On (2-d)-Kernels in the Tensor Product of Graphs

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Abstract: In this paper, we study the existence, construction and number of (2-d)-kernels in the tensor product of paths, cycles and complete graphs. The symmetric distribution of (2-d)-kernels in these products helps us to characterize them. Among others, we show that the existence of (2-d)-kernels in the tensor product does not require the existence of a (2-d)-kernel in their factors. Moreover, we determine the number of (2-d)-kernels in the tensor product of certain factors using Padovan and Perrin numbers.

Keywords: domination; independence; (2-d)-kernel number; tensor product

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1. Introduction and Preliminary Results

In general, we use the standard terminology and notation of graph theory; see [1]. Only undirected, simple graphs are considered. By $P_n$, $C_n$ and $K_n$ we mean a path, cycle and complete graph on $n$ vertices, respectively. Moreover, $K_{n,m}$ is a complete bipartite graph on $n + m$ vertices. A vertex of degree one is called a leaf, and the set of all leaves of a graph $G$ is denoted by $L(G)$.

Let $G$ and $H$ be two disjoint graphs. The tensor product of two graphs $G$ and $H$ is the graph $G \times H$ such that $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(x_i,y_j)(x_j,y_q); x_i \in E(G)$ and $y_p,y_q \in E(H)\}$. The tensor product is also called a direct product or a categorical product; see [2] for the details.

We say that $D \subseteq V(G)$ is a dominating set of $G$, if every vertex of $G$ is either in $D$ or is adjacent to at least one vertex of $D$. A subset $S$ of vertices is an independent set of $G$ if no two vertices of $S$ are adjacent in $G$.

A subset $J$ is a kernel of $G$ if $J$ is independent and dominating. The topic of kernels was introduced by von Neumann and Morgenstern in digraphs in the 1930s in the context of game theory; see [3].

Since then, kernels have been intensively studied in graph theory for their relations to various problems, such as list colourings and perfectness. C. Berge made a great contribution to the theory of kernels in digraphs, being one of the pioneers studying the problem of the existence of kernels in digraphs and using a kernel for solving problems in other areas of mathematics; see [4–6]. In the literature, we can find many types and generalizations of kernels in digraphs, where the existence problems are studied, and some interesting results can be found in [7–11].

In an undirected graph, every maximal independent set is a kernel, so the existence problem in an undirected graph is trivial. The problem becomes more complicated when restrictions related to the domination or the independence are added. In that way, many interesting types of kernels in undirected graphs were introduced and studied (for example, (k,l)-kernels [12–14], efficient dominating sets [15], secondary independent dominating sets [16,17], restrained independent dominating sets [18], strong (1,1,2)-kernels [19] and others). Among many types of kernels in undirected graphs, there are kernels related to multiple domination. The concept of multiple domination was introduced by J. F. Fink...
and M. S. Jacobson in [20]. Recall that for an integer \( p \geq 1 \), a subset \( D \subseteq V(G) \) is called a \( p \)-dominating set of \( G \) if every vertex from \( V(G) \setminus D \) has at least \( p \) neighbours in \( D \). If \( p = 1 \), then we get the classical definition of the dominating set. If \( p = 2 \), then we get the two-dominating set.

Based on this definition in [21], A. Włoch introduced and studied the concept of a two-dominating kernel ((2-d)-kernel). A subset \( J \subseteq V(G) \) is a (2-d)-kernel of \( G \) if \( J \) is independent and two-dominating. The number of (2-d)-kernels in a graph \( G \) is denoted by \( \sigma(G) \). Let \( G \) be a graph with a (2-d)-kernel. The minimum (resp. maximum) cardinality of a (2-d)-kernel of \( G \) is called a lower (resp. upper) (2-d)-kernel number, and it is denoted by \( j(G) \) (resp. \( J(G) \)).

Existence problems of (2-d)-kernels were next considered in [22–24]. Among others, in [24], it was proven that the problem of the existence of (2-d)-kernels is \( \mathcal{NP} \)-complete for general graphs. This paper is a continuation of those considerations.

Referring to the definition of the (2-d)-kernel and papers [21–24], Z. L. Nagy extended the concept of (2-d)-kernels to \( k \)-dominating kernels (named as \( k \)-dominating independent sets) by considering a \( k \)-dominating set instead of the two-dominating set, and that type of kernels was studied in [25–27].

However, it is worth noting that the existence of \( k \)-dominating kernels requires a sufficiently large degree of vertices, significantly reducing the number of graphs with \( k \)-dominating kernels. Consequently, it is interesting to limit the requirements for the degrees of vertices if existing problems are considered.

Graph products are useful tools for obtaining new classes of graphs that can be described based on factor properties. In kernel theory, the existence problems are very often studied in products of graphs; see for example [17,19]. For (2-d)-kernels in the Cartesian product, some results were obtained in [23].

This paper considers the problem of the existence of (2-d)-kernels and their number in the tensor product of graphs. The problem of counting various sets (dominating, independent, kernels) in graphs and their relations to the numbers of the Fibonacci type were studied in [11,24,28]. Our results are as follows.

- We partially solve the problem of the existence of (2-d)-kernels in the tensor product, in particular giving the complete solutions for cases where factors are paths, cycles or complete graphs.
- We obtain new interpretations of the Padovan and Perrin numbers related to (2-d)-kernels in the tensor product of basic classes of graphs, i.e., paths, cycles and complete graphs.

2. (2-d)-Kernels in the Tensor Product of Graphs

In this section, we consider the problem of the existence and the number of (2-d)-kernels in the tensor product of two graphs. We will use the following results of the tensor product of two graphs, which has been obtained in [2,29].

