Safe Online Bid Optimization with Return-On-Investment and Budget Constraints subject to Uncertainty

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Abstract

In online marketing, the advertisers’ goal is usually a tradeoff between achieving high volumes and high profitability. The companies’ business units customarily address this tradeoff by maximizing the volumes while guaranteeing a lower bound to the Return On Investment (ROI). Technically speaking, such a task can be naturally modeled as a combinatorial optimization problem subject to ROI and budget constraints to be solved online since the parameter values are uncertain and need to be estimated during the sequential arrival of data. In this picture, the uncertainty over the constraints’ parameters plays a crucial role. Indeed, these constraints can be arbitrarily violated during the learning process due to an uncontrolled algorithms’ exploration, and such violations represent one of the major obstacles to the adoption of automatic techniques in real-world applications as often considered unacceptable risks by the advertisers. Thus, to make humans trust online learning tools, controlling the algorithms’ exploration so as to mitigate the risk and provide safety guarantees during the entire learning process is of paramount importance. In this paper, we study the nature of both the optimization and learning problems. In particular, when focusing on the optimization problem without uncertainty, we show that it is inapproximable within any factor unless $P = NP$, and we provide a pseudo-polynomial-time

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algorithm that achieves an optimal solution. When considering uncertainty, we prove that no online learning algorithm can violate the (ROI or budget) constraints during the learning process a sublinear number of times while guaranteeing a sublinear pseudo-regret. Thus, we provide an algorithm, namely GCB, guaranteeing sublinear regret at the cost of a potentially linear number of constraints violations. We also design its safe version, namely GCBsafe, guaranteeing w.h.p. a constant upper bound on the number of constraints violations at the cost of a linear pseudo-regret. More interestingly, inspired by the previous two algorithms, we provide an algorithm, namely GCBsafe(ψ, φ), guaranteeing both sublinear pseudo-regret and safety w.h.p. at the cost of accepting tolerances ψ and φ in the satisfaction of the ROI and budget constraints, respectively. This algorithm actually mitigates the risks due to the constraints violations without precluding the convergence to the optimal solution. Finally, we experimentally compare our algorithms in terms of pseudo-regret/constraint-violation tradeoff in settings generated from real-world data, showing the importance of adopting safety constraints in practice and the effectiveness of our algorithms.

Keywords: Regret Minimization, Online Learning, Safe Online Learning, Advertising

1. Introduction

Nowadays, Internet advertising is de facto the leading advertising medium. Notably, while the expenditure on physical ads, radio, and television has been stable for a decade, that on Internet advertising is increasing with an average ratio of 20% per year, reaching the considerable amount of 124 billion USD in 2019 only in the US [1]. Internet advertising has two main advantages over traditional advertising channels. The former is to provide precise ad targeting, and the latter is to allow an accurate, real-time evaluation of investment performance. On the other hand, the amount of data provided by the platforms and the plethora of parameters to be set make its optimization impractical without adopting of artificial intelligence (AI) tools.
The advertisers’ goal is usually to set bids in the attempt to balance the tradeoff between achieving high volumes, corresponding to maximizing the sales of the products to advertise, and high profitability, corresponding to maximizing ROI. The companies’ business units need simple ways to address this tradeoff, and, customarily, they maximize the volumes while constraining ROI to be above a given threshold. The importance of ROI constraints, in addition to standard budget constraints, is remarked by several empirical studies. We mention, e.g., the data analysis on the auctions performed on Google’s AdX by Golrezaei et al. [2], showing that many advertisers take into account ROI constraints, particularly in hotel booking. However, no platform provides features to force the satisfaction of ROI constraints, and some platforms (e.g., TripAdvisor and Trivago) do not even allow the setting of daily budget constraints. Thus, the problem of satisfying those constraints is a challenge that the advertisers need to address by designing suitable bidding strategies. In this picture, uncertainty plays a crucial role as the revenue and cost of the advertising campaigns are unknown beforehand and need to be estimated online by learning algorithms during the sequential arrival of data. As a result, the constraints are subject to uncertainty, and wrong estimations of the parameters can make the ROI and budget constraints be arbitrarily violated when using an uncontrolled exploration like that used by classical online learning algorithms. Such violations represent today the major obstacles to adopting AI tools in real-world applications as often considered unacceptable risks by the advertisers. Remarkably, this issue is particularly crucial in the early stages of the learning process as adopting algorithms with an uncontrolled exploration when a small amount of data is available can make the advertising campaigns’ performance oscillate with a large magnitude. Therefore, to make humans trust online artificial intelligence algorithms, controlling their exploration accurately to mitigate risks and provide safety guarantees during the entire learning process is of paramount importance.
1.1. Related Works

Many works study Internet advertising, both from the publisher perspective (e.g., Vazirani et al. [3] design auctions for ads allocation and pricing) and from the advertiser perspective (e.g., Feldman et al. [4] study the budget optimization problem in search advertising). Few works deal with ROI constraints, and, to the best of our knowledge, they only focus on the design of auction mechanisms. In particular, Szymanski and Lee [5] and Borgs et al. [6] show that ROI-based bidding heuristics can lead to cyclic behavior and reduce the allocation’s efficiency, while Golrezaei et al. [2] propose more efficient auctions with ROI constraints. The learning algorithms for daily bid optimization available in the literature address only budget constraints in the restricted case in which the platform allows the advertisers to set a daily budget limit (notice that some platforms such as, e.g., TripAdvisor and Trivago, do not even allow the setting of the daily budget limit). For instance, Zhang et al. [7] provide an offline algorithm that exploits accurate models of the campaigns’ performance based on low-level data rarely available to the advertisers. Nuara et al. [8] propose an online learning algorithm that combines combinatorial multi-armed bandit techniques [9] with regression by Gaussian Processes [10]. This work provides no guarantees on ROI. More recent works also present pseudo-regret bounds [11] and study subcampaigns interdependencies offline [12]. Thomaidou et al. [13] provide a genetic algorithm for budget optimization of advertising campaigns. Ding et al. [14] and Trovò et al. [15] address the bid optimization problem in a single subcampaign scenario when the budget constraint is cumulative over time.

A research field strictly related to our work is learning with safe exploration with constraints subject to uncertainty, and the goal is to guarantee w.h.p. their satisfaction during the entire learning process. The only known results on safe exploration in multi-armed bandits address the case with continuous, convex arm spaces and convex constraints. The learner can converge to the optimal solution in these settings without violating the constraints [16, 17]. Conversely, the case with discrete and/or non-convex arm spaces or non-convex constraints,
such as ours, is unexplored in the literature so far. We remark that some bandit algorithms address uncertain constraints where the goal is their satisfaction on average \cite{18, 19}. However, the per-round violation can be arbitrarily large (particularly in the early stages of the learning process), not fitting with our setting as humans could be alarmed and, thus, give up on adopting the algorithm. We also notice that several other works in reinforcement learning \cite{20, 21, 22} and multi-armed bandit \cite{23, 24} investigate safe exploration, providing safety guarantees on the revenue provided by the algorithm, but not on the satisfaction w.h.p. of uncertain constraints.

1.2. Original Contributions

As customary in the online advertising literature, see, \textit{e.g.}, Devanur and Kakade \cite{25}, we make the assumption of stochastic (\textit{i.e.}, non-adversarial) clicks, and we adopt Gaussian Processes (GPs) to model the problem. Let us remark that the assumption that clicks are generated stochastically is reasonable in practice as the advertising platforms can limit manipulation due to malicious bidders. For instance, Google Ads can identify invalid clicks and exclude them from the advertisers’ spending. In this paper, we study the nature of both the optimization and learning problems. Initially, we focus on studying our optimization problem without uncertainty, showing that no approximation within any strictly positive factor is possible with ROI and budget constraints unless $P = NP$, even in simple, realistic instances. However, when dealing with a discretized space of the bids as it happens in practice, the problem admits an exact pseudo-polynomial time algorithm based on dynamic programming. Most importantly, when the problem is with uncertainty, we show that no online learning algorithm can violate the ROI and/or budget constraints a sublinear number of times while guaranteeing a sublinear pseudo-regret. Notably, this result holds in general bandit settings beyond advertising when the constraints are subject uncertainty and the arm space or constraints are not convex. We provide an algorithm, namely GCB, providing pseudo-regret sublinear in the time horizon $T$ at the cost of a linear number of violations of the constraints. We also pro-
vide its safe version, namely $\text{GCB}_{\text{safe}}$, guaranteeing w.h.p. a constant upper bound on the number of constraints’ violations at the cost of a regret linear in $T$. Inspired by the two previous algorithms, we design a new algorithm, namely $\text{GCB}_{\text{safe}}(\psi, \phi)$, guaranteeing both the violation w.h.p. of the constraints for a constant number of times and a pseudo-regret sublinear both in $T$ and the maximum information gain of the GP when accepting tolerances $\psi$ and $\phi$ in the satisfaction of the ROI and budget constraints, respectively. We experimentally compare our algorithms in terms of pseudo-regret/constraint-violation tradeoff in settings generated from real-world data, showing the importance of adopting safety constraints in practice and the effectiveness of our algorithms. In particular, using small tolerances $\psi$ and $\phi$ with $\text{GCB}_{\text{safe}}(\psi, \phi)$ guarantees very smooth dynamics and a negligible loss in reward.

1.3. Structure of the Paper

The remainder of this paper is structured as follows. Section 2 formally states the problem we study. Section 3 introduces a meta-algorithm whose different flavours are investigated in the subsequent sections. Section 4 studies the optimization problem without uncertainty, whereas Section 5 investigates the online learning problem. Section 6 provides an experimental evaluation of our algorithms. Section 7 concludes the paper and discusses possible future works. In the appendices, we report the proofs of the theorems presented in the main paper and details on the experimental activity to ease their reproducibility.

2. Problem Formulation

We are given an advertising campaign $C = \{C_1, \ldots, C_N\}$, with $N \in \mathbb{N}$ and where $C_j$ is the $j$-th subcampaign, and a finite time horizon of $T \in \mathbb{N}$ rounds (each corresponding to one day in our application). In this work, as common in the literature on ad allocation optimization, we refer to a subcampaign as a single ad or a group of homogeneous ads requiring to set the same bid. For every round $t \in \{1, \ldots, T\}$ and every subcampaign $C_j$, an advertiser needs to
specify the bid $x_{j,t} \in X_j$, where $X_j \subset \mathbb{R}^+$ is a finite set of values that can be set for subcampaign $C_j$. For every round $t \in \{1,\ldots,T\}$, the goal is to find the values of bids maximizing the overall cumulative expected revenue while keeping the overall ROI above a fixed value $\lambda \in \mathbb{R}^+$ and the overall budget below a daily value $\beta \in \mathbb{R}^+$. Formally, the resulting constrained optimization problem at round $t$ is as follows:

$$\max_{(x_1,t,\ldots,x_N,t) \in X_1 \times \ldots \times X_N} \sum_{j=1}^N v_j n_j(x_{j,t})$$

s.t. $$\frac{\sum_{j=1}^N v_j n_j(x_{j,t})}{\sum_{j=1}^N c_j(x_{j,t})} \geq \lambda,$$

$$\sum_{j=1}^N c_j(x_{j,t}) \leq \beta,$$

where $n_j(x_{j,t})$ and $c_j(x_{j,t})$ are the expected number of clicks and the expected cost given the bid $x_{j,t}$ for subcampaign $C_j$, respectively, and $v_j$ is the value per click for subcampaign $C_j$. Moreover, Constraint (1b) is the ROI constraint, forcing the revenue to be at least $\lambda$ times the costs, and Constraint (1c) keeps the daily spend under a predefined overall budget $\beta$.

We focus on the customary case in which $n_j(\cdot)$ and $c_j(\cdot)$ are unknown functions whose values need to be estimated online. Our problem can be naturally modeled as a multi-armed bandit where the available arms are the different values of the bid $x_{j,t} \in X_j$ satisfying the combinatorial constraints of the optimization problem. A super-arm is an arm profile specifying one bid per subcampaign. A learning policy $\Pi$ solving such a problem is an algorithm returning, for every round $t$, a set of bid $\{\hat{x}_{j,t}\}_{j=1}^N$. Policy $\Pi$ can only use estimates of the unknown number-of-click and cost functions built during the learning process. Therefore, the solutions returned by policy $\Pi$ may not be optimal and/or vio-

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1In economic literature, it is also used an alternative definition of ROI: $\frac{\sum_{j=1}^N [v_j n_j(x_{j,t}) - c_j(x_{j,t})]}{\sum_{j=1}^N c_j(x_{j,t})}$. We can capture this case by substituting the right hand side of Constraint (1b) with $\lambda + 1$.

2Here, we assume that the value per click $v_j$ is known. In the case it is unknown, we point an interested reader to Nuara et al. [8] for details.
late Constraints \((1b)\) and \((1c)\) when evaluated with the true values. Notice that, even if this setting is closely related to the one studied by Badanidiyuru et al. \cite{20}, the specific non-matroidal nature of the constraints does not allow to cast the bid allocation problem above into the bandit-with-knapsack framework.

We are interested in evaluating learning policies \(U\) in terms of both loss of revenue (a.k.a. pseudo-regret) and violation of the ROI and budget constraints. The pseudo-regret and safety of a learning policy \(U\) are defined as follows.

**Definition 1** (Learning policy pseudo-regret). Given a learning policy \(U\), we define the pseudo-regret as:

\[
R_T(U) := T G^* - \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{j=1}^{N} v_j n_j(\hat{x}_{j,t}) \right],
\]

where \(G^* := \sum_{j=1}^{N} v_j n_j(x_j^*)\) is the expected revenue provided by a clairvoyant algorithm, the set of bids \(\{x_j^*\}_{j=1}^{N}\) is the optimal clairvoyant solution to the problem in Equations \((1a)-(1c)\), and the expectation \(\mathbb{E}[\cdot]\) is taken w.r.t. the stochasticity of the learning policy \(U\).

Our goal is the design of algorithms that minimize the pseudo-regret \(R_T(U)\). In particular, we are interested in no-regret algorithms guaranteeing a regret that increases sublinearly in \(T\). Now, we focus on the notion of safety.

