Abstract

There is a class of models for pol/mil/econ bargaining and conflict that is loosely based on the Median Voter Theorem which has been used with great success for about 30 years. However, there are fundamental mathematical limitations to these models. They apply to issues which can be represented on a single one-dimensional continuum, like degree of centralization of a government, or cartel members negotiating the price to ask for their commodity. They represent fundamental group decision process by a deterministic Condorcet Election. There has been some extension to multidimensional issue sets, but they are not well documented and are limited to cases where the difference between policies is well approximated by a Euclidean distance and each actor’s utility is monotonically declining in distance. This work provides a methodology for addressing a broader class of problems.

The first extension is to continuous issue sets where the consequences of policies are not well-described by a distance measure or utility is not monotonic in distance. Simple one-dimensional examples are discussed. The difficulty is more acute in multidimensional policy spaces. An example is the negotiations over national economic policies, where the effects differ by region, by industrial sector, and by social group. Further, even a weighted sum over the effects can be non-monotonic or multi-peaked.

The second fundamental extension is to inherently discrete issue sets. The discussion will focus on subset selection problems, though the methodology is not limited to them. Two examples are the selection of which subset of competing parties will form a parliament (and which will be excluded), or the selection of a portfolio of defense projects. In the parliament formation case, the utility of a potential parliament to each actor can be modeled as a look-ahead by that actor in order to consider the various policies which that parliament might choose. This models a two-stage process, where the uncertainty in the second stage (choices on issues) is an important factor in the first stage (choices of parliaments). Because there are generally more issues than parties, the discrete choices in the first stage embody trade-offs in the second stage. Because the options cannot easily be mapped into a multidimensional space so that the utility depends on distance, we refer to it as a non-spatial issue set.

The third, but most fundamental, extension is to represent the negotiating process as a probabilistic Condorcet election. This provides the flexibility to make the first two extensions possible; this flexibility comes at the cost of less precise predictions and more complex validation. Because the analyses are inherently probabilistic, this provides a smooth "response surface" for expected utility, thus simplifying strategy optimization even in discrete issue sets. Some common bargaining algorithms are inapplicable in the more general issue sets addressed here. For example, power-weighted interpolation between the positions of two actors is often inapplicable in continuous but multimodal issue sets. More fundamentally, interpolation is unusable in discrete issue sets because continuous interpolation is not even defined between discrete options.

We provide motivation and overview of the general methodology followed by mathematical details. The methodology has been implemented in two proof-of-concept prototypes which address the subset

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1 Problem and State of the Art

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Many political, military, and/or economic disputes involve bargaining and conflict between powerful groups. They can be modeled as thousands to millions of small actors, or as a small number of larger actors; we consider the latter class of models. Game theory, economics, and public choice theory all play important roles in different contexts, but one common theme is the use of models loosely based on the Median Voter Theorem (MVT). There is a vast amount of academic analysis, formal modeling tools, and popular political commentary based on the same concepts, which we will collectively refer to as spatial political models (SPM).

We present a methodology called Non-spatial Probabilistic Condorcet Elections (NPCE) and describe some advantages and disadvantages compared to SPM. This is not the only way to approach the problem of CE which lack winners. Some approaches avoid the CE concept entirely, some retain the idea of a CE but add some probabilistic elements. The usual probabilistic approach is to model the CE as a deterministic function with stochastic parameters. As explained below, our approach treats the CE itself as stochastic, i.e. a stochastic function even with deterministic parameters.

[PT02] gives a succinct motivation: "the probabilistic voting approach was developed in the spatial voting model to guarantee the existence of equilibrium in situations, such as multidimensional policy space, in which a Condorcet winner fails to exist; see Coughlin 1992 for an overview of probabilistic voting and Osborne (1995) for an overview of spatial voting theory" (the two citations are [PC92] and [MO95] respectively).

The SPM analyze the behavior of “actors”, which can stand for formal political parties, informal factions, prominent families, nation states, ethnic groups, social classes, industry groups, and so on. The issues over which they struggle and/or negotiate are represented as a continuum of choices, e.g. the degree of pro-US or anti-US alignment, price of a cartel’s commodity, level of social services, location of facilities, level of troop commitment to a war, and so on. The general ability of each actor to influence an outcome is their “capability”, which could reflect religious authority, available campaign funds, armed soldiers, size of an ethnic voting bloc, GDP, or whatever is relevant to the situation being analyzed. Voting is generalized from a formal process of counting ballots to the more general concept of exerting capability in order to influence outcomes: “voting with their feet”, “voting with their guns”, expending campaign funds to defeat a political opponent, expending strike funds to coerce an employer, and so on. The set of actors, the issues, and their capabilities are generally estimated by careful expert analysis before the SPM are applicable. The actors’ capabilities would then summarize the results of those analyses. Expert interpretation of the model output is also necessary, as the true meaning of the output depends on how the inputs were defined and estimated.

A great many studies support the idea that SPM are accurate descriptive models of US domestic politics, especially spending issues. [PD85] analyzed all the recorded votes in the US Congress and concluded that 80%-90% can be explained by the uni-dimensional SPM.

The SPM predict that there will be a clear order of public choice, without cyclic preferences, which was confirmed analysis of Congressional votes over district specific grants [TS90]. Several studies [CS91, CB95] support the idea that SPM are accurate descriptive models of US domestic politics.
found strong evidence that SPM are better models of legislative decisions on large scale public programs than are other interest group models.

Nevertheless, there are well-known situations, both theoretical and practical, in which the SPM could be improved upon. Some issues are normative and some are descriptive; this work considers only the descriptive. Many overall discussions of the problems exist, including [CR84, RH89, PE12]. Some of the issues will be discussed below in the context of explaining the NPCE. To see how the NPCE methodology is fundamentally different from the SPM, we examine an even older analytic construct on which they are both based: the concept of a Condorcet winner (CW). Just before the French Revolution, [MdC85] introduced the CW in the context of designing group decision procedures (especially elections) which were fair according to some unarguable “gold standard”. A deterministic “Condorcet election” (DCE) is the infeasible procedure of pairing every option against every other in one-on-one contests; the CW is that option which would win against every other option. Any feasible choice procedure which always chooses the CW is Condorcet consistent.

Even in theory, not every election has a CW, though several special kinds of elections do. In 1950, Duncan Black published a special case of Condorcet’s approach, the MVT, which assumes the following [DB48]:

- Policy options (positions advocated and outcomes experienced) can be expressed as a single number (e.g. level of troop commitment) which is one-dimensional, bounded, and continuous.
- While each actor may have a different preference function, each function must be unimodal and monotonically declining in distance from their most-preferred option
- In any choice between two options, each actor exerts all their capability toward whichever is preferable to them. Because each utility curve slopes down away from the actor’s position, this means that they vote for whichever option is closer to their own.
- The group vote is the sum of the individual votes between options.
- Group choices between options are deterministic.

The MVT states that under these conditions, a CW does exist and is the median voter position, where half the total capability is on either side. Notice that if there are gaps between the positions of different groups, the actor holding the CW can shift its position and thus shift the group outcome. Thus, they have the power to control the outcome, as long as they do not move so far that some other actor becomes the CW. While some actors are inflexibly committed to hold their position, some actors are motivated by the desire to hold the decision power and enjoy its benefits. For the latter, there is a strong incentive to occupy the CW position.

