OPTIMAL CURVES OF GENUS 3 OVER FINITE FIELDS WITH DISCRIMINANT $-19$

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Abstract. In this work we study the properties of maximal and minimal curves of genus 3 over finite fields with discriminant $-19$. We prove that any such curve can be given by an explicit equation of certain form (see Theorem 5.1). Using these equations we obtain a table of maximal and minimal curves over finite fields with discriminant $-19$ of cardinality up to 997. We also show that existence of a maximal curve implies that there is no minimal curve and vice versa.

1. Introduction

The number of rational points of an irreducible non-singular projective curve $C/F_q$ of genus $g$ satisfy the Hasse-Weil-Serre bound:

$$|\# C(F_q) - q - 1| \leq g[2\sqrt{q}].$$

In case of equality, i.e. the number of rational points of the curve is $q + 1 + g[2\sqrt{q}]$ (resp. $q + 1 - g[2\sqrt{q}]$), then the curve is called maximal over $F_q$ (resp. minimal over $F_q$). We will call such curves optimal over $F_q$.

Let $C$ be an optimal curve of genus $g$ over $F_q$. Then the Frobenius endomorphism induces a homomorphism

$$F : T_l \text{Jac}(C) \otimes \mathbb{Z} \mathbb{Q} \rightarrow T_l \text{Jac}(C) \otimes \mathbb{Z} \mathbb{Q},$$

where $T_l \text{Jac}(C)$ is the projective limit $\lim \rightarrow \text{Jac}(C)[l^n]$. Moreover, if the characteristic polynomial of $\text{Jac}(C)$ splits

$$P_{\text{Jac}(C)}(T) = \prod_{i=1}^{2g}(T - \alpha_i),$$

then the number of rational points on $C$ equals to

$$\# C(F_q) = q + 1 - \sum_{i=1}^{g} \alpha_i = q + 1 - \sum_{i=1}^{g} (\alpha_i + \overline{\alpha_i}),$$

The research the fourth author was supported by the Claude Shannon Institute, Science Foundation Ireland Grant 07/RFP/ENM123.
with $\alpha_{i+g} = \bar{\alpha}_i$. The eigenvalues of the endomorphism Frobenius $F$ have following property: $\alpha_i + \bar{\alpha}_i = -\lfloor 2\sqrt{q} \rfloor$ when $C$ is maximal and $\alpha_i + \bar{\alpha}_i = [2\sqrt{q}]$ when $C$ is minimal. Therefore, if a curve is optimal, then the $L$-polynomial of this curve is determined by

$$L(t) = \prod_{i=1}^{2g}(1 - \alpha_it) = \prod_{i=1}^{g}(1 + [2\sqrt{q}]t + qt^2)$$

for minimal (resp. maximal) case. Then the theory of Honda-Tate shows that the Jacobian $\text{Jac}(C)$ of a maximal (resp. minimal) curve $C$ is isogenous to a product of copies of a maximal (resp. minimal) elliptic curve, i.e. $\text{Jac}(C) \sim E^g$, where $E$ is a maximal (resp. minimal) elliptic curve over a finite field $\mathbb{F}_q$. The isogeny class of $E$ over a finite field $\mathbb{F}_q$ is characterized by the characteristic polynomial of the Frobenius endomorphism of $E$.

