Abstract

We present a gentle approach to the justification of effective media approximations, for PDE’s set outside the union of \( n \gg 1 \) spheres with low volume fraction. To illustrate our approach, we consider three classical examples: the derivation of the so-called \textit{strange term}, made popular by Cioranescu and Murat, the derivation of the Brinkman term in the Stokes equation, and a scalar analogue of the effective viscosity problem. Under some separation assumption on the spheres, valid for periodic and random distributions of the centers, we recover effective models as \( n \to +\infty \) by simple arguments.

1 Introduction

Let \( B_{i,n} = B(x_{i,n}, r_n), 1 \leq i \leq n \), a collection of disjoint balls, included in a compact subset \( K \) of \( \mathbb{R}^3 \). We assume convergence of the empirical measure

\[
\frac{1}{n} \sum_{i=1}^{n} \delta_{x_{i,n}} \to \rho(x) dx, \quad n \to +\infty \tag{H0}
\]

where \( \rho \) is a bounded density with support \( K_\rho \subset K \). Moreover, we assume that the volume fraction of the balls

\[
\lambda_n = \frac{4\pi n r_n^3}{3 |K_\rho|} \tag{1}
\]

is small uniformly in \( n \) (and in some cases vanishes as \( n \) goes to infinity).

Our concern is the asymptotics of elliptic PDE’s of Laplace or Stokes type, in the domain \( \Omega_n = \mathbb{R}^3 \setminus \bigcup_i B_{i,n} \):

\[
Lu_n = g \quad \text{in } \Omega_n, \tag{2}
\]

completed with boundary conditions at each ball \( B_i \) and a decay condition at infinity. There are numerous examples from various areas: electrostatics, optics, fluid mechanics, heat conduction and many more. Our main three will be:

i) Diffusion in domains with holes:

\[
-\Delta u_n = g \quad \text{in } \Omega_n, \quad u_n|_{B_{i,n}} = 0, \quad 1 \leq i \leq n. \tag{3}
\]

ii) Drag in fluid flows around obstacles:

\[
-\Delta u_n + \nabla p_n = g, \quad \text{div } u_n = 0 \quad \text{in } \Omega_n, \quad u_n|_{B_{i,n}} = 0, \quad 1 \leq i \leq n. \tag{4}
\]
iii) Permittivity in composites with perfectly conducting inclusions:

\[-\Delta u_n = g \quad \text{in } \Omega_n, \quad u_n|_{B_{i,n}} = u_{i,n}, \quad u_{i,n} \in \mathbb{R}, \quad \int_{\partial B_i} \partial_n u_n d\sigma = -\int_{B_i} g, \quad 1 \leq i \leq n,\]

(5)

with \( \nu \) the unit normal vector pointing outward.

Just like (3) has (4) for fluid counterpart, (5) is the scalar version of the effective viscosity problem

\[-\Delta u_n + \nabla p_n = g, \quad \text{div } u_n = 0 \quad \text{in } \Omega_n, \quad u_n|_{B_{i,n}} = u_{i,n} + \omega_{i,n} \times (x - x_{i,n}) \quad u_{i,n}, \omega_{i,n} \in \mathbb{R}^3, \]

\[
\int_{\partial B_i} (2D(u_n)\nu - p\nu) d\sigma = 0, \quad \int_{\partial B_i} (2D(u_n)\nu - p\nu) \times (x - x_i) d\sigma = 0, \quad 1 \leq i \leq n,
\]

(6)

that could be included in our analysis as well. Other problems fit a similar framework, such as the propagation of waves in bubbly fluids [3].

Back to the general formulation (2), the point is to determine if, for large \( n \) and small volume fraction, the solution \( u_n \) is close to the solution \( u \) of an effective model:

\[Lu + L_e u = 0 \quad \text{in } \mathbb{R}^3\]

(7)

where the extra effective operator \( L_e \) reflects some average effect of the spheres. Moreover, the hope is to express \( L_e \) in terms of macroscopic characteristics of the spheres distribution, like the limit density \( \rho \) and the limit volume fraction \( \lambda = \lim_n \lambda_n \) (when non zero). This hope is supported by the following formal reasoning. If the volume fraction of spherical inclusions is small, one can expect the average distance between the spheres to be large compared to their radius. Therefore, the interaction of the spheres should be negligible, and their contribution to the effective operator should be additive. Furthermore, if the volume fraction vanishes as \( n \) grows, the spheres should be well approximated by points. By combining both arguments, the leading effect of the spheres at large \( n \) should be through the empirical measure. In the limit \( n \to +\infty \), one should then recover \( L_e = L_e[\rho] \).

In this spirit, early formal calculations of effective models were done by Maxwell Garnett [29], Clausius [9], Mossotti [31], to describe heterogeneous electromagnetic media. Similar computation of the effective viscosity of dilute suspensions was done by Einstein [16]. For the drag generated by obstacles in a fluid flow, see Brinkman [5]. Transposed to our three examples, these calculations lead to the following loose statements:

i) For system (3), the critical scaling is \( r_n = \frac{1}{n} \), with convergence of \( u_n \) to the solution \( u \) of

\[-\Delta u + 4\pi \rho u = g \quad \text{in } \mathbb{R}^3.\]

(8)

ii) For system (4), the critical scaling is \( r_n = \frac{1}{n} \), with convergence of \( u_n \) to the solution \( u \) of

\[-\Delta u + \nabla p + 6\pi \rho u = g, \quad \text{div } u = 0 \quad \text{in } \mathbb{R}^3.\]

(9)

iii) For system (5), the critical scaling is when \( \lambda = \lim_n \lambda_n \) is small but non zero. One has in this case \( u_n = u + o(\lambda) \) uniformly in \( n \), with

\[-\text{div} \left( (1 + 3\lambda|K_\rho|\rho)\nabla u \right) = g \quad \text{in } \mathbb{R}^3.\]

(10)
By critical scaling, we mean that other scalings would lead to trivial limits: typically, in system (3), one has \( u = 0 \) if \( r_n \gg \frac{1}{n} \), while it solves the original Laplace equation if \( r_n \ll \frac{1}{n} \).

Turning these assertions into rigorous mathematical statements has attracted a lot of attention since 1970. Pioneering results on (3) are due to Hruslov and Marchenko [28], Cioranescu and Murat [8], Papanicolaou and Varadhan [34], or Ozawa [33]. A more abstract approach through \( \Gamma \)-convergence was completed in [10, 11]. For the most up to date statements, we refer to the nice study by Giunti, Höfer and Velázquez [20], and to the bibliography therein. On the asymptotics of (4), one can mention the work of Allaire [1], as well as improvements by Desvillettes, Golse and Ricci [14], Hillairet [23], or Höfer [25]. As regards the analysis of (5) and of the more involved fluid problem (6), rigorous results were obtained in Levy and Sanchez-Palencia [35, 27], Ammari et al [2], Haines and Mazzucato [22], as well as in the recent papers by Hillairet and Wu [24], Niethammer and Schubert [32], or the author and Hillairet [18]. These references are by no mean exhaustive: one could further cite [26, 6, 7, 30, 15, 21] and many more on related problems.

As explained before, the derivation of the limit systems (8), (9) and (10) is based on neglecting the interaction between the spheres. It requires some separation assumption on the centers \( x_{i,n} \) of the spheres. The challenge is to show convergence under the mildest possible separation assumption. This is what differentiates the works mentioned above. Three types of assumptions emerge from the literature:

- the most stringent one is that the centers are periodically distributed over a mesh of typical length \( n^{-1/3} \). This is assumed for instance in [8], [1] or in the studies [35, 27] as well as in [22]. Note that in such a case, the limit density \( \rho \) is constant over its support: \( \rho = |K_\rho|^{-1}K_\rho \).

- Other studies relax the periodicity assumption, but keep in addition to (H0) a condition on the minimal distance between the centers:

\[
\inf_{i \neq j} [x_{i,n} - x_{j,n}] \geq cn^{-1/3}.
\]  

This is the case in [33], or in [14], and as far as we know in all works on the effective permittivity problem (5) or on the effective viscosity problem (6). Although it is an improvement to the periodic case, such assumption is not fully satisfactory: in particular, it is not satisfied if the centers are drawn randomly and independently with law \( \rho(x)dx \). Indeed, in this latter case, it is known that the typical minimal distance scales like \( n^{-2/3} \).

- Eventually, the asymptotics of system (3) or (4) may be established under weaker assumptions, which apply when the centers \( x_{i,n} \) are given by i.i.d. random variables, or derived from reasonable stationary point processes. A big step in that direction was made in [34]: roughly\(^1\), the solution \( u_n \) of (3) tends to the solution \( u \) of (8) under the weak separation assumption

\[
\frac{1}{n^2} \sum_{i \neq j} \frac{1}{|x_{i,n} - x_{j,n}|} \to \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{|x - y|} \rho(x)\rho(y)dxdy, \quad n \to +\infty
\]  

It is shown in [34] that this convergence holds in probability in the i.i.d. case. Recently, almost sure convergence of \( u_n \) to \( u \) was proved in [20] for a large class of stationary point processes, allowing for random radii of the spheres. See also [19] for the analogue in the Stokes case.

\(^1\)actually, [34] treats the time dependent version of the equations
The works mentioned above may rely on very different techniques. Still, key arguments are often difficult and based on explicit constructions. For instance, the work of Cioranescu and Murat, as well as the refined analyses [1, 20], are based on the construction of so-called correctors, which may be quite technical notably in the Stokes case. Another instance is [34], where the proof relies on the explicit representation of \( u_n \) with the Feynman-Kac formula, and on non-trivial manipulations of this formula. In particular, none of these methods really exploits the fact that the formal limit equation is already known. The goal of this paper is to present a softer and shorter approach to these asymptotic problems. This approach relies on separation hypotheses of the same type as (12), although slightly stronger. In the context of problems (3) and (4), our assumption covers the periodic and random i.i.d. cases. In the context of (5), we derive the effective permittivity formula under a more general condition than (11), which to our knowledge is new. Still, we insist that the point of this paper is not the novelty of our statements, for which much more refined versions often exist. Our goal is rather to give a less constructive method than in former works.