**Definition 2** (\(\eta\)-safe learning policy). Given \(\eta \in (0, T]\), a learning policy \(U\) is \(\eta\)-safe if \(\{\hat{x}_{j,t}\}_{j=1}^{N}\), i.e., the expected number of times at least one of the Constraints \((1b)\) and \((1c)\) is violated from \(t = 1\) to \(T\) is less than \(\eta\) or, formally:

\[
\sum_{t=1}^{T} \sum_{j=1}^{N} \left( \frac{v_j n_j(\hat{x}_{j,t})}{\sum_{j=1}^{N} c_j(\hat{x}_{j,t})} < \lambda \lor \sum_{j=1}^{N} c_j(\hat{x}_{j,t}) > \beta \right) \leq \eta.
\]

Our goal is the design of safe algorithms that minimize \(\eta\). In particular, we are interested in safe algorithms guaranteeing that \(\eta\) increases sublinearly in (or is independent of) \(T\).
Algorithm 1 Meta-algorithm

Input: sets of bid values $X_1, \ldots, X_N$, ROI threshold $\lambda$, daily budget $\beta$

1: Initialize the GPs for the number of clicks and costs
2: for $t \in \{1, \ldots, T\}$ do
3:     for $j \in \{1, \ldots, N\}$ do
4:         for $x \in X_j$ do
5:             Call the estimation subroutine to estimate $\hat{n}_{j,t-1}(x)$, $\hat{\sigma}_{j,t-1}^n(x)$ using the GP on the number of clicks
6:             Call the estimation subroutine to estimate $\hat{c}_{j,t-1}(x)$, $\hat{\sigma}_{j,t-1}^c(x)$ using the GP on the costs
7:         Compute $\mu$ using the GPs estimates
8:         Call the optimization subroutine $\text{Opt}(\mu, \lambda)$ to get a solution $\{\hat{x}_{j,t}\}_{j=1}^N$
9:     Set the prescribed allocation during round $t$
10:    Get revenue $\sum_{j=1}^N v_j \hat{n}_j(\hat{x}_{j,t})$
11:    Update the GPs using the new information $\hat{n}_j(\hat{x}_{j,t})$ and $\hat{c}_j(\hat{x}_{j,t})$

3. Meta-algorithm

We provide the pseudo-code of our meta-algorithm in Algorithm 1 which solves the problem in Equations (1a)–(1c) online. Algorithm 1 is based on three components: Gaussian Processes (GPs) to model the parameters whose values are unknown (details are provided below), an estimation subroutine to generate estimates of the parameters from the GPs, and an optimization subroutine to solve the optimization problem once given the estimates.

In Algorithm 1, GPs are used to model functions $n_j(\cdot)$ and $c_j(\cdot)$, describing the expected number of clicks and the costs, respectively. The employment of GPs to model these functions provides several advantages w.r.t. other regression techniques, such as the provision of a probability distribution over the possible values of the functions for every bid value $x \in X_j$ relying on a finite set of samples. GPs use the noisy realization of the actual number of clicks $\hat{n}_{j,h}(\hat{x}_{j,h})$ collected from each subcampaign $C_j$ for every previous round $h \in \{1, \ldots, t-1\}$.
to generate, for every bid $x \in X_j$, the estimates for the expected value $\hat{n}_{j,t-1}(x)$ and the standard deviation of the number of clicks $\hat{\sigma}_{j,t-1}(x)$. Analogously, using the noisy realizations of the actual cost $\tilde{c}_{j,h}(\hat{x}_{j,h})$, with $h \in \{1, \ldots, t-1\}$, GPs generate, for every bid $x \in X_j$, the estimates for the expected value $\hat{c}_{j,t-1}(x)$ and the standard deviation of the cost $\hat{\sigma}_{c,j,t-1}(x)$. Formally, we compute the above values as follows:

$$\hat{n}_{j,t-1}(x) := k_{j,t-1}(x) \top (K_{j,t-1} + \sigma^2 I)^{-1} k_{j,t-1}(x),$$

$$\hat{\sigma}_{n,j,t-1}(x) := k_j(x,x) - k_{j,t-1}(x) \top (K_{j,t-1} + \sigma^2 I)^{-1} k_{j,t-1}(x),$$

$$\hat{c}_{j,t-1}(x) := h_{j,t-1}(x) \top (H_{j,t-1} + \sigma^2 I)^{-1} h_{j,t-1}(x),$$

$$\hat{\sigma}_{c,j,t-1}(x) := h_j(x,x) - h_{j,t-1}(x) \top (H_{j,t-1} + \sigma^2 I)^{-1} h_{j,t-1}(x),$$

where $k_j(\cdot, \cdot)$ and $h_j(\cdot, \cdot)$ are the kernels for the GPs over the number of clicks and costs, respectively, $K_{j,t-1}$ and $H_{j,t-1}$ are the Gram matrix over the training bids for the two GPs, $\sigma^2$ is the variance of the noise of the GPs, $k_{j,t-1}(x)$ and $h_{j,t-1}$ are vectors built computing the kernel between the training bids and the current bid $x$, and $I$ is the identity matrix of order $t-1$. For further details on the use of GPs, we point an interested reader to Rasmussen and Williams [10]. We recall that the asymptotic running time of the GP estimation procedure is $\Theta(\sum_{j=1}^N |X_j| t^2)$, where $t$ is the number of samples (corresponding to the rounds), and the asymptotic space complexity is $\Theta(Nt^2)$, i.e., the space required to store the Gram matrix. A better, linear dependence on the number of days $t$ can be obtained by using the recursive formula for the GP mean and variance computation (see Chowdhury and Gopalan [27] for details).

The estimation subroutine returns the vector $\mu$ composed of the estimates generated from the GPs. In the following sections, we investigate two different procedures to compute $\mu$. Then, the vector $\mu$ is given as input to the optimization subroutine, namely $\text{Opt}(\mu, \lambda)$, solving the problem stated in Equations (1a)–(1c) and returns the bid strategy $\{\tilde{x}_{j,t}\}_{j=1}^N$ to play the next round $t$. Finally, once the strategy has been applied, the revenue $\sum_{j=1}^N v_j \tilde{n}_j(\tilde{x}_{j,t})$ is obtained, and the stochastic realization of the number of clicks $\tilde{n}_{j,t}(\tilde{x}_{j,t})$ and
costs $\hat{c}_{j,t}(\hat{x}_{j,t})$ are observed and provided to the GPs to update the models that will be used at round $t + 1$. For the sake of presentation, we first describe the optimization subroutine $\text{Opt}(\mu, \lambda)$ and, then, some estimation subroutines together with the theoretical guarantees provided by Algorithm 1 when these subroutines are adopted.

4. Optimization Subroutine

At first, we show that, even if all the values of the parameters of the optimization problem are known, the optimal solution cannot be approximated in polynomial time within any strictly positive factor (even depending on the size of the instance), unless $P = NP$. We reduce from SUBSET-SUM that is an $NP$-hard problem. Given a set $S$ of integers $u_i \in \mathbb{N}$ and an integer $z \in \mathbb{N}$, SUBSET-SUM requires to decide whether there is a set $S^* \subseteq S$ with $\sum_{i \in S^*} u_i = z$.

Theorem 1 (Inapproximability). For any $\rho \in (0, 1]$, there is no polynomial-time algorithm returning a $\rho$-approximation to the problem in Equations (1a)–(1c), unless $P = NP$.

It is well known that SUBSET-SUM is a weakly $NP$-hard problem, admitting an exact algorithm whose running time is polynomial in the size of the problem and the magnitude of the data involved rather than the base-two logarithm of their magnitude. The same can be showed for our problem. Indeed, we can design a pseudo-polynomial-time algorithm to find the optimal solution in polynomial time w.r.t. the number of possible values of revenues and costs. In real-world settings, the values of revenue and cost are in limited ranges and rounded to the nearest cent, allowing the problem to be solved in a reasonable time. For simplicity, in the following we assume the discretization of the ranges of the values of the daily cost $Y$ and revenue $R$ be evenly spaced.

The pseudo-code of the $\text{Opt}(\mu, \lambda)$ subroutine, solving the problem in Equations (1a)–(1c) with a dynamic programming approach, is provided in Algorithm 2. It takes as input the set of the possible bid values $X_j$ for each
Algorithm 2 Opt(\(\mu, \lambda\)) subroutine

Input: sets of bid values \(X_1, \ldots, X_N\), set of cumulative cost values \(Y\), set of revenue values \(R\), vector \(\mu\), ROI threshold \(\lambda\)

1: Initialize \(M\) empty matrix with dimension \(|Y| \times |R|\)
2: Initialize \(x_{y,r}^{\text{next}} = [\ ]\), \(\forall y \in Y, r \in R\)
3: \(S(y,r) = \bigcup \{x \in X_1 | \tau_1(x) \leq y \wedge w_1(x) \geq r\}\) \(\forall y \in Y, r \in R\)
4: \(x_{y,r}^y = \arg\max_{x \in S} w_1(x)\) \(\forall y \in Y, r \in R\)
5: \(M(y,r) = \max_{x \in S} w_1(x)\) \(\forall y \in Y, r \in R\)
6: for \(j \in \{2, \ldots, N\}\) do
7:   for \(y \in Y\) do
8:     for \(r \in R\) do
9:       Update \(S(y,r)\) according to Equation (2)
10:  \(x_{y,r}^{\text{next}} = \arg\max_{s \in S(y,r)} \sum_{i=1}^{j} \overline{w}_i(s_i)\)
11:  \(M(y,r) = \max_{s \in S(y,r)} \sum_{i=1}^{j} \overline{w}_i(s_i)\)
12: \(x_{y,r} = x_{y,r}^{\text{next}}\)
13: Choose \((y^*, r^*)\) according to Equation (3)

Output: \(x_{y^*, r^*}\)

subcampaign \(C_j\), the set of the possible cumulative cost values \(Y\) such that \(\max_{y \in Y} y = \beta\), the set of the possible revenue values \(R\), a ROI threshold \(\lambda\), and a vector \(\mu\) of parameters characterizing the specific instance of the optimization problem that is defined as follows:

\[
\mu := [\overline{w}_1(x_1), \ldots, \overline{w}_N(x_{|X_N|}), \overline{w}_1(x_1), \ldots, \overline{w}_N(x_{|X_N|}), -\overline{\tau}_1(x_1), \ldots, -\overline{\tau}_N(x_{|X_N|})],
\]

where \(w_j(x_j) := v_j n_j(x_j)\) denotes the revenue for a subcampaign \(C_j\). We use \(\overline{h}\) and \(\underline{h}\) to denote potentially different estimated values of a generic function \(h\) used by the learning algorithms in the next sections. In particular, if the functions are known beforehand, then it holds \(\overline{h} = \underline{h} = h\) for both \(h = w_j\) and \(h = c_j\). For the sake of clarity, \(\overline{w}_j(x)\) is used in the objective function, while \(\overline{w}_j(x)\) and \(\overline{\tau}_j(x)\) are used in the constraints. At first, the subroutine initializes a matrix \(M\) in which it stores the optimal solution for each combination of
values $y \in Y$ and $r \in R$, and it initializes the vectors $\mathbf{x}^{y,r} = \mathbf{x}_{\text{next}}^{y,r} = [ ]$, $\forall y \in Y, \forall r \in R$ (Lines 1 and 2, respectively). Then, the subroutine generates the set $S(y,r)$ of the bids for subcampaign $C_1$ (Line 3). More precisely, the set $S(y,r)$ contains only the bids $x$ that induce the overall costs to be lower than or equal to $y$ and the overall revenue to be higher than or equal to $r$. The bid in $S(y,r)$ that maximizes the revenue calculated with parameters $\pi_j$ is included in the vector $\mathbf{x}^{y,r}$, while the corresponding revenue is stored in the matrix $M$ (Lines 4–5). Then, the subroutine iterates over each subcampaign $C_j$, with $j \in \{2, \ldots, N\}$, all the values $y \in Y$, and all the values $r \in R$ (Lines 9–11). At each iteration, for every pair $(y,r)$, the subroutine stores in $\mathbf{x}^{y,r}_{\text{next}}$ the optimal set of bids for subcampaigns $C_1, \ldots, C_j$ that maximizes the objective function and stores the corresponding optimum value in $M(y,r)$. At every $j$-th iteration, the computation of the optimal bids is performed by evaluating a set of candidate solutions $S(y,r)$, computed as follows:

$$S(y,r) := \bigcup \left\{ s = [x^{y',r'},x] \text{ s.t. } y' + \pi_j(x) \leq y \land r' + w_j(x) \geq r \land x \in X_j \land y' \in Y \land r' \in R \right\}.$$  

This set is built by combining the optimal bids $\mathbf{x}^{y',r'}$ computed at the $(j-1)$-th iteration with one of the bids $x \in X_j$ available for the $j$-th subcampaign, such that these combinations satisfy the ROI and budget constraints. Then, the subroutine assigns the element of $S(y,r)$ that maximizes the revenue to $\mathbf{x}^{y,r}_{\text{next}}$ and the corresponding revenue to $M(y,r)$. At the end, the subroutine computes the optimal pair $(y^*,r^*)$ as follows:

$$(y^*,r^*) = \left\{ y \in Y, r \in R \text{ s.t. } \frac{r}{y} \geq \lambda \land M(y,r) \geq M(y',r'), \forall y' \in Y, \forall r' \in R \right\},$$

and the corresponding set of bids $\mathbf{x}^{y^*,r^*}$, containing one bid for each subcampaign. We can state the following:

**Theorem 2 (Optimality).** The $\text{Opt}(\mu, \lambda)$ subroutine returns the optimal solution to the problem in Equations (1a)–(1c) when $\pi_j(x) = \overline{w}_j(x) = v_j n_j(x)$ and
\[ \tau_j(x) = c_j(x) \text{ for each } j \in \{1, \ldots, N\} \text{ and the values of revenues and costs are in } R \text{ and } Y, \text{ respectively.} \]

The asymptotic running time of the Opt procedure is \( \Theta \left( \sum_{j=1}^{N} |X_j| |Y|^2 |R|^2 \right) \), where \( |X_j| \) is the cardinality of the set of bids \( X_j \), since it cycles over all the subcampaigns and, for each one of them, finds the maximum bids and compute the values in the matrix \( S(y, r) \). Moreover, the asymptotic space complexity of the Opt procedure is \( \Theta \left( \max_{j=\{1, \ldots, N\}} |X_j| |Y| |R| \right) \) since it stores the values in the matrix \( S(y, r) \) and finds the maximum over the possible bids \( x \in X_j \).

5. Estimation Subroutine

Initially, we focus on the nature of our learning problem, and we show that no online learning algorithm can provide a sublinear pseudo-regret while guaranteeing safety.

**Theorem 3** (Pseudo-regret/safety tradeoff). For every \( \epsilon > 0 \) and time horizon \( T \), there is no algorithm with pseudo-regret smaller than \( (1/2 - \epsilon)T \) that violates (in expectation) the constraints less than \( (1/2 - \epsilon)T \) times.

This impossibility result is crucial in practice, showing that no online learning algorithm can theoretically guarantee both a sublinear regret and a sublinear number of violations of the constraints. Therefore, in real-world applications, advertisers have necessarily to accept a tradeoff between the two requirements.

The following sections describe three estimation subroutines, each providing theoretical guarantees for a different, relaxed version of the optimization problem. More precisely, in Section 5.1 we relax the safety requirement and provide an algorithm, namely \( \text{GCB} \), guaranteeing a sublinear regret. In Section 5.2 we relax the no-regret requirement and provide an algorithm, namely \( \text{GCB}_{\text{safe}} \), guaranteeing safety. In Section 5.3 we accept a fixed tolerance \( (\psi, \phi) \) in the safety requirements and provide an algorithm, namely \( \text{GCB}_{\text{safe}}(\psi, \phi) \), guaranteeing both a sublinear regret and a sublinear number of violations of the constraints.
5.1. Guaranteeing Sublinear Pseudo-regret: GCB

Accabi et al. \cite{28} propose the GCB algorithm to face general combinatorial bandit problems where the arms are partitioned in subsets and the payoffs of the arms belonging to the same subset are modeled with a Gaussian Process (GP). To obtain theoretical sublinear guarantees on the regret for our online learning problem, we use a specific definition of $\mu$ vector, making Algorithm 1 be an extension of GCB to the case in which the payoffs and constraints are functions whose parameters are modeled by multiple independent GPs. With a slight abuse of terminology, we refer to this extension as GCB.

