Non-standard symmetries and Killing tensors

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Abstract. Higher order symmetries corresponding to Killing tensors are investigated. The intimate relation between Killing-Yano tensors and non-standard supersymmetries is pointed out. The gravitational anomalies are absent if the hidden symmetry is associated with a Killing-Yano tensor. In the Dirac theory on curved spaces, Killing-Yano tensors generate Dirac type operators involved in interesting algebraic structures as dynamical algebras or even infinite dimensional algebras or superalgebras. The general results are applied to space-times which appear in modern studies. The 4-dimensional Euclidean Taub-NUT space and its generalizations introduced by Iwai and Katayama are analyzed from the point of view of hidden symmetries. One presents the infinite dimensional superalgebra of Dirac type operators on Taub-NUT space that can be seen as a twisted loop algebra. The axial anomaly, interpreted as the index of the Dirac operator, is computed for the generalized Taub-NUT metrics. The existence of the conformal Killing-Yano tensors is investigated for some spaces with mixed Sasakian structures.

1. Introduction

One of the key concepts in physics is that of symmetries, Noether’s theorem giving a correspondence between symmetries and conserved quantities. For the geodesic motions on a space-time the usual conserved quantities are related to the isometries which correspond to Killing vectors. Sometimes a space-time could admit higher order symmetries described by Killing tensors. These are known as hidden symmetries and the typical example is the Runge-Lenz vector in the Kepler/Coulomb problem. The corresponding conserved quantities are quadratic, or, more general, polynomial in momenta. Their existence guarantees the integrability of the geodesic motions and is intimately related to separability of Hamilton-Jacobi (see, e. g. [1]) and the Klein-Gordon equation [2] at the quantum level.

In the case of the gravitational interactions, a consistent perturbative quantization is not available, even if there are no fermions. The implementation of fermions on curved space-times represent an additional difficulty. Fermions are essentially quantum objects while the space-time backgrounds are classic. Sometimes it can appear anomalies representing discrepancies between the conservation laws at the classical level and the corresponding ones at the quantum level.

The above two problems are correlated. First of all the symmetric Killing tensors, called Stäckel-Killing (S-K), are used to construct the conserved quantities, polynomials in momenta. The next most simple objects that can be studied in connection with the symmetries of a manifold after the S-K tensors are the Killing-Yano tensors (K-Y) [3]. Their physical interpretation remained obscure until Floyd [4] and Penrose [5] showed that the Killing tensor $K_{\mu\nu}$ of the 4-dimensional Kerr-Newman space-time admits a certain square-root which defines a K-Y tensor. Subsequently it was observed [6] that a K-Y tensor generates additional supercharges
in the dynamics of pseudo-classical spinning particles being the natural geometrical objects to be coupled with the fermionic degrees of freedom [6, 7]. In this way it was realized the significant connection between K-Y tensors and non-standard supersymmetries. Passing to quantum Dirac equation it was discovered [8] that K-Y tensors generate conserved non-standard Dirac operators which commute with the standard one.

The conformal extension of the Killing tensor equation determines the conformal Killing tensors [9] which define first integrals of the null geodesic equation. Investigation of the properties of higher-dimensional space-times has pointed out the role of the conformal K-Y (CKY) tensors to generate background metrics with black-hole solutions (see, e. g. [10]).

The aim of this paper is to investigate a few examples of curved spaces endowed with special structures admitting K-Y tensors which could be relevant in the theories of modern physics [7, 6, 11, 12].

The first example is represented by the 4-dimensional Euclidean Taub-Newman-Unti-Tamburino (Taub-NUT) space. The motivation to carry out this example is twofold. First of all, in the Taub-NUT geometry there are known to exist four K-Y tensors [13]. From this point of view the Taub-NUT manifold is an exceedingly interesting space to exemplify the effective construction of the conserved quantities in terms of geometric ones. On the other hand, the Taub-NUT geometry is involved in many modern studies in physics. For example the Kaluza-Klein monopole of Gross and Perry [14] and of Sorkin [15] was obtained by embedding the Taub-NUT gravitational instanton [16] into five-dimensional Kaluza-Klein theory. Remarkably the same object has re-emerged in the study of monopole scattering. In the long distance limit, neglecting radiation, the relative motion of slow Bogomolny-Prasad-Sommerfield monopoles is described by the geodesics of this space [17, 18]. The dynamics of well-separated monopoles is completely soluble and has a Kepler type symmetry [13].

In the second example we investigate the existence of CKY tensors in higher dimensional space-times [19]. Investigations of the properties of space-times of higher dimensions (\(D > 4\)) have recently attracted considerable attention as a result of their appearance in theories of unification such as string and \(M\) theories.

Versions of \(M\)–theory could be formulated in space-times with various number of time dimensions giving rise to exotic space-time signatures. The \(M\)–theory in \(10 + 1\) dimensions is linked via dualities to a \(M^*\) theory in \(9 + 2\) dimensions and a \(M'\) theory in \(6 + 5\) dimensions. Various limits of these will give rise to \(IIA\)– and \(IIB\)–like string theories in many variants of dimensions and signatures [20].

The paraquaternionic structures arise in a natural way in theoretical physics, both in string theory and integrable systems ([21], [22], [23], [24]). The counterpart in odd dimension of a paraquaternionic structure was introduced in [25]. It is called mixed 3-structure, which appears in a natural way on lightlike hypersurfaces in paraquaternionic manifolds. A compatible metric with a mixed 3-structures is necessarily semi-Riemann and mixed 3-Sasakian manifolds are Einstein [26, 27], hence the possible importance of these structures in theoretical physics.

