Abstract. We theoretically study the collective excitations of an ideal gas confined in an isotropic harmonic trap. We give an exact solution to the Boltzmann-Vlasov equation; as expected for a single-component system, the associated mode frequencies are integer multiples of the trapping frequency. We show that the expressions found by the scaling ansatz method are a special case of our solution. Our findings, however, are most useful in case the trap contains more than one phase: we demonstrate how to obtain the oscillation frequencies in case an interface is present between the ideal gas and a different phase.

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1 Introduction

Since the experimental realization of quantum degenerate gases, the role of particle interactions has been thoroughly studied. While weak interactions characterize most ultracold systems due to their diluteness, the use of Feshbach resonances allows the study of very strongly interacting systems. Although interactions are generally important, many experimentally realized gases may be well-modelled by ideal gases. For example, collisions are suppressed in fully-polarized Fermi gases at temperatures well below their Fermi temperature because s-wave interactions are forbidden due to Pauli blocking. Such gases appear for instance in the experiments on imbalanced fermion gases \[1,2,3] and in phase-segregated Bose-Fermi mixtures \[4,5,6]. A description in terms of ideal gases is also appropriate in ultracold two-component Fermi systems when the interspecies scattering length is tuned to zero by means of a Feshbach resonance \[7]. Moreover, it is well-known that sufficiently dilute gases become collisionless: as a result, collisionless behavior also sets in as the temperature of trapped gases well above the temperature of quantum degeneracy \[8].

For a hydrodynamic gas (or superfluid) under experimentally relevant conditions, the variables relevant for studying the collective excitations are, for example, the local density and velocity; on the other hand, for a collisionless fluid, one must, in general, find the entire distribution function in momentum- and coordinate-space. In some cases, one may extract dynamical solutions using a scaled form of the equilibrium distribution function \[9,10,11,12,13]. However, the scaling approach fails for a two-component system with an interface \[3,4,5]. Other methods to obtain the collective mode frequencies include the method of averaging, the sum-rule approach, and the random-phase approximation \[14,15,16,17,18,19,20].

We present here a solution for the Boltzmann-Vlasov equation of an ideal gas in an isotropic harmonic trap. For the single-component system, we recover the spectrum \(\nu_0\) with \(n\) integer and \(\omega_0\) the trapping frequency. We then show how our solution may be used to study a two-component system with an interface by writing down the appropriate boundary conditions at the interface. For the monopole modes, we give analytical expressions which may be straightforwardly used to calculate their frequencies in isotropic and highly-elongated elliptical traps. Finally, we discuss damping of these modes.

2 Boltzmann-Vlasov Equation

An ideal gas, confined by an isotropic potential \(m\omega_0^2 r^2/2\), is described by a distribution function \(f(\mathbf{r}, \mathbf{p}, t)\), evolving according to the Boltzmann-Vlasov equation:

\[
m\partial_t f + \mathbf{p} \cdot \nabla f - m^2 \omega_0^2 \mathbf{r} \cdot \nabla_p f = 0.
\]

In the following \(f_0\) denotes the equilibrium solution, for which \(\partial_t f_0 = 0\). Collective oscillations may be studied by looking at periodic deviations from \(f_0\). Writing \(f = f_0 + \nu e^{-i\omega t}\) we obtain the result

\[
\nu = C \langle \{L_j, E_j\} \rangle \sum_{j=x,y,z} a_j (p_j + im\omega_0 r_j)^2 \omega/2\omega_0,
\]

with the \(a_j\) arbitrary dimensionless constants and \(C\) an arbitrary function of the angular momentum components.
$L_j = (r \times p)_j$ and of the $E_j = (p_j^2/m + m\omega_0^2 r_j^2)/2$ (with $j = x, y, z$). Note that solution (2) is not obtained by linearization so $\nu$ may be large compared to $f_0$.

### 3 Single-component System

Consider a trap containing a single particle component. Due to the absence of boundary conditions, the function $C$ in Eq. (2) is arbitrary; physically, $\nu/C$ must then be continuous everywhere. Hence, $\omega$ has to be chosen such that no discontinuities appear when the coordinates $r_j$ cross zero. This is guaranteed if $\omega = n\omega_0$ (from now on, $n$ is an integer), which is the well-known spectrum of a single-component trapped collisionless gas [3]. We show now that indeed all experimentally relevant collective modes can be obtained from Eq. (3).

