On piecewise smooth cohomology of Lie groupoids and Lie algebroids

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Abstract
A. Mishchenko [4] proved that piecewise smooth and Lie algebroid cohomology of a transitive Lie algebroid defined over a combinatorial manifold are isomorphic. In this paper, we describe two applications of that result. The first application consists in the relationship between piecewise de Rham cohomology of a Lie groupoid and piecewise smooth cohomology of its Lie algebroid. For the second application, we combine the classical result dealing with invariant cohomology in Lie algebroids ([1]) with the Mishchenko’s theorem stated in [4].

Notation. For definitions and notations used in this paper, we follow the Mackenzie’s book [3].

1 Piecewise de Rham cohomology of Lie groupoids
Mishchenko in [4] has defined the notion of piecewise smooth cohomology of a transitive Lie algebroid over a combinatorial manifold and it was shown that its Lie algebroid cohomology is isomorphic to piecewise smooth cohomology of the same algebroid. We call this result Mishchenko’s theorem. In this section, we establish a basic relationship between piecewise de Rham cohomology of left invariants forms of a Lie groupoid and piecewise smooth cohomology of its Lie algebroid. Constructions concerning differential forms in Lie groupoids are quite analogous to constructions in Lie algebroids. We recall now the definition of de Rham cohomology of left invariants forms of a Lie groupoid. Let $M$ be a smooth manifold and $G$ a Lie groupoid on $M$ with source projection $\alpha$ and target projection $\beta$. Following the notation of [3], we denote by $G_x$ the $\alpha$-fibre of $x$, for each $x \in M$. Since the source projection $\alpha$ is a surjective submersion, it follows that $\alpha$ induces a foliation $\mathcal{F}$ on $G$ where the leaves are defined to be the connected components of $G_x$, for each $x \in M$. We denote by $T\mathcal{F}$ the tangent bundle of $\mathcal{F}$. For each $p \geq 0$, let $\Omega^p_e(G)$ be the $C^\infty(G)$-module of the smooth sections of the exterior vector bundle $\bigwedge^p(T^*\mathcal{F};\mathbb{R}_M)$, in which
\( \mathbb{R}_M \) denotes the trivial vector bundle on \( M \). A smooth \( \alpha \)-form of degree \( p \) on the Lie groupoid \( G \) is, by definition, an element of \( \Omega^p_\alpha(G) \). Thus, a smooth \( \alpha \)-form \( \omega \in \Omega^p_\alpha(G) \) is defined on \( G \) such that, for each \( g \in G \), one has

\[
\omega_g \in \Lambda^p( \bigsqcup_{g \in G} T^*_g G_{\alpha(g)}; \mathbb{R} )
\]

The usual exterior derivative along the \( \alpha \)-fibres is defined by

\[
(d^p \omega)(X_1, X_2, \cdots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} X_j \cdot (\omega(X_1, \cdots, \widehat{X_j}, \cdots, X_{p+1})) + \sum_{i<k} (-1)^{i+k} \omega([X_i, X_k], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_k}, \cdots, X_{p+1})
\]

in which \( \omega \in \Omega^p_\alpha(G) \) and \( X_1, X_2, \cdots, X_{p+1} \) are smooth vector fields on \( G \). The complex \( (\Omega^\ast_\alpha(G), d^\ast_\alpha) \) is a commutative differential graded algebra defined on \( \mathbb{R} \). The set \( \Omega^p_{\alpha,L}(G) \) consisting of all \( \alpha \)-forms on \( G \) which are invariant under the groupoid left translations is a subcomplex of \( (\Omega^\ast_\alpha(G), d^\ast_\alpha) \). Its cohomology is denoted by \( H^\ast_{\alpha,L}(G) \).

Denote by \( 1 : M \to G \) the object inclusion map of \( G \). We recall that the Lie algebroid of \( G \) is \((\mathcal{A}(G), [\cdot, \cdot], \gamma)\), in which

\[
\mathcal{A}(G) = \bigsqcup_{x \in M} T_x G_x \quad (\text{disjoint union})
\]

the anchor \( \gamma : \mathcal{A}(G) \to TM \) is defined by \( \gamma(a) = D\beta_1(a) \) and the Lie bracket is defined by \([\xi, \eta] = [\xi', \eta']_0 \) (cf. \[2\] , [3]).

