Constant factor approximations for Lower and Upper bounded Clusterings

Neelima Gupta¹, Sapna Grover² and Rajni Dabas³
Department of Computer Science, University of Delhi, India.

1. ngupta@cs.du.ac.in
2. sgrover@cs.du.ac.in, sapna.grover5@gmail.com
3. rajni@cs.du.ac.in

Abstract

Clustering is one of the most fundamental problem in Machine Learning. Researchers in the field often require a lower bound on the size of the clusters to maintain anonymity and upper bound for the ease of analysis. Specifying an optimal cluster size is a problem often faced by scientists. In this paper, we present a framework to obtain constant factor approximations for some prominent clustering objectives, with lower and upper bounds on cluster size. This enables scientists to give an approximate cluster size by specifying the lower and the upper bounds for it. Our results preserve the lower bounds but may violate the upper bound a little.

We also reduce the violation in upper bounds for a special case when the gap between the lower and upper bounds is not too small.

Keywords: Facility Location, k-Median, k-Center, Lower Bound, Upper Bound, Approximation.

1 Introduction

Clustering is one of the most fundamental problem in Machine Learning: given a set of data points, we wish to group them into clusters so that the points within a cluster are most similar to each other and those in different clusters are most dissimilar. Various objective functions have been used in literature to capture the notion of similarity. Three objectives that have been extensively studied in combinatorial optimization are captured in k-Center (kC)/k-Median (kM) and facility location (FL) problems. In the k-Center (/k-Median) problem, we wish to identify a set of k centers and assign points to it so as to minimise the maximum distance(dissimilarity) of any point from its assigned center(/minimise
the sum of distances of the points from the assigned center). In FL, instead of a bound on the number of centers to open, we have center/facility opening cost and we wish to form clusters so that sum of facility opening cost and the sum of distances of the points/clients from the assigned center is minimised. 

$k$-Facility Location problem (kFL) is a common generalization of kM and FL, where-in we have a bound $k$ on the maximum number of facilities to be opened and facility opening costs associated with each facility. And, the objective now is to minimize the total cost of opening the selected facilities and assigning clients to them.

The problems are well known to be NP-hard. Several approximation results have been developed for these problems in the basic version. However, constraints occur naturally in clustering. Adding constraints makes the problem harder. One such constraint requires every center to have a minimum number of data points assigned to it. An application requiring lower bound on the cluster size is data privacy [60] that demands every point in the cluster to be "alike" and indistinguishable from each other. In the FL inspired clustering, lower bounds are required to ensure profitability.

Another constraint imposes an upper bound on the maximum number of data points that can be assigned to a center. In the FL inspired clustering, capacities occur naturally on the facilities. In case of cluster analysis, researchers often do not want the clusters to be too big for the ease of analysis.

In this paper, we study the problems with both the lower as well as the upper bounds on the minimum and the maximum number, respectively, of data points that are assigned to a center. Researchers often face the problem of determining the appropriate size of clusters and, several heuristics are used to estimate the same [11, 30]. In this paper, we handle this situation somewhat giving them an opportunity to give a rough estimate of the size in terms of lower and upper bound on the size. Heuristics provide no performance guarantee whereas our solution generates clusters that respect the lower bounds and violate the upper bounds a little. In FL inspired clustering, limitation of capacities and the requirement of scale for profitability occur together in several applications like market-place, transportation problem etc. We present our results when one of the bounds is uniform. As upper bound has been popularly called as capacity in the literature, we will use the two terms inter-changeably in the paper. In the $k$-center problem, the set of data points is same as the set of centers i.e., the centers can be opened at the data points. $k$- supplier ($kS$) is a generalization of the $k$-center problem in which the two sets are different. Our results are applicable to the $k$-supplier problem.

**Theorem 1.** Given an $\alpha$-approximation for Upper bounded $k$-Median problem (UkM)/$k$-Facility Location problem (UkFL)/Facility Location problem (UFL)/$k$-Supplier problem (UkS)/$k$-Center problem (UkC) violating the upper bound by a factor of $\beta$ and a $\gamma$-approximation for uniform Lower bounded Facility Location problem (LFL)/$k$-Supplier problem (LkS)/$k$-Center problem (LkC), an $O(\alpha + \gamma)$-approximation can be obtained for Lower and Upper bounded $k$-Median problem (LUkM)/$k$-Facility Location problem (LUkFL)/Facility Location problem (LUkC).
Table 1: Our results for uniform lower bound.

| Problem | $\alpha$ | $\beta$ | $\gamma$ | Factor | Capacity violation |
|---------|----------|---------|----------|--------|-------------------|
| LUkM    | $O(1/\epsilon^2)$ | $1 + \epsilon$ | $82.6$ | $O(1/\epsilon^3)$ | $(2 + \epsilon)$ |
| LUFL    | $5$ | $1$ | $82.6$ | $O(1)$ | $2$ |
| LUkS    | $3$ | $1$ | $3$ | $O(1)$ | $2$ |
| LUkC    | $2$ | $1$ | $2$ | $O(1)$ | $2$ |

Table 2: Our results for uniform upper bound.

| Problem | $\alpha$ | $\beta$ | $\gamma$ | Factor | Capacity violation |
|---------|----------|---------|----------|--------|-------------------|
| LUkM    | $O(1/\epsilon^2)$ | $1 + \epsilon$ | $O(1)$ | $O(1/\epsilon^2)$ | $(2 + \epsilon)$ |
| LUkFL   | $O(1/\epsilon^2)$ | $2 + \epsilon$ | $O(1)$ | $O(1/\epsilon^2)$ | $(3 + \epsilon)$ |
| LUFL    | $3$ | $1$ | $O(1)$ | $O(1)$ | $2$ |
| LUkS    | $3$ | $1$ | $3$ | $O(1)$ | $2$ |
| LUkC    | $2$ | $1$ | $3$ | $O(1)$ | $2$ |

Theorem 2. Given an $\alpha$-approximation for uniform Upper bounded $k$-Median problem (UkM)/$k$-Facility Location problem (UkFL)/$k$-Supplier problem (UkS)/$k$-Center problem (UkC) violating the upper bound by a factor of $\beta + 1$.

Several studies have been devoted to the lower bounds and the upper bounds separately for the problems. The only works that deal with both the bounds together is due to Gupta et al. [36] and Friggstad et al. [29] for FL and, Ding et al. [28] and, Rösner and Schmidt [59] for kS and kC. Friggstad et al. [29] violates bounds on both the sides for non-uniform lower and uniform upper bounds whereas Gupta et al. [36] violates only the upper bound by a factor of 5/2 when both the bounds are uniform. For LkS/LukC, Ding et al. [28] and, Rösner and Schmidt [59] independently gave true constant factor approximations for uniform lower bounds and general upper bounds.

Thus, we present first approximations for LkM and LukFL. For LkFL, we reduce the 5/2 factor violation in upper bounds obtained by Gupta et al. [36] to 2. Our result for LukC and LukS are the first results with general lower bounds and uniform upper bounds.

Tables 1 and 2 summarize our results after applying the current best known
results for the underlying problems. Though no results are known for UkFL with
genral upper bounds and general facility opening costs, to plug into the result
of Theorem 1, the result will be useful in case we get one in future. As a special
case of LUKFL, we obtain first true constant factor approximation for kFL with
lower bounds (LkFL). The only result known for LkFL by Han et al. [37] for
general lower bounds, violates the lower bounds. The following corollary follows
from the theorem:

Corollary 1. There is a polynomial time algorithm that approximates Lower
bounded k-Facility Location within a constant factor.

Next, we improve upon the violation in the upper bound for a particular
scenario when the gap between the lower and the upper bounds is not too small
(a reasonable scenario to occur in real applications). In particular, if \( \mathcal{L}_i \) and \( \mathcal{U}_i \)
represent the lower and the upper bounds respectively of a facility \( i \), we present
the following results when \( 2\mathcal{L}_i \leq \mathcal{U}_i \forall i \in \mathcal{F} \) and either of the bounds is uniform.

Theorem 3. For \( 2\mathcal{L}_i \leq \mathcal{U}_i \forall i \in \mathcal{F} \), given an \( \alpha \)-approximation for Upper bounded
k-Median problem (UkM)/k-Facility Location problem (UkFL)/Facility Location
problem (UFL)/k-Supplier problem (UkS)/k-Center problem (UkC) violating the upper bound by a factor of \( \beta \) and a \( \gamma \)-approximation for uniform
Lower bounded Facility Location problem (LFL)/k-Supplier problem (LkS)/k-
Center problem (LkC), an \( (O(\alpha + \gamma)/\epsilon) \)-approximation can be obtained for
Lower and Upper bounded k-Median problem (LUkM)/k-Facility Location problem
(LUkFL)/Facility Location problem (LUkFL)/k-Supplier problem (LUkS)/k-
Center problem (LUkC) with uniform lower bounds in polynomial time that
violates the upper bound by a factor of \( \beta + \epsilon \) for a fixed \( \epsilon > 0 \).