**Theorem 1** ([29]). Let \( G \) and \( H \) be connected non-trivial graphs. If at least one of them has an odd cycle, then \( G \times H \) is connected. If both \( G \) and \( H \) are bipartite, then \( G \times H \) has exactly two connected components.

**Theorem 2** ([2]). Let \( G \) and \( H \) be graphs. If \( G \) is bipartite, then \( G \times H \) is bipartite.

The following theorems, concerning the problem of the existence of (2-d)-kernels in paths and bipartite graphs, were proven in [21]. We use them in the rest of the paper.

**Theorem 3** ([21]). The graph \( P_n \) has a (2-d)-kernel \( J \) if and only if \( n = 2p + 1 \), \( p \geq 1 \), and this kernel is unique. If it holds, then \( |J| = p + 1 \).
Theorem 4 ([21]). Let $G = G(V_1, V_2)$ be a bipartite graph. If for each two vertices $x, y \in L(G)$, $d_G(x, y) \equiv 0 \pmod{2}$, then $G$ has a $(2,d)$-kernel.

The simple observation of bipartite graphs leads us to the following result.

Corollary 1. Let $G = G(V_1, V_2)$ be a bipartite graph. If $\delta(G) \geq 2$, then $G$ has a $(2,d)$-kernel.

We present results concerning the existence problems of $(2,d)$-kernels, as well as their number in the tensor product of paths, cycles and complete graphs. Note that tensor products of these graphs have a symmetric structure. This property is very useful in finding $(2,d)$-kernels in these products. Moreover, we can notice that the kernels themselves are distributed symmetrically in tensor products.

It turns out that even in the simplest cases of the tensor product of graphs, the problem of the existence of $(2,d)$-kernels is not an easy one. Theorem 3 tells us that there are not any $(2,d)$-kernels in the path with an even number of vertices. When studying the tensor product of two paths, a similar result was found, namely the $(2,d)$-kernel exists if and only if both paths do not have even number of vertices simultaneously. We prove this in the following Theorem.

Theorem 5. Let $n, m \geq 1$ be integers. The graph $P_n \times P_m$ has a $(2,d)$-kernel if and only if at least one of the numbers $n$ or $m$ is odd.

Proof. Let $n, m \geq 1$ be integers. Suppose that $V(P_n) = \{x_1, x_2, \ldots, x_n\}$, $V(P_m) = \{y_1, y_2, \ldots, y_m\}$ with the numbering of vertices in the natural order. If $n = 1$, then the graph $P_n \times P_m$ is an edgeless graph, so $V(P_n \times P_m)$ is a $(2,d)$-kernel. Let $n, m \geq 2$. Suppose that $n, m$ are odd. If $n, m = 3$, then the graph $P_3 \times P_3$ is disconnected and contains two connected components $C_4$ and $K_{1,4}$. Graphs $C_4$ and $K_{1,4}$ have a $(2,d)$-kernel; thus, the graph $P_3 \times P_3$ has a $(2,d)$-kernel. Let $n \geq 3$, $m > 3$. Since $P_n$ and $P_m$ are bipartite, by Theorem 1, the graph $P_n \times P_m$ has two connected components $G_1, G_2$. By the definition of the graph $P_n \times P_m$, it follows that $L(G_1) \neq \emptyset$, $G_2$ is bipartite and $\delta(G_2) \geq 2$. By Corollary 1, it follows that $G_2$ has a $(2,d)$-kernel. We will show that $G_1$ has a $(2,d)$-kernel. By the definition of the tensor product, it follows that $L(G_1) = \{(x_1, y_1), (x_1, y_m), (x_m, y_1), (x_m, y_m)\}$. By the definition of the graph $P_n \times P_m$ and the assumption that $n, m$ are odd, it follows that the distance between vertices from $L(G_1)$ is even. Moreover, by Theorem 2, it follows that the graph $G_1$ is bipartite, and by Theorem 4, we obtain that $G_1$ has a $(2,d)$-kernel. Suppose that $n$ is odd and $m$ is even. If $n \geq 3$, $m = 2$, then the graph $P_n \times P_2$ is disconnected and contains two connected components isomorphic to a path $P_n$. Since $n$ is odd, by Theorem 3, it follows that the graph $P_n$ has a $(2,d)$-kernel. Hence, the graph $P_n \times P_2$ has a $(2,d)$-kernel. Let $n \geq 3$, $m > 2$. By Theorem 1, it follows that the graph $P_n \times P_m$ is disconnected and contains two connected components $H_1, H_2$. Moreover, by the definition of the tensor product, we obtain that the graph $H_1 \cong H_2$ and $L(H_1) \neq \emptyset$. Then, proving analogously as for $G_1$, we obtain that $H_1$, $H_2$ have a $(2,d)$-kernel. Hence, the graph $P_n \times P_m$ has a $(2,d)$-kernel.

