A2-Planar Algebras II:
Planar Modules

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Abstract

Generalizing Jones’s notion of a planar algebra, we have previously introduced an A2-planar algebra capturing the structure contained in the SU(3) ADE subfactors. We now introduce the notion of modules over an A2-planar algebra, and describe certain irreducible Hilbert A2-TL-modules. A partial decomposition of the A2-planar algebras for the ADE nimrep graphs associated to SU(3) modular invariants is achieved.

1 Introduction

We introduced in [7] the notion of an A2-planar algebra. This was useful to understand the double complexes of finite dimensional algebras which arise in the context of SU(3) subfactors and modular invariants. Here we begin a study of their planar modules.

These planar algebras are direct generalization of the planar algebras of Jones [15]. To avoid too much confusion one could refer to these planar algebras of Jones here as A1-planar algebras which naturally contain the Temperley-Lieb algebra which encodes the representation theory of quantum SU(2).

Our A2-planar algebras naturally encode the representation theory of quantum SU(3) or in the dual Hecke picture, the finite dimensional algebras which appear from the representations of the deformation of the symmetric group with generators the self-adjoint operators 1, U1, U2, . . . , Un−1 and relations:

H1: \[ U_i^2 = \delta U_i, \]
H2: \[ U_iU_j = U_jU_i, \quad |i - j| > 1, \]
H3: \[ U_iU_{i+1}U_i - U_i = U_{i+1}U_iU_{i+1} - U_{i+1}, \]
where $\delta = q + q^{-1}$, $q \in \mathbb{C}$, and the extra relation
\[
(U_i - U_{i+2}U_{i+1}U_i + U_{i+1}) (U_{i+1}U_{i+2}U_{i+1} - U_{i+1}) = 0.
\]

(1)

In [5] we computed the numerical values of the Ocneanu cells, announced by Ocneanu (e.g. [23, 24]), and consequently representations of the Hecke algebra, for the $SU(3)$ ADE graphs. These cells give numerical weight to Kuperberg’s [20] diagram of trivalent vertices – corresponding to the trivial representation is contained in the triple product of the fundamental representation of $SU(3)$ through the determinant. They will yield in a natural way, representations of an $A_2$-Temperley-Lieb or Hecke algebra. For $SU(2)$ or bipartite graphs, the corresponding weights (associated to the diagrams of cups or caps), arise in a more straightforward fashion from a Perron-Frobenius eigenvector, giving a natural representation of the Temperley-Lieb algebra or Hecke algebra.

The bipartite theory of the $SU(2)$ setting has to some degree become a three-colourable theory in our $SU(3)$ setting. This theory is not completely three-colourable since some of the graphs are not three-colourable- namely the graphs $A^{(n)}, n \geq 4$, and $D^{(n)}, n \not\equiv 0 \mod 3$. The figures for the complete list of the ADE graphs are given in [1, 5].

2 Preliminaries on Jones’ planar algebras and planar modules

Let us briefly review the basic construction of Jones’ $A_1$-planar algebras, and the notion of planar modules over these algebras. A planar $k$-tangle consists of a disc $D$ in the plane with $2k$ vertices on its boundary, $k \geq 0$, and $n \geq 0$ internal discs $D_j$, $j = 1, \ldots, n$, where the disc $D_j$ has $2k_j$ vertices on its boundary, $k_j \geq 0$. One vertex on the boundary of each disc (including the outer disc $D$) is chosen as a marked vertex, and the segment of the boundary of each disc between the marked vertex and the vertex immediately adjacent to it as we move around the boundary in an anti-clockwise direction is labelled $\ast$. Inside $D$ we have a collection of disjoint smooth curves, called strings, where any string is either a closed loop, or else has as its endpoints the vertices on the discs, and such that every vertex is the endpoint of exactly one string. Any tangle must also allow a checkerboard colouring of the regions inside $D$.

The planar operad is the collection of all diffeomorphism classes of such planar tangles, with composition of planar tangles defined. A planar algebra $P$ is then defined to be an algebra over this operad, i.e. a family $P = (P^+_0, P^-_0, P_k, k > 0)$ of vector spaces with $P^+_0 \subset P_k \subset P^-_k$, for $0 < k < k'$, and such that for every $k$-tangle $T$ with $n$ internal discs $D_j$ labelled by elements $x_j \in P_k$, $j = 1, \ldots, n$, there is an associated linear map $Z(T) : \otimes_{j=1}^n P_k \to P_k$, which is compatible with the composition of tangles and re-ordering of internal discs.

A planar module over $P$ is a graded vector space $V = (V^+_0, V^-_0, V_k, k > 0)$ with an action of $P$. Given a planar $m$-tangle $T$ in $P$ with distinguished (“$V$ input”) internal disc $D_1$ with with $2k$ vertices on its boundary, $k \geq 0$, and other (“$P$ input”) internal discs $D_p$, $p = 2, \ldots, n$, with $2k_p$ vertices on its boundary, $k_p \geq 0$, there is a linear map $Z(T) : V_k \otimes (\otimes_{p=2}^n P_k) \to V_m$, where $Z(T)$ satisfies the same compatibility conditions as for $P$ itself.
2.1 $P^G$ as a TL-module for an ADE Dynkin diagram $\mathcal{G}$

In the case of $SU(2)$, Jones [14] determined all Hilbert modules $H^{k,\omega}$ of lowest weight $k > 0$, $k \in \mathbb{N}$, and $H^\mu$ of lowest weight 0. We will give a brief overview of these modules. For $k, m \in \mathbb{N}$, let $ATL_{m,k}$ denote the space of all annular $(m,k)$-tangles (having $2m$ vertices on the outer disc and $2k$ vertices on the (distinguished) inner disc, where the vertices have alternating orientations) with no other internal discs, where composition of tangles is defined by inserting one annular $(m,k)$-tangle inside the internal disc of an annular $(k,n)$-tangle. For $1 \leq k \leq m$, $k, m \in \mathbb{N}$, let $T_m^k$ denote the set of annular $(m,k)$-tangles with no internal discs and $2k$ through strings. If $\widetilde{ATL}_{m,k}$ denotes the quotient of $ATL_{m,k}$ by the ideal generated by all annular $(m,k)$-tangles with no internal discs and strictly less than $2k$ through strings, then the equivalence classes of the elements of $T_m^k$ form a basis for $\widetilde{ATL}_{m,k}$. The group $\mathbb{Z}_k$ acts by an internal rotation, which permutes the basis elements. The action of $ATL$ on $\widetilde{ATL}_{m,k}$ is given as follows. Let $T$ be an annular $(p,m)$-tangle in $ATL_{p,m}$ and $R \in T_m^k$. Define $T(R)$ to be $\delta^r TR$ if the $(p,k)$-tangle $TR$ has $2k$ through strings and 0 otherwise, where $TR$ contains $r$ contractible circles and $\tilde{T}R$ is the tangle $TR$ with all the contractible circles removed. Since the action of $ATL$ commutes with the action of $\mathbb{Z}_k$, as a TL-module $\widetilde{ATL}_{m,k}$ splits as a direct sum, over the $k^{th}$ roots of unity $\omega$, of TL-modules $V^{k,\omega}$ which are the eigenspaces for the action of $\mathbb{Z}_k$ with eigenvalue $\omega$. For each $k$ one can choose a faithful trace $tr$ on the abelian $C^*$-algebra $\widetilde{ATL}_{k,k}$, which extends to $ATL_{k,k}$ by composition with the quotient map. The inner-product on $\widetilde{ATL}_{m,k}$ is then defined to be $\langle S,T \rangle = tr(T^*S)$ for $S,T \in \widetilde{ATL}_{m,k}$.

We now turn to the zero-weight case ($k = 0$). The algebras $ATL_{\pm}$, which have the regions adjacent to both inner and outer boundaries shaded $\pm$, are generated by elements $\sigma_\pm \sigma_\mp$, where $\sigma_\pm$ is the $(\pm, \mp)$-tangle which is just a single non-contractible circle, with the region which meets the outer boundary shaded $\pm$ and the region which meets the inner boundary shaded $\mp$. Then the dimensions on $V_+ \pm V_-$ must be 1 or 0 for any TL-module $V$. Then in $V$, the maps $\sigma_\pm \sigma_\mp$ must contribute a scalar factor $\mu^2$, where $0 \leq \mu \leq \delta$. If $\mu = \delta$, $V^\delta$ is simply the ordinary Temperley-Lieb algebra. When $0 \leq \mu < \delta$, $V^\mu$ is the TL-module such that $V^{\mu}_m$, $m \geq 0$, has as basis the set of $(m, \pm)$-tangles with no internal discs and at most one non-contractible circle. The action of $ATL$ on $V^\mu$, $0 \leq \mu \leq \delta$, is given as follows. Let $T$ be an annular $(p,m)$-tangle in $ATL_{p,m}$ and $R$ be a basis element of $V^\mu$. Define $T(R)$ to be $\delta^r \mu^{2d} \tilde{T}R$, where $TR$ contains $r$ contractible circles and $2d + i$ non-contractible circles, where $i \in \{0,1\}$, and $\tilde{T}R$ is the tangle $TR$ with all the contractible circles removed and $2d$ of the non-contractible circles removed. The inner product on $V^\mu$ is defined by $\langle S,T \rangle = \delta^r \mu^{2d}$, where $T^*S$ contains $r$ contractible circles and $2d$ non-contractible circles. When $\mu = 0$, we have TL-modules $V^{0,+}$ and $V^{0,-}$, where $V^{0,\pm}_m$ has as basis the set of $(m, \pm)$-tangles with no internal discs and no contractible circles. The action of $ATL$ on $V^{0,\pm}$ is given as follows. Let $T$ be an annular $(p,m)$-tangle in $ATL_{p,m}$ and $R$ be a basis element of $V^{0,\pm}$. Define $T(R)$ to be $\delta^r \tilde{T}R$, where $TR$ contains $r$ contractible circles. Now $\tilde{T}R$ is zero if $TR$ contains any non-contractible circles, and is the tangle $TR$ with all the contractible circles removed otherwise. The inner product on $V^{0,\pm}$ is defined by $\langle S,T \rangle = 0$ if $T^*S$ contains any non-contractible circles, and $\langle S,T \rangle = \delta^r$ otherwise, where $r$ is the number of contractible circles in $T^*S$.

In the generic case, $\delta > 2$, it was shown that the inner-product is always positive.
definite, so that \( H = V \) is a Hilbert TL-module, for the irreducible lowest weight \( k \) TL-module \( V \). In the non-generic case, if the inner product is positive semi-definite, \( H \) is defined to be the quotient of \( V \) by the vectors of zero-length with respect to the inner product.

Let \( G \) be a bipartite graph. Then the vertex set of \( G \) is given by \( \mathcal{V} = \mathcal{V}_+ \cup \mathcal{V}_- \), where there are no connecting a vertex in \( \mathcal{V}_+ \) to another, and similarly for \( \mathcal{V}_- \). We call the vertices in \( \mathcal{V}_+, \mathcal{V}_- \) the even respectively vertices of \( G \), and the distinguished vertex * of \( G \), which has the highest Perron-Frobenius weight, is an even vertex. The adjacency matrix of \( G \) can thus be written in the form \( \begin{pmatrix} 0 & A_G \\ A_G^T & 0 \end{pmatrix} \). We let \( r_\pm = |\mathcal{V}_\pm| \). The planar algebra \( P^G \) of a bipartite graph \( G \) was constructed in [13], which is the path algebra on \( G \) where paths may start at any of the even vertices of \( G \), and where the \( m \)th graded part \( P^G_m \) is given by all pairs of paths of length \( m \) on \( G \) which start at the same even vertex and have the same end vertex. Let \( \mu_j, j = 1, \ldots, r_+ \), denote the eigenvalues of \( A_G A_G^T \).

Then the following result is given in [26, Prop. 13], which motivated Proposition 5.4: The irreducible weight-zero submodules of \( P^G \) are \( H^{\mu_j}, j = 1, \ldots, r_- \), and \( r_+ - r_- \) copies of \( H^0 \), and these can be assumed to be mutually orthogonal.

Reznikoff [26] computed the decomposition of \( P^G \) as a TL-module into irreducible TL-modules for the ADE Dynkin diagrams. For the graphs \( A_m \), \( m \geq 3 \),

\[
P^{A_m} = \bigoplus_{j=1}^{s} H^{\mu_j},
\]

where \( s = \lfloor (m + 1)/2 \rfloor \) is the number of even vertices of \( A_m \) and \( \mu_j = 2 \cos(j\pi/(m + 1)), j = 1, \ldots, s \). For \( D_m, m \geq 3 \),

\[
P^{D_m} = \bigoplus_{j=1}^{t} H^{\mu_j} \oplus (s - t) H^0 \oplus \bigoplus_{j=1}^{s-2} H^{2j-1},
\]

where \( s = \lfloor (m + 2)/2 \rfloor \), \( t = \lfloor (m - 1)/2 \rfloor \) are the number of even, odd vertices respectively of \( D_m \), and \( \mu_j = 2 \cos((2j - 1)\pi/(2m - 2)), j = 1, \ldots, t \). For the exceptional graphs the results are

\[
P^{E_6} = H^{\mu_1} \oplus H^{\mu_4} \oplus H^{\mu_5} \oplus H^{2,-1} \oplus H^{3,\omega} \oplus H^{3,\omega^{-1}},
\]

\[
P^{E_7} = H^0 \oplus H^{\mu_1} \oplus H^{\mu_5} \oplus H^{\mu_7} \oplus H^{2,-1} \oplus H^{3,\omega} \oplus H^{3,\omega^{-1}} \oplus H^{4,-1} \oplus H^{8,-1},
\]

\[
P^{E_8} = H^{\mu_1} \oplus H^{\mu_7} \oplus H^{\mu_11} \oplus H^{\mu_{13}} \oplus H^{2,-1} \oplus H^{3,\omega} \oplus H^{3,\omega^{-1}} \oplus H^{4,-1} \\
\oplus H^{5,\zeta} \oplus H^{5,\zeta^{-1}} \oplus H^{5,\zeta^2} \oplus H^{5,\zeta^{-2}},
\]

where \( \omega = e^{2\pi i/3}, \zeta = e^{2\pi i/5} \), and \( \mu_j = 2 \cos(\pi j/h) \) where \( h \) is the Coxeter number.

