x, y)$ such that $y \in t(x)$. A relation $\rho \subseteq A^* \times A^*$ is called rational if it is realized by a transducer. If every input and every output label of $t$ is in $A \cup \{\varepsilon\}$, then we say that $t$ is in standard form. The domain of the transducer $t$ is the set of words $x$ such that $t(x) \neq \emptyset$. The transducer is called input-preserving if $x \in t(x)$, for all words $x$ in the domain of $t$. The inverse of $t$, denoted by $t^{-1}$, is the transducer that is simply obtained by making a copy of $t$ and changing each transition $(s, x/y, t)$ to $(s, y/x, t)$. Then

$$x \in t^{-1}(y)$$

if and only if $y \in t(x)$.

We note that every transducer can be converted (in linear time) to one in standard form realizing the same relation.

In our objective to model channels $\er$ as transducers, we require that a transducer $\er$ is a channel if it allows error-free communication, that is, $\er$ is input-preserving.

**Definition 2.** An error specification $\er$ is an input-preserving transducer. The (combinatorial) channel specified by $\er$ is $R(\er)$, that is, the relation realized by $\er$. For the purposes of this paper, however, we simply identify the concept of channel with that of error specification.

A piece of notation that is useful in this work is the following, where $W$ is any set of words,

$$
er(W) = \bigcup_{w \in W} \er(w)
$$

Thus, $\er(W)$ is the set of all possible outputs of $\er$ when the input is any word from $W$. For example, if $\er = 1d_1 = \{0\}$, the channel that allows to 1 symbol to be deleted or inserted in the input word, then $1d_1(\{00, 11\}) = \{00, 000, 100, 010, 001, 11, 1, 011, 101, 110, 111\}$.

Fig. 4 considers examples of channels that have been defined in past research when designing error control codes. Here these channels are shown as transducers, which can be used as inputs to algorithms for computing error control codes. For the channel $\sub_2$, we have $00101 \in \sub_2(00000)$ because on input 00000, the channel $\sub_2$ can read the first two input 0’s at state $s$ and output 0, 0; then, still at state $s$, read the 3rd 0 and output 1 and go to state $t_1$; etc.

The concepts of error-detection and -correction mentioned in the introduction are phrased below more rigorously.

**Definition 3.** Let $C$ be a block code and let $\er$ be a channel. We say that $C$ is $\er$-detecting if

$$v \in C, w \in C \text{ and } w \in \er(v) \text{ imply } v = w.$$

We say that $C$ is $\er$-correcting if

$$v \in C, w \in C \text{ and } x \in \er(v) \cap \er(w) \text{ imply } v = w.$$

An $\er$-detecting block code $C$ is called maximal $\er$-detecting if $C \cup \{w\}$ is not $\er$-detecting for any word of length $\ell$ that is not in $C$. The concept of a maximal $\er$-correcting code is similar.

From a logical point of view (see Lemma 4 below) error-detection subsumes the concept of error-correction. This connection is stated already in [7] but without making use of it there. Here we add the fact that maximal error-detection subsumes maximal error-correction. Due to this observation, in this paper we focus only on error-detecting codes.

**Note:** The operation $\circ$ between two transducers $t$ and $s$ is called composition and returns a new transducer $s \circ t$ such that $z \in (s \circ t)(x)$ if and only if $y \in t(x)$ and $z \in s(y)$, for some $y$.

**Lemma 4.** Let $C \subseteq A^\ell$ be a block code and $\er$ be a channel. Then $C$ is $\er$-correcting if and only if it is $(\er^{-1} \circ \er)$-detecting. Moreover, $C$ is maximal $\er$-correcting if and only if it is maximal $(\er^{-1} \circ \er)$-detecting.

**Proof:** The first statement is already in [7]. For the second
statement, first assume that $C$ is maximal er-correcting and consider any word $w \in A^I \setminus C$. If $C \cup \{w\}$ were $(\text{er} \circ \text{er}^{-1})$-detecting then $C \cup \{w\}$ would also be er-correcting and, hence, $C$ would be non-maximal; a contradiction. Thus, $C$ must be maximal $(\text{er} \circ \text{er}^{-1})$-detecting. The converse can be shown analogously.