We consider the equivalence between the category of ordinary abelian varieties $\text{Jac}(C)$ over $\mathbb{F}_q$ which are isogenous to $E^g$ (hence $E$ is ordinary) and the category of $R$-modules, where $R$ is the ring defined by the Frobenius endomorphism of $E$. In our case let $C$ be a smooth irreducible projective algebraic curve over a finite field $\mathbb{F}_q$ with discriminant $-19$. Therefore $R = \mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{-19})$. Let $\text{Jac}(C)$ be the principal polarized Jacobian variety of $C$ with Theta-divisor $\theta$. By Torelli’s Theorem, the curve $C$ is completely defined by $(\text{Jac}(C), \theta)$, up to a unique isomorphism over an algebraic closure of $\mathbb{F}_q$. Consider the Hermitian module $(\mathcal{O}_K^g; h)$, where $\mathcal{O}_K^g$ is a $\mathcal{O}_K$-module, and $h : \mathcal{O}_K^g \times \mathcal{O}_K^g \to \mathcal{O}_K$ is a Hermitian form. The equivalence of categories is defined by the functor $\mathcal{F} : \text{Jac}(C) \to \text{Hom}(E, \text{Jac}(C))$ and its inverse $\mathcal{V} : \mathcal{O}_K^g \to \mathcal{O}_K^g \otimes_{\mathcal{O}_K} E$. Under this equivalence the principal polarisation of Jacobian $\text{Jac}(C)$ corresponds to an irreducible Hermitian $\mathcal{O}_K$-form $h$. Therefore we can use the classification of unimodular irreducible Hermitian forms in order to study the isomorphism classes of $\text{Jac}(C)$. For a detailed description of this equivalence of categories see the Appendix by J.-P. Serre in [3].

Deligne’s Theorem [1] yields that the number of isomorphism classes of abelian varieties isogenous to $A$ equals the number of isomorphism classes of $R$-modules, which may be embedded as lattices in the $K$-vector space $K^g$, where $K = \text{Quot}(R)$. Since in our case there exists one isomorphism class of such $R$-modules, then there exists a unique isomorphism class of abelian varieties. Therefore Deligne’s Theorem together with [2, 1] show that $\text{Jac}(C)$ is actually isomorphic to $E^g$.

The main result of this paper is putting this theory to practical use. We give a characterization of isomorphism classes of optimal curves of
genus 3 over finite fields with discriminant $-19$ in such a way that we are able to give an explicit description of all such curves. In particular, we produce maximal and minimal curves of genus 3 over finite fields with discriminant $-19$ of cardinality up to 997.

2. Optimal Curves of Genus 1 and 2

2.1. Optimal Elliptic Curves. In this subsection we explore optimal elliptic curves over $\mathbb{F}_q$ and produce concrete calculations for the finite fields $\mathbb{F}_q$ of the discriminant $-19$ and $q \leq 1000$.

The endomorphism ring $\text{End}(E)$ of an elliptic curve $E$ is the set of all isogenies $\phi : E(\mathbb{F}_q) \to E(\mathbb{F}_q)$, with multiplication corresponding to composition. If a curve $E$ has complex multiplication, then by Deuring’s theory [7] the endomorphism ring $\text{End}(E)$ is an order in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$. The theory of complex multiplication and Deuring’s lifting theory give us the following: given a quadratic field $K$, the number of isomorphism classes of elliptic curves over $\mathbb{F}_q$ whose endomorphism rings are isomorphic to the maximal order $\mathcal{O}_K$ is equal the number of ideal classes $h_K$ of $K$.

**Proposition 2.1.** Let $\mathbb{F}_q$ be a finite field with discriminant $-19$. There exist exactly two $\mathbb{F}_q$-isomorphism classes of optimal elliptic curves $E$ over $\mathbb{F}_q$, namely the class of maximal and the class of minimal elliptic curves over $\mathbb{F}_q$.

**Proof.** Deuring’s Theorem provides the existence of maximal and minimal elliptic curves over a finite field with discriminant $-19$. Let $E$ be such a curve. Then $\text{End}_{\mathbb{F}_q}(E)$ contains the ring $\mathbb{Z}[x]/(x^2 + mx + q)$, where $m = \pm 2\sqrt{-19}$ is the trace of the Frobenius endomorphism of $E$. Therefore we have $\text{End}_{\mathbb{F}_q}(E) \cong \mathcal{O}_K \cong \mathbb{Z}[x]/(x^2 + mx + q)$, where $K$ is the imaginary quadratic field $\mathbb{Q}(\sqrt{-19})$ with discriminant $-19$. We have $\mathcal{O}_K = \mathbb{Z}[\frac{-19 + \sqrt{-19}}{2}]$ and Minkowski’s bound is $B_K \approx 2.77$. Then any non-principal ideal class must be representable by an ideal of norm $\leq 2.77$. We verify that $2\mathcal{O}_K$ is a principal ideal to conclude that $h_K = 1$. From the class number and the mass formula (see [6]) it follows that there exists a unique class of isomorphic elliptic curves over $\mathbb{F}_q$. \qed

