HEXAGONAL GEOMETRIC TRIANGULATIONS

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Abstract. It is well-known that the Euclidean plane has a standard 6-regular geodesic triangulation, and the unit sphere has a 5-regular geodesic triangulation, which is induced from the regular Dodecahedron, and the hyperbolic plane has an $n$-regular geodesic triangulation for any $n > 6$. Here we constructed a 6-regular geodesic triangulation of the hyperbolic plane.

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1. INTRODUCTION

Here we consider the problem on the existence of special classes of geometric triangulations on the Euclidean plane and the hyperbolic plane.

A triangulation $T$ on a surface $S$ is $k$-regular for some integer $k > 0$ if the degrees of all its vertices are $k$, and it is geometric if all the edges are embedded as geodesic arcs with respect to the metric. The most familiar geometric 6-regular triangulation is the standard hexagonal triangulation on the plane, and the 7-regular equilateral triangulation of the hyperbolic plane. The triangular groups \[ \Delta(3,k) = \{a,b,c | a^3 = b^k = c^2 = 1\} \]
produce $k$-regular equilateral triangulations for any $k > 6$.

We ask whether we can construct 6-regular geometric triangulations on the hyperbolic plane. Moreover, we can consider the general question as follows.

Question: Does there exist $k$-regular geometric triangulations on the Euclidean plane $\mathbb{E}^2$, the round 2-sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$ for any $k \geq 3$? Furthermore, can we construct such a geometric triangulation $T$ with a uniform bound on the edge lengths of $T$?

If $k = 6$, the number of vertices grows linearly with respect to the combinatorial distance from one point. If $k > 6$, the number of vertices grows exponentially. Intuitively, they correspond to the Euclidean and hyperbolic geometry. We show that we can construct 6-regular geometric triangulation on $\mathbb{H}^2$. 
Figure 1. Degree-regular geometric triangulations.

The semi-regular tiling of the Euclidean and hyperbolic plane by polygons has been studied by Datta and Gupta [2].

2. PRELIMINARY CASE: CLOSED SURFACES

The basic combinatorial formulas below of Euler characteristics of closed surfaces show that there is no $k$-regular triangulation on $\mathbb{E}^2$ and $\mathbb{H}^2$ if $k = 3, 4, 5$, and there is no $k$-regular triangulation on $\mathbb{S}^2$ if $k \geq 6$.

From the combinatorial formula
\[ 12(1 - g) = 6\chi(S) = \sum_{i \in V} (6 - d_i) = |V|(6 - k), \]
we have a necessary condition for a closed orientable surface $S$ to admit a $k$-regular triangulation $T$, namely
\[ (k - 6)|12(g - 1). \]
It implies immediately that there is no $k$-regular triangulation on $\mathbb{S}^2$ if $k \geq 6$, and the only $k$-regular triangulation on tori is when $k = 6$. This condition turns out to be also sufficient for the existence of $k$-regular triangulation on $S$ with genus $g$.

**Lemma 2.0.1.** Let $S$ be a surface with genus $g$. Then there exists a $k$-regular triangulation on $S$ if and only if
\[ (k - 6)|12(g - 1). \]

**Proof.** The necessity is proved above. The sufficiency follows from the result by Jucovic and Trenkler [9]. First notice that if a $k$-regular triangulation $T$ exists on $S$, then set $V_0$, $E_0$, and $F_0$ be the number of vertices, edges, and faces of $T$. Then $V_0 = 6\chi(S)/(6 - k)$, $E_0 = 3k\chi(S)/(6 - k)$, and $F_0 = 2k\chi(S)/(6 - k)$.

