Metric geometry of nonregular weighted Carnot-Carathéodory spaces

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Abstract

We investigate local and metric geometry of weighted Carnot-Carathéodory spaces which are a wide generalization of sub-Riemannian manifolds and arise in nonlinear control theory, subelliptic equations etc. For such spaces the intrinsic Carnot-Carathéodory metric might not exist, and some other new effects take place. We describe the local algebraic structure of such a space, endowed with a certain quasimetric (first introduced by A. Nagel, E.M. Stein and S. Wainger), and compare local geometries of the initial C-C space and its tangent cone at some fixed (possibly nonregular) point. The main results of the present paper are new even for the case of sub-Riemannian manifolds. Moreover, they yield new proofs of such classical results as the Local approximation theorem and the Tangent cone theorem, proved for Hörmander vector fields by M. Gromov, A.Bellaiche, J.Mitchell etc.

Key words: Carnot-Carathéodory space, weighted vector fields, local tangent cone, quasimetric, nonregular points

MSC: Primary 53C17, 49J20; Secondary 51F99, 35H20.

1 Introduction

We investigate local and metric geometry of a general class of Carnot-Carathéodory spaces (see Definition 1) which generalize classical sub-Riemannian manifolds (see e.g. [5, 25, 28, 23, 9, 38, 41] and references therein) and naturally arise in different areas, in particular, geometric control theory, harmonic analysis and subelliptic equations.

As it is well-known, if \( X_1, X_2, \ldots, X_m \) are smooth “horizontal” vector fields on a smooth connected manifold \( \mathbb{M} (\dim \mathbb{M} = N, m \leq N) \), a necessary and sufficient condition for a system

\[
\dot{x} = \sum_{i=1}^{m} a_i X_i(x)
\]  

(1)

to be locally controllable is that \( X_1, X_2, \ldots, X_m \) span, together with their commutators up to some finite order \( M \), the tangent space \( T_v \mathbb{M} \) at any point \( v \in \mathbb{M} \) (Hörmander’s condition [26]), i.e. define a sub-Riemannian geometry on \( \mathbb{M} \). The existence of a controllable “horizontal” path, joining two arbitrary points \( v, w \in \mathbb{M} \), is equivalent to the Rashevsky-Chow connectivity theorem [13, 47]. This theorem implies existence of an intrinsic Carnot-Carathéodory metric \( d_c(v, w) \) defined as the infimum of lengths of all horizontal curves (with their tangent vectors belonging to the subbundle \( H\mathbb{M} = \text{span}\{X_1, X_2, \ldots, X_m\} \)) joining \( v \) and \( w \). Investigation of local geometry of sub-Riemannian manifolds is important e.g. for constructing optimal motion planning algorithms for (1) and studying their

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complexity \([5, 6, 25, 28, 29, 57]\). In particular, investigation of the algebraic structure of the tangent cone (in Gromov’s sense \([11, 22, 23, 24]\)) \((\mathbb{M}, d^u)\) to the metric space \((\mathbb{M}, d_c)\) plays here a crucial role, as well as obtaining estimates on comparison of the metrics \(d_c\) and \(d^u\). In contrast to the Riemannian case they are not bilipschitz-equivalent, but the following estimate holds:

**Theorem** (Local approximation theorem \([5, 23, 28, 61]\)). If, for \(u, v \in \mathbb{M}\), \(d_c(u, v) = O(\varepsilon)\) and \(d_c(u, w) = O(\varepsilon)\), then \(|d_c(v, w) - d^u(v, w)| = O(\varepsilon^{1+\frac{1}{M}})\), where \(M\) is the depth of the sub-Riemannian manifold \(\mathbb{M}\).

If the dependence of the right-hand part of a control system is nonlinear on the control functions (see \([2, 15]\) and references therein):

\[
\begin{align*}
\dot{x} &= f(x, a), \\
x(0) &= x_0,
\end{align*}
\]

where \(x \in \mathbb{M}, a \in \mathbb{R}^m\), then a sufficient (but not necessary) controllability condition is that

\[
\text{span}\{h(0) : h \in \text{Lie} \frac{\partial^{|\alpha|}}{\partial a^\alpha} f(0, \cdot), \alpha \in \mathbb{N}^M\} = T_{x_0}\mathbb{M}
\]

for some \(M \in \mathbb{N}\). Letting

\[
F_\nu = \left\{ \frac{\partial^\alpha}{\partial a^\alpha} f(0, \cdot) : |\alpha| \leq \nu \right\}
\]

and

\[
H_k(q) = \text{span}\{[f_1, [f_2, \ldots, [f_{i-1}, f_i] \ldots](q) : f_j \in F_{\nu_j}, \nu_1 + \nu_2 + \ldots + \nu_i \leq k, i > 0\},
\]

one obtains a weighted filtration

\[
\{0\} \subseteq H_1 \subseteq H_2 \subseteq \ldots \subseteq H_M = T\mathbb{M},
\]

such that \([H_i, H_j] \subseteq H_{i+j}\), of the tangent bundle. The condition of having such a filtration is obviously weaker than the Hörmander’s condition, and in this case it may happen that not all points can be joined by a horizontal path (see Example 2), i.e. the Rashevsky-Chow theorem fails to hold and the intrinsic metric \(d_c\) might not exist.

Other examples, where weighted Carnot-Carathéodory spaces appear, stem from the theory of subelliptic equations \([7, 14, 36, 42]\). Besides weakening the Hörmander’s condition, an important line of generalization of sub-Riemannian geometry is minimizing the smoothness assumptions on the vector fields \(X_i\) generating the space (see e.g. \([7, 8, 21, 31, 33, 32, 34, 39, 40, 49, 45, 46, 55, 56, 61, 62]\)).

In this paper we consider the following notion of a weighted Carnot-Carathéodory space (this definition is close to the one of the paper \([14]\)). A smooth manifold \(\mathbb{M}\) will be called a (weighted) Carnot-Carathéodory space (shortly, C-C space) if there are \(C^{2M+1}\)-smooth vector fields \(X_1, X_2, \ldots, X_q\) given on an area \(U \subseteq \mathbb{M}\) (the number \(M\) is defined below), endowed with formal weights \(\text{deg}(X_i) = d_i, 1 \leq d_1 \leq d_2 \leq \ldots \leq d_q, d_j \in \mathbb{N}\), with the following properties. It is assumed that \(\text{span}\{X_I(v)\} = T_v\mathbb{M}\) for all \(v \in U\) and some \(M \in \mathbb{N}\), where

\[
\text{deg} X_I = |I|_h = d_{i_1} + \ldots + d_{i_k}
\]
is the homogeneous degree of the commutator \( X_I = [X_{i_1}, \ldots, [X_{i_{k-1}}, X_{i_k}]] \ldots \). Letting 
\[ H_j = \text{span}\{X_I\}_{|I|_h \leq j} \] we get a weighted filtration of the tangent bundle \( H_1 \subseteq H_2 \subseteq \ldots \subseteq H_M = TM \), which meets the property \( [H_i, H_j] \subseteq H_{i+j} \). A point \( u \in U \) is called regular if there is a neighborhood of \( u \) in which the dimensions of all \( H_k \) are constant; otherwise this point is called nonregular.

This notion of a C-C space is suitable to describe nonlinear control systems (2). One of the peculiarities stemming from the presence of a formal degree structure is that different choices of weights may lead to different distributions of regular and nonregular points on the space (see Example 1).

Because of the mentioned difficulties, new methods for studying local geometry of such spaces are needed. In particular, since the metric \( d_c \) might not exist, we obtain all estimates w.r.t. the following distance function, first introduced in [42], which is actually not a metric, but a quasimetric, i.e. the triangle inequality holds only in the generalized sense, with some constant.

\[ \rho(v, w) = \inf \{ \delta > 0 | \text{there is a curve } \gamma : [0, 1] \to U \text{ such that} \]
\[ \gamma(0) = v, \gamma(1) = w, \dot{\gamma}(t) = \sum_{|I|_h \leq M} w_I X_I(\gamma(t)), |w_I| < \delta |I|_h \} \].

A crucial result on local geometry, which we prove in Section 5, is the estimate on comparison of this quasimetric w.r.t. the initial vector fields and the quasimetric \( \rho^u \) (see Section 3), induced in by their nilpotentizations \( \hat{X}_I^u \) at a point \( u \), which is possibly nonregular.

**Theorem** (Theorem on divergence of integral lines) If \( u, v \in U, \rho(u, v) = O(\varepsilon) \) and \( r = O(\varepsilon) \), then we have
\[ R(u, v, r) = O(\varepsilon^{1+\frac{1}{M}}), \]
where
\[ R(u, v, r) = \max \{ \sup_{\hat{y} \in B^{\rho^u}(v, r)} \{ \rho^u(y, \hat{y}) \}, \sup_{y \in B^\rho(v, r)} \{ \rho(y, \hat{y}) \} \}. \]

Here the points \( y \) and \( \hat{y} \) are defined as follows. Let \( \gamma(t) \) be an arbitrary curve such that
\[ \begin{cases} 
\dot{\gamma}(t) = \sum_{|I|_h \leq M} b_I \hat{X}_I^u(\gamma(t)), \\
\gamma(0) = v, \gamma(1) = \hat{y}, 
\end{cases} \]
and
\[ \rho^u(v, \hat{y}) \leq \max_{|I|_h \leq M} \{|b_I|^{1/|I|_h}\} \leq r. \]

Define \( y = \exp(\sum_{|I|_h \leq M} b_I X_I)(v) \). In this way, the supremum is taken not only over \( \hat{y} \in B^{\rho^u}(v, r) \), but also over the infinite set of admissible \( \{b_I\}_{|I|_h \leq M} \).

This theorem allows construction of motion planning algorithms for the system (2) like it was done for (1) in [5, 6, 25, 28, 29], and to prove an analog of the local approximation theorem, as well as to study the algebraic structure of the tangent cone.
Theorem (Local approximation theorem). For any points \( u \in U \) and \( v, w \in U \), such that \( \rho(u, v) = O(\varepsilon) \), \( \rho(u, w) = O(\varepsilon) \), we have
\[
|\rho(v, w) - \rho^u(v, w)| = O(\varepsilon^{1+1/\mu}).
\]

Theorem (Tangent cone theorem). The quasimetric space \((U, \rho^u)\) is a local tangent cone at the point \( u \) to the quasimetric space \((U, \rho)\). The tangent cone is a homogeneous space \( G/H \), where \( G \) is a nilpotent graded group with a weight structure.

This theorem, see Section 6, generalizes an analogous fact for sub-Riemannian manifolds, known as Mitchell’s cone theorem. Namely, it is known that, at a regular point, the tangent cone to a sub-Riemannian manifold is a nilpotent stratified group \([23, 38]\), while at a nonregular point it is a homogeneous space \([5, 28]\).

The notion of the tangent cone to a quasimetric space, extending the Gromov’s notion for metric spaces, was introduced and studied recently in \([49, 50]\). Note that a straightforward generalization of the Gromov-Hausdorff convergence theory would make no sense for quasimetric spaces, since the Gromov-Hausdorff distance between any two quasimetric spaces would be equal to zero. However, such generalization can be done for particular classes of compact quasimetric spaces \([24]\).

All of the mentioned three results are new even for the case of “classical” sub-Riemannian manifolds; moreover, methods of their proofs allow to prove in a new way the classical results for sub-Riemannian manifolds (see Section 7). In particular, in contrast to the proof of the Local approximation theorem in \([5]\), we do not need special polynomial “privileged” coordinates and do not use Newton-type approximation methods.

The proofs of the main results of this paper heavily rely on results of \([34, 61]\) for the case of regular C-C spaces, see Definition 5 and on methods of submersion of a C-C space into a regular one, \([18, 5, 14, 25, 28]\), as well as on obtaining new geometric properties for the quasimetrics \( \rho \) and \( \rho^u \) (Section 4).

This paper is essentially an extended version of the short notes \([49, 52]\).

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2 Basic definitions, examples and known facts

Recall that locally any vector field \( X_i \) on a manifold \( \mathbb{M} \) can be viewed as a first-order differential operator \( X_i = \sum_{j=1}^{N} a_{ij}(x) \frac{\partial}{\partial x_j} \) acting on a function \( f \in C^\infty(\mathbb{M}) \), and its smoothness coincides with the smoothness of the coordinate functions \( a_{ij}(x) \). A commutator of two vector fields is a vector field defined as \([X_i, X_j] = X_iX_j - X_jX_i\).

In this paper we will use the following definition of a weighted Carnot-Carathéodory space (this definition is very close to the one formulated in \([14]\)). It is easy to see that this definition can be reformulated in such a way that it involves only first-order (not higher-order) commutators of the vector fields \( X_1, X_2, \ldots, X_q \) and thus can be applied to...
the case of $C^1$-smooth vector fields.

