Thermodynamics and Evaporation of the 2+1-D Black Hole

BENNI REZNIK

School of Physics and Astronomy
Beverly and Raymond Sackler Faculty of Exact Sciences
Tel Aviv University, Tel-Aviv 69978, Israel.

Abstract

The properties of canonical and microcanonical ensembles of a black hole with thermal radiation and the problem of black hole evaporation in 3-D are studied. In 3-D Einstein-anti-de Sitter gravity we have two relevant mass scales, \( m_c = 1/G \), and \( m_p = (\hbar^2 \Lambda/G)^{1/3} \), which are particularly relevant for the evaporation problem. It is argued that in the ‘weak coupling’ regime \( \Lambda < (\hbar G)^{-2} \), the end point of an evaporating black hole formed with an initial mass \( m_0 > m_p \), is likely to be a stable remnant in equilibrium with thermal radiation. The relevance of these results for the information problem and for the issue of back reaction is discussed. In the ‘strong coupling’ regime, \( \Lambda > (\hbar G)^{-2} \) a full fledged quantum gravity treatment is required. Since the total energy of thermal states in anti-de Sitter space with reflective boundary conditions at spatial infinity is bounded and conserved, the canonical and microcanonical ensembles are well defined. For a given temperature or energy black hole states are locally stable. In the weak coupling regime black hole states are more probable than pure radiation states.

\(^1\text{e-mail: reznik@tauvm.tau.ac.il}\)
1 Introduction

Three dimensional Einstein’s gravity has no local degrees of freedom and also no long range Newtonian-like interaction. Local spacetime curvature exits only in the presence of a local matter source.\(^1\) Notwithstanding this apparent simplicity of the theory, Bañados Teitelboim and Zanelli\(^2\) have managed, by adding a negative cosmological constant source, and by a particular identification of points in anti-de Sitter spacetime, to find a black hole solution that resembles in many features the 4-D black hole. Their solution is locally indistinguishable from anti-de Sitter spacetime. The curvature is constant, but globally the solution has the topology of a black hole.\(^3\)

The metric of a 3-D black hole, of mass \(m\) and vanishing angular momentum and charge, in a static coordinate system is\(^4\)

\[
ds^2 = \left( \Lambda r^2 - 8Gm \right) dt^2 - \left( \Lambda r^2 - 8Gm \right)^{-1} dr^2 - r^2 d\theta^2
\]

(1)

where \(\Lambda\) is the negative of the cosmological constant, and the location of the horizon is given by \(r_h = \sqrt{8Gm/\Lambda}\).

The entropy of the black hole is given by \(S = L/4\hbar G + C_d\) (\(L = 2\pi r_h\)), where \(C_d\) is an unknown additive constant.\(^5\) It can be shown that for any classical evolution this entropy satisfies \(\delta S \geq 0\), provided that suitable energy positivity conditions hold.\(^6\) The entropy is functionally related to the Hawking temperature\(^7\) of the black hole \(T_H = \hbar \kappa / 2\pi\) (\(\kappa = \Lambda r_h\) is the surface gravity on evaluated on the horizon), via \(dm = T dS\), in a complete analogy to the first law of thermodynamic. Therefore, the 3-D black hole satisfies all the usual mechanical laws analogous to the laws of thermodynamics.\(^1\), \(^8\)

In spite of these similarities, the 3-D black hole differs significantly from the ordinary 4-D Schwarzschild black hole. We note that the Hawking temperature decreases like \(m^{1/2}\) as the black hole evaporates. Naively, by Stefan’s law, the time required for a complete evaporation is therefore infinite. However at the ‘end point’ the spacetime metric does not yield empty anti-de Sitter spacetime but rather a throat of zero radius that can be regarded as a sort of ‘extremal’ black hole.\(^2\) The empty anti-de Sitter spacetime is separated by a negative energy gap from the spectra of black hole states. Furthermore, the metric of the black hole is asymptotically anti-de Sitter, and spatial infinity is time-like. Therefore, as is well known, spacetime is not globally hyperbolic\(^12\) and one has to impose reflective boundary conditions at infinity in order to have a well defined Cauchy problem.\(^13\) As we shall show, these features significantly modify the thermodynamics of the black holes and the problem of the final state.

