Galilean Superconformal Symmetries

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Abstract

We consider the non-relativistic \( c \to \infty \) contraction limit of the \((N = 2k)\)-extended \( D = 4 \) superconformal algebra \( su(2,2;N) \), introducing in this way the non-relativistic \((N = 2k)\)-extended Galilean superconformal algebra. Such a Galilean superconformal algebra has the same number of generators as \( su(2,2|2k) \). The \( usp(2k) \) algebra describes the non-relativistic internal symmetries, and the generators from the coset \( \frac{su(2k)}{usp(2k)} \) become central charges after contraction.

1 Introduction

There are two ways of enlarging the standard Galilei algebra by additional generators related with the conformal symmetries (see e.g. [1])

i) One can add to the Galilean symmetries the dilatations and the one-parameter conformal transformations which determine the invariance group of free Schrödinger equation [2–6]. In such a way we obtain the so-called Schrödinger algebra, which is the Galilean algebra enlarged by two generators \( D \) (dilatations) and \( K \) (time expansions). Recently, some authors have referred to the Schrödinger symmetry as Galilean conformal symmetry (see
e.g. \[2, 7\]), but this approach does not lead to the typical features of conformal systems such as vanishing masses, the presence of conformal space translations etc.

ii) Following the contraction of the Poincaré algebra to the Galilei one, there was performed as well the non-relativistic contraction of relativistic conformal algebra (in \(D\) dimensions \(o(D, 2)\) to Galilean conformal algebra \[8, 9\].

Denoting by \(P_\mu = (P_0, P_i)\), \(M_{\mu\nu} = (M_{ij}, M_{i0})\) \((\mu, \nu = 0, 1 \ldots D - 1; i, j = 1, 2 \ldots d = D - 1)\) the Poincaré generators, the relativistic conformal algebra also includes the generators of dilatations \(D\) and those of the special conformal transformations \(K_\mu = (K_0, K_i)\). If we rescale the relativistic generators in the following way \[8\]

\[
\begin{align*}
P_0 &= \frac{H}{c}, & M_{i0} &= cB_i, \\
K_0 &= cK, & K_i &= c^2F_i,
\end{align*}
\]

\((P_i, M_{ij}\) and \(D\) remaining unchanged) we obtain, after performing the non-relativistic contraction limit \(c \to \infty\), the following \(\frac{1}{2}(D+1)(D+2)\)-dimensional Galilean conformal algebra \[8\]–\[10\]:

\[
\begin{align*}
[H, P_i] &= 0, & [H, B_i] &= P_i, & [H, F_i] &= 2B_i, \\
[K, P_i] &= -2B_i, & [K, B_i] &= F_i, & [K, F_i] &= 0, \\
[D, P_i] &= -P_i, & [D, B_i] &= 0, & [D, F_i] &= F_i,
\end{align*}
\]

and

\[
\begin{align*}
[D, H] &= -H, & [K, H] &= -2D, & [D, K] &= K,
\end{align*}
\]

\(\delta_{ij} A_i - \delta_{ik} A_j\),

where the subalgebra \(A_i = (P_i, B_i, F_i)\) describes the generators of the maximal Abelian subgroup. The algebra of rotations \(o(d)\) is described by the commutators

\[1\] In comparison with \[8\] we denote here the Galilean boosts by \(B_i\), and the relativistic conformal translations generators by \(K_\mu\).

\[2\] It should be added that a family of non-relativistic conformal algebras was introduced in \[14\], with one member providing the relations \[24\]–\[4\].
\[ [M_{ij}, M_{kl}] = \delta_{ik} M_{jl} - \delta_{il} M_{jk} + \delta_{jl} M_{ik} - \delta_{jk} M_{il}, \quad (7) \]

\[ [M_{ij}, \mathcal{A}] = 0, \quad \mathcal{A} = (H, D, K). \quad (8) \]

The generators \((P_i, B_i, H, M_{ij})\) define the \(D\)-dimensional Galilean algebra, where \(B_i\) are the Galilean boosts and \(H\), the non-relativistic energy operator, generates the Galilean time translations. We see that one can treat the Galilean conformal algebra as the result of adding the generators \(F_i\) of constant accelerations to the Schrödinger algebra \[10\]. We note that the subalgebra \((5)\) is the one-dimensional conformal algebra \(o(2, 1)\).

