The Wave Front Set of the Wigner Distribution and Instantaneous Frequency

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Abstract We prove a formula expressing the gradient of the phase function of a function $f : \mathbb{R}^d \to \mathbb{C}$ as a normalized first frequency moment of the Wigner distribution for fixed time. The formula holds when $f$ is the Fourier transform of a distribution of compact support, or when $f$ belongs to a Sobolev space $H^{d/2+1+\varepsilon}(\mathbb{R}^d)$ where $\varepsilon > 0$. The restriction of the Wigner distribution to fixed time is well defined provided a certain condition on its wave front set is satisfied. Therefore we first need to study the wave front set of the Wigner distribution of a tempered distribution.

Keywords Wigner distribution · Microregularity · Wave front set · Restriction of distributions · Instantaneous frequency

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1 Introduction

This paper treats a time-frequency version of the following trivial observation in Fourier analysis. Let $f(t) = C e^{2\pi i \xi_0 \cdot t}$, $C \in \mathbb{C} \setminus 0$, $t, \xi_0 \in \mathbb{R}^d$, be a nonzero complex multiple of a character on $\mathbb{R}^d$, $\xi_0 \cdot t$ denoting the inner product on $\mathbb{R}^d$. Its Fourier
transform is \( \hat{f} = C \delta_{\xi_0} \) so the frequency \( \xi_0 \) may be expressed using the Fourier transform as the normalized first order moment formula

\[
\xi_0 = \frac{\langle \hat{f}, \xi \rangle}{\langle \hat{f}, 1 \rangle}
\]  

(1.1)

where \( \langle \hat{f}, \xi \rangle \) is the vector \( \langle \hat{f}, \xi \rangle = (\langle \hat{f}, \xi_j \rangle)_{j=1}^d \in \mathbb{R}^d \) and \( \xi_j : \mathbb{R}^d \mapsto \mathbb{R} \) is coordinate function \( j, 1 \leq j \leq d \).

We will deduce a time-frequency version of this formula for more general functions, which looks like

\[
\frac{1}{2\pi} \nabla \arg f(t) = \frac{\langle W_f(t, \cdot), \xi \rangle}{\langle W_f(t, \cdot), 1 \rangle}, \quad \forall t \in \mathbb{R}^d : f(t) \neq 0, \ f \in \mathcal{FE}'(\mathbb{R}^d).
\]  

(1.2)

In the formula (1.2) \( W_f \) denotes the Wigner distribution, defined by

\[
W_f(t, \xi) = \int_{\mathbb{R}^d} f(t + \tau/2)f(t - \tau/2)e^{-2\pi i \tau \cdot \xi} d\tau
\]

for \( f \in \mathcal{S}'(\mathbb{R}^d) \). \( \mathcal{FE}' \) is the Fourier image of the compactly supported distributions (cf. Sect. 2).

For functions in \( \mathcal{FE}'(\mathbb{R}^d) \) which are not multiples of characters \( e^{2\pi i \xi_0 \cdot t} \), the frequency is not well defined. Therefore it is replaced in (1.2) by the natural generalization

\[
\frac{1}{2\pi} \nabla \arg f(t) = \frac{1}{2\pi} \left( \partial_j \arg f(t) \right)_{j=1}^d,
\]

that is, a normalized gradient of the phase function. We use the term instantaneous frequency, taken from the engineering literature [1, 5, 11], for this quantity. Thus (1.2) may be seen as a time-frequency version of the observation (1.1). In Sect. 5 we shall also prove a version of (1.2) for functions \( f \) that belong to a Sobolev space \( H^{d/2+1+\varepsilon}(\mathbb{R}^d) \) (see Sect. 2) where \( \varepsilon > 0 \). Then the distribution actions \( \langle W_f(t, \cdot), \xi \rangle \) and \( \langle W_f(t, \cdot), 1 \rangle \) are Lebesgue integrals.

In order to prove (1.2) we need to restrict the Wigner distribution as \( W_f \mapsto W_f(t, \cdot) \) to fixed time \( t \in \mathbb{R}^d \). For \( f \in \mathcal{S}'(\mathbb{R}^d) \), we have \( W_f \in \mathcal{S}'(\mathbb{R}^{2d}) \) and the restriction is a map \( \mathcal{S}'(\mathbb{R}^{2d}) \mapsto \mathcal{S}'(\mathbb{R}^d) \), provided it is well defined. Restriction of a distribution to a submanifold is possible under certain conditions on the wave front set (cf. [17]). More precisely, the restriction defines a well defined distribution provided the normal bundle of the submanifold has empty intersection with the wave front set of the distribution. Thus we are led to study the wave front set of the Wigner distribution first. We pursue this study in somewhat greater generality than actually needed in order to prove formula (1.2).

In Sect. 3 we define the space \( WFW^\perp \) of tempered distributions such that the wave front set of the Wigner distribution is directed purely in the frequency direction, and the space \( WFW^\neq \) of tempered distributions such that the wave front set of the Wigner distribution is nowhere parallel to the time direction. The latter space admits restriction \( W_f \mapsto W_f(t, \cdot) \) for all \( t \in \mathbb{R}^d \). We show the inclusions \( C_{\infty}^\text{slow} \subseteq WFW^\perp \) and \( C_{\text{slow}} \subseteq V_{\text{con}} \subseteq WFW^\neq \). Here \( C_{\text{slow}}^\infty \) (cf. Definition 3.4) denotes the space of