Model uncertainty is a type of inevitable financial risk. Mistakes on the choice of pricing model may cause great financial losses. In this paper we investigate financial markets with mean-volatility uncertainty. Models for stock markets and option markets with uncertain prior distribution are established by Peng’s G-stochastic calculus. The process of stock price is described by generalized geometric G-Brownian motion in which the mean uncertainty may move together with or regardless of the volatility uncertainty. On the hedging market, the upper price of an (exotic) option is derived following the Black-Scholes-Barenblatt equation. It is interesting that the corresponding Barenblatt equation does not depend on the risk preference of investors and the mean-uncertainty of underlying stocks. Hence under some appropriate sublinear expectation, neither the risk preference of investors nor the mean-uncertainty of underlying stocks pose effects on our super and subhedging strategies. Appropriate definitions of arbitrage for super and subhedging strategies are presented such that the super and subhedging prices are reasonable. Especially the condition of arbitrage for sub-hedging strategy fills the gap of the theory of arbitrage under model uncertainty. Finally we show that the term $K$ of finite-variance arising in the super-hedging strategy is interpreted as the max Profit&Loss of being short a delta-hedged option. The ask-bid spread is in fact the accumulation of summation of the superhedging $P&L$ and the subhedging $P&L$.

**Keywords:** mean-volatility uncertainty, no arbitrage, option pricing, risk-neutral valuation, P&L, overestimation, uncertainty volatility model, G-expectation, G-Brownian motion

**Mathematics Subject Classification 2010:** 91G20, 91B24, 91B26, 91B28, 91G80, 60H05, 60H10, 60H30

**JEL classification codes:** G13, D81, C61.

### 1. INTRODUCTION

Mathematical models have come to play an important role in pricing and hedging derivative instruments since Black and Scholes’ seminal work ([Black and Scholes (1973)]). The Black-Scholes option pricing formula has been used extensively, even to evaluate options whose underlying asset (e.g. the stock) is known to not satisfy the Black-Scholes hypothesis of a constant volatility. We go about our work as if we are correct, we often
treat parameters as if we think they are. And yet in the strict sense of word we do not know how much we do not know. Unknown parameters, typically, mean and volatility uncertainty lead to model risk\footnote{Model risk is the risk of error in our estimated risk measure due to inadequacies in our risk models (Dowd (2005)). Model uncertainty leads to a kind of model risk. Ambiguity on volatility is a typical case of model uncertainty.} or model uncertainty. Model risk is an inescapable consequence of model use. It is often hidden or glossed over and is often overlooked. A failure to consider model risk can lead a firm to disaster, and sometimes has, as pointed in Cont (2006).

A typical case of model risk is the choice of probabilistic models. Often a decision maker or a risk manager is not able to attribute a precise probability to future outcomes. This situation has been called uncertainty by Knight (1921). Knight uncertainty sometimes is used to designate the situation where probabilities are unknown. Alternatively, we speak of \textit{ambiguity} when we are facing several possible specifications $P_1, P_2, \ldots$ for probabilities on future outcomes (Epstein (1999)). Ambiguity aversion has shown to have important consequences in macroeconomics (Hansen, Sargent and Tallarini (1999)) and for price behavior in capital markets (Chen and Epstein (2002); Epstein and Wang (1995); Routledge and Zin (2009)). In this circumstance, fair option values and perfectly replicating hedges cannot be determined with certainty. The existence of volatility risk in derivative trading is a concrete manifestation of market incompleteness.

The problem of model uncertainty has long been recognized in economics and finance. Dow and Werlang (1992) studied a single period portfolio choice problem employing the uncertainty averse preference model developed by Schmeidler (1989). Epstein and Wang (1994) and Chen and Epstein (2002) studied the implications for equilibrium asset prices in the representative agent economics. Cash-subadditive risk measures with interest ambiguity was studied in El Karoui, and Ravanelli (2009). Xu (2014) investigated multidimensional risk measures under multiple priors. See also Epstein and Wang (1995), Gundel (2005), Riedel (2009) and references therein for more papers on model uncertainty and multi-prior model. We do not list them all here. Note that in existing works on model uncertainty (Chen and Epstein (2002); El Karoui, and Ravanelli (2009); Epstein and Wang (1995); Gundel (2005); Xu (2014)), all probability measures $P \in \mathcal{P}$ are assumed to be equivalent to a reference probability $P_0$. This technical requirement is actually quite restrictive: it means all model agree on a universe scenario and only differ on their probabilities. An example of diffusion model with uncertain volatility (Avellaneda, Levy and Paras (1995); Cont (2006); Lyons (1995)) does not verify this hypothesis. Recent explorations include Vorbrink (2010), Nutz and Soner (2010), Epstein and Ji (2011, 2013) and Eberlein, Madan, Pistorius and Yor (2014); Eberlein, Madan, Pistorius, Schoutens and Yor (2014); Madan (2012); Madan and Schoutens (2012).

Volatility of a financial market is difficult to predict. Although we have lots of historical data within hand, the volatility might move as large as she wants and seems to be quite sensitive to new information. One could approximate short-period volatility but never the long-term one. There are too many factors determining volatility. Sometimes we assume that the volatility is driven by stochastic elements, e.g. itself is a diffusion process. Such a model is called stochastic volatility model (Heston (1993)). It often has
several parameters which can be chosen either to fit historical data or calibrate to the market. A robust choice to the problem of modeling the unknown volatility is to treat it as uncertain as it actually is. We just stand on two bounds $\sigma$ and $\bar{\sigma}$ to deduce prices representing worst-case scenario and best-case scenario respectively. The interval $[\sigma, \bar{\sigma}]$ characterizes the uncertain level of volatility. Larger interval, larger fluctuation of volatility. Also this interval depends on investor’s preference or aversion of risk. A conservative investor may establish a large interval and choose the minimal superstrategy. However a too large interval yields such a high superstrategy that it is meaningless.

We now recall the uncertain volatility model introduced in Avellaneda, Levy and Paras (1995). For simplicity, we only consider derivative securities based on a single liquidly traded stock which pays no dividends over the contract’s lifetime. The paths followed by future stock prices are assumed to be Itô process,

$$dS_t = S_t (u_t dt + \sigma_t dW_t).$$

where $(u_t)$ and $(\sigma_t)$ are adapted processes such that

$$\sigma \leq \sigma_t \leq \bar{\sigma},$$

where $(W_t)$ is the standard Brownian motion under a given probability space $(\Omega, \mathcal{F}, P)$. The constants $\bar{\sigma}$ and $\sigma$ represent upper and lower bounds of the volatility that should be input to the model according to the investor’s expectation and uncertainty about future price fluctuations. These two bounds could be statistically obtained from peaks of volatility in historical stock or option-implied volatilities. They can be viewed as determining a confidence interval for future volatility values, as pointed in Avellaneda, Levy and Paras (1995).

Note that two different volatility processes will typically yield mutually singular probability measures on the set of possible paths. So volatility ambiguity leads to model uncertainty with a set of risk-neutral probabilities $\mathcal{P}$, each of them corresponding to a volatility process with value at each time in $[\sigma, \bar{\sigma}]$. Naturally we look for the cheapest superhedging price at which we can sell and manage an option in such environment. A convenient framework is the stochastic control framework, in which the managing volatility is interpreted as a control variable. It turns out that the value function in such an optimal control will yield the cheapest superstrategy price. Nevertheless, the connection between superstrategy problem and stochastic control is not that obvious. Recall that a stochastic control problem is to maximize an expectation over a set of processes, whereas a superstrategy is over a set of probabilities, i.e., $\sup_{P \in \mathcal{P}} E_P$. This issue is avoided in Avellaneda, Levy and Paras (1995), handled partially in Lyons (1995), and more formally in Martini (1997) and Frey (2000). A significant progress toward a general framework is available in Denis and Martini (2006), which can be viewed as a quasi-sure stochastic analysis. See also Soner, Touzi and Zhang (2010a) addressing conditioning or updating which is a crucial ingredient in modeling dynamic pricing.

Peng (2006, 2007b, 2008a) established a path analysis, called G-stochastic analysis, which extends the classical Wiener analysis to a framework of sublinear expectation on
events field $\Omega = C_0([0, +\infty), \mathbb{R}^d)$, the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$ with $\omega_0 = 0$, equipped with a uniform norm on compact subspaces. Notions such as G-normal distribution, G-Brownian motion, G-expectation were introduced (see Appendix A or Peng’s review paper, Peng (2009), and summative book, Peng (2010a)).

The representation for G-expectation (Hu and Peng (2009)),

$$\mathbb{E} [\cdot] = \sup_{P \in \mathcal{P}} \mathbb{E}_P [\cdot]$$

tells us that G-expectation induces a set of probabilities $\mathcal{P}$ naturally.\(^2\) It is shown that $(B_t)$ is a martingale under every $P \in \mathcal{P}$ (Nutz and Soner (2010); Soner, Touzi and Zhang (2011a)) and there exists a unique adapted process $(\sigma^P_t)$ such that $\sigma^2 \leq |\sigma^P_t|^2 \leq \sigma^2$, a.a.\(^3\) $t$, $P$-a.s. and

$$B_t = \int_0^t \sigma^P_s dW^P_s, \forall t \geq 0, P$-a.s.$$

where $(W^P_t)$ is a standard $E_P$–Brownian motion. Therefore an interesting phenomenon comes up: the quadratic variance of $(B_t)$ under any $P \in \mathcal{P}$,

$$\langle B \rangle^P_t = \int_0^t |\sigma^P_s|^2 ds, \forall t \geq 0, P - a.s.$$

is no longer a deterministic function of time $t$.

All results in G-stochastic analysis work in a model-free way: They hold under all probabilities $P \in \mathcal{P}$ or quasi-surely (q.s.), i.e. a property holds outside a polar set $A$ with $P(A) = 0$ for all $P \in \mathcal{P}$. As pointed in Peng’s ICM\(^4\) lecture (Peng (2010b)), G-expectation may appear as a natural candidate to measure volatility risk. In this direction, initial work has been done by Vorbrink (2010) in which the main focus is on the no-arbitrage argument. Based on the work of Karatzas and Kou (1996), Vorbrink adapted the notion of absence of arbitrage from the market with constraints on portfolio choice to the framework of uncertain volatility model. The risk premium of portfolio is not considered when modeling the wealth process. Thus technical difficulties to change the subjective risk preference to a risk-neutral world are avoided. More recently, recursive utility is studied by Epstein and Ji (2011, 2013) accommodating mean and volatility ambiguity. They applied the model to a representative agent endowment economy to study equilibrium asset returns in both Arrow-Debreu style and sequential Radner-style economies. Madan (2012) presents an equilibrium model for two-price economics in which the market clearing condition is defined. See also Eberlein, Madan, Pistorius and Yor (2014); Eberlein, Madan, Pistorius, Schoutens and Yor (2014); Madan and Schoutens (2012) for related results using the theory of G-expectation.

The present paper considers mean-volatility uncertainty simultaneously. As pointed later in the next section, mean-uncertainty occurs often with volatility uncertainty. The

---

\(^2\) $\mathcal{P}$ is a weekly compact set. Recently Bion-Nadal and Kervarec (2012) showed that there is a numerable weakly relatively compact set $\{P_n, n \in \mathbb{N}\} \subset \mathcal{P}$ such that the above representation still holds.

\(^3\) a.a.: almost all; a.s.: almost surely; a.e.: almost everywhere.

\(^4\) International Congress of Mathematicians
stock price is modeled as a generalized geometric G-Brownian motion in which mean-uncertainty may move without regarding to the volatility-uncertainty. Section 3 derives the superhedging PDEs for both state-dependent and discrete-path-dependent options. What is interesting is that, neither the preference of investors nor the mean-uncertainty appear in the superhedging PDEs, which demonstrates that risk-neutral measures exist indeed in such ambiguous environment. Section 4 extends the classical Black-Scholes-Merton model to the uncertain volatility case. A superhedging strategy is just a solution of a backward stochastic differential equation driven by G-Brownian motion (G-BSDE for short). It is shown that the solution is the minimal superstrategy with no-arbitrage. In particular, conditions of arbitrage for substrategy are given which are essentially different from Vorbink’s work. The finite-variance term $K$ is interpreted as Profit&Loss (P&L for short) of an investor. Recall of Peng’s G-stochastic analysis and some technical results are arranged in the appendix.