2 Strategy and hypotheses for the convergence

Let us explain our idea to go rigorously from (2) to (7). Given a smooth and compactly supported test field \( \varphi \) on \( \mathbb{R}^3 \), the point is to introduce a solution \( \phi_n = \phi_n(\varphi) \) of

\[
L^* \phi_n = -L^*_e \varphi \quad \text{in } \Omega_n
\]

completed with appropriate boundary conditions at \( \partial \Omega_n \). By appropriate, we mean that for natural extensions of \( u_n \) and \( \phi_n \) inside \( \bigcup_i B_{i,n} \), still denoted \( u_n \) and \( \phi_n \), one has the identity

\[
\int_{\Omega_n} (Lu_n - g) (\varphi - \phi_n) = \int_{\mathbb{R}^3} (Lu_n + L_e u_n - g) \varphi + \int_{\mathbb{R}^3} g \phi_n \tag{14}
\]

which in the case without boundaries would come from the formal calculations:

\[
\int_{\mathbb{R}^3} Lu_n (\varphi - \phi_n) = \int_{\mathbb{R}^3} Lu_n \varphi - \int_{\mathbb{R}^3} u_n L^* \phi_n = \int_{\mathbb{R}^3} Lu_n \varphi + \int_{\mathbb{R}^3} u_n L^*_e \varphi = \int_{\mathbb{R}^3} (L + L_e) u_n \varphi.
\]

By (2), it will result from (14) that

\[
\int_{\mathbb{R}^3} (Lu_n + L_e u_n - g) \varphi + \int_{\mathbb{R}^3} g \phi_n = 0.
\]

Hence, any accumulation point \( u \) of \( u_n \) satisfies

\[
\left| \int_{\mathbb{R}^3} (Lu + L_e u - g) \varphi \right| \leq \limsup_n \left| \int_{\mathbb{R}^3} g \phi_n \right|
\]

To obtain (7) in the limit \( n \to +\infty \) (or at least up to \( o(\lambda) \) terms), it is then enough to show that \( \phi_n \) converges to 0 weakly (or at least up to \( o(\lambda) \) term). This can be done using relatively soft arguments, in contrast with more involved constructions of correctors.

Roughly, in the case where \( L_e = L_e(\rho) \), the idea is that

\[
\phi_n \approx G^* * \left( -L^*_e(\rho) \varphi \right) - G^* * \left( -L^*_e [n^{-1} \sum \delta_x_i] \varphi \right) \quad \text{in } \Omega_n,
\]

with \( G^* \) the fundamental solution of \( L^* \). The first term at the right-hand side takes care of the source term in (13), while the second term takes care of the boundary conditions at the spheres, neglecting the interaction between them. This r.h.s goes (formally) to zero. Hence,
the main part of the reasoning is to prove that it indeed provides a good approximation of \( \phi_n \), which is true under separation assumptions of type (12).

More precisely, in the context of system (3), with \( r_n = \frac{1}{n} \), our hypotheses will be
\[
\inf |x_i - x_j| \geq \frac{c}{n}, \quad i \neq j, \quad c > 2 \quad (H1)
\]
\[
\limsup_{n \to +\infty} \sum_i \int_{B_i} \left( n^2 \left| \frac{1}{n} \sum_{j \neq i} G(x - x_j) - \int G(x - y) \rho(y) dy \right|^2 + \frac{1}{n^2} \left| \sum R_{St}(x - x_j) \right|^2 \right) dx = 0. \quad (H2)
\]

We have noted \( x_i = x_{i,n} \), \( B_i = B_{i,n} \) for short, and \( G(x) = \frac{1}{4\pi |x|^3} \) the kernel of \(-\Delta\). Settings in which (H1)-(H2) are satisfied (including periodic and random i.i.d. cases) will be discussed in the last section 4. We state

**Theorem 1.** Let \( g \in L^6(\mathbb{R}^3) \), \( r_n = \frac{1}{n} \). Under (H0)-(H1)-(H2), the solution \( u_n \) of (3) converges weakly in \( \dot{H}^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \) to the solution \( u \) of (8).

In the context of the Stokes equation (4), assumption (H2) must be modified into
\[
\limsup_{n \to +\infty} \sum_i \int_{B_i} \left( n^2 \left| \frac{1}{n} \sum_{j \neq i} G_{St}(x - x_j) - \int G_{St}(x - y) \rho(y) dy \right|^2 + \frac{1}{n^2} \left| \sum R_{St}(x - x_j) \right|^2 \right) dx = 0 \quad (H2')
\]

with \( G_{St}(x) = \frac{1}{8\pi} \left( \frac{1}{|x|^3} + \frac{x \otimes x}{|x|^5} \right) \) the kernel of \(-\mathbb{P}\Delta\), and \( R_{st}(x) = \frac{1}{8\pi} \left( \frac{1}{|x|^3} - \frac{x \otimes x}{|x|^5} \right) \).

**Theorem 2.** Let \( g \in L^6(\mathbb{R}^3)^3 \), \( r_n = \frac{1}{n} \). Under (H0)-(H1)-(H2'), the solution \( u_n \) of (4) converges weakly in \( \dot{H}^1(\mathbb{R}^3)^3 \cap L^6(\mathbb{R}^3)^3 \) to the solution \( u \) of (9).

Eventually, system (5) can be analyzed in the regime where the volume fraction \( \lambda_n = \lambda \) is small but independent of \( n \). We will rely this time on assumptions
\[
\inf |x_i - x_j| \geq cr_n, \quad i \neq j, \quad c > 2, \quad \text{(A1)}
\]

and for all smooth \( \varphi \),
\[
\limsup_{n \to +\infty} \frac{1}{n^2} \sum_i \int_{B_i} \left| \sum_{j \neq i} \nabla V(x - x_j) \nabla \varphi(x_j) \right|^2 dx \leq \eta(\lambda) \|
\nabla \varphi\|_{L^\infty}, \quad \eta(\lambda) \xrightarrow{\lambda \to 0} 0, \quad \text{(A2)}
\]

where \( V(x) = \nabla \frac{1}{4\pi |x|^3} \). This last assumption will be discussed further in Section 4. It will be notably shown to be implied by the usual condition (11). We shall prove

**Theorem 3.** Let \( g \in L^6(\mathbb{R}^3) \). Assume \( \lambda_n = \lambda \) for all \( n \). Let \( u_{n,\lambda} \) the solution of (5). Under (H0)-(A1)-(A2), any accumulation point \( u_\lambda \) of \( u_{n,\lambda} \) in \( \dot{H}^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \) solves
\[
-\text{div} \left( (1 + 3\lambda |K_\rho|) \nabla u \right) = g + R_\lambda \quad \text{in } \mathbb{R}^3, \quad \text{(15)}
\]

with a remainder \( R_\lambda \) satisfying
\[
\langle R_\lambda, \varphi \rangle \leq o(\lambda) \|
\nabla \varphi\|_{L^\infty}, \quad \forall \varphi \in \dot{H}^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \cap \text{Lip}(\mathbb{R}^3). \quad \text{(16)}
\]
3 Proofs

3.1 Proof of Theorem 1

We extend $u_n$ by zero inside the balls, and still denote $u_n$ this extension. A simple energy estimate of (3) together with Sobolev imbedding yields

$$\|\nabla u_n\|_{L^2(\mathbb{R}^3)} \leq \|g\|_{L^6(\mathbb{R}^3)} \|u_n\|_{L^6(\mathbb{R}^3)} \leq C\|g\|_{L^6(\mathbb{R}^3)} \|\nabla u_n\|_{L^2(\mathbb{R}^3)}$$

so that $u_n$ is bounded in $\dot{H}^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$. Up to a subsequence in $n$, it has a weak limit $u$. Now, given $\varphi \in C_c^\infty(\mathbb{R}^3)$, we introduce $\phi_n$ the solution of

$$-\Delta \phi_n = -4\pi \rho \varphi \quad \text{in } \Omega_n, \quad \phi_n\big|_{\cup B_i} = \varphi. \quad (17)$$

Taking $\varphi - \phi_n$ as a test function in (3), we get by Green’s formula:

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla \varphi + 4\pi \int_{\mathbb{R}^3} \rho u_n \varphi = \int_{\mathbb{R}^3} g \varphi - \int_{\mathbb{R}^3} g \phi_n.$$

We have used here that $u_n$ and $\varphi - \phi_n$ vanish inside the balls to replace integrals over $\Omega_n$ by integrals over $\mathbb{R}^3$. Now, if we prove that the last term at the right-hand side goes to zero, then $u \in \dot{H}^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$ will be a variational solution of (8). As this solution is unique, the full sequence $u_n$ will converge to $u$, proving the theorem. As $g$ is arbitrary in $L^6(\mathbb{R}^3)$, we need to prove that $\phi_n$ converges weakly to zero in $L^6(\mathbb{R}^3)$. By a standard energy estimate on $\phi_n - \varphi$, which vanishes at the balls, it is easily seen that $\phi_n$ is bounded in $\dot{H}^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$, so that it is enough to show convergence of $\phi_n$ to zero in the sense of distributions.

