A Common $q$-Analogue of Two Supercongruences

Victor J. W. Guo and Wadim Zudilin

Abstract. We give a $q$-congruence whose specializations $q = -1$ and $q = 1$ correspond to supercongruences (B.2) and (H.2) on Van Hamme’s list (in: $p$-Adic Functional Analysis (Nijmegen, 1996), Lecture Notes in Pure and Applied Mathematics, vol 192. Dekker, New York, pp 223–236, 1997):

\[
\sum_{k=0}^{(p-1)/2} (-1)^k (4k + 1)A_k \equiv p(-1)^{(p-1)/2} \pmod{p^3}
\]

and

\[
\sum_{k=0}^{(p-1)/2} A_k \equiv a(p) \pmod{p^2},
\]

where $p > 2$ is prime,

\[
A_k = \prod_{j=0}^{k-1} \left( \frac{1/2 + j}{1 + j} \right)^3 = \frac{1}{2^6k} \binom{2k}{k}^3 \quad \text{for } k = 0, 1, 2, \ldots,
\]

and $a(p)$ is the $p$th coefficient of the modular form $q \prod_{j=1}^{\infty} (1 - q^{4j})^6$ (of weight 3). We complement our result with a general common $q$-congruence for related hypergeometric sums.

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1. Introduction

The formula of Bauer [1] from 1859,

$$\sum_{k=0}^{\infty} (-1)^k (4k+1)A_k = \frac{2}{\pi}, \quad \text{where } A_k = \frac{1}{2^{6k}} \binom{2k}{k}^3$$

for \( k = 0, 1, 2, \ldots \), (1.1)

is one of traditional targets for different methods of proofs of hypergeometric identities. Its special status is probably linked to the fact that it belongs to a family of series for \( 1/\pi \) of Ramanujan type, after Ramanujan [21] brought to life in 1914 a long list of similar looking equalities for the constant but with a faster convergence. Identity (1.1) is a particular instance of \( 4F3 \) hypergeometric summation (known to Ramanujan) but there are several proofs of it, including the original one [1] of Bauer, that do not require any knowledge of hypergeometric functions. One notable—computer—proof of (1.1) was given in 1994 by Ekhad and Zeilberger [2] using the Wilf–Zeilberger (WZ) method of creative telescoping.

It was observed in 1997 by Van Hamme [28] that many Ramanujan’s and Ramanujan-like evaluations have nice \( p \)-adic analogues; for example, the congruence

$$\sum_{k=0}^{(p-1)/2} (-1)^k (4k+1)A_k \equiv p(-1)^{(p-1)/2} \mod p^3$$

(tagged (B.2) on Van Hamme’s list) is valid for any prime \( p > 2 \) and corresponds to the equality (1.1). The congruence (1.2) was first proved by Mortenson [19] using a \( 6F5 \) hypergeometric transformation; it later received another proof by one of these authors [29] via the WZ method [in fact, using the very same ‘WZ certificate’ as in [2] for (1.1)]. Notice that (1.2) is an example of supercongruence meaning that it holds modulo a power of \( p \) greater than 1.

Another entry on Van Hamme’s 1997 list [28], tagged (H.2), is the congruence

$$\sum_{k=0}^{(p-1)/2} A_k \equiv \begin{cases} -\Gamma_p(1/4)^4 \mod p^3 & \text{if } p \equiv 1 \mod 4, \\ 0 \mod p^2 & \text{if } p \equiv 3 \mod 4, \end{cases}$$

again for any \( p > 2 \) prime, and \( \Gamma_p(x) \) is the \( p \)-adic Gamma function. Van Hamme not only observed but also proved (1.3) in [28], and it was later generalized by Sun [23,24, Theorem 2.5], Guo and Zeng [12, Corollary 1.2], Long and Ramakrishna [17], Liu [15,16, Theorem 1.5] in different ways. For example, Long and Ramakrishna [17, Theorem 3] gave the following generalization of (1.3):

$$\sum_{k=0}^{(p-1)/2} A_k \equiv \begin{cases} -\Gamma_p(1/4)^4 \mod p^3 & \text{if } p \equiv 1 \mod 4, \\ \frac{p^2}{16} \Gamma_p(1/4)^4 \mod p^3 & \text{if } p \equiv 3 \mod 4. \end{cases}$$