2. MEAN-VOLATILITY UNCERTAINTY OF STOCKS

2.1. Volatility-uncertainty brings mean-uncertainty

We assume the price process of a stock satisfies the following linear stochastic differential equation (SDE for short):

$$dS_t = S_t (\mu dt + dB_t),$$

where $(B_t)$ is a G-Brownian motion. Define the continuously compounded rate of return per annum realized between 0 and $T$ as $\zeta$. It follows that

$$S_T = S_0 e^{\zeta T}$$

and

$$\zeta = \frac{1}{T} \ln \frac{S_T}{S_0} = \frac{1}{T} \left\{ BT + \mu T - \frac{1}{2} \langle B \rangle T \right\}.$$ 

So the mean of expected continuously compounded rate of return will fluctuate within $[\mu - \frac{1}{2}\sigma^2, \mu - \frac{1}{2}\sigma^2]$. We do not consider any ambiguity of stock appreciation. However the mean or the expected rate of return is uncertain. $S_t$ is not a symmetric random variable at each time $t$, since we do not necessarily have $E[S_t] = -E[-S_t]$.

Volatility ambiguity leads to model uncertainty, i.e., multi-prior model. Naturally the expected value of stock price $S_t$ may be ambiguous under a set of probabilistic models. This paper will take into account mean-volatility uncertainty simultaneously by Peng’s G-stochastic analysis.

---

5 See Martini and Jacquier (2010), Jacquier and Slaoui (2010) for the notion of P&L.
6 For convenience of writing, in the following of the paper, we will always consider models driven by one dimensional $G$-Brownian motion.
2.2. The process for stock prices

In the classical Black-Scholes-Merton option-pricing model, the price process of a stock is assumed to be Itô process

\[
    dS_t = S_t (\mu_t dt + \sigma_t dW_t),
\]

where \( W \) is the standard Brownian motion under a given linear probability space \((\Omega, \mathcal{F}, P)\); \( \sigma_t \) is the volatility of the stock price; \( \mu_t \) is the expected rate of return.

An application of Itô formula yields

\[
    S_t = S_0 \exp \left\{ \int_0^t \sigma_s dW_s + \int_0^t (\mu_s - \frac{1}{2} \sigma_s^2) ds \right\}.
\]

which is called geometric Brownian motion.

We now consider a stock market with mean-uncertainty and volatility-uncertainty together. We do not have confidence in which direction the expected rate of return \( \mu \) and the volatility \( \sigma \) will move or even their distribution in future but they are sure to change within \([\mu, \mu]\) and \([\sigma, \sigma]\). This uncertain model could be described by finite-variance \( G_{[\mu, \mu]} \)-Brownian motion and zero-mean \( G_{[\sigma^2, \sigma^2]} \)-Brownian motion together. Let \((\beta_t)\) be a finite-variance \( G_{[\mu, \mu]} \)-Brownian motion and \((B_t)\) a zero-mean \( G_{[\sigma^2, \sigma^2]} \)-Brownian motion under a given sublinear expectation \( \mathbb{E} \). Then model (2.1) could be rewritten as

\[
    dS_t = S_t (r dt + d\beta_t + dB_t).
\]

However, we prefer the following modification about the expected rate of returns: let \( r \) be the riskless interest rate. If \( \mu \) varies in \([\mu, \mu]\), then \( \mu - r \) varies in \([\mu - r, \mu - r]\). Let \((\beta_t)\) be a finite-variance \( G_{[\mu - r, \mu - r]} \)-Brownian motion, then (2.2) is in form of

\[
    dS_t = S_t (r dt + d\beta_t + dB_t).
\]

It is important to keep in mind that we do not assume a risk-neutral world in advance in model (2.3). Of course we will see later that there does exist a risk-neutral world in which even the uncertainty of expected returns does not influence our super and sub-hedging strategies.

Particularly taking \( \beta_t = \langle B \rangle_t \) means that the expected returns and volatility move together. See Xu, Shang and Zhang (2011), Osuka (2011) and Beißner (2012), they consider mean-uncertainty of this type. An example is referred in Epstein and Ji (2011) Example 2.4 and Epstein and Schneider (2003) Section 3.1.2 by specifying

\[
    \mu = \mu + z, \sigma^2 = \sigma^2 + \frac{2z}{\gamma},
\]

where \( 0 \leq z \leq \bar{z} \) and \( \mu, \sigma, \bar{z} \) and \( \gamma \) are fixed and known parameters, which means that \( \mu = \mu + \frac{\gamma}{2} (\sigma^2 - \bar{\sigma}^2) \) and yields

\[
    dS_t = S_t \left[ \left( \mu - \frac{\gamma}{2} \sigma^2 \right) dt + \frac{\gamma}{2} d\langle B \rangle_t + dB_t \right]
\]
or equivalently
\[ dS_t = S_t \left( r dt + d\beta_t + dB_t \right), \text{where } \beta_t = \left( \mu - r - \frac{\gamma}{2} \sigma^2 \right) t + \frac{\gamma}{2} \langle B \rangle_t. \]

Illeditsch (2010) showed that such models exist when agents receive bad news of ambiguous precision since bad news lowers both the conditional mean and the conditional variance of returns.

2.3. Approximate evaluation for stocks

The classical price process of stock yields
\[ \ln S_t \sim N \left( \ln S_0 + (r - \frac{1}{2} \sigma^2)T, \sigma \sqrt{T} \right) \]
(see Hull (2009)) where \( N \) is the distribution function of normal distribution. There is a 95% probability that a normally distributed variable has a value with 1.96 standard deviation of its mean. Hence, with 95% confidence under a single \( P \) we have
\[ \ln S_0 + (r - \frac{1}{2} \sigma^2)T - 1.96\sigma \sqrt{T} < \ln S_T < \ln S_0 + (r - \frac{1}{2} \sigma^2)T + 1.96\sigma \sqrt{T}. \]

Typical values of the volatility of a stock are in the range of 20% to 40% per annum and usually we take \( T \leq 1 \). If we define
\[ f_1(\sigma) = -\frac{1}{2} \sigma^2 T - 1.96\sigma \sqrt{T}, \]
\[ f_2(\sigma) = -\frac{1}{2} \sigma^2 T + 1.96\sigma \sqrt{T}, \]
it is easy to check that when \( \sigma \) takes values in \( [\sigma, \bar{\sigma}] \subseteq [0.2, 0.4] \), the function \( f_1 \) is decreasing and \( f_2 \) increasing. If we take the maximum of volatility \( \bar{\sigma} \), we have that
- For any \( P \in \mathcal{P} \), with at least 95% confidence we have
\[ \ln S_0 + (r - \frac{1}{2} \sigma^2)T - 1.96\bar{\sigma} \sqrt{T} < \ln S_T < \ln S_0 + (r - \frac{1}{2} \sigma^2)T + 1.96\bar{\sigma} \sqrt{T}. \]

3. Risk-neutral & Mean-certain valuation

This section derives the superhedging PDEs for both state-dependent and discrete-path-dependent options, which shows the existence of a risk-neutral & mean-certain world in which all investors are hedging without the influence of risk preference and mean-uncertainty.
3.1. State-dependent payoffs

The Black-Scholes equation is derived for state-dependent European options. Now we derive the superhedging PDE within this framework which is easy to understand and comparable with Black-Scholes-Merton’s model. We assume the price process of the stock satisfies the following SDE:

\[ dS_t = S_t (\mu_t dt + \sigma_t dW_t), \]

where \( W \) is the standard Brownian motion under a given linear probability space \((\Omega, \mathcal{F}, P)\); \( \mu \) is the expected rate of return varying in \([\mu, \bar{\mu}]\); \( \sigma \) is the volatility of the stock price varying in \([\sigma, \bar{\sigma}]\). \((\mu_t), (\sigma_t)\) and the riskless interest rate \((r_t)\) are assumed to be deterministic functions of \(t\). Note that we do not assume any relation between \(\mu\) and \(\sigma\). The mean uncertainty may move together with or regardless of the volatility uncertainty, while in Xu, Shang and Zhang (2011), Osuka (2011) and Beißner (2012), they in fact consider the case \(\mu_t = \sigma_t^2\). Note also that we do not assume in advance a risk-neutral world which is different from Jiang, Xu, Ren and Li (2008) and Meyer (2004).

Let \( V(t, S_t) \) be the price of the option with payoff \( \Phi(S_T) \), where \( V \) and \( \Phi \) are both deterministic functions. Assume also that \( V \in C^{1,2}([0,T] \times \mathbb{R}) \).

By Itô’s formula,

\[ dV(t, S_t) = \frac{\partial V}{\partial t} dt + S_t \frac{\partial V}{\partial S} (\mu_t dt + \sigma_t dW_t) + \frac{1}{2} \sigma_t^2 S_t^2 \frac{\partial^2 V}{\partial S^2} dt. \]

The discrete versions of equations (3.1) and (3.2) are

\[ \Delta S_t = S_t (\mu_t \Delta t + \sigma_t \Delta W_t) \]

and

\[ \Delta V_t = \frac{\partial V}{\partial t} \Delta t + S_t \frac{\partial V}{\partial S} (\mu_t \Delta t + \sigma_t \Delta W_t) + \frac{1}{2} \sigma_t^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \Delta t. \]

For a delta-hedging portfolio \( \Pi \), the holder of this portfolio is short one derivative and long an amount \( \frac{\partial V}{\partial S} \) of shares of stocks and \( (V - \frac{\partial V}{\partial S} S) \) cash left in a bank account. Then the P&L variance of the portfolio is

\[ \Delta \Pi_t = \frac{\partial V}{\partial S} \Delta S_t - \Delta V_t + \left(V_t - \frac{\partial V}{\partial S} S_t \right) r_t \Delta t. \]

The first part corresponds to the stock price movements, of which we hold \( \frac{\partial V}{\partial S} \) units, the second one to the price variation of the option, and the third part is the risk-free return of the amount of cash to make the portfolio have zero value. Now, substituting equations (3.3) and (3.4) into (3.5) yields

\[ \Delta \Pi_t = -\frac{\partial V}{\partial t} \Delta t - \frac{1}{2} \sigma_t^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \Delta t + \left(V_t - \frac{\partial V}{\partial S} S_t \right) r_t \Delta t. \]

\(^7\) \( C^{j,k}([0,T] \times \mathbb{R}) \) denotes the set of functions defined on \((0,T) \times \mathbb{R}\) which are \(j\) times differentiable in \(t \in (0,T)\) and \(k\) times differentiable in \(x \in \mathbb{R}\) such that all these derivatives are continuous.
Observe that neither the random noise nor the stock appreciation arise in $\Delta \Pi$ explicitly. If the managing volatility\(^8\) of the option coincides with the realised volatility of stocks, of course, by the principle of no-arbitrage, $\Delta \Pi_t = 0$. However it is unclear which is the realised volatility. The seller of the option wishes to find a cheapest managing policy yielding a non-negative P&L, at least no loss. More precisely, we want to have

$$\inf_{\sigma \leq \sigma_t \leq \bar{\sigma}} \Delta \Pi_t = 0.$$  

Consequently, we deduce that

\[
\frac{\partial V}{\partial t}(t, x) + \frac{1}{2} \sigma^2 \sup_{\sigma \leq \sigma_t \leq \bar{\sigma}} \left\{ \sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t, x) \right\} + r_t x \frac{\partial V}{\partial x}(t, x) - r_t V(t, x) = 0,
\]

$$V(T, x) = \Phi(x).$$

Then by the comparison theorem of PDEs, $V(t, x)$ is the minimal upper price outperforming all $\mu_t$ varying in $[\mu, \bar{\mu}]$ and $\sigma_t$ varying in $[\sigma, \bar{\sigma}]$. There is no novelty in equation (3.6) which is the so called Black-Scholes-Barenblatt (BSB) equation (Avellaneda, Levy and Paras (1995); Barenblatt (1979)). What is new is that, although we put risk preference and uncertainty into stock appreciation $\mu$, the BSB equation does not involve any variables that are affected by the risk preference of investors. $\mu$ depends on risk preference and interval $[\mu, \bar{\mu}]$ determines mean-uncertainty. The higher the level risk and ambiguity aversion by investors, the higher $\mu$ and the larger the uncertain interval will be for any given stock. It is fortunate that $\mu$ happens to drop out in the differential equation. So the risk preference of investors and mean-uncertainty do not pose effects on our superhedging strategy. Thus it is possible to consider risk-neutral & mean-certain valuation under model uncertainty.

