Sobolev functions on closed subsets of the real line

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Abstract

For each \( p > 1 \) and each positive integer \( m \) we use divided differences to give intrinsic characterizations of the restriction of the homogeneous Sobolev space \( L^m_p(\mathbb{R}) \) to an arbitrary closed subset of the real line.

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1. Introduction.

In this paper we characterize the restrictions of Sobolev functions of one variable to an arbitrary closed subset of the real line. Given $m \in \mathbb{N}$ and $p \in [1, \infty]$, we let $L^m_p(\mathbb{R})$ denote the homogeneous Sobolev space of all real valued functions $f$ on $\mathbb{R}$ such that $f^{(m-1)}$ is absolutely continuous on $\mathbb{R}$ and $f^{(m)} \in L^p(\mathbb{R})$. $L^m_p(\mathbb{R})$ is seminormed by

$$
\|f\|_{L^m_p(\mathbb{R})} = \|f^{(m)}\|_{L^p(\mathbb{R})}.
$$

In this paper we study the following

**Problem 1.1** Let $p \in (1, \infty)$, $m \in \mathbb{N}$, and let $E$ be a closed subset of $\mathbb{R}$. Let $f : E \to \mathbb{R}$ be a function on $E$. We ask two questions:

1. How can we decide whether there exists a function $F \in L^m_p(\mathbb{R})$ such that the restriction $F|_E$ of $F$ to $E$ coincides with $f$?

2. Consider the $L^m_p(\mathbb{R})$-norms of all functions $F \in L^m_p(\mathbb{R})^n$ such that $F|_E = f$. How small can these norms be?

We denote the infimum of all these norms by $\|f\|_{L^m_p(\mathbb{R}) |_E}$; thus

$$
\|f\|_{L^m_p(\mathbb{R}) |_E} = \inf\{\|F\|_{L^m_p(\mathbb{R})} : F \in L^m_p(\mathbb{R}), F|_E = f\}. \tag{1.1}
$$

We refer to $\|f\|_{L^m_p(\mathbb{R}) |_E}$ as the trace norm of the function $f$ in $L^m_p(\mathbb{R})$. This quantity provides the standard quotient space seminorm in the trace space $L^m_p(\mathbb{R}) |_E$ of all restrictions of $L^m_p(\mathbb{R})$-functions to $E$, i.e., in the space

$$
L^m_p(\mathbb{R}) |_E = \{f : E \to \mathbb{R} : \text{there exists } F \in L^m_p(\mathbb{R}) \text{ such that } F|_E = f\}.
$$

Whitney [57] completely solved an analog of the part 1 of Problem [1.1] for the space $C^m(\mathbb{R})$. Whitney’s extension construction [57] provides a certain extension operator

$$
F^{(W)}_{m,E} : C^m(\mathbb{R})|_E \to C^m(\mathbb{R}) \tag{1.2}
$$

which linearly and continuously maps the trace space $C^m(\mathbb{R})|_E$ into $C^m(\mathbb{R})$. An important ingredient of this construction is the classical Whitney’s extension method for $C^m$-jets [56]. (See also Merrien [39].)

The extension method developed by Whitney in [57] also provides a complete solution to Problem [1.1] for the space $L^m_{\infty}(\mathbb{R})$. (Note that $L^m_{\infty}(\mathbb{R})$ can be identified with the space $C^{m-1,1}(\mathbb{R})$ of all $C^{m-1}$-functions on $\mathbb{R}$ whose derivatives of order $m-1$ satisfy the Lipschitz condition.) In particular, the method of proof and technique developed in [57] and [39] lead us to the following well known description of the trace space $L^m_{\infty}(\mathbb{R})|_E$: A function $f \in L^m_{\infty}(\mathbb{R})|_E$ if and only if the following quantity

$$
\mathcal{W}_{m,\infty}(f : E) = \sup_{S \subset E, \#S = m+1} |A^m f[S]| \tag{1.3}
$$

is finite. Furthermore,

$$
\|f\|_{L^m_{\infty}(\mathbb{R})|_E} \sim \mathcal{W}_{m,\infty}(f : E) \tag{1.4}
$$

with the constants of equivalence depending only on $m$. 

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Here given \((m+1)\)-point set \(S = \{x_0, x_1, ..., x_m\} \subset \mathbb{R}\), we let \(\Delta^m f[S]\) denote the divided difference of \(f\) on \(S\):
\[
\Delta^m f[S] = \Delta^m f[x_0, x_1, ..., x_m] = \sum_{i=0}^{m} \frac{f(x_i)}{\prod_{0 \leq j \leq m, j \neq i} (x_i - x_j)}.
\]

We refer the reader to [32] for further results in this direction.

A special case of Problem [1.1] for a strictly increasing sequence \(E = \{x_i\}_{i=k}^{n}, \ k \leq i \leq n\),
\[
E = \{x_i\}_{i=k}^{n}, \ k \leq i \leq n, \quad (1.5)
\]
of points in \(\mathbb{R}\) (finite or infinite) has been intensively studied (mainly in 60th-70th) in papers of Favard [17], Chui, Smith, Ward [6–8, 54, 55], de Boor [9–12], Fisher and Jerome [26, 27], Golomb [30], Jakimovski and Russell [32], Kunkle [36, 37], Pinkus [40, 41] and Schoenberg [43–45].

In particular, Favard [17] developed an extension method for the space \(L_\infty^m(\mathbb{R})\) distinct from the Whitney’s method. Favard’s approach relies on a certain delicate duality argument which leads us to the following refinement of equivalence (1.4):
\[
\|f\|_{L_\infty^m(\mathbb{R})|E} \sim \sup_{i = k, ..., n-m} |\Delta^m f[x_i, ..., x_{i+m}]| .
\]

Basing on the Favard’s extension method, de Boor [10] characterized the trace space \(L_\infty^p(\mathbb{R})|E\) for sets \(E\) determined by (1.5).

**Theorem 1.2 (10)** Let \(m \in \mathbb{N}, \ p \in (1, \infty), \) and let \(k, n \in \mathbb{Z} \cup \{\pm \infty\}, \ k + m \leq n. \) Let \(E = \{x_i\}_{i=k}^{n} \subset \mathbb{R}\) be a strictly increasing sequence in \(\mathbb{R}\). A function \(f \in L_\infty^m(\mathbb{R})|E\) if an only if the following quantity
\[
L_{m,p}(f : E) = \left( \sum_{i=k}^{n-m} (x_{i+m} - x_i) |\Delta^m f[x_i, ..., x_{i+m}]|^p \right)^{\frac{1}{p}}
\]
is finite. Furthermore, \(\|f\|_{L_\infty^m(\mathbb{R})|E} \sim L_{m,p}(f : E)\) with constants depending only on \(m\) and \(p\).

For certain classes of sequences \(E\) satisfying some global mesh ratio restrictions, the criterion (1.6) was proved by Golomb [30]. See also Estévez [16].

Using a certain limiting argument, Golomb [30] showed that the general Problem [1.1] for an arbitrary set \(E \subset \mathbb{R}\) can be reduced to the same problem, but for arbitrary finite sets \(E\). More specifically, it is proved in [30] that a function \(f \in L_\infty^m(\mathbb{R})|E\) if and only if for every finite subset \(E' \subset E\) its restriction \(f|_{E'}\) belongs to \(L_\infty^p(\mathbb{R})|E'\). Furthermore, the method of proof of this statement given in [30] enables us to show that
\[
\|f\|_{L_\infty^m(\mathbb{R})|E} = \sup \{ \|f\|_{L_\infty^m(\mathbb{R})|E'} : E \subset E, \#E' < \infty \}.
\]

Combining this result with Theorem 1.2 we obtain a solution to Problem [1.1] for every \(p \in (1, \infty)\) and every closed set \(E \subset \mathbb{R}\).

**Theorem 1.3** Let \(p \in (1, \infty)\) and let \(m\) be a positive integer. Let \(E \subset \mathbb{R}\) be a closed set containing at least \(m + 1\) points. A function \(f : E \rightarrow \mathbb{R}\) can be extended to a function \(F \in L_\infty^m(\mathbb{R})\) if and only if the following quantity
\[
W_{m,p}(f : E) = \sup_{i=0}^{n-m} \left( \sum_{i=0}^{n-m} (x_{i+m} - x_i) |\Delta^m f[x_i, ..., x_{i+m}]|^p \right)^{\frac{1}{p}}
\]

\[\boxend\]
is finite. Here the supremum is taken over all finite strictly increasing sequences \( \{x_0, \ldots, x_n\} \subset E \) with \( n \geq m \). Furthermore,

\[
\|f\|_{L^p_m(\mathbb{R})_E} \sim W_{m,p}(f : E). \tag{1.8}
\]

The constants in equivalence (1.8) depend only on \( m \) and \( p \).

We refer to this result as a variational criterion for the traces of \( L^p_m(\mathbb{R}) \)-functions.

In the present paper we give a direct and explicit proof of Theorem 1.3 which does not use any limiting argument. Actually we show, perhaps surprisingly, that the very same Whitney extension operator (1.2) \( F^{(W)}_{m,E} \) which was introduced in [57] for characterization of the trace space \( C^m(\mathbb{R})_E \), provides an almost optimal extension of functions belonging to \( L^p_m(\mathbb{R})_E \) for every \( p \in (1, \infty) \).

Applying (1.8) to \( E = \mathbb{R} \), we obtain that for every \( f \in L^p_m(\mathbb{R}) \) and every \( p \in (1, \infty) \) the following equivalence

\[
\|f\|_{L^p_m(\mathbb{R})} \sim W_{m,p}(f : \mathbb{R}) \tag{1.9}
\]

holds (with constants depending only on \( m \) and \( p \)). In other words, the quantity \( W_{m,p}(\cdot : \mathbb{R}) \) provides an equivalent seminorm on \( L^p_m(\mathbb{R}) \). This characterization of the space \( L^p_m(\mathbb{R}) \) was obtained by Riesz [42] for \( m = 1, 1 < p < \infty \), by Schoenberg [44] for \( p = 2, m \in \mathbb{N} \), and by Jerome and Schumaker [33] for all \( m \in \mathbb{N} \) and \( p \in (1, \infty) \). Of course, equivalence (1.9) implies the necessity part of Theorem 1.3.

Our proof of the sufficiency part of Theorem 1.3 relies on a combination of a certain modification of the extension construction given in [57], and an extension theorem for \( m \)-jets generated by Sobolev functions which we prove in Section 4 (see Theorem 4.1).

In Section 3 we give another characterization of the trace space \( L^p_m(\mathbb{R})_E \) expressed in terms of \( L^p \)-norms of certain maximal functions. For each \( m \in \mathbb{N} \), each closed set \( E \subset \mathbb{R} \) with \( \#E > m \), and each function \( f : E \to \mathbb{R} \) we let \( (\Delta^m f)_E^\# \) denote a certain kind of “sharp maximal function” associated with \( f \) which is defined by

\[
(\Delta^m f)_E^\#(x) = \sup_{\{x_0, \ldots, x_m\} \subset E} \frac{|\Delta^{m-1} f[x_0, \ldots, x_{m-1}] - \Delta^{m-1} f[x_1, \ldots, x_m]|}{|x - x_0| + |x - x_m|}, \quad x \in \mathbb{R}. \tag{1.10}
\]

Note that

\[
(\Delta^m f)_E^\#(x) \leq \sup_{S \subset E, \#S = m+1} \frac{|\Delta^m f[S]| \operatorname{diam} S}{\operatorname{diam}((x) \cup S)} \leq 2 (\Delta^m f)_E^\#(x), \quad x \in \mathbb{R}. \tag{1.11}
\]

(See below property (2.2) of the divided differences.)

**Theorem 1.4** Let \( p > 1 \), \( m \in \mathbb{N} \), and let \( f \) be a function defined on a closed set \( E \subset \mathbb{R} \). Then \( f \in L^p_m(\mathbb{R})_E \) if and only if \( (\Delta^m f)_E^\# \in L^p(\mathbb{R}) \). Furthermore,

\[
\|f\|_{L^p_m(\mathbb{R})_E} \sim \| (\Delta^m f)_E^\# \|_{L^p(\mathbb{R})}
\]

with the constants in this equivalence depending only on \( m \) and \( p \).
Remark 1.5 Thus Theorem 1.6, (1.10) and (1.11) provide two explicit formulae for the trace norm of a function $f$ defined on $E$:

$$
\|f\|_{L^p(E)} \sim \left\{ \int_{\mathbb{R}} \sup_{x_0 < x_1 < \ldots < x_m} \frac{\left| \Delta^{m-1} f[x_0, \ldots, x_{m-1}] - \Delta^{m-1} f[x_1, \ldots, x_m] \right|^p}{|x - x_0|^p + |x - x_m|^p} \, dx \right\}^{\frac{1}{p}}.
$$

The next theorem states that there is a solution to Problem 1.1 which depends linearly on the initial data, i.e., the functions defined on $E$.

Theorem 1.6 For every closed subset $E \subset \mathbb{R}$, every $p > 1$ and every $m \in \mathbb{N}$ there exists a linear continuous extension operator which maps the trace space $L^m_p(E)$ into $L^m_p(\mathbb{R})$. Its operator norm is bounded by a constant depending only on $m$ and $p$.

Remark 1.7 As we have noted above, for every $p \in (1, \infty)$ the Whitney extension operator $\mathcal{F}^{(W)}_{m,E}$ (see (1.2)) provides an almost optimal extension of functions from $L^m_p(E)$ to functions from $L^m_p(\mathbb{R})$. Since $\mathcal{F}^{(W)}_{m,E}$ is linear, it has the properties described in Theorem 1.6.

Let us recall something of the history of Theorem 1.6. This theorem and Theorem 1.4 have been announced in [31]. If $E = \{x_i\}_{i=1}^n$ is a sequence of points in $\mathbb{R}$ (finite or infinite), the statement of this theorem is immediate from the approach suggested in [12]. Indeed, it is shown in [12] that, similar to the Whitney extension operator, the Favard’s extension method [17] provides an almost optimal extension operator for the space $L^m_p(\mathbb{R})$. Since the Favard’s method is linear, it produces a continuous linear extension operator from $L^m_p(E)$ to $L^m_p(\mathbb{R})$ for any sequence $E$.

Luli [38] gave an alternative proof of Theorem 1.6 for a finite set $E$. In the multidimensional case the existence of corresponding linear continuous extension operators for the Sobolev spaces $L^m_p(\mathbb{R}^n), n < p < \infty$, was proven in [50] ($m = 1, n \in \mathbb{N}, E \subset \mathbb{R}^n$ is arbitrary), [31] and [52] ($m = 2, n = 2, E \subset \mathbb{R}^2$ is finite), and [24] (arbitrary $m, n \in \mathbb{N}$ and an arbitrary $E \subset \mathbb{R}^n$). For the case $p = \infty$ see [4] ($m = 2$) and [20, 21] ($m \in \mathbb{N}$).

In Section 5 we discuss the constants of the equivalence (1.4). We interpret this equivalence as a particular case of the Finiteness Principle for traces of smooth functions. (See Theorem 5.1 and Theorem 5.10 below). We refer the reader to [8, 15, 18, 19, 22, 49] and references therein for numerous results related to the Finiteness Principle.

For the space $L^m_\infty(\mathbb{R})$, the Finiteness Principle implies the following statement: there exists a constant $\gamma = \gamma(m)$ such that for every closed set $E \subset \mathbb{R}$ and every $f \in L^m_\infty(\mathbb{R})$ the following inequality

$$
\|f\|_{L^m_\infty(E)} \leq \gamma \sup_{S \subset E, \#S = m+1} \|\mathcal{F}[S]\|_{L^m_\infty(\mathbb{R})}.
$$

(1.12)

holds.

We refer to the constant $N = m + 1$ appearing in this inequality as the finiteness constant for the space $L^m_\infty(\mathbb{R})$. We refer to any constant $\gamma$ from (1.12) as a multiplicative finiteness constant for the space $L^m_\infty(\mathbb{R})$. By $\gamma^\sharp(L^m_\infty(\mathbb{R}))$ we denote the sharp value of the multiplicative finiteness constant for $L^m_\infty(\mathbb{R})$. 

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One can easily see that $\gamma^d(L^1_\infty(\mathbb{R})) = 1$. We prove that

$$\gamma^d(L^2_\infty(\mathbb{R})) = 2 \quad \text{and} \quad (\pi/2)^m < \gamma^d(L^m_\infty(\mathbb{R})) < (m - 1)9^m \quad \text{for every} m > 2. \quad (1.13)$$

See Theorem 5.3 below. The proof of (1.13) relies on results of Favard [17] and de Boor [12,13] devoted to calculation of certain extension constants for the space $L^m_\infty(\mathbb{R})$.

In particular, (1.13) implies the following version of the Finiteness Principle for $L^m_\infty(\mathbb{R})$, $m > 2$:

Let $f$ be a function defined on a closed set $E \subset \mathbb{R}$. Suppose that for every $(m+1)$-point subset $E' \subset E$ there exists a function $F_{E'} \in L^m_\infty(\mathbb{R})$ with $\|F_{E'}\|_{L^m_\infty(\mathbb{R})} \leq 1$, such that $F_{E'} = f$ on $E'$. Then there exists a function $F \in L^m_\infty(\mathbb{R})$ with $\|F\|_{L^m_\infty(\mathbb{R})} \leq (m - 1)9^m$ such that $F = f$ on $E$.

Furthermore, there exists a closed set $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$ such that for every $(m+1)$-point subset $E' \subset E$ there exists a function $F_{E'} \in L^m_\infty(\mathbb{R})$ with $\|F_{E'}\|_{L^m_\infty(\mathbb{R})} \leq 1$, such that $F_{E'} = f$ on $E'$, but nevertheless, $\|F\|_{L^m_\infty(\mathbb{R})} \geq (\pi/2)^m$ for every $F \in L^m_\infty(\mathbb{R})$ such that $F = f$ on $E$.

See Section 5 for more details.

2. Main Theorems: necessity.

Let us fix some notation. Throughout the paper $C, C_1, C_2, \ldots$ will be generic positive constants which depend only on parameters determining function spaces $(m, p, \text{etc})$. These symbols may denote different constants in different occurrences. The dependence of a constant on certain parameters is expressed, for example, by the notation $C = C(m, p)$. We write $A \sim B$ if there is a constant $C \geq 1$ such that $A/C \leq B \leq CA$.

Given a measurable set $A \subset \mathbb{R}$, we let $|A|$ denote the Lebesgue measure of $A$. If $A \subset \mathbb{R}$ is finite, by $\#A$ we denote the number of elements of a $A$.

Let $A, B \subset \mathbb{R}$. We put $\text{diam } A = \sup \{|a - a' | : a, a' \in A \}$ and

$$\text{dist}(A, B) = \inf \{|a - b | : a \in A, b \in B \}.$$ 

For $x \in \mathbb{R}$ we also set $\text{dist}(x, A) = \text{dist}([x], A)$.

Given a function $g \in L^1_{1,\infty}(\mathbb{R})$ we let $M[g]$ denote the Hardy-Littlewood maximal function of $g$:

$$M[g](x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |g(y)|dy, \quad x \in \mathbb{R}. \quad (2.1)$$

Here the supremum is taken over all closed intervals $I$ in $\mathbb{R}$ containing $x$.

By $\mathcal{P}_m$ we denote the space of all polynomials of degree at most $m$ defined on $\mathbb{R}$. Finally, given a nonnegative integer $k$, a $(k+1)$-point set $S \subset \mathbb{R}$ and a function $f$ on $S$, we let $L_S[f]$ denote the Lagrange polynomial of degree at most $k$ interpolating $f$ on $S$; thus $L_S[f] \in \mathcal{P}_k$ and $L_S[f](x) = f(x)$ for every $x \in S$.

2.1. Divided differences: main properties.

Let us recall several useful properties of the divided differences of functions which we use in the proofs of the main theorems. We refer the reader to monographs [15], Ch. 4, §7, and [28] for the proofs of these properties.

Let $k \in \mathbb{N}$ and let $S = \{x_0, \ldots, x_k\}, x_0 < x_1 < \ldots < x_k$, be a subset of $\mathbb{R}$ consisting of $k+1$ distinct points:
(1) Let \( f \) be a function on \( S \). Then
\[
\Delta^k f[S] = \Delta^k f[x_0, x_1, \ldots, x_k] = \left( \Delta^{k-1} f[x_1, \ldots, x_k] - \Delta^{k-1} f[x_0, \ldots, x_{k-1}] \right) / (x_k - x_0). \tag{2.2}
\]

(2) Let \( f \in C^k[x_0, x_1] \). Then there exists \( \xi \in [x_0, x_1] \) such that
\[
\Delta^k f[x_0, x_1, \ldots, x_k] = \frac{1}{k!} f^{(k)}(\xi). \tag{2.3}
\]

(3) Let \( f : S \to \mathbb{R} \). Then
\[
\Delta^k f[S] = \frac{1}{k!} L^k_S[f]. \tag{2.4}
\]

Recall that \( L_S[f] \) is the Lagrange polynomial of degree \( \leq k \) interpolating \( f \) on \( S \). In other words, \( \Delta^k f[S] = A_k \) where \( A_k \) is the coefficient of \( x^k \) of the polynomial \( L_S[f] \).

