MORE ON SIGNED GRAPHS WITH AT MOST THREE EIGENVALUES

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Abstract

We consider signed graphs with just 2 or 3 distinct eigenvalues, in particular (i) those with at least one simple eigenvalue, and (ii) those with vertex-deleted subgraphs which themselves have at most 3 distinct eigenvalues. We also construct new examples using weighing matrices and symmetric 3-class association schemes.

Keywords: adjacency matrix, simple eigenvalue, strongly regular signed graph, vertex-deleted subgraph, weighing matrix, association scheme.

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1. Introduction

A signed graph  \( \hat{G} \) is defined to be a pair \((G, \sigma)\), in which \( G = (V, E) \) is an unsigned graph, called the underlying graph, and \( \sigma : E \rightarrow \{1, -1\} \) is the sign function, also known as the signature. The order of a signed graph, denoted by \( n \), is the number of its vertices. The edge set of \( \hat{G} \) consists of the subsets of positive and negative edges. We interpret an unsigned graph as a signed graph in which all edges are positive.

The adjacency matrix \( A_{\hat{G}} \) of \( \hat{G} \) is the \( n \times n \) \((0, 1, -1)\)-matrix which is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1s which correspond to negative edges. By the spectrum of \( \hat{G} \), we mean the spectrum of \( A_{\hat{G}} \). An eigenvalue of \( \hat{G} \) is called a main eigenvalue if the corresponding eigenspace is not orthogonal to the all-1 vector. Throughout the paper, by the statement ‘\( \hat{G} \) has \( k \) eigenvalues’ we mean that \( \hat{G} \) has exactly \( k \) distinct eigenvalues.

In Section 2 we give some terminology and notation, and prove some auxiliary results. The problem of classifying graphs with a comparatively small number of eigenvalues has attracted a great deal of attention in the last 70 years; some recent results can be found in [4, 5, 7, 16, 17]. In [14] we considered regular and non-regular signed graphs with at most 3 eigenvalues; here we continue this research and pay more attention to non-regular signed graphs. In Section 3, we consider connected signed graphs with 3 eigenvalues at least one of which is simple. The number of simple eigenvalues governs our investigation in Section 4 of vertex-deleted subgraphs which themselves have 3 eigenvalues. In Section 5 we construct some signed graphs with 2 or 3 eigenvalues using weighing matrices or symmetric 3-class association schemes, and note the implications for vertex-deleted subgraphs.

2. Preliminaries

We write \( I, O, J, 0 \) and \( j \) for an identity matrix, an all-0 matrix, an all-1 matrix, an all-0 vector and an all-1 vector, respectively. Subscripts indicate size as necessary.

A signed graph \( \hat{G} \) is said to be connected, complete, regular or bipartite if the same holds for its underlying graph. The degree of a vertex in \( \hat{G} \) is the degree of the same vertex in \( G \). The net-degree of a vertex \( i \), denoted by \( d^+_i \), is the difference between the numbers of positive and negative edges incident with \( i \). A signed graph in which vertex net-degrees are equal is called net-regular. Similarly, a net-biregular signed graph is a signed graph which has 2 distinct net-degrees. It is known that \( \hat{G} \) is net-regular if and only if \( j \) is an eigenvector of \( \hat{G} \), and then \( j \) belongs to the eigenspace of the net-degree \([20]\).

A signed graph is said to be homogeneous if all its edges have the same sign.
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(in particular, if its edge set is empty). Otherwise, it is said to be inhomogeneous. The negation \(-\hat{G}\) is obtained by reversing the sign of every edge of \(\hat{G}\).

We say that signed graphs \(\hat{G}\) and \(\hat{H}\) are isomorphic if there is a permutation matrix \(P\) such that \(A_{\hat{H}} = P^{-1}A_{\hat{G}}P\). In this case we write \(\hat{G} \cong \hat{H}\). We say that \(\hat{G}\) and \(\hat{H}\) are switching equivalent if there is a vertex subset \(S \subseteq V(\hat{G})\), such that \(\hat{H}\) is obtained by reversing the sign of every edge with one vertex in \(S\) and the other in \(V(\hat{G}) \setminus S\).

If the vertex labelling is transferred from the underlying graph common to \(\hat{G}\) and \(\hat{H}\), then \(\hat{G}\) and \(\hat{H}\) are switching equivalent if and only if there is a diagonal matrix \(D\) with \(\pm 1\) on the diagonal such that \(A_{\hat{H}} = D^{-1}A_{\hat{G}}D\). Clearly, isomorphism and switching equivalence preserve the spectrum.

An equitable partition of a signed graph \(\hat{G}\) is a partition of the vertex set \(V(\hat{G})\) into non-empty cells \(C_1, C_2, \ldots, C_s\), such that each cell induces a net-regular signed graph and for \(1 \leq i < j \leq s\) the edges between \(C_i\) and \(C_j\) induce a net-biregular or net-regular signed graph, in which vertices from each of \(C_i, C_j\) are equal in net-degree.

We say that a signed graph \(\hat{G}\) is strongly regular (for short, \(\hat{G}\) is a SRSG) with parameters \(r, a, b, c\) if the entries of \(A^2_{\hat{G}}\) satisfy

\[
a^{(2)}_{ij} = \begin{cases} 
    r & \text{if } i = j, \\
    a & \text{if } i \not\sim j, \\
    b & \text{if } i \sim j, \\
    c & \text{if } i \not\sim j \text{ and } i \neq j.
\end{cases}
\]

Note that \(a^{(2)}_{ij}\) is the difference between the numbers of positive and negative \(i\-j\) walks of length 2 in \(\hat{G}\). Accordingly, this definition generalizes the definition of strongly regular graphs. We mostly deal with SRSGs in Subsection 5.2.

