THE POINCARÉ INEQUALITY DOES NOT IMPROVE WITH BLOW-UP

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ABSTRACT. For each \( \beta > 1 \) we construct a family \( F_\beta \) of metric measure spaces which is closed under the operation of taking weak-tangents (i.e. blow-ups), and such that each element of \( F_\beta \) admits a \((1, P)\)-Poincaré inequality if and only if \( P > \beta \).

Contents

1. Introduction 1
2. Construction of doubling graphs 4
3. Construction of good walks 10
4. The exponents for which the Poincaré inequality holds 16
5. Stability under blow-up 33
References 37

1. Introduction

Background. The abstract Poincaré inequality was introduced in [HK98] in the study of quasiconformal homeomorphisms of metric measure spaces where points can be connected by good families of rectifiable curves. The investigation of PI-spaces, i.e. metric measure spaces equipped with doubling measures and which admit a \((1, P)\)-Poincaré inequality for some \( P \in [1, \infty) \), has been object of intensive research.

One trend of investigation has focused on the infinitesimal structure of such spaces. For example, Cheeger [Che99] formulated a generalization of the classical Rademacher Differentiation Theorem which holds for PI-spaces and showed that in such spaces the infinitesimal geometry of Lipschitz maps is rather constrained. Moreover, this result has allowed to extend notions of differential geometry, like tangent and cotangent bundles, to a large class of nonsmooth spaces which includes Carnot groups [Jer86], spaces with synthetic Ricci lower bounds [Raj12], some inverse limit systems of cube complexes [CK15], and boundaries of certain Fuchsian buildings [BP99]. There are also more complicated examples which involve gluing constructions [HH00, HK98] and passing to subsets [MTW13]. However, the infinitesimal geometry of all these examples is rather special, in the sense that a generic tangent/blow-up is biLipschitz equivalent to a product of Carnot groups with an

2010 Mathematics Subject Classification. 31E05, 28A80.

The author was supported by the “ETH Zurich Postdoctoral Fellowship Program and the Marie Curie Actions for People COFUND Program”. 
inverse limit systems of cube complexes as in [CK15]. In general, little is thus known about the infinitesimal structure of PI-spaces; nevertheless, recent progress on the topic has been achieved in [CKS15], whose results imply that a version of metric differentiation holds of PI-spaces, and that for a typical blow-up \((Y, \nu)\) of a PI-space the measure \(\nu\) admits a Fubini-like representation in terms of unit speed geodesics in \(Y\).

Another line of investigation has focused on the study of the properties of the Poincaré inequality that depend on the exponent \(P\). For \(\Delta > 0\), a \((1, P)\)-Poincaré inequality is stronger than a \((1, P + \Delta)\)-Poincaré inequality in the sense that the former implies the latter; moreover, one can use gluing constructions to produce examples of spaces which admit a \((1, P)\)-Poincaré inequality but not a \((1, P - \Delta)\)-Poincaré inequality for some \(\Delta > 0\). Intuitively, in a space admitting a \((1, P)\)-Poincaré inequality any pair of points can be connected by a nice family of rectifiable curves, and the quality of these connections improves as \(P\) decreases.

We mention two areas of research where understanding the exponent \(P\) is important. One is the study of quasiconformal maps. For example in [KM98] it is shown that if \(\varphi : X \to Y\) is a quasiconformal, where \(X\) and \(Y\) are metric measure spaces satisfying some regularity assumptions (in particular \(X\) is assumed to be \(Q\)-Ahlfors regular), if \(X\) admits a \((1, P)\)-Poincaré inequality for \(P \in [1, Q]\), so does \(Y\). However, in [KM98] it is also shown that this is not the case if \(P > Q\).

A second area is the study of the regularity of minimizers and quasiminimizers of the \(P\)-Dirichlet energy (see for instance [KS01, KM02]); in this setting it is usually necessary to assume a \((1, P - \Delta)\)-Poincaré inequality for some \(\Delta > 0\).

Given a doubling metric measure space \((X, \mu)\) we denote by \(I_{PI}(X, \mu)\) the largest range of exponents \(P \geq 1\) such that \((X, \mu)\) admits a \((1, P)\)-Poincaré inequality. An open question in analysis, even for metric spaces which can be isometrically embedded in some Euclidean space, was whether \(I_{PI}(X, \mu)\) is an open ray of the form \((\beta, \infty)\). This question was answered in the affirmative in [KZ08].

**Main Result.** As remarked above, as of today there is only one known class of models for the infinitesimal geometry of PI-spaces, i.e. biLipschitz deformations of products of Carnot groups and inverse limit systems of cube complexes as in [CK15]. These examples always admit a \((1, 1)\)-Poincaré inequality; while at a conference at IPAM (2013) we learned from Le Donne of a question of Keith about whether a \((1, P)\)-Poincaré inequality improves to a \((1, 1)\)-Poincaré inequality by taking tangents. We answer this in the negative and produce new models for the infinitesimal geometry of a PI-space.

**Theorem 1.1.** There is a doubling metric space \(X\) such that, for each \(P_c \in (1, \infty)\) there exists a doubling measure \(\mu_{P_c}\) on \(X\) such that \((X, \mu_{P_c})\) and any of its weak tangents admit a \((1, P)\)-Poincaré inequality if and only if \(P > P_c\). The space \(X\) has Assouad-Nagata dimension 1, and there is a Lipschitz function \(\pi : X \to \mathbb{R}\) such that \((X, \mu_{P_c})\) has a unique differentiability chart \((X, \pi)\) (i.e. the analytic dimension is 1).

An interesting feature of this example is that the measures \(\{\mu_{P_c}\}_{P_c}\) can be taken mutually singular. The existence of \((1, 1)\)-Poincaré inequalities for mutually singular measures was observed recently [Sch15] in connection with the fact that Cheeger’s differentiation theorem is not sharp.
Recent examples of spaces which admit $(1, P)$-Poincaré inequalities but not $(1, P - \Delta)$-Poincaré inequalities have been constructed in [MTW13, DS13]. However, such examples are rectifiable, and so do not provide new infinitesimal geometries. Of particular interest are the examples of [DS13] which show that Cheeger’s minimal upper gradient depends on the choice of the exponent $P$ (i.e. if one has a $(1, P)$-Poincaré inequality but not a $(1, P - \Delta)$-Poincaré, the $P$-minimal upper gradient and the $(P - \Delta)$-minimal upper gradient can be different). One may check that this is not the case for our examples; this is unavoidable in the context of taking blow-ups: this will be discussed in a forthcoming paper where we show that blow-ups of differentiability spaces are differentiability spaces.

Overview. We observed that to produce new examples for the infinitesimal geometry of PI-spaces one might consider an inverse limit of square complexes where the gluing locus has 0 1-capacity [CK15, Example 11.13]. However, such examples would have analytic and Assouad-Nagata dimension 2, and would not give access to the full range of exponents $P_c$. Moreover, the arguments in [CK15] would not carry over and one would have to resort to modulus estimates.

We thus decided to obtain $X$ as an asymptotic cone of a metric graph $G$ so that the stability under blow-up would be already built in the model. Note that one might also realize $X$ as an inverse limit of a system of metric measure graphs, but it would not satisfy the same axioms as the inverse systems in [CK15]. Specifically, Axiom (2) in [CK15], i.e. the requirement that simplicial projections are open, would fail and the analysis in [CK15] would not carry over.

In Section 2 we construct the graph $G$ and the corresponding measure $\mu_G$ in function of some parameters. The choices for the weights on the measure will produce the different measures $\mu_{P_c}$. We then make a study of the shape of balls.

In Section 3 we construct good quasigeodesics that connect pairs of points in $G$. For convenience, we focus on the construction of good walks.

Section 4 contains the technical part of the paper. We establish modulus estimates to prove/disprove the Poincaré inequality in $G$ for a given choice of $P$. In this section we also recall the definition of modulus and a “geometric” characterization of the Poincaré inequality in terms of random curves.

In Section 5 we define asymptotic cones and complete the proof of Theorem 1.1. In passing information from $G$ to $X$ we take advantage of a discretization procedure in [GL14].

A future line of investigation will involve the study of examples where in $G$ we have only one set $\Theta$ of labels: these possess worse connectivity properties than the examples discussed here.

Notational conventions. We use the convention $a \approx b$ to say that $a/b, b/a \in [C^{-1}, C]$ where $C$ is a universal constant; when we want to highlight $C$ we write $a \approx_C b$. We similarly use notations like $a \lesssim b$ and $a \gtrsim_C b$. In the following $C$ often denotes an unspecified universal constant (that can change from line to line) which can be explicitly estimated. We use the notation $E[\varphi]$ to denote the expectation of the random variable $\varphi$. The notation $B(A, r)$ denotes a ball of radius $R$ centred on the set $A$, i.e. the set of points $p$ at distance $< r$ from the set $A$.

Acknowledgements. We thank E. Le Donne for bringing Keith’s question to our awareness, and NYU/UCLA for funding the trip to IPAM.
2. Construction of doubling graphs

2.1. Choice of parameters. We choose some parameters to construct the metric space $X$:

(P1): An integer $N \geq 2$;
(P2): A sequence of integers $\{m_k\}_k$: $m_k \in \{2, \cdots, N\}$ for each $k$;
(P3): Two finite sets of symbols $\text{Symb}_1$, $\text{Symb}_2$ with $\# \text{Symb}_1 \geq 3$ and $\# \text{Symb}_2 \geq 2$. The sets $\text{Symb}_1$ and $\text{Symb}_2$ share one symbol $\emptyset$ which we will call the end symbol; the set $\text{Symb}_1$ contains another symbol $\spadesuit$ which we will call the gluing symbol.

The space $X$ will be kind of self-similar in the sense that in order to analyze its geometry we will use only a sequence of scales. We thus introduce the scales $\sigma_k = \prod_{j=1}^k m_j$.

To construct the metric space $X$ we will take an asymptotic cone of an infinite graph $G$, see Section 5. For most of the paper we will work with $G$, which is obtained as follows. We let $\Lambda$ (resp. $\Theta$) denote the set of labels on $\text{Symb}_1$ (resp. $\text{Symb}_2$), i.e. the infinite strings $\lambda = \{\lambda(n)\}$ (resp. $\theta = \{\theta(n)\}$) where $\lambda(n) \in \text{Symb}_1$ (resp. $\theta(n) \in \text{Symb}_2$) and $\lambda(n)$ (resp. $\theta(n)$) is eventually the end symbol. We now regard $\mathbb{R}$ as a graph whose vertices are the elements of $\mathbb{Z}$; using the scales $\sigma_k$ we associate to each $m \in \mathbb{Z}$ an order $\text{ord}(m)$ by the formula:

\begin{equation}
\text{ord}(m) = \begin{cases} 
0 & \text{if } m = 0 \\
\max\{k : \sigma_k \text{ divides } |m|\} & \text{otherwise.}
\end{cases}
\end{equation}

**Definition 2.2.** Consider the disconnected graph $\mathbb{R} \times \Lambda \times \Theta$ and a vertex $v = (m, \lambda, \theta)$. We say that $v$ is a gluing point of order $t$ if $\text{ord}(m) = t > 1$ and at least one symbol in $\{\lambda(j)\}_{j<t}$ is not the gluing symbol. We say that $v$ is a socket point of order $t$ if $\text{ord}(m) = t$ and $\lambda(j)$ is the gluing symbol for $j < t$. Note that a vertex with $\text{ord}(m) = 1$ is always a socket point.

The graph $G$ is obtained from $\mathbb{R} \times \Lambda \times \Theta$ by gluing pairs of vertices $(m_1, \lambda_1, \theta_1), (m_2, \lambda_2, \theta_2) \in (\mathbb{Z} \times \Lambda \times \Theta)^2$ if either:

- $(m_1, \lambda_1, \theta_1), (m_2, \lambda_2, \theta_2)$ are gluing points;
- $m_1 = m_2$ and $\theta_1 = \theta_2$;
- $\lambda_1(j) = \lambda_2(j)$ for $j \neq \text{ord}(m_1)$.

or:

- $(m_1, \lambda_1, \theta_1), (m_2, \lambda_2, \theta_2)$ are socket points;
- $m_1 = m_2$;
- $\lambda_1(j) = \lambda_2(j)$ and $\theta_1(j) = \theta_2(j)$ for $j \neq \text{ord}(m_1)$.

We consider on $G$ the length metric where each edge has length 1. Points in $G$ are then equivalence classes $[(t, \lambda, \theta)]$ of points $(t, \lambda, \theta) \in \mathbb{R} \times \Lambda \times \Theta$. The quotient map $\mathbb{R} \times \Lambda \times \Theta \to G$ will be denoted by $Q$. The $Q$-image of a gluing point (resp. a socket point) will be called a gluing point (resp. a socket point) of $G$. Note that the projection $\mathbb{R} \times \Lambda \to \mathbb{R}$ induces a 1-Lipschitz map $\pi : G \to \mathbb{R}$.

To analyze the shape of balls in $G$ the following definitions are useful.
Definition 2.3. To the sequence of scales \( \{\sigma_k\} \) we associate the discretized logarithm \( \log : [0, \infty) \to \mathbb{N} \) as follows:

\[
\log(p) = \begin{cases} 
0 & \text{if } |p| < \sigma_1 \\
\max k : \sigma_k \leq |p| & \text{otherwise.}
\end{cases}
\]

Note that each vertex \( v \in G \) has the form \([k, \lambda]\) where \( k \in \mathbb{Z} \), and \( \text{ord}(k) \) will be called the order of \( v \).

2.2. Construction of walks. A walk on \( G \) is a finite string on vertices and edges \( W = \{w_0 e_1 w_1 \cdots e_l w_l\} \) where \( w_i = 1 \) and \( w_i \) are the endpoints of \( e_i \) for \( 1 \leq i \leq l \). In the following we will often suppress the edges from the notation, i.e. simply write \( W = \{w_0 w_1 \cdots w_l\} \); we will also say that \( W \) is a walk from \( w_0 \) to \( w_l \) and that \( l \) is the length of \( W \), which we will denote by \( \text{len}(W) \). The starting point \( \text{str}(W) \) of \( W \) is \( w_0 \) and the end point \( \text{end}(W) \) of \( W \) is \( w_l \). Two walks \( W_1, W_2 \) with \( \text{end}(W_1) = \text{str}(W_2) \) can be concatenated to obtain a walk \( W_1 \ast W_2 \).

We say that a walk \( W \) from \( x \) to \( y \) is geodesic if \( \text{len}(W) = d(x, y) \). This notion can be also extended to the case in which \( x \) and / or \( y \) are not vertices of \( G \). In this case a geodesic walk from \( x \) to \( y \) is a geodesic walk from a vertex \( w_x \) to a vertex \( w_y \) such that:

\[
\begin{align*}
\text{(2.5)} & \quad d(x, w_x) < 1 \\
\text{(2.6)} & \quad d(y, w_y) < 1 \\
\text{(2.7)} & \quad d(x, y) = d(x, w_x) + \text{len}(W) + d(y, w_y); 
\end{align*}
\]

note that (2.7) implies \( \text{len}(W) = d(w_x, w_y) \). A walk \( W = \{w_0 w_1 \cdots w_l\} \) is monotone increasing (resp. decreasing) if for \( 0 \leq i \leq l - 1 \) one has \( \pi(w_{i+1}) > \pi(w_i) \) (resp. \( \pi(w_{i+1}) < \pi(w_i) \)).

We have preferred to introduce walks because they are more convenient than parametrized paths to describe the construction of quasigeodesics and random curves that we present later in the paper. In working with walks, it is important to keep track of the labels of their vertices and edges. Specifically, except for countably many points of \( G \), the fibre \( Q^{-1}(x) \) is a singleton; the points \( x \) for which \#\( Q^{-1}(x) > 1 \) are either gluing points or socket points. Note that if \( x \) is neither a gluing point nor a socket point, the labels \( \lambda_x \in \Lambda \) and \( \theta_x \in \Theta \) are well-defined as \( x = [(\pi(x), \lambda, \theta)] \) for unique \( \lambda = \lambda_x \) and \( \theta = \theta_x \). In particular, if \( e \) is an edge, all points in \( e \), except possibly one of the vertices, have the same labels \( \lambda_e \) and \( \theta_e \).

On the other hand, if \( x \) is a gluing point of order \( k \), \( x \) is a vertex of \( G \) of the form \( [(\pi(x), \lambda, \theta)] \) where: \( \theta \) is uniquely defined, and \( \lambda(l) \) is uniquely defined for \( l \neq k \). If \( x \) is a socket point of order \( k \), then it is a vertex of \( G \) of the form \( [(\pi(x), \lambda, \theta)] \) where: \( \lambda(l) \) is the gluing symbol for \( l < k \), \( \lambda(l) \) is uniquely defined for \( l > k \), and \( \theta(l) \) is uniquely defined for \( l \neq k \). Therefore, if \( x \) is either a gluing point or a socket point, at most one entry of each label \( \lambda(l) \) and / or \( \theta(l) \) is not uniquely defined; in this case we will sometimes make an arbitrary choice and still write \( \lambda_x(l) \) or \( \theta_x(l) \).

Note also that if \( x \) is a gluing point \( Q^{-1}(x) \) has cardinality \#\( \text{Symb}_1 \), and if \( x \) is a socket point \( Q^{-1}(x) \) has cardinality \#\( \text{Symb}_1 \times \#\text{Symb}_2 \). Sometimes we will say that \( \lambda \) is the \( \Lambda \)-label of an edge or vertex and that \( \theta \) is the \( \Theta \)-label of an edge or vertex.

In discussing walks that pass through socket points of \( G \), it will convenient to have defined a partial order on the set of labels \( \Lambda \) as follows: \( \lambda < \tilde{\lambda} \) if there
are $1 \leq k_1 \leq k_2$ such that: $\lambda(j) = \tilde{\lambda}(j)$ for $j < k_1$ and $j > k_2$, and for some $j \in [k_1, k_2]$ $\tilde{\lambda}(j)$ is not the gluing symbol, and $\lambda(j) = \{\bullet\}$ for $j \in [k_1, k_2]$. A walk $W = \{w_0, e_1, w_1, \ldots, e_l w_l\}$ is label nondecreasing (resp. nonincreasing) if for $1 \leq i \leq l - 1$ one has $\lambda_{e_i+1} \geq \lambda_{e_i}$ (resp. $\lambda_{e_i+1} \leq \lambda_{e_i}$).

The following Lemmas 2.8, 2.9, and 2.14 will be used to build quasigeodesics in Section 3 and to prove the Poincaré inequality in Section 4.

**Lemma 2.8.** Let $(p, k) \in G \times \mathbb{N}$, and let $(\lambda, \theta)$ denote the labels of one of the edges $e$ incident to $p$. Then there is a constant $C$ depending only on (P1)–(P3) such that there are monotone walks $W_+$ and $W_-$ satisfying:

1. $W_\pm$ is a walk from $p$ to $v_\pm$, where either $v_\pm$ is a gluing point if some $\lambda(j) \in [\lambda, C \sigma_k]$;
2. $\lambda(p) \in [\lambda, C \sigma_k]$ and $\text{len} W_\pm \in [\lambda, C \sigma_k]$;
3. All edges in $W_\pm$ have the same labels $(\lambda, \theta)$.

