The hybrid spectral test

Peter Hellekalek

Abstract. The starting point of this paper is the interplay between the construction principle of a sequence and the characters of the compact abelian group that underlies the construction. In case of the Halton sequence in base \( b = (b_1, \ldots, b_s) \) in the \( s \)-dimensional unit cube \([0, 1)^s\), which is an important type of a digital sequence, this kind of duality principle leads to the so-called \( b \)-adic function system and provides the basis for the \( b \)-adic method, which we present in connection with hybrid sequences. This method employs structural properties of the compact group of \( b \)-adic integers as well as \( b \)-adic arithmetic to derive tools for the analysis of the uniform distribution of sequences in \([0, 1)^s\).

We first clarify the point which function systems are needed to analyze digital sequences. Then, we present the hybrid spectral test in terms of trigonometric-, Walsh-, and \( b \)-adic functions. Various notions of diaphony as well as many figures of merit for rank-1 quadrature rules in Quasi-Monte Carlo integration and for certain linear types of pseudo-random number generators are included in this measure of uniform distribution. Further, discrepancy may be approximated arbitrarily close by suitable versions of the spectral test.

Keywords. Uniform distribution of sequences, discrepancy, diaphony, Fourier series, Walsh series, spectral test, pseudorandom number generation, hybrid sequences.

AMS classification. 11K06, 11K38, 11K45, 11K70.

1 Introduction

This paper is about certain mathematical concepts to analyze the uniform distribution behaviour of sequences on the \( s \)-dimensional torus \([0, 1)^s\). We discuss qualitative aspects, by which we understand the study of properties of an infinite sequence that induce its uniform distribution in \([0, 1)^s\), and also quantitative aspects, i.e. the question how to measure the uniform distribution of a finite or infinite sequence on the \( s \)-torus.

A general setting to generate a (finite or infinite) sequence \( \omega = (x_n)_{n \geq 0} \) in some output space \( \mathcal{O} \), like \( \mathcal{O} = [0, 1)^s \), is the following. We consider a nonempty set \( \mathcal{S} \),
the so-called state space, and a map $T : S \to S$, the state update transformation. We employ $T$ to generate a (finite or infinite) sequence of states $(s_n)_{n \geq 0}$ in $S$. This sequence is then mapped to a sequence $\omega = (x_n)_{n \geq 0}$ in $O$ by an output map $\varphi : S \to O$, by letting $x_n = \varphi(s_n)$, $n \geq 0$.

In numerous applications of this concept, the sequence $(s_n)_{n \geq 0}$ is constructed by iterating $T$. We start with some initial state $s_0$ and put $s_n = T^n(s_0)$, $n \geq 0$. Illustrative examples are linear or inversive congruential pseudorandom number generators (see [29]), the well known $(n\alpha)_{n \geq 0}$ sequences (see [24]), or the block cipher AES in Output Feedback mode (see [2] p. 28).

In some other cases, we employ a ring $(R, +, \cdot)$ with unity 1 and a function $\psi : R \to S$ to map the sequence $(n1)_{n \geq 0}$ (the so-called “counter”) first to a sequence of states $(\psi(n1))_{n \geq 0}$ in $S$. Then, this sequence is “encrypted” by $T$ to give the sequence $(T \circ \psi(n1))_{n \geq 0}$ in $S$. Finally, this sequence of states is mapped into the output space $O$. This gives $\omega = (\varphi \circ T \circ \psi(n1))_{n \geq 0}$. Examples of such constructions are explicit inversive congruential pseudorandom number generators (see [11, 30, 31, 36]), certain digital sequences (see [3]), or AES in Counter mode (see [2] p. 28). In these examples, we let $R = \mathbb{Z}$, the integral domain of integers.

In all of the cases exhibited above, we employ some arithmetical operation like addition on the state space $S$. If $S$ is a compact abelian group with respect to the chosen operation, then we have the arsenal of abstract harmonic analysis at our disposition (see [22]). The choice of the compact group $S$ determines which function system is suitable for the analysis of the equidistribution behavior of the sequence $\omega$ in $O$, because the group operation on $S$ is intrinsically related to the dual group $\hat{S}$ of $S$.

An example of such a suitable match between sequences and function systems based on this duality principle is given by Kronecker sequences or, in the discrete version, good lattice points, and the trigonometric functions. Here, $S = O = [0, 1)^s$, and the construction method uses addition modulo one on the $s$-torus $[0, 1)^s$, $T(s) = s + \alpha$, with $\alpha \in \mathbb{R}^s$ (see [4], [29] Ch. 5) and [34]). The dual of the compact group $S$ may be interpreted as the trigonometric function system. For the background of these group rotations in ergodic theory, we refer to the monographs [32, 37].

A second example of this duality principle is given by digital nets and sequences and the Walsh functions. Here, addition without carry of digit vectors comes into play (see [29] Ch. 4] and [3]). For example, for nets and sequences in base 2, the underlying group $S$ is the compact group $\mathbb{F}_2^\infty$ and its dual group is the Walsh function system in base 2. The same relation holds for general integer bases $b \geq 2$.

An important type of a digital sequence, the Halton sequence in integer base $b = (b_1, \ldots, b_s)$ on the $s$-torus $[0, 1)^s$, is generated by addition with carry of digit vectors. For this reason, it makes sense to choose as the underlying group $S$ the compact group of $b$-adic integers $\mathbb{Z}_{b_1} \times \cdots \times \mathbb{Z}_{b_s}$. Considering the dual group then leads to the $b$-adic function system, which is the main tool of the $b$-adic method introduced in [14, 15, 17]. It will be discussed in Section 3.

The general duality principle presented before also accommodates for hybrid se-
quences, which are sequences \( \omega \) of points in \([0, 1)^s\) where certain coordinates of the points stem from one lower-dimensional sequence \( \omega_1 \), with state space \( \mathcal{S}_1 \), and the remaining coordinates from a second lower-dimensional sequence \( \omega_2 \), with state space \( \mathcal{S}_2 \). The duality principle leads us to consider hybrid function systems, which arise from the product group \( \hat{\mathcal{S}}_1 \times \hat{\mathcal{S}}_2 \). Of course, this idea may be generalized to mixing more than two subsequences into a hybrid sequence.

This paper is structured as follows. In Section 2 we consider the question which function systems will suffice to analyze digital sequences. In Section 3 we introduce the necessary notation and in Section 4 we define the hybrid spectral test and show that it is a measure of the uniform distribution of a sequence in \([0, 1)^s\). Section 5 deals with special cases of the spectral test, like diaphonies, the spectral test for pseudorandom number generators, various figures of merit for integer lattice points, like they appear in the context of good lattice points and rank-1 lattice rules. Finally, we show that the extreme as well as the star discrepancy can be approximated arbitrarily close by special cases of the hybrid spectral test.

2 Adding digit vectors

Digital sequences on the \( s \)-torus are sequences that arise from operations with digit vectors in some given integer bases \( b_i \geq 2, 1 \leq i \leq s \). For the sake of simplicity, we restrict the following discussion to the one-dimensional case.

As we have seen in Section 1 and as the block ciphers IDEA [25] and AES [2] illustrate in cryptography, two types of addition of digit vectors are in use in applications, addition without carry and addition with carry. By the duality principle, the first addition leads to Walsh functions and the second type to \( b \)-adic functions as the proper tools for the analysis of such sequences.

Are these two types of function systems sufficient to study digital sequences? In algebraic terms, are there any other possibilities to add digit vectors than addition with or without carry? If yes, then this would lead to additional types of groups and dual groups and, hence, to additional function systems. If not, then the Walsh functions and the \( b \)-adic functions suffice to analyze digital sequences.

For details, in particular for the proofs in the following considerations and a refinement using compositions of positive integers instead of partitions and automorphisms of certain groups to define additions, we refer the reader to [16].

Let \( b \geq 2 \) be a fixed integer and let \( \mathcal{A}_b = \{0, 1, \ldots, b - 1\} \) denote the set of \( b \)-ary digits. For \( m \in \mathbb{N}, m \geq 2 \), let \( \mathcal{A}_b^m \) stand for the \( m \)-fold cartesian product of the set \( \mathcal{A}_b \) with itself. We study the following question: What are the binary operations “*” on the set \( \mathcal{A}_b^m \) of digit vectors of length \( m \) such that the pair \( (\mathcal{A}_b^m, +) \) is an abelian group?

