INITIAL BOUNDARY VALUE PROBLEM FOR
TWO-DIMENSIONAL VISCOS BOUSSINESQ EQUATIONS FOR
MHD CONVECTION

DONGFEN BIAN

School of Mathematics and Statistics, Beijing Institute of Technology
Beijing 100081, China

and

Beijing Key Laboratory on MCAACI, Beijing Institute of Technology
Beijing 100081, China

Dedicated to Professor Boling Guo on the occasion of his 80th birthday

ABSTRACT. This paper is concerned with the initial boundary value problem for two-dimensional viscous Boussinesq equations for MHD convection. We show that the system has a unique classical solution for $H^3$ initial data, and the non-slip boundary condition for velocity field and the perfectly conducting wall condition for magnetic field. In addition, we show that the kinetic energy is uniformly bounded in time.

1. Introduction. We consider in this paper the following 2-D incompressible Boussinesq equations for magnetohydrodynamics (MHD) convection

\[
\begin{aligned}
\begin{cases}
    \text{u_t} + \text{u} \cdot \nabla \text{u} - \mu \Delta \text{u} + \nabla \Pi = g \theta e_2 + J \text{B}^\perp, \\
    \theta_t + \text{u} \cdot \nabla \theta = 0, \\
    \text{B_t} + \text{u} \cdot \nabla \text{B} - \gamma \Delta \text{B} = \text{B} \cdot \nabla \text{u}, \\
    \nabla \cdot \text{u} = 0, \quad \nabla \cdot \text{B} = 0,
\end{cases}
\end{aligned}
\]

in a bounded domain $\Omega$ with prescribed initial conditions:

\[
(\theta, \text{u}, \text{B})(x, 0) = (\theta_0, \text{u}_0, \text{B}_0), \quad x \in \Omega,
\]

and boundary conditions:

\[
\begin{aligned}
    \text{u}|_{\partial \Omega} &= 0, \\
    \text{B} \cdot \text{n}|_{\partial \Omega} &= 0, \quad (\nabla \times \text{B}) \times \text{n}|_{\partial \Omega} = 0,
\end{aligned}
\]

where $\text{n}$ denotes the unit outward normal on $\partial \Omega$. The condition (3) is the so-called non-slip boundary condition, and the condition (4) is known as the perfectly conducting wall condition which describes the case where the wall of container is
made of perfectly conductive materials. This boundary condition (4) is classical in the theory of magnetohydrodynamics.

The unknowns in (1) are the solenoidal velocity field \( u = (u_1, u_2) \), temperature \( \theta \) (or the density in the modeling of geophysical fluids), the magnetic field \( B = (B_1, B_2) \), and the pressure \( \Pi \). The second equation of (1) means the momentum conservation law of the fluid with the effect of the gravity and the Lorentz force \( JB^\perp \). We denote here by \( \mu \) its viscosity, and \( \gamma \) the electrical resistivity, all of them being constants, \( e_2 = (0,1) \) the direction of gravitational acceleration, the gravity unit \( g = 1 \) for simplicity, \( B^\perp = (-B_2, B_1) \), and the current density \( J = J(B) = \nabla^\perp \cdot B = \partial_1 B_2 - \partial_2 B_1 \).

The system (1) is a combination of the incompressible Boussinesq equations of fluid dynamics and Maxwell’s equations of electromagnetism, where the displacement current can be neglected [24].

If the fluid is not affected by the temperature, then the equations (1) becomes the MHD system, which govern the dynamics of the velocity and the magnetic field in electrically conducting fluids such as plasmas and reflect the basic physics conservation laws. There have been a lot of studies on MHD by physicists and mathematicians. For instance, G. Duvaut and J.-L. Lions [12] established the local existence and uniqueness of the solution in the Sobolev spaces \( H^s(\mathbb{R}^d) \), \( s \geq d \), and proved global existence of the solution for small initial data, and then M. Sermange and R. Temam [30] examined some properties of these solutions. In particular, the 2-D local strong solution has been proved to be global and unique. Recent work on the MHD equations developed regularity criteria in terms of the velocity field and dealt with the MHD equations with dissipation and magnetic diffusion (see, e.g. [9, 17]). Also the issue of the global regularity problem on the MHD equations with partial dissipation, has been extensively studied (see, e.g., [7, 26, 27, 29]). Further background and motivation for the MHD system may be found in [7, 9, 11, 12, 15, 16, 17, 27, 30] and references therein.

On the other hand, if the fluid is not affected by the Lorentz force, that is, \( B \equiv 0 \), then the equations (1) becomes the classical Boussinesq system without diffusion. The Cauchy problem for the 2-D Boussinesq system with full viscosity has been well studied. Cannon-Dibenedetto [6] and Wang-Zhang [32] proved the global well-posedness for the Cauchy problem for the Boussinesq system with full viscosity for the smooth initial data. Recently, there are many works devoted to the study of the Boussinesq system with partial constant viscosity. For instance, D. Chae [8], and T. Y. Hou and C. Li [21] independently proved the global well-posedness of the Cauchy problem for the 2-D Boussinesq system without diffusion, and D. Chae [8] also considered the case of zero viscosity (that is, \( \mu = 0 \)). Abidi-Hmidi [2] established the global well-posedness of the Cauchy problem for the 2-D Boussinesq system in the critical spaces without diffusion, and Hmidi-Keraani [18] showed the global well-posedness of the Cauchy problem for the 2-D Boussinesq system in the critical spaces in the case of zero viscosity. Lai-Pan-Zhao [23] considered the case of bounded domain and proved the global well-posedness for the initial boundary value problem of the 2-D Boussinesq system without diffusion. R. Danchin and M. Paicu [10] investigated the global existence of weak solution for \( L^2 \) data and the global well-posedness for small smooth data in 3-D case for the Cauchy problem of the Boussinesq system. T. Hmidi and F. Rousset [19, 20] proved the global well-posedness for the Cauchy problem of the 3-D axisymmetric Boussinesq system without swirl. Recently, Li and Xu [25] proved global well-posedness
of strong solutions for the Cauchy problem of the 2-D inviscid Boussinesq system with temperature-dependent diffusion with large initial data in Sobolev spaces. The global regularity/singularity question for the Boussinesq system with zero viscosity and zero diffusion still remains an outstanding open problem in mathematical fluid mechanics.

For Boussinesq-MHD system (1) with its viscosity $\mu$, the diffusion $\kappa$, and the electrical resistivity $\gamma$ being smooth, positive functions of the temperature $\mu = \mu(\theta)$, $\kappa = \kappa(\theta)$, $\gamma = \gamma(\theta)$, Bian-Guo-Gui [3] and Bian-Guo-Gui-Xin [4] rigorously justified the stability and instability in a fully nonlinear, dynamical setting from mathematical point of view. In this paper, we are interested in the initial boundary value problem for the system (1) with $\mu$, $\gamma$ two positive constants. For the global existence of smooth solutions, we assume the following compatibility conditions

$$
\begin{cases}
\nabla \cdot u_0 = 0, \quad \nabla \cdot B_0 = 0, \\
u_0|_{\partial \Omega} = 0, \quad B_0 \cdot n|_{\partial \Omega} = 0, \quad (\nabla \times B_0) \times n|_{\partial \Omega} = 0, \\
\mu \Delta u_0 - \nabla \Pi_0 + g \theta_0 e_2 + J_0 B_0^\perp = 0, \quad x \in \partial \Omega, \quad t = 0, \\
\gamma \Delta B_0 + B_0 \cdot \nabla u_0 = 0, \quad x \in \partial \Omega, \quad t = 0,
\end{cases}
$$

(5)

where $\Pi_0(x) = \Pi(x, 0)$ is the solution to the Neumann boundary problem

$$
\begin{cases}
\Delta \Pi_0 = \nabla \cdot [g \theta e_2 - u_0 \cdot \nabla u_0 + J_0 B_0^\perp], \quad x \in \Omega, \\
\nabla \Pi_0 \cdot n|_{\partial \Omega} = [\mu \Delta u_0 + g \theta_0 e_2] \cdot n|_{\partial \Omega}.
\end{cases}
$$

(6)

Our main results are stated in the following theorem.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. If $(\theta_0, u_0, B_0) \in H^3(\Omega)$ satisfies the compatibility conditions (5)-(6), then there exists a unique solution $(\theta, u, B)$ of (1)-(4) globally in time such that $\theta \in C([0, T]; H^3(\Omega))$ and $(u, B) \in C([0, T]; H^3(\Omega)) \cap L^2([0, T]; H^1(\Omega)) \times C([0, T]; H^3(\Omega)) \cap L^2([0, T]; H^1(\Omega))$ for any $T > 0$. Moreover, there exists a constant $C > 0$ independent of $t$ such that

$$
\|(u, B)\|^2_{L^2} \leq \max \left\{ \|(u_0, B_0)\|^2_{L^2}, \frac{C^2}{\sigma \mu} \|\theta_0\|^2_{L^2} \right\}, \quad \forall \ t \geq 0,
$$

(7)

with $\sigma = \min \{\mu, 2\gamma\}$.

**Remark 1.** Because of the perfectly conducting wall condition for magnetic field, we need the Poincaré inequality revised in Lemma 2.2 to deal with the term $\|\nabla B\|_{L^2(\Omega)}$ in (62). We also use several techniques to manipulate the nonlinear terms from the strong coupling of the velocity $u$, magnetic $B$ and temperature $\theta$.

**Remark 2.** To our knowledge, the question of global regularity/finite time singularity for the cases of partial viscosity, such as (1) with either $\gamma = 0$ or $\mu = 0$ is still open. In a separate paper, we will consider these cases.

The proof of Theorem 1.1 consists of three main parts. First, we show the global existence of weak solutions to (1)-(4), that is, solutions satisfying the following definition:

**Definition 1.2.** $(\theta, u, B)$ is said to be a global weak solution of (1)-(4), if for any $T > 0$, $(u, B) \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H^1_0(\Omega)) \times C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H^1(\Omega))$.
\[ H^1(\Omega), \theta \in C([0, T]; L^p(\Omega)), \forall \ 1 \leq p < \infty, \text{ it holds that} \]
\[
\int_\Omega u_0 \cdot \Phi(x, 0)dx \\
+ \int_0^T \int_\Omega \left( (u \cdot \Phi_t + (u \cdot \nabla \Phi) \cdot u - (B \cdot \nabla \Phi) \cdot B + \theta \phi_2 - \mu \nabla u \cdot \nabla \Phi) \right) dxdt = 0, \\
\int_\Omega \theta_0 \cdot \zeta(x, 0)dx + \int_0^T \int_\Omega \left( \theta \zeta_t + \theta (u \cdot \nabla \zeta) \right) dxdt = 0, \\
\int_\Omega B_0 \cdot \Psi(x, 0)dx \\
+ \int_0^T \int_\Omega \left( (B \cdot \Psi_t + (u \cdot \nabla \Psi) \cdot B - (B \cdot \nabla \Psi) \cdot u - \gamma \nabla B \cdot \nabla \Psi) \right) dxdt = 0,
\]
for any \( \Phi = (\phi_1, \phi_2) \in C_0^\infty(\Omega \times [0, T])^2 \) satisfying \( \Phi(x, T) = 0 \) and \( \nabla \cdot \Phi = 0 \), for any \( \Psi = (\psi_1, \psi_2) \in C_0^\infty(\Omega \times [0, T])^2 \) satisfying \( \Psi(x, T) = 0 \) and \( \nabla \cdot \Psi = 0 \), and for any \( \zeta \in C_0^\infty(\Omega \times [0, T]) \) satisfying \( \zeta(x, T) = 0 \).