Proof. We just build $W_+$. Because $p$ is incident to an edge with label $(\lambda, \theta)$, we have $p \in Q(\mathbb{R} \times \{\lambda\} \times \{\theta\})$, and thus we can find a monotone increasing walk $W_0 \subset Q(\mathbb{R} \times \{\lambda\} \times \{\theta\})$ in the label $(\lambda, \theta)$ which starts at $p$, has length $\text{len} W_0 = 2 \sigma_k$, and ends at a vertex $v_\theta$ with $\text{ord}(v_\theta) = 0$. There is a uniform constant $C \geq 1$ such that the set $\mathbb{R} \cap \pi(v_\theta) \in [\lambda, C \sigma_k]$ contains an integer $t$ with $\text{ord}(t) = k$. Let $v_+ \in Q(\mathbb{R} \times \{\lambda\} \times \{\theta\})$ be a monotone increasing walk starting at $v_\theta$ and ending at $v_+$. Then $W_+$ is obtained by concatenating $W_0$ and $W_1$.

**Lemma 2.9.** Let $(p, k) \in G \times \mathbb{N}$ and let $(\lambda, \theta)$ be the labels of an edge incident to $p$. Then there is a constant $C$ depending on (P1)–(P3) such that there are label nonincreasing monotone walks $W_+$ and $W_-$ satisfying:

1. $W_\pm$ is a walk from $p$ to $v_\pm$, where $v_\pm$ is a socket point of order $k$ such that $\lambda(v_\pm) = \lambda(p)$ for $l > k$;
2. $(\Theta \lambda(p), \pi(p)) \in [\lambda, C \sigma_k]$ and $\text{len} W_\pm \in [\lambda, C \sigma_k]$;
3. The $\Theta$-label equals $\theta$ along all the edges of $W_\pm$;
4. All the edges in $W_\pm[0, 3 \sigma_k/2]$ have the same label $(\lambda, \theta)$;
5. There are $\tau_i \leq k - 1 \subset \mathbb{N} \cap [1, \text{len} W_\pm]$ such that the map $i \mapsto \tau_i$ is strictly decreasing, $\tau_{k-1} \in \frac{3 \sigma_k}{2}, C \sigma_k]$;
6. The point $w_{\tau_i}$ is either a gluing point or a socket point of order $i$;
7. $\text{len} W_\pm - \tau_i \in [\lambda, C \sigma_i]$;
8. Let $e_i$ be an edge of $W_\pm$; if $l \in [0, \tau_{k-1}]$, $\lambda_{e_i} = \lambda_{p}$; if $l \in (\tau_{i+1}, \tau_i) \lambda(e_i; j) = \lambda(p; j)$ for $j < i$ or $j > k - 1$ and $\lambda(e_i; j) = \{\bullet\}$ for $i + 1 \leq j \leq k - 1$; if $\lambda \in (\tau_i, \text{len} W_\pm] \lambda(e_i; j) = \{\bullet\}$ for $1 \leq j \leq k - 1$ and $\lambda(e_i; j) = \lambda(p_0; j)$ for $j \geq k$.

Proof. We focus on building $W_+$ which will be built as a concatenation of walks $W, W_{k-1}, W_{k-2}, \ldots, W_0$. Because $p$ is incident to an edge with label $(\lambda, \theta)$ we have $p \in Q(\mathbb{R} \times \{\lambda\} \times \{\theta\})$, and thus we can find a monotone increasing walk $W \subset Q(\pi(p), \infty) \times \{\lambda\} \times \{\theta\}$ of length $\text{len} W \in [\frac{3 \sigma_k}{2}, 2 \sigma_k]$ which starts at $p$ and ends at a vertex $v$ with $\text{ord}(v) = 0$.

Let $I = [\pi(v), \infty)$; in $I$ we can find a sequence of integers:

$$t_{k-1} \leq t_{k-2} \leq \cdots \leq t_1 \leq t_0$$

(2.10)
such that \( \text{ord}(t_i) = i \) for \( i \geq 1 \) and \( \text{ord}(t_0) = k \), and for some universal constant \( C \) one has \( 0 < t_0 - t_{k-1} \leq C \sigma_k \). To be explicit, let \( t_0 \) be an integer of order \( k \) in \([\pi(\bar{v}) + \sigma_k, \pi(\bar{v}) + 3\sigma_k]\) and let \( t_i = t_0 - \sigma_i \) for \( i \geq 1 \).

In the following we will let:

\[
(2.11) \quad \tau_i = t_i - \pi(p),
\]

and we will introduce the auxiliary notation \( \lambda^{(i)} \) for the label:

\[
(2.12) \quad \lambda^{(i)}(j) = \begin{cases} 
\lambda(j) & \text{if } j \geq k \text{ or } j \leq i \\
\{\clubsuit\} & \text{otherwise}. 
\end{cases}
\]

Let \( v_{k-1} \) be the vertex of \( Q([\pi(p), \infty) \times \{\lambda\} \times \{\theta\}) \) with \( \pi(v_{k-1}) = t_{k-1} \); then we let \( W_{k-1} \subset Q([\pi(p), \infty) \times \{\lambda\} \times \{\theta\}) \) be a monotone increasing walk which starts at \( \bar{v} \) and ends in \( v_{k-1} \). We let \( w_{n_{k-1}} = v_{k-1} \) and note that \( v_{k-1} \) is either a gluing or a socket point of order \( k - 1 \).

For \( i \geq 1 \) the walk \( W_i \) is obtained from \( W_{i+1} \) as follows. The (backward) inductive assumption is that the last edge of \( W_{i+1} \) has label \( (\lambda^{(i+1)}, \theta) \) and that the last vertex \( v_{i+1} \) of \( W_{i+1} \) is either a gluing or a socket point of order \( i + 1 \). Note that then \( v_{i+1} \in Q([\pi(p), \infty) \times \{\lambda^{(i)}\} \times \{\theta\}) \); we now let \( v_i \) denote the vertex of \( Q([\pi(p), \infty) \times \{\lambda^{(i)}\} \times \{\theta\}) \) with \( \pi(v_i) = t_i \). Therefore, by (2.12) \( v_i \) is either a gluing or a socket point of order \( i \). The walk \( W_i \subset Q([\pi(p), \infty) \times \{\lambda^{(i)}\} \times \{\theta\}) \) is then defined as a monotone increasing walk starting at \( v_{i+1} \) and ending in \( v_i \). We then let \( w_{n_i} = v_i \).

We complete the construction by producing \( W_0 \) as follows; we let \( \lambda^{(0)} \) be the label such that:

\[
(2.13) \quad \lambda^{(0)}(j) = \begin{cases} 
\lambda(j) & \text{if } j \geq k \\
\{\clubsuit\} & \text{otherwise}. 
\end{cases}
\]

We then let \( v_0 = v_0 \) be the vertex of \( Q([\pi(p), \infty) \times \{\lambda^{(0)}\} \times \{\theta\}) \) such that \( \pi(v_0) = t_0 \). The walk \( W_0 \) is then a monotone increasing walk joining \( v_1 \) to \( v_0 \).

We now explain how each property in the statement of this Lemma holds:

1. because \( v_0 = v_0 \) is a socket point of order \( k \) as \( \text{ord}(t_0) = k \) and the label \( \lambda^{(0)} \) has its first \( k - 1 \) entries equal to \( \{\clubsuit\} \);

2. because we have \( \text{len} \bar{W} \leq \sigma_k \), \( \text{len} W_i \leq \sigma_i \) for \( i \geq 1 \) and \( \text{len} W_0 \leq \sigma_1 \);

3. because the walks \( \bar{W}, W_{k-1}, W_{k-2}, \ldots, W_0 \) lie in \( Q(\mathbb{R} \times \Lambda \times \{\theta\}) \);

4. because of how \( \bar{W} \) was constructed;

5–7: because of how the \( t_i \) where chosen;

8: because of how the labels \( \lambda^{(i)} \) were chosen.

\[ \square \]

The next Lemma 2.14 is proven like Lemma 2.9; the proof is omitted as it looks like the specular image of the previous one.

**Lemma 2.14.** Let \( v \in G \) be a socket point of order \( k_0 \) and let \( \lambda \) be a label in \( \Lambda \) such that for \( k \leq k_0 \) one has \( \lambda(l) = \lambda_{v}(l) \) for \( l > k, l \neq k_0 \). Let \( \theta \) be a label in \( \Theta \) such that \( \theta_{v}(j) = \theta(j) \) for \( j \neq k_0 \). Then there is a constant \( C \) depending on \( (P1)-(P3) \) such that there are label non-decreasing monotone walks \( W_+ \) and \( W_- \) satisfying:

1. \( W_\pm \) is a walk from \( v \) to a vertex \( p_\pm \) of order 0 such that \( \lambda_{p_\pm} = \lambda \) and \( \theta_{p_\pm} = \theta \).
Lemma 2.16. Let the following lemma shows that boxes and balls are uniformly comparable.

(2) $\pm (\pi(p_\pm) - \pi(v)) \in [\sigma_k, C\sigma_k]$ and $\text{len } W_\pm \in [\sigma_k, C\sigma_k]$;

(3) All edges of $W_\pm$ have $\Theta$-label $\theta$;

(4) All the edges in $W_\pm [\text{len } W_\pm - \sigma_k/2, \text{len } W_\pm]$ have the same labels;

(5) There are $(\tau_i)_{1 \leq i \leq k-1} \subset \mathbb{N} \cap [0, \text{len } W_k]$ such that the map $i \mapsto \tau_i$ is strictly increasing, and $\tau_{k-1} \in [0, \text{len } W_\pm - \frac{\sigma_k}{2}]$;

(6) The point $w_{\tau_i}$ is either a gluing point or a socket point of order $i$;

(7) $\tau_i \in [\sigma_i, C\sigma_i]$;

(8) Let $e_i$ be an edge of $W_\pm$; if $l \in [0, \tau_1]$, $\lambda(e_i;j) = \lambda(v;j)$ for $j \neq k_0$ and $\lambda(e_i;k_0) = \lambda(k_0)$; if $l \in (\tau_i, \tau_{i+1}]$ $\lambda(e_i;j) = \lambda(j)$ for $j \leq i$ or $j > k-1$ and $\lambda(e_i;j) = \{\bullet\}$ for $i < j \leq k-1$; if $\lambda \in (\tau_{k-1}, \text{len } W_\pm]$ $\lambda(e_i) = \lambda$.

2.3. Comparison of balls and boxes. In the following it will be useful to replace balls by boxes because it is easier to estimate the measure of a box; given a Borel set $I \subset \mathbb{R}$, $k \in \mathbb{N} \cup \{0\}$ and a finite set $S_1 \times S_2 \subset \Lambda \times \Theta$, we define the box $\text{Box}(I, S_1 \times S_2, k)$ as follows:

$$(2.15) \quad \left\{ [(t, \lambda, \theta)] \in G : t \in I \text{ and } \exists (\tilde{\lambda}, \tilde{\theta}) \in S_1 \times S_2 : \forall l > k \ (\lambda(l), \theta(l)) = (\tilde{\lambda}(l), \tilde{\theta}(l)) \right\}.$$ 

The following lemma shows that boxes and balls are uniformly comparable.

Lemma 2.16. Let $x = [(t, \lambda, \theta)] \in G$ and $R > 0$. Let $M$ be the highest order of an integer $m \in [t - R, t + R]$. If $M \leq \lg(2R)$ let $S(x, R) = \{(\lambda, \theta)\}$. If $M > \lg(2R)$ let $\Omega_M$ be the set of those labels $(\lambda', \theta')$ obtained from $(\lambda, \theta)$ by making $(\lambda(M), \theta(M))$ arbitrary, and let $S(x, R) = \Omega_M$. Then there is a universal constant $C$ depending only on (P1)–(P3) such that:

$$(2.17) \quad \text{Box}(\mathcal{I}(x) - R/2, \mathcal{I}(x) + R/2, \{(\lambda, \theta)\}, \lg(R/C)) \subset \tilde{B}(x, R) \ \subset \ \text{Box}(\mathcal{I}(x) - R, \mathcal{I}(x) + R, S(x, R), \lg(2R)).$$

Proof. If $C$ is sufficiently large, using Lemmas 2.9, 2.14 we can find, for any label $(\lambda, \theta)$ such that:

$$(2.18) \quad (\tilde{\lambda}(j), \tilde{\theta}(j)) = (\lambda, \theta) \quad \text{ (for } j > \lg(R/C)), \quad \text{ a path of length at most } R/2 \text{ from } x \text{ to a point } \tilde{x} \text{ such that:}$$

$$(2.19) \quad \tilde{x} = Q(R \times \{\tilde{\lambda}\} \times \{\tilde{\theta}\});$$

this implies the inclusion:

$$(2.21) \quad \text{Box}(\mathcal{I}(x) - R/2, \mathcal{I}(x) + R/2, \{(\lambda, \theta)\}, \lg(R/C)) \subset \tilde{B}(x, R).$$

Let $\gamma$ be a geodesic from $x$ to $p \in \tilde{B}(x, R)$; note that $\text{len } \pi(\gamma) = \text{len } \gamma$ and thus $\pi(\gamma(t)) \in [\pi(x) - R, \pi(x) + R]$ for each $t \in \text{dom } \gamma$. Therefore, if $(\lambda(p; k), \theta(p; k)) \neq (\lambda(x; k), \theta(x; k))$, then $\pi(\gamma)$ passes through an integer $t_k$ of order $k$. Assume that $k < M$ and let $t_M \in [\pi(x) - R, \pi(x) + R]$ have order $M$; as:

$$(2.22) \quad |t_k - t_M| \geq \sigma_k,$$

we conclude that $k \leq \lg(2R)$. Therefore the inclusion

$$(2.23) \quad \tilde{B}(x, R) \subset \text{Box}([\pi(x) - R, \pi(x) + R], S(x, R), \lg(2R))$$

follows. \qed
2.4. Construction of measures. We now turn to the construction of the measure \( \mu \) on \( G \). One possibility is to take the pushforward under the quotient map \( Q : \mathbb{R} \times \Lambda \times \Theta \to G \) of the measure which coincides with Lebesgue measure on each \( \mathbb{R} \times \{ \lambda \} \).

For extra flexibility we choose a finite set of weights \( \text{Weight} = \{ w_s \}_{s \in \text{Symb}_1 \cup \text{Symb}_2} \) subject to the restrictions \( w_s > 0 \) and \( w(\emptyset) = 1 \). For each \( \lambda \in \Lambda \) and \( \theta \in \Theta \) we denote by \( w(\lambda), w(\theta) \) the associated weights:

\[
\begin{align*}
    w(\lambda) &= \prod_{n=1}^{\infty} w_{\lambda(n)}, \\
    w(\theta) &= \prod_{n=1}^{\infty} w_{\theta(n)},
\end{align*}
\]

where the products in (2.24–2.25) are actually finite. We also use the notation \( w((\lambda, \theta)) \) for the product \( w(\lambda)w(\theta) \).

**Definition 2.26.** We denote by \( \mu \) the measure on \( G \) which is the pushforward of the measure on \( \mathbb{R} \times \Lambda \times \Theta \) which coincides with \( w((\lambda, \theta))L^1 \) on each \( \mathbb{R} \times \{ (\lambda, \theta) \} \).

Note that different choices of the weights in Weight will produce mutually singular measures on the asymptotic cone \( X \), compare [Sch15].

**Lemma 2.27.** Let \( S \) be a set of pairs of labels and \( k \geq 1 \); assume that whenever \( (\lambda, \theta), (\lambda', \theta') \in S \) and \( (\lambda, \theta) \neq (\lambda', \theta') \), then \( (\lambda', \theta') \) cannot be obtained from \( (\lambda, \theta) \) by modifying some of the first \( k \)-entries of \( \lambda \) and/or \( \theta \). For \( i = 1, 2 \) let \( C_{gw,i} = \sum_{s \in \text{Symb}_i} w_s \); then the measure of a box is given by:

\[
\mu(\text{Box}(I, S, k)) = \mathcal{L}^1(I) \times C_{gw,1}^k C_{gw,2}^k \sum_{(\lambda, \theta) \in S} \prod_{n=k+1}^{\infty} w(\lambda(n), \theta(n)).
\]

In particular, if \( x = [(t, \lambda, \theta)] \):

\[
\mu(\hat{B}(x, R)) \approx R(C_{gw,1}C_{gw,2})^{\log R} \sum_{\lambda \in S(x, R)} \prod_{n=1}^{\infty} w(\lambda(n), \theta(n)).
\]

In particular, if the \( m_k \) are all equal to some \( m \) and if \( R \geq 1 \), we have:

\[
\mu(\hat{B}(x, R)) \approx R^{1+\log m} C_{gw,1}^{\log R} C_{gw,2} \sum_{\lambda \in S(x, R)} \prod_{n=1}^{\infty} w(\lambda(n), \theta(n));
\]

moreover, if all the weights are equal to \( 1 \) one has:

\[
\mu(\hat{B}(x, R)) \approx R^{1+\log m} C_{gw,1}^{\log R} C_{gw,2}.
\]

**Proof.** For each pair of labels \( (\lambda, \theta) \) let \( T_{\lambda, \theta}^{(k)} \) be the set of labels that can be obtained from \( (\lambda, \theta) \) by making the first \( k \) entries of \( \lambda \) and/or \( \theta \) arbitrary. We then compute
as follows:

\begin{align}
(2.32) \\
\mu(\text{Box}(I, S, k)) &= \sum_{(\lambda, \theta) \in S} \sum_{(\tilde{\lambda}, \tilde{\theta}) \in T^{(k)}_{\lambda, \theta}} \mu(\text{Box}(I, S, k) \cap Q(\mathbb{R} \times \{\tilde{\lambda}\} \times \{\tilde{\theta}\})) \\
&= \sum_{(\lambda, \theta) \in S} \sum_{(\tilde{\lambda}, \tilde{\theta}) \in T^{(k)}_{\lambda, \theta}} L^1(I) w(\tilde{\lambda}, \tilde{\theta}) \\
&= L^1(I) \sum_{(\lambda, \theta) \in S} \sum_{(\tilde{\lambda}, \tilde{\theta}) \in T^{(k)}_{\lambda, \theta}} \prod_{n=1}^{k} w((\tilde{\lambda}(n), \tilde{\theta}(n))) \cdot \prod_{n=k+1}^{\infty} w((\lambda(n), \theta(n))) \\
&= L^1(I) \times C_{gw, 1}^k C_{gw, 2}^k \sum_{(\lambda, \theta) \in S} \prod_{n=k+1}^{\infty} w(\lambda(n), \theta(n)),
\end{align}

which gives (2.28).

