Three-dimensional QCD in the adjoint representation and random matrix theory

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Abstract

In this paper we complete the derivations of finite volume partition functions for QCD using random matrix theories by calculating the effective low-energy partition function for three-dimensional QCD in the adjoint representation from a random matrix theory with the same global symmetries. As expected, this case corresponds to Dyson index $\beta = 4$, that is, the Dirac operator can be written in terms of real quaternions. After discussing the issue of defining Majorana fermions in Euclidean space, the actual matrix model calculation turns out to be simple. We find that the symmetry breaking pattern is $O(2N_f) \to O(N_f) \times O(N_f)$, as expected from the correspondence between symmetric (super)spaces and random matrix universality classes found by Zirnbauer. We also derive the first Leutwyler–Smilga sum rule.
1 Introduction

The recent strong interest in the low-energy limit of the QCD spectrum originates in the Banks-Casher relation [1]

\[ \Sigma = \frac{\pi \rho(0)}{V} \]  

relating the density of eigenvalues of the Dirac operator at the origin, \( \rho(0) \), to the order parameter for spontaneous breaking of chiral symmetry, the quark condensate \( \Sigma \), in the thermodynamic and chiral limits (\( V \) here is the size of the box). It is expected on the basis of lattice QCD simulations that chiral symmetry will be restored above some critical temperature and/or chemical potential. By studying the Dirac spectrum in the infrared limit, one hopes to obtain analytical information about the chiral phase transition.

The presence of gauge fields alters the distribution of the Dirac eigenvalues. For free quarks in 4d the spacing between eigenvalues is \( \sim V^{-1/4} \), whereas for interacting quarks it is \( \sim V^{-1} \). Moreover, it is known that the Dirac eigenvalues derived from QCD in a finite volume are constrained by sum rules first discovered by Leutwyler and Smilga [2]. These sum rules are obtained by expanding the partition function in powers of the quark mass \( m \) before and after averaging over the gauge field configurations, and matching powers of \( m \).

The same sum rules can also be obtained from a random matrix theory with the same global symmetries as the QCD partition function. This was first noticed in [3]. In the random matrix theory (where the size of the random matrices \( N \) is identified with the space-time volume \( V \)), the average over gauge field configurations is substituted by an average over Gaussian distributed random matrices with the same structure of matrix elements as the Euclidean space Dirac operator \( \gamma_\mu D_\mu \). The matrix elements can be real, complex or quaternion real, corresponding to Dyson indices \( \beta = 1, 2, 4 \) respectively. The corresponding matrix models are called the (chiral) Gaussian orthogonal, unitary and symplectic ensembles.

In fact, the extreme infrared limit of the QCD partition function maps onto the same effective partition function (usually called the finite volume partition function) as the random matrix theory [3, 4, 5, 6, 7, 8, 9]. In retrospect this is not too surprising, since the chiral Lagrangian to lowest order in the quark momenta is completely determined by the pattern of chiral symmetry breaking and Lorentz invariance. In the so-called mesoscopic range \( \Lambda_{QCD}^{-1} \ll L \ll \lambda_G \), where \( L \) is the side of the box and \( \lambda_G \) is the Compton wavelength of the Goldstone modes, this partition function expresses the quark mass dependence in the static limit and in a finite volume. It is expressed as an integral over the Goldstone manifold of the composite variables corresponding to the pion fields and it is a function of one scaling variable \( N \Sigma \mathcal{M} \), where \( \mathcal{M} \) is a mass matrix.
in flavor space. In substituting the complicated average over gauge field configurations with an average over random matrices, we have the advantage that we can utilize a well-established mathematical framework that had its beginnings with the pioneering works in the field by Wigner, Dyson and Mehta.

The sum rules can be expressed using the so-called microscopic spectral density $\rho_S(\lambda)$, defined by magnifying the distribution of eigenvalues in the vicinity of the origin ($\lambda = 0$) on the scale of the average eigenvalue spacing. $\rho_S(\lambda)$ is a highly universal quantity \cite{3,6,10,11,12,13}. It is completely determined by the global symmetries of the partition function, and does not depend on the matrix potential, for example.

However, the sum rules are not sufficient for determining the spectral density itself. Fortunately, the Dirac spectrum can be obtained using the so-called partially quenched partition function for QCD \cite{14}, in which one introduces bosonic and fermionic valence quark species in addition to the ordinary quarks. When the bosonic and fermionic valence quark masses coincide the partially quenched partition function coincides with the original QCD partition function. For a nice review of this approach see also \cite{15}.