We then establish the regularity and uniqueness of the solution by energy estimate under the initial and boundary conditions (2)-(4).

The present paper is structured as follows. Section 2 is devoted to give some basic facts which will be used in the following sections and the global existence of weak solutions to (1)-(4). In Section 3, we improve the regularity of the solution obtained in Section 2 by energy estimate. The last Section is dedicated to establish the uniqueness of the solution, and then we prove Theorem 1.1.

2. Preliminaries and weak solutions. In this section, we first list several facts which will be used in the proof of Theorem 1.1, and then give the global existence of weak solutions of (1)-(4).

**Lemma 2.1.** ([1, 22, 23]) Let \( \Omega \in \mathbb{R}^2 \) be any bounded domain with \( C^1 \) smooth boundary. Then the following embeddings and inequalities hold:

(i) \( H^1(\Omega) \hookrightarrow L^p(\Omega), \forall \ 1 \leq p < \infty; \)

(ii) \( W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega), \forall \ 2 < p < \infty; \)

(iii) \( \|f\|_{L^2}^2 \leq C(\|f\|_{L^2}^2 \|\nabla f\|_{L^2}^2), \forall \ f : \Omega \to \mathbb{R} \text{ and } f \in H^1_0(\Omega); \)

(iv) \( \|f\|_{L^2}^2 \leq C(\|f\|_{L^2}^2 \|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2), \forall \ f : \Omega \to \mathbb{R} \text{ and } f \in H^1(\Omega). \)

We also need the Poincaré inequality revised.

**Lemma 2.2.** Let \( \Omega \) be any open bounded domain in \( \mathbb{R}^2 \) with smooth boundary \( \partial \Omega \). Assume that \( u \) satisfies \( u \in W^{1,p}(\Omega) \) and \( u \cdot n = 0 \) on \( \partial \Omega \). Then there exists a constant \( C = C(\Omega, p) \) such that

\[
\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, 
\]

**Proof.** Assume that \( u \in W^{1,p}(\Omega) \) and \( u \cdot n = 0 \) on \( \partial \Omega \). By contradiction suppose (8) is not true. This means that for \( \forall \ j \in \mathbb{N}, \exists \ u_j \) satisfies the same hypotheses such that

\[
\|u_j\|_{L^p(\Omega)} \geq j\|\nabla u_j\|_{L^p(\Omega)}. 
\]

Normalize \( u_j \) in \( L^p(\Omega) \) by setting

\[
w_j = \frac{u_j}{\|u_j\|_{L^p(\Omega)}}. 
\]
Then, from (9),
\[ \|w_j\|_{L^p(\Omega)} = 1 \quad \text{and} \quad \|\nabla w_j\|_{L^p(\Omega)} < \frac{1}{j} \leq 1. \]
Thus, \{w_j\} is bounded in \( W^{1,p}(\Omega) \) and by Rellich’s theorem there exists a sequence \( w_{jk} \) and \( w \) also satisfies the same hypotheses such that
\[ w_j \to w \text{ strongly in } L^p(\Omega), \]
\[ \nabla w_j \to \nabla w \text{ weakly in } L^p(\Omega). \]
The continuity of the norm gives
\[ \|w\|_{L^p(\Omega)} = \lim_{j \to \infty} \|w_j\|_{L^p(\Omega)} = 1. \]
On the other hand, the weak semicontinuity of the norm yields
\[ \|\nabla w\|_{L^p(\Omega)} \leq \liminf_{j \to \infty} \|\nabla w_j\|_{L^p(\Omega)} = 0, \]
so that \( \nabla w = 0 \). Since \( \Omega \) is connected, \( w \) is constant and since \( w \) satisfies \( w \cdot n = 0 \) on \( \partial \Omega \), we infer \( w = 0 \), in contradiction to \( \|w\|_{L^p(\Omega)} = 1 \).

Now, we establish the global existence of weak solutions of (1)-(4).

\section*{Proposition 1}
Under the assumptions in Theorem 1.1, there exists a global weak solution \((\theta, u, B)\) of (1)-(4), such that, for any \( T > 0 \), \((u, B) \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H^1_0(\Omega)) \times C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H^1(\Omega))\), and \( \theta \in C([0, T]; L^p(\Omega)) \), \( 1 \leq p < \infty \).

\textbf{Proof.} Following [23, 28], we can show easily the proposition by a fixed point argument. This proposition can also be proved by the method in [5, 13, 14]. For the sake of completeness, we give the proof using the method of [23, 28] here.

For that, denote \( \tilde{H}^1_0(\Omega) = \{ u \in H^1(\Omega) \colon u|_{\partial \Omega} = 0 \text{ or } u \cdot n = 0, \ (\nabla \times u) \times n = 0 \} \).

Let \( A \) be the closed convex set in \( C([0, T]; L^2(\Omega)) \cap L^2([0, T]; \tilde{H}^1_0(\Omega)) \) defined by
\[ A = \left\{ (v, H) \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; \tilde{H}^1_0(\Omega)) \mid \nabla \cdot v = 0, \ \nabla \cdot H = 0 \ \text{a.e. on } \Omega \times (0, T), \right\}. \]

\[ \| (v, H) \|_{C([0, T]; L^2(\Omega))} + \| (v, H) \|_{L^2(0,T;\tilde{H}^1_0(\Omega))} \leq R_0 \}
\]
where \( v = (v_1, v_2), \ H = (H_1, H_2) \) and \( R_0 \) will be determined later. For fixed \( \varepsilon \in (0, 1) \) and any \((v, H) \in A\), we first mollify \( v \) and \( H \) by the standard procedure (see [23, 28]) to get
\[ v_{\varepsilon} = \bar{v}_{\varepsilon} * \frac{\eta_{\varepsilon}}{2}, \ H_{\varepsilon} = \bar{H}_{\varepsilon} * \frac{\eta_{\varepsilon}}{2}, \]
where \( \bar{v}_{\varepsilon} \) and \( \bar{H}_{\varepsilon} \) are the truncation of \( v \) and \( H \) in \( \Omega_{\varepsilon} = \{ x \in \Omega, \ \text{dist}(x, \partial \Omega) > \varepsilon \} \) (extended by 0 to \( \Omega \)), and \( \frac{\eta_{\varepsilon}}{2} \) is the standard mollifier. Then \( v_{\varepsilon} \) and \( H_{\varepsilon} \) satisfy
\[ (v_{\varepsilon}, H_{\varepsilon}) \in C([0, T]; C^0_0(\Omega)), \ \nabla \cdot v_{\varepsilon} = 0, \ \nabla \cdot H_{\varepsilon} = 0, \]
\[ \| (v_{\varepsilon}, H_{\varepsilon}) \|_{C([0, T]; L^2(\Omega))} \leq C \| (v, H) \|_{C([0, T]; L^2(\Omega))}, \]
\[ \| (v_{\varepsilon}, H_{\varepsilon}) \|_{L^2(0,T;\tilde{H}^1_0(\Omega))} \leq C \| (v, H) \|_{L^2(0,T;\tilde{H}^1_0(\Omega))}, \]
for some constant \( C > 0 \) which is independent of \( \varepsilon \). Similarly, we regularize the initial data to obtain the smooth approximation \( \theta_{\varepsilon}^0 \) for \( \theta_0 \), \( u_{\varepsilon}^0 \) for \( u_0 \) and \( B_{\varepsilon}^0 \) for \( B_0 \).
respectively, such that
\[ \theta^0_\varepsilon \in C^0_0(\Omega), \quad \| \theta^0_\varepsilon - \theta_0 \|_{H^1(\Omega)} \leq \varepsilon, \]
\[ u^0_\varepsilon \in C^0_0(\Omega), \quad \nabla \cdot u^0_\varepsilon = 0, \quad \| u^0_\varepsilon - u_0 \|_{H^1(\Omega)} \leq \varepsilon, \]
\[ B^0_\varepsilon \in C^0_0(\Omega), \quad \nabla \cdot B^0_\varepsilon = 0, \quad \| B^0_\varepsilon - B_0 \|_{H^1(\Omega)} \leq \varepsilon. \]
Then we solve the transport equation with smooth initial data
\[
\begin{cases}
\theta_t + v_\varepsilon \cdot \nabla \theta = 0, \\
\theta(x, 0) = \theta^0_\varepsilon(x),
\end{cases}
\tag{12}
\]
and we denote the solution by \( \theta^\varepsilon \). Next, we solve the nonhomogeneous (Linearized) MHD equations with smooth initial data
\[
\begin{cases}
\nabla \cdot u = 0, \quad \nabla \cdot B = 0, \\
u_t + v_\varepsilon \cdot \nabla u + \nabla p = \mu \Delta u + \theta^\varepsilon e_2 + H_\varepsilon \cdot \nabla B, \\
B_t + v_\varepsilon \cdot \nabla B = \gamma \Delta B + H_\varepsilon \cdot \nabla u, \\
u|_{\partial \Omega} = 0, \quad B \cdot n|_{\partial \Omega} = 0, \quad (\nabla \times B) \times n|_{\partial \Omega} = 0, \\
u(x, 0) = u^0_\varepsilon(x), \quad B(x, 0) = B^0_\varepsilon(x),
\end{cases}
\tag{13}
\]
and denote the solution by \((u^\varepsilon, B^\varepsilon)\) and the corresponding pressure by \(p^\varepsilon\). We then define the mapping \(F_\varepsilon(v_\varepsilon, H_\varepsilon) = (u^\varepsilon, B^\varepsilon)\). The solvabilities of (12) and (13) follow easily from [23, 28]. Next, using the energy method to prove that \(F_\varepsilon\) satisfies the conditions of the schauder fixed point theorem, that is, \(F_\varepsilon : A \to A\) is continuous and compact.