The lowest dipole or Kohn mode can be found by taking, for example, $a = (0, 0, 1)$: it is then clear that $\omega = \omega_0$ ensures that $\nu/C$ converges to the same value if $r_z \to 0^+$ or $r_z \to 0^-$ when $p_z < 0$. It is also easy to see that, when $C$ is rotationally invariant in coordinate space, the density deviation $\delta \rho \propto \int d^3p \nu$ is simply proportional to $r_z$, which, as expected for the dipole mode, is proportional to the spherical harmonic $Y_{0}^{0}$.

Due to rotational symmetry in space the monopole modes can be described by taking $a = (1, 1, 1)$ such that

$$\nu = C \left( L^2, E \right) \left[ p^2 - m^2 \omega_0^2 r^2 + 2i m \omega_0 r \cdot p \right]^{\omega/2\omega_0},$$

with $E = \sum_j E_j$. Arguments similar to those for the dipole mode lead to the conclusion that $\omega = 2n\omega_0$, which is the well-known spectrum for monopole modes. Moreover, for the lowest excitation ($n = 1$), the density deviation $\delta \rho$ is proportional to the spherical harmonic $Y_{0}^{0}$.

Finally, for the quadrupole mode, two of the $a_j$ must be equal while the third opposite in sign. Choosing $a = (1, 1, -2)$, one finds that $\omega = 2\omega_0$ and that $\delta \rho \propto x^2 + y^2 - 2z^2 \propto Y_{2}^{0}$, as must $\nu$.

A highly-elongated elliptical trap may be modelled by an isotropic 2D confinement $U(r) = m\omega_0^2 (r_x^2 + r_y^2)/2$. This results in a cylindrical configuration with the $x$-axis along the axial direction and rotational symmetry in the $y - z$ plane. The angular dependencies of $\delta \rho$ for the lowest monopole, dipole and quadrupole modes are known to vary as $e^{i\theta} \propto (y + iz)^{\ell}$ with $\theta$ the angle in the $y - z$ plane and $\ell = 0, 1, 2$. The Kohn mode ($\ell = 1$) can therefore be recovered by an appropriate superposition of Eq. (2) with $a = (0, 1, 0)$ and Eq. (2) with $a = (0, 0, 1)$, together with a rotational-invariant function $C$. In an analogous manner, and using a $C$-function which depends on $L_x$, one may obtain the quadrupole mode ($\ell = 2$).

We now explain the relation between Eq. (2) and the scaling ansatz method, widely used in the literature [9][10][11][12].

### 4 Relation to the Scaling Ansatz

Consider a confined single-component gas with equilibrium distribution $f_0$. Due to the radial symmetry, $f_0$ can be written as a function of $L^2$ and $E$ only; we assume that $f_0$ is a function of $E$ only. According to the scaling ansatz, the distribution function $f$ of a trapped single-component system at a certain time $t$ may be written in terms of the equilibrium distribution $f_0$ as follows:

$$f(r, p, t) = f_0(r(t), p(t)).$$

In order to find the monopole and the quadrupole modes one continues by taking the scaling functions $r_j(t) = r_j/\alpha_j$ and $p_j(t) = \alpha_j p_j - m\dot{a}_j r_j$ where the dot denotes the derivative with respect to time and $\alpha_j$ is a periodic function of time [9][10][11][12]. For small deviations from equilibrium, one writes $\alpha_j = 1 + \varepsilon a_j e^{-i\omega t}$ with $\varepsilon \ll 1$ and $a_j$ dimensionless constants. Linearization of $f$ in terms of $\varepsilon$ yields:

$$f = f_0 + \varepsilon e^{-i\omega t}m^{-1}(\partial_E f_0)$$

$$\times \left( \sum_{j=x,y,z} [a_j(p_j + im\omega_0 r_j)^2 + ima_j(\omega - 2\omega_0)r_j p_j] \right).$$

Clearly $f$ reduces to Eq. (2) when $\omega = 2\omega_0$, which, as we have seen, is indeed the frequency associated with the lowest monopole and quadrupole modes of an ideal gas.

The Kohn mode, on the other hand, may be recovered by substitution of $r_z(t) = r_z - \beta$ and $p_z(t) = p_z - \beta$ while for $j = x, y$, one takes $r_j(t) = r_j$ and $p_j(t) = p_j$. Assuming a time-dependent function $\beta(t) = \varepsilon e^{-i\omega t}$ and linearizing in terms of $\varepsilon$ yields:

$$f = f_0 + i\varepsilon e^{-i\omega t}(\partial_E f_0) [\omega p_z + ima_0^2 r_z^2].$$

Again this reduces to our solution (2) in case $a = (0, 0, 1)$ and $\omega = \omega_0$.

