A TORSION THEORY IN THE CATEGORY OF COCOMMUTATIVE HOPF ALGEBRAS

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Abstract. The purpose of this article is to prove that the category of cocommutative Hopf $K$-algebras, over a field $K$ of characteristic zero, is a semi-abelian category. Moreover, we show that this category is action representable, and that it contains a torsion theory whose torsion-free and torsion parts are given by the category of groups and by the category of Lie $K$-algebras, respectively.

1. Introduction

The starting point of this article on Hopf algebras is a well-known result due to A. Grothendieck, as outlined in [26], saying that the category of finite-dimensional, commutative and cocommutative Hopf $K$-algebras over a field $K$ is abelian. This result was extended by M. Takeuchi to the category of commutative and cocommutative Hopf $K$-algebras, not necessarily finite-dimensional [27]. The category $\text{Hopf}_{K,coc}$ of cocommutative Hopf $K$-algebras is not additive, thus it can not be abelian. In the present article we investigate some of its fundamental exactness properties, showing that it is a homological category (Section 3), and that it is Barr-exact (Section 5), leading to the conclusion that the category $\text{Hopf}_{K,coc}$ is semi-abelian [18] when the field $K$ is of characteristic zero (Theorem 5.1). This result establishes a new link between the theory of Hopf algebras and the more recent one of semi-abelian categories, both of which can be viewed as wide generalizations of group theory. Since a category $C$ is abelian if and only if both $C$ and its dual $C^{op}$ are semi-abelian, this observation can be seen as a "non-commutative" version of Takeuchi’s theorem mentioned above. The fact that the category $\text{Hopf}_{K,coc}$ is semi-abelian was independently obtained by Clemens Berger and Stephen Lack.

A recent article of Christine Vespa and Marc Wambst shows that the abelian core of $\text{Hopf}_{K,coc}$ is the category of commutative and cocommutative Hopf $K$-algebras [28].

In the present work we also prove the existence of a non-abelian torsion theory $(T, F)$ in $\text{Hopf}_{K,coc}$ (Theorem 4.3), where the torsion subcategory $T$ is the category of primitive Hopf $K$-algebras, which is isomorphic to the category of Lie $K$-algebras, and the torsion-free subcategory $F$ is the category of group Hopf $K$-algebras, which is isomorphic to the category of groups.

The categories of groups and of Lie $K$-algebras are two typical examples of semi-abelian categories: this shows again that the theories of cocommutative Hopf algebras and of semi-abelian categories are strongly intertwined. The category $\text{Hopf}_{K,coc}$ inherits some fundamental exactness properties from groups and Lie algebras thanks to the well-known canonical decomposition of a cocommutative Hopf algebra into a semi-direct product of a group Hopf algebra and a primitive Hopf algebra (a result associated with the names Cartier-Gabriel-Kostant-Milnor-Moore). The present work opens the way to some new applications of categorical Galois theory [17] in the category of cocommutative Hopf $K$-algebras, since the reflection from this category to the torsion-free subcategory of group Hopf algebras...
enjoys all the properties needed for this kind of investigations, as we briefly explain in Section 4. We conclude the article by observing that the semi-abelian category \( \text{Hopf}_{K,coc} \) is also an action representable category \([4]\) (Corollary 5.2).

2. Preliminaries

2.1. Semi-abelian categories.

Semi-abelian categories \([18]\) are finitely complete, pointed, exact in the sense of M. Barr \([2]\), protomodular in the sense of D. Bourn \([5]\), with finite coproducts. These categories have been introduced to capture some typical algebraic properties valid for non-abelian algebraic structures such as groups, Lie algebras, rings, crossed modules, varieties of \( \Omega \)-groups in the sense of P. Higgins \([16]\) and compact groups. As already mentioned in the introduction, every abelian category is in particular semi-abelian.

Although protomodularity is a property that can be expressed in any category with finite limits, in the pointed context, i.e. when there is a zero object \(0\) in \(C\), protomodularity amounts to the fact that the following formulation of the \textit{Split Short Five Lemma} holds: given a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & K & \to & A & \to & B & \to & 0 \\
& & \gamma & & \alpha & & \beta \\
0 & \to & K' & \to & A' & \to & B' & & \\
& & k' & & f' & & & \\
& & & & s' & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\]

where \(k = \ker(f), \ k' = \ker(f')\), \(f \circ s = 1_B\), and \(f' \circ s' = 1_{B'}\) (i.e. \(f\) and \(f'\) are split epimorphisms with sections \(s\) and \(s'\)), if both \(\kappa\) and \(\beta\) are isomorphisms, then so is \(\alpha\).