Now (2.29) follows from (2.28) and Lemma 2.16 by observing that for any \( C_0 \) there is a \( C(C_0) \) such that:

\begin{equation}
(2.33) \quad \lg(C_0 R) \leq \lg R + C(C_0).
\end{equation}

If we assume that all the \( m_k \) are equal to \( m \), then the discretized logarithm \( \lg \) is just \( \log_m \) up to a bounded additive error, and hence (2.30), (2.31) follow. \( \square \)

### 3. Construction of good walks

In this section we prove the existence of good walks between points in \( G \). These walks correspond to quasigeodesics which are used to build the families of curves used to prove Poincaré inequalities.

Let \( x, y \in G \); choose labels \((\lambda_x, \theta_x), (\lambda_y, \theta_y)\) such that \( x = [(\pi(x), \lambda_x, \theta_x)] \), \( y = [(\pi(y), \lambda_y, \theta_y)] \) and the cardinality of the set:

\begin{equation}
(3.1) \quad \mathbb{N}(x, y) = \{k : (\lambda_x(k), \theta_x(k)) \neq (\lambda_y(k), \theta_y(k))\}
\end{equation}

is minimal.

In the following \( C \) will denote a universal constant that can change from line to line and that can be explicitly estimated.

**Definition 3.2.** Given \( x, y \in G \) with \( d(x, y) > 1 \) a good walk \( W = \{w_0 e_1 w_1 \cdots e_L w_L\} \) from \( x \) to \( y \) is a walk having the following properties:

- **(GW1):** \( \text{len} W \leq C d(x, y); \)
- **(GW2):** \( d(w_0, x), d(w_L, y) \in [0, 1]; \)
- **(GW3):** for \( i > 0 \) one has \( d(w_i, x) \geq i/C. \)

**Lemma 3.3.** If \( \lg d(x, y) < \max \mathbb{N}(x, y) \) then for each \( k \in \mathbb{N}(x, y) \setminus \{\max \mathbb{N}(x, y)\} \) one has \( \lg d(x, y) \geq k. \)

**Proof.** Let \( w_0, w_1 \) be vertices of \( G \) with \( \text{ord}(w_0) \neq \text{ord}(w_1) \), then:

\begin{equation}
(3.4) \quad d(w_0, w_1) \geq |\pi(w_0) - \pi(w_1)| \geq \sigma_{\min(\text{ord}(w_0), \text{ord}(w_1))}.
\end{equation}

Take a geodesic walk \( W \) from \( x \) to \( y \). Then there are \( w_{j_0}, w_{j_1} \in W \) such that \( w_{j_0} \) is either a gluing or a socket point of order \( \max \mathbb{N}(x, y) \) and \( w_{j_1} \) is either a gluing or a
socket point of order \(k\); let \(\tilde{W}\) be a subwalk of \(W\) joining \(w_{j_0}\) and \(w_{j_1}\), and observe that:
\[
\text{len}(W) \geq \text{len}(\tilde{W}) \geq d(w_{j_0}, w_{j_1}) \geq \sigma_k. 
\]

Theorem 3.6. If \(\lg d(x, y) \geq k_{\text{max}} = \max N(x, y)\) there is a good walk \(W\) from \(x\) to \(y\) which has the following additional properties:

\((\text{GWA1})\): If \(k \in N(x, y)\) is such that \(\theta_x(k) = \theta_y(k)\), there is a distinguished gluing or socket point \(w_{s(k)}\) such that each edge \(e\) preceding \(w_{s(k)}\) satisfies \(\lambda_e(k) = \lambda_e(k)\), and each edge \(e\) following \(w_{s(k)}\) satisfies either \(\lambda_e(k) = \lambda_y(k)\) or \(\lambda_e(k) = \{\dagger\}\). Moreover, in this case all edges \(e\) satisfy \(\theta_e(k) = \theta_x(k)\). If \(k \in N(x, y)\) is such that \(\theta_x(k) \neq \theta_y(k)\), there is a distinguished socket point \(w_{s(k)}\) such that each edge \(e\) preceding \(w_{s(k)}\) satisfies \(\theta_e(k) = \theta_x(k)\) and \(\lambda_e(k) = \lambda_x(k)\), and each edge \(e\) following \(w_{s(k)}\) satisfies \(\theta_e(k) = \theta_y(k)\) and either \(\lambda_e(k) = \lambda_y(k)\) or \(\lambda_e(k) = \{\dagger\}\). Moreover, the map \(k \mapsto s(k)\) is monotone increasing and the subwalk \(W_k\) from \(w_{s(k)}\) to \(w_{s(k+1)}\) satisfies:
\[
\text{len}(W_k) \approx \sigma_{k+1} \approx d(w_{s(k)}, w_{s(k+1)});
\]

\((\text{GWA2})\): The walk \(W\) satisfies:
\[
\text{len}(W) \approx \max \{|\pi(x) - \pi(y)|, \sigma_{k_{\text{max}}}|\}.
\]

Proof. Without loss of generality we can assume \(\pi(x) \leq \pi(y)\). If \(N(x, y) = 0\) then \(x, y\) lie in some \(Q(\mathbb{R} \times \{\lambda\} \times \{\theta\})\) and the construction of the walk is immediate. Let \(w_0\) be the vertex of \(G\) satisfying \(\pi(w_0) \in [\pi(x), \pi(x) + 1]\), \((\lambda_{w_0}, \theta_{w_0}) = (\lambda_x, \theta_x)\) (if the labels for \(w_0\) or \(x\) are not unique, one can choose them so that equality holds. Note that for a non-unique label \((\lambda_p, \theta_p)\) only one entry \((\lambda_p(m), \theta_p(m))\) is not uniquely determined). Order the elements of \(N(x, y)\) increasingly:
\[
k_0 < k_1 < \cdots < k_q.
\]

Now either \(\theta_x(k_0) = \theta_y(k_0)\) or \(\theta_x(k_0) \neq \theta_y(k_0)\). The goal is to construct a walk \(W_{k_0}\) of length comparable to \(\sigma_{k_0}\) which allows to change the \(k_0\)-th entries of the labels. We build \(W_{k_0}\) in two parts \(W_{k_0}^{(-)}\) and \(W_{k_0}^{(+)}\).

We now consider the first case \(\theta_x(k_0) = \theta_y(k_0)\) which implies \(\lambda_x(k_0) \neq \lambda_y(k_0)\); by Lemma 2.8 we can find a monotone increasing walk \(W_{k_0}^{(-)}\) from \(w_0\) to a gluing or a socket point \(v_{k_0}^{(-)}\) of order \(k_0\) such that:
\[
(1)\quad \pi(v_{k_0}^{(-)}) \in [\pi(w_0) + \sigma_{k_0}, \pi(w_0) + C\sigma_{k_0}];
(2)\quad \text{all edges of } W_{k_0}^{(-)} \text{ have the same labels } (\lambda_{w_0}, \theta_{w_0});
(3)\quad \text{len } W_{k_0}^{(-)} \in [\sigma_{k_0}, C\sigma_{k_0}] .
\]

\(W_{k_0}^{(-)}\) is the first part of the walk \(W_{k_0}\) and we let \(w_{s(k_0)} = v_{k_0}^{(-)}\). Let \(\hat{\lambda}_{w_0}\) be the label which agrees with \(\lambda_{w_0}\) except at the \(k_0\)-th entry \(\hat{\lambda}_{w_0}(k_0) = \lambda_y(k_0)\). The second part of the walk \(W_{k_0}^{(+)}\) is a monotone walk of length \(\text{len } W_{k_0}^{(+)} \in [1, \sigma_{k_0}]\) which terminates at a vertex of order 0 and whose edges have the same label \((\hat{\lambda}_{w_0}, \theta_{w_0})\). We now consider the second case \(\theta_x(k_0) \neq \theta_y(k_0)\) which is slightly more complicated. By Lemma 2.9 we can find a label-nonincreasing monotone walk \(W_{k_0}^{(-)}\) from \(w_0\) to a socket point \(v_{k_0}^{(-)}\) such that:
We then let \( w_{k_0} = v_{k_0}^{(-)} \).

By Lemma 2.14 we find a label-nonincreasing monotone walk \( W_{k_0}^{(+)} \) from \( v_{k_0}^{(-)} \) to a vertex \( v_{k_0}^{(+)} \) of order zero satisfying:

1. \( \pi(v_{k_0}^{(+)}) \in [\pi(w_0) + \sigma_{k_0}, \pi(w_0) + C\sigma_{k_0}] \).
2. For \( l \leq k_0 \) one has \( (\lambda(v_{k_0}^{(-)}; l), \theta(v_{k_0}^{(-)}; l)) = (\lambda(w_0; l), \theta(w_0; l)) \).
3. \( \text{len} W_{k_0}^{(-)} \in [\sigma_{k_0}, C\sigma_{k_0}] \).
4. All edges of \( W_{k_0}^{(+)} \) satisfy \( (\lambda_c(k_0), \theta_c(k_0)) \).

\( W_{k_0}^{(-)} \) is the first part of the walk \( W_{k_0} \) and we let \( w_{s(k_0)} = v_{k_0}^{(-)} \).

The construction continues by induction on \( k_j \), i.e. suppose we have constructed the subwalks \( \{W_{k_0}, \cdots, W_{k_j}\} \) which form the first part of \( W \). The first part \( W_{k_{j+1}}^{(-)} \) of \( W_{k_{j+1}} \) is a label-nonincreasing monotone walk \( W_{k_{j+1}}^{(-)} \) from \( v_{k_{j+1}}^{(+)} \) to a socket point \( v_{k_{j+1}}^{(-)} \) of order \( k_{j+1} \) such that:

1. \( \pi(v_{k_{j+1}}^{(-)}) \in \left[ \pi(v_{k_{j+1}}^{(+)}), \pi(v_{k_{j+1}}^{(-)}), \pi(v_{k_{j+1}}^{(-)}) + C\sigma_{k_{j+1}} \right] \).
2. \( v_{k_{j+1}}^{(-)} \) has order \( k_{j+1} \) and for \( l > k_{j+1} \) one has \( (\lambda(v_{k_{j+1}}^{(-)}; l), \theta(v_{k_{j+1}}^{(-)}; l)) = (\lambda(v_{k_{j+1}}^{(+)}; l), \theta(v_{k_{j+1}}^{(+)}; l)) \).
3. \( \text{len} W_{k_{j+1}}^{(-)} \in [\sigma_{k_{j+1}}, C\sigma_{k_{j+1}}] \).

We then let \( w_{s(k_{j+1})} = v_{k_{j+1}}^{(-)} \).

By Lemma 2.14 we complete \( W_{k_{j+1}} \) by finding a label-nonincreasing monotone walk \( W_{k_{j+1}}^{(+)} \) from \( v_{k_{j+1}}^{(-)} \) to a vertex \( v_{k_{j+1}}^{(+)} \) such that:

1. \( \pi(v_{k_{j+1}}^{(+)}) \in \left[ \pi(v_{k_{j+1}}^{(-)}), \pi(v_{k_{j+1}}^{(-)}), \pi(v_{k_{j+1}}^{(-)}) + C\sigma_{k_{j+1}} \right] \).
2. For \( l \leq k_{j+1} \) one has \( (\lambda(v_{k_{j+1}}^{(+)}; l), \theta(v_{k_{j+1}}^{(+)}; l)) = (\lambda(v_{k_{j+1}}^{(-)}; l), \theta(v_{k_{j+1}}^{(-)}; l)) \) and for \( l > k_{j+1} \) \( (\lambda(v_{k_{j+1}}^{(+)}; l), \theta(v_{k_{j+1}}^{(+)}; l)) = (\lambda(v_{k_{j+1}}^{(-)}; l), \theta(v_{k_{j+1}}^{(-)}; l)) \).
3. \( \text{len} W_{k_{j+1}}^{(+)} \in [\sigma_{k_{j+1}}, C\sigma_{k_{j+1}}] \).
4. All edges of \( W_{k_{j+1}}^{(+)} \) satisfy \( (\lambda_c(k_{j+1}), \theta_c(k_{j+1})) \).

When we reach \( j = q \) we have constructed the first part \( W^{(1)} \) of the walk \( W \).

Property \((GW3)\) is satisfied because \( W^{(1)} \) is monotone increasing and the part of \((GW2)\) concerning \( w_0 \) is also satisfied; the additional condition \((GWA1)\) is also satisfied on \( W^{(1)} \), and needs only to be checked there because of the way in which we construct the second part \( W^{(2)} \) of the walk.

There are two cases to consider to complete the proof.

\textbf{(Case 1):} \( \pi(v_{k_q}^{(+)} \leq \pi(y) \); then \( v_{k_q}^{(+)} \) and \( y \) belong to \( Q(\mathbb{R} \times \{\lambda_y\} \times \{\theta_y\}) \). Therefore, \( W^{(2)} \) is constructed by taking a geodesic walk in \( Q(\mathbb{R} \times \{\lambda_y\} \times \{\theta_y\}) \) from \( v_{k_q}^{(+)} \) to
y. We need only to prove (GW1) which is a consequence of (GWA2):

\[
\text{len } W = \sum_{j=0}^{q} (\text{len } W_{k_j}^{(-)} + \text{len } W_{k_j}^{(+)}) + \pi(y) - \pi(v_{k_q}^{(+)})
\]

(3.10)

\[
\leq C \sum_{j=0}^{q} \sigma_{k_j} + \pi(y) - \pi(v_{k_q}^{(+)})
\]

\[
\leq C \sigma_{k_q} + \pi(y) - \pi(v_{k_q}^{(+)})
\]

(3.11)

however, \(\pi(x) \leq \pi(v_{k_q}^{(-)}) \leq \pi(y)\) and so \(\sigma_{k_q} \leq \pi(y) - \pi(x)\) which implies:

As \(\pi\) is 1-Lipschitz and as \(W\) is monotone increasing, we have \(\text{len } W \geq \pi(y) - \pi(x)\) which completes the proof of (GWA2).

(Case 2): \(\pi(y) < \pi(v_{k_q}^{(+)})\); then \(v_{k_q}^{(+)}\) and \(y\) belong to \(Q(\mathbb{R} \times \{\lambda_{g_y}\} \times \{\theta_{y}\})\) and \(W^{(2)}\) is constructed by taking a geodesic walk in \(Q(\mathbb{R} \times \{\lambda_{g_y}\} \times \{\theta_{y}\})\) from \(v_{k_q}^{(+)}\) to \(y\); note that \(W^{(2)}\) is monotone decreasing. Let:

\[
W^{(2)} = \{z_0, \cdots, z_m = v_y\},
\]

where \(v_y\) is the unique vertex satisfying \((\lambda_y, \theta_y) = (\lambda_{v_y}, \theta_{v_y})\) and \(\pi(v_y) \in [\pi(y), \pi(y) + 1]\). Note that:

\[
\pi(x) \leq \pi(y) \leq \pi(v_y) \leq C \sigma_{k_q} + \pi(x),
\]

and so:

\[
\sigma_{k_q} = \text{len } W = \sum_{j=0}^{q} (\text{len } W_{k_j}^{(-)} + \text{len } W_{k_j}^{(+)}) + \pi(v_{k_q}^{(+)}) - \pi(y)
\]

(3.13)

\[
\leq C \sigma_{k_q} \leq C d(x, y),
\]

which establishes (GW1), (GWA2) and the part of (GW2) concerning \(w_L\).

If \(\pi(z_m) \geq \pi(x) + \sigma_{k_q}/2\) then (GW3) holds for some universal constant \(C\). Otherwise, let \(m_0 \leq m\) denote the first integer so that:

\[
\pi(z_{m_0}) < \pi(x) + \sigma_{k_q}/2;
\]

(3.14)

for \(\tilde{m} \geq m_0\) we have \(d(z_{\tilde{m}}, z_m) < \sigma_{k_q}/2\) as \(W^{(2)}\) is a monotone decreasing geodesic walk; thus:

\[
d(z_{\tilde{m}}, x) \geq d(z_{\tilde{m}}, x) - \sigma_{k_q}/2 \geq \frac{\sigma_{k_q}}{2} - 1,
\]

(3.15)

and so (GW3) holds for some universal constant \(C\) (recall that \(k_q = k_{\text{max}}\)).

**Theorem 3.17.** If \(\log d(x, y) < k_{\text{max}} = \max N(x, y)\) then there is a good walk \(W\) from \(x\) to \(y\) which has the following additional property:

(GWA3): If \(\theta(x, k_{\text{max}}) = \theta(y, k_{\text{max}})\) there is a distinguished gluing or socket point \(u_{k_{\text{max}}}'\) in \(W\) of order \(k_{\text{max}}\) such that each edge \(e\) preceding \(u_{k_{\text{max}}}'\) satisfies \(\lambda(e; k_{\text{max}}) = \lambda(x; k_{\text{max}})\) and each edge following \(u_{k_{\text{max}}}'\) satisfies \(\lambda(e; k_{\text{max}}) = \lambda(y; k_{\text{max}})\). Moreover, in this case all edges \(e\) of \(W\) satisfy \(\theta(e; k_{\text{max}}) = \theta(x; k_{\text{max}})\). On the other hand, if \(\theta(x, k_{\text{max}}) \neq \theta(y, k_{\text{max}})\) there is a distinguished socket point \(u_{k_{\text{max}}}'\) such that each edge preceding \(u_{k_{\text{max}}}'\) satisfies \(\lambda(e; k_{\text{max}}), \theta(e; k_{\text{max}}) \neq (\lambda(x; k_{\text{max}}), \theta(x; k_{\text{max}}))\) and each edge \(e\)
following \( u_{k_{\text{max}}} \) satisfies \((\lambda(e; k_{\text{max}}), \theta(e; k_{\text{max}})) = (\lambda(y; k_{\text{max}}), \theta(y; k_{\text{max}}))\). Moreover, \( W \) can be decomposed into consecutive walks \( W_x \) and \( W_y \) where \( W_x \) is a good walk from \( x \) to \( u_{k_{\text{max}}} \) satisfying the conclusion of Theorem 3.6, and \( W_y \) is a good walk from \( u_{k_{\text{max}}} \) to \( y \) satisfying the conclusion of Theorem 3.6.

**Proof.** The construction in the cases \( \theta_x(k_{\text{max}}) = \theta_y(k_{\text{max}}) \) and \( \theta_x(k_{\text{max}}) \neq \theta_y(k_{\text{max}}) \) is essentially the same, and we thus discuss only the latter case. The properties of the labels \((\lambda(e; k_{\text{max}}), \theta(e; k_{\text{max}}))\) follow from the construction and Theorem 3.6.