In the forthcoming sections we frequently use the following result.

**Proposition 1** [14]. A connected signed graph \(\hat{G}\) has exactly one positive eigenvalue if and only if \(\hat{G}\) is switching equivalent to a non-trivial complete multipartite graph. If \(\hat{G}\) has exactly one non-negative eigenvalue, then \(\hat{G}\) is switching equivalent to a complete graph.

We now transfer the following two results from the domain of unsigned graphs.

**Proposition 2.** If \(A\) is a real symmetric matrix with distinct eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_k\) such that \(\lambda_1\) is a simple eigenvalue, then

\[
\prod_{i=2}^{k}(A - \lambda_i I) = \left(\prod_{i=2}^{k} (\lambda_1 - \lambda_i)\right) xx^T,
\]
where \( x \) is a unit eigenvector associated with \( \lambda_1 \). For \( k = 3 \), there exists an eigenvector \( a \) for \( \lambda_1 \), such that

\[
(A - \lambda_2 I)(A - \lambda_3 I) = paa^T,
\]

where

\[
p = \begin{cases} 
1 & \text{if } \lambda_1 \not\in (\lambda_2, \lambda_3), \\
-1 & \text{if } \lambda_1 \in (\lambda_2, \lambda_3).
\end{cases}
\]

**Proof.** Considering the spectral decomposition of \( A \), we see that there exists an orthogonal matrix \( X \) such that

\[
\prod_{i=2}^{k} (A - \lambda_i I) = X \begin{pmatrix} \prod_{i=2}^{k} (\lambda_1 - \lambda_i) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{pmatrix} X^\top = \left( \prod_{i=2}^{k} (\lambda_1 - \lambda_i) \right) xx^\top,
\]

where \( x \) is a unit eigenvector of \( \prod_{i=2}^{k} (A - \lambda_i I) \) afforded by \( \prod_{i=2}^{k} (\lambda_1 - \lambda_i) \). The result follows since \( Ax = \lambda_1 x \).

For \( k = 3 \), by taking

\[
a = \sqrt{p(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} x,
\]

we arrive at the desired result.

The previous theorem is a slight extension of the result in which \( A \) is the adjacency matrix of an unsigned graph, \( k = 3 \) and \( \lambda_1 \) is the largest eigenvalue

\[4, 7, 16\]. Our formulation is more general in order to embrace signed graphs, with the possibility that \( \lambda_1 \) is not the largest eigenvalue.

**Proposition 3.** Let \( \hat{G} \) be obtained from a signed graph \( \hat{H} \) of order \( n \) by adding a new vertex whose neighbourhood in \( \hat{H} \) is determined by the characteristic \((0, 1, -1)\)-vector \( r \). The characteristic polynomial of \( \hat{G} \) is given by

\[
P_{\hat{G}}(x) = P_{\hat{H}}(x) \left( x - \sum_{i=1}^{m} \frac{||Q_i r||^2}{x - \mu_i} \right),
\]

where \( \mu_1, \mu_2, \ldots, \mu_m \) are the distinct eigenvalues of \( \hat{H} \) and \( Q_1, Q_2, \ldots, Q_m \) are the matrices of the orthogonal projections of \( \mathbb{R}^n \) onto the eigenspaces of \( \hat{H} \) with respect to the canonical basis.

**Proof.** Using the Schur matrix decomposition in conjunction with the known identity \( \text{adj}(xI - A_H) = \det(xI - A_H)(xI - A_H)^{-1} \), we obtain

\[
P_{\hat{G}}(x) = \det \begin{pmatrix} x & -r^\top \\
r & xI - A_H \end{pmatrix} = xP_{\hat{H}}(x) - r^\top \text{adj} (xI - A_H) r
\]

\[
= P_{\hat{H}}(x) (x - r^\top(xI - A_H)^{-1} r) .
\]
Since \((xI - A\dot{H})^{-1}\) has spectral decomposition \(\sum_{i=1}^{m} \frac{1}{x - \mu_i} Q_i\), we have

\[
P_{\dot{G}}(x) = P_{\dot{H}}(x) \left( x - \mathbf{r}^T \left( \sum_{i=1}^{m} \frac{1}{x - \mu_i} Q_i \right) \mathbf{r} \right).
\]

Now (1) follows since \(\mathbf{r}^T Q_i \mathbf{r} = \mathbf{r}^T Q_i Q_i \mathbf{r} = \mathbf{r}^T Q_i^T Q_i \mathbf{r} = (Q_i \mathbf{r})^T Q_i \mathbf{r} = ||Q_i \mathbf{r}||^2\). □

The ‘unsigned’ version of the previous theorem is well-known, see [6, Theorem 2.2.8]. The cone over a signed graph \(\dot{G}\) is obtained by adding a vertex \(v\) along with positive edges between \(v\) and every vertex of \(\dot{G}\). We denote this cone by \(K_1 \nabla \dot{G}\). The following result is a direct consequence of the previous one.

