The Rate of Stellar Mass Black Hole Scattering in Galactic Nuclei

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Abstract
We consider a black hole (BH) density cusp in a nuclear star cluster (NSC) hosting a supermassive back hole (SMBH) at its center. Assuming the stars and BHs inside the SMBH sphere of influence are mass-segregated, we calculate the number of BHs that sink into this region under the influence of dynamical friction. We find that the total number of BHs increases significantly in this region due to this process for lower-mass SMBHs by up to a factor of 5, but there is no increase in the vicinity of the highest mass SMBHs. Due to the high BH number density in the NSC, BH–BH binaries form during close approaches due to gravitational wind (GW) emission. We update the previous estimate of O’Leary et al. for the rate of such GW capture events by estimating the $(n^2)/(n)^2$ parameter where $n$ is the number density. We find a BH merger rate for this channel to be in the range $\sim 0.002-0.04$ Gpc$^{-3}$ yr$^{-1}$. The total merger rate is dominated by the smallest galaxies hosting SMBHs, and the number of heaviest BHs in the NSC. It is also exponentially sensitive to the radial number density profile exponent, reaching $> 100$ Gpc$^{-2}$ yr$^{-1}$ when the BH mass function is $m^{-2.3}$ or shallower, and the heaviest BH radial number density is close to $r^{-2}$. Even if the rate is much lower than the range constrained by the current LIGO detections, the GW captures around SMBHs can be distinguished by their high eccentricity in the LIGO band.

Key words: black hole physics – galaxies: nuclei – gravitational waves

1. Introduction

Ten stellar black-hole–black-hole (BH–BH) detections of binary mergers have been announced to date by Advanced LIGO and Virgo, which imply a merger rate density of 24–112 yr$^{-1}$ Gpc$^{-3}$ in the universe for a power-law BH mass function prior (The LIGO Scientific Collaboration et al. 2018a, 2018b). Several astrophysical channels have been proposed to explain these rates, including isolated binary evolution in the galactic field (Belczynski et al. 2016) and dynamically formed binaries in globular clusters (Rodriguez et al. 2016; Fragione & Kocsis 2018). All these processes are consistent with being approximately circular, which is expected if the binaries form with a sufficiently large periastron, as gravitational wave (GW) emission circularizes the orbit as it shrinks (Peters 1964).

However, a few channels are predicted to produce eccentric BH binaries that retain significant nonzero eccentricity in the Advanced LIGO frequency range ($\gtrsim 10$ Hz for design sensitivity). First, the Kozai–Lidov effect can enhance the BH binary eccentricity in hierarchical triples. The tertiary component can be a star or another stellar mass BH. Such triples can form in the galactic field (Antonini et al. 2017; Silsbee & Tremaine 2017) or in globular clusters as a result of a binary–binary interaction (Antonini et al. 2016). Alternatively, the tertiary component may be a supermassive BH (SMBH) in a galactic center (Antonini & Perets 2012; Hamers et al. 2018; Hoang et al. 2018). Furthermore, “resonant” binary-single scattering interactions may lead to highly eccentric binaries in globular clusters where the tertiary increases the eccentricity of the inner binary during close pericenter passages (Rodriguez et al. 2018; Samsing 2018). Eccentric BH binary GW sources can also be created from non-hierarchical triples (Arca-Sedda et al. 2018).

The focus of this paper is on another way to produce highly eccentric BH–BH binaries, the so-called “GW captures”, in which two single BHs undergo a close encounter and lose a sufficient amount of energy due to GW emission to become bound (O’Leary et al. 2009). For a sufficiently low impact parameter, the newly formed binary has a sufficiently small semimajor axis and high eccentricity to merge quickly before it is disrupted by an interaction with another star or BH. These events are most frequent in dense stellar clusters, e.g., galactic nuclei and globular clusters. However, the low relative velocity of BHs in globular clusters implies that most GW captures will form binaries on a wide orbit, and the binary eccentricity will typically become low due to GW emission when reaching the LIGO band (O’Leary et al. 2009, Figure 6). In contrast, BHs in galactic nuclei sink to the inner regions close to the SMBH due to dynamical friction (DF), and form a mass-segregated steep density cusp, where the velocity dispersion is much higher (Bahcall & Wolf 1977; Freitag et al. 2006; Hopman & Alexander 2006; O’Leary et al. 2009). In these environments, GW capture binaries typically form in the LIGO band with high eccentricities (Gondán et al. 2018b). As was shown in Gondán et al. (2018a), the aLIGO-adVirgo-KAGRA detector network will be able to measure the merging BH binary’s eccentricity with high accuracy and therefore potentially distinguish the GW capture events from other astrophysical formation channels.

The purpose of this paper is to refine the rate estimate of GW capture events in galactic nuclei, highlight the main sources of uncertainties, and to calculate the distribution of their total masses and mass ratios. Previously, O’Leary et al. (2009) estimated the event rates by solving the Fokker–Planck equations for isotropic multimass BH distributions. The results were quite different for models with a limited mass range of stellar BHs and models with a BH mass function that extends to higher masses (e.g., $m_{\text{max}} = 15 M_\odot$ versus 45 $M_\odot$). Importantly, the rates were found to be proportional to a parameter $\xi$ defined as the mean squared number density over the square of the mean number density of galactic nuclei. The value of this parameter was not estimated, its fiducial value was assumed to be $\xi = 30$. Kocsis & Levin (2012) extended O’Leary et al. (2009) using post-Newtonian simulations, and showed with
simple analytical estimates that rare galaxies with a high $\xi$ and BH mass fractions may dominate the rates, but did not determine these quantities. Tsang (2013) estimated $\xi$, but did not calculate BH–BH merger rates, but rather focused on NS–NS binaries which do not form highly mass-segregated density cusps. Gondán et al. (2018b) derived the mass and eccentricity distribution of GW capture binaries, but also did not estimate their total merger rate.

In this paper we fill in the missing pieces in the puzzle to determine the GW capture rate using simple analytical estimates, and examine the dependence on various model parameters. We estimate $\xi$ based on the observed scatter of the $M-\sigma$ relation. Further, we calculate the BH number density taking into account the DF bringing BHs (predominantly heavier ones) into the galactic center (Miralda-Escudé & Gould 2000). Given the number density, the event rate does not depend on any additional parameters (O'Leary et al. 2009; Gondán et al. 2018b). We consider various assumptions for the initial BH mass function and, based on the heaviest BH detected by LIGO (The LIGO Scientific Collaboration et al. 2018a), assume it extends up to $50 M_\odot$. We also briefly consider the effect of the steepness of the BH density cusp (Kesht et al. 2009).

The paper is organized as follows. In Section 2, we calculate the number of BHs around the SMBH using the results of previous papers about their mass-segregated density profile. Then, in Section 3 we utilize this result to calculate the rate of GW captures in a galactic nucleus. Finally, in Section 4 we integrate over all galaxies and calculate the event rate per unit volume. In Section 5 we summarize our conclusions and briefly discuss other eccentric BH merger mechanisms. Several details about the calculations are given in the appendix.

2. Number of BHs in a Galactic Center

In this section we calculate the increase in the number of stellar mass BHs within the SMBH radius of influence $r_0$ due to the sinking of BHs from larger radii caused by DF.