Because the CW cannot be defeated by any other proposal, it is the predicted outcome of a group bargaining process. Any actor proposing an option closer to the CW has a winning proposal, power-seekers always have a motive to do so, and once the CW is proposed, nothing can defeat it. Thus, with free introduction of proposals, the CW is the predicted outcome of a group bargaining process. In particular, when the “options” are different candidates running for election, this means first appealing to the median voter of their party’s primary election, then appealing to the median voter of the overall multiparty election. The fact that those two medians usually differ appears to be the origin of the popular political maxim that politicians must always “run to the center” after the primary.

2 A great deal of political commentary and analysis deals with the issues of how to effectively stake out a position near the median of the primary electorate even though primary voters expect a run-to-the-middle later, and how to effectively stake out a position near the median of the general electorate even after the general voters saw the primary position. The tactics of how best to adopt positions or to exert influence are not considered in this paper.
Some of the SPM generalize this approach somewhat to a multidimensional space, but they still require unimodal, monotonically declining preferences on a continuous set of options. The status of multidimensional SPM is quite complicated, largely because Generalized Mean Voter Theorem states that the multidimensional analog of the MVT holds only under restrictive symmetry conditions.

### 1.1 Normative vs. Descriptive

In our context, the CW is taken as purely descriptive, not normative. While the CW may have some normative content in some contexts, it is important to understand that it cannot be assumed that the CW represents any sort of collegial agreement or compromise pleasing to both sides. It might represent an explicit agreement between negotiators, or it might represent the outcome of a group process in which no party explicitly negotiates.

Similarly, the “election” and “bargaining” process could be formal, explicit, and professional, or it could be the implicit outcome of group dynamics. We do not assume the absence of coercion, so the relative capability of parties will greatly affect the bargains and compromises reached. Where coercion is possible, the dynamic is similar to an out-of-court settlement or an armed threat: If actor A is much stronger than B, he can unilaterally create a situation of potential conflict. In this case, the “bargain” to avoid a clash will likely be a very harsh one which B would have avoided if he were stronger. Consider the example of an armed robbery. The “reference situation” of open conflict is that the victim dies and the robber takes the money. For the victim to surrender the wallet is the outcome of Nash’s model of bargaining because each is better off than the reference situation (see [JN50]): the victim loses their money but stays alive, and the robber gets the money and avoids a murder charge. Where coercion is not possible, the reference situation is simply the status quo in case of no agreement, or what the parties can achieve unilaterally; this is what negotiators refer to as “best alternative to a negotiated agreement” or BATNA.

As discussed in [JW95], Nash emphasized that his model was applicable to conscious agents with explicit foresight (the “rational actor” model) as well as more unconscious, evolutionary processes (the “mass action” model). While bargains could be explicitly negotiated as written agreements, they also could implicitly evolve without communication, such as the evolution of tacit truces between artillery on opposite sides of the trenches in WWI. An example implicit process would be the continual adjustment of each side’s behavior and propaganda to the other side’s behavior and propaganda, even if neither side thought of itself as explicitly bargaining; see section (4).

A simple example was given (from the Northern perspective) in Lincoln’s Second Inaugural address:

> Both parties deprecated war; but one of them would make war rather than let the nation survive, and the other would accept war rather than let it perish, and the war came.

This can be seen as the CW between the North and the South, where each preferred peace to war, but each preferred war to the other’s peace. Hence, war is the CW option, even though neither party alone would have chosen it. Thus, the issue set must be constructed to include a wider set of options than either actor would choose alone, as it must also include both desirable and undesirable “compromises”, the results of failure to formally agree (e.g. the coming or continuation of civil war, the continuation of status quo if no legislation is passed), and so on. In the SPM, all possible positions can be represented by numerical coordinates, but non-spatial models raise subtle issues.

Some authors argue a desirable property of normative decision rules is join-consistency: if group $A$ alone chooses option $X$, and group $B$ alone chooses option $X$, then groups $A$ and $B$ together should also choose option $X$. Some argue that the mere fact of merger might in some cases change the context, so that join-consistency cannot be specified as an iron-clad rule for all situations. From a descriptive perspective, it is
easy to cite examples of join-inconsistency, such as two small, effective organizations which merge to form one larger organization that takes ineffective actions which neither component would choose. This is illustrated by the proverb that “a camel is a race horse designed by a committee”: each member might design their own kind of race horse, but because they are different the group can not agree on a single good design and incorporates contributions from each member, even when they are inconsistent with each other.

Deliberately echoing the Civil War example, each side preferred a horse to a camel, but each preferred a camel to the other’s horse. This can happen because any complex option has a large set of possible attributes. When considering the infinite space of designs, group A’s design of a “fast racer” might be quite different in detail from group B’s design of a “fast racer”. The fact that both are denoted as “option X” obscures the fact that they might diverge in very significant ways, though they share the abstract characterization of “fast”. Again, North would have chosen “peace” and the South would also have chosen “peace”, but their ideas of peace were so divergent that the CW of both together was war. From a purely non-normative perspective, the requirement of join-consistency appears to be violated by actual examples.

The design example raises the issue of explicit enumeration of small issue sets versus implicit definition of infinite issue sets. More discussion can be found in section 3.7.

2 Overview of NPCE

The "non-spatial probabilistic Condorcet election" (NPCE) methodology starts with the same concept of a Condorcet Election, but diverges even before the MVT. The NPCE methodology treats the group choices between options as probabilistic, rather than deterministic. It generalizes the deterministic Condorcet approach to handle a more general set of cases at the cost of providing less specific predictions and requiring somewhat more computational resources. Given modern computing infrastructure, e.g. on-demand provisioning via cloud computing, this is not a serious limitation.

The NPCE methodology is open to specialization for problems with unique structure, analogous to the situation with linear programming. Linear programming (LP) is a very general and powerful methodology, but many problems have special structures that are utilized by specialized algorithms to compute the same result more efficiently. Classic LP examples include transshipment algorithms which assume a network of balanced, positive flows. It is well-known that deterministic CW exist and can be efficiently computed for certain network location problems [DR78, HT81]. Some macroeconomic problems also have CW, under restrictive assumptions about the actors’ utility functions (e.g. all identical Cobb-Douglas); examples are discussed in [OR04]. More directly, there is a large literature on exploiting matrix structure to compute various kinds of eigenvectors, which is quite similar to the core NPCE problem of computing a limiting distribution over possible CW; we fully expect that analogous special structures can be utilized for NPCE.

The new NPCE methodology extends the state of the art in several ways:

- Non-deterministic: The methodology is designed to identify likely outcomes and interactions, as well as estimate the associated risk, opportunities, and variability.

3It could be objected that this is a false counter example, because there were two distinct options, “Northern-peace” and “Southern-peace”, not a single option “peace” which was selected by both North and South alone. By this logic, join-consistency is preserved, because there never was a single option which each alone would select. However, this same logic calls into question the entire rationale of join-consistency, because real world agreements on a single option almost always mask differences in detail of definition and implementation, so that the two parties never choose exactly the same option down to the last detail: there are essentially always distinct options, and the pre-conditions of join-consistency are almost never met. Apparently, the most that can be said is that join-consistency applies in those cases where the differences are not important and it does not apply in those cases where the differences do matter.
2.1 Deterministic vs. Non-deterministic

The SPM appear to be built on the assumption that political processes are largely deterministic. In a legislative vote where one position gets 11 votes and the other get 10, then the former is the unambiguous winner. The published analytical models do not appear to include any provision for uncertain outcomes and appear to use expected-value computations throughout, ultimately producing a single point prediction of a single forecast outcome, given particular input parameters. Much of the literature on CW and probability appears to focus upon the probability that a CW will result from a deterministic CE, for various models of stochastic voter preferences. That is, the CW is considered as a deterministic function with stochastic parameters.