**Remark 3.1** Suppose that function $\Phi$ is a bounded continuous function. Assume that $\sigma > 0$. By Krylov (1987) Theorem 6.4.3 or Wang (1992), equation (3.6) has a $C^{1+\frac{\alpha}{2},2+\alpha}(0, T) \times \mathbb{R}$-solution $u(t, x)$ for some $\alpha \in (0, 1)$. The uniqueness can be obtained from Ishii (1989). See also Vargiolu (2001) for smooth solutions with locally Lipschitz terminal condition.

**Remark 3.2** If there is uncertainty for the riskless interest rate, i.e., $r \in [\underline{\tau}, \bar{\tau}]$, then the superhedging PDE should be

\[
\frac{\partial V}{\partial t}(t, x) + \frac{1}{2} \sup_{(r, \sigma) \in [\underline{\tau}, \bar{\tau}] \times [\sigma, \bar{\sigma}]} \left\{ \sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t, x) + r \left( x \frac{\partial V}{\partial x}(t, x) - V(t, x) \right) \right\} = 0.
\]

### 3.2. Discrete-path-dependent payoffs

We now consider the case of discrete-path-dependent payoff $\Phi(X_{t_1}, \ldots, X_{t_N})$ with $\Phi : \mathbb{R}^N \mapsto \mathbb{R}$ for a partition $0 = t_0 < t_1 < \cdots < t_N = T$. A typical example is the

\(^8\)The managing volatility is the volatility at which the option is sold.
arithmetic-mean Asian call $\Phi(X^{(N)}) = \left( \frac{1}{N} \sum_{i=0}^{N} X_{t_i} - K \right)^+, K$ is the fixed strike price. See Shreve (2004) for other kinds of path-dependent options, such as lookback option, barrier option. For any $x := (x_1, \ldots, x_N) \in \mathbb{R}^N$ and $k = 1, \ldots, N$, we use the following notations:

\[ x^{(k)} := (x_1, \ldots, x_k), \quad X_t^{(k)} := (X_{t_1 \wedge t}, \ldots, X_{t_k \wedge t}), \quad X^{(k)} := (X_{t_1}, \ldots, X_{t_k}). \]

Let $X_t$ denote the following stock price

\[ dX_t = X_t (\mu_t dt + \sigma_t dW_t), \tag{3.8} \]

where $W$ is the standard Brownian motion under a given linear probability space $(\Omega, \mathcal{F}, P)$; $\mu_t, \sigma_t : [0, +\infty) \to \mathbb{R}$ valued in $[\mu, \overline{\mu}]$ and $[\sigma, \overline{\sigma}]$ respectively. We now derive the superhedging PDE on each $[t_{k-1}, t_k]$. Let $V^k(t, X^{(k-1)}), t \in [t_{k-1}, t_k]$ be the price of the option with payoff $\Phi(X^{(N)})$. Assume also that $V^k(\cdot, x^{(k-1)}, \cdot) \in C^{1,2}([t_{k-1}, t_k] \times \mathbb{R})$. By Itô’s formula, we have, $\forall t \in [t_{k-1}, t_k]$,

\[ dV^k(t, x^{(k-1)}, X_t) = \frac{\partial V^k}{\partial t}(t, x^{(k-1)}, X_t) dt + X_t \frac{\partial V^k}{\partial x}(t, x^{(k-1)}, X_t) (\mu_t dt + \sigma_t dW_t) + \frac{1}{2} \sigma_t^2 X_t^2 \frac{\partial^2 V^k}{\partial x^2}(t, x^{(k-1)}, X_t) dt. \]

By properties of integrals $dt$ and $dW_t$, we can replace $x^{(k-1)}$ by $X^{(k-1)}$ and get

\[ dV^k(t, X^{(k-1)}, X_t) = \frac{\partial V^k}{\partial t}(t, X^{(k-1)}, X_t) dt + X_t \frac{\partial V^k}{\partial x}(t, X^{(k-1)}, X_t) (\mu_t dt + \sigma_t dW_t) + \frac{1}{2} \sigma_t^2 X_t^2 \frac{\partial^2 V^k}{\partial x^2}(t, X^{(k-1)}, X_t) dt. \]

By an analogous procedure as in Section 3.1, the superhedging price should satisfy

\[ \frac{\partial V^k}{\partial t}(t, x^{(k-1)}, x) + \frac{1}{2} \sup_{2 \leq \sigma \leq \overline{\sigma}} \left\{ \sigma^2 x^2 \frac{\partial^2 V^k}{\partial x^2}(t, x^{(k-1)}, x) \right\} + r_t x \frac{\partial V^k}{\partial x}(t, x^{(k-1)}, x) - r_t V^k(t, x^{(k-1)}, x) = 0 \tag{3.9} \]

The sequence of PDEs $V^k, k = 1, \ldots, N$, is defined recursively in a backward manner. The terminal conditions are defined respectively by

\[ V^N(T, x^{(N-1)}, x) = \Phi(x^{(N-1)}, x), \]

\[ \vdots \]

\[ V^k(t_k, x^{(k-1)}, x) = V^{k+1}(t_k, x^{(k-1)}, x, x). \tag{3.10} \]

As we see, the stock appreciation $\mu$ does not appear in (3.9) due to delta-hedging. (3.9) can be used to super-hedge discrete-path-dependent options. The existence and uniqueness of smooth solutions for (3.9) and (3.10) can be guaranteed by Krylov (1987) and Vargiolu (2001). The randomness of $\mu_t$ and $\sigma_t$ does not influence PDE (3.9).
Remark 3.3 (3.9) has another form:

\[
\frac{\partial V}{\partial t}(t, x^{(k-1)}, x) + \frac{1}{2}\sigma^2 \left( \frac{\partial^2 V}{\partial x^2} (t, x^{(k-1)}, x) \right) x^2 \frac{\partial^2 V}{\partial x^2} (t, x^{(k-1)}, x)
+ r_t x \frac{\partial V^k}{\partial x} (t, x^{(k-1)}, x) - r_t V(t, x^{(k-1)}, x) = 0.
\]

where \( \sigma^2 \left( \frac{\partial^2 V}{\partial x^2} (t, x^{(k-1)}, x) \right) = \sigma, \) if \( \frac{\partial^2 V}{\partial x^2} (t, x^{(k-1)}, x) \geq 0; \sigma^2 \left( \frac{\partial^2 V}{\partial x^2} (t, x^{(k-1)}, x) \right) = \sigma, \) if \( \frac{\partial^2 V}{\partial x^2} (t, x^{(k-1)}, x) < 0. \) As we see above, the form of \( \sigma_t \) does not pose effect on (3.9).

Hence for the uncertain volatility model, we could just stand on the bounds of the interval \([\bar{\sigma}, \bar{\sigma}]\).

3.3. General payoffs

From section 3.2, for discrete-path-dependent payoffs, we can consider risk-neutral valuation. Consider a stock price process whose differential is

\[
dX_t = X_t (r_t dt + dB_t), 0 \leq t \leq T,
\]

where \((B_t)\) a \(G_{\bar{\sigma}^2, \bar{\sigma}^2}\)-Brownian motion, \(r_t\) is the interest rate from \([0, T]\) to \(\mathbb{R}\). For the solution \(V^k(t, x^{(k-1)}, x) \in C^{1,2}([t_{k-1}, t_k] \times \mathbb{R})\) of (3.9), applying Itô’s formula to \(V^k(t, x^{(k-1)}, x)\), and substituting \(x^{(k-1)}\) by \(X^{(k-1)}\) we deduce that for \([t_{k-1}, t_k]\),

\[
\begin{align*}
Y_t^k &= V^k(t, X_t^{(k)}), \\
Z_t^k &= X_t \frac{\partial V^k}{\partial x} (t, X_t^{(k)}), \\
K_t^k &= \frac{1}{2} \int_0^t \sup_{2 \leq \sigma \leq \bar{\sigma}} \left\{ \sigma^2 X_t^2 \frac{\partial^2 V^k}{\partial x^2} (t, X_t^{(k)}) \right\} dt - \frac{1}{2} \int_0^t X_t^2 \frac{\partial^2 V^k}{\partial x^2} (t, X_t^{(k)}) d \langle B \rangle_t
\end{align*}
\]

satisfy the following risk-neutral BSDE:

\[
Y_t = Y_{t_k} - \int_t^{t_k} r_s Y_s ds - \int_t^{t_k} Z_s dB_s + \int_t^{t_k} dK_s, \ t \in [t_{k-1}, t_k],
\]

which coincides with the following BSDE:

\[
Y_t = \Phi(X^{(N)}) - \int_t^T r_s Y_s ds - \int_t^T Z_s dB_s + \int_t^T dK_s, \ t \in [0, T]
\]

on \([t_{k-1}, t_k]\). For a general payoff \(\xi \in L_{\mathbb{G}}^1(\Omega_T)\), it can be approximated by a sequence of \(\Phi^n(X^{(n)}), n = 1, 2, \ldots\), for appropriate diffusion process \((X_t)\). We can check that the following sequence

\[
Y_t^n = \Phi^n(X^{(n)}) - \int_t^T r_s Y_s^n ds - \int_t^T Z_s^n dB_s + \int_t^T dK_s^n, \ t \in [0, T]
\]
converges to

\[
Y_t = \xi - \int_t^T r_s Y_s ds - \int_t^T Z_s dB_s + \int_t^T dK_s, \quad t \in [0, T]
\]

in the space $\mathcal{M}_G^2(0, T)$. So for a general payoff, its risk-neutral superhedging price exists and can be calculated by

\[
V_t = D_t^{-1} \mathbb{E}[D_T \xi | \mathcal{F}_t]
\]

where $D_t = \exp \left\{ -\int_0^t r_s ds \right\}$.

**Remark 3.4** We can replace $r_t$ in (3.11) by the random interest rate $r(X_t^{(k)})$, then by approximation, $(r_t)$ in (3.12) can be an adapted stochastic process.

4. **Superhedging and Subhedging under Volatility Uncertainty and Arbitrage Ambiguity**

Now we consider the hedging problem by Peng’s G-stochastic analysis in the risk-neutral & mean-certain world. Let $\{P\}$ be the set of risk-neutral measures and $\mathbb{E}$ the corresponding risk-neutral sublinear expectation. Let $(B_t)$ denote the $G[\sigma_2, \sigma_2]$-Brownian motion under $\mathbb{E}$. Let $\mathcal{F}_t$ be the minimal $\sigma$-algebra $\cap_{t>t'} \{ B_s, s \leq r \}$. Here $T$ is a fixed time. We consider a financial market with two assets. One of them is a locally riskless asset (the bank account) with price per unit ($C_t$) governed by the equation

\[
dC_t = C_t r_t dt,
\]

where $(r_t)$ is the nonnegative short rate. In addition to the bank account, consider a stock price process whose differential is

\[
dS_t = S_t (r_t dt + dB_t), \quad 0 \leq t \leq T,
\]

where $(B_t)$ a $G[\sigma_2, \sigma_2]$-Brownian motion. Stochastic process $(r_t)$ is allowed to be bounded $\mathcal{F}_t$-progressively measurable process in $\mathcal{M}_G^2(0, T)$.

The market can not be complete because of the uncertain volatility. So investors could not expect to replicate exactly any general contingent claim and have to choose some criterion to hedge the claim.

4.1. **Superhedging for the option seller**

Let $\xi$ be an $\mathcal{F}_T$-measurable random variable which represents the payoff at time $T$ of a derivative security. We allow this payoff to be path-dependant, i.e., to depend on anything that occurs between times 0 and $T$. We now give the definition of superstrategy under model uncertainty.