2.1. Initial Conditions

First, we assume that all stars and BHs formed in the galactic center early in the galactic lifetime ($T = 12$ Gyr ago), and that every star heavier than a certain mass $m_{\text{cr}}$, produced a BH remnant. We will also consider the case of continuous star formation in the next subsection. The initial stellar mass function is taken from Kroupa (2001):

$$f_{\text{IMF}}(m) = \begin{cases} 
 25 \left( \frac{m}{M_\odot} \right)^{-0.3}, & m < 0.08 M_\odot, \\
 2 \left( \frac{m}{M_\odot} \right)^{-1.3}, & 0.08 M_\odot < m < 0.5 M_\odot, \\
 \left( \frac{m}{M_\odot} \right)^{-2.3}, & m > 0.5 M_\odot,
\end{cases}$$

(1)

where $C$ is a normalization parameter. The total stellar mass is then

$$M_\star = \int_0^\infty m f_{\text{IMF}}(m) \, dm = 5.58 M_\odot C,$$

(2)

and the total (initial) number of BHs is

$$N_{\text{init}} = \int_{m_{\text{cr}}}^\infty f_{\text{IMF}}(m) \, dm = k M_\star,$$

(3)

where

$$k = 2.86 \times 10^{-3} M_\odot^{-1} \left( \frac{m_{\text{cr}}}{20 M_\odot} \right)^{-1.3}.$$  

(4)

Equation (3) allows us to calculate the initial number of BHs in any region where we know the total stellar mass; for example, inside the influence radius $r_0$ defined where $M_\star = 2M_{\text{SMBH}}$

$$N_{\text{init}}(r_0) = 2k M_{\text{SMBH}},$$

(5)

Thus, the initial BH mass fraction is

$$\kappa_{\text{init}} = \frac{N_{\text{init}}(m_{\text{BH}})}{M_\star} = k \langle m_{\text{BH}} \rangle,$$

(6)

where $\langle m_{\text{BH}} \rangle$ is the average BH mass. Given a power-law BH mass distribution

$$f_{\text{BH,init}}(m) \propto m_{\text{BH}}^{-\beta}, \quad m_{\text{min}} < m_{\text{BH}} < m_{\text{max}},$$

(7)

the average BH mass is

$$\langle m_{\text{BH}} \rangle = \frac{\int_{m_{\text{min}}}^{m_{\text{max}}} m_{\text{BH}}^{-\beta} \, dm}{\int_{m_{\text{min}}}^{m_{\text{max}}} m_{\text{BH}}^{-\beta} \, dm} = \frac{\beta - 1}{\beta - 2} \left( \frac{m_{\text{min}}^{1-\beta} - m_{\text{max}}^{1-\beta}}{m_{\text{min}}^{2-\beta} - m_{\text{max}}^{2-\beta}} \right).$$

(8)

For example, $m_{\text{min}} = 5 M_\odot$ and $m_{\text{max}} = 40 M_\odot$ give $\kappa_{\text{init}} \approx 0.03$ for $2 < \beta < 3$.

The mass distribution of BHs born inside $r_0$ is

$$\frac{dN_{\text{init}}}{dm} = f_{\text{BH,init}}(m) \, M_\star(r_0) = f_{\text{BH,init}}(m) k \int_0^{r_0} \rho_\star(r) 4\pi r^2 \, dr,$$

(9)

where $\rho_\star(r)$ is stellar density

$$\rho_\star(r) = \begin{cases} 
 \rho_0 \left( \frac{r}{r_0} \right)^{-\gamma_1}, & r \leq r_0, \\
 \rho_0 \left( \frac{r}{r_0} \right)^{-\gamma_2}, & r > r_0,
\end{cases}$$

(10)

where

$$\rho_0 = \frac{(3 - \gamma_1)}{4\pi \gamma_2} \frac{2M_{\text{SMBH}}}{r_0^3}.$$  

(11)

This gives

$$\frac{dN_{\text{init}}}{dm} = \frac{4\pi}{3 - \gamma_1} k \rho_0 r_0^3 f_{\text{BH,init}}(m).$$

(12)

As we consider the Milky Way (MW) NSC to be relaxed (Bahcall & Wolf 1977), we assume $\gamma_1 = 1.5$, which is consistent with observed deep star counts (Gallego-Cano et al. 2018). For the density profile outside $r_0$, we assume $\gamma_2 = 3.2$, which is consistent with both star counts and diffuse light measurements (Gallego-Cano et al. 2018; Schödel et al. 2018).

1 However, diffuse light measurements of Schödel et al. (2018) give a lower value $\gamma_1 = 1.13 \pm 0.08$. The difference in the BH number between $\gamma_1 = 1.5$ and $\gamma_1 = 1.1$ is only $\sim 15\%$. 

2.2. The Effect of DF on the BH Number Density

The black hole mass function in the NSC is affected by DF, which delivers BHs into this region. The total number of BHs with a given mass within \( r_0 \) at present is defined by the maximum radius \( r_{DF} \) from where a BH sinks to within \( r_0 \) in a Hubble time:

\[
\frac{dN_{\text{tot}}}{dm} = f_{\text{BH, init}}(m) \int_0^{r_{DF}(m)} n(r) 4\pi r^2 dr,
\]

where \( n(r) \) is the BH number density. Here \( r_{DF}(m) \) can be defined as the initial orbital radius of a BH given the final radius \( r_0 \) and BH mass \( m \). The evolution of a BH orbital radius can be approximated as (Binney & Tremaine 2008)

\[
\frac{dr}{dt} = \frac{-r}{t_{DF}} = -r \cdot \ln\Lambda \frac{4\pi G^2 \rho_s m}{v^3} \int_0^r 4\pi u^2 F(u) du,
\]

where

\[
\ln\Lambda \approx \ln(M_*/m) \approx 13,
\]

\( M_{\text{SMBH}} \) is the central SMBH mass, \( v \) is the BH velocity, and \( F(u) \) is the velocity distribution of ambient stars. For a Maxwellian velocity distribution the value of the integral is 0.54 for \( v^2 = \langle u^2 \rangle \). Assuming a circular BH orbit,

\[
v = \sqrt{\frac{GM(r)}{r}},
\]

where \( M(r) \) is the total mass inside of radius \( r \)

\[
M(r) = 3M_{\text{SMBH}} + \int_0^r \rho_0 \left( \frac{r}{r_0} \right)^{-\gamma_2} 4\pi r^2 dr
\]

\[
= M_{\text{SMBH}} \left[ 3 + 2 \frac{3 - \gamma_1}{\gamma_2 - 3} \left( 1 - \left( \frac{r}{r_0} \right)^{3 - \gamma_2} \right) \right].
\]

As a result, the dependence of DF timescale on radius is the following

\[
t_{DF} = t_{DF,0} x^{\gamma_2 - 3/2} \left[ 1 + 2 \frac{3 - \gamma_1}{\gamma_2 - 3} (1 - x^{3 - \gamma_2}) \right]^{3/2},
\]

\[
x \equiv \frac{r}{r_0},
\]

\[
t_{DF,0} \equiv \frac{(3r_0)^{1/2} M_{\text{SMBH}}^{1/2}}{1.08(3 - \gamma_1) \ln\Lambda G^{1/2} m}
\]

\[
= 3.8 \text{ Gyr} \left( \frac{m}{10 M_\odot} \right)^{-1} \left( \frac{r_0}{3 \text{ pc}} \right)^{3/2}
\]

\[
\times \left( \frac{M_{\text{SMBH}}}{4 \times 10^6 M_\odot} \right)^{1/2}.
\]

The equation of motion (14) then becomes

\[
\frac{dx}{d\tau} = -x \left( \frac{t_{DF}}{t_{DF,0}} \right)^{-1}
\]

\[
= -x^{5/2 - \gamma_2} \left[ 1 + 2 \frac{3 - \gamma_1}{\gamma_2 - 3} \left( 1 - x^{3 - \gamma_2} \right) \right]^{-3/2},
\]

\[
\tau \equiv \frac{t}{t_{DF,0}}.
\]

The radius \( r_{DF}(m) \) from which objects of mass \( m \) sink to within the radius of influence \( r_0 \) within time \( T \) satisfies

\[
x \left( \frac{T}{t_{DF,0}} \right) = 1,
\]

\[
x(\tau = 0) = \frac{r_{DF}}{r_0}.
\]

The numerical solution of Equation (19) with boundary conditions (20) for \( \gamma_1 = 1.5, \gamma_2 = 3.2 \) can be approximated (with 3% accuracy for \( 5 M_\odot < m < 40 M_\odot \)) as

\[
r_{DF} = r_0 \left[ 1 + k_1 \left( \frac{T}{t_{DF,0}} \right)^{k_2} \right]^{k_3},
\]

where \( k_1 = 1.016, k_2 = 0.740, k_3 = 0.654 \). Note that \( r_{DF} \) depends on \( m \) through \( t_{DF,0} \), as shown on Figure 1, e.g. in an MW-like galaxy, for \( m = 40 M_\odot \), \( r_{DF} \approx 11 \text{ pc} \), which is about the distance where the NSC starts dominating over the galactic background in the MW (if we take the MW stellar density from e.g., Gnedin et al. 2014).