In this paper, when we speak of an “addition on \( \mathcal{A}_b^m \)”, we mean a binary operation “*” on the set \( \mathcal{A}_b^m \) of digit vectors in base \( b \) such that the pair \( (\mathcal{A}_b^m, +) \) is an abelian group. The reader should note that the term “binary” has two different meanings here, which will become clear from the context. A binary operation on a set \( G \) is a map from
the cartesian product $G \times G$ into $G$. Referring to the representation of real numbers in base $b = 2$, the elements of the set $A_2^m$ are called binary vectors, and for $m = 1$ one speaks of binary digits or bits.

Let us consider the case $b = 2$ first. There are two well known examples for addition of digit vectors, which will be discussed below.

For $n \in \mathbb{N}$, $n \geq 2$, let $\mathbb{Z}/n\mathbb{Z}$ denote the additive group of residue classes modulo $n$. We identify this cyclic group with the set of integers \{0, 1, \ldots, n - 1\} equipped with addition modulo $n$.

**Example 2.1.** We identify $A_2$ with $\mathbb{Z}/2\mathbb{Z}$. For $x, y \in A_2^m$, $x = (x_0, \ldots, x_{m-1})$ and $y = (y_0, \ldots, y_{m-1})$, we define

$$x + y = (x_0 \oplus y_0, \ldots, x_{m-1} \oplus y_{m-1}),$$

where ‘$\oplus$’ denotes addition on $\mathbb{Z}/2\mathbb{Z}$, $0 \oplus 0 = 1 \oplus 1 = 0$, and $0 \oplus 1 = 1 \oplus 0 = 1$. The pair $(A_2^m, +)$ is an abelian group. In fact, it is isomorphic to the product group $(\mathbb{Z}/2\mathbb{Z})^m$. We call this binary operation *addition without carry*, or XOR-addition of digit vectors.

Every nonnegative integer $k$, $0 \leq k < 2^m$, has a unique representation in base 2 of the form $k = k_0 + k_1 2 + \cdots + k_{m-1} 2^{m-1}$ with digits $k_j \in A_2$, $0 \leq j \leq m - 1$.

**Example 2.2.** We identify $A_2^m$ with the group $\mathbb{Z}/2^m\mathbb{Z}$ as follows. For $x \in A_2^m$, $x = (x_0, \ldots, x_{m-1})$, we define the map $\text{int}_2 : A_2^m \to \mathbb{Z}/2^m\mathbb{Z}$,

$$\text{int}_2(x) = x_0 + x_1 2 + \cdots + x_{m-1} 2^{m-1}.$$ 

Further, let $\text{dig}_2 : \mathbb{Z}/2^m\mathbb{Z} \to A_2^m$.

$$\text{dig}_2(k) = (k_0, k_1, \ldots, k_{m-1}),$$

where $k = k_0 + k_1 2 + \cdots + k_{m-1} 2^{m-1}$ is the representation of $k$ in base 2. Finally, for $x, y \in A_2^m$, we define

$$x + y = \text{dig}_2(\text{int}_2(x) + \text{int}_2(y) \pmod{2^m}).$$

With this binary operation, the pair $(A_2^m, +)$ is an abelian group. Clearly, it is isomorphic to the additive group $\mathbb{Z}/2^m\mathbb{Z}$. We call this type of binary operation *addition with carry* or *integer addition* of digit vectors.

For $m \geq 2$, our two examples of addition act on non-isomorphic groups, because $\mathbb{Z}/2^m\mathbb{Z}$ is cyclic and $(\mathbb{Z}/2\mathbb{Z})^m$ is not. Apart from these two examples, are there any other possibilities to define addition on the set $A_2^m$? From the Fundamental Theorem for Finite Abelian Groups (see, for example, [21, Sec. 10]) we obtain the following lemma. In this context, a *partition* of a positive integer $m$ is a finite sequence $(t_i)_{i=1}^r$, $r \in \mathbb{N}$, of positive integers $t_i$ with the two properties (i) $t_1 \geq t_2 \geq \cdots \geq t_r$, and (ii) $t_1 + t_2 + \cdots + t_r = m$. 

P. Hellekalek
Lemma 2.3. The non-isomorphic groups of order $2^m$, $m \in \mathbb{N}$, are given by the product groups

$$(\mathbb{Z}/2^{t_1}\mathbb{Z}) \times (\mathbb{Z}/2^{t_2}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/2^{t_r}\mathbb{Z}),$$

where $(t_i)_{i=1}^r$ is a partition of $m$.

Hence, in view of Lemma 2.3, an addition on the set $A_2^m$ is defined if we choose a partition $m = t_1 + t_2 + \cdots + t_r$ of $m$ and put

$$(A_2^m, +) \cong (\mathbb{Z}/2^{t_1}\mathbb{Z}) \times (\mathbb{Z}/2^{t_2}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/2^{t_r}\mathbb{Z}). \tag{2.1}$$

Here, the symbol "\cong" denotes that the two groups are isomorphic. From the structure of the factors in (2.1) we obtain the following information.

Theorem 2.4. The only two types of binary operations on (sub)vectors of digits that may appear in the group law of the abelian group $(A_2^m, +)$ are the following:

- addition given by finite product groups of the form $(\mathbb{Z}/2^i\mathbb{Z}) \times \cdots \times (\mathbb{Z}/2^i\mathbb{Z})$, which is what we have called addition without carry, or
- addition in groups of residues of the form $\mathbb{Z}/2^t\mathbb{Z}$, $t \geq 2$, which we have called addition with carry.

Theorem 2.4 directly generalizes from base 2 to arbitrary prime bases $p$. In the case of a composite base $b$, we also have the equivalent of (2.1), as well as some additional direct products arising from the factorization of $b$ into prime powers. Even in the latter cases, only two types of addition of (sub)vectors of digits appear, addition with or without carry.

Remark 2.5. We have seen that there exist only two types of addition for digit vectors, while there exist many different variants of this binary operation, for example by changing the underlying partition sequence. As a consequence, from the duality principle only two types of function systems arise, the Walsh functions and the $b$-adic functions. For those digits that are added without carry, the Walsh system applies, and for those digits that are added with carry, the $b$-adic system of functions is appropriate. These two function systems cover all possible cases.

3 Notation

Throughout this paper, $b$ denotes a positive integer, $b \geq 2$, and $b = (b_1, \ldots, b_s)$ stands for a vector of not necessarily distinct integers $b_i \geq 2$, $1 \leq i \leq s$. $\mathbb{N}$ represents the positive integers, and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The underlying space is the $s$-dimensional torus $\mathbb{R}^s/\mathbb{Z}^s$, which will be identified with the half-open interval $[0, 1)^s$. Haar measure on the $s$-torus $[0, 1)^s$ will be denoted by $\lambda_s$. We put $e(y) = e^{2\pi iy}$ for $y \in \mathbb{R}$, where $i$ is the imaginary unit.
We will use the standard convention that empty sums have the value 0 and empty products value 1.

For a nonnegative integer \( k \), let \( k = \sum_{j \geq 0} k_j b^j, k_j \in \{0, 1, \ldots, b - 1\} \), be the unique \( b \)-adic representation of \( k \) in base \( b \). With the exception of at most finitely many indices \( j \), the digits \( k_j \) are equal to 0.

Every real number \( x \in [0, 1) \) has a representation in base \( b \) of the form \( x = \sum_{j \geq 0} x_j b^{-j-1} \), with digits \( x_j \in \{0, 1, \ldots, b - 1\} \). If \( x \) is a \( b \)-adic rational, which means that \( x = ab^{-g}, a \) and \( g \) integers, \( 0 \leq a < b^g \), \( g \in \mathbb{N} \), and if \( x \neq 0 \), then there exist two such representations.

The \( b \)-adic representation of \( x \) is uniquely determined under the condition that \( x_j \neq b - 1 \) for infinitely many \( j \). In the following, we will call this particular representation the regular \( (b \)-adic) representation of \( x \).