While deterministic outcomes seem quite realistic when analyzing formal elections with ballots, it becomes questionable when comparing the generalized “voting” mentioned earlier. For example, if two equally strong interest groups try to influence legislators to get their preferred outcome, then either one is equally likely to prevail. This can be taken as the operational definition of “equal capability”, as is often done in systems to rank the capability of chess players, sports teams, and so on. Thus, expert opinion or prior statistics on relative odds can be used to derive relative capabilities via paired comparison techniques. When they are slightly mismatched, the odds can be expected to shift slightly, rather than snapping to a deterministic all-or-nothing outcome. This suggests that the CW could be considered as a stochastic function with deterministic parameters.

The simplest model is that the probability of success in each one-on-one contest depends on the relative capabilities of the actors involved:

\[ P[i > j] = \frac{c_i}{c_i + c_j} \]  

Zermelo’s original chess ranking function [EZ28], the Bradley-Terry model [BT52], the Glicko model [MG99] for sports teams and many others [WL11] are examples. While many alternative variations are discussed in the literature, most are transformations of equation (1). However, the core of the NPCE methodology is the technique of using probabilistic contests; equation (1) is a sub-model which can be varied based on the problem at hand.

Unlike methodologies based on DCE and the MVT, the NPCE methodology is designed from the foundations to represent this stochastic aspect of political/military/economic struggles. The standard “forecast” is a probability distribution over the possible outcomes. A disadvantage of the NPCE is that while the MVT provides the strong prediction of a single outcome, the NPCE provides a weaker prediction of probabilities.\footnote{When the CW exists, it is usually ranked as the most probable outcome. When the CW defeats other options by wide margins, it is much more likely than other options. When the CW just barely defeats other options, it may be slightly more or less likely than the alternatives.}

Adopting a probabilistic perspective is a fundamental change from the DCE and MVT and work built upon them. In particular, the strict problem structure necessary to make the strong prediction of a single
outcome is no longer necessary. This one change opens the door to a broad range of new problems areas including (but not limited to) multidimensional choices, discrete choices, and domain-specific utility models. This breadth comes at a price. First, less precise conclusions can be drawn. Second, with only statistical predictions, validation is more challenging. Third, models of dynamics and bargaining tailored to DCE and MVT, in SPM, are no longer applicable.

2.1.1 Strategy Optimization
Deterministic SPM often exhibit the following properties. Because a small shift in the CW actor shifts the predicted outcome, while small shifts in all other actors has no effect, efforts to use SPM to plan strategies often focus on actors at the median position, especially when they are powerful. With discrete options, strategy optimization with deterministic models has the difficulty that the response surface (predicted outcome as a function of the actor capabilities) is discontinuous, as the prediction is either one discrete outcome or another: the response is everywhere perfectly flat except where it is discontinuous.

With NPCE, the prediction is not a single outcome but a distribution over outcomes. Even for the case of discrete options, the distribution is a continuous function of the input parameters, so the expected value of the result is a continuous function of the input parameters. The availability of continuous slope information makes it possible to bring many powerful optimization techniques to bear in order to suggest novel strategies.

In the case of one-dimensional models, the NPCE can be used to evaluate strategies that do not necessarily focus immediately on the CW actor. While influencing the first-tier actors is always an option to be analyzed, the NPCE makes it easier to analyze the potential for shifting the odds in one’s favor by influencing second-tier actors.

The NPCE methodology also facilitates analyzing the robustness of outcomes and strategies, for both continuous and discrete issue sets. For example, suppose an analyst is looking for strategies to influence a group so as to select option X. The predicted distribution for strategy A might indicate option X was the most likely, but only by a slim margin. With a different strategy B, option X might still be the most likely, but by a large margin. While a deterministic analysis might simply indicate option X in both cases, the NPCE will provide the additional information that strategy A is more risky, in the sense of having higher RMS deviation from one’s goal.

2.2 Continuous vs. Discrete Options
The SPM represent multiple political choices as points in a continuous, multidimensional space, where each dimension represents an independent sub-issue. Each point defines an overall policy which an actor could advocate and/or a policy compromise among several actors. A large class of problems which do not fit the SPM paradigm are subset selection problems (SSP). We will consider two: interest groups selecting parties to form a government, and managers choosing projects to form a research program.

The composition of a government coalition is discrete, not continuous: each party is either in or out. Similarly, the assignments of cabinet seats to coalition members are inherently discrete: it is impossible to assign 2.3 cabinet seats to a party. Thus, neither of these problems can be represented well in SPM. The problem can be represented as multistage, because the choice of government in the first stage strongly influences the policy outcomes in the second stage.

Because each party has a slate of policy preferences, a choice between one ruling party and another might have implications for several issues at once: free markets vs. central planning, extensive or minimal rights for women, orientation toward America or Russia, and so on. Because there are generally more policy issues in play than there are parties, choosing which party to back (or which party to get a cabinet seat) inevitably
involves tradeoffs between those issues. Notice that this is a completely different mechanism to address policy tradeoffs than is used in multidimensional SPM.

Once in power, the parties may cooperate via “logrolling” pairs of issues where each does a favor for the other. Finally, all these interactions are anticipated in the struggles over how to form a government, as every actor is trying to shape the final outcome. The NPCE methodology can be used by analysts to examine how the actors involved might address the discrete choices in forming a government, in light of their understanding of the multistage nature of the process.

There are fundamental reasons why the discrete structure of a government cannot be easily summarized into a single number, greatly limiting the ability of SPM to address these tradeoffs. When there are a half-dozen parties competing to get into the government, then each alternative outcomes is simply a subset specifying which get in and which are left out. This can be represented by a simple list of 1’s and 0’s, which is more like the evenly spaced corners of a unit hypercube than a continuous line segment. Any scheme to reduce it to a simple continuum (e.g. Hamming distance from a given corner) inevitably maps significantly different points onto each other, thus obscuring the difference between alternatives. With \( m \) parties, there are \( 2^m \) options for which subset is “in” and which is “out”, with the widest diversity of options at the level of 50% in and 50% out. An analogy would be to categorize every point on the globe by its distance from the Entebbe airport and then placing them on a line: places as dissimilar as the North Pole, Vietnam, the Indian Ocean, the South Pole, and the North Atlantic would all collapse to the point representing “halfway around the world”.

While policy options can be mapped to \([0,1]\) vectors and thus to the corners a hypercube, the utilities common in SSP are poorly represented by a weighted Euclidean distance; an example is discussed below. A generalization of the SSP is to determine a matching. Given a dozen factions and six cabinet seats, each possible matching can be represented by a six-element list of which factions get to control each seat. With \( n \) seats and \( m \) factions, there are \( n^m \) possible ways to match them up. Because neither seats nor factions are naturally ordered, (unlike the 0/1 alternatives for whether a party is in the coalition or not) there is no obvious way to place all the matchings on the corners a hypercube. Again, there is (in general) no way to reduce a matching to a single number without potentially losing critical information. For example, a large pro-US faction might be forced to cooperate with a small but very anti-US faction by granting them one out of five seats in the cabinet, so that every possible cabinet is 80% pro-US and 20% anti-US. However, it is important whether that seat controls farm policy or the entire military: we would like to disaggregate the single 80% number into a matching of factions with seats.

### 2.3 Policies vs. Outcomes

SPM postulate that the attractiveness of a policy option to a decision maker depends on the weighted Euclidean distance between that policy option and the decision maker’s preferred option: their own option is best, and desirability declines as “distance in policy space” increases. Technically, their utility function is unimodal and monotonically declining with distance. Thus, to fit a problem into the SPM framework, one must define a single continuous space (and weighted distance measures on it), so that every actor’s preference function becomes unimodal and monotonically declining. As described below, there are many important cases when this is not possible even with continuous options.

