STABILITY AND TAIL LIMITS OF TRANSPORT-BASED QUANTILE CONTOURS

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Abstract. We extend Robert McCann’s treatment of the existence and uniqueness of an optimal transport map between two probability measures on a Euclidean space to a class of possibly infinite measures, finite outside neighbourhoods of the origin. For convergent sequences of pairs of such measures, we study the stability of the multivalued transport maps and associated quantile contours, defined as the images of spheres under these maps. The measures involved in the coupling are not required to be absolutely continuous, there are no restrictions on their supports, and no moment assumptions are needed. Weakly convergent sequences of probability measures forming a special case, our results apply to the recently introduced Monge–Kantorovich depth contours. The set-up involving infinite limit measures is applied to regularly varying probability measures: we derive tail limits of transport maps and of quantile contours defined with respect to a judiciously chosen spherical reference measure. Examples are discussed in detail.

Key words. Cyclic monotonicity. Graphical convergence. Maximal monotone mapping. Monge–Kantorovich quantile. Optimal transport. Regular variation.

1. Introduction

Euclidean space of dimension higher than one lacks a natural ordering. This is why there is no clear-cut definition of quantiles or of quantile contours for multivariate probability distributions. Several definitions exist, usually via the level curves of, for instance, the multivariate cumulative distribution function, the Lebesgue density, or some statistical depth function. Each choice comes with its own advantages and drawbacks.

Chernozhukov et al. [7] proposed a definition based on the theory of optimal transport, the roots of which go back to Gaspard Monge and Leonid V. Kantorovich. Let \( \mu \) be a spherically symmetric, absolutely continuous probability measure on \( \mathbb{R}^d \), for instance the uniform distribution on the unit ball. For any other probability measure \( \nu \) on \( \mathbb{R}^d \), there exists a convex function \( \psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \) which is finite and differentiable \( \mu \)-almost everywhere (a.e.) and such that, if \( X \) denotes a random vector with distribution \( \mu \), the distribution of the random vector \( \nabla \psi(X) \) is \( \nu \). Moreover, according to the Main Theorem in McCann [20], extending work of Brenier [6], the gradient \( \nabla \psi \) is uniquely determined \( \mu \)-a.e. Furthermore, if \( \mu \) and \( \nu \) possess finite second moments, then \( \nabla \psi \) minimizes the expected squared distance \( \mathbb{E}[(X - T(X))^2] \) over all measurable transformations \( T \) of \( \mathbb{R}^d \) that are defined \( \mu \)-a.e. and such that the law of \( T(X) \) is \( \nu \). The map \( \nabla \psi \) thus represents the smallest distortion of space needed to push the reference measure \( \mu \) forward to the target measure \( \nu \).

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Since $\mu$ is spherically symmetric, its half-space depth contours are spheres centred at the origin. Chernozhukov et al. [7] therefore define the Monge–Kantorovich depth contours of $\nu$ as the images of such spheres by $\nabla\psi$. The precise definition must take into account the fact that $\psi$ need not be differentiable everywhere and involves the subdifferential, $\partial\psi$, of $\psi$ rather than its gradient, and we will come back to this in Section 5. Rather than of depth contours, we prefer to speak of quantile contours, but this is obviously a matter of taste. For spherical distributions, these amount to the same thing anyway.

In Hallin [13] and del Barrio et al. [11], the measure transportation approach in [7] is reconsidered from the geometric perspective of McCann [20], relying on the monotonicity properties of gradients of convex functions as explained in Section 2. In this way, moment assumptions can be avoided completely, which is clearly desirable when studying quantiles. Still, rather strong regularity assumptions are imposed on the target probability measure in order to apply a result by Figalli [12] ensuring continuity and smoothness properties of the transport plan $\nabla\psi$ and its inverse: it needs to be absolutely continuous with a density bounded away from zero and infinity on compacta, implying, among others, that the measure has full support.

The question which motivated this article concerns the limiting shape of the Monge–Kantorovich contours as the depth or tail probability approaches zero, or equivalently, as the reference spheres grow large. Knowing the asymptotic behaviour of Monge–Kantorovich contours opens the way to their estimation, possibly beyond the range of the data. Extreme quantile contours represent large but not quite impossible values of the random vector of interest. In an engineering context, for example, they may provide load conditions for reliability testing of a system or structure in a numerical or physical scale model.

For the Monge–Kantorovich quantile contours to have an interesting tail limit, it is inevitable to assume some kind of regularity condition on the multivariate tail of the target measure $\nu$. Here we rely on the framework of multivariate regular variation [25, 26], common in the asymptotic theory of affinely normalized sample maxima and sums to max-stable and sum-stable distributions, respectively. Because regularly varying distributions may not have finite second moments, the moment-free set-up in McCann [20] is ideal for our purpose.

For the reference measure $\mu$, we choose the spherically symmetric measure which has the same radial measure as the target measure $\nu$. This problem-specific choice of reference measure is different from the uniform distribution on the unit ball proposed in [7] and [13], and is motivated by our search for stability of the (multivalued) transport mapping $\partial\psi$ between $\mu$ and $\nu$ as we move further out into the tail: it ensures that $\mu$ is regularly varying as well, and as regular variation involves the convergence of certain finite intensity measures built from $\mu$ and $\nu$ to intensity measures $\bar{\mu}$ and $\bar{\nu}$, one might suspect that a basic stability result along the lines of Theorem 5.20 in Villani [30] applies to optimal transport maps between the intensity measures.

This is where we encounter a major issue: the limiting measures $\bar{\mu}$ and $\bar{\nu}$ are not even finite; convergence takes place in the space $\mathcal{M}_0(\mathbb{R}^d)$ of measures that are finite on complements of neighbourhoods of the origin [16]. This is why we extend in Sections 3
and 4 the results on existence, uniqueness and representations of optimal couplings between probability measures in McCann [20] to the possibly infinite measures in the space $\mathcal{M}_0(\mathbb{R}^d)$. For two such measures, we show existence of a coupling measure with cyclically monotone support, and if one of the measures vanishes on Borel sets of Hausdorff dimension $d - 1$, then the coupling is induced by the gradient of a convex function, the gradient being essentially unique.

The space $\mathcal{M}_0(\mathbb{R}^d)$ comes equipped with a topology able to accommodate convergence of possibly infinite measures. For sequences of pairs of measures converging in $\mathcal{M}_0(\mathbb{R}^d)$, we show in Section 4 the stability of the sequence of coupling measures with cyclically monotone supports and the subdifferentials that support those coupling measures. Weak convergence of probability measures being equivalent to their convergence in $\mathcal{M}_0(\mathbb{R}^d)$, our stability result complements those in Cuesta-Albertos et al. [8, Section 3] and Villani [30, Chapter 5]. As in Hallin [13] and del Barrio et al. [11], no moment assumptions are needed, but, in contrast to these papers, the measures to be coupled do not need to be absolutely continuous either.

In our stability result, the subdifferentials of the convex potentials supporting the coupling measures converge graphically. Thanks to the maximal monotonicity of such mappings, their graphical convergence implies a form of local uniform convergence. In Section 5, we exploit this property to show Hausdorff metric convergence of quantile contours generated by cyclically monotone transport plans. For Monge–Kantorovich quantile contours, this further sets our work apart from [7, Appendix A] as we do not assume that the limit measures are absolutely continuous nor that they have compact support.

In Section 6, we finally establish the asymptotic shape of tail quantile contours of regularly varying distributions. Here, it is essential to have allowed for possibly infinite measures, motivating our choice for the space $\mathcal{M}_0(\mathbb{R}^d)$. Section 7 treats two examples: regularly varying elliptical distributions and linear transformations of vectors of independent regularly varying random variables. Section 8 concludes. Statements and proofs of some auxiliary results are gathered in the Appendix.

2. Preliminaries

We will be needing concepts and results from variational analysis and measure transportation. This section serves to recall some basic facts and to fix some notation. The interior, closure and boundary of a subset $A$ of a topological space are denoted by $\text{int}(A)$, $\text{cl}(A)$ and $\text{bnd}(A)$, respectively. Let $\text{conv}(A)$ denote the convex hull of a subset $A$ of a real vector space. The indicator function of $A$ is $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ if $x \notin A$; sometimes we also write $\mathbb{I}(x \in A)$. The identity function on a space clear from the context is denoted by I. The scalar product and the Euclidean norm on Euclidean space are denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively. The unit sphere in $\mathbb{R}^d$ is denoted by $S_{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$, while the open and closed unit balls in Euclidean space are $B^o = \{v : |v| < 1\}$ and $B = \{v : |v| \leq 1\}$, respectively, where the dimension will be clear from the context. For a point $x$ and a scalar $r > 0$, the sets $B^o(x, r) = x + rB^o = \{x + rv : |v| < 1\}$ and $B(x, r) = x + rB = \{x + rv : |v| \leq 1\}$ are the open and closed balls with centre $x$ and radius $r$. More generally, for subsets $A$ and $B$ of some Euclidean space, we put $A + B = \{a + b : a \in A, b \in B\}$. The Hausdorff
distance between two non-empty bounded subsets $K$ and $L$ of Euclidean space is

$$d_H(K, L) = \inf\{ \varepsilon \geq 0 : K \subset L + \varepsilon B \text{ and } L \subset K + \varepsilon B \}.$$ 

It is a metric when restricted to the collection of non-empty compact subsets. The set of all Borel probability measures on $\mathbb{R}^k$ is denoted by $\mathcal{P}(\mathbb{R}^k)$.

2.1. Convex functions and their subdifferentials. We consider convex functions $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ whose domain, $\text{dom} \, \psi = \{ x \in \mathbb{R}^d : \psi(x) < +\infty \}$, is not empty; such convex functions are called proper [27], a property which we will assume throughout. A convex function $\psi$ is said to be closed if its epigraph, $\{(x, \lambda) \in \mathbb{R}^d \times \mathbb{R} : \psi(x) \leq \lambda\}$, is closed, or equivalently, if $\psi$ is lower semicontinuous. A convex function can be closed by taking its lower semicontinuous minorant. This operation amounts to closing the function’s epigraph in $\mathbb{R}^d \times \mathbb{R}$ and only affects its values on the boundary of its domain.

The subdifferential of a convex function $\psi$ at a point $x \in \mathbb{R}^d$ is the set $\partial \psi(x)$ of all points $y \in \mathbb{R}^d$ such that

$$\forall z \in \mathbb{R}^d, \quad \psi(z) \geq \psi(x) + \langle y, z - x \rangle.$$ 

The domain of $\partial \psi$, notation $\text{dom} \partial \psi$, is the set of all $x$ at which $\partial \psi(x)$ is not empty; at such points, we say that $\psi$ is subdifferentiable. The function $\psi$ is differentiable at $x$ if and only $\partial \psi(x)$ is a singleton, and then $\partial \psi(x) = \{ \nabla \psi(x) \}$. This happens at all $x$ in the interior of $\text{dom}(\psi)$ minus a set of Hausdorff dimension at most $d - 1$ [2] and thus of zero Lebesgue measure [27, Theorem 25.5]. The subdifferential $\partial \psi(x)$ is empty if $\psi(x) = +\infty$ and nonempty if $x \in \text{int}(\text{dom} \, \psi)$. Combining Rockafellar [27, Theorem 6.3, Theorem 23.4, and Corollary 25.1.1], we have in fact

$$\begin{align*}
\text{dom} \nabla \psi &\subset \text{int}(\text{dom} \partial \psi) = \text{int}(\text{dom} \psi), \\
\text{dom} \partial \psi &\subset \text{dom} \psi, \\
\text{cl}(\text{dom} \partial \psi) &\subset \text{cl}(\text{dom} \psi).
\end{align*}$$

We view $\partial \psi$ as a multivalued mapping (or mapping in short), that is, a map from $\mathbb{R}^d$ into the power set of $\mathbb{R}^d$; notation $\partial \psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$. The graph of a mapping $S : \mathcal{X} \rightrightarrows \mathcal{Y}$ is

$$\text{gph}(S) = \{ (x, y) \in \mathcal{X} \times \mathcal{Y} : y \in S(x) \}.$$ 

Note that the graph of $S$ is a subset of $\mathcal{X} \times \mathcal{Y}$ rather than of the Cartesian product of $\mathcal{X}$ and the power set of $\mathcal{Y}$. The domain of $S$ is $\text{dom} \, S = \{ x \in \mathcal{X} : S(x) \neq \emptyset \}$. For two mappings $S, T : \mathcal{X} \rightrightarrows \mathcal{Y}$, we say that $S \subset T$ if $S(x) \subset T(x)$ for all $x \in \mathcal{X}$, or equivalently, if $\text{gph}(S) \subset \text{gph}(T)$.

The derivative of a convex function on a real interval is non-decreasing, and non-decreasing functions constitute the cheapest way for moving masses on the real line with respect to squared distance as cost function. On Euclidean space, subdifferentials of convex functions play a similar role.

A mapping $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is called monotone if, for all pairs $(x_1, y_1)$ and $(x_2, y_2)$ such that $y_1 \in S(x_1)$ and $y_2 \in S(x_2)$, we have

$$\langle x_2 - x_1, y_2 - y_1 \rangle \geq 0. \quad (2)$$

Inequality (2) is equivalent to

$$|x_1 - y_1|^2 + |x_2 - y_2|^2 \leq |x_1 - y_2|^2 + |x_2 - y_1|^2, \quad (3)$$

where $\langle \cdot , \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$. The set of all closed convex functions on $\mathbb{R}^d$ is denoted by $\mathcal{C}(\mathbb{R}^d)$. We say that $\psi$ is proper if $\psi(x) = +\infty$ for all $x \not\in \text{dom} \psi$. A mapping $S : \mathcal{X} \rightrightarrows \mathcal{Y}$ is proper if $S(x) = \emptyset$ for all $x \not\in \text{dom} \, S$. A closed proper convex function $\psi$ takes finite values on its domain and can be recovered from its subdifferential by taking its lower semicontinuous minorant. In particular, $\psi(x) = \inf_{y \in \text{dom} \, \psi} \{ \langle y, x \rangle - \psi(y) \}$ for all $x \in \text{dom} \, \psi$.
stating that transporting a unit mass from $x_1$ to $y_1$ and another one from $x_2$ to $y_2$ is cheaper than when the destinations are switched. Monotone mappings enjoy many boundedness and continuity properties reviewed in Alberti and Ambrosio [1] and in [28, Chapter 12], some of which we will use later on.

Subdifferentials of convex functions enjoy an even stronger property. A multivalued mapping $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is called cyclically monotone if, for every finite collection of pairs $(x_1, y_1), \ldots, (x_n, y_n)$ such that $y_i \in S(x_i)$ for all $i = 1, \ldots, n$, we have, writing $y_{n+1} = y_1$,

$$\sum_{i=1}^n \langle x_i, y_i \rangle \geq \sum_{i=1}^n \langle x_i, y_{i+1} \rangle. \quad (4)$$

Inequality (4) extends inequality (2) from two to $n$ pairs $(x_i, y_i)$, and it implies that, rather than (3), we have

$$\sum_{i=1}^n |x_i - y_i|^2 \leq \sum_{i=1}^n |x_i - y_{i+1}|^2, \quad (5)$$

stating that moving a unit point mass from location $x_i$ to location $y_i$ for all $i = 1, \ldots, n$ is the cheapest possible transport plan among all plans moving unit point masses from locations $x_1, \ldots, x_n$ to locations $y_1, \ldots, y_n$ (since any permutation of $\{1, \ldots, n\}$ can be decomposed into cycles on disjoint subsets). A cyclically monotone mapping is clearly monotone, but in dimension $d \geq 2$, the converse is not true.

A mapping $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is called maximal (cyclically) monotone if it is (cyclically) monotone and if it is not strictly contained in another (cyclically) monotone mapping. The graphs of such mappings are necessarily closed. The relevance of subdifferentials of convex functions for optimal measure transport comes from the following characterization [27, Theorem 24.8 and 24.9; 28, Theorem 12.25].

**Theorem 2.1** (Rockafellar’s Theorem). A mapping $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is cyclically monotone if and only if it is contained in the subdifferential of a closed convex function. It is maximal cyclically monotone if it is equal to the subdifferential of a closed convex function.

In addition, subdifferentials of closed convex functions are maximal monotone, see [27, Corollary 31.5.2] or [28, Theorem 12.17]. This property does not follow immediately from the fact that they are maximal cyclically monotone and is greatly helpful when studying graphical convergence of sequences of subdifferentials (Appendix B).

A subset $T$ of $\mathbb{R}^d \times \mathbb{R}^d$ can be identified with the multivalued mapping sending $x \in \mathbb{R}^d$ to $\{y \in \mathbb{R}^d : (x, y) \in T\}$; clearly, $T$ is the graph of this mapping. Such a set $T$ will be said to enjoy one of the monotonicity properties above if the associated mapping possesses the property.

To study the asymptotic properties of a sequence of subdifferentials of closed convex functions, we will rely on the concept of graphical convergence. A sequence of mappings $S_n : \mathbb{R}^k \rightrightarrows \mathbb{R}^\ell$ converges graphically to a mapping $S : \mathbb{R}^k \rightrightarrows \mathbb{R}^\ell$ if the graphs of $S_n$ converge to the graph of $S$ as subsets of $\mathbb{R}^k \times \mathbb{R}^\ell$ in the sense of Painlevé–Kuratowski. The graphical limit of a sequence of subdifferentials of closed convex functions is again the subdifferential of a closed convex function, and the convergence takes place locally
uniformly at points in the domain of the gradient of the limit function. Graphical convergence and relevant properties thereof, some of which we believe are novel, are reviewed in Appendix B.

Cyclically monotone mappings provide solutions to optimal assignment problems via (5). They also play a key role in the theory of optimal transport, as explained next.

2.2. Measure transportation. For two Borel measures \( \mu \) and \( \nu \) on \( \mathbb{R}^d \), not necessarily finite, a coupling measure is a Borel measure \( \pi \) on \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying \( \pi(A \times \mathbb{R}^d) = \mu(A) \) and \( \pi(\mathbb{R}^d \times A) = \nu(A) \) for every Borel set \( A \subset \mathbb{R}^d \). The collection of coupling measures of \( \mu \) and \( \nu \) is denoted by \( \Pi(\mu, \nu) \). We call \( \mu \) and \( \nu \) the (left and right) marginals of \( \pi \).

Suppose \( \mu \) and \( \nu \) are finite Borel measures on \( \mathbb{R}^d \) with equal, non-zero mass. An optimal transport plan with respect to the squared distance as cost function is a coupling measure \( \pi \in \Pi(\mu, \nu) \) solving the Kantorovich problem

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\pi(x, y) = \inf_{\pi' \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\pi'(x, y). \tag{6}
\]

An optimal transport plan exists if \( \mu \) and \( \nu \) have finite second-order moments, i.e., \( \int_{\mathbb{R}^d} |x|^2 \, d\mu(x) < \infty \) and \( \int_{\mathbb{R}^d} |y|^2 \, d\nu(y) < \infty \) [29, Theorem 2.12]. The infimum in (6) is the square of the (quadratic) Wasserstein distance between \( \mu \) and \( \nu \). The case of more general cost functions is extensively covered in Villani [30].

The condition that \( \mu \) and \( \nu \) have finite second-order moments can be relaxed if we allow the optimal transport cost in (6) to be infinite. In this case, all transport plans have “optimal” cost, but a meaningful generalisation of an optimal transport plan can still be defined as a coupling measure \( \pi \) with cyclically monotone support; see, e.g., [29, Section 2.3]. Recall that the support \( \text{spt}(\mu) \) of a Borel measure \( \mu \) on a metric space is the collection of points in the space with the property that every neighbourhood of the point receives positive mass. The support is necessarily a closed set, and it is in fact the smallest closed subset of the space of which the complement is a null set by the given measure. Recall that \( \mathcal{P}(\mathbb{R}^k) \) denotes the set of Borel probability measures on \( \mathbb{R}^k \).

**Theorem 2.2** (McCann [20], Theorem 6). Let \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \). There exists \( \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) having \( \mu \) and \( \nu \) as its marginals and with cyclically monotone support.

A variation of the Kantorovich optimal transport problem (6) is the Monge problem: if \( X \) and \( Y \) are random vectors with distributions \( \mu \) and \( \nu \), respectively, then we seek a \( \mu \)-almost everywhere defined Borel measurable map \( T \) that minimizes \( \mathbb{E}[|X - T(X)|^2] \) under the constraint that \( T(X) \) is equal in distribution to \( Y \). The distribution of \( T(X) \) is the push-forward, \( T_{\#} \mu \), of \( \mu \) by \( T \), defined as

\[
T_{\#} \mu(A) = \mu \left( T^{-1}(A) \right) = \mu(\{x \in \mathbb{R}^d : T(x) \in A\})
\]

for all Borel sets \( A \subset \mathbb{R}^d \). The joint distribution, \( \pi \), of \( (X, T(X)) = (\text{Id} \times T)(X) \) is \( (\text{Id} \times T)_{\#} \mu \), with \( \text{Id} \) the identity map. Its support is a subset of the closure in \( \mathbb{R}^d \times \mathbb{R}^d \) of the graph of \( T \). It is a coupling of \( \mu \) and \( \nu \) provided \( T_{\#} \mu = \nu \).

