UNIQUENESS OF THE FOURIER TRANSFORM ON THE EUCLIDEAN MOTION GROUP

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Dedicated to Prof. Gadadhar Misra on the occasion of his 60th birthday.

Abstract. In this article, we prove that if the Fourier transform of a certain integrable function on the Euclidean motion group is of finite rank, then the function has to vanish identically. Further, we explore a new variance of the uncertainty principle, the Heisenberg uniqueness pairs on the Euclidean motion group as well as on the product group $\mathbb{R}^n \times K$, where $K$ is a compact group.

1. Introduction

In a fundamental article, M. Benedicks [4] had generalized the Euclidean Paley-Wiener theorem to the class of integrable functions. That is, support of an integrable function $f$ on $\mathbb{R}^n$ and its Fourier transform $\hat{f}$ both cannot be of finite measure simultaneously.

Thereafter, a series of analogous results to the Benedicks theorem and related problems had been explored in various set ups, including the Heisenberg group and the Euclidean motion groups (see [18–24]). In particular, Narayanan and Ratnakumar [18] had worked out an analogous result to the Benedicks theorem for the partially compactly supported functions on the Heisenberg group in terms of the finite rank of Fourier transform of the function. Further, in a recent article [34], Vemuri has relaxed the compact support condition on the function by the finite support. Since the group Fourier transform on the Heisenberg group is operator valued, the latter result seems close to the classical Benedicks theorem. However, it would be a good question to consider the case when the spectrum of the Fourier transform of an integrable function would be supported on a thin uncountable set.

In the path-breaking article [11], a major variation of the uncertainty principle has been observed by Hedenmalm and Montes-Rodríguez, in terms of the measures supported on the curves. If the Fourier transform of a finitely supported Borel measure vanishes on a thin set, then the measure can be determined.

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In the article [11], Hedenmalm and Montes-Rodríguez have shown that the pair (hyperbola, some discrete set) is a Heisenberg uniqueness pair. As a dual problem, a weak* dense subspace of $L^\infty(\mathbb{R})$ has been constructed to solve the Klein-Gordon equation. Further, in the same article, a complete characterization of the Heisenberg uniqueness pairs corresponding to any two parallel lines has been established.

Let $\Gamma$ be a finite disjoint union of smooth curves in $\mathbb{R}^2$. Let $X(\Gamma)$ be the space of all finite complex-valued Borel measure $\mu$ in $\mathbb{R}^2$ which is supported on $\Gamma$ and absolutely continuous with respect to the arc length measure on $\Gamma$. The Fourier transform of $\mu$ is defined by
\[
\hat{\mu}(\xi,\eta) = \int_{\Gamma} e^{-i\pi(x\cdot\xi+y\cdot\eta)} d\mu(x,y),
\]
where $(\xi,\eta) \in \mathbb{R}^2$. Let $\Lambda$ be a set in $\mathbb{R}^2$. The pair $(\Gamma, \Lambda)$ is called a Heisenberg uniqueness pair for $X(\Gamma)$ if any $\mu \in X(\Gamma)$ satisfies $\hat{\mu}|_{\Lambda} = 0$, implies $\mu$ is identically zero. For more details, we refer the article [11].

Now, we state the main result on the Heisenberg uniqueness pairs due to Hedenmalm and Montes-Rodríguez [11].

**Theorem 1.1.** [11] Let $\Gamma$ be the hyperbola $x_1x_2 = 1$ and $\Lambda_{\alpha,\beta}$ a lattice-cross defined by
\[
\Lambda_{\alpha,\beta} = (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),
\]
where $\alpha,\beta$ are positive reals. Then $(\Gamma, \Lambda_{\alpha,\beta})$ is a Heisenberg uniqueness pair if and only if $\alpha\beta \leq 1$.

Further, the questions pertaining to Heisenberg uniqueness pair have been studied in the plane as well as in the Euclidean spaces by several authors. We skip writing more histories and details about the Heisenberg uniqueness pairs, however, we would like to refer [2, 7, 9, 10, 13, 17, 25, 26, 28, 35, 36].

The question of Heisenberg uniqueness pairs in the higher dimension has been first taken up by Gonzalez Vieli [35]. We worked out an analogous result to Theorem 1.2 for the Euclidean motion groups.

**Theorem 1.2.** [35] Let $\Gamma = S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ and $\Lambda$ be a sphere of radius $r$. Then $(\Gamma, \Lambda)$ is a HUP if and only if $J_{(n+2k-2)/2}(r) \neq 0$ for all $k \in \mathbb{Z}_+$. 

In this article, we emphasize on an analogue of the Benedicks theorem to the Euclidean motion group $M(n)$. We prove that if the Fourier transform of certain integrable functions is of finite rank, then the function has to vanish identically. Further, we explore the possibility of the Heisenberg uniqueness pairs for the Fourier transform on $M(n)$ as well as on the product group $\mathbb{R}^n \times K$. 
2. Notation and preliminaries

Euclidean motion group $G = M(n)$ is the group of isometries of $\mathbb{R}^n$ that leaves invariant the Laplacian. Since the action of the special orthogonal group $K = SO(n)$ defines a group of automorphisms on $\mathbb{R}^n$ via $y \mapsto ky + x$, where $x \in \mathbb{R}^n$ and $k \in K$, the group $M(n)$ can be identified as the semidirect product of $\mathbb{R}^n$ and $K$. Hence the group law on $G$ can be expressed as

$$(x, s) \cdot (y, t) = (x + sy, st).$$

Since a right $K$-invariant function on $G$ can be thought as a function on $\mathbb{R}^n$, we infer that the Haar measure on $G$ can be written as $dg = dxdk$, where $dx$ and $dk$ are the normalized Haar measures on $\mathbb{R}^n$ and $K$ respectively.

Let $\mathbb{R}_+ = (0, \infty)$ and $M = SO(n - 1)$ be the subgroup of $K$ that fixes the point $e_n = (0, \ldots, 0, 1)$. Let $\hat{M}$ be the unitary dual group of $M$. Given a unitary irreducible representation $\sigma \in \hat{M}$ realized on the Hilbert space $H_\sigma$ of dimension $d_\sigma$, we consider the space $L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$ consisting of $d_\sigma \times d_\sigma$ complex matrices valued functions $\varphi$ on $K$ such that $\varphi(uk) = \sigma(u)\varphi(k)$, where $u \in M$, $k \in K$ and satisfying

$$\int_K \|\varphi(k)\|^2 dk = \int_K \text{tr}(\varphi(k)^*\varphi(k))dk.$$  

It is easy to see that $L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$ is a Hilbert space under the inner product

$$\langle \varphi, \psi \rangle = \int_K \text{tr}(\varphi(k)\psi(k)^*)dk.$$  

For each $(a, \sigma) \in \mathbb{R}_+ \times \hat{M}$, defines a unitary representation $\pi_{a,\sigma}$ of $G$ by

$$(2.1) \quad \pi_{a,\sigma}(g)(\varphi)(k) = e^{-ia(x,k-e_n)}\varphi(s^{-1}k),$$

where $\varphi \in L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$. Let $\varphi = (\varphi_1, \ldots, \varphi_{d_\sigma})$, where $\varphi_j$ are the column vectors of $\varphi$. Then $\varphi_j(uk) = \sigma(u)\varphi_j(k)$. Now, consider the space

$$H(K, \mathbb{C}^{d_\sigma}) = \left\{ \varphi : K \to \mathbb{C}^{d_\sigma}, \int_K |\varphi(k)|^2 dk < \infty, \varphi(uk) = \sigma(u)\varphi(k), u \in M \right\}.$$  