The GCB algorithm relies on the idea to build the $\mu$ vector so that the parameters corresponding to the reward are statistical upper bounds to the expected values of the random variables, and those corresponding to the costs are statistical lower bounds. The rationale is that this choice satisfies the optimism vs. uncertainty principle. Formally, we have:

$$w_j(x) = \nu_j := \hat{n}_{j,t-1}(x) + \sqrt{b_{t-1} \sigma^2_{j,t-1}(x)}, \quad (4)$$

$$c_j(x) := \hat{c}_{j,t-1}(x) - \sqrt{b_{t-1} \sigma^2_{j,t-1}(x)}, \quad (5)$$

where $b_t := 2 \ln \left( \frac{\pi^2 N Q T^2}{3d} \right)$ is an uncertainty term used to guarantee the confidence level required by GCB. \footnote{GCB algorithm was presented at EWRL 2018 without any archival version of the paper.}

In what follows we bound the GCB pseudo-regret in terms of the maximum information gain $\gamma_{j,t}$ of the GP modeling the number of clicks of subcampaign $C_j$ at round $t$, formally defined as:

$$\gamma_{j,t} := \frac{1}{2} \max_{(x_{j,1}, \ldots, x_{j,t}), x_{j,h} \in X_j} \left| I_t + \frac{\Phi(x_{j,1}, \ldots, x_{j,t})}{\sigma^2} \right|,$$

where $I_t$ is the identity matrix of order $t$, $\Phi(x_{j,1}, \ldots, x_{j,t})$ is the Gram matrix of

\footnote{For the sake of simplicity, we assume that the values of the bounds correspond to values in $R$ and $Y$, respectively. If the bound values for $w_j(x)$ are not in the set $R$, we need to round them up to the nearest value belonging to $R$. Instead, if $c_j(x)$ are not in the set $Y$, a rounding down should be performed to the nearest value in $Y.$}
the GP computed on the vector \((x_{j,1}, \ldots, x_{j,t})\), and \(\sigma \in \mathbb{R}^+\) is the noise standard deviation. Thanks to this definition, we can state the following.

**Theorem 4 (GCB pseudo-regret).** Given \(\delta \in (0, 1)\), GCB applied to the problem in Equations (1a)–(1c), with probability at least 1 – \(\delta\), suffers from a pseudo-regret of:

\[
R_T(GCB) \leq \sqrt{\frac{8Tv_{\max}^2N^3bT}{\ln(1 + \sigma^2)} \sum_{j=1}^{N} \gamma_{j,T}},
\]

where \(b_T := 2 \ln \left(\frac{\pi^2NQT^2}{36}\right)\) is an uncertainty term used to guarantee the confidence level required by GCB, \(v_{\max} := \max_{j \in \{1, \ldots, N\}} v_j\) is the maximum value per click over all subcampaigns, and \(Q := \max_{j \in \{1, \ldots, N\}} |X_j|\) is the maximum number of bids in a subcampaign.

We remark that the upper bound provided in the above theorem is expressed in terms of the maximum information gain \(\gamma_{j,T}\) of the GPs over the number of clicks. The problem of bounding \(\gamma_{j,T}\) for a generic GP has been already addressed by [29], where the authors present the bounds for the squared exponential kernel \(\gamma_{j,T} = O((\ln T)^2)\) for 1-dimensional GPs. Notice that, thanks to the previous result, the GCB algorithm using squared exponential kernels suffers from a sublinear pseudo-regret since the terms \(\gamma_{j,T}\) is bounded by \(O((\ln T)^2)\), and the bound in Theorems 4 is \(O(N^{3/2}(\ln T)^{5/2})\sqrt{T})\).

On the other hand, the GCB algorithm violates (in expectation) the constraints a linear number of times in \(T\) as stated by the following theorem.

**Theorem 5 (GCB safety).** Given \(\delta \in (0, 1)\), GCB applied to the problem in Equations (1a)–(1c) is \(\eta\)-safe where \(\eta \geq T - \frac{\delta}{2NQT}\) and, therefore, the number of constraints violations is linear in \(T\).

### 5.2. Guaranteeing Safety: GCB\textsubscript{safe}

We propose GCB\textsubscript{safe}, a variant of GCB relying on different values to be used in the vector \(\mu\). More specifically, we employ optimistic estimates for the parameters used in the objective function and pessimistic estimates for the
parameters used in the constraints. Formally, in \( \text{GCB}_{\text{safe}} \), we set:

\[
\begin{align*}
\overline{w}_j(x) &:= v_j \left[ \hat{n}_{j,t-1}(x) + \sqrt{b_{t-1}\hat{s}_{j,t-1}^n(x)} \right], \\
\underline{w}_j(x) &:= v_j \left[ \hat{n}_{j,t-1}(x) - \sqrt{b_{t-1}\hat{s}_{j,t-1}^n(x)} \right], \\
\overline{c}_j(x) &:= \hat{c}_{j,t-1}(x) + \sqrt{b_{t-1}\hat{s}_{j,t-1}^c(x)},
\end{align*}
\]

Furthermore, \( \text{GCB}_{\text{safe}} \) needs a default set of bids \( \{x_{d,j,t}\}_{j=1}^{N} \), that is known a priori to be feasible for the problem in Equations (1a)-(1c) with the actual values of the parameters.\(^5\) The pseudo-code of \( \text{GCB}_{\text{safe}} \) is provided in Algorithm [1] with the above definition of the parameters of vector \( \mu \), except that it returns \( \{\hat{x}_{j,t}\}_{j=1}^{N} = \{x_{d,j,t}\}_{j=1}^{N} \) if the optimization problem does not admit any feasible solution with the current estimates. We can show the following.

Theorem 6 (\( \text{GCB}_{\text{safe}} \) safety). Given \( \delta \in (0, 1) \), \( \text{GCB}_{\text{safe}} \) applied to the problem in Equations (1a)-(1c) is \( \delta \)-safe and, therefore, the number of constraints violations is constant in \( T \).

The safety property comes at the cost that \( \text{GCB}_{\text{safe}} \) may suffer from a much larger pseudo-regret than \( \text{GCB} \) as stated by the following theorem.

Theorem 7 (\( \text{GCB}_{\text{safe}} \) pseudo-regret). Given \( \delta \in (0, 1) \), \( \text{GCB}_{\text{safe}} \) applied to the problem in Equations (1a)-(1c) suffers from a pseudo-regret \( R_t(\text{GCB}_{\text{safe}}) = \Theta(T) \).

5.3. Guaranteeing Sublinear Pseudo-regret and Safety with Tolerance: \( \text{GCB}_{\text{safe}}(\psi, \phi) \)

In what follows, we show that, when a tolerance in the violation of the constraints is accepted, an adaptation of \( \text{GCB}_{\text{safe}} \) can be exploited to obtain a sublinear pseudo-regret. Given an instance of the problem in Equations (1a)-(1c) that we call original problem, we build an auxiliary problem in which we slightly relax the ROI and budget constraints. Formally, the \( \text{GCB}_{\text{safe}}(\psi, \phi) \) is the \( \text{GCB}_{\text{safe}} \) applied to the auxiliary problem in which the parameters \( \lambda \) and

\(^5\)A trivial default feasible bid allocation is \( \{x_{d,j,t} = 0\}_{j=1}^{N} \).
β have been substituted with \( \lambda - \psi \) and \( \beta + \phi \), respectively. Thanks to the results provided in Section 5.2, \( GCB_{\text{safe}}(\psi, \phi) \), w.h.p., does not violate the ROI constraint of the original problem by more than the tolerance \( \psi \) and the budget constraint of the original problem by more than the tolerance \( \phi \). In the following, we distinguish three cases depending on the \textit{a priori} information available to the advertisers. Indeed, the advertisers may know that a constraint is not active at the optimal solution of the given instance thanks to the observation of old data. In these cases, we just need a milder relaxation of the problem than in the general case in which the advertisers has no \textit{a priori} information. At first, we focus on the case in which we \textit{a priori} know that the budget constraint is not active at the optimal solution. We show the following:

\textbf{Theorem 8} \( (GCB_{\text{safe}}(\psi, 0) \) pseudo-regret and safety with tolerance). When:

\[
\psi \geq 2 \frac{\beta_{\text{opt}} + n_{\text{max}}}{\beta_{\text{opt}}^2} \sum_{j=1}^{N} v_j \sqrt{2 \ln \left( \frac{\pi^2 NQT^3}{3\delta'} \right) \sigma}
\]

and

\[
\beta_{\text{opt}} < \beta \frac{\sum_{j=1}^{N} v_j}{\beta_{\text{opt}} + n_{\text{max}}} + \sum_{j=1}^{N} v_j
\]

where \( \delta' \leq \delta \), \( \beta_{\text{opt}} \) is the spend at the optimal solution of the original problem, and \( n_{\text{max}} := \max_{j,x} n_j(x) \) is the maximum over the sub-campaigns and the admissible bids of the expected number of clicks, \( GCB_{\text{safe}}(\psi, 0) \) provides a pseudo-regret w.r.t. the optimal solution to the original problem of \( O \left( \sqrt{\frac{T}{\sum_{j=1}^{N} \gamma_{j,T}}} \right) \) with probability at least \( 1 - \delta - \frac{\delta'}{NQT} \), while being \( \delta \)-safe w.r.t. the constraints of the auxiliary problem.

The above result states that, if we allow a violation of at most \( \psi \) of the ROI constraint, the result provided in Theorem 1 can be circumvented for a class of instances of the optimization problem. In this case, \( GCB_{\text{safe}}(\psi, 0) \) guarantees sublinear regret and a number of constraints violations that is constant in \( T \).

Notice that the magnitude of the violation \( \psi \) increases linearly in the maximum number of clicks \( n_{\text{max}} \) and \( \sum_{j=1}^{N} v_j \), that, in its turn, increases linearly in the number of sub-campaigns \( N \). This suggests that in large instances this value
may be large. However, in practice, the maximum number of clicks of a sub-campaign $n_{\text{max}}$ is a sublinear function in the optimal budget $\beta_{opt}$, and usually it goes to a constant as the budget spent goes to infinity. Moreover, the number of sub-campaigns $N$ usually depends on the budget, i.e., the budget planned by the business units is linear in the number of sub-campaigns. As a result, $\beta_{opt}$ is of the same order of $\sum_{j=1}^{N} v_j$, and therefore, since $n_{\text{max}}$ is sublinear in $\beta_{opt}$ and $\sum_{j=1}^{N} v_j$ is of the order of $\beta_{opt}$, the final expression of $\psi$ is sub-linear in $\beta_{opt}$. This means that the lower bound to $\psi$ to satisfy the assumption needed by Theorem 8 goes to zero as $\beta_{opt}$ increases.

We can derive a similar result in the case in which we a priori know that the ROI constraint is not active at the optimal solution. In particular, we state the following.

**Theorem 9** (GCB_{safe}$(0, \phi)$ pseudo-regret and safety with tolerance). When

$$\phi \geq 2N \sqrt{2 \ln \left( \frac{\pi^2 N^2 T^3}{3\delta'} \right)}$$

and

$$\lambda_{opt} > \lambda + \frac{(\beta + n_{\text{max}}) \phi \sum_{j=1}^{N} v_j}{N \beta^2},$$

where $\delta' \leq \delta$, and $n_{\text{max}} := \max_{j,x} n_j(x)$ is maximum expected number of clicks, $\text{GCB}_{\text{safe}}(0, \phi)$ provides a pseudo-regret w.r.t. the optimal solution to the original problem of $O \left( \sqrt{T \sum_{j=1}^{N} \gamma_{j,T}} \right)$ with probability at least $1 - \delta - \frac{6\delta'}{\pi^2 T^2}$, while being $\delta$-safe w.r.t. the constraints of the auxiliary problem.

On the satisfaction of the assumption needed by the above theorem, we can produce a consideration similar to that done before for the case in which the budget constraint is not active at the optimal solution. As previously, the lower bound to $\phi$ to satisfy the assumption needed by Theorem 9 goes to zero as $\beta_{opt}$ increases.

Finally, we focus on the case in which the advertiser has no information on which constrain is active. In this case, we can state the following which generalizes the two results provided above.
Theorem 10 (GCB\textsubscript{safe}(\psi, \phi) pseudo-regret and safety with tolerance). Setting

$$\psi = 2 \beta_{\text{opt}} + n_{\text{max}} \sum_{j=1}^{\mathcal{N}} v_j \sqrt{2 \ln \left( \frac{\pi^2 Q T^3}{3 \delta'} \right)} \sigma$$

and

$$\phi = 2 N \sqrt{2 \ln \left( \frac{\pi^2 Q T^3}{3 \delta'} \right)} \sigma,$$

where $\delta' \leq \delta$, GCB\textsubscript{safe}(\psi, \phi) provides a pseudo-regret w.r.t. the optimal solution to the original problem of $O \left( \sqrt{T \sum_{j=1}^{\mathcal{N}} \gamma_{j,T}} \right)$ with probability at least $1 - \delta - \frac{\delta'}{Q T^2}$, while being $\delta$-safe w.r.t. the constraints of the auxiliary problem.

6. Experimental Evaluation

We experimentally evaluate our algorithms in terms of pseudo-regret and safety in synthetic settings generated from real-world data. The adoption of synthetic settings allows us to evaluate our algorithms in many different realistic scenarios and, for each of them, to find the optimal clairvoyant solution necessary to measure the algorithms’ regret and safety. The real-world dataset is provided by AdsHotel (https://www.adshotel.com/), an Italian media agency working in the hotel booking market.

Our experimental activity is structured as follows. In Section 6.1, we evaluate how GCB and GCB\textsubscript{safe} violate the constraints. In Section 6.2, we evaluate how the performances of GCB\textsubscript{safe}(\psi, \phi) vary as $\psi$ varies when the budget constraint is known to be active. In Section 6.3, we evaluate the performances of all the algorithms when the ROI constraint is known to be active. Finally, in Section 6.4, we run our algorithms with multiple, heterogeneous settings and evaluate the average performances.

Additional details useful for the complete reproducibility of our results are provided in Appendix B.1, while the code used in our experimental activity is available at: https://github.com/oi-tech/safe_bid_opt.
6.1. Experiment #1: evaluating constraint violation with GCB and GCBsafe

In this experiment, we show that GCB suffers from significant violations of both ROI and budget constraints even in simple settings, while GCBsafe does not.

Setting. We simulate $N = 5$ subcampaigns, with $|X_j| = 201$ bid values evenly spaced in $[0, 2]$, $|Y| = 101$ cost values evenly spaced in $[0, 100]$, and $|R| = 151$ revenue values evenly spaced in $[0, 1200]$. For a generic subcampaign $C_j$, at every $t$, the daily number of clicks is returned by function $n_j(x) := \theta_j(1 - e^{-x/\delta_j}) + \xi_j^n$ and the daily cost by function $c_j(x) = \alpha_j(1 - e^{-x/\gamma_j}) + \xi_j^c$, where $\theta_j \in \mathbb{R}^+$ and $\alpha_j \in \mathbb{R}^+$ represent the maximum achievable number of clicks and cost for subcampaign $C_j$ in a single day, $\delta_j \in \mathbb{R}^+$ and $\gamma_j \in \mathbb{R}^+$ characterize how fast the two functions reach a saturation point, and $\xi_j^n$ and $\xi_j^c$ are noise terms drawn from a $N(0, 1)$ Gaussian distribution (these functions are customarily used in the advertising literature, e.g., by Kong et al. [30]). We assume a unitary value for each click, i.e., $v_j = 1$ for each $j \in \{1, \ldots, N\}$. The values of the parameters of cost and revenue functions of the subcampaigns are specified in Table B.2 reported in Appendix B.2. We set a daily budget $\beta = 100$, $\lambda = 10$ in the ROI constraint, and a time horizon $T = 60$. The peculiarity of this setting is that, at the optimal solution, the budget constraint is active, while the ROI constraint is not.