The aim of this paper is to introduce the supersymmetry of Galilean conformal symmetry. There have been several proposals to derive a supersymmetric Galilei algebra \[12\]–\[16\], but among the conformal generalizations only the Schrödinger symmetry has been enlarged to a superSchrödinger symmetry \[17\]–\[20\]. These supersymmetric Schrödinger algebras have not been obtained by performing the non-relativistic limit \(c \to \infty\) of relativistic superconformal algebra \(su(2, 2|N)\), but rather by considering a suitable projection of the relativistic odd generators (see e.g. \[19, 20\]). In this paper we shall use the natural procedure of the non-relativistic contraction \(c \to \infty\) limit applied to the \(N\)-extended Wess-Zumino superconformal algebra \(su(2, 2|2k)\) \((k = 1, 2, \ldots)\). For odd \(N\) (in particular for \(N = 1\)) the method of non-relativistic contraction presented in this paper does not work.

The paper is organized as follows. In Sect. 2 we describe the superconformal algebra \(su(2, 2|N)\), which we shall rewrite further for \(N = 2k\) by using suitably projected supercharges. In Sect. 3 we consider our non-relativistic contraction limit \(c \to \infty\), which leads to the Galilean superconformal algebra. From the relativistic superconformal internal sector \(u(2k)\) we obtain the non-relativistic superconformal internal symmetries \(usp(2k) \simeq sp(k; H)\) \((k(2k + 1)\) generators) and a set of \(k(2k - 1)\) Abelian central charges.

Finally in Sect. 4 we present an outlook.
2 \( \mathbf{N\text{-extended } D = 4 \text{ superconformal algebra } su(2, 2|N)} \)

We describe the \( su(2, 2|N) \) superalgebra of antiHermitean generators\(^\text{3}\) using the \( D = 4 \) Majorana representation for the \( 4 \times 4 \) real Dirac matrices \( \gamma_\mu \equiv (\gamma_\mu)^\alpha_\beta (\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}; \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)) \) satisfying the properties

\[ \gamma_i = \gamma_i^T, \quad C = \gamma_0 = -\gamma_0^T, \quad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = -\gamma_5^T, \]

(9)

where \( C \equiv C_{\alpha\beta} \) describes the symplectic metric in the space of real Majorana spinors.

a) bosonic sector \((M_\mu\nu, P_\rho, D, K_\mu, T^{ab} \in u(N))\)

\[
\begin{align*}
[M_\mu\nu, M_\rho\tau] &= \eta_{\mu\tau}M_\nu\rho - \eta_{\nu\rho}M_\mu\tau + \eta_{\nu\rho}M_\mu\tau - \eta_{\mu\tau}M_\nu\rho, \\
[M_\mu\nu, P_\rho] &= \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu, \\
[M_\mu\nu, K_\rho] &= \eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu, \\
[P_\mu, P_\nu] &= [K_\mu, K_\nu] = 0, \\
[D, P_\mu] &= -P_\mu, \quad [D, K_\mu] = K_\mu, \quad [D, M_\mu\nu] = 0, \\
[P_\mu, K_\nu] &= 2(\eta_{\mu\nu}D - M_\mu\nu).
\end{align*}
\]

(10)

The internal superconformal symmetries (sometimes called \( R \)-symmetries) are given by the \( u(N) \) generators \((T_S^{ab}, iT_A^{ab})\) where \( T_S^{ab} = T_S^{ba} \) and \( T_A^{ab} = -T_A^{ba} \), besides \( T_S^{ab} \in \text{su}(N)_{\text{o}(N)} \) and \( T_A^{ab} \in \text{o}(N) \). The internal \( u(N) \) Lie algebra can be described by the generators \( T_S^{ab}, T_A^{ab} \) as follows

\[
\begin{align*}
[T_S^{ab}, T_S^{cd}] &= \delta^{bc}T_A^{ad} + \delta^{ac}T_A^{bd} + \delta^{ad}T_A^{bc} + \delta^{bd}T_A^{ac}, \\
[T_S^{ab}, T_A^{cd}] &= \delta^{ad}T_S^{bc} + \delta^{bd}T_S^{ac} - \delta^{ac}T_S^{bd} - \delta^{bc}T_S^{ad}, \\
[T_A^{ab}, T_A^{cd}] &= \delta^{bc}T_A^{ad} - \delta^{ac}T_A^{bd} + \delta^{ad}T_A^{bc} - \delta^{bd}T_A^{ac},
\end{align*}
\]