#### 2.3.1 Multiple Peaks

It frequently turns out that the desirability of a policy choice is not single-peaked in any distance measure. As we will demonstrate, this can be a problem even for unidimensional issues.
2.3 Policies vs. Outcomes

One classic example is a policy choice over the level of troops to commit to a war. One actor’s best option might be to commit a high number of troops in order to win decisively, next-best to commit a low number as non-provocative observers, and least desirable to commit a middle amount: enough to suffer large casualties but not enough for a significant chance of victory. For this actor, the preference ordering of policies, based on military outcome, be \( H > L > M \). When this actor’s utility is graphed versus number of troops, there are peaks at both extremes and a dip in the middle (the proverbial “worst of both worlds”).

The case of public bads has been analyzed in [GA12]. This situation is characterized by a unidimensional scale, where each actor’s utility function has a single triangular dip, with perfectly flat indifference regions on either side. The situation of multiple dips or peaks in each utility function, or having multiple dimensions or discrete options, is not analyzed.

Because actors have different utility functions, each actor might have different number of dips and peaks, located in different places. The SPM are inherently unable to represent actor’s preferences in such situations where benefit has multiple peaks. The fundamental problem is that actors often care about outcomes that are non-monotonic in policy.

2.3.2 Multiple Orders

Frequently, actors will value options by different criteria, because they value the outcomes by different criteria. This is just one of several reasons why it is generally impossible to map discrete options into a single distance measure for all actors: while Hamming distance might be appropriate for one actor’s valuation of the consequences, it is not therefore automatically appropriate for another actor who values the consequences differently. In the example of troop commitments, one actor, \( A \), might order the desirability of alternative by the military outcome, giving \( H > L > M \) discussed earlier. Another actor, \( B \), might be concerned with cost, giving the \( L > M > H \) ordering. Of course, for any given actor, the alternative policies could be sorted according to how good the outcome was for that actor (with some overlaps). It is simple to enumerate all six orders of \( L, M, H \) to verify that none make \( A \)’s utility function unimodal and \( B \)’s unimodal at the same time.

Because actors can have arbitrary preferences over outcomes, there is in no way to define any distance measure which gives the correct preference ordering for all actors when combined with a unimodal, monotonically declining utility measure. Even in the uni-dimensional continuous case, there is not always an embedding of the options so that they are placed in one order for one actor, while appearing to be in a completely different order for another actor; the troop deployment problem provides an example. The NPCE methodology completely avoids this problem by working directly with actor’s individual utility of outcomes, without trying to collapse them all into a single measure of distance in a common policy space.

2.3.3 Domain Specific Utilities

The SPM treat options basically as points in continuous space, where some measure of distance determines utility. Even when actors embed policy options in space differently, the basic spatial concept applies. However, there are many cases where the spatial analogy is quite strained. A simple example is the SSP of which research projects to fund and which not to fund. Each option can be represented as a discrete choice of which subset of projects to fund. Relative to a set of research goals, the utility of the top-level discrete choices depends not on a measure of “distance in policy space” but on an estimate of the real-world consequences. Because of the potentially complex precedence and complementarity relations between projects, utility can

\footnote{This situation is superficially similar to Arrow’s Impossibility Theorem [KA50]. However, Arrow’s theorem applies when each individual’s preferences are defined by a total order without numeric weights, so it is not applicable to situations where choices have numerical weights, such as range voting or the generalized voting considered in this work.}
be highly non-monotonic in any total or partial order (rather like a combinatorial auction). For selection of defense projects, complex non-monotonic considerations of tactics, available industrial base, strategy, and geography are necessary: detailed modeling and analysis are often required before the summary assessments of utility can be used as actors’ estimates of the utility.

Of course, it is possible that a more bureaucratically minded actor might evaluate options purely by how many of their agency’s favored programs are included and how many of rival agencies’ programs are excluded. Because the NPCE methodology does not assume actors have the same utility function, including both research oriented and bureaucratically oriented actors presents no difficulty.

The NPCE methodology represents this situation by using sub-models which mimic the process by which actors anticipate the consequences of their choices, and value their choices according to their expected value of the consequences. In the case of funding research tasks, one plausible sub-model would be a probabilistic PERT chart for the tasks. It not assumed that each actor has the same model, that any are objectively correct, or even that they agree on what the consequences are likely to be. The model of their expectations is termed a domain specific utility model (DSUM).

As mentioned earlier, the choice of what government to form (or which cabinet positions to assign to which coalition members) can be modeled as turning on an estimate of the various policies which might result from that government. The example DSUM considered in this case assigns values to each top-level choice of government by invoking sub-models for each issue; each sub-model is another copy of the NPCE for the corresponding one-dimensional issue, parametrized for that particular government’s composition. The RMS deviations from that actor’s ideal result on each issue are separately estimated (i.e. no logrolling across issues by different members of the government), then combined with the actor’s salience for each issue to derive a final utility for that government. Thus, the DSUM mimics the actor’s foresight as to the likely range of outcomes from their choice of government.

Another example is the design of tax and subsidy policies or economic stimulus packages in a multiregional country: there are multiple groups affected by each policy choice, groups are often affected in different ways, regions are affected in different ways, and effects can easily be non-monotonic in the size or targeting of a policy. Political actors are likely to weigh the benefits to their own constituency heavily, while weighting the effect on others’ constituencies quite differently. Similar considerations apply to multinational models.

3 Details of NPCE

Rather like linear programming, the core NPCE methodology is simple and abstract enough to be stated in just a few equations: \( (13) \), \( (17) \), and \( (19) \). This section is devoted to examining the implications and usage of the equations, as well as presenting various options for some of the sub-models which support the equations. Like linear programming, there is an “art and science” of building a model within the framework, and another science to configuring and solving the models efficiently.

3.1 Model Variations

There are many choices at each stage of the model-building process. Basic structure of the issue set, domain specific utility functions, voting rules, third-party commitment, and search strategy are just some of those available. Almost all combinations yield a model in which no deterministic CE exists, necessitating the use of probabilistic voting models as explained in section \( (1) \). The basic NPCE methodology has been implemented in several prototypes whose main purpose is to explore the effect of varying those assumptions in both continuous and discrete issue sets, so that in each problem the important assumptions can be identified for further study. With its “Chinese menu” of options, more model structures can be created than can be
discussed here: most variants are not discussed in the literature, some new variants are likely to be useful, but many will not.

For each of several sub-models, each prototype supports several sub-models (e.g., voting rules from equations (2), (3), (7) and more), but we will generally limit discussion in this paper to sub-models whose behavior can be described in fairly simple equations (e.g., binary or proportional voting). Thus, many of the equations which follow describe the behavior of the generic numeric methods when applied with particular sub-models (e.g., voting rule, third party commitment, etc.) but do not appear in the software: they only describe the behavior for that particular set of modeling options. The software uses only the top-level equations and generic numeric methods, which can be applied to a wider range of options than admit simple analytic solutions.

3.2 Issue sets

The set of possible positions on an issue define the issue set. It may be explicitly given, such as the set of all integer values from 0% to 100%, or it could be an implicit definition of an enormous combinatorial set, such as the set of possible organizational charts.

