CHERN NUMBERS OF MANIFOLDS WITH TORUS ACTION.

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Abstract. We show that every set of numbers that occurs as the set of Chern numbers of an almost complex manifold $M^n$, $n \geq 6$, may be realized as the set of Chern numbers of a connected almost complex manifold $M^n$ with an almost complex action of a two-dimensional compact torus.

1. Introduction

There is a well-known problem (posed by Hirzebruch) asking which combinations of Chern numbers of almost complex manifolds (or, equivalently, which complex cobordism classes) may be realized by a smooth connected complex projective variety. As shown by Milnor, any complex cobordism class may be realized by a disjoint union of smooth projective varieties, so the connectivity condition in the formulation of Hirzebruch problem is crucial. As shown in ([4]), any complex cobordism class may be realized by a connected smooth almost complex manifold.

Chern numbers of compact almost complex and symplectic manifolds with compact Lie group action have been an object of intensive research in recent years ([6], [7], [9], [12], [13]). Any smooth manifold $M$ with an almost complex action having only isolated fixed points necessarily has $c_n(M) \geq 0$; many other restrictions are shown to exist in this case. In this paper we show that if fixed points are allowed to be non-isolated, then all restrictions on Chern numbers vanish.

Theorem 1.1. Every complex cobordism class of dimension $n \geq 6$ may be realized by a connected almost complex manifold $M^n$ with an almost complex action of a two-dimensional compact torus.

Moreover, the set of fixed points of the action consists only of isolated fixed points and two-dimensional oriented closed surfaces.

Definition 1.2. A non-singular complex projective variety $M = M^n$ will be called good if it is a product of one or several varieties belonging to following classes:

- compact complex curves of genus $g > 1$,
- non-singular compact projective toric varieties $X^n$,
- Milnor hypersurfaces $H_{i,j}$ with $4 \leq i \leq j$,

such that there is at most one complex curve in the product.
Lemma 1.3. Any complex cobordism class may be represented by a disjoint union of good varieties.

We will obtain the proof of theorem 1.1 by applying the construction of fiber connected sum (see lemma 3.2, cf. [5] for contact structures) after establishing lemma 1.3.

2. Proof of lemma 1.3

Recall that the complex cobordism ring $\Omega_U^*$ is generated by smooth stably complex manifolds modulo the equivalence relation: $M_1 \sim M_2$ if and only if there exists stably complex manifold $N$ such that $\partial N = M_1 \cup (-M_2)$ where $(-M_2)$ is $M_2$ with reversed orientation and restriction of stably complex structure of $N$ to $M_1$ and $M_2$ is equivalent to stably complex structures on $M_1$ and $M_2$ respectively.

The operations of taking disjoint union and product of manifolds turn $\Omega_U^*$ into a graded commutative ring; stably complex manifold $M^n$ determines a class $[M^n] \in \Omega_U^{-n}$. 

As shown by Milnor and Novikov ([10], [11]), the ring $\Omega_U^*$ is isomorphic to the polynomial ring $\mathbb{Z}[a_1, a_2, \ldots ]$, where $\deg a_i = -2i$. In particular, the manifold is zero in $\Omega_U^*$ if and only if all of its Chern numbers are zero. Moreover, the set of stably complex manifolds $\{M^{2n}\}, n \geq 1$, may solve as the set of polynomial generators for $\Omega_U^*$ if and only if for any $n$ we have $s_n([M^{2n}]) = \eta(n)$, where

$$\eta(n) = p, \quad \text{if } n + 1 = p^k \text{ for some prime } p,$$

and $s_n$ is a Milnor number which is a Chern number corresponding to Newton symmetric polynomial of dimension $n$ (e.g. $s_1 = c_1$, $s_2 = c_1^2 - 2c_2$, $s_3 = c_1^3 - 3c_1c_2 + 3c_3$, etc.).

We proceed with the proof of lemma 1.3 by induction. Any complex cobordism class of dimension two is represented by a disjoint union of complex curves, since any two-dimensional almost complex manifold is automatically a complex curve.

Suppose that the statement of theorem 1.3 is true for all complex cobordism classes of dimension $\leq 2n$.

Lemma 2.1. Any complex cobordism class $[M] \in \Omega_U^{-2n}$ satisfying $s_n([M]) = 0$ may be represented by a good variety.

