Contractions in Persistence and Metric Graphs

Žiga Virk

Abstract
We prove that the existence of a 1-Lipschitz retraction (a contraction) from a space $X$ onto its subspace $A$ implies the persistence diagram of $A$ embeds into the persistence diagram of $X$. As a tool we introduce tight injections of persistence modules as maps inducing the said embeddings. We show contractions always exist onto shortest loops in metric graphs and conjecture the existence of contractions in planar metric graphs onto all loops of a shortest homology basis. Of primary interest are contractions onto loops in geodesic spaces. These act as ideal circular coordinates. Furthermore, as the Theorem of Adamaszek and Adams describes the pattern of persistence diagram of $S^1$, a contraction $X \to S^1$ implies the same pattern appears in persistence diagram of $X$.

Keywords Persistent homology · Persistence diagram · Metric graph · Rips complex · Lipschitz map · Persistence module

Mathematics Subject Classification 55N31 · 55U05 · 51F30

1 Introduction

Persistent homology is a parameterized version of homology that has received a lot of attention in the past two decades. The inherent nature of the scale parameter results in features of persistent homology that are absent in standard homology: persistent...
homology is stable and also encodes geometric information about the underlying space. Nowadays, the study of persistent homology encompasses a multitude of aspects including algorithmic, geometric, algebro-topological, stochastic, and data analytical points of view. However, despite all the progress little is known about the way persistent homology encodes geometry of the underlying space or how to interpret persistence diagrams, both of which are fundamental questions.

The purpose of this paper is to provide a new interpretation of parts of persistent homology generated via Rips filtrations in terms of the geometry of an underlying space. Given a compact metric space $X$ and a 1-Lipschitz retraction (called a contraction) $X \to A$ onto a subspace $A \subset X$, we show how the persistent homology of $A$ appears within the persistent homology of $X$ itself via the inclusion induced map, see Proposition 4.6. As the main result, we prove that in such a case even the persistence diagram of $A$ appears as a subset of the persistence diagram of $X$ (Corollary 4.8). For the purpose of the latter statement, we introduce tight embeddings of persistence modules and show they induce inclusions on persistence diagrams (Theorem 3.5). This is a property a generic embedding of persistence modules does not possess. We conclude the paper by demonstrating the existence of contractions onto shortest loops in metric graphs (Theorem 5.7) and conjecture contractions on loops of shortest homology basis always exist on planar metric graphs (Conjecture 5.6).

The importance of our results stems from their interpretative capacity. Suppose we are given an elementary subspace, say a simple geodesically closed loop (i.e., a geodesic circle) $\alpha \subset X$ in a Riemannian manifold $X$. If there is a contraction $X \to \alpha$ our results show that the persistence diagram of $\alpha$ is contained in the persistence diagram of $X$, see Fig. 1 for an example. As the persistence diagram of a geodesic $S^1$ is known [1] to consist of odd-dimensional points, the same points also appear in the persistence diagram of $X$ and thus we are able to deduce parts of the latter. Going in the opposite direction, if we are given a persistence diagram of $X$ (potentially as an approximation, via the stability result, arising from a computation of a sample of $X$) which contains the pattern of a persistence diagram of $S^1$, we might expect to find a geodesic circle within $X$. Analogous conclusions may be made for other subspaces of $X$ for which at least a part of persistent homology is known such as certain ellipses and regular polygons, see Related work below.

Besides the mentioned interpretative capacity, our results raise new questions and connections. The first of them is an existence of contractions: it would be of interest to know when such maps $X \to A$ exist, especially onto a simple loop $A = \alpha$. Proposition 5.2 gives a simple required condition in geodesic spaces: $\alpha$ should be a member of a shortest homology basis. Example 5.4 demonstrates that this condition is not sufficient and leads to Conjecture 5.6. Contractions $X \to \alpha$ actually represent 1-Lipschitz cohomology classes in dimension 1 (inspiring Example 5.5) and would represent ideal circular coordinates in the sense of [9]. A related question would be to determine optimal Lipschitz constants of cohomology classes as maps to Eilenberg–MacLane spaces. On the other hand, existence of contractions could be rephrased as extension problems and in fact a particular case of the Kirszbraun theorem [12, 21]: find all $A \subseteq X$ for which the identity on $A$ extends to a 1-Lipschitz map on $X$. 

Springer
The upper left side represents the homotopy type of the Rips filtration of a circle (i.e., odd-dimensional spheres) equipped with a geodesic metric by [1]. The black shape on the right is a two-dimensional torus $X$ and below it is an excerpt from its barcode in dimensions up to 3. There exist contractions $X \to a$ and $X \to b$, yielding the thick odd-dimensional bars corresponding to these two loops by Corollary 4.8. The homology of a torus appears at $r = 0$ by [11, 19] while the later 2- and 3-dimensional bars arising from a geodesic loop $c$ appear by [18]. Note that there is no contraction $X \to c$. The results of [18] require a wide enough neighborhood of a geodesic loop and cannot be used in the case of $b$.

The treatment of this paper is tailored for (open) Rips filtrations and induced persistent homology and homotopy groups. Analogous arguments could be made for closed Rips filtrations, any Cech filtration, and more general settings of persistence modules.