The operation ‘∪’ between any two transducers $t$ and $s$ is obtained by simply taking the union of their five corresponding components (states, alphabet, initial states, transitions, final states) after a renaming, if necessary, of the states such that the two transducers have no states in common. Then

\[(t \lor s)(x) = t(x) \cup s(x)\]

Let $\text{er}$ be a channel, let $C \subseteq A^\ell$ be an er-detecting block code, and let $w \in A^I \setminus C$. In [3], the authors show that

\[C \cup \{w\} \text{ is er-detecting iff } w \notin (\text{er} \lor \text{er}^{-1})(C)\]  \hspace{1cm} (2)

**Definition 5.** Let $C \subseteq A^\ell$ be an er-detecting block code. We say that a word $w$ can be added into $C$ if $w \notin (\text{er} \lor \text{er}^{-1})(C)$.

**Statement (2) above implies that**

\[C \text{ is maximal er-detecting iff } A^I \setminus (\text{er} \lor \text{er}^{-1})(C) = \emptyset\]  \hspace{1cm} (3)

**Definition 6.** The maximality index of a block code $C \subseteq A^\ell$ w.r.t. a channel $\text{er}$ is the quantity

\[\text{maxind}(C, \text{er}) = \frac{|A^\ell \setminus (\text{er} \lor \text{er}^{-1})(C)|}{|A^\ell|}\]

Let $f$ be a real number in $[0, 1]$. An er-detecting block code $C$ is called $f$-maximal er-detecting if $\text{maxind}(C, \text{er}) \geq f$.

The maximality index of $C$ is the proportion of the ‘used up’ words of length $\ell$ over all words of length $\ell$. One can verify the following useful lemma.

**Lemma 7.** Let $\text{er}$ be a channel and let $C \subseteq A^\ell$ be an er-detecting block code.

1) $\text{maxind}(C, \text{er}) = 1$ if and only if $C$ is maximal er-detecting.

2) Assuming that words are chosen uniformly at random from $A^\ell$, the maximality index is the proportion that a randomly chosen word $w$ of length $\ell$ cannot be added into $C$ preserving its being er-detecting, that is,

\[\text{maxind}(C, \text{er}) = \Pr[w \text{ cannot be added into } C]\]

**Proof:** The first statement follows from Definition 6 and condition (3). The second statement follows when we note that the event that a randomly chosen word $w$ from $A^\ell$ cannot be added into $C$ is the same as the event that $w \in A^\ell \setminus (\text{er} \lor \text{er}^{-1})(C)$.

### III. GENERATING ERROR CONTROL CODES

We turn now our attention to algorithms processing channels and sets of words. A set of words is called a language, with a block code being a particular example of language. A powerful method of representing languages is via finite automata [17]. A (finite) automaton $a$ is a 5-tuple $(S, A, I, T, F)$ as in the case of a channel, but each transition has only an input label, that is, it is of the form $(s, x, t)$ with $x$ being one alphabet symbol or the empty word $\varepsilon$. The language accepted by $a$ is denoted by $L(a)$ and consists of all words formed by concatenating the labels in any path from an initial to a final state. The automaton is called deterministic, or DFA for short, if $I$ consists of a single state, there are no transitions with label $\varepsilon$, and there are no two distinct transitions with same labels going out of the same state. Special cases of automata are constraint systems in which normally all states are final (pg 1635–1764 of [16]), and trellises. A trellis is an automaton accepting a block code, and has one initial and one final state (pg 1989–2117 of [16]). In the case of a trellis we talk about the code represented by $a$, and we denote it as $C(a)$, which is equal to $L(a)$.