Theorem 4. For \( 2\mathcal{L}_i \leq \mathcal{U}_i \forall i \in \mathcal{F} \), given an \( \alpha \)-approximation for uniform Upper bounded
k-Median problem (UkM)/k-Facility Location problem (UkFL)/Facility Location
problem (UFL)/k-Supplier problem (UkS)/k-Center problem (UkC) violating the upper bound by a factor of \( \beta \) and a \( \gamma \)-approximation for Lower bounded
Facility Location problem (LFL)/k-Supplier problem (LkS)/k-Center problem (LkC), an \( (O(\alpha + \gamma)/\epsilon) \)-approximation can be obtained for Lower and Upper bounded k-Median problem (LUkM)/k-Facility Location problem
(LUkFL)/Facility Location problem (LUkFL)/k-Supplier problem (LUkS)/k-
Center problem (LUkC) with uniform upper bounds in polynomial time that violates the upper bound by a factor of \( \beta + \epsilon \) for a fixed \( \epsilon > 0 \).

Tables 3 and 4 present our results after applying Theorem 3 and 4 to different
problems for \( 2\mathcal{L}_i \leq \mathcal{U}_i \forall i \in \mathcal{F} \).

1.1 Related Work
The classic k-center problem has been approximated with constant factor ratio
(2) by Gonzalez [31] and, Hochbaum and Shmoys [39]. The ratio is tight, i.e., it is
NP-hard to approximate it with a factor less than 2 unless P=NP [40]. Hochbaum
and Shmoys [39] also approximated k-supplier to 3 factor, which is also tight.
Upper bounded $k$-center problem was first studied by Bar-Ilan et al. [13] in 1993 for uniform capacities. The authors gave a 10-factor approximation algorithm. The factor was subsequently improved to 6 by Khuller and Sussmann [45]. For non-uniform capacities, Cygan et al. [26] gave the first constant factor approximation algorithm. Their factor was large, which was later improved to 9 by An et al. [7]. An 11 factor approximation algorithm for $UkS$ problem was given by An et al. [7]. Aggarwal et al. [4] introduced and gave the first true constant (2) factor approximation for $k$-center with uniform lower bounds.

For $k$-supplier and $k$-center with non-uniform lower bounds, Ahmadian and Swamy [6] gave the first true constant factor (3) approximation algorithm. $k$-Median has been extensively studied in past [9, 11, 16, 18, 20, 21, 43, 53] with the best approximation ratio of 2.611 + $\epsilon$ by Byrka et al. [16]. Several constant factor approximations are known for upper bounded $kM$ using LP rounding [2, 15, 18, 20, 25, 32, 46, 48] that violate upper bounds or the cardinality constraint by a factor of 2 or more. Barrier of 2 was broken to obtain $(1 + \epsilon)$ violation in upper bounds for uniform and non-uniform bounds by Byrka et al. [17] and Demirci et al. [27] respectively by strengthening the LP. Li gave constant factor approximations using at most $(1 + \epsilon)k$ facilities for uniform upper bounds. For $kM$ with lower bounds (LkM), Guo et al. [35], Han et al. [38], and Arutyunova and Schmidt [10] gave true constant factor approximations. For LkFL, Han et al. [37] gave a constant factor approximation that violates the lower bounds.

A wide range of approximation algorithms have been developed for the uncapacitated facility location problem [11, 14, 19, 22, 23, 34, 43, 46, 49, 54, 56, 61, 62, 65]. Li [49] gave a 1.488 factor algorithm almost closing the gap between the best known and best possible of 1.463 by Guha et al. [34] for the problem. UFL has been extensively studied in past [8, 15, 32, 47, 61]. For

---

| Problem | $\alpha$ | $\beta$ | $\gamma$ | Factor | Capacity Violation |
|---------|---------|---------|---------|--------|--------------------|
| LUkM    | 5       | 1       | 82.6    | $O(1/\epsilon^6)$ | $(1 + \epsilon)$ |
| LUFL    | 3       | 1       | 3       | $O(1)$ | $(1 + \epsilon)$ |
| LUkC    | 2       | 1       | 2       | $O(1)$ | $(1 + \epsilon)$ |

Table 3: Our results for uniform lower bounds when $2L_i \leq U_i \forall i \in F$.

| Problem | $\alpha$ | $\beta$ | $\gamma$ | Factor | Capacity Violation |
|---------|---------|---------|---------|--------|--------------------|
| LUkFL   | 3       | 1       | 82.6    | $O(1/\epsilon^2)$ | $(2 + \epsilon)$ |
| LUkM    | 2       | 1       | 2       | $O(1)$ | $(1 + \epsilon)$ |
| LUFL    | 2       | 1       | 2       | $O(1)$ | $(1 + \epsilon)$ |

Table 4: Our results for uniform upper bounds when $2L_i \leq U_i \forall i \in F$. 
uniform bounds, Shmoys et al. [61] gave the first constant factor (7) approximation violating the upper bounds by a factor of 7/2 using LP rounding technique. An $O(1/\epsilon^2)$-approximate solution, with $(2 + \epsilon)$ violation in upper bounds was given by Byrka et al. [15] as a special case of UkFL. Grover et al. [32] further reduced the violation to $(1 + \epsilon)$. For non-uniform capacities, Levi et al. [47] gave a 5-factor approximation algorithm with uniform facility opening costs. An et al. [3] strengthened the LP to obtain a true constant factor approximation for the problem. Local search technique has been particularly successful in dealing with UFL for uniform capacities [3, 24, 46] and non-uniform capacities [12, 55, 57, 64]. The best known results are 3-factor and 5-factor approximation by Aggarwal et al. [3] and Bansal et al. [12] for uniform and non-uniform capacities respectively. Facility location with lower bounds was first introduced by Karger and Minkoff [44] and, Guha et al. [33] independently in 2000. Both the works violate the lower bounds. Svitkina [63] gave the first true constant factor (448) approximation with uniform lower bounds, which was subsequently improved to 82.6 by Ahmadian and Swamy [5]. For non-uniform lower bounds, Li [52] gave the first true constant (4000) factor approximation for the problem.

Several constant factor approximation algorithms have been developed for the uncapacitated k-facility location problem [20, 41, 42, 43, 65] with the best approximation ratio being $2 + \sqrt{3} + \epsilon$ due to Zhang [65]. For UkFL problem, Aardal et al. [2] gave a constant factor approximation for the problem with uniform opening costs, using at most $2k$ facilities for non-uniform capacities and $2k - 1$ facilities for uniform capacities. Byrka et al. [15] gave an $O(1/\epsilon^2)$ factor approximation for uniform capacities violating the capacities by a factor of $(2 + \epsilon)$. Grover et al. [32] also gave two constant factor results on the problem with a trade-off between the capacity and the cardinality violations: (i) $(1 + \epsilon)$ violation in capacity with $2/(1 + \epsilon)$ factor cardinality violation and (ii) $(2 + \epsilon)$ violation in capacity and no cardinality violation. To the best of our knowledge, no result is known for the problem with non-uniform capacities and general opening costs.

Results for bounds on both the sides were first obtained by Friggstad et al. [29] for facility location problem. The paper considers general lower bounds and uniform upper bounds. They gave the first constant factor approximation using LP rounding techniques, violating both, the lower and the upper bounds. Gupta et al. [36] gave the first constant factor approximation when both the bounds are uniform, violating the upper bounds by a factor of 5/2 while respecting the lower bounds. Constant factor results were obtained for LUkC and LUkS with uniform lower bounds and general upper bounds independently by Ding et al. [28] and, Rössner and Schmidt [59].

