Avoiding superluminal propagation of higher spin waves via projectors onto $W^2$ invariant subspaces.

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Abstract  
We propose to describe higher spins as invariant subspaces of the Casimir operators of the Poincaré Group, $P^2$, and the squared Pauli-Lubanski operator, $W^2$, in a properly chosen representation, $\psi(p)$ (in momentum space), of the Homogeneous Lorentz Group. The resulting equation of motion for any field with $s \neq 0$ is then just a specific combination of the respective covariant projectors. We couple minimally electromagnetism to this equation and show that the corresponding wave fronts of the classical solutions propagate causally. Furthermore, for $(s,0) \oplus (0,s)$ representations, the formalism predicts the correct gyromagnetic factor, $g_s = \frac{1}{s}$. The advocated method allows to describe any higher spin without auxiliary conditions and by one covariant matrix equation alone. This master equation is only quadratic in the momenta and its dimensionality is that of $\psi(p)$. We prove that the suggested master equation avoids the Velo-Zwanziger problem of superluminal propagation of higher spin waves and points toward a consistent description of higher spin quantum fields.
I. INTRODUCTION.

The field theoretical description of interacting particles with spin \( > 1 \) is a long standing problem. The interaction of a spin \( \frac{3}{2} \) Rarita-Schwinger (RS) field minimally coupled to an external electromagnetic field was shown to be inconsistent more than forty years ago [1]. Later on, Velo and Zwanziger observed superluminal propagation of the RS wave front in the presence of a minimally coupled electromagnetic field [2] and studied also the conditions under which the Proca field interacting with an external electromagnetic field propagates causally [3]. After these works many authors have addressed above problem from different perspectives and for different interactions [4] and the general feeling seems to be that it is not possible to construct a consistent quantum theory for massive particles with \( s > 1 \).

At several decades of distance in looking afresh onto the equations of motion can lead to different understanding of this fundamental problem. Weinberg emphasizes in his textbook on quantum field theory [5] that the equation of motion satisfied by the Dirac field is nothing but the record about the way how one puts together the two irreducible representations, \((1/2,0)\) and \((0,1/2)\), of the proper orthochronous Lorentz group to form a field that transforms invariantly under parity. In a wider understanding, this means that the equations of motion satisfied by a field are just a consequence of the properties of the representations of the Homogeneous Lorentz Group (HLG) chosen by us to accommodate the field and the discrete symmetries we require to be realized in this space. Closely related arguments can be found, among others in [6], [7], [8], and [9].

More recently, Refs. [10, 11] studied covariant projectors onto invariant subspaces of the squared Pauli-Lubanski operator in the representation space of the four-vector-spinor and showed that the associated equations are free from the Velo-Zwanziger problem. The corresponding projectors for the \((s,0) \oplus (0,s)\) representation space were studied in [12] where it was shown that under minimal coupling a particle in this representation has the correct value for the spin gyromagnetic factor, \( g_s = \frac{s}{s} \), thus proving Belinfante’s conjecture [13] from 1953.

In this work we explore the projectors onto the invariant subspaces of the Poincaré Casimir operators, the squared four-momentum and the squared Pauli-Lubanski operator, for any \( s \), and study propagation of the corresponding wave fronts along the lines of Refs. [2, 3]. The paper is organized as follows. In the next Section we recall in brief current description of
higher spins and its relation to the Poincaré group. In Section III we suggest to describe higher spins as invariant subspaces of the Poincaré Casimirs. In Section IV we show that particles within this framework propagate causally in the presence of an electromagnetic field, thus avoiding the classical Velo-Zwanziger problem. The paper closes with a brief Summary.

