SMOOTHNESS OF LOEWNER SLITS

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Abstract. In this paper, we show that the chordal Loewner differential equation with $C^\delta$ driving function generates a $C^{\delta/2}$ slit for $\frac{1}{2} < \delta \leq 2$, except when $\delta = \frac{3}{2}$ the slit is only proved to be weakly $C^{1,1}$.

1. Introduction

The Loewner differential equation is a classical tool in complex analysis which has been successfully applied to various extremal problems, including the famous de Branges theorem (see [2]). In recent years, the Schramm-Loewner Evolution (SLE) has been extensively studied by mathematicians and physicists. One can think of SLE as a random curve in the upper half-plane, which is generated via the Loewner differential equation with a random driving function. Currently, some natural questions on the deterministic side of an SLE are still open. In this paper, we investigate the smoothness of slits generated by $C^\delta$ driving functions.

Given a slit (Definition 2.1 in §2) $\gamma: [0, T] \to \mathbb{H}$ in the upper half-plane $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \}$, the region $H_t := \mathbb{H} \setminus \gamma([0, t])$ is simply connected for each $t$. There is a unique conformal map $g_t: H_t \to \mathbb{H}$ satisfying the hydrodynamic normalization

$$g_t(z) = z + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \cdots$$

as $z \to \infty$. Figure 1 illustrates the situation. The coefficient $a_1(t)$ is called the half-plane capacity of the set $K_t = \gamma([0, t])$. See [16] for a geometric interpretation of half-plane capacity and its relation to conformal radius, and see [7] for a probabilistic approach. Although it is not immediate from the definition, it is routine to show that $a_1(t)$ is a continuous and strictly increasing real-valued function with $a_1(0) = 0$. If the slit is reparametrized so that $a_1(t) = 2t$, then $g_t(z)$ is differentiable in $t$ and satisfies the chordal Loewner differential equation

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} g(t, z) = \frac{2}{g(t, z) - \lambda(t)}, \\
g(0, z) = z
\end{array} \right.$$  

(1)

for $z \in H_t$, where $\lambda: [0, T] \to \mathbb{R}$ is a continuous function called the driving function for the slit. Moreover, $\lambda(t) = g_t(\gamma(t))$ is the image of the tip under the conformal map.

The foregoing procedure can be reversed. Suppose we are given some continuous function $\lambda: [0, T] \to \mathbb{R}$. For each $t \in [0, T]$, let $H_t$ be the set of points $z \in \mathbb{H}$ for which the solution of (1) is well-defined up to time $t$, i.e. $g(s, z) \neq \lambda(s)$ for $s \in [0, t]$. One can show that $H_t$ is a simply connected region and $z \mapsto g(t, z)$...
A slit $\gamma(t)$ and its driving function $\lambda(t)$ are related by a conformal map. The set $K_t := \mathbb{H} \setminus H_t$ is in general not a slit. Kufarev [6] constructed an example, in the classical (radial) setting, for which a continuous driving function does not generate a slit. The example can also be found in [3, §3.4]. Even if the driving function is in Lip($\frac{1}{2}$), also known as $\frac{1}{2}$-Holder continuous, the set $K_t$ may not be locally connected (see [12] for an example).

Throughout this paper, we assume $\lambda: [0, T] \to \mathbb{R}$ is Lip($\frac{1}{2}$), i.e.

$$
\|\lambda\|_{\text{Lip}(\frac{1}{2})} := \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda(t_1) - \lambda(t_2)|}{|t_1 - t_2|^\frac{1}{2}} < \infty.
$$

In 2005, Marshall and Rohde [12] showed that there is an absolute constant $c_0 > 0$ so that for any $\lambda: [0, T] \to \mathbb{R}$ with $\|\lambda\|_{\text{Lip}(\frac{1}{2})} < c_0$, the Loewner equation (1) generates a quasi-slit in the upper half-plane $\mathbb{H}$ and the slit meets $\mathbb{R}$ non-tangentially. Lind [11] proved that all these statements hold for $c_0 = 4$, and this constant is the largest possible. In an unpublished paper [15] Steffen Rohde, Huy Vo Tran and Michel Zinsmeister give a sufficient condition for the driving function to generate a rectifiable curve.

What more can we say if $\lambda: [0, T] \to \mathbb{R}$ is more regular (smooth) than Lip($\frac{1}{2}$)? In [1, page 59] it was proved that if $\lambda: [0, T] \to \mathbb{R}$ has bounded first derivative, then its slit is $C^1$. (The original statement was a radial version. Here we formulate it in the chordal setting.) As of the writing of this paper and to the author’s knowledge, it is the only result in the literature concerning the smoothness of a slit generated by a driving function more regular than Lip($\frac{1}{2}$). Marshall and Rohde [12] implicitly suggest the following.

**Main Statement** (heuristic version). $\lambda \in C^\beta$ implies $\gamma \in C^{\beta + \frac{1}{2}}$ for $\beta > \frac{1}{2}$.

In this paper, we will prove this statement for $\frac{1}{2} < \beta \leq 2$, except when $\beta = \frac{3}{2}$ the slit $\gamma$ is only proved to be weakly $C^{1,1}$. The precise statements are in Theorem 4.8 and Theorem 5.2 and Theorem 6.2 corresponding to the cases $\beta \in (\frac{1}{2}, 1]$, $\beta \in (1, \frac{3}{2}]$ and $\beta \in (\frac{3}{2}, 2]$. One of the key ingredients of our method is the Lipschitz continuity (Theorem 3.4 below) of the map $\lambda \mapsto \gamma^\lambda$, which was only known to be continuous [10, Theorem 4.1]. Another ingredient is an integral representation of $\gamma'(t)$; see Corollary 4.3.

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1. The theorems in [12] were proved in the radial case. In a private communication, Don Marshall translated (with rigorous proof) the results to the chordal case.

2. By definition, a *quasi-slit* is a slit satisfying the Ahlfors three-point condition, i.e. there is a constant $L \geq 1$ such that for all points $z_1, z_2, z_3$ on the slit in that order, $|z_1 - z_2| + |z_2 - z_3| \leq L |z_1 - z_3|$.
Theorem 3.4 (Lipschitz continuity). Suppose $\lambda_1, \lambda_2 : [0, T] \to \mathbb{R}$ satisfy $\|\lambda_j\|_{\text{Lip}(\frac{1}{2})} \leq 1$ for $j = 1, 2$. Then $\|\gamma^{\lambda_1} - \gamma^{\lambda_2}\|_\infty \leq c \|\lambda_1 - \lambda_2\|_\infty$, where $c > 0$ is an absolute constant.

There is another natural and interesting question which we won’t discuss in this paper but we mention for the sake of completeness. If we know that a slit $\gamma$ is $C^n$, how smooth is its driving function? Earle and Epstein [4] answered this question in 2001 for the radial case (and the chordal case follows). Suppose $0 \in \Omega \subseteq \mathbb{C}$ is a simply connected region and $\gamma : (0, T] \to \Omega$ is a slit avoiding the origin with base point $\gamma(0) \in \partial \Omega$. Let $R(t)$ be the conformal radius of $\Omega_t := \Omega \setminus \gamma((0, T])$ with respect to the origin. Earle and Epstein showed that if $\gamma$ is $C^n$ regular on $(0, T]$ for some integer $n \geq 2$, then the radial capacity $a(t) := -\log R(t)$ is $C^{n-1}$ on $(0, T]$. Moreover, if the slit is reparametrized so that $a(t) = a(0) + t$, then its driving function $\lambda$ is $C^{n-1}$ on $(a(0), a(T))$. See [4] for the precise definitions and statements. In the same paper, it was also proved that real analytic slits generate real analytic driving functions.

2. Definitions, notation and preliminaries

General notation/convention.

(i) $a(\varepsilon) \lesssim b(\varepsilon)$ means $a(\varepsilon) \leq C b(\varepsilon)$ for some constant $C > 0$ (independent of $\varepsilon$).

(ii) $a(\varepsilon) \asymp b(\varepsilon)$ means $a(\varepsilon) \lesssim b(\varepsilon)$ and $b(\varepsilon) \lesssim a(\varepsilon)$.

(iii) $a(\varepsilon) \preccurlyeq b(\varepsilon)$ as $\varepsilon \to 0$ if $\frac{a(\varepsilon)}{b(\varepsilon)}$ has a positive and finite limit.

(iv) In this paper, the lowercase $c$ is reserved to denote an absolute constant which may vary even in a single chain of equalities.

Definition 2.1. A slit in $\mathbb{H}$ is a simple curve $\gamma : [0, T] \to \mathbb{H}$ with $\gamma(0) \in \mathbb{R}$ and $\gamma(t) \in \mathbb{H}$ for $0 < t \leq T$.

All driving functions $\lambda : [0, T] \to \mathbb{R}$ in this paper satisfy

$$\|\lambda\|_{\text{Lip}(\frac{1}{2})} := \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda(t_1) - \lambda(t_2)|}{|t_1 - t_2|^{\frac{1}{2}}} < 4$$

(at least locally) and therefore generate slits by [12] and [11]. We will use the following notation frequently.

Notation 2.2.

(i) The Lip($\frac{1}{2}$)-norm\footnote{Strictly speaking, $\|\lambda\|_{\text{Lip}(\frac{1}{2})}$ is only a semi-norm.} of $\lambda : [0, T] \to \mathbb{R}$ is denoted by

$$\|\lambda\|_{\text{Lip}(\frac{1}{2}, [0, T])} := \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda(t_1) - \lambda(t_2)|}{|t_1 - t_2|^{\frac{1}{2}}}.$$ Usually, we write $\|\lambda\|_{\text{Lip}(\frac{1}{2})}$ instead of $\|\lambda\|_{\text{Lip}(\frac{1}{2}, [0, T])}$.

