THE GEOMETRY OF A-GRADED ALGEBRAS

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Abstract
We study algebras \( k[ x_1, \ldots, x_n ] / I \) which admit a grading by a subsemigroup of \( \mathbb{N}^d \) such that every graded component is a one-dimensional \( k \)-vector space. V.I. Arnold and coworkers proved that for \( d = 1 \) and \( n \leq 3 \) there are only finitely many isomorphism types of such \( \mathcal{A} \)-graded algebras, and in these cases \( I \) is an initial ideal (in the sense of Gröbner bases) of a toric ideal. In this paper it is shown that Arnold’s finiteness theorem does not extend to \( n = 4 \). Geometric conditions are given for \( I \) to be an initial ideal of a toric ideal. The varieties defined by \( \mathcal{A} \)-graded algebras are characterized in terms of polyhedral subdivisions, and the distinct \( \mathcal{A} \)-graded algebras are parametrized by a certain binomial scheme.

1. Introduction
What are the graded algebras that have the simplest possible Hilbert function? This question was raised and partially answered by V.I. Arnold [1] and his coworkers E. Korkina, G. Post and M. Roelofs [9],[10],[11]. They considered finitely generated \( \mathbb{N} \)-graded \( k \)-algebras such that each non-trivial graded component has dimension 1 over the ground field \( k \). We propose the following multigraded version of their definition: Let \( \mathcal{A} = \{ a_1, a_2, \ldots, a_n \} \) be a subset of \( \mathbb{N}^d \setminus \{0\} \), and let \( \mathbb{N} \mathcal{A} \) denoted the sub-semigroup of \( \mathbb{N}^d \) spanned by \( \mathcal{A} \). An \( \mathcal{A} \)-graded algebra is a \( \mathbb{N}^d \)-graded \( k \)-algebra \( R = \bigoplus_b R_b \) with homogeneous generators \( X_1, X_2, \ldots, X_n \) in degrees \( a_1, a_2, \ldots, a_n \) such that

\[
\dim_k( R_b ) = \begin{cases} 
1 & \text{if } b \in \mathbb{N} \mathcal{A} \\
0 & \text{otherwise}
\end{cases}
\quad \text{for all } b \in \mathbb{N}^d. \tag{1.1}
\]

Every \( \mathcal{A} \)-graded algebra has a natural presentation as a quotient of a polynomial ring:

\[
0 \rightarrow I \rightarrow k[ x_1, x_2, \ldots, x_n ] \rightarrow R \rightarrow 0.
\]

The presentation ideal \( I = \ker( x_i \mapsto X_i ) \) is called \( \mathcal{A} \)-graded as well. It is easy to see (cf. [5, Proposition 1.11]) that \( I \) is generated by polynomials with at most two terms, that is, \( I \) is a binomial ideal. In the following we use the abbreviation \( k[ x ] := k[ x_1, x_2, \ldots, x_n ] \).

Arnold, Korkina, Post and Roelofs studied the case \( d = 1 \), and they proved that for \( n \leq 3 \) there is only a finite number of non-isomorphic \( \mathcal{A} \)-graded algebras. To state their
main result (Theorem 1.1), we need some definitions. Two $\mathcal{A}$-graded algebras $R$ and $R'$ are isomorphic if there exists an algebra isomorphism of degree 0. This holds if and only if, for the corresponding ideals $I$ and $I'$, there exists $\lambda = (\lambda_1, \ldots, \lambda_n) \in (k^*)^n$ such that

$$I' = \lambda \cdot I := \{ f(\lambda_1 x_1, \ldots, \lambda_n x_n) : f \in I \}.$$  

(1.2)

Gröbner basis theory suggests taking limits with respect to one-parameter subgroups $\omega$ of the torus $(k^*)^n$. Each such limit is a new $\mathcal{A}$-graded ideal: the initial ideal in $\omega(I)$, which is spanned by the highest forms in $\omega(f)$ of all polynomials $f$ in $I$. Here $\omega$ is identified with a vector in $\mathbb{Z}^n$, and, if $\text{in}_\omega(I)$ is a monomial ideal, then $\omega$ is called a term order for $I$.

The paradigm of an $\mathcal{A}$-graded algebra is the semigroup algebra

$$k[\mathbb{N} \mathcal{A}] = k[t^{a_1}, t^{a_2}, \ldots, t^{a_n}] = k[x]/I_\mathcal{A}.$$  

The prime ideal $I_\mathcal{A}$ is called the toric ideal of $\mathcal{A}$. It is generated by all binomials $x_1^{u_1} \cdots x_n^{u_n} - x_1^{v_1} \cdots x_n^{v_n}$ such that $u_1 a_1 + \cdots + u_n a_n = v_1 a_1 + \cdots + v_n a_n$. We call an $\mathcal{A}$-graded algebra $R = k[x]/I$ coherent if there exists $\omega \in \mathbb{Z}^n$ such that $I$ is isomorphic to $\text{in}_\omega(I_\mathcal{A})$.

**Theorem 1.1.** (Arnold, Korkina, Post and Roelofs)
If $d = 1$ and $n = 3$ then every $\mathcal{A}$-graded algebra is coherent.

Arnold [1] expressed the number of isomorphism classes of $\mathcal{A}$-graded algebras for $\mathcal{A} = \{1, p_2, p_3\}$ in terms of the continued fraction expansion for the rational number $p_3/p_2$. A proof of this result appeared in [9], and its extension to the case $\mathcal{A} = \{p_1, p_2, p_3\}$ was given in [10],[11]. We propose a reformulation of Theorem 1.1 using the concept of the state polytope of a toric ideal (see [2],[8],[16]). Recall that the dimension of the state polytope is $n - d$. Hence for $n = 3, d = 1$ it is a lattice polygon. The edge directions of this state polygon are perpendicular to the “star” as defined in [9, Def. 2.9], [11, §2.3].

**Corollary 1.2.** If $d = 1$ and $n = 3$ then the isomorphism classes of $\mathcal{A}$-graded algebras are in bijection with the faces of the state polytope of the toric ideal $I_\mathcal{A}$.

A question left open in [11] was whether these results hold for $d = 1$ and arbitrary $n$. The answer is “no”. A first counterexample with $n = 7$ was constructed by D. Eisenbud (personal communication). One contribution of this paper is a new counterexample for $n = 4$, and hence a proof that Theorem 1.1 is best possible. Incoherent monomial graded algebras for $d = 1$ and $n = 4$ are the topic of Section 2 below. In Section 3 we introduce two necessary geometric conditions for coherence, and we construct examples of incoherent algebras which violate these conditions. In Section 4 we characterize the radicals of $\mathcal{A}$-graded ideals in terms of polyhedral subdivisions. A special case of our characterization is the main theorem in [15] which relates triangulations and Gröbner bases. In Section 5 we construct the parameter space $P_\mathcal{A}$ whose points are the distinct $\mathcal{A}$-graded ideals in $k[x]$. A list of open problems and conjectures is given in Section 6.
2. One-dimensional mono-AGAs with four generators

An $A$-graded algebra $R = k[x]/I$ is called monomial (or a mono-AGA) if its ideal $I$ is generated by monomials. The non-zero monomials of a mono-AGA $R$ are called standard. They constitute a $k$-vector space basis for $R$. The following is our first main result.