Take a geodesic walk \( W \) from \( x \) to \( y \). Note that there must be a socket point \( \tilde{u} \in W \) of order \( k_{\text{max}} \) so that:

\[
d(x, \tilde{u}) + d(\tilde{u}, y) = d(x, y);
\]

moreover, let \( U \) denote the set of socket points of order \( k_{\text{max}} \) and let \( u_{k_{\text{max}}} \) be an element of \( U \) at minimal distance from \( x \) so that \( d(x, u_{k_{\text{max}}}) \leq d(x, \tilde{u}) \leq d(x, y) \). Let \( k \in \mathbb{N}(x, u_{k_{\text{max}}}) \); then if \( k > k_{\text{max}} \) a geodesic walk \( W \) from \( x \) to \( u_{k_{\text{max}}} \) would pass through either a gluing or a socket point of order \( k \) and by Lemma 3.3 we would have:

\[
d(x, u_{k_{\text{max}}}) = \text{len } W \geq \sigma_{k_{\text{max}}} > d(x, y),
\]

yielding a contradiction. Hence \( k \leq k_{\text{max}} \); note that \((\lambda(u_{k_{\text{max}}}; k_{\text{max}}), \theta(u_{k_{\text{max}}}; k_{\text{max}}))\) can take any value, and hence \( k < k_{\text{max}} \); we can then take a geodesic walk from \( x \) to \( u_{k_{\text{max}}} \) which must pass through either a gluing or a socket point of order \( k \), and we apply Lemma 3.3 to conclude that:

\[
d(x, u_{k_{\text{max}}}) = \text{len } W \geq \sigma_k.
\]

Thus we can apply Theorem 3.6 to obtain a good walk \( W_x \) from \( x \) to \( u_{k_{\text{max}}} \). Note that (3.19) implies that \((\lambda(u_{k_{\text{max}}}; l), \theta(u_{k_{\text{max}}}; l)) = (\lambda(x; l), \theta(x; l))\) for \( l > k_{\text{max}} \); in particular, as \( k_{\text{max}} = \max \mathbb{N}(x, y) \), if \( k \in \mathbb{N}(u_{k_{\text{max}}}, y) \) we have \( k < k_{\text{max}} \). Let \( W \) be a geodesic walk from \( u_{k_{\text{max}}} \) to \( y \); then it must pass through either a gluing or a socket point of order \( k \) and Lemma 3.3 implies:

\[
d(y, u_{k_{\text{max}}}) = \text{len } W \geq \sigma_k;
\]

therefore, we can apply Theorem 3.6 to obtain a good walk \( W_y \) from \( u_{k_{\text{max}}} \) to \( y \). For later reference, we also note here that:

\[
d(x, u_{k_{\text{max}}}) + d(y, u_{k_{\text{max}}}) \in [d(x, y), 3d(x, y)].
\]

The walk \( W \) is obtained by concatenating \( W_x \) and \( W_y \) so that it satisfies \( \text{GW}_A3 \). Property \( \text{GW1} \) follows observing that:

\[
\text{len } W = \text{len } W_x + \text{len } W_y \leq C (d(x, u_{k_{\text{max}}}) + d(u_{k_{\text{max}}}, y)),
\]

and using (3.22) to conclude that:

\[
\text{len } W \leq Cd(x, y).
\]

Property \( \text{GW2} \) holds because it holds for \( W_x \) and \( W_y \). We discuss property \( \text{GW3} \) in some cases. We will denote by \( C_1 \geq 2 \) the constant in \( \text{GW3} \) provided by Theorem 3.6. In the following we use the notations \( k^{(x)}_{\text{max}} = \max \mathbb{N}(x, u_{k_{\text{max}}}) \) and \( k^{(y)}_{\text{max}} = \max \mathbb{N}(u_{k_{\text{max}}}, y) \).

\( \text{Case 1}: \pi(x) \leq \pi(u_{k_{\text{max}}}) \leq \pi(y). \)
(Case 1,1): $W_z$ and $W_y$ are both monotone. Then $W$ is monotone and \((GW3)\) holds.

(Case 1,2): $W_z$ is not monotone and $W_y$ is monotone. As in Theorem 3.6 we decompose $W_z$ is a first part $W_z^{(m)}$ which is monotone, and a second part $\{z_0, \cdots, z_m = u_{k_{\text{max}}}\}$. Then $\text{len} \ W_x \approx \sigma_{k_{\text{max}}}^{(x)}$ and $d(z_i, x) \geq j_x(i)/C$ there $j_x(i)$ is the index / position of $z_i$ in the walk $W_z$, and $C$ is a universal constant. Let $w \in W_y$ and $j_y(w)$ denote the position of $w$ in $W_y$ and $j(w)$ the position in $W$. If $j_y(w) < 2 \text{len} \ W_z$, then $j(w) \leq 3 \text{len} \ W_x$ and so:

$$d(w, x) \geq d(w, z_m = u_{k_{\text{max}}}) \geq \frac{\text{len} \ W_x}{C_1} \geq \frac{j(w)}{3C_1}.$$

If $j_y(w) > 2 \text{len} \ W_z$, then $d(w, x) \geq d(w, u_{k_{\text{max}}}) - d(u_{k_{\text{max}}}, x)$; as $W_y$ is monotone we have $d(w, u_{k_{\text{max}}}) \geq j_y(w)$ and so:

$$d(w, x) \geq j_y(w) - \text{len} \ W_x \geq \frac{j_y(w)}{2};$$

thus

$$j(w) = j_y(w) + \text{len} \ W_x \leq \frac{3}{2} j_y(w),$$

and so

$$d(w, x) > \frac{j(w)}{3}.$$

(Case 1,3): Suppose that $W_z$ is monotone but $W_y$ is not. As in Theorem 3.6 we decompose $W_y$ in a first part $W_y^{(m)}$ which is monotone and a second part $\{z_0, \cdots, z_m = v_y\}$. On $W_y^{(m)}$ we obtain \((GW3)\) as in (Case 1,1).

Note that:

$$\text{len} \ W_x \approx \pi(u_{k_{\text{max}}}) - \pi(x)$$

$$\text{len} \ W_y \approx \text{len} \ W_y^{(m)} + m \approx \sigma_{k_{\text{max}}}^{(y)} \approx d(u_{k_{\text{max}}}, y).$$

Note that for each $i$ we have $d(z_i, u_{k_{\text{max}}}) \geq \sigma_{k_{\text{max}}}^{(y)} / C_1$. If $\pi(z_i) \geq \pi(u) + \sigma_{k_{\text{max}}}^{(y)} / 2$ we conclude that:

$$d(z_i, x) \geq \pi(u_{k_{\text{max}}}) - \pi(x) + \frac{\sigma_{k_{\text{max}}}^{(y)}}{2} \geq d(x, u_{k_{\text{max}}}) + d(u_{k_{\text{max}}}, y)$$

where in $(*)$ we used (3.22) and where the constant in the lower bound can be explicitly estimated in terms of $C_1$.

Suppose that $\pi(z_i) \in \left[ \pi(u_{k_{\text{max}}}), \pi(u_{k_{\text{max}}}) + \frac{\sigma_{k_{\text{max}}}^{(y)}}{2} \right]$. Then any geodesic walk from $x$ to $z_i$ must pass through some socket point $u \in U$, and we would also have $k_{\text{max}} \in N(u, y)$ so that:

$$d(x, z_i) \geq d(u, x) + \frac{\sigma_{k_{\text{max}}}^{(y)}}{2} \geq d(u, u_{k_{\text{max}}}) + \sigma_{k_{\text{max}}}^{(y)} \geq d(x, u_{k_{\text{max}}}) + d(u_{k_{\text{max}}}, y) \geq d(x, y).$$

The bounds (3.31), (3.32) imply that \((GW3)\) holds on $\{z_0, \cdots, z_m\}$ with a constant that can be computed in terms of $C_1$.\]
(Case 1,4): $W_x$ and $W_y$ are both not monotone. The argument for (Case 1,3) can be adapted noting that $d(x, u_{k_{\max}}) \approx \sigma_{l(x)}$.

(Case 2): $\pi(u_{k_{\max}}) \leq \pi(x) \leq \pi(y)$. After reaching $u_{k_{\max}}$, the walk $W$ starts to move in the direction of increasing values of $\pi$.

(Case 2,1): $W_y$ is monotone. There is a $\theta > 0$ depending only on $(P2)$ so that $\sigma_{l+\theta} > 3\sigma_l$ for each $l$, and there is a $C_\theta$ depending on $(P2)$ so that $\sigma_{l+\theta} \leq C_\theta \sigma_l$ for each $l$. Let $l = \lceil \lg d(x, u_{k_{\max}}) \rceil$ and fix $w \in W_y$. If $j(w) \leq \sigma_{l+\theta}$ we have that any walk from $x$ to $w$ must pass through a socket point of order $k_{\max}$ and so:

\begin{equation}
\begin{aligned}
d(w, x) &\geq d(x, u_{k_{\max}}) \geq \sigma_l \geq \sigma_{l+\theta} \\
&\geq j(w).
\end{aligned}
\end{equation}

Let $j(w) > \sigma_{l+\theta}$; then $d(w, x) \geq d(w, u_{k_{\max}}) - d(u_{k_{\max}}, x)$; as $W_y$ is monotone, $d(w, u_{k_{\max}}) \geq j_y(w)$ and so:

\begin{equation}
\begin{aligned}
d(w, x) &\geq j_y(w) - \sigma_l \geq j_y(w); \\
\end{aligned}
\end{equation}

but:

\begin{equation}
\begin{aligned}
j(w) &= j_y(w) + \text{len } W_x \lesssim j_y(w) + \sigma_l \\
&\lesssim j_y(w),
\end{aligned}
\end{equation}

and so $d(w, x) \gtrsim j(w)$ where the constant in the lower bound can be estimated in terms of $C_1$, $C_\theta$ and $\theta$.

(Case 2,2): $W_y$ is not monotone. We decompose $W_y$ as $W_y^{(m)} \cup \{z_0, \ldots, z_m = v_y\}$ and note that we can use (Case 2,1) on $W_y$. For $\{z_0, \ldots, z_m = v_y\}$ one can adapt the argument used in (Case 1,3).

(Case 3): $\pi(x) \leq \pi(y) \leq \pi(u_{k_{\max}})$. This case can be dealt with along the lines of (Case 2) except in the case in which $W_y$ is not monotone, where a different estimate is required on the terminal part $\{z_0, \ldots, z_m = v_y\}$. Any walk from $x$ to $z_i$ must pass through socket points of orders $k_{\max}$ and $k(y)$ so that:

\begin{equation}
\begin{aligned}
d(z_i, x) &\geq d(x, u_{k_{\max}}) + \sigma_{l(x)}; \\
\end{aligned}
\end{equation}

but $W_y$ is not monotone, which implies $\sigma_{l(x)} \approx \text{len } W_y$ which gives:

\begin{equation}
\begin{aligned}
d(z_i, x) &\gtrsim d(x, u_{k_{\max}}) + j_y(z_i); \\
\end{aligned}
\end{equation}

but $d(x, u_{k_{\max}}) \gtrsim \text{len } W_x$ and $j(z_i) = \text{len } W_x + j_y(w_i)$ so that $d(z_i, x) \gtrsim j(z_i)$. □

4. The exponents for which the Poincaré inequality holds

4.1. Geometric characterizations of the Poincaré inequality. The proof of the Poincaré inequality will involve the construction of families of curves joining points in $G$. Overall, we have preferred to avoid using the language of pencils of curves employed by [Sem96, Hei01], and preferred a probabilistic language. The rationale is that our construction is naturally modelled by Markov chains, a fact that also occurs in the examples [CK15]. Specifically, we will deal with measurable functions defined on a probability space which take value in the set of (Lipschitz) curves on a metric space $X$; such maps will be called random curves. To a random curve $\Gamma$ one can associate a measurable function defined on the same probability
space and which takes values in the space of Radon measures on \( X \) by \( \Gamma \mapsto \| \Gamma \| \) (the length measure); such a map will be called a **random measure**. Finally, the maps to the end and starting points of \( \Gamma \), \( \Gamma \mapsto \text{end } \Gamma \) and \( \Gamma \mapsto \text{str } \Gamma \), produce **random points** in \( X \). The support \( \text{spt } \Gamma \) of a random curve \( \Gamma \) is the set of edges that \( \Gamma \) crosses in positive measure with positive probability:

\[
\text{spt } \Gamma = \{ e : P_\Gamma (\| \Gamma \| (e) > 0) > 0 \}.
\]

To disprove the Poincaré inequality we will use the notion of modulus of families of curves, which we now recall.

**Definition 4.2.** Let \( P \geq 1 \) and \( A \) be a family of locally rectifiable curves in the metric space \( X \). We say that a Borel function \( g : X \to [0, \infty] \) is admissible for \( A \) if for each \( \gamma \in A \) one has:

\[
\int g \, d\| \gamma \| \geq 1.
\]

Having fixed a background measure \( \nu \) on \( X \), we define the \( P \)-modulus of \( A \), \( \text{mod}_P(A) \), as the infimum of:

\[
\int g^P \, d\nu
\]

where \( g \) ranges over the set of functions admissible for \( A \). We will be mainly interested in modulus when \( A \) is the family \( A_{p,q} \) of locally rectifiable curves connecting two points \( p, q \), and when \( \nu \) is of the form:

\[
\mu_{p,q}^{(C)} = \left( \frac{d(p, \cdot)}{\mu(B(p, d(p, \cdot)))} \chi_{B(p, C d(p, q))} \right) \left( \frac{d(q, \cdot)}{\mu(B(q, d(q, \cdot)))} \chi_{B(q, C d(p, q))} \right) \mu,
\]

where \( \mu \) is a doubling measure on \( X \) and \( C > 0 \). In this case we will use the notation \( \text{mod}_P(p, q; \mu_{p,q}^{(C)}) \) for the modulus of \( A_{p,q} \) when the background measure is \( \mu_{p,q}^{(C)} \).

We finally recall the definition of the **Riesz potential centred on** \( p \):

\[
\mu_p = \frac{d(p, \cdot)}{\mu(B(p, d(p, \cdot)))} \mu.
\]

The following Theorem summarizes a geometric characterization of \((1, P)\)-Poincaré inequalities. It combines results of Heinonen-Koskela [HK98], Hajłasz-Koskela [HK95], Keith [Kei03], and Ambrosio, Di Marino and Savaré [ADS13], and the proof is included just for the sake of completeness. Note that we will take Theorem 4.7 as the working definition of the Poincaré inequality, and so we will not need to recall the usual definition of the Poincaré inequality.

**Theorem 4.7.** Let \((X, \mu)\) be a complete doubling metric measure space; then \( P \in \text{Ip}_1(X, \mu) \) if and only if one of the following equivalent conditions holds:

1. There is a universal constant \( C \) such that for each pair of points \( p, q \in X \) one has:

\[
d(p, q)^{P-1} \text{mod}_P(p, q; \mu_{p,q}^{(C)}) \geq C;
\]

2. There is a universal constant \( C \) such that any pair of points \( p, q \) can be joined by a random curve \( \Gamma \) satisfying:

\[
\left\| \frac{dE[\| \Gamma \|]}{d\mu_{p,q}^{(C)}} \right\|_{L^Q(\mu_{p,q}^{(C)})} \leq C d(p, q).
\]
Proof. The characterization of the Poincaré inequality in terms of (4.8) is due to Keith [Kei03], who built on previous results of Heinonen-Koskela [Hei01, HK98], and Hajlasz-Koskela [HK95].

Step 1: (1) implies (2).

Consider the set $A$ of locally rectifiable curves joining $p$ to $q$; fix $M$ large to be determined later and write $A = A_{\text{exit}} \cup A_{\text{long}} \cup A_{\text{good}}$, where:

(1) $A_{\text{exit}}$ consists of the locally rectifiable curves in $A$ which meet $X \setminus \overline{B}([p, q], C d(p, q))$ in positive length;
(2) $A_{\text{long}}$ are the locally rectifiable curves in $A \setminus A_{\text{exit}}$ which have length $\geq M d(x, y)$;
(3) $A_{\text{good}}$ are the rectifiable curves in $A \setminus (A_{\text{exit}} \cup A_{\text{long}})$.

We will now fix $\mu^{(C)}_{p, q}$ as the background measure with respect to which we compute moduli; using the test functions $g_{\text{exit}} = 0$ on $\overline{B}([p, q], C d(p, q))$ and $g_{\text{exit}} = \infty$ elsewhere, and $g_{\text{long}} = M d(p, q)$ on $B([p, q], C d(p, q))$ and 0 elsewhere, we see that:

\begin{align}
\text{mod}_P(A_{\text{exit}}) &= 0 \quad (4.10) \\
\text{mod}_P(A_{\text{long}}) &\lesssim \frac{d(p, q)}{(M d(p, q))^P}; \quad (4.11)
\end{align}

thus for $M$ sufficiently large,

\begin{equation}
\frac{d(p, q)^{P-1}}{\nu} \text{mod}_P(A_{\text{good}}) \geq C/2. \quad (4.12)
\end{equation}

Instead of computing modulus on $A_{\text{good}}$ we can compute it on the family of measures:

\begin{equation}
\Sigma_{\text{good}} = \{ H_1^\gamma : \gamma \in A_{\text{good}} \} \quad (4.13)
\end{equation}

Applying the main result of [ADS13] we get a probability $\pi$ on $\Sigma_{\text{good}}$ such that, denoting by $\nu = \int_{\Sigma_{\text{good}}} \eta \, d\pi(\eta)$, we get:

\begin{equation}
\left\| \frac{d\nu}{d\mu^{(C)}_{p, q}} \right\|_{L^Q(\mu^{(C)}_{p, q})} = \text{mod}_P(\Sigma_{\text{good}})^{-1/P}; \quad (4.14)
\end{equation}

using (4.12) we conclude that:

\begin{equation}
\left\| \frac{d\nu}{d\mu^{(C)}_{p, q}} \right\|_{L^Q(\mu^{(C)}_{p, q})} \lesssim d(p, q)^{1/Q}. \quad (4.15)
\end{equation}

Now, to each $\eta \in \Sigma_{\text{good}}$ we can associate a unique unit-speed curve $\gamma : [0, \text{len} \gamma] \to X$ such that $H_1^\gamma = \eta$. Thus $\pi$ becomes the law of a random curve $\Gamma$ with $E[\|\Gamma\|] = \nu$ and then (4.9) follows from (4.15).

Step 2: (2) implies (1).
Take a random curve $\Gamma$ satisfying (4.9) and let $g$ be admissible for the curves joining $p$ to $q$. Then:
\[
1 \leq \mathbb{E} \left[ \int g \, d||\Gamma|| \right] = \int g \, d\mathbb{E}[||\Gamma||]
\]
(4.16)
\[
\leq ||g||_{L^p(p,q)} \left\| \frac{d\mathbb{E}[||\Gamma||]}{d\mathbb{P}(\pi)} \right\|_{L^q(p,q)}
\]
\[
\leq C ||g||_{L^p(p,q)} \cdot d(p,q)^{1/q},
\]
and (4.8) follows minimizing in $g$. $\square$

4.2. Construction of Random curves. In this subsection we construct the ingredients to build the random curves used to verify the Poincaré inequality. This is the subsection where most of the technical work takes place. As we work with walks but need to produce random curves, we define the Lipschitz path associated to a walk as follows.

**Definition 4.17.** To a walk $W = \{w_0, e_1, w_1 \cdots e_l, w_l\}$ we can canonically associate a 1-Lipschitz map $\Gamma_W : [0, \text{len} W] \to G$ by letting $\Gamma_W[p, l+1]$ be a unit speed parametrization of the edge $e_1$.

We now define a notion of lift for walks used in the subsequent constructions.