**Definition 1.** Let $X_1, X_2, \ldots, X_q \in C^{2M+1}$-smooth vector fields given on an area $U$ in a connected $C^\infty$-smooth manifold $\mathbb{M}$ (the number $M$ is defined below) and associated with formal weights $\deg(X_i) = d_i$, $1 \leq d_1 \leq d_2 \leq \ldots \leq d_q$, $d_j \in \mathbb{N}$. To the commutator $X_I = [X_{i_1}, [\ldots, [X_{i_{k-1}}, X_{i_k}]]\ldots]$ a weight equal to its homogeneous degree is assigned:

$$\deg X_I = |I|_h = d_{i_1} + \ldots + d_{i_k}.$$  

(3)

It is assumed that $\text{span}\{X_I(v)\}_{|I|_h \leq M} = T_v \mathbb{M}$ for all $v \in U$. Letting $H_j = \text{span}\{X_I\}_{|I|_h \leq j}$ we get a weighted filtration of the tangent bundle

$$H_1 \subseteq H_2 \subseteq \ldots \subseteq H_M = T\mathbb{M},$$

(4)

which meets the property

$$[H_i, H_j] \subseteq H_{i+j}.$$  

(5)

A manifold $\mathbb{M}$ endowed with the described structure will be called a *(weighted)* Carnot-Carathéodory space (shortly, C-C space).

The minimal number $M$ of the elements $H_i$ in the filtration (4) is called the depth of the given Carnot-Carathéodory space.

Note that (3), (5) relate the natural algebraic structure, induced by commutators of the vector fields $X_1, X_2, \ldots, X_q$, with the additional formal degree structure.

If $X_j \in C^\infty(U)$ and $d_1 = \cdots = d_q = 1$ then Definition 1 coincides with the classical definition of a sub-Riemannian manifold. The subbundle $H_1$ is then called horizontal and generates, by commutation, the whole tangent bundle (Hörmander’s condition).

**Remark 1.** For simplicity of notation we will carry out all computations for the basic case when $d_1 = 1$, $d_q = M$. All results of this paper hold in the framework of Definition 1 replacing $\max\{d_q, M\}$ by $d_1 \max\{d_q, M\}$.

**Definition 2.** A point $u \in U$ of a Carnot-Carathéodory space is called regular if there is a neighborhood of $u$ in which the dimensions of all $H_k$ are constant; otherwise this point is called nonregular or singular.

**Definition 3.** Let us consider a distance function on $U$ defined as

$$\rho(v, w) = \inf\left\{\delta > 0 \mid \text{there is a curve } \gamma : [0, 1] \to U \text{ such that}
\right.$$

$$\gamma(0) = v, \gamma(1) = w, \dot{\gamma}(t) = \sum_{|I|_h \leq M} w_I X_I(\gamma(t)), |w_I| < \delta^{|I|_h}\}.$$  

(6)

The distance function (6) was first introduced in [42] where it was proved that it is continuous and, for “classical” sub-Riemannian manifolds, equivalent to the intrinsic Carnot-Carathéodory metric $d_c$ (Ball-Box theorem, see also [51, 28, 32, 34, 39, 40]).

**Definition 4** ([54]). A quasimetric space $(X, d_X)$ is a topological space $X$ endowed with a quasimetric $d_X$. A quasimetric is a mapping $d_X : X \times X \to \mathbb{R}^+$ meeting the following properties
(1) \(d_X(u, v) \geq 0; d_X(u, v) = 0 \iff u = v;\)
(2) \(d_X(u, v) \leq c_Xd_X(v, u),\) where \(1 \leq c_X < \infty\) is a constant independent of \(u, v \in X\) (generalized symmetry property);
(3) \(d_X(u, v) \leq Q_X(d_X(u, w) + d_X(w, v)),\) where \(1 \leq Q_X < \infty\) is a constant independent of \(u, v, w \in X\) (generalized triangle inequality);
(4) \(d_X(u, v)\) is upper-semicontinuous on the first argument.

If \(c_X = Q_X = 1,\) then \((X, d_X)\) is a metric space.

**Proposition 1.** \((U, \rho)\) is a quasimetric space.

*Proof.* Properties (1), (2) and (4) immediately follow from the properties of solutions of ordinary differential equations (and we have \(\rho(v, w) = \rho(w, v)\)). The generalized triangle inequality will be proved below (Proposition 16). □

The simplest examples of (regular) weighted Carnot-Carathéodory spaces are Carnot groups endowed with an additional degree structure.

**Example 1** ([16] [17]). Consider the Heisenberg group \(\mathbb{H}^n:\) let \(M = \mathbb{R}^N,\ N = 2n + 1,\) with the coordinates \((x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, t) \in \mathbb{R}^N.\) Consider the vector fields
\[
X_j = \partial_{x_j} - \frac{y_j}{2} \partial_t, \quad Y_j = \partial_{y_j} + \frac{x_j}{2} \partial_t, \quad \partial_t
\]
with commutator relations
\[
[X_j, Y_j] = T.
\]

Let us first assign to all of these vector fields the weights naturally defined by their commutator table:
\[
\deg(X_j) = \deg(Y_j) = 1, \quad \deg(T) = 2,
\]
then for \(w = \exp(x_jX_j + y_jY_j + tT)(v)\) we have
\[
\rho(v, w) = \max\{|x_1|, \ldots, |x_n|, |y_1|, \ldots, |y_n|, |t|^{1/2}\}.
\]

Now let
\[
\deg(X_j) = a_j, \quad \deg(Y_j) = b_j, \quad \deg(T) = c, \quad \text{where} \quad a_j + b_j = c
\]
for all \(j = 1, \ldots, N.\) Then
\[
\rho(v, w) = \max\{|x_j|^{1/a_j}, |y_j|^{1/b_j}, |t|^{1/c}\}
\]
is a quasimetric not equivalent to the previous one. In both cases all points of \(\mathbb{R}^N\) are regular.

The next example illustrates that, for the C-C spaces from Definition 11 the Rashevsky-Chow theorem may fail to hold, i.e. the intrinsic C-C metric might not exist.

**Example 2** ([54]). Consider the Euclidean space \(\mathbb{R}^N\) with the standard basis
\[
\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N}
\]
and let \(\deg(\partial_{x_i}) = d_j \text{ for } i = k_j + 1, k_j + 2, \ldots, k_{j+1},\) where \(k_1 \leq k_2 \leq \ldots \leq k_M = N.\)
Obviously, the subbundles
\[ H_i = \text{span}\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_i}\} \]
meet the condition \([H_i, H_j] \subseteq H_{i+j}\), since \([H_i, H_j] = \{0\}\). At the same moment, none of the subsets of the set of vector fields \(\{\partial_{x_i}\}\) meets the Hörmander’s condition, and, for any “horizontal” subbundle, there are points of \(\mathbb{R}^N\) which can not be joined by a horizontal curve.

In the considered example all points of \(\mathbb{R}^N\) are regular. If \(v, w \in \mathbb{R}^N\) and \(w - v = (x_1, x_2, \ldots, x_N)\), then
\[ \rho(v, w) = \max_{i=1,\ldots,N} \{|x_i|^{1/d_i}\}. \]

A further peculiarity of the considered weighted C-C spaces is that different choices of weights \(d_i\) may lead to different combinations of regular and nonregular points.

**Example 3.** Consider on the Euclidean space \(\mathbb{M} = \mathbb{R}^3\) the vector fields
\[ \{X_1 = \partial_y, X_2 = \partial_x + y \partial_t, X_3 = \partial_x\} \]
with the only nontrivial commutator relation \([X_1, X_2] = \partial_t\).

Let first \(\deg(X_i) := 1, i = 1, 2, 3\). Then \(\deg([X_1, X_2]) = 2\) and
\[ H_1 = \text{span}\{X_1, X_2, X_3\}, H_2 = H_1 \cup \text{span}\{[X_1, X_2]\}. \]

In this case \(\{y = 0\}\) is a plane consisting of nonregular points. Really, for \(y \neq 0\) we have \(\dim(H_1) = 3\), while for \(y = 0\) we have \(\dim(H_1) = 2\).

Now assume that \(\deg(X_1) := a, \deg(X_2) := b\) and \(\deg(X_3) := a + b\), where \(a \leq b\). Then \(\deg([X_1, X_2]) = a + b\), hence
\[ H_a = \text{span}\{X_1\}, H_b = H_a \cup \text{span}\{X_2\}, H_{a+b} = H_a \cup H_b \cup \text{span}\{X_3, [X_1, X_2]\}. \]

In this case all points of \(\mathbb{R}^3\) are regular.

Let us now briefly recall the approach of the papers of S. Vodopyanov and M. Karmanova \[31, 33, 34, 61, 62\], devoted to regular C-C spaces (they are particular cases of weighted C-C spaces from Definition 1) in minimal smoothness assumptions, and some main results of those papers, on which the proofs of the main results of the present paper heavily rely.

**Definition 5** \([34, 61, 62, 21, 31, 33, \text{ cf. } 5, 23, 9, 36, 42, 48\] etc.). Let \(\mathbb{M}\) be a connected \(C^\infty\)-smooth Riemannian manifold of dimension \(N\). The manifold \(\mathbb{M}\) is called a regular Carnot-Carathéodory space, if there is a filtration of its tangent bundle \(T\mathbb{M}\)
\[ HM = H_1 \subset \ldots \subset H_i \subset \ldots H_M = T\mathbb{M}, \tag{7} \]
such that in some area \(U \subset \mathbb{M}\) there are \(C^p\)-smooth vector fields \(X_1, \ldots, X_N\), where \(p > 1\), meeting the following conditions.

For all \(u \in U\) we have
\( i \) \( H_i(u) = \text{span}\{X_1(u), \ldots, X_{\dim H_i}(u)\} \) is a subbundle of \(T_u\mathbb{M}\) of dimension \(\dim H_i\), \(i = 1, \ldots, M\);
(ii) The following decomposition holds

\[ [X_i, X_j](v) = \sum_{k: \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v), \]  

where \( \deg X_k = \min \{m | X_k \in H_m\} \) is the degree of the vector field \( X_k \).

The number \( M \) is again called the depth of the C-C space \( \mathbb{M} \).

The condition (i) is equivalent to (5) in Definition 1.

Remark 2. In the present paper it suffices to have, for regular C-C spaces, smoothness \( p = M + 1 \), but most of the results of this section are true for \( C^{1,\alpha} \)-smooth vector fields \( X_1, \ldots, X_N \), where \( \alpha > 0 \) is the Hölder constant of the first-order derivatives. In this case, the expression \( \frac{1}{M} \) in the estimates below is replaced by \( \frac{\alpha}{M} \).

Consider on \( U \subseteq \mathbb{M} \) canonical first-order coordinates defined in a neighborhood of a point \( g \in \mathbb{M} \) as

\[ \theta_g(v_1, \ldots, v_N) = \exp \left( \sum_{i=1}^{N} v_i X_i \right)(g). \]  

From theorems on continuous dependence of the solutions of ODE on the initial data (see e.g. [43]) it follows that \( \theta_g \) is a \( C^1 \)-diffeomorphism of the Euclidean ball \( B_E(0, r) \subset \mathbb{R}^N \) to \( \mathbb{M} \), where \( 0 \leq r < r_g \) for a sufficiently smooth \( r_g > 0 \). Let \( U_g = \theta_g(B_E(0, r_g)) \). The tuple \( (v_1, v_2, \ldots, v_N) = \theta_g^{-1}(v) \in B_E(0, r_g) \) is called the first-order coordinates of the point \( v \in U_g \). Further we assume that \( U \subseteq \bigcup_{g \in U} U_g \).

In the regular case, the tuple \( (v_1, v_2, \ldots, v_N) \) is uniquely defined, thus the quasimetric (6), denoted in the above-mentioned papers as \( d_\infty \), is defined for points \( w, v \in U \), such that

\[ d_\infty(w, v) = \rho(w, v) = \max_i \{|v_i|^{\deg X_i}\}. \]

The generalized triangle inequality for \( d_\infty \) is proved in [31, 34, 62] in minimal smoothness assumptions and in [42] for sufficiently smooth vector fields (in the general case, not just near regular points).

The balls w. r. t. the quasimetric \( d_\infty \) will be denoted as \( \text{Box}(u, r) = \{v \in U \mid d_\infty(v, u) < r\} \).

The vector fields \( \widehat{X}_i^u \), obtained from the commutator table (8) replacing the inequality by equality, i. e.

\[ [\widehat{X}_i^u, \widehat{X}_j^u](v) = \sum_{k: \deg X_k = \deg X_i + \deg X_j} c_{ijk}(u) \widehat{X}_k(v), \]

define a graded nilpotent Lie algebra \( V = V_1 \oplus \ldots \oplus V_M \), where \( [V_i, V_i] \subseteq V_{i+1}, i = 1, \ldots, M - 1 \), due to the following result.

**Theorem 1 (34).** For a fixed point \( u \in U \) consider a family of coefficients

\[ \{c^k_{ij}\} = \{c_{ijk}(u)\}^{\deg X_k = \deg X_i + \deg X_j}. \]
In the present paper we will use the following results.