(1.4)
Recently, these authors [14, Theorem 2] proved that, for any positive odd integer \( n \), modulo \( \Phi_n(q)^2 \),

\[
\sum_{k=0}^{(n-1)/2} \frac{(q^2; q^4)^2_k (q^2; q^4)_k^2}{(q^2; q^4)^2_k (q^4; q^4)_k^2} q^{2k} = \begin{cases} (q^2; q^4)^2_{(n-1)/4} (q^{n-1}/4)^2 & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}. \end{cases}
\]

(1.5)

Here and in what follows, \( \Phi_n(q) \) denotes the nth cyclotomic polynomial; the \( q \)-shifted factorial is given by \( (a; q)_0 = 1 \) and \( (a; q)_n = (1 - a)(1 - aq)\ldots(1 - aq^{n-1}) \) for \( n \geq 1 \) or \( n = \infty \), while \([n] = [n]_q = 1 + q + \cdots + q^{n-1} \) stands for the \( q \)-integer. Van Hamme [27, Theorem 3] also proved that

\[
\left( -\frac{1}{2} \right)_{(p-1)/4} \equiv -\frac{\Gamma_p(1/4)^2}{\Gamma_p(1/2)} \pmod{p^2};
\]

in view of \( \Gamma_p(1/2)^2 = -1 \) for \( p \equiv 1 \pmod{4} \), by letting \( q \to 1 \) in (1.5) for \( n = p \) we immediately obtain (1.3).

One feature of (1.3) (not highlighted in [28]) is its connection with the coefficients

\[
a(p) = \begin{cases} 2(a^2 - b^2) & \text{if } p = a^2 + b^2, a \text{ odd,} \\ 0 & \text{if } p \equiv 3 \pmod{4}, \end{cases}
\]

(1.6)

of CM modular form \( q \prod_{j=1}^{\infty} (1 - q^{4j})^6 \) of weight 3, namely, the congruence

\[
a(p) \equiv -\Gamma_p(1/4)^4 \pmod{p^2} \quad \text{for primes } p \equiv 1 \pmod{4}.
\]

This served as a main motivation in [14] for not only establishing (1.5) but also speculating on possible \( q \)-deformation of modular forms.

For some other recent progress on \( q \)-analogues of supercongruences, the reader is referred to [4,5,7–11,13,20,22,26,29]. In particular, the authors [13] introduced and executed a new method of creative microscoping to prove (and reprove) many \( q \)-analogues of classical supercongruences and also raised some problems on \( q \)-congruences. Using this method, the first author [6] gave a refinement of (1.5) modulo \( \Phi_n(q)^3 \) for \( n \equiv 3 \pmod{4} \), in other words, a \( q \)-analogue of (1.4) for \( p \equiv 3 \pmod{4} \).

A goal of this note is to present the following new \( q \)-analogue of Van Hamme’s supercongruence (1.3).

**Theorem 1.1.** Let \( n \) be a positive odd integer. Then

\[
\sum_{k=0}^{(n-1)/2} \frac{(1 + q^{4k+1}) (q^2; q^4)_k^3}{(1 + q) (q^4; q^4)_k} q^k \equiv \begin{cases} [n]_q^2 (q^3; q^4)_{(n-1)/2} (q^{(1-n)/2}) & \text{if } n \equiv 1 \pmod{4}, \\ (q^5; q^4)_{(n-1)/2} (q^{(1-n)/2}) & \text{if } n \equiv 3 \pmod{4}, \end{cases}
\]

(1.7)
Note that $\Phi_n(q)\Phi_n(-q) = \Phi_n(q^2)$ for odd indices $n$.