In this work we study the canonical and microcanonical ensembles of a black hole and thermal radiation and the problem of black hole evaporation in 3-D. The total energy of a thermal state in anti-de Sitter spacetime is bounded, and with reflective boundary

\(^1\)For further study of the black hole see references [3 – 8].
\(^2\)In the literature Newton’s constant is sometimes taken as \(G = 1/8\). We chose the usual action \(1/16\pi G \int R\).
conditions at spatial infinity it is also conserved. Therefore, the canonical partition function $Z(\beta)$ and the microcanonical density function $N(E)$ are well defined. Some aspects of the problem resembles those of a 4-D Schwarzschild black hole in an anti-de Sitter spacetime. However, contrary to the 4-D case, a black hole of mass $m > (\hbar^2 \Lambda/G)^{1/3}$ is always in local equilibrium with a thermal radiation state. When tunneling between states is also considered, the value of the cosmological constant determines whether a black hole with radiation or pure radiation without a black hole, is more probable and determines the final state of the system.

Contrary to 4-D gravity, in 3-D Einstein-anti-de Sitter gravity we have two relevant mass scales, $1/G$ and $(\hbar^2 \Lambda/G)^{1/3}$, that dictate the physical content of of the model. The relation of these two scales is determined by the value of the cosmological constant. Only for $\Lambda = (G\hbar)^{-2}$ do the two scales coincide. The existence of two mass scales is particularly relevant for the final state of an evaporating black hole. It is argued that for $\Lambda < (hG)^{-2}$ the end point of a black hole formed with an initial mass $m_0 > (\hbar^2 \Lambda/G)^{1/3}$ is likely to be a stable remnant. For the particular case $m_0 > 1/G > (\hbar^2 \Lambda/G)^{1/3}$ the system will wind up as a black hole of mass $m_{bh}(m_0)$ in a state of equilibrium with thermal radiation of total energy $m_{\text{rad}} = m_0 - m_{bh}$. When the mass of the initially formed black hole $1/G > m > (\hbar^2 \Lambda/G)^{1/3}$, it is argued that the black hole will radiate at a low rate, and reach gradually equilibrium. For $m < (\hbar^2 \Lambda/G)^{1/3}$ the semi-classical approximation breaks down. In the regime of ‘large curvature’ $\Lambda > (hG)^{-2}$ things are more complicated, a full fledged quantum gravity treatment is required.

The paper proceeds as follows. In section 2, we discuss the basic units which are relevant to the problem of a black hole in anti-de Sitter spacetime. The canonical and microcanonical ensembles for a black hole and radiation, are studied in sections 3 and 4 respectively. The problem of black hole evaporation is examined in section 5. Finally, we discuss the relevance of our results to the information problem and to the issue of back reaction in section 6. Throughout the paper we adopt the units $k_B = c = 1$.

## 2 Basic Units

In 4-dimensions, the Planck mass provides the basic scale of the theory. As the mass of the Schwarzschild black hole reaches the Planck scale, the fluctuations of the geometry, and the back reaction of the emitted radiation, becomes large. The Hawking temperature also reaches the Planck scale.

The situation is different for the 2+1 black hole. One can not construct a basic Planckian mass scale out of Newton’s constants $G$ and $\hbar$ alone since $G$ has a dimension of $(mass)^{-1}$. The cosmological constant (of dimension $(length)^{-2}$) is needed to fix the radius of the black hole’s horizon. Combining $\Lambda$, $G$ and $\hbar$ yields various possible definitions of a basic mass unit.