For NPCE, no more structure is required. The key steps in defining an issue set for SPM are the following:

- Define a distance metric, \( d \), on the set of alternatives, \( \{\theta_i\} \). The most common way to do this is to associate one or a few numerical coordinates with each option, then use a weighted Euclidean distance metric on those coordinates. Any distance metric must obey the standard definition:
  - Non-negative: \( d(\theta_i, \theta_j) \geq 0 \)
  - Identity of indiscernibles: \( d(\theta_i, \theta_j) = 0 \iff \theta_i = \theta_j \)
  - Symmetry: \( d(\theta_i, \theta_j) = d(\theta_j, \theta_i) \)
  - Triangle inequality: \( d(\theta_i, \theta_k) \leq d(\theta_i, \theta_j) + d(\theta_j, \theta_k) \).

- Define an indexed family of utility functions which are declining in distance: \( d(\theta_i, \theta_j) < d(\theta_i, \theta_k) \iff U_i(\theta_j) > U_i(\theta_k) \)

Notice that, taken together, these conditions imply that each actor ranks their own position higher than any other distinct position: \( \forall \theta_i \neq \theta_j U_i(\theta_i) > U_i(\theta_j) \)

As discussed earlier, it is not always possible to construct the \( (d, U_i) \) required for the SPM.

3.3 Voting

Given a choice between two alternatives, \( \alpha : \beta \), the effort an actor will exert to promote one or the other is called his “vote”; it is determined by his “capability”, \( c_i \) and the difference in utility to him of the two alternatives.\(^6\) As mentioned earlier, “voting” is generalized to mean the exertion of capability in order to affect outcomes. Positive favors option \( \alpha \), while negative favors option \( \beta \). Voting can be over many kinds of alternatives: positions currently held by actors, alliances with other actors, bargains struck between actors to define new positions, and more. We follow the von Neumann convention of bounding utility by \( 0 \leq U_i(\alpha) \leq 1 \).

\(^6\)Note that the models in [BDMFS94, BDMNR85, BDM84] do not fit this model, because they assign utility not to the positions of actors but to changes in those positions. We will denote the particular formula to calculate utility to actor \( i \) of a change in state from \( S_1 \) to \( S_2 \) as \( U_i(S_1, S_2) \). It can not be represented as the difference between utilities of individual states, because the utility for no change at all is assigned a positive value; see [BDMNR85] page 52. While \( U_i(S_1, S_1) > 0 \) for every state \( S_1 \), there is no function \( u \) so that \( u(S_1) - u(S_1) > 0 \). Therefore, the work described in this paper does not apply to those models or to models based upon them.
3.3 Voting

3.3.1 Voting Rules

The binary voting rule is as follows:

\[ v_i(\alpha : \beta) = \begin{cases} +c_i & \text{if } U_i(\alpha) > U_i(\beta) \\ -c_i & \text{if } U_i(\alpha) < U_i(\beta) \\ 0 & \text{if } U_i(\alpha) = U_i(\beta) \end{cases} \]  

(2)

This all-or-nothing behavior is a good model of casting yes/no votes (weighted or not), but in many cases of exerting informal influence, a more nuanced exertion of effort is observed. One simple model of nuanced voting is that actors exert effort in strict proportion to what is at stake for them between the two alternatives:

\[ v_i(\alpha : \beta) = c_i[U_i(\alpha) - U_i(\beta)] \]  

(3)

For example, environmental interest groups might spend only a small amount of effort on unimportant issues, while expending a great deal of time, money, and attention on issues they view as critical. As derived in [BW10a, BW10b] and published in [EJ11], proportional voting leads to CW under extremely general conditions. The proof is quite simple:

\[ V(\alpha : \beta) = \sum_i v_i(\alpha : \beta) = \sum_i c_i[U_i(\alpha) - U_i(\beta)] = \sum_i c_iU_i(\alpha) - \sum_i c_iU_i(\beta) = \omega(\alpha) - \omega(\beta) \]  

(4)

where \( \omega(x) = \sum_i c_iU_i(x) \)  

(5)

Similar to Black’s original proof of the MVT, actors seeking office always have a motive to propose options with higher \( \omega \) values, leading the group to the “Central Position” as the maximum; we call this the Central Position Theorem (CPT). The difficulty of maximizing \( \omega \) can range from very easy to very difficult, depending on the properties of the utility functions and set of options. A special case is analyzed in Corollary 4.4 of [PC92] assuming a probabilistic Luce model of weighted binary voting, over a continuous, convex, compact set of options with a particular exponential form for the probabilities.

With the von-Neumann scaled utilities of this paper, the resulting probability of the i-th actor voting for option \( \alpha \) over \( \beta \) is simply the following:

\[ p_i(\alpha : \beta) = \frac{1 + U_i(\alpha) - U_i(\beta)}{2} \]  

(6)

When the actual votes between options are set equal to their expected values under these assumptions of [PC92], then the position maximizing equation (4) is the voting equilibrium.

While proportional voting is a common modeling choice, there are well-known cases where the proportional rule does not appear to hold. There is a well-known lobbying rule to “focus benefits and diffuse costs” which suggests that equation (3) does not hold in this case. Suppose the US Congress creates a program which draws 1.5 billion dollars from general tax revenue to confer benefits on a constituency of ten thousand people. The benefit to the constituents is $150,000 to each of 10,000 people. If there are effectively 150 million taxpayers, then the cost is only about $10 each. Assuming for now that utility is proportional to dollars and that each affected individual has equal capability to exert influence, the strict linearity of equation (3) implies that the
influence of the few but highly motivated supporters would be exactly neutralized by the net influence of
the numerous but weakly motivated opponents. However, it is well known that such diffuse costs are largely
ignored by those effected, leading to the lobbying advice. To model this situation, the cubic rule is designed
to have a shallow slope for small changes, but steep slope for large ones. This reflects the lobbying rule so
that voters harmed are in the shallow region while voters helped are in the steep region.

\[ v_i(\alpha : \beta) = c_i[U_i(\alpha) - U_i(\beta)]^3 \] (7)

One can imagine many other alternative voting rules, such as linear mixture of cubic, proportional, and
cubic. Most such rules have not been extensively studied; this may be because they are not algebraically
tractable, do not reliably produce CW with DCE in common cases, or both. Though it seems to reflect the
empirical experience of lobbyists, this author has found no discussion of the cubic voting rule, or its CE/CW
properties, in the literature.

Regardless of the individual voting rule, the net vote of the group between the alternatives is just the
sum of the individual votes. Note that inter-subjective comparisons of utility is never done; only capability
to exert power is compared. This is why it is important that the changes in utilities be in a fixed range (e.g.
\([-1, +1]\] models with utility on a [0, 1] scale): the limits of actions by actors should be determined by their
relative capability to exert power, not by the scale on which their utilities happen to be measured.

\[ V(\alpha : \beta) = \sum_i v_i(\alpha : \beta) \] (8)

If this quantity is positive, then we say that the group prefers \(\alpha\) over \(\beta\), though weaker players may
strongly disagree. In this sense, the stronger actors can force through outcomes against the wishes of weaker
actors. This is normal and expected in democratic elections: if one interest group (aka “actor”) gets 70% of
the vote, then the majority wins, even if the remaining 30% still would have preferred the other candidate.
On the other hand, the CW might not be very desirable to any actors, even though it is the best the group
can decide upon.

As mentioned above, the strong CW is defined to be that option \(\alpha\) which defeats all others:

\[ \forall \beta \neq \alpha \quad V(\alpha : \beta) > 0 \] (9)

The weak CW loses to none:

\[ \forall \beta \quad V(\alpha : \beta) \geq 0 \] (10)

Because this allows the possibility of very weak “winners” (e.g. all options are tied), some authors extend
this to require that a weak CW is never defeated and defeats at least one:

\[ \forall \beta \quad V(\alpha : \beta) \geq 0 \land \exists \gamma \quad V(\alpha : \gamma) > 0 \] (11)

This allows only non-trivial ties, such as A defeats C and ties B, while B defeats C and ties A, where both
A and B are weak CW. Further refinements of the CW criteria exist but are not necessary for our purposes.

