SIMULTANEOUS UNITARY EQUIVALENCE TO BI-CARLEMAN OPERATORS WITH ARBITRARILY SMOOTH KERNELS OF MERCER TYPE

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ABSTRACT. In this paper, we characterize the families of those bounded linear operators on a separable Hilbert space which are simultaneously unitarily equivalent to integral bi-Carleman operators on $L_2(\mathbb{R})$ having arbitrarily smooth kernels of Mercer type. The main result is a qualitative sharpening of an earlier result of [7].

1. INTRODUCTION. MAIN RESULT

Throughout, $\mathcal{H}$ will denote a separable Hilbert space with the inner product $\langle \cdot , \cdot \rangle_{\mathcal{H}}$ and the norm $\| \cdot \|_{\mathcal{H}}$, $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and $\mathbb{C}$, and $\mathbb{N}$, and $\mathbb{Z}$, the complex plane, the set of all positive integers, the set of all integers, respectively. For an operator $A$ in $\mathcal{B}(\mathcal{H})$, $A^*$ will denote the Hilbert space adjoint of $A$ in $\mathcal{B}(\mathcal{H})$. Given an operator $T \in \mathcal{B}(\mathcal{H})$, define an operator set

$$\mathcal{M}(T) = (T \mathcal{B}(\mathcal{H}) \cup T^* \mathcal{B}(\mathcal{H})) \cap (\mathcal{B}(\mathcal{H}) T^* \cup \mathcal{B}(\mathcal{H}) T)$$

where $S \mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{H}) S$ stand for the sets

$$\{ SA \mid A \in \mathcal{B}(\mathcal{H}) \}, \quad \{ AS \mid A \in \mathcal{B}(\mathcal{H}) \},$$

respectively.

Throughout, $C(X, B)$, where $B$ is a Banach space (with norm $\| \cdot \|_B$), denote the Banach space (with the norm $\| f \|_{C(X, B)} = \sup_{x \in X} \| f(x) \|_B$) of continuous $B$-valued functions defined on a locally compact space $X$ and vanishing at infinity (that is, given any $f \in C(X, B)$ and $\varepsilon > 0$, there exists a compact subset $X(\varepsilon, f) \subset X$ such that $\| f(x) \|_B < \varepsilon$ whenever $x \notin X(\varepsilon, f)$).

Let $\mathbb{R}$ be the real line $(-\infty, +\infty)$ with the Lebesgue measure, and let $L_2 = L_2(\mathbb{R})$ be the Hilbert space of (equivalence classes of) measurable

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complex-valued functions on \( \mathbb{R} \) equipped with the inner product

\[
\langle f, g \rangle = \int_{\mathbb{R}} f(s) \overline{g(s)} \, ds
\]

and the norm \( \| f \| = \langle f, f \rangle^{\frac{1}{2}} \).

A linear operator \( T : L_2 \to L_2 \) is said to be integral if there exists a measurable function \( T \) on the Cartesian product \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \), a kernel, such that, for every \( f \in L_2 \),

\[
(Tf)(s) = \int_{\mathbb{R}} T(s, t) f(t) \, dt
\]

for almost every \( s \) in \( \mathbb{R} \). A kernel \( T \) on \( \mathbb{R}^2 \) is said to be Carleman if \( T(s, \cdot) \in L_2 \) for almost every fixed \( s \) in \( \mathbb{R} \). An integral operator with a kernel \( T \) on \( \mathbb{R}^2 \) is said to be Carleman if \( T(s, \cdot) \in L_2 \) for almost every fixed \( s \) in \( \mathbb{R} \). An integral operator with a kernel \( T \) is called Carleman if \( T \) is a Carleman kernel, and it is called bi-Carleman if both \( T \) and \( T^* \) (\( T^*(s, t) = T(t, s) \)) are Carleman kernels. Every Carleman kernel, \( T \), induces a Carleman function \( t \) from \( \mathbb{R} \) to \( L_2 \) by \( t(s) = T(s, \cdot) \) for all \( s \) in \( \mathbb{R} \) for which \( T(s, \cdot) \in L_2 \).