Theorem 2.1. Let $d = 1, n = 4$ and $A = \{1, 3, 4, 7\}$, and suppose $k$ is an infinite field.

(a) There exists a monomial $A$-graded algebra which is not coherent.

(b) There exists an infinite family of pairwise non-isomorphic $A$-graded algebras.

Proof: In the polynomial ring $k[x_1, x_2, x_3, x_4]$ we consider the monomial ideal

$$I := \langle x_1^3, x_1x_2, x_2^2, x_2x_3, x_1x_4, x_1^2x_3, x_1x_3^2, x_2x_4, x_3x_4, x_3^2, x_1x_3^3, x_4^2 \rangle.$$  \hfill (2.1)

The quotient algebra $R = k[x_1, x_2, x_3, x_4]/I$ is $A$-graded. To verify this, one must compute the Hilbert series of $I$ with respect to the grading $\deg(x_1) = 1, \deg(x_2) = 3, \deg(x_3) = 4, \deg(x_4) = 7$. (This can be done easily using the command hilb-numer in the computer algebra system MACAULAY [3].) We list the standard monomials of low degrees:

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\hline
x_1 & x_1^2 & x_2 & x_3 & x_1x_3 & x_1^2x_3 & x_4 & x_3 & x_1x_3^2 & x_2x_4 & x_3x_4 & x_3^2 & x_1x_3^3 & x_4^2 \\
15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\
x_2x_4 & x_3^3 & x_2x_4 & x_3x_4 & x_3^3 & x_4^2 & x_3x_4 & x_3^3 & x_4^2 & x_5 & x_3x_4^3 & x_3x_4 & x_3x_4^3 & x_3x_4 & x_3x_4^3 & x_4^2 & x_3x_4 & x_5 \\
\end{array}
\]

The proof is by contradiction. Suppose that $R$ is coherent. Then there exists a rational vector $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ such that $I = \text{in}_\omega(I_A)$.

(i) In degree 6 we have $x_2^2 \in I$ but $x_2x_3 \notin I$. This implies $2\omega_2 > 2\omega_1 + \omega_3$.

(ii) In degree 17 we have $x_1x_3^4 \in I$ but $x_2x_3^4 \notin I$. This implies $\omega_1 + 4\omega_3 > \omega_2 + 2\omega_4$.

(iii) In degree 28 we have $x_4^4 \in I$ but $x_4^5 \notin I$. This implies $4\omega_4 > 7\omega_3$.

Combining these three inequalities we get

$$(2\omega_2) + 2 \cdot (\omega_1 + 4\omega_3) + (4\omega_4) > (2\omega_1 + \omega_3) + 2 \cdot (\omega_2 + 2\omega_4) + (7\omega_3).$$ \hfill (2.2)

The left hand side and the right hand side are both equal to $2\omega_1 + 2\omega_2 + 8\omega_3 + 4\omega_4$. This is a contradiction, and we conclude that $R$ is not coherent. This proves part (a).

To prove part (b) of Theorem 2.1 we consider the following family of ideals:

$$\langle x_1^2x_3 - c_1x_2^2, x_1x_3^4 - c_2x_2x_4^2, x_7^2 - c_3x_4^4, x_1^3, x_1x_2, x_1x_4, x_2^3, x_2x_4, x_2x_3, x_2x_4^3 \rangle,$$ \hfill (2.3)

where $c_1, c_2, c_3$ are indeterminate parameters over $k$. For every value of $c_1, c_2, c_3$ this is an $A$-graded ideal in $k[x_1, x_2, x_3, x_4]$. In other words, the given three-dimensional family of ideals is flat over $k^3$. To see this, we note that the given generators in (2.3) are a Gröbner
basis with respect to the lexicographic term order induced from \( x_1 > x_2 > x_3 > x_4 \). Note that the three first generators in (2.3) correspond to the three cases (i),(ii) and (iii) above. The ideal \( I \) in (2.1) is obtained from (2.3) by a deformation of the form \( c_1, c_3 \to \infty, c_2 \to 0 \).

Two ideals in this family define isomorphic \( A \)-graded algebras if and only if they can be mapped into each other by an element in the torus \( (k^*)^4 \) (acting naturally on the four variables). This is the case if and only if the invariant \( c_1 c_3 / c_2^2 \) has constant value. We conclude that the ideals in (2.3) define a one-dimensional family of non-isomorphic \( A \)-graded algebras. In particular, this family is infinite, since \( k \) is infinite. \( \blacksquare \)

The incoherent mono-AGA in Theorem 2.1 was found through a systematic search of \( A \)-graded monomial algebras. For this search we used computational techniques refining the ones presented in [11, §7]. Our point of departure was the following lemma which restricts the degrees of minimal generators of an \( A \)-graded ideal. A binomial \( x^u - x^v \) in the toric ideal \( I_A \) is called primitive if there are no proper monomial factors \( x^{u'} \) of \( x^u \) and \( x^{v'} \) of \( x^v \) such that \( x^{u'} - x^{v'} \in I_A \). This nomenclature is consistent with [4]. Primitive binomials were called star relations in [9] and Graver basis elements in [16]. We say that \( b \in N_A \) is a primitive degree if there exists a primitive binomial of degree \( b \) in \( A \). The following lemma for \( d = 1 \) appears in [9, Proposition 2.1].

**Lemma 2.2.** The degree of every minimal generator of an \( A \)-graded ideal is primitive.

Proof: Let \( I \) be an \( A \)-graded ideal, let \( f \) be a homogeneous minimal generator of \( I \), and let \( b := \text{deg}(f) \in N_A \). We must find a primitive binomial \( x^u - x^v \) of degree \( b \) in \( I_A \). By the defining property (1.1), there exists a monomial \( x^v \) of degree \( b \) which is non-zero modulo \( I \). We may assume that \( f \) has a minimal number of monomials distinct from \( x^v \). Clearly, this number is at least one, that is, \( f \) contains a monomial \( x^u \) distinct from \( x^v \).

We claim that \( x^u - x^v \) is a primitive relation in \( I_A \). Suppose not, and let \( x^{u'} \) be a proper factor of \( x^u \) and \( x^{v'} \) a proper factor of \( x^v \) such that \( x^{u'} - c_1 x^{v'} \in I \). Since \( x^{v'} \) is standard, there exists \( c_1 \in k \), such that \( x^{u'} - c_1 x^{v'} \in I \). By the same reasoning, there exists \( c_2 \in k \) such that \( x^{u'-u} - c_2 x^{v-v'} \in I \). This implies \( x^u - c_1 c_2 x^v \in I \). We may now replace the occurrence of \( x^u \) in \( f \) by \( c_1 c_2 x^v \). This is a contradiction to our minimality assumption, and we are done. \( \blacksquare \)

**Proposition 2.3.** Let \( d = 1 \) and \( A = \{a_1 < a_2 < \cdots < a_n\} \subset N \). Then every minimal generator of an \( A \)-graded ideal has degree at most \( a_{n-1} \cdot a_n \).