By the Main Theorem in [9], if three positive numbers $V$, $E$, and $F$ on $S$ satisfies two conditions
1. $F - (4 - k)V = 8(1 - g)$;
2. $kV = 3F = 2E$,
then a $k$-regular triangulation with $V$ vertices, $E$ edges and $F$ faces exists on $S$. It is straightforward to check that $V_0$, $E_0$, and $F_0$ satisfies the two conditions. \qed

**Lemma 2.0.2.** There is no $k$-regular triangulation on $\mathbb{E}^2$ and $\mathbb{H}^2$ if $k = 3, 4, 5$. 


Proof. If such a triangulation $T$ exists, take an arbitrary subcomplex $T'$ which is homeomorphic disk. A combinatorial formula for a triangulated disk is given by

$$\sum_{i \in V_I} (6 - d_i) = 6 + \sum_{i \in V_B} (d_i - 4)$$

where $V_I$ and $V_B$ are the index sets for interior vertices and boundary vertices. Since $T$ is $k$-regular for $k < 6$, if $v_i$ is a boundary vertex of $T'$, then $d_i < 5$ since $T'$ is part of $T$. Then the left side is no larger than 6. Hence the disk $T'$ can only contains at most 6 vertices. This leads to a contradiction because we can take $T'$ as large as we want. \qed

For $k = 3, 4, 5$, we have $k$-regular geometric triangulations on $\mathbb{S}^2$, given by projecting regular tetrahedron, octahedron, and icosahedron to the unit sphere from the center. It is straightforward to construct a 6-regular triangulation on a torus using the standard hexagonal triangulation on $\mathbb{E}^2$.

![Figure 2. Degree-regular geometric triangulations on the unit 2-sphere.](image)

3. General facts about regular geometric triangulations

The goal is to construct $k$-regular geometric triangulations on $\mathbb{E}^2$ with $k > 6$, and 6-regular geometric triangulations on $\mathbb{H}^2$. We first point out that such triangulations contain skinny triangles.

**Lemma 3.0.1.** Let $T$ be a $k$-regular geometric triangulations on $\mathbb{E}^2$ with $k > 6$, or a 6-regular geometric triangulations on $\mathbb{H}^2$. Then the infimum of the angles in $T$ is zero.

**Proof.** If not, in the first case for $\mathbb{E}^2$ we can construct a quasiconformal homeomorphism from $\mathbb{E}^2$ to $\mathbb{H}^2$ by sending a triangle in $T$ to the corresponding triangle in the standard geometric triangulation of $\mathbb{H}^2$ generated by reflecting equilateral triangles with inner angle $2\pi/k$. Similarly, in the second case we can construct a quasiconformal homeomorphism from $\mathbb{H}^2$ to $\mathbb{E}^2$ sending $T$ to the standard equilateral hexagonal triangulation. This contradicts to the fact that there is no quasiconformal map from $\mathbb{E}^2$ to $\mathbb{H}^2$. \qed

In this sense, the “best” $k$-regular geometric triangulation we can expect is a geometric triangulation with a uniform low bound $c > 0$ and a uniform upper $C$ on the lengths of edges.

**Conjecture 3.0.2.** There is no $k$-regular geometric triangulation on $\mathbb{E}^2$ with a uniform low bound $c > 0$ and a uniform upper $C$ on the lengths of edges for $k \geq 7$. Similarly, there is no 6-regular geometric triangulation on $\mathbb{H}^2$ with a uniform low bound $c > 0$ and a uniform upper $C$ on the lengths of edges.
4. k-Regular geometric triangulations on the Euclidean plane

It is known that one can construct $k$-regular geometric triangulation on $\mathbb{E}^2$ for $k > 6$. The idea is to construct chains of circles with radius $n$ and distribute the points on the circle evenly. The similar computation is given in [2]. We can show that these triangulations have the desired uniform bounds.

Lemma 4.0.1. For any $k \geq 6$, there exist $k$-regular geometric triangulations on $\mathbb{E}^2$ with uniform upper bounds on the lengths of edges.

Proof. The construction of a $k$-regular geometric triangulation is given above. Let $C_n$ be circles with radius $r_n = n$ centered at the origin of $\mathbb{E}^2$. Let $a_n$ be the number of vertices on $C_n$ on this construction. These vertices have combinatorial distance $n$ from the vertex at the origin.