For both GCB and GCBsafe, the kernels for the number of clicks GPs $k(x, x')$ and for the costs GPs $h_j(x, x')$ are squared exponential kernels of the form $\sigma_j^2 \exp \left\{ -\frac{(x-x')^2}{l} \right\}$ for every $x, x' \in X_j$, where the parameters $\sigma_j \in \mathbb{R}^+$ and $l \in \mathbb{R}^+$ are estimated from data, as suggested by Rasmussen and Williams [10]. The confidence for the algorithms is $\delta = 0.2$.

Results. We evaluate the algorithms in terms of:

- daily revenue: $P_t(\Omega) := \sum_{j=1}^{N} v_j n_j(\hat{x}_{j,t})$;
- daily ROI: $ROI_t(\Omega) := \frac{\sum_{j=1}^{N} v_j n_j(\hat{x}_{j,t})}{\sum_{j=1}^{N} c_j(\hat{x}_{j,t})}$.
Figure 1: Results of Experiment #1: daily revenue (a), ROI (b), and spend (c) obtained by GCB and GCBsafe. Dashed lines correspond to the optimal values for the revenue and ROI, while dash-dotted lines correspond to the values of the ROI and budget constraints.

- daily spend: $S_t(\Omega) := \sum_{j=1}^{N} c_j(\hat{x}_{j,t})$.

We perform 100 independent runs for each algorithm.
the budget constraint over the entire time horizon and the ROI constraint in the first 7 days in more than 50\% of the runs. This happens because, in the optimal solution, the ROI constraint is not active, while the budget constraint is. Conversely, $\text{GCB}_{\text{safe}}$ satisfies the budget and ROI constraints over the time horizon for more than 90\% of the runs, and has a slower convergence to the optimal revenue. If we focus on the median revenue, $\text{GCB}_{\text{safe}}$ has a similar behaviour to that of $\text{GCB}$ for $t > 15$. This makes $\text{GCB}_{\text{safe}}$ a good choice even in terms of overall revenue. However, it is worth to notice that, in the 10\% of the runs, $\text{GCB}_{\text{safe}}$ does not converge to the optimal solution before the end of the learning period. These results confirm our theoretical analysis showing that limiting the exploration to safe regions might lead the algorithm to get a large regret. Furthermore, let us remark that the learning dynamics of $\text{GCB}_{\text{safe}}$ are much smoother than those of $\text{GCB}$, which present, instead, oscillations.

6.2. Experiment #2: evaluating $\text{GCB}_{\text{safe}}(\psi,0)$ when the budget constraint is active

In real-world scenarios, the business goals in terms of volumes-profitability tradeoff are often blurred, and sometimes it can be desirable to slightly violate the constraints (usually, the ROI constraint) in favor of a significant volume increase. However, analyzing and acquiring information about these tradeoff curves requires exploring volumes opportunities by relaxing the constraints. In this experiment, we show how our approach can be adjusted to address this problem in practice.

Setting. We use the same setting of Experiment #1, except that we evaluate $\text{GCB}_{\text{safe}}$ and $\text{GCB}_{\text{safe}}(\psi,\phi)$ algorithms. More precisely, we relax the ROI constraint by a tolerance $\psi \in \{0, 0.05, 0.1, 0.15\}$ (while keeping $\phi = 0$). Notice that $\text{GCB}_{\text{safe}}(0,0)$ corresponds to the use of $\text{GCB}_{\text{safe}}$ in the original problem. As a result, except for the case $\phi = 0$, we allow $\text{GCB}_{\text{safe}}(\psi,\phi)$ to violate the ROI constraint, but, with high probability, the violation is bounded by at most 0.5\%, 1\%, 1.5\% of $\lambda$, respectively. Instead, we do not introduce any tolerance for the daily budget constraint $\beta$. 

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Results. In Figures 2, we show the median values, on 100 independent runs, of the performance in terms of daily revenue, ROI, and spend of GCBsafe(ψ, 0) for every value of ψ. The 10% and 90% quantiles are reported in Figure B.4, B.5, B.6, and B.7 in Appendix B.3. The results show that allowing a small tolerance in the ROI constraint violation, we can improve the exploration and, therefore, lead to faster convergence. We note that if we set a value of ψ ≥ 0.05, we achieve significantly better performance in the first learning steps (t < 20) still maintaining a robust behavior in terms of constraints violation. Most importantly, the ROI constraint is always satisfied by the median and also by
the 10% and 90% quantiles. Furthermore, the few violations are concentrated in the early stages of the learning process.

6.3. Experiment #3: comparing GCB, GCB\textsubscript{safe}, and GCB\textsubscript{safe}(\psi, 0) when the ROI constraint is active

We study a setting in which the ROI constraint is active at the optimal solution, \( i.e., \lambda = \lambda_{\text{opt}} \), while the budget constraint is not. This means that, at the optimal solution, the advertiser would have an extra budget to spend. However, such budget is not spent, the ROI constraint would be violated otherwise.
Setting. The experimental setting is the same of Experiment #1, except that we set the budget constraint as $\beta = 300$. The optimal daily spend is $\beta_{opt} = 161$.

Results. In Figure 3, we show the median values of the daily revenue, the ROI, and the spend of GCB, GCB$_{safe}$, GCB$_{safe}(0.05, 0)$ obtained with 100 independent runs. The 10% and 90% of the quantities provided by GCB, GCB$_{safe}$, and GCB$_{safe}(0.05, 0)$ are reported in Figures B.8, B.9, and B.10 in Appendix B.4.

We notice that, even in this setting, GCB violates the ROI constraint for the entire time horizon, and the budget constraint in $t = 6$ and $t = 7$. However, it achieves a revenue larger than that of the optimal constrained solution. On the other side, GCB$_{safe}$ and always satisfies both the constraints, but it does not perform enough exploration to quickly converge to the optimal solution. We observe that it is sufficient to allow a tolerance in the ROI constraint violation by slightly perturbing the input value $\lambda (\psi = 0.05$, corresponding to a violation of the constraint by at most 0.5%) to make GCB$_{safe}(\psi, \phi)$ capable of approaching the optimal solution while satisfying both constraints for every $t \in \{0, \ldots, T\}$.

This suggests that, in real-world applications, GCB$_{safe}(\psi, \phi)$ with a small tolerance represents an effective solution, providing guarantees on the violation of the constraints while returning high values of revenue.

6.4. Experiment #4: comparing GCB, GCB$_{safe}$, and GCB$_{safe}(\psi, \phi)$ with multiple, heterogeneous settings

In this experiment we extend the experimental activity we conduct in Experiments #1 and #3 to other multiple, heterogeneous settings.

Setting. We simulate $N = 5$ subcampaigns with a daily budget $\beta = 100$, with $|X_j| = 201$ bid values evenly spaced in $[0, 2]$, $|Y| = 101$ cost values evenly spaced in $[0, 100]$, being the daily budget $\beta = 100$, and $|R|$ evenly spaced revenue values depending on the setting. We generate 10 scenarios that differ in the parameters defining the cost and revenue functions, and in the ROI parameter $\lambda$. Recall that the number-of-click functions coincide with the revenue functions since $v_j = 1$ for each $j \in \{1, \ldots, N\}$. Parameters $\alpha_j \in \mathbb{N}^+$ and $\theta_j \in \mathbb{N}^+$ are sampled.
from discrete uniform distributions $\mathcal{U}\{50, 100\}$ and $\mathcal{U}\{400, 700\}$, respectively. Parameters $\gamma_j$ and $\delta_j$ are sampled from the continuous uniform distributions $\mathcal{U}\{0.2, 1.1\}$. Finally, parameters $\lambda$ are chosen such that the ROI constraint is active at the optimal solution. Table B.3 in Appendix B.5 specifies the values of such parameters.

**Results.** We compare the GCB, GCB$_{safe}$, GCB$_{safe}(0.05, 0)$, and GCB$_{safe}(0.10, 0)$ algorithms in terms of:

- $W_t := \sum_{h=1}^{t} P_t(\Omega)$: average (over 100 runs) cumulative revenue at round $t$ (and the corresponding standard deviation $\sigma_t$);
- $M_t$: median (over 100 runs) of the cumulative revenue at round $t$;
- $U_t$: 90-th percentile (over 100 runs) of the cumulative revenue at round $t$;
- $L_t$: 10-th percentile (over 100 runs) of the cumulative revenue at round $t$;
- $V_{ROI}$: the fraction of days in which the ROI constraint is violated;
- $V_B$: the fraction of days in which the budget constraint is violated.

Table 1 reports the algorithms performances at $\lceil T/2 \rceil = 28$ and at the end of the learning process $t = T = 57$. As already observed in the previous experiments, GCB violates the ROI constraint at every round, run, and setting. More surprisingly, GCB violates the budget constraint most of the time (60% on average) even if that constraint is not active at the optimal solution. Interestingly, GCB$_{safe}(\psi, 0)$ never violates the budget constraints (for every $\psi$). As expected, the violation of the ROI constraint is close to zero with GCB$_{safe}$, while it increases as $\psi$ increases. In terms of average cumulative revenue, at $T$, we observe that GCB$_{safe}$ gets about 56% of the revenue provided by GCB, while the ratio related to GCB$_{safe}(0.05, 0)$ is about 66% and that related to GCB$_{safe}(0.10, 0)$ is about 78%. At $T/2$, we the ratios are about 52% for GCB, 61% for GCB$_{safe}(0.05, 0)$, and 73% for GCB$_{safe}(0.10, 0)$, showing that those ratios increase as $T$ increases. The rationale is that in the early stages of the
Table 1: Results of Experiment #4.

| Setting #2 | Setting #3 | Setting #4 | Setting #5 | Setting #6 | Setting #7 | Setting #8 | Setting #9 | Setting #10 |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| GCB        | GCB        | GCB        | GCB        | GCB        | GCB        | GCB        | GCB        | GCB        |
| \(W_T\)    | \(W_{T/2}\) | \(\sigma_T\) | \(\sigma_{T/2}\) | \(M_T\)    | \(M_{T/2}\) | \(U_T\)    | \(U_{T/2}\) | \(L_T\)    | \(L_{T/2}\) | \(V_{10\%}\) | \(V_0\) |
| GCB        | 57481      | 30706      | 556        | 476        | 57497      | 30811      | 58084      | 31239      | 56758      | 32828      | 1.00    | 0.62 |
| GCB        | 44419      | 21549      | 4766       | 2474       | 45348      | 21972      | 46783      | 23163      | 42287      | 28324      | 0.02    | 0.00 |
| GCB        | 30539      | 23524      | 4902       | 2487       | 48620      | 23831      | 50388      | 24827      | 46307      | 23506      | 0.21    | 0.00 |
| GCB        | 62327      | 25659      | 829        | 611        | 52388      | 25887      | 53324      | 26605      | 51316      | 25104      | 0.94    | 0.00 |
| GCB        | 63964      | 35566      | 1049       | 679        | 63701      | 35573      | 64984      | 36524      | 62229      | 34675      | 1.04    | 0.14 |
| GCB        | 34075      | 16290      | 8541       | 4448       | 37928      | 17647      | 39594      | 19473      | 27748      | 11141      | 0.03    | 0.00 |
| GCB        | 90626      | 19564      | 6013       | 3122       | 41233      | 20352      | 44468      | 21698      | 38640      | 17645      | 0.04    | 0.00 |
| GCB        | 66904      | 23999      | 6382       | 3112       | 43749      | 22433      | 51564      | 24776      | 44099      | 19929      | 0.72    | 0.00 |
| GCB        | 71404      | 37383      | 351        | 262        | 71399      | 37878      | 71877      | 37732      | 70930      | 37021      | 0.98    | 0.98 |
| GCB        | 29101      | 13417      | 7052       | 3646       | 32032      | 14680      | 35062      | 17256      | 20509      | 19562      | 0.00    | 0.00 |
| GCB        | 19610      | 18270      | 3932       | 1945       | 38296      | 17994      | 53375      | 21992      | 30415      | 12678      | 0.07    | 0.00 |
| GCB        | 21814      | 19993      | 9653       | 4856       | 58540      | 21574      | 47659      | 20118      | 36570      | 14450      | 0.75    | 0.00 |
| GCB        | 13430      | 37575      | 327        | 260        | 67130      | 35795      | 67536      | 36111      | 66726      | 35424      | 0.98    | 0.98 |
| GCB        | 14448      | 7707       | 6006       | 3095       | 15019      | 3075      | 18581      | 3908      | 6781      | 3926      | 0.02    | 0.00 |
| GCB        | 14908      | 7710       | 6174       | 2974       | 15161      | 8157      | 19548      | 10351      | 7954      | 3860      | 0.02    | 0.00 |
| GCB        | 34716      | 15597      | 16131      | 1720       | 37499      | 14601      | 55236      | 25066      | 9895      | 5188      | 0.19    | 0.00 |
| GCB        | 63038      | 35380      | 873        | 401        | 63088      | 5867      | 64226      | 35795      | 61754      | 34825      | 1.00    | 0.41 |
| GCB        | 41062      | 14806      | 5651       | 3890       | 33009      | 15570      | 35004      | 16922      | 28296      | 11338      | 0.04    | 0.00 |
| GCB        | 37744      | 17006      | 4173       | 2619       | 38321      | 18161      | 41184      | 19905      | 33914      | 15276      | 0.03    | 0.00 |
| GCB        | 42528      | 20464      | 7497       | 4824       | 42765      | 20683      | 47187      | 22301      | 38988      | 18314      | 0.70    | 0.00 |
| GCB        | 70280      | 37363      | 672        | 347        | 70275      | 37352      | 71123      | 37811      | 69379      | 36942      | 1.00    | 0.34 |
| GCB        | 60116      | 18895      | 5524       | 3047      | 40673      | 19357      | 43859      | 21161      | 37310      | 21722      | 0.03    | 0.00 |
| GCB        | 51338      | 23683      | 3110       | 2036       | 50984      | 23375      | 54545      | 26174      | 47405      | 21385      | 0.03    | 0.00 |
| GCB        | 63574      | 29675      | 818        | 3423       | 64011      | 30112      | 66658      | 32559      | 69970      | 27280      | 0.80    | 0.00 |
| GCB        | 80570      | 41073      | 435        | 444       | 80568      | 42019      | 81127      | 42388      | 80823      | 41496      | 1.00    | 0.98 |
| GCB        | 38685      | 28785      | 3097       | 1465       | 60033      | 28817      | 62353      | 30535      | 54590      | 26931      | 0.02    | 0.00 |
| GCB        | 63865      | 31004      | 3787       | 1876       | 62573      | 31550      | 67364      | 33105      | 57860      | 28349      | 0.02    | 0.00 |
| GCB        | 68480      | 33358      | 4224       | 2181       | 70888      | 33998      | 72739      | 35838      | 61971      | 30317      | 0.65    | 0.00 |
learning process, safe algorithms learn more slowly than non-safe algorithms. Similar performances can be observed when focusing on the other indices. Summarily, the above results show that our algorithms provide advertisers with a wide spectrum of effective tools to address the revenue/safety tradeoff. A small value of $\psi$ (and $\phi$) represents a good tradeoff. By the way, the choice of the specific configuration to adopt in practice depends on the advertiser’s aversion to the violation of the constraints.