**Remark 2.2.** Alternatively, we can find the number of $\mathbb{F}_q$-isomorphism classes of elliptic curves over $\mathbb{F}_q$ within a given isogeny class by using the following properties. If two elliptic curves are given by $E : y^2 = x^3 + ax + b$ and $E' : y'^2 = x^3 + a'x + b'$, then $E \cong E'$ over $\mathbb{F}_q$ if and only if the following relations on the coefficients hold: $a' = ac^4$, $b' = bc^6$ for a some $c \in \mathbb{F}_q$. 


Example 2.3. We give examples of maximal and minimal elliptic curves over finite fields over \( \mathbb{F}_q \) with discriminant \(-19\) for all \( q < 1000 \).

| \( q \) | Maximal | Minimal |
|-------|---------|---------|
| 47    | \( y^2 = x^3 + x + 38 \) | \( y^2 = x^3 + 32x + 27 \) |
| 61    | \( y^2 = x^3 + 6x + 29 \) | \( y^2 = x^3 + 32x + 57 \) |
| 137   | \( y^2 = x^3 + x + 36 \) | \( y^2 = x^3 + 61x + 47 \) |
| 277   | \( y^2 = x^3 + 2x + 61 \) | \( y^2 = x^3 + 61x + 47 \) |
| 311   | \( y^2 = x^3 + x + 50 \) | \( y^2 = x^3 + 18x + 308 \) |
| 347   | \( y^2 = x^3 + 2x + 96 \) | \( y^2 = x^3 + 174x + 12 \) |
| 467   | \( y^2 = x^3 + 2x + 361 \) | \( y^2 = x^3 + 234x + 337 \) |
| 557   | \( y^2 = x^3 + 3x + 132 \) | \( y^2 = x^3 + 140x + 295 \) |
| 761   | \( y^2 = x^3 + x + 82 \) | \( y^2 = x^3 + 592x + 454 \) |
| 997   | \( y^2 = x^3 + 6x + 493 \) | \( y^2 = x^3 + 500x + 934 \) |

2.2. Optimal Curves of Genus 2. We start with a proposition which was proved in [8].

Proposition 2.4. Up to an isomorphism over the field \( \mathbb{F}_q \) there exists exactly one maximal (resp. minimal) optimal curve \( C \) of genus 2 over \( \mathbb{F}_q \), viz., the fibered product over \( \mathbb{P}^1 \) of the two maximal (resp. minimal) optimal elliptic curves

\[
E_1 : y^2 = f(x) \quad \text{and} \quad E_2 : y^2 = f(x)(\alpha x + \beta).
\]

Example 2.5. Here we produce examples of elliptic curves \( E_2 \) from the proposition above and maximal curve of genus 2 over the finite field \( \mathbb{F}_q \) of the discriminant \(-19\) and \( q < 1000 \).

| \( q \) | Maximal elliptic curve | Maximal curve of genus two |
|-------|------------------------|---------------------------|
| 47    | \( y^2 = (x^3 + x + 38)(x + 30) \) | \( z^2 = x^6 + 4x^4 + 22x^2 + 33 \) |
| 61    | \( y^2 = (x^3 + 6x + 29)(x + 2) \) | \( z^2 = x^6 + 55x^4 + 18x^2 + 9 \) |
| 137   | \( y^2 = (x^3 + x + 36)(x + 18) \) | \( z^2 = x^6 + 83x^4 + 14x^2 + 77 \) |
| 277   | \( y^2 = (x^3 + 2x + 61)(2x + 80) \) | \( z^2 = 104x^6 + 247x^4 + 185x^2 + 245 \) |
| 311   | \( y^2 = (x^3 + x + 50)(x + 134) \) | \( z^2 = x^6 + 220x^4 + 66x^2 + 19 \) |
| 347   | \( y^2 = (x^3 + 2x + 96)(x + 166) \) | \( z^2 = x^6 + 196x^4 + 84x^2 + 316 \) |
| 467   | \( y^2 = (x^3 + 2x + 361)(x + 47) \) | \( z^2 = x^6 + 326x^4 + 91x^2 + 118 \) |
| 557   | \( y^2 = (x^3 + 3x + 132)(2x + 266) \) | \( z^2 = 209x^6 + 318x^4 + 356x^2 + 421 \) |
| 761   | \( y^2 = (x^3 + 3x + 132)(x + 257) \) | \( z^2 = x^6 + 751x^4 + 288x^2 + 98 \) |
| 997   | \( y^2 = (x^3 + 3x + 132)(x + 760) \) | \( z^2 = x^6 + 711x^4 + 20x^2 + 30 \) |