It is obvious that $L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$ is the direct sum of $d_\sigma$ copies of the Hilbert space $H(K, \mathbb{C}^{d_\sigma})$ equipped with the inner product

$$\langle \varphi, \psi \rangle = d_\sigma \int_K (\varphi(k), \psi(k)) dk.$$  

Now, it can be shown that an infinite dimensional unitary irreducible representation of $G$ is the restriction of $\pi_{a,\sigma}$ to $H(K, \mathbb{C}^{d_\sigma})$. In other words, each of $(a, \sigma) \in \mathbb{R}_+ \times \hat{M}$ defines a principal series representation $\pi_{a,\sigma}$ of $G$ via (2.1).

In addition to the principal series representations, there are finite-dimensional unitary irreducible representations of $G$ which can be identified with $\hat{K}$, though these unitary representations do not take part in the Plancherel formula. For more details, we refer to Kumahara [15] and Sugiura [30].
Now, we define the group Fourier transform of a function \( f \in L^1(G) \) by
\[
\hat{f}(a, \sigma) = \int_G f(g) \pi_{a, \sigma}(g^{-1}) dg
\]
and
\[
\hat{f}(\delta) = \int_G f(x, k) \delta(k^{-1}) dxdk,
\]
where \( \delta \in \hat{K} \). Further, the operator \( \hat{f}(a, \sigma) \) can be explicitly written as
\[
(\hat{f}(a, \sigma) \varphi)(k) = \int_{\mathbb{R}^n} \int_K f(x, s) e^{-i(x, ak - s)} \varphi(s^{-1}k) dx ds,
\]
where \( \mathcal{F}_1 \) stands for the usual Fourier transform in the first variable and \( \varphi \in H(\hat{K}, \mathbb{C}^{d_\delta}) \). For more details, we refer to [5, 8, 16].

Now, if \( f \in L^1 \cap L^2(G) \), then the operator \( \hat{f}(a, \sigma) \) will be a Hilbert-Schmidt operator on \( H(\hat{K}, \mathbb{C}^{d_\delta}) \). Since the Plancherel measure \( \mu \) on \( \hat{G} \) can be expressed as \( d\mu = c_n a^{n-1} da \), where \( c_n \) depends only on \( n \), the corresponding Plancherel formula is given by
\[
\int_0^\infty \left( \sum_{\sigma \in \hat{M}} d_\sigma \| \hat{f}(a, \sigma) \|_{HS}^2 \right) d\mu(a) = \| f \|_2^2.
\]

We would like to mention that our main result Theorem 3.1 has a close relation with the following Wiener’s theorem on motion group due to R. Gangolli [6]. For a function \( f \) on \( G \), defining the two-sided translate by \( g_1 f g_2 : g_j \in G; j = 1, 2 \),
\[
{g_1 f g_2 : g_j \in G; j = 1, 2}. \]

**Theorem 2.1.** [6] Let \( f \in L^1(G) \) and \( S = \text{span} \{g_1 f g_2 : g_j \in G; j = 1, 2\} \). Then the space \( S \) is dense in \( L^1(G) \) if and only if \( \hat{f}(a, \sigma) \neq 0 \) and \( \hat{f}(\delta) \neq 0 \) for all \( (a, \sigma) \in \mathbb{R}_+ \times \hat{M} \) and \( \delta \in \hat{K} \).

A close observation of Theorem 2.1 shows that if \( \hat{f}(a, \sigma) \) is a finite rank operator, then \( \overline{S} \) can be a proper subspace of \( L^1(G) \). Hence, it opens a window to look at the determining properties of \( \hat{f} \).

Next, we recall certain facts about the spherical harmonics. Let \( \hat{K}_M \) denote the set of all equivalence classes of irreducible unitary representations of \( K \) which have a nonzero \( M \)-fixed vector. It is well known that each representation in \( \hat{K}_M \) has, in fact, a unique nonzero \( M \)-fixed vector, up to a scalar multiple.

For a \( \delta \in \hat{K}_M \), which is realized on \( V_\delta \), let \( \{e_1, \ldots, e_{d_\delta}\} \) be an orthonormal basis of \( V_\delta \), with \( e_1 \) as the \( M \)-fixed vector. Let \( \varphi_{ij}^\delta(k) = \langle e_i, \delta(k)e_j \rangle, k \in K \). Then by the Peter-Weyl theorem, it follows that \( \{\sqrt{d_\delta} \varphi_{ij}^\delta : 1 \leq j \leq d_\delta, \delta \in \hat{K}_M\} \) is an orthonormal basis of \( L^2(K/M) \).
We would further need a concrete realization of the representations in $\hat{K}_M$, which can be done in the following way.

For $l \in \mathbb{Z}_+$, denote the set of all non-negative integers, let $P_l$ denote the space of all homogeneous polynomials $P$ in $n$ variables of degree $l$.

Let $H_l = \{ P \in P_l : \Delta P = 0 \}$, where $\Delta$ is the standard Laplacian on $\mathbb{R}^n$. The elements of $H_l$ are called solid spherical harmonics of degree $l$. It is easy to see that the natural action of $K$ leaves the space $H_l$ invariant. In fact, the corresponding unitary representation $\delta_l$ is in $\hat{K}_M$. Moreover, $\hat{K}_M$ can be identified, up to unitary equivalence, with the collection $\{ \delta_l : l \in \mathbb{Z}_+ \}$.

Define $Y_{lj}(\omega) = \sqrt{d_l} \phi_{ij}(k)$, where $\omega = k \cdot e_n \in S^{n-1}$, $k \in K$, and $d_l$ is the dimension of $H_l$. Then the set $\tilde{H}_l = \{ Y_{lj} : 1 \leq j \leq d_l \}$ forms an orthonormal basis for $L^2(S^{n-1})$. Thus, a suitable function $g$ on $S^{n-1}$ can be expanded as

\[
g(\omega) = \sum_{l=0}^{\infty} \sum_{j=1}^{d_l} a_{lj} Y_{lj}(\omega).
\]

These spherical functions $Y_{lj}$ are called the spherical harmonics on the unit sphere $S^{n-1}$. For more details, see [33], p. 11.

Next, we consider an orthogonality relation among the matrix coefficients of the irreducible unitary representations in $\hat{K}_M$.

