Dynamic concurrent van Emde Boas array

Konrad Kulakowski

AGH University of Science and Technology, al. Mickiewicza 30, Kraków, Poland, konrad.kulakowski@agh.edu.pl

Abstract. The growing popularity of shared-memory multiprocessor machines has caused significant changes in the design of concurrent software. In this approach, the concurrently running threads communicate and synchronize with each other through data structures in shared memory. Hence, the efficiency of these structures is essential for the performance of concurrent applications. The need to find new concurrent data structures prompted the author some time ago to propose the cvEB array modeled on the van Emde Boas Tree structure as a dynamic set alternative.

This paper describes an improved version of that structure - the dcvEB array (Dynamic Concurrent van Emde Boas Array). One of the improvements involves memory usage optimization. This enhancement required the design of a tree which grows and shrinks at both: the top (root) and the bottom (leaves) level. Another enhancement concerns the successor (and predecessor) search strategy. The tests performed seem to confirm the high performance of the dcvEB array. They are especially visible when the range of keys is significantly larger than the number of elements in the collection.

1 Introduction

The rapid rise in the popularity of multi-core shared-memory processor systems makes concurrent programs increasingly common and desirable. Following the growing market for concurrent software, an increasing demand for the use of concurrent data structures can be observed. Such a situation makes the search for new concurrent data structures particularly important. One of the attempts to find such a structure is the work [15] in which the author proposed the early version of the concurrent van Emde Boas (cvEB) array. The structure presented here is an example of concurrent dynamic set implementation providing, in addition to the standard methods insert(), remove() and find(), also the method successor(), which allows users to determine the first greater element from the specified one. It has very good theoretical and practical properties, as confirmed by tests and analyses carried out. Unfortunately, one of the shortcomings of that solution is the need to allocate all the required memory at the very beginning, as in the case of a regular array. Another limitation of that structure is the implementation of successor(), which in the case of massive interference with remove() operating in different threads might be delayed or failed due to the search repetition. These deficiencies led the author to propose a new dynamic concurrent van Emde Boas (dcvEB) array, which, on the one hand, retains the good properties of its antecedent, and on the other hand is deprived of its shortcomings. Hence, the new structure presented in this article allocates and deallocates memory dynamically, depending on the amount of data stored in it. In addition, a new strategy for the successor() and predecessor() methods has been adopted. The new structure, rather than repeating the successor or predecessor search, continues searching until an appropriate element is found or the absence of such an element is decided. The assumed strategy is more robust and less susceptible to interference. It also seems to be more intuitive and justifiable in the context of user expectation.

The article consists of several sections, where, except for introductory ones (Sec. 1-3), the dcvEB array (Subsection 4.2) and its implementation (Subsection 4.3) are discussed. Next, the mechanisms of concurrent expanding and shrinking are explained (Subsection 5.4). Other enhancements, such as dynamic memory allocation and the new search strategy, are explained (Subsections: 5.2 and 5.3). Then, the successor search running time and the structure correctness are discussed (Section 5). The experimental results are examined in Section 6. The comments and discussion (Section 7) and a brief summary (Section 8) close the article.

2 Background

A dynamic set is one of the basic data structures in computer science. Usually, it is assumed that a dynamic set supports the following operations: insert(), delete(), search(), minimum(), maximum(), successor() and predecessor() [5, p. 230]. The first two of them are included in the category of modifying operations, while others are queries, which do not modify the structure. Due to increasing demands for data format, different structures support dynamic set operations to varying degrees. In particular, good dynamic set operation performance is provided by bal-
anced search trees. For instance, all the dynamic set operations can be handled by RB-Trees [1] in a sequential running time $O(\lg \alpha)$, whilst van Emde Boas trees [25] need barely $O(\lg \lg \alpha)$ time to complete any of the mentioned operations [5]. Unfortunately, transition from the sequential to the concurrent objects is not easy [24]. Hence, many concurrent dynamic set implementations (e.g. [6,13]) do not support all the dynamic set operations and instead focus on dictionary operations.

The early works on the concurrent balanced search trees with dictionary operations began to emerge in the 70s [23,2]. In the subsequent years, the topic was studied in [7,10,19]. The studies, initially focusing on lock strategy [2] and lock coupling [19,17], began to deal with the relaxed (delayed) re-balancing [21,19] and the non-blocking synchronization schemes [3,6,13].

Skip List, proposed by Pugh [22], is an alternative to balanced search trees. It provides several linked lists arranged in a hierarchy, so that the single list corresponds to the set of nodes at the same depth in a search tree. The structure avoids additional re-balancing due to the randomized fashion of the insertion algorithm. SkipList is suitable for both the sequential and concurrent applications. Very efficient SkipList implementation [10], based on Fraser [8], is part of a standard Java API 5. The Java SkipList implementation as one of the few (the second is a SnapTree Map by Bronson [3]) supports all the dynamic set operations including successor() and predecessor().

3 van Emde Boas tree

The tree structure proposed by van Emde Boas [25] is not a typical search tree. It supports all the dynamic set operations, such as insert(), delete(), search(), minimum(), maximum(), successor() and predecessor() [3, p. 230] in $O(\ln \ln \alpha)$. This tremendous speed involves the requirement that the keys must be unique integers in the range 0 to $\alpha - 1$. Thus, from a practical point of view, the van Emde Boas tree is something between an array and a search tree. Assuming that the number of stored elements is essentially smaller than $\alpha$, the vEB tree is better than the array as regards the speed of successor(), predecessor(), minimum() and maximum().

Of course, the efficiency of the array operations insert(), delete() and search() remains unchallenged regardless of the stored data size. In general, the vEB tree operates faster than the other search trees. However, the strong constraint on the key values makes it unusable if the stored objects cannot be represented as unique integers.

The key to the efficiency of the vEB tree operations is the uneven number of subtrees on different levels of the vEB tree. Thus, the root node has $\alpha^{1/2}$ of subtrees, whereas each next level of the vEB tree shrinks the number of children in the nodes by the square root. Assuming that an operation over the vEB tree performs $O(1)$ work at each level of the hierarchy, the running time of a method is $O(h)$, where $h$ is the height of the vEB tree. Reducing the number of subtrees can not be carried out indefinitely. Thus, at the last but one level of the tree, the nodes have at most two single-element subtrees, i.e. $\alpha^{1/2h-1} = 2$. Hence, we obtain $\ln \alpha = 2^{h-1}$, and finally $h = \ln \ln \alpha + 1$. Thus, the asymptotic running time of an exemplary operation is $O(\ln \ln \alpha + 1) = O(\ln \ln \alpha)$.

To be able to traverse each level of a tree in $O(1)$ the vEB tree methods use the arrays of references to the subtrees. For this reason the root node $T_{\text{root}}$ needs to store $T_{\text{root.arr}} - \alpha^{1/2}$-element array of subtrees, their children $T, T_{\text{arr}} - \alpha^{1/4}$-element arrays of their subtrees, and so on. With this construction, every method can calculate in which subtree the given key can be found. For example, in the case of the root node, the key $x$ is expected to be in $[x/\alpha^{1/2}]$ subtree etc.

To achieve $O(1)$ level traversing time, the more complex methods like successor() and predecessor() need further information about the subtrees. Thus, with every node $T$ the next three variables are assigned: $T_{\text{max}}, T_{\text{min}}$ and $T_{\text{summary}}$, where $T_{\text{max}}, T_{\text{min}}$ denote correspondingly the maximal and the minimal value of a key in the subtree rooted in $T$. The summary is an auxiliary search structure. Intuitively speaking, the search() method traverses down the vEB tree along a well-defined path from the root to the given key. The successor() must deviate from this path to the right (predecessor() to the left). The decision whether to go down into the subtree according to the predetermined path or go to the right at the same level is taken on the basis of the value $T_{\text{max}}$. Thus, if $T$ is a subtree in which, according to the path calculation, the key $x$ should be stored, then the successor() goes down into $T$ only if $x < T_{\text{max}}$, i.e. when the maximal key in $T$ is greater than $x$. If $x \geq T_{\text{max}}$ the successor() method needs to move horizontally to the right in search of the first non-empty subtree. Of course, such a horizontal search might be time consuming. For instance, the linear browsing $T_{\text{root.arr}}$ may take up to $O(\alpha^{1/2})$. In order to shorten the horizontal search, the same mechanism as in the case of the whole structure is used. $T_{\text{summary}}$ is an auxiliary tree that holds informa-
tion about the occupancy of the array $T.arr$ in the same manner as the main tree holds the keys. Thus, traversing $T.summary$ takes at most $O(\ln \ln |T.arr|)$. In the results, the overall asymptotic running time of $successor()$ and $predecessor()$ is $O(\ln \ln \alpha)$.

A good and systematic introduction into the vEB trees theory can be found in [5].