1.1 Related Work

The following are known results about the geometric information encoded in persistent homology.

At small scales, the Rips complexes of tame spaces attain the homotopy type of the underlying space [11, 13, 19]. The entire homotopy type of a Rips filtration is essentially only known in one non-trivial case: $S^1$ [1]. The methods of [1] can be used to extract some further results on ellipses [2] and regular polygons [3]. The entire 1-dimensional persistent homology (and fundamental group) of geodesic spaces has been completely classified in [15, 16]. Paper [18] (and also [20]) contains a local version of the result of this paper: if a subset $A \subset X$ has a sufficiently nice neighborhood, then parts of its persistent homology embed into persistent homology of $X$. The technical assumptions of these results hold for loops $a$ and $c$ of Fig. 1, but not $b$. The assumptions of our main results of this paper are much easier to verify and in some cases hold more...
generally. Overall, persistent homology in dimensions 1, 2, and 3 is known to encode some geodesic circles and shortest 1-homology basis by [15, 18, 20] (and now also
by results of this paper), properties of thick-thin decomposition [4], and filling radius
[14]. On a similar note, the systole of a geodesic space is detected as the first critical
scale of persistent fundamental group [15]. Parts of the persistent homology of certain
spheres have been detected via the stability theorem yielding a counterexample to
Hausmann’s conjecture [17].

2 Preliminaries

We briefly recall the notions used throughout the paper. For extended background
see [10] for persistent homology, [15] for persistent fundamental group of geodesic
spaces, and [6] for persistence diagrams and barcodes.

Given a metric space $X$ and $x \in X$, the open ball around $x$ of radius $r > 0$ is
denoted by $B(x, r)$, while the closed $r$-ball is denoted by $\overline{B}(x, r)$. A map $f : X \to Y$
between metric spaces is $L$-Lipschitz for $L > 0$ if

$$d_Y(f(x), f(y)) \leq L \cdot d_X(x, y), \quad \forall x, y \in X.$$  

A metric space $X$ is geodesic, if for each $x, y \in X$ there is an isometric embedding
$[0, d(x, y)] \to X$ mapping $0 \mapsto x$ and $d(x, y) \mapsto y$. Given a closed subspace
$A \subset X$ of a topological space $X$, a retraction $X$ to $A$ is a map $f : X \to A$ satisfying
$f|_A = id_A$.

Given a metric space $X$ and $r > 0$, the Rips complex $\text{Rips}(X, r)$ is an abstract
simplicial complex with the vertex set being $X$, and a finite $\sigma \subset X$ being a simplex
iff $\text{Diam}(\sigma) < r$.

Given a metric space $X$ and an interval $J \subseteq \mathbb{R}$, the Rips filtration over $J$ is a
collection $\{\text{Rips}(X, r)\}_{r \in J}$ of simplicial complexes along with the simplicial inclusion
maps

$$\rho_{r, r'} : \text{Rips}(X, r) \hookrightarrow \text{Rips}(X, r'),$$

which are identities on vertices for each $r \leq r'$. When $J = (0, \infty)$ we refer to the
filtration simply as the Rips filtration. Given an Abelian group $G$, $n \in \{0, 1, \ldots\}$ and
a basepoint $\bullet \in X$, we apply the homology $H_n(\cdot; G)$ or homotopy group $\pi_n(\cdot, \bullet)$ func-
tor to a filtration to obtain persistent homology groups $\{H_n(\text{Rips}(X, r); G)\}_{r \in J}$ and
persistent homotopy groups $\{\pi_n(\text{Rips}(X, r), \bullet)\}_{r \in J}$. Each of these is also equipped
with induced (and consequently commuting) homomorphisms. These are denoted
by $\rho_{r, r'}^G : H_n(\text{Rips}(X, r); G) \hookrightarrow H_n(\text{Rips}(X, r'); G)$ for persistent homology and
$\rho_{r, r'}^\pi : \pi_n(\text{Rips}(X, r), \bullet) \hookrightarrow \pi_n(\text{Rips}(X, r'), \bullet)$ for persistent homotopy groups.

Given a field $\mathbb{F}$ and an interval $J \subseteq \mathbb{R}$, a persistence module $\mathcal{M}$ over $J$ is a
collection of vector spaces $\{M_r\}_{r \in J}$ and commuting linear bonding maps $\rho_{r, r'} : M_r \to M_{r'}$. Given a field $\mathbb{F}$ and an interval $J' \subset J \subseteq \mathbb{R}$, the interval module $\mathbb{F}_{J'}$ is a
collection of vector spaces $\{V_r\}_{r \in J}$ with
\[ \begin{align*}
\bullet & \quad V_r = \mathbb{F} \text{ for } r \in J'; \\
\bullet & \quad V_r = 0 \text{ for } r \notin J',
\end{align*} \]