Let \( \mathbb{Z}_b \) denote the compact group of the \( b \)-adic integers. We refer the reader to \([22, 23]\) for details. An element \( z \) of \( \mathbb{Z}_b \) will be written as a formal sum \( z = \sum_{j \geq 0} z_j b^j \), with digits \( z_j \in \{0, 1, \ldots, b - 1\} \). The set \( \mathbb{Z} \) of integers is embedded in \( \mathbb{Z}_b \). If \( z \in \mathbb{N}_0 \), then at most finitely many digits \( z_j \) are different from 0. If \( z \in \mathbb{Z}_b, z < 0 \), then at most finitely many digits \( z_j \) are different from \( b - 1 \). In particular, \( -1 = \sum_{j \geq 0} (b - 1) b^j \).

We recall the following concepts from \([15, 17, 20]\).

**Definition 3.1.** The map \( \varphi_b: \mathbb{Z}_b \to [0, 1) \), given by \( \varphi_b(\sum_{j \geq 0} z_j b^j) = \sum_{j \geq 0} z_j b^{-j-1} \) \( \mod 1 \), will be called the regular \( (b \)-adic) radical-inverse function in base \( b \). The Monna map is surjective, but not injective. It may be inverted in the following sense.

The restriction of \( \varphi_b \) to \( \mathbb{N}_0 \) is often called the regular \( (b \)-adic) radical-inverse function in base \( b \). The Monna map is surjective, but not injective. It may be inverted in the following sense.

**Definition 3.2.** We define the pseudoimverse \( \varphi_b^+ \) of the regular \( (b \)-adic) radical-inverse function \( \varphi_b \) by

\[
\varphi_b^+[0, 1) \to \mathbb{Z}_b, \quad \varphi_b^+(\sum_{j \geq 0} x_j b^{-j-1}) = \sum_{j \geq 0} x_j b^j,
\]

where \( \sum_{j \geq 0} x_j b^{-j-1} \) stands for the regular \( (b \)-adic) representation of the element \( x \in [0, 1) \).

The image of \([0, 1)\) under \( \varphi_b^+ \) is the set \( \mathbb{Z}_b \setminus \{-N\} \). Furthermore, \( \varphi_b \circ \varphi_b^+ \) is the identity map on \([0, 1)\), and \( \varphi_b^+ \circ \varphi_b \) the identity on \( \mathbb{N}_0 \subset \mathbb{Z}_b \). In general, \( z \neq \varphi_b^+(\varphi_b(z)) \), for \( z \in \mathbb{Z}_b \). For example, if \( z = -1 \), then \( \varphi_b^+(\varphi_b(-1)) = \varphi_b^+(0) = 0 \neq -1 \).

It has been shown in \([20]\) that the dual group \( \hat{\mathbb{Z}}_b \) can be written in the form \( \hat{\mathbb{Z}}_b = \{\chi_k : k \in \mathbb{N}_0\} \), where \( \chi_k : \mathbb{Z}_b \to \{c \in \mathbb{C} : |c| = 1\} \), \( \chi_k(\sum_{j \geq 0} z_j b^j) = c(\varphi_b(k)(z_0 + z_1 b + \cdots)) \). We note that \( \chi_k \) depends only on a finite number of digits of \( z \) and, hence, this function is well defined.

As in \([15]\), we employ the function \( \varphi_b^+ \) to lift the characters \( \chi_k \) to the torus.

**Definition 3.3.** For \( k \in \mathbb{N}_0 \), let \( \gamma_k : [0, 1) \to \{c \in \mathbb{C} : |c| = 1\}, \gamma_k(x) = \chi_k(\varphi_b^+(x)), \) denote the \( k \)th \( (b \)-adic) function. We put \( \Gamma_b = \{\gamma_k : k \in \mathbb{N}_0\} \) and call it the regular \( (b \)-adic) function system on \([0, 1)\).
There is an obvious generalization of the preceding notions to the higher-dimensional case. Let $b = (b_1, \ldots, b_s)$ be a vector of not necessarily distinct integers $b_i \geq 2$, let $x = (x_1, \ldots, x_s) \in [0,1)^s$, let $z = (z_1, \ldots, z_s)$ denote an element of the compact product group $\mathbb{Z}_b = \mathbb{Z}_{b_1} \times \cdots \times \mathbb{Z}_{b_s}$ of $b$-adic integers, and let $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$. We define $\varphi_b(z) = (\varphi_{b_1}(z_1), \ldots, \varphi_{b_s}(z_s))$, and $\varphi_b^+(x) = (\varphi_{b_1}^+(x_1), \ldots, \varphi_{b_s}^+(x_s))$.

Let $\chi_k(z) = \prod_{i=1}^s \chi_{k_i}(z_i)$, where $\chi_{k_i} \in \mathbb{Z}_{b_i}$, and define $\gamma_k(x) = \prod_{i=1}^s \gamma_{k_i}(x_i)$, where $\gamma_{k_i} \in \Gamma_{b_i}$, $1 \leq i \leq s$. Then $\gamma_k = \chi_k \circ \varphi_b^+$. Let $\Gamma_b^{(s)} = \{ \gamma_k : k \in \mathbb{N}_0^s \}$ denote the $b$-adic function system in dimension $s$.

The dual group $\hat{\mathbb{Z}}_b$ is an orthonormal basis of the Hilbert space $L^2(\mathbb{Z}_b)$. A rather elementary proof of this result is given in [20, Theorem 2.12].

For the Walsh functions defined below, we refer the reader to [3, 10, 12] for elementary properties of these functions and to [33] for the background in harmonic analysis.

**Definition 3.4.** For $k \in \mathbb{N}_0$, $k = \sum_{j \geq 0} k_j b^j$, and $x \in [0,1)$, with regular $b$-adic representation $x = \sum_{j \geq 0} x_j b^{-j-1}$, the $k$th Walsh function in base $b$ is defined by $w_k(x) = e((\sum_{j \geq 0} k_j x_j)/b)$. For $k \in \mathbb{N}_0^s$, $k = (k_1, \ldots, k_s)$, and $x \in [0,1)^s$, $x = (x_1, \ldots, x_s)$, we define the $k$th Walsh function $w_k$ in base $b = (b_1, \ldots, b_s)$ on $[0,1)^s$ as the following product: $w_k(x) = \prod_{i=1}^s w_{k_i}(x_i)$, where $w_{k_i}$ denotes the $k_i$th Walsh function in base $b_i$, $1 \leq i \leq s$. The Walsh function system in base $b$ in dimension $s$ is denoted by $\mathcal{W}_b^{(s)} = \{ w_k : k \in \mathbb{N}_0^s \}$.

The trigonometric function system defined below is the classical function system in the theory of uniform distribution of sequences (see the monograph [24]).

**Definition 3.5.** Let $k \in \mathbb{Z}$. The $k$th trigonometric function $e_k$ is defined as $e_k : [0,1) \to \mathbb{C}$, $e_k(x) = e(kx)$. For $k = (k_1, \ldots, k_s) \in \mathbb{Z}^s$, the $k$th trigonometric function $e_k$ is defined as $e_k : [0,1)^s \to \mathbb{C}$, $e_k(x) = \prod_{i=1}^s e(k_i x_i)$, $x = (x_1, \ldots, x_s) \in [0,1)^s$. The trigonometric function system in dimension $s$ is denoted by $\mathcal{T}^{(s)} = \{ e_k : k \in \mathbb{Z}^s \}$.

