Nonsmooth Optimization over Stiefel Manifold: Riemannian Subgradient Methods

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Abstract

We consider a class of nonsmooth optimization problems over Stiefel manifold, which are ubiquitous in engineering applications but still largely unexplored. We study this type of nonconvex optimization problems under the settings that the function is weakly convex in Euclidean space and locally Lipschitz continuous, where we propose to address these problems using a family of Riemannian subgradient methods. First, we show iteration complexity $O(\varepsilon^{-4})$ for these algorithms driving a natural stationary measure to be smaller than $\varepsilon$. Moreover, local linear convergence can be achieved for Riemannian subgradient and incremental subgradient methods if the optimization problem further satisfies the sharpness property and the algorithms are initialized close to the set of weak sharp minima. As a result, we provide the first convergence rate guarantees for a family of Riemannian subgradient methods utilized to optimize nonsmooth functions over Stiefel manifold, under reasonable regularities of the functions. The fundamental ingredient for establishing the aforementioned convergence results is that any weakly convex function in Euclidean space admits an important property holding uniformly over Stiefel manifold which we name Riemannian subgradient inequality. We then extend our convergence results to a broader class of compact Riemannian manifolds embedded in Euclidean space. Finally, we discuss the sharpness property for robust subspace recovery and orthogonal dictionary learning, and demonstrate the established convergence performance of our algorithms on both problems via numerical simulations.

Key words — Manifold optimization, nonconvex optimization, orthogonal constraint, iteration complexity, linear convergence, robust subspace recovery, dictionary learning.

1 Introduction

In this paper, we consider the following (possibly nonsmooth) optimization problem over Stiefel manifold $\text{St}(n,r) := \{ X \in \mathbb{R}^{n \times r} : X^\top X = I_r \}$ embedded in $\mathbb{R}^{n \times r}$,

$$\begin{align*}
\text{minimize} \quad & f(X) = \frac{1}{m} \sum_{i=1}^{m} f_i(X) \\
\text{subject to} \quad & X \in \text{St}(n,r).
\end{align*}$$

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We assume each component $f_i : \mathbb{R}^{n \times r} \mapsto \mathbb{R}$ being $\tau$-weakly convex in Euclidean space\(^1\) and Lipschitz continuous with parameter $L$ on some subset of $\mathbb{R}^{n \times r}$ whose relative interior contains $\text{St}(n, r)$\(^2\). Variants of (1) are ubiquitous in many engineering problems, such as representation learning and imaging science.

1.1 Motivations

As we discuss in the following, many engineering problems can be naturally cast as separable nonsmooth optimization over Stiefel manifold. Taking $\ell^1$ norm loss as an example, one may argue that the nonsmoothness of $\ell^1$ norm can be avoided by considering its smooth variants such as Huber loss \([41, 57]\) or log cosh($\cdot$) function \([31, 62, 66]\). However, in practice nonsmooth optimization formulation is found to have several clear advantages than its smooth counterpart: (i) it better promotes the robustness of the solution against outliers \([15, 24, 47]\), and requires fewer samples for exact recovery \([5]\) than its smooth loss variant \([66]\); (ii) solving it can directly return the exact solutions \([5, 47, 79]\), while optimizing its smoothing variants only produces approximate solutions \([48, 57, 65, 66]\).

Application 1: robust subspace recovery. Fitting a linear subspace to dataset corrupted by outliers is a fundamental problem in machine learning and statistics, primarily known as robust principal component analysis (PCA) \([50]\), or robust subspace recovery (RSR) \([43]\). More specifically, given measurements $\tilde{Y}$ of the form

$$
\tilde{Y} = \begin{bmatrix} Y_{\text{inliers}} & O_{\text{outliers}} \end{bmatrix} \Gamma_{\text{permutation}} \in \mathbb{R}^{n \times m},
$$

where the columns of $Y \in \mathbb{R}^{n \times m_1}$ form inlier points spanning a $d$-dimensional subspace $S$, the columns of $O \in \mathbb{R}^{n \times m_2}$ are outlier points with no linear structure, and $\Gamma \in \mathbb{R}^{m \times m}$ is an unknown permutation, the goal is to recover the subspace $S$. It is well-known that the presence of outliers can severely affect the quality of the solutions obtained by the classical PCA approach \([50]\), when one minimizes a smooth least squares loss.

In contrast, recent works in \([43, 44, 52]\) suggest that optimizing the nonsmooth least absolute deviations (LAD) better promotes robustness of the solution against outliers,

$$
\begin{align*}
\min_{X \in \mathbb{R}^{n \times d}} & \quad f(X) := \frac{1}{m} \sum_{i=1}^{m} \left\| (I_n - XX^\top) \tilde{y}_i \right\|_2 \\
\text{subject to} & \quad X \in \text{St}(n, d),
\end{align*}
$$

(2)

so that columns of a global optimizer to (2) are expected to form an orthonormal basis of the subspace $S$. The weak convexity and local Lipschitz continuity of the objective function in (2) can be verified by following \([47, \text{Proposition 3}, \text{Proposition 4}]\). On the other hand, the authors in \([67, 78, 79]\) consider a dual form of the problem, leading to the so-called dual principal component pursuit (DPCP) formulation,

$$
\begin{align*}
\min_{X_d \in \mathbb{R}^{n \times r}} & \quad f(X) =: \frac{1}{m} \sum_{i=1}^{m} \left\| \tilde{y}_i^\top X \right\|_2 \\
\text{subject to} & \quad X \in \text{St}(n, r).
\end{align*}
$$

(3)

In contrast to the primal formulation (2), the dual version (3) is to find the orthogonal basis of $S^\perp$ (the orthogonal complement to $S$) with dimension $r = n - d$.

---

\(^1\)A function $h$ is said to be weakly convex with parameter $\tau$ if $h + \frac{\tau}{2} \left\| \cdot \right\|^2$ is convex \([68]\). $\tau = 0$ indicates $h$ is convex. Note that differentiability is not required for weak convexity.

\(^2\)Indeed, any weakly convex function is locally Lipschitz continuous. The Lipschitz parameter $L$ plays an important role in our analysis. Thus, we state this assumption explicitly.
Application 2: learning sparsely-used dictionaries. Dictionary learning (DL) aims to learn the underlying compact representation from the data \( Y = [y_1, \ldots, y_m] \in \mathbb{R}^{n \times m} \), which finds many applications in machine learning, and computer vision [3,27,51,60,74]. Mathematically speaking, the problem is to factorize the data into a dictionary \( A \) and a sparse code matrix \( S = [s_1, \ldots, s_m] \),

\[
Y \approx A \text{ dictionary sparse code}
\]

When the dictionary \( A \in \mathbb{R}^{n \times n} \) is orthogonal, the authors in [58, 66] reduce the problem to finding the sparsest vector in the row space \( S = \text{row}(Y) \) of \( Y \) based on the sparsity assumption on \( s_i \). This task can be formulated as a nonsmooth optimization problem over the sphere [5],

\[
\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{m} \| Y^\top x \|_1 = \frac{1}{m} \sum_{i=1}^m |y_i^\top x| \quad \text{subject to } x \in \text{St}(n, 1), \quad (4)
\]

leading to the first provable methods for globally solving DL with orthogonal dictionaries [5,65]. However, as the solution of (4) only returns one column of \( A \), extra refinement technique is needed to fully solve the DL problem, such as deflation [65] or repetitive independent trials [5]. Usually it makes this type of approach sensitive to noise. Recent work in [72,76] attempts to directly recover the orthogonal dictionary \( A \) by

\[
\min_{X \in \mathbb{R}^{n \times n}} f(X) := \frac{1}{m} \| Y^\top X \|_1 = \frac{1}{m} \sum_{i=1}^m \| y_i^\top X \|_1 \quad \text{subject to } X \in \text{St}(n,n), \quad (5)
\]

Moreover, it should be noted that the approach can be easily extended to handle any complete (i.e., square and invertible) dictionaries via preconditioning [66,76].

Application 3: sparse blind deconvolution with multiple inputs. Given measurements in the form

\[
y_i \text{ measurements} = a \text{ kernel } \circ \text{ sparse signal} \quad 1 \leq i \leq m,
\]

where \( a, y_i, s_i \in \mathbb{R}^n \) for all \( i \) and \( \circ \) represents the circulant convolution. The sparse blind deconvolution problem is to simultaneously recover the ground truth kernel \( a \) and the sparse signals \( \{s_i\}_{i=1}^m \), which finds many applications in computational imaging [7,61], neuroscience [26,32] and other fields. When the kernel \( a \) is invertible (here, it means the circulant matrix \( C_a \) of kernel \( a \) is invertible), the problem can be solved by optimizing [57,70],

\[
\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{m} \sum_{i=1}^m \| C_{y_i} P x \|_1 \quad \text{subject to } x \in \text{St}(n,1), \quad (6)
\]

where \( C_{y_i} \) denotes the circulant matrix of \( y_i \) and \( P \) is a preconditioning matrix that whitens the data (see [48,57,62] for more details). Although smooth variants of (6) have been considered in [48,57,62], as aforementioned, experimental results in [57] suggest that directly optimizing the nonsmooth objective (6) via Riemannian subgradient method demonstrates much superior performances.

1.2 Main contributions

For solving nonsmooth optimization problem (1) over Steifel manifold, in this work we develop a family of \textit{Riemannian subgradient methods} including Riemannian subgradient method, Riemannian incremental subgradient method, and Riemannian stochastic subgradient method (see Section 2.2).
We first show that the family of Riemannian subgradient methods globally converge to an $\varepsilon$-nearly stationary point of problem (1) within $O(\varepsilon^{-4})$ iterations\(^3\) (see Section 4). Second, when the function $f$ further satisfies the so-called sharpness property with respect to the set of weak sharp minima $\mathcal{X}$ (see Definition 3), the Riemannian subgradient and incremental subgradient methods locally converge to $\mathcal{X}$ at a linear rate (see Section 5), with a properly designed geometrically diminishing stepsize and a good initialization. We also extend the above convergence results to a wider class of compact Riemannian manifolds embedded in Euclidean space. To the best of our knowledge, this is the first convergence result with rate guarantees of Riemannian subgradient methods for optimizing nonsmooth functions over Stiefel manifold.

The guiding principle for establishing these convergence results is an algorithmically independent property that we discovered for weakly convex functions constrained over Stiefel manifold, which we term Riemannian subgradient inequality (see Section 3). This is one of main contributions of this work and could be of independent interest for other Riemannian optimization problems.

Finally, we discuss about the sharpness properties for the aforementioned applications, such as RSR and orthogonal DL. We demonstrate the established convergence results on these problems via numerical simulations (see Section 7). For reproducible research, our code can be found via the following link:

https://github.com/lixiao0982/Riemannian-subgradient-methods

Our results closed the gap between nonsmooth optimization over Euclidean space and Stiefel manifold. Moreover, we believe our results could have broad implications for understanding convergence behaviors of algorithms for solving more general nonsmooth optimization problems over a large family of Riemannian manifolds embedded in Euclidean space.

1.3 Connections with prior arts

Nonsmooth optimization in Euclidean space. It should be noted that our results over the Stiefel manifold are related to those of minimizing nonsmooth weakly convex functions in Euclidean space [12, 18–20, 22, 24, 33, 46, 47, 59]. The main algorithms for nonsmooth weakly convex optimization in Euclidean space includes subgradient type methods [18, 20, 46, 47] and proximal point type methods [22, 24], where their convergence analysis rely on a function regularity property called weakly convex inequality. However, their results can only be applied to problems with convex constraint, while the Stiefel manifold is a nonconvex constraint that is beyond the scope of these methods. In this work, when the function is weakly convex in Euclidean space but constrained over Stiefel manifold, we establish a notion called Riemannian subgradient inequality over Stiefel manifold which has similar structure to the weakly convex inequality. Therefore, we can resort to the analysis for nonsmooth weakly convex optimization in Euclidean space, and prove new convergence results for our Riemannian subgradient methods for solving nonsmooth optimization problem (1) over Steifel manifold.

Smooth optimization over Riemannian manifold. Riemannian optimization for smooth $f$ has been studied extensively and well-understood in [2, 9, 11, 38, 49]. Recently, the global sublinear convergence results for Riemannian gradient descent and Riemannian trust region are presented in [11]. Their analysis relies on the assumption that the pullbacks of objective function $f$ to the tangent spaces of manifold $\mathcal{M}$ has Lipschitz continuous gradient, which allows one to follow the analysis of gradient descent method in Euclidean space. However, these results break down and cannot be extended to the case that $f$ is nonsmooth, because gradient of the pullbacks of $f$ do not exist everywhere.

Nonsmooth optimization over Riemannian manifold. In contrast to smooth problems, nonsmooth Riemannian optimization problems are largely open [1]. In the following, we briefly review some state-of-the-art results in nonsmooth regime, explaining their limitations and connections to our results.