We have the following recursive relation:

$$a_{n+1} = (k - 4)a_n - a_{n-1}$$

with $a_0 = 0$ and $a_1 = k$. Solve this series and we have the formula for $a_n$ with $n \geq 1$:

$$a_n = k\frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where $\alpha = \cosh^{-1}((k - 4)/2)$ and $\beta = 1/\alpha$. Thus, asymptotically $a_n$ grows as $\alpha^{n-1}$ to infinity. But the length of $C_n$ is given by $l_n = 2\pi n$. So the arclength of each small arc goes to zero as $n$ goes to infinity, hence the length between two consecutive vertices on $C_n$ goes to zero.

By the construction above, if $v$ is a vertex in $C_n$, then the $k - 4$ or $k - 3$ vertices $v_i$ in $C_{n-1}$ connecting to $v$ forms an arc $\gamma$ in $C_{n+1}$, and the ray starting from the origin passing through $v$ intersects with $\gamma$. As the length of $\gamma$ goes to zero, all the lengths of the edges connecting $v$ to $v_i$ goes to $r_{n+1} - r_n = 1$.

So as $n \to \infty$, the lengths of edges approaches either 1 or 0. Hence we can take the maximal length of a compact part of $T$ bounded by $C_n$ with $n$ large enough so that its complement contains edges no longer than 2. Then we can find a uniform bound.

□

Using the same idea, we can construct a $k$-regular geometric triangulation of $\mathbb{E}^2$ whose edges has a uniform lower bound. We can pick $r_n = 2^n$ and follow the construction above.
5. 6-regular geometric triangulations on the Hyperbolic Plane

We now give a construction of the desired 6-regular geometric triangulations with uniform bound in the hyperbolic plane using the Klein disk model.

(1) Put one vertex $v_0$ in the origin, and shoot out two rays $R_1$ and $R_2$ along the positive $x$-axis and along the direction $e^{i\pi/3}$.

(2) Let $r_n = \alpha \log n$ with $0 < \alpha < 1/2$. Construct a sequence of points $a_n$ and $b_n$ on $R_1$ and $R_2$ with distance $r_n$ from $v_0$ in hyperbolic metric. Connect $a_n$ with $a_{n+1}$ and $b_n$ with $b_{n+1}$ using geodesics. Call these edges of type 0.

(3) Connect $a_n$ with $b_n$ by circle arcs $C_n$, called the $n$-th layer in this construction. Equally distribute $n-1$ points on $C_n$. Then each $C_n$ are divided to $n$ arcs. Connect the consecutive vertices on $C_n$ to generate $n$ edges. Call these edges of type 1. By symmetry, these edge have the same length.
(4) Connect points on $C_n$ and $C_{n+1}$ based on the combinatorics of the hexagonal triangulation to generate $2(n+1)$ edges. Call these edges of type 2.

(5) Rotate the configuration by $\pi/3$ to generate the full hexagon geometric triangulation.

We can give explicit coordinates of the vertices on $C_n$ in this triangulation in the plane

$$v_n^k = \tanh r_n e^{i(k \pi)/3n}, \quad k = 0, 1, \ldots, n.$$ 

And the vertex $v_n^k$ is connected with vertices $v_{n+1}^k$ and $v_{n+1}^{k+1}$ in this triangulation.

**Proposition 5.0.1.** The construction above generates a hexagonal geometric triangulation of $\mathbb{H}^2$ with uniform bound $K > 0$ on the lengths of edges.

The proof of this proposition consists of two lemmas.

**Lemma 5.0.2.** The construction above generates a valid geometric triangulations with no intersection of edges and degenerate triangles.