As discussed in section (2.1), discrete jumps between options are appropriate to actual legislative votes,
while smoothly changing probabilities are more appropriate to generalized voting. An advantage of the
NPCE over DCE is that NPCE distinguishes between a strong CW which just barely defeats the other
options (marginal) and a strong CW which soundly defeats all others (robust). In the marginal case, the CW
is estimated to be just barely more likely than the second-most likely, while in the robust case the CW is
estimated to be much more likely than the second-most. For our purposes, the distinction between marginal
and robust CW is more important than the distinction between strong and weak CW. Consider the case that
option A barely defeats B and soundly defeats C, while B barely loses to A and soundly defeats C. In the DCE methodology, option A is ranked as the strong CW no matter how small the margin of victory, even when they are insignificant round-off errors, until the discrete transition from “strong” to “weak” when the tiny margin becomes exactly zero. When there is even a tiny margin of defeat (even round-off error), it is classified as a Condorcet loser. With NPCE methodology, there are no discrete jumps between options as the strength of their support varies, just smoothly declining probability from very high odds to nearly even.

When the context is clear, we often refer to the current positions of particular actors, \( \theta_i \), by just the index \( i \). The effort which actor \( k \) will exert to support option \( \theta_i \) over option \( \theta_j \) is \( v_k(i : j) \). We do not consider framing effects, so swapping \( i \) and \( j \) flips the sign:

\[
v_k(i : j) + v_k(j : i) = 0 \tag{12}
\]

We will consider different models of third-party voting in section 3.6; for now we simply use \( v_k(i : j) \). The set of \( k \) with \( v_k(i : j) > 0 \) are the coalition supporting \( i \) against \( j \). While each model of the third-party choice gives a different estimate of \( v_k(i : j) \), the total capability of \( i \)'s coalition against \( j \) is always the sum of votes for \( i \) against \( j \):

\[
C_{i:j} = \sum_{v_k(i:j) > 0} v_k(i : j) \tag{13}
\]

Equations (12) and (13) imply that

\[
C_{j:i} = \sum_{v_k(i:j) < 0} |v_k(i : j)| \tag{14}
\]

Note that this includes the bilateral contributions of \( i \) and \( j \) themselves, which may be large or small depending on how high are the stakes. By the symmetry of equation (12),

\[
C_{j:i} = \sum_{v_k(j:i) > 0} v_k(j : i) \tag{15}
\]

Because every non-zero term in the sum for \( V(i : j) \) appears in either the sum for \( C_{i:j} \) or \( C_{j:i} \), we have

\[
V(i : j) = C_{i:j} - C_{j:i} \tag{16}
\]

In a political election, or a legislative vote, the winner is uniquely determined by the sign of the difference in capabilities, e.g. 11 votes to 10. When the concept of exerting influence is extended from legislative votes to more general contests of capability, then the outcome is not so clear-cut. For example, if one legislative pressure group has $11 million to spend on lobbying efforts and the other has $10 million, then \textit{ceterus paribus} the probability of the former prevailing in a legislative vote should be only slightly better than 50%. The probability that \( i \) will defeat \( j \) depends on the relative capability of the coalition supporting each, via the simplest generalization of equation (1):

\[
P[i > j] = \frac{C_{i:j}}{C_{i:j} + C_{j:i}} = P_{ij} \tag{17}
\]

To avoid division by zero errors and give 50 : 50 odds in the case of exact ties, a small positive constant is added to each \( C \) before applying equation (17). A reasonable heuristic would be to make it a tiny fraction of the root mean square of the \( C_{i:j} \) values. This also has the useful result that \( P_{ii} = \frac{1}{2} \), which means that the purely formal contest \( i : i \) has 50% probability of each side winning, or 100% chance that \( i \) will remain if no other actor challenges it.

Note that \( P_{ij} + P_{ji} = 1 \) and that \( V(i : j) \geq 0 \) if and only if \( P_{ij} \geq \frac{1}{2} \).
3.4 Limiting Distribution

The MVT and derived models use the heuristic of repeated deterministic contests in Condorcet Election to demonstrate that group decision making should converge to the median voter. The NPCE methodology used the heuristic of a Markov process where repeated probabilistic contests yield a limiting distribution. As mentioned earlier, this is not the only Markov model of probabilistic voting. Two examples of using NPCE combined with a Markov model to estimate a limiting probability distribution are [STW06] and Zapal’s PhD thesis, [JZ12]; several seminar presentations extracted from the thesis explore “simple Markovian equilibria in dynamic spatial legislative bargaining”.

The deterministic CW is defined by the heuristic of a full $n^2$ deterministic Condorcet election. The limiting distribution of a probabilistic Condorcet election can be computed by a heuristic of “fictitious play”. Suppose $p^t_i$ is the probability of $i$ at turn $t$. Then there are three ways that $i$ could be chosen at turn $t + 1$:

- Option $i$ was chosen at time $t$, challenged by option $j$, and $i$ defeated $j$
- Option $i$ was chosen at time $t$, and not challenged.
- Option $k$ was chosen at time $t$, challenged by option $i$, and $i$ defeated $k$

As mentioned above, the situation of $i$ not being challenged can be formally represented as $i$ challenging $i$, which results with certainty in $i$. We denote the probability of a challenge from $j$ to $i$ as $c_{ji}$, so the three conditions can be combined to give the Markov transition probabilities as follows:

$$p^{t+1}_i = p^t_i \left( \sum_{j \neq i} c_{ji} p_j \right) + p^t_i \left( 1 - \sum_{j \neq i} c_{ji} \right) + c_{ik} \left( \sum_{k \neq i} P_{ik} p^t_k \right) \quad (18)$$

Because $P_{ij} + P_{ji} = 1$, the assumption that $c_{ji} = 1/n$ yields the following:

$$p^{t+1}_i = \frac{1}{n} \sum_{j=1}^n P_{ij} (p^t_i + p^t_j) \quad (19)$$

This can be expressed more concisely as the following vector equation:

$$p^{t+1} = T \left( P, p^t \right) \quad (20)$$

Equation (19) implies that each term of $p^t_i$ appears in $\sum p^{t+1}_i$ with the coefficient $\frac{1}{n} \sum_{j=1}^n (P_{ij} + P_{ji})$, which is 1. This implies that

$$\sum_i p^t_i = 1 \Rightarrow \sum_i p^{t+1}_i = 1 \quad (21)$$

as required.