Actually, we can further simplify the analysis by introducing the solution $\Phi_n$ of

$$-\Delta \Phi_n = -4\pi \rho \quad \text{in } \Omega_n, \quad \Phi_n\big|_{\cup B_i} = 1. \quad (18)$$

Indeed, let $R_n = \phi_n - \Phi_n \varphi$, that satisfies

$$-\Delta R_n = 2\nabla \Phi_n \cdot \nabla \varphi + \Phi_n \Delta \varphi \quad \text{in } \Omega_n, \quad R_n\big|_{\cup B_i} = 0. \quad (19)$$

If we prove that $\Phi_n \to 0$ strongly in $L^2_{loc}$, then, by a standard energy estimate, $R_n \to 0$ strongly in $\dot{H}^1_{loc}$, and eventually $\phi_n \to 0$ strongly in $L^2_{loc}$.

For $\eta > 0$, we denote $\delta_n^1 = \frac{1}{4\pi n^2} \sum_i s_n(\cdot - x_i)$ where $s_n$ is the surface measure on the sphere of radius $\eta$. To prove strong convergence of $\Phi_n$ to zero in $L^2_{loc}$, we split $\Phi_n = \Phi_n^1 + \Phi_n^2$, where

$$\Phi_n^1 = 4\pi \sum_i G(n(x - x_i)) - 4\pi G \ast \rho$$

with $G(x) = \frac{1}{4\pi|x|}$ the fundamental solution of $-\Delta$, $G(x) = G(x)$ if $|x| \geq 1$, $G(x) = \frac{1}{4\pi}$ if $|x| \leq 1$. We compute

$$-\Delta \Phi_n^1 = 4\pi \delta_n^{1/n} - 4\pi \rho \quad \text{in } \mathbb{R}^3 \quad (20)$$

and $\Phi_n^1(x) = 1 + \frac{4\pi}{n} \sum_{j \neq i} G(x - x_j) - 4\pi \int_{\mathbb{R}^3} G(x - y) \rho(y) dy$ for $x \in B_i$, for all $i$. Hence,

$$-\Delta \Phi_n^2 = 0 \quad \text{in } \Omega_n, \quad \Phi_n^2|_{B_i}(x) = -\frac{4\pi}{n} \sum_{j \neq i} G(x - x_j) + 4\pi \int_{\mathbb{R}^3} G(x - y) \rho(y) dy \quad (21)$$
Now, we use that under assumption (H1), there is $C > 0$ such that for all $w = (w_i)_{1 \leq i \leq n}$ in $\prod_i H^1(B_i)$, the solution $u[w] \equiv -\Delta u[w] = 0$ in $\Omega_n$, $u[w]|_{B_i} = w_i$ for all $i$ satisfies

$$\|\nabla u[w]\|_{L^2(\mathbb{R}^3)} \leq C \sum_i \left(n^2\|w_i\|_{L^2(B_i)}^2 + \|\nabla w_i\|_{L^2(B_i)}^2\right)$$

(22)

We refer to Lemma 5 for a proof in the slightly more difficult Stokes case. Note that the factor $n^2$ at the right-hand side of the first inequality is consistent with scaling considerations. By applying this inequality with $w_i = -\frac{4\pi}{n} \sum_j \delta_{ij} G(x - x_j) + 4\pi \int_{\mathbb{R}^3} G(x - y) \rho(y) dy$, noticing that $\sum_i \int_{B_i} |\int_{\mathbb{R}^3} \nabla G(x - y) \rho(y) dy|^2 \|x\| = O(\frac{1}{n^2})$, and combining with (H2), we find that $\|\nabla \Phi_n\|_{L^2(\mathbb{R}^3)}$ goes to zero, so that $\Phi_n$ goes to zero strongly in $H^1_{loc}$.

The last step is to show that $\Phi_n$ goes strongly to zero in $L^2_{loc}$. As the right-hand side of (20) is bounded in $W^{-1,p}(\mathbb{R}^3)$ for any $p < \frac{3}{2}$, $\Phi_n$ is bounded in $W^{1,p}$ for any $p < \frac{3}{2}$, with compact embedding in $L^q_{loc}$ for any $q < 3$. Hence, it is enough to prove convergence to zero in the sense of distributions. Let $\psi \in C_c^{\infty}(\mathbb{R}^3)$. Clearly the function $\Psi = \Delta^{-1}\psi$ is well-defined and smooth. We get

$$\int_{\mathbb{R}^3} \Phi_n \psi = \int_{\mathbb{R}^3} \Phi_n \Delta \Psi = \langle \Delta \Phi_n, \psi \rangle = 4\pi \langle \rho - \delta_1^{1/n}, \Psi \rangle \to 0$$

by the assumption (H0). This concludes the proof.

### 3.2 Proof of Theorem 2

Standard estimates show that the sequence $u_n$ of solutions of (4) is bounded in $H^1(\mathbb{R}^3)^3 \cap L^6(\mathbb{R}^3)^3$. Let $\varphi \in C_c^{\infty}(\mathbb{R}^3)^3$ a divergence-free vector field. We introduce the solution $\phi_n$ of

$$-\Delta \phi_n + \nabla q_n = -6\pi \rho \varphi, \; \; \text{div} \phi_n = 0 \text{ in } \Omega_n, \; \; \phi_n|_{B_i} = \varphi. \quad (23)$$

Arguing exactly as in Paragraph 3.1, to show the weak convergence of $u_n$ to the solution $u$ of (9), it is enough to prove the convergence of $\phi_n$ to zero in the sense of distributions. We consider this time the matrix-valued solution $\Phi_n$ of

$$-\Delta \Phi_n + \nabla Q_n = -6\pi \rho I, \; \; \text{div} \Phi_n = 0 \text{ in } \Omega_n, \; \; \Phi_n|_{B_i} = I, \quad (24)$$

where $I$ denotes the identity matrix. With $R_n = \phi_n - \Phi_n \varphi$, and $S_n = q_n - Q_n \cdot \varphi$, we find

$$-\Delta R_n + \nabla S_n = 2\partial_i \Phi_n \partial_i \varphi + \Phi_n \Delta \varphi + (\nabla \varphi) Q_n, \; \; \text{div} R_n = \Phi_n : \nabla \varphi \; \text{ in } \Omega_n, \; \; R_n|_{B_i} = 0.$$

We admit for the time being

**Lemma 4.** Let $q > 3$, and $U$ a smooth bounded domain containing $K$. There exists a family of operators $B_n : L^q(\Omega_n \cap U) \to H^1_0(\Omega_n \cap U)$ such that $\text{div} B_n h = h$ for all $h \in L^q(\Omega_n \cap U)$ and such that $\|B_n\|_{L(H^1_0, L^q)}$ is bounded uniformly in $n$.

Let $U$ containing both $K$ and the support of $\varphi$. One can check easily that $\Phi_n : \nabla \varphi$ has zero average in both $U$ and $\Omega_n \cap U$. Extending $B_n(\Phi_n : \nabla \varphi)$ by zero outside $U$, the field $R_n = R_n - B_n(\Phi_n : \nabla \varphi)$ satisfies

$$-\Delta R_n + \nabla S_n = 2\partial_i \Phi_n \partial_i \varphi + \Phi_n \Delta \varphi + (\nabla \varphi) Q_n + \Delta B_n(\Phi_n : \nabla \varphi) \; \text{ in } \Omega_n,$$
plus divergence-free and homogeneous Dirichlet conditions. By the previous lemma and a standard estimate, if we prove that $\Phi_n \to 0$ strongly in $L_{\text{loc}}^q$ for some $q > 3$ and that $Q_n \to 0$ strongly in $H^{-1}(U)$ (for some appropriate extension in $\cup_i B_i$), then $R_n' \to 0$ strongly in $H_{\text{loc}}^1$, and so does $R_n$. Eventually, $\phi_n$ will go to zero.

Therefore, we decompose $\Phi_n = \Phi_1^n + \Phi_2^n$, $Q_n = Q_1^n + Q_2^n$. This time, 
\[ \Phi_1^n(x) = 6\pi \sum_i g_{SI}(n(x - x_i)) - 6\pi G_{SI} \ast \rho. \]

Here, $G_{SI}(x) = \frac{1}{4\pi} \left( \frac{1}{|x|} + \frac{x^2}{|x|^3} \right)$ is the kernel of $-\nabla\Delta$, and for $R_{SI}(x) = \frac{1}{4\pi} \left( \frac{1}{|x|} - \frac{x^2}{|x|^3} \right)$, we have set 
\[ g_{SI}(x) = G_{SI}(x) + R_{SI}(x) \text{ if } |x| \geq 1, \quad g_{SI}(x) = \frac{1}{6\pi} \text{ if } |x| \leq 1. \]

A tedious calculation shows that 
\[ -\Delta \Phi_1^n + \nabla Q_1^n = 6\pi(\delta_{n}^{1/n} - \rho)I, \quad \text{div} \Phi_1^n \quad \text{in} \quad \mathbb{R}^3. \tag{25} \]

while
\[ -\Delta \Phi_2^n + \nabla Q_2^n = 0, \quad \text{div} \Phi_2^n = 0 \quad \text{in} \quad \Omega_n \]
with boundary conditions 
\[ \Phi_2^n|_{B_i}(x) = -\frac{6\pi}{n} \sum_{j \neq i} G_{SI}(x - x_j) + 6\pi G_{SI} \ast \rho(x) - \frac{6\pi}{n} \sum_{j \neq i} R_{SI}(x - x_j), \quad x \in B_i. \]

To estimate $\Phi_2^n$, we use the following (see below for a proof):

**Lemma 5.** Under (H1), inequality (22) is true for all (matrix or vector-valued) family $(w_i)_{1 \leq i \leq n}$ in $\prod_{i} W^{1,\infty}(B_i)$ satisfying the compatibility condition $\int_{\partial B_i} w_i \cdot n = 0$ for all $i$, and $u[w]$ the solution of 
\[ -\Delta u[w] + \nabla p[w] = 0, \quad \text{div} u[w] = 0 \quad \text{in} \quad \Omega_n, \quad u[w]|_{B_i} = w_i \quad \text{for all} \quad i. \]

