THIRD HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF $p$-VALENT ANALYTIC FUNCTIONS

D. VAMSHEE KRISHNA

This paper is dedicated to Professor T. RAMREDHY on his 72nd birthday.

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Abstract. The objective of this paper is to obtain an upper bound to the third Hankel determinant for certain subclass of $p$-valent functions, using Toeplitz determinants.

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1. INTRODUCTION

Let $A_p$ denote the class of functions $f$ of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + \cdots,$$

in the open unit disc $E = \{z : |z| < 1\}$ with $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Let $S$ be the subclass of $A_1 = A$, consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture, i.e., for a univalent function its $n^{th}$ coefficient is bounded by $n$ (see [3]). The bounds for the coefficients of these functions give information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of $f$ for $q \geq 1$ and $n \geq 1$ (when $p = 1$) was defined by Pommerenke [10] as follows and has been extensively studied.

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$  \hspace{1cm} (1.2)

One can easily observe that the Fekete-Szegő functional is $H_2(1) = a_3 - a_2^2$. Fekete and Szegő then further generalized the estimate $|a_3 - \mu a_2^2|$ with $\mu$ real and $f \in S$. Further, sharp upper bounds for the functional $H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$, the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant.
(functional), were obtained for various subclasses of univalent and multivalent analytic functions. Janteng et al. [6] have considered the functional \(|a_2a_4 - a_3^2|\) and found a sharp upper bound for the function \(f\) in the subclass \(R\) of \(S\), consisting of functions whose derivative has a positive real part (also called bounded turning functions) studied by MacGregor [9] and have showed that |\(H_2(2)| \leq \frac{q}{2^q}\). For our discussion in this paper, we consider the Hankel determinant in the case of \(q = 3\) and \(n = p\), denoted by \(H_3(p)\), given by

\[
H_3(p) = \begin{vmatrix}
    a_p & a_{p+1} & a_{p+2} \\
    a_{p+1} & a_{p+2} & a_{p+3} \\
    a_{p+2} & a_{p+3} & a_{p+4}
\end{vmatrix}. \tag{1.3}
\]

For \(f \in A_p\), \(a_p = 1\), so that, we have

\[
H_3(p) = a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2) - a_{p+3}(a_{p+3} - a_{p+1}a_{p+2}) + a_{p+4}(a_{p+2} - a_{p+1}^2)
\]

and by applying the triangle inequality, we obtain

\[
|H_3(p)| \leq |a_{p+2}| |a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}| |a_{p+1}a_{p+2} - a_{p+3}| + |a_{p+4}| |a_{p+2} - a_{p+1}^2|. \tag{1.4}
\]

Incidentally, all of the functionals on the right hand side of the inequality (1.4) have known (and sharp) upper bounds except \(|a_{p+1}a_{p+2} - a_{p+3}|\). It was known that if \(f \in \mathcal{R}_p\), the class of \(p\)-valent bounded turning functions, then \(|a_k| \leq \frac{2^p}{k}\), where \(k \in \{p + 1, p + 2, \ldots\}\) and \(|a_{p+2} - a_{p+1}^2| \leq \frac{2^p}{p+2}\), with \(p \in \mathbb{N}\).

Motivated by the result obtained by Babalola [1] in finding the sharp upper bound to the Hankel determinant \(|H_3(1)|\) for the class \(\mathcal{R}\), in this paper we obtain an upper bound to the functional \(|a_{p+1}a_{p+2} - a_{p+3}|\) and hence for \(|H_3(p)|\), for the function \(f\) given in (1.1), belonging to certain subclass of \(p\)-valent analytic functions, as follows.

**Definition 1 ([13]).** A function \(f \in A_p\) is said to be in the class \(I_p(\bar{\beta})(\bar{\beta}\text{ is real}), if it satisfies the condition

\[
\text{Re} \left\{ (1 - \bar{\beta}) \frac{f(z)}{z^p} + \bar{\beta} \frac{f'(z)}{pzz^{p-1}} \right\} > 0, \quad z \in E. \tag{1.5}
\]

(1) Choosing \(\bar{\beta} = 1\) and \(p = 1\), we obtain \(I_1(1) = \mathcal{R}\).

