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CERTAIN PROPERTIES OF THE MODIFIED DEGENERATE GAMMA FUNCTION

KWARA NANTOMAH

ABSTRACT. In this paper, we prove some inequalities satisfied by the modified degenerate gamma function which was recently introduced. The tools employed include the Holder’s inequality, the mean value theorem, the Hermite-Hadamard’s inequality and the Young’s inequality. By some parameter variations, the established results reduce to the corresponding results for the classical gamma function.

1. Introduction

In recent times, degenerate special functions and polynomials have been a subject of intense discussion. See for example [4], [5], [7], [8], [10] and the related references therein.

In 2017, Kim and Kim [6] introduced the degenerate gamma function as

$$\Gamma_{\lambda}(x) = \int_0^\infty t^{x-1} (1 + \lambda t)^{-\frac{1}{\lambda}} dt,$$  

(1)

where $\lambda \in (0, \infty)$ and $0 < \Re(x) < \frac{1}{\lambda}$. This was motivated by the degenerate exponential function which is defined as [6]

$$e^{\lambda}_t = (1 + \lambda t)^{\frac{1}{\lambda}},$$  

(2)

where $\lambda \in (0, \infty)$. It is clear that $\lim_{\lambda \to 0} e^{\lambda}_t = e^t$ and $\lim_{\lambda \to 0} \Gamma_{\lambda}(x) = \Gamma(x)$, where $\Gamma(x)$ is the classical gamma function.

In 2018, Kim and Kim et al. [9] introduced the modified degenerate gamma function which is defined as

$$\Gamma_{\lambda}(x) = \int_0^\infty t^{x-1} (1 + \lambda)^{-\frac{1}{\lambda}} dt,$$  

(3)

where $\lambda \in (0, 1)$ and $\Re(x) > 0$. This definition is equivalent to

$$\Gamma_{k}(x) = \int_0^\infty t^{x-1} (1 + 1/k)^{-kt} dt,$$  

(4)

where $1 < k < \infty$ and $\Re(x) > 0$. Here, $\lim_{k \to \infty} \Gamma_{k}(x) = \Gamma(x)$. The modified degenerate gamma function (3) satisfies the following properties [9].

$$\Gamma_{\lambda}(1) = \frac{\lambda}{\ln(1 + \lambda)};$$  

(5)

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\[ \Gamma_\lambda(x + 1) = \frac{\lambda x}{\ln(1 + \lambda)} \Gamma_\lambda(x), \quad (6) \]
\[ \Gamma_\lambda(m + 1) = \frac{\lambda^{m+1}m!}{(\ln(1 + \lambda))^{m+1}}, \quad m \in \mathbb{N}. \quad (7) \]

Derivatives of the modified degenerate gamma function are given as
\[ \Gamma_\lambda^{(r)}(x) = \int_0^\infty (\ln t)^r t^{x-1} (1 + \lambda)^{-\frac{x}{1-t}} dt, \quad (8) \]
where \( r \in \mathbb{N}_0. \)

In a recent work, He et al. [2] introduced the modified degenerate digamma function which is defined as
\[ \psi_\lambda(x) = \frac{d}{dx} \ln \Gamma_\lambda(x) = \frac{\Gamma'_\lambda(x)}{\Gamma_\lambda(x)}, \quad (9) \]
and has the following representations among others.
\[ \psi_\lambda(x) = -\gamma + \ln \left( \frac{\lambda}{\ln(1 + \lambda)} \right) + \sum_{k=0}^{\infty} \frac{x - 1}{(k + 1)(k + x)}, \quad (10) \]
\[ = -\gamma + \ln \left( \frac{\lambda}{\ln(1 + \lambda)} \right) - \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x}{k(k + x)}, \quad (11) \]
\[ = -\gamma + \ln \left( \frac{\lambda}{\ln(1 + \lambda)} \right) + \int_0^1 \frac{1 - t^{x-1}}{1 - t} dt, \quad (12) \]
where \( \gamma \) is the Euler-Mascheroni constant. It also satisfies the following basic properties
\[ \psi_\lambda(1) = -\gamma + \ln \left( \frac{\lambda}{\ln(1 + \lambda)} \right), \quad (13) \]
\[ \psi_\lambda(x + 1) = \psi_\lambda(x) + \frac{1}{x}, \quad (14) \]
and similarly, it is clear that \( \lim_{\lambda \to 0} \psi_\lambda(x) = \psi(x) \) where \( \psi(x) \) is the classical digamma function. For further properties of the function \( \psi_\lambda(x) \), one may refer to [2].