We prove now that the tensor product $P_n \times P_m$ has no $(2,d)$-kernels if $n, m$ are even. If $n = 2$ and $m = 2k$, $k \geq 1$, then the graph $P_2 \times P_{2k}$ is disconnected and contains two connected components isomorphic to a path $P_{2k}$. By Theorem 3, it follows that the graph $P_{2k}$ does not have a $(2,d)$-kernel. Let $n, m > 2$. Then, by Theorem 1, it follows that the graph $P_n \times P_m$ is disconnected and contains two connected components $A_1, A_2$. Moreover, by the definition of the tensor product, we obtain that $A_1 \cong A_2$ and $L(A_1) \neq \emptyset$. Assume, to the contrary, that $P_n \times P_m$ has a $(2,d)$-kernel $J$. Thus, the graph $A_i, i = 1, 2$ has a $(2,d)$-kernel $J_i$. Let us consider the graph $A_1$. Without loss of generality, assume that $L(A_1) = \{(x_1, y_1), (x_n, y_m)\}$. Hence, vertices $(x_1, y_1), (x_n, y_m) \in J_1$ and $(x_1, y_2), (x_{n-1}, y_{m-1}) \notin J_1$. Since $N_{P_n \times P_m}(x_1, y_2) = \{(x_2, y_2), (x_{2s-1}, y_{2t-1})\}, t = 2, 3, \ldots, \frac{n}{2}$ and $(x_2, y_2) \notin J_1$, we have $(x_1, y_2) \in J_1$. Analogously we show that $(x_{2j-1}, y_{2t-1}) \in J_1, j = 2, 3, \ldots, \frac{n}{2}$, $s = 1, 2, \ldots, \frac{m}{2}$. This means that $(x_{n-1}, y_{m-1}) \in J_1$. Since the vertex $(x_n, y_m) \in J_1$ is adjacent
to the vertex \((x_{n-1}, y_{m-1}) \in J_1\), we obtain a contradiction with the independence of \(J_1\), which ends the proof. \(\Box\)

Based on the proof of Theorem 5, the following corollary is obtained. It concerns the number of \((2d)\)-kernels in the tensor product of paths, as well as the lower and upper \((2d)\)-kernel numbers.

**Corollary 2.** Let \(n, m \geq 2\) be integers. If the graph \(P_n \times P_m\) has a \((2d)\)-kernel, then:

(i) \(\sigma(P_n \times P_m) = \begin{cases} 1 & \text{if } m \text{ is even and } n \text{ is odd}, \\
2 & \text{if both } n \text{ and } m \text{ are odd,} \end{cases} \)

(ii) \(j(P_n \times P_m) = \begin{cases} m \cdot \left\lceil \frac{n}{2} \right\rceil & \text{if } m \text{ is even and } n \text{ is odd}, \\
\min\{m \cdot \left\lceil \frac{n}{2} \right\rceil, n \cdot \left\lceil \frac{m}{2} \right\rceil\} & \text{if both } n \text{ and } m \text{ are odd,} \end{cases} \)

(iii) \(J(P_n \times P_m) = \begin{cases} m \cdot \left\lceil \frac{n}{2} \right\rceil & \text{if } m \text{ is even and } n \text{ is odd}, \\
\max\{m \cdot \left\lceil \frac{n}{2} \right\rceil, n \cdot \left\lceil \frac{m}{2} \right\rceil\} & \text{if both } n \text{ and } m \text{ are odd.} \end{cases} \)

**Proof.** Let \(n, m \geq 2\) be integers. Suppose that \(V(P_n) = \{x_1, x_2, \ldots, x_n\}, V(P_m) = \{y_1, y_2, \ldots, y_m\}\) with the numbering of vertices in the natural order.

(i) Consider the case where \(n\) is odd and \(m\) is even. If \(n \geq 3, m = 2\), then \(\sigma(P_n \times P_2) = \sigma(P_n)\sigma(P_2) = 1\). Let \(n \geq 3, m > 2\). Then, the graph \(P_n \times P_m\) is disconnected and contains two isomorphic connected components \(H_1, H_2\). Hence, \(\sigma(P_n \times P_m) = \sigma(H_1)\sigma(H_2) = \sigma^2(H_1)\). Proving analogously as in the first paragraph of the proof of Theorem 5, we conclude that \(J = \{(x_{2i-1}, y_{2i-1}); s = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil, t = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil\}\) is the unique \((2d)\)-kernel of the graph \(H_1\). Thus, \(\sigma(P_n \times P_m) = 1\).

Suppose that \(n\) and \(m\) are odd. If \(n = m = 3\), then \(\sigma(P_3 \times P_3) = \sigma(C_4)\sigma(K_14) = 2\). Let \(n \geq 3, m > 3\). Then, the graph \(P_n \times P_m\) is disconnected and contains two connected components \(G_1, G_2\). Hence, \(\sigma(P_n \times P_m) = \sigma(G_1)\sigma(G_2)\). Assume that \(L(G_1) \neq \emptyset, G_2\) is bipartite and \(\delta(G_2) \geq 2\). Proving analogously as in Theorem 5, we conclude that \(J = \{(x_{2i-1}, y_{2i-1}); s = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil, t = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil\}\) is the unique \((2d)\)-kernel of the graph \(G_1\). Thus, \(\sigma(G_1) = 1\). We will show that the graph \(G_2\) has exactly two \((2d)\)-kernels. Let \(J_2\) be a \((2d)\)-kernel of the graph \(G_2\). We consider the following cases.

1. \((x_1, y_2) \notin J_2\).

Then, \((x_2, y_{2k-1}) \in J_2, k = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil\). Otherwise, \((x_1, y_2), i = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil\) are not two-dominated by \(J_2\). This means that \((x_3, y_{2i}) \notin J_2, i = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil\). Then, by analogous reasoning, we obtain \(J_2 = \{(x_{2i}, y_{2i-1}); a = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil, b = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil\}\) is the unique \((2d)\)-kernel of the graph \(G_2\) not containing the vertex \((x_1, y_2)\).

2. \((x_1, y_2) \in J_2\).

Proving analogously as in case 1, we obtain that \(J_2 = \{(x_{2i-1}, y_{2j}); a = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil, b = 1, 2, \ldots, \left\lceil \frac{m}{2} \right\rceil\}\) is the unique \((2d)\)-kernel of the graph \(G_2\) containing the vertex \((x_1, y_2)\).