(11)

where we assume that \((T_S^{ab})^\dagger = -T_S^{ba}\). One can introduce the \( N \times N \) matrix representation of the algebra \((\{x_S^{ab}\}, x_A^{ab})\) which can be deduced

\(^3\text{In the special } N = 4 \text{ case the axial charge becomes central, and the } N = 4 \text{ superconformal algebra is often denoted as } \text{psu}(2, 2|4)\)
from a $N \times N$ generalization of the Pauli matrices $\sigma_i$, supplemented by the unit matrix. For $N = 2$ we get

$$\tau_{ab}^S = (1^{ab}, \sigma_1^{ab}, \sigma_3^{ab}), \quad \tau_A^{ab} = \varepsilon^{ab} = -i\sigma_2^{ab}. \quad (12)$$

b) fermionic sector ($Q^a_{\alpha}, S^a_{\alpha}; \alpha, \beta = 1 \ldots 4, a, b = 1 \ldots N$)

$$\{Q^a_{\alpha}, Q^b_{\beta}\} = 2\delta^{ab}(\gamma^\mu C)_{\alpha\beta}P_\mu, \quad (13a)$$

$$\{S^a_{\alpha}, S^b_{\beta}\} = -2\delta^{ab}(\gamma^\mu C)_{\alpha\beta}K_\mu, \quad (13b)$$

$$\{Q^a_{\alpha}, S^b_{\beta}\} = \delta^{ab}[2C_{\alpha\beta}D - (\sigma^{\mu\nu} C)_{\alpha\beta}M_{\mu\nu} - 4(\gamma_5 C)_{\alpha\beta}A]$$

$$+ 2C_{\alpha\beta}T^b_A + 2(\gamma_5 C)_{\alpha\beta}T^b_S, \quad (13c)$$

where $(\gamma^\mu C)_{\alpha\beta} = (\gamma^\mu)_{\alpha}C_{\gamma\beta}$ and the axial charge generator $A$ corresponds to the trace of $T^b_S$ (in formula (13c) and further below we assume that $tr T_S \equiv T^a_S = 0$).

c) mixed bosonic-fermionic sector (covariance relations)

$$[P_\mu, Q^a_{\alpha}] = 0, \quad [P_\mu, S^a_{\alpha}] = -(\gamma_\mu)^\beta_{\alpha}Q^a_{\beta},$$

$$[K_\mu, Q^a_{\alpha}] = -(\gamma_\mu)^\beta_{\alpha}S^a_{\beta}, \quad [K_\mu, S^a_{\alpha}] = 0,$$

$$[M_{\mu\nu}, Q^a_{\alpha}] = -\frac{1}{2}(\sigma_{\mu\nu})^\beta_{\alpha}Q^a_{\beta}, \quad [M_{\mu\nu}, S^a_{\alpha}] = -\frac{1}{2}(\sigma_{\mu\nu})^\beta_{\alpha}S^a_{\beta},$$

$$[D, Q^a_{\alpha}] = -\frac{1}{4}Q^a_{\alpha}, \quad [D, S^a_{\alpha}] = \frac{1}{2}S^a_{\alpha},$$

$$[A, Q^a_{\alpha}] = -\frac{1}{4}(1 - \frac{4}{N})(\gamma^5)^\beta_{\alpha}Q^a_{\beta},$$

$$[A, S^a_{\alpha}] = \frac{1}{4}(1 - \frac{4}{N})(\gamma^5)^\beta_{\alpha}S^a_{\beta}, \quad (14)$$

$$[T^a_{S}, Q^C_{\alpha}] = (\gamma_5)^\beta_{\alpha}(\tau^a_{S})^{cd}Q^d_{\beta},$$

$$[T^a_{S}, S^C_{\alpha}] = -(\gamma_5)^\beta_{\alpha}(\tau^a_{S})^{cd}S^d_{\beta},$$

$$[T^a_{A}, Q^C_{\alpha}] = (\tau^a_{A})^{cd}Q^d_{\alpha},$$

$$[T^a_{A}, S^C_{\alpha}] = (\tau^a_{A})^{cd}S^d_{\alpha}. \quad (15)$$

Further we introduce the following matrix projector operators
\[ (P_{\pm})_{\alpha\beta}^{ab} = \frac{1}{2}(\delta_{\alpha\beta} \delta^{ab} \pm C_{\alpha\beta} \Omega^{ab}) , \]  