Proof. Since $s_n([M]) = 0$, the class $[M]$ is decomposable and we have $[M] = \sum_{k=1}^{K} [N_k \times N'_k]$ where $N_k$ and $N'_k$ are good varieties. If for some $k$ both $N_k$ and $N'_k$ contain complex curves of genus $g$ and $g'$ respectively, we replace these curves with disjoint unions of $(g - 1)$ and $(g' - 1)$ copies of $\mathbb{C}P^1$ correspondingly. $\square$
Lemma 2.2. If the class $[M] \in \Omega_{\U}^{-2n}$ may be represented by a good variety, the same is true for $(-[M])$.

Proof. Recall that blowing up a point on a compact complex variety $X$ lowers its Milnor number $s_n([X])$ by the value $(n+(-1)^i)^i$. This implies that in every dimension $n > 1$ there exist toric varieties with positive and negative Milnor numbers – namely, the complex projective space $\mathbb{C}P^n$ and the variety obtained from $\mathbb{C}P^n$ by blowing up three different fixed points of torus action.

This argument shows that there exists a toric variety $N$ such that $s_n(a[M] + b[N]) = 0$, where $a, b \in \mathbb{Z}_{>0}$. By lemma 2.1, the class $(-a[M] - b[N])$ may be represented by a disjoint union of good varieties. Then the same is true for the class $-[M] = (-a[M] - b[N]) + b[N] + (a - 1)[M]$. \(\square\)

Lemma 2.3. For every $n \geq 1$ there exists a good variety $G_n$ (of complex dimension $n$) with $s_n([G_n]) = \pm \eta(n)$.

Proof. We will use the following result on toric generators in complex cobordism.

Theorem 2.4. [14] For every odd $n$ and every even $n \leq 100001$ there exists a compact non-singular toric variety $G_n$ (of complex dimension $n$) satisfying $s_n([G_n]) = \pm \eta(n)$.

Therefore, it is enough to show that for every even $n > 100001$ the complex cobordism generator $a_n \in \Omega_{\U}^{-2n}$ may be realized by a disjoint union of good varieties.

Recall that we have $s_n(\mathbb{C}P^n) = n + 1$ and $s_n(H_{i,j}) = -\binom{i+j}{i}$ for $2 \leq i \leq j$. By Kummer theorem, the maximum degree $k$ for which $p^k$ divides $\binom{i+j}{i}$ is equal to the number of carries when $i$ is added to $j$ in base $p$. We need to show that $\gcd\left(\{((n+1), \binom{i+j}{i}) : i + j = n + 1, 2 \leq i \leq j\}\right) = \eta(n)$ for every even $n$ large enough.

Suppose first that $n + 1 = p^k$ and $\eta(n) = p$ for some prime $p$. Using 2.4 we may only consider the case $p > 2$. If $k = 1$, then $s_n([\mathbb{C}P^n]) = n + 1 = p = \eta(n)$, so we may also assume that $k > 1$. Adding up numbers $p^{k-1}$ and $p^k - p^{k-1}$ in base $p$ has only one carry and by Kummer theorem we see that $\binom{p^k}{p^{k-1}}$ is divisible by $p$ and not by $p^2$. So the statement of the lemma is true unless $p^{k-1} \leq 3$. But in this case $p = 3$ and $k = 2$, so $n = 8$ and toric generator $G_8$ exists by 2.4.

Next, let us assume that $(n+1)$ is not a degree of a prime $p$. If $p$ does not divide $(n+1)$, then there is nothing to prove since $s_n([\mathbb{C}P^n]) = n+1$. The representation of $(n+1)$ in base $p$ has at least two non-zero digits. Let $l > 0$ be the maximum number such that $p^l < (n+1)$. Then $\binom{n+1}{p^l}$ does not divide by $p$. Recalling that $p$ divides $(n+1)$, we see that if
(n + 1) − p^l ≤ 3, then either n + 1 = 2^l + 2 or n + 1 = 3^l + 3, so n is odd and we may again apply Lemma 1.4 □

3. FIBER CONNECTED SUMS OF ALMOST COMPLEX MANIFOLDS

In this section we apply lemma 1.3 to prove the main result, theorem 1.1.

Lemma 3.1. Any disjoint union of good varieties admits a faithful action of compact torus $T^{\min(4, n-1)}$ with the set of fixed points being a disjoint union of nonsingular curves and isolated fixed points.

Proof. Recall that Milnor hypersurface $H_{i,j} \subset \mathbb{C}P^i \times \mathbb{C}P^j, i \leq j$, is given by the equation $\{z_0w_0 + \ldots + z_iw_i = 0\}$ and has the natural faithful action of compact torus $T^i$ given by the formula

$$(t_1, \ldots, t_i) \cdot (z_0, w_0, \ldots, z_i, w_i) = (z_0, w_0, t_1z_1, t_1^{-1}w_1, \ldots, t_i z_i, t_i^{-1} w_i).$$

The action is faithful and has only isolated fixed points. Furthermore, every complex projective toric variety $X^n$ has a natural faithful action of compact torus $T^n$ and any curve of genus $g > 1$ has a trivial torus action.