Consequently, \(\sigma(G_2) = 2\), and finally, \(\sigma(P_n \times P_m) = 2\).

From the construction of \((2d)\)-kernels of the graph \(P_n \times P_m\), it follows that \(j(P_n \times P_m) = \max\{m \cdot \left\lceil \frac{n}{2} \right\rceil, n \cdot \left\lceil \frac{m}{2} \right\rceil\}\), for both \(n\) and \(m\) odd, and also, \(j(P_n \times P_m) = \min\{m \cdot \left\lceil \frac{n}{2} \right\rceil, n \cdot \left\lceil \frac{m}{2} \right\rceil\}\), for both \(n\) and \(m\) even, which proves (ii) and (iii). \(\Box\)

By analysing all other cases of the tensor product of paths, cycles and complete graphs, it turns out that all of them have a \((2d)\)-kernel, regardless of the number of vertices.

**Theorem 6.** Let \(n, m\) be integers. Then, the graphs

(i) \(P_n \times C_m\), for \(n \geq 1, m \geq 3\),

(ii) \(C_n \times C_m\), for \(n, m \geq 3\),

(iii) \(P_n \times K_m\), for \(n \geq 1, m \geq 3\),

(iv) \(C_n \times K_m\), for \(n, m \geq 3\),

(v) \(K_n \times K_m\), for \(n \geq 1, m \geq 3\),
have a \((2,d)\)-kernel.

**Proof.** Consider the following cases.

(i) Let \(n \geq 1, m \geq 3\) be integers. We will show that the graph \(P_n \times C_m\) has a \((2,d)\)-kernel. Suppose that \(V(P_n) = \{x_1, x_2, \ldots, x_n\}\), \(V(C_m) = \{y_1, y_2, \ldots, y_m\}\) with the numbering of vertices in the natural order. If \(n = 1\), then the graph \(P_1 \times C_m\) is an edgeless graph, so \(V(P_1 \times C_m)\) is a \((2,d)\)-kernel. Let \(n = 2\). If \(m\) is odd, then \(P_2 \times C_m \cong C_{2m}\), so it has a \((2,d)\)-kernel. If \(m\) is even, then \(P_2 \times C_m = C_m \cup C_m\). Thus, \(P_2 \times C_m, m \geq 3\) has a \((2,d)\)-kernel. Let \(n \geq 3\). Suppose that \(V_i = \{x_i\} \times V(C_m)\), \(i = 1, 2, \ldots, n\). We will show that the set \(J = \bigcup_{j=1}^{\lfloor \frac{n}{2} \rfloor} V_{2j-1}\) is a \((2,d)\)-kernel of the graph \(P_n \times C_m\). By the definition of the tensor product, it follows that \(V_i\) is an independent set, for \(i = 1, 2, \ldots, n\). Moreover, the factor \(P_n\) ensures that \(J\) is independent. We will show that \(J\) is two-dominating. Let \((x_a, y_b) \in V(P_n \times C_m) \setminus J\), \(a = 1, 2, \ldots, 2j, j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\), \(b = 1, 2, \ldots, m\). Then, \((x_a, y_b) \in \bigcup_{j=1}^{\lfloor \frac{n}{2} \rfloor} V_{2j}\). Since for all \(b = 1, 2, \ldots, m\), it holds that \(\deg_{C_m}(y_b) = 2\), and each vertex from \(V_{2j}\), \(j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\) is two-dominated by \(V_{2j-1}\). This means that \(J\) is a \((2,d)\)-kernel of the graph \(P_n \times C_m\).

(ii) Let \(n, m \geq 3\). Proving analogously as in Case (i), we can show that \(J = \bigcup_{j=1}^{\lfloor \frac{n}{2} \rfloor} V_{2j-1}\) is a \((2,d)\)-kernel of the graph \(P_n \times C_m\).

(iii) Let \(n \geq 1, m \geq 3\). Proving analogously as in Case (i), we can show that \(J = \bigcup_{j=1}^{\lfloor \frac{n}{2} \rfloor} V_{2j-1}\) is a \((2,d)\)-kernel of the graph \(P_n \times C_m\).

(iv) Let \(n, m \geq 3\). Proving analogously as in Case (i), we can show that \(J = V(C_m) \times \{y_1\}\) is a \((2,d)\)-kernel of the graph \(P_n \times C_m\).

(v) Let \(n \geq 1, m \geq 3\). Proving analogously as in Case (i), we can show that \(J = V(C_n) \times \{y_1\}\) is a \((2,d)\)-kernel of the graph \(P_n \times C_m\).

Let \(\mathcal{J}(G)\) be a family of all \((2,d)\)-kernels of a graph \(G\). Then, \(\sigma(G) = |\mathcal{J}(G)|\). For any vertex \(x \in V(G)\), let \(\mathcal{J}_x(G) = \{J \in \mathcal{J}(G); x \in J\}\) and \(\mathcal{J}_{-x}(G) = \{J \in \mathcal{J}(G); x \notin J\}\). Thus, \(|\mathcal{J}(G)| = |\mathcal{J}_x(G)| + |\mathcal{J}_{-x}(G)|\). Let \(\sigma_x(G) = |\mathcal{J}_x(G)|\) and \(\sigma_{-x}(G) = |\mathcal{J}_{-x}(G)|\). Then, \(\sigma(G) = \sigma_x(G) + \sigma_{-x}(G)\).