(ii) For a positive integer $n \in \mathbb{N}$ and $0 < \alpha \leq 1$, the $C^{n, \alpha}$-norm of $\lambda : [0, T] \to \mathbb{R}$ is

$$\|\lambda\|_{C^{n, \alpha}} = \|\lambda\|_{C^{n, \alpha}([0, T])} := \sum_{k=0}^{n} \sup_{t \in [0, T]} |\lambda^{(k)}(t)| + \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda^{(n)}(t_1) - \lambda^{(n)}(t_2)|}{|t_1 - t_2|^{\alpha}}.$$
For a slit \( \gamma \), its \( C^{n,\alpha} \)-norm \( \| \gamma \|_{C^{n,\alpha}} \) is defined similarly. If \( \beta > 1 \) is not an integer, the notation \( C^\beta \) refers to \( C^{[\beta],\beta-[\beta]} \), where \([\beta]\) is the integer part of \( \beta \). For example, \( C^{1,\frac{1}{2}} \) is the same as \( C^{1,\frac{1}{2}} \).

(iii) \( \gamma^\lambda: [0, T] \to \mathbb{H} \) denotes the slit generated by \( \lambda: [0, T] \to \mathbb{R} \). When no confusion can occur, we write \( \gamma \) instead of \( \gamma^\lambda \) for the sake of notation. The base of \( \gamma \) is \( \gamma(0) = \lambda(0) \in \mathbb{R} \).

(iv) For each \( t \in [0, T] \), \( g_t: H_t \to \mathbb{H} \) denotes the (unique) conformal map from \( H_t = \mathbb{H} \setminus \gamma([0, t]) \) onto the upper half-plane \( \mathbb{H} \) satisfying the normalization

\[
g_t(z) = z + \frac{a_1(t)}{z} + \cdots
\]

as \( z \to \infty \). All slits in this paper are parametrized by half-plane capacity, i.e. \( a_1(t) = 2t \). Alternative notation such as \( g(t, z) \) or \( g_t^\lambda(z) \) may be used interchangeably.

(v) \( f_t: \mathbb{H} \to H_t \) is the inverse function of \( g_t \), i.e. \( g_t(f_t(z)) = z \) for all \( z \in \mathbb{H} \). We sometimes write \( f(t, z) \) or \( f_t^\lambda(z) \).

(vi) For \( 0 \leq s < t \leq T \), we define \( \gamma(s, t) := g_s(\gamma(t)) - \lambda(s) \). To be flexible it may also be written as \( \gamma(s, t) \) or \( \gamma^\lambda(s, t) \). The normalized version of \( \gamma(s, t) \) is \( \tau(s, t) := \gamma(s, t) - \sqrt{\frac{s-t}{s}} \). Note that \( \tau(s, t) = 2i\) if and only if \( \lambda \) is constant on \([s, t]\).

(vii) In \( \mathbb{H} \) we introduce the notation

\[
L(s) = L^\lambda(s) = \int_0^s \left[ \frac{1}{2} + \frac{2}{\tau(s-u, s)^2} \right] du
\]

and show that \( \gamma'(s) = \frac{i}{\sqrt{s}} e^{L(s)} \) under appropriate assumptions.

**Definition 2.3.**

(i) We say that \( \lambda: [0, T] \to \mathbb{R} \) satisfies the \( \sigma - \text{Lip}(\frac{1}{2}) \) condition if \( \sigma \geq 0 \) and

\[
|\lambda(t_1) - \lambda(t_2)| \leq \sigma |t_1 - t_2|^{\frac{1}{2}}
\]

for all \( t_1, t_2 \in [0, T] \).

(ii) We say that \( \lambda: [0, T] \to \mathbb{R} \) satisfies the \( (M, T, \delta) - \text{Lip}(\frac{1}{2} + \delta) \) condition if \( \| \lambda \|_{\text{Lip}(\frac{1}{2} + \delta)} \leq 1 \) and

\[
|\lambda(t_1) - \lambda(t_2)| \leq M |t_1 - t_2|^{\frac{1}{2} + \delta}
\]

for all \( t_1, t_2 \in [0, T] \), where \( M, T, \delta > 0 \).

(iii) We say that \( \lambda: [0, T] \to \mathbb{R} \) satisfies the \( (M, T, n, \alpha) - C^{n,\alpha} \) condition if \( \| \lambda \|_{\text{Lip}(\frac{1}{2} + \delta)} \leq 1 \), \( \lambda \in C^{n,\alpha} \) on \([0, T]\) and \( \| \lambda \|_{C^{n,\alpha}} \leq M \), where \( n \in \mathbb{N} \), \( 0 < \alpha \leq 1 \) and \( M > 0 \).

We will not consider \( \text{Lip}(\frac{1}{2} + \delta) \) driving functions until \( \mathbb{H} \). As a remark on the terminologies, careful readers may see that in (i) we do not make an explicit reference to \( T \) but we do in (ii). This is because \( \text{Lip}(\frac{1}{2} + \delta) \)-norm is not invariant under Brownian scaling. The terminologies reflect that all quantitative estimates in \( \mathbb{H} \) depend only on \( M, T \) and \( \delta \), while in \( \mathbb{R} \) our estimates are mostly in terms of \( \sigma \).

In this paper, we use the diagram in Figure 2 to represent a situation that \( \gamma(t) \) and \( \lambda(t) \) are related by the Loewner equation.

\[\text{For any } r > 0, \text{ the function } \lambda(t) = r^{-2}\lambda(rt) \text{ satisfies } \| \lambda \|_{\text{Lip}(\frac{1}{2})} = \| \lambda \|_{\text{Lip}(\frac{1}{2})}. \] The analog statement is not true for the \( \text{Lip}(\frac{1}{2} + \delta) \)-norm.
For any continuous driving function $\lambda : [0, T] \to \mathbb{R}$, the solution $g_t(z)$ of (1) satisfies

$$\log g_t'(z) = -\int_0^t \frac{2}{g_u(z) - \lambda(u)}^2 du,$$

(2)

$$g''_t(z) = 4g'_t(z) \int_0^t \frac{g_u(z)}{[g_u(z) - \lambda(u)]^3} du,$$

(3)

for all $z \in H_t$. Equality (2) can be derived easily if we differentiate (1) with respect to $z$, which gives

$$\frac{\partial}{\partial t} g_t'(z) = -\frac{2g_t'(z)}{[g_t(z) - \lambda(t)]^2}.$$

To prove (3), we differentiate (2) with respect to $z$. We comment that (2) can be used to estimate the size of $|g'_s(\gamma(s + \varepsilon))|$ as $\varepsilon \downarrow 0$, and this kind of estimate is crucial in our work as well as in other SLE problems. Equality (3) will be useful if one wants to obtain second derivative estimates near the tip.

Equalities (2) and (3) hold for any continuous driving function $\lambda : [0, T] \to \mathbb{R}$. So far we haven’t made any smoothness assumption on $\lambda$. We are going to do it in the coming sections.

3. When $\lambda \in \text{Lip}(\frac{1}{2})$

We begin by stating some useful facts.

Fact 3.1. Suppose $\lambda : [0, T] \to \mathbb{R}$ satisfies $\|\lambda\|_{\text{Lip}(\frac{1}{2})} < 4$.

(a) (Scaling property) If we define $\tilde{\lambda} : [0, 1] \to \mathbb{R}$ by $\tilde{\lambda}(s) := \frac{1}{\sqrt{T}} [\lambda(sT) - \lambda(0)]$, then $\|\tilde{\lambda}\|_{\text{Lip}(\frac{1}{2})} = \|\lambda\|_{\text{Lip}(\frac{1}{2})}$, and for all $s \in [0, 1],

$$\gamma^{\tilde{\lambda}}(s) = \frac{1}{\sqrt{T}} [\gamma^{\lambda}(sT) - \gamma^{\lambda}(0)].$$

For example, suppose a slit $\gamma$ is parametrized by half-plane capacity and $\lambda$ is the driving function for $\gamma$. The half-plane capacity reparametrization of the slit $3\gamma(t)$ is $\tilde{\gamma}(t) = 3\gamma^{\frac{1}{6}}(t)$. The scaling property says that the driving function for $\tilde{\gamma}$ is $\lambda(t) = 3\lambda^{\frac{1}{6}}(t)$.

(b) (Stationary property) For any $s \in (0, T)$, the time shift $\lambda_s : [0, T - s] \to \mathbb{R}$ of $\lambda$ is the function $\lambda_s(u) := \lambda(s + u)$. The corresponding slit is

$$\gamma^{\lambda_s}(u) = g_s(\gamma(s + u)).$$

See Figure 3 for an illustration.

(c) For any $t \in [0, T],

$$\gamma^{\lambda}(t) = \lim_{y \downarrow 0} f_t(\lambda(t) + iy) = \lambda(t) - \int_0^t \frac{2}{\gamma(t - u, t)} du.$$
The stationary property states that the above diagram commutes.

The scaling property is extremely useful; in many situations it suffices to work only on the case $T = 1$. The proofs of the scaling property and stationary property are elementary exercises. A proof of (4) is given below. We remark that the first equality in (4) is a non-trivial result for SLE curves, whose driving functions are random and almost surely not Lip($\frac{1}{2}$) (see [14], [9]).

**Proof of (4).** We know from [12] that $\gamma([0, t])$ is a slit (in particular, locally connected), and it follows from the Carathéodory continuity theorem (see, for example, [13, Theorem 2.1]) that the conformal map $f_t: \mathbb{H} \to H_t$ is continuous at the boundary point $\lambda(t)$. This proves the first equality in (4). The second equality is an immediate consequence of the Loewner differential equation (1) and the fundamental theorem of calculus applied to the function $u \mapsto g_u(\gamma(t))$ ($0 \leq u \leq t$).

For $0 \leq \sigma < 4$, let $X_\sigma$ be the space of all (continuous) functions $\lambda: [0, 1] \to \mathbb{R}$ satisfying $\lambda(0) = 0$ and $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq \sigma$. Under the supremum norm $\|\cdot\|_{\infty}$, the metric space $X_\sigma$ is compact. It is known [10] that the map $X_\sigma \to \mathbb{H}$ defined by $\lambda \mapsto \gamma^\lambda(1)$ is continuous. It follows that $E_\sigma = \{\gamma^\lambda(1): \lambda \in X_\sigma\}$ is a compact subset of $\mathbb{H}$.