**Corollary 4.** The cone over \(\dot{H}\) has the characteristic polynomial

\[
P_{K_1 \nabla \dot{H}}(x) = P_{\dot{H}}(x) \left( x - \sum_{i=1}^{m} \frac{n\beta_i^2}{x - \mu_i} \right),
\]

where \(\mu_1, \mu_2, \ldots, \mu_m\) are distinct eigenvalues of \(\dot{H}\) and \(\beta_1, \beta_2, \ldots, \beta_m\) are the corresponding main angles defined by \(\beta_i = ||Q_i \mathbf{j}||/\sqrt{n}\).

### 3. Signed Graphs with 3 Eigenvalues, at Least One of which is Simple

In this section we give some characterizations of signed graphs described in the section title. We start with the following lemma.

**Lemma 5.** If \(\dot{G}\) is a connected signed graph with 3 eigenvalues such that at least 2 of them are simple, then \(\dot{G}\) is switching equivalent to a complete bipartite graph.

**Proof.** If every eigenvalue of \(\dot{G}\) is simple, then \(\dot{G}\) is switching equivalent to (the complete bipartite graph) \(K_{1,2}\). Now suppose that \(\lambda\) is the unique non-simple eigenvalue. If \(\lambda = 0\) then \(\dot{G}\) has exactly one positive eigenvalue, hence is switching equivalent to a complete multipartite graph by Proposition 2.1. Moreover, since its spectrum has the form \([-\rho, 0^{n-2}, \rho]\), \(\dot{G}\) is switching equivalent to a complete bipartite graph [6, p. 47].

If \(\lambda \neq 0\) then \(\dot{G}\) has a connected subgraph without \(\lambda\) as an eigenvalue, namely \(K_1\). Since the eigenspace of \(\lambda\) has codimension 2, \(K_1\) can be extended to a connected induced subgraph \(\dot{H}\) of order 2 without \(\lambda\) as an eigenvalue (see [6, Theorem 5.1.6], which can be extended to the framework of signed graphs with slight modifications in the proof). Since \(\dot{H} \cong \pm K_2\) we have \(\lambda \notin \{1, -1\}\). Since also \(\lambda \neq 0\), we know from [13, Theorem 3.3] that \(\dot{G}\) has at most 4 vertices, and this case is resolved by inspection. □
Now we consider signed graphs with 3 eigenvalues, exactly one of which is simple. Accordingly, we assume that a connected signed graph \( \hat{G} \) has spectrum \([\rho, \mu^m, \lambda^l]\), with \( m, l \geq 2 \) and \( \mu > \lambda \). By Proposition 2, there is a non-zero vector \( a = (a_1, a_2, \ldots, a_n)^\top \) such that \( Aa = \rho a \) and

\[
(A - \mu I)(A - \lambda I) = paa^\top,
\]

where \( A = A_{\hat{G}} \), \( p = -1 \) if \( \mu > \rho > \lambda \), and \( p = 1 \) otherwise. By equating the diagonal entries of both sides, we get

\[
d_i = pa_i^2 - \mu \lambda,
\]

where \( d_i \) is degree of the vertex \( i \).

**Lemma 6.** If \( \hat{G} \) is a connected net-regular signed graph with spectrum \([\rho, \mu^m, \lambda^l]\), where \( \rho \) is its net-degree, then \( \hat{G} \) is regular.

**Proof.** Every eigenvector afforded by \( \rho \) is constant, which, by (3), means that \( \hat{G} \) is regular. □

We note in passing that, by \cite{8}, the signed graph \( \hat{G} \) mentioned in the previous result is strongly regular. Moreover, we have the following result, which will be used in our last section.

**Lemma 7.** A connected inhomogeneous non-complete regular signed graph \( \hat{G} \) is net-regular with spectrum \([\rho, \mu^m, \lambda^l]\) (\( \rho \) being the net-degree) if and only if \( \hat{G} \) is strongly regular and its parameters satisfy \( a + b = 2c \neq 0 \).

**Proof.** Let \( \hat{G} \) have spectrum as in the statement of the lemma. Since every eigenvector afforded by \( \rho \) is constant, by (2) we have

\[
A^2 = (\mu + \lambda)A - \mu \lambda I + kJ,
\]

for some \( k \neq 0 \). Comparing the entries of the left and the right hand side, we conclude that \( \hat{G} \) is strongly regular with \( a + b = 2k \) and \( c = k \neq 0 \). The converse follows directly from \cite[Theorem 4.2]{8}. □

Next we deal with the case in which an eigenvalue other than \( \rho \) is the only non-main eigenvalue.

**Theorem 8.** If \( \hat{G} \) is a connected signed graph with spectrum \([\rho, \mu^m, \lambda^l]\) (\( m, l \geq 2 \)) such that only \( \lambda \) is non-main, then there is a non-zero constant \( \alpha \) such that

\[
d_i = \alpha(d_i^\pm - \mu)^2 - \mu \lambda,
\]

where \( d_i \) and \( d_i^\pm \) are the degree and the net-degree of the vertex \( i \), respectively.
In particular, if $\hat{G}$ is regular then it is net-biregular, the sum of the corresponding net-degrees is $2\mu$ and $\hat{G}$ is switching equivalent to a net-regular signed graph. If $\hat{G}$ is biregular then it is switching equivalent to a net-biregular signed graph.