1.2 High Level Idea

Let $I$ be an input instance. For LUkM/LUkFL/LUFL, we create two instances: $I_1$ of LFL and $I_2$ of UkM/UkFL/UFL from $I$ by dropping the upper bounds

---

1 A careful analysis of the result shows that the violation is $2/(1 + \epsilon)$
and the cardinality constraint for $I_1$ and, dropping the lower bound for $I_2$. For \( \text{LUkS/LUkC}, \) \( \text{LFL} \) does not work as the component of service cost in \( \text{kS/kC} \) is different from the one in FL. Thus, for \( \text{LUkS/LUkC}, \) we create two instances: $I_1$ of \( \text{LkS/LkC} \) and $I_2$ of \( \text{UkS/UkC} \) from \( I \) in a similar way. Since, any solution to \( I \) is feasible for \( I_1 \) and \( I_2 \), costs of \( I_1 \) and \( I_2 \) are bounded. We obtain approximate solutions \( AS_1 \) and \( AS_2 \) to \( I_1 \) and \( I_2 \) respectively and, combine them to obtain a solution for \( I \). If the solution of \( I_2 \) violates the upper bound by a factor of \( \beta \), then our solution violates it by a factor of \( \beta + 1 \). If there were no upper bounds, we could do the following: For every facility \( i \) opened in \( AS_1 \), we open its nearest facility \( \eta(i) \) in \( AS_2 \) and assign its clients to \( \eta(i) \). The cost of doing this can be bounded. However, we can not do this in the presence of upper bounds as \( i \) could be serving any number of clients in \( AS_1 \) and hence \( \eta(i) \) may be assigned arbitrary number of clients. To take care of the upper bounds, we do the following: for every facility \( i' \) in \( AS_2 \), we identify its nearest facility \( \eta(i') \) in \( AS_1 \). We treat all such facilities \( i' \)'s, whose nearest facility \( \eta(i') \) is same, along with \( \eta(i') \), together and open at most all but one of them. Some of the facilities in \( AS_2 \) that violate the lower bound are closed transferring their clients to other facilities. When we arrive at a facility that has received sufficient number of clients, we open it and assign all the clients to it. Clearly such a facility receives no more clients than \( \beta + 1 \) times its capacity.

1.3 Organisation of the Paper

Section 2 presents our framework for the LU(lower and upper bounded) variants of the various clustering problems under consideration. In Section 3, we improve our results for the special case when \( 2\mathcal{L}_i \leq \mathcal{U}_i \ \forall \ i \in \mathcal{F} \). We finally conclude in Section 4 giving scope of future work.

2 Framework for Lower and Upper Bounded Clustering

In this section, we present our framework for lower and upper bounded variants of the clustering problems under consideration.

**Problem Definitions:** Given a set \( \mathcal{C} \) of clients, a set \( \mathcal{F} \) of facilities, opening cost \( f_i \) associated with facility \( i \), metric cost \( c(i, j) \) (called the service cost) of serving a client \( j \) from a facility \( i \), lower bounds \( \mathcal{L}_i \) and upper bounds \( \mathcal{U}_i \) on the minimum and the maximum number of clients, respectively, an open facility \( i \) should serve, cardinality bound \( k \) on the maximum number of facilities that can be opened, the objective is to open a subset \( \mathcal{F}' \subseteq \mathcal{F} \) and compute an assignment function \( \sigma: \mathcal{C} \rightarrow \mathcal{F}' \) (where \( \sigma(j) \) denotes the facility that serves client \( j \) in the solution) such that \( |\mathcal{F}'| \leq k \) and \( \mathcal{L}_i \leq |\sigma^{-1}(i)| \leq \mathcal{U}_i \ \forall \ i \in \mathcal{F}' \). We consider the following variants of the clustering problems:

- **Lower and Upper bounded \( k \)-Median (LUkM):** \( f_i = 0 \ \forall i \in \mathcal{F}, \mathcal{F} \cap \mathcal{C} = \emptyset \). We wish to minimise \( \sum_{i \in \mathcal{F}'} \sum_{j \in \mathcal{C}} c(j, \sigma(j)) \).
Thus, we need to select which facilities in $F$ violate the upper bounds and, the facilities in $i$ nearest to $F$ facilities in $F$. To facilitate this, we construct a directed graph $G$ and the cardinality constraint, while violating the upper bounds by a factor of $\alpha$. Obtain a solution $\alpha$ in this section, we combine solutions $\alpha$ and $\alpha$ to obtain $\alpha$.

Algorithm 1 presents the main steps of the framework where $Cost_2(S)$ denotes the cost of a solution $S$ to an instance $I$. Steps 1–4 are self explanatory. We present the main step, step 5 in detail in Section 2.1.

**Algorithm 1:**

**Input:** Instance $I$

1. For $\alpha$ and $\alpha$, create an instance $I_1$ of LFL from $I$ by dropping the upper bounds and the cardinality constraint and for $\alpha$, create $I_1$ of $\alpha$ by dropping the upper bounds.

   Since any solution to $I$ is feasible for $I_1$ as well, we have $\text{Cost}_{I_1}(O_1) \leq \text{Cost}_{I_1}(O)$, where $O$ and $O_1$ denote optimal solution to $I$ and $I_1$ respectively.

2. Create an instance $I_2$ of $\alpha$ by dropping the lower bounds from $I$. Since any solution to instance $I$ is feasible for $I_2$ as well, we have $\text{Cost}_{I_2}(O_2) \leq \text{Cost}_{I_2}(O)$, where $O_2$ denotes an optimal solution to $I_2$.

3. Obtain a $\gamma$-approximate solution $\alpha_1 = (F_1, \sigma_1)$ to instance $I_1$.

4. Obtain an $\alpha$-approximate solution $\alpha_2 = (F_2, \sigma_2)$ to instance $I_2$. Let $\beta$ denote the violation in upper bounds, if any, in $\alpha_2$.

5. Combine solutions $\alpha_1$ and $\alpha_2$ to obtain a solution $\alpha = (F_1, \sigma_1)$ to $I$ such that $\text{Cost}_{I_1}(\alpha_1) \leq \Delta_1 \text{Cost}_{I_1}(\alpha_1) + \Delta_2 \text{Cost}_{I_2}(\alpha_2)$ with $(\beta + 1)$ violation in upper bound, where $\Delta_1$ and $\Delta_2$ are positive constants.

### 2.1 Combining solutions $\alpha_1$ and $\alpha_2$ to obtain $\alpha_1$

In this section, we combine solutions $\alpha_1 = (F_1, \sigma_1)$ and $\alpha_2 = (F_2, \sigma_2)$ to obtain a solution $\alpha = (F_1, \sigma_1)$ to $I$ respecting the lower bounds on facilities and the cardinality constraint, while violating the upper bounds by a factor of $(\beta + 1)$.

We will open some facilities in $F_1 \cup F_2$. Note that the facilities in $F_1$ may violate the upper bounds and, the facilities in $F_2$ may violate the lower bounds. Thus, we need to select which facilities in $F_1 \cup F_2$ to open closing the remaining facilities. To facilitate this, we construct a directed graph $G_1$ on the set of facilities in $F_1 \cup F_2$. For a facility $i \in F_2$, let $\eta(i)$ denote the facility in $F_1$ nearest to $i$ (assuming that the distances are distinct). Add an edge $(i, \eta(i))$ in
the graph. In order to avoid self-loops when \( i = \eta(i) \), we denote the occurrence of \( i \) in \( F_2 \) by \( i_c \) so that \( \eta(i_c) = i \). Thus, we obtain a forest of trees where-in each tree is a star. Formally, we define a star \( S_i \) to be a collection of nodes in \( \{ i \cup \eta^{-1}(i) \} \) with \( i \in F_1 \) as the center of the star and \( \eta^{-1}(i) \subseteq F_2 \). See Figure 1.

Though every facility in \( F_2 \) belongs to some star, there may be facilities in \( F_1 \) that do not belong to any star. We process the stars to decide the set of facilities to open in \( F_1 \cup F_2 \). Consider a star \( S_i \) centered at facility \( i \). Clearly the total assignments on \( i \) in \( AS_1 \) satisfy the lower bound but may violate the upper bound arbitrarily. On the other hand, the total assignments on a facility \( i' \in \eta^{-1}(i) \) in \( AS_2 \) satisfy the upper bound (within \( \beta \) factor) but may violate the lower bound arbitrarily. We close some facilities in \( \eta^{-1}(i) \) by transferring their clients (assigned to them in \( AS_2 \)) to other facilities in \( \eta^{-1}(i) \) (and to \( i \), if required) and open those who have got at least \( \mathcal{L} \) clients. We may also have to open \( i \) in the process. We make sure that upper bound is violated within the claimed bounds at \( i \) and the total number of facilities opened in \( S_i \) is at most \( |\eta^{-1}(i)| \). Thus, we open no more than \( k \) facilities in all.