(4) For each function \( f : [x_0, x_k] \to \mathbb{R} \)
\[
f(x) = L_S[f](x) + (x - x_0) \cdots (x - x_k) \Delta^{k+1} f[x_0, \ldots, x_k, x] \quad \text{for every} \; x \in [x_0, x_k]. \tag{2.5}
\]

(5) Let \( M_k = M_k[S](t), t \in \mathbb{R} \), be the B-spline (basis spline) associated with the set \( S = \{x_0, \ldots, x_k\} \). See, e.g., \([14]\). Recall that \( M_k = M_k[S](t) \) is the divided difference of the function \( g(u) = k \cdot (u - t)^{k-1}_+ \) over the set \( S \); more precisely,
\[
M_k = M_k[S](t) = \Delta^k g[S] = k \sum_{i=0}^{k} \frac{(x_i - t)^{k-1}_+}{\omega'(x_i)} \tag{2.6}
\]
where \( \omega(x) = (x - x_0) \cdots (x - x_k) \).

Note that \( M_k[S](t) \geq 0 \) and \( \text{supp} \, M_k[S] \subset [x_0, x_k] \). Furthermore,
\[
M_k[S](t) \leq k / (x_k - x_0) \quad \text{for every} \; t \in \mathbb{R}. \tag{2.7}
\]

Let \( F : [x_0, x_k] \to \mathbb{R} \) be a function such that \( F^{(k-1)} \) is absolutely continuous on \([x_0, x_k] \). Then
\[
\Delta^k F[S] = \frac{1}{k!} \int_{x_0}^{x_k} M_k[S](t) F^{(k)}(t) \, dt. \tag{2.8}
\]

See \([14]\) or \([15]\), p. 137. Putting here \( F(t) = t^k \), we obtain
\[
\int_{x_0}^{x_k} M_k[S](t) \, dt = 1. \tag{2.9}
\]

Furthermore, by \((2.7)\),
\[
|\Delta^k F[S]| \leq \frac{1}{(k-1)!} \cdot \frac{1}{x_k - x_0} \int_{x_0}^{x_k} |F^{(k)}(t)| \, dt. \tag{2.10}
\]
Hence, for every \( p \in [1, \infty) \),

\[
|\Delta^k F[S]| \leq \frac{1}{(k-1)!} \left( \frac{1}{x_k - x_0} \int_{x_0}^{x_k} |F^{(k)}(t)|^p \, dt \right)^{\frac{1}{p}}.
\]  

(2.11)

Combining this inequality with (2.5), we obtain the following important statement:

Let \( k \) be a non-negative integer, \( p \in (1, \infty) \), and let \( S = \{x_0, \ldots, x_k\} \), \( x_0 < \ldots < x_k \). Let \( F \in L_p^{k+1}(\mathbb{R}) \). Then for every \( x \in [x_0, x_k] \) the following inequality holds.

\[
|F(x) - L_S[F](x)| \leq \frac{1}{k!} |x - x_0| \cdots |x - x_k| \left( \frac{1}{x_k - x_0} \int_{x_0}^{x_k} |F^{(k+1)}(t)|^p \, dt \right)^{\frac{1}{p}}.
\]

2.2. Proofs of the necessity part of Theorem 1.3 and Theorem 1.4

(Theorem 1.3: Necessity) As we have noted in Introduction, the necessity part of Theorem 1.3 directly follows from the description of \( L_p^n \)-spaces given in [33]. See (1.9). For the reader’s convenience, in this section we present a proof of this statement.

Let \( 1 < p < \infty \) and let \( f \in L_p^n(\mathbb{R})|_E \). Let \( F \in L_p^n(\mathbb{R}) \) be an arbitrary function such that \( F|_E = f \). Let \( n \geq m \) and let \( \{x_0, \ldots, x_n\} \subset \mathbb{E} \), \( x_0 < \ldots < x_n \). Then, by (2.11), for every \( i, 0 \leq i \leq n - m \), we have

\[
(x_{i+m} - x_i) |\Delta^m f[x_i, \ldots, x_{i+m}]|^p = (x_{i+m} - x_i) |\Delta^m F[x_i, \ldots, x_{i+m}]|^p \\
\leq (x_{i+m} - x_i) \frac{1}{((m-1)!)^p} \frac{1}{x_{i+m} - x_i} \int_{x_i}^{x_{i+m}} |F^{(m)}(t)|^p \, dt \\
= \frac{1}{((m-1)!)^p} \int_{x_i}^{x_{i+m}} |F^{(m)}(t)|^p \, dt.
\]

Hence,

\[
A = \sum_{i=0}^{n-m} (x_{i+m} - x_i) |\Delta^m f[x_i, \ldots, x_{i+m}]|^p \leq \sum_{i=0}^{n-m} \frac{1}{((m-1)!)^p} \int_{x_i}^{x_{i+m}} |F^{(m)}(t)|^p \, dt.
\]

Clearly, the covering multiplicity of the family \( \{(x_i, x_{i+m}) : i = 0, \ldots, n-m\} \) of sets is bounded by \( m \), so that

\[
A \leq \frac{m}{((m-1)!)^p} \int_{x_0}^{x_n} |F^{(m)}(t)|^p \, dt \leq \frac{m}{((m-1)!)^p} \|F\|_{L_p^n(\mathbb{R})}^p.
\]

Taking the supremum over all \((n + 1)\)-point subsets \( \{x_0, \ldots, x_n\} \subset \mathbb{E} \) with \( n \geq m \), we obtain:

\[
W_{m,p}(f : E) \leq \frac{m^\frac{1}{p}}{(m-1)!} \|F\|_{L_p^n(\mathbb{R})}.
\]
See (1.7). Finally, taking in this inequality the infimum over all functions \( F \in L^m_p(\mathbb{R}) \), \( F|_E = f \), we get

\[
\mathcal{W}_{m,p}(f : E) \leq \frac{m^\frac{1}{p}}{(m - 1)!} \|f\|_{L^m_p(\mathbb{R})}\|E\|
\]

proving the necessity part of Theorem 1.3. \( \square \)

(Theorem 1.4: Necessity) Let \( f \in L^m_p(\mathbb{R})|_E \) and let \( F \in L^m_p(\mathbb{R}) \), \( F|_E = f \). Let \( S = \{x_0, \ldots, x_m\}, \) \( x_0 < \ldots < x_m \), be a subset of \( E \) and let \( x \in \mathbb{R} \). Then, by (2.10),

\[
B = \frac{|\Delta^{m-1}f[x_0, \ldots, x_{m-1}] - \Delta^{m-1}f[x_1, \ldots, x_m]|}{|x - x_0| + |x - x_m|} = \frac{|\Delta^{m}F[x_0, \ldots, x_{m-1}] - \Delta^{m}F[x_1, \ldots, x_m]|}{|x - x_0| + |x - x_m|} \\
\leq \frac{1}{(m - 1)! (|x - x_0| + |x - x_m|)} \int_{x_0}^{x_m} |\Delta F^{(m)}(t)| \, dt.
\]

Let \( I \) be the smallest closed interval containing \( S \) and \( x \). Clearly,

\[ |x - x_0| + |x - x_m| \geq |I|. \]

Hence,

\[
B \leq \frac{1}{(m - 1)!} \frac{1}{|I|} \int_I |\Delta F^{(m)}(t)| \, dt \leq \frac{1}{(m - 1)!} \mathcal{M}[\Delta^{(m)}F](x).
\]

(Recall that \( \mathcal{M} \) denotes the Hardy-Littlewood maximal function, see (2.1).) Taking in this inequality the supremum over all \((m + 1)\)-point subsets \( S \subset E \), we obtain

\[
(\Delta^{m}f)^\#_E(x) \leq \frac{1}{(m - 1)!} \mathcal{M}[\Delta^{(m)}F](x), \quad x \in \mathbb{R}.
\]

See (1.10). Hence,

\[
\| (\Delta^{m}f)^\#_E \|_{L^p(\mathbb{R})} \leq \frac{1}{(m - 1)!} \|\mathcal{M}[\Delta^{(m)}F]\|_{L^p(\mathbb{R})}
\]

so that, by the Hardy-Littlewood maximal theorem,

\[
\| (\Delta^{m}f)^\#_E \|_{L^p(\mathbb{R})} \leq \frac{C(p)}{(m - 1)!} \|F^{(m)}\|_{L^p(\mathbb{R})} = \frac{C(p)}{(m - 1)!} \|f\|_{L^m_p(\mathbb{R})}\|E\|.
\]

Taking the infimum in this inequality over all functions \( F \in L^m_p(\mathbb{R}) \) such that \( F|_E = f \), we finally obtain the required inequality

\[
\| (\Delta^{m}f)^\#_E \|_{L^p(\mathbb{R})} \leq \frac{C(p)}{(m - 1)!} \|f\|_{L^m_p(\mathbb{R})}|_E.
\]

The proof of the necessity part of Theorem 1.4 is complete. \( \square \)
3. The Whitney extension method in $\mathbb{R}$ and traces of Sobolev functions.

In this section we prove the sufficiency part of Theorem 1.4.

Let us introduce several notions related to polynomial jets generated by Sobolev functions. Given a function $F \in C^m(\mathbb{R})$ and $x \in \mathbb{R}$, we let

$$T^m_x[F](y) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha F(x)(y-x)^\alpha, \quad y \in \mathbb{R},$$

denote the Taylor polynomial of $F$ of degree $m$ at $x$.

Let $E$ be a closed subset of $\mathbb{R}$ and let $P = \{P_x : x \in E\}$ be a family of polynomials of degree at most $m$ indexed by points of $E$. (Thus $P_x \in P_m$ for every $x \in E$.) Following [23] we refer to $P$ as a Whitney $m$-field defined on $E$.

We say that a function $F \in C^m(\mathbb{R})$ agrees with the Whitney $m$-field $P = \{P_x : x \in E\}$ on $E$, if $T^m_x[F] = P_x$ for each $x \in E$. In that case we also refer to $P$ as the Whitney $m$-field on $E$ generated by $F$ or as the $m$-jet generated by $F$. We define the $L^p_m$-"norm" of the $m$-jet $P = \{P_x : x \in E\}$ by

$$||P||_{m,p,E} = \inf \{||F||_{L^p_m(\mathbb{R})} : F \in L^p_m(\mathbb{R}), T^{m-1}_x[F] = P_x \text{ for every } x \in E\}. \quad (3.1)$$

We prove the sufficiency part of Theorem 1.4 in two steps. At the first step, given $m \in \mathbb{N}$ we construct a linear operator which assigns to every function $f : E \to \mathbb{R}$ a certain Whitney $(m-1)$-field $P^{m,E}(f) = \{P_x \in P_{m-1} : x \in E\}$ such that $P_x(x) = f(x)$ for all $x \in E$. We obtain $P^{m,E}(f)$ using a slight modification of an extension construction suggested by Whitney in [57]. See also [39], [28], [29], [35] where similar constructions have been used for characterization of traces of $L^m(\mathbb{R})$-functions.

At the second step of the proof we show that for every $p \in (1, \infty)$ and every function $f : E \to \mathbb{R}$ such that $(\Delta^m f)_E^\sharp \in L_p(\mathbb{R})$ (see (1.10)) the following inequality

$$||P^{m,E}(f)||_{m,p,E} \leq C(m, p)|| (\Delta^m f)_E^\sharp ||_{L_p(\mathbb{R})} \quad (3.2)$$

holds. One of the main ingredients of the proof of (3.2) is a trace criterion for jets generated by Sobolev functions proven in [53].

3.1. Interpolating knots and their properties.

Let $E \subset \mathbb{R}$ be a closed subset and let $k$ be a non-negative integer such that $k < \#E$. Given a point $x \in E$ we construct a certain finite set $S_{k,E}(x) \subset E$. This object is an important ingredient of the extension algorithm: it provides interpolating knots for Lagrange (Hermite) polynomials which we use in our modification of the Whitney extension method.

**Definition 3.1** Let $A$ be a finite subset of $E$.

(i) Suppose that $A \neq E$ and each point of $A$ is an isolated point of $E$. By $N(A)$ we denote the family of all nearest points to $A$ on the set $E \setminus A$.

Thus, dist$(y, A) = \text{dist}(A, E \setminus A)$ for every $y \in N(A)$. (Clearly, $N(A)$ is a finite subset of $E$.)

We let $\text{RNP}(A) = \max\{a : a \in N(A)\}$ denote the right nearest point to $A$ on $E \setminus A$.

(ii) Suppose that $A$ contains a limit point of $E$. In this case, we let $\text{RNP}(A)$ denote the maximal limit point of $E$ belonging to $A$. 

Let us construct a family of points \( \{y_0(x), y_1(x), \ldots, y_{n_k(x)} \} \) in \( E \), \( 0 \leq n_k(x) \leq k \), using the following inductive procedure.

First we put \( y_0(x) = x \) and \( Y_0(x) = \{y_0(x)\} \). If \( k = 0 \), we put \( n_k(x) = 0 \), and stop.

Suppose that \( k > 0 \). If \( y_0(x) = x \) is a limit point of \( E \), we again put \( n_k(x) = 0 \), and stop. If \( y_0(x) \) is an isolated point of \( E \), we continue the procedure.

We define a point \( y_1(x) \in E \) by \( y_1(x) = \text{RNP}(Y_0(x)) \), and set \( Y_1(x) = \{y_0(x), y_1(x)\} \). (Thus \( y_1(x) \) is the right nearest point to \( y_0(x) \) on \( E \setminus \{y_0(x)\} \).)

Let \( k > 1 \) and \( y_1(x) \) is an isolated point of \( E \). In this case we put \( y_2(x) = \text{RNP}(Y_1(x)) \) and \( Y_2(x) = \{y_0(x), y_1(x), y_2(x)\} \). If \( k = 2 \), we set \( n_k(x) = 2 \), and stop. But if \( k > 2 \), we continue the procedure, defining \( y_3 \) or stopping, etc.

At the \( j \)-th step of this algorithm we obtain a \( j + 1 \)-point set \( Y_j(x) = \{y_0(x), \ldots, y_j(x)\} \). If \( j = k \) or \( y_j(x) \) is a limit point of \( E \) we put \( n_k(x) = j \) and stop. But if \( j < k \) and \( y_j(x) \) is an isolated point of \( E \), we define a point \( y_{j+1}(x) \) and a set \( Y_{j+1}(x) \) by the formulae

\[
y_{j+1}(x) = \text{RNP}(Y_j(x)) \quad \text{and} \quad Y_{j+1}(x) = \{y_0(x), \ldots, y_j(x), y_{j+1}(x)\}.
\]

Clearly, for a certain

\[
n = n_k(x), \quad 0 \leq n \leq k, \quad \text{the procedure stops}.
\]

This means that either \( n = k \) or, whenever \( n < k \), the points \( y_0(x), \ldots, y_{n-1}(x) \) are isolated points of \( E \), but

\[
y_n(x) \quad \text{is a limit point of} \quad E.
\]

We put

\[
y_j(x) = y_{n_k(x)}(x) \quad \text{and} \quad Y_j(x) = Y_{n_k(x)}(x) \quad \text{provided} \quad n_k(x) \leq j \leq k.
\]

Note that, given \( x \in E \) the definitions of points \( y_j(x) \) and the sets \( Y_j(x) \) does not depend on \( k \), i.e., \( y_j(x) \) is the same point and is the same set \( Y_j(x) \) and set for every \( k \geq j \).

In the next lemma we present several important properties of the points \( y_j(x) \) and the sets \( Y_j(x) \).

**Lemma 3.2** Given \( x \in E \) the sets \( y_j(x), 0 \leq j \leq k \), have the following properties:

(a) \( y_0(x) = x \) and \( y_j(x) = \text{RNP}(Y_{j-1}(x)) \) for every \( j, 1 \leq j \leq k \);

(b) Let \( n = n_k(x) \geq 1 \). Then \( y_0(x), \ldots, y_{n-1}(x) \) are isolated points of \( E \);

(c) For every non-negative integers \( i, j, 0 \leq j < i \), the following inequality

\[
| x - y_j(x) | \leq | x - y_i(x) |
\]

holds. In particular, for every \( j \geq 0 \),

\[
| x - y | \leq | x - y_j(x) | \quad \text{for all} \quad y \in Y_j(x).
\]

(d) Let \( j \geq 0 \) and let \( y \in E \). If \( | x - y | < | x - y_j(x) | \), then \( y \in Y_j(x) \). In particular,

\[
[ \min Y_j(x), \max Y_j(x) ] \cap E = Y_j(x).
\]

Furthermore, every point \( y \in E \) such that \( | x - y | < | x - y_{n_k(x)}(x) | \) is an isolated point of \( E \);

(e) \( \# Y_j(x) = \min\{j, n_k(x)\} + 1 \) for every \( j, 0 \leq j \leq k \).
Proof. Properties (b)-(e) are immediate from the definitions of the points $y_j(x)$ and the sets $Y_j(x)$ and their constructions. Let us prove (a).

We know that $y_0(x) = x$ and, by (3.3), $y_j(x) = \text{RNP}(Y_{j-1}(x))$ for every $j = 1, \ldots, n_k(x)$.

Let $n_k(x) < j \leq k$. Then, by (3.6), $y_j(x) = Y_{n_k(x)}(x)$ for every $j, n_k(x) \leq j \leq k$. On the other hand, since $n_k(x) < k$, the point $y_{n_k(x)}$ is a unique limit point of $E$. See (3.5). Hence, by Definition 3.1 and (3.6), for every $j, n_k(x) < j \leq k$,

$$\text{RNP}(Y_{j-1}(x)) = \text{RNP}(Y_{n_k(x)}(x)) = y_{n_k(x)} = y_j(x)$$

proving property (a) for the case under consideration. \qed

Let us note the following useful property of the sets $Y_j(x)$, $x \in E$, $0 \leq j \leq k$.

**Lemma 3.3** Let $x_1, x_2 \in E$, $0 \leq j \leq k$. If $x_1 \leq x_2$ then

$$\min Y_j(x_1) \leq \min Y_j(x_2) \quad (3.9)$$

and

$$\max Y_j(x_1) \leq \max Y_j(x_2) \quad (3.10)$$

*Proof.* Since $Y_0(x_1) = \{x_1\}$ and $Y_0(x_2) = \{x_2\}$, inequalities (3.9) and (3.10) hold for $j = 0$.

Suppose that these inequalities are true for some $j, 0 \leq j \leq k - 1$. Let us prove that

$$\min Y_{j+1}(x_1) \leq \min Y_{j+1}(x_2) \quad (3.11)$$

and

$$\max Y_{j+1}(x_1) \leq \max Y_{j+1}(x_2) \quad (3.12)$$

First we prove (3.11). Recall that, by (3.3), for each $\ell = 1, 2$, we have $y_{j+1}(x_{\ell}) = \text{RNP}(Y_j(x_{\ell}))$ and

$$Y_{j+1}(x_{\ell}) = Y_j(x_{\ell}) \cup \{y_{j+1}(x_{\ell})\} \quad (3.13)$$

If $Y_j(x_2)$ contains a limit point of $E$, then, by Definition 3.1, $y_{j+1}(x_2) \in Y_j(x_2)$ so that, by (3.13), $Y_{j+1}(x_2) = Y_j(x_2)$. Hence, by (3.9),

$$\min Y_{j+1}(x_1) \leq \min Y_j(x_1) \leq \min Y_j(x_2) = \min Y_{j+1}(x_2)$$

proving (3.11) in the case under consideration.

Thus, later on we may assume that all points of $Y_j(x_2)$ are isolated points of $E$. In particular, this implies that $0 \leq j \leq n_k(x_2)$, see the part (b) of Lemma 3.2 and definitions (3.5) and (3.6). Hence, by the part (e) of Lemma 3.2, $\#Y_j(x_2) = j + 1$.

Now let us suppose that

$$\min Y_j(x_1) < \min Y_j(x_2) \quad (3.14)$$

Then for each point $a \in \mathbb{R}$ nearest to $Y_j(x_2)$ on the set $E \setminus Y_j(x_2)$ we have $a \geq \min Y_j(x_1)$. Hence, by Definition 3.1,

$$\text{RNP}(Y_j(x_2)) = y_{j+1}(x_2) \geq \min Y_j(x_1).$$
Combining this inequality with \((3.13)\) and \((3.14)\), we obtain the required inequality \((3.11)\).

Consider the remaining case: \(\min Y_j(x_1) = \min Y_j(x_2)\). This inequality and inequality \((3.10)\) imply the following inclusion:

\[ Y_j(x_1) \subset I = [\min Y_j(x_2), \max Y_j(x_2)] . \]

Note that, by inequality \((3.8)\), \(I \cap E = Y_j(x_2)\). Hence, \(Y_j(x_1) \subset Y_j(x_2)\). Recall that in the case under consideration all points of \(Y_j(x_2)\) are isolated points of \(E\). Therefore, all points of \(Y_j(x_1)\) are isolated points of \(E\) as well. Now using the same argument as for the set \(Y_j(x_2)\), we conclude that \(#Y_j(x_1) = j + 1 = #Y_j(x_2)\).

Thus \(Y_j(x_1) \subset Y_j(x_2)\) and \(#Y_j(x_1) = #Y_j(x_2)\) so that \(Y_j(x_1) = Y_j(x_2)\) which implies \((3.11)\). This completes the proof of the inequality \((3.11)\).

