Problem statement

Problem

- A channel that does not have a mathematical or probabilistic model
- A feedback link exists
- What can we say about achievable rates (with/without feedback)?
Unknown channels - the traditional approach

AVC (Arbitrarily Varying Channel) approach

- Binary example: \( \{e_i\} \) arbitrary, but \( \hat{\epsilon} = \frac{1}{n} \sum_{i=1}^{n} e_i \leq \epsilon_0 \leq \frac{1}{2} \)

- The maximum rate that can be transmitted reliably (with common randomness):

\[
C_{AVC} = 1 - h_b(\epsilon_0)
\]

- Without a prior guarantee on \( \epsilon_0 \): \( C_{AVC} = 0 \)

- Worst case approach - no gain when channel is better
Unknown channels - the individual noise sequence approach

Shayevitz and Feder [1] showed that with feedback, the following rate is achievable without prior knowledge of $\hat{\epsilon}$:

$$R = 1 - h_b(\hat{\epsilon})$$

Eswaran et al [2] extended to general discrete channels.

Conclusion

Feedback $\Rightarrow$ Rate adaptation $\Rightarrow$ No prior constraint required!
Back to our problem - main result

Discrete case
The empirical mutual information \( \hat{I}(x; y) \) is asymptotically achievable without assuming a channel model (adaptively, when there is feedback)

Continuous (real valued) case
\[
R_{\text{emp}} = \frac{1}{2} \log \left( \frac{1}{1 - \hat{\rho}^2(x, y)} \right)
\]
is asymptotically adaptively achievable
Definition of a rate adaptive system with feedback

- The message is an infinite bit sequence
- The decoder decides on a rate $R$ and decodes the first $nR$ bits
- Error is measured only on these bits
- A noiseless (but possibly rate limited) feedback channel exists
- Common randomness is assumed
A rate function $R_{emp}(x, y)$ is a function of the input and output sequences.

- **Fixed rate, no feedback:**
  - Fixed transmission rate $R$
  - Reliable communication whenever $R_{emp}(x, y) \geq R$

- **Variable rate, with feedback:**
  - Variable transmission rate $R \geq R_{emp}(x, y)$
  - Reliable communication for all $x, y$. 
The proposed rate adaptive scheme

- Iterative rateless coding scheme
- Varying number of blocks $B$, varying block size, each encoding $K$ bits
- The codebook $C_{M \times n}$ is drawn i.i.d. $\sim Q$
- End of block determined by the decoder (through feedback).

\[
\begin{array}{cccccc}
\text{time} = 1 & \text{Block 1} & \text{Block 2} & \text{Block 3} & \text{Block 4} & \text{Block 5} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
n & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]
The proposed rate adaptive scheme (2)

- The decoder announces the end of the block if any codeword $x_i$ has empirical rate exceeding the decoding threshold:

$$\exists i : \hat{R}_{\text{emp}}((x_i)_{k_b}, y_{k_b}^k) \geq \mu^*_k - k_b + 1$$

- Where the decoding threshold $\mu^*_m$:

$$\mu^*_m = \begin{cases} 
\frac{K + \log\left(\frac{n}{P_e}\right) + |X||Y| \log(m+1)}{m} & \text{discrete} \\
\frac{K + \log\left(\frac{2n}{P_e}\right)}{m-1} & \text{continuous}
\end{cases}$$

- The decoder chooses an $\hat{i}_b$ fulfilling the condition

- Last block potentially not decoded
The proposed rate adaptive scheme (3)
The scheme

Proof techniques

Two main properties are used:

▶ Concentration properties:
  For fixed $y$ and a random i.i.d. sequence $x$

$$\Pr\{\hat{I}(x; y) \geq t\} \approx \exp(-nt)$$

Relate the decoding threshold and error probability

▶ “Likely convexity”:
  For subsets $A_i \subset \{1, \ldots, n\}$, for most $x$ sequences:

$$\sum_{i} \frac{|A_i|}{n} \hat{I}(x_{A_i}; y_{A_i}) \geq \hat{I}(x; y) - \Delta$$

where $\Delta$ can be made small for large $n$.

These properties have analogs in the continuous case.

Yuval Lomnitz, “Communication over Individual Channels”, ACC 2011
Main observations

- Rates can be defined based on the *empirical channel behavior*, without knowing the “true” underlying model.
- Using feedback we can adapt the rate and avoid outage.
Advantages of the proposed approach

- Not relying on probabilistic models to construct the communication system
  - Probabilistic models may be inaccurate / non existent
- Stronger achievability theorems:
  - Rates and error probabilities are obtained per sequence & message
  - Achievable rates for various models follow by applying LLN (AVC-s, individual noise sequences, probabilistic models, compound channels)
- Adaptivity (when FB exists), as opposed to worst case design
- Small amount of feedback required to adapt the rate (asymptotically “zero rate”)
What’s missing?

The framework is not satisfactory in the following senses:

- It does not answer “how to select the rate function”?
- When channel is individual, cannot compare systems
  - Do not know what will be the effect of different input
  - Fundamental difference from universal compression/prediction
  - No comparability ⇒ no competitive universality!

We have to assume a fixed prior (no framework for prior adaptation).
Further results and questions

The end goal:

- With feedback, achieve universally the same rates that a system optimized to the channel would achieve, with minimal assumptions (e.g. no “stationarity”)

Further results obtained under individual channel model

- Characterization of the set of achievable rate functions (upper and lower bounds)
- Extensions to MIMO and higher order statistics (in time)
Further results and questions (2)

Unknown vector channels
- Channel is an unknown probabilistic relation between input and output
- This framework that allows a definition of competition, and hence competitive universality.
- Shown competitive universality for limited classes of channels:
  - Modulo additive channel with an individual noise sequence (fixed input prior)
  - Channels memoryless in the input (arbitrarily varying memoryless channel) - adaptive input prior.
- The comparison class is finite block length encoders and decoders.
Discussion and Summary

Summary

- Communication rates approaching the empirical mutual information $\hat{I}(x; y)$ can be attained adaptively with feedback, without prior assumptions on the channel.
- May be useful when there is no natural model.
- May be a step toward “universal” communication.

Thank you!

The material is available on my website http://www.eng.tau.ac.il/~yuvall
[1] O. Shayevitz and M. Feder, “Communicating using feedback over a binary channel with arbitrary noise sequence,” in *IEEE International Symposium on Information Theory (ISIT)*, Sep. 2005.

[2] K. Eswaran, A. Sarwate, A. Sahai, and M. Gastpar, “Zero-rate feedback can achieve the empirical capacity,” *IEEE Transactions on Information Theory*, vol. 58, no. 1, Jan. 2010.