For any \(2 \leq p < \infty\), multiplying the first equation of (12) by \(\theta_0^{p-1}\) and integrating the resulting equation over \(\Omega\) by parts, one gets
\[
\| \theta^\varepsilon \|_{L^p} = \| \theta^0_\varepsilon \|_{L^p} \leq \| \theta_0 \|_{L^p} + \varepsilon c(\Omega, p), \quad \forall \ 0 \leq t \leq T, \quad \forall \ 0 < \varepsilon < 1,
\]
that is,
\[
\| \theta^\varepsilon \|_{L^p} \leq \| \theta_0 \|_{L^p} + \varepsilon c(\Omega, p), \quad \forall \ 0 \leq t \leq T, \quad \forall \ 0 < \varepsilon < 1,
\tag{14}
\]
where \(c(\Omega, p)\) is a constant depending only on \(\Omega\) and \(p\). We then estimate \(\|(u^\varepsilon, B^\varepsilon)\|_{L^2(\Omega; H^1_0(\Omega))}^2\)

Recall the boundary condition (13)_4 and the identity \(\Delta B = \nabla(\nabla \cdot B) - \nabla \times (\nabla \times B)\), it follows from integrating by parts that
\[
- \int_{\Omega} \Delta B \cdot B dx = \int_{\Omega} \nabla \times (\nabla \times B) - \nabla(\nabla \cdot B) \cdot B dx
\]
\[
= \int_{\partial \Omega} \nabla \times (\nabla \times B) \cdot B dS + \int_{\Omega} |\nabla \times B|^2 dx
\]
\[
= - \int_{\partial \Omega} n \times (\nabla \times B) \cdot B dS + \int_{\Omega} |\nabla \times B|^2 dx
\tag{15}
\]
Taking the \(L^2\) inner product of (13)_2 with \(u\) and (13)_3 with \(B\) respectively, integrating the resulting equations over \(\Omega\) by parts, one gets
By Young’s inequality, the right-hand side of (16) can be estimated as
\[
\int_{\Omega} [(H_\varepsilon \cdot \nabla B) \cdot u + (H_\varepsilon \cdot \nabla u) \cdot B] dx + \int_{\Omega} \theta^\varepsilon e_2 \cdot u dx.
\]
which together with the result in [31] and (16), implies that
\[
\|\nabla \phi\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \gamma \|\nabla B\|_{L^2}^2 \leq C(\delta) \|\theta^\varepsilon\|_{L^2}^2 + \delta \|u\|_{L^2}^2,
\]
where \(\delta > 0\) is a constant to be determined. By Lemma 2.2, one gets \(\|u\|_{L^2} \leq C\|\nabla u\|_{L^2}\) for some constant \(C\) depending only on \(\Omega\). Choosing \(\delta = \frac{\varepsilon}{27}\) in (16), one obtains
\[
\frac{1}{2} \frac{d}{dt} \|(u, B)\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 + \gamma \|\nabla B\|_{L^2}^2 \leq C(\varepsilon) \|\theta^\varepsilon\|_{L^2}^2 + \delta \|u\|_{L^2}^2,
\]
which together with (14) yields, after integration over \([0, T]\), that
\[
\|(u, B)\|_{C([0, T]; L^2(\Omega))}^2 + \|\nabla u\|_{L^2(0, T; L^2(\Omega))}^2 + \gamma \|\nabla B\|_{L^2(0, T; L^2(\Omega))}^2 \leq C(\|\theta_0\|_{L^2}^2 + \|\varepsilon\| T + (\|(u_0, B_0)\|_{L^2}^2 + \varepsilon).
\]
Since \(0 < \varepsilon < 1\), one has
\[
\|(u, B)\|_{C([0, T]; L^2(\Omega))}^2 + \|(u, B)\|_{L^2(0, T; H_0^1(\Omega))}^2 \leq C(T, \theta_0, u_0, B_0, \mu, \gamma, \Omega),
\]
that is,
\[
\|(u^\varepsilon, B^\varepsilon)\|_{C([0, T]; L^2(\Omega))}^2 + \|(u^\varepsilon, B^\varepsilon)\|_{L^2(0, T; H_0^1(\Omega))}^2 \leq C(T, \theta_0, u_0, B_0, \mu, \gamma, \Omega). \tag{17}
\]
Choosing \(R_0\) such that \(R_0 \geq C(T, \theta_0, u_0, B_0, \mu, \gamma, \Omega)\), we know that \(F_\varepsilon\) maps \(A\) into \(A\) for any \(0 < \varepsilon < 1\).

Next, we prove the compactness of \(F_\varepsilon\). For that, we continue to find estimates of \(\|\nabla u^\varepsilon\|_{C([0, T]; L^2(\Omega))}^2\) and \(\|u^\varepsilon\|_{L^2(0, T; L^2(\Omega))}\). Taking \(L^2\) inner product of (13)_2 and (13)_3 with \(u_t\) and \(B_t\) respectively, one gets
\[
\frac{\mu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \gamma \frac{d}{dt} \|\nabla B\|_{L^2}^2 + \|(u_t, B_t)\|_{L^2}^2 \leq \int_{\Omega} |v_\varepsilon| |\nabla u| |u_t| dx + \int_{\Omega} |H_\varepsilon| |\nabla B| |u_t| dx + \int_{\Omega} \theta^\varepsilon e_2 \cdot u_t dx \tag{18}
\]
We estimate the right-hand side of (18) as follows
\[
\int_{\Omega} |v_\varepsilon| |\nabla u| |u_t| dx \leq C \|v_\varepsilon\|_{L^\infty} \|\nabla u\|_{L^2}^2 + \frac{1}{6} \|u_t\|_{L^2}^2,
\]
\[
\int_{\Omega} |H_\varepsilon| |\nabla B| |u_t| dx \leq C \|H_\varepsilon\|_{L^\infty} \|\nabla B\|_{L^2}^2 + \frac{1}{6} \|u_t\|_{L^2}^2,
\]
It follows from Gronwall’s inequality and (11) that
\[
\int_\Omega \theta^e \cdot u_t dx \leq \frac{1}{6} ||u_t||_{L^2}^2 + C ||\theta^e||_{L^2}^2 \leq \frac{1}{6} ||u_t||_{L^2}^2 + C,
\]
\[
\int_\Omega |v_e| \cdot |\nabla B||B_t| dx \leq C |v_e|_{L^\infty} \cdot |\nabla B||_{L^2}^2 + \frac{1}{4} ||B_t||_{L^2}^2,
\]
\[
\int_\Omega |H_e| \cdot |\nabla u||B_t| dx \leq C |H_e|_{L^\infty} \cdot |\nabla u||_{L^2}^2 + \frac{1}{4} ||B_t||_{L^2}^2,
\]
which together with (18), gives
\[
\frac{\mu}{2} \frac{d}{dt} ||u||_{L^2}^2 + \frac{\gamma}{2} \frac{d}{dt} ||\nabla B||_{L^2}^2 + \frac{1}{2} ||(u_t, B_t)||_{L^2}^2 \leq C ||(v_e, H_\varepsilon)||_{L^\infty} \cdot ||(\nabla u, \nabla B)||_{L^2}^2 + C.
\]
(19)

It follows from Gronwall’s inequality and (11) that
\[
|| (\nabla u, \nabla B) ||_{L^2(0, T; L^2(\Omega))}^2 + || (u_t, B_t) ||_{L^2(0, T; L^2(\Omega))}^2 \leq C.
\]
(20)

Using \(H^2\) estimates to Stokes equations and elliptic equations, one obtains from (13)\_2 and (13)\_3 that
\[
|| (u, B) ||_{H^2}^2 \leq C (|| (u_t, B_t) ||_{L^2}^2 + || \theta^e ||_{L^2}^2 + || v_e \cdot \nabla u ||_{L^2}^2 + || H_e \cdot \nabla B ||_{L^2}^2
\]
\[
+ || v_e \cdot \nabla B ||_{L^2}^2 + || H_e \cdot \nabla u ||_{L^2}^2)
\]
\[
\leq C || (u_t, B_t) ||_{L^2}^2 + C ||(v_e, H_\varepsilon)||_{L^\infty} \cdot ||(\nabla u, \nabla B)||_{L^2}^2 + C,
\]
(21)

which together with (20) yields
\[
|| (u^e, B^e) ||_{L^2(0, T; H^2(\Omega))} \leq C.
\]
(22)

Obviously, (20) and (22) imply that \(F_\varepsilon\) is compact by the Sobolev embedding theorem.

Now, we prove the continuity of \(F_\varepsilon\). Let \(F_\varepsilon(v, \lambda) = (u^e_\varepsilon, B^e_\varepsilon)\), by definition, we see
\[
\left\{
\begin{array}{l}
\theta^e_{it} + v_e \cdot \nabla \theta^e_i = 0, \\
u^e_{it} + v_e \cdot \nabla u^e_i + \nabla p^e_i = \mu \Delta u^e_i + \theta^e_i e_2 + H_e \cdot \nabla B^e_i,
\end{array}
\right.
\]
\[
\left\{
\begin{array}{l}
B^e_{it} + v_e \cdot \nabla B^e_i = \gamma \Delta B^e_i + H_e \cdot \nabla u^e_i,
\end{array}
\right.
\]
\[
\left\{
\begin{array}{l}
\nabla \cdot u^e_i = 0, \nabla \cdot B^e_i = 0,
\end{array}
\right.
\]
\[
\left\{
\begin{array}{l}
u^e_i |_{\partial \Omega} = 0, \ B^e_i |_{\partial \Omega} = 0, \ (\nabla \times B^e_i) \times n |_{\partial \Omega} = 0,
\end{array}
\right.
\]
\[
\left\{
\begin{array}{l}(\theta^e_i, u^e_i, B^e_i)(x, 0) = (\theta_0^e, u_0^e, B_0^e)(x).
\end{array}
\right.
\]

Let \(\theta^e = \theta^e_1 - \theta^e_2\), \(W_\varepsilon = v_1 - v_2\), \(J_\varepsilon = H_1 - H_2\), \(\chi^e = u^e_1 - u^e_2\), \(Z_\varepsilon = B^e_1 - B^e_2\) and \(Q^e = p_1 - p_2\), then \((\theta^e, \chi^e, Z^e, Q^e)\) satisfy
\[
\left\{
\begin{array}{l}
\theta^e_t + v_1 \cdot \nabla \theta^e + W_\varepsilon \cdot \nabla \theta^e_2 = 0, \\
\chi^e_t + v_1 \cdot \nabla \chi^e + W_\varepsilon \cdot \nabla u^e_2 + \nabla Q^e = \mu \Delta \chi^e + \theta^e e_2 + H_1 \cdot \nabla Z^e + J_\varepsilon \cdot \nabla B^e_2,
\end{array}
\right.
\]
\[
\left\{
\begin{array}{l}
Z^e_t + v_1 \cdot \nabla Z^e + W_\varepsilon \cdot \nabla B^e_2 = \gamma \Delta Z^e + H_1 \cdot \nabla \chi^e + J_\varepsilon \cdot \nabla u^e_2,
\end{array}
\right.
\]
\[
\left\{
\begin{array}{l}
\nabla \cdot \chi^e = 0, \nabla \cdot Z^e = 0, \\
\chi^e |_{\partial \Omega} = 0, \ Z^e \cdot n |_{\partial \Omega} = 0, \ (\nabla \times Z^e) \times n |_{\partial \Omega} = 0,
\end{array}
\right.
\]
\[
\left\{
\begin{array}{l}(\theta^e, \chi^e, Z^e)(x, 0) = (0, 0, 0, 0)(x).
\end{array}
\right.
\]

Multiplying the first three equations of the above system with \(\theta^e\), \(\chi^e\) and \(Z^e\) respectively, one has
\[
\frac{1}{2} \frac{d}{dt} ||\theta^e||_{L^2}^2 = \int_\Omega (W_\varepsilon \cdot \nabla \theta^e_2) \theta^e dx,
\]
(23)
\begin{equation}
\frac{1}{2} \frac{d}{dt} \| (\chi^\varepsilon, Z^\varepsilon) \|^2_{L^2} + \mu \| \nabla \chi^\varepsilon \|^2_{L^2} + \gamma \| \nabla Z^\varepsilon \|^2_{L^2} = - \int_{\Omega} [(W_\varepsilon \cdot \nabla u_2^\varepsilon) \cdot \chi^\varepsilon + (W_\varepsilon \cdot \nabla B_2^\varepsilon) \cdot Z^\varepsilon] \, dx + \int_{\Omega} \theta^\varepsilon \chi^\varepsilon \, dx \\
+ \int_{\Omega} [(J_\varepsilon \cdot \nabla B_2^\varepsilon) \cdot \chi^\varepsilon + (J_\varepsilon \cdot \nabla u_2^\varepsilon) \cdot Z^\varepsilon] \, dx.
\end{equation}