On physical grounds, we note that the theory singles out two basic mass units. The fluctuations of the black hole geometry become important when $r_{\text{horizon}} \sim r_{\text{compton}}$, i.e.,
the radius of the black hole becomes comparable to the Compton wave length. This yields a 3-D analog of the Planck mass

$$m_{p}^{(3)} = \left( \frac{\hbar^2 \Lambda}{G} \right)^{1/3} \equiv m_p.$$  \tag{2}

However, the Hawking temperature (or the wavelength of the radiation) of a black hole with mass $m_p$ does not coincide in general with this ‘Planck scale’.

Newton’s constant yields a second basic mass unit

$$m_c = \frac{1}{G}.$$ \tag{3}

The physical significance of this ‘classical’ mass unit is that for $m > m_c$ ($m < m_c$) the wave length $\lambda$ of the Hawking radiation satisfies $\lambda < r_h$ ($\lambda > r_h$). The relation of the two mass units is determined by the value of the cosmological constant. For $\Lambda = (\hbar G)^{-2}$, $m_c = m_p$. Notice that in the ‘strong coupling’ regime $\Lambda > (\hbar G)^{-2}$, the classical action

$$\frac{1}{8\pi G} \int (R + 2\Lambda) \sqrt{g} d^3x$$

of anti-de Sitter spacetime is smaller than the quantum action $\hbar$. Therefore, it is likely that quantum fluctuations are significant and classical considerations may not be adequate.

## 3 The Canonical Ensemble

Thermal states in anti-de Sitter space can be constructed by a periodic identification of the Euclidean time coordinate $\tau_e = it$ with a period $\beta = T^{-1}$. The thermal radiation will be in equilibrium, with local temperature $T/\sqrt{g_{tt}}$, in the static coordinate system ($\tilde{t}$), and hence its presence breaks the $SO(2,2)$ invariance of the ‘empty’ anti-de Sitter spacetime to $SO(2) \times SO(2)$. Due to the red shift with respect to the preferred origin, the local temperature decreases as $1/r$ at infinity and the local energy is expected to decrease like $1/r^3$. The total energy of the thermal radiation is therefore bounded. Neglecting the back-reaction of the radiation on the geometry the local energy of a conformally coupled scalar field in a thermal state can be approximated by

$$T_t = \frac{a_0}{\hbar^2 g_{tt}(r)^{3/2}} + O(T^2) \tag{4}$$

$a_0$ is a dimensionless Stefan-Boltzmann constant. The total energy of the state is approximated by

$$E_{\text{rad}} \simeq \frac{a_0}{\hbar^2 \Lambda} T^3 + O(T^2). \tag{5}$$

The entropy and free energy of the radiation can be easily derived,

$$S_{\text{rad}} \simeq \frac{3a_0}{2} \frac{1}{\Lambda \hbar^2} T^2 + O(T), \tag{6}$$
and
\[ F_{rad} \simeq -\frac{a_0}{2} \frac{1}{\Lambda \hbar^2} T^3 + O(T^2). \] (7)

The partition function is given by
\[ Z_{rad}(\beta) \simeq \exp\left(\frac{a_0}{2} \frac{1}{\Lambda \hbar^2} \beta^{-2} + O(\beta^{-1})\right). \] (8)

A state of pure thermal radiation is unstable against collapse to a black at sufficiently high temperature. This can be seen by noting that for \( r << \Lambda^{-1/2} \) the total energy of the radiation in a sphere of radius \( r \) increases as \( r^2 T^3 \), while for the 3-D black hole we have \( m_{bh} \sim r^2 \). This yields an upper bound \( T < T_{\text{critical}} \) on the maximal value of the temperature,
\[ T < T_{\text{critical}} = \left(\frac{\Lambda \hbar^2}{G}\right)^{1/3} = m_p, \] (9)

which corresponds to a state of total energy \( E_{rad} < m_c \). Alternatively, we may look at a solution of Einstein’s equation with a negative cosmological constant and a spherically symmetric source. We have \[ g^{rr}(r) = \Lambda r^2 - 16\pi G \int_0^r \rho r dr + c. \] (10)

For \( c=1 \) and \( \rho = 0 \) the metric corresponds to an anti-de Sitter space. With \( \rho = a_0 T^3/\hbar^2 \) we note that a horizon forms unless (9) is maintained.

