COMBINATORICS OF MULTIVARIATE CHROMATIC POLYNOMIALS FOR ROOTED GRAPHS

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Abstract. Richard Stanley defined the chromatic symmetric function $X_G$ of a graph $G$ and conjectured that trees $T$ and $U$ are isomorphic if and only if $X_T = X_U$. We study a variation of the chromatic symmetric function for rooted graphs, where we require the root vertex to have a specified color. In addition to presenting various combinatorial identities and recursions satisfied by these rooted chromatic polynomials, this work contains three main results. The first is that our polynomials satisfy the analogue of Stanley’s conjecture: two rooted trees are isomorphic as rooted graphs if and only if their rooted chromatic polynomials are equal. We give two proofs of this theorem, one based on unique factorization, the other based on algebraic transformations to pointed chromatic functions and rooted $U$-polynomials. The second result, which is used in the factorization-based proof, is that for each tree $T$, $X_T(x_1, \ldots, x_N)$ is irreducible in $\mathbb{Q}[x_1, \ldots, x_N]$ for large enough $N$. We prove this by a novel application of Eisenstein’s Criterion. The third result answers a question of Pawlowski by providing a combinatorial interpretation of the monomial expansion of pointed chromatic functions.

1. Introduction

1.1. Stanley’s Chromatic Symmetric Functions. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A proper coloring of $G$ is a coloring of the vertices in $V(G)$ such that adjacent vertices receive different colors. The classical chromatic polynomial $\chi_G$ counts proper colorings: $\chi_G(k)$ is the number of proper colorings of $G$ when there are $k$ available colors. In 1995, Richard Stanley [33] defined the chromatic symmetric function $X_G$, which is a multivariable generalization of $\chi_G$. For fixed $N \geq |V(G)|$, $X_G(x_1, x_2, \ldots, x_N)$ is a weighted sum of proper colorings of $G$ using colors in $\{1, 2, \ldots, N\}$. A proper coloring that colors $e_i$ vertices with color $i$ (for $1 \leq i \leq N$) contributes the monomial $\prod_{i=1}^n x_i^{e_i}$ to $X_G(x_1, \ldots, x_N)$. Each $X_G$ is a symmetric polynomial in $N$ variables. Taking a formal limit as $N$ goes to infinity produces the chromatic symmetric function $X_G$ using variables $(x_n : n > 0)$. We recover $\chi_G(k)$ from $X_G$ by setting $x_i = 1$ for $1 \leq i \leq k$ and $x_i = 0$ for all $i > k$.

The chromatic symmetric functions have been a rich source of combinatorial insights and problems. The seminal paper [33, pg. 170] contains an open problem that we call Stanley’s Chromatic Conjecture for Trees:

Conjecture 1. Trees $T$ and $U$ are isomorphic (as graphs) if and only if $X_T = X_U$.

This conjecture remains unsettled, although progress has been made in several directions. Using computer investigations, Heil and Ji [22] verified that the conjecture is true for all trees with at most 29 vertices. The conjecture also holds for certain restricted classes of trees: spiders [26], 2-spiders [23], caterpillars [5, 21, 27], trivially perfect graphs [34], and trees with diameter at most

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1.2. **Irreducibility Questions.** Suppose a graph $G$ has connected components $C_1, C_2, \ldots, C_k$. Then $X_G = X_{C_1}X_{C_2}\cdots X_{C_k}$, since each proper coloring of $G$ arises by independently choosing a proper coloring for each component. This means that the chromatic symmetric polynomial for a disconnected graph has a nontrivial factorization into smaller polynomials. Conversely, one might ask if $X_G$ is irreducible when $G$ is a connected graph or a tree. As we see later, irreducibility is a key technical property used (here and in earlier literature) to prove analogues of Stanley’s conjecture for rooted trees and related structures.

To formulate precise questions about irreducibility, we must specify the ring in which the factorization occurs. The polynomial $X_G(x_1, \ldots, x_N)$ belongs to the polynomial ring $\mathbb{Q}[x_1, \ldots, x_N]$ and also to the subring $\Lambda_N$ of symmetric polynomials in $N$ variables. Both rings are unique factorization domains (UFDs); in fact, we can view $\Lambda_N$ as a polynomial ring $\mathbb{Q}[p_1, \ldots, p_N]$ in the algebraically independent power-sum polynomials $p_1, \ldots, p_N$.

Using work of Cho and van Willigenburg [10], Tsujie [34] proved that for any tree $T$ and $N \geq |V(T)|$, $X_T(x_1, \ldots, x_N)$ is irreducible in the ring $\Lambda_N$. In Theorem 7, we prove that for any tree $T$, $X_T(x_1, \ldots, x_N)$ is irreducible in $\mathbb{Q}[x_1, \ldots, x_N]$ for large enough $N$. This is a stronger statement for such $N$, since there are more potential irreducible factors in the full polynomial ring compared to its subring $\Lambda_N$. The core of our proof (Lemma 13) invokes Eisenstein’s Criterion to show the specialization $X_T(q, q, 1, \ldots, 1)$ is irreducible in $\mathbb{Q}[q]$ for suitable $N$. The proof succeeds because of a surprising affinity between the hypotheses of Eisenstein’s Criterion and the enumerative properties of proper colorings of trees.

1.3. **Variations of Chromatic Symmetric Polynomials.** Stanley’s chromatic symmetric function is just one member of an entire ecosystem of polynomials and algebraic constructs based on graph colorings. Some important variations of $X_G$ include the noncommutative chromatic symmetric functions of Gebhard and Sagan [19], the chromatic quasisymmetric functions of Shareshian and Wachs [31], the strict order quasisymmetric functions of Hasebe and Tsujie [21], the pointed chromatic functions of Pawlowski [30], and the rooted $U$-polynomials of Aliste-Prieto, de Mier, and Zamora [4], among many others (see Section 5 for a fuller discussion).

In this paper, we start with a rooted graph $G_*$ and study some multivariable chromatic polynomials $X_i(G_*)$ and $X_{\neq i}(G_*)$ that are weighted sums of certain proper colorings of $G_*$. For example, $X_0(G_*; x_0, x_1, \ldots, x_N)$ is the weighted sum of proper colorings of $G_*$ using available colors $\{0, 1, \ldots, N\}$ such that the root vertex of $G$ must receive color 0. $X_0(G_*)$ is a polynomial in $x_0$ where the coefficient of each $x_0^k$ is a symmetric polynomial in $x_1, \ldots, x_N$. We prove various combinatorial and algebraic properties of these polynomials, including: a recursion for rooted trees (Proposition 6), a deletion-contraction recursion for rooted graphs (Theorem 17), combinatorial interpretations of the coefficients of $x_0^k$ in $X_0(G_*)$ and $X_{\neq 0}(G_*)$ (Proposition 20), and a formula for the power-sum expansion of $X_0(G_*)$ (Proposition 27). Using the power-sum formula, we show how to obtain $X_0(G_*)$ as a certain algebraic transformation of pointed chromatic functions or the rooted $U$-polynomials (Sections 5.1 and 5.2).

A key fact (Theorem 15) is that the analogue of Stanley’s Conjecture holds for the rooted chromatic polynomials: two rooted trees $T_*$ and $U_*$ are isomorphic (as rooted graphs) if and only if $X_0(T_*) = X_0(U_*)$. We give two proofs of this theorem. The first proof uses our main irreducibility result (Theorem 7) to relate recursive decompositions of rooted trees to unique factorizations of polynomials into irreducible factors. This proof strategy has been used before by various authors; see [4, Theorem 9], [20, Theorem 2.8], and [21, Theorem 1.3]. The second proof, given in Section 5.2, uses an algebraic transformation to reduce Theorem 15 to an analogous theorem for rooted $U$-polynomials (see [4, Theorem 9]).
Previous authors have found other polynomial invariants characterizing isomorphism classes of rooted trees. In addition to the rooted \( U \)-polynomials already mentioned, each of the following is a complete invariant for rooted trees: the two-variable Chaudhury-Gordon polynomials \cite{8}, the polychromates of Bollabás and Riordan \cite{7}, the greedoid polynomials of Gordon and McMahon \cite{20}, and the strict order quasisymmetric functions of Hasebe and Tsujie \cite{21}. Related results appear in \cite{16, 19, 31}.

**Remark 2.** The polynomials \( Z^*_G(c) \) defined by Heil and Ji \cite[Def. 3.1]{22} are essentially the same as our polynomials \( X_r(G^*_r) \). However, given Heil and Ji’s algorithmic focus, there is no overlap with this paper beyond the decomposition of \cite{14}. We note for completeness that Lemma 3 and the following corollaries in \cite{22} must be modified slightly to account for the fact that the \( F_i \) are symmetric functions in all the variables except \( x_i \).

Given the many prior variants of Stanley’s chromatic symmetric functions, we should highlight some particular benefits of the polynomials \( X_0(G_*) \) studied here. First, \( X_0(G_*) \) is combinatorially very close to the original polynomial \( X_G \) — the only new restriction on proper colorings is that the root vertex must get color 0. Second, we can easily recover \( X_G \) from \( X_0(G_*) \) by the symmetry operations specified in \cite{5} and \cite{9}. Proposition \cite{21} shows that being able to solve the reverse problem (recovering all \( X_0(G_*) \) that arise from \( X_G \) by varying the choice of root) is equivalent to Stanley’s Conjecture \cite{1}. Third, the deletion-contraction recursion for \( X_0(G_*) \), which is not directly available for \( X_G \), lets us efficiently compute both polynomials. Fourth, the simple restriction on the color of the root vertex suffices to yield a short proof of the desired invariance property for rooted trees. The proof makes compelling contact between a combinatorial property (isomorphism of rooted trees) and an algebraic property (unique factorization of polynomials into irreducible factors).

Some previously studied relatives of \( X_G \) achieve similar invariance results by incorporating additional information in other ways. The Gebhard-Sagan noncommutative chromatic symmetric function \cite{19} remembers which vertices receive which colors by using non-commuting variables. The Shareshian–Wachs chromatic quasisymmetric polynomial \cite{31} uses a new variable \( t \) to record an ascent statistic for each proper coloring. Hasebe and Tsujie \cite{21} adopt a related approach where only proper colorings satisfying additional ascent constraints are allowed. Section \ref{section:related-work} contains a more detailed discussion.

1.4. **Outline of Paper.** The outline of this paper is as follows. Section \ref{section:definitions} introduces needed definitions, notation, and background. Section \ref{section:irreducible-polynomials} proves that for all trees \( T \), the polynomials \( X_T(x_1, \ldots, x_N) \) are irreducible in \( \mathbb{Q}[x_1, \ldots, x_N] \) for large \( N \), along with a similar result for rooted trees. We use these results to prove the analogue of Stanley’s conjecture holds for \( X_0(T_*) \). Section \ref{section:deletion-contraction} proves the deletion-contraction recursion for \( X_0(G_*) \) and describes the coefficients of \( X_0(G_*) \) in various bases. Section \ref{section:related-work} examines some previously studied variants of Stanley’s chromatic symmetric function and their relations to the rooted chromatic polynomials.