Any protomodular category \(C\) is a \textit{Mal’tsev} category \([8]\): this means that every (internal) reflexive relation \(C\) is an (internal) equivalence relation. Recall that a reflexive relation on an object \(X\) is a diagram of the form

\[
(2.1)
\]

\[
R \xrightarrow{p_1} X,
\]

where \(p_1\) and \(p_2\) are jointly monic, and \(p_1 \circ \delta = 1_X = p_2 \circ \delta\); such a reflexive relation \(R\) is an equivalence relation when, moreover, there exist \(\sigma: R \to R\) and \(\tau: R \times X R \to R\) as in the following diagram

\[
R \times X R \xrightarrow{\tau} R \xrightarrow{\sigma} R \xrightarrow{p_1} X
\]

such that \(p_1 \circ \sigma = p_2\) and \(p_2 \circ \sigma = p_1\) (symmetry), and \(p_1 \circ \tau = p_1 \circ \pi_1\) and \(p_2 \circ \tau = p_2 \circ \pi_2\) (transitivity), where \(\pi_1\) and \(\pi_2\) are the projections in the following pullback:

\[
\begin{array}{ccc}
R \times X R & \to & R \\
\downarrow & & \downarrow \\
R & \to & X
\end{array}
\]

In the present article, by a \textit{regular} category is meant a finitely complete category where every morphism can be factorized as a regular epimorphism followed by a monomorphism, and where regular epimorphisms are pullback stable. A regular category \(C\) is said to be \textit{Barr-exact} if, moreover, every equivalence relation
is effective, i.e. every equivalence relation is the kernel pair of a morphism in \( C \). A category which is pointed, protomodular and regular is said to be homological \([3]\). In this context several basic diagram lemmas of homological algebra hold true (such as the snake lemma, the 3-by-3-Lemma, etc.).

We end these preliminaries with the following diagram indicating some implications between the different contexts recalled above:

\[ \text{semi-abelian} \quad \Rightarrow \quad \text{homological} \quad \Rightarrow \quad \text{Barr-exact} \quad \Rightarrow \quad \text{protomodular} \quad \Rightarrow \quad \text{Mal’tsev} \]

\[ \text{regular} \]

2.2. The category \( \text{Hopf}_{K,coc} \) of cocommutative Hopf \( K \)-algebras.

The category we study in this article is the category of Hopf \( K \)-algebras over a field \( K \), denoted by \( \text{Hopf}_K \). The objects in \( \text{Hopf}_K \) are Hopf \( K \)-algebras, i.e. sextuples \((H, M, u, \Delta, \epsilon, S)\) where \((H, M, u)\) is a \( K \)-algebra and \((H, \Delta, \epsilon)\) is a \( K \)-coalgebra, such that these two structures are compatible, i.e. maps \( M \) and \( u \) are \( K \)-coalgebras morphisms, making \((H, M, u, \Delta, \epsilon)\) a \( K \)-bialgebra. The linear map \( S \) is called the antipode, and makes the following diagram commute:

\[
\begin{array}{ccc}
H & \overset{\Delta}{\longrightarrow} & H \otimes H \\
\downarrow & & \downarrow \\
K & \overset{\epsilon}{\longrightarrow} & H \\
\end{array}
\]

\[
\begin{array}{ccc}
H \otimes H & \overset{S \otimes \text{id}}{\longrightarrow} & H \otimes H \\
\downarrow & & \downarrow \\
\text{id} \otimes S & \overset{\sigma \otimes \text{id}}{\longrightarrow} & H \otimes H \\
\end{array}
\]

Morphisms in \( \text{Hopf}_K \) are exactly morphisms of \( K \)-bialgebras (i.e. morphisms that are both morphisms of \( K \)-algebras and \( K \)-coalgebras), as morphisms of \( K \)-bialgebras always preserve antipodes.