### 7. Conclusions and Future Works

In this paper, we propose a novel framework for Internet advertising campaigns. While previous works available in the literature focus only on the maximization of the revenue provided by the campaign, we introduce the concept of safety for the algorithms choosing the bid allocation each day. More specifically, we aim that the bidding satisfies, with high probability, some daily ROI and budget constraints fixed by the business units of the companies. The constraints are subject to uncertainty, as their parameters are not a priori known (some platforms do not allow the bidders to set daily budget constraints, while no platform allows the bidders to set daily constraints on ROI). Our goal is to maximize the revenue satisfying w.h.p. the uncertain constraints (a.k.a. safety).

We model this setting as a combinatorial optimization problem, and we prove that such a problem is inapproximable within any strictly positive factor, unless P = NP, but it admits an exact pseudo-polynomial-time algorithm. Most interestingly, we prove that no online learning algorithm can provide sublinear pseudo-regret while guaranteeing a sublinear number of violations of the uncertain constraints. We show that the GCB algorithm suffers from a sublinear pseudo-regret, but it may violate the constraints a linear number of times. Thus, we design $\text{GCB}_{\text{safe}}$, a novel algorithm that guarantees safety at the cost of a linear pseudo-regret. Remarkably, a simple adaptation of $\text{GCB}_{\text{safe}}$, namely $\text{GCB}_{\text{safe}}(\psi, \phi)$, guarantees a sublinear pseudo-regret and safety at the cost of tolerances $\psi$ and $\phi$ on the ROI and budget constraints, respectively. Finally, we
evaluate the empirical performance of our algorithms with synthetically advertising problems generated from real-world data. These experiments show that \( \text{GCB}_{\text{safe}}(\psi, \phi) \) provides good performance in terms of safety while suffering from a smaller cumulative revenue w.r.t. \( \text{GCB} \).

An interesting open research direction is the design of an algorithm which adopts constraints changing during the learning process, so as to identify the active constraint and relax those that are not active. Moreover, understanding the relationship between the relaxation of one of the constraints and the increase of the revenue constitutes an interesting line of research. We are also interested in extending our algorithms to more general problems with constraints subject to uncertainty beyond advertising.

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Appendix A. Omitted Proofs

Appendix A.1. Proofs Omitted from Section 4

Theorem 1 (Inapproximability). For any \( \rho \in (0, 1] \), there is no polynomial-time algorithm returning a \( \rho \)-approximation to the problem in Equations (13)–(16), unless \( P = NP \).

Proof. We restrict to the instances of SUBSET-SUM such that \( z \leq \sum_{i \in S} u_i \). Solving these instances is trivially NP-hard, as any instance with \( z > \sum_{i \in S} u_i \) is not satisfiable, and we can decide it in polynomial time. Given an instance of SUBSET-SUM, let \( \ell = \frac{\sum_{i \in S} u_i + 1}{\rho} \). Let us notice that, the lower the degree of approximation we aim, the larger the value of \( \ell \). For instance, when studying the problem of computing an exact solution, we set \( \rho = 1 \) and therefore \( \ell = \sum_{i \in S} u_i + 1 \), whereas, when we require a 1/2-approximation, we set \( \rho = 1/2 \) and therefore \( \ell = 2(\sum_{i \in S} u_i + 1) \). We have \( |S| + 1 \) subcampaigns, each denoted with \( C_j \). The available bid values belong to \{0, 1\} for every subcampaign \( C_j \).

The parameters of the subcampaigns are set as follows.

- Subcampaign \( C_0 \): we set \( v_0 = 1 \), and

\[
c_0(x) = \begin{cases} 2\ell + z & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}, \quad n_0(x) = \begin{cases} \ell & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}.
\]

- Subcampaign \( C_j \) for every \( j \in S \): we set \( v_j = 1 \), and

\[
c_j(x) = \begin{cases} u_i & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}, \quad n_j(x) = \begin{cases} u_i & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}.
\]
We set the daily budget $\beta = 2(z + \ell)$ and the ROI limit $\lambda = \frac{1}{2}$.

We show that, if a SUBSET-SUM instance is satisfiable, then the corresponding instance of our problem admits a solution with a revenue larger than $\ell$, while, if a SUBSET-SUM instance is not satisfiable, the maximum revenue in the corresponding instance of our problem is at most $\rho \ell - 1$. Thus, the application of any polynomial-time $\rho$-approximation algorithm to instances of our problem generated from instances of SUBSET-SUM as described above would return a solution whose value is not smaller than $\rho \ell$ when the SUBSET-SUM instance is satisfiable, and it is not larger than $\rho \ell - 1$ when the SUBSET-SUM instance is not satisfiable. As a result, whenever such an algorithm returns a solution with a value that is not smaller than $\rho \ell$, we can decide that the corresponding SUBSET-SUM instance is satisfiable. Analogously, whenever such an algorithm returns a solution with a value that is not smaller than $\rho \ell$, we can decide that the corresponding SUBSET-SUM instance is satisfiable. Hence, such an algorithm would decide in polynomial time whether or not a SUBSET-SUM instance is satisfiable, but this is not possible unless $P = NP$. Since this holds for every $\rho \in (0, 1]$, then no $\rho$-approximation to our problem is allowed in polynomial time unless $P = NP$.

If. Suppose that SUBSET-SUM is satisfied by the set $S^* \subseteq S$ and that its solution assigns $x_i = 1$ if $i \in S^*$ and $x_i = 0$ otherwise, and it assigns $x_0 = 1$. The total revenue is $\ell + z + z = 2(\ell + z)$, while ROI = $\frac{\ell + z}{2\ell + 2z} = \frac{1}{2}$.

Only if. Assume by contradiction that the instance of our problem admits

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7For the sake of clarity, the proof uses simple instances. The adoption of these instances is crucial to identify the most basic settings in which the problem is hard, and it is customarily done in the literature. Let us notice that it is possible to prove the theorem using more realistic instances. For example, we can build a reduction in which the costs are smaller than the values, i.e., $c_i(x) < n_i(x)u_i$. In particular, the reduction holds even if we set $c_0(1) = \epsilon(2\ell + z)$, $c_j(1) = ru_i$, $\beta = 2\epsilon(z + \ell)$, and $\lambda = 1/(2\epsilon)$ for an arbitrary small $\epsilon$. 

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a solution with a revenue strictly larger than $\rho \ell - 1$ and that SUBSET-SUM is not satisfiable. Then, it is easy to see that we need $x_0 = 1$ for campaign $C_0$ as the maximum achievable revenue is $\sum_{i \in S} u_i = \rho \ell - 1$ when $x_0 = 0$. Thus, since $x_0 = 1$, the budget constraint forces $\sum_{i \in S: x_i = 1} c_i(x_i) \leq z$, thus implying $\sum_{i \in S: x_i = 1} u_i \leq z$. By the satisfaction of the ROI constraint, i.e., \( \frac{\sum_{i \in S: x_i = 1} u_i + 1}{\sum_{i \in S: x_i = 1} u_i + 2 \ell + z} \geq \frac{1}{2} \), it must hold $\sum_{i \in S: x_i = 1} u_i \geq z$. Therefore, the set $S^* = \{ i \in S : x_i = 1 \}$ is a solution to SUBSET-SUM, thus reaching a contradiction. This concludes the proof. 

**Theorem 2** (Optimality). The $\text{Opt}(\mu, \lambda)$ subroutine returns the optimal solution to the problem in Equations (1a)–(1c) when $\overline{m}_j(x) = \overline{w}_j(x) = v_j n_j(x)$ and $\overline{c}_j(x) = c_j(x)$ for each $j \in \{1, \ldots, N\}$ and the values of revenues and costs are in $R$ and $Y$, respectively.

*Proof.* Since all the possible values for the revenues and costs are taken into account in the subroutine, the elements in $S(y, r)$ satisfy the two inequalities in Equation (2) with the equal sign. Therefore, all the elements in $S(y, r)$ would contribute to the computation of the final value of the ROI and budget constraints, i.e., the ones after evaluating all the $N$ subcampaigns, with the same values for revenue and costs, being their overall revenue equal to $r$ and their overall cost equal to $y$. Notice that Constraint (1c) is satisfied as long as it holds $\max(Y) = \beta$. The maximum operator in Line 11 excludes only solutions with the same costs and a lower revenue, therefore, the subroutine excludes only solutions that would never be optimal (and, for this reason, said dominated). The same reasoning holds also for the subcampaign $C_1$ analysed by the algorithm. Finally, after all the dominated allocations have been discarded, the solution is selected by Equation (3), i.e., among all the solutions satisfying the ROI constraints the one with the largest revenue is selected. \( \square \)

**Appendix A.2. Proofs Omitted from Section 5**

**Theorem 3** (Pseudo-regret/safety tradeoff). For every $\epsilon > 0$ and time horizon $T$, there is no algorithm with pseudo-regret smaller than $(1/2 - \epsilon) T$ that violates
Proof. In what follows, we provide an impossibility result for the optimization problem in Equations (1a)–(1c). For the sake of simplicity, our proof is based on the violation of (budget) Constraint (1c), but its extension to the violation of (ROI) Constraint (1b) is direct.

Initially, we show that an algorithm satisfying the two conditions of the theorem can be used to distinguish between \( N(1, 1) \) and \( N(1 + \delta, 1) \) with an arbitrarily large probability using a number of samples independent from \( \delta \). Consider two instances of the bid optimization problem defined as follows. Both instances have a single subcampaign with \( x \in \{0, 1\} \), \( c(0) = 0 \), \( r(0) = 0 \), \( r(1) = 1 \), \( \beta = 1 \), and \( \lambda = 0 \). The first instance has cost \( c^1(1) = N(1, 1) \), while the second one has cost \( c^2(1) = N(1 + \delta, 1) \). With the first instance, the algorithm must choose \( x = 1 \) at least \( T(1/2 + \epsilon) \) times in expectation, otherwise the pseudo-regret would be strictly greater than \( T(1/2 - \epsilon) \), while, with the second instance, the algorithm must choose \( x = 1 \) at most than \( T(1/2 - \epsilon) \) times in expectation, otherwise the constraint on the budget would be violated strictly more than \( T(1/2 - \epsilon) \) times. Standard concentration inequalities imply that, for each \( \gamma > 0 \), there exists a \( n(\epsilon, \gamma) \) such that, given \( n(\epsilon, \gamma) \) runs of the learning algorithm, with the first instance the algorithm plays \( x = 1 \) strictly more than \( Tn(\epsilon, \gamma)/2 \) times with probability at least \( 1 - \gamma \), while with the second instance it is played strictly less than \( Tn(\epsilon, \gamma)/2 \) times with probability at least \( 1 - \gamma \). This entails that the learning algorithm can distinguish with arbitrarily large success probability (independent of \( \delta \)) between the two instances using (at most) \( n(\epsilon, \gamma)T \) samples from one of the normal distributions.

However, the Kullback-Leibler divergence between the two normal distributions is \( KL(\mathcal{N}(1, 1), \mathcal{N}(1 + \delta, 1)) = \delta^2/2 \) and each algorithm needs at least \( \Omega(1/\delta^2) \) samples to distinguish between the two distributions with arbitrarily large probability. Since \( \delta \) can be arbitrarily small, we have a contradiction.
Thus, such an algorithm cannot exist. This concludes the proof. □

Appendix A.3. Omitted Proofs from Section 5.1

Theorem 4 (GCB pseudo-regret). Given $\delta \in (0, 1)$, GCB applied to the problem in Equations (1a)-(1c), with probability at least $1 - \delta$, suffers from a pseudo-regret of:

$$R_T(GCB) \leq \sqrt{\frac{8Tv^2_{max}N^3b_T}{\ln(1 + \sigma^2)} \sum_{j=1}^{N} \gamma_jT},$$

where $b_t := 2\ln \left( \frac{\pi^2NQT_t^2}{3\delta} \right)$ is an uncertainty term used to guarantee the confidence level required by GCB, $v_{max} := \max_{j \in \{1, \ldots, N\}} v_j$ is the maximum value per click over all subcampaigns, and $Q := \max_{j \in \{1, \ldots, N\}} |X_j|$ is the maximum number of bids in a subcampaign.

Proof. This proof extends the proof provided by Accabi et al. [28] to the case in which multiple independent GPs are present in the optimization problem.

Let us define $r_\mu(x)$ as the expected reward provided by a specific allocation $x = (x_1, \ldots, x_N)$ under the assumption that the parameter vector of the optimization problem is $\mu$. Moreover, let

$$\eta := [w_1(x_1), \ldots, w_N(x_{1|X_N}), w_1(x_1), \ldots, w_N(x_{|X_N}), -c_1(x_1), \ldots, -c_N(x_{X_N})],$$

be the vector characterizing the optimization problem in Equations (1a)-(1c), $x_t$ be the allocation chosen by the GCB algorithm at round $t$, $x^*_\eta$ the optimal allocation—i.e., the one solving the discrete version of the optimization problem in Equations (1a)-(1c) with parameter $\eta$—, and $r^*_\eta$ the corresponding expected reward.

8Notice that the theorem can be modified to hold even with instances that satisfy real-world assumptions, e.g., with costs much smaller than the budget. Indeed, we can apply the same reduction in which the costs are arbitrary, e.g., $c(0) = c(1) = q$ with an arbitrary small $q$ and $\beta = 1$, while the utilities are $r(0) = 0$, $r(1) = N(1, 1)$ or $r(1) = N(1 - \delta, 1)$, and the ROI limit is $\lambda = 1/q$. 37
To guarantee that GCB provides a sublinear pseudo-regret, we need that a few assumptions are satisfied. More specifically, we need a monotonicity property, stating that the value of the objective function increases as the values of the elements in $\mu$ increase and a Lipschitz continuity assumption between the parameter vector $\mu$ and the value returned by the objective function in Equation (1a). Formally:

**Assumption 1 (Monotonicity).** The expected reward $r_\mu(S) := \sum_{j=1}^{N} v_j n_j(x_{j,i})$, where $S$ is the bid allocation, is monotonically non decreasing in $\mu$, i.e., given $\mu, \eta$ s.t. $\mu_i \leq \eta_i$ for each $i$, we have $r_\mu(S) \leq r_\eta(S)$ for each $S$.

**Assumption 2 (Lipschitz continuity).** The expected reward $r_\mu(S)$ is Lipschitz continuous in the infinite norm w.r.t. the expected payoff vector $\mu$, with Lipschitz constant $\Lambda > 0$. Formally, for each $\mu, \eta$ we have $|r_\mu(S) - r_\eta(S)| \leq \Lambda ||\mu - \eta||_\infty$, where the infinite norm of a payoff vector is $||\mu||_\infty := \max_i |\mu_i|$.

Trivially, we have that the Lipschitz continuity holds with constant $\Lambda = N$ (number of subcampaigns). Instead, the monotonicity property holds by definition of $\mu$, as the increase of a value of $w_j(x)$ would increase the value of the objective function, and the increase of the values of $w_j(x)$ or $\pi_j(x)$ would enlarge the feasibility region of the problem, thus not excluding optimal solutions.

