ON THE SECOND KAHN–KALAI CONJECTURE

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ABSTRACT. For any given graph \( H \), we are interested in \( p_c(H) \), the minimal \( p \) such that the Erdős–Rényi graph \( G(n, p) \) contains a copy of \( H \) with probability at least 1/2. Kahn and Kalai (2007) conjectured that \( p_c(H) \) is given up to a logarithmic factor by a simpler “subgraph expectation threshold” \( \tilde{p}_E(H) \), which is the minimal \( p \) such that for every subgraph \( H' \subseteq H \), the Erdős–Rényi graph \( G(n, p) \) contains in expectation at least 1/2 copies of \( H' \). It is trivial that \( \tilde{p}_E(H) \leq p_c(H) \), and the so-called “second Kahn–Kalai conjecture” states that \( p_c(H) \leq \tilde{p}_E(H) \log e(H) \) where \( e(H) \) is the number of edges in \( H \).

In this article we present a natural modification \( \tilde{p}_E(H) \) of the Kahn–Kalai subgraph expectation threshold, which we show is sandwiched between \( p_c(H) \) and \( \tilde{p}_E(H) \). The new definition \( \tilde{p}_E(H) \) is based on the simple observation that if \( G(n, p) \) contains a copy of \( H \) and \( H \) contains \( m \) copies of \( H' \), then \( G(n, p) \) must also contain \( m \) copies of \( H' \). We then show that \( p_c(H) \leq \tilde{p}_E(H) \log e(H) \), thus proving a modification of the second Kahn–Kalai conjecture. The bound follows by a direct application of the set-theoretic “spread” property, which led to recent breakthroughs in the sunflower conjecture by Alweiss, Lovett, Wu and Zhang and the first fractional Kahn–Kalai conjecture by Frankston, Kahn, Narayanan and Park.

1. INTRODUCTION

In this work, we are interested in the following fundamental question. Given a graph \( H \), what is the smallest value \( p = p_c(H) \) for which the Erdős–Rényi graph \( G = G(n, p) \) contains an isomorphic copy of \( H \) with probability at least 1/2?\(^1\) The value \( p_c(H) \) is often referred to as the “critical threshold” for the appearance of \( H \). A well-known conjecture from [KK07] posits that \( p_c(H) \) is given up to a logarithmic factor by a simpler “subgraph expectation threshold” \( \tilde{p}_E(H) \), which is the maximum first-moment threshold among all subgraphs of \( H \).\(^2\) More precisely, for any (labelled) graphs \( H \) and \( H' \), let \( M_{H',H} \) denote the number of subgraphs of \( H \) which are isomorphic copies of \( H' \), and define

\[
\tilde{p}_E(H) = \min \left\{ p : \mathbb{E}M_{H',G(n,p)} \geq \frac{1}{2} \text{ for all } H' \subseteq H \right\}.
\]

It is a trivial consequence of Markov’s inequality (see §2.1) that \( \tilde{p}_E(H) \) lower bounds \( p_c(H) \). Kahn and Kalai proposed that this easy lower bound may not be far from the truth:

**Conjecture ([KK07, Conjecture 2]).** It holds that \( p_c(H) \leq L \tilde{p}_E(H) \log e(H) \) for a universal constant \( L \).

The factor \( \log e(H) \) is necessary in some important examples, such as when \( H \) is a perfect matching or Hamiltonian cycle. In both cases \( \tilde{p}_E(H) \approx 1/n \) but \( p_c(H) \approx (\log n)/n \) ([ER60, ER66, Pós76, Kor76]); for more details see the discussion in [KK07]. The above conjecture remains open, although related conjectures of [KK07, Tal10] were proved recently ([FKNP21, PP22]), as we review below.

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\(^{1}\)The graph \( H \) is allowed to depend on \( n \). Indeed, when \( e(H) \) does not grow with \( n \) the value of \( p_c(H) \) is known [ER60, Bol81], so the main contribution of the present paper is in the setting where \( e(H) \) grows with \( n \).