To denote the comultiplication map of a Hopf algebra \( H \), we will use the Sweedler notation: \( \forall h \in H, \Delta(h) = h_1 \otimes h_2 \) by omitting the summation sign. A Hopf algebra \( H \) is said to be cocommutative if its comultiplication map \( \Delta \) makes the following diagram commute, where \( \sigma \) is the linear map such that \( \sigma(x \otimes y) = y \otimes x, \forall x, y \in H \)

\[
\begin{array}{ccc}
H & \overset{\Delta}{\longrightarrow} & H \otimes H \\
\downarrow & & \downarrow \\
H & \overset{\sigma}{\longrightarrow} & H \otimes H \\
\end{array}
\]

The category of cocommutative Hopf \( K \)-algebras will be denoted by \( \text{Hopf}_{K,coc} \). In the category \( \text{Hopf}_{K,coc} \) there are two full subcategories which will be of importance for our work: the category \( \text{GrpHopf}_K \) of group Hopf algebras, and the category \( \text{PrimHopf}_K \) of primitive Hopf algebras, whose definitions we are now going to recall.

(1) The group Hopf algebra on a group \( G \), denoted by \( K[G] \), is the free vector space on \( G \) over the field \( K \), i.e. \( K[G] = \{ \sum_{g \in G} \alpha_g g \} \) where \( (\alpha_g)_{g \in G} \) is a family of scalars with only a finite number being non zero) and \( \{ g \mid g \in G \} \) is a basis of \( K[G] \). The group Hopf algebra \( K[G] \) can be equipped with a structure of cocommutative Hopf algebra, by taking the multiplication induced
Remark 2.1. As can be seen from the formula for comultiplication, both group Hopf algebras and primitive Hopf algebras are cocommutative. Therefore the categories $\text{Grp} \times K[-], \text{H}^{\text{Prim}} \times K[-], \text{GrpK}$ and $\text{PrimK}$ are also full subcategories of $\text{HopfK,coc}$. The functors $\text{U}, \text{P}, K[-], \text{G}$ and their adjunctions are represented in the following diagram:

\[
\begin{array}{c}
\text{Grp} \xrightarrow{K[-]} \text{HopfK,coc} \xrightarrow{P} \text{LieAlgK} \\
\text{G} \xrightarrow{} \text{U}
\end{array}
\]
3. The category of cocommutative Hopf algebras over a field of characteristic zero is homological

The category $\mathbf{Hopf}_{K,coc}$ is certainly pointed, with the zero object $K$, that will be denoted by $0$, from now on. $\mathbf{Hopf}_{K,coc}$ is complete and cocomplete, since it is locally presentable [23]. We will now establish its protomodularity and regularity.

3.1. Protomodularity of the category $\mathbf{Hopf}_K$. Let us consider the following commutative diagram of short exact sequences in the category $\mathbf{Hopf}_{K,coc}$:

\[
\begin{array}{ccccccccc}
0 & \to & A & \to & C_1 & \to & B & \to & 0 \\
\downarrow{id_A} & & \downarrow{\theta} & & \downarrow{id_B} & & \downarrow{} & & \\
0 & \to & A & \to & C_2 & \to & B & \to & 0 \\
\end{array}
\]

Thanks to the explicit descriptions of equalizers and coequalizers given in [1] one can easily prove that the kernel and the cokernel of $\theta$ are the zero object. This proves that $\theta$ is a monomorphism and an epimorphism of $\mathbf{Hopf}_K$-algebras. Since monomorphisms are injections and epimorphisms are surjections in the category $\mathbf{Hopf}_{K,coc}$ [10, 21], this shows that $\theta$ is an isomorphism of $\mathbf{Hopf}_K$-algebras and so the category $\mathbf{Hopf}_{K,coc}$ is protomodular.

The category $\mathbf{Hopf}_{K,coc}$ is actually even strongly protomodular [6], since it has finite limits and it can be viewed as the category of internal groups in the category of cocommutative $K$-coalgebras (see [25], for instance).

3.2. Semi-direct products of cocommutative Hopf algebras. Let $B$ be a cocommutative Hopf algebra. A $B$-module Hopf algebra is a Hopf algebra $A$ that is at the same time a left $B$-module with action $\rho : B \otimes A \to A, \rho(b \otimes a) = b \cdot a$ such that $\rho$ is a morphism of bialgebras. The semi-direct product (also known as smash product) of $B$ and $A$, denoted by $A \rtimes B$, is the Hopf algebra whose underlying vector space is the tensor product $A \otimes B$ and with the following structure: the unit is $u_{A \rtimes B} = u_A \otimes u_B$ and multiplication given by

\[
(a \otimes b)(a' \otimes b') = a(b_1 \cdot a') \otimes b_2 b',
\]

for all $a, a' \in A$ and $b, b' \in B$. The coalgebra structure is given by the tensor product coalgebra, i.e. $\Delta_{A \rtimes B} = (id_A \otimes \sigma \otimes id_B)(\Delta_A \otimes \Delta_B)$ and $\epsilon_{A \rtimes B} = \epsilon_A \otimes \epsilon_B$. The antipode is given by $S_{A \rtimes B}(a \otimes b) = S_B(b_1) \cdot S_A(a) \otimes S_B(b_2)$.