The cases corresponding to fundamental fermions with gauge group $SU(2)$, fundamental fermions with gauge group $SU(N_c)$ ($N_c \geq 3$), and adjoint fermions with gauge group $SU(N_c)$ ($N_c \geq 2$) (labeled respectively by $\beta = 1, 2, 4$) in four dimensions, and the corresponding cases labeled by $\beta = 1, 2$ in three dimensions have been analyzed in \cite{5,7,9}. In this paper we will treat the only “missing” case, namely three-dimensional QCD in the adjoint representation. We take the color gauge group to be $SU(N_c)$ with $N_c \geq 2$. This case will correspond to a matrix model labeled by $\beta = 4$, as we will find that the Dirac operator is quaternion real, that is, that its matrix elements can be written in the form $Q_{kl} = a_{kl}^0 + i\vec{a}_{kl} \cdot \vec{\sigma}$ where the $a_{kl}^\mu$ are real numbers, $1$ is the $2 \times 2$ unit matrix and $\vec{\sigma}$ the triplet of Pauli matrices. Like for all the other cases, we will propose a Gaussian random matrix theory corresponding to the low-energy partition function by substituting the integral over gauge fields with an integral over a random, quaternion real matrix. By expressing the fermion determinant as an integral over Grassmann variables (utilizing the supersymmetric formalism developed in \cite{16}), we will then perform the Gaussian integration corresponding in the field theory to the average over gauge fields. After manipulating the partition function further and eventually performing the Grassmann integration, we will obtain the finite volume partition function. From this partition function the pattern of spontaneous flavor symmetry breaking (assuming that such breaking takes place) will emerge. We will assume from the outset that there is a nonzero condensate $\Sigma$. The discrete parity symmetry discussed below and in \cite{7,9,17} remains unbroken. (For a more detailed discussion of this issue see these references.)
2 The parity-invariant Dirac operator in three dimensions

To begin, let us consider the Minkowski space Lagrangian for QCD in the adjoint representation

\[ L = -\frac{1}{4} \text{tr} F^2 + \sum_{f=1}^{2N_f} \bar{\psi}_f (iD - m_f) \psi_f \]  

(2.2)

where \( F \) is the gauge field tensor, \( D \equiv \gamma^\mu D_\mu \), \( D_\mu \) is the covariant derivative for the adjoint representation given explicitly in (4.12), and \( m_f \) is the quark mass corresponding to flavor \( f \). For reasons that will become evident, we call the total number of flavors \( 2N_f \). \( \psi_f \) are quark spinors in the adjoint representation and \( f \) is the flavor index (the indices corresponding to color and spin are suppressed).

The lowest-dimensional representation of \( \gamma^\mu \) is given by the Pauli matrices \( \gamma^0 = \sigma_3 \), \( \gamma^1 = i\sigma_1 \), \( \gamma^2 = i\sigma_2 \). In this 2d representation, there is no chiral symmetry, since there is no \( 2 \times 2 \) matrix that anticommutes with the \( \sigma_k \).

For zero masses \( m_f \), the above Lagrangian is invariant under parity \( P \), defined in three dimensions by

\[
\begin{align*}
\psi(t, x_1, x_2) &\rightarrow \gamma_1 \psi(t, -x_1, x_2) \\
A_0(t, x_1, x_2) &\rightarrow A_0(t, -x_1, x_2) \\
A_1(t, x_1, x_2) &\rightarrow -A_1(t, -x_1, x_2) \\
A_2(t, x_1, x_2) &\rightarrow A_2(t, -x_1, x_2)
\end{align*}
\]

(2.3)

The mass term breaks this \( P \) invariance. However, by choosing half of the masses equal to \(+m\) and half equal to \(-m\), we can achieve a \((P, Z_2)\)-invariant Lagrangian \([4, 18]\). In terms of 2–spinors, this choice corresponds to \( N_f \) 2–spinors \( \phi_f \) with mass \(+m\), and \( N_f \) 2–spinors \( \chi_f \) with mass \(-m\). Under \( P \) the mass terms for the 2–spinors change sign, so that if the two sets of two-spinors transform into each other in a \( Z_2 \) transformation \( \phi_f \leftrightarrow \chi_f \), \( f = 1, 2, ..., N_f \), the total Lagrangian is invariant under the combined transformations \( P \) and \( Z_2 \). We can use this choice of mass term to write down a \((P, Z_2)\)-invariant Lagrangian in the adjoint representation:
\[ L = -\frac{1}{4} \text{tr} F^2 + \sum_{f=1}^{2N_f} \bar{\psi}_f iD \psi_f - \sum_{f=1}^{N_f} m \bar{\psi}_f \psi_f + \sum_{f=N_f+1}^{2N_f} m \bar{\psi}_f \psi_f \] (2.4)

As we will see in section 4, the given representation of \( D_\mu \) – in this case the adjoint representation – uniquely defines (in Euclidean space) the anti-unitary operator \( Q \) that commutes with the Dirac operator \( iD \). We will see that for the adjoint representation, the condition \( Q \psi = \psi \) leads in the fermionic partition function to an integral over only half of the number of fermionic degrees of freedom with respect to the fundamental representation. The condition \( Q \psi = \psi \) is therefore called the Majorana condition.