The plan of the paper is as follows: in Section 2 we present the Killing tensors which generalize the Killing vectors, while Section 3 explains how the hidden symmetries associated with Killing tensors do not generate gravitational anomalies. In Section 4 we describe the Dirac-type operators generated by K-Y tensors. In Section 5 the general results are applied to the 4-dimensional Euclidean Taub-NUT space [28, 29] and its generalizations [30, 31, 32]. The CKY tensors on manifolds with mixed 3–structures are presented in the last Section. Some details concerning the geometrical properties of these metrics are summarized in the Appendices.

2. Killing vector fields and their generalizations

Let \((M, g)\) be a semi-Riemannian manifolds. A vector field \(X\) on \(M\) is said to be a Killing vector field if the Lie derivative with respect to \(X\) of the metric \(g\) vanishes.
It is clear that a Killing vector field $X$ on a semi-Riemannian manifold $(M, g)$ preserves the metric. Killing vector fields can be generalized to conformal Killing vector fields [3], i.e. vector fields with a flow preserving a given conformal class of metrics.

A natural generalization of conformal Killing vector fields is given by the CKY tensors [33].

**Definition 1** A CKY tensor of rank $p$ on a semi-Riemannian manifold $(M, g)$ is a $p$-form $f$ which satisfies:

$$\nabla_X f = \frac{1}{p + 1} X f - \frac{1}{n - p + 1} X^* \wedge d^* f,$$

for any vector field $X$ on $M$.

Here $\nabla$ is the Levi-Civita connection of $g$, $n$ is the dimension of $M$, $X^*$ is the 1-form dual to the vector field $X$ with respect to the metric $g$, $\wedge$ is the operator dual to the wedge product and $d^*$ is the adjoint of the exterior derivative $d$. If $f$ is co-closed in (1), then we obtain the definition of a K-Y tensor (introduced by Yano [3]). We can easily see that for $p = 1$, they are dual to Killing vector fields.

A K-Y tensor can be characterized in several ways. In an equivalent manner a differential $p$-form $f$ is called a K-Y tensor if its covariant derivative $\nabla_X f_{\mu_1...\mu_p}$ is totally antisymmetric. As a consequence of the antisymmetry a K-Y tensor satisfy the equation

$$\nabla_{(\lambda} f_{\mu_1...\mu_p)} = 0,$$

(2)

Let us remark that for covariantly constant K-Y tensors each term of the l. h. s. of (2) vanishes. The covariantly constant K-Y tensors represent a particular class of K-Y tensors and they play a special role in the theory of Dirac operators as it will be seen in Section 4.

We mention that K-Y tensors are also called Yano tensors or Killing forms, and CKY tensors are sometimes referred as conformal Yano tensors, conformal Killing forms or twistor forms [34, 35, 36].

For generalizations of the Killing vectors one might also consider higher order symmetric tensors.

**Definition 2** A symmetric tensor of $K_{\mu_1...\mu_r}$ of rank $r > 1$ satisfying a generalized Killing equation

$$\nabla_{(\lambda} K_{\mu_1...\mu_r)} = 0,$$

(3)

is called a S-K tensor.

The relevance in physics of the S-K tensors is given by the following proposition which could be easily proved:

**Proposition 1** A symmetric tensor $K$ on $M$ is a S-K tensor if and only if the quantity

$$K = K_{\mu_1...\mu_r} s^{\mu_1} \cdots s^{\mu_r},$$

(4)

is constant along every geodesic $s$ in $M$.

Here the over-dot denotes the ordinary proper time derivative and the proposition ensures that $K$ is a first integral of the geodesic equation.

These two generalizations (3) and (2) of the Killing vector equation could be related. Let $f_{\mu_1...\mu_p}$ be a K-Y tensor, then the tensor field

$$K_{\mu\nu} = f_{\mu_2...\mu_p} f^{\mu_2...\mu_p}_{\mu\nu},$$

(5)

is a S-K tensor and it sometimes refers to this S-K tensor as the associated tensor with $f$. However, the converse statement is not true in general: not all S-K tensors of rank 2 are associated with a K-Y tensor. That is the case of the generalized Taub-NUT spaces (Appendix A) which admit S-K tensors but no K-Y tensors.
3. Gravitational anomalies

Let us consider a non-trivial S-K tensor of rank 2 with a quadratic constant along the geodesic flow constructed as in (4). The generalized Killing equation (3) represents the necessary and sufficient condition for the existence of a quadratic constant of motion as follows from the Poisson bracket of $K$ with the Hamiltonian. Passing from the classical motion to the hidden symmetries of a quantized system, the corresponding quantum operator analog of the quadratic function (4) is \[ K = \Delta \mu K^{\mu \nu} \Delta_\nu, \]

where $\Delta_\mu$ is the covariant differential operator on the manifold $M$. Working out the commutator of (6) with the scalar Laplacian

\[ \mathcal{H} = \Delta_\mu \Delta^\mu, \]

we get after some calculations

\[ [\mathcal{H}, K] = -\frac{4}{3} \{ K^{[\mu}^{\nu]} R^{\nu]}_{\mu} \Delta_\mu, \]

which means that in general the quantum operator $\mathcal{K}$ does not define a genuine quantum mechanical symmetry [7]. On a generic curved space-time there appears a gravitational quantum anomaly proportional to a contraction of the S-K tensor $K^{\mu \nu}$ with the Ricci tensor $R^{\mu \nu}$.