\(^3\)Here, $X$ is called an $\varepsilon$-nearly stationary point of problem (1) if $\Theta(X) \leq \varepsilon$, where $\Theta(X)$ is a surrogate stationary measure for problem (1); see Section 4.1 for details.
Nonsmooth Riemannian optimization with geodesic convexity. Recently, the authors in [6, 28, 30, 77] studied Riemannian subgradient method when the objective function is geodesically convex over Riemannian manifold. Thanks to the availability of a geodesic version of convex subgradient inequality, one can carry the conventional analysis of convex optimization in Euclidean space to Riemannian optimization. In particular, asymptotic convergence result is first established in [28]; later on, authors in [6, 30] established $O(k^{-\frac{2}{3}})$ global sublinear convergence rate; when $f$ is geodesically strongly convex over Riemannian manifold, the convergence rate is shown to be $O(k^{-1})$ in [77]. Unfortunately, these results are not directly useful to our problem (1), where it is constrained over a compact manifold. This is because every continuous function that is geodesically convex on a compact Riemannian manifold can only be a constant [8, Proposition 2.2].

Riemannian gradient sampling algorithms. For general nonsmooth optimization over Riemannian manifold, recent works in [10, 34, 35, 37] proposed Riemannian gradient sampling algorithms, which originate from gradient sampling algorithms for nonconvex optimization in Euclidean space [14]. As introduced in [34, 35, 37], given the current iterate $X_k$, the Riemannian gradient sampling algorithm first samples some points $\{X_k^j\}_{j=1}^J$ in the neighborhood of $X_k$ at which the objective functions $\{f(X_k^j)\}_{j=1}^J$ are differentiable and $J$ usually needs to be larger than the dimension of manifold $M$. Then to build a descent direction, it solves the following quadratic program

$$\xi_k = \arg\min_{G \in \text{conv}(W)} \|G\|^2. \tag{7}$$

Here, conv($W$) denotes the convex hull of the set $W := \{\text{grad } f(X_k), \text{grad } f(X_k^1), \cdots, \text{grad } f(X_k^J)\}$, and grad $f(X)$ is the Riemannian gradient at differentiable point $X$. The update can then be performed via classical retractions on manifold using the descent direction $\xi_k$. This type of algorithms can be potentially utilized for solving a large class of nonsmooth Riemannian optimization problems. However, the Riemannian gradient sampling algorithms are only known to converge asymptotically without rate guarantee [34, 35, 37]. On the other hand, for tackling our problem (1) with large $n$ and $r$ using this sampling algorithm, one has to sample $J$ Riemannian gradients ($J$ should be larger than the dimension of Stiefel manifold) so that solving the subproblem (7) in each iteration can be very expensive. In contrast, our result requires extra mild conditions of the function such as weak convexity and sharpness, but our proposed algorithms are proven to converge nonasymptotically with explicit rate guarantees. Also, the execution of our algorithm in each iteration only involves computation of one Riemannian subgradient which is much cheaper.

Two types of proximal point methods. Another classical approach to tackle nonsmooth Riemannian optimization is proximal point methods, whose idea is to iteratively compute the proximal mapping over Riemannian manifold [4, 21, 29]. These methods are shown to converge globally with sublinear rate, based on the so-called sufficient decrease property. However, the main issue is that each subproblem of this method is as difficult as the original problem which renders it not practical. When specialized to Stiefel manifold, this difficulty has been alleviated by some recent advances [16, 17, 39, 40]. They proposed to compute the proximal mapping over the tangent space instead of over the Stiefel manifold, which results in a linearly constrained convex subproblem and hence is much easier than the original problem. The new algorithm is also proven to converge globally with sublinear rate. Nonetheless, its subproblem still needs to be solved via an iterative algorithm. In contrast, our methods do not need to solve expensive subproblems except the computation of one Riemannian subgradient, so that the overall computational complexity is much lower.

Nonsmooth optimization over Stiefel manifold for specific problems. Finally, we close this subsection by pointing out several problem specific convergence analysis results for Riemannian subgradient method [5, 78, 79]. In particular, the authors in [5] and [78, 79] propose to solve the orthogonal DL problem (4) and the RSR problem (3) with Riemannian subgradient method, respectively. Similar to our results, they also proved local linear convergence guarantees. However, their proofs are based on the notion of Riemannian regularity condition, instead of the sharpness property studied here. We will discuss in details about the limitation of the Riemannian regularity condition in the last paragraph of Section 5 in comparison to ours. On the other hand, their analysis [5, 78, 79] critically depends on the specific model structure of each individual problem, which cannot be generalized. In contrast, we developed a more general framework
that can tackle a family of nonsmooth nonconvex optimization problems over a class of compact Riemannian submanifolds with both global and local convergence guarantees.

1.4 Notations

In this paper, $k$ only reads iteration count of iterative algorithms. We use $T_X \text{St} := \{\xi \in \mathbb{R}^{n \times r} : \xi^\top X + X^\top \xi = 0\}$ to represent the tangent space to Stiefel manifold $\text{St}(n,r)$ at point $X \in \text{St}(n,r)$. Let $\langle A, B \rangle = \text{trace}(A^\top B)$ denote the Euclidean inner product of two matrices $A, B$ with proper size. We utilize the Riemannian metric on Stiefel manifold $\text{St}(n,r)$ that is induced from the inner product in Euclidean space. That being said, we have $\langle X, Y \rangle = \text{trace}(X^\top Y)$ for any $X, Y \in T_Z \text{St}$. grad $h(X)$ denotes the Riemannian gradient of $h$ if it is differentiable at $X$. $\partial h(X)$ is the Fréchet subdifferential of $h$ at $X$ and $\overline{\partial h}(X) \in \partial h(X)$ represents any Euclidean subgradient. While $\partial_R h(X)$ denotes Riemannian subdifferential of $h$ at $X \in \text{St}(n,r)$ with respect to the Stiefel manifold and $\overline{\partial}_R h(X) \in \partial_R h(X)$ represents any Riemannian subgradient. That being said, we use notations $\overline{\nabla}$ and $\overline{\partial}_R$ to distinguish between Euclidean subgradient and Riemannian subgradient. $\mathcal{P}_C$ denotes the orthogonal projection onto closed set $C$. $h_{\lambda} : \text{St}(n,r) \mapsto \mathbb{R}$ and $P_{\lambda h} : \text{St}(n,r) \mapsto \text{St}(n,r)$ represent the Moreau envelope and proximal mapping of function $h$ over Stiefel manifold, respectively. The distance function $\text{dist}(X, C) := \inf_{Y \in C} \|X - Y\|_F$.

2 Preliminaries and Algorithms

In this section, we first introduce some basic knowledge about Riemannian optimization for manifolds embedded in Euclidean space. Based on this, we state the proposed Riemannian subgradient algorithms for optimizing the problem (1).

2.1 Optimization over Stiefel manifold

First, we introduce several fundamental elements of manifold optimization which will be used in the sequel: (i) Riemannian subgradient, (ii) first order optimality, (iii) retraction on Stiefel manifold.

**Riemannian subgradient and first order optimality condition.** Towards characterizing the first order optimality condition of problem (1), we introduce the following notion of Riemannian subdifferential which is valid for any embedded manifold in Euclidean space.

**Definition 1** (Riemannian subdifferential [36]). For a locally Lipschitz continuous function $h$ on any embedded submanifold $\mathcal{M}$ in Euclidean space, the generalized directional derivative of $h$ at $X \in \mathcal{M}$ along the direction $V$ is defined by

$$h^\circ(X; V) = \limsup_{Y \to X, t \downarrow 0} \frac{h\left(\phi^{-1}\left(\phi(Y) + t \cdot D\phi(X)[V]\right)\right) - h(\phi^{-1}(\phi(Y)))}{t},$$

where $(\phi, U)$ is a coordinate chart at $X$ and $D\phi(X)[V]$ denotes the differential of $\phi(X)$ along direction $V$.

The Riemannian subdifferential of $h$ at $X \in \mathcal{M}$, denoted as $\partial_R h(X)$, is given by

$$\partial_R h(X) = \left\{\overline{\partial}_R h(X) \in T_X \mathcal{M} : \langle \overline{\partial}_R h(X), V \rangle \leq h^\circ(X; V), \forall V \in T_X \mathcal{M}\right\}.$$ 

and the element $\overline{\partial}_R h(X) \in \partial_R h(X)$ is called Riemannian subgradient at $X \in \mathcal{M}$.

Note that generalized directional derivative $h^\circ(X; V)$ and $\partial_R h(X)$ are independent of the choice of the coordinate chart $(\phi, U)$. Since we consider the Euclidean inner product as the Riemannian metric, for any embedded submanifold $\mathcal{M}$ in Euclidean space, [75, Theorem 5.1] asserts that $\partial_R h(X) = \mathcal{P}_{T_X \mathcal{M}}(\partial h(X))$.
for any locally Lipschitz function $h$ that is regular\textsuperscript{4} at $X$ along $T_X\mathcal{M}$, where $\mathcal{P}_{T_X\mathcal{M}}$ denotes orthogonal projection onto the tangent space $T_X\mathcal{M}$. We now let $\mathcal{M} = \text{St}(n,r)$, a direct consequence of [75, Lemma 5.1] says the objective function $f$ (or each $f_i$) in (1) is regular at any $X \in \text{St}(n,r)$ along $T_X\text{St}$. Hence, the Riemannian subdifferential of the objective function $f$ in (1) is given by
\begin{equation}
\partial_R f(X) = \mathcal{P}_{T_X\text{St}}(\partial f(X)).
\end{equation}
Correspondingly, we have the Riemannian subgradient $\tilde{\nabla}_R f(X) = \mathcal{P}_{T_X\text{St}}(\tilde{\nabla} f(X))$, with $\tilde{\nabla} f(X) \in \partial f(X)$.

The first order optimality condition of problem (1) is given by [75, Theorem 4.1]
\begin{equation}
0 \in \partial_R f(X).
\end{equation}
Thus, we call $X \in \text{St}(n,r)$ a stationary point of problem (1) if $X$ satisfies (9).

**Retraction on Stiefel manifold.** Let us start with the formal definition of retraction on smooth manifold $\mathcal{M}$ embedded in Euclidean space.

**Definition 2** (Retraction, see, e.g., Definition 4.1.1 in [2]). A retraction operator, $\text{Retr}(\cdot) : T\text{St} \rightarrow \mathcal{M}$ on differentiable manifold $\mathcal{M}$ embedded in the Euclidean space, is a smooth mapping from the tangent bundle $T\text{St}$ onto $\mathcal{M}$ such that (here $\text{Retr}_X$ denotes the restriction of $\text{Retr}$ onto $T_X\mathcal{M}$)
\begin{enumerate}
\item $\text{Retr}_X(0) = X$ for all $X \in \mathcal{M}$, where $0$ denotes the origin of $T_X\mathcal{M}$ and
\item $D \text{Retr}_X(0) = \text{Id}$, where $D \text{Retr}_X$ is the differential of the retraction and $\text{Id}$ is the identity mapping over $T_X\mathcal{M}$.
\end{enumerate}

From the inverse function theorem, we know that retraction is a local diffeomorphism. Thus, there exists a neighborhood $\mathcal{U}$ such that $(\text{Retr}_X^{-1}, \mathcal{U})$ is a coordinate chart at $X$, where $\text{Retr}_X^{-1}$ is the inverse of $\text{Retr}_X$ and $\text{Retr}_X^{-1}$ is also smooth in $\mathcal{U}$. Define the pullback function
\begin{equation}
\hat{h}(\xi) := h(\text{Retr}_X(\xi)), \quad \forall \xi \in T_X\mathcal{M},
\end{equation}
which lifts the function from $\mathcal{M}$ to tangent space. Then, the definition of generalized directional derivative in Definition 1 gives $h^\circ(X;V) = \hat{h}^\circ(0;V)$, where
\begin{equation}
\hat{h}^\circ(0;V) = \limsup_{\xi \rightarrow 0, t \downarrow 0} \frac{\hat{h}(\xi + tV) - \hat{h}(\xi)}{t}.
\end{equation}
And we also have $\partial_R h(X) = \hat{h}(0)$, where $\partial h(0) = \{\tilde{\nabla}_R h(0) \in T_X\mathcal{M} : \langle \tilde{\nabla}_R h(0), V \rangle \leq h^\circ(0;V), \forall V \in T_X\mathcal{M}\}$. We note that the Riemannian subdifferential of $h$ on manifold $\mathcal{M}$ is defined in the way $\partial_R h(X) = \hat{h}(0)$ in [35].