Again, relaxing the condition that \( \mu \) and \( \nu \) have finite second-order moments, we can seek a cyclically monotone function \( T \) such that \( T_{\#} \mu = \nu \). If \( \mu \) vanishes on certain small sets, then any measure \( \pi \) in Theorem 2.2 turns out to be exactly of the form \( (\text{Id} \times T)_{\#} \mu \), with \( T \) the gradient of a convex function. Recall Rockafellar’s Theorem 2.1, by which
the cyclically monotone support of $\pi$ in Theorem 2.2 must be a subset of the graph of the subdifferential of a closed convex function.

**Proposition 2.3** (McCann [20], Proposition 10). Suppose $\pi \in P(\mathbb{R}^d \times \mathbb{R}^d)$ is supported on the graph of the subdifferential $\partial \psi$ of a convex function $\psi$. Let $\mu$ and $\nu$ denote the marginals of $\pi$. If $\mu$ vanishes on (Borel) sets of Hausdorff dimension $d - 1$, then $\pi = (\text{Id} \times \nabla \psi)\#\mu$ and thus $\nu = \nabla \psi\#\mu$.

Not only do Theorem 2.2 and Proposition 2.3 guarantee the existence of a convex function $\psi$ such that $\nabla \psi$ pushes $\mu$ forward to $\nu$, the map $\nabla \psi$ is actually uniquely defined up to $\mu$-null sets.

**Theorem 2.4** (McCann [20], Main Theorem). Let $\mu, \nu \in P(\mathbb{R}^d)$ and suppose $\mu$ vanishes on (Borel) subsets of $\mathbb{R}^d$ having Hausdorff dimension $d - 1$. Then there exists a convex function $\psi$ on $\mathbb{R}^d$ whose gradient $\nabla \psi$ pushes $\mu$ forward to $\nu$. Although $\psi$ is not unique, the map $\nabla \psi$ is uniquely determined $\mu$-almost everywhere.

The uniqueness of the gradient $\nabla \psi$ in Theorem 2.4 together with the description of coupling measures with cyclically monotone support in Proposition 2.3 implies the uniqueness of a coupling measure $\pi$ with cyclically monotone support.

**Corollary 2.5** (McCann [20], Corollary 14). Suppose $\mu, \nu \in P(\mathbb{R}^d)$, and that one of these measures vanishes on all sets of Hausdorff dimension $d - 1$. Then the joint measure $\pi \in \Pi(\mu, \nu)$ with cyclically monotone support is unique.

McCann’s theorems are formulated for probability measures, but obviously, they remain true for finite Borel measures with equal, finite, non-zero mass.

### 3. Cyclically monotone transports between infinite measures

Motivated by the study of tail quantile contours of regularly varying probability measures in Section 6, we seek to extend McCann’s theory as sketched in Section 2.2 to a certain space of Borel measures on $\mathbb{R}^d$ with possibly infinite mass. Let $\mathcal{M}_0(\mathbb{R}^d)$ be the set of all Borel measures $\mu$ on $\mathbb{R}^d \setminus \{0\}$ that are finite on complements of neighbourhoods of the origin, that is, such that $\mu(\mathbb{R}^d \setminus rB)$ for all $r > 0$. The existence of the gradient of a convex function pushing one such measure to another one will follow from an approximation argument involving a sequence of finite measures, see Theorem 4.3 below. We first establish the uniqueness of such a gradient. Note that McCann’s arguments do not and cannot readily extend to arbitrary infinite measures, not even to $\sigma$-finite ones: the Lebesgue measure on Euclidean space, for instance, is invariant under translations, violating uniqueness of the gradient.

**Theorem 3.1** (Uniqueness). Let $\mu \in \mathcal{M}_0(\mathbb{R}^d)$ be a non-zero measure and suppose that $\mu$ vanishes on all sets of Hausdorff dimension at most $d - 1$. Let $\psi$ and $\phi$ be convex functions $\mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ that are finite $\mu$-almost everywhere. If $\nabla \psi\#\mu = \nabla \phi\#\mu \in \mathcal{M}_0(\mathbb{R}^d)$, then $\nabla \psi = \nabla \phi$ $\mu$-almost everywhere.

The proof of Theorem 3.1 requires several steps and makes extensive use of the property that measures in $\mathcal{M}_0(\mathbb{R}^d)$ are finite on sets bounded away from the origin in order to construct objects with certain properties. In contrast, the following result only requires $\sigma$-finiteness, and its proof remains close to the one of Proposition 10 in [20].
Proposition 3.2 (Representation). Let the (possibly infinite) Borel measure $\pi$ on $\mathbb{R}^d \times \mathbb{R}^d$ be supported on the graph of the subdifferential $\partial \psi$ of a closed convex function $\psi$. Let $\mu$ and $\nu$ denote the marginals of $\pi \in \Pi(\mu, \nu)$. If $\mu$ is $\sigma$-finite and vanishes on (Borel) sets of Hausdorff dimension $d-1$, then $\nabla \psi$ is defined $\mu$-almost everywhere and $\pi = (\Id \times \nabla \psi)\# \mu$, implying $\nabla \psi \# \mu = \nu$.

Remark 3.3. In Theorem 3.1, the condition that the measure $\nabla \psi \# \mu = \nabla \phi \# \mu$ belongs to $\mathcal{M}_0(\mathbb{R}^d)$ is used only to show via Lemma 3.4 that $\psi(0)$ and $\phi(0)$ are finite in case $\mu$ has infinite mass. A weaker condition was thus possible, but since the focus in our paper is on the space $\mathcal{M}_0(\mathbb{R}^d)$, we have opted for the current formulation.

Proofs.

Proof of Theorem 3.1. If $0 \notin \spt \mu$, then $\spt \mu$, being closed, is bounded away from 0. But as $\mu \in \mathcal{M}_0(\mathbb{R}^d)$, we must have $0 < \mu(\mathbb{R}^d) < +\infty$ and we can resort to the uniqueness proof in McCann [20, pp. 318–319]. Therefore, consider the case that $\mu(\mathbb{R}^d) = +\infty$ and thus $0 \in \spt \mu$. By Lemma 3.4, $\psi$ and $\phi$ attain their minima at 0, so that $\psi(0)$ and $\phi(0)$ are certainly finite. Therefore, without losing generality, take $\psi(0) = \phi(0) = 0$.

Step 1. — Since $\psi$ and $\phi$ are finite $\mu$-a.e., $\spt \mu$ must lie in $\cl(\dom \psi) \cap \cl(\dom \phi)$. Both closures are convex. Therefore, their boundaries have Hausdorff dimension at most $d-1$, so they are $\mu$-null sets. Consequently, we may take closed convex functions for $\psi$ and $\phi$, since closure only affects their values on the boundaries of their domains. For the same reason, we can restrict attention to

$$D := \text{int}(\dom \psi) \cap \text{int}(\dom \phi) = \text{int}(\dom \psi \cap \dom \phi)$$

and its subsets [see (1)]

$$V := \dom(\nabla \psi) \cap \dom(\nabla \phi),$$

$$W := \{ y \in V : \nabla \psi(y) \neq \nabla \phi(y) \},$$

$$A := \{ y \in D : \psi(y) - \phi(y) \neq 0 \}.$$  

By Alberti and Ambrosio [1, Theorem 2.2] (see also [2]), $D \setminus V$ is contained in the union of two sets with Hausdorff dimension at most $d-1$, so

$$\mu(D \setminus V) = 0.$$  

Our claim is equivalent to $\mu(W) = 0$. To prove it, we examine the consequences of its failure, so henceforth, we suppose that

$$\mu(W) > 0,$$

and we will construct a subset of $\mathbb{R}^d$ that receives different masses from the two push-forwards $\nabla \psi \# \mu$ and $\nabla \phi \# \mu$, in contradiction to the assumption of the theorem.

Step 2. — By Lemma 3.5, the set

$$W \setminus A = \{ y \in V : \psi(y) - \phi(y) = 0 \text{ and } \nabla \psi(y) \neq \nabla \phi(y) \}$$

has Hausdorff dimension at most $d-1$. Therefore, $\mu(W \setminus A) = 0$. By (11), we obtain

$$\mu(W \cap A) > 0.$$
Step 3. — Recall $D$ in (7). For $t \in [-\infty, +\infty]$, define
\[
A^-_t := \{ y \in D : \psi(y) - \phi(y) < t \}, \\
A^+_t := \{ y \in D : \psi(y) - \phi(y) > t \}, \\
A^0_t := \{ y \in D : \psi(y) - \phi(y) = t \}.
\]
Then $A^-_0 \cup A^+_0$ is a partition of $A$ in (9) and thus, by (12),
\[
\mu(W \cap A^-_0) + \mu(W \cap A^+_0) = \mu(W \cap A) > 0.
\]
Therefore, $\mu(W \cap A^-_0) > 0$ or $\mu(W \cap A^+_0) > 0$. Without loss of generality, assume
\[
\mu(W \cap A^-_0) > 0.
\]
In the other case, just switch the roles of $\psi$ and $\phi$ in the next steps.

Step 4. — For every $t < 0$, the closure of the set $A^-_t$ does not contain the origin, since $\psi$ and $\phi$ are continuous and $\psi(0) - \phi(0) = 0 > t$. As $A \in M_d(\mathbb{R}^d)$, we must have
\[
\forall t < 0, \quad \mu(A^-_t) < +\infty.
\]

Step 5. — Recall $V$ in (8). We claim that there exists $z \in V$ with the following three properties:
\begin{itemize}
  \item $t := \psi(z) - \phi(z) < 0$;
  \item $\nabla \psi(z) \neq \nabla \phi(z)$;
  \item for all $\delta > 0$,
  \[
  \mu(A^-_t \cap B^\circ(z, \delta)) > 0,
  \]
\end{itemize}
where $B^\circ(\cdot, \cdot)$ denotes the open ball with given centre and radius.

To prove the claim, define, for every $t < 0$, the sets
\[
\Omega_t := A^0_t \cap W \cap \text{spt } \mu \\
= \{ x \in V : \psi(x) - \phi(x) = t \text{ and } \nabla \psi(x) \neq \nabla \phi(x) \} \cap \text{spt } \mu, \\
\Xi_t := \{ x \in \Omega_t : \exists \delta > 0, \mu(A^-_t \cap B^\circ(x, \delta)) = 0 \}.
\]

We need to show that there exists $t < 0$ such that $\Omega_t \setminus \Xi_t \neq \emptyset$. To this end, define
\[
T_\Omega := \{ t \in (-\infty, 0) : \Omega_t \neq \emptyset \},
\]
\[
T_\Xi := \{ t \in (-\infty, 0) : \Xi_t \neq \emptyset \}.
\]

We will show that $T_\Omega$ is uncountable while $T_\Xi$ is at most countable, so that there must exist (uncountable many) $t < 0$ such that $\Omega_t$ is nonempty and $\Xi_t$ is empty.

By Lemma 3.5, for every $t < 0$, the set $\Omega_t$ has Hausdorff measure at most $d - 1$ and is therefore a $\mu$-null set. Still, their union over all $t$ less than $0$ is
\[
\bigcup_{t < 0} \Omega_t = \bigcup_{t < 0} A^0_t \cap W \cap \text{spt } \mu = A^0_0 \cap W \cap \text{spt } \mu,
\]
a set receiving positive mass from $\mu$ by (13). Therefore, $T_\Omega$ must be uncountable.

We have $\Xi_t = \bigcup_{n \in \mathbb{N}} \Xi_{t, n}$ and thus $T_\Xi = \bigcup_{n \in \mathbb{N}} T_n$, where, for $n \in \mathbb{N}$,
\[
\Xi_{t, n} := \{ x \in \Omega_t : \mu(A^-_t \cap B^\circ(x, 2/n)) = 0 \},
\]
\[
T_n := \{ t \in (-\infty, 0) : \Xi_{t, n} \neq \emptyset \}.
\]

To show that $T_\Xi$ is countable, it is sufficient to show that each $T_n$ is countable.
Fix $n \in \mathbb{N}$. Let $s, t \in \mathcal{T}_n$ with $s < t$ and let $x_s \in \Xi_{s,n}$ and $x_t \in \Xi_{t,n}$. As $\Xi_{s,n} \subset \Omega_s \subset \text{spt} \mu$, the point $x_s$ belongs to the support of $\mu$ and can therefore not belong to the open $\mu$-null set $A^-_t \cap B^o(x_t, 2/n)$. But as $\psi(x_s) = \phi(x_s) = s < t$, we do have $x_s \in A^-_t$. Therefore, $x_s$ cannot be an element of $B(x_t, 2/n)$, that is, $|x_s - x_t| > 2/n$. By the triangle inequality, the balls $B^o(x_s, 1/n)$ and $B^o(x_t, 1/n)$ must be disjoint. Define
\[ \forall t \in \mathcal{T}_n, \quad G_t := \bigcup_{x \in \Xi_{t,n}} B^o(x, 1/n). \]
Each $G_t$ is non-empty and open, and by the argument just given, $G_s \cap G_t = \emptyset$ whenever $s, t \in \mathcal{T}_n$ with $s < t$. Since $\mathbb{R}^d$ is separable, $\mathcal{T}_n$ is at most countable, as required. The claim made at the beginning of this step is proved.

**Step 6.** — For $z$ from Step 5, define the closed convex functions $\varphi := \phi - \phi(z)$ and $\bar{\varphi} := \psi - \psi(z)$. Then $\varphi(z) = \bar{\varphi}(z)$ and $\nabla \varphi(z) \neq \nabla \bar{\varphi}(z)$. Let $t = \psi(z) - \phi(z) < 0$ and consider the open set $\mathcal{M} := A^-_t = \{ y \in D : \varphi(y) > \bar{\varphi}(y) \}$. By Steps 4 and 5, we have $0 < \mu(M) < +\infty$.

At this point, we can apply the reasoning of McCann [20, pp. 318–319]. It shows that $\partial \varphi(M)$ is a Borel set and that by Aleksandrov’s lemma [20, Lemma 13], the set $Z := [\nabla \bar{\varphi}]^{-1}(\partial \varphi(M))$ is at a nonzero distance from $z$ and satisfies $Z \subset M$. Therefore, (14) implies that $\mu(M \setminus Z) > 0$, and since $\mu(M)$ is finite, that $\mu(Z) < \mu(M)$. Hence,
\[ [\nabla \bar{\varphi}]^{-1}(\partial \varphi(M)) = [\nabla \varphi]^{-1}(\partial \varphi(M)) = \mu(Z) < \mu(M) \]
\[ \leq \mu ([\nabla \varphi]^{-1}(\partial \varphi(M))) = |\nabla \varphi|^{-1}(\partial \varphi(M)), \]
the last inequality deriving from (10) and
\[ M \subset (\mathcal{D} \setminus V) \cup [\nabla \varphi]^{-1}(\partial \varphi(M)) \subset (\mathcal{D} \setminus V) \cup [\nabla \varphi]^{-1}(\partial \varphi(M)). \]
As a consequence, $\nabla \bar{\varphi} \# \mu \neq \nabla \varphi \# \mu$, in contradiction to the assumption of the theorem. The inequality (11) must therefore be false, and the proof is complete. □

**Lemma 3.4.** Let $\mu, \nu \in \mathcal{M}_0(\mathbb{R}^d)$ with $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d) = +\infty$ and let $\pi \in II(\mu, \nu)$ be supported by $\text{gph}(\partial \psi)$ for some closed convex function $\psi$. Then $\psi(0) = \inf_{x \in \mathbb{R}^d} \psi(x)$. This holds in particular if $\mu \in \mathcal{M}_0(\mathbb{R}^d)$ has infinite mass, $\psi$ is a closed convex function that is $\mu$-a.e. differentiable, and $\nu = \nabla \psi \# \mu \in \mathcal{M}_0(\mathbb{R}^d)$.

**Proof.** The coupling measure $\pi$ must have infinite mass but still belong to $\mathcal{M}_0(\mathbb{R}^d \times \mathbb{R}^d)$, see Lemma A.1(a). Therefore, necessarily $\pi(\varepsilon B \times \varepsilon B) = +\infty$ for every $\varepsilon > 0$, which means that $\text{spt} \pi$ meets $\varepsilon B \times \varepsilon B$ for every $\varepsilon > 0$. Since moreover $\pi$ does not charge the origin, there must exist points $(x_n, y_n) \in \text{spt} \pi \subset \text{gph} \partial \psi$ such that $0 \neq x_n \to 0$ and $0 \neq y_n \to 0$ as $n \to \infty$. In particular $x_n \in \text{dom} \psi$ for every $n \in \mathbb{N}$. For every $z \in \mathbb{R}^d$, we have
\[ \forall n \in \mathbb{N}, \quad \psi(z) \geq \psi(x_n) + \langle z - x_n, y_n \rangle \geq \psi(x_n) - |z - x_n| |y_n|. \]
Let $n \to \infty$ and use the lower semicontinuity of $\psi$ to conclude that
\[ \psi(0) \leq \liminf_{n \to \infty} \psi(x_n) \leq \liminf_{n \to \infty} \left( \psi(z) + |z - x_n| |y_n| \right) = \psi(z). \]

If $\psi$ is $\mu$-a.e. differentiable and if $\nu = \nabla \psi \# \mu$, then $\pi = (\text{Id} \times \nabla \psi) \# \mu$ belongs to $II(\mu, \nu)$ and the support of $\pi$ is contained in $\text{cl}(\{(x, \nabla \psi(x)) : x \in \text{dom} \nabla \psi\})$, which is in turn a
subset of \( \text{gph}(\partial \psi) \). Note that \( \text{gph}(\partial \psi) \) is closed since \( \psi \) was assumed to be closed and thus \( \partial \psi \) is maximal cyclically monotone by Rockafellar’s theorem (Section 2.1).

**Lemma 3.5.** Let \( \psi \) and \( \phi \) be convex functions on \( \mathbb{R}^d \). For every \( t \in \mathbb{R} \), the set
\[
\{ x \in \text{dom}(\nabla \psi) \cap \text{dom}(\nabla \phi) : \psi(x) - \phi(x) = t \text{ and } \nabla \psi(x) \neq \nabla \phi(x) \}
\]
has Hausdorff dimension at most \( d - 1 \).

**Proof.** Let \( X \) be the set in (15). By McCann’s Implicit Function Theorem [20, Theorem 17 and Corollary 19], every point \( x \) in \( X \) has an open neighbourhood \( N_x \) such that the \((d - 1)\)-dimensional Hausdorff measure of the set
\[
\{ y \in N_x : \psi(y) - \phi(y) = t \}
\]
is finite. The collection of sets \( \{N_x : x \in X\} \) forms an open cover of \( X \). By the Lindelöf property of \( \mathbb{R}^d \), there exists a countable subcover, i.e., a countable set \( Q \subset X \) such that \( X \) is a subset of \( \bigcup_{x \in Q} N_x \) and then also of
\[
\bigcup_{x \in Q} \{ y \in N_x : \psi(y) - \phi(y) = t \}.
\]
The latter set is a countable union of sets of Hausdorff dimension at most \( d - 1 \) and has therefore Hausdorff dimension at most \( d - 1 \) as well.

**Proof of Proposition 3.2.** The proof is almost identical to the one of Proposition 10 in McCann [20]. The first paragraph of that proof yields that \( \psi \) is differentiable on a Borel set whose complement is a \( \mu \)-null set and that \( \nabla \psi \) is Borel measurable.

In the second paragraph of the cited proof, it is verified that \( \pi \) and \( (\text{Id} \times \nabla \psi)\#\mu \) coincide on rectangles \( M \times N \) of Borel sets \( M, N \subset \mathbb{R}^d \). This property implies that \( \pi \) and \( (\text{Id} \times \nabla \psi)\#\mu \) are equal, even if they are not necessarily finite, thanks to the hypothesis that \( \mu \) is \( \sigma \)-finite. Indeed, let \( (B_n)_n \) be a sequence of Borel sets of \( \mathbb{R}^d \) whose union is \( \mathbb{R}^d \) and such that \( \mu \) is finite on each \( B_n \). Then \( \pi(B_n \times \mathbb{R}^d) = \mu(B_n) = ((\text{Id} \times \nabla \psi)\#\mu)(B_n \times \mathbb{R}^d) \) are finite too. The restrictions of \( \pi \) and \( (\text{Id} \times \nabla \psi)\#\mu \) to \( B_n \times \mathbb{R}^d \) coincide on Borel measurable rectangles, and thus they must be equal. As the union of the sets \( B_n \times \mathbb{R}^d \) over all \( n \) is equal to \( \mathbb{R}^d \times \mathbb{R}^d \), we obtain the stated equality.

Finally, from \( \pi = (\text{Id} \times \nabla \psi)\#\mu \) it follows that
\[
\nu(B) = \pi(\mathbb{R}^d \times B) = \mu((\text{Id} \times \nabla \psi)^{-1}(\mathbb{R}^d \times B)) = \mu((\nabla \psi)^{-1}(B)) = (\nabla \psi\#\mu)(B)
\]
for every Borel set \( B \subset \mathbb{R}^d \).