It is obvious that for a Ricci-flat manifold this quantum anomaly is absent. However, a more interesting situation is represented by the manifolds in which the S-K tensor $K^{\mu \nu}$ can be written as a product of K-Y tensors [8] as in (5). On the other hand the integrability condition for any solution of (2) implies the vanishing of the commutator (8) for S-K tensors which admit a decomposition in terms of K-Y tensors. That is the case of the standard Euclidean Taub-NUT space, but not for its generalizations [30, 31, 32]. In the case of generalized Taub-NUT metrics, there are S-K tensors, but no K-Y tensors [37] and consequently for these spaces there are quantum gravitational anomalies [38].

4. Dirac-type operators

For a quantum relativistic description of a spin-1/2 particle on a curved space we use the standard Dirac operator

\[ D_s = i \gamma^\mu \nabla_\mu, \]

where $\nabla_\mu$ are the spin covariant derivatives including spin-connection, while $\gamma^\mu$ are the standard Dirac matrices carrying natural indices.

We note that for any isometry with a Killing vector $R^\mu$ there is an appropriate operator [8]

\[ X_k = -i(R^\mu \nabla_\mu - \frac{1}{4} \gamma^{[\mu} \gamma^{\nu} R_{\nu]}^\mu), \]

which commutes with $D_s$.

Moreover each K-Y tensor $f_{\mu \nu}$ produces a non-standard Dirac operator of the form [8]

\[ D_f = i \gamma^\mu (f^\nu_{\mu} \nabla_\nu - \frac{1}{6} \gamma^{[\nu} \gamma^\rho f_{\mu \nu; \rho}^\rho), \]

which anticommutes with the standard Dirac operator $D_s$ and can be involved in new types of genuine or hidden (super)symmetries.
4.1. Covariantly constant K-Y tensors

Remarkable superalgebras of Dirac-type operators can be produced by special second-order K-Y tensors that represent square roots of the metric tensor [39, 40, 41].

**Definition 3** The non-singular real or complex-valued K-Y tensor \( f \) of rank 2 defined on \( M \) which satisfies

\[
 f^\mu_\alpha f^\mu_\beta = g_{\alpha\beta},
\]  

(12)

is called an unit root of the metric tensor of \( M \), or simply an unit root of \( M \).

Let us observe that (12) is a particular case of (3) with the metric tensor as an ordinary S-K tensor. It was shown that any K-Y tensor that satisfy (12) is covariantly constant [39], i.e. \( f_{\mu\nu;\sigma} = 0 \).

In what follows we look for manifolds admitting families of unit roots \( f = \{ f^i | i = 1, 2, ... N_f \} \) having supplementary properties which should guarantee that: (I) the linear space \( L_f = \{ \rho | \rho = \rho_i f^i, \rho_i \in \mathbb{R} \} \) is isomorphic with a real Lie algebra, and (II) each element of \( L_f - 0 \) is a root (of arbitrary norm). In these circumstances we have the following theorem [41]:

**Theorem 1** The unique type of family of unit roots with \( N_f > 1 \) having the properties (I) and (II) are the triplets \( f = \{ f^1, f^2, f^3 \} \) which satisfy

\[
 \langle f^i \rangle \langle f^j \rangle = -\delta_{ij} 1_n + \varepsilon_{ijk} \langle f^k \rangle, \quad i, j, k... = 1, 2, 3.
\]  

(13)

**Proof:** We have denoted by \( \langle f \rangle \) the matrix form of the K-Y tensor \( f \). Taking into account that \( \varepsilon_{ijk} \) is the antisymmetric tensor with \( \varepsilon_{123} = 1 \) we recognize that equations (13) are the well-known multiplication rules of the quaternion units or similar algebraic structures (e.g. the Pauli matrices). Consequently, the matrices \( \langle f^i \rangle \) and \( 1_n \) generate a matrix representation of the quaternionic algebra \( \mathbb{H} \). Other choices are forbidden by the Frobenius theorem.

If the unit roots \( f^i \) have only real-valued components we recover the hypercomplex structures that obey (13) and these are connected with the hyper-Kähler structure of the space. An example of hyper-Kähler manifold is the Euclidean Taub-NUT space (see Appendix A) which is equipped with only one family of real unit roots.

The main geometric feature of all manifolds admitting a triplet of unit roots is given by

**Theorem 2** If a manifold \( M \) allows a triplet of unit roots then this must be Ricci flat (i.e. \( R_{\mu\nu} = 0 \)).

**Proof:** As in the case of the hyper-Kähler manifolds, we start with the identity

\[
 0 = f_{\mu\nu;\alpha;\beta} - f_{\mu\nu;\beta;\alpha} = f_{\mu\sigma} R_{\nu\alpha\beta}^\sigma + f_{\sigma\nu} R_{\mu\alpha\beta}^\sigma,
\]

and calculate

\[
 R_{\mu\nu\alpha\beta} f^1_{\alpha\beta} = R_{\mu\nu\sigma\alpha} f^3_{\sigma\alpha} (\langle f^3 \rangle \langle f^1 \rangle)_{\alpha\beta} = R_{\mu\nu\sigma\alpha} f^3_{\sigma\alpha} f^2_{\alpha\beta} = -R_{\mu\nu\alpha\beta} f^1_{\alpha\beta} = 0.
\]

Then, permutating the first three indices of \( R \) we find the identity

\[
 2 R_{\mu\nu\alpha\beta} f^1_{\alpha\beta} = R_{\mu\nu\alpha\beta} f^1_{\alpha\beta} = 0.
\]
Finally, writing
\[ R_{\mu
u} = R_{\mu\alpha\beta} f^1_\alpha f^1_\beta = -R_{\mu\alpha\beta} f^1_\sigma f^1_\alpha \beta = 0, \]
we draw the conclusion that the manifold is Ricci flat. The same procedure holds for \( f^{(2)} \) or \( f^{(3)} \) leading to similar identities.