Let us now focus on the per-step expected regret, defined as:

$$r_{\text{reg}_t} := r^*_\eta - r_\eta(x_t).$$

Let us recall a property of the Gaussian distribution which will be useful in what follows. Be $r \sim \mathcal{N}(0, 1)$ and $c \in \mathbb{R}^+$, we have:

$$\mathbb{P}[r > c] = \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}} \int_{c}^\infty e^{-\frac{(r-c)^2}{2}} dr \leq e^{-\frac{c^2}{2}} \mathbb{P}[r > 0] = \frac{1}{2} e^{-\frac{c^2}{2}},$$

since $e^{-c(r-c)} \leq 1$ for $r \geq c$. For the symmetry of the Gaussian distribution, we have:

$$\mathbb{P}[|r| > c] \leq e^{-\frac{c^2}{2}}. \quad (A.1)$$
Let us focus on the GP modeling the number of clicks. Given a generic sequence of elements \((x_{j,1}, \ldots, x_{j,1})\), with \(x_{j,h} \in X_j\), and the corresponding sequence of number of clicks \((\hat{n}_{j,1}(x_{j,1}), \ldots, \hat{n}_{j,t}(x_{j,t}))\), we have that:

\[ n_{j,t}(x) \sim N(\hat{n}_{j,t}(x), (\hat{\sigma}_{j,t}^n(x))^2), \]

for all \(x \in X_j\). Thus, substituting \(r = \frac{\hat{n}_{j,t}(x) - n_{j,t}(x)}{\hat{\sigma}_{j,t}^n(x)}\) and \(c = \sqrt{b_t}\) in Equation (A.1), we obtain:

\[ P \left[ \left| \hat{n}_{j,t}(x) - n_{j,t}(x) \right| > \sqrt{b_t} \hat{\sigma}_{j,t}^n(x) \right] \leq e^{-\frac{b_t}{2}}. \tag{A.2} \]

Recall that, after \(n\) rounds, each arm can be chosen a number of times from 1 to \(n\). Applying the union bound over the rounds \((h \in \{1, \ldots, T\})\), the sub-campaigns \(C_j\) \((C_j\) with \(j \in \{1, \ldots, N\})\), the number of times the arms in \(C_j\) are chosen \((t \in \{1, \ldots, n\})\), and the available arms in \(C_j\) \((x \in X_j)\), and exploiting Equation (A.2), we obtain:

\[ P \left[ \bigcup_{h,j,t,x} \left( \left| \hat{n}_{j,t}(x) - n_{j,t}(x) \right| > \sqrt{b_t} \hat{\sigma}_{j,t}^n(x) \right) \right] \leq T \sum_{h=1}^{N} \sum_{j=1}^{n} \sum_{t=1}^{T} |X_j| e^{-\frac{b_t}{2}}. \tag{A.3} \]

Thus, choosing \(b_t = 2 \ln \left( \frac{\pi^2 NQT^2}{3\delta} \right)\), we obtain:

\[
\sum_{h=1}^{T} \sum_{j=1}^{N} \sum_{t=1}^{n} |X_j| e^{-\frac{b_t}{2}} \leq T \sum_{h=1}^{N} \sum_{j=1}^{n} \sum_{t=1}^{T} Q \frac{3\delta}{\pi^2 NQT^2} \sum_{n=1}^{\infty} \frac{2\delta}{\pi^2 n^2} = \frac{\delta}{2},
\]

where we used the fact that \(Q \geq |X_j|\) for each \(j \in \{1, \ldots, N\}\).

Using the same proof on the GP defined over the costs leads to:

\[ P \left[ \bigcup_{h,j,t,x} \left( \left| \hat{c}_{j,t}(x) - \hat{c}_{j,t}(x) \right| > \sqrt{b_t} \hat{\sigma}_{j,t}^c(x) \right) \right] \leq \frac{\delta}{2}. \]

The above proof implies that the union of the event that all the bounds used in the GCB algorithm holds with probability at least \(1 - \delta\). Formally, for each
\( t \geq 1 \), we know that with probability at least \( 1 - \delta \) the following holds for all \( x_j \in X_j, j \in \{1,...,N\} \), and number of times the arm \( x_j \) has been pulled over \( t \) rounds:

\[
|\hat{n}_j(x_j) - n_j(x_j)| \leq \sqrt{b_t \hat{\sigma}^n_n(x_j)}, \quad (A.5)
\]

\[
|\hat{c}_j(x_j) - c_j(x_j)| \leq \sqrt{b_t \hat{\sigma}^c_n(x_j)}. \quad (A.6)
\]

From now on, let us assume we are in the clean event that the previous bounds hold.

Let us focus on the term \( r_{\mu}(x_t) \). The following holds:

\[
r_{\mu}(x_t) \geq r^*_\mu \geq r_{\mu}(x^*_\mu) \geq r^*_\eta = r^*_n, \quad (A.7)
\]

where we use the definition of \( r^*_\mu \), and the monotonicity property of the expected reward (Assumption 1), being \( (\mu)_i \geq (\eta)_i \), \( \forall i \). Using Equation (A.7), the instantaneous expected pseudo-regret \( \text{reg}_t \) at round \( t \) satisfies the following inequality:

\[
\text{reg}_t = r^*_\eta - r_{\eta}(x_t) \leq r_{\mu}(x_t) - r_{\eta}(x_t) =
\leq \underbrace{r_{\mu}(x_t) - r_{\mu}(x_t)}_{r_a} + \underbrace{r_{\mu}(x_t) - r_{\eta}(x_t)}_{r_b} \quad (A.9)
\]

where

\[
\hat{\mu} := \left[ \hat{w}_{1,t-1}(x_1), \ldots, \hat{w}_{N,t-1}(x_{1|X_N}), \hat{c}_{1,t-1}(x_1), \ldots, \hat{c}_{N,t-1}(x_{1|X_N}) \right], \quad (A.10)
\]

is the vector composed of the estimated average payoffs for each arm \( x \in X_j \) and each campaign \( C_j \), where \( \hat{w}_{j,t-1}(x) := v_j \hat{n}_{j,t-1}(x) \).

We use the Lipschitz property of the expected reward function (see Assumption 2) to bound the terms in Equation (A.9) as follows:

\[
r_a \leq \Lambda ||\mu - \hat{\mu}||_\infty = \Lambda \max_{j \in \{1,...,N\}} \left( v_{\max} \sqrt{\max_{x \in X_j} \hat{\sigma}^n_n(x)} \right) \quad (A.11)
\]

\[
\leq N v_{\max} \sqrt{\max_{j \in \{1,...,N\}} \left( \max_{x \in X_j} \hat{\sigma}^n_n(x) \right)} \quad (A.12)
\]
\[ N_{\text{max}} \sqrt{b_t \sum_{j=1}^{N} \left( \max_{x \in X_j} \hat{\sigma}_{j,t}^n(x) \right) }, \quad (A.13) \]

\[ r_b \leq A\|\hat{\mu} - \eta\|_{\infty} \leq N_{\text{max}} \sqrt{b_t \sum_{j=1}^{N} \left( \max_{x \in X_j} \hat{\sigma}_{j,t}^n(x) \right) }, \quad (A.14) \]

where Equation (A.11) holds by the definition of \( \mu \), Equation (A.13) holds since the maximum over a set is not greater than the sum of the elements of the set, if they are all non-negative, and Equation (A.14) directly follows from Equation (A.5). Plugging Equations (A.13) and (A.14) into Equation (A.9), we obtain:

\[ \text{reg}_{t} \leq 2 N_{\text{max}} \sqrt{b_t \sum_{j=1}^{N} \left( \max_{x \in X_j} \hat{\sigma}_{j,t}^n(x) \right) }. \quad (A.15) \]

We need now to upper bound \( \hat{\sigma}_{j,t}^n(x) \). Consider a realization \( n_j(\cdot) \) of a GP over \( X_j \) and recall that, thanks to Lemma 5.3 in [29], under the Gaussian assumption we can express the information gain \( IG_{j,t} \) provided by \( (\hat{n}_j(\hat{x}_{j,1}), \ldots, \hat{n}_j(\hat{x}_{j,|X_j|})) \) corresponding to the sequence of arms \( (\hat{x}_{j,1}, \ldots, \hat{x}_{j,|X_j|}) \) as:

\[ IG_{j,t} = \frac{1}{2} \sum_{h=1}^{t} \log \left( 1 + \sigma^{-2} (\hat{\sigma}_{j,t}^n(\hat{x}_{j,h}))^2 \right). \quad (A.16) \]

We have that:

\[ (\hat{\sigma}_{j,t}^n(\hat{x}_{j,h}))^2 = \sigma^2 \left[ \sigma^{-2}(\hat{\sigma}_{j,t}^n(\hat{x}_{j,h}))^2 \right] \leq \frac{\log \left[ 1 + \sigma^{-2}(\hat{\sigma}_{j,t}^n(\hat{x}_{j,h}))^2 \right]}{\log(1+\sigma^{-2})}, \quad (A.17) \]

since \( s^2 \leq \frac{\sigma^{-2} \log(1+s^2)}{\log(1+\sigma^{-1})} \) for all \( s \in [0, \sigma^{-1}] \), and \( \sigma^{-2}(\hat{\sigma}_{j,t}^n(\hat{x}_{j,h}))^2 \leq \sigma^{-2} k(\hat{x}_{j,h}, \hat{x}_{j,h}) \leq \sigma^{-2} \), where \( k(\cdot, \cdot) \) is the kernel of the GP. Since Equation (A.17) holds for any \( x \in X_j \) and for any \( j \in \{1, \ldots, N\} \), then it also holds for the arm \( \hat{x}_{\text{max}} \) maximizing the variance \( (\hat{\sigma}_{j,t}^n(\hat{x}_{j,h}))^2 \) over \( X_j \). Thus, setting \( \bar{c} = \frac{8N^2}{\log(1+\sigma^{-1})} \) and exploiting the Cauchy-Schwarz inequality, we obtain:

\[ R^2_{T}(GCB) \leq T \sum_{t=1}^{T} \text{reg}_{t}^2 \leq T \sum_{t=1}^{T} 4N^2 v_{\text{max}}^2 b_t \left[ \sum_{j=1}^{N} \left( \max_{x \in X_j} \hat{\sigma}_{j,t}^n(x) \right) \right]^2 \]

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\[
\leq 4N^2v_{\text{max}}^2TB_T \sum_{t=1}^{T} \left[ N \sum_{j=1}^{N} \max_{x \in X_j} (\hat{\sigma}^n_{j,t}(x))^2 \right]
\]
\[
\leq \bar{c}Nv_{\text{max}}^2TB_T \sum_{j=1}^{N} \frac{1}{2} \sum_{t=1}^{T} \max_{x \in X_j} \left( 1 + \sigma^{-2} (\hat{\sigma}^n_{j,t}(\hat{x}_{j,h}))^2 \right)
\]
\[
\leq \bar{c}Nv_{\text{max}}^2TB_T \sum_{j=1}^{N} \gamma_{j,T}.
\]

We conclude the proof by taking the square root on both the r.h.s. and the l.h.s. of the last inequality.

**Theorem 5** (GCB safety). Given \( \delta \in (0, 1) \), GCB applied to the problem in Equations (1a)–(1c) is \( \eta \)-safe where \( \eta \geq T - \frac{\delta}{2NQT} \) and, therefore, the number of constraints violations is linear in \( T \).

**Proof.** Let us focus on a specific day \( t \). Consider the case in which Constraints (1b) and (1c) are active, and, therefore, the left side equals the right side: \( \sum_{j=1}^{N} w_j(x_{j,t}) - \lambda \sum_{j=1}^{N} c_j(x_{j,t}) = 0 \) and \( \sum_{j=1}^{N} c_j(x_{j,t}) = \beta \). For the sake of simplicity, we focus on the costs \( c_j(x_{j,t}) \), but similar arguments also apply to the revenues \( w_j(x_{j,t}) \). A necessary condition for which the two constraints are valid also for the actual (non-estimated) revenues and costs is that for at least one of the costs it holds \( c_j(x_{j,t}) \leq c_j(x_{j,t}) \). Indeed, if the opposite holds, i.e., \( c_j(x_{j,t}) < c_j(x_{j,t}) \) for each \( j \in \{1, \ldots, N\} \) and \( x_{j,t} \in X_j \), the budget constraint would be violated by the allocation since \( \sum_{j=1}^{N} c_j(x_{j,t}) > \sum_{j=1}^{N} c_j(x_{j,t}) = \beta \). Since the event \( c_j(x_{j,t}) \leq c_j(x_{j,t}) \) occurs with probability at most \( \frac{3\delta}{\pi NQT} \), over the \( t \in \mathbb{N} \), formally:

\[
P \left( \frac{\sum_{j=1}^{N} v_j n_j(\hat{x}_{j,t})}{\sum_{j=1}^{N} c_j(\hat{x}_{j,t})} < \lambda \lor \sum_{j=1}^{N} c_j(\hat{x}_{j,t}) > \beta \right) \geq 1 - \frac{3\delta}{\pi^2 NQT T^2}.
\]

Finally, summing over the time horizon \( T \) the probability that the constraints are not violated is at most \( \frac{\delta}{2NQT} \), formally:

\[
\sum_{t=1}^{T} P \left( \frac{\sum_{j=1}^{N} v_j n_j(\hat{x}_{j,t})}{\sum_{j=1}^{N} c_j(\hat{x}_{j,t})} < \lambda \lor \sum_{j=1}^{N} c_j(\hat{x}_{j,t}) > \beta \right) \geq T - \frac{\delta}{2NQT}
\]

This concludes the proof. \( \square \)
Appendix A.4. Proofs Omitted from Section 5.2

**Theorem 6** (GCB\textsubscript{safe} safety). Given $\delta \in (0, 1)$, GCB\textsubscript{safe} applied to the problem in Equations (1a–1c) is $\delta$-safe and, therefore, the number of constraints violations is constant in $T$.