Using this lemma and (H2'), we find that $\Phi_2^n$ converges strongly to zero in $H^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$. Thanks to Lemma 4, we can then show that for any $p < \frac{3}{2}$, the pressure term $Q_2^n$, normalized to be mean free in $\Omega_n \cap U$ and extended by zero in $\cup_i B_i$, converges strongly in $L^p(U)$. Indeed, for any $q > 3$ and any $h \in L^2(U)$, denoting $h_0 = h - \int_{\Omega_n \cap U} h$, we find
\[ \int_{U} Q_2^n h = \int_{\Omega_n \cap U} Q_2^n h_0 = \int_{\Omega_n \cap U} Q_2^n \text{div} B_n h_0 = -\int_{\Omega_n \cap U} \nabla Q_2^n B_n h = -\int_{\Omega_n \cap U} \Delta \Phi_2^n B_n h_0 \]
\[ = \int_{\Omega_n \cap U} \nabla \psi_2^n \cdot \nabla B_n h_0 \leq C\|\nabla \Phi_2^n\|_{L^2(\Omega_n \cap U)} \|\nabla B_n h\|_{L^2(\Omega_n \cap U)} \]
\[ \leq C\|\nabla \Phi_2^n\|_{L^2(\mathbb{R}^3)} \|h_0\|_{L^2(U)} \leq C'\|\nabla \Phi_2^n\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(U)}. \]

The strong convergence of $Q_2^n$ in $L^p$, $p = q'$, follows.

We finally have to look at the convergence of $(\Phi_1^n, Q_1^n)$ solving (25). The source $6\pi(\delta_{n}^{1/n} - \rho)I$ is compactly supported, and converges to zero in the sense of measures, in particular weakly in $W^{-1,p}(\mathbb{R}^3)$ for any $p < \frac{3}{2}$. Moreover, for an appropriate normalization of the pressure $Q_1^n$, we have the estimate 
\[ \|\nabla \Phi_1^n\|_{L^p(\mathbb{R}^3)} + \|\Phi_1^n\|_{L^{3p/(3-p)}(\mathbb{R}^3)} + \|Q_1^n\|_{L^p(\mathbb{R}^3)} \leq C\|6\pi(\delta_{n}^{1/n} - \rho)I\|_{W^{-1,p}(\mathbb{R}^3)}. \]
Indeed, assumption \((\text{radius } 1, \text{application of usual Bogovski operators yield for each } 1)\)

Proof of Lemma 4. Let \(q > 3\), and \(h \in L^q_\text{loc}(\Omega \cap U)\). We extend \(h\) by zero in \(\cup B_i\). First, we introduce a standard Bogovski operator \(B : L^q(U) \to W^{1,q}_0(U)\) which is continuous and satisfies \(\text{div } Bh = h\). The next step is to find a field \(w_n = w_n[h] \in H^1_0(U)\) such that

\[
\text{div } w_n = 0 \quad \text{in } U \cap \Omega, \quad w_n|_{B_i} = Bh|_{B_i}, \quad \|w_n\|_{H^1_0} \leq C\|Bh\|_{W^{1,q}}. \quad C \text{ independent of } n.
\]

Indeed, \(B_n h = Bh - w_n\) will have the required properties. As the balls \(n B_i = B(nx_i, 1)\) have radius 1, application of usual Bogovski operators yield for each \(1 \leq i \leq n\) a divergence-free function \(W_i \in H^1_0(B(nx_i, 2))\) such that

\[
W_i|_{nB_i} = Bh(n^{-1} \cdot)|_{nB_i}, \quad \|W_i\|_{H^1_0(B(nx_i, 2))} \leq C\|Bh(n^{-1} \cdot)\|_{H^1(nB_i)}, \quad C \text{ independent of } n.
\]

We set \(w_n(x) = \sum_i W_i(nx)\). Clearly, \(w_n\) is divergence-free, and belongs to \(H^1_0(U)\). Moreover, assumption (H1) implies that the balls \(B(nx_i, 2)\) are disjoint, so that \(w_n|_{B_i} = W_i(n^{-1} \cdot)|_{B_i} = Bh|_{B_i}\), and

\[
\|w_n\|^2_{H^1_0(U)} \leq C\|
abla w_n\|^2_{L^2(U)} = \sum_i \|\nabla (W_i(n^{-1} \cdot))\|^2_{L^2(B(nx_i, 2/n))}
\leq C \sum_i \left(n^2 \|Bh\|^2_{L^2(B_i)} + \|\nabla Bh\|^2_{L^2(B_i)}\right)
\leq \frac{4\pi C}{3} \|Bh\|_{L^\infty(U)} + C\|\nabla Bh\|^2_{L^2(U)} \leq C'\|Bh\|_{W^{1,q}}(U).
\]

where the last inequality involves the Sobolev embedding \(W^{1,q} \hookrightarrow L^\infty\). This ends the proof.

Proof of Lemma 5. By the classical variational characterization of the Stokes solution,

\[
\|
abla u[w]\|^2_{L^2(\mathbb{R}^3)} \leq \|
abla v[w]\|^2_{L^2(\mathbb{R}^3)}
\]

for any divergence-free vector field \(v = v[w]\) such that \(v|_{B_i} = w_i\) for all \(i\). Hence, it is enough to prove that there exists such a \(v\) satisfying the bound (22). One proceeds as in the proof of Lemma 4: one looks for \(v\) under the form \(v = \sum_i W_i(nx)\), where \(W_i \in H^1_0(B(nx_i, 2))\) is provided through the use of Bogovski operator:

\[
W_i|_{nB_i} = w_i(n^{-1} \cdot)|_{nB_i}, \quad \|W_i\|_{H^1_0(B(nx_i, 2))} \leq C\|w_i(n^{-1} \cdot)\|_{H^1(nB_i)}, \quad C \text{ independent of } n.
\]

Estimate (22) follows as in the proof of Lemma 4.
3.3 Proof of Theorem 3

We remind that here, \( r_n \) is of order \( n^{-1/3} \), namely with a volume fraction \( \lambda = n^{4/3} r_n^3 \) positive and independent of \( n \). As in other examples, the energy estimate and the Sobolev imbedding give that the sequence \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( H^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \). After extraction, it converges weakly to some \( u_\lambda \). Let \( \varphi \in \dot{H}^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3) \). We consider \( \phi_{n,\lambda} \) satisfying

\[
-\Delta \phi_{n,\lambda} = \text{div} (3\lambda |K_\rho| \rho \nabla \varphi) \text{ in } \Omega_n, \quad \int_{\partial B_i} \partial_\nu \phi_{n,\lambda} = -\int_{\partial B_i} 3\lambda |K_\rho| \rho \partial_\nu \varphi, \phi_{n,\lambda}|_{B_i} = \varphi|_{B_i} + \text{cst}, \forall i. \tag{26}
\]

Testing \( \varphi - \phi_{n,\lambda} \) in (5), we obtain after a few integrations by parts

\[
\int_{\mathbb{R}^3} \nabla u_{n,\lambda} \cdot \left( 1 + 3\lambda |K_\rho| \rho \right) \nabla \varphi = \int_{\mathbb{R}^3} g \varphi - \int_{\mathbb{R}^3} g \varphi_{n,\lambda}.
\]

As \( n \) goes to infinity, the left-hand side converges to \( \int_{\mathbb{R}^3} \nabla u_{\lambda} \cdot (1 + 3\lambda f) \nabla \varphi \), which implies that the last term at the right-hand side, that can be written in the abstract form \( \langle R_{n,\lambda}, \varphi \rangle \) for an element \( R_{n,\lambda} \) in the dual of \( \dot{H}^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \), converges to some \( \langle R_{\lambda}, \varphi \rangle \), with \( R_{\lambda} \) in this same dual. To prove the theorem, it is enough to show that

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^3} g \varphi_{n,\lambda} \leq \eta \lambda \| \nabla \varphi \|_{L^\infty}, \quad \eta \lambda \to 0, \tag{27}
\]

Let us stress that restricting to smooth test functions \( \varphi \) instead of Lipschitz is no problem: indeed, if (27) holds for smooth functions, we can apply it to \( \rho_\delta \ast \varphi \) with \( \rho_\delta \) a mollifier. We deduce

\[
\langle R_{\lambda}, \rho_\delta \ast \varphi \rangle \leq \lambda \eta \lambda \| \nabla \rho_\delta \ast \varphi \|_{L^\infty} \leq \lambda \eta \lambda \| \nabla \varphi \|_{L^\infty}
\]

As \( \rho_\delta \ast \varphi \) converges in \( H^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \), the l.h.s. converges to \( \langle R_{\lambda}, \varphi \rangle \), which yields (16).