After we substitute Equation (21) into (13), we find that the BH mass function within distance \( r_0 \) after a Hubble time is

\[
\frac{dN_{\text{tot}}}{dm} = 4\pi \rho_0 r_0^3 f_{\text{BH, init}}(m)
\]

\[
\times \left\{ \frac{1}{3 - \gamma_1} + \frac{1}{\gamma_2 - 3} \left[ 1 - \left( \frac{r_{DF}(m)}{r_0} \right)^{3 - \gamma_2} \right]^{3/2} \right\}.
\]

As \( f_{\text{BH, init}}(m) \) and \( k \) (Equation (4)) are highly uncertain, it is useful to calculate the relative increase in the number of BHs
due to DF:

\[
\zeta = 1 + \frac{3 - \gamma_1}{\gamma_2 - 3} \left[ 1 - \left\{ 1 + k_i \left( \frac{T}{t_{DF,0}} \right)^{k_i} \right\}^{(3 - \gamma_2)k_i} \right].
\]

(23)

So far we have assumed that all of the BHs in the Galactic center were born \( T = 12 \) Gyr ago. Additionally, we also consider the possibility in which the BHs are produced continuously in time with a constant rate. The true BH formation history may be expected to lie between these two extreme cases, and the corresponding estimates for the number of BHs or their merger rate may represent upper and lower limits, respectively. The corresponding expressions for \( \zeta \) are derived in Appendix A.

Figure 2 shows that the value of \( \zeta \) ranges between 1.5 and 3 in MW-like galaxies depending on the BH mass and the assumed BH formation history. However, as shown in the next section, \( \zeta \) is higher or lower in more or less massive galaxies, respectively.

### 3. Merger Rate Per Galaxy

The rate of GW captures in a single NSC, \( \Gamma \), may be calculated by adding up the contribution of different radial shells around the SMBH within its radius of influence, accounting for the local flux of objects and the cross section to form binaries by GW emission for given BH masses. To do that, we adopt the formula from Gondán et al. (2018b, Equation (125)):

\[
\frac{\partial^2 \Gamma_0}{\partial m_A \partial m_B} \approx \frac{G^{17/4}}{c^{10/7}} \frac{N_B^{2/7}}{N_{SMBH}^{11/14}} \left( \frac{m_{SMBH}^{3/4}}{r_0} \right)^{11/14 - \beta} \left( m_{max} \right)^{11/4 - \beta} \left( m_{min} \right)^{-3/4} m_{tot} \frac{2 \rho_0 m_{tot} m_{max} - 4 m_{max}^2 + 9 m_{max}^2 - 6 \rho_0 m_{max} m_{tot} + 4 \rho_0^2 m_{tot}}{16 m_{max}^2}. \]

(24)

Here \( m_{A,B} \) are the BH masses, \( r_{min} \) and \( r_0 \) are the minimum and maximum radii of the BH density distribution, \( N_B \) is the total number of BHs within \( r_{min} < r < r_0 \), \( \beta \) is the BH mass function power law (as in Equation (7)) and \( m_{min,max} \) are the minimum and maximum BH masses (in this paper we assume \( m_{min} = 5 M_\odot \), \( m_{max} = 40 M_\odot \)). Also,

\[
m_{tot} \equiv m_A + m_B, \quad \mu \equiv \frac{m_A m_B}{m_A + m_B}, \quad \eta \equiv \frac{\mu}{m_{tot}} \quad (25)
\]

are the total mass, reduced mass, and symmetric mass ratio, respectively, and \( a_q \equiv (340 \pi / 3) \eta \).

![Figure 2](image-url)

Figure 2. The relative increase in the number of BHs in the center of a Milky Way-like galaxy due to dynamical friction (DF) depending on the BH mass \( m \). All BHs are assumed to form 12 Gyr ago (solid) or with a constant rate during the last 12 Gyr (dashed). The influence radius is \( r_0 = 2.45 \) pc (Equation (31)) and the inner and outer stellar density slopes are \( \gamma_0 = 1.5 \), \( \gamma_2 = 3.2 \), respectively.

For illustrative purposes, we also derive the formulae for the rates of mergers between the smallest and the heaviest BHs as

\[
\begin{align*}
\Gamma_{min} &= \frac{\partial^2 \Gamma_0}{\partial \ln m_A \partial \ln m_B} \bigg|_{m_A=m_B=m_{min}} \\
&= \frac{\partial^2 \Gamma_0}{\partial m_A \partial m_B} m_{min}^2 \\
&\approx C \cdot \frac{63}{22} \beta - 1 \left[ \frac{\beta - 1}{1 - \left( \frac{m_{min}}{m_{max}} \right)^{\beta - 1}} \right] m_{min}^3, \quad (26a)
\end{align*}
\]

\[
\begin{align*}
\Gamma_{max} &= \frac{\partial^2 \Gamma_0}{\partial \ln m_A \partial \ln m_B} \bigg|_{m_A=m_B=m_{max}} \\
&= \frac{\partial^2 \Gamma_0}{\partial m_A \partial m_B} m_{max}^2 \\
&\approx C \left[ \frac{\beta - 1}{1 - \left( \frac{m_{min}}{m_{max}} \right)^{\beta - 1}} \right] \frac{14}{3} \left( \frac{r_{min}}{r_0} \right)^{-3/4} m_{max}^2, \quad (26b)
\end{align*}
\]

\[
C \equiv \frac{G^{17/4} N_B^{2/7}}{c^{10/7} N_{SMBH}^{11/14}}. \quad (26c)
\]

Here we assume \( p_0 = 0.5 \) and \( m_{min} \ll m_{max} \). As we can see from those expressions, the majority of mergers happen between the smallest BHs and the heaviest BHs when \( \beta \geq 2 \) and \( \beta \leq 2 \), respectively (which is later shown more rigorously on Figure 4):

\[
\begin{align*}
\frac{\Gamma_{min}}{\Gamma_{max}} &\approx \frac{27}{44} \left( \frac{m_{min}}{m_{max}} \right)^{4 - 2\beta} \left( \frac{r_{min}}{r_0} \right)^{3/4}. \quad (27)
\end{align*}
\]

Equation (24) assumes a steady-state, mass-segregated radial 3D number density profile derived by O’Leary et al. (2009) using the Fokker–Planck equation:

\[
\begin{align*}
n(m, r) &\propto r^{-3 - p_0 \eta} m^\eta. \quad (28)
\end{align*}
\]
Table 1

| Reference                  | O’Leary et al. (2009) | Kocsis & Levin (2012) | Gondán et al. (2018b) |
|----------------------------|-----------------------|-----------------------|-----------------------|
| \( t_{\text{min}} \)     | \( t_{\text{GW}}(r_{\text{min}}) = t_H \) | \( N_{\text{BH}}(r_{\text{min}}) = 1 \) | \( t_{\text{GW}}(r_{\text{min}}) = t_{\text{GW}}(r_{\text{min}}) \) |
|                           | \( r_{\text{min}} \propto M_{\text{SMBH}}^{1/2} \) | \( r_{\text{min}} \propto M_{\text{SMBH}}^{1/2} \) | \( r_{\text{min}} \propto M_{\text{SMBH}}^{1/16} \) |
| \( \Gamma \)             | \( \propto M_{\text{SMBH}}^{3/28} \) | \( \propto M_{\text{SMBH}}^{2/28} \) | \( \propto M_{\text{SMBH}}^{1/28} \) |
|                           | \( m_{\text{tot}} < \frac{1}{4} m_{\text{max}} \) | \( m_{\text{tot}} < \frac{1}{2} m_{\text{max}} \) | \( m_{\text{tot}} > \frac{1}{16} m_{\text{max}} \) |

where \( p_0 \approx 0.5 \) (O’Leary et al. 2009), i.e., the radial power-law index varies from \(-1.5\) for the lightest BHs to \(-2\) for the heaviest ones. However, we also examine different assumptions in Section 4.