II. CURRENT DESCRIPTION OF FIELDS AND ITS RELATION TO POINCARÉ GROUP REPRESENTATIONS.

The primary classification of elementary systems is usually done by identifying them (up to form factors) with the irreducible representations (irreps) of the Poincaré group \( (PG) \). If so, then one necessarily has to consider particles as invariant spaces of the Casimir operators of this group– the squared four-momentum \( P^2 \), on the one side, and the squared Pauli-Lubanski operator \( W^2 \), on the other side and label them by their respective eigenvalues, \( p^2 \), and \( -p^2s(s+1) \), as \( |p^2, s(s+1)\rangle \). Further quantum numbers can be associated with the Casimir invariants of the underlying Homogeneous Lorentz Group (HLG), \( SO(1,3) \), and are approached by the reduction chain \( PG \supset SO(1,3) \). For finite dimensional representations, the Casimir invariants of \( SO(1,3) \) are frequently expressed in terms of two \( SU(2) \) Casimirs, in turn denoted by \( S^2_L \) and \( S^2_R \) of \( SU(2)_L \otimes SU(2)_R \), a group that is locally isomorphic to \( SL(2,C) \), the universal covering of HLG. The two additional quantum labels gained in this manner are the well known left– and right handed ”angular momenta”, \( s_L \), and \( s_R \), respectively. Therefore, a covariant state labeling can be introduced as: \( |p^2, s(s+1); s_L, s_R\rangle \), with \( s = |s_L - s_R|, ..., s_L + s_R \). In so doing one encounters essentially two types of finite dimensional HLG representations.

1. The first ones contain just one \( W^2 \) invariant subspace, and correspond to the case when one of the \( s_L, s_R \) labels vanishes (i.e. either \((s_L,0)\) or \((0,s_R)\)), and \( s_R = s_L \). In such a case, \( s_{L/R}(s_{L/R} + 1) = s(s+1) \), equals the \( \left(-\frac{1}{m^2}W^2\right) \) eigenvalue in the space under consideration (see Eq. (20) below) and \( W^2 \) – and \( S^2_{L/R} \) invariant spaces coincide. Irreps of the above type are suggestive of replacing \( W^2 \)– by \( SU(2) \) spin labels.

As long as the basic fields in physics are precisely of the above type (the Dirac field is \((1/2,0) \oplus (0,1/2)\), the electromagnetic field strength tensor is \((1,0) \oplus (0,1)\), and
scalars are just $(0,0)$ identifying Poincaré labels with $SU(2)$ spins works out without any harm.

2. The second ones are HLG irreps containing several $W^2$ invariant subspaces. In this case, both $s_L$, and $s_R$ are non-vanishing, and the irreps are of the type $(s_L, s_R)$ with $s_L \neq 0$, and $s_R \neq 0$. Examples are the vector-- and tensor gauge fields, $(1/2,1/2)$, and $(1,1)$, respectively. In the rest frame, $W^2 = -\frac{1}{m^2} S^2$ hence $W^2$ and $S^2$ invariant sub-spaces coincide. However, beyond rest frame, in flight, $W^2$ and $S^2$ invariant sub-spaces are no longer identical, a situation caused by the property of the boost to mix up $SU(2)$ spins differing by one unit.

Often, Lorentz representations that contain as building blocks irreps of the second type, appear attractive for the description of higher spins, the classical examples being the totally symmetric $K$ rank Lorentz tensors with Dirac spinor components, generically denoted by $\psi_{\mu_1...\mu_K}$. They are exploited for the description of fields that have been labeled in the rest frame by the highest spin $J = K + 1/2$. The separation between Lorentz and spinor indices inherent to such tensors makes them especially appealing for the construction of covariant fermion-boson vertices. However, one has to face the problem how to pick up the favored degrees of freedom and exclude interference with the unwanted ones. It seems inevitable to return back to the Poincaré invariants, if one wishes to distinguish all the degrees of freedom contained in $\psi_{\mu_1...\mu_K}$ in a covariant and transitionally invariant fashion. Yet, for one reason or the other, this is not the path tenaciously pursued by the theory. Rather, one still prefers to stay within the elaborated scheme of substituting $W^2$ by $SU(2)$ labels, but, yes, modify the latter scheme to account for the new situation in introducing constraints, considered as appropriate.

To be specific, in order to select out of $\psi_\mu$ (a field belonging to $[(1/2,0) \oplus (0,1/2)] \otimes (1/2,1/2)$) the $W^2$ invariant subspace that relates to spin $3/2$ at rest, one requires

$$
(i\partial^\mu \gamma_\mu - m)\psi_\mu = 0, \\
\partial^\mu \psi_\mu = 0, \\
\gamma^\mu \psi_\mu = 0.
$$

Exploiting constraints (some times termed to as auxiliary, or, supplementary, conditions) in place of $W^2$ quantum numbers brings the advantage to remain within the framework of
equations linear in the momenta, and to work with four-dimensional Dirac spinors. However, these advantages reveal themselves as deceptive at the moment one has to face grave worries about compatibility of constraints and dynamics. Recall, that the constraints change upon gauging and one has to make sure that the modification is preserved in time by the equation of motion and the latter does not violate causality. Notice that covariance alone is indeed a necessary but not a sufficient condition for special relativity. For example, space-like intervals are doubtlessly covariant objects, but they are unacceptable for the description of free physical fields as they prescribe the particle to violate causality during propagation. Precisely a flaw of that very type was revealed by Velo and Zwanziger in Ref. \cite{2} regarding the $\gamma^\mu \psi_\mu = 0$ constraint onto the four–vector spinor. Velo and Zwanziger showed that above constraint triggers acausal propagation of Rarita-Schwinger particles crossing an electromagnetic field.