and commuting linear bonding maps \( V_r \to V_{r'} \) which are identities whenever possible (i.e., for \( r, r' \in J' \)) and zero elsewhere. Each persistent homology (with coefficients in \( \mathbb{F} \)) of a Rips filtration over \( J \) built upon a compact metric space is a persistence module that decomposes (uniquely up to permutation of the summands) as a direct sum of interval modules (see q-tameness condition in Proposition 5.1 of [7], the property of being radical in [6], and the main result in [6] along with its corollaries for details). The underlying intervals of the said collection of interval modules are called bars and form a multiset called barcode of the persistence module. For each bar, its endpoints form a pair of numbers from \((0, \infty) \cup \{\infty\}\), with the left endpoint being smaller than \( \infty \). These pairs form a multiset called a persistence diagram. For each element of a barcode or a persistence diagram, its multiplicity is the number of repetitions of the element in the said multiset. The persistence diagram of \( n \)-dimensional homology with coefficients in \( \mathbb{F} \) of a compact metric space \( X \) built via open Rips complexes on an open interval \( J \subseteq \mathbb{R} \) is denoted by \( \text{PD}(\{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in J}) \), while the corresponding barcode is \( B(\{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in J}) \). A barcode also encodes the nature of the endpoints of its bars and hence contains more information than a persistence diagram. However, in our setting the nature of the endpoints is “fixed,” see Lemma 3.2, and hence both structures contain the same information.

### 3. Tight Inclusions of Persistence Modules

Fix an open interval \((a, b) \subseteq \mathbb{R}\) and a field \( \mathbb{F} \). Given persistence modules \( \mathcal{M} = \{M_r\}_{r \in J} \) with bonding maps \( \rho^\mathcal{M}_{r, r'} \) and \( \mathcal{N} = \{N_r\}_{r \in J} \) with bonding maps \( \rho^\mathcal{N}_{r, r'} \), an inclusion \( \varphi \) of \( \mathcal{M} \) into \( \mathcal{N} \) is a collection of injective linear maps \( \varphi_r : M_r \to N_r \) commuting with the bonding maps \( \rho^\mathcal{N}_{s, s} \). Inclusions of persistence modules do not induce inclusions of barcodes or persistence diagrams. For example, over \( \mathbb{R}, \mathbb{F}[2,3] \) can be included into \( \mathbb{F}[1,3] \) yet the persistence diagrams are disjoint. Inclusions of persistence diagrams have been shown to only prolong the “embedding bars” to the left (and not to the right) in case of pointwise finite-dimensional persistence modules [5]. Contractions, on the other hand, will be shown to induce embeddings on persistence diagrams. Working toward the proof of this statement, we introduce a particular kind of inclusions of persistence modules that induce inclusions on persistence diagrams.

**Definition 3.1** Fix an open interval \( J \subseteq \mathbb{R} \) and a field \( \mathbb{F} \). Inclusion \( \varphi = \{\varphi_r\}_{r \in J} \) of persistence module \( \mathcal{M} = \{M_r\}_{r \in J} \) into persistence module \( \mathcal{N} = \{N_r\}_{r \in J} \) is tight, if for all \( r' < r \) we have

\[ \text{Im} \rho^\mathcal{M}_{r', r} = M_r \cap \text{Im} \rho^\mathcal{N}_{r', r}. \]

Informally speaking, tight inclusions do not bring back the emergence of homology classes to an earlier scale, but rather include their emergence “tightly.”
Proof

Throughout the proof we consider Assumption $X$ is a compact metric space, $n \in \{0, 1, \ldots\}$, and $\mathbb{F}$ is a field. Then, each bar of $B(\{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in \mathbb{R}})$ is of the form $(a, b]$ or $(a, \infty)$ for some $0 \leq a < b < \infty$.

**Proof**

Take a cycle $\alpha$ from Rips($X, r$) representing a bar. Due to the strict inequality appearing in the definition of Rips complexes, there exists $r' < r$ such that $\alpha$ is also a cycle in Rips($X, r'$). Hence, the bar is open at $a$. The same argument for a nullhomology implies that if $\alpha$ is nullhomologous in some Rips($X, r$), it is also nullhomologous in some Rips($X, r'$) for some $r' < r$. \hfill \square

**Corollary 3.3**

Assume $X$ is a compact metric space, $n \in \{0, 1, \ldots\}$, $\mathbb{F}$ is a field, and $J = (j_1, j_2) \subset \mathbb{R}$ is an open interval. Then, each bar of $B(\{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in J})$ is of the form $(a, b)$ or $(a, j_2)$ for some $j_1 \leq a < b < j_2$.

**Proof**

We can decompose $B(\{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in J})$ into interval modules. Restricting these interval modules to scale span $r \in J$, we obtain a decomposition of $B(\{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in J})$. The proof now follows from Lemma 3.2. \hfill \square

For the sake of clarity of the argument of Theorem 3.5, we state the following simple algebraic lemma. It can be proved straight from the definitions.

**Lemma 3.4**

Let $\mathbb{F}$ be a field and suppose $W \leq V$ are finite dimensional vector spaces over $\mathbb{F}$. Let $q : V \rightarrow V/W$ denote the natural quotient map. Then for each subspace $V' \leq V$, we have:

- $\ker(q|_{V'}) = W \cap V'$, and
- $\dim q(V') = \dim V' - \dim(W \cap V')$.