### 3 A₂-Planar Algebras

We will now review the basic construction of our \( A_2 \)-planar algebras. An \( A_2 \)-planar \( i,j \)-tangle consists of a disc \( D = D_0 \) in the plane together with a finite (possibly empty) set of disjoint sub-discs \( D_1, D_2, \ldots, D_n \) in the interior of \( D \). Each disc \( D_k, k \geq 0 \), will have
an even number $2(i_k + j_k) \geq 0$ of vertices on its boundary $\partial D_k$ ($i_0 = i, j_0 = j$). A vertex will be called a source vertex if the string attached to it has orientation away from the vertex and a sink vertex if the string attached has orientation towards the vertex. The first $j_k$ vertices are restricted to be sources, the next $2i_k$ vertices alternate between sources and sinks (with vertex $j_k + 1$ a source), and finally the last $j_k$ vertices are all sinks. We will use the convention of numbering the vertices along the bottom edge (the last $i_k + j_k$ vertices) by $1, \ldots, i_k + j_k$ in reverse order, so that the $2(i_k + j_k)$-th vertex is called the first vertex along the bottom edge. We say that $D_k$ has pattern $i_k, j_k$. Inside $D$ we have a tangle where the endpoint of any string is either a trivalent vertex (see Figure 1) or one of the vertices on the boundary of a disc $D_k$ for $k = 0, \ldots, n$, or else the string forms a closed loop. Each vertex on the boundaries of the $D_k$ is the endpoint of exactly one string, which meets $\partial D_k$ transversally.

The regions inside $D$ have as boundaries segments of the $\partial D_k$ or the strings. These regions are labelled $\overline{0}$, $\overline{1}$ or $\overline{2}$, called the colouring, such that if we pass from a region $R$ of colour $\overline{a}$ to an adjacent region $R'$ by passing to the right over a vertical string with downwards orientation, then $R'$ has colour $\overline{a + 1}$ (mod 3). The segment of each $\partial D_k$ between the last and first vertices is marked with $\ast_{b_k}$, $b_k \in \{0, 1, 2\}$, so that the region inside $D$ which meets $\partial D_k$ at this segment is of colour $\overline{b_k}$, and the choice of these $\ast_{b_k}$ must give a consistent colouring of the regions. For the outer boundary $\partial D$ we impose the restriction $b_0 = 0$. For $i, j = 0, 0$ we have three types of tangle, depending on the colour $\overline{b}$ of the region near $\partial D$.

An $A_2$-planar $i, j$-tangle $T$ with an internal disc $D_l$ with $i_l, j_l = i', j'$ vertices on its boundary can be composed with another $A_2$-planar $i', j'$-tangle $S$, which has external disc $D'$ such that the orientations of the vertices on its boundary are consistent with those of $D_l$, giving a new $i, j$-tangle $T \circ_l S$, by inserting the $A_2$-tangle $S$ inside the inner disc $D_l$ of $T$ such that the vertices on the outer disc of $S$ coincide with those on the disc $D_l$ and the regions marked by $\ast$ also coincide. The boundary of the disc $D_l$ is then removed, and the strings are smoothed if necessary. We let $\tilde{P}$ be the collection of all diffeomorphism classes of such $A_2$-planar tangles, with composition defined as above. The $A_2$-planar operad $P$ is the quotient of $\tilde{P}$ by the Kuperberg relations K1-K3 below, which are relations on a local part of the diagram:

\begin{align*}
\text{K1:} & \quad \begin{array}{c}
\phi \\
\downarrow \\
\alpha
\end{array} = \alpha \\
\text{K2:} & \quad \begin{array}{c}
\phi \\
\downarrow \\
\delta
\end{array} = \delta
\end{align*}
An $A_2$-planar algebra is then defined to be an algebra over this operad, i.e. a family
\[ P = \{ P^a_{i,j}, a \in \{0, 1, 2\}, P_{i,j}, i, j > 0, i, j \neq 0, 0 \} \]
of vector spaces with $P^a_{i,j} \subset P_{i,j} \subset P_{i',j'}$ for $0 < i \leq i'$, $0 < j \leq j'$, $a \in \{0, 1, 2\}$, and with the following property: for every labelled $i,j$-tangle $T \in \mathcal{P}_{i,j}$ with internal discs $D_1, D_2, \ldots, D_n$, where $D_k$ has pattern $i_k, j_k$, there is associated a linear map $Z(T) : \bigotimes_{k=1}^n P_{i_k,j_k} \rightarrow P_{i,j}$ which is compatible with the composition of tangles in the following way. If $S$ is an $i_k,j_k$-tangle with internal discs $D_{n+1}, \ldots, D_{n+m}$, where $D_k$ has pattern $i_k, j_k$, then the composite tangle $T \circ_i S$ is an $i,j$-tangle with $n + m - 1$ internal discs $D_k$, $k = 1, 2, \ldots, l-1, l+1, l+2, \ldots, n + m$. From the definition of an operad, associativity means that the following diagram commutes:
\[
\begin{array}{c}
\bigotimes_{k \neq l}^n P_{i_k,j_k} \otimes \bigotimes_{k=n+1}^{n+m} P_{i_k,j_k} \\
\downarrow \text{id} \otimes Z(S) \\
\bigotimes_{k=1}^n P_{i_k,j_k}
\end{array}
\bigotimes Z(T)
\]

so that $Z(T \circ_i S) = Z(T')$, where $T'$ is the tangle $T$ with $Z(S)$ used as the label for disc $D_l$. We also require $Z(T)$ to be independent of the ordering of the internal discs, that is, independent of the order in which we insert the labels into the discs. If $i = j = 0$, we adopt the convention that the empty tensor product is the complex numbers $\mathbb{C}$.

4 $A_2$-Planar Modules and $A_2$-ATL

We now extend Jones’s notion of planar algebra modules and the annular Temperley algebra to our $A_2$-planar algebras (cf. Section 2 of [14]).

**Definition 4.1** (cf. [14] Def. 2.1) An $A_2$-annular tangle $T$ will be a tangle in $\mathcal{P}$ with the choice of a distinguished internal disc, which we will call the inner disc. In particular, $T$ will be called an $A_2$-annular $(m_1, m_2 : k_1, k_2)$-tangle if it is an $A_2$-annular tangle with pattern $m_1, m_2$ on its outer disc and pattern $k_1, k_2$ on its inner disc. If $m_1 = m_2 = 0$ or $k_1 = k_2 = 0$, we replace the 0,0 with $\overline{a}$, $a \in \{0, 1, 2\}$, corresponding to the colour of the region which meets the outer or inner disc respectively. When $m_1 = k_1$ and $m_2 = k_2$ we will call $T$ an $A_2$-annular $m_1, m_2$-tangle.

Note, this annular tangle is different to those defined in [7]- here more than one internal disc is allowed, but one of those is chosen to be the distinguished disc (the inner boundary of the "annulus").
Figure 2: The m-tangle \( W_i, i = 1, \ldots, m - 1 \).

Definition 4.2 (cf. [14] Def. 2.2) If \( P \) is an \( A_2 \)-planar algebra, a \textit{module} over \( P \), or \textit{\( P \)-module}, will be a graded vector space \( V = (V_{i,j}, i, j \geq 0, i, j \neq 0, 0, V_{0,0}) \) with an action of \( P \). Given an \( A_2 \)-annular \((i,j : i',j')\)-tangle \( T \) in \( P \) with distinguished (“\( V \) input”) internal disc \( D_i \) with pattern \( i',j' \) and other (“\( P \) input”) internal discs \( D_p, p = 2, \ldots, n \), with patterns \( i_p,j_p \), there is a linear map \( Z(T) : V_{i',j'} \otimes (\otimes_{p=2}^{n} P_{i_p,j_p}) \rightarrow V_{i,j} \). The map \( Z(T) \) satisfies the same compatibility condition [7] for the composition of tangles as \( P \) itself.

An \( A_2 \)-planar algebra is always a module over itself- we will call it the \textit{trivial module}. Any relation (i.e. linear combination of labelled \( A_2 \)-planar tangles) that holds in \( P \) will hold in \( V \), e.g. K1-K3 hold in \( V \) where \( \alpha, \delta \) have the same values as in \( P \).

A module over an \( A_2 \)-planar algebra \( P \) can be understood as a module over the \( A_2 \)-annular algebra \( A_2-AP \), defined as follows. We define the associated annular category \( A_2-AnnP \) to have three objects \( \pi \) for \( i = j = 0, a \in \{0,1,2\} \), and one object for each \( i,j \geq 0 \) with \( i,j \) not both equal to zero, and whose morphisms are \( A_2 \)-annular labelled tangles with labelling set all of \( P \). Let \( A_2-FAP \) be the linearization of \( A_2-AnnP \)- it has the same objects, but the set of morphisms from object \( i,j \) to object \( i',j' \) is the vector space having as basis the morphisms in \( A_2-AnnP \) from \( i,j \) to \( i',j' \). Composition of morphisms is \( A_2-FAP \) is by linear extension of composition in \( A_2-AnnP \). The \textit{\( A_2 \)-annular algebra} \( A_2-AP = \{A_2-AP(i,j : i',j')\} \) is the quotient of \( A_2-FAP \) by relations K1-K3.

Definition 4.3 (cf. [14] Def. 2.6) We define \( A_2-AP_{i,j} \) to be the algebra \( A_2-AP(i,j : i,j) \) for \( i,j \geq 0 \) with \( i,j \) not both zero, and \( A_2-AP_{\pi}, a \in \{0,1,2\} \), to be the algebras spanned by \( A_2 \)-annular tangles with no vertices on the outer and inner boundaries, and with the regions which meet the boundaries coloured \( \pi \).

We apply this procedure to the \( A_2 \)-Temperley-Lieb algebra \( \mathcal{V}^{A_2} = \text{alg}(1, W_i, i \geq 1) \) for fixed \( \delta \in \mathbb{C} \), where \( W_i \) are the tangles illustrated in Figure 2. This algebra was shown in [7] to be isomorphic to the \( A_2 \) Temperley-Lieb algebra. The labels for the internal discs are now just \( A_2 \)-annular tangles. For \( m_1, m_2, n_1, n_2 \geq 0 \) let \( A_2-AnnTL(m_1, m_2 : n_1, n_2) \) be the set of all basis \( A_2 \)-annular \((m_1, m_2 : n_1, n_2)\)-tangles. Elements of \( A_2-AnnTL(m_1, m_2 : n_1, n_2) \) define elements of \( A_2-ATL(m_1, m_2 : n_1, n_2) \) by passing to the quotient of \( A_2-ATL \) by relations K1-K3. The objects of \( A_2-ATL \) are \( \emptyset, \mathbb{T} \) and \( \mathbb{Z} \) for \( m_1 = m_2 = 0 \). When \( m_1 \) and \( m_2 \) are not both equal to zero, the objects are the sets of \( 2(m_1 + m_2) \) points with pattern \( m_1, m_2 \). \( A_2-ATL_{m_1,m_2}(\delta) \) has as basis the set of \( A_2 \)-annular \( m_1,m_2 \)-tangles with no contractible circles, or embedded circles or squares. However, non-contractible circles are allowed, which make each algebra \( A_2-ATL_{m_1,m_2} \) infinite dimensional. Multiplication in \( A_2-ATL_{m_1,m_2}(\delta) \) is by composition of tangles, then reducing the resulting tangle using
relations K1-K3 to remove closed loops (K1), embedded circles (K2) or embedded squares (K3).

For all $m_1, m_2 \geq 0$ such that $m_1 + m_2 \geq 2$, the algebras $A_2$-$ATL_{m_1,m_2}$ are also infinite dimensional due to the possibility of an infinite number of embedded hexagons in basis tangles in the annular picture, as illustrated in Figure 3.

We have a notion of the rank of a tangle. A minimal cut loop $\gamma$ in an annular $(i,j : i',j')$-tangle $T$ will be a clockwise closed path which encloses the distinguished internal disc and crosses the least number of strings. We associate a weight $w_\gamma = (t_1, t_2)$ to a minimal cut loop $\gamma$, where $t_1$ is the number of strings of $T$ that cross $\gamma$ with orientation from left to right, and $t_2$ the number of strings that have orientation from right to left, as we move along $\gamma$ in a complete clockwise loop. For a weight $(t_1, t_2)$, let $t_{\text{max}} = \max\{t_1, t_2\}$ and $t_{\text{min}} = \min\{t_1, t_2\}$. We will say $(t_1', t_2')$ is less than $(t_1, t_2)$, and write $(t_1', t_2') < (t_1, t_2)$, if $t_1' + t_2' < t_1 + t_2$, and if $t_1' + t_2' = t_1 + t_2$ then $(t_1', t_2') < (t_1, t_2)$ if $2t_{\text{max}} + t_{\text{min}} < 2t_{\text{max}} + t_{\text{min}}$. The rank of $T$ is then given by the smallest weight $w_\gamma = (t_1, t_2)$ associated to a minimal cut loop, such that $(t_1, t_2) \leq w_{\gamma'}$ for all other minimal cut loops $\gamma'$.