There are mainly four ways to perform retraction on Stiefel manifold, including exponential mapping [25], $QR$ decomposition, Cayley transformation [73] and polar decomposition based methods. The authors in [17] pointed out that among which the polar decomposition based retraction is the most efficient one among the other three in terms of computational complexity. Therefore, in this work we focus on polar decomposition based retraction on Stiefel manifold\textsuperscript{5}, which can be implemented as
\begin{equation}
\text{Retr}_X(\xi) = (X + \xi)(I_r + \xi^\top \xi)^{-\frac{1}{2}}.
\end{equation}
It is easy to verify that $\text{Retr}_X(\xi) \in \text{St}(n,r)$. Let $X^+ = X + \xi$. In the following lemma, we show that the polar decomposition based retraction can be reviewed as projecting $X^+$ from tangent bundle $T\text{St}$ to the Stiefel manifold and this projection is 1-Lipschitz and hence non-expansive.

\textsuperscript{4}The notion ‘regular’ is defined in [71, Definition 5.1].

\textsuperscript{5}Actually, our convergence results also apply to the other three retractions which we will discuss in Section 6.
Lemma 1. Suppose $X \in \text{St}(n,r)$ and $\xi \in T_X \text{St}$. Consider the polar decomposition based retraction (11) and denote $X^+ = X + \xi$. We then have $\text{Retr}_X(\xi) = X^+ (X^+ X^+)^{-\frac{1}{2}} = P_{\text{St}}(X^+)$ and
\[
\|\text{Retr}_X(\xi) - \bar{X}\|_F \leq \|X^+ - \bar{X}\|_F = \|X + \xi - \bar{X}\|_F.
\]
for any $\bar{X} \in \text{St}(n,r)$.

Proof of Lemma 1. It is well known that the convex hull of Stiefel manifold $\text{St}(n,r)$ is given by $H(n,r) := \{Y \in \mathbb{R}^{n \times r} : \|Y\|_2 \leq 1\}$ where $\|Y\|_2$ represents the spectral norm (i.e. the largest singular value) of $Y$. Thus, we have $\bar{X} \in H(n,r)$. The remaining task is to show that $\text{Retr}_X(\xi) = P_{\text{St}}(X^+) = P_H(X^+)$. Towards that end, denote the SVD of $X^+ = UV^\top$. Since $\xi \in T_X \text{St}$, we have $X^+ X^+ = I_r + \xi^\top \xi$ which directly implies that $\sigma_k(X^+) \geq 1, \forall k \in \{1, \cdots, r\}$. This, together with the Hoffman-Wielandt Theorem for singular values, implies that $P_{\text{St}}(X^+) = P_H(X^+) = UV^\top$. One the other hand, one can recognize that $\text{Retr}_X(\xi) = X^+ (X^+ X^+)^{-\frac{1}{2}} = UV^\top$, hence $\|\text{Retr}_X(\xi) - \bar{X}\|_F = \|P_H(X^+) - P_H(\bar{X})\|_F$. Finally, the non-expansiveness property of convex projections ends the proof.

2.2 A family of Riemannian subgradient methods

In this section, we introduce Riemannian subgradient method, Riemannian incremental subgradient method, and Riemannian stochastic subgradient method for solving problem (1).

2.2.1 Riemannian subgradient method

We begin with revisiting the Riemannian gradient descent for smooth optimization over Stiefel manifold. Suppose $h$ is a smooth function and consider
\[
\text{minimize} \quad h(X).
\]
For solving the problem above, a generic Riemannian gradient descent method is [2, Section 4.2]
\[
X_{k+1} = \text{Retr}_{X_k}(\xi_k), \quad \text{with} \quad \xi_k = -\gamma_k \text{grad} h(X_k),
\]
where $\text{grad} h(X_k)$ denotes the Riemannian gradient of $h$ at $X_k$ and $\gamma_k > 0$ is the stepsize and $\text{Retr}_{X_k}(\cdot)$ is the retraction on Stiefel manifold which can be implemented according to (11).

In this paper, we consider problem (1) in which the objective function $f$ is nonsmooth. For tackling it, we need to generalize the above Riemannian gradient descent to the following Riemannian subgradient method,
\[
X_{k+1} = \text{Retr}_{X_k}(\xi_k), \quad \text{with} \quad \xi_k = -\gamma_k \nabla_R f(X_k),
\]
(12)
with the only difference being replacing the Riemannian gradient $\text{grad} h(X_k)$ by the Riemannian subgradient $\nabla_R f(X_k)$ which we introduced in Section 2.1. Recall from (8), one has $\nabla_R f(X) = P_{T_X \text{St}}(\nabla f(X))$ with $\nabla f(X) \in \partial f(X)$ being the subgradient of $f$ at $X$ and projecting any $B \in \mathbb{R}^{n \times r}$ onto $T_X \text{St}$ can be computed as $P_{T_X \text{St}}(B) = B - \frac{1}{2} X (B^\top X + X^\top B), [2, \text{Example 3.6.2}].$

Intuitively, Riemannian subgradient method first move $X_k$ along the negative Riemannian subgradient direction, and then project the resultant point back to Stiefel manifold to obtain a new iterate $X_{k+1}$; see Figure 1 for an illustration.

2.2.2 Riemannian incremental and stochastic subgradient methods

We now introduces two variants to the Riemannian subgradient method in (12) by utilizing the finite sum structure in problem (1). For a separable function $f = \frac{1}{m} \sum_{i=1}^m f_i$ with very large $m$ like modern machine learning tasks, it may be intractable to evaluate its full Riemannian subgradient. Therefore, in this situation
it is more desirable to have algorithms that use fewer samples but having similar convergence behaviors to the Riemannian subgradient method. This comes to Riemannian incremental and stochastic subgradient methods that we will introduce below.

In these two algorithms, updating $X_k$ to $X_{k+1}$ generates $m$ inner iterates $X_{k,1}, \ldots, X_{k,i}, \ldots, X_{k,m}$. The algorithms start with

$$X_{k,0} = X_k,$$

the $X_{k,i}$ is updated through

$$X_{k,i} = \text{Retr}_{X_{k,i-1}}(\xi_{k,i-1}), \quad \text{with} \quad \xi_{k,i-1} = -\gamma_k \tilde{\nabla}_R f_{\ell_i}(X_{k,i-1}),$$

with $f_{\ell_i}$ selected from $\{f_1, \ldots, f_m\}$ according to certain rule. After $m$ inner iterations, the next iterate $X_{k+1}$ is obtained by

$$X_{k+1} = X_{k,m}.$$

The difference between incremental and stochastic methods lies in the selection of the component function $f_{\ell_i}$. In particular, we have

- **Riemannian incremental subgradient method** picks the component function $f_{\ell_i}$ sequentially from $f_1$ to $f_m$. In other words, $\xi_{k,i-1} = -\gamma_k \tilde{\nabla}_R f_{\ell_i}(X_{k,i-1})$;

- **Riemannian stochastic subgradient method** picks the component function $f_{\ell_i}$ independently and uniformly from $\{f_1, \ldots, f_m\}$ in each inner iteration (14). That is, $\xi_{k,i-1} = -\gamma_k \tilde{\nabla}_R f_{\ell_i}(X_{k,i-1})$ with $\ell_i \sim \text{i.i.d. Uniform} (\{1, \ldots, m\})$.

### 3 Riemannian Subgradient Inequality over Stiefel Manifold

In this section, we will present an useful inequality about Riemannian subgradient with respect to Stiefel manifold for locally Lipschitz continuous functions in Euclidean space. The function class we consider in this section can either be convex or $\tau$-weakly convex (see Footnote 1). This inequality plays an fundamental role in our subsequent convergence analysis for our Riemannian subgradient methods for solving problem (1).

The main motivation for showing the Riemannian subgradient inequality in Theorem 1 is from the convex (weakly convex) inequality for convex (weakly convex) function in Euclidean space which is the core ingredient of the convergence analysis of subgradient methods. Let us take the weakly convex functions in Euclidean space as an example. If $h$ is $\tau$-weakly convex, then equivalently we have the following weakly convex inequality [68, Proposition 4.8],

$$h(Y) \geq h(X) + \left\langle \tilde{\nabla} h(X), Y - X \right\rangle - \frac{\tau}{2} \|Y - X\|^2_F, \quad \forall \tilde{\nabla} h(X) \in \partial h(X),$$

(16)
for all \( X, Y \in \mathbb{R}^{n \times r} \). Note that (16) holds uniformly in the whole Euclidean space. This inequality has been extensively used in [18–20, 46, 47] to show the convergence of subgradient methods for weakly convex optimization in Euclidean space. Thus, it motivates us to show a similar form of inequality involving Riemannian subgradient.

To begin with, we give some intuitions. We consider the pullback function \( \hat{h}(\xi) \) defined in (10) with \( h \) being convex and Lipschitz continuous. Note that \( \hat{h}(\cdot) \) is the composition of convex function \( h \) and a smooth mapping \( \text{Retr}_X(\cdot) \) and is defined on the vector space \( T_X \text{St} \), it follows that \( \hat{h}(\xi) \) satisfies the following inequality [23, Lemma 4.2]

\[
\hat{h}(\xi) \geq h(0) + \langle \partial h(0), \xi \rangle - \frac{\tau_0}{2} \|\xi\|_F^2,
\]

where \( \tau_0 > 0 \) is a constant. Note that \( \partial h(0) \) is a set, we provisionally abuse \( \langle \partial h(0), \xi \rangle \) to denote the inner product between any element of \( \partial h(0) \) and \( \xi \). From the Definition 2, we have \( \lim_{t \to 0^+} \frac{\text{Retr}_X(t\xi) - (X + t\xi)}{t} = 0 \), which implies there exist small enough \( t > 0 \) and \( \epsilon > 0 \) such that

\[
\| \text{Retr}_X(t\xi) - X \|_F \geq (1 - \epsilon)t\|\xi\|_F.
\]

It follows that (17) can be deduced to

\[
\begin{align*}
\hat{h}(\text{Retr}_X(t\xi)) &\geq h(X) + \langle \partial_R h(X), t\xi \rangle - \frac{\tau_0}{2(1 - \epsilon)^2} \| \text{Retr}_X(t\xi) - X \|_F^2 \\
&= h(X) + \langle \partial_R h(X), \text{Retr}_X(t\xi) - X \rangle - \frac{\tau_0}{2(1 - \epsilon)^2} \| \text{Retr}_X(t\xi) - X \|_F^2 \\
&\quad + \langle \partial_R h(X), t\xi + X - \text{Retr}_X(t\xi) \rangle \\
&\geq h(X) + \langle \partial_R h(X), \text{Retr}_X(t\xi) - X \rangle - \frac{c}{2} \| \text{Retr}_X(t\xi) - X \|_F^2,
\end{align*}
\]

for small enough \( t \) and some constant \( c > 0 \), where the last inequality comes from Lipschitz continuity of \( h \) and the second order boundedness of retraction on compact Riemannian manifold \( \| t\xi + X - \text{Retr}_X(t\xi) \|_F \leq M\|\xi\|_F^2 \) for some constant \( M > 0 \) [11]. However, the inequality (18) can hardly be used for analyzing algorithms, since it only holds in a small enough neighborhood of \( X \). Moreover, the radius of neighborhood is affected by the retraction used.

Fortunately, by combining with the specific structure of Stiefel manifold, an uniform Riemannian subgradient inequality holds as follows.

**Theorem 1.** Suppose function \( h : \mathbb{R}^{n \times r} \to \mathbb{R} \) is Lipschitz continuous with parameter \( L \) on a subset of \( \mathbb{R}^{n \times r} \) whose relative interior contains the Stiefel manifold \( \text{St}(n, r) \).

(a) If \( h \) is convex, we have:

\[
h(Y) \geq h(X) + \langle \nabla_R h(X), Y - X \rangle - \frac{L}{2} \| Y - X \|_F^2,
\]

for all \( X, Y \in \text{St}(n, r) \), and \( \nabla_R h(X) \in \partial_R h(X) \).

(b) If \( h \) is \( \tau \)-weakly convex, we have:

\[
h(Y) \geq h(X) + \langle \nabla_R h(X), Y - X \rangle - \frac{\tau + L}{2} \| Y - X \|_F^2,
\]

for all \( X, Y \in \text{St}(n, r) \), and \( \nabla_R h(X) \in \partial_R h(X) \).