In the same fashion we prove inequality \((3.12)\).

The proof of the lemma is complete. \(\Box\)

The next lemma shows that, in a certain sense, each set \(Y_j(x)\) is “well concentrated” around the point \(x\). For various versions of this lemma we refer the reader to \[57\], Sections 8 and 9, \[39\], Section 3.3, and \[35\], Section 22.12.

**Lemma 3.4** Let \(x_1, x_2 \in E\) and let \(j\) be a non-negative integer. Suppose that \(Y_j(x_1) \neq Y_j(x_2)\). Then

\[
\max\{ |x_1 - y_j(x_1)|, |x_2 - y_j(x_2)| \} \leq j |x_1 - x_2| .
\]

\((3.15)\)

**Proof.** Let \(n' = n_k(x_1), n'' = n_k(x_2)\), and let \(n'' \leq n'\). By \((3.6)\), it suffices to prove inequality \((3.15)\) for \(0 \leq j \leq n'\).

We do this by induction on \(j\). Since \(Y_0(x_i) = \{x_i\}, i = 1, 2\), see property (a) of Lemma \[3.2\], inequality \((3.15)\) is trivial for \(j = 0\). Let us assume that given \(j \in \{1, ..., n' - 1\}\) the statement of the lemma is true. Let us prove this statement for \(j + 1\).

Suppose that \(Y_{j+1}(x_1) \neq Y_{j+1}(x_2)\). Then, by \((3.3)\),

\[
Y_j(x_1) \neq Y_j(x_2)
\]

\((3.16)\)

as well so that, by the induction assumption, inequality \((3.15)\) holds.

Let us prove that

\[
\max\{ |x_1 - y_{j+1}(x_1)|, |x_2 - y_{j+1}(x_2)| \} \leq (j + 1) |x_1 - x_2| .
\]

Suppose that this inequality it is not true, i.e.,

\[
\max\{ |x_1 - y_{j+1}(x_1)|, |x_2 - y_{j+1}(x_2)| \} > (j + 1) |x_1 - x_2| ,
\]

\((3.17)\)

and show that this leads us to a contradiction. Consider three cases.

**Case 1:** \(|x_2 - y_{j+1}(x_2)| \leq |x_1 - y_{j+1}(x_1)|\).

Then, by \((3.17)\),

\[
|x_1 - y_{j+1}(x_1)| > (j + 1) |x_1 - x_2| .
\]

\((3.18)\)

We prove that \(Y_j(x_2) \subset Y_j(x_1)\). Indeed, let \(y \in Y_j(x_2)\). Then, by \((3.7)\) and \((3.15)\),

\[
|x_1 - y| \leq |x_1 - x_2| + |x_2 - y| \leq |x_1 - x_2| + |x_2 - y_j(x_2)| \leq |x_1 - x_2| + j |x_1 - x_2| .
\]
Hence, by (3.18),
\[ |x_1 - y| \leq (1 + j)|x_1 - x_2| < (1 + j)(j + 1)^{-1}|x_1 - y_{j+1}(x_1)|. \]
Thus \(|x_1 - y| < |x_1 - y_{j+1}(x_1)|\) so that, by property (d), \(y \in Y_j(x_1)\), proving the required imbedding \(Y_j(x_2) \subset Y_j(x_1)\).

Note that \(j \leq n' - 1\) so that, by property (b), the set \(Y_j(x_1)\) consists of isolated points of \(E\). Furthermore, by property (e), \(\#Y_j(x_1) = j + 1\).

Since \(Y_j(x_2) \subset Y_j(x_1)\), the set \(Y_j(x_2)\) consists of isolated points of \(E\) as well. Hence we conclude that \(j \leq n'' = n_k(x_2)\). Indeed, otherwise \(n'' < j \leq k\) so that the point \(y_{n''}(x_2) \in Y_j(x_2)\), see (3.6). But in this case \(y_{n''}(x_2)\) is a limit point of \(E\), a contradiction.

Thus, \(j \leq n''\) so that, by property (e), \(\#Y_j(x_2) = j + 1\). We have obtained that \(Y_j(x_1)\) is a finite set which contains \(Y_j(x_2)\) and has the same cardinality as \(Y_j(x_2)\). Hence, \(Y_j(x_1) = Y_j(x_2)\) which contradicts (3.16).

Case 2: \(|x_1 - y_{j+1}(x_1)| < |x_2 - y_{j+1}(x_2)| and \(j + 1 \leq n''\).
A similar proof shows that in this case \(Y_j(x_1) = Y_j(x_2)\), and we again have a contradiction.

Case 3: \(|x_1 - y_{j+1}(x_1)| < |x_2 - y_{j+1}(x_2)| and \(j + 1 > n''\).
In this case, by (3.6), \(y_{j+1}(x_2) = y_j(x_2)(= y_{n''}(x_2))\). Now, if (3.17) is true, then
\[ |x_2 - y_{j+1}(x_2)| > (j + 1)|x_1 - x_2|. \]
On the other hand, by (3.15), \(|x_2 - y_j(x_2)| \leq j| x_1 - x_2 |. But \(y_{j+1}(x_2) = y_j(x_2)\) so that \(|x_2 - y_j(x_2)| > |x_2 - y_j(x_2)|\), a contradiction.

The proof of the lemma is complete. \(\square\)

3.2. Lagrange polynomials and divided differences on interpolating knots.

In this subsection we describe main properties of the Lagrange polynomials on finite subsets of the set \(E\).

Lemma 3.5 Let \(P \in \mathcal{P}_k\) where \(k\) is a nonnegative integer. Suppose that \(P\) has \(k\) real roots which lie in a set \(S \subset \mathbb{R}\). Let \(I \subset \mathbb{R}\) be a closed interval.

Then for every \(i, 0 \leq i \leq k\), the following inequality
\[ \max_i |P^{(i)}| \leq (\text{diam}(I \cup S))^{k-i} |P^{(k)}| \]
holds.

Proof. Let \(S = \{x_1, ..., x_k\}\) be the family of the roots of \(P\) so that
\[ P(x) = \frac{P^{(k)}}{k!} \prod_{i=1}^{k} (x - x_i), \quad x \in \mathbb{R}. \]

Hence, for every \(i, 0 \leq i \leq k\),
\[ P^{(i)}(x) = \frac{i!}{k!} \sum_{S' \subset S, \#S' = k-i} \prod_{y \in S'} (x - y), \]
so that for every \(x \in I\)
\[ \max_i |P^{(i)}| \leq \frac{i!}{k!} \frac{k!}{i!(k-i)!} (\text{diam}(I \cup S))^{k-i} |P^{(k)}| \]
14
proving the lemma. □

We recall that given $S \subset \mathbb{R}$ with $\#S = k + 1$ and a function $f : S \to \mathbb{R}$, by $L_\delta[f]$ we denote the Lagrange polynomial of degree at most $k$ interpolating $f$ on $S$.

**Lemma 3.6** Let $S_1, S_2 \subset \mathbb{R}, S_1 \neq S_2$, and let $\#S_1 = \#S_2 = k + 1$ where $k$ is a nonnegative integer. Let $I \subset \mathbb{R}$ be a closed interval. Then for every function $f : S_1 \cup S_2 \to \mathbb{R}$ and every $i, 0 \leq i \leq k$,

$$
\max_i |L_{S_1}^{(i)}[f] - L_{S_2}^{(i)}[f]| \leq (k + 1)! (\text{diam}(I \cup S_1 \cup S_2))^{k-i} A
$$

where

$$
A = \max_{S' \subset S_1 \cup S_2 \atop \#S' = k+2} |\Delta^{k+i} f[S']| \text{ diam } S';
$$

**Proof.** Let $n = k + 1 - \#(S_1 \cap S_2)$ and let $\{Y_j : j = 0, ..., n\}$ be a family of $(k + 1)$-point subsets of $S$ such that $Y_0 = S_1, Y_n = S_2$, and $\#(Y_j \cap Y_{j+1}) = k$ for every $j = 0, ..., n - 1$.

Let $P_j = L_{Y_j}[f], j = 0, ..., n$. Then

$$
\max_i |L_{S_1}^{(i)}[f] - L_{S_2}^{(i)}[f]| = \max_i |P_0^{(i)} - P_n^{(i)}| \leq \sum_{i=0}^{n-1} \max_i |P_j^{(i)} - P_{j+1}^{(i)}|.
$$

Note that each point $y \in Y_j \cap Y_{j+1}$ is a root of the polynomial $P_j - P_{j+1} \in \mathcal{P}_k$. Thus, if the polynomial $P_j - P_{j+1}$ is not identically 0, it has precisely $k$ distinct real roots which belong to the set $S_1 \cup S_2$. Then, by Lemma 3.5

$$
\max_i |P_j^{(i)} - P_{j+1}^{(i)}| \leq (\text{diam}(I \cup S_1 \cup S_2))^{k-i} |P_j^{(k)} - P_{j+1}^{(k)}|.
$$

But, by (2.4),

$$
|P_j^{(k)} - P_{j+1}^{(k)}| = |L_{Y_j}^{(k)}[f] - L_{Y_{j+1}}^{(k)}[f]| = k! |\Delta^k f[Y_j] - \Delta^k f[Y_{j+1}]|
$$

so that, by (2.2),

$$
|P_j^{(k)} - P_{j+1}^{(k)}| \leq k! |\Delta^{k+1} f[Y_j \cup Y_{j+1}]| \text{ diam}(Y_j \cup Y_{j+1}) \leq k! \max_{S' \subset S_1 \cup S_2 \atop \#S' = k+2} |\Delta^{k+1} f[S']| \text{ diam } S' = k! A.
$$

Combining this inequality with inequalities (3.21) and (3.22), we obtain the required inequality (3.19). □

**Lemma 3.7** Let $k$ be a nonnegative integer, $\ell \in \mathbb{N}, k < \ell$, and let $\mathcal{Y} = \{y_j\}_{j=0}^\ell$ be a strictly increasing sequence in $\mathbb{R}$. Let $I = [y_0, y_\ell], S_1 = \{y_0, ..., y_k\}, S_2 = \{y_{\ell-k}, ..., y_\ell\},$ and let

$$
S^{(j)} = \{y_j, ..., y_{k+j+1}\}, \quad j = 0, ..., \ell - k - 1.
$$

Then for every function $f : \mathcal{Y} \to \mathbb{R}$, every $i, 0 \leq i \leq k$, and every $p \in [1, \infty)$ the following inequality

$$
\max_i |L_{S_1}^{(i)}[f] - L_{S_2}^{(i)}[f]|^p \leq ((k + 3)!)^p (\text{diam } I)^{(k+1)p-1} \sum_{j=0}^{\ell-k-1} |\Delta^{k+1} f[S^{(j)}]|^p \text{ diam } S^{(j)}
$$

holds.
Proof. Let \( Y_j = \{y_j, ..., y_{j+k}\}, j = 0, ..., \ell - k \), so that \( S_1 = Y_0, S_2 = Y_{\ell-k}, \) and \( S^{(j)} = Y_j \cup Y_{j+1}, j = 0, ..., \ell - k - 1 \). Let \( P_j = L_{Y_j}[f], j = 0, ..., \ell - k \). Then

\[
\max_i |L^{(i)} [f] - L^{(i)} [f]| = \max_i |P^{(i)}_0 - P^{(i)}_{\ell-k}| \leq \sum_{i=0}^{\ell-k-1} \max_i |P^{(i)}_j - P^{(i)}_{j+1}|. \tag{3.25}
\]

Note that every \( y \in Y_j \cap Y_{j+1} \) is a root of the polynomial \( P_j - P_{j+1} \in P_k \). Thus, if the polynomial \( P_j - P_{j+1} \) is not identically 0, it has precisely \( k \) distinct real roots on \( I \). Then, by Lemma \ref{lem3.5}

\[
\max_i |P^{(i)}_j - P^{(i)}_{j+1}| \leq (\text{diam} \ I)^{k-i} |P^{(k)}_j - P^{(k)}_{j+1}|. \tag{3.26}
\]

By \ref{eq2.4},

\[
|P^{(k)}_j - P^{(k)}_{j+1}| = |L^{(k)} [f] - L^{(k)} [f]| = k!|\Delta^k f[Y_j] - \Delta^k f[Y_{j+1}]|
\]

so that, by \ref{eq2.2},

\[
|P^{(k)}_j - P^{(k)}_{j+1}| = k!|\Delta^k f[Y_j \cup Y_{j+1}]| \text{ diam}(Y_j \cup Y_{j+1}) = k!|\Delta^k f[S^{(j)}]| \text{ diam} S^{(j)}.
\]

This inequality and inequalities \ref{eq3.25} and \ref{eq3.26} imply the following:

\[
\max_i |L^{(i)} [f] - L^{(i)} [f]| \leq k!(\text{diam} I)^{k-i} \sum_{j=0}^{\ell-k-1} |\Delta^k f[S^{(j)}]| \text{ diam} S^{(j)}. \tag{3.27}
\]

Let \( I_j = \{y_j, y_{k+j+1}\} \). Then, by \ref{eq3.28}, \( \text{diam} I_j = \text{diam} S_j = y_{k+j+1} - y_j \). Furthermore, since \( \{y_j\}_{j=0}^\ell \) is a strictly increasing sequence and \( \#S_j = k + 2 \), the covering multiplicity of the family \( \{I_j : j = 0, ..., \ell - k - 1\} \) is bounded by \( k + 2 \). Hence,

\[
\sum_{j=0}^{\ell-k-1} \text{ diam} S^{(j)} = \sum_{j=0}^{\ell-k-1} \text{ diam} I_j \leq (k + 2) \text{ diam} I.
\]

Finally, applying the Hölder inequality and the above inequality to the right hand side of \ref{eq3.27}, we obtain

\[
\max_i |L^{(i)} [f] - L^{(i)} [f]|^p \leq (k!)^p (\text{diam} I)^{(k-i)p} \left( \sum_{j=0}^{\ell-k-1} |\Delta^k f[S^{(j)}]| \text{ diam} S^{(j)} \right)^p
\]

\[
\leq (k!)^p (\text{diam} I)^{(k-i)p} \left( \sum_{j=0}^{\ell-k-1} \text{ diam} S^{(j)} \right)^{p-1} \sum_{j=0}^{\ell-k-1} |\Delta^k f[S^{(j)}]|^p \text{ diam} S^{(j)}
\]

\[
\leq (k + 2)^{p-1} (k!)^p (\text{diam} I)^{(k-i+1)p-1} \sum_{j=0}^{\ell-k-1} |\Delta^k f[S^{(j)}]|^p \text{ diam} S^{(j)}
\]

proving inequality \ref{eq3.24}. \( \square \)

**Lemma 3.8** Let \( k \) be a nonnegative integer and let \( 1 < p < \infty \). Let \( f \) be a function defined on a closed set \( E \subset \mathbb{R} \) with \( \#E > k + 1 \). Suppose that

\[
\lambda = \sup_{S \subset E, \#S = k+2} |\Delta^k f[S]|(\text{diam} S)^\frac{1}{p} < \infty. \tag{3.28}
\]
Then for every limit point $x$ of $E$ and every $i, 0 \leq i \leq k$, there exists a limit

$$f_i(x) = \lim_{S \to x, S \subseteq E, \#S = k+1} L_S^{(i)}[f](x).\quad (3.29)$$

Here the notation $S \to x$ means $\text{diam}(S \cup \{x\}) < \delta$.

Furthermore, let $P_x \in \mathcal{P}_k$ be a polynomial such that

$$P_x^{(i)}(x) = f_i(x) \quad \text{for every} \quad i, 0 \leq i \leq k. \quad (3.30)$$

Then for every $\delta > 0$ and every set $S \subset E$ such that $\#S = k + 1$ and $\text{diam}(S \cup \{x\}) < \delta$ the following inequality

$$\max_{[x-\delta,x+\delta]} |P_x^{(i)} - L_S^{(i)}[f]| \leq C(k) \lambda \delta^{k+1-i-1/p}, \quad 0 \leq i \leq k, \quad (3.31)$$

holds.

**Proof.** Let $\delta > 0$ and let $S_1, S_2$ be two subsets of $E$ such that $\#S_j = k + 1$ and $\text{diam}(S_j \cup \{x\}) < \delta$, $j = 1, 2$. Hence, $S = S_1 \cup S_2 \subseteq I = [x - \delta, x + \delta]$. Then, by Lemma [3.6]

$$|L_{S_1}^{(i)}[f](x) - L_{S_2}^{(i)}[f](x)| \leq (k + 1)! (\text{diam} I)^{k-i} \max_{S' \subseteq S, \#S' = k+2} |\Delta^{k+1} f[S']| \text{ diam} S'.$$

By (3.28),

$$|\Delta^{k+1} f[S']| \leq \lambda (\text{diam} S')^{-\frac{1}{p}} \quad (3.32)$$

for every $(k + 2)$-point subset $S' \subseteq E$. Hence,

$$|L_{S_1}^{(i)}[f](x) - L_{S_2}^{(i)}[f](x)| \leq (k + 1)! \lambda (2\delta)^{k-i} \max_{S' \subseteq S, \#S' = k+2} (\text{diam} S')^{1-1/p}$$

$$\leq (k + 1)! \lambda (2\delta)^{k-i} (2\delta)^{1-1/p}$$

so that

$$|L_{S_1}^{(i)}[f](x) - L_{S_2}^{(i)}[f](x)| \leq C(k) \lambda \delta^{k+1-i-1/p}. \quad (3.33)$$

Thus

$$|L_{S_1}^{(i)}[f](x) - L_{S_2}^{(i)}[f](x)| \to 0 \quad \text{as} \quad \delta \to 0$$

proving the existence of the limit in (3.29).

Let us prove inequality (3.31). By (3.33), for every two sets $S, \tilde{S} \in E$, with $\#S = \#\tilde{S} = k + 1$ such that $\text{diam}(S \cup \{x\}), \text{diam}(\tilde{S} \cup \{x\}) < \delta$, the following inequality

$$|L_{\tilde{S}}^{(i)}[f](x) - L_{S}^{(i)}[f](x)| \leq C(k) \lambda \delta^{k+1-i-1/p}$$

holds. Passing to the limit in this inequality whenever the set $\tilde{S} \to x$ (i.e., $\text{diam}(\tilde{S} \cup \{x\}) \to 0$) we obtain the following:

$$|P_{x}^{(i)}(x) - L_{\tilde{S}}^{(i)}[f](x)| \leq C(k) \lambda \delta^{k+1-i-1/p}.$$

See (3.29) and (3.30). Hence, for each $y \in [x - \delta, x + \delta]$, we have

$$|P_{x}^{(i)}(y) - L_{\tilde{S}}^{(i)}[f](y)| = \left| \sum_{j=0}^{k-i} \frac{1}{j!} (P_{x}^{(i+j)}(x) - L_{\tilde{S}}^{(i+j)}[f](x)) (y - x)^j \right| \leq \sum_{j=0}^{k-i} \frac{1}{j!} |P_{x}^{(i+j)}(x) - L_{\tilde{S}}^{(i+j)}[f](x)| \delta^j$$

$$\leq C(k) \lambda \sum_{j=0}^{k-i} \delta^{k+1-i-j-1/p} \delta^j \leq C(k) \lambda \delta^{k+1-i-1/p}.$$

The proof of the lemma is complete. \qed
Lemma 3.9 Let \( k, p, E, f, \lambda \) and \( x \) be as in the statement of Lemma 3.8. Then for every \( i, 0 \leq i \leq k, \)
\[
\lim_{S \to x, S \subset E, \#S = i+1} i! \Delta^i f[S] = f_i(x).
\]

Proof. Let \( \delta > 0 \) and let \( S \subset E \) be a finite set such that \( \#S = i+1 \) and \( \text{diam}([x], S) < \delta \). Since \( x \) is a limit point of \( E \), there exists a set \( Y \subset E \cap [x - \delta, x + \delta] \) with \( \#Y = k + 1 \) such that \( S \subset Y \). Then, by (3.19),
\[
\max_{|x - \delta, x + \delta|} |P^{(i)}_{x} - L_{Y}^{(i)}[f]| \leq C(k) \lambda \delta^{k+1-i-1/p}.
\]

(3.34)

Since the Lagrange polynomial \( L_{Y}[f] \) interpolates \( f \) on \( S \), we have \( \Delta^i f[S] = \Delta^{i}(L_{Y}[f])[S] \) so that, by (2.3), there exists \( \xi \in [x - \delta, x + \delta] \) such that \( i! \Delta^i f[S] = L_{Y}^{(i)}(\xi) \). Then, by (3.34),
\[
|P^{(i)}_{x}(\xi) - i! \Delta^i f[S]| = |P^{(i)}_{x}(\xi) - L_{Y}^{(i)}[f](\xi)| \leq C(k) \lambda \delta^{k+1-i-1/p} \to 0 \text{ as } \delta \to 0.
\]

(3.35)

Hence,
\[
|f_i(x) - i! \Delta^i f[S]| = |P^{(i)}_{x}(x) - i! \Delta^i f[S]| \leq |P^{(i)}_{x}(x) - P^{(i)}_{x}(\xi)| + |P^{(i)}_{x}(\xi) - i! \Delta^i f[S]| \to 0 \text{ as } \delta \to 0
\]

because of the continuity of \( P^{(i)}_{x} \) and (3.35). \( \square \)

Lemma 3.10 Let \( p \in (1, \infty), k \in \mathbb{N}, \) and let \( f \) be a function defined on a closed set \( E \subset \mathbb{R} \) with \( \#E > k+1 \). Suppose that \( f \) satisfies condition (3.28).