For computational complexity considerations, the size $|m|$ of a finite state machine (automaton or transducer) $m$ is the number of states plus the sum of the sizes of the transitions. The size of a transition is 1 plus the length of the label(s) on the transition. We assume that the alphabet $A$ is small so we do not include its size in our estimates.

An important operation between an automaton $a$ and a transducer $t$, here denoted by ‘$\triangleright$’, returns an automaton $(a\triangleright t)$ that accepts the set of all possible outputs of $t$ when the input is any word from $L(a)$, that is,

\[L(a \triangleright t) = t(L(a))\]

**Remark 8.** We recall here the construction of $(a \triangleright t)$ from given $a = (S_1, A, I_1, T_1, F_1)$ and $t = (S_2, A, I_2, T_2, F_2)$, where we assume that $a$ contains no transition with label $\varepsilon$. First, if necessary, we convert $t$ to standard form. Second, if $t$ contains any transition whose input label is $\varepsilon$, then we add into $T_1$ transitions $(q, \varepsilon, q)$, for all states $q \in S_1$. Let $T_1$ denote now the updated set of transitions. Then, we construct the automaton $b = (S_1 \times S_2, A, I_1 \times I_2, T, F_1 \times F_2)$ such that $((p_1, x), (q_1, q_2)) \in T$, exactly when there are transitions $(p_1, x, q_1) \in T_1$ and $(p_2, x/y, q_2) \in T_2$. The above construction can be done in time $O(|a||t|)$ and the size of $b$ is $O(|a||t|)$. The required automaton $(a\triangleright t)$ is the trim version of $b$, which can be computed in time $O(|b|)$. The trim version of an automaton $m$ is the automaton resulting when we remove any states of $m$ that do not occur in some path from an initial to a final state of $m$.

**Figure 2.** Algorithm nonMax—see Theorem 9.
Next we present our randomized algorithms—we use [14] as reference for basic concepts. We assume that we have available to use in our algorithms an ideal method pickFrom\((A, \ell)\) that chooses uniformly at random a word in \(A^\ell\). A randomized algorithm \(R(\cdots)\) with specific values for its parameters can be viewed as a random variable whose value is whatever value is returned by executing \(R\) on the specific values.

**Theorem 9.** Consider the algorithm \(\text{nonMax}\) in Fig. 2, which takes as input a channel \(\text{er}\), a trellis \(a\) accepting an \(\text{er}\)-detecting code, and two numbers \(f, \varepsilon \in \{0, 1\}\).

1) The algorithm either returns a word \(w \in A^\ell \setminus C(a)\) such that the code \(C(a) \cup \{w\}\) is \(\text{er}\)-detecting, or it returns None.

2) If \(C(a)\) is not \(f\)-maximal \(\text{er}\)-detecting, then
\[
\Pr[\text{nonMax returns None}] < \varepsilon.
\]

3) The time complexity of \(\text{nonMax}\) is
\[
O\left(\ell |a| |\text{er}| / (\varepsilon (1 - f)^2)\right).
\]

**Proof:** The first statement follows from statement (2) in the previous section, as any \(w\) returned by the algorithm is not in \((\text{er} \lor \text{er}^{-1})(C(a))\). For the second statement, suppose that the code \(C(a)\) is not \(f\)-maximal \(\text{er}\)-detecting. Let \(\text{Cnt}\) be the random variable whose value is the value of \(\text{tr} - 1\) at the end of execution of the randomized algorithm \(\text{nonMax}\). Then, \(\text{Cnt}\) counts the number of variables whose value is the value of \(\text{tr} - 1\) at the end of execution of the randomized algorithm \(\text{nonMax}\). Hence, \(\text{Cnt}\) is binomial: the number of successes (words \(w\) in \(L(b)\)) in \(n\) trials. So \(E(\text{Cnt}) = np\), where \(p = \Pr[w \in L(b)]\).