2. **Definitions and Background**

2.1. **Definition of Polynomials for Rooted Graphs.** The notation \( G = (V(G), E(G)) \) means \( G \) is a graph with vertex set \( V(G) \) and edge set \( E(G) \). A *rooted graph* \( G_* \) is a nonempty graph \( G \) with one vertex \( r \) of \( G \) marked as the *root*. When we need to display the root, we write \( G^*_r \) for the rooted graph obtained from \( G \) with root vertex \( r \). The *color set* is \( C = \{0, 1, 2, \ldots, N\} \), where \( N \) is a fixed positive integer with \( N + 1 \geq |V(G)| \). A *proper coloring* of \( G \) is a function \( \kappa : V(G) \to C \) such that for all \( v, w \in V(G) \), if an edge joins \( v \) to \( w \), then \( \kappa(v) \neq \kappa(w) \). Let \( \text{COL}(G) \) be the set of proper colorings of \( G \). For each color \( i \in C \), let \( \text{COL}_i(G_*) \) be the set of proper colorings of \( G \) where
\( \kappa(r) = i \) (the root gets color \( i \)). Let \( \text{col}_{\neq i}(G_*) \) be the set of proper colorings of \( G \) where \( \kappa(r) \neq i \) (the root’s color is not \( i \)). The weight of a coloring \( \kappa : V(G) \to C \) is \( \text{wt}(\kappa) = \prod_{v \in V(G)} x_{\kappa(v)} \).

We now introduce several versions of chromatic polynomials for a rooted graph \( G_* \). These are polynomials in \( \mathbb{Q}[x_0, x_1, \ldots, x_N] \) with nonnegative integer coefficients. Define:

\[
\begin{align*}
(1) \quad X(G_*; x_0, x_1, \ldots, x_N) &= \sum_{\kappa \in \text{col}(G)} \text{wt}(\kappa); \\
(2) \quad X_i(G_*; x_0, x_1, \ldots, x_N) &= \sum_{\kappa \in \text{col}_i(G_*)} \text{wt}(\kappa); \\
(3) \quad X_{\neq i}(G_*; x_0, x_1, \ldots, x_N) &= \sum_{\kappa \in \text{col}_{\neq i}(G_*)} \text{wt}(\kappa).
\end{align*}
\]

We omit the variable list from the notation when it is clear from context. Note \( X(G_*; x_0, x_1, \ldots, x_N) \) is Stanley’s chromatic symmetric function \( X_G \) specialized to the given variable set. This polynomial is symmetric in \( x_0, \ldots, x_N \), since applying any permutation of the color set \( C \) to a proper coloring of \( G \) produces another proper coloring of \( G \). For each \( i \in C \), \( X_i(G_*) \) and \( X_{\neq i}(G_*) \) are polynomials in \( x_0, \ldots, x_N \) that are symmetric in all the variables except \( x_i \). This follows since any permutation of \( \{0, 1, \ldots, N\} \) fixing \( i \) induces bijections from \( \text{col}_i(G_*) \) to itself and from \( \text{col}_{\neq i}(G_*) \) to itself. We refer to \( X_0(G_*) \) as the rooted chromatic polynomial for \( G_* \) in \( N + 1 \) variables.

Let \( \Lambda_N \) be the ring of symmetric polynomials in variables \( x_1, \ldots, x_N \), let \( \Lambda \) be the ring of symmetric functions in \( x_k : k \geq 1 \), and let \( z = x_0 \). Then \( X_0(G_*) \) and \( X_{\neq 0}(G_*) \) are in \( \Lambda_N[z] \), the ring of polynomials in \( z \) with coefficients in \( \Lambda_N \). The constant coefficient of \( X_{\neq 0}(G_*) \), namely the specialization upon setting \( z = 0 \), is Stanley’s chromatic symmetric polynomial \( X_G \) in variables \( x_1, \ldots, x_N \). Thus, we may view \( X_{\neq 0}(G_*) \) as a refinement of the original chromatic symmetric function.

The coefficient of \( z^k \) in \( X_0(G_*) \) is a symmetric polynomial in \( x_1, \ldots, x_N \) that is homogeneous of degree \( n - k \), where \( n = |V(G_*)| \). For all \( N \) and \( M \) with \( N > M \geq n \), setting the last \( N - M \) variables equal to 0 in \( X_0(G_*; x_0, x_1, \ldots, x_N) \) produces \( X_0(G_*; x_0, x_1, \ldots, x_M) \). Because of this stability property, we can let the number of variables tend to infinity to obtain a version of \( X_0(G_*) \) in \( \Lambda[z] \). Informally, this polynomial in \( z \) (with symmetric function coefficients) contains exactly the same information as each finite version \( X_0(G_*; x_0, x_1, \ldots, x_N) \) where \( N \geq |V(G_*)| \). Similar comments are true for \( X_{\neq 0}(G_*) \).

**Example 3.** Suppose \( N = 2 \) and \( G \) is a three-vertex path. Then

\[
(4) \quad X_{\bullet-\circ-\circ}(x_0, x_1, x_2) = 6x_0x_1x_2 + x_0^2x_1 + x_0^2x_2 + x_0x_1^2 + x_1^2x_2 + x_0x_2^2 + x_1x_2^2.
\]

For instance, the coefficient of \( x_0x_2^2 \) is 1 because there is only 1 proper coloring of \( G \) using color 2 twice and color 0 once (the middle vertex must receive color 0). Now consider rooted graphs with underlying graph \( G \). Choosing the root to be either endpoint of the path gives

\[
\begin{align*}
(5) \quad X_0(\bullet-\circ-\circ) &= X_0(\circ-\bullet-\circ) = 2x_0x_1x_2 + x_0^2x_1 + x_0^2x_2; \\
(6) \quad X_{\neq 0}(\bullet-\circ-\circ) &= X_{\neq 0}(\circ-\bullet-\circ) = 4x_0x_1x_2 + x_0x_1^2 + x_0x_2^2 + x_1^2x_2 + x_1x_2^2.
\end{align*}
\]

On the other hand, choosing the root to be the middle vertex yields

\[
\begin{align*}
(7) \quad X_0(\circ-\bullet-\circ) &= x_0(2x_1x_2 + x_1^2 + x_2^2); \quad X_{\neq 0}(\circ-\bullet-\circ) = 4x_0x_1x_2 + x_0^2x_1 + x_0^2x_2 + x_1x_2^2 + x_1^2x_2.
\end{align*}
\]

2.2. Basic Identities. We now establish some identities relating the various chromatic polynomials just defined. Let \( \mathfrak{S} \) be the symmetric group on the color set \( C = \{0, 1, \ldots, N\} \). For \( i \neq j \) in \( C \), \( (i, j) \) is the transposition in \( \mathfrak{S} \) that interchanges \( i \) and \( j \). The group \( \mathfrak{S} \) acts on \( \mathbb{Q}[x_0, x_1, \ldots, x_N] \) by permuting the variables: \( \sigma \cdot x_i = x_{\sigma(i)} \) for \( \sigma \in \mathfrak{S} \) and \( i \in C \). The following identities follow directly from the definitions and symmetry arguments.
Proposition 4. For all rooted graphs $G_*$ and all $k \in \{0, 1, \ldots, N\}$,

\[ X(G_*) = \sum_{i=0}^{N} X_i(G_*) = X_k(G_*) + X_{\neq k}(G_*); \]  

\[ X_k(G_*) = (0, k) \bullet X_0(G_*); \]  

\[ X_{\neq 0}(G_*) = \sum_{i=1}^{N} X_i(G_*) = \sum_{i=1}^{N} (0, i) \bullet X_0(G_*); \]  

\[ X_{\neq k}(G_*) = (0, k) \bullet X_{\neq 0}(G_*). \]

Hence, $X(G_*) = X_G$, $X_{\neq 0}(G_*)$, $X_k(G_*)$, and $X_{\neq k}(G_*)$ can all be recovered from the rooted chromatic polynomial $X_0(G_*)$.

A more interesting fact is that knowledge of $X_{\neq 0}(G_*)$ is also sufficient to recover $X_0(G_*)$ and all other polynomials listed here. Given $X_{\neq 0}(G_*)$, we first obtain $X_{\neq i}(G_*) = (0, i) \bullet X_{\neq 0}(G_*)$ for all $i$ between 1 and $N$. The key observation is:

\[ \sum_{i=0}^{N} X_{\neq i}(G_*) = \sum_{i=0}^{N} \sum_{j=0}^{N} X_j(G_*) = \sum_{j=0}^{N} X_j(G_*) \sum_{i=0}^{N} 1 = N \sum_{j=0}^{N} X_j(G_*) = NX(G_*). \]

So $X(G_*) = \frac{1}{N} \sum_{i=0}^{N} (0, i) \bullet X_{\neq 0}(G_*)$. Combining this with (8) yields the following.

Proposition 5. For any rooted graph $G_*$,

\[ X_0(G_*) = \frac{\sum_{i=0}^{N} (0, i) \bullet X_{\neq 0}(G_*)}{N} - X_{\neq 0}(G_*). \]

2.3. Isomorphisms of Graphs and Rooted Graphs. Given graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, a graph isomorphism from $G$ to $H$ is a bijection $f : V(G) \to V(H)$ such that for all $v, w \in V(G)$, $E(G)$ contains the edge joining $v$ and $w$ if and only if $E(H)$ contains the edge joining $f(v)$ and $f(w)$. Suppose $G_*$ is a rooted graph with underlying graph $G$ and root $r$, and $H_*$ is a rooted graph with underlying graph $H$ and root $s$. A rooted graph isomorphism from $G_*$ to $H_*$ is a graph isomorphism $f : V(G) \to V(H)$ with $f(r) = s$. When such an isomorphism exists, we write $G_* \cong H_*$. The concept of rooted graph isomorphism for rooted trees is closely related to the following recursive construction of rooted trees. For any rooted tree $T_*$, either $T_*$ is a one-vertex graph consisting of the root $r$ and no edges; or $T_*$ has root $r$ joined by edges to the roots of $c \geq 1$ principal subtrees $T_{1*}, T_{2*}, \ldots, T_{c*}$. Note that these subtrees can be listed in any order (in contrast to subtrees in an ordered plane tree, where the order of the children of the root is significant). Studying isomorphism classes of rooted trees amounts to erasing all vertex labels. Upon doing this, the principal subtrees of $T_*$ form a multiset of isomorphism classes of rooted trees, in which order does not matter, but repetitions are allowed.