In many counting problems, the solution is given by the Padovan numbers \(S_1(n)\) and Perrin numbers \(S_2(n)\), which are the well-known numbers of the Fibonacci type. They are defined by the linear recurrence equation \(S_1(n) = S_1(n-2) + S_1(n-3)\), \(i = 1, 2\), with initial values \(S_1(0) = S_1(1) = S_1(2) = 1\) and \(S_2(0) = S_2(1) = S_2(2) = 2\). In [30], Z. Füredi observed that the total number of maximal independent sets of \(P_n\) and \(C_n\) is given by \(S_1(n+1)\) and \(S_2(n)\), respectively.

The next theorem shows the relation between the number of \((2,d)\)-kernels in the tensor product \(P_n \times C_m\) and the Padovan numbers.

**Theorem 7.** Let \(n \geq 1, m \geq 3\) be integers. Then:

\[
\begin{align*}
\text{(i)} & \quad \sigma(P_n \times C_m) = \begin{cases} S_1(n+1) & \text{if } m \text{ is odd}, \\ (S_1(n+1))^2 & \text{if } m \text{ is even}, \end{cases} \\
\text{(ii)} & \quad j(P_n \times C_m) = m \cdot \lfloor \frac{n}{2} \rfloor, \\
\text{(iii)} & \quad J(P_n \times C_m) = m \cdot \lceil \frac{n}{2} \rceil.
\end{align*}
\]

**Proof.** Let \(n \geq 1, m \geq 3\) be integers. Assume that \(V(P_n) = \{x_1, x_2, \ldots, x_n\}\) and \(V(C_m) = \{y_1, y_2, \ldots, y_m\}\) with the numbering of vertices in the natural order, and let \(V_i = \{x_i\} \times V(C_m), i = 1, 2, \ldots, n\).

(i) Consider the following cases.

1. \(m\) is odd.

Then, the graph \(P_n \times C_m\) is connected for each \(n \geq 2\). If \(n = 1, 2\), then \(\sigma(P_1 \times C_m) = 1 = S_1(2)\) and \(\sigma(P_2 \times C_m) = 2 = S_1(3)\), respectively. If \(n = 3\), then the family
\( \mathcal{J}(P_3 \times C_m) \) of all \((2d)\)-kernels has the form \( \mathcal{J}(P_3 \times C_m) = \{ V_1 \cup V_3, V_2 \} \). This means that \( \sigma(P_3 \times C_m) = 2 = S_1(4) \). Let \( n \geq 4 \). We will show that \( \sigma(P_n \times C_m) = S_1(n+1) \).

Let \( J \subset V(P_n \times C_m) \) be a \((2d)\)-kernel of the graph \( P_n \times C_m \). We have two possibilities.

1.1. \( (x_0, y_1) \in J \).

Then, \( (x_{n-1}, y_2) \notin J \). Since \( \text{deg}_{P_n \times C_m}(x_{n-1}, y_2) = 2 \) and vertices \( (x_{n-1}, y_2) \), \( (x_n, y_3) \) are adjacent, the vertex \( (x_n, y_3) \) has to belong to a \((2d)\)-kernel \( J \). Otherwise, \( (x_n, y_3) \) is not two-dominated by \( J \). Since \( m \) is odd, by analogous reasoning, we obtain \( (x_n, y_k) \in J, k = 1, 2, \ldots, m \). Thus, \( V_n \subset J \). Then, \( V_{n-1} \cap J = \emptyset \), and moreover, every vertex from \( V_{n-1} \) is two-dominated by \( J \). Hence, \( J = J' \cup V_n \), where \( J' \) is any \((2d)\)-kernel of the graph \( P_{n-2} \times C_m \). Thus, \( \sigma_{(x_0,y_1)}(P_n \times C_m) = \sigma(P_{n-2} \times C_m) \).

1.2. \( (x_0, y_1) \notin J \).

Then, \( (x_{n-1}, y_2), (x_{n-1}, y_3) \in J \). Proving analogously as in Case 1.1, we obtain that \( V_{n-1} \subset J \) and \( V_n \cap J = \emptyset = V_{n-2} \cap J \). Then, every vertex from \( V_{n-2} \cup V_n \) is two-dominated by \( J \). Hence, \( J = J'' \cup V_{n-1} \), where \( J'' \) is an \((2d)\)-kernel of the graph \( P_{n-3} \times C_m \). Thus, \( \sigma_{(x_0,y_1)}(P_n \times C_m) = \sigma(P_{n-3} \times C_m) \).

Consequently, \( \sigma(P_n \times C_m) = \sigma(P_{n-2} \times C_m) + \sigma(P_{n-3} \times C_m), n \geq 4 \). By the initial conditions, it follows that \( \sigma(P_n \times C_m) = S_1(n+1), n \geq 1 \).

2. \( m \) is even.

Then, by Theorem 1, it follows that the graph \( P_n \times C_m, n \geq 2 \) is disconnected and contains two isomorphic connected components \( G_1, G_2 \). Thus, \( \sigma(P_n \times C_m) = \sigma(G_1) \sigma(G_2) = (\sigma(G_1))^2 \). Let \( V_i^1 = \{ x_i \} \times \{ y_{2i-1}, t = 1, 2, \ldots, \frac{m}{2} \}, V_i^2 = \{ x_i \} \times \{ y_{2i}, t = 1, 2, \ldots, \frac{m}{2} \}, \) for \( i = 1, 2, \ldots, n \). Without loss of generality, assume that \( (x_1, y_1) \in V(G_1) \). If \( n = 1, 2 \), then \( \sigma(P_1 \times C_m) = 1 = (S_1(2))^2 \) and \( \sigma(P_2 \times C_m) = 4 = (S_1(3))^2 \), respectively. If \( n = 3 \), then the family \( \mathcal{J}(P_3 \times C_m) \) of all \((2d)\)-kernels has the form \( \mathcal{J}(P_3 \times C_m) = \{ V_1^1 \cup V_3^2, V_2^3 \} \). This means that \( \sigma(P_3 \times C_m) = 2^2 = (S_1(4))^2 \). Let \( n \geq 4 \). Proving analogously as in Case 1, we obtain that \( \sigma(G_1) = S_1(n+1) \). Hence, \( \sigma(P_n \times C_m) = (S_1(n+1))^2, n \geq 1 \).