By the definition of \(n\) in \(\text{nonMax}\), we get \(1/(4n(1 - f)^2) < \varepsilon\). Now consider the Chebyshev inequality, \(\Pr[|X - E(X)| \geq a] \leq \sigma^2/a^2\), where \(a > 0\) is arbitrary and \(\sigma^2\) is the variance of some random variable \(X\). For \(X = \text{Cnt}\) the variance is \(np(1-p)\), and we get
\[
\Pr[|\text{Cnt} - np| \geq 1 - f] < \varepsilon,
\]
where we used \(a = n(1 - f)\) and the fact that \(p(1-p) \leq 1/4\).

Using Lemma 7 and the assumption that \(C(a)\) is not \(f\)-maximal, we have that \(\text{maxind}(C(a), \text{er}) < f\), which implies \(\Pr[w \in L(b)] < f\); hence, \(p < f\). Then
\[
\Pr[\text{nonMax returns None}] = \Pr[\text{Cnt} = n] = \Pr[\text{Cnt} = n - 1] \leq \Pr[|\text{Cnt} - np| \geq 1 - p] \leq \Pr[|\text{Cnt} - np| \geq 1 - f] < \varepsilon,
\]
as required.

For the third statement, we use standard results from automaton theory, [17], and Remark 8. In particular, computing \(b\) can be done in time \(O(|a| \cdot |\text{er}|)\) such that \(|b| = O(|a| \cdot |\text{er}|)\). Testing whether \(w\) is \(\text{er}\)-detecting can be done in time \(O(|w|/|b|) = O(\ell|b|)\). Thus, the algorithm works in time \(O(\ell |a| |\text{er}| / (\varepsilon (1 - f)^2))\).}

**Remark 10.** We mention the important observation that one can modify the algorithm \(\text{nonMax}\) by removing the construction of \(b\) and replacing the ‘if’ line in the loop with

\[
\text{if } (C(a) \cup \{w\} \text{ is } \text{er}\text{-detecting}) \text{ return } w;
\]

While with this change the output would still be correct, the time complexity of the algorithm would increase to \(O\left(|a||\text{er}| / (\varepsilon (1 - f)^2)\right)\). This is because testing whether \(L(v)\) is \(\text{er}\)-detecting, for any given automaton \(v\) and channel \(\text{er}\), can be done in time \(O(|v|^2 |\text{er}|)\), and in practice \(|v|\) is much larger than \(\ell\).

In Fig. 3, we present the main algorithm for adding new words into a given deterministic trellis \(a\).

**Algorithm makeCode**—see Theorem 12. The trellis \(a\) can be omitted so that the algorithm would start with an empty set of codewords. In this case, however, the algorithm would require as extra input the codeword length \(\ell\) and the desired alphabet \(A\). We used the fixed values 0.95 and 0.05, as they seem to work well in practical testing.

**Algorithm makeCode**\( er, a, N \)

\[
W := \text{empty list}; \quad c := a
\]

\[
\text{cnt} := 0; \quad \text{more} := \text{True};
\]

while (\(\text{cnt} < N\) and more)

\[
\text{w} := \text{nonMax}(\text{er, c, 0.95, 0.05});
\]

if (\(\text{w}\) is None) more := False;

else \{\(\text{add w to c}\) and to \(W\); \(\text{cnt} := \text{cnt} + 1\);\}

return \(c, W\);

Figure 3. Algorithm makeCode—see Theorem 12. The trellis \(a\) can be omitted so that the algorithm would start with an empty set of codewords. In this case, however, the algorithm would require as extra input the codeword length \(\ell\) and the desired alphabet \(A\). We used the fixed values 0.95 and 0.05, as they seem to work well in practical testing.

**Remark 11.** In some sense, algorithm makeCode generalizes to arbitrary channels the idea used in the proof of the well-known Gilbert-Varshamov bound [12] for the largest possible block code \(M \subseteq A^\ell\) that is \(\text{sub}_{k}\)-correcting, for some number \(k\) of substitution errors. In that proof, a word can be added into the code \(M\) if the word is outside of the union of the “balls” \(\text{sub}_{k}(u)\), for all \(u \in M\). In that case, we have that \(\text{sub}_{k-1}(u) \subseteq \text{sub}_{k}(u)\) and \(\text{sub}_{k-1} \circ \text{sub}_{k}(u) = \text{sub}_{2k}(u)\). The present algorithm adds new words \(w\) to the constructed trellis \(c\) such that each new word \(w\) is outside of the “union-ball” \((\text{er} \lor \text{er}^{-1})(C(c))\).