Note that the manifolds possessing only single unit roots (as the Kähler ones) are not forced to be Ricci flat.

It is worthy to be noted that the covariantly constant K-Y tensors give rise to Dirac-type operators of the form (11) connected with the standard Dirac operators as follows:

**Theorem 3** The Dirac-type operator \( D_f \) produced by the K-Y tensor \( f \) satisfies the condition
\[ (D_f)^2 = D_s^2, \] (14)
if and only if \( f \) is an unit root.

*Proof:* The arguments of Ref. [39] show that the condition (14) is equivalent with (12) \( f \) being a covariantly constant K-Y tensor.

5. Dirac operators on Taub-NUT space
To make things more specific let us consider the Taub-NUT space which is hyper-Kähler and possesses many non-standard symmetries expressed in terms of four K-Y tensors and three S-K tensors.

From the covariantly constant K-Y tensors \( f^i \) (A.5), using prescription (11), we can construct three Dirac-type operators \( D^{(i)} \) which anticommute with standard Dirac operator \( D_s \) (9). It is convenient to define [42]
\[ Q_i = iH^{-1}D^{(i)}, \] (15)
where \( H = -\gamma^0D_s \) is the massless Hamiltonian operator. These operators form a representation of the quaternionic units:
\[ Q_i Q_j = \delta_{ij}I + i\varepsilon_{ijk}Q_k. \] (16)

On the other hand Dirac-type operator constructed from the K-Y tensor \( f^Y \) (A.6) is \( D^Y \) and again it is convenient to define a new operator \( Q^Y = HD^Y \).

The conserved Runge-Lenz operator of the Dirac theory is
\[ K_i = \frac{i}{4}(Q^Y, Q_i) + \frac{1}{2}(B - P_4)Q_i - J_iP_4, \] (17)
where \( B^2 = P_4^2 - H^2 \), \( J_i, (i = 1, 2, 3) \) are the components of the total angular momentum, while \( P_4 = -i\partial_4 \) corresponding to the fourth Cartesian coordinate \( x^4 = -4m(\chi + \varphi) \).

The operators \( J_i \) and \( K_i \) are involved in the following system of commutation relations:
\[
\begin{align*}
[J_i, J_j] &= i\varepsilon_{ijk}J_k, \\
[J_i, K_j] &= i\varepsilon_{ijk}K_k, \\
[K_i, K_j] &= i\varepsilon_{ijk}B^2,
\end{align*}
\] (18)
and commute with the operators \( Q_i \) (15)
\[
[J_i, Q_j] = i\varepsilon_{ijk}Q_k, \quad [K_i, Q_j] = i\varepsilon_{ijk}Q_kB. \] (19)
The algebra (18) does not close as a finite Lie algebra because of the factor $B^2$. In the standard treatment one concentrates on individual subspaces of the whole Hilbert space which belong to definite eigenvalues of $B^2$. This is similar to the dynamical algebra of the hydrogen atom which can be identified in a natural way with an infinite dimensional twisted loop algebra [43].

The dynamical algebras of the Dirac theory have to be obtained by replacing this operator $B^2$ with its eigenvalue $q^2 - E^2$ and rescaling the operators $K_i$. The same kind of problems appears for the anticommutators involving the fermionic operators $Q_i$ and $Q^Y$. In what follows, in order to keep the presentation as simple as possible, we shall only give the briefest account of the algebra of operators connected with hidden symmetries in the bosonic sector. For the algebra of operators from the fermionic sector the reader should consult [44].

In the bosonic sector of conserved operators let us define the new operators ”absorbing” the operator $B$ [45]:

$$J^i_n = J_i B^n, \quad K^i_n = K_i B^n,$$

for any $n = 0, 1, 2, ...$

Non-trivial commutators of the bosonic sector (18) become

$$[J^i_n, J^j_m] = i\varepsilon_{ijk} J^k_{n+m},$$
$$[J^i_n, K^j_m] = i\varepsilon_{ijk} K^k_{n+m},$$
$$[K^i_n, K^j_m] = i\varepsilon_{ijk} J^k_{n+m+2}.\quad (21)$$

We should like to show that this algebra can be seen as an infinite dimensional twisted loop algebra. The simplest way to achieve a Lie algebra of the Kac-Moody type is to assign grades to each operator

$$A^i_{2n} := J_i B^n, \quad B^i_{2n+2} := K_i B^n.\quad (22)$$

In this way the commutation relations of the bosonic sector are

$$[A^i_{2n}, A^j_{2m}] = i\varepsilon_{ijk} A^k_{2(n+m)},$$
$$[A^i_{2n}, B^j_{2m+2}] = i\varepsilon_{ijk} B^k_{2(n+m+1)},$$
$$[B^i_{2n+2}, B^j_{2m+2}] = i\varepsilon_{ijk} A^k_{2(n+m+2)}.\quad (23)$$

and in this Kac-Moody type algebra the sum of the grades is conserved under commutation as it is expected.