In the present article we shall avoid above inconsistencies in developing a different view on form and content of wave equations for higher spins. Namely, we take the position that the equation of motion for whatever free particle has to be (i) a function of $P^2$ and $W^2$, the Casimir invariants of the Poincaré group, (ii) operates immediate, i.e. without any supplementary constraints, on the HLG representation chosen to embed the field as one of its covariant sectors\cite{16}.

Within this context, there are two primordial equations of motion to be satisfied by any field. One of them searches for $P^2$ invariant subspaces. It is nothing more but the Klein-Gordon equation. The other one secures in addition invariance under pseudo–rotations and pins-down $W^2$ invariant subspaces by means of appropriately constructed covariant projectors. It is that very latter type of equations on which we focus attention here. For the sake of self-sufficiency of the presentation, the subsequent Section opens with a brief review of the basics of space-time symmetries.
III. COVARIANT WAVE EQUATIONS FOR HIGHER SPINS FROM $W^2$ INVARIANT SUBSPACES.

A. Basics of Space-Time Transformations.

A general Poincaré transformation in space time can be written in the factorized form

$$g(b, \Lambda) = T(b) \Lambda,$$

where $T(b) = g(b, E)$ (E denotes the unit matrix) is a translation and $\Lambda = g(0, \Lambda)$ is a proper Lorentz transformation. In the standard convention, the generators of the translation group in 1+3 time-space dimensions, $T_{1,3}$, are $P_\mu$ in $T(b)$, which are commuting,

$$[P_\mu, P_\nu] = 0. \quad (3)$$

The HLG transformation in coordinate space,

$$x'_\mu = \Lambda_\mu \nu x_\nu, \quad \Lambda_\mu \nu = \exp \left[ -\frac{i}{2} \theta^{\mu\nu} L_{\mu\nu} \right], \quad L_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu, \quad (4)$$

induces the following transformation for a field $\psi(x)$,

$$\psi'(x) = \exp \left[ -\frac{i}{2} \theta^{\mu\nu} M_{\mu\nu} \right] \psi(\Lambda^{-1} x). \quad (5)$$

Here, $\theta^{\mu\nu}$ are continuous parameters, while the $n \times n$ matrices $M_{\mu\nu}$ represent a totally anti-symmetric 2nd rank Lorentz tensor. They are the generators of the homogeneous Lorentz group in the representation space of interest, and satisfy the commutation relations of the associated algebra:

$$[M_{\mu\nu}, M_{\alpha\beta}] = -i(g_{\mu\alpha} M_{\nu\beta} - g_{\mu\beta} M_{\nu\alpha} + g_{\nu\beta} M_{\mu\alpha} - g_{\nu\alpha} M_{\mu\beta}). \quad (6)$$

Their commutators with the generators of the translation group read

$$[P_\mu, M_{\alpha\beta}] = i(g_{\mu\alpha} P_\beta - g_{\mu\beta} P_\alpha), \quad (7)$$

where $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the metric tensor. The $M_{\mu\nu}$ generators consist of

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \quad [L_{\mu\nu}, S_{\mu\nu}] = 0, \quad (8)$$
where \( L_{\mu\nu} \), and \( S_{\mu\nu} \) in turn generate rotations in external coordinate– and internal representation spaces. The generators of boosts \((\mathcal{K}_x, \mathcal{K}_y, \mathcal{K}_z)\) and rotations \((J_x, J_y, J_z)\) are related to \( M_{\mu\nu} \) via

\[
\mathcal{K}_i = M_{0i}, \quad J_i = \frac{1}{2} \epsilon_{ijk} M_{jk}, \quad (9)
\]

respectively.