Note that the corresponding minimal curves of genus 2 can be obtained by twisting of maximal curves.
3. A DEGREE OF A PROJECTION

We can calculate the degree of the maps $C \to E$, obtained via the embedding of $C$ into $\text{Jac}(C) \cong E^g$ and projections onto $E$.

The following result can be found in [8], we include it here with the proof for the sake of completeness. Note that proof relies on the fact that the hermitian lattice corresponding to $\text{Jac}(C)$ is a free $\mathcal{O}_K$-module, which holds in the case when $\mathbb{F}_q$ has discriminant $-19$.

**Proposition 3.1.** Let $C$ be an optimal curve over $\mathbb{F}_q$. Fix an isomorphism $\text{Jac}(C) \cong E^g$ such that the theta divisor corresponds to the hermitian form $(h_{ij})$ on $\mathcal{O}_K^g$ on the canonical lift of $\text{Jac}(C)$. Then degree of the $k$-th projection

$$f_k : C \hookrightarrow \text{Jac}(C) \cong E^g \xrightarrow{pr_k} E$$

equals $\det(h_{ij})_{i,j \neq k}$.

**Proof.** We enumerate factors of the abelian variety $E^g$ by $E_1 \times \ldots \times E_g$, where $E_i = E$, and consider the first projection. The degree of the map $f_1$ equals the intersection number $[C] \cdot [E_2 \times \ldots \times E_g]$. The cohomology class $[C]$ of $C$ in an appropriate cohomology theory is $[\Theta^{g-1}/(g - 1)!]$. Recall that if $L$ is a line bundle on an abelian variety $A$ of dimension $g$ then by the Riemann-Roch theorem one has $(L^g/g!)^2 = \deg(\varphi_L)$, and $\deg(\varphi_L) = \det(r_{ij})^2$, where the matrix $(r_{ij})$ gives the hermitian form corresponding to the first Chern class of the line bundle $L$. Since the hermitian form $(h_{ij})_{i,j \neq 1}$ corresponds to the line bundle $\Theta|_{E_2 \times \ldots \times E_g}$ on the abelian variety $E_2 \times \ldots \times E_g$ the degree of $f_1$ is given by

$$[C] \cdot [E_2 \times \ldots \times E_g] = \frac{1}{(g - 1)!}(\Theta|_{E_2 \times \ldots \times E_g})^{g-1} = \det((h_{ij})_{i,j \neq 1}).$$

□

4. PROPERTIES OF THE AUTOMORPHISM GROUP

In this section we prove that an optimal curve of genus 3 over a finite field with the discriminant is $-19$ is not hyperelliptic. Furthermore, we prove that there exists either a maximal or a minimal curve.

From the table of classification of hermitian modules with discriminant $-19$ along with the Lemma [12] proved that an order of an automorphism group of an optimal curve of genus 3 over a finite field with the discriminant $-19$ is 6.