**Definition 4.18.** Let $W = \{w_0, e_1, w_1 \cdots e_l, w_l\}$ and $w_0'$ a point such that $\pi(w_0') = \pi(w_0)$. We construct a new walk $\{w'_0, e'_1, w'_1 \cdots e'_l, w'_l\}$ as follows. The vertex $w'_{l+1}$ is adjacent to $w'_l$ and is determined as follows. If $w'_l$ is not a socket point the requirement $\pi(w'_{l+1}) = \pi(w_{l+1})$ uniquely determines $w'_{l+1}$. Otherwise, assume that $w'_l$ is a socket point of order $k$ and let $e'_{l+1}$ denote the edge between $w'_l$ and $w'_{l+1}$. We require that $\lambda(e'_{l+1}; k) = \lambda(e_{l+1}; k)$ and $\theta(e'_{l+1}; k) = \theta(e_{l+1}; k)$ for all $k$. We say that $W'$ is the lift of $W$ starting at $w'_0$ and we will denote it by $w'_0 \cdot W$.

We now add some auxiliary definitions used in the constructions, e.g. when concatenating random curves.

**Definition 4.19.** Let $p \in G$ a vertex with $\text{ord}(p) = 0$ and $k \in \mathbb{N}$. Let $F(p, k)$ denote the set of those $p' \in G$ satisfying $\pi(p') = \pi(p)$ and $(\lambda_{p'}(l), \theta_{p'}(l)) = (\lambda_{p}(l), \theta_{p}(l))$ for $l > k$. For $k = 0$ we let $F(p, 0) = \{p\}$. To $F(p, k)$ we can associate a canonical probability measure $P$, i.e. the law of a random point in $F(p, k)$. The probability $P$ satisfies:
\[
P(p') = \frac{w((\lambda_{p'}, \theta_{p'}))}{w((\lambda_{p''}, \theta_{p''}))} (\forall p, p' \in F(p, k)).
\]
(4.20)

For $p' \in F(p, k)$ denote by $s(p')$ the finite string of pairs $\{(\lambda_p(j), \theta_p(j))\}_{j \leq k}$; then:
\[
P(p') = (C_{gw,1}C_{gw,2})^{-k}w(s(p')).
\]
(4.21)

Given $F(p_0, k)$, $F(p_1, k)$ we define a canonical map $\tau : F(p_0, k) \to F(p_1, k)$ so that $\tau(p'_0)$ is the unique point $p'_1 \in F(p_1, k)$ such that $s(p'_0) = s(p'_1)$. Note that $\tau_0 \cdot P_0 = P_1$.

Let $p \in G$ a vertex and $k \in \mathbb{N}$. We denote by $F_{\theta}(p, k)$ the set of those $p' \in G$ satisfying $\pi(p') = \pi(p)$, $\lambda_{p'} = \lambda_p$ and $\theta_{p'}(l) = \theta_{p}(l)$ for $l > k$. As above, to $F_{\theta}(p, k)$
we associate a canonical probability $P$ by requiring:

$$
\frac{P(p')}{P(p'')} = \frac{\omega(\theta_{p'})}{\omega(\theta_{p''})} \quad (\forall p, p' \in F_{\Theta}(p, k)).
$$

We now present the construction of a random curve which goes through a socket point $\xi$ in $G$ if one has a walk that passes through $\xi$. In the following, given a walk $W = \{w_0, e_1, w_1 \cdots e_i, w_i\}$ we denote by $W^{-1}$ the reversed walk $\{w_i, e_i, w_{i-1}, \cdots, e_1, w_0\}$.

**Theorem 4.23.** Let $W_0$ be a monotone walk. Let $p_0 = \text{str} \ W_0$, $\xi = \text{end} \ W_0$. Assume that:

1. $\text{ord}(p_0) = 0$ and $\xi$ is a socket point of order $K \geq k$;
2. $\text{len} \ W_0 \in [\sigma_k, C_0 \sigma_k]$ and all edges of $W_0$ have the same $\Theta$-label $\theta$;
3. There are $(\tau_i)_{1 \leq i \leq k-1} \subset \mathbb{N} \cap [0, \text{len} \ W_0]$ such that the map $i \mapsto \tau_i$ is strictly decreasing, $\text{len} W_0 - \tau_i \in [\sigma_i, C_0 \sigma_i]$, $\tau_i$ is either a gluing or a socket point of order $i$, and if $l \geq \tau_i + 1$ one has $\lambda(e_i; j) = \{\star\}$ for $i \leq j \leq k - 1$;
4. If $w_s \in W$ satisfies $\text{ord}(w_s) \geq k$, then $\lambda_{w_s} = \lambda_{w_{s+1}}$;
5. For an edge $e_i$ of $W_0$ one has the following: if $t \in [1, \tau_{k-1}]$ then $\lambda_{p_0} = \lambda_i$; if $t \in (\tau_{i+1}, \tau_i]$ then $\lambda_i(l) = \lambda_{p_0}(l)$ for $l < i$ or $l \geq k$; if $t \in [\tau_i, \text{len} \ W_0]$ then $\lambda_i(l) = \lambda_{p_0}(l)$ for $l \geq k$.

Fix $J_{\text{cut}} \in \mathbb{N} \cup \{0\}$ and let $p_0'$ be the canonical probability on $F(p_0; k - J_{\text{cut}})$. Construct a random curve $\Gamma$ as follows: choose $p_0' \in F(p_0; k - J_{\text{cut}})$ according to $P_0$ and let $\Gamma = \Gamma_{p_0'}$.

- **(C1):** end $\Gamma$ has law $P_1$, where $P_1$ is the canonical probability on $F(\xi; k - J_{\text{cut}})$;
- **(C2):** spt $\Gamma \subset B(\Gamma_W, C_1 \sigma_{k - J_{\text{cut}}})$;
- **(C3):** To each $e \in \text{spt} \Gamma$ there is associated a unique $\text{in}(e)$ such that $\pi(e) = \pi(\text{in}(e))$, where $\text{in}(e)$ is the $\text{in}(e)$-th edge of $W_0$, and one has:

$$
\frac{d\mathbb{E}[\|\Gamma\|_p]}{d\mu}(\mathbb{E}) \approx C_1 C_{gw,1}^{-\text{lg}(\text{len} W_0 - \text{in}(e))} C_{gw,2}^{k-e - J_{\text{cut}}} \wedge \prod_{j=k}^{\infty} \omega(\lambda(e; j), \theta(e; j))^{-1},
$$

where $C_1$ depends on $J_{\text{cut}}$, $C_0$, (P1)–(P3) and Weight.

**Proof.** We prove (C1). Let $\xi' = \text{end}(p_0' \cdot W_0)$; we use the notation $w_i, e_i$ for the vertices, respectively the edges of $W_0$; we use the notation $w'_i, e'_i$ for the corresponding edges and vertices of $p_0' \cdot W_0$. We note that if $t \geq \tau_i + 1$ (H3) implies that $\lambda(e'_i; l) = \{\star\}$ for $i \leq l \leq k - 1$. We thus conclude that $\lambda(\xi'; l) = \{\star\}$ for $l \leq k - 1$; for $l \geq k$ the label $\lambda_e$ coincides with that of $\lambda_{e_i}$ and so we conclude that $\lambda(\xi'; l) = \lambda(\xi; l)$ for $l \geq k$. Therefore, $\xi'$ is a socket point of order $K$. By (H2) all edges of $W_0$ have the same label $\theta$, and this implies that all edges of $p_0' \cdot W_0$ have the same label $\theta_{p_0}$. As $\pi(\xi') = \pi(\xi)$, we conclude that $\xi'$ is the point of $F_{\Theta}(\xi; k - J_{\text{cut}})$ with label $\theta_{p_0}$ and thus (C1) follows.

We now prove (C2). Note that the $i$-th vertices $w_i, w'_i$ of $W_0$ and $p_0' \cdot W_0$ have $\pi(w_i) = \pi(w'_i)$, and the labels $(\lambda(w_i), \theta(w_i))$, $(\lambda(w'_i), \theta(w'_i))$ and can differ only in the first $k - J_{\text{cut}}$ entries. Hence (C2) follows from Lemma 2.16.
We now prove (C3). First let \( e \in \text{spt} \Gamma \) and assume that \( e = e'_i \in p'_0 \cdot W_0 \), \( e = e''_j \in p''_0 \cdot W_0 \). As the path \( W_0 \) is monotone, \( l = l' \) and there is a unique edge \( e_s \) of \( W_0 \) such that \( \pi(e) = \pi(e_s) \). We can thus associate to \( e \) the unique integer \( \text{in}(e) = s \). We now turn to the proof of (4.24). For \( p'_0 \in \Lambda(p_0, k - J_{\text{cut}}) \) we will denote by \( e(p'_0) \) the \( l \)-th edge of \( p'_0 \cdot W_0 \).

We now fix \( e \in \text{spt} \Gamma \) and assume that \( \text{in}(e) = s \). We first consider the case \( s \in [1, \tau_{k-1}] \). Then by (H5) there is a unique \( p'_0 \in F(p_0; k - J_{\text{cut}}) \) such that \( e \) is the \( s \)-th edge of \( p'_0 \cdot W_0 \). In this case by (H2) - (H3) \( \text{lg}(\text{len} W_0 - \text{in}(e)) \) is comparable to \( k \) up to a multiplicative constant depending on \( C_0 \). Assume now that \( s \in (\tau_i, \tau_{i+1}] \); then \( e \) is the \( s \)-th edge of \( p'_0 \cdot W_0 \) if and only if:

\[
\begin{align*}
\theta_{p'_0} &= \theta_{e_s} \\
\lambda(p'_0; j) &= \lambda(e; j) \quad (1 \leq j < i);
\end{align*}
\]

note also that in this case \( \text{lg}(\text{len} W_0 - \text{in}(e)) \) is comparable to \( i \). Finally by (H3) if \( e \in \text{spt} \Gamma \) is the \( s \)-th edge of \( p'_0 \cdot W_0 \) whenever \( p'_0 \in F(p_0; k - J_{\text{cut}}) \) satisfies \( \theta_{p'_0} = \theta_e \). Note that in this case \( \text{lg}(\text{len} W_0 - \text{in}(e)) \) is comparable to \( 1 \). We can now put all this information together:

\[
(4.25) \quad \theta_{p'_0} = \theta_{e_s} \\
(4.26) \quad \lambda(p'_0; j) = \lambda(e; j) \quad (1 \leq j < i);
\]

and so (4.24) follows by taking the quotient of (4.27) and (4.28).

**Corollary 4.29.** Suppose that \( W_0 \) satisfies the assumptions of Theorem 4.23 and let \( p \in G \). Assume that for some \( C_0 > 0 \) one has:

\[
(4.30) \quad \text{dist}(p, \text{spt} \Gamma) \approx_{C_0} \sigma_k.
\]

Then there is a \( C_1 = C_1(C_0, J_{\text{cut}}) \) such that:

\[
(4.31) \quad \left\| \frac{dE[||\Gamma||]}{d\mu_p} \right\|_{L^Q(\mu_p)} \approx_{C_1} \sum_{k=1}^{C_2} (w^{-1}(\mu_p) C_{w,1})^{(Q-1)\sigma_k}.
\]

**Proof.** By assumption (4.30) we have that on the edges of \( \text{spt} \Gamma \):

\[
(4.32) \quad \frac{d\mu_p}{d\mu} \approx_{C(C_0)} (C_{w,1} C_{w,2})^{-k} \prod_{n=k+1}^{\infty} w((\lambda(p; n), \theta(p; n)))^{-1}.
\]
We now obtain the following estimate using that $W_0[\tau_1, \text{len } W_0]$ has a number of edges $\lesssim \sigma_i$:

\begin{align}
(4.33) \quad \left\| \frac{dE[|\Gamma|]}{d\mu_p} \right\|_{L^Q(\mu_p)}^Q = & \left( \sum_{i \in (\tau_0, \tau_{k-1})} \left( \sum_{i=k-1}^1 \sum_{\substack{\text{in}(c) \in [\tau_1, \tau_i]}} \frac{1}{|G|} \right) \right) \\
& \quad \times \text{weight}(\mu; e) \\
\approx & \sum_{\text{in}(c) \in [0, \tau_{k-1})} (C_{gw,1}C_{gw,2})^{-k} \prod_{n=k+1}^\infty w(\lambda(p; n), \theta(p; n))^{-1} \text{weight}(\mu; e) \\
& \quad + \sum_{i=1}^{k-1} \sum_{\text{in}(c) \in [\tau_i, \tau_{i+1}]} (w^{-1}_{\text{in}(c)}C_{gw,1})^{(k-i)Q} (C_{gw,1}C_{gw,2})^{-k} \\
& \quad \times \prod_{n=k+1}^\infty w(\lambda(p; n), \theta(p; n))^{-1} \text{weight}(\mu; e) \\
& \quad + \sum_{\text{in}(c) \in [\tau_k, \text{len } W]} (w^{-1}_{\text{in}(c)}C_{gw,1})^{Q} (C_{gw,1}C_{gw,2})^{-k} \\
& \quad \times \prod_{n=k+1}^\infty w(\lambda(p; n), \theta(p; n))^{-1} \text{weight}(\mu; e) \\
\approx & \sum_{l=1}^k (w^{-1}_{\text{in}(c)}C_{gw})^{(Q-1)} \sigma_{k-l}. \quad \qedhere
\end{align}

In the following theorem we construct a random curve which moves “parallel” to a given walk $W$.

**Theorem 4.34.** Let $W = \{w_0 e_1 w_1 \cdots e_i w_i \}$ be a monotone walk joining $p_0$ to $p_1$ where $\text{ord}(p_1) = 0$. Let $P_i$ denote the canonical probability measure on $F(p_i; k)$.

To each $p_0' \in F(p_0; k)$ we associate a walk $W_{p_0}'$ as follows. We let $w_0' = p_0$. Then, $e_1'$ and (hence) $w_{i+1}'$ are determined by $w_i'$ and $e_i'$ as follows. First $\pi(e_i') = \pi(e_i)$. If $\text{ord}(w_i') = 0$ or $w_i'$ is not a gluing or a socket point the previous requirement uniquely determines $e_i'$. If $w_i'$ is either a gluing or a socket point of order $> k$ we take the edge $e_i'$ satisfying the additional requirement $(\lambda(e_i'; \text{ord}(w_i')), \theta(e_i'; \text{ord}(w_i')))$ = $(\lambda(e_i; \text{ord}(w_i)), \theta(e_i; \text{ord}(w_i)))$. If $w_i'$ is a socket point of order $\leq k$ then $e_i'$ is determined by the additional requirement that $(\lambda(e_i'; \theta(e_i')) = (\lambda(e_i'; \text{ord}(w_i')), \theta(e_i'; \text{ord}(w_i')))$.

Let $\Gamma$ be the random curve determined by choosing $p_0'$ according to $P_0$ and letting $\Gamma = W_{p_0}'$. Then the following holds:

(C1): end $\Gamma$ has law $P_1$;

(C2): $\text{spt } \Gamma \subset B(\Gamma_W, C\sigma_k)$;
(C3): For $e \in \text{spt } \Gamma$ one has:

\[
\frac{dE[\|\Gamma\|]}{d\mu}|_e \approx_{C_1} (C_{gw,1}C_{gw,2})^{-k} \prod_{j=k}^{\infty} w((\lambda_e(j), \theta_e(j)))^{-1},
\]

where $C_1$ depends on (P1)–(P3) and Weight.

Proof. Fix $\mu$ and let $e \in \text{spt } \Gamma$ imply that:

Thus, for $e \in \text{spt } \Gamma$ there are a unique $t \in \mathbb{N}$ and a unique $p_0 \in F(p_0; k)$ such that $e$ is the $t$-th edge of $W_{p_0}$. We now prove (C1). Observe that the end point $p_1$ of $W_{p_0}$ satisfies:

\[
\lambda(p_1') = \lambda(p_0) \quad \text{and} \quad \theta(p_1') = \theta(p_0).
\]

Then, using the definition of the map $\tau$ in Definition 4.19, we get $p_1' = \tau(p_0)$ and so (C1) follows.

Statement (C2) is proven like in Theorem 4.23. We now show statement (C3). Let $e \in \text{spt } \Gamma$ and let $(t, p_0')$ be the unique pair such that $e$ is the $t$-th edge of $W_{p_0}$. Then:

\[
\text{weight } (E[\|\Gamma\|]; e) = P(p_0) = (C_{gw,1}C_{gw,2})^{-k} \prod_{j=1}^{k} w((\lambda(p_0'; j), \theta(p_0'; j)))
\]

and the result follows dividing (4.39) by weight($\mu; e$). \hfill \blacksquare

Corollary 4.40. Let $W$ satisfy the assumptions of Theorem 4.34 and let $p \in G$. Assume that for some $C_0 > 0$ one has:

\[
\text{dist}(p, \text{spt } \Gamma) \approx_{C_0} \sigma_k,
\]

and that $\text{len } W \leq C_0 \sigma_k$. Then there is a $C_1 = C_1(C_0)$ such that:

\[
\| \frac{dE[\|\Gamma\|]}{d\mu_p} \|_{C^Q(\mu_p)} < C_1 \sigma_k.
\]

Proof. By assumption (4.41) we have

\[
\frac{d\mu_p}{d\mu} \approx_{C(C_0)} (C_{gw,1}C_{gw,2})^{-k} \prod_{n=k+1}^{\infty} (w((\lambda(p; n), \theta(p; n))))^{-1}.
\]

on the edges of $\text{spt } \Gamma$. Then for $e \in \text{spt } \Gamma$ one has:

\[
\frac{dE[\|\Gamma\|]}{d\mu_p} \approx 1.
\]
On the other hand, \( \text{len } W \lesssim \sigma_k \) and so:

\[
(4.45) \quad \left\| \frac{dE[\Gamma]}{d\mu_p} \right\|_{L^Q(\mu_p)}^Q = \sum_{t=1}^{\text{len } W} \sum_{p_0' \in F(\rho_0;k)} \left( \frac{dE[\Gamma]}{d\mu_p} \right)_{\text{t-th edge of } W_{\rho_0}} \times \frac{d\mu_p}{d\mu} \text{weight}(\mu;e) \\
\approx C(C_0) \sum_{t=1}^{\text{len } W} \sum_{p_0' \in F(\rho_0;k)} (C_{gw,1}C_{gw,2})^{-k} \prod_{j=1}^{k} w((\lambda(p_0';j),\theta(p_0';j))) \\
\lesssim \sigma_k.
\]

\[
\blacksquare
\]

In the following theorem we assume that the walk is monotone increasing for concreteness; the same result holds if the walk is monotone decreasing. The goal is to build a random curve which “expands” gaining access to new labels. This is needed to get the estimate (4.9).

**Theorem 4.46.** Let \( W \) be a monotone increasing walk joining \( p_0 \) to \( p_1 \) where \( \text{ord}(p_i) = 0 \) and \( \text{len } W \in [\sigma_k/2, \sigma_k] \). Assume that all edges in \( W \) have the same label. Then there is a \( C_0 \) which depends only on (P1)-(P3) such that the following holds whenever \( J_{\text{cut}} \geq C_0 \). Let:

\[
(4.47) \quad (\lambda(p_0;k - J_{\text{cut}} + 1), \theta(p_0;k - J_{\text{cut}} + 1)) = (s_0,t_0),
\]

and choose \((s_1,t_1) \in \text{Symb}_1 \times \text{Symb}_2 \setminus \{(s_0,t_0)\} \).