Also for \( \beta = 1 \) (fundamental fermions with \( SU(2) \) color symmetry) we have an anti-unitary symmetry \([iD, \tilde{Q}] = 0\) (with the difference that \( \tilde{Q}^2 = +1 \), whereas in the adjoint case we have \( Q^2 = -1 \); see below), which leads to the same kind of relation \( \tilde{Q} \psi = \psi \). In the case of the fundamental \( SU(2) \) representation this relation defines a basis in which the Dirac operator has real matrix elements. In the adjoint case, because \( Q^2 = -1 \), we will find that we can write the Dirac operator in terms of real quaternions (see section 5). In contrast to the adjoint case, in the fundamental \( SU(2) \) case we integrate in the partition function over both \( \bar{\psi} \) and \( \psi \), which are independent degrees of freedom. This was shown explicitly in section 3 of ref. [9]. For the gauge groups \( SU(N_c) \) \((N_c \geq 3)\) and fundamental fermions, we have no anti-unitary symmetry.

To discuss the theory (2.4) in Euclidean space, we first discuss the general issue of defining Majorana fermions in Euclidean space.

3 Defining Majorana fermions in Euclidean space

It is not in general straightforward to define Majorana fermions in Euclidean space. In 4d, this is due to the fact that the would-be Majorana fermions do not transform like Dirac spinors under Euclidean Lorentz-transformations. This in turn is due to the absence, in Euclidean space, of an equation relating the right- and lefthanded Lorentz-transformations. For Minkowski space there is such a relation:

\[ \sigma_2 \Lambda_R \sigma_2 = \Lambda_L^* \] (3.5)

This relation guarantees, in 4d Minkowski space, that the upper and lower components of the usual Majorana fermions, \( \xi_L \) and \(-\sigma_2 \xi_L^*\), transform like lefthanded and righthanded components, respectively. In [3] it was shown that one can nevertheless construct a fermionic partition function
for Majorana fermions in 4d (or 2d) Euclidean space. To begin with, we will here briefly recall how this was done.

Given the Euclidean Dirac operator for adjoint fermions, $i\slashed{D}$ (defined as usual by $\slashed{D} = \gamma_\mu D_\mu$ where $\gamma_\mu$ are Euclidean gamma matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and $D_{\mu ab} = \partial_\mu \delta_{ab} + f_{abc} A_c^\mu$), an anti-unitary operator $Q$ defined by

$$[i\slashed{D}, Q] = 0$$

was identified. Since in 4d it turns out that $Q^2 = -1$, the Majorana condition

$$Q\psi = \psi$$

is contradictory (since it implies that $\psi = -\psi$ and thus $\psi = 0$), unless we define

$$\psi^{**} = -\psi$$

This condition is called conjugation of the second kind. It is common in the literature on Grassmann variables and in calculations involving supersymmetric random matrix theories. If $Q$ commutes with Euclidean Lorentz-transformations, a partition function for Euclidean Majorana fermions can now be defined if we can find an operator $\mathcal{O}$ such that

$$Z = \sqrt{\det(i\slashed{D})} = \int D\psi \ e^{-\psi^T \mathcal{O} i\slashed{D} \psi}$$

where $\psi$ are Majorana fermions satisfying $Q\psi = \psi$, $\det \mathcal{O} = 1$ and $\mathcal{O} i\slashed{D}$ is an antisymmetric operator. This last condition guarantees that the square root is well-defined. Lorentz-invariance of this partition function is guaranteed because in addition, we demand that $\mathcal{O}$ by construction satisfies

$$\psi^T \mathcal{O} = \psi^T \equiv (Q\psi)^\dagger$$

(we recall that in Euclidean space the Lorentz-invariant quantity is $\psi^T \psi$). In (3.10) we integrate over only half of the degrees of freedom with respect to the partition function for the usual Dirac fermions.
Conversely, the Majorana condition can be identified by the condition (3.10), once we have written down the partition function, by demanding Lorentz-invariance of the latter.

4  The fermion determinant

Following these hints, we can now construct a partition function for Euclidean Majorana fermions in 3d. As it turns out, the problems with defining Euclidean Majorana fermions that we encounter in four dimensions, are not present in 3d because the Majorana fermions defined by (3.7) transform correctly under 3d Euclidean Lorentz-transformations. This is because in 3d, there is just one kind of 2-spinor and just one two-dimensional representation of the Lorentz group, which is equivalent to the rotation group.

We start by defining Hermitian gamma matrices for our 3d Euclidean space. They can simply be taken to be the Pauli matrices

\[ \gamma_0 = \sigma_3, \quad \gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_2 \]  

(4.11)
satisfying \( \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \). The Dirac operator in the adjoint representation is

\[ \gamma_\mu D_{\mu ab} = \gamma_\mu (\partial_\mu \delta_{ab} + iA_\mu^c(T_c)_{ab}) = \gamma_\mu (\partial_\mu \delta_{ab} + f_{abc}A_\mu^c) \]  

(4.12)

where \( f_{abc} \) are real structure constants for the gauge group. \( D_\mu \) is antisymmetric under transposition and \( D \) is antihermitean. It is easy to find the antiunitary operator that satisfies (3.6) in this case. It is given by \( Q = i\gamma_2 K \) where \( i\gamma_2 \equiv C \) is the charge conjugation operator satisfying \( C\gamma_\mu C^{-1} = -\gamma_\mu \) and \( K \) denotes complex conjugation. We see that like in the 4d case outlined above, \( Q^2 = -1 \) and the Majorana condition

\[ \psi = CK\psi \]  