From the construction of \((2d)\)-kernels of the graph \( P_n \times C_m \), it follows that \( m \cdot \left\lfloor \frac{n}{2} \right\rfloor \leq |J| \leq m \cdot \left\lceil \frac{n}{2} \right\rceil \). Hence, \( j(P_n \times C_m) = m \cdot \left\lceil \frac{n}{2} \right\rceil \) and \( J(P_n \times C_m) = m \cdot \left\lfloor \frac{n}{2} \right\rfloor \), which proves (ii) and (iii).

Proving analogously as in Theorem 7, we can give similar equalities for the tensor product of a path and a complete graph.

**Theorem 8.** Let \( n, m \geq 3 \) be integers. Then:

(i) \( \sigma(P_n \times K_m) = S_1(n+1) \),

(ii) \( j(P_n \times K_m) = m \cdot \left\lceil \frac{n}{2} \right\rceil \),

(iii) \( J(P_n \times K_m) = m \cdot \left\lfloor \frac{n}{2} \right\rfloor \).

Now, we give the relation between the Padovan and Perrin numbers, which will be useful to prove the next theorem concerning parameters \( \sigma, j, J \) in the tensor product of specific cycles.

**Lemma 1.** For \( n \geq 5 \),

\[
S_2(n) = S_1(n-2) + S_1(n-4) + 2S_1(n-5). \tag{1}
\]

**Proof** (induction on \( n \)). If \( n = 5, 6, 7 \), then the result holds by the definitions of the Padovan and Perrin numbers. Let \( n \geq 8 \), and suppose that (1) is true for all \( k < n \). We will show that \( S_2(n) = S_1(n-2) + S_1(n-4) + 2S_1(n-5) \). By the definition of Perrin numbers and the induction hypothesis, we have \( S_2(n) = S_2(n-2) + S_2(n-3) = S_1(n-4) + S_1(n-6) + S_1(n-7) + S_1(n-5) + S_1(n-7) + 2S_1(n-8) = S_1(n-2) + S_1(n-4) + 2S_1(n-5) \), which ends the proof.
The next theorem presents the relation between the number of $(2d)$-kernels in the tensor product $C_n \times C_3$ and the Perrin numbers.

**Theorem 9.** Let $n \geq 3$ be an integer. Then:

(i) $\sigma(C_n \times C_3) = 3 + S_2(n)$,

(ii) $j(C_n \times C_3) = n$,

(iii) $J(C_n \times C_3) = 3 \cdot \lceil \frac{n}{2} \rceil$.

**Proof.** Let $n \geq 3$ be an integer. Assume that $V(C_3) = \{y_1, y_2, y_3\}$ and $V(C_n) = \{x_1, x_2, \ldots, x_n\}$ with the numbering of vertices in the natural order. Let $X_i = \{x_i\} \times V(C_3)$, $i = 1, 2, \ldots, n$ and $Y_j = V(C_n) \times \{y_j\}$, $j = 1, 2, 3$.

(i) Let $n = 3, 4, 5, 6$. It is easy to check that the set of $(2d)$-kernels of $C_n \times C_3$ is $\{x_1, x_2, x_3, y_1, y_2, y_3\}$, $\{x_1 \cup x_3, x_2 \cup x_4, y_1, y_2, y_3\}$, $\{x_1 \cup x_3, x_1 \cup x_4, x_2 \cup x_4, x_2 \cup x_5, x_3 \cup x_5, y_1, y_2, y_3\}$, $\{x_1 \cup x_3 \cup x_5, x_1 \cup x_4, x_2 \cup x_4 \cup x_6, x_2 \cup x_5, x_3 \cup x_6, y_1, y_2, y_3\}$ for $n = 3, 4, 5, 6$, respectively; hence, the proposition holds for these cases. Let $n \geq 7$. We will show that $\sigma(C_n \times C_3) = 3 + S_2(n)$. Let $J \subset V(C_n \times C_3)$ be a $(2d)$-kernel of the graph $C_n \times C_3$. We have two possibilities:

1. $(x_{n, y_1}) \notin J$. Then, $(x_{n-1, y_2}), (x_{n-1, y_3}), (x_{1, y_1}), (x_{1, y_2}) \notin J$. Consider the vertex $(x_{n, y_2})$. If $(x_{n, y_2}) \notin J$, then $(x_{n, y_2}) \notin J$. Otherwise, $(x_{n, y_2})$ cannot be two-dominated by $J$. This means that $X_n \subset J$ and $X_{n-1} \cap J = \emptyset = X_1 \cap J$. Moreover, every vertex from $X_1 \cup X_{n-1}$ is two-dominated by $J$. Hence, $J = J^* \cup X_n$, where $J^*$ is any $(2d)$-kernel of the graph $P_{n-3} \times C_3$. Thus, $\sigma(P_{n-3} \times C_3)$ is the number of $(2d)$-kernels $J$ containing vertices $(x_{n, y_1})$ and $(x_{n, y_2})$. If $(x_{n, y_2}) \notin J$, then $(x_{n-1, y_1}), (x_{1, y_1}) \notin J$. Otherwise, $(x_{n, y_2})$ is not two-dominated by $J$. Then, $(x_{n, y_2}) \notin J$, and moreover, $(x_{n-2, y_k}), (x_{2, y_k}) \notin J$, $k = 2, 3$. By analogous reasoning, we obtain $Y_1 \subset J$ and $Y_2 \cap J = \emptyset = Y_3 \cap J$. This means that $Y_1$ is the unique $(2d)$-kernel of the graph $C_n \times C_3$ containing the vertex $(x_{n, y_1})$ and not containing the vertex $(x_{n, y_2})$. Hence, $\sigma(x_{n, y_1})(C_n \times C_3) = \sigma(P_{n-3} \times C_3) + 1$.