(16)

where

\[ \Omega^{ab} = -\Omega^{ba} , \quad \Omega^2 = -1 . \]  

(17)

The definition (16) leads to the following projection operator properties

\[ (P_{\pm})_{\alpha\beta}^{ab} (P_{\pm})_{\beta\gamma}^{bc} = (P_{\pm})_{\alpha\beta}^{ac} , \quad (P_{\pm})_{\alpha\beta}^{ab} (P_{\pm})_{\beta\gamma}^{bc} = 0 . \]  

(18)

We define the projected supercharges as follows

\[ Q_{\pm\alpha}^a = (P_{\pm})_{\alpha\beta}^{ab} Q_{\beta}^b , \quad S_{\pm\alpha}^a = (P_{\pm})_{\alpha\beta}^{ab} S_{\beta}^b . \]  

(19)

Using the symmetry property

\[ (P^T_{\pm})_{\alpha\beta}^{ab} \equiv (P_{\pm})_{\beta\alpha}^{ba} = (P_{\pm})_{\alpha\beta}^{ab} , \]  

(20)

one gets from (13a–13c)

\[ \{Q_{\pm\alpha}^a, Q_{\pm\beta}^b\} = 2(P_{\pm})_{\alpha\beta}^{ab} P_0 , \]

\[ \{Q_{+\alpha}^a, Q_{-\beta}^b\} = 2(P_+)^{ab}_{\alpha\beta} (\gamma_i C)^{\gamma\beta} P_i , \]  

(21)

\[ \{S_{\pm\alpha}^a, S_{\pm\beta}^b\} = -2(P_{\pm})_{\alpha\beta}^{ab} K_0 , \]

\[ \{S_{+\alpha}^a, S_{-\beta}^b\} = -2(P_+)^{ab}_{\alpha\beta} (\gamma_i C)^{\gamma\beta} K_i . \]  

(22)

In order to rewrite the relation (13c) in terms of the projected supercharges (19) we split the generators \( T_{S_{\pm}}^{ab}, T_{A_{\pm}}^{ab} \) into the following four sectors

\[ T_{S_{\pm}}^{ab} = T_{S_{+}}^{ab} + T_{S_{-}}^{ab} , \quad T_{A_{\pm}}^{ab} = T_{A_{+}}^{ab} + T_{A_{-}}^{ab} , \]  

(23)

where

\[ [\Omega, T_{S_{+}}^{ab}] = 0 , \quad \{\Omega, T_{S_{-}}^{ab}\} = 0 . \]  

(24)

Then, we obtain
\begin{align}
\{Q_{\pm\alpha}^a, S_{\pm\beta}^b\} &= (P_{\pm\alpha})^{ac}_{\alpha\gamma} [\delta^b_{\gamma\delta} (-\sigma^{ij} C)_{\gamma\beta} M_{ij} \\
&\quad + 2C_{\gamma\beta} D] + 2C_{\gamma\beta} T^a_{\alpha+} + 2(\gamma_5 C)_{\gamma\beta} T^a_{S-}], \quad (25a) \\
\{Q_{\pm\alpha}^a, S_{\pm\beta}^b\} &= (P_{\pm\alpha})^{ac}_{\alpha\gamma} [\delta^b_{\gamma\delta} (-\sigma^{ij} C)_{\gamma\beta} M_{ij} \\
&\quad - 4(\gamma_5 C)_{\gamma\beta} A] + 2C_{\gamma\beta} T^a_{\alpha-} + 2(\gamma_5 C)_{\gamma\beta} T^a_{S+}]. \quad (25b)
\end{align}

The relations (24) can be expressed also in the following way

\[ \Omega \cdot H = -H^T \cdot \Omega, \quad H = (T_{S-}, T_{A+}) \quad (26a) \]
\[ \Omega \cdot K = K^T \cdot \Omega, \quad K = (T_{S+}, T_{A-}) \quad (26b) \]

Recalling that the $N \times N$ ($N = 2k$) matrix representation of the generators of $u(2k)$ when it is additionally constrained by the symplectic condition (26a) define the matrix subalgebra $usp(2k)$ [21], we obtain

\[ H = usp(2k), \quad K = \frac{u(N)}{usp(2k)}. \quad (27) \]