(4.13)
makes sense only if we introduce conjugation of the second kind, \( \psi^{**} = -\psi \). The Majorana condition is then consistent with the explicit form

\[ \psi = \left( \begin{array}{c} \chi \\ -\chi^* \end{array} \right) \]  

(4.14)
of the spinors. The generators of the Euclidean Lorentz-group corresponding to the representation \( (8,56) \) of \( \gamma_\mu \) are given by \[ S^\mu_\nu = \frac{i}{4} [\gamma_\mu, \gamma_\nu] \] (4.15)

We find \( S_{01} = -\sigma_2/2 \), \( S_{02} = \sigma_1/2 \) and \( S_{12} = -\sigma_3/2 \). These are equivalent to the generators of the rotation group and one can easily verify that the corresponding Lorentz-transformations commute with the anti-unitary operator \( Q \) defining the Majorana condition in our case.

To write down the fermionic action we now look for an operator \( \mathcal{O} \) such that \( \det \mathcal{O} = 1 \) and \( \psi^T \mathcal{O} = \psi^\dagger = (Q\psi)^\dagger \). We immediately arrive at

\[ \mathcal{O} = -i\sigma_2 \] (4.16)

In addition, \( \mathcal{O}i\mathcal{D} \) is antisymmetric so it is now straightforward to write down a partition function for Majorana fermions satisfying (4.13) in 3d Euclidean space

\[ \sqrt{\det(i\mathcal{D})} = \sqrt{\det(\mathcal{O}i\mathcal{D})} = \int D\psi e^{-\psi^T \mathcal{O}i\mathcal{D}\psi} \] (4.17)

5 The Dirac operator for Majorana fermions

In analogy with 4d, we expect that there is a basis in which \( i\mathcal{D} \) can be written in the form of real quaternions \[ i\mathcal{D} = \mathcal{D}(\mathcal{H}) \] (5.18)

where \( \hat{\phi}_k \) are arbitrary c-number 2-spinors and \( \chi_k \) are Grassmann variables. Since \( (CK)^2 = -1 \), \( \hat{\phi}_k \) and \( C\hat{\phi}_k^* \) are linearly independent.

We then see that the expression in the fermionic action involving the Dirac operator can be written
\[ \psi^\dagger i \mathcal{D} \psi = \sum_{kl} \left( \hat{\phi}_k^\dagger \chi_k^* \left( C \hat{\phi}_k^\dagger \right)^\dagger \chi_k \right) i \mathcal{D} \left( \hat{\phi}_l \chi_l + C \hat{\phi}_l^* \chi_l^* \right) \]
\[ = \sum_{kl} \left( \chi_k \chi_k^* \right)^* \left( \begin{array}{cc}
\hat{\phi}_k^\dagger i \mathcal{D} \hat{\phi}_l \\
\hat{\phi}_k^T C^\dagger i \mathcal{D} \hat{\phi}_l \\
\hat{\phi}_k^\dagger C^\dagger i \mathcal{D} C \hat{\phi}_l^* 
\end{array} \right) \left( \begin{array}{c} 
\chi_l \\
\chi_l^* 
\end{array} \right) \] (5.19)

The matrix

\[ Q_{kl} = \begin{pmatrix}
\hat{\phi}_k^\dagger i \mathcal{D} \hat{\phi}_l & \hat{\phi}_k^\dagger i \mathcal{D} C \hat{\phi}_l^* \\
\hat{\phi}_k^T C^\dagger i \mathcal{D} \hat{\phi}_l & \hat{\phi}_k^T C^\dagger i \mathcal{D} C \hat{\phi}_l^* 
\end{pmatrix} \] (5.20)

has the form

\[ Q_{kl} = \begin{pmatrix}
A & B \\
-B^* & A^* 
\end{pmatrix} \] (5.21)

This can be verified using the properties of \( C \) and eq. (3.6) from which we derive

\[ C^\dagger i \mathcal{D} C = (i \mathcal{D})^* \]
\[ C^\dagger i \mathcal{D} = -(i \mathcal{D})^* C \] (5.22)

\( Q_{kl} \) can be written as a real quaternion:

\[ Q_{kl} = a_{kl}^0 + i a_{kl}^i \mathbf{\hat{\sigma}} \cdot \mathbf{\hat{\sigma}} \] (5.23)

where \( a_{kl}^0, a_{kl}^i \) are real numbers.