To determine which facilities in \( \eta^{-1}(i) \) to open and which facilities to close, we consider the facilities in \( \eta^{-1}(i) \) in the order of decreasing distance from \( i \). Let the order be \( y_1, y_2, ..., y_l \). We collect the clients assigned to them, by \( AS_2 \), in a bag looking for a facility \( t \) at which we would have collected at least \( \mathcal{L}_{y_t} \) clients. We open such a facility and empty the bag by assigning all the clients in the bag to the facility just opened and start the process again with the next facility in the order. The problem occurs when at the last facility \( (y_l) \), in the order, the bag has less than \( \mathcal{L}_{y_{y_l}} \) clients. In this case, we would like to assign these clients to the star center \( i \) making use of the fact that \( i \) was assigned at least \( \mathcal{L}_i \) clients in \( AS_1 \). The problem here is that the clients assigned to \( i \) in \( AS_1 \) might have been assigned to the facilities in \( \eta^{-1}(i') \) for some star \( S_{y'} \) processed earlier or to the facilities in \( \eta^{-1}(i) \) itself. Figure 2 explains the situation. Thus, we need to process the stars in a carefully chosen sequence so as to avoid this kind of dependency amongst them. That is, we process the stars in a such a way that if

![Figure 1: Graph $G_1$: $I_1$ is an instance of LFL/LkS/LkC and $I_2$ that of UkM/UkFL/UFL/UkS/UkC.](image)
we are processing star \( S_i \), then the clients assigned to \( i \) in \( AS_1 \) are not assigned to facilities in \( \eta^{-1}(i') \) for a star \( S_i \) processed earlier. For this, we construct a weighted directed (dependency) graph \( G_2 \) on stars. We will denote the graph by \( G_2(\sigma_1, \sigma_2) \) to show that it is a function of the assignments in \( AS_1 \) and \( AS_2 \). The graph changes as any of these assignments change.

The graph \( G_2(\sigma_1, \sigma_2) \) has the stars \( (S_i, i : |\eta^{-1}(i)| > 0) \) as the vertices and we include the directed edge \((S_{i_1}, S_{i_2})\) from star \( S_{i_1} \) to \( S_{i_2} \) if a client \( j \in \sigma_1^{-1}(i_1) \) is served (in \( AS_2 \)) by some facility \( i' \in \eta^{-1}(i_2) \) in star \( S_{i_2} \) i.e., \( |\sigma_1^{-1}(i_1) \cap (\cup_{i'' \in \eta^{-1}(i_2)} \sigma_2^{-1}(i''))| > 0 \). Let \( w(S_{i_1}, S_{i_2}) = |\sigma_1^{-1}(i_1) \cap (\cup_{i'' \in \eta^{-1}(i_2)} \sigma_2^{-1}(i''))| \) be the weight on the edge \((S_{i_1}, S_{i_2})\). See Figure 3. If the resulting graph is a directed acyclic graph (DAG), a topological ordering of its vertices gives us a sequence in which the stars can be processed. However, if there are directed cycles in the graph, we redefine the assignments in \( AS_1 \) to obtain another solution \( AS_1 = \sigma_1, \sigma_2 > \) to break the cycles. The dependency graph for \((\sigma_1, \sigma_2)\) will then be an almost-DAG. We say that a directed graph is an almost-DAG, if the only directed cycles in it are self loops.

**Breaking the cycles** (see Algorithm 2 and Figure 4): let \( SC = \sigma_1, \sigma_2, \ldots, \sigma_{i_l} > \) be a directed cycle in the graph \( G_2(\sigma_1, \sigma_2) \). Wlog, let \((S_{i_1}, S_{i_2})\) be the minimum weight edge in the cycle. We reassign any \( \kappa = w(S_{i_1}, S_{i_2}) \) clients in \( \sigma_1^{-1}(i_r) \cap (\cup_{i'' \in \eta^{-1}(i_{r+1})} \sigma_2^{-1}(i'')) \) from \( i_r \) to \( i_{r+1} \) and reduce the weight of the edge \( w(S_{i_r}, S_{i_{r+1}}) \) by \( \kappa \) for \( r = 1 \ldots l - 1 \) and reassign any \( \kappa \) clients in \( \sigma_1^{-1}(i_l) \cap (\cup_{i'' \in \eta^{-1}(i_1)} \sigma_2^{-1}(i'')) \) from \( i_l \) to \( i_1 \) and reduce the weight of the edge \( w(S_{i_1}, S_{i_l}) \) by \( \kappa \). Let \( \sigma_1 \) denote the new assignments in \( AS_1 \). Note that \( |\sigma_1^{-1}(i)| = |\sigma_1^{-1}(i)| \) and hence \( |\sigma_1^{-1}(i)| \geq L_i \) is maintained for all \( i \in F_i \) after the reassignments. The weight of the edge \((S_{i_1}, S_{i_2})\) becomes zero and we remove it thereby breaking the cycle.
Figure 3: (a) Stars $S_{i_1}, S_{i_2}$ and $S_{i_3}$, (b) $\sigma_1^{-1}(i_1) \cap (\cup_{i' \in \eta^{-1}(i_2)}\sigma_2^{-1}(i')) = \{j_1\}$, $\sigma_1^{-1}(i_2) \cap (\cup_{i' \in \eta^{-1}(i_3)}\sigma_2^{-1}(i')) = \{j_2, j_3, j_4\}$, $\sigma_1^{-1}(i_3) \cap (\cup_{i' \in \eta^{-1}(i_1)}\sigma_2^{-1}(i')) = \{j_5, j_6\}$, (c) Directed cycle in $G_2(\sigma_1, \sigma_2)$.

Figure 4: Breaking a cycle: (b) Assign $j_1$ to $i_2$, $j_4$ to $i_3$ and $j_5$ to $i_1$, that is, $\hat{\sigma}_1(j_1) = i_2$, $\hat{\sigma}_1(j_4) = i_3$ and, $\hat{\sigma}_1(j_5) = i_1$ (c) The subgraph of $G_2(\hat{\sigma}_1, \sigma_2)$ after breaking the cycle.
Lemma 1. Let \( j \in \mathcal{C} \). The service cost paid by \( j \) in solution \( \hat{\mathcal{S}}_1 \), \( c(j, \hat{\sigma}_1(j)) \), is bounded by \( c(j, \sigma_1(j)) + 2c(j, \sigma_2(j)) \).

Proof. Let \( j \in \mathcal{C} \). We have \( c(j, \hat{\sigma}_1(j)) \leq c(j, \sigma_1(j)) + c(\sigma_2(j), \hat{\sigma}_1(j)) \leq c(j, \sigma_2(j)) + c(\sigma_2(j), \sigma_1(j)) \leq c(j, \sigma_2(j)) + (c(\sigma_2(j), j) + c(j, \sigma_1(j))) = c(j, \sigma_1(j)) + 2c(j, \sigma_2(j)) \), where the second inequality follows because \( \eta(\sigma_2(j)) = \hat{\sigma}_1(j) \). See Figure 5.

Now that we have an almost-DAG on the stars, we process the stars in the sequence \( < S_{i_1}, S_{i_2}, \ldots, S_{i_t} > \) defined by a topological ordering of the vertices.
in $G_2(\hat{\sigma}_1, \sigma_2)$ (ignoring the self-loops). While processing the stars, we maintain partition of our clients into two sets $C_s$ and $C_u$ of settled and unsettled clients respectively. We say that a client is settled if it has been assigned to an open facility in $S_l$ and unsettled otherwise. Initially $C_s = \emptyset$ and $C_u = C$. As we process the stars, more and more clients get settled.

Consider star $S_l$. Algorithm 3 gives the processing of $S_l$ in detail. For $i' \in \eta^{-1}(i)$, let $N_{i'}$ be the set of unsettled clients, assigned to $i'$ in $AS_2$. We open some facilities in $\eta^{-1}(i)$ and assign these clients to them. Consider the facilities in $\eta^{-1}(i)$ in decreasing order of distance from $i$, i.e., $y_1, y_2, ..., y_l$. We collect the unsettled clients assigned to them, by $AS_2$, in a bag looking for a facility $t$ at which we have collected at least $L_i$ clients. We open such a facility and empty the bag by assigning all the clients in the bag to the facility just opened (Type-I assignment) and start the process again with the next facility in the order (lines 10–16). To avoid the problem mentioned earlier and exhibited in Figure 2 before we do this, we reserve some clients in $\hat{\sigma}_1^{-1}(i)$ so that the total number of unsettled clients assigned (in $AS_2$) to the last facility $y_l$ in the order, along with the reserved clients is at least $L_i$. For this, we reserve $\max\left\{0, L_i - |N_{y_l}| \right\}$ clients from $\hat{\sigma}_1^{-1}(i) \setminus N_{y_l}$ at $i$ (line 7). Note that $\hat{\sigma}_1^{-1}(i)$ and hence $\hat{\sigma}_1^{-1}(i) \setminus N_{y_l}$ may contain clients from $\cup_{i' \in \eta^{-1}(i)} N_{i'}$. Thus, we delete the reserved clients from $N_{i'}$, $i' \in \eta^{-1}(i)$ (lines 8–9) before processing the facilities in $\eta^{-1}(i)$.