Indeed using the table of $[A, B]$ commutators, obtained from the relations (11) and (24)

\begin{center}
\begin{tabular}{c|cccc}
$B \setminus A$ & $T_{S+}$ & $T_{S-}$ & $T_{A+}$ & $T_{A-}$ \\
\hline
$T_{S+}$ & $T_{A+}$ & $T_{A-}$ & $T_{S+}$ & $T_{S-}$ \\
$T_{S-}$ & $T_{A-}$ & $T_{A+}$ & $T_{S-}$ & $T_{S+}$ \\
$T_{A+}$ & $T_{S+}$ & $T_{S-}$ & $T_{A+}$ & $T_{A-}$ \\
$T_{A-}$ & $T_{S-}$ & $T_{S+}$ & $T_{A-}$ & $T_{A+}$ \\
\end{tabular}
\end{center}

Table 1: Fourfold split of $u(N)$ algebra

one can check the relations

\[ [H, H] \subset H, \quad [H, K] \subset K, \quad [K, K] \subset H, \quad (28) \]

that describe the symmetric Riemannian space $(H, K)$. The subalgebra $H$ has $k(2k+1)$ generators, and the coset $K$ contains $k(2k-1)$ generators. For low values of $k$ one gets:
Using the relations (14) and the projected supercharges (19) one gets the following set of non-zero commutators for the mixed bosonic-fermionic sector

\[
\begin{align*}
[P_i, S_{\pm\alpha}] &= - (\gamma_i)_{\alpha} \beta Q_{\mp\beta}^a, \\
[P_0, S_{\pm\alpha}] &= Q_{\pm\alpha}^a, \\
[K_i, Q_{\pm\alpha}] &= - (\gamma_i)_{\alpha} \beta S_{\mp\beta}^a, \\
[K_0, Q_{\pm\alpha}] &= S_{\pm\alpha}^a, \\
[M_{ij}, Q_{\pm\alpha}] &= - \frac{1}{2} (\sigma_{ij})_{\alpha} \beta Q_{\mp\beta}^a, \\
[M_{0i}, Q_{\pm\alpha}] &= - \frac{1}{2} (\sigma_{0i})_{\alpha} \beta Q_{\mp\beta}^a, \\
[M_{ij}, S_{\pm\alpha}] &= - \frac{1}{2} (\sigma_{ij})_{\alpha} \beta S_{\mp\beta}^a, \\
[M_{0i}, S_{\pm\alpha}] &= - \frac{1}{2} (\sigma_{0i})_{\alpha} \beta S_{\mp\beta}^a, \\
[D, Q_{\pm\alpha}] &= - \frac{1}{2} Q_{\pm\alpha}^a, \\
[D, S_{\pm\alpha}] &= \frac{1}{2} S_{\pm\alpha}^a, \\
[A, Q_{\pm\alpha}] &= - \frac{1}{4} (1 - \frac{4}{N}) (\gamma_5)_{\alpha} \beta Q_{\mp\beta}^a, \\
[A, S_{\pm\beta}] &= \frac{1}{4} (1 - \frac{4}{N}) (\gamma_5)_{\alpha} \beta S_{\mp\beta}^a.
\end{align*}
\]

The projected supercharges (19) permit to rewrite the relations (15) as follows:

Table 2: Lower-dimensional symplectic cosets of $u(2k)$
The action of the generators \((T_{S-}, T_{A+}) \in H\) produces linear transformations on the projected supercharges, and the generators \((T_{S+}, T_{A-}) \in K\) transform \((Q^a_{\alpha \pm}, S^a_{\alpha \pm})\) into the complementary projections \((Q^a_{\alpha \mp}, S^a_{\alpha \mp})\).