There is an isomorphism \( \psi : \Omega^\ast_{\alpha,L}(G) \to \Omega^\ast(\mathcal{A}(G)) \) of differential graded algebras defined by \( \psi(\omega)_x = \omega_1 \). Consequently, we can state the following proposition.

**Proposition 1.1.** Keeping the same hypothesis and notations as above, we have

\[
H^\ast_{\alpha,L}(G) \simeq H^\ast(\mathcal{A}(G); \mathbb{R})
\]

Let us to introduce the notion of piecewise smooth cohomology of Lie groupoids. Let \((M, K)\) be a combinatorial manifold and \( G \) a locally trivial Lie groupoid on \( M \) with source projection \( \alpha \) and target projection \( \beta \). For each simplex \( \Delta \in K \), we denote by \( G^\Delta \) the Lie groupoid restriction of \( G \) to \( \Delta \) and \( \mathcal{A}(G^\Delta) \) its Lie algebroid. Since \( G \) is locally trivial its Lie algebroid \( \mathcal{A}(G) \) is transitive and we know that \( \mathcal{A}(G^\Delta) \simeq \mathcal{A}_\Delta \). Similarly to piecewise smooth forms on Lie algebroids, we give now the notion of piecewise smooth form on \( G \).

**Definition.** A piecewise smooth \( \alpha \)-form of degree \( p \) (\( p \geq 0 \)) on \( G \) is a family \((\omega_\Delta)_{\Delta \in K}\) such that the following conditions are satisfied.
• For each $\Delta \in K$, $\omega_\Delta \in \Omega^p_{\alpha,L}(G^c_\Delta)$ is a $\alpha$-smooth form of degree $p$ on $G^c_\Delta$.

• If $\Delta$ and $\Delta'$ are two simplices of $K$ such that $\Delta' \prec \Delta$, one has $(\omega_\Delta)_{\Delta'} = \omega_{\Delta'}$.

The $C^\infty(G)$-module of all piecewise $\alpha$-smooth forms of degree $p$ on $G$ will be denoted by $\Omega^p_{\alpha,L,ps}(G)$.

A wedge product and an exterior derivative can be defined on $\Omega^\ast_{\alpha,L,ps}(G)$ by the corresponding operations on each submanifold $G^c_\Delta$, giving to $\Omega^\ast_{\alpha,L,ps}(G)$ a structure of differential graded algebra defined over $R$. The cohomology space of this complex will be denoted by $H^\ast_{\alpha,L,ps}(G)$.

Our aim is to relate the cohomology space $H^\ast_{\alpha,L,ps}(G)$ of $G$ to the cohomology space $H^\ast_{ps}(A(G))$ of its Lie algebroid $A(G)$. For that, we have to consider a map $\phi$ from the complex $\Omega^\ast_{\alpha,L,ps}(G)$ to the complex $\Omega^\ast_{ps}(A(G); K)$. In order to obtain such map $\phi$, we are going to use the isomorphism $\psi_\Delta : \Omega^p_{\alpha,L}(G^c_\Delta) \rightarrow \Omega^p(A(G^c_\Delta))$ which induces the isomorphism of the proposition 1.

Consider now a piecewise smooth $\alpha$-form $\omega = (\omega_\Delta)_{\Delta \in K} \in \Omega^p_{\alpha,L,ps}(G)$ of degree $p$. For each simplex $\Delta \in K$, take the smooth $\xi_\Delta = \psi_\Delta(\omega_\Delta) \in A(G^c_\Delta)$. Next proposition shows us that this process gives a piecewise smooth form on $A(G)$.