We shall also recall a characterization of bi-Carleman representable operators. Its version for self-adjoint operators was first obtained by von Neumann [10] and was later extended by Korotkov to the general case (see [4, p. 100], [2, p. 103]). The assertion says that a necessary and sufficient condition that an operator \( S \in \mathcal{B}(\mathcal{H}) \) be unitarily equivalent to a bi-Carleman operator is that there exist an orthonormal sequence \( \{e_n\} \) such that

(1) \[ \| Se_n \|_{\mathcal{H}} \to 0, \quad \| S^* e_n \|_{\mathcal{H}} \to 0 \quad as \quad n \to \infty \]

(or, equivalently, that \( 0 \) belong to the essential spectrum of \( SS^* + S^* S \)).

**Definition 1.** Given any non-negative integer \( m \), we say that a function \( K \) on \( \mathbb{R}^2 \) is a \( K^m \)-kernel (see [7], [6]) if

(i) the function \( K \) and all its partial derivatives on \( \mathbb{R}^2 \) up to order \( m \) are in \( C(\mathbb{R}^2, \mathbb{C}) \),

(ii) the Carleman function \( k, k(s) = \overline{K(s, \cdot)} \), and all its (strong) derivatives on \( \mathbb{R} \) up to order \( m \) are in \( C(\mathbb{R}, L_2) \),

(iii) the conjugate transpose function \( K^*, K^*(s, t) = \overline{K(t, s)} \), satisfies Condition (ii), that is, the Carleman function \( k^*, k^*(s) = \overline{K^*(s, \cdot)} \), and all its (strong) derivatives on \( \mathbb{R} \) up to order \( m \) are in \( C(\mathbb{R}, L_2) \).

In addition, we say that a function \( K \) is a \( K^\infty \)-kernel (see [8], [9]) if it is a \( K^m \)-kernel for each non-negative integer \( m \).

**Definition 2.** Let \( K \) be a \( K^m(\mathbb{R}^2) \)-kernel and let \( T \) be the integral operator it induces. We say that the \( K^m(\mathbb{R}^2) \)-kernel \( K \) is of Mercer type if every operator \( A \in \mathcal{M}(T) \) is an integral operator having \( K^m(\mathbb{R}^2) \)-kernel.

The concept of Mercer type \( K^m \)-kernels for finite \( m \) was first introduced in our paper [7] where there is a motivation of the reason why this subclass of \( K^m \)-kernels deserves the qualification “of Mercer type”.
Given any non-negative integer \( m \), the following result both gives a characterization of all bounded operators whose unitary orbits contain a bi-Carleman operator having \( K^m \)-kernel of Mercer type and describes families of those operators that can be simultaneously unitarily represented as bi-Carleman operators having \( K^m \)-kernels of Mercer type (cf. (1)).

**Proposition (7).** If for an operator family \( \{ S_\alpha \mid \alpha \in \mathcal{A} \} \subset \mathcal{R}(\mathcal{H}) \) there exists an orthonormal sequence \( \{ e_n \} \) such that

\[
\lim_{n \to \infty} \sup_{\alpha \in \mathcal{A}} \| S_\alpha^* e_n \|_{\mathcal{H}} = 0, \quad \lim_{n \to \infty} \sup_{\alpha \in \mathcal{A}} \| S_\alpha e_n \|_{\mathcal{H}} = 0,
\]

then there exists a unitary operator \( U_m : \mathcal{H} \to L^2 \) such that all the operators \( U_m S_\alpha U_m^{-1} \) (\( \alpha \in \mathcal{A} \)) and their linear combinations are bi-Carleman operators having \( K^m \)-kernels of Mercer type.

The construction of the unitary operator \( U_m \) given in the proof of Proposition depends on the preassigned order \( m < \infty \) of smoothness (see [7]). The purpose of the present paper is to show that Proposition is true with \( K^\infty \)-kernels in the conclusion, that is, to prove the following qualitative sharpening of Proposition.