Proof: It was proved in [4] that every primitive binomial has degree at most \( a_{n-1} \cdot a_n \). Now apply Lemma 2.2. \( \blacksquare \)

Proposition 2.3 improves the bound in [9, Proposition 2.11]. If \( a_{n-1} \) and \( a_n \) are relatively prime, then the bound \( a_{n-1} \cdot a_n \) is best possible. To see this note that the
binomial $x_n^{a_n-1} - x_{n-1}^{a_n-1}$ appears in the reduced Gröbner basis of $I_A$ with respect to the lexicographic term order induced by $x_1 > \cdots > x_n$. The initial ideal of $I_A$ for this term order is an $A$-graded ideal which has a minimal generator of degree $a_{n-1} \cdot a_n$.

The following table comprises a complete catalogue of all incoherent mono-AGAs for $A = \{a_1, a_2, a_3, a_4\}$ with $1 \leq a_1 < a_2 < a_3 < a_4 \leq 9$. We write the set $A$ as a bracket $[a_1a_2a_3a_4]$. The three integers listed immediately after each bracket are:

(i) the number of primitive binomials in $I_A$.
(ii) the total number of all $A$-graded monomial ideals,
(iii) the number of incoherent $A$-graded monomial ideals.

If a quadruple does not appear in this list, then all mono-AGAs are coherent for that $A$.

| Quadruple   | Primitive Binomials | Total Monomial Ideals | Incoherent Monomial Ideals |
|-------------|---------------------|-----------------------|----------------------------|
| [1347]      | 27                  | 53                    | 2                          |
| [1459]      | 37                  | 90                    | 10                         |
| [1578]      | 33                  | 79                    | 2                          |
| [1789]      | 52                  | 174                   | 42                         |
| [2359]      | 24                  | 58                    | 8                          |
| [2579]      | 45                  | 168                   | 42                         |
| [2789]      | 41                  | 113                   | 10                         |
| [3578]      | 35                  | 88                    | 2                          |
| [4579]      | 40                  | 120                   | 6                          |
| [6789]      | 37                  | 94                    | 6                          |

Table 1. Incoherent one-dimensional mono-AGAs with $n = 4$ and degrees $\leq 9$.

3. Polyhedral conditions for coherence

The computational results in Section 2 raise the question whether there exist structural features of coherent AGAs which are not shared by all AGAs. Here we identify two such features: standard monomials (Theorem 3.3) and degrees of minimal generators (Theorem 3.6) are subject to certain geometric restrictions in the coherent case. Our presentation assumes familiarity with the language of combinatorial convexity (see e.g. [6],[17]).

For a fixed set $A = \{a_1, \ldots, a_n\} \subset \mathbb{N}^d \setminus \{0\}$ we consider the linear map

$$deg : \mathbb{N}^n \to \mathbb{N}A, \quad u = (u_1, \ldots, u_n) \mapsto \sum_{i=1}^{n} u_i \cdot a_i.$$ 

Each inverse image $deg^{-1}(b)$ consists of (the exponent vectors of) the monomials of degree $b$. Following [16] we form their convex hull $P[b] := conv(deg^{-1}(b))$. This polytope is called the fiber of $A$ over $b$. By the fiber of a monomial $x^u$ we mean $P[deg(u)]$. 

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Observation 3.1. Every standard monomial \( x^u \) of a coherent mono-AGA corresponds to a vertex \( u \) of its fiber \( P[\deg(u)] \).

Proof: Let \( x^u \) be standard in \( k[x]/\text{in}_\omega(I_A) \). Then \( u \) is the unique point in \( \deg^{-1}(\deg(u)) \) at which the linear functional \( \omega \) attains its minimum. Hence \( u \) is a vertex of \( P[\deg(u)] \). \( \blacksquare \)

The theory of \( A \)-graded algebras provides an abstract setting for the study of integer programming problems with respect to a fixed matrix (cf. [16]). This was a main motivation for writing the present paper. The subsequent remark makes it more precise.

Polyhedral Remark 3.2. Consider the following alternative definition for the object of study in Section 2: A mono-AGA is a rule which selects one lattice point (called standard monomial) from each fiber, subject to the axiom that the set of standard monomials is closed under divisibility. This selection is coherent if it is induced by a linear functional \( \omega \). Our choice of the term “coherent” parallels the notions of coherent subdivisions and coherent triangulations. We refer to [6],[17] and the references given there (and to Section 3 below). Also our usage of the letter “\( A \)” is consistent with that of [6], [15].

Does there exist an incoherent mono-AGA which has a standard monomial that is not a vertex of its own fiber? The answer was found to be “no” for all 218 incoherent mono-AGAs listed in Table 1. We do not know the answer for \( d = 1 \) and \( n = 4 \) in general. For \( d = 1 \) and \( n = 5 \) we can show that the answer is “yes”.

Theorem 3.3. Let \( d = 1, n = 5 \) and \( A = \{3, 4, 5, 13, 14\} \). There exists a monomial \( A \)-graded algebra with a standard monomial \( x^u \) such that \( u \) is not a vertex of its fiber \( P[\deg(u)] \).

Proof. In the polynomial ring \( k[x_1, x_2, x_3, x_4, x_5] \) we consider the ideal

\[
I = \langle x_1^3, x_2^2, x_3^2, x_1x_5, x_2x_5, x_3x_5, x_5^2 \rangle.
\]

This ideal is \( A \)-graded. Indeed, an easy MAPLE or MACAULAY computation shows that \( R = k[x]/I \) has the correct Hilbert series, namely,

\[
\frac{1}{1-t-t^2} = \sum_{m \in \mathbb{N}_A} t^m.
\]

The monomial \( x_1^2x_2x_3 \) does not lie in \( I \): it is a standard monomial of degree 15. Its fiber is the Newton polytope of the coefficient \( c_{15}(x) \) in the expansion of the formal power series

\[
1/(1-x_1^2)(1-x_2^4)(1-x_3^5)(1-x_4^{13})(1-x_5^{14}) = \sum_{m=0}^{\infty} c_m(x_1, x_2, x_3, x_4, x_5) t^m.
\]

We find that \( c_{15}(x) = x_1^5 + x_1x_3^3 + x_1^2x_2x_3 + x_3^3 \). The point \( (2, 1, 1, 0, 0) \) lies in the relative interior of the triangle \( P[(2, 1, 1, 0, 0)] = \text{conv} \{ (5, 0, 0, 0, 0), (1, 3, 0, 0, 0), (0, 0, 3, 0, 0) \} \). \( \blacksquare \)
After this discussion of vertices of fibers, we now turn our attention to edges of fibers. An element \( \mathbf{b} \) of \( \mathcal{N}_A \) is said to be a Gröbner degree if a binomial of degree \( \mathbf{b} \) appears in some reduced Gröbner basis of \( I_A \). Equivalently, the Gröbner degrees are precisely the degrees of the minimal generators of all coherent \( \mathcal{A} \)-graded ideals. Lemma 2.2 implies that every Gröbner degree is also a primitive degree. The following geometric characterization of Gröbner degrees was proved in the joint work [16] with R. Thomas.