The same kind of construction can be done in the fermionic sector for the anticommutation relations of the operators $Q_i, Q^Y$.

5.1. Index formulas and axial anomalies

Atiyah, Patodi and Singer [46] (APS) discovered an index formula for first-order differential operators on manifolds with boundary with a non-local boundary condition. Their index formula contains two terms, none of which is necessarily an integer, namely a bulk term (the integral of a density in the interior of the manifold) and a boundary term defined in terms of the spectrum of the boundary Dirac operator. Endless trouble is caused in this theory by the condition that the metric and the operator be of ”product type” near the boundary.

In [38] we computed the index of the Dirac operator on annular domains and on disk, with the non-local APS boundary condition. For the generalized Taub-NUT metrics [30, 31, 32], we found that the index is a number-theoretic quantity which depends on the metrics. In particular, our formula shows that the index vanishes on balls of sufficient large radius, but can be non-zero for some values of the parameters $c, d$ (see Appendix A).
Theorem 4 If $c > -\frac{\sqrt{15}}{2}$ then the extended Taub-NUT metric does not contribute to the axial anomaly on any annular domain (i.e., the index of the Dirac operator with APS boundary condition vanishes).

Proof: The proof is given in [38].

The result is natural since the index of an operator is unchanged under continuous deformations of that operator. In our case this would amount to a continuous change in the metric. The absence of axial anomalies is due to the fact there exists an underlying structure that does not depend on the metric. However for larger deformations of the metric there could appear discontinuities in the boundary conditions and therefore the index could present jumps. Our formula for the index involves a computable number-theoretic quantity depending on the parameters of the metric.

We also examined the Dirac operator on the complete Euclidean space with respect to this metric, acting in the Hilbert space of square-integrable spinors. We found that this operator is not Fredholm, hence even the existence of a finite index is not granted.

We mentioned in [38] some open problems in connection with unbounded domains. The paper [47] brings new results in this direction. First we showed that the Dirac operator on $\mathbb{R}^4$ with respect to the standard Taub-NUT metric does not have $L^2$ harmonic spinors. This follows rather easily from the Lichnerowicz formula, since the standard Taub-NUT metric has vanishing scalar curvature. In particular, the index vanishes.

Theorem 5 For the standard Taub-NUT metric on $\mathbb{R}^4$ the Dirac operator does not have $L^2$ solutions.

Proof: Recall that the standard Taub-NUT metric is hyper-Kähler, hence its scalar curvature $\kappa$ vanishes.

By the Lichnerowicz formula,

$$D^2 = \nabla^* \nabla + \frac{\kappa}{4} = \nabla^* \nabla.$$ 

Let $\phi \in L^2$ be a solution of $D$ in the sense of distributions. Then, again in distributions, $\nabla^* \nabla \phi = 0$. The operator $\nabla^* \nabla$ is essentially self-adjoint with domain $C_c^\infty(\mathbb{R}^4, \Sigma)$, which implies that its kernel equals the kernel of $\nabla$. Hence $\nabla \phi = 0$. Now a parallel spinor has constant pointwise norm, hence it cannot be in $L^2$ unless it is 0, because the volume of the metric $ds^2_K$ is infinite. Therefore $\phi = 0$.

6. CKY tensors on manifolds with mixed 3-structures

An almost para-hypercomplex structure on a smooth manifold $M$ is a triple $H = (J_\alpha)_{\alpha = 1, 3}$, where $J_1$ is an almost complex structure on $M$ and $J_2, J_3$ are almost product structures on $M$, satisfying: $J_1 J_2 J_3 = -Id$. In this case $(M, H)$ is said to be an almost para-hypercomplex manifold.

A semi-Riemannian metric $g$ on $(M, H)$ is said to be para-hyperhermitian if it satisfies:

$$g(J_\alpha X, J_\alpha Y) = \epsilon_\alpha g(X, Y), \ \alpha \in \{1, 2, 3\},$$

for all $X, Y \in \Gamma(TM)$, where $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$. In this case, $(M, g, H)$ is called an almost para-hyperhermitian manifold. Moreover, if each $J_\alpha$ is parallel with respect to the Levi-Civita connection of $g$, then $(M, g, H)$ is said to be a para-hyper-Kähler manifold.
**Theorem 6** Let $(M, g)$ be a semi-Riemannian manifold. Then the following five assertions are mutually equivalent:

1. $(M, g)$ admits a mixed 3-Sasakian structure.
2. The cone $(C(M), \overline{g}) = (M \times \mathbb{R}_+, dr^2 + r^2g)$ admits a para-hyper-Kähler structure.
3. There exists three orthogonal Killing vector fields $\{\xi_1, \xi_2, \xi_3\}$ on $M$, with $\xi_1$ unit space-like vector field and $\xi_2, \xi_3$ unit time-like vector fields satisfying
   $$[\xi_\alpha, \xi_\beta] = -2\epsilon_{\alpha\beta\gamma}\xi_\gamma,$$
   (25)
   where $(\alpha, \beta, \gamma)$ is an even permutation of $(1,2,3)$ and $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$, such that the tensor fields $\phi_\alpha$ of type $(1,1)$, defined by: $\phi_\alpha X = \nabla_X \xi_\alpha, \alpha \in \{1,2,3\}$, satisfies the conditions (B.7), (B.8) and (B.9).
4. There exists three orthogonal Killing vector fields $\{\xi_1, \xi_2, \xi_3\}$ on $M$, with $\xi_1$ unit space-like vector field and $\xi_2, \xi_3$ unit time-like vector fields satisfying (25), such that:
   $$R(X, \xi_\alpha)Y = g(\xi_\alpha, Y)X - g(X, Y)\xi_\alpha, \alpha \in \{1,2,3\},$$
   (26)
   where $R$ is the Riemannian curvature tensor of the Levi-Civita connection $\nabla$ of $g$.
5. There exists three orthogonal Killing vector fields $\{\xi_1, \xi_2, \xi_3\}$ on $M$, with $\xi_1$ unit space-like vector field and $\xi_2, \xi_3$ unit time-like vector fields satisfying (25), such that the sectional curvature of every section containing $\xi_1, \xi_2$ or $\xi_3$ equals 1.