**Theorem.** If for an operator family \( \{ S_\alpha \mid \alpha \in \mathcal{A} \} \subset \mathcal{R}(\mathcal{H}) \) there exists an orthonormal sequence \( \{ e_n \} \) such that

\[
\lim_{n \to \infty} \sup_{\alpha \in \mathcal{A}} \| S_\alpha^* e_n \|_{\mathcal{H}} = 0, \quad \lim_{n \to \infty} \sup_{\alpha \in \mathcal{A}} \| S_\alpha e_n \|_{\mathcal{H}} = 0,
\]

then there exists a unitary operator \( U_\infty : \mathcal{H} \to L^2 \) such that all the operators \( U_\infty S_\alpha U_\infty^{-1} \) (\( \alpha \in \mathcal{A} \)) and their linear combinations are bi-Carleman operators having \( K^\infty \)-kernels of Mercer type.

\[\text{2. PROOF OF THEOREM}\]

The proof is broken up into three steps. The first step is to find suitable orthonormal bases \( \{ u_n \} \) in \( L^2 \) and \( \{ f_n \} \) in \( \mathcal{H} \) on which the construction of \( U_\infty \) will be based. The next step is to define a certain unitary operator that sends the basis \( \{ f_n \} \) onto the basis \( \{ u_n \} \). This operator is suggested as \( U_\infty \) in the theorem, and the rest of the proof is a straightforward verification that it is indeed as desired. Thus, the proof yields more than just existence of the unitary equivalence; it yields an explicit construction of the unitary operator. From the point of view of the applications to operator equations, the explicit computability of \( U_\infty \) is an important side issue.

**Step 1.** For the proof, it will be convenient to have the following notation: if an equivalence class \( f \in L^2 \) contains a function belonging to \( C(\mathbb{R}, \mathbb{C}) \), then we shall use \( [f] \) to denote that function.

Let \( \{ S_\alpha \mid \alpha \in \mathcal{A} \} \subset \mathcal{R}(\mathcal{H}) \) be a family satisfying \( \text{(2)} \) with the orthonormal sequence \( \{ e_n \}_{n=1}^\infty \). Take orthonormal bases \( \{ f_n \} \) for \( \mathcal{H} \) and \( \{ u_n \} \) for \( L^2 \) which satisfy the conditions:
Remark. Let \( \{u_n\} \) of derivatives be in \( C(\mathbb{R}, \mathbb{C}) \),
for each \( i \) (here and throughout, the letter \( i \) is reserved for integers
in \( [0, +\infty) \)),
\( \{u_n\} = \{g_k\}_{k=1}^{\infty} \cup \{h_k\}_{k=1}^{\infty} \), where \( \{g_k\}_{k=1}^{\infty} \cap \{h_k\}_{k=1}^{\infty} = \emptyset \), and,
for each \( i \),
\[
\sum_k H_{k,i} < \infty \quad \text{with} \quad H_{k,i} = \left\| [h_k]^{(i)} \right\|_{C(\mathbb{R}, \mathbb{C})} \quad (k \in \mathbb{N})
\]
(the sum notation \( \sum_k \) will always be used instead of the more detailed
symbol \( \sum_{k=1}^{\infty} \)).
\( \{f_n\} = \{x_k\}_{k=1}^{\infty} \cup \{y_k\}_{k=1}^{\infty} \) where \( \{x_k\}_{k=1}^{\infty} \cap \{y_k\}_{k=1}^{\infty} = \emptyset \), \( \{x_k\}_{k=1}^{\infty} \subset
\{e_n\}_{n=1}^{\infty} \) and, for each \( i \),
\[
\sum_k d_k (G_{k,i} + 1) < \infty
\]
with \( d_k = 2 \left( \sup \alpha \|S_{\alpha} x_k\|_{L^1} + \sup \alpha \|S_{\alpha}^* x_k\|_{L^1} \right) \leq 1 \), and \( G_{k,i} = \left\| [g_k]^{(i)} \right\|_{C(\mathbb{R}, \mathbb{C})} \) \( (k \in \mathbb{N}) \).
The proof uses the bases just described to construct the desired unitary oper-
ator \( U_{\infty} \).