Note that, if the Hörmander’s condition holds, \(G^g = (U, \star)\) at the point \(g\). The group operation \(\star\) is defined as follows: if 
\[
x = \exp\left(\sum_{i=1}^{N} x_i \hat{X}^g_i\right)(g), \quad y = \exp\left(\sum_{i=1}^{N} y_i \hat{X}^g_i\right)(g),
\]
then 
\[
x \star y = \exp\left(\sum_{i=1}^{N} z_i \hat{X}^g_i\right) \circ \exp\left(\sum_{i=1}^{N} x_i \hat{X}^g_i\right)(g) = \exp\left(\sum_{i=1}^{N} z_i \hat{X}^g_i\right)(g),
\]
where \(z_i\) are calculated by means of the Campbell-Hausdorff formula.

Note that, if the Hörmander’s condition holds, \(G^g\) is a local Carnot group, i.e. \(V = V_1 \oplus \ldots \oplus V_M\), where \([V_i, V_i] = V_{i+1}, i = 1, \ldots M - 1\).

The quasimetric on \(G^g\) is defined in a similar way as \(d_{\infty}\): for \(u, v \in G^g\) such that \(v = \exp\left(\sum_{i=1}^{N} u_i \hat{X}^g_i\right)(u)\) let
\[
d^g_{\infty}(u, v) = \rho^g(u, v) = \max_i \{|v_i|_{\deg X_i}\}.
\]

In the present paper we will use the following results.

**Theorem 2** ([34]). For all \(x \in U\), such that \(|x_j| \leq \varepsilon |I_j|\), the following decompositions hold:
\[
X_j(x) = \sum_{i=1}^{N} a_{j,k}(x) \hat{X}_k(x), \quad (11)
\]
where
\[
a_{j,k} = \begin{cases} 
\delta_{j,k} + O(\varepsilon), & \deg(X_j) = \deg(X_k), \\
O(\varepsilon), & \deg(X_j) < \deg(X_k), \\
O(\varepsilon |I_k| - |I_j|), & \deg(X_k) > \deg(X_j).
\end{cases}
\]

Note that Theorem 2 implies Gromov’s nilpotentization theorem, which is proved in [23, 5, 48, 34] for smooth vector fields, in [34] for \(C^1\) vector fields and depth \(M = 2\), in [21] for \(C^2\) vector fields, in [33] for \(C^1, \alpha\) vector fields, where \(\alpha > 0\).

**Theorem 3** (Theorem on divergence of integral lines [61]). Let \(u, v \in U\), \(d_{\infty}(u, v) = C\varepsilon, C < \infty\). Consider the curves \(\gamma(t), \hat{\gamma}(t)\) in \(\Box(v, K\varepsilon)\), satisfying the equations
\[
\begin{cases} 
\dot{\gamma}(t) = \sum_{i=1}^{N} b_i(t) X_i(\gamma(t)), \\
\gamma(0) = v,
\end{cases} \quad \text{and} \quad \begin{cases} 
\hat{\gamma}(t) = \sum_{i=1}^{N} b_i(t) \hat{X}^\mu_i(\hat{\gamma}(t)), \\
\hat{\gamma}(0) = v,
\end{cases}
\]
where

\[ \int_0^1 |b_i(t)| dt < S \varepsilon^{\deg X_i}, S < \infty. \]

Then

\[ \max \{d_\infty(w, \hat{w}), d^u_\infty(w, \hat{w})\} = O(\varepsilon^{1+\frac{1}{M}}) \]

uniformly on \( U \).

Note that in [31, 33] an analog of this result (with constant coefficients \( b_i \)) is proved without using the Campbell-Hausdorff formula and Gromov’s nilpotentization theorem, for the case of \( C^{1,\alpha} \)-smooth vector fields, by means of estimates obtained in [31, 34].

Theorem 3 and its analogs have many important corollaries, in particular, each of them allows to prove the local approximation theorem, in the smoothness assumptions considered in each case, and also the Ball-Box theorem in the framework of the following definition.

**Definition 7** ([34]). If in Definition 5 the following assumption (3) holds, then \( \mathbb{M} \) is called a Carnot manifold.

(3) The factor-mapping \([\cdot, \cdot]: H_1 \times H_j/H_{j-1} \to H_{j+1}/H_j\), induced by the Lie bracket, is an epimorphism for all \( 1 \leq j < M \) (here it is assumed that \( H_0 = \{0\} \)).

In this case, the subbundle \( HM = H_1 \) is called horizontal.

By means of Theorem 3 an analog of the Rashevsky-Chow theorem is proved in [31, 34] for Carnot manifolds defined by \( C^{1,\alpha} \)-smooth vector fields. Thus it is possible to define the intrinsic C-C metric

\[ d_c(u, v) = \inf_{\gamma \in \mathcal{C}, \gamma(0) = u, \gamma(1) = v} \{ L(\gamma) \}. \]

(12)

The following assertion is formulated and proved in [61], in the proof of the local approximation theorem.

**Theorem 4.** Consider the curves \( \gamma \) and \( \hat{\gamma} \), satisfying the equations

\[
\begin{align*}
\dot{\gamma}(t) &= \sum_{i=1}^m a_i(t) \xi_i(\gamma(t)), \\
\gamma(0) &= \bar{v},
\end{align*}
\]

and

\[
\begin{align*}
\dot{\hat{\gamma}}(t) &= \sum_{i=1}^m a_i(t) \hat{\xi}_i(\hat{\gamma}(t)), \\
\hat{\gamma}(0) &= v.
\end{align*}
\]

Denote \( \gamma(1) = w, \hat{\gamma}(1) = \hat{w} \). If we have \( d_c(u, v) = O(\varepsilon) \) and \( d_c(v, w) = O(\varepsilon) \), then

\[ \max \{d_c(w, \hat{w}), d^u_c(w, \hat{w})\} = O(\varepsilon^{1+\frac{1}{M}}). \]

(13)

**Theorem 5** (Local approximation theorem [61]). Uniformly on \( u \in U, v, w \in B_{d_c}(u, \varepsilon) \) the following estimate holds

\[ |d_c(v, w) - d^u_c(v, w)| = O(\varepsilon^{1+\frac{1}{M}}). \]
3 Choice of basis, nilpotent approximation and a homogeneous quasimetric

Definition 8. Among the vector fields \( \{X_I\}_{|I|_h \leq M} \) we choose a basis

\[
\{Y_1, Y_2, \ldots, Y_N\}
\]

as follows:

(i) the vector fields \( Y_1, Y_2, \ldots, Y_N \) are linearly independent at the point \( u \) (hence, in some neighborhood of \( u \));

(ii) the sum of their weights \( \sum_{i=1}^{N} \deg Y_i \) is minimal;

(iii) the sum of orders \( \sum_{j=1}^{N} |I_j| \) of the commutators \( X_{I_j} \), corresponding to \( Y_j \), is minimal.

We say that the basis meeting conditions (i), (ii), (iii) is associated with the filtration \( (4) \) at the point \( u \).

Denote the dimension of the \( k \)-th element \( H_k \) of filtration \( (4) \) at the point \( u \) as \( n_k = \dim H_k(u) \). Then items (i), (ii) of Definition 8 are equivalent to the fact that the vectors \( \{Y_1(u), \ldots, Y_{n_k}(u)\} \) form bases of \( H_k(u) \) for all \( k = 1, \ldots, M \).

Remark 3. Bases satisfying (i), (iii) were considered for “classical” sub-Riemannian geometry in [5, 28, 41] and other papers (“normal” or “minimal” frame), when (ii) and (iii) coincide. In our case the necessity of considering both (ii) and (iii) can be seen from the Example 3: having only (i), (ii) we can choose both the basis \( \{X_1, X_2, X_3\} \) and \( \{X_1, X_2, [X_1, X_2]\} \); these bases define a different algebraic structure. Adding both conditions excludes such examples.

Proposition 2. For any vector field \( X \in H_s \) we have

\[
X(v) = \sum_{i=1}^{N} \xi_i(v)Y_i(v), \quad \text{where } \xi_i(u) = 0 \text{ for } \deg Y_i > s.
\]

Proof. Really, by choice of the basis \( (14) \) the vectors \( Y_1(u), \ldots, Y_{n_s}(u) \) constitute a basis of \( H_s(u) \), hence \( \xi_i(u) = 0 \) for \( i > n_s \). Consequently, \( \xi_i(u) = 0 \) for \( \deg Y_i > s \).

Proposition 3. At a fixed point \( u \in U \) the following identity holds:

\[
[Y_i, Y_j](u) = \sum_{\deg Y_k \leq \deg Y_i + \deg Y_j} c_{ijk}(u)Y_k(u).
\]

If the point \( u \) is regular, this identity holds not just in \( u \), but in some neighborhood of \( u \).

Proof. The identity \( (16) \) follows from the fact that \( [H_m, H_l] \subseteq H_{m+l} \).

In some neighborhood of a regular point we can choose the same basis, satisfying (i), (ii), (iii), for all points, by definition of regularity.
Definition 9. Consider second-kind canonical coordinates $\Phi^u : \mathbb{R}^N \to U$ on $U$ defined as
\[
\Phi^u(x_1, \ldots, x_N) = \exp(x_1 Y_1) \circ \exp(x_2 Y_2) \circ \cdots \circ \exp(x_N Y_N)(u) \quad (17)
\]

Due to the smoothness assumptions of the Definition 1 and theorems on continuous dependence of solutions of ordinary differential equations on parameters [43], the mapping $\Phi^u$ is a $C^{M+1}$-diffeomorphism onto some neighborhood of zero $V \subseteq \mathbb{R}^N$.

We will construct nilpotent approximations in these coordinates [17] in the same way as it was done in [5, 25]. Dilations are defined like in [5, 17, 25]: on $\mathbb{R}^N$ let $\delta_\varepsilon(x_1, x_2, \ldots, x_N) = (x_1^{\varepsilon \deg Y_1}, x_2^{\varepsilon \deg Y_2}, \ldots, x_N^{\varepsilon \deg Y_N})$. The function $f : \mathbb{R}^N \to \mathbb{R}$ is homogeneous of order $l$, if $f(\delta_\varepsilon x) = \varepsilon^l f(x)$.

Definition 10. A vector field $X$ on $\mathbb{R}^N$ is homogeneous of order $s$, if $\delta_\varepsilon^s X = \varepsilon^s X$, where the action of dilations on a vector field is defined as $\delta_\varepsilon^s X(f \circ \delta_\varepsilon) = (X f) \circ \delta_\varepsilon$.

The proofs of the next Proposition 4 and Corollary 2 follow the scheme of [25] for $C^\infty$ vector fields meeting the Hörmander’s condition. We recall briefly main steps of these proofs.

Proposition 4. In coordinates $\Phi^u$ for the $C^{M+1}$-smooth vector field $X_I$ the following decomposition holds:
\[
X_I'(x) := (\Phi^u)^{-1}_* X_I(\Phi^u(x)) = \sum_{j=1}^N a_j(x) \frac{\partial}{\partial x_j} = \\
= \sum_{j=1}^N \left( \sum_{|\alpha|_h \geq \deg Y_j - \deg X_I, |\alpha| \leq M} f_{j,\alpha} x^\alpha + o(||x||^M) \right) \frac{\partial}{\partial x_j} \text{ for } ||x|| \to 0,
\]
where $\alpha = (\alpha_1, \ldots, \alpha_N)$, $|\alpha|_h = \sum_{i=1}^N \alpha_i \deg Y_i$, $|\alpha| = \sum_{i=1}^N \alpha_i$, $f_{j,\alpha} \in \mathbb{R}$, $||x||$ is the Euclidean norm in $\mathbb{R}^N$.