By the scaling property,

$$\gamma^\lambda(t) \in \sqrt{t} E_\sigma$$

for any $0 \leq t \leq 1$, $\lambda \in X_\sigma$, $0 \leq \sigma < 4$. On the other hand, it is easy to show (using the compactness argument) that $E_\sigma$ shrinks to a singleton $\{2i\}$ as $\sigma \to 0$. Our first question is: at what rate does the diameter of $E_\sigma$ go to zero?

**Lemma 3.2.** Suppose $\lambda: [0, 1] \to \mathbb{R}$ satisfies the $\sigma - \text{Lip}(\frac{1}{2})$ condition with $0 \leq \sigma < 4$ and $\lambda(0) = 0$. Then

$$|\text{Re} \gamma^\lambda(1)| \leq \sigma \quad \text{and} \quad 4 - \sigma^2 \leq |\text{Im} \gamma^\lambda(1)|^2 \leq 4.$$

In particular, $\text{diam}(E_\sigma) \leq c \sigma$ for all $0 \leq \sigma < 4$, where $c > 0$ is an absolute constant.
Proof. Write $\gamma(t) = \lambda^\gamma t$ for the sake of notation. The estimate $|\text{Re} \gamma(1)| \leq \sigma$ follows from the simple observation that if $a \leq \lambda(t) \leq b$ for all $t \in [0, 1]$, $z_0 \in \mathbb{H}$ and $\text{Re}(z_0) > b$ (respectively $\text{Re}(z_0) < a$), then the Loewner differential equation (1) implies that $g_t(z_0)$ is defined and satisfies $\text{Re} g_t(z_0) > b$ (respectively $\text{Re} g_t(z_0) < a$) for all $t \in [0, 1]$.

To estimate $\text{Im} \gamma(1)$, we use the fact that $
abla_t^\gamma(t) = \lim_{y \downarrow 0} h_t^\gamma(t + iy),$ where $h_t^\gamma(z)$ is the solution to the initial value problem

\begin{equation}
\begin{cases}
\dot{h}_t(z) = -\frac{2}{h_t(z) - \lambda(1-t)}, \\
h_0(z) = z.
\end{cases}
\end{equation}

Fix any $y > 0$ and write $h_t^\gamma(t + iy) = x_t + iy_t$ ($x_t, y_t \in \mathbb{R}$). Let $A_t = (x_t - \lambda(1-t))^2$ and $B_t = y_t^2$. By scaling, or by our argument in the beginning of the proof, $A_t \leq \sigma^2 t$ for all $0 \leq t \leq 1$. Comparing the imaginary parts of (5), we have

$$\dot{B}_t = \frac{4B_t}{A_t + B_t}.$$ 

Since $A_t \geq 0$, $\dot{B}_t \leq 4$, and therefore $B_t \leq 4 + y$. We have $\text{Im} h_t^\gamma(t + iy) \leq \sqrt{4 + y}$. Letting $y \downarrow 0$ gives $\text{Im} \gamma(1) \leq 2$.

For the lower bound of $B_1$, we assume without loss of generality that $a := 4 - \sigma^2 > 0$. (Otherwise, the lower bound is trivial.) Suppose for the sake of contradiction that $B_t < at$ for some $t \in [0, 1)$. Let $T = \inf\{t \geq 0 : B_t < at\}$. We have $0 < T < 1$, $B_T = aT$, and, for $0 \leq t \leq T$,

$$\dot{B}_t = \frac{4B_t}{A_t + B_t} \geq \frac{4at}{\sigma^2 t + at} = a.$$ 

This shows that $B_T \geq B_0 + aT > aT$, which is a contradiction. We have proved that $B_1 \geq 4 - \sigma^2$. \hfill $\square$

Consider the example $\lambda(t) = \sigma (1 - \sqrt{1-t})$. When $0 < \sigma < 4$, this driving function satisfies the $\sigma - \text{Lip}(\frac{1}{2})$ condition and generates a logarithmic spiral with tip

$$\gamma(1) = \frac{\sigma}{2} + i\sqrt{4 - \frac{\sigma^2}{4}}.$$
This example together with Lemma 3.2 show
\[ c_1 \sigma \leq \text{diam}(E_\sigma) \leq c_2 \sigma \]
for all \(0 < \sigma < 4\), where \(c_1, c_2 > 0\) are absolute constants.

The compactness of \(E_\sigma\) has a simple geometric consequence. If \(\lambda(0) = 0\) and
\(\sigma = \|\lambda\|_{\text{Lip}(\frac{1}{2})} < 4\), by scaling we see that \(\gamma(t) \in \sqrt{t}E_\sigma\) and the slit \(\gamma\) is contained
in a cone whose angle depends on \(\sigma\). If \(\lambda \in \text{Lip}(\frac{1}{2} + \delta)\), one has \(\gamma(t) \in \sqrt{t}E_{\sigma(t)}\) with \(\sigma(t) \lesssim t\delta\) as \(t \to 0\). The slit grows vertically. See Figure 5.

**Figure 5.** One important difference between the slits of \(\text{Lip}(\frac{1}{2})\)
and \(\text{Lip}(\frac{1}{2} + \delta)\) driving functions.

In [14] we will show that
\[
\gamma'(t) = \lim_{\varepsilon \downarrow 0} \frac{i}{\sqrt{\varepsilon}g'_{t-\varepsilon}(\gamma(t))} = \lim_{\varepsilon \downarrow 0} \frac{i}{\sqrt{\varepsilon}g'_{t-\varepsilon}(\gamma(t + \varepsilon))}
\]
under an appropriate smoothness assumption on \(\lambda\). We will see that \(|g'_{t-\varepsilon}(\gamma(t + \varepsilon))| \simeq \varepsilon^{-\frac{1}{2}}\) as \(\varepsilon \downarrow 0\) is more or less equivalent to the existence of \(\gamma'(t)\). Of course, in this section we are still in the \(\text{Lip}(\frac{1}{2})\) case and do not expect \(\gamma'(t)\) to be differentiable. The next lemma says that \(|g'_{t-\varepsilon}(\gamma(t))| \asymp \varepsilon^{-\frac{1}{2} + O(\sigma)}\), with an error term in the exponent.

**Lemma 3.3.** If \(\lambda: [0, T] \to \mathbb{R}\) satisfies the \(\sigma - \text{Lip}(\frac{1}{2})\) condition for some \(\sigma \in [0, 1]\),
then for any \(0 < s < t \leq T\),
\[
\left(\frac{t}{t-s}\right)^{\frac{1}{2} - c\sigma} \leq |g'(s)| \leq \left(\frac{t}{t-s}\right)^{\frac{1}{2} + c\sigma},
\]
where \(c > 0\) is an absolute constant.

**Proof.** Without loss of generality, we assume \(s = 1\) and \(\lambda(0) = 0\). Let \(w = \gamma(t)\). Then [24] gives
\[ \log g_1(w) = \int_0^1 \frac{2}{\tau(u,t)^2} \frac{du}{t-u}, \]
where \(\tau(u,t) = \frac{g_s(w) - \lambda(u)}{\sqrt{t-u}}\) was defined in Notation 2.2 [16] in [24]. If the driving function \(\lambda\) is identically zero, \(\tau^0(u,t)\) becomes \(\tau^0(u,t) \equiv 2i\) and the above equality reduces to
\[ \frac{1}{2} \log \frac{t}{t-1} = \int_0^1 \frac{1}{2} \frac{du}{t-u}. \]
Subtracting the two equalities gives

\begin{equation}
\log g'_1(w) - \frac{1}{2} \log \frac{t}{t-1} = - \int_0^1 \left[ \frac{1}{2} + \frac{2}{\tau(u,t)w^2} \right] \frac{du}{t-u}.
\end{equation}

By Lemma 3.2 \[ \frac{1}{2} \log \frac{t}{t-1} \leq c\sigma, \] where \( c > 0 \) is an absolute constant. Here we have implicitly used the condition \( \sigma \leq 1 \), which guarantees that \( \tau(u,t) \) stays in a fixed compact set \( E_1 \subseteq \mathbb{H} \). The absolute constant \( c \) in our last estimate is related to the derivative bound of the map \( z \mapsto \frac{1}{2} \) on the compact set \( E_1 \).

Finally, equation (6) gives

\[ \left| \log g'_1(w) - \frac{1}{2} \log \frac{t}{t-1} \right| \leq \int_0^1 \frac{c\sigma}{u-t} \, du = c\sigma \log \frac{t}{t-1}. \]

\[ \square \]

Lemma 3.2 and the following Theorem 3.4 will serve as two fundamental tools for the rest of this paper.

**Theorem 3.4** (Lipschitz continuity). Suppose \( \lambda, \widetilde{\lambda}: [0,T] \rightarrow \mathbb{R} \) satisfy the \( \sigma - \text{Lip}(\frac{1}{2}) \) condition for \( \sigma = 1 \). Then, \( \| \gamma^\lambda - \gamma^{\widetilde{\lambda}} \|_\infty \leq c \| \lambda - \widetilde{\lambda} \|_\infty \), where \( c > 0 \) is an absolute constant.

Fix any \( T > 0 \). Let \( \widetilde{X}_\sigma \) be the space of all (continuous) \( \lambda: [0,T] \rightarrow \mathbb{R} \) with \( \| \lambda \|_{\text{Lip}(\frac{1}{2})} \leq \sigma \). Recently, Joan Lind, Don Marshall and Steffen Rohde proved [10] Theorem 4.1 that the map \( \lambda \mapsto \gamma^\lambda \) is a continuous map from \( (\widetilde{X}_\sigma, \| \cdot \|_\infty) \) into \( (C([0,T]), \| \cdot \|_\infty) \) for every \( 0 \leq \sigma < 4 \). Their proof uses the theory of quasi-conformal maps. When \( \sigma \leq 1 \), Theorem 3.4 says the map is Lipschitz continuous.