**Proposition 4.1.** There exists an optimal curve $C$ of genus 3 over $\mathbb{F}_q$, namely the double covering of a maximal or minimal elliptic curve respectively.
Proof. The equivalence of categories as described in the Introduction tells us that a polarization of the Jacobian corresponds to a class of irreducible unimodular hermitian forms. According to the classification \cite{5} of unimodular hermitian modules, there is a unique class of irreducible unimodular hermitian forms. This class can be represented by the unimodular hermitian matrix below.

\[
\begin{pmatrix}
2 & 1 & -1 \\
1 & 3 & \frac{-3 + \sqrt{-19}}{2} \\
-1 & \frac{-3 - \sqrt{-19}}{2} & 3
\end{pmatrix}
\]

Therefore by the Theorem of Oort and Ueno \cite{4}, there exists a unique \(\mathbb{F}_q\)-isomorphism class of optimal curves over \(\mathbb{F}_q\). By Proposition 3.1 the degree of \(f_1 : C \to E\) is equal to the determinant

\[
\text{det} \left( \begin{pmatrix}
3 & -\frac{3 + \sqrt{-19}}{2} \\
-\frac{-3 - \sqrt{-19}}{2} & 3
\end{pmatrix} \right)
\]

which is 2. Hence \(C\) is a double covering of an optimal elliptic curve, as desired. \[\square\]

Now we show an optimal curve of genus 3 is not hyperelliptic.

**Lemma 4.2.** Let \(C\) be an optimal curve of genus 3 over a finite field \(\mathbb{F}_q\) with discriminant \(-19\). Then \(C\) is non-hyperelliptic.

**Proof.** For the sake of contradiction suppose that \(C\) is a hyperelliptic curve. Then there are two involutions, the first involution \(\tau\) is the hyperelliptic involution and the second involution \(\sigma\) corresponds to the double cover \(f_1 : C \to E\) from the previous proposition. So \(C/\langle \sigma \rangle\) is an optimal elliptic curve and \(C/\langle \sigma \tau \rangle\) is a projective line. The subgroup \(\langle \sigma, \tau \rangle\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) and we have the following diagram of coverings

\[
\begin{array}{c}
\text{C/} \langle \sigma \rangle \cong E \\
\text{C/} \langle \sigma \tau \rangle \cong \mathbb{P}^1
\end{array}
\]

Furthermore the formal relation of groups

\[
2 \cdot \frac{1}{4} \{id, \tau, \sigma, \sigma \tau\} + \{id\} = \frac{1}{2} \{id, \sigma\} + \frac{1}{2} \{id, \tau\} + \frac{1}{2} \{id, \sigma \tau\}
\]
implies the relation between idempotents in \( \text{End}(\text{Jac}(C)) \) (see [2]) and therefore we have an isogeny

\[
(4.1) \quad \text{Jac}(C) \sim \text{Jac}(C/\langle \sigma \rangle) \times \text{Jac}(C/\langle \sigma \circ \tau \rangle).
\]

From the isogeny above and Hurwitz’ formula, it follows that \( C \to C/\langle \sigma \rangle \) is an unramified double covering. Therefore the number of rational points \( #C(F_q) \) is even. On other hand \( #C(F_q) = q + 1 \pm 3m \) is odd since \( m \) is odd. \( \square \)

The next lemma shows the relation between minimal and maximal curves.

**Lemma 4.3.** Let \( F_q \) be a finite field with discriminant \(-19\). Then \( F_q \) cannot admit minimal and maximal curves simultaneously.

**Proof.** Suppose there exist a maximal curve \( C_M \) and a minimal curve \( C_m \) over \( F_q \). Then \( \text{Jac}(C_M \times_{F_q} F_{q^2}) \cong \text{Jac}(C_m \times_{F_q} F_{q^2}) \) and hence we have an \( F_{q^2} \)-isomorphism \( (C_M \times_{F_q} F_{q^2}) \cong (C_m \times_{F_q} F_{q^2}) \).

We denote \( C_M \times_{F_q} F_{q^2} \) by \( C \). Then there are automorphisms \( F_M \) and \( F_m \) on \( C \) which are induced by corresponding Frobenius endomorphisms. In other words if \( F_q(C_M) \cong F_q(x, y) \) and \( F_q(C_m) \cong F_q(u, w) \subset F_{q^2}(C) \) then \( F_{q^2}(C) = F_q(x, y) \),

\[
F_M : \begin{cases}
F_{q^2}(C) &\longrightarrow F_{q^2}(C) \\
\sum_{\alpha_j} x_i y_j &\longmapsto \sum_{\alpha_j q^i} x_i y_j \\
\sum_{\beta_j} x^i y^m &\longmapsto \sum_{\beta_j q^i} x^i y^m,
\end{cases}
\]

and

\[
F_m : \begin{cases}
F_{q^2}(C) = F_{q^2}(u, w) &\longrightarrow F_{q^2}(u, w) \\
\sum_{\alpha_j} u_i w_j &\longmapsto \sum_{\alpha_j q^i} u_i w_j \\
\sum_{\beta_j} w^i y^m &\longmapsto \sum_{\beta_j q^i} w^i y^m.
\end{cases}
\]