**Proof of Theorem 1.** The proof of argument (a) can be derived directly from that of argument (b), hence we only provide the detailed proof for argument (b). We start with the weak convexity of \( h \) in Euclidean
where the last equality is due to (25). Combing (23), (24) and (26) and plugging them into (21) gives

\[ h(Y) \geq h(X) + \left< \nabla h(X), Y - X \right> - \frac{\tau}{2} \|Y - X\|_F^2, \quad \forall \nabla h(X) \in \partial h(X). \]

for all \( X, Y \in \mathbb{R}^{n \times r} \). Now, taking two arbitrary points \( X, Y \in \text{St}(n, r) \) and plugging them into the above inequality, one has

\[ h(Y) \geq h(X) + \left< \nabla h(X), Y - X \right> - \frac{\tau}{2} \|Y - X\|_F^2 = h(X) + \left< \mathcal{P}_{\text{St}X}(\nabla h(X)) + \mathcal{P}_{\text{St}X}^\perp(\nabla h(X)), Y - X \right> - \frac{\tau}{2} \|Y - X\|_F^2, \]

where [2, Example 3.6.2]

\[ \mathcal{P}_{\text{St}X}^\perp(B) = \frac{1}{2} X \left( B^\top + X^\top B \right), \quad \text{for all } B \in \mathbb{R}^{n \times r}. \]

We can proceed as

\[
\begin{align*}
\left< \mathcal{P}_{\text{St}X}^\perp(\nabla h(X)), Y - X \right> &= \left< \mathcal{P}_{\text{St}X}^\perp(\nabla h(X)), \mathcal{P}_{\text{St}X}(Y - X) + \mathcal{P}_{\text{St}X}(Y - X) \right> \\
&= \left< \mathcal{P}_{\text{St}X}^\perp(\nabla h(X)), \mathcal{P}_{\text{St}X}(Y - X) \right> \\
&= \left< \nabla h(X), \mathcal{P}_{\text{St}X}(Y - X) \right> \\
&= \frac{1}{2} \left< \nabla h(X), X (Y^\top X + X^\top Y - 2I_r) \right> \\
&\geq -\frac{1}{2} \|\nabla h(X)\|_F \|Y^\top X + X^\top Y - 2I_r\|_F \\
&\geq -\frac{L}{2} \|Y^\top X + X^\top Y - 2I_r\|_F,
\end{align*}
\]

where the last equality comes from (22), both the second last two lines utilized the fact that \( X \in \text{St}(n, r) \), the last line is because of the Lipschitz continuity of \( h \). Denote \( Q = Y^\top X + X^\top Y \in \mathbb{R}^{r \times r} \) which is a symmetric matrix. Hence, \( Q \) must admit an eigenvalue decomposition of the form \( Q = U\Lambda U^\top \) with \( U \in \mathbb{R}^{r \times r} \) being a real orthogonal matrix and \( \Lambda \in \mathbb{R}^{r \times r} \) is a real diagonal matrix. Then, one can see that

\[ \|Y^\top X + X^\top Y - 2I_r\|_F = \sqrt{\sum_{i=1}^r (2 - \lambda_i)^2} \leq \sum_{i=1}^r |2 - \lambda_i|. \]

According to variational characterization of the eigenvalues of \( Q \), for the largest eigenvalue, we have

\[ \lambda_{\text{max}} := \max_{\|v\|_1 = 1} \{ v^\top Qv = (Yv)^\top Xv + (Xv)^\top Yv : v \in \mathbb{R}^r \} \leq 2, \]

by Cauchy-Schwarz inequality and the fact that \( X, Y \in \text{St}(n, r) \). Hence, we have

\[ \|X - Y\|_F^2 = 2r - 2 \text{trace}(X^\top Y) = \sum_{i=1}^r (2 - \lambda_i) = \sum_{i=1}^r |2 - \lambda_i|. \]

where the last equality is due to (25). Combing (23), (24) and (26) and plugging them into (21) gives

\[ h(Y) \geq h(X) + \left< \mathcal{P}_{\text{St}X}(\nabla h(X)), Y - X \right> - \frac{\tau + L}{2} \|Y - X\|_F^2. \]

Finally, recognizing \( X, Y \in \text{St}(n, r) \) and \( \nabla h(X) \in \partial h(X) \) are taken arbitrarily and the fact that \( \partial h(X) = \mathcal{P}_{\text{St}X}(\partial h(X)) \) (see (8)) ends the proof. \( \square \)
The Riemannian subgradient inequality in Theorem 1 has a similar form to (16) but with differences in: 1) It involves the Riemannian subgradient rather than the usual Euclidean subgradient. 2) Inequalities (19)-(20) are only valid uniformly over the Stiefel manifold. 3) As a price paid for the Stiefel manifold constraint, an extra compensation term $\frac{L_2}{2} \| Y - X \|^2_F$ is needed.

Notice that the implementation of our Riemannian subgradient methods contains Riemannian subgradient and all the iterates generated by the algorithms will stay on the Stiefel manifold due to the retraction procedure. Thus, it is natural to ask if the Riemannian subgradient inequality in Theorem 1 can be used to develop the convergence analysis of Riemannian subgradient methods for tackling problem (1), as it has similar structure to the weakly convex inequality (16) which serves as the fundamental basis for convergence analysis of subgradient methods for weakly convex optimization in Euclidean space. Fortunately, the answer is affirmative. We will show in the subsequent sections that the Riemannian subgradient inequality plays a similar role to the weakly convex inequality and it connects the analysis of Riemannian subgradient methods with their Euclidean counterparts. In particular, equipped with Theorem 1, we can obtain the iteration complexity of our Riemannian subgradient methods for addressing problem (1). Moreover, if the problem (1) further has the sharpness property defined in Definition 3, local linear convergence can be achieved for our algorithms using a geometrically diminishing stepsize, provided a proper initialization.

In Section 6, we will extend the Riemannian subgradient inequality to a class of compact Riemannian manifolds embedded in Euclidean space.

4 Global Convergence

In this section, we study the iteration complexity of Riemannian subgradient methods for solving problem (1). Our analysis in this section relies on the Riemannian subgradient inequality in Theorem 1.

4.1 Surrogate stationary measure

In classical nonsmooth convex optimization, the iteration complexity is presented in terms of functional suboptimality gap $f(X_k) - \min f$; see, e.g., [55, Theorem 3.2.2], [53, Proposition 2.3]. More relevantly, when each $f_i$ in problem (1) is smooth, one can derive the iteration complexity based on the continuous stationary measure $\| \nabla f(X_k) \|_F$ with $\nabla f(X)$ being the Riemannian gradient of $f$ [11].

Recall that our goal in this paper is to tackle the nonconvex and nonsmooth optimization problem (1). Neither the functional suboptimality gap $f(X_k) - \min f$ (due to nonconvexity) nor the smallest norm of Riemannian subgradients $\text{dist}(0, \partial R f(X_k))$ (due to nonsmoothness) can be chosen as an appropriate stationary measure. Therefore, towards obtaining iteration complexity of Riemannian subgradient methods, we have to choose a surrogate stationary measure for tracking the progress of the algorithms.

Our choice of surrogate stationary measure and the corresponding arguments for showing global convergence are motivated by the recent seminal papers [18,23] in which the authors focus on minimizing nonsmooth weakly convex functions in Euclidean space. They propose to utilize the gradient of the Moreau envelope of the weakly convex function as the surrogate stationary measure. Then, in [18] they conduct analysis to characterize the iteration complexity for stochastic subgradient method based on this surrogate stationary measure. However, in the presence of Stiefel manifold constraint, the smoothing feature of Moreau envelope does not hold any longer, i.e., the Moreau envelope of $f$ in (1) over Stiefel manifold is still not differentiable. Fortunately, we will discuss in the following that the principle of finding a surrogate stationary measure remains valid.

To start, we define the Moreau envelope and proximal mapping of problem (1) for some $\lambda > 0$ as [29]

$$
\begin{align*}
    f_\lambda(X) &= \min_{Y \in \text{St}(n,r)} f(Y) + \frac{1}{2\lambda} \| Y - X \|^2_F, \quad X \in \text{St}(n,r), \\
    P_\lambda f(X) &= \arg\min_{Y \in \text{St}(n,r)} f(Y) + \frac{1}{2\lambda} \| Y - X \|^2_F, \quad X \in \text{St}(n,r).
\end{align*}
$$

(27)

Note that $f_\lambda(X)$ and $P_\lambda f(X)$ are well defined since optimizing continuous functions over compact set always has solution. We have changed the geodesic distance in [29] to the Euclidean distance in (27) in order to be compatible with our later analysis.
where

According to (27), we have

The proof of Proposition 1.

Let \( \{x_k\} \) be the sequence generated by Riemannian incremental subgradient method (14) for solving problem (1) with arbitrary initialization. For any \( \lambda < \frac{1}{2(L+\tau)} \) in (27), we have

\[
m\gamma_k \Theta^2(x_k) \leq \frac{2(f_\lambda(x_k) - f_\lambda(x_{k+1})) + m^2\gamma_2^2L^2/\lambda + C(m)\gamma_2^2L^2(L + \tau)/\lambda}{2\lambda (\frac{1}{2\lambda} - (L + \tau))}, \quad \forall \; k \geq 0.
\]

where \( C(m) = \frac{2}{3}m(m-1)m(2m-1) \).

Proof of Proposition 1. According to (27), we have

\[
f_\lambda(x_{k+1}) = f(P_\lambda f(x_{k+1})) + \frac{1}{2\lambda} \|P_\lambda f(x_{k+1}) - x_{k+1}\|^2_F
\]

the inequality is from the optimality of \( P_\lambda f(x_{k+1}) \) and the fact that \( P_\lambda f(x_k) \in St(n,r) \). According to the update of Riemannian incremental subgradient method, we claim that, for any \( l \in \{1, \cdots, m\} \), one has

\[
\|P_\lambda f(x_k) - x_{k,l}\|^2_F \leq \|x_k - P_\lambda f(x_k)\|^2_F - 2\gamma_k \sum_{i=1}^l (f_i(x_{k,i-1}) - f_i(P_\lambda f(x_k)))
\]

\[+ \gamma_k(L + \tau) \sum_{i=1}^l \|x_{k,i-1} - P_\lambda f(x_k)\|^2_F + l\gamma_2^2L^2.\]

We conduct an induction to establish this claim. We first verify it for \( l = 1 \),

\[
\|P_\lambda f(x_k) - x_{k,1}\|^2_F \leq \|x_k + \xi_k - P_\lambda f(x_k)\|^2_F
\]

\[\leq \|x_k - P_\lambda f(x_k)\|^2_F - 2\gamma_k (f_1(x_k) - f_1(P_\lambda f(x_k)))
\]

\[+ \gamma_k(L + \tau) \|x_k - P_\lambda f(x_k)\|^2_F + \gamma_2^2L^2.\]

\[\text{In nonsmooth Riemannian optimization, surrogate stationary measure such as the norm of the search direction computed by Riemannian proximal gradient method is previously used for convergence rate analysis in [17,39].} \]
where we have used Lemma 1 and the fact that $X_{k,1} = \text{Retr}_{X_k}(\xi_k)$ in the first inequality, while in the second inequality we have applied Theorem 1 and the fact that $\left\| \nabla R f_i (X_{k,i-1}) \right\|_F \leq L$. We now suppose (30) is true for $l = j$. The verification for $l = j + 1$ follows from the same steps in (31). The claim (30) is established. Invoking $l = m$ in (30) and plugging it into (29) and rearranging yields

$$f_\lambda (X_{k+1}) \leq f_\lambda (X_k) + \frac{\gamma_k}{\lambda} \sum_{i=1}^m (f_i (P_{\lambda f} (X_k)) - f_i (X_{k,i-1}))$$

$$+ \frac{\gamma_k (L + \tau)}{2\lambda} \sum_{i=1}^m \|X_{k,i-1} - P_{\lambda f} (X_k)\|_F^2 + \frac{m\gamma_k^2 L^2}{2\lambda},$$

(32)
in which we utilized the relation that $f_\lambda (X_k) = f (P_{\lambda f} (X_k)) + \frac{1}{2\lambda} \|X_k - P_{\lambda f} (X_k)\|_F^2$ (since $X_k \in \text{St}(n,r)$).

To proceed, we first claim that

$$\|X_{k,i-1} - X_k\|_F \leq (i - 1)\gamma_k L,$$

(33)

for all $i \in \{1, \ldots , m\}$. When $i = 1$, it holds trivially. We now suppose this claim is true for $i = j$, it remains to check the case when $i = j + 1$. To this end, we have

$$\|X_{k,j} - X_k\|_F \leq \|X_{k,j-1} + \xi_{k,j-1} - X_k\|_F$$

$$\leq j\gamma_k L,$$

where we have used the Lemma 1 and the fact that $X_{k,j} = \text{Retr}_{X_{k,j-1}} (\xi_{k,j-1})$ in the first inequality.