Remark. Let \( \{u_n\} \) be an orthonormal basis for \( L_2 \) such that, for each \( i \),
\[
[u_n]^{(i)} \in C(\mathbb{R}, \mathbb{C}) \quad (n \in \mathbb{N}),
\]
\[
\left\| [u_n]^{(i)} \right\|_{C(\mathbb{R}, \mathbb{C})} \leq D_n A_i \quad (n \in \mathbb{N}),
\]
\[
\sum_k D_{n_k} < \infty,
\]
where \( \{D_n\}_{n=1}^{\infty}, \{A_i\}_{i=0}^{\infty} \) are sequences of positive numbers, and \( \{n_k\}_{k=1}^{\infty} \) is
a subsequence of \( \mathbb{N} \) such that \( \mathbb{N} \setminus \{n_k\}_{k=1}^{\infty} \) is a countable set. Since \( d(e_n) \to 0 \) as \( n \to \infty \), it follows that there exists a subset \( \{x_k\}_{k=1}^{\infty} \subset \{e_n\}_{n=1}^{\infty} \) for
which Condition (4) holds with \( \{g_k\}_{k=1}^{\infty} = \{u_n\} \setminus \{u_{nk}\}_{k=1}^{\infty} \). Moreover, the
properties (6) and (7) imply Condition (8) for \( h_k = u_{nk} \) \( (k \in \mathbb{N}) \). Complete
the set \( \{x_k\}_{k=1}^{\infty} \) to an orthonormal basis, and let \( y_k \) \( (k \in \mathbb{N}) \) denote the new
elements of that basis. Then the bases \( \{f_n\} = \{x_k\}_{k=1}^{\infty} \cup \{y_k\}_{k=1}^{\infty} \) and \( \{u_n\} \)
satisfy Conditions (a)-(c).

A good example of the basis satisfying (5)-(7) is a basis generated by the
Lemarié-Meyer wavelet
\[
u(s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(\frac{1}{2} + s)} \text{sign} \xi b(|\xi|) \, d\xi \quad (s \in \mathbb{R}),
\]
with the bell function \( b \) belonging to \( C^\infty(\mathbb{R}) \) (for construction of the Lemarié-
Meyer wavelets we refer to [5], [11, § 4], [3, Example D, p. 62]). In this case,
\( u \) belongs to the Schwartz class \( \mathcal{S}(\mathbb{R}) \), and hence all the derivatives \( [u]^{(i)} \) are
in $\mathcal{C}(\mathbb{R}, \mathbb{C})$. The corresponding orthonormal basis for $L_2$ is given by

$$u_{jk}(s) = 2^j s (2^j s - k) \quad (j, k \in \mathbb{Z}).$$

Rearrange, in a completely arbitrary manner, the orthonormal set $\{u_{jk}\}_{j,k \in \mathbb{Z}}$ into a simple sequence, so that it becomes $\{u_n\}_{n \in \mathbb{N}}$. Since, in view of this rearrangement, to each $n \in \mathbb{N}$ there corresponds a unique pair of integers $j_n, k_n$, and conversely, we can write, for each $i$,

$$\left\| [u_n]^{(i)} \right\|_{\mathcal{C}(\mathbb{R}, \mathbb{C})} = \left\| [u_{j_n k_n}]^{(i)} \right\|_{\mathcal{C}(\mathbb{R}, \mathbb{C})} \leq D_n A_i,$$

where

$$D_n = \begin{cases} 2^j n & \text{if } j_n > 0, \\ \left(\frac{1}{\sqrt{2}}\right)^{|j_n|} & \text{if } j_n \leq 0. \end{cases} \quad A_i = 2^{(i+\frac{1}{2})^2} \left\| [u]^{(i)} \right\|_{\mathcal{C}(\mathbb{R}, \mathbb{C})}.$$

Whence it follows that if $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ is a subsequence such that $j_{n_k} \to -\infty$ as $k \to \infty$, then

$$\sum_k D_{n_k} < \infty.$$

Thus, the basis $\{u_n\}$ satisfies Conditions (5)-(7).

**Step 2.** In this step our intention is to construct a candidate for the desired unitary operator $U_\infty$ in the theorem. Define such a unitary operator $U_\infty : \mathcal{H} \to L_2$ on the basis vectors by setting

$$(8) \quad U_\infty x_k = g_k, \quad U_\infty y_k = h_k \quad \text{for all } k \in \mathbb{N},$$

in the harmless assumption that $U_\infty f_n = u_n$ for all $n \in \mathbb{N}$.

**Step 3.** The verification that $U_\infty$ in (8) has the desired properties is straightforward. Fix an arbitrary $\alpha \in \mathcal{A}$ and put $T = U_\infty S_\alpha U_\infty^{-1}$. Once this is done, the index $\alpha$ may be omitted for $S_\alpha$.