We therefore conclude that, in a linearized form, the expressions for the distribution functions used by the scaling ansatz method reduce to our solution (2) in case $\omega = \omega_0$ and $\omega = 2\omega_0$. Note that, while our solution (2) solves the Boltzmann-Vlasov equation, Eqs. (4) and (5) do not, except for specific values of $\omega$. Therefore, the reason why, for a single-component system, the correct spectrum arises from Eq. (4), and from Eqs. (4) and (5), is different. Note further that, although the scaling ansatz does not give the most general solution for the Boltzmann-Vlasov equation, it may be generalized to find oscillation frequencies using a Boltzmann equation which includes a mean-field and collision term [9][10][11][12].

### 5 Two-Component System with an Interface

Assume in the following the presence of two phases, separated in the trap by an interface at radial position $r$. We investigate the boundary conditions at the interface for the ideal-gas phase and leave the nature of the other phase.
unspecified. Experimentally relevant cases include the imbalanced fermion gases \([12, 3, 21]\) and phase-segregated Bose-Fermi mixtures \([4, 5, 6, 22, 23]\).

Under the assumption that particles are specularly reflected, conservation of energy and particle number relate the distribution function of incoming and outgoing particles at the interface \([24]\):

\[
\nu(\zeta, \chi) - \nu(\zeta, -\chi) = 2m\chi \dot{\zeta}(\partial_\nu f_0),
\]

with \(\chi\) given by \(\mathbf{r} \cdot \mathbf{p} = \chi r p\) where \(r = |\mathbf{r}|\) and \(p = |\mathbf{p}|\). Consider the point \(\mathbf{r} = (0, 0, \zeta)\) on the interface. The sole result of a particle reflecting there is its change of velocity component \(p_z\) to \(-p_z\). Therefore, aside from \(L_z, E_x\) and \(E_y\), also \(L_z, L_y\) and \(E_x\) are invariant under reflection, such that \(\mathcal{C}(\chi) = \mathcal{C}(-\chi)\). The solution of Eq. (2) satisfying the boundary condition \([6]\) at position \(\mathbf{r} = (0, 0, \zeta)\) is then:

\[
\nu = 2m\chi \dot{\zeta}(\partial_\nu f_0) [1 - e^{-i\omega t}],
\]

where,

\[
\tau = \omega^{-1}\text{Arg}\left[a_x p_x^2 + a_y p_y^2 + a_z (\chi p + im\omega_0 \zeta)^2\right].
\]

A general solution for \(\nu\) at \(\mathbf{r} = (0, 0, \zeta)\) can be obtained by taking superpositions of Eq. (7) with each different values of \(a\). In the following, we show how one can study breathing modes for gases confined in isotropic and highly-elongated harmonic traps. Multipole modes may then be studied in an analogous manner.

**Spherical trap** — The breathing modes can be found by imposing the condition of rotational symmetry in Eq. (8), such that:

\[
\tau = \omega^{-1}\text{Arg}\left[p^2 - m^2 \omega_0^2 \zeta^2 + 2im\omega_0 \chi p\right].
\]

Physically, \(\tau\) is the time for a particle departing radially from the interface, to return to it. For a fully-degenerate Fermi gas, the values for \(\zeta\) and \(\omega\) for which damping occurs are depicted in Fig.2 of Ref. [6]. Note that, due to the rotational invariance Eq. (7) is valid at any point on the interface.

**Highly Elongated Traps** — In case of highly-elongated harmonic traps, one can approximate the trapping potential by \(U(\mathbf{r}) = m\omega_0 (r_y^2 + r_z^2)/2\). The breathing modes can be studied assuming \(a = (1, 1, 0)\) and considering \(\nu\) in the spatial point \((0, 0, \zeta)\) where Eq. (7) is valid with:

\[
\tau = \omega^{-1}\text{Arg}\left[p^2 (\chi^2 + (1 - \chi^2) \cos^2 \phi) -m^2 \omega_0^2 \zeta^2 + 2im\omega_0 \chi p\right].
\]