The following presentation complements the concepts discussed in [17]. As will become clear, any finite number of factors can be accommodated. For given dimensions $s_1, s_2$, and $s_3$, with $s_i \in \mathbb{N}_0$, not all equal to 0, put $s = s_1 + s_2 + s_3$ and write a point $y \in \mathbb{R}^s$ in the form $y = (y^{(1)}, y^{(2)}, y^{(3)})$ with components $y^{(j)} \in \mathbb{R}^{s_j}$, $j = 1, 2, 3$. Let us fix two vectors of bases $b^{(1)} = (b_1, \ldots, b_{s_1})$, and $b^{(2)} = (b_{s_1+1}, \ldots, b_{s_1+s_2})$ with not necessarily distinct integers $b_i \geq 2$. Let $k = (k^{(1)}, k^{(2)}, k^{(3)})$, with components $k^{(1)} \in \mathbb{N}_0^{s_1}$, $k^{(2)} \in \mathbb{N}_0^{s_2}$, and $k^{(3)} \in \mathbb{Z}^{s_3}$. The tensor product $\xi_k = w_{k^{(1)}} \otimes \gamma_{k^{(2)}} \otimes e_{k^{(3)}}$, where $w_{k^{(1)}} \in \mathcal{W}_{b^{(1)}}^{(s_1)}$, $\gamma_{k^{(2)}} \in \Gamma_{b^{(2)}}^{(s_2)}$, and $e_{k^{(3)}} \in \mathcal{T}^{(s_3)}$, defines a function $\xi_k$ on the $s$-dimensional unit cube,

$$\xi_k : [0,1)^s \to \mathbb{C}, \quad \xi_k(x) = w_{k^{(1)}}(x^{(1)}) \gamma_{k^{(2)}}(x^{(2)}) e_{k^{(3)}}(x^{(3)}),$$

where $x = (x^{(1)}, x^{(2)}, x^{(3)}) \in [0,1)^s$. 


Definition 3.6. Let \( s_1, s_2, s_3 \in \mathbb{N}_0 \), not all \( s_i \) equal to 0, and put \( s = s_1 + s_2 + s_3 \). The family of functions

\[
W^{(s_1)}_b(1) \otimes \Gamma^{(s_2)}_b(2) \otimes T^{(s_3)} = \{ \xi_k, k = (k^{(1)}, k^{(2)}, k^{(3)}) \in \mathbb{N}^{s_1}_0 \times \mathbb{N}^{s_2}_0 \times \mathbb{Z}^{s_3} \},
\]

is called a hybrid function system on \([0, 1)^s\).

Remark 3.7. It follows from [17, Theorem 1 and Corollary 4] and the techniques exhibited in [20] that such hybrid function systems are an orthonormal basis of \( L^2([0, 1)^s) \).

4 The hybrid spectral test

Remark 4.1. All of the following results remain valid if we change the order of the factors in the hybrid function system, as it will become apparent from the proofs below. In particular, out of the given \( s \) coordinates, we may select arbitrary \( s_1 \) coordinates and assign to them the Walsh system \( W^{(s_1)}_b(1) \) in some base \( b^{(1)} \), treat \( s_2 \) of the remaining \( s - s_1 \) coordinates with a \( b^{(2)} \)-adic system \( \Gamma^{(s_2)}_b(2) \), and use the system \( T^{(s_3)} \) for the final \( s_3 \) coordinates.

Definition 4.2. Let \( s \in \mathbb{N} \). By an \( s \)-dimensional index set \( \Lambda \) we understand one of the additive semigroups \((\mathbb{Z}^s, +)\), \((\mathbb{N}^s_0, +)\), and \((\mathbb{N}^s, +)\), or finite direct products of these semigroups such that the dimensions of the factors add up to \( s \).

Let \( \Lambda^* \) denote the index set \( \Lambda \setminus \{0\} \).

Examples of \( s \)-dimensional index sets are direct products of the form \( \mathbb{N}^{s_1}_0 \times \mathbb{N}^{s_2}_0 \times \mathbb{Z}^{s_3} \), where \( s = s_1 + s_2 + s_3 \), with \( s_1, s_2, s_3 \in \mathbb{N}_0 \), not all \( s_i \) equal to 0, as they appear in hybrid function systems.

If \( \omega = (x_n)_{n \geq 0} \) is a -possibly finite- sequence in \([0, 1)^s\) with at least \( N \) elements, and if \( f : [0, 1)^s \to \mathbb{C} \), we define

\[
S_N(f, \omega) = \frac{1}{N} \sum_{n=0}^{N-1} f(x_n).
\]

Definition 4.3. Let \( \Lambda \) be an \( s \)-dimensional index set. A subclass \( \mathcal{F} = \{ \xi_k : k \in \Lambda \} \) of the class of Riemann integrable functions on \([0, 1)^s\) is called a uniform distribution determining (u.d.d.) function system on \([0, 1)^s\) if the functions \( \xi_k \) are normalized in the sense that \( ||\xi_k||_\infty = \sup\{ |\xi_k(x)| : x \in [0, 1)^s \} \leq 1 \) for all \( k \), and \( \int_{[0,1)^s} \xi_k d\lambda_s = 0 \) for all \( k \in \Lambda^* \), and if, for any sequence \( \omega \) in \([0, 1)^s\), the property

\[
\forall k \in \Lambda^* : \lim_{N \to \infty} S_N(\xi_k, \omega) = 0
\]

implies the uniform distribution of \( \omega \) in \([0, 1)^s\).
Examples of u.d.d. classes of functions on \([0, 1)^s\) are the hybrid function systems \(W_{b_1}^{(s_1)} \otimes \Gamma_{b_2}^{(s_2)} \otimes \mathcal{T}^{(s_3)}\). This follows from the hybrid Weyl Criterion \([17, \text{Theorem 1}]\), whose proof is easily adapted to provide for the non-prime bases \(b_i\) we allow here in the factor \(\Gamma_{b_2}^{(s_2)}\).

**Definition 4.4.** Let \(\| \cdot \|\) be an arbitrary norm on \(\mathbb{R}^s\) and let \(\Lambda\) be an \(s\)-dimensional index set. We call a real-valued function \(\rho\) a *weight function* on \(\Lambda\) if, for all \(k \in \Lambda\), \(\rho(k) > 0\), and if for all \(\epsilon > 0\), there exists a positive real number \(K_0 = K_0(\epsilon)\) such that \(\rho(k) < \epsilon\) for all \(k \in \Lambda\) with \(\|k\| > K_0\).

With a u.d.d. function system \(\mathcal{F} = \{\xi_k : k \in \Lambda\}\) on \([0, 1)^s\) and a weight function \(\rho\) on \(\Lambda\) we may associate the array of weighted functions

\[
(\rho(k)\xi_k)_{k \in \Lambda}.
\]

The operator \(S_N(\cdot, \omega)\) is linear, hence we may write

\[
S_N((\rho(k)\xi_k)_{k \in \Lambda}, \omega) = (\rho(k)S_N(\xi_k, \omega))_{k \in \Lambda}.
\]

**Definition 4.5.** Let \(\mathcal{F} = \{\xi_k : k \in \Lambda\}\) be a u.d.d. function system on \([0, 1)^s\), and let \(\rho\) be a weight function on \(\Lambda\). For a given sequence \(\omega = (x_n)_{n \geq 0}\) in \([0, 1)^s\), the *spectral test* \(\sigma_N(\omega)\) of the first \(N\) elements of \(\omega\), with respect to \(\mathcal{F}\) and \(\rho\), is defined as

\[
\sigma_N(\omega) = \| (\rho(k))_{k \in \Lambda^*} \|_\infty^{-1} \| (\rho(k)S_N(\xi_k, \omega))_{k \in \Lambda^*} \|_\infty
= \sup_{k \in \Lambda^*} \{\rho(k)\}^{-1} \sup_{k \in \Lambda^*} \{\rho(k) |S_N(\xi_k, \omega)|\}.
\]

Let \(\alpha > 1\) be a given real number. If the weight function \(\rho\) fulfills the additional condition

\[
\sum_{k \in \Lambda} \rho(k)^\alpha < \infty,
\]

then the \(L^\alpha\)-diaphony \(F_N^{(\alpha)}(\omega)\) of the first \(N\) elements of \(\omega\), with respect to \(\mathcal{F}\) and \(\rho\), is defined as

\[
F_N^{(\alpha)}(\omega) = \| (\rho(k))_{k \in \Lambda^*} \|_\alpha^{-1} \| (\rho(k)S_N(\xi_k, \omega))_{k \in \Lambda^*} \|_\alpha
= \left(\sum_{k \in \Lambda^*} \rho(k)^\alpha\right)^{-1/\alpha} \left(\sum_{k \in \Lambda^*} \rho(k)^\alpha |S_N(\xi_k, \omega)|^\alpha\right)^{1/\alpha}.
\]

We note that the fact \(|S_N(\xi_k, \omega)| \leq 1\) implies that the spectral test as well as diaphony are normalized: \(0 \leq \sigma_N(\omega) \leq 1\), and \(0 \leq F_N^{(\alpha)}(\omega) \leq 1\).