The following theorem is the main result of this section. It states that tight inclusions of persistence modules induce inclusions of barcodes and persistence diagrams. Its formulation is tailored to our setting although it holds more generally.

**Theorem 3.5**

Fix an open interval $J = (j_1, j_2) \subset \mathbb{R}$ and a field $\mathbb{F}$. Assume inclusion $\varphi = \{\varphi_r\}_{r \in J}$ of persistence module $\mathcal{M} = \{M_r\}_{r \in J}$ into persistence module $\mathcal{N} = \{N_r\}_{r \in J}$ is tight. If both persistence modules arise as the persistent homology of compact metric spaces via open Rips filtrations (and hence admit the interval decompositions), then $B(\mathcal{M}) \subseteq B(\mathcal{N})$ and $P(\mathcal{M}) \subseteq P(\mathcal{N})$.

**Proof**

Throughout the proof we consider $M_r$ to be a subspace of $N_r$ via the inclusion $\varphi$, $\forall r$. By Corollary 3.3, we only have to consider two types of bars. First let us assume $I = (a, b) \subset J$ is a bar we consider.

Choose $t \in (a, b)$. For each $* \in \{\mathcal{M}, \mathcal{N}\}$ define (see [8] or [6] for background) $W^*_t \leq V^*_t \leq N_t$ as follows:

$$V^*_t = \bigcap_{s \in (a, t)} \text{Im} \rho^*_s \cap \bigcap_{s > b} \ker \rho^*_s,$$

\hfill 1 The author has been kindly notified that the lemma already appears as Theorem 8 in [14].

\hfill Springer
\[
W_t^* = \left( \text{Im} \rho_{a,t}^* \cap \bigcap_{s > b} \ker \rho_{t,s}^* \right) + \left( \bigcap_{s \in (a,t)} \text{Im} \rho_{s,t}^* \cap \ker \rho_{t,b}^* \right)
\]

These expressions provide the number of bars of the form \((a',b']\) containing \(t\) as follows:

- \(\dim V_t^*\) is the number of bars with \(a' \leq a\) and \(b' \leq b\);
- The dimension of the first term of \(W_t^*\) is the number of bars with \(a' < a\) and \(b' < b\);
- The dimension of the second term of \(W_t^*\) is the number of bars with \(a' \leq a\) and \(b' < b\).

The multiplicity of bar \((a,b)\) in a barcode of \(* \in \{\mathcal{M},\mathcal{N}\}\) is \(\mu^* = \dim(V_t^*/W_t^*) = \dim V_t^* - \dim W_t^*\). Within this setup we state two claims:

Claim 1 \(V_t^\mathcal{M} \leq V_t^\mathcal{N}\). This claim follows from our assumptions.

Claim 2 \(W_t^\mathcal{N} \cap V_t^\mathcal{M} \subseteq W_t^\mathcal{M}\). Let us prove this claim. As \(V_t^\mathcal{M}\) and \(W_t^\mathcal{N}\) both contain \(W_t^\mathcal{M}\) we have \(W_t^\mathcal{N} \cap V_t^\mathcal{M} \supseteq W_t^\mathcal{M}\). In order to prove the other inclusion, choose \(v \in V_t^\mathcal{M} \cap W_t^\mathcal{N}\):

- For each \(t'\), containment \(v \in \ker \rho_{t', t'}^\mathcal{N}\) implies \(v \in \ker \rho_{t', t'}^\mathcal{M}\);
- For each \(t'\), containment \(v \in \text{Im} \rho_{t', t}^\mathcal{N}\) implies \(v \in \ker \rho_{t', t}^\mathcal{M}\) by the tight inclusion assumption as \(v \in V_t^\mathcal{M} \subseteq M_t\).

Combining these two implications with the conditions that \(v \in W_t^\mathcal{N}\) implies \(v \in W_t^\mathcal{M}\), which proves \(W_t^\mathcal{N} \cap V_t^\mathcal{M} \subseteq W_t^\mathcal{M}\) and thus Claim 2.

Now let \(\mu^\mathcal{M} = \dim V_t^\mathcal{M}/W_t^\mathcal{M}\) and \(\mu^\mathcal{N} = \dim V_t^\mathcal{N}/W_t^\mathcal{N}\) denote the multiplicity of \(I\) in \(\mathcal{M}\) and \(\mathcal{N}\), respectively. Using the above claims and Lemma 3.4 for the quotient map \(q: V_t^\mathcal{N} \to V_t^\mathcal{N}/W_t^\mathcal{N}\), we conclude

\[
\mu^\mathcal{N} = \dim V_t^\mathcal{N}/W_t^\mathcal{N} = \dim q(V_t^\mathcal{N}) \geq \dim q(V_t^\mathcal{M}) = \dim V_t^\mathcal{M} - \dim(W_t^\mathcal{N} \cap V_t^\mathcal{N}) = \dim V_t^\mathcal{M} - \dim W_t^\mathcal{M} = \mu^\mathcal{M}.
\]

hence \(q\) induces an injection on bars of form \(I = (a,b)\).