With (33), one can see

$$f_i (P_{\lambda f} (X_k)) - f_i (X_{k,i-1}) = f_i (P_{\lambda f} (X_k)) - f_i (X_k) + f_i (X_k) - f_i (X_{k,i-1})$$

$$\leq (i - 1)\gamma_k L^2 + f_i (P_{\lambda f} (X_k)) - f_i (X_k),$$

(34)

and

$$\|X_{k,i-1} - P_{\lambda f} (X_k)\|_F^2 = \|X_{k,i-1} - X_k + X_k - P_{\lambda f} (X_k)\|_F^2$$

$$\leq 2(i - 1)^2 \gamma_k L^2 + 2 \|X_k - P_{\lambda f} (X_k)\|_F^2,$$

(35)

Thus, plugging (34) and (35) into (32) yields

$$f_\lambda (X_{k+1}) \leq f_\lambda (X_k) + \frac{m\gamma_k}{\lambda} (f (P_{\lambda f} (X_k)) - f (X_k))$$

$$+ \frac{m\gamma_k (L + \tau)}{\lambda} \|X_k - P_{\lambda f} (X_k)\|_F^2 + \frac{m^2 \gamma_k^2 L^2 + \frac{1}{3}(m-1)m(2m-1)\gamma_k^2 L^2 (L + \tau)}{2\lambda},$$

(36)

By the definition of Moreau envelope and proximal mapping in (27), we have

$$- \left[ f(X_k) - f (P_{\lambda f} (X_k)) - (L + \tau) \|X_k - P_{\lambda f} (X_k)\|_F^2 \right]$$

$$= - \left[ f(X_k) - \left( f (P_{\lambda f} (X_k)) + \frac{1}{2\lambda} \|X_k - P_{\lambda f} (X_k)\|_F^2 \right) + \left( \frac{1}{2\lambda} - (L + \tau) \right) \|X_k - P_{\lambda f} (X_k)\|_F^2 \right]$$

$$\leq - \left( \frac{1}{2\lambda} - (L + \tau) \right) \|X_k - P_{\lambda f} (X_k)\|_F^2,$$

(37)

the last inequality is due to $f_\lambda (X_k) = f (P_{\lambda f} (X_k)) + \frac{1}{2\lambda} \|X_k - P_{\lambda f} (X_k)\|_F^2$ (since $X_k \in \text{St}(n,r)$) and $f_\lambda (X_k) \leq f (X_k)$. Invoking (37) into (36) provides

$$\frac{m\gamma_k}{\lambda} \left( \frac{1}{2\lambda} - (L + \tau) \right) \|P_{\lambda f} (X_k) - X_k\|_F^2 \leq f_\lambda (X_k) - f_\lambda (X_{k+1}) + \frac{m^2 \gamma_k^2 L^2 + C(m)\gamma_k^3 L^2 (L + \tau)}{2\lambda},$$

(38)

where $C(m) = \frac{1}{2}(m-1)m(2m-1)$. Recognizing $\Theta (X_k) = \lambda^{-1} \|P_{\lambda f} (X_k) - X_k\|_F$ in (28) and rearranging this recursion yields the desired result.

\[ \square \]
Based on Proposition 1, we now provide the iteration complexity for Riemannian subgradient and incremental subgradient methods.

**Theorem 2.** Let \( \{X_k\} \) be the sequence generated by Riemannian incremental subgradient method (14) for solving problem (1) with arbitrary initialization. For any \( \lambda < \frac{1}{\tau(L+\tau)} \) in (27), we have:

(a) If one chooses the constant stepsize \( \gamma_k = \frac{1}{m\sqrt{1+T}} \) with \( T \) being the total number of iterations, then
\[
\min_{0 \leq k \leq T} \Theta^2(X_k) \leq \frac{2(f_{\lambda}(X_0) - \min f_{\lambda}) + \frac{L^2}{\lambda} + \frac{L^2(L+\tau)}{\lambda m^2} C(m)}{2\lambda(\frac{1}{\lambda^2} - (L + \tau)) \sqrt{T + 1}}.
\]

(b) If the stepsize \( \gamma_k = \frac{1}{m\sqrt{k+1}} \) is diminishing along \( k \), then
\[
\min_{0 \leq k \leq T} \Theta^2(X_k) \leq \frac{2(f_{\lambda}(X_0) - \min f_{\lambda}) + \left(\frac{L^2}{\lambda} + \frac{L^2(L+\tau)}{\lambda m^2} C(m)\right)(\ln(T + 1) + 1)}{2\lambda(\frac{1}{\lambda^2} - (L + \tau)) \sqrt{T + 1}}.
\]

where \( C(m) = \frac{1}{2}(m-1)m(2m-1) \), \( L \) is the Lipschitz continuity parameter, \( \tau \) is the weakly convex parameter, \( f_{\lambda}(X) \) is the Moreau envelope of problem (1) defined in (27), and \( \Theta(X) \) is the surrogate stationary measure defined in (28).

**Proof of Theorem 2.** Unrolling the recursion obtained in Proposition 1 from \( k = 0 \) to \( k = T \) gives
\[
\min_{0 \leq k \leq T} \Theta^2(X_k) \leq \frac{2(f_{\lambda}(X_0) - \min f_{\lambda}) + \frac{L^2}{\lambda} m^2 \sum_{k=0}^{T} \gamma_k^2 + \frac{L^2(L+\tau)}{\lambda} C(m) \sum_{k=0}^{T} \gamma_k^3}{2\lambda(\frac{1}{\lambda^2} - (L + \tau)) m \sum_{k=0}^{T} \gamma_k},
\]
invoking the constant stepsize \( \gamma_k = \frac{1}{m\sqrt{1+T}} \) into the above inequality provides the desired result in argument (a). Invoking the diminishing stepsize \( \gamma_k = \frac{1}{m\sqrt{k+1}} \) into the above inequality, together with \( \sum_{k=0}^{T} \frac{1}{\sqrt{k+1}} > \sqrt{T + 1} \) and \( \sum_{k=0}^{T} \frac{1}{k+1} < \ln(T + 1) + 1 \), gives the desired result in argument (b). \( \Box \)

We notice that for Riemannian subgradient method, the denominator of recursion in Proposition 1 is \( 2\lambda(\frac{1}{\lambda^2} - (L + \tau)) \). Thus, the condition \( \lambda < \frac{1}{\tau(L+\tau)} \) in Theorem 2 can be relaxed as \( \lambda < \frac{1}{\tau} \).

As can be observed from Theorem 2, if one set \( \lambda = \frac{1}{4(L+\tau)} \), then the convergence rate for constant stepsize case \( \gamma_k = \frac{1}{m\sqrt{1+T}} \) is
\[
\min_{0 \leq k \leq T} \Theta(X_k) \leq \frac{2\sqrt{(f_{\lambda}(X_0) - \min f_{\lambda}) + 2L^2(1 + L + \tau)}}{(T + 1)^{\frac{1}{4}}}
\]
which implies \( O(k^{-\frac{1}{4}}) \) sublinear rate of convergence. In other words, to achieve an \( \varepsilon \)-nearly stationary point, i.e., \( \Theta(X_k) \leq \varepsilon \), one needs at most \( O(\varepsilon^{-4}) \) iterations.

### 4.3 Nonsmooth stochastic optimization over Stiefel manifold

In this subsection, we provide global sublinear convergence result for Riemannian stochastic subgradient method. Instead of considering the finite sum structure in problem (1), we turn to work on the following more general stochastic optimization over Stiefel manifold,

\[
\minimize_{X \in \mathbb{R}^{n \times r}} f(X) = E_{\zeta \sim D}[g(X, \zeta)]
\]

subject to \( X \in \text{St}(n, r) \).
each \(g(X, \zeta)\) is \(\tau\)-weakly convex. When the distribution \(D\) contains \(m\) discrete data samples and each data sample contributes equally to \(f\), the stochastic optimization problem (40) reduces to (1). To execute Riemannian stochastic subgradient method for (40), we impose the following standard assumptions on the Riemannian stochastic subgradient oracle; see assumptions (A1), (A2), Equation (2.5) in [54]:

(a) It is possible to generate i.i.d. samples \(\zeta_0, \zeta_1, \cdots \sim D\).

(b) For an input \((X, \zeta)\), there is an oracle which returns Riemannian stochastic subgradient \(\nabla_R g(X, \zeta)\) such that \(E_{\zeta \sim D}[\nabla_R g(X, \zeta)] \in \partial_R f(X)\) for any \(X \in \text{St}(n, r)\).

(c) There exists an \(L > 0\) such that \(E_{\zeta \sim D}\left[ \big\| \nabla_R g(X, \zeta) \big\|_F^2 \right] \leq L^2\) for any \(X \in \text{St}(n, r)\).

At \(k + 1\)-th iteration, the Riemannian stochastic subgradient oracle generates a sample \(\zeta_k\) that is independent of \(\zeta_0, \cdots, \zeta_{k-1}, X_0, \cdots, X_k\) and a Riemannian stochastic subgradient \(\nabla_R g(X_k, \zeta_k)\). By replacing \(f_t\) by \(g(\cdot, \zeta_k)\) in (14), like (12), the algorithm is defined as

\[X_{k+1} = \text{Retr}_X(\xi_k),\] (41)

where the search direction \(\xi_k = -\gamma_k \nabla_R g(X_k, \zeta_k)\) with stepsize \(\gamma_k > 0\).

**Proposition 2.** Let \(\{X_k\}\) be the sequence generated by Riemannian stochastic subgradient method (41) for solving problem (40) with arbitrary initialization. For any \(\lambda < \frac{1}{L + \tau}\) in (27), we have

\[\gamma_k E[\Theta^2(X_k)] \leq \frac{2\left(E[f_k(X_k)] - E[f_k(X_{k+1})]\right)}{\lambda} + \frac{\gamma_k^2 L^2/\lambda}{2\lambda}, \quad \forall \, k \geq 0.\]

**Proof of Proposition 2.** Based on the update of Riemannian stochastic subgradient method,

\[E_{\zeta_k} [f_{\lambda}(X_{k+1})] \leq f(P_{\lambda f}(X_k)) + \frac{1}{2\lambda} E_{\zeta_k} \left[ \|P_{\lambda f}(X_k) - X_{k+1}\|_F^2 \right]
\]

\[\leq f(P_{\lambda f}(X_k)) + \frac{1}{2\lambda} E_{\zeta_k} \left[ \|X_k - \gamma_k \nabla_R g(X_k, \zeta_k) - P_{\lambda f}(X_k)\|_F^2 \right] \]

\[\leq f_{\lambda}(X_k) + \frac{\gamma_k}{\lambda} \left( \nabla_R g(X_k, \zeta_k), P_{\lambda f}(X_k) - X_k \right) + \frac{\gamma_k^2 L^2}{2\lambda}. \] (42)

where the first inequality is from the optimality of \(P_{\lambda f}(X_{k+1})\), the second inequality comes from the fact that \(P_{\lambda f}(X_k) \in \text{St}(n, r)\) and Lemma 1. Since \(E_{\zeta_k} \left[ \nabla_R g(X_k, \zeta_k) \right] \in \partial_R f(X_k)\), Theorem 1 implies

\[E_{\zeta_k} [f_{\lambda}(X_{k+1})] \leq f_{\lambda}(X_k) - \frac{\gamma_k}{\lambda} \left( f(X_k) - f(P_{\lambda f}(X_k)) - \frac{L + \tau}{2} \|X_k - P_{\lambda f}(X_k)\|_F^2 \right) + \frac{\gamma_k^2 L^2}{2\lambda},\]

\[\leq f_{\lambda}(X_k) - \frac{\gamma_k}{2\lambda} \left( \frac{1}{\lambda} - (L + \tau) \right) \|X_k - P_{\lambda f}(X_k)\|_F^2 + \frac{\gamma_k^2 L^2}{2\lambda}, \] (43)

in which the last line is due to (37). We now take the total expectation with respect to all the previous realizations \(\zeta_0, \cdots, \zeta_k\), it gives

\[E[f_{\lambda}(X_{k+1})] \leq E[f_{\lambda}(X_k)] - \frac{\gamma_k}{2\lambda} \left( \frac{1}{\lambda} - (L + \tau) \right) E \left[ \|X_k - P_{\lambda f}(X_k)\|_F^2 \right] + \frac{\gamma_k^2 L^2}{2\lambda}. \] (44)

Recognizing \(\Theta(X_k) = \lambda^{-1} \|P_{\lambda f}(X_k) - X_k\|_F\) in (28) and rearranging the above recursion provides the desired result. \(\square\)

We now give our global convergence result for Riemannian stochastic subgradient method (41) used to optimize problem (40).
Theorem 3. Let \( \{X_k\} \) be the sequence generated by Riemannian stochastic subgradient method (41) for solving problem (40) with arbitrary initialization. Suppose one chooses the constant stepsize \( \gamma_k = \frac{1}{\sqrt{T+1}} \) with \( T \) being the total iteration number and the algorithm returns \( X_k \) with \( \bar{k} \) sampled from \( \{1, \cdots, T\} \) uniformly at random. For any \( \lambda < \frac{1}{L+\tau} \) in (27), we have

\[
E[\Theta^2(X_{\bar{k}})] \leq \frac{1}{\lambda \left(\frac{1}{\lambda} - (L + \tau)\right)} \frac{2(f_{\lambda}(X_0) - \min f_{\lambda}) + L^2/\lambda}{\sqrt{T+1}}.
\]

where \( L \) is the Lipschitz constant, \( \tau \) is the weakly convex parameter, \( f_{\lambda}(X) \) is the Moreau envelope of problem (1) defined in (27), and \( \Theta(X) \) is the surrogate stationary measure defined in (28).