In this paper, we continue to investigate the modified degenerate gamma function. Precisely, we prove some inequalities satisfied by this generalized function. The techniques we employed are analytical in nature.

2. Results and Discussion

**Theorem 2.1.** For \( s \in (0, 1] \) and \( x > 0 \), the inequality
\[ \left( \frac{\lambda x}{\ln(1 + \lambda)} \right)^{1-s} \leq \frac{\Gamma_\lambda(x + 1)}{\Gamma_\lambda(x + s)} \leq \left( \frac{\lambda(x + s)}{\ln(1 + \lambda)} \right)^{1-s}, \quad (15) \]
holds.
Proof. The case for \( s = 1 \) is obvious. So, let \( s \in (0,1) \) and \( x > 0 \). Then by applying the Holder’s inequality for integrals, we have

\[
\Gamma_\lambda(x + s) = \int_0^\infty t^{x+s-1}(1 + \lambda)^{-\frac{x}{s}}dt
\]

\[
= \int_0^\infty t^{(1-s)(x-1)}(1 + \lambda)^{-\frac{(1-s)x}{s}}t^s(1 + \lambda)^{-\frac{x}{s}}dt
\]

\[
\leq \left( \int_0^\infty t^{x-1}(1 + \lambda)^{-\frac{x}{s}}dt \right)^{1-s} \left( \int_0^\infty t^x(1 + \lambda)^{-\frac{x}{s}}dt \right)^s
\]

\[
= [\Gamma_\lambda(x)]^{1-s}[\Gamma_\lambda(x + 1)]^s,
\]

and by using (6), we obtain

\[
\Gamma_\lambda(x + s) \leq \left( \frac{\lambda x}{\ln(1 + \lambda)} \right)^s \Gamma_\lambda(x). \tag{16}
\]

By replacing \( s \) with \( 1 - s \) in (16), followed by substituting \( x \) by \( x + s \), we obtain

\[
\Gamma_\lambda(x + 1) \leq \left( \frac{\lambda(x + s)}{\ln(1 + \lambda)} \right)^{1-s} \Gamma_\lambda(x + s). \tag{17}
\]

Now, combining (16) and (17) gives

\[
\left( \frac{\ln(1 + \lambda)}{\lambda(x + s)} \right)^{1-s} \Gamma_\lambda(x + 1) \leq \Gamma_\lambda(x + s) \leq \left( \frac{\lambda x}{\ln(1 + \lambda)} \right)^s \Gamma_\lambda(x), \tag{18}
\]

and by using (6), we obtain the desired results (15).

\[\square\]

Remark 2.2. Inequality (18) can also be rearranged as

\[
\left( \frac{x}{x + s} \right)^{1-s} \leq \frac{\Gamma_\lambda(x + s)}{\left( \frac{\lambda x}{\ln(1 + \lambda)} \right)^s \Gamma_\lambda(x)} \leq 1, \tag{19}
\]

which is the degenerate form of the Wendel’s inequality (see (7) of [14]). Furthermore, by Squeezes theorem, (19) implies that

\[
\lim_{x \to \infty} \frac{\Gamma_\lambda(x + s)}{\left( \frac{\lambda x}{\ln(1 + \lambda)} \right)^s \Gamma_\lambda(x)} = 1 \tag{20}
\]

which is the degenerate form of the Wendel’s asymptotic relation (see (1) of [14]). The limit (20) also implies that

\[
\lim_{x \to \infty} \left( \frac{\lambda x}{\ln(1 + \lambda)} \right)^{r-s} \frac{\Gamma_\lambda(x + s)}{\Gamma_\lambda(x + r)} = 1. \tag{21}
\]