Let $A_2$-$AnnTL(m_1, m_2 : m_1,m_2)_{(t_1,t_2)}$ denote the set of tangles in $A_2$-$AnnTL(m_1, m_2 : m_1, m_2)$ with rank $(t_1, t_2)$. Since the rank cannot increase under composition of tangles, the linear span of $A_2$-$AnnTL(m_1, m_2 : m_1, m_2)_{(t_1,t_2)}$ for all $(t_1, t_2) < (t_1', t_2')$ for any fixed $t_1', t_2'$ is an ideal in $A_2$-$ATL_{m_1,m_2}$.

**Lemma 4.4** (cf. [14, Lemma 2.10]) Let $P$ be an $A_2$-planar algebra and let $t_1, t_2$ satisfy $2t_{\text{max}} + t_{\text{min}} = 3m$. For any $t_1', t_2'$ such that $2t_{\text{max}}' + t_{\text{min}}' \leq 3m$, denote by $A_2$-$AP_{(t_1,t_2)}^{(t_1',t_2')}$ the linear span in the algebra $A_2$-$AP_{t_1,t_2}$ of all labelled $A_2$-annular $t_1, t_2$-tangles with rank $(s_1, s_2) < (t_1', t_2')$. Then $A_2$-$AP_{(t_1,t_2)}^{(t_1',t_2')}$ is a two-sided ideal.

**Remark:** For $A_2$-ATL the quotient of $A_2$-$ATL_{t_1,t_2}$ by the ideal $A_2$-$AP_{t_1,t_2}^{(t_1',t_2')}$ is not in general finite dimensional, for $2t_{\text{max}}' + t_{\text{min}}' \leq 2t_{\text{max}} + t_{\text{min}}$. For example, consider the quotient of $A_2$-$ATL_{t_1,t_2}$ by $A_2$-$AP_{t_1,t_2}^{(3k,0)}$ (or $A_2$-$AP_{t_1,t_2}^{(0,3k)}$), for $3 \leq 3k \leq 2t_{\text{max}} + t_{\text{min}}$. The elements $\varphi_{(3k,0)}$ and $\varphi_{(0,3k)}$ (see Figure 4) have ranks $(3k, 0)$ and $(0, 3k)$ respectively, and can be composed an infinite number of times, but the resulting tangle cannot be reduced using K1-K3.

**Lemma 4.5** (cf. [14, Lemma 2.11]) Let $V = (V_{i,j})$ be a $P$-module. Then $V$ is indecomposable if and only if $V_{i,j}$ is an indecomposable $A_2$-$AP_{t_i,t_j}$-module for each $i, j \geq 0$.  

Figure 3: A basis $A_2$-annular $(2,0 : 2,0)$-tangle containing hexagons, and the possibility of an infinite number of hexagons
Definition 4.6 (cf. [14, Def. 2.12]) The weight \( \text{wt}(V) \) of a \( P \)-module \( V \) is the smallest integer \( i + j \) for which \( V_{i,j} \) is non-zero. If \( V_{i,j} \) is non-zero for \( a \in \{0, 1, 2\} \) we say \( V \) has weight zero. Elements of \( V_{i',j'} \) for \( i' + j' = \text{wt}(V) \) will be called lowest weight vectors in \( V \), and \( V_{i',j'} \) is an \( A_2-AP_{i',j'} \)-module which we call a \textit{lowest weight module}.

Note that for \( i' + j' = \text{wt}(V) \), all \( V_{i+k,j-k}, -i' \leq k \leq j' \), are lowest weight modules for \( V \).

Definition 4.7 (cf. [14, Def. 2.13]) The Hilbert series (called the dimension in [14]) of a \( P \)-module \( V \) is the formal power series

\[
\Phi_V(z_1, z_2) = \frac{1}{3} \dim(V_0 \oplus V_1 \oplus V_2) + \sum_{i, j=0}^{\infty} \dim(V_{i,j}) z_1^i z_2^j,
\]

where \( i, j \) not both \( = 0 \).

4.1 Hilbert \( P \)-modules

If \( P \) is a \( C^*-A_2 \)-planar algebra, the \( * \)-algebra structure on \( P \) induces a \( * \)-structure on \( A_2-AP \), where the involution \( * \) is defined by reflecting an \( A_2 \)-annular \( (m_1, m_2 : k_1, k_2) \)-tangle \( T \) about a circle halfway between the inner and outer disc, and reversing the orientation. \( T^* \) will be an \( A_2 \)-annular \( (k_1, k_2 : m_1, m_2) \)-tangle. If \( P \) is a \( C^*-A_2 \)-planar algebra this defines an antilinear involution \( * \) on \( A_2-FAP \) by taking the \( * \) of the underlying unlabelled tangle for a labelled tangle \( T \), replacing the labels of \( T \) by their \( * \)'s, and extending by antilinearity. Since \( P \) is an \( A_2 \)-planar \( * \)-algebra, all the \( A_2 \)-planar relations are preserved under \( * \) on \( A_2-FAP \), so \( * \) passes to an antilinear involution on the algebra \( A_2-AP \). In particular, all the \( A_2-AP_{i,j} \) are \( * \)-algebras.

Definition 4.8 (cf. [14, Def. 3.1]) Let \( P \) be a \( C^*-A_2 \)-planar algebra. A \( P \)-module \( H \) will be called a \textit{Hilbert \( P \)-module} if each \( H_{i,j} \) is a finite dimensional Hilbert space with inner-product \( \langle \cdot, \cdot \rangle \) satisfying

\[
\langle av, w \rangle = \langle v, a^*w \rangle,
\]

for all \( v, w, a \in H \) and \( a \in A_2-AP \).

As in the \( SU(2) \) situation, a \( P \)-submodule of a Hilbert \( P \)-module is a Hilbert \( P \)-module. Also, the orthogonal complement of a \( P \)-submodule is a \( P \)-module, so that indecomposability and irreducibility are the same for Hilbert \( P \)-modules. The following Lemmas are \( SU(3) \) versions of Lemmas in [14], the proofs of which are very similar to those in the \( SU(2) \) situation.
Lemma 4.9 (cf. [14, Lemma 3.4]) Let $P$ be an $A_2$-C*-planar algebra and $H$ a Hilbert $P$-module. If $W \subseteq H_{i,j}$ is an irreducible $A_2$-$AP_{i,j}$-submodule of $H_{i,j}$ for some $i, j$, then $A_2$-$AP(W)$ is an irreducible $P$-submodule of $H$.

Lemma 4.10 (cf. [14, Lemma 3.5]) Let $P$ be an $A_2$-C*-planar algebra and $H$ a Hilbert $P$-module. Let $V$ and $W$ be orthogonal $A_2$-$AP_{i,j}$ invariant subspaces of $H_{i,j}$ for some $i, j$. Then $A_2$-$AP(V)$ is orthogonal to $A_2$-$AP(W)$.

As in the proof of [7, Lemma 4.16], there is a bijection $g_k : V_{i,j} \rightarrow V_{i+k,j-k}, -i \leq k \leq j$. Then $\dim(V_{i,j}) = \dim(V_{i+k,j-k})$, since if $v \neq 0$ in $V_{i,j}$ but $g_k(v) = 0$ in $V_{i+k,j-k}$ then $v = g_k^{-1}g_k(v) = g_k^{-1}(0) = 0$ which is a contradiction. The following Lemma shows that an irreducible Hilbert $P$-module $H$ is determined by its lowest weight modules, and in particular $H$ is determined by its lowest weight module $H_{0,\wt(H)}$, since for all other $i + j = \wt(H)$, $H_{i,j} = g_i(H_{0,\wt(H)})$.

Lemma 4.11 (cf. [14, Lemma 3.7]) Let $P$ be an $A_2$-C*-planar algebra and let $H^{(1)}$, $H^{(2)}$ be Hilbert $P$-modules with $H^{(1)}$ irreducible. Suppose there is a non-zero $A_2$-$AP_{i,j}$ homomorphism $\theta : H^{(1)}_{i,j} \rightarrow H^{(2)}_{i,j}$. Then $\theta$ extends to an injective homomorphism $\Theta$ of $P$-modules.

We will now determine which $A_2$-$AP_{i,j}$-modules can be lowest weight modules.

Lemma 4.12 Let $P$ be an $A_2$-C*-planar algebra and $H$ a Hilbert $P$-module of lowest weight $k$ and rank $(t_1, t_2)$. With $i + j = k$, any element $w \in H_{i,j}$ can be written, up to a scalar, as $aw$ for some $a \in A_2$-$AP_{i,j}$ with $\rank(a) = \rank(w)$.

Proof: First form $wv^* \in A_2$-$AP_{i,j}$. Then dividing out by the relations K1-K3 we obtain a linear combination of elements in $A_2$-$AP_{i,j}$, and we remove those elements that have rank $< (t_1, t_2)$. Ignoring the scalar factor we are left with a single element $a \in A_2$-$AP_{i,j}$ with $\rank(a) = (t_1, t_2)$. If we form $aw$, then dividing out by K1-K3 we obtain $aw = \mu w + \sum_i \mu_i w_i$, where $\mu, \mu_i \in \mathbb{C}$ and $w_i \in H_{i,j}$ with $\rank(w_i) < (t_1, t_2)$ for each $i$. Then in $H$ the $w_i$ are all zero, so that $\mu^{-1}aw = w$. □

Lemma 4.13 (cf. [14, Lemma 3.8]) Let $P$ be an $A_2$-C*-planar algebra and $H$ a Hilbert $P$-module. For $3(i+j-1) < 2t_{\max}^2 + t_{\min}^2 \leq 3(i+j)$, let $H^{(t_1,t_2)}_{i,j}$ be the $A_2$-$AP_{i,j}$-submodule of $H_{i,j}$ spanned by the $i,j$-graded pieces of all $P$-submodules with rank $< (t_1, t_2)$. Then

$$(H^{(t_1,t_2)}_{i,j})^\perp = \bigcap_{a \in A_2$-$AP^{(t_1,t_2)}_{i,j}} \ker(a).$$

Corollary 4.14 (cf. [14, Cor. 3.10]) The lowest weight modules of an irreducible $P$-module of rank $(t_1, t_2)$ are $A_2$-$AP_{i,j}/A_2$-$AP^{(t_1,t_2)}_{i,j}$-modules, where $2t_{\max} + t_{\min} = 3(i+j)$.

Then for an $A_2$-C*-planar algebra $P$, we can determine all Hilbert $P$-modules by first determining the algebras $A_2$-$AP_{0,j}/A_2$-$AP^{(t_1,t_2)}_{0,j}$ and their irreducible modules, for $2t_{\max} + t_{\min} = 3j$, and then determining which of these modules extend to $P$-modules.
4.2 Irreducible $A_2$-STL-modules

We can easily determine certain irreducible $A_2$-STL-modules. We will describe some zero-weight modules. However we have not determined all irreducible $A_2$-STL-modules, even for the zero-weight case, since it is not clear that elements of the from $\sigma^{(i)}_{(k_1,k_2)}$ defined below must necessarily contribute scalar factor, as $A_2$-STL$_\pi$ is not one-dimensional (and hence not isomorphic to $\mathbb{C}$).

**Proposition 4.15** (cf. [14, Prop. 5.9]) The algebra $A_2$-STL$_\pi$, for any $a \in \{0, 1, 2\}$, is generated by the 0-tangles $\sigma_{j,j\pm 1}$ illustrated in Figure 5, $j \in \{0, 1, 2\}$.

\[ \sigma_{j,j+1} = \] \hspace{1cm} \[ \sigma_{j,j-1} = \]

**Figure 5**: $\sigma_{j,j+1}$ and $\sigma_{j,j-1}$

Let $H$ be an irreducible Hilbert $A_2$-STL-module of lowest weight zero. For $i \in \{0, 1, 2\}$ and non-negative integers $k_1 \equiv k_2 \mod 3$, let

\[ \sigma^{(i)}_{(k_1,k_2)} = (\sigma_{i,i+1}(i+1,i+2) \cdots \sigma_{i+k_1-1,i+k_1})(\sigma_{i+k_1,i+k_1-1,i+k_1-2} \cdots \sigma_{i+k_1-k_2+1,i+k_1-k_2}). \]

If the maps $\sigma^{(i)}_{(k_1,k_2)}$ for any $k_1, k_2$ just give the complex number $\beta^{k_1} \overline{\beta}^{k_2}$ for some fixed $\beta \in \mathbb{C}$, i.e. $\sigma^{(i)}_{(k_1,k_2)} = \beta^{k_1} \overline{\beta}^{k_2} 1_\pi$, then the dimensions of $H_\pi$ are at most 5, for $a \in \{0, 1, 2\}$. To see this consider an arbitrary element given by a product of elements $\sigma_{j,j\pm 1}$. Whenever the product $\sigma_{l,l+1}\sigma_{l+1,l}$ appears, for some $l \in \{0, 1, 2\}$, we get a factor of $|\beta|^2$. Removing all such products we will be left with an element which contains only non-contractible circles with the same orientation. Any three consecutive such circles contribute a factor of $\beta^3$ or $\overline{\beta}^3$. Then up to some scalar, any element will have at most two non-contractible circles, with each circle having the same orientation.