**Proof.** Observe first that $\hat{G}$ is not net-regular, since it has more than one main eigenvalue. We retain the notation introduced in Lemma 6. Since $\rho$ and $\mu$ are main, we have

\[
(A - \rho I)(A - \mu I)\mathbf{j} = \mathbf{0},
\]

as proved in [19] on the basis of the result for unsigned graphs which can be found in [15]. It follows that $A^2\mathbf{j} \in \text{span}(\mathbf{d}, \mathbf{j})$, where $A\mathbf{j} = \mathbf{d} = (d_1^+, d_2^+, \ldots, d_n^+)^T$. Moreover, since $a^\top \mathbf{j} \neq 0$, (2) shows that $a \in \text{span}(\mathbf{d}, \mathbf{j})$. Hence, we may write $a = r\mathbf{d} + sj$, where $r \neq 0$ as $\hat{G}$ is not net-regular.

By (5), we have $A^2\mathbf{j} = (\rho + \mu)A\mathbf{j} - \rho\mu\mathbf{j}$, which together with (2), gives

\[
(\rho - \lambda)\mathbf{d} - \mu(\rho - \lambda)\mathbf{j} = p(a^\top \mathbf{j}) = p(a^\top \mathbf{j})(r\mathbf{d} + sj).
\]

By equating the coefficients of $\mathbf{d}$ and $\mathbf{j}$, we find that $s = -\mu r$, and so

\[
a = r(\mathbf{d} - \mu \mathbf{j}).
\]

Using (3), we obtain $d_i = pr^2(d_i^+ - \mu)^2 - \mu\lambda$, and by setting $\alpha = pr^2$, we arrive at (4).

Now, if $\hat{G}$ is $d$-regular, then $d = \alpha(d_i^+ - \mu)^2 - \mu\lambda$ for every vertex $i$. Evidently, this equation has 2 solutions in $d_i^+$ and since $\hat{G}$ is not net-regular, both solutions appear as net-degrees; hence, $\hat{G}$ is net-biregular. The sum of the corresponding net-degrees follows from the previous equation for $d$. Lastly, from (3) we see that the coordinates of $a$ are equal in absolute value. If $D$ is the diagonal matrix of $\pm 1$s with 1 in the $i$th position precisely when $a_i$ is positive, then $D^{-1}AD$ is the adjacency matrix of a switching equivalent signed graph, say $\hat{H}$. Moreover, $Da$ is a constant eigenvector associated with $\rho$ in $\hat{H}$, which means that $\hat{H}$ is net-regular.

Finally, suppose that $\hat{G}$ is biregular with degrees $d_1$ and $d_2$, and assume that $\hat{G}$ is not net-biregular. Then, for at least one $j$ ($j \in \{1, 2\}$), there are vertices of degree $d_j$ which differ in net-degree. By (6), the corresponding coordinates of $a$ are different, while by (3), they are equal in absolute value. Using $D$ formed exactly as before, we obtain a signed graph $\hat{H}$ for which we have $A_{\hat{H}}Da = \rho Da$ where $Da$ has 2 different coordinates. By (6), this means that $\hat{H}$ has 2 net-degrees, and so it is net-biregular.

Of course, there is an analogous statement with $\mu$ in the role of the unique non-main eigenvalue. Here is a closer description of $\hat{G}$ being net-biregular.
Corollary 9. If the signed graph $\hat{G}$ of Theorem 8 is net-biregular, then $\hat{G}$ is biregular and its net-degrees determine an equitable vertex bipartition.

**Proof.** From (4) we see that $\hat{G}$ must be biregular. In addition, since $\hat{G}$ is non-regular, its vertices are equal in degree if and only if they are equal in net-degree. By (6), the eigenvector $a$ of (2) has 2 different coordinates, say $a_u$ and $a_w$, which correspond to different net-degrees and determine the vertex set partition $V = U \cup W$. It remains to show that this partition is equitable. For $v \in V$, let $d^\pm_{vu}$ and $d^\pm_{vw}$ denote its net-degree in $U$ and $W$, respectively. Then we also have $d^\pm_v = d^\pm_{vu} + d^\pm_{vw}$. If $v \in U$, since $a$ is associated with $\rho$, we have $d^\pm_{vu} a_u + d^\pm_{vw} a_w = \rho a_u$, i.e., $d^\pm_{vu} a_u + (d^\pm_v - d^\pm_{vu}) a_w = \rho a_u$ and $(d^\pm_v - d^\pm_{vu}) a_u + d^\pm_{vw} a_w = \rho a_u$. The last two equalities lead to

$$d^\pm_{vu} = \frac{\rho a_u - d^\pm_v a_w}{a_u - a_w} \quad \text{and} \quad d^\pm_{vw} = a_u \frac{d^\pm_v - \rho}{a_u - a_w}.$$  

In a very similar way, we obtain

$$d^\pm_{vu} = a_w \frac{d^\pm_v - \rho}{a_w - a_u} \quad \text{and} \quad d^\pm_{vw} = a_w \frac{\rho a_w - d^\pm_v a_u}{a_w - a_u},$$

for $v \in W$. In other words, the net-degrees determine an equitable vertex bipartition.

It is not difficult to construct some examples. For instance, by making a switch with respect to 3 mutually adjacent vertices of the Paley graph with 9 vertices, we obtain a regular and net-biregular signed graph with spectrum $[4, 1^4, (-2)^4]$. Also, the cone over the complete bipartite signed graph $\hat{K}_{4,4}$, in which negative edges form a perfect matching, is biregular and net-biregular, while its spectrum is $[4, 2^3, (-2)^5]$.