**Lemma 2.2.** For $\delta \in \hat{K}$, define $\phi_{ij}^\delta(k) = \left( e_i, \delta(k) e_j \right)$. Then for $\delta_1, \delta_2 \in \hat{K}$, there exists $\alpha \in \mathbb{Z}_+$ such that

\[
\int_M \phi_{ij}^{\delta_1}(km) \overline{\phi_{ij}^{\delta_2}(km)} dm = \sum_{v=0}^{\alpha} c_v Y_v(k \cdot e_n).
\]

**Proof.** Since we know that the matrix coefficients of $\delta \in \hat{K}_M$ satisfy the functional relation

\[
\phi_{ij}^\delta(km) = \sum_{p=1}^{d_\delta} \phi_{ip}^\delta(k) \phi_{pj}^\delta(m)
\]

and $M$ is a compact subgroup of $K$, it follows that each of $\delta \in \hat{K}$ will be the direct sum of finitely many irreducible unitary representations of $M$. Hence each of $\phi_{ij}^\delta$ satisfies

\[
\phi_{ij}^\delta = \sum_{q=1}^{d_{\delta,ij}} \phi_{ij}^{\delta_q},
\]

where $\delta_q \in \hat{M}$. Using the orthogonality of the coefficients $\phi_{ij}^{\delta_q}$’s and the fact that the left-hand side of (2.5) is $M$-invariant, we infer that it is a finite sum of the product of some spherical harmonics. Further, a homogeneous polynomial
can be uniquely decomposed in terms of homogeneous harmonics polynomials, it follows that (2.5) holds.

For a fixed $\xi \in S^{n-1}$, we define a linear functional on $H_l$ by $\xi \mapsto Y_i(\xi)$. Then there exists a unique spherical harmonic, say $Z^{(l)}_\xi \in H_l$ such that

\begin{equation}
Y_i(\xi) = \int_{S^{n-1}} Z^{(l)}_\xi(\eta)Y_i(\eta)d\sigma(\eta).
\end{equation}

The spherical harmonic $Z^{(l)}_\xi$ is a $K$ bi-invariant real-valued function which is constant on the geodesics those are orthogonal to the line joining the origin and $\xi$. The spherical harmonic $Z^{(l)}_\xi$ is called the zonal harmonic of the space $\tilde{H}_l$ for the above and various other peculiar reasons. For more details, see [29].

Since the zonal harmonic $Z^{(l)}_\xi(\eta)$ is $K$ bi-invariant, there exists a reasonable function $F$ on $(-1, 1)$ such that $Z^{(l)}_\xi(\eta) = F(\xi \cdot \eta)$. Hence, the extension of the formula (2.7) is inevitable. For $F \in L^1(-1, 1)$, the Funk-Hecke identity is

\begin{equation}
\int_{S^{n-1}} F(\xi \cdot \eta)Y_i(\eta)d\sigma(\eta) = c_l Y_i(\xi),
\end{equation}

where the constant $c_l$ is given by

\[ c_l = \alpha_l \int_{-1}^1 F(t)G^{n-2}_l(t)(1 - t^2)^{n-3} dt \]

and $G^{\beta}_l$ stands for the Gegenbauer polynomial of order $\beta$ and degree $l$.

Further, using the Funk-Hecke identity, it can be shown that

\begin{equation}
\int_{S^{n-1}} e^{-ix \cdot \eta}Y_j(\eta)d\sigma(\eta) = i^j \frac{J_j+(n-2)/2}{r^{(n-2)/2}} Y_j(\xi),
\end{equation}

whenever $Y_j \in \tilde{H}_l$. For a proof of the identity (2.9), we refer [11], p. 464.

Let $f$ be a function in $L^1(S^{n-1})$. For each $l \in \mathbb{Z}_+$, we define the $l$th spherical harmonic projection of the function $f$ by

\begin{equation}
\Pi_l f(\xi) = \int_{S^{n-1}} Z^{(l)}_\xi(\eta)f(\eta)d\sigma(\eta).
\end{equation}

Then $\Pi_l f$ is a spherical harmonic of degree $k$. Now, for $\delta > (n - 2)/2$, if we denote $A^p_l = (p-l+\delta)(p+\delta)^{-1}$, then the spherical harmonic expansion $\sum_{l=0}^{\infty} \Pi_l f$ is $\delta$- Cesaro summable to $f$. In other words,

\begin{equation}
f = \lim_{p \to \infty} \sum_{l=0}^{p} A^p_l \Pi_l f,
\end{equation}

where the limit on the right-hand side of (2.11) exists in $L^1(S^{n-1})$. For more details, we refer to [14, 27].
3. Uniqueness results for the Fourier transform on $G$

In this section, we work out some of the uniqueness results for the Fourier transform on the motion group $G$ as an analogue to the Benedick’s theorem. We prove that if the Fourier transform of the function $f \in L^1 \cap L^2(G)$ is a finite rank operator on $H(K, \mathbb{C}^{d_r})$, then the function has to vanish identically.

**Theorem 3.1.** Let $f \in L^1 \cap L^2(G)$ be such that $\hat{f}(a, \sigma)$ is a finite rank operator for each $a > 0$ and some $\sigma \in \hat{M}$. Then $f = 0$ if and only if $\hat{f}(\delta) \neq 0$ except for finitely many $\delta \in \hat{K}$.

*Proof.* Let $h = f * f^*$, where $f^*(g) = \overline{f(g^{-1})}$. Then it can be shown that $\hat{h}(a, \sigma) = \hat{f}(a, \sigma)^* \hat{f}(a, \sigma)$, (see [30], p. 170). Hence $\hat{h}(a, \sigma)$ is a positive, finite rank operator on $H(K, \mathbb{C}^{d_r})$. By the spectral theorem, it follows that

$$
\hat{h}(a, \sigma)\varphi = \sum_{j=1}^{m} \lambda_j \langle \varphi, \varphi_j \rangle \varphi_j,
$$

where the set $\{\varphi_j \in H(K, \mathbb{C}^{d_r}) : j = 1, \ldots, m\}$ forms an orthonormal basis for the range space of $\hat{h}(a, \sigma)$ which satisfies $\hat{h}(a, \sigma)\varphi_j = \lambda_j \varphi_j$ with $\lambda_j \geq 0$. Let $\varphi_j = (\varphi_{j,1}, \ldots, \varphi_{j,d_r})$. Then in view of (2.2), we can express

$$
\int_{K} \mathcal{F}_{1} h(ak \cdot e_n, s) \varphi_{j,q}(s^{-1}k) ds = \lambda_j \varphi_{j,q}(k),
$$

where $q \in \{1, \ldots, d_r\}$. Since $h \in L^1(G)$, we can write the spherical harmonic decomposition of $h$ in the first variable $x = |x| t$, $t \in S^{n-1}$ as

$$
h(x, s) = \lim_{p \to \infty} \sum_{l=0}^{p} A^p_l h_l(|x|, s) \Pi_l h(t, s),
$$

where the series on the right-hand side is $\delta$- Cesaro summable. Now, by the Hecke-Bochner identity, we obtain

$$
\mathcal{F}_1 (h_l(., s) \Pi_l h(., s))(a\omega) = i^{-l} a^l (\mathcal{F}_{n+2l} H_l)(a, s) \Pi_l h(t, s),
$$

where $\mathcal{F}_{n+2l}$ is the $(n + 2l)$-dimensional Fourier transform of $H_l = \frac{h}{|x|^l}$. In view of (3.3) and (3.4), we can rewrite (3.2) as

$$
\int_{K} \lim_{p \to \infty} \sum_{l=0}^{p} A^p_l i^{-l} a^l (\mathcal{F}_{n+2l} H_l)(a, s) \Pi_l h(t, s) \varphi_{j,q}(s^{-1}k) ds = \lambda_j \varphi_{j,q}(k)
$$