The $n \equiv 3 \pmod{4}$ case of Theorem 1.1 confirms a conjecture of these authors [13, Conjecture 4.13], which states that, for $n \equiv 3 \pmod{4}$,

$$\sum_{k=0}^{(n-1)/2} \frac{(1 + q^{4k+1}) (q^2; q^4)_k^3}{(1 + q) (q^4; q^4)_k^3} q^k \equiv 0 \pmod{\Phi_n(q)\Phi_n(-q)}.$$ 

It is not difficult to verify that

$$\frac{(3/4)(p-1)/2}{(5/4)(p-1)/2} \equiv - \frac{p}{16} \Gamma_p (1/4)^4 \pmod{p^2}$$

for $p \equiv 3 \pmod{4}$, where $(a)_n = a(a+1)\ldots(a+n-1)$ denotes the rising factorial (also known as Pochhammer’s symbol). Therefore, the $q$-congruence (1.7) reduces to (1.4) for $p \equiv 3 \pmod{4}$ when $n = p$ and $q \to 1$, and it reduces to (1.3) for $p \equiv 1 \pmod{4}$ when $n = p$ and $q \to 1$. Moreover, letting $n = p$ and $q \to -1$ in (1.7), we immediately get (1.2). Thus, Theorem 1.1 presents a generalization of (1.2) for $p \equiv 3 \pmod{4}$ when $n \equiv 1 \pmod{4}$.

The $n \equiv 1 \pmod{4}$ case of Theorem 1.2 also confirms a conjecture of the first author and Schlosser [11, Conjecture 10.2].

For $n$ prime, letting $q \to 1$ in Theorem 1.2 we obtain the following generalization of (1.8).

**Theorem 1.2.** Let $n > 1$ be an odd integer. Then

$$\sum_{k=0}^{(n+1)/2} \frac{(1 + q^{4k-1}) (q^{-2}; q^4)_k}{(1 + q) (q^4; q^4)_k^3} q^k \equiv \begin{cases} \frac{[n]q^2(q^4)_n^{-1/2}}{(q^2; q^4)_{n-1/2}^2} q^{(n-3)/2} \pmod{\Phi_n(q^3)\Phi_n(-q^3)} & \text{if } n \equiv 1 \pmod{4}, \\ \pmod{\Phi_n(q^2)\Phi_n(-q^3)} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

The $n \equiv 1 \pmod{4}$ case of Theorem 1.2 also confirms a conjecture of the first author and Schlosser [11, Conjecture 10.2].

For $n$ prime, letting $q \to 1$ in Theorem 1.2 we obtain the following generalization of (1.8).

**Corollary 1.3.** Let $p$ be an odd prime. Then

$$\sum_{k=0}^{(p+1)/2} \frac{(-1/2)_k^3}{k!^3} \equiv \begin{cases} \frac{p}{(4/4)(p-1)/2} (\pmod{p^3}) & \text{if } p \equiv 1 \pmod{4}, \\ \frac{p}{(7/4)(p-1)/2} (\pmod{p^2}) & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

On the other hand, for $n$ prime and $q \to -1$ in Theorem 1.2, we are led to the following result:

$$\sum_{k=0}^{(p+1)/2} (-1)^k (4k-1) \frac{(-1/2)_k^3}{k!^3} \equiv p(-1)^{(p+1)/2} (\pmod{p^3}).$$
It should be mentioned that a different $q$-analogue of (1.9) was given in [13, Theorem 4.9] with $r = -1$, $d = 2$ and $a = 1$ (see also [11, Section 5]). Moreover, for the summation formula
\[
\sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^3}{k!^3} = 12\frac{\Gamma(3/4)^4}{\pi^3},
\]
we have the following $q$-analogue.