Proof. Applying to both parts of the obvious equality
\[
\Phi^u_* X_I'(x) = X_I(\Phi^u(x)), \ x \in V \subseteq \mathbb{R}^N
\]
the mapping $\exp(-x_N Y_N)_* \cdots \exp(-x_1 Y_1)_*$ and carrying out all the differentiations [25] in the obtained equality
\[
\sum_{j=1}^N a_j(x) \exp(-x_N Y_N)_* \cdots \exp(-x_1 Y_1)_* \Phi^u_* \left( \frac{\partial}{\partial x_j} \right) = \\
= \exp(-x_N Y_N)_* \cdots \exp(-x_1 Y_1)_* X_I(\Phi^u(x)),
\]
we get the identity
\[
\sum_{|\nu|=0}^M \frac{(x_1)^{\nu_1}}{\nu_1 !} \frac{(x_2)^{\nu_2}}{\nu_2 !} \cdots \frac{(x_N)^{\nu_N}}{\nu_N !} (\text{ad}^{\nu_N} Y_N \cdots \text{ad}^{\nu_2} Y_2 \text{ad}^{\nu_1} Y_1, X_I)(u) + o(||x||^M) =
\]
\[
\sum_{|\nu|=0}^{M} a_j(x) \frac{(-x_{j+1})^{\nu_{j+1}}}{\nu_{j+1}!} \cdots \frac{(-x_N)^{\nu_N}}{\nu_N!} (\text{ad}^{\nu_N} Y_N \cdots \text{ad}^{\nu_{j+1}} Y_{j+1}, Y_j)(u) + o(||x||^M),
\]
where
\[
(\text{ad} Z, Y) = [Z, Y]; \ (\text{ad}^{\nu+1} Z, Y) = [Z, (\text{ad}^\nu Z, Y)]; \ [\text{ad}^0 Z, Y] = Y.
\]
According to Proposition 2, the following decomposition holds:
\[
(\text{ad}^{\nu_N} Y_N \cdots \text{ad}^{\nu_2} Y_2 \text{ad}^{\nu_1} Y_1, X_I)(u) = \sum_{k=1}^{N} \beta^k \nu Y_k(u),
\]
where
\[
\beta^k \nu = 0 \text{ for } |\nu|_h = \sum_{j=1}^{N} \nu_j \deg Y_j < \deg Y_k - \deg X_I.
\]
Denoting
\[
b_k(x) = \sum_{|\nu|=0}^{M} \beta^k \nu \frac{(-x_1)^{\nu_1}}{\nu_1!} \frac{(-x_2)^{\nu_2}}{\nu_2!} \cdots \frac{(-x_N)^{\nu_N}}{\nu_N!}
\]
and
\[
c_{jk}(x) = \frac{(-x_{j+1})^{\nu_{j+1}}}{\nu_{j+1}!} \cdots \frac{(-x_N)^{\nu_N}}{\nu_N!} \gamma^k_{\nu,j},
\]
where
\[
(\text{ad}^{\nu_N} Y_N \cdots \text{ad}^{\nu_{j+1}} Y_{j+1}, Y_j)(u) = \sum_{k=1}^{N} \gamma^k_{\nu,j} Y_k(u),
\]
we derive
\[
\sum_{j=1}^{N} a_j(x) \left[ Y_j(u) + \sum_{k=1}^{N} c_{jk}(x) Y_k(u) \right] = \sum_{k=1}^{N} b_k(x) Y_k(u) + o(||x||^M) \text{ for } ||x|| \to 0,
\]
Here \(b_k(x)\) is a polynomial function beginning from terms \(x_1^{\nu_1} \cdots x_N^{\nu_N}\) of order \(|\nu|_h \geq \deg Y_k - \deg X_I\), while \(||(c_{jk}(x))|| < 1\) in some neighborhood of zero. Denoting
\[
a(x) = (a_1(x), a_2(x), \ldots, a_N(x)),
\]
\[
b(x) = (b_1(x), b_2(x), \ldots, b_N(x)),
\]
\[
C(x) = (c_{jk}(x))_{j,k=1}^{N},
\]
we finally obtain
\[
a(x) = (I + C(x))^{-1}(b(x) + o(||x||^M)) = b(x) - C(x)b(x) + o(||x||^M) \text{ for } ||x|| \to 0,
\]
from where, according to the properties of \(b_k(x)\), the proposition follows. \(\square\)

Since \(\deg X_I = |I|_h\) and the vector field \(\frac{\partial}{\partial x_j}\) is homogeneous of order \(- \deg Y_j\), we have

**Corollary 1.** The vector field \(X'_I \in C^M, |I|_h \leq M,\) can be written as
\[
X'_I(x) = (X'_I)^{(-|I|_h)}(x) + (X'_I)^{(-|I|_h+1)}(x) + \ldots + (X'_I)^{(-|I|_h+M)}(x) + o(||x||^M) \text{ for } ||x|| \to 0,
\]
where the \(C^\infty\)-smooth vector field \((X'_I)^{(-j)}\) is homogeneous of order \(-j\).
Corollary 2. The $C^{M+1}$-smooth vector fields $\{\tilde{X}_I^u\}_{|I|_h \leq M}$ on $\mathcal{M}$, where $\tilde{X}_I^u = \Phi^u((X_I^u)^{|I|_h})$, constitute a nilpotent Lie algebra

$$L = \text{Lie}\{\tilde{X}_1^u, \ldots, \tilde{X}_q^u\}$$

and we have

$$H_I(u) = \tilde{H}_I(u), \quad \text{where } \tilde{H}_I = \text{span}\{\tilde{X}_I^u\}_{|I|_h \leq t}.$$

The vector fields $\{\tilde{Y}_1^u, \tilde{Y}_2^u, \ldots, \tilde{Y}_N^u\}$, chosen from the commutators $\tilde{X}_I^u$ in the same way as the basis (14) from the commutators $X_I$, form a basis, associated with the filtration (4) on some neighborhood $\tilde{U}$ of the point $u$.

Proof. The smoothness assertion follows from the fact that $\Phi^u$ is a $C^{M+1}$-diffeomorphism and that $\Phi^u[X, Y] = [\Phi^uX, \Phi^uY]$. The Lie algebra is nilpotent since for $|I|_h > M$ we have $(X_I^u)^{|I|_h} = 0$.

To prove the second part of the corollary it is sufficient to note that $(\tilde{Y}_i^u)'(0) = \frac{\partial}{\partial x_i}$ due to differentiation rules and homogeneity of the vector fields $(\tilde{Y}_i^u)'$. Thus the vector fields $\tilde{Y}_i^u$ are linearly independent at the point $u$ and hence in some its neighborhood. Moreover, if

$$X_I(v) = \sum_{i=1}^N \xi_i(v)Y_i(v), \quad \tilde{X}_I(v) = \sum_{i=1}^N \eta_i(v)Y_i(v),$$

then $\xi_i(u) = \eta_i(u)$ for $n_{|I|_h-1} + 1 \leq i \leq N$. Indeed, in coordinates (17) we have

$$X_I'(0) = (\Phi^u)^{-1}X_I(u) = \sum_{i=1}^N \xi_i(u)\frac{\partial}{\partial x_i}; \quad \tilde{X}_I'(0) = (\Phi^u)^{-1}\tilde{X}_I(u) = \sum_{i=1}^N \eta_i(u)\frac{\partial}{\partial x_i};$$

$$X_I'(x) = \tilde{X}_I'(x) + Z(x),$$

where the vector field $Z(x) = \sum_{i=1}^N z_i(x)\frac{\partial}{\partial x_i}$ consists of summands having order of homogeneity bigger than $-|I|_h$, hence $z_i(0) = 0$ for $n_{|I|_h-1} + 1 \leq i \leq N$. \hfill \Box

W.l.o.g. assume that $U = \tilde{U}$.

Definition 11. The vector fields $\{\tilde{X}_I^u\}_{|I|_h \leq M}$ are called nilpotent approximations of the vector fields $\{X_I\}_{|I|_h \leq M}$.

Definition 12. Define a dilation group, associated with the basis (14), $\Delta^u_\varepsilon = \Phi^u(\Phi^u)^{-1}$ on $U$: if

$$w = \exp(w_1Y_1) \circ \exp(w_2Y_2) \circ \ldots \circ \exp(w_NY_N)(v),$$

then

$$\Delta^u_\varepsilon w = \exp(w_1\varepsilon^{\deg Y_1}Y_1) \circ \exp(w_2\varepsilon^{\deg Y_2}Y_2) \circ \ldots \circ \exp(w_N\varepsilon^{\deg Y_N}Y_N)(v).$$

(23)

From Proposition 14 it follows immediately

Corollary 3. On $U$ the following convergence takes place:

$$\lim_{\varepsilon \to 0} (\Delta^u_{\varepsilon^{-1}})^{|I|_h}X_I(\Delta^u_\varepsilon(v)) \to \tilde{X}_I^u(v) \text{ for } \varepsilon \to 0, |I|_h \leq M.$$
Proof. Really, in coordinates (17) we have

$$(\delta_{\varepsilon^{-1}})_{*}\varepsilon^{||I||h}X^\prime_I(\delta_{\varepsilon}(x))$$

$$= \delta_{\varepsilon}^{*}\varepsilon^{||I||h} \sum_{j=1}^{N} \left( \sum_{\deg Y_{j} - |I||h| \leq |\alpha||h| \leq M} f(j,\alpha)(\varepsilon x)^{\alpha} + o(||\varepsilon x||^{M}) \right) \frac{\partial}{\partial x_j}$$

$$= \varepsilon^{||I||h} \sum_{j=1}^{N} \varepsilon^{\deg Y_{j}} \left( \sum_{\deg Y_{j} - |I||h| \leq |\alpha||h| \leq M} f(j,\alpha)\varepsilon^{||\alpha||h}x^{\alpha} + o(||\varepsilon x||^{M}) \right) \frac{\partial}{\partial x_j} \to$$

$$\to \sum_{j=1}^{N} \left( \sum_{|\alpha||h| = \deg Y_{j} - |I||h|} f(j,\alpha)x^{\alpha} \right) \frac{\partial}{\partial x_j} = (X^\prime_I)^{(-||I||h)}$$

for $\varepsilon \to 0$. 

Introduce a distance function on $U$, generated by nilpotent approximations, in a similar way as in (6):

$$\rho^{u}(v, w) = \inf \{\delta > 0 \mid \text{there is a curve } \gamma : [0, 1] \to U, \text{ such that}$$

$$\gamma(0) = v, \gamma(1) = w, \dot{\gamma}(t) = \sum_{|I||h| \leq M} w_{I}X^u_I(\gamma(t)), |w_{I}| < \delta^{||I||h} \}.$$  

(24)

Actually, $\rho^{u}$ is again a quasimetric; the generalized triangle inequality will be proved in the next subsection.

Proposition 5. The quasimetric $\rho^{u}$ meets the conical property

$$\rho^{u}(\Delta^{u}_{\varepsilon}v, \Delta^{u}_{\varepsilon}w) = \varepsilon \rho^{u}(v, w).$$  

(25)

Proof. By definition, $\rho^{u}(v, w)$ is the infimum of $\max \{|a_{I}|^{1/||I||h}\}$ over all curves $\gamma$ such that

$$\begin{cases}
\dot{\gamma}(t) = \sum_{|I||h| \leq M} a_{I}X^u_I(\gamma(t)), \\
\gamma(0) = v, \gamma(1) = w.
\end{cases}$$

Consider the curve

$$\gamma_{\varepsilon}(t) = \Delta^{u}_{\varepsilon}\gamma(t).$$

(26)

Due to homogeneity of the vector fields $\hat{X}^u_I$ we have

$$\begin{cases}
\dot{\gamma}_{\varepsilon}(t) = \sum_{|I||h| \leq M} a_{I}\varepsilon^{||I||h}\hat{X}^u_I(\gamma_{\varepsilon}(t)), \\
\gamma_{\varepsilon}(0) = \Delta^{u}_{\varepsilon}v, \gamma_{\varepsilon}(1) = \Delta^{u}_{\varepsilon}w.
\end{cases}$$

Note that all curves connecting the points $\Delta^{u}_{\varepsilon}v$ and $\Delta^{u}_{\varepsilon}w$ have the form (26): really, let $\kappa(t)$ be an arbitrary curve connecting the points $\Delta^{u}_{\varepsilon}v$ and $\Delta^{u}_{\varepsilon}w$, then the curve $\gamma(t) = \Delta^{u}_{\varepsilon}\kappa(t)$ connects the points $v$ and $w$. Hence $\rho^{u}(\Delta^{u}_{\varepsilon}v, \Delta^{u}_{\varepsilon}w)$ is the infimum of $\varepsilon \max \{|a_{I}|^{1/||I||h}\}$ over $\gamma$, from where the proposition follows. 

$\square$
4 The lifting construction and further properties of the quasi-metrics $\rho$ and $\rho^u$

In this section we first recall the lifting construction proposed in by L. Rotshild and E. M. Stein [48] and developed in many other papers ([19, 27, 14, 28, 8] etc.). We present this construction in the form suitable for our purposes, making essentially a synthesis of the ideas of papers [14] and [28], in order to get a (quasi)metric-decreasing embedding of our C-C space into a regular one.

Using this embedding and results for regular quasimetric C-C spaces [34, 61] we will derive some important geometric properties of the quasimetrics $\rho$ and $\rho^u$, in particular prove the generalized triangle inequality for both of them. Crucial for proving main theorems of the next section is the “rolling-of the-box lemma” (Proposition 10).

Let us recall the construction of a free nilpotent Lie algebra $N^M_{d_1, \ldots, d_q}$ with $q$ generators $X_1, \ldots, X_q$ of weights $\{d_i\}_{i=1}^q$ and depth $M$ [14].

