Subgraph posets and graph reconstruction

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Abstract

We consider only finite simple undirected graphs in this paper. Let $G$ be an arbitrary graph. Let $\mathcal{P}(G)$ be the set consisting of $K_1$ and the distinct unlabelled nonempty induced subgraphs of $G$. The abstract induced subgraph poset of $G$ is the isomorphism class of the weighted poset $(\mathcal{P}(G), \leq_v, w_v: \mathcal{P}(G) \times \mathcal{P}(G) \to \mathbb{N})$, where for $G_i, G_j \in \mathcal{P}(G)$ we define $G_i \leq_v G_j$ if $G_i$ is an induced subgraph of $G_j$, and $w_v(G_i, G_j)$ is the number of induced subgraphs of $G_j$ that are isomorphic to $G_i$. We write $\overline{\mathcal{P}}(G)$ for the isomorphism class of the weighted poset defined above. In an earlier paper, we showed that several invariants of $G$ can be computed from the abstract poset $\overline{\mathcal{P}}(G)$, i.e., the deck of $G$ is not required. In this paper, we study reconstruction questions on two analogously defined posets: the abstract weighted lattice $\overline{\Omega}(G)$ of distinct unlabelled connected partitions of $G$, which we call the abstract bond lattice of $G$, and the abstract weighted poset $\overline{\mathcal{P}}(G)$ of distinct unlabelled edge-subgraphs of $G$, which we call the abstract edge-subgraph poset of $G$.

We show that $\overline{\Omega}(G)$ can be constructed from $\overline{\mathcal{P}}(G)$, and that $\overline{\mathcal{P}}(G)$ can be constructed from $\overline{\Omega}(G)$ if $G$ is not a star or a disjoint union of edges and has no isolated vertices. The first construction implies that if a graph invariant can be computed from $\overline{\Omega}(G)$, then it can also be computed from $\overline{\mathcal{P}}(G)$. An examples of such an invariant is the chromatic symmetric function. Since every tree $T$ on 2 or more vertices can be reconstructed up to isomorphism from $\overline{\mathcal{P}}(T)$, the second construction implies that every tree $T$ on 2 or more vertices that is not a star can be reconstructed up to isomorphism from $\overline{\Omega}(T)$. We also give simple proofs that the chromatic symmetric function $X_G(x)$ and the symmetric Tutte polynomial $X_G(x,t)$ of $G$ can be computed from $\overline{\mathcal{P}}(G)$. The main tools that we use to prove these results are a generalisation to abstract induced subgraph posets of a lemma of Kocay in graph reconstruction theory and other related subgraph counting identities.

Stanley has asked if every tree $T$ is determined up to isomorphism by its chromatic symmetric function $X_T(x)$. Analogously, Noble and Welsh have asked if every tree $T$ is determined up to isomorphism by its symmetric Tutte polynomial $X_T(x,t)$. We show that the two questions are equivalent by showing that, for every tree $T$, $X_T(x,t)$ is determined by $X_T(x)$.

In Section 5, we consider the problem of reconstructing an arbitrary graph $G$ up to isomorphism from its abstract edge-subgraph poset $\overline{\mathcal{P}}(G)$, which we call the $Q$-reconstruction problem, and study its relation to the edge reconstruction conjecture of Harary. We present an infinite family of graphs that are not $Q$-reconstructible, and show that the edge reconstruction conjecture is true if and only if the graphs in the family are the only graphs that are not $Q$-reconstructible.

Let $\mathcal{G}$ be the set of all graphs, and let $\mathcal{G}/\cong$ be the set of all unlabelled graphs (isomorphism classes). Let $\text{hom}(G, H)$ denote the number of homomorphisms from $G$ to $H$. Let $f: \mathcal{G}/\cong \to \mathcal{G}/\cong$ be a bijection such that for all $G, H \in \mathcal{G}/\cong$, we have $\text{hom}(f(G), f(H)) = \text{hom}(f(G), f(H))$. We conjecture that $f(G) = G$ for all $G \in \mathcal{G}/\cong$. Our conjecture is motivated by Lovász’s homomorphism cancellation laws. We prove that the conjecture stated above is weaker than the edge reconstruction conjecture.
1 Introduction

We consider only finite undirected simple graphs in this paper. A well-known conjecture of Ulam [21] and Kelly [6], known as the vertex reconstruction conjecture or Ulam’s conjecture, states that every graph on 3 or more vertices is determined up to isomorphism by its deck (the collection or multiset of its unlabelled vertex-deleted subgraphs). An analogous conjecture, known as the edge reconstruction conjecture, was proposed by Harary [5]. It states that every graph with at least 4 edges is determined up to isomorphism by its edge-deck (the collection of its unlabelled edge-deleted subgraphs). These are some of the foremost unsolved problems in graph theory. We refer the reader to a survey of these conjectures by Bondy [2].

1.1 Notation

1.1.1 Miscellaneous notation

We denote the set of integers, the set of positive integers, and the set of natural numbers (including 0) by \( \mathbb{Z} \), \( \mathbb{Z}^+ \), and \( \mathbb{N} \), respectively. We denote the family of \( k \)-element subsets of a set \( S \) by \( \binom{S}{k} \), the set of \( k \)-element tuples (or the set of sequences of length \( k \)) from \( S \) by \( S^k \), and the powerset of \( S \) by \( 2^S \). When the range of an index is unspecified, e.g., as in \( \sum_i a_i \) or \( \bigcup_i H_i \) or in expressions such as “... for all \( i \)”, we understand that full range of the index over which the objects in the context are defined is implied. We use this convention especially outside displayed mathematics.
1.1.2 Graphs

We denote the set of all graphs by \( \mathcal{G} \) and the set of connected graphs by \( \mathcal{G}^c \). Throughout this paper, we take \( G \) and \( H \) to be arbitrary graphs. We denote the vertex set of \( G \) by \( V(G) \), its edge set by \( E(G) \), number of vertices in \( G \) by \( \nu(G) \), the number of edges in \( G \) by \( \epsilon(G) \), and the number of components of \( G \) by \( c(G) \). An empty graph is a graph with empty edge set. A null graph \( \Phi \) is a graph with no vertices. For \( X \subseteq V(G) \), we denote the subgraph of \( G \) induced by \( X \) by \( G[X] \), the subgraph of \( G \) induced by \( V(G) \setminus X \) by \( G - X \), or simply \( G - u \) if \( X = \{u\} \).

For \( E \subseteq E(G) \), we denote the subgraph of \( G \) induced by \( E \) by \( G[E] \), the spanning subgraph of \( G \) with edge set \( E \) by \( G_E \), and the spanning subgraph of \( G \) with edge set \( E(G) \setminus E \) by \( G - E \) (or just \( G - e \) if \( E = \{e\} \)).

By an induced subgraph, we always mean a subgraph induced by a vertex set; a subgraph induced by an edge set is called an edge-subgraph. We write \( H \subseteq G \) when \( H \) is a subgraph of \( G \), \( H \subseteq_v G \) when \( H \) is an edge-subgraph of \( G \), \( H \subseteq_e G \) when \( H \) is an induced subgraph of \( G \), \( H \leq G \) when \( H \) is isomorphic to a subgraph of \( G \), \( H \leq_v G \) when \( H \) is isomorphic to an edge-subgraph of \( G \), and \( H \leq_e G \) when \( H \) is isomorphic to an induced subgraph of \( G \). We denote the number of subgraphs (induced subgraphs, edge-subgraphs, components) of \( G \) that are isomorphic to \( H \) by \( \text{sub}(H,G) \) (respectively, \( \text{ind}(H,G), \text{esub}(H,G), c(H,G) \)).

1.1.3 Unlabelled graphs

If \( H \) is isomorphic to \( G \), then we write \( H \cong G \). Isomorphism is an equivalence relation on \( \mathcal{G} \). An unlabelled graph is an isomorphism class. A class of graphs is a set of graphs closed under isomorphism. We use the quotient notation \( \mathcal{G} / \cong \) to denote the set of isomorphism classes, and take \( I \) to be the quotient map of isomorphism; thus \( I(G) : = \{ H \in \mathcal{G} | H \cong G \} \) is the isomorphism class of \( G \). For \( S \subseteq \mathcal{G} \), we write \( I(S) : = \{ I(G) | G \in S \} \). Given an isomorphism class (unlabelled graph) \( G \), we denote a representative labelled graph in \( G \) by \( G^* \).

A graph-invariant is a function \( f \) on \( \mathcal{G} \) that is constant over each isomorphism class. If \( f \) is a graph invariant, we define \( f(S) : = f(S^*) \) for all \( S \in \mathcal{G}/ \cong \). Definitions of many terms (e.g., deck, edge-deck, etc.) and parameters (e.g., \( \text{ind}(\ldots), \text{esub}(\ldots), \text{hom}(\ldots), \ldots \)) naturally extend to and are well-defined for unlabelled graphs if they depend on invariant properties of graphs. For example, if \( S, T, \in \mathcal{G} \), then \( \text{ind}(S,T) = \text{ind}(S',T') \) for all \( S' \in I(S), \text{for all} \ T' \in I(T) \), which allows us to define \( \text{ind}(S,T) : = \text{ind}(S',T') \) if \( S \) and \( T \) are unlabelled graphs. Similarly, for unlabelled graphs \( S \) and \( T \), we say that \( S \) is an induced subgraph (or an edge-subgraph) of \( T \) if \( S^* \) is isomorphic to an induced subgraph (or an edge-subgraph) of \( T^* \).

We denote a path on \( n \) vertices by \( P_n \), a cycle on \( n \) vertices by \( C_n \), a complete graph on \( n \) vertices by \( K_n \), a complete bipartite graph with \( n \) and \( m \) vertices in the two partitions by \( K_{n,m} \), and the graph \( K_4 \) minus an edge by \( K_4 \setminus e \); here \( P_k, C_k, \text{and} K_4 \setminus e \) are unlabelled graphs. Similarly, \( K_{n,m} \) and unlabelled graphs. We write \( G \in K_n \) to refer to a (labelled) graph in \( K_n \).

Let \( H_i, i = 1, \ldots, m \) be distinct unlabelled graphs. Let \( \mathcal{F} : = \{ F_1, \ldots, F_n \} \subseteq \mathcal{G} \) be a collection of mutually vertex-disjoint graphs. If \( G \subseteq \biguplus_{i=1}^n F_i \), \( x_i := | \mathcal{F} \cap H_i |, \text{for} \ i = 1, \ldots, m \), then we write \( G \in \sum_i x_i H_i \) and \( I(G) = \sum_i x_i H_i \).

1.1.4 Reconstruction terminology

Most of the following notions are standard in the reconstruction literature (see, e.g., Bondy [2]), so we define them concisely below.

The deck of a labelled graph \( G \) is the set \( \text{deck}(G) : = \{ (I(G-u), \text{ind}(G-u,G)) | u \in V(G) \} \). We write \( \text{deck}(G) : = \text{deck}(G^*) \) when \( G \) is an unlabelled graph. We say that \( H \) is a
reconstruction of $G$ if deck$(G) = $ deck$(H)$; and that $G$ is reconstructible if it is determined up to isomorphism by deck$(G)$ (i.e., every reconstruction of $G$ is isomorphic to $G$). A set $S$ of unlabelled graphs is a counter example to Ulam's conjecture if deck$(G) = $ deck$(H)$ for all $G, H \in S$ and $|S| \geq 2$. Let $C$ be a class of graphs and let $f$ be a graph invariant. We say that $C$ is reconstructible if each graph in $C$ is reconstructible: $f(G)$ is reconstructible if deck$(G)$ determines $f(G)$; and $f$ is reconstructible for $C$ if it is reconstructible for all graphs in $C$.

Similar definitions may be given for other reconstruction problems. In particular, by replacing deck by edge-deck or by abstract induced subgraph poset (to be defined in Section 1.2) or by abstract bond lattice (to be defined in Section 1.3) or by abstract edge-subgraph poset (to be defined in Section 1.4), we define the corresponding notions of edge reconstructibility, $P$-reconstructibility, $\Pi$-reconstructibility, $Q$-reconstructibility, respectively.

### 1.1.5 Partially ordered sets

We follow Stanley [18] for terminology on partially ordered sets. Let $(S, \leq)$ be a partially ordered set. For $x, y \in S$, we say that $y$ covers $x$ if $x \leq y$, $x \neq y$, and there is no $z \in S \setminus \{x, y\}$ such that $x \leq z \leq y$. A partially ordered set (poset) is called ranked if it admits a rank function $\rho : S \to \mathbb{N}$ such that for all $x, y \in S$, $y$ covers $x$ implies $\rho(y) = \rho(x) + 1$. The down-set of an element $x$ is the set $S(x) := \{y \in S : y \leq x\}$.

The posets in this paper are weighted. We say that weighted posets $(S, \leq, w)$ and $(S', \leq', w')$ are isomorphic if there is a bijection $f : S \to S'$, called an isomorphism, such that for all $x, y \in S$, we have $x \leq y$ if and only if $f(x) \leq' f(y)$ and $w(x, y) = w'(f(x), f(y))$. An automorphism of a poset $(S, \leq, w)$ is an isomorphism from $(S, \leq, w)$ to itself.

### 1.1.6 Partitions

A partition of a positive integer $n$ is a tuple $\lambda := (\lambda_1, \ldots, \lambda_k)$ of integers, where $\lambda_1 \geq \cdots \geq \lambda_k > 0$, and $\sum \lambda_i = n$. The length of a partition $\lambda$, denoted by $l(\lambda)$, is the number of elements in $\lambda$. Let $\lambda := (\lambda_1, \ldots, \lambda_k)$ and $\mu := (\mu_1, \ldots, \mu_l)$ be two partitions of $n$. We say that $\lambda$ refines $\mu$, and write $\lambda \models \mu$, if there is an onto map $f : [1, k] \to [1, l]$ such that $\sum_{j \in f^{-1}(i)} \lambda_j = \mu_i$ for all $i \in [1, l]$. The refinement relation makes the set of partitions of $n$ a lattice. We also write $\lambda \models n$ to say that $\lambda$ is a partition of $n$.

Let $\pi := \{X_1, \ldots, X_k\}$ be a family of mutually disjoint non-empty subsets of $V$. We write $\pi \vdash V$. If $\pi$ is a partition of $V$, we write $\pi \models V$; we associate with $\pi$ an integer partition $\lambda(\pi)$ of $n$ obtained by ordering $|X_i|$ in a non-increasing order. Let $\pi$ and $\sigma$ be partitions of $V$. We say that $\pi$ refines $\sigma$, and write $\pi \models \sigma$, if each block of $\pi$ is a subset of some block of $\sigma$. The refinement relation makes the set of partitions of $V$ a lattice. It is called the partition lattice of $V$, and is denoted by $\Pi(V)$.

### 1.2 Induced subgraph posets and Ulam’s conjecture

**Definition 1.1.** Define a partial order $\leq_v$ on $\mathcal{G}/\cong$ as follows: for all $F_i, F_j \in \mathcal{G}/\cong$, $F_i \leq_v F_j$ if and only if $F_i$ is an induced subgraph of $F_j$. Define a weight function $w_v : (\mathcal{G}/\cong) \times (\mathcal{G}/\cong) \to \mathbb{N}$ as follows: for all $F_i, F_j \in \mathcal{G}/\cong$, $(F_i, F_j) := \text{ind}(F_i, F_j)$. Let $\mathcal{P} := (\mathcal{G}/\cong, \leq_v, w_v)$. For $G \in \mathcal{G}/\cong$, let $\mathcal{P}(G) := \{F \in \mathcal{G}/\cong \mid F \leq_v G, \text{ and } n(F) = 1 \text{ or } \epsilon(F) > 0\}$. The concrete induced subgraph poset of $G$ is the restriction of $\mathcal{P}$ to $\mathcal{P}(G)$; it is denoted by just $\mathcal{P}(G)$. The abstract induced subgraph poset is the isomorphism class of $(\mathcal{P}(G), \leq_v, w_v)$. We take $\mathcal{P}(G) := \{G_1, \ldots, G_M\}$, where $G_1, \ldots, G_M$ are distinct unlabelled graphs, and $\overline{\mathcal{P}}(G) := (\{g_1, \ldots, g_M\}, \leq_v, w_v)$ to be a representative poset isomorphic to $\mathcal{P}(G)$. Moreover, we assume that there is an isomorphism $f : \overline{\mathcal{P}}(G) \to \mathcal{P}(G)$ such that $f(g_i) = G_i$ for all $i$. We assume that the minimum element in $\overline{\mathcal{P}}(G)$
is $g_1$ (thus $G_1 = K_1$), and that $g_2$ covers $g_1$ (thus $G_2 = K_2$). Since $\epsilon(G_i) = w_v(g_2, g_i)$, we define $\epsilon(g_i) := w_v(g_2, g_i)$. We assume that the maximal element in $\mathcal{P}(G)$ is $g_M$ (thus $G_M = G$). We define a rank function $\nu$ on $\mathcal{P}(G)$ so that $\nu(g_1) := 1$; thus $\nu(g_i) = \nu(G_i)$ for all $i$.

**Remark.** When a graph $G$ is labelled, we define $\mathcal{P}(G) := \mathcal{P}(I(G))$, and $\mathcal{P}(G) := \mathcal{P}(I(G))$, and so on, where $I(G)$ is the isomorphism class of $G$. The set $\mathcal{P}(G)$ contains only unlabelled graphs.

**Definition 1.2.** An unlabelled graph $G$ is said to be $P$-reconstructible if it is determined by $\mathcal{P}(G)$. A labelled graph $G$ is said to be $P$-reconstructible if its isomorphism class is determined by $\mathcal{P}(G)$. A class of graphs is said to be $P$-reconstructible if each graph in the class is $P$-reconstructible.

**Example 1.3.** An unlabelled graph $G$, the graphs in its concrete induced subgraph poset along with their multiplicities, and the induced subgraph poset of $G$ (concrete and abstract) are shown in Figure 1. Note that the Hasse diagram is only for illustration; it does not display all weights. But the weights on all related pairs can be computed from the weights shown in the Hasse diagram; see Lemma 2.3 in [19].

![Diagram](image)

Figure 1: An unlabelled graph $G$ and the graphs in its concrete induced subgraph poset along with their multiplicities are shown in the top two rows. In the bottom row, the concrete induced subgraph poset is shown on the left; the abstract induced subgraph poset is shown in the middle; a consistent labelling of its elements by formal sums of indeterminates (see Definition 3.1) is shown on the right.

In [19], we considered the $P$-reconstruction problem, i.e., the problem of reconstructing $G$
up to isomorphism or computing some of its invariants from \( \mathcal{P}(G) \). The following theorem summarises our results from [19].

**Theorem 1.4.**

1. Ulam’s conjecture is true if and only if every non-empty graph can be reconstructed up to isomorphism from its abstract induced subgraph poset.