By using the orthogonality relations of the spherical harmonics, we infer that

$$
\int_{K} (\mathcal{F}_{n+2l} H_l)(a, s) \varphi_{j,q}(s^{-1}k) ds = \begin{cases} 
\lambda_q \varphi_{j,q}(k), & \text{if } l = 0 \\
0, & \text{if } l \neq 0.
\end{cases}
$$

Let $G_a(s) = \mathcal{F}_n h_0(a, s)$. Then from (3.5), it follows that

$$
G_a \ast \varphi_{j,q} = \lambda_j \varphi_{j,q}.
$$
For a function $\phi \in L^1(K)$ and $\delta \in \hat{K}$, define $\phi^\delta = \phi \ast \chi_\delta$, where $\chi_\delta$ is the character of the representation $\delta$. Then $\phi^\delta$ is class function and hence

$$\hat{\phi}^\delta(\eta) = \frac{1}{d_\eta} \langle \phi^\delta, \chi_\eta \rangle I = \frac{1}{d_\eta} \langle \phi, \chi_\delta \rangle \langle \chi_\delta, \chi_\eta \rangle I.$$ 

Thus, from (3.6) we get

$$\hat{\varphi}_{j,q}^\delta(\delta) \left( \hat{G}_{a}^\delta(\delta) - \lambda_j \right) = 0. \tag{3.7}$$

Then $\hat{G}_{a}^\delta(\delta) = \lambda_j$, for finitely many $\delta \in \hat{K}$, otherwise, by the Riemann-Lebesgue lemma, it follows that $\lambda_j = 0$, whenever $j = 1, \ldots, m$. Hence from (3.1) we get $\hat{h}(a, \sigma) = 0$. In view of (2.2), we infer that $F_{1} h(ak \cdot e_n, s) = 0$ for almost all $s, k \in K$ and all $a > 0$. This, in turn, by the uniqueness of the Fourier transform $F_1$, it follows that $h = 0$. Since $h = f \ast f^*$ is continuous, we can write $h(o) = \|f\|^2_2$. Thus, $f = 0$.

Now, we need to resolve the case when $\hat{G}_{a}^\delta(\delta) = \lambda_j \neq 0$ for the only finitely many $\delta \in \hat{K}$. Thus, each of $\varphi_{j,q}$ is a trigonometric polynomial in $L^2(K)$. By trigonometric polynomial we mean a finite linear combination of matrix coefficients $\varphi_{ij}$.

Notice that, $\delta$ and $j$ are independent and the the fact that all of $\lambda_j$ cannot be zero simultaneously, it follows that $\lambda_j$‘s are equal. Since $\varphi_{j,q}$ is a trigonometric polynomial, $\pi_{a,\sigma}(g) \hat{h}(a, \sigma)$ will be a trace class operator on $H(K, \mathbb{C}^{d_\sigma})$.

Let $\psi_{lu}^\delta = \sqrt{d_\delta} \varphi_{lu}^\delta$ denotes the matrix coefficients of $\delta \in \hat{K}$. Then by the Peter-Weyl theorem, the set $\{\psi_{lu}^\delta : 1 \leq l, u \leq d_\delta, \delta \in \hat{K}\}$ forms an orthonormal basis for $L^2(K)$. Let $\{\tau_\nu : \nu \in \mathbb{N}\}$ be an orthonormal basis for $H(K, \mathbb{C}^{d_\sigma})$. Then

$$\text{tr} \left( \pi_{a,\sigma}(x, s) \hat{h}(a, \sigma) \right) = \sum_{\nu \in \mathbb{N}} \left\langle \pi_{a,\sigma}(x, s) \hat{h}(a, \sigma), \tau_\nu \right\rangle$$

$$= \sum_{\nu \in \mathbb{N}} \sum_{j=1}^{m} \lambda \left\langle \tau_\nu, \varphi_j \right\rangle \left\langle \pi_{a,\sigma}(x, s) \varphi_j, \tau_\nu \right\rangle$$

$$\tag{3.8} = \sum_{\nu \in \mathbb{N}} \sum_{j=1}^{m} \alpha_j \sum_{q=1}^{d_\sigma} \int_{K} e^{-i(x,ak \cdot e_n)} \varphi_{j,q}(s^{-1}k) \overline{\tau_{\nu,q}(k)} dk,$$

where $\alpha_j = \lambda d_\sigma \left\langle \tau_\nu, \varphi_j \right\rangle$. Since $\varphi_{j,q}$ is a trigonometric polynomial, there exists a finite set $F_o$ in $\hat{K}$ such that

$$\tag{3.9} \varphi_{j,q} = \sum_{\delta \in F_o} \sum_{l,u=1}^{d_\delta} c_{lu}^\delta \psi_{lu}^\delta,$$
where \( c_{lu}^\delta \)'s are constant. Now, in view of (2.10), we can express
\[
\psi_{lu}^\delta(s^{-1}k) = \sum_{p=1}^{d_\delta} \psi_{lp}^\delta(s^{-1})\psi_{pu}^\delta(k).
\]

On the other hand, since \( \tau_{\nu,q} \in L^2(K) \), by the Peter-Weyl theorem, we get
\[
(3.10) \quad \tau_{\nu,q} = \sum_{\delta \in K, \eta=1}^{d_\delta} \kappa_{\nu,\eta}^\delta \psi_{\nu,\eta}^\delta,
\]
where \( \kappa_{\nu,\eta}^\delta \)'s are constants. In view of the fact that \( e^{-i(x,ak \cdot e_n)} \) is an \( M \)-invariant function, the followings identities hold.
\[
\int_K e^{-i(x,ak \cdot e_n)}\psi_{pu}^\delta(k)\overline{\psi_{\nu,\eta}^\delta(k)}dk = \int_M \int_K e^{-i(x,a \cdot km \cdot e_n)}\psi_{pu}^\delta(k)\overline{\psi_{\nu,\eta}^\delta(k)}dk dm = \int_K e^{-i(x,ak \cdot e_n)} \int_M \psi_{pu}^\delta(km^{-1})\overline{\psi_{\nu,\eta}^\delta(km^{-1})}dm dk.
\]

By using Lemma 2.2 and the Funk-Hecke identity, we infer that
\[
\int_K e^{-i(x,ak \cdot e_n)}\psi_{pu}^\delta(k)\overline{\psi_{\nu,\eta}^\delta(k)}dk = \sum_{j=0}^{\beta} c_j \int_K e^{-i(x,ak \cdot e_n)}Y_j(k \cdot e_n)dk = \sum_{j=0}^{\beta} c_j^j \phi_j(a|x|)Y_j(\omega)
\]
where \( x = |x|\omega, \ \omega \in S^{n-1} \) and \( \phi_j(a|x|) = \frac{J_{j+(n-2)/2}(a|x|)}{(a|x|)^{(n-2)/2}} \). Now, in view of the Fourier inversion formula ( [30], p. 175) for function in \( L^2(M(2)) \), an inversion formula for the function \( h \in L^2(M(n)) \) can be deduced in a similar way and hence we omit its proof here. Thus, the function \( h \) can be recovered at \( (x,s) \) by
\[
h(x,s) = c_n \sum_{\sigma \in \mathcal{M}} \int_0^\infty \text{tr} \left( \pi_{a,\sigma}(g)\hat{h}(a,\sigma) \right) a^{n-1} da,
\]
Since \( J_{j+(n-2)/2}(a|x|) \approx (a|x|)^{-(n-2)/2} \) as \( |x| \to \infty \), (see [31], p. 15), it follows that \( \phi_j \in L^p(\mathbb{R}^n) \) if and only if \( p > \frac{2n}{n-1} \). This contradicts the hypothesis that \( h \in L^1(G) \) and hence \( h = 0 \).