**Proposition 4.16** (cf. [14, Theorem 5.12]) An irreducible Hilbert $A_2$-STL-module $H$ of weight zero in which the maps $\sigma^{(i)}_{(k_1,k_2)}$, $i \in \{0, 1, 2\}$, are given by the complex number $\beta^{k_1} \overline{\beta}^{k_2}$ for some fixed $\beta \in \mathbb{C}$ is determined up to isomorphism by the dimensions of $H_\pi$, $a \in \{0, 1, 2\}$, and the number $\beta$, where we require $|\beta| \leq \alpha$.

**Proof**: The uniqueness of the $A_2$-STL-module is a consequence of Lemma 4.11 since at least one of $H_\pi, H_\tau$ and $H_\omega$ is non-zero. Let $E_1, E_2$ be the tangles

\[ E_1 = \] \hspace{1cm} \[ E_2 = \]
so that $\alpha^{-1}E_1 \cdot \alpha^{-1}E_2$ are projections. Then since $E_1 E_2 E_1 = |\beta|^2 E_1$, we have $||\alpha E_1 \cdot \alpha^{-1}E_2|| = |\beta|^2 |\alpha^{-2}| |\alpha^{-1}E_1||$ so that $1 \geq |\beta|^2 |\alpha^{-2}$. Hence $|\beta| \leq \alpha$. 

For $\beta = \alpha$, $V^\alpha_{i,j} = STL_{i,j}$ (since when $\beta = \alpha$ there is no distinction between contractible and non-contractible circles). For $\alpha > 3$ (which corresponds to $\delta > 2$), the inner product is positive definite by [7, Lemma 3.10] and [27, Theorem 3.6], and $H^\alpha_{i,j} = V^\alpha_{i,j}$ is a Hilbert $A_2-STL$-module. For $0 < \alpha \leq 3$, if the inner product is positive semi-definite on $V^\alpha_{i,j}$ we let $H^\alpha_{i,j}$ be the quotient of $V^\alpha_{i,j}$ by the subspace of vectors of length zero; otherwise $H^\alpha_{i,j}$ does not exist.

Now consider the case when $0 < |\beta| < \alpha$. We define for each $i,j \geq 0$ (with 0 replaced by $\overline{0}$, $a \in \{0,1,2\}$, as usual), the set $TH_{i,j}$ to be the set of all $(i,j : 0)$-tangles with no contractible circles and at most two non-contractible circles. Now for each $\beta$ we form the graded vector space $V^\beta$, where $V^\beta_{i,j}$ has basis $TH_{i,j}$, and we equip it with an $A_2-STL$-module structure as follows. Let $T \in A_2-ATL(i',j' : i,j)$ and $R \in A_2-ATL_{i,j}$. We from the tangle $TR$ and reduce it using K1-K3, so that $TR = \sum_{j} \delta^b \alpha^c TR_j$, for some basis $A_2$-annular $(i',j' : 0)$-tangles $TR_j$, where $b, c$ are non-negative integers. Let $\sharp^a$, $\sharp_j$ denote the number of non-contractible circles in the tangle $TR_j$ which have anti-clockwise orientation removed. We define integers $d_j$, $f_j$ and $g_j$ as follows: $d_j = \min(\sharp^a, \sharp_j)$, $f_j = \sharp^a - \sharp_j - \gamma f_j$ if $\sharp^a \geq \sharp_j$ and $f_j = 0$ otherwise, and $g_j = \sharp^a - \sharp_j - \gamma g_j$ if $\sharp^a \leq \sharp_j$ and $g_j = 0$ otherwise, where $\gamma f_j, \gamma g_j \in \{0,1,2\}$ such that $f_j, g_j \equiv 0 \mod 3$.

Then we set $T(R) = \sum_j \delta^b \alpha^c \beta d_j + f_j \beta^{d_j + f_j} TR_j$, where $TR_j$ is the tangle $TR_j$ with $d_j + f_j$ anti-clockwise non-contractible circles removed, and $d_j + g_j$ clockwise ones removed.

**Proposition 4.17** The above definition make $V^\beta$ into an $A_2-STL$-module of weight zero in which $\sigma^{(a)}_{k_1,k_2} = \beta^{k_1} \beta^{k_2}$ for $a = 0,1,2$.

As with the $A_1$ situation, the choice of $(i,j : 0)$-tangles rather than $(i,j : 1)$- or $(i,j : 2)$-tangles to define $V^\beta$ was arbitrary. For these other two choices, the maps $T \rightarrow \beta^{-1}T \sigma_01$, $T \rightarrow \overline{\beta}^{-1}T \sigma_02$ respectively would define isomorphisms from those modules to the one defined above.

**Definition 4.18** (cf. [47, Def. 5.17]) Given $S, T \in TH_{i,j}$, we reduce $T^*S$ using K1-K3 so that $T^*S = \sum_j \delta^b \alpha^c (T^*S)_j$ for basis $(0 : 0)$-tangles $(T^*S)_j$. Define $d_j$, $f_j$ and $g_j$ for each $(T^*S)_j$ as above. We define an inner-product by $\langle S,T \rangle = \sum_j \delta^b \alpha^c \beta d_j + f_j \beta^{d_j + f_j}$.

Invariance of this inner-product follows from the fact that $T^*S = \langle S,T \rangle T_0$ where $T_0$ is the annular $(0 : 0)$-tangle with no strings at all. When the above inner-product is positive semi-definite, we define the Hilbert $A_2-STL$-module $H^\beta$ of weight zero to be the quotient of $V^\beta$ by the subspace of vectors of length zero. Otherwise $H^\beta$ does not exist.

**Proposition 4.19** For the above Hilbert $A_2-STL$-module $H^\beta$ of weight zero, the dimension of $H^\beta_{\overline{\alpha}}$ is either 0 or 1 for any $\beta \in \mathbb{C} \setminus \{0\}$.

**Proof:** For $a = 0$ the result is trivial since $V^\alpha_{\overline{\alpha}}$ is the linear span of the empty tangle $T_0$ given in Definition 4.18. For $a = 1$, $V^\beta_{\overline{1}} = \text{span}(\sigma_{10}, \sigma_{12} \sigma_{20})$. Let $w = |\beta|^2 \sigma_{12} \sigma_{20} - \beta^3 \sigma_{10}$. Then

$$\langle w,w \rangle = |\beta|^4 \langle \sigma_{12} \sigma_{20}, \sigma_{12} \sigma_{20} \rangle - |\beta|^2 \beta^3 \langle \sigma_{12} \sigma_{20}, \sigma_{10} \rangle - \beta^3 |\beta|^2 \langle \sigma_{10}, \sigma_{12} \sigma_{20} \rangle + |\beta|^6 \langle \sigma_{10}, \sigma_{10} \rangle$$

$$= |\beta|^4 (|\beta|^4) - |\beta|^2 \beta^3 \beta^3 - \beta^3 |\beta|^2 \beta^3 + |\beta|^6 |\beta|^2 = 0.$$
Then $\sigma_{10} = |\beta|^2 \beta^{-2} \sigma_{12} \sigma_{20} = \overline{\beta} \beta^{-2} \sigma_{12} \sigma_{20}$ in $H_1^0$. Similarly when $a = 2$, $\sigma_{21} \sigma_{10} = \overline{\beta} \beta^{-1} \sigma_{20}$.

So we may define $H^0$ so that it does not contain any clockwise non-contractible circles, where we replace every $\sigma_{10}$ by $\overline{\beta} \beta^{-2} \sigma_{12} \sigma_{20}$ and every $\sigma_{21} \sigma_{10}$ by $\overline{\beta} \beta^{-1} \sigma_{20}$.

**Proposition 4.20** (cf. [14, Cor. 5.8]) The Hilbert $A_2$-STL-module $H^0$, $|\beta| < \alpha$, is irreducible.

**Proof:** Since $H^0$ is at most one-dimensional it must be irreducible, for each $a \in \{0, 1, 2\}$. The maps $\sigma_{j,j+1}$ moves a non-zero element in $H^0_j$ to an element in $H^0_{j+1}$, and hence the lowest weight module $H_0^0 = H_0^0 \oplus H_1^0 \oplus H_2^0$ is irreducible as an $A_2$-ATL$_0$-module. Since $H^0 = A_2$-ATL($H_0^0$), the result follows from Lemma 4.9.

Now we consider the case when $\beta = 0$. For each $i, j \geq 0$ (with $0, 0$ replaced by $\pi$, $a \in \{0, 1, 2\}$, as usual), the set $Th^0_{i,j}$ is defined to be the set of all $(i, j : \overline{\alpha})$-tangles with no contractible or non-contractible circles at all. The cardinality of $Th^0_{i,j}$ is $\delta_{a,b}$. We form the graded vector space $V^{0,\pi}_i$, where $V^{0,\pi}_i$ has basis $Th_{i,j}^\pi$. We equip it with an $A_2$-STL-module structure of lowest weight zero as follows. Let $T \in A_2$-ATL($i, j' : i, j$) and $R \in Th^\pi_{i,j'}$. We form $TR$ and reduce it using K1-K3, so that $TR = \sum_j \delta^b \alpha^c$ as in the case $0 < |\beta| < \alpha$. We define $T(R)_j$ to be zero if there are any non-contractible circles in $TR_j$, and $TR_j$ otherwise. Then $T(R) = \sum_j \delta^b \alpha^c T(R)_j$.

**Proposition 4.21** The above definition make $V^{0,\pi}_i$ into an $A_2$-STL-module of weight zero in which $\sigma_{j,j+1} = 0$ for $j = 0, 1, 2$ mod 3.

**Definition 4.22** (cf. [14, Def. 5.22]) Given $S, T \in Th^\pi_{i,j}$, we reduce $T^*S$ using K1-K3 so that $T^*S = \sum_j \delta^b \alpha^c (T^*S)_j$ for basis $(\overline{\pi} : \overline{\alpha})$-tangles $(T^*S)_j$. We define $\langle S, T \rangle_j$ to be 0 if there are any non-contractible circles in $(T^*S)_j$, and 1 otherwise. Then we define an inner-product by $\langle S, T \rangle = \sum_j \delta^b \alpha^c \langle S, T \rangle_j$.

This inner-product is invariant as in the case $0 < |\beta| < \alpha$. Again, if the inner product is positive semi-definite we define $H^{0,\pi}$ to be the quotient of $V^{0,\pi}$ by the subspace of vectors with length zero; otherwise $H^{0,\pi}$ does not exist.

**Proposition 4.23** The Hilbert $A_2$-STL-module $H^{0,\pi}$, $a \in \{0, 1, 2\}$, is irreducible.

Proof is as for $H^0$.

5 The $A_2$-planar algebra of an $ADE$ graph

Let $G$ be any finite $SU(3)$ $ADE$ graph (not necessarily one for which there exists a flat connection) with vertex set $\mathfrak{V}_G = \mathfrak{V}_0^G \cup \mathfrak{V}_1^G \cup \mathfrak{V}_2^G$, where $\mathfrak{V}_a^G$ is the set of $a$-coloured vertices of $G$, $a = 0, 1, 2$. Let $n_a = |\mathfrak{V}_a^G|$ denote the number of $a$-coloured vertices and $n = |\mathfrak{V}^G| = n_0 + n_1 + n_2$ the total number of vertices of $G$. Note that $n_1 = n_2$ due to the conjugation property of the $SU(3)$ $ADE$ graphs. For the non-three-colourable graphs $n_1 = n_2 = 0$ and $n = n_0$. Let $\alpha = [3]^q$, $q = e^{i\pi/n}$, be the Perron-Frobenius eigenvalue of $G$ and let $\phi = (\phi_a)$ be the corresponding eigenvector.
We define a double sequence

\[
C_{0,0} \subset C_{0,1} \subset C_{0,2} \subset \cdots \\
\cap \quad \cap \quad \cap \\
C_{1,0} \subset C_{1,1} \subset C_{1,2} \subset \cdots \\
\cap \quad \cap \quad \cap \\
C_{2,0} \subset C_{2,1} \subset C_{2,2} \subset \cdots \\
\cap \quad \cap \quad \cap \\
\vdots \quad \vdots \quad \vdots 
\]

where \(C_{0,0} = \mathbb{C}^{m_0}\). The Bratteli diagrams for horizontal inclusions \(C_{i,j} \subset C_{i,j+1}\) are given by \(\mathcal{G}\). If \(\mathcal{G}\) is three-colourable, the vertical inclusions \(C_{i,j} \subset C_{i+1,j}\) are given by its \(j,j+1\)-part \(\mathcal{G}_{j,j+1}\), where \(\overline{p} = \tau(p)\) is the colour of \(p\) for \(p = j,j+1\), whilst if \(\mathcal{G}\) is not three-colourable we use the graph \(\mathcal{G}\) for all the vertical inclusions \(C_{i,j} \subset C_{i+1,j}\). Then for the inclusions

\[
C_{i,j} \subset C_{i,j+1} \\
\cap \quad \cap \quad \cap \\
C_{i+1,j} \subset C_{i+1,j+1}
\]

with \(i\) even, we define a connection by

\[
X^{\sigma_1,\sigma_2}_{\sigma_3,\sigma_4} = \frac{\sigma_1}{\sigma_4} \downarrow \sigma_2 = q^{\frac{2}{3}} \delta_{\sigma_1,\sigma_2} \delta_{\sigma_2,\sigma_4} - q^{-\frac{1}{3}} U^{\sigma_1,\sigma_2}_{\sigma_3,\sigma_4}, \quad (10)
\]

