Supersymmetry requires $g = 2$ for vector bosons

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Relativistic arbitrary spin Hamiltonians are shown to exhibit a SUSY structure under a certain condition. This condition is identical to that required for the exactness of the Foldy-Wouthuysen transformation. Applied to the charged spin-1 particle in a constant magnetic field, supersymmetry necessarily requires a gyromagnetic factor $g = 2$.

I. INTRODUCTION

There is some common agreement that the gyromagnetic ratio $g$ of charged elementary particles when coupled to an electromagnetic field is $g = 2$. There are several reasonable arguments for this. The equation of motion for the spin vector, as shown by Bargmann, Michel and Telegdi [1], takes a very simple form when $g = 2$. As argued by Weinberg [2] and much later by Ferrara, Porrati and Telegdi [3], the requirement to have a good high-energy behaviour of scattering amplitudes, one must choose $g = 2$ for any spin. This is of course in agreement with the standard model. Here the currently known electrically charged elementary particles are either spin-$\frac{1}{2}$ fermions or spin-1 bosons. The standard model indeed requires for all these charged particles a value $g = 2$ at the tree level. Whereas for the elementary fermions ($s = 1/2$) this assertion is consistent with the Belifante [4] conjecture $g = 1/s$, it obviously disagrees with the case of vector bosons where $s = 1$ and the Belifante conjecture would imply $g = 1$. In fact, precision measurements at the Tevatron [5] resulted in the bounds $1.944 \leq g \leq 2.080$ at 95% C.L. for the W boson. Hence, the case of spin-1 elementary particles is of particular interest as no charged higher-spin elementary particles are known yet.

In 1994 Jackiw [6] showed that the value $g = 2$ also follows from a gauge symmetry in the case of a massless spin-1 field coupled to an electromagnetic field. In this brief report we like to provide yet another symmetry argument based on a supersymmetric structure of the relativistic spin-1 Hamiltonian. For this we first generalise the concept of a supersymmetric (SUSY) Dirac Hamiltonian to relativistic Hamiltonians for arbitrary spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Then we consider the case of a charged particle with $s = 1$ interacting with an external constant magnetic field. Here the coupling to the spin-degree of freedom is a prior considered with arbitrary gyromagnetic ratio $g$. It is shown that SUSY will require $g = 2$.

II. SUPERSYMMETRIC RELATIVISTIC HAMILTONIANS

The Hamiltonian of an arbitrary spin-$s$ Hamiltonian can be put into the form

$$H = \beta \mathcal{M} + \mathcal{O},$$

which acts on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2s+1}$ whose elements are $2(2s+1)$-dimensional spinors. The matrix $\beta$ obeys the relation $\beta^2 = 1$ and may be represented as a block-diagonal matrix

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where here the 1 stands for the $2s+1$-dimensional unit matrix. Let us note that the two parts of the Hamiltonian (1) are chosen such that $\mathcal{M}$ accommodates all the even elements and $\mathcal{O}$ all the odd elements of $H$ with respect to $\beta$. That is, we have the commutation and anti-commutation relations

$$[\beta, \mathcal{M}] = 0, \quad \{\beta, \mathcal{O}\} = 0.$$  

Note that the Hamiltonian (1) is hermitian, i.e. $H = H^\dagger$, only for the case of fermions, and hence for half-odd-integer $s$. For bosons, where $s$ is integer, the Hamiltonian is pseudo-hermitian, i.e. $H = \beta H^\dagger \beta$. Having this in mind it is straightforward to show that in the representation (2) both parts of $H$ are necessarily of the form

$$\mathcal{M} = \begin{pmatrix} M^+ & 0 \\ 0 & M^- \end{pmatrix}, \quad \mathcal{O} = \begin{pmatrix} 0 & A \\ (-1)^{(2s+1)} A^\dagger & 0 \end{pmatrix},$$

where $M^\pm : \mathcal{H}^\pm \mapsto \mathcal{H}^\pm$ with $M^\pm_\pm = M^\pm$, $A : \mathcal{H}_- \mapsto \mathcal{H}_+$ and $A^\dagger : \mathcal{H}_+ \mapsto \mathcal{H}_-$. Here we have introduced the positive and negative energy subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$, which are also eigenspaces of $\beta$ for eigenvalue $+1$ and $-1$, respectively. Note that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{H}_+ = L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2s+1}$. Well-know examples are the Klein-Gordon ($s = 0$) and the Dirac ($s = 1/2$) particle in a magnetic field [8].