The following Lemma is a reformulation of Theorem 4.1 in [20]:

**Lemma 3.1.** Every split short exact sequence in $\mathbf{Hopf}_{K,coc}$

\[
\begin{array}{cccccc}
0 & \to & A & \xrightarrow{k} & H & \xrightarrow{p} & B & \to & 0 \\
\end{array}
\]

is canonically isomorphic to the semi-direct product exact sequence

\[
\begin{array}{cccccc}
0 & \to & A & \xrightarrow{i_1} & A \rtimes B & \xrightarrow{p_2} & B & \to & 0 \\
\end{array}
\]

where $i_1 = id_A \otimes u_B$, $i_2 = u_A \otimes id_B$ and $p_2 = \epsilon_A \otimes id_B$. 

Proof. The arrow \( h: A \times B \to H \) in the diagram below is given by \( h(a \otimes b) = k(a)s(b) \) for all \( a \otimes b \in A \times B \).

\[
\begin{array}{ccccccccc}
0 & \to & A & \xrightarrow{i_1} & A \times B & \xrightarrow{\pi_2} & B & \xrightarrow{p} & 0 \\
\downarrow & & \downarrow h & & \downarrow & & \downarrow & & \downarrow 0 \\
0 & \to & A & \xrightarrow{k} & H & \xrightarrow{p_H} & K[G_H] & \to & 0
\end{array}
\]

This is a morphism of split short exact sequences, and therefore \( h \) is an isomorphism by protomodularity of \( \text{Hopf}_{K,\text{coc}} \).

We use this lemma to reformulate the well-known structure theorem for cocommutative Hopf algebras over an algebraically closed field of characteristic zero (see for instance [26], page 279 in combination with Lemma 8.0.1(c)) in terms of split exact sequences.

**Theorem 3.2** (Cartier-Gabriel-Moore-Milnor-Kostant). Every cocommutative Hopf \( K \)-algebra \( H \), over an algebraically closed field \( K \) of characteristic 0, is isomorphic to the semi-direct product

\[
H \cong U(L_H) \rtimes K[G_H]
\]

of the universal enveloping algebra of a Lie algebra \( U(L_H) \) with the group Hopf algebra \( K[G_H] \), where \( L_H \) and \( G_H \) are given respectively by the space of primitive elements and the group of group-like elements of \( H \). Consequently, with notations of Lemma 3.1 for each \( H \) there exists a canonical split exact sequence of cocommutative Hopf algebras of the following form

\[
0 \to U(L_H) \xrightarrow{i_H} H \xrightarrow{p_H} K[G_H] \to 0
\]

where \( i_H = h \circ i_1 \), \( s_H = h \circ i_2 \) and \( p_H = p_2 \circ h^{-1} \).

**Remark 3.3.** Every morphism \( f : H_1 \to H_2 \) of cocommutative Hopf \( K \)-algebras gives rise to a morphism of split exact sequences of following form

\[
\begin{array}{ccccccccc}
0 & \to & U(L_{H_1}) & \xrightarrow{i_{H_1}} & H_1 & \xrightarrow{s_{H_1}} & K[G_{H_1}] & \to & 0 \\
\downarrow f_1 := U(P(f)) & & \downarrow f & & \downarrow & & \downarrow f_2 := K[G(f)] & & \downarrow 0 \\
0 & \to & U(L_{H_2}) & \xrightarrow{i_{H_2}} & H_2 & \xrightarrow{s_{H_2}} & K[G_{H_2}] & \to & 0
\end{array}
\]