We denote by \(\tilde{\mathcal{G}}\) the reverse graph of \(\mathcal{G}\), which is the graph obtained by reversing the direction of every edge of \(\mathcal{G}\). For the inclusions \((\tilde{\mathcal{G}})\) with \(i\) odd, let \(\sigma_1, \sigma_4\) be edges on \(\mathcal{G}\) and let \(\tilde{\sigma}_2, \tilde{\sigma}_3\) be edges on the reverse graph \(\tilde{\mathcal{G}}\) (so that \(\sigma_2, \sigma_3\) are edges on \(\mathcal{G}\)). We define the connection by

\[
X^{\sigma_1,\tilde{\sigma}_2}_{\sigma_3,\sigma_4} = \frac{\sigma_1}{\sigma_4} \downarrow \tilde{\sigma}_2 = \frac{\phi_s(\sigma_4)\phi_r(\sigma_2)}{\phi_r(\sigma_3)\phi_s(\sigma_2)} \frac{\sigma_3}{\sigma_1} \downarrow \sigma_2. \quad (11)
\]

It was shown in \([5]\) that these connections satisfy the unitarity axiom

\[
\sum_{\sigma_3,\sigma_4} X^{\sigma_1,\sigma_2}_{\sigma_3,\sigma_4} X^{\sigma_1',\sigma_2'}_{\sigma_3,\sigma_4} = \delta_{\sigma_1,\sigma_1'} \delta_{\sigma_2,\sigma_2'}. \quad (12)
\]

We make the double sequence \((C_{i,j})\) into an \(A_2\)-\(C^*\)-planar algebra as follows. Let \(P^{\mathcal{G}}_i = \mathfrak{Y}_a^i\) for \(i = j = 0\), and \(P^{\mathcal{G}}_{i,j} = C_{i,j}\) for all other \(i, j \geq 0\). We define a presenting map \(Z : \mathcal{P}(P^{\mathcal{G}}) \rightarrow P^{\mathcal{G}_{i,j}}\) in the same way as we did for the double sequence \((B_{i,j})\) of finite dimensional algebras with a flat connection in \([7]\). First, convert all the discs \(D_k\) to rectangles, with the first \(i_k + j_k\) vertices along one edge, and the next \(i_k + j_k\) vertices along the opposite edge, and rotate each rectangle so that those edges are horizontal with the first vertex on the top edge. Next, isotope the strings of \(T\) so that each horizontal strip only contains one of the following elements: a rectangle with label \(x_k\), a cup, a cap, a \(Y\)-fork, or an inverted \(Y\)-fork. Let \(C\) be the set of all strips containing one of these
elements except for a labelled rectangle. We will use the following notation for elements of $C$, as shown in Figures 6, 7 and 8. A strip containing a cup, cap will be $\cup^{(i)}$, $\cap^{(i)}$ respectively, where there are $i-1$ vertical strings to the left of the cup or cap. Strips containing an incoming Y-fork, inverted Y-fork will be $\gamma^{(i)}$, $\lambda^{(i)}$ respectively, where there are $i-1$ vertical strings to the left of the (inverted) Y-fork. A bar will denote that it is an outgoing (inverted) Y-fork.

For an element $c \in C$ with $n_1$, $n_2$ strings having endpoints (we will call these endpoints vertices) along the top, bottom edge respectively of the strip, let the orientations of these vertices along the top, bottom edge respectively of the strip be given by the sequences $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ respectively, where for $i = 1, 2$, $\mathbf{v}^{(i)} = (v_0^{(i)}, v_1^{(i)}, v_2^{(i)}, \ldots, v_{l_i}^{(i)})$, where $v_0^{(i)} \in \mathbb{N} \cup \{0\}$ and $v_k^{(i)} \in \mathbb{N}$ for $k \geq 1$, with $\sum_{k=1}^{l_i} v_k^{(i)} = n_i$. The numbers $v_k^{(i)}$ denote the number of consecutive vertices with downwards, upwards orientation for $k$ even, odd respectively. Note that if the first vertex along the top, bottom of the strip has upwards orientation, then $v_0^{(i)} = 0$ for $i = 1, 2$ respectively. The leftmost region of the strip $c$ corresponds to the vertex $*$ of $\mathcal{G}$, and each vertex along the top (or bottom) with downwards, upwards orientation respectively, corresponds to an edge on $\mathcal{G}$, $\tilde{\mathcal{G}}$ respectively ($\tilde{\mathcal{G}}$ is the graph $\mathcal{G}$ with all orientations reversed). Then the top, bottom edge of the strip corresponds is labelled by all paths on $\mathcal{G}$ and $\tilde{\mathcal{G}}$ which start at $*$ and have the form given by $\mathbf{v}^{(i)}$. These paths are uniquely described by the sequence of edges they pass along. Let $H_1$, $H_2$ be the Hilbert spaces corresponding to all paths of the form $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ respectively. Then $Z(c)$ defines an operator $M_c \in \text{End}(H_1, H_2)$ as follows.
For a cup $\cup^{(i)}$, and paths $\alpha = \alpha_1 \cdot \alpha_2 \cdots \alpha_j$, $\beta = \beta_1 \cdots \beta_{j+2}$,

$$(M_{\cup^{(i)}})_{\alpha,\beta} = \delta_{\alpha_1,\beta_1} \delta_{\alpha_2,\beta_2} \cdots \delta_{\alpha_{i-1},\beta_{i-1}} \delta_{\alpha_i,\beta_i+2} \cdots \delta_{\alpha_m,\beta_{m+2}} \delta_{\beta_i,\beta_i+1} \frac{\sqrt{\phi_r(\beta_i)}}{\sqrt{\phi_s(\beta_i)}}. \tag{13}$$

For a cap $\cap^{(i)}$,

$$M_{\cap^{(i)}} = M_{\cup^{(i)}}^*. \tag{14}$$

For an incoming (inverted) Y-fork $\gamma^{(i)}$ or $\lambda^{(i)}$,

$$(M_{\gamma^{(i)}})_{\alpha,\beta} = \delta_{\alpha_1,\beta_1} \cdots \delta_{\alpha_{i-1},\beta_{i-1}} \delta_{\alpha_i,\beta_i+1} \cdots \delta_{\alpha_m,\beta_m+1} \frac{1}{\sqrt{\phi_s(\alpha_i) \phi_r(\alpha_i)}} W(\Delta(\alpha_i,\beta_i)),$$  

$$(M_{\lambda^{(i)}})_{\alpha,\beta} = \delta_{\alpha_1,\beta_1} \cdots \delta_{\alpha_{i-1},\beta_{i-1}} \delta_{\alpha_i,\beta_i+1} \cdots \delta_{\alpha_m,\beta_m+1} \frac{1}{\sqrt{\phi_s(\beta_i) \phi_r(\beta_i)}} W(\Delta(\beta_i,\alpha_i)),$$ \hspace{1cm} \tag{15}

where $W$ is a cell system on $G$ (see [23, 5]).

For an outgoing (inverted) Y-fork $\nu^{(i)}$ or $\nu^{(i)}$,

$$M_{\nu^{(i)}} = M_{\nu^{(i)}}^*, \tag{17}$$

$$M_{\nu^{(i)}} = M_{\nu^{(i)}}^*. \tag{18}$$

For a strip containing a rectangle with label $x_k = \sum_{\gamma, \gamma'} \lambda_{\gamma, \gamma'}(\gamma, \gamma')$ where $\lambda_{\gamma, \gamma'} \in \mathbb{C}$ and $(\gamma, \gamma') \in P_{ik,jk}$ are matrix units indexed by paths $\gamma$, $\gamma'$, we define the operator $M_{bk} = Z(b_k)$ as follows. Let $p_k, p_k'$ be the number of vertical strings to the left, right of the rectangle in strip $b_k$ respectively, with orientations given by the sequences $v^{(p_k)} = (v_0^{(p_k)}, v_1^{(p_k)}, \ldots, v_{p_k}^{(p_k)})$, $v^{(p_k')} = (v_0^{(p_k')}, v_1^{(p_k')}, \ldots, v_{p_k'}^{(p_k')})$ respectively. Then $\sum_{\gamma, \gamma', \mu_1, \gamma \cdot \mu_2, \mu_1 \cdot \gamma' \cdot \mu_2} \lambda_{\gamma, \gamma'}(\mu_1 \cdot \gamma \cdot \mu_2, \mu_1 \cdot \gamma' \cdot \mu_2)$ defines a matrix $M_{bk}$, where the summation is over all paths $\mu_1$ on $G$, $\tilde{G}$ of length $p_k$ of the form $v^{(p_k)}$, and paths $\mu_2$ on $G$, $\tilde{G}$ of length $p_k'$ of the form $v^{(p_k')}$. For a tangle $T \in \mathcal{P}_{i,j}$ with $l$ horizontal strips $s_i$, where $s_1$ is the lowest strip, $s_2$ the strip immediately above it, and so on, we define $Z(T) = Z(s_1)Z(s_2) \cdots Z(s_l)$, which will be an element of $P^G_{i,j}$.

**Theorem 5.1** The above definition of $Z(T)$ for any $A_2$-planar tangle $T$ makes the double sequence $(C_{i,j})$ into an $A_2$-$C^*$-planar algebra $P^G$, with $\dim(P^G_{i,j}) = n_a$, $a = 0, 1, 2$, and parameter [3].

**Proof:** This follows as in the proof of Theorem 5.4 in [17], where the only small difference occurs for isotopies of the tangle which involve rectangles. However the invariance is simpler here as the connection is not used. □

The partition functions $Z : \mathcal{P}_\pi \rightarrow \mathbb{C}$ are defined as the linear extensions of the function which takes the basis path $v$ to $\phi_v^2$. There is an extra multiplicative factor of $\phi_v^2$ for the external region. This is required for spherical isotopy.

**Proposition 5.2** (cf. [13, Prop. 3.4]) The partition function of a closed labelled tangle $T$ depends only on $T$ up to isotopies of the 2-sphere.
We normalize \((\phi_v)\) so that the partition function of an empty closed tangle is equal to one. We will say that the SU(3)-planar algebra of a graph \(G\) is **normalized** if

\[
\sum_{v \in V_G} \phi_v^2 = 1.
\]

**Theorem 5.3** (cf. [13, Theorem 3.6]) Let \(P^G\) be the normalized \(A_2\)-planar algebra of an \(ADE\) graph \(G\), with (normalized) Perron-Frobenius eigenvector \((\phi_v)\). Then for \(x \in P^{G}_{i,j}\),

\[
\text{tr}(x) = [3]^{-i-j} Z(\hat{x})
\]

defines a normalized trace on the union of the \(P\)'s, where \(\hat{x}\) is any 0-tangle obtained from \(x\) by connecting the first \(i+j\) boundary points to the last \(i+j\). The scalar product

\[
\langle x, y \rangle = \text{tr}(x^* y)
\]

is positive definite.

**Proof:** The normalization makes the definition of the trace consistent with the inclusions. The property \(\text{tr}(ab) = \text{tr}(ba)\) is a consequence of planar isotopy when all the strings added to \(x\) to get \(\hat{x}\) go round \(x\) in the same direction, as in Figure 9. Spherical isotopy reduces the general case to the one above. Positive definiteness follows from the fact that the matrix units \(e = (\gamma, \gamma') \in P^G_{i,j}\) are mutually orthogonal elements of positive length: \(\langle e, e \rangle = [3]^{-i-j} \phi_{v_1} \phi_{v_2} > 0\), where \(e \in P^G_{i,j}\) is a pair of paths of length \(i+j\) starting at vertex \(v_1\) and ending at vertex \(v_2\), and \(\phi_v > 0\) for all \(v\) since \(\phi\) is a Perron-Frobenius eigenvector.

\[\blacksquare\]

### 5.1 \(P^G\) as an \(A_2\)-STL-module

Let \(\Delta_G\) denote the adjacency matrix of the graph \(G\). If \(G\) is three-colourable then \(\Delta_G\) may be written in the form

\[
\Delta_G = \begin{pmatrix}
0 & \Delta_{01} & 0 \\
0 & 0 & \Delta_{12} \\
\Delta_{20} & 0 & 0
\end{pmatrix},
\]

where \(\Delta_{01}, \Delta_{12}\) and \(\Delta_{20}\) are matrices which give the number of edges between each 0,1,2-coloured vertex respectively of \(G\) to each 1,2,0-coloured vertex respectively. By a suitable ordering of the vertices the matrix \(\Delta_{12}\) may be chosen to be symmetric. These matrices satisfy the conditions \(\Delta_{01}^T \Delta_{01} = \Delta_{20} \Delta_{20} = \Delta_{12}^2\), \(\Delta_{01} \Delta_{01}^T = \Delta_{20}^T \Delta_{20}\), which follow from the fact that \(\Delta_G\) is normal [6]. For non-three-colourable \(G\), we define \(\Delta_{01} = \Delta_{12} = \Delta_{20} = \Delta_G\). Let \(\Lambda^1_{i,j}, \Lambda^2_{i,j}\) be the product of \(j\), \(i\) matrices respectively, given by

\[
\Lambda^1_{i,j} = \Delta_{01} \Delta_{12} \Delta_{20} \Delta_{01} \cdots \Delta_{j-1,j}, \quad \Lambda^2_{i,j} = \Delta_{j,j+1} \Delta_{j,j+1}^T \Delta_{j,j+1} \Delta_{j,j+1}^T \cdots \Delta^T.
\]
where $\Delta'$ is $\Delta_{j+1,j}$ if $i$ is odd, $\Delta_{j,j+1}$ if $i$ is even, and $\bar{p}$ is the colour of $p$. Note that $\Lambda_{1,j}^1$ is a normal operator since $\Lambda_{1,j}^1(\Lambda_{1,j}^1)^* = \Lambda_{1,j}^1(\Lambda_{1,j}^1)^T = (\Delta_{01}\Delta_{01}^T)^j$ and $(\Lambda_{1,j}^1)^*(\Lambda_{1,j}^1)^T = (\Delta_{01}\Delta_{01}^T)^j$. Similarly $\Lambda_{2,j}^2$ is a normal operator.