As shown in Gondán et al. (2018b), for any \( M_{\text{SMBH}} \lesssim 10^7 M_\odot \), the relaxation time inside \( r_0 \) is shorter than the Hubble time. Given that most of the merger events come from the low-mass galaxies (O’Leary et al. 2009), this justifies our assumption that the SBH sphere of influence is fully relaxed. And because the BH density outside \( r_0 \) falls down quickly (\( \propto r^{-3/2} \)), we assume we can ignore the contribution of those BHs to the total merger rate.

Following Gondán et al. (2018b), we define \( r_{\text{min}} \) as the radius where the GW inspiral time becomes shorter than the relaxation time

\[
t_{\text{rel}} = 0.34 \frac{\sigma^3(r_{\text{min}})}{G^2 n(r_{\text{GW}})(m_{\text{BH}})^2 \ln \Lambda} = t_{\text{GW}} = \frac{5c^5 r_{\text{min}}^4}{64 G^2 m_{\text{BH}} M_{\text{SMBH}}},
\]

\[
r_{\text{min}} = \frac{82}{\zeta \ln \Lambda} r_{0,12}^{1.12} \left( \frac{G M_{\text{SMBH}}}{c^2} \right)^{2.5} \left( \frac{m_{\text{BH}}}{M_\odot} \right)^{0.69} \left( \frac{m_{\text{BH}}}{20 M_\odot} \right)^{1.31/3.62} = 6.9 \times 10^{-5} \text{ pc} \left( \frac{M_{\text{SMBH}}}{4 \times 10^6 M_\odot} \right)^{0.69} \left( \frac{r_0}{3 \text{ pc}} \right)^{0.31} \times \left( \frac{m_{\text{BH}}}{20 M_\odot} \right)^{0.28} \left( \frac{\zeta}{4} \right)^{-0.28} \left( \frac{m_{\text{BH}}}{20 M_\odot} \right)^{0.36},
\]

(29b)

(See Appendix B for the derivation). This value is in a good agreement with Gondán et al. (2018b, Figure 11).

However, previous papers (O’Leary et al. 2009; Kocsis & Levin 2012) have assumed different definitions of \( r_{\text{min}} \), as summarized in Table 1.\(^2\) In Kocsis & Levin (2012), \( r_{\text{min}} \) is the radius with only one BH inside it (as determined by \( n(r) \)). However, even the region containing <1 BH can on average still make a nonnegligible contribution to the total merger rate due to its very high average BH density and orbital velocity. We extrapolate the number density Equation (28) into this region given that \( t_{\text{GW}} \ll t_{\text{GW}} \), but warn the reader that the assumptions used to derive that function (phase space distribution function is smooth and correlations are negligible)

break there. And in O’Leary et al. (2009), \( r_{\text{min}} \) is the radius where \( t_{\text{GW}} = t_{\text{GW}} \), which is a more conservative assumption than ours given that the relaxation time is smaller than \( t_{\text{GW}} \). In any case, the dependence of the merger rate on \( r_{\text{min}} \) is rather weak (\( \Gamma \propto r_{\text{min}}^{-3/14} \)).

3.1. The Effect of DF on the Merger Rate

As the merger rate defined by Equation (24) is proportional to the total numbers of BHs with masses \( m_\alpha \) and \( m_\beta \), to account for the effects of DF we only have to multiply it by the corresponding BH number increase coefficients,

\[
\frac{\partial^2 \Gamma}{\partial m_\alpha \partial m_\beta} = \zeta(m_\alpha) \zeta(m_\beta) \frac{\partial^2 \Gamma_0}{\partial m_\alpha \partial m_\beta}.
\]

We present the two-dimensional mass distributions of the GW capture rate as a function of total BH mass and mass ratio following Gondán et al. (2018b) and also calculate the marginalized 1D total mass distribution, as discussed in Appendix E.

To make a prediction for the total observed merger rate, we add up the local merger rates \( \Gamma \) for every galaxy within the observable volume. For that purpose, it is useful to express \( r_0 \) in terms of the central SMBH mass using the \( M-\sigma \) relation (Kormendy & Ho 2013):

\[
M_{\text{SMBH}} = M_0 \left( \frac{\sigma}{\sigma_0} \right)^{\gamma_0},
\]

\[
M_0 = 3.097 \times 10^6 M_\odot, \quad \sigma = 200 \text{ km s}^{-1}, \quad \alpha_0 = 4.384, \quad r_0 = \frac{G M_{\text{SMBH}}}{4 \times 10^6 M_\odot} 0.543.
\]

The value of \( r_0 \) in this formula for a MW mass galaxy matches the measured value (\( \approx 3 \text{ pc} \)). As for \( \gamma_{1,2} \), we assume they have MW values \( \gamma_1 = 1.5, \gamma_2 = 3.2 \) for all galaxies. Under these assumptions, Equation (18(c)) takes the form

\[
t_{\text{DF,0}} = 4.1 \text{ Gyr} \left( 5 - \gamma_1 \right) \left( \frac{M_{\text{SMBH}}}{10 M_\odot} \right)^{-1} \left( \frac{M_{\text{SMBH}}}{4 \times 10^6 M_\odot} \right)^{1.31},
\]

(34)

which is to be used in Equations (21)–(23), instead of Equation (18(c)). This shows that the DF time in more massive galaxies is longer, implying that \( \zeta \), the increase in BH number due to DF, is less, as shown in Figure 3 (top left).

In analogy with \( \zeta \), we calculate the relative increase in merger rate due to the DF effects, which is shown in Figure 3 for two different values of \( M_{\text{SMBH}} \). In accordance with Equation (34), we can see that the enhancement is stronger for heavier BHs and lower-mass galaxies. As also shown in O’Leary et al. (2009), Kocsis & Levin (2012), and Gondán et al. (2018b), the dependence of merger rate per galaxy on the SMBH mass (without explicitly taking into account DF effects) is very weak—which implies that the total merger rate is dominated by numerous low-mass galaxies. And our results show that in these galaxies, the merger rates per galaxy are up to 20 times higher due to DF. This increases the overall contribution of low-mass galaxies even further.

4. Total Merger Rate

The total merger rate per unit volume may be calculated from the average merger rate per galaxy with a given SMBH mass and the distribution of the number density of galaxies...
with respect to the SMBH mass:

\[ \frac{\partial^2 R}{\partial \rho_0 \partial m_b} = \int_{M_{\text{SMBH,min}}}^{M_{\text{SMBH,max}}} \left( \frac{\partial^2 \Gamma}{\partial \rho_0 \partial m_b} \right) \times \frac{dn_{\text{gal}}}{dM_{\text{SMBH}}} dM_{\text{SMBH}}. \]

(35)

Previously, we assumed that all galaxies follow the $M$–$\sigma$ relation (31) exactly. In practice there may be significant variations in the model parameters between galaxies so that

(1) \[ M_{\text{SMBH}} = C_{\text{M}_\sigma} M_0 (\sigma / \sigma_0)^{\alpha}, \quad \langle C_{\text{M}_\sigma} \rangle = 1, \]

(2) the $r_0$ containing the stellar mass $M_* = 2M_{\text{SMBH}}$ satisfies

\[ r_0 = C_{\text{inf}} \frac{G M_{\text{SMBH}}}{\sigma^2}, \]

(37)

(3) the parameters of BH distribution ($m_{\text{min, max}}, \beta$ and $m_{\text{cr}}$) as well as $\gamma_{1,2}$ may also vary over different galaxies, which also affects the total number of black holes in the NSC, $N_{\text{BH}}$.