2. $(x_{n, y_1}) \notin J$. Consider four cases.

2.1. $(x_{n-1, y_1}), (x_{1, y_1}) \notin J$. It is obvious that $Y_2, Y_3$ are the only $(2d)$-kernels of the graph $C_n \times C_3$.

2.2. $(x_{n-1, y_1}) \notin J$ and $(x_{1, y_1}) \notin J$. We will show that $(x_{n, y_1})$ is two-dominated by $(x_{n-1, y_1}), (x_{n-1, y_3}) \in J$. Otherwise, at least one of the vertices $(x_{1, y_2}), (x_{1, y_3})$ belongs to a $(2d)$-kernel $J$. If $(x_{1, y_2}) \notin J$, then $(x_{1, y_2})$ cannot be two-dominated by $J$. If $(x_{1, y_2}) \notin J$ or $(x_{1, y_3}) \notin J$, then $(x_{1, y_1})$ is not dominated by $J$. This means that $X_n \cap J = \emptyset = X_1 \cap J$ and $X_{n-1} \subset J$. Then, $X_2 \subset J$, otherwise vertices from $X_1$ are not two-dominated by $J$. Moreover, $X_{n-2} \cap J = \emptyset = X_3 \cap J$, and the vertices from $X_1 \cup X_3 \cup X_{n-2} \cup X_n$ are two-dominated by $J$. Hence, $J = J^{*} \cup X_3 \cup X_{n-1}$, where $J^{*}$ is any $(2d)$-kernel of the graph $P_{n-6} \times C_3$. This means that $\sigma(P_{n-6} \times C_3)$ is the number of $(2d)$-kernels $J$ containing $(x_{n-1, y_1})$ and not containing $(x_{1, y_1}), (x_{n, y_1})$.

2.3. $(x_{n-1, y_1}) \notin J$ and $(x_{1, y_1}) \notin J$. Proving analogously as in Subcase 2.2, we obtain that $\sigma(P_{n-6} \times C_3)$ is the number of $(2d)$-kernels $J$ containing $(x_{1, y_1})$ and not containing $(x_{n-1, y_1}), (x_{n, y_1})$.

2.4. $(x_{n-1, y_1}), (x_{1, y_1}) \notin J$. Then, $X_n \cap J = \emptyset$, $X_1 \subset J$ and $X_{n-1} \subset J$. Otherwise, $J$ is not two-dominated. This means that $X_2 \cap J = \emptyset = X_{n-2} \cap J$. Moreover, every vertex from $X_2 \cup X_{n-2} \cup X_n$ is two-dominated by $J$. Thus, $J = J \cup X_1 \cup X_{n-1}$, where $J$ is any $(2d)$-kernel of the graph $P_{n-5} \times C_3$. This means that $\sigma(P_{n-5} \times C_3)$ is the number of $(2d)$-kernels $J$ containing $(x_{n-1, y_1}), (x_{1, y_1})$ and not containing $(x_{n, y_1})$. 


Hence, \( \sigma_{-(x_n,y_j)}(C_n \times C_3) = 2 + 2\sigma(P_{n-6} \times C_3) + \sigma(P_{n-5} \times C_3) \).

Finally, from the above cases, we obtain that \( \sigma(C_n \times C_3) = 3 + \sigma(P_{n-3} \times C_3) + 2\sigma(P_{n-6} \times C_3) + \sigma(P_{n-5} \times C_3) \).

By Theorem 7, we have that \( \sigma(C_n \times C_3) = 3 + S_1(n-2) + 2S_1(n-5) + S_1(n-4) \). Moreover, by Lemma 1, it follows that \( \sigma(C_n \times C_3) = 3 + S_2(n), \quad n \geq 3 \). From the construction of \((2,d)\)-kernels of the graph \( C_n \times C_3 \), it follows that \( n \leq |J| \leq 3 \cdot \lfloor \frac{n}{2} \rfloor \). Hence, \( j(C_n \times C_3) = n \) and \( J(C_n \times C_3) = 3 \cdot \lfloor \frac{n}{2} \rfloor \), which proves (ii) and (iii). \( \square \)

Using the same method as in Theorem 9, we are able to show similar equalities for the tensor product of a cycle and a complete graph.

**Theorem 10.** Let \( n, m \geq 3 \) be integers. Then:

(i) \( \sigma(C_n \times K_m) = S_2(n) + m \),

(ii) \( j(C_n \times K_m) = n \),

(iii) \( J(C_n \times K_m) = m \cdot \lfloor \frac{n}{2} \rfloor \).

Finally, let us consider the tensor product of two complete graphs.

**Theorem 11.** Let \( n, m \geq 3 \) be integers. Then:

(i) \( \sigma(K_n \times K_m) = n + m \),

(ii) \( j(K_n \times K_m) = \min\{n,m\} \),

(iii) \( J(K_n \times K_m) = \max\{n,m\} \).