4. Existence and stability of optimal coupling measures

4.1. Convergence of infinite measures. To study the stability of coupling measures and transport plans between measures in \( \mathcal{M}_0(\mathbb{R}^d) \), a notion of convergence is required that can handle sequences of measures with infinite mass. To this end, Hult and Lindskog [16] consider the set \( \mathcal{C}_0(\mathbb{R}^d) \) of continuous, bounded, real functions \( f \) on \( \mathbb{R}^d \) for which there is \( r > 0 \) such that \( f(x) = 0 \) whenever \( |x| \leq r \). They endow \( \mathcal{M}_0(\mathbb{R}^d) \) with the topology generated by open sets of the form
\[
\left\{ \nu \in \mathcal{M}_0(\mathbb{R}^d) : |\int f_i \, d\nu - \int f_i \, d\mu| < \varepsilon, \, i = 1, \ldots, k \right\}
\]
for $\mu \in \mathcal{M}_0(\mathbb{R}^d)$, positive integer $k$, functions $f_1, \ldots, f_k$ in $C_0(\mathbb{R})$, and $\varepsilon > 0$. The resulting topology on $\mathcal{M}_0(\mathbb{R}^d)$ is metrizable [16, Theorem 2.3]. Convergence in $\mathcal{M}_0(\mathbb{R}^d)$ is denoted by $\overset{w}{\rightharpoonup}$ and, in contrast to weak convergence, allows mass to accumulate near the origin (but not at infinity). A convenient criterion for $\nu_n \overset{w}{\rightharpoonup} \nu$ is that $\int f d\nu_n \to \int f d\nu$ for all $f \in C_0(\mathbb{R}^d)$. In many respects, $\mathcal{M}_0$-convergence is similar to weak convergence of finite Borel measures, with versions of the Portmanteau lemma, the continuous mapping theorem, and criteria for relative compactness, see for instance [16, 19].

Although all results will be formulated in terms of sequences of measures converging in $\mathcal{M}_0(\mathbb{R}^d)$, they also apply to weakly convergent sequence of probability measures. Clearly, the restriction of a probability measure on $\mathbb{R}^d$ to the complement of $\{0\}$ belongs to $\mathcal{M}_0(\mathbb{R}^d)$. Moreover, weak convergence of probability measures, denoted by $\overset{w}{\rightharpoonup}$, is equivalent to convergence in $\mathcal{M}_0(\mathbb{R}^d)$ of their restrictions to $\mathbb{R}^d \setminus \{0\}$.

**Lemma 4.1.** For a sequence of measures $(\mu_n)_n$ in $\mathcal{P}(\mathbb{R}^d)$, $\mu_n \overset{w}{\rightharpoonup} \mu \in \mathcal{P}(\mathbb{R}^d)$ if and only if $\mu_n(\cdot \setminus \{0\}) \overset{w}{\rightharpoonup} \mu' \in \mathcal{M}_0(\mathbb{R}^d)$. Furthermore, $\mu' = \mu(\cdot \setminus \{0\})$.

**Proof.** On the one hand, weak convergence implies $\mathcal{M}_0$-convergence, since the former involves a larger class of test functions of which the integrals should converge, namely all continuous, bounded real functions on $\mathbb{R}^d$, which includes $C_0(\mathbb{R}^d)$; $\mu' = \mu(\cdot \setminus \{0\})$ follows immediately.

On the other hand, $\mu_n(\cdot \setminus \{0\}) \overset{w}{\rightharpoonup} \mu' \in \mathcal{M}_0(\mathbb{R}^d)$ implies tightness of the sequence of probability measures $(\mu_n)_n$, so by Prokhorov’s theorem, every subsequence has a further subsequence converging weakly to some $\mu \in \mathcal{P}(\mathbb{R}^d)$. But then $\mu_n(\cdot \setminus \{0\}) \overset{w}{\rightharpoonup} \mu(\cdot \setminus \{0\})$ along this subsequence, so $\mu(\cdot \setminus \{0\}) = \mu'$. This determines $\mu$ uniquely, as it is a probability measure. Therefore, $\mu_n \overset{w}{\rightharpoonup} \mu$ along the full sequence. $\square$

### 4.2. Existence and stability

Theorem 3.1 treated the uniqueness of the gradient of a convex function pushing one measure in $\mathcal{M}_0(\mathbb{R}^d)$ forward to another one. Here, we will treat both the existence of such transport plans as well as their stability with respect to convergence in $\mathcal{M}_0(\mathbb{R}^d)$. The proof of the existence in Theorem 4.3 below is based on an approximation argument involving sequences of finite measures $\mu_n$ and $\nu_n$, for which McCann’s Theorem 2.2 already guarantees the existence of coupling measures $\pi_n$ with cyclically monotone support, contained in the graph of the subdifferential of a closed convex function $\psi_n$. To make the argument work, we need to be assured of the stability of the construction with respect to convergence in $\mathcal{M}_0(\mathbb{R}^d)$, and this is guaranteed by Theorem 4.2 below, which also applies to possibly infinite measures $\mu_n$ and $\nu_n$. Using relative compactness arguments, we show convergence of the coupling measures $\pi_n$ and the subdifferentials $\partial \psi_n$ along subsequences. Different subsequences may have different limits, but under a weak smoothness assumption on $\mu$, the limits are essentially unique, yielding convergence along the full sequence. For the subdifferentials, we consider graphical convergence (possibly relative to an open set) as in Appendix B, while for the coupling measures, we consider convergence in $\mathcal{M}_0(\mathbb{R}^d \times \mathbb{R}^d)$. Recall that $d_H$ denotes Hausdorff distance.

**Theorem 4.2** (Stability). Let $\mu_n, \nu_n, \bar{\mu}, \bar{\nu} \in \mathcal{M}_0(\mathbb{R}^d)$ be such that $\mu_n(\mathbb{R}^d) = \nu_n(\mathbb{R}^d) \in (0, +\infty]$ and $\bar{\mu}(\mathbb{R}^d) = \bar{\nu}(\mathbb{R}^d) \in (0, +\infty]$. Suppose that

$$
\mu_n \overset{w}{\rightharpoonup} \bar{\mu}, \quad \nu_n \overset{w}{\rightharpoonup} \bar{\nu}, \quad n \to \infty.
$$

(16)
Let \((\psi_n)_n\) be a sequence of closed convex functions such that for each \(n \in \mathbb{N}\), there exists \(\pi_n \in \Pi(\mu_n, \nu_n)\) such that \(\text{spt}(\pi_n) \subset \text{gph}(\partial \psi_n)\).

(a) Every subsequence contains a further subsequence for which there exists \(\bar{\pi} \in \Pi(\mu, \nu)\) and a closed convex function \(\bar{\psi}\) on \(\mathbb{R}^d\) such that \(\text{spt} \bar{\pi} \subset \text{gph}(\partial \bar{\psi})\) and, along the subsequence, \(\pi_n \rightharpoonup \bar{\pi}\) and \(\partial \psi_n \rightharpoonup \partial \bar{\psi}\). For \(x \in \text{dom}(\nabla \bar{\psi})\), we have \(d_H(\partial \psi_n(x_n), \{\nabla \psi(x)\}) \to 0\) whenever \(x_n \to x\) along the subsequence.

(b) If, in addition, \(\bar{\mu}\) vanishes on sets of Hausdorff dimension at most \(d - 1\), then \(\pi_n\) converges in \(\mathcal{M}_0(\mathbb{R}^d \times \mathbb{R}^d)\) along the full sequence to a unique limit measure \(\bar{\pi} \in \Pi(\bar{\mu}, \nu)\) of the form \(\bar{\pi} = (\text{Id} \times \nabla \bar{\psi}) \# \bar{\mu}\), and \(\nu = \nabla \bar{\psi} \# \bar{\mu}\), with the map \(\nabla \bar{\psi}\) determined uniquely \(\mu\)-almost everywhere.\(^1\)

(c) If, in addition, \(V := \text{int} \text{spt}(\bar{\mu})\) is nonempty, then the subdifferentials \(\partial \bar{\psi}\) of the functions \(\psi\) in (a) all coincide on \(V\). Along the full sequence, \(\partial \psi_n\) converges graphically to \(\partial \bar{\psi}\) relative to \(V\). For \(x \in V \cap \text{dom}(\nabla \bar{\psi})\), we have \(d_H(\partial \psi_n(x_n), \{\nabla \psi(x)\}) \to 0\) whenever \(x_n \to x\) as \(n \to \infty\).

**Theorem 4.3** (Existence). Let \(\mu, \nu \in \mathcal{M}_0(\mathbb{R}^d)\) have equal, non-zero mass. There exists a coupling measure \(\pi \in \Pi(\mu, \nu)\) with cyclically monotone support contained in the graph of the subdifferential of a closed convex function \(\psi\) on \(\mathbb{R}^d\). If \(\mu\) vanishes on all sets of Hausdorff dimension at most \(d - 1\), then \(\psi\) is differentiable \(\mu\)-almost everywhere, \(\nabla \psi \# \mu = \nu\), and the map \(\nabla \psi\) is uniquely determined \(\mu\)-almost everywhere.

If \(\mu\) and \(\nu\) in Theorem 4.3 have infinite mass, then by Lemma 3.4, the convex potential \(\psi\) contains its minimum at the origin.

The following corollary extends Theorem 4.2 to the case of weakly converging probability measures.

**Corollary 4.4.** Theorem 4.2 continues to hold if \(\mu_n, \nu_n, \mu, \nu\) are probability measures on \(\mathbb{R}^d\) and if \(\mathcal{M}_0\)-convergence in (16) is replaced by weak convergence.

### 4.3. Proofs.

**Proof of Theorem 4.2.** (a) The sequences \((\mu_n)_n\) and \((\nu_n)_n\) converge in \(\mathcal{M}_0(\mathbb{R}^d)\) and are thus relatively compact. By Lemma A.1, the sequence \((\pi_n)_n\) is then relatively compact in \(\mathcal{M}_0(\mathbb{R}^d \times \mathbb{R}^d)\). Every subsequence of \((\pi_n)_n\) has therefore a further subsequence, with indices in \(N \subset \mathbb{N}\), say, that converges in \(\mathcal{M}_0(\mathbb{R}^d \times \mathbb{R}^d)\) to some measure \(\bar{\pi}\), possibly depending on \(N\).

By Lemma A.3, \(\text{spt}(\bar{\pi})\) is a subset of the Painlevé–Kuratowski inner limit of \(\text{spt}(\pi_n)\) as \(n \to \infty\) in \(N\). Since \(\text{spt}(\pi_n) \subset \text{gph}(\partial \psi_n)\), also

\[
\text{spt}(\bar{\pi}) \subset \liminf_{n \to \infty, n \in N} \text{gph}(\partial \psi_n),
\]

so that \(\partial \psi_n\) cannot escape to the horizon as \(n \to \infty\) in \(N\). Therefore, there is a further subsequence with indices in \(M \subset N\) such that \(\partial \psi_n\) converges graphically to some multivalued mapping \(T\) with nonempty domain as \(n \to \infty\) in \(M\) [28, Theorem 5.36]. By Lemma B.2, \(T\) must be equal to the subdifferential of some closed convex function \(\psi\). As \(\text{gph}(\partial \psi)\) is the Painlevé–Kuratowski limit of \(\text{gph}(\partial \psi_n)\) as \(n \to \infty\) in \(M\), it is also equal

\(^1\)In the sense that any two functions \(\bar{\psi}\) that can appear in (a) have gradients that are equal \(\bar{\mu}\)-almost everywhere.
to the inner limit of those sets, which contains the inner limit of $\text{gph} (\partial \psi_n)$ as $n \to \infty$ in the larger subsequence $N$. By (17), we find $\text{spt} (\tilde{\pi}) \subset \text{gph} (\partial \tilde{\psi})$.

For $x \in \text{dom}(\nabla \tilde{\psi})$, we have $\partial \tilde{\psi}(x) = \{\nabla \tilde{\psi}(x)\}$, and the convergence $\partial \psi_n(x_n) \to \{\nabla \tilde{\psi}(x)\}$ along the subsequence follows from Proposition B.5(d), slightly strengthening [28, Exercise 12.40(a)]. To apply the latter result, the condition that $x \in \text{int}(\text{dom}(\partial \tilde{\psi}))$ follows from $\text{dom}(\nabla \tilde{\psi}) \subset \text{int}(\text{dom}(\partial \tilde{\psi}))$, see (1).

(b) By Lemma A.5 and by (1), $\text{spt} \mu \subset \text{cl}(\text{dom} \partial \tilde{\psi}) = \text{cl}(\text{dom} \tilde{\psi})$. Since $\tilde{\psi}$ is differentiable everywhere on $\text{cl}(\text{dom} \tilde{\psi})$ except for a set of Hausdorff dimension at most $d - 1$ [2], $\tilde{\psi}$ is differentiable and finite $\mu$-almost everywhere.

Measures in $\mathcal{M}_0(\mathbb{R}^d)$ are obviously $\sigma$-finite. Proposition 3.2 in combination with (a) then implies that for every subsequence limit $(\tilde{\pi}, \partial \tilde{\psi})$ in (a), $\tilde{\pi} = (\text{Id} \times \nabla \tilde{\psi})_\# \tilde{\mu}$ and $\tilde{\nu} = \nabla \tilde{\psi} \# \tilde{\mu}$, with $\nabla \tilde{\psi}$ defined $\tilde{\mu}$-almost everywhere. By Theorem 3.1, the map $\nabla \tilde{\psi}$ is unique $\tilde{\mu}$-almost everywhere.

(c) For any pair of proper lsc functions $\tilde{\psi}_a$ and $\tilde{\psi}_b$ whose subdifferentials can occur as graphical limits of (possibly different) subsequences in (a), we can find a $\tilde{\mu}$-null set $S$ such that $\tilde{\psi}_a$ and $\tilde{\psi}_b$ are differentiable on $V \setminus S$ and their gradients coincide on that set. As a consequence, also $\partial \tilde{\psi}_a(x) = \{\nabla \tilde{\psi}_a(x)\} = \{\nabla \tilde{\psi}_b(x)\} = \partial \tilde{\psi}_b(x)$ for all $x \in V \setminus S$.

For every open ball $B$ in $V$, the set $B \setminus S$ is dense in $B$ (Lemma A.4). Therefore, by Corollary 1.5 of Alberti and Ambrosio [14], $\partial \tilde{\psi}_a(x) = \partial \tilde{\psi}_b(x)$ for every $x \in B$ for every open ball $B$ in $V$ and therefore, for every $x \in V$. Therefore, all functions $\tilde{\psi}$ occurring in (a) have the same subdifferential $\partial \tilde{\psi}(x)$ and thus the same gradient $\nabla \tilde{\psi}(x)$ for all $x \in V$.

This implies that $\partial \psi_n$ converges graphically to $\partial \tilde{\psi}$ at each $x \in V$ along the full sequence, for any function $\psi$ appearing in (a); see Lemma B.4. Since $V$ is open and nonempty, $\partial \psi_n \xrightarrow{\text{w}} \partial \tilde{\psi}$ relative to $V$ along the full sequence.

Finally, convergence $d_H(\partial \psi_n(x_n), \{\nabla \psi(x)\}) \to 0$ whenever $x_n \to x \in V \cap \text{dom}(\nabla \psi)$ as $n \to \infty$ follows from Proposition B.5(d). \qed

**Proof of Theorem 4.3.** If the common value of $\mu(\mathbb{R}^d)$ and $\nu(\mathbb{R}^d)$ is finite, we can normalize $\mu$ and $\nu$ to become probability measures and apply McCann’s results in Section 2.2. So assume that $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d) = +\infty$.

For $n \in \mathbb{N}$, let $\nu_n$ be the restriction of $\nu$ to the complement of $n^{-1}B$. Then $a_n := \nu_n(\mathbb{R}^d) = \nu(\mathbb{R}^d \setminus n^{-1}B)$ is finite but grows to infinity as $n \to \infty$. Further, $\nu_n \stackrel{\text{w}}{\to} \nu$ as $n \to \infty$, since for every $f \in C_0(\mathbb{R}^d)$, we have $\nu_n(f) = \nu(f)$ for every $n \in \mathbb{N}$ that is sufficiently large such that $f$ vanishes on $n^{-1}B$.

Let $\varepsilon_n = \inf \{\varepsilon > 0 : \mu(\mathbb{R}^d \setminus \varepsilon B) \leq a_n\}$, for sufficiently large $n$ such that the set in the definition of the infimum is not empty. Necessarily $\mu(\mathbb{R}^d \setminus \varepsilon_n B) \leq a_n$ and $\mu(\mathbb{R}^d \setminus \varepsilon_n B) \to +\infty$ and thus $\varepsilon_n \to 0$ as $n \to \infty$. Put $m_n = a_n - \mu(\mathbb{R}^d \setminus \varepsilon_n B)$, let $\kappa_n$ denote the Lebesgue-uniform distribution on $\varepsilon_n B$ and let $\mu_n$ be the sum of $(\cdot \setminus \varepsilon_n B)$ and $m_n \kappa_n$. Then $\nu_n(\mathbb{R}^d) = a_n$ by construction and $\mu_n \overset{\text{w}}{\to} \mu$ as $n \to \infty$, again because for every $f \in C_0$ we have $\mu_n(f) = \mu(f)$ for all sufficiently large $n$.

Apply Theorem 2.2 to the probability measures $\mu_n = a_n^{-1} \mu_n$ and $\nu_n = a_n^{-1} \nu_n$ to find $\tilde{\pi}_n \in H(\mu_n, \nu_n)$ with cyclically monotone support. Then $\pi_n = a_n \tilde{\pi}_n \in H(\mu_n, \nu_n)$ and has the same, cyclically monotone support as $\tilde{\pi}_n$. By Rockafellar’s theorem, there exists a closed convex function $\psi_n$ on $\mathbb{R}^d$ such that $\text{spt} \pi_n$ is contained in $\text{gph} \partial \psi_n$. Apply Theorem 4.2(a) to extract a subsequence along which $\pi_n \overset{\text{w}}{\to} \pi$ and $\partial \psi_n \overset{\text{w}}{\to} \partial \tilde{\psi}$ as $n \to \infty$, \qed
respectively, for some \( \pi \in \Pi(\mu, \nu) \) and some closed convex function \( \psi \) on \( \mathbb{R}^d \) with the property that \( \text{spt} \pi \subseteq \text{gph}(\partial \psi) \).

If \( \mu \) vanishes on sets with Hausdorff dimension not larger than \( d - 1 \), we can apply Theorem 3.1 and Proposition 3.2 to find the representation in terms of \( \nabla \psi \) and the uniqueness \( \mu \)-a.e. of the latter.

**Proof of Corollary 4.4.** Lemma 4.1 provides a close connection between weak convergence and \( \mathcal{M}_0 \)-convergence of probability measures; the remaining task is to ensure that the restriction of the measures and their couplings to the complement of the origin does not pose problems.

We can always find a point \( x_0 \in \mathbb{R}^d \) which is not an atom of any of the probability measures \( \mu_n, \nu_n, \bar{\mu} \) and \( \bar{\nu} \). Therefore, replacing these measures by their push-forwards under the translation \( x \mapsto x - x_0 \), all coincide with their restrictions to \( \mathbb{R}^d \setminus \{0\} \). Thus, by Lemma 4.1, weak convergence is equivalent to \( \mathcal{M}_0 \)-convergence in (16) to the same limits, and Theorem 4.2 applies. Since these translated measures do not charge the origin in \( \mathbb{R}^d \), couplings between them do not charge the origin in \( \mathbb{R}^d \times \mathbb{R}^d \), and by Lemma 4.1, \( \mathcal{M}_0 \)-convergence of a sequence of these couplings implies its weak convergence to the same limit, which is a probability measure. Therefore, all claims of Theorem 4.2 with weak convergence instead of \( \mathcal{M}_0 \)-convergence verify for the translated measures \( \mu_n, \nu_n, \bar{\mu} \) and \( \bar{\nu} \). Since the validity of these claims is unaffected by a translation of all measures and mappings involved, the inverse translation \( x \mapsto x + x_0 \) verifies them for the original measures. \( \square \)

5. **Stability of Monge–Kantorovitch quantile contours**

Let \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \). Suppose \( \mu \) vanishes on sets of Hausdorff dimension \( d - 1 \) and let \( \psi \) be a convex function such that \( \nabla \psi \) pushes \( \mu \) forward to \( \nu \), see Theorem 2.4. In Chernozhukov et al. [7, eq. (5)], a quantile contour at “level” \( r \geq 0 \) for the target distribution \( \nu \) with respect to the reference distribution \( \mu \) is defined as \( \bigcup_{x|\|x\|=r} y_x \) with \( y_x \) chosen arbitrarily in \( \partial \psi(x) \). They name this a Monge–Kantorovich (M-K) quantile contour. The label \( r \) of the contour can be replaced by a probability, e.g., \( \mu(\{x \in \mathbb{R}^d : |x| > r\}) \) or, if \( \mu \) is spherically symmetric, the halfspace depth \( \mu(\{x \in \mathbb{R}^d : \langle x, u \rangle \geq r\}) \), with \( u \) any unit vector, but in this paper, there is no need for such relabelling.

Their definition of M-K quantile contours inspires us to define, for a closed convex function \( \psi \), the set

\[
Q_{\psi}(R) := \partial \psi(\{x \in \mathbb{R}^d : |x| \in R\}), \quad R \subset [0, \infty).
\]

From Proposition B.5(c), it follows that for compact \( K \) contained in the interior of \( \text{dom} \partial \psi \), the set \( \partial \psi(K) \) is compact too. Specializing to the case where \( R \) is a singleton, we define the quantile contour

\[
Q_{\psi}(r) = Q_{\psi}(\{r\}) = \partial \psi(rS_{d-1}) = \partial \psi(\{x \in \mathbb{R}^d : |x| = r\}), \quad r \geq 0.
\]

Rather than selecting a point \( y_x \in \partial \psi(x) \) arbitrarily as in [7], we include the whole set \( \partial \psi(x) \). The difference is minor, since \( \partial \psi(x) \) is a singleton if and only if \( \psi \) is differentiable at \( x \), which holds for almost all \( x \) anyway.