It is routine to check the following criterion for two rooted trees $T_*$ and $U_*$ (with more than one vertex) to be isomorphic as rooted graphs. If $T_*$ has principal subtrees $T_{1*}, \ldots, T_{c*}$ and $U_*$ has principal subtrees $U_{1*}, \ldots, U_{d*}$, then:

\[ T_* \cong U_* \text{ if and only if } c = d \text{ and (after suitable reordering) } T_{is} \cong U_{is} \text{ for } 1 \leq i \leq c. \]

2.4. Recursion for $X_0(T_*)$. There is a simple recursion for computing $X_0(T_*)$ when $T_*$ is a rooted tree. A version of this result was used for computational purposes by Heil and Ji [22] (cf. [7, 8]).
Proposition 6. Let $T_*$ be a rooted tree with principal rooted subtrees $T_{1*}, T_{2*}, \ldots, T_{c*}$. Then

\begin{equation}
X_0(T_*) = x_0 \prod_{j=1}^{c} X_{\neq 0}(T_{j*}).
\end{equation}

Proof. The left side $X_0(T_*)$ is the weighted sum of all colorings in $\text{COL}_0(T_*)$. We build each such coloring (uniquely) by making the following choices. First, color the root vertex $r$ with color 0 (the weight monomial for this step is $x_0$). Next, color the subtree $T_{1*}$ using any coloring in $\text{COL}_{\neq 0}(T_{1*})$; color the subtree $T_{2*}$ using any coloring in $\text{COL}_{\neq 0}(T_{2*})$; and so on. The colorings of the subtrees can be chosen independently since the subtrees do not connect with each other except through the root $r$. The recursion \cite{14} follows by the Product Rule for Weighted Sets \cite{25} §5.8.

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3. Irreducibility Results

The main result proved in this section is the following theorem.

Theorem 7. For each integer $n > 0$, there exists $M$ such that for all $N \geq M$:

(a) For all trees $T$ with $|V(T)| \leq n$, $X_T(x_1, \ldots, x_N)$ is irreducible in $\mathbb{Q}[x_1, \ldots, x_N]$.

(b) For all rooted trees $T_*$ with $|V(T_*)| \leq n$, $X_{\neq 0}(T_*; x_0, x_1, \ldots, x_N)$ is irreducible in $\mathbb{Q}[x_0, x_1, \ldots, x_N]$.

3.1. Background on Irreducibility and Specializations. This subsection collects some basic facts about irreducible polynomials needed for our proof of Theorem 7. The key tool used is Eisenstein’s Criterion for irreducibility of polynomials in $\mathbb{Q}[q]$.

Theorem 8 (Eisenstein’s Criterion \cite{13}). Suppose $f = a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n \in \mathbb{Q}[q]$ is a polynomial of degree $n > 0$ with integer coefficients, $p$ is a prime, $p$ divides $a_k$ for $0 \leq k < n$, $p$ does not divide $a_n$, and $p^2$ does not divide $a_0$. Then $f$ is irreducible in the polynomial ring $\mathbb{Q}[q]$.

Lemma 9. Suppose $f \in \mathbb{Q}[x_0, \ldots, x_N]$ is reducible and homogeneous of degree $n > 0$. Then $f$ has a proper factorization $f = gh$ for some non-constant homogeneous polynomials $g, h \in \mathbb{Q}[x_0, \ldots, x_N]$.

Proof. By irreducibility of $f$, there is a proper factorization $f = gh$ where $g, h \in \mathbb{Q}[x_0, \ldots, x_N]$ are not initially known to be homogeneous. Write $g = g^{(d)} + \text{(lower terms)}$, where $d > 0$ is the maximum degree of any monomial in $g$, and $g^{(d)}$ is the sum of all monomials in $g$ having degree $d$. Write $h = h^{(e)} + \text{(lower terms)}$, where $e > 0$ is the maximum degree of any monomial in $h$, and $h^{(e)}$ is the sum of all monomials in $h$ having degree $e$. Then $gh = g^{(d)}h^{(e)} + \text{(lower terms)}$. Since $\mathbb{Q}[x_0, \ldots, x_N]$ is an integral domain, the product $g^{(d)}h^{(e)}$ (which is homogeneous of degree $d + e$) cannot be zero. Since $f$ is homogeneous, this product must equal $f$. Replacing $g$ by $g^{(d)}$ and $h$ by $h^{(e)}$, we therefore have $f = gh$ with both factors homogeneous and non-constant.

Write $f|_{x_i=0}$ for the specialization of a polynomial $f$ obtained by setting variable $x_i$ equal to 0.

Lemma 10. Suppose $f \in \mathbb{Q}[x_0, x_1, \ldots, x_N]$ is homogeneous of degree $n > 0$. If $f|_{x_i=0}$ is an irreducible polynomial in $\mathbb{Q}[x_0, x_1, \ldots, x_N]$ (where $x_i$ means variable $x_i$ is omitted), then $f$ is irreducible in $\mathbb{Q}[x_0, x_1, \ldots, x_N]$.

Proof. Let $f' = f|_{x_i=0}$. To get a contradiction, assume $f'$ is irreducible (hence non-constant and nonzero) and $f$ is reducible. By the previous lemma, there is a proper factorization $f = gh$ where $g$ and $h$ are both homogeneous of positive degree. Letting $g' = g|_{x_i=0}$ and $h' = h|_{x_i=0}$, we have $f' = g'h'$. Since $f' \neq 0$, $g'$ and $h'$ are nonzero. It follows at once that $g'$ and $h'$ are homogeneous of positive degree. Thus $f' = g'h'$ is a proper factorization of $f'$, which is a contradiction.

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Lemma 11. Fix a rooted graph $G_*$. For all $N \geq M \geq n = |V(G_*)|$:

1. If $X_G(x_1, \ldots, x_M)$ is irreducible, then $X_{\neq 0}(G_*; x_0, x_1, \ldots, x_M)$ is irreducible.
2. If $X_G(x_1, \ldots, x_M)$ is irreducible, then $X_G(x_1, \ldots, x_N)$ is irreducible.

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(3) If \(X_{\neq 0}(G_s;x_0,\ldots,x_M)\) is irreducible, then \(X_{\neq 0}(G_s;x_0,\ldots,x_N)\) is irreducible.

**Proof.** All polynomials mentioned in the lemma are homogeneous of degree \(n = |V(G_s)|\) and are nonzero, since the hypothesis \(N \geq M \geq n\) ensures there are enough colors to produce at least one coloring satisfying all needed conditions. Part (1) follows from Lemma 10 since setting \(x_0 = 0\) in \(X_{\neq 0}(G_s;x_0,\ldots,x_M)\) produces \(X_G(x_1,\ldots,x_M)\). Parts (2) and (3) follow from Lemma 10 by setting \(x_N = 0,\ldots,x_{M+1} = 0\) one at a time. \(\square\)

**Lemma 12.** If \(P = P(x_1,\ldots,x_N) \in \mathbb{Q}[x_1,\ldots,x_N]\) is a nonzero polynomial and \(S \subseteq \mathbb{Q}\) is an infinite set, then there exist \(s_1,\ldots,s_N \in S\) such that \(P(s_1,\ldots,s_N)\) is a nonzero element of \(\mathbb{Q}\).

**Proof.** See, for instance, [55, Thm. 14, pg. 38]. \(\square\)

3.2. **Proof of Theorem 7.** This section proves Theorem 7(a). Theorem 7(b) follows at once from part (a) and Lemma 11. We begin with a lemma containing the core idea of the proof.

**Lemma 13.** Let \(T\) be any \(n\)-vertex tree \((n > 0)\). Take any \(M \geq n\) such that \(p = M - 2\) is an odd prime. Let \(f(q) \in \mathbb{Q}[q]\) be the specialization of \(X_T(x_1,\ldots,x_M)\) obtained by setting \(x_1 = x_2 = q\) and \(x_3 = \cdots = x_M = 1\). Then \(f\) is irreducible in \(\mathbb{Q}[q]\).

**Proof.** Since \(X_T(x_1,\ldots,x_M)\) is a sum of monic monomials \(w_t(\kappa)\), we can write \(f(q) = \sum_{k \geq 0} a_k q^k\) with all \(a_k \in \mathbb{Z}_{\geq 0}\). We claim \(a_k\) is the number of proper colorings \(\kappa : V(T) \rightarrow \{1,2,\ldots,M\}\) such that the number of \(v\) with \(\kappa(v) \in \{1,2\}\) is \(k\). This holds since the weight monomial \(x_1^{e_1} x_2^{e_2} \cdots x_M^{e_M}\) of \(\kappa\) specializes to \(q^{e_1+e_2}\), where the power of \(q\) is the number of vertices that \(\kappa\) colors 1 or 2.

Since \(T\) has \(n\) vertices, we have \(a_k = 0\) for \(k > n\). Recall the one-variable chromatic polynomial for trees: there are \(y(y - 1)^{s-1}\) ways to color any \(s\)-vertex tree using \(y\) available colors. Taking \(y = 2\) and \(s = n\), we see that there are exactly 2 colorings of \(T\) where all \(n\) vertices receive color 1 or color 2. So \(a_n = 2\) and \(\deg(f) = n\). Note \(p\) does not divide \(a_n\) since \(p\) is odd.

We next show that \(p\) divides \(a_k\) for \(0 \leq k < n\). Fix \(k\) in this range, and consider the colorings counted by \(a_k\). Let \(A_k\) be the set of pairs \((S,\kappa')\) such that \(S \subseteq V(T), |S| = k,\) and \(\kappa' : S \rightarrow \{1,2\}\) is a proper coloring of the subgraph of \(T\) with vertex set \(S\). Let \(U\) be the subgraph of \(T\) with vertex set \(V(T) \setminus S\). We obtain a coloring \(\kappa\) counted by \(a_k\) by choosing a pair \((S,\kappa') \in A_k\), choosing a proper coloring \(\kappa'' : V(U) \rightarrow \{3,\ldots,M\}\), and letting \(\kappa\) be the union of the functions \(\kappa'\) and \(\kappa''\). The graph \(U\) must consist of some trees \(T_1,\ldots,T_{\ell}\), where \(T_i\) has \(s_i\) vertices and \(s_1 + \cdots + s_{\ell} = n - k > 0\). By the Product Rule, the number of colorings \(\kappa\) that arise from a fixed choice of \((S,\kappa') \in A_k\) is

\[
\prod_{i=1}^{\ell} [(M - 2)(M - 3)^{s_i - 1}] = p^\ell(p - 1)^{n - k - \ell}.
\]

Here \(\ell = \ell(S,\kappa')\) depends on \((S,\kappa')\) and is always positive. Adding over all choices of \((S,\kappa')\), we get

\[
a_k = \sum_{(S,\kappa') \in A_k} p^\ell(S,\kappa')(p - 1)^{n - k - \ell(S,\kappa')}.
\]

This integer is divisible by \(p\).