To make sure that we do not open more than $|\eta^{-1}(i)|$ facilities in $S_l$, we open only one of $i$ and $y_l$ for the remaining $(Bag \cup N_{y_l} \cup reserved(i))$ clients. This also ensures that we do not open $i$ more than once. We open $i$ and give all the remaining clients to $i$ because (as we will show later) the cost of assigning clients from $Bag \cup N_{y_l}$ to $i$ is bounded whereas we do not know how to bound the cost of assigning clients in $reserved(i)$ to $y_l$. Clearly, we open no more than $k$ facilities in all.

**Claim 1.** At line 7, we have sufficient clients in $\hat{\sigma}_1^{-1}(i) \setminus N_{y_l}$, i.e., $|\hat{\sigma}_1^{-1}(i) \setminus N_{y_l}| \geq L_i - |N_{y_l}|$.

**Proof.** Note that the set of clients that get settled while processing star $S_l$ is a subset of $\hat{\sigma}_1^{-1}(i) \cup (\cup_{i' \in \eta^{-1}(i)} \sigma_2^{-1}(i'))$. To prove the claim, we only need to prove that $\hat{\sigma}_1^{-1}(i) \cap C_s = \emptyset$. The claim then follows by using the fact that $|\hat{\sigma}_1^{-1}(i)| \geq L_i$. Let $S_i$ be a star that was processed before the star $S_l$, $i \neq i$. Then since $G_2(\hat{\sigma}_1, \sigma_2)$ is an almost-DAG, there is no edge from $S_i$ to $S_l$, i.e., $|\hat{\sigma}_1^{-1}(i) \cap (\cup_{i' \in \eta^{-1}(i)} \sigma_2^{-1}(i'))| = 0$. Obviously, $|\hat{\sigma}_1^{-1}(i) \cap \hat{\sigma}_1^{-1}(i)| = 0$. Thus, the set of clients that got settled while processing $S_l$ has no intersection with $\hat{\sigma}_1^{-1}(i)$. Hence the claim follows.
Algorithm 3: Process($S_i$)

**Input**: $S_i: i \in F_1$

1. `reserved(i) ← ∅`
2. `Bag ← ∅`
3. **for** $i' \in \eta^{-1}(i)$ **do**
   4. `N_{i'} ← C_u \cap \sigma_2^{-1}(i')`
5. Arrange the facilities in $\eta^{-1}(i)$ in the sequence $<y_1, \ldots, y_l>$ such that $c(y_l, i) \geq c(y_{l+1}, i)$ for all $l' = 1 \ldots l - 1$
6. **if** $|N_{y_l}| < \mathcal{L}_i$ **then**
   7. `reserved(i) ← set of any $\mathcal{L}_i - |N_{y_l}|$ clients from $\hat{\sigma}_1^{-1}(i) \setminus N_{y_l}$`
8. **for** $i' \in \eta^{-1}(i)$ **do**
9. `N_{i'} ← N_{i'} \setminus reserved(i)`
10. **for** $l' = 1$ to $l - 1$ **do**
11. `Bag ← Bag \cup N_{y_{l'}}`
12. **if** $|Bag| \geq \mathcal{L}_{y_{l'}}$ **then**
13. Open facility $y_{l'}$
14. **for** $j \in Bag$ **do**
15. Assign $j$ to $y_{l'}$, $C_u ← C_u \cup \{j\}$, $C_u ← C_u \setminus \{j\}$
16. `Bag ← ∅`
17. $t ← i$
18. **if** $|Bag \cup N_{y_l} \cup reserved(i)| > (\beta + 1)\mathcal{U}_t$ **then** $t ← y_l$
19. Open $t$
20. **for** $j \in Bag \cup N_{y_l} \cup reserved(i)$ **do**
21. Assign $j$ to $t$, $C_u ← C_u \cup \{j\}$, $C_u ← C_u \setminus \{j\}$

Note that when one of the bounds is uniform, we have $\mathcal{U}_i \geq \mathcal{L}_{i'}$ for all $i$ and $i'$. Clearly, the facilities opened by the above algorithm in line 13 (Type-I assignment) satisfy the lower bounds. Let the assignment of clients to facility $i$ when $t = i$ in lines 20 - 21 be called as Type-II assignments and those to facility $y_l$ when $t = y_l$ be called as Type-III assignments. In Type-II assignments, the star center $i$ satisfies the lower bound [if opened at at line 19] as $|Bag \cup N_{y_l} \cup reserved(i)| \geq \mathcal{L}_i$ where the inequality follows because $|reserved(i)| = \max\{0, \mathcal{L}_i - N_{y_l}\}$. In Type-III assignments, facility $y_l$ (if opened at line 19) also satisfies the lower bound as $|Bag \cup N_{y_l} \cup reserved(i)| > (\beta + 1)\mathcal{U}_t \geq 2\mathcal{L}_{y_l}$ when one of the bounds is uniform.

We next bound the violations in the upper bound. Consider the facilities in $\eta^{-1}(i)$. These facilities receive clients only in Type-I assignments (lines 14 - 15). Note that for $l' = 2, \ldots, l - 1$, we have $|Bag| < \mathcal{L}_{y_{l'-1}}$ just before line 11 and hence $|Bag| < \mathcal{L}_{y_{l'-1}} + \beta\mathcal{U}_{y_{l'}}$ (just after line 11) $\leq (1 + \beta)\mathcal{U}_{y_{l'}}$ when one of the bounds is uniform. For $l' = 1$, $|Bag| = 0$ just before line 11 and hence $|Bag| \leq \beta\mathcal{U}_{y_1}$ (just after line 11). For Type-II assignments, the bound holds trivially because the center $i$ receives clients only when $|Bag| + |N_{y_l}| + |reserved(i)| \leq (\beta + 1)\mathcal{U}_t$. The maximum number of clients received by facility $y_l$ in Type-III assignments is no more than $|Bag| + |N_{y_l}| + |reserved(i)| = |Bag| + |N_{y_l}| + \max\{0, \mathcal{L}_i - N_{y_l}\} = |Bag| + \max\{\mathcal{L}_i, N_{y_l}\} \leq \mathcal{L}_{y_{l'-1}} + \beta\mathcal{U}_{y_l} \leq (\beta + 1)\mathcal{U}_{y_l}$ when either of the bounds is uniform.
Next, we bound the service cost. Consider a client $i$ assigned to a facility $i_2 \in \eta^{-1}(i)$ in Type-I assignments where $j$ was assigned to $i_1 \in \eta^{-1}(i)$ in $AS_2$ i.e., $i_1 = \sigma_2(j)$ and $i_2 = \sigma_1(j)$. See Figure 6(a). The cost paid by $j$ is $c(i_2, j) \leq c(i_1, j) + c(i_1, i) + c(i, i_2) \leq c(i_1, j) + 2c(i_1, i) \leq c(i_1, j) + 2c(i_1, \hat{j}_1(j)) \leq c(i_1, j) + 2c(i_1, j) + c(j, \hat{j}(j)) = 3c(i_1, j) + 2c(j, \hat{j}(j)) \leq 2c(j, \sigma_1(j)) + 7c(j, \sigma_2(j))$ where the third inequality follows because $\eta(i_1) = i$ and last follows by Lemma 1. Next, consider Type-II assignments. Let $j \in reserved(i)$ be assigned to $i$. Then since $\hat{j}_1(j) = i$, the service cost $c(i, j) = c(\hat{j}_1(j), j))$ is bounded by $c(j, \sigma_1(j)) + 2c(j, \sigma_2(j))$ by Lemma 1. Next, let $j \in N_{\gamma'}$ i.e. $i' \in \eta^{-1}(i)$ be a client assigned to $i$ i.e. $i' = \sigma_2(j)$ and $i = \sigma_1(j)$. See Figure 6(b). Then, the service cost $c(i, j) = c(\sigma_2(j), j) + c(\sigma_2(j), i) = c(\sigma_2(j), j) + c(\sigma_2(j), \eta(\sigma_2(j))) \leq c(\sigma_2(j), j) + c(\sigma_2(j), \hat{j}(j)) \leq c(\sigma_2(j), j) + c(\sigma_2(j), j) + c(\sigma_2(j), \hat{j}(j)) = 2c(j, \sigma_2(j)) + c(j, \hat{j}(j)) \leq 4c(j, \sigma_2(j)) + c(j, \sigma_1(j))$ where the last inequality follows by Lemma 1. For Type-III assignments, note that $|Bag \cup N_{\gamma} \cup reserved(i)| > (\beta + 1)|\mathcal{U}_i| \Rightarrow |reserved(i)| = 0$, for otherwise $|N_{\gamma} \cup reserved(i)| = \mathcal{L}_i$ and thus $|Bag \cup N_{\gamma} \cup reserved(i)| < \mathcal{L}_{prev} + \mathcal{L}_i \leq 2\mathcal{U}_i$. When either of the bounds is uniform, hence, the cost of assigning $|Bag \cup N_{\gamma}|$ clients to $y_1$ is bounded in the same manner as the cost of Type-I assignments.