**Theorem 12.** Algorithm makeCode in Fig. 3 takes as input a channel \(\text{er}\), a deterministic trellis \(a\) of some length \(\ell\), and an integer \(N > 0\) such that the code \(C(a)\) is \(\text{er}\)-detecting, and returns a deterministic trellis \(c\) and a list \(W\) of words such that the following statements hold true:

1) \(C(c) = C(a) \cup W\) and \(C(c)\) is \(\text{er}\)-detecting,

2) If \(W\) has less than \(N\) words, then either \(\text{maxind}(C(c), \text{er}) \geq 0.95\) or the probability that a randomly chosen word from \(A^\ell\) can be added in \(C(c)\) is \(< 0.05\).

3) The algorithm runs in time \(O\left(\ell N |\text{er}| |a| + \ell^2 N^2 |\text{er}|\right)\).

**Proof:** Let \(c_i\) be the value of the trellis \(c\) at the end of the \(i\)-th iteration of the while loop. The first statement follows from Theorem 9: any word \(w\) returned by \(\text{nonMax}\) is such that \(C(c_i) \cup \{w\}\) is \(\text{er}\)-detecting. For the second statement, assume that, at the end of execution, \(W\) has \(< N\) words and \(C(c)\) is not \(95\%\)-maximal. By the previous theorem, this means that the random process \(\text{nonMax}(\text{er, c, 0.95, 0.05})\) returns None
with probability < 0.05, as required. For the third statement, as
the loop in the algorithm nonMax performs a fixed number
of iterations (≈2000), we have that the cost of nonMax is
$O(\ell(c_i|er|) \cdot |a| + |a| + i \ell)$. The cost of adding a new word
$w$ of length $\ell$ to $c_{i-1}$ is $O(\ell)$ and increases its size by
$O(\ell)$, so each $c_i$ is of size $O(|a| + i \ell)$. Thus, the cost of the
$i$-th iteration of the while loop in makeCode is $O(\ell(\ell|er|(|a| + i \ell))$. As there are
up to $N$ iterations the total cost is
\[ \sum_{i=1}^{N} O(\ell|er| \cdot (|a| + i \ell)) = O(\ell N|er| |a| + \ell^2 N^2|er|) \]

\[ \blacksquare \]

Remark 13. In the algorithm makeCode, attempting to add
only one word into $C(a)$ (case of $N = 1$), requires time
$O(\ell|er| |a| + \ell^2|er|)$, which is of polynomial magnitude. This
case is equivalent to testing whether $C(a)$ is maximal er-
detecting, which is shown to be a hard decision problem
in Theorem 15.

Remark 14. In the version of the algorithm makeCode
where the initial trellis $a$ is omitted, the time complexity is
$O(\ell^2 N^2|er|)$. We also note that the algorithm would work
with the same time complexity if the given trellis $a$ is not
deterministic. In this case, however, the resulting trellis would
not be (in general) deterministic either.

IV. WHY NOT USE A DETERMINISTIC ALGORITHM

Our motivation for considering randomized algorithms is
that the embedding problem is computationally hard: given
a deterministic trellis $d$ and a channel $er$, compute (using a
deterministic algorithm) a trellis that represents a maximal er-
detecting code containing $C(d)$. By computationally hard, we
mean that a decision version of the embedding problem is
coNP-hard. This is shown next.

Theorem 15. The following decision problem is coNP-hard.

Instance: deterministic trellis $d$ and channel $er$.
Answer: whether $C(d)$ is maximal er-detecting.