2. Ulam’s conjecture is true if and only if the abstract induced subgraph poset of every non-empty graph has only the trivial automorphism.

3. Every tree can be reconstructed up to isomorphism from its abstract induced subgraph poset.

4. The following invariants of a graph \( G \) can be computed from its abstract induced subgraph poset:
   
   (a) the number of spanning trees in \( G \); and hence whether \( G \) is connected,
   
   (b) the number of unicyclic subgraphs of \( G \) containing a cycle of a specified length; and hence the number of Hamiltonian cycles in \( G \),
   
   (c) the number of spanning subgraphs of \( G \) having specified numbers of vertices and edges in their components,
   
   (d) the characteristic polynomial of \( G \), the chromatic polynomial of \( G \), and the rank polynomial of \( G \).

The invariants listed above were originally proved to be reconstructible by Tutte [20]. The above results are slightly stronger than the results of Tutte in the sense that to compute the invariants listed above for a graph, we do not need to know its deck - its abstract induced subgraph poset is sufficient.

### 1.3 The connected partition lattice

Computing the graph invariants listed in Theorem 1.4-(4) (in our proofs as well as in the proofs by Tutte [20] and Kocay [8]) requires first counting certain disconnected spanning subgraphs. When a deck is given, counting disconnected spanning subgraphs is made easier by a lemma of Kocay [8]. In the proof of Theorem 1.4, since the deck is not given, counting disconnected spanning subgraphs with a given number of components, and a given number of vertices and edges in each component, is more difficult. We can nevertheless imitate Kocay’s lemma by using minimal information about the graphs in the induced subgraph poset (e.g., \( \nu(G_i) \) and \( \epsilon(G_i) \) for each \( G_i \) in \( \mathcal{P}(G) \)). Counting disconnected spanning subgraphs would be easier if we were given the lattice of connected partitions.

In the proof of Theorem 1.4-(4), we implicitly constructed and used partial information about the connected partition lattice. We commented in [19, Section 5] that understanding the relationship between the induced subgraph poset of a graph and its connected partition lattice would be interesting. One of the objectives of this paper is to clarify this relationship.

First we define the connected partition lattice of a labelled graph. Recall that when \( \pi := \{X_1, \ldots, X_m\} \) is a family of disjoint non-empty subsets of a set \( V \), we write \( \pi \vdash V \); and when \( \pi := \{X_1, \ldots, X_m\} \) is a partition of \( V \), we write \( \pi \models V \).

**Definition 1.5.** Let \( G \) be a labelled graph. Let \( \pi := \{X_1, \ldots, X_m\} \) be a family of disjoint non-empty subsets of \( V(G) \). We say that \( \pi \) is a connected family (or \( \pi \) is a connected partition in case \( \pi \) is a partition of \( V(G) \)) if subgraphs \( G[X_k], k \in [m] \) induced by the blocks of \( \pi \) are all
connected; in this case we write \( \pi \vdash_c V(G) \) (or \( \pi \models_c V(G) \) in case \( \pi \) is a partition of \( V(G) \)). We define \( G[\pi] := \bigcup_{k \in [m]} G[X_k] \), and call it the subgraph of \( G \) induced by \( \pi \). Let \( \vdash_c \) be a partial order on \( G \) defined by: \( H \vdash_c G \) if there exists a connected family \( \pi \vdash_c V(G) \) such that \( G[\pi] = H \). Let \( \Pi^c_G \) denote the set of subgraphs of \( G \) induced by its connected partitions, i.e., \( \Pi^c_G := \{ G[\pi] \mid \pi \vdash_c V(G) \} \). The connected partition lattice of \( G \) is the restriction of \( (G, \vdash_c) \) to \( \Pi^c_G \); it is denoted by just \( \Pi^c_G \). The unique minimal element \( \hat{0} \) of \( \Pi^c_G \) is the finest partition of \( V(G) \) consisting of only singletons.

Next we define analogous notions for an unlabelled graph. Recall that given an unlabelled graph \( H \), we denote a representative labelled graph in \( H \) by \( H^* \).

**Definition 1.6.** The partial order \( \vdash_c \) on the set of labelled graphs induces a partial order \( \vdash_c \) on the set of unlabelled graphs. We define it by considering labelled representatives of unlabelled graphs. For unlabelled graphs \( H_j, H_k \), we define \( H_j \vdash_c H_k \) if there exists \( \pi \vdash_c V(H_k^*) \) such that \( H_k^*[\pi] \cong H_j^* \). We define a *weight function* \( w_e : (\mathcal{G}/\cong) \times (\mathcal{G}/\cong) \to \mathbb{N} \) by \( w_e(H_j, H_k) := |\{ \pi \vdash_c V(H_k^*) \mid H_k^*[\pi] \in H_j \}| \). Thus we have a weighted partially ordered set \( \Omega := (\mathcal{G}/\cong, \vdash_c, w_e) \). The *folded connected partition lattice* of a labelled graph \( G \) is the restriction of \( \Omega \) to \( I(\Pi^c_G) \). (Recall that \( I(\Pi^c_G) \) is the set of distinct isomorphism classes of graphs in \( \Pi^c_G \).) We denote the set \( I(\Pi^c_G) \) as well as the folded connected partition lattice of \( G \) by just \( \Omega(G) \). The *abstract folded connected partition lattice* of \( G \) is the isomorphism class of \( \Omega(G) \). We take \( \Omega(G) := \{ H_1, \ldots, H_N \} \), where \( H_1, \ldots, H_N \) are distinct unlabelled graphs, and \( \Omega(G) := ((h_1, \ldots, h_N), \vdash_c, w_e) \) to be a representative poset isomorphic to \( \Omega(G) \). Moreover, we assume that there is an isomorphism \( f : \overline{\Omega}(G) \to \Omega(G) \) such that \( f(h_i) = H_i \) for all \( i \). We assume that the minimal element in \( \overline{\Omega}(G) \) is \( h_1 \) (thus \( H_1 = \nu(G)K_1 \)), and the maximal element in \( \overline{\Omega}(G) \) is \( h_N \) (thus \( H_N = I(G) \)). We define a rank function \( \rho \) on \( \overline{\Omega}(G) \) so that \( \rho(h_1) = 0 \); hence \( c(H_i) = \nu(G) - \rho(h_i) \) for all \( i \).

**Remark.** When \( G \) is an unlabelled graph, we define \( \Omega(G) \) and \( \overline{\Omega}(G) \) in terms of a representative labelled graph; i.e., \( \Omega(G) := \Omega(G^*) \) and \( \overline{\Omega}(G) := \overline{\Omega}(G^*) \). Regardless of whether \( G \) is labelled or unlabelled, the set \( \Omega(G) \) contains only unlabelled graphs.

**Definition 1.7.** An unlabelled graph \( G \) is said to be *\( \Pi \)-reconstructible* if it is determined by \( \overline{\Omega}(G) \). A labelled graph \( G \) is said to be *\( \Pi \)-reconstructible* if its isomorphism class is determined by \( \overline{\Omega}(G) \). A class of graphs is said to be *\( \Pi \)-reconstructible* if each graph in the class is \( \Pi \)-reconstructible.

**Remark.** The connected partition lattice of a graph \( G \) forms a geometric sub-lattice of the partition lattice \( \Pi(V(G)) \). Elsewhere in the literature, it has been called the lattice of contractions or the *bond lattice* of \( G \); see Stanley [16, 18], Chow [3]. The lattice \( \Omega(G) \) (without its weights) is obtained by identifying those partitions in \( \Pi^c_G \) that induce isomorphic graphs; hence we think of \( \Omega(G) \) as the *folded connected partition lattice of \( G \). We use the term *abstract bond lattice* only for the abstract folded connected partition lattice, and the term *concrete connected partition lattice* for the labelled lattice \( \Pi^c_G \) defined above.

**Example 1.8.** An unlabelled graph \( G \) and its connected partitions along with their multiplicities are shown in Figure 2. The graphs \( G_1, G_2, \ldots, G_6 = G \) are the same graphs as in Figure 1. The bond lattice of \( G \) (at different levels of abstraction) is shown in Figure 3. To be precise, the lattice in the middle is abstract folded connected partition lattice or the abstract bond lattice. A consistent labelling of its elements by formal sums of indeterminates, indicating the component structure of graphs induced by connected partitions, is shown on the right. The notion of consistent labelling is formalised in Definition 3.1. As in the case of the induced subgraph poset, the Hasse diagram is only for illustration; it does not display the weights on all related pairs of graphs in \( \overline{\Omega}(G) \).
Example 1.9. The graphs $K_{1,n}$ and $nK_2$ have the same abstract bond lattice, which is a total order with appropriate weights. Moreover, adding isolated vertices to a graph does not change its abstract bond lattice. But no non-empty graph other than graphs isomorphic to $nK_2 + mK_1$ or $K_{1,n} + mK_1$ for some $n,m \in \mathbb{N}$ has a totally ordered abstract bond lattice. We do not know any other non-trivial pairs of nonisomorphic graphs that have the same abstract bond lattice. On the other hand, the graphs $K_{1,n}$ and $nK_2$ have distinct induced subgraph posets.

\[ \begin{array}{c}
1 \times (G = G_6) & 4 \times (G_5 + G_1) & 2 \times (G_4 + G_2) & 2 \times (G_4 + 2G_1) \\
5 \times (G_3 + 2G_1) & 6 \times (G_2 + 3G_1) & 1 \times (5G_1) \\
\end{array} \]

Figure 2: An unlabelled graph $G$ and its distinct connected partitions along with their multiplicities.

In Section 2, we introduce Kocay’s lemma (Lemma 2.2) and two variants of it, namely, Lemma 2.4 for induced subgraphs and Lemma 2.7 for abstract induced subgraph posets. Given a list of graphs $(F_1, \ldots, F_k) \in \mathcal{G}^k$ and the deck of $G$, Kocay’s lemma (see Bondy [2] or Kocay [8]) shows how to compute the number of covers of $G$ by $(F_1, \ldots, F_k)$, i.e., the number of tuples $(H_1, \ldots, H_k)$ of subgraphs of $G$ such that for all $i = 1, \ldots, k$ we have $H_i \cong F_i$ and $\bigcup_i H_i = G$.

Our generalisation in Lemma 2.7 demonstrates how we may compute the number of covers by a list of unknown induced subgraphs of $G$ (i.e., the graphs $I(F_i)$ are specified only as elements
of the abstract induced subgraph poset). We illustrate Lemma 2.7 by giving simple proofs that for every nonempty graph $G$, the chromatic symmetric function of $G$ and the symmetric Tutte polynomial of $G$ (both introduced by Stanley [16, 17]) are $P$-reconstructible.

In Section 3, we give two constructions: we show, using counting arguments based on Kocay’s lemma that are developed in Section 2, that the abstract bond lattice of every graph can be constructed from its abstract induced subgraph poset (Theorem 3.3); and that the abstract induced subgraph poset of every graph that is not a star or a disjoint union of edges and that has no isolated vertices can be constructed from its abstract bond lattice (Theorem 3.6). Theorem 3.3 and Theorem 1.4-(3) together imply Corollary 3.7-(2) that every tree (or forest) on 2 or more vertices that is not a star or a disjoint union of edges and that has no isolated vertices can be reconstructed up to isomorphism from its abstract bond lattice. In Section 4, we give short proofs of the reconstructibility of the symmetric Tutte polynomial and the chromatic symmetric function. In particular, we give another expansion of the chromatic symmetric function based on the abstract bond lattice (Corollary 4.7).

Another motivation of this paper is a question of Stanley regarding the chromatic symmetric function: can two non-isomorphic trees have the same chromatic symmetric function? Corollary 3.7-(2) suggests that one way to approach Stanley’s question may be to try to construct the abstract bond lattice of a tree from its chromatic symmetric function. In Section 4.3, we give a few preliminary results, which we summarise below.

Let $T$ be a tree with vertex set $V(T)$. Let $v := (v_1, \ldots, v_r)$ and $e := (e_1, \ldots, e_r)$ be two integer vectors. Let $\theta(v, e; T)$ be the number of ordered partitions $(V_1, \ldots, V_r)$ of $V(T)$ such that $|V_i| = v_i$, and $e(V_i) = e_i$. We show in Lemma 4.11 that for all trees $T$ and for all vectors $v$, $e$, we can compute $\theta(v, e; T)$ from the chromatic symmetric function $X_T(x)$ of $T$. We apply the lemma to prove that the symmetric Tutte polynomial of every tree can be obtained from its chromatic symmetric function, which is not the case for graphs in general. Noble and Welsh [14] have asked if trees are distinguished by their U-polynomial, which, for trees, is equivalent to the symmetric Tutte polynomial. Our result shows that the questions of Noble & Welsh and Stanley are in fact equivalent. The collection of invariants $\theta(v, e; T)$ may have sufficient information to construct the abstract bond lattice, and hence the tree by Corollary 3.7-(2).

### 1.4 The edge-subgraph poset

**Definition 1.10.** Define a partial order $\leq_e$ on $G/\cong$ as follows: for all $F_i, F_j \in G/\cong$, $F_i \leq_e F_j$ if and only if $F_i$ is an edge-subgraph of $F_j$. Define a weight function $w_e : (G/\cong) \times (G/\cong) \to \mathbb{N}$ as follows: for all $F_i, F_j \in G/\cong$, $w_e(F_i, F_j) := e(F_i, F_j)$. Let $Q := (G/\cong, \leq_e, w_e)$.

For $G \in G/\cong$, let $Q(G) := \{F \in G/\cong \mid F \leq_e G \text{ and } e(F) > 0\}$. The concrete edge-subgraph poset of $G$ is the restriction of $Q$ to $Q(G)$; it is denoted by just $Q(G)$. The abstract edge-subgraph poset of $G$ is the isomorphism class of $(Q(G), \leq_e, w_e)$. We take $Q(G) := \{G_1, \ldots, G_M\}$, where $G_1, \ldots, G_M$ are distinct unlabelled graphs, and $\overline{Q}(G) := (\{g_1, \ldots, g_M\}, \leq_e, w_e)$ to be a representative poset isomorphic to $Q(G)$. We assume that there is an isomorphism $f : \overline{Q}(G) \to Q(G)$ such that $f(g_i) = G_i$ for all $i$; hence $\overline{Q}(g_i) \cong Q(G_i)$ for $i = 1$ to $M$. We assume that the minimum element in $Q(G)$ is $g_1$ (thus $G_1 = K_2$), and that the maximal element in $Q(G)$ is $g_M$ (thus $G_M = G$). We define a rank function $\rho$ on $\overline{Q}(G)$ so that $\rho(g_1) := 1$; thus $\rho(g_i) = e(G_i)$ for all $i$.

**Remark.** When a graph $G$ is labelled, we define $Q(G) := Q(I(G))$, and $\overline{Q}(G) := \overline{Q}(I(G))$, where $I(G)$ is the isomorphism class of $G$. The set $Q(G)$ contains only unlabelled graphs.

**Definition 1.11.** An unlabelled graph $G$ is said to be $Q$-reconstructible if it is determined by $\overline{Q}(G)$. A labelled graph $G$ is said to be $Q$-reconstructible if its isomorphism class is determined...
A class of graphs is said to be $Q$-reconstructible if each graph in the class is $Q$-reconstructible. For all $G \in \mathcal{G} / \cong$, we have $\overline{Q}(G) = \overline{Q}(G + K_1)$. Therefore, we understand $Q$-reconstructibility to mean $Q$-reconstructibility modulo isolated vertices. A $Q$-set is a maximal set of cardinality at least 2 of unlabelled graphs that have no isolated vertices, and have isomorphic edge-subgraph poset. A $Q$-pair is 2-element subset of a $Q$-set.

We ask which graphs are $Q$-reconstructible, and if there is a relation between the problem of $Q$-reconstructibility and the edge reconstruction conjecture of Harary (analogous to Theorem 1.4-(4) for Ulam’s conjecture). The question of $Q$-reconstructibility is not quite similar to the analogous question for induced subgraphs. If a graph is not edge reconstructible, then it is also not $Q$-reconstructible. It turns out that there many graphs that are edge reconstructible but not $Q$-reconstructible.

Example 1.12. Figure 4 shows part of an edge-subgraph poset: its elements are labelled by the corresponding graphs on the left; a representative isomorphic poset is shown on the right. Graphs $K_{1,2}$, $2K_2$, $K_{1,3}$, and $K_3$ are not $Q$-reconstructible, since they are not edge reconstructible. There are many more graphs that are not $Q$-reconstructible. For example, the down-sets of $g_4$, $g_7$, and $g_8$ are isomorphic, and correspond to the $Q$-set $\{3K_2, K_{1,3}, K_3\}$. In general, for all $m > 1$, $\{K_{1,m}, mK_2\}$ is a $Q$-set. The down-sets of $mK_2$ and $K_{1,m}$ are totally ordered, with suitable edge weights. The down-sets of $g_{10}$ and $g_{11}$ are isomorphic, and correspond to the $Q$-set $\{P_4 + K_2, T_4\}$, where the graph $T_4$ is defined in Figure 6.

Figure 4: Part of an edge-subgraph poset: the poset on the left is labelled by the graphs corresponding to its elements; the poset on the right is an isomorphic representative poset.

In Theorem 5.1, which is the main result of Section 5, we construct an infinite family of graphs that are not $Q$-reconstructible, and show that the edge reconstruction conjecture is true if and only if these are the only graphs that are not $Q$-reconstructible.

Section 6 is motivated by edge reconstruction as well as two results of Lovász [9] known as homomorphism cancellation laws. Let $\text{hom}(G, H)$ denote the number of homomorphisms from $G$ to $H$. Lovász proved that if $G$ and $H$ are any two finite simple graphs such that $\text{hom}(G, F) = \text{hom}(H, F)$ for all simple graphs $F$, then $G$ and $H$ are isomorphic. Lovász also proved an analogous complimentary result (replacing $\text{hom}(G, F)$ by $\text{hom}(F, G)$, and $\text{hom}(H, F)$ by $\text{hom}(F, H)$ in the statement given above). Analogous to $\overline{P}(G)$, $\overline{U}(G)$ and $\overline{Q}(G)$, we may consider the isomorphism class of a weighted complete binary relation $\mathcal{R}$ on the set of unlabelled graphs, defined by $\mathcal{R} := ((\mathcal{G} / \cong) \times (\mathcal{G} / \cong), \text{hom})$, which assigns each pair $(G, H)$ of unlabelled graphs a weight $\text{hom}(G, H)$. We now ask a reconstruction-type question: does the isomorphism
class of \( \mathcal{R} \) determine uniquely the unlabelled graphs corresponding to its elements? Equivalently, does \( \mathcal{R} \) have only the trivial automorphism? We conjecture that it does not have non-trivial automorphisms; or equivalently, if \( f : (\mathcal{G}/\cong) \to (\mathcal{G}/\cong) \) is a bijection such that \( \text{hom}(G, H) = \text{hom}(f(G), f(H)) \) for all \( G \) and \( H \) in \( \mathcal{G}/\cong \), then \( f \) is the identity map. Similarly, for labelled graphs, we conjecture that if \( f : \mathcal{G} \to \mathcal{G} \) is a bijection such that \( \text{hom}(G, H) = \text{hom}(f(G), f(H)) \) for all \( G, H \in \mathcal{G} \), then \( G \cong H \). We show in Proposition 6.4 that this conjecture is weaker than the edge reconstruction conjecture. Note also that the statement of the conjecture is analogous to Theorem 1.4-(2).