(2) Selecting \(\bar{\beta} = 1\), we get \(I_p(1) = \mathcal{R}_p\).

2. Preliminary Results

In this section some preliminary lemmas are stated which are required for proving our results.

Let \(\mathcal{P}\) denote the class of functions consisting of \(p\), such that

\[
p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \ldots = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2.1}
\]

which are analytic in the open unit disc \(E\) and satisfy \(\text{Re} p(z) > 0\) for any \(z \in E\). Here \(p(z)\) is called Carathéodory function [4].
**Lemma 1** ([11, 12]). If \( p \in \mathcal{P} \), then \(|c_k| \leq 2\), for each \( k \geq 1 \) and the inequality is sharp for the function \( p(z) = \frac{1 + z}{1 - z} \).

**Lemma 2** ([5]). The power series for \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) given in (2.1) converges in the open unit disc \( E \) if and only if the Toeplitz determinants

\[
D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \ldots
\]

and \( c_k = \overline{c_k} \), are all non-negative. They are strictly positive except for \( p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{i\alpha}z) \), with \( \sum_{k=1}^{m} \rho_k = 1 \), \( t_k \) real and \( t_k \neq t_j \), for \( k \neq j \), where \( p_0(z) = \frac{1 + z}{1 - z} \); in this case \( D_n > 0 \) for \( n < (m - 1) \) and \( D_n = 0 \) for \( n \geq m \).

We may assume without restriction that \( c_1 \geq 0 \). Using Lemma 2, for \( n = 2 \) and \( n = 3 \), for some complex values \( x \) and \( z \) with \( |x| \leq 1 \) and \( |z| \leq 1 \) respectively, we have

\[
2c_2 = c_1^2 + x(4 - c_1^2)
\]

and

\[
4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z.
\]

To obtain our results, we refer to the classical method devised by Libera and Zlotkiewicz [8], which is used by many authors in the literature.

### 3. Main Results

**Theorem 1.** If \( f \in \mathcal{I}_p(\beta) \) (0 < \( \beta \leq 1 \)) with \( p \in \mathbb{N} \), then

\[
|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{2p}{p + 3\beta}.
\]

**Proof.** For \( f = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{I}_p(\beta) \), by virtue of Definition 1, there exists an analytic function \( p \in \mathcal{P} \) in the open unit disc \( E \) with \( p(0) = 1 \) and \( \text{Re} p(z) > 0 \) such that

\[
(1 - \beta) \frac{f(z)}{z^p} + \beta \frac{f'(z)}{pz^{p-1}} = p(z) \Leftrightarrow (1 - \beta)pf(z) + \beta f'(z) = pz^p p(z). \tag{3.1}
\]

\[
(1 - \beta)p \left\{ z^p + \sum_{n=p+1}^{\infty} a_n z^n \right\} + \beta \left\{ pz^{p-1} + \sum_{n=p+1}^{\infty} n a_n z^{n-1} \right\} = pz^p \left\{ 1 + \sum_{n=1}^{\infty} c_n z^n \right\}.
\]

Upon simplification, we obtain

\[
(p + \beta)a_{p+1}z^{p+1} + (p + 2\beta)a_{p+2}z^{p+2} + (p + 3\beta)a_{p+3}z^{p+3} + (p + 4\beta)a_{p+4}z^{p+4} + \ldots
\]