### 3.3. Regularity of the category \( \text{Hopf}_{K,\text{coc}} \)

#### 3.3.1. The regular epimorphism/monomorphism factorization in \( \text{Hopf}_{K,\text{coc}} \)

Let \( f : A \to B \) be a morphism of cocommutative Hopf \( K \)-algebras. By the protomodularity of \( \text{Hopf}_{K,\text{coc}} \), it is well known that regular epimorphisms are the same as cokernels, i.e. normal epimorphisms. Thus, to construct the regular epimorphism/monomorphism factorization of the morphism \( f \), we consider the kernel \( i : H\ker(f) \to A \) of \( f \) and the cokernel \( p : A \to H\text{Coker}(i) \) of \( i \), both computed in the category of \( \text{Hopf}_{K,\text{coc}} \) :
The existence of this factorization $m$ such that $m \circ p = f$ follows from the universal property of the cokernel $p$ of $i$. It remains to prove that $m$ is a monomorphism, which is equivalent in $\text{Hopf}_{K,coc}$ to showing that $m$ is an injection. In the category $\text{Hopf}_{K,coc}$ the above factorization is obtained as in the category of vector spaces since $H\text{Coker}(i) = \frac{A}{A(H\text{ker}(f)) + A}$ (by [1]), and $\text{ker}(f) = A(H\text{ker}(f)) + A$ (by [24, 22]).

Note that any epimorphism of cocommutative Hopf $K$-algebras is then a normal epimorphism, and the following classes of epimorphisms coincide in $\text{Hopf}_{K,coc}$:

- normal epis = regular epis = epis = surjective morphisms.

3.3.2. Pullback stability of regular epimorphisms in the category $\text{Hopf}_{K,coc}$. To prove the pullback stability of regular epimorphisms in the category $\text{Hopf}_{K,coc}$, the approach we follow is to apply the pullback stability of regular epimorphisms in the two full subcategories $\text{GrpHopf}_{K}$ and $\text{PrimHopf}_{K}$ of $\text{Hopf}_{K,coc}$, which are both semi-abelian, and closed under pullbacks and quotients in $\text{Hopf}_{K,coc}$.

From the regularity of these two categories and the decomposition Theorem 3.2 we shall deduce the regularity of $\text{Hopf}_{K,coc}$.

**Lemma 3.4.** The two full subcategories $\text{GrpHopf}_{K}$ and $\text{PrimHopf}_{K}$ of $\text{Hopf}_{K,coc}$ are semi-abelian categories. Both these categories are closed under quotients and pullbacks in $\text{Hopf}_{K,coc}$.

**Proof.** As recalled in Preliminaries 2.2, the two full subcategories $\text{GrpHopf}_{K}$ and $\text{PrimHopf}_{K}$ are isomorphic to the category $\text{Grp}$ of groups and to the category $\text{LieAlg}_{K}$ of Lie $K$-algebras, respectively. The fact that $\text{Grp}$ and $\text{LieAlg}_{K}$ are semi-abelian is well-known (see [3], for instance).

The categories $\text{GrpHopf}_{K}$ and $\text{PrimHopf}_{K}$ are closed under quotients since morphisms of Hopf $K$-algebras preserve both group-like and primitive elements. These categories are closed under products in the category $\text{Hopf}_{K,coc}$ as explained in [19]. To see that $\text{PrimHopf}_{K}$ is closed under subobjects in $\text{Hopf}_{K,coc}$, let us consider a monomorphism $m : A \rightarrow U(L)$ in $\text{Hopf}_{K,coc}$ with codomain a primitive Hopf algebra. The morphism $m$, being a Hopf algebra morphism, preserves group-like elements and the group of group-like elements is trivial in the primitive Hopf algebra $U(L)$. Since $m$ is injective, we see that the group of group-like elements of $A$ is trivial as well. By applying Theorem 3.2 we conclude that $A$ has to be a primitive Hopf algebra as well. Similar arguments show that $\text{GrpHopf}_{K}$ is closed under subobjects in $\text{Hopf}_{K,coc}$. It follows that the subcategories $\text{GrpHopf}_{K}$ and $\text{PrimHopf}_{K}$ are closed under pullbacks in $\text{Hopf}_{K,coc}$. □

**Remark 3.5.** In the following we shall assume that $K$ is an algebraically closed field. It can be checked that this is not a restriction: indeed, given a field $K$ and $\phi : K \rightarrow \overline{K}$ an embedding of $K$ in an algebraic closure $\overline{K}$, one has the adjunction $\overline{\text{Hopf}}_{\overline{K},\text{coc}} \cong \overline{\text{Hopf}}_{K,\text{coc}}$ where $R_{\phi}$ is the “restriction of scalars functor” and $L_{\phi} = - \otimes_{K} \overline{K}$ its left adjoint, the “extension of scalars” functor. Being a left adjoint, $L_{\phi}$ preserves regular epimorphisms and moreover $L_{\phi}$ reflects regular epimorphisms and preserves finite limits. Accordingly, knowing that $\text{Hopf}_{\overline{K},\text{coc}}$ is regular (respectively, exact), one can deduce from this that $\text{Hopf}_{K,\text{coc}}$ is regular (resp. exact) as well.