**Theorem 4.6.** Let \(\mathcal{F}\), \(\rho\) and \(\sigma_N(\omega)\) be as in Definition 4.5. Then
(i) The quantity $\sigma_N(\omega)$ is a maximum.

(ii) The sequence $\omega$ is uniformly distributed modulo one if and only if

$$\lim_{N \to \infty} \sigma_N(\omega) = 0.$$ 

Proof. The proof generalizes the arguments in [12, Sec. 5.2]. For an arbitrary positive integer $K$, let

$$A_{N,K} = \sup \{ \rho(k) \mid S_N(\xi_k, \omega) : 0 < ||k|| \leq K \},$$

and

$$B_{N,K} = \sup \{ \rho(k) \mid S_N(\xi_k, \omega) : ||k|| > K \}.$$ 

Clearly,

$$\sigma_N(\omega) = \max \{ A_{N,K}, B_{N,K} \}. \quad (4.1)$$

We have $\sigma_N(\omega) > 0$. Otherwise, all terms $S_N(\xi_k, \omega)$, $k \in \Lambda^*$, would be equal to zero, which is impossible. Hence, there exists $\delta > 0$ such that $\delta < \sigma_N(\omega)$, and an index $K_0 = K_0(\delta) \in \mathbb{N}$ such that for all $k$ with $||k|| > K_0$ we have $\rho(k) < \delta$. This implies

$$B_{N,K_0} \leq \sup \{ \rho(k) : ||k|| > K_0 \} \leq \delta < \sigma_N(\omega).$$

Hence, $\sigma_N(\omega) = A_{N,K_0}$. We observe that the set $\{ k : ||k|| \leq K_0 \}$ is finite. In this context, we recall that all norms on $\mathbb{R}^s$ are equivalent. This proves (i).

In order to prove (ii), suppose first that $\lim_{N \to \infty} \sigma_N(\omega) = 0$. This implies that $\lim_{N \to \infty} S_N(\xi_k, \omega) = 0$ for all $k \in \Lambda^*$. The class $\mathcal{F}$ is u.d.d., hence $\omega$ is uniformly distributed in $[0, 1)^s$.

To prove the converse, assume that $\omega$ is uniformly distributed in $[0, 1)^s$. For any $\epsilon > 0$, there exists a positive integer $K_0 = K_0(\epsilon)$ such that $\rho(k) < \epsilon$ for all $k$ with $||k|| > K_0$. As in the proof of part (i), this gives $B_{N,K_0} \leq \epsilon$, and, due to (4.1),

$$\sigma_N(\omega) \leq A_{N,K_0} + \epsilon.$$ 

The number $A_{N,K_0}$ is a maximum and the class $\mathcal{F}$ is u.d.d.. This implies the existence of $N_0 = N_0(\epsilon) \in \mathbb{N}$ with the property

$$\forall N \geq N_0(\epsilon) : A_{N,K_0} = \max \{ \rho(k) \mid S_N(\xi_k, \omega) : 0 < ||k|| \leq K_0 \} < \epsilon.$$ 

We deduce the relation

$$\forall N \geq N_0(\epsilon) : \sigma_N(\omega) < 2\epsilon.$$ 

This proves the theorem. \qed
Corollary 4.7. Let $\mathcal{F}$, $\rho$ and $\sigma_N(\omega)$ be as in Definition 4.5 and let $K$ denote an arbitrary positive integer. Then we have the following inequality of Erdös-Turán-Koksma for the spectral test:

$$\sigma_N(\omega) \leq \max \left\{ \max_{0 \leq ||k|| \leq K} \{ \rho(k) \ |S_N(\xi_k, \omega)| \}, \sup_{||k|| > K} \{ \rho(k) \} \right\}.$$

Proof. This follows directly from (4.1).

Theorem 4.8. Let $\mathcal{F}$, $\rho$ and $F_N^{(\alpha)}(\omega)$ be as in Definition 4.5 and suppose that

$$\sum_{k \in \Lambda^s} \rho(k)^\alpha < \infty.$$

Then sequence $\omega$ is uniformly distributed in $[0, 1)^s$ if and only if

$$\lim_{N \to \infty} F_N^{(\alpha)}(\omega) = 0.$$

Proof. We adapt the proof of [17, Theorem 2] and the splitting technique used in the proof of Theorem 4.6 to the case of diaphony.

5 Examples

5.1 Examples I: integration lattices

Definition 4.5 generalizes various known notions of the spectral test and of diaphony. The spectral test has its origin in pseudorandom number generation. It measures the “coarseness” of lattices that can be associated with certain linear types of generators. We refer to [23, Ch. 3.3.4.] for a seminal discussion, and to the surveys [26, 29] as well as to [5].

We recall some notions from the theory of lattices (for details see [29, 34]). An $s$-dimensional lattice is a discrete subset of $\mathbb{R}^s$ which is closed under addition and subtraction. For any $s$-dimensional lattice $L$, there exists a lattice basis, by which we understand $s$ independent vectors $g_1, \ldots, g_s$ in $\mathbb{R}^s$ such that

$$L = \left\{ \sum_{i=1}^s t_i g_i : t_i \in \mathbb{Z} \right\}.$$

Lattice bases are not unique.

A point $g \in L \setminus \{0\}$ is called a primitive point of $L$ if the line segment joining the origin $0$ and $g$ does not contain any other point of $L$.

The dual lattice $L^\perp$ of $L$ is defined as

$$L^\perp = \{ z \in \mathbb{R}^s : z \cdot x \in \mathbb{Z}, \text{ for all } x \in L \}.$$
where $\mathbf{z} \cdot \mathbf{x}$ denotes the usual inner product.

An $s$-dimensional integration lattice is an $s$-dimensional lattice that contains $\mathbb{Z}^s$ as a sublattice. An $s$-dimensional $N$-point lattice rule is an $s$-dimensional integration lattice $L$ of the form

$$L = \bigcup_{n=0}^{N-1} (\mathbf{x}_n + \mathbb{Z}^s),$$

(5.1)

where $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$ are the $N$ distinct points of $L$ that belong to $[0, 1)^s$.

The spectral test for lattices is defined as follows.

**Definition 5.1.** Let $L$ be an $s$-dimensional lattice in $\mathbb{R}^s$. We will call a family $\mathcal{H}_C$ of parallel hyperplanes $H_c$, $c \in C$, in $\mathbb{R}^s$ a cover of $L$ if (i) $L \subseteq \bigcup_{c \in C} H_c$, and (ii) $C$ is the smallest set (in the sense of set-inclusion) with this property.

Let $\mathcal{H}_C$ denote an arbitrary cover of $L$ in $\mathbb{R}^s$. The spacing $d(\mathcal{H}_C)$ of $\mathcal{H}_C$ will denote the minimal distance between adjacent hyperplanes in this family, $d(\mathcal{H}_C) = \inf \{d(H_c, H_d) : c \neq d, c, d \in C\}$.

We define the spectral test of $L$ as the number

$$\sigma(L) := \sup \{d(\mathcal{H}_C) : \mathcal{H}_C \text{ is a cover of } L\}.$$ 

The following theorem is well known in the theory of pseudorandom number generation (see [6, 12, 23, 27]). Computational aspects are discussed in [1].

**Theorem 5.2.** Let $L$ be an $s$-dimensional lattice. Then

$$\sigma(L) = 1 / \min\{||g||_2 : g \in L^\perp, g \text{ primitive }\},$$

where $||g||_2 = (g_1 + \cdots + g_s)^{1/2}$ is the Euclidean norm on $\mathbb{R}^s$.

In the field of quasi-Monte Carlo integration, similar concepts have been developed within the context of good lattice points.

**Definition 5.3.** For a given integer $N \geq 2$ and for a given dimension $s \geq 2$, we call an integer point $a = (a_1, \ldots, a_s) \in \mathbb{Z}^s$, $\gcd(a_i, N) = 1$, $1 \leq i \leq s$, a good lattice point modulo $N$ if the finite sequence

$$\omega_a = \left(\left\{\frac{n}{N} a\right\}\right)_{n=0}^{N-1}$$

(5.2)

is “very uniformly” distributed in $[0, 1)^s$ in the sense that its discrepancy is low.