\[
= pc_1z^{p+1} + pc_2z^{p+2} + pc_3z^{p+3} + pc_4z^{p+4} + \ldots \tag{3.2}
\]
Equating the coefficients of \(z^{p+1}, z^{p+2}, z^{p+3}\) and \(z^{p+4}\) respectively in (3.2), we have
\[
a_{p+1} = \frac{pc_1}{p+\beta}; \quad a_{p+2} = \frac{pc_2}{p+2\beta}; \quad a_{p+3} = \frac{pc_3}{p+3\beta} \quad \text{and} \quad a_{p+4} = \frac{pc_4}{p+4\beta}. \tag{3.3}
\]
Substituting the values of \(a_{p+1}, a_{p+2}\) and \(a_{p+3}\) from (3.3) in the functional \(\{a_{p+1}a_{p+2} - a_{p+3}\}\), after simplifying, we get
\[
|a_{p+1}a_{p+2} - a_{p+3}| = \frac{p}{(p+\beta)(p+2\beta)(p+3\beta)}|p(p+3\beta)c_1c_2 - (p+\beta)(p+2\beta)c_3|.
\]
The above expression is equivalent to
\[
|a_{p+1}a_{p+2} - a_{p+3}| = \frac{p}{(p+\beta)(p+2\beta)(p+3\beta)}|d_1c_1c_2 + d_2c_3|, \tag{3.4}
\]
where \(d_1 = p(p+3\beta); \quad d_2 = -(p+\beta)(p+2\beta). \tag{3.5}\)
Substituting the values of \(c_2\) and \(c_3\) from (2.2) and (2.3) respectively from Lemma 2 on the right-hand side of (3.4), we have
\[
|d_1c_1c_2 + d_2c_3| = |d_1c_1 \times \frac{1}{2}\{c_1^2 + x(4 - c_1^2)\} + d_2 \\
\times \frac{1}{4}\{c_1^2 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\}|
\]
Using the facts \(|z| \leq 1\) and \(|xa + yb| \leq |x||a| + |y||b|\), where \(x, y, a\) and \(b\) are real numbers, which simplifies to
\[
4|d_1c_1c_2 + d_2c_3| \leq \[(2d_1 + d_2)(|c_1|^3 + 2|d_2|(4-c_1^2) + 2(|d_1| + d_2)|c_1|(4-c_1^2)|||x| + |d_2||(c_1 + 2)||(4-c_1^2)||x|^2]. \tag{3.6}
\]
From (3.5), we can write
\[
2d_1 + d_2 = p^2 + 3p\beta - 2\beta^2; \quad d_1 + d_2 = -2\beta^2. \tag{3.7}
\]
Substituting the calculated values from (3.7) along with (3.5) on the right-hand side of (3.6), we have
\[
4|d_1c_1c_2 + d_2c_3| \leq \[(p^2 + 3p\beta - 2\beta^2)c_1^3 + 2(p + \beta)(p + 2\beta)(4-c_1^2) \\
+ 4\beta^2c_1(4-c_1^2)|x| + (c_1 + 2)(p + \beta)(p + 2\beta)(4-c_1^2)||x|^2].
\]
Since \(c_1 = c \in [0,2]\), noting that \(c_1 - a \leq c_1 + a\), where \(a \geq 0\) and replacing \(|x|\) by \(\mu\) on the right-hand side of the above inequality, we get
\[
4|d_1c_1c_2 + d_2c_3| \leq \[(p^2 + 3p\beta - 2\beta^2)c_1^3 + 2(p + \beta)(p + 2\beta)(4-c_1^2) + 4\beta^2c(4-c^2)\mu \\
+ (c - 2)(p + \beta)(p + 2\beta)(4-c^2)\mu^2] = F(c, \mu), \tag{3.8}
\]
for \(0 \leq \mu = |x| \leq 1\) and \(0 \leq c \leq 2\), where
\[
F(c, \mu) = (p^2 + 3p\beta - 2\beta^2)c_1^3 + 2(p + \beta)(p + 2\beta)(4-c_1^2) + 4\beta^2c(4-c^2)\mu \\
+ (c - 2)(p + \beta)(p + 2\beta)(4-c^2)\mu^2. \tag{3.9}
\]
Next, we need to find the maximum value of the function $F(c, \mu)$ on the closed region $[0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ given in (3.9) partially with respect to $\mu$ and $c$ respectively, we obtain

$$\frac{\partial F}{\partial \mu} = 4\beta^2c(4 - c^2) + 2(p + \beta)(p + 2\beta)(4c - c^3 - 8 + 2c^2)\mu. \quad (3.10)$$

and

$$\frac{\partial F}{\partial c} = 3(p^2 + 3p\beta - 2\beta^2)c^2 - 4c(p + \beta)(p + 2\beta) + 16\beta^2\mu - 12\beta^2c^2\mu$$

$$+ (p + \beta)(p + 2\beta)(4 - 3c^2 + 4c)\mu^2. \quad (3.11)$$

For the extreme values of $F(c, \mu)$, consider

$$\frac{\partial F}{\partial \mu} = 0 \quad \text{and} \quad \frac{\partial F}{\partial c} = 0. \quad (3.12)$$