\footnote{$\mathcal{M}_G^2$ is the space consisting of square-integrable random variables such that the G-stochastic integral is well defined. See Appendix A for details.}
DEFINITION 4.1 A K-financing superstrategy against a contingent claim \( \xi \) under model uncertainty is a vector process \((V, \pi, K)\), where \( V \) is the managing price, \( \pi \) is the portfolio process, and \( K \) is the pricing error, such that

\[
dV_t = r_t V_t dt + \pi_t dB_t - dK_t, \text{ q.s.}
\]

\[V_T = \xi, \text{ and } \int_0^T |\pi_t|^2 dt < \infty, \text{ q.s.}\]

where \( K \) is an increasing, right-continuous adapted process q.s. with \( K_0 = 0 \).

REMARK 4.1 Any superstrategy defined by Definition 4.1 satisfies

\[V_t \geq v_t^P, \forall t \in [0, T], \forall P \in \mathcal{P}, P - a.s.\]

due to the comparison of BSDEs (El Karoui, Peng and Quenez (1997)), where \((v_t^P)\) solves

\[
dv_t^P = r_t v_t^P dt + \pi_t^P \sigma_t^P dW^P_t, \quad v_T^P = \xi, \quad P - a.s.
\]

with \( \sigma_t^P \) being \( \mathcal{F}_t^P \)-adapted process valued in \([\underline{\sigma}, \overline{\sigma}]\) and \((W_t^P)\) the standard Brownian motion in a linear expectation space \((\Omega, (\mathcal{F}_t^P)_{t \geq 0}, E^P)\).

DEFINITION 4.2 There is an arbitrage for a superstrategy \((V_t, \pi_t, K_t)\) if the value process \((V_t)\) satisfies \(V_0 = 0\) and

\[
V_T \geq 0, \text{ q.s. and } P[V_T > 0] > 0, \text{ for at least one } P \in \mathcal{P}.
\]

THEOREM 4.1 The solution triple \((V_t, \pi_t, K_t)\) to BSDE (4.3) is the minimal superstrategy with no-arbitrage. The “minimal” means that for any other superstrategy \((V_t', \pi_t', K_t')\), we have \(V_t \leq V_t', \forall t, q.s.\).

PROOF: Let \((V_t, \pi_t, K_t)\) be the unique triple satisfying the BSDE (4.3) and \(V_T = \xi\) with \((-K_t)\) being a continuously nonincreasing G-martingale. Obviously \((V_t, \pi_t, K_t)\) is a superstrategy according Definition 4.1. Furthermore, by Theorem B.1, we have

\[
V_t = D_t^{-1} \mathbb{E} [D_T \xi | \mathcal{F}_t]
\]

where \(D_t = \exp \left\{ -\int_0^t r_s ds \right\} \).

Let \((V_t', \pi_t', K_t')\) be another superstrategy defined by Definition 4.1 with \((K_t')\) being an increasing, right-continuous adapted process q.s. and \(K_0' = 0\). Applying Itô’s formula to \(D_t V_t'\), we obtain that

\[
d(D_t V_t') = D_t \left[ r_t V_t' dt + \pi_t' dB_t - dK_t' \right] - V_t' D_t r_t dt
\]

\[= D_t \pi_t' dB_t - D_t dK_t'.\]
Note that \((D_t V'_t + \int_0^t D_s dK'_s)\) is a G-martingale. Therefore
\[
V'_t = D_t^{-1} \left( \mathbb{E} \left[ D_T \xi + \int_0^T D_s dK'_s | \mathcal{F}_t \right] - \int_0^t D_s dK'_s \right)
\]
(4.7)

Since \(\int_t^T D_{t,s} dK'_s \geq 0, \forall t \in [0, T], q.s.\), then by the monotonicity of conditional G-expectation, we obtain
\[
V'_t \geq V_t, \forall t \in [0, T], q.s.
\]

Hence \((V_t, \pi_t, K_t)\) is the minimal superstrategy covering every probabilistic model.

If the terminal position
\[
V_T \geq 0, q.s. \text{ and } \exists P \in \mathcal{P}, \text{ such that } P[V_T > 0] > 0,
\]
then
\[
V_0 = \mathbb{E} [D_T V_T] = \sup_{P \in \mathcal{P}} \mathbb{E}_P [D_T V_T] > 0.
\]

So the superstrategy \((V, \pi, K)\) is arbitrage-free. □

By Theorem 4.1, we know that a hedging strategy is a minimal superstrategy under model uncertainty if and only if \((-K_t)\) is a G-martingale with finite variance such that (4.3) holds and \(V_T = \xi\). We will give more explicit explanation for \(K\) in the language of P&L, see Section 5.1.

**Remark 4.2 (K-financing and self-financing)** The solution triple \((V_t, \pi_t, K_t)\) to BSDE (4.3) is a K-financing superstrategy with \((-K_t)\) being a continuously nonincreasing G-martingale. Clearly \((V_t, \pi_t, K_t)\) is not necessary a self-financing strategy because the cumulative consumption \((K_t)\) is a nonnegative increasing process q.s. with \(K_0 = 0\). Since \((-K_t)\) is a G-martingale, there exists a probability measure \(P\) such that
\[
0 = K_T = \mathbb{E} [-K_T] = \mathbb{E}_P [-K_T].
\]
Thus \(K_T \equiv 0\) and \(K_t \equiv 0, P\text{-a.s.}, \text{ for each } t\). So a K-financing superstrategy \((V_t, \pi_t, K_t)\) is a self-financing strategy under some \(P \in \mathcal{P}\). Certainly if any other \(P' \in \mathcal{P}\) is equivalent to \(P\), then \(K_t \equiv 0, \mathcal{P}\text{-q.s.}, \text{ for each } t\) and \((V_t, \pi_t)\) is a self-financing strategy under each \(P \in \mathcal{P}\). So for a set probabilities \(\mathcal{P}\) which consists of mutually singular probability measures, in general, we can not find a universal self-financing hedging strategy, which leads to the incompleteness of a financial market.

**4.2. Subhedging for the option buyer**

Usually an option buyer puts more attention on substrategies, in particular the maximal substrategy which can be viewed as the maximal amount that the buyer of the option is willing to pay at time 0 such that he/she is sure to cover at time \(T\), the debt he/she incurred at time 0.
**Definition 4.3** A K-financing substrategy against a contingent claim \( \xi \) under model uncertainty is a vector process \((\widetilde{V}, \widetilde{\pi}, \widetilde{K})\), where \( \widetilde{V} \) is the market value, \( \widetilde{\pi} \) is the portfolio process, and \( \widetilde{K} \) is the pricing error, such that

\[
d\widetilde{V}_t = r_t \widetilde{V}_t dt + \widetilde{\pi}_t dB_t + d\widetilde{K}_t, \quad \text{q.s.} \quad \text{and} \quad \widetilde{V}_T = \xi, \quad \int_0^T |\widetilde{\pi}_t|^2 dt < \infty, \quad \text{q.s.}
\]

where \( \widetilde{K} \) is an increasing, right-continuous \( F_t \)-progressively measurable process q.s. with \( \widetilde{K}_0 = 0 \).

**Remark 4.3** Any substrategy defined by Definition 4.3 satisfies

\[
\widetilde{V}_t \leq v^P_t, \quad \forall t \in [0, T], \quad \forall P \in \mathcal{P}, \quad \text{P-a.s.}
\]

where \((v^P_t)\) solves BSDE (4.4).

Let \( \overline{\mathbb{E}} \cdot |\mathcal{F}_t| := -\mathbb{E} \cdot |\mathcal{F}_t| \). Then one can easily check that \( \overline{\mathbb{E}} \) satisfies the following super-additivity:

\[
\overline{\mathbb{E}}[X + Y | \mathcal{F}_t] \geq \overline{\mathbb{E}}[X | \mathcal{F}_t] + \overline{\mathbb{E}}[Y | \mathcal{F}_t]
\]

and shares all other properties of \( \mathbb{E} \).

**Theorem 4.2** The maximal substrategy \((\widehat{V}, \widehat{\pi}, \widehat{K})\) satisfying

\[
d\widehat{V}_t = r_t \widehat{V}_t dt + \widehat{\pi}_t dB_t + d\widehat{K}_t, \quad \widehat{V}_T = \xi
\]

where \((\widehat{K}_t)\) is a continuous, increasing process with \( \widehat{K}_0 = 0 \) and \((\widehat{K}_t)\) being a martingale under \( \overline{\mathbb{E}} \). More explicitly we have for any \( t \in [0, T] \),

\[
\widehat{V}_t = D_t^{-1} \overline{\mathbb{E}}[D_T \xi | \mathcal{F}_t], \quad \text{q.s.}
\]

The “maximal” means that for any other substrategy \((\tilde{V}'_t, \tilde{\pi}'_t, \tilde{K}'_t)\), we have \( \tilde{V}_t \geq \tilde{V}'_t, \forall t, \text{q.s.} \).

**Proof:** Let \((\widehat{V}, \widehat{\pi}, \widehat{K})\) be the unique triple satisfying BSDE (4.9) with \((\widehat{K}_t)\) being a continuous, increasing martingale under \( \overline{\mathbb{E}} \cdot |\mathcal{F}_t| \). Obviously \((\widehat{V}, \widehat{\pi}, \widehat{K})\) is a substrategy according Definition 4.3. Applying \( \text{It\'o's formula to } D_t \widehat{V}_t \), we get that

\[
\widehat{V}_t = D_t^{-1} \overline{\mathbb{E}}[D_T \xi | \mathcal{F}_t], \quad \text{q.s.}
\]

Let \((\tilde{V}'_t, \tilde{\pi}'_t, \tilde{K}'_t)\) be another substrategy defined by Definition 4.3 with \( \tilde{K}' \) being an increasing, right-continuous adapted process q.s. and \( \tilde{K}'_0 = 0 \). By direct calculation similarly to equation (4.7), we get

\[
\tilde{V}'_t = D_t^{-1} \overline{\mathbb{E}} \left[ D_T \xi - \int_t^T D_s d\tilde{K}'_s | \mathcal{F}_t \right].
\]

Since \( -\int_t^T D_s d\tilde{K}'_s \leq 0, \forall t \in [0, T], \text{q.s.} \), then by the monotonicity of conditional expectation \( \overline{\mathbb{E}} \cdot |\mathcal{F}_t| \), we obtain

\[
\tilde{V}'_t \leq \tilde{V}_t, \forall t \in [0, T], \text{q.s.}
\]

Therefore \((\widehat{V}, \widehat{\pi}, \widehat{K})\) is the maximal substrategy under every probabilistic model. \( \square \)
Remark 4.4. For a substrategy \((\tilde{V}, \tilde{\pi}, \tilde{K})\) satisfying (4.9), condition (4.5) does not guarantee no-arbitrage. Even condition (4.5) of arbitrage is replaced by

\begin{equation}
\tilde{V}_T \geq 0, \text{ q.s. and for all } P \in \mathcal{P}, \ P[\tilde{V}_T > 0] > 0,
\end{equation}

then still there may be an arbitrage opportunity. In fact if (4.10) holds, we have \(\forall P \in \mathcal{P}, \ P[\tilde{V}_T > 0] > 0\), But after taking infimum, perhaps

\[ \tilde{V}_0 = \inf_{P \in \mathcal{P}} E_P [D_T \xi] = 0. \]

So we have to redefine the notion of arbitrage for sub-hedging strategies.

Definition 4.4. There is an arbitrage for a substrategy \((\tilde{V}, \tilde{\pi}, \tilde{K})\) satisfying (4.9), if the value process \((\tilde{V}_t)\) satisfies \(\tilde{V}_0 = 0\) and

\begin{equation}
\tilde{V}_T \geq 0, \text{ q.s. and } \inf_{P \in \mathcal{P}} P[\tilde{V}_T > 0] > 0.
\end{equation}

Under the above definition, we have,

Theorem 4.3. The substrategy \((\tilde{V}, \tilde{\pi}, \tilde{K})\) is arbitrage-free.