Theorem 2.3. For \( 0 < u \leq v \), the inequality

\[
\exp \{(v - u)\psi_\lambda(u)\} \leq \frac{\Gamma_\lambda(v)}{\Gamma_\lambda(u)} \leq \exp \{(v - u)\psi_\lambda(v)\}, \tag{22}
\]

holds.
Proof. The case for \( u = v \) is trivial. So, consider the function \( \ln \Gamma_\lambda(x) \) on the interval \( 0 < u < v \). Then by the mean value theorem, there exist a \( k \in (u, v) \) such that
\[
\frac{\ln \Gamma_\lambda(v) - \ln \Gamma_\lambda(u)}{v - u} = \psi_\lambda(k).
\]
Since \( \psi_\lambda(x) \) is increasing, then
\[
\psi_\lambda(u) < \psi_\lambda(k) < \psi_\lambda(v),
\]
which yields
\[
(v - u)\psi_\lambda(u) < \ln \frac{\Gamma_\lambda(v)}{\Gamma_\lambda(u)} < (v - u)\psi_\lambda(v),
\]
and by exponentiation, we obtain the desired result (22).

Corollary 2.4. For \( s \in (0, 1] \) and \( x > 0 \), the inequality
\[
\exp \left\{ (1 - s)\psi_\lambda(x + s) \right\} \leq \frac{\Gamma_\lambda(x + 1)}{\Gamma_\lambda(x + s)} \leq \exp \left\{ (1 - s)\psi_\lambda(x + 1) \right\},
\]
holds.

Proof. Let \( v = x + 1 \) and \( u = x + s \) in Theorem 2.3.

Corollary 2.5. For \( x \geq 0 \), the inequality
\[
\frac{\lambda}{\ln(1 + \lambda)} \exp \left\{ x\psi_\lambda(x + 1) \right\} \leq \Gamma_\lambda(x + 1) \leq \frac{\lambda}{\ln(1 + \lambda)} \exp \left\{ x\psi_\lambda(1) \right\},
\]
holds.

Proof. Let \( v = x + 1 \) and \( u = 1 \) in Theorem 2.3.

Remark 2.6. Inequality (23) is the degenerate form of inequality (3.4) of [11].

Theorem 2.7. For \( 0 < u \leq v \), the inequality
\[
\exp \left\{ \frac{(v - u)\psi_\lambda(u) + \psi_\lambda(v)}{2} \right\} \leq \frac{\Gamma_\lambda(v)}{\Gamma_\lambda(u)} \leq \exp \left\{ (v - u)\psi_\lambda \left( \frac{u + v}{2} \right) \right\},
\]
holds.

Proof. Let \( 0 < u \leq v \) and consider the function \( \psi_\lambda(x) \) on the interval \([u, v]\). Since \( \psi_\lambda(x) \) is concave, then by the classical Hermite-Hadamard inequality, we have
\[
\frac{\psi_\lambda(u) + \psi_\lambda(v)}{2} \leq \frac{1}{v - u} \int_u^v \psi_\lambda(t) dt \leq \psi_\lambda \left( \frac{u + v}{2} \right),
\]
which translates to
\[
\frac{(v - u)\psi_\lambda(u) + \psi_\lambda(v)}{2} \leq \ln \frac{\Gamma_\lambda(v)}{\Gamma_\lambda(u)} \leq (v - u)\psi_\lambda \left( \frac{u + v}{2} \right),
\]
and by exponentiation, we obtain the desired result (25).
Corollary 2.8. For $s \in (0, 1]$ and $x > 0$, the inequality
\[
\exp \left\{ (1 - s) \psi_{\lambda}(x + 1) + \psi_{\lambda}(x + s) \right\} \leq \frac{\Gamma_{\lambda}(x + 1)}{\Gamma_{\lambda}(x + s)} \leq \exp \left\{ (1 - s) \psi_{\lambda} \left( x + \frac{s + 1}{2} \right) \right\},
\]
holds.