The definition (18) of a quantile contour of \( \nu \) is of most value when spheres are the natural quantiles for \( \mu \). In particular, this is the case if \( \mu \) is spherical (i.e., spherically
symmetric) as proposed in Chernozhukov et al. [7] and which will be the case in the next section. The cyclically monotone mapping \( \partial \psi \) represents a “minimal distortion of the space” required to push \( \mu \) forward to \( \nu \), which is the rationale for calling the transformed sphere \( Q_\psi(r) \) a quantile contour of \( \nu \).

For regularly varying \( \nu \in \mathcal{P}(\mathbb{R}^d) \), we wonder what the quantile contours \( Q_\psi(r) \) with respect to a well-chosen reference measure \( \mu \in \mathcal{P}(\mathbb{R}^d) \) look like when \( r \) grows large. Since the support of \( \nu \) is unbounded in this case, we expect those contours to grow indefinitely as well. Some sort of scaling is thus needed, and for that, we rely on the measure convergence in (22). For this approach to work, the contours generated by \( \psi_n \) in Theorem 4.2 should converge to those generated by \( \bar{\psi} \). This is the question we treat in Theorem 5.1 before studying the regularly varying case in Section 6.

Both \( Q_{\psi_n}(r) \) and \( Q_\psi(r) \) depend on the choice of the convex potential, but under the assumptions of Theorem 4.2, \( Q_\psi(r) \) actually does not depend on this choice if \( rS_{d-1} \) is included in the interior of the support of \( \bar{\mu} \). Furthermore, certain notions of uniform convergence of \( Q_{\psi_n}(r) \) to \( Q_\psi(r) \) apply. Recall that \( d_H \) denotes the Hausdorff distance.

**Theorem 5.1.** In Theorem 4.2, assume that \( \bar{\mu} \in \mathcal{M}_0(\mathbb{R}^d) \) vanishes on Borel sets of Hausdorff dimension at most \( d - 1 \) and that its support contains an open set \( V \). For every compact \( R \subset [0, \infty) \) such that \( RS_{d-1} := \bigcup_{\epsilon \in R} rS_{d-1} \subset V \), the sets \( Q_\psi(R) \) and, for sufficiently large \( n \), \( Q_{\psi_n}(R) \) are compact. For every \( r \in R \), we have

\[
\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \sup_{\epsilon} d_H(Q_{\psi_n}([r - \epsilon, r + \epsilon]), Q_\psi(r)) = 0. \tag{19}
\]

For every \( \epsilon > 0 \), there exists a positive integer \( n_{\epsilon,R} \) such that, for every \( n \geq n_{\epsilon,R} \) and every \( r \in R \), we have

\[
Q_{\psi_n}(r) \subset Q_\psi([r - \epsilon, r + \epsilon]) + \epsilon B, \tag{20}
\]

\[
Q_\psi(r) \subset Q_{\psi_n}([r - \epsilon, r + \epsilon]) + \epsilon B. \tag{21}
\]

**Remark 5.2.** For spherically symmetric measures \( \bar{\mu} \), the assumption in Theorem 5.1 that \( RS_{d-1} \subset V \) is equivalent to the assumption that \( R \) belongs to some open subset of the support of the radial measure of \( \bar{\mu} \). For spherical measures, the assumption that sets of Hausdorff dimension \( d - 1 \) are not charged implies that the radial measure is diffuse, i.e., has no atoms, for otherwise we would find a sphere centred at the origin receiving non-zero measure. However, starting from an uncountable subset of the real line with zero Hausdorff measure, examples can be constructed with a diffuse radial measure which generates a spherical measure \( \bar{\mu} \) supported on a set of Hausdorff dimension \( d - 1 \).

By Corollary 4.4, Theorem 5.1 has an important counterpart for weakly converging sequences of probability measures. The following result is similar in spirit to Theorems A.1 and A.2 and their corollaries in [7], but requires much less restrictive assumptions.

**Corollary 5.3.** Theorem 5.1 continues to hold if \( \mu_n, \nu_n, \bar{\mu}, \bar{\nu} \) are probability measures on \( \mathbb{R}^d \) and if \( \mathcal{M}_0 \)-convergence is replaced by weak convergence.

**Proof of Theorem 5.1.** The measure \( \bar{\mu} \) satisfies the conditions of Theorem 4.2(b,c). As a consequence, \( \partial \psi_n \) converges graphically to \( \partial \bar{\psi} \) relative to \( V \). We apply Proposition B.5 to the mappings \( T_n = \partial \psi_n \) and \( T = \partial \bar{\psi} \), which are maximal monotone [27, Corollary 31.5.2].
In Proposition B.5(b), put \( K = R S_{d-1} \) and \( A = r S_{d-1} \) with \( r \in \mathbb{R} \), and note that
\[
r S_{d-1} + \varepsilon \mathbb{B} = \{ x \in \mathbb{R}^d : |x| \in [r - \varepsilon, r + \varepsilon] \}.
\]
Equations (20) and (21) then follow from the inclusion statements in Proposition B.5(b).

In Proposition B.5(c), put \( K = r S_{d-1} \) and see that (19) follows from (42). \( \square \)

6. Regular variation and tail quantile contour convergence

6.1. Regularly varying probability measures. In probability, regular variation underpins asymptotic theory for sample maxima and sums [5, 9, 25] as it provides a coherent framework to describe the behaviour of functions at infinity and thus for the tails of random variables and vectors. A Borel measurable function \( f \) defined in a neighbourhood of infinity and taking positive values is regularly varying with index \( \alpha \) where
\[
f(x) = cx^\alpha + o(x^\alpha) \quad \text{as} \quad x \to \infty
\]
for some non-zero \( c \in \mathbb{R} \). The latter statement means that, in the space \( M_0(\mathbb{R}) \), we have \( t \mathbb{P}[R/b(t) > \lambda] \to \lambda^{-\alpha} \) as \( t \to \infty \) for all \( \lambda \in (0, \infty) \). The latter statement means that, in the space \( M_0(\mathbb{R}) \), we have \( t \mathbb{P}[R/b(t) > \lambda] \to \lambda^{-\alpha} \) as \( t \to \infty \) for all \( \lambda \in (0, \infty) \). The latter statement means that, in the space \( M_0(\mathbb{R}) \), we have \( t \mathbb{P}[R/b(t) > \lambda] \to \lambda^{-\alpha} \) as \( t \to \infty \) for all \( \lambda \in (0, \infty) \). The latter statement means that, in the space \( M_0(\mathbb{R}) \), we have \( t \mathbb{P}[R/b(t) > \lambda] \to \lambda^{-\alpha} \) as \( t \to \infty \) for all \( \lambda \in (0, \infty) \). The latter statement means that, in the space \( M_0(\mathbb{R}) \), we have \( t \mathbb{P}[R/b(t) > \lambda] \to \lambda^{-\alpha} \) as \( t \to \infty \) for all \( \lambda \in (0, \infty) \). The latter statement means that, in the space \( M_0(\mathbb{R}) \), we have \( t \mathbb{P}[R/b(t) > \lambda] \to \lambda^{-\alpha} \) as \( t \to \infty \) for all \( \lambda \in (0, \infty) \).

More generally, a probability measure \( \nu \in \mathcal{P}(\mathbb{R}^d) \) is regularly varying if there exists a Borel measurable function \( b \) defined in a neighbourhood of infinity and taking positive values such that
\[
t \nu(b(t) \cdot) \xrightarrow{\alpha} \tilde{\nu}, \quad t \to \infty
\]
for some non-zero \( \tilde{\nu} \in M_0(\mathbb{R}^d) \). The function \( b \) must be regularly varying with index \( 1/\alpha \in (0, \infty) \), say, and the limit measure \( \tilde{\nu} \) must be homogeneous:
\[
\tilde{\nu}(\cdot) = \lambda^{-\alpha} \tilde{\nu}(.), \quad \lambda > 0.
\]
We call \( \alpha \in (0, \infty) \) the index of regular variation of \( \nu \).

The radial measure of a Borel measure \( \nu \) on \( \mathbb{R}^d \) is the Borel measure \( \nu_0 \) on \([0, \infty)\) given by \( \nu_0(\cdot) = \cdot \# \nu \) \( \nu = \nu_0(\{ x \in \mathbb{R}^d : |x| \in \cdot \}) \). For \( \nu \in \mathcal{P}(\mathbb{R}^d) \), it is the distribution of \( |Y| \) if the random vector \( Y \) has distribution \( \nu \). The radial measure \( \nu_0 \) of a regularly varying \( \nu \in \mathcal{P}(\mathbb{R}^d) \) as in (22) satisfies
\[
t \nu_0(|rb(t), \infty)) \to \tilde{\nu}(\{ x \in \mathbb{R}^d : |x| \geq r \}) = cr^{-\alpha}, \quad t \to \infty,
\]
where \( c = \tilde{\nu}(\{ x \in \mathbb{R}^d : |x| \geq 1 \}) \). We find that \( t \nu_0(b(t) \cdot) \) converges in \( M_0(\mathbb{R}) \) to the measure \( c \alpha^{-\alpha} \nu \mathbb{1}_{(0, \infty)}(r) dr \). The radial measure \( \nu_0 \) is thus regularly varying with auxiliary function \( b \), index \( \alpha > 0 \), and limit measure as stated. Replacing the auxiliary function \( b \) by \( t \to b(ct) \), we can suppose without loss of generality that \( c = 1 \).

Regularly varying distributions can be constructed as positively homogeneous transformations of spherically symmetric distributions with regularly varying radial measure. In Section 6.2 we will see that, asymptotically, all regularly varying distributions can be constructed in this way, and that we can moreover assume that the transformation is given by the gradient of a convex function which is positively homogeneous of order two; see in particular Remark 6.9.
**Example 6.1** (Transformations of spherically symmetric distributions). Let $\mu$ be the distribution of the random vector $RU$, where $R$ is a nonnegative random variable, $U$ is uniformly distributed on the unit sphere in $\mathbb{R}^d$, and $R$ and $U$ are independent. Assume that $R$ is regularly varying with index $\alpha \in (0, \infty)$ and scale function $b$ as in the beginning of this section.

Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be measurable, positively homogeneous of order one [i.e., $T(\lambda x) = \lambda T(x)$ for $x \in \mathbb{R}^d$ and $\lambda \in (0,\infty)$], bounded on the unit sphere, and not identically zero. The probability measure $\nu = T#\mu$, which is the law of $T(RU) = RT(U)$, is regularly varying with index $\alpha$ and auxiliary function $b$: for $f \in \mathcal{C}_b(\mathbb{R}^d)$, the function $r \mapsto \mathbb{E}[f(rT(U))]$ is in $\mathcal{C}_0(\mathbb{R})$, so that

$$t \int_{\mathbb{R}^d} f(z) \, d\nu(b(t)z) = t \mathbb{E}[f((R/b(t)) T(U))] \to \int_0^\infty \mathbb{E}[f(rT(U))] \alpha r^{-\alpha-1} \, dr, \quad t \to \infty.$$ 

We find that $t \nu(b(t) \cdot) \xrightarrow{\circ} T#\mu = \bar{\nu}$ in $\mathcal{M}_0(\mathbb{R}^d)$ as $t \to \infty$, where $\bar{\mu}$ is the spherically symmetric measure with radial measure $\bar{\mu}(\{x \in \mathbb{R}^d : |x| \geq r\}) = r^{-\alpha}$ for all $r \in (0,\infty)$. By Fubini’s theorem, $\bar{\nu}(\{x \in \mathbb{R}^d : |x| \geq r\}) = r^{-\alpha} \mathbb{E}[|T(U)|^\alpha]$ for $r \in (0,\infty)$, so that $\bar{\nu}$ and $\bar{\mu}$ share the same radial measure if $T$ is scaled in such a way that $\mathbb{E}[|T(U)|^\alpha] = 1$. Viewing $T(U)$ as a random $d \times 1$ matrix, regular variation of $RT(U)$ also follows from Basrak et al. [3, Proposition A.1].

The following result states that the spherically symmetrized version of a regularly varying distribution is still regularly varying. We will make use of this property when choosing the reference distribution in the construction of quantile contours.

**Lemma 6.2** (Symmetrization). Let $\nu \in \mathcal{P}(\mathbb{R}^d)$ be regularly varying with auxiliary function $b$, limit measure $\bar{\nu} \in \mathcal{M}_0(\mathbb{R}^d)$ and index $\alpha > 0$. Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be spherically symmetric and with radial measure equal to the one of $\nu$. Then $\mu$ is regularly varying too, with the same auxiliary function and index as $\nu$ and with spherically symmetric limit measure $\bar{\mu} \in \mathcal{M}_0(\mathbb{R}^d)$ having the same radial measure as $\bar{\nu}$.

The proof of the lemma is standard and is omitted for brevity. It combines the regular variation of the radial part as in (23) with Example 6.1, setting $R = |Y|$ for $Y$ a random vector with distribution $\nu$ and with $T$ the identity mapping.

6.2. **Tail quantile contours.** Let $\nu \in \mathcal{P}(\mathbb{R}^d)$ be regularly varying with auxiliary function $b$, index $\alpha > 0$, and limit measure $\bar{\nu} \in \mathcal{M}_0(\mathbb{R}^d)$. Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\bar{\mu} \in \mathcal{M}_0(\mathbb{R}^d)$ be spherically symmetric and have the same radial measures as $\nu$ and $\bar{\nu}$, respectively. Recall from Lemma 6.2 that $\mu$ is regularly varying too, with the same index and auxiliary function as $\nu$, and with limit measure $\bar{\mu}$. Further, note that $\bar{\mu}$ and the Lebesgue measure on $\mathbb{R}^d$ are equivalent in the sense that they vanish on the same sets: $\bar{\mu}$ has a homothetic nonvanishing density w.r.t. the Lebesgue measure on $\mathbb{R}^d \setminus \{0\}$ and vice-versa, and both measures vanish on $\{0\}$.

By Theorem 2.2, there exists a coupling measure $\pi \in \Pi(\mu,\nu)$ with cyclically monotone support contained in the graph of the subdifferential $\partial \psi$ of some closed convex function $\psi$ on $\mathbb{R}^d$. We may thus define quantile contours $Q_\psi(r)$ of $\nu$ with respect to $\mu$ as in (18).
Motivated by the study of tail quantiles, we study how $Q_\psi(r)$ grows as $r \to \infty$. This growth can be described by investigating the asymptotic behavior of $\partial \psi(x)$ when $|x|$ is large, and for this, we determine the graphical limit, $\bar{\partial} \psi$, of $u^{-1} \partial \psi(u \cdot)$ as $u \to \infty$. This limit turns out to be positively homogeneous and to generate the unique coupling of $\bar{\mu}$ and $\bar{\nu}$ with cyclically monotone support.

**Theorem 6.3.** Let $\nu \in P(\mathbb{R}^d)$ be regularly varying with auxiliary function $b$, index $\alpha > 0$ and limit measure $\bar{\nu} \in M_0(\mathbb{R}^d)$. Let $\mu \in P(\mathbb{R}^d)$ and $\bar{\mu} \in M_0(\mathbb{R}^d)$ be the spherical measures having the same radial measures as $\nu$ and $\bar{\nu}$, respectively. Let $\psi$ be a closed convex function such that the graph of $\partial \psi$ contains the cyclically monotone support of some $\pi \in \Pi(\mu, \nu)$.

(a) We have $u^{-1} \partial \psi(u \cdot) \xrightarrow{u \to \infty} \bar{\partial} \psi$ as $u \to \infty$, where $\bar{\psi}$ is a closed convex function, finite everywhere and unique up to vertical shifts, such that $(\nabla \bar{\psi})_{\bar{\mu}} = \bar{\nu}$. Moreover, sup$_x \sup_{y \in \mathbb{R}^d} \{|y| : y \in \partial \psi(x)\}/(1 + |x|) < +\infty$.

(b) The coupling $\pi$ is regularly varying and $t \pi(b(t \cdot)) \xrightarrow{t \to \infty} \bar{\pi}$ in $M_0(\mathbb{R}^d \times \mathbb{R}^d)$ as $t \to \infty$ with $\bar{\pi} = (1d \times \nabla \bar{\psi})_{\bar{\mu}}\bar{\mu}$ the unique coupling measure between $\bar{\mu}$ and $\bar{\nu}$ having cyclically monotone support. In particular, $\bar{\pi}(\lambda \cdot) = \lambda^{-\alpha} \bar{\pi}(\cdot)$ for $\lambda > 0$.

**Theorem 6.4.** Let $\bar{\psi}$ be the limit function in Theorem 6.3. Its subdifferential is positively homogeneous of order one:

$$\partial \bar{\psi}(\lambda x) = \lambda \partial \bar{\psi}(x), \quad \lambda \geq 0, \ x \in \mathbb{R}^d. \tag{24}$$

Assuming that $\bar{\psi}(0) = 0$ without loss of generality, $\bar{\psi}$ is positively homogeneous of order two:

$$\bar{\psi}(\lambda x) = \lambda^2 \bar{\psi}(x), \quad \lambda \geq 0, \ x \in \mathbb{R}^d.$$  

As a consequence, there exists a compact, convex set $C \subset \mathbb{R}^d$ strictly containing the origin and such that

$$\bar{\psi}(x) = \frac{1}{2} \sigma_C(x)^2 \text{ and } \partial \bar{\psi}(x) = \sigma_C(x) \partial \sigma_C(x), \quad x \in \mathbb{R}^d, \tag{25}$$

where $\sigma_C(x) = \sup\{\langle x, y \rangle : y \in C\}$ is the support function of $C$ and $\partial \sigma_C(x)$ is the set of $y \in C$ where the supremum in the definition of $\sigma_C(x)$ is attained.

We turn to quantile contours as in (18). By homogeneity of $\partial \bar{\psi}$ in (24), we have

$$Q_\bar{\psi}(r) = \partial \bar{\psi}(rS_{d-1}) = r \partial \bar{\psi}(S_{d-1}) = rQ_\bar{\psi}(1)$$

for all $r \in (0, \infty)$. In view of (25), all quantile contours of $\bar{\psi}$ are thus homothetic to $Q_\bar{\psi}(1) = \bigcup_{x: |x| = 1} \sigma_C(x) \partial \sigma_C(x)$.

**Theorem 6.5.** Consider the set-up of Theorem 6.3. The sets $Q_\psi(R)$ and $Q_\psi(R)$ are compact for every compact $R \subset [0, \infty)$, and for every set $\Lambda \subset (0, \infty)$ which is bounded away from 0 and which has non-empty interior, we have

$$\lim_{r \to \infty} d_H \left( \bigcup_{\lambda \in \Lambda} (r\lambda)^{-1}Q_\psi(r\lambda), Q_\psi(1) \right) = 0.$$  

Let $\| \cdot \|$ be an arbitrary norm on $\mathbb{R}^d$, not necessarily the Euclidean norm $| \cdot |$. The angular or spectral measure of $\bar{\nu}$ with respect to $\| \cdot \|$ is defined as the finite Borel measure $H$ on $\{x \in \mathbb{R}^d : \|x\| = 1\}$ given by

$$H(\cdot) = \bar{\nu}(\{x \in \mathbb{R}^d : \|x\| \geq 1, x/\|x\| \in \cdot\}). \tag{26}$$
In extreme-value theory, the name spectral measure is usually reserved for the situation where $\hat{\nu}$ is concentrated on the positive quadrant $[0, \infty)^d$ and the index of regular variation is $\alpha = 1$; see for instance [26, Theorem 6.1] and [9, Section 6.1.4]. The limit measure $\hat{\nu}$ is homogeneous and therefore entirely determined by the index $\alpha$ and the spectral measure $H$ via the identity $\hat{\nu}(\{x \in \mathbb{R}^d : |x| \geq r, x/|x| \in B\}) = r^{-\alpha}H(B)$ for every Borel set $B$ contained in $\{x : \|x\| = 1\}$. We now give a formula expressing $H$ in terms of $\nabla \hat{\psi}$.

**Theorem 6.6.** Let the random vector $U$ be uniformly distributed on the Euclidean unit sphere $S_{d-1}$. For $\hat{\mu}, \hat{\nu}$ and $\hat{\psi}$ as in Theorem 6.3, the spectral measure $H$ in (26) is

$$H(\cdot) = c \mathbb{E} \left[ \|\nabla \hat{\psi}(U)\|^{\alpha} \mathbf{1} \{ \nabla \hat{\psi}(U)/\|\nabla \hat{\psi}(U)\| \in \cdot \} \right], \quad (27)$$

where $c = \hat{\nu}(\{x \in \mathbb{R}^d : |x| \geq 1\})$. In particular, the gradient map satisfies

$$\mathbb{E} \left[ \|\nabla \hat{\psi}(U)\|^{\alpha} \right] = 1. \quad (28)$$

**Remark 6.7.** In addition to the assertions of Theorem 6.3, also $d_H(\partial \psi_t(x_t), \{\nabla \hat{\psi}(x)\}) \to 0$ whenever $x_t \to x$, for all $x \in \mathbb{R}^d \setminus N$ where $N$ is a cone of Hausdorff dimension at most $d-1$. This is a consequence of Theorem 4.2, the property that a convex function is differentiable everywhere except on a set of Hausdorff dimension at most $d-1$, and the homogeneity of $\partial \psi$.