**Theorem 3.4.** An element \( \mathbf{b} \) of \( \mathcal{N}_A \) is a Gröbner degree if and only if its fiber \( P[\mathbf{b}] \) has an edge which is not parallel to any edge of a different fiber \( P[\mathbf{b}'] \) with \( \mathbf{b}' \leq \mathbf{b} \).

*Proof:* This is a reformulation of Theorem 5.1 in [16]. ■

Theorem 3.4 implies that a primitive binomial lies in some reduced Gröbner basis of \( I_A \) if and only if its two monomials are connected by an edge in their fiber.

It was shown in [16, §5] that there may exist primitive degrees which are not Gröbner. Here is an example for \( d = 1 \). It is derived from Korkina’s example in [9, Remark 3.3].

**Example 3.5.** *(A primitive degree which is not a Gröbner degree)*

Let \( \mathcal{A} = \{15, 20, 23, 24\} \). The binomial \( x_1^2x_2^3x_4^2 - x_3^6 \) is the unique primitive binomial of degree 138. (This can be verified easily using [16, Algorithm 4.3].) We shall prove that it is not an element of any reduced Gröbner basis of \( I_A \). Consider the convex combination

\[
\frac{1}{4}(0, 0, 6, 0) + \frac{3}{4}(2, 3, 0, 2) = \frac{1}{4}(5, 2, 1, 0) + \frac{1}{4}(1, 5, 1, 0) + \frac{1}{2}(0, 1, 2, 3).
\]

This shows that \( \text{conv}\{(2, 3, 0, 2), (0, 0, 6, 0)\} \), the segment corresponding to our binomial, is not an edge of its fiber \( P[138] \). Using Theorem 3.4 we conclude that the given binomial does not lie in any reduced Gröbner basis. Therefore 138 is a primitive degree which is not a Gröbner degree. ■

This raises the question whether there exists some (necessarily incoherent) \( \mathcal{A} \)-graded algebra which has a minimal generator of non-Gröbner degree. The answer is “no” for the set \( \mathcal{A} = \{15, 20, 23, 24\} \) above, but it becomes “yes” after adding sufficiently many new generators. Clearly, the number \( n = 145 \) below is not best possible.

**Theorem 3.6.** For \( d = 1 \) and \( n = 145 \) there exists an \( \mathcal{A} \)-graded ideal \( I \) which has a minimal generator whose degree is not a Gröbner degree.

*Proof:* Let \( \mathcal{A}' = \{15, 20, 23, 24, 107, 109\} \). Let \( \mathcal{S}' \) be a polynomial ring in six variables \( x_{15}, x_{20}, x_{23}, x_{24}, x_{107}, x_{109} \). We grade \( \mathcal{S} \) by setting \( \text{deg}(x_i) = i \). Let \( M' \) be the ideal generated by the six variables, and let \( M'_{\geq 139} \) be the ideal generated by all monomials of degree \( \geq 139 \) in \( \mathcal{S}' \). Let \( I \) be the binomial ideal generated by

\[
\begin{align*}
x_{15}^4, & \quad x_{20}^2x_{23}, \quad x_{15}^3x_{23}, \quad x_{15}^2x_{24}, \quad x_{24}^2x_{23}, \quad x_{15}x_{23}^3, \quad x_{20}x_{24}^3, \quad x_{15}x_{20}^4, \quad x_{15}^2x_{20}x_{23}^2, \\
x_{24}^5, & \quad x_{15}x_{93}, \quad x_{15}x_{109}, \quad x_{24}x_{109}, \quad x_{15}^2x_{107}, \quad \text{and} \quad x_{15}^3x_{20}x_{24}^2 - x_6^7.
\end{align*}
\]
The binomial \(x_{15}^2 x_{20}^3 x_{24}^2 - x_{23}^6\) has degree 138, and this degree is primitive but not Gröbner (by Example 3.5). The ideal \(I\) is constructed to have the following property: the Artinian ring \(S'/(I + M_{\geq 139}')\) is \(A\)-graded up to degree 138. In other words, its Hilbert series equals \(\sum \{t^b : b \in \mathbb{N}A \text{ and } b \leq 138\}\).

Let \(A'' = \{139, 140, \ldots, 277\}\) and introduce the corresponding polynomial ring \(S'' = k[x_{139}, x_{140}, \ldots, x_{277}]\). We write \(M''\) for the ideal generated by all 139 variables in \(S''\), and we let \(J''\) be any \(A''\)-graded ideal in \(S''\). Finally, we set \(A'' = A'' \cup A'\), and we introduce the corresponding 145-variate polynomial ring \(S = S' \otimes_k S''\). In this ring we form the ideal \(J = \langle M' \cdot M'' \rangle + \langle I + M_{\geq 139}' \rangle + \langle J'' \rangle\). By construction, the ideal \(J\) is \(A\)-graded in \(S\). It has the primitive binomial \(x_{15}^2 x_{20}^3 x_{24}^2 - x_{23}^6\) among its minimal generators. However, its degree 138 is not a Gröbner degree for \(A\). This completes the proof.

4. The radical of an \(A\)-graded ideal

In this section a polyhedral construction is presented for the radical of any \(A\)-graded ideal. The positive hull of \(A\) is the closed convex polyhedral cone 
\[
\text{pos}(A) := \left\{ \sum_{i=1}^{n} \lambda_i \cdot a_i : (\lambda_1, \ldots, \lambda_n) \geq 0 \right\}.
\]

If \(\sigma\) is any subset of \(A\) then we similarly write \(\text{pos}(\sigma)\) for the positive hull of \(\sigma\). By a face of \(\sigma\) we mean a subset \(\tau\) such that the cone \(\text{pos}(\tau)\) is a face of the cone \(\text{pos}(\sigma)\) and \(\tau = \text{pos}(\tau) \cap \sigma\). We identify the toric ideal \(I_{\sigma}\) corresponding to a subset \(\sigma\) with the prime ideal \(I_{\sigma} + \langle x_i : a_i \notin \sigma \rangle\) in \(k[x]\). A polyhedral subdivision of \(A\) is a collection \(\Delta\) of subsets of \(A\) such that \(\{\text{pos}(\sigma) : \sigma \in \Delta\}\) is a polyhedral fan with support equal to the cone \(\text{pos}(A)\).

A basic construction due to R. Stanley associates to any integral polyhedral complex a radical binomial ideal. (See [5, Example 4.7] for an algebraic discussion.) If \(\Delta\) is a polyhedral subdivision of \(A\), then its Stanley ideal is \(I_{\Delta} := \cap_{\sigma \in \Delta} I_{\sigma}\). We remark that \(R = k[x]/I_{\Delta}\) is also graded by the semigroup \(\mathbb{N}A\), but it is generally not \(A\)-graded because, for some \(b \in \mathbb{N}A\), the graded component \(R_b\) may be zero. Finally, we call two arbitrary ideals \(I\) and \(I'\) in \(k[x]\) torus isomorphic if there exists \(\lambda \in (k^*)^n\) such that (1.2) holds. The following is the main result in this section.