The following result concerning split short exact sequences in $\text{Hopf}_{K,\text{coc}}$ will be useful in the proof of the regularity of this category:
Lemma 3.6. Given the following commutative diagram of split short exact sequences in \( \text{Hopf}_{K,coc} \):

\[
\begin{array}{ccccccccc}
0 & \to & A_1 & \to & H_1 & \to & B_1 & \to & 0 \\
\downarrow{h_A} & & \downarrow{h} & & \downarrow{s_{H_1}} & & \downarrow{h_B} & & \downarrow{0} \\
0 & \to & A_2 & \to & H_2 & \to & B_2 & \to & 0
\end{array}
\]

We have that \( h \) is surjective if and only if both \( h_A \) and \( h_B \) are surjective.

Proof. We apply Lemma 3.1 to the exact sequences in the statement of the Lemma, we obtain the following commutative diagram which is canonically isomorphic to the previous one:

\[
\begin{array}{ccccccccc}
0 & \to & A_1 & \to & A_1 \rtimes B_1 & \to & B_1 & \to & 0 \\
\downarrow{h_A} & & \downarrow{h_A \otimes h_B} & & \downarrow{p_1} & & \downarrow{h_B} & & \downarrow{0} \\
0 & \to & A_2 & \to & A_2 \rtimes B_2 & \to & B_2 & \to & 0
\end{array}
\]

Hence, the morphism \( h \) is surjective if and only if \( h_A \otimes h_B : A_1 \rtimes B_1 \to A_2 \rtimes B_2 \) is surjective.

If \( h_A \) and \( h_B \) are surjective, then \( h_A \otimes h_B \) is surjective by considering this morphism on its underlying vector space. For the converse implication, if \( h_A \otimes h_B \) is surjective, let us note that for each semi-direct product \( A_i \rtimes B_i \), the underlying coalgebra is exactly the categorical product of the coalgebras \( A_i \) and \( B_i \); we denote \( \xi_i = id_{A_i} \otimes \epsilon_{B_i} \) for the coalgebra-projection of \( A_i \rtimes B_i \) onto \( A_i \) (which is not a Hopf algebra morphism).

It is clear that \( h_A \circ \xi_1 = \xi_2 \circ (h_A \otimes h_B) \), as coalgebra morphisms. Since \( \xi_2 \) is a split epimorphism and \( h_A \otimes h_B \) is surjective, we conclude that \( h_A \) is surjective. It is clear that \( h_B \) is surjective whenever \( h \) is.

\[\square\]

Theorem 3.7. Consider the following pullback \((P, \pi_A, \pi_B)\) in the category \( \text{Hopf}_{K,coc} \):

\[
\begin{array}{ccc}
P & \xrightarrow{\pi_B} & B \\
\pi_A \downarrow & & \downarrow g \\
A & \xrightarrow{f} & C
\end{array}
\]

if \( f \) is a regular epimorphism then \( \pi_B \) is a regular epimorphism.

Proof. Thanks to Lemma 3.4, regular epimorphisms are pullback stable whenever the Hopf algebras \( A \), \( B \) and \( C \) in diagram (3.2) belong to \( \text{GrpHopf}_K \), or to \( \text{PrimHopf}_K \), respectively.

Let us now consider \( A \), \( B \) and \( C \) cocommutative Hopf \( K \)-algebras over a field \( K \) of characteristic zero. By Theorem 3.2 we have: \( A \cong U(L_A) \rtimes K[G_A] \), \( B \cong U(L_B) \rtimes K[G_B] \) and \( C \cong U(L_C) \rtimes K[G_C] \).

Thanks to Remark 3.3 we can build the following commutative diagram.
where \((P_1, \pi_{A_1}, \pi_{B_1})\) is the pullback of \(f_1\) and \(g_1\), \((P_2, \pi_{A_2}, \pi_{B_2})\) is the pullback of \(f_2\) and \(g_2\).