The conditions $\gcd(a_i, N) = 1$, $1 \leq i \leq s$, ensure that all points in $\omega_a$ and in its projections to lower dimensions are distinct.

We may view $\omega_a$ as the node set of an $s$-dimensional $N$-point lattice rule. Let $L(\omega_a)$ denote the integration lattice associated with $\omega_a$ defined by (5.1). The dual lattice is given by $L(\omega_a)^\perp = \{k \in \mathbb{Z}^s : k \cdot a \equiv 0 \pmod{N}\}$. 

With an integer point \( a \in \mathbb{Z}^s \) subject to the conditions of Definition 5.3, we may associate several figures of merit. In view of the spectral test for lattices, define

\[
\sigma(a, N) = \sigma(L(\omega_a)).
\]

Then

\[
\sigma(a, N) = \frac{1}{\min \{|k|_2 : k \in \mathbb{Z}^s \setminus \{0\}, \ k \cdot a \equiv 0 \pmod{N}\}}.
\]

For a real number \( \alpha > 1 \), define

\[
P_\alpha(a, N) = \sum_{k \neq 0} \frac{1}{r(k)^\alpha},
\]

where summation is over integer vectors \( k \), and

\[
r(k) = \prod_{i=1}^{s} \max\{1, |k_i|\}, \ k = (k_1, \ldots, k_s) \in \mathbb{Z}^s.
\]

Another important figure of merit is the Babenko-Zaremba index

\[
\kappa(a, N) = \frac{1}{\min \{r(k) : k \in \mathbb{Z}^s \setminus \{0\}, \ k \cdot a \equiv 0 \pmod{N}\}},
\]

see the monographs [29, 34].

The three quantities \( \sigma(a, N), P_\alpha(a, N), \) and \( \kappa(a, N) \) are special cases of the spectral test introduced in Definition 4.5 applied to the sequence \( \omega_a \). In order to establish this connection, we employ the following result of Sloan and Kachoyan [35] (for the proof see [29, Lemma 5.21] and [34, Lemma 2.7]):

Lemma 5.4. Let \( \omega = (x_n)_{n=0}^{N-1} \) be the sequence of nodes of an \( s \)-dimensional \( N \)-point lattice rule \( L \). Then

\[
S_N(e_k, \omega) = \begin{cases} 
1, & \text{if } k \in L^\perp, \\
0, & \text{if } k \not\in L^\perp.
\end{cases}
\]

Example 5.5. Let \( F = T^{(s)} \) (hence \( \Lambda = \mathbb{Z}^s \)), and put \( \omega_a = ((n/N) a)_{n=0}^{N-1} \), where \( a \in \mathbb{Z}^s \) is subject to the conditions in Definition 5.3. Then,

(i) for the choice \( \rho(k) = ||k||_2^{-1} \) for \( k \neq 0 \), the hybrid spectral test \( \sigma_N(\omega_a) \) introduced in Definition 4.5 with respect to \( F \) and \( \rho \), is equal to the classical spectral test of Definition 5.1 applied to the integration lattice \( L(\omega_a) \).

(ii) for \( \rho(k) = r(k)^{-1} \) for \( k \neq 0 \), we obtain \( \sigma_N(\omega_a) = \kappa(a, N) \), the Babenko-Zaremba index defined in (5.1).

(iii) for \( \rho(k) = r(k)^{-1} \) for \( k \neq 0 \), and \( \alpha > 1 \), we get

\[
\left( \sum_{k \neq 0} r(k)^{-\alpha} \right)^{1/\alpha} = F_N^{(\alpha)}(\omega_a) = (P_\alpha(a, N))^{1/\alpha},
\]

for the \( L^\alpha \)-diaphony of the sequence \( \omega_a \).
Example 5.6. By obvious choices of $F$ and $\rho$ and with $\alpha = 2$, we obtain the classical diaphony of Zinterhof [38], the dyadic (Walsh) diaphony of Hellekalek and Leeb [19], the $b$-adic (Walsh) diaphony versions of Grozdanov et al. [7,8,9], the $p$-adic diaphony of Hellekalek [15], and the more general notion of hybrid diaphony that was introduced in [17].

Example 5.7. By obvious choices for $F$ and $\rho$, we obtain the Walsh spectral test of Hellekalek [13].

5.2 Examples II: extreme and star discrepancy

The hybrid spectral test introduced in Definition 4.5 does not include the extreme discrepancy and the star discrepancy, but we may approximate these two measures of uniform distribution arbitrarily close by suitable versions of the hybrid spectral test.

We recall the definition of discrepancy. Let $\mathcal{J}$ denote the class of all subintervals of $[0,1)^s$ of the form $\prod_{i=1}^s [u_i,v_i)$, $0 \leq u_i < v_i \leq 1$, $1 \leq i \leq s$, and let $\mathcal{J}^*$ denote the subclass of $\mathcal{J}$ of intervals of the type $\prod_{i=1}^s [0,v_i)$ anchored at the origin. The extreme discrepancy and the star discrepancy of a sequence are defined as follows (see [24,29]).

Definition 5.8. Let $\omega = (x_n)_{n \geq 0}$ be a sequence in $[0,1)^s$. The (extreme) discrepancy $D_N(\omega)$ of the first $N$ elements of $\omega$ is defined as

$$D_N(\omega) = \sup_{J \in \mathcal{J}} |S_N(1_J - \lambda_s(J),\omega)|.$$ 

The star discrepancy $D_N^*(\omega)$ of the first $N$ elements of $\omega$ is defined as

$$D_N^*(\omega) = \sup_{J \in \mathcal{J}^*} |S_N(1_J - \lambda_s(J),\omega)|.$$ 

We first approximate $D_N(\omega)$ and $D_N^*(\omega)$ by discrete discrepancies.

Definition 5.9. Let $b = (b_1,\ldots,b_s)$, with not necessarily distinct integers $b_i \geq 2$. A $b$-adic interval in the resolution class defined by $g = (g_1,\ldots,g_s) \in \mathbb{N}_0^s$ (or with resolution $g$) is a subinterval of $[0,1)^s$ of the form

$$\prod_{i=1}^s \left[a_ib_i^{-g_i},d_ib_i^{-g_i}\right), \quad 0 \leq a_i < d_i \leq b_i^{g_i}, \quad a_i,d_i \in \mathbb{N}_0, \quad 1 \leq i \leq s.$$ 

We denote the class of all $b$-adic intervals with resolution $g$ by $\mathcal{J}_{b,g}$. The subclass of those $b$-adic intervals anchored at the origin will be denoted by $\mathcal{J}^*_{b,g}$. Further, let

$$\mathcal{J}_b = \bigcup_{g \in \mathbb{N}_0^s} \mathcal{J}_{b,g}$$
denote the class of all $b$-adic intervals in $[0, 1)^s$ and put

$$\mathcal{J}_b^* = \bigcup_{g \in \mathbb{N}_0^s} \mathcal{J}_{b,g}^*.$$ 

For a given resolution $g \in \mathbb{N}_0^s$, we define the domains

$$\Delta_b(g) = \{ k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s : 0 \leq k_i < b_i^g, 1 \leq i \leq s \},$$

$$\Delta_b^*(g) = \Delta_b(g) \setminus \{0\},$$

and

$$\nabla_b(g) = \{ k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s : 1 \leq k_i \leq b_i^g, 1 \leq i \leq s \}.$$ 

We note that $\Delta_b(0) = \{0\}$ and $\nabla_b(0) = \emptyset$. Further, we observe that we may write the intervals in $\mathcal{J}_{b,g}$ in the form

$$\mathcal{J}_{b,g} = \{ I_{a,d,g} : (a, d) \in \Delta_b(g) \times \nabla_b(g) \},$$

where $I_{a,d,g} = \prod_{i=1}^s [\varphi_{b_i}(a_i), \varphi_{b_i}(d_i)]$, and $a = (a_1, \ldots, a_s)$, and $d = (d_1, \ldots, d_s)$. The intervals in $\mathcal{J}_{b,g}^*$ are of the form $I_{0,d,g}$, with $d \in \nabla_b(g)$.