Since \( \theta_2^\varepsilon \in C([0, T]; C^\infty(\bar{\Omega})) \), it follows from (23) that

\begin{equation}
\frac{1}{2} \frac{d}{dt} \| \theta^\varepsilon \|^2_{L^2} \leq \| \nabla \theta_2^\varepsilon \|^2_{L^\infty} \| W_\varepsilon \|^2_{L^2} \| \theta^\varepsilon \|^2_{L^2} \leq C (\| W_\varepsilon \|^2_{L^2} + \| \theta^\varepsilon \|^2_{L^2}),
\end{equation}

which implies that

\begin{equation}
\| \theta^\varepsilon \|^2_{L^2} \leq e^{CT} \left( \| W_\varepsilon \|^2_{C([0, T]; L^2(\Omega))} \right).
\end{equation}

Since \( (u_2^\varepsilon, B_2^\varepsilon) \in L^2(0, T; H^2(\Omega)) \), one derives from (24) that

\begin{equation}
\frac{1}{2} \frac{d}{dt} \| (\chi^\varepsilon, Z^\varepsilon) \|^2_{L^2} + \mu \| \nabla \chi^\varepsilon \|^2_{L^2} + \gamma \| \nabla Z^\varepsilon \|^2_{L^2} \leq \| W_\varepsilon \|_{L^2} \| \nabla u_2^\varepsilon \|_{L^4} \| \chi^\varepsilon \|_{L^4} + \mu \| \nabla \chi^\varepsilon \|^2_{L^2} + \gamma \| \nabla Z^\varepsilon \|^2_{L^2} \\
+ \| \theta^\varepsilon \|_{L^2} \| \nabla \chi^\varepsilon \|^2_{L^2} + \| J_\varepsilon \|_{L^2} \| \nabla B_2^\varepsilon \|_{L^4} \| \chi^\varepsilon \|_{L^4} + \| J_\varepsilon \|_{L^2} \| \nabla u_2^\varepsilon \|_{L^4} \| \chi^\varepsilon \|_{L^4} + \| J_\varepsilon \|_{L^2} \| \nabla B_2^\varepsilon \|_{L^4} \| \chi^\varepsilon \|_{L^4} \\
+ \| \theta^\varepsilon \|_{L^2} \| \nabla \chi^\varepsilon \|^2_{L^2} + \| J_\varepsilon \|_{L^2} \| B_2^\varepsilon \|_{H^2} \| \chi^\varepsilon \|_{H^1} + \| J_\varepsilon \|_{L^2} \| u_2^\varepsilon \|_{H^2} \| Z^\varepsilon \|_{H^1} \\
\leq C \| W_\varepsilon \|_{L^2} \| u_2^\varepsilon \|_{H^2} \| \nabla \chi^\varepsilon \|_{L^2} + \| W_\varepsilon \|_{L^2} \| B_2^\varepsilon \|_{H^2} \| \nabla Z^\varepsilon \|_{L^2} + \frac{1}{2} \| \theta^\varepsilon \|^2_{L^2} \\
+ \frac{1}{2} \| \chi^\varepsilon \|^2_{L^2} + C \| J_\varepsilon \|_{L^2} \| B_2^\varepsilon \|_{H^2} \| \nabla \chi^\varepsilon \|_{L^2} + C \| J_\varepsilon \|_{L^2} \| u_2^\varepsilon \|_{H^2} \| \nabla Z^\varepsilon \|_{L^2} \\
\leq C \| W_\varepsilon \|^2_{L^2} \| u_2^\varepsilon \|^2_{H^2} + \frac{\mu}{2} \| \nabla \chi^\varepsilon \|^2_{L^2} + C \| W_\varepsilon \|^2_{L^2} \| B_2^\varepsilon \|^2_{H^2} + \frac{\gamma}{2} \| \nabla Z^\varepsilon \|^2_{L^2} \\
+ \frac{1}{2} \| \theta^\varepsilon \|^2_{L^2} + \frac{1}{2} \| \chi^\varepsilon \|^2_{L^2} + C \| J_\varepsilon \|^2_{L^2} \| B_2^\varepsilon \|^2_{H^2} + C \| J_\varepsilon \|^2_{L^2} \| u_2^\varepsilon \|^2_{H^2}.
\end{equation}

Combining (25) with (26), one arrives at

\begin{equation}
\frac{1}{2} \frac{d}{dt} \| (\chi^\varepsilon, Z^\varepsilon) \|^2_{L^2} + \mu \| \nabla \chi^\varepsilon \|^2_{L^2} + \frac{\gamma}{2} \| \nabla Z^\varepsilon \|^2_{L^2} \leq \frac{1}{2} \| \chi^\varepsilon \|^2_{L^2} + C(\| W_\varepsilon \|_{L^2})^2 \| (W_\varepsilon, J_\varepsilon) \|^2_{C([0, T]; H^2(\Omega))},
\end{equation}

where \( \int_0^T C(t) \, dt \leq C \). Applying Gronwall’s inequality to (27), one gets

\begin{equation}
\| (\chi^\varepsilon, Z^\varepsilon) \|^2_{L^2} \leq C \| (W_\varepsilon, J_\varepsilon) \|^2_{C([0, T]; L^2(\Omega))}.
\end{equation}

Integrating (27) over \([0, T]\) and using (28), one obtains

\begin{equation}
\int_0^T \| (\nabla \chi^\varepsilon, \nabla Z^\varepsilon) \|^2_{L^2} \, dt \leq C \| (W_\varepsilon, J_\varepsilon) \|^2_{C([0, T]; L^2(\Omega))}.
\end{equation}

This estimate together with (28), gives

\begin{equation}
\| (\chi^\varepsilon, Z^\varepsilon) \|^2_{C([0, T]; L^2(\Omega))} + \| (\chi^\varepsilon, Z^\varepsilon) \|^2_{L^2(0, T; H^2(\Omega))} \leq C \| (v_1 - v_2, H_1 - H_2) \|^2_{C([0, T]; L^2(\Omega))},
\end{equation}

that is,

\begin{equation}
\| u_1^\varepsilon - u_2^\varepsilon, B_1^\varepsilon - B_2^\varepsilon \|^2_A \leq C \| (v_1 - v_2, H_1 - H_2) \|^2_A.
\end{equation}
where \( \| \cdot \|_A := \| \cdot \|_{C([0,T];L^2(\Omega))} + \| \cdot \|_{L^2(0,T;H^1_0(\Omega))} \). By the definition, we know that

\[
\| F_\varepsilon(v_1, H_1) - F_\varepsilon(v_2, H_2) \|_A \leq C \| (v_1 - v_2, H_1 - H_2) \|_A^2,
\]

which implies that \( F_\varepsilon : A \to A \) is continuous. Therefore, the Schauder theorem implies that for any fixed \( \varepsilon \in (0,1) \), there exist \( u^\varepsilon \in A \) and \( B^\varepsilon \in A \) such that

\[
F_\varepsilon(u^\varepsilon, B^\varepsilon) = (u^\varepsilon, B^\varepsilon),
\]

where \( u_\varepsilon \) and \( B_\varepsilon \) are the regularization of \( u^\varepsilon \) and \( B^\varepsilon \), respectively. By a bootstrap argument (c.f. [23, 28]) we know that \( (\theta^\varepsilon, u^\varepsilon, B^\varepsilon) \in C^\infty(\Omega \times [0,T]) \). Then, it is obvious that \( (\theta^\varepsilon, u^\varepsilon, B^\varepsilon) \) satisfy the following integral identities:

\[
\begin{align*}
0 &= \int_\Omega \theta^\varepsilon_0 \cdot \zeta(x,0)dx + \int_0^T \int_\Omega [\theta^\varepsilon \cdot \zeta_t + \theta^\varepsilon (u_\varepsilon \cdot \nabla \zeta)]dxdt, \\
0 &= \int_\Omega u^\varepsilon_0 \cdot \Phi(x,0)dx + \int_0^T \int_\Omega [u^\varepsilon \cdot \Phi_t + (u_\varepsilon \cdot \nabla \Phi) \cdot u^\varepsilon]dxdt \\
&\quad + \int_0^T \int_\Omega [\theta^\varepsilon e_2 \cdot \Phi - (B_\varepsilon \cdot \nabla \Phi) \cdot B^\varepsilon - \mu \nabla u^\varepsilon \cdot \nabla \Phi]dxdt, \\
0 &= \int_\Omega B^\varepsilon_0 \cdot \Psi(x,0)dx + \int_0^T \int_\Omega [B^\varepsilon \cdot \Psi_t + (u_\varepsilon \cdot \nabla \Psi) \cdot B^\varepsilon]dxdt \\
&\quad - \int_0^T \int_\Omega [(B_\varepsilon \cdot \nabla \Psi) \cdot u^\varepsilon + \gamma \nabla B^\varepsilon \cdot \nabla \Psi]dxdt
\end{align*}
\]

for any \( \varepsilon > 0 \), test functions \( \zeta \in C^\infty(\Omega \times [0,T]) \), \( \Phi = (\Phi_1, \Phi_2) \in C^\infty(\Omega \times [0,T])^2 \) and \( \Psi = (\Psi_1, \Psi_2) \in C^\infty(\Omega \times [0,T])^2 \) satisfying

\[
\zeta(x,T) = 0, \quad \Phi(x,T) = 0, \quad \Psi(x,T) = 0, \quad \nabla \cdot \Phi = 0, \quad \nabla \cdot \Psi = 0.
\]

In view of (14), (17) and the definition of \( u_\varepsilon \) and \( B_\varepsilon \), we know that there exist functions \( u \in A, B \in A \) and \( \theta \in C([0,T];L^p(\Omega)), \forall 2 \leq p < \infty \), such that as \( \varepsilon \to 0_+ \),

\[
\begin{align*}
&u_\varepsilon \rightharpoonup u, \quad B_\varepsilon \rightharpoonup B, \quad \text{weakly in} \quad C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)), \\
u^\varepsilon \rightharpoonup u, \quad B^\varepsilon \rightharpoonup B, \quad \text{weakly in} \quad C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)), \\
&\theta^\varepsilon \to \theta, \quad \text{weakly in} \quad C([0,T];L^p(\Omega)), \quad \forall 2 \leq p < \infty,
\end{align*}
\]

and

\[
\begin{align*}
\| (u, B) \|_{C^0([0,T];L^2(\Omega))} + \| (u, B) \|_{L^2(0,T;H^1_0(\Omega))}^2 \leq C(T, \theta_0, u_0, B_0, \mu, \gamma, \Omega), \\
\| \theta \|_{C^0([0,T];L^p(\Omega))} + \| \theta \|_{C^0([0,T];L^p(\Omega))}^2 \leq C(T, \theta_0, u_0, B_0, \mu, \gamma, \Omega), \quad \forall 2 \leq p < \infty.
\end{align*}
\]

(30)

Since \( u \cdot \nabla \zeta \) belongs to \( C([0,T];L^2(\Omega)) \), one has

\[
\left| \int_0^T \int_\Omega [\theta^\varepsilon(u_\varepsilon \cdot \nabla \zeta) - \theta(u \cdot \nabla \zeta)]dxdt \right|
\]
Proposition 1 satisfies the following estimates:

3. Global regularity.

Under the assumptions in Theorem 1.1, the solution obtained in Proposition 1.

Lemma 3.1.