Proof. Let $v = x + 1$ and $u = x + s$ in Theorem 2.7.

Corollary 2.9. For $x \geq 0$, the inequality
\[
\frac{\lambda}{\ln(1 + \lambda)} \exp \left\{ \frac{1}{2} \left[ \psi_{\lambda}(x + 1) + \psi_{\lambda}(1) \right] \right\} \leq \frac{\Gamma_{\lambda}(x + 1)}{\Gamma_{\lambda}(x + s)} \leq \frac{\lambda}{\ln(1 + \lambda)} \exp \left\{ \psi_{\lambda} \left( \frac{x}{2} + 1 \right) \right\}
\]
holds.

Proof. Let $v = x + 1$ and $u = 1$ in Theorem 2.7.

Remark 2.10. Inequalities (25), (26) and (27) are respectively better than (22), (23) and (24).

Remark 2.11. By letting $\lambda \to 0$, inequality (26) reduces to
\[
\exp \left\{ (1 - s) \psi_{\lambda}(x + 1) + \psi_{\lambda}(x + s) \right\} \leq \frac{\Gamma_{\lambda}(x + 1)}{\Gamma_{\lambda}(x + s)} \leq \exp \left\{ (1 - s) \psi_{\lambda} \left( x + \frac{s + 1}{2} \right) \right\}.
\]
The upper bound of (28) coincides with the upper bound of inequality (1.2) in the work [3] which was obtained by a different procedure. However, the lower bound of (28) is better than the lower bound of inequality (1.2) in [3] since
\[
\frac{\psi(x + 1) + \psi(x + s)}{2} \geq \sqrt{\psi(x + 1)\psi(x + s)} \geq \psi(x + s) \geq \psi(x + \sqrt{s}).
\]
This is by virtue of the arithmetic-geometric mean inequality and the monotonicity property of $\psi(x)$.

Theorem 2.12. For $x > 0$ and $s \in (0, 1]$, the inequality
\[
\left( \frac{\ln(\lambda + 1)}{\lambda} \right)^{1-s} (x + s)^{s-1} \leq \frac{\Gamma_{\lambda}(x + s)}{\Gamma_{\lambda}(x + 1)} \leq \frac{\ln(\lambda + 1)}{\lambda} s^{1-s} \Gamma_{\lambda}(s)(x + s)^{s-1}
\]
holds.

Proof. Let $\Delta(x) = (x + s)^{1-s} \frac{\Gamma_{\lambda}(x+s)}{\Gamma_{\lambda}(x+1)}$ for $x > 0$ and $s \in (0, 1]$. Then
\[
\lim_{x \to 0} \Delta(x) = \frac{\ln(\lambda + 1)}{\lambda} s^{1-s} \Gamma_{\lambda}(s).
\]
Also, inequality (19) implies that
\[
\lim_{x \to \infty} \Delta(x) = \left( \frac{\ln(\lambda + 1)}{\lambda} \right)^{1-s}.
\]
Furthermore,
\[
\frac{\Delta'(x)}{\Delta(x)} = \frac{1-s}{x + s} + \psi_{\lambda}(x + s) - \psi_{\lambda}(x + 1) \leq 0
\]
which shows that $\Delta(x)$ is decreasing. Hence $\Delta(\infty) \leq \Delta(x) \leq \Delta(0)$ which yields (29).

\begin{remark}
As a particular case, by letting $s = \frac{1}{2}$ and $\lambda \to 0$, we obtain
\begin{equation}
\sqrt{\frac{2}{\pi}} \left( x + \frac{1}{2} \right) < \frac{\Gamma(x + 1)}{\Gamma(x + \frac{1}{2})} < \sqrt{\left( x + \frac{1}{2} \right)}.
\end{equation}
\end{remark}