When \(f\) is surjective, the surjectivity of \(f_1\) and \(f_2\) follows both from Lemma 3.6 applied to the lower part of the diagram. The front and back squares of the diagram are in \(\text{GrpHopf}_K\) and in \(\text{PrimHopf}_K\), respectively, thus both \(\pi_{B_1}\) and \(\pi_{B_2}\) are surjective (by Lemma 3.4). Applying Lemma 3.6 again (in converse direction), we obtain that \(\pi_B\) is also surjective.

\[
\square
\]

4. A torsion theory in the category \(\text{Hopf}_{K,coc}\)

In the non-abelian context of homological categories it is natural to define and study a general notion of torsion theory, that extends the one introduced by S.E. Dickson in the frame of abelian categories [11]. This study was first initiated in [7], and further developed in [12], [13], also in relationship with semi-abelian homology theory.

Let us recall the definition of a torsion theory in the homological context:

**Definition 4.1.** In a homological category \(C\), a torsion theory is given by a pair \((T, F)\) of full and replete (i.e., isomorphism closed) subcategories of \(C\) such that:

i. For any object \(X\) in \(C\), there exists a short exact sequence:

\[
0 \longrightarrow T \xrightarrow{i_X} X \xrightarrow{\eta_X} F \longrightarrow 0
\]

where \(0\) is the zero object in \(C\), \(T \in T\) and \(F \in F\).

ii. The only morphism \(f : T \longrightarrow F\) from \(T \in T\) to \(F \in F\) is the zero morphism.

When \((T, F)\) is a torsion theory, \(T\) is called the torsion subcategory of \(C\), and \(F\) its torsion-free subcategory. Among the many examples in the homological context, let us just mention the following ones:

**Example 4.2.**

1. Every torsion theory in an abelian category \(C\). For instance: the pair \((\text{Ab}_t, \text{Ab}_{tf})\) in the category \(\text{Ab}\) of abelian groups, where \(\text{Ab}_t\) and \(\text{Ab}_{tf}\) denote the full and replete subcategories of the category of abelian groups whose objects are torsion and torsion-free abelian groups, respectively.

2. The pair \((\text{NilCRng}, \text{RedCRng})\) in the category \(\text{CRng}\) of commutative rings, where \(\text{NilCRng}\) and \(\text{RedCRng}\) denote the full subcategories of nilpotent commutative rings, and of reduced commutative rings (i.e., commutative rings with no nontrivial nilpotent elements), respectively.
(3) The pair \((\text{Grp}(\text{Ind}), \text{Grp}(\text{Haus}))\) in the category \(\text{Grp}(\text{Top})\) of topological groups, where \(\text{Grp}(\text{Ind})\) and \(\text{Grp}(\text{Haus})\) denote the full subcategories of indiscrete groups and of Hausdorff groups, respectively.

From now on, in the homological category of cocommutative Hopf \(K\)-algebras, \(T\) will always denote the (full) subcategory \(\text{PrimHopf}_K\) of primitive Hopf algebras, and \(F\) the (full) subcategory \(\text{GrpHopf}_K\) of group Hopf algebras.

**Theorem 4.3.** The pair \((\text{PrimHopf}_K, \text{GrpHopf}_K)\) is a hereditary torsion theory in \(\text{Hopf}^\text{K,coc}\).

**Proof.** Thanks to Theorem 3.2, we know that we can associate the following short exact sequence with any cocommutative Hopf \(K\)-algebra \(H\):

\[
0 \rightarrow U(L_H) \xrightarrow{\iota_H} H \xrightarrow{PH} K[G_H] \rightarrow 0
\]

Any morphism \(f : U(L) \rightarrow K[G]\) from a primitive Hopf algebra \(U(L) \in \text{PrimHopf}_K\) to a group Hopf algebra \(K[G] \in \text{GrpHopf}_K\) is the zero morphism in \(\text{Hopf}^\text{K,coc}\). Indeed, any primitive Hopf algebra \(U(L)\) is generated by its primitive elements, which are preserved by any morphism of Hopf algebras. Since a group Hopf algebra \(K[G]\) does not contain any non-zero primitive element, it follows that \(f\) is the zero morphism. It follows that \((\text{PrimHopf}_K, \text{GrpHopf}_K)\) is a torsion theory, which is actually hereditary thanks to Lemma 3.4.