4 Construction of the dcvEB array

4.1 From the vEB Tree to the dcvEB array

One of the reasons why vEB trees are not so popular in practice are space requirements [5]. The need to allocate one continuous block of memory in the root of a structure capable of holding $\alpha^{1/2}$ element array might be inconvenient. The problem can be addressed in different ways [20,5]. One of them implemented in the dcvEB array proposes the use of a fixed number of subtrees per node. It results in a worse theoretical time complexity, however, in many practical applications the achieved speed appears to be quite sufficient. For the same reason, the summary structure is simplified to a bit-vector aligned to the length of a machine-word. The use of high-speed non-blocking bitwise operations on the summary vector allows users to avoid the use of $T.min$ and $T.max$. The logic behind some methods of the dcvEB array is also changed. For example, in the vEB tree, the delete() method performs one single pass from the top to bottom. Due to synchronization issues in the dcvEB array [15] the delete() method proceeds bottom-up. Similarly, successor() and predecessor() first reach the bottom of the tree, then start to traverse the tree moving up and down in search of the appropriate element. The dcvEB array tries to use the non-blocking synchronization mechanisms as often as possible. For example, the get() method uses only the lock-free synchronization mechanisms, which results in its very good performance in the tests (Sec. [9]). The only exception is the mutual synchronization of insert() and delete(). In this case, in order to ensure data consistency [15] p. 373 the readers-writer lock [11] is used.

Despite the fact that the creation of the dcvEB array was inspired by the vEB tree, the differences between these two structures seem to be fundamental. Therefore the dcvEB array should be treated, not as a concurrent extension of the sequential vEB tree but, as the new and original data structure.

4.2 Structure organization

The dcvEB and cvEB arrays can be seen as a tree of arrays [15]. Each array’s cell holds the reference to ArrayHolder (AH) - a tree node structure, which wraps the lower-level array or stores a specific value if AH is a leaf. The leaves are kept at the lower level of the tree. Each array corresponds to an associated summary - a bit vector, in which the $i-th$ bit is enabled only if the appropriate array’s cell holds the lower level $AH$. Besides an array and the associated bit vector, every $AH$ also contains the readers-writers lock object [11]. The leaf $AH$ instead of an array reference holds an element and an integer index value as its key. The value of the key determines the path from the root to the leaf understood as a sequence of positions on the various levels of the tree. The path positions are calculated according to the following recurrent formula: $i_k = i_{k-1} - lp_{k-1} \times n^{h-k-1}$, $lp_k = \lfloor i_k / n^{h-k-1} \rfloor$ where $n$ is the length of a bit vector, $h$ is the height of the tree, $lp_k$ is the path position on the $k-th$ level, $i_0$ is a key of the element. A position at the root level $lp_0$ is defined as $lp_0 = \lfloor i_0 / n^{h-1} \rfloor$. The dcvEB array of the height $h$ can hold elements within the range $[0, \ldots, n^h - 1]$. If there is a need to store an element with a key greater than $n^h - 1$ or by removing the item there are no elements with the keys within the range $[n^{h-1}, \ldots, n^h - 1]$ the tree has to be vertically resized. The concurrent tree resizing algorithms as integral parts of the insert() and remove() procedures are discussed later. If the dcvEB array does not contain a particular element, and its key fits the current key range, the insert procedure recreates the missing $AH$ along the path from the root to the leaf. Similarly, the remove procedure deletes $AH$s if the appropriated summaries are 0.

4.3 dcvEB array methods

The dcvEB array is designed to support all the dynamic set methods as specified in [15] p. 230. Not all of them are extensively discussed in the article, although all of them are implemented1. In particular, the basics of the missing $predecessor()$ method are very similar to $successor()$, which is discussed below, whilst the methods $minimum()$ and $maximum()$ have straightforward implementation using $successor()$ and $predecessor()$. It is assumed that the stored objects are uniquely identified by integer keys. Thus, the key appears in most of the dcvEB array methods as an input parameter, whilst the return value of all the query methods is the pair consisting of the key and the stored element. The presented implementation uses locks as well as lock-free synchronization mechanisms. Hence, wherever an atomic, lock-free element is used, an appropriate object or variable is declared

\footnote{The minimum can be determined by the call $successor(0)$, whilst maximum by the call $predecessor(n^h - 1)$}
as atomic. The main purpose of this section is to allow the reader to understand the general idea behind the presented algorithms and the data structures they use. For this reason some issues connected with synchronization and concurrency are only indicated, and will be discussed later.

The methods presented above use two additional structures: \textit{ArrayHolder} (Listing: 1), \textit{ArrayParam}, and one atomic common variable \textit{ap}, which holds the current \textit{ArrayParam} value. The \textit{ArrayHolder} contains five fields: \textit{array} - atomic array of references to the lower-level \textit{AH}s, \textit{summary} - an atomic bit vector implemented as any integer type available on the current hardware platform, \textit{index} - an atomic key value of the stored object, \textit{data} - an atomic reference to the stored object, and \textit{lock} - reader-writer lock object associated with the given \textit{AH}.

\begin{lstlisting}[language=C]
1 ArrayHolder
2 AtomicRefArray array
3 AtomicInt summary
4 AtomicInt index
5 AtomicRef data
6 RWLock lock
\end{lstlisting}

\textbf{Listing 1:} Array Holder structure

The second structure \textit{ArrayParam} (\textit{AP}) contains the fields: \textit{size} - the number of indices assignable at the moment in the \textit{dcvEB} array (i.e. the maximal object stored in the \textit{dcvEB} array cannot have a key greater than \textit{size} – 1), \textit{height} - the number of levels of a tree implementing the \textit{dcvEB} array structure except the last leaf level, and \textit{root} - a root’s \textit{AH} (Listing: 2).

\begin{lstlisting}[language=C]
7 ArrayParam
8 int size
9 int height
10 ArrayHolder root
\end{lstlisting}

\textbf{Listing 2:} Array Parameters structure

The current value of \textit{ArrayParam} is stored in the common atomic variable \textit{ap}. Except for initialization, the fields of AP are read-only, hence they do not need to be synchronized.

The first presented method discussed in this section is \textit{insert()} (Listing: 3). At the very beginning it locks the common atomic variable \textit{ap} (in order to prevent altering the current \textit{ArrayParam} reference by \textit{remove()}), then it makes the local copy of the current array parameters (Line: 12). Next, it locks the root, (Line: 13), and unlocks \textit{ap} (Line: 13).

Then, it checks whether the key value fits the current array size and, if not, it tries to extend the array (Listing: 3 Lines: 15 - 17). Array growing is implemented by adding successive levels above the current root (Listing: 4). When the new top of the \textit{dcvEB} array tree is ready, the algorithm tries to set it as the new root within the newly created \textit{ArrayParam} record (Listing: 4 Line: 17). Then, irrespectively of the result of the \textit{CAS} invoke, it unlocks \textit{AP}’s root (Listing: 3 Line: 18). If, due to concurrent interference with other threads, \textit{CAS} fails and the common array parameters are not changed, the root locking guarded by the \textit{ap} lock is repeated (Listing: 3 Lines: 20 - 21), and the loop condition is re-evaluated (Listing: 3 Line: 14). If \textit{CAS} succeeds, then \textit{AP} is updated (Line: 22), and the loop is interrupted.

After the size of the array has been adapted to the size of a key, the algorithm traverses the tree structure starting from the current root (Listing: 3 Line: 24) to the leaf. On every step of the loop while (Listing: 3 Lines: 25 - 38) a subsequent level of the tree is visited. The loop starts from calculating the level position \textit{lp}, then, if it is not the top level, \textit{cAH} becomes read locked, and the previous node \textit{pAH} is unlocked (Listing: 3 Line: 28). Next, \textit{pAH} is set to \textit{cAH}, and \textit{cAH} is atomically updated (Listing: 3 Lines: 29 - 31). This update is to set the \textit{n} – \textit{lp} bit in \textit{summary} corresponding to the \textit{lp} cell of the \textit{cAH}’s \textit{array} field. After setting the bit indicating that at \textit{lp} position in \textit{cAH}’s array there is a subtree, iteration moves to the lower level of the tree, i.e. the current value of \textit{cAH}

\begin{lstlisting}[language=C]
2 CAS(a,b,c) - compare and swap atomic action operating under the scheme: if \textit{a} = \textit{b} then \textit{a} ← \textit{c} and return true. Return false otherwise.
\end{lstlisting}
is replaced by the reference to its $lp$ children (Listing: 3, Line: 32).