Proof: If (4.11) holds, then by the strict comparison theorem in Li (2010)\(^{10}\), we have \(\tilde{V}_0 = \mathbb{E} [D_T \xi] = \inf_{P \in \mathcal{P}} E_P [D_T \xi] > 0\). Thus there is no arbitrage for the substrategy. \(\square\)

4.3. Put-call parity

In a complete financial market, there is a parity relation between a pair of European call option and European put option underlying the same stock \(S\) and with the same expiration date and strike price. We now consider similar parity relation for superhedging strategies in an incomplete market. The superhedging prices of a European call option and a European put option underlying the same stock \(S\) and sharing the same strike price \(L\) are given by

\[
c_t = (S_T - L)^+ - \int_t^T r_s c_s ds - \int_t^T \pi_s^c dB_s + \int_t^T dK_s^c, \ t \in [0, T],
\]

and

\[
p_t = (L - S_T)^+ - \int_t^T r_s p_s ds - \int_t^T \pi_s^p dB_s + \int_t^T p_s dK_s^p, \ t \in [0, T],
\]

where \(L \in \mathbb{R}^+\) is the strike price and \((S_t)\) is the stock price following

\[ dS_t = S_t (r_t dt + dB_t), t \geq 0, \]

where \(r_t\) is \(\mathcal{F}_t\)-measurable bounded processes belonging to \(\mathcal{M}^2_{\mathbb{G}}\).

\(^{10}\)The strict comparison theorem says that: for \(\xi^1, \xi^2 \in L^1_\mathbb{G}(\Omega)\), if \(\xi^1 \geq \xi^2\) and \(\inf_{P \in \mathcal{P}} P[\xi^1 > \xi^2] > 0\), then \(E[\xi^1] > E[\xi^2]\) and \(\bar{E}[\xi^1] > \bar{E}[\xi^2]\).
Theorem 4.4 Let \( c_t \) and \( p_t \) be the superhedging prices of a European call option and a European put option underlying the same stock \( S \) and sharing the same strike price \( L \). Then
\[
c_t + L \cdot \exp \left\{ - \int_t^T r_s ds \right\} = p_t + S_t, \text{ q.s.}
\]
Similarly the parity relation also holds for subhedging prices.

Proof: Let \( L_t = L \cdot \exp \left\{ - \int_t^T r_s ds \right\} \). Then
\[
L_t = L - \int_t^T r_s L_s ds.
\]
By doing summation, we get
\[
c_t + L_t = (S_T - L)^+ + L - \int_t^T r_s (c_s + L_s) ds - \int_t^T \pi_s^c dB_s + \int_t^T dK^c_s,
\]
and
\[
p_t + S_t = (L - S_T)^+ + S_T - \int_t^T r_s (p_s + S_s) ds - \int_t^T (\pi_s^p + S_s) dB_s + \int_t^T dK^p_s.
\]
Observing that \((S_T - L)^+ + L = (L - S_T)^+ + S_T = \max \{L, S_T\}\) and the uniqueness of solution (See Theorem B.1) of the following BSDE
\[
y_t = \max \{L, S_T\} - \int_t^T r_s y_s ds - \int_t^T z_s dB_s + \int_t^T dK_s, \quad t \in [0, T],
\]
we deduce that the put-call parity
\[
c_t + L_t = p_t + S_t
\]
holds. □

4.4. Asset with strictly non-zero upper price and generalized geometric G-Brownian motion

Definition 4.5 A sublinear expectation \( \mathbb{E} \) is said to be risk-neutral if the discounted stock price \((D_t S_t)\) (paying no dividend) is a symmetric G-martingale under \( \mathbb{E} \).

Proposition 4.1 Let \( \mathbb{E} \) be a risk-neutral sublinear expectation in a market model. Then the upper price of every discounted portfolio is a G-martingale (not necessarily symmetric) under \( \mathbb{E} \).

Proof: Let \((B_t)\) be the G-Brownian motion under \( \mathbb{E} \). Assume that the stock price follows \( \frac{dS_t}{S_t} = r_t dt + dB_t \). Then the upper price of a portfolio follows
\[
\begin{align*}
   dV_t &= r_t (V_t - \pi_t) dt + \pi_t \frac{dS_t}{S_t} - dK_t \\
         &= r_t V_t dt + \pi_t dB_t - dK_t,
\end{align*}
\]
where \((-K_t)\) is a continuous nonincreasing G-martingale under \(\mathbb{E}\). Then the differential of the discounted upper price is
\[
d(D_t V_t) = D_t dV_t + V_t dD_t = D_t \left[ r_t (V_t - \pi_t) dt + \pi_t \frac{dS_t}{S_t} - dK_t \right] + V_t dD_t
\]
\[
= D_t \left[ r_t (V_t - \pi_t) dt + \pi_t (r_t dt + dB_t) - dK_t \right] - r_t D_t V_t dt
\]
\[
= \pi_t d(D_t S_t) - D_t dK_t.
\]
Under the risk-neutral sublinear expectation \(\mathbb{E}\), \((D_t S_t)\) is a symmetric G-martingale, \((-\int_0^t D_s dK_s)\) is a G-martingale with finite variance. Hence the process \((D_t V_t)\) must be a G-martingale. \(\square\)

**Definition 4.6** A process \((V_t)\) is called a geometric G-Brownian motion if it follows

\[(4.12)\]
\[
dV_t = V_t (r_t dt + \alpha_t dK_t + \theta_t dB_t)
\]

where \((B_t)\) is a G-Brownian motion, \((K_t)\) is a right-continuous increasing process, \(r_t \in M^1_G\), \(\alpha_t \in M^1_G\) and \(\left( \sup_{0 \leq t \leq T} |\alpha_t| \right) \cdot K_T < \infty\), \(\theta_t \in M^2_G\). Or equivalently

\[
V_t = V_0 \exp \left\{ \int_0^t \theta_s dB_s + \int_0^t r_s ds + \int_0^t \alpha_s dK_s - \frac{1}{2} \int_0^t \theta_s^2 d\langle B \rangle_s \right\}.
\]

An asset with strictly non-zero upper price is a security paying \(V_T\) at time \(T\) whose upper price \(V_t \neq 0\), q.s. for each \(t \in [0,T]\).

**Theorem 4.5** The upper price of an asset is strictly non-zero if and only if the upper price is a generalized geometric G-Brownian motion with \(V_0 \neq 0\).

**Proof:** Let \(\mathbb{E}\) be the unique risk-neutral sublinear expectation. Then \(\mathbb{E}[D_T V_T | \mathcal{F}_t] = D_t V_t\), q.s. for each \(t \in [0,T]\). By the Martingale Representation Theorem, there exists an adapted process \((Z_t)\) and nonincreasing \(\mathbb{E}\)-martingale \((-K_t)\) such that

\[
D_t V_t = \mathbb{E}[D_T V_T | \mathcal{F}_t] = V_0 + \int_0^t Z_s dB_s - K_t,
\]

where \((B_t)\) is a G-Brownian motion under \(\mathbb{E}\). Thus the differential of \((V_t)\) is

\[
dV_t = r_t V_t dt + D_t^{-1} Z_t dB_t - D_t^{-1} dK_t
\]

Set \(\theta_t = \frac{D_t^{-1} Z_t}{V_t}, \alpha_t = \frac{D_t^{-1}}{V_t} \). Then

\[
dV_t = V_t (r_t dt + \alpha_t dK_t + \theta_t dB_t)
\]

Or

\[
V_t = V_0 \exp \left\{ \int_0^t \theta_s dB_s + \int_0^t r_s ds + \int_0^t \alpha_s dK_s - \frac{1}{2} \int_0^t \theta_s^2 d\langle B \rangle_s \right\}.
\]

The sufficiency is obvious. \(\square\)
Corollary 4.1 Every asset with strictly positive payoff is a generalized geometric G-Brownian motion.

Proof: Since the payoff $V_T > 0$, q.s., by the risk-neutral pricing formula, for each $t \in [0, T]$,

$$V_t = D_t^{-1} \mathbb{E}_Q[D_T V_T | \mathcal{F}_t] > 0, \text{q.s.}$$

Then this corollary is obtained by Theorem 4.5. □

5. RESULTS IN MARKOVIAN SETTING

In this section, we consider some results using the state-dependent BSB equation.

5.1. Interpretation of $\eta$ and $K$

Why do $K$ and $\eta$ arise when we super-hedge under volatility uncertainty? Do they have certain sound financial meaning? We have given a rough explanation of the finite-variance term $K$ in BSDE (4.3). In Markovian setting, $K$ has a concrete decomposition:

$$K_t = \int_0^t \left[ 2G(\eta_s) ds - \eta_s d\langle B \rangle_s \right],$$

where $\eta_t = \frac{1}{2} S_t^2 \Gamma_t$, $\Gamma_t = \frac{\partial^2}{\partial S^2} u(S_t)$ is the Gamma of the option with payoff $\Phi(S_T)$. Obviously

- $\eta$ corresponds to Gamma $\Gamma$ of the option, while we have known that $Z$ corresponds to Delta $\Delta$ of the option.

In the classical Black-Scholes-Merton model, when a trader uses the Black-Scholes formula to sell and dynamically hedge a call option at managing volatility $\sigma_t$, if the realized volatility is lower than the managing volatility, the corresponding P&L will be non negative. An application of Itô formula shows us that the instantaneous P&L of being short a delta-hedged option reads

$$P&L(t, t + dt) = \frac{1}{2} S_t^2 \Gamma_t \left[ \sigma_t^2 dt - \left( \frac{dS_t}{S_t} \right)^2 \right]$$

where $\sigma_t$ is the managing volatility, i.e. the volatility at which the option is sold and $\left( \frac{dS_t}{S_t} \right)^2$ represents the realized variance over the period $[t, t + dt]$. $\Gamma$ is positive for a call option and an upper bound of the realized volatility is enough to grant a profit (conversely, a lower bound for option buyers).

For an option with payoff $\Phi(S_T)$ and with volatility fluctuating in interval $[\sigma, \overline{\sigma}]$ at each time $t$, investors seek for a managing policy yielding a non negative P&L whatever the realized path. So investors sell the option at maximal volatility in some sense such that the maximal instantaneous P&L of being short a delta-hedged option should be in form of

$$P&L(t, t + dt) = \frac{1}{2} \sup_{\sigma_t \leq \sigma_t \leq \overline{\sigma}} \{ \sigma_t^2 S_t^2 \Gamma_t \} dt - \frac{1}{2} S_t^2 \Gamma_t d\langle B \rangle_t = dK_t.$$  

See Martini and Jacquier (2010), Jacquier and Slaoui (2010) for the definition and derivation of P&L.
Theorem 5.1 For state-dependent payoffs, the maximal instantaneous P&L of being short a delta-hedged option is of the form (5.2).

Proof: We consider the risk-neutral & mean-certain world. The stock price follows

\[ dS_t = S_t \left( r_t dt + dB_t \right), \]

where \( r_t \) is assumed to be a bounded function. Let \( V \) be the unique smooth solution of Barenblatt equation (3.9). Then by Itô’s formula,

\[ dV(S_t) = \frac{\partial V}{\partial t} dt + S_t \frac{\partial V}{\partial S} (r_t dt + dB_t) + \frac{1}{2} S_t^2 \frac{\partial^2 V}{\partial S^2} d\langle B \rangle_t. \]

The discrete versions of equations (5.3) and (5.4) are

\[ \Delta S_t = S_t (r_t \Delta t + \Delta B_t) \]

and

\[ \Delta V = \frac{\partial V}{\partial t} \Delta t + S_t \frac{\partial V}{\partial S} (r_t \Delta t + \Delta B_t) + \frac{1}{2} S_t^2 \frac{\partial^2 V}{\partial S^2} \Delta \langle B \rangle_t. \]

For a delta-hedging portfolio \( \Pi \), the holder of this portfolio is short one derivative and long an amount \( \frac{\partial V}{\partial S} \) of shares of stocks and \( (V - \frac{\partial V}{\partial S} S) \) cash left in a bank account. Namely the P&L variance of the portfolio is

\[ \Delta \Pi_t = \frac{\partial V}{\partial S} \Delta S_t - \Delta V + \left( V - \frac{\partial V}{\partial S} S_t \right) r_t \Delta t. \]

Now, substituting equations (5.5) and (5.6) into (5.7) yields

\[ \Delta \Pi_t = -\frac{\partial V}{\partial t} \Delta t - \frac{1}{2} S_t^2 \frac{\partial^2 V}{\partial S^2} \Delta \langle B \rangle_t + \left( V - \frac{\partial V}{\partial S} S_t \right) r_t \Delta t. \]

Moreover, as the superhedging price of the option follows the Barenblatt equation (3.9), we get

\[ \Delta \Pi_t = \frac{1}{2} \sup_{\sigma \leq \sigma_t \leq \sigma} \left\{ \sigma^2 S^2_t \Gamma_t \right\} \Delta t - \frac{1}{2} S_t^2 \Gamma_t \Delta \langle B \rangle_t. \]

Hence the final P&L on \((t, t + dt)\) reads

\[ P&L(t, t + dt) = \frac{1}{2} \sup_{\sigma \leq \sigma_t \leq \sigma} \left\{ \sigma^2 S^2_t \Gamma_t \right\} dt - \frac{1}{2} S_t^2 \Gamma_t d\langle B \rangle_t. \]

\[ \square \]

Therefore \( K_t \) over \((t, t + dt)\) coincides with the maximal P&L of being short a delta-hedged option. That is, by choosing appreciate managing volatility \( \sigma \), we obtain a non-negative P&L (or \( K \)) for a robust strategy. Then we come back to equality (4.3) in section 4, which now has a clear meaning that:
• The minimal superstrategy satisfies: changes of values of the portfolio minus the instantaneous P&L, equals to the change of the managing price of the option. That is to say, we can withdraw money $P\&L_{(t,t+dt)}$ along the way and end up with the terminal payoff.