**Remark 3.2.** We would like to mention the necessity of the non-vanishing of the Fourier coefficients in Theorem 3.1. Since \( M(2) \) is the semidirect product of \( \mathbb{R}^2 \) and \( SO(2) \), each of \( a \in \mathbb{R}_+ \) defines a unitary irreducible representation \( \pi^a \) of \( M(2) \) on \( L^2([0,2\pi]) \). In other words, for \( (x,\theta) \in \mathbb{R}^2 \times [0,2\pi] \), the action of \( \pi^a \) can be realized by
\[
(\pi^a(x,\theta)\phi)(\omega) = e^{-i\langle x,a\omega \rangle}\phi(\omega - \theta),
\]
where $\varphi \in L^2([0, 2\pi])$. For $g \in L^1(\mathbb{R}^2)$, consider $f(x, \theta) = g(x)e^{i\theta}$. Then

$$
(\hat{f}(a)\varphi)(\omega) = \int_{\mathbb{R}^2} \int_0^{2\pi} f(x, \theta)e^{-i\langle x, ae^{i\omega}\rangle}\varphi(\omega - \theta)dxd\theta = \hat{g}(\omega)\varphi(\omega).
$$

Hence we infer that, if the Fourier coefficients

$$
\delta_n(f) = \int_{\mathbb{M}_2(2)} f(x, \theta)e^{i\theta}dxd\theta \neq 0
$$

for finitely many $n \in \mathbb{Z}$, then $\hat{f}(a)$ is a finite rank operator, however, $f$ need not be the zero function.

Next, we prove a uniqueness result for the Fourier transform on $G$ which has a sharp contrast with the Benidicks theorem. That is, the group Fourier transform of a non-zero bounded Borel measurable function in $L^1(G)$ cannot be compactly supported in $(0, \infty)$.

In order to prove this result, we need the following result from [3]. Let $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_j \geq 0; j = 1, \ldots, n\}$. The following result had appeared in the article [3] by Bagchi and Sitaram, p. 421, as a part of the proof of Proposition 2.1.

**Proposition 3.3.** Let $h$ be a non-zero bounded Borel measurable function which is supported on $\mathbb{R}_+^n$. Then supp $\hat{h} = \mathbb{R}^n$.

**Theorem 3.4.** Let $f \in L^1(G)$ be a bounded Borel measurable function and supported away from the origin in the first variable. If $\hat{f}(., \sigma)$ is compactly supported in $\mathbb{R}_+$, then $f = 0$.

**Proof.** Suppose $f$ is a radial function in the first variable. Then $\mathcal{F}_1 f(., s)$ will be radial and hence

$$
(\hat{f}(a, \sigma)\varphi)(k) = \int_{\mathcal{M}_2(2)} \mathcal{F}_1 f(a, s)\varphi(s^{-1}k)ds,
$$

where $\varphi \in H(K, \mathbb{C}^d)$. Since $\hat{f}(., \sigma)$ is compactly supported in $\mathbb{R}_+$, it follows from (3.11) that $\mathcal{F}_1 f(., s)$ is compactly supported in $\mathbb{R}_+$, for almost all $s \in K$. This, in turn, contradicts Proposition 3.3. Thus, we conclude that $f = 0$.

Since $f \in L^1(G)$, in view of (2.11), we can write the spherical harmonic decomposition of $f$ in the first variable $x = |x|t$, $t \in S^{n-1}$ as

$$
f(x, s) = \lim_{p \to \infty} \sum_{l=0}^{p} A^p_l f_l(|x|, s)\Pi_l f(t, s),
$$

where the series on the right-hand side is $\delta$-Cesaro summable. Now, an application of the Hecke-Bochner identity to (3.11) yields

$$
(\hat{f}(a, \sigma)\varphi)(k) = \int_{\mathcal{K}} \lim_{p \to \infty} \sum_{l=0}^{p} A^p_l i^{-l}a^l \mathcal{F}_{n+2l} H_l(a, s)\Pi_l f(t, s)\varphi(s^{-1}k)ds,
$$

where $\varphi \in H(K, \mathbb{C}^d)$. Since $\hat{f}(., \sigma)$ is compactly supported in $\mathbb{R}_+$, it follows from (3.11) that $\mathcal{F}_1 f(., s)$ is compactly supported in $\mathbb{R}_+$, for almost all $s \in K$. This, in turn, contradicts Proposition 3.3. Thus, we conclude that $f = 0$.
where \( \mathcal{F}_{n+2l} \) is the \((n + 2l)\)-dimensional Fourier transform of \( H_t = \frac{f}{|z|^l} \). Since \( \hat{f} \) is compactly supported, it follows that

\[
\lim_{p \to \infty} \sum_{l=0}^{p} A_l^p i^{-l} a^l \mathcal{F}_{n+2l} H_t(a, s) \Pi_l f(t, s) = 0.
\]

We know that the set \( \{ \Pi_l f(\cdot, s) : l \in \mathbb{Z}_+ \} \) form an orthogonal set in \( L^2(S^{n-1}) \), from (3.13), it is easy to see that

\[
\mathcal{F}_{2+2l} H_t(a, s) \| \Pi_l f(\cdot, s) \|^2 = 0.
\]

Since \( f \) is supported away from the origin in the first variable, \( H_t \) must be a bounded Borel measurable function. If \( \mathcal{F}_{2+2l} H_t(a, s) = 0 \), then by the radial case, we infer that \( H_t = 0 \). Otherwise, \( \| \Pi_l f(\cdot, s) \|_2 = 0 \). Thus, it follows from (3.12) that \( f = 0 \). □

Further, we prove that a radial function on \( G \) can be determined by its Fourier transform at a single point.

**Proposition 3.5.** Let \( f \in L^1(G) \) be a radial function in the first variable such that \( \text{sign}(J_0 f) \geq 0 \). If \( \hat{f}(a_o, \sigma) = 0 \) for some \( a_o \in \mathbb{R}_+ \) and a fixed \( \sigma \in \hat{M} \), then \( f = 0 \).