4. **Vertex-Deleted Subgraphs with 3 Eigenvalues**

In this section we consider the question of whether a vertex-deleted subgraph of a connected signed graph $\hat{G}$ with 3 eigenvalues also has 3 eigenvalues. We distinguish 3 cases depending on the number of simple eigenvalues of $\hat{G}$. First, if all of them are simple then all vertex-deleted subgraphs have fewer than 3 eigenvalues; this case is trivial. If $\hat{G}$ has 2 simple eigenvalues, then $\hat{G}$ is switching equivalent to a complete bipartite graph, by Lemma 5. If so, then every vertex-deleted subgraph is also switching equivalent to a complete bipartite graph; such a subgraph has 3 eigenvalues unless $\hat{G}$ is the star $\hat{K}_{1,n-1}$ and the degree of the deleted vertex is $n - 1$. The remaining case is more complicated and it is considered in the following two theorems.
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Theorem 10. Let \( \hat{G} \) be a connected signed graph with spectrum \([\rho, \mu, \lambda]\), with \( m, l \geq 2 \) and \( \mu > \lambda \). Let \( \hat{H} = \hat{G} - v \) and let \( r \) be the characteristic \((0, 1, -1)\)-vector that determines the neighbourhood of \( v \) in \( \hat{H} \). If \( \hat{H} \) has 3 eigenvalues, then \( r \) is an eigenvector associated with an eigenvalue of \( \hat{H} \) distinct from \( \mu \) and \( \lambda \), and:

(i) for \( \rho > \mu \), the spectrum of \( \hat{H} \) is \( [\mu^m, \rho + \lambda, \lambda^{l-1}] \) with \( ||r||^2 = -\rho\lambda \);

(ii) for \( \rho < \lambda \), the spectrum of \( \hat{H} \) is \( [\mu^{m-1}, \rho + \mu, \lambda] \) with \( ||r||^2 = -\rho\mu \);

(iii) for \( \rho \in (\lambda, \mu) \), the spectrum of \( \hat{H} \) is \( [\mu^{m-1}, \rho^2, \lambda^{l-1}] \) with \( \rho = \mu + \lambda \), \( ||r||^2 = -\mu\lambda \) or \( [\mu^m, \rho + \lambda, \lambda^{l-1}] \) with \( ||r||^2 = -\rho\lambda \), or \( [\mu^{m-1}, \rho + \mu, \lambda] \) with \( ||r||^2 = -\rho\mu \).

Conversely, if \( r \) is an eigenvector of \( \hat{H} \) associated with an eigenvalue distinct from \( \mu \) and \( \lambda \), then \( \hat{H} \) has 3 eigenvalues when either \( \rho \) is an eigenvalue of multiplicity 2 in \( \hat{H} \) or \( \hat{H} \) does not have \( \rho \) as an eigenvalue.

Proof. Computing \( \text{tr}(A_{\hat{G}}) \) and \( \text{tr}(A_{\hat{G}}^2) \), we obtain

\[
\begin{align*}
\text{(7)} & \quad \rho + m\mu + l\lambda = 0, \\
\text{(8)} & \quad \rho^2 + m\mu^2 + l\lambda^2 = 2e,
\end{align*}
\]

where \( e \) denotes the number of edges of \( \hat{G} \). Let \( g \) be the number of edges in \( \hat{G} \) but not in \( \hat{H} \), so that \( g = ||r||^2 \).

Suppose first that \( \rho > \mu \). Observe that \( \lambda < 0 \), since \( \hat{G} \) must have at least one negative eigenvalue (see Proposition 1), and then we also have \( \mu \geq 0 \). By eigenvalue interlacing, the eigenvalues of \( \hat{H} \) are \( \nu, \mu^{m-1}, \theta, \lambda^{l-1} \), where \( \rho \geq \nu \geq \mu \geq \theta \geq \lambda \). Now, since \( \hat{H} \) has 3 eigenvalues we have one of the following situations:

(a) \( \nu = \rho \) and \( \theta \in \{\mu, \lambda\} \),
(b) \( \nu = \mu \) and \( \theta \notin \{\mu, \lambda\} \),
(c) \( \nu \notin \{\rho, \mu\} \) and \( \theta = \mu \),
(d) \( \nu \notin \{\rho, \mu\} \) and \( \theta = \lambda \).

For (a), when \( \theta = \mu \) we have \( \rho + m\mu + (l - 1)\lambda = 0 \), which together with (7) leads to the contradiction \( \lambda = 0 \). When \( \theta = \lambda \) in a similar way we get \( \mu = 0 \), but then \( \hat{H} \) and \( \hat{G} \) have the same number of edges (by (8)), a contradiction since \( \hat{G} \) is connected.

For (b), we use the equalities (7) and (8) along with \( \text{tr}(A_{\hat{H}}) = 0 \) and \( \text{tr}(A_{\hat{H}}^2) = 2(e - g) \) to obtain \( \theta = \rho + \lambda \) and \( g = -\rho\lambda \). To complete the proof of (i) it remains to prove that \( r \) is an eigenvector of \( \hat{H} \) associated with \( \rho + \lambda \).

Let \( Q_\xi \) denote the matrix of the orthogonal projection of \( \mathbb{R}^{n-1} \) onto the eigenspace of an eigenvalue \( \xi \) of \( \hat{H} \). By Proposition 3, we have

\[
P_{\hat{G}}(x) = P_H(x) \left( x - \frac{||Q_\rho r||^2}{x - \mu} - \frac{||Q_\sigma r||^2}{x - \theta} - \frac{||Q_\lambda r||^2}{x - \lambda} \right).
\]
Since the multiplicity of $\mu$ and $\lambda$ in $\hat{G}$ is not less than the multiplicity of the same eigenvalue in $\hat{H}$, we have $Q_\mu r = Q_\lambda r = 0$, which means that $r$ is orthogonal to the eigenspaces of $\mu$ and $\lambda$, equivalently $r$ belongs to the eigenspace of $\theta = \rho + \lambda$.