**Theorem 1.4.** We have
\[
\sum_{k=0}^{\infty} \frac{(1 + q^{4k-1})(q^{-2}; q^4)^3}{(1 + q^{-1})(q^4; q^4)^3} q^{7k} = \frac{(q^2; q^4)_{\infty}(q^5; q^4)^2(q^6; q^4)_{\infty}}{(q^3; q^4)_{\infty}(q^4; q^4)^2(q^7; q^4)_{\infty}}.
\]

Both Theorems 1.1 and 1.2 are particular cases of a more general result, which we state and prove in the next section, while Theorem 1.4 follows from a classical $q$-identity.

### 2. A Family of $q$-Congruences from the $q$-Dixon Sum

In this section we establish the following family of one-parameter $q$-congruences.

**Theorem 2.1.** Let $n \geq 1$ be an odd integer and $\ell$ an integer with $0 \leq \ell \leq (n-1)/2$. Then
\[
\sum_{k=0}^{n-1} \frac{(1 + q^{4k-2\ell+1})(q^{2\ell-4}\ell; q^4)^3}{(1 + q^{-1-2\ell})(q^4; q^4)^3} q^{(6\ell+1)k} = \begin{cases} 
(1 - q^{2n})(q^{3-6\ell}; q^4)_{(n-1)/2+\ell} q^{(2\ell-1)((n-1)/2+\ell)} & \text{if } n + 2\ell \equiv 1 \pmod{4}, \\
(1 - q^{2-4\ell})(q^{5-2\ell}; q^4)_{(n-1)/2+\ell} & \text{if } n + 2\ell \equiv 3 \pmod{4}. 
\end{cases}
\]

(2.1)

Note that the $q$-congruence (2.1) remains true when the sum is over $k$ from 0 to $(n - 1)/2 + \ell$, since $(q^{2-4\ell}; q^4)_{k}/(q^4; q^4)_{k} \equiv 0 \pmod{\Phi_n(q^2)}$ for $(n - 1)/2 + \ell < k \leq n - 1$. Furthermore, when $\ell = 0$ and $\ell = 1$ (hence $n \geq 3$) the theorem reduces to Theorems 1.1 and 1.2, respectively.

The following easily proved $q$-congruence (see [11, Lemma 3.1]) is necessary in our derivation of Theorem 2.1.

**Lemma 2.2.** Let $n$ be a positive odd integer. Then, for $0 \leq k \leq (n-1)/2$, we have
\[
\frac{(aq; q^2)^{(n-1)/2-k}}{(q^2/a; q^2)^{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k} \frac{(aq; q^2)^k}{(q^2/a; q^2)^k} q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}.
\]

Like the proofs given in [13], we start with the following generalization of (1.7) with an extra parameter $a$. 
Theorem 2.3. Let $n > 1$ be an odd integer and $0 \leq \ell \leq (n-1)/2$. Then

$$\sum_{k=0}^{n-1} \frac{(1 + q^{4k-2\ell+1}) (aq^{2-4\ell}; q^4)_k (q^{2-4\ell}/a; q^4)_k (q^{2-4\ell}; q^4)_k}{(1 + q^{1-2\ell}) (aq^4; q^4)_k (q^4/a; q^4)_k (q^4; q^4)_k} q^{(6\ell+1)k}$$

$$= \frac{(1 - q^{2n}) (q^{3-6\ell}; q^4)_{(n-1)/2+\ell}}{(1 - q^{2-4\ell}) (q^{5-2\ell}; q^4)_{(n-1)/2+\ell}} \begin{cases} \mod \Phi_n(-q)(1 - aq^{2n})(a - q^{2n}) & \text{if } n + 2\ell \equiv 1 \pmod{4}, \\ \mod \Phi_n(q^2)(1 - aq^{2n})(a - q^{2n}) & \text{if } n + 2\ell \equiv 3 \pmod{4}. \end{cases} \quad (2.2)$$