**Proof:**

1) $\Rightarrow$ 2) If $M^{4n+3}$ is a manifold endowed with a mixed 3-Sasakian structure (see Appendix B) $((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}, g)$, then we can define a para-hyper-Kähler structure $\{J_\alpha\}_{\alpha=1,3}$ on the cone $(C(M), \overline{g}) = (M \times \mathbb{R}_+, dr^2 + r^2g)$, by:
   $$J_\alpha X = \phi_\alpha X - \eta_\alpha (X)\Phi, \ J_\alpha \Phi = \xi_\alpha,$$
   (27)
   for any $X \in \Gamma(TM)$ and $\alpha \in \{1,2,3\}$, where $\Phi = r\partial_r$ is the Euler field on $C(M)$.

2) $\Rightarrow$ 1) If the cone $(C(M), \overline{g}) = (M \times \mathbb{R}_+, dr^2 + r^2g)$ admits a para-hyper-Kähler structure $\{J_\alpha\}_{\alpha=1,3}$, then we can identify $M$ with $M \times \{1\}$ and we have a mixed 3-Sasakian structure $((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}, g)$ on $M$ given by:
   $$\xi_\alpha = J_\alpha (\partial_r), \ \phi_\alpha X = \nabla_X \xi_\alpha, \ \eta_\alpha (X) = g(\xi_\alpha, X),$$
   (28)
   for any $X \in \Gamma(TM)$ and $\alpha \in \{1,2,3\}.$

(2) $\iff$ (3) This equivalence is clear (see also [48]).

(3) $\iff$ (4) This equivalence follows from direct computations.

(4) $\iff$ (5) This equivalence follows using the formula of the sectional curvature. $\blacksquare$

From the above Theorem we can easily obtain the next properties (see also [36]).

**Corollary 1** Let $M^{4n+3}$ be a manifold endowed with a mixed 3-Sasakian structure $((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}, g)$. Then:

1. $\xi_1$ is unit space-like Killing vector field and $\xi_2, \xi_3$ are unit time-like Killing vector fields on $M$;
2. $\eta_1, \eta_2, \eta_3$ are CKY tensors of rank 1 on $M$;
3. $d\eta_1, d\eta_2, d\eta_3$ are CKY tensors of rank 2 on $M$.
4. $M$ admits K-Y tensors of rank $(2k + 1)$, for $k \in \{0,1,\ldots,2n + 1\}$.

**Corollary 2** Let $M^{4n+3}$ be a manifold endowed with a mixed 3-Sasakian structure $((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}, g)$. Then the distribution spanned by $\{\xi_1, \xi_2, \xi_3\}$ is integrable and defines a 3-dimensional Riemannian foliation on $M$, having totally geodesic leaves of constant curvature 1.
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Appendix A. Euclidean Taub-NUT space

Let us consider the Taub-NUT space \cite{28, 29} and the chart with Cartesian coordinates \( x^\mu (\mu, \nu = 1, 2, 3, 4) \) having the line element

\[
    ds^2 = g_{\mu\nu} dx^\mu dx^\nu = f(r)(d\vec{x})^2 + \frac{g(r)}{16m^2} (dx^4 + A_i dx^i)^2, \tag{A.1}
\]

where \( \vec{x} \) denotes the three-vector

\[
    \vec{x} = (r, \theta, \varphi), \quad (d\vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2,
\]

and \( A_i \) is the gauge field of a monopole

\[
    \text{div} A = 0, \quad \vec{B} = \text{rot} \vec{A} = 4m \frac{\vec{x}}{r^3}.
\]

The real number \( m \) is a parameter of the theory which enter in the form of the functions

\[
    f(r) = g^{-1}(r) = V^{-1}(r) = \frac{4m + r}{r}, \tag{A.2}
\]

and the so called NUT singularity is absent if \( x^4 \) is periodic with period \( 16\pi m \). Sometimes it is convenient to make the coordinate transformation \( x^4 = -4m(\chi + \varphi) \), with \( 0 \leq \chi < 4\pi \).

In the Taub-NUT geometry there are four Killing vectors \cite{13, 49}

\[
    D_A = R^\mu_A \partial_\mu, \quad A = 1, 2, 3, 4. \tag{A.3}
\]

Three Killing vectors correspond to the invariance of the metric (A.1) under spatial rotations \((A = 1, 2, 3)\), obeying an \( SU(2) \) algebra with \([D_1, D_2] = -D_3, etc. \). \( D_4 \) generates the \( U(1) \) of \( \chi \) translations, commuting with the other Killing vectors.