Let us now assume that the odd and even parts of $H$ commute, that is, $[\mathcal{M}, \mathcal{O}] = 0$. This assumption implies

$$M^+ A = A M^-, \quad A^\dagger M^+ = M^- A^\dagger.$$
From this condition follows that the squared Hamiltonian (1) becomes block diagonal as the off-diagonal blocks vanish due to (5). That is

\[ H^2 = \begin{pmatrix} M_0^2 + (-1)^{2s+1} A A^\dagger & 0 \\ 0 & M_0^2 + (-1)^{2s+1} A A^\dagger \end{pmatrix}, \]

which allows us to define a SUSY structure in analogy to the Dirac case [9, 10]. To be more explicit let us introduce the non-negative SUSY Hamiltonian

\[ H_{\text{SUSY}} := \frac{(-1)^{2s+1}}{2mc^2} (H^2 - M^2) = \frac{1}{2mc^2} \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} \geq 0 \]

and the corresponding complex SUSY charge

\[ Q := \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} A \\ 0 \end{pmatrix}, \quad Q^\dagger := \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 \\ A^\dagger \end{pmatrix}. \]

It is now obvious that these operators together with the \( Z_2 \)-grading (or Witten) operator \( W := \beta \) obey the SUSY algebra

\[ H_{\text{SUSY}} = \{ Q, Q^\dagger \}, \quad \{ Q, W \} = 0, \quad Q^2 = 0 = (Q^\dagger)^2, \]

\[ [W, H_{\text{SUSY}}] = 0, \quad W^2 = 1. \]

Hence, an arbitrary spin Hamiltonian of the form (1) obeying the condition (5) may be called a \textit{supersymmetric arbitrary-spin Hamiltonian}. This is consistent with the usual definition [8, 10] of a supersymmetric Dirac Hamiltonian in the case \( s = \frac{1}{2} \). Let us note that condition (5) also implies that \( \mathcal{M} \) commutes with all operators of the algebra (9).

It is interesting to note that the condition \( [\mathcal{M}, \mathcal{O}] = 0 \) in addition implies that there exists an exact Foldy-Wouthuysen transformation [11, 13]

\[ U := \frac{[H] + \beta H}{\sqrt{2H^2 + 2M[H]}} = \frac{1 + \beta \text{sgn}H}{\sqrt{2 + \{\text{sgn}H, \beta\}}}, \]

\[ \text{sgn}H := H/\sqrt{H^2}, \]

which brings the Hamiltonian into a block diagonal form, cf. eq. (6).

\[ H_{\text{FW}} := U H U^\dagger = \beta \sqrt{H^2} = \beta |H|. \]

As a side remark we mention that the two operators

\[ B_\pm := \frac{1}{2} [1 \pm \beta], \quad \Lambda_\pm := \frac{1}{2} [1 \pm \text{sgn}H] \]

are projection operators onto the subspaces \( H^\pm \) of positive and negative eigenvalues of \( \beta \) and \( H \), respectively, and they are related to each other via the unitary relation [14]

\[ B_\pm = U \Lambda_\pm U^\dagger. \]

That is, the positive and negative energy eigenspaces are transformed via \( U \) into eigenspaces of positive and negative eigenvalues of \( \beta = W \), cf. eq. (11). Note that we may express \( U \) in terms of \( B_\pm \) and \( \Lambda_\pm \) as follows

\[ U = \frac{B_+ \Lambda_+ + B_- \Lambda_-}{\sqrt{(B_+ \Lambda_+ + B_- \Lambda_-)(\Lambda_+ B_+ + \Lambda_- B_-)}}, \]

\[ \text{III. SUPERSYMMETRIC VECTOR BOSONS} \]

Let us now consider the case of a vector boson with charge \( e \) and mass \( m \) interacting with a constant external magnetic field \( \vec{B} \) characterised via the vector potential \( \vec{A} = \frac{1}{2} \vec{B} \times \vec{r} \). The corresponding Hamiltonian is given by

\[ H = \begin{pmatrix} M_+ & A \\ -A & -M_- \end{pmatrix}, \]

where

\[ M_\pm := mc^2 + \frac{\vec{p}^2}{2m} - \frac{ge}{2mc}(\vec{S} \cdot \vec{B}), \]

\[ A := \frac{\vec{p}^2}{2m} - \frac{1}{m}(\vec{S} \cdot \vec{B})^2 + \frac{(g-2)e}{2mc}(\vec{S} \cdot \vec{B}) = A^\dagger \]