Intervals of the form \((a, j_2)\) are treated in the same way by choosing \(t \in (a, j_2)\) and defining

\[
V_t^* = \bigcap_{s \in (a,t)} \text{Im} \rho_{s,t}^* ,
\]

\[
W_t^* = \left( \text{Im} \rho_{a,t}^* \right) + \left( \bigcap_{s \in (a,t)} \text{Im} \rho_{s,t}^* \cap \ker \rho_{t,s}^* \right).
\]

\(\square\)
4 Contractions in Persistence

Throughout this section, let $A \subseteq X$ be a closed subspace of a metric space $X$ and let $i : A \hookrightarrow X$ be the associated inclusion.

Definition 4.1 Let $r > 0$. An $r$-contraction of $X$ to $A$ is a retraction $f : X \to A$ for which

$$d(x, y) < r \implies d(f(x), f(y)) < r.$$ 

A map $X \to A$ is a contraction if it is an $r$-contraction for each $r > 0$.

Remark 4.2 Contractions are 1-Lipschitz retractions. It should be apparent that the property of being a contraction is much stronger than the property of being an $r$-contraction for some $r$.

Proposition 4.3 (Contractions induce retractions at single scale) Suppose $f : X \to A$ is an $r$-contraction for some $r > 0$. Then:

1. The induced map $\tilde{f} : \text{Rips}(X, r) \to \text{Rips}(A, r)$ is a simplicial retraction.
2. Map $i$ induces injection on all homology and homotopy groups.

Proof Part (1) follows straight from the definition. In order to prove (2), choose $n \in \{0, 1, \ldots\}$ and a homology element in $H_n(\text{Rips}(A, r); G)$ represented by an $n$-cycle $\alpha$. If $\alpha = \partial \beta$ for some $(n + 1)$-chain in $\text{Rips}(X, r)$, then $\tilde{f}(\beta)$ (obtained by applying $f$ to vertices involved in $\beta$) is an $(n + 1)$-chain in $\text{Rips}(A, r)$ demonstrating that $[\alpha] = 0$, hence the statement holds for homology groups. The proof for homotopy groups is the same using simplicial representatives of maps. \qed

Remark 4.4 Statement (1) of Proposition 4.3 implies that $\tilde{f} \circ \tilde{i} = id|_{\text{Rips}(A, r)}$. This is a particular case of homotopy dominance. Recall that a topological space $W$ is homotopy dominated by a topological space $Z$ if there exist maps $f : Z \to W$ and $g : W \to Z$ such that $f \circ g \simeq 1_W$.

Remark 4.5 Contractions induce retractions on Rips complexes at all scales. In an analogous way, [11] introduced crushings as maps which behave like deformation retraction on Rips complexes. Crushings were used under the name deformation contractions in [18] to prove local variant of the main result of this paper.

In a similar manner, $r$-contractions induce retractions on Rips complexes at scale $r$. Analogous maps are $r$-crushings of [13], which induce deformation retractions on Rips complexes at scale $r$.

Proposition 4.6 (Contractions induce retractions at multiple scales) Let $J \subseteq \mathbb{R}$ be an open interval. Suppose $f : X \to A$ is an $r$-contraction for all $r \in J$. Then:

1. Map $i$ induces injection on all persistent homology and persistent homotopy groups on the interval $r \in J$.
(2) For any $\mathbb{F}$ and $n \in \{0, 1, \ldots\}$, the inclusion

$$\{H_n(\text{Rips}(A, r); \mathbb{F})\}_{r \in J} \hookrightarrow \{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in J}$$

of persistence modules induced by $i$ is tight.

**Proof** Statement (1) is apparent from Proposition 4.3 and definitions. In order to prove statement (2), choose an $n$-cycle $\alpha$ representing an element in $H_n(\text{Rips}(A, r); \mathbb{F})$.

- For each $r' > r$, we have $\rho_{r,r'}^n(\alpha) \in H_n(\text{Rips}(A, r'); \mathbb{F})$ hence $H_n(\text{Rips}(A, r'); \mathbb{F})$ is invariant for $\rho_{r,r'}^n$. In the terminology of Definition 3.1:

  $$\text{Im} \rho_{r,r'}^n(\text{Rips}(A, \ast); \mathbb{F}) \subseteq H_n(\text{Rips}(A, r'); \mathbb{F}) \cap \text{Im} \rho_{r,r'}^n(\text{Rips}(X, \ast); \mathbb{F}).$$

- If $[\alpha] = \rho_{r,r'}^n[\beta]$ for some $n$-cycle $\beta$ representing an element in $H_n(\text{Rips}(X, r'); \mathbb{F})$, then (as in Proposition 4.3), $[\alpha] = \rho_{r,r'}^n[\tilde{f}(\beta)]$ with $\tilde{f}(\beta)$ being an $n$-cycle in $H_n(\text{Rips}(A, r'); \mathbb{F})$. Indeed: if $\alpha - \beta = \partial \gamma$ for some $(n+1)$-chain in $\text{Rips}(X, r)$, then

  $$\partial \tilde{f}(\gamma) = \tilde{f}(\partial \gamma) = \tilde{f}(\alpha - \beta) = \alpha - \tilde{f}(\beta).$$