Let $FCost_I(S)$ and $SCost_I(S)$ denote the facility opening cost and service cost of a solution $S$ of instance $I$ respectively. Then, $FCost_I(S) \leq FCost_{I_1}(AS_1) + FCost_{I_2}(AS_2)$ and $SCost_I(S) \leq 2SCost_{I_1}(AS_1) + 7SCost_{I_2}(AS_2)$.

**Proof of Theorem 1.** Using a $\gamma$-approximation for instance $I_1$ of LFL or LkS/LkC, as the case may be, with uniform lower bounds and an $\alpha$-approximation for instance $I_2$ of UkM/UkFL/UFL/UkS/UkC, we get the desired claims.

**Proof of Theorem 2.** Using a $\gamma$-approximation for instance $I_1$ of LFL or LkS/LkC, as the case may be, and an $\alpha$-approximation for instance $I_2$ of UkM/UkFL/UFL/UkS/UkC with uniform upper bounds, we get the desired claims.
Algorithm 4 summarizes Step 5 of Algorithm 1.

**Algorithm 4: Constructing $AS_I$**

**Input:** $<AS_1 = F_1, \sigma_1>, <AS_2 = F_2, \sigma_2>$

**Output:** $AS_I$

1. Construct graph $G_1 = <F_1 \cup F_2, E>$ where $E = \{(i', \eta(i')) : i' \in F_2\}$

2. Construct an almost-DAG $G_2(\hat{\sigma}_1, \sigma_2)$ from $G_1$ using algorithm 2

3. Obtain a topological ordering $<S_{i_1}, S_{i_2}, \ldots, S_{i_t}>$ of stars in the almost-DAG $G_2(\hat{\sigma}_1, \sigma_2)$.

4. for $r = 1$ to $t$ do

5. Process $S_{i_r}$ using Algorithm 3

### 3 Reducing violation in upper bounds when $2L_t \leq U_t$

In this section, assuming $2L_t \leq U_t \forall t \in F$, we modify Algorithm 3 to obtain Algorithm 5 that reduces the violation in upper bounds from $(\beta + 1)$ to $(\beta + \epsilon)$ for a given $\epsilon > 0$ when one of the bounds is uniform. We do the following modifications to the algorithm: (i) on arriving at a facility, say $t$, at which $|\text{Bag}| \geq L_t$, we open $t$ and instead of emptying the bag, we assign only $\beta U_t$ clients to $t$. Remaining clients are carried forward to the next facility in the order. (ii) We keep account of the last facility (in $\text{Prev}$), if any, that is not opened, and the number of clients in the bag at that instant (in $\text{Prev\_count}$) i.e., $\text{Prev}$ is the facility $y_l$ for which $|\text{Bag}| < L_{y_l}$, immediately after line 11 (hence at line 18) and $\text{Prev\_count} = |\text{Bag}|$ at that time. We open $\text{Prev}$ at the end, if required. This is done as follows: if $|\text{Bag} \cup \text{reserved}(i) \cup N_{y_l}| \leq (\beta + \epsilon)U_t/(U_{y_l})$, we are done (we open $i(/y_l)$ and assign all clients to it). Else, we open both $\text{Prev}$ and $y_l$ (at line 31) (note that $\text{Prev} \neq y_l$ must exist in this case) and, assign the clients in $\text{Bag} \cup \text{reserved}(i) \cup N_{y_l}$ to $\text{Prev}$ and $y_l$, so that they receive at least $L_{\text{Prev}}$ and $L_{y_l}$ clients respectively. When $2L_t \leq U_t$ and at least one of the bounds is uniform, it is possible to do so as $U_t \geq L_r + L_s$ for all $r, s$ and $t$. We will show that the violation in upper bound and the service cost are bounded in
Algorithm 5: Process($S_i$) when $2\mathcal{L}_i \leq \mathcal{U}_i \forall t \in \mathcal{F}$

\textbf{Input}: $S_i : i \in \mathcal{F}$ \\
1 reserved($i$) ← $\emptyset$, Bag ← $\phi$ \\
2 for $i' \in \eta^{-1}(i)$ do \\
3 \hspace{1em} $N_i \leftarrow C_u \cap \sigma_2^{-1}(i')$ \\
4 \hspace{1em} Arrange the facilities in $\eta^{-1}(i)$ in the sequence $<y_1, \ldots, y_l>$ such that \\
5 \hspace{1em} $c(y_l, i) \geq c(y_{l+1}, i) \forall l' = 1 \ldots l - 1$ \\
6 if $|N_{y_l}| < \mathcal{L}_i$ then \\
7 \hspace{1em} reserved($i$) ← set of any $\mathcal{L}_i - |N_{y_l}|$ clients from $\sigma_1^{-1}(i) \setminus N_{y_l}$ \\
8 \hspace{1em} for $i'' \in \eta^{-1}(i)$ do \\
9 \hspace{2em} $N_{i''} \leftarrow N_{i''} \setminus \text{reserved}(i)$ \\
10 Prev ← null, Prev\_count = 0 \\
11 for $l' = 1$ to $l - 1$ do \\
12 \hspace{1em} Bag ← Bag $\cup$ $N_{y_l}$, \\
13 \hspace{1em} if $|Bag| \geq \mathcal{L}_{y_l}$ then \\
14 \hspace{2em} Open facility $y_l$ \\
15 \hspace{2em} Count ← 0 \\
16 \hspace{2em} for $j \in \text{Bag}$ do \\
17 \hspace{3em} if Count $\leq \beta\mathcal{U}_{y_l}$ then \\
18 \hspace{4em} Assign $j$ to $y_l$, $C_s \leftarrow C_s \cup \{j\}$, $C_u \leftarrow C_u \setminus \{j\}$, \\
19 \hspace{4em} Bag $\leftarrow$ Bag $\setminus \{j\}$, Count $\leftarrow$ Count $+$, \\
20 else \\
21 \hspace{3em} Prev $\leftarrow$ $y_l$ \\
22 \hspace{3em} // Prev denotes the last unopened facility in $\eta^{-1}(i)$ \\
23 \hspace{3em} Prev\_count $\leftarrow$ |Bag| \\
24 \hspace{2em} if $|Bag \cup N_{y_l} \cup \text{reserved}(i)| \leq (\beta + \epsilon)\mathcal{U}_i$ then \\
25 \hspace{3em} Open $i$ \\
26 \hspace{3em} for $j \in Bag \cup N_{y_l} \cup \text{reserved}(i)$ do \\
27 \hspace{4em} Assign $j$ to $i$, $C_s \leftarrow C_s \cup \{j\}$, $C_u \leftarrow C_u \setminus \{j\}$ \\
28 return \\
29 if $|Bag \cup N_{y_l} \cup \text{reserved}(i)| \leq (\beta + \epsilon)\mathcal{U}_i$ then \\
30 Open $y_l$ \\
31 \hspace{1em} for $j \in Bag \cup N_{y_l} \cup \text{reserved}(i)$ do \\
32 \hspace{2em} Assign $j$ to $y_l$, $C_s \leftarrow C_s \cup \{j\}$, $C_u \leftarrow C_u \setminus \{j\}$ \\
33 return \\
34 \hspace{1em} Open Prev and $y_l$ \\
35 // reserved($i$) = 0 when $|Bag \cup N_{y_l} \cup \text{reserved}(i)| > (\beta + \epsilon)\mathcal{U}_i$/\mathcal{U}_i) \\
36 Count ← 0 \\
37 \hspace{1em} Bag ← Bag $\cup$ $N_{y_l}$ $\cup$ reserved($i$) \\
38 \hspace{1em} for $j \in \text{Bag}$ do \\
39 \hspace{2em} if Count $\leq \mathcal{L}_{\text{prev}}$ then \\
40 \hspace{3em} Assign $j$ to $\text{prev}$, $C_s \leftarrow C_s \cup \{j\}$, $C_u \leftarrow C_u \setminus \{j\}$, \\
41 \hspace{3em} Bag $\leftarrow$ Bag $\setminus \{j\}$, Count $\leftarrow$ Count $+$ \\
42 else \\
43 \hspace{3em} Assign all remaining clients in Bag to $y_l$ and Break
Let the assignment of clients to facility $\text{Prev}$ in line 36 be called as Type-IV assignments. The assignments in line 17, line 24 and lines 29 & 38 are Type-I, Type-II and Type-III assignments respectively. Before we proceed to prove our claims, note that we open at most one of $y_l$ and $i$: if $i$ is opened at line 22, we return at line 25 and thus $y_l$ is never opened in this case. As before, this ensures that $i$ is not opened more than once.