Let $F_q$ be a free (infinite-dimensional) Lie algebra with $q$ generators, i.e. the only interrelation between commutators of vector fields $\{X_i\}$ are the skewcommutativity and the Jacobi identity. Introduce on $F_q$ dilations acting as

$$\delta_\varepsilon(\sum_{j=1}^q c_j X_j) = \sum_{j=1}^q c_j \varepsilon^{d_j} X_j, \quad \delta_\varepsilon(X_I) = \varepsilon^{|I|}\varepsilon X_I.$$  \hfill (27)

Consider subspaces $F^l_q$, invariant of order $l$ under dilations (27). Then $F_q = \bigoplus_{l=1}^{\infty} F^l_q$. Let

$$N = N^M_{d_1, \ldots, d_q} = F_q/I_M, \quad \text{where } I_M = \bigoplus_{l>M} F^l_q$$  \hfill (28)

is an Lie algebra ideal in $F_q$. Note that $F_q/I_M$ isomorphic to the direct sum $\bigoplus_{l\leq M} F^l_q$.

Let $\psi : N \rightarrow \bigoplus_{l\leq M} F^l_q$ be a Lie algebra isomorphism and $X_j = \psi(X_j)$. Denote

$$\tilde{N} = \tilde{N}(d_1, \ldots, d_q, M) = \dim N^M_{d_1, \ldots, d_q}. \hfill (29)$$

**Definition 13.** The vector fields $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_q$ on $\tilde{U} \subseteq \tilde{M}$, defining a filtration of the form (4), are called free up to the order $s$ at the point $u \in \tilde{U}$, if $\dim H_s(u) = \tilde{N}(d_1, \ldots, d_q, s)$.

**Remark 4 ([48, 14]).** If the vector fields $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_q$ on $\tilde{U} \subseteq \tilde{M}$ are free up to the order $M$ at the point $u \in \tilde{U}$, where $M$ is the depth of the C-C space $\tilde{M}$, then the point $u$ is regular.

The proof of the next proposition follows the same lines as the proof of a similar assertion in [28] for the case of smooth vector fields meeting the Hörmander’s condition. We recall this proof, since some of its details are needed below.

**Proposition 6.** Let all conditions of Definition 1 be satisfied and $\tilde{N}$ be the dimension defined by (29) of the corresponding free Lie algebra. Consider the manifold $\tilde{M} = \tilde{M} \times$
\[ \mathbb{R}^{\hat{N}-N} \] of the dimension \( \hat{N} \). Then there are a neighborhood \( \tilde{U} \) of the point \((u, 0)\) in \( \hat{M} \), a neighborhood \( U \) of the point \( u \), where \( U \times \{0\} \subseteq \tilde{U} \), coordinates \((y, z)\) on \( \tilde{U} \) and two systems of \( C^M \)-smooth vector fields

\[
\dot{X}_k(y, z) = X_k(y) + \sum_{j=N+1}^{\hat{N}} b_{kj}(y, z) \frac{\partial}{\partial z_j} \quad \text{and} \quad \dot{\tilde{X}}_k(y, z) = \tilde{X}_k^u(y) + \sum_{j=N+1}^{\hat{N}} b_{kj}(y, z) \frac{\partial}{\partial z_j}, \quad (30)
\]

\( k = 1, 2, \ldots, q \), defining a \( C-C \) structure of depth \( M \) on \( \tilde{U} \subseteq \hat{M} \) and, hence, free up to order \( M \) on \( \tilde{U} \). Here \( b_{jk}(y, z) \) are polynomial functions on \( \tilde{U} \), such that the vector fields

\[
\sum_{j=N+1}^{\hat{N}} b_{kj}(y, z) \frac{\partial}{\partial z_j}
\]

are homogeneous of order \(-d_k\), \( k = 1, 2, \ldots, q \).

All points of some neighborhood \( \tilde{V} = \tilde{V}(\tilde{u}) \subseteq \tilde{U} \) are regular.

**Proof.** Consider canonical vector fields \( \{\dot{X}_I\}_{|I|_h \leq M} \in C^\infty(\mathbb{R}^{\hat{N}}) \) which generate the Lie algebra \( \mathcal{N} \) defined in (28) in such way that \( \mathcal{F}_l = \text{span}\{\dot{X}_I\}_{|I|_h \leq l}, \dot{X}_{lI}(0) = e_j, j = 1, \ldots, \hat{N} \) [44, 17, 9, 34].

By definition of a free algebra, there is a surjective homomorphism of nilpotent Lie algebras \( \Psi : \mathcal{N} \to \text{Lie}\{\dot{X}_1^u, \dot{X}_2^u, \ldots, \dot{X}_q^u\} \) such that \( \Psi(\dot{X}_I') = \dot{X}_I^u, |I|_h \leq M \).

Let \( G = \exp(\mathcal{N})(0) \) be the corresponding Lie group \( \mathbb{R}^{\hat{N}} \). Define the action of \( G \) on \( \hat{M} \) by means of the homomorphism \( \Psi \): for \( g = \exp\left(\sum_{j=1}^{\hat{N}} c_j \dot{X}_j\right) \in G, v \in U \) let

\[
g(v) = \exp\left(\sum_{j=1}^{\hat{N}} c_j \Psi(\dot{X}_j)(v)\right) = \exp\left(\sum_{j=1}^{\hat{N}} c_j \dot{X}_j^u(v)\right).
\]

The isotropy subgroup \( H = \{g \in G \mid g(u) = u\} \subseteq G \) is connected and invariant under dilations

\[
\tilde{\delta}_\varepsilon(x_1, x_2, \ldots, x_N) = (\varepsilon^{|I|_h} x_1, \varepsilon^{|I|_h} x_2, \ldots, \varepsilon^{|I|_h} x_N), \quad (31)
\]

due to homogeneity of the vector fields. Moreover,

\[
\mathcal{H} = \text{span}\left\{ \sum_j c_j \dot{X}_j \mid \sum_j c_j \dot{X}_j^u(u) = 0 \right\} \quad (32)
\]

Denote by \( \dot{\mathcal{Z}}_{N+1}, \ldots, \dot{\mathcal{Z}}_{\hat{N}} \) the basis of the subalgebra \( \mathcal{H} \) consisting of vector fields homogeneous under dilations.

The mapping \( \varphi_u : G \to U \subseteq \hat{M} \) defined as \( \varphi_u(g) = g(u) \) induces a diffeomorphism from the homogeneous space \( G/H = \{Hg \mid g \in G\} \) onto the neighborhood \( U \): \( \varphi_u(Hg) = g(u) \). Consider on \( G/H \) left-invariant vector fields

\[
\dot{X}_i^h(\widetilde{H}g) = \frac{d}{dt}\bigg|_{t=0} [\widetilde{H}g \exp(t \dot{X}_i)](0), \quad i = 1, \ldots, q.
\]
By the diffeomorphism \( \varphi_u \) identify them with the vector fields \( \hat{X}^u_i \):
\[
(\varphi_u)_*(\hat{X}^h_i)(Hg) = \left. \frac{d}{dt}[\varphi(Hg \exp(t\hat{X}'_i)(0))] \right|_{t=0} = \\
= \left. \frac{d}{dt}[\exp(t\hat{X}'_i)(g(u))] \right|_{t=0} = \hat{X}^u_i(g(u)).
\]

Consider on \( U \) the basis
\[
\hat{Y}^u_1, \hat{Y}^u_2, \ldots, \hat{Y}^u_N, \tag{33}
\]
consisting of the same commutators of the vector fields \( \{\hat{X}^u_i\}_{i|h\leq M} \), as the basis \( \{X^I_i\}_{i|h\leq M} \) of the commutators of \( \{X^I_i\}_{i|h\leq M} \).

Taking in account \( (32) \), we see that the family of vector fields \( \{\hat{Y}^u_i\}_{i=1}^N \) is a basis of the algebraic complement to \( \mathcal{H} \) in the Lie subalgebra \( N_{M,m} \), consisting of homogeneous vector fields.

Introduce on \( G \) coordinates
\[
(y, z) \in \mathbb{R}^\tilde{N} \mapsto g = \exp \left( \sum_{k=N+1}^{\tilde{N}} z_k \hat{Z}_k \right) \exp(y_N \hat{Y}_N) \ldots \exp(y_1 \hat{Y}_1) \tag{34}
\]

In these coordinates it holds
\[
\hat{X}'^i_k(y, z) = \hat{X}^h_i(y) + \sum_{j=N+1}^{\tilde{N}} b_{kj}(y, z) \frac{\partial}{\partial z_j}, \quad j = 1, 2, \ldots, q. \tag{35}
\]

Indeed,
\[
\hat{X}'_i(g) = \left. \frac{d}{dt}[g \exp(t\hat{X}'_i)](0) \right|_{t=0};
\]
in coordinates \( (34) \) we have
\[
g \exp(t\hat{X}'_i)(0) = \exp \left( \sum_{k=N+1}^{\tilde{N}} z_k \hat{Z}_k \right) \exp(y_N \hat{Y}_N) \ldots \exp(y_1 \hat{Y}_1) \exp(t\hat{X}'_i)(0) = \\
= \exp \left( \sum_{k=N+1}^{\tilde{N}} z_k \hat{Z}_k \right) h(t) \exp(c_N(y, t)\hat{Y}_N) \ldots \exp(c_1(y, t)\hat{Y}_1),
\]
where \( h(t) \in H; \)
\[
Hg \exp(t\hat{X}'_i)(0) = H \exp \left( \sum_{k=N+1}^{\tilde{N}} z_k \hat{Z}_k \right) \exp(y_N \hat{Y}_N) \ldots \exp(y_1 \hat{Y}_1) \exp(t\hat{X}'_i)(0) = \\
= H \exp(c_N(y, t)\hat{Y}_N) \ldots \exp(c_1(y, t)\hat{Y}_1).
\]
Thus the coordinates of the vector fields \( \hat{X}^h_i \) and \( \hat{X}'_i \) by \( \frac{\partial}{\partial y_k} \) coincide and are equal to \( \frac{d}{dt}c_k(y, 0) \). Hence, we have \( (35) \).
Now define the vector fields $\hat{X}_k, \hat{X}_k'$ by formulas (30). Since the vector fields $\hat{X}_k'$ are homogeneous of order $-d_k$, then the vector fields $\sum_{j=N+1}^{\hat{N}} b_{kj}(y, z) \frac{\partial}{\partial z_j}$, $k = 1, 2, \ldots, q$ are homogeneous of the same order. By construction, we have that $\hat{X}_k = \hat{X}_k^{(-d_k)}$ w.r.t. the dilations (31). Thus the vector fields $\{\hat{X}_k\}_{k=1}^q$ define a C-C structure of depth $M$ on $\hat{M}$ and are free of order $M$ on $U$. The point $\hat{u}$ and hence all points in some of its neighborhoods are regular, according to Remark 4.

**Proposition 7.** For all multiindices $I$, such that $|I|_h \leq M$, the following decompositions hold:

$$\hat{X}_I(y, z) = X_I(y) + \sum_{j=N+1}^{\hat{N}} b_{Ij}(y, z) \frac{\partial}{\partial z_j} \quad \text{and} \quad \hat{X}_I(y, z) = \hat{X}_I^u(y) + \sum_{j=N+1}^{\hat{N}} \hat{b}_{Ij}(y, z) \frac{\partial}{\partial z_j},$$

where $b_{Ij}(y, z), \hat{b}_{Ij}(y, z) \in C^{M+1}(\hat{U})$.

**Proof.** Let us prove the first decomposition of (36) by induction on the length of $I$ (the second decomposition is proved in a similar way). Let (36) be true for all $J$, such that $|J|_h \leq l$. By the Jacobi identity, any vector field $\hat{X}_I$, where $|I|_h \leq l + \min\{d_1, \ldots, d_q\}$, can be represented as $\hat{X}_I = [\hat{X}_i, \hat{X}_j]$, where $i = 1, \ldots, q$ and $|J|_h \leq l$. By induction and taking into account the identity (30), we get

$$\hat{X}_I(y, x) = [X_i, X_j](y) + [X_i, \sum_{j=N+1}^{\hat{N}} b_{Ij}(y, z) \frac{\partial}{\partial z_j}] +$$

$$+ \left[ \sum_{j=N+1}^{\hat{N}} b_{Ij}(y, z) \frac{\partial}{\partial z_j}, X_J \right] + \left[ \sum_{j=N+1}^{\hat{N}} b_{Ij}(y, z) \frac{\partial}{\partial z_j}, \sum_{j=N+1}^{\hat{N}} b_{Ij}(y, z) \frac{\partial}{\partial z_j} \right]$$

$$= X_I(y) + \sum_{j=N+1}^{\hat{N}} \left( X_i b_{Ij} - X_j b_{Ii} + \frac{\partial}{\partial z_j} b_{Ij} - \frac{\partial}{\partial z_j} b_{Ii} \right) \frac{\partial}{\partial z_j}.$$ 

Thus the vector field $\hat{X}_I$ has the desired form. The rest of the proposition follows from the smoothness assumptions of Definition 1.