**B. Pauli Lubanski Vector and Associated Casimir Invariant.**

The Pauli–Lubanski (PL) vector is now defined as

\[
W_\mu = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} M_{\nu\alpha} P_{\beta}, \quad (10)
\]

where \( \epsilon_{0123} = 1 \). This operator can be shown to satisfy the commutators

\[
[W_\alpha, M_{\mu\nu}] = i(g_{\alpha\mu} W_\nu - g_{\alpha\nu} W_\mu), \quad [W_\alpha, P_\mu] = 0, \quad (11)
\]

i.e. it transforms as a four-vector under Lorentz transformations. The remarkable point is that the external coordinate part of \( M_{\mu\nu} \), namely the ”orbital” momentum \( L_{\mu\nu} \), does not contribute to \( W_\mu \) due to the anti-symmetric Levi-Civita tensor. As a result, \( W_\mu \) restricts to

\[
W_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho\tau} S_{\nu\rho} P_{\tau}, \quad (12)
\]

and its squared (in covariant form) is calculated to be

\[
W^2 = -\frac{1}{2} S_{\mu\nu} S^{\mu\nu} P^2 + G^2, \quad G_\mu := S_{\mu\nu} P^\nu. \quad (13)
\]

The operators \( S_{\mu\nu} \) act exclusively in the internal spin space and commute like

\[
[S_{\mu\nu}, S_{\alpha\beta}] = -i(g_{\mu\alpha} S_{\nu\beta} - g_{\mu\beta} S_{\nu\alpha} + g_{\nu\beta} S_{\mu\alpha} - g_{\nu\alpha} S_{\mu\beta}). \quad (14)
\]

As long as Eq. (14) has same form as Eq. (6), one may view \( S_{\mu\nu} \) as generators of Lorentz transformations in the intrinsic space. However, in contrast to Eq. (7), \( S_{\mu\nu} \) commute with the operators of translations

\[
[P_\alpha, S_{\mu\nu}] = 0. \quad (15)
\]

In effect, one does not find precisely Poincaré transformations in the internal space but rather a contracted form of them. Hereafter we will refer to the group generated by \( S_{\mu\nu} \) as
the “Internal Homogeneous Lorentz Group” (IHLG) to distinguish it from the HLG spanned by $M_{\mu\nu}$. In summary, one can write down generators of boosts and rotations in the internal space as

$$K_i = S_{0i}, \quad S_i = \frac{1}{2} \epsilon_{ijk} S_{jk}.$$  \hspace{1cm} (16)

The internal HLG has by itself two Casimir invariants, in turn given by $C_1 = \frac{1}{4} S_{\mu\nu} S^{\mu\nu}$, and $C_2 = S_{\mu\nu} \tilde{S}^{\mu\nu}$, with $\tilde{S}_{\mu\nu} = \epsilon_{\mu\nu\rho\tau} S^{\rho\tau}$. In terms of $K_i$ and $S_i$ one finds

$$C_1 = \frac{1}{2} (S^2 - K^2), \quad C_2 = iS \cdot K.$$ \hspace{1cm} (17)

The latter equation allows to cast $W^2$ into the form

$$W^2 = -2C_1 P^2 + G^2.$$ \hspace{1cm} (18)

For irreps of the type $(s,0) \oplus (0,s)$ where $K_i = \mp i S_i$, one finds the insightful relation \[11\]

$$G^2 = -W^2.$$ \hspace{1cm} (19)

Insertion of Eq. (19) into Eq. (18) amounts to

$$W^2 = -S^2 P^2.$$ \hspace{1cm} (20)

The latter relation explains the privileged position of $(s,0) \oplus (0,s)$ states to carry unique SU(2) spin both at rest (where $W^2$ any way reduces to $-S^2 m^2$ in accord with Eq. (20)) and in flight. However, for all the other types of Lorentz representations, $W^2 \neq G^2$ and the $(\frac{1}{m^2}W^2)$ labels for particles in flight do not have the interpretation of ordinary SU(2) spin. In the following we label $W^2$ invariant sub-spaces by $s$ but in general without any reference to SU(2) spin.