As $\{ C_{\text{M}_\sigma}, C_{\text{inf}}, m_{\text{min}}, m_{\text{max}}, \beta, m_{\text{cr}}, \gamma_1, \gamma_2 \}$ may vary from galaxy to galaxy, this variance can significantly change (usually increase) the average merger rate compared to its value calculated using the average parameter values:

\[ \left\{ \frac{\partial^2 \Gamma}{\partial \rho_0 \partial m_b} \right\} \left( C_{\text{M}_\sigma}, C_{\text{inf}}, \ldots \right) \]

\[ = \xi_{\text{M}_\sigma} \xi_{\text{inf}} \xi_{\text{other}} \frac{\partial^2 \Gamma}{\partial \rho_0 \partial m_b} \left( \langle C_{\text{M}_\sigma} \rangle, \langle C_{\text{inf} \rangle, \ldots \rangle, \right), \]

(38)

where $\xi_{\text{M}_\sigma}, \xi_{\text{inf}},$ and $\xi_{\text{other}}$ are the enhancement coefficients due to the variance in $C_{\text{M}_\sigma}, C_{\text{inf}}$ and all the other factors, respectively

\[ \xi_x = \left\{ \frac{\partial^2 \Gamma}{\partial \rho_0 \partial m_b} \right\} \left( C_x \right). \]

(39)
Here we have assumed there are no correlations between different galaxy parameters and that the dependence of $\Gamma$ on them is separable:

$$\frac{\partial^2 \Gamma}{\partial m_A \partial m_B} = f_1(C_{M\sigma}) f_2(C_{\text{inf}}).$$  
(40)

To obtain the parameter dependencies we eliminate $\sigma$ from the definition of $r_0$ in (37) using (36), and substitute the result in Equations 29(b) and 18(c) to obtain the scaling of $r_{\text{min}}$ and $t_{\text{DF}}$ with $C_{M\sigma}$ and $C_{\text{inf}}$:

$$r_0 = C_{\text{inf}} G M_{\text{SMBH}} \left( \frac{M_{\text{SMBH}}}{C_{M\sigma} M_{\bullet}} \right)^{-2/\alpha_0}$$  
(41a)

$$r_{\text{min}} \propto r_0^{0.31} \propto C_{\text{inf}}^{-0.14} C_{M\sigma}^{0.14},$$  
(41b)

$$t_{\text{DF,0}} \propto r_0^{3/2} \propto C_{\text{inf}}^{3/2} C_{M\sigma}^{-0.68}.$$  
(41c)

According to Kormendy & Ho (2013), the intrinsic scatter of $M-\sigma$ relation is 0.29 dex. This implies $1.3 \lesssim \xi_{M\sigma} \lesssim 1.5$ depending on $m$ and $M_{\text{SMBH}}$ (Appendix D). As for $C_{\text{inf}}$, given the stellar density profile, it only depends on the velocity anisotropy for relaxed NSCs (Binney & Tremaine 2008, Section 8.4.1). We assume all galactic nuclei to be isotropic, which gives $C_{\text{inf}} = 1$ and $\xi_{\text{inf}} = 1$.

Tsang (2013) made an estimate $C_{\text{inf}} = 6.1$ based on the observed relation between $\rho_0$ and $\sigma$ and its scatter (Merritt et al. 2007). However, that is likely an upper limit to $C_{\text{inf}}$ as they have ignored the possible observational errors in both $\rho_0$ and $\sigma$ and also overestimated the spread in $\rho_0$ at fixed $\sigma$. Using the same plot from Merritt et al. (2007), O’Leary et al. (2009) arrived at the rough estimate of $\xi = 30$; however, they only accounted for the variance in $\rho_0$ and ignored the variance in $r_0$ which is in fact related to $\rho_0$ at a given $M_{\text{SMBH}}$ (Equation (11)).

The mass distribution of SMBHs is taken from Shankar et al. (2004):\footnote{It is consistent within uncertainties with the other SMBH mass estimates in the literature (e.g., Hopkins et al. 2007) as well as the SMBH masses inferred from the galaxy bulge mass distribution (Thajovarr et al. 2016) with $M_{\text{SMBH}}/M_{\text{b Bulg}} = 0.003$.}

\[
\frac{dn_{\text{gal}}}{dM_{\text{SMBH}}} = \frac{1}{M_{\text{SMBH}} \ln 10 \, d \log M_{\text{SMBH}}} = \frac{\Phi_*}{M_{\text{SMBH}} \ln 10} \left( \frac{M_{\text{SMBH}}}{M_*} \right)^{0.1+1} \times \exp \left[ - \left( \frac{M_{\text{SMBH}}}{M_*} \right)^{\beta} \right].
\]  
(42a)

\[
\Phi_* = 7.7 \times 10^{-3} \text{Mpc}^{-3},
\]

\[
M_* = 6.4 \times 10^7 M_\odot,
\]

\[
\alpha = -1.11,
\]

\[
\beta = 0.49.
\]  
(42b)

Instead of merger rate per unit $m_A$ and $m_B$, we calculate a more illustrative merger rate per unit log total mass and mass ratio:

\[
\frac{\partial^2 R}{\partial \log m_{\text{tot}} \partial q} = \frac{m_{\text{tot}}^2}{(1 + q)^2} \frac{\partial^2 R}{\partial m_A \partial m_B}.
\]  
(43a)

\[
\frac{\partial^2 R}{\partial m_{\text{tot}}^2} = \frac{8.4 \times 10^{-16} \text{yr}^{-1}}{1.4} \left( \frac{m_{\text{tot}}/M_\odot}{1 + q} \right)^2 
\times \left( 9 - 6\rho_0 \frac{m_{\text{tot}}}{m_{\text{max}}} + 4\rho_0^2 \frac{\mu m_{\text{tot}}}{m_{\text{max}}^2} \right) \left( \frac{m_{\text{tot}}}{20 M_\odot} \right)^{-2.6}
\times \left( M_{\text{SMBH}} \right)^{0.328}
\times \left( \frac{M_{\text{SMBH}}}{4 \times 10^6 M_\odot} \right)^{3.28}
\times \left[ 1 - \left( \frac{r_{\text{inf}}}{r_0} \right)^{1 - \frac{m_{\text{max}}}{m_{\text{min}}}} \frac{dn_{\text{gal}}}{dM_{\text{SMBH}}},
\right] \frac{dn_{\text{gal}}}{dM_{\text{SMBH}}}
\]  
(43b)

\[
\frac{r_{\text{min}}}{r_0} = 2.6 \times 10^{-5} \left( \frac{M_{\text{SMBH}}}{4 \times 10^6 M_\odot} \right)^{0.35} \left( \frac{m_{\text{tot}}/(1 + q)}{20 M_\odot} \right)^{0.28},
\]  
(43c)

\[
\xi = \xi_{M\sigma} \xi_{\text{inf}} \xi_{\text{other}}.
\]  
(43d)

The results of the calculation are shown in Figure 4. Compared to Figure 9 of Gondán et al. (2018b), the distribution is shifted toward higher BH masses due to DF. The SMBH mass function is almost log-uniform at low SMBH masses, and low-mass galaxies actually contribute more events due to enhanced DF (see Figure 3, top left). Therefore, the total merger rate depends on how far into the low SMBH masses the distribution (42) extends, which is illustrated in Figure 5 (left). This figure shows the total merger rate integrated over all BH masses and mass ratios:

\[
\mathcal{R} = \int_{2m_{\text{min}}}^{2m_{\text{max}}} dm_{\text{tot}} \int_{m_{\text{min}}/m_{\text{max}}}^{1} dq \frac{\partial^2 R}{\partial \log m_{\text{tot}} \partial q}.
\]  
(44)

Figure 5 (left panel) also shows that the merger rate is a decreasing function of $\beta$ (i.e., an increasing function of the average BH mass). Given $\xi = 3$, $\rho_0 = 0.5$, $m_{\text{crr}} = 20 M_\odot$, and ranges $M_{\text{SMBH,min}} = 10^{-3} - 10^8 M_\odot$ and $\beta = 1-3$, the overall merger rate is 0.002–0.04 Gpc$^{-3}$ yr$^{-1}$.