2 Kocay’s lemma and its generalisations

Computations of many interesting invariants of a graph require knowing certain spanning subgraphs. For example, to compute the characteristic polynomial of a graph, we need to know the number of subgraphs of each isomorphism type in the class of graphs whose connected components are cycles or edges; see Biggs [1]. Kocay [8] gave a counting argument for counting some types of spanning subgraphs of a graph, given its deck. In this section we prove variants Kocay’s lemma and develop other counting arguments that use similar ideas as in Kocay’s lemma. The main objective here is to do similar computations on the abstract induced subgraph poset.

**Definition 2.1.** Let \( H \in \mathcal{G} \) and \( \mathcal{F} := (F_1, \ldots, F_k) \in \mathcal{G}^k \). A cover of \( H \) by \( \mathcal{F} \) is a tuple \((H_1, \ldots, H_k)\) of subgraphs of \( H \) such that \( H_i \cong F_i \) for all \( i \), and \( \bigcup_i H_i = H \). Let \( \text{cov}(\mathcal{F} \to H) \) denote the number of covers of \( H \) by \( \mathcal{F} \). When \( \mathcal{F} \) is a tuple of unlabelled graphs or \( H \) is an unlabelled graph, we define \( \text{cov}(\mathcal{F} \to H) := \text{cov}(\mathcal{F}^* \to H^*) \), where \( \mathcal{F}^* := (F_1^*, \ldots, F_k^*) \).

**Lemma 2.2.** (Kocay’s Lemma [8]) Let \( \mathcal{F} := (F_1, \ldots, F_k) \in \mathcal{G}^k \). We have

\[
\prod_{i=1}^k \text{sub}(F_i, G) = \sum_{H \subseteq G} \text{cov}(\mathcal{F} \to H) = \sum_{H \in \mathcal{G}/\cong} \text{cov}(\mathcal{F} \to H) \text{sub}(H, G). \tag{1}
\]

Equivalently,

\[
\sum_{H \in \mathcal{G}/\cong, \nu(H) = \nu(G)} \text{cov}(\mathcal{F} \to H) \text{sub}(H, G) = \prod_{i=1}^k \text{sub}(F_i, G) - \sum_{H \subseteq G, \nu(H) < \nu(G)} \text{cov}(\mathcal{F} \to H) \text{sub}(H, G). \tag{2}
\]

Moreover, if \( \nu(F_i) < \nu(G) \) for all \( i \), then the left side of Equation (2) is reconstructible.

Next we prove Kocay’s lemma for induced subgraphs.

**Definition 2.3.** Let \( H \in \mathcal{G} \) and \( \mathcal{F} := (F_1, \ldots, F_k) \in \mathcal{G}^k \). A vertex cover of \( H \) by \( \mathcal{F} \) is a tuple \((H_1, \ldots, H_k)\) of induced subgraphs of \( H \) such that \( H_i \cong F_i \) for all \( i \), and \( \bigcup_i V(H_i) = V(H) \). Let \( \text{cov}(\mathcal{F} \rightarrow H) \) denote the number of vertex covers of \( H \) by \( \mathcal{F} \). When \( \mathcal{F} \) is a tuple of unlabelled graphs or \( H \) is an unlabelled graph, we define \( \text{cov}(\mathcal{F} \rightarrow H) := \text{cov}(\mathcal{F}^* \rightarrow H^*) \), where \( \mathcal{F}^* := (F_1^*, \ldots, F_k^*) \).

Note that a vertex cover of \( H \) does not necessarily cover all edges of \( H \).

**Lemma 2.4.** (Kocay’s Lemma for induced subgraphs) Let \( \mathcal{F} := (F_1, \ldots, F_k) \in \mathcal{G}^k \). We have

\[
\prod_{i=1}^k \text{ind}(F_i, G) = \sum_{H \subseteq \mathcal{G}/\cong} \text{cov}(\mathcal{F} \rightarrow H) = \sum_{H \in \mathcal{G}/\cong} \text{cov}(\mathcal{F} \rightarrow H) \text{ind}(H, G). \tag{3}
\]

Moreover, if \( \nu(F_i) < \nu(G) \) for all \( i \), then \( \text{cov}(\mathcal{F} \rightarrow G) \) is reconstructible.
Proof. Equation (3) is self-explanatory. To prove the second part, we write Equation (3) in the following form:

$$\text{cov}(F \toup G) = \prod_{i=1}^{k} \text{ind}(F_i, G) - \sum_{H \in \mathcal{G}^\sigma} \text{cov}(F \toup H) \text{ind}(H, G),$$

which is similar to Equation (2), except that now there is only one term on the left side. \hfill \blacksquare

In Lemma 2.5 and its proof, we use the following notation. Let $S := \bigsqcup_{i=1}^{m} S_i$ be a graph with connected components $S_i, i = 1, \ldots, m$. Let $S := (S_1, \ldots, S_m)$. For each $i \in \sigma([1, m])$, we write $S_{\sigma^{-1}(i)} := \bigsqcup_{j \in \sigma^{-1}(i)} S_j$ and $S_{\sigma^{-1}(i)} := (S_i, i \in \sigma^{-1}(k))$ (the restricted tuple).

We say that two maps $\sigma: [1, m] \rightarrow [1, n]$ and $\pi: [1, m] \rightarrow [1, n]$ are equivalent if $\sigma([1, m]) = \pi([1, m])$ and $S_{\sigma^{-1}(i)} \cong S_{\pi^{-1}(i)}$ for all $i \in \sigma([1, m])$, and inequivalent otherwise. Let $\Sigma$ be a set of mutually inequivalent maps such that exactly one representative of each equivalence class of maps is in $\Sigma$.

**Lemma 2.5.** Let $S := \bigsqcup_{k=1}^{m} S_k$ and $T := \bigsqcup_{k=1}^{m} T_k$ be two graphs with connected components $S_i, i \in [1, m]$ and $T_i, i \in [1, n]$, respectively. Let $S := (S_1, \ldots, S_m)$. Then

$$\text{cov}(S \toup T) = \sum_{\sigma: [1, m] \rightarrow [1, n]} \prod_{k=1}^{n} \text{cov}(S_{\sigma^{-1}(k)} \toup T_k),$$

and

$$\text{ind}(S, T) = \sum_{\sigma \in \Sigma} \prod_{k \in \sigma([1, m])} \text{ind}(S_{\sigma^{-1}(k)}, T_k).$$

Proof. Each vertex cover $(H_1, \ldots, H_m)$ of $T$ by $S$ corresponds to a unique onto map $\sigma: [1, m] \rightarrow [1, n]$ such that $\sigma(i) = j$ if and only if $H_i \subseteq T_j$. Thus the set of vertex covers is partitioned into subsets indexed by onto maps $\sigma: [1, m] \rightarrow [1, n]$. The number of vertex covers that correspond to a fixed onto map $\sigma$ is the product term in the summation in Equation (5).

Each induced subgraph $H$ isomorphic to $S$ corresponds to a unique $\sigma \in \Sigma$ such that for all $k \in \sigma([1, m])$ we have $H[V(T_k)] = S_{\sigma^{-1}(k)}$. Thus the set of induced subgraphs of $T$ that are isomorphic to $S$ is partitioned into subsets indexed by maps in $\Sigma$. The number of induced subgraphs that correspond to a fixed $\sigma \in \Sigma$ is given by the product term in Equation (6). \hfill \blacksquare

**Lemma 2.6.** Let $\mathcal{H} \subseteq \mathcal{G}$ be a class of connected graphs that is closed under connected induced subgraphs (i.e., if a connected graph $F$ is in $\mathcal{H}$, then every connected induced subgraph of $F$ is also in $\mathcal{H}$). Let $I(\mathcal{H}) := \{H_1, H_2, \ldots \}$. Let $S := \bigsqcup_{k=1}^{m} S_k$ and $T := \bigsqcup_{k=1}^{m} T_k$ be two graphs such that for all $i \in \mathcal{H}$ and for all $j \in \mathcal{H}$. Let $S := (S_1, \ldots, S_m)$. Then $\text{ind}(S, T)$ and $\text{cov}(S \toup T)$ are functions of $\text{ind}(H_i, H_j), i, j \geq 0$.

Proof. If $\nu(S) = \nu(T)$, then $\text{ind}(S, T) = 1$ or 0 depending, respectively, on whether $S$ and $T$ are isomorphic or not. If $\nu(S) < \nu(T)$, then by Kelly’s lemma [6], we have

$$\text{ind}(S, T) = \frac{\sum_{v \in V(T)} \text{ind}(S, T - v)}{\nu(T) - \nu(S)}.$$

Since each component of each induced subgraph $T - v$ is in $\mathcal{H}$, an inductive argument on $\nu(T)$ implies that $\text{ind}(S, T)$ is a function of $\text{ind}(H_i, H_j), i, j \geq 0$. 

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We prove by induction on $\nu(T)$ that $\text{cov}(S \xrightarrow{v} T)$ is a functions of $\text{ind}(H_i, H_j), i, j \geq 0$. When $\nu(T) = 1$, we have $\text{ind}(S, T) = \text{cov}(S \xrightarrow{v} T) = 1$ if $S \in K_1$, and $\text{ind}(S, T) = \text{cov}(S \xrightarrow{v} T) = 0$ otherwise. Let the result be true when $1 \leq \nu(T) < k$. Let $\nu(T) = k$. By Lemma 2.4, we have

$$\text{cov}(S \xrightarrow{v} T) = \prod_{i=1}^{m} \text{ind}(S_i, T) - \sum_{H \in \mathcal{G} / \geq \nu} \text{cov}(S \xrightarrow{v} H) \text{ind}(H, T).$$

Each factor $\text{ind}(S_i, T)$ in the first term is a function of $\text{ind}(H_i, H_j), i, j \geq 0$. If there is an unlabelled graph $H$ such that $\nu(H) < \nu(T)$ that contributes to the summation, then $\text{ind}(H, T) \neq 0$: hence all connected components of $H$ are in $\mathcal{H}$; now by the induction hypothesis the factor $\text{cov}(S \xrightarrow{v} H)$ is a function of $\text{ind}(H_i, H_j), i, j \geq 0$, and $\text{ind}(H, T)$ is a function of $\text{ind}(H_i, H_j), i, j \geq 0$ as shown above. This completes the induction step for $\text{cov}(S \xrightarrow{v} T)$. \hfill \blacksquare

In the rest of this section we show how most of the computations done above are also possible given the abstract induced subgraph poset of a graph. For example, to compute various invariants of a graph $G$, given $\overline{\mathcal{P}}(G)$, we would need a lemma analogous to Lemma 2.4 for tuples of elements $f_i$ of $\overline{\mathcal{P}}(G)$ that only implicitly specify unlabelled graphs $F_i$. Since the induced subgraph poset does not include empty graphs, we consider tuples $(f_1, \ldots, f_k)$, where each $f_i$ is either an empty graph $F_i = r_iK_1$ or an element of $\overline{\mathcal{P}}(G)$, in which case it represents an unknown unlabelled graph $F_i$. Let $\mathcal{F} := (F_1, \ldots, F_k)$. If $\nu(f_i) < \nu(G)$ for all $i \in [1, k]$, then $\text{cov}(\mathcal{F} \xrightarrow{v} G)$ can be computed from $\overline{\mathcal{P}}(G)$.

**Lemma 2.7. (Kocay’s Lemma for abstract induced subgraph posets)** Let $(f_1, \ldots, f_k)$ be a $k$-tuple, where each $f_i$ is either an empty graph $F_i = r_iK_1$ or an element of $\overline{\mathcal{P}}(G)$, in which case it represents an unknown unlabelled graph $F_i$. Let $\mathcal{F} := (F_1, \ldots, F_k)$. If $\nu(f_i) < \nu(G)$ for all $i \in [1, k]$, then $\text{cov}(\mathcal{F} \xrightarrow{v} G)$ can be computed from $\overline{\mathcal{P}}(G)$.

**Proof.** Let $H \leq_v G$. If it is non-empty, it corresponds to an element $h$ of $\overline{\mathcal{P}}(G)$. In this case, $\nu(H) = \nu(h)$ (which is the rank of $h$ in $\overline{\mathcal{P}}(G)$, as defined earlier). For any $i$, we have

$$\text{ind}(F_i, H) = \begin{cases} w_v(f_i, h) - \sum_{g \in \overline{\mathcal{P}}(G): \nu(g) = r_i} w_v(g, h) & \text{if } F_i \text{ is non-empty,} \\ (r_i) & \text{if } F_i = r_iK_1. \end{cases}$$

On the other hand, $H$ itself could be empty, say $H = rK_1$, in which case

$$\text{ind}(F_i, H) = \begin{cases} 0 & \text{if } F_i \text{ is non-empty,} \\ (r_i) & \text{if } F_i = r_iK_1. \end{cases}$$

If $\text{cov}(\mathcal{F} \xrightarrow{v} H)$ is $P$-reconstructible for each $H <_v G$, then $\text{cov}(\mathcal{F} \xrightarrow{v} G)$ is $P$-reconstructible by Equation (4). We show how $\text{cov}(\mathcal{F} \xrightarrow{v} H)$ may be computed recursively on $\overline{\mathcal{P}}(G)$, given $(f_1, \ldots, f_k)$.

Let $r := \max\{\nu(F_1), \ldots, \nu(F_k)\}$. If $\nu(H) < r$, then $\text{cov}(\mathcal{F} \xrightarrow{v} H) = 0$. Therefore, we take $\nu(H) = r$ as the base case. There are two possibilities. If $H$ is an empty graph, then $\text{cov}(\mathcal{F} \xrightarrow{v} H) = 0$ if there are non-empty graphs in $\mathcal{F}$; if there are no non-empty graphs in $\mathcal{F}$, then $\text{cov}(\mathcal{F} \xrightarrow{v} H)$ can be calculated from $r_i, i = 1, \ldots, k$. If $H$ is non-empty, and $h$ is the corresponding element of $\overline{\mathcal{P}}(G)$, then

$$\text{cov}(\mathcal{F} \xrightarrow{v} H) = \prod_{i=1}^{k} \text{ind}(F_i, H),$$

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where the factors on the right are computed in Equations (7) and (8); and since $\nu(H) = r = \max\{\nu(F_1), \ldots, \nu(F_k)\}$, there are no other terms as in Equation (4). Now we proceed by induction on $\nu(H)$ (with Equation (4)) to compute $\text{cov}(F \to G)$.

\textbf{Remark.} We can now define $\text{cov}((f_1, \ldots, f_k) \to G)$ to mean $\text{cov}(F \to G)$ whenever $(f_1, \ldots, f_k)$ is a tuple whose elements $f_i$ are elements of the abstract induced subgraph poset $\mathcal{P}(G)$ that correspond to unknown unlabelled graphs $F_i$, respectively, or empty graphs.

\textbf{Lemma 2.8.} \textit{Connectedness of graphs is a $P$-reconstructible property.}

\textbf{Proof.} We prove the claim by induction on the number of vertices of $G$. The base case is when $\nu(G) = 2$, in which case $G$ is connected if and only if $\mathcal{P}(G)$ has exactly two elements. Let the claim be true for all graphs on at most $k$ vertices. Let $\nu(G) = k + 1$. By induction hypothesis, we construct $S := \{g_i \in \mathcal{P}(G) \mid (\nu(g_i) \leq k) \land (c(G_i) = 1)\}$; that is, we mark elements of $\mathcal{P}(G)$ of rank at most $k$ that correspond to connected graphs. Now $G$ is connected if and only if at least two of its vertex-deleted subgraphs are connected, i.e., if and only if $g_M$ covers at least two distinct elements of $S$.

With Lemma 2.8, we assume without loss of generality that $g_i \in \mathcal{P}(G)$ are ordered so that for some $c$ the graphs $G_i, i = 1, \ldots, c$ are connected, and the remaining graphs are disconnected, and that $\nu(g_i) \leq \nu(g_{i+1})$ for $i \leq c - 1$.

\textbf{Lemma 2.9.} Let $(s_1, \ldots, s_m) \in \{g_1, \ldots, g_c\}^m$ such that, for $i = 1, \ldots, c$, the element $g_i$ occurs $m_i$ times in $(s_1, \ldots, s_m)$. It represents a graph $S = \sum_{i=1}^c m_i G_i$. Suppose that $\sum_{i=1}^c \nu(s_i) = \nu(G)$. Then

$$w_\pi(S, G) = \frac{\text{cov}(S \to \nu G)}{\prod_{i=1}^k m_i!},$$

and hence it is $P$-reconstructible.

Lemmas 2.8 and 2.9 are the basis of the construction of the abstract bond lattice from the abstract induced subgraph poset.

\section{Relating the abstract induced subgraph poset and the abstract bond lattice}

In this section, we show that the abstract induced subgraph poset of any graph and its abstract bond lattice can be constructed from each other except when the graph is either a star or a disjoint union of edges. The constructions are based on Lemma 2.7 and Theorem 1.4-(4a). In fact, we use only the $P$-reconstructibility of connectedness, for which we gave a very short proof in Lemma 2.8.

\textbf{Definition 3.1.} Let $X := \{x_i, i \in \mathbb{N}\}$ be a set of indeterminates. Let $\alpha: (G^c/\sim) \to X$ be a bijection. Then an unlabelled graph $G := \sum_{j \in J} k_j G_j$, where $G_j, j \in J$ are connected unlabelled graphs, corresponds to the formal sum $\sum_{j \in J} k_j \alpha(G_j)$. In the notation of free abelian groups, let $\mathbb{Z}(X)$ denote the set of finite formal sums of elements of $X$.

Let $f_a: \mathcal{P}(G) \to \mathcal{P}(G)$ be an isomorphism. An $X$-labelling of $\mathcal{P}(G)$ is a map $a: \mathcal{P}(G) \to \mathbb{Z}(X)$. An $X$-labelling $a$ of $\mathcal{P}(G)$ is \textit{consistent} with $f_a$ and $\alpha$ if for all $G_i \in \mathcal{P}(G) \cap G^c/\sim$, we have $(a f_a)(G_i) = \alpha(G_i)$; and for all $G_i \in \mathcal{P}(G)$, we have $(a f_a)(G_i) = \sum_{j \in J} k_j \alpha(G_j)$, for some indexing set $J$, if and only if $G_i = \sum_{j \in J} k_j G_j$ for connected unlabelled graphs $G_j, j \in J$. An
X-labelling \( a \) of \( \overline{\mathcal{P}}(G) \) is consistent if there exist \( \alpha \) and \( f_a \) as described above with which \( a \) is consistent.