Let \( x \in E \) be a limit point of \( E \), and let \( x \in S \) where \( S \subset E, \#S \leq k \). Then for every \( i, 0 \leq i \leq k+1 - \#S, \)
\[
\lim_{S \setminus S \to x} \lim_{S \subset E, \#S' = k+1} L_{S'}^{(i)}[f](x) = f_i(x).
\]

Proof. Let \( I_0 = [x-1/2, x+1/2] \) so that \( \text{diam} I_0 = 1 \). We prove that for every \( i, 0 \leq i \leq k, \) a family of functions \( \{L_{Y}^{(i)}[f] : Y \subset I_0 \cap E, \#Y = k+1\} \) is uniformly bounded on \( I_0 \) provided condition (3.28) holds. Indeed, fix a subset \( Y_0 \subset I_0 \cap E \) with \( \#Y_0 = k+1 \). Then for arbitrary \( Y \subset I_0 \cap E \) with \( \#Y = k+1 \), by (3.19),
\[
\max_{l_0} |L_{Y}^{(i)}[f] - L_{Y_0}^{(i)}[f]| \leq (k+1)! (\text{diam} (I_0 \cup Y \cup Y_0))^{k-i} A = (k+1)! A
\]

where \( A \) is defined by (3.20). Hence, by (3.27),
\[
\max_{l_0} |L_{Y}^{(i)}[f] - L_{Y_0}^{(i)}[f]| \leq (k+1)! A \max_{S' \subset Y_0 \cup Y, \#S' = k+2} (\text{diam} S')^{1-1/p} \leq (k+1)! A.
\]

By this inequality, for an arbitrary set \( Y \subset I_0 \cap E \) with \( \#Y = k+1 \) and each \( i, 0 \leq i \leq k, \)
\[
\max_{l_0} |L_{Y}^{(i)}[f]| \leq B_i
\]

(3.36)

where \( B_i = \max_{l_0} |L_{Y_0}^{(i)}[f]| + (k+1)! A \). Note that the constant \( B_i \) is independent of \( Y \).

Now let \( \varepsilon > 0 \). By Lemma 3.9 there exists \( \delta \in (0, 1] \) such that
\[
|i! \Delta^i f[V] - f_i(x)| \leq \varepsilon/2
\]

(3.37)

for an arbitrary set \( V \subset E \) with \( \#V = i+1 \) such that \( \text{diam}([x], V) < \delta \).
Let $S'$ be an arbitrary subset of $E$ such that $S \subset S'$, $\#S' = k + 1$ and
\[
\text{diam}(\{x\}, S' \setminus S) < \delta = \min\{\delta, \varepsilon/(2B_1)\}.
\] (3.38)

Recall that $B_i$ is the constant from inequality (3.36).

Choose a subset $V \subset S' \setminus S$ with $\#V = i + 1$. By (2.3), there exists $\xi \in [x - \delta, x + \delta]$ such that
\[
i! \Delta^i(L_{S'}[f])(V) = L_{S'}^{(i)}[f](\xi).
\]

On the other hand, since the polynomial $L_{S'}[f]$ interpolates $f$ on $V$, we have $\Delta^i f[V] = \Delta^i(L_{S'}[f])[V]$ so that, $i!\Delta^i f[V] = L_{S'}^{(i)}[f](\xi)$. Hence, by (3.37),
\[
|L_{S'}^{(i)}[f](\xi) - f_i(x)| \leq \varepsilon/2.
\]

Clearly, by (3.38) and (3.36),
\[
|L_{S'}^{(i)}[f](\xi) - L_{S'}^{(i)}[f](x)| \leq \max_{[x-1,x+1]}|L_{S'}^{(i+1)}[f]| |x - \xi| \leq B_i \delta \leq (\varepsilon/(2B_1)) = \varepsilon/2.
\]

Finally, we obtain
\[
|f_i(x) - L_{S'}^{(i)}[f](x)| \leq |f_i(x) - L_{S'}^{(i)}[f](\xi)| + |L_{S'}^{(i)}[f](\xi) - L_{S'}^{(i)}[f](x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon
\]
proving the lemma. \qed

3.3. Whitney $m$-fields and the Hermite polynomials.

Let $m \in \mathbb{N}$ and let $p \in (1, \infty)$. In this subsection, given a function $f : E \to \mathbb{R}$ whose divided differences on $E$ satisfy a certain Lipschitz-like condition (see (3.39)), we construct a Whitney $(m-1)$-field
\[
P^{(m,E)}[f] = \{P_x \in \mathcal{P}_{m-1} : x \in E\}
\]
such that
\[
P_x(x) = f(x) \quad \text{for all} \quad x \in E.
\]

In the next section we apply to $P^{(m,E)}[f]$ a criterion for extensions of Sobolev jets given in [53]. This criterion enables us to show that $f \in L^m_p(\mathbb{R})|_E$ provided $\Delta^m f|_E \in L^p(\mathbb{R})$ completing the proof of the sufficiency part of Theorem 1.4.

We turn to constructing the Whitney field $P^{(m,E)}[f]$.

In this section we assume that the function $f$ satisfies the following condition:
\[
\sup_{S \subset E, \#S = m+1} |\Delta^m f[S]| (\text{diam } S)^{\frac{1}{2}} < \infty.
\] (3.39)

Let $k = m - 1$. Given $x \in E$ let
\[
S_x = Y_k(x) = \{y_0(x), \ldots, y_{n_k(x)}(x)\}
\] (3.40)

and let
\[
s_x = y_{n_k(x)}.
\] (3.41)

Recall that the points $y_j(x)$ and the sets $Y_j(x)$ are defined by formulae (3.3) and (3.6).

In the next two propositions we describe the main properties of the sets $\{S_x : x \in E\}$ and the points $\{s_x : x \in E\}$. These properties are immediate from the results of Lemmas 3.2, 3.3 and 3.4.
Proposition 3.11  (i) \( x \in S_x \) and \( \#S_x \leq m \) for every \( x \in E \);
(ii) For every \( x_1, x_2 \in E \) such that \( S_{x_1} \neq S_{x_2} \) the following inequality
\[
\text{diam } S_{x_1} + \text{diam } S_{x_2} \leq 2m |x_1 - x_2|
\]  \hfill (3.42)
holds;
(iii). If \( x_1, x_2 \in E \) and \( x_1 < x_2 \) then
\[
\min S_{x_1} \leq \min S_{x_2} \quad \text{and} \quad \max S_{x_1} \leq \max S_{x_2}.
\]  \hfill (3.43)

Proposition 3.12  (i) For every \( x \in E \) the point \( s_x \in S_x \) and
\[
(x - |x - s_x|, x + |x - s_x|) \cap E = S_x \setminus \{s_x\}.
\]  \hfill (3.44)
Thus \( s_x \) is either minimal or maximal point of \( S_x \). In particular,
\[
[\min S_x, \max S_x] \cap E = S_x.
\]  \hfill (3.45)
(ii) All points of the set \( S_x \setminus \{s_x\} \) are isolated points of \( E \). If \( y \in S_x \) and \( y \) is a limit point of \( E \), then \( y = s_x \);
(iii) If \( \#S_x < m \) then \( s_x \) is a limit point of \( E \).

Remark 3.13  Let \( n \geq m \) and let \( E = \{x_i\}_{i=1}^n \) be a strictly increasing sequence of points in \( \mathbb{R} \). In this case, for each \( i, 1 \leq i \leq n \), the set \( S_{x_i} \) consists of \( m \) consecutive elements of the sequence \( E \). In other words, there exists \( \nu \in \mathbb{N}, 1 \leq \nu \leq n \), such that
\[
S_{x_i} = \{x_\nu, \ldots, x_{\nu+m-1}\}.
\]  \hfill (3.46)
Indeed, let \( k = m - 1 \). Since all points of \( E \) are isolated, \( n_k(x) = k \) for every \( x \in E \). See \((\text{3.4})\). In particular, in this case \( s_x = y_k(x) \).
Thus, by \((\text{3.40})\), \( S_x = Y_k(x) = \{y_0(x), \ldots, y_k(x)\} \) so that \( \#S_x = k + 1 = m \). On the other hand, by \((\text{3.45})\), \( S_x = [\min S_x, \max S_x] \cap E \) which proves \((\text{3.46})\). \( \triangleq \)

Given a function \( f : E \to \mathbb{R} \) satisfying condition \((\text{3.39})\), we define the Whitney \((m - 1)\)-field
\[
P^{(m,E)}[f] = \{P_x \in \mathcal{P}_{m-1} : x \in E\}
\]
as follows:

(\(\bullet\)) If \( \#S_x < m \), then, by part (iii) of Proposition \((\text{3.12})\), \( s_x \) is a limit point of \( E \). Then, by \((\text{3.39})\) and Lemma \((\text{3.8})\) for every \( i, 0 \leq i \leq m - 1 \), there exists a limit
\[
f_i(s_x) = \lim_{S \to s_x} L_i^0[f](s_x).
\]  \hfill (3.47)
We define a polynomial \( P_x \in \mathcal{P}_{m-1} \) as the Hermite polynomial satisfying the following conditions:
\[
P_x(y) = f(y) \quad \text{for every} \quad y \in S_x,
\]  \hfill (3.48)
and

\[ P^{(i)}_x(s_x) = f_i(s_x) \quad \text{for every } i, 1 \leq i \leq m - \#S_x. \]  \hspace{1cm} (3.49)

(\bullet) If \#S_x = m, we put \( P_x = L_S[f] \). Thus in this case (3.48) holds as well. Note that, since \( x \in S_x \), by (3.48),

\[ P_x(x) = f(x) \quad \text{for every } x \in E. \]  \hspace{1cm} (3.50)

Let us describe main properties of the the Hermite polynomial \( P_x \) determined by (3.48) and (3.49). See, e.g., [1], Ch. 2, Section 11.

First of all, we note that this polynomial exists and unique. We describe its structure in more details. Consider three cases.

**CASE 1.** Suppose that \#S_x = m. In this case \( P_x \) is the Lagrange polynomial determined by the set \( S_x \), i.e., \( P_x = L_{S_x}[f] \).

**CASE 2.** \#S_x = 1, i.e., \( s_x = x \) and \( S_x = \{x\} \). In this case \( x \) is a limit point of \( E \) so that, by (3.48) and (3.49),

\[ P_x(y) = \sum_{i=0}^{m-1} \frac{f_i(x)}{i!} (y-x)^i, \quad y \in \mathbb{R} \]

where \( f_i(x) \) is defined by (3.47).

**CASE 3.** Let now \( m > 1 \) and let \#S_x < m. Let \( n = \#S_x - 1 \) and \( y_i = y(x), i = 0, ..., n \), so that \( S_x = \{y_0, ..., y_n\} \). See (3.50). (Recall also that \( s_x = y(n) \).) In this case the Hermite polynomial \( P_x \) determined by (3.48) and (3.49), can be represented as a linear combination of polynomials \( H_0, ..., H_n, \bar{H}_1, ..., \bar{H}_{m-n-1} \in \mathcal{P}_{m-1} \). The polynomials \( H_0, ..., H_n \) are determined by the following conditions:

\[ H_i(y_i) = 1 \quad \text{for every } i, 0 \leq i \leq n, \quad \text{and} \quad H_j(y_i) = 0 \quad \text{for every } i, j, 0 \leq i, j \leq n, i \neq j, \]

and

\[ H'_i(y_n) = ... = H^{(m-n-1)}_i(y_n) = 0 \quad \text{for every } i, 0 \leq i \leq n. \]

In turn, the polynomials \( \bar{H}_1, ..., \bar{H}_{m-n-1} \) are determined as follows:

\[ \bar{H}_j(y_i) = 0 \quad \text{for every } i, j, 0 \leq i \leq n, 1 \leq j \leq m - n - 1, \]

and for every \( j, 1 \leq j \leq m - n - 1, \)

\[ \bar{H}^{(j)}_j(y_n) = 1 \quad \text{and} \quad \bar{H}^{(j)}_j(y_n) = 0 \quad \text{for every } \ell, 1 \leq \ell \leq m - n - 1, \ell \neq j. \]

The existence and uniqueness of the polynomials \( H_i \) and \( \bar{H}_j, 0 \leq i \leq n, 1 \leq j \leq m - n - 1, \) are shown in [1], Ch. 2, Section 11. Thus, for every \( P \in \mathcal{P}_{m-1} \) the following unique representation

\[ P(y) = \sum_{i=0}^{n} P(y_i) H_i(y) + \sum_{j=1}^{m-n-1} P^{(j)}(y_n) \bar{H}_j(y), \quad y \in \mathbb{R}. \] \hspace{1cm} (3.51)

holds. In particular,

\[ P_x(y) = \sum_{i=0}^{n} f(y_i) H_i(y) + \sum_{j=1}^{m-n-1} f_j(y_n) \bar{H}_j(y), \quad y \in \mathbb{R}. \] \hspace{1cm} (3.52)
Clearly, $P_x$ meets conditions (3.48) and (3.49).

Let $I \subset \mathbb{R}$ be a bounded closed interval. We supply the space $C^m(I)$ of $m$-times continuously differentiable functions on $I$ with the norm

$$\|f\|_{C^m(I)} = \sum_{i=0}^{m} \max_{j} |f^{(i)}|.$$  

Let us note the following important property of the polynomials $\{P_x : x \in E\}$.

**Lemma 3.14** Let $f$ be a function defined on a closed set $E \subset \mathbb{R}$ with $\#E > m + 1$, and satisfying condition (3.39). Let $I$ be a bounded closed interval in $\mathbb{R}$. Then for every $x \in E$

$$\lim_{S' \ni S_x \to x} \|L_{S'}[f] - P_x\|_{C^m(I)} = 0.$$  

*Proof.* The lemma is obvious whenever $\#S_x = m$ because in this case $L_{S_x}[f] = P_x$.

Let now $\#S_x < m$. In this case the polynomial $P_x$ can be represented in the form (3.52). Since in this case $S_x$ is a limit point of $E$, see part (iii) of Proposition 3.12 by Lemma 3.10

$$\lim_{S' \ni S_x \to x} L_{S'}(i)[f](s_x) = f_i(s_x) = P_x(s_x) \quad \text{for every} \quad i, 1 \leq i \leq m - n - 1.$$  

See (3.49). By (3.51), for every set $S' \subset E$ with $\#S' = m$ such that $S_x \subset S'$, the polynomial $L_{S'}[f]$ has the following representation:

$$L_{S'}[f](y) = \sum_{i=0}^{n} f(y) H_i(y) + \sum_{j=1}^{m-n-1} L_{S'}(j)[f](s_x) \tilde{H}_j(y), \quad y \in \mathbb{R}.$$  

This representation, (3.52) and (3.53) imply the following:

$$\max_{i} |L_{S'}[f] - P_x| = \max_{i} \left| \sum_{j=1}^{m-n-1} (L_{S'}(j)[f](s_x) - P_x(j)(s_x)) \tilde{H}_j \right|$$

$$\leq \sum_{j=1}^{m-n-1} \left| L_{S'}(j)[f](s_x) - P_x(j)(s_x) \right| \max_{i} |\tilde{H}_j| \to 0$$

as $S' \ni S_x \to x$, provided $S_x \subset S' \subset E$ and $\#S' = m$.

Since $P_x$ and $L_{S'}[f]$ for each $S'$ are elements of the finite dimensional space $\mathcal{P}_{m-1}$, this implies the convergence of $L_{S'}[f]$ to $P_x$ in $C^m(I)$-norm proving the lemma.  \[\square\]

### 3.4. Proof of the sufficiency part of Theorem 1.4

Let $f$ be a function on $E$ such that

$$\left(\Delta^m f\right)_E \in L_p(\mathbb{R}).$$  

See (1.10) and (1.11).

Let us prove that $f$ satisfies condition (3.39). Indeed, let $S = \{x_0, ..., x_n\} \subset E$, $x_0 < ... < x_m$. Then $\text{diam}(\{x\} \cup S) = \text{diam} S$ for every $x \in I = [x_0, x_m]$. Hence,

$$\text{diam} S = \frac{|\Delta^m f[S]|}{(\text{diam}(\{x\} \cup S))^p} \leq \left(\|\Delta^m f\|_{L_p}\right)^p (x_m - x_0) \leq \left(\frac{\text{diam} S}{\text{diam}(\{x\} \cup S)}\right)^p (x_m - x_0)$$
for every \( x \in [x_0, x_m] \). Integrating this inequality (with respect to \( x \)) over the interval \([x_0, x_m]\) we obtain
\[
|\Delta^m f[S]|^P \text{ diam } S \leq \int_{x_0}^{x_m} ((\Delta^m f)_E^p(x))^p \, dx \leq \| (\Delta^m f)_E^p \}_{L_p(\mathbb{R})}^p
\]

Hence,
\[
\sup_{S \subset E, \# S = m + 1} |\Delta^m f[S]| (\text{diam } S)^\frac{p}{m} \leq \| (\Delta^m f)_E^p \|_{L_p(\mathbb{R})} < \infty
\]
proving (3.39).

This condition guarantees that the Whitney \((m - 1)\)-field \( P^{(m,E)}[f] = \{ P_x : x \in E \} \) determined by formulae (3.48) and (3.49) is well defined.

Let us to show that inequality (3.2) holds. We prove this inequality with the help of Theorem 3.15 below which provides a criterion for the restrictions of Sobolev jets.

For each family \( P = \{ P_x : x \in E \} \) of polynomials we let \( P^\sharp \) denote a certain kind of “sharp maximal function” associated with \( P \) which is defined by
\[
P^\sharp_{m,E}(x) = \sup_{a_1, a_2 \in E, a_1 \neq a_2} \frac{|P_{a_1}(x) - P_{a_2}(x)|}{|x - a_1|^m + |x - a_2|^m}, \quad x \in \mathbb{R}.
\]

**Theorem 3.15** (3.53) Let \( m \in \mathbb{N} \), \( p \in (1, \infty) \), and let \( E \) be a closed subset of \( \mathbb{R} \). Suppose we are given a family \( P = \{ P_x : x \in E \} \) of polynomials of degree at most \( m - 1 \) indexed by points of \( E \).

Then there exists a \( C^{m-1} \)-function \( F \in L^p_p(\mathbb{R}) \) such that \( T^{m-1}_f = P_x \) for every \( x \in E \) if and only if \( P^\sharp_{m,E} \in L^p_{p}(\mathbb{R}) \). Furthermore,
\[
\| P \|_{m,p,E} \sim \| P^\sharp_{m,E} \|_{L^p_{p}(\mathbb{R})}
\]
with the constants in this equivalence depending only on \( m \) and \( p \).

We recall that the quantity \( \| P \|_{m,p,E} \) is defined by (3.1).

**Lemma 3.16** For every \( x \in \mathbb{R} \) the following inequality
\[
(P^{(m,E)}[f]^{\sharp}_{m,E}(x) \leq C(m) (\Delta^m f)_E^p(x)
\]
holds.

**Proof.** Let \( x \in \mathbb{R}, a_1, a_2 \in E, a_1 \neq a_2, \) and let \( r = |x - a_1| + |x - a_2|. \) Let \( S_j = S_{a_j} \) and let \( s_j = s_{a_j}, j = 1, 2. \) See (3.40) and (3.41). Note that by (3.42),
\[
\text{diam } S_1 + \text{diam } S_2 \leq 2m |a_1 - a_2|.
\]

Fix an \( \varepsilon > 0. \) By Lemma 3.14 there exist \( m \)-point subsets \( S_j \subset E, j = 1, 2, \) such that \( S_j \subset S_j, \)
\[
\text{diam } (S_j \cup (S_j \setminus S_j)) \leq r
\]
and
\[
|P_{a_j}(x) - L_{S_j}[f](x)| \leq \varepsilon r^m/2^{m+1}.
\]

In particular, if \( \# S_j = m \) the set \( S_j \) coincides with \( S_j. \) Furthermore, in this case \( P_{a_j} = L_{S_j}[f]. \)
Since \( s_j \in \tilde{S}_j \), by (3.57),
\[
\text{diam } S_j \leq \text{diam } \tilde{S}_j + \text{diam}((s_j) \cup (S_j \setminus \tilde{S}_j)) \leq \text{diam } \tilde{S}_j + r,
\]
so that, by (3.56),
\[
\text{diam } S_j \leq 2m |a_1 - a_2| + r, \quad j = 1, 2.
\]  
(3.59)

Let \( I \) be the smallest closed interval containing \( S_1 \cup S_2 \cup \{x\} \). Since \( a_j \in S_j, \ j = 1, 2, \)
\[
\text{diam } I \leq |x - a_1| + |x - a_2| + \text{diam } S_1 + \text{diam } S_2
\]
so that, by (3.59),
\[
\text{diam } I \leq |x - a_1| + |x - a_2| + 4m |a_1 - a_2| + 2r \leq (4m + 3) r.
\]  
(3.60)

(Recall that \( r = |x - a_1| + |x - a_2| \).)