**Theorem 4.1.** If \(I\) is any \(A\)-graded ideal, then there exists a polyhedral subdivision \(\Delta\) of \(A\) such that \(\text{Rad}(I) = \cap_{\sigma \in \Delta} J_{\sigma}\) where each ideal \(J_{\sigma}\) is prime and torus isomorphic to \(I_{\sigma}\).

We shall make a two remarks before presenting the proof. First, as a special case of Theorem 4.1, we can recover the main result in [15]. Namely, this is the case where
• the given ideal \( I \) is a coherent mono-AGA, and
• the semigroup \( \mathbb{N}A \) is graded (or, equivalently, the toric ideal \( I_A \) is homogeneous).

If these two hypotheses are met, then \( pos(A) \) is the cone over a polytope \( conv(A) \), and \( \Delta \) is (the complex of cones over) a \textit{regular triangulation} of this polytope.

Our second remark is to explain the mysterious appearance of the ideals \( J_\sigma \). What is the point of replacing \( I_\sigma \) by a torus isomorphic ideal \( J_\sigma \), for each maximal cell \( \sigma \) of \( \Delta \)? The answer is that the Stanley ideal \( I_\Delta = \bigcap_{\sigma \in \Delta} I_\sigma \) itself is generally \textit{not} torus isomorphic to the radical of \( I \). We present an example where this happens.

**Example 4.2.** An \( \mathcal{A} \)-graded ideal \( I \) such that \( \text{Rad}(I) \) is not torus isomorphic to any \( I_\Delta \).

Let \( d = 3, n = 6 \) and \( \mathcal{A} = \{(4,0,0),(0,4,0),(0,0,4),(2,1,1),(1,2,1),(1,1,2)\} \). For every choice of non-zero constants \( c_1, c_2, c_3 \in k^* \), the following ideal is \( \mathcal{A} \)-graded:

\[
I_{c_1, c_2, c_3} = \langle x_1x_2x_3, x_1x_5x_6, x_2x_4x_6, x_3x_4x_5, x_1x_2x_6^2, x_1x_3x_5^2, x_2x_3x_4^2, \\
x_1x_5^4 - c_1x_2x_4^4, x_2x_6^4 - c_2x_3x_5^4, x_3x_4^4 - c_3x_1x_6^4 \rangle.
\]

The underlying subdivision \( \Delta \) of \( A \) consists of three quadrangular cones and one triangular cone. This can be seen from the prime decomposition \( \text{Rad}(I_{c_1, c_2, c_3}) = \langle x_1, x_2, x_3 \rangle \cap \langle x_1x_5^4 - c_1x_2x_4^4, x_3, x_6 \rangle \cap \langle x_2x_6^4 - c_2x_3x_5^4, x_1, x_4 \rangle \cap \langle x_3x_4^4 - c_3x_1x_6^4, x_2, x_5 \rangle \).

From this decomposition we can see that \( \text{Rad}(I_{c_1, c_2, c_3}) \) is torus isomorphic to the Stanley ideal \( I_\Delta \) if and only if the invariant \( c_1c_2c_3 \) attains the value 1. The reader will not fail to note a certain analogy between this example and the three-dimensional family in (2.3).

The reason for the phenomenon in Example 4.2 is the existence of moduli (infinite families) of isomorphism classes of \( \mathcal{A} \)-graded algebras. These moduli stem from “extraneous components” in the parameter space \( P_A \) (see Section 5 and Problem 6.4). For the proof of Theorem 4.1 we shall need the following lemma.

**Lemma 4.3.** Let \( I \subset k[x] \) be an \( \mathcal{A} \)-graded ideal which contains no monomials. Then \( I \) is isomorphic to the toric ideal \( I_A \).

**Proof:** This follows from the characterization of Laurent binomial ideals in [5, §2].

**Proof of Theorem 4.1:** The polynomial ring \( S = k[x_1, \ldots, x_n] \) is graded by the semigroup \( \mathbb{N}A \) via \( \text{deg}(x_i) = a_i \). For any \( b \in \mathbb{N}A \) we define the subalgebra \( S(b) := \bigoplus_{m=0}^{\infty} S_{mb} \). This algebra is generated by a finite set of monomials. Inside it we consider the binomial ideal \( I(b) := I \cap S(b) \). The corresponding subalgebra \( R(b) := S(b)/I(b) \) of our given \( \mathcal{A} \)-graded algebra \( R = S/I \) is a finitely generated \( k \)-algebra of Krull dimension 1. It is not possible that all elements in such an algebra are nilpotent. We conclude that there exists a monomial \( x^u \) in \( S(b) \) which is not nilpotent modulo \( I(b) \).
Let $x^u$ and $x^v$ be two such non-nilpotent monomials in $R_{(b)}$. We claim that their product $x^u x^v \in R_{(b)}$ is not nilpotent either. To see this, we choose integers $m_1$ and $m_2$ such that $x^{m_1 u}$ and $x^{m_2 v}$ have the same degree. There exists a non-zero constant $c \in k^*$ such that $x^{m_1 u} = c \cdot x^{m_2 v}$ in $R_{(b)}$. This implies $(x^{m_1 u} x^{m_2 v})^m = c^m (x^v)^{2m_{m_2}} = c^{-m} (x^u)^{2m_{m_1}} \neq 0$ in $R$ for all $m > 0$, and consequently $(x^u x^v)^m \neq 0$ in $R_{(b)}$ for all $m > 0$. We have shown that the set of non-nilpotent monomials in $R_{(b)}$ is multiplicatively closed.

This multiplicativity property allows us to synthesize the polyhedral subdivision $\Delta$. The support of a monomial is defined as $supp(x^u) := \{ a_i \in A : u_i \neq 0 \}$. Clearly, we have $supp(x^u x^v) = supp(x^u) \cup supp(x^v)$. This implies that the set of supports of non-nilpotent monomials in $R_{(b)}$ has a unique maximal element. This subset of $A$ is denoted by cell$(b)$. We define $\Delta$ to be the collection of all subsets cell$(b)$ as $b$ ranges over $NA$.