For (c), as in the previous case, we obtain $\nu = \rho + \lambda$ and $g = -\rho\lambda$, along with the conclusion that $r$ belongs the eigenspace of $\nu$.

For (d), we find that $\nu = \rho + \mu$, which is impossible as $\mu \geq 0$ and $\nu \neq \rho$.

This completes the proof of (i), while (ii) follows analogously.

Now suppose that $\rho \in (\lambda, \mu)$. Here $\mu > 0, \lambda < 0$ and the possible eigenvalues of $\hat{H}$ are $\mu^{m-1}, \nu, \theta, \lambda^{l-1}$. The cases that arise are considered in the same way as before, and yield the results summarized in (iii).

Assume now that $r$ belongs to the eigenspace of an eigenvalue of $\hat{H}$ distinct from $\mu$ and $\lambda$. If $\rho$ is an eigenvalue of $\hat{H}$ with multiplicity 2, then (by eigenvalue interlacing) $\hat{H}$ has 3 eigenvalues, and so it remains to consider the case in which $\rho$ does not belong to the spectrum of $\hat{H}$. In this case $\hat{H}$ has at most 4 eigenvalues. If $\lambda, \mu$ are the only eigenvalues of $\hat{H}$ then $\rho = 0$ and we obtain the contradiction $g = 0$. Now suppose that $\hat{H}$ has 4 eigenvalues, $\mu, \lambda, \nu$ and $\theta$. By Proposition 3, we have

$$P_G(x) = P_H(x)\left(x - \frac{\|Q_\mu r\|^2}{x - \mu} - \frac{\|Q_\lambda r\|^2}{x - \lambda} - \frac{\|Q_\nu r\|^2}{x - \nu} - \frac{\|Q_\theta r\|^2}{x - \theta}\right).$$

If $r$ belongs to the eigenspace of (say) $\nu$, then we obtain

$$(9) \quad (x - \nu)(x - \rho)(x - \mu)^m(x - \lambda)^l = (x(x - \nu) - \|Q_\nu r\|^2)P_H(x).$$

Observe that the multiplicities of $\mu$ and $\lambda$ in $\hat{H}$ are $m - 1$ and $l - 1$, respectively, which implies $(x - \mu)(x - \lambda) = (x(x - \nu) - \|Q_\nu r\|^2)$; but then since $(x - \rho)$ must appear on the right hand side of (9), we conclude that $\rho$ is an eigenvalue of $\hat{H}$. This contradiction completes the proof.

We note a consequence of Theorem 10 in the case that $\hat{G}$ is connected and switching equivalent to its underlying graph. Then $\rho$ is the largest eigenvalue of $\hat{G}$, and so $\rho$ is not an eigenvalue of $\hat{H}$. Hence, if $r$ is an eigenvector of $\hat{H}$ associated with an eigenvalue other than $\mu$ or $\lambda$ then $\hat{H}$ has 3 eigenvalues. Plenty of examples can be found among unsigned graphs; for instance $\hat{H}$ can be the Petersen graph, with $\hat{G}$ the cone over $\hat{H}$. We further consider cones by setting $\hat{G} \cong K_1 \nabla \hat{H}$ in Theorem 10.

**Corollary 11.** Suppose that $K_1 \nabla \hat{H}$ has spectrum $[\rho, \mu^m, \lambda^l]$, where $m, l \geq 2$ and $\mu > \lambda$. If $\hat{H}$ is connected with 3 eigenvalues, then $\hat{H}$ is net-regular and either:

(i) $\hat{H}$ has spectrum $[\rho + \lambda, \mu^m, \lambda^{l-1}]$, with $\rho > \rho + \lambda > \mu > \lambda$

(ii) $\hat{H}$ has spectrum $[\mu^{m-1}, \lambda^l, \rho + \mu]$, with $\rho < \rho + \mu < \lambda < \mu$. 
Proof. By setting $r = j$ in Theorem 10 we see that $j$ is an eigenvector of $H$, and so $H$ is net-regular. Let $A_H j = \nu j$, $\hat{G} \cong K_1 \nabla H$ and $n = 1 + l + m$. From Theorem 10 we see that $\nu \notin \{\mu, \lambda\}$ and there are five possible scenarios:

(a) $\rho > \mu$, $\nu = \rho + \lambda$, $n-1 = -\rho \lambda$ and $H$ has spectrum $[\mu^{m-1}, \rho + \lambda, \lambda^{l-1}]$;
(b) $\rho < \lambda$, $\nu = \rho + \mu$, $n-1 = -\rho \mu$ and $H$ has spectrum $[\mu^{m-1}, \rho + \mu, \lambda^{l-1}]$;
(c) $\lambda < \rho < \mu$, $\nu = \rho + \lambda$, $n-1 = -\rho \lambda$ and $H$ has spectrum $[\mu^{m-1}, \rho + \lambda, \lambda^{l-1}]$;
(d) $\lambda < \rho < \mu$, $\nu = \rho + \mu$, $n-1 = -\rho \mu$ and $H$ has spectrum $[\mu^{m-1}, \rho + \mu, \lambda^{l-1}]$;
(e) $\lambda < \rho < \mu$, $\nu = \rho$, $n-1 = -\lambda \mu$, $\rho = \mu + \lambda$ and $H$ has spectrum $[\mu^{m-1}, \rho^2, \lambda^{l-1}]$.