For option buyers, to guarantee a profit, he/she has to choose the minimal volatility such that his/her P&L on $(t,t+dt)$

$$P\&L_{(t,t+dt)} = \frac{1}{2} S_t^2 \tilde{\Gamma}_t d\langle B \rangle_t - \frac{1}{2} \inf_{\sigma \leq \sigma_t \leq \sigma_c} \{\sigma_t^2 S_t^2 \tilde{\Gamma}_t\} dt$$

will always be nonnegative.

5.2. Estimating the spread

Considering the following minimal superstrategy

$$dV_t = r_t V_t dt + \pi_t dB_t - (2G(\eta_t) dt - \eta_t d\langle B \rangle_t),$$

$$V_T = \Phi(S_T)$$

and maximal substrategy

$$d\tilde{V}_t = r_t \tilde{V}_t dt + \tilde{\pi}_t dB_t + (2G(\tilde{\eta}_t) dt - \tilde{\eta}_t d\langle B \rangle_t),$$

$$\tilde{V}_T = \Phi(S_T),$$

where $S$ is defined by (5.3), $\eta_t = \frac{1}{2} S_t^2 \Gamma_t$, and $\tilde{\eta}_t = -\frac{1}{2} S_t^2 \tilde{\Gamma}_t$. In an incomplete market the superhedging price and subhedging price (also called ask/bid price) do not usually equal to each other and a set of hedging prices exist. Cont (2006) proposed to measure the impact of model uncertainty on the value of a contingent claim $\xi$ by

$$e^P(\xi) := V_0[\xi] - \tilde{V}_0[\xi].$$

Define $D_t = \exp \left\{ -\int_0^t r_s ds \right\}$. Since $e^P(\xi) = E[D_T \xi] + E[-D_T \xi]$, then $e^P(\cdot)$ satisfies

(i) $e^P(D_T^{-1} c) = 0$, $\forall c \in \mathbb{R}$,

(ii) $e^P(\xi + \eta) \leq e^P(\xi) + e^P(\eta), \forall \xi, \eta \in L_1 \cap L_\infty$, $p > 1$,

(iii) $e^P(\xi) \geq 0$.

The following result shows that $e^P(\cdot)$ depends on closely the volatility uncertainty and gamma risk.

**Theorem 5.2** For all $\xi = \Phi(S_T)$, $\Phi$ is a Lipschitz function of $S_T$, we have

$$e^P(\xi) \leq (\sigma^2 - \sigma^2) \cdot L,$$

where $L = E\left[ \int_0^T D_t S_t^2 \max(|\Gamma_t|, |\tilde{\Gamma}_t|) dt \right]$. 

PROOF: We denote $V_t = V_t - \overline{V}_t$, $\pi_t = \pi_t - \overline{\pi}_t$, $K_t = \int_0^t 2G(\eta_s) \, ds - \int_0^t \eta_s d\langle B \rangle_s$. Then

$$V_t = 0 - \int_t^T r_s V_s \, ds - \int_t^T \pi_s dB_s - \int_t^T dK_s.$$ 

Note that in general $(K_s)$ is not a G-martingale since $G$ is a subadditive function.

Applying Itô’s formula to $(D_t V_t)$ we get

$$V_t = D_t^{-1} \mathbb{E} \left[ \int_t^T D_s d\overline{K}_s | \mathcal{F}_t \right], q.s. \quad (5.10)$$

Therefore

$$e_P(\xi) = V_0 = \mathbb{E} \left[ \int_0^T D_s d\overline{K}_s \right]$$

$$\leq (\sigma^2 - \sigma^2) \cdot \mathbb{E} \left[ \int_0^T D_s (|\eta_s| + |\overline{\eta}_s|) \, dt \right]$$

$$\leq \frac{1}{2} (\sigma^2 - \sigma^2) \cdot \mathbb{E} \left[ \int_0^T D_s S_t^2 (|\Gamma_t| + |\overline{\Gamma}_t|) \, dt \right]$$

$$\leq (\sigma^2 - \sigma^2) \cdot \mathbb{E} \left[ \int_0^T D_s S_t^2 \max(|\Gamma_t|, |\overline{\Gamma}_t|) \, dt \right].$$

$\square$

Theorem 5.2 and results in Section 5.1 also hold for discrete-path-dependent payoffs.

Remark 5.1 Observing from (5.10) that, the ask-bid spread is in fact the accumulation of summation of the superhedging P&L and the subhedging P&L.

6. CONCLUSION

We consider mean-volatility uncertainty by Peng’s G-stochastic analysis in this paper. All results can be applied to path-dependent options. Price of stock is assumed to be generalized geometric G-Brownian motion in which the mean-uncertainty is not necessarily related to the volatility-uncertainty. A neat formulation of superhedging problem is given by BSDE driven by G-Brownian motion. For subhedging we have to impose strong conditions to guarantee no-arbitrage, which is essentially different from Vorbink’s work.

Another phenomenon deserving mention is that the mean-uncertainty does not influence pricing a security. When we deriving the superhedging PDEs, the stock appreciation disappears after delta-hedging, which shows that there is a risk-neutral world under which all investors price and hedge in a risk-neutral & mean-certain way.

In Markovian setting, we give a precise and practical explanation of the finite-variance term in the minimal superstrategy in the language of P&L. The control of price fluctuations by volatility interval are also discussed.

All shows that G-stochastic analysis is a convenient tool to measure model uncertainty. Although, in the eloquent words of Derman (1997): even the finest model is only a model of the phenomena, and not the real thing, we believe we are modeling in a more efficient way to solve problems of the real thing.
APPENDIX A: PENG’S G-STOCHASTIC CALCULUS

In this section we recall some necessary notions and lemmas of Peng’s G-stochastic calculus needed in this paper. Readers could refer to Peng (2010a) for more systematic information.

For two stochastic processes \((X_t)\) and \((Y_t)\), let \((X,Y)_t\) denote their mutual variance. We denote by \(S(n)\) the collection of \(n \times n\) symmetric matrices, \(S_+(d)\) the positive-semidefinite elements of \(S(d)\). We observe that \(S(n)\) is a Euclidean space with the scalar product \(\langle A,B \rangle = tr(AB)\). Let \(\Omega\) be a complete nonempty measure space. Typically we can take \(\Omega = C([0, +\infty), \mathbb{R}^d)\) with the topology of uniform convergence on compact subspaces. \(\mathcal{B}(\Omega)\) denotes the Borel \(\sigma\)-algebra of \(\Omega\). Let \(\mathcal{H}\) be a linear space of real functions defined on \(\Omega\) such that if \(X_1, \ldots, X_n \in \mathcal{H}\) then \(\varphi(X_1, \ldots, X_n) \in \mathcal{H}\) for each \(\varphi \in C_{\text{l.Lip}}(\mathbb{R}^n)\) where \(C_{\text{l.Lip}}(\mathbb{R}^n)\) denotes the linear space of (local Lipschitz) functions \(\varphi\) satisfying

\[
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \ \forall x, y \in \mathbb{R}^n,
\]

for some \(C > 0, m \in \mathbb{N}\) depending on \(\varphi\). \(\mathcal{H}\) is considered as a space of ‘random variables’. In this case \(X = (X_1, \ldots, X_n)\) is called an \(n\)-dimensional random vector, denoted by \(X \in \mathcal{H}^n\). We also denote by \(C_{\text{l.Lip}}^k(\mathbb{R}^n)\) the space of bounded and \(k\)-time continuously differentiable functions with bounded derivatives of all orders less than or equal to \(k\); \(C_{\text{l.Lip}}(\mathbb{R}^n)\) the space of Lipschitz continuous functions.

**Definition A.1** A sublinear expectation \(\mathbb{E}\) on \(\mathcal{H}\) is a functional \(\mathbb{E} : \mathcal{H} \rightarrow \mathbb{R}\) satisfying the following properties: for all \(X, Y \in \mathcal{H}\), we have

(a) Monotonicity: If \(X \geq Y\), then \(\mathbb{E}[X] \geq \mathbb{E}[Y]\).

(b) Constant preserving: \(\mathbb{E}[c] = c, \forall c \in \mathbb{R}\).

(c) Sub-additivity: \(\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]\).

(d) Positive homogeneity: \(\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0\).

**Definition A.2** Let \(X_1\) and \(X_2\) be two \(n\)-dimensional random vectors defined on nonlinear expectation spaces \((\Omega_1, \mathcal{H}_1, \mathbb{E}_1)\) and \((\Omega_2, \mathcal{H}_2, \mathbb{E}_2)\) respectively. They are called identically distributed, denoted by \(X_1 \overset{d}{=} X_2\), if

\[
\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)], \ \forall \varphi \in C_{\text{l.Lip}}(\mathbb{R}^n).
\]

**Definition A.3** In a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) a random vector \(Y \in \mathcal{H}^n\) is said to be independent of another random vector \(X \in \mathcal{H}^m\) under \(\mathbb{E}\) if for each test function \(\varphi \in C_{\text{l.Lip}}(\mathbb{R}^{m+n})\) we have

\[
\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)]_{x=X}].
\]

**Remark A.1** It is interesting that \(Y\) is independent of \(X\) does not necessarily imply \(X\) is independent of \(Y\). See Chapter I, Example 3.13 in Peng (2010a).

**Definition A.4** (G-normal distribution). A \(d\)-dimensional random vector \(X = (X_1, \ldots, X_d)\) in a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) is called G-normal distributed if for each \(a, b > 0\) we have

\[
aX + bX \overset{d}{=} \sqrt{a^2 + b^2}X
\]

where \(X\) is an independent copy of \(X\).

**Remark A.2** It is easy to check that \(\mathbb{E}[X] = \mathbb{E}[-X] = 0\). The so called ‘G’ is related to \(G : S(d) \rightarrow \mathbb{R}\) defined by

\[
G(A) = \frac{1}{2} \mathbb{E}[(AX, X)],
\]

Hu and Peng (2009) proved that for a sublinear expectation \(\mathbb{E}\) on \((\Omega, \mathcal{H})\), there exists a family of linear expectation \(\{E_P; P \in \mathcal{P}\}\) on \((\Omega, \mathcal{H})\) such that \(\mathbb{E}[\cdot] = \sup_{P \in \mathcal{P}} E_P[\cdot]\).
DEFINITION A.5  For a given set of probability measures \( P \), we introduce the natural Choquet capacity

\[
C(A) := \sup_{P \in P} P(A), \quad A \in \mathcal{B}(\Omega).
\]

A property holds quasi-surely (q.s.) if it holds outside a polar set \( A \), i.e., \( C(A) = 0 \). A mapping \( X \) on \( \Omega \) with values in a topological space is said to be quasi-continuous (q.c.) if \( \forall \varepsilon > 0 \), there exists an open set \( O \) with \( C(O) < \varepsilon \) such that \( X|_O \) is continuous.