Let $E$ be the orthogonal projection onto the closed linear span of the vectors $x_k$ ($k \in \mathbb{N}$). Split the operator $S$ as follows:

$$(9) \quad S = (1 - E) S + ES, \quad S^* = (1 - E) S^* + ES^*.$$  

The operators $J = SE$ and $\tilde{J} = S^* E$ are nuclear operators and, therefore, are Hilbert–Schmidt operators; these properties are almost immediate consequences of (4).

Write the Schmidt decompositions

$$J = \sum_n s_n \langle \cdot, p_n \rangle_\mathcal{H} q_n, \quad \tilde{J} = \sum_n \tilde{s}_n \langle \cdot, \tilde{p}_n \rangle_\mathcal{H} \tilde{q}_n,$$

where the $s_n$ are the singular values of $J$ (eigenvalues of $(JJ^*)^{\frac{1}{2}}$), $\{p_n\}$, $\{q_n\}$ are orthonormal sets (the $p_n$ are eigenvectors for $J^* J$ and $q_n$ are eigenvectors for $JJ^*$). The explanation of the notation for $\tilde{J}$ is similar.

Now introduce auxiliary operators $B, \tilde{B}$ by

$$(10) \quad B = \sum_n s_n^{\frac{1}{2}} \langle \cdot, p_n \rangle_\mathcal{H} q_n, \quad \tilde{B} = \sum_n \tilde{s}_n^{\frac{1}{2}} \langle \cdot, \tilde{p}_n \rangle_\mathcal{H} \tilde{q}_n.$$
The Schwarz inequality yields

\[
\|B^* x_k\|_\mathcal{H} + \|B x_k\|_\mathcal{H} + \|\tilde{B}^* x_k\|_\mathcal{H} + \|\tilde{B} x_k\|_\mathcal{H} \\
= \|(J J^*)^{1/2} x_k\|_\mathcal{H} + \|(J^* J)^{1/2} x_k\|_\mathcal{H} \\
+ \|(\tilde{J} \tilde{J}^*)^{1/2} x_k\|_\mathcal{H} + \|(\tilde{J}^* \tilde{J})^{1/2} x_k\|_\mathcal{H} \\
\leq \|J^* x_k\|_\mathcal{H}^{1/2} + \|J x_k\|_\mathcal{H}^{1/2} + \|\tilde{J}^* x_k\|_\mathcal{H}^{1/2} + \|\tilde{J} x_k\|_\mathcal{H}^{1/2} \leq d_k.
\]

It follows that all the operators \(B, \tilde{B}\) are nuclear operators (see (4)) and hence

\[
\sum_n s_n^{1/2} < \infty, \quad \sum_n \tilde{s}_n^{1/2} < \infty.
\]

Define \(Q = (1 - E)S^*, \widetilde{Q} = (1 - E)S\). Then Condition (c) provides the representations

\[
Qf = \sum_k \langle Qf, y_k \rangle_\mathcal{H} y_k = \sum_k \langle f, S y_k \rangle_\mathcal{H} y_k,
\]

\[
\tilde{Q}f = \sum_k \langle \tilde{Q}f, y_k \rangle_\mathcal{H} y_k = \sum_k \langle f, S^* y_k \rangle_\mathcal{H} y_k,
\]

for all \(f \in \mathcal{H}\).

Using the decompositions (9), which now look like \(S = \tilde{Q} + J^*, S^* = Q + J^*\), we shall prove presently that \(T^*\) is an integral operator having \(K^\infty\)-kernel of Mercer type.

From (13) and (8), it follows that, for each \(f \in L_2\),

\[
Pf = U_\infty Q U_\infty^{-1} f = \sum_k \langle f, T h_k \rangle_\mathcal{H} h_k,
\]

\[
\tilde{P}f = U_\infty \tilde{Q} U_\infty^{-1} f = \sum_k \langle f, T^* h_k \rangle_\mathcal{H} h_k.
\]