We now examine states that contain a black hole. The Hawking temperature of a black hole of mass \( m \) is \( \hbar \Lambda r_h(m)/2\pi \), therefore
\[ m(T) = \left(\frac{2\pi}{\hbar}\right)^2 \frac{1}{\Lambda G} T^2. \] (11)

This semi-classical approximation safely holds only when \( m > m_p \), this yields
\[ T > (h^4 \Lambda^2 G)^{1/3} \equiv T_1 \] (12)

The heat capacity of the black holes \( \partial m/\partial T \) is always positive. Therefore, for a given temperature \( T \) there exists a black hole of mass \( m(T) \) in a state of local equilibrium with the radiation.

Contrary to a Schwarzschild black hole in an asymptotically flat space the canonical partition function is well defined. The entropy of the black hole is given by
\[ S_{bh} = \frac{\pi}{2hG} r_h + C_d = \frac{\pi}{2h} \left(\frac{8}{\Lambda G}\right)^{1/2} m^{1/2} + C_d \] (13)

Assuming the integration constant \( C_d \) is a finite number\[ \text{13} \], the density of states of the black holes is
\[ N(m) \sim e^S \sim e^{m^{1/2}}. \] (14)

4
The integral that defines the partition function

\[ Z(\beta) = \int_0^\infty N(m)e^{-m\beta}dm \]  

therefore converges.

\( Z(\beta) \) can be computed directly from (15) or from the the Euclidean path integral approach by a saddle point approximation.\[^7\] In either of the two ways we find

\[ \log Z = -I_e = \frac{F}{T} = \left(\frac{2\pi}{\hbar^2}\right)^2 \frac{\beta^{-1}}{8G\Lambda} \]  

and the free energy

\[ F_{bh} = -\left(\frac{2\pi}{\hbar^2}\right)^2 \frac{T^2}{8G\Lambda}. \]  

Comparing (7) and (17) we note that for \( T > m_c \), \( F_{rad} < F_{bh} \). In this case an initial state with a black hole will ultimately tunnel to a configuration with pure radiation. The properties of the ensemble depend on the value of \( \Lambda \). For \( \Lambda < (\hbar G)^{-2} \), we have \( m_p < m_c \). Since a pure state is unstable for \( T > m_p \), in this case only black hole configurations are stable. Note however that by (12) our semi-classical consideration strictly hold only for \( T > T_1 > m_c \).

On the other hand, in the strong coupling regime, \( \Lambda > (\hbar G)^{-2} \), we have \( m_p > m_c \). If the large fluctuations of the metric can be disregarded, then for \( T_1 < T < m_p \) (\( T_1 < m_c \)) a configuration with pure thermal radiation is stable, and for \( T > m_p \) only black hole states are stable. For \( T < T_1 \) (which corresponds to \( m < m_p \)) our semi-classical approximation for the black hole breaks.

## 4 The Microcanonical Ensemble

The boundary conditions at infinity\[^{13}\] insure that all the outgoing flux to infinity is reflected back and the total energy is conserved. Therefore, one can consider the microcanonical ensemble and evaluate the number of states \( N(E)dE \) of the system between \( E \) to \( E + dE \). Given the canonical partition function \( Z(\beta) \), \( N(E) \) is expressed by the inverse Laplace transform

\[ N(E) = \int_{-i\infty}^{+i\infty} Z(\beta)e^{\beta E}d\beta. \]  