11 insert(key, data)
12 apLock.rLock(); cAP ← ap;
13 cAP.root.rLock(); apLock.rUnlock();
14 while (key > cAP.size)
15 ArrayParam newAP ← grow(key);
16 newAP.rLock();
17 tmp ← CAS(ap,cAP,newAP);
18 cAP.root.rUnlock();
19 if not tmp then
20 apLock.rLock(); cAP ← ap;
21 cAP.root.rLock(); apLock.rUnlock();
22 else cAP ← newAP; break;
23 end while;
24 cAH ← cAP.root; cl ← 0; pAH ← nil;
25 while (cl < cAP.height)
26 lp ← lvlPos(cl, key);
27 if cl ≠ 0 then
28 cAH.rLock(); pAH.rUnlock();
29 pAH ← cAH; cAH.summary ←
30 $0_n ... 0_{n−lp+1}1_{n−lp}0_{n−lp−1}... 0_1$;
31 ORbit cAH.summary;
32 cAH ← cAH.array[lp];
33 if (cAH = nil) then
34 cAH ← createAH();
35 CAS(pAH.array[lp],nil,cAH)
36 cAH ← pAH.array[lp];
37 cl ← cl+1;
38 end while;
39 cAH.data ← data; cAH.index ← key;
40 pAH.rUnlock();

Listing 3: Insert method

Of course, it is possible that the subtree has not yet been initialized (Listing: 3, Line: 33). In such a case the new $cAH$ is created, atomically assigned to the parent $AH$'s array when possible (Listing: 3, Line: 35), then due to the possible interference with another insert thread (but not remove thread) the final value of $cAH$ is read from the parent $cAH$'s array (Listing: 3, Line: 36). At the end of the loop, the variable determining the current level of iteration is incremented (Listing: 3, Line: 37). The loop ends when $cAP$ is pointing at some leaf $AH$. Hence, at the end of the method both leaf $AH$'s fields: data and index, are updated. In the last line of insert() the leaf’s parent node lock is released (Listing: 3, Line: 40).

An important routine used within the insert() method is grow(). It is responsible for extending the dcvEB array, when it is too small to hold an element with the given key. Enlarging the array relies on adding additional levels above the existing root so that the total height $h$ of the dcvEB array tree increases. Hence, the dcvEB array becomes capacious enough to encompass the key i.e. it requires $n^h > key$.

As a result of this operation, a new AP record is created (Listing: 4, Line: 12). Then, the grow() procedure calculates the appropriate new height and size (Listing: 4, Lines: 13 - 14). The number of levels to create is determined as the difference between the previous height and the new height of the tree (Listing: 4, 46). Then, the procedure starts the loop while (Listing: 4, Lines: 18 - 57), and within every turn of the loop the new $AH$ is generated. The first generated $AH$ becomes a new root of the tree (Listing: 4, Line: 52), each further one becomes the leftmost child of its predecessor (Listing: 4, Line: 54), and finally the last generated $AH$ takes the previous root $AH$ as its leftmost child (Listing: 4, Line: 56). The sequential running time of grow() is $O(\log_n\alpha)$. Since the number of iterations of the loop while (Listing: 3, Lines: 14 - 23) depends on the interferences with the concurrently operating delete threads, whilst the number of iterations of while (Listing: 3, Lines: 25 - 40) is limited by the height $h = \log_n\alpha$ of the tree, then the overall sequential running time of insert() is $O(\log_n\alpha)$.

41 grow(key)
42 nAP ← createAP();
43 nAP.height ← 1\lfloor \log_k key \rfloor
44 nAP.size ← \alpha nAP.height
45 cAP ← ap;
46 topSize ← nAP.height - cAP.height
47 cl ← 0
48 while (cl < topSize)
49 cAH ← createAH();
50 cAH.summary ← 1_{n}0_{n−1}... 0_{1};
51 if cl = 0
52 nAP.root ← cAH
53 else
54 pAH.array[0] ← cAH
55 if cl = topSize - 1
56 cAH.array[0] ← cAP.root
57 pAH ← cAH
58 return nAP

Listing 4: Array growing

The next method get(), similarly to insert(), first retrieves the current snapshot of the dcvEB array parameters (Listing: 5, Line: 60), then traverses the structure down from the root to the leaf following the subsequent level positions. The main difference between get() in the dcvEB array and get() from the previous version of the structure [15] is that currently the enabled bit in a summary does not guarantee the existence of the corresponding lower level array holder. Hence, the additional check whether the next $AH$ is not actually nil is necessary (Listing: 5, Lines: 67 - 68). As can be seen, the sequential running time
of get() is determined by the loop (Listing: 5 Lines: 61-70) and is \(O(\log_n \alpha)\).

59 get(key)
60 cAP ← ap; cAH ← cAP.root; cl ← 0;
61 while (cl < cAP.height)
62 lp ← lvlPos(cl, key)
63 if \(\langle 0_0, 0_{n-1}...0_{n-lp+1}1_{n-lp}0_{n-lp-1}...0_1 \rangle AND_{hit} cAH.summary\) = 0
64 return nil
65 cAH ← cAH.array[lp]
66 if cAH = nil
67 return nil
68 cl++
69 end while
70 return (cAH.data, cAH.index)

Listing 5: Get method

Changes resulting from the introduction of dynamic memory allocation also affected the delete() method. Since insert() is able to expand top of the dcvEB array tree and to generate missing lower level AHs, then delete() needs to be able to trim the top of the tree and to remove redundant nodes. The delete() method implementation can be logically divided into three stages: preparing a path towards a leaf, deleting the leaf with the deletion propagation and cleaning, and the dcvEB array top trimming. Like almost all presented dynamic set methods, delete() also starts from fetching the snapshot of the current dcvEB array parameters (Listing: 6 Lines: 73-74). Then, after the creation of the two empty tables ahol and pos for holding the path between the root and the node for disposal, the method makePath is invoked (Listing: 6 Line: 77). The purpose of this method is to fill these tables with the subsequent AHs and their positions along the way from the root to the leaf node being removed according to the formula for \(lp_k\). During the iteration, similarly to in get(), the presence of the child must be checked twice. Firstly, by checking a summary bit vector (Listing: 7 Lines: 87-88), the second time by checking whether the retrieved subsequent AH is not nil (Listing: 6 Line: 91). It is assumed that the arguments of makePath are in-out, which means that the changes made inside the method are visible outside.

It is noteworthy that makePath may not contain a complete path between the root and the leaf designed to be disposed. This happens when there is no such path i.e. because the desired element has just been removed. In such a case the procedure stops, and leaves the arrays ahol and pos partially filled. In the case of delete(), not entirely filled arrays indicate that there is no element to delete (Listing: 5 Line: 78). Hence the method can finalize its operation (Listing: 6 Line: 79).

72 delete(key)
73 cAP ← ap; cAH ← cAP.root;
74 pAH ← nil; cl ← 0;
75 ahol ← makeEmptyArray(cAP.height)
76 pos ← makeEmptyArray(cAP.height)
77 makePath(key, cAP, cAH, pAH, cl, ahol, pos)
78 if (!is_fielded(ahol, cAP.height)) then
79 return;
80 delIntern(key, cAP, cAH, pAH, cl, ahol, pos);
81 while (cAP != ap and rep < maxRep)
82 deleteClean(key); rep ← rep + 1;
83 topTrim();

Listing 6: Delete method

84 makePath(key, cAP, cAH, pAH, cl, ahol, pos)
85 while (cl < cAP.height)
86 lp ← lvlPos(cl, key);
87 if \(\langle 0_0...0_{n-lp+1}1_{n-lp}0_{n-lp-1}...0_1 \rangle AND_{hit} cAH.summary\) = 0 then break;
88 pos[cl] ← lp; ahol[cl] ← cAH;
90 pAH ← cAH; cAH ← cAH.array[lp];
91 if (cAH = nil) then break;
92 cl ← cl + 1;
93 end while

Listing 7: makePath - delete auxiliary method

The second and the major subroutine of delete() is delIntern(). In terms of the synchronization structure, it is similar to the original delete() method presented in [15]. The need, however, for effective array holder removal caused the necessity to introduce a few new elements into the code of the algorithm. The delIntern() method is executed only if the array holder structure is correctly filled, which takes place only if makePath() (Listing: 7) does not break its while loop. Hence, at the very beginning of delIntern(), it is assumed that the variable pAH of AHs from the last but one (cAP.height-1) level, whilst cAH points at the element from the last level containing pairs (index, data).

Thus, after locking appropriate array holders (Listing: 8 Line: 95), the stored data are overwritten by nil (Listing: 8 Line: 96). Afterwords delIntern() begins its arduous journey towards the root iterating within the loop while (Listing: 8 Lines: 98-101). It starts from the last but one level (Listing: 8 Line: 100). First, it sets an appropriate bit in cAH’s summary to 0 (Listing: 8 Line: 102). Therefore, the data was logically removed from the structure (data field is set to nil, index to −1, and AH is not by the parent’s summary), although an appropriate array holder still exists. Such an array holder will be physically removed only when the whole cAH’s summary is 0 (Listing: 8 Line: 107). Otherwise, if cAH’s summary is not 0, the
previously locked nodes are released and the method exits (Listing: 8 Lines: 103 - 105).