Represent the equivalence classes \(T h_k, T^* h_k (k \in \mathbb{N})\) by the Fourier expansions

\[
T h_k = \sum_n \langle y_k, S^* f_n \rangle_\mathcal{H} u_n, \quad T^* h_k = \sum_n \langle y_k, S f_n \rangle_\mathcal{H} u_n,
\]

where the series converge in the \(L_2\) sense. But more than that can be said about convergence, namely that, for each fixed \(i\), the series

\[
\sum_n \langle y_k, S^* f_n \rangle_\mathcal{H} [u_n]^{(i)}(s), \quad \sum_n \langle y_k, S f_n \rangle_\mathcal{H} [u_n]^{(i)}(s) \quad (k \in \mathbb{N})
\]

converge in the norm of \(C(\mathbb{R}, \mathbb{C})\). Indeed, all the series are everywhere pointwise dominated by one series

\[
\sum_n (\|S^* f_n\|_\mathcal{H} + \|S f_n\|_\mathcal{H}) \left| [u_n]^{(i)}(s) \right|,
\]
which is uniformly convergent on \( \mathbb{R} \) for the following reason: its subseries
\[
\sum_k \left( \|Sx_k\|_H + \|S^*x_k\|_H \right) |g_k|^{(i)}(s),
\]
\[
\sum_k \left( \|Sy_k\|_H + \|S^*y_k\|_H \right) |h_k|^{(i)}(s)
\]
are uniformly convergent on \( \mathbb{R} \) because they in turn are dominated by the convergent series
\[
\sum_k d_k G_{k,i}, \quad \sum_k 2\|S\| H_{k,i},
\]
respectively (see (4), (3)).

It is now evident that the pointwise sums in (15) define functions that belong to \( C(\mathbb{R}, \mathbb{C}) \). Moreover, the above arguments prove that, for each fixed \( i \), the derivative sequences \( \{[Th_k]^{(i)}\}, \{[T^*h_k]^{(i)}\} \) are uniformly bounded in \( C(\mathbb{R}, \mathbb{C}) \) in the sense that there exists a positive constant \( C_i \) such that
\[
\|T h_k\|^{(i)}_{C(\mathbb{R}, \mathbb{C})} < C_i, \quad \|T^* h_k\|^{(i)}_{C(\mathbb{R}, \mathbb{C})} < C_i,
\]
for all \( k \). Hence, by (3), it is possible to infer that, for all non-negative integers \( i, j \), both
\[
\sum_k [h_k]^{(i)}(s)[Th_k]^{(j)}(t) \quad \text{and} \quad \sum_k [h_k]^{(i)}(s)[T^*h_k]^{(j)}(t)
\]
converge in the norm of \( C(\mathbb{R}^2, \mathbb{C}) \). This makes it obvious that both
\[
P(s, t) = \sum_k [h_k](s)[Th_k](t)
\]
and
\[
\tilde{P}(s, t) = \sum_k [h_k](s)[T^*h_k](t),
\]
(17)
satisfy Condition (i) for each \( m \).

Now we prove that the (Carleman) functions
\[
p(s) = \overline{P}(s, \cdot) = \sum_k [h_k](s)Th_k,
\]
(18)
\[
\overline{p}(s) = \overline{P}(s, \cdot) = \sum_k [h_k](s)T^*h_k
\]
satisfy Condition (ii) for all \( m \). Indeed, the series displayed converge absolutely in the \( C(\mathbb{R}, L_2) \) sense, because those two series whose terms are \( \|h_k\|(s)\|Th_k\| \) and \( \|h_k\|(s)\|T^*h_k\| \), respectively, are dominated by the second series in (16) for \( i = 0 \). For the remaining \( i \), a similar reasoning implies the same conclusion for the series
\[
\sum_k [h_k]^{(i)}(s)Th_k, \quad \sum_k [h_k]^{(i)}(s)T^*h_k.
\]
The asserted property of both \( p \) and \( \overline{p} \) to satisfy (ii) for each \( m \) then follows from the termwise differentiation theorem. Now observe that, by (3) and
The series in (14) (viewed, of course, as ones with terms belonging to \(C(\mathbb{R}, \mathbb{C})\)) converge (absolutely) in \(C(\mathbb{R}, \mathbb{C})\)-norm to the functions
\[
[Pf](s) \equiv \langle f, p(s) \rangle \equiv \int_{\mathbb{R}} P(s, t) f(t) \, dt,
\]
\[
[\tilde{P}f](s) \equiv \langle f, \tilde{p}(s) \rangle \equiv \int_{\mathbb{R}} \tilde{P}(s, t) f(t) \, dt,
\]
respectively. Thus, both \(P\) and \(\tilde{P}\) are Carleman operators with \(P\) and \(\tilde{P}\)
their kernels, respectively, satisfying Conditions (i), (ii) for each \(m\).