\(^{2}\)Throughout, a “subgraph” of a graph \( H \) refers to the edge-induced subgraph associated with a subset of the edges of \( H \).
1.1. Main result. In this article we introduce a natural variant \( \tilde{p}_E(H) \) of \( p_E(H) \), and show that it captures \( p_c(H) \) up to a logarithmic factor, thus proving a modification of [KK07, Conjecture 2]. The modification is based on the simple observation that if \( G = G(n, p) \) contains a copy of \( H \), then we must have \( M_{H', G} \geq M_{H', H} \) for any subgraph \( H' \) of \( H \) — in contrast with the weaker bound \( M_{H', G} \geq 1 \), which is used to show \( p_E(H) \leq p_c(H) \). Thus, if we define the “modified subgraph expectation threshold”

\[
\tilde{p}_E(H) = \min \left\{ p : \mathbb{E}M_{H', G(n, p)} \geq \frac{M_{H', H}}{2} \text{ for all } H' \subseteq H \right\} ,
\]

then it is easy to see that \( p_E(H) \leq \tilde{p}_E(H) \leq p_c(H) \) (see also §2.1 below). Our main result is that the new lower bound \( \tilde{p}_E(H) \leq p_c(H) \) is tight up to a logarithmic factor:

**Theorem 1.** It holds that \( p_c(H) \leq L \tilde{p}_E(H) \log e(H) \) for a universal constant \( L \).

A straightforward but tedious calculation gives that \( \tilde{p}_E(H) = 1/n \) when \( H \) is a Hamiltonian cycle. Therefore, as with [KK07, Conjecture 2], the factor \( \log e(H) \) is indeed necessary for Theorem 1 to hold.

1.2. Comparison with previous work. The works most closely related to ours arise from the study of [KK07, Conjecture 1]. This “first Kahn–Kalai conjecture” applies more broadly to the setting of all monotone properties, but is weaker than the second Kahn–Kalai conjecture ([KK07, Conjecture 2]) in the setting of graph inclusion properties (which are the focus of this article). The first Kahn–Kalai conjecture states that for any monotone property \( \mathcal{F} \subseteq \{0, 1\}^X \), we have

\[
p_c(\mathcal{F}) \leq q(\mathcal{F}) \log \ell(\mathcal{F})
\]

where \( q(\mathcal{F}) \) is the maximum first moment threshold among all covers of \( \mathcal{F} \), and \( \ell(\mathcal{F}) \) is the size of a largest minimal element of \( \mathcal{F} \). Talagrand [Tal10] proposed a relaxation of the above, the so-called “fractional Kahn–Kalai conjecture”

\[
p_c(\mathcal{F}) \leq q_f(\mathcal{F}) \log \ell(\mathcal{F})
\]

where \( q(\mathcal{F}) \) is the maximum first moment threshold among all fractional covers of \( \mathcal{F} \). It is trivial that \( q(\mathcal{F}) \leq q_f(\mathcal{F}) \leq p_c(\mathcal{F}) \). Both conjectures were long-standing open problems, which were resolved only recently in two notable works [FKNP21, PP22].

In comparison, the second Kahn–Kalai conjecture is an even stronger conjecture in the particular setting of graph inclusion properties; Kahn and Kalai referred to it as their “starting point” in formulating their first conjecture. If \( \mathcal{F}_H \) is the property of containing a copy of some graph \( H \), then the threshold \( p_E(H) \) of (1) is the maximum first moment threshold among all “subgraph containment covers” of \( \mathcal{F}_H \). Therefore \( p_E(H) \leq q(\mathcal{F}) \), and in the graph inclusion setting the first Kahn–Kalai conjecture may be viewed as a relaxation of the second. To the best of our knowledge, beyond the results on the first conjecture, no further progress has been made on the second one; and it has been reiterated in various places [FHHH+14, FKNP21]. In this work, we modify and prove the second Kahn–Kalai conjecture (Theorem 1). It is an interesting question whether Theorem 1 can be used to prove (or disprove) the second Kahn–Kalai conjecture.

Interestingly, for graph inclusion properties, our result slightly improves on the fractional Kahn–Kalai conjecture (posed by [Tal10] and proved by [FKNP21] for general monotone properties). Indeed, our modified subgraph expectation threshold \( \tilde{p}_E(H) \) can be interpreted as the maximum first moment threshold among certain “subgraph containment fractional covers” of \( \mathcal{F}_H \). To make the correspondence, using the notation of [FKNP21], for any subgraph \( H' \) of \( H \) one can assign weight \( g_{H'}(S) = 1/M_{H', H} \) to any subgraph \( S \subseteq K_n \) that is a copy of \( H' \). This leads to a fractional cover of \( \mathcal{F}_H \) whose first-moment threshold is the unique \( p \) satisfying \( \mathbb{E}M_{H', G(n, p)} = M_{H', H}/2 \). It follows that \( \tilde{p}_E(H) \leq q_f(\mathcal{F}_H) \). Thus, our result implies the fractional Kahn–Kalai bound for graph inclusion properties, in fact with an explicit choice of fractional
covers. Whether our result implies the original first Kahn–Kalai conjecture remains an interesting open problem.