Clearly, lower bound is satisfied by the Type-I and Type-IV assignments done in line 17 and 36 for the facilities opened in lines 13 and 31 respectively. Also, since $|\text{Bag} \cup N_{y_l} \cup \text{reserved}(i)| \geq L_i$, lower bound is satisfied by the Type-II assignments done in line 24 for the facility $i$ opened in line 22. At line 29, $|\text{Bag} \cup N_{y_l} \cup \text{reserved}(i)| > (\beta + \epsilon)\mathcal{U}_l \geq (\beta + \epsilon)\mathcal{L}_{y_l}$ when either of the bounds is uniform. Clearly, the upper bound is violated at most by a factor of $(\beta + \epsilon)$ at all the facilities opened in lines 17, 24, 29 and 36. To bound the assignments done in line 38, we look at the status at line 31. At line 31, $|\text{reserved}(i)| = 0$, for otherwise $|N_{y_l} \cup \text{reserved}(i)| = L_i$, hence $|\text{Bag} \cup \text{reserved}(i) \cup N_{y_l}| < L_{\text{Prev}} + L_i \leq \mathcal{U}_l$ (when one of the bounds is uniform). Thus, $|\text{Bag} \cup N_{y_l} \cup \text{reserved}(i)| = |\text{Bag} \cup N_{y_l}| < L_{\text{Prev}} + \beta \mathcal{U}_y$. Also, $|\text{Bag} \cup N_{y_l} \cup \text{reserved}(i)| > (\beta + \epsilon)\mathcal{U}_l > L_{\text{Prev}} + \beta \mathcal{U}_y$ (when one of the bounds is uniform). Thus, $L_{y_l} < |\text{Bag} \cup N_{y_l} \cup \text{reserved}(i)| - L_{\text{Prev}} < \beta \mathcal{U}_y$, i.e., at line 38, $L_{y_l} < |\text{Bag}| < \beta \mathcal{U}_y$.

Costs of Type-I assignments, Type-II assignments and Type-III assignments are bounded in the same manner as in Section 2.1. To bound the service cost of Type-IV assignments (line 36), observe that $|\text{Bag} \cup \text{reserved}(i) \cup N_{y_l}| > (\beta + \epsilon)\mathcal{U}_y$, implies that $|\text{Bag}| > d\mathcal{U}_y$ as $|\text{reserved}(i)| = 0$ and $N_{y_l} \leq \beta \mathcal{U}_y$; hence $\text{Prev}_i \geq |\text{Bag}|$ (at line 20) $> d\mathcal{U}_y > \epsilon L_{\text{Prev}}$ (last inequality holds when at least one of the bounds is uniform). Note that $\text{Prev}$ and $\text{Prev}_i$ do not change after exiting the for-loop at line 20. Thus, $\text{Prev}_i > \epsilon L_{\text{Prev}}$ after line 31 also. Thus, the cost of assigning at most $L_{\text{Prev}}$ clients from $N_{y_l}$ to $\text{Prev}$ is bounded by $(1/\epsilon)$ times the cost of assigning $\epsilon L_{\text{Prev}}$ clients from $\cup_i \mathcal{U}_l$ occurs before $\text{Prev}$ in the order $N_{y_l} \cup N_{\text{Prev}}$ to $y_l$. Hence, the total cost of Type-IV assignments is bounded by $(1/\epsilon)$ total cost of Type-III assignments.

Theorem 3 and 4 then follow in the same manner as Theorem 1 and 2 respectively.

## 4 Conclusion and Future Work

In this paper, we presented first constant factor approximations for lower and upper bounded $k$-median and $k$-facility location problems violating the upper bounds by a factor of $\beta + 1$ where $\beta$ is the violation in upper bounds in the solutions of the underlying problems with upper bounds. We studied the problems when one of the bounds is uniform. Any improvement in $\beta$ in future will lead to improved results for our problems as well. Our approach also gives a constant factor approximation for lower and upper bounded facility location; we improve upon the upper bound violation of 5/2 obtained by Gupta et al. [36] to 2. We also presented first constant factor approximations for lower and uniform upper bounded $k$-center and its generalization, $k$-supplier problem.
For the special case when $2L_i \leq U_i \forall i \in F$, an improvement in upper bound violation to $\beta + \epsilon$ was also obtained for a given $\epsilon > 0$.

**Additional results:** Our framework also provides a polynomial time algorithm that approximates LUkFL within a constant factor violating the upper bounds by a factor of $(2 + \epsilon)$ and cardinality by a factor of $\frac{2}{1+\epsilon}$ for a given $\epsilon > 0$ using approximation of Grover et al. [32] that violates capacities by $(1 + \epsilon)$ factor and cardinality by $(\frac{2}{1+\epsilon})$. We also get a result that violates the upper bound by 2 factor and cardinality by a factor of 2 when the facility costs are uniform by using Aardal et al. [2].

**Future Work:** One direction for future work would be to obtain true constant factor approximations for the problems. For kM and kFL, this would be challenging as no true approximations are known for the upper bounded variants of the underlying problems. Can we obtain a true approximation for LUFL? For LUkS/LUkC, true approximations are known for uniform lower bounds and general upper bounds. However, to obtain a true approximation with general lower bounds and uniform upper bounds is open for the problem.

**References**

[1] Data clustering: 50 years beyond k-means. *Pattern Recognition Letters*, 31(8):651–666, 2010.

[2] Karen Aardal, Pieter L. van den Berg, Dion Gijswijt, and Shanfei Li. Approximation algorithms for hard capacitated k-facility location problems. *European Journal of Operations Research*, 242(2):358–368, 2015.

[3] Ankit Aggarwal, Anand Louis, Manisha Bansal, Naveen Garg, Neelima Gupta, Shibham Gupta, and Surabhi Jain. A 3-approximation algorithm for the facility location problem with uniform capacities. *Journal of Mathematical Programming*, 141(1-2):527–547, 2013.

[4] Gagan Aggarwal, Rina Panigrahy, Tomás Feder, Dilys Thomas, Krishnaram Kenthapadi, Samir Khuller, and An Zhu. Achieving anonymity via clustering. *ACM Trans. Algorithms*, 6(3):49:1–49:19, 2010.

[5] Sara Ahmadian and Chaitanya Swamy. Improved approximation guarantees for lower-bounded facility location. In *Approximation and Online Algorithms*, pages 257–271, 2013.

[6] Sara Ahmadian and Chaitanya Swamy. Approximation algorithms for clustering problems with lower bounds and outliers. In *ICALP*, volume 55, pages 69:1–69:15, 2016.

[7] Hyung-Chan An, Aditya Bhaskara, Chandra Chekuri, Shalmoli Gupta, Vivek Madan, and Ola Svensson. Centrality of trees for capacitated k-center. *Journal of Mathematical Programming*, 154:29–53, 2015.
[8] Hyung-Chan An, Mohit Singh, and Ola Svensson. LP-based algorithms for capacitated facility location. In *Proceedings of FOCS, 2014*, pages 256–265.

[9] Aaron Archer, Ranjithkumar Rajagopalan, and David B. Shmoys. Lagrangian relaxation for the $k$-median problem: new insights and continuity properties. In *Proceedings of Symposium on Algorithms*, pages 31–42, 2003.

[10] Anna Arutyunova and Melanie Schmidt. Achieving Anonymity via Weak Lower Bound Constraints for k-Median and k-Means. In *STACS 2021*, pages 7:1–7:17, 2021.

[11] Vijay Arya, Naveen Garg, Rohit Khandekar, Adam Meyerson, Kamesh Munagala, and Vinayaka Pandit. Local search heuristics for $k$-median and facility location problems. *SIAM Journal of Computing*, 33(3):544–562, 2004.

[12] Manisha Bansal, Naveen Garg, and Neelima Gupta. A 5-approximation for capacitated facility location. In *Proceedings of ESA*, pages 133–144, 2012.

[13] Judit Bar-Ilan, Guy Kortsarz, and David Peleg. How to allocate network centers. *J. Algorithms*, 15(3):385–415, 1993.

[14] Jaroslaw Byrka. An optimal bifactor approximation algorithm for the metric uncapacitated facility location problem. In *Proceedings of RANDOM-APPROX*, pages 29–43, 2007.

[15] Jaroslaw Byrka, Krzysztof Fleszar, Bartosz Rybicki, and Joachim Spoerhase. Bi-factor approximation algorithms for hard capacitated $k$-median problems. In *Proceedings of SODA, 2015*, pages 722–736.

[16] Jaroslaw Byrka, Thomas Pensyl, Bartosz Rybicki, Aravind Srinivasan, and Khoa Trinh. An improved approximation for $k$-median, and positive correlation in budgeted optimization. In *Proceedings of SODA*, pages 737–756, 2015.