Now consider the (integral) Hilbert–Schmidt operators \(F = U_\infty J^* U_\infty^{-1}\)
and \(\tilde{F} = U_\infty \tilde{J}^* U_\infty^{-1}\). Prove that both \(F\) and \(\tilde{F}\) have kernels satisfying (i)
for each \(m\). Starting from the Schmidt decompositions for \(F\) and \(\tilde{F}\), define
their kernels by
\[
F(s, t) = \sum_n s_n^m \left[ U_\infty B^* q_n \right](s) \left[ U_\infty B p_n \right](t),
\]
\[
\tilde{F}(s, t) = \sum_n \tilde{s}_n^m \left[ U_\infty \tilde{B}^* \tilde{q}_n \right](s) \left[ U_\infty \tilde{B} \tilde{p}_n \right](t),
\]
for all \(s, t \in \mathbb{R}\), in the tacit assumption that the square brackets are everywhere permissible (for the auxiliary operators \(\tilde{B}, \tilde{\tilde{B}}\) see (10)). In view of
the desired conclusion that the kernels so defined satisfy (i) for each \(m\)
can be inferred as soon as it is known that for each fixed \(i\) the terms of the sequences
\[
\left\{ [U_\infty B p_k]^{(i)} \right\}, \quad \left\{ [U_\infty B^* q_k]^{(i)} \right\}, \quad \left\{ [U_\infty \tilde{B}^* \tilde{q}_k]^{(i)} \right\}, \quad \left\{ [U_\infty \tilde{B} \tilde{p}_k]^{(i)} \right\}
\]
make sense, are in \(C(\mathbb{R}, \mathbb{C})\), and are uniformly bounded in \(C(\mathbb{R}, \mathbb{C})\).

To see the validity of the properties indicated, observe that all the series
\[
\sum_n \langle p_k, B^* f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \langle q_k, B f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s),
\]
\[
\sum_n \langle \tilde{p}_k, \tilde{B}^* f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \langle \tilde{q}_k, \tilde{B} f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s) \quad (k \in \mathbb{N})
\]
(which in the case where \(i = 0\) are just the Fourier expansions for \(U_\infty B p_k\),
\(U_\infty B^* q_k\), \(U_\infty \tilde{B} \tilde{p}_k\), \(U_\infty \tilde{B}^* \tilde{q}_k\)) are dominated by one series
\[
\sum_n c(f_n) \left| [u_n]^{(i)}(s) \right|,
\]
where \(c(g) = \| B^* g \|_{\mathcal{H}} + \| B g \|_{\mathcal{H}} + \| \tilde{B}^* g \|_{\mathcal{H}} + \| \tilde{B} g \|_{\mathcal{H}}\) whenever \(g \in \mathcal{H}\).

The last series is uniformly convergent, because it consists of the two dominatedly and uniformly convergent subseries
\[
\sum_n c(x_k) \left| [g_k]^{(i)}(s) \right|, \quad \sum_n c(y_k) \left| [h_k]^{(i)}(s) \right|
\]
the corresponding dominant series are
\[
\sum_k d_k G_{k,i}, \quad \sum_k 2 \left( \| B \| + \| \tilde{B} \| \right) H_{k,i}
\]
the last series is uniformly convergent, because it consists of the two dominatedly and uniformly convergent subseries
\[
\sum_n c(x_k) \left| [g_k]^{(i)}(s) \right|, \quad \sum_n c(y_k) \left| [h_k]^{(i)}(s) \right|
\]
the corresponding dominant series are
\[
\sum_k d_k G_{k,i}, \quad \sum_k 2 \left( \| B \| + \| \tilde{B} \| \right) H_{k,i}
\]
(see (14), (4), (3)).