Let \( f_b: \Omega(G) \rightarrow \overline{\Omega}(G) \) be an isomorphism. An X-labelling of \( \overline{\Omega}(G) \) is a map \( b: \overline{\Omega}(G) \rightarrow \mathbb{Z}^{(X)} \). An X-labelling \( b \) of \( \overline{\Omega}(G) \) is consistent with \( f_b \) and \( \alpha \) if for all \( H_i \in \Omega(G) \), \( (b_{f_b}(H_i)) = \sum_{j \in J} k_j \alpha(G_j) \), for some indexing set \( J \), if and only if \( H_i = \sum_{j \in J} k_j G_j \) for connected unlabelled graphs \( G_j, j \in J \). An X-labelling \( b \) of \( \overline{\Omega}(G) \) is consistent if there exist \( \alpha \) and \( f_b \) as described above with which \( b \) is consistent.

**Example 3.2.** The labelling of the abstract induced subgraph poset on the right in Figure 1 is consistent with the map \( \alpha(G_i) = x_i, i = 1, \ldots, 5 \), where the graphs \( G_i \) are shown at the top of Figure 1. In this example, there is a unique isomorphism \( f_a: \mathcal{P}(G) \rightarrow \mathcal{P}(G) \). Similarly, a consistent X-labelling of the abstract bond lattice in Figure 3 is shown on the right. In this example also, the isomorphism \( f_b: \Omega(G) \rightarrow \overline{\Omega}(G) \) is unique.

### 3.1 From the abstract induced subgraph poset to the abstract bond lattice

**Theorem 3.3.** The abstract bond lattice of \( G \), along with a consistent \( X \)-labelling, can be constructed from its abstract induced subgraph poset.

**Proof.** By Lemma 2.8, we construct \( S := \{ g_i \in \mathcal{P}(G) \mid c(G_i) = 1 \} \); and without loss of generality assume that \( S = \{ g_1, \ldots, g_c \} \). We define \( \alpha(G_i) := x_i, i = 1, \ldots, c \). We compute a set \( B \subseteq \mathbb{Z}^{(X)} \) of formal sums of \( x_1, \ldots, x_c \) that are possible labels of \( h_i \in \overline{\Omega}(G) \), and then we construct a weighted lattice on \( B \) that is isomorphic to \( \overline{\Omega}(G) \).

**Computing \( B \).**

Let \( F := (G_1^{k_1}, \ldots, G_c^{k_c}) \), where \( k_i \in \mathbb{N} \) are such that \( \sum_i k_i \nu(G_i) = \nu(G) \). We add the formal sum \( \sum_{i=1}^c k_i x_i \) to \( B \) if and only if \( \text{cov}(F \xrightarrow{\nu} G) \neq 0 \). By Lemma 2.7, \( \text{cov}(F \xrightarrow{\nu} G) \) can be computed given \( \overline{\mathcal{P}}(G) \) and the tuple \((g_1^{k_1}, \ldots, g_c^{k_c})\) that corresponds to \( F \). In this manner, we compute all formal sums of \( x_1, \ldots, x_c \) that must be in \( B \).

**Constructing a weighted lattice structure on \( B \) that is isomorphic to \( \overline{\Omega}(G) \).**

Let \( h_a := \sum_i a_i x_i \) and \( h_b := \sum_{i=1}^c b_i x_i \) be arbitrary formal sums in \( B \). They define graphs \( H_a = \sum_i a_i G_i \) and \( H_b = \sum_i b_i G_i \). By Lemma 2.6, \( \text{cov}(H_a \xrightarrow{\nu} H_b) \) is a function of \( w_{\nu}(g_i, g_j), i, j \in [1, c] \) only; and \( w_{\nu}(g_i, g_j), i, j \in [1, c] \) are known once the elements \( g_i \) in \( \overline{\mathcal{P}}(G) \) that correspond to connected graphs are marked as in Lemma 2.8. Then by Lemma 2.9 we calculate \( w_{\nu}(H_a, H_b) \). Repeating this calculation for all \( h_a, h_b \in B \) we obtain a weighted partially ordered set with ground set \( B \), that is isomorphic to \( \Omega(G) \), along with a consistent \( X \)-labelling on it. \[ \square \]

### 3.2 From the abstract bond lattice to the abstract induced subgraph poset

In this section, we construct the abstract induced subgraph poset of a graph a from its abstract bond lattice in two steps: we first construct a consistent \( X \)-labelling of the abstract bond lattice (Lemma 3.4); then we construct the abstract induced subgraph poset, along with a consistent \( X \)-labelling (Lemma 3.5). In view of Example 1.9, we exclude \( K_{1,n} \) and \( nK_2 \) and graphs with isolated vertices in the following result.

**Lemma 3.4.** If \( G \) has no isolated vertices, and \( G \not\in K_{1,n} \) for any \( n > 1 \), and \( G \not\in nK_2 \) for any \( n > 1 \), then \( \overline{\Omega}(G) \) has a unique consistent \( X \)-labelling (up to automorphisms of \( \overline{\Omega}(G) \) and permutations of the indeterminates).
Proof. We construct an $X$-labelling $b : \overline{\Omega}(G) \rightarrow \mathbb{Z}(X)$ in two passes. In the beginning, we do not know the number of distinct connected induced subgraphs of $G$; we compute it in the first pass. At the same time, we construct $b(h_i)$ for each $h_i \in \overline{\Omega}(G)$, modulo the number of isolated vertices. We assign the minimal element $h_1$ the label $nx_1$, and assign the only element $h_2$ of rank 1 the label $(n-2)x_1 + x_2$, where $n := \nu(G)$ is unknown. We introduce new indeterminates when needed. At the end of the first pass, we obtain the label $b(h_N)$ (where $h_N$ is the maximal element), which, together with the assumption that $G$ has no isolated vertices, determines $n$. In the second pass, we calculate the exact labels of other elements whose labels have an $x_1$-term.

Assigning labels to elements $h \in \overline{\Omega}(G)$ of rank $\rho(h) = 2$.

Since $G \notin \mathcal{K}_{1,n}$ for any $n > 1$ and $G \notin n \mathcal{K}_2$ for any $n > 1$, both $2 \mathcal{K}_2$ and $\mathcal{K}_{1,2}$ are subgraphs of $G$; hence there exist $H_3, H_4 \in \Omega(G)$ of rank 2 such that $\{H_3, H_4\} = \{n - 4\mathcal{K}_1 + 2\mathcal{K}_2, (n - 3)\mathcal{K}_1 + \mathcal{K}_{1,2}\}$; therefore, there exist $h_3, h_4 \in \overline{\Omega}(G)$ of rank 2 such that $\{b(h_3), b(h_4)\} = \{(n - 4)x_1 + 2x_2, (n - 3)x_1 + x_3\}$. (Since we know that there is a connected induced subgraph from the isomorphism class $\mathcal{K}_{1,2}$, we introduce a new indeterminate $x_3$ and define $\alpha(\mathcal{K}_{1,2}) = x_3$). Since $\Omega(\mathcal{K}_{1,2}) \cong \Omega(2\mathcal{K}_2)$, we do not know $b(h_3)$ and $b(h_4)$ immediately.

Let $\mathcal{F} := \{S_4, C_4, \mathcal{K}_3 + \mathcal{K}_2, \mathcal{K}_{1,3} + \mathcal{K}_2, P_4 + \mathcal{K}_2, 2\mathcal{K}_{1,2}, \mathcal{K}_{1,2} + 2\mathcal{K}_2, \mathcal{K}_4\}$, where $S_4$ is the graph shown in Figure 6. If there exist $\pi \vdash_c \mathcal{V}(G)$ such that $I(G[\pi]) \in \mathcal{F}$, then $b(h_3)$ and $b(h_4)$ are uniquely determined. This is proved by verifying that

1. each graph in the above list is uniquely determined by its bond lattice;

2. the abstract bond lattice of each graph in the list has only a trivial automorphism, which implies that the abstract bond lattice of each graph in the list has a unique consistent labelling up to choice and permutations of indeterminates.

We demonstrate the argument for $C_4$. Suppose that $C_4 \not\subseteq G$. Since $C_4$ is $\Pi$-reconstructible, there is a unique $h \in \overline{\Omega}(G)$ such that $\overline{\Omega}(h) \cong \Omega(C_4)$. Hence $b(h) = (n - 4)\mathcal{K}_1 + C_4$. Observe that $h$ covers exactly two elements, $h_3$ and $h_4$. Now $w_x((n - 4)\mathcal{K}_1 + 2\mathcal{K}_2, (n - 4)\mathcal{K}_1 + C_4) = 2$ and $w_x((n - 3)\mathcal{K}_1 + \mathcal{K}_{1,2}, (n - 4)\mathcal{K}_1 + C_4) = 4$. Therefore, for $h_i, i \in \{3, 4\}$, we have $b(h_i) = (n - 4)x_1 + 2x_2$ if $w_x(h_i, h) = 2$ and $b(h_i) = (n - 3)x_1 + x_3$ if $w_x(h_i, h) = 4$. A similar argument works for all the graphs in $\mathcal{F}$.

If there is no $F \in \mathcal{F}$ such that $F \subseteq G$, then $I(G) \in \{K_4 \setminus e, P_3, C_3\}$, in which case $G$ is $\Pi$-reconstructible; therefore, an $X$-labelling of $\overline{\Omega}(G)$ is determined up to isomorphism and choice of indeterminates. Therefore, in the following, we assume that there exists $F \in \mathcal{F}$ such that $F \subseteq G$, and (without loss of generality) that $b(h_3) = (n - 3)x_1 + x_3$ and $b(h_4) = (n - 4)x_1 + 2x_2$.

Assigning labels to elements $h \in \overline{\Omega}(G)$ of rank $\rho(h) > 2$ in the case when the corresponding graph $H$ has at most one non-trivial component.

For each connected graph $G_i \in \mathcal{P}(G)$, there is a distinct graph $H_i \in \Omega(G)$ such that $H_i = m_i\mathcal{K}_1 + G_i$. Hence we define $\Gamma := \{h_i \in \overline{\Omega}(G) \mid H_i = m_i\mathcal{K}_1 + G_i \text{ for some connected graph } G_i\}$, and $\Gamma_r := \{h \in \Gamma \mid \rho(h) \leq r\}$, and construct $\Gamma$ by constructing $\Gamma_r$ recursively.

We have $\Gamma_2 = \{h_1, h_2, h_3\}$. Suppose that we have constructed $\Gamma_k$ for all $k \in \{2, \ldots, r\}$. Let $h_i \in \overline{\Omega}(G)$ such that $\rho(h_i) = r + 1$. We use the fact that a graph on two or more vertices is connected if and only if at least 2 of its vertex-deleted subgraphs are connected. If $h_i$ is in $\Gamma_{r+1}$, then $h_i$ must cover at least two distinct elements $h_j, h_k$ in $\Gamma_r$. But the converse is not true. If $h_i$ covers two distinct elements $h_j, h_k$ in $\Gamma_r$, then either $h_i = a\mathcal{K}_1 + \mathcal{K}_2 + F$ for some $a \geq 0$ and a connected unlabelled graphs $F$, or $h_i = a\mathcal{K}_1 + F$ for some $a \geq 0$ and a connected unlabelled graph $F$. We want to add $h_i$ to $\Gamma_{r+1}$ only in the latter case. The necessary and sufficient condition for the former case is that there exists an element $h_{i\ell}$ covered by $h_i$ such
that \( \epsilon(H_t) = \epsilon(H_i) - 1 \) and \( w_x((n-4)K_1 + 2K_2, H_i) = w_x((n-4)K_1 + 2K_2, H_t) + \epsilon(H_t) \). These conditions are recognised from \( \overline{\Omega}(G) \) and the induction hypothesis since \( \epsilon(H_t) = w_x(h_2, h_t) \). If such an element \( h_t \) does not exist, then we add \( h_t \) to \( \Gamma_{r+1} \) and define \( b(h_i) := (n - r - 2)x_1 + x_i \), where we have introduced a new indeterminate \( x_i \) for the (unknown) graph \( F \) such that \( H_t = (n - r - 2)K_1 + F \).

Without loss of generality, we assume that \( b(h_i) = (n - \rho(h_i) - 1)x_1 + x_i \), for \( i = 1, \ldots, c \), and that \( h_1, \ldots, h_c \in \Gamma \) are ordered so that \( \rho(h_i) \leq \rho(h_{i+1}) \) for all \( i \in [1, c - 1] \).

Assigning labels to elements \( h_i \in \overline{\Omega}(G) \), \( i > c \) (i.e., to \( h_i \in \overline{\Omega}(G) \backslash \Gamma \)) in terms of \( x_1, \ldots, x_c \).

Recall the notation \( c(G_i, G_j) \) for the number of components of \( G_j \) that are isomorphic to \( G_i \). We have

\[
\text{ind}(G_j, H_i) = c(G_j, H_i) + \sum_{r=j+1}^{c} \text{ind}(G_j, G_r) c(G_r, H_i) \quad \text{for all } j \leq c \text{ and for all } i > c
\]

\[
\therefore w_x(h_j, h_i) = c(G_j, H_i) + \sum_{r=j+1}^{c} w_x(h_j, h_r) c(G_r, H_i) \quad \text{for all } j \leq c \text{ and for all } i > c. \tag{9}
\]

Now \( b(h_i) = \sum_{i=1}^{c} x_i c(G_i, H_i) \) is obtained for each \( h_i, i > c \) by solving the system of equations (9) for \( c(G_j, H_i) \) for all \( j \leq c \). Indeed, for all \( i > c \), \( c(G_i, H_i) = w_x(h_c, h_i) \), and for all \( j < c \), for all \( i > c \), \( c(G_j, H_i) \) is expressed in terms of \( c(G_r, H_i) \), \( r > j \).

Inferring \( n \) from the label of the maximum element \( h_n \), and fixing labels that contain \( n \).

Once \( b(h_n) = \sum_{i=2}^{c} k_i x_i \) has been calculated, we calculate \( n = \nu(G) = \rho(G) + c(G) = \rho(h_n) + \sum_{i=2}^{c} k_i \); then for all \( h \in \overline{\Omega}(G) \), the \( x_1 \) term in \( b(h) \) is determined exactly.

**Lemma 3.5.** The abstract induced subgraph poset of \( G \) along with a consistent \( X \)-labelling of \( G \) can be constructed from its consistently \( X \)-labelled abstract bond lattice.

**Proof.** Let \( b: \overline{\Omega}(G) \to \mathbb{Z}^{(X)} \) be a consistent \( X \)-labelling of \( \overline{\Omega}(G) \). Without loss of generality, we assume that \( b(h_i) = (n - n_i)x_1 + x_i \), for \( i = 1, \ldots, c \), and \( b(h_i), i > c \) are other distinct formal sums of \( x_i, i = 1, \ldots, c \), and that \( n - n_i \) are in non-increasing order. We first construct a set \( A \subseteq \mathbb{Z}^{(X)} \) of possible formal sums that can be in an \( X \)-labelling \( a \) of \( \overline{\Omega}(G) \). Then we make \( A \) a weighted partially ordered set that is isomorphic to \( \overline{\Omega}(G) \).

We have \( \{x_1, \ldots, x_c\} \subseteq A \); and \( b(h_n) \in A \) since \( a(g_m) = b(h_n). \) If \( b(h_1) = nx_1 \), then \( \nu(G) = n \); hence \( \nu(g_i) := n_i \) is determined by \( b(h_i) \) for all \( i \in [1, c] \). Moreover, for all \( i, j \) such that \( 1 \leq i \leq j \leq c \),

\[
w_x(x_i, x_j) := \text{ind}(G_i, G_j) = \begin{cases} n_j & \text{if } i = 1, \\ w_x(h_i, h_j) & \text{otherwise}. \end{cases} \tag{10}
\]

Let \( \mathcal{H} := \cup_{i=1}^{c} G_i \) (where \( G_i \) are isomorphism classes). Lemma 2.6 may be applied to \( \mathcal{H} \) since it is closed under connected induced subgraphs, and \( \text{ind}(G_i, G_j) \) are calculated above. Let \( F = \sum_{i=1}^{c} k_i G_i \) for some \( k_i \in \mathbb{N} \). By Lemma 2.6, \( \text{ind}(F, G) \) is a function of \( w_x(x_i, x_j), a(g_m), \) and \( k_i, i = 1, \ldots, c \). So we can determine if any given linear combination \( \sum_{i=1}^{c} k_i x_i \) is in \( A \). Thus we list all linear combinations \( \sum_{i=1}^{c} k_i x_i \) that must be in \( A \). Then again by Lemma 2.6 we calculate \( \text{ind}(G_j, G_k) \) for any two graphs \( G_j, G_k \in \mathcal{P}(G) \) represented, respectively, by linear combinations \( \sum_{i=1}^{c} k_i x_i \) and \( \sum_{i=1}^{c} l_i x_i \) in \( \overline{\Omega}(G) \). This completes the construction of a consistently \( X \)-labelled \( \overline{\Omega}(G) \).

**Theorem 3.6.** If \( G \) has no isolated vertices, and \( G \not\cong K_{1,n} \) for any \( n > 1 \), and \( G \not\cong nK_2 \) for any \( n > 1 \), then \( \overline{\Omega}(G) \) can be constructed from \( \overline{\Omega}(G) \).

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Proof. Either $G$ itself is II-reconstructible (as in the case of $K_4\setminus e$, $P_5$ and $C_5$), or the claim follows from Lemmas 3.4 and 3.5. □

Corollary 3.7.

1. Ulam’s conjecture is true if and only if every graph $G \in \mathcal{G}$ such that $G$ has no isolated vertices, and $G \notin K_{1,n}$ for any $n > 1$, and $G \notin nK_2$ for any $n > 1$, can be constructed up to isomorphism from its abstract bond lattice $\overline{\Pi}(G)$.

2. Every tree or forest $T$ such that $T \notin K_{1,n}$ for any $n > 1$, $T \notin nK_2$ for any $n > 1$, and $T$ has no isolated vertices, can be constructed up to isomorphism from its abstract bond lattice $\overline{\Pi}(T)$.

Proof. The first part follows from Theorem 3.6 and Theorem 1.4-(1). The second part of the theorem now follows from Theorem 3.6 and Theorem 1.4-(3). □

4 Colouring polynomials of graphs

We define two symmetric polynomial invariants of graphs, namely, the chromatic symmetric function and a stronger invariant called the symmetric Tutte polynomial, which were introduced by Stanley [16, 17].