**Proof.** Let us focus on a specific day $t$. Constraints (1b) and (1c) are satisfied by the solution of $\text{Opt}(\mu, \lambda)$ for the properties of the optimization procedure. Define $n_j(x_{j,t}) := \hat{n}_j(x_{j,t}) - \sqrt{b_{t-1} \sigma^2(x_{j,t})}$. Thanks to the specific construction of the upper bounds, we have that $c_j(x_{j,t}) \leq \hat{c}_j(x_{j,t})$ and $n_j(x_{j,t}) \geq \hat{n}_j(x_{j,t})$, each holding with probability at least $1 - 3\delta \pi^2 N Q T_l^2$. Therefore, we have:

$$
\sum_{j=1}^N v_j n_j(x_{j,t}) / \sum_{j=1}^N c_j(x_{j,t}) > \sum_{j=1}^N v_j \hat{n}_j(x_{j,t}) / \sum_{j=1}^N \hat{c}_j(x_{j,t}) \geq \lambda
$$

and

$$
\sum_{j=1}^N c_j(x_{j,t}) < \sum_{j=1}^N \hat{c}_j(x_{j,t}) \leq \beta.
$$

Using a union bound over:

- the two GPs (number of clicks and costs);
- the time horizon $T$;
- the number of times each bid is chosen in a subcampaign (at most $t$);
- the number of arms present in each subcampaign ($|X_j|$);
- the number of subcampaigns ($N$);

we have:

$$
\sum_{t=1}^T \mathbb{P} \left( \frac{\sum_{j=1}^N v_j n_j(\hat{x}_{j,t})}{\sum_{j=1}^N c_j(\hat{x}_{j,t})} < \lambda \lor \sum_{j=1}^N c_j(\hat{x}_{j,t}) > \beta \right) \leq 2 \sum_{j=1}^N |X_j| \sum_{t=1}^T \sum_{l=1}^\infty \frac{3\delta}{\pi^2 N Q T_l^2} = \delta.
$$

(A.18)

$$
\leq 2 \sum_{j=1}^N \sum_{k=1}^Q \sum_{h=1}^T \sum_{l=1}^\infty \frac{3\delta}{\pi^2 N Q T_l^2} = \delta.
$$

(A.19)

This concludes the proof. \qed
Theorem 7 (GCB\textsubscript{safe} pseudo-regret). Given $\delta \in (0,1)$, GCB\textsubscript{safe} applied to the problem in Equations (1a)–(1c) suffers from a pseudo-regret $R_t(\text{GCB}\textsubscript{safe}) = \Theta(T)$.

Proof. At the optimal solution, at least one of the constraints is active, i.e., it has the left-hand side equal to the right-hand side. Assume that the optimal clairvoyant solution $\{x^*_j\}_{j=1}^N$ to the optimization problem has a value of the ROI $\lambda_{opt}$ equal to $\lambda$. We showed in the proof of Theorem 6 that for any allocation, with probability at least $1 - \frac{3\delta}{\pi^2 N QT^2}$, it holds that $\sum_{j=1}^N v_j n_j(x_j) > \sum_{j=1}^N c_j(x_j)$. This is true also for the optimal clairvoyant solution $\{x^*_j\}_{j=1}^N$, for which $\lambda = \sum_{j=1}^N v_j n_j(x^*_j) > \sum_{j=1}^N c_j(x^*_j)$, implying that the values used in the ROI constraint make this allocation not feasible for the Opt($\mu, \lambda$) procedure.

As shown before, this happens with probability at least $1 - \frac{3\delta}{\pi^2 N QT^2}$ at day $t$, and $1 - \delta$ over the time horizon $T$. To conclude, with probability $1 - \delta$, not depending on the time horizon $T$, we will not choose the optimal arm during the time horizon and, therefore, the regret of the algorithm cannot be sublinear. Notice that the same line of proof is also holding in the case the budget constraint is active, therefore, the previous result holds for each instance of the problem in Equations (1a)–(1c).

Appendix A.5. Proofs Omitted from Section 5.3

Theorem 8 (GCB\textsubscript{safe}(\psi, 0) pseudo-regret and safety with tolerance). When:

$$\psi \geq 2 \frac{\beta_{opt} + n_{max}}{\beta_{opt}} \sum_{j=1}^N v_j \sqrt{2 \ln \left( \frac{\pi^2 NT^{3/2}}{3\delta'} \right) \sigma}$$

and

$$\beta_{opt} < \frac{\sum_{j=1}^N v_j}{\beta_{opt} n_{max} + \sum_{j=1}^N v_j}$$

where $\delta' \leq \delta$, $\beta_{opt}$ is the spend at the optimal solution of the original problem, and $n_{max} := \max_{j,x} n_j(x)$ is the maximum over the sub-campaigns and the admissible bids of the expected number of clicks, GCB\textsubscript{safe}(\psi, 0) provides a pseudo-regret w.r.t. the optimal solution to the original problem of $O \left( \sqrt{T \sum_{j=1}^N \gamma_j T} \right)$.
with probability at least $1 - \delta - \frac{\delta'}{\sqrt{T}}$, while being $\delta$-safe w.r.t. the constraints of the auxiliary problem.

**Proof.** In what follows, we show that, at a specific day $t$, since the optimal solution of the original problem $\{x_j^*\}_{j=1}^N$ is included in the set of feasible ones, we are in a setting analogous to the one of GCB, in which the regret is sublinear.

Let us assume that the upper bounds on all the quantities (number of clicks and costs) holds. This has been shown before to occur with overall probability $\delta$ over the whole time horizon $T$. Moreover, notice that combining the properties of the budget of the optimal solution $\beta_{opt}$ and using $\psi = 2\frac{\beta_{opt} + n_{max}}{\beta_{opt}} \sum_{j=1}^N v_j \sqrt{2 \ln \left( \frac{2^N NQT^3}{3\delta'} \right) \sigma}$, we have:

$$\beta_{opt} < \beta \frac{\sum_{j=1}^N v_j}{\frac{N \beta_{opt} \psi}{\beta_{opt} + n_{max}} + \sum_{j=1}^N v_j}$$  \hspace{1cm} (A.20)

$$\left( \frac{N \beta_{opt} \psi}{\beta_{opt} + n_{max}} + \sum_{j=1}^N v_j \right) \beta_{opt} < \beta \sum_{j=1}^N v_j$$  \hspace{1cm} (A.21)

$$2N \sum_{j=1}^N v_j \sqrt{2 \ln \left( \frac{\pi^2 NQT^3}{3\delta'} \right) \sigma} + \sum_{j=1}^N v_j \beta_{opt} < \beta \sum_{j=1}^N v_j$$  \hspace{1cm} (A.22)

$$\beta > \beta_{opt} + 2N \sqrt{2 \ln \left( \frac{\pi^2 NQT^3}{3\delta'} \right) \sigma}.$$  \hspace{1cm} (A.23)

First, let us evaluate the probability that the optimal solution is not feasible. This occurs if its bounds are either violating the ROI or budget constraints. First, we show that analysing the budget constraint, the optimal solution of the original problem is feasible with high probability. Formally, it is not feasible with probability:

$$\mathbb{P} \left( \sum_{j=1}^N \tau_j(x_j^*) > \beta \right) \leq \mathbb{P} \left( \sum_{j=1}^N \tau_j(x_j^*) > \beta_{opt} + 2N \sqrt{2 \ln \left( \frac{\pi^2 NQT^3}{3\delta'} \right) \sigma} \right)$$  \hspace{1cm} (A.24)

$$= \mathbb{P} \left( \sum_{j=1}^N \tau_j(x_j^*) > \sum_{j=1}^N \tau_j(x_j^*) + 2N \sqrt{2 \ln \left( \frac{\pi^2 NQT^3}{3\delta'} \right) \sigma} \right)$$  \hspace{1cm} (A.25)
\[
\sum_{j=1}^{N} \mathbb{P}\left( c_j(x_j^*) > c_j(x_j^*) + 2 \sqrt{2 \ln \frac{\pi^2 N Q T^3}{3 \delta'}} \sigma \right) \leq \sum_{j=1}^{N} \mathbb{P}\left( \hat{c}_{j,t-1}(x_j^*) - c_j(x_j^*) > -\sqrt{b_t} \hat{\sigma}_{j,t-1}(x_j^*) + 2 \sqrt{2 \ln \frac{\pi^2 N Q T^3}{3 \delta'}} \sigma \right)
\]

(A.26)

\[
= \sum_{j=1}^{N} \mathbb{P}\left( \hat{c}_{j,t-1}(x_j^*) - c_j(x_j^*) > -\sqrt{b} \hat{\sigma}_{j,t-1}(x_j^*) + 2 \sqrt{2 \ln \frac{\pi^2 N Q T^3}{3 \delta'}} \sigma \right)
\]

(A.27)

\[
\leq \sum_{j=1}^{N} \mathbb{P}\left( \hat{c}_{j,t-1}(x_j^*) - c_j(x_j^*) > \sqrt{2 \ln \frac{\pi^2 N Q T^3}{3 \delta'}} \sigma \right)
\]

(A.28)

\[
\leq \sum_{j=1}^{N} \frac{3 \delta'}{\pi^2 N Q T^3} = \frac{3 \delta'}{\pi^2 Q T^3},
\]

(A.29)

where, in the inequality in Equation (A.24) we used Equation (A.23), in Equation (A.29) we used the fact that \( \pi^2 N Q T^3 \leq \pi^2 N Q T^3 \) for each \( t \in \{1, \ldots, T\} \), \( \hat{\sigma}_{j,t-1}(x_j^*) \leq \sigma \) for each \( j \) and \( t \), and the inequality in Equation (A.30) is from Srinivas et al. [29]. Summing over the time horizon \( T \), we get that the optimal solution of the original problem \( \{x_j^*\}_{j=1}^{N} \) is excluded from the set of feasible ones with probability at most \( \frac{3 \delta'}{\pi^2 Q T^3} \).

Second, we derive a bound over the probability that the optimal solution of the original problem is feasible due to the newly defined ROI constraint. Let us notice that since the ROI constraint is active we have \( \lambda = \lambda_{\text{opt}} \). The probability that \( \{x_j^*\}_{j=1}^{N} \) is not feasible due to the ROI constraint is:

\[
P \left( \frac{\sum_{j=1}^{N} v_j \hat{n}_j(x_j^*)}{\sum_{j=1}^{N} \hat{n}_j(x_j^*)} < \lambda - \psi \right)
\]

(A.31)

\[
\leq P \left( \frac{\sum_{j=1}^{N} v_j \hat{n}_j(x_j^*)}{\sum_{j=1}^{N} \hat{c}_j(x_j^*)} < \lambda_{\text{opt}} - 2 \beta_{\text{opt}} + \max_{j} \frac{n_{\text{max}}}{\beta_{\text{opt}}^2} \sum_{j=1}^{N} v_j \sqrt{2 \ln \frac{\pi^2 N Q T^3}{3 \delta'}} \sigma \right)
\]

(A.32)

\[
= P \left( \frac{\sum_{j=1}^{N} v_j \hat{n}_j(x_j^*)}{\sum_{j=1}^{N} \hat{c}_j(x_j^*)} < \frac{\sum_{j=1}^{N} v_j n_j(x_j^*)}{\sum_{j=1}^{N} c_j(x_j^*)} - 2 \beta_{\text{opt}} + \max_{j} \frac{n_{\text{max}}}{\beta_{\text{opt}}^2} \sum_{j=1}^{N} v_j \sqrt{2 \ln \frac{\pi^2 N Q T^3}{3 \delta'}} \sigma \right)
\]

(A.33)
\[
\begin{align*}
&= \mathbb{P} \left( \sum_{j=1}^{N} c_j(x_j^*) + \sum_{j=1}^{N} \tau_j(x_j^*) \geq \sum_{j=1}^{N} \pi_j n_j(x_j^*) \right) \\
&\quad \leq \mathbb{P} \left( \sum_{j=1}^{N} c_j(x_j^*) + \sum_{j=1}^{N} \tau_j(x_j^*) \geq \sum_{j=1}^{N} \pi_j n_j(x_j^*) \right) - \frac{2 \beta_{opt} + n_{max}}{\beta_{opt}^2} \sum_{j=1}^{N} c_j(x_j^*) + \sum_{j=1}^{N} \tau_j(x_j^*) \\
&\quad + \frac{2}{\beta_{opt}} \sum_{j=1}^{N} c_j(x_j^*) \sum_{j=1}^{N} \tau_j(x_j^*) \sum_{j=1}^{N} v_j \sqrt{2 \ln \frac{\pi^2 NQT^3}{3\delta'}} \sigma < 0 \right) \quad (A.34) \\
&\leq \mathbb{P} \left( \sum_{j=1}^{N} c_j(x_j^*) + \sum_{j=1}^{N} \tau_j(x_j^*) \geq \sum_{j=1}^{N} \pi_j n_j(x_j^*) \right) - \frac{2 \beta_{opt} + n_{max}}{\beta_{opt}^2} \sum_{j=1}^{N} c_j(x_j^*) + \sum_{j=1}^{N} \tau_j(x_j^*) \\
&\quad + \frac{2}{\beta_{opt}} \sum_{j=1}^{N} c_j(x_j^*) \sum_{j=1}^{N} \tau_j(x_j^*) \sum_{j=1}^{N} v_j \sqrt{2 \ln \frac{\pi^2 NQT^3}{3\delta'}} \sigma < 0 \right) \quad (A.35) \\
&\leq \sum_{j=1}^{N} \mathbb{P} \left( u_j(x_j^*) - n_j(x_j^*) + 2 \sqrt{2 \ln \frac{\pi^2 NQT^3}{3\delta'}} \sigma \leq 0 \right) \\
&\quad + \sum_{j=1}^{N} \mathbb{P} \left( c_j(x_j^*) - \tau_j(x_j^*) + 2 \sqrt{2 \ln \frac{\pi^2 NQT^3}{3\delta'}} \sigma < 0 \right) \quad (A.37) \\
&\leq \sum_{j=1}^{N} \mathbb{P} \left( \hat{u}_{j,t-1}(x_j^*) - \sqrt{b_t \sigma_{j,t-1}(x_j^*)} - n_j(x_j^*) + 2 \sqrt{2 \ln \frac{\pi^2 NQT^3}{3\delta'}} \sigma < 0 \right)
\end{align*}
\]
\[
\sum_{j=1}^{N} P\left( c_j(x_j^*) - \hat{c}_{j,t-1}(x_j^*) - \sqrt{b_t \hat{\sigma}_{j,t-1}^2(x_j^*)} + 2 \sqrt{\frac{2 \ln \pi^2 N^3 T}{3 \delta'}} \sigma < 0 \right)
\]

\[
\leq \sum_{j=1}^{N} P\left( n_j(x_j^*) < \hat{n}_{j,t-1}(x_j^*) + \sqrt{2 \ln \pi^2 N^3 T / 3 \delta'} \hat{\sigma}_{j,t-1}^n(x_j^*) \right)
\]

\[
+ \sum_{j=1}^{N} P\left( c_j(x_j^*) - \hat{c}_{j,t-1}(x_j^*) - \sqrt{2 \ln \pi^2 N^3 T / 3 \delta'} \hat{\sigma}_{j,t-1}(x_j^*) \right)
\]

\[
= \sum_{j=1}^{N} P\left( \frac{n_j(x_j^*) - \hat{n}_{j,t-1}(x_j^*)}{\hat{\sigma}_{j,t-1}^n(x_j^*)} > \sqrt{2 \ln \pi^2 N^3 T / 3 \delta'} \right)
\]

\[
+ \sum_{j=1}^{N} P\left( \frac{\hat{c}_{j,t-1}(x_j^*) - c_j(x_j^*)}{\hat{\sigma}_{j,t-1}(x_j^*)} > \sqrt{2 \ln \pi^2 N^3 T / 3 \delta'} \right)
\]

\[
\leq 2 \sum_{j=1}^{N} \frac{3 \delta'}{\pi^2 N^3 T^3} = \frac{6 \delta'}{\pi^2 QT^3},
\]

(A.38)

where in Equation (A.39) we used the fact that \( \frac{\pi^2 N^2 T}{3 \delta'} \leq \frac{\pi^2 N^3 T}{3 \delta'} \) for each \( t \in \{1, \ldots, T\} \), \( \hat{\sigma}_{j,t-1}^n(x_j^*) < \sigma \) for each \( j \) and \( t \), and the inequality in Equation (A.41) is from Srinivas et al. [29]. Summing over the time horizon \( T \) ensures that the optimal solution of the original problem \( \{x_j^*\}_{j=1}^{N} \) is excluded from the feasible solutions at most with probability \( \frac{6 \delta'}{\pi^2 QT^3} \). Finally, using a union bound, we have that the optimal solution can be chosen over the time horizon with probability at least \( 1 - \frac{2 \delta'}{\pi^2 QT^3} - 6 \delta' \leq 1 - \frac{\delta'}{QT^2} \).