Proof of Theorem 3. Unrolling the recursion in Proposition 2 yields

\[
\sum_{k=0}^{T} \gamma_k E[\Theta^2(X_{k+1})] \leq 2(f_{\lambda}(X_0) - \min f_{\lambda}) + L^2/\tau \sum_{k=0}^{T} \gamma_k^2,
\]

dividing \( \sum_{k=0}^{T} \gamma_k^2 \) on both sides in (45) provides

\[
\sum_{k=0}^{T} \gamma_k^2 \sum_{k=0}^{T} \gamma_k E[\Theta^2(X_k)] \leq \frac{1}{\lambda \left(\frac{1}{\lambda} - (L + \tau)\right)} \frac{2(f_{\lambda}(X_0) - \min f_{\lambda}) + L^2}{\lambda} \sum_{k=0}^{T} \gamma_k^2,
\]

note that the LHS is exactly \( E[\Theta^2(X_{\bar{k}})] \) in which we also take expectation with respect to \( \bar{k} \). Invoking the constant stepsize \( \gamma_k = \frac{1}{\sqrt{T+1}} \) into the above inequality gives the desired result.

5 Local Linear Convergence for Sharp Functions

In this section, we consider the fast local convergence of Riemannian subgradient and incremental subgradient methods with the help of sharpness property defined in Definition 3. As a counterpart to incremental algorithms, it should be noted that showing local linear convergence for stochastic algorithms is much more challenging. The difficulty comes from the fact that linear convergence result for nonconvex optimization is usually local whereas the randomness in stochastic algorithms renders ensuring all iterates stay deterministically within this local convergence region hard.

5.1 Sharpness: Weak sharp minima

To start, we present the sharpness property. Given a set of weak sharp minima \( \mathcal{X} \), it basically states that the function grows linearly from the set \( \mathcal{X} \).

Definition 3 (Sharpness; see, e.g., [13]). \( \mathcal{X} \) is said to be the set of weak sharp minima for \( h \) with sharpness parameter \( \alpha \) if there exists \( \rho > 0 \) such that for any \( X \in \mathcal{B} := \{X : \text{dist}(X, \mathcal{X}) < \rho\} \cap \text{St}(n, r) \), one has

\[
h(X) - h(Y) \geq \alpha \text{dist}(X, \mathcal{X}),
\]

for all \( Y \in \mathcal{X} \), where \( \text{dist}(X, \mathcal{X}) := \min_{Y \in \mathcal{X}} \|Y - X\|_F \).

Clearly, \( \mathcal{X} \) is the set of minimizer for \( h \) over \( \mathcal{B} \). When \( \mathcal{B} = \text{St}(n, r) \), the above definition is a global version of sharpness as \( \mathcal{X} \) is the set of global minima to \( h \) over \( \text{St}(n, r) \).

The sharpness condition is tailored for nonsmooth functions. Interestingly, we note that a sharp function need not necessarily to be convex; see, e.g., [15, 24, 47] for nonsmooth nonconvex optimization problems in Euclidean space which possess the sharpness property. The notion of weak sharp minima has also been generalized to functions defined on general Riemannian manifold in [42, 45].
Sharpness property plays a fundamental role in showing linear convergence for subgradient methods for nonsmooth optimization in Euclidean space; see, e.g., [19,20,33,46]. In particular, the author in [33] proves that the subgradient method with a geometrically diminishing stepsize converges linearly to the optimal solution set for minimizing sharp convex functions. In [20], the authors extend the linear convergence result in [33] to the sharp weakly convex optimization. In [46], the authors show a linear convergence result for incremental subgradient, proximal point, and prox-linear methods for minimizing sharp weakly convex functions. In this paper, we show a similar linear convergence result for Riemannian subgradient methods for minimizing nonsmooth weakly convex functions over Stiefel manifold, provided that the sharpness condition in Definition 3 is valid.

5.2 Linear convergence result

When problem (1) has the sharpness property, we will utilize geometrically diminishing stepsize of the form $\gamma_k = \beta^k \gamma_0$ in our Riemannian subgradient methods in order to get local linear convergence result. The motivation comes from optimizing sharp functions in Euclidean space [20,33,46,53,56,63,79].

We present the proofs for Riemannian incremental subgradient method and they automatically apply to Riemannian subgradient method by assigning $m = 1$ in all the results, as Riemannian subgradient method can be regarded as a special case of the incremental method with only one component function. The following proposition is essential for establishing local linear convergence.

**Proposition 3.** Let $\{X_k\}$ be the sequence generated by Riemannian incremental subgradient method (14) for solving problem (1), then, for any $X \in St(n,r)$ and $k \geq 0$, we have,

$$
\|X_{k+1} - X\|_F^2 \leq \left(1 + 2m\gamma_k(L + \tau)\right)\|X_k - X\|_F^2 - 2m\gamma_k \left(f(X_k) - f(\bar{X})\right) + m^2\gamma_k^2 L^2 + C(m)\gamma_k^3 L^2(L + \tau),
$$

where $C(m) = \frac{(m-1)m(2m-1)}{3}$.

**Proof of Proposition 3.** According to Lemma 1, for any $X \in St(n,r)$, we have

$$
\|X_{k+1, i} - X\|_F^2 \leq \|X_{k,i-1} + \xi_{k,i-1} - X\|_F^2 \leq (i) \|X_{k,i-1} - X\|_F^2 - 2\gamma_k \left(\nabla_R f_i(X_{k,i-1}), X_{k,i-1} - X\right) + \gamma_k^2 L^2
$$

$$
\leq \|X_{k,i-1} - X\|_F^2 - 2\gamma_k \left(f_i(X_{k,i-1}) - f_i(\bar{X})\right) + \gamma_k(L + \tau) \|X_{k,i-1} - X\|_F^2 + \gamma_k^2 L^2.
$$

where $(i)$ follows from the fact that $\|\nabla_R f_i(X_{k,i-1})\| \leq L$, and $(ii)$ is from Theorem 1. From (34), (35), and (33), one has

$$
\begin{cases}
\{ f_i(\bar{X}) - f_i(X_{k,i-1}) \leq (i - 1)\gamma_k L^2 - (f_i(X_k) - f_i(\bar{X})), \\
\|X_{k,i-1} - X\|_F^2 \leq 2(i - 1)^2\gamma_k^2 L^2 + 2\|X_k - X\|_F^2,
\end{cases}
$$

Invoking the above two upper bounds into (47) gives

$$
\|X_{k+1, i} - X\|_F^2 \leq \|X_{k,i-1} - X\|_F^2 - 2\gamma_k \left(f_i(X_k) - f_i(\bar{X})\right) + 2\gamma_k(L + \tau) \|X_k - X\|_F^2 + (2i - 1)^2\gamma_k^2 L^2 + 2(i - 1)^2\gamma_k^2 L^2(L + \tau).
$$

Summing up the above inequality from $i = 1$ to $i = m$ yields

$$
\|X_{k+1} - X\|_F^2 \leq \left(1 + 2m\gamma_k(L + \tau)\right)\|X_k - X\|_F^2 - 2m\gamma_k \left(f(X_k) - f(\bar{X})\right) + m^2\gamma_k^2 L^2 + \frac{(m-1)m(2m-1)}{3}\gamma_k^3 L^2(L + \tau),
$$

and this ends the proof. \[\square\]
Based on Proposition 3, we are going to show the local linear convergence result in the following theorem. The argument in Theorem 4 adopts the idea from [46, Theorem 1] and [20, Theorem 5.1], though in [20, 46] the theme is considering weakly convex optimization in Euclidean space and they assume a global version of sharpness property. Indeed, this type of argument for showing linear convergence for subgradient methods used to optimize sharp convex functions in Euclidean space even dates back to [33].

**Theorem 4.** Suppose the sharpness property in Definition 3 is valid for problem (1). Let \( \{X_k\} \) be the sequence generated by Riemannian incremental subgradient method (14) for solving problem (1) with the initialization satisfying \( \text{dist}(X_0, X) < \left( \frac{\alpha}{L + \tau} \right) \) and \( \text{dist}(X_0, X) < \rho \) (i.e., \( X_0 \in B \)). Suppose further the stepsize \( \gamma_k \) is diminishing in a geometric rate, i.e., \( \gamma_k = \beta^k \gamma_0 \), where \( \gamma_0 < \min \left\{ \frac{2(m \alpha_0 - m(L + \tau) \epsilon_0)}{d(m)L^2}, \frac{\gamma_0}{2m(\alpha - (L + \tau) \epsilon_0)} \right\} \) and \( 1 > \beta \geq \beta_{\min} := \sqrt{1 + \left( \frac{2m(L + \tau) - 2m\alpha}{\epsilon_0} \right) \gamma_0 + \frac{d(m)L^2 \gamma_0^2}{\epsilon_0^2}} \) with \( d(m) = \frac{5}{8} m^2 - m + \frac{1}{3} \) and \( \epsilon_0 = \min \left\{ \max \left\{ \text{dist}(X_0, X), \frac{\alpha}{2(L + \tau)} \right\}, \rho \right\} \). Then, we have: \( \text{dist}(X_k, X) \leq \epsilon_0 \) and \( X_k \in B \) for all \( k \geq 0 \), and

\[
\text{dist}(X_k, X) \leq \beta^k \cdot \epsilon_0, \quad \forall \ k \geq 0.
\]

In the above statement, \( L \) is the Lipschitz continuity parameter, \( \tau \) is the weakly convex parameter, \( \alpha \) is the sharpness parameter in Definition 3, \( \rho > 0 \), \( B \in \mathbb{R}^{n \times r} \) and \( X \in \mathbb{R}^{n \times r} \) are defined in Definition 3 with \( X \) being the set of weak sharp minima relative to \( B \).

**Proof of Theorem 4.** We first show that \( \beta_{\min} \in (0, 1) \) and \( \gamma_0 > 0 \) are well defined. Towards that end, we have

\[
\beta_{\min} = \sqrt{1 + v(\gamma_0)} \quad \text{with} \quad v(\gamma_0) = \left( 2m(L + \tau) - \frac{2m\alpha}{\epsilon_0} \right) \gamma_0 + \frac{d(m)L^2 \gamma_0^2}{\epsilon_0^2} \quad \text{being a quadratic function on} \ \gamma_0.
\]

Thus, it is equivalent to show \( v(\gamma_0) \in (-1, 0) \). Firstly, \( v(\gamma_0) < 0 \) can be guaranteed by \( \gamma_0 < \frac{2(m \alpha_0 - m(L + \tau) \epsilon_0)}{d(m)L^2} \) which is the definition of \( \gamma_0 \). Then, \( v(\gamma_0) \) attains its minimum value at \( \gamma_0 = \frac{m \alpha_0 - m(L + \tau) \epsilon_0}{d(m)L^2} \) with function value

\[
v(\gamma_0) = \frac{m^2 \alpha^2 - m^2(L + \tau) \epsilon_0^2}{d(m)L^2} \quad \text{due to} \quad 7L \geq \alpha.
\]

On the other hand, \( \epsilon_0 < \frac{\alpha}{L + \tau} \) indicates the upper bound \( \min \left\{ \frac{2(m \alpha_0 - m(L + \tau) \epsilon_0)}{d(m)L^2}, \frac{\gamma_0}{2m(\alpha - (L + \tau) \epsilon_0)} \right\} \) of initial stepsize \( \gamma_0 \) is positive and well defined.

We now prove the theorem by induction. When \( k = 0 \), this is true because of the definition of \( \epsilon_0 \). Now, we suppose \( \text{dist}(X_k, X) \leq \beta^k \cdot \epsilon_0 \). Then, Step 1 is true for \( k \)-th iteration. It remains to show \( \text{dist}(X_{k+1}, X) \leq \beta^{k+1} \cdot \epsilon_0 \). To continue, we derive the key recursion based on Proposition 3 and the sharpness property of problem (1).