94 dellIntern(key,cAP,cAH,pAH,cl,ahol,pos)
95 pAH.wLock(); cAH.wLock();
96 cAH.data ← nil; cAH.index ← -1;
97 pAH ← cAH;
98 while (cl ≥ 0)
99 cAH ← ahol[cl]; lp ← pos[cl];
100 if (cl = cAP.height - 1)
101 cAH.summary ← 1\(\ldots\)0\(\ldots\)1
102 ANDn\(\ldots\)cAH.summary;
103 if (cAH.summary ≠ 0)
104 pAH.wUnlock(); cAH.wUnlock();
105 return;
106 else
107 cAH.array ← {nil\(\ldots\)nil\(\ldots\)}
108 else
109 cAH.wLock(); pAH.wLock();
110 isSummaryAltered ← false
111 if (pAH.summary = 0)
112 cAH.summary ← 1\(\ldots\)0\(\ldots\)1
113 ANDn\(\ldots\)pAH.summary;
114 if (cAH.summary ≠ 0)
115 cAH.array ← {nil\(\ldots\)nil\(\ldots\)}
116 cAH.wUnlock(); pAH.wUnlock();
117 if (not isSummaryAltered)
118 return;
119 cl ← cl - 1; pAH ← cAH;
120 end while

Listing 8: Delete internal - delete auxiliary method

This, "my brother keeps me alive", lazy strategy aims to reduce the amount of memory allocation performed during the course of the algorithm. If dellIntern() processes the element on the level cAP.height = 2 or higher, then it first locks the current and previous AH’s node, and next alters the cAH summary by removing the bit corresponding to the removed children (Listing: 8 Line: 113). As in the previous case, if cAH summary is 0 then the child node is dereferenced (Listing: 8 Lines: 115 - 116). Then, after unlocking cAH and pAH (Listing: 8 Line: 117) and checking whether it makes sense to propagate a delete action towards the root (Listing: 8 Lines: 118 - 119) the else block ends. At the end of the procedure the variables cl (current level) and pAH (previous array holder) are updated (Listing: 8 Line: 120).

The next subroutine of delete() is deleteClean(). It is called from delete() just after dellIntern() (Listing: 9 Lines: 81 - 82). The main reason for which it is introduced is the danger of not removing all the required AH when the delete action interferes with the insert action. The idea of deleteClean() implementation and further explanations are in Subsection 5.1.

122 topTrim()
123 cAP ← ap;
124 while (cAP.root.summary = 1\(\ldots\)0\(\ldots\)0)
125 if (cAP.height = 1) then break;
126 nAP ← createAP();
127 nAP.height ← cAP.height - 1;
128 nAP.size ← cAP.size - 1;
129 theLonelyChild ← cAP.root.array[0];
130 if (theLonelyChild = nil) then
131 return nil;
132 nAP.root ← theLonelyChild;
133 nAP.root.array[0] ← nil;
134 nAP.root.wLock(); cAP.root.wLock();
135 if (cAP.root.summary = 1\(\ldots\)0\(\ldots\)0)
136 then CAS(ap,cAP,nAP);
137 cAP.root.wUnlock();apLock.wUnlock();
138 cAP ← ap;

Listing 9: TopTrim - delete auxiliary method

The purpose of topTrim() - the last auxiliary method involved in delete() implementation is to cut the top of the dcvEB array tree if it is reduced to the list (Listing: 9). It is possible that, after the internalDelete() call, the root and a few nodes below have only one, the leftmost, child. In such a case, the sequence of such vertices starting from the root needs to be safely removed. The topTrim() reduces the top of the tree iteratively. It removes only one node (root) in every course of the loop while (Listing: 9 Lines: 124 - 138). If the loop while condition is met, i.e. the root has only one child at the leftmost cell in the array, then the new AP candidate is prepared (Listing: 9 Lines: 126 - 128). Next the “lonely” child is retrieved (Listing: 9 Line: 130). If it is not nil (it might be nil due to another delete thread), it is promoted to a new root candidate of the whole dcvEB array tree (Listing: 9 Line: 133). Finally, if the array properties are not changed during the course of the topTrim() routine (i.e. the assertion that the root has only one leftmost child still holds) the newly prepared nAP becomes the main array parameters reference. The topTrim() method does not trim the trees shallower than the ones composed of the root and leaves (Listing: 9 Line: 125). The CAS call (Listing: 9 Line: 136) responsible for the ArrayParam altering is guarded by two locks (Listing: 9 Line: 134). They prevent a situation in which insert() adds the new element into the subtree rooted in the node, which is subject to removal by topTrim(). The sequential running time of delete() depends on the complexity of their subroutines. The first of them makePath() (Listing: 7) comprises one loop while. Due to the loop condition (Listing: 7 Line: 85) it is clear that the sequential running time of makePath() is limited by the height of the tree i.e. \(O(\log_{n} \alpha)\). The methods dellIn-
term(), deleteClean() and topTrim() also need at most to visit all the nodes on a single path between the root and a leaf. Therefore, their sequential running time is \(O(\log_2 n)\). Furthermore, if only one thread is up and running, the loop while (Listing: 6, Line: 51) executes only one. Thus the overall sequential running time of delete() equals the maximum of the running time of all their subcomponents, and is \(O(\log_2 n)\).

The pseudo code of the last method successor() was divided into two parts. The first (Listing: 10, Lines 140) one is responsible for the attempt to reach the leaf \(AH\) holding the data indexed by a key. If such a leaf exists, it will be returned as its own successor. The second part (Listing: 11) contains a loop which consists of two other loops, where the first internal loop is responsible for traversing the \(devEB\) array tree up, whilst the second traverses the tree down. Such a structure of the code in the second part corresponds to the successor()’s searching strategy. In other words, first the method tries to go a little bit higher to check where a successor leaf could be (the first internal loop), then tries to go towards the leaf in order to retrieve the stored data and key (the second internal loop). Of course, sometimes during the gliding down the tree the successor candidate might be removed. In such a case, the second loop must be aborted and the method once again starts to follow up the tree in order to find another potential successor candidate.

Listing 10: Successor method (part 1)

At the very beginning, the successor() sets its own local copy of the array parameters (Listing: 10, Line: 140), then it prepares a pair of holders, \(cAH\) and \(pAH\), used to traverse the structure. Then the \(cl\) variable indicating the visited level and two other variables, \(ahol\) and \(pos\), referring to arrays holding \(AHs\) and their level positions along the path from the root to the visited node, are defined (Listing: 10, Lines 140 - 144). All the newly introduced variables, including \(cAH\), \(pAH\), \(cl\), \(ahol\) and \(pos\) are initiated within the makePath() auxiliary method. If makePath() reaches the leaf level, (the condition \(cl = cAP.height\) is true, see Fig. 1) this means that the element indexed by the key exists. Hence, if only the successor() procedure manages to fetch the stored data, then the appropriate \((data, key)\) pair is returned by the method (Listing: 10, Lines 146 - 151). Of course, makePath() may not reach the leaf level (the condition \(cl = cAP.height\) does not hold) or even if the leaf is reached, its removal might start before the leaf data are extracted (one of the following three conditions is true: \(cAH = nil\), \(data = nil\), \(index = -1\)). In such a case, the method control goes to the while loop (Listing: 11) and the algorithm starts to explore other successor candidates.

Listing 11: Successor method (part 2)

The second part of the successor() method (Listing: 11) is responsible for traversing the structure up and down looking for the next successor candidate. The first inner loop (Listing: 11, Lines 153 - 159) is responsible for traversing the structure up until the node with the non-empty subtree further to the right is found or the root level is achieved, i.e. \(cl = 0\). At the very beginning, it initiates \(cAH\) and \(lp\) using the values stored in \(pos\) and \(ahol\) (Listing: 11, Line: 154). Then, it prepares the \(tmpSum\) vector so that the value \(tmpSum\) is non-zero, only if there are some bits enabled to the right of the \(lp\) position (Listing: 11, Lines 155 - 156). In other words, the \(tmpSum\) is non zero only if there exists some successor...
the nodes of the 
during their execution, the 
then down again (Listing: 11, Lines: 161 - 177). Thus, 
begins the journey up the tree.
i.e. go to the beginning of the outer loop 
up the tree and look for another successor candidate 
starts from checking the bit vector tmpSum. The 0 = tmpSum 
indicates that the successor candidate is removed after 
it has been checked (Listing: 11, Line: 162), hence 
the loop is interrupted (Listing: 11, Line: 163), and 
the control goes back to the beginning of the outer loop while 
(Listing: 11, Line: 166). If the successor is 
still there, the leftmost bit of tmpSum corresponding to 
the position of the first non-empty subtree is calculated 
(Listing: 11, Line: 164). Then, the position in the bit vector bp 
to the level position lp is transformed (Listing: 11, Line: 165), and the variables pos, 
ahl and cAH are appropriately updated (Listing: 11 
Lines: 165 - 166). In particular, the value of cAH 
is set to the subtree reference. Hence, if the only 
element in the subtree was a successor candidate, and 
unfortunately it was removed during the course of 
the second inner loop, then the value of cAH is nil 
(List: 167). In such a case, the algorithm must go 
up the tree and look for another successor candidate 
i.e. go to the beginning of the outer loop while (List-
ing: 11, Line: 168). Otherwise, the tmpSum 
is set to cAH’s bit vector (Line: 169) and the current level cl 
is incremented. The second inner loop ends when the 
current level reaches the leaf level (Listing: 11, Line: 
170). At the leaf level, the algorithm tries to fetch 
the key and data and, if it succeeds, the successor el-
ment is returned. If not, the control goes back to 
the beginning of the outer loop while, and the algorithm 
begins the journey up the tree.