**C. Covariant projectors onto $W^2$ invariant subspaces.**

To begin with we recall that the interpretation of elementary systems as Poincaré group irreducible representations requires any field to transform invariantly under the action of both $P^2$ and $W^2$. In the following we work with massive fields. The former invariance leads to the Klein-Gordon equation for any arbitrary field

$$ (P^2 - m^2) \psi(p) = 0.$$ \hspace{1cm} (21)
Invariance under the action of $W^2$ results into the new condition

$$\Pi^*(\mathbf{p})\psi(\mathbf{p}) = \psi(\mathbf{p}),$$  \hspace{1cm} (22)

where $\Pi^*(\mathbf{p})$ stands for an appropriately constructed covariant projector onto the $(-p^2s(s+1))$ invariant subspace of $W^2$ in $\psi(\mathbf{p})$. To be specific, for the case of the four-vector spinor, such projectors have been presented in Ref. \[10\]. In general, equations of the type (22) are equivalent to

$$[W^2 + P^2s(s+1)]\psi(\mathbf{p}) = 0.$$  \hspace{1cm} (23)

Next, it is necessary to account for the mass shell condition in Eq. (21). For this purpose, we sum up Eqs. (23) and (21) to obtain

$$\left[\frac{1}{s}W^2 + sp^2 + m^2\right]\psi(\mathbf{p}) = 0,$$  \hspace{1cm} (24)

and cast the latter equation into the explicitly covariant form

$$[t_{\mu\nu}P^\mu P^\nu - m^2]\psi(\mathbf{p}) = 0.$$  \hspace{1cm} (25)

Here $t_{\mu\nu}$ stands for

$$t_{\mu\nu} = \frac{1}{s}(2C_1g_{\mu\nu} - S_\alpha S^\alpha_{\mu\nu}) - s\ g_{\mu\nu},$$

$C_1$ denotes the first Casimir in Eq. (17) and $S^{\beta\rho}$ are the IHLG generators in the particular representation chosen for $\psi(\mathbf{p})$. Their construction as solutions of the algebra of the Lorentz group for the representation space under consideration is straightforward \[7, 9, 10\].

Using now the gauge principle for electromagnetism in this equation we obtain

$$\left[\left(\frac{1}{s}(2C_1g_{\mu\nu} - S_{\alpha\beta}S^\alpha_{\mu\nu}) - s\ g_{\mu\nu}\right)\pi^\mu\pi^\nu - m^2\right]\psi(\mathbf{p}) = 0,$$  \hspace{1cm} (26)

with $\pi^\mu = P^\mu + eA^\mu$, and $e$ denoting the charge of the field $\psi(\mathbf{p})$. Notice that Eq. (26) is a covariant matrix equation that operates in the vector space of the dimensionality of $\psi(\mathbf{p})$. For example, when $\psi(\mathbf{p})$ stands for the four-vector– spinor, $W^2$ is represented by a $16 \times 16$ matrix. For the sake of illustration, we here bring the Lagrangian density for the lowest Rarita-Schwinger representation. It reads

$$\mathcal{L}(x) = \overline{\psi}(x)\ t_{\mu\nu}\ \pi^\mu\pi^\nu\ \psi(x) - m^2\overline{\psi}(x)\psi(x),$$  \hspace{1cm} (27)
where $\bar{\psi}(x) = \psi^\dagger(x)(\gamma^0 \otimes g)$ where $g$ is the matrix of the metric tensor. The definition of $\bar{\psi}(x)$ has to be performed for each representation individually. When applied to the Dirac representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, Eq. (26) also has the great advantage to yield the correct value of the gyromagnetic factor, $g_s = 2$. This is not fortuitous but reflects the general property of our master equation (26) to predict the correct value for the gyromagnetic ratio as $g_s = \frac{1}{s}$ for fields in $(s, 0) \oplus (0, s)$ [12]. Had we used instead Eq. (23) alone, we would have found the problematic case of $g_s = \frac{1}{s(s+1)}$.

With respect to $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, Eq. (24) is nothing more but the Klein-Gordon equation for each field component. This is due to the fact that the squared Pauli-Lubanski vector for all $(s, 0) \oplus (0, s)$ fields is just $-s(s+1)P^2 \mathbf{1}_{(2s+1) \times (2s+1)}$. The $W^2$ Casimir invariant identifies only the spin content and remains indifferent to the discrete $C$, $P$, or $T$ properties of the representation of interest. Recall that one has different options to stick together, say, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ in depending on whether one wants $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ to diagonalize the parity–, $\gamma^0 \mathcal{R}$ or, the charge conjugation–, $i\gamma_2 K$, operator. For parity eigenstates one ends up with the standard Dirac equation, while for $C$– parity states one finds again the Dirac equation but with a Majorana mass term, respectively. Notice however that, under gauging, this equation gives the right magnetic properties for $(s, 0) \oplus (0, s)$ fields. This means that solutions to Dirac equation are solutions to Eq. (26) although the converse is not necessarily true since our equation specifies just the value of the spin.