Since \( X_k \in B \), taking \( \bar{X} \) as the projection of \( X_k \) onto set \( X \) in Proposition 3, the sharpness condition of \( f \) implies

\[
f(X_k, X) - f(\bar{X}) \geq \alpha \text{ dist}(X_k, X).
\]

This, together with the facts that \( \text{dist}(X_k, X) = \|X_k - X\|_F \) and \( \text{dist}(X_{k+1}, X) \leq \|X_{k+1} - X\|_F \), yields

\[
dist^2(X_{k+1}, X) \leq (1 + 2m\gamma_0(L + \tau)) \text{ dist}^2(X_k, X) - 2m\gamma_0 \alpha \text{ dist}(X_k, X)
\]

\[
+ m^2\gamma_0^2 L^2 + C(m)\gamma_0^2 \beta^2 L^2(L + \tau).
\]

(51)

where we also used the fact that \( \gamma_k \leq \gamma_0 \). Note that the RHS of the above key recursion is a quadratic function on \( \text{dist}(X_k, X) \). By definition of \( \gamma_0 \), we have \( \frac{2m\gamma_0 \alpha}{1 + 2m\gamma_0(L + \tau)} \leq \epsilon_0 \). This indicates that the RHS in (51) achieves its maximum at \( \text{dist}(X_k, X) = \beta^k \epsilon_0 \) (recall that \( 0 \leq \text{dist}(X_k, X) \leq \beta^k \epsilon_0 \) by the inductive hypothesis). Plugging \( \gamma_k = \gamma_0 \beta^k \) and \( \text{dist}(X_k, X) = \beta^k \epsilon_0 \) into (51) yields

\[
dist^2(X_{k+1}, X) \leq \beta^{2k} \epsilon_0^2 + 2m\gamma_0(L + \tau) \beta^{2k} \epsilon_0^2 - 2m\gamma_0 \alpha \beta^{2k} \epsilon_0^2
\]

\[
+ m^2\gamma_0^2 L^2 \beta^{2k} + C(m)\gamma_0^3 \beta^2 L^2(L + \tau)
\]

\[
= \beta^{2k} \epsilon_0^2 \left[ 1 + \left( \frac{2m(L + \tau) - 2m\alpha}{\epsilon_0} \right) \gamma_0 + \left( \frac{m^2 L^2 + C(m)\gamma_0^2 L^2(L + \tau)}{\epsilon_0^2} \right) \gamma_0^2 \right].
\]

(52)

\footnote{This relation can be observed from Lipschitz continuity and sharpness of \( f \), \( \alpha \|X - \bar{X}\|_F \leq f(X) - f(\bar{X}) \leq L \|X - \bar{X}\|_F \) for certain \( X \in \mathbb{R}^{n \times r} \) and \( \bar{X} \in \mathcal{P}_X(X) \).}
Recall that $\gamma_0 < \frac{2(m_0 e_0 - m(L + \tau)e_0^2)}{d(m) L^2} \leq \frac{m e_0^2}{2d(m) L^2(L + \tau)} < \frac{1}{m(L + \tau)}$. (52) further implies

$$\text{dist}^2(X_{k+1}, X) \leq \beta^2\epsilon_0^2 \left[1 + \left(2m(L + \tau) - \frac{2m\alpha}{\epsilon_0}\right) \gamma_0 + \frac{d(m)L^2}{\epsilon_0^2}\gamma_0^2\right]$$

(53)

The proof is completed.

Our linear convergence result requires the stepsize $\gamma_k$ in Riemannian subgradient and incremental subgradient methods to diminish geometrically in terms of iteration number $k$. Once the geometrical decay factor $\beta$ and initial stepsize $\gamma_0$ satisfy certain conditions, the algorithm will have linear rate of convergence to the target set $X$ and the factor of linear convergence is exactly $\beta$. Thus, it is supposed to choose an appropriate $\gamma_0$ such that $\beta_{\text{min}}$ is as small as possible. It is easy to see that if one can choose $\gamma_0 = \beta_0 = \frac{m e_0 (r(L + \tau) e_0)}{d(m) L^2}$, $\beta_{\text{min}}$ attains its minimum value $\sqrt{1 - \frac{m^2 (\alpha (L + \tau) e_0)^2}{d(m) L^2}}$. Hence, under the requirement on the initial stepsize $\gamma_0$ in Theorem 4, we suggest setting $\gamma_0$ as close as possible to $\beta_0$.

When the sharpness property holds globally over Stiefel manifold (it is discussed in Section 7 that this is the case for DPCP (3)), we have $B = \mathbb{S}(n, r)$ in Definition 3 and the parameter $\rho$ can be set as large as possible. In this situation, we have $e_0 = \max \{\text{dist}(X_0, X), \frac{\alpha}{2(L + \tau)}\}$ and hence $\gamma_0 < \frac{2(m e_0 - m(L + \tau)e_0^2)}{d(m) L^2}$ in Theorem 4, which implies we can choose $\gamma_0 = \beta_0$ to obtain the smallest possible $\beta_{\text{min}}$.

We end this section by comparing the ‘Riemannian regularity condition’ used in [5] and [78] for orthogonal dictionary learning and robust subspace recovery, respectively. In [78], they show local linear convergence results for Riemannian subgradient method using the Riemannian regularity condition. For the target set $X$, this condition says

$$\langle \nabla_R f(X), X - Y \rangle \geq \kappa \text{dist}(X, X),$$

for $X$ in a small neighborhood of $X$ and some constant $\kappa > 0$, and for all $Y \in \mathcal{P}_X(X)$. This condition comes from the necessity for bounding the inner product term in the proof of Riemannian subgradient method, see (47) with $f_i = f$ and $X_{k, i-1} = X_k$. Thus, Riemannian regularity condition can be regarded as the combination of Riemannian subgradient inequality (Theorem 1) of $f$ and sharpness property of $f$. However, this reduction could potentially hurt the usage of the two properties: (i) Since the Riemannian regularity condition can only hold locally, it cannot be used for establishing global convergence and obtaining iteration complexity for Riemannian subgradient methods. (ii) It cannot be valid for individual $f_i$ which blocks its usage in the analysis of Riemannian incremental subgradient and stochastic subgradient methods.

### 6 Extension

In this section, we briefly study the extension of our convergence results for Riemannian subgradient methods to other compact Riemannian manifolds embedded in Euclidean space. The extension builds on generalizing the Riemannian subgradient inequality in Theorem 1 and the second order boundedness of retractions.

**Extension of Theorem 1** We first generalize the Riemannian subgradient inequality to other smooth compact embedded manifolds in Euclidean space. We consider a special case when $\mathcal{M}$ is defined by $\mathcal{M} = \{X \mid F(X) = 0\}$ where $F : \mathbb{R}^p \to \mathbb{R}^q$ is a smooth mapping with $DF(X)$ has full row rank for all $X \in \mathcal{M}$.\(^8\) We state it in the following corollary.

\(^8\)This representation could potentially involve manifolds such as generalized Stiefel manifold, Oblique manifold and Symplectic manifold, etc [2].
Corollary 1. Suppose $\mathcal{M}$ is a compact Riemannian manifold which is embedded in Euclidean space. And $\mathcal{M} = \{X \mid F(X) = 0\}$ where $F : \mathbb{R}^p \to \mathbb{R}^q$ is a smooth mapping with $DF(X)$ has full row rank for all $X \in \mathcal{M}$. Then for any weakly convex function $h$ defined in Euclidean space, we have
\[
h(Y) \geq h(X) + \langle \nabla_R h(X), Y - X \rangle - c\|Y - X\|_F^2,
\]
for some constant $c > 0$.

Proof. The proof directly follows that of Theorem 1. It follows from [2, Equation (3.19)] that $T_X\mathcal{M} = \ker(DF(X))$, where $\ker(B)$ means the null space of $B$. Thus, the projection matrix of $P_{T_X\mathcal{M}}$ is given by $DF(X)^\top(DF(X)DF(X)^\top)^{-1}DF(X)$. The key step in the proof of Theorem 1 lies in the inequality (23). While for $\mathcal{M}$ defined by $F$, the inner product term $\langle \nabla h(X), P_{T_X\mathcal{M}}(Y - X) \rangle$ in (23) can be bounded as follows
\[
\langle \nabla h(X), P_{T_X\mathcal{M}}(Y - X) \rangle \leq \|\nabla h(X)\|_F\|P_{T_X\mathcal{M}}(Y - X)\|_F
\]
\[
= \|\nabla h(X)\|_F\|DF(X)^\top(DF(X)DF(X)^\top)^{-1}DF(X)(Y - X)\|_F
\]
\[
\leq \|\nabla h(X)\|_F\max_{X \in \mathcal{M}}\|DF(X)^\top(DF(X)DF(X)^\top)^{-1}\|_F\|DF(X)(Y - X)\|_F.
\]

Notice that $F(Y) = F(X) + DF(X)(Y - X) + O(\|Y - X\|_F^2)$, we have $\langle \nabla h(X), P_{T_X\mathcal{M}}(Y - X) \rangle = O(\|Y - X\|_F^2)$, if $\mathcal{M}$ is compact.

For general manifolds not embedded in Euclidean space, the inequality is not well-defined since the inner product $\langle \nabla h(X), Y - X \rangle$ does not make any sense.

Extension to other retractions In our convergence proofs, we restricted ourselves to polar decomposition based retraction due to its efficiency and non-expansiveness property. In order to obtain convergence results for other manifolds $\mathcal{M}$, we have to deal with other retractions. Despite the non-expansiveness property might not hold any more, we remark that our convergence proofs apply to any retractions on manifold $\mathcal{M}$ defined in Corollary 1. In this case, we have the following result for any $X \in \mathcal{M}$
\[
\|X_{k,i} - X\|_F = \|\text{Retr}_{X_{k,i-1}}(\xi_{k,i-1}) - X\|_F
\]
\[
= \|(X_{k,i-1} + \xi_{k,i-1}) - X + \text{Retr}_{X_{k,i-1}}(\xi_{k,i-1}) - (X_{k,i-1} + \xi_{k,i-1})\|_F
\]
\[
\leq \|(X_{k,i-1} + \xi_{k,i-1}) - X\|_F + b\|\xi_{k,i-1}\|_F^2
\]
\[
\leq \|(X_{k,i-1} + \xi_{k,i-1}) - X\|_F + b\gamma_k^2L^2,
\]
for some constant $b > 0$, where we used the second order boundedness property [11] of retractions in the first inequality. Compared to Lemma 1, we have the extra term $O(\gamma_k^2)$ in the above inequality which fortunately will not break down the convergence analysis. The price to pay is that some constants in the convergence results will become slightly worse.

Equipped with the generalized Riemannian subgradient inequality in Corollary 1 and the property (54) for retractions, one can follow our analysis in Section 4 and Section 5 to obtain: 1) Iteration complexity $O(\varepsilon^{-4})$ and 2) local linear convergence, for Riemannian subgradient methods for optimizing weakly convex functions over any manifold $\mathcal{M}$ defined in Corollary 1.

7 Applications and Experimental Results

In this section, we apply the Riemannian subgradient methods for solving the robust subspace recovery and dictionary learning problems. As described in Section 1, the objective functions of both problems (which
will be restated soon) are weakly convex. Thus Theorem 2 and Theorem 3 ensure global convergence with rate $O(k^{-1/4})$ for Riemannian subgradient methods when utilized to solve those problems with arbitrary initialization. Moreover, as guaranteed by Theorem 4, our Riemannian subgradient and incremental subgradient methods converge in a linear rate if the problems further satisfy the sharpness property which will be discussed.

### 7.1 Robust subspace recovery (RSR)

We begin with the robust subspace recovery problem using the DPCP formulation (3) which has a relatively simpler form than (2). For convenience, we first restate the optimization formulation (3) as follows:

$$
\begin{align*}
\min_{X \in \mathbb{R}^{n \times r}} & \quad f(X) = \frac{1}{m} \sum_{i=1}^{m} \| \bar{y}_i^T X \|_2 \\
\text{subject to} & \quad X \in \text{St}(n, r),
\end{align*}
$$

in which $\bar{Y} = [Y \ O] \in \mathbb{R}^{n \times m}$ are data instances and the columns $y_i$’s of $Y \in \mathbb{R}^{n \times m_1}$ are inlier points spanning a $d$-dimensional subspace $S$ of $\mathbb{R}^n$ and $r = n - d$, the columns $O_i$’s of $O \in \mathbb{R}^{n \times m_2}$ are outlier points, and $\Gamma$ is an unknown permutation. Note that the objective function in (55) is rotation invariant in the sense that $f(X) = f(XR)$ for any $X \in \text{St}(n, r)$ and $R \in \text{St}(r, r)$.

**Sharpness** Let $S^\perp \in \text{St}(n, r)$ be an orthonormal basis of $S^\perp$. Since the goal of DPCP is to find an orthonormal basis (but not necessary $S^\perp$) for $S^\perp$, we define the target set as

$$
\mathcal{C} = \{ S^\perp R \in \mathbb{R}^{n \times r} : R \in \text{St}(r, r) \}.
$$

Due to rotation invariant, $f$ is constant for any $X \in \mathcal{C}$. To analyze (55), we recall two quantities that reflect how well distributed the inliers and outliers are:

$$
\begin{align*}
c_{O,\text{max}} & := \frac{1}{m_2} \max_{b \in S^{D-1}} \| O^T b \|_1, \\
c_{Y,\text{min}} & := \frac{1}{m_1} \min_{b \in S^\perp} \| Y^T b \|_1.
\end{align*}
$$

Here $c_{Y,\text{min}}$ is also referred to as the *permeance statistic* in [44]. In a nutshell, larger values of $c_{Y,\text{min}}$ (respectively, smaller values of $c_{O,\text{max}}$) correspond to a more uniform distributions of inliers (respectively, outliers). For detailed explanations, we refer to [44, 79] for the concentration inequalities concerning these quantities derived for certain statistical models.\footnote{Under mild statistical assumptions on the model such as the Haystack model used in [43] (with both inliers and outliers have unit energy in expectation) and the random spherical model in [70], the quantity $c_{Y,\text{min}}$ and $c_{O,\text{max}}$ respectively concentrate around $\frac{1}{\sqrt{n-r}}$ and $\frac{1}{\sqrt{r}}$, with high probability.}

**Proposition 4.** Suppose $m_2 c_{O,\text{max}} \leq m_1 c_{Y,\text{min}}/2r$. Then the problem (55) satisfies sharpness condition with respect to the target set $\mathcal{C}$ (56) with parameter $\alpha = \frac{1}{m} \left( \frac{m_2 c_{Y,\text{min}}}{2r} - m_2 \sqrt{r} c_{O,\text{max}} \right) > 0$, i.e.,

$$
| f(X) - f(S^\perp) | \geq \alpha \text{dist}(X, \mathcal{C}), \quad \forall X \in \text{St}(n, r).
$$

**Proof of Proposition 4.** Let $S \in \mathbb{R}^{n \times d}$ be an orthonormal basis of the subspace $S$. For any $X \in \text{St}(n, r)$, we rewrite it as

$$
X = SS^\top X + S^\perp (S^\perp)^\top X,
$$

where $SS^\top X$ and $S^\perp (S^\perp)^\top X$ respectively represent the projection of $X$ onto the subspace $S$ and $S^\perp$.