Proof. Performing the parameter substitutions $q \mapsto q^4$, $a \mapsto q^{2-4\ell}$, $b \mapsto bq^{2-4\ell}$ and $c \mapsto cq^{2-4\ell}$ in the $q$-Dixon sum [3, Appendix (II.13)], we obtain

$$\sum_{k=0}^{\infty} \frac{(1 + q^{4k-2\ell+1}) (q^{2-4\ell}; q^4)_k (q^{2-4\ell}; q^4)_k (cq^{2-4\ell}; q^4)_k}{(1 + q^{1-2\ell}) (q^4/b; q^4)_k (q^4/c; q^4)_k (q^4; q^4)_k} q^{(6\ell+1)k}$$

$$= \frac{(q^{6-4\ell}; q^4)_\infty (q^{2\ell+3}/b; q^4)_\infty (q^{2\ell+3}/c; q^4)_\infty (q^{4\ell+2}/bc; q^4)_\infty}{(q^4/b; q^4)_\infty (q^4/c; q^4)_\infty (q^{5-2\ell}; q^4)_\infty (q^{6\ell+1}/bc; q^4)_\infty}. \quad (2.3)$$

Since $n$ is odd, putting $b = q^{2n}$ and $c = q^{2n}$ in (2.3) we see that the left-hand side terminates and is equal to

$$\sum_{k=0}^{(n-1)/2+\ell} \frac{(1 + q^{4k-2\ell+1}) (q^{2-4\ell-2n}; q^4)_k (q^{2-4\ell+2n}; q^4)_k (q^{2-4\ell}; q^4)_k}{(1 + q^{1-2\ell}) (q^{2n}/q^4)_k (q^{4n+2n}/q^4)_k (q^4; q^4)_k} q^{(6\ell+1)k}$$

$$= \sum_{k=0}^{n-1} \frac{(1 + q^{4k-2\ell+1}) (q^{2-4\ell-2n}; q^4)_k (q^{2-4\ell+2n}; q^4)_k (q^{2-4\ell}; q^4)_k}{(1 + q^{1-2\ell}) (q^{4n-2n}/q^4)_k (q^{4n+2n}/q^4)_k (q^{4n}; q^4)_k} q^{(6\ell+1)k},$$

while the right-hand side becomes

$$\frac{(q^{2\ell-2n+3}; q^4)_{(n-1)/2+\ell} (q^{6-4\ell}; q^4)_{(n-1)/2+\ell}}{(q^{4n}; q^4)_{(n-1)/2+\ell} (q^{5-2\ell}; q^4)_{(n-1)/2+\ell}}$$

$$= \frac{(1 - q^{2n}) (q^{3-6\ell}; q^4)_{(n-1)/2+\ell}}{(1 - q^{2-4\ell}) (q^{5-2\ell}; q^4)_{(n-1)/2+\ell}} q^{(2\ell-1)((n-1)/2+\ell)}.$$

This proves that the $q$-congruence (2.2) holds modulo $1 - aq^{2n}$ or $a - q^{2n}$.

On the other hand, by Lemma 2.2, for $0 \leq k \leq (n-1)/2 + \ell$, modulo $\Phi_n(q)$ we have
\[
\frac{(aq^{1-2\ell}; q^2)_{(n-1)/2+\ell-k}}{(q^2/a; q^2)_{(n-1)/2+\ell-k}} = \frac{(aq^{1-2\ell}; q^2)_{(aq; q^2)_{(n-1)/2-k}}}{(q^{n+1-2k}/a; q^2)^{\ell}(q^2/a; q^2)_{(n-1)/2-k}} \\
\equiv (-a)^{(n-1)/2-2k} \frac{(aq^{1-2\ell}; q^2)_{(aq; q^2)_{k}}}{(q^{n+1-2k}/a; q^2)^{\ell}(q^2/a; q^2)_{k}} q^{(n-1)/2+4k} \\
= (-a)^{(n-1)/2-2k} \frac{(aq^{1-2\ell}; q^2)_{k}}{(q^2/a; q^2)_{k}} q^{(n-1)/2+4k} \\
= (-a)^{(n-1)/2+\ell-2k} \frac{(aq^{1-2\ell}; q^2)_{k}}{(q^2/a; q^2)_{k}} q^{(n-1)/2+4k+(2k-\ell)},
\]