**Proof.** For \( \varphi \in H(K, \mathbb{C}^{d_r}) \), we have

\[
(\hat{f}(a_o, \sigma) \varphi)(k) = \int_K \mathcal{F}_1 f(a_o, s) \varphi(s^{-1}k) ds.
\]

By the hypothesis, \( \hat{f}(a_o, \sigma) \varphi = 0 \), it follows that \( \mathcal{F}_1 f(a_o, \cdot) = 0 \). Hence

\[
\mathcal{F}_1 f(a_o, s) = \int_0^\infty \int_{S^{n-1}} f(|t\omega|, s) e^{-ia_o t \omega} d\omega t^{n-1} dt = \int_0^\infty J_0(a_o t) f(t, s) t^{n-1} dt = 0.
\]

Since \( \text{sign}(J_0 f) \geq 0 \) and the Bessel function \( J_0 \) can vanish only at the countably many points, we conclude that \( f = 0 \). □

4. **Some auxiliary results on compact group**

In this section, we observe some of the properties of a Weyl type transform on the space \( L^1(K) \) as analogous to the Weyl transform on the Heisenberg group, (see [32]). We use it to work out some uniqueness results for the Fourier transform on the motion group \( G \).

Let \( K \) be a compact group. For a function \( g \in L^1(K) \), we define an operator \( W \) on \( H(K, \mathbb{C}^{d_r}) \) by

\[
W(g) = \int_K g(t) \pi(t) dt,
\]
where $\pi$ is the left regular representation of $K$. Then $W(g)$ maps $H(K, \mathbb{C}^{d_e})$ into $H(K, \mathbb{C}^{d_e})$. Now, we derive the Plancherel formula and the Fourier inversion formula for the transform $W$.

**Plancherel formula.** For $g \in L^2(K)$ and $\varphi \in H(K, \mathbb{C}^{d_e})$, we have

$$(W(g)\varphi)(k) = \int_K g(t)(\pi(t)\varphi)(k)dt = \int_K g(t)\varphi(t^{-1}k)dt = \int_K g(s^{-1}k)\varphi(s)ds.$$

Write $K_g(s,k) = g(s^{-1}k)$. Then $W(g)$ is an integral operator with the kernel $K_g \in L^2(K \times K)$. Hence $W(g)$ is a Hilbert-Schmidt operator that satisfying

$$\|W(g)\|_{HS}^2 = \int_{K \times K} |K_g(s,k)|^2 dsdk = \int_{K \times K} |g(s^{-1}k)|^2 dsdk = \|g\|_2^2.$$

In other words, $W$ maps $L^1(K)$ onto $S_2$, the space of Hilbert-Schmidt operators on $H(K, \mathbb{C}^{d_e})$.

Next, we prove the Fourier inversion formula for the transform $W$.

**Proposition 4.1.** If $g \in C^2(K)$, then the transform $W$ satisfies the inversion formula $g(t) = \text{tr}(\pi(t)^*W(g))$.

**Proof.** Given that $g \in C^2(K)$,

$$(\pi(t))^*W(g) = \int_K g(s)(\pi(t))^*\pi(s)ds = \int_K g(s)\pi(t^{-1})\pi(s)ds$$

$$= \int_K g^t(p)\pi(p)dp = W(g^t),$$

where $g^t(p) = g(tp)$. That is, $W(g)$ is an integral operator with kernel $K_g$. Since the kernel $K_{g^t}$ satisfies $K_{g^t}(s,k) = g^t(s^{-1}k)$, we obtain $K_{g^t}(s,s) = g(t)$ and hence

$$\text{tr}[\pi(t)^*W(g)] = \text{tr}(W(g^t)) = \int_K K_{g^t}(s,s)ds$$

$$= \int_K g(t)ds = g(t).$$

Further, by using the Peter-Weyl theorem, we prove that if $g \in L^1(K)$, then the operator $W(g)$ has finite rank as long as $g$ is a trigonometric polynomial.

**Proposition 4.2.** Let $g \in L^1(K)$. Then the operator $W(g)$ is of finite rank if and only if $g$ is a trigonometric polynomial on $K$. 

Proof. Consider the function $h = g \ast g^*$, where $g^*(t) = \overline{g(t^{-1})}$. Now, we show that $W(h) = W(g) \ast W(g)$. For this, we have

$$W(h) = \int_K h(t)\pi(t)dt = \int_K (g \ast g^*)(t)\pi(t)dt$$

$$= \int_K \int_K g(s)g^*(ts^{-1})\pi(t)dt\,ds$$

$$= \int_K g(s) \left( \int_K g^*(ts^{-1})\pi(t)dt \right)\,ds.$$ 

By the change of variables $ts^{-1} = p$ in the inner integral, we get

$$W(h) = \int_K g(s) \left( \int_K g^*(p)\pi(ps)dp \right)\,ds = W(g^*)W(g).$$

Further, we require proving $W(g)^* = W(g^*)$. For $\varphi, \psi \in H(K, \mathbb{C}^{d_\sigma})$, consider

$$\langle W(g^*)\varphi, \psi \rangle = \int_K g(t^{-1})\langle \pi(t)\varphi, \psi \rangle dt = \int_K g(s)\langle \pi(s^{-1})\varphi, \psi \rangle ds.$$ 

Since $\pi$ is the left regular representation of $K$, the operator $\pi(s)$ will be unitary. Hence

$$\langle W(g^*)\varphi, \psi \rangle = \int_K \langle \varphi, g(s)\pi(s)\psi \rangle\,ds = \langle \varphi, W(g)\psi \rangle = \langle W(g)^*\varphi, \psi \rangle.$$ 

This, in turn, implies that $W(h) = W(g)^*W(g)$ is a positive finite rank operator. Thus, by the spectral theorem, there exists an orthonormal set $\{\varphi_j \in H(K, \mathbb{C}^{d_\sigma}) : j = 1, \ldots, m\}$ and scalars $\lambda_j \geq 0$ such that

$$(4.1) \quad W(h)\varphi = \sum_{j=1}^m \lambda_j \langle \varphi, \varphi_j \rangle \varphi_j,$$

whenever $\varphi \in H(K, \mathbb{C}^{d_\sigma})$. Let $\varphi_j = (\varphi_{j,1}, \ldots, \varphi_{j,d_\sigma})$. Then by (4.1), it follows that $h \ast \varphi_{j,q} = \lambda_j \varphi_{j,q}$. By taking Fourier coefficient of both the sides, we get

$$\hat{\varphi}_{j,q}(\hat{h} - \lambda_j) = 0,$$

where $\delta \in \hat{K}$. Then $\hat{h}^{\delta} = \lambda_j$, for finitely many $\delta \in \hat{K}$, otherwise, by the Riemann-Lebesgue lemma $\lambda_j = 0$. Hence $\hat{\varphi}_{j,q}(\delta) \neq 0$ at most for finitely many $\delta \in \hat{K}$. Thus, by the Peter-Weyl theorem, we infer that $\varphi_{j,q}$ is a trigonometric polynomial. Since $h = g^* \ast g$, it follows that $\hat{h}(\delta) = |\hat{g}(\delta)|^2$. Thus, from (4.1), we conclude that $g$ is a trigonometric polynomial.