Based on fits to the isotropic Fokker–Planck results of O’Leary et al. (2009), which imply $p_0 \approx 0.5–0.6$, we have assumed $p_0 = 0.5$, i.e., a power-law BH radial density distribution given by Equation (28) which has a slope 1.5 for the lightest BHs and 2 for the heaviest ones. However, using different mass functions in isotropic Fokker–Planck models, (Keshet et al. 2009, Figure 3) has shown that for $\beta \geq 4$ and $m_{\text{max}}/m_{\text{min}} \gtrsim 10$ the heaviest BH density profile can be steeper, up to $r^{-3}$, which would correspond to $p_0 = 1.5$ in Equations (28) and (24). There is indeed some observational evidence that the surface density distribution of massive O-stars (i.e., BH progenitors) in the Galactic center is $\propto R^{-1.4}$ (Bartko et al. 2009), which implies 3D density $\propto R^{-2.4}$, i.e., $p_0 \approx 0.9$ (see, however, Støstad et al. 2015, who claim that the young star density distribution is better described by a broken power law with the inner slope $\propto R^{-0.9}$). Another possible reason for the density cusp to be steeper than $r^{-2}$ is binary disruption by
the SMBH’s tidal field (Fragione & Sari 2018). Figure 5 (right) shows that such an increase in $p_0$ could increase the merger rate by orders of magnitude and therefore deserves further study.

Up to this point, we have assumed the BH density distribution to be spherical. However, the MW NSC is observed to be flattened with mean axis ratio 0.7–0.8 (Schödel et al. 2014; Fritz et al. 2016). Feldmeier-Krause et al. (2017) found it to be decreasing toward the center within $r < 1$ pc, reaching 0.4 when $r \to 0$. According to their orbit-based modeling, that corresponds to a triaxial density profile with axial ratios $c/a = 0.28$, $b/a = 0.64$ in the center. The triaxiality leads to an increase in the average BH density $n_{BH}$ within $r_0$ and, consequently, to an increase in the event rate $\mathcal{R} \propto n_{BH}^2$. A crude upper limit estimate for this increase in $\mathcal{R}$ may be obtained as $(a^2/bc)^2 \approx 30$. In addition, from a theoretical point of view, vector resonant relaxation in a multimass system causes the heaviest objects (BHs) to segregate from an initially spherical stellar distribution into a disk (Szőlgyén & Kocsis 2018). The final distribution of heavy BH angular momenta in Figure 2 of Szölgyén & Kocsis (2018) implies the $\mathcal{R}$ increase by a factor of 2.8. How effective that phenomenon is in nonspherical systems is currently unknown and deserves further study.

In addition to the merger rate, we also calculate the universal dimensionless parameter

$$\alpha = -m_{tot}^2 \frac{\partial^2}{\partial m_A \partial m_B} \ln \frac{\partial^2 \mathcal{R}}{\partial m_A \partial m_B},$$

which is independent of the BH mass function and is sensitive to the astrophysical process leading to the BH merger (Kocsis et al. 2018). Its value varies from 1.4 for the smallest BHs to $-6.3$ for the heaviest ones which is in good agreement with Gondán et al. (2018b); its dependence on $m_{tot}$ is shown in Figure 6 (top). It turns out to be practically independent of $q$. This is different from, e.g., $\alpha = 1$ for primordial BH binaries formed in the early universe (Kocsis et al. 2018) or $\alpha = 1.43$ for BHs in dark matter halos (Bird et al. 2016).

5. Conclusions

We have calculated the number density of stellar mass BHs in a galactic nucleus with a SMBH in the center using

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Total merger rate per unit mass ratio per unit log total mass for BH mass distribution slope $\beta = 1$, 2.35, and 3, respectively, assuming the minimal SMBH mass $M_{SMBH,\text{min}} = 10^5 M_\odot$.}
\end{figure}
simplified isotropic models, where we assumed the BHs and stars reach a mass-segregated, steady-state distribution inside the influence radius and took into account the DF bringing BHs inside that region. We have shown that DF can increase the BH number up to \( \sim 5 \) times and also that its effect is much more pronounced in small galaxies.

We used this information to calculate the rate of GW captures in galactic nuclei taking into account the observed SMBH mass distribution and the scaling relations between the influence radius, SMBH mass, and velocity dispersion. The total event rate turns out to be dominated by small galaxies, due to both the event rate per galaxy being weakly dependent on mass and the DF effect being more pronounced in smaller galactic nuclei.

The event rate is determined, on one hand, by the BH mass distribution parameters (their total mass fraction \( \kappa \) and mass distribution slope \( \beta \)) and also by the SMBH mass function below \( \sim 10^7 M_\odot \): the SMBH number density per unit log \( M_{\text{SMBH}} \) per unit comoving volume \( N_{\text{SMBH}} \) and the SMBH mass lower limit \( M_{\text{SMBH,min}} \). The approximate dependence of the total event rate on all these parameters is

\[
R \approx 0.02 \text{ Gpc}^{-3} \text{ yr}^{-1} \left( \frac{N_{\text{SMBH}}}{0.015 \text{ Gpc}^{-3} \text{ dex}^{-1}} \right) \left( \frac{\kappa}{2.9 \times 10^{-3}} \right)^2 \times \left( \frac{M_{\text{SMBH,min}}}{10^5 M_\odot} \right)^{0.32} e^{1.06(1-\beta)},
\]

(46)

which is similar to a previously published estimate of Tsang (2013) and below the rates cited in O’Leary et al. (2009), 0.6(\xi/30)...45(\xi/30) yr\(^{-1}\) Gpc\(^{-3}\) for different galaxy models. However, Equation (46) assumes a certain BH number density distribution (Equation (28) with \( p_0 = 1/2 \)); a steeper (\( p_0 > 1/2 \)) and/or nonspherical distribution could increase \( R \) by orders of magnitude, as discussed in Section 4.

The GW capture rates we calculated are much lower than the current estimates by LIGO (24–112 yr\(^{-1}\) Gpc\(^{-3}\); The LIGO Scientific Collaboration et al. 2018a), therefore they are unlikely to be the dominant source of BH mergers.
number of predicted aLIGO detections per year in our model is determined by the event rate per unit comoving volume $\mathcal{R}$ and the accessible volume:

$$N_{GW} = \mathcal{R} \int_{0}^{z_{\text{max}}} \frac{1}{1 + z} dV_c dz,$$

(47a)

$$\frac{dV_c}{dz} = 4\pi c \frac{z^2}{H_0 \sqrt{\Omega_M (1 + z)^3 + \Omega_\Lambda}},$$

(47b)

$$d(z) = \frac{c}{H_0} \int_{0}^{z} \frac{dz'}{\sqrt{\Omega_M (1 + z')^3 + \Omega_\Lambda}}.$$  

(47c)

Here, $z_{\text{max}}$ is the maximum accessible redshift (we ignore its dependence on the BH masses and other parameters of the binary), $dV_c/dz$ is the comoving volume per unit redshift and $d(z)$ is the comoving distance. Figure 6 (bottom) illustrates that dependence. In this equation we have not accounted for the possibility of $p_0$ in Equation (28) being higher than 0.5 that could potentially increase the event rate up to a few orders of magnitude (Figure 5, right).