For product representation spaces of the type $\psi_{\mu_1, \mu_2, ..., \mu_K}$, the most interesting representation space for applications in hadron physics, the situation is different provided, one is tracking the highest spin. As long as the highest spins are non-degenerate, there is no confusion with parity doubling, as would be the case for the lower spins. For these reasons, Eq. (26) has its major merits with respect to the highest spins in the representations.

Next we study wave front propagation of particles described by means of Eq. (26) along the line of Refs. [2, 3].

IV. AVOIDING SUPERLUMINAL PROPAGATION OF HIGHER SPIN WAVES.

Wave propagation is associated with a hyperbolic system of partial differential equations [14]. For such a class of differential equations the initial value problem can be posed on a class of surfaces ( ”space like” surfaces with respect to the equation of motion). The
equations possess solutions with wave fronts traveling along rays at finite velocities. At any point on the surface, the rays form a cone that is entirely determined by the coefficients of the highest derivatives in the equation of motion \[14\]. The wave front can be characterized by \( n^\mu = (n^0, \mathbf{n}) \), the vectors normal to the characteristic surface. The system of equations is hyperbolic if \( n^0 \) is real for any \( n \). To find the normal vectors it is sufficient to first replace in the highest derivatives of the equation of motion \( P_\mu \) by \( n_\mu \) and then calculate the determinant \( D(n) \) (so called ”characteristic determinant” \[3\]) of the matrix given by the corresponding coefficients.

A. Wave front propagation of the Klein-Gordon, Dirac and Rarita-Schwinger equations.

In cases when the coupling to external fields is carried by the lower derivatives in the equation of motion, such as, say, the Klein–Gordon equation, the ray cones for interacting and free fields coincide. Indeed, in the latter case and under minimal coupling one finds

\[
\left[ \pi^\mu \pi_\mu - m^2 \right] \psi(p) = \left[ P^\mu P_\mu + e(P^\mu A_\mu + A^\mu P_\mu) + e^2 A^\mu A_\mu - m^2 \right] \psi(p) = 0. \tag{28}
\]

The vanishing of the characteristic determinant in this case yields

\[
D(n) = \text{Det}(n^2) = n^2 = 0, \tag{29}
\]

which has real \( n^0 \) for any \( n \). Same is true for Dirac particles, though not as obvious. As is well known, a Dirac particle coupled minimally to the electromagnetic field is described by

\[
[\gamma^\mu (P_\mu + eA_\mu) - m] \psi(p) = 0. \tag{30}
\]

Now, the resulting characteristic determinant is found to be the squared of Eq. \[29\]

\[
D(n) = \text{Det}(\gamma^\mu n_\mu) = (n^2)^2. \tag{31}
\]

The vanishing of this determinant results once again into a ray cone that coincides with the light cone.

The wave front propagation of the solution of the Rarita-Schwinger set of equations was studied in great detail in Ref. \[2\]. To understand the essence of the latter work recall that Eqs. \[1\] or the analogous equation in the interacting case, can be derived from a
Lagrangian, a method suggested by Fierz and Pauli \[15\]. Within the latter framework not all the Euler-Lagrange equations appear as genuine equations of motion, meaning that some of them may not involve time derivatives, a property that qualifies them only as constraints onto the fields. Precisely this is the case for the Rarita-Schwinger framework discussed in Section II. As a consequence, any surface in space-time is a characteristic surface \[14\]. The Rarita-Schwinger system of coupled equations turns to be equivalent to a system of hyperbolic equations supplemented by constraints that are conserved in time. In this case, the wave fronts of the constrained system are no longer given by the characteristic determinant of the Euler-Lagrange equations. Rather, it is necessary to find the genuine equation of motion, i.e. the one which (i) contains all the higher order derivatives needed for the complete characterization of the system, (ii) preserves the constraints in time. Finding such an equation in general introduces, in addition to the derivatives already present in the system of coupled equations, also new ones which as a rule spoil causal propagation, a result due to \[2, 3\].

B. Wave front propagation of $W^2$ invariant subspaces.

In the present work we suggested an alternative formalism to the Rarita-Schwinger framework. Our proposal was to pin down the degrees of freedom of interest by means of Eq. (26). This equation was built upon the covariant projector onto the $W^2$ invariant vector spaces in the representation under consideration, and did not invoke any supplementary conditions. In this concern, it is worth to remark that the formalism does not deal with the whole representation space but only with one of its $W^2$ invariant subspaces. Below we prove that equations of the latter type do not suffer the Velo-Zwanziger problem upon gauging.