### 6 Random matrix theory

We are now ready to define the random matrix theory for massive, parity-invariant three-dimensional QCD with fermions in the adjoint representation. In analogy with all the other
cases in 3d [4, 5] and in 4d [3, 4, 5, 10], we substitute the integral over gauge field configurations in the partition function for QCD with an integral over a Hermitian random matrix $T$. We take this matrix to have the quaternion structure (5.20). Using the same notation as in [3] we therefore set

$$T_{ij} = \sum_{\mu=0}^{3} a_{ij}^{\mu} i\sigma_{\mu}$$  \hspace{1cm} (6.24)

where $a_{ij}^{\mu}$ are real numbers and we have defined $\sigma_0 = -i$ and $\sigma_1, \sigma_2, \sigma_3$ are the usual Pauli matrices. This means the matrix elements of $T$ are themselves $2 \times 2$ matrices. The operator $D$ is antihermitian, so we can substitute $D$ in the Euclidean fermion determinant with the matrix $iT$ where $T$ is Hermitian:

$$Z(m_1, ..., m_{2N_f}) = \int DA e^{-S[A]} \prod_{f=1}^{2N_f} \sqrt{\det(D + m_f)}$$

$$\rightarrow \int DT e^{-N\Sigma^2 i(T^T T)} \prod_{f=1}^{2N_f} \sqrt{iT + m_f}$$  \hspace{1cm} (6.25)

This random matrix theory has the same global flavor symmetry as the QCD partition function. In addition, $T$ is here taken to be a matrix of $N \times N$ real quaternions, so that the anti-unitary symmetry of adjoint QCD is reproduced, and as we will see, the flavor symmetry breaking pattern will appear from the random matrix partition function. $DT$ is the invariant (Haar) measure. As usual [3, 4, 5, 7, 9, 10] $N$ is identified with the space-time volume. We will see that $\Sigma$ is the value of the condensate (the order parameter for spontaneous symmetry breaking) defined by

$$\Sigma = - \lim_{m_f \to 0} \lim_{N \to \infty} \frac{1}{N} \frac{\partial}{\partial m_f} \ln Z(m_1, ..., m_{2N_f})$$  \hspace{1cm} (6.26)

We are assuming that this condensate is non-zero. As discussed in Section 3, we choose the masses in pairs of opposite sign, so that the corresponding Minkowski space theory is parity-invariant. We call the total number of flavors $2N_f$, and we choose $m_f = +m$ for $f = 1, ..., N_f$ and $m_f = -m$ for $f = N_f + 1, ..., 2N_f$, where $m$ is a real positive number.

In order to evaluate the partition function (6.25) we rewrite the square root of the fermion determinant as an integral over Grassmann variables (cf. (5.13)):
\[ \prod_f \sqrt{\det(iT + m_f)} = \int \prod_f D\phi_f \exp \left[ -i \sum_f \phi^*_f (T - im_f)_{ij} \phi^i_f \right] \]  

(6.27)

where the indices \( i, j \) are summed over from 1 to \( N \). Here we are using the formalism developed in [16] for supersymmetric matrix integrals. Our matrix integral is pure fermionic and involves no bosonic variables, so we will apply this formalism using only the fermion-fermion block. Our integration measure is

\[ \prod_f D\phi_f = \prod_{f=1}^{2N_f} \prod_{i=1}^{N} d\chi^i_f d\chi^i_f \]  

(6.28)

Recall that \( T_{ij} \) are quaternions, so the \( \phi^i_f \) are 2-component vectors like in (5.19):

\[ \phi^i_f = \left( \chi^i_f, \chi^i_f^* \right) \]  

(6.29)

We use conjugation of the second kind \( \chi^{**} = -\chi \) for the Grassmann variables.

The first step is to perform the Gaussian integral over the random matrix \( T \) in the integral

\[
Z(m) = \int DT \int \prod_f D\phi_f \exp \left[ -N \Sigma^2 \text{tr}(T^\dagger T) - i \sum_f \phi^*_f (T - im_f)\phi_f \right] = \int Da^{\mu} \prod_f D\phi_f \exp \left[ -N \Sigma^2 a^{\mu}_{ij} a^\mu_{ij} + \sum_f \phi^*_f \sigma^{\mu}_j \phi^j_f - \sum_f m_f \phi^*_f \phi_f \right]
\]  

(6.30)

where we sum over repeated indices \( \mu \) and \( i, j \). In the second step of (6.30) we have used that \( \sigma^\mu \sigma^\nu + \sigma^\nu \sigma^\mu = 2\delta_{\mu\nu} \). To perform the Gaussian integral we complete the square in the exponent of (6.30) by setting

\[ a^{\mu}_{ij} \rightarrow \tilde{a}^{\mu}_{ij} \equiv a^{\mu}_{ij} - \frac{1}{2\Sigma^2 N} \sum_f \phi^*_f \sigma^\mu \phi^i_f \]  

(6.31)
The symmetry properties of the $\tilde{a}_{ij}^\mu$ are the same as the symmetry properties of $a_{ij}^\mu$, namely $a_{ij}^0 = a_{ji}^0$, $a_{ij}^k = -a_{ji}^k$ ($k = 1,2,3$). These symmetry properties follow from the hermiticity of the matrix $T$ and (6.24). The integral over $T$ is uniformly convergent in the fermionic variables $\phi_f^i$, so we can interchange the two integrals and perform the Gaussian integration. We then arrive at

$$Z(m) \sim \int \prod_f D\phi_f \exp \left[ \frac{1}{4\Sigma^2 N} \sum_{fg} \phi_f^i \bar{\sigma}_\mu \phi_g^j \phi_g^i \bar{\sigma}_\mu \phi_f^j - \sum_f m_f \phi_f^i \phi_f^i \right]$$