Let $\beta_l^i$, $l \in \mathcal{G}_l^0$, be the eigenvalues of $\Lambda_{1,3}$, and $v^{(l)}$ their corresponding eigenvectors. Then $(\Lambda_{1,j}^1)^T v^{(l)} = \beta_l^i v^{(l)}$ and $(\Delta_{01}\Delta_{01}^T)^3 v^{(l)} = \Lambda_{1,j}^1(\Lambda_{1,j}^1)^T v^{(l)} = |\beta_l^i|^6 v^{(l)}$. Then if $\lambda_l$ are the eigenvalues of $\Delta_{01}\Delta_{01}^T$ with corresponding eigenvectors $v^{(l')}$, $l \in \mathcal{G}_l^0$, we have $(\Delta_{01}\Delta_{01}^T)^3 v^{(l')} = \lambda_l^3 v^{(l')}$ so that $v^{(l')} = v^{(l)}$ and $\lambda_l = |\beta_l|^2$.

Let $n' = \min\{n_0, n_1\}$. The dimension of $P_{i,j}^G$ is given by the trace of $\Lambda\Lambda^T$ where $\Lambda = (\Lambda_{1,j}^1)(\Lambda_{2,j}^2)^j$, which counts the number of pairs of paths on $\mathcal{G}$, $\overline{\mathcal{G}}$. Since $\Lambda\Lambda^T = (\Delta\Delta^T)^{i+j}$, the trace of $\Lambda\Lambda^T$ is given by the sum $\sum_l v_{i,j}^{(l)}$ of its eigenvalues, $l = 1, 2, \ldots, n'$. The eigenvalues $v_{i,j}^{(l)}$ are given by $|\beta_l|^{2(i+j)}$, where $\beta_l^i$ are the eigenvalues of $\Lambda_{1,3}^1$. The Hilbert series for $P_G^0$ is then given by

$$
\Phi_{P_G^0}(z_1, z_2) = \frac{1}{3}(n_0 + 2n_1 - 3n') + \sum_{i=1}^{n'} \frac{1}{(1 - |\beta_l|^2 z_1)(1 - |\beta_l|^2 z_2)}.
$$

**Proposition 5.4 (cf. [20] Prop. 13)** Let $\mathcal{G}$ be one of the finite SU(3) $ADE$ graphs, let $\zeta_l$, $l = 1, 2, \ldots, n'$, be the non-zero eigenvalues of $\Lambda_{0,3}$, counting multiplicity, and let $\beta_l$ be any cubic root of $\zeta_l$, $l = 1, 2, \ldots, n'$. For all the three-colourable graphs except $\mathcal{E}^{(12)}_5$, we have $n_0 \geq n_1$, all the irreducible weight-zero $A_2$-ATL-submodules of $P_G^0$ are $H^\beta_l$, $l = 1, 2, \ldots, n_1$, and $(n_0 - n_1)$ copies of $H^0$, and these can be assumed to be mutually orthogonal. For $\mathcal{E}^{(12)}_5$ we have $n_1 > n_0$, and all the irreducible weight-zero $A_2$-ATL-submodules of $P_G^{(12)}$ are $H^\beta_l$, $l = 1, 2, \ldots, n_0$, and $2(n_1 - n_0)$ copies of $H^0$, which can again be assumed to be mutually orthogonal. If $\mathcal{G}$ is not three-colourable, all the irreducible weight-zero $A_2$-ATL-submodules of $P_G^0$ are $H^\beta_l$, $l = 1, 2, \ldots, n_0$, where $n_0$ is the total number of vertices of $\mathcal{G}$.

**Proof:** Consider the case where $n_0 > n_1$ (the case for $\mathcal{E}^{(12)}_5$ where $n_1 > n_0$ is similar). Each $\beta_l$-eigenvector $v^{(l)} = (v_w^{(l)})$, $w \in \mathcal{G}_w^0$, of $\Delta_{01}\Delta_{01}^T$ spans a one-dimensional subspace of $P_{i,j}^G$ that is invariant under $A_2$-ATL. To see this, first consider the element $\sigma_0 \sigma_{12} \sigma_{20}$:

$$
\sigma_0 \sigma_{12} \sigma_{20} v^{(l)} = \sigma_{01} \sigma_{12} \sigma_{20} \sum_{w \in \mathcal{G}_w^0} v_w^{(l)} = \sum_{w',w} (\Delta_{01}\Delta_{12}\Delta_{20})_{w',w} v_{w}^{(l)},
$$

which, by the $\beta_l$ eigenvalue gives

$$
\sigma_0 \sigma_{12} \sigma_{20} v^{(l)} = \sum_{w'} \beta_l^3 v_{w'}^{(l)} = \beta_l^3 v^{(l)}.
$$

Similarly for $\sigma_{20}^* \sigma_{12}^* \sigma_{01}^*$. Next consider the general element $\sigma$ given by the composition of $2k$ elements $\sigma_0 \sigma_{12} \sigma_{20}^* \sigma_1 \sigma_0 \sigma_{12}^* \sigma_{20} \cdots \sigma_{k-1,k}^* \sigma_{k-1,k} \cdots \sigma_{12} \sigma_{01}^*$:

$$
\sigma v^{(l)} = \sum_{w',w} (\Delta_{01}\Delta_{12}\cdots \Delta_{k-1,k} \Delta_{k-1,k}^T \cdots \Delta_{01}^T)_{w',w} v^{(l)}
$$

$$
= \sum_{w',w} ((\Delta_{01}^T)_{w',w} v^{(l)}_w = \sum_{w'} |\beta_l|^{2k} v^{(l)}_w = |\beta_l|^{2k} v^{(l)}.
$$

(20)
Any element of $A_2-ATL_{\overline{5}}$ is a linear combination of products of elements $\sigma_{j,j\pm 1}$ such that the regions which meet the outer and inner boundaries have colour 0. Let $\sigma$ be such an element. Then the action of $\sigma$ on the $\beta_l$-eigenvector $v^{(l)}$ is given by $\sigma v^{(l)} = \sum_{w,w'} M(w',w) v^{(l)}$, where $M$ is the product of matrices $\Delta, \Delta^T$ given by replacing every $\sigma_{j,j+1}, \sigma_{j',j'-1}$ in $\sigma$ by $\Delta, \Delta^T$ respectively. Then by (19) and (20), this gives some scalar multiple of $v^{(l)}$. Then each $\beta_l$-eigenvector $v^{(l)}$ generates the submodule $H^{\beta_l}$ by Proposition 4.16. The inner product on $H^{\beta_l}$ coincides with the inner product on $P^G$. To see this we only need to check its restriction to the zero-weight part because of (8). For any element $A \in A_2-ATL_{\overline{5}}$, $\langle Av, v \rangle_{H^{\beta_l}} = c(v,v)_{H^{\beta_l}}$ whilst for $\langle Av^{(l)}, v^{(l)} \rangle_{P^G} = d(v^{(l)}, v^{(l)})_{P^G}$. The element $A$ is necessarily a combination of non-contractible circles, which gives the same contribution in $P^G$ as in $H^{\beta_l}$ by (19), (20). So $c = d$. This shows that the inner product on the $H^{\beta_l}$ is positive definite, since the inner product on $P^G$ is.

Similarly, a 0-eigenvector generates the submodule $H^0$, where for $n_0 > n_1$, dim$(H^0_{\overline{5}}) = 1$ and dim$(H^0_T) = 1$, whilst for $E^{(12)}_6$ we have dim$(H^0_T) = 1$ and dim$(H^0_{\overline{5}}) = 0$. As in the $SU(2)$ case, in order to make the resulting submodules orthogonal we take an orthogonal set of eigenvectors.

For an $ADE$ graph $G$ with Coxeter number $n$, let $\beta_{(l_1,l_2)}$ be the eigenvalue of $G$ given by

$$\beta_{(l_1,l_2)} = \exp\left(\frac{2i\pi}{3n}(l_1 + 2l_2 + 3)\right) + \exp\left(-\frac{2i\pi}{3n}(2l_1 + l_2 + 3)\right) + \exp\left(\frac{2i\pi}{3n}(l_1 - l_2)\right)$$

for exponent $(l_1, l_2)$. Then for the graphs $A^{(n)}$, we have for $n \not\equiv 0 \mod 3$,

$$P^{A^{(n)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}},$$

whilst for $n = 3k, k \geq 2$,

$$P^{A^{(3k)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}} \oplus H^0_{\overline{5}},$$

where in both cases the summation is over all $(l_1, l_2) \in \{(m_1, m_2) | 3m_2 \leq n - 3, 3m_1 + 3m_2 \leq 2n - 6\}$, i.e. each $\beta_{(l_1,l_2)}$ is a cubic root of an eigenvalue of $\Lambda_{0,3}$. We believe that we in fact have equality here, so that $P^{A^{(n)}} = \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}}$. In the $SU(2)$ case this was achieved by a dimension count of the left and right hand sides [26, Theorem 15]. However, we have not yet been able to determine a similar result for the $SU(3)$ $A$ graphs.

For the other $ADE$ graphs, Proposition $\Lambda_4$ gives the following results for the zero-weight part of $P^G$. For the $D$ graphs, we have

$$P^{D^{(3k)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}} \oplus 3H^0_{\overline{5}},$$

for $k \geq 2$, where the summation is over all $(l_1, l_2) \in \{(m_1, m_2) | m_2 \leq k - 1, m_1 + m_2 \leq 2k - 2, m_1 - m_2 \equiv 0 \mod 3\}$, whilst for $n \not\equiv 0 \mod 3$,

$$P^{D^{(n)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}},$$

for $k \geq 2$, where the summation is over all $(l_1, l_2) \in \{(m_1, m_2) | m_2 \leq k - 1, m_1 + m_2 \leq 2k - 2, m_1 - m_2 \equiv 0 \mod 3\}$, whilst for $n \not\equiv 0 \mod 3$,
where the summation is over all \((l_1, l_2) \in \{(m_1, m_2) | 3m_2 \leq n - 3, 3m_1 + 3m_2 \leq 2n - 6\}\). The path algebras for \(A_l\) for \(2 \leq t_1^0 = \tilde{\varphi}(l_1, l_2)\),

\[
P^{A^{(n)*}} = P^{D^{(n)*}} \supset \bigoplus_{(l_1, l_2)} H^{(l_1, l_2)},
\]

where the summation is over all \((l_1, l_2) \in \{(m, m) | m = 0, 1, \ldots, [(n - 3)/2]\}\). Similarly, the path algebras for \(E^{(8)}\) and \(E^{(8)*}\) are identified, and

\[
P^{E^{(8)}} = P^{E^{(8)*}} \supset H^{(4, 0)} \oplus H^{(3, 0)} \oplus H^{(3, 0)} \oplus H^{(2, 2)}.
\]

For the graphs \(E_i^{(12)}\), \(i = 1, 2, 3\), we have

\[
P^{E_i^{(12)}} \supset H^{(4, 0)} \oplus 2H^{(2, 2)} \oplus H^{(1, 4)}.
\]

For the remaining exceptional graphs we have

\[
P^{E_4^{(12)}} \supset H^{(6, 0)} \oplus H^{(2, 2)} \oplus H^{(4, 4)} \oplus 2H^{0, 0},
\]

\[
P^{E_5^{(12)}} \supset H^{(6, 0)} \oplus H^{(3, 0)} \oplus H^{(3, 0)} \oplus H^{(2, 2)} \oplus H^{(4, 4)} \oplus H^{(4, 0)} \oplus H^{0, 0} \oplus H^{0, 0},
\]

\[
P^{E^{(24)}} \supset H^{(6, 0)} \oplus H^{(2, 2)} \oplus H^{(4, 4)} \oplus H^{(4, 0)} \oplus H^{(3, 0)} \oplus H^{(4, 4)} \oplus H^{(4, 0)} \oplus H^{(6, 0)} \oplus H^{(10, 10)}.
\]

The \(A_2\)-planar algebra \(P \cong STL\) (for the graphs \(A^{(n)}\)) clearly have decomposition \(P = H^\alpha\) as an \(A_2\)-\(ATL\)-module, since \(STL\) is equal to the \(A_2\)-\(ATL\)-module \(H^\alpha\) (see Section 4.2). Since every \(A_2\)-planar algebra contains \(STL\), the \(A_2\)-planar algebra for all the \(A\)-\(D\)-\(E\) graphs with a flat connection will contain the zero-weight module \(H^\alpha\).

### 5.2 Irreducible modules with non-zero weight

We will now present some conjectured irreducible \(A_2\)-\(ATL\)-modules with non-zero weight. It is not known whether the inner-products on these modules are positive definite. Our construction of these modules is based on the following assumption. Let \(\varphi(t_1, t_2)\), \(\tilde{\varphi}(t_1, t_2)\) be the tangles illustrated in Figure 10. Note that \(\tilde{\varphi}(t_2, t_1)\) is the rotation of \(\varphi(t_1, t_2)\) by \(\pi\). These tangles can be viewed as some sort of “rotation by one”. They have rank \((t_1, t_2)\). It appears that the infinite dimensional algebra \(\hat{A}_k = A_2\)-\(ATL\) is generated by the two tangles \(\varphi(k, k)\) and \(\tilde{\varphi}(k, k)\), \(k \geq 1\). From now on will assume that this is true.