and \( \vec{p} := \vec{p} - e\vec{A}/c \). Here \( \vec{S} = (S_1, S_2, S_3)^T \) is a vector whose components are \( 3 \times 3 \) matrices obeying the SO(3) algebra \( [S_i, S_j] = i\epsilon_{ijk}S_k \) representing the spin-one-degree of freedom of the particle, that is \( S^2 = 2 \) as the spin \( s = 1 \) for a vector boson. In above Hamiltonian we have introduced an arbitrary gyromagnetic factor \( g \) which describes the coupling of this spin-degree of freedom to the magnetic field \( \vec{B} \). Note that above Hamiltonian was, to the best of our knowledge, first derived in 1940 by Taketani and Sakata [15] with \( g = 1 \). At the same time Corben and Schwinger [16] had studied the electromagnetic properties of mesotrons and concluded that \( g = 2 \) is required to have a singularity free theory. For later work using above Hamiltonian with both \( g_2 = 2 \) as well as arbitrary values for \( g \) see, for example, refs. [17–22].

Let us now investigate if above Hamiltonian (15) together with (10) does form a supersymmetric relativistic spin-1 Hamiltonian. For this we recall the relation [21]

\[ \left[ \vec{p}^2, (\vec{S} \cdot \vec{p})^2 \right] = \frac{2e \hbar}{c} \left[ (\vec{S} \cdot \vec{B}), (\vec{S} \cdot \vec{p})^2 \right] \]

which allows us to explicitly calculate the commutator

\[ [M_\pm, A] = (g-2) \frac{e \hbar}{2mc^2} \left[ (\vec{S} \cdot \vec{B}), (\vec{S} \cdot \vec{p})^2 \right]. \]

Obviously for a non-vanishing magnetic field the SUSY condition (5) is only fulfilled when \( g = 2 \). In other words, when we require that the relativistic Hamiltonian for a massive charged spin-1 particle in a magnetic field is a \textit{supersymmetric} Hamiltonian only \( g = 2 \) is allowed. This is indeed similar to the argument [10] that the phenomenological non-relativistic Pauli-Hamiltonian for a charged spin-\( \frac{1}{2} \) fermion exhibits a SUSY structure only when \( g = 2 \).
IV. CONCLUDING REMARKS

In a final remark let us note that the above SUSY structure allows to reduce the eigenvalue problem of a supersymmetric arbitrary-spin Hamiltonian to that of a non-relativistic Hamiltonian $H_s$. As we will show elsewhere [23], for a charged particle in the constant magnetic field $\vec{B}$ the FW-transformed Hamiltonian (11) for the cases $s = 0$, $s = \frac{1}{2}$ and $s = 1$ takes the form

$$H_{\text{FW}} = \beta mc^2 \sqrt{1 + \frac{2H_s}{mc^2}}, \quad (19)$$

where

$$H_0 := \frac{1}{2m}(\vec{p} - e\vec{A}/c)^2,$$
$$H_{\frac{1}{2}} := \frac{1}{2m}(\vec{p} - e\vec{A}/c)^2 - \frac{e\hbar}{mc}(\vec{\sigma} \cdot \vec{B}), \quad (20)$$
$$H_1 := \frac{1}{2m}(\vec{p} - e\vec{A}/c)^2 - \frac{e\hbar}{mc}(\vec{\sigma} \cdot \vec{B}).$$

Obviously, $H_0$ and $H_{\frac{1}{2}}$ represent the well-known non-relativistic Landau and Pauli-Hamiltonian, respectively, and $H_1$ is the correct non-relativistic Hamiltonian for a spin-1 particle in a magnetic field with $g = 2$. In above $\vec{\sigma}$ is a vector whose components consist of Pauli’s $2 \times 2$ matrices representing the spin-$\frac{1}{2}$-degree of freedom.

The purpose of this short note is two-fold. First we generalised the concept of supersymmetric relativistic Hamiltonians known from the Dirac Hamiltonian with $s = \frac{1}{2}$ to the general case of arbitrary $s$. More explicitly, under the condition (5) it was shown that a SUSY structure, cf. eqs. (7)-(9), can be accommodated. Second, by considering a massive charged vector boson, i.e. $s = 1$, in the presents of a constant magnetic field, this system resembles a SUSY structure if and only if its gyromagnetic factor $g = 2$.

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