Moreover,

\[
\int_0^T \int_\Omega \left[ (u_\varepsilon \cdot \nabla \Phi) \cdot u_\varepsilon - (u \cdot \nabla \Phi) \cdot u \right] dx dt \\
= \int_0^T \int_\Omega \left[ ((u_\varepsilon - u) \cdot \nabla \Phi) \cdot u_\varepsilon + (u \cdot \nabla \Phi) \cdot (u_\varepsilon - u) \right] dx dt
\]

\[
\leq C \int_0^T \int_\Omega \left( |u_\varepsilon - u| |u_\varepsilon| + |u||u_\varepsilon - u| \right) dx dt
\]

\[
\leq C \left( |u_\varepsilon - u|_{L^2(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;L^2(\Omega))} \right) \to 0, \quad \varepsilon \to 0_+	ext{,}
\]

and similarly,

\[
\int_0^T \int_\Omega \left[ (B_\varepsilon \cdot \nabla \Phi) \cdot B_\varepsilon - (B \cdot \nabla \Phi) \cdot B \right] dx dt \to 0, \quad \varepsilon \to 0_+,
\]

\[
\int_0^T \int_\Omega \left[ (u_\varepsilon \cdot \nabla \Phi) \cdot B_\varepsilon - (u \cdot \nabla \Phi) \cdot B \right] dx dt \to 0, \quad \varepsilon \to 0_+.
\]

Thus, using the above relations and letting \( \varepsilon \to 0_+ \) in (29), we verify that \((\theta, u, B)\) is a weak solution to (1)-(4) in \( \Omega \times [0, T] \). We conclude the argument by noticing that \( T \) is arbitrary, and then complete the proof of Proposition 1 by combining with (30).

\[\square\]

3. Global regularity. This section is devoted to giving the regularity of the solution obtained in Proposition 1.

Proposition 2. Under the assumptions in Theorem 1.1, the solution obtained in Proposition 1 satisfies the following estimates:

\[
\|(u, B)\|_{C([0,T];H^3(\Omega))} + \|(u, B)\|_{L^2([0,T];H^4(\Omega))} + \|\theta\|_{C([0,T];H^4(\Omega))} \leq C,
\]

for any \( T > 0 \). Moreover, there exists a constant \( \bar{C} > 0 \) depends only on \( \Omega \) such that

\[
\|(u, B)\|_{L^2}^2 \leq \max \left\{ \|(u_0, B_0)\|_{L^2}^2, \frac{\bar{C}}{\gamma \sigma} \|\theta_0\|_{L^2}^2 \right\}, \quad \forall \ t \geq 0,
\]

with \( \sigma = \min\{\mu, 2\gamma\} \).

The proof of Proposition 2 is based on a priori estimates which are stated as a sequence of lemmas.

Lemma 3.1. Under the assumptions in Theorem 1.1, it holds that

\[
\|\theta\|_{L^p} = \|\theta_0\|_{L^p}, \quad \forall \ 1 \leq p \leq \infty, \quad \forall \ t \geq 0.
\]

Proof. For any \( 2 \leq p < \infty \), multiplying the temperature equation of (1) by \( \theta^{p-1} \), integrating the resulting equation over \( \Omega \) by parts, one gets

\[
\frac{d}{dt} \int_\Omega \theta^p dx = 0,
\]
which implies that
$$\|\theta\|_{L^p} = \|\theta_0\|_{L^p}.$$  
Furthermore, letting $p \to \infty$ in above estimate, one shows
$$\|\theta\|_{L^\infty} = \|\theta_0\|_{L^\infty}, \ \forall \ t \geq 0.$$  
This completes the proof of Lemma 3.1.  

\[\text{Lemma 3.2.}\] Under the assumptions in Theorem 1.1, it holds that
$$\|(u, B)\|^2_{C([0, T]; L^2(\Omega))} \leq C, \quad \|(\nabla u, \nabla B)\|^2_{L^2([0, T]; L^2(\Omega))} \leq C. \quad (33)$$

\textbf{Proof.} Multiplying the momentum and the magnetic equations of (1) by $u$ and $B$ respectively, integrating the resulting equations over $\Omega$ by parts, one has
$$\frac{1}{2} \frac{d}{dt} \|(u, B)\|^2_{L^2} + \mu \|\nabla u\|^2_{L^2} + \gamma \|\nabla \times B\|^2_{L^2}$$
$$= \int_{\Omega} [(B \cdot \nabla B) \cdot u + (B \cdot \nabla u) \cdot B] dx + \int_{\Omega} \theta e_2 \cdot u dx,$$
where we have used the boundary condition (4) and (15). By Cauchy-Schwarz inequality, the right-hand side of (34) can be estimated as
$$\int_{\Omega} [(B \cdot \nabla B) \cdot u + (B \cdot \nabla u) \cdot B] dx = \int_{\Omega} B \cdot \nabla (uB) dx = 0,$$
$$\int_{\Omega} \theta e_2 \cdot u dx \leq \frac{1}{2} \|\theta\|^2_{L^2} + \frac{1}{2} \|u\|^2_{L^2},$$
which together with the result in [31], (34) and Lemma 3.1, implies that
$$\frac{1}{2} \frac{d}{dt} \|(u, B)\|^2_{L^2} + \mu \|\nabla u\|^2_{L^2} + \gamma \|\nabla B\|^2_{L^2} \leq \frac{1}{2} \|\theta_0\|^2_{L^2} + \frac{1}{2} \|u\|^2_{L^2}. \quad (35)$$
By Gronwall’s inequality, one gets
$$\|(u, B)\|^2_{L^2} \leq e^t \|(u_0, B_0)\|^2_{L^2} + \int_0^t \|\theta_0\|^2_{L^2} dt \leq e^T \|(u_0, B_0)\|^2_{L^2} + T \|\theta_0\|^2_{L^2} \leq C,$$
for any $t \in [0, T]$, which together with (35), gives
$$\mu \int_0^T \|\nabla u\|^2_{L^2} dt + \gamma \int_0^T \|\nabla B\|^2_{L^2} dt \leq C.$$
This completes the proof of Lemma 3.2.  

\[\text{Lemma 3.3.}\] Under the assumptions of Theorem 1.1, it holds that
$$\|(\nabla u, \nabla B)\|^2_{C([0, T]; L^2(\Omega))} \leq C, \quad \|(u_t, B_t)\|^2_{L^2([0, T]; L^2(\Omega))} \leq C.$$

\textbf{Proof.} Multiplying the momentum and magnetic equations by $u_t$ and $B_t$ respectively, integrating the resulting equations over $\Omega$ by parts, one gets
$$\frac{\mu}{2} \frac{d}{dt} \|\nabla u\|^2_{L^2} + \frac{\gamma}{2} \frac{d}{dt} \|\nabla B\|^2_{L^2} + \|(u_t, B_t)\|^2_{L^2}$$
$$\leq \int_{\Omega} |u| |\nabla u| u_t dx + \int_{\Omega} |B| |\nabla B| u_t dx + \int_{\Omega} \theta e_2 \cdot u_t dx$$
$$+ \int_{\Omega} |u| |\nabla B| B_t dx + \int_{\Omega} |B| |\nabla u| B_t dx. \quad (36)$$
The right-hand side of (36) can be estimated as

\[ \int_{\Omega} |u| \nabla u|u_t| dx \leq C||u||_{L^2}^2 \nabla u||_{L^2}^2 + \frac{1}{8} ||u_t||_{L^2}^2, \]

\[ \int_{\Omega} |B| \nabla B|u_t| dx \leq C||B||_{L^2} ||\nabla B||_{L^2}^2 + \frac{1}{8} ||u_t||_{L^2}^2, \]

\[ \int_{\Omega} \theta \epsilon_2 \cdot u_t dx \leq \frac{1}{8} ||u_t||_{L^2}^2 + C||\theta_0||_{L^2}^2, \]

\[ \int_{\Omega} |u| \nabla B|B_t| dx \leq C||u||_{L^2} ||\nabla B||_{L^2}^2 + \frac{1}{8} ||B_t||_{L^2}^2, \]

\[ \int_{\Omega} |B| \nabla u|B_t| dx \leq C||B||_{L^2} ||\nabla u||_{L^2}^2 + \frac{1}{8} ||B_t||_{L^2}^2. \]

It follows from Lemma 2.1 (iii) and (iv) that

\[ ||u||_{L^2} ||\nabla u||_{L^2}^2 \leq C||u||_{L^2} ||\nabla u||_{L^2}^2 + ||\nabla u||_{L^2}^2 + ||\nabla u||_{L^2}^2 \leq C(||\nabla u||_{L^2}^2 ||\nabla u||_{L^2}^2 + ||\nabla u||_{L^2}^2 + \delta ||u||_{H^2}^2, \]

\[ ||B||_{L^2} ||\nabla B||_{L^2}^2 \leq C(||B||_{L^2} ||\nabla B||_{L^2}^2 + ||B||_{L^2}^2)(||\nabla B||_{L^2} ||\nabla B||_{L^2}^2 + ||\nabla B||_{L^2}^2 \leq C(||\nabla B||_{L^2}^2 ||\nabla B||_{L^2}^2 + ||\nabla B||_{L^2}^2 + ||\nabla B||_{L^2}^2 \leq C(\delta)(||\nabla B||_{L^2}^2 + ||\nabla B||_{L^2}^2 + \delta ||B||_{H^2}^2, \]

where we have used Lemma 3.2 and \( \delta > 0 \) is a small number to be determined. Hence, (36) can be rewritten as

\[ \frac{\mu}{2} \frac{d}{dt} ||\nabla u||_{L^2}^2 + \frac{\gamma}{2} \frac{d}{dt} ||\nabla B||_{L^2}^2 + \frac{5}{8} ||(u_t, B_t)||_{L^2}^2 \leq C + C(\delta)(||\nabla u, \nabla B||_{L^2}^2 + ||(\nabla u, \nabla B)||_{L^2}^2 + \delta ||(u, B)||_{H^2}^2. \]

Applying \( H^2 \) estimates to Stokes equations and elliptic equations, it follows from (1)_1, (1)_3 and (37)-(40), one obtains

\[ ||u||_{H^2}^2 \leq C(||u_t||_{L^2}^2 + ||\theta||_{L^2}^2 + ||u \cdot \nabla u||_{L^2}^2 + ||B \cdot \nabla B||_{L^2}^2 \leq C(||u_t||_{L^2}^2 + 1) + C||u||_{L^2} ||\nabla u||_{L^2}^2 + C||B||_{L^2} ||\nabla B||_{L^2}^2 \leq \bar{C}(1 + ||u_t||_{L^2}^2 + ||(\nabla u, \nabla B)||_{L^2}^2 + ||(\nabla u, \nabla B)||_{L^2}^2)

+ \frac{1}{4} ||u||_{H^2}^2 + \frac{1}{4} ||B||_{H^2}^2. \]
Under the assumptions of Theorem 1.1, it holds that
\[ \|B\|_{H^2}^2 \leq C(\|B_t\|_{L^2}^2 + \|u \cdot \nabla B\|_{L^2}^2 + \|B \cdot \nabla u\|_{L^2}^2) \]
\[ \leq C(\|B_t\|_{L^2}^2 + \|u\|_{L^2}^2 \|\nabla B\|_{L^2}^2 + \|B\|_{L^2}^2 \|\nabla u\|_{L^2}^2) \]
\[ \leq \tilde{C}(\|B_t\|_{L^2}^2 + \|(\nabla u, \nabla B)\|_{L^2}^2 + \|(\nabla u, \nabla B)\|_{L^2}^2) \] (43)
\[ + \frac{1}{4}\|u\|_{H^2}^4 + \frac{1}{4}\|B\|_{H^2}^4. \]
Choosing \( \delta = \frac{1}{8\tilde{C}} \), combining (41)-(43), one gets
\[ \frac{\mu}{2} \frac{d}{dt}\|\nabla u\|_{L^2}^2 + \frac{\gamma}{2} \frac{d}{dt}\|\nabla B\|_{L^2}^2 + \frac{3}{8}\|(u_t, B_t)\|_{L^2}^2 \]
\[ \leq C(\|(\nabla u, \nabla B)\|_{L^2}^2 + \|(\nabla u, \nabla B)\|_{L^2}^2) + C. \] (44)
By Young inequality and Gronwall’s inequality, one shows
\[ \|(\nabla u, \nabla B)\|_{L^2}^2 \leq C, \quad \forall \ t \in [0, T], \]
which together with (44), implies that
\[ \int_0^T \|(u_t, B_t)\|_{L^2}^2 dt \leq C. \]
This completes the proof of Lemma 3.3.