**Remark 6.8.** An argument similar to the one in Example 6.1 or to the proof of equation (27) shows that for bounded, measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ that vanish in a neighbourhood of the origin, we have

$$\hat{\nu}(f) = c \int_0^\infty \mathbb{E}[f(r\nabla \hat{\psi}(U))] r^{-\alpha-1} \, dr.$$ 

This expresses $\hat{\nu}$ in terms of the tail index $\alpha > 0$, the values of $\nabla \hat{\psi}$ on the unit sphere, and the normalizing constant $c = \hat{\nu}(\{x \in \mathbb{R}^d : |x| \geq 1\})$.

**Remark 6.9.** Up to scaling, every compact, convex set $C \subset \mathbb{R}^d$ strictly containing the origin can appear in Theorem 6.4. Indeed, if $f : \mathbb{R}^d \to [0, \infty)$ is convex, then so is $f^2/2$. It follows that for every compact, convex set $C \subset \mathbb{R}^d$ containing the origin, $\tilde{\psi} = \frac{\sigma_C^2}{2}$ is a convex function with subdifferential $\partial \tilde{\psi} = \sigma_C \partial \sigma_C$, see (the proof of) Theorem 6.4. If $C \neq \emptyset$, then the set of $u \in S_{d-1}$ such that $\partial \tilde{\psi}(u)$ is non-zero contains at least the half-sphere $\{u \in S_{d-1} : \langle u, v \rangle > 0\}$, for every $v \in C \setminus \{0\}$, which implies that $\mathbb{E}[\|\nabla \tilde{\psi}(U)\|^\alpha]$ is positive for every $\alpha > 0$. The equality constraint (28) can then be made to hold by replacing $C$ by $\lambda C$ for a suitable $\lambda \in (0, \infty)$. In view of Example 6.1, the probability measure $\nu = (\nabla \tilde{\psi})_#\mu_{\alpha} \in \mathcal{P}(\mathbb{R}^d)$, with $\mu_{\alpha} \in \mathcal{P}(\mathbb{R}^d)$ the spherically symmetric distribution with radial measure $\mu_{\alpha}(\{x \in \mathbb{R}^d : |x| \geq r\}) = r^{-\alpha}$ for $r \in [1, \infty)$, satisfies the conditions in Theorem 6.3 with $\psi = \tilde{\psi}$.

**Remark 6.10.** By Rockafellar and Wets [28, Proposition 11.21 on p. 492] applied to the self-conjugate function $t \mapsto \frac{1}{2}t^2$ on $\mathbb{R}$, the convex conjugate [28, p. 473] of $\psi$ is

$$\psi^*(x) = \frac{1}{2} \gamma_C(x)^2 = \frac{1}{2} \sigma_D(x)^2,$$

where $\gamma_C(x) = \inf\{\lambda \geq 0 : x \in \lambda C\}$ is the gauge function of $C$ and $D = \{x \in \mathbb{R}^d : \sigma_C(x) \leq 1\}$ is the polar of $C$; see the proof of Theorem 6.4. The convex conjugate is
potentially interesting because, for instance, \((\partial \bar{\psi})^{-1} = \partial \bar{\psi}^*\) and vice versa as multivalued mappings [28, Proposition 11.3 on p. 476]. Moreover, duality is greatly helpful in describing and calculating optimal couplings [23, 24, 30].

6.3. Scaled subdifferentials and their graphical limits.

**Lemma 6.11** (Scaling). Let \(\mu, \nu \in \mathcal{P}(\mathbb{R}^d)\) and let \(\pi \in \Pi(\mu, \nu)\) have support contained in the graph of \(\partial \psi\) for some closed convex function \(\psi\). For \(b > 0\), consider the measures \(\mu_b = \mu(b \cdot), \nu_b = \nu(b \cdot), \) and \(\pi_b = \pi(b \cdot).\) Then \(\pi_b \in \Pi(\mu_b, \nu_b)\) and its support is contained in the graph of the subdifferential of the closed convex function \(\psi_b = b^{-2} \psi(b \cdot),\) the subdifferential being given by \(\partial \psi_b(x) = b^{-1} \partial \psi(bx)\) for all \(x\).

**Proof.** It is clear that \(\pi_b\) is a coupling measure of \(\mu_b\) and \(\nu_b\). The point \((x, y)\) belongs to the support of \(\pi_b\) if and only if the point \((bx, by)\) belongs to the support of \(\pi\), that is, \(\text{spt} \pi_b = b^{-1} \text{spt} \pi\).

Further, \(\psi_b\) is a closed convex function since this is true for \(\psi\). We have \(y \in \partial \psi_b(x)\) if and only if

\[
\psi_b(z) \geq \psi_b(x) + \langle y, x - z \rangle, \quad z \in \mathbb{R}^d.
\]

In view of the definition of \(\psi_b\), this is the same as

\[
\psi(bz) \geq \psi(bx) + \langle by, bx - bz \rangle, \quad z \in \mathbb{R}^d,
\]

and thus to \(by \in \partial \psi(bx)\). We find that \(\partial \psi_b(x) = b^{-1} \partial \psi(bx)\).

The previous argument also shows that \((x, y) \in \text{gph} \partial \psi_b\) if and only if \((bx, by) \in \text{gph} \partial \psi\), so that \(\text{gph} \partial \psi_b = b^{-1} \text{gph} \partial \psi\). Since \(\text{spt} \pi_b = b^{-1} \text{spt} \pi\), we conclude that the support of \(\pi_b\) is included in the graph of \(\partial \psi_b\). \(\square\)

**Proposition 6.12** (Linear growth). Let the mapping \(S : \mathbb{R}^d \Rightarrow \mathbb{R}^d\) be maximal monotone and suppose that there exists a sequence \(0 < u_n \to \infty\) with \(\limsup_n u_{n+1} / u_n < \infty\) such that \(u_n^{-1} S(u_n \cdot) \overset{s}{\rightharpoonup} T\) as \(u \to \infty\), where the limit mapping \(T : \mathbb{R}^d \rightharpoonup \mathbb{R}^d\) is maximal monotone and has \(\text{dom}(T) = \mathbb{R}^d\). Then

\[
\sup_{x \in \mathbb{R}^d} \frac{\sup \{|y| : y \in S(x)\}}{1 + |x|} < \infty.
\]

**Proof.** By passing to a subsequence if necessary, we may assume that \(u_n\) is non-decreasing. Further, by modifying the head of the sequence \((u_n)_n\) if needed, we can suppose that \(u_n \geq 1\) for all \(n\) and thus \(\sup\{u_{n+1} / u_n : n = 1, 2, \ldots\} < \infty\).

Define \(S_n = u_n^{-1} S(u_n \cdot)\). By Proposition B.5, the sets \(S_n(B)\) are compact for all sufficiently large \(n\). This implies that there exists a positive integer \(\bar{n}\) and a scalar \(k > 0\) such that \(|z| \leq k\) for all \(z \in S_n(B) = u_n^{-1} S(u_n B)\) and all integer \(n \geq \bar{n}\). Writing \(z = u_n^{-1} y\), we find

\[
\sup_{n \geq \bar{n}} \sup_{x \in u_n B} \sup_{u_n} \left\{ |y| : y \in S(x) \right\} \leq k.
\]

We obtain

\[
\sup_{x : |x| \leq u_{\bar{n}}} \sup_{|y|} \left\{ |y| : y \in S(x) \right\} \leq ku_{\bar{n}} \tag{29}
\]
as well as
\[
\sup_{x:|x|>u_n} \sup\{|y| : y \in S(x)\} = \sup_{n\geq n} \sup_{x:|x|\in\{u_n,u_{n+1}\}} \sup\{|y| : y \in S(x)\} \leq \sup_{n\geq n} \frac{k\bar{u}_{n+1}}{u_n}.
\]
(30)

We conclude that the supremum in the statement of the proposition is bounded by the maximum of the bounds in (29) and (30).

\[\square\]

**Proposition 6.13** (Homogeneous subdifferentials). Let \( \psi : \mathbb{R}^d \to \mathbb{R} \) be a convex function with \( \psi(0) = 0 \). If the graph of \( \partial \psi \) is a multiplicative cone in \( \mathbb{R}^d \times \mathbb{R}^d \), i.e.,
\[
\lambda \text{gph}(\partial \psi) = \text{gph}(\partial \psi), \quad \lambda > 0,
\]
(31)
then
\[
\psi(\lambda x) = \lambda^2 \psi(x) \quad \text{and} \quad \partial \psi(\lambda x) = \lambda \partial \psi(x)
\]
(32)
for all \( \lambda > 0 \) and \( x \in \mathbb{R}^d \). Moreover, we have the representation
\[
\psi(x) = \frac{1}{2} \sigma_C(x)^2 \quad \text{and} \quad \partial \psi(x) = \sigma_C(x) \partial \sigma_C(x)
\]
(33)
for all \( x \in \mathbb{R}^d \), where \( C \) is a compact, convex subset of \( \mathbb{R}^d \) containing the origin, \( \sigma_C(x) = \sup\{|x,y| : y \in C\} \) is its support function, and \( \partial \sigma_C(x) \) is the set of \( y \in C \) where the supremum in the definition of \( \sigma_C(x) \) is attained.

**Proof.** If \((x,y)\) belongs to the graph of \( \partial \psi \), then so does \((\lambda x, \lambda y)\) for \( \lambda > 0 \) by (31). In other words, \( y \in \partial \psi(x) \) implies \( \lambda y \in \partial \psi(\lambda x) \). We find that \( \partial \psi(\lambda x) = \lambda \partial \psi(x) \), which is the second equality in (32).

If \( \psi \) is differentiable in \( x \), then \( \partial \psi(x) = \{\nabla \psi(x)\} \) and thus \( \partial \psi(\lambda x) = \lambda \partial \psi(x) = \{\lambda \nabla \psi(x)\} \) for \( \lambda > 0 \), implying that \( \psi \) is differentiable in \( \lambda x \) as well and that \( \nabla \psi(\lambda x) = \lambda \nabla \psi(x) \). Define \( f(t) = \psi(tx) \) for \( t \in [0,1] \). The function \( f \) is continuous on \([0,1]\) and is continuously differentiable on \((0,1)\) with derivative \( f'(t) = \langle x, \nabla \psi(tx) \rangle = t \langle x, \nabla \psi(x) \rangle \).

We get
\[
\psi(x) = f(1) - f(0) = \int_0^1 f'(t) \, dt = \frac{1}{2} \langle x, \nabla \psi(x) \rangle.
\]
It follows that \( \psi(\lambda x) = \lambda^2 \psi(x) \) for all \( x \) at which \( \psi \) is differentiable, and since this covers almost every \( x \), the equality extends to all \( x \) by continuity of \( \psi \). This proves the first equality of (32).

Since \( \psi \) is positively homogeneous of degree two, Corollary 15.3.1 in [27] states that \( \psi(x) = \frac{1}{2} \{\gamma_D(x)\}^2 \) for all \( x \in \mathbb{R}^d \), where \( D \) is a closed convex set containing the origin and \( \gamma_D \) is its gauge function, i.e.,
\[
\gamma_D(x) = \inf\{\lambda \geq 0 : x \in \lambda D\}, \quad x \in \mathbb{R}^d.
\]
Let \( C \) be the polar of \( D \), that is,
\[C = \{v \in \mathbb{R}^d : \forall x \in D, \langle v, x \rangle \leq 1\}.
\]
The set \( C \) is closed and convex and it contains the origin too. By [27, Theorem 14.5], the gauge function of \( D \) is equal to the support function of \( C \), that is, \( \gamma_D = \sigma_C \). Since \( \psi \) and thus \( \sigma_C \) are finite everywhere, \( C \) is compact. This yields the first identity in (33).
The subdifferential $\partial \sigma_C(x)$ consists of the points $y \in C$ where the function $y \mapsto \langle x, y \rangle$ attains its maximum [27, Corollary 23.5.3]. If $\psi(x) = 0$, then also $\sigma_C(x) = 0$, and since $\sigma_C$ locally Lipschitz, we have $\psi(z) = \frac{1}{2}\{\sigma_C(z) - \sigma_C(x)\}^2 = O(|z - x|^2)$ as $z \to x$, whence $\partial \psi(x) = \{0\}$. If $\psi(x) > 0$, then $\psi$ is differentiable in $x$ if and only if $\sigma_C$ is differentiable in $x$, and for such $x$, by the chain rule, $\nabla \psi(x) = \sigma_C(x)\nabla \sigma_C(x)$. Theorem 25.6 in [27] then implies the second equality in (33). □

6.4. Proofs of Theorems of Section 6.

Proof of Theorem 6.3. Choose a sequence $1 \leq t_n \to \infty$ and consider the Borel measures $\mu_n = t_n \mu(b(t_n) \cdot )$ and $\nu_n = t_n \nu(b(t_n) \cdot )$, both with mass $t_n$. In view of Lemma 6.11, the function $\psi_n = b(t_n)^{-2}\psi(b(t_n) \cdot )$ is closed and convex, and the graph of $\partial \psi_n$ contains the support of the measure $\pi_n = t_n \pi(b(t_n) \cdot )$ that couples $\mu_n$ and $\nu_n$.

By regular variation of $\nu$ and by Lemma 6.2, we have $\mu_n \overset{\circ}{\to} \bar{\mu}$ and $\nu_n \overset{\circ}{\to} \bar{\nu}$ in $\mathcal{M}_0(\mathbb{R}^d)$ as $n \to \infty$. Note that $\bar{\mu}$ and $\bar{\nu}$ do not depend on the sequence $t_n$. The limit measure $\bar{\mu}$ is spherically symmetric and its radial measure has a strictly positive Lebesgue density on $(0, +\infty)$, so that $\bar{\mu}$ is equivalent to the $d$-dimensional Lebesgue measure.

We apply Theorem 4.2 to obtain that $\pi_n \overset{\circ}{\to} \bar{\pi}$ in $\mathcal{M}_0(\mathbb{R}^d \times \mathbb{R}^d)$ and $\partial \psi_n \overset{\circ}{\to} \bar{\psi}$ as $n \to \infty$, with $\bar{\pi} = (\text{Id} \times \nabla \bar{\psi})\# \bar{\mu}$ a unique coupling measure for $\bar{\mu}$ and $\bar{\nu}$, and $\bar{\psi}$ a closed convex function, unique up to vertical shifts, such that the support of $\bar{\pi}$ is contained in the graph of $\partial \bar{\psi}$. The uniqueness of the limits $\partial \bar{\psi}$ and $\bar{\pi}$ (which must extend to the limits for different sequences $t_n$) implies that we can redefine $\psi_t = b(t)^{-2}\psi(b(t) \cdot )$ and $\pi_t = t \pi(b(t) \cdot )$ for $t \in [1, \infty)$ and assert that $\partial \psi_t \overset{\circ}{\to} \bar{\psi}$ and $\pi_t \overset{\circ}{\to} \bar{\pi}$ as $t \to \infty$, so $\pi$ is regularly varying.

The bound on the subdifferentials in (a) follows from Proposition 6.12 and the choice $u_n = b(n)$. Regular variation of $b$ implies that $u_{n+1}/u_n \to 1$ as $n \to \infty$.

So far, we have shown that $b(t)^{-1}\partial \psi(b(t) \cdot ) \overset{\circ}{\to} \partial \bar{\psi}$ as $t \to \infty$. Recall that graphical convergence is defined as Painlevé–Kuratowski set convergence of the associated graphs. Since the graph of $b(t)^{-1}\partial \psi(b(t) \cdot )$ is equal to $b(t)^{-1}\text{gph}(\partial \psi)$, we get

$$b(t)^{-1}\text{gph}(\partial \psi) \to \text{gph}(\partial \bar{\psi}), \quad t \to \infty.$$

Let $b^*$ be an asymptotic inverse to $b$, i.e., a (Borel measurable) function defined in a neighbourhood of infinity such that $b(b^*(u))/u \to 1$ as $u \to \infty$ [5, Section 1.5.7]. The function $b^*$ is regularly varying with index $1/\alpha$ and $b^*(u) \to \infty$ as $u \to \infty$. Therefore

$$\frac{1}{b(b^*(u))}\text{gph}(\partial \psi) \overset{\circ}{\to} \text{gph}(\partial \psi), \quad u \to \infty.$$

By continuity of the scalar multiplication on Fell space (Proposition B.1), we get

$$\frac{1}{u}\text{gph}(\partial \psi) = \frac{b(b^*(u))}{u}\frac{1}{b(b^*(u))}\text{gph}(\partial \psi) \to \text{gph}(\partial \psi), \quad u \to \infty. \quad (34)$$

Since $u^{-1}\text{gph}(\partial \psi)$ is the graph of the mapping $u^{-1}\partial \psi(u \cdot )$, we conclude that $u^{-1}\partial \psi(u \cdot )$ $\overset{\circ}{\to} \bar{\psi}$ as $u \to \infty$. □
We find $\psi(x) = \lambda u/\lambda$, because then $\hat{\psi}$ would be constant and thus $\hat{\nu}$ would assign zero mass to $\mathbb{R}^d \setminus \{0\}$.

**Proof of Theorem 6.4.** Graphical convergence $u^{-1}\partial \psi(u \cdot) \xrightarrow{\varepsilon} \hat{\psi}$ as $u \to \infty$ is equivalent to (34). Clearly, it follows that for $\lambda \in (0, \infty)$,

$$\frac{\lambda}{u} \text{gph}(\partial \psi) \to \lambda \text{gph}(\partial \hat{\psi}), \quad u \to \infty.$$ But since $\lambda/u = 1/(u/\lambda)$ and since $u/\lambda \to \infty$ as $u \to \infty$, equation (34) also implies

$$\frac{\lambda}{u} \text{gph}(\partial \psi) \to \text{gph}(\partial \hat{\psi}), \quad u \to \infty.$$ We conclude that the set $\text{gph}(\partial \hat{\psi})$ is a multiplicative cone, i.e.,

$$\forall \lambda \in (0, \infty), \quad \lambda \text{gph}(\partial \hat{\psi}) = \text{gph}(\partial \hat{\psi}).$$ Apply Proposition 6.13 to see that $\hat{\psi}$ and $\partial \hat{\psi}$ are positively homogeneous of orders two and one, respectively, and to arrive at the representations in (25). Finally, note that $C$ cannot be equal to $\{0\}$, because then $\hat{\psi}$ would be constant and thus $\hat{\nu}$ would assign zero mass to $\mathbb{R}^d \setminus \{0\}$.

**Proof of Theorem 6.5.** For compact $R \subset [0, \infty)$, compactness of $Q_\psi(R)$ follows from Theorem 5.1(b). Compactness of $Q_\psi(R) = \partial \psi(\{x : |x| \in R\})$ is a special case of the compactness of $\partial \psi(K)$ for every compact $K \subset \mathbb{R}^d$, and the latter is a consequence of the fact that $\partial \psi(K)$ is the projection onto the second coordinate of $\text{gph}(\partial \psi) \cap (K \times \mathbb{R}^d)$, a set which is closed since $\text{gph}(\partial \psi)$ is closed and which is bounded by Theorem 6.3(a).

By Theorem 5.1(b), we have

$$\lim_{\varepsilon \downarrow 0} \limsup_{r \to \infty} d_H\left( r^{-1}Q_\psi([r(1-\varepsilon), r(1+\varepsilon)]), Q_\psi(1) \right) = 0. \quad (35)$$

Let $m = \sup_{x \in \mathbb{R}^d}[\sup\{|y| : y \in \hat{\psi}(x)\}/(1 + |x|)]$, which is finite by Theorem 6.3(a). Then $|y| \leq (1 + r)m \leq 2rm$ for every $r \geq 1$ and every $y \in Q_\psi(r)$. By Lemma 6.14 below, for every $r \geq 2$ and every $\varepsilon \in (0, 1/2]$,

$$d_H\left( \bigcup_{h \in [1-\varepsilon, 1+\varepsilon]}(r\bar{h})^{-1}Q_\psi(r\bar{h}), r^{-1}Q_\psi([r(1-\varepsilon), r(1+\varepsilon)]) \right)$$

$$= d_H\left( \bigcup_{h \in [1-\varepsilon, 1+\varepsilon]}(r\bar{h})^{-1}Q_\psi(r\bar{h}), \bigcup_{h \in [1-\varepsilon, 1+\varepsilon]}r^{-1}Q_\psi(r\bar{h}) \right)$$

$$\leq \sup_{h \in [1-\varepsilon, 1+\varepsilon]} d_H\left( (r\bar{h})^{-1}Q_\psi(r\bar{h}), h(r\bar{h})^{-1}Q_\psi(r\bar{h}) \right)$$

$$\leq \sup_{h \in [1-\varepsilon, 1+\varepsilon]} \left( |h - 1| \sup\{|z| : z \in (r\bar{h})^{-1}Q_\psi(r\bar{h})\} \right)$$

$$\leq 2m\varepsilon.$$ In view of (35), the triangle inequality for the Hausdorff distance then yields

$$\lim_{\varepsilon \downarrow 0} \limsup_{r \to \infty} d_H\left( \bigcup_{h \in [1-\varepsilon, 1+\varepsilon]}(r\bar{h})^{-1}Q_\psi(r\bar{h}), Q_\psi(1) \right) = 0.$$

If $\Lambda \subset (0, \infty)$ is non-empty and bounded away from 0, then for every function $f$,

$$\limsup_{r \to \infty} f(r) = \inf_{r > 0} \sup_{s \geq r} f(s) = \inf_{r > 0} \sup_{s \geq r} \sup_{\lambda \in \Lambda} f(s\lambda) = \limsup_{r \to \infty} \sup_{\lambda \in \Lambda} f(r\lambda).$$

We find

$$\lim_{\varepsilon \downarrow 0} \limsup_{r \to \infty} \sup_{\lambda \in \Lambda} d_H\left( \bigcup_{h \in [1-\varepsilon, 1+\varepsilon]}(r\lambda h)^{-1}Q_\psi(r\lambda h), Q_\psi(1) \right) = 0,$$
and thus, by Lemma 6.14(a),
\[
\lim_{\varepsilon \downarrow 0} \limsup_{r \to \infty} d_H \left( \bigcup_{x \in \Lambda} \bigcup_{h \in [1-\varepsilon,1+\varepsilon]} (r\lambda h)^{-1}Q_{\psi}(r\lambda h), Q_{\psi}(1) \right) = 0. \tag{36}
\]

Let \( \lambda_0 \in \text{int}(\Lambda) \). Equation (36) remains true when \( \Lambda \) is replaced by \( \{\lambda_0\} \). For sufficiently small \( \varepsilon > 0 \), we have
\[
\{\lambda_0 h : h \in [1-\varepsilon,1+\varepsilon]\} \subset \Lambda \subset \{\lambda h : \lambda \in \Lambda, h \in [1-\varepsilon,1+\varepsilon]\}.
\]
The conclusion then follows from Lemma 6.14(c).