Equation (19) is essentially an eigenvector problem, so we can use similar solution techniques. One simple, general method is the iterative Newton procedure:

$$p^{t+1} = \frac{p^t + T (P, p^t)}{2} \quad (22)$$
3.5 Utility of a Contest

Starting from a uniform \( p_i^0 = \frac{1}{n} \) distribution, it efficiently converges on the limiting distribution of the Markov process, \( p = p^\infty \):

\[
p_i^\infty = \frac{1}{n} \sum_{j=1}^{n} P_{ij} (p_i^\infty + p_j^\infty)
\]

A little algebra shows that for the two-options case, we get exactly the intuitive expected result:

\[
\begin{align*}
p_1 &= \frac{C_{1:2}}{C_{1:2} + C_{2:1}} \\
p_2 &= \frac{C_{2:1}}{C_{1:2} + C_{2:1}}
\end{align*}
\]

With more than two options, there is no simple closed form for the limiting distribution; the following numerical example is purely for reference. For illustration, we have assumed that \( c_{ij} = \frac{1}{5} \) though other assumptions could reasonably be made. The capabilities \( C_{ij} \) were generated uniformly from \([0,1]\), those on the middle row were raised by 0.5 to make it the CW, and equation (17) was used to compute each probability.

\[
P = \begin{bmatrix}
0.5000 & 0.4192 & 0.1814 & 0.8272 & 0.5211 \\
0.5808 & 0.5000 & 0.3326 & 0.7129 & 0.1856 \\
0.8186 & 0.6674 & 0.5000 & 0.7674 & 0.5043 \\
0.1728 & 0.2871 & 0.2326 & 0.5000 & 0.1777 \\
0.4789 & 0.8144 & 0.4957 & 0.8223 & 0.5000 
\end{bmatrix}
\]

(23)

\[
p = \begin{bmatrix}
0.1597 \\
0.1400 \\
0.3401 \\
0.0638 \\
0.2964 
\end{bmatrix}
\]

(24)

3.5 Utility of a Contest

When advantageous, any actor \( i \) could initiate a one-on-one contest in order to get actor \( j \) to adopt their position. Even if the contest does not actually occur, the possibility can be anticipated by both parties as they consider what bargain to reach. If it were a strictly bilateral contest, where both sides exerted their full capability, \( c_i \), we would expect the probability of winning to be determined by their relative capabilities, similar to equation (1). However, it is unrealistic to model every conflict between actors as the most intense conflict of which they are capable: if the difference in utilities were quite low, then one would expect little effort to impose or resist. Because the actors have different utility functions, it is entirely possible for the stakes to be very high for one actor and very low for the other.

The proportional voting rule of equation (3) leads to the following effort by each actor.

\[
\begin{align*}
v_i(i:j) &= c_i (U_i (\theta_i) - U_i (\theta_j)) &\geq& 0 \\
v_j(i:j) &= c_j (U_j (\theta_i) - U_j (\theta_j)) &\leq& 0
\end{align*}
\]

(25)

(26)

Thus, the probability of \( i \) winning depend on both the relative capabilities and relative stakes. We modify equation (17) to use not the total capability available but only that amount justified by the stakes, as follows:

\[
p[i > j] = \frac{\max(v_i(i:j), 0)}{v_i(i:j) + v_j(i:j)}
\]

(27)

(28)
For this particular model, we will assume that the utility to an actor $i$ of a state where all other positions are $\theta_j$ is simply the sum of the utilities of positions:

$$S = \{\theta_j\}$$

$$U_i(S) = \sum_j U_i(\theta_j)$$

(29)

For brevity, we will omit terms which cancel. Suppose actor $i$ considers challenging actor $j$. Considered as a bilateral contest, only $\theta_i$ and $\theta_j$ could change, so all other terms will be omitted from status quo utility:

$$U_i(S) = U_i(\theta_i) + U_i(\theta_j)$$

(30)

The expected utility for party $i$ in a contest against $j$ rests on the assumption that the loser must adopt the winner’s position. If $i$ defeats $j$, then the new positions are $\theta_i' = \theta_i$ and $\theta_j' = \theta_j$; the converse holds if $j$ defeats $i$. Thus, the expected utility of the outcome is as follows:

$$U_i(i : j) = \frac{v_i}{v_i + v_j} (U_i(\theta_i') + U_i(\theta_j')) + \frac{v_j}{v_i + v_j} (U_i(\theta_i'') + U_i(\theta_j''))$$

$$= \frac{v_i}{v_i + v_j} (U_i(\theta_i) + U_i(\theta_j)) + \frac{v_j}{v_i + v_j} (U_i(\theta_j) + U_i(\theta_j))$$

$$= \frac{2v_i U_i(\theta_i) + v_j U_i(\theta_j)}{v_i + v_j}$$

(31)

Comparing $U_i(i : j)$ to the status quo utility, we can see that $i$’s incentive to initiate a one-on-one contest $j$ is the following:

$$U_i(i : j) - U_i(S) = \frac{v_i - v_j}{v_i + v_j} (U_i(\theta_i) - U_i(\theta_j))$$

(32)

Taken by itself, equation (32) implies that each actor has an incentive to initiate a contest which depends on the interaction of capability and stakes. Notice that it can be advantageous for a weak actor, $i$, to confront a strong actor, $j$, if the stakes are small for the large actor yet high for the weak actor: $c_i < c_j$ yet $v_i > v_j$ so $U_i(i : j) - U_i(S) > 0$. This dynamic is often cited as an explanation of how weak countries can defeat strong countries, e.g. when the former fight without end on their home territory and the latter easily tire of overseas expeditions.

However, all the third parties also have the opportunity to exert influence, so contests cannot be expected to always remain bilateral between just two actors.

### 3.6 Third Party Support

As mentioned earlier, third parties are likely to gain or lose from a contest and hence have an incentive to vote with $v_k(i : j)$. Those supporting $i$ make up one coalition while those supporting $j$ make another. In this way, we can view the multilateral contest as a two-sided contest between two coalitions. The composition and strength of the coalitions depend on the votes of third parties.

Third parties always have at least three options: support $i$, support $j$, or abstain. Again, one would expect them to exert effort to support the party whose victory is most favorable to them. We assume that the third parties have limited rationality in that they look at the first-order, local choices and consequences, without complex game-theoretic consideration of others’ choices (which might be based on similarly complex consideration of others’ choices, and so on.)

We model third party support as uncommitted, semi-committed, or fully committed.

---

7 Other models might take the utility of a state to be the utility to $i$ of the CW of the state. While this is intuitively appealing, it introduces much more complexity.
3.6 Third Party Support

3.6.1 Support When Uncommitted

In the uncommitted case, third parties are not committed to adopt the position of whichever side they support, but the loser of the bilateral contest is still forced to adopt the position of the victor. The expected utility to k of supporting i would be simply the expected value of the two outcomes. If i and k together defeat j, then i remains at their initial position $θ_i$, j must adopt position $θ_j$, and k can remain at position $θ_k$. If i and k together lose to j, then i must adopt $θ_j$. When multiple actors form a coalition, the probability of victory is determined by the strength of each coalition, which is taken to be the sum of their individual capabilities. Because positions for i, j, and k are involved, we carry the terms of $U(S')$ and $U(S'')$ for all three positions:

$$U_k(i, k : j) = \frac{c_i + c_k}{c_i + c_k + c_j} (U_k(\theta_i') + U_k(\theta_j') + U_k(\theta_k')) + \frac{c_j}{c_i + c_k + c_j} (U_k(\theta_i'') + U_k(\theta_j'') + U_k(\theta_k''))$$

Similarly for k supporting j:

$$U_k(j, k : i) = \frac{c_i}{c_i + c_k + c_j} (U_k(\theta_i) + U_k(\theta_i) + U_k(\theta_k)) + \frac{c_j + c_k}{c_i + c_k + c_j} (U_k(\theta_j) + U_k(\theta_j) + U_k(\theta_k))$$

As mentioned earlier, there are multiple voting rules discussed in the literature, our prototypes use several, and expert judgment will be required to determine which is most appropriate for any given case. Because it is analytically clearer, we will present the results only for proportional voting:

$$v_k(i : j) = w_k \left( U_k(i, k : j) - U_k(j, k : i) \right) = w_k \frac{2c_i}{c_i + c_k + c_j} (U_k(\theta_i) - U_k(\theta_j))$$

The particular case of equation (35) (uncommitted, proportional voting) is similar to the formula described in [BDMNR85], albeit with different notation.