Let \(\rho_{0,k}\) be the \(0, k\)-tangle given by the image of \(\varphi(k, k)\tilde{\varphi}(k, k)\) in \(\hat{A}_k\), illustrated in Figure 11 and let \(\rho_{i,k} -1\) be the image of \(\rho_{0,k}\) under the map \(\rho_k : A_2\)-\(ATL_{0,k} \to A_2\)-\(ATL_{i,k-1}\) as in Section 4.1. Then \(\rho_{i,j}\) is some sort of “rotation by two”. Indeed, it can be shown for \(2t_{\text{max}} + t_{\text{min}} = 3(i + j)\), for any \(i, j \geq 0\), that \(\rho_{i,j}\) is a rotation of order \(i + j\) in \(A_2\)-\(ATL_{i,j}\), i.e. \((\rho_{i,j})^{i+j} = 1_{i,j}\).

By drawing pictures it is easy to see that \(\varphi(k, k) \tilde{\varphi}(k, k) = \rho_{0,k}\) in \(\hat{A}_k\), and hence that \(\varphi(k, k)\), \(\tilde{\varphi}(k, k)\) commute in \(\hat{A}_k\). It is also easy to check that

\[
\varphi(k, k) \tilde{\varphi}(k, k) = \varphi(k, k) \tilde{\varphi}(k, k) = \tilde{\varphi}(k, k) \varphi(k, k) = \tilde{\varphi}(k, k) \tilde{\varphi}(k, k) = 1_{0,k}.
\]
so that \( \varphi^*_{(k,k)} = \varphi^{-1}_{(k,k)} \) and \( \tilde{\varphi}^*_{(k,k)} = \tilde{\varphi}^{-1}_{(k,k)} \) are inverse elements in \( \tilde{A}_k \). Again, by drawing pictures it is clear that \( (\varphi_{(k,k)} \tilde{\varphi}^*_{(k,k)})^k = \varphi^*_{(k,k)} \). Then we have

\[
\varphi^k_{(k,k)} = (\varphi^*_{(k,k)} \varphi_{(k,k)})^k = (\varphi^*_{(k,k)})^k \varphi_{(k,k)} = (\varphi^*_{(k,k)})^k (\varphi_{(k,k)} \tilde{\varphi}^*_{(k,k)})^k = (\varphi^*_{(k,k)})^k,
\]

and so \( \tilde{\varphi}^k_{(k,k)} = \varphi^k_{(k,k)} \tilde{\varphi}^{k-1}_{(k,k)} \) and \( \varphi^*_{(k,k)} = \varphi^{-1}_{(k,k)} \tilde{\varphi}^{k}_{(k,k)}. \)

The algebras \( \tilde{A}_k \) are infinite dimensional, since \( \varphi^l_{(k,k)}, l = 1,2, \ldots \), are all distinct and non-zero in \( \tilde{A}_k \), as are \( \tilde{\varphi}^l_{(k,k)}, l = 1,2, \ldots \). One way to obtain a finite-dimensional \( A_2 \)-\( ATL \)-module \( V^{(k),\gamma} \) is to consider the element \( \varphi_{(k,k)} \tilde{\varphi}^*_{(k,k)} \) as acting as a scalar \( \gamma^2 \) in the lowest weight module \( V_{0,k}^{(k),\gamma} \), i.e. \( \varphi_{(k,k)} \tilde{\varphi}^*_{(k,k)} = \gamma^2 1_{0,k} \) in \( V_{0,k}^{(k),\gamma} \), for some \( \gamma \in \mathbb{C} \). By drawing the element \( \varphi^2_{(k,k)} \) we see that \( \varphi^2_{(k,k)} = \gamma^2 1_{0,k}. \) Then we have \( \varphi^*_{(k,k)} = \gamma^{-2} \varphi^2_{(k,k)} \), and by (32),

\[
\tilde{\varphi}^k_{(k,k)} = (\varphi^*_{(k,k)})^k = (\gamma^{-2k} \varphi^2_{(k,k)})^k = \gamma^{-2k^2} \varphi^2_{(k,k)} = \gamma^{-2k^2} \gamma^{2k(k-1)} \varphi^k_{(k,k)} = \gamma^{-2k} \varphi^k_{(k,k)},
\]

so that \( \tilde{\varphi}^2_{(k,k)} = \gamma^{-4k} \varphi^2_{(k,k)} = \gamma^{-2k} 1_{0,k}. \) Now \( \varphi^{k+1}_{(k,k)} \tilde{\varphi}^{k-1}_{(k,k)} = \varphi_{(k,k)} \tilde{\varphi}^*_{(k,k)} = \gamma^2 1_{0,k}, \) so that

\[
\tilde{\varphi}^{k+1}_{(k,k)} \varphi^{k+1}_{(k,k)} \tilde{\varphi}^{k-1}_{(k,k)} = \gamma^2 \varphi^{k+1}_{(k,k)}.
\]

But we also have

\[
\tilde{\varphi}^{k+1}_{(k,k)} \varphi^{k+1}_{(k,k)} \tilde{\varphi}^{k-1}_{(k,k)} = \varphi^{k+1}_{(k,k)} \tilde{\varphi}^{2k}_{(k,k)} = \gamma^{-2k} \varphi^{k+1}_{(k,k)}.
\]

Comparing (33) and (34) we find that \( \gamma^2 \tilde{\varphi}^{k+1}_{(k,k)} = \gamma^{-2k} \varphi^{k+1}_{(k,k)} \), which gives

\[
\tilde{\varphi}^{k+1}_{(k,k)} = \gamma^{-2(k+1)} \varphi^{k+1}_{(k,k)}.
\]
Then by (32), (35) we have
\[ \tilde{\varphi}(k,k) = \tilde{\varphi}(k,k) \varphi^k(\tilde{\varphi}(k,k))^k = \varphi^{k+1}(\varphi^*(k,k))^k = \gamma^{-2(k+1)} \varphi^{k+1}(\gamma^{2k}(\varphi^*(k,k))^k) = \gamma^{-2} \varphi(k,k). \]

Then
\[ V_{0,k}^{(k,k),\gamma} = \text{span}(\varphi_{(k,k)}^l | l = 0, 1, \ldots, 2k - 1), \]
where \( \varphi_{(k,k)}^{2k} = \gamma^{2k} \mathbf{1}_{0,k}. \) We see that \( \varphi_{(k,k)} \) acts on \( V_{0,k}^{(k,k),\gamma} \) as \( \mathbb{Z}_{2k} \), by permuting the 2k basis elements \( \varphi_{(k,k)}^l \), and so the \( A_2-ATL \)-module \( V^{(k,k),\gamma} \) decomposes as a direct sum over the \( 2k \) roots of unity \( \omega \) of \( A_2-ATL \)-modules \( V^{(k,k),\gamma} \), where \( V^{(k,k),\gamma}\omega \) is the \( \omega \)-eigenspace for the action of \( \mathbb{Z}_{2k} \) with eigenvalue \( \omega \).

For each \( k \), we can choose a faithful trace \( tr' \) on \( \tilde{A}_k \), which we extend to a trace \( tr \) on \( A_2-ATL_{0,k} \) by \( tr = tr' \circ \pi \), where \( \pi \) is the quotient map \( A_2-ATL_{0,k} \rightarrow \tilde{A}_k \). We can define an inner product on \( A_2-ATL(i,j : 0, k) \) by \( \langle S, T \rangle = tr(T^*S) \) for any \( S, T \in A_2-ATL(i, j : 0, k) \). Since \( \varphi_{(k,k)}^* \varphi_{(k,k)} = \mathbf{1}_{0,k} \), the decomposition into \( V^{(k,k),\gamma}\omega \) is orthogonal. If we let \( \psi_{\gamma,\omega}^{(k)} \) be the vector in \( V_{0,k}^{(k,k),\gamma}\omega \) which is orthogonal to \( \sum_{j=0}^{2k-1} (\omega^j)^{-j} \varphi_{(k,k)}^j \) such that \( \langle \psi_{\gamma,\omega}^{(k)}, \psi_{\gamma,\omega}^{(k)} \rangle = 1 \), then \( \varphi_{(k,k)} \psi_{\gamma,\omega}^{(k)} = \omega^\gamma \psi_{\gamma,\omega}^{(k)} \). We see that \( \dim(V_{0,k}^{(k,k),\gamma}\omega) = 1 \), and \( V_{0,k}^{(k,k),\gamma}\omega \) is the span of \( \psi_{\gamma,\omega}^{(k)} \). We define the Hilbert \( A_2-ATL \)-module \( H^{(k,k),\gamma}\omega \) to be the quotient of \( V^{(k,k),\gamma}\omega \) by the zero-length vectors with respect to this inner product.

We can also construct a finite-dimensional \( A_2-ATL \)-module \( V^{(3,0)} \) with lowest weight 2 and minimum rank \((3,0)\) as follows. Let \( W_{i,j}^{(3,0)} \) be the vector space of all linear combinations of tangles with one inner disc, where the outer disc has pattern \( i, j \), the inner disc has 3 sink vertices, with one of these vertices chosen as a distinguished vertex, and such that as we pass along the string that has this distinguished vertex as its endpoint, the region to its right must be coloured \( \overline{0} \). Let \( V_{i,j}^{(3,0)} \) be the quotient of \( W_{i,j}^{(3,0)} \) by the ideal generated by the Kuperberg relations K1-K3. The vector space \( V_{i,j}^{(3,0)} \) is infinite dimensional due to the possibility of composing the elements \( \varphi_{(3,0)} \) an infinite number of times, where each \( \varphi_{(3,0)}^l, l = 1, 2, \ldots \), is a tangle which has rank \((3,0)\) and does not contain any closed circles, or embedded circles or squares. If however, we let \( \varphi_{(3,0)} \varphi_{(3,0)}^* \) count as some scalar in \( V^{(3,0)} \), i.e. \( \varphi_{(3,0)} \varphi_{(3,0)}^* = \gamma^3 \in \mathbb{C} \), then \( V_{i,j}^{(3,0)} \) is finite-dimensional since

\[ \varphi_{(3,0)}^* = \begin{array}{c} \varphi_{(3,0)}^2 = \end{array} \]

and hence \( \varphi_{(3,0)}^3 = \gamma^3 \in \mathbb{C} \) (and similarly \( \varphi_{(3,0)}^3 \) is also a scalar). Since the elements \( \varphi_{(3,0)}^* \) and \( \varphi_{(3,0)} \) are the same, \( \gamma^3 = \varphi_{(3,0)}^3 = \varphi_{(3,0)}^3 = \gamma^3 \), so \( \gamma^3 \in \mathbb{R} \). We will denote the module \( V^{(3,0)} \) where \( \varphi_{(3,0)} \varphi_{(3,0)}^* = \gamma^3 \in \mathbb{C} \) by \( V^{(3,0),\gamma} \), where \( \gamma \in \mathbb{R} \).

Let \( U_l, \bar{U}_l \in A_2-ATL_{i,j}, l = 1, \ldots, j \), be the annular \( i, j \)-tangles illustrated in Figure 12. From drawing pictures, it appears that the lowest weight module \( V_{0,2}^{(3,0),\gamma} \) is the span of \( v_l, l = 1, \ldots, 6 \), where \( v_1 \) is the tangle in Figure 13: \( v_{2l} = \varphi_{(2,2)} v_{2l-1}, l = 1, 2, 3, \) and \( v_{2l+1} = \bar{U}_1 v_{2l}, l = 1, 2 \). These are the only tangles we can find that have rank no smaller
than (3, 0), do not contain any closed circles or embedded circles or squares, and which cannot be written as a linear combination of tangles of the form \( v'\varphi^{3p}_{(3,0)} \) for some \( p \in \mathbb{N} \), where \( v' \) is one of the elements \( v_l \) above, and the tangle \( \varphi^{3p}_{(3,0)} \) is inserted in the inner disc of \( v' \).

The action of \( A_2-\text{ATL}_{0,2} \) on \( V^{(3,0),\gamma}_{0,2} \) is given as follows. For a tangle \( T \in A_2-\text{ATL}(i, j : 0, 2) \) and one of the elements \( v_l \) above, we form \( Tv_l \) and divide out by the relations \( K1-K3 \) to obtain a linear combination of tangles with pattern \( i, j \) on the outer disc and three sink vertices on the inner disc. Any tangle that has rank < (3, 0) is equal to zero. For the remaining tangles, any tangle that is of the form \( v'\varphi^p_{(3,0)} \) (\( p \) must necessarily by some integer multiple of 3 to respect the colouring at the inner disc) becomes \( \gamma^p v' \in V^{(3,0),\gamma}_{i,j} \).