\textbf{Lemma 6.14 (Hausdorff distance).} (a) Let \((K_i)_{i \in I}\) and \((L_i)_{i \in I}\) be arbitrary collections of non-empty bounded sets in some Euclidean space. If the sets \(\bigcup_i K_i\) and \(\bigcup_i L_i\) are bounded, then
\[
d_H(\bigcup_i K_i, \bigcup_i L_i) \leq \sup_i d_H(K_i, L_i).
\]
(b) Let \(K\) be a non-empty bounded set in some Euclidean space. For every \(\lambda > 0\), we have
\[
d_H(\lambda K, K) \leq |\lambda - 1| \sup\{|x| : x \in K\}.
\]
(c) Let \(K_1, K_2, K_3\) and \(L\) be non-empty bounded sets in some Euclidean space. If \(K_1 \subset K_2 \subset K_3\), then
\[
d_H(K_2, L) \subset \max\{d_H(K_1, L), d_H(K_3, L)\}.
\]

\textbf{Proof.} (a) Let \(h > \sup_i d_H(K_i, L_i)\). Then \(K_i \subset L_i + hB\) for all \(i\), and thus
\[
\bigcup_i K_i \subset \bigcup_i (L_i + hB) = \bigcup_i L_i + hB.
\]
By symmetry, also \(\bigcup_i L_i \subset \bigcup_i K_i + hB\). It follows that \(d_H(\bigcup_i K_i, \bigcup_i L_i) \leq h\). Since \(h > \sup_i d_H(K_i, L_i)\) was arbitrary, we obtain the inequality stated in the lemma.

(b) Put \(m = \sup\{|x| : x \in K\}\). For every \(x \in K\), we have \(|\lambda x - x| \leq |\lambda - 1| m = h\).

But then \(\lambda K \subset K + hB\) and \(K \subset \lambda K + hB\), and thus \(d_H(\lambda K, K) \leq h\).

(c) Let \(\varepsilon > \max\{d_H(K_1, L), d_H(K_3, L)\}\). On the one hand, \(L \subset K_1 + \varepsilon B \subset K_2 + \varepsilon B\) and on the other hand, \(K_2 \subset K_3 \subset L + \varepsilon B\). As a consequence, \(\varepsilon \geq d_H(K_2, L)\).

\textbf{Proof of Theorem 6.6.} Let \(B\) be a Borel set in \(\{x \in \mathbb{R}^d : ||x|| = 1\}\); recall that \(\|\cdot\|\) is an arbitrary norm on \(\mathbb{R}^d\), not necessarily the Euclidean one. Since \(\psi = \nabla \bar{\psi} \bar{\mu}\), since \(\bar{\mu}\) is spherically symmetric and since \(\nabla \bar{\psi}\) is positively homogeneous of order one, we have, using Fubini’s theorem,
\[
H(B) = \bar{\psi}(\{y \in \mathbb{R}^d : ||y|| \geq 1, y/||y|| \in B\})
\]
\[
= (\nabla \bar{\psi} \bar{\mu})(\{y \in \mathbb{R}^d : ||y|| \geq 1, y/||y|| \in B\})
\]
\[
= \bar{\mu}(\{x \in \mathbb{R}^d : ||\nabla \bar{\psi}(x)|| \geq 1, \nabla \bar{\psi}(x)/||\nabla \bar{\psi}(x)|| \in B\})
\]
\[
= c \int_0^\infty \mathbb{E} \left[ \mathbb{1}\{||\nabla \bar{\psi}(rU)|| \geq 1, \nabla \bar{\psi}(rU)/||\nabla \bar{\psi}(rU)|| \in B\} \right] \alpha r^{-\alpha-1} dr
\]
\[
= c \mathbb{E} \left[ \left( \int_0^\infty \mathbb{1}\{r||\nabla \bar{\psi}(U)|| \geq 1\} \alpha r^{-\alpha-1} dr \right) \mathbb{1}\{||\nabla \bar{\psi}(U)|| \in B\} \right]
\]
\[
= c \mathbb{E} \left[ ||\nabla \bar{\psi}(U)||^\alpha \mathbb{1}\{||\nabla \bar{\psi}(U)|| \in B\} \right].
\]

Equation (28) follows if we let \(\|\cdot\|\) be the Euclidean norm \(|\cdot|\) and if we set \(B = \mathbb{S}_{d-1}\), since then \(H(\mathbb{S}_{d-1}) = \bar{\psi}(\{x \in \mathbb{R}^d : |x| \geq 1\}) = c\). \(\square\)
7. Examples

The limiting tail quantile contours of an elliptical distribution with regularly varying radial component are ellipsoids, and the boundary of the convex set whose support function generates the limiting cyclically monotone transport plan is an ellipsoid too. Regular variation of elliptical distributions was first studied in [14, Theorem 4.3]. A well-known example is the multivariate Student $t$ distribution, for which the square of the random variable $R$ in Proposition 7.1 has a Fisher $F$ distribution.

Proposition 7.1 (Elliptical distributions). Let $\nu \in \mathcal{P}(\mathbb{R}^d)$ be the distribution of $Y = RAU$, where $R$ is a positive random variable, $A$ is a (deterministic) symmetric, positive definite $d \times d$ matrix, and the random vector $U$ is uniformly distributed on the unit sphere $\mathbb{S}_{d-1}$ and is independent of $R$. Assume that $\lim_{r \to \infty} \mathbb{P}(R > \lambda r)/\mathbb{P}(R > r) = \lambda^{-\alpha}$ for all $\lambda > 0$.

(a) The distribution $\nu$ is regularly varying with index $\alpha$, auxiliary function $b$ such that $\lim_{t \to \infty} t \mathbb{P}[R > b(t)] = 1$, and limit measure $\tilde{\nu} \in \mathcal{M}_0(\mathbb{R}^d)$ given by

$$\tilde{\nu}(f) = \int_0^\infty \mathbb{E}[f(rAU)] \alpha r^{-\alpha} dr$$

for bounded, measurable $f : \mathbb{R}^d \to \mathbb{R}$ that vanish in a neighbourhood of the origin.

(b) Letting $\tilde{\mu} \in \mathcal{M}_0(\mathbb{R}^d)$ denote the spherically symmetric measure with the same radial measure as $\tilde{\nu}$, we have $(\nabla \tilde{\psi})_A \tilde{\mu} = \tilde{\nu}$, where $\tilde{\psi}(x) = (\beta/2)x'Ax$ and thus

$$\nabla \tilde{\psi}(x) = \beta Ax, \quad x \in \mathbb{R}^d,$$

with $\beta = (\mathbb{E}[|AU|^\alpha])^{-1/\alpha}$.

(c) The limiting quantile contour is

$$Q_{\tilde{\psi}}(1) = \{ \beta Au : u \in \mathbb{S}_{d-1} \}$$

while $\tilde{\psi} = \frac{1}{2} \sigma_n^2$, with $C = \sqrt{\beta}B'v : v \in \mathbb{S}_{d-1}$ and $B \in \mathbb{R}^{d \times d}$ any matrix such that $B'B = A$. If $A$ is invertible, then also $C = \{ z \in \mathbb{R}^d : z'A^{-1}z \leq \beta \}$.

A second model we consider is when large values of the components of $Y$ are generated through $q$ independent random variables $Z_1, \ldots, Z_q$, to be thought of as factors. A stylized model is the linear one, $Y_j = \sum_{k=1}^d a_{jk}Z_k$ for all $j = 1, \ldots, d$, where $a_{jk}$ are scalars. Focusing on large positive values only, we could also consider the max-linear model $Y_j = \max_{k=1, \ldots, q} a_{jk}Z_k$, with similar asymptotic properties. Writing $Y = AZ$ where $A = (a_{jk})_{j,k} \in \mathbb{R}^{d \times d}$ is deterministic and $Z = (Z_1, \ldots, Z_q)'$ has independent components, regular variation of $Y$ follows from regular variation of $Z$ [3, Proposition A.1]. Here, we are interested in how the limit measure $\tilde{\nu}$ of $Y$ can be obtained as the push-forward of a spherically symmetric measure by a cyclically monotone mapping. The target measure $\tilde{\nu}$ has discrete spectral measure and is supported on the union of $q$ rays determined by the coefficient vectors. The quantile contour is to be constructed from the faces of the set $C$ in Theorem 6.4, which turns out to be polyhedral.

Proposition 7.2 (Factor model). Let $Y = u_1Z_1 + \cdots + u_qZ_q$ where $u_1, \ldots, u_q$ are distinct unit vectors in $\mathbb{R}^d$ and where $Z_1, \ldots, Z_q$ are independent, nonnegative random variables.
Suppose that there exists $\alpha > 0$ and a function $b : (0, \infty) \to (0, \infty)$, regularly varying at infinity with index $1/\alpha$, such that
\[
\lim_{t \to \infty} t \mathbb{P}[Z_k > b(t)] = \gamma_k \in (0, \infty), \quad k = 1, \ldots, q.
\]
(a) The distribution $\nu$ of $Y$ is regularly varying with index $\alpha$, auxiliary function $b$, and limit measure $\bar{\nu} \in \mathcal{M}_0(\mathbb{R}^d)$ given by
\[
\bar{\nu}(f) = \sum_{k=1}^q \gamma_k \int_0^\infty f(ru_k) a r^{-\alpha-1} \, dr
\]
for bounded, measurable $f : \mathbb{R}^d \to \mathbb{R}$ vanishing in a neighbourhood of the origin. The spectral measure $H$ of $\bar{\nu}$ with respect to the Euclidean norm is discrete with atoms $u_1, \ldots, u_q$ and masses $\gamma_1, \ldots, \gamma_q$.
(b) Letting $\bar{\mu} \in \mathcal{M}_0(\mathbb{R}^d)$ denote the spherically symmetric measure with the same radial measure as $\bar{\nu}$, we have $(\nabla \bar{\nu}) \# \bar{\mu} = \bar{\nu}$, with $\bar{\nu} = \frac{1}{2} \bar{\sigma}_C^2$ and
\[
\sigma_C(x) = \max \{0, \langle x, \beta_1 u_1 \rangle, \ldots, \langle x, \beta_q u_q \rangle\}, \quad x \in \mathbb{R}^d,
\]
the support function of the convex hull, $C$, of $\{0, \beta_1 u_1, \ldots, \beta_q u_q\}$, with $\beta_1, \ldots, \beta_q \in (0, \infty)$ determined by the equations
\[
\beta_k^p \mathbb{E} \left[ \langle U, \beta_k u_k \rangle^p \mathbf{1} \{ (U, \beta_k u_k) = \sigma_C(U) \} \right] = \frac{\gamma_k}{\gamma_1 + \cdots + \gamma_q}, \quad k = 1, \ldots, q,
\]
and $U$ uniformly distributed on $S_{d-1}$.

Remark 7.3. The subdifferential of $\sigma_C$ is [27, Corollary 23.5.3]
\[
\partial \sigma_C(x) = \{ y \in C : \forall z \in C, \langle x, y \rangle \geq \langle x, z \rangle \}
\]
\[
= \{ y \in C : \langle x, y \rangle = \sigma_C(x) \}, \quad x \in \mathbb{R}^d.
\]
For non-zero $x$, the set $\partial \sigma_C(x)$ is a convex subset of the boundary of $C$, and for polyhedral $C$ as in Proposition 7.2, it is actually the face of the boundary of $C$ given by the convex hull of the vertices where the maximum in the definition of $\sigma_C$ is attained. The limiting quantile contour is
\[
Q_{\bar{\psi}}(x) = \bigcup_{x : |x|=1} \sigma_C(x) \partial \sigma_C(x)
\]
\[
= \bigcup_{x : |x|=1} \{ \langle x, y \rangle y : y \in \partial \sigma_C(x) \}.
\]
For $y \in C$, let $K_C(y)$ be the normal cone of $C$ at $y$; this is the set of all $x \in \mathbb{R}^d$ such that $\langle z - y, x \rangle \leq 0$ for all $z \in C$. For $y$ in the interior of $C$, we have $K_C(y) = \{0\}$, while for $y$ on the boundary of $C$, we have $x \in K_C(y)$ if and only if $y \in \partial \sigma_C(x)$. It follows that the quantile contour can also be written as
\[
Q_{\bar{\psi}}(1) = \bigcup_{y \in \text{bnd} C} \{ \langle x, y \rangle y : x \in K_C(y), |x| = 1 \}.
\]
For polyhedral $C$ as in Proposition 7.2, this formula allows to draw the quantile contour by considering each of the faces of the boundary of $C$. Perhaps it is more natural to just consider the intersection of $Q_{\bar{\psi}}(1)$ with the support of $\bar{\nu}$, which in Proposition 7.2 is given by the union of the rays $\{ ru_k : r \in [0, \infty) \}$ over $k = 1, \ldots, q$. The intersection will then be equal to the union of certain line segments on those rays and, possibly, the origin.
Remark 7.4. The conditions in Proposition 7.2 are less restrictive than they may seem.

- If some of the vectors $u_k$ would coincide, we could get back to the hypotheses of the proposition by aggregating the corresponding variables $Z_j$ and the constants $\gamma_j$.
- The proposition remains true if some, but not all, $\gamma_j$ are zero.
- If we have instead $Y = a_1 Z_1 + \cdots + a_q Z_q$ with non-zero vectors $a_k$, then we can put $u_k = a_k / |a_k|$, provided these vectors are all different, and then replace $Z_k$ by $|a_k| Z_k$ and $\gamma_k$ by $\gamma_k |a_k|^\alpha$, respectively.
- The case where the variables $Z_k$ can also have a regularly varying lower tail can be covered by replacing $u_k Z_k$ by $u_k \tilde{Z}_{k,+} - u_k \tilde{Z}_{k,-}$, where $\tilde{Z}_{k,\pm}$ are independent copies of the positive and negative parts of $Z_k$. This modified random vector has the same regular variation properties as the original one.

Proofs.

Proof of Proposition 7.1. (a) Nonnegativity of $R$, regular variation of $r \mapsto \mathbb{P}(R \geq r)$ and the choice of the function $b$ imply that, in $\mathcal{M}_0(\mathbb{R})$, we have
\[
 t \mathbb{P}[R/b(t) \in \cdot] \overset{\alpha r^{-\alpha-1}}{\longrightarrow} \mathbb{1}_{(0,\infty)}(r) dr, \quad t \to \infty.
\]
For $f \in C_0(\mathbb{R}^d)$, the function $r \mapsto \mathbb{E}[f(r A U)]$ belongs to $C_0(\mathbb{R})$, so that
\[
t \mathbb{E}[f(Y/b(t))] = t \int_{(0,\infty)} \mathbb{E}[f((r/b(t)) A U)] d\mathbb{P}(R \leq r)
\]
\[
\to \int_{0}^{\infty} \mathbb{E}[f(r A U)] \alpha r^{-\alpha} dr = \bar{\nu}(f), \quad t \to \infty. \tag{37}
\]

(b) The radial component of $\nu$ is the distribution of $|Y| = R |A U|$, which is in general different from what is commonly called the radial part $R$ of the elliptically distributed random vector $Y$. Let $V$ be another random vector that is uniformly distributed on the unit sphere, independent of $(R, U)$. The distribution, $\mu$, of $X = R |A U| V$ is spherically symmetric and its radial component is the distribution of $|X| = R |A U| = |Y|$, which is the radial component of $\nu$. The probability measure $\mu$ is regularly varying too and, for $f \in C_0(\mathbb{R}^d)$, we have
\[
t \mathbb{E}[f(X/b(t))] = t \int_{(0,\infty)} \mathbb{E}[f((r/b(t)) |A U| V)] d\mathbb{P}(R \leq r)
\]
\[
\to \int_{0}^{\infty} \mathbb{E}[f(r |A U| V)] \alpha r^{-\alpha} dr = \bar{\mu}(f), \quad t \to \infty.
\]

We identify the convex potential $\tilde{\psi}$ on $\mathbb{R}^d$ such that $\nabla \tilde{\psi}^\# \bar{\mu} = \bar{\nu}$. Intuitively, we expect that the optimal transport plan is given by $\nabla \tilde{\psi}(x) = \beta A x$ for some constant $\beta \in (0, \infty)$, and thus $\tilde{\psi}(x) = (\beta/2) x' A x$ for $x \in \mathbb{R}^d$, which is convex since the Hessian matrix $\beta A$ is positive semi-definite by assumption. For such $\tilde{\psi}$, the transported measure is
\[
\int_{\mathbb{R}^d} f(x) d((\nabla \tilde{\psi})^\# \bar{\mu})(x) = \int_{\mathbb{R}^d} f(\beta A x) d\bar{\mu}(x) = \int_{0}^{\infty} \mathbb{E}[f(\beta r |A U| AV)] \alpha r^{-\alpha} dr
\]
Just like $\tilde{\mu}$ and $\tilde{\nu}$, this measure is homogeneous with index $-\alpha$. To determine $\beta$, we ensure that the spectral measures of $(\nabla \tilde{\psi})_{\#} \tilde{\mu}$ and $\tilde{\nu}$ with respect to the Euclidean norm are the same. Let $g : \mathbb{S}_{d-1} \to \mathbb{R}$ be continuous. Applying the previous formula with $f(x) = g(x/|x|) I\{|x| \geq 1\}$, we get, by Fubini’s theorem,

$$
\int_{x:|x| \geq 1} g(x/|x|) d((\nabla \tilde{\psi})_{\#} \tilde{\mu})(x) = \int_0^\infty E \left[ g \left( \frac{AV}{|AV|} \right) I\{|\beta AU| |AV| \geq 1\} \right] \alpha r^{-\alpha-1} dr
$$

$$
= E \left[ g \left( \frac{AV}{|AV|} \right) |\beta AU| |AV| \right] r^{-\alpha-1} dr
$$

$$
= \beta^\alpha E |AU| \alpha E \left[ g \left( \frac{AV}{|AV|} \right) |AV| \right],
$$

since $U$ and $V$ are independent.

On the other hand, in view of (37), the spectral measure of $\tilde{\nu}$ is determined by

$$
\int_{x:|x| \geq 1} g(x/|x|) d\tilde{\nu}(x) = \int_0^\infty E \left[ g \left( \frac{AU}{|AU|} \right) I\{|AU| \geq 1\} \right] \alpha r^{-\alpha-1} dr
$$

$$
= E \left[ g \left( \frac{AU}{|AU|} \right) |AU| \right].
$$

As $U$ and $V$ have the same distribution, we find that $(\nabla \tilde{\psi})_{\#} \tilde{\mu}$ is equal to $\tilde{\nu}$ when $\nabla \tilde{\psi}(x) = \beta Ax$ for all $x \in \mathbb{R}^d$ with $\beta$ such that $\beta^\alpha E |AU| \alpha = 1$, i.e., $\beta = (E |AU|^\alpha)^{-1/\alpha}$.