3.6.2 Support When Semi-committed

Of course, many small nations have learned that it matters greatly whether their powerful allies win or lose, because they win or lose with their patrons. In the semi-committed case, third party k can keep $θ_k$ if on the winning side but must adopt winner’s position otherwise:

$$U_k(i, k : j) = \frac{c_i + c_k}{c_i + c_k + c_j} (U_k(\theta_i) + U_k(\theta_i) + U_k(\theta_k)) + \frac{c_j}{c_i + c_k + c_j} (U_k(\theta_j) + U_k(\theta_j) + U_k(\theta_j))$$

Similarly for k semi-committed to j:

$$U_k(j, k : i) = \frac{c_i}{c_i + c_k + c_j} (U_k(\theta_i) + U_k(\theta_i) + U_k(\theta_k)) + \frac{c_j + c_k}{c_i + c_k + c_j} (U_k(\theta_j) + U_k(\theta_j) + U_k(\theta_k))$$

With proportional voting,

$$v_k(i : j) = w_k \left( U_k(i, k : j) - U_k(j, k : i) \right) = w_k \frac{2c_i}{c_i + c_k + c_j} (U_k(\theta_i) - U_k(\theta_j)) + \frac{c_i(U_k(\theta_k) - U_k(\theta_i) - c_i(U_k(\theta_k) - U_k(\theta_j)))}{c_i + c_k + c_j}$$

This is simply the uncommitted case, plus a term corresponding to the danger of joining a losing coalition.
3.6.3 Support When Fully Committed

In the fully-committed case, third party $k$ must always adopt the winner’s position:

$$U_k^f(i, k : j) = \frac{c_i + c_k}{c_i + c_k + c_j} (U_k(\theta_i) + U_k(\theta_i) + U_k(\theta_i)) + \frac{c_j}{c_i + c_k + c_j} (U_k(\theta_j) + U_k(\theta_j) + U_k(\theta_j)) \quad (39)$$

Similarly for $k$ fully committed to $j$:

$$U_k^f(j, k : i) = \frac{c_i}{c_i + c_k + c_j} (U_k(\theta_i) + U_k(\theta_i) + U_k(\theta_i)) + \frac{c_j + c_k}{c_i + c_k + c_j} (U_k(\theta_j) + U_k(\theta_j) + U_k(\theta_j)) \quad (40)$$

With proportional voting,

$$v_k^f(i : j) = w_k \left( U_k^f(i, k : j) - U_k^f(j, k : i) \right) = w_k \frac{3c_i}{c_i + c_k + c_j} (U_k(\theta_i) - U_k(\theta_j)) \quad (41)$$

Taken together, equations (35) and (11) imply that modeling third parties as fully committed will increase their effort by a factor of $3/2$. As the behavior of the main parties is the same in either case, the choice of voting rule can change the result of equation (17) because the purely bilateral contributions might dominate in the uncommitted case while the larger third-party contributions might dominate in the fully committed case.

Regardless of how committed the third parties might be if they support one side or another, they always have the third option of abstaining with $v_k(i : j) = 0$. In that case, the utility to actor $k$ is just their own expected utility of the bilateral contest:

$$U_k^n(i : j) = \frac{c_i}{c_i + c_j} (U_k(\theta_i) + U_k(\theta_i) + U_k(\theta_i)) + \frac{c_j}{c_i + c_j} (U_k(\theta_j) + U_k(\theta_j) + U_k(\theta_j)) \quad (42)$$

If $U_k^n(i : j) \geq U_k(j, k : i)$ and $U_k^n(i : j) \geq U_k(i, k : j)$ then the utility of abstaining is greater than the utility of supporting either side, and that particular third party will abstain. This represents the real-world situation of “sitting on the fence” until it becomes clear that the dangers of full commitment or even semi-commitment are worthwhile, i.e. until a “strong horse” emerges.

Again, equations like (35), (38), (41), (42) do not actually appear in our prototype implementations. They implement the general NPCE method, and use the abstract computation. However, under each specific parameterization, the behavior of the abstract method can be described by each specific equation, even though the same abstract computation is used in each case. The situation is entirely analogous to linear programming, where the general simplex algorithm can be applied to data sets with many different structures. For some structures (e.g. multi-commodity flow), there exists more specific algebraic descriptions of how the general algorithm behaves with that specific structure.

3.7 Search for a Condorcet Winner

Much of the literature uses an abstract model inspired by elections: the group simply selects one from a small set of predefined options. The unidimensional issue set in the SPM is often characterized by the integers from 0% to 100%, or even just the positions of a dozen or so actors: it is easy to check each option for the CW. Even in multidimensional SPM, the positions currently held by actors is a small set that can be
easily searched. With discrete, combinatorial issue sets, one quickly encounters enormous sets of options: the “combinatorial explosion” of options is unavoidable.

In infinite (or just astronomically large) issue sets, it is generally impossible to analytically solve for the CW, so any effective procedure - either for a modeling toolkit or a legislative process - must do some sort of incremental exploration of the issue set, looking for local improvements until no more can be found. Both hill-climbing and genetic search are examples. Each alternates between two steps: generating new local options, and selecting which to discard and which to further improve. Because the complicated interactions of the generation and search procedure are impossible to predict in detail (otherwise analytic solution for the CW would be possible), the search for a CW in a large combinatorial issue set is similar to the “preference construction” process described in [RD83] and discussed in [PE12]:

Alternatives are processed in pairs, with the values of the two alternatives compared on each attribute, and the alternative with a majority of winning (better) attribute values is retained. The retained option is then compared with the next alternative from the choice set, and this process of pairwise comparison continues until all the alternatives have been evaluated and one option remains.

This can be interpreted as a hill-climbing procedure to maximize the $\omega$ function of equation (4). The CPT suggested that the observed convergence to definite legislative outcomes can be explained by the existence of a CW, as well as by appeals to institutional structures. Modified for a two-sided comparison process, this could just as well describe the classic “drift to war” (see section (1.1)) where each side observes the behavior of the other side and responds as seems best at the time, the other side reacts similarly, and the process continues. Each comparison of alternative responses can be seen as another one-on-one comparison of options in an implicit Condorcet election, similar to the legislative process of voting on successive proposed amendments.

In general, optimization algorithms can settle on different local optima based on their starting point and search procedure; political scientists often refer to the real-world manifestation of this phenomenon as “path dependence”. Therefore, researchers could pursue two different paths: model the particular incremental, path-dependent search process applicable to their particular problem, or try to design a search algorithm which is likely to find the true global optimum within reasonable time limits. However, the optimization issues involved are not unique to the NPCE methodology, so they will not be discussed here.

4 Summary and Conclusions

This paper has presented a mathematical framework that significantly extends the very successful voting models inspired by the Median Voter Theorem (MVT). Two broad classes of generalization are described. The limitation of previous methods to representing outcomes via uni-dimensional and continuous scales is generalized to allow discontinuous and multidimensional spaces that can allow complex structured policies to be analyzed. And by representing negotiation and bargaining as a non-deterministic processes this approach will allow the exploration of possible outcomes in situations that are sufficiently complex and uncertain that relying on deterministic forecasts could be misleading.

Increased computational power means that simplifications previously made for analytic tractability are no longer required. The combination of the more general mathematical framework presented here and modern computational resources now allow applications utilizing this framework where earlier simpler formulations would not be suitable. Issues such as the formulation of legislation, the formation of coalition governments, the influence of stakeholder groups on direct parties to negotiations and deliberations can now be analyzed using
this framework. Further, the more general approach can serve to avoid the hazard of excluding important information from analyses in order to make them feasible. Deterministic models have the potential to mislead by portraying best estimate outcomes as certain, and the non-deterministic framework described here can serve statistical properties of outcome distributions, which can be important when there are a large number of complex policy options.
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