A vertex colouring of a graph $G$ is a map $\kappa : V(G) \to \mathbb{Z}^+$. Let $\kappa$ be a vertex colouring of $G$, and let $U \subseteq V(G)$. We say that $U$ (or a subgraph on $U$) is monochromatic if $\kappa$ is constant on $U$. We say that $\kappa$ is proper if there are no monochromatic edges.

Let $x_1, x_2, \ldots$ and $t$ be commuting indeterminates. We denote by $1^n$ the assignment $x_1 = x_2 = \cdots = x_n = 1$ and $x_{n+1} = x_{n+2} = \cdots = 0$. We write $x_1, x_2, \ldots$ collectively as just $x$.

Definition 4.1. The chromatic symmetric function $X_G(x)$ of $G$ is defined by

$$X_G(x) := \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)},$$

where the summation is over all proper colourings $\kappa$ of $G$.

Definition 4.2. The symmetric Tutte polynomial $X_G$ of $G$ is defined by

$$X_G(x; t) := \sum_{\kappa} (1 + t)^{m(\kappa)} \prod_{v \in V(G)} x_{\kappa(v)},$$

where the summation is over all vertex colourings $\kappa$ of $G$, and $m(\kappa)$ is the number of monochromatic edges of $\kappa$.

Stanley [16] and Chow [3] have noted that $X_G(x; -1) = X_G(x)$; and $X_G(1^n)$ and $X_G(1^n; t)$ are equivalent to the chromatic polynomial and the Tutte polynomial of $G$, respectively, where $n := \nu(G)$. Thus the symmetric Tutte polynomial $X_G(x; t)$ specialises to the other three invariants mentioned above.

Theorem 4.3 (Stanley [16, 17] and Chow [3]). We have

$$X_G(x; t) = \sum_{S \subseteq E(G)} t^{|S|} \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)},$$

and

$$X_G(x) = X_G(x; -1) = \sum_{S \subseteq E(G)} (-1)^{|S|} \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)},$$

where the inner summation in each equation is over all vertex colourings $\kappa$ that are monochromatic on the connected components of the spanning subgraph $G_S$ of $G$ with edge set $S$. 

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4.1 Reconstructing the symmetric Tutte polynomial

The reconstructibility of \( X_G(x) \) and \( X_G(x;t) \) is not immediately obvious from their expansions given in Theorem 4.3. In the following, we prove, using Kocay’s lemma for abstract induced subgraph poset (Lemma 2.7), that \( X_G(x;t) \) is reconstructible (in fact \( P \)-reconstructible), which implies that the chromatic symmetric function is reconstructible as well.

**Theorem 4.4.** The symmetric Tutte polynomial is \( P \)-reconstructible.

**Proof.** Each colouring \( \kappa : V(G) \to \mathbb{Z}^+ \) with \( k \) colours defines two \( k \)-tuples: a tuple \((m_1, \ldots, m_k)\) of positive integers, where \( m_1 < \cdots < m_k \) are the \( k \) colours, and a tuple \((F_1, \ldots, F_k)\) of vertex-disjoint subgraphs \( F_i \) of \( G \), which are induced by the colour classes \( m_i \), respectively; hence \( \sum_i \nu(F_i) = \nu(G) \). Therefore, by the definition of \( X_G(x;t) \), we have

\[
X_G(x;t) = \sum_{k \in \mathbb{Z}^+} \left( \sum_{(m_1, \ldots, m_k) \in \mathbb{Z}^+} \sum_{m_1 < \cdots < m_k} (1 + t) \sum_i \epsilon(F_i) \prod_i x_{m_i}^{\nu(F_i)}. \right)
\]

We say that tuples \((F_1, \ldots, F_k)\) and \((F'_1, \ldots, F'_k)\) are equivalent if there is a bijection \( f : [1, k] \to [1, k] \) such that \( F_i \cong F'_{f(i)} \) for all \( i \); else they are inequivalent. Similarly, we say that tuples \((f_1, \ldots, f_k)\) and \((f'_1, \ldots, f'_k)\), where \( f_i \) and \( f'_i \) are elements of \( \overline{P}(G) \) or empty graphs, are equivalent if there is a bijection \( g : [1, k] \to [1, k] \) such that \( f_i = f'_{g(i)} \) for all \( i \); else they are inequivalent.

We write the inner summation over mutually inequivalent tuples \((F_1, \ldots, F_k)\). There are \( \text{cov}((F_1, \ldots, F_k) \Rightarrow G) \) tuples in the equivalence class of any given tuple \((F_1, \ldots, F_k)\). Hence

\[
X_G(x;t) = \sum_{k \in \mathbb{Z}^+} \left( \sum_{(m_1, \ldots, m_k) \in \mathbb{Z}^+} \sum_{m_1 < \cdots < m_k} \text{cov}((F_1, \ldots, F_k) \Rightarrow G) (1 + t) \sum_i \epsilon(F_i) \prod_i x_{m_i}^{\nu(F_i)}. \right)
\]

Finally, by Lemma 2.7, we compute the inner summation over mutually inequivalent tuples \((f_1, \ldots, f_k)\), where each \( f_i \) is an element of \( \overline{P}(G) \) or is an empty graph, and \( \sum_i \nu(f_i) = \nu(G) \). Hence

\[
X_G(x;t) = \sum_{k \in \mathbb{Z}^+} \left( \sum_{(m_1, \ldots, m_k) \in \mathbb{Z}^+} \sum_{m_1 < \cdots < m_k} \text{cov}((f_1, \ldots, f_k) \Rightarrow G) (1 + t) \sum_i \epsilon(f_i) \prod_i x_{m_i}^{\nu(f_i)}. \right)
\]

This implies the \( P \)-reconstructibility of \( X_G(x;t) \).

In fact, \( X_G(x;t) \) may be generalised as follows. Define a map \( \kappa : V(G) \to 2^{\mathbb{Z}^+} \) to be a set-colouring of \( G \) when each colour set \( \kappa(v) \) is finite. The invariant \( Y_G(x;y) \) over indeterminates \( x_1, x_2, \ldots, y_1, y_2, \ldots \) defined below generalises the symmetric Tutte polynomial in a natural way:

\[
Y_G(x;y) := \sum_{\kappa} \prod_{\{u,v\} \in E(G)} \prod_{i \in \kappa(u) \cap \kappa(v)} (1 + y_i) \prod_{w \in V(G)} \prod_{j \in \kappa(w)} x_j.
\]

We call it the set-colouring symmetric Tutte polynomial of \( G \). We do not know if the invariant \( Y_G(x;y) \) is strictly stronger than \( X_G(x;t) \) (i.e., if there are graphs distinguished by this stronger invariant but not by their symmetric Tutte polynomial). In this paper we do not study it beyond observing (without proof) that \( Y_G(x;y) \) is \( P \)-reconstructible; the proof is similar to that of Theorem 4.4, except that we do not require that \( \sum_i \nu(f_i) = \nu(G) \) since the colour classes of a colouring do not partition \( V(G) \).
For other known proofs of the reconstructibility of the various colouring polynomials, we refer to the induced subgraph expansion of the chromatic polynomial in [1], the reconstructibility of the chromatic symmetric function in [3], and the reconstructibility of the chromatic and Tutte polynomials in [2]. As far as the author is aware, the reconstructibility of the symmetric Tutte polynomial presented above is a new result.

4.2 Reconstructing the chromatic symmetric function

The reconstructibility of the chromatic symmetric function follows from the reconstructibility of the symmetric Tutte polynomial. But here we given another proof based on its expansion given by Stanley [16, Theorem 2.6] (which is Theorem 4.6 in this section) and our Theorem 3.3.

Let $\zeta$ and $\mu$ be, respectively, the zeta function and the Möbius function of $\Pi^c_G$ (which is the (unfolded) connected partition lattice of $G$ defined in Section 1.3).

By definition, for any two connected partitions $\pi$ and $\tau$ of $V(G)$, we have

$$\sum_{\sigma} \mu(\pi, \sigma) \zeta(\sigma, \tau) = \begin{cases} 1 & \text{if } \tau = \pi \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

Moreover, if $\sigma_1$ and $\sigma_2$ are any two connected partitions of $V(G)$ such that $G[\sigma_1] \cong G[\sigma_2]$, then $\mu(0, \sigma_1) = \mu(0, \sigma_2)$. Therefore, a function $\psi: \overline{\Omega}(G) \to \mathbb{Z}$ given by $\psi(h_i) := \mu(0, \sigma)$, where $\sigma$ is any connected partition of $V(G)$ such that $G[\sigma] \cong H_i$, is well defined. Therefore, when $\pi = 0$, the system of equations (11) may be re-written as

$$\sum_{h_j} \psi(h_i) w_\sigma(h_i, h_j) = \begin{cases} 1 & \text{if } h_j = h_1 \\ 0 & \text{otherwise}, \end{cases} \quad (12)$$

for all $h_j \in \overline{\Omega}(G)$.

**Lemma 4.5.** The abstract bond lattice $\overline{\Omega}(G)$ uniquely determines $\psi(h_i), h_i \in \overline{\Omega}(G)$.

**Proof.** The weight function $w_\sigma$ is invertible in the incidence algebra of $\overline{\Omega}(G)$. Therefore, the system of equations (12) has a unique solution for $\psi(h_i), h_i \in \overline{\Omega}(G)$.

For an integer partition $\lambda := (\lambda_1, \ldots, \lambda_l)$, let $p_\lambda$ denote its power sum symmetric function given by $p_\lambda = \prod_{k=1}^{l} \sum_i x_i^{\lambda_k}$, see Stanley [18].

**Theorem 4.6** (Stanley [16]). We have

$$X_G(x) = \sum_{\pi \in \Pi^c_G} \mu(0, \pi) p_{\lambda(\pi)}. \quad (13)$$

**Corollary 4.7.** If $G$ has no isolated vertices, and $G \notin K_{1,n}$ for any $n > 1$, and $G \notin nK_2$ for any $n > 1$, then $X_G(x)$ can be computed from $\overline{\Omega}(G)$.

**Proof.** Equation (13) may be re-written as

$$X_G(x) = \sum_{h_i \in \overline{\Omega}(G)} \psi(h_i) w_\sigma(h_i, h_N) p_{\lambda(h_i)}, \quad (14)$$

where $\lambda(h_i) := \lambda(H_i)$. For each $h_i$ in $\overline{\Omega}(G)$, $\lambda(H_i)$ is determined by Theorem 3.3 and $\psi(h_i)$ is determined by Lemma 4.5. 

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4.3 Colouring polynomials of trees

Let $T$ be a tree. Let $\Omega(T) := \{h_1, \ldots, h_N\}$, where $h_i$ are enumerated so that the number of components $c(H_i)$ are non-increasing. Let $\mu$ be the Möbius function of $\Pi_T^c$, and let $\psi: \Omega(T) \to \mathbb{Z}$ be as defined in Section 4.2.

**Lemma 4.8.** For all $h_i$ in $\Omega(T)$,

$$\psi(h_i) = (-1)^{c(H_i)} = (-1)^{\nu(T) - c(H_i)}. \quad (15)$$

**Proof.** The connected partition lattice of $T$ is isomorphic to the power set lattice of $E(T)$. The Möbius function of the power set lattice of $E(T)$ is given by $\mu(E_1, E_2) = (-1)^{|E_2| - |E_1|}$, for $E_1, E_2 \subseteq E(T)$. Now the result follows from the definition of $\psi$.

Given a graph $G$ and a partition $\lambda$ of $\nu(G)$, let $k(\lambda, G) := |\{\pi \models V(G) \mid \lambda(\pi) = \lambda\}|$, and let $\Lambda(G) := \{(\lambda, k(\lambda, G)) \mid \lambda \models \nu(G)\}$.

**Lemma 4.9.** The invariants $\Lambda(T)$ and $X_T(x)$ are equivalent, i.e., they can be computed from each other.

**Proof.** By Equation (14) and Lemma 4.8, we have

$$X_T(x) = \sum_{\lambda \models \nu(T)} (-1)^{\nu(T) - \ell(\lambda)} k(\lambda, T) p_\lambda, \quad (16)$$

which shows that $X_T(x)$ can be computed from $\Lambda(T)$.

Given $X_T(x)$, equation (16) can be solved for $k(\lambda, T)$ as follows. Let $\lambda := (\lambda_1, \ldots, \lambda_l)$ be a partition of $\nu(T)$, and let $b_T(\lambda)$ be the coefficient of $\prod_{i=1}^l x_{\lambda_i}$ in $X_T(x)$. The numbers $b_T(\lambda)$ and $k(\lambda, T)$ satisfy the equation

$$b_T(\lambda) = \sum_{\lambda' \models \lambda} (-1)^{\nu(T) - \ell(\lambda')} k(\lambda', T) a(\lambda', \lambda), \quad (17)$$

where $a(\lambda', \lambda)$ is the coefficient of $\prod_{i=1}^l x_{\lambda_i}$ in $p_{\lambda'}$. Moreover, $a(\lambda', \lambda) = 0$ if $\lambda'$ is not a refinement of $\lambda$, and $a(\lambda', \lambda) = 1$ if $\lambda' = \lambda$. The system of equations (17) can be recursively solved for $k(\lambda, T)$ (the initial condition being $k(\lambda, T) = 1$ when $\lambda$ is the finest partition of $\nu(T)$). Thus we can construct $\Lambda(T)$ given $X_T(x)$.

**Remark.** The above result is not valid for graphs in general; e.g., the graphs $G$ and $H$ shown in Figure 5 have the same chromatic symmetric function (see Stanley [16]), but we can verify that $\Lambda(G)$ and $\Lambda(H)$ are different.

![Figure 5: Non-isomorphic graphs with identical chromatic symmetric function.](image)

**Lemma 4.10.** Whether $G$ is a tree or not can be recognised from $X_G(x)$ and from $\Lambda(G)$. 

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Proof. We obtain \( \nu(G) \) and \( \epsilon(G) \) by observing that \( \nu(G) \) is the degree of each monomial in \( X_G(x) \); it is the length of the finest partition \((1,1,\ldots,1)\) that appears in a pair in \( \Lambda(G) \); the coefficient of \( x_1^2x_2x_3 \cdots x_{\nu(G)} - 1 \) in \( X_G(x) \) is \((\nu(G))^2 - \epsilon(G); \) and for \( \lambda := (2,1,1,\ldots,1) \), we have \( k(\lambda,G) = \epsilon(G) \).

Now \( G \) is a tree if and only it \( \epsilon(G) = \nu(G) - 1 \) and it has no cycles; and \( G \) is acyclic if and only if the number of acyclic orientations of \( G \) is \( 2^{\nu(G)} \). Stanley [15] has shown that the number of acyclic orientations of a graph \( G \) can be calculated from its chromatic polynomial \( \chi_G(x) \), which is given by \( X_G(1^n) \).

The number of components in \( G \) is \( \min(\ell(\lambda)) \) over \((\lambda,k(\lambda,G))\) in \( \Lambda(G) \). Now \( G \) is a tree if and only if \( \epsilon(G) = \nu(G) - 1 \) and \( G \) has a single component.

Notation. Let \( v := (v_1,\ldots,v_r) \) and \( e := (e_1,\ldots,e_r) \) and \( \lambda := (\lambda_1,\ldots,\lambda_s) \) be integer vectors.

Let
\[
\phi(v,e,\lambda) := \left\{ f : [1,i] \to [1,r] \mid (\forall j \in [1,r]) \left( \sum_{j \in f^{-1}(j)} \lambda_i = v_j, \sum_{j \in f^{-1}(j)} (\lambda_i - 1) = e_j \right) \right\}.
\]

Let \( \theta(v,e;T) \) be the number of ordered partitions \((V_1,\ldots,V_r)\) of \( V(T) \) such that \( |V_i| = v_i \) and \( \epsilon(T[V_i]) = e_i \); we call such partitions \((v,e)\)-partitions of \( V(T) \). For vectors \( v \) and \( e \) of unequal length, we define \( \phi(v,e,\lambda) = 0 \) and \( \theta(v,e;T) = 0 \). Given integer vectors \( e := (e_1,\ldots,e_r) \) and \( f := (f_1,\ldots,f_s) \), we define \( e \leq f \) if and only if \( r = s \) and \( e_i \leq f_i \) for all \( i \); the relation \( \leq \) makes the set of integer vectors a locally finite partially ordered set.

Lemma 4.11. For all integer vectors \( v \) and \( e \), the parameter \( \theta(v,e;T) \) is determined by \( X_T(x) \).

Proof. For all vectors \( v \) and \( e \), we have
\[
\theta(v,f;T) \prod_{i=1}^r \frac{f_i}{e_i} = \sum_{\pi \models V(T)} \phi(v,e,\lambda(\pi)) = \sum_{\lambda \models \nu(T)} \phi(v,e,\lambda)k(\lambda,T).
\]

Equation (18) is trivially true if vectors \( v \) and \( e \) have unequal lengths, or \( v \) has some non-positive entries, or \( e \) has some negative entries. Otherwise, we prove the equation by double counting as follows. Let vectors \( v \) and \( e \), both of length \( r \), be fixed. We define a matrix \( A := A_{v,e} \) with rows indexed by connected partitions of \( V(T) \), and columns indexed by \((v,f)\)-partitions of \( V(T) \) such that \( f \geq e \). For a connected partition \( \pi := \{X_1,\ldots,X_s\} \) of \( V(T) \) and a \((v,f)\)-partition \( \sigma := \{Y_1,\ldots,Y_t\} \) of \( V(T) \), an entry \( A(\pi,\sigma) \) is 1 if \( \pi \) refines \( \sigma \), and for all \( j \in [1,r] \), \( \sum_{X_i \subseteq Y_j} |X_i| = v_j \) and \( \sum_{X_i \subseteq Y_j} (|X_i| - 1) = e_j \); and \( A(\pi,\sigma) = 0 \) otherwise. For a fixed vector \( f \), the number of 1s in a column of \( A \) indexed by a \((v,f)\)-partition is \( \prod_{i=1}^r (\frac{f_i}{e_i}) \), and the number of \((v,f)\)-partitions is \( \theta(v,f;T) \). The number of 1s in a row indexed by a connected partition \( \pi := \{X_1,\ldots,X_r\} \) is \( \phi(v,e,\lambda(\pi)) \). Now counting the number of 1s in \( A \) by columns and rows gives the first equality. The second equality follows from the definition of \( k(\lambda,T) \).

By Lemma 4.9, \( \Lambda(T) \) can be constructed from \( X_T(x) \); therefore, the right side of Equation (18) is known. Now Equation (18) is solved for \( \theta(v,e;T) \) for all \( v \) and \( e \) by Möbius inversion on the partially ordered set of integer vectors.

Theorem 4.12. For trees, the chromatic symmetric function and the symmetric Tutte polynomial are equivalent.