(c) The formula for $Q_{\tilde{\psi}}(1) = \bigcup_{x \in \mathbb{S}_{d-1}} \partial \tilde{\psi}(x)$ is immediate from the expression for $\tilde{\psi}$. The support function of $C = \{\sqrt{\beta} B^tv : |v| \leq 1\}$ is

$$
\sigma_C(x) = \sup \{x, \sqrt{\beta} B^tv : v \in \mathbb{R}^d, |v| \leq 1\}
$$

$$
= \sqrt{\beta} \sup \{|Bx, v| : v \in \mathbb{R}^d, |v| \leq 1\}
$$

$$
= \sqrt{\beta} |Bx|, \quad x \in \mathbb{R}^d,
$$

and thus $2^{-1} \{\sigma_C(x)\}^2 = (\beta/2) |Bx|^2 = (\beta/2) x^t B^t B x = \tilde{\psi}(x)$ for $x \in \mathbb{R}^d$. The formula for $C$ in case $A$ and thus $B$ are invertible follows from simple algebra. \(\square\)

**Proof of Proposition 7.2.** (a) The assumptions on $Z_k$ imply that $t P[Z_k > b(t) \lambda] \to \gamma_k \lambda^{-\alpha}$ as $t \to \infty$ for $\lambda \in (0, \infty)$. Therefore, the random variable $Z_k$ is regularly varying with index $\alpha$, scale function $b$, and limit measure in $\mathcal{M}_0(\mathbb{R})$ given by

$$
t P[Z_k/b(t) \in \cdot] \overset{\alpha}{\to} \gamma_k \alpha r^{-\alpha-1} \mathbb{I}_{(0,\infty)}(r), \quad t \to \infty.
$$

Regular variation of $Y = u_1 Z_1 + \cdots + u_q Z_q$ follows from the “single-shock heuristic”: the most likely way for a sum of independent, heavy-tailed components to be large in norm is for one of the terms $u_k Z_k$ to be large in norm, and in that case, the sum is dominated by that component. For $f \in C_0(\mathbb{R}^d)$, the function $r \mapsto f(u_k r)$ belongs to
\(C_0(\mathbb{R})\) for all \(k = 1, \ldots, d\), and we have
\[
 t \mathbb{E}[f((u_1Z_1 + \cdots + u_qZ_q)/b(t))] = t \sum_{k=1}^{q} \mathbb{E}[f(u_kZ_k/b(t))] + o(1)
\]
\[
\rightarrow \sum_{k=1}^{q} \gamma_k \int_{0}^{\infty} f(ru_k) \alpha r^{-\alpha-1} \, dr = \bar{n}u(f), \quad t \to \infty.
\]
(38)

The proof is similar to the one of [21, Proposition 7.1]; see also [17, Corollary 3.1] in case the variables are independent and identically distributed. The resulting expression determines \(\bar{\nu}(f)\) for \(f \in C_0(\mathbb{R})\) and thus also for general bounded, measurable functions vanishing in a neighbourhood of the origin.

The limit measure \(\bar{\nu}\) is concentrated on the \(q\) rays \(\{ru_k : r \in (0, \infty)\}\). To find its spectral measure \(H\) with respect to the Euclidean norm, let \(g : S_{d-1} \to \mathbb{R}\) be measurable and bounded. Setting \(f(y) = g(y/|y|) \, \mathbb{1}\{|y| \geq r\}\) in (38) yields
\[
\int_{S_{d-1}} g(u) \, dH(u) = \int_{\mathbb{R}^d} g(y/|y|) \, \mathbb{1}\{|y| \geq 1\} \, d\bar{\nu}(y) = \sum_{k=1}^{q} g(u_k) \gamma_k.
\]
(39)

(b) The radial component of \(\bar{\nu}\) follows from setting \(f(y) = \mathbb{1}\{|y| \geq r\}\) in (38), giving
\[
\bar{\nu}(\{y \in \mathbb{R}^d : |y| \geq r\}) = cr^{-\alpha}, \quad r \in (0, \infty),
\]
with \(c = \gamma_1 + \cdots + \gamma_q\). Let \(\bar{\mu} \in \mathcal{M}_0(\mathbb{R}^d)\) be the spherically symmetric measure having the same radial component as \(\bar{\nu}\): for bounded, measurable \(f : \mathbb{R}^d \to \mathbb{R}\) that vanish in the neighbourhood of the origin, we have
\[
\bar{\mu}(f) = c \int_{0}^{\infty} \mathbb{E}[f(rU)] \alpha r^{-\alpha-1} \, dr,
\]
where \(U\) is uniformly distributed on the unit sphere.

We look for a convex function \(\bar{\psi} : \mathbb{R}^d \to [0, \infty)\) such that \((\nabla \bar{\psi})\#\bar{\mu} = \bar{\nu}\). In view of the form of \(\bar{\nu}\), we try \(\bar{\psi} = \frac{1}{2} \bar{\sigma}_C^2\), with
\[
\sigma_C(x) = \max_{k=0,1,\ldots,q} \langle x, a_k \rangle, \quad x \in \mathbb{R}^d,
\]
with \(a_0 = 0\) and \(a_k = \beta_k u_k\) for \(k = 1, \ldots, q\), for \(\beta_k \in (0, \infty)\) to be determined. The gradient of \(\bar{\psi}\) is
\[
\nabla \bar{\psi}(x) = \langle x, a_k \rangle a_k, \quad x \in K_k, \ k = 0, 1, \ldots, q,
\]
where \(K_k\) is the interior of the normal cone to \(C\) at \(a_k\), that is,
\[
K_k = \left\{ x \in \mathbb{R}^d : \langle x, a_k \rangle > \max_{i \in \{0,1,\ldots,q\}\setminus\{k\}} \langle x, a_i \rangle \right\}.
\]
Let \( g : \mathcal{S}_{d-1} \to \mathbb{R} \) be bounded and measurable and let the random vector \( U \) be uniformly distributed on \( \mathcal{S}_{d-1} \). The spectral measure of \( \nabla \psi \# \mu \) follows from

\[
\int_{\mathbb{R}^d} g(y/|y|) \mathbb{1}\{|y| \geq 1\} \, d(\nabla \psi \# \mu)(y) = \int_{\mathbb{R}^d} g(\nabla \psi(x)/\|\nabla \psi(x)\|) \mathbb{1}\{|\nabla \psi(x)| \geq 1\} \, d\mu(x) = c \int_0^\infty \mathbb{E}[g(\nabla \psi(rU)/\|\nabla \psi(rU)\|) \mathbb{1}\{|\nabla \psi(rU)| \geq 1\}] \, \alpha r^{-\alpha - 1} \, dr
\]

\[
= c \sum_{k=1}^q g(u_k) |a_k|^\alpha \mathbb{E}[(U, a_k)^\alpha \mathbb{1}\{U \in K_k\}].
\]

(40)

Recall that \( a_k = \beta_k u_k \) for \( k = 1, \ldots, q \), with \( \beta_k \) to be determined. Equating (39) and (40), we find that the scalars \( \beta_1, \ldots, \beta_q \) must solve the \( q \) equations

\[
\beta_k^{2\alpha} \mathbb{E}[(U, u_k)^\alpha \mathbb{1}\{U \in K_k\}] = \gamma_k/c, \quad j = 1, \ldots, q.
\]

[Note that the coefficients \( \beta_1, \ldots, \beta_q \) also appear in the definition of the cone \( K_k \), which makes the above system of equations potentially hard to solve.] The boundary of \( K_k \) is a union of cones of dimension \( d-1 \) or lower, the intersection of which with the unit sphere is a null set for \( U \). Therefore, we may replace \( K_k \) by its closure, which is the set of \( x \) such that \( \langle x, a_k \rangle = \sigma_C(x) \).

\[ \square \]

8. Discussion

In this paper, we have treated existence, uniqueness and stability of cyclically monotone transport plans between measures in \( \mathcal{M}_0(\mathbb{R}^d) \), containing measures which are finite on complements of neighbourhoods of the origin. Our contribution extends the results in McCann [20] to the infinite measures in this class, and raises the question what other spaces of infinite measures might allow a unique optimal transport mapping under similar conditions, notably, the vanishing of one of the measures on sets of Hausdorff dimension not larger than \( d-1 \).

In particular, we have showed that for sequences of pairs of measures \( (\mu_n, \nu_n) \) converging in \( \mathcal{M}_0(\mathbb{R}^d) \), the graphs of subdifferentials \( \partial \psi_n \) of convex potentials \( \psi_n \) supporting an optimal coupling measures are stable, and converge to a unique limit relative to the interior of the support of one of the limit measures. For quantile contours, defined as images of spheres by such subdifferentials, we could then deduce set convergence in the Hausdorff metric. Throughout, we were able to avoid smoothness conditions, moment conditions, and conditions on the supports of the measures involved. Moreover, our results cover weak convergence of probability measures on \( \mathbb{R}^d \) and therefore reinforce stability results on Monge–Kantorovich depth or quantile contours studied in Chernozhukov et al. [7] and Hallin [13].

For regularly varying distributions, we considered quantile contours via a spherical reference distribution \( \mu \) having the same radial measure as the target distribution \( \nu \). For this choice, with \( \psi \) a convex potential satisfying that the graph of its subdifferential \( \partial \psi \) supports an optimal coupling between \( \mu \) and \( \nu \), the tail limit \( \partial \psi \) of suitably scaled copies of \( \partial \psi \) is itself a positively homogeneous subdifferential of a convex potential \( \psi \). As a
consequence, all limiting quantile contours are homothetic to a single shape, \(Q_\varphi(r) = rQ_\varphi(1)\), which results from a deformation of the boundary of a compact convex body. For regularly varying elliptical distributions, the latter set as well as the limiting contours are elliptical, while for statistical models where large values are generated through a finite number of independent, heavy-tailed components, the quantile contours originate from convex polyhedral sets.

Apart from the two examples just mentioned, the computation of the limiting shape \(Q_\varphi(1)\) is a challenging topic: the convex gradient, for instance, is the solution to a Monge–Ampère partial differential equation \([10, 18]\), although methods based on computational fluid mechanics \([4]\) may be more suitable to exploit the special structure of the present problem. If the target distribution is unknown, and only a sample from it is available, the tail quantile contour will need to be estimated, a task which may be handled by extending the ideas presented in this paper to random measures. To be practically useful, our choice of reference measure might then require a form of prior centring of the target measure or its sample.

Possibly, the present article may be extended with a similar analysis assuming a regularly varying spherical reference measure \(\mu\) chosen independently from \(\nu\). However, in applications, one may also choose to decouple radial tail from tail dependence, and normalise the radial measures of \(\nu\) and \(\mu\) to the same regularly varying standard measure before determining quantile contours. This case is already covered by our results.

**Appendix A. Infinite measures: supports and convergence**

In this section, we collect some results on possibly infinite Borel measures on Euclidean space, with particular attention to \(\mathcal{M}_0\)-convergence and to properties of the supports of such measures.

**Lemma A.1** (Coupling in \(\mathcal{M}_0(\mathbb{R}^d)\)). (a) If \(\mu, \nu \in \mathcal{M}_0(\mathbb{R}^d)\), then every \(\pi \in \Pi(\mu, \nu)\) belongs to \(\mathcal{M}_0(\mathbb{R}^d \times \mathbb{R}^d)\).

(b) If \(M\) and \(N\) are relative compact sets of measures in \(\mathcal{M}_0(\mathbb{R}^d)\), then \(\bigcup\{\Pi(\mu, \nu) : \mu \in M, \nu \in N\}\) is relatively compact in \(\mathcal{M}_0(\mathbb{R}^d \times \mathbb{R}^d)\).

**Proof.** (a) If \((x, y) \in \mathbb{R}^d \times \mathbb{R}^d\) is such that \(|(x, y)| \geq r\) for some \(r > 0\), then necessarily \(|x| \geq r/2\) or \(|y| \geq r/2\). For \(\mu, \nu \in \mathcal{M}_0(\mathbb{R}^d)\) and \(\pi \in \Pi(\mu, \nu)\), we thus have

\[
\pi(\{(x, y) : |(x, y)| \geq r\}) \leq \mu(\{x : |x| \geq r/2\}) + \nu(\{y : |y| \geq r/2\}),
\]

which is finite by assumption.

(b) A necessary and sufficient criterion for relative compactness of a subset \(K\) in \(\mathcal{M}_0(\mathbb{R}^d)\) is \(\sup_{\mu \in K} \mu(S_r) < \infty\) for all \(r > 0\) and \(\lim_{r \to \infty} \sup_{\mu \in K} \mu(S_r) = 0\) \([16, \text{Section 4.1}]\). Apply this criterion, twice in \(\mathcal{M}_0(\mathbb{R}^d)\) and once in \(\mathcal{M}_0(\mathbb{R}^d \times \mathbb{R}^d)\), in combination with (41), to conclude the proof. \(\square\)

**Lemma A.2** (From joint to marginal convergence). Let \(\mu_n, \nu_n \in \mathcal{M}_0(\mathbb{R}^d)\) and let \(\pi_n \in \Pi(\mu_n, \nu_n)\) for all \(n\). If \(\pi_n \overset{\text{d}}{\rightarrow} \pi\) as \(n \to \infty\), then also \(\mu_n \overset{\text{d}}{\rightarrow} \mu\) and \(\nu_n \overset{\text{d}}{\rightarrow} \nu\) as \(n \to \infty\) and \(\pi \in \Pi(\mu, \nu)\).

**Proof.** This is a consequence of Theorem 2.5 in Hult and Lindskog \([15]\) applied to the projection mappings \((x, y) \mapsto x\) and \((x, y) \mapsto y\) from \(\mathbb{R}^d \times \mathbb{R}^d\) into \(\mathbb{R}^d\), each of which is continuous and maps the origin in \(\mathbb{R}^d \times \mathbb{R}^d\) to the origin in \(\mathbb{R}^d\). \(\square\)
The inner limit of a sequence of sets \( A_n \subset \mathbb{R}^d \) is the set \( \liminf_{n \to \infty} A_n \) of points \( x \in \mathbb{R}^d \) every neighbourhood of which intersects all but finitely many \( A_n \) (Section B.1). Note that this topological inner limit is in general larger than the set-theoretic one, defined as \( \bigcup_{n \geq 1} \cap_{k \geq n} A_k \).

**Lemma A.3** (Support of a limit measure). If \( \mu_n \) converges to \( \mu \) in \( \mathcal{M}_0(\mathbb{R}^d) \), then \( \text{spt}(\mu) \subset \liminf_{n \to \infty} \text{spt}(\mu_n) \).

**Proof.** Let \( U \) be an open neighbourhood of \( x \in \text{spt}(\mu) \). Then \( \mu(U) > 0 \). By the Portmanteau theorem for \( \mathcal{M}_0 \) [16, Theorem 2.4],

\[
\liminf_{n \to \infty} \mu_n(U) \geq \mu(U) > 0.
\]

Hence \( U \) has a non-empty intersection with \( \text{spt} \mu_n \) for all sufficiently large \( n \), for otherwise, we could find a subsequence such that \( \mu_n(U) = 0 \) along the subsequence. Since \( U \) was arbitrary, we conclude that \( x \) belongs to the inner limit of \( \text{spt}(\mu_n) \). \( \square \)

**Lemma A.4** (Support of a Borel measure). Let \( \mu \) be a non-zero Borel measure on \( \mathbb{R}^d \). If \( U \subset \text{spt}(\mu) \) is non-empty and open and if \( S \subset \mathbb{R}^d \) is a Borel set with \( \mu(S) = 0 \), then \( U \setminus S \) is dense in \( U \).

**Proof.** Let \( x \in U \cap S \) and let \( V \) be a neighbourhood of \( x \). We need to show that \( V \) intersects \( U \setminus S \). Since \( x \in \text{spt}(\mu) \) and since \( U \cap V \) is a neighbourhood of \( x \), we have \( \mu(U \cap V) > 0 \). Since \( \mu(S) = 0 \), we find \( \mu((U \cap V) \setminus S) > 0 \) too. As a consequence, \( (U \cap V) \setminus S = (U \setminus S) \cap V \) cannot be empty. \( \square \)

Let \( \text{proj}_1 \) be the projection map \( (x, y) \mapsto x \) from \( \mathbb{R}^d \times \mathbb{R}^d \) into \( \mathbb{R}^d \).

**Lemma A.5** (Support of a margin). If \( \mu = (\text{proj}_1)_\# \pi \) is the left marginal of the Borel measure \( \pi \) on \( \mathbb{R}^d \times \mathbb{R}^d \), then \( \text{spt} \mu \subset \text{cl}(\text{proj}_1(\text{spt} \pi)) \). As a consequence, if \( \text{spt} \pi \subset \text{gph}(T) \) for some multivalued mapping \( T : \mathbb{R}^d \rightrightarrows \mathbb{R}^d \), then \( \text{spt} \mu \subset \text{cl}(\text{dom} T) \).

**Proof.** Put \( G = \text{cl}(\text{proj}_1(\text{spt} \pi)) \). We need to show that \( \mu(G^c) = 0 \). We have \( \mu(G^c) = \pi(G^c \times \mathbb{R}^d) \) and

\[
G^c \times \mathbb{R}^d \subset (\text{proj}_1(\text{spt} \pi))^c \times \mathbb{R}^d \subset (\text{spt} \pi)^c,
\]

which is a \( \pi \)-null set. For the second statement, it suffices to notice that

\[
\text{proj}_1(\text{spt} \pi) \subset \text{proj}_1(\text{gph} T) \subset \text{dom} T.
\]

\( \square \)

**Appendix B. Graphical convergence of multivalued mappings**

Subdifferentials of convex functions are multivalued mappings. To study the asymptotic properties of a sequence of such mappings, we employ the notion of graphical convergence, which is in turn based on the theory of Painlevé–Kuratowski set convergence. We review the basic definitions and facts and we state some results that we use in the paper and that we have not been able to find in the literature. Proofs are collected at the end. Our exposition leans heavily on [28, Chapters 4, 5 and 12] and [22, Appendix B].
B.1. Painlevé–Kuratowski set convergence. The inner limit of a sequence of sets \( C_n \subset \mathbb{R}^d \) is the set \( \liminf_{n \to \infty} C_n \) of points \( x \in \mathbb{R}^d \) for which there exists a sequence of points \( x_n \in C_n \) such that \( x_n \to x \) as \( n \to \infty \). The outer limit of the sequence \( C_n \) is the set \( \limsup_{n \to \infty} C_n \) of points \( x \in \mathbb{R}^d \) for which there exists an infinite set \( N \subset \mathbb{N} \) such that \( x_n \in C_n \) for all \( n \in N \) and such that \( x_n \to x \) as \( n \to \infty \) in \( N \). Equivalently, the inner limit is the set of all points \( x \) every neighbourhood of which intersects all but a finite number of sets \( C_n \), while the outer limit is the set of points \( x \) every neighbourhood of which intersects an infinite number of sets \( C_n \). The inner and outer limits are both closed, and the inner limit is contained in the outer limit.

A sequence of sets \( C_n \subset \mathbb{R}^d \) is said to converge to \( C \subset \mathbb{R}^d \) if the inner and outer limits agree and are equal to \( C \); this is Painlevé–Kuratowski set convergence. When restricted to the space \( \mathcal{F} = \mathcal{F}(\mathbb{R}^d) \) of closed subsets of \( \mathbb{R}^d \), Painlevé–Kuratowski convergence is equivalent to convergence in the Fell hit-and-miss topology \cite[Theorem B.6]{PainlevéKuratowski}. Proposition D.5 in \cite{PainlevéKuratowski} asserts that the map \( \mathcal{F} \to \mathcal{F} : (\lambda, C) \mapsto \lambda C \) is continuous, for every \( \lambda \in \mathbb{R} \). Here, we need a slightly stronger statement.

**Proposition B.1.** Scalar multiplication \((\mathbb{R} \setminus \{0\}) \times \mathcal{F} \to \mathcal{F} : (\lambda, C) \mapsto \lambda C\) is continuous.

B.2. Graphical convergence. The graphical inner limit of a sequence of mappings \( S_n : \mathbb{R}^k \rightrightarrows \mathbb{R}^\ell \) is the mapping \( \text{g-liminf}_{n \to \infty} S_n \) whose graph is equal to the inner limit of the sequence of graphs of \( S_n \). Similarly, the graphical outer limit of the sequence \( S_n \) is the mapping \( \text{g-limsup}_{n \to \infty} S_n \) whose graph is equal to the outer limit of the graphs of \( S_n \). The sequence \( S_n \) converges graphically to \( S : \mathbb{R}^k \rightrightarrows \mathbb{R}^\ell \), notation \( S_n \rightarrow^*_S S \), if the graphical inner and outer limits coincide and are equal to the graph of \( S \). In other words, we have \( S_n \rightarrow^*_S S \) as \( n \to \infty \) if the inner and outer limits of \( \text{gph}(S_n) \) in \( \mathbb{R}^k \times \mathbb{R}^\ell \) are equal to \( \text{gph}(S) \).

We will apply the concept of graphical convergence to subdifferentials of convex functions. A useful fact is that the graphical limit of a sequence of subdifferentials of closed convex functions is again the subdifferential of a closed convex function.

**Lemma B.2** (Graphical limits of subdifferentials). Let \( \psi_n \) be a sequence of closed convex functions on \( \mathbb{R}^d \) and suppose that \( \partial \psi_n \) converges graphically to some mapping \( T : \mathbb{R}^d \rightrightarrows \mathbb{R}^d \) with non-empty domain. Then \( T = \partial \psi \) for some closed convex function \( \psi \) on \( \mathbb{R}^d \).

To show the existence of graphically converging subsequences of a sequence of mappings, there exists a simple criterion. A sequence of mappings \( S_n : \mathbb{R}^k \rightrightarrows \mathbb{R}^\ell \) is said to escape to the horizon if for every bounded set \( B \subset \mathbb{R}^k \times \mathbb{R}^\ell \), there exists an infinite set \( N \subset \mathbb{N} \) such that \( \text{gph}(S_n) \) does not intersect \( B \) for all \( n \in N \). In other words, for all bounded sets \( C \subset \mathbb{R}^k \) and \( D \subset \mathbb{R}^\ell \), there exists an infinite set \( N \subset \mathbb{N} \) such that \( S_n(x) \cap D = \emptyset \) for all \( x \in C \) and \( n \in N \). Now if \( S_n \) does not escape to the horizon, then it necessarily contains a subsequence that converges graphically to a mapping \( S : \mathbb{R}^k \rightrightarrows \mathbb{R}^\ell \) with non-empty domain \cite[Theorem 5.36]{EscobarValderramaSegers}.