Consider the neighborhood $\hat{U}$ and the vector fields $\hat{X}_I$ from Propositions 6, 7. Let $\pi : \hat{U} \to U$ be a canonical projection acting on an arbitrary point $\pi(\tilde{v}) = (v, y)$, such that $v \in U$, $y \in \mathbb{R}^{\hat{N}-N}$, as $\pi(\tilde{v}) = v$. The next proposition states that the projection is distance-decreasing (cf. [5, 28]).

**Proposition 8.** For any $v, w \in U$ and $p, q \in \mathbb{R}^{\hat{N}-N}$ the following inequalities hold:

$$\rho(v, w) \leq \hat{\rho}((v, p), (w, q)), \quad \rho^a(v, w) \leq \hat{\rho}^a((v, p), (w, q)),$$

where the quasimetrics $\hat{\rho}, \hat{\rho}^a$ on the regular C-C space $\hat{U}$ are defined in a similar way as $\rho, \rho^a$ on the initial neighborhood $U \subseteq \hat{M}$.
Proof. Show the inequality (37). Denote \( \tilde{v} = (v, p), \tilde{w} = (w, q) \). There is a unique curve \( \tilde{\gamma}(t) \) such that

\[
\begin{cases}
\tilde{\gamma}(t) = \sum_{|I|_h \leq M} w_I \tilde{X}_I(\tilde{\gamma}(t)) = \sum_{|I|_h \leq M} w_I(X_I(\pi(\tilde{\gamma}(t))) + \sum_{k=N+1}^{\tilde{N}} b_{ik}(\tilde{\gamma}(t)) \frac{\partial}{\partial z_k}), \\
\tilde{\gamma}(0) = \tilde{v}, \tilde{\gamma}(1) = \tilde{w}.
\end{cases}
\]

By definition, \( \tilde{\rho}(\tilde{v}, \tilde{w}) = \max \{|w_I|^{1/|I|_h} \} \). Let \( \gamma(t) = \pi(\tilde{\gamma}(t)) \), then

\[
\begin{cases}
\dot{\gamma}(t) = \sum_{|I|_h \leq M} w_I X_I(\gamma(t)), \\
\gamma(0) = v, \gamma(1) = w.
\end{cases}
\]

Thus, the curve \( \gamma(t) \) lies in \( U \) and joins the points \( v \) and \( w \), from where (37) follows:

\[
\rho(v, w) \leq \max \{|w_I|^{1/|I|_h} \} = \tilde{\rho}(\tilde{v}, \tilde{w}).
\]

The inequality (38) is proved in the same way. □

**Proposition 9 (Generalized triangle inequalities).** For any point \( g \in U \) there are constants \( Q, Q_g > 0 \) such that, for all \( u, v, w \in U \), we have

\[
\begin{align*}
\rho(v, w) &\leq Q(\rho(u, v) + \rho(u, w)), \\
\rho^2(v, w) &\leq Q_g(\rho^2(u, v) + \rho^2(u, w)).
\end{align*}
\]

**Proof.** For any (arbitrarily small) \( \zeta > 0 \) consider

\[
\{a_I\}_{|I|_h \leq M} \text{ and } \{b_I\}_{|I|_h \leq M},
\]

such that

\[
v = \exp\left( \sum_{|I|_h \leq M} a_I X_I(u) \right), \quad w = \exp\left( \sum_{|I|_h \leq M} b_I X_I(u) \right)
\]

and

\[
\max_{|I|_h \leq M} \{|a_I|^{1/|I|_h}\} \leq \rho(u, v) + \zeta, \quad \max_{|I|_h \leq M} \{|b_I|^{1/|I|_h}\} \leq \rho(u, w) + \zeta.
\]

Let \( \tilde{u} = (u, 0) \) and consider on \( \tilde{U} \) points

\[
\tilde{v} = \exp\left( \sum_{|I|_h \leq M} a_I \tilde{X}_I(\tilde{u}) \right) \text{ and } \tilde{w} = \exp\left( \sum_{|I|_h \leq M} b_I \tilde{X}_I(\tilde{u}) \right).
\]

Then we have \( v = \pi(\tilde{v}), w = \pi(\tilde{w}) \) and

\[
\tilde{\rho}(\tilde{u}, \tilde{v}) = \max_{|I|_h \leq M} \{|a_I|^{1/|I|_h}\}, \quad \tilde{\rho}(\tilde{u}, \tilde{w}) = \max_{|I|_h \leq M} \{|b_I|^{1/|I|_h}\}.
\]

According to Proposition [15] and the generalized triangle inequality for \( \tilde{\rho} \) (in the neighborhood of a regular point [34]) we have

\[
\rho(v, w) \leq \tilde{\rho}(\tilde{u}, \tilde{v}) \leq Q(\tilde{\rho}(\tilde{u}, \tilde{v}) + \tilde{\rho}(\tilde{u}, \tilde{w})) \leq Q(\rho(u, v) + \rho(u, w) + 2\zeta),
\]

from where (39) follows; (40) is proved in a similar way. □
Proposition 10 ("Rolling-of-the-box" lemma). For all points \( u, v \in U \) and \( r, \xi > 0 \), for which both parts of the following inclusions make sense (i.e. lie in \( U \)), we have

\[
\bigcup_{x \in B^0(v, r)} B^\rho(x, \xi) \subseteq B^\rho(v, r + C\xi), \tag{41}
\]

\[
\bigcup_{x \in B^\rho(v, r)} B^0(x, \xi) \subseteq B^\rho(v, r + C\xi + O(r^{1+\frac{1}{M}}) + O(\xi^{1+\frac{1}{M}})). \tag{42}
\]

Proof. Let us prove (42). Fix points \( x, z \), such that \( \rho(v, x) < r \), \( \rho(x, z) < \xi \), and show that \( \rho(v, z) < r + C\xi + O(r^{1+\frac{1}{M}}) + O(\xi^{1+\frac{1}{M}}) \). For arbitrarily small \( \zeta > 0 \) consider two curves \( \gamma_1, \gamma_2 \), such that

\[
\begin{align*}
\gamma_1(t) &= \sum_{|I_h| \leq M} x_I X_I(\gamma_1(t)), \\
\gamma_1(0) &= v, \gamma_1(1) = x,
\end{align*}
\]

\[
\begin{align*}
\gamma_2(t) &= \sum_{|I_h| \leq M} z_I X_I(\gamma_2(t)), \\
\gamma_2(0) &= x, \gamma_2(1) = z,
\end{align*}
\]

and

\[
\max_{|I_h| \leq M} \{|x_I|^{1/|I_h|}\} \leq \rho(v, x) + \zeta, \quad \max_{|I_h| \leq M} \{|z_I|^{1/|I_h|}\} \leq \rho(x, z) + \zeta.
\]

Consider a point \( \tilde{v} = (v, 0) \in U \) and a curve \( \tilde{\gamma}_1 \) such that

\[
\begin{align*}
\dot{\tilde{\gamma}}_1(t) &= \sum_{|I_h| \leq M} x_I X_I(\tilde{\gamma}_1(t)), \\
\tilde{\gamma}_1(0) &= \tilde{v}.
\end{align*}
\]

Since \( \gamma_1(t) = \pi(\tilde{\gamma}_1(t)) \), we have \( \tilde{\gamma}_1(1) = (x, p) =: \tilde{x} \in \tilde{U} \), where \( p \in \mathbb{R}^{\tilde{N}-N} \). However,

\[
\tilde{\rho}(\tilde{v}, \tilde{x}) = \max_{|I_h| \leq M} \{|x_I|^{1/|I_h|}\} < r + \zeta.
\]

In a similar way, for a curve \( \tilde{\gamma}_2 \), such that

\[
\begin{align*}
\dot{\tilde{\gamma}}_2(t) &= \sum_{|I_h| \leq M} x_I X_I(\tilde{\gamma}_2(t)), \\
\tilde{\gamma}_2(0) &= \tilde{x}.
\end{align*}
\]

We have \( \gamma_2(t) = \pi(\tilde{\gamma}_2(t)) \), and hence \( \tilde{\gamma}_2(1) = (z, q) =: \tilde{z} \in \tilde{U} \), where \( q \in \mathbb{R}^{\tilde{N}-N} \), and

\[
\tilde{\rho}(\tilde{x}, \tilde{z}) = \max_{|I_h| \leq M} \{|z_I|^{1/|I_h|}\} < \xi + \zeta.
\]

According to Remark 4, all points of \( \tilde{U} \) are regular w. r. t. the C-C structure induced by the vector fields \( \{X_I\}_{|I_h| \leq M} \).

By the Campbell-Hausdorff formula [9], for any vector fields \( X, Y \in C^{k_0+1} \) the following decomposition is true:

\[
\begin{align*}
\exp(sY) \circ \exp(tX)(v) &= \exp(sY + tX + \frac{st}{2}[X, Y] + \sum_{2 \leq k+j \leq k_0} s^k t^j C_{kj}(X, Y) + O(s^{k_0+1}) + O(t^{k_0+1})), \tag{43}
\end{align*}
\]
where \( C_{kj}(X,Y) \) are linear combinations of \((k + j - 1)\)-order commutators of \( X \) and \( Y \).

Applying (43), by simple computations, we get

\[
\exp \left( \sum_{|I|_h \leq M} z_I \tilde{X}_I \right) \circ \exp \left( \sum_{|I|_h \leq M} x_I \tilde{X}_I \right) = \exp \left( \sum_{|I|_h \leq M} v_I \tilde{X}_I \right) (v),
\]

where

\[
v_I = x_I + y_I + \sum_{|\alpha + \beta| \leq M \atop |\alpha + \beta|_h > |I|_h} F^I_{\alpha, \beta} x^\alpha z^\beta + O(||x||^{M+1}) + O(||z||^{M+1}).
\]

Consequently,

\[
|v_I| \leq |x_I| + |z_I| + \sum_{|\alpha + \beta| = |I|_h} |F^I_{\alpha, \beta} x^\alpha z^\beta + \\
+ \sum_{|\alpha + \beta| \leq M \atop |\alpha + \beta|_h > |I|_h} |F^I_{\alpha, \beta} x^\alpha z^\beta + O(||x||^{M+1}) + O(||z||^{M+1}) \leq \\
\leq (\tilde{r} + C \tilde{\xi})|I|_h + O(\tilde{r}^{\nu_{|I|h+1}}) + O(\tilde{\xi}^{|I|_h+1}) + O(\tilde{r}^{M+1}) + O(\tilde{\xi}^{M+1}),
\]

from where it follows, that

\[
\tilde{\rho}(\tilde{v}, \tilde{z}) = \max \{ |v_I|^{1/|I|_h} \} \leq \tilde{r} + C \tilde{\xi} + O(\tilde{r}^{1+\frac{1}{M}}) + O(\tilde{\xi}^{1+\frac{1}{M}}).
\]

Applying (37), we finally obtain

\[
\rho(v, z) \leq \tilde{\rho}(\tilde{v}, \tilde{z}) \leq r + C \xi + O(r^{1+\frac{1}{M}}) + O(\xi^{1+\frac{1}{M}}) + O(\zeta),
\]

from where (42) follows. The inclusion (41) can be proved in a similar way.

\[\square\]

5 Main theorems on local geometry

Proposition 11. Consider on \( \tilde{U} \) bases \( \{\tilde{X}_I\}_{|I|_h \leq M} \) and \( \{\hat{X}_I\}_{|I|_h \leq M} \), consisting of commutators of the vector fields defined in (30).

Then, in coordinates \( x = (y, z) \) defined in (34), for all \( x \in \tilde{U} \), such that \( |x_j| \leq \epsilon^{|J|} \), the following decompositions hold:

\[
\hat{X}_I(x) = \sum_{i=1}^{\tilde{N}} a_{I,J}(x) \tilde{X}_J(x), \tag{44}
\]

where

\[
a_{I,J} = \begin{cases} \\
\delta_{I,J} + O(\epsilon), & |J|_h = |I|_h, \\
o(\epsilon^{|J|_h-|I|_h}), & |J|_h > |I|_h, \\
O(1), & |J|_h < |I|_h.
\end{cases}
\]
Proof. From Propositions \([7]\) and \([4]\) it follows that
\[
\hat{X}_I(x) = \tilde{X}_I(x) + R_I(x),
\]
where \(x = (y, z) \in \mathbb{R}^{\tilde{N}}\), while the vector field \(R_I\) consists of summands of homogeneity order, w.r.t. the dilations \((31)\), bigger than \(-|I|_h\). Since the vector fields \(\hat{X}_J\) are homogeneous of order \(|J|_h\), we have
\[
R_I(x) = \sum_{|J|_h \leq M} \sum_{|\alpha|_h > |J|_h - |I|_h} c_{\alpha I} x^\alpha \hat{X}_J =
\]
\[
= \sum_{|J|_h > |I|_h} \varepsilon^{|J|_h - |I|_h + 1} (O(1) + O(\varepsilon)) \hat{X}_J +
\]
\[
+ \sum_{|J|_h = |I|_h} \varepsilon (O(1) + O(\varepsilon)) \hat{X}_J + \sum_{|J|_h < |I|_h} (O(1) + O(\varepsilon)) \hat{X}_J = \sum a_{I,J} \hat{X}_J,
\]
from where the proposition follows.