2. Proofs

2.1. Basic notations and calculations. Recall that \( M_{H,H} \) denotes the number of (labelled) subgraphs of \( H \) which are copies of \( H' \). We abbreviate \( M_H \equiv M_{H,K_n} \) where \( K_n \) is the complete graph on \( n \) vertices. We also abbreviate \( Z_H \equiv M_{H,G} \) where \( G \) is the Erdös–Rényi graph \( G(n,p) \). Writing \( \mathbb{P}_p \) for the law of \( G(n,p) \), recall that

\[
p_c(H) = \inf \left\{ p : \mathbb{P}_p(Z_H \geq 1) \geq \frac{1}{2} \right\}.
\]

If \( p \geq p_c(H) \), then Markov’s inequality gives

\[
\frac{1}{2} \leq \mathbb{P}_p(Z_H \geq 1) \leq \mathbb{P}_p\left(Z_{H'} \geq 1 \forall H' \subseteq H\right) \leq \min \left\{ \mathbb{E}Z_{H'} : H' \subseteq H \right\},
\]

which implies \( p_E(H) \leq p_c(H) \) for \( p_E(H) \) as defined by (1). Our definition (2) of \( \tilde{p}_E(H) \) is based on the simple observation that in fact \( p \geq p_c(H) \) together with Markov’s inequality implies

\[
\frac{1}{2} \leq \mathbb{P}_p(Z_H \geq 1) \leq \mathbb{P}_p\left(Z_{H'} \geq M_{H,H'} \forall H' \subseteq H\right) \leq \min \left\{ \frac{\mathbb{E}Z_{H'}}{M_{H,H'}} : H' \subseteq H \right\},
\]

therefore \( \tilde{p}_E(H) \leq p_c(H) \). It is clear that \( p_E(H) \leq \tilde{p}_E(H) \); moreover, if \( \mathbb{E}_p \) denotes expectation with respect to \( \mathbb{P}_p \), then \( \mathbb{E}_p Z_{H'} = M_{H,H'} p_H(H') \), so we can rewrite (1) as

\[
\rho_E(H) = \max \left\{ \left( \frac{1}{2M_{H'}} \right)^{1/e(H')} : H' \subseteq H \right\},
\]

and likewise we can rewrite (2) as

\[
\tilde{p}_E(H) = \max \left\{ \left( \frac{M_{H,H'}}{2M_{H'}} \right)^{1/e(H')} : H' \subseteq H \right\}.
\]

2.2. The spread lemma. The proof of Theorem 1 is an application of a powerful tool, which we refer to as the “spread lemma.” Various forms of this lemma have played a key role in establishing recent breakthrough results on the sunflower theorem [ALWZ21] (see also [Rao20, Tao20]) and the proof of the fractional Kahn–Kalai conjecture [FKNP21].

To state the lemma in our setting, let \( \pi \) be an arbitrary distribution over subgraphs of \( K_n \) (e.g., the copies of particular subgraph of \( K_n \)). For \( R > 1 \), we say that \( \pi \) is \( R \)-spread if for all (without loss of generality, nonempty) subgraphs \( J_0 \subseteq K_n \), if \( H \sim \pi \) then

\[
\pi(J_0 \subseteq H) \leq R^{-e(J_0)}.
\]

Then the spread lemma as stated in [FKNP21, Theorem 1.6] applied to graph inclusion properties implies the following result.

**Lemma 2.** Fix integers \( k, M \geq 1 \). Let \( G_1, \ldots, G_M \) be subgraphs of \( K_n \) with \( e(G_i) \leq k, i \in [M] \). For some universal constant \( C > 0 \), if the uniform distribution \( \pi \) over \( \{G_1, \ldots, G_M\} \) is \( R \)-spread and \( p > C \frac{\log k}{n} \), then a sample from \( G(n,p) \) contains one of the \( G_i \)'s with probability at least \( 1/2 \).