Finally, we show that \(p^2\) does not divide \(a_0\). The coefficient \(a_0\) is the number of colorings of \(T\) using only colors in \(\{3,\ldots,M\}\), so \(a_0 = p(p - 1)^{n-1}\). Since \(p\) is prime, \(a_0\) is divisible by \(p\) but not by \(p^2\). We conclude that \(f\) is irreducible in \(\mathbb{Q}[q]\) by Eisenstein’s Criterion (Theorem 8). \(\square\)

Lemma 13 suggests that \(X_T(x_1,\ldots,x_M)\) is likely to be irreducible, since its specialization \(f(q)\) is irreducible. However, we need some technical modifications to rule out the possibility that a proper factorization of \(X_T\) happens to specialize to a trivial factorization of \(f(q)\). (As a simple example of this phenomenon, note \((x_1 + x_2)(x_3 + x_4 + x_5)\) is reducible in \(\mathbb{Q}[x_1,\ldots,x_5]\), but its specialization \(6q\) is irreducible in \(\mathbb{Q}[q]\).)
Lemma 14. Suppose $T$ is any $n$-vertex tree ($n > 0$), $M \geq n$ is an integer such that $p = M - 2$ is an odd prime, and $c_3, \ldots, c_M$ are any integers. Let $g(q) \in \mathbb{Q}[q]$ be the specialization of $X_T(x_1, \ldots, x_M)$ obtained by setting $x_1 = x_2 = q$ and $x_{\ell} = 1 + cp^2$ for $3 \leq \ell \leq M$. Then $g$ is irreducible in $\mathbb{Q}[q]$.

Proof. We have $g(q) = \sum_{k=0}^{n} b_k q^k$ for certain integers $b_k$. Let $f(q) = \sum_{k=0}^{n} a_k q^k$ be the polynomial from Lemma 13. We first show $a_k \equiv b_k \pmod{p^2}$ for all $k$ between 0 and $n$. Let $\text{COL}^k(T)$ be the set of proper colorings of $T$ such that exactly $k$ vertices receive a color in $\{1,2\}$. These colorings, and no others, contribute to the coefficient of $q^k$ when computing $f(q)$ or $g(q)$. On one hand, $a_k = |\text{COL}^k(T)| = \sum_{\kappa \in \text{COL}^k(T)} 1$. On the other hand, a coloring $\kappa \in \text{COL}^k(T)$ with weight monomial $\text{wt}(\kappa) = x_1^{s_1} \cdots x_M^{s_M}$ specializes to $q^k \prod_{\ell=3}^{M} (1 + cp^2)^{e_\ell}$ when computing $g$. The coefficient of $q^k$ is an integer congruent to 1 (mod $p^2$). Therefore, $b_k \equiv \sum_{\kappa \in \text{COL}^k(T)} 1 = a_k \pmod{p^2}$, as needed. It follows that all the divisibility hypotheses for Eisenstein’s Criterion hold for the coefficients $b_0, \ldots, b_n$, since they are known to hold for $a_0, \ldots, a_n$. (For example, $b_0$ is divisible by $p$ but not by $p^2$ since $b_0 \equiv a_0 \pmod{p^2}$ and $a_0$ is divisible by $p$ but not by $p^2$.) So $g(q)$ is irreducible in $\mathbb{Q}[q]$ by Eisenstein’s Criterion.

We now prove Theorem 7(a).

Proof. Fix an integer $n > 0$. Choose the least $M$ such that $M \geq n$ and $p = M - 2$ is an odd prime. Let $T$ be any nonempty tree with $n_1 \leq n$ vertices. We must prove: for all $N \geq M$, $X_T(x_1, \ldots, x_N)$ is irreducible in $\mathbb{Q}[x_1, \ldots, x_N]$. It suffices to do this for $N = M$, by Lemma 11(2). Assume, to get a contradiction, that $X_T(x_1, \ldots, x_M)$ is reducible. Since $X_T$ is homogeneous of degree $n_1 > 0$, we have $X_T = gh$ for some $g,h \in \mathbb{Q}[x_1, \ldots, x_M]$, where $g$ is homogeneous of degree $d > 0$ and $h$ is homogeneous of degree $n_1 - d > 0$ (Lemma 9). Our strategy is to choose a judicious specialization of these polynomials that prevents the proper factorization of $X_T$ from degenerating into a trivial factorization, and then to invoke Lemma 14 with this specialization to obtain a contradiction.

To begin, let $[M] = \{1,2,\ldots, M\}$. We claim there exist $i \neq j$ in $[M]$ such that $x_i$ appears in $g$ and $x_j$ appears in $h$. Note that every variable $x_1, \ldots, x_M$ does appear in $X_T = gh$, since colorings of $T$ exist using any particular color. So $[M]$ is the union of $I = \{i : x_i \text{ appears in } g\}$ and $J = \{j : x_j \text{ appears in } h\}$. $I$ and $J$ are nonempty sets since $g$ and $h$ have positive degree. If $|I| = 1$, then $|J| \geq M - 1 \geq 2$, so we can take $i$ to be the unique element of $I$ and take $j$ to be something in $J$ other than $i$. If $|I| = 1$, then $|I| \geq M - 1 \geq 2$, so we can take $i$ to be the unique element of $J$ and take $i$ to be something in $I$ other than $j$. If $|I| \geq 2$ and $|J| \geq 2$, then we can pick any $i \in I$ and any $j \in J$ different from $i$. So the claim holds. Since $X_T$ is symmetric, we may assume $i = 1$ and $j = 2$ in the rest of the proof, by applying $(1,i)(2,j)$ to the factorization $X_T = gh$.

Specialize $X_T$, $g$, and $h$ by setting $x_1 = q$ and $x_2 = q$, where $q$ is a new formal variable distinct from $x_1, \ldots, x_M$, to get a specialized factorization $X_T' = g'h'$. $X_T$ does not become 0 when we specialize (since it is a positive sum of monomials), so neither $g$ nor $h$ can become 0 either. Also, $g'$ and $h'$ are still homogeneous (in the list of variables $q,x_3,\ldots,x_M$) of degrees $d > 0$ and $n_1 - d > 0$, respectively. So the factorization $X_T' = g'h'$ is still a proper factorization.

We next claim there exist $c_3, \ldots, c_M \in \mathbb{Z}_{\geq 0}$ such that the specialization $E : \mathbb{Q}[x_1, \ldots, x_M] \to \mathbb{Q}[q]$ with $E(x_1) = E(x_2) = q$ and $E(x_{\ell}) = 1 + cp^2$ (for $3 \leq \ell \leq M$) yields a proper factorization $E(X_T') = E(g)E(h)$ in $\mathbb{Q}[q]$. Let $R$ be the polynomial ring $\mathbb{Q}[x_3, \ldots, x_M]$. By the choice of $i$ and $j$, $g'$ and $h'$ are polynomials in $R[q]$ of positive degree in $q$. Say $g' = g_st^q$ + (lower) and $h' = h_st^q$ + (lower) for some $s,t > 0$ and nonzero polynomials $g_s, h_t \in R$. Apply Lemma 12 (with $\mathbb{Q}[x_1, \ldots, x_N]$ replaced by $R$), taking $P = g_s h_t$ and $S = \{1 + cp^2 : c \in \mathbb{Z}_{\geq 0}\}$. By the lemma, there exist $c_3, \ldots, c_M \in \mathbb{Z}_{\geq 0}$ such that $g_s(c_3, \ldots, c_M)h_t(c_3, \ldots, c_M) \neq 0$. Defining $E$ using this choice of $c_3, \ldots, c_M$, it follows that $q^s$ still appears in $E(g)$ with nonzero coefficient, and $q^t$ still appears in $E(h)$ with nonzero coefficient. So the claim holds, which means that $E(X_T)$ is reducible.
in $\mathbb{Q}[q]$. This contradicts Lemma 14. We conclude that $X_T(x_1, \ldots, x_M)$ must be irreducible, as required. \hfill \Box

3.3. Analogue of Stanley’s Conjecture for Rooted Trees. As an application of Theorem 7, this section gives our first proof of the following analogue of Stanley’s Conjecture for rooted trees.

**Theorem 15.** Rooted trees $T_r$ and $U_r$ are isomorphic (as rooted graphs) if and only if $X_0(T_r) = X_0(U_r)$.

We require one preliminary lemma.

**Lemma 16.** Suppose $T_r$ and $U_r$ are rooted trees such that $X_{\neq 0}(T_r)$ and $X_{\neq 0}(U_r)$ are associate ring elements in $\mathbb{Q}[x_0, \ldots, x_N]$. Then $X_{\neq 0}(T_r) = X_{\neq 0}(U_r)$.

**Proof.** We know $X_{\neq 0}(T_r)$ is homogeneous of degree $d > 0$, where $d = |V(T_r)|$. Since $X_{\neq 0}(U_r) = \alpha X_{\neq 0}(T_r)$ for some nonzero $\alpha \in \mathbb{Q}$, we also have $d = |V(U_r)|$. The coefficient of $x_1 x_2 \cdots x_d$ in $X_{\neq 0}(T_r)$ is $d!$, since we obtain all colorings in $\mathrm{col}_{\neq 0}(T_r)$ with weight $x_1 x_2 \cdots x_d$ by choosing a bijection from $V(T_r)$ to the color set $\{1, 2, \ldots, d\}$. Similarly, the coefficient of $x_1 x_2 \cdots x_d$ in $X_{\neq 0}(U_r)$ is also $d!$. This forces $\alpha = 1$ and $X_{\neq 0}(T_r) = X_{\neq 0}(U_r)$. \hfill \Box

We now prove Theorem 15. The forward implication (if $T_r \cong U_r$, then $X_0(T_r) = X_0(U_r)$) is routine. For the converse implication, we use induction on the number $n$ of vertices in each tree. Fix $n$-vertex rooted trees $T_r$ and $U_r$ with $X_0(T_r) = X_0(U_r)$ (note that $n$ is the degree of each monomial in this polynomial, so $T_r$ and $U_r$ must have the same number of vertices). If $n = 1$, then $T_r \cong U_r$ since both trees consist of a single root vertex. Assume $n > 1$ from now on. Use Theorem 7(b) to find an $N \geq n$ such that for every rooted tree $S_r$ with at most $n$ vertices, $X_{\neq 0}(S_r; x_0, \ldots, x_N)$ is irreducible in $\mathbb{Q}[x_0, \ldots, x_N]$. Let the root $r$ of $T_r$ have principal subtrees $T_{r_1}, \ldots, T_{r_c}$, and let the root $s$ of $U_r$ have principal subtrees $U_{r_1}, \ldots, U_{r_d}$. Applying Proposition 6 to $T_r$ and to $U_r$, the assumption $X_0(T_r) = X_0(U_r)$ becomes

\begin{equation}
X_0(T_r) = x_0 \prod_{j=1}^c X_{\neq 0}(T_{r_j}) = x_0 \prod_{j=1}^d X_{\neq 0}(U_{r_j}) \quad \text{in } \mathbb{Q}[x_0, \ldots, x_N].
\end{equation}

By our choice of $N$, each side of (15) is a factorization of the polynomial $X_0(T_r) = X_0(U_r)$ into irreducible factors. Because $\mathbb{Q}[x_0, x_1, \ldots, x_N]$ is a unique factorization domain, we can conclude that $c = d$ and (after reordering factors appropriately) $X_{\neq 0}(T_{r_j})$ is an associate of $X_{\neq 0}(U_{r_j})$ for $1 \leq j \leq c$. By Lemma 16, we have $X_{\neq 0}(T_{r_j}) = X_{\neq 0}(U_{r_j})$ for all $j$ between 1 and $c$. By (12), we deduce $X_0(T_{r_j}) = X_0(U_{r_j})$ for $1 \leq j \leq c$. Each principal subtree $T_{r_j}$ and $U_{r_j}$ has fewer than $n$ vertices. By the induction hypothesis, $T_{r_j} \cong U_{r_j}$ for $1 \leq j \leq c$. Finally, $T_r \cong U_r$ follows from (13).