We split \( \phi_{n,\lambda} = \phi^1_n + \phi^2_n \), with

\[
\phi^1_n = 3\lambda |K_\rho| V \ast (\rho \nabla \varphi) - 4\pi r_n \sum \nabla \varphi(x_i) \cdot V \left( \frac{x - x_i}{r_n} \right),
\]

where \( V(x) = \nabla \frac{1}{4\pi |x|}, V(x) = V(x) \text{ for } |x| \geq 1, V(x) = -\frac{1}{4\pi} \text{ for } |x| \leq 1. \)

We compute \( \Delta V = -\frac{n}{4\pi r_n^2} \), with \( s^n \) the surface measure on the sphere of radius \( \eta \). Hence,

\[
-\Delta \phi^1_n = \text{div} \left( 3\lambda |K_\rho| \rho \nabla \varphi \right) + \frac{3}{r_n^2} \sum \nabla \varphi(x_i) \cdot (x - x_i) s^n(x - x_i)
\]

\[
= \text{div} \left( 3\lambda |K_\rho| \rho \nabla \varphi \right) - 3 \sum \nabla \varphi(x_i) \cdot \nabla 1_{B_i} \text{ in } \mathbb{R}^3 \tag{28}
\]

\[
= \text{div} \left( 3\lambda |K_\rho| \rho \nabla \varphi - 3\lambda |K_\rho| \left( \frac{1}{n} \sum |B_i| 1_{B_i} \nabla \varphi(x_i) \right) \right) \text{ in } \mathbb{R}^3.
\]

One also checks that \( \int_{\partial B_i} \partial_\nu \phi^1_n = -\int_{\partial B_i} 3\lambda \rho \partial_\nu \varphi \). We further decompose \( \phi^2_n = \psi^2_n + \tilde{\psi}^2_n \), with

\[
-\Delta \psi^2_n = -\Delta \tilde{\psi}^2_n = 0, \quad \int_{\partial B_i} \partial_\nu \psi^2_n = \int_{\partial B_i} \partial_\nu \tilde{\psi}^2_n = 0,
\]

and for all \( i \):

\[
\psi^2_{n,i}|_{B_i}(x) = -3\lambda |K_\rho| V \ast (\rho \nabla \varphi)(x) + 4\pi r_n \sum_{j \neq i} \nabla \varphi(x_j) \cdot V \left( \frac{x - x_j}{r_n} \right) + \text{cst}
\]

\[
\tilde{\psi}^2_{n,i}|_{B_i}(x) = \varphi(x) - \nabla \varphi(x_i) \cdot (x - x_i) + \text{cst}.
\]

10
By classical variational characterization, the function \( \psi \), resp. \( \tilde{\psi} \), minimizes the Dirichlet integral among all functions \( \psi \in H^{1}(\mathbb{R}^3) \) satisfying \( \psi|_{B_i} = \psi_i|_{B_i} + c_i \), resp. \( \psi|_{B_i} = \tilde{\psi}_i|_{B_i} + c_i \), on \( B_i \), for some \( c_i \in \mathbb{R} \), for all \( i \). From there, one can proceed as in the proof of (22), cf. Lemma 5, and show that under (A1):

\[
\|\nabla \psi \|^2_{L^2(\mathbb{R}^3)} \leq C \sum_i \int_{B_i} |\nabla \psi_i|^2, \quad \|\nabla \tilde{\psi} \|^2_{L^2(\mathbb{R}^3)} \leq C \sum_i \int_{B_i} |\nabla \tilde{\psi}_i|^2.
\]  

(29)

Note that, in contrast to (22), the right-hand side involves only gradients, consistently with the additional degree of freedom provided by possible addition of constants \( c_i \) on \( B_i \). In the case of \( \tilde{\psi}_i \), we get

\[
\|\nabla \tilde{\psi}_i \|^2_{L^2(\mathbb{R}^3)} \leq C \|\nabla \varphi \|^2_{L^\infty} \sum_i \int_{B_i} |x - x_i|^2 dx = O(n^{-2/3}).
\]

(30)

As regards \( \psi \), we compute

\[
\nabla \psi_i|_{B_i}(x) = -3\lambda|K_\rho| \int \rho(y) \nabla V(x - y) \nabla \varphi(y) dy + \frac{3\lambda|K_\rho|}{n} \sum_{j \neq i} \nabla V(x - x_j) \nabla \varphi(x_j)
\]

\[
= -b(x) + a_i(x),
\]

where \( \nabla V = -\frac{1}{|x|} \nabla |x|^{-1} \) is a Calderon-Zygmund kernel. Hence, \( b \) satisfies for all \( q > 1 \):

\[
\|b\|_{L^q(\mathbb{R}^3)} \leq C_q \lambda \|\rho \nabla \varphi\|_{L^q(\mathbb{R}^3)} \leq C_q \lambda \|\rho \nabla \varphi\|_{L^q(K)}
\]

It follows by Hölder inequality that for all \( q > 2 \)

\[
\sum_i \|b_i\|^2_{L^q(B_i)} = \|b\|^2_{L^q(\cup B_i)} \leq \frac{\lambda^{q/2}}{\lambda^{q-2}} \|b\|^2_{L^q(\mathbb{R}^3)} \leq C \lambda^{3-\frac{q}{2}} \|\nabla \varphi\|^2_{L^\infty} .
\]

(31)

Eventually, by (29), we find for all \( \varepsilon > 0 \):

\[
\limsup_{n \to +\infty} \|\nabla \psi_n\|^2_{L^2(\mathbb{R}^3)} \leq C \varepsilon \lambda^{3-\varepsilon} \|\nabla \varphi\|^2_{L^\infty} + \limsup_{n \to +\infty} \sum_i \|a_i\|^2_{L^2(B_i)}
\]

\[
\leq (C \varepsilon \lambda^{3-\varepsilon} + \eta(\lambda) \lambda^2) \|\nabla \varphi\|^2_{L^\infty(\mathbb{R}^3)} , \quad \eta(\lambda) \longrightarrow 0
\]

where the last inequality comes from assumption (A2). Together with (30), we deduce

\[
\limsup_{n \to +\infty} \|\nabla \phi_n\|^2_{L^2(\mathbb{R}^3)} \leq \eta(\lambda) \lambda \|\nabla \varphi\|_{L^\infty}, \quad \eta(\lambda) \longrightarrow 0.
\]

To show (27) and conclude in this way the proof of the theorem, it is enough to show that \( \phi_n \) goes weakly to zero as \( n \to +\infty \) in \( L^6(\mathbb{R}^3) \). Standard estimates on \( \phi - \phi_{n,\lambda} \) show that \( \phi_{n,\lambda} \), and from there \( \phi_n \), is bounded uniformly in \( n \) in \( H^1(\mathbb{R}^3) \) \( \cap L^6(\mathbb{R}^3) \), so that convergence to zero in the sense of distributions is enough. It follows from a duality argument: for \( h \) any smooth and compactly supported function, we introduce \( H \) the solution of \( \Delta H = h \) in \( \mathbb{R}^3 \), and thanks to (28), we write

\[
\langle \phi_n, h \rangle = \langle \Delta \phi_n, H \rangle = 3\lambda|K_\rho| \langle \rho \nabla \varphi - \frac{1}{n} \sum_{i} \frac{1}{|B_i|} 1_{B_i} \nabla \varphi(x_i), \nabla H \rangle
\]

which is easily seen to go to zero as \( n \to +\infty \), by (H0).
4 Further discussion of the hypotheses

We conclude this paper with further comments on the assumptions of our theorems. First, we show that the assumptions (H1) and (H2) in Theorem 1 are implied by the conditions

\[
\lim_{n \to +\infty} nd_n = +\infty, \quad d_n = \inf_{i \neq j} |x_i - x_j| \quad (H1^2)
\]

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{i} \left( \frac{1}{n} \sum_{j \neq i} \frac{1}{|x_i - x_j|} - \int_{\mathbb{R}^3} \frac{\rho(y)}{|x_i - y|} dy \right)^2 = 0 \quad (H2^2)
\]

Clearly, (H1^2) is stronger than (H1). Then, (H2) follows from (H2^2) if we prove that

\[
\lim_{n} \sum_{i} \int_{B_i} \left( \left| \sum_{j \neq i} (G(x - x_j) - G(x_i - x_j)) \right|^2 + n^2 \left| \int_{\mathbb{R}^3} (G(x - y)\rho(y)dy - \int_{\mathbb{R}^3} G(x_i - y)\rho(y)dy \right|^2 + \frac{1}{n} \sum_{j \neq i} \nabla G(x - x_j) \right|^2 \right) dx = 0.
\]

We focus on the first term, as the second one is simpler and the third one similar. We write:

\[
\sum_{i} \int_{B_i} \left| \sum_{j \neq i} (G(x - x_j) - G(x_i - x_j)) \right|^2 \leq C \sum_{i} \int_{B_i} \left| \sum_{j \neq i} \sup_{z \in [x_i, x]} |\nabla G(z - x_j)||x - x_i| \right|^2 \leq \frac{C'}{n^3} \sum_{i} \left| \sum_{j \neq i} \frac{1}{|x_i - x_j|} \right|^2 \leq \frac{C''}{(nd_n)^2}
\]

where we deduced from (H2^2) that \( \frac{1}{n} \sum_{i} \left| \sum_{j \neq i} \frac{1}{|x_i - x_j|} \right|^2 \) is bounded uniformly in \( n \). Convergence to zero follows then from (H1^2).

Note that (H2^2) is in the same spirit as the hypothesis (12) used in [34], although a bit stronger. Both (H1^2) and (H2^2) apply to many contexts. First, they are satisfied when the points are well-separated, cf. (11). Let us deduce (H2^2) from (11). For all \( j \), we set \( B_j = B(y_j, \frac{c}{n} n^{-1/3}) \), where \( c \) is the constant appearing in (11). As \( \frac{1}{|x|} \) is harmonic, we apply the mean value formula to write for any \( i \):

\[
\frac{1}{n} \sum_{j \neq i} \frac{1}{|x_i - x_j|} = \frac{1}{n} \sum_{j \neq i} \frac{1}{|B_j|} \int_{B_i} \frac{1}{|x_i - y|} dy = \frac{1}{n} \sum_{j} \frac{1}{|B_j|} \int_{B_j} \frac{1}{|x_i - y|} dy + O(n^{-2/3})
\]

\[
= \int_{\mathbb{R}^3} \frac{1}{|x_i - y|} g_n(y) dy + O(n^{-2/3})
\]

where \( \rho_n(y) = \frac{1}{n} \sum_{j} \frac{1}{|B_j|} 1_{B_j}(y) \). By (H2^1), the sequence \( (\rho_n)_{n \in \mathbb{N}} \) is bounded in \( L^\infty \), and by (H0) is easily seen to converge weakly * to \( \rho \). Hence, for all \( x \in K \), \( f_n(x) = \int_{\mathbb{R}^3} \frac{1}{|x - y|} \rho_n(y) dy \) converges to \( f(x) = \int_{\mathbb{R}^3} \frac{1}{|x - y|} \rho(y) dy \). It is also easily seen that \( (f_n) \) is equicontinuous over \( K \), from which we deduce that \( \sup_K |f_n - f| \to 0 \), and from there (H2^2).