### 3 Galilean \((N = 2k)\)-extended superconformal algebra as the non-relativistic contraction of \(su(2, 2|2k)\)

In this Section we perform the contraction of the \(N\)-extended \((N = 2k)\) \(D = 4\) relativistic conformal superalgebras in a way that provides the supersymmetrization of Galilean conformal algebra, (see Eqs. (2–8)). To achieve this contraction with finite contraction limits we supplement the bosonic rescalings (1) by the following ones

\[
Q^a_{\alpha \pm} = \frac{1}{c\sqrt{c}} \tilde{Q}^a_{\alpha \pm}, \quad Q^a_{\alpha \mp} = \sqrt{c} \tilde{Q}^a_{\alpha \mp}, \quad (31a)
\]
\[
S^a_{\alpha \pm} = \sqrt{c} \tilde{S}^a_{\alpha \pm}, \quad S^a_{\alpha \mp} = (c)^{3/2} \tilde{S}^a_{\alpha \mp}, \quad (31b)
\]

where the tilde denotes the Galilean algebra supercharges. Further we introduce the rescaling (see (26a))

\[4\] The relative factor \(c\) between the two projections \(Q^a_{\pm \alpha}, S^a_{\pm \alpha}\) can be justified if we observe that the projector \(P_{\pm} = \frac{1}{2}(1 \pm \gamma^0)\) separates the “large” and “small” components of a Dirac spinor, a fact that has been used before [13, 15]. The global factor \(c\) between the \(Q\) and the \(S\) rescalings follows from dimensional considerations.
\[ h_{ab} \equiv (T_{S-}^{ab}, T_{A+}^{ab}) = (\tilde{T}_{S-}^{ab}, \tilde{T}_{A+}^{ab}) , \]

\[ k_{ab} \equiv (A, T_{S+}^{ab}, T_{A-}^{ab}) = (c\tilde{A}, c\tilde{T}_{S+}^{ab}, c\tilde{T}_{A-}^{ab}) , \]

where \( h_{ab} \in \mathbb{H}, k_{ab} \in \mathbb{K}, \) and \( \tilde{T}_{S-}^{ab}, \tilde{T}_{A+}^{ab}, \tilde{A} \) denote the Galilean internal symmetry generators or, more briefly,

\[ (\mathbb{H}, \mathbb{K}) = (\tilde{\mathbb{H}}, c\tilde{\mathbb{K}}) . \]

From (28) we get

\[ [\tilde{\mathbb{H}}, \tilde{\mathbb{H}}] \subset \tilde{\mathbb{H}}, \quad [\tilde{\mathbb{H}}, \tilde{\mathbb{K}}] \subset \tilde{\mathbb{K}}, \]

and

\[ [\tilde{\mathbb{K}}, \tilde{\mathbb{K}}] \subset \frac{1}{c^2} \mathbb{H} \quad \text{as} \quad c \to \infty \]

i.e. the generators \( \tilde{k}_{ab} \in \tilde{\mathbb{K}} \) become Abelian. As a result of the contraction we see that the Galilean internal symmetry algebra

\[ \tilde{T} = \tilde{\mathbb{K}} \supset \tilde{\mathbb{H}} = T^{2k(2k-1)} \supset \text{usp}(2k) \]

is non-semisimple.

We perform now the contraction \( c \to \infty \) of the relations (21–22), (25a–25b) and (29–30). The algebraic relations that describe the \( 2k \)-extended Galilean superconformal algebra are:

a) Fermionic sector

\[ \{ \tilde{Q}_{a+}^{\alpha}, \tilde{Q}_{b+}^{\beta} \} = 2(P_{+})^{ab}_{\alpha\beta} H , \quad \{ \tilde{Q}_{a-}^{\alpha}, \tilde{Q}_{b-}^{\beta} \} = 0 , \]

\[ \{ \tilde{Q}_{a+}^{\alpha}, \tilde{Q}_{b-}^{\beta} \} = 2(P_{+})^{ab}_{\alpha\gamma}(\gamma \gamma C)_{\gamma\beta} P_{i} , \]

\[ \{ \tilde{S}_{a+}^{\alpha}, \tilde{S}_{b+}^{\beta} \} = -2(P_{+})^{ab}_{\alpha\beta} K , \quad \{ \tilde{S}_{a-}^{\alpha}, \tilde{S}_{b-}^{\beta} \} = 0 , \]

\[ \{ \tilde{S}_{a+}^{\alpha}, \tilde{S}_{b-}^{\beta} \} = -2(P_{+})^{ab}_{\alpha\gamma}(\gamma \gamma C)_{\gamma\beta} F_{i} , \]

\[ \{ \tilde{Q}_{a+}^{\alpha}, \tilde{S}_{b+}^{\beta} \} = (P_{+})^{ac}_{\alpha\gamma}[\delta^{cb} \{ -(\sigma^{ij}C)_{\gamma\beta} M_{ij} + 2C_{\alpha\beta D} + 2C_{\gamma\beta \tilde{T}_{A+}} + (2\gamma_{5}C)_{\gamma\beta \tilde{T}_{S-}} \} , \]