(6.32)

To evaluate $\sum_\mu \phi_f^i \bar{\sigma}_\mu \phi_g^i \phi_g^i$, we use the Fierz' identity [5]

$$\sum_\mu \bar{\sigma}_\mu \sigma_\mu = 2(\delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\beta} \delta_{\gamma\delta})$$

(6.33)

and insert the explicit form for the 2-component Grassmann variables $\phi_f^i$

$$\phi_f^i = \begin{pmatrix} \chi_f^i \\ \chi_f^{i*} \end{pmatrix}$$

(6.34)

to arrive at

$$\sum_\mu \phi_f^i \bar{\sigma}_\mu \phi_g^i \phi_g^i \phi_f^i = 2 F_{fg}^2$$

(6.35)

where we have set

$$F_{fg} \equiv \chi_f^i \chi_g^i + \chi_g^i \chi_f^i$$

(6.36)

(a sum over $i$ is understood). Disregarding for a moment the mass term, we have rewritten the exponent in $Z(m)$ as a square. We can therefore use the Hubbard-Stratonovitch transformation [16]

$$\exp \left[ -\alpha F_{fg} F_{fg} \right] \sim \int d\sigma_{fg} \exp \left[ - \frac{1}{4\alpha} \sigma_{fg} \sigma_{fg} - i \sigma_{fg} F_{fg} \right]$$

(6.37)
where $\sigma_{fg}$ is a real variable, to rewrite the partition function as

$$Z(m) \sim \int \prod_f D\chi_f D\sigma \exp \left[ -\frac{\Sigma^2 M^2}{2} + \sigma_{fg} F_{fg} - 2 \sum_f m_f \chi^*_f \chi_f^f \right]$$  \hspace{1cm} (6.38)$$

In (6.38), $D\sigma$ is the Haar measure for the real symmetric matrix $\sigma$. In order to preserve the flavor symmetry of $Z(m)$, $\sigma$ is chosen symmetric like the matrix $F$. Interchanging again the Grassmann integral with the integral over $\sigma$, and writing out the Grassmann components of $F_{fg}$ we see that the partition function is proportional to

$$Z(M) \sim \int D\sigma \prod_f D\chi_f \exp \left[ -\frac{\Sigma^2 M^2}{2} \text{tr}(\sigma \sigma^T) + 2 \chi^*_i (\sigma + M) \chi^i \right]$$  \hspace{1cm} (6.39)$$

where $M$ is the $2N_f \times 2N_f$ mass matrix in flavor space,

$$M = \begin{pmatrix} m & \cdots \\ \vdots & \ddots \\ \cdots & \cdots & m \\ -m & \cdots & \cdots & -m \end{pmatrix}$$  \hspace{1cm} (6.40)$$

Performing the Grassmann integrals we get

$$Z(M) \sim \int DS e^{-\frac{\Sigma^2 M^2}{2} \text{tr}(SS^T) \det N(S + M)}$$  \hspace{1cm} (6.41)$$

with $S$ a symmetric real matrix and $DS$ the Haar measure. In the next section we will evaluate this expression using a saddle point analysis to find the low-energy effective partition function for three-dimensional QCD with adjoint fermions.

7 The effective partition function

We now decompose the real symmetric matrix $S$ in (6.41) into “polar” coordinates [21]:
\[ S = O \Lambda O^T \]  \hspace{1cm} (7.42)

where \( O \) is a real orthogonal matrix and \( \Lambda \) is the real diagonal matrix

\[ \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{2N_f} \end{pmatrix}, \]  \hspace{1cm} (7.43)

One can always choose the polar coordinates such that

\[ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{2N_f} \]  \hspace{1cm} (7.44)

The integration variables in \( S \) and those in \( O \Lambda O^T \) will be in one-to-one correspondence if the integral over \( DS \) is taken to be

\[ \int DS = \int_{O \in [O]} DO \Lambda D \Lambda J(\Lambda) \]  \hspace{1cm} (7.45)

where \([O]\) denotes the set of left cosets of the group of \( 2N_f \times 2N_f \) real orthogonal matrices with respect to the subgroup consisting of matrices of the form

\[ \left( \begin{array}{cccc} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{array} \right) \]  \hspace{1cm} (7.46)

and \( J(\Lambda) \) is the Jacobian corresponding to (7.42). This Jacobian was given in [2] and is proportional to

\[ J(\Lambda) \propto \prod_{i<j} |\lambda_i - \lambda_j| \]  \hspace{1cm} (7.47)

We are now ready to determine the saddle point of the partition function at zero mass. Setting
\( M = 0 \) in (6.41) and making the substitution (7.42), (7.45) in \( Z(0) \) the partition function takes the form

\[
Z(0) \sim \int D\Lambda \exp \left[ -\frac{\Sigma^2 N}{2} \sum_f \lambda_f^2 + N \sum_f \ln \lambda_f - \ln J(\lambda_1, \ldots \lambda_{2N_f}) \right]
\]  