Proof. For all graphs, \( X_G(x) \) is a specialisation of \( X_G(x;t) \). We show how \( X_G(x;t) \) is determined by \( X_G(x) \) when \( G \) is a tree. We have
\[
X_T(x;t) = \sum_{k \in \mathbb{Z}^+} \sum_{v_k \in E_k, k_k \in E_k} \theta(v_k,e_k;T)(1 + t)^{\sum_i e_i \prod_{i=1}^k x_i^v_i},
\]

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implies that Stanley’s question (whether the chromatic symmetric function distinguishes trees) and the question of Noble and Welsh (whether their weighted chromatic function with unit weights, which is equivalent to the symmetric Tutte polynomial, distinguishes trees) are equivalent.

Lemma 4.11 may be a useful tool to study Stanley’s question. We illustrate one simple application of the lemma. Let the degree of a subtree \( F \) of a tree \( T \) be the number of edges in \( T \) with one end in \( F \) and one end outside \( F \).

**Corollary 4.13.** The number of subtrees of \( T \) with a given number of vertices and a given degree is determined by \( X_T(x) \); in particular, the degree sequence of \( T \) is determined by \( X_T(x) \).

**Proof.** Let \( v := (k, \nu(T) - k) \) and \( e := (k - 1, \nu(T) - k - d) \). Now the number \( \theta(v, e; T) \) counts the number of subtrees on \( k \) vertices having degree \( d \). Setting \( k = 1 \), we get the degree sequence. ■

## 5 Edge-subgraph posets and edge reconstruction

In this section, we classify graphs that are not \( Q \)-reconstructible. We show that if the edge reconstruction conjecture is true, then graphs that are not \( Q \)-reconstructible, except finitely many, have a simple structure: if \( G \) and \( H \) are distinct unlabelled graphs that have isomorphic edge subgraph poset, then except in finitely many cases, \( \{G, H\} = \{K_{1,m}, mK_2\} \) for some \( m \geq 2 \), or \( \{G, H\} = \{pK_3 + qK_{1,3} + F, qK_{1,3} + pK_{1,3} + F\} \), where \( p \neq q \) and \( F \) itself is a graph with quite simple structure.

**Notation.** Graphs \( B_1, \ldots, B_4 \), and \( m \)-edge graphs \( S_m \) and \( T_m \), where \( (m \geq 3) \), which are frequently referred to in the proofs, are as shown in Figure 6. We denote by \( K_4^{+e} \) the isomorphism class of a graph obtained by deleting an edge from a copy of \( K_4 \). Let

\[
\mathcal{F} := \{P_n \mid n \geq 2\} \cup \{C_n, n \geq 4\} \cup \{S_4, K_4^{+e}, K_4\},
\]

Let \( \mathbb{N}(\mathcal{F}) \) be the set of all finite unlabelled graphs (including the null graph) with components from \( \mathcal{F} \).

For an unlabelled graph \( F \), define \( \{F^{+e}\} \) to be the set of unlabelled graphs that can be obtained by adding a new edge to a copy of \( F \), where the added edge may have 0, 1, or 2 end-vertices in the copy of \( F \); formally, \( \{F^{+e}\} := \{I(H_1 \cup H_2) \mid H_1 \in F, H_2 \in K_2\} \). For example, \( \{K_{1,3}^{+e}\} = \{K_{1,3} + K_2, K_{1,4}, T_4, S_4\} \).

![Figure 6: Some graphs referenced in Theorem 5.1](image)

**Theorem 5.1.**
1. The graphs in each of the following sets have the same abstract edge-subgraph poset:

(a) \{K_3, K_{1,3}, 3K_2\}, \{P_4, K_{1,2}+K_2\}, \{P_4+K_2, T_4\}, \{C_4, 2K_{1,2}\}, \{C_4+K_2, B_1\}, \{P_6, B_2\}, and \{B_3, B_1\};

(b) \{K_{1,m}, mK_2\}, for all \(m > 1\);

(c) \{pK_3+qK_{1,3}+F, qK_3+pK_{1,3}+F\}, where \(p \neq q\) and \(F \in \mathbb{N}^{(F)}\).

2. The edge reconstruction conjecture is true if and only if all graphs, except the graphs listed above, are \(Q\)-reconstructible.

**Lemma 5.2.** If \(G \in pK_3+qK_{1,3}+F\) for some \(p, q \in \mathbb{N}\) and \(F \in \mathbb{N}^{(F)}\), and \(G_i\) is an edge-subgraph of \(G\), then \(G_i \in p_iK_3+q_iK_{1,3}+F_i\) for some \(p_i, q_i \in \mathbb{N}\) and \(F_i \in \mathbb{N}^{(F)}\).

**Proof.** The claim is proved by verifying that all proper edge-subgraphs of \(K_3\), all proper edge-subgraphs of \(K_{1,3}\), and all edge-subgraphs of each graph in \(F\) are in \(\mathbb{N}^{(F)}\). □

In the following, we say that labelled graphs \(G\) and \(H\) are **conjugates** if there exist \(p, q \in \mathbb{N}\) (possibly equal) and \(F \in \mathbb{N}^{(F)}\) such that \(G \in pK_3+qK_{1,3}+F\) and \(H \in pK_{1,3}+qK_3+F\).

**Lemma 5.3.** Let the graphs \(G\) and \(H\) be conjugates. Then there exists a bijection \(f: 2^{E(G)} \rightarrow 2^{E(H)}\) such that for all \(E \subseteq E(G)\), the edge-subgraphs \(G[E]\) and \(H[f(E)]\) are conjugates.

**Proof.** Order the components \(G^i\) of \(G\) and components \(H^i\) of \(H\) such that \(G^i \in K_3\) if and only if \(H^i \in K_{1,3}\), and \(G^i \in K_{1,3}\) if and only if \(H^i \in K_3\), and \(G^i \cong H^i\) otherwise.

For each \(i \in \{1, \ldots, p + q + r\}\), define a bijection \(f_i: 2^{E(G^i)} \rightarrow 2^{E(H^i)}\) such that

\[
H^i[f_i(E)] = \begin{cases} 
K_{1,3} & \text{if } G^i[E] \in K_3, \\
K_3 & \text{if } G^i[E] \in K_{1,3}, \\
G^i[E] & \text{otherwise.}
\end{cases}
\]

Such a bijection always exists due to the chosen ordering of the components of \(G\) and \(H\), and since for each graph \(F \in \mathcal{F}\), we have \(\text{esub}(K_3, F) = \text{esub}(K_{1,3}, F);\) and if \(G^i \in K_3\) or \(G^i \in K_{1,3}\), then any bijection \(f_i\) such that \(|f_i(E)| = |E|\) serves the purpose.

Define \(f: 2^{E(G)} \rightarrow 2^{E(H)}\) by extending the component-wise maps \(f_i\) such that for all \(E \subseteq E(G)\),

\[f(E) = \bigcup_i f_i(E \cap E(G^i)).\]

The bijection \(f\) has the desired property. □

**Lemma 5.4.** Let \(G^i, H^i\) and \(G^j, H^j\) be pairs of conjugates. Then \(\text{esub}(G^i, G^j) = \text{esub}(H^i, H^j)\).

**Proof.** Let \(f: 2^{E(G^j)} \rightarrow 2^{E(H^j)}\) be a bijection as defined in the statement of Lemma 5.3. Let \(E \subseteq E(G^j)\). We have \(G^j[E] \cong G^i\) if and only if \(H^j[f(E)] \cong H^i\); hence \(\text{esub}(G^i, G^j) = \text{esub}(H^i, H^j)\). □

**Proof of Theorem 5.1–1.** For the graphs listed in Theorem 5.1–1a and 5.1–1b, the claim is proved by constructing the abstract edge-subgraph poset for each graph and verifying that the graphs within each set have the same abstract edge-subgraph poset.

Let \(G = pK_3+qK_{1,3}+F\) and \(H = pK_{1,3}+qK_3+F\), where \(p, q \in \mathbb{N}\), \(p \neq q\), and \(F \in \mathbb{N}^{(F)}\). Lemmas 5.2, 5.3, and 5.4 imply that there exists an isomorphism from \(\overline{G}\) to \(\overline{H}\) that maps an edge-subgraph \(p_iK_3+q_iK_{1,3}+F_i\) of \(G\) to an edge-subgraph \(p_iK_{1,3}+q_iK_3+F_i\) of \(H\), where \(p_i, q_i \in \mathbb{N}\) and \(F_i \in \mathbb{N}^{(F)}\). But \(G \neq H\) since \(p \neq q\). This proves the result for graphs listed in Theorem 5.1–1c. □
Proof of Theorem 5.1–2 (the ‘if’ part). The assertion follows from the following facts.

1. All the graphs listed in Theorem 5.1–1 that have at least 4 edges are edge reconstructible since disconnected graphs on 4 or more edges, trees on 4 or more edges, and unicyclic graphs on 4 or more edges are edge reconstructible; see Kelly [7], Greenwell and Henninger [4], and Manvel [11].

2. All Q-reconstructible graphs are edge reconstructible since the abstract edge-subgraph poset of a graph can be constructed given its edge-deck.

Proof of Theorem 5.1–2 (outline of the ‘only if’ part). The ‘only if’ part is proved in 3 steps.

1. In Proposition 5.6, the result is proved for graphs with at least 4 edges, containing both $K_{1,2} + 2K_2$ and $T_4$ as subgraphs.

2. In Proposition 5.9, a proof of the result is sketched for graphs with at most 7 edges.

3. In Proposition 5.10, the result is proved for graphs with at least 7 edges that do not have at least one of the graphs $K_{1,2} + 2K_2$ and $T_4$ as a subgraph.

In the rest of this section, we prove Propositions 5.6, 5.9 and 5.10, followed by some corollaries and questions.

A legitimate labelling of $\overline{Q}(G)$ is a one-to-one map $\pi : \overline{Q}(G) \to G/\cong$ such that $\pi(g_1) = K_2$ and $w_i(g_i, g_j) = \text{esub}(\pi(g_i), \pi(g_j))$ for all $g_i, g_j \in \overline{Q}(G)$. We say that an element $g_i$ (or a subset $S$) of $\overline{Q}(G)$ is uniquely labelled if for all legitimate labelling maps $\pi$ and $\sigma$ from $\overline{Q}(G)$ to $G/\cong$, we have $\pi(g_i) = \sigma(g_i)$ (or $\pi(g_i) = \sigma(g_i)$ for all $g_i$ in $S$). Thus $G$ is Q-reconstructible if and only if the maximal element of $\overline{Q}(G)$ is uniquely labelled. The Q-reconstructibility of $G$ does not imply that $\overline{Q}(G)$ is uniquely labelled; for example, $S_4$ is Q-reconstructible, but given a legitimate labelling of $\overline{Q}(S_4)$, the labels $K_{1,3}$ and $K_3$ may be interchanged keeping all other labels fixed to obtain another legitimate labelling.

Lemma 5.5. The graph $K_{1,2} + 2K_2$ is Q-reconstructible, and the abstract edge-subgraph poset of $K_{1,2} + 2K_2$ is uniquely labelled.

Proof. In Figure 4, we have $\overline{Q}(g_9) \cong Q(K_{1,2} + 2K_2)$. Let $\pi$ be a legitimate labelling of $\overline{Q}(g_9)$. We show that $\pi$ is unique. If $\pi(g_2) = K_{1,2}$ and $\pi(g_3) = 2K_2$, then $\pi(g_5) = P_4$, and $\pi(g_4) \in \{K_{1,3}, K_3\}$. Such a labelling cannot be extended so as to assign a legitimate label to $g_9$, since it would imply that $\pi(g_9)$ has edge-deck $\{P_4, P_3, K_3, K_3\}$ or $\{P_4, P_4, K_{1,3}, K_{1,3}\}$. But neither $\{P_4, P_4, K_3, K_3\}$ nor $\{P_4, P_3, K_{1,3}, K_{1,3}\}$ is a legitimate edge-deck. Hence $g_9$ does not have a legitimate labelling $\pi$ in which $\pi(g_2) = K_{1,2}$ and $\pi(g_3) = 2K_2$. If $\pi(g_2) = 2K_2$ and $\pi(g_3) = K_{1,2}$, then the edge reconstructibility of each of $3K_2$, $K_{1,2} + 2K_2$, and $K_{1,2} + 2K_2$ determines that $\pi(g_4) = 3K_2$, $\pi(g_5) = K_{1,2} + 2K_2$, and $\pi(g_9) = K_{1,2} + 2K_2$.

Proposition 5.6. If the edge reconstruction conjecture is true, then all graphs with at least 4 edges that contain both $K_{1,2} + 2K_2$ and $T_4$ as subgraphs are Q-reconstructible.

Proof. Given $\overline{Q}(G)$, where $|G| \geq 4$, we recognise that $K_{1,2} + 2K_2$ and $T_4$ are subgraphs of $G$; then we uniquely label all elements of $\overline{Q}(G)$ of rank 3; then, assuming that the edge reconstruction conjecture is true, we extend the unique labelling to all of $\overline{Q}(G)$, thereby implying the Q-reconstructibility of $G$.

Recognising that $K_{1,2} + 2K_2$ is a subgraph of $G$: By Lemma 5.5, $K_{1,2} + 2K_2$ is a subgraph of $G$ if and only if there exists $g_i \in \overline{Q}(G)$ such that $\overline{Q}(g_i) \cong Q(K_{1,2} + 2K_2)$; and if such a $g_i$
exists, then \( g \) is unique and \( \overline{Q}(g) \) is uniquely labelled. Suppose that there exists such a \( g \). Let \( \pi \) be an arbitrary labelling of \( \overline{Q}(G) \); it is unique on \( \overline{Q}(g) \). Let \( \pi(g_1) = K_2, \pi(g_2) = 2K_2, \pi(g_3) = K_{1,2} \), \( \pi(g_4) = 3K_2 \) and \( \pi(g_5) = K_{1,2} + K_2 \).

Recognizing that \( T_4 \) is a subgraph of \( G \) given that \( K_{1,2} + 2K_2 \) is a subgraph of \( G \): We have \( Q(T_4) \cong Q(P_4 + K_2) \), and there is no other graph \( H \) such that \( Q(H) \cong Q(T_4) \). Let \( g_j \in \overline{Q}(G) \) such that \( \overline{Q}(g_j) \cong Q(T_4) \). Now \( 3K_2 \) is a subgraph of \( P_4 + K_2 \) but not of \( T_4 \), hence \( \pi(g_j) = P_4 + K_2 \) if \( g_j \leq g_3 \), else \( \pi(g_j) = T_4 \). Suppose that it is the latter case.

**Uniquely labelling all elements of rank 3:** The unique partial labelling constructed above extends to all elements of \( \overline{Q}(G) \) of rank 3 as follows. We have \( Q(3K_2) \cong Q(K_{1,3}) \cong Q(K_3) \), but the labels \( 3K_2 \) and \( T_3 \) have been assigned uniquely, and \( K_{1,3} \) is a subgraph of \( T_4 \) while \( K_3 \) is not a subgraph of \( T_4 \); thus the labels \( K_{1,3} \) and \( K_3 \) are uniquely assigned. Other graphs with 3 edges are uniquely labelled since they are edge reconstructible, and graphs with 2 edges are uniquely labelled.

**Reconstruction:** If the edge reconstruction conjecture is true, then for all \( k \geq 3 \), a unique labelling of all elements of rank \( k \) extends to a unique labelling of all elements of rank \( k + 1 \). Since all elements of rank 3 are uniquely labelled, an induction on the number of edges implies the result.

**Lemma 5.7.** For all \( m \geq 4 \), if \( G \in \{K_{1,m}^+\} \setminus \{K_{1,m+1}\} \), then \( G \) is \( Q \)-reconstructible, and the label \( K_{1,3} \) and the label \( K_3 \) (if \( K_3 \) is a subgraph of \( G \)) are uniquely assigned to elements of \( \overline{Q}(G) \).

**Proof.** We have \( \{K_{1,m}^+\} = \{K_{1,m+1}, S_{m+1}, T_{m+1}, K_{1,m} + K_2\} \), where the graphs \( S_{m+1} \) and \( T_{m+1} \) are as shown in Figure 6. The affirmation is true for \( m = 4 \), i.e., the graphs \( S_5 \), \( T_5 \) and \( K_{1,4} + K_2 \) are \( Q \)-reconstructible. Hence we assume that \( m > 4 \). Now \( G = S_{m+1} \) (or \( G = T_{m+1} \), or \( G = K_{1,m} + K_2 \) if and only if the following conditions are satisfied:

1. \( \overline{Q}(G) \) is not totally ordered
2. there exists \( g_i \in \overline{Q}(G) \) such that \( \rho(g_i) = m \), and \( \overline{Q}(g_i) \) is totally ordered
3. there exists \( g_j \in \overline{Q}(G) \) such that \( \overline{Q}(g_j) \cong Q(S_5) \) (or \( \overline{Q}(g_j) \cong Q(T_5) \) or \( \overline{Q}(g_j) \cong Q(K_{1,4} + K_2) \), respectively).

The necessity is directly verified for each graph in \( \{K_{1,m}^+\} \setminus \{K_{1,m+1}\} \). For sufficiency, the first two conditions imply that either \( G \in \{K_{1,m}^+\} \setminus \{K_{1,m+1}\} \) or \( G \in \{(mK_2)^+\} \setminus \{(m + 1)K_2\} \); the third condition implies that \( G \not\in \{(mK_2)^+\} \setminus \{(m + 1)K_2\} \); finally, \( S_5 \) is a subgraph of \( S_{m+1} \) but not of \( T_{m+1} \) or \( K_{1,m} + K_2 \) (and \( T_5 \) is a subgraph of \( T_{m+1} \) but not of \( S_{m+1} \) or \( K_{1,m} + K_2 \), and \( K_{1,4} + K_2 \) is a subgraph of \( K_{1,m} + K_2 \) but not of \( T_{m+1} \) or \( S_{m+1} \)).

**Assigning the labels \( K_{1,3} \) and \( K_3 \):** If \( G = S_{m+1} \), then \( K_3 \) and \( K_{1,3} \) are subgraphs of \( G \), and \( 3K_2 \) is not a subgraph of \( G \). But \( esub(K_3, S_{m+1}) = 1 \) and \( esub(K_{1,3}, S_{m+1}) > 1 \). Hence the labels \( K_{1,3} \) and \( K_3 \) are uniquely assigned. If \( G = T_{m+1} \) or \( G = K_{1,m} + K_2 \), then neither \( K_3 \) nor \( 3K_2 \) is a subgraph of \( G \), hence the label \( K_{1,3} \) is uniquely assigned.

**Lemma 5.8.** For all \( m \geq 4 \), if \( G \in \{(mK_2)^+\} \setminus \{(m + 1)K_2\} \), then \( G \) is \( Q \)-reconstructible. A unique element in \( G \) is labelled \( 3K_3 \).