In the sequential case the successor() method first 
traverses the dcvEB array down (Listing: 10, Line: 
145), next a bit up (Listing: 11, Lines: 153 - 159), and 
then down again (Listing: 11, Lines: 161 - 177). Thus, 
during their execution, the successor() method visits 
the nodes of the dcvEB array at most 3lognα times. Thus, 
their sequential running time is O(3 ⋅ lognα) = 
O(lognα).

5 Concurrency and dynamism

5.1 Expanding and trimming

It is widely accepted that the trees in computer 
science are usually drawn with the root at the top, and 
grow downwards instead of upwards. Donald Knuth 
in [14] writes: "There is an overwhelming tendency 
to make hand-drawn charts grow downwards instead 
of upward (...) even the word 'subtree' (as opposed 
to 'supertree') tends to connote a downward relation-
ship". In this context, the solution adopted in the 
dcvEB array might be a little unintuitive. The tree 
of arrays, which in fact is the dcvEB array, on the one 
second preserves the rule of the-root-at-the-top, on the 
other hand, it grows in both directions: downwards and upwards. The dcvEB array’s tree grows down-
wards if the key of the inserted element is smaller 
than its current maximal capacity, and upwards if the 
key of the inserted element exceeds the current 
capacity and the tree needs to be expanded. Adding 
a new root above the current one makes the old tree 
the leftmost subtree of the new root. Hence, the new 
root summary, just after tree expansion, has the left-
most bit enabled. Inserting the new element, which 
was the cause of expansion, enables another bit in 
the root summary. The tree expansion increases the 
size of the dcvEB array (understood as the range of 
the allowable keys) exponentially. Hence, adding k 
levels above the current root increases the size of a 
dcvEB array Cnew = Cold ⋅ nhk times, where n is a 
summary size, Cold - the old capacity and Cnew - the 
new capacity. The dcvEB array tree expansion does 
ot reorganize the old tree. Hence, all the existing el-
ements remain in place. Because the structure of the 
subtree does not change the threads, which started to 
use the subtree before expansion, we do not have to 
worry about the changed size of the tree. The variable 
size of the tree does not affect the method of deter-
moving the position of an element with the given key 
in a tree. Following the changed size of a dcvEB array 
the height of the tree), the algorithm also adapts 
to the new size of the array. Hence, lp k(h) - the level 
position computed for the tree of height h and the 
fixed key ih < nh - 1 equals lp k(h+r) - the level posi-
tion computed after the dcvEB array tree expansion 
of r levels, where the level position lp at the newly 
added levels is 0 i.e. lpo = ... = lpr−1 = 0. Since the 
expansion does not change the structure of the exist-
ing tree, there is no reason to make any additional 
blocking synchronization mechanisms due to the 
expansion itself. The threads that start working before 
expansion also finish their work within the old struc-
ture using the synchronization scheme as is presented 
in [15]. The new threads that start working after the

3 Of course, in Mathematics, a “tree” is just an acyclic 
and connected graph. Thus, it can expand in all direc-
tions. From this perspective, adding a new root over 
the old one is not unusual.
expansion use the new structure, however, within the subtree resulting from the previous tree, they synchronize with the “old” threads using the locks available within the subtree. Hence, the main role of synchronization in the case of the tree expansion is to provide the new threads with the latest consistent information about the dcEB array tree. For this reason, the vital dcEB array information, such as root reference, height, and the associated size, are kept in the atomic global variable ap (ArrayParam structure). That is why every interface method starts from fetching the current array parameters (Listings: 3, 5, 6, 10 Lines: 12, 60, 73, 140). The dcEB array expansion is implemented as part of the insert() implementation (Listing: 2). The auxiliary method grow() first prepares the new top of the dcEB array tree, then the insert() method tries to atomically replace the current ArrayParam set with the new one prepared by grow(). The insert() method managed to set the new top of the tree. The gray triangle represents the tree before expansion.

Most methods do not interfere with insert() as regards extending the tree. This group includes queries such as get(), successor(), predecessor(), min() and max(). If the expansion occurs after they fetch their own AP copy, the result they return will just not take into account the new elements, for which the key is greater than the size of the structure before expansion. Hence, the synchronization with respect to these methods can be limited to the nonblocking operations on atomic variables. Unfortunately, in the case of delete() a dcEB array tree expansion would be easily disrupted by the trimming procedure.

It is possible that, if there were no additional synchronization mechanism a global read-write lock apLock, (Listings: 3, 9). In the absence of this mechanism it is possible that between fetching the new array parameters (Listing: 3, Line: 12) and read-locking the root (Listing: 5, Line: 13) the top of the array would be trimmed. In such a case, insert() would use the root AH that was in fact removed from the tree, thus an insertion would be ineffective. Hence, the aim of the lock combinations (apLock read lock and AH read lock) used in both insert() and topTrim() is to prevent removal of the node when it is examined due to the insertion procedure. Both locks support reader-writer semantics. Hence, many different inserts but only one delete can be handled at the same time.

Calling delete() during the expansion of the tree might also cause another problem that may lead (if not handled) in the long time perspective to performance deterioration. Namely, delete() first goes towards the leaf, then removes it, and next it tries to propagate the delete information to the higher levels of the tree. If the tree is extended after delete() fetches the current AP snapshot (Listing: 6, Line: 73), then the highest reachable node for delete is the root of the tree before expansion. Hence, delIntern() is not able to propagate delete information up to the new root and ends earlier within a subtree. In such a case, delete() may leave the path starting from the new root (and ending in the old root), which is composed of nodes containing false information indicating that their subtrees are non-empty. This issue can be solved in a few different ways. One of them could be to relax the delete operation and use query methods like get() or successor() to clean up the tree. Another, proposed in this article, is to repeat deleteClean() (Listing: 6, Line: 62) - the procedure, which discovers an undeleted path’s residues and removes them when needed. The method deleteClean() is very similar in implementation to delete() itself. However, other than delete(), it does not remove the element, but instead it tries to confirm that there is no element with the given key in the tree. If there is no such element, but there are some AHs in the tree leading to them and only to them, then such nodes are removed and appropriate bits are disabled. The implementation of the functionality of deleteClean() boils down to small changes in the code of the makePath() and delIntern() methods. Hence, taking into account that the Java code of the presented solution is publicly available, the pseudocode of deleteClean() is not thoroughly analyzed in the paper. An optimal number of deleteClean() calls is discussed later, when the progress condition is considered.

As the dcEB array grows upwards, it is trimmed from the top. The trimming routine is located at the end of the delete() method. It checks whether the current root has only one child at position 0, and if so, it attempts to remove the root. Hence, the root can be trimmed if the maximal element stored within the structure is smaller than size/n, where size means the current size of the structure (the largest key that can be inserted into the dcEB array without triggering its extension), and n is the length of the summary bit vector. Trimming is implemented as the inverse of expansion. The only difference is that the insert() method tries to insert all the required nodes at once, whilst delete() trims elements one by one. The root element after trimming is removed from the tree, and its only child is promoted to the new root element. The trimming mechanism is completely neutral for get(). This is partly a merit of Java, where, of course, nodes can be detached from the tree, but they are not destroyed, just as in C++ or C. Hence, as only get() obtains its own reference to the AP structure (Listing: 6, Line: 60), and thus, the reference to the
root, then regardless of whether any visited node is detached from the tree, \textit{get()} may always continue traversing the structure. Similarly, the other query methods are not affected by the trimming. For instance, \textit{successor()}, when it obtains the root snapshot, creates the path snapshot from the root to the lowest existing node on the path leading to the element with the given key (Listing: 10 Line: 145). The path is kept in the two auxiliary arrays \textit{ahol} and \textit{pos}. Thus, if some of the vertices stored in \textit{ahol} are removed from the tree, they are not removed from \textit{ahol}, thus the \textit{successor()} algorithm still has access to them. Hence, if there is a need to traverse the tree upwards, \textit{successor()} is always able to do that. On the other hand, if some of the visited nodes have been actually removed from the tree, this also implies that there are no further successor candidates available, and \textit{successor()} may finish its task earlier. For the same reason, trimming does not affect \textit{delete()} either. Since, during the passage down, \textit{delete()} creates its own \textit{ahol} array, it has no problems with propagation of the deletion status up. Of course, if some of the nodes referenced in \textit{ahol} are removed (trimmed) from the tree, \textit{delete()} might stop status propagation immediately after encountering such a node. This observation, as well as an analogous situation in the case of the successor search, can also be a subject for further optimization.