Lemma 3.4. Under the assumptions of Theorem 1.1, it holds that
\[ \|(u_t, B_t)\|_{L^2(0,T;L^2(\Omega))}^2 \leq C, \quad \|(\nabla u_t, \nabla B_t)\|_{L^2(0,T;L^2(\Omega))}^2 \leq C. \] (45)

Proof. Derivating the momentum and the magnetic equations of (1) with respect to \( t \), one gets
\[ u_{tt} + u_t \cdot \nabla u + u \cdot \nabla u_t + \nabla \Pi_t = \mu \Delta u_t + \theta_t e_2 + B_t \cdot \nabla B + B \cdot \nabla B_t, \] (46)
\[ B_{tt} + u_t \cdot \nabla B + u \cdot \nabla B_t = \gamma \Delta B_t + B_t \cdot \nabla u + B \cdot \nabla u_t. \] (47)
Multiplying (46) by \( u_t \) and (47) by \( B_t \), and then integrating the resulted equations over \( \Omega \) by parts, one has
\[ \frac{1}{2} \frac{d}{dt}\|u_t\|_{L^2}^2 + \|B_t\|_{L^2}^2 + \mu\|\nabla u_t\|_{L^2}^2 + \gamma\|\nabla B_t\|_{L^2}^2 \]
\[ = \int_{\Omega} (u_t \cdot \nabla u) u_t dx + \int_{\Omega} \theta_t e_2 \cdot u_t dx + \int_{\Omega} (B_t \cdot \nabla B) \cdot u_t dx \]
\[ - \int_{\Omega} (u_t \cdot \nabla B) B_t dx + \int_{\Omega} (B_t \cdot \nabla u) B_t dx \]
\[ \leq \|u_t\|_{L^2}^2 \|\nabla u\|_{L^2}^2 - \int_{\Omega} (u_t \cdot \nabla \theta) e_2 \cdot u_t dx \]
\[ + (\|B_t\|_{L^2}^2 + \|u_t\|_{L^2}^2) \|\nabla B\|_{L^2} + \|B_t\|_{L^2} \|\nabla u\|_{L^2} \]
\[ \leq C(\|B_t\|_{L^2}^2 + \|u_t\|_{L^2}^2) + \|\nabla u_t\|_{L^2} \|\theta\|_{L^\infty} \|u\|_{L^2} \]
\[ \leq C(\|u_t\|_{L^2} \|\nabla u_t\|_{L^2} + \|B_t\|_{L^2} \|\nabla B_t\|_{L^2} + \|B_t\|_{L^2}^2) + \frac{\mu}{4}\|u_t\|_{L^2}^2 + C \]
\[ \leq \frac{\mu}{2}\|u_t\|_{L^2}^2 + \frac{\gamma}{2}\|B_t\|_{L^2}^2 + C\|(u_t, B_t)\|_{L^2}^2 + C. \]
Therefore, we arrive at
\[ \frac{d}{dt}(\|u_t\|_{L^2}^2 + \|B_t\|_{L^2}^2) + \mu\|\nabla u_t\|_{L^2}^2 + \gamma\|\nabla B_t\|_{L^2}^2 \leq C(\|(u_t, B_t)\|_{L^2}^2 + 1). \]
Using Gronwall’s inequality, one obtains (45). Thus, the proof of Lemma 3.4 is finished.
Lemma 3.5. Under the assumptions of Theorem 1.1, it holds that
\[ \int_0^T \|(u_t, B_t)\|_{L^p}^2 dt \leq C, \quad \forall \ 1 \leq p < \infty. \]

Proof. From Lemma 2.1 (i), Lemma 3.3 and Lemma 3.4, one gets easily
\[ \int_0^T \|(u_t, B_t)\|_{L^p}^2 dt \leq \int_0^T \|(u_t, B_t)\|_{H^1}^2 dt \leq C, \quad \forall \ 1 \leq p < \infty. \]
Thus, the proof of Lemma 3.5 is completed. \qed

Lemma 3.6. Under the assumptions of Theorem 1.1, it holds that
\[ \|(u, B)\|_{L^2([0,T];L^\infty(\Omega))}^2 \leq C, \quad \|(\nabla u, \nabla B)\|_{L^2(0,T;L^\infty(\Omega))}^2 \leq C. \]

Proof. It follows from (42), (43), Lemma 3.3 and Lemma 3.4 that
\[ \|(u, B)\|_{H^2}^2 \leq C((\|(u_t, B_t)\|_{L^2}^2 + \|(\nabla u, \nabla B)\|_{L^2}^2 + \|(\nabla u, \nabla B)\|_{L^2})^2 + C) \leq C, \quad \forall \ t \in [0,T]. \]
which implies, by Sobolev embedding $H^2 \hookrightarrow L^\infty$,
\[ \|(u, B)\|_{L^\infty}^2 \leq C, \quad \forall \ t \in [0,T]. \]

Hence,
\[ \|u \cdot \nabla u\|_{H^1}^2 + \|B \cdot \nabla B\|_{H^1}^2 + \|u \cdot \nabla B\|_{H^1}^2 + \|B \cdot \nabla u\|_{H^1}^2 \leq C((\|u\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 + \|(u, B)\|_{H^2}^2))\|(u, B)\|_{H^2}^2 \leq C, \quad \forall \ t \in [0,T], \]
which, together with Lemma 2.1 (i), gives
\[ \|u \cdot \nabla u\|_{L^p}^2 + \|B \cdot \nabla B\|_{L^p}^2 + \|u \cdot \nabla B\|_{L^p}^2 + \|B \cdot \nabla u\|_{L^p}^2 \leq C, \quad \forall \ 1 \leq p < \infty, \ t \in [0,T]. \]
Therefore, applying $W^{2,p}$ estimates to Stokes equations and elliptic equations, and using Lemma 3.1 and Lemma 3.5, one gets from (1)_1 and (1)_3 that
\[ \int_0^T \|u\|_{W^{2,p}}^2 dt + \int_0^T \|B\|_{W^{2,p}}^2 dt \]
\[ \leq C \int_0^T \left( \|(u_t, B_t)\|_{L^p}^2 + \|u \cdot \nabla u\|_{L^p}^2 + \|B \cdot \nabla B\|_{L^p}^2 + \|u \cdot \nabla B\|_{L^p}^2 + \|B \cdot \nabla u\|_{L^p}^2 + \|\theta\|_{L^p}^2 \right) dt \]
\[ \leq C, \quad \forall \ 1 \leq p < \infty. \]
Applying Lemma 2.1 (ii) to $\nabla u$, we get
\[ \int_0^T \|(\nabla u, \nabla \theta)\|_{L^\infty} dt \leq C. \]
This finishes the proof of Lemma 3.6. \qed

Lemma 3.7. Under the assumptions of Theorem 1.1, it holds that
\[ \|\nabla \theta\|_{L^\infty} \leq C, \quad \forall \ t \in [0,T]. \]

Proof. For any $p \geq 2$, applying the operator $\nabla$ to the both sides of (1)_2, multiplying the resulted equation with $|\nabla \theta|^{p-2} \nabla \theta$, and then integrating by parts over $\Omega$, one obtains
\[ \frac{1}{p} \frac{d}{dt} \|\nabla \theta\|_{L^p}^p \leq \|\nabla u\|_{L^\infty} \|\nabla \theta\|_{L^p}^p. \]
Using Gronwall’s inequality and Lemma 3.6, one has
\[ \|
\nabla \theta \|_{L^\infty} \leq \|
\nabla \theta_0 \|_{L^p} e^{\int_0^t \|
\nabla u \|_{L^\infty} dt} \leq C, \quad \forall \ p \geq 2, \ \forall \ t \in [0, T]. \] (52)

Letting \( p \to \infty \), one obtains (51), which completes the proof of Lemma 3.7.

**Lemma 3.8.** Under the assumptions of Theorem 1.1, it holds that
\[ \| (\nabla u_t, \nabla B_t) \|_{L^2(0, T; L^2(\Omega))}^2 \leq C, \quad \| (u_{tt}, B_{tt}) \|_{L^2(0, T; L^2(\Omega))}^2 \leq C. \] (53)

**Proof.** Multiplying (46) and (47) by \( u_{tt} \) and \( B_{tt} \), respectively, and integrating the resulting equations over \( \Omega \) by parts, one obtains
\[
\frac{\mu}{2} \frac{d}{dt} \| \nabla u_t \|_{L^2}^2 + \frac{\gamma}{2} \frac{d}{dt} \| \nabla B_t \|_{L^2}^2 + \| (u_{tt}, B_{tt}) \|_{L^2}^2 \leq \int_\Omega \left[ (|u_{tt}| |\nabla u| + |u_{tt}| |\nabla u| |\nabla u|) + |u_{tt}| |\theta_t| + |u_{tt}| |\nabla B_t| + |u_{tt}| |\nabla B_t| \right] dx \\
+ \int_\Omega \left[ |B_{tt}| |\nabla B| + |B_{tt}| |\nabla B_t| + |B_{tt}| |B_t| |\nabla u| + |B_{tt}| |B_t| |\nabla u_t| \right] dx = \sum_{i=1}^9 I_i.
\]

For any \( t \in [0, T] \), the right-hand side of the above inequality can be estimated as in the following
\[
|I_1| \leq \frac{1}{12} \| u_{tt} \|_{L^2}^2 + C \| \nabla u \|_{L^\infty} \| u_{tt} \|_{L^2} \leq \frac{1}{12} \| u_{tt} \|_{L^2}^2 + C \| \nabla u \|_{L^\infty}^2,
\]
\[
|I_2| \leq \frac{1}{12} \| u_{tt} \|_{L^2}^2 + C \| u \|_{L^\infty} \| \nabla u_t \|_{L^2} \leq \frac{1}{12} \| u_{tt} \|_{L^2}^2 + C \| \nabla u_t \|_{L^2}^2,
\]
\[
|I_3| \leq \frac{1}{12} \| u_{tt} \|_{L^2}^2 + C \| \theta_t \|_{L^2} \leq \frac{1}{12} \| u_{tt} \|_{L^2}^2 + C \| \theta_t \|_{L^2}^2,
\]
\[
|I_4| \leq \frac{1}{12} \| u_{tt} \|_{L^2}^2 + C \| \nabla \theta \|_{L^\infty} \| B_t \|_{L^2} \leq \frac{1}{12} \| u_{tt} \|_{L^2}^2 + C \| \nabla \theta \|_{L^\infty} \| B_t \|_{L^2}^2,
\]
\[
|I_5| \leq \frac{1}{12} \| u_{tt} \|_{L^2}^2 + C \| B \|_{L^\infty} \| \nabla B_t \|_{L^2} \leq \frac{1}{12} \| u_{tt} \|_{L^2}^2 + C \| B \|_{L^\infty} \| \nabla B_t \|_{L^2}^2,
\]
\[
|I_6| \leq \frac{1}{12} \| B_t \|_{L^2}^2 + C \| u_t \|_{L^\infty} \| \nabla B_t \|_{L^2} \leq \frac{1}{12} \| B_t \|_{L^2}^2 + C \| u_t \|_{L^\infty} \| \nabla B_t \|_{L^2}^2,
\]
\[
|I_7| \leq \frac{1}{12} \| B_t \|_{L^2}^2 + C \| u \|_{L^\infty} \| \nabla B_t \|_{L^2} \leq \frac{1}{12} \| B_t \|_{L^2}^2 + C \| u \|_{L^\infty} \| \nabla B_t \|_{L^2}^2,
\]
\[
|I_8| \leq \frac{1}{12} \| B_t \|_{L^2}^2 + C \| \nabla u \|_{L^\infty} \| B_t \|_{L^2} \leq \frac{1}{12} \| B_t \|_{L^2}^2 + C \| \nabla u \|_{L^\infty} \| B_t \|_{L^2}^2,
\]
\[
|I_9| \leq \frac{1}{12} \| B_t \|_{L^2}^2 + C \| \nabla u_{tt} \|_{L^\infty} \| B_t \|_{L^2} \leq \frac{1}{12} \| B_t \|_{L^2}^2 + C \| \nabla u_{tt} \|_{L^\infty} \| B_t \|_{L^2}, \ \forall \ t \in [0, T].
\]

