Geographical Coarsegraining of Complex Networks

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We perform the renormalization-group-like numerical analysis of geographically embedded complex networks on the two-dimensional square lattice. At each step of coarsegraining procedure, the four vertices on each $2 \times 2$ square box are merged to a single vertex, resulting in the coarsegrained system of the smaller sizes. Repetition of the process leads to the observation that the coarsegraining procedure does not alter the qualitative characteristics of the original scale-free network, which opens the possibility of subtracting a smaller network from the original network without destroying the important structural properties. The implication of the result is also suggested in the context of the recent study of the human brain functional network.

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peat the following geographic coarsegraining, which is in parallel to the Kadanoff block spin renormalization group procedure in standard statistical mechanical systems (see Fig. 1): Four vertices on each square box of the size $2 \times 2$ is merged to a single vertex and accordingly the edges connecting intra-box vertices (dashed lines in Fig. 1) are disregarded, but the inter-box connections are kept (thin solid line in Fig. 1). We also keep track of how strong the edges are by assigning the weight $w_{vw}$ that is simply the number of edges connecting two merged vertices $v$ and $w$. For example, in Fig. 1 there exist two edges (thin solid line) connecting the two square boxes before the merging, which gives rise to the weight $w = 2$ for the edge (thick solid line) connecting the two vertices after the merging.

If we keep all the edges in the coarsegrained network, the average degree increases as the procedure is iterated, resulting in the fully-connected network eventually. To remedy this, we fix the average degree at each step of coarsegraining by removing weaker edges with smaller values of the weight. Suppose that we have to remove $M_r$ edges to keep the average degree the same, and that there are $M_w$ edges of the weight $w$. For example, for $M_r < M_{w=1}$, randomly picked $M_r$ edges of $w = 1$ is removed. If $M_{w=1} < M_r < M_{w=2}$, all $w = 1$ edges removed and $M_r - M_{w=1}$ edges with $w = 2$ are randomly deleted. The above procedure makes sense since in real situations, it is common that the coarsegraining is often accompanied by the change of the sensitivity of the measurement: When the system is looked at a far distance, we only have interest in large scale structures.

Figure 2 shows the result of the coarsegraining. Original networks of the size $64 \times 64$ are generated following Ref. 2 for $\gamma = 2.5, 3.0, 4.0, 5.0$ and the above explained coarsegraining process is iterated. The network size in this work is smaller than the cutoff length scale beyond which the network ceased to be scale free, which is also seen in Fig. 2 a) where the cutoff degree scale is absent (see Ref. 6). One sees clearly that the coarsegraining process does not change the degree exponent $\gamma$. In the terminology of the renormalization group (RG) formalism, the scale-free network with any value of $\gamma$ is the stable fixed point of the RG flow. This observation implies that the scale-free network in Ref. 2 possesses neither the degree scales nor the length scales up to the cutoff length.
neurons also argue that since each voxel contains large number of elements through which biochemical signal transfers. One can envision of two separate voxels, not the actual path of voxels through which biochemical signal transfers. One can also argue that since each voxel contains large number of neurons [about $O(10^5)$], the observed scale-free distributions can be the artifact of the coarsegraining, considering the recent study in Ref. [4] that scale-free distributions can emerge from merging. Our main results [7] in the present study implies that this is not the case and that the scale-free distribution in human brain functional network is expected to be the genuine property of the brain, not the artifact of the coarsegrained information.