4. Further Properties of Chromatic Polynomials for Rooted Graphs

In this section and the next, certain standard facts regarding various bases of the ring of symmetric functions $\Lambda$ are needed. See [25] for background including definitions of the monomial basis $\{m_\lambda\}$, the elementary basis $\{e_\lambda\}$, and the power-sum basis $\{p_\lambda\}$.

4.1. Deletion-Contraction Recursion for $X_0(G_r)$. The classical one-variable chromatic polynomials satisfy the deletion-contraction recursion $\chi_G = \chi_{G-e} - \chi_{G_e}$, where $G$ is a graph, $e$ is an edge of $G$, $G - e$ is the graph $G$ with edge $e$ deleted, and $G_e$ is the graph obtained from $G$ by contracting the edge $e$. There is no such recursion for Stanley’s chromatic symmetric function $X_G$, although deletion-contraction recursions are known for the pointed chromatic symmetric functions $[30]$ Lemma 3.5, the W-polynomials and U-polynomials $[28]$ pg. 1059, and the noncommutative chromatic functions $Y_G$ $[19]$ Prop. 3.5. Related recursions can be found in $[29]$. We now show that the rooted version $X_0(G_r)$ does satisfy a simple generalization of the recursion for $\chi_G$. 
Theorem 17. Suppose $G_*$ is a rooted graph with root vertex $r$, and $\varepsilon$ is an edge from $r$ to $s$. Let $G_* - \varepsilon$ be $G_*$ with edge $\varepsilon$ deleted (using the same root $r$). Let $G_{\varepsilon s}$ be $G_*$ with edge $\varepsilon$ contracted, meaning that we identify vertices $r$ and $s$ in $G_{\varepsilon}$ and use the identified vertex as the new root. Then

$$X_0(G_*) = X_0(G_* - \varepsilon) - x_0 X_0(G_{\varepsilon s}).$$

An initial condition occurs when no edge of $G$ touches the root $r$. In that case,

$$X_0(G_*) = x_0 X(G - r; x_0, \ldots, x_N),$$

where $G - r$ is the unrooted graph obtained from $G$ by deleting the isolated root vertex.

Proof. The theorem can be deduced from [30, Lemma 3.5], using the algebraic transformation explained in Section 5.1, but we prefer to give a direct bijective proof based on colorings. Let $S = \text{col}_0(G_*)$, $T = \text{col}_0(G_* - \varepsilon)$, and $U = \text{col}_0(G_{\varepsilon s})$. Evidently, $T$ is the disjoint union of the two subsets

$$S = \{\kappa \in T : 0 = \kappa(r) \neq \kappa(s)\} \quad \text{and} \quad U' = \{\kappa \in T : 0 = \kappa(r) = \kappa(s)\}.$$

There is a bijection from $U'$ to $U$ sending $\kappa$ to $\bar{\kappa}$, where the two colorings agree on all vertices other than $r$ and $s$, and $\bar{\kappa}$ colors the new root vertex 0. It follows at once that $wt(\kappa) = x_0 wt(\bar{\kappa})$. Since $T$ is the disjoint union of $S$ and $U'$, the Weighted Sum Rule gives $X_0(G_* - \varepsilon) = X_0(G_*) + x_0 X_0(G_{\varepsilon s})$, as needed. The initial condition is immediate from the Weighted Product Rule. \hfill $\square$

Example 18. Continuing Example 3 we use Theorem 17 to compute (for $N = 2$)

$$X_0(\bullet - \bullet) = X_0(\bullet - \bullet) - x_0 X_0(\bullet - \bullet) = x_0(2x_0 x_1 + 2x_0 x_2 + 2x_1 x_2) - x_0(x_0 x_1 + x_0 x_2)$$

$$\quad = 2x_0 x_1 x_2 + x_0^2 x_1 + x_0^2 x_2,$$

in agreement with [5].

Recursions analogous to (16) hold for any specializations of $X_0(G_*)$ we care to compute. For example, if we use the principal specialization given by $x_0 \to 1, x_1 \to q, \ldots, x_N \to q^N$, then the $x_0$ multiplying the subtracted term in (16) becomes 1.

Conjecture 19. For all trees $T, U$ with at most $N + 1$ vertices,

$$T \cong U \quad \text{if and only if} \quad X_T(1, q, q^2, \ldots, q^N) = X_U(1, q, q^2, \ldots, q^N).$$

We have confirmed this conjecture by computer calculations for all trees with at most 17 vertices.

4.2. Interpretation of Coefficients of $x_0^k$ in $X_0(G_*)$. Let $G_*$ be a rooted graph. Writing $z = x_0$, we have

$$X_0(G_*; z, x_1, \ldots, x_N) = \sum_{k \geq 1} c_k(x_1, \ldots, x_N) z^k,$$

where each $c_k(x_1, \ldots, x_N)$ is a symmetric polynomial in $\Lambda_N$. We now give combinatorial interpretations for these coefficients. Recall that an independent set in a graph $G$ is a subset $A$ of $V(G)$ such that no two vertices in $A$ are joined by an edge in $G$. Given such an $A$, let $G - A$ be the graph with vertex set $V(G) - A$ and edge set obtained from $E(G)$ by deleting all edges with a vertex in $A$ as one endpoint.

Proposition 20. Let $G_*$ be a rooted graph.

(a) For $k > 0$, let $I(G_*, k)$ be the set of $k$-element independent subsets of $G$ that contain the root. Then

$$X_0(G_*)|_{x_0^k} = \sum_{A \in I(G_*, k)} X_{G - A}(x_1, \ldots, x_N).$$
(b) For $k \geq 0$, let $I'(G_s, k)$ be the set of $k$-element independent subsets of $G$ that do not contain the root. Then

$$X_{\neq 0}(G_s)|_{x_0^k} = \sum_{A \in I'(G_s, k)} X_{G-A}(x_1, \ldots, x_N).$$

**Proof.** We build the proper colorings $\kappa \in \text{COL}_0(G_s)$ contributing to the coefficient of $x_0^k$ in $X_0(G_s)$ as follows. First, choose a $k$-element independent set $A \in I(G_s, k)$ and color the vertices of $A$ (including the root $r$) with color 0. Second, choose any proper coloring of the graph $G - A$ using color set $\{1, 2, \ldots, N\}$. The formula in (a) follows from the Sum and Product Rules for Weighted Sets. Part (b) is proved in the same way, but now we choose $A \in I'(G_s, k)$ to ensure the root does not receive color 0. \hfill \square

4.3. Reformulation of Stanley’s Conjecture. The following proposition illuminates the relationship between Stanley’s original conjecture for unrooted trees (Conjecture 1) and the analogous result for rooted trees (Theorem 15).

**Proposition 21.** Conjecture 1 holds if and only if for every tree $T$, the multiset $[X_0(T_r^*) : r \in V(T)]$ is uniquely determined by the chromatic symmetric function $X_T$.

**Proof.** Assume the condition on multisets is true; we prove Conjecture 1. Let $T$ and $U$ be any $n$-vertex trees with $X_T = X_U$. By the multiset condition, $[X_0(T_r^*) : r \in V(T)] = [X_0(U_r^*) : s \in V(U)]$. So there exist $r \in V(T)$ and $s \in V(U)$ with $X_0(T_r^*) = X_0(U_s^*)$. By Theorem 15, $T_r^*$ and $U_s^*$ are isomorphic as rooted graphs. So $T$ and $U$ are isomorphic as graphs.

We prove the converse implication by contradiction. Assume Conjecture 1 is true, but the condition on multisets is false. Then there exist trees $T$ and $U$ such that $X_T = X_U$, but $[X_0(T_r^*) : r \in V(T)] \neq [X_0(U_r^*) : s \in V(U)]$. Since $X_T = X_U$, Conjecture 1 says $T \cong U$, so there is a graph isomorphism $f : V(T) \to V(U)$. For each $r \in V(T)$, $f$ is a rooted graph isomorphism from $T_r^*$ to $U_s^*(r)$. But then $X_0(T_r^*) = X_0(U_s^*(r))$ for all $r \in V(T)$, which means the two multisets are equal. This gives the required contradiction. \hfill \square

At present, we do not know how to recover the multiset $[X_0(T_r^*) : r \in V(T)]$ from $X_T$. In fact, this cannot be done for general graphs $G$, as the following example shows. This example uses the augmented monomial symmetric functions $\tilde{m}_\lambda$, defined by $\tilde{m}_\lambda = r_1!r_2! \cdots m_\lambda$ if $\lambda$ has $r_1$ parts equal to 1, $r_2$ parts equal to 2, and so on.

**Example 22.** Stanley [33, pg. 170] gave this example of two non-isomorphic graphs with equal chromatic symmetric functions:

$$X_{\mathcal{G}} = X_{\mathcal{H}} = 2\tilde{m}_{221} + 4\tilde{m}_{2111} + \tilde{m}_{111111}.$$ 

Let $z = x_0$. For the first graph $G$, the multiset $[X_0(G_r^*)]$ consists of

$$X_0\left(\mathcal{G}\right) = X_0\left(\mathcal{H}\right) = X_0\left(\mathcal{I}\right) = z(2\tilde{m}_{211} + \tilde{m}_{1111}) + z^2(2\tilde{m}_{21} + 2\tilde{m}_{111}),$$

$$X_0\left(\mathcal{J}\right) = z(2\tilde{m}_{22} + 4\tilde{m}_{211} + \tilde{m}_{1111}).$$
For the second graph $H$, the multiset $[X_0(H^*_s)]$ consists of

$$X_0 \left( \mathcal{G} \right) = X_0 \left( \mathcal{G} \right) = z(\bar{m}_{22} + 3\bar{m}_{211} + \bar{m}_{1111}) + z^2(\bar{m}_{21} + \bar{m}_{111}),$$

$$X_0 \left( \mathcal{G} \right) = z(2\bar{m}_{211} + \bar{m}_{1111}) + z^2(2\bar{m}_{21} + 2\bar{m}_{111}),$$

$$X_0 \left( \mathcal{G} \right) = z(3\bar{m}_{211} + \bar{m}_{1111}) + z^2(2\bar{m}_{21} + \bar{m}_{111}),$$

$$X_0 \left( \mathcal{G} \right) = z(\bar{m}_{21} + \bar{m}_{1111}) + z^2(2\bar{m}_{21} + 3\bar{m}_{111}).$$

Despite this example, there is a simple way to recover the sum of all $X_0(G^*_s)$ from $X_G$.