Two or more different \textit{topTrim()} calls are synchronized with each other by using \textit{apLock} writer lock. On the other hand, for the purpose of mutual \textit{topTrim()} synchronization, a nonblocking mechanism seems to be sufficient. Hence, the \textit{apLock} usage in this place is caused only by the necessity of protecting \textit{insert()} against trimming. The writer locks used here (Listing: 9 Line: 134) provide \textit{topTrim()} exclusive access to the top of the structure, which could be potentially dangerous for the concurrent performance of the structure. However, in practice, the need to trim the top of the \textit{dcvEB} array tree is not frequent, thus with the performance issue in mind, it is better first to check whether the trimming is needed (Listing: 7 Lines: 124 - 125), and if so, lock the top of the structure, and before trimming, check once again (Listing: 9 Line: 135) whether the trimming is actually needed.

5.2 Dynamic adding and removing elements

In addition to extending and trimming the root, the \textit{dcvEB} array tree also adds and removes elements below the root. This is similar to the behaviour known from the classical trees, where inserting or removing data causes the creation or deletion of appropriate nodes. In the presented solution, it is assumed (due to the desire to avoid separate handling of the uninitialized root case) that there is always at least one node within the tree. This means that the root node exists, even if there are no other elements in the tree. The \textit{insert()} method creates the new element when it turns out that there is no \textit{AH} at the specified position (Listing: 3 Line: 34). Adding a child \textit{AH} into the parent array may interfere with another insert activity. Hence, to avoid overwriting one \textit{AH} by another \textit{AH}, the parent’s array update is implemented as a \textit{CAS} instruction (Listing: 3 35). Then, if the first thread wins and successfully updates the parent \textit{AH}’s array, the second thread just retrieves the winning \textit{AH} (Listing: 3 Line: 36) and continues the insertion procedure with them. It is worth noting that the whole operation is protected in the same way as the summary bit-vector update, i.e. by the \textit{AH}’s reader lock (Listing: 3 Line: 25). Thus, there is no risk that the newly updated \textit{AH} would be removed by \textit{delete()} (after execution Line: 35), and \textit{insert()} could fetch a \textit{nil} value (Listing: 3 Line: 36). A more detailed analysis of the synchronization scheme used here can be found in \[15\].

Since operations related to the memory allocation and deallocation involve operating system function call, they are usually time-consuming. Therefore, for the purpose of this algorithm a kind of lazy approach has been adopted. Therefore, the \textit{AH}’s are not removed from the tree immediately after they become empty (leaves are considered empty when the fields’ data and index are set to \textit{nil}, whilst other nodes are empty if their summary bit-vector equals 0). Instead, if some non-leaf node becomes empty, it actually removes all its children \textit{AH}’s from the memory. Therefore, the operation of deleting elements can be considered as composed of two phases: the first one - logical removing - when data and index are set to \textit{nil} or appropriate position in the parent’s summary bit-vector is zeroed, and the second one - physical removing - when the \textit{AH}’s reference is physically removed from the parent’s array and the \textit{AH} record is actually removed from the memory. Since the \textit{AH} record is physically removed (Listing: 8 Lines: 107 - 116) only when it has no siblings, the physical removing is likely to occur less frequently than the logical removing. Both logical and physical removing use the same synchronization scheme as presented in \[15\]. Thus, the modified elements are always exclusively held by the thread performing deletion.
5.3 Successor search strategy

The working scheme of the successor() method is composed of three phases. During the first one (Listing: [7]), the algorithm goes as far as possible towards the element indexed by the given key. If the element exists, it returns them. If not, the control goes to the second phase (Listing: [11] Lines: [153 - 159]) in which the algorithm retracts until the next nonempty subtree is found. Then, during the third phase (Listing: [11] Lines: [161 - 171]), the algorithm goes down towards the element which is minimal within the detected nonempty subtree. If the third phase fails (the desired element can be removed in the meantime), then the control goes to the second phase and the algorithm starts to go upwards. The second and third phases are repeated as many times as needed. The main difference in comparison with [15] is that, in the case of the necessity to repeat the second and the third phase, the second phase starts exactly from the same point where the third phase has been stopped. Thus, with each search failure the successor() method tries to look further for the next possible successor candidate. Therefore, the number of repetitions of phase two and three is naturally limited by the size of the dcvEB array.

Thanks to the adopted strategy, the successor element will always be found if it remains in a dcvEB array long enough. Thus, let key be the index of element x whose successor we are looking for, and let succ be the index of y - some successor of x, such that key < succ. In such a case, if during processing the second part of the successor() method (Listing: [11]) y is not removed from the dcvEB array, then the maximal possible key of the next successor of x is succ. In other words, when the search algorithm detects the subtree (phase 2) containing y as the minimal element, then the algorithm seamlessly (i.e. without failures) reaches y. The current algorithm always returns the successor if it is available during the whole course of the successor() method. In the previous implementation [10] there was a small chance that the successor would not be found and successor() would return nil. On the other hand, the previous implementation limits the number of failures that can be safely handled by the successor(), whilst currently, the allowable number of failures is limited only by the current size of the structure. This raises the question of the actual concurrent running time of the current successor() implementation. As will be shown in the next section (Sec. 5.4), the concurrent running time estimation is worse than in the sequential case. Fortunately, the tests carried out indicate rather high overall efficiency of the structure rather than its susceptibility to the interferences and thereby performance deterioration.

5.4 Successor search concurrent running time

The sequential running time of all the methods presented in the article is the same as in [15], and equals $O(\log_2 n)$, where n is the size of the summary bit vector, whilst $\alpha$ is the current size of the dcvEB array. The detailed arguments presented previously, with only minor amendments, also fit the dcvEB array. The concurrent running time estimation is much more complex, because, besides the code structure, different kinds of concurrent interactions, such as blocking synchronization, need to be taken into account. Fortunately, some of the dcvEB array methods discussed in this article use only non-blocking synchronization mechanisms. These methods are: get() and successor(). Since the get() method only checks the presence of one, well-defined element in the array, if the check fails (no matter when the element has been removed), it returns nil. Hence, its concurrent running time estimation is not affected by the interferences with other concurrently executed methods. Thus, the concurrent running time of get() does not change and is $O(\log_2 n)$.

In contrast, in the case of successor(), despite the non-blocking synchronization, every deletion of a successor candidate may increase the overall concurrent running time of the method. In the worst case scenario, the value of the successor() method is called for input argument 0, and the next greater element has the key 1. Then, if the element indexed by 1 is deleted just after successor() reaches the end of the loop (Listing: [11] Line: 159), then the deletion has been discovered (Listing: [11] Line: 173) and the next iteration will be initiated (Listing: [11] Line: 176). This may lead to alteration of the subsequent removal and search attempts. The worst case scenario described above may occur $\alpha - 1$ times, forcing the successor() method to check every single position in the dcvEB array. Hence, in the worst case, the concurrent running time of successor() is $O(\alpha)$. A natural question arises whether this slightly disappointing result can be improved. The easiest solution (but without any guarantee that the successor will be found even if it exists in the tree) is to limit the possible number of iterations of phases 2 and 3 (the loop while Listing: [11] Lines: [152 - 177]) by some empirically chosen constant. In such a case, the user must accept that (probably) very rarely the successor() method would
fail and never return the correct value. Despite the moderately good theoretical concurrent running time estimation, the current solution performs very well in practice. Conducted tests for the random data (Section 6) seem to suggest that the successor() average running time is closer to $O(\log_n \alpha)$ than $O(\alpha)$.

There are also some theoretical arguments that may indicate in favour of $O(\log_n \alpha)$. For simplicity, let us assume that the dcvEB array is full (without nil values) and every deleted element is reinserted into the table in a short time after removal. Hence, without making a big mistake, it can be assumed that the successor of $s_k$ is $s_{k+1}$ for $k = 1, \ldots, m$. Thus, the collision may happen, if at roughly the same time, successor() processes $s_k$ and delete() removes $s_{k+1}$. Assuming that both: successor() and delete() process $m$ randomly selected indices: $(s_1, \ldots, s_m)$ and $(d_1, \ldots, d_m)$ at the same and equal time intervals, the likelihood of the simultaneous execution of delete($d_k$) and successor($s_k$) where $d_k = s_k + 1$ is $1/\alpha$. Hence, the execution of $m$ consecutive successor() and delete() calls may result in $m/\alpha$ collisions. Since every collision entails additional tree traversal by successor(), the total expected concurrent running time of $m$ successor() calls is $T(\alpha, m) = mO(\log_n \alpha) + (m/\alpha)O(\log_n \alpha)$. Thus, the amortized concurrent running time of a single successor() call is $T(\alpha, m)/m = (1 + 1/\alpha)O(\log_n \alpha)$. In most cases $\alpha$ is large, for that it is safe to assume that $T(\alpha, m)/m \approx O(\log_n \alpha)$. Similarly, the amortized concurrent running time of a single successor call for $h$ delete threads running in parallel is $T(\alpha, m)/m = (1 + h/\alpha)O(\log_n \alpha)$. Thus, as long as $h$ is significantly smaller than $\alpha$ it still holds that $T(\alpha, m)/m \approx O(\log_n \alpha)$. At the expense of increasing the complexity, the above reasoning can be adapted to the successor() algorithm as presented in the article.