Combining these estimates, one gets
\[
\frac{\mu}{2} \frac{d}{dt} \| \nabla u_t \|_{L^2}^2 + \frac{\gamma}{2} \frac{d}{dt} \| \nabla B_t \|_{L^2}^2 + \frac{1}{2} \| (u_{tt}, B_{tt}) \|_{L^2}^2 \leq C(\| (\nabla u_t, \nabla B_t) \|_{L^2}^2 + \| (\nabla u, \nabla B) \|_{L^\infty}^2) + C,
\]
which together with Lemma 3.4 and Lemma 3.6 gives that
\[ \frac{\mu}{2} \| \nabla u_t \|_{L^2}^2 + \frac{\gamma}{2} \| \nabla B_t \|_{L^2}^2 + \frac{1}{2} \int_0^t \| (u_{tt}, B_{tt}) \|_{L^2}^2 ds \leq C, \forall \ t \in [0, T]. \]
This completes the proof of Lemma 3.8.

Lemma 3.9. Under the assumptions of Theorem 1.1, it holds that
\[ \|(\theta, u, B)\|_{C([0,T];H^3(\Omega))}^2 \leq C, \quad \|(u, B)\|_{L^2(0,T;H^4(\Omega))}^2 \leq C. \] (54)

Proof. Using $H^3$ estimates to Stokes equations and elliptic equations, it follows from (1)\textsubscript{1}, (1)\textsubscript{3}, (50), (52) and (53) that
\[ \|(u, B)\|_{H^5} \leq C(\|\theta\|_{H^2}^2 + \|u \cdot \nabla u\|_{H^1} + \|B \cdot \nabla B\|_{H^1} + \|u \cdot \nabla B\|_{H^1}) + C(\|B \cdot \nabla u\|_{H^1} + \|(u, B_t)\|_{H^1}) \]
\[ \leq C, \forall \ t \in [0, T], \] (55)
which together with Lemma 2.1 (i), gives
\[ \|(u, B)\|_{W^{2,p}} \leq C(\|(u, B)\|_{H^5}) \leq C, \forall \ t \in [0, T], \forall \ 1 \leq p < \infty. \] (56)
Thus, by Lemma 2.1 (ii), one has
\[ \|(\nabla u, \nabla B)\|_{L^\infty} \leq C, \forall \ t \in [0, T]. \] (57)

Notice that
\[ \|u_t \cdot \nabla u\|_{L^2}^2 \leq \|u_t\|_{L^2}^2 \|\nabla u\|_{L^\infty}^2 \leq C, \]
\[ \|u \cdot \nabla u_t\|_{L^2}^2 \leq \|\nabla u_t\|_{L^2}^2 \|u\|_{L^\infty}^2 \leq C, \]
\[ \|\theta_t\|_{L^2}^2 = \|u \cdot \nabla \theta\|_{L^2}^2 \leq \|\nabla \theta\|_{L^2}^2 \|u\|_{L^\infty}^2 \leq C, \]
\[ \|B_t \cdot \nabla B\|_{L^2} \leq \|B_t\|_{L^2} \|\nabla B\|_{L^\infty} \leq C, \]
\[ \|B \cdot \nabla B_t\|_{L^2} \leq \|B\|_{L^2} \|\nabla B_t\|_{L^\infty} \leq C, \]
\[ \|u_t \cdot \nabla B\|_{L^2} \leq \|u_t\|_{L^2} \|\nabla B\|_{L^\infty} \leq C, \]
\[ \|u \cdot \nabla B_t\|_{L^2} \leq \|\nabla B_t\|_{L^2} \|u\|_{L^\infty} \leq C, \]
\[ \|B_t \cdot \nabla u\|_{L^2} \leq \|B_t\|_{L^2} \|\nabla u\|_{L^\infty} \leq C, \]
\[ \|B \cdot \nabla u_t\|_{L^2} \leq \|\nabla u_t\|_{L^2} \|B\|_{L^\infty} \leq C. \]

Applying $H^2$ estimates to Stokes equations and elliptic equations, then from (46), (47) and Lemma 3.8, one can show that
\[ \int_0^T \|(u_t, B_t)\|_{L^2}^2 dt \leq C \int_0^T (\|u_{tt}\|_{L^2}^2 + \|B_{tt}\|_{L^2}^2 + \|u_t \cdot \nabla u\|_{L^2}^2 + \|\theta_t\|_{L^2}^2) dt \]
\[ + C \int_0^T (\|u \cdot \nabla u_t\|_{L^2}^2 + \|B_t \cdot \nabla B\|_{L^2}^2 + \|B \cdot \nabla B_t\|_{L^2}^2) dt \]
\[ + C \int_0^T (\|u_t \cdot \nabla B\|_{L^2}^2 + \|u \cdot \nabla B_t\|_{L^2}^2 + \|B_t \cdot \nabla u\|_{L^2}^2) dt \]
\[ + C \int_0^T (\|B \cdot \nabla u_t\|_{L^2}^2) dt \]
\[ \leq C, \forall \ 1 \leq p < \infty. \] (58)
By Sobolev inequality and (55), one gets
\[
\|u \cdot \nabla u\|_{L^2}^2 + \|B \cdot \nabla B\|_{L^2}^2 + \|u \cdot \nabla B\|_{H^2}^2 + \|B \cdot \nabla u\|_{H^2}^2 \\
\leq C(\|u\|_{L^2}^2 \|u\|_{H^2}^2 + \|B\|_{L^2}^2 \|B\|_{H^3}^2 + \|u\|_{H^2}^2 \|u\|_{L^\infty}^2) \\
+ C(\|B\|_{H^3}^2 \|\nabla B\|_{L^2}^2 + \|u\|_{L^2}^2 \|B\|_{H^3}^2 + \|u\|_{L^2}^2 \|B\|_{L^\infty}^2) \\
+ C(\|u\|_{H^3}^2 \|B\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2 \|B\|_{H^2}^2) \\
\leq \|(u, B)\|_{H^2}^2 \|(u, B)\|_{H^3} \\
\leq C, \ \forall \ t \in [0, T].
\]

Now, it remains to give a higher order estimate on \(\theta\) to complete the proof of this lemma. For that, applying the operator \(\nabla^2\) to (1.2), multiplying the resulting equation by \(p|\nabla^2\theta|^{p-2}\nabla^2\theta\), and then integrating over \(\Omega\) by parts, one obtains
\[
\frac{d}{dt} \int_{\Omega} |\nabla^2 \theta|^p dx \leq C \int_{\Omega} (|\nabla^2 u||\nabla \theta||\nabla^2 \theta|^{p-1} + |\nabla u||\nabla^2 \theta|^p) dx \\
\leq C(\|\nabla^2 \theta\|_{L^p}^{p-1} \|\nabla^2 u\|_{L^p} \|\nabla \theta\|_{L^\infty} + \|\nabla^2 \theta\|_{L^p} \|\nabla u\|_{L^\infty}) \\
\leq C(\|\nabla^2 \theta\|_{L^p}^{p-1} + \|\nabla^2 \theta\|_{L^p}),
\]
where we have used (51), (56) and (57). Thus,
\[
\frac{d}{dt} \|\nabla^2 \theta\|_{L^p} \leq C(1 + \|\nabla^2 \theta\|_{L^p}).
\]
Gronwall’s inequality gives that
\[
\|\nabla^2 \theta\|_{L^p} \leq C, \ \forall \ 2 \leq p < \infty, \ \forall \ t \in [0, T].
\]
Similarly, one can obtain that
\[
\frac{d}{dt} \|\nabla^3 \theta\|_{L^2}^2 \leq C(\|\nabla^3 \theta\|_{L^2}^2 \|\nabla u\|_{L^\infty} + \|\nabla^3 u\|_{L^2} \|\nabla^2 \theta\|_{L^\infty} \|\nabla \theta\|_{L^\infty} \\
+ \|\nabla^3 u\|_{L^4} \|\nabla^2 \theta\|_{L^4} \|\nabla^3 \theta\|_{L^2}) \\
\leq C(\|\nabla^3 \theta\|_{L^2}^2 + \|\nabla^3 \theta\|_{L^2}) \\
\leq C(\|\nabla^3 \theta\|_{L^2}^2 + 1),
\]
which implies
\[
\|\theta\|_{H^3} \leq C, \ \forall \ t \in [0, T]. \quad (60)
\]
Applying \(H^4\) estimates to Stokes equations and elliptic equations, one can prove from (1)_1, (1)_3 and (58)-(60) that
\[
\int_0^T \|(u, B)\|_{H^4}^2 dt \leq C \int_0^T \|(u_t, B_t)\|_{H^2}^2 + \|u \cdot \nabla u\|_{H^2}^2 + \|B \cdot \nabla B\|_{H^2}^2 dt \\
+ C \int_0^T \|u \cdot \nabla B\|_{H^2}^2 + \|B \cdot \nabla u\|_{H^2}^2 + \|\theta\|_{H^2}^2 dt \\
\leq C.
\]
Thus, the proof of Lemma 3.9 is finished.