Note first that if $\lambda$ has multiplicity $l - 1$ in $\hat{H}$ then $l > 2$, for otherwise $\hat{H}$ has 2 simple eigenvalues and, by Lemma 5, is switching equivalent to a complete bipartite graph, say $K_{r,s}$. Then $\mu = 0$ and $\{\nu, \lambda\} = \{-\sqrt{rs}, \sqrt{rs}\}$. Moreover, $r + s = n - 1 = -\rho \lambda = (\lambda - \nu)\lambda = 2rs$, whence $\hat{H} \cong K_2$, a contradiction. Similarly, if $\mu$ has multiplicity $m - 1$ in $\hat{H}$, then $m > 2$. Now we may apply [14, Theorem 5.5] to $\hat{H}$, because $\hat{H}$ is net-regular, $\nu$ is a simple eigenvalue of $\hat{H}$ and each of $\lambda, \mu$ has multiplicity $\geq 2$ in $\hat{H}$.

For (a), by [14, Theorem 5.5] either $\mu(\mu - \nu) = n - 1$ and $\mu > 0 > \lambda > \nu$ or $\lambda(\lambda - \nu) = n - 1$ and $\nu > \mu > 0 > \lambda$. In the former case, we have $\nu > \mu + \lambda$ and so $\lambda > \nu - \mu > \nu$, a contradiction. In the latter case, we have part (i) of this corollary.

For (b), we again get either $\mu(\mu - \nu) = n - 1$ and $\mu > 0 > \lambda > \nu$ or $\lambda(\lambda - \nu) = n - 1$ and $\nu > \mu > 0 > \lambda$. In the former case, we have part (ii) of this corollary. In the latter case, $\mu$ is the largest eigenvalue of $\hat{G}$ because $\rho < \lambda$. Hence $\nu \leq \mu$, a contradiction.

For (c), we have $\lambda < 0$ and $\mu > 0$, because $\lambda$ and $\mu$ are respectively the least and largest eigenvalues of $\hat{G}$. By [14, Theorem 5.5], either $\mu(\mu - \lambda) = n - 1$ or $\lambda(\lambda - \mu) = n - 1$. Since $\mu > \rho$, we have $-\lambda \mu > -\lambda \rho$, and so in the former case $\mu(\mu - \lambda) = n - 1$, $\nu > \mu > 0 > \lambda$. In the latter case, we have $\lambda(\lambda - \mu) = n - 1 = -\rho \lambda$, whence $\rho = \mu > \lambda > \mu$, a contradiction.

For (d), we have $\lambda < 0$, $\mu > 0$ and either $\mu(\mu - \lambda) = n - 1$ or $\lambda(\lambda - \mu) = n - 1$. In the former case, $\mu(\mu - \lambda) = n - 1 = -\rho \mu$, whence $\rho = \mu - \lambda < \lambda$, a contradiction. In the latter case, $\rho > \lambda$ and so $\lambda(\lambda - \mu) = -\rho \mu < -\lambda \rho$, whence $\lambda - \mu > -\mu$ and the contradiction $\lambda > 0$.

For (e), we have $\nu = \lambda + \mu$, and so by [14, Theorem 5.5], $\hat{G}$ has just 2 eigenvalues, a contradiction.

It remains to consider signed graphs with 3 eigenvalues such that none of them is simple.
Theorem 12. Let $\hat{G}$ be a connected signed graph with spectrum $[\rho^r, \mu^m, \lambda^l]$, with $r, m, l \geq 2$. Let $\hat{H} = \hat{G} - v$ and let $r$ denote the characteristic $(0, 1, -1)$-vector that determines the neighbourhood of $v$ in $\hat{H}$.

If $\hat{H}$ has 3 eigenvalues then, to within a permutation of the eigenvalues, its spectrum is $[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}]$; moreover, $\rho = \mu + \lambda$, $||r||^2 = -\mu \lambda$, and $r$ belongs to the eigenspace of $\rho$ in $\hat{H}$.

Conversely, if $r$ belongs to the eigenspace of an eigenvalue of $\hat{H}$, then $\hat{H}$ has at most 4 eigenvalues, with 3 eigenvalues precisely when the eigenvalue associated with $r$ also belongs to the spectrum of $\hat{G}$.

Proof. Assume that $\hat{H}$ has 3 eigenvalues and suppose first that the multiplicities of two of them (say $\mu$ and $\lambda$) are transferred from $\hat{G}$. Since $\text{tr}(A_G) = \text{tr}(A_H)$ we have $\rho = 0$, and since $\text{tr}(A_G^2) = \text{tr}(A_H^2)$ we see that $\hat{G}$ and $\hat{H}$ have the same number of edges, a contradiction. By eigenvalue interlacing, the remaining case is the one in which every eigenvalue changes its multiplicity: one increases and two decrease. If the multiplicity of $\rho$ increases, then as before we have $\rho = \mu + \lambda$.

Using Proposition 3 and following the proof of Theorem 10, we find that $\rho = \mu + \lambda$. Using Proposition 3 and following the proof of Theorem 10, we find that $\rho = \mu + \lambda$. Using Proposition 3 and following the proof of Theorem 10, we find that $\rho = \mu + \lambda$. Using Proposition 3 and following the proof of Theorem 10, we find that $\rho = \mu + \lambda$. Using Proposition 3 and following the proof of Theorem 10, we find that $\rho = \mu + \lambda$.