The convergences in (H1^2) and (H2^2) hold also in probability when the \( x_i \) are random i.i.d variables with law \( \rho \). More precisely, it is well-known that in such setting, \( n^\alpha d_n \to +\infty \) in probability for any \( \alpha > \frac{2}{3} \), which is of course stronger than (H1^2). Moreover,

\[
\lim_{n \to +\infty} \mathbb{E} \frac{1}{n} \sum_{i} \left( \frac{1}{n} \sum_{j \neq i} \frac{1}{|x_i - x_j|} - \int_{\mathbb{R}^3} \frac{\rho(y)}{|x_i - y|} dy \right)^2 = 0
\]
which implies again convergence in probability. To establish this last limit is very classical: we write
\[ E\left( \frac{1}{n} \sum_{j \neq i} \frac{1}{|x_i - x_j|} - \int_{\mathbb{R}^3} \frac{\rho(y)}{|x - y|} dy \right)^2 = E\left( \frac{1}{n} \sum_{j \neq i} \left( \frac{1}{|x_i - x_j|} - \int_{\mathbb{R}^3} \frac{\rho(y)}{|x - y|} dy \right)^2 \right) + O(n^{-2}) \]
\[ = \frac{1}{n^2} \sum_{j, j' \neq i} \mathbb{E}X_{ij}X_{ij'} + O(n^{-2}) = \frac{1}{n^2} \sum_{j \neq j' \neq i} \mathbb{E}X_{ij}X_{ij'} + O(n^{-1}) \]
where \( X_{ij} := \frac{1}{|x_i - x_j|} - \int_{\mathbb{R}^3} \frac{\rho(y)}{|x - y|} dx \) has mean zero and variance
\[ \mathbb{E}X_{ij}^2 = \int \rho(x) \rho(z) \left| \frac{1}{|x - z|} - \int \frac{\rho(y)}{|x - y|} dy \right|^2 dx dz. \]
Eventually, for \( j \neq j' \neq i \), we compute using the independence of the \( x_k \)'s:
\[ \mathbb{E}X_{ij}X_{ij'} = \int_{\mathbb{R}^3} \rho(z) \rho(x) \rho(x') \left( \int \frac{\rho(y)}{|z - x|} dy \right) \left( \int \frac{\rho(y')}{|z - y'|} dy' \right) dz dx dx' \]
\[ = \int_{\mathbb{R}^3} \rho(z) \left( \int_{\mathbb{R}^3} \frac{\rho(x)}{|z - x|} dx - \int_{\mathbb{R}^3} \frac{\rho(y)}{|z - y|} dy \right) \left( \int_{\mathbb{R}^3} \frac{\rho(x')}{|z - x'|} dx' - \int_{\mathbb{R}^3} \frac{\rho(y')}{|z - y'|} dy' \right) dz = 0. \]
Hence, (H1\textsuperscript{2})-(H2\textsuperscript{2}) holds in probability when the \( x_i \) are i.i.d. random variables. We remind that a sequence is converging in probability if and only if any subsequence has itself a subsequence that converges almost surely. Using this characterization, and applying the proof of Theorem 1, we find that \( u_n \) converges to \( u \) solution of (8) in probability, for any distance metrizing the weak topology of the ball of \( H^1 \cap L^3 \) to which all \( u_n \) belong.

Similar considerations apply to the Stokes case. We leave to the reader to check that assumptions (H1)-(H2) are implied by (H1\textsuperscript{2})-(H2\textsuperscript{2}), with
\[ \lim_{n \to +\infty} \frac{1}{n} \sum_{i} \left( \frac{1}{n} \sum_{j \neq i} \frac{(x_i - x_j) \otimes (x_i - x_j)}{|x_i - x_j|^3} - \int_{\mathbb{R}^3} \frac{(x_i - y) \otimes (x_i - y)}{|x_i - y|^3} \rho(y) dy \right)^2 = 0. \ (H2\textsuperscript{2}) \]
These stronger hypotheses are again verified under the assumption (11), or in the i.i.d. case. To this respect, the only real change is in showing that (11) implies (H2\textsuperscript{2}). We use this time that \( \frac{1}{|x|^3} = -\nabla |x|^{-2} \) is harmonic and obtain
\[ \frac{1}{n} \sum_{j \neq i} \frac{(x_i - x_j) \otimes (x_i - x_j)}{|x_i - x_j|^3} = \int_{\mathbb{R}^3} \frac{x_i - y}{|x_i - y|^3} \otimes (x_i - x_j) dy \]
\[ = \int_{\mathbb{R}^3} \frac{x_i - y}{|x_i - y|^3} \otimes g_n(x_i, y) dy + O(n^{-2/3}) \]
where \( g_n(x, y) = \frac{1}{n} \sum_{j} \frac{1}{|B_j|} (x - x_j) 1_{B_j}(y) \) converges for all \( x \) weakly * to \( g(x, y) = (x - y) \rho(y) \) in \( L^\infty_y \). Thus, \( f_n(x) = \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \otimes g_n(x, y) dy \) converges to \( f(x) = \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \otimes g(x, y) dy \) and one can check that \( f_n \) is equicontinuous over \( K \). We conclude as in the case of the Laplacian.

We now turn to the discussion of assumptions (A1)-(A2). We shall see that they are satisfied under the strong assumption (11) on the minimal distance. The point is to check (A2). For all \( i \) and smooth \( \varphi \), we set \( R_i(x) = \frac{1}{n} \sum_{j \neq i} \nabla V(x - x_j) \nabla \varphi(x_j) \). We shall prove the stronger statement:
\[ \limsup_n \sum_i \| R_i \|^2_{L^2(B_i)} \leq C \lambda^{1-\varepsilon} \| \nabla \varphi \|^2_{L^\infty} \quad \forall \varepsilon > 0 \]
We have for all $x \in B_i$,

$$|R_i(x)| \leq |R_i(x) - R_i(x_i)| + |R_i(x_i)| \leq \|\nabla R_i\|_{L^\infty} |x - x_i| + |R_i(x_i)|$$

$$\leq \frac{C}{n} \sum_{j \neq i} \frac{r_n}{|x_i - x_j|^q} \|\nabla \varphi\|_{L^\infty} + |R_i(x_i)|$$

By assumption (11), setting $y_k = n^{1/3} x_k$, we find

$$\frac{1}{n} \sum_{j \neq i} \frac{r_n}{|x_i - x_j|^q} \|\nabla \varphi\|_{L^\infty} \leq \frac{1}{n} \sum_{j \neq i} \frac{1}{|y_i - y_j|^q} \|\nabla \varphi\|_{L^\infty} \leq C \lambda^{1/3} \|\nabla \varphi\|_{L^\infty}. \quad (32)$$

Hence,

$$\sum_{i} \|a_i\|_{L^2(B_i)}^2 \leq C \lambda^{\frac{2}{3}} \|\nabla \varphi\|_{L^\infty}^2 + r_n^3 \sum_{i} |R_i(x_i)|^2.$$  

From a slight variation of [18, Lemma 2.4], see also [24], we get for all $q \geq 2$, and $p$ the conjugate exponent of $q$:

$$\sum_{i} |R_i(x_i)|^q \leq \frac{C}{\lambda^q} \frac{\lambda^{q/p}}{n} \sum_{i} |\nabla \varphi(x_i)|^q = \frac{C}{\lambda} \sum_{i} |\nabla \varphi(x_i)|^q$$

Then, by Hölder inequality:

$$r_n^3 \sum_{i} |R_i(x_i)|^2 \leq r_n^3 n^\frac{q-2}{q} \left(\sum_{i} |R_i(x_i)|^q\right)^{\frac{2}{q}} \leq C \lambda \left(\sum_{i} |\nabla \varphi(x_i)|^q\right)^{\frac{2}{q}} \leq C \lambda^{1-\frac{2}{q}} \|\nabla \varphi\|_{L^\infty}^2$$

The result follows.

Eventually, it is interesting to understand the meaning of (A2) when the centers $x_{i,n}$ of the balls are given by a stationary point process [12, 13]. More precisely, we will discuss under what conditions on the process we have: for all smooth $\varphi$,

$$\limsup_{n \to +\infty} \frac{1}{n^2} \sum_{i,j} \int_{B_i} \left| \sum_{j \neq i} \nabla V(x - x_j) \nabla \varphi(x_j) \right|^2 \, dx \to 0, \quad \lambda \to 0. \quad (A2_2)$$

This assumption is slightly weaker than (A2): it only implies that for $\varphi \in \dot{H}^1 \cap L^6 \cap C^\infty$, $\langle R_\lambda, \varphi \rangle = o(\lambda)$, with $R_\lambda$ the remainder in (15).