Conversely, suppose $g$ is a trigonometric polynomial. Then without loss of generality, we can assume that $g = \varphi_{ij}^\delta$. Now, we can write

$$\varphi_{ij}^\delta(t^{-1}s) = \langle \delta(t^{-1}s)e_j, e_i \rangle = \langle \delta(s)e_j, \delta(t)e_i \rangle.$$
Since $H_\delta$ is $\pi$-invariant, it follows that
\[ \varphi^\delta_{ij}(t^{-1}s) = \sum_{l=1}^{d_\delta} \varphi^\delta_{lj}(s) \overline{\varphi^\delta_{il}(t)}. \]
A straightforward calculation leads to $W(g)\varphi(s) = d_\delta \left< \varphi, \varphi^\delta_{ij} \right> \varphi^\delta_{ij}$. Thus, we conclude that $W(g)$ is of finite rank. \(\square\)

**Remark 4.3.** In view of the Minkowski integral inequality, it can be easily seen that $\|W(g)\| \leq \|g\|_1$. Hence, the spectral radius of the operator $W(g)$ will satisfy the condition $\lambda[\sigma(W(g))] \leq \|g\|_1$.

Next, by using Proposition 4.2, we prove that a radial function on the motion group $G$ can be determined by its group Fourier transform at a single point. However, for a sake of simplicity, we prove the result for $G = M(2)$. For proving this result, we need the following lemma.

**Lemma 4.4.** Let $f \in L^1(G)$ be such that $f(x, s) = f(|x|, s)$. If $\hat{f}(a_o, \sigma)$ is of finite rank for some $a_o \in \mathbb{R}_+$, then
\[ \int_0^\infty J(a_o t)F_1(s)tdt = \sum_{|n| \leq a_o} \int_0^\infty J(a_o t)\hat{F}_1(n)Y_n(s)tdt, \]
where $F_1(s) = f(t, s)$.

**Proof.** We know that for $\varphi \in L^2(K, \mathbb{C}^{d_\sigma})$ we have
\[ (\hat{f}(a_o, \sigma)\varphi)(k) = \int_{\mathbb{R}^n} \int_K f(x, s)e^{-i(x,a_o k)}\varphi(s^{-1}k)dxds \]
\[ = \int_K F_1 f(a_o k, s)\varphi(s^{-1}k)ds. \]
Since $f$ is radial in the first variable, then it follows that
\[ (\hat{f}(a_o, \sigma)\varphi)(k) = (W(F_1 f(a_o, \cdot)) \varphi)(k). \]
By the hypothesis, $\hat{f}(a_o, \sigma)$ is a finite rank operator, $W(F_1 f(a_o, \cdot))$ must be of finite rank. From Proposition 4.2, we conclude that $F_1 f(a_o, \cdot)$ is a trigonometric polynomial. That is,
\[ (4.2) \quad F_1 f(a_o, s) = \sum_{|m| \leq a_o} \hat{G}_{a_o}(m)\chi_m(s), \]
where $G_{a_o}(s) = F_1 f(a_o, s)$. On the other hand, we have
\[ (4.3) \quad F_1 f(a_o, s) = \int_0^\infty \int_K f(|t\omega|, s)e^{-iap\omega}d\omega dt \]
\[ = \int_0^\infty J_0(a_o t)f(t, s)tdt. \]
Now, we have
\[ \hat{G}_a(m) = \int_K \hat{F}_1 f(a_o, k) \chi_{-m}(k) dk = \int_0^\infty J_0(a_o t) \left( \int_K f(t, k) \chi_{-m}(k) dk \right) t dt = \int_0^\infty J_0(a_o t) \hat{F}_t(m) t dt, \]
where \( F_t(k) = f(t, k) \). Hence from (4.2) we get
\[ \mathcal{F}_1 f(a_o, s) = \sum_{|m| \leq a_o} \int_0^\infty J_0(a_o t) \hat{F}_t(m) \chi_m(s) t dt. \]
By comparing (4.3) with (4.4), we get the required identity. \( \square \)

**Remark 4.5.** Notice that by taking inverse Fourier transform in both the sides of (4.2), we can assume \( f \) is trigonometric polynomial as long as \( \hat{f}(a_o, \sigma) \) is a finite rank operator for some \( a_o \in \mathbb{R}_+ \) and \( \sigma \in \hat{M} \).

**Theorem 4.6.** Let \( f \in L^1(G) \) be a radial function in the first variable which integrates zero in the second variable. If \( \hat{f}(a_o, \sigma) \) is a finite rank operator for some \( a_o \in \mathbb{R}_+ \) and \( \text{sign}(J_0 f) \geq 0 \), then \( f = 0 \).

**Proof.** In view of Remark 4.5, from Lemma 4.4 we infer that
\[ \int_0^\infty \hat{F}_t(o) t dt = \int_0^\infty f(t, s) t dt. \]
This, in turn, implies that
\[ \int_0^\infty \left( \int_K f(t, k) dk \right) t dt = \int_0^\infty f(t, s) t dt. \]
Since \( f \) integrates zero on \( K \) and \( \text{sign}(J_0 f) \geq 0 \), we conclude that \( f = 0 \). \( \square \)

5. **Some results on the Heisenberg uniqueness pairs**

In this section, we explore the Heisenberg uniqueness pairs for the Fourier transform on the motion group \( G \) as well as on the product group \( G' = \mathbb{R}^n \times K \), where \( K \) is a compact group. Further, we observed a one to one correspondence between the class of HUP’s on \( \mathbb{R}^n \) and the class of HUP’s on \( G' \).

Let \( \Gamma \) be a smooth surface (or a finite union of smooth surfaces) in \( \mathbb{R}^n \) and \( \Gamma' = \Gamma \times K \). Let \( X(\Gamma') \) be the space of all finite complex-valued Borel measures \( \mu \) in the motion group \( G \) which is supported on \( \Gamma' \) and absolutely continuous with respect to the surface measure on \( \Gamma' \).

We define the Fourier transform of \( \mu \) on \( G \) by
Hence \( F = \) where \( \mu \in X(\Gamma') \) is such that \( \hat{\mu}(a_o) = 0 \) for some \( a_o \not\in J_{(n+2l-2)/2}(0) \); \( \forall \ l \in \mathbb{Z} \) and \( \sigma \in \hat{M} \), then \( \mu = 0 \).

Proof. Since \( \mu \) is absolutely continuous with respect to the surface measure on \( \Gamma' \), by Radon-Nikodym theorem, there exists a function \( f \in L^1(\Gamma') \) such that \( d\mu = f ds dt \). By hypothesis, we have

\[
(\hat{\mu}(a_o) \varphi)(k) = \int_{\Gamma'} \int_K f(x, s)e^{-i(x,ak \cdot e_n)} \varphi(s^{-1}k) d\mu(x) ds,
\]

where \( a \in \mathbb{R}^+ \) and \( \varphi \in H(K, \mathbb{C}^{d_x}) \).

**Theorem 5.1.** Let \( \Gamma' = S^{n-1} \times K \), where \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \). Suppose \( \mu \in X(\Gamma') \) is such that \( \hat{\mu}(a_o) = 0 \) for some \( a_o \not\in J_{(n+2l-2)/2}(0) \); \( \forall \ l \in \mathbb{Z} \) and \( \sigma \in \hat{M} \), then \( \mu = 0 \).