For \( T < T_c \) the partition function of a thermal state is given by (8). The integral (18) has a saddle point at

\[ \beta \simeq \left(\frac{1}{a_0 \hbar^2} \frac{1}{\Lambda}\right)^{1/3} E^{-1/3} \]  

and in the stationary phase approximation \( N(E) \) is given by

\[ N_{rad}(E) \simeq \exp\left(\frac{3}{2} \left(\frac{a_0}{\Lambda\hbar^2}\right)^{1/3} E^{2/3}\right) \]  

5
This expression for $N(E)$ holds for $E < m_c$, which corresponds to a saddle point at $\beta > m_p^{-1}$.

From the partition function of the black hole (16) we find a saddle point at

$$
\beta = \left(\frac{2\pi}{\hbar}\right) \left(\frac{1}{8G\Lambda}\right)^{1/2} E^{-1/2}
$$

Equation (22) holds provided that $E > m_p$. Comparing equations (21) and (22) we find that for $E > E_1 = G^{-3}\Lambda^{-1}\hbar^{-2}$, $N_{bh} < N_{rad}$. $E_1$ corresponds to the total energy of a thermal distribution with temperature $m_c$. For $\Lambda < (G\hbar)^{-2}$ $E_1 > m_c$ and hence black hole states in equilibrium are always more probable. Any configuration with initial energy $E > m_p$ will settle finally to a state of a black hole in equilibrium with thermal radiation.

At the stationary point, the temperature of the system will be determined by the equation

$$
E \simeq a_0 \frac{1}{\hbar^2\Lambda} \beta^{-3} + \left(\frac{2\pi}{\hbar}\right)^2 \frac{1}{\Lambda G} \beta^{-2}
$$

Since $E, T > 0$, (23) has only one solution for $\beta$.

For $\Lambda > (G\hbar)^{-2}$, and $m > m_p$, black hole states are more probable. Thermal radiation states are more likely at $E_1 < E < m_c$, but in this regime our semi-classical approximation breaks.

## 5 Black Hole Evaporation

We have seen in the previous section that a configuration of given total energy $m_0 > m_p$ evolves finally to a state of a black hole in equilibrium with thermal radiation. This suggests that even in a dynamical situation of matter collapsing to a black hole the final state may be a black hole of a mass $m(m_0)$.

Coming to the dynamical problem of black hole evaporation it is essential to ask whether the black hole can really lose mass. During the evaporation the geometry of space remains asymptotically anti-de Sitter. Any massive particle emitted by the black hole that travels along a geodesic path will not reach spatial infinity. Rather it will be captured back by the black hole on a time scale of $\Lambda^{-1/2}$. Massless particles do reach spatial infinity. However, since anti-de Sitter spacetime is not globally hyperbolic, one must impose appropriate boundary conditions at spatial infinity in order to get a well defined Cauchy problem. This can be obtained by imposing either von-Neumann or Dirichlet boundary conditions on the matter fields. As a result energy is conserved and the energy flux is reflected back from infinity. Therefore, even massless particles return to the black hole on the same time scale. (Since space is not empty the motion for both cases, $m = 0$ and $m \neq 0$, will not necessarily be geodesic but this should not alter this basic picture.)
Let a matter distribution of total mass $m_0$ in an anti-de Sitter space collapse and form a black hole. We first consider the possibility that $\Lambda < (\hbar G)^{-2}$, i.e., $m_c > m_p$. If the initial mass satisfies $m_0 > m_c$ then the typical wave length related to Hawking’s temperature is smaller than the radius of the black hole. Therefore, the process of energy emission can be well approximated since $\lambda < r_h$ by Stefan’s radiation law, at least until the black hole’s mass reduces to $m \sim m_c$. A simple integration yields the result

$$\Delta t = \frac{1}{\hbar G^2 \Lambda} \left( m_c^{-1} - m_0^{-1} \right) > \Lambda^{-1/2}.$$  \hspace{1cm} (24)