5.5 Correctness

As with other concurrent data structures, the dcvEB array should be considered while bearing in mind the concurrent objects specificity. In particular,

In fact, this approach provides a "limited warranty" to find a successor, i.e. if the iteration number is limited by e.g. 1000 and the successor() method is called with $\alpha$ on its input, then (even in the worst case scenario) there is a guarantee that if the successor exists during the successor() call, then it will be found if only $\beta - \alpha \leq 1000$, where $\beta$ is the key of successor.

It is enough to assume that at the certain moment of time $d_k$ is fixed, whilst the index $s_k$ is selected as one out of $1, \ldots, N$.
Long maxRep to an array indexed by sufficient height is then assuming the length of the bit-vector as key. For instance, if the array key is delIntern() the that the how many times in a row there may be a situation tent. Hence, the question about the optimal value of fields summary and the requested element, information contained in all the nodes on the path from the (new) root and It starts from the new root and checks whether for

3, Line: 13), hence the method delete() is not able to trim the dcvEB array in the meantime. Secondly, let us note that in every turn of the loop cAP.height increases by at least one. Although within the current loop insert() tries to increase the array as far as necessary, it is possible that it loses the race with another thread, which increases the array height only by one.

Another important loop with an unobvious end-condition is connected with the cleaning after deletion (Listing: 3 Lines: 81 - 82). Although the number of possible iterations is limited by the maxRep constant, the question arises as to how many iterations are sufficient, i.e. how large the maxRep constant should be.

To answer this question, let us recall that the loop is introduced to prevent incomplete deletion, which may happen if, during the element removal, the tree is expanded up by the insert() method. In such a case, since delete() knows only the root before expansion, then it is able to propagate the delete information only up to this old root. Hence, if some nodes placed above the old root also need to be removed, the solution proposes a re-run of the deleteClean() method. It starts from the new root and checks whether for all the nodes on the path from the (new) root and the requested element, information contained in AH’s fields summary and AH’s array are mutually consistent. Hence, the question about the optimal value of maxRep (Listing: 3 Line: 81) is the question about how many times in a row there may be a situation that the dcvEB array tree will be expanding up when the delIntern() or deleteClean() operations are ongoing. The answer depends on the domain of the array’s key. For instance, if the array key is Java’s Integer then assuming the length of the bit-vector as 64, the height of the dcvEB array tree that could hold an object with the key equal to the maximal representable integer is $6 = \lfloor \log_{64}(2^{31} - 1) \rfloor$. For Java’s Long, the sufficient height is $11 = \lfloor \log_{64}(2^{63} - 1) \rfloor$. Hence, for an array indexed by Java’s Integer, it is enough to set maxRep to 6, or to 11 if an array is indexed by Java’s Long. Of course, the exact value of maxRep depends on the specific data types and hardware platform and needs to be re-calculated for every specific implementation.

There is also one “hidden” loop, mentioned in [15], which is not explicitly shown in pseudo-code, although it is important for practical implementation. This loop is connected with atomic update of the summary bit-vector within the insert() method (Listing: 3 Line: 31). Since, in most of the program-

6 Experimental results

The experimental implementation of the test application together with the dcvEB array was written in Java 7 and has been tested on an isolated test station Intel® Core™ i7-3930K (6 cores, 12 threads, 3.8 GHz) processor with 8 GB of operating memory. As in the case of the cveEB array, the results achieved by its successor, the dcvEB array, are very promising. In many cases, the new structure turns out to be faster than the compared alternatives. Of course, the presented results are indicative and do not pretend to be a ranking or review.

Since one of the advantages of the dcvEB array is to support all the dynamic set’s methods mentioned in [5 p. 230], one of the major challenges was to find an appropriate concurrent dynamic set implementation, which in addition to the standard get(), delete() and insert() also provides successor() and predecessor(). An experimental implementation of the dcvEB array has been written in Java. Since sometimes even small
implementation details may affect the overall application performance, it was equally important to find such dynamic set solutions written in Java that have been identified by the authors as reference solutions.

Besides the dcevEB array, SnapTree Map \([8]\), ConcurrentSkipListMap \([11]\), non-blocking k-ary search tree \([9]\) and the synchronized java.util.TreeMap were selected for the tests. The structure proposed by Bronson et al. was selected for testing because of the publicly available Java implementation provided by the authors and the presence of support for successor() and predecessor() (via Java’s ConcurrentNavigableMap interface). The second structure, ConcurrentSkipListMap, as it is implemented in recent Java distributions was a natural candidate for comparison. In \([10]\) Herlihy et al. wrote about ConcurrentSkipListMap that “(...) written by Doug Lea based on work by Fraser and Harris \([12]\) and released as part of the JavaTM SE 6 platform, is the most effective concurrent SkipList implementation that we are aware of”\(^8\). The structure also implements successor() and predecessor(). Moreover, its inclusion into the standard Java 6 platform indicates its optimality.

The non-blocking k-ary search tree \((\text{LockFree9ST class})\) \([3]\) is a new efficient tree structure modeled on \([6]\) with the publicly available Java implementation. It does not support the successor() and predecessor() operations. Thus, for the test purposes, the successor operation has been implemented as the iterative get() method calls for subsequent indices. The iteration ends when get() returns a non-null element or the iteration reaches the largest possible index that can be stored in the tree. In general, such an approach is not effective, especially when the indices are distributed sparsely. Fortunately, during the tests the indices are distributed fairly densely, thus, one may expect that the performance deterioration resulting from using this “naive” successor search strategy is not too high.

In contrast to the ConcurrentSkipListMap structure, TreeMap is not even a concurrent object implementation. It is a globally synchronized sequential structure, which is included into the tests in order to show the difference between concurrent and sequential approaches.

In testing, there are four groups of threads: getters(), inserters(), removers(), and successors(). Each getter thread repeatedly calls the method get(), inserter thread - insert() etc. Every thread has to perform the same \(z\) number of calls. The input parameter \(\text{key}\) is randomly chosen from \(K \subseteq [0, \ldots, x-1]\), where \(x\) is a test run parameter. The number of threads in groups is denoted by the letters \(g, i, r\) and \(s\). For instance, \(g = 3, s = 5\) means that in the given test run three getters and five successor searchers are involved. Since the number of operations per thread is fixed, the method performance is measured as an execution time according to the principle, the lower the value, the better the result. The execution time is measured per thread, thus the time of the whole test run is the arithmetic mean of the execution time of each thread involved. In other words, the thread execution time is the time needed by the thread to make \(z\) calls of the given method.

As before, the first test (Fig. 2) measures how the overall thread execution time grows, when the number of threads in each group is the same and increases. In the first test run there are four threads \(g = i = r = s = 1\), in the second one eight threads \(g = i = r = s = 2\) and, finally, in the last one 100 threads \(g = i = r = s = 25\). Every thread performs \(z = 500000\) method calls. The results are averaged, so that the bad test result of one method can be compensated by the good result of another method.

![Fig. 2](image.jpg)

The best result has been achieved by the dcevEB array. Its averaged thread execution time gets 1104 ms. and is approximately 1.8 times better than the second result belonging to LockFree9ST. Both TreeMap and SnapTree are far behind the two previous structures. Such a structure of results demonstrates that the improvements made to the original cvEB array do not affect its overall performance, and, on the contrary, they seem to be even better than in \([12]\).

In addition to the general performance comparison of the structures, it is interesting to compare the performance of the particular methods. As in \([15]\), the results were averaged, so that the particular number in milliseconds denotes the average time required by a getter, an inserter, a deleter and a successor

\[\text{https://github.com/nbronson/snaptree}\]

\[\text{www.cs.utoronto.ca/~labrown/ksts/}\]

\(^9\) Due to the symmetrical similarity with successor(), the predecessor() method has not been subject to testing.
The obtained results (Fig. 3) show that the performance of each method varies. The very fast get() method and quite fast delete() and successor() are contrasted with the average insert(). The relationships between the results are similar to those of [15], except insert(), which seems to be slower than before. The need for memory allocation, as expected, results in performance degradation. Fortunately, however, despite the decrease in performance, the insert() method still has a comparable speed to the reference ConcurrentSkipListMap. Noteworthy is the very good result of insert() achieved by LockFree9ST. Other than for the successor() method, the dcvEB array turns out to be the fastest.