\[ \{ \tilde{Q}_{a+}^{\alpha}, \tilde{S}_{b-}^{\beta} \} = 0 , \]

\[ \{ \tilde{Q}_{a+}^{\alpha}, \tilde{S}_{b-}^{\beta} \} = (P_{+})^{ac}_{\alpha\gamma}[-\delta^{cb} \{ (\sigma^{0j}C)_{\gamma\beta} B_{j} - 4(\gamma_{5}C)_{\gamma\beta} A \} + 2C_{\gamma\beta \tilde{T}_{A-}} + 2(\gamma_{5}C)_{\gamma\beta \tilde{T}_{S+}}] . \]
where in Eq. (37c) only remain the generators which belong to the subalgebra $\mathbb{H}$, and in Eq. (37d) appear only the central charges obtained from the generators belonging to $\mathbb{K}$ (see Eqs. (32) and (35)).

b) Mixed bosonic-fermionic sector

After the rescalings (11), (31a) we obtain from (29) in the limit $c \to \infty$

\[
[P_i, \tilde{S}^a_{\pm \alpha}] = -(\gamma_i)_\alpha^\beta \tilde{Q}^a_{\pm \beta}, \quad [P_i, \tilde{S}^a_{\mp \alpha}] = 0,
\]

\[
[H, \tilde{S}^a_{\pm \alpha}] = \tilde{Q}^a_{\pm \alpha},
\]

\[
[F_i, \tilde{Q}^a_{\pm \alpha}] = -(\gamma_i)_\alpha^\beta \tilde{S}^a_{\pm \beta}, \quad [F_i, \tilde{Q}^a_{\mp \alpha}] = 0,
\]

\[
[K, \tilde{Q}^a_{\pm \alpha}] = \tilde{S}^a_{\pm \alpha},
\]

\[
[M_{ij}, \tilde{Q}^a_{\pm \alpha}] = -\frac{1}{2}(\sigma_{ij})_\alpha^\beta \tilde{Q}^a_{\pm \beta},
\]

\[
[B_i, \tilde{Q}^a_{\pm \alpha}] = 0,
\]

\[
[M_{ij}, \tilde{S}^a_{\pm \alpha}] = -\frac{1}{2}(\sigma_{ij})_\alpha^\beta \tilde{S}^a_{\pm \beta},
\]

\[
[B_i, \tilde{S}^a_{\pm \alpha}] = 0,
\]

\[
[D, \tilde{Q}^a_{\pm \alpha}] = -\frac{1}{2} \tilde{Q}^a_{\pm \alpha},
\]

\[
[D, \tilde{S}^a_{\pm \alpha}] = \frac{1}{2} \tilde{S}^a_{\pm \alpha},
\]

\[
[\tilde{A}, \tilde{Q}^a_{\pm \alpha}] = -\frac{1}{4}(1 - \frac{4}{N})(\gamma_5)_\alpha^\beta \tilde{Q}^a_{\pm \beta}, \quad [\tilde{A}, \tilde{S}^a_{\pm \alpha}] = 0,
\]

\[
[\tilde{A}, \tilde{S}^a_{\pm \alpha}] = \frac{1}{4}(1 - \frac{4}{N})(\gamma_5)_\alpha^\beta \tilde{S}^a_{\pm \beta},
\]

(38)

The relations (30) that give the transformation properties under the internal symmetry generators have the following non-relativistic limits:

i) for the generators $\hat{h}^{ab} = (\tilde{T}_{S_-}, \tilde{T}_{A_+})$

\[
[T_{S_-}^{ab}, \tilde{Q}^c_{\pm \alpha}] = -(\gamma_5)_\alpha^\beta (\tau_{S_-}^{ab})^{cd} \tilde{Q}^d_{\pm \beta},
\]

\[
[T_{A_+}^{ab}, \tilde{Q}^c_{\pm \alpha}] = (\tau_{A_+}^{ab})^{cd} \tilde{Q}^d_{\pm \alpha},
\]

\[
[T_{S_-}^{ab}, \tilde{S}^c_{\pm \alpha}] = -(\gamma_5)_\alpha^\beta (\tau_{S_-}^{ab})^{cd} \tilde{S}^d_{\pm \beta},
\]

\[
[T_{A_+}^{ab}, \tilde{S}^c_{\pm \alpha}] = (\tau_{A_+}^{ab})^{cd} \tilde{S}^d_{\pm \alpha}.
\]