The upper bound of (30) agrees with the upper bound of (3) in [13]. Comparing the lower bounds of (30) and (3) in [13] reveals that, the lower bound of (30) is stronger if $0 < x < \frac{1}{\pi - 2}$ and is weaker if $x > \frac{1}{\pi - 2}$.

\begin{theorem}
Let $u > 1$ and $\frac{1}{u} + \frac{1}{v} = 1$. Then
\begin{equation}
\Gamma_\lambda(x + y) \leq \left[ \Gamma_\lambda(ux) \right]^\frac{1}{u} \left[ \Gamma_\lambda(vy) \right]^\frac{1}{v},
\end{equation}
holds for $x > 0$ and $y > 0$.
\end{theorem}

\begin{proof}
By the Holder’s inequality for integrals, we have
\begin{align*}
\Gamma_\lambda(x + y) &= \int_0^\infty t^{x+y-1}(1 + \lambda)^{-\frac{x}{\lambda}} dt \\
&= \int_0^\infty t^{x-\frac{1}{2}}(1 + \lambda)^{-\frac{x}{\lambda}} t^{y-\frac{1}{2}}(1 + \lambda)^{-\frac{y}{\lambda}} dt \\
&\leq \left( \int_0^\infty t^{ux-1}(1 + \lambda)^{-\frac{x}{\lambda}} dt \right)^\frac{1}{u} \left( \int_0^\infty t^{vy-1}(1 + \lambda)^{-\frac{y}{\lambda}} dt \right)^\frac{1}{v} \\
&= \left[ \Gamma_\lambda(ux) \right]^\frac{1}{u} \left[ \Gamma_\lambda(vy) \right]^\frac{1}{v},
\end{align*}
which concludes the proof.
\end{proof}

\begin{remark}
Applying Young’s inequality on the right hand side of (31) reveals that
\begin{equation}
\Gamma_\lambda(x + y) \leq \frac{\Gamma_\lambda(ux)}{u} + \frac{\Gamma_\lambda(vy)}{v},
\end{equation}
\end{remark}

\begin{theorem}
Let $r_1, r_2 \in \{2n : n \in \mathbb{N}_0\}$, $k_1 > 1$, $\frac{1}{k_1} + \frac{1}{k_2} = 1$ and $\frac{r_1}{k_1} + \frac{r_2}{k_2} \in \mathbb{N}_0$. Then inequality
\begin{equation}
\Gamma_\lambda^{\left( \frac{r_1}{k_1} + \frac{r_2}{k_2} \right)} \left( \frac{x}{k_1} + \frac{y}{k_2} \right) \leq \left[ \Gamma^{(r_1)}_\lambda(x) \right]^\frac{1}{k_1} \left[ \Gamma^{(r_2)}_\lambda(y) \right]^\frac{1}{k_2},
\end{equation}
holds for $x > 0$ and $y > 0$.
\end{theorem}
**Proof.** By using (8) and the Holder’s inequality, we have
\[
\Gamma_{r_1^2 + r_2^2} \left( \frac{x}{k_1} + \frac{y}{k_2} \right) = \int_0^\infty (\ln t)^{\frac{r_1}{k_1} + \frac{r_2}{k_2}} t^{\frac{r_1}{k_1} + \frac{r_2}{k_2} - 1} (1 + \lambda)^{-\frac{x}{r}} dt
\]
\[
= \int_0^\infty (\ln t)^{\frac{r_1}{k_1} + \frac{r_2}{k_2}} (1 + \lambda)^{-\frac{x}{r}}(\ln t)^{\frac{r_2}{k_2}} t^{\frac{r_2}{k_2} - 1} (1 + \lambda)^{-\frac{x}{r}} dt
\]
\[
\leq \left( \int_0^\infty (\ln t)^{\frac{r_1}{k_1} t^{r_1 - 1}} (1 + \lambda)^{-\frac{x}{r}} dt \right)^{\frac{1}{k_1}} \left( \int_0^\infty (\ln t)^{r_2 t^{r_2 - 1}} (1 + \lambda)^{-\frac{x}{r}} dt \right)^{\frac{1}{k_2}}
\]
\[
= \left[ \Gamma (r_1) (x) \right]^{\frac{1}{k_1}} \left[ \Gamma (r_2) (y) \right]^{\frac{1}{k_2}},
\]
which concludes the proof. \(\square\)