**Proposition 23.** For any graph $G$,

$$x_0 \frac{\partial}{\partial x_0} X_G(x_0, \ldots, x_N) = \sum_{r \in V(G)} X_0(G^*_s; x_0, \ldots, x_N).$$

**Proof.** The proof is an application of the pointing construction to the generating function $X_G$ (see [17] Theorem I.4). Each coloring $\kappa \in \text{col}(G)$ contributes a monomial $x_0^{e_0} \cdots x_N^{e_N}$ to $X_G$. Applying the operator $x_0 \frac{\partial}{\partial x_0}$ multiplies this monomial by $e_0$, which is the number of vertices colored 0 by $\kappa$. The coloring $\kappa$ belongs to exactly $e_0$ of the sets $\text{col}_0(G^*_s)$ as $r$ varies through $V(G)$, since the root $r$ must be colored 0. Thus, each coloring $\kappa$ makes the same contribution (namely $e_0 x_0^{e_0} \cdots x_N^{e_N}$) to both sides of (18).

**Example 24.** Continuing Example 3 the following calculation illustrates Proposition 23

$$X_0(\bullet\circ\circ\circ) + X_0(\circ\bullet\circ\circ) + X_0(\circ\circ\bullet\bullet)$$

$$= (2x_0x_1x_2 + x_0^2(x_1 + x_2)) + (2x_0x_1x_2 + x_0(x_1^2 + x_2^2)) + (2x_0x_1x_2 + x_0(x_1^2 + x_2^2))$$

$$= 6x_0x_1x_2 + 2x_0^2(x_1 + x_2) + x_0(x_1^2 + x_2^2) = x_0 \frac{\partial}{\partial x_0} X_{\circ\circ\circ\circ}(x_0, x_1, x_2).$$

4.4. Analogue of the $(3+1)$-conjecture. Much of the research pertaining to $X_G$ has focused on characterizing the coefficients of $X_G$ when expressed in various bases of the space $\Lambda$ of symmetric functions. One outstanding conjecture by Stanley and Stembridge is the following $e$-positivity conjecture (recent research pertaining to this conjecture can be found in places such as [18, 31, 31, 1]).

For $a \geq 1$, let $a$ denote the totally ordered poset on $a$ elements. Given posets $(P, \leq_P)$ and $(Q, \leq_Q)$, where we can choose notation so that $P \cap Q = \emptyset$, the disjoint union $P + Q$ is the poset $(P \cup Q, \leq_P \cup \leq_Q)$ (meaning that $x \leq_{P+Q} y$ iff $x, y \in P$ and $x \leq_P y$ or $x, y \in Q$ and $x \leq_Q y$). Given any posets $(P, \leq_P)$ and $(Q, \leq_Q)$, we say that $P$ is $Q$-free if $Q$ is not isomorphic to an induced subposet of $P$. The incomparability graph $G(P)$ of a poset $(P, \leq_P)$ is the undirected graph with vertex set $P$ and edges $\{x, y\}$ for all $x, y \in P$ such that $x$ and $y$ are incomparable under $\leq_P$.

**Conjecture 25.** [33] Conj. 5.1, [32] Conj. 5.5 If $(P, \leq_P)$ is $(3+1)$-free, then $X_G(P)$ is $e$-positive.

If we view $X_0(G^*_s; x_0, \ldots, x_N)$ as an element of $\Lambda[x_0]$, the coefficient of $x_0$ is $X_{G-r}(x_1, \ldots, x_N)$. For $f \in \Lambda[z]$, we say that $f$ is $e$-positive if the coefficient of $z^k$ is an $e$-positive element of $\Lambda$ for each $k \geq 0$. Here is a refinement of Conjecture 25 which we have checked (by computer) for posets with at most 8 vertices.
Conjecture 26. Let \((P, \leq_P)\) be a poset. For all \(r \in P\), if \(P - r\) is \((3 + 1)\)-free, then \(X_0(G(P)^*_r)\) is e-positive.

When \((P, \leq_P)\) is \((3 + 1)\)-free, \(G(P)\) is claw-free (i.e., has no vertex-induced subgraph isomorphic to the complete bipartite graph \(K_{1,3}\)). One might hope that we could strengthen Conjecture 25 to assert that \(X_0(G^*_r)\) is e-positive whenever \(G - r\) is claw-free. However, as discussed in [33], claw-free graphs do not have e-positive chromatic symmetric functions in general (see also [12]).

4.5. **Power-Sum Formula.** Stanley [33] Thm. 2.5] used an inclusion–exclusion argument to find the power-sum expansion of the chromatic symmetric function \(X_G\). For any edge \(e \in E(G)\), define \(A_e\) to be the set of (non-proper) colorings \(\kappa : V(G) \to \mathbb{Z}_{\geq 0}\) such that the endpoints of edge \(e\) receive the same color. For any \(S \subseteq E(G)\), let \(G_S\) denote the graph with vertex set \(V(G)\) and edge set \(S\). Let \(\lambda(G_S)\) be the integer partition whose parts are the sizes of the connected components of \(G_S\) in weakly decreasing order. Let \(p_\lambda\) be the power-sum symmetric function indexed by \(\lambda\). A coloring \(\kappa : V(G) \to \mathbb{Z}_{\geq 0}\) is proper if and only if it does not belong to the union \(\bigcup_{e \in E(G)} A_e\). Hence, the Inclusion–Exclusion Formula leads to

\[
X_G = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\lambda(G_S)}.
\]

We can adapt Stanley’s argument to obtain an analogous expansion of \(X_0(G^*_v; x_0, x_1, \ldots, x_N)\). (This formula is closely related to the power-sum expansion of Pawlowski’s pointed chromatic symmetric functions, as we explain in Section 5.1.) Given a rooted graph \(G^*_v\) with root \(v\) and \(S \subseteq E(G)\), let \(\lambda^+_e(G_S)\) be the size of the component of \(G_S\) containing \(v\), and let \(\lambda^-_e(G_S)\) be the partition \(\lambda(G_S)\) with a single part of size \(\lambda^+_e(G_S)\) deleted.

**Proposition 27.** For all rooted graphs \(G^*_v\) and \(N \geq |V(G)|\),

\[
X_0(G^*_v; x_0, x_1, \ldots, x_N) = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\lambda^+_e(G_S)}(x_0, x_1, \ldots, x_N) x_0^{|\lambda^-_e(G_S)|}.
\]

**Proof.** Let \(A^v\) be the set of all (not necessarily proper) colorings \(\kappa : V(G) \to \{0, 1, \ldots, N\}\) where \(\kappa(v) = 0\). For each edge \(e\), let \(A^v_e\) be the set of \(\kappa \in A^v\) that assign the same color to the two endpoints of edge \(e\). For each \(S \subseteq E(G)\), we can count the weighted set \(\bigcap_{e \in S} A^v_e\) as follows. Write \(\lambda^+_e(G_S) = k\), where \(k\) is the size of the component \(C^v_e(G_S)\) of \(G_S\) containing the root \(v\). Write \(\lambda^-_e(G_S) = (\lambda_1, \ldots, \lambda_m)\), where each \(\lambda_i\) is the size of another component \(C_i\) of \(G_S\). We build a coloring in \(\bigcap_{e \in S} A^v_e\) as follows. Color all \(k\) vertices in \(C^v\) with color 0 (as we must), giving a weight contribution of \(x_0^k\). For each remaining component \(C_i\), choose the common color for all \(\lambda_i\) vertices in that component. The weight contribution for that choice is \(x_0^{\lambda_1} + x_1^{\lambda_1} + \cdots + x_N^{\lambda_1} = p_{\lambda_1}(x_0, x_1, \ldots, x_N)\). By the Product Rule, the generating function for \(\bigcap_{e \in S} A^v_e\) is \(p_{\lambda^+_e(G_S)}(x_0, x_1, \ldots, x_N) x_0^{|\lambda^-_e(G_S)|}\). Equation (20) follows at once from the Inclusion–Exclusion Formula, since \(X_0(G^*_v)\) counts colorings in \(A^v\) outside \(\bigcup_{e \in E(G)} A^v_e\). □

5. Other Variants of Stanley’s Chromatic Symmetric Functions

In this section, we review some of the previously studied polynomials and algebraic constructs related to the chromatic symmetric function. When applicable, we explain the connection between these objects and the polynomials \(X_0(G^*_v)\).

5.1. **Pawlowski’s Pointed Chromatic Symmetric Functions.** Given a rooted graph \(G^*_v\) with root \(v\) and \(S \subseteq E(G)\), Pawlowski’s pointed chromatic symmetric function is defined in [30] Def. 3.1] as

\[
P_{G,v} = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\lambda^+_e(G_S)} x_0^{\lambda^+_e(G_S) - 1} \in \Lambda[z],
\]
where \( p_\lambda = 1 \) if \( \lambda \) has no positive parts. The polynomial version using variables \( x_1, \ldots, x_N \) is

\[
P_{G,v}(x_1, \ldots, x_N) = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\lambda_S}(G_S)(x_1, \ldots, x_N) z^{\lambda_v(G_S)-1} \in \Lambda_N[z].
\]

Let \( x_0 = z \). Comparing (21) and (20), we see that \( X_0(G^*_v; z, x_1, \ldots, x_N) \) is the transformation of \( P_{G,v} \) obtained by replacing each abstract power-sum \( p_k \) by \( z^k + x_1^k + \cdots + x_N^k \) (as opposed to \( x_1^k + \cdots + x_N^k \)) and multiplying by \( z \). This extra \( z \) accounts for the subtracted 1 in the exponent of \( z \) in (21). We state this result more formally as follows.

**Proposition 28.** Regard \( X_0(G^*_v) \) and \( P_{G,v} \) as elements of \( \Lambda[z] \), where \( \Lambda = \mathbb{Q}[p_k : k \geq 1] \) is the ring of symmetric functions. Define evaluation homomorphisms \( \phi, \psi \) on \( \Lambda \) by setting \( \phi(p_k) = z^k + p_k \) and \( \psi(p_k) = -z^k + p_k \) for all \( k \). Then \( X_0(G^*_v) = \phi(z P_{G,v}) \) and \( z P_{G,v} = \psi(X_0(G^*_v)) \).