Let $\Lambda$ a random point process on a probability space $\Omega$. In particular, for $\omega \in \Omega$, $\Lambda(\omega)$ is a discrete subset of $\mathbb{R}^3$. We assume that the process is stationary, of mean intensity $\lambda_0$, and ergodic. Note that we allow $\lambda_0$ to depend on $\lambda$. Then, given a small parameter $\varepsilon$, we set $\{x_{1,n}, \ldots, x_{n,n}\} = \varepsilon \Lambda \cap O$, where the labeling of the centers is arbitrary. Note that $n$ depends on $\varepsilon$, and is random: by the ergodic theorem, $n \varepsilon^3 \to \lambda_0 |O|$ almost surely as $\varepsilon \to 0$. It implies that $r_n \varepsilon^{-1} \to (\frac{2}{3} \lambda / \lambda_0)^{1/3}$ almost surely as $\varepsilon \to 0$. It allows to reformulate condition (A2$_2$): for all smooth $\varphi$,

$$\limsup_{\varepsilon \to 0} \frac{\varepsilon^6}{\lambda_0^2} \sum_{i} \int_{B_i} \left| \sum_{j \neq i} \nabla V(x - x_j) \nabla \varphi(x_j) \right|^2 dx \to 0, \quad \lambda \to 0.$$  

We want to understand under what conditions

$$\mathbb{E} \limsup_{\varepsilon \to 0} \frac{\varepsilon^6}{\lambda_0^2} \sum_{i} \int_{B_i} \left| \sum_{j \neq i} \nabla V(x - x_j) \nabla \varphi(x_j) \right|^2 dx \to 0, \quad \lambda \to 0.$$
which implies (A2₂) in probability. First, we want to reverse the limsup and the expectation. This will follow from the dominated convergence theorem if we show an $L^\infty$ bound on

$$I_\varepsilon = \frac{\varepsilon^6}{\lambda_0^2} \sum_i \int_{B_i} \left| \sum_{j \not= i} \nabla V(x - x_j) \nabla \varphi(x_j) \right|^2 \, dx$$

that is uniform in $\varepsilon$ and $\omega$ (but not necessarily on $\lambda$). We claim that such bound holds if for instance the process satisfies the condition

$$|y_i - y_j| \geq c(\lambda/\lambda_0)^{1/3}, \quad c > 2 \quad \forall i \neq j$$

which ensures that for $\varepsilon$ small enough, assumption (A1) holds. Indeed, using that $\nabla V$ is a harmonic function, we can write

$$I_\varepsilon = \frac{\varepsilon^6}{\lambda_0^2} \sum_i \int_{B_i} \left| \sum_{j \not= i} \nabla V(x - x_j) dy \nabla \varphi(x_j) \right|^2 \, dx = \frac{\varepsilon^6}{\lambda_0^2} \sum_i \int_{B_i} \left| \nabla^2 \Delta^{-1} \sum_j 1_{B_j} \nabla \varphi(x_j) \right|^2 \, dx.$$

Note that $|\varepsilon^6 n| \leq \frac{C_0}{\lambda}$ for an absolute constant $C_0$. We also remind that $\lambda_0$ possibly depends on $\lambda$. Hence, we find that

$$I_\varepsilon \leq C_\lambda \sum_i \int_{B_i} \left| \nabla^2 \Delta^{-1} 1_{B_i} \nabla \varphi(x_i) \right|^2 + C_\lambda \int_{\mathbb{R}^3} \left| \nabla^2 \Delta^{-1} \sum_j 1_{B_j} \nabla \varphi(x_j) \right|^2 \leq C_\lambda'$$

where we use the well-know fact that $\nabla^2 \Delta^{-1} 1_{B_i}$ is bounded in $L^\infty$ to control the first term, while we use the continuity of $\nabla^2 \Delta^{-1}$ over $L^2$ and the fact that the balls $B_j$ are disjoint to control the second term. Hence, it remains to understand under what conditions one has

$$\limsup_{\varepsilon \to 0} \mathbb{E} \frac{\varepsilon^6}{\lambda_0^2} \sum_i \int_{B_i} \left| \sum_{j \not= i} \nabla V(x - x_j) \nabla \varphi(x_j) \right|^2 \, dx \to 0, \quad \lambda \to 0.$$

Let $B_i = B(x_i, c_0(\lambda/\lambda_0)^{1/3} \varepsilon)$, with $c_0 = (\frac{3}{4\pi})^{1/3}$, so that almost surely $r_n \sim c_0(\lambda/\lambda_0)^{1/3} \varepsilon$ as $\varepsilon \to 0$. Introducing

$$\tilde{I}_\varepsilon = \frac{\varepsilon^6}{\lambda_0^2} \sum_i \int_{B_i} \left| \sum_{j \not= i} \nabla V(x - x_j) \nabla \varphi(x_j) \right|^2 \, dx$$

we show as in the case of $I_\varepsilon$ that it is bounded uniformly in $\varepsilon$ and $\omega$, and moreover that

$$|I_\varepsilon - \tilde{I}_\varepsilon| \leq C_\lambda' \sum_i \int_{B_i \Delta \tilde{B}_i} \left| \nabla^2 \Delta^{-1} 1_{B_i} \nabla \varphi(x_i) \right|^2 + C_\lambda \int_{\cup(B_i \Delta \tilde{B}_i)} \left| \nabla^2 \Delta^{-1} \sum_j 1_{B_j} \nabla \varphi(x_j) \right|^2$$

$$\leq C_\lambda' \left[ 1 - \frac{c_0 \lambda^{1/3} \varepsilon}{\lambda_0^{1/3} r_n} \right] + C_\lambda \left| \sum_i 1_{B_i \Delta \tilde{B}_i} \| L^2 \| \sum_j 1_{B_j} \nabla \varphi(x_j) \| L^4 \right|^2$$

$$\leq C_\mu \left[ 1 - \frac{c_0 \lambda^{1/3} \varepsilon}{\lambda_0^{1/3} r_n} \right] + \left| 1 - \frac{c_0 \lambda^{1/3} \varepsilon}{\lambda_0^{1/3} r_n} \right|^{1/2} \to 0, \quad \varepsilon \to 0.$$

We used the continuity of $\nabla^2 \Delta^{-1}$ over $L^4$ in the second inequality. By the dominated convergence theorem, the final step is to understand when

$$\limsup_{\varepsilon \to 0} \mathbb{E} \frac{\varepsilon^6}{\lambda_0^2} \sum_i \int_{B_i} \left| \sum_{j \not= i} \nabla V(x - x_j) \nabla \varphi(x_j) \right|^2 \, dx \to 0, \quad \lambda \to 0.$$
The advantage of $\tilde{B}_i$ over $B_i$ is that its radius is not random anymore. We write

$$
\begin{align*}
E\frac{\varepsilon^6}{\lambda_0^2} \sum_i \int_{\tilde{B}_i} \left| \sum_{j \neq i} \nabla V(x - x_j) \nabla \phi(x_j) \right|^2 dx = E\frac{\varepsilon^3}{\lambda_0^2} \sum_i \int_{\varepsilon^{-1}\tilde{B}_i} \left| \sum_{j \neq i} \nabla V(y - y_j) \nabla \phi(x_j) \right|^2 dy \\
= E\frac{\varepsilon^3}{\lambda_0^2} \sum_{i \neq j} \int_{\varepsilon^{-1}\tilde{B}_i} \left| \nabla V(y - y_j) \nabla \phi(x_j) \right|^2 dy \\
+ E\frac{\varepsilon^3}{\lambda_0^2} \sum_{i \neq j \neq k} \int_{\varepsilon^{-1}\tilde{B}_i} \nabla V(y - y_j) \nabla \phi(x_j) \cdot \nabla V(y - y_j') \nabla \phi(x_j') dy.
\end{align*}
$$

Using the definition of the $k$-point correlation functions $\rho_k$, see [4, p18], we find

$$
E\frac{\varepsilon^3}{\lambda_0^2} \sum_{i \neq j} \int_{\varepsilon^{-1}\tilde{B}_i} \left| \nabla V(y - y_j) \nabla \phi(x_j) \right|^2 dy
$$

$$
= \frac{\varepsilon^3}{\lambda_0^2} \int_{(\varepsilon^{-1}O)^2} dzdz' \rho_2(z, z') \int_{B(x, \varepsilon \lambda/\lambda_0)^{1/3}} \left| \nabla V(y - z') \nabla \phi(\varepsilon z') \right|^2 dy
$$

$$
\leq C \frac{\varepsilon^3}{\lambda_0^2} \int_{(\varepsilon^{-1}O)^2} dzdz' \rho_2(z, z') \int_{B(x, \varepsilon \lambda/\lambda_0)^{1/3}} \frac{1}{|y - z'|^6} dy.
$$

By stationarity, $\rho_2(z, z') = \rho_2(0, z' - z)$. After a change of variable, we get

$$
E\frac{\varepsilon^3}{\lambda_0^2} \sum_{i \neq j} \int_{\varepsilon^{-1}\tilde{B}_i} \left| \nabla V(y - y_j) \nabla \phi(x_j) \right|^2 dy \leq C \frac{|O|}{\lambda_0^2} \int_{\mathbb{R}^3} \int_{B(0, \varepsilon \lambda/\lambda_0)^{1/3}} \rho_2(0, z') dy dz'.
$$

Under assumption (33) we find eventually that the r.h.s. goes to zero if and only if

$$
\lambda \frac{\lambda_0}{\lambda_0^2} \int_{\mathbb{R}^3} \frac{\rho_2(0, z')}{z'^6 + (\lambda/\lambda_0)^2} dz' \to 0.
$$

(35)