Proof. Since \( \mu \) is absolutely continuous with respect to the surface measure on \( \Gamma' \), by Radon-Nikodym theorem, there exists a function \( f \in L^1(\Gamma') \) such that \( d\mu = f ds dt \). By hypothesis, we have

\[
(\hat{\mu}(a_o) \varphi)(k) = \int_{S^{n-1}} \int_K f(t, s)e^{-i(t,a_o k \cdot e_n)} \varphi(s^{-1}k) dt ds = 0,
\]

whenever \( \varphi \in C(K, \mathbb{C}^{d_x}) \). Now, by Fubini’s theorem, we can write

\[
\int_K \int_{S^{n-1}} f(t, s)e^{-i(t,a_o k \cdot e_n)} dt \varphi(s^{-1}k) ds = \int_K F_a \varphi(a_o k \cdot e_n, s) \varphi(s^{-1}k) ds = 0.
\]

Hence \( F_a \varphi(a_o k \cdot e_n, s) = 0 \) for almost all \( s, k \in K \). Since \( SO(n) \) can be identified with \( S^{n-1} \) via \( k \rightarrow k \cdot e_n \), it follows that \( F_a \varphi(y, s) = 0 \) for almost all \( y \in S^{n-1}(o) \) and \( s \in K \). Since we know from Theorem [5.2] that the pair \( (S^{n-1}, S^{n-1}(o)) \) is a HUP as long as \( J_{(n+2l-2)/2}(a_o) \neq 0 \) for all \( l \in \mathbb{Z}, \) we conclude that \( \mu = 0 \).

**Remark 5.2.** Let \( (\Gamma, K) \) be a HUP in \( \mathbb{R}^n \) and suppose \( \mu \in X(\Gamma') \) is such that \( \hat{\mu}(a_o) = 0 \) for some \( a_o \not\in J_{(n+2l-2)/2} \) and \( \forall \ l \in \mathbb{Z} \), then \( \mu = 0 \).

The Haar measure on the product group \( G' \) is given by \( dq = dx dk \), where \( dx \) is Lebesgue measure on \( \mathbb{R}^n \) and \( dk \) is normalized Haar measure on \( K \). Since the unitary dual of \( G' \) can be parameterized by \( G' = \mathbb{R}^n \times K \), for each \( (y, \delta) \in G \), the map \( (x, k) \rightarrow e^{-2\pi i x y} \delta(k) \) is a unitary operator on the Hilbert space \( \mathcal{H}_\delta \). Hence, we can define the Fourier transform of the function \( f \in L^1(G') \) by

\[
\hat{f}(y, \delta) = \int_{\mathbb{R}^n} \int_K f(x, k)e^{-2\pi i x y} \delta(k^{-1}) dx dk.
\]

Let \( \Gamma' = \Gamma \times K \), where \( \Gamma \) is a smooth surface (or a finite union of smooth surfaces) in \( \mathbb{R}^n \). Let \( X(\Gamma') \) be the space of all finite complex-valued Borel measure \( \mu \) in \( G' \) which is supported on \( \Gamma' \) and absolutely continuous with respect to the surface measure on \( \Gamma' \). Then by the Radon-Nikodym theorem, there exists a function \( f \in L^1(\Gamma') \) such that \( d\mu = f dv dk \), where \( v \) is the surface measure on \( \Gamma \).

Now, the Fourier transform of the measure \( \mu \) can be defined by
\[
\hat{\mu}(y, \delta) = \int_{\Gamma} \int_{K} e^{-2\pi i x \cdot y} \delta(k^{-1}) d\mu(x, k)
\]
(5.3)

\[
= \int_{\Gamma} \int_{K} f(x, k) e^{-2\pi i x \cdot y} \delta(k^{-1}) d\nu(x) dk.
\]

**Theorem 5.3.** The pair \((\Gamma, \Lambda)\) is a Heisenberg uniqueness pairs in \(\mathbb{R}^n\) if and only if \((\Gamma', \Lambda \times \hat{K})\) is a Heisenberg uniqueness pairs in \(G'\).

**Proof.** Suppose \((\Gamma, \Lambda)\) is a Heisenberg uniqueness pair in \(\mathbb{R}^n\) and \(\mu \in X(\Gamma')\). Then by Fubini’s theorem, the map \(x \mapsto f(x, k)\) belongs to \(L^1(\Gamma, d\nu)\) for almost all \(k \in K\). Hence for \((k, \sigma) \in K \times \hat{K}\), we can define the projection \(f_{k, \sigma}\) of \(f\) by

\[
f_{k, \sigma}(x) = \int_{K} f(x, kh^{-1}) \chi_{\sigma}(h) dh,
\]
(5.4)

where \(\chi_{\sigma} = \text{tr} \sigma(\cdot)\), the character of the representation \(\sigma\). Thus, the Euclidean Fourier transform of the projection \(f_{k, \sigma}\) gives

\[
\hat{f}_{k, \sigma}(y) = \int_{\Gamma} \int_{K} f(x, kh^{-1}) e^{-2\pi i x \cdot y} \chi_{\sigma}(h) dh d\nu(x)
\]
\[
= \text{tr} \int_{\Gamma} \int_{K} f(x, kh^{-1}) \sigma(h) dh e^{-2\pi i x \cdot y} d\nu(x)
\]
\[
= \text{tr} \int_{\Gamma} \int_{K} f(x, h) \sigma(h^{-1}) dh e^{-2\pi i x \cdot y} d\nu(x) \sigma(k)
\]
(5.5)

\[
= \text{tr} \left(\hat{\mu}(y, \sigma) \sigma(k)\right).
\]

Suppose \(\hat{\mu}|_{\Lambda \times \hat{K}} = 0\). Since \((\Gamma, \Lambda)\) is a Heisenberg uniqueness pair in \(\mathbb{R}^n\), from (5.3), it follows that \(f_{k, \sigma} = 0\). Hence by the uniqueness of the Fourier series

\[
f(x, k) = \sum_{\sigma \in \hat{K}} d_{\sigma} f_{k, \sigma}(x)
\]

we conclude that \(f = 0\).

Conversely, suppose \((\Gamma', \Lambda \times \hat{K})\) is a Heisenberg uniqueness pair in \(G'\). Then for \(\mu \in X(\Gamma')\), there exists a function \(f \in L^1(\Gamma')\) such that \(d\mu = f d\nu\). Suppose \(\hat{\mu}|_\Lambda = 0\). Then

\[
\int_{\Gamma} f(x) e^{-2\pi i x \cdot y} d\nu(x) = 0
\]
for each \(y \in \Lambda\). This, in turn, implies

\[
\int_{\Gamma} \int_{K} f(x) e^{-2\pi i y \cdot x} \delta(k^{-1}) dk d\nu(x) = 0.
\]
(5.6)

Now, if we write \(d\rho = f d\nu dk\), then \(\rho \in X(\Gamma')\). Since \((\Gamma', \Lambda \times \hat{K})\) is a Heisenberg uniqueness pair, by (5.6), it follows that \(\rho = 0\). Thus, using the fact that group compact group \(K\) is unimodular, we conclude that the measure \(\mu = 0\). \(\square\)
Concluding remarks:

We have shown that if the Fourier transform of a function in $L^1 \cap L^2(M(n))$ lands into the space of finite rank operators, then the function has to vanish. However, it would be an interesting question to consider the case when the spectrum of the group Fourier transform of a function $L^1(M(n))$ is supported on a thin uncountable set. We leave this question open for the time being.

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