A possible way to distinguish GW captures from the other channels is their high eccentricity in the LIGO frequency range ($e > 0.1$ at $f > 10$ Hz). However, eccentric mergers can also be produced in triple systems where a BH binary achieves extreme eccentricity through interaction with with a third body. Those systems could be:

1. A binary BH orbiting around a SMBH in a galactic center can experience variations of inclination and eccentricity due to the Kozai–Lidov effect, sometimes reaching $e > 0.9999$, which quickly leads to coalescence through GW energy loss (Antonini & Perets 2012; Fragione et al. 2018; Hamers et al. 2018; Hoang et al. 2018).
2. The Lidov–Kozai effect can also cause the inner BH binary coalescence in hierarchical BH triples that form through binary–binary interactions in globular clusters (Antonini et al. 2016) or through the evolution of massive star triples in galactic field (Antonini et al. 2017; Silsbee & Tremaine 2017).
3. Triple BHs can also form via binary BH—single BH encounters in globular clusters (Samsing et al. 2018; Zevin et al. 2019). A merger subsequently happens as a result of GW emission during a close encounter between two of the three BHs while they temporarily form a bound three-body state (Samsing 2018).

Table 2 shows the merger rates for all those different mechanisms in different environments. The rates turn out to be comparable ($\sim 0.01$–0.1 Gpc$^{-3}$ yr$^{-1}$) but it still might be possible to distinguish them by their eccentricity distribution.

Apart from the NSCs with an SMBH in the center that we considered in this paper, other kinds of star clusters could also contribute to the GW capture merger rate, e.g., NSCs without SMBHs, globular clusters, and open clusters. However, in those other systems the velocity dispersion is not as high, which means the GW capture mergers in them are much less eccentric (Figure 6 in O’Leary et al. 2009). It is also worth noticing that the number density of NSCs with SMBHs in them can be up to 40% higher due to ultracompact dwarf galaxies, some of which are believed to be stripped NSCs of low-mass galaxies (Voggel et al. 2019). Finally, we assumed all BHs to be formed in situ and have not accounted for the BHs brought into the Galactic Center via globular cluster inspiral (Arca-Sedda & Gualandris 2018; Arca-Sedda & Capuzzo-Dolcetta 2019).

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**Appendix A**

**Increase in BH Number in the Assumption of Continuous BH Formation**

Let $t(m, r)$ be the time it takes for a BH of mass $m$ to reach $r_0$ starting from $r > r_0$. From Equation (21) we know that

$$t = t_{DF,0} \left( \frac{r/r_0}{k_1} \right)^{1/k_3} - 1, \quad \frac{1}{k_1},$$

(48)

where $k_{1,2,3}$ are the same as in Equation (21). Assuming BHs are formed with a constant rate, only a fraction $1 - t/T$ of all the BHs formed at radius $r$ will be born early enough to reach $r_0$ by present time. Therefore, Equation (13) for the total number of BHs at present now reads as follows:

$$\frac{dN_{\text{tot}}}{dm} = f_{\text{BH,init}} \int_0^{r_0} n(r) 4\pi r^2 dr$$

$$+ f_{\text{BH,init}} \int_{r_0}^{r_{\text{tot}}} n(r) 4\pi r^2 dr \left( 1 - \frac{t(m, r)}{t_H} \right).$$

(49)
and the BH number increase coefficient is
\[
\zeta = 1 + (3 - \gamma_0) \int_1^{r_{\text{GW}}/r_0} x^{-\gamma_2} (1 - \frac{\rho_0}{r_0} \frac{x^{1/\kappa_3} - 1}{k_1})^{1/\kappa_2} \ dx.
\] (50)

**Appendix B**

**Minimum Radius of BH Distribution**

As mentioned in Section 3, the BHs are depleted due to their inspiral into the SMBH below a minimum radius \(r_{\text{min}}\):

\[
0.34 \frac{\sigma^3(r_{\text{GW}})}{G^2 n(r_{\text{GW}}) \langle m^2 \rangle \ln \Lambda} = \frac{5 \varepsilon^5 r_{\text{GW}}^4}{64 \pi^3 r_{\text{GW}} M_{\text{MBH}}^2},
\] (51)

where \(\sigma(r) = \sqrt{G M_{\text{MBH}}/r}\) and \(n\) is the total number density of objects (both BHs and stars) and \(\langle m^2 \rangle\) is their average squared mass:

\[
n(r) \langle m^2 \rangle = \int_0^\infty \frac{dn_r(m, m_a) m_a^2 dm_a}{dm_a} + \int_{m_{\text{min}}}^{m_{\text{max}}} \frac{dn_{\text{BH}}(r, m_{\text{BH}}) m_{\text{BH}}^2 dm_{\text{BH}}}{dm_{\text{BH}}} = n_s(r) \langle m_s^2 \rangle + n_{\text{BH}}(r) \langle m_{\text{BH}}^2 \rangle.
\] (52)

At \(r \ll r_0\), which is a reasonable assumption given Equation (51), the BH term dominates the right-hand side of Equation (52). Indeed,

\[
\frac{dn_{\text{BH}}}{dm} = \left( \frac{r}{r_0} \right)^{-3/2 - p_0 m/m_{\text{max}}} C(m),
\] (53)

where \(C(m)\) is a function of \(m\) that can be found from

\[
\int_0^{m_{\text{BH}}} \frac{dn_{\text{BH}}}{dm} 4 \pi r^2 dr = \frac{dN_{\text{BH}}(r_0)}{dm} = \zeta(m) \frac{dN_{\text{init}}(r_0)}{dm},
\] (54)

\(N_{\text{init}}(r_0)\) and \(N_{\text{BH}}(r_0)\) being the initial and final numbers of BHs within the sphere of influence (Equation (3)), so that

\[
\frac{dN_{\text{init}}(r_0)}{dm} = N_{\text{init}}(r_0) (1 - \beta) m^{-\beta} m_{\text{max}}^{-1 - \beta} - m_{\text{min}}^{-1 - \beta}. \tag{55}
\]

Equations (53)–(55) yield

\[
\frac{dn_{\text{BH}}}{dm} = \chi N_{\text{init}}(r_0) \left( \frac{3}{2} - p_0 m/m_{\text{max}} \right) \left( \frac{r}{r_0} \right)^{-3/2 - p_0 m/m_{\text{max}}} m_{\text{max}}^{-1 - \beta} - m_{\text{min}}^{-1 - \beta}.
\] (56)

which implies that

\[
\frac{dn_{\text{BH}}}{dm} \approx \chi N_{\text{init}}(r_0) \left( \frac{3}{2} - p_0 m/m_{\text{max}} \right) \left( \frac{r}{r_0} \right)^{-3/2 - p_0 m/m_{\text{max}}} m^2 - \beta \ dm.
\] (57)

Here we have ignored the dependence of \(\zeta\) on BH mass. This is justified by the limited range of \(\zeta\) (Figure 2), weak dependence of \(r_{\text{min}}\) on \(\zeta\) (Equation (29b)), and also weak dependence of the merger rate on \(r_{\text{min}}\) (Equation (24)). For example, for an SBH mass \(10^6 M_\odot\) the assumption of constant \(\zeta = 4\) results in <10% error in the value of \(n_{\text{BH}}(r) \langle m_{\text{BH}}^2 \rangle\). At the default values of \(p_0 = 0.5\), \(m_{\text{max}} = 40 M_\odot\), \(m_{\text{min}} = 5 M_\odot\), and \(\beta = 2.3\), the dependence of the integral on \(r/r_0\) can be approximated by a power law (with 3% accuracy for \(r/r_0 < 10^{-3}\), which turns out to be a safe assumption):

\[
n_{\text{BH}}(r) \langle m_{\text{BH}}^2 \rangle \approx \frac{\chi N_{\text{init}}(r_0)}{4 \pi r_0^3} \cdot 116 M_\odot^2 \left( \frac{r}{r_0} \right)^{-1.88}.
\] (58)