In view of (3.12), on solving the equations in (3.10) and (3.11), we obtain the only critical point for the function $F(c, \mu)$ which lies in the closed region $[0, 2] \times [0, 1]$ is $(0, 0)$. At the critical point $(0, 0)$, we observe that

$$\frac{\partial^2 F}{\partial \mu^2} = -4(p + \beta)(p + 2\beta) < 0;$$

$$\frac{\partial^2 F}{\partial c^2} = -16(p + \beta)(p + 2\beta) < 0;$$

$$\frac{\partial^2 F}{\partial c \partial \mu} = 16\beta^2;$$

$$\left[ \left( \frac{\partial^2 F}{\partial \mu^2} \right) \left( \frac{\partial^2 F}{\partial c^2} \right) - \left( \frac{\partial^2 F}{\partial c \partial \mu} \right)^2 \right] = 64[(p + \beta)^2(p + 2\beta)^2 - 4\beta^4] > 0,$$

with $p \in \mathbb{N}$ and $0 < \beta \leq 1$.

Therefore, the function $F(c, \mu)$ has maximum value at the point $(0, 0)$, from (3.9), it is given by

$$G_{\text{max}} = F(0, 0) = 8(p + \beta)(p + 2\beta). \quad (3.13)$$

Simplifying the expressions (3.4) and (3.8) together with (3.13), we obtain

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{2p}{p + 3\beta}. \quad (3.14)$$

This completes the proof of our theorem. □

**Remark 1.** Choosing $p = 1$ and $\beta = 1$ in (3.14), we obtain $|a_2a_3 - a_4| \leq \frac{1}{2}$, this inequality is sharp and coincides with the result of Bansal et al. [2].
**Theorem 2.** If \( f \in L_p(\beta) \ (0 < \beta \leq 1) \) with \( p \in \mathbb{N} \) then
\[
|a_{p+2} - a_{p+1}^2| \leq \frac{2p}{p + 2\beta}
\]
and the inequality is sharp for the values \( c_1 = c = 0, \ c_2 = 2, \text{ and } x = 1.\)

**Proof.** On substituting the values of \( a_{p+1} \) and \( a_{p+2} \) from (3.3) in the functional
\[
|a_{p+2} - a_{p+1}^2| = \frac{p}{(p+i\beta)^2(p+2\beta)} |(p+i\beta)^2c_2 - p(p+2\beta)c_1^2|.
\]
(3.15)
The above expression is equivalent to
\[
|a_{p+2} - a_{p+1}^2| = \frac{p}{(p+i\beta)^2(p+2\beta)} |d_1c_2 + d_2c_1^2|,
\]
(3.16)
where \( d_1 = (p+i\beta)^2 \) and \( d_2 = -p(p+2\beta). \)

Substituting the value of \( c_2 \) from (2.2) of Lemma 2, applying the triangle inequality on the right-hand side of (3.16), after simplifying, we get
\[
2|d_1c_2 + d_2c_1^2| \leq \left[(d_1 + 2d_2)||c_1||^2 + |d_1||(4-c_1^2)||x||\right].
\]
(3.18)
From (3.17), we can write
\[
d_1 + 2d_2 = -(p^2 + 2p\beta - \beta^2); \ d_1 = (p+i\beta)^2.
\]
(3.19)
Substituting the calculated values from (3.19), taking \( c_1 = c \in [0,2], \) replacing \( ||x|| \) by \( \mu \) on the right-hand side of (3.18), we obtain
\[
2|d_1c_2 + d_2c_1^2| \leq \left[(p^2 + 2p\beta - \beta^2)c_2 + (p+i\beta)^2(4-c_1^2)\mu\right]
\]
\[
= F(c,\mu), \ 0 \leq \mu = ||x|| \leq 1 \text{ and } 0 \leq c \leq 2,
\]
(3.20)
where \( F(c,\mu) = (p^2 + 2p\beta - \beta^2)c_2 + (p+i\beta)^2(4-c_1^2)\mu. \)