We shall prove that $\Delta$ is indeed a polyhedral subdivision of $A$. Let $\tau$ be any face of $\sigma = cell(b)$ (possibly $\tau = \sigma$), and let $b'$ be any lattice point in the relative interior of $pos(\tau)$. It suffices to show that $cell(b') = \tau$. By the property of being a face, $\tau$ is the unique maximal subset of $\sigma$ which is the support of any monomial $x^u$ in $S_{(b')}$. Such a monomial $x^u$ is not nilpotent modulo $I$, since there exists a monomial $x^u$ of degree $b$ whose support equals $\sigma \supset \tau = supp(x^u')$. Suppose there exists a non-nilpotent monomial $x^u''$ in $R_{(b')}$ whose support $\rho := supp(x^u'')$ properly contains $\tau$. Then $\rho \setminus \sigma = \rho \setminus \tau \neq \emptyset$. Choose integers $m_1, m_2$ and a non-zero constant $c \in k^*$ such that $x^{m_1 u} - c \cdot x^{m_2 u''} \in I$. Let the degree of this binomial be $m_3 \cdot b'$. Choose an integer $m_4 \gg 0$ such that $m_4 \cdot b - b'$ lies in the relative interior of $pos(\sigma)$, and let $x^w$ be a monomial having degree $m_4 \cdot b - b'$ and support $\sigma$. We conclude that $x^{m_4 w + m_1 u} - c \cdot x^{m_4 w + m_2 u''}$ lies in the degree $m_3 m_4 \cdot b$ component of the ideal $I$. The first monomial $x^{m_3 w + m_1 u}$ is not nilpotent modulo $I$ since it has support $\sigma$. The second monomial $x^{m_3 w + m_2 u''}$ is nilpotent modulo $I$ since its support $\sigma \cup \rho$ strictly contains $\sigma$. This is a contradiction, and we conclude that $cell(b') = \tau$. This completes the proof that $\Delta$ is a polyhedral subdivision of $A$.

We next compute the radical of $I$. Let $\sigma$ be a maximal cell in $\Delta$. By construction, the elimination ideal $I \cap k[x_i : i \in \sigma]$ is a $\sigma$-graded ideal which contains no monomials. Lemma 4.3 implies that its natural embedding into $k[x]$, $J_{\sigma} := (I \cap k[x_i : i \in \sigma]) + \langle x_j : a_j \notin \sigma \rangle$, is torus isomorphic to the toric prime $I_{\sigma}$. We claim that $Rad(I) = \bigcap_{\sigma \in \Delta} J_{\sigma}$.

We first show the inclusion $I \subseteq \bigcap_{\sigma \in \Delta} J_{\sigma}$. (This automatically implies $Rad(I) \subseteq \bigcap_{\sigma \in \Delta} J_{\sigma}$ because the right hand side is radical.) If $x^u$ is any monomial not contained in $\bigcap_{\sigma \in \Delta} \langle x_j : j \notin \sigma \rangle$, then $supp(x^u) \subseteq \sigma$ for some $\sigma \in \Delta$, and hence $x^u$ is not nilpotent modulo $I$. This shows that all monomials which are nilpotent modulo $I$ lie in $\bigcap_{\sigma \in \Delta} J_{\sigma}$. Consider any binomial $f := x^u - c \cdot x^v$ in $I$ with both terms not nilpotent modulo $I$. Let $b = deg(x^u) = deg(x^v)$. Fix $\sigma \in \Delta$. If $cell(b)$ is a face of $\sigma$, then $f \in I_{(b)} \cap k[x_i : i \in \Delta]$. If $cell(b)$ is a maximal cell, then $f \in J_{\sigma}$. Otherwise, $f$ lies in the relative interior of a face of $\sigma$, and hence $f \in stable(I)$. Therefore, $f \in \bigcap_{\sigma \in \Delta} J_{\sigma}$. This completes the proof.
\[ \sigma \subseteq J_\sigma. \] If \( \text{cell}(b) \) is not a face of \( \sigma \), then the supports of \( x^u \) and \( x^v \) are not subsets of \( \sigma \). Therefore both \( x^u \) and \( x^v \) lie in \( \langle x_j : j \notin \sigma \rangle \), and hence \( f \in J_\sigma \).

For the reverse inclusion \( \bigcap_{\sigma \in \Delta} J_\sigma \subseteq \text{Rad}(I) \) we use the Nullstellensatz: it suffices to prove that the variety \( V(I) \) is contained in \( \bigcup_{\sigma} V(J_\sigma) \). Let \( u \in \bar{k}^n \) be any zero of \( I \), where \( \bar{k} \) is the algebraic closure of \( k \). Abbreviate \( \rho := \text{supp}(u) \). Consider the monomial \( \prod_{i \in \rho} x_i \) and let \( b \) be its degree. Let \( \sigma \) be any maximal cell of \( \Delta \) which has \( \text{cell}(b) \) as a face. By construction, no power of \( \prod_{i \in \rho} x_i \) vanishes at \( u \). Hence the monomial \( \prod_{i \in \rho} x_i \) is not nilpotent modulo \( I \), and its support \( \rho \) is a subset of \( \text{cell}(b) \subseteq \sigma \). In other words, \( u \) is a zero of the ideal \( \langle x_j : j \notin \sigma \rangle \). Therefore \( u \) is a zero of \( J_\sigma \). This completes the proof. \( \blacksquare \)

5. The parameter space.

In this section we aim to answer the question posed in the first sentence of the introduction. We construct the parameter space \( P_\mathcal{A} \) whose points are in bijection with the distinct \( \mathcal{A} \)-graded ideals in \( k[x] \). The torus \( (k^*)^n \) acts naturally on the space \( P_\mathcal{A} \), and its orbits are in bijection with the isomorphism types of \( \mathcal{A} \)-graded algebras.

Let \( \mathcal{A} = \{a_1, \ldots, a_n\} \subseteq \mathbb{N}^d \setminus \{0\} \) as before, and fix the \( \mathbb{N} \mathcal{A} \)-grading of \( S = k[x] \) given by \( \text{deg}(x_i) = a_i \). For any integer \( r > 0 \) we define the following zonotope in \( \mathbb{R}^d \):

\[
Z_r(\mathcal{A}) := \left\{ \sum_{i=1}^{n} \lambda_i \cdot a_i : 0 \leq \lambda_i \leq r \text{ for } i = 1, \ldots, n \right\}. \tag{5.1}
\]

Let \( M_r \) be the ideal in \( S = k[x] \) spanned by all monomials \( x^u \) such that \( \text{deg}(x^{u+v}) \notin Z_r(\mathcal{A}) \) for all \( v \in \mathbb{N}^n \). We write \( S^{(r)} \) for the quotient algebra \( S/M_r \). The algebra \( S^{(r)} \) is graded by \( \mathbb{N} \mathcal{A} \), and it is artinian because each variable \( x_i \) has a power lying in \( M_r \). An ideal \( J \subseteq S^{(r)} \) is called \( \mathcal{A} \)-graded if \( \dim_k((S^{(r)}/J)_b) = 1 \) for all \( b \in Z_r(\mathcal{A}) \cap \mathbb{N} \mathcal{A} \). If \( I \) is any \( \mathcal{A} \)-graded ideal in \( S \), then its image \( I^{(r)} := (I + M_r)/M_r \) is an \( \mathcal{A} \)-graded ideal in \( S^{(r)} \).

**Proposition 5.1.** There exists an integer \( r \gg 0 \) such that the assignment \( I \mapsto I^{(r)} \) defines a bijection between the \( \mathcal{A} \)-graded ideals in \( S \) and the \( \mathcal{A} \)-graded ideals in \( S^{(r)} \).