Notice that here we want to compute the regret of the \texttt{GCBsafe} algorithm w.r.t. \( \{x_j^*\}_{j=1}^{N} \) which is not optimal for the analysed relaxed problem. Nonetheless, the proof on the pseudo-regret provided in Theorem 4 is valid also for suboptimal solutions in the case it is feasible with high probability. This can be trivially shown using the fact that the regret w.r.t. a generic solution cannot be larger than the one computed w.r.t. the optimal one. Thanks to that, using a union bound over the probability that the bounds hold and that \( \{x_j^*\}_{j=1}^{N} \) is feasible, we conclude that with probability at least \( 1 - \delta - \frac{\delta'}{QT^2} \) the regret
GCB\textsubscript{safe} is of the order of $\mathcal{O}\left(\sqrt{T\sum_{j=1}^{N} \gamma_{j,T}}\right)$. Finally, thanks to the property of the GCB\textsubscript{safe} algorithm shown in Theorem 6, the learning policy is $\delta$-safe for the relaxed problem.

**Theorem 9 (GCB\textsubscript{safe}(0, $\phi$) pseudo-regret and safety with tolerance).** When

$$\phi \geq 2N\sqrt{2\ln \left(\frac{\pi^2 N QT^3}{3\delta^2}\right)} \sigma$$

and

$$\lambda_{opt} > \lambda + \frac{N\beta}{\sum_{j=1}^{N} v_j},$$

where $\delta' \leq \delta$, and $n_{\text{max}} := \max_{j, x} n_j(x)$ is maximum expected number of clicks, GCB\textsubscript{safe}(0, $\phi$) provides a pseudo-regret w.r.t. the optimal solution to the original problem of $\mathcal{O}\left(\sqrt{T\sum_{j=1}^{N} \gamma_{j,T}}\right)$ with probability at least $1 - \delta - \frac{6\delta'}{\pi^2 T^2}$, while being $\delta$-safe w.r.t. the constraints of the auxiliary problem.

**Proof.** We show that at a specific day $t$ since the optimal solution of the original problem $\{x^*_j\}_{j=1}^{N}$ is included in the set of feasible ones, we are in a setting analogous to the one of GCB, in which the regret is sublinear. Let us assume that the upper bounds to all the quantities (number of clicks and costs) holds. This has been shown before to occur with overall probability $\delta$ over the whole time horizon $T$.

First, let us evaluate the probability that the optimal solution is not feasible. This occurs if its bounds are either violating the ROI or budget constraints. From the fact that the ROI of the optimal solution satisfies $\lambda_{opt} > \lambda + \frac{(\beta + n_{\text{max}})\phi \sum_{j=1}^{N} v_j}{N\beta^2}$, we have:

\begin{equation}
\mathbb{P}\left(\sum_{j=1}^{N} v_j \frac{n_j(x^*_j)}{\bar{c}_j(x^*_j)} < \lambda\right) \geq \mathbb{P}\left(\sum_{j=1}^{N} v_j \frac{n_j(x^*_j)}{\bar{c}_j(x^*_j)} < \lambda_{opt} - \frac{(\beta + n_{\text{max}})\phi \sum_{j=1}^{N} v_j}{N\beta^2}\right) \tag{A.42}
\end{equation}

\begin{equation}
\mathbb{P}\left(\sum_{j=1}^{N} v_j \frac{n_j(x^*_j)}{\bar{c}_j(x^*_j)} < \frac{\beta_{opt} + n_{\text{max}}}{\beta_{opt}} \sum_{j=1}^{N} v_j \sqrt{\ln \frac{\pi^2 N QT^3}{3\delta^2} \sigma}\right) \tag{A.43}
\end{equation}

\begin{equation}
\mathbb{P}\left(\sum_{j=1}^{N} v_j \frac{n_j(x^*_j)}{\bar{c}_j(x^*_j)} < \frac{\beta_{opt} + n_{\text{max}}}{\beta_{opt}} \sum_{j=1}^{N} v_j \sqrt{\ln \frac{\pi^2 N QT^3}{3\delta^2} \sigma}\right) \tag{A.44}
\end{equation}

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\[
\leq \frac{3\delta'}{\pi^2QT^3}, \quad (A.45)
\]

where the derivation uses arguments similar to the ones applied in the proof for the ROI constraint in Theorem 8. Summing over the time horizon \(T\) ensures that the optimal solution of the original problem \(\{x_j^*\}_{j=1}^N\) is excluded from the feasible solutions at most with probability \(\frac{3\delta'}{\pi^2QT^3}\).

Second, let us evaluate the probability for which the optimal solution of the original problem \(\{x_j^*\}_{j=1}^N\) is excluded due to the budget constraint, formally:

\[
P\left(\sum_{j=1}^N \bar{c}_j(x_j^*) > \beta + \phi\right) \quad (A.46)
\]

\[
\leq P\left(\sum_{j=1}^N \bar{c}_j(x_j^*) > \beta + 2N\sqrt{2 \ln \frac{\pi^2NQT^3}{3\delta'}}\right) \quad (A.47)
\]

\[
= P\left(\sum_{j=1}^N \bar{c}_j(x_j^*) > \sum_{j=1}^N c_j(x_j^*) + 2N\sqrt{2 \ln \frac{\pi^2NQT^3}{3\delta'}}\right) \quad (A.48)
\]

\[
\leq \sum_{j=1}^N P\left(\bar{c}_j(x_j^*) > c_j(x_j^*) + 2\sqrt{2 \ln \frac{12N^3}{\pi^2\delta'}}\right) \quad (A.49)
\]

\[
= \sum_{j=1}^N P\left(\hat{c}_{j,t-1}(x_j^*) - c_j(x_j^*) \geq -\sqrt{b_t \delta_{j,t-1}(x_j^*)} + 2\sqrt{2 \ln \frac{\pi^2NQT^3}{3\delta'}}\right) \quad (A.50)
\]

\[
\leq \sum_{j=1}^N P\left(\hat{c}_{j,t-1}(x_j^*) - c_j(x_j^*) \geq -\sqrt{b_t \delta_{j,t-1}(x_j^*)} \right) \quad (A.51)
\]

\[
\leq \sum_{j=1}^N \frac{3\delta'}{\pi^2NQT^3} = \frac{3\delta'}{\pi^2QT^3}, \quad (A.53)
\]

where we use the fact that \(\beta = \beta_{opt}\), and the derivation uses arguments similar to the ones applied in the proof for the budget constraint in Theorem 8. Summing over the time horizon \(T\), we get that the optimal solution of the original problem \(\{x_j^*\}_{j=1}^N\) is excluded from the set of the feasible ones with probability at most \(\frac{3\delta'}{\pi^2QT^3}\). Finally, using a union bound, we have that the optimal solution can be
chosen over the time horizon with probability at least $1 - \frac{3\delta'}{\pi Q T^2}$.

Notice that here we want to compute the regret of the $\text{GCB}_{\text{safe}}$ algorithm w.r.t. $\{x^*_j\}_{j=1}^N$ which is not optimal for the analysed relaxed problem. Nonetheless, the proof on the pseudo-regret provided in Theorem 4 is valid also for suboptimal solutions in the case it is feasible with high probability. This can be trivially shown using the fact that the regret w.r.t. a generic solution cannot be larger than the one computed on the optimal one. Thanks to that, using a union bound over the probability that the bounds hold and that $\{x^*_j\}_{j=1}^N$ is feasible, we conclude that with probability at least $1 - \delta - \frac{6\delta'}{\pi Q T^2}$ the regret $\text{GCB}_{\text{safe}}$ is of the order of $O\left(\sqrt{T \sum_{j=1}^N \gamma_{j,T}}\right)$. Finally, thanks to the property of the $\text{GCB}_{\text{safe}}$ algorithm shown in Theorem 6 the learning policy is $\delta$-safe for the relaxed problem.

**Theorem 10 ($\text{GCB}_{\text{safe}}(\psi, \phi)$ pseudo-regret and safety with tolerance).** Setting

$$\psi = 2\frac{\beta_{\text{opt}} + n_{\text{max}}}{\beta_{\text{opt}}^2} \sum_{j=1}^N v_j \sqrt{2 \ln \left(\frac{\pi^2 N Q T^3}{3\delta'}\right)} \sigma$$

and

$$\phi = 2N \sqrt{2 \ln \left(\frac{\pi^2 N Q T^3}{3\delta'}\right)} \sigma,$$

where $\delta' \leq \delta$, $\text{GCB}_{\text{safe}}(\psi, \phi)$ provides a pseudo-regret w.r.t. the optimal solution to the original problem of $O\left(\sqrt{T \sum_{j=1}^N \gamma_{j,T}}\right)$ with probability at least $1 - \delta - \frac{\delta'}{Q T^2}$, while being $\delta$-safe w.r.t. the constraints of the auxiliary problem.

**Proof.** The proof follows from combining the arguments about the ROI constraint used in Theorem 4 and those about the budget constraint used in Theorem 9. \qed
Appendix B. Additional Details on the Experimental Activity

Appendix B.1. Additional Information for Reproducibility

In this section, we provide additional information for the full reproducibility of the experiments provided in the main paper.

The code has been run on a Intel(R) Core(TM) i7-4710MQ CPU with 16 GiB of system memory. The operating system was Ubuntu 18.04.5 LTS, and the experiments have been run on Python 3.7.6. The libraries used in the experiments, with the corresponding version were:

- matplotlib==3.1.3
- gpflow==2.0.5
- tikzplotlib==0.9.4
- tf_nightly==2.2.0.dev20200308
- numpy==1.18.1
- tensorflow==2.3.0

On this architecture, the average execution time of each algorithm takes an average of \( \approx 30 \text{ sec} \) for each day \( t \) of execution.

Appendix B.2. Parameters and Setting of Experiment #1

Table B.2 specifies the values of the parameters of cost and number-of-click functions of the subcampaigns used in Experiment #1.
Table B.2: Parameters of the synthetic settings used in Experiment #1.

|          | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|----------|-------|-------|-------|-------|-------|
| $\theta_j$ | 60    | 77    | 75    | 65    | 70    |
| $\delta_j$ | 0.41  | 0.48  | 0.43  | 0.47  | 0.40  |
| $\alpha_j$ | 497   | 565   | 573   | 503   | 536   |
| $\gamma_j$ | 0.65  | 0.62  | 0.67  | 0.68  | 0.69  |
| $\sigma_f$ GP revenue | 0.669 | 0.499 | 0.761 | 0.619 | 0.582 |
| $l$ GP revenue | 0.425 | 0.469 | 0.471 | 0.483 | 0.386 |
| $\sigma_f$ GP cost | 0.311 | 0.443 | 0.316 | 0.349 | 0.418 |
| $l$ GP cost | 0.76  | 0.719 | 0.562 | 0.722 | 0.727 |
Appendix B.3. Additional Results of Experiment #2

In Figures B.4, B.5, B.6, and B.7 we report the 90% and 10% of the quantities related to Experiment #2 provided by the GCB\text{safe}, GCB\text{safe}_{(0, 0.05)}, GCB\text{safe}_{(0, 0.10)}, and GCB\text{safe}_{(0, 0.15)}, respectively.

Figure B.4: Results of Experiment #2: daily revenue (a), ROI (b), and spend (c) obtained by GCB\text{safe}. The dash-dotted lines correspond to the optimum values for the revenue and ROI, while the dashed lines correspond to the values of the ROI and budget constraints.

Figure B.5: Results of Experiment #2: daily revenue (a), ROI (b), and spend (c) obtained by GCB\text{safe}_{(0, 0.05)}. The dash-dotted lines correspond to the optimum values for the revenue and ROI, while the dashed lines correspond to the values of the ROI and budget constraints.
Figure B.6: Results of Experiment #2: daily revenue (a), ROI (b), and spend (c) obtained by and GCB_{safe}(0, 0.10). The dash-dotted lines correspond to the optimum values for the revenue and ROI, while the dashed lines correspond to the values of the ROI and budget constraints.

Figure B.7: Results of Experiment #2: daily revenue (a), ROI (b), and spend (c) obtained by and GCB_{safe}(0, 0.15). The dash-dotted lines correspond to the optimum values for the revenue and ROI, while the dashed lines correspond to the values of the ROI and budget constraints.
Appendix B.4. Additional Results of Experiment #3

In Figures B.8, B.9, and B.10 we report the 90% and 10% of the quantities analysed in the experimental section for Experiment #3 provided by the GCB, GCB_{safe}, and GCB_{safe}(0.05, 0), respectively. These results show that the constraints are satisfied by GCB_{safe}, and GCB_{safe}(0.05, 0) also with high probability. While for GCB_{safe} this is expected due to the theoretical results we provided, the fact that also GCB_{safe}(0.05, 0) guarantees safety w.r.t. the original optimization problem suggests that in some specific setting GCB_{safe} is too conservative. This is reflected in a lower cumulative revenue, which might be negative from a business point of view.

Figure B.8: Results of Experiment #3: daily revenue (a), ROI (b), and spend (c) obtained by GCB. The dash-dotted lines correspond to the optimum values for the revenue and ROI, while the dashed lines correspond to the values of the ROI and budget constraints.
Figure B.9: Results of Experiment #3: daily revenue (a), ROI (b), and spend (c) obtained by $\text{GCB}_{\text{safe}}$. The dash-dotted lines correspond to the optimum values for the revenue and ROI, while the dashed lines correspond to the values of the ROI and budget constraints.

Figure B.10: Results of Experiment #3: daily revenue (a), ROI (b), and spend (c) obtained by $\text{GCB}_{\text{safe}}(0.05, 0)$. The dash-dotted lines correspond to the optimum values for the revenue and ROI, while the dashed lines correspond to the values of the ROI and budget constraints.
Table B.3: Values of the parameters used in the 10 different settings of Experiment #4.

| Setting | C1 | C2 | C3 | C4 | C5 | λ |
|---------|----|----|----|----|----|---|
| Setting 1 | 390 | 417 | 548 | 271 | 550 | 10.0 |
| Setting 2 | 399 | 597 | 488 | 698 | 498 | 14.0 |
| Setting 3 | 242 | 0.3 | 1.37 | 3.39 | 0.69 | 10.5 |
| Setting 4 | 89 | 597 | 488 | 698 | 498 | 14.0 |
| Setting 5 | 399 | 597 | 488 | 698 | 498 | 14.0 |
| Setting 6 | 399 | 597 | 488 | 698 | 498 | 14.0 |
| Setting 7 | 399 | 597 | 488 | 698 | 498 | 14.0 |
| Setting 8 | 399 | 597 | 488 | 698 | 498 | 14.0 |
| Setting 9 | 399 | 597 | 488 | 698 | 498 | 14.0 |
| Setting 10 | 399 | 597 | 488 | 698 | 498 | 14.0 |

Appendix B.5. Parameters of Settings of Experiment #4

We report in Table B.3 the values of the parameters of cost and number-of-click functions of the subcampaigns used in Experiment #4.