(3.21)
Now, we maximize the function \( F(c,\mu) \) on the closed region \([0,2] \times [0,1]. \) Let us suppose that there exists a maximum value for \( F(c,\mu) \) at any point in the interior of the closed region \([0,2] \times [0,1]. \) Differentiating \( F(c,\mu) \) given in (3.21) partially with respect to \( \mu, \) we obtain
\[
\frac{\partial F}{\partial \mu} = (p+i\beta)^2(4-c_1^2).
\]
(3.22)
For \( 0 < \beta \leq 1, \) for fixed values of \( c \) with \( 0 < c < 2 \) and \( p \in \mathbb{N}, \) from (3.22), we observe that \( \frac{\partial F}{\partial \mu} > 0. \) Therefore, \( F(c,\mu) \) which is independent of \( \mu \) becomes an increasing function of \( \mu \) and hence it cannot have a maximum value at any point in the interior of the closed region \([0,2] \times [0,1]. \) The maximum value of \( F(c,\mu) \) occurs only on the boundary i.e., when \( \mu = 1. \) Therefore, for fixed \( c \in [0,2], \) we have
\[
\max_{0 \leq \mu \leq 1} F(c,\mu) = F(c,1) = G(c).
\]
(3.23)
In view of (3.23), replacing $\mu$ by 1 in (3.21), it simplifies to

$$G(c) = -2\beta^2 c^2 + 4(p + \beta)^2,$$  \hfill (3.24)

$$G'(c) = -4\beta^2 c.$$  \hfill (3.25)

From the expression (3.25), we observe that $G'(c) \leq 0$ for each $c \in [0, 2]$ and for every $\beta$ with $0 < \beta \leq 1$. Therefore, $G(c)$ becomes a decreasing function of $c$, whose maximum value occurs at $c = 0$ only and from (3.24), it is given by

$$G_{\text{max}} = G(0) = 4(p + \beta)^2.$$  \hfill (3.26)

Simplifying the expressions (3.16), (3.20) along with (3.26), we obtain

$$|a_{p+2} - a_{p+1}^2| \leq \frac{2p}{p+2\beta}.$$  \hfill (3.27)

This completes the proof of our theorem. \hfill \Box

Remark 2. If $p = 1$ and $\beta = 1$ in (3.27) then $|a_3 - a_2^2| \leq \frac{2}{3}$, this result coincides with that of Babalola [1].

Theorem 3. If $f \in I_p(\beta)$ ($0 < \beta \leq 1$) then

$$|a_{p+k}| \leq \frac{2p}{p+k\beta}, \text{ for } p, k \in \mathbb{N}.$$  \hfill (3.28)

Proof. Using the fact that $|c_n| \leq 2$, for $n \in \mathbb{N}$, with the help of $c_2$ and $c_3$ values given in (2.2) and (2.3) respectively, together with the values obtained in (3.3), we get $|a_{p+k}| \leq \frac{2p}{p+k\beta}$, with $p, k \in \mathbb{N}$. This completes the proof of our theorem. \hfill \Box

Substituting the results of Theorems 1, 2, 3 together with the known inequality

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{2p}{p+2\beta}\right]^2$$ (see [7]) in the inequality given in (1.4), we obtain the following Corollary.

Corollary 1. If $f \in I_p(\beta)$ ($0 < \beta \leq 1$) with $p \in \mathbb{N}$ then

$$|H_3(p)| \leq 4p^2 \left[\frac{2p}{(p+2\beta)^3} + \frac{1}{(p+3\beta)^2} + \frac{1}{(p+2\beta)(p+4\beta)}\right].$$  \hfill (3.29)

Remark 3. In particular for the values $p = 1$ and $\beta = 1$ in (3.29), which simplifies to $|H_3(1)| \leq \frac{49}{540}$. This result coincides with that of Bansal et al. [2].

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Author’s address

D. Vamshee Krishna
Department of Mathematics, Gitam Institute of Science, GITAM University, Visakhapatnam, 530 045, A.P., India

E-mail address: vamsheekrishna1972@gmail.com