In the scalar case these invariances would correspond to the conservation of angular momentum and the so called “relative electric charge”:

\[
    \vec{J} = \vec{r} \times \vec{p} + q \frac{\vec{r}}{r}, \quad q = g(r)(\dot{\theta} + \cos \theta \dot{\varphi}), \tag{A.4}
\]

where \( \vec{p} = m \dot{r} \vec{r} \) is the mechanical momentum.

On the other hand in the Taub-NUT geometry there are known to exist four K-Y tensors of valence 2. The first three are covariantly constant

\[
    f_i = 8m(d\chi + \cos \theta d\varphi) \wedge dx_i - \epsilon_{ijk}(1 + \frac{4m}{r})dx_j \wedge dx_k,
\]

\[
    \nabla_\mu f^\nu_{i\lambda} = 0, \quad i, j, k = 1, 2, 3. \tag{A.5}
\]
The $f^i$ define three anticommuting complex structures of the Taub-NUT manifold, their components realizing the quaternion algebra

$$f^i f^j + f^j f^i = -2\delta_{ij}, \quad f^i f^j f^k = -2\varepsilon_{ijk} f^k.$$

The existence of these K-Y tensors is linked to the hyper-Kähler geometry of the manifold and shows directly the relation between the geometry and the $N=4$ supersymmetric extension of the theory [6, 50].

The fourth K-Y tensor is

$$f_Y = 8m(d\chi + \cos \theta d\phi) \wedge dr + 4r(r + 2m)(1 + \frac{r}{4m}) \sin \theta d\theta \wedge d\varphi,$$

having a non-vanishing covariant derivative

$$f_{Y\mu
u} = 2(1 + \frac{r}{4m}) r \sin \theta.$$

In Taub-NUT space there is a conserved vector analogous to the Runge-Lenz vector of the Kepler-type problem [13, 51, 52]

$$\vec{K} = \frac{1}{2} K_{\mu
u} \dot{x}^{\mu}\dot{x}^{\nu} = \vec{p} \times \vec{j} + \left(\frac{\vec{q}^2}{4m} - 4mE\right) \frac{\vec{r}}{r},$$

where the $E$ is the conserved energy

$$E = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu}\dot{x}^{\nu}.$$

The components $K_{\mu\nu}$ involved with the Runge-Lenz vector (A.7) are S-K tensors satisfying (3) and they can be expressed as symmetrized products of the K-Y tensors $f_i, f_Y$ and Killing vectors $R_A$ [12]

$$K_{\mu\nu} - \frac{1}{8m}(R_{4\mu}R_{4\nu} + R_{4\nu}R_{4\mu}) = m\left(f_{Y\mu\nu} f_{i,\lambda}^{\lambda} \right).$$

In what follows we restrict ourselves to the generalized Taub-NUT manifolds whose metrics are defined on $\mathbb{R}^4 \setminus \{0\}$ by the line element [30, 31, 32]:

$$ds_K^2 = g_{\mu\nu}(x) dx^{\mu} dx^{\nu} = f(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + g(r)(d\chi + \cos \theta d\phi)^2,$$

where the angle variables ($\theta, \phi, \chi$) parametrize the sphere $S^3$ as mentioned above, while the functions

$$f(r) = \frac{a + br}{r}, \quad g(r) = \frac{ar + br^2}{1 + cr + dr^2}.$$  

depend on the arbitrary real constants $a, b, c$ and $d$.

Here it is worth pointing out that the above metrics are related to the Berger family of metrics on 3-spheres [53]. These are introduced starting with the Hopf fibration $\pi_H: S^3 \to S^2$ that defines the vertical subbundle $V \subset TS^3$ and its orthogonal complement $H \subset TS^3$ with respect to the standard metric $g_{S^3}$ on $S^3$. Denoting with $g_H$ and $g_V$ the restriction of $g_{S^3}$ to the horizontal, respectively the vertical bundle, one finds that the corresponding line elements are $ds_H^2 = \frac{1}{4} ds_{S^3}^2$ and $ds_V^2 = \frac{1}{4}(ds_{S^3}^2 - ds_{S^3}^2)$. For each constant $\lambda > 0$ the Berger metric on $S^3$ is defined by the formula

$$g_\lambda = g_H + \lambda^2 g_V."
This line element can be written in terms of the Berger metrics as
\[ ds_K^2 = (ar + br^2) \left( \frac{dr^2}{r^2} + 4ds_{\lambda(r)}^2 \right), \]
where \( ds_{\lambda(r)}^2 = (g_{\lambda(r)})_{\mu\nu}dx^\mu dx^\nu \) and
\[ \lambda(r) = \frac{1}{\sqrt{1 + cr + dr^2}}. \]

If one takes the constants
\[ c = \frac{2b}{a}, \quad d = \frac{b^2}{a^2}, \tag{A.11} \]
the generalized Taub-NUT metric becomes the original Euclidean Taub-NUT metric (A.1), (A.2) up to a constant factor.

By construction, the spaces with the metric (A.9) have four Killing vectors. The corresponding constants of motion in generalized Taub-NUT backgrounds consist of a conserved quantity for the cyclic variable \( \chi \)
\[ q = g(r)(\dot{\chi} + \cos \theta \dot{\phi}), \]
and the angular momentum vector
\[ \vec{J} = \vec{x} \times \vec{p} + \frac{q}{r} \vec{x}, \quad \vec{p} = f(r)\dot{\vec{x}}. \]

The generalized Taub-NUT space admits the conserved quantities (A.4) involving the functions \( f \) and \( g \) given by (A.10).