**Proof.** We have \( \{(mK_2)^+\} = \{(m + 1)K_2, K_{1,2} + (m - 1)K_2, P_4 + (m - 2)K_2\} \) for \( m \geq 2 \). The affirmation is true for \( m = 4 \), i.e., the graphs \( K_{1,2} + 3K_2 \) and \( P_4 + 2K_2 \) are \( Q \)-reconstructible. Hence we assume that \( m > 4 \). Now \( G = K_{1,2} + (m - 1)K_2 \) (or \( G = P_4 + (m - 2)K_2 \) if and only if the following conditions are satisfied:

1. \( \overline{Q}(G) \) is not totally ordered
2. there exists $g_i \in \mathcal{Q}(G)$ such that $\rho(g_i) = m$, and $\mathcal{Q}(g_i)$ is totally ordered \\
3. there exists $g_j \in \mathcal{Q}(G)$ such that $\mathcal{Q}(g_j) \cong \mathcal{Q}(K_{1,2} + 3K_2)$ (or, respectively, $\mathcal{Q}(g_j) \cong \mathcal{Q}(P_4 + 2K_2)$).

The necessity is directly verified for each of the graphs $K_{1,2} + (m - 1)K_2$ and $P_4 + (m - 2)K_2$. For sufficiency, the first two conditions imply that either $G \in \{K_{1,m}^{+}\} \setminus \{K_{1,m+1}\}$ or $G \in \{(mK_2)^{+}\} \setminus \{(m + 1)K_2\}$; the third condition implies that $G \not\in \{K_1^{+}\} \setminus \{K_{1,m+1}\}$, hence $g_i$ must be labelled $mK_2$; finally, $G = K_{1,2} + (m - 1)K_2$ if $\text{esub}(mK_2, G) = \text{esub}(g_i, g_m) = 2$, and $G = P_4 + (m - 1)K_2$ if $\text{esub}(mK_2, G) = \text{esub}(g_i, g_m) = 1$.

Assigning the label $3K_2$: Neither $K_3$ nor $K_{1,3}$ is a subgraph of any graph $G$ in $\{(mK_2)^{+}\} \setminus \{(m + 1)K_2\}$, hence the label $3K_2$ is uniquely assigned.

**Proposition 5.9.** Graphs with at most 7 edges, except the ones listed in Theorem 5.1–1, are $Q$-reconstructible.

**A sketch of the proof.** The proof requires looking at several straightforward cases. Therefore, we only indicate two techniques, besides Lemmas and Propositions 5.5 to 5.8, that we use to prove the result efficiently.

Let $\{G, H\}$ be a $Q$-pair of graphs with $m$ edges, where $m \geq 4$. Let $f$ be an isomorphism from $\mathcal{Q}(G)$ to $\mathcal{Q}(H)$. Then there exists a $Q$-pair $\{G_i, H_i\}$ of edge-deleted subgraphs $G_i \leq_G G$ and $H_i \leq_H H$, such that $f(G_i) = H_i$; otherwise the edge reconstruction conjecture would imply that $G$ and $H$ are isomorphic. Thus once all $Q$-pairs $\{G_i, H_i\}$ of graphs with $m - 1$ edges are enumerated, we only need to consider pairs $\{G, H\}$ such that $G \in \{G_i^{+}\}$ and $H \in \{H_i^{+}\}$ as probable candidates for $Q$-pairs on $m$ edges. Many graphs in the sets $\{G_i^{+}\}$ and $\{H_i^{+}\}$ are proved to be $Q$-reconstructible using Lemmas 5.5 to 5.8. This significantly reduces the number of cases that need to be analysed.

The second technique is assigning unique labels to some elements of $\mathcal{Q}(G)$. If $G_i$ is a $Q$-reconstructible graph, and $\mathcal{Q}(g_i) \cong \mathcal{Q}(G_i)$, then only $g_i$ can be assigned the label $G_i$. Such an assignment of a label may unify labels $G_j$ of graphs which may not otherwise be $Q$-reconstructible. We used this idea in Lemmas 5.5 to 5.8; for example, presence of a subgraph $K_{1,2} + 2K_2$, which is $Q$-reconstructible, unifies labels $K_{1,2}, 2K_2, 3K_2, K_{1,2} + K_2$, even though neither of these graphs is $Q$-reconstructible; similarly, if a graph in $\{K_{1,4}^{+}\}$ is a subgraph of $G$, then it unifies labels $K_{1,3}, K_3, 3K_2$ even though neither of these graphs is $Q$-reconstructible. Once many labels have been fixed, the $Q$-reconstructibility may follow quickly.

Let the graphs listed in Theorem 5.1–1 be grouped into 3 families defined below:

\[
\begin{align*}
C_1 & := \{P_4, K_{1,2} + K_2, P_4 + K_2, T_4, C_4, 2K_{1,2}, C_4 + K_2, B_1, P_6, B_2, B_3, B_4\}, \\
C_2 & := \{K_{1,m} \mid m > 1\} \bigcup \{mK_2 \mid m > 1\}, \text{ and} \\
C_3 & := \{pK_3 + qK_{1,3} + F \mid p \neq q \text{ and } F \in \mathbb{N}^{(F)}\}.
\end{align*}
\]

**Proposition 5.10.** If the edge reconstruction conjecture is true, then graphs with at least 7 edges, except the ones listed in Theorem 5.1–1, are $Q$-reconstructible.

**Proof.** We assume that the edge reconstruction conjecture is true. Let $G$ be a graph to be $Q$-reconstructed and $\epsilon(G) \geq 7$. By Propositions 5.6, we assume that $G$ does not contain at least one of the graphs $K_{1,2} + 2K_2$ and $T_4$ as a subgraph. The graphs in $C_1$ have 6 or fewer edges. Hence we show by induction on $\epsilon(G)$, that if $G$ is not $Q$-reconstructible, then $G \in C_2 \cup C_3$.

We take $\epsilon(G) = 7$ as the base case, for which the result follows from Theorem 5.1–1 and Proposition 5.9 since the graphs in $C_1$ all have at most 6 edges. Suppose that the affirmation
is true when \(7 \leq e(G) \leq m\). Let \(G\) be an \((m+1)\)-edge graph that is not \(Q\)-reconstructible. We show that \(G \in C_2 \cup C_3\).

There must exist an \(m\)-edge subgraph \(G_i\) of \(G\) that is not \(Q\)-reconstructible; otherwise we would be able to construct the edge-deck of \(G\) from its abstract edge-subgraph poset, and then the edge reconstruction conjecture would imply the \(Q\)-reconstructibility of \(G\). By induction hypothesis, each \(m\)-edge subgraph of \(G\) that is not \(Q\)-reconstructible is in \(C_2 \cup C_3\).

**Claim 1.** Let \(G_i\) be an \(m\)-edge subgraph \(G\) that is not \(Q\)-reconstructible. If \(G_i \in C_2\), then \(G \in C_2\).

**Proof.** If \(G_i = K_{1,m}\), then \(G \in \{K_{1,m}^{\pm e}\}\). By Lemma 5.7, \(G = K_{1,m+1} \in C_2\). If \(G_i = mK_2\), then \(G \in \{(mK_2)^{\pm e}\}\). By Lemma 5.8, \(G = (m+1)K_2 \in C_2\). \(\square\)

Therefore, in the following, we assume that all \(m\)-edge subgraphs of \(G\) that are not \(Q\)-reconstructible are in \(C_3\), and show that \(G\) is in \(C_3\).

**Claim 2.** The graph \(K_{1,2} + 2K_2\) is a subgraph of \(G\), and \(T_4\) is not a subgraph of \(G\).

**Proof.** Let \(G_i\) be an \(m\)-edge subgraph of \(G\) that is not \(Q\)-reconstructible. The assumptions \(G_i \in C_3\) and \(e(G_i) \geq 7\) imply that \(K_{1,2} + 2K_2 \leq_e G_i \leq_e G\). But it follows from Proposition 5.6 that both \(K_{1,2} + 2K_2\) and \(T_4\) cannot be subgraphs of \(G\), which implies the claim. \(\square\)

**Claim 3.** The parameters \(esub(S_4, G), esub(K_4^{\pm e}, G), \) and \(esub(K_4, G)\) are \(Q\)-reconstructible. The graphs \(S_4, K_4^{\pm e}\) and \(K_4\) can appear as subgraphs of \(G\) only within components of \(G\) on 4 vertices.

**Proof.** The graphs \(S_4, K_4^{\pm e}\) and \(K_4\) are \(Q\)-reconstructible, implying the first part. If any of the graphs \(S_4, K_4^{\pm e}\) and \(K_4\) is in a component of \(G\) on 5 or more vertices, then \(T_4\) is a subgraph of \(G\); but \(T_4\) has been eliminated by Claim 2. \(\square\)

**Claim 4.** All elements of \(\overline{Q}(G)\) of rank 3 and 4, except possibly the ones corresponding to \(K_{1,3}, K_3, K_{1,3} + K_2\) and \(K_3 + K_2,\) are uniquely labelled. The parameters \(esub(K_3, G) + esub(K_{1,3}, G)\), and \(esub(K_3 + K_2, G) + esub(K_{1,3} + K_2, G)\), and \(esub(T, G)\), are \(Q\)-reconstructible, where \(T\) is any graph with at most 4 edges.

**Proof.** The claim follows from the following two statements. By Claim 2, \(K_{1,2} + 2K_2\) is a subgraph of \(G\). By Lemma 5.5, \(K_{1,2} + 2K_2\) is \(Q\)-reconstructible, and the elements of \(Q(K_{1,2} + 2K_2)\) are uniquely labelled (in particular, a unique element is assigned the label 3\(K_2)\). \(\square\)

**Claim 5.** If \(G\) contains \(K_4\), then \(G \in C_3\).

**Proof.** Since \(K_4\) is \(Q\)-reconstructible, \(G\) contains \(K_4\) if and only if there is a unique element \(g_j \in \overline{Q}(G)\) such that \(\overline{Q}(g_j) \cong Q(K_4)\). Assume that that is the case. By Claim 3, \(K_4\) can only occur as a component of \(G\). Let \(g_k\) be an element of \(\overline{Q}(G)\) such that \(\rho(g_k) = m,\) and \(w_i(g_j, g_k) \neq w_i(g_j, g_M)\); i.e., \(esub(K_4, G_k) \neq esub(K_4, G)\). Such an element must exist since deleting an edge from a component isomorphic to \(K_4\) gives a graph such as \(G_k\). The graph \(G_k\) cannot be \(Q\)-reconstructible; otherwise \(G\) would be obtained by adding an edge to any component of \(G_k\) that is isomorphic to \(K_4^{\pm e}\), and hence \(G\) would be \(Q\)-reconstructible as well. Hence \(G_k \in C_3;\) adding an edge to a component of \(G_k\) that is isomorphic to \(K_4^{\pm e}\) results in a graph in \(C_3\). \(\square\)

Therefore, we assume that \(K_4\) is not a subgraph of \(G\).

**Claim 6.** If \(G\) contains \(K_4^{\pm e}\), then \(G \in C_3\).

**Proof.** Since \(K_4^{\pm e}\) is \(Q\)-reconstructible, \(G\) contains \(K_4^{\pm e}\) if and only if there is a unique element \(g_j \in \overline{Q}(G)\) such that \(\overline{Q}(g_j) \cong Q(K_4^{\pm e})\). Assume that that is the case. Since we have assumed
that $G$ does not contain $K_4$, by Claim 3, the graph $K_4^{-\epsilon}$ can only occur as a component of $G$. Let $g_4$ be an element of $\overline{Q}(G)$ such that $\rho(g_4) = m$, and $\text{esub}(K_4^{-\epsilon}, G_k) = \text{esub}(K_4^{-\epsilon}, G) - 1$, and $\text{esub}(K_3, G_k) + \text{esub}(K_1, G_k) = \text{esub}(K_3, G) + \text{esub}(K_1, G) - 4$. Such an element $g_4$ must exist since there is a unique edge in $K_4^{-\epsilon}$ that belongs to two 3-stars and two triangles. Now, as in Claim 5, the graph $G_k$ cannot be $Q$-reconstructible; otherwise $G$ would be obtained by adding an edge to any component of $G_k$ that is isomorphic to a 4-cycle, and hence $G$ would be $Q$-reconstructible as well. Hence $G_k \in C_3$; and adding an edge to a component 4-cycle of $G_k$, results in a graph in $C_3$. □

Therefore, we assume that $K_4^{-\epsilon}$ is not a subgraph of $G$.

**Claim 7.** If $S_4$ is a component of $G$, then $G \in C_3$.

**Proof.** Since $S_4$ is $Q$-reconstructible, we recognise that $S_4$ is a component of $G$ if and only if there exists $g_4 \in \overline{Q}(G)$ such that $\overline{Q}(g_4) \cong Q(S_4)$. Assume that that is the case. Since we have assumed that $G$ does not contain $K_4$ or $K_4^{-\epsilon}$, the graph $S_4$ can only occur as a component of $G$. Claim 4 implies that there is a unique $g_4 \in \overline{Q}(G)$ that is labelled $P_4$ (even though $P_4$ itself is not $Q$-reconstructible). There is an edge in $S_4$ that belongs to exactly one $P_4$, therefore, there exists $g_4 \in \overline{Q}(G)$ of rank $m$ such that $w_e(g_4, g_4) = w_e(g_4, g_4M) - 1$, and $w_e(g_k, g_k) = w_e(g_k, g_kM) - 1$. The two conditions mean, respectively, $\text{esub}(S_4, G_k) = \text{esub}(S_4, G) - 1$ and $\text{esub}(P_4, G_k) = \text{esub}(P_4, G) - 1$. Hence $G$ is obtained from $G_k$ by adding an edge in a component isomorphic to $P_4$ so as to create a component isomorphic to $S_4$. As in Claims 5 and 6, the graph $G_k$ cannot be $Q$-reconstructible; otherwise $G$ would be $Q$-reconstructible as well. Hence then $G_k \in C_3$, implying that $G \in C_3$. □

Therefore, we assume that $S_4$ is not a subgraph of $G$.

**Claim 8.** If $G$ contains a cycle, then the cycle is a component. If $G$ contains a path on 4 or more vertices, then the path is either a component or a subgraph of a component that is either a cycle or a path.

**Proof.** Both parts follows from the assumption that neither $S_4$ nor $T_4$ is a subgraph of $G$. □

**Claim 9.** All elements of $\overline{Q}(G)$ corresponding to paths and cycles, except possibly $K_3$, are uniquely labelled.

**Proof.** By Claim 4, the elements of $\overline{Q}(G)$ corresponding to $P_2, P_3, \text{ and } P_4$ are uniquely labelled. For $n \geq 4$, the graphs $P_n$ and $C_n$ are edge reconstructible. Now the claim is proved by induction on $n \geq 4$. □

Therefore, we assume that $G$ itself is not a path or a cycle.

**Claim 10.** If $P_n, n \geq 2$ is a component of $G$, then $G \in C_3$.

**Proof.** Let $g_j$ be an element of rank $m$ in $\overline{Q}(G)$. The graph $G_j$ is obtained from $G$ by deleting an edge at the end of a component path $P_n$ if and only if $\text{esub}(P_k, G) = \text{esub}(P_k, G_j) + 1$ for all $k \leq n$, and $\text{esub}(P_{n+1}, G) = \text{esub}(P_{n+1}, G_j)$. These conditions are recognised from $\overline{Q}(G)$ by Claim 9. Assume that $g_j$ is such an element; hence $G$ is obtained by adding an edge at the end of a path isomorphic to $P_{n-1}$ in $G_j$. The graph $G_j$ is not $Q$-reconstructible, since otherwise $G$ would be $Q$-reconstructible also. Hence both $G_j$ and $G$ must be in $C_3$. □

Therefore, we assume that $G$ does not contain a path on 2 or more vertices as a component.

**Claim 11.** If $G$ has a component isomorphic to a cycle $C_n, n \geq 4$, then $G \in C_3$.

**Proof.** Let $g_j$ be an element of rank $m$ in $\overline{Q}(G)$. The graph $G_j$ is obtained from $G$ by deleting an edge of a component cycle $C_n$ if and only if $\text{esub}(C_n, G_j) = \text{esub}(C_n, G) - 1$. This condition can
be recognised from $\overline{\mathcal{Q}}(G)$ by Claim 9. Assume that $g_j$ is such an element; hence $G$ is obtained by adding an edge to $G_j$ joining the end vertices of a component $P_n$ in $G_j$. As in earlier claims, the graph $G_j$ cannot be $Q$-reconstructible; otherwise $G$ would be $Q$-reconstructible also. Hence both $G_j$ and $G$ must be in $\mathcal{C}_3$.

Therefore, we assume that $G$ does not contain a cycle on 4 or more vertices. Now the only remaining graphs are the graphs in which all components are isomorphic to $K_3$ or $K_{1,3}$, and their total number is $Q$-reconstructible by Claim 4, completing the proof of Proposition 5.10. $\blacksquare$

The method of the proof of Proposition 5.10 may be applied to any class of graphs that is closed under edge-deletion.

Corollary 5.11. Acyclic graphs (i.e., trees and forests) that are not in class $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ are $Q$-reconstructible.

Proof. Acyclic graphs with four or more edges are edge reconstructible. The class of acyclic graphs is closed under edge-deletion. Hence the proof of Proposition 5.10 may be restricted to the class of acyclic graphs to show that acyclic graphs that are not $Q$-reconstructible belong to $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. $\blacksquare$

Proposition 5.12 (Müller [12]). All graphs $G$ such that $2^{\epsilon(G)} - 1 > \nu(G)!$ are edge reconstructible.

The following result is a weaker version of Müller’s result for the $Q$-reconstruction problem.

Corollary 5.13. The edge reconstruction conjecture is true if and only if all graphs $G$ such that $2^{\epsilon(G)} - 1 \leq \nu(G)!$, except the ones in the class $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, are $Q$-reconstructible.

Proof. Theorem 5.1 and Proposition 5.12 imply the result. $\blacksquare$

Remark. Corollary 5.13 implies that if all graphs $G$ such that $G \not\in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ and $2^{\epsilon(G)} - 1 \leq \nu(G)!$ are $Q$-reconstructible, then graphs $G$ such that $G \not\in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ and $2^{\epsilon(G)} - 1 > \nu(G)!$ are $Q$-reconstructible as well.

We end the section with a few open problems.

Problem 5.14. Prove that all graphs $G$ such that $\epsilon(G) > \binom{\nu(G)}{2}/2$ are $Q$-reconstructible.

Problem 5.15. Prove that if the edge reconstruction conjecture is false, then there are infinitely many graphs $G$ such that $2^{\epsilon(G)} - 1 > \nu(G)!$ (preferably with $\epsilon(G) = \binom{\nu(G)}{2}/2$) that are not $Q$-reconstructible.