DEFINITION A.6  (G–Brownian motion). A \( d \)-dimensional process \( (B_t)_{t \geq 0} \) on a sublinear expectation space \( (\Omega, \mathcal{H}, \mathbb{E}) \) is called a G–Brownian motion if the following properties are satisfied:

(i) \( B_0(\omega) = 0 \);

(ii) For each \( t, s \geq 0 \), the increment \( B_{t+s} - B_t \) is independent from \( (B_t, B_{t_1}, \ldots, B_{t_n}) \), for each \( n \in \mathbb{N} \) and \( 0 \leq t_1 \leq \cdots \leq t_n \leq t \);

(iii) \( B_{t+s} - B_t \overset{d}{=} \sqrt{s}X \), where \( X \) is G-normal distributed.

DEFINITION A.7  (Maximal distribution). A \( d \)-dimensional random vector \( X = (X_1, \ldots, X_d) \) in a sublinear expectation space \( (\Omega, \mathcal{H}, \mathbb{E}) \) is called maximal distributed if for each \( a, b > 0 \) we have

\[
aX + b\overline{X} \overset{d}{=} (a + b)X
\]

where \( \overline{X} \) is an independent copy of \( X \).

REMARK A.3  For a maximal distributed random variable \( X \), there exists a bounded, closed and convex subset \( \Gamma \subset \mathbb{R}^d \) such that

\[
\mathbb{E}[\varphi(X)] = \max_{a \in \Gamma} \varphi(a), \quad \forall \varphi \in C_{l.lip}(\mathbb{R}^d).
\]

DEFINITION A.8  (Finite-variance G–Brownian motion). A \( d \)-dimensional process \( (\beta_t)_{t \geq 0} \) on a sublinear expectation space \( (\Omega, \mathcal{H}, \mathbb{E}) \) is called a finite-variance G–Brownian motion if the following properties are satisfied:

(i) \( \beta_0(\omega) = 0 \);

(ii) For each \( t, s \geq 0 \), the increment \( \beta_{t+s} - \beta_t \) is independent from \( (\beta_t, \beta_{t_1}, \ldots, \beta_{t_n}) \), for each \( n \in \mathbb{N} \) and \( 0 \leq t_1 \leq \cdots \leq t_n \leq t \);

(iii) \( \beta_{t+s} - \beta_t \overset{d}{=} sX \), where \( X \) is maximal distributed.

Typically, \( (B_t) \), the quadratic variance process of \( (B_t) \), is a finite-variance G–Brownian motion. We conclude properties of finite-variance G–Brownian motion as following

PROPOSITION A.1  Let \( \beta_t \) be a one-dimensional finite-variance G–Brownian motion. Then

(i) \( \beta_t \) is a continuous process with finite variance, independent and stationary increments under \( \mathbb{E} \).

(ii)

\[
\mathbb{E}[\varphi(\beta_{t+s} - \beta_s)|\mathcal{F}_s] = \max_{\mu \leq \rho \leq \overline{\rho}} \varphi(\mu t), \quad \forall \varphi \in C_{l.lip}(\mathbb{R}).
\]

where we denote the usual parameters \( \overline{\mathbb{E}} = \mathbb{E}[\beta_1], \mu = \mathbb{E}[-\beta_1]. \)

(iii) For each \( 0 \leq t \leq T < \infty \), we have q.s.

\[
\mu(T - t) \leq \beta_T - \beta_t \leq \overline{\mathbb{E}}(T - t).
\]

PROOF: See Peng (2010a) for (i), (ii) and (iii). \( \square \)

In the sequence, let \( \Omega = C_0([0, +\infty), \mathbb{R}^d) \) denote the space of all \( \mathbb{R}^d \)-valued continuous paths \( (\omega_t)_{t \in \mathbb{R}^+} \) with \( \omega_0 = 0 \), by \( C_0(\Omega) \) all bounded and continuous functions on \( \Omega \). For each fixed \( T \geq 0 \), we consider the following space of random variables:

\[
L_{l.lip}(\Omega_T) := \{ X(\omega) = \varphi(\omega_{t_1 \wedge T}, \ldots, \omega_{t_m \wedge T}), \forall m \geq 1, \forall \varphi \in C_{l.lip}(\mathbb{R}^m) \}.
\]
We also denote
\[ L_{ip}(\Omega) := \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n). \]

We will consider the canonical space and set \( B_t(\omega) = \omega_t \). For a given sublinear function \( G(A) = \frac{1}{2}\sup_{\gamma \in \Gamma} \{ tr[A \gamma] \} \), where \( A \in \mathbb{S}(d) \), \( \Gamma \) is a given nonempty, bounded and closed convex subset of \( \mathbb{S}_+(d) \), by the following
\[ \partial_t u(t, x) - G(D^2_t u) = 0, \quad u(0, x) = \varphi(x), \]

Peng (2006) defined \( G \)-expectation \( \mathbb{E} \) as \( \mathbb{E}[\varphi(x + B_t)] = u(t, x) \). For each \( p \geq 1 \), \( X \in L_{ip}(\Omega) \), \( \|X\|_p = (\mathbb{E}[|X|^p])^{\frac{1}{p}} \) forms a norm and \( \mathbb{E} \) can be continuously extended to a Banach space, denoted by \( L^p_{ip}(\Omega) \).

We will consider the canonical space and set \( \Omega \). We also denote \( \mathcal{M}^p_{\mathcal{G}}(\Omega) \) as the collection of processes \( \eta(\omega) = \sum_{j=0}^{N} \xi_j(\omega) \cdot \mathbb{1}_{(t_j,t_{j+1})}(t) \), where \( \xi_j \in L^p_{ip}(\Omega), j = 0, 1, \ldots, N \);
\[ \mathcal{M}^p_{\mathcal{G}}(0, T) : \text{the completion of } \mathcal{M}^p_{\mathcal{G}}(0, T) \text{ under norm } \|\eta\|_{\mathcal{M}} = \left( \mathbb{E} \left[ \int_0^T |\eta|^p \, dt \right] \right)^{\frac{1}{p}}; \]
\[ \mathcal{H}^p_{\mathcal{G}}(0, T) : \text{the completion of } \mathcal{M}^{p,1}_{\mathcal{G}}(0, T) \text{ under norm } \|\eta\|_{\mathcal{H}} = \left( \mathbb{E} \left( \int_0^T |\eta|^2 \, dt \right) \right)^{\frac{1}{2}}. \]

It is easy to prove that \( \mathcal{H}^p_{\mathcal{G}}(0, T) = \mathcal{M}^p_{\mathcal{G}}(0, T) \).

For any \( \eta \in \mathcal{M}^p_{\mathcal{G}} \), \( G \)-Itô integral is well defined in Peng (2006, 2008a) and extended to \( \mathcal{H}^p_{\mathcal{G}} \) by Song (2010a).

**APPENDIX B: BSDE WITH LINEAR GENERATOR AND DRIVEN BY G-BROWNIAN MOTION**

We define \( D_t = \exp \left\{ -\int_0^t r_s \, ds \right\} \). Then \( D_t \) satisfies
\[ dD_t = -r_t D_t \, dt, \quad D_{t=0} = 1. \]

Consider the following one dimensional BSDE with linear generator and driven by one dimensional \( G \)-Brownian motion:
\[ dY_t = (r_t Y_t - \phi_t) \, dt + Z_t dB_t - dK_t, \]
\[ Y_T = \xi, \]
where \( \xi \in L^p_{ip}(\Omega_T) \), \( p > 1 \), \( r_t \) and \( \phi_t \) are \( \mathcal{F}_t \)-measurable bounded processes belonging to \( \mathcal{M}^p_{\mathcal{G}} \).

**DEFINITION B.1** A solution to BSDE (B.1) is a triple of adapted processes \((Y_t, Z_t, K_t)\) where \((K_t)\) is a continuous, increasing process with \( K_0 = 0 \) and \((-K_t)\) being a \( G \)-martingale.

For BSDE (B.1), we have

**THEOREM B.1** There is a unique triple \((Y_t, Z_t, K_t)\) satisfying (B.1) with \( Y \in \mathcal{M}^p_{\mathcal{G}} \), \( Z \in \mathcal{H}^p_{\mathcal{G}} \) and \( K_T \in L^p_{ip}(\Omega_T) \), \( 1 < \alpha < p, p > 1 \). Furthermore we have q.s.,
\[ Y_t = D_t^{-1} \mathbb{E} \left[ D_T \xi + \int_t^T \sigma_s \phi_s \, ds \mid \mathcal{F}_t \right]. \]
PROOF: Consider the following BSDE under sublinear expectation $\mathbb{E}$:

\begin{equation}
\text{(B.3)}
Y_t = \mathbb{E} \left[ \xi - \int_t^T (r_s Y_s - \phi_s) \, ds \bigg| \mathcal{F}_t \right].
\end{equation}

By the technique of contracting mapping principle employed in Peng (2010a), Ch.V, Sec. 2, one can similarly prove that there is a unique solution $Y \in \mathcal{M}_T^\alpha$ to BSDE (B.3). Applying martingale representation theorem established in Song (2010a), there is a unique pair $(Z, K)$ with $Z \in \mathcal{M}_T^\alpha$ and $K_t \in L_\alpha^2(\Omega_T)$, $1 \leq \alpha < p$ such that

\[
\mathbb{E} \left[ \xi - \int_0^T (r_s Y_s - \phi_s) \, ds \bigg| \mathcal{F}_t \right] = Y_0 + \int_0^t Z_s dB_s - K_t, \quad \mathcal{P}-q.s.
\]

Hence

\[
Y_t = \mathbb{E} \left[ \xi - \int_0^T (r_s Y_s - \phi_s) \, ds \bigg| \mathcal{F}_t \right] + \int_0^t (r_s Y_s - \phi_s) \, ds
\]

\[
= Y_0 + \int_0^t (r_s Y_s - \phi_s) \, ds + \int_0^T Z_s dB_s - K_t, \quad \mathcal{P}-q.s.
\]

or in backward form

\[
Y_t = \xi - \int_t^T (r_s Y_s - \phi_s) \, ds - \int_t^T Z_s dB_s + \int_t^T K_s, \quad \mathcal{P}-q.s.
\]

Thus the triple $(Y_t, Z_t, K_t)$ constructed by above procedure is a solution of (B.1).

Applying Itô’s formula to $D_t Y_t$, we have

\[
d(D_t Y_t) = D_t \left[ (r_t Y_t - \phi_t) \, dt + Z_t dB_t - dK_t \right] - Y_t D_t r_t \, dt
\]

\[
= -D_t \phi_t \, dt + D_t Z_t dB_t - D_t dK_t.
\]

Note that $(D_t Y_t + \int_0^t D_s \phi_s \, ds)$ is a G-martingale. Hence

\[
D_t Y_t = \mathbb{E} \left[ D_T \xi + \int_0^T D_s \phi_s \, ds \bigg| \mathcal{F}_t \right] - \int_0^t D_s \phi_s \, ds
\]

\[
= \mathbb{E} \left[ D_T \xi + \int_t^T D_s \phi_s \, ds \bigg| \mathcal{F}_t \right].
\]

Therefore the solution of (B.1) has the following unique form:

\[
Y_t = D_t^{-1} \mathbb{E} \left[ D_T \xi + \int_t^T D_s \phi_s \, ds \bigg| \mathcal{F}_t \right].
\]

□

Let $Y$ be the solution of (B.1) with parameters $(\xi^i, \phi^i)$, $i = 1, 2$. It is interesting that $(Y^1 + Y^2)$ is no longer a solution of (B.1) with parameters $(\xi^1 + \xi^2, \phi^1 + \phi^2)$, though BSDE (B.1) has a linear generator. All attributes to the sublinearity. $(-K^1 - K^2)$ is no more a G-martingale. We have the following

**Corollary B.1** Let $\tilde{Y}$ be the solution of (B.1) with parameters $(\xi^1 + \xi^2, \phi^1 + \phi^2)$. Then

\[
Y^1 + Y^2 \geq \tilde{Y}.
\]

**Proof:** It is just a sequence of (B.2) and the sublinearity of G-expectation $\mathbb{E}$. □

This property reflects that if two agent cooperate with each other, then superhedging the whole might yield less pricing error.
REFERENCES

AVELLANEDA, M., A. LEVY AND A. PARAS (1995): “Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities,” Appl Math Finance, 2, 73–88.