For \( \sigma = 1 \), the slit \( \gamma^\lambda \) of \( \lambda \in X_\sigma \) is contained in the cone \( V = \{ z \in \mathbb{H} : \frac{1}{\pi} < \arg(z) < \frac{3\pi}{4} \} \). Theorem 3.4 remains true (with a slightly larger absolute constant \( c \)) if the constant 1 is replaced by a slightly larger number where the slit is still contained in \( V \). We do not know whether Theorem 3.4 holds for \( \sigma = 4 - \varepsilon \) when \( \varepsilon > 0 \) is small.

**Proof.** By scaling we can assume \( T = 1 \). Let \( \varepsilon := \sup_{0 \leq t \leq 1} |\lambda(t) - \widetilde{\lambda}(t)| \). Without loss of generality, we can further assume \( \lambda(1) = \widetilde{\lambda}(1) \). (If not, translate one of the slits by \( \lambda(1) - \widetilde{\lambda}(1) \), which has absolute value at most \( \varepsilon \).) We extend \( \lambda \) so that \( \lambda(t) = \lambda(1) \) for all \( t \geq 1 \). Fix any small \( \delta > 0 \). The tip \( \gamma^\lambda(1 + \delta) \) is equal to \( h_1 \), where \( h_1: [0,1] \rightarrow \mathbb{C} \) is the solution of the backward Loewner differential equation

\[ \begin{cases} 
\partial_t h_t = -\frac{2}{h_t - \xi(t)}, \\
h_0 = \xi(0) + 2i\sqrt{\delta},
\end{cases} \]

and \( \xi(t) := \lambda(1 - t) \). Similarly, we extend \( \widetilde{\lambda} \), define \( \widetilde{\xi}(t) \), \( \widetilde{h}_t \) and let \( Y(t) = h_t - \widetilde{h}_t \). Note that \( Y(1) = \gamma^\lambda(1 + \delta) - \gamma^{\widetilde{\lambda}}(1 + \delta) \) and \( Y(0) = 0 \) since \( \lambda(1) = \widetilde{\lambda}(1) \). By direct computation,

\begin{equation}
\partial_t Y(t) = A(t) \left[ Y(t) + (\widetilde{\xi}(t) - \xi(t)) \right],
\end{equation}
Lemma 4.1. If $improve the exponent in Lemma 3.3.

Notice that $A(t) = 2(h_t - \xi(t))^{-1}(h_t - \bar{\xi}(t))^{-1}$. We view (7) as a first order linear ODE in $Y(t)$ and solve it using the method of integrating factor. Let $\mu(t) = \exp\left(-\int_0^t A(s) \, ds\right)$. One has $\frac{d}{dt} [\mu(t)Y(t)] = \mu(t)A(t) \left(\bar{\xi}(t) - \xi(t)\right)$ and

$$Y(1) = \int_0^1 \frac{\mu(s)}{\mu(1)} A(s) \left(\bar{\xi}(s) - \xi(s)\right) \, ds.$$ 

We know $|\bar{\xi}(s) - \xi(s)| \leq \varepsilon$ for all $s \in [0, 1]$. To complete the proof, it remains to estimate the size of the integrating factor $\mu(t)$.

Notice that $A(t) \in \frac{1}{1+\delta} K$ for all $t \in [0, 1]$, where

$$K = \left\{ \frac{2}{|z_1 - z_2|} : z_1, z_2 \in E_1 \right\}$$

and $E_1$ is the compact set defined right before Lemma 3.2 (see Figure 4). For the convenience of the readers, we recall the definition

$$E_1 := \left\{ \gamma^\lambda(1) : \lambda(0) = 0 \text{ and } \|\lambda\|_{\text{Lip}(\frac{1}{2},[0,1])} \leq 1 \right\}.$$ 

By Lemma 3.2 and the assumption $\sigma = 1$, $K$ is contained in the left half-plane $\{z \in \mathbb{C} : \text{Re}(z) < 0\}$. Let

$$\beta = \inf\{-\text{Re}(z) : z \in K\} > 0.$$ 

Since $A(t) \in \frac{1}{1+\delta} K$, we have $-\text{Re}A(t) \geq \frac{\beta}{1+\delta}$ for all $t \in [0, 1]$. For any $s \in [0, 1],

$$\frac{\mu(s)}{\mu(1)} = \exp\left[\int_s^1 A(u) \, du\right] \quad \text{and} \quad \left|\frac{\mu(s)}{\mu(1)}\right| \leq (s + \delta)^\beta.$$ 

Finally,

$$\left|\gamma^\lambda(1+\delta) - \gamma^\bar{\lambda}(1+\delta)\right| = |Y(1)| \leq \varepsilon \int_0^1 \left|\frac{\mu(s)}{\mu(1)}\right| |A| \, ds \leq c \varepsilon \int_0^1 (s + \delta)^{-1+\beta} \, ds,$$

where $c = \sup_{z \in K} |z| < \infty$ is an absolute constant. The result follows by letting $\delta \to 0$. 

4. When $\lambda \in \text{Lip}(\frac{1}{2} + \delta)$ with $0 < \delta \leq \frac{1}{2}$

In this section, $\lambda : [0, T] \to \mathbb{R}$ satisfies the $(M, T, \delta)$-$\text{Lip}(\frac{1}{2} + \delta)$ condition, i.e. $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$ and

$$|\lambda(t_1) - \lambda(t_2)| \leq M |t_1 - t_2|^{\frac{1}{2} + \delta}$$

for all $t_1, t_2 \in [0, T]$, where $M, T, \delta > 0$. The extra smoothness allows us to improve the exponent in Lemma 3.2.

**Lemma 4.1.** If $\lambda : [0, T] \to \mathbb{R}$ satisfies the $(M, T, \delta)$-$\text{Lip}(\frac{1}{2} + \delta)$ condition for some $0 < \delta \leq \frac{1}{2}$, then for any $0 < s < t \leq T,$

$$(8) \quad \frac{1}{C} \leq \sqrt[\varepsilon]{\frac{t - s}{t} |g_\varepsilon'(\gamma(t))|} \leq C,$$

where $C = C(M, T, \delta) > 0$. Moreover, for all $s \in (0, T)$, the limit

$$\lim_{\varepsilon \downarrow 0} \sqrt{\varepsilon} g_\varepsilon'(\gamma(s + \varepsilon)) = \sqrt{s} \exp\left\{- \int_0^s \left[\frac{1}{2u} + \frac{2}{\gamma(s - u, s)}\right] \, du\right\}$$

exists and is non-zero.
Proof. As in the proof of Lemma 3.3, 
\[
\log g_s'(\gamma(t)) - \frac{1}{2} \log \frac{t}{t-s} = - \int_0^s \left[ \frac{1}{2} + \frac{2}{\tau(u,t)^2} \right] \frac{du}{t-u}.
\]
The \((M,T,\delta)\)-Lip\((\frac{1}{2}+\delta)\) condition implies \(\|\lambda\|_{\text{Lip}(\frac{1}{2}+\delta)} \leq M(t-u)^\delta\). Lemma 3.2 gives an estimate of our integral kernel:
\[
\left| \frac{1}{2} + \frac{2}{\tau(u,t)^2} \right| \leq c |\tau(u,t) - 2i| \leq cM(t-u)^\delta
\]
for some absolute constant \(c > 0\). (Again, we have implicitly used the condition \(\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1\), which guarantees that \(\tau^\lambda(u,t)\) stays in a fixed compact set \(E \subseteq \mathbb{H}\).) Therefore,
\[
\left| \log g_s'(\gamma(t)) - \frac{1}{2} \log \frac{t}{t-s} \right| \leq cM \int_0^s (t-u)^\delta - 1 \, du = \frac{cM}{\delta} (t^\delta - (t-s)^\delta) \leq \frac{cMs^\delta}{\delta}
\]
and (8) follows. Taking \(t = s + \varepsilon\) with \(\varepsilon > 0\) gives
\[
\log \frac{\sqrt{\varepsilon} g_s'(\gamma(s+\varepsilon))}{\sqrt{s+\varepsilon}} = - \int_0^s \left[ \frac{1}{2} + \frac{2}{\tau(u,s+\varepsilon)^2} \right] \frac{du}{s+\varepsilon - u}.
\]
The existence of \(\lim_{\varepsilon \downarrow 0} \sqrt{\varepsilon} g_s'(\gamma(s+\varepsilon))\) follows from the Lebesgue dominated convergence theorem. \(\Box\)

**Lemma 4.2.** Let \(\lambda_1, \lambda_2 : [0,T] \rightarrow \mathbb{R}\) satisfy the \((M,T,\delta)\)-Lip\((\frac{1}{2}+\delta)\) condition for some \(0 < \delta \leq \frac{1}{2}\). Suppose \(\lambda_1 = \lambda_2\) on \([0,s]\) for some \(s \in (0,T)\). Then, for any \(\varepsilon \in (0,T-s]\),
\[
|\gamma^{\lambda_1}(s+\varepsilon) - \gamma^{\lambda_2}(s+\varepsilon)| \leq C \varepsilon^{1+\delta},
\]
where \(C = C(M,T,\delta,s) > 0\).

**Proof.** With \(M, T, \delta\) fixed, let \(X\) be the space of all functions \(\lambda : [0,T] \rightarrow \mathbb{R}\) satisfying the \((M,T,\delta)\)-Lip\((\frac{1}{2}+\delta)\) condition and \(\lambda = \lambda_1\) on \([0,s]\). For each \(\varepsilon \in (0,T-s]\), consider the compact set
\[
K_\varepsilon = \{ \gamma^{\lambda}(s+\varepsilon) \in \mathbb{H} : \lambda \in X \}.
\]
It suffices to show \( \text{diam}(K_\varepsilon) \leq C\varepsilon^{1+\delta} \). Let \( g_s = g_s^{\lambda_i} \) be the hydrodynamically normalized conformal map from \( \mathbb{H} \setminus \gamma^{\lambda_i}([0, s]) \) onto \( \mathbb{H} \), and let \( f_s = g_s^{-1} \). Note that
\[
\text{diam}(K_\varepsilon) \leq \text{diam}(g_s(K_\varepsilon)) \cdot \sup_{z \in E} |f'_s(z)|,
\]
where \( E \) is the convex hull of \( g_s(K_\varepsilon) \). By Lemma 3.2 \( \text{diam}(g_s(K_\varepsilon)) \leq cM\varepsilon^{\frac{1}{2}+\delta} \).