**Proof:** For the injectivity of the map \( I \mapsto I^{(r)} \) it suffices to choose \( r \) such that the zonotope \( Z_r(\mathcal{A}) \) contains all primitive degrees. For instance, if \( a \) is the maximum of the Euclidean norms \( ||a_i|| \), then \( r = (n-d) \cdot a^d \) has this property by \([15, \S 2] \). If \( I \) and \( J \) are two distinct \( \mathcal{A} \)-graded ideals in \( S \), then, by Lemma 2.2, there exists a primitive degree \( b \) such that \( I_b \) and \( J_b \) are distinct subspaces of \( S_b \). But \( S_b = S_b^{(r)} \), hence \( I_b^{(r)} = I_b \neq J_b = J_b^{(r)} \), and therefore \( I^{(r)} \) and \( J^{(r)} \) are distinct \( \mathcal{A} \)-graded ideals in \( S^{(r)} \).

To prove surjectivity we choose \( r \gg 0 \) to have the following property: If \( I \) is any binomial ideal in \( k[x] \) whose generators have primitive degrees, and \( \prec \) is any term order, then the initial ideal \( \text{in}_{\prec}(I) \) is generated by monomials of degree \( r \). For instance, combining
the above bound with the known doubly-exponential degree bounds for Gröbner bases [14], we see that the choice \( r = (n - d)2^n \cdot a^{2^n} \) will surely suffice.

Let \( J = \langle f_1, \ldots, f_m \rangle \) be any \( A \)-graded ideal in \( S^{(r)} \). Here the \( f_i \) are binomials of primitive degree, so they have a unique preimage in \( S \). We consider the ideal in \( S \) generated by these preimages \( I := \langle f_1, \ldots, f_m \rangle \). Since the conclusion \( I^{(r)} = J \) is automatic, we only have to show that \( I \) is \( A \)-graded. Equivalently, we must show that \( I \) has the same \( \mathbb{N}A \)-graded Hilbert series as the toric ideal \( I_A \). Let \( \prec \) be any term order. Since the Hilbert series is preserved under passing to the initial monomial ideal, it suffices to show that \( in_{\prec}(I_A) \) and \( in_{\prec}(I) \) have the same Hilbert series. Let \( x^{u_1}, \ldots, x^{u_s} \) be the minimal generators of \( in_{\prec}(I) \). The Hilbert series of interest equals the rational function

\[
H(I; t) = H(in_{\prec}(I); t) = \frac{\sum_{\nu \subseteq \{1, \ldots, s\}} (-1)^{|\nu|} \cdot \deg(lcm(\{x^{u_j} : j \in \nu\}))}{\prod_{i=1}^{n}(1 - t_1^{a_{i1}} \cdots t_d^{a_{id}})}. \tag{5.2}
\]

By construction, the degree of each term in the numerator polynomial lies in \( \mathbb{Z}_r(A) \), and the same is true for \( I_A \). Therefore \( H(I, t) - H(I_A, t) \) is a rational function of the form \( p(t)/\prod_{i=1}^{n}(1 - t^{a_i}) \), where all monomials appearing in \( p(t) \) lie in \( \mathbb{Z}_r(A) \). Moreover, since \( I \) is \( A \)-graded in \( S^{(r)} \), no multiple of a monomial appearing in the power series expansion of \( H(I, t) - H(I_A, t) \) can lie in \( \mathbb{Z}_r(A) \). In other words, the image of \( H(I, t) - H(I_A, t) \) in the artinian ring \( k[t_1, \ldots, t_d]/\langle t^b : b \notin Z_r(A) \rangle \) is zero. Indeed, in this ring the product \( \prod_{i=1}^{n}(1 - t^{a_i}) \) is invertible, and we can conclude that \( p(t) \) is the zero polynomial.

Proposition 5.1 is somewhat unsatisfactory in that the doubly-exponential lower bound for \( r \) used in its proof seems too big. We conjecture that the choice \( r = (n - d) \cdot a^d \) is large enough. In fact, our argument shows that this choice is large enough to give the desired bijection for \( A \)-graded monomial ideals. However, even for monomial ideals, it is not enough to require “\( A \)-gradedness” only up to the primitive degrees.

Example 5.2. (One-dimensional in all primitive degrees does not imply \( A \)-graded)

Let \( A = \{(3, 0), (2, 1), (1, 2), (0, 3)\} \). The toric ideal \( I_A \) is ideal of the twisted cubic curve in \( \mathbb{P}^3 \). It contains precisely five primitive binomials (cf. [15, §4]):

\[
x_1x_3 - x_2^2, \ x_1x_4 - x_2x_3, \ x_2x_4 - x_3^2, \ x_1^2x_4 - x_2^3, \text{ and } x_1x_4^2 - x_3^3.
\]

The set of primitive degrees is \( D = \{(4, 2), (3, 3), (2, 4), (6, 3), (3, 6)\} \). Consider the monomial ideal \( I := \langle x_1x_4, x_2^2, x_3^2 \rangle \). The quotient algebra \( k[x]/I \) is one-dimensional in all degrees \( b \) with \( b \leq d \) for some \( d \in D \). But \( I \) is not \( A \)-graded since it contains all monomials of degrees \((4, 5)\) and \((5, 4)\).

We are now prepared to construct the parameter space of \( A \)-graded ideals. Let \( b \in \mathbb{N}A \). Consider the vector space \( k^{\deg^{-1}(b)} \) of all \( k \)-valued functions on the fiber \( \deg^{-1}(b) \),
and let \( P_b \) denote the projectivization of \( k^{deg^{-1}(b)} \). Choose \( r \gg 0 \) as in Proposition 5.1 and form the product of projective spaces \( P := \Pi_b P_b \), where \( b \) runs over all points in \( Z_r(A) \cap N A \). If \( f \) is an element of the product \( P \), then we write \( f^b \in P_b \) for its \( b \)-th component, and, if \( u \in N^n \) with \( deg(u) = b \), then \( f^b_u \) denotes the homogeneous coordinate of \( f^b \) indexed by \( u \). We define a closed subscheme \( P_A \) of \( P \) by the equations

\[
    f^b_u \cdot f^b_{v+w} = f^b_v \cdot f^b_{u+w} \quad \text{whenever } deg(u) = deg(v) = b \text{ and } deg(w) = c. \tag{5.3}
\]

The following is the main result in this section.

**Theorem 5.3.** There exists a natural bijection between the set of \( A \)-graded ideals in the polynomial ring \( k[x] \) and the set of closed points of the scheme \( P_A \).

**Proof:** With every point \( f \in P_A \) we associate the following binomial ideal in \( k[x] \):

\[
    I_f := \langle f^b_u \cdot x^v - f^b_v \cdot x^u : u, v \in N^n, b \in N A \text{ s.t. } deg(u) = deg(v) = b \rangle. \tag{5.4}
\]

Fix \( b \in Z_r(A) \cap N A \). There exists \( u \in deg^{-1}(b) \) such that \( f^b_u \neq 0 \). Every monomial \( x^v \) of degree \( b \) is a scalar multiple of \( x^u \) modulo the relations in \( I_f \). Therefore \( dim_k((S/I_f)_b) \leq 1 \). We must show that equality holds. The equations (5.3) imply that the \( b \)-th graded component of \( I_f \) is spanned as a \( k \)-vector space by the binomials \( f^b_u \cdot x^v - f^b_v \cdot x^u \) where \( u, v \in deg^{-1}(b) \). This space is a proper subspace of \( S_b \). We conclude that \( I_f \) is an \( A \)-graded ideal in \( S(r) \), and by Proposition 5.1, it lifts to a unique \( A \)-graded ideal in \( S \).