Problem 5.16. Counter examples to the edge reconstruction conjecture, if they exist, are characterised by a lemma of Nash-Williams [13]; see also Bondy [2]. Is there a characterisation, analogous to the lemma of Nash-Williams, of graphs that are not $Q$-reconstructible?

Problem 5.17. Let $\mathcal{G}_E$ be the class of graphs that are not edge reconstructible, and let $\mathcal{G}_Q$ be the class of graphs that are not $Q$-reconstructible. We have shown that $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \subseteq \mathcal{G}_Q$, where equality holds if the edge reconstruction conjecture is true. If the reconstruction conjecture is false, then we only know that $\mathcal{G}_E \subseteq \mathcal{G}_Q$ (ignoring isolated vertices in graphs in $\mathcal{G}_E$), but we do not know if there are graphs in $\mathcal{G}_Q \setminus (\mathcal{G}_E \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$. We showed in [19] that if Ulam’s conjecture is false, and if $G$ and $H$ are non-isomorphic graphs with the same deck, then $\overline{\mathcal{P}}(2G) = \overline{\mathcal{P}}(2H)$. Is there an analogous result for the edge reconstruction problem?
6 Homomorphism cancellation

Let $G, H \in \mathcal{G}$. A homomorphism from $G$ to $H$ is a map $f : V(G) \to V(H)$ such that if $\{x, y\}$ is an edge in $G$ then $\{f(x), f(y)\}$ is an edge in $H$. A one-to-one homomorphism is called a monomorphism. Let $\text{hom}(G, H)$ denote the number of homomorphisms from $G$ to $H$, and let $\text{mon}(G, H)$ denote the number of monomorphisms from $G$ to $H$. Both these parameters are well-defined even when one or both of $G$ and $H$ is unlabelled, since $\text{hom}(G, H) = \text{hom}(G', H')$ and $\text{mon}(G, H) = \text{mon}(G', H')$ whenever $G \cong G'$ and $H \cong H'$. Given an unlabelled graph $G$, we denote by $G^\ast$ a representative labelled graph in $G$. Lovász [9] proved the following result.

**Theorem 6.1** (Lovász [9]; see also Problem 20, Chapter 13 in Lovász [10]). Let $G_1, G_2 \in \mathcal{G}$.

1. If $\text{hom}(G_1, H) = \text{hom}(G_2, H)$ for all $H \in \mathcal{G}$, then $G_1 \cong G_2$.
2. If $\text{hom}(H, G_1) = \text{hom}(H, G_2)$ for all $H \in \mathcal{G}$, then $G_1 \cong G_2$.

We propose the following conjecture, which in a sense generalises the idea of homomorphism cancellation in Theorem 6.1.

**Conjecture 6.2.** Let $\pi : (\mathcal{G}/\cong) \to (\mathcal{G}/\cong)$ be a bijection such that $\text{hom}(G, H) = \text{hom}(\pi(G), \pi(H))$ for all $G, H \in \mathcal{G}/\cong$. Then $G = \pi(G)$ for all $G \in \mathcal{G}/\cong$.

We show in Proposition 6.4 that Conjecture 6.2 is weaker than the edge reconstruction conjecture.

**Lemma 6.3.** Let $\pi : (\mathcal{G}/\cong) \to (\mathcal{G}/\cong)$ be a bijection such that $\text{hom}(G, H) = \text{hom}(\pi(G), \pi(H))$ for all $G, H \in \mathcal{G}/\cong$. Then $\nu(G) = \nu(\pi(G))$ and $\epsilon(G) = \epsilon(\pi(G))$, and $\pi(G) = G$ for all $G$ such that $\epsilon(G) \leq 3$.

**Proof.** Claim 1. $\pi(\Phi) = \Phi$ (where $\Phi$ denotes the null graph).

Proof. Let $\pi(G) = \Phi$ and $\pi(\Phi) = H$ for some graphs $G$ and $H$. Therefore, $\text{hom}(G, \Phi) = \text{hom}(\pi(G), \pi(\Phi)) = \text{hom}(\Phi, H)$. We have $\text{hom}(\Phi, H) = 1$, and $\text{hom}(G, \Phi) = 1$ if and only if $\nu(G) = 0$, i.e., $G = \Phi$. Therefore, $\pi(\Phi) = \Phi$. □

Claim 2. $\pi(K_1) = K_1$.

Proof. Let $\pi(G) = K_1$ and $\pi(K_1) = H$ for some graphs $G$ and $H$. Since, $\pi(\Phi) = \Phi$, the graphs $G$ and $H$ are non-null. Now, $\text{hom}(G, K_1) = \text{hom}(\pi(G), \pi(K_1)) = \text{hom}(K_1, H) = \nu(H) \geq 1$. We have $\text{hom}(G, K_1) = 1$ if $\epsilon(G) = 0$, and $\text{hom}(G, K_1) = 0$ otherwise. Therefore, $\nu(H) = 1$ and $\epsilon(G) = 0$. Moreover, since $H$ is simple, we have $\pi(K_1) = H = K_1$. □

Claim 3. For all $G$, we have $\nu(G) = \nu(\pi(G))$.

Proof. We have $\nu(G) = \text{hom}(K_1, G) = \text{hom}(\pi(K_1), \pi(G)) = \text{hom}(K_1, \pi(G)) = \nu(\pi(G))$. □

Claim 4. $\pi(K_2) = K_2$ and $\pi(2K_1) = 2K_1$.

Proof. If the claim is not true, then by Claim 3 we must have $\pi(K_2) = 2K_1$ and $\pi(2K_1) = K_2$, but that is not possible since $\text{hom}(K_2, K_2) \neq \text{hom}(2K_1, 2K_1)$. □

Claim 5. For all $G$, we have $\epsilon(G) = \epsilon(\pi(G))$.

Proof. We have $2\epsilon(G) = \text{hom}(K_2, G) = \text{hom}(K_2, \pi(G)) = 2\epsilon(\pi(G))$. □

Claim 6. $\pi(K_3) = K_3$ and $\pi(K_{1,2}) = K_{1,2}$.

Proof. The claim follows from Claims 3 and 5, and that every graph $G$ on at most 3 vertices is determined by the pair $(\nu(G), \epsilon(G))$. □

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Claim 7. If $\epsilon(G) \leq 3$ then $\pi(G) = G$.

Proof. If $G$ contains a triangle, then $\pi(G) = G$, which follows from Claim 3 and that $\text{hom}(K_3, G) = \text{hom}(K_3, \pi(G))$. If $G$ does not contain a triangle and has at most 3 edges, then $G$ is uniquely determined by the triple $\nu(G), \epsilon(G), \text{hom}(K_{1,2}, G)$. □

Proposition 6.4. The edge reconstruction conjecture implies Conjecture 6.2.

Proof. Let $\pi : (\mathcal{G}/\cong) \rightarrow (\mathcal{G}/\cong)$ be a bijection such that $\text{hom}(G, H) = \text{hom}(\pi(G), \pi(H))$ for all $G, H \in \mathcal{G}/\cong$. We assume the edge reconstruction conjecture to be true, and prove by induction on the number of edges that $\pi(G) = G$ for all $G$. In Lemma 6.3, we proved that $\pi(G) = G$ for all $G$ such that $\epsilon(G) \leq 3$. Suppose that $\pi(G) = G$ for all graphs $G$ such that $3 \leq \epsilon(G) \leq m$.

Let $G \in \mathcal{G}/\cong$ be a graph with $m + 1$ edges.

For an unlabelled graph $H$ and an equivalence relation $\Theta$ on $V(H^*)$, let $H^*/\Theta$ denote the graph obtained by identifying vertices in each equivalence class of $\Theta$. Each homomorphism from $H^*$ to $G^*$ is a monomorphism from $H^*/\Theta$ to $G^*$ for some equivalence relation $\Theta$ on $V(H^*)$. Therefore,

$$\text{hom}(H^*, G^*) = \sum_{\Theta} \text{mon}(H^*/\Theta, G^*),$$

where the summation is over all equivalence relations on $V(H^*)$. In general, for all equivalence relations $\Theta$ on $V(H^*)$ we have

$$\text{hom}(H^*/\Theta, G^*) = \sum_{\Theta' \mid \Theta \leq \Theta'} \text{mon}(H^*/\Theta', G^*),$$

where $\Theta \leq \Theta'$ means $\Theta$ is a refinement of $\Theta'$. Following Lovász [10, Chapter 15, Problem 20], we solve the system of Equations (22) for $\text{mon}(H^*, G^*)$ in terms of $\text{hom}(H^*/\Theta, G^*)$, and write

$$\text{mon}(H, G) = \text{mon}(H^*, G^*) = \sum_{\Theta} \alpha_{H^*/\Theta} \text{hom}(H^*/\Theta, G^*),$$

where $\alpha_{H^*/\Theta}$ are constants (that do not depend on $G$). (Another way to look at the solutions of the system of equations is via Möbius inversion.)

For all $H \in \mathcal{G}/\cong$ such that $\epsilon(H) \leq m$, and for all equivalence relations $\Theta$ on $V(H^*)$, we have, by induction hypothesis, $\text{hom}(H^*/\Theta, G) = \text{hom}(H^*/\Theta, \pi(G))$. Hence $\text{mon}(H, G) = \text{mon}(H, \pi(G))$ (by Equation 23). In other words, $G$ and $\pi(G)$ have the same edge-deck. Now the edge reconstruction conjecture implies that $G = \pi(G)$, completing the induction step, and the result.

The following statement is analogous to Conjecture 6.2, but for labelled graphs.

Conjecture 6.5. Let $\pi : \mathcal{G} \rightarrow \mathcal{G}$ be a bijection such that $\text{hom}(G, H) = \text{hom}(\pi(G), \pi(H))$ for all $G, H \in \mathcal{G}$. Then $G \cong \pi(G)$ for all $G \in \mathcal{G}$.

It is unclear if Conjecture 6.2 and Conjecture 6.5 are equivalent, although it is tempting to believe that they are. The edge reconstruction conjecture implies Conjecture 6.5 as well; the proof of this fact is similar to the proof of Proposition 6.4, and we skip it.

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**Nomenclature**

\[ \cong \] isomorphic to; used for graphs and posets ......................... 3

\[ \leq \] less than or equal; isomorphic to a subgraph of ................. 3

\[ \leq_e \] isomorphic to an edge-subgraph of .......................... 3

\[ \leq_v \] isomorphic to an induced subgraph of ..................... 3

\[ \subseteq \] subset of; subgraph of ........................................ 3

\[ \subseteq_e \] edge-subgraph of ................................ ........... 3

\[ \subseteq_v \] induced subgraph of ................................ ........... 3

\[ \hat{0} \] minimal element of the connected partition lattice ......... 7

\[ 1^n \] assignment \( x_i = 1 \) for \( i \leq n \), and \( x_i = 0 \) for \( i > n \) ................. 18

\[ 2^S \] powerset of \( S \) ................................ .................. 2

\[ B_1, \ldots, B_4 \] certain special graphs ........................................ 23

\[ C_n \] cycle on \( n \) vertices ........................................... 3

\[ c(G) \] number of components of \( G \) ................................ ...... 3

\[ c(H, G) \] number of components of \( G \) that are isomorphic to \( H \) .......... 3

\[ \text{cov}(\mathcal{F} \to H) \] number of covers of graph \( H \) by a tuple \( \mathcal{F} \) of graphs ........... 11
cov(\(F \rightarrow H\)) \hspace{1cm} \text{number of vertex covers of graph } H \text{ by a tuple } F \text{ of graphs} \hspace{1cm} 11

cov((f_1, \ldots, f_k) \rightarrow H) \hspace{1cm} \text{similar to covers; } f_i \text{ elements of } \overline{\mathcal{F}}(G) \hspace{1cm} 14

E(G) \hspace{1cm} \text{edge set of } G \hspace{1cm} 3

\{F^+e\} \hspace{1cm} \text{set of unlabelled graphs obtained by adding an edge to } F \hspace{1cm} 23

\mathcal{G} \hspace{1cm} \text{set of all labelled graphs} \hspace{1cm} 3

\mathcal{G}/\cong \hspace{1cm} \text{set of all unlabelled graphs} \hspace{1cm} 3

\mathcal{G}^c \hspace{1cm} \text{set of all labelled connected graphs} \hspace{1cm} 3

\mathcal{G}_E \hspace{1cm} \text{class of graphs that are not edge reconstructible} \hspace{1cm} 30

\mathcal{G}_Q \hspace{1cm} \text{class of graphs that are not } Q\text{-reconstructible} \hspace{1cm} 30

G^*, G \in \mathcal{G}/\cong \hspace{1cm} \text{representative labelled graph in an isomorphism class } G \hspace{1cm} 3

G - E, E \subseteq E(G) \hspace{1cm} \text{spanning subgraph of } G \text{ with edge set } E(G) \setminus E \hspace{1cm} 3

G - e, e \in E(G) \hspace{1cm} \text{spanning subgraph of } G \text{ with edge set } E(G) \setminus \{e\} \hspace{1cm} 3

G[E], E \subseteq E(G) \hspace{1cm} \text{subgraph of } G \text{ induced by } E \hspace{1cm} 3

G_E, E \subseteq E(G) \hspace{1cm} \text{spanning subgraph of } G \text{ with edge set } E \hspace{1cm} 3

G - u, u \in V(G) \hspace{1cm} \text{subgraph of } G \text{ induced by } V(G) \setminus \{u\} \hspace{1cm} 3

G - X, X \subseteq V(G) \hspace{1cm} \text{subgraph of } G \text{ induced by } V(G) \setminus X \hspace{1cm} 3

G[X], X \subseteq V(G) \hspace{1cm} \text{subgraph of } G \text{ induced by } X \hspace{1cm} 3

g_1, \ldots, g_m \hspace{1cm} \text{elements in an abstract edge-subgraph poset} \hspace{1cm} 9

g_1, \ldots, g_m \hspace{1cm} \text{elements in an abstract induced subgraph poset} \hspace{1cm} 4

h_1, \ldots, h_N \hspace{1cm} \text{elements in an abstract bond lattice} \hspace{1cm} 7

\text{hom}(G, H) \hspace{1cm} \text{number of homomorphisms from } G \text{ to } H \hspace{1cm} 31

I(G), G \in \mathcal{G} \hspace{1cm} \text{isomorphism class of a labelled graph } G \hspace{1cm} 3

\text{ind}(H, G) \hspace{1cm} \text{number of induced subgraphs of } G \text{ that are isomorphic to } H \hspace{1cm} 3

K_4^-e \hspace{1cm} K_4 \text{ minus an edge} \hspace{1cm} 3

K_n \hspace{1cm} \text{complete graph on } n \text{ vertices} \hspace{1cm} 3

K_{n,m} \hspace{1cm} \text{complete bipartite graph} \hspace{1cm} 3

k(\lambda, G) \hspace{1cm} \text{number of connected partitions of } G \text{ of type } \lambda \hspace{1cm} 21

\ell(\lambda) \hspace{1cm} \text{length of an integer partition} \hspace{1cm} 4

m(\kappa) \hspace{1cm} \text{number of monochromatic edges in a vertex colouring } \kappa \hspace{1cm} 18

\text{mon}(G, H) \hspace{1cm} \text{number of monomorphisms from } G \text{ to } H \hspace{1cm} 31

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| Symbol | Description |
|--------|-------------|
| $\mathbb{N}$ | set of natural numbers (including 0) |
| $\mathbb{N}(F)$ | set of all unlabelled graphs with components from $F$ |
| $\mathcal{P}$ | induced subgraph poset with ground set $\mathcal{G}/\cong$ |
| $\mathcal{P}(G)$ | concrete induced subgraph poset of $G$ |
| $\overline{\mathcal{P}}(G)$ | abstract induced subgraph poset of $G$ |
| $P_n$ | path on $n$ vertices |
| $p_\lambda$ | power sum symmetric function of integer partition $\lambda$ |
| $\mathcal{Q}$ | weighted edge-subgraph poset with ground set $\mathcal{G}/\cong$ |
| $\mathcal{Q}(G)$ | concrete edge-subgraph poset |
| $\overline{\mathcal{Q}}(G)$ | abstract edge-subgraph poset |
| $S_{m, m \geq 3}$ | certain special graphs |
| $S^k$ | set of $k$-element tuples of elements in $S$ |
| $(\hat{S})_k$ | family of $k$-element subsets of $S$ |
| $\text{sub}(H, G)$ | number of subgraphs of $G$ that are isomorphic to $H$ |
| $T_{m, m \geq 3}$ | certain special graphs |
| $V(G)$ | vertex set of $G$ |
| $w_e$ | weight function of the edge-subgraph poset |
| $w_\pi$ | weight function of the bond lattice |
| $w_v$ | weight function of the inducted subgraph poset |
| $X_G(x)$ | chromatic symmetric function of a graph $G$ |
| $X_G(x; t)$ | symmetric Tutte polynomial of a graph $G$ |
| $x_1, x_2, \ldots$ and $t$ | commuting indeterminates |
| $\mathbb{Z}$ | set of integers |
| $\mathbb{Z}^+$ | set of positive integers |
| $\mathbb{Z}^{(X)}$ | set of finite formal sums of elements of $X$ |
| $\epsilon(G)$ | number of edges of $G$ |
| $\zeta(\cdot, \cdot)$ | zeta function of a poset |
| $\kappa$ | vertex colouring function on a graph |
| $\lambda \models \mu$ | integer partition $\lambda$ refines integer partition $\mu$ |
| $\lambda \models n$ | $\lambda$ is a partition of $n$ |
| Symbol | Definition | Page |
|--------|------------|------|
| $\lambda(\pi)$ | integer partition associated with a set partition $\pi$ | 4 |
| $\mu(.,.)$ | Möbius function of a poset | 20 |
| $\nu(G)$ | number of vertices of $G$ | 3 |
| $\Pi^c_G$ | connected partition lattice of $G$ | 7 |
| $\Pi(V)$ | partition lattice of $V$ | 4 |
| $\pi \models \sigma$ | set partition $\pi$ refines set partition $\sigma$ | 4 |
| $\pi \models V$ | $\pi$ is partition of $V$ | 4 |
| $\pi \vdash V$ | $\pi$ is a family of mutually disjoint subsets of $V$ | 4 |
| $\rho(x)$ | rank of element $x$ in a poset | 7 |
| $\theta(v, e; T)$ | number of certain partitions defined for a tree | 22 |
| $\Phi$ | null graph | 3 |
| $\phi(v, e, \lambda)$ | certain function of integer vectors $v$, $e$ and an integer partition $\lambda$ | 22 |
| $\psi(.)$ | a function related to the Möbius function of a bond lattice | 20 |
| $\Omega(G)$ | concrete bond lattice of $G$ | 7 |
| $\overline{\Omega}(G)$ | abstract bond lattice of $G$ | 7 |