The dcvEB array concurrent processing efficiency comes from the local nature of the synchronization mechanisms (lock objects are spread over all the AH). However, it can be assumed that this “locality” may not be beneficial when the requested indices are close to each other. Hence, it is natural to test how the structure can handle situations when the key set is small. For the test purposes it was adopted that \( K = [0 \ldots m] \), where \( m \) increases starting from 10 and ending at 776000.

As before, for the narrow key ranges ConcurrentSkipListMap runs similarly fast to the dcvEB array. The result of LockFree9ST does not differ significantly from those two structures. For the larger values of \( m \) the dominance of the dcvEB array becomes apparent. The slow increase in execution time is caused mainly by the insert() and successor() methods. With the increasing key range, insert() needs to allocate more and more memory and the distances scanned by successor() get bigger. A very good result is also achieved by LockFree9ST. It is almost two times better than ConcurrentSkipListMap, although it is worse than the dcvEB array. As in the previous test, the results achieved by the other two structures are worse than the results of the two ranking leaders.

The results achieved, although encouraging, have to be interpreted carefully. In particular, it is difficult to prejudge the performance of the dcvEB array relative to LockFree9ST, ConcurrentSkipListMap and SnapTree. The last two of these structures have a very powerful interface providing users with much more functionality than the dcvEB array. Moreover, ConcurrentSkipListMap and LockFree9ST are not optimized to integer keys, hence, in return, they can handle any object as the key. LockFree9ST does not support natively the successor() and predecessor() operations. Hence, its actual results can be better than those observed during the tests. Finally, SnapTree adopts an optimistic locking strategy, which in some cases can cause performance degradation. Despite the better results achieved by the dcvEB array than by the cvEB array [15] and the similar testing procedure, the performance of both structures cannot be directly compared. The presented tests were carried out on another (faster) machine than the previous tests, the code of the testing procedure has been improved, and another (new) version of Java has been used. It should also be noted that the dcvEB array used in the experiment is “flatter” than the cvEB array used in [15], since every non-leaf node in the dcvEB array has 64 children, whilst in the cvEB array used in [15] it was less than half of that.

7 Comments and discussion

The presented dcvEB array is the result of improvements introduced into its previous version [15].
contrast to the cvEB array, the presented structure is able to grow when more data need to be stored in it, and to shrink when the data are removed. Hence, the amount of occupied memory (RAM) is in relation to the amount of stored data. Of course, this relationship is not as simple as in the case of an ordinary binary search tree or a heap tree [5]. In those structures, the amount of memory occupied depends more or less linearly on the amount of stored data. In the case of the dcvEB array, it also depends on the data layout. More precisely, if the smallest key is 0 and the largest key is \( n^h - 1 \) (where \( n \) is the length of the AHs’ summary bit vector), then the dcvEB array can have up to \( 1 + n + \ldots + n^{h-1} = \frac{n^h - 1}{n-1} \) non-leaf AHs, and up to \( n^h - 1 \) leaf AHs. Whilst the number of leaf AHs depends only on the amount of data actually stored in the dcvEB array, the number of non-leaf AHs also depends on data distribution and the value of the maximal key. If the stored keys are scattered throughout the array, so that at all 64 positions (assuming that \( n = 64 \)) at least one element can be found, then the memory consumption by the dcvEB array is relatively high. In fact, in the worst case scenario even 64 times more items could be stored in the dcvEB array without increasing the number of non-leaf AHs. Reversely, if the data is indexed by the keys arranged one after the other, memory utilization is optimal and any increase in the amount of stored data by at least 64 keys, triggers the creation of at least one non-leaf AH.

Scattered keys, although undesirable due to the use of extra RAM, may be beneficial to the methods using blocking synchronization, such as delete() or insert(). The keys’ dispersion increases the chance that the threads operating on the dcvEB array less frequently block each other, which results in an increase in the speed of the structure.

8 Summary

In this paper, the author proposes the dcvEB array - the new concurrent implementation of the dynamic set structure based on the synchronization scheme proposed in [15]. The preliminary tests conducted using a prototype Java-based implementation of the structure seem to confirm that it maintains the high performance of the concurrent operations. The introduced extensions are of great practical importance. Thanks to them, the structure is able to dynamically adapt its size to the amount of stored data and increase or decrease the range of indices stored. There are also some smaller improvements, such as a new successor search strategy.

The variable amount of memory used, and no limit on the size of the key in combination with the high performance of concurrent applications, can make this structure useful for a wide range of professionals involved in concurrent or parallel programming.

Acknowledgment

The author would like to thank Prof. Jacek Kitowski and Piotr Matyasik, Ph.D. (AGH) for providing the test environments, and Paweł Salata, for helping the author in launching multiple variants of tests.

References

1. R. Bayer, “Symmetric binary B-Trees: Data structure and maintenance algorithms,” Acta Informatica, 1972.
2. R. Bayer and M. Schkolnick, “Concurrency of Operations on B-Trees,” Acta Informatica, 1977.
3. N. G. Bronson, J. Casper, H. Chaﬁ, and K. Ohkotum, “A practical concurrent binary search tree,” in Proceedings of 15th ACM SIGPLAN, 2010.
4. T. Brown and J. Helga, “Non-blocking K-ary Search Trees,” in Proceedings of the 15th International Conference on Principles of Distributed Systems, 2011.
5. T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, Introduction to Algorithms, 3rd ed. MIT Press, 2009.
6. F. Ellen, P. Fatourou, E. Ruppert, and F. van Breugel, “Non-blocking binary search trees,” in Proceeding of the 29th ACM SIGACT-SIGOPS symposium, 2010.
7. C. S. Ellis, “Concurrent Search and Insertion in 2-3 Trees,” Acta Informatica, vol. 14, no. 1, pp. 63–86, 1980.
8. K. Fraser, “Practical lock-freedom,” University of Cambridge, Tech. Rep. UCAM-CL-TR-579, 2004.
9. S. Hanke, T. Ottmann, and E. Soisalon-Soininen, “Relaxed balanced red-black trees,” Algorithms and Complexity, 1997.
10. M. Herlihy, L. Lev, V. Luchangco, and N. Shavit, “A simple optimistic skiplist algorithm,” in SIROCCO’07: Proceedings of the 14th international conference on Structural information and communication complexity. Springer-Verlag, Jun. 2007.
11. M. Herlihy and N. Shavit, “The Art of Multiprocessor Programming”. Elsevier, 2008.
12. P. M. Herlihy and J. M. Wing, “Linearizability: a correctness condition for concurrent objects,” ACM Transactions on Programming Languages and Systems, vol. 12, 1990.
13. S. V. Howley and J. Jones, “A non-blocking internal binary search tree,” in Proceedings of the 24th ACM symposium on Parallelism in algorithms and architectures, 2012.
14. D. E. Knuth, *The Art of Computer Programming, Volume I: Fundamental Algorithms, 2nd Edition*. Addison-Wesley, 1973.

15. K. Kułakowski, “A concurrent van Emde Boas array as a fast and simple concurrent dynamic set alternative,” *Concurrency and Computation: Practice and Experience*, vol. 26, no. 2, pp. 360–379, 2014. [Online]. Available: http://dx.doi.org/10.1002/cpe.2995

16. H. T. Kung and P. L. Lehman, “Concurrent manipulation of binary search trees,” *Trans. on Dat. Systems*, 1980.

17. V. Lanin and D. Shasha, “A symmetric concurrent b-tree algorithm,” in *Proceedings of 1986 ACM Fall joint computer conference*, 1986.

18. D. Lea, *Concurrent Programming in Java, Second Edition: Design Principles and Patterns*, 2nd ed. Boston, MA, USA: Addison-Wesley Longman Publishing Co., Inc., 1999.

19. P. L. Lehman and S. Bing Yao, “Efficient locking for concurrent operations on B-trees,” *Trans. on Dat. Systems*, 1981.

20. K. Mehlhorn and S. Näher, “Bounded ordered dictionaries in o(loglogn) time and o(n) space,” *Inf. Process. Lett.*, vol. 35, no. 4, pp. 183–189, Aug. 1990.

21. O. Nurmi, E. Soisalon-Soininen, and D. Wood, “Relaxed AVL Trees, Main-Memory Databases, and Concurrency,” HKUST, Tech. Rep., 1996.

22. W. Pugh, “Concurrent maintenance of skip lists,” College Park, MD, USA, Tech. Rep., 1990.

23. B. Samadi, “B-trees in a system with multiple users,” *Information Processing Letters*, pp. 107–112, 1976.

24. N. Shavit, “Data structures in the multicore age,” *Communication of the ACM*, 2011.

25. P. van Emde Boas, “Preserving order in a forest in less than logarithmic time and linear space,” *Inf. Process. Lett.*, vol. 6, no. 3, pp. 80–82, 1977.