We shall construct the inverse to the map \( f \mapsto I_f \). Let \( J \) be any \( A \)-graded ideal in \( S \). For each \( b \in Z_r(A) \cap N A \) there exists a monomial \( x^u \) which does not lie in \( J \). We define \( f = f(J) \in P \) as follows: for \( v \in deg^{-1}(b) \) let \( f^b_v \) be the unique scalar in \( k \) such \( x^v - f^b_v \cdot x^u \) lies in \( J \). Note \( f^b_u = 1 \). We see that \( f^b = (f^b_v : v \in deg^{-1}(b)) \) is a well-defined point in the projective space \( P_b \), independent of the choice of \( u \). Our assumption that \( J \) is \( A \)-graded implies that the two binomials

\[
    x^w \cdot (f^b_u \cdot x^v - f^b_v \cdot x^u) \quad \text{and} \quad f^b_{u+w} \cdot x^{v+w} - f^b_{v+w} \cdot x^{u+w}
\]

are non-zero constant multiples of each other, whenever \( x^{u+w} \) is a monomial not in \( J \). This proves that \( f \) satisfies the equations (5.3) and hence lies in \( P_A \). It is now obvious that \( I_f = J \), and we are done. \( \blacksquare \)

**Corollary 5.4.** All irreducible components of \( P_A \) are rational varieties.

**Proof:** This follows from the decomposition theorem for arbitrary binomial schemes in [5]. Indeed, under the Segre embedding of the product \( P \), the equations (5.3) translate into linear equations with two terms. This shows that \( P_A \) is a binomial scheme. \( \blacksquare \)
There is a natural action of the torus \((k^*)^n\) on the product of projective spaces \(\mathbf{P}\). If \(f \in \mathbf{P}\) and \(\lambda \in (k^*)^n\), then \(\lambda f\) has coordinates \((\lambda f)^b_u := \lambda^n \cdot f^b_u\). Clearly, this action preserves the subscheme \(\mathcal{P}_A\), and we have the relation \(I_{\lambda f} = \lambda^{-1} \cdot I_f\).

**Remark 5.5.** The bijection \(f \mapsto I_f\) between \(\mathcal{P}_A\) and \(A\)-graded ideals is \((k^*)^n\)-equivariant.

**Corollary 5.6.** The set of isomorphism classes of \(A\)-graded algebras in \(k[x]\) is in bijection with the set of \((k^*)^n\)-orbits in \(\mathcal{P}_A\).

In Section 3 we have seen that for \(n = 4\) and \(d = 1\) there may be infinitely many \((k^*)^n\)-orbits on \(\mathcal{P}_A\). In this situation it is desirable to construct a moduli space \(\mathcal{M}_A = \mathcal{P}_A/(k^*)^n\) of isomorphism classes of \(A\)-graded algebras. However, such an enterprise would immediately face the usual intricacies of geometric invariant theory [13], such as:

- The GIT-quotient is not unique but depends on the choice of linearization. Is there a best linearization? (Or perhaps the Chow quotient of [7] is more useful here?)
- Which are the semi-stable orbits? And which ones get necessarily lost under the quotient construction?

Leaving these questions for future studies (see Problem 6.5), we close this section with a corollary about the “coherent component” of \(\mathcal{P}_A\). Let \(e\) denote the point in \(\mathcal{P}_A \subset \mathbf{P}\) all of whose coordinates \(e^b_u\) are equal to one. Then \(I_e\) equals the toric ideal \(I_A\).

**Corollary 5.7.** The map \(f \mapsto I_f\) defines a bijection between the closure of the \((k^*)^n\)-orbit of \(e\) in \(\mathcal{P}_A\) and the set of coherent \(A\)-graded ideals in \(k[x]\). This orbit closure \((k^*)^n \cdot e\) equals the projective toric variety defined by the state polytope of \(A\).

**6. Open problems.**

Starting from the examples in Section 2, it is easy to construct incoherent \(A\)-graded algebras for all choices of \(n\) and \(d = \text{dim}(A)\) with \(n \geq d + 3\). The case \(n \leq d + 1\) being trivial, and the case \(n = 3, d = 1\) being answered by Theorem 1.1, the question remains what happens for \(n = d + 2 \geq 4\). In view of C. Lee’s result [12] that all triangulations of \((d - 1)\)-polytopes with \(d + 2\) vertices are coherent, we venture the following conjecture.

**Conjecture 6.1.** If \(n - d \leq 2\), then all \(A\)-graded algebras are coherent.

We also conjecture the following converse to Theorem 3.1.

**Conjecture 6.2.** For every polyhedral subdivision \(\Delta\) of a finite set \(A \subset \mathbb{N}^d\) there exists an \(A\)-graded ideal \(I\) whose radical \(\text{Rad}(I)\) equals the Stanley ideal \(I_\Delta\).

Theorem 4.1 and Conjecture 6.2 (if true) would completely characterize the reduced schemes defined by \(A\)-graded algebras: they are precisely the (algebraic sets associated with) polyhedral subdivisions of \(A\). Another obvious question we left open is the following.
Problem 6.3. Find an optimal bound for the integer $r$ in Proposition 5.1.

The parameter space $\mathcal{P}_A$ has the following general structure. It has one nice component consisting of all coherent AGAs (cf. Corollary 5.7), and it may have many other components about which we know very little. For instance, we do not know whether there can be embedded components. One line of attack is suggested by the geometry of Example 3.2. Here the extraneous component corresponds to a family of incoherent subdivisions of $\mathcal{A}$. On the other hand, such incoherent subdivisions give rise to extraneous components in the inverse limit of toric GIT-quotients introduced in [7, §4]. Here we make the assumption that all vectors in $\mathcal{A}$ have the same coordinate sum, so that $\mathcal{A}$ defines an action of the torus $(k^*)^d$ on projective space $P^{n-1}$ (see [7] for details).

Problem 6.4. Does there exist a natural morphism from the parameter space $\mathcal{P}_A$ onto the inverse limit of all toric GIT-quotients $P^{n-1}/(k^*)^n$ with respect to the action

$$(x_1 : x_2 : \cdots : x_n) \mapsto (t^{a_1}x_1 : t^{a_2}x_2 : \cdots : t^{a_n}x_n)$$

The restriction of the desired morphism to the coherent component is well-known in combinatorial algebraic geometry: it is the contraction from the toric variety of the state polytope onto the toric variety of the secondary polytope [8]. The latter is the Chow quotient $P^{n-1}/(k^*)^n$ which appears as the distinguished component in the inverse limit of GIT-quotients. Our last question was already asked in the end of Section 5.

Problem 6.5. Construct and study the moduli space $\mathcal{M}_A$ of $(k^*)^n$-orbits on $\mathcal{P}_A$.

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