From the construction of the automorphisms \( F_M, F_m \) it follows that the quotient curves \( C/\langle F_M \rangle \) and \( C/\langle F_m \rangle \) are defined over \( F_q \) and

\[
F_q(C/\langle F_M \rangle) = F_{q^2}(C)^{(F_M)} = F_q(C_M),
\]

\[
F_q(C/\langle F_m \rangle) = F_{q^2}(C)^{(F_m)} = F_q(C_m).
\]

The automorphisms \( F_m \) and \( F_M \) induce automorphisms on \( \text{Jac}(C) \) which we, by abuse of notation, denote by \( F_m \) and \( F_M \), respectively. In \( \text{End}_{F_{q^2}}(\text{Jac}(C)) \) we have the relation \( F_m^2 = F_M^2 \) and hence \( F_m = -F_M \), since the two are distinct. On the other hand the automorphism \( F_m \) and \( F_M \) induce two different automorphisms of \( C \). Therefore by Torelli’s Theorem \( C \) must be a hyperelliptic curve. But we showed that this is impossible in Lemma 4.2. \( \square \)
In this section we combine the theoretical results which we derived in order to produce optimal curves of genus 3 over finite fields with discriminant $-19$.

**Theorem 5.1.** Let $C$ be an optimal curve over $\mathbb{F}_q$. Then $C$ can be written in one of the following forms:

\[
\begin{cases}
  z^2 = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \beta_0 y,
  \\
y^2 = x^3 + ax + b,
\end{cases}
\]

\[
\begin{cases}
  z^2 = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + (\beta_0 + \beta_1 x)y,
  \\
y^2 = x^3 + ax + b,
\end{cases}
\]

\[
\begin{cases}
  z^2 = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + (\beta_0 + \beta_1 x)y,
  \\
y^2 = x^3 + ax + b,
\end{cases}
\]

with coefficients in $\mathbb{F}_q$ and the equation $y^2 = x^3 + ax + b$ corresponding to an optimal elliptic curve.

**Proof.** Let $C$ be an optimal curve of genus 3 over a finite field $\mathbb{F}_q$ and let $f : C \to E$ be a double covering of $C$ with the equation $y^2 = x^3 + ax + b$. Set $D = f^{-1}(\infty') = \sum_{P|\infty'} e(P|\infty') \cdot P \in \text{Div}(C)$, where $\infty' \in E$ lies over $\infty \in \mathbb{P}^1$ by the action $E \to \mathbb{P}^1$, $\deg D = 2$.

By Riemann-Roch Theorem

$$\dim D = \deg D + 1 - g + \dim(W - D) = \dim(W - D),$$

where $W$ is a canonical divisor of the curve $C$. Consequently, $D$ is equivalent to the positive divisor $W - D_1$, where $\deg D_1 = 2$. Conclude $\dim D = \dim(W - D) < \dim W = 3$. Taking into account that $C$ is a non-hyperelliptic curve and $\deg D = 2$, we conclude $\dim D = 1$.

Consider the divisor $2D$. By Clifford’s Theorem

$$\dim 2D \leq 1 + \frac{1}{2}\deg 2D.$$ 

Therefore, $\dim 2D \leq 3$.

We separate the proof into three cases.

(1) Suppose $\dim 2D = 3$.

Then there exist linearly independent elements $1, x, z' \in L(2D)$.