\[\square\]

As it follows from the results in [7] the reflector \(I\) in the adjunction

\[
\begin{array}{ccc}
F & \xleftarrow{\l} & \text{Hopf}^\text{K,coc} \\
\downarrow & & \downarrow \iota \\
H & & \text{Hopf}^\text{K,coc}
\end{array}
\]

is semi-left-exact in the sense of Cassidy-Hebert-Kelly [9], i.e. it preserves all pullbacks of the form

\[
\begin{array}{ccc}
P & \xrightarrow{p_2} & Y \\
\downarrow \quad \quad & & \downarrow \eta_Y \\
H(X) & \xrightarrow{H(f)} & H(Y)
\end{array}
\]

where \(\eta_Y\) is the \(Y\)-component of the unit of the adjunction and \(f\) lies in the subcategory \(F\). The adjunction is then *admissible* in the sense of categorical Galois theory [17]: this opens the way to further investigations in the direction of semi-abelian homology [12]. The fact that the torsion theory is hereditary and \(\text{Hopf}^\text{K,coc}\) a homological category implies that the corresponding Galois coverings are precisely those regular epimorphisms \(f : A \rightarrow B\) in \(\text{Hopf}^\text{K,coc}\) with the property that the kernel \(\text{Hker}(f)\) is in \(F\) (by applying Theorem 4.5 in [14]). This fact is crucial to describe generalized Hopf formulae for homology, as explained in [13].

5. The category of cocommutative Hopf algebras over a field of characteristic zero is semi-abelian

In order to prove that \(\text{Hopf}^\text{K,coc}\) is semi-abelian, it remains to show that equivalence relations are effective. For this, we shall show that any equivalence relation \(R\) as in diagram (2.1) in \(\text{Hopf}^\text{K,coc}\) is the kernel pair of its coequalizer \(q : X \rightarrow \overline{X}\). We first apply Theorem 3.2 to the equivalence relation \(R\),
obtaining the following commutative diagram, where the morphisms \( q_1, q_2 \) and \( q \) are the coequalizers of \( p_{11} \) and \( p_{21}, p_{12} \) and \( p_{22}, p_1 \) and \( p_2 \), respectively, and \( (Eq(q), \pi_1, \pi_2) \) is the kernel pair of \( q \).

Thanks to Lemma 5.1, the left column is exact, i.e. \( U(L_R) = Eq(q_1) \). On the other hand, let us explain why the right column is also exact. First observe that \( (K[G_R], p_{12}, p_{22}) \) is a reflexive relation on \( K[G_X] \), while the category \( \text{GrpHopf}_K \) of group Hopf algebras is exact Mal'tsev and closed under pullbacks and quotients in \( \text{Hopf}_{K,coc} \). It follows that \( K[G_R] \) is the kernel pair of \( q_2 \) in \( \text{Hopf}_{K,coc} \).

The universal property of \( (Eq(q), \pi_1, \pi_2) \) gives the unique arrow \( \theta: R \to Eq(q) \) with \( \pi_1 \circ \theta = p_1 \) and \( \pi_2 \circ \theta = p_2 \). One can check that the arrow \( i_X \) is a monomorphism, by using the fact that the coequalizers of equivalence relations in \( \text{Hopf}_{K,coc} \) are computed as in \( \text{Coalg}_{K,coc} \). Consequently, the lower row in the diagram above is an exact sequence as is then the lower row in the following diagram.

By applying the Split Short Five Lemma to the above commutative diagram, it follows that the morphism \( \theta \) is an isomorphism, and the equivalence relation \( R \) is effective. One accordingly has the following:

**Theorem 5.1.** For any field \( K \) of characteristic 0, the category \( \text{Hopf}_{K,coc} \) of cocommutative Hopf \( K \)-algebras is semi-abelian.

This result has the following interesting

**Corollary 5.2.** For any field \( K \) of characteristic 0, the category \( \text{Hopf}_{K,coc} \) is action representable (i.e. it has representable object actions in the sense of [4]).

**Proof.** This follows from Theorem 4.4 in [4], the exactness of \( \text{Hopf}_{K,coc} \) (Theorem 5.1), and the fact that \( \text{Hopf}_{K,coc} \) can be viewed as the category of internal groups in the cartesian closed category of cocommutative \( K \)-coalgebras (see 2.3 in [15]).

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