Proof: Let us call the decision problem in question
MAXED, and let FULLBLOCK be the problem of deciding
whether a given trellis over the alphabet $A_2$ with no
$\varepsilon$-labeled transitions accepts $A_2^\ell$, for some $\ell$. The statement is a
logical consequence of the following claims.

Claim 1: FULLBLOk is coNP-complete.
Claim 2: FULLBLOk is polynomially reducible to
MAXED.

The first claim follows from the proof of the following fact
on page 329 of [10]: Deciding whether two given star-free
regular expressions over $A_2$ are inequivalent is an NP-complete
problem. Indeed, in that proof the first regular expression can
be arbitrary, but the second regular expression represents the
language $A_2^\ell$, for some positive integer $\ell$. Moreover, converting
a star-free regular expression to an acyclic automaton with no
$\varepsilon$-labeled transitions is a polynomial time problem.

For the second claim, consider any trellis $a =
(S, A_2, s, T, F)$ with no $\varepsilon$-labeled transitions in $T$. We need to
construct in polynomial time an instance $(d, er)$ of MAXED
such that $a$ accepts $A_2^\ell$ if and only if $C(d)$ is a maximal er-
detecting block code of length $\ell$. The rest of the proof consists
of 5 parts: construction of deterministic trellis $d$ accepting
words of length $\ell$, construction of $er$, facts about $d$ and $er$,
proving that $C(d)$ is $er$-detecting, proving that $a$ accepts $A_2^\ell$
if and only if $C(d)$ is maximal $er$-detecting.

Construction of $d$: Let $A$ be the alphabet $A_2 \cup T$, where $T$
is the set of transitions of $a$. The required deterministic trellis
d is any deterministic trellis accepting $A^\ell \setminus A_2^\ell$, that is,
$C(d) = A^\ell \setminus A_2^\ell$.

This can be constructed, for instance, by making deterministic
trellises $d_1$ and $d_2$ accepting, respectively, $A^\ell$ and $A_2^\ell$, and
then intersecting $d_1$ with the complement of $d_2$. Note that
any word in $C(d)$ contains at least one symbol in $T$.

Construction of $er$: This is of the form $(er_1 \lor er_2)$ as
follows. The transducer $er_2$ has only one state $s$ and transitions
$(s, a/\alpha, s)$, for all $\alpha \in A$, and realizes the identity relation
$\{(x, x) | x \in A^*\}$. Thus, we have that $er_2(x) = \{x\}$, for all
words $x \in A$. The transducer $er_1 = (S, A, s, T''', F,)$ is such that
$T'''$ consists of exactly the transitions $(p, (p, a, q)/a, q)$ for
which $(p, a, q)$ is a transition of $a$.

Facts about $d$ and $er$: The following facts are helpful in
the rest of the proof. Some of these facts refer to the determinant
trellis $d_1 = (S, T, s, T_1, F)$ resulting by omitting the output
parts of the transition labels of $er_1$, that is, $(p, (p, a, q), T_1)$
everything exactly when $(p, (p, a, q)/a, q) \in T'''$. Then, $C(d_1) \subseteq (A \setminus A_2)^\ell \subseteq C(d)$.

F0: $L(d_1 \triangleright er_1) = L(a)$.
F1: The domain of $er_1$ is $C(d_1)$, a subset of $(A \setminus A_2)^\ell$.
F2: If $v \in er_1(u)$ then $v \in A_2^\ell$ and $v \neq u$.
F3: $er_1(C(d)) = L(a)$.
F4: $er^{-1}_1(C(d)) = \emptyset$.