In the terminology of Definition 3.1, this means

$$\text{Im} \rho_{r,r'}^n(\text{Rips}(A, \ast); \mathbb{F}) \supseteq H_n(\text{Rips}(A, r); \mathbb{F}) \cap \text{Im} \rho_{r,r'}^n(\text{Rips}(X, \ast); \mathbb{F}).$$

The mentioned cases together imply the condition of Definition 3.1. □

**Theorem 4.7** Let $J \subseteq \mathbb{R}$ be an open interval, $X$ a compact metric space, $\mathbb{F}$ a field, and $n \in \{0, 1, \ldots\}$. Suppose $f : X \rightarrow A$ is an $r$-contraction for all $r \in J$. Then, there are inclusions of barcodes and persistence diagrams:

- $\mathcal{B}(\{H_n(\text{Rips}(A, r); \mathbb{F})\}_{r \in J}) \subseteq \mathcal{B}(\{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in J})$ and
- $\mathcal{P}(\{H_n(\text{Rips}(A, r); \mathbb{F})\}_{r \in J}) \subseteq \mathcal{P}(\{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in J}).$

The conclusion of Theorem 4.7 can be stated as follows: the persistence module $\{H_n(\text{Rips}(A, r); \mathbb{F})\}_{r \in J}$ arising from $A$ is a direct summand of the persistence module $\{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in J}$ arising from $X$.

**Proof** By (2) of Proposition 4.6, the inclusion of persistence modules exists and is tight. By Theorem 3.5, the interval decompositions exist and the tight inclusion induces inclusions of barcodes and persistence diagrams. □

**Corollary 4.8** Let $X$ be a compact metric space, $\mathbb{F}$ a field, and $n \in \{0, 1, \ldots\}$. Suppose $f : X \rightarrow A$ is a contraction. Then, there are inclusions of barcodes and persistence diagrams:

- $\mathcal{B}(\{H_n(\text{Rips}(A, r); \mathbb{F})\}_{r \in \mathbb{R}}) \subseteq \mathcal{B}(\{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in \mathbb{R}})$ and
- $\mathcal{P}(\{H_n(\text{Rips}(A, r); \mathbb{F})\}_{r \in \mathbb{R}}) \subseteq \mathcal{P}(\{H_n(\text{Rips}(X, r); \mathbb{F})\}_{r \in \mathbb{R}})$.

With these results, we are able to justify the interpretation of the example provided in Fig. 1.
5 Contractions in Metric Graphs

In this section, we discuss existence of contractions on metric graphs. We are particularly interested in contractions onto loops isometric to a geodesic $S^1$ as these are essentially the only spaces for which we know the entire persistent homology.

**Definition 5.1** Given a geodesic space $X$, a **geodesic circle** is a simple closed loop in $X$, whose subspace metric makes it a geodesic space.

Geodesic circles have been discussed in our context in [15, 18].

**Proposition 5.2** Suppose $X$ is a compact, locally contractible geodesic space. If there exists a contraction $f : X \rightarrow \alpha$ onto a simple closed curve $\alpha \subset X$, then $\alpha$ is a geodesic circle and $[\alpha]$ is a member of a lexicographically shortest homology basis of $H_1(X; G)$ in any coefficients $G$.

**Proof** Let $\ell$ be the length of $\alpha$ and choose an Abelian group $G$. Assume $[\alpha] = [\beta] + [\gamma]$, with $\beta$ and $\gamma$ being loops in $X$ of lengths shorter than $\ell$. Let $\tilde{f} : H_1(X; G) \rightarrow H_1(\alpha; G)$ be the map induced by $f$. Then, computing in $H_1(\alpha; G)$, we have

$$1 = [\alpha] = [f(\alpha)] = \tilde{f}[\alpha] = \tilde{f}[\beta] + \tilde{f}[\gamma] = [f(\beta)] + [f(\gamma)].$$

hence at least one of the last two terms, say $[f(\beta)]$, is non-trivial. This means that the winding number of $f(\beta)$ in $\alpha$ is non-trivial and thus $f(\beta)$ is of length at least $\ell$. But this is a contradiction as a contraction decreases the length and $\beta$ was assumed to be of length less than $\ell$. Hence, $\alpha$ is a member of a lexicographically shortest homology base of $H_1(X; G)$.

By [15], a compact locally contractible geodesic space has a finite lexicographically shortest homology basis of $H_1(X; G)$ in any coefficients $G$, and all members of any such basis are geodesic circles. \hfill \Box

A natural follow-up question is whether the required condition of Proposition 5.2 for the existence of a contraction onto a simple loop is sufficient. We will answer that question in a negative way in the context of metric graphs by Example 5.4.

**Definition 5.3** A **metric graph** is a geodesic space homeomorphic to a finite 1-dimensional simplicial complex.

The following example was suggested by Arseniy Akopyan.