Choose by Lemma 2.9 a monotone increasing walk \( W_{0}^{(\text{new})} \) from \( p_0 \) to a socket point \( \xi \) of order \( k - J_{\text{cut}} + 1 \), and which satisfies \( \text{len } W_{0}^{(\text{new})} \leq \text{len } W \). Let \( \hat{p}_1 \in F(p_1;k - J_{\text{cut}} + 1) \) be the point satisfying:

\[
(4.48) \quad (\lambda(\hat{p}_1;j),\theta(\hat{p}_1;j)) = \begin{cases} 
(\lambda(p_1;j),\theta(p_1;j)) & \text{for } j \neq k - J_{\text{cut}} + 1 \\
(s_1,t_1) & \text{for } j = k - J_{\text{cut}} + 1.
\end{cases}
\]

Using Lemma 2.14 obtain a monotone increasing walk \( W_{1/2}^{(\text{new})} \) from \( \xi \) to a point \( \hat{p}_{1/2} \) such that \( \lambda_{\hat{p}_{1/2}} = \lambda_{\hat{p}_1}, \theta_{\hat{p}_{1/2}} = \theta_{\hat{p}_1} \) and:

\[
(4.49) \quad \text{len } W_{1/2}^{(\text{new})} \leq \text{len } W - \text{len } W_{0}^{(\text{new})}.
\]

Finally concatenate \( W_{1/2}^{(\text{new})} \) with a monotone increasing walk whose edges have constant label \( (\lambda_{\hat{p}_1},\theta_{\hat{p}_1}) \) to obtain a walk \( W_{1}^{(\text{new})} \) joining \( \xi \) to \( \hat{p}_1 \) and satisfying:

\[
(4.50) \quad \text{len } W_{0}^{(\text{new})} + \text{len } W_{1}^{(\text{new})} = \text{len } W.
\]

Construct a random curve as follows. Choose \( p_0' \in F(p_0;k - J_{\text{cut}}) \) using the probability \( P_0 \). Then with probability:

\[
(4.51) \quad (C_{gw,1}C_{gw,2})^{-1}w((s_0,t_0)) \quad (\text{event } E^{(\text{old})})
\]
let \( \Gamma \) be the canonical path \( \Gamma_{p'_0; W} \) associated to \( p'_0 \cdot W \). For \((s, t) \neq (s_0, t_0)\) let \( \hat{p}'_{1, s, t} \) be the point in \( F(p_1; k - J_{\text{cut}} + 1) \) such that:

\[
(4.52) \quad (\lambda(\hat{p}'_{1, s, t}; j), \theta(\hat{p}'_{1, s, t}; j)) = \begin{cases} 
(\lambda(p'_0; j), \theta(p'_0; j)) & \text{if } j \neq k - J_{\text{cut}} + 1 \\
(s, t) & \text{if } j = k - J_{\text{cut}} + 1.
\end{cases}
\]

Then with probability:

\[
(4.53) \quad (C_{gw,1}C_{gw,2})^{-1} w((s, t)) \quad \text{(event } E_{s,t}^{(\text{new})})
\]

let \( \Gamma \) be the canonical path associated to the walk:

\[
(4.54) \quad p'_0 \cdot W_{0}^{(\text{new})} \ast (\hat{p}'_{1, s, t} \cdot W_{1}^{(\text{new})})^{-1}.
\]

Then the following hold:

- (C1): end \( \Gamma \) has law \( P_1 \) on \( F(p_1; k - J_{\text{cut}} + 1) \);
- (C2): \( \text{spt} \Gamma \subset B(\Gamma_W, C\sigma_{k-J_{\text{cut}}+1}) \);
- (C3): Let

\[
(4.55) \quad E^{(\text{new})} = \bigcup_{(s, t) \neq (s_0, t_0)} E_{s,t}^{(\text{new})};
\]

let \( \Gamma^{(\text{old})} \) denote \( \Gamma \) conditioned on \( E^{(\text{old})} \) and \( \Gamma^{(\text{new})} \) denote \( \Gamma \) conditioned on \( E^{(\text{new})} \). Then for each \( e \in \text{spt} \Gamma \) there is a unique \( e_{\in(e)} \in \mathbb{N} \) such that \( \pi(e) = \pi(e_{\in(e)}) \) where \( e_{\in(e)} \) is the \( \in(e) \)-th edge of \( W \). If \( e \in \text{spt} \Gamma^{(\text{new})} \) one has:

\[
(4.56) \quad \frac{dE[\|\Gamma^{(\text{new})}\|]}{d\mu}(e) \approx_{C_1} C_{gw,1}^{-T(e)} w^{-k+T(e)} C_{gw,2}^{-k} \prod_{j=k}^{\infty} w((\lambda(e; j), \theta(e; j)))^{-1},
\]

where

\[
(4.57) \quad T(e) = \begin{cases} 
\lg(\text{len } W_0^{(\text{new})} - \in(e)) & \text{if } \max(\pi(e)) \leq \pi(\xi) \\
\lg(\in(e) - \text{len } W_0^{(\text{new})}) & \text{otherwise};
\end{cases}
\]

and if \( e \in \text{spt} \Gamma^{(\text{old})} \) then:

\[
(4.58) \quad \frac{dE[\|\Gamma^{(\text{old})}\|]}{d\mu}(e) \approx_{C_1} (C_{gw,1} C_{gw,2})^{-k} \prod_{j=k}^{\infty} w((\lambda(e; j), \theta(e; j)))^{-1},
\]

where \( C_1 \) depends on \( J_{\text{cut}}, \text{(P1)}-\text{(P3)} \) and Weight.

Proof. We first explain why the construction of the walks \( W_0^{(\text{new})} \), \( W_1^{(\text{new})} \) and \( W_1^{(\text{new})} \) can be carried out. If \( C_0 \) is sufficiently large, one can ensure that whenever \( J_{\text{cut}} \geq C_0 \), and if \( C \) is the constant appearing in Lemmas 2.9, 2.14, one has:

\[
(4.59) \quad 2C\sigma_{k-J_{\text{cut}}} \leq \text{len } W,
\]

and thus one can construct \( W_0^{(\text{new})} \) and \( W_1^{(\text{new})} \) satisfying:

\[
(4.60) \quad \text{len } W_0^{(\text{new})} + \text{len } W_1^{(\text{new})} \leq \text{len } W.
\]

We now explain why the concatenation in (4.54) is well-defined. Note that \( W_0^{(\text{new})} \) and \( (W_1^{(\text{new})})^{-1} \) satisfy the assumptions of Theorem 4.23; referring to the notation of Theorem 4.23, we have to set \( K = k \) where \( k \) is now given by the integer \( k - J_{\text{cut}} + 1 \) used in this Theorem; for \( W_0^{(\text{new})} \) the value of \( J_{\text{cut}} \) now used in
Theorem 4.23 is 0, while for \((W_1^{(new)})^{-1}\) the value of \(J_{cut}\) now used in Theorem 4.23 is 1. Now, Theorem 4.23 ensures that both \(p_0' \cdot W_0^{(new)}\) and \((p_1', t, W_1^{(new)})^{-1}\) end at the point \(\xi' \in F_\theta(\xi; k - J_{cut} + 1)\) such that \(\theta(p_0'; l) = \theta(\xi', l)\) for \(l \neq k - J_{cut} + 1\). Therefore, the concatenation in (4.54) is well-defined.

We now turn to the proof of (C1). Let \(p_0' = \text{str} \Gamma\); conditional on the event \(E^{(old)}\) one has that end \(\Gamma = p_1'\) where \(p_1'\) is the point of \(F(p_1'; k - J_{cut} + 1)\) satisfying \((\lambda_{p_0'}, \theta_{p_0'}) = (\lambda_{p_1'}, \theta_{p_1'})\). The probability of the event:

\[(4.61)\]

\(\{\text{str} \Gamma = p_0'\} \cap E^{(old)}\)

is:

\[(4.62)\]

\[
(C_{gw,1}C_{gw,2})^{-k + J_{cut}} \prod_{n=1}^{k - J_{cut}} w ((\lambda(p_0'; n), \theta(p_0'; n))) \cdot (C_{gw,1}C_{gw,2})^{-1} w ((s_0, t_0))
\]

\[
= (C_{gw,1}C_{gw,2})^{-k + J_{cut} - 1} \prod_{n=1}^{k - J_{cut} + 1} w ((\lambda(p_0'; n), \theta(p_0'; n))).
\]

Conditional on the event \(E^{(new)}_{s,t}\) one has end \(\hat{\Gamma} = p_{1,s,t}^{(new)}\), and the probability of the event

\[(4.63)\]

\(\{\text{str} \Gamma = p_0'\} \cap E^{(new)}_{s,t}\)

is:

\[(4.64)\]

\[
(C_{gw,1}C_{gw,2})^{-k + J_{cut}} \prod_{n=1}^{k - J_{cut}} w ((\lambda(p_0'; n), \theta(p_0'; n))) \cdot (C_{gw,1}C_{gw,2})^{-1} w ((s, t))
\]

\[
= (C_{gw,1}C_{gw,2})^{-k + J_{cut} - 1} \prod_{n=1}^{k - J_{cut} + 1} w ((\lambda(p_{1,s,t}^{(new)}; n), \theta(p_{1,s,t}^{(new)}; n))).
\]

We thus conclude that (C1) holds.

For (C2) we can apply the same argument as in Theorem 4.23.

We now prove (C3). The fact that \(\text{in}(e)\) is well-defined follows from the monotonicity of the walks \(W, W_0^{(new)}\) and \(W_1^{(new)}\). As all edges of \(W\) have the same label, for \(p_0' \in F(p_0'; k - J_{cut})\) one has that \(p_0' : W = W_{p_0'}\), where \(W_{p_0'}\) is defined as in Theorem 4.34. Therefore, the estimate (4.58) on the Radon-Nikodym derivative of \(E \left[ \| \Gamma^{(old)} \| \right]\) can be obtained from (4.35). Let now \(t_\xi = \text{in}(e_\xi)\) where \(e_\xi\) is the last edge of \(W_0^{(new)}\). As remarked above, the walk \(W_0^{(new)}\) satisfies the assumptions of Theorem 4.23. Thus, if \(e \in \text{spt} \Gamma^{(new)}\) and \(\text{in}(e) \leq t_\xi\) we can apply (4.24) to get (4.56) with \(T(e) = \log(\text{len}W_0 - \text{in}(e))\). On the other hand, also the path \((W_1^{(new)})^{-1}\) satisfies the assumptions of Theorem 4.23. In this case the point end \(\Gamma^{(new)}\) avoids the sets of points \(p_1' \in F(p_1'; k - J_{cut} + 1)\) such that:

\[(4.65)\]

\[
(\lambda(p_1'; k - J_{cut} + 1), \theta(p_1'; k - J_{cut} + 1)) = (s_0, t_0);
\]

in applying Theorem 4.23 this can only introduce a multiplicative error lying in \([\|C_{gw,1}C_{gw,2}\|^{-1}, C_{gw,1}C_{gw,2}]\) in the estimate (4.24). Note also that if \(\text{in}(e) \geq t_\xi\), considering the reverse walk \((W_1^{(new)})^{-1}\), the integer \(\text{in}(e)\) in (4.24) must be replaced with \(\text{len}W - \text{in}(e)\) and thus the proof of (4.56) is complete. \(\square\)
Corollary 4.66. Let $W$ be as in Theorem 4.46 and let $p \in G$. Assume that for some $C_1 > 0$ one has:

\begin{equation}
\text{dist}(p, \text{spt} \Gamma) \approx C_1 \sigma_k.
\end{equation}

Then there is a $C_2 = C_2(C_1, J_{\text{cut}})$ such that:

\begin{equation}
\left\| \frac{dE[\|\Gamma\|]}{d\mu_p} \right\|^{Q}_{L^Q(\mu_p)} \approx C_2 \sum_{i=1}^{k}(w_0^{-1}C_{gw})^{(Q-1)}\sigma_{k-i}.
\end{equation}

Proof. We first apply convexity of the $Q$-th power of the $L^Q(\mu_p)$ norm to get:

\begin{equation}
\left\| \frac{dE[\|\Gamma\|]}{d\mu_p} \right\|^{Q}_{L^Q(\mu_p)} = \left\| P(E^{(\text{new})}) \frac{dE[\|\Gamma\|]}{d\mu_p} + P(E^{(\text{old})}) \frac{dE[\|\Gamma\|]}{d\mu_p} \right\|^{Q}_{L^Q(\mu_p)}
\end{equation}

\begin{align*}
\leq & P(E^{(\text{new})}) \left\| \frac{dE[\|\Gamma\|]}{d\mu_p} \right\|^{Q}_{L^Q(\mu_p)} + P(E^{(\text{old})}) \left\| \frac{dE[\|\Gamma\|]}{d\mu_p} \right\|^{Q}_{L^Q(\mu_p)};
\end{align*}

let $t_\xi = \text{in}(\epsilon_\xi)$ where $\epsilon_\xi$ is the last edge of $W_0^{(\text{new})}$. By assumption (4.67) we can apply Corollary 4.29 to $\Gamma^{(\text{new})}[0, t_\xi]$ and $\Gamma^{(\text{new})}[t_\xi, \text{len} W]$. Similarly, by assumption (4.67) we can apply Corollary 4.40 to $\Gamma^{(\text{old})}$. Thus, (4.68) follows substituting (4.31), and (4.42) in (4.69).

4.3. Proof of the Poincaré inequality. In this subsection we join the random curves constructed in Subsection 4.2 to prove the Poincaré inequality.

Definition 4.70. Given $P \geq 1$ we denote by $Q$ the conjugate exponent $P/(P-1)$. Let $I_{\text{neck}}$ denote the range of exponents $P \geq 1$ such that there is a $C = C(P)$ such that for each $k \in \mathbb{N}$ one has:

\begin{equation}
\sum_{i=1}^{k}(w_0^{-1}C_{gw})^{(Q-1)}\frac{\sigma_{k-i}}{\sigma_k} \leq C.
\end{equation}

As $m_k \geq 2$ one has that:

\begin{equation}
\left\{ \log_2(w_0^{-1}C_{gw}) + 1, \infty \right\} \subset I_{\text{neck}};
\end{equation}

on the other hand, if all $m_k$ are equal to some $m$ one has:

\begin{equation}
I_{\text{neck}} = \left\{ \log_m(w_0^{-1}C_{gw}) + 1, \infty \right\}.
\end{equation}

Theorem 4.74. For $P \in I_{\text{neck}}$ the metric measure space $(G, \mu)$ satisfies a $(1, P)$-Poincaré inequality, i.e. $I_{\text{neck}} \subset I_{\text{pt}}(G, \mu)$.

Proof. We apply Theorem 4.7, i.e. for any pair of points $(x, y)$ we show the existence of a random curve $\Gamma$ satisfying:

\begin{equation}
\text{spt} \Gamma \subset B(\{x, y\}, Cd(x, y)),
\end{equation}

\begin{equation}
\left\| \frac{dE[\|\Gamma\|]}{d(\mu_x + \mu_y)} \right\|^{Q}_{L^Q(\mu_x + \mu_y)} \lesssim C_0 d(x, y),
\end{equation}

where $C$ does not depend on $x, y$, and $C_0$ does not depend on $x, y$ but depends on $Q$. $\Gamma$ is built by concatenating curves obtained by using Theorems 4.23, 4.34, 4.46.
We observe that if end $\Gamma_0 = \text{str} \Gamma_1$ the random curves $\Gamma_0, \Gamma_1$, up to translating their domains, can be concatenated to obtain a random curve $\Gamma_0 \ast \Gamma_1$.

**Step 1:** First part of building ‘‘half’’ of a random curve joining $x$ to $y$.

Fix points $x, y$ and assume that $\max \mathbb{N}(x, y) \leq \lg d(x, y)$. This assumption will be removed in Step 2. Using Theorem 3.6 we can choose a good walk from $x$ to $y$ satisfying (GWA1) and (GWA2). We let $K = \lg d(x, y)$. We thus have a uniform constant $C_0$ such that:

\begin{align}
(4.77) \quad & C_0 \sigma_k \geq \text{len } W \\
(4.78) \quad & d(x, w_i) \geq C_0^{-1} i \quad (w_i \in W \text{ is the } i\text{-th vertex}).
\end{align}

For the moment let $C$ be the maximum of the constants occurring at points (C2) of Theorems 4.23, 4.34, 4.46. We can find $C_1 = C(C_0)$, $J_1 = J(C_0)$ such that, if $J \geq J_1$ and $\hat{w}$ satisfies:

\begin{equation}
(4.79) \quad d(\hat{w}, w_i) \leq C \sigma_{\lg i - J},
\end{equation}

then one has:

\begin{equation}
(4.80) \quad d(\hat{w}, x) \geq C^{-1}_1 i.
\end{equation}

We now subdivide $W$ into subwalks $\{W_\alpha\}_{\alpha \in I}$ ($I$ is a finite set of integers), the idea being that $W$ can be thought of as a concatenation of the $\{W_\alpha\}$. More precisely, this can be formalized by using a strictly increasing map $\alpha \rightarrow m_\alpha$, and letting $W_\alpha$ denote the part of $W$ starting at the $m_\alpha$-th vertex $w_{m_\alpha}$ and ending at the $m_{\alpha+1}$-th vertex $w_{m_{\alpha+1}}$. Note that we obtain an order relation $< \text{ on } \{W_\alpha\}_{\alpha \in I}$ where $W_\alpha < W_{\alpha+1}$.

Using the properties of the good walk constructed in Theorem 3.6 we obtain a $J_2$ such that there is a decomposition of $W$ into monotone subwalks $\{W_\alpha\}_{\alpha \in I}$ having the following properties:

- **(Dec1):** For each $k \in \{J_2, \cdots, K\}$ there is a $W_\alpha = W_k^{(\exp)}$ satisfying the assumptions of Theorem 4.46 and:

\begin{equation}
(4.81) \quad \text{dist}(W_\alpha, x) \approx_{C} \sigma_k.
\end{equation}

- **(Dec2):** For each $k \in \mathbb{N}(x, y)$ such that $\theta_x(k) \neq \theta_y(k)$, there is a $W_\alpha = W_k^{(\text{neck})}$ which can be decomposed into subwalks $\tilde{W}_0, \tilde{W}_1$ which satisfy the following: one has $\text{end } \tilde{W}_0 = w_{s(k)} = \text{str } \tilde{W}_1$; moreover, for $J_{\text{cut}} \geq J_2$ the walks $\tilde{W}_0$ and $\tilde{W}_1^{-1}$ satisfy the assumptions of Theorem 4.23 where $\xi = w_{s(k)}$.