For fact F0, note that the product construction described in
Remark 8 produces in $(d_1 \triangleright er_1)$ exactly the transitions
$((p, q), a, (q), q)$, where $(p, a, q)$ is a transition in $a$, by matching
any transition $(p, (p, a, q), q)$ of $d_1$ only with the transition
$(p, (p, a, q)/a, q)$ of $er_1$. Fact F1 follows by the construction
of $er_1$ and the definition of $d_1$: any accepting computation
of $er_1$, the input labels appear in an accepting computation
of $d_1$ that uses the same sequence of states. F3 is shown as follows:
As the domain of $er_1$ is $C(d_1)$ and $C(d_1) \subseteq C(d)$, we have that
$er_1(C(d)) = er_1(C(d_1))$, which is $L(a)$ by F0. Fact F4 follows by noting that the domain of $er^{-1}_1$ is a subset of
$A_2^\ell$ but $C(d)$ contains no words in $A_2^\ell$.

$C(d)$ is $er$-detecting: Let $u, v \in C(d)$ such that $v \in er(u) = er_1(u) \cup \{u\}$. We need to show that $v = u$, that is,
to show that $v \notin er_1(u)$. Indeed, if $v \notin er_1(u)$ then $v \in A_2^\ell$
which contradicts $v \in C(d) = A^\ell \setminus A_2^\ell$.

$a$ accepts $A_2^\ell$ if and only if $C(d)$ is maximal $er$-detecting:
By statement (3) we have that $C(d)$ is maximal $er$-detecting,
if and only if $(er \lor er^{-1})(C(d)) = A^\ell$. We have:

$(er \lor er^{-1})(C(d)) = C(d) \cup er_1(C(d)) \cup er^{-1}_1(C(d))$
$= (A^\ell \setminus A_2^\ell) \cup L(a) \cup \emptyset$
$= (A^\ell \setminus A_2^\ell) \cup L(a)$.

Thus, $C(d)$ is maximal $er$-detecting, if and only if $L(a) = A_2^\ell$, as required.
\[ \blacksquare \]
V. Implementation and Use

All main algorithmic tools have been implemented over the years in the Python package FAdo [1], [4], [6]. Many aspects of the new module FAdo.codes are presented in [6]. Here we present methods of that module pertaining to generating codes.

Assume that the string d1 contains a description of the transducer del1 in FAdo format. In particular, d1 begins with the type of FAdo object being described, the final states, and the initial states (after the character ‘*’). Then, d1 contains the list of transitions, with each one of the form “s x y t | n”, where ‘n’ is the new-line character. This shown in the following Python script.

```python
import FAdo.codes as codes
d1 = '@Transducer 0 2 * 0\n' 
'0 0 0 0\n0 0 0 @epsilon 1\n' 
'0 1 @epsilon 1\n0 0 1\n1 1 1\n' 
'1 @epsilon 0 2\n1 @epsilon 1 2\n'
pd1 = codes.buildErrorDetectPropS(d1)
a = pd1.makeCode(100, 8, 2)
print(pd1.notSatisfiesW(a))
print(pd1.nonMaximalW(a, m))
s2 = ...string for transducer sub_2
ps2 = codes.buildErrorDetectPropS(s2)
pdls2 = pd1 & ps2
b = pdls2.makeCode(100, 8, 2)
```

The above script uses the string d1 to create the object pd1 representing the del1-detection property over the alphabet {0,1}. Then, it constructs an automaton a representing a del1-detecting block code of length 8 with up to 100 words over the 2-symbol alphabet {0,1}. The method notSatisfiesW(a) tests whether the code C(a) is del1-detecting and returns a witness of non-error-detection (= pair of codewords u, v with v ∈ del1(u)), or (None, None)—of course, in the above example it would return (None, None). The method nonMaximalW(a, m) tests whether the code C(a) is maximal del1-detecting and returns either a word v ∈ L(m) \ C(a) such that C(a) ∪ {v} is del1-detecting, or None if C(a) is already maximal. The object m is any automaton—here it is the trellis representing A'. This method is used only for small codes, as in general the maximality problem is algorithmically hard (recall Theorem 15), which motivated us to consider the randomized version nonMax in this paper. For any channel er and trellis a, the method notSatisfiesW(a) can be made to work in time $O(|C(a)|^2)$, which is of polynomial complexity. The operation ‘&’ combines error-detection properties. Thus, the second call to makeCode constructs a code that is del1-detecting and sub2-detecting (= sub1-correcting).