**Example 5.4** Consider two concentric circles of lengths 1000 and 999, along with additional connections as shown by Fig. 2. The inner circle $A$ of length 999 induces a member of a shortest homology basis $[A]$ in homology. However, there is no contraction of this metric graph onto $A$, as any contraction would have to map a loop of length 1993 that goes around most of the inner loop once and around most of the outer loop once, changing between them at the cross at the bottom, twice around $A$. 
Example 5.5 A similar example disproving the converse of Proposition 5.2 can be designed by treating a standard flat Klein bottle $K$ obtained by identifying sides of a square. Recall that $H_1(K; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, meaning that the shortest homology basis has at two elements (corresponding to “horizontal” and “vertical” lines on the defining square). However, due to the shift in torsion we have $H_1(K; \mathbb{Z}) \cong \mathbb{Z}$ and as the elements of the later cohomology groups correspond to homotopy classes of maps to $K(\mathbb{Z}, 1) = S^1$, there is not even a continuous retraction of $K$ onto a loop generating torsion in $H_1(K; \mathbb{Z})$.

In both these examples, we appear to have used the fact that the space is not planar. This motivates the following conjecture.

Conjecture 5.6 Suppose $X$ is a planar metric graph and $\alpha \subset X$ is a simple closed loop such that $[\alpha]$ is a member of a shortest homology basis of $H_1(X; G)$. We conjecture that then there exists a contraction $X \rightarrow \alpha$.

A positive answer to a conjecture could be combined with the main results of this paper and [1] to show that for each member of a shortest homology basis of $H_1(X; G)$ of a planar metric graph $X$ the barcodes of $\{H_\bullet(\text{Rips}(X, r); \mathbb{F})\}_{r > 0}$ contain corresponding odd-dimensional bars in all odd dimensions (similarly to Corollary 5.8).

We end this section by proving that a contraction onto a shortest loop in a metric graph always exists. Throughout the forthcoming proof, we will be using the following simple fact: given an injective path $\gamma$ in a metric graph and another path $\gamma'$ with the same endpoints, if the interiors of $\gamma$ and $\gamma'$ are disjoint, then the paths form a non-contractible loop.

Theorem 5.7 Suppose $X$ is a metric graph and $\alpha \subset X$ is a shortest (non-contractible) loop in $X$. Then, there exists a contraction $X \rightarrow \alpha$.

Proof By Proposition 5.2, $\alpha$ is a geodesic circle. Let $2\ell$ be the length of $\alpha$. Fix a point $a \in \alpha$ and let $a' \in \alpha$ be the point opposite to $a$, i.e., $d(a, a') = \ell$. We may assume
that neither $a$ nor $a'$ is a vertex of $X$. Define

$$U = \bigcup_{x \in a} B(x, d(a, x)) .$$

Furthermore, for each $p \in \alpha$ let $T_p$ denote the set of all points of $U$, whose closest point on $\alpha$ is $p$. Observe that $U = \bigcup_{p \in \alpha} T_p$. Let $\mathcal{P} \subset \alpha$ denote the (finite) subset of all points $p \in \alpha$, for which $T_p$ is not a singleton. Observe that $a' \notin \mathcal{P}$. We proceed by two claims.

Claim 1 $T_p \cap T_q = \emptyset, \forall p \neq q$. If this was not the case there would exist $p \neq q \in \alpha$, and $v \in T_p \cap T_q$. We could then choose geodesics $\gamma_p$ from $v$ to $p$ and $\gamma_q$ from $v$ to $q$. Let $\gamma'$ be a geodesic from $p$ to $q$ along $\alpha$. Concatenating $\gamma_p, \gamma_q$, and $\gamma'$ we obtain a loop $\gamma$. As the interior of $\gamma'$ is disjoint from $\gamma_p$ and $\gamma_q$, the loop $\gamma$ is not contractible. Its length is

$$d(v, p) + d(v, q) + d(p, q) < d(a, p) + d(a, q) + d(p, q) \leq 2 \ell,$$

which contradicts the assumption of $\alpha$ being a shortest non-contractible loop. This proves Claim 1.

Claim 2 For each $p \in \mathcal{P}, T_p$ is a tree. Working toward the proof of the claim we again assume the conclusion does not hold, i.e., we assume there exists a simple closed loop $\beta$ in $T_p$ for some $p \in \alpha$. Let $b \in \beta$ be a point closest to $p$ and let $\gamma_b$ be a geodesic between the two points. As the length of $\beta$ is larger than $2\ell$, we can choose a point $c \in \beta$ such that the length of $\gamma'$, which is defined as a shortest segment along $\beta$ from $b$ to $c$, equals $d(p, a) - d(p, b)$. Let $\gamma_c$ denote a geodesic from $c$ to $p$. As $c \in T_p$, its length is less than $d(a, p)$. Paths $\gamma_b, \gamma'_b$, and $\gamma_c$ form a loop $\gamma$ in $T_p$. Paths $\gamma_b$ and $\gamma'$ only intersect at $a$ by definition and their concatenation is of length $d(p, a)$. Path $\gamma_c$ is shorter and thus $\gamma$ is not contractible. The length of $\gamma$ is

$$d(p, b) + (d(p, a) - d(p, b)) + d(c, p) < d(p, a) + d(p, a) \leq 2 \ell.$$ 

Again, this is a contradiction with our assumptions and thus Claim 2 holds.