Here \(\phi\) is the azimuth of \(\mathbf{p}\) in the \(x\)-\(y\)-\(z\) coordinate system. Apart from the boundary condition \([6]\), which by itself ensures that the mean velocity of the collisionless gas at the interface is equal to \(\dot{\zeta}\), another boundary condition applies, ensuring local mechanical equilibrium at the interface. This condition is the Laplace equation and it relates the radial pressure tensors \(\Pi_{rr}\) of the inner phase A and the outer phase B at the interface:

\[
\Pi^A_{rr} - \Pi^B_{rr} = \gamma_{AB}(1/R_1 + 1/R_2).
\]

Here \(R_1\) and \(R_2\) are the principal radii of curvature of the interface and \(\gamma_{AB}\) is the interface tension of phase A and B.

Knowledge of the distribution function \(\nu\) from Eq. (8) allows a direct calculation of the pressure tensor \(\Pi_{rr}\) in Eq. (11) for the ideal-gas phase. Assuming one knows the pressure tensor for the other phase, all the necessary ingredients are present to find the collective mode frequencies, as shown in Refs. [3, 21].

### 6 Work Done by a Moving Interface

We now show that the total energy transfer through the interface over one oscillation period vanishes unless a pole is present in the pressure tensor, in which case the collective mode gets damped. We determine also a criterion for this damping to occur.

The work done by a moving interface onto a collisionless gas is most simply expressed using the pressure tensor normal to that interface, \(\Pi_{rr}\), and the average velocity of the gas in the same direction, \(u_r\) (here \(r\) denotes the radial coordinate). At \(\mathbf{r} = (0, 0, \zeta)\), the functions \(\delta \Pi_{rr}\) and \(u_r\) vary as:

\[
\delta \Pi_{rr} \propto e^{-i\omega t} \int_0^\infty dp \int_0^\pi d\chi \chi^3 p^4 (\partial_\nu f_0)\cot\left(\frac{\omega t}{2}\right)
\]

and

\[
u = \int_0^\infty dp \int_0^\pi d\chi \chi^3 p^4 (\partial_\nu f_0),
\]

where we have used that \(\tau(\chi) = -\tau(-\chi)\). The work done per unit area during one oscillation period \(T = 2\pi/\omega\) is given by

\[
\Delta W = \int_0^T dt \text{Re} \left(\delta \Pi_{rr} \cdot u_r\right).
\]

If there appears no pole in the integrand of \(\delta \Pi_{rr}\), then \(\delta \Pi_{rr}\) and \(u_r\) are \(\pi\) out of phase and the total work done during a single cycle averages to zero. If, on the other hand, the integrand of \(\delta \Pi_{rr}\) has a pole for some \(\chi\), then one must pass below that pole when integrating over \(\chi\) \([23]\); as a result, \(\delta \Pi_{rr}\) acquires an imaginary part and is no longer \(\pi\) out of phase with \(u_r\). This results in net work being done on the collisionless gas during one period. In other words, it corresponds to damping of the collective modes, with their energy transferred to single-particle modes in the collisionless gas (in analogy to Landau damping).

One may ask now when a pole appears in the integrand of \(\delta \Pi_{rr}\). From Eq. (7), \(\nu(\zeta, \chi)\) diverges if:

\[
\tau(\zeta, \chi) = 2\pi n/\omega,
\]

with \(n\) again an integer. This gives rise to the following criterion: damping at the interface can only happen when the collective mode frequency exceeds \(2\omega_0\). Indeed, since \(|\tau| \leq \pi/\omega_0\), Eq. (12) can only be satisfied when \(\omega \geq 2\omega_0\).

Finally, note that it can be shown that, if no pole appears in the integrand of \(\delta \Pi_{rr}\) at the interface, then there is no pole anywhere in the trap.
7 Conclusions

We have presented a solution (see Eq. (2)) for the Boltzmann-Vlasov equation, describing the collective excitations of an ideal gas in an isotropic harmonic trap. If the trap is entirely filled by the ideal gas, the associated mode frequencies are simply integer multiples of the trapping frequency. In that case, we proved that the expressions for the distribution function used by the scaling ansatz method, are a special case of our solution. We further applied our solution to a trap consisting of an ideal gas in contact with a different phase by means of an interface. Taking the particles of the ideal-gas phase to specularly reflect on the interface, we show how the breathing mode frequencies may be obtained in isotropic and highly-elongated elliptical traps. Finally, we discuss how, in the presence of an interface, damping may arise.

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