**Example 29.** For \( G^*_v = \bigcirc \bigcirc \bigcirc \), we use Equation (21) to compute

\[
P_{G,v} = (p_{1111} - 2p_{2111} + p_{31}) z^0 + (-2p_{1111} + 2p_{21} - p_3) z^1 + (3p_{11}) z^2 + (-3p_{1}) z^3 + z^4.
\]

Replacing each \( p_k \) by \( z^k + p_k \) and simplifying, we obtain

\[
X_0(G^*_v) = (p_{1111} - 2p_{2111} + p_{31}) z + (2p_{1111} - 2p_{21}) z^2 + p_{11} z^3.
\]

Notice that the coefficient of \( z = x_0 \) in \( X_0(G^*_v) \) and \( z P_{G,v} \) is the same; this holds in general.

Pawlowski proved [30, Cor. 3.6] that replacing \( z \) by \( -z \) in \( P_{G,v} \) gives a positive sum of monomials. He asked for a combinatorial interpretation of the coefficient of \( (-z)^k \) in \( P_{G,v} \). We give one such interpretation in Theorem 31 below. First we need the following definitions. Suppose \( H \) is a connected graph with \( k + 1 \) vertices. Define

\[
f(H) = \sum_{S \subseteq E(H); H_S \text{ is connected}} (-1)^{|S|-k},
\]

which is a signed sum of all connected spanning subgraphs of \( H \). (Recall that \( H_S \) has vertex set \( V(H) \) and edge set \( S \).) The following combinatorial interpretation of \( f(H) \) shows that \( f(H) \) is always a positive integer.

**Lemma 30.** Let \( H \) be a connected graph with \( k+1 \) vertices. Fix a total ordering \( e_1 < e_2 < \cdots < e_m \) of the edge set \( E(H) \). Say that a spanning tree \( T \) of \( H \) is special iff for every cycle in \( H \) such that all but one edge \( e_j \) of the cycle belongs to \( T \), the missing edge is not the largest edge in the cycle. Then \( f(H) \) is the number of special spanning trees of \( H \).

**Proof.** Let \( Z = \{ S \subseteq E(H) : H_S \text{ is connected} \} \), and let the sign of \( S \) be \( (-1)^{|S|-k} \) for each \( S \in Z \). It suffices to define a sign-reversing involution \( I : Z \rightarrow Z \) whose fixed points are positive and correspond to the special spanning trees of \( H \). Given \( S \in Z \), compute \( I(S) \) by the following algorithm. Process the edges \( e_1, e_2, \ldots, e_m \) in the given order. When edge \( e_j \) is reached, do the following. If \( e_j \in S \) and \( e_j \) is the largest edge in some cycle of \( H_S \), then return \( I(S) = S \setminus \{e_j\} \). If \( e_j \notin S \) and adding \( e_j \) to \( S \) creates a cycle with largest edge \( e_j \), then return \( I(S) = S \cup \{e_j\} \). Otherwise, proceed to the next edge. If no output is returned after processing all edges, then return \( I(S) = S \).

Since \( I \) acts by deleting or adding one edge from \( S \), \( I \) reverses signs when acting on a non-fixed point. To see that \( I \) is an involution, suppose we compute \( I(S) = S' \) by returning an answer when edge \( e_j \) is processed. The decisions made when processing edges \( e_1, \ldots, e_{j-1} \) do not depend on whether \( e_j \) is or is not in \( S \). So, when we compute \( I(S') \), the same decisions will be made for edges \( e_1, \ldots, e_{j-1} \) (namely, continue without returning an output). When the algorithm processes edge \( e_j \), it returns \( I(S') = S \), as needed.
Now consider the fixed points of $I$. On one hand, let $S \in Z$ correspond to a spanning subgraph of $H$ that is not a tree. This subgraph has a cycle, so there exists an edge $e_j$ that is the largest edge in some cycle of $H_S$. Then $I$ matches $S$ to some $S'$ of opposite sign when processing $e_j$ (or perhaps earlier). On the other hand, by comparing the definition of special spanning tree to the definition of $I$, we see that the edge set $S$ of a spanning tree of $H$ is a fixed point of $I$ iff this spanning tree is special. All such trees have $k$ edges and therefore have positive sign. □

The next theorem shows that the coefficient of $(-z)^k$ in $P_G,v$ is the weighted sum of pairs $(T,\kappa)$, where $T$ is a special spanning subtree of a $(k + 1)$-vertex induced subgraph of $G$ containing the root $v$, and $\kappa$ is a proper coloring of the vertices of $G$ not in this subgraph.

**Theorem 31.** Let $G$ be a graph with root vertex $v$. For all $k$, the coefficient of $(-z)^k$ in $P_G,v$ is

\[
\sum_{H} f(H) X_{G \setminus H},
\]

where we sum over all $H = (V(H), E(H))$ satisfying the following conditions: $H$ is a connected graph, $|V(H)| = k + 1$, $v \in V(H)$, and $E(H)$ consists of all edges in $E(G)$ joining vertices in $V(H)$. Here, $G \setminus H$ is the graph obtained from $G$ by deleting all vertices in $H$ and all edges incident to those vertices.

**Proof.** Fix $k$, and let $\mathcal{H}$ be the set of all graphs $H$ satisfying the conditions in the theorem statement. Let $S$ be the set of all $S \subseteq E(G)$ such that $\lambda^+_1(G_S) - 1 = k$. By the defining formula \([21]\), the coefficient of $(-z)^k$ in $P_G,v$ is

\[
(-1)^{|S| - k} p_{\lambda^-_1(G_S)},
\]

Define a function $h : S \to \mathcal{H}$ by letting $h(S) = H$, where $V(H)$ is the connected component of $v$ in $G_S$. For each $H \in \mathcal{H}$, we can build all $S \in S$ with $h(S) = H$ as follows. Choose $S_1 \subseteq E(H)$ such that $H_{S_1}$ is connected, choose $S_2 \subseteq E(G \setminus H)$ arbitrarily, and let $S = S_1 \cup S_2$. By definition, $p_{\lambda^-_1(G_S)} = p_{\lambda((G \setminus H),S_2)}$. Therefore, equation \((25)\) can be rewritten as

\[
\sum_{H \in \mathcal{H}} \left( \sum_{S \subseteq E(G \setminus H) : H_{S_1} \text{ is connected}} (-1)^{|S_1| - k} \right) \cdot \left( \sum_{S_2 \subseteq E(G \setminus H)} (-1)^{|S_2|} p_{\lambda((G \setminus H),S_2)} \right).
\]

By Lemma \([30]\), the sum over $S_1$ equals $f(H)$. By \([19]\), the sum over $S_2$ equals $X_{G \setminus H}$. □

When $G$ itself is a tree, the subgraphs $H$ in Theorem \([31]\) are also trees. In this case, all the multipliers $f(H)$ are 1. We see that for trees $T$, the coefficient of $z^k$ in each of $X_0(T^w)$ and $P_{T,v}(-z)$ is a weighted sum of proper colorings of certain subgraphs of $T$. For $X_0(T^w)$, these subgraphs are complements of independent sets containing $v$ (Proposition \([20]\)(a)). For $P_{T,v}$, these subgraphs are complements of connected subgraphs containing $v$.

### 5.2. Rooted $U$-Polynomials

The pointed chromatic symmetric functions are closely related to another family of polynomials called rooted $U$-polynomials \([4]\). To describe these, we first review the $W$-polynomials and $U$-polynomials of Noble and Welsh \([28]\). A weighted graph is a pair $(G,\omega)$ where $G$ is a graph (possibly containing loop edges or multiple edges with the same endpoints) and $\omega : V(G) \to \mathbb{Z}_{>0}$ is a weight function on the vertex set of $G$. Writing $x$ for the sequence of commuting indeterminates $x_1, x_2, \ldots$, the $W$-polynomials $W_{(G,\omega)}(x, y)$ are defined recursively using a version of the deletion-contraction recursion. As an initial condition, if $G$ has no edges, then $W_{(G,\omega)}(x, y) = \prod_{v \in V(G)} x_{\omega(v)}$. If $G$ has a loop edge $e$, then $W_{(G,\omega)}(x, y) = y W_{(G-e,\omega)}(x, y)$, where $G-e$ is the graph $G$ with loop $e$ deleted. If $G$ has an edge $e$ with distinct endpoints $v_i$ and $v_j$, then

\[
W_{(G,\omega)}(x, y) = W_{(G-e,\omega)}(x, y) + W_{(G,e,\omega_e)}(x, y),
\]
where $G_e$ is the contraction of $G$ along $e$ (i.e., delete $e$ and identify $v_i$ and $v_j$ as a new vertex $v_{ij}$), $\omega_e(v_{ij}) = \omega(v_i) + \omega(v_j)$, and $\omega_e(v) = \omega(v)$ for all $v \not\in \{v_i, v_j\}$. Note that contraction of an edge could lead to a graph with a loop edge or multiple edges with the same endpoint. When $G$ is a forest, the variable $y$ does not appear in $W_{(G, \omega)}$.

**Remark 32.** In 1994, Chmutov, Duzhin, and Lando [9] independently defined functions on weighted graphs (in the context of Vassiliev knot invariants) that are essentially equivalent to the $W$-polynomials of Noble and Welsh with $y = 0$. Equation (19) is a specialization of an equation appearing at the beginning of Section 2.2 in [9].

The $U$-polynomial for a graph $G$ is $U_G(x, y) = W_{(G, 1)}(x, y)$, where $1$ is the weight function sending each $v \in V(G)$ to 1.

**Example 33.** Let $\omega$ be the weight function that assigns 3 to the leftmost vertex of $T = \circ \circ \circ \circ$ and 1 to the other two vertices. One can check that $W_{(T, \omega)}(x, y) = x_1^2x_3 + x_2x_3 + x_1x_4 + x_5$ and $U_T(x, y) = x_1^3 + 2x_1x_2 + x_3$.

The $U$-polynomials can also be defined directly on unweighted graphs $G$ as an alternating sum over subsets of the edge set, as in (19). Define the rank of $S \subseteq E(G)$ to be $r(S) = |V(G)| - k(G_S)$, where $k(G_S)$ is the number of connected components of $G_S$. Then

$$U_G(x, y) = \sum_{S \subseteq E(G)} x_{\lambda(G_S)}(y - 1)^{|S| - r(S)}, \tag{26}$$

where $x_{\lambda} = \prod_i x_{\lambda_i}$. When $G$ is a forest, $k(G_S) = |V(G)| - |S|$, so $r(S) = |S|$ and the power of $y - 1$ disappears. For any loopless graph $G$, we can recover $X_G$ from $U_G$ by setting $y = 0$ and $x_i = -p_i$ for all $i > 0$ and multiplying by $(-1)^{|V(G)|}$ (see [28, Theorem 6.1]).