For any two elements \( S, T \in V^{(3,0),\gamma}_{i,j} \), the tangle \( T^*S \) will have three (source) vertices on its outer disc and three (sink) vertices on its inner disc. We use relations \( K1-K3 \) on \( T^*S \) to obtain a linear combination of tangles \( \sum_j c_j(T^*S)_j \) which do not contain any closed circles, or embedded circles or squares, where \( c_j \in \mathbb{C} \). We let \( \langle S, T \rangle_l \) be zero if \( \text{rank}((T^*S)_l) < (3, 0) \). Otherwise, \( (T^*S)_l \) will be equal to \( \varphi^p_{(3,0)} \) for some \( p = 0, 1, 2, \ldots \), and we let \( \langle S, T \rangle_l \) be \( \gamma^p \). We then define an inner product on \( V^{(3,0),\gamma} \) by \( \langle S, T \rangle = \sum_j c_j \langle S, T \rangle_j \). We define the Hilbert \( A_2-\text{ATL} \)-module \( H^{(3,0),\gamma} \) to be the quotient of \( V^{(3,0),\gamma} \) by the zero-length vectors with respect to this inner product.
For $\gamma \neq \pm 1$, $H_{0,2}^{(3,0),\gamma}$ has dimension 6, and the action of $A_2$-ATL$_{0,2}$ on $H_{0,2}^{(3,0),\gamma}$ is given explicitly by

\[
\begin{align*}
\varphi(2,2)v_{2l-1} &= v_{2l}, & \varphi(2,2)v_{2l} &= v_{2l-1}, & l &= 1, 2, 3, \\
\tilde{U}_1v_{2l-1} &= \delta v_{2l-1}, & \tilde{U}_1v_{2l} &= v_{2l+1}, & l &= 1, 2, \\
\tilde{\varphi}(2,2)v_l &= v_l, & \tilde{U}_1v_l &= 0, & \text{for all } l.
\end{align*}
\]

For $\gamma = \pm 1$, the dimension of $H_{0,2}^{(3,0),\pm 1}$ is 2, and $H_{0,2}^{(3,0),\pm 1}$ is the span of the elements $v_1$, $v_2$ above. The action of $A_2$-ATL$_{0,2}$ on $H_{0,2}^{(3,0),\pm 1}$ is given by

\[
\begin{align*}
\varphi(2,2)v_1 &= v_2, & \varphi(2,2)v_2 &= v_1, \\
\tilde{U}_1v_1 &= \delta v_1, & \tilde{U}_1v_2 &= \gamma v_1, \\
\tilde{\varphi}(2,2)v_l &= v_l, & \tilde{U}_1v_l &= 0, & l &= 1, 2.
\end{align*}
\]

There is a similar description of modules $H^{(0,3),\gamma}$ of minimum rank $(0, 3)$, where there are now three source vertices on the inner disc. The roles of $U_l$ and $\tilde{U}_l$ are interchanged for $H^{(0,3),\gamma}$.

We were able to conjecture certain irreducible modules of non-zero weight that the $A_2$-planar algebra $P^\mathcal{E}$ for the graphs $\mathcal{E}^{(8)}$ and $\mathcal{D}^{(6)}$ should contain, since the action of the rotation $\rho_{0,2}$ on the $A_2$-planar algebras for these graphs was much easier to write down than for the other graphs.

For the graph $\mathcal{E}^{(8)}$, its zero-weight irreducible modules are $H^{\beta_{(0,0)}}, H^{\beta_{(3,0)}}, H^{\beta_{(0,3)}}$ and $H^{\beta_{(2,2)}}$. By computing the inner-products $\langle v_i, v_j \rangle$ of the elements $v_l \in H^{\beta_0}_{0,1}$ explicitly, and using Mathematica to compute the rank of the matrix $(\langle v_i, v_j \rangle)_{i,j}$, we computed the dimension of $H^{\beta_{(0,0)}}_{0,1}$, $H^{\beta_{(3,0)}}_{0,1}$, $H^{\beta_{(0,3)}}_{0,1}$ and $H^{\beta_{(2,2)}}_{0,1}$ and found that $P^{\mathcal{E}^{(8)}}$ did not contain any irreducible modules of lowest weight 1. It should be noted that Mathematica is not an open-source software, and the users have no way of knowing the reliability of results obtained using it. Similarly, by computing the dimensions of $W = H^{\beta_{(0,0)}}_{0,2} \oplus H^{\beta_{(3,0)}}_{0,2} \oplus H^{\beta_{(2,2)}}_{0,2} \oplus H^{\beta_{(0,3)}}_{0,2}$, we find that $\dim(W) = 30$ whilst $\dim(P^{\mathcal{E}^{(8)}}) = 36$, so that the dimension of $W^\perp \cap P^{\mathcal{E}^{(8)}}$ is 6. Then for modules of lowest weight 2, we conjecture

\[
\begin{align*}
P^{\mathcal{E}^{(8)}} &= H^{\beta_{(0,0)}}_{0,2} \oplus H^{\beta_{(3,0)}}_{0,2} \oplus H^{\beta_{(0,3)}}_{0,2} \oplus H^{\beta_{(2,2)}}_{0,2} \oplus H^{(3,0),\varepsilon_1}_{0,2} \oplus H^{(0,3),\varepsilon_1}_{0,2} \oplus H^{(2,2),\gamma_1,\varepsilon_2\varepsilon_1}_{0,2} \oplus H^{(2,2),\gamma_2,\varepsilon_2\varepsilon_1}_{0,2},
\end{align*}
\]

where $\varepsilon_i \in \{\pm 1\}$, $i = 1, 2, 3$, and $\gamma_1, \gamma_2 \in \mathbb{T}$, where the exact values of these six parameters has not yet been determined. This conjecture arises from computing the eigenvalues of the actions of $\rho_{0,2}$, $U_1$ and $\tilde{U}_1$ on $W^\perp \cap P^{\mathcal{E}^{(8)}}$. Each action is a linear transformation, which we computed by hand, and then computed using Mathematica the eigenvalues of the matrix which gives this linear transformation. These eigenvalues are

\[
\begin{align*}
\rho_{0,2} : & \quad 1 \text{ twice, } -1 \text{ four times,} \\
U_1, \tilde{U}_1 : & \quad [4]_0 \delta^{-2} \text{ once, } 0 \text{ five times.}
\end{align*}
\]

The eigenvalues of the actions of these elements on $H^{(2,2),\gamma_{\omega}}, H^{(3,0),\gamma}$ and $H^{(0,3),\gamma}$ are given in the Table \[] 1.

Then we see that $W^\perp \cap P^{\mathcal{E}^{(8)}}$ should contain one copy of both of $H^{(3,0),\varepsilon_1}$ and $H^{(0,3),\varepsilon_1'}, \varepsilon_1, \varepsilon_1' \in \{\pm 1\}$, and since $P^{\mathcal{E}^{(8)}}$ is invariant under conjugation of the graph $\mathcal{E}^{(8)}$, we should
have $\varepsilon_1 = \varepsilon'_1$. Then we need to rank (2, 2) modules of $H_{0,2}^{(2,2),\gamma_1,\omega}$, $H_{0,2}^{(2,2),\gamma_2,\omega}$ such that the action of $\rho_{0,2}$ on both has an eigenvalue $\omega^2 = -1$, i.e. $\omega = \pm i$. Since $P^{E(8)}$ is invariant under complex conjugation, we would either have $\gamma_1, \gamma_2 \in \mathbb{R}$ or else $\gamma_1 = \overline{\gamma}_2$. However, to determine the exact values of $\varepsilon_i$, $i = 1, 2, 3$, and $\gamma_1, \gamma_2$, we would need to consider the action of $\varphi_{(2,2)}$ on $W \perp \cap P_0^{E(8)}$, the computation of which would take many weeks to write down. So we have

$$P^{E(8)} \supset H^{(0,0)} + H^{(3,0)} + H^{(2,2)} + H^{(3,0),\varepsilon_1} + H^{(0,3),\varepsilon_1} + H^{(2,2),\gamma_1,\varepsilon_2} + H^{(2,2),\gamma_2,\varepsilon_3}.$$

Similarly for the graph $D^{(6)}$, we found that $P^{D(6)}$ also contains no irreducible modules of lowest weight 1. Computing the dimensions of $P_{0,2}^{D(6)}$ and $W = H^{(0,0)}_{0,2} + H^{(3,0)}_{0,2}$ as for the $E(8)$ case, we find $\dim(P_{0,2}^{D(6)}) = 16$ and $\dim(W) = 14$. Then the dimension of $W \perp \cap P_{0,2}^{D(6)}$ is 2, and hence $P_{0,2}^{D(6)}$ must either contain one copy of $H^{(3,0),\gamma}$ or else $H^{(2,2),\gamma_1,\omega_1} + H^{(2,2),\gamma_2,\omega_2}$. By considering the action of $\rho_{0,2}$ on $W \perp \cap P_{0,2}^{D(6)}$, we have the eigenvalue 1 twice. Then $W = H^{(2,2),\gamma_1,\omega_1} + H^{(2,2),\gamma_2,\omega_2}$, where $\omega_i^2 = 1$, $i = 1, 2$. Then we see that

$$P^{D(6)} \supset H^{(0,0)} + H^{(3,0)} + H^{(2,2),\gamma_1,\varepsilon_1} + H^{(2,2),\gamma_2,\varepsilon_2},$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$, and either $\gamma_1, \gamma_2 \in \mathbb{R}$ or else $\gamma_1 = \overline{\gamma}_2$. Again, to determine the values of $\varepsilon_i, \gamma_i$, $i = 1, 2$, explicitly requires considering the eigenvalues of the action of $\varphi_{(2,2)}$ on $W \perp \cap P_{0,2}^{D(6)}$.

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**References**

[1] R. E. Behrend, P. A. Pearce, V. B. Petkova and J.-B. Zuber, Boundary conditions in rational conformal field theories, Nuclear Phys. B 579 (2000), 707–773.

[2] D. Bisch, P. Das and S. K. Ghosh, The Planar Algebra of Group-Type Subfactors, 2008. arXiv:0807.4134 [math.OA].
[3] D. Bisch, P. Das and S. K. Ghosh, The Planar Algebra of Diagonal Subfactors, 2008. arXiv:0811.1084 [math.OA].

[4] H. Burgos Soto, The Jones Polynomial and the Planar Algebra of Alternating Links, 2008. arXiv:0807.2600 [math.OA].

[5] D. E. Evans and M. Pugh, Ocneanu Cells and Boltzmann Weights for the SU(3) \(\mathcal{ADE}\) Graphs, Münster J. Math. (to appear).

[6] D. E. Evans and M. Pugh, SU(3)-Goodman-de la Harpe-Jones subfactors and the realisation of SU(3) modular invariants. Rev. Math. Phys. (to appear).

[7] D. E. Evans and M. Pugh, \(A_2\)-Planar Algebras I. Preprint.

[8] D. E. Evans and M. Pugh, Spectral Measures and Generating Series for Nimrep Graphs in Subfactor Theory, Comm. Math. Phys. (to appear).

[9] S. K. Ghosh, Representations of group planar algebras, J. Funct. Anal. 231 (2006), 47–89.

[10] S. K. Ghosh, Planar Algebras: A Category Theoretic Point of View, 2008. arXiv:0712.2904 [math.OA].

[11] A. Guionnet, V. F. R. Jones and D. Shlyakhtenko, Random Matrices, Free Probability, Planar Algebras and Subfactors, 2007. arXiv:0712.2904 [math.OA].

[12] V. P. Gupta, Planar Algebra of the Subgroup-Subfactor, 2008. arXiv:0806.1791 [math.OA].

[13] V. F. R. Jones, The planar algebra of a bipartite graph, in Knots in Hellas ’98 (Delphi), Ser. Knots Everything 24, 94–117, World Sci. Publ., River Edge, NJ, 2000.

[14] V. F. R. Jones, The annular structure of subfactors, in Essays on geometry and related topics, Vol. 1, 2, Monogr. Enseign. Math. 38, 401–463, Enseignement Math., Geneva, 2001.

[15] V. F. R. Jones, Planar algebras. I, New Zealand J. Math., to appear.

[16] V. F. R. Jones and S. A. Reznikoff, Hilbert space representations of the annular Temperley-Lieb algebra, Pacific J. Math. 228 (2006), 219–249.

[17] V. F. R. Jones, D. Shlyakhtenko and K. Walker, An Orthogonal Approach to the Subfactor of a Planar Algebra, 2008. arXiv:0807.4146 [math.OA].

[18] V. Kodiyalam and V. S. Sunder, On Jones’ planar algebras, J. Knot Theory Ramifications 13 (2004), 219–247.

[19] V. Kodiyalam and V. S. Sunder, From Subfactor Planar Algebras to Subfactors, 2008. arXiv:0807.3704 [math.OA].

[20] G. Kuperberg, Spiders for rank 2 Lie algebras, Comm. Math. Phys. 180 (1996), 109–151.
[21] S. Morrison, E. Peters and N. Snyder, Skein Theory for the $D_{2n}$ Planar Algebras, 2008. arXiv:0808.0764 [math.OA].

[22] A. Ocneanu, Paths on Coxeter diagrams: from Platonic solids and singularities to minimal models and subfactors. (Notes recorded by S. Goto), in Lectures on operator theory, (ed. B. V. Rajarama Bhat et al.), The Fields Institute Monographs, 243–323, Amer. Math. Soc., Providence, R.I., 2000.

[23] A. Ocneanu, Higher Coxeter Systems (2000). Talk given at MSRI. http://www.msri.org/publications/ln/msri/2000/subfactors/ocneanu

[24] A. Ocneanu, The classification of subgroups of quantum $SU(N)$, in Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000), Contemp. Math. 294, 133–159, Amer. Math. Soc., Providence, RI, 2002.

[25] M. Pugh, The Ising Model and Beyond, PhD thesis, Cardiff University, 2008.

[26] S. A. Reznikoff, Temperley-Lieb planar algebra modules arising from the $ADE$ planar algebras, J. Funct. Anal. 228 (2005), 445–468.

[27] H. Wenzl, Hecke algebras of type $A_n$ and subfactors, Invent. Math. 92 (1988), 349–383.