where we used \(q^n \equiv 1 \pmod{\Phi_n(q)}\) in the last step. Using the above \(q\)-congruence we can easily check that, for odd \(n > 1\) and \(0 \leq k \leq (n-1)/2+\ell\), sum of the \(k\)th and \(((n-1)/2 + \ell - k)\)th summands on the left-hand side of (2.2) is congruent to 0 modulo \(\Phi_n(q)\) (or modulo \(\Phi_n(q^2)\) if \(n \equiv 3 - 2\ell \pmod{4}\)). It follows that

\[
\sum_{k=0}^{(n-1)/2+\ell} \frac{(1+q^{4k-2\ell+1})}{(1+q^{1-2\ell})} \frac{(aq^{2-4\ell}; q^4)_{k}(q^{2-4\ell}/a; q^4)_{k}(q^{2-4\ell}; q^4)_{k}}{(aq^4; q^4)_{k}(q^4/a; q^4)_{k}(q^4; q^4)_{k}} q^{(6\ell+1)k} \equiv 0 \\
\frac{(\text{mod } \Phi_n(q))}{(\text{mod } \Phi_n(q^2))} \begin{cases} 
\text{if } n+2\ell \equiv 1 \pmod{4}, \\
\text{if } n+2\ell \equiv 3 \pmod{4}.
\end{cases}
\]

Clearly, the right-hand side of (2.1) is congruent to 0 modulo \(\Phi_n(q)\) if \(n+2\ell \equiv 1 \pmod{4}\) and modulo \(\Phi_n(q^2)\) if \(n+2\ell \equiv 3 \pmod{4}\). Therefore, the \(q\)-congruence (2.2) holds modulo \(\Phi_n(q)\) if \(n+2\ell \equiv 1 \pmod{4}\) and modulo \(\Phi_n(q^2)\) if \(n+2\ell \equiv 3 \pmod{4}\). Since the polynomials \(1-aq^{2n}\), \(a-q^{2n}\) and \(\Phi_n(q)\) (or \(\Phi_n(q^2)\)) are pairwise coprime, we complete the proof of (2.2). \(\square\)

**Proof of Theorem 2.1.** We assume that \(n > 1\), since the \(n=1\) case (making \(\ell = 0\) only possible) is trivial. The limits of the denominators on both sides of (2.2) as \(a \rightarrow 1\) are relatively prime to \(\Phi_n(q^2)\), since \(k\) is in the range \(0 \leq k \leq (n-1)/2 + \ell\). On the other hand, the limit of \((1-aq^{2n})(a-q^{2n})\) as \(a \rightarrow 1\) contains the factor \(\Phi_n(q^2)^2\). \(\square\)

**Proof of Theorem 1.4.** Take \(b = c = \ell = 1\) in Eq. (2.3). \(\square\)

### 3. Discussion

The method of creative microscoping used in our proofs indicates the origin of \(q\)-congruences from infinite \(q\)-hypergeometric identities; for example, the \(q\)-congruence (1.7) corresponds to the identity

\[
\sum_{k=0}^{\infty} \frac{(1+q^{4k+1})}{(1+q)(q^4; q^4)_k} q^k = \frac{(q^2; q^4)_{\infty}^2(q^3; q^4)_{\infty}^2}{(1+q)(q^4; q^4)_{\infty}^2(q^4; q^4)_{\infty}^2},
\]
which is just a particular instance of (2.3). Note that the limiting cases as $q \to -1$ and $q \to 1$ of (3.1) give the formulas (1.1) and

$$\sum_{k=0}^{\infty} \frac{(1/2)_k^3}{k!^3} = \frac{\Gamma(1/4)^4}{4\pi^3} = \frac{8L(f,1)}{\pi} = \frac{16L(f,2)}{\pi^2} \quad (3.2)$$

where

$$f(\tau) = q \prod_{j=1}^{\infty} (1 - q^{4j})^6 = \sum_{n=1}^{\infty} a(n)q^n,$$  

is the CM modular form from the introduction and $L(f,s)$ denotes its L-function. This means that the $q$-identity (3.1) presents a common $q$-extension of evaluations (1.1) and (3.2)—the fact that makes it less surprising that the $q$-congruence (1.7) simultaneously extends (1.2) and (1.3).