Seven elements $1, x, x^2, y, z', (z')^2, zx$ lie in the vector space $L(4D)$. Since $\dim 4D = 6$, then there exists relation

$$a_1 z'^2 + a_2 z' + a_3 z'x = a_4 + a_5 x + a_6 x^2 + a_7 y,$$
where \(a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \mathbb{F}_q\). Recall that \(a_1 \neq 0\), otherwise the equation for \(z'\) over \(k(x, y)\) will be of degree 1, which is a contradiction, since \([k(C) : k(x, y)] = 2\). Dividing both parts of the equation by \(a_1\) and making the substitution \(z = z' + (\frac{a_2}{a_1} + \frac{a_3}{a_1}x)/2\), we obtain the equation
\[
z^2 = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \beta_0 y.
\]

(2) Suppose \(\dim(2D) = 2\) and \(D = Q_1 + Q_2\), where \(Q_1 \neq Q_2\), \(Q_1, Q_2 \in C(\mathbb{F}_q)\).

Then we have \(\dim(2D + Q_1) = 3\), by Riemann-Roch Theorem. The elements \(1, x, x^2, y, z, z^2, xz, yz, xz^2 \in L(4D + 2Q_1)\) are linearly dependent since \(\dim(4D + 2Q_1) = 8\) and \(x \in L(2D), z \in L(2D + Q_1), y \in L(3D)\). Therefore,
\[
z^2(\alpha_0 + \alpha_1 x) + z(\beta_0 + \beta_1 x + \beta_2 y) + (\gamma_0 + \gamma_1 x + \gamma_2 x^2 + \delta y) = 0.
\]

Denoting the expressions in brackets by \(\varphi_1, \varphi_2, \varphi_3\) respectively, we rewrite the expression above as
\[
z^2 \varphi_1 + z \varphi_2 + \varphi_3 = 0.
\]

Knowing that \(\varphi_1 = \alpha_0 + \alpha_1 x \neq 0\) (otherwise \(v_{P_1}(x) = 0\)) and the equation above can be rewritten as
\[
(z + \frac{\varphi_2}{2\varphi_1})^2 + \frac{\varphi_3}{\varphi_1} - \frac{\varphi_2^2}{4\varphi_1^2} = 0.
\]

After appropriate substitutions we get the desired equation
\[
z^2 = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + (\beta_0 + \beta_1 x)y.
\]

(3) Suppose \(\dim(2D) = 2\) and \(D = Q_1 + Q_2 = 2Q\), where \(Q_1 = Q_2 = Q \in C(\mathbb{F}_q)\).

In order to manage this case we prove that the elements \(1, x, z, y, x^2, z^2, xy, xz\) are linearly dependent. As a corollary of this fact we obtain the equation of the second type.

In this case the functions \(x \in L(2D), y \in L(3D)\) have pole divisors \((x)_\infty = 4Q, (y)_\infty = 6Q\), and there is a function \(z \in L(2D + Q)\) such that \((z)_\infty = 5Q\).

The element \(z\) is an integral element over \(\mathbb{F}_q[x, y]\). Indeed, either
\[
1, x, z, y, x^2, z^2, xy, xz \in L(10D)
\]
or
\[
1, x, y, z, x^2, zx, xy, z^2, zy, x^3, zz^2, xyz, z^3 \in L(15Q)
\]
are linearly dependent and in both cases we have relations with nonzero leading coefficients at $z$. This yields that $z$ is integral over $\mathbb{F}_q[x, y]$.

It is clear that $z \not\in \mathbb{F}_q(x, y)$ (otherwise $2$ divides $v_Q(z) = 5$).

The minimal polynomial of $z$ has degree $2$ and coefficients in $\mathbb{F}_q[x, y]$, since the degree of extension $[\mathbb{F}_q(C) : \mathbb{F}_q(x, y)]$ is $2$. Therefore we have that

$$z^2 + \sum_{i \geq 0} a_i z y x^i + \sum_{j \geq 0} b_j z x^j + \sum_{l \geq 0} c_l x^l + \sum_{s \geq 0} d_s y x^s = 0,$$

and hence

$$(5.1)$$

$$z^2 + c_0 + c_1 x + c_2 x^2 + d_0 y + b_0 z + b_1 z x + d_1 y x =$$

$$= -z(b_2 x^2 + \ldots) + zy(a_0 + a_1 x + \ldots) + (c_4 x^4 + \ldots) + y(d_2 x^2 + \ldots).$$

Furthermore, we have

- $v_Q(z x^i) = -5 - 4i \equiv 3 \mod 4$
- $v_Q(z y x^i) = -5 - 6 - 4i \equiv 1 \mod 4$
- $v_Q(x^l) = -4l \equiv 0 \mod 4$
- $v_Q(y x^i) = -6 - 4i \equiv 2 \mod 4$.