The rooted $U$-polynomial for the graph $G$ rooted at $v$ is defined in [4] as

$$U^r(G, v; x, y, z) = \sum_{S \subseteq E(G)} x_{\lambda^r(G_S)}z^{|\lambda^r(G_S)|}y^{-1}z^{-r(S)}, \tag{27}$$

Note that (27) modifies (26) in the same way that (21) modifies (19), up to a factor of $z^{-1}$. In particular, Pawlowski's pointed chromatic symmetric function can be recovered from the rooted $U$-polynomial by setting $y = 0$ and $x_i = -p_i$ for all $i > 0$ and multiplying by $(-1)^{|V(G)| + 1}z^{-1}$. (An extra sign is needed since $\lambda^r(G_S)$ has one fewer part than $\lambda(G_S)$.)

**Example 34.** For the rooted graph $\circ \circ \circ$, we compute $U^r(G, v; x, y, z) = x_1^2z + 2x_1z^2 + z^3$.

Corollary 10 of [4] proves that the rooted $U$-polynomials distinguish isomorphism classes of rooted trees. Combining this with Proposition 28 leads to a second proof of Theorem 15. In detail, suppose rooted trees $T_s$ and $S_s$ have $X_0(T_s) = X_0(S_s)$. Then $P_{T_s} = P_{S_s}$ by Proposition 28. Since $y$ does not appear in the rooted $U$-polynomial for a tree, we can reverse the transformation above to conclude that $U^r(T, v; x, y, z) = U^r(S, w; x, y, z)$. Hence, $T_s$ and $S_s$ are isomorphic by [4, Cor. 10].

5.3. **Noncommutative Chromatic Symmetric Functions.** In 2001, Gebhard and Sagan introduced a noncommutative version of the chromatic symmetric function [19] (see also the more recent [11]). Let $x_1, x_2, \ldots$ be noncommuting indeterminates. Given a graph $G$, let $v_1 < v_2 < \cdots < v_n$ be a fixed total ordering of $V(G)$. The weight of a proper coloring $\kappa : V(G) \to \mathbb{Z}_{>0}$ is $x_{\kappa(v_1)}x_{\kappa(v_2)}\cdots x_{\kappa(v_n)}$. The noncommutative chromatic symmetric function $Y_G$ is the sum of the weights of all proper colorings. Clearly, $Y_G$ reduces to $X_G$ when the variables are allowed to commute. A primary motivation for introducing $Y_G$ is that $Y_G$ satisfies a deletion-contraction recursion, which is based on an operation called induction. Proposition 8.2 of [19] shows that two simple graphs $G$ and $H$ are isomorphic if and only if $Y_G = Y_H$. 

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Example 35. Using a left-to-right ordering of the vertices, we have 
\[ Y_{\text{o-o-o}} = \sum_{i,j,k} x_i x_j x_k + \sum_{i,j} x_i x_j x_i, \]
where the indices \(i, j, k\) in the two sums are required to be distinct.

Let \( G_* \) be a rooted graph with root \( r \), and suppose the chosen total ordering \( v_1 < \cdots < v_n \) of \( V(G) \) satisfies \( v_1 = r \). We can recover \( X_0(G_*; x_0, x_1, \ldots, x_N) \) from \( Y_G(x_0, x_1, \ldots, x_N) \) as follows: discard all monomials in \( Y_G \) that do not begin with \( x_0 \), and then allow all indeterminates \( x_i \) to commute.

Each of the functions \( X_G, X_0(G_*), \) and \( Y_G \) reveals certain information about the proper colorings of \( G \). Each monomial in \( X_G \) tells us the multiset of colors used in the associated coloring \( \kappa \). The monomials in \( X_0(G_*) \) contain similar information, but now we are also told that one specific vertex (the root) has color 0. The monomial for \( \kappa \) in \( Y_G \) tells us much more, since we can recover \( \kappa \) itself from the monomial \( x_{\kappa(v_1)} \cdots x_{\kappa(v_n)} \). For any graph \( G \), it is easy to reconstruct the edge set \( E(G) \) from the information in \( Y_G \) [19, Prop. 8.2]. Stanley’s Conjecture (still open) asks whether a tree \( T \) can be reconstructed from the information in \( X_T \). Theorem [15] shows that a rooted tree \( T_* \) can be recovered from \( X_0(T_*) \). This result is tantalizing because \( X_0(T_*) \) seems so much closer to \( X_T \) compared to \( Y_T \).

5.4. Chromatic Quasisymmetric Functions. Shareshian and Wachs [31] studied another variation of \( X_G \) involving \( \text{QSym} \), the ring of quasisymmetric functions in commuting indeterminates \( x_1, x_2, \ldots \). Fix a total ordering \( v_1 < v_2 < \cdots < v_n \) of \( V(G) \). Given a proper coloring \( \kappa \) of \( G \), define \( \text{asc}(\kappa) \) to be the number of \( i < j \) such that there is an edge in \( G \) from \( v_i \) to \( v_j \) and \( \kappa(v_i) < \kappa(v_j) \).

The Shareshian–Wachs chromatic quasisymmetric function is
\[
X_G(x, t) = \sum_{\kappa} t^{\text{asc}(\kappa)} \text{wt}(\kappa),
\]
where we sum over all proper colorings \( \kappa : V(G) \to \mathbb{Z}_{\geq 0} \). It can be shown that \( X_G(x, t) \in \text{QSym}[t] \), meaning that the coefficient of each \( t^m \) is a quasisymmetric function. The specialization \( X_G(x, 1) \) is Stanley’s symmetric function \( X_G \).

Example 36. Let \( M_{\alpha} \) be the monomial quasisymmetric function indexed by the composition \( \alpha \). Using a left-to-right ordering of the vertices, we find
\[
X_{\text{o-o-o}}(x, t) = M_{111} + (M_{21} + M_{12} + 4M_{111})t + M_{111}t^2.
\]
Setting \( t = 1 \) gives \( X_G = m_{21} + 6m_{111} \).

In a related construction, Hasebe and Tsujie [21] define the strict order quasisymmetric function of a poset \( P \), as follows. Let \( \text{Hom}^< (P, \mathbb{Z}_{\geq 0}) \) be the set of functions \( f : P \to \mathbb{Z}_{\geq 0} \) such that for all \( u, v \in P \), \( u < v \) in \( P \) implies \( f(u) < f(v) \) in \( \mathbb{Z}_{\geq 0} \). Define
\[
\Gamma^<(P, x) = \sum_{f \in \text{Hom}^< (P, \mathbb{Z}_{\geq 0})} \prod_{v \in P} x_{f(v)}.
\]
A rooted tree \( T_* \) has an associated poset \( P(T_*) \), where \( u \leq v \) in \( P(T_*) \) iff the unique path in \( T_* \) from the root to \( v \) passes through \( u \). \( \Gamma^<(P(T_*), x) \) sums over the subset of proper colorings of \( T_* \) where the colors strictly increase following any path away from the root.

Example 37. We have \( \Gamma^< (\text{o-o-o}, x_0, x_1, x_2) = x_0 x_1^2 + 2x_0 x_1 x_2 + x_0 x_2^2 = X_0 (\text{o-o-o}) \). Equality holds here only because every path from the root has length 1 and the number of colors equals the number of vertices. Note that \( \Gamma^< (\text{o-o-o}, x_0, x_1, x_2) = x_0 x_1 x_2 \), which does not equal \( X_0 (\text{o-o-o}; x_0, x_1, x_2) \) as given in [5].

Hasebe and Tsujie show that \( \Gamma^< \) is a complete invariant for rooted trees [21, Theorem 1.3]. Their proof is structurally similar to our proof of Theorem [15] relying on \( \text{QSym} \) being a unique factorization domain and on the irreducibility of various polynomials appearing in the recursion. In
a related vein, Elzey [15, 16] extended the chromatic symmetric function to directed graphs. In [6], Aval, Djenabou, and McNamara build on this work and generalize some of the work of Hasebe and Tsujie.

Let us informally compare the information encoded in \(X_0(T_*)\), \(X_T(x; t)\), and \(\Gamma^<(P(T_*))\) for a rooted tree \(T_*\) with root \(r\). Assume \(X_T(x; t)\) is computed using a total ordering of \(V(G)\) compatible with the poset structure on \(T_*\) (so \(r\) must be \(v_1\)). Each monomial in \(\Gamma^<(P(T_*))\) records the multiset of colors used by a proper coloring, but we also know that colors must strictly increase along all paths leading away from the root. The restriction on which colorings are allowed for \(X_0(T_*)\) is much weaker: we only require that the root itself receive the smallest available color (color 0). \(X_T(x; t)\) contains even more information than \(\Gamma^<(P(T_*))\), since \(\Gamma^<(P(T_*))\) is none other than the coefficient of \(t^{|E(G)|}\) in \(X_T(x; t)\) given our assumption on the total ordering of \(V(G)\).

We can deduce that \(X_T(x; t)\) distinguishes rooted trees, in the following sense. Suppose \(T_*\) and \(U_*\) are two \(n\)-vertex rooted trees with unlabeled vertices. In each tree, label the vertices 1, 2, \ldots, \(n\) in a way that is compatible with the poset structure on each tree. Suppose \(X_T(x; t) = X_U(x; t)\) using the vertex order 1 < 2 < \cdots < \(n\). Then \(\Gamma^<(P(T_*)) = \Gamma^<(P(U_*))\) by taking the coefficient of \(t^{n-1}\) on both sides. By Hasebe and Tsujie’s result, \(T_*\) is isomorphic to \(U_*\).

### 5.5. Other Variations for Weighted Graphs

For completeness, we mention a few other variants of \(X_G\) for weighted graphs. In [11], the authors define extended chromatic functions for weighted graphs (cf. earlier overlapping results of [9]). Given a graph \(G\) and a weight function \(\omega : V(G) \to \mathbb{Z}_{\geq 0}\), define the weight of a proper coloring \(\kappa\) of \(G\) (relative to \(\omega\)) to be \(\prod_{v \in V(G)} \omega^v(\kappa(v))\). The extended chromatic function \(X_{(G, \omega)}(x)\) is the sum of the weights of all proper colorings of \(G\). These functions satisfy a deletion-contraction recursion and can be recovered from the \(W\)-polynomial. They generalize Stanley’s chromatic symmetric function \(X_G\) as well as the Shareshian–Wachs chromatic quasisymmetric functions. In [3], the authors work with a more general notion of vertex-weighting and define \(M\)-polynomials for marked graphs. They also consider a specialization of \(M\)-polynomials called \(D\)-polynomials. Another generalization, the \(V\)-polynomial, is considered in [4].

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