**Proof.** We choose a sufficiently large constant \( C > K \) where \( K \) is the universal constant from [FKNP21, Theorem 1.6]. From standard concentration results, when \( C > K \) is large enough, a sample from \( G(n,p) \) contains, with probability at least \( 2/3 \), a uniformly random undirected graph on \( n \) vertices and \( K \frac{\log k}{n} \choose 2 \) edges. The result then follows directly from [FKNP21, Theorem 1.6] applied to \( X = K_n \) and \( \kappa = R \). □
2.3. The proof. We now show that Theorem 1 follows easily from Lemma 2:

Proof of Theorem 1. Let \( \pi \equiv \pi_H \) denote the uniform distribution over all the copies of \( H \) in the complete graph \( K_n \), and let \( H \) denote a sample from \( \pi \). For a nonempty \( J \subseteq H \), let \( \pi_J \) denote the uniform distribution over all the copies of \( J \) in \( K_n \), and let \( J \) denote a sample from \( \pi_J \). Let \( J_0, H_0 \) be any fixed copies of \( J, H \) respectively in \( K_n \). Then we have

\[
\pi_H(J_0 \subseteq H) = \pi_J(J \subseteq H_0) = \frac{M_{J,H}}{M_J}.
\]

(The first equality is by symmetry. The second holds because there are \( M_J \) possibilities for \( J \), of which \( M_{J,H} \) are contained in \( H_0 \).) By combining (5) with the definition (3) of \( \tilde{p}_E(H) \), we find

\[
\pi_H(J_0 \subseteq H) \overset{(5)}{=} \frac{M_{J,H}}{M_J} \overset{(3)}{=} 2 \tilde{p}_E(H)^{e(J)} \overset{(5)}{\leq} \left( \frac{1}{2 \tilde{p}_E(H)} \right)^{-e(J)},
\]

where the last inequality uses that \( e(J) \geq 1 \). Since \( J_0 \) is arbitrary, we conclude that \( \pi \) is \( R \)-spread with

\[
R = \frac{1}{2 \tilde{p}_E(H)}.
\]

An appeal to Lemma 2 with \( k = e(G) \) concludes the proof. \( \square \)

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References

[ALWZ21] R. Alweiss, S. Lovett, K. Wu, and J. Zhang. Improved bounds for the sunflower lemma. *Ann. of Math. (2)*, 194(3):795–815, 2021.

[Bol81] B. Bollobás. Random graphs. *London Math. Soc. Lec. Note Series*, 52:80–102, 1981.

[ER60] P. Erdős and A. Rényi. On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 5:17–61, 1960.

[ER66] P. Erdős and A. Rényi. On the existence of a factor of degree one of a connected random graph. *Acta Math. Acad. Sci. Hungar*, 17:359–368, 1966.

[FHH`14] Y. Filmus, H. Hatami, S. Heilman, E. Mossel, R. O’Donnell, S. Sachdeva, A. Wan, and K. Wimmer. Real analysis in computer science: A collection of open problems. *Preprint available at https://simons.berkeley.edu/sites/default/files/openprobsmerged.pdf*, 2014.

[FKNP21] K. Frankston, J. Kahn, B. Narayanan, and J. Park. Thresholds versus fractional expectation-thresholds. *Ann. of Math. (2)*, 194(2):475–495, 2021.

[KK07] J. Kahn and G. Kalai. Thresholds and expectation thresholds. *Combin. Probab. Comput.*, 16(3):495–502, 2007.

[Kor76] A. D. Koršunov. Solution of a problem of P. Erdős and A. Rényi on Hamiltonian cycles in undirected graphs. *Dokl. Akad. Nauk SSSR*, 228(3):529–532, 1976.

[Póp76] L. Pósa. Hamiltonian circuits in random graphs. *Discrete Math.*, 14(4):359–364, 1976.

[PP22] J. Park and H. T. Pham. A proof of the Kahn–Kalai conjecture. *arXiv preprint arXiv:2203.17207*, 2022.

[Rao20] A. Rao. Coding for sunflowers. *Discrete Analysis*, (2), 2020.

[Tal10] M. Talagrand. Are many small sets explicitly small? In *Proc. 42nd STOC*, pages 13–35. ACM, New York, 2010.

[Tao20] T. Tao. The sunflower lemma via Shannon entropy. Blog post, https://terrytao.wordpress.com/2020/07/20/the-sunflower-lemma-via-shannon-entropy, 2020.