Firstly, we have to check that for all the degrees of freedom of $\psi(p)$, the second order time-derivatives enter Eq. (26) with non-vanishing coefficients. This can be done in full generality in momentum space where

\[
t_{00} = \frac{1}{s} (2C_1 g_{00} - S_{\alpha0} S^{\alpha0}) - sg_{00} = 1.
\]

(32)

Therefore, for all $W^2$ invariant subspaces, the time derivative of each field component in Eq. (26) does not vanish. This equation will be hyperbolic if the solutions $n^0$ to $D(n) = 0$
are real for any \( n \). In this case we must solve
\[
\text{Det} \left[ -\frac{1}{s} W^2(n) - s \, n^2 \right] = 0. \tag{33}
\]

In order to demonstrate that (26) is a hyperbolic equation in the HLG representation space chosen for \( \psi(p) \) we here exploit decomposition of the latter into invariant subspaces of \( W^2 \).

The most transparent representation of \( W^2 \) is obtained in the basis of \( p \)-dependent \( W^2 \) eigenstates where \( W^2 \) is block diagonal and equal to
\[
W^2(P) = -P^2 \text{Diag} \left[ s_1(s_1 + 1)1_{s_1}, \, s_2(s_2 + 1)1_{s_2}, \ldots s_N(s_N + 1)1_{s_N} \right]. \tag{34}
\]

Here \( \{s_1, s_2, \ldots s_N\} \) label the different eigensubspaces of \( W^2 \) (one of them being \( s \)) in the representation of interest, while \( 1_{s_j} \) denotes the unit matrix of dimensionality \((2s_j + 1) \times (2s_j + 1)\). Notice that the dimensionality, \( d \), of the representation space \( \psi(p) \) relates to the \( W^2 \) quantum numbers via \( d = \sum_i m_i(2s_i + 1) \), where \( m_i \) is the multiplicity of \( s_i \). The latter accounts for possible degeneracies of the \( W^2 \) invariant subspaces in \( \psi(p) \) with respect to further symmetries such like, say, one of the discrete space–time symmetries.

The determinant (33) is calculated as
\[
\text{Det} \left[ -\frac{1}{s} W^2(n) - s \, n^2 \right] = \prod_{k=0}^{N} \left( n^2 \left( \frac{s_k(s_k + 1)}{s} \right) - s \right)^{2s_k + 1}. \tag{35}
\]

As long as for the integer/half-integer \( s \) under consideration, there are no positive integers and half-integers \( s_k \) satisfying
\[
\frac{s_k(s_k + 1)}{s} - s = 0, \tag{36}
\]
the roots of the characteristic determinant are \( n^2 = 0 \). Thus the solutions have \( n^0 \) real for any \( n \), and Eq.(26) is a set of hyperbolic equations for the \( \psi(p) \) components. The characteristic surfaces are same for free and interacting particles, and the ray cone coincides with the light cone. In other words, the wave front propagation of \( W^2 \) invariant subspaces is free from the Velo–Zwanziger problem.

V. CONCLUSIONS AND PERSPECTIVES.

In the present article we advocate the idea to consider higher spins as invariant subspaces of the Casimir operators of the Poincaré group, the squared four-momentum and the squared
Pauli-Lubanski vector, in a properly chosen representation of the HLG, $\psi(p)$. In executing the idea we demonstrated that any higher spin is described in terms of one covariant matrix equation that (i) is determined exclusively by the HLG generators in $\psi(p)$, (ii) is of the dimensionality of $\psi(p)$, (iii) is always of second order in the momenta. We gauged this equation minimally and found the resulting particle propagation to be causal, thus avoiding the classical Velo-Zwanziger problem. Moreover, for the single spin valued $(s,0) \oplus (0,s)$ representations, our master equation (26) has the great advantage to predict the correct value for the gyromagnetic ratio, $g_s = \frac{1}{s}$, thus proving Belinfante’s conjecture [13] from 1953.

The development of a calculation scheme for interacting particles of higher spins from the perspective of the present work is underway.

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[16] We shall design argumentation very general and in reference to any one Lorentz representation with more but one $W^2$ invariant subspace. The particular case of $\psi_{\mu_1,\ldots,\mu_K}$ is then automatically accounted for.