For stellar number density we have

\[
n_s(r) \langle m_s^2 \rangle = \frac{n_s \langle m_s^2 \rangle}{\langle m_s^2 \rangle} = \frac{3 M_{\text{MBH}} \langle m_{\text{sBH}}^2 \rangle}{4 \pi r_0^3} \left( \frac{r}{r_0} \right)^{-3/2}.
\] (59)

From Equation (1) \(\langle m_s \rangle = 0.30 M_\odot\) and \(\langle m_{\text{sBH}}^2 \rangle = 0.79 M_\odot^2\), so that

\[
\frac{n_{\text{BH}}(r) \langle m_{\text{sBH}}^2 \rangle}{n_s(r) \langle m_s^2 \rangle} \approx \frac{\chi N_{\text{init}}(r_0) \langle m_s \rangle}{3 M_{\text{MBH}} \langle m_{\text{sBH}}^2 \rangle} \left( \frac{r}{r_0} \right)^{-0.38} \left( \frac{m_{\text{max}}}{20 M_\odot} \right)^{1.3}.
\] (60)

We can conclude that BHs dominate the relaxation process everywhere inside \(0.05 r_0\) (similarly to what was reported in O’Leary et al. 2009, Figure 2) and the stellar term can indeed be ignored in Equation (51). Combined with Equations (58) and (15), Equation (51) yields Equation 29(b).

**Appendix C**

**Contribution of BHs Inside \(r_{\text{min}}\) to the Event Rate**

Inside the sphere of radius \(r_{\text{min}}\) where relaxation is ineffective compared to the GW-induced orbital radius shrinking (the orbits are assumed circular) the BH number density is defined by

\[
\frac{dN}{dr} = 4 \pi r^2 n(r) \propto \frac{dt}{dr} \propto r^5,
\] (61)

so that \(n(r) \propto r^3\). From Kocsis & Levin (2012) we know that the merger rate per unit radius is

\[
\frac{d\Gamma}{dr} \propto r^{39/14} n^2(r).
\] (62)

Here we consider mergers between heaviest BHs that have \(n(r) \propto r^{-2}\), \(r > r_{\text{min}}\). Then

\[
\frac{d\Gamma}{dr} = C \left( \frac{r}{r_{\text{GW}}} \right)^{-17/14}, \quad r > r_{\text{GW}}
\]

\[
= C \left( \frac{r}{r_{\text{GW}}} \right)^{-123/14}, \quad r < r_{\text{GW}},
\] (63)
From this we can conclude that the contribution of BHs inside $r_{\text{min}}$ to the total event rate is negligible

$$\frac{\Gamma(r < r_{\text{min}})}{\Gamma(r > r_{\text{min}})} \approx \frac{\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dv}{dr} dr}{\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dv}{dr} dr}$$

$$= \frac{(14/137) \varphi_{\text{GW}}}{(14/3) \varphi_{\text{GW}}} \approx 0.02. \quad (64)$$

**Appendix D**

The Impact of the Intrinsic Scatter of $M-\sigma$ Relation

The increase in the merger rate $\Gamma$ due to scatter in $C_{M\sigma}$ (distributed log-normally with mean 1 and standard deviation $\delta$) is

$$\xi_{M\sigma} = \left[ \int_0^{\infty} \frac{\partial^2 \Gamma(C_{M\sigma})}{\partial m_t \partial m_B} \exp \left( -\frac{(\ln C_{M\sigma})^2}{2\delta^2} \right) \right] \times \frac{\partial C_{M\sigma}}{\sqrt{2\pi \delta C_{M\sigma}}} \left[ \frac{\partial^2 \Gamma}{\partial m_t \partial m_B}(1) \right]. \quad (65)$$

As we see from Equations (24) and (30), $\Gamma$ depends on $C_{M\sigma}$ through $r_0$, $r_{\text{min}}$, and $\zeta$:

$$\frac{\partial^2 \Gamma(C_{M\sigma})}{\partial m_t \partial m_B} \propto \zeta(m_t) \zeta(m_B) r_0^{-31/14} \left( 1 - \frac{r_{\text{min}}}{r_0} \right)^{11/14 - \frac{\rho_m}{m_{\text{max}}} m_{\text{max}}}$$

$$\propto \frac{\zeta(m_t) \zeta(m_B)}{C_{M\sigma}^{1.26 + 0.16 m_{\text{max}}/m_{\text{max}}} m_{\text{tot}} > \frac{11}{7} m_{\text{max}}}$$

$$\propto \frac{\zeta(m_t) \zeta(m_B)}{C_{M\sigma}^{1.01} m_{\text{tot}} < \frac{11}{7} m_{\text{max}}}$$

$$\zeta = 1 + \frac{3 - \gamma_1}{\gamma_2 - 3} \left( 1 - \frac{1 + k_1 \left( \frac{m}{10 M_\odot} \right)}{2.9} \right) \left( \frac{M_{\text{SMBH}}}{4 \times 10^6 M_\odot} \right)^{-1.31} C_{M\sigma}^{-0.68} \left( 3^{-\gamma_1} k_1 \right). \quad (66a)$$

$$\propto \left( \frac{M_{\text{SMBH}}}{4 \times 10^6 M_\odot} \right)^{-1.31} C_{M\sigma}^{-0.68} \left( 3^{-\gamma_1} k_1 \right). \quad (66b)$$

Here we used Equations (41), (23), and (34) to determine the dependence of $r_0$, $r_{\text{min}}$, and $\zeta$ on $C_{M\sigma}$.

**Appendix E**

Manipulation of Merger Rate Distributions

As derived in Gondán et al. (2018b), the merger rate distributions per unit $m_{\text{tot}}$, $q = m_B/m_A$, $\mathcal{M}$ and $\eta$ can be calculated as

$$\frac{\partial^2 \Gamma}{\partial m_{\text{tot}} \partial q} = \frac{m_{\text{tot}}}{(1 + q)^2 \partial m_A \partial m_B}, \quad (67a)$$

$$\frac{\partial^2 \Gamma}{\partial \mathcal{M} \partial \eta} = \mathcal{M} \eta^{-6/5} (1 - 4\eta)^{-1/2} \frac{\partial^2 \Gamma}{\partial m_A \partial m_B}. \quad (67b)$$

The merger rate distribution as a function of only one variable can be given by marginalizing one of these equations over the other variable, e.g.,

$$\frac{\partial^2 \Gamma}{\partial m_{\text{tot}} \partial q} = \int_{m_{\text{min}}}^{m_{\text{tot}}} \frac{\partial^2 \Gamma}{\partial m_{\text{tot}} \partial q} \frac{m_{\text{tot}}}{(1 + q)^2} \left( \frac{m_{\text{tot}}}{1 + q} \right) \left( q m_{\text{tot}} \right) dq,$$

$$\propto \int_{m_{\text{min}}}^{m_{\text{tot}}} \frac{\partial^2 \Gamma}{\partial m_{\text{tot}} \partial q} \frac{m_{\text{tot}}}{(1 + q)^2} \left( \frac{m_{\text{tot}}}{1 + q} \right) \left( q m_{\text{tot}} \right) dq, \quad (68)$$

where $\Gamma_0$ is the merger rate calculated without taking into account the effects of DF (Equation (24)).
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