Next we introduce an important characteristic of the C-C space \(\mathcal{M}\).

**Definition 14.** Let \(u, v \in U, r > 0\). The divergence of integral lines with nilpotentions centered at \(u\) over a box of radius \(r\) centered at \(v\) is the value
\[
R(u, v, r) = \max \left\{ \sup_{\tilde{y} \in B^\rho(u, y)} \{ \rho^u(y, \tilde{y}) \}, \sup_{y \in B^\rho(v, y)} \{ \rho(y, \tilde{y}) \} \right\}.
\]
(45)

Here the points \(y\) and \(\tilde{y}\) are defined as follows. Let \(\gamma(t)\) be an arbitrary curve, defined as a solution of the system of ODE
\[
\begin{cases}
\dot{\gamma}(t) = \sum_{|I|_h \leq M} b_I \hat{X}_I^u(\gamma(t)), \\
\gamma(0) = v, \gamma(1) = \tilde{y},
\end{cases}
\]
and
\[
\rho^u(v, \tilde{y}) \leq \max \left\{ \|b_I\|_{|I|_h} \right\} \leq r.
\]
(46)

Define \(y = \exp(\sum_{|I|_h \leq M} b_I X_I)(v)\). In this way, the supremum in the first expression of (45) is taken not only over \(\tilde{y} \in B^\rho(u, v, r)\), but also over the infinite set of the possible \(\{b_I\}_{|I|_h \leq M}\), satisfying (46). The second expression is understood in a similar way.

**Proposition 12.** Let \(u, v \in U\) and \(r > 0\). Then the following inclusions are true:
\[
B^\rho(v, r) \subseteq B^\rho (v, r + CR(u, v, r)),
\]
(47)
\[
B^\rho(v, r) \subseteq B^\rho (v, r + CR(u, v, r) + O(r^{1+\frac{1}{M}}) + O(R(u, v, r)^{1+\frac{1}{M}})),
\]
(48)
where \(R(u, v, r)\) is defined by (45).
Proof. Let \( y \in B^\rho(v, r) \), i.e. \( \rho(v, y) < r \), and show that \( \rho^u(v, y) < r + CR(u, v, r) \) for some constant \( C \).

By definition of the quasimetric \( \rho \), for arbitrarily small \( \zeta > 0 \) there are \( \{a_I\}_{|I|_h \leq M} \), such that

\[
y = \exp \left( \sum_{|I|_h \leq M} a_I X_I(v) \right)
\]

and

\[
\max_{|I|_h \leq M} \{|a_I|^{1/|I|_h}\} \leq \rho(v, y) + \zeta \leq r.
\]

Consider a point \( \hat{y} = \exp \left( \sum_{|I|_h \leq M} a_I X_I(v) \right) \). Then \( \hat{y} \in B^{\rho^u}(v, r) \), since

\[
\rho^u(v, \hat{y}) \leq \max_{|I|_h \leq M} \{|a_I|^{1/|I|_h}\} \leq r.
\]

Obviously, \( \rho^u(y, \hat{y}) < R(u, v, r) \). Hence, by (41),

\[
y \in \bigcup_{x \in B^{\rho^u}(v, r)} B^{\rho^u}(x, R(u, v, r)) \subseteq B^{\rho^u}(v, r + CR(u, v, r)),
\]

and (47) is proved.

The inclusion (48) is proved in the same way with the application of (42). \( \square \)

**Theorem 6** (Theorem on divergence of integral lines). *Let \( u, v \in U, \rho(u, v) = O(\varepsilon), r = O(\varepsilon) \) and \( B^\rho(v, r) \cup B^{\rho^u}(v, r) \subseteq U \). Then we have the following estimate on divergence of integral lines from Definition 14:

\[
R(u, v, r) = O(\varepsilon^{1+\frac{1}{\lambda}}).
\]

Proof. For a fixed point \( \hat{y} \in B^{\rho^u}(v, r) \) and \( \zeta > 0 \) we consider arbitrary \( \{b_I\}_{|I|_h \leq M} \) such that

\[
\hat{y} = \exp \left( \sum_{|I|_h \leq M} b_I \hat{X}_I^u(v) \right) \quad \text{and} \quad \max_{|I|_h \leq M} \{|b_I|^{1/|I|_h}\} \leq \rho^u(v, \hat{y}) + \zeta \leq r.
\]

Let \( y = \exp \left( \sum_{|I|_h \leq M} b_I X_I(v) \right) \) and \( v = \exp \left( \sum_{|I|_h \leq M} v_I X_I(v) \right) \in U \). Consider points

\[
\hat{v} = \exp \left( \sum_{|I|_h \leq M} v_I \hat{X}_I(v) \right) \in \hat{U} \quad \text{and} \quad \hat{\hat{y}} = \exp \left( \sum_{|I|_h \leq M} b_I \hat{X}_I(v) \right) \in \hat{\hat{U}}.
\]

Then

\[
\hat{\rho}(\hat{v}, \hat{\hat{y}}) = \max_{|I|_h \leq M} \{|b_I|^{1/|I|_h}\} = O(\varepsilon).
\]

Let \( \hat{\hat{y}} := \exp \left( \sum_{|I|_h \leq M} b_I \hat{X}_I(v) \right) \). Since all points of \( \hat{\hat{U}} \) are regular, from Theorem 3 it follows that

\[
\max \{\hat{\rho}(\hat{y}, \hat{\hat{y}}), \hat{\rho}(\hat{\hat{y}}, \hat{\hat{y}})\} = O(\varepsilon^{1+\frac{1}{\lambda}}),
\]

from where, taking into account Proposition 8, the proposition follows. The application of this theorem is possible due to Proposition 11. \( \square \)
Remark 5. In the paper [61], where Theorem 3 was proved, the nilpotentized vector fields satisfy estimates (11) which are stronger than (44), namely, with $O(\varepsilon)$ in place of $O(1)$ in the last estimate. Here we can not guarantee $O(\varepsilon)$ because, in contrast to the case of regular points, not all of the values of commutators $\hat{X}_I(u)$ at $u$ might coincide with the values $X_I(u)$ (see [25] and references therein). Nevertheless, a revision of the proof of Theorem 3 shows that it holds also with these weaker estimates. Note also that this theorem 3 is true in any coordinates, in which the decomposition (11) or (44) is true.

Theorem 7 (Local approximation theorem). For any points $u \in U$ and $v, w \in U$, such that $\rho(u, v) = O(\varepsilon)$, $\rho(u, w) = O(\varepsilon)$, we have
$$|\rho(v, w) - \rho^u(v, w)| = O(\varepsilon^{1+\frac{1}{M}}).$$

Proof. In Proposition 12 let $r := \rho(v, w)$. Then $w \in B^{\rho}(v, r)$, hence
$$\rho^u(v, w) \leq \rho(v, w) + CR(u, v, r).$$

In the same way, setting $r := \rho^u(v, w)$, we obtain
$$\rho(v, w) \leq \rho^u(v, w) + CR(u, v, r) + O(r^{1+\frac{1}{M}}) + O(R(u, v, r)^{1+\frac{1}{M}}).$$

Due to Proposition 16 (generalized triangle inequality for $\rho$) we have $r = O(\varepsilon)$, since from Theorem 6 the proposition follows.

6 The tangent cone theorems

First we briefly recall the notion and basic properties of convergence of a sequence of quasimetric spaces, as well as the notion of the tangent cone to a quasimetric space, introduced in [49, 50] as an extension of Gromov’s theory for metric spaces.

The distortion (see e.g. [11]) of a mapping $f : (X, d_X) \to (Y, d_Y)$ is the value
$$\text{dis}(f) = \sup_{u, v \in X} |d_Y(f(u), f(v)) - d_X(u, v)|,$$

which is a measure of difference of $f$ from an isometry.

Definition 15 ([49, 50]). The distance $d_{qm}(X, Y)$ between quasimetric spaces $(X, d_X)$ and $(Y, d_Y)$ is defined as the infimum taken over $\rho > 0$ for which there exist (not necessarily continuous) mappings $f : X \to Y$ and $g : Y \to X$ such that
$$\max \left\{ \text{dis}(f), \text{dis}(g), \sup_{x \in X} d_X(x, g(f(x))), \sup_{y \in Y} d_Y(y, f(g(y))) \right\} \leq \rho.$$

Note that for bounded quasimetric spaces the introduced distance is obviously finite.

Proposition 13. The distance $d_{qm}$ possesses the following properties:
1) if quasimetric spaces $X$ and $Y$ are isometric, then $d_{qm}(X, Y) = 0$; if $X$ and $Y$ are compact and $d_{qm}(X, Y) = 0$, then $X$ and $Y$ are isometric (nondegeneracy).
2) $d_{qm}(X, Y) = d_{qm}(Y, X)$ (symmetry).
3) $d_{qm}(X, Y) \leq (Q_Z + 1)(d_{qm}(X, Z) + d_{qm}(Z, Y))$ (analog of the generalized triangle inequality).
Note that the constant in 3) depends on the constant $Q_Z$.

By means of the (quasi)distance $d_{qm}$ a convergence, the limit by which is unique up to isometry, for compact quasimetric spaces can be introduced, in a similar way as it was done for metric spaces. Namely, for a sequence $\{X_n\}$ of compact quasimetric spaces, we say that $X_n \to X$, if $d_{qm}(X_n, X) \to 0$, when $n \to \infty$. Note that a straightforward generalization of Gromov’s definition of the distance $d_{GH}$ between two metric spaces is possible only for a particular class of quasimetric spaces [20].

For noncompact spaces we use the following more general notion of convergence. A pointed (quasi)metric space is a pair $(X, p)$ consisting of a (quasi)metric space $X$ and a point $p \in X$. Whenever we want to emphasize what kind of (quasi)metric is on $X$, we shall write the pointed space as a triple $(X, p, d_X)$.

**Definition 16.** A sequence $(X_n, p_n, d_{X_n})$ of pointed quasimetric spaces converges to the pointed space $(X, p, d_X)$, if there exists a sequence of reals $\delta_n \to 0$ such that for each $r > 0$ there exist mappings $f_{n,r} : B^{d_X}(p, r + \delta_n) \to X$, $g_{n,r} : B^{d_X}(p, r + 2\delta_n) \to X_n$ such that

1) $f_{n,r}(p_n) = p$, $g_{n,r}(p) = p_n$;
2) $\text{dis}(f_{n,r}) < \delta_n$, $\text{dis}(g_{n,r}) < \delta_n$;
3) $\sup_{x \in B^{d_X}(p, r+\delta_n)} d_{X_n}(x, g_{n,r}(f_{n,r}(x))) < \delta_n$.

Recall that a quasimetric space $X$ is boundedly compact, if all closed bounded subsets of $X$ are compact. Two pointed quasimetric spaces $(X, p)$ and $(Y, q)$ are called isometric, if there exists an isometry $\eta : Y \to X$ such that $\eta(q) = p$. The following theorem (see [49, 50] for details) informally states that, for boundedly compact spaces, the limit is unique up to isometry.

**Theorem 8.** 1) Reduced to the case of metric spaces, the convergence of Definition 16 is equivalent to the Gromov-Hausdorff convergence.

2) Let $(X, p)$, $(Y, q)$ be two complete pointed quasimetric spaces obtained as limits (in the sense of definition 16) of the same sequence $(X_n, p_n)$ such that $|Q_{X_n}| \leq C$ for all $n \in \mathbb{N}$. If $X$ is boundedly compact then $(X, p)$ and $(Y, q)$ are isometric.

The tangent cone is then defined as usual:

**Definition 17.** Let $X$ be a boundedly compact (quasi)metric space, $p \in X$. If the limit of pointed spaces $\lim_{\lambda \to \infty} (\lambda X, p) = (T_pX, e)$ (in the sense of definition 16) exists, then $T_pX$ is called the tangent cone to $X$ at $p$. Here $\lambda X = (X, \lambda \cdot d_X)$; the symbol $\lim_{\lambda \to \infty} (\lambda X, p)$ means that, for any sequence $\lambda_n \to \infty$, there exists $\lim_{\lambda_n \to \infty} (\lambda_n X, p)$ which is independent of the choice of sequence $\lambda_n \to \infty$ as $n \to \infty$. A local tangent cone is an arbitrary neighborhood $U(e) \subseteq T_pX$ of fixed point $e \in T_pX$.