**Definition 5.10.** Let $\omega = (x_n)_{n \geq 0}$ be a sequence in $[0, 1)^s$, let $b = (b_1, \ldots, b_s)$, with not necessarily distinct integers $b_i \geq 2$, and let $g \in \mathbb{N}_0^s$ be a given resolution vector. The discrete (extreme) discrepancy in base $b$, for resolution $g$, of the first $N$ elements of $\omega$ is defined as

$$D_{N;b,g}(\omega) = \max_{I \in \mathcal{J}_{b,g}} |S_N(1_I - \lambda_s(I), \omega)|.$$ 

The discrete star discrepancy in base $b$, for resolution $g$, of the first $N$ elements of $\omega$ is defined as

$$D_{N;b,g}^*(\omega) = \max_{I \in \mathcal{J}_{b,g}^*} |S_N(1_I - \lambda_s(I), \omega)|.$$ 

**Theorem 5.11.** Let $b = (b_1, \ldots, b_s)$ be a vector of $s$ not necessarily distinct integers $b_i \geq 2$. Then, for all $g = (g_1, \ldots, g_s) \in \mathbb{N}_0^s$,

$$D_{N;b,g}(\omega) \leq D_N(\omega) \leq \epsilon_b(g) + D_{N;b,g}(\omega),$$

$$D_{N;b,g}^*(\omega) \leq D_N^*(\omega) \leq \epsilon_b^*(g) + D_{N;b,g}^*(\omega),$$

where the error terms $\epsilon_b(g)$ and $\epsilon_b^*(g)$ are given by

$$\epsilon_b(g) = 1 - \prod_{i=1}^s (1 - 2b_i^{-g_i}), \quad \epsilon_b^*(g) = 1 - \prod_{i=1}^s (1 - b_i^{-g_i}).$$
Proof. The inequalities $D_{N; b, g}(\omega) \leq D_N(\omega)$ and $D^*_N; b, g(\omega) \leq D^*_N(\omega)$ are trivial.

In order to show $D_N(\omega) \leq \epsilon_b(g) + D_{N; b, g}(\omega)$, we proceed as in the proof of Theorem 3.12 in [18]. For an arbitrary subinterval $J$ of $[0, 1)^s$ we obtain the following bound (see [18, Inequality (6))]:

$$|S_N(1_J - \lambda_s(J), \omega)| \leq \epsilon_b(g) + \max_{I \in \mathcal{J}_{b, g}} \{|S_N(1_I - \lambda_s(I), \omega)|\}. \quad (5.3)$$

This bound is independent of the choice of $J$. As a consequence,

$$D_N(\omega) = \sup_{J \in \mathcal{J}} |S_N(1_J - \lambda_s(J), \omega)| \leq \epsilon_b(g) + D_{N; b, g}(\omega).$$

In the case of the star discrepancy $D^*_N(\omega)$, the intervals $J$ are anchored at the origin. It is easy to see from the proof of Theorem 3.12 in [18] that this fact allows us to replace the error term $\epsilon_b(g)$ by $\epsilon^*_b(g)$. This finishes the proof. \(\square\)

Remark 5.12. An elementary analytic argument shows that $\epsilon_b(g) \leq 2s\delta_g$, and $\epsilon^*_b(g) \leq s\delta_g$, where $\delta_g = \max_{1 \leq i \leq s} b_i^{-g_i}$.

Corollary 5.13. We have the following discretization:

$$D_N(\omega) = \sup_{I \in \mathcal{J}_b} |S_N(1_I - \lambda_s(I), \omega)|,$$

$$D^*_N(\omega) = \sup_{I \in \mathcal{J}_b^*} |S_N(1_I - \lambda_s(I), \omega)|.$$

We recall that a sequence $\omega$ is called uniformly distributed in $[0, 1)^s$ if and only if

$$\forall J \in \mathcal{J} : \lim_{N \to \infty} S_N(1_J - \lambda_s(J), \omega) = 0. \quad (5.4)$$

Corollary 5.14. It follows from Inequality (5.3) that a sequence $\omega$ is uniformly distributed in $[0, 1)^s$ if and only if

$$\forall I \in \mathcal{J}_b : \lim_{N \to \infty} S_N(1_I - \lambda_s(I), \omega) = 0.$$

The argument to approximate the discrepancies $D_N$ and $D^*_N$ by a hybrid spectral test goes as follows. Let $b = (b_1, \ldots, b_s)$ be a vector of $s$ not necessarily distinct integers $b_i \geq 2$. As index set, we choose $\Lambda = \mathbb{N}^s_0 \times \mathbb{N}^s$ and observe that

$$\Lambda = \bigcup_{g \in \mathbb{N}^s_0} (\Delta_b(g) \times \nabla_b(g)).$$

An index point $(a, d) \in \Lambda$ is called admissible if there exists $g \in \mathbb{N}^s_0$ such that $(a, d) \in \Delta_b(g) \times \nabla_b(g)$ and the interval $I_{a, d, g}$ belongs to $\mathcal{J}_{b, g}$. The point $(a, d) \in \Lambda$ is called
non-admissible} otherwise. The reader should note that different admissible points may produce the same $b$-adic interval.

As the elements of the functions system $\mathcal{F} = \{\xi_{(a,d)} : (a, d) \in \Lambda\}$, we choose the functions $1_I - \lambda_s(I)$, with $I \in \mathcal{J}_b$, or the constant function 0, subject to the following parametrization. For an admissible index $(a, d) \in \Lambda$, let

$$\xi_{(a,d)} = 1_{I_{a,d}} - \lambda_s(I_{a,d})$$

It follows that $\mathcal{F}$ contains all functions $1_I - \lambda_s(I)$, with $I \in \mathcal{J}_b$. If $(a, d)$ is non-admissible, define $\xi_{(a,d)}$ to be identically 0. Corollary 5.14 implies that $\mathcal{F}$ is u.d.d.

For a given $g \in \mathbb{N}^s$, we define the weight function $\rho_g$ on $\Lambda$ in the following manner. If $k \in \mathbb{N}_0$, with $b$-adic representation $k = k_0 + k_1b + \cdots$, we put

$$v_b(k) = \begin{cases} 0, & \text{if } k = 0, \\ 1 + \max\{j : k_j \neq 0\}, & \text{if } k \geq 1. \end{cases}$$

For $(a, d) \in \Lambda$, $a = (a_1, \ldots, a_s)$, $d = (d_1, \ldots, d_s)$, let

$$\rho_g((a, d)) = \begin{cases} 1, & \text{if } (a, d) \in \Delta_b(g) \times \nabla_b(g), \\ \prod_{i=1}^s b_i^{-v_b(a_i)+v_b(d_i)}, & \text{otherwise}. \end{cases}$$

Further, we choose the maximum norm on $\mathbb{R}^s$. Then $\rho_g$ is a weight function in the sense of Definition 4.4.

**Theorem 5.15.** Let $\Lambda$ and $\mathcal{F}$ be as above. For every $\epsilon > 0$, there exists an integer vector $g \in \mathbb{N}^s$ such that for any sequence $\omega$ in $[0, 1]^s$, the spectral test $\sigma_N(\omega)$ of the first $N$ elements of $\omega$, with respect to $\mathcal{F}$ and $\rho_g$, has the property

$$|\sigma_N(\omega) - D_N(\omega)| < \epsilon.$$ 

**Proof.** Let $\epsilon > 0$ be given. We choose $g \in \mathbb{N}^s$ such that

$$\max_{1 \leq i \leq s} b_i^{-g_i} < \epsilon/(4s).$$

For the function system $\mathcal{F}$ and for the weight function $\rho_g$ defined in (5.5), we have

$$\sigma_N(\omega) = \max \left\{ \max \{ |S_N(\xi_{(a,d)}, \omega)| : (a, d) \in \Delta_b(g) \times \nabla_b(g) \} , \sup \{ \rho_g((a, d)) |S_N(\xi_{(a,d)}, \omega)| : (a, d) \notin \Delta_b(g) \times \nabla_b(g) \} \right\} .$$

We have

$$D_{N,b,g}(\omega) = \max \{ |S_N(\xi_{(a,d)}, \omega)| : (a, d) \in \Delta_b(g) \times \nabla_b(g) \} ,$$

and

$$\sup \{ \rho_g((a, d)) |S_N(\xi_{(a,d)}, \omega)| : (a, d) \notin \Delta_b(g) \times \nabla_b(g) \} \leq \max_{1 \leq i \leq s} b_i^{-1-g_i}.$$
The choice of $g$ implies

$$D_{N;b,g}(\omega) \leq \sigma_N(\omega) \leq D_{N;b,g}(\omega) + \epsilon/(4s).$$

On the other hand, Theorem 5.11 and Remark 5.12 yield

$$D_{N;b,g}(\omega) \leq D_N(\omega) \leq D_{N;b,g}(\omega) + \epsilon/2.$$

The result follows.