Hence the emitted radiation has sufficient time to reach back the black hole. Since the whole process takes place for black hole much above the Planck scale this result strongly indicates that the final state is that of a macroscopic black hole in a state of equilibrium with thermal radiation at a temperature given by Eq. (23). When the initial mass is $m_p < m < m_c$ the situation is drastically altered. The wavelength of Hawking’s radiation $\lambda$ satisfies $\lambda > r_h$. We argue that the black hole can not emit radiation at such a long wavelength. Since $\lambda \sim \Lambda^{-1/2}$, the reflected flux of a single wave reaches back to the black hole before it can get away and interferes destructively. In other words real particle can not materialize. The situation is very similar to that of an exited atom inside a cavity. If the wavelength related to the energy gap between the energy levels of the atom is of the order of the size of the box or larger, the probability to emit a photon will be significantly smaller. Therefore, in this case, we expect a much slower rate of emission, and the black hole will have sufficient time to reach an equilibrium.

The state of affairs is more complicated for $\Lambda > (\hbar G)^{-2}$. As noted in section 2., a semiclassical description of spacetime may not be appropriate for this ‘strong coupling’ regime. Nevertheless, let us examine the evaporation assuming a classical spacetime background. In this case the typical wavelength of Hawking radiation will satisfy $\lambda < r_h$ all the way to the Planck mass. Since $m_p > m_c$, the black hole will emit radiation, and the rate of emission can be described adequately up to the Planck scale by Stefan’s law. It is easily verified that the time required for the black hole to reach the Planck mass in this case satisfies $\Delta t < \Lambda^{-1/2}$. Hence, the black hole may lose its mass much before the arrival of the return flux. If Stefan’s law is extrapolated beyond $m_p$, $\Delta t \to \infty$. However, in this case, without a quantum model for the black hole, it is not clear at what rate the black hole radiates if it does at all, or what is the cross section for absorption.

### 6 Discussion

We have seen that in the small coupling regime, $\Lambda < (\hbar G)^{-2}$, the end point of an evaporating 3-D black hole is likely to be a macroscopic black hole whose mass is determined by the initial configuration. Consequently, there should not be an information problem in this case. In general the entropy $S_0$ of the initial state of collapsing matter with mass $m_0 > m_c$ can not exceed the entropy of thermal radiation with the same mass. We have from eq. (3) $S_0 < S_{rad} \sim (\hbar^2 \Lambda)^{-1/3} m_0^{2/3} > 1$. On the other hand, the entropy of the final
black hole state is given by eq. (13), and it is easily verified to be of the same order of magnitude of $S_0$ above when the integration constant $C_d$ is bounded and small. Therefore, there is sufficient room for storing the information in the ‘surface’ states alone without the introduction of a large degeneracy constant $C_d$. Our conclusion agrees with the suggestion made in reference [8] (where the relation of the entropy to the process of tunneling from an initial matter configuration to a black hole was studied), that the degeneracy of the 3-D black hole is likely to be finite or zero, i.e., $C_d \sim 0$, and that the entropy is given simply by the length of the horizon. Our semi-classical arguments may not apply to the strong coupling regime ($\Lambda > (\hbar G)^{-2}$).

It is also interesting to note that for the case $m > m_c$, $\lambda < r_h$, i.e., during the evaporation process photons are emitted from a small region of the horizon. As indicated in the following this fact may seem at first to be paradoxical. Having two scales of length, $\lambda$ and $r_h$, one may average the stress tensor on an intermediate scale $l$, $\lambda < l < r_h$, and compute its expectation value: $\langle \int T_{ab} \rangle$. The domains of size $l^2$ are larger than the scale where quantum fluctuations are important, but still these domains are sufficiently close to the horizon. Since also a large number of photons is emitted from such a domain in a time interval $l/c$, we could expect that $\langle \int T_{ab} \rangle$ behaves as a classical stress tensor. However, a classical stress tensor should satisfy the positivity requirement and hence, by the second law of black hole mechanics, the radius of the black hole’s horizon can not decrease and therefore, radiation can not be emitted. On the other hand, we could compute the stress tensor on small region of scale $\lambda$ near the horizon and find negative energy, and therefore conclude that the black hole can radiate and lose mass. Somehow the first averaged, classically behaved, stress tensor must also contain a negative contribution that accounts for Hawking’s radiation. A similar ‘classical’ limit does not exist in 4-D since the wavelength of the Hawking radiation is always of the size of the horizon. It is possible that by studying the interpolation between the two cases, one could understand better the problem of back reaction, at least in the 3-D context.