**Proof.** Let \( n, m \geq 3 \) be integers. Assume that \( V(K_n) = \{x_1, x_2, \ldots, x_n\} \) and \( V(K_m) = \{y_1, y_2, \ldots, y_m\} \). Let \( X_i = \{x_i\} \times V(K_m), \quad i = 1, 2, \ldots, n \) and \( Y_j = V(K_n) \times \{y_j\}, \quad j = 1, 2, \ldots, m \).

(i) Let \( J \subseteq V(K_n \times K_m) \) be a \((2,d)\)-kernel of the graph \( K_n \times K_m \). Without loss of generality, assume that \( (x_s, y_t) \in J, \quad 1 \leq s \leq n, \quad 1 \leq t \leq m \). Then, \( (x_q, y_t) \in J, \quad 1 \leq q \leq n, \quad q \neq s \) or \( (x_s, y_r) \in J, \quad 1 \leq r \leq m, \quad r \neq t \). Otherwise, \( J \) is not independent. If \( (x_q, y_t) \in J \), then \( N_{K_n \times K_m}(x_s, y_t) \subseteq V(K_n \times K_m) \setminus J \). Thus, \( J = Y_t \). If \( (x_s, y_r) \in J \), then, by analogous reasoning, we obtain \( J = X_s \). This means that the family \( J \) of \((2,d)\)-kernels has the form \( J(K_n \times K_m) = \{X_i; i = 1, 2, \ldots, n\} \cup \{Y_j; j = 1, 2, \ldots, m\} \). Thus, \( \sigma(K_n \times K_m) = n + m \). Moreover, \( |X_i| = m, \quad i = 1, 2, \ldots, n \) and \( |Y_j| = n, \quad j = 1, 2, \ldots, m \); hence, \( j(K_n \times K_m) = \min\{n,m\} \) and \( J(K_n \times K_m) = \max\{n,m\} \), which proves (ii) and (iii). \( \square \)

### 3. Concluding Remarks

In this paper, we discussed the problem of the existence of \((2,d)\)-kernels in the tensor product of two graphs, in particular paths, cycles and complete graphs. Moreover, special graph parameters were determined for these cases. It was shown that if both factors of the tensor product are either paths, cycles or complete graphs, then their tensor product has a \((2,d)\)-kernel almost in all cases, except for the case of two paths with an even number of vertices. Moreover, the new interpretation of the Padovan and Perrin numbers related to the numbers of \((2,d)\)-kernels in a certain tensor product of graphs was given. The obtained results are a starting point to studying and counting \((2,d)\)-kernels in the tensor product where one of the factors is an arbitrary graph. It seems interesting to complement these results by obtaining sufficient conditions for the existence of \((2,d)\)-kernels in tensor products \( G \times P_n, G \times C_n, G \times K_n \). The results presented in this paper should be valuable to obtain such conditions.

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References
1. Diestel, R. *Graph Theory*; Springer: Heidelberg, Germany; New York, NY, USA, 2005.
2. Hammack, R.; Imrich, W.; Klavžar, S. *Handbook of Product Graphs*, 2nd ed.; CRC Press, Inc.: Boca Raton, FL, USA, 2011.
3. Morgenstern, O.; Von Neumann, J. *Theory of Games and Economic Behavior*; Princeton University Press: Princeton, NJ, USA, 1944.
4. Berge, C. *Graphs and Hypergraphs*; North-Holland: Amsterdam, The Netherlands, 1973.
5. Berge, C.; Duchet, P. Recent problems and results about kernels in directed graphs. *Discret. Math.* 1990, 86, 27–31.
6. Galeana-Sánchez, H.; Hernández-Cruz, C.; Arumugam, S. On the existence of (k, l)-kernels in digraphs with a given circumference. *AKCE Int. J. Graphs Comb.* 2013, 10, 15–28.
7. Galeana-Sánchez, H.; Ramírez, L. Monochromatic kernel-perfectness of special classes of digraphs. *Discuss. Math. Graph Theory* 2007, 27, 389–400.
8. Kucharska, M. On (k, l)-kernels of special superdigraphs of $P_n$ and $C_n$. *Discuss. Math. Combin. Probab. Stat.* 2001, 21, 95–109.
9. Bednarz, U. Strong (1;1;2)-kernels in the corona of graphs and some realization problems. *Iran. J. Sci. Technol. Trans. Sci.* 2020, 44, 401–406.
10. Fink, J.F.; Jacobson, M.S. $n$-domination in graphs. In *Graph Theory with Applications to Algorithms and Computer Science*; John Wiley & Sons: New York, NY, USA, 1985; pp. 283–300.
11. Gerbner, D.; Keszegh, B.; Methuku, A.; Patkós, B.; Vizer, M. An improvement on the maximum number of k-Dominating Independent Sets. *arXiv* 2017, arXiv:1709.04720.
12. Nagy, Z.L. Generalizing Erdős, Moon and Moser’s result—The number of k-dominating independent sets. *Electron. Notes Discret. Math.* 2017, 61, 909–915.
13. Nagy, Z.L. On the Number of k-Dominating Independent Sets. *J. Graph Theory* 2017, 84, 566–580.
14. Dosal-Trujillo, L.A.; Galeana-Sánchez, H. The Fibonacci numbers of certain subgraphs of circulant graphs. *AKCE Int. J. Graphs Comb.* 2015, 12, 94–103.
15. Weichsel, P.M. The Kronecker product of graphs. *Proc. Am. Math. Soc.* 1962, 13, 47–52.
16. Füredi, Z. The number of maximal independent sets in connected graphs. *J. Graph Theory* 1987, 11, 463–470.