**Remark 2.17.** If \(r_1 = r_2 = r\), then (33) reduces to
\[
\Gamma_{r} \left( \frac{x}{k_1} + \frac{y}{k_2} \right) \leq \left[ \Gamma_r (x) \right]^{\frac{1}{k_1}} \left[ \Gamma_r (y) \right]^{\frac{1}{k_2}}, \tag{34}
\]
which implies that the function (8) is log-convex for any even order derivative. Moreover, if \(r = 0\) in (34), we obtain
\[
\Gamma_r \left( \frac{x}{k_1} + \frac{y}{k_2} \right) \leq \left[ \Gamma_r (x) \right]^{\frac{1}{k_1}} \left[ \Gamma_r (y) \right]^{\frac{1}{k_2}}, \tag{35}
\]
which shows that the modified degenerate gamma function is log-convex.

**Remark 2.18.** If \(r_1 = r, r_2 = r + 2, k_1 = k_2 = 2\) and \(x = y\), then (33) reduces to the Turan type inequality
\[
\left( \Gamma_{r+1} (x) \right)^2 \leq \Gamma_r (x) \Gamma_{r+2} (x).
\]

**Remark 2.19.** If \(r_1 = s - 1, r_2 = s + 1, k_1 = k_2 = 2\) and \(x = y\), then (33) reduces to the Turan type inequality
\[
\left( \Gamma_{s} (x) \right)^2 \leq \Gamma_{s-1} (x) \Gamma_{s+1} (x),
\]
where \(s \in \{2n + 1 : n \in \mathbb{N}_0\}\). This is the degenerate version of main result of [1].

**Theorem 2.20.** For \(x \geq 1\), the inequality
\[
\Gamma_r (x) \geq \frac{\lambda}{\ln(1 + \lambda)} - 1 + \frac{1}{x}, \tag{36}
\]
holds.

**Proof.** Let \(\phi(x) = \Gamma_r (x) - \frac{1}{x}\). Then
\[
\phi(x + 1) - \phi(x) = \Gamma_r (x + 1) - \Gamma_r (x) + \frac{1}{x} - \frac{1}{x + 1}
\]
\[
= \Gamma_r (x) \left( \frac{\lambda x}{\ln(1 + \lambda)} - 1 \right) + \frac{1}{x} - \frac{1}{x + 1} > 0.
\]
Thus, \(\phi(x)\) is increasing and for \(x \geq 1\), we have \(\phi(x) \geq \phi(1)\) which completes the proof. \(\square\)
Lemma 2.21. The function $\beta(x) = x\Gamma'_\lambda(x)$ is increasing for all $x > 0$.

Proof. By using (6) and (9) and the monotonicity property of $\psi_\lambda(x)$, we have
\[\beta(x) = x\Gamma'_\lambda(x)\psi_\lambda(x) = \frac{\ln(\lambda + 1)}{\lambda} \Gamma_\lambda(x + 1)\psi_\lambda(x),\]
and consequently, we obtain
\[\frac{\lambda}{\ln(\lambda + 1) \Gamma_\lambda(x + 1)} \beta'(x) = \psi_\lambda(x + 1)\psi_\lambda(x) + \psi'_\lambda(x) > 0,\]
which completes the proof. \qed

Theorem 2.22. The inequalities
\[\Gamma_\lambda(x)\Gamma_\lambda(1/x) \geq \left(\frac{\lambda}{\ln(\lambda + 1)}\right)^2,\]  
(37)
\[\Gamma_\lambda(x) + \Gamma_\lambda(1/x) \geq \frac{2\lambda}{\ln(\lambda + 1)},\]  
(38)
hold for $x > 0$.