This inequality and inequality (3.58) enables us to estimate \( |P_{a_1}(x) - P_{a_2}(x)| \), We have
\[
|P_{a_1}(x) - P_{a_2}(x)| \leq \left| P_{a_1}(x) - L_{S_1}[f](x) \right| + \left| L_{S_1}[f](x) - L_{S_2}[f](x) \right| + \left| P_{a_2}(x) - L_{S_2}[f](x) \right| = J + \varepsilon r^m / 2^m
\]
where \( J = |L_{S_1}[f](x) - L_{S_2}[f](x)| \).

Let us estimate \( J \). (We may assume that \( S_1 \neq S_2 \); otherwise \( J = 0 \).) By (3.19) (with \( k = m - 1 \) and \( i = 0 \),
\[
J \leq \max_J |L_{S_1}[f] - L_{S_2}[f]| \leq m! (\text{diam } I)^{m-1} \max_{S' \subseteq S, \ #S' = m+1} |\Delta^m f|_{S'} \text{ diam } S'
\]
where \( S = S_1 \cup S_2 \). Hence, by (3.60) and (1.11),
\[
J \leq C(m) r^{m-1} (\Delta^m f)_{E}^\sharp (x) \max_{S' \subseteq S, \ #S' = m+1} \text{ diam } (x) \cup S' \leq C(m) r^m (\Delta^m f)_{E}^\sharp (x).
\]

Hence,
\[
\frac{|P_{a_1}(x) - P_{a_2}(x)|}{|x - a_1|^m + |x - a_2|^m} \leq 2^m r^{-m} |P_{a_1}(x) - P_{a_2}(x)| \leq 2^m r^{-m} (J + \varepsilon r^m / 2^m)
\]
\[
\leq C(m) 2^m r^{-m} r^m (\Delta^m f)_{E}^\sharp (x) + \varepsilon = C(m) (\Delta^m f)_{E}^\sharp (x) + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, inequality (3.55) follows. \( \square \)

We finish the proof of Theorem 1.4 as follows.
Let \( P^{(m,E)}[f] = \{P_x : x \in E\} \). By Lemma 3.16,
\[
\|P^{(m,E)}[f]\|_{m,E}^{\#} \|L_p(\mathbb{R}) \leq C(m) \| (\Delta^m f)_{E}^\sharp \|_{L_p(\mathbb{R})}.
\]

Combining this inequality with the equivalence (3.54), we obtain inequality (3.2).

This inequality and definition (3.1) imply the existence of a function \( F \in L_p^m(\mathbb{R}) \) such that \( T_x^{m-1}[F] = P_x \) on \( E \) and
\[
\|F\|_{L_p^m(\mathbb{R})} \leq 2 \|P^{(m,E)}[f]\|_{m,p,E} \leq C(m, p) \| (\Delta^m f)_{E}^\sharp \|_{L_p(\mathbb{R})}.
\]  
(3.61)

We also note that \( P_x(x) = f(x) \) on \( E \), see (3.50), so that
\[
F(x) = T_x^{m-1}[F](x) = P_x(x) = f(x), \quad x \in E.
\]

Thus \( F \in L_p^m(\mathbb{R}) \) and \( F|_E = f \) proving that \( f \in L_p^m(\mathbb{R})|_E \). Furthermore, by (1.1) and (3.61),
\[
\|f\|_{L_p^m(\mathbb{R})|_E} \leq \|F\|_{L_p^m(\mathbb{R})} \leq C(m, p) \| (\Delta^m f)_{E}^\sharp \|_{L_p(\mathbb{R})}.
\]

The proof of Theorem 1.4 is complete. \( \square \)
4. A variational criterion for Sobolev traces: the sufficiency part of Theorem 1.3

4.1. A variational criterion for Sobolev jets.

Our proof of the sufficiency part of Theorem 1.3 relies on the following extension theorem for Sobolev jets.

**Theorem 4.1** Let \( m \in \mathbb{N} \), \( p \in (n, \infty) \), and let \( E \) be a closed subset of \( \mathbb{R} \). Suppose we are given a Whitney \((m - 1)\)-field \( \mathbf{P} = \{ P_x : x \in E \} \) defined on \( E \).

Then there exists a \( C^{m-1} \)-function \( F \in L^p_m(\mathbb{R}) \) such that

\[
T_x^{m-1}[F] = P_x \quad \text{for every } x \in E \tag{4.1}
\]

if and only if the following quantity

\[
\mathcal{N}_{m,p,E}(\mathbf{P}) = \sup \left\{ \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{|P_{x_j}^{(i)}(x_j) - P_{x_{j+1}}^{(i)}(x_j)|^p}{(x_{j+1} - x_j)^{m-1}p-1} \right\}^{1/p} \tag{4.2}
\]

is finite. Here the supremum is taken over all finite strictly increasing sequences \( \{x_j\}_{j=1}^k \subset E \).

Furthermore,

\[
\|\mathbf{P}\|_{m,p,E} \sim \mathcal{N}_{m,p,E}(\mathbf{P}) \tag{4.3}
\]

The constants of equivalence in (4.3) depend only on \( m \) and \( p \).

Theorem 4.1 is a refinement (for the one dimensional case) of a general extension criterion for extensions of \( L^p(\mathbb{R}^n) \)-jets given in [53].

**Proof. (Necessity.)** Let \( \mathbf{P} = \{ P_x : x \in E \} \) be a Whitney \((m - 1)\)-field on \( E \), and let \( F \in L^p_m(\mathbb{R}) \) be a function satisfying condition (4.1). We recall that, by the Taylor formula with the reminder in the integral form, for every \( x \in \mathbb{R} \) and every \( a \in E \) the following equality

\[
F(x) - T_a^{m-1}[F](x) = \frac{1}{(m - 1)!} \int_a^x F^{(m)}(t)(x - t)^{m-1} \, dt
\]

holds. Hence, by (4.1),

\[
F(x) - P_a(x) = \frac{1}{(m - 1)!} \int_a^x F^{(m)}(t)(x - t)^{m-1} \, dt.
\]

Differentiating this equality \( i \) times (with respect to \( x \)) we obtain the formula

\[
F^{(i)}(x) - P_a^{(i)}(x) = \frac{1}{(m - 1 - i)!} \int_a^x F^{(m)}(t)(x - t)^{m-1-i} \, dt
\]

which implies the equality

\[
P_{x_j}^{(i)}(x) - P_a^{(i)}(x) = \frac{1}{(m - 1 - i)!} \int_a^{x_j} F^{(m)}(t)(x - t)^{m-1-i} \, dt.
\]

Let \( \{x_j\}_{j=1}^k \subset E \) be an arbitrary strictly increasing sequence in \( E \). Then, by the last formula,

\[
P_{x_j}^{(i)}(x_j) - P_{x_{j+1}}^{(i)}(x_j) = \frac{1}{(m - 1 - i)!} \int_{x_j}^{x_{j+1}} F^{(m)}(t)(x_{j+1} - t)^{m-1-i} \, dt
\]
so that
\[ \left| P^{(i)}_{x_j}(x_j) - P^{(i)}_{x_{j+1}}(x_j) \right|^p \leq \frac{(x_{j+1} - x_j)^{(m-1-i)p}}{(m-1-i)!^p} \left( \int_{x_j}^{x_{j+1}} \left| F^{(m)}(t) \right|^p dt \right). \]

Hence,
\[ \frac{\left| P^{(i)}_{x_j}(x_j) - P^{(i)}_{x_{j+1}}(x_j) \right|^p}{(x_{j+1} - x_j)^{(m-1-i)p-1}} \leq \frac{(x_{j+1} - x_j)^{1-p}}{(m-1-i)!^p} \left( \int_{x_j}^{x_{j+1}} \left| F^{(m)}(t) \right|^p dt \right) \leq \frac{1}{(m-1-i)!^p} \int_{x_j}^{x_{j+1}} \left| F^{(m)}(t) \right|^p dt \]

so that
\[ \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{\left| P^{(i)}_{x_j}(x_j) - P^{(i)}_{x_{j+1}}(x_j) \right|^p}{(x_{j+1} - x_j)^{(m-1-i)p-1}} \leq \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{1}{(m-1-i)!^p} \int_{x_j}^{x_{j+1}} \left| F^{(m)}(t) \right|^p dt \]

proving that
\[ \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{\left| P^{(i)}_{x_j}(x_j) - P^{(i)}_{x_{j+1}}(x_j) \right|^p}{(x_{j+1} - x_j)^{(m-1-i)p-1}} \leq e^p \| F \|_{L^p}^p. \]

Taking the supremum in this inequality over all finite strictly increasing sequences \( \{x_j\}_{j=1}^k \subset E \), and the infimum over all function \( F \in L^p_{p}(\mathbb{R}) \) satisfying (4.1), we obtain
\[ N_{m,p,E}(P) \leq e \| P \|_{m,p,E}. \]

The proof of the necessity is complete.

(Sufficiency.) Let \( P = \{P_x : x \in E\} \) be a Whitney \((m-1)\)-field defined on \( E \) such that \( N_{m,p,E}(P) < \infty \). See (4.2). Thus there exists a constant \( \lambda > 0 \) such that for every strictly increasing sequence \( \{x_j\}_{j=1}^k \subset E \) the following inequality

\[ \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{\left| P^{(i)}_{x_j}(x_j) - P^{(i)}_{x_{j+1}}(x_j) \right|^p}{(x_{j+1} - x_j)^{(m-1-i)p-1}} \leq \lambda^p \]  \[ (4.4) \]

holds.

We prove the existence of a function \( F \in L^p_{p}(\mathbb{R}) \) such that \( T^{m-1}_x[F] = P_x \) for every \( x \in E \) and \( \| F \|_{L^p_{p}(\mathbb{R})} \leq C(m, p) \lambda \).

We construct \( F \) with the help of the classical Whitney extension method [56]. It is proven in [53] that this method provides an almost optimal extension of the restrictions of Whitney \((m-1)\)-fields generated by Sobolev functions. In this paper we will make use of a simpler version of this method suggested by Whitney for the one dimensional case. See [57], Section 4.

Since \( E \) is a closed set, the set \( \mathbb{R} \setminus E \) is open so that it can be represented as a union of a family

\[ \mathcal{J}_E = \{J_k = (a_k, b_k) : k \in \mathcal{K}\} \]

a finite or countable number of pairwise disjoint open intervals (bounded or unbounded). Thus

\[ \mathbb{R} \setminus E = \bigcup \{J_k = (a_k, b_k) : k \in \mathcal{K}\} \quad \text{and} \quad J_k \cap J_{k'} = \emptyset \quad \text{for every} \quad k', k'' \in \mathcal{K}, k' \neq k''. \]
To each interval $J \in \mathcal{J}_E$ we assign a polynomial $H_J \in \mathcal{P}_{2m-1}$ as follows:

(1) Let $J$ be an unbounded open interval, i.e., $J = (-\infty, a)$ or $J = (b, \infty)$ for some $a, b \in E$. In this case we put $H_J = P_a$ or $H_J = P_b$ respectively.

(2) Let $J = (a, b) \in \mathcal{J}_E$ be a bounded interval so that $a, b \in E$. In this case we define the polynomial $H_J \in \mathcal{P}_{2m-1}$ as the Hermite polynomial satisfying the following conditions:

$$H_J^{(0)}(a) = P_a^{(0)}(a) \quad \text{and} \quad H_J^{(0)}(b) = P_b^{(0)}(b) \quad \text{for all} \quad i = 0, \ldots, m - 1.$$ (4.5)

Finally, we define the extension $F$ by the formula:

$$F(x) = \begin{cases} P_a(x), & x \in E, \\ \sum_{J \in \mathcal{J}_E} H_J(x) \chi_J(x), & x \in \mathbb{R} \setminus E. \end{cases}$$ (4.6)

Note that, by (4.4), for every $x, y \in E$ and every $i, 0 \leq i \leq m - 1$, the following inequality

$$|P_x^{(i)}(x) - P_y^{(i)}(x)| \leq \lambda |x - y|^{m-i-1/p}$$

holds. This proves that

$$P_x^{(i)}(x) - P_y^{(i)}(x) = o(|x - y|^{m-1-i}), \quad x, y \in E, \quad 0 \leq i \leq m - 1,$$

so that the $(m-1)$-field $\mathbf{P} = \{P_x : x \in E\}$ satisfies the hypothesis of the Whitney extension theorem [56]. In [57], Section 4, Whitney proved that the extension $F : \mathbb{R} \to \mathbb{R}$ defined by formula (4.6) is a $C^{m-1}$-function which agrees with the Whitney $(m-1)$-field $\mathbf{P}$, i.e., $F^{(i)}(x) = P_x^{(i)}(x)$ for every $x \in E$ and every $i, 0 \leq i \leq m - 1$.

Let us show that

$$F \in L_p^m(\mathbb{R}) \quad \text{and} \quad \|F\|_{L_p^m(\mathbb{R})} \leq C(m, p) \lambda.$$ (4.7)

Our proof of these facts relies on the following description of $L_p^1(\mathbb{R})$-functions.

**Theorem 4.2** Let $p > 1$ and let $\tau > 0$. Let $G$ be a continuous function on $\mathbb{R}$ satisfying the following condition: There exists a constant $A > 0$ such that for every finite family $I = \{I \in [u_I, v_I]\}$ of pairwise disjoint closed intervals of diameter $\text{diam} I = v_I - u_I \leq \tau$ the following inequality

$$\sum_{I = [u_I, v_I] \in I} \frac{|G(u_I) - G(v_I)|^p}{(v_I - u_I)^{p-1}} \leq A$$

holds. Then $G \in L_p^1(\mathbb{R})$ and $\|G\|_{L_p^1(\mathbb{R})} \leq C(p) A^{1/p}$.

For $\tau = \infty$ this result follows from the Riesz theorem [42]. For the case $0 < \tau < \infty$ see [53], Section 7. We will also need the following auxiliary lemmas.

**Lemma 4.3** Let $J = (a, b) \in \mathcal{J}_E$ be a bounded interval. Then for every $x \in [a, b]$ the following inequality

$$|H_J^{(m)}(x)| \leq C(m) \min \left\{ \sum_{i=0}^{m-1} \frac{|P_b^{(i)}(b) - P_a^{(i)}(b)|}{(b-a)^{m-i}}, \sum_{i=0}^{m-1} \frac{|P_b^{(i)}(a) - P_a^{(i)}(a)|}{(b-a)^{m-i}} \right\}$$

holds. Here $H_J \in \mathcal{P}_{2m-1}$ is the Hermite polynomial defined by equalities (4.5).
Proof. By (4.5), there exist $\gamma_m, \gamma_{m+1}, \ldots, \gamma_{2m-1} \in \mathbb{R}$ such that

$$H_j(x) = P_a(x) + \sum_{k=m}^{2m-1} \frac{1}{k!} \gamma_k (x-a)^k \text{ for every } x \in [a,b].$$

Hence, for every $i = 0, \ldots, m - 1$,

$$H^{(i)}_j(x) = P_{a}^{(i)}(x) + \sum_{k=m}^{2m-1} \frac{1}{(k-i)!} \gamma_k (x-a)^{k-i}, \quad x \in [a,b]. \quad (4.8)$$

In particular,

$$H^{(i)}_j(b) = P_{a}^{(i)}(b) + \sum_{k=m}^{2m-1} \frac{1}{(k-i)!} \gamma_k (b-a)^{k-i}$$

so that, by (4.5),

$$\sum_{k=m}^{2m-1} \gamma_k \frac{(b-a)^{k-i}}{(k-i)!} = P_{b}^{(i)}(b) - P_{a}^{(i)}(b), \quad \text{for all } i = 0, \ldots, m - 1.$$

Thus $(\gamma_m, \gamma_{m+1}, \ldots, \gamma_{2m-1})$ is a solution to the above system of $m$ linear equations with respect to $m$ unknowns. Following [57] we solve this linear system and obtain a solution in the following form:

$$\gamma_k = \sum_{i=0}^{m-1} K_{k,i} \frac{P_{b}^{(i)}(b) - P_{a}^{(i)}(b)}{(b-a)^{k-i}}, \quad k = m, \ldots, 2m - 1, \quad (4.9)$$

where $K_{k,i}$ are constants depending only on $m$.

This representation enables us to estimate $H^{(m)}_j(x)$ for every $x \in [a, (a+b)/2]$ as follows. By (4.9),

$$|\gamma_k| \leq C(m) \sum_{i=0}^{m-1} \frac{|P_{b}^{(i)}(b) - P_{a}^{(i)}(b)|}{(b-a)^{k-i}}, \quad k = m, \ldots, 2m - 1.$$

On the other hand, by (4.8),

$$H^{(m)}_j(x) = \sum_{k=m}^{2m-1} \frac{1}{(k-m)!} \gamma_k (x-a)^{k-m},$$

so that

$$|H^{(m)}_j(x)| \leq C(m) \sum_{k=m}^{2m-1} \sum_{i=0}^{m-1} \frac{|P_{b}^{(i)}(b) - P_{a}^{(i)}(b)|}{(b-a)^{k-i}} (x-a)^{k-m} = C(m) \sum_{i=0}^{m-1} \sum_{k=m}^{2m-1} \frac{|P_{b}^{(i)}(b) - P_{a}^{(i)}(b)|}{(b-a)^{k-i}} (x-a)^{k-m}.$$

Since $x-a \leq (b-a)/2$ for every $x \in [a, (a+b)/2]$,

$$|H^{(m)}_j(x)| \leq C(m) \sum_{i=0}^{m-1} \frac{|P_{b}^{(i)}(b) - P_{a}^{(i)}(b)|}{(b-a)^{m-i}} \left( \sum_{k=m}^{2m-1} (x-a)^{k-m} \right) \frac{2m-1}{2^{m-k}}$$

$$\leq C(m) \sum_{i=0}^{m-1} \frac{|P_{b}^{(i)}(b) - P_{a}^{(i)}(b)|}{(b-a)^{m-i}} \left( \sum_{k=m}^{2m-1} 2^{m-k} \right).$$
proving that
\[
|H^{(m)}_J(x)| \leq C(m) \sum_{i=0}^{m-1} \frac{|P^{(i)}(b) - P^{(i)}_a(b)|}{(b-a)^{m-i}} \quad \text{for every } x \in [a,(a+b)/2]. \quad (4.10)
\]

In the same fashion we prove that
\[
|H^{(m)}_J(x)| \leq C(m) \sum_{i=0}^{m-1} \frac{|P^{(i)}(a) - P^{(i)}_a(a)|}{(b-a)^{m-i}} \quad \text{for all } x \in [(a+b)/2,b]. \quad (4.11)
\]

This inequality enables us to show that inequality (4.10) holds for all \(x \in [a,b]\). Indeed,
\[
P^{(i)}_b(a) - P^{(i)}_a(a) = \sum_{k=i}^{m-1} \frac{1}{(k-i)!} (P^{(k)}_b(b) - P^{(k)}_a(b))(b-a)^{k-i}
\]
so that for every \(x \in [(a+b)/2,b]\) the following inequality
\[
|H^{(m)}_J(x)| \leq C(m) \sum_{i=0}^{m-1} \sum_{k=i}^{m-1} \frac{|P^{(k)}_b(b) - P^{(k)}_a(b)|}{(b-a)^{m-k}} = C(m) \sum_{i=0}^{m-1} \frac{m!}{(m-i)!} \frac{|P^{(i)}_a(a) - P^{(i)}_b(b)|}{(b-a)^m}
\]
holds. Hence
\[
|H^{(m)}_J(x)| \leq m C(m) \sum_{k=0}^{m-1} \frac{|P^{(k)}_b(b) - P^{(k)}_a(b)|}{(b-a)^{m-k}} \quad \text{provided } x \in [(a+b)/2,b].
\]

In the same way we show that inequality (4.11) holds for all \(x \in [a,b]\).

The proof of the lemma is complete. \(\square\)

**Lemma 4.4** Let \(I = [I = [u_1, v_1]]\) be a finite family of pairwise disjoint closed intervals such that \((u_1, v_1) \subset \mathbb{R} \setminus E\) for every \(I \in \mathcal{I}\). Then
\[
\sum_{I=[u_1,v_1] \in \mathcal{I}} \frac{|F^{(m-1)}(u_1) - F^{(m-1)}(v_1)|^p}{(v_1 - u_1)^{p-1}} \leq C(m, p) \lambda^p. \quad (4.12)
\]

**Proof.** Let \(I = [u_1, v_1] \in \mathcal{I}\) and let \((u_1, v_1) \subset \mathbb{R} \setminus E\). Then there exist an interval \(J = (a, b) \in \mathcal{F}_E\) containing \((u_1, v_1)\). Then, by the extension formula (4.6), \(F|_J = H_J\) so that, by Lemma 4.3
\[
|F^{(m-1)}(u) - F^{(m-1)}(v)| = |H_J^{(m-1)}(u) - H_J^{(m-1)}(v)| \leq \max_I |H^{(m)}_I|(v-u)
\]
\[
\leq C(m) (v-u) \sum_{i=0}^{m-1} \frac{|P^{(i)}_a(a) - P^{(i)}_b(b)|}{(b-a)^{m-i}}.
\]

Hence,
\[
\frac{|F^{(m-1)}(u) - F^{(m-1)}(v)|^p}{(v-u)^{p-1}} \leq C (v-u)^{1-p} (v-u)^p \sum_{i=0}^{m-1} \frac{|P^{(i)}_a(a) - P^{(i)}_b(b)|^p}{(b-a)^{(m-i)p}}
\]
so that
\[
\frac{|F^{(m-1)}(u) - F^{(m-1)}(v)|^p}{(v-u)^{p-1}} \leq C (v-u) \sum_{i=0}^{m-1} \frac{|P^{(i)}_a(a) - P^{(i)}_b(b)|^p}{(b-a)^{(m-i)p}}. \quad (4.13)
\]
where $C = C(m, p)$ is a constant depending only on $m$ and $p$.