Taking products of torus actions, we see that every good variety $X^n$ possesses a faithful holomorphic action of a torus of dimension equal to at least $\min(4, n - 1)$ □

Let $X_1 = X_1^{2n}$ and $X_2 = X_2^{2n}$ be two smooth manifolds with a smooth faithful action of torus $T^k$. Any principal orbit of the action has an equivariant tubular neighborhood diffeomorphic to $T^k \times D^{2n-k}$, where $D^{2n-k}$ is an open unit ball in $\mathbb{R}^{2n-k}$ centered at the origin. Let us choose two principal orbits $O_1 \subset X_1$ and $O_2 \subset X_2$ equipped with equivariant tubular neighborhoods $U_1$ and $U_2$ respectively. One can now construct a fiber connected sum $Y = X_1 \#_{T^k} X_2$ by gluing together diffeomorphic parts $U_1 - O_1$ and $U_2 - O_2$ of open manifolds $X_1 - O_1$ and $X_2 - O_2$ by the diffeomorphism $(t, r, \varphi) \sim (t, \frac{1}{r}, \varphi)$, where $t \in T^k$, $(r, \varphi) \in D^{2n-k}, r > 0, \varphi \in S^{2n-k-1}$.

There are smooth embeddings $(X_1 - U_1) \hookrightarrow Y$ and $(X_2 - U_2) \hookrightarrow Y$. If $X_1$ and $X_2$ are almost complex manifolds, there is a natural question whether we can extend a $T^k$-invariant almost complex structure from a disjoint union of manifolds $(X_1 - U_1) \cup (X_2 - U_2)$ to the entire manifold $Y$.

Lemma 3.2. Suppose that $X_1 = X_1^{2n}$ and $X_2 = X_2^{2n}$ are compact manifolds with a smooth faithful almost complex action of compact torus $T^k$. If the stable homotopy group $\pi_{2n-k-1}(SO/U)$ is trivial, then the invariant almost complex structure can be extended from a disjoint union $(X_1 - U_1) \cup (X_2 - U_2)$ to an invariant almost complex structure on $Y$. Moreover, $[Y] = [X_1] + [X_2]$ in $\Omega_U$. 
Proof. The space \( U_1 - O_1 \approx U_2 - O_2 \) is a trivial fibration over the torus \( T^k \). We denote by \( A_t \) its fiber (an open annulus) over the point \( t \in T^k \). Let us fix some point \( t_0 \in T^k \). Suppose that we managed to extend somehow the almost complex structure operator \( J \) from \( (X_1 - U_1) \cup (X_2 - U_2) \) to the restriction of the tangent bundle \( \tau_R(Y)|_{A_{t_0}} \). Then this new structure may be automatically extended by the action of torus \( T^k \) to an invariant structure on the space \( U_1 - O_1 \approx U_2 - O_2 \), and therefore, on the entire space \( Y \).

The obstruction to extending \( J \) to \( \tau_R(Y)|_{A_{t_0}} \) lies in a homotopy group \( \pi_{2n-k-1}(SO(2n)/U(n)) \). Since \( k > 0 \), this group is equal to the stable homotopy group \( \pi_{2n-k-1}(SO/U) \).

The equality \([Y] = [X_1] + [X_2]\) now follows from the localization theorem ([1], [2], [8]), which implies that Chern numbers of a manifold with an almost complex torus action are determined by the behaviour near fixed points. □

Proof of the theorem [11] By Bott periodicity, we have \( \pi_{j-1}(SO/U) = \pi_j(O) \) and the stable homotopy group \( \pi_j(O) \) is trivial if and only if \( j \equiv 2, 4, 5, 6 \mod 8 \).

Consider the class \([M^n] \in \Omega_U^{-2n}, n \geq 3 \). By lemma 1.3 it can be realized by a disjoint union of good varieties and by lemma 3.1 every of these varieties admits an action of torus \( T^l \), where \( l \geq \min(4, n-1) \). Let \( k = 4 \), if \( n \equiv 1 (\mod 4) \), and \( k = 2 \) otherwise. We see that

- \( k \leq l \) and therefore every good variety admits a faithful almost complex action of torus \( T^k \);
- the stable homotopy group \( \pi_{2n-k-1}(SO/U) = \pi_{2n-k}(O) \) is trivial.

Applying lemma 3.2 finishes the proof of 1.1. □

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