\begin{lemma}
Under the assumptions of Theorem 1.1, it holds that
\[
\|(u, B)\|_{L^2}^2 \leq \max \left\{ \|(u_0, B_0)\|_{L^2}^2 \frac{C^2}{\sigma \mu} \|\theta_0\|_{L^2}^2 \right\}, \ \forall \ t \geq 0, \quad (61)
\]
with \(\sigma = \min\{\mu, 2\gamma\} \).
\end{lemma}
Proof. From (1) and (1), one can show that
\[
\frac{1}{2} \frac{d}{dt} \|(u, B)\|_{L^2}^2 + \mu \|
abla u\|_{L^2}^2 + \gamma \|
abla B\|_{L^2}^2 = \int_{\Omega} \theta e_2 \cdot u dx \\
\leq \frac{1}{2\delta \mu} \|	heta\|_{L^2}^2 + \frac{\delta \mu}{2} \|u\|_{L^2}^2,
\]
for any positive \(\delta\). It follows from Lemma 2.2 that there is a constant \(\bar{C} = \bar{C}(\Omega)\) such that
\[
\|u\|_{L^2} \leq \bar{C} \|
abla u\|_{L^2}, \\
\|B\|_{L^2} \leq \bar{C} \|
abla B\|_{L^2}.
\]
Choosing \(\delta = \bar{C}^{-1}\), it holds from (62) that
\[
\frac{d}{dt} \|(u, B)\|_{L^2}^2 + \frac{\mu}{\bar{C}} \|u\|_{L^2}^2 + \frac{2\gamma}{\bar{C}} \|B\|_{L^2}^2 \leq \frac{\bar{C}}{\mu} \|	heta\|_{L^2}^2.
\]
Set \(\sigma = \min\{\mu, 2\gamma\}\), then the above estimate can be rewritten as
\[
\frac{d}{dt} \|(u, B)\|_{L^2}^2 + \frac{\sigma}{\bar{C}} \|(u, B)\|_{L^2}^2 \leq \frac{\bar{C}}{\mu} \|	heta\|_{L^2}^2,
\]
which implies that
\[
\frac{d}{dt} \left(e^{\bar{C} t} \|(u, B)\|_{L^2}^2\right) \leq \frac{\bar{C}^2}{\sigma \mu} \|\theta_0\|_{L^2}^2 e^{\bar{C} t}.
\]
Integrating this estimate with respect to \(t\), leads to
\[
e^{\bar{C} t} \|(u, B)\|_{L^2}^2 - \|(u_0, B_0)\|_{L^2}^2 \leq \frac{\bar{C}^2}{\sigma \mu} \|\theta_0\|_{L^2}^2 (e^{\bar{C} t} - 1),
\]
that is, for any \(t > 0\),
\[
\|(u, B)\|_{L^2}^2 \leq e^{-\bar{C} t} \left(\|(u_0, B_0)\|_{L^2}^2 - \frac{\bar{C}^2}{\sigma \mu} \|\theta_0\|_{L^2}^2\right) + \frac{\bar{C}^2}{\sigma \mu} \|\theta_0\|_{L^2}^2,
\]
which concludes (61). Therefore, the proof of Lemma 3.10 is completed. \(\square\)

**Proof of Proposition 2.** Obviously, Lemma 3.9 and Lemma 3.10 imply Proposition 2. \(\square\)

4. **Uniqueness.** In this section, we will use the global regularity established in Lemmas 3.1-3.9 to prove the uniqueness of the solution.

**Proposition 3.** Under the assumptions in Theorem 1.1, the solution of (1)-(4) is unique.

**Proof.** Suppose the system (1)-(4) has two solutions \((\theta_1, u_1, B_1)\) and \((\theta_2, u_2, B_2)\).
Setting \(\theta = \theta_1 - \theta_2\), \(\tilde{u} = u_1 - u_2\), \(\tilde{B} = B_1 - B_2\), and \(\tilde{P} = \Pi_1 - \Pi_2\), then \((\tilde{\theta}, \tilde{u}, \tilde{B}, \tilde{P})\) satisfy
\[
\begin{align*}
\frac{d}{dt} \left(\begin{array}{c}
\tilde{u} \\
\tilde{\theta} \\
\tilde{B} \\
\tilde{P}
\end{array}\right) &= \left(\begin{array}{c}
\tilde{u} \\
\tilde{\theta} \\
\tilde{B} \\
\tilde{P}
\end{array}\right), \\
\tilde{u}(x, 0) &= 0, \quad \tilde{B}(x, 0) = 0, \quad \tilde{\theta}(x, 0) = 0, \quad x \in \Omega.
\end{align*}
\]
Multiplying the first three equations by \(\tilde{u}, \tilde{\theta}\) and \(\tilde{B}\) respectively, and then integrating the resulting equations over \(\Omega\) by parts, gives that

\[
\frac{1}{2} \frac{d}{dt} (\|\tilde{\theta}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{B}\|_{L^2}^2) + \mu \|\nabla \tilde{u}\|_{L^2}^2 + \gamma \|\nabla \tilde{B}\|_{L^2}^2 \\
= \int_{\Omega} (\tilde{\theta} \tilde{e}_2 + B_1 \nabla \tilde{B} + \tilde{B} \cdot \nabla B_2 - \tilde{u} \cdot \nabla u_2) \cdot \tilde{u} dx - \int_{\Omega} (\tilde{u} \cdot \nabla \tilde{\theta}_2) \tilde{\theta} dx \\
+ \int_{\Omega} (B_1 \cdot \nabla \tilde{u} + \tilde{B} \cdot \nabla u_2 - \tilde{u} \cdot \nabla B_2) \cdot \tilde{B} dx \\
\leq C (\|\tilde{\theta}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|B_1\|_{L^\infty}^2 \|\tilde{u}\|_{L^2}^2 + \frac{\gamma}{2} \|\nabla \tilde{B}\|_{L^2}^2 + \|\nabla B_2\|_{L^\infty} (\|\tilde{u}\|_{L^2}^2 + \|\tilde{B}\|_{L^2}^2) \\
+ \|\nabla u_2\|_{L^\infty} \|\tilde{u}\|_{L^2}^2 + \|\nabla \tilde{\theta}_2\|_{L^\infty} (\|\tilde{u}\|_{L^2}^2 + \|\tilde{\theta}\|_{L^2}^2) + \|B_1\|_{L^\infty} \|\tilde{B}\|_{L^2}^2 + \frac{\mu}{2} \|\nabla \tilde{u}\|_{L^2}^2 \\
+ \|\nabla u_2\|_{L^\infty} \|\tilde{B}\|_{L^2}^2) \\
\leq C (\|\tilde{\theta}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{B}\|_{L^2}^2) + \frac{\mu}{2} \|\nabla \tilde{u}\|_{L^2}^2 + \frac{\gamma}{2} \|\nabla \tilde{B}\|_{L^2}^2.
\]

Thus,

\[
\frac{1}{2} \frac{d}{dt} (\|\tilde{\theta}, \tilde{u}, \tilde{B}\|_{L^2}^2) \leq C (\|\tilde{\theta}, \tilde{u}, \tilde{B}\|_{L^2}^2),
\]

which by Gronwall’s inequality, implies that

\[
\|\tilde{\theta}, \tilde{u}, \tilde{B}\|_{L^2}^2 \leq e^{2Ct} \|\tilde{\theta}(0), \tilde{u}(0), \tilde{B}(0)\|_{L^2}^2 = 0.
\]

So, the solution of (1)-(4) is unique. This completes the proof of Proposition 3. \(\square\)

Proof of Theorem 1.1. Our main result, Theorem 1.1, can be proved from Proposition 2 and Proposition 3 immediately. \(\square\)

REFERENCES

[1] R. A. Adams, Sobolev Spaces, Academic, New York, 1975.
[2] H. Abidi and T. Hmidi, On the global well-posedness for Boussinesq system, J. Differential Equations, 233 (2007), 199–220.
[3] D. Bian and G. Gui, On 2-D Boussinesq equations for MHD convection with stratification effects, J. Differential Equations, 261 (2016), 1669–1711.
[4] D. Bian, G. Gui, B. Guo and Z. Xin, On the stability for the incompressible 2-D Boussinesq system for magnetohydrodynamics convection, preprint, 2015.
[5] D. Bian and B. Guo, Global existence and large time behavior of solutions to the electric-magnetohydrodynamic equations, Kinetic and Related Models, 6 (2013), 481–503.
[6] J. R. Cannon and E. Dibenedetto, The initial value problem for the Boussinesq with data in \(L^p\). In: Approximation Methods for Navier-Stokes Problems, Lecture Notes in Mathematics, vol. 771, pp. 129–144, Springer, Berlin, 1980.
[7] C. Cao and J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, Adv. Math., 226 (2011), 1803–1822.
[8] D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms, Adv. Math., 203 (2006), 497–513.
[9] Q. Chen, C. Miao and Z. Zhang, The Beale-Kato-Majda criterion for the 3D magnetohydrodynamics equations, Comm. Math. Phys., 275 (2007), 861–872.
[10] R. Danchin and M. Paicu, Les théorèmes de Leray et de Fujita-Kato pour le système de Boussinesq partiellement visqueux, Bull. Soc. Math. France, 136 (2008), 261–309.
[11] B. Desjardins and C. Le Bris, Remarks on a nonhomogeneous model of magnetohydrodynamics, Differential Integral Equations, 11 (1998), 379–394.
[12] G. Duvaut and J.-L. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, Arch. Ration. Mech. Anal., 46 (1972), 241–279.
[13] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford University Press, Oxford, 2004.
[14] E. Feireisl, A. Novotný and H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, *J. Math. Fluid Mech.*, 3 (2001), 358–392.

[15] J. F. Gerbeau and C. Le Bris, Existence of solution for a density-dependent magnetohydrodynamic equation, *Adv. Differential Equations*, 2 (1997), 427–452.

[16] G. Gui, Global well-posedness of the two-dimensional incompressible magnetohydrodynamics system with variable density and electrical conductivity, *J. Functional Analysis*, 267 (2014), 1488–1539.

[17] C. He and Z. Xin, Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations, *J. Functional Analysis*, 227 (2005), 113–152.

[18] T. Hmidi and S. Keraani, On the global well-posedness of the Boussinesq system with zero viscosity, *Indiana Univ. Math. J.*, 58 (2009), 1591–1618.

[19] T. Hmidi and F. Rousset, Global well-posedness for the Navier-Stokes-Boussinesq system with axisymmetric data, *Ann. I. H. Poincare-A.N.*, 27 (2010), 1227–1246.

[20] T. Hmidi and F. Rousset, Global well-posedness for the Euler-Boussinesq system with axisymmetric data, *J. Functional Analysis*, 260 (2011), 745–796.

[21] T. Y. Hou and C. Li, Global well-posedness of the viscous Boussinesq equations, *Discrete Contin. Dyn. Syst.*, 12 (2005), 1–12.

[22] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, 1968.

[23] M.-J. Lai, R. Pan and K. Zhao, Initial boundary value problem for two-dimensional viscous Boussinesq equations, *Arch. Ration. Mech. Anal.*, 199 (2011), 739–760.

[24] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed., Pergamon, New York, 1984.

[25] D. Li and X. Xu, Global wellposedness of an inviscid 2D Boussinesq system with nonlinear thermal diffusivity, *Dyn. Partial Differ. Equ.*, 10 (2013), 255–265.

[26] F. Lin, L. Xu and P. Zhang, Global small solutions of 2-D incompressible MHD system, *J. Differential Equations*, 259 (2015), 5440–5485, arXiv: 1302.5877v2.

[27] F. Lin and P. Zhang, Global small solutions to an MHD-type system: The three-dimensional case, *Comm. Pure Appl. Math.*, 67 (2014), 531–580.

[28] P. L. Lions, *Mathematical Topics in Fluid Mechanics*, vol. I, II. Oxford University Press, New York, 1996, 1998.

[29] X. Ren, J. Wu, Z. Xiang and Z. Zhang, Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion, *J. Functional Analysis*, 267 (2014), 503–541.

[30] M. Sermange and R. Temam, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.*, 36 (1983), 635–664.

[31] W. von Wahl, Estimating $\nabla u$ by $\text{div} u$ and $\text{curl} u$, *Math. Methods Appl. Sci.*, 15 (1992), 123–143.

[32] C. Wang and Z. Zhang, Global well-posedness for 2-D Boussinesq system with the temperature-density viscosity and thermal diffusivity, *Adv. Math.*, 228 (2011), 43–62.

Received July 2015; revised September 2016.

E-mail address: biandongfen@bit.edu.cn