Acknowledgments
I would like to thank Yakir Aharonov and Aharon Casher for very helpful discussions. The research was supported in part by grant 425-91-1 of the Basic Research Foundation, administered by the Israel Academy of Sciences and Humanities.
References

[1] A. Staruszkiewicz, Acta. Phys. Polon. 24, 739 (1963).
   S. Giddings, J. Abbott and K. Kuchar, Gen. Rel. Grav., 16, 751 (1984).
   S. Deser, R. Jackiw and G. t’Hooft, Ann. Phys., 152, 220 (1984).

[2] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett., 69, 1849 (1992).
   M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48, 1506 (1993).

[3] D. Cangemi, M. Leblanc and R. B. Mann, Phys. Rev. D48, 3606 (1993).
   A. Achúcarro and M. E. Otriz, Phys. Rev. D48, 3600 (1993).
   N. Kaloper, Phys. Rev. D48, 2598 (1993).
   G. Horowitz and D. Welch, Phys. Rev. Lett. 71, 328 (1993).

[4] The gravitational collapse of dust to a 3-D black hole was studied by: S. F. Ross, and
   R. B. Mann, Phys. Rev. D47, 3319 (1993).

[5] The geodesic motion around the 2+1-D black hole was studied by:
   C. Farina, J. Gamboa and A. J. Seguí-Sntonja, Class. Quantum Grav. 10, 193 (1993).
   N. Cruz, C. Mrtínez and L. Peña,”Geodesic structure of the 2+1 black hole”, gr-qc/9401025.

[6] For a calculation of the stress tensor in the background of the black hole see:
   A. R. Steif, Phys. Rev. D59, 585, (1994).
   G. Lifschytz and M. Ortiz,”Scalar Field Quantization on the 2+1 Dimensional Black
   Hole Background”, Preprint CPT-2243.

[7] The thermodynamic functions of the black hole where computed from the action
   in the saddle point approximation by: M. Bañados, C. Teitelboim and J. Zanelli,
   “Dimensionally Continued Black Holes”, gr-qc/9307033.

[8] It was argued that for the in 2+1 black hole the constant $C_d$ is probably finite or zero
   in: F. Englert and B. Reznik, “Entropy Generation by Tunneling in 2+1-Gravity”,
   TAUP-2102-93, gr-qc/9401010.

[9] For a discussion on the thermodynamics of outer event horizons in 2+1-D gravity see:
   B. Reznik, Phys. Rev. D 45, 2151 (1992).

[10] S. W. Hawking, Commun. Math. Phys., 43, 199 (1975).

[11] J.M. Bardeen, B. Carter and S.W. Hawking, Cummun. Math. Phys. 31, 161 (1973).

[12] S. W. Hawking and G. F. R. Ellis, The Large Scale of Space-Time , Cambridge
   University Press, Cambridge, England, 1973.
[13] S. J. Avis, C. J. Isham and D. Storey, Phys. Rev. D18, 3565 (1978).

[14] S. W. Hawking and D. N. Page, Commun. Math. Phys. 87, 577 (1983).

[15] The stress tensor of a thermal state in 4-D anti-de Sitter space was obtained by: B. Allen, A. Folacci and G. W. Gibbons, Phys. Lett. B 189, 304 (1987).