**Proposition 1.2.** Keeping the same hypothesis and notations as above, if $\Delta$ and $\Delta'$ are two simplices of $K$ such that $\Delta'$ is a face of $\Delta$ then $(\xi_\Delta)_{\Delta'} = \xi_{\Delta'}$ and so $\xi = (\xi_\Delta)_{\Delta \in K}$ is a piecewise smooth on $A(G)$. Moreover, the map

$$\Phi : \Omega^\ast_{\alpha,L,ps}(G) \rightarrow \Omega^\ast_{ps}(A(G))$$

defined by $\Phi(\omega_\Delta)_{\Delta \in K} = \psi_\Delta(\omega_\Delta)$ is an isomorphism of differential graded algebras.

We can state now the main proposition of this section. Let $r_G$ be the restriction map

$$r_G : \Omega^\ast_{\alpha,L}(G) \rightarrow \Omega^\ast_{\alpha,L,ps}(G)$$

defined $r_G(\omega) = \omega_\Delta$ for each simplex $\Delta$ of $K$. Our proposition is the following.

**Proposition 1.3.** Let $(M, K)$ be a combinatorial manifold and $G$ a locally trivial Lie groupoid on $M$ with source projection $\alpha$ and target projection $\beta$. Then, the map $r_G : \Omega^\ast_{\alpha,L}(G) \rightarrow \Omega^\ast_{\alpha,L,ps}(G)$ given by restriction induces an isomorphism in cohomology.

Proof. The diagram
Our last proposition below says that the piecewise de Rham cohomology of a locally trivial Lie groupoid over a combinatorial manifold doesn’t depend on the triangulation of the base. For the proof of this result, we need a corollary from Mishchenko’s theorem [4], which we state now.

**Proposition 1.4.** Let $M$ be a smooth manifold smoothly triangulated by a simplicial complex $K$ and $\mathcal{A}$ a transitive Lie algebroid on $M$. Let $L$ be other simplicial complex and assume that $L$ a subdivision of $K$. Then, the piecewise smooth cohomology of the sheaf $(\mathcal{A}_\Delta)_{\Delta \in K}$ is isomorphic to the one of the sheaf $(\mathcal{A}_{\Delta})_{\Delta \in L}$. Thus, the morphism from $\Omega^p_{ps}(\mathcal{A}; K)$ to $\Omega^p_{ps}(\mathcal{A}; L)$ which induces that isomorphism in cohomology is given by restriction.

**Proof.** The result follows from the commutativity of the following diagram

\[
\begin{array}{ccc}
\Omega^p(A; M) & \xymatrix{ & \Omega^p_{ps}(A; K) } \\
\Omega^p_{ps}(A; L) & \xymatrix{ & \Omega^p_{ps}(A; L) }
\end{array}
\]

where $\Phi$ is also given by restriction. □

Lastly, our proposition follows.

**Proposition 1.5.** Let $M$ be a smooth manifold smoothly triangulated by a simplicial complex $K$ and $L$ other simplicial complex which a subdivision of $K$. Let $G$ be a locally trivial Lie groupoid on $M$. Then, the piecewise de Rham cohomology of $G$ obtained by the combinatorial manifold $(M, K)$ is isomorphic to the the piecewise de Rham cohomology of $G$ obtained by the combinatorial manifold $(M, L)$. Thus, this isomorphism is induced by the restriction map.

**Proof.** Denote by $\phi : \Omega^*_{\alpha, L, ps}(G; K) \longrightarrow \Omega^*_{\alpha, L, ps}(G; L)$ the map given by restriction. The diagram

\[
\begin{array}{ccc}
\Omega^*_{\alpha, L}(G) & \xymatrix{ & \Omega^*(A; M) } \\
\Omega^*_{\alpha, L, ps}(G) & \xymatrix{ & \Omega^*_{ps}(A; K) }
\end{array}
\]

is commutative, where $r_A$ is the restriction map given at Mishchenko’s theorem [4]. By this theorem, the map $r_A$ is an isomorphism in cohomology and so the proof is done. □
is commutative. By propositions 2, 3 and 4, the non labeled maps are isomorphisms in cohomology and so the map \( \phi \) also is isomorphism in cohomology. \( \square \)

## 2 Piecewise invariant cohomology

In this section we will note some remarks regarding to piecewise invariant cohomology of a transitive Lie algebroid over a combinatorial manifold when a Lie group acts on the Lie algebroid. We recall the basic definitions and the main result regarding to invariant cohomology, following the paper [1] by Kubarski. We will begin first with a general result concerned to natural transformations between functors.