**Corollary 5.16.** In the case of the star discrepancy, put $\Lambda = \mathbb{N}^s$ and let $\mathcal{F}$ be defined accordingly, such that it contains all the functions $1_I - \lambda_s(I)$, $I \in \mathcal{F}_b^*$. For every $\epsilon > 0$, there exists $g \in \mathbb{N}^s$ such that for any sequence $\omega$ in $[0,1]^s$, the spectral test $\sigma_N(\omega)$ of the first $N$ elements of $\omega$, with respect to $\mathcal{F}$ and $\rho_g$, has the property

$$|\sigma_N(\omega) - D_N^*(\omega)| < \epsilon.$$

**Acknowledgments.** My special thanks are due to Roswitha Hofer, University of Linz, for several helpful and most enjoyable discussions.

**Bibliography**

[1] R. Couture and P. L’Ecuyer, Lattice computations for random numbers, *Math. Comp.* 69 (2000), 757–765.

[2] J. Daemen and V. Rijmen, *The Design of Rijndael*, Springer Verlag, New York, 2002.

[3] J. Dick and F. Pillichshammer, *Digital Nets and Sequences: Discrepancy Theory and Quasi-Monte Carlo Integration*, Cambridge University Press, Cambridge, 2010.

[4] M. Drmota and R.F. Tichy, *Sequences, Discrepancies and Applications*, Lecture Notes in Mathematics 1651, Springer, Berlin, 1997.

[5] K. Entacher, P. Hellekalek and P. L’Ecuyer, Quasi-Monte Carlo node sets from linear congruential generators, in: *Monte Carlo and Quasi-Monte Carlo Methods 1998* (H. Niederreiter and J. Spanier, eds.), Springer Lectures Notes in Computational Science and Engineering, Springer, New York, 1999.

[6] G.S. Fishman, *Monte Carlo: Concepts, Algorithms, and Applications*, Springer, New York, 1996.

[7] V. Grozdanov, The weighted $b$-adic diaphony, *J. Complexity* 22 (2006), 490–513.

[8] V. Grozdanov, E. Nikolova and S. Stoilova, Generalized $b$-adic diaphony, *C. R. Acad. Bulgare Sci.* 56 (2003), 23–30.

[9] V. S. Grozdanov and S. S. Stoilova, On the theory of $b$-adic diaphony, *C. R. Acad. Bulgare Sci.* 54 (2001), 31–34.
[10] P. Hellekalek, General discrepancy estimates: the Walsh function system, *Acta Arith.* 67 (1994), 209–218.

[11] ———, Inversive pseudorandom number generators: concepts, results, and links, in: *Proceedings of the 1995 Winter Simulation Conference* (C. Alexopoulos, K. Kang, W.R. Lilegdon and D. Goldsman, eds.), pp. 255–262, 1995.

[12] ———, On the assessment of random and quasi-random point sets, in: *Random and Quasi-Random Point Sets* (P. Hellekalek and G. Larcher, eds.), Lecture Notes in Statistics 138, pp. 49–108, Springer, New York, 1998.

[13] ———, Digital \((t, m, s)\)-nets and the spectral test, *Acta Arith.* 105 (2002), 197–204.

[14] ———, A general discrepancy estimate based on \(p\)-adic arithmetics, *Acta Arith.* 139 (2009), 117–129.

[15] ———, A notion of diaphony based on \(p\)-adic arithmetic, *Acta Arith.* 145 (2010), 273–284.

[16] ———, *Adding digit vectors*, http://arxiv.org/abs/1209.3585, 2012.

[17] ———, Hybrid function systems in the theory of uniform distribution of sequences, in: *Monte Carlo and Quasi-Monte Carlo Methods 2010* (L. Plaskota and H. Woźniakowski, eds.), Springer Proceedings in Mathematics and Statistics 25, pp. 435–449, Springer, Berlin, Heidelberg, 2012.

[18] ———, A hybrid inequality of Erdős-Turán-Koksma for digital sequences, (2013), to appear in *Mh. Math.*

[19] P. Hellekalek and H. Leeb, Dyadic Diaphony, *Acta Arith.* 80 (1997), 187–196.

[20] P. Hellekalek and H. Niederreiter, Constructions of uniformly distributed sequences using the \(b\)-adic method, *Unif. Distrib. Theory* 6 (2011), 185–200.

[21] I.N. Herstein, *Abstract Algebra*, 3rd ed, Wiley, New York, 1999.

[22] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. I*, second ed, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 115, Springer-Verlag, Berlin, 1979.

[23] D.E. Knuth, *The Art of Computer Programming, Vol. 2*, third ed, Addison-Wesley, Reading, Mass., 1998.

[24] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, John Wiley, New York, 1974, reprint, Dover Publications, Mineola, NY, 2006.

[25] X. Lai and J. L. Massey, *A proposal for a new block encryption standard*, Advances in Cryptology—EUROCRYPT ’90 (Aarhus, 1990), Lecture Notes in Comput. Sci. 473, Springer, Berlin, 1991, pp. 389–404.

[26] P. L’Ecuyer, Random number generation, in: *The Handbook of Simulation* (Jerry Banks, ed.), pp. 93–137, Wiley, New York, 1998.

[27] H. Leeb, *Random numbers for computer simulation*, Master’s thesis, Institut für Mathematik, Universität Salzburg, Austria, 1995.
[28] K. Mahler, *p-adic Numbers and Their Functions*, second ed, Cambridge Tracts in Mathematics 76, Cambridge University Press, Cambridge, 1981.

[29] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, SIAM, Philadelphia, 1992.

[30] ______, A discrepancy bound for hybrid sequences involving digital explicit inversive pseudorandom numbers, *Unif. Distrib. Theory* 5 (2010), 53–63.

[31] H. Niederreiter and A. Winterhof, Discrepancy bounds for hybrid sequences involving digital explicit inversive pseudorandom numbers, *Unif. Distrib. Theory* 6 (2011), 33–56.

[32] W. Parry, *Topics in Ergodic Theory*, Cambridge Tracts in Mathematics 75, Cambridge University Press, Cambridge, 2004, Reprint of the 1981 original.

[33] F. Schipp, W.R. Wade and P. Simon, *Walsh Series. An Introduction to Dyadic Harmonic Analysis. With the collaboration of J. Pál.*, Adam Hilger, Bristol and New York, 1990.

[34] I. H. Sloan and S. Joe, *Lattice Methods for Multiple Integration*, Clarendon Press, Oxford, 1994.

[35] I. H. Sloan and P. J. Kachoyan, Lattice methods for multiple integration: theory, error analysis and examples, *SIAM J. Numer. Anal.* 24 (1987), 116–128.

[36] A. Topuzoğlu and A. Winterhof, *Pseudorandom sequences*, Topics in geometry, coding theory and cryptography, Algebr. Appl. 6, Springer, Dordrecht, 2007, pp. 135–166.

[37] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics 79, Springer-Verlag, New York, 1982.

[38] P. Zinterhof, Über einige Abschätzungen bei der Approximation von Funktionen mit Gleichverteilungsmethoden, *Sitzungsber. Österr. Akad. Wiss. Math.-Natur. Kl. II* 185 (1976), 121–132.

**Author information**

Peter Hellekalek, Dept. of Mathematics, University of Salzburg, Hellbrunner Strasse 34, 5020 Salzburg, Austria.

E-mail: peter.hellekalek@sbg.ac.at