Proof. By letting $k_1 = k_2 = 2$ and replacing $x$ and $y$ with $1 + x$ and $1 + \frac{1}{x}$ in (35), we obtain
\[\ln \Gamma_\lambda\left(1 + \frac{x}{2} + \frac{1}{2x}\right) \leq \frac{\ln \Gamma_\lambda(1 + x)}{2} + \frac{\ln \Gamma_\lambda(1 + \frac{1}{x})}{2}.\]  
(39)
Also, since $x + \frac{1}{x} \geq 2$ for $x > 0$, then $1 + \frac{x}{2} + \frac{1}{2x} \geq 2$. Now, let $\phi(x) = \Gamma_\lambda(x)\Gamma_\lambda(1/x)$ for $x > 0$. Then by using (6), we have
\[\phi(x) = \left(\frac{\ln(\lambda + 1)}{\lambda}\right)^2 \Gamma_\lambda(1 + x)\Gamma_\lambda(1 + 1/x).\]  
(40)
Next, by using (5), (39) and (40), we obtain
\[\ln \phi(x) \geq 2 \ln \left(\frac{\ln(\lambda + 1)}{\lambda}\right) + 2 \ln \Gamma_\lambda \left(1 + \frac{x}{2} + \frac{1}{2x}\right)
    \geq 2 \ln \left(\frac{\ln(\lambda + 1)}{\lambda}\right) + 2 \ln \Gamma_\lambda(2)
    = \ln \left(\frac{\lambda}{\ln(\lambda + 1)}\right)^2,\]
which gives (37). Next, let $\theta(x) = \Gamma_\lambda(x) + \Gamma_\lambda(1/x)$ for $x > 0$. Then
\[x\theta'(x) = x\Gamma'_\lambda(x) - \frac{1}{x} \Gamma'_\lambda \left(\frac{1}{x}\right).\]
It follows from Lemma 2.21 that, $\theta(x)$ is increasing if $x > 1$ and decreasing if $0 < x < 1$. For both cases, we have $\theta(x) > \theta(1) = \frac{2\lambda}{\ln(\lambda + 1)}$ which gives inequality (38). \qed
Remark 2.23. Inequality (38) can be obtained from inequality (37) by applying the arithmetic-geometric mean inequality.

Theorem 2.24. Let $r, s \in \{2n : n \in \mathbb{N}_0\}$ and $r \geq s$. Then inequality

$$\left(\exp \Gamma^{(r)}_\lambda(x)\right)^2 \leq \exp \Gamma^{(r-s)}_\lambda(x) \cdot \exp \Gamma^{(r+s)}_\lambda(x)$$

holds for $x > 0$.

Proof. We adopt the technique of Mortici [12] to estimate the function

$$\frac{\Gamma^{(r-s)}_\lambda(x) + \Gamma^{(r+s)}_\lambda(x)}{2} - \Gamma^{(r)}_\lambda(x)$$

$$= \frac{1}{2} \int_0^\infty (\ln t)^{r-s} t^{x-1}(1 + \lambda)^{-\frac{x}{\lambda}} dt + \frac{1}{2} \int_0^\infty (\ln t)^{r+s} t^{x-1}(1 + \lambda)^{-\frac{x}{\lambda}} dt$$

$$- \int_0^\infty (\ln t)^{s} t^{x-1}(1 + \lambda)^{-\frac{x}{\lambda}} dt$$

$$= \frac{1}{2} \int_0^\infty \left[ \frac{1}{(\ln t)^s} + (\ln t)^s - 2 \right] (\ln t)^r t^{x-1}(1 + \lambda)^{-\frac{x}{\lambda}} dt$$

$$= \frac{1}{2} \int_0^\infty [1 - (\ln t)^s]^2 (\ln t)^r t^{x-1}(1 + \lambda)^{-\frac{x}{\lambda}} dt$$

$$\geq 0.$$
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