The remarkable result of Iwai and Katayama [30, 31, 32] is that the generalized Taub-NUT space (A.9) admits a hidden symmetry represented by a conserved vector, quadratic in 4-velocities, analogous to the Runge-Lenz vector of the following form
\[ \vec{K} = \vec{p} \times \vec{J} + \kappa \frac{\vec{x}}{r}. \tag{A.12} \]

The constant \( \kappa \) involved in the Runge-Lenz vector (17) is \( \kappa = -aE + \frac{1}{2}c q^2 \) where the conserved energy \( E \) is
\[ E = \frac{\vec{p}^2}{2f(r)} + \frac{q^2}{2g(r)}. \]

As it is expected the components \( K_i = k_i^{\mu\nu}p_\mu p_\nu \) of the vector \( \vec{K} \) (17) involve three S-K tensors \( k_i^{\mu\nu} \), \( i = 1, 2, 3 \) satisfying (3).

However, the generalized Taub-NUT space does not admit K-Y tensors [37]. Therefore a decomposition of the S-K tensors \( k_i^{\mu\nu} \) in terms of K-Y tensors is not possible, in contrast to (A.8) which is applicable only to the standard Taub-NUT space.

**Appendix B. Manifolds with mixed 3-structures**

Let \( M \) be a differentiable manifold equipped with a triple \((\phi, \xi, \eta)\), where \( \phi \) is a field of endomorphisms of the tangent spaces, \( \xi \) is a vector field and \( \eta \) is a 1-form on \( M \) such that:
\[ \phi^2 = -\epsilon I + \eta \otimes \xi, \quad \eta(\xi) = \epsilon. \tag{B.1} \]

If \( \epsilon = 1 \) then \((\phi, \xi, \eta)\) is said to be an almost contact structure on \( M \) (see [54]), and if \( \epsilon = -1 \) then \((\phi, \xi, \eta)\) is said to be an almost paracontact structure on \( M \) (see [55]).
Definition 4 [25] Let $M$ be a differentiable manifold which admits an almost contact structure $(\varphi_1, \xi_1, \eta_1)$ and two almost paracontact structures $(\varphi_2, \xi_2, \eta_2)$ and $(\varphi_3, \xi_3, \eta_3)$, satisfying the following conditions:

\begin{align}
\eta_\alpha (\xi_\beta) &= 0, \forall \alpha \neq \beta, \tag{B.2} \\
\varphi_\alpha (\xi_\beta) &= -\varphi_\beta (\xi_\alpha) = \epsilon_\gamma \xi_\gamma, \tag{B.3} \\
\eta_\alpha \circ \varphi_\beta &= -\eta_\beta \circ \varphi_\alpha = \epsilon_\gamma \eta_\gamma, \tag{B.4} \\
\varphi_\alpha \varphi_\beta - \eta_\beta \otimes \xi_\alpha &= -\varphi_\beta \varphi_\alpha + \eta_\alpha \otimes \xi_\beta = \epsilon_\gamma \varphi_\gamma, \tag{B.5}
\end{align}

where $(\alpha, \beta, \gamma)$ is an even permutation of $(1,2,3)$ and $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$.

Then the manifold $M$ is said to have a mixed 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}$.

Definition 5 If a manifold $M$ with a mixed 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}$ admits a semi-Riemannian metric $g$ such that:

\begin{align}
g(\varphi_\alpha X, \varphi_\alpha Y) &= \epsilon_\alpha g(X, Y) - \eta_\alpha (X) \eta_\alpha (Y), \tag{B.6} \\
(\nabla_X \varphi_\alpha)Y &= g(X, Y) \xi_1 - \eta_1 (Y) X, \tag{B.7}
\end{align}

for all $X, Y \in \Gamma(TM)$ and $\alpha = 1, 2, 3$, then we say that $M$ has a metric mixed 3-structure and $g$ is called a compatible metric. Moreover, if $(\varphi_1, \xi_1, \eta_1, g)$ is a Sasakian structure, i.e. (see [54]):

\begin{align}
(\nabla_X \varphi_1)Y &= g(X, Y) \xi_1 - \eta_1 (Y) X, \tag{B.7} \\
(\nabla_X \varphi_2)Y &= g(\varphi_2 X, \varphi_2 Y) \xi_2 + \eta_2 (Y) \varphi_2 X, \tag{B.8} \\
(\nabla_X \varphi_3)Y &= g(\varphi_3 X, \varphi_3 Y) \xi_3 + \eta_3 (Y) \varphi_3 X, \tag{B.9}
\end{align}

and $(\varphi_2, \xi_2, \eta_2, g), (\varphi_3, \xi_3, \eta_3, g)$ are LP-Sasakian structures, i.e. (see [55]):

then $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,3}, g)$ is said to be a mixed Sasakian 3-structure on $M$.

It is easy to see that any manifold $M$ with a mixed 3-structure admits a compatible semi-Riemannian metric $g$. Moreover, the signature of $g$ is $(2n + 1, 2n + 2)$ and the dimension of the manifold $M$ is $4n + 3$. The main property of a manifold endowed with a mixed 3-Sasakian structure is the following (see [26, 27]):

Theorem 7 Any $(4n+3)$-dimensional manifold endowed with a mixed 3-Sasakian structure is an Einstein space with Einstein constant $\lambda = 4n + 2$. 