As regards the other term, we get

$$
E\frac{\varepsilon^3}{\lambda_0^2} \sum_{i \neq j \neq k} \int_{\varepsilon^{-1}\tilde{B}_i} \nabla V(y - y_j) \nabla \phi(x_j) \cdot \nabla V(y - y_j') \nabla \phi(x_j') dy
$$

$$
= E\frac{\varepsilon^3}{\lambda_0^2} \int_{(\varepsilon^{-1}O)^3} dzdz'dz'' \rho_3(z, z', z'') \int_{B(x, \varepsilon \lambda/\lambda_0)^{1/3}} \nabla V(y - z') \nabla \phi(z) \cdot \nabla V(y - z'') \nabla \phi(z'') dy
$$

$$
= E\frac{\varepsilon^3}{\lambda_0^2} \int_{(\varepsilon^{-1}O)^3} dzdz'dz'' \rho_3(0, z', z'') \int_{B(0, \varepsilon \lambda/\lambda_0)^{1/3}} \nabla V(y - z') \nabla \phi(z + z') \cdot \nabla V(y - z'') \nabla \phi(z + z'') dy
$$

where we have used stationarity: $\rho_3(z, z', z'') = \rho_3(0, z' - z, z'' - z)$ and a change of variable to obtain the last line. Finally,

$$
\left| E\frac{\varepsilon^3}{\lambda_0^2} \sum_{i \neq j \neq k} \int_{\varepsilon^{-1}\tilde{B}_i} \nabla V(y - y_j) \nabla \phi(x_j) \cdot \nabla V(y - y_j') \nabla \phi(x_j') dy \right|
$$

$$
\leq |O| \sup_{x \in \mathbb{R}^3} \int_{B(0, \varepsilon \lambda/\lambda_0)^{1/3}} \left( \frac{\partial}{\partial z'_k} \frac{\partial}{\partial z''_k} \Delta_{z'}^{-1} \left( \frac{\partial}{\partial z''_k} \Delta_{z''}^{-1} A_{z,k,j}(z' = y, z'' = y) \right) \right) dy
$$

where

$$
A_{z,k,j}(z', z'') = 1_{\varepsilon^{-1}\mathcal{O}(z')} 1_{\varepsilon^{-1}\mathcal{O}(z'')} \rho_3(0, z', z'') \partial_k \phi(x + \varepsilon z') \partial_j \phi(x + \varepsilon z'').
$$

Although this last quantity is a bit intricate, we may expect that it goes to zero with $\lambda$ under reasonable assumptions on the process. For instance, if the process is Poisson of
intensity $\lambda_0$ (neglecting previous assumptions on the minimal distance), then $\rho_3 = \lambda_0^3$, and $|A_{x,k_j}^\varepsilon(z',z'')| \leq C\lambda_0^3$, resulting in

$$|E_{\beta \lambda_0^3}^3 \sum_{i \neq j \neq j'} \int_{\varepsilon^{-1} B_i} \nabla V(y - y_j) \nabla \varphi(x_j) \cdot \nabla V(y - y_{j'}) \nabla \varphi(x_{j'}) dy| = O(\lambda).$$

The condition (35) is more stringent: for the Poisson process of intensity $\lambda_0$, where $\rho_2 = \lambda_0^2$, we find that the quantity at the left-hand side of (35) is $O(1)$ as $\lambda \to 0$, but non-vanishing. Hence, our result does not cover this case: to cover it, we would need a weaker criterion than (A2), in the same way as the criterion (12) derived in [34] is weaker than (H2) or (H2'). Still, (35) is fulfilled by much more configurations than those satisfying (11) almost surely. Indeed, this latter case corresponds to $\rho_2(0,z') = 0$ for $|z'| \geq c\lambda_0^{1/3}$, so that the quantity at the left-hand side of (35) is $O(\lambda)$, much stronger than the $o(1)$ asked in (A2).

Acknowledgements

The author acknowledges the support of the Institut Universitaire de France, and of the SingFlows project, grant ANR-18-CE40-0027 of the French National Research Agency (ANR).

References

[1] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes. Arch. Rational Mech. Anal., 113(3):209–259, 1990.

[2] H. Ammari, P. Garapon, H. Kang, and H. Lee. Effective viscosity properties of dilute suspensions of arbitrarily shaped particles. Asymptot. Anal., 80(3-4):189–211, 2012.

[3] H. Ammari and H. Zhang. Effective medium theory for acoustic waves in bubbly fluids near Minnaert resonant frequency. SIAM J. Math. Anal., 49(4):3252–3276, 2017.

[4] A. Borodin and S. Serfaty. Renormalized energy concentration in random matrices. Comm. Math. Phys., 320(1):199–244, 2013.

[5] H. C. Brinkman. A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. Flow, Turbulence and Combustion, 1(1):27, Dec 1949.

[6] L. A. Caffarelli and A. Mellet. Random homogenization of an obstacle problem. Ann. Inst. H. Poincaré Anal. Non Linéaire, 26(2):375–395, 2009.

[7] C. Calvo-Jurado, J. Casado-Díaz, and M. Luna-Laynez. Homogenization of nonlinear Dirichlet problems in random perforated domains. Nonlinear Anal., 133:250–274, 2016.

[8] D. Cioranescu and F. Murat. A strange term coming from nowhere. In Topics in the mathematical modelling of composite materials, volume 31 of Progr. Nonlinear Differential Equations Appl., pages 45–93. Birkhäuser Boston, Boston, MA, 1997.

[9] R. Clausius. Die mechanische Behandlung der Elektricität. Vieweg, Braunshweig, 1879.

[10] G. Dal Maso. Asymptotic behaviour of minimum problems with bilateral obstacles. Ann. Mat. Pura Appl. (4), 129:327–366 (1982), 1981.

[11] G. Dal Maso and P. Longo. F-limits of obstacles. Ann. Mat. Pura Appl. (4), 128:1–50, 1981.

[12] D. J. Daley and D. Vere-Jones. An introduction to the theory of point processes. Vol. I. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2003. Elementary theory and methods.

[13] D. J. Daley and D. Vere-Jones. An introduction to the theory of point processes. Vol. II. Probability and its Applications (New York). Springer, New York, second edition, 2008. General theory and structure.

[14] L. Desvillettes, F. Golse, and V. Ricci. The mean-field limit for solid particles in a Navier-Stokes flow. J. Stat. Phys., 131(5):941–967, 2008.

[15] M. Duerinckx and A. Gloria. Analyticity of homogenized coefficients under Bernoulli perturbations and the Clausius-Mossotti formulas. Arch. Ration. Mech. Anal., 220(1):297–361, 2016.
[16] A. Einstein. Eine neue bestimmung der moleküldimensionen. *Ann. Physik.*, 19:289–306, 1906.

[17] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. Linearized steady problems.

[18] D. Gérard-Varet and M. Hillairet. Analysis of the viscosity of dilute suspensions beyond Einstein’s formula. arXiv:1905.08208, May 2019.

[19] A. Giunti and R. Höfer. Homogenization for the Stokes equations in randomly perforated domains under almost minimal assumptions on the size of the holes. arXiv:1809.04491, September 2018.

[20] A. Giunti, R. Höfer, and J. J. L. Velázquez. Homogenization for the Poisson equation in randomly perforated domains under minimal assumptions on the size of the holes. *Comm. Partial Differential Equations*, 43(9):1377–1412, 2018.

[21] A. Gloria. A scalar version of the Caflisch-Luke paradox. arXiv:1907.08182, July 2019.

[22] B. M. Haines and A. L. Mazzucato. A proof of Einstein’s effective viscosity for a dilute suspension of spheres. *SIAM J. Math. Anal.*, 44(3):2120–2145, 2012.

[23] M. Hillairet. On the homogenization of the Stokes problem in a perforated domain. *Arch. Ration. Mech. Anal.*, 230(3):1179–1228, 2018.

[24] M. Hillairet and D. Wu. Effective viscosity of a polydispersed suspension. Preprint arXiv:1905.12306, 2019.

[25] R. M. Höfer. The inertialess limit of particle sedimentation modeled by the Vlasov-Stokes equations. *SIAM J. Math. Anal.*, 50(5):5446–5476, 2018.

[26] P.-E. Jabin and F. Otto. Identification of the dilute regime in particle sedimentation. *Comm. Math. Phys.*, 250(2):415–432, 2004.

[27] T. Lévy and E. Sánchez-Palencia. Einstein-like approximation for homogenization with small concentration. II. Navier-Stokes equation. *Nonlinear Anal.*, 9(11):1255–1268, 1985.

[28] V. A. Marčenko and E. J. Hruslov. Estimation of the accuracy of approximation of solutions of boundary value problems with a fine-grained boundary. In *Problems in mechanics and mathematical physics (Russian)*, pages 208–223, 298. 1976.

[29] J. Maxwell Garnett. Colours in metal glasses and in metallic films. *Philosophical Transactions of the Royal Society of London. Series A*, 203, 1904.

[30] A. Mecherbet. Sedimentation of particles in Stokes flow. arXiv:1806.07795, June 2018.

[31] O. Mossotti. Discussione analitica sul’influenza che l’azione di un mezzo dielettrico ha sulla distribuzione dell’elettricità alla superficie di più corpi elettrici diseminati in esso. *Mem. Mat. Fis. della Soc. Ital. di Sci. in Modena*, 24:49–74, 1850.

[32] B. Niethammer and R. Schubert. A local version of Einstein’s formula for the effective viscosity of suspensions. arXiv:1903.08554.

[33] S. Ozawa. Point interaction potential approximation for $(-\Delta + U)^{-1}$ and eigenvalues of the Laplacian on wildly perturbed domain. *Osaka J. Math.*, 20(4):923–937, 1983.

[34] G. C. Papanicolaou and S. R. S. Varadhan. Diffusion in regions with many small holes. In *Stochastic differential systems (Proc. IFIP-WG 7/1 Working Conf., Vilnius, 1978)*, volume 25 of *Lecture Notes in Control and Information Sci.*, pages 190–206. Springer, Berlin-New York, 1980.

[35] E. Sánchez-Palencia. Einstein-like approximation for homogenization with small concentration. I. Elliptic problems. *Nonlinear Anal.*, 9(11):1243–1254, 1985.