On the other hand, Lemma 4.1 implies
\[
\int_0^s \frac{1}{2u} + \frac{2}{\gamma(s - u, s)^2} \, du \leq C\varepsilon,
\]
for all \( s \). By Lemma 3.2, \( \text{diam}(g_s(K_\varepsilon)) \leq cM\varepsilon^{\frac{1}{2}+\delta} \).

By formula (4), proving the smoothness of \( \gamma \) is equivalent to proving the smoothness of the integral, which we call \( L(s) \) from now on.

**Notation 4.4.** Let
\[
L(s) = L^\lambda(s) = \int_0^s \left[ \frac{1}{2u} + \frac{2}{\gamma(s - u, s)^2} \right] \, du.
\]
This integral makes sense provided \( \lambda: [0, T] \to \mathbb{R} \) satisfies the \( (M, T, \delta)\)-Lip\((\frac{1}{2} + \delta) \) condition for some \( 0 < \delta \leq \frac{1}{2} \). In the coming sections, when we impose more regularity assumptions on \( \lambda \), we will keep this notation.
In the proof of Lemma 4.1, we have implicitly proved an upper bound of \(|L(s)|\). By (11), controlling the size of \(|L(s)|\) gives an upper bound of \(\left| \gamma'(s) - \frac{i}{\sqrt{s}} \right|\). This estimate will be useful later and we now state it explicitly.

**Lemma 4.5.** Under the \((M, T, \delta)\)-Lip\(\left(\frac{1}{2} + \delta\right)\) condition with \(0 < \delta \leq \frac{1}{2}\),

\[
|L(s)| \leq \frac{cM s^{\delta}}{\delta}
\]

for all \(s \in [0, T]\), where \(c > 0\) is an absolute constant.

We will show that \(L \in \text{Lip}(\delta)\). For any \(s \in (0, T)\) and \(\varepsilon \in [0, T - s]\),

\[
L(s + \varepsilon) - L(s) = \int_0^s \left[ \frac{2}{\tau(s + \varepsilon - u, s + \varepsilon)^2} - \frac{2}{\tau(s - u, s)^2} \right] \frac{du}{u} + \int_s^{s + \varepsilon} \left[ \frac{1}{2} + \frac{2}{\tau(s + \varepsilon - u, s + \varepsilon)^2} \right] \frac{du}{u}.
\]

(11)

Since \(\|\lambda\|_{\text{Lip}\left(\frac{1}{4}; [a, b]\right)} \leq M(b - a)^{\delta}\), Lemma 3.2 gives \(|\tau(s + \varepsilon - u, s + \varepsilon) - 2i| \leq cMu^{\delta}\), and therefore

\[
\int_s^{s + \varepsilon} \left( \frac{1}{2} + \frac{2}{\tau(s + \varepsilon - u, s + \varepsilon)^2} \right) \frac{du}{u} \leq \int_s^{s + \varepsilon} cMu^{\delta - 1} du \leq \frac{cM}{\delta} \varepsilon^{\delta}.
\]

The second integral in (11) is under control. The first integral can be estimated using the quantity

\[
\omega(s, u, \varepsilon) := \sup_{0 \leq u \leq \varepsilon} |\lambda(s + \varepsilon - v) - \lambda(s + \varepsilon - u) - \lambda(s - v) + \lambda(s - u)|.
\]

(13)

Note that \(\omega(s, u, \varepsilon)\) can be expressed as \(\|\lambda_1 - \lambda_2\|_{\infty}\), where \(\lambda_1, \lambda_2 : [0, u] \to \mathbb{R}\) are driving functions whose tips are \(\gamma(s + \varepsilon - u, s + \varepsilon)\) and \(\gamma(s - u, s)\). Theorem 3.4 implies

\[
|\gamma(s + \varepsilon - u, s + \varepsilon) - \gamma(s - u, s)| \leq c \omega(s, u, \varepsilon)
\]

for some absolute constant \(c > 0\). This gives an estimate of the first integral in (11). We have proved the following lemma.

**Lemma 4.6.** If \(\lambda : [0, T] \to \mathbb{R}\) satisfies the \((M, T, \delta)\)-Lip\(\left(\frac{1}{2} + \delta\right)\) condition for some \(0 < \delta \leq \frac{1}{2}\), then for any \(0 \leq s \leq s + \varepsilon \leq T\),

\[
|L(s + \varepsilon) - L(s)| \leq c \int_0^s u^{-\frac{1}{2}} \omega(s, u, \varepsilon) du + \frac{cM}{\delta} \left[ (s + \varepsilon)^{\delta} - s^{\delta} \right]
\]

(14)

where \(c > 0\) is an absolute constant and \(\omega(s, u, \varepsilon)\) is defined in (13).

The first inequality in (12) will be used in (15) and we use the second estimate in this section. When \(\delta = \frac{1}{2}\), we will soon see that (12) gives \(|L(s + \varepsilon) - L(s)| = O(\sqrt{\varepsilon})\) as \(\varepsilon \downarrow 0\). We achieve this by controlling the size of \(\omega(s, u, \varepsilon)\). The estimate depends on the regularity of \(\lambda\). In this section, \(\lambda \in \text{Lip}(\frac{1}{2} + \delta)\) and the following estimate of \(\omega(s, u, \varepsilon)\) is what we should expect. The estimate will be improved in (15) under the assumption \(\lambda \in C^{1, \delta}\).
Lemma 4.7. Let $\lambda: [0, T] \to \mathbb{R}$ satisfy the $(M, T, \delta)$-Lip($\frac{1}{\delta} + \delta$) condition. For any $0 \leq s < s + \varepsilon \leq T$ and $0 < u < s$

$$\omega(s, u, \varepsilon) \leq \begin{cases} 2Mu^{\frac{1}{2}+\delta}, & u \leq \varepsilon, \\ 2M\varepsilon^{\frac{1}{2}+\delta}, & u \geq \varepsilon, \end{cases}$$

and $|L(s + \varepsilon) - L(s)| \leq C\varepsilon^\delta$, where $C = C(M, T, \delta) > 0$.

Proof. When $u \leq \varepsilon$,

$$|\lambda(s + \varepsilon - v) - \lambda(s + \varepsilon - u) - \lambda(s - v) + \lambda(s - u)| \leq 2Mu^{\frac{1}{2}+\delta}$$

for any $0 \leq v \leq u$. When $u \geq \varepsilon$, we rearrange terms as

$$|\lambda(s + \varepsilon - v) - \lambda(s + \varepsilon - u) - \lambda(s - v) + \lambda(s - u)| \leq 2M\varepsilon^{\frac{1}{2}+\delta}$$

for any $0 \leq v \leq u$. We have proved the desired estimates of $\omega(s, u, \varepsilon)$.

To prove $|L(s + \varepsilon) - L(s)| \leq C\varepsilon^\delta$, we split the integral in (12):

$$\left| \int_0^s u^{-\frac{3}{2}} \omega(s, u, \varepsilon) \, du \right| \leq \int_0^\varepsilon 2Mu^{-1+\delta} \, du + \int_\varepsilon^s 2Mu^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}+\delta} \, du \leq C\varepsilon^\delta.$$

\[\square\]

We have all the ingredients for proving our first main result.

Theorem 4.8. Let $\lambda: [0, T] \to \mathbb{R}$ be such that

$$|\lambda(t_1) - \lambda(t_2)| \leq M|t_1 - t_2|^{\frac{1}{2}+\delta}$$

for all $t_1, t_2 \in [0, T]$, where $M > 0$ and $\delta \in (0, \frac{1}{2}]$ are constants. Then

(a) $\Gamma(t) := \gamma(t^2)$ is $C^{1,\delta}$ regular\footnote{Whose derivative is non-zero everywhere.} on $[0, \sqrt{T}]$ and

(b) the slit $\gamma(t)$ grows vertically at $t = 0$.

With an extra assumption that $\|\lambda\|_{Lip(\frac{1}{\delta})} \leq 1$, these statements are quantitative:

\begin{equation}
\|\Gamma\|_{C^{1,\delta}(0, T)} \leq N \quad \text{and} \quad \inf_{t \in [0, T]} |\Gamma'(t)| \geq \frac{1}{N},
\end{equation}

where $N = N(M, T, \delta) > 0$ depends only on $M$, $T$ and $\delta$. Furthermore,

\begin{equation}
|\Gamma'(t) - 2i| \leq Nt^{2\delta} \quad (0 < t \leq \sqrt{T}).
\end{equation}

Proof of Theorem 4.8. We first assume $\|\lambda\|_{Lip(\frac{1}{\delta})} \leq 1$. By Corollary 4.3, $\Gamma(t) = \gamma(t^2)$ is differentiable and $\Gamma'(t) = 2ie^{L(t^2)}$. With this formula of $\Gamma'(t)$, we claim that $\Gamma'(t)$ is Lip($\delta$). To see this, we first note from Lemma 4.5 that $\sup_{0 \leq t \leq \sqrt{T}} |L(t^2)| \leq R$ for some constant $R = \frac{cMT^\delta}{\delta}$ depending only on $M$, $T$ and $\delta$. This tells us

$$|\Gamma'(t_1) - \Gamma'(t_2)| \leq 2 \left( \sup_{|z| \leq R} |ze^z| \right) |L(t_1^2) - L(t_2^2)| \leq C|t_1 - t_2|^\delta$$
for some \( C = C(M, T, \delta) > 0 \) by Lemma 4.7. This proves (a). The estimates (15) and (16) follow from Lemma 4.5 and other estimates we have proved. For example, 

\[
|\Gamma'(t)| = 2 |e^{L(t^2)}| = 2 e^{Re{L(t^2)}} \geq 2 e^{-R}
\]

and (16) can be derived from 

\[
|\Gamma'(t) - 2i| = 2 |e^{L(t^2)} - 1| \leq 2 \left( \sup_{|z| \leq R} |e^z| \right) |L(t^2)|.
\]

If \( \|\lambda\|_{\text{Lip}(\frac{1}{2})} > 1 \), we pick a partition \( 0 = t_0 < t_1 < \cdots < t_n = T \) for which \( M(t_j+2-t_j)^\delta < 1 \) for all \( j = 0, 1, \ldots, n-2 \). This guarantees that \( \|\lambda\|_{\text{Lip}(\frac{1}{2}, [t_j, t_{j+2}])} < 1 \) for all \( j \). It suffices to show 

(i) \( \Gamma(t) = \gamma(t^2) \) is \( C^{1,\delta} \) regular on \( [t_0, t_2] \) and 

(ii) \( \gamma(t) \) is \( C^{1,\delta} \) regular on \( [t_j, t_{j+2}] \) for \( j = 1, 2, \ldots, n-2 \).