**Remark 6.** According to Theorem 8, the tangent cone from Definition 17 is unique up to isometry, i.e. one should treat it as a class of pointed quasimetric spaces isometric to each other. Note also that the tangent cone is isometric to $(\lambda T_pX, e)$ for all $\lambda > 0$ and is completely defined by any (arbitrarily small) neighborhood of the point.
**Theorem 9.** Let $\mathbb{M}$ be a C-C space from Definition 4. Then the quasimetric space $(U, \rho^u)$ is a local tangent cone at the point $u$ to the quasimetric space $(U, \rho)$, where the quasimetrics $\rho$ and $\rho^u$ are defined by (6) and (24), respectively. The tangent cone is a homogeneous space $G/H$, constructed in the proof of the Proposition 6 (here $G$ is a nilpotent graded group).

**Proof.** We have to verify Definition 17 for the spaces $X_n = (U, u, \lambda_n \cdot \rho)$, $X = (U, u, \rho^u)$, where $\lambda_n \to \infty$, $\lambda_n \geq 0$ is an arbitrary sequence of reals (w.l.o.g. we assume $\lambda_n \geq 1$). It is sufficient to take $f_{n,r} = \Delta^u_{\lambda_n}$, $g_{n,r} = \Delta^{u-1}_{\lambda_n}$. Due to the conical property (25) and Theorem 7 we have the first assertion.

To verify the second assertion, we have to verify the left-invariance of $\rho^u$, i.e. to prove that

$$\rho^u(g(v), g(w)) = \rho^u(v, w),$$

(49)

where $g$ is defined in Proposition 6.

Consider a curve $\gamma(t)$ such that

$$\begin{cases} 
\dot{\gamma}(t) = \sum_{|I| \leq M} b_I \tilde{X}^u_I(\gamma(t)), \\
\gamma(0) = v, \gamma(1) = w.
\end{cases}$$

Due to the left-invariance of the vector fields $\{\tilde{X}^u_I\}_{|I| \leq M}$, introduced in the proof of Proposition 6, and the existence of the homomorphism $\Psi(\tilde{X}^u_I) = \tilde{X}^u_I$, the curve $\gamma_g(t) = g(\gamma(t))$ is a solution of the system of equations

$$\begin{cases} 
\dot{\gamma}_g(t) = \sum_{|I| \leq M} b_I \tilde{X}^u_I(\gamma_g(t)), \\
\gamma(0) = g(v), \gamma(1) = g(w).
\end{cases}$$

By definition of the quasimetric $\rho^u$, we get the required assertion.

**Corollary 4.** At a regular point, the tangent cone to a weighted C-C space is a nilpotent graded group.

### 7 The case of Hörmander vector fields

**Definition 18.** The vector fields $\{X_1, \ldots, X_m\} \in C^p$ on $U \subseteq \mathbb{M}$, $m \leq N$, meet Hörmander’s condition of depth $M$, if they span, by their commutators up to the order $M - 1$, the whole tangent space $T_u \mathbb{M}$ at any point $u \in U$, and $M$ is the minimal number with such property.

Obviously, for the case of regular points, $\mathbb{M}$ is an example of a Carnot manifold, see Definition 7. In this paper we assume that $p = 2M + 1$.

The homogeneous degree of the vector field $X_I$ is now equal to its commutator order

$$\text{deg}(X_I) = \text{degalg}(X_I) = |I| = i_1 + \ldots + i_k,$$
and the conditions (ii) and (iii) for the basis \( \{ \gamma \} \) coincide. Introduce the same local coordinates on \( U \) as in \( \{ \, 17 \} \) and construct the nilpotent approximations \( \{ \hat{X}_i \} \mid u \leq M \), as in Proposition \( \{ \, 14 \} \). The lifting construction is also carried out in a similar way as before, see Proposition \( \{ \, 6 \} \). Here we have \( q = m \) and the Lie group of the free algebra \( N \) is a Carnot group. These constructions and results of \( \{ \, 12 \} \) for regular points allow to prove an analog of the Rashevsky-Chow theorem for spaces from Definition \( \{ \, 18 \} \). This result is, however, not new, in particular, the existence of \( d_c \) for the case when \( p = M - 1 \), \( \alpha \) was proved in \( \{ \, 7 \} \) with other methods.

**Theorem 10.** On \( U \) there are finite metrics

\[
d_c(v, w) = \inf_{\gamma(0) = v, \gamma(1) = w} \{ L(\gamma) \} \quad \text{and} \quad d^u_c(v, w) = \inf_{\hat{\gamma}(0) = v, \hat{\gamma}(1) = w} \{ L(\hat{\gamma}) \}. \tag{50}
\]

**Proof.** Consider the manifold \( \bar{M} \) and the vector fields \( \bar{X}_i, \hat{X}_i \) constructed in Proposition \( \{ \, 9 \} \). Due to Remark \( \{ \, 4 \} \) and to the results of \( \{ \, 31 \} \) for regular points, on the neighborhood \( \bar{U} \) there are finite metrics \( \bar{d}_c \) and \( \hat{d}_c \), defined by the horizontal vector fields \( \bar{X}_i \) and \( \hat{X}_i \), respectively.

Denote as \( \pi : \bar{M} \to M \) the canonical projection, i.e. \( \pi(v, z) = v \), where \( v \in M, z \in \mathbb{R}^{N - N} \).

Assume \( \bar{\gamma}(t) : [0, 1] \to \bar{U} \) be a geodesic of the distance \( \bar{d}_c((v, 0), (w, 0)) \). Consider the curve \( \gamma : [0, 1] \to U \) defined as \( \gamma(t) = \pi(\bar{\gamma}(t)) \). Then, in coordinates \( \{ \, 17 \} \), we have

\[
\begin{align*}
\dot{\hat{\gamma}}(t) &= \sum_{i=1}^m a_i(t)\hat{X}_i(\hat{\gamma}(t)) = \sum_{i=1}^m a_i(t) \left[ X_i(\gamma(t)) + \sum_{i=N+1}^\bar{N} b_{ij}(\bar{\gamma}(t)) \frac{\partial}{\partial z_j} \right], \\
\hat{\gamma}(0) &= (v, 0), \quad \bar{\gamma}(0) = (w, 0),
\end{align*}
\]

hence the curve \( \gamma(t) \) connects the points \( v, w \in U \) and is horizontal w.r.t. the vector fields \( X_1, \ldots, X_m \).

The proof for \( \{ \, 50 \} \) is carried out in a similar way, with help of the existence of the metric \( \hat{d}_c \).

Since the vector fields \( \{ \hat{X}_i \} \) are homogeneous of order \(-1\), the metric \( \{ \, 50 \} \) meets the conical property:

\[
d^u_c(\Delta^u_x v, \Delta^u_x w) = \varepsilon d^u_c(v, w). \tag{52}
\]

The next two propositions are proved in the same way as in the “classical” \( C^\infty \)-smooth case \( \{ \, 48 \} \); we write down the proofs for the convenience of the reader.

**Proposition 14.** The projections of the balls w.r.t. the metric \( d_c \) onto the initial neighborhood \( U \subseteq M \) coincide with the balls w.r.t. the metric \( d_c \), i.e.

\[
B^d_c(v, r) = \pi \left( B^{\hat{d}_c}((u, z), r) \right), \tag{53}
\]

where \( u \in U, z \in \mathbb{R}^{N - N}, \pi : \bar{M} \to M \) is a canonical projection \( \pi(v, z) = v \).
Proof. Let \( \tilde{\gamma}(t) : [0, 1] \to \tilde{U} \) be any horizontal curve starting from \((v, z)\). Then
\[
\begin{aligned}
\dot{\tilde{\gamma}}(t) &= \sum_{i=1}^{m} a_i(t)\xi_i(\tilde{\gamma}(t)) + \sum_{i=1}^{N} b_{ij}(\tilde{\gamma}(t)) \frac{\partial}{\partial x_j}, \\
\tilde{\gamma}(0) &= (v, z).
\end{aligned}
\] (54)

Denote \( \gamma(t) = \pi(\tilde{\gamma}(t)) \),
\[
\gamma(t) = \pi(\tilde{\gamma}(t)),
\]
then
\[
\tilde{\gamma}(t) = \begin{pmatrix}
\gamma(t) \\
\tilde{\gamma}_{N+1}(t) \\
\tilde{\gamma}_{N+2}(t) \\
\vdots \\
\tilde{\gamma}_{N+p}(t)
\end{pmatrix}
\] (56)
and
\[
\begin{aligned}
\dot{\gamma}(t) &= \sum_{i=1}^{m} a_i(t)X_i(\gamma(t)), \\
\gamma(0) &= v,
\end{aligned}
\] (57)
i. e. the curve \( \gamma(t) = \pi(\tilde{\gamma}(t)) \) is horizontal w. r. t. the vector fields \( X_1, X_2, \ldots, X_m \) and is of the same length as \( \tilde{\gamma}(t) \), i. e. the projections of horizontal curves on \( \tilde{M} \) are horizontal curves on \( M \).

Conversely, if \( \gamma(t) \) is a horizontal curve on \( M \), a horizontal curve \( \tilde{\gamma}(t) \) on \( \tilde{M} \) can be defined in such way that (55) holds. Indeed, it is sufficient to define \( \tilde{\gamma}(t) \) by (56), where the last \( N - N \) components are computed as the solutions of the Cauchy problem
\[
\begin{aligned}
\dot{\tilde{\gamma}}_{N+j}(t) &= \sum_{i=1}^{m} a_i(t)b_{ij}(\tilde{\gamma}(t)), \\
\tilde{\gamma}_{N+j}(0) &= z_j.
\end{aligned}
\]
In this way, the set of horizontal curves on \( M \) coincides with the set of projections of the horizontal curves on \( \tilde{M} \), hence the equality of balls (53) is true. \( \blacksquare \)

Proposition 15. The projection \( \pi \) is distance-decreasing, i. e. for any points \( v, w \in U \), \( p, q \in \mathbb{R}^{N-N} \) the following inequalities hold:
\[
\begin{aligned}
d_c(v, w) &\leq \tilde{d}_c((v, p), (w, q)), \\
d^n_c(v, w) &\leq \tilde{d}^n_c((v, p), (w, q)).
\end{aligned}
\] (58) (59)

Proof. Denote \( \tilde{v} = (v, p) \), \( \tilde{w} = (w, q) \), \( r = \tilde{d}_c(\tilde{v}, \tilde{w}) \). Obviously, \( \tilde{w} \in \tilde{B}^{d}_c(\tilde{v}, r) \). Since \( w = \pi(\tilde{w}) \), then \( w \in B^{d}_c(v, r) \) due to Proposition [14] from where (58) follows. The inequality (59) is proved in a similar way. \( \blacksquare \)

The sketch of proof of the next theorem is similar to the proof of its analog in [3]; the main difference lies in the method of proof of the divergence of integral lines. In particular, we do not need special polynomial “privileged” coordinates (though the second-order coordinates, as well as coordinates constructed in Proposition [6] are privileged as well) and do not use Newton-type approximation methods.

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**Theorem 11** (Local approximation theorem). For the points $u, v, w \in U$, such that $d_c(u, v) = O(\varepsilon)$ and $d_c(u, w) = O(\varepsilon)$, the following estimate is true

$$|d_c(v, w) - d_c^u(v, w)| = O(\varepsilon^{1+\frac{1}{M}}).$$

**Proof.** Let $\gamma : [0, 1] \to \mathbb{M}$ be a geodesic for the distance $d_c$, i.e.

$$\begin{align*}
\dot{\gamma}(t) &= \sum_{i=1}^{m} a_i(t) X_i(\gamma(t)), \\
\gamma(0) &= v, \quad \gamma(1) = w
\end{align*}$$

and $L(\gamma) = d_c(v, w)$. Consider a curve $\hat{\gamma}(t)$ such that

$$\begin{align*}
\dot{\hat{\gamma}}(t) &= \sum_{i=1}^{m} a_i(t) X_i(\hat{\gamma}(t)), \\
\hat{\gamma}(0) &= v
\end{align*}$$

and denote $\hat{w} = \hat{\gamma}(1)$. Note that the lengths of the curves $\gamma$ and $\hat{\gamma}$ differ on a value of order $O(\varepsilon^2)$ [61]. Consequently,

$$d_c(v, w) = L(\gamma) = L(\hat{\gamma}) + O(\varepsilon^2) \geq d_c^u(v, \hat{w}) \geq d_c^u(v, w) - d_c^u(w, \hat{w}) + O(\varepsilon^2).$$

In a similar way,

$$d_c^u(v, w) \geq d_c(v, w) - d_c(w, \hat{w}) + O(\varepsilon^2).$$

Taking into account Theorem [11] and the estimates (58) we get the required assertion. \[ \square \]

The following tangent cone result is proved in a similar way as Theorem 12 with the help of Theorem 5 and the homogeneity of the vector fields $\tilde{X}_i^u$.

**Theorem 12** ([52]). The metric space $(U, d_c^u)$ is a local tangent cone at $g$ to the metric space $(U, d_c)$. The tangent cone has the structure of a homogeneous space $G/H$, where $G$ is a Carnot group.

If $u$ is a regular point, the tangent cone is isomorphic to a Carnot group.

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