(39)

ii) for the generators $\hat{k}^{ab} = (\tilde{T}_{S_+}, \tilde{T}_{A_-})$

\[
[\tilde{T}_{S_+}^{ab}, \tilde{Q}^c_{\alpha}] = -(\gamma_5)_\alpha^\beta (\tau_{S_+}^{ab})^{cd} \tilde{Q}^d_{\beta -}, \quad [\tilde{T}_{S_+}^{ab}, \tilde{Q}^c_{\alpha -}] = 0,
\]

\[
[\tilde{T}_{A_-}^{ab}, \tilde{Q}^c_{\alpha}] = (\tau_{A_-}^{ab})^{cd} \tilde{Q}^d_{\alpha -}, \quad [\tilde{T}_{A_-}^{ab}, \tilde{Q}^c_{\alpha -}] = 0,
\]

\[
[\tilde{T}_{S_+}^{ab}, \tilde{S}^c_{\alpha}] = -(\gamma_5)_\alpha^\beta (\tau_{S_+}^{ab})^{cd} \tilde{S}^d_{\beta -}, \quad [\tilde{T}_{S_+}^{ab}, \tilde{S}^c_{\alpha -}] = 0,
\]

\[
[\tilde{T}_{A_-}^{ab}, \tilde{S}^c_{\alpha}] = (\tau_{A_-}^{ab})^{cd} \tilde{S}^d_{\alpha -}, \quad [\tilde{T}_{A_-}^{ab}, \tilde{S}^c_{\alpha -}] = 0.
\]

(40)
c) Bosonic sector of Galilean superconformal algebra

The spacetime bosonic sector is described by the $D = 4$ Galilean conformal algebra (see (2)–(8)) and the internal bosonic sector is the inhomogeneous $T^{k(2k-1)} \supset usp(2k)$ algebra (36). These two sectors commute in agreement with the Coleman-Mandula theorem [22].

4 Outlook

In this paper we have introduced a contraction prescription that provides a supersymmetrization of the Galilean conformal algebra, containing the non-relativistic conformal translations that describe constant accelerations. There are some further questions to be studied:

i) The contraction presented in this paper which selects as distinguished translation generator the single Hamiltonian is well adapted to the description of point particle dynamics. If we move to considering $p$-brane dynamics before performing the non-relativistic limit we should split the relativistic translation generators into the ones that describe the shift of world volume coordinates and those which correspond to the directions transversal to the $p$-brane [23, 24]. There have been already some proposals to describe the non-relativistic superconformal symmetries in such a framework (see e.g. [25, 26]). In this paper the rescaling is given by Eq. (31a), which is the way to obtain a standard non-relativistic superconformal limit $c \to \infty$.

ii) Other important question is the description of the quantum-mechanical realizations of our Galilean superconformal algebra. In [8] a realization of the Galilean conformal algebra in $D = 2 + 1$-dimensional non-relativistic phase space was proposed, but the corresponding action [26] is defined by a Lagrangian with higher order time derivatives. It would be interesting to see using e.g. the techniques presented in [27] how the appearance of central charges in our scheme can be used for the description of new classes of non-relativistic superconformal mechanics.

iii) An interesting question is to understand in a more fundamental way whether, as follows from our method, there are no $N$-extended Galilean superconformal symmetries for $N$ odd (e.g. $N = 1$).

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5Such superconformal mechanics should be not confused with the models providing $N$-extended conformal supersymmetry on the world line, described by $osp(N|2)$ superalgebra.
Note added:
The Galilean superconformal algebra was found independently by M. Sakaguchi [29], where a generalization of Galilean conformal and superconformal algebras for non-relativistic systems with finite space curvature described by Newton-Hooke geometries were also discussed. There, a rescaling of $P_i$ leaving $P_0$ unchanged was used, whereas in this paper we have supersymmetrically extended the standard nonrelativistic rescaling of $P_0$ (Eq. (1)) leaving $P_i$ unchanged. The authors would like to thank M. Sakaguchi for spotting some misprints in our paper.

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