Item (i) follows from the argument of the previous paragraph, since we are back to the case \( \|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1 \). To show (ii), we pick \( \varepsilon > 0 \) for which \( M(t_{j+2}-t_j+\varepsilon)^\delta \leq 1 \). Again, by our earlier argument for the case \( \|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1 \), the function 

\[
u \mapsto \gamma(t_j - \varepsilon, t_j + \nu)
\]

is \( C^{1,\delta} \) regular on \( [0, t_{j+2} - t_j] \), and therefore 

\[
u \mapsto \gamma(t_j + \nu) = f_{t_j - \varepsilon}(\lambda(t_j - \varepsilon) + \gamma(t_j - \varepsilon, t_j + \nu))
\]

is also \( C^{1,\delta} \) regular on \( [0, t_{j+2} - t_j] \). \hfill \Box

5. When \( \lambda \in C^{1,\delta} \) with \( 0 < \delta \leq \frac{1}{2} \)

In this section, our driving function \( \lambda : [0, T] \to \mathbb{R} \) satisfies the \( (M, T, 1, \delta)-C^{1,\delta} \) condition with \( 0 < \delta \leq \frac{1}{2} \). That is, \( \|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1 \), \( \lambda \in C([0, T]) \) and 

\[
\|\lambda\|_{C^{1,\delta}} = \sup_{t \in [0, T]} \|\lambda(t)\| + \sup_{t \in [0, T]} \|\lambda'(t)\| + \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda'(t_1) - \lambda'(t_2)|}{|t_1 - t_2|^\delta} \leq M.
\]

Our goal is to improve the estimate \( |L(s + \varepsilon) - L(s)| = O(\varepsilon^\delta) \) given in (14) and we are expecting \( O(\varepsilon^{\frac{1}{2}+\delta}) \). Any \( C^{1,\delta} \) function is Lipschitz, i.e. \( \text{Lip}(\frac{1}{2} + \frac{1}{2}) \). Applying Lemma 4.5 and the first inequality in (12) with \( \delta = \frac{1}{2} \) yields 

\[
(17) \quad |L(s)| \leq cM \sqrt{s}
\]

and 

\[
(18) \quad |L(s + \varepsilon) - L(s)| \leq c \int_0^s u^{-\frac{3}{2}} \omega(s, u, \varepsilon) \, du + \frac{cM \varepsilon}{\sqrt{s}},
\]

for \( 0 < s < s + \varepsilon \leq T \). We now improve the estimate of \( \omega(s, u, \varepsilon) \) in Lemma 4.7 to the following. Recall the definition 

\[
\omega(s, u, \varepsilon) := \sup_{0 \leq v \leq u} |\lambda(s + \varepsilon - v) - \lambda(s + \varepsilon - u) - \lambda(s - v) + \lambda(s - u)|.
\]

**Lemma 5.1.** Let \( \lambda : [0, T] \to \mathbb{R} \) satisfy the \( (M, T, 1, \alpha)-C^{1,\alpha} \) condition with \( 0 < \alpha \leq 1 \). For any \( 0 \leq s < s + \varepsilon \leq T \) and \( 0 < u < s \), 

\[
\omega(s, u, \varepsilon) \leq \begin{cases} 
M \varepsilon^\alpha u, & u \leq \varepsilon, \\
M u^\alpha \varepsilon, & u \geq \varepsilon.
\end{cases}
\]
If $\alpha = \delta < \frac{1}{2}$, for any $0 < s < s + \epsilon \leq T$, we have

\begin{equation}
|L(s + \epsilon) - L(s)| \leq \frac{cM}{1 - 2\delta} \cdot \left( \epsilon^{1/2 + \delta} + \frac{\epsilon}{\sqrt{s}} \right),
\end{equation}

where $c > 0$ is an absolute constant. When $\delta = \frac{1}{2}$,

\begin{equation}
|L(s + \epsilon) - L(s)| \leq cM \epsilon \cdot \left[ 1 + \log^+ \left( \frac{s}{\epsilon} \right) + \frac{1}{\sqrt{s}} \right],
\end{equation}

where $\log^+ x = \max\{\log(x), 0\}$.

**Proof.** The equalities

\[ \lambda(s + \epsilon - v) - \lambda(s + \epsilon - u) - \lambda(s - v) + \lambda(s - u) \]

\[ = \int_{s-u}^{s-v} \lambda'(w + \epsilon) - \lambda'(w) \, dw \]

\[ = \int_{s-u}^{s+\epsilon-u} \lambda'(w + u - v) - \lambda'(w) \, dw \]

hold for all $v \in [0, u]$. Since $\lambda' \in \text{Lip}(\alpha)$, we deduce the desired estimate of $\omega(s, u, \epsilon)$. If $\alpha = \delta \in (0, \frac{1}{2})$, equation (18) and our estimates of $\omega(s, u, \epsilon)$ give the following. (We assume $s > \epsilon$ in the computation below. When $s \leq \epsilon$, the integral $\int_{\epsilon}^{s} \epsilon$ is not present and our estimate still holds.)

\[ |L(s + \epsilon) - L(s)| \leq cM \int_{0}^{\epsilon} u^{-\frac{3}{2}} \epsilon \delta \, du + cM \int_{\epsilon}^{s} u^{-\frac{3}{2}} \epsilon \delta \, du + \frac{cM \epsilon}{\sqrt{s}} \]

\[ \leq cM \epsilon^{1/2 + \delta} + \frac{cM \epsilon^{1/2 + \delta}}{1 - \delta} + \frac{cM \epsilon}{\sqrt{s}}. \]

This proves (19). When $\delta = \frac{1}{2}$, the second term in the last expression should be replaced by $cM \epsilon \log^+ \frac{\epsilon}{s}$, and (20) follows. \qed

Combining Theorem 4.8 and Lemma 5.1 we have the following theorem.

**Theorem 5.2.** Suppose $\lambda: [0, T] \to \mathbb{R}$ satisfies the $(M, T, 1, \delta)$-$C^{1, \delta}$ condition with $0 < \delta < \frac{1}{2}$. Then the curve $\Gamma(t) := \gamma(t^2)$ is $C^{1, \frac{1}{2} + \delta}$ regular on $[0, \sqrt{T}]$. In fact,

\[ \| \Gamma \|_{C^{1, \frac{1}{2} + \delta}([0, T])} \leq N \quad \text{and} \quad \inf_{t \in [0, T]} |\Gamma'(t)| \geq \frac{1}{N}, \]

where $N = N(M, T, \delta) > 0$. When $\delta = \frac{1}{2}$, $\Gamma'(t)$ is weakly Lipschitz in the sense that

\begin{equation}
|\Gamma'(t_1) - \Gamma'(t_2)| \leq N |t_1 - t_2| \max \left\{ 1, \log \left( \frac{1}{|t_1 - t_2|} \right) \right\} \quad (0 \leq t_1 < t_2 \leq \sqrt{T}),
\end{equation}

where $N = N(M, T) > 0$.

We do not know whether or not $\limsup_{\delta \uparrow \frac{1}{2}} N(M, T, \delta) < \infty$. The (non-optimal) constant $C = \frac{cM}{1 - 2\delta}$ in inequality (19) blows up when $\delta \uparrow \frac{1}{2}$. When $\delta = \frac{1}{2}$, inequality (20) implies $|\Gamma'(s + \epsilon) - \Gamma'(s)| = O(\epsilon \log^+ \frac{1}{\epsilon})$ as $\epsilon \downarrow 0$. In particular, $\Gamma(t)$ is weakly $C^{1, \alpha}$ in the sense that it is $C^{1, \alpha}$ for every $\alpha < 1$. 

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Proof. We know from Theorem 4.8 that $\Gamma \in C^1$. Since $\Gamma'(s) = 2ie^{L(s^2)}$, all we need to show is that $s \mapsto L(s^2)$ is $\text{Lip}(\frac{1}{2} + \delta)$. Suppose $0 < s < s + \varepsilon \leq \sqrt{T}$. If $s \leq \varepsilon$, then (17) gives

$$|L((s + \varepsilon)^2) - L(s^2)| \leq |L((s + \varepsilon)^2)| + |L(s^2)| \leq C\varepsilon$$

for some constant $C > 0$. If $s > \varepsilon$, Lemma 5.1 implies

$$|L((s + \varepsilon)^2) - L(s^2)| \leq C(2s\varepsilon + \varepsilon^2)^{\frac{1}{2} + \delta} + \frac{C(2s\varepsilon + \varepsilon^2)}{s} \leq C(2s + \varepsilon)^{\frac{1}{2} + \delta} \varepsilon^{\frac{1}{2} + \delta} + 3C\varepsilon \leq C\varepsilon^{\frac{1}{2} + \delta}.$$

Suppose $\delta = \frac{1}{2}$. To prove (21), it suffices to show

$$|L((s + \varepsilon)^2) - L(s^2)| \leq C\varepsilon (1 + |\log \varepsilon|)$$

for $0 \leq s < s + \varepsilon \leq \sqrt{T}$ and some constant $C > 0$. As before, when $s \leq 2\varepsilon$ the desired estimate follows from (17). When $s > 2\varepsilon$, (20) implies

$$|L((s + \varepsilon)^2) - L(s^2)| \leq C(2s + \varepsilon)\varepsilon \left(1 + \log \frac{s^2}{(2s + \varepsilon)^2} + \frac{1}{s}\right) \leq Cs\varepsilon \left(1 + \log \frac{s}{2\varepsilon} + \frac{1}{s}\right) \leq C\varepsilon (1 + |\log \varepsilon|),$$

where $C = C(M, T) > 0$. \hfill \Box

Corollary 5.3. Under the assumptions of Theorem 5.2, the slit $\gamma(t) = \gamma^\lambda(t)$ is $C^{1,\frac{1}{2} + \delta}$ regular on $[a, T]$ (or weakly $C^{1,1}$ when $\delta = \frac{1}{2}$) for every $a > 0$. When $0 < \delta < \frac{1}{2}$,

$$\|\gamma\|_{C^{1,\frac{1}{2} + \delta}([a, T])} \leq N \quad \text{and} \quad \inf_{t \in [a, T]} |\gamma'(t)| \geq \frac{1}{N},$$

where $N = N(M, T, \delta, a) > 0$.