Consider the category of all transitive Lie algebroids over combinatorial manifolds and the category of all cochain algebras. Suppose that \( F \) and \( G \) are two functors from the category of transitive Lie algebroids over combinatorial manifolds to the category of cochain algebras and let \( t \) be a natural transformation between the functors \( F \) and \( G \). For each transitive Lie algebroid \( A \) over a combinatorial manifold, denote by \( t_A \) the cochain morphism \( t_A : F(A) \to G(A) \). Our next proposition is the following.

**Proposition 2.1.** Keeping the same hypothesis and notations as above, suppose yet that the following conditions hold.

- For each finite dimensional real Lie algebra \( g \) and each contractible combinatorial manifold \( M \),
  
  \[ H(F(TM \times g)) \cong H(G(TM \times g)) \]

- For each transitive Lie algebroid on a combinatorial manifold \( M \), if \( U \) and \( V \) are regular open subsets in \( M \) and \( t_{A_U} \), \( t_{A_V} \) and \( t_{A_{U \cap V}} \) induce isomorphism in cohomology, then \( t_{A_{U \cup V}} \) also induces isomorphism in cohomology.

Then, in these conditions, \( t_A \) induces isomorphism in cohomology for all transitive Lie algebroids \( A \) over a combinatorial manifold \((M, K)\).

Proof. The proof is essentially the same as the proof of Mishchenko’s theorem done in [2]. Let \( A \) be a transitive Lie algebroid over a combinatorial manifold \((M, K)\) and \( U = St\Delta \), for some simplex \( \Delta \in K \). It can be
seen in [3] that $A_U$ is isomorphic, by a Lie algebroid isomorphism, to the trivial Lie algebroid $TU \times g$ defined over $U$. Hence, by the first hypothesis and properties of the functor homology, we have

$$H(F(A_U)) \simeq H(F(TU \times g)) \simeq H(G(TU \times g)) \simeq H(G(A_U))$$

and so $t_{A_U}$ induces isomorphism in cohomology. Now, by using the open covering of $M$ made from the stars of all vertices of $K$, the arguments given for the proof of Mishchenko's theorem in [4] are valid mutatis-mutandis in this case and therefore $t_A : F(A) \rightarrow G(A)$ is a quasi-isomorphism. □

We will introduce now the invariant cohomology. Fix then a transitive Lie algebroid $A$ on a combinatorial manifold $(M, K)$ and consider the corresponding sheaf $\{A^!_\Delta\}_{\Delta \in K}$. Denote by $\gamma : A \rightarrow TM$ the anchor of $A$ and by $\pi : A \rightarrow M$ the projection of the vector bundle $A$. Let $G$ be a Lie group and $(T, t)$ a left action of $G$ on the Lie algebroid $A$. That consists of left actions $T : G \times A \rightarrow A$ and $t : G \times M \rightarrow M$

such that $\pi$ is equivariant and, for each $g \in G$, the left translations $(T_g, t_g)$ are morphisms of Lie algebroids. Suppose yet that, for each $g \in G$, the family $\lambda_g = (T_g \Delta)_{\Delta \in K}, t_g$ is a morphism of sheaves of Lie algebroids from $\{A^!_\Delta\}_{\Delta \in K}$ into $\{A^!_\Delta\}_{\Delta \in K}$. Under these conditions, we give the definition of piecewise invariant form.

**Definition.** Given a piecewise smooth form $\omega = (\omega_\Delta)_{\Delta \in K} \in \Omega^*_\text{ps}(A; K)$ we say that $\omega$ is piecewise invariant if, for each $g \in G$,

$$\lambda_g^*(\omega) = \omega$$

The space of all piecewise invariant forms on $A$ will be denoted by $\Omega^*_\text{ps}(A; K)$ or $\Omega^*_\text{ps}(A; M)$. Thus, since $d(\lambda_g^* \omega) = \lambda_g^*(d\omega)$, the space $\Omega^*_\text{ps}(A; K)$ is stable under the exterior differential and so it is a complex where the differential is the restriction of the exterior derivative to each simplex of $K$. The cohomology space of the differential complex $\Omega^*_\text{ps}(A; K)$ will be denoted by $H^*_\text{ps}(A; K)$.