- **(Dec3):** For each of the remaining walks $W_\alpha$ there is a $k$ such that:

\begin{align}
(4.82) \quad & \text{len } W_\alpha \leq C \sigma_k \\
(4.83) \quad & \text{dist}(W_\alpha, x) \geq C^{-1}_1 \sigma_k.
\end{align}

$\Gamma$ is constructed by concatenating curves $\Gamma_\alpha$ for each $\alpha \in I$. This is done inductively, and one starts by letting $\Gamma_1 = \Gamma_{W_1}$ with probability 1. The next step depends on which of the conditions **(Dec)** is satisfied by $W_{\alpha+1}$:

- **Case of (Dec1).** We have $W_{\alpha+1} = W_k^{(\exp)}$ and we know that end $\Gamma_\alpha$ is a random point in $F(w_{m_{\alpha+1}}; k - J_{\text{cut}})$ whose law is the canonical probability. We obtain $\Gamma_{\alpha+1}$ applying Theorem 4.46, so that end $\Gamma_{\alpha+1}$ is a random point.
in \( F(w_{m_{a+1}}; k - J_{\text{cut}} + 1) \) whose law is the canonical probability. Moreover, by (4.81) we can apply Corollary 4.66 to conclude that:

\[
\left\| \frac{dE[\Gamma_{\alpha+1}]}{d\mu_x} \right\|_Q^{L^Q(\mu_x)} \lesssim C_2 \sum_{l=1}^k (w_l^{-1} C_{gw})^l(Q-1)\sigma_l,
\]

where \( C_2 \) is a uniform constant depending on the constants \( C_0, C_1, C, J_0, J_1, J_{\text{cut}} \).

Moreover, by the assumption on \( P \) we have that there is a uniform constant \( C_3 \) depending on \( C_2 \) and \( Q \) such that:

\[
\left\| \frac{dE[\Gamma]}{d\mu_x} \right\|_Q^{L^Q(\mu_x)} \lesssim C_3 \sigma_k.
\]

**Case of (Dec2).** We have \( W_{\alpha+1} = W_k^{(\text{neck})} \) and we know that \( \text{end} \Gamma_\alpha \) is a random point in \( F_\Theta(w_{m_{a+1}}; k - J_{\text{cut}}) \) whose law is the canonical probability.

We apply Theorem 4.23 to build \( \Gamma_0 \) from \( W_0 \). We then take the canonical probability on \( F_\Theta(w_{m_{a+2}}; k - J_{\text{cut}}) \) and use again Theorem 4.23 to build \( \Gamma_1 \) from \( W_1^{-1} \). We obtain \( \Gamma_{\alpha+1} \) by concatenating \( \Gamma_0 \) and \( \Gamma_1^{-1} \) subject to the following additional prescription; suppose that \( \text{str} \Gamma_0 = p_0' \); then one takes \( \text{str} \Gamma_1 = \tau(p_0') \) where \( \tau : F(w_{m_{a+1}}; k - J_{\text{cut}}) \to F(w_{m_{a+2}}; k - J_{\text{cut}}) \) is the canonical map of Definition 4.19. Note that:

\[
\text{spt} \Gamma_0 \cap \text{spt} \Gamma_1 = \{ w_{s(k)} \},
\]

as the labels of the edges in \( \text{spt} \Gamma_0 \) and \( \text{spt} \Gamma_1 \) have different \( k \)-th entries. Moreover, as \( \xi = w_{s(k)} \) and \( d(x, w_{s(k)}) \approx C_\sigma_k \), we can apply Corollary 4.29 to obtain the estimate:

\[
\left\| \frac{dE[\Gamma_{\alpha+1}]}{d\mu_x} \right\|_Q^{L^Q(\mu_x)} \approx C_2 \sum_{l=1}^k (w_l^{-1} C_{gw})^l(Q-1)\sigma_{k-l},
\]

where \( C_2 \) is a uniform constant depending on the constants \( C_0, C_1, C, J_0, J_1, J_{\text{cut}} \).

Moreover, by the assumption on \( P \) we have that there is a uniform constant \( C_3 \) depending on \( C_2 \) and \( Q \) such that:

\[
\left\| \frac{dE[\Gamma]}{d\mu_x} \right\|_Q^{L^Q(\mu_x)} \lesssim C_3 \sigma_k.
\]

**Case of (Dec3).** We know that \( \text{end} \Gamma_\alpha \) is a random point in \( F(w_{m_{a+1}}; k - J_{\text{cut}}) \) and that \( \text{len} W_{\alpha+1} \leq C\sigma_k \). We build \( \Gamma_{\alpha+1} \) by applying Theorem 4.34. In particular, the assumptions of Corollary 4.40 are also met an so we have:

\[
\left\| \frac{dE[\Gamma_{\alpha+1}]}{d\mu_x} \right\|_Q^{L^Q(\mu_x)} \lesssim C_2 \sigma_k,
\]

where \( C_2 \) is a uniform constant depending on the constants \( C_0, C_1, C, J_0 \) and \( J_1 \).

Note that by the choice of \( C_1 \), if \( \text{spt} \Gamma_\alpha \cap \text{spt} \Gamma_\beta \neq \emptyset \), then \( |\alpha - \beta| \leq C_4 \), where \( C_4 \) is a uniform constant. We thus obtain that:

\[
\left\| \frac{dE[\Gamma]}{d\mu_x} \right\|_Q^{L^Q(\mu_x)} \lesssim C_4 \, d(x, y)
\]
and that for some uniform $C$:

\[(4.91)\quad \text{spt } \Gamma \subset B(x, Cd(x, y)).\]

**Step 2: Modifying Step 1 if $\max N(x, y) > \lg d(x, y)$.**

In this case $W$ is given by Theorem 3.17. If $\theta_x(k_{\text{max}}) = \theta_y(k_{\text{max}})$ the construction can proceed as in **Step 1** because at $u_{k_{\text{max}}}$ there is no change of the $\theta$-label.

We now discuss the modifications for the case $\theta_x(k_{\text{max}}) \neq \theta_y(k_{\text{max}})$. We first enlarge $W$ at $w_i = u_{k_{\text{max}}}$ by inserting $4$ subwalks $\{W_i^i\}_{i=0}^3$ between $w_i$ and $w_{i+1}$. Let $M = \lg d(x, u_{k_{\text{max}}})$, and let $e$ denote the edge of $W$ before $u_{k_{\text{max}}}$. We take $\tilde{W}_0$ to be a monotone geodesic walk whose edges have all the same label $(\lambda_e, \theta_e)$, with length $\tilde{W}_0 = \sigma_M$ and $d(\tilde{W}_0, x) \geq C_1^{1-\sigma_M}$. For $W_1$ we take $W_0^{-1}$. Let now $e$ denote the edge of $W$ after $u_{k_{\text{max}}}$. Then $\tilde{W}_2$ is a monotone geodesic walk whose edges have all the same label $(\lambda_e, \theta_e)$, with length $\tilde{W}_2 = \sigma_M$ and $d(\tilde{W}_2, x) \geq C_1^{-1}\sigma_M$. For $W_3$ we take $W_2^{-1}$.

One then proceeds as in **Step 1**, by subdividing $W$. The subdivision must satisfy the additional requirement that the $\{W_i^i\}_{i=0}^3$ are subwalks of the subdivision, and we have only to specify how to construct the corresponding $\{\tilde{\Gamma}_i\}_{i=0}^3$. On $W_0$ we apply Theorem 4.34 and Corollary 4.40 and obtain the estimate:

\[(4.92)\quad \left\| \frac{d E[\|\tilde{\Gamma}_0\|]}{d\mu_x} \right\|_{L^Q(\mu_x)} \lesssim C_2 \sigma_M.\]

Then $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are built by applying Theorem 4.23 and Corollary 4.29 to $\tilde{W}_1$ and $\tilde{W}_2^{-1}$ respectively. Note that $\text{str } \tilde{\Gamma}_2$ is taken to be a random point in $F(\text{str } \tilde{W}_2^{-1}; M - J_{\text{cut}})$ whose law is the canonical probability. We build $\tilde{\Gamma}_1$ by concatenating $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2^{-1}$ with the additional prescription that if $\text{str } \tilde{\Gamma}_1 = p_1'$ then $\text{str } \tilde{\Gamma}_2 = \tau(p_1')$ where

\[(4.93)\quad \tau : F(\text{str } \tilde{W}_1; M - J_{\text{cut}}) \to F(\text{str } \tilde{W}_2^{-1}; M - J_{\text{cut}})\]

is the canonical map of Definition 4.19. We thus obtain the estimate:

\[(4.94)\quad \left\| \frac{d E[\|\tilde{\Gamma}_1\|]}{d\mu_x} \right\|_{L^Q(\mu_x)} \lesssim C_3 \sigma_M.\]

Finally, $\tilde{\Gamma}_3$ is obtained by applying Theorem 4.34 and Corollary 4.40 to $\tilde{W}_3$. We then have the estimate:

\[(4.95)\quad \left\| \frac{d E[\|\tilde{\Gamma}_3\|]}{d\mu_x} \right\|_{L^Q(\mu_x)} \lesssim C_2 \sigma_M.\]

With these modifications, one obtains (4.90), (4.91) where the constants have possibly worsened compared to **Step 1**.

**Step 3: building a random curve satisfying (4.75), (4.76).**

Fix $x, y \in G$ at distance $> 1$. We choose a vertex $z$ of order 0 satisfying:

\[(4.96)\quad \left| d(z, x) - \frac{d(x, y)}{2} \right| \leq 1\]

\[(4.97)\quad \left| d(z, y) - \frac{d(x, y)}{2} \right| \leq 1;\]
we then choose $J_{\text{cut},x}$ and $J_{\text{cut},y}$ larger than $J_2$ of Step (a) 1, 2 such that:

\begin{align}
|J_{\text{cut},x} - J_2| &\leq 3 \\
|J_{\text{cut},y} - J_2| &\leq 3 \\
\lg d(z,x) - J_{\text{cut},x} &\leq \lg d(z,y) - J_{\text{cut},y}.
\end{align}

We then construct random curves $\Gamma_x$ connecting $x$ to $F(z; \lg d(z,x) - J_{\text{cut},x})$, and $\Gamma_y$ connecting $y$ to $F(z; \lg d(z,y) - J_{\text{cut},y})$ using Steps 1, 2. Note that (4.100) implies that $\end \Gamma_x$ and $\end \Gamma_y$ have the same law. We can thus obtain $\Gamma$ by concatenating $\Gamma_x$ and $\Gamma_y^{-1}$. Now (4.75) follows from (4.91) and (4.96), (4.97). On the other hand, (4.76) follows from (4.90) and:

\begin{equation}
\left\Vert \frac{dE[[\Gamma]]}{d(\mu_x + \mu_y)} \right\Vert^Q_{L^Q(\mu_x + \mu_y)} \leq \left\Vert \frac{dE[[\Gamma_x]]}{d\mu_x} \right\Vert^Q_{L^Q(\mu_x)} + \left\Vert \frac{dE[[\Gamma_y]]}{d\mu_y} \right\Vert^Q_{L^Q(\mu_y)} \lesssim C_Q d(x,y).
\end{equation}

\[ \square \]

4.4. Lack of the Poincaré inequality. To show that a $(1,P)$ Poincaré inequality does not hold if $P$ is sufficiently small, we produce pairs of points such that the modulus estimate (4.8) does not hold.

**Lemma 4.102.** Fix a constant $C_0 \geq 1$; then there are constants $M = M(C_0), l = l(C_0)$ such that the following holds. Let $(\lambda, \theta)$ be labels such that $\lambda(j) = \{♠\}$ for $j \leq k + M$. Let $m \in \mathbb{Z}$ have order $M + k$ and let $R = 3C_0\sigma_k$. In the box

\begin{equation}
B_{\text{bad}} = \text{Box}([m - R, m + R], (\lambda, \theta), k + l)
\end{equation}

select two points $p_0, p_1$ such that:

1. $\pi(p_0) = m - \sigma_k$ and $\pi(p_1) = m + \sigma_k$;
2. $\lambda_{p_0} = \lambda_{p_1}$;
3. $\theta(p_0; j) = \theta(p_1; j)$ if $j \neq k + M$ and $\theta(p_0; k + M) \neq \theta(p_1; k + M)$.

Then there is a constant $C_1(C_0, P)$ such that:

\begin{equation}
d(p_0, p_1)^{P-1} \mod P(p_0, p_1; \mu_{p_0, p_1}^{(C_0)}) \leq \frac{C_1}{(k-1)^P} \sum_{i=1}^{k-1} \left( \frac{\sigma_k}{\sigma_i} \right)^{P-1} (\theta^0(\bullet) C_{w,1}^{(C_0,1)})^{k-1-i}.
\end{equation}

**Proof.** Let $\xi \in B_{\text{bad}}$ denote the socket point of label $(\lambda, \theta)$ such that $\pi(\xi) = m$. Let $\gamma$ be a continuous curve joining $p_0$ to $p_1$. Note that by possibly enlarging $C_0$ we have $d(p_0, p_1) \approx C_0 \sigma_k$ and so for $l(C_0)$ sufficiently large, by Lemma 2.16 we have:

\begin{equation}
B(p_0, p_1) \subset B_{\text{bad}}.
\end{equation}

If $M(C_0)$ is sufficiently large, the only integer of order $k + M$ contained in $\pi(B_{\text{bad}})$ is $m$. To estimate $\mod P(p_0, p_1; \mu_{p_0, p_1}^{(C_0)})$ we need to produce an appropriate Borel function $g$. For the moment we let $g = \infty$ on $B_{\text{bad}}^c$ and then the case of interest becomes when $\gamma$ stays in $B_{\text{bad}}$; in particular, $\gamma$ must pass through a socket point $\xi' \in F_0(\xi; k + l)$.
Let \( s \in \text{dom}\, \gamma \) be the first time when \( \gamma(s) \in F_\Theta(\xi; k + l) \) and let \( \gamma_1 = \gamma|[0, s] \). Note that:

\[
(4.106) \quad [m - \sigma_{k-1}, m] \subset \pi \circ \gamma([0, s]);
\]

for \( i < k \) let \( t_i = m - \sigma_i \) and let \( \phi_i \) be the last time such that \( \pi \circ \gamma_1(\phi_i) = t_i \). Let \( E(i) \) denote the set of edges \( e \in B_{\text{bad}} \) such that \( \pi(e) \subset [t_i, t_{i-1}] \). As there are no integers of order \( i \) in \([t_i, m]\) we conclude that the curve \( \gamma_1[\phi_i, \phi_{i-1}] \) passes through edges \( \{e_1, \ldots, e_i\} \subset E(i) \) such that:

\[
(4.107) \quad \int g \, d\Sigma^1(\gamma) \geq \sum_{i=1}^{k-1} \int_{\Theta_i} \chi_{E(i)}(\gamma) g d\Sigma^1(\gamma)
\]

\[
\geq \sum_{i=1}^{k-1} \int_{\Theta_i} \chi_{E(i)}(\gamma(\tau)) g(\gamma(\tau)) \, d\tau
\]

\[
\geq \sum_{i=1}^{k-1} \frac{\sigma_i - \sigma_{i-1}}{(k-1)(\sigma_i - \sigma_{i-1})} = 1,
\]

where we let \( \sigma_0 = 0 \).

Note now that \( \gamma_1[\phi_{k-1}, s] \) is at distance \( \approx C_2 \sigma_k \) from \( p_0, p_1 \), where \( C_2 \) is a uniform constant. Therefore, we have:

\[
(4.108) \quad \frac{d\mu(C_{\phi_0, p_1})}{d\mu} \big| (\gamma_1[\phi_{k-1}, s]) \approx C(C_2) (C_{gw,1} C_{gw,2})^{-k-1} \prod_{j \geq k-1} w(\lambda(p_0; j), \theta(p_0; j))^{-1};
\]

note that \( g \neq 0 \) in \( B_{\text{bad}} \) only on \( \bigcup_{i=1}^{k-1} E(i) \), and let \( \bar{E}(i) \) denote the set of edges of \( E(i) \) satisfying \((E(i), 3)\) and \((E(i), 4)\); as \( g \) vanishes on \( E(i) \setminus \bar{E}(i) \), we have for some \( C_1(C_0, C_2, P, M, l) \):

\[
\int g^P \, d\mu(C_{\phi_0, p_1}) \lesssim C_1 \sum_{i=1}^{k-1} \frac{1}{(k-1)^P \sigma_i^P} (C_{gw,1} C_{gw,2})^{-k+1}
\]

\[
\times \sum_{e \in \bar{E}(i)} \prod_{j \leq k-1} w(\lambda_e(j), \theta_e(j))
\]

\[
\lesssim C_1 \sum_{i=1}^{k-1} \frac{\sigma_i}{(k-1)^P \sigma_i^P} (C_{gw,1} C_{gw,2})^{-k+1}
\]

\[
\times w^{k-1-i} C_{gw,1}^{k+i} C_{gw,2}^{k+i}
\]

\[
\lesssim C_1 \sum_{i=1}^{k-1} \frac{\sigma_i}{(k-1)^P \sigma_i^P} (w(\dot{\gamma}) C_{gw,1}^{-1})^{k-1-i},
\]

from which (4.104) follows. \qed
Theorem 4.110. If $P \leq 1 + \log_N(w^{-1}(\{A\})C_{gw,1})$ then $P \not\in I_{p1}(G, \mu)$. Moreover, if all the $m_k$ are equal to some $m$, $I_{\text{neck}} = I_{p1}(G, \mu)$.

Proof. We show that for any value of $C$, (1) in Theorem 4.7 fails. For any $k \geq 1$ we can find a bad box $B_{\text{bad}}$ satisfying the assumptions of Lemma 4.102. Hence we find sequences of pairs of points $(p_0^{(k)}, p_1^{(k)}) \in G^2$ such that:

$$(4.111)$$

$$d \left( p_0^{(k)}, p_1^{(k)} \right) \left( P_0^{(k)}, P_1^{(k)} ; \mu^{(C)} \right) \leq \left( \frac{C}{k-1} \right)^{P-1} \sum_{i=1}^{k-1} \left( \frac{\sigma_i}{\sigma_i} \right)^{P-1} \left( w^{(i)} C_{gw,1}^{-1} \right)^{k-1-i}.$$

As $P \leq 1 + \log_N(w^{-1}(\{A\})C_{gw,1})$ and as $\sigma_i/\sigma_i \leq N^{k-i}$, the rhs. of (4.111) goes to 0 as $k \to \infty$.

If all the $m_k$ are equal to some $m$, then the rhs. of (4.111) goes to 0 exactly when $P \leq 1 + \log_m(w^{-1}(\{A\})C_{gw,1})$. □

Remark 4.112. Note that as $w^{-1}(\{A\})C_{gw,1} \not\to \infty$ one has $\min I_{p1}(G, \mu) \to \infty$, i.e. the range of exponents for which a Poincaré inequality holds gets narrower and narrower. On the other hand, as $w^{-1}(\{A\})C_{gw,1} \to 1$, $\min I_{p1}(G, \mu) \to 1$ and thus the range of exponents for which the Poincaré inequality holds can be arbitrarily prescribed. However, as either $w^{-1}(\{A\})C_{gw,1}$ goes to 0 or $\infty$, the doubling constant of $\mu_G$ blows up.