We proceed by defining a contraction $f : X \to \alpha$. The map can informally be described as “combing $U$ toward $a$ along $\alpha$,” see Fig. 3. In particular, for $x \in T_p$ define $f(x)$ as the point on the geodesic segment from $p$ to $a$ along $\alpha$ with $d(p, f(x)) = d(p, x)$. By the claims above this defines a continuous map on each $T_p$ and also on $U$. For $x \notin U$, we define $f(x) = a$. We next show $f$ is a contraction. Let $\gamma$ be a geodesic between $x, y \in X$. We can decompose $\gamma$ into segments such that each segment is contained in a single edge of $X$ and also either in $U$ or $X \setminus U$. By definition $f$ maps each segment either via an isometric embedding to $\alpha$ or to a constant map at $a$. Hence, the length of $f(\gamma)$ does not exceed the length of $\gamma$. As $X$ is geodesic this implies $f$ is a contraction (and in particular, continuous).

\[ \square \]

Corollary 5.8 Suppose $X$ is a metric graph and $\alpha \subset X$ is a shortest loop in $X$. Let $\ell$ be the length of $\alpha$. Then for each $k \in \{0, 1, 2, \ldots\}$, the barcode

$$\mathcal{B}([H_{2k+1}(\text{Rips}(X, r); \mathbb{F})]_{r > 0})$$

\[ \copyright \text{Springer} \]
contains a bar \( \left( \frac{\ell}{2k+1}, \frac{(k+1)\ell}{2k+3} \right) \) induced by inclusion \( \alpha \hookrightarrow X \).

**Proof** The statement holds by Theorem 5.7 and Corollary 4.8.

---

**References**

1. Adamaszek, M., Adams, H.: The Vietoris–Rips complexes of a circle. Pac. J. Math. **290**, 1–40 (2017)
2. Adamaszek, M., Adams, H., Reddy, S.: On Vietoris–Rips complexes of ellipses. J. Topol. Anal. **11**, 661–690 (2019)
3. Adams, H., Chowdhury, S., Quinn Jaffe, A., Sibanda, B.: Vietoris–Rips Complexes of Regular Polygons, arXiv:1807.10971
4. Adams, H. Coskunuzer, B.: Geometric Approaches on Persistent Homology, arXiv:2103.06408
5. Bauer, U., Lesnick, M.: Induced matchings and the algebraic stability of persistence barcodes. J. Comput. Geom. **6**(2), 162–191 (2015)
6. Chazal, F., Crawley-Boevey, W., de Silva, V.: The observable structure of persistence modules. Homol. Homot. Appl. **18**(2), 247–265 (2016)
7. Chazal, F., de Silva, V., Oudot, S.: Persistence stability for geometric complexes. Geom. Dedicata **173**, 193 (2014)
8. Crawley-Boevey, W.: Decomposition of pointwise finite-dimensional persistence modules. J. Algebra Appl. **14**(05), 1550066 (2015)
9. de Silva, V., Morozov, D., Vejdemo-Johansson, M.: Persistent cohomology and circular coordinates. Discret. Comput. Geom. **45**, 737–759 (2011)
10. Edelsbrunner, H., Harer, J.L.: Computational Topology. American Mathematical Society Providence, Rhode Island, An Introduction (2010)
11. Hausmann, Jean-Claude.: On the Vietoris–Rips complexes and a cohomology theory for metric spaces. Ann. Math. Stud. 138, 175–188 (1995)
12. Kirszbraun, M.D.: Über die zusammenziehenden und Lipschitzsche Transformationen. Fund. Math. 22, 77–108 (1934)
13. Latschev, J.: Vietoris–Rips complexes of metric spaces near a closed Riemannian manifold. Archiv der Mathematik 77(6), 522–528 (2001)
14. Lim, S., Mémoli, F., Okutan, O.B.: Vietoris–Rips Persistent Homology, Injective Metric Spaces, and the Filling Radius, arXiv:2001.07588, (2020)
15. Virk, Ž: 1-dimensional intrinsic persistence of geodesic spaces. J. Topol. Anal. 12, 169–207 (2020)
16. Virk, Ž: Approximations of 1-dimensional intrinsic persistence of geodesic spaces and their stability. Revista Matemática Complutense 32, 195–213 (2019)
17. Virk, Ž.: A counter-example to Hausmann’s conjecture, Found Comput Math (2021)
18. Virk, Ž: Footprints of geodesics in persistent homology. Mediterr. J. Math. 19(4), 1–29 (2022)
19. Virk, Ž: Rips complexes as nerves and a functorial Dowker-Nerve diagram. Mediterr. J. Math. 18(2), 1–24 (2021)
20. Virk, Ž.: Persistent Homology with Selective Rips Complexes Detects Geodesic Circles, arXiv:2108.07460
21. Wells, J.H., Williams, L.R.: Embeddings and extensions in analysis. Ergebnisse der Mathematik und ihrer Grenzgebiete 84, Springer-Verlag, Berlin, Germany, (1975)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.