Let now $J = (a, b) \in \mathcal{J}_E$. By $I_J$ we denote a subfamily of $I$ defined by

$$I_J = \{I \in I : I \subset [a, b]\}.$$ 

Let $\mathcal{J} = \{J \in \mathcal{J} : I_J \neq \emptyset\}$. Then, by (4.13), for every $J = (a_J, b_J) \in \mathcal{J}$

$$S_J = \sum_{I=[u_I, v_I] \in I_J} \frac{|F^{(m-1)}(u_I) - F^{(m-1)}(v_I)|^p}{(v_I - u_I)^{p-1}} \leq C \sum_{I \in I_J} \frac{\sum_{i=0}^{m-1} |F^{(i)}(a_I) - F^{(i)}(b_I)|}{(b_I - a_I)^{(m-i)p}}.$$ 

Since the intervals of the family $I_J$ are pairwise disjoint (because the intervals of the family $I$ are pairwise disjoint),

$$S_J \leq C (b_J - a_J) \sum_{I \in I_J} \sum_{i=0}^{m-1} \frac{|F^{(i)}(a_J) - F^{(i)}(b_J)|^p}{(b_J - a_J)^{(m-i)p}} = C \sum_{I \in I_J} \sum_{i=0}^{m-1} \frac{|F^{(i)}(a_J) - F^{(i)}(b_J)|^p}{(b_J - a_J)^{(m-i)p}}.$$ 

Finally,

$$S = \sum_{I=[u_I, v_I] \in I} \frac{|F^{(m-1)}(u_I) - F^{(m-1)}(v_I)|^p}{(v_I - u_I)^{p-1}} = \sum_{J=(a_J, b_J) \in \mathcal{J}} \sum_{I=[u_I, v_I] \in I_J} \frac{|F^{(m-1)}(u_I) - F^{(m-1)}(v_I)|^p}{(v_I - u_I)^{p-1}}.$$ 

Since the open intervals from the family $\mathcal{J}$ are pairwise disjoint, by (4.4), $S \leq C \lambda^p$ proving the lemma.

\[\square\]

**Lemma 4.5** Let $I = \{I = [u_I, v_I]\}$ be a finite family of intervals such that $u_I, v_I \in E$ for each $I \in I$. Suppose that the open intervals $((u_I, v_I) : I \in I)$ are pairwise disjoint. Then inequality (4.12) holds.

**Proof.** Since $F$ agrees with the Whitney $(m-1)$-field $\mathbf{P} = \{P_x : x \in E\}$, we have $F^{(m-1)}(x) = P_x^{(m-1)}(x)$ for every $x \in E$. Hence,

$$S = \sum_{I=[u_I, v_I] \in I} \frac{|F^{(m-1)}(u_I) - F^{(m-1)}(v_I)|^p}{(v_I - u_I)^{p-1}} = \sum_{I=[u_I, v_I] \in I} \frac{|P_x^{(m-1)}(u_I) - P_x^{(m-1)}(v_I)|^p}{(v_I - u_I)^{p-1}}.$$ 

Since the intervals $((u_I, v_I) : I \in I)$ are pairwise disjoint, by (4.4), $S \leq \lambda^p$ proving the lemma. \[\square\]

We are in a position to finish the proof of the sufficiency. Let $I$ be a finite family of closed intervals. We introduce the following notation: given an interval $I = [u, v], u \neq v$, we put

$$Y(I; F) = \frac{|F^{(m-1)}(u_I) - F^{(m-1)}(v_I)|^p}{(v_I - u_I)^{p-1}}.$$ 

We put $Y(I; F) = 0$ whenever $u = v$, i.e., $I = [u, v]$ is a singleton.

Let $I = [u_I, v_I] \in I$ be an interval such that

$$I \cap E \neq \emptyset \quad \text{and} \quad \{u_I, v_I\} \notin E.$$ 

Thus either $u_I$ or $v_I$ belongs to $\mathbb{R} \setminus E$.

Let $u_I'$ and $v_I'$ be the points of $E$ nearest to $u_I$ and $v_I$ on $I \cap E$ respectively. Then $[u_I', v_I'] \subset [u_I, v_I]$. Let

$$I^{(1)} = [u_I, u_I'], \quad I^{(2)} = [u_I', v_I'] \quad \text{and} \quad I^{(3)} = [v_I', v_I].$$

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Note that \((u_I, u_I') \subset \mathbb{R} \setminus E\) provided \(u_I \notin E\) and \(v_I \notin E\), and \(u_I', v_I' \in E\). Furthermore,

\[
Y(I; F) \leq C \left\{ Y(I^{(1)}; F) + Y(I^{(2)}; F) + Y(I^{(3)}; F) \right\},
\]

with \(C = C(m, p)\). See (4.14).

If \(I \in \mathcal{I}\) and \((u_I, v_I) \subset \mathbb{R} \setminus E\), we put \(I^{(1)} = I^{(2)} = I^{(3)} = I\). We have:

\[
S = \sum_{I \in \mathcal{I}} Y(I; F) \leq C \sum_{I \in \mathcal{I}} \left\{ Y(I^{(1)}; F) + Y(I^{(2)}; F) + Y(I^{(3)}; F) \right\}
\]

proving that

\[
S \leq C \sum_{I \in \mathcal{I}} Y(I; F) \quad \text{where} \quad \mathcal{I} = \left\{ I^{(1)}, I^{(2)}, I^{(3)} : I \in \mathcal{I} \right\}.
\]

We know that for each \(I = [u_I, v_I] \subset \mathcal{I}\) either \((u_I, v_I) \in \mathbb{R} \setminus E\), or \(u_I, v_I \in E\), or \(u_I = v_I\) (and so \(Y(I; F) = 0\)). Furthermore, the sets \(\{(u_I, v_I) : I \in \mathcal{I}\}\) are pairwise disjoint.

We apply to the family \(\mathcal{I}\) the results of Lemmas 4.4 and 4.5 and obtain that \(S \leq C(m, p) \lambda^p\). Then, by Theorem 4.2, \(F^{(m-1)} \in L^p_0(\mathbb{R})\) and \(\|F^{(m-1)}\|_{L^p_0(\mathbb{R})} \leq C(m, p) \lambda\) proving (4.7).

The proof of Theorem 4.4 is complete. \(\square\)

**Remark 4.6** Let \(E = \{x_j\}_{j=1}^n\) be a strictly increasing sequence of points in \(\mathbb{R}\) (finite or infinite). In this case the criterion (4.3) can be slightly simplified. Namely, obvious changes in the proof of the sufficiency part lead us to the following formula:

\[
\|P\|_{m,p,E} \sim \left\{ \sum_{j=1}^{n} \sum_{i=0}^{m-1} \frac{|p_{i,j}(x_j) - p_{i,j+1}(x_j)|^p}{(x_{j+1} - x_j)^{(m-1)p-1}} \right\}^{1/p}
\]

with the constants in this equivalence depending only on \(m\) and \(p\).

### 4.2. Proof of the sufficiency part of Theorem 1.3

Let \(f\) be a function on \(E\). Let us assume the following:

**Assumption 4.7** Suppose that there exists a constant \(\lambda > 0\) such that for every finite strictly increasing sequence of points \(\{x_0, ..., x_n\} \subset E\), \(n \geq m\), the following inequality

\[
\sum_{i=0}^{n-m} (x_{i+m} - x_i) |\Delta^m f[x_i, ..., x_{i+m}]|^p \leq \lambda^p
\]

holds.

Let us prove that under Assumption 4.7 there exists a function \(F \in L^m_0(\mathbb{R})\) such that

\[
F|_E = f \quad \text{and} \quad \|F\|_{L^m_0(\mathbb{R})} \leq C(m, p) \lambda.
\]

First let us note that, by (4.15),

\[
\sup_{S \subset E, \#S = m+1} |\Delta^m f[S]| (\text{diam } S)^{\frac{1}{p}} \leq \lambda,
\]

so that inequality (3.39) holds. In Section 3 we have proved that for any function \(f : E \to \mathbb{R}\) satisfying this inequality the Whitney \((m - 1)\)-field

\[
P^{(m,E)}(f) = \{P_x \in P_{m-1} : x \in E\}
\]

determined by formulae (3.47)-(3.49)
is well defined.

We apply the variational criterion for Sobolev jets given in Theorem \[ \text{Proposition 4.8} \] to the Whitney \((m - 1)\)-field \(P^{m,E}(\mathcal{J})\). More specifically, we show that for every finite strictly increasing sequence \(\{x_j\}_{j=1}^k \subset E\) the following inequality

\[
\sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{|P^{(i)}_{x_j}(x_j) - P^{(i)}_{x_j}(x_{j+1})|^p}{|x_{j+1} - x_j|^{(m-i)p-1}} \leq C(m, p) \lambda^p
\]

holds. See Lemma 4.9 below.

Let \(X = \{x_1, ..., x_k\}\) and let \(x_0 = x_1\) and \(x_{k+1} = x_k\). Our proof of inequality 4.17 relies on the following statement.

**Proposition 4.8** Let \(\varepsilon > 0\). There exist a finite strictly increasing sequence \(V = \{v_1, ..., v_\ell\}\) of points in \(E\) and a mapping \(H : X \to 2^V\) which to every \(x \in X\) assigns \(m\) consecutive points \(H(x) = \{v_{j_1(x)}, ..., v_{j_\ell(x)}\}\) (4.18) of \(V\), \(1 \leq j_1(x) \leq j_2(x) = j_1(x) + m - 1 \leq \ell\), such that the following conditions are satisfied:

1. \(x \in H(x)\) for each \(x \in X\);
2. Let \(x', x'' \in X\), \(x' < x''\). Then
   \[
   \min H(x') \leq \min H(x'').
   \]
3. For every \(x', x'' \in X\) such that \(H(x') \neq H(x'')\) the following inequality
   \[
   \text{diam } H(x') + \text{diam } H(x'') \leq 2(m + 1)|x' - x''|\]
   holds;
4. For every \(i, 0 \leq i \leq m - 1\),
   \[
   |P^{(i)}_{x_j}(x_j) - L^{(i)}_{H(x_j)}[f](x_j)| + |P^{(i)}_{x_j}(x_{j-1}) - L^{(i)}_{H(x_j)}[f](x_{j-1})| < \varepsilon \quad \text{for all } j = 1, ..., k.
   \]

**Proof.** We recall that, given \(x \in E\) by \(S_x\) and \(s_x\) we denote a subset of \(E\) and a point in \(E\) whose properties are described in Propositions 3.11 and 3.12 respectively.

Let \(S = \bigcup_{x \in X} S_x\). (4.21)

Since \(\#S_x \leq m\), see part (i) of Proposition 3.12 and \(\#X = k\), the set \(S\) is a finite subset of \(E\). Let us enumerate its point in the increasing order. In other words, we represent \(S\) as a strictly increasing sequence \(S = \{u_1, u_2, ..., u_k\}\). (4.22)

Here \(u_j \in E\), \(j = 1, ..., k\), \(u_i < u_j\) whenever \(1 \leq i < j \leq k\), and \(k \in [m, km]\) is a positive integer.

Now for each \(x \in S\) the set \(S_x\) is an enumerated subset (subsequence) of \(S\). Let us note two important properties of this enumeration:

1. For every \(x \in X\) the set \(S_x\) consists of at most \(m\) consecutive points of \(S\). In other words, if \(\min S_x = u_{j_1}\) and \(\max S_x = u_{j_2}\) for some \(j_1, j_2, 1 \leq j_1 \leq j_2 \leq k\), then
   \[
   S_x = \{u_j : j_1 \leq j \leq j_2\}.
   \]
2. Let \(x', x'' \in X\), \(x' \leq x''\), and let
   \[
   S_{x'} = \{u_{i_1}, ..., u_{j_1}\} \quad \text{and} \quad S_{x''} = \{u_{i_2}, ..., u_{j_2}\}.
   \]

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Then $i_1 \leq i_2$ and $j_1 \leq j_2$.

These properties of the sets $S_x$ are immediate follow from part (i) of Proposition 3.12 and part (iv) of Proposition 8.11.

Note that the sets $\{S_x : x \in X\}$ have properties similar to properties (1)-(3) of Proposition 4.8. This enables us to put

$$H(x) = S_x \text{ whenever } \#S_x = m. \quad (4.23)$$

Let us define the set $H(x)$ for $x \in S$ such that $\#S_x < m$. Let

$$A = \{x \in X : \#S_x < m\}. \quad (4.25)$$

We know that for each $x \in A$ the set $S_x$ contains a point $s_x$ which is a limit point of $E$ possessing properties (i)-(ii) of Proposition 3.12. In particular, by part (i) of this proposition,

either $s_x = \min S_x$ or $s_x = \max S_x$. \quad (4.24)

Let

$$L = \{z \in X : z \text{ is a limit point of } E\}. \quad (4.26)$$

Given $z \in L$ let

$$K_z = \{x \in A : s_x = z\}. \quad (4.27)$$

Clearly, $z \in K_z$ because $s_x = z$ for every limit point $z$ of $E$.

Let us prove the following important property of the sets $\{K_z\}$: for every $z \in L$

either $\max K_z \leq z$ or $\min K_z \geq z$. \quad (4.28)

Indeed, let $x', x'' \in K_z$ and let $x', x'' \neq z$. Thus $z = s_{x'} = s_{x''}$. Suppose that $x' < x''$ and prove that $z \notin [x', x'']$. Since $z = s_{x''}$, by (3.44), whenever $x' < z$, the interval $(x', z)$ contains only a finite number of points of $E$ (because all these points belong to $S_{x'}$). In the same way we show that the interval $(z, x'')$ contains a finite number of points of $E$ provided $z < x''$. Thus if $z \in [x', x'']$ and $z \neq x', x''$, the point $z$ is an isolated point of $E$, a contradiction.

Hence we conclude that either $\min(z_1, z_2) > s_z$ or $\max(z_1, z_2) < z$ which implies the property (4.24).

Now let us fix a point $z \in L$ and define a set $H(x)$ for every $x \in K_z$. By property (4.28), it suffices to consider the following three cases:

**Case 1:** Suppose that $\#K_z > 1$ and

$$x > z \text{ for every } x \in K_z, \ x \neq z. \quad (4.29)$$

Note that, by (3.45),

$$[z, \max S_x] \cap E = S_x \text{ for every } x \in K_z,$$

and, by (3.43),

$$\max S_{x_1} \leq \max S_{x_2} \text{ whenever } x_1 \leq x_2, \ x_1, x_2 \in K_z.$$

Hence

$$S_{x_1} \subset S_{x_2} \text{ provided } x_1 \leq x_2, \ x_1, x_2 \in K_z.$$

This inclusion implies the following: Let $\tilde{x} = \max K_z$. Then

$$S_{\tilde{x}} \subset \max S_{\tilde{x}} \text{ for every } x \in K_z.$$

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K_x = [z, \bar{x}] \cap S_{\bar{x}}. \tag{4.27}

Hence \#K_x \leq \#S_{\bar{x}} \leq m.

Recall that \( X = \{x_1, \ldots, x_k\} \) where \( \{x_j\}_{j=1}^k \) is a strictly increasing sequence of points in \( E \). Let \( x_0 = x_1 \) and \( x_{k+1} = x_k \).

Fix a point \( x \in K_x \). Let \( j = j(x) \in \{1, \ldots, k\} \) be the index corresponding to \( x \) in the sequence \( \{x_j\} \), i.e., \( x = x_{j(x)} \). Let

\[
I_x = [x_{j-1}, x_j] \quad \text{provided} \quad j = j(x), 2 \leq j \leq k.
\]

Recall that inequality (3.39) holds which enables us to apply Lemma 3.14 to the interval \( I = I_x \) and the point \( x \in K_x \). By this lemma,

\[
\lim_{\delta_x \to z} \|L_{S'}[f] - P_x\|_{C^0(I_x)} = 0.
\]

Thus there exists a constant \( \delta_x = \tilde{\delta}_x(\varepsilon) > 0 \) satisfying the following condition: for every \( m \)-point set \( S' \) such that \( S_x \subset S' \subset E \) and \( S' \setminus S_x \subset (z - \tilde{\delta}_x, z + \tilde{\delta}_x) \) we have

\[
|P_x^{(j)}(y) - L_{S'}^{(j)}[f](y)| < \varepsilon/2 \quad \text{for every} \quad i, 0 \leq i \leq m - 1, \quad \text{and every} \quad y \in I_x.
\]

Let

\[
\tau = \frac{1}{4} \min_{i=1,\ldots,k-1} (x_{i+1} - x_i). \tag{4.28}
\]

Clearly,

\[
|x - y| > 2\tau \quad \text{provided} \quad x, y \in X, x \neq y. \tag{4.29}
\]

Let

\[
\delta_x = \min_{x \in K_x} (\tilde{\delta}_x, \tau). \tag{4.30}
\]

Clearly, since \( K_x \) is a finite set, the constant \( \delta_x > 0 \).

Such choice of \( \delta_x > 0 \) implies the following property: Let \( x = x_j \in K_x \) and let \( \bar{x} = x_{j-1} \). Then for an arbitrary \( m \)-point set \( S' \) such that

\[
S_x \subset S' \subset E \quad \text{and} \quad S' \setminus S_x \subset (z - \delta_x, z + \delta_x) \tag{4.31}
\]

and for every \( i, 0 \leq i \leq m - 1 \), the following inequality

\[
|P_x^{(i)}(y) - L_{S'}^{(i)}[f](y)| + |P_x^{(i)}(\bar{x}) - L_{S'}^{(i)}[f](\bar{x})| < \varepsilon
\]

holds.

Recall that \( \#K_x > 1 \) so that, by (4.27), the interval \([z, \bar{x}]\) contains a finite number of points of \( E \). Since \( z \) is a limit point of \( E \), the interval \((z - \delta_x, z)\) contains the infinite number of points of \( E \). Let us choose \( m - 1 \) points of \( E \) in this interval and denote this family of points by \( W(z) \). Let us enumerate \( W(z) \) in the increasing order, i.e., represent \( W(z) \) as

\[
W(z) = \{a_1, a_2, \ldots, a_{m-1}\}
\]

where \( \{a_j\}_{j=1}^{m-1} \) is a strictly increasing sequence. Thus

\[
a_1 < a_2 < \ldots < a_{m-1} < z \quad \text{and} \quad z - a_j < \delta_x, \quad j = 1, \ldots, m-1. \tag{4.33}
\]
Now let $x \in K_z$ and let $\ell_x = \#S_x$. We introduce a set $\overline{H}(x)$ associated to $x$ by letting

$$\overline{H}(x) = \{a_{\ell_x}, a_{\ell_x+1}, \ldots, a_{m-1}\}. \quad (4.34)$$

Clearly, $\#\overline{H}(x) = m - \#S_x$. Finally, we put

$$H(x) = \overline{H}(x) \cup S_x. \quad (4.35)$$

**Case 2:** $\#K_z > 1$ and $x < z$ for every $x \in K_z$, $x \neq z$.

In this case we construct the sets $\{H(x) : x \in K_z\}$ in a way similar to the preceding case. The only difference is that in this case $\{a_j\}_{j=1}^{m-1}$ is a sequence satisfying the following conditions:

$$z < a_1 < a_2 < \ldots < a_{m-1} \text{ and } a_j - z < \delta_z, \quad j = 1, \ldots, m-1. \quad (4.36)$$

In addition, for each $x \in K_z$ we put

$$\overline{H}(x) = \{a_1, a_2, \ldots, a_{m-\ell_x}\}. \quad (4.37)$$

The set $H(x)$ we define by the formula $(4.35)$.

**Case 3:** $\#K_z = 1$, i.e., $K_z = S_z = \{z\}$.

In this case $z \in X$ is a limit point of $E$ so that there exists an $m - 1$ point subset $W(z) = \{a_1, \ldots, a_{m-1}\} \subset E$ satisfying either $(4.33)$ or $(4.36)$. Then we define the set $H(z)$ using the same formula $(4.35)$. More specifically, in this case we put

$$H(z) = \{z, a_1, a_2, \ldots, a_{m-1}\}. \quad (4.38)$$

Thus the set $H(x)$ is well defined for all $x \in X$.

Let

$$V = \bigcup \{H(x) : x \in X\}. \quad (4.39)$$

Clearly, $V$ is a finite subset of $E$. Let us enumerate this set in the increasing order: thus we represent $V$ in the form

$$V = \{v_1, v_2, \ldots, v_\ell\}$$

where $\ell$ is a positive integer and $\{v_j\}_{j=1}^\ell$ is a strictly increasing sequence of points of $E$.