**Proof.** The proof is based on induction on $n$. The base case $n = 1$ is trivial. Assume it is true for the triangulation generated up to the $n$-th layer. We will show that the triangles between $C_n$ and $C_{n+1}$ lie in the ring $R_n$ bounded $C_n$ and $C_{n+1}$. There are two types of triangles:

1. **Type 1:** two vertices on $C_n$ and one vertex on $C_{n+1}$. The three vertices are

$$v_n^k = \tanh r_n e^{i(k \pi)/3n}, \quad v_{n+1}^k = \tanh r_{n+1} e^{i(k+1) \pi/3n}, \quad \text{and} \quad v_{n+1}^{k+1} = \tanh r_{n+1} e^{i(k+1) \pi/3(n+1)}.$$ 

Notice that

$$\frac{k}{n} < \frac{k+1}{n+1} < \frac{k+1}{n}, \quad \text{and} \quad r_n \leq r_{n+1},$$

2. **Type 2:** one vertex on $C_n$ and two vertices on $C_{n+1}$. In this case, the three vertices are

$$v_n^k = \tanh r_n e^{i(k \pi)/3n}, \quad v_{n+1}^k = \tanh r_{n+1} e^{i(k+1) \pi/3n}, \quad \text{and} \quad v_{n+1}^{k+1} = \tanh r_{n+1} e^{i(k+1) \pi/3(n+1)}.$$ 

Notice that

$$\frac{k}{n+1} \leq \frac{k}{n} \leq \frac{k+1}{n+1}, \quad \text{and} \quad r_n \leq r_{n+1}.$$ 

We want to show that the edges of type 2 lie in the ring bounded by two consecutive $C_n$ and $C_{n+1}$, so it does not intersect with previous layers. If $r_n$ increase too slowly, then it is possible that the edge determined by $v_n^k$ and $v_{n+1}^{k+1}$ might intersect with $v_n^k$. We need to show that if $r_n = \alpha \log n$ with $0 < \alpha < 1/2$, this will not occur.

The critical figures for triangles of type 1 and type 2 are given below, where the angles at $v_n^k$ or $v_{n+1}^{k+1}$ are right angles.
In this case, we can show that for sufficiently large \( n > N \),
\[
\tanh r_n < \tanh r_{n+1} \cos \frac{\pi}{6(n + 1)},
\]
and
\[
\tanh r_n < \tanh r_{n+1} \cos \frac{\pi}{6n},
\]
which is equivalent to
\[
\sinh (r_{n+1} - r_n) > \cosh r_n \sinh r_{n+1} (1 - \cos \frac{\pi}{6n}).
\]
One can show that the left side has order $O(1/n)$ and right side has order $O(1/n^{2-2\alpha})$, so the inequality holds for large $n$ if $0 < \alpha < 1/2$.

This means that for large $n > N$, the triangles of type 2 will not intersect with the previous layers. For $0 \leq n \leq N$, we can treat it as a polygon in the Euclidean plane, and adjust the radius to avoid the intersections. \hfill \Box

**Lemma 5.0.3.** The lengths of edges of the geometric triangulation above are uniformly bounded.

**Proof.** The length of the edge of type 1 is bounded by the length of the arc of the circle connecting $v_n^k$ and $v_n^{k+1}$ given by

$$\sinh |v_n^k v_n^{k+1}| < \frac{\pi}{3n} \sinh r_n \sim O\left(\frac{1}{n^{1-\alpha}}\right) \to 0.$$  

From the previous lemma, all the edges of type 0 and type 2 in the $n$-th layer lie in the ring $R_n$. The length of the edges in $R_n$ with vertices on $C_n$ is bounded by the following situation

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6}
\caption{Estimate for Edges of Type 2.}
\end{figure}

The half length $L$ of this edge is given by hyperbolic Pythagorean theorem

$$L = 2 \cosh^{-1} \left( \frac{\cosh r_{n+1}}{\cosh r_n} \right) \leq 2 \cosh^{-1} (2(1 + \frac{1}{n})^\alpha) < \infty.$$  

Hence we can bound the edges of all the three types for large $n > 0$. The remaining part is naturally bounded by compactness. \hfill \Box

6. **Related work**

The existence of certain types of geodesic triangulations is a fundamental question. Recently various theory upon geodesic triangle meshes were developed, where people discuss problems about rigidity, convergence, variational principles, discrete maps and discrete geometric structures. See [3] [4] [14] [15] [16] [10] [1] [7] [6] [18] [5] [11] [20] [19] [21] [12] [13]. for example.
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