6. When $\lambda \in C^{1,\frac{1}{2} + \delta}$ with $0 < \delta \leq \frac{1}{2}$

In this section, $\lambda: [0, T] \to \mathbb{R}$ satisfies the $(M, T, n, \alpha)$-continuous $C^{n,\alpha}$ condition for $n = 1$ and $\alpha = \frac{1}{2} + \delta$, where $0 < \delta \leq \frac{1}{2}$. That is to say, $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$ and

$$\|\lambda\|_{C^{1,\frac{1}{2} + \delta}} = \sup_{t \in [0, T]} |\lambda(t)| + \sup_{t \in [0, T]} |\lambda'(t)| + \sup_{t_1 \neq t_2 \in [0, T]} \frac{|\lambda'(t_1) - \lambda'(t_2)|}{|t_1 - t_2|^{\frac{1}{2} + \delta}} \leq M.$$

Our goal is to show $\gamma \in C^{2,\delta}$ on $[a, T]$ for every $a > 0$, which is equivalent to proving $L \in C^{1,\delta}$ on the same interval.

Since $\lambda \in C^{1,\frac{1}{2} + \delta}$, it is in particular $C^{1,\frac{1}{2}}$ and we know from Lemma 5.1 that $L^\lambda(s)$ is weakly Lipschitz on $[a, T]$ for every $a > 0$. We claim that $L(s)$ is differentiable on $(0, T)$ and $L'(s) \in \text{Lip}(\delta, [a, T])$. By (11), at least formally one has

$$L'(s) = \frac{1}{2s} + \frac{2}{\gamma(s)^2} + \int_0^s \partial_s \left[ \frac{2}{\gamma(s-u, s)^2} \right] du = \frac{1}{2s} + \frac{2}{\gamma(s)^2} - 4 \int_0^s \partial_s \gamma(s-u, s) \frac{2}{\gamma(s-u, s)^3} du.$$
To see that this formula is valid, we must show that $\partial_s \left[ \frac{2}{\gamma(s-u,s)^2} \right]$ is integrable over $u \in [0, s]$.  

**Lemma 6.1.** Let $\lambda: [0, T] \to \mathbb{R}$ satisfy the $(M, T, 1, \alpha)\cdot C^{1, \alpha}$ condition with $0 < \alpha \leq 1$. For any $0 < u \leq s < s + \varepsilon \leq T$, we have

\begin{align}
|\gamma(s + \varepsilon - u, s + \varepsilon) - \gamma(s - u, s)| &\leq c M \min\left(u \varepsilon^\alpha, \varepsilon u^\alpha\right), \\
|\partial_s \gamma(s - u, s)| &\leq c M u^\alpha, \\
|\partial_s \gamma(s + \varepsilon - u, s + \varepsilon) - \partial_s \gamma(s - u, s)| &\leq C \varepsilon^\alpha,
\end{align}

where $C = C(M, T) > 0$ and $c > 0$ is an absolute constant. When $0 < \alpha = \frac{1}{2} + \delta$ with $0 < \delta \leq \frac{1}{2}$, $L(s)$ is differentiable for $s \in (0, T]$ and $L'(s)$ is given by (22). Moreover, $L'(s) \in \text{Lip}(\delta)$ on $[a, T]$ for every $a > 0$.

**Proof.** By Theorem 3.4 $\gamma(s + \varepsilon - u, s + \varepsilon) - \gamma(s - u, s) \leq c \omega(s, u, \varepsilon)$. Inequality (23) follows immediately from Lemma 5.1 and it implies that

\[ \left| \frac{\gamma(s + \varepsilon - u, s + \varepsilon) - \gamma(s - u, s)}{\varepsilon} \right| \leq c M u^\alpha. \]

Letting $\varepsilon \to 0$ gives (24). To prove (25), we differentiate $\gamma(s - u, s) = g_{s - u}(\gamma(s)) - \lambda(s - u)$:

\[ \partial_s \gamma(s - u, s) = \frac{2}{g_{s - u}(\gamma(s)) - \lambda(s - u)} + g'_{s - u}(\gamma(s)) \gamma'(s) - \lambda'(s - u) \]

\[ = \frac{2}{\gamma(s - u, s)} + \gamma'_{s - u}(s) - \lambda'(s - u). \]

The last term $\lambda'(s - u)$ is Lip$(\alpha)$ in $s$ by assumption. The term $\frac{2}{\gamma(s - u, s)}$ is also Lip$(\alpha)$ in $s$ by (23):

\[ \frac{2}{\gamma(s + \varepsilon - u, s + \varepsilon)} - \frac{2}{\gamma(s - u, s)} = \frac{2}{\gamma(s + \varepsilon - u, s + \varepsilon)} \left( \frac{1}{\gamma(s - u, s)} - \frac{1}{\gamma(s + \varepsilon - u, s + \varepsilon)} \right) \]

\[ \leq \frac{c M u \varepsilon^\alpha}{\sqrt{u} \cdot \sqrt{u}} = c M \varepsilon^\alpha. \]

The remaining term $\gamma'_{s - u}(s)$ is given by $\gamma'_{s - u}(s) = \frac{1}{\sqrt{u}} \exp L(s - u, s)$, where

\[ L^\lambda(s - u, s) = L(s - u, s) := \int_0^u \left[ \frac{1}{2v} + \frac{2}{\gamma(s - v, s)^2} \right] dv. \]

Note that $L^\lambda(s - u, s) = L^\lambda(u)$, where $\lambda(t) = \lambda(s - u + t)$ is a time shift of $\lambda$. From Lemma 4.5 we know that $|L(s - u, s)| \leq c M \sqrt{u} \leq c M T$. On the ball $\{ z \in \mathbb{C}: |z| \leq c M T \}$ the function $e^z$ has bounded derivative, so

\[ \left| \gamma'_{s+\varepsilon-u}(s+\varepsilon) - \gamma'_{s-u}(s) \right| = \frac{1}{\sqrt{u}} \left| e^{L(s+\varepsilon-u,s+\varepsilon)} - e^{L(s-u,s)} \right| \]

\[ \leq \frac{C}{\sqrt{u}} |L(s + \varepsilon - u, s + \varepsilon) - L(s - u, s)|,
\]
where \( C = e^{cMT} \). On the other hand,

\[
(27) \quad |L(s + \varepsilon - u, s + \varepsilon) - L(s - u, s)| \leq \int_0^u \left| \frac{2}{\gamma(s + \varepsilon - v, s + \varepsilon)^2} - \frac{2}{\gamma(s - v, s)^2} \right| dv \\
\leq c \int_0^u v^{-\frac{3}{2}} \omega(s, v, \varepsilon) dv \\
\leq c \int_0^u v^{-\frac{3}{2}} M \varepsilon^\alpha dv \\
\leq cM \sqrt{\varepsilon^\alpha}.
\]

By (26) and (27), \( |\gamma'_{s+\varepsilon-u}(s+\varepsilon) - \gamma'_{s-u}(s)| \leq C \varepsilon^\alpha \) with \( C = C(M, T) > 0 \), and (25) holds.

If \( \frac{1}{2} < \alpha \leq 1 \), we will show \( L(s) \) is differentiable on \((0, T]\) and \( L'(s) \) is given by (22). By extending \( \lambda \), we can assume without loss of generality that \( 0 < s < T \).

For small \( \varepsilon > 0 \),

\[
\frac{L(s + \varepsilon) - L(s)}{\varepsilon} = \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \left[ \frac{1}{2u} + \frac{2}{\gamma(s + \varepsilon - u, s + \varepsilon)^2} \right] du \\
+ \int_0^s \frac{1}{\varepsilon} \left[ \frac{2}{\gamma(s + \varepsilon - u, s + \varepsilon)^2} - \frac{2}{\gamma(s - u, s)^2} \right] du.
\]

It is not hard to see that the integrand in the last term is dominated by \( Cu^{\alpha-\frac{3}{2}} \), which is integrable since \( \alpha > \frac{1}{2} \). By the Lebesgue dominated convergence theorem, the second integral converges as \( \varepsilon \downarrow 0 \). Convergence of the first integral follows from continuity and does not require \( \alpha > \frac{1}{2} \). Since \( L(s) \) has a continuous right derivative, it is differentiable on \((0, T]\).

We now prove the main result in this section.

**Theorem 6.2.** Suppose \( \lambda \colon [0, T] \to \mathbb{R} \) satisfies the \((M, T, 1, \alpha)\)-\(C^{1, \alpha}\) condition with \( \alpha = \frac{1}{2} + \delta \) and \( 0 < \delta \leq \frac{1}{2} \). Then the slit \( \gamma(t) = \gamma^\lambda(t) \) is \( C^{2, \delta} \) regular on \([a, T]\) for every \( a > 0 \). The statement is quantitative in the sense that

\[
\|\gamma\|_{C^{2, \delta}([a, T])} \leq N \quad \text{and} \quad \inf_{t \in [a, T]} |\gamma'(t)| \geq \frac{1}{N},
\]

where \( N = N(M, T, \delta, a) > 0 \) depends only on \( M, T, \delta \) and \( a \).