First of all, let us note that for every $x \in X$ the set $\overline{H}(x)$ defined by $(4.34)$, $(4.37)$ and $(4.38)$ possesses the following property:

$$\overline{H}(x) \subset (s_x - \tau, s_x + \tau). \quad (4.40)$$

This follows from $(4.33)$, $(4.36)$ and the inequality $\delta_z \leq \tau$, see $(4.30)$.

But, by definition $(4.28)$ of $\tau$ and by $(4.29)$,

$$(s_x - \tau, s_x + \tau) \cap (\cup \{H(y) : y \in X, y \neq x\}) = \emptyset \quad (4.41)$$

which implies the following representation of the set $V$:

$$V = S \cup (\cup \{W(z) : z \in L\}).$$

See $(4.21)$ and $(4.22)$.  

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Furthermore, the point \( s_x \) separates the sets \( \overline{H}(x) \) and \( S_x \), i.e., these sets lie on distinct sides of the real line with respect to \( s_x \). More specifically, by definitions (4.23), (4.34), (4.37) and (4.38), precisely one of the following conditions is satisfied:

(a) \( #S_x = m \) and \( \overline{H}(x) = \emptyset \). In this case either \( s_x = \min S_x \) or \( s_x = \max S_x \). See part (i) of Proposition 3.12.

(b) \( #S_x < m \) and \( \max \overline{H}(x) < s_x = \min S_x \). We refer to this case as Case 1.

(c) \( #S_x < m \) and \( s_x = \max S_x < \min \overline{H}(x) \). (This is Case 2).

(d) \( #S_x = 1 \) so that \( H(x) = \{x\} = s_x \). In this case either \( s_x > \max \overline{H}(x) \) or \( s_x < \min \overline{H}(x) \) (Case 3).

These properties enables us to prove the required properties of the sets \( \{H(x) : x \in X\} \) from Proposition 4.8. In particular, by (●1), each set \( S_x \) consists of at most \( m \) consecutive points of the sequence \( S = \{u_j\}_{j=1}^k \). See (4.22). Since for every \( x \in X \) the point \( s_x \) separates \( \overline{H}(x) \) and \( S_x \), by (4.40) and (4.41), the set \( H(x) \) consists of consecutive points of the sequence \( V = \{v_j\}_{j=1}^\ell \). See (4.39).

We prove that the family of sets \( \{H(x) : x \in X\} \) satisfies conditions (1)-(4) of Proposition 4.8.

**Proof of property (1).** Since \( x \in S_x \) for every \( x \in E \), by (4.35), \( x \in S_x \subset H(x) \) proving (1).

**Proof of property (2).** Let \( x', x'' \in X, x' < x'' \). Let us show that the inequality

\[
\min H(x'') < \min H(x')
\]  

leads to a contradiction.

Let

\[
M' = \min S_{x'} \quad \text{and} \quad M'' = \min S_{x''}.
\]

By part(iii) of Proposition 3.11,

\[
M' \leq M''.
\]

Since \( H(x) = \overline{H}(x) \cup S_{x'} \), see (4.35), \( \min H(x') \leq M' \) so that, by assumption (4.42),

\[
\min H(x'') < \min H(x') \leq M'.
\]

Hence,

\[
\min H(x'') < M''.
\]  

Analyzing the construction of the sets \( H(x), x \in X \) we see that inequality (4.44) holds only in Case 1 and Case 3 of this construction. See properties (a)-(d). More specifically, if (4.44) is true then \( #S_x < m \), \( s_{x''} \) is a limit point of \( E \) such that

\[
s_{x''} = M'' = \min S_{x''},
\]

and

\[
\overline{H}(x'') \neq \emptyset, \quad \overline{H}(x'') \subset (s_{x''} - \tau, s_{x''}).
\]

See (4.33) and (4.38).

Note that, by (3.43),

\[
M' = \min S_{x'} \leq M'' = \min S_{x''} = s_{x''}.
\]

See (4.45). We also note that, by (4.43), \( \min H(x'') < M' \).

On the other hand, by (4.46),

\[
s_{x''} - \tau < \min \overline{H}(x') \quad \text{so that} \quad s_{x''} - \tau < M'.
\]

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Since $M'$ and $s_{x''}$ belong to $X$, we have $|M' - s_{x''}| > \tau$ provided $M' \neq s_{x''}$. See (4.29). Hence, $s_{x''} \leq M'$ so that, by (4.47), $M' = s_{x''}$.

In particular, $M'$ is a limit point of $E$ (because $s_{x''}$ is) so that, by part (iii) of Proposition 3.11,

$$M' = \min S_{x'} = s_{x'}.$$

Thus $s_{x'} = s_{x''}$. Let $z = s_{x'} = s_{x''}$. Since $x' \in S_{x'}$, 

$$z = \min S_{x'} \leq x' < x''$$

so that, by (3.44), $S_{x'} \subset S_{x''}$. Hence,

$$\ell_{x'} = \#S_{x'} \leq \ell_{x''} = \#S_{x''} < m.$$

Thus $x', x'' \in K_x$, see (4.25). In particular, #K_x > 1 so that it suffices to consider Case 1 only.

By formula (4.34), in Case 1 for every $x \in K_x$

$$\min H(x) = \min \bar{H}(x) = a_{\ell_x}.$$ 

Since $\{a_j\}_{j=1}^{m-1}$ is a strictly increasing sequence, see (4.33), and $\ell_{x'} \leq \ell_{x''}$, we conclude that $a_{\ell_{x'}} \leq a_{\ell_{x''}}$. Hence we obtain an inequality

$$\min H(x') \leq \min H(x'')$$

which contradicts assumption (4.42). The proof of property (2) is complete.

**Proof of property (3).** Let $x', x'' \in X$, $x' \neq x''$, and let $H(x') \neq H(x'')$.

Note that, by (4.29), $|x' - x''| > 2\tau$. We also note that, by (4.35), $H(x) = \bar{H}(x) \cup S_x$ for every $x \in X$ so that 

$$\text{diam } H(x) \leq \text{diam } \bar{H}(x) + \text{diam } S_x.$$

Furthermore, the set $\bar{H}(x) = \emptyset$ whenever $\#S_x = m$, see (4.23). If $\#S_x < m$, in Cases 1-3 of our construction $\bar{H}(x) \subset (s_x - \tau, s_x + \tau)$. See (4.33) and (4.34). Hence,

$$\max(\text{diam } \bar{H}(x'), \text{diam } \bar{H}(x'')) \leq 2\tau < |x' - x''|$$

so that

$$\text{diam } H(x') \leq \text{diam } S_{x'} + |x' - x''| \quad \text{and} \quad \text{diam } H(x'') \leq \text{diam } S_{x''} + |x' - x''|.$$ 

(4.48)

We prove that the condition $H(x') \neq H(x'')$ implies that $S_{x'} \neq S_{x''}$ as well. Indeed, suppose that $S_{x'} = S_{x''} =: \tilde{S}$. If $\#\tilde{S} = m$ then $S_{x'} = H_{\tilde{x}'}$ and $S_{x''} = H_{\tilde{x}''}$, see (4.23). Thus $H(x') = H(x'')$, a contradiction.

Let now $\#\tilde{S} < m$. In this case $s_{x'}$ is the unique limit point of $E$ which belongs to $S_{x'}$, see part (ii) of Proposition 3.12. A similar statement is true for $s_{x''}$ and $S_{x''}$. Hence $s_{x'} = s_{x''} =: s$.

Note that in this situation the conditions of one of the Cases 1-3 are satisfied. Also note that in each of these cases the set $\bar{H}(x)$ depends only on the set $\tilde{S}$ and the point $s$ so that 

$$\bar{H}(x') = \bar{H}(x'') =: \bar{H}.$$ 

But this equality leads to a contradiction because in this case

$$H(x') = \bar{H}(x') \cup S_{x'} = \bar{H} \cup \tilde{S} = \bar{H}(x'') \cup S_{x''} = H(x'').$$

This contradiction proves that $S_{x'} \neq S_{x''}$ so that, by part (ii) of Proposition 3.12

$$\text{diam } S_{x'} + \text{diam } S_{x''} \leq 2m|x' - x''|.$$
Combining this inequality with inequalities (4.48) we obtain the required inequality (4.19).

- **Proof of property (4).** Let \( x = x_j \), \( \tilde{x} = x_{j-1} \), and \( S' = H(x) \). Then, by (4.35) and (4.56),

\[
S_x \subset S' \subset E \quad \text{and} \quad S' \setminus S_x = \overline{H(x)} \subset (z - \delta_z, z + \delta_z)
\]

where \( z = s_x \). Thus the condition (4.31) is satisfied so that inequality (4.32) holds. This inequality coincides with inequality (4.20) proving property (4).

The proof of Proposition 4.8 is complete. \( \square \)

**Lemma 4.9** Let \( f : E \to \mathbb{R} \) be a function satisfying Assumption 4.7 and let \( P^{(m,E)}[f] \) be the Whitney \((m - 1)\)-field determined by (4.16). Then for every finite strictly increasing sequence \( \{x_j\}^{k}_{j=1} \subset E \) inequality (4.20) holds.

**Proof.** We put \( x_0 = x_1 \) and \( x_{k+1} = x_k \). Let us fix \( \varepsilon > 0 \) and apply Proposition 4.8 to the set \( X = \{x_1, \ldots, x_k\} \) and the Whitney \((m - 1)\)-field \( P^{(m,E)}[f] \). By this proposition there exist a finite strictly increasing sequence \( V = \{v_j\}^{k}_{j=1} \subset E \) and a mapping \( H : X \to 2^V \) which to every \( x \in X \) assigns \( m \) consecutive points

\[
H(x) = \{v_{j_1}(x), \ldots, v_{j_3}(x)\}, \quad 1 \leq j_1(x) \leq j_2(x) = j_1(x) + m - 1 \leq \ell,
\]

of \( V \) such that conditions (1)-(4) of this proposition are satisfied.

Using property (4) let us replace the Hermite polynomials \( \{P_{x_j} : j = 1, \ldots, k\} \) in the left hand side of inequality (4.17) by corresponding Lagrange polynomials \( L_{H(x)} \). For every \( j = 1, \ldots, k \) and \( i = 0, \ldots, m-1 \) we have

\[
|P^{(i)}_{x_j}(x_j) - P^{(i)}_{x_{j+1}}(x_j)| \leq |P^{(i)}_{x_j}(x_j) - L^{(i)}_{H(x)}[f](x_j)| + |L^{(i)}_{H(x)}[f](x_j) - L^{(i)}_{H(x_{j+1})}[f](x_j)|
\]

so that, by the property (4),

\[
|P^{(i)}_{x_j}(x_j) - P^{(i)}_{x_{j+1}}(x_j)| \leq |L^{(i)}_{H(x)}[f](x_j) - L^{(i)}_{H(x_{j+1})}[f](x_j)| + 2\varepsilon.
\]

See (4.20). Hence,

\[
|P^{(i)}_{x_j}(x_j) - P^{(i)}_{x_{j+1}}(x_j)|^p \leq 2^p |L^{(i)}_{H(x)}[f](x_j) - L^{(i)}_{H(x_{j+1})}[f](x_j)|^p + 4^p \varepsilon^p
\]

so that

\[
A_1 \leq 2^p A_2 + \varepsilon^p A_3 \tag{4.49}
\]

where

\[
A_1 = \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{|P^{(i)}_{x_{j+1}}(x_j) - P^{(i)}_{x_j}(x_j)|^p}{(x_{j+1} - x_j)^{(m-i)p-1}}, \quad A_2 = \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{|L^{(i)}_{H(x_{j+1})}[f](x_j) - L^{(i)}_{H(x)}[f](x_j)|^p}{(x_{j+1} - x_j)^{(m-i)p-1}}
\]

and

\[
A_3 = 4^p \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} (x_{j+1} - x_j)^{(1-(m-i)p)}.
\]

Now let \( 1 \leq j \leq k - 1 \) and let \( H(x_j) \neq H(x_{j+1}) \). Since

\[
x_j \in H(x_j) \quad \text{and} \quad x_{j+1} \in H(x_{j+1}) \tag{4.50}
\]

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see property (1) of Proposition [4.8] by (4.19),
\[
\text{diam}(H(x_j) \cup H(x_{j+1})) \leq \text{diam } H(x_j) + \text{diam } H(x_{j+1}) + (x_{j+1} - x_j)
\leq 2(m + 1)(x_{j+1} - x_j) + (x_{j+1} - x_j) = (2m + 3)(x_{j+1} - x_j).
\]
Thus for every \( j = 1, \ldots, k - 1 \), and every \( i = 0, \ldots, m - 1 \),
\[
\frac{|L_{H(x_{j+1})}^{(i)}[f](x_j) - L_{H(x_j)}^{(i)}[f](x_j)|^p}{(x_{j+1} - x_j)^{(m-i)p-1}} \leq (2m + 3)^{(m-i)p-1} \frac{|L_{H(x_{j+1})}^{(i)}[f](x_j) - L_{H(x_j)}^{(i)}[f](x_j)|^p}{(\text{diam}(H(x_j) \cup H(x_{j+1})))^{(m-i)p-1}}
\]
so that, by (4.50),
\[
A_2 \leq C(m, p) \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \max_{i} |L_{H(x_{j+1})}^{(i)}[f] - L_{H(x_j)}^{(i)}[f]|^p (\text{diam}(H(x_j) \cup H(x_{j+1})))^{(m-i)p-1}
\]
where
\[
I_j = [\min(H(x_j) \cup H(x_{j+1})), \max(H(x_j) \cup H(x_{j+1}))]
\]
is the smallest closed interval containing \( H(x_j) \cup H(x_{j+1}) \). Clearly,
\[
\text{diam } I_j = \text{diam}(H(x_j) \cup H(x_{j+1})).
\]
Without loss of the generality we may assume that
\[
H(x_j) \neq H(x_{j+1}) \text{ for all } j = 1, \ldots, k - 1.
\]
Recall that, by (4.18), for every \( j = 1, \ldots, k - 1 \), the set \( H(x_j) \) consists of \( m \) consecutive points of the sequence \( V = \{v_n\}_{n=1}^\ell \subset E \), see Proposition [4.8]. Let \( n_j \) be the index of the minimal point of \( H(x_j) \) in the sequence \( V \). Thus
\[
H(x_j) = \{v_{n_j}, \ldots, v_{n_j+m-1}\}, \quad j = 1, \ldots, k - 1.
\]
Note that, by (4.53) and property (2) of Proposition [4.8], \( \{v_{n_j}\}_{j=1}^{k-1} \) is a strictly increasing subsequence of the sequence \( V \).
Let us apply Lemma [3.7] with \( k = m - 1 \) to the sets
\[
S_1 = H(x_j) = \{v_{n_j}, \ldots, v_{n_j+m-1}\}, \quad S_2 = H(x_{j+1}) = \{v_{n_j+1}, \ldots, v_{n_j+m-1}\},
\]
and the closed interval \( I = I_j = [v_{n_j}, v_{n_j+m-1}] \). See (4.51).
Let
\[
S_j^{(n)} = \{v_{n_j}, \ldots, v_{n+m-1}\}, \quad n_j \leq n \leq n_{j+1}.
\]
Then, by this lemma,
\[
\max_{I_j} |L_{H(x_{j+1})}^{(i)}[f] - L_{H(x_j)}^{(i)}[f]|^p \leq ((m + 2)!)^p (\text{diam } I_j)^{(m-i)p-1} \sum_{n=n_j}^{n_{j+1}} |\Delta^m f[S_j^{(n)}]|^p \text{ diam } S_j^{(n)}.
\]
This inequality and (4.52) imply the following estimate of \( A_2 \):
\[
A_2 \leq C(m, p) \sum_{j=1}^{k-1} \sum_{n=n_j}^{n_{j+1}} |\Delta^m f[S_j^{(n)}]|^p \text{ diam } S_j^{(n)}.
\]
Corollary 4.10 Let \( f \) be a function satisfying Assumption 4.7. Then

\[
\mathcal{N}_{m,p,E}(\mathbf{P}^{(m,E)}[f]) \leq C \lambda
\]

where \( C \) is a constant depending only on \( m \) and \( p \).

Proof of the sufficiency part of Theorem 4.3 Let \( f \) be a function on \( E \) such that the \( \mathcal{W}_{m,p}(f : E) < \infty \). See (1.7). Let \( \lambda = \mathcal{W}_{m,p}(f : E) \); thus, \( \lambda < \infty \).

By (1.7), Assumption 4.7 holds for \( f \) and \( \lambda \). Therefore, by Corollary 4.10

\[
\mathcal{N}_{m,p,E}(\mathbf{P}^{(m,E)}[f]) \leq C(m,p) \lambda = C(m,p) \mathcal{W}_{m,p}(f : E).
\]

By Theorem 4.1

\[
\|\mathbf{P}^{(m,E)}[f]\|_{m,p,E} \sim \mathcal{N}_{m,p,E}(\mathbf{P}^{(m,E)}[f])
\]

so that

\[
\|\mathbf{P}^{(m,E)}[f]\|_{m,p,E} \leq C(m,p) \mathcal{W}_{m,p}(f : E).
\]

We recall that the quantity \( \|f\|_{m,p,E} \) is defined by (4.2). This definition implies the existence of a function \( F \in L_{p}^{m}(\mathbb{R}) \) such that \( T_{x}^{m-1}[F] = P_{x} \) on \( E \) and

\[
\|F\|_{L_{p}^{m}(\mathbb{R})} \leq 2 \|\mathbf{P}^{(m,E)}[f]\|_{m,p,E} \leq C(m,p) \mathcal{W}_{m,p}(f : E).
\]

Since \( P_{x}(x) = f(x) \) on \( E \), see (4.54), we have

\[
F(x) = T_{x}^{m-1}[F](x) = P_{x}(x) = f(x) \quad \text{for all} \quad x \in E.
\]

Thus \( F \in L_{p}^{m}(\mathbb{R}) \) and \( F|_{E} = f \) proving that \( f \in L_{p}^{m}(\mathbb{R})|_{E} \). Furthermore, by (4.55) and (1.1),

\[
\|f\|_{L_{p}^{m}(\mathbb{R})|_{E}} \leq \|F\|_{L_{p}^{m}(\mathbb{R})} \leq C(m,p) \mathcal{W}_{m,p}(f : E).
\]

The proof of Theorem 4.3 is complete. \( \square \)
5. The Finiteness Principal for $L^m_\infty(\mathbb{R})$ traces: multiplicative finiteness constants.

5.1. Multiplicative finiteness constant of the space $L^m_\infty(\mathbb{R})$.

Let $m \in \mathbb{N}$. Everywhere in this section we assume that $E$ is a closed subset of $\mathbb{R}$ with $\#E \geq m + 1$.

We will discuss equivalence (1.4) which states that

$$\|f\|_{L^m_\infty(\mathbb{R})|_E} \sim \sup_{S \subset E, \#S = m + 1} |\Delta^m f[S]|$$

for every function $f \in L^m_\infty(\mathbb{R})|_E$. The constants in this equivalence depend only on $m$.

We interpret this equivalence as a special case of the Finiteness Principle which we mentioned in the Introduction. Let us formulate this principle for the space $L^m_\infty(\mathbb{R})$.

**Theorem 5.1** Let $m \in \mathbb{N}$. There exists a constant $\gamma = \gamma(m) > 0$ depending only on $m$, such that the following holds: Let $E \subset \mathbb{R}$ be a closed set, and let $f : E \to \mathbb{R}$.

For every subset $E' \subset E$ with at most $N = m + 1$ points, suppose there exists a function $F_{E'} \in L^m_\infty(\mathbb{R})$ with the seminorm $\|F_{E'}\|_{L^m_\infty(\mathbb{R})} \leq 1$, such that $F_{E'} = f$ on $E'$.

Then there exists a function $F \in L^m_\infty(\mathbb{R})$ with the seminorm $\|F\|_{L^m_\infty(\mathbb{R})} \leq \gamma$ such that $F = f$ on $E$.

**Proof.** Applying equivalence (5.1) to an arbitrary set $E = S$ with $\#S = m + 1$ we conclude that

$$\|f\|_{L^m_\infty(\mathbb{R})|_S} \sim |\Delta^m f[S]|.$$

Hence, by the theorem’s hypothesis, $|\Delta^m f[S]| \leq C_1(m)$ for every $S \subset E$ with $\#S = m + 1$. By this inequality, $\mathcal{W}_{m,\infty}(f : E) \leq C_1(m)$. See (1.3). Therefore, by the Whitney’s extension theorem [57], $f$ belongs to the trace space $L^m_\infty(\mathbb{R})|_E$. Furthermore, by (1.4),

$$\|f\|_{L^m_\infty(\mathbb{R})|_E} \leq C_2(m)\mathcal{W}_{m,\infty}(f : E) \leq C_2(m)C_1(m) = \gamma(m),$$

and the proof of Theorem 5.1 is complete. □

We refer to the constant $N = m + 1$ as the finiteness constant of the space $L^m_\infty(\mathbb{R})$. Clearly, the value $N(m) = m + 1$ in the finiteness Theorem 5.1 is sharp; in other words, Theorem 5.1 is false in general if $N = m + 1$ is replaced by some number $N < m + 1$.