VI. More on Channel Modelling, Testing

In this section, we consider further examples of channels and show how operations on channels can result in new ones. We also show the results of testing our codes generation algorithm for several different channels.

Remark 16. We note that the definition of error-detecting (or error-correcting) block code C is trivially extended to any language L, that is, one replaces in Definition 3 ‘block code C’ with ‘language L’. Let $er, er_1, er_2$ be channels. By Definition 3 and using standard logical arguments, it follows that

1) $L$ is $er_1$-detracting and $er_2$-detracting, if and only if $L$ is $(er_1 \lor er_2)$-detracting;
2) $L$ is $er^{-1}$-detracting, if and only if it is $er$-detracting, and the same as the $(er \lor er^{-1})$-detracting ones—this is shown in [15] as well. The method of using transducers to model channels is quite general and one can give many more examples of past channels as transducers, as well as channels not studied before. Some further examples are shown in the next figures, Fig. 4-6.

One can go beyond the classical error control properties and define certain synchronization properties via transducers. Let OF be the set of all overlap-free words, that is, all words w such that a proper and nonempty prefix of w cannot be a suffix of w. A block code $C \subseteq OF$ is a solid code if any proper and nonempty prefix of a C-word cannot be a suffix of a C-word. For example, $\{0100, 1001\}$ is not a block code, as 01 is a prefix and a suffix of some codewords and 01 is nonempty and a proper prefix (shorter than the codewords). Solid codes can also be non-block codes by extending appropriately the above definition [19] (they are also called codes without overlaps in [9]). The transducer ov in Fig. 6 is such that any block code $C \subseteq OF$ is a solid code, if and only if C is an ‘ov-detecting’ block code. We note that solid codes have instantaneous synchronization capability (in particular all solid codes are comma-free codes) as well as synchronization in the presence of noise [5].

![Figure 4](image_url)

Figure 4. The channel specified by $bsid_2$ allows up to two errors in the input word. Each of these errors can be a deletion, an insertion, or a bit shift: a 10 becomes 01, or a 01 becomes 10. The alphabet is \{0, 1\}.

For $\epsilon = 0.05$ and $f = 0.95$, the value of $n$ in nonMax is 2000. We performed several executions of the algorithm...
error-detection which is equivalent to 1-synchronization-
error-correction. Here the Levenshtein code [8] of length 8 has 30
codewords. We recall that a maximal code is not necessarily
maximum, that is, having the largest possible number of
codewords, for given \( \epsilon \) and \( \ell \). It seems maximum codes are
rare, but there are many random maximal ones having lower
rates. The \( \text{del}_1 \)-detecting code of [15] has higher rate than all
the random ones generated here.

For the case of block solid codes (last column of the
table), we note that the function \text{pickFrom} in the algorithm
\text{nonMax} has to be modified as the randomly chosen word \( w \)
should be in \( \text{OF} \).

## VII. Conclusions

We have presented a unified method for generating error
control codes, for any rational combination of errors. The
method cannot of course replace innovative code design, but
should be helpful in computing various examples of codes.
The implementation \text{codes.py} is available for anyone for
download and use [4]. In the implementation for generating
codes, we allow one to specify that generated words only come
from a certain desirable subset \( M \) of \( A^\ell \), which is represented
by a deterministic trellis. This requires changing the function
\text{pickFrom} in \text{nonMax} so that it chooses randomly words
from \( M \). There are a few directions for future research. One
is to work on the efficiency of the implementations, possibly
allowing parallel processing, so as to allow generation of
block codes having longer block length. Another direction is
to somehow find a way to specify that the set of generated
codewords is a ‘systematic’ code so as to allow efficient
encoding of information. A third direction is to do a systematic
study on how one can map a stochastic channel \( sc \), like the
binary symmetric channel or one with memory, to a channel
\( er \) (representing a combinational channel), so as the available
algorithms on \( er \) have a useful meaning on \( sc \) as well.

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