The graph of the graphical limit of a sequence of mappings is closed, and mappings with closed graphs are outer semicontinuous \cite[Theorem 5.7]{EscobarValderramaSegers}, which is a property of the values of the mapping in the neighbourhood of a given point. Here, we need to extend this to neighbourhoods of a compact set.
Lemma B.3. If the graph of $T : \mathbb{R}^k \Rightarrow \mathbb{R}^\ell$ is closed and if the compact set $K \subset \mathbb{R}^k$ is such that $T(G)$ is bounded for some open set $G \supset K$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$T(K + \varepsilon B) \subset T(K) + \varepsilon B.$$  

B.3. Graphical convergence at a point or relative to an open set. A sequence of mappings $S_n : \mathbb{R}^k \Rightarrow \mathbb{R}^\ell$ is said to converge graphically to $S : \mathbb{R}^k \Rightarrow \mathbb{R}^\ell$ at a point $x \in \mathbb{R}$ if $(g\liminf_{n \to \infty} S_n)(x)$ and $(g\limsup_{n \to \infty} S_n)(x)$ are equal to $S(x)$ [28, Section 5.E]. Graphical convergence of $S_n$ to $S$ is thus equivalent to graphical convergence of $S_n$ to $S$ at every $x \in \mathbb{R}^k$. If $V \subset \mathbb{R}^k$ is open, then $S_n$ is said to converge graphically to $S$ relative to $V$ if $S_n$ converges graphically to $S$ at every point $x \in V$.

Lemma B.4 (Graphical convergence at a point). Let $S_n : \mathbb{R}^k \Rightarrow \mathbb{R}^\ell$ be such that no subsequence escapes to the horizon, i.e., each subsequence contains a graphically converging subsequence. Suppose that there exist $x \in \mathbb{R}^k$ and $C \subset \mathbb{R}^\ell$ such that $S(x) = C$ for any $S : \mathbb{R}^k \Rightarrow \mathbb{R}^\ell$ which is the graphical limit of some subsequence of $S_n$. Then $(g\liminf_{n \to \infty} S_n)(x) = C = (g\limsup_{n \to \infty} S_n)(x)$, i.e., $S_n$ converges graphically at $x$ to $S : \mathbb{R}^k \Rightarrow \mathbb{R}^\ell$ provided $S(x) = C$.

For a sequence of maximal monotone mappings $S_n : \mathbb{R}^d \Rightarrow \mathbb{R}^d$, graphical convergence implies locally uniform convergence at points in the interior of the domain of the limit map where the latter is single-valued [28, Exercise 12.40(a)]. More precisely, for such $S_n$, if $S_n \rightharpoonup S$ as $n \to \infty$ and if $S(x)$ is a singleton for some $x \in \text{int}(\text{dom} S)$, then $S_n(x_n) \to S(x)$ as $n \to \infty$ (set convergence) for every sequence $x_n \in \mathbb{R}^d$ such that $x_n \to x$ as $n \to \infty$. This property can be extended and strengthened, in particular to cover images of sets rather than of points, and to cover graphical convergence relative to an open set rather than full graphical convergence.

Proposition B.5 (Uniformity in graphical convergence relative to an open set). Let $T_n, T : \mathbb{R}^d \Rightarrow \mathbb{R}^d$ be maximal monotone mappings and suppose that $T_n$ converges graphically to $T$ relative to an open subset $V$ of $\text{dom} T$. Let $K \subset V$ be compact.

(a) There exists $n_K \in \mathbb{N}$ and an open set $U \subset V$ such that $K \subset U \subset \text{int}(\text{dom} T_n)$ for all integer $n \geq n_K$.

(b) For every $\varepsilon > 0$, there exists $n_{\varepsilon,K} \in \mathbb{N}$ such that for all $n \geq n_{\varepsilon,K}$ and all $A \subset K$,

$$T(A) \subset T_n(A + \varepsilon B) + \varepsilon B,$$

$$T_n(A) \subset T(A + \varepsilon B) + \varepsilon B.$$

(c) The sets $T(K)$ and $T_n(K)$ are compact, the latter for sufficiently large $n$, and

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} d_H (T_n(K + \varepsilon B), T(K)) = 0. \tag{42}$$

(d) For every $x \in V$ for which $T(x)$ is a singleton and every sequence $(x_n)_n$ in $\mathbb{R}^d$ such that $x_n \to x$, we have $d_H(T_n(x_n), T(x)) \to 0$ as $n \to \infty$.  

\footnote{For not necessarily open $X \subset \mathbb{R}^d$, graphical convergence of $S_n$ to $S$ relative to $X$ as defined in [28, p. 168] entails restricting the point sequences in the definition of set convergence to sequences $(x_n, y_n) \in X \times \mathbb{R}^d$ with $x_n \to x$ as $n \to \infty$. For open $V$, this restriction does not make a difference, since $x_n \to x \in V$ as $n \to \infty$ implies that $x_n \in V$ for all sufficiently large $n$ anyway.}
B.4. Proofs.

Proof of Proposition B.1. As the Fell topology is metrizable [22, Appendix B], it suffices to show that if \((\lambda_n)_n\) and \((C_n)_n\) are sequences in \((0, \infty)\) and \(C_n \to C \in \mathcal{F}\), respectively, then also \(\lambda_n C_n \to \lambda C\) as \(n \to \infty\) too. We check that \(\lambda_n C_n \to \lambda C\) in the sense of Painlevé–Kuratowski. First, let \(N\) be an infinite subset of \(\mathbb{N}\) and let \(y_n \in \lambda_n C_n\) be such that \(y_n \to y \in \mathbb{R}^d\) as \(n \to \infty\) in \(N\). Then \(y_n = \lambda_n x_n\) with \(x_n = \lambda_n^{-1} y_n \to \lambda^{-1} y = x\) as \(n \to \infty\) in \(N\). Necessarily, \(x \in C\) and thus \(y = \lambda x \in \lambda C\).

Second, let \(y \in \lambda C\). Then \(x = \lambda^{-1} y \in C\), and thus there exist points \(x_n \in C_n\) such that \(x_n \to x\) as \(n \to \infty\). But then also \(y_n = \lambda_n x_n \in \lambda_n C_n\) and \(y_n \to \lambda x = y\) as \(n \to \infty\) \(\Box\).

Proof of Lemma B.2. Since \(\text{gph}(\partial \psi_n)\) is cyclically monotone, its limit \(\text{gph}(T)\) is cyclically monotone too [30, proof of Theorem 5.20]. Furthermore, since \(\text{gph}(\partial \psi_n)\) is maximal monotone [27, Corollary 31.5.2], its limit is maximal monotone too [28, Theorem 12.32]. Because cyclic monotonicity implies monotonicity, the graph of \(T\) must be maximal cyclically monotone. Rockafellar’s Theorem 2.1 then implies that \(T = \partial \psi\) for some closed convex function \(\psi\) \(\Box\).

Proof of Lemma B.3. Let \(K \subset \mathbb{R}^k\) be compact, let \(G \subset \mathbb{R}^k\) be open, and suppose that \(K \subset G\) and that \(T(G)\) is bounded. For sufficiently small \(\delta > 0\), we have \(K + \delta B \subset G\).

Suppose the statement does not hold. Then we can find \(\varepsilon_0 > 0\), a sequence \(\delta_n > 0\) tending to zero and points \(x_n \in K + \delta_n B\) and \(y_n \in T(x_n)\) such that \(y_n \not\in T(K) + \varepsilon_0 B\). For every \(n\), there exists \(x'_n \in K\) such that \(|x'_n - x_n| < \delta_n\). Passing to a subsequence if necessary we may assume that \(x_n \to x \in K\) and thus also \(x_n \to x \in K\). Further, since \(y_n \in T(G)\) for all sufficiently large \(n\) and since \(T(G)\) is bounded, we can, upon passing to a further subsequence, assume that \(y_n \to y\). The limit point \(y \) cannot belong to \(T(K)\) since \(y_n \not\in T(K) + \varepsilon_0 B\) for all \(n\). We find that \((x_n, y_n) \in \text{gph}(T)\) converges to a point \((x, y)\) not in \(\text{gph}(T)\), in contradiction to the assumption that \(\text{gph}(T)\) is closed \(\Box\).

Proof of Lemma B.4. Write \(\overline{S} = \text{g-lim sup}_n S_n\) and \(\underline{S} = \text{g-lim inf}_n S_n\). Since \(\text{gph}(\overline{S}) = \liminf_{n \to \infty} \text{gph}(S_n) \supset \limsup_{n \to \infty} \text{gph}(S_n) = \text{gph}(\underline{S})\), we have \(\overline{S}(x) \subset \underline{S}(x)\). It is thus sufficient to show that \(\overline{S}(x) \subset C \subset \underline{S}(x)\).

First, let \(u \in \overline{S}(x)\), i.e., \((x, u) \in \text{gph}(\overline{S}) = \limsup_n \text{gph}(S_n)\). Then there exists a subsequence \(N\) and points \((x_n, u_n) \in \text{gph}(S_n)\) for \(n \in N\) such that \((x_n, u_n) \to (x, u)\) as \(n \to \infty\) in \(N\). Extract a graphically converging subsequence \(M \subset N\) and let \(S\) denote the graphical limit of \(S_n\) as \(n \to \infty\) in \(M\). By the construction of \((x_n, u_n)\), we have \((x, u) \in \liminf_{n \in M} \text{gph}(S_n) = \text{gph}(S)\), whence \(u \in S(x) = C\) by assumption. We conclude that \(\overline{S}(x) \subset C\).

Second, let \(u \notin \underline{S}(x)\), i.e., \((x, u) \notin \text{gph}(\underline{S}) = \liminf_n \text{gph}(S_n)\). Then there exists a neighbourhood \(V\) of \((x, u)\) in \(\mathbb{R}^k \times \mathbb{R}^\ell\) and a subsequence \(N\) such that \(\text{gph}(S_n) \cap V = \emptyset\) for all \(n \in N\). Extract a graphically converging subsequence \(M \subset N\) and let \(S\) denote the graphical limit of \(S_n\) as \(n \to \infty\) in \(M\). Then there cannot exist \((x_n, u_n) \in \text{gph}(S_n)\), \(n \in M\), such that \((x_n, u_n) \to (x, u)\) as \(n \to \infty\) in \(M\), given that \(\text{gph}(S_n) \cap V = \emptyset\) for \(n \in N\) and \(V\) is a neighbourhood of \((x, u)\). Hence \((x, u) \notin \text{gph}(S)\) and thus \(u \notin \underline{S}(x) = C\) by assumption. We conclude that \((\overline{S}(x))^c \subset C^c\), whence \(C \subset \overline{S}(x)\) \(\Box\)
Proof of Proposition B.5. (a) Let $x \in V$. We will show that there exists $n_x \in \mathbb{N}$ and an open neighbourhood $U_x$ of $x$ in $V$ such that $U_x \subset \text{dom} T_n$ for all integer $n \geq n_x$. For compact $K \subset V$, apply the reasoning to each $x \in K$ separately and extract a finite cover of $K$ from the collection of open sets $U_x$; the union of the selected sets and the maximum of the corresponding integers meets the requirements.

To prove the statement, we make use of the simplex technique in [28, Exercise 2.28]. Find $d + 1$ affinely independent points $v_0, v_1, \ldots, v_d$ in $V$ such that the simplex $\Delta = \text{conv}\{v_0, v_1, \ldots, v_d\}$ is a neighbourhood of $x$ in $V$. For $j = 0, 1, \ldots, d$, let $w_j \in T(v_j)$. By graphical convergence of $T_n$ to $T$ at $v_j$, we can find sequences $(v_{j,n}, w_{j,n}) \in gph T_n$ such that $(v_{j,n}, w_{j,n}) \to (v_j, w_j)$ as $n \to \infty$, for all $j$. In particular, $v_{j,n}$ is an element of $\text{dom} T_n$ for all $j$ and all sufficiently large $n$. Moreover, we can find an open neighbourhood $U_x$ of $x$ that is contained in the (interior of the) simplex $\Delta_n = \text{conv}\{v_{0,n}, \ldots, v_{d,n}\}$, for all sufficiently large $n$. As $T_n$ is maximal monotone, its domain is nearly convex [28, Theorem 12.41], i.e., there exists a convex set $C_n$ such that $C_n \subset \text{dom} T_n \subset \text{cl} C_n$. But then $\Delta_n \subset \text{conv}(\text{dom} T_n) \subset \text{cl} C_n$ and thus, by convexity, $U_x \subset \text{int} \Delta_n \subset \text{int} C_n \subset \text{int}(\text{dom} T_n)$ for all sufficiently large $n$.

(b) Let $\overline{\text{gph}}$ and $\overline{T}$ denote the graphical inner and outer limits, respectively, of the sequence $T_n$: by definition, their graphs are

$$gph \overline{\text{gph}} = \liminf_{n \to \infty} gph T_n \subset \limsup_{n \to \infty} gph T_n = \text{gph} \overline{T}.$$  \hfill (43)

By assumption, we have $\overline{\text{gph}}(x) = \overline{T}(x) = T(x)$ for all $x \in V$.

Let $\eta > 0$ be small enough such that $K + \eta \mathbb{B} \subset V$. Since $K + \eta \mathbb{B}$ is compact and since a maximal monotone mapping is locally bounded on the interior of its domain [28, Corollary 12.38], there exists $\rho > 0$ such that $K + \eta \mathbb{B}$ and $T(K + \eta \mathbb{B}) + \eta \mathbb{B}$ are both contained in $\rho \mathbb{B}$. Further, by part (a), there exists $n_K \in \mathbb{N}$ such that $K + \eta \mathbb{B}$ is contained in the interior of $\text{dom} T_n$ for all $n \geq n_K$, and therefore, $T_n(K + \eta \mathbb{B})$ is bounded for all such $n$ too and thus contained in some compact set $B_n$.

Let $\varepsilon \in (0, \eta/2]$ and write $\sigma := \rho + \eta$. From [28, Theorem 4.10], the inner and outer limits in (43) guarantee the existence of $n_\varepsilon \geq n_K$ such that for all $n \geq n_\varepsilon$,

$$gph \overline{\text{gph}} \cap (\sigma \mathbb{B} \times \sigma \mathbb{B}) \subset gph T_n + (\varepsilon \mathbb{B} \times \varepsilon \mathbb{B}),$$  \hfill (44)

and

$$gph T_n \cap (\sigma \mathbb{B} \times \sigma \mathbb{B}) \subset gph \overline{T} + (\varepsilon \mathbb{B} \times \varepsilon \mathbb{B}).$$  \hfill (45)

Let $A \subset K + \varepsilon \mathbb{B}$ and let $x \in A$ and $y \in T(x)$. Then $(x, y) \in gph \overline{\text{gph}} \cap (\sigma \mathbb{B} \times \sigma \mathbb{B})$. For $n \geq n_\varepsilon$ we can therefore invoke (44) to find $(x_n, y_n) \in gph T_n$ such that $|x - x_n| \leq \varepsilon$ and $|y - y_n| \leq \varepsilon$. But then $x_n \in x + \varepsilon \mathbb{B} \subset A + \varepsilon \mathbb{B}$ and thus $y \in y_n + \varepsilon \mathbb{B} \subset T_n(x_n) + \varepsilon \mathbb{B} \subset T_n(A + \varepsilon \mathbb{B}) + \varepsilon \mathbb{B}$. We find that, for $n \geq n_\varepsilon$ and all $A \subset K + \varepsilon \mathbb{B}$, we have

$$T(A) \subset T_n(A + \varepsilon \mathbb{B}) + \varepsilon \mathbb{B}.$$  \hfill (46)

A similar argument based on (45) yields that, for $n \geq n_\varepsilon$ and $A \subset K + \varepsilon \mathbb{B}$, we have

$$T_n(A) \cap \sigma \mathbb{B} \subset T(A + \varepsilon \mathbb{B}) + \varepsilon \mathbb{B}.$$  \hfill (47)

If we can moreover show that $T_n(K) \subset \sigma \mathbb{B}$ for all sufficiently large $n \geq n_\varepsilon$, then in the above inclusion and for $A \subset K$, we can omit the intersection with $\sigma \mathbb{B}$. 

In (47) with $A = K + \varepsilon B$ we have $A + \varepsilon B = K + 2\varepsilon B$ and thus, by the choice of $\rho$,
\[
T_n(K + \varepsilon B) \cap \sigma B \subset T(K + \eta B) + \eta B
\subset \rho B.
\]

As a consequence, $T_n(K + \varepsilon B)$ does not meet $\{y \in \mathbb{R}^d : \rho < |y| \leq \sigma\}$. For every $x \in \text{dom} T_n$, the set $T_n(x)$ is convex [28, Exercise 12.8(c)]. It follows that for $x \in K + \varepsilon B$, the set $T_n(x)$ is either contained in $\rho B$ or in $\mathbb{R}^d \setminus \sigma B$. Moreover, we had already found a compact $B_n$ containing $T_n(K + \eta B)$. It follows that we can partition $K + \varepsilon B$ into
\[
L_{n,1} := \{x \in K + \varepsilon B : T_n(x) \subset \rho B\},
\]
\[
L_{n,2} := \{x \in K + \varepsilon B : T_n(x) \subset \text{cl}(B_n \setminus \sigma B)\}.
\]

Both sets are compact, for they are the projections onto the first coordinate of the compact sets $\text{gph} T_n \cap [(K + \varepsilon B) \times \rho B]$ and $\text{gph} T_n \cap [(K + \varepsilon B) \times \text{cl}(B_n \setminus \sigma B)]$, respectively.

We will show that $K \cap L_{n,2}$ is empty. If $x \in K$ belongs to $L_{n,2}$, then $x + \varepsilon B$ is contained in $K + \varepsilon B$ and can therefore be partitioned into its intersections with $L_{n,1}$ and $L_{n,2}$. As both intersections are closed, they cannot both be non-empty, for the ball $x + \varepsilon B$ is connected. But $x \in L_{n,2}$ by assumption, and thus $x + \varepsilon B \subset L_{n,2}$. By definition of $L_{n,2}$ this means that $T_n(x + \varepsilon B) \subset \text{cl}(B_n \setminus \sigma B)$. However, by (46) we also have $T(x) \subset T_n(x + \varepsilon B) + \varepsilon B$. This is leads to a contradiction, as the already knew that $T(x)$ is non-empty and is contained in $\rho B$, which is disjoint from $\text{cl}(B_n \setminus \sigma B) + \varepsilon B$.

As $K \cap L_{n,2} = \emptyset$ and $K \subset K + \varepsilon B = L_{n,1} \cup L_{n,2}$, we find $K \subset L_{n,1}$ and thus, by definition of $L_{n,1}$, also $T_n(K) \subset \rho B$. For $A \subset K$, we can therefore omit the intersection with $\sigma B$ on the left-hand side of (47). Together with (46), this completes the proof of (b).

(c) Since $K$ is a subset of the interior of $\text{dom} T$ and, for sufficiently large $n$, of $\text{dom} T_n$ [see (a)], the sets $T(K)$ and $T_n(K)$ are compact: they are the projections onto the second coordinate of the sets $\text{gph} T \cap (K \times \mathbb{R}^d)$ and $\text{gph} T_n \cap (K \times \mathbb{R}^d)$, which are closed and, as shown in (b), bounded, and thus compact.

Let $\eta > 0$. By Lemma B.3, there exists $\varepsilon \in (0, \eta/4)$ satisfying
\[
T(K + 2\varepsilon B) \subset T(K) + (\eta/2)B.
\]
Applying part (b) to the set $K + \varepsilon B$, we obtain that for all sufficiently large $n$,
\[
T(K) \subset T_n(K + \varepsilon B) + \varepsilon B
\subset T(K + 2\varepsilon B) + 2\varepsilon B
\subset T(K) + \eta B.
\]
Ultimately, the Hausdorff distance between $T_n(K + \varepsilon B)$ and $T(K)$ must be bounded by $\eta$. This implies (42).

(d) Let $\varepsilon_0 > 0$ and $n_0 \in \mathbb{N}$ be such that $x + \varepsilon_0 B$ is contained in both $V$ and $\text{int}(\text{dom} T_n)$ for all $n \geq n_0$; see part (a).

For $\eta > 0$, there exists $\varepsilon \in (0, \varepsilon_0]$ and integer $n_1 \geq n_0$ such that
\[
\forall n \geq n_1, \quad d_H(T_n(x + \varepsilon B), T(x)) \leq \eta.
\]
Let $x_n \to x$ as $n \to \infty$ and write $T(x) = \{y\}$. There exists $n_2 \geq n_1$ such that $x_n \in x + \varepsilon B \subset \text{int}(\text{dom} T_n)$ for $n \geq n_2$. We get
\[
\forall n \geq n_2, \quad T_n(x_n) \subset T_n(x + \varepsilon B) \subset T(x) + \eta B.
\]
For \( y_n \in T_n(x_n) \), the latter inclusion implies \( |y_n - y| \leq \eta \) and thus
\[
\forall n \geq n_2, \quad T(x) = \{ y \} \subset y_n + \eta B \subset T_n(x_n) + \eta B.
\]
As a consequence, \( d_H(T_n(x_n), T(x)) \leq \eta \) for sufficiently large \( n \). As \( \eta > 0 \) was arbitrary, we find that \( d_H(T_n(x_n), T(x)) \to 0 \) as \( n \to \infty \). \( \square \)

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