Theorem 5.1 implies the following inequality: For every $f \in L^m_\infty(\mathbb{R})|_E$ we have

$$\|f\|_{L^m_\infty(\mathbb{R})|_E} \leq \gamma(m) \sup_{S \subset E, \#S = m + 1} \|f\|_{L^m_\infty(\mathbb{R})|_S}$$

(5.2)

(Clearly, the converse inequality is trivial and holds with $\gamma(m) = 1$.) Inequality (5.2) motivates us to call the constant $\gamma = \gamma(m)$ the multiplicative finiteness constant for the space $L^m_\infty(\mathbb{R})$.

The following natural question arises:

**Question 5.2** What is the sharp value of the multiplicative finiteness constant for $L^m_\infty(\mathbb{R})$?

We denote this sharp value of $\gamma(m)$ by $\gamma^\sharp(L^m_\infty(\mathbb{R}))$. Thus,

$$\gamma^\sharp(L^m_\infty(\mathbb{R})) = \sup_{\|f\|_{L^m_\infty(\mathbb{R})|_E}} \frac{\|f\|_{L^m_\infty(\mathbb{R})|_E}}{\sup\|f\|_{L^m_\infty(\mathbb{R})|_S} : S \subset E, \#S = m + 1}$$

(5.3)

where the supremum is taken over all closed sets $E \subset \mathbb{R}$ with $\#E \geq m + 1$, and all functions $f \in L^m_\infty(\mathbb{R})|_E$.

The next theorem answers to Question 5.2 for $m = 1, 2$ and provides lower and upper bounds for $\gamma^\sharp(L^m_\infty(\mathbb{R}))$ for $m > 2$. These estimates show that $\gamma^\sharp(L^m_\infty(\mathbb{R}))$ grows exponentially as $m \to +\infty$. 

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Theorem 5.3 We have:

(i) \( \gamma^2(L^1_\infty(\mathbb{R})) = 1 \) and \( \gamma^2(L^2_\infty(\mathbb{R})) = 2 \).

(ii) For every \( m \in \mathbb{N}, m > 2 \), the following inequalities

\[
\left( \frac{\pi}{2} \right)^{m-1} < \gamma^2(L^m_\infty(\mathbb{R})) < (m - 1)9^m
\]  

(5.4)

hold.

The proof is based on works \([17], [12]\) and \([13]\) devoted to calculation of a certain constant related to optimal extensions of \( L^m_\infty(\mathbb{R}) \)-functions. This constant is defined by

\[
K(m) = \sup_n \frac{\|f\|_{L^m_\infty(\mathbb{R})}}{\max \{ m! |\Delta^m f(x_i, ..., x_{i+n})| : i = 1, ..., n \}}
\]  

(5.5)

where the supremum is taken over all \( n \in \mathbb{N} \), all finite strictly increasing sequences \( X = \{x_1, ..., x_{m+n}\} \subset \mathbb{R} \), and all functions \( f \in L^m_\infty(\mathbb{R}) \).

The constant \( K(m) \) was introduced by Favard \([17]\). (See also \([12], [13]\).) Favard \([17]\) proved that \( K(2) = 2 \), and de Boor found efficient lower and upper bounds for \( K(m) \).

We prove that \( \gamma^2(L^m_\infty(\mathbb{R})) = K(m) \), see Proposition \( 5.29 \) below. This enables us to use the results of Favard and de Boor, and obtain in this way the required lower and upper bounds for \( \gamma^2(L^m_\infty(\mathbb{R})) \).

We will need three auxiliary lemmas.

Lemma 5.4 Let \( S \subset \mathbb{R}, \#S = m + 1 \), and let \( f : S \to \mathbb{R} \). Then

\[
\|f\|_{L^m_\infty(\mathbb{R})} = m! |\Delta^m f[S]|.
\]

Proof. Let \( A = \min S \), \( B = \max S \). Let \( F \in L^m_\infty(\mathbb{R}) \) be an arbitrary function such that \( F|_S = f \). Then, by (2.8) and (2.9),

\[
m! |\Delta^m f[S]| = m! |\Delta^m F[S]| = \int_A^B M_m[S](t)E^{(m)}(t)dt \leq \left( \int_A^B M_m[S](t)dt \right) \|F\|_{L^m_\infty(\mathbb{R})} = \|F\|_{L^m_\infty(\mathbb{R})}.
\]

Taking the infimum in this inequality over all functions \( F \in L^m_\infty(\mathbb{R}) \) such that \( F|_S = f \), we obtain:

\[
m! |\Delta^m f[S]| \leq \|f\|_{L^m_\infty(\mathbb{R})}.
\]

Let us prove the converse inequality. Let \( F = L_S[f] \) be the Lagrange polynomial of degree at most \( m \) interpolating \( f \) on \( S \). Then, by (2.4),

\[
m! |\Delta^m f[S]| = |L_S^{(m)}[f]| = \|F\|_{L^m_\infty(\mathbb{R})}.
\]

Since \( F|_S = f \), we obtain

\[
\|f\|_{L^m_\infty(\mathbb{R})} \leq \|F\|_{L^m_\infty(\mathbb{R})} = m! |\Delta^m f[S]|
\]

proving the lemma. \( \Box \)

Lemma 5.5 (\([28]\), p. 15) Let \( E = \{x_1, ..., x_{m+n}\} \subset \mathbb{R}, n \in \mathbb{N} \), be a strictly increasing sequence, and let \( f : E \to \mathbb{R} \). Then for every subset \( S \subset E \) with \( \#S = m + 1 \) there exist numbers \( \alpha_i \in \mathbb{R}, \alpha_i \geq 0, i = 1, ..., n \), such that \( \alpha_1 + ... + \alpha_n = 1 \) and

\[
\Delta^m f[S] = \sum_{i=1}^n \alpha_i \Delta^m f[x_i, ..., x_{i+n}].
\]

This lemma and Lemma 5.4 imply two corollaries.
**Corollary 5.6** Let \( E = \{x_j\}_{j=-\infty}^\infty \) be a strictly increasing sequence in \( \mathbb{R} \), and let \( f : E \to \mathbb{R} \). Then
\[
\sup\{|f|_{L^\infty_m(\mathbb{R})_E} : S \subset E, \#S = m + 1\} = \sup_{i} m! |\Delta^m f(x_i, ..., x_{i+m})|.
\]

**Corollary 5.7** Let \( E = \{x_1, ..., x_{m+n}\} \subset \mathbb{R}, \ n \in \mathbb{N}, \) be a strictly increasing sequence, and let \( f : E \to \mathbb{R} \). Then
\[
\max_{S \subset E, \#S = m+1} |\Delta^m f[S]| = \max_{i=1, ..., n} |\Delta^m f[x_i, ..., x_{i+m}]|.
\]

**Lemma 5.8** Let \( E \subset \mathbb{R} \) be a closed set, and let \( f \in L^m_{\infty}(\mathbb{R})_E \). Then
\[
||f||_{L^m_{\infty}(\mathbb{R})_E} = \sup_{E' \subset E, \#E' < \infty} ||f||_{L^m_{\infty}(\mathbb{R})_{E'}}.
\]

**Proof.** The lemma readily follows from the following well known fact: for every closed bounded interval \( I \subset \mathbb{R} \) a ball in the space \( L^m_{\infty}(I) \) is a precompact subset in the space \( C(I) \). We leave the details to the interested reader. \( \Box \)

**Proposition 5.9** For every \( m \in \mathbb{N} \) the following equality
\[
\gamma^\sharp(L^\infty_{\infty}(\mathbb{R})) = K(m)
\]
holds.

**Proof.** By Corollary 5.7 Lemma 5.4 and definition (5.5),
\[
K(m) = \sup_{E \subset \mathbb{R}} \frac{||f||_{L^m_{\infty}(\mathbb{R})_E}}{\sup\{|f|_{L^\infty_m(\mathbb{R})_E} : S \subset E, \#S = m + 1\}} \leq \gamma^\sharp(L^\infty_{\infty}(\mathbb{R})).
\]

where the supremum is taken over all finite subsets \( E \subset \mathbb{R} \) and all functions \( f \) on \( E \). Comparing (5.6) with (5.3), we conclude that \( K(m) \leq \gamma^\sharp(L^\infty_{\infty}(\mathbb{R})). \)

Let us prove the converse inequality. By Lemma 5.8 and (5.6), for every closed set \( E \subset \mathbb{R} \) and every function \( f \in L^m_{\infty}(\mathbb{R})_E \) we have the following:
\[
||f||_{L^m_{\infty}(\mathbb{R})_E} = \sup_{E' \subset E, \#E' < \infty} ||f||_{L^m_{\infty}(\mathbb{R})_{E'}} \leq \sup_{E' \subset E, \#E' < \infty} K(M) \sup_{S \subset E', \#S = m+1} ||f||_{L^\infty_m(\mathbb{R})_E} = K(M) \sup_{S \subset E, \#S = m+1} ||f||_{L^\infty_m(\mathbb{R})_E}.
\]

This inequality and definition (5.3) imply the required inequality \( \gamma^\sharp(L^\infty_{\infty}(\mathbb{R})) \leq K(m) \) proving the proposition. \( \Box \)

**Proof of Theorem 5.7** The equality \( \gamma^\sharp(L^1_{\infty}(\mathbb{R})) = 1 \) is immediate from the well known fact that a function satisfying the Lipschitz condition on a subset of \( \mathbb{R} \) can be extended to all of \( \mathbb{R} \) with preservation of the Lipschitz constant.

As we have mentioned above,
\[
K(2) = 2
\]
due to a result of Favard [17], de Boor [12, 13] proved that
\[
\left( \frac{\pi}{2} \right)^{m-1} < c_m \leq K(m) \leq C_m < (m - 1) 9^m \quad \text{for each} \quad m > 2.
\]
\[ c_m = \left( \frac{\pi}{2} \right)^{m+1} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^{m+1}} \]
and
\[ C_m = 2^{m-2}/m + \sum_{i=1}^{m} \binom{m}{i} \binom{m-1}{i-1} 4^{m-i}. \]

But, by Proposition 5.9, \( \gamma^\#(L_\infty^m(\mathbb{R})) = K(m) \), so that, by (5.7) and (5.8), the statements (i) and (ii) of the theorem hold.

The proof of Theorem 5.1 is complete. \( \square \)

5.2. Further remarks and comments.

- An extremal function for the lower bound in Theorem 5.1

First we note that, by (5.8) and Proposition 5.9,
\[ c_m \leq \gamma^\#(L_\infty^m(\mathbb{R})) \leq C_m \quad \text{for every} \quad m \in \mathbb{N} \] (5.9)

which slightly improves the lower and upper bounds for \( \gamma^\#(L_\infty^m(\mathbb{R})) \) given in (5.4).

Since
\[ \lim_{m \to \infty} \frac{c_m}{(\pi/2)^{m+1}} = 1/2, \]
see [12], we obtain the following asymptotic lower bound for \( \gamma^\#(L_\infty^m(\mathbb{R})) \):
\[ \lim_{m \to \infty} \frac{\gamma^\#(L_\infty^m(\mathbb{R}))}{(\pi/2)^{m+1}} \geq 1/2. \]

We also note that, the proof of the inequality \( c_m \leq K(m) (= \gamma^\#(L_\infty^m(\mathbb{R}))) \) given in [12], is constructive, i.e., this proof provides an explicit formula for a function on \( \mathbb{R} \) and a set \( E \) for which this lower bound for \( K(m) \) is attained.

More specifically, let \( E = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \) be the set of all integers, and let \( f : E \to \mathbb{R} \) be a function on \( E \) defined by
\[ f(i) = (-1)^i \quad \text{for every} \quad i \in E. \]

We let \( E_m : \mathbb{R} \to \mathbb{R} \) denote the Euler spline introduced by Schoenberg [46]:
\[ E_m(t) = c_m \sum_{i \in \mathbb{Z}} (-1)^i M_{m+1}[S_i](t + (m + 1)/2), \quad t \in \mathbb{R}. \]

Here \( S_i = \{i, \ldots, i + m + 1\} \), and \( M_k[S] \) is the B-spline defined by formula (2.6).

We refer the reader to the monograph [46] for various remarkable properties of this spline. In particular, \( E_m \in C^{m-1}(\mathbb{R}) \). If \( m \) is odd, \( E_m \) is a polynomial of degree at most \( m \) on every interval \([i, i + 1], i \in \mathbb{Z}\); if \( m \) is even, the same true on every interval \([i - 1/2, i + 1/2]\).

Furthermore, \( \|E_m\|_{L_\infty^m(\mathbb{R})} = c_m 2^m \), and \( E_m(i) = (-1)^i, i \in \mathbb{Z} \), so that
\[ \|E_m\|_E = f, \]
and
\[ m! |\Delta^m f[i, \ldots, i + m]| = 2^m \quad \text{for every} \quad i \in \mathbb{Z}. \] (5.10)
It is also proven in [12] that
\[ \|E_m\|_{L^\infty(R^n)} \leq \|F\|_{L^\infty(R^n)} \]
for every function \( F \in L^m_{\infty}(R^n) \) such that \( F|_E = f \). This enables us to calculate the trace norm of \( f \) in \( L^m_{\infty}(R^n) \):
\[ \|f\|_{L^m_{\infty}(R^n)|_E} = \|E_m\|_{L^m_{\infty}(R^n)} = c_m 2^m. \]

This equality, (5.10), definition (5.3) and Corollary 5.6 imply the following inequality
\[ \gamma(L^m_{\infty}(R^n)) \geq \frac{\|f\|_{L^m_{\infty}(R^n)|_E}}{\sup\{||f||_{L^m_{\infty}(R^n)} : S \subset E, \#S = m + 1\}} = \frac{\|f\|_{L^m_{\infty}(R^n)|_E}}{\sup_i m!\|\Delta^m f[x_i, \ldots, x_{i+m}]\|} = c_m \]
proving the first inequality in (5.9).

Note that \( c_2 = 2 \), therefore \( \gamma(L^2_{\infty}(R^n)) \geq 2 \). Favard [17] proved that \( K(2) \leq 2 \), so that, by Lemma 5.9, \( \gamma(L^2_{\infty}(R^n)) \leq 2 \). Hence, \( \gamma(L^2_{\infty}(R^n)) = 2 \) proving the second equality in part (i) of Theorem 5.3.

- Finiteness constants and multiplicative finiteness constants for the space \( L^m_{\infty}(R^n) \).
  
  We identify the homogeneous Sobolev space \( L^m_{\infty}(R^n) \) with the space \( C^{m-1,1}(R^n) \) of all \( C^{m-1} \)-functions on \( R^n \) whose partial derivatives of order \( m - 1 \) satisfy the Lipschitz condition on \( R^n \). \( L^m_{\infty}(R^n) \) is seminormed by

\[ \|F\|_{L^m_{\infty}(R^n)} = \left\| \nabla^m F \right\|_{L^\infty(R^n)} \text{ where } \nabla^m F = \left( \sum_{|\alpha| = m} |D^\alpha F|^2 \right)^{1/2}. \]

We let \( W^m_{\infty}(R^n) \) denote the corresponding Sobolev space of all functions \( F \in L^m_{\infty}(R^n) \) equipped with the norm

\[ \|F\|_{W^m_{\infty}(R^n)} = \left\| \max_{k=0, \ldots, m} \nabla^k F \right\|_{L^\infty(R^n)}. \]

We recall the Finiteness Principle for the space \( L^m_{\infty}(R^n) \):

**Theorem 5.10** There exist a positive integer \( N \) and a constant \( \gamma > 0 \) depending only on \( m \) and \( n \), such that the following holds: Let \( E \subset R^n \) be a closed set, and let \( f : E \to R \).
  
  For every subset \( E' \subset E \) with at most \( N \) points, suppose there exists a function \( F_{E'} \in L^m_{\infty}(R^n) \) with \( \|F_{E'}\|_{L^m_{\infty}(R^n)} \leq 1 \), such that \( F_{E'} = f \) on \( E' \). Then there exists a function \( F \in L^m_{\infty}(R^n) \) with \( \|F\|_{L^m_{\infty}(R^n)} \leq \gamma \) such that \( F = f \) on \( E \).

See [48] for the case \( m = 2 \), and [13] for the general case \( m \in \mathbb{N} \). Note that the Finiteness Principle for the space \( W^m_{\infty}(R^n) \) holds as well. See [18].

We say that a number \( N \in \mathbb{N} \) is a finiteness constant (for the space \( L^m_{\infty}(R^n) \)) if there exists a constant \( \gamma > 0 \) such that the Finiteness Principle formulated above holds with \( N \) and \( \gamma \). In this case for every function \( f \in L^m_{\infty}(R^n)|_E \) the following inequality

\[ \|f\|_{L^m_{\infty}(R^n)|_E} \leq \gamma \sup_{S \subset E, \#S \leq N} \|f||_{L^m_{\infty}(R^n)|_S} \quad (5.11) \]
holds.

We let \( N^\sharp(L^m_{\infty}(R^n)) \) denote the sharp finiteness constant for the space \( L^m_{\infty}(R^n) \). In the same way we introduce the notion of the finiteness constant and the sharp finiteness constant for the space \( W^m_{\infty}(R^n) \).

As we have noted above, \( N^\sharp(L^2_{\infty}(R^n)) = m + 1 \). It was proven in [48] that \( N^\sharp(L^2_{\infty}(R^n)) = 3 \cdot 2^{n-1} \). We also know that \( N^\sharp(L^m_{\infty}(R^n)) \leq 2^{m-1} \), see [2] and [49]. (This is the smallest upper bound for \( N^\sharp(L^m_{\infty}(R^n)), n > 2 \), known to the moment.) For a certain conjecture related to the sharp finiteness constant \( N^\sharp(L^m_{\infty}(R^n)) \) we refer the reader to [49].
Given a finiteness constant $N$ (for $L^m_{\infty}(\mathbb{R}^n)$) we let $\gamma^d(N; L^m_{\infty}(\mathbb{R}^n))$ denote the infimum of constants $\gamma$ from inequality (5.11). We refer to $\gamma^d(N; L^m_{\infty}(\mathbb{R}^n))$ as the multiplicative finiteness constant associated with the finiteness constant $N$. Thus,

$$\gamma^d(N; L^m_{\infty}(\mathbb{R}^n)) = \sup \frac{\|f\|_{L^m_{\infty}(\mathbb{R}^n)|_E}}{\sup\|f\|_{L^m_{\infty}(\mathbb{R}^n)} : S \subset E, \#S \leq N}$$

where the supremum is taken over all closed sets $E \subset \mathbb{R}^n$ and all functions $f \in L^m_{\infty}(\mathbb{R}^n)|_E$. Clearly, $\gamma^d(N; L^m_{\infty}(\mathbb{R}^n))$ is a non-increasing function of $N$.

If $N = N^d(L^m_{\infty}(\mathbb{R}^n))$, we write $\gamma^d(L^m_{\infty}(\mathbb{R}^n))$ rather than $\gamma^d(N^d(L^m_{\infty}(\mathbb{R}^n)); L^m_{\infty}(\mathbb{R}^n))$. We refer to $\gamma^d(L^m_{\infty}(\mathbb{R}^n))$ as the sharp multiplicative finiteness constant (for $L^m_{\infty}(\mathbb{R}^n)$).

In the same fashion we introduce the constants $\gamma^d(N; W^m_{\infty}(\mathbb{R}^n))$ and $\gamma^d(W^m_{\infty}(\mathbb{R}^n))$.

Our knowledge about the constants $\gamma^d(N; \cdot)$ and $\gamma^d(\cdot)$ is very restricted. Actually, besides the results related to $\gamma^d(L^m_{\infty}(\mathbb{R}))$ which we present in Section 5.1, nothing is known about the behaviour of the sharp multiplicative finiteness constants for $n > 1$.

Concerning the behaviour of the multiplicative finiteness constants as functions of $N$, we note the following interesting and surprising result obtained by Fefferman and Klartag [25]:

**Theorem 5.11** For any positive integer $N$, there exists a finite set $E \subset \mathbb{R}^2$ and a function $f : E \to \mathbb{R}$ with the following properties

1. For any $W^2_{\infty}$-function $F : \mathbb{R}^2 \to \mathbb{R}$ with $F|_E = f$ we have that $\|F\|_{W^2_{\infty}(\mathbb{R}^2)} > 1 + c_0$.
2. For any subset $S \subset E$ with $\#S \leq N$, there exists an $W^2_{\infty}$-function $F_S : \mathbb{R}^2 \to \mathbb{R}$ with $F_S|_S = f$ and $\|F_S\|_{W^2_{\infty}(\mathbb{R}^2)} \leq 1$.

Here, $c_0 > 0$ is a universal constant.

This result admits the following equivalent reformulation in terms of multiplicative finiteness constants for the space $W^2_{\infty}(\mathbb{R}^2)$: The following inequality

$$\inf_{N \in \mathbb{N}} \gamma^d(N; W^2_{\infty}(\mathbb{R}^2)) > 1$$

holds.

Since $\gamma^d(N; W^2_{\infty}(\mathbb{R}^2))$ is a non-increasing function of $N$, the above inequality implies the following:

$$\lim_{N \to \infty} \gamma^d(N; W^2_{\infty}(\mathbb{R}^2)) > 1.$$

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