5. Stability under blow-up

In this section we show how to use $G$ to construct a metric measure space $X$ which satisfies the conclusion of Theorem 1.1. From now on the measure on $G$ that we constructed in Section 2 will be denoted by $\mu_G$. In this section we often deal with balls of different spaces, and so at times we add a subscript to them to distinguish the space to which they belong. Given a metric space $X$, we use $\lambda X$ to denote $X$ with the metric rescaled by the factor $\lambda > 0$.

5.1. Asymptotic cones. In this subsection we define asymptotic cones and construct the example $X$.

Definition 5.1. An asymptotic cone of a metric measure space $(X, \mu)$ is a measured pointed Gromov-Hausdorff limit of a sequence of rescalings:

$$(5.2)$$

$$\left( \frac{\lambda_n^{-1} X}{\mu(B_X(p_n, \lambda_n))}, p_n \right)$$

where $\lim_{n \to \infty} \lambda_n = \infty$. Note that $B_X(p_n, \lambda_n)$ denotes a ball of radius $\lambda_n$ in $X$, that is a ball of radius 1 in $\lambda_n^{-1} X$. The set of asymptotic cones of $(X, \mu)$ will be denoted by as-Con$(X, \mu)$. Note that it would be more appropriate to say that as-Con$(X, \mu)$ is a set of equivalence classes of metric spaces under measure-preserving isometries, but we will avoid such subtleties in the following discussion.

Definition 5.3. A weak tangent $(Y, \nu, q)$ of a metric measure space $(X, \mu_X)$ is a measured pointed Gromov-Hausdorff limit of a sequence of rescalings:

$$(5.4)$$

$$\left( \frac{\lambda_n X}{\mu_X(B_X(p_n, \lambda_n^{-1}))}, p_n \right)$$
where \( \lim_{n \to \infty} \lambda_n = \infty \). The set of weak tangents of \((X, \mu_X)\) will be denoted by \( \text{w-Tan}(X, \mu_X) \).

In the case of \((G, \mu_G)\) the fact that asymptotic cones exist and that the corresponding measures are doubling with uniformly bounded doubling constants follows from a standard compactness argument.

**Lemma 5.5.** The set of asymptotic cones \( \text{as-Con}(G, \mu_G) \) is closed under the operation of taking weak tangents, i.e. whenever \((X, \mu_X, p) \in \text{as-Con}(G, \mu_G)\) one has \( \text{w-Tan}(X, \mu_X) \subset \text{as-Con}(G, \mu_G) \).

**Proof.** On the metric level, the proof is straightforward using that one can approximate a weak tangent \((Y, \mu_Y, q) \in \text{w-Tan}(X, \mu_X)\) by rescaling an approximating sequence for \((X, \mu_X, p)\). There is, however, an issue with normalization of balls which is addressed in the following lemma. \( \square \)

**Lemma 5.6.** Let \((X, \mu, p) \in \text{as-Con}(G, \mu_G)\) and consider a sequence of rescalings:

\[
\left( \lambda_n^{-1} G, \frac{\mu_G}{\mu_G(B_X(p_n, \lambda_n))} p_n \right) \to (X, \mu, p).
\]

Then for each \( t \geq 0 \) one has:

\[
\lim_{n \to \infty} \nu_n(B_G(p_n, \lambda_n t)) = \mu(B_X(p, t)).
\]

**Proof.** Using that \( n \mapsto \nu_n(G) \) is lower semicontinuous if \( G \) is open and upper semicontinuous if \( G \) is compact, it suffices to show that one has, uniformly in \( p_n, \lambda_n \):

\[
\frac{\mu_G(B(p_n, \lambda_n t) \setminus B(p_n, \lambda_n(t - \varepsilon)))}{\mu_G(B(p_n, \lambda_n t))} \leq O(\varepsilon^{1/2}).
\]

For \( s \in (0, 1) \) let \( L(s) \) denote the set of labels \((\lambda, \theta)\) of edges intersecting \( \partial B(p_n, \lambda_n(1 - s)t) \). Note that \( s_1 < s_2 \) implies \( L(s_2) \supset L(s_1) \). However, as:

\[
\frac{\lambda_n(1 - s)t}{\lambda_n(1 - \varepsilon^{1/2})t} \leq \frac{3}{2}
\]

for \( \varepsilon \) sufficiently small and \( s \geq \varepsilon \), any label \((\lambda, \theta) \in L(s) \setminus L(\varepsilon^{1/2})\) can differ from a label in \( L(\varepsilon^{1/2}) \) only at the \( j \)-th entry, where either:

\[
j \in \left\{ \lfloor 2\lambda_n(1 - \varepsilon^{1/2})t \rfloor, \lfloor 2\lambda_n(1 - \varepsilon^{1/2})t \rfloor + 1 \right\},
\]

or \( j = j_0 \), where \( j_0 \) is some fixed integer \( > \lfloor 2\lambda_n(1 - \varepsilon^{1/2})t \rfloor + 1 \) (this can occur if the ball \( B(p_n, \lambda_n t) \) contains a socket point of order greater than \( \lfloor 2\lambda_n t \rfloor \)). We thus obtain:

\[
\frac{\mu_G(B(p_n, \lambda_n t) \setminus B(p_n, \lambda_n(t - \varepsilon)))}{\mu_G(B(p_n, \lambda_n t))} \leq (C_{gw,1}C_{gw,2})^3 \frac{\lambda_n \varepsilon t}{\lambda_n \varepsilon^{1/2} t},
\]

from which (5.9) follows. \( \square \)

We will use a discretization procedure of Gill and Lopez [GL14] that allows to compare PI spaces and graphs. We rephrase their result in a slightly more general context, where there is more freedom in the choice of the approximating graph; the proof is omitted being a straightforward generalization of their argument.
**Theorem 5.13.** Let $H$ be a connected graph whose metric is a constant multiple of the length metric. For $\varepsilon > 0$ and $C_0 > 0$ consider a subset $V$ of vertices of $H$ which is an $\varepsilon$-separated net and $C_0\varepsilon$-dense. Assume that for some $C_1 \geq 0$ there is a $C_1$-biLipschitz embedding $F: V \to X$ such that $F(V)$ is $C_1\varepsilon$-dense in $X$. Let $\mu_X$ be a doubling measure on $X$ with constant $C_2$. Let $\mu_H$ be a doubling measure on $H$ which restricts to a multiple of arclength on each edge and such that one has, for some $C_3 > 0$:

\begin{equation}
\mu_H(B_H(v, r)) \approx_{C_3} \mu_X(B_X(F(v), r)) \quad (\forall (v, r) \in V \times [\varepsilon, \infty)).
\end{equation}

Then $I_{\text{Pt}}(X, \mu_X) \subset I_{\text{Pt}}(H, \mu_H)$; moreover, if $C_X(P)$ denotes the constant of the $(1, P)$-Poincaré inequality in $(X, \mu_X)$, then the corresponding constant $C_H(P)$ in $(H, \mu_H)$ satisfies:

\begin{equation}
C_H(P) \leq C(C_0, C_1, C_2, C_3, C_X(P), \varepsilon).
\end{equation}

Since we work with pointed measured Gromov-Hausdorff convergence we need a local version of Theorem 5.13.

**Corollary 5.16.** In Theorem 5.13, assume that $V$ is not $C_0\varepsilon$-dense in the whole of $H$, but that $V$ now lies in a ball $B_H(h, R)$ with $R > 0$ in which it is $C_0\varepsilon$-dense. Assume also that $F(V)$ contains a $C_1\varepsilon$-dense set in a ball $B_X(x, C_1^{-1}R)$. Furthermore, assume that $X$ is geodesic. Then the conclusion of Theorem 5.13 holds replacing $(H, \mu_H)$ with:

\begin{equation}
\left(\hat{B}_H(h, \hat{C}^{-1}R), \mu_H \ll B_H(h, \hat{C}^{-1}R)\right),
\end{equation}

where $\hat{C}$ depends only on $C_0, C_1, C_2$, and $\varepsilon$.

**Proof.** One can reduce this local case to the global one, Theorem 5.13, by recalling that if $(X, \mu)$ is geodesic and admits a $(1, P)$-Poincaré inequality with exponent $C(P)$, there is a $C_1(C(P))$ such that each for each $R > 0$ the metric measure space $(B(x, R), \mu \ll B(x, R))$ admits a $(1, P)$-Poincaré inequality with constant $C_1$ (see [HK95]).

**Theorem 5.18.** Let $(X, \mu_X, p) \in \text{as-Con}(G, \mu_G)$; then:

\begin{equation}
I_{\text{Pt}}(X, \mu_X) = I_{\text{Pt}}(G, \mu_G).
\end{equation}

**Proof.** Step 1: $I_{\text{Pt}}(X, \mu_X) \subset I_{\text{Pt}}(G, \mu_G)$.

Let

\begin{equation}
\left(\lambda_n^{-1}G, \mu_G(B_G(p_n, \lambda_n)), p_n\right) \to (X, \mu_X, p)
\end{equation}

and assume that $P \in I_{\text{Pt}}(X, \mu_X)$, $C(P)$ being the corresponding constant. Choose $N(n)$ such that:

\begin{equation}
1 \leq \frac{\lambda_n}{\sigma_{N(n)}} \leq N
\end{equation}
and pass to a subsequence such that \( \lim_{n \to \infty} \frac{\lambda_n}{\sigma_{N(n)}} \) exists. Therefore, up to rescaling the metric on \( X \) by a factor in \([1/N, 1]\) we can assume that:

\[
(5.22) \quad \left( \sigma^{-1}_{N(n)} G, \nu_n, p_n \right) \to (X, \mu_X, p);
\]

note also that \((X, \mu_X)\) is geodesic being a limit of geodesic metric spaces. Fix \( \varepsilon, R > 0 \); for \( n \geq D_0(R, \varepsilon) \) we can assume that the Gromov-Hausdorff distance between \( B_{G_n}(p_n, R) \) and \( B_X(p, R) \) is at most \( \frac{1}{2} \). Now the vertices of order \( \geq l \) in \( G \) form a maximal \( \sigma_l \)-net which becomes a maximal \( \sigma_l \sigma_{N(n)}^{-1} \)-net in \( G_n \); for each \( n \) we choose \( N_\varepsilon(n) \leq N(n) \) such that:

\[
(5.23) \quad \varepsilon \leq \sigma_{N_\varepsilon(n)} \sigma_{N(n)}^{-1} \leq N_\varepsilon.
\]

Let \( V(n; \varepsilon) \) be the set of vertices of \( G_n \) whose order in \( G \) is at least \( N_\varepsilon(n) \) and which are contained in \( B_{G_n}(p_n, R) \). Then \( V(n; \varepsilon) \) is an \( \varepsilon \)-separated net in \( B_{G_n}(p_n, R) \) and is also \( N_\varepsilon \)-dense there. Thus the cardinality of \( V(n; \varepsilon) \) is uniformly bounded in \( n \) and \( V(n; \varepsilon) \to W \) in the Hausdorff sense where \( W \) is a \( \frac{3}{6} \)-separated net in \( B_X(p, R) \) in which it is also \( \frac{1}{4} N_\varepsilon \)-dense. Therefore for \( n \geq D_0(R, \varepsilon) \) we find an \( L \)-biLipschitz map:

\[
(5.24) \quad F_n : V(n; \varepsilon) \to W;
\]

where \( L \) does not depend on \( \varepsilon \) or \( n \). Now, as the cardinalities of \( V(n; \varepsilon) \) and \( W \) are uniformly bounded, for \( n \geq D_1(R, \varepsilon) \) we can assume that the sets \( V(n; \varepsilon) \) and \( W \) have the same cardinality and write \( V(n; \varepsilon) = \{ v^{(n)}_\alpha \}_{\alpha \in A} \) and \( W = \{ w_\alpha \}_{\alpha \in A} \) so that \( F_n(v^{(n)}_\alpha) = w_\alpha \) for each \( \alpha \in A \). We now use a variation on the argument of Lemma 5.6 (where we take balls not centred on the basepoints) to conclude that for each \( r \in [\varepsilon, R] \) one has:

\[
(5.25) \quad \nu_n \left( B_{G_n}(v^{(n)}_\alpha, R) \right) \to \mu_X \left( B_X(w_\alpha, R) \right);
\]

so for \( n \geq D_2(R, \varepsilon) \) we can assume that:

\[
(5.26) \quad \nu_n \left( B_{G_n}(v^{(n)}_\alpha, R) \right) \approx_{1+\varepsilon} \mu_X \left( B_X(w_\alpha, R) \right).
\]

We now apply Corollary 5.16 and find \( C_{\text{cut}} = C_{\text{cut}}(\varepsilon) \) such that

\[
(5.27) \quad \left( \bar{B}_{G_n}(p_n, R/C_{\text{cut}}), \nu_n, \bar{B}_{G_n}(p_n, R/C_{\text{cut}}) \right)
\]

admits a \((1, P)\)-Poincaré inequality with constant \( C_{P1} = C(C(P), \varepsilon) \). By rescaling back we conclude that:

\[
(5.28) \quad \left( \bar{B}_G(p_n, \sigma_{N(n)} R/C_{\text{cut}}), \mu_G, \bar{B}_G(p_n, \sigma_{N(n)} R/C_{\text{cut}}) \right)
\]

admits a \((1, P)\)-Poincaré inequality with constant \( C_{P1} \). Fix a basepoint \( q \in G \). For each \( s > 0 \) we can find \( n \geq D_3(s) \) such that \( \bar{B}_G(p_n, \sigma_{N(n)} R/C_{\text{cut}}) \) contains an isometric copy \( B_s \) of \( \bar{B}_G(q, s) \) and such that the measures \( \mu_G B_s \) and \( \mu_G \bar{B}_G(q, s) \) agree up to a multiple. Thus

\[
(5.29) \quad \left( \bar{B}_G(q, s), \mu_G, B(G, q, s) \right)
\]

admits a \((1, P)\)-Poincaré inequality with constant \( C_{P1} \); as \( C_{P1} \) does not depend on \( s \) we conclude by letting \( s \to \infty \).

**Step 2:** \( \text{I}_{P1}(G, \mu_G) \subset \text{I}_{P1}(X, \mu_X) \).
This follows from the stability of the Poincaré inequality under measured pointed Gromov-Hausdorff convergence, see [Kei03].

5.2. Putting all together. In this subsection we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The existence of the measures \( \{ \mu_{P_c} \} \) follows combining Theorems 5.18, 4.74, 4.110 and Remark 4.112.

The projection map \( \pi : G \to \mathbb{R} \) passes to the limit giving a 1-Lipschitz map \( \pi : X \to \mathbb{R} \). The geodesic lines of the form \( \mathbb{R} \times \{ \lambda \} \times \{ \theta \} \) pass to the limit and give a Fubini-like representation of the measure \( \mu_{P_c} \). To this Fubini representation one can associate a Weaver derivation \( D \), i.e. a horizontal vector field as in [Sch13].

The verification that \((X, \pi)\) is a chart is standard and can be carried out in two ways. The first way uses a Sobolev-space argument like Sec. 9 in [CK15]. The second uses \( D \) and the Stone-Weierstrass Theorem for Lipschitz Algebras as in Example[Wea00, Example 5E].

The claim about the Assouad-Nagata dimension follows because the graph \( G \) has Assouad-Nagata dimension 1 and the Assouad-Nagata dimension is stable in passing to asymptotic cones.

References

[ADS13] L. Ambrosio, S. Di Marino, and G. Savaré. On the duality between \( p \)-Modulus and probability measures. ArXiv e-prints, November 2013.

[BP99] Marc Bourdon and Hervé Pajot. Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings. Proc. Amer. Math. Soc., 127(8):2315–2324, 1999.

[Che99] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. Geom. Funct. Anal., 9(3):428–517, 1999.

[CK15] Jeff Cheeger and Bruce Kleiner. Inverse limit spaces satisfying a Poincaré inequality. Anal. Geom. Metr. Spaces, 3:15–39, 2015.

[CKS15] J. Cheeger, B. Kleiner, and A. Schioppa. Infinitesimal structure of differentiability spaces, and metric differentiation. ArXiv e-prints, March 2015.

[DS13] S. Di Marino and G. Speight. The \( p \)-Weak Gradient Depends on \( p \). ArXiv e-prints, November 2013.

[GL14] J. T. Gill and M. Lopez. Discrete Approximations of Metric Measure Spaces of Controlled Geometry. ArXiv e-prints, August 2014.

[Hei01] Juha Heinonen. Lectures on analysis on metric spaces. Universitext. Springer-Verlag, New York, 2001.

[HH00] Bruce Hanson and Juha Heinonen. An \( n \)-dimensional space that admits a Poincaré inequality but has no manifold points. Proc. Amer. Math. Soc., 128(11):3379–3390, 2000.

[HK95] Piotr Hajłasz and Pekka Koskela. Sobolev meets Poincaré. C. R. Acad. Sci. Paris Sér. I Math., 320(10):1211–1215, 1995.

[HK98] Juha Heinonen and Pekka Koskela. Quasiconformal maps in metric spaces with controlled geometry. Acta Math., 181(1):1–61, 1998.

[Jer86] David Jerison. The Poincaré inequality for vector fields satisfying Hörmander’s condition. Duke Math. J., 53(2):503–523, 1986.

[Kei03] Stephen Keith. Modulus and the Poincaré inequality on metric measure spaces. Math. Z., 245(2):255–292, 2003.

[KM98] Pekka Koskela and Paul MacManus. Quasiconformal mappings and Sobolev spaces. Studia Math., 131(1):1–17, 1998.

[KM02] Juha Kinnunen and Olli Martio. Nonlinear potential theory on metric spaces. Illinois J. Math., 46(3):857–883, 2002.

[KS01] Juha Kinnunen and Nageswari Shanmugalingam. Regularity of quasi-minimizers on metric spaces. Manuscripta Math., 105(3):401–423, 2001.
Stephen Keith and Xiao Zhong. The Poincaré inequality is an open ended condition. *Ann. of Math. (2)*, 167(2):575–599, 2008.

John M. Mackay, Jeremy T. Tyson, and Kevin Wildrick. Modulus and Poincaré inequalities on non-self-similar Sierpiński carpets. *Geom. Funct. Anal.*, 23(3):985–1034, 2013.

Tapio Rajala. Local Poincaré inequalities from stable curvature conditions on metric spaces. *Calc. Var. Partial Differential Equations*, 44(3-4):477–494, 2012.

Andrea Schioppa. Derivations and Alberti representations. *ArXiv e-prints*, November 2013.

Andrea Schioppa. Poincaré inequalities for mutually singular measures. *Anal. Geom. Metr. Spaces*, 3:40–45, 2015.

S. Semmes. Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities. *Selecta Math. (N.S.)*, 2(2):155–295, 1996.

Nik Weaver. Lipschitz algebras and derivations. II. Exterior differentiation. *J. Funct. Anal.*, 178(1):64–112, 2000.

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