The intermediate use of *parametric* $q$-hypergeometric identities in our proof of Theorem 2.1 based on the $q$-Dixon sum suggests that different $q$-congruences underlying (3.1) are possible. This is indeed the case when we analyze the formula (3.1) as the $a = 1$ specialization of

$$\sum_{k=0}^{\infty} \frac{(1 + q^{4k+1}) (aq; q^2)_k(q/a; q^2)_k(-q; q^2)_k^2(q^4; q^4)_k}{(1 + q)(q^2; q^2)_k^2(-aq^2; q^2)_k(-q^2/a; q^2)_k^2(q^4; q^4)_k} q^k = \frac{(-q; q^2)_\infty^2(q^{4m+4}; q^4)_\infty^2(q^{2-4m}; q^4)_\infty^2}{(1 + q)(-aq^2; q^2)_\infty^2(-q^2/a; q^2)_\infty^2(q^2; q^4)_\infty^2} \quad (3.3)$$

which originates from a $q$-analogue of Watson’s $3 \ F_2$ sum [3, Appendix (II.16)]. When we choose $a = q^n$ (or $a = q^{-n}$) in (3.3), for $n > 1$ odd, we get the sum terminating after $(n - 1)/2$ terms on the left-hand side of (3.3), while the right-hand side vanishes if $n$ is of the form $4m + 3$ and it becomes equal to

$$\frac{(-q; q^2)_\infty^2(q^{4m+4}; q^4)_\infty^2(q^{2-4m}; q^4)_\infty^2}{(1 + q)(-aq^2; q^2)_\infty^2(-q^2/a; q^2)_\infty^2(q^2; q^4)_\infty^2} = [4m + 1] \frac{(q^2; q^4)_m^2}{(q^4; q^4)_m^2}$$

if $n = 4m + 1$. This means that modulo $(a - q^n)(1 - aq^n)$ we have

$$\sum_{k=0}^{N} \frac{(1 + q^{4k+1}) (aq; q^2)_k(q/a; q^2)_k(-q; q^2)_k^2(q^4; q^4)_k}{(1 + q)(q^2; q^2)_k^2(-aq^2; q^2)_k(-q^2/a; q^2)_k^2(q^4; q^4)_k} q^k = \begin{cases} [4m + 1] \frac{(q^2; q^4)_m^2}{(q^4; q^4)_m^2} & \text{if } n = 4m + 1, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

for any $N \geq (n - 1)/2$. The limiting $a \to 1$ case of the congruences can be shown to be

$$\sum_{k=0}^{(n-1)/2} \frac{(1 + q^{4k+1}) (q^2; q^4)_k^2}{(1 + q)(q^4; q^4)_k^2} q^k = \begin{cases} [4m + 1] \frac{(q^2; q^4)_m^2}{(q^4; q^4)_m^2} & \text{if } n = 4m + 1, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \end{cases} \quad (3.4)$$
modulo $\Phi_n(q)^2\Phi_n(-q)$. This is quite similar in spirit to (1.5), though still far from constructing $q$-analogues for the coefficients $a(p)$ in (1.6) of the modular form $f(\tau)$. The latter means that a hunt for $q$-rational functions, which equal the left-hand side of (1.5) or (3.4) modulo $\Phi_n(q)^2$ and specialize to $a(n)$ as $q \to 1$ (at least for $n$ prime), is still on its way. Such $q$-rational functions are also expected to be self-reciprocal, that is, invariant under the involution $q \mapsto 1/q$, as all the left- and right-hand sides in (1.5), (1.7), (3.4) and also (2.1) are.

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