(7.48)

The matrix \( \Lambda \) is diagonal and the saddle point is given by

\[
\lambda_f = \pm \frac{1}{|\Sigma|}
\]  

(7.49)

since the Jacobian drops out in the large \( N \) limit. It follows from the derivation of the Banks-Casher formula that if the flavor symmetry is broken spontaneously, the condensate for each flavor has to have the same sign as the mass \([1, 7, 9]\), so we choose, in accordance with (7.44) and (6.40)

\[
\Lambda_{sp} = \frac{1}{|\Sigma|} \begin{pmatrix}
1 & \cdots & \cdots \\
\vdots & 1 & -1 \\
\cdots & \cdots & -1
\end{pmatrix} \equiv \frac{1}{|\Sigma|} J
\]  

(7.50)

We now expand the determinant in (6.41) at the saddle point, \( \Lambda = \Lambda_{sp} \) for a small mass matrix \( M \neq 0 \). To first order in \( M \) we find

\[
Z(M) \sim \int DO \det^N (O\Lambda_{sp}O^T + M)
\]

\[
\sim \int DO \det^N (O\Lambda_{sp}O^T) e^{N \text{tr} \ln (1 + O\Lambda_{sp}^{-1}O^T M)}
\]

\[
\propto \int DO e^{N \Sigma \text{tr}(OJOT)M}
\]  

(7.51)

The matrix \( J \) was defined in (7.50). This is our expression for the low-energy effective partition function. As usual, it is a function of the scaling variable \( N\Sigma M \). The matrix \( J \) is invariant under the subgroup \( O(N_f) \times O(N_f) \), so in our final expression for \( Z(M) \),
\[ Z(\mathcal{M}) \sim \int_{O(2N_f)/\left(O(N_f) \times O(N_f)\right)}^{} DO e^{N \Sigma \text{tr}(OJO^T \mathcal{M})} \]  

(7.52)

the integral goes over the coset space \( O(2N_f)/(O(N_f) \times O(N_f)) \). The flavor symmetry breaking pattern is thus \( O(2N_f) \rightarrow O(N_f) \times O(N_f) \). This is also what we expect from the fermionic action,

\[ S_F = \int d^3x \sum_{f=1}^{2N_f} \psi_f^T \mathcal{D} \psi_f + m_f \psi_f \]  

(7.53)

It is evident that \( \sum_f \psi_f^T \mathcal{D} \psi \) is invariant under \( O(2N_f) \) transformations in flavor space (note that the operator \( \mathcal{D} \) is diagonal in flavor space), while the mass term \( \sum_f m_f \psi_f^T \psi_f \) is invariant under \( O(N_f) \times O(N_f) \), which is the unbroken subgroup. The dimension of the coset is

\[ M = \frac{2N_f(2N_f - 1)}{2} - 2 \frac{N_f(N_f - 1)}{2} = N_f^2 \]  

(7.54)

8 Sum rules

We can easily derive the first Leutwyler-Smilga like sum rule for the eigenvalues of the Dirac operator. We will use the same method as in [5, 9].

The sum rules are obtained by expanding the expression for \( Z(\mathcal{M}) \), eq. (7.52) and comparing the coefficients order by order in \( m^2 \) to the (normalized) expectation value of the fermion determinant:

\[ \langle \prod_f \prod_{\lambda_k > 0} \left( 1 + \frac{m^2}{\lambda_k^2} \right) \rangle \]  

(8.55)

(Note that because of our choice of \((P, Z_2)\)-invariant Lagrangian, there is an effective chiral symmetry that makes the spectrum symmetric around \( \lambda = 0 \). This chiral symmetry applies to the 4 \( \times \) 4 gamma matrices given in eq. (2.4) of ref. [9]:

\[ \gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \]  

(8.56)
or equivalently to the corresponding Euclidean gamma matrices. We could just as well write \( \mathcal{L} \) in terms of these.) Here the expectation value is defined as

\[
\langle f(\lambda, m) \rangle = \frac{\int DA e^{-S[A]} \prod f_k^2 f(\lambda, m)}{\int DA e^{-S[A]} \prod f_k^2 f(\lambda, 0)}
\] (8.57)

where \( A \) is the gauge field and \( S[A] \) the Euclidean Yang-Mills action. Expanding the integrand in (7.52) the surviving group integrals at order \( m^2 \) have the form

\[
\zeta(X) = \int_{O \in G/H} DO \text{tr}(OJO^T X)\text{tr}^*(OJO^T X) \equiv \int_{O \in G/H} DO \text{tr}^2(OJO^T X)
\] (8.58)

where \( G/H \) is the coset and \( X \equiv N\Sigma M = N\Sigma mJ \). The first order term is killed by the group integration. We note that the matrices \( OJO^T \) are \( 2N_f \times 2N_f \) symmetric unimodular matrices. We now choose real, symmetric and traceless generators \( t_k, k = 1, ..., M_s \) for these. \( M_s \) is the number of such generators:

\[
M_s = \frac{2N_f(2N_f + 1)}{2} - 1
\] (8.59)