BARENBLATT, G. I. (1979): “Similarity, Self-Similarity and Intermediate Asymptotics,” Consultants Bureau, New York.

BEISSNER, P. (2012): “Coherent Price Systems and Uncertainty-Neutral Valuation,” arXiv:1202.6632v1 [q-fin.GN].

Bingham, N. H. and R. Kiesel (2004) : “Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives,” 2nd ed., Springer.

BION-NADAL, J., AND M. KERVAREC (2012): “Risk Measuring under Model Uncertainty,” Ann. Appl. Probab., 22(1), 213–238.

BLACK, F. AND M. SCHOLES (1973): “The Pricing of Options and Corporate Liabilities,” Journal of Political Economy, 81, 637–654.

CHEN, Z., AND L. EPSTEIN (2002): “Ambiguity, Risk and Asset Returns in Continuous Time,” Econometrica, 70(4), 1403–1443.

CONT, R. (2006): “Model Uncertainty and its Impact on the Pricing of Derivative Instruments,” Mathematical Finance, 16, 519-547.

DENIS, L., M. HU AND S. PENG (2011): “Function Spaces and Capacity Related to a Sublinear Expectation: Application to G-Brownian Motion Paths,” Potential Analysis, 34, 139-161.

DENIS, L., AND C. MARTINI (2006): “A Theoretical Framework for the Pricing of Contingent Claims in the Presence of Model Uncertainty,” The Annals of Applied Probability, 16 (2), 827–852.

DERMAN, E. (1997): “Model risk,” pp. 83–88 in S. Grayling (ed.) VaR- Understanding and Applying Value-at-Risk. London: Risk Publications.

DOW, J. AND S. WEIRLANG (1992): “Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio,” Econometrica, 60, 197–204.

DOWD, K. (2005): “Measuring Market Risk,” Second Edition, John Wiley & Sons Ltd.

E. EBERLEIN, MADAN, D. B., PISTORIUS, M. AND M. YOR (2014): “Bid and ask prices as non-linear continuous time G-expectations based on distortions,” Math Financ Econ, 8(3), 265–289.

E. EBERLEIN, MADAN, D. B., PISTORIUS, M., SCHOUTENS, W. AND M. YOR (2014): “Two price economies in continuous time,” Ann Financ, 10, 71–100.

EL KAROUI, N., S. PENG, AND M. C. QUENEZ, (1997): “Backward Stochastic Differential Equation in Finance,” Math. Financ., 1:1-71.

EL KAROUI, N., AND C. RAVANELLI, (2009): “Cash Sub-additive Risk Measures and Interest Rate Ambiguity,” Math. Financ, 19(4), 561–590.

EPSTEIN, L.G. (1999): “A Definition of Uncertainty Aversion,” Review of Economic Studies, 65, 579–608.

EPSTEIN, L.G. AND S. JT (2011): “Ambiguous Volatility, Possibility and Utility in Continuous Time,” arXiv:1105.1652v7 [q-fin.GN].

EPSTEIN, L.G. AND S. JT (2013): “Ambiguous Volatility and Asset Pricing in Continuous Time ,” The Review of Financial Studies, 26 (7), 1740-1786.

EPSTEIN, L.G. AND M. SCHNEIDER (2003): “Recursive Multiple Priors,” Journal of Economic Theory, 113(1), 1–31.

EPSTEIN, L.G. AND L. WANG (1994): “Intertemporal Asset Pricing under Knightian Uncertainty,” Econometrica, 62, 283–322.

EPSTEIN, L.G. AND L. WANG (1995): “Uncertainty, Risk Neutral Measures and Asset Price Booms and Crashes,” Journal of Economic Theory, 67, 40–80.

FÖLLMEL, H., AND M. SCHWEIZER (1990): “Hedging of Contingent Claims under Incomplete Information,” in Applied Stochastic Ananlysis, eds. M. H. A. Davis and R. J. Elliot. London: Gordon and Breach.

FÖLLMEL, H., AND SONDERMANN (1986): “Hedging of Non-redundant Contingent Claims,” in contributions to Mathematics Economics. In honor of Gerard Debreu, eds. W. Hildentrand. A. Mas-Colell. Amsterdam: North-Holland.

FREY, R. (2000): “Superreplication in Stochastic Volatility Models and Optimal Stopping,” Finance Stochast., 4 (2), 2000.
Gao, F. (2009): “Pathwise Properties and Homeomorphic Flows for Stochastic Differential Equations Driven by G-Brownian Motion,” Stochastic Processes and their Applications, 119, 3356-3382.

Gundel, A. (2005): “Robust Utility Maximization for Complete and Incomplete Market models,” Finance Stochast., 9, 151–176.

Hansen L., Th. Sargent and T. Tallarini (1999): “Robust Permanent Income and Pricing,” Review of Economic Studies, 66, 872–907.

Heston, S. (1993): “A Closed-Form Solution for Options with Stochastic Volatility with Application to Bond and Currency Options,” Review of Financial Studies, 6 327-343

Hu M. and S. Peng (2009): “On Representation Theorem of G-expectations and Paths of G-Brownian Motion,” Acta Math Appl Sinica English Series, 25, 1–8.

Hull, J. C. (2009) “Options, Futures, and Other Derivatives” (6th Edition), Pearson education international.

Illeditsch P. (2010): “Ambiguous Information, Risk Aversion and Asset Pricing,” Unpublished manuscript, Wharton.

Ishii, H. (1989): “On Uniqueness and Existence of Viscosity Solutions of Fully Nonlinear Second-Order Elliptic PDE’s,” Comm. Pure Appl. Math., 42, 15-45.

Jacquier A. and S. Slaoui (2010): “Variance Dispersion and Correlation Swaps,” arXiv:1004.0125v1 [q-fin.PR].

Jiang, L. S., C. L. Xu, X. M. Ren and S. H. Li (2008): “Mathematical Models and Case Study on Pricing Financial Derivatives,” 1nd ed.(Chinese version), High Education Press.

Karatzas, I. and S. G. Kou (1996); “On the Pricing of Contingent Claims under Constraints,” The Annals of Applied Probability, 6, 321-369.

Knight, F. (1921): “Risk, Uncertainty and Profit,” Boston: Houghton Mifflin.

Krylov, N. V. (1987) “Nonlinear Parabolic and Elliptic Equations of the Second Order,” Reidel Publishing Company (Original Russian version by Nauka, Moscow, 1985).

Li X. (2010): “On the Strict Comparison Theorem for G-expectations,” arXiv:1002.1765v2 [math.PR].

Li X. and S. Peng (2011): “Stopping Times and Related Ito’s Calculus with G-Brownian Motion,” Stochastic Processes and their Applications, 121, 1492–1508.

Lyons, T. (1995): “Uncertain Volatility and the Risk Free Synthesis of Derivatives,” Applied Mathematical Finance, 2, 117–133.

Madan, D. B. (2012): “A two price theory of financial equilibrium with risk management implications,” Ann Finance, 8(4), 489-505.

Madan, D. B., Schoutens,W. (2012): “Structured products equilibria in conic two price markets,” Math Finance Econ, 6, 37–57.

Martini, C. (1997): “Superreplications and Stochastic Control,” 3rd Italian Conference on Mathematical Finance, Trento.

Martini, C. and Jacquier, A. (2010): “Uncertain Volatility Model,” Encyclopedia of Quantitative Finance.

Merton, R. C. (1990): “Continuous Time Finance,” Blackwell Publishing.

Meyer, G. (2006): “The Black Scholes Barenblatt Equation for Options with Uncertain Volatility and Its Application to Static Hedging,” Int. J. Theoretical and Appl. Finance, 9, 673–703.

Nutz, M., and M. Soner (2010): “Superhedging and Dynamic Risk Measures under Volatility Uncertainty,” arXiv:1011.2958v1 [q-fin.RM].

Osuka, E. (2011): “Girsanov’s Formula for G-Brownian Motion,” arXiv:1106.2387v1 [math.PR].

Peng, S. (1992): “A Generalized Dynamic Programming Principle and Hamilton-Jacobi-Bellman Equations,” Stochastics and Stochastics reports, 38, 119-134.

Peng, S. (2006): “G-expectation, G-Brownian Motion and Related Stochastic Calculus of Ito’s Type,” In: Stochastic Analysis and Applications, The Abel Symposium 2005, Abel Symposia 2. New York: Springer-Verlag, 2006, 541-567.

Peng, S. (2007a): “Law of Large Numbers and Central Limit Theorem under Nonlinear Expectations,” arXiv:0702358v1.

Peng, S. (2007b): “G-Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty,” arXiv:0711.2834 v1 [math.PR].

Peng, S. (2008a): “Multi-dimentional G-Brownian Motion and Related Stochastic Calculus under G-
expectation,” *Stochastic Processes and their Applications*, 118, 2223-2253.

Peng, S. (2008b) “A New Central Limit Theorem under Sublinear Expectations,” arXiv:0803.2656v1 [math.PR].

Peng, S. (2009): “Survey on Normal Distributions, Central Limit Theorem, Brownian Motion and the Related Stochastic Calculus under Sublinear Expectations,” Science in China Series A: Mathematics, 52, 1391-1411.

Peng, S. (2010a): “Nonlinear Expectations and Stochastic Calculus under Uncertainty,” arXiv:1002.4546v1 [math.PR].

Peng, S. (2010b): “Backward Stochastic Differential Equation, Nonlinear Expectations and Their Applications,” *Proceedings of the International Congress of Mathematicians*. Hyderabad, India, 2010.

Riedel, F. (2009): “Optimal Stopping with Multiple Priors,” Econometrica, 77, 857–908.

Routledge, B. and S. Zin (2009): “Model Uncertainty and Liquidity,” *Review of Economic Dynamics*, 12(4), 543-566.

Schmeidler, D. (1989): “Subjective Probability and Expected Utility without Additivity,” *Econometrica*, 57, 571–587.

Schweizer, M. (1991): “Option Hedging for Semimartingale,” *Stochastic Processes and their Applications* 37, 339-363

Shreve, S. E. (2004): “Stochastic Calculus for Finance II: Continuous Time Models,” Springer.

Soner, H. M., N. Touzi, and J. Zhang (2010a): “Quasi–sure Stochastic Analysis through Aggregation,” arXiv:1003.4431v1.

Soner, H. M., N. Touzi, and J. Zhang (2011a): “Martingale Representation Theorem under G-expectation.” *Stochastic Processes and their Applications*, 121, 265-287.

Song, Y. (2010a): “Some Properties on G-evaluation and Its Applications to G-martingale Decomposition,” arXiv:1001.2802v2.

Song, Y. (2010b): “Characterizations of Processes with Stationary and Independent Increments under G-expectation,” arXiv:1009.0109v1 [math.PR].

Vargiolu, T. (2001) “Existence, Uniqueness and Smoothness for the Black-Scholes-Barenblatt Equation,” Università di Padova, July 23, 2001.

Vorbrink, J. (2010): “Financial Markets with Volatility Uncertainty,” arxiv.org/abs/1012.1535.

Wang, L. (1992) “On the Regularity of Fully Nonlinear Parabolic Equations: II,” *Comm. Pure Appl. Math.*, 45, 141-178.

Xu, J., H. Shang, and B. Zhang (2011): “A Girsanov Type Theorem under G-Framework,” *Stochastic Analysis and Applications*, 29(3): 386-406.

Xu, Y. H. (2013): “Probabilistic solutions for a class of path-dependent HJB equations,” *Stochastic Analysis and Applications*, 31(3), 440-459.

Xu, Y. H. (2014): “Multidimensional Dynamic Risk Measure via Conditional g-Expectation,” *Mathematical Finance*, 1-37, DOI: 10.1111/mafi.12062 2014 Wiley Periodicals, Inc.

Xu Y. H. (2014): “Stochastic maximum principle for optimal control with multiple priors,” *Systems and Control Letters*, 64, 114-118.