Define the principal angles between the subspaces spanned by $X$ and $S^\perp$ as $[64] \phi_i = \arccos(\sigma_i(S^\perp X))$ for
are orthogonal matrices. With this, we connect the distance between $X \in S$ and $B$ where the last line follows because for any $i$, the RHS of the above equation as 

$$\text{dist}^2(X, C) = \min_{R \in \text{Rot}(r, r)} \|X - S^\perp R\|_F^2 = \|X - S^\perp UW^\top\|_F^2$$

(59)

$$= 2r - 2 \text{trace}(\cos(\Phi)) = 2 \sum_{i=1}^r (1 - \cos(\phi_i))$$

$$\leq 2r(1 - \cos(\phi_r)) = 4r \sin^2(\phi_{\max}/2) \leq 4r \sin^2(\phi_r),$$

where the second equality follows because the optimal rotation matrix that solves the well-known orthogonal Procrustes problem $\min_{R \in \text{Rot}(r, r)} \|X - S^\perp R\|_F^2$ is $UW^\top$.

Without loss of generality, we assume $r \leq d$.\(^\text{10}\) In this case, we can then rewrite $S^\top X = V\sin(\Phi)W^\top$, where $V \in \mathbb{R}^{d \times r}$ is an orthogonal matrix. Thus, we have

$$X = SV\sin(\Phi)W^\top + S^\perp U\cos(\Phi)W^\top.$$  

(60)

Taking an arbitrary point $X^* \in \mathcal{P}_C(X)$, we have

$$f(X) - f(S^\perp) = \frac{1}{m} \sum_{i=1}^m \|y_i^\top X\|_2 - \frac{1}{m} \sum_{i=1}^m \|y_i^\top S^\perp\|_2$$

(61)

$$= \frac{1}{m} \sum_{i=1}^m \|y_i^\top X\|_2 + \frac{1}{m} \left[ \sum_{i=1}^{m_1} \|y_i^\top X\|_2 + \sum_{i=1}^{m_2} \|y_i^\top (X - S^\perp)\|_2 \right],$$

where the second line follows because the inliers $\{y_i\}_{i=1}^{m_1}$ are orthogonal to $S^\perp$. We bound the first term in the RHS of the above equation as

$$\sum_{i=1}^{m_1} \|y_i^\top X\|_2 = \sum_{i=1}^{m_1} \|y_i^\top SV\sin(\Phi)W^\top\|_2 = \sum_{i=1}^{m_1} \|y_i^\top SV\sin(\Phi)\|_2$$

(62)

$$= \sum_{i=1}^{m_1} \left[ \sum_{i=1}^r \sin^2(\phi_i)\|y_i^\top Sv_i\|_2 \right] \geq \sum_{i=1}^{m_1} \sin(\phi_r) \|y_i^\top Sv_r\|_2 \geq m_1 c_{Y, \text{min}} \sin(\phi_r),$$

where $c_{Y, \text{min}}$ is defined in (58). On the other hand, the second term in the RHS of (61) can be upper bounded by

$$\sum_{i=1}^{m_2} \|o_i^\top X\|_2 - \|o_i^\top S^\perp\|_2 \leq \sum_{i=1}^{m_2} \|o_i^\top X\|_2 - \|o_i^\top S^\perp\|_2 \leq \sum_{i=1}^{m_2} \|o_i^\top (X - S^\perp)\|_2$$

(63)

$$\leq \|X - X^*\|_F \sum_{i=1}^{m_2} \|o_i^\top (X - S^\perp)\|_2 \leq m_2 \sqrt{r} c_{O, \text{max}} \|X - S^\perp\|_F,$$

where the last line follows because for any $B \in \mathbb{R}^{n \times r}$ satisfying $\|B\|_F = 1$, we have

$$\sum_{i=1}^{m_2} \|o_i^\top B\|_2 \leq \sum_{i=1}^{m_2} \sum_{j=1}^r |o_i^\top b_j| + \sum_{j=1}^r \sum_{i=1}^{m_2} |o_i^\top b_j| \leq m_2 c_{O, \text{max}} \sum_{j=1}^r |b_j| \leq m_2 \sqrt{r} c_{O, \text{max}},$$

\(^\text{10}\)For the case $r > d$, we have at least $\phi_1 = \cdots = \phi_{d-r} = 0$. Similar to the case $r \leq d$, we can also rewrite $S^\top B = V\sin(\Phi)R^\top$, where $V = [0 \; \mathbf{V}]$ with $\mathbf{V} \in \mathbb{R}^{d \times d}$ an orthogonal matrix. Thus, we also have (60) and the following proofs are the same.
where $c_{O,\max}$ is defined in (57).

We complete the proof by plugging (62) and (63) into (61) to obtain

$$f(X) - f(S^\perp) \geq \frac{1}{m} m_1 c_{Y,\min} \sin(\phi_r) - \frac{1}{m} m_2 \sqrt{r} c_{O,\max} \text{dist}(X, C)$$

$$\geq \frac{1}{m} \left( \frac{m_1 c_{Y,\min}}{2\sqrt{r}} - m_2 \sqrt{r} c_{O,\max} \right) \text{dist}(X, C),$$

where the last line utilizes (59).

The requirement $m_2 c_{O,\max} \leq m_1 c_{Y,\min}/2r$ in Proposition 4 determines the number of outliers that can be tolerated. Under the popular statistical models described in Footnote 9, Proposition 4 requires the number of outliers to be at most the same order of inliers, which matches the bound in [44] but is not optimal compared to the result in [79]. We believe a better upper bound on the number of outliers can be obtained with a more sophisticated analysis for the term in (63). Despite of the suboptimality of the number of tolerated outliers, Proposition 4 provides new insight on DPCP that is absent from previous works [78, 79]. In particular, an immediate consequence implied by Proposition 4 is that the set of global minima of (55) is exactly the collection of all orthonormal basis for $S^\perp$, i.e., $C$. This observation establishes the global identifiability of DPCP. Finally, we note that by using the same arguments, we can obtain similar global sharpness property for the other RSR formulation (4). A local sharpness property for (2) is also established in [52, Equation 28].

Proposition 4 together with Theorem 4 implies a linear convergence result of the Riemannian subgradient and incremental subgradient methods using geometrically diminishing stepsize provided proper initialization, which we compute as the bottom eigenvectors of $\tilde{Y} \tilde{Y}^\top$ as in [52, 79]. We show several experimental results illustrating the convergence property of our Riemannian subgradient methods in the following.

**Experiments** We first randomly sample a subspace $S$ with co-dimension $r = 10$ in ambient dimension $n = 100$. We then generate $m_1 = 1500$ inliers uniformly at random from the unit sphere in $S$ and $m_2 = 3500$ outliers uniformly at random from the unit sphere in $\mathbb{R}^n$. We initialize all the algorithm at the same point whose entries follow standard Gaussian distribution as it provides comparable performance to the carefully designed good initialization in [52, 79]. The experimental results are displayed in Figure 2. Sublinear convergence can be observed from Figure 2a in which we use polynomially diminishing stepsize as suggested in Theorem 2 and Theorem 3. In Figure 2b, we use geometrically diminishing stepsize of the form $\gamma_k = \gamma_0 \beta^k$. 

\[ \text{dist}(X_k, C) \]
We fix $\gamma_0 = 0.1$ and tune the best geometrical decay factor $\beta$ for each algorithm. As one can see, linear rate of convergence can be observed which corroborates our theoretical results.

### 7.2 Orthogonal dictionary learning (ODL)

We now consider ODL that aims to learn a concise representation for the data. Formally, given $Y = AS \in \mathbb{R}^{n \times m}$, where $A \in \text{St}(n, n)$ is an unknown orthonormal dictionary and each column of $S \in \mathbb{R}^{n \times m}$ is sparse, we attempt to learn $A$ by noting that $A^\top Y = S$ is sparse. For convenience, we restate the ODL formulations (4) and (5) in the following:

\[
\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{m} \|Y^\top x\|_1 = \frac{1}{m} \sum_{i=1}^{m} |y_i^\top x|
\]

subject to $x \in \text{St}(n, 1), \quad (64)$

or in a matrix form

\[
\min_{X \in \mathbb{R}^{n \times n}} f(X) := \frac{1}{m} \|Y^\top X\|_1 = \frac{1}{m} \sum_{i=1}^{m} \|y_i^\top X\|_1
\]

subject to $X \in \text{St}(n, n). \quad (65)$

**Sharpness** The sharpness property for recovering one column of the dictionary by (64) has been studied in [5]. For the approach (65) that recovers the entire dictionary in one time, it has been analyzed in [72] which shows that the underlying orthogonal dictionary $A$ and its permutations are the only sharp local minimum to (65) when the number of samples $m \to \infty$. Although we will not formally verify the sharpness property for (65) for finite samples case, the following experiments suggest that it is very possible that (65) has the sharpness property since our Riemannian subgradient methods converge in a linear rate with geometrically diminishing stepsize, even with random initialization. We leave the study of formal analysis as future work.

**Experiments** We generate the synthetic data for ODL application according to [5]. We first generate a random underlying orthogonal dictionary $A \in \text{St}(n, n)$ with $n = 30$. We set the number of samples $m = 1643 \approx 10 \times n^{1.5}$. The sparse coefficient matrix $S \in \mathbb{R}^{n \times m}$ is generated with each entry satisfying Bernoulli-Gaussian distribution with parameter 0.3 (sparsity), i.e., each entry $S_{i,j}$ is independently drawn
from a standard Gaussian with probability 0.3 and zero otherwise. Then the observation $Y$ is generated as $Y = AS$. We initialize all the algorithms at the same point whose entries follow standard Gaussian distribution. The results are shown in Figure 3. Figure 3a displays the sublinear convergence result with polynomially diminishing stepsize as suggested in Theorem 2 and Theorem 3. One can observe from Figure 3b that our Riemannian subgradient methods converge linearly with geometrically diminishing stepsize, where $\gamma_0 = 0.1$ and the geometrical decay factor $\beta$ is tuned as the best one for each algorithm.

8 Conclusion

In this work, we introduced a family of Riemannian subgradient methods for optimizing nonsmooth functions over Stiefel manifold. We showed nonasymptotic convergence rate of these algorithms based on a Riemannian subgradient inequality which plays a fundamental role in our analysis. In particular, when the function is weakly convex in Euclidean space, we proved our algorithms converge globally with a sublinear rate; when the function further has the sharpness property, we showed that Riemannian subgradient and incremental subgradient methods converge linearly to the set of weak sharp minima $\mathcal{X}$ of problem (1) with geometrically diminishing stepsize, provided the algorithms initialized properly. Moreover, our results can be naturally extended to nonsmooth optimization problems over other compact Riemannain manifolds embedded in Euclidean space. Finally, we presented the existence of the sharpness property in the robust subspace recovery and orthogonal dictionary learning problems, and verified the convergence performance of our methods on both problems via numerical simulations.

We believe our work has opened several interesting questions to be investigated in the future. First, one can readily generalize our result to nonsmooth optimization over the product of a series of Stiefel manifolds, which find applications in $\ell_1$-PCA [43] and robust phase synchronization [69]. On the other hand, our results are specific to Stefiel manifold, it would be very interesting to see if one can extend our convergence results to nonsmooth optimization over a broad class of Riemannian manifolds. We believe this is quite promising based on the analytical framework developed in this work. Finally, we suspect our global convergence rate $O(k^{-\frac{1}{4}})$ is not tight for solving problem (1) with Riemannian subgradient methods. This is because the Riemannian proximal point method for solving problem (1) has global convergence rate $O(k^{-\frac{1}{2}})$ [16], and in smooth optimization gradient descent has the same global convergence rate to its proximal point method counterpart. Hence, how to improve this rate can be an interesting question.

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