**Proof.** Any \( C^{1, \frac{1}{2}+\delta} \) driving function \( \lambda \) is in particular \( C^{1, \frac{1}{2}} \). Applying Theorem 5.2 we know that \( \gamma \) is weakly \( C^{1,1} \) regular on every \([a, T]\). All we need to show is that \( \gamma'' \) exists and is \( \text{Lip}(\delta) \) on \([a, T]\).

By Lemma 6.1 \( L \) is differentiable and therefore \( \gamma'' \) exists on \((0, T]\) and is given by

\[
(28) \quad \gamma''(s) = \frac{2\gamma'(s)}{\gamma(s)^2} - 4\gamma'(s)Q(s),
\]

where

\[
Q(s) := \int_0^s \partial_u \frac{\gamma(s-u, s)}{\gamma(s-u, s)^3} du.
\]

The first term \( \frac{2\gamma'(s)}{\gamma(s)^2} \) in \( (21) \) is \( \text{Lip}(\delta) \) on \([a, T]\) because both \( \gamma' \) and \( \gamma \) are \( \text{Lip}(\delta) \) and the size of the denominator \( |\gamma(s)|^2 \asymp s \) is bounded below by positive constant.
It remains to prove \( Q \in \text{Lip}(\delta, [0, T]) \). The integral kernel of \( Q \) has the form 
\[
K(x, y) = \frac{2}{y^2}.
\]
We have
\[
\begin{align*}
|Q(s + \varepsilon) - Q(s)| &\leq \int_0^s |K(x + \Delta x, y + \Delta y) - K(x, y)| \, du \\
&+ \int_s^{s + \varepsilon} |K(x + \Delta x, y + \Delta y)| \, du,
\end{align*}
\]
where \( x = \partial_s \gamma(s - u, s), \ y = \gamma(s - u, s), \ \Delta x = \partial_s \gamma(s + \varepsilon - u, s + \varepsilon) - \partial_s \gamma(s - u, s) \) and \( \Delta y = \gamma(s + \varepsilon - u, s + \varepsilon) - \gamma(s - u, s) \). By Lemma 6.1 and scaling,
\[
|K(x + \Delta x, y + \Delta y)| \leq \frac{c Mu^{\frac{1+\delta}{2}}}{u^\frac{3}{2}} = cMu^{-1+\delta},
\]
and the last term in (29) is bounded by
\[
\int_s^{s + \varepsilon} |K(x + \Delta x, y + \Delta y)| \, du \leq \frac{cM}{\delta} \left((s + \varepsilon)^\delta - s^\delta\right) \leq \frac{cM}{\delta} \varepsilon^\delta,
\]
whenever \( 0 \leq s < s + \varepsilon \leq T \). On the other hand, we split the first integral in (29) into two terms and handle them separately. If \( s \leq \varepsilon \), by triangle inequality and Lemma 6.1,
\[
|K(x + \Delta x, y + \Delta y) - K(x, y)| \leq |K(x + \Delta x, y + \Delta y)| + |K(x, y)| \leq cMu^{-1+\delta}.
\]
Integrating gives
\[
\int_0^\varepsilon |K(x + \Delta x, y + \Delta y) - K(x, y)| \, du \leq \frac{cM}{\delta} \varepsilon^\delta.
\]
If \( s \geq \varepsilon \), we still need to estimate the integral from \( \varepsilon \) to \( s \).
\[
\begin{align*}
|K(x + \Delta x, y + \Delta y) - K(x, y)| &\leq |K(x + \Delta x, y) - K(x, y)| + |K(x + \Delta x, y + \Delta y) - K(x + \Delta x, y)| \\
&\leq |\Delta x| \cdot \sup |\partial_x K| + |\Delta y| \cdot \sup |\partial_y K| \\
&\leq C \varepsilon^{\frac{1+\delta}{2}} u^{-\frac{3}{2}} + C \varepsilon u^{-1+2\delta}
\end{align*}
\]
by Lemma 6.1 again. Integrating gives
\[
\int_\varepsilon^s |K(x + \Delta x, y + \Delta y) - K(x, y)| \, du \leq C \varepsilon^\delta
\]
with \( C = C(M, T, \delta) > 0 \). \( \square \)

For \( \lambda(t) \in C^{1, \frac{1+\delta}{2}} \) with \( 0 < \delta \leq \frac{1}{2} \), Theorem 6.2 shows that \( \gamma(t) \) is \( C^{2,\delta}([a, T]) \) for every \( a > 0 \). Certainly \( \|\gamma\|_{C^{1,\alpha}([a, T])} \to \infty \) as \( a \downarrow 0 \). To obtain smoothness up to \( t = 0 \), one has to reparametrize the slit, and a natural candidate is \( \Gamma(t) = \gamma(t^2) \). We do not know whether \( \Gamma(t) \) is \( C^{2,\delta} \) up to \( t = 0 \), but assuming this we have a quadratic approximation
\[
\Gamma(t) = \gamma(t^2) = 2it + \frac{2}{3} \lambda'(0)t^2 + O(t^{2+\delta})
\]
as \( t \to 0 \). The heuristic reason is that on a small interval close to the origin any \( C^{1,\alpha} \) driving function \( \lambda(t) \) can be approximated by a fixed linear function \( \lambda'(0)t \). For driving functions of the form \( \lambda(t) = at \ (a > 0) \) the quadratic approximation of \( \gamma(t^2) \) can be explicitly computed.
Example 6.3. Let \( \lambda(t) = t \). Since a linear function is invariant under time shift, 
\( \gamma(s-u,s) = \gamma(u) \) does not depend on \( s \), and we have
\[
L(s) = \int_0^s \left[ \frac{1}{2u} + \frac{2}{\gamma(u)^2} \right] du \quad \text{and} \quad L'(s) = \frac{1}{2s} + \frac{2}{\gamma(s)^2}.
\]
For this case it is possible to explicitly compute the series expansion of \( L(s) \) near \( s = 0 \). The reader may refer to [5] for the computation. As \( s \to 0 \), one has \( \gamma(s) = 2i\sqrt{s} + \frac{2}{3}s + O(s^{3/2}) \) and \( L'(s) = -\frac{i}{3\sqrt{s}} + O(1) \). Note that \( [L(s^2)]' = 2sL'(s^2) \to \frac{-2i}{3} \). The function \( s \to L(s^2) \) is \( C^1 \) up to \( s = 0 \). Since \( \Gamma'(s) = 2i \exp L(s^2) \), the curve \( \Gamma(s) = \gamma(s^2) \) is \( C^2 \) up to \( s = 0 \) and has a quadratic approximation \( \Gamma(s) = 2is + \frac{2}{3}s^2 + O(s^3) \). Note that this agrees with (30).

For any constant \( b > 0 \), the driving function \( \lambda_b(t) = bt \) can be obtained from \( \lambda_1(t) = t \) and a Brownian scaling: \( \lambda_b(t) = \frac{1}{b}\lambda_1(b^2t) \). The computation in the above example gives
\[
\gamma_{\lambda_b}(s) = \frac{1}{b}\gamma_{\lambda_1}(b^2s) = 2i\sqrt{s} + \frac{2b}{3}s + O(s^{3/2})
\]
as \( s \to 0 \). We have just verified (30) for all driving functions of the form \( \lambda_b(t) = bt \) with \( b \in \mathbb{R} \). (The case \( b < 0 \) follows from symmetry.)

Proposition 6.4. Suppose \( \lambda: [0, T] \to \mathbb{R} \) satisfies the \((M,T,1,\alpha)-C^{1,\alpha}\) condition with \( \alpha = \frac{1}{2} + \delta \) and \( 0 < \delta \leq \frac{1}{2} \). Then \( \Gamma(t) = \gamma(t^2) \) is twice differentiable everywhere on \([0, \sqrt{T}]\) and \( \Gamma''(0) = \frac{4}{3}\lambda'(0) \).

Proof. We already know \( \Gamma(t) \) is \( C^2 \) on \([0, \sqrt{T}]\) (Theorem 6.2) and we still need to show the existence of \( \Gamma''(0) \). By comparing \( \lambda(t) \) with the linear driving function \( \lambda(t) = \lambda'(0)t \), we will show that \( s \to L^\lambda(s^2) \) is differentiable at \( s = 0 \). To simplify the notation, we write \( L(\cdot) = L^\lambda(\cdot) \) and \( \tilde{\tau}(\cdot, \cdot) = \tau^\lambda(\cdot, \cdot) \). Notice that
\[
L(s^2) - \tilde{L}(s^2) \leq \int_0^{s^2} \left| \frac{2}{\tau(s^2-u,s^2)^2} - \frac{2}{\tilde{\tau}(s^2-u,s^2)^2} \right| \frac{du}{u}
\]
\[
\leq c \int_0^{s^2} \left| \tau(s^2-u,s^2) - \tilde{\tau}(s^2-u,s^2) \right| \frac{du}{u}
\]
Using the condition \( \lambda \in C^{1,1/2+\delta} \), we can estimate the \( \| \cdot \|_\infty \) distance between the two driving functions which generate \( \tau(s^2-u,s^2) \) and \( \tilde{\tau}(s^2-u,s^2) \). The Lipschitz continuity Theorem 3.4 implies that
\[
L(s^2) - \tilde{L}(s^2) = O(s^{1+2\alpha}).
\]
For the purpose of computing \( \lim_{s \to 0} \frac{L(s^2)}{s} \), we can replace \( \lambda(t) \) by \( \lambda'(0)t \) without affecting the existence of the limit and its value. Since we are able to compute this limit for linear driving functions, it follows that
\[
\frac{dL(s^2)}{ds} \bigg|_{s=0} = \lim_{s \to 0} \frac{L(s^2)}{s} = \lim_{s \to 0} \frac{\tilde{L}(s^2)}{s} = -\frac{2i}{3} \lambda'(0).
\]
From the formula \( \Gamma'(s) = 2i \exp L(s^2) \) and the above computation, we have \( \Gamma''(0) = \frac{4}{3}\lambda'(0) \). \(\square\)
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