If the right part of the equation $5.1$ is non-zero, then we can apply the strict triangle inequality. As a consequence we get that on the one hand

$$v_Q(z^2 + c_0 + c_1 x + c_2 x^2 + d_0 y + b_0 z + b_1 z x + d_1 y x) \leq -11$$

and on the other hand

$$v_Q(z^2 + c_0 + c_1 x + c_2 x^2 + d_0 y + b_0 z + b_1 z x + d_1 y x) \geq -10.$$ 

Therefore the right part of the equation above is zero, i.e. the elements $1, x, z, y, x^2, z^2, xy, xz$ are linearly dependent.

□

Example 5.2. We produce examples of optimal curves over finite fields with discriminant $-19$. It suffices to find either a maximal or a minimal curve as their existence is mutually exclusive.
| $q$ | Maximal optimal curve | Minimal optimal curve |
|-----|-----------------------|-----------------------|
| 47  | $y^2 = x^3 + x + 38$, $z^2 = 10x^2 + 46x + 39 + y$ | - |
| 61  | $y^2 = x^3 + 6x + 29$, $z^2 = x^2 + 54x + 38 + 3y$ | - |
| 137 | $y^2 = x^3 + x + 36$, $z^2 = 3x^2 + 95x + 92 + 10y$ | - |
| 277 | $y^2 = x^3 + 2x + 61$, $z^2 = x^2 + 33x + 212 + 5y$ | - |
| 311 | $y^2 = x^3 + 18x + 308$, $z^2 = 11x^2 + 222x + 32 + 65y$ | - |
| 347 | - $y^2 = x^3 + 174x + 12$, $z^2 = 2x^2 + 310x + 219 + 94y$ | - |
| 467 | $y^2 = x^3 + 2x + 361$, $z^2 = 2x^2 + 381x + 242 + 159y$ | - |
| 557 | - $4y^2 = x^3 + 2x + 151$, $z^2 = 439 + 322x + 5x^2 + 122y$ | - |
| 761 | $y^2 = x^3 + 4x + 105$, $z^2 = 406 + 131x + 3x^2 + 247y$ | - |
| 997 | - $y^2 = x^3 + 500x + 934$, $z^2 = x^2 + 336x + 564 + 196y$ | - |

References

[1] Pierre Deligne. Variétés abéliennes ordinaires sur un corps fini. *Invent. Math.*, 8:238–243, 1969.
[2] E. Kani and M. Rosen. Idempotent relations and factors of Jacobians. *Math. Ann.*, 284(2):307–327, 1989.
[3] Kristin Lauter. The maximum or minimum number of rational points on genus three curves over finite fields. *Compositio Math.*, 134(1):87–111, 2002. With an appendix by Jean-Pierre Serre.
[4] Frans Oort and Kenji Ueno. Principally polarized abelian varieties of dimension two or three are Jacobian varieties. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 20:377–381, 1973.
[5] Alexander Schiemann. Classification of Hermitian forms with the neighbour method. *J. Symbolic Comput.*, 26(4):487–508, 1998.
[6] Gerard van der Geer and Marcel van der Vlugt. Supersingular curves of genus 2 over finite fields of characteristic 2. *Math. Nachr.*, 159:73–81, 1992.
[7] William C. Waterhouse. Abelian varieties over finite fields. *Ann. Sci. École Norm. Sup. (4)*, 2:521–560, 1969.
[8] A. Zaytsev. Optimal curves of low genus over finite fields, 2007. [http://arxiv.org/abs/0706.4203](http://arxiv.org/abs/0706.4203)

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