In view of (12) and the uniform boundedness in \( C(\mathbb{R}, \mathbb{C}) \) of the sequences 
\[
\{ [U_\infty B^* q_n]^{(i)} \}, \quad \{ [U_\infty \tilde{B}^* q_n]^{(i)} \}
\]
each operators, and their kernels 
\[
K^{(i)} = 0
\]
for each \( i \), the series
\[
\sum_n \frac{1}{n!} [U_\infty B^* q_n]^{(i)}(s) U_\infty B p_n, \quad \sum_n \frac{1}{n!} [U_\infty \tilde{B}^* q_n]^{(i)}(s) U_\infty \tilde{B} p_n
\]
are absolutely convergent in the \( C(\mathbb{R}, L_2) \) sense, and hence their sums belong to \( C(\mathbb{R}, L_2) \). Observe by (19) that two of them, namely those for \( i = 0 \), represent the Carleman functions \( f(s) = F(s, \cdot), \tilde{f}(s) = \tilde{F}(s, \cdot) \).

Thus, both Carleman functions \( f \) and \( \tilde{f} \) satisfy Condition (ii) for every \( m \).

In accordance with (7), the operator \( T \), which is the transform by \( U_\infty \) of \( S \), has the decompositions \( T = \tilde{P} + \tilde{F}, T^* = P + F \) where all the terms are the Carleman operators already described. So both \( T \) and \( T^* \) are Carleman operators, and their kernels \( K \) and \( \tilde{K} \), which are defined by
\[
K(s, t) = \tilde{P}(s, t) + \tilde{F}(s, t), \quad \tilde{K}(s, t) = P(s, t) + F(s, t),
\]
for all \( s, t \in \mathbb{R} \), inherit the two properties (i), (ii) from their terms, for each \( m \). Since (cf. [2, p. 37]) it is possible to write \( K(s, t) = \tilde{K}(t, s) \) and \( \tilde{K}(s, t) = \tilde{K}(t, \cdot) \) for all \( s, t \in \mathbb{R} \), the kernel \( K \) satisfies Conditions (i), (ii), (iii) for each \( m \), so that it is a \( K^\infty \)-kernel.

As the preceding proof shows, the major condition that an operator \( A \in \mathfrak{R}(\mathcal{H}) \) must satisfy in order that \( U_\infty A U_\infty^{-1} \) be an integral operator having \( K^\infty \)-kernel is that, for each \( k \),
\[
2 \left( \|Ax_k\|_{\mathcal{H}} + \|A^*x_k\|_{\mathcal{H}} \right) \leq d_k.
\]
If \( A \in \mathcal{M}(S) \) then there exist operators \( V, W \in \mathfrak{R}(\mathcal{H}) \) such that at least one of the relations \( A = SV = WS, A = S^*V = WS^*, A = SV = WS^*, A = VS = S^*W \) holds. In any event, whatever its decomposition may be, the operator \( A \) satisfies the inequalities
\[
2 \left( \|Ax_k\|_{\mathcal{H}} + \|A^*x_k\|_{\mathcal{H}} \right) \leq 2c \left( \|Ax_k\|_{\mathcal{H}} + \|A^*x_k\|_{\mathcal{H}} \right) \leq cd_k,
\]
where \( c^d = \max \{ \|V\|, \|W\| \} \). This implies, by the above remark, that \( U_\infty \) automatically carries every \( A \in \mathcal{M}(S) \) onto an integral operator \( U_\infty A U_\infty^{-1} \) having \( K^\infty \)-kernel so that \( K \) in (20) is a \( K^\infty \)-kernel of Mercer type.

The fact that those \( K^\infty \)-kernels which induce finite linear combinations of \( U_\infty S_\alpha U_\infty^{-1} \) are of Mercer type remains to be proved. The result can be inferred from the result for \( S \), which has just been obtained. Indeed, consider any finite linear combination \( G = \sum z_\alpha S_\alpha \) with \( \sum |z_\alpha| \leq 1 \). It is seen easily that, for each \( n \),
\[
\left\| \sum z_\alpha S_\alpha e_n \right\|_{\mathcal{H}} \leq \sup_\alpha \|S_\alpha e_n\|_{\mathcal{H}}, \quad \left\| \sum z_\alpha S^*_\alpha e_n \right\|_{\mathcal{H}} \leq \sup_\alpha \|S^*_\alpha e_n\|_{\mathcal{H}}.
\]
There is, therefore, no barrier to assuming that \( G \) was, from the start, in \( \{S_\alpha \mid \alpha \in \mathcal{A}\} \) and even equal to \( S \). The proof of Theorem is complete.
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