The generators can be normalized as follows:

\[
\text{tr}(t_k t_l) = 2N_f \delta_{kl}
\] (8.60)

This normalization is natural since the generators are \( 2N_f \times 2N_f \) matrices. We wish to choose \( t_1 \) such that \( t_1 = J \). In doing this we can use the relation

\[
\sum_{k=1}^{M_s} \text{tr}(At_k)\text{tr}(Bt_k) = 2N_f \text{tr}(AB)
\] (8.61)

to calculate the value of \( \zeta(X) \). Equation (8.61) is valid for any two symmetric unimodular matrices \( A \) and \( B \). Because of the invariance of the measure in the integral defining \( \zeta(X) \), one finds that \( \zeta(X) \) is proportional to \( \text{tr}(X^\dagger X) \). Due to eq (8.61) and the choice of \( t_1 \), we find \( \zeta(t_1) = \zeta(t_2) = \ldots = \zeta(t_{M_s}) \). Therefore we can set
\[
\zeta(t_1) = \frac{1}{M_s} \sum_{k=1}^{M_s} \int D\theta \, \text{tr}^2(O\Omega O^T t_k) \tag{8.62}
\]

Using (8.61) and tr\((J^2) = 2N_f\) we now immediately see that

\[
\zeta(X) = \frac{1}{M_s} \text{vol}(G/H)(N\Sigma m)^2(2N_f)^2 \tag{8.63}
\]

Inserting this into the expansion we get

\[
\frac{Z(m)}{Z(0)} = \left\langle 1 + m^22N_f \sum_{\lambda_k > 0} \frac{1}{\lambda_k^2} + \ldots \right\rangle = 1 + \frac{1}{2}(N\Sigma m)^2 \frac{1}{M_s}(2N_f)^2 + \ldots \tag{8.64}
\]

where the volume of the coset cancels in the ratio. Inserting the value of \(M_s\) we therefore arrive at the sum rule

\[
\left\langle \sum_{\lambda_k > 0} \frac{1}{(N\Sigma \lambda_k)^2} \right\rangle = \frac{2N_f}{2(2N_f - 1)(N_f + 1)} \tag{8.65}
\]

where \(2N_f\) is the original number of flavors. Considering also the sum rules found in [7, 9] for Dyson index \(\beta = 2\), 1 in 3d, the sum rules for three dimensions can be summarized as follows:

\[
\left\langle \sum_{\lambda_k > 0} \frac{1}{(N\Sigma \lambda_k)^2} \right\rangle = \frac{2N_f}{2(2N_f - 1)(4N_f/\beta + 1)} \tag{8.66}
\]

where \(2N_f\) is the number of flavors and \(\beta\) the Dyson index. The corresponding formula for the 4d cases was given in [4, 5]. In the published version of this paper, as well as in the previous electronic version, the wrong sum rule was given due to an error in (8.61).

9 Conclusion

We have obtained the mass dependence of the finite volume partition function for three-dimensional QCD with quarks in the adjoint representation and gauge group \(SU(N_c)\) \((N_c \geq 2)\). We chose the
quark masses such that the corresponding Minkowski space QCD Lagrangian is \((P, Z_2)\)-invariant, where \(P\) denotes the parity transformation defined in three dimensions by eq. (2.3).

As a starting point we used a random matrix theory with the same global symmetries as the gauge theory. This is possible since the gauge theory and the random matrix theory are equivalent in the mesoscopic regime.

The effective partition function describes the static limit of the Goldstone modes resulting from the spontaneous breaking of global flavor symmetry. It is determined by the symmetry breaking pattern. We assumed that flavor symmetry breaking occurs, and found that the global flavor symmetry \(O(2N_f)\) is broken by the vacuum state to \(O(N_f) \times O(N_f)\) symmetry.

Although the author does not claim to understand the classification of symmetric superspaces, it is interesting to note that the result obtained here for the Goldstone manifold, together with the other Goldstone manifolds obtained for the \(\beta = 1, 2, 4\) in 3d and 4d, exhaust the scheme of Zirnbauer [21] with respect to the fermionic symmetric spaces \(M_F\). (I thank Poul Damgaard and Jac Verbaarschot for this remark.) We can see this by looking at Table 3 of [21] and matching the various random matrix ensembles with the fermionic symmetric spaces given by Zirnbauer. These spaces, together with the corresponding bosonic symmetric spaces \(M_B\), make up the Riemannian symmetric superspaces corresponding to the various supersymmetric generating functions. In simple terms, each symmetric superspace corresponds to a random matrix theory. For us the \(M_F\) define the respective Goldstone manifolds, since we do not have any bosonic sector. From the fermionic symmetric spaces given by Zirnbauer [21] we confirm that our case, the Gaussian symplectic ensemble, corresponds to the random matrix theory denoted AII in Table 3 of [21], while the coset spaces obtained in [5, 7, 9] nicely fit the compact symmetric spaces belonging to the classes labelled by A, AI, AIII, BDI, and CII. The classes A, AI, and AII correspond to the non-chiral ensembles while AIII, BDI, and CII correspond to the chiral ones.

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