We next investigate other important structural properties of networks. Many real networks including Internet, World Wide Web, and the actor network, are characterized by the existence of the hierarchical structure [8, 11], which can be usually detected by the negative correlation between the clustering coefficient (see Ref. [2]) and the degree [8]. For example, the Barabási-Albert network [2], which does not possess hierarchical structure, is known to have the clustering coefficient $C_v$ of the vertex $v$ independent of its degree $k_v$ [i.e., $C(v) \sim k^0$], see Ref. [8], while Holme-Kim model [9] has been shown to have $C(k) \sim k^{-1}$ [10], in accord with the observations of many real networks [8]. In Fig. 4 we plot $C(k)$ at the $n$-th iteration step of the coarsegraining for the initial network of the size $128 \times 128$ with $\gamma = 3$. The geographically embedded network in Ref. [8] is found to be somehow special since $C(k)$ is better described by $C(k) \sim k^{-2}$ rather than the abundantly found $C(k) \sim k^{-1}$. But this feature remains the same upon the coarsegraining procedure, implying that the coarsegraining does not change the hierarchical structure of the network.
We next study the assortative mixing characteristics\cite{12,13} of the network. For the assortative network, vertices with the higher degree tend to have high-degree neighbor vertices, while for the disassortative network, higher degree vertices favor to have lower degree neighbors. The degrees $k_v$ and $k_w$ of the two vertices $v$ and $w$ connecting each edge is measured and then the histogram is computed by using $20 \times 20$ bins in log-log scales in $k_v - k_w$ plane. The brightness of the region in Fig. 5 (a)-(c) is chosen in proportion to the logarithm of the height of the histogram in that region. Again found is that the coarsegraining procedure does not change the disassortative mixing property of the network, i.e., at any iteration step, the high-degree vertices in the network tend to have low-degree neighbors. This behavior of the disassortative mixing can also be detected by the assortativity coefficient $r$ (see Ref. 12 for the definition). If $r$ has positive value, the network has assortative mixing property while it is disassortative otherwise. In Fig. 5 (d), $r$ is shown to have negative values at the $n = 0, 1, 2$ coarsegraining steps. The decrease of $r$ with $n$ is not completely understood, although this dependence of $r$ versus the network size $N$ appears to be consistent with Ref. 12, where $r$ tends to approach zero from below as the larger disassortative network is considered.

So far we have introduced a geographical coarsegraining procedure and applied it to the geographically embedded scale-free network in Ref. 3. Although the network sizes become smaller as the coarsegraining procedure proceeds, it has been found that several key features of the initial networks do not change qualitatively. In particular, the degree exponent $\gamma$ does not change, and hierarchical structure [detected by the negative correlation between the clustering coefficient $C(k)$ versus degree $k$] and the disassortativity [detected by that more edges connect high-degree vertices to low-degree vertices than to high-degree vertices] are remain qualitatively the same. Our geographic coarsegraining procedure can be useful when the initial network is of a huge size since one can then systematically reduce the network size without destroying important characteristics of the network. Modification of the present coarsegraining method to apply for the network which is not geographical embedded can be an interesting extension. The main results also suggest that the scale-free distribution found recently for the human brain functional network may not be the artifact due to the large voxel size (each contains $O(10^5)$ neuron cells), but the genuine property of the brain.

We finally study the two-dimensional Watts-Strogatz (WS) network, built similarly to Ref. 2. Vertices are put on the two-dimensional square lattice points and every vertex is connected to its nearest and next-nearest neighbor vertices. Each edge is visited once, and with the rewiring probability $P$, is rewired to a randomly chosen vertex. The resulting network belongs to the so-called exponential network since the tail in the degree distribution is exponentially small. We then iterate our geographic coarsegraining procedure with the average degree kept constant at each iteration. In Fig. 6 (a), the initial network of the size $128 \times 128$ at the rewiring probability $P = 0.1$ is coarsegrained $n$ times. As $n$ becomes larger, the degree distribution remains to be exponential and tends to saturate. In Fig. 6 (b), initial two-dimensional WS networks of various sizes $L = 128, 256, 512$ and $1024$ at $P = 0.1$ are coarsegrained and the clustering coefficient $C(n)$ is plotted as a function of the number $n$ of iterations. As the coarsegraining process proceeds the clustering coefficient is shown to decrease towards zero, which indicates that the RG stable fixed point of the WS network is close to the random network of Erdős and Rényi.

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