[17] Jaroslaw Byrka, Bartosz Rybicki, and Sumedha Uniyal. An approximation algorithm for uniform capacitated $k$-median problem with $(1 + \epsilon)$ capacity violation. In *Proceedings of IPCO*, pages 262–274, 2016.

[18] Moses Charikar and Sudipto Guha. Improved combinatorial algorithms for the facility location and $k$-median problems. In *Proceedings of FOCS*, pages 378–388, 1999.

[19] Moses Charikar and Sudipto Guha. Improved combinatorial algorithms for facility location problems. *SIAM Journal on Computing*, 34(4):803–824, 2005.

[20] Moses Charikar, Sudipto Guha, Éva Tardos, and David B. Shmoys. A constant-factor approximation algorithm for the $k$-median problem (extended abstract). In *Proceedings of STOC, 1999*, pages 1–10.
[21] Moses Charikar and Shi Li. A dependent lp-rounding approach for the k-median problem. In *Proceedings of ICALP, 2012*, pages 194–205.

[22] Fabián A. Chudak. Improved approximation algorithms for uncapitated facility location. In *Proceedings of IPCO*, pages 180–194, 1998.

[23] Fabián A. Chudak and David B. Shmoys. Improved approximation algorithms for the uncapitated facility location problem. *SIAM Journal of Computing*, 33(1):1–25, 2003.

[24] Fabián A. Chudak and David P. Williamson. Improved approximation algorithms for capacitated facility location problems. In *Proceedings of IPCO*, pages 99–113, 1999.

[25] Julia Chuzhoy and Yuval Rabani. Approximating k-median with non-uniform capacities. In *Proceedings of SODA*, pages 952–958, 2005.

[26] Marek Cygan, MohammadTaghi Hajiaghayi, and Samir Khuller. LP rounding for k-centers with non-uniform hard capacities. In *Proceedings of FOCS*, pages 273–282, 2012.

[27] H. Gökalp Demirci and Shi Li. Constant approximation for capacitated k-median with (1 + ϵ) capacity violation. In *Proceedings of ICALP*, pages 73:1–73:14, 2016.

[28] Hu Ding, Lunjia Hu, Lingxiao Huang, and Jian Li. Capacitated center problems with two-sided bounds and outliers. In *WADS*, pages 325–336, 2017.

[29] Zachary Friggstad, Mohsen Rezapour, and Mohammad R. Salavatipour. Approximating Connected Facility Location with Lower and Upper Bounds via LP Rounding. In *SWAT*, volume 53, pages 1:1–1:14, 2016.

[30] Jonatan Gómez, Elizabeth León-Guzmán, and Olfa Nasraoui. Minimum cluster size estimation and cluster refinement for the randomized gravitational clustering algorithm. In *IBERAMIA*, pages 51–60, 2012.

[31] Teofilo F. Gonzalez. Clustering to minimize the maximum intercluster distance. *Theor. Comput. Sci.*, 38:293–306, 1985.

[32] Sapna Grover, Neelima Gupta, Samir Khuller, and Aditya Pancholi. Constant factor approximation algorithm for uniform hard capacitated knapsack median problem. In *FSTTCS*, pages 23:1–23:22, 2018.

[33] S. Guha, A. Meyerson, and K. Munagala. Hierarchical placement and network design problems. In *FOCS*, page 603, 2000.

[34] Sudipto Guha and Samir Khuller. Greedy strikes back: Improved facility location algorithms. *Journal of Algorithms*, 31(1):228–248, 1999.
[35] Yutian Guo, Junyu Huang, and Zhen Zhang. A constant factor approximation for lower-bounded k-median. In Theory and Applications of Models of Computation, pages 119–131, 2020.

[36] Neelima Gupta, Sapna Grover, and Rajni Dabas. Respecting lower bounds in uniform lower and upper bounded facility location problem. In COCOON, 2021.

[37] Lu Han, Chunlin Hao, Chenchen Wu, and Zhenning Zhang. Approximation algorithms for the lower-bounded k-median and its generalizations. In COCOON, pages 627–639, 2020.

[38] Lu Han, Chunlin Hao, Chenchen Wu, and Zhenning Zhang. Approximation algorithms for the lower-bounded knapsack median problem. In AAIM, volume 12290, pages 119–130, 2020.

[39] Dorit S. Hochbaum and David B. Shmoys. A unified approach to approximation algorithms for bottleneck problems. J. ACM, 33(3):533–550, 1986.

[40] Wen-Lian Hsu and George L. Nemhauser. Easy and hard bottleneck location problems. Discret. Appl. Math., 1(3):209–215, 1979.

[41] Kamal Jain, Mohammad Mahdian, Evangelos Markakis, Amin Saberi, and Vijay V. Vazirani. Greedy facility location algorithms analyzed using dual fitting with factor-revealing LP. Journal of ACM, 50:795–824, 2003.

[42] Kamal Jain, Mohammad Mahdian, and Amin Saberi. A new greedy approach for facility location problems. In Proceedings of STOC, 2002, pages 731–740.

[43] Kamal Jain and Vijay V. Vazirani. Approximation algorithms for metric facility location and k-median problems using the primal-dual schema and lagrangian relaxation. Journal of ACM, 48(2):274–296, March 2001.

[44] D. R. Karger and M. Minkoff. Building steiner trees with incomplete global knowledge. In FOCS, pages 613–623, 2000.

[45] Samir Khuller and Yoram J. Sussmann. The capacitated k-center problem. SIAM Journal on Discrete Mathematics, pages 403–418, 2000.

[46] Madhu Sudan, C. Greg Plaxton, and Rajmohan Rajaraman. Analysis of a local search heuristic for facility location problems. Journal of Algorithms, 37(1):146–188, 2000.

[47] Retsef Levi, David B. Shmoys, and Chaitanya Swamy. Lp-based approximation algorithms for capacitated facility location. Journal of Mathematical Programming, 131(1-2):365–379, 2012.

[48] Shanfei Li. An Improved Approximation Algorithm for the Hard Uniform Capacitated k-median Problem. In Proceedings of APPROX/RANDOM, pages 325–338, 2014.
[49] Shi Li. A 1.488 approximation algorithm for the uncapacitated facility location problem. *Journal of Information and Computation*, 222:45–58, 2013.

[50] Shi Li. On uniform capacitated $k$-median beyond the natural LP relaxation. In *Proceedings of SODA*, pages 696–707, 2015.

[51] Shi Li. Approximating capacitated $k$-median with $(1 + \epsilon)k$ open facilities. In *Proceedings of SODA*, pages 786–796, 2016.

[52] Shi Li. On facility location with general lower bounds. In *SODA*, pages 2279–2290, 2019.

[53] Shi Li and Ola Svensson. Approximating $k$-median via pseudo-approximation. In *Proceedings of STOC*, pages 901–910, 2013.

[54] Mohammad Mahdian, Evangelos Markakis, Amin Saberi, and Vijay Vazirani. A greedy facility location algorithm analyzed using dual fitting. In *Proceedings of RANDOM-APPROX*, pages 127–137. 2001.

[55] Mohammad Mahdian and Martin Pál. Universal facility location. In *Proceedings of ESA*, pages 409–421, 2003.

[56] Mohammad Mahdian, Yinyu Ye, and Jiawei Zhang. Improved approximation algorithms for metric facility location problems. In *Proceedings of APPROX*, pages 229–242, 2002.

[57] M. Pál, É. Tardos, and T. Wexler. Facility location with nonuniform hard capacities. In *Proceedings of FOCS*, pages 329–338, 2001.

[58] Martin Pál and Éva Tardos. Group strategyproof mechanisms via primal-dual algorithms. In *Proceedings of FOCS*, pages 584–593, 2003.

[59] Clemens Rösner and Melanie Schmidt. Privacy preserving clustering with constraints. In *ICALP*, pages 96:1–96:14, 2018.

[60] Pierangela Samarati. Protecting respondents’ identities in microdata release. *IEEE Trans. Knowl. Data Eng.*, 13(6):1010–1027, 2001.

[61] David B. Shmoys, Éva Tardos, and Karen Aardal. Approximation algorithms for facility location problems (extended abstract). In *Proceedings of STOC*, pages 265–274, 1997.

[62] Maxim Sviridenko. An improved approximation algorithm for the metric uncapacitated facility location problem. In *Proceedings of IPCO*, pages 240–257, 2002.

[63] Zoya Svitkina. Lower-bounded facility location. In *SODA*, pages 1154–1163, 2008.
[64] Jiawei Zhang, Bo Chen, and Yinyu Ye. A multiexchange local search algorithm for the capacitated facility location problem. *Journal of Mathematics of Operations Research*, 30(2):389–403, 2005.

[65] Peng Zhang. A new approximation algorithm for the k-facility location problem. *Journal of Theoretical Computer Science*, 384(1):126 – 135, 2007.