Keeping the same hypothesis, we have a natural map $\Phi$ given by restriction

$$\Phi : \Omega_t(A; M) \rightarrow \Omega_{I_{ps}}(A; K)$$

$$\Phi(\omega) = (\omega_\Delta)_{\Delta \in K}$$

**Proposition 2.2.** Keeping the same hypothesis and notations as above, the map $\Phi$ induces an isomorphism in cohomology.
Proof. Let $d_{dR}$ and $d_{ps}$ denote the differential of the complexes $\Omega^*_{dR}(A; M)$ and $\Omega^*_{ps}(A; K)$ respectively and $\Psi$ the restriction map given at the Mishchenko’s theorem \[4\]. We are going to apply the proposition 2.1. Let $g$ be a finite dimensional real Lie algebra, $(M, K)$ a contractible combinatorial manifold and $A = TM \times g$ the trivial Lie algebroid on $M$. We want to check that

$$H^*_I(A; M) \simeq H^*_{I_{ps}}(A; K)$$

The steps used in the proof of the proposition 3.1.2 are valid here provide we check that, if $\xi = (\xi_\Delta)_{\Delta \in K} \in \Omega^*_{I_{ps}}(M)$ is a piecewise invariant form, then $(\gamma_\Delta \xi_\Delta)_{\Delta \in K}$ is a piecewise invariant form, which is true since $\gamma$ commutes with $t_g$. The second condition of the proposition 2.1 is the analogous to the diagram used in the proof of Mishchenko’s in \[4\]. □

For next proposition, we recall that Kubarski showed us that, if a Lie group $G$ acts on a transitive Lie algebroid $A$ and the action extends to a Lie algebroid morphism from $TG \times A \to A$, then the inclusion induces a monomorphism in cohomology in the case $G$ to be compact and moreover the inclusion induces an isomorphism in cohomology in the case of $G$ to be compact and connected \[1\].

**Proposition 2.3.** Let $A$ be a transitive Lie algebroid on a combinatorial manifold $(M, K)$ and $(T, \ell)$ a left action of a Lie group $G$ on $A$. Suppose that the action $T$ extends to a morphism of Lie algebroids $\widehat{T}: TG \times A \to A$

Denote by $i_{ps}$ the inclusion map

$$\begin{array}{ccc}
\Omega^*_{I_{ps}}(A_M) & \xrightarrow{i_{ps}} & \Omega^*_{ps}(A_M) \\
\Phi & & \Psi \\
\Omega^*_{I_{ps}}(A_M) & \xrightarrow{i_{ps}} & \Omega^*_{ps}(A_M)
\end{array}$$

Then, if $G$ is compact, the inclusion induces a monomorphism in cohomology. Thus, if $G$ is compact and connected, then the inclusion induces isomorphism in cohomology.

Proof. It is clear that the following diagram

$$\begin{array}{ccc}
\Omega^*_T(A_M) & \xrightarrow{i} & \Omega^*_T(A_M) \\
\Phi & & \Psi \\
\Omega^*_{I_{ps}}(A_M) & \xrightarrow{i_{ps}} & \Omega^*_{ps}(A_M)
\end{array}$$

is commutative, where $i$ and $i_{ps}$ are the inclusion and $\Psi$ is the restriction map given at the Mishchenko’s theorem. So, it is also commutative the following diagram.
\[ H^i_\ast(A_M) \xrightarrow{H(i)} H^\ast(A_M) \]
\[ H(\Phi) \downarrow \downarrow H(\Psi) \]
\[ H^i_{ps}(A_M) \xrightarrow{H(i_{ps})} H^\ast_{ps}(A_M) \]
where \( H(i) \) and \( H(\Psi) \) are isomorphisms by Kubarski’s and Mishchenko’s theorems respectively. The map \( H(\Phi) \) is isomorphism by last proposition. So, the result is proved. □

References

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