Conversely, suppose that $r$ belongs to the eigenspace of the eigenvalue $\xi$ of $\hat{H}$. By Proposition 3, we have

\begin{equation}
(x - \xi)P_{\hat{G}}(x) = P_{\hat{H}}(x)(x(x - \xi) - ||Q_\xi t||^2).
\end{equation}

If $\hat{H}$ has 5 eigenvalues (more than this is impossible), those transferred from $\hat{G}$ along with (say) $\nu$ and $\theta$, then at least one of the factors $(x - \nu)$ and $(x - \theta)$ occurs only on the right hand side of (10), which is impossible. Therefore, the number of eigenvalues of $\hat{H}$ is 3 or 4. If their number is 3, then we see immediately that $\xi$ is an eigenvalue of $\hat{G}$. Conversely, if $\xi$ is an eigenvalue of $\hat{G}$, and $\nu$ is the fourth eigenvalue of $\hat{H}$ (the one distinct from $\rho, \mu, \lambda$), then as before $(x - \nu)$ occurs only on the right hand side of (10), which is impossible, and we are done.

Example 13. To obtain an example for Theorem 12 it is convenient to start from $\hat{H}$ as a signed graph with spectrum $[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}]$, and set $r = j$. In this case, $\hat{H}$ is net-regular, and $\hat{G}$ is a cone over $\hat{H}$. Moreover, $\rho = \mu + \lambda$ and $\hat{H}$ has $-\mu \lambda$ vertices. By inspecting some known net-regular signed graphs with 3 eigenvalues, we arrive at a signed graph which can be found in [1] and satisfies all the numerical constraints. This is the signed graph obtained by reversing the sign of every edge belonging to a fixed Hamiltonian cycle of the Paley graph with 9 vertices. Its spectrum is $[3^2, 0^5, (-3)^2]$ (with $\rho = 0$). The corresponding cone has the spectrum $[3^3, 0^4, (-3)^3]$.

Motivated by the previous example, in which $\hat{G}$ is a cone over $\hat{H}$, we give a closer description of both signed graphs in this particular case. The first statement of the next theorem is a general one.
Theorem 14. The following statements hold.

(i) Let $\dot{G}$ be a signed graph with $\rho$ as an eigenvalue of multiplicity $r \geq 2$, let $\dot{H} = G - v$ and let $r$ be the characteristic vector that determines the neighbourhood of $v$ in $\dot{H}$. If the eigenspace of $\rho$ in $\dot{H}$ has orthogonal basis $r, x_1, x_2, \ldots, x_k$ and $v$ is vertex 1 in $\dot{G}$, then $(0, x_1^T), (0, x_2^T), \ldots, (0, x_k^T)$ are linearly independent eigenvectors associated with $\rho$ in $\dot{G}$.

(ii) In particular, if $\dot{G} \cong K_1 \nabla \dot{H}$ with spectrum $[\rho^r, \mu^m, \lambda^l]$ $(r, m, l \geq 2)$, if $j$ belongs to the eigenspace of $\rho$ in $\dot{H}$ and if $\dot{H}$ has spectrum $[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}]$, then $\rho$ is the unique main eigenvalue of $\dot{H}$ and the unique non-main eigenvalue of $\dot{G}$.

Proof. We have

$$A_{\dot{G}} \begin{pmatrix} 0 \\ x_i \end{pmatrix} = \begin{pmatrix} r^T x_i \\ \rho x_i \end{pmatrix} = \rho \begin{pmatrix} 0 \\ x_i \end{pmatrix},$$

for $1 \leq i \leq k$. Linear independence follows directly, and we have (i).

For (ii), first note that $\rho$ is the unique main eigenvalue of $\dot{H}$ because $j$ is an eigenvector of $\rho$ in $\dot{H}$. Taking $k = r$ in (i) we obtain a basis for the eigenspace of $\rho$ in $\dot{G}$ consisting of vectors orthogonal to $j$. Thus $\rho$ is non-main in $\dot{G}$. If $\dot{G}$ has another non-main eigenvalue, then the remaining one is main, but this means that the cone $\dot{G}$ is net-regular and hence the complete graph with just 2 eigenvalues, a contradiction.

5. Constructions of Signed Graphs with at Most 3 Eigenvalues

Here we give some examples of signed graphs with 3 eigenvalues, along with applications of some results from Section 3. The first construction is based on weighing matrices, while the second one is based on symmetric 3-class association schemes.

5.1. Weighing matrices

Let $W = W(n, \alpha)$ be a weighing matrix of order $n$ with weight $\alpha$, i.e., an $n \times n$ $(0, 1, -1)$-matrix such that $W^T W = \alpha I$. For $1 \leq m \leq n$, we call the submatrix of $W$ indexed by rows 1, 2, $\ldots$, $m$ and columns 1, 2, $\ldots$, $n$ a partial weighing matrix with weighing extension $W$.

Theorem 15. Let $W_1'$ and $W_2'$ be two partial weighing matrices of size $m \times n$ with weighing extensions $W_1, W_2$ of weight $\alpha$. The following block matrix has spectrum $[-\sqrt{\alpha^{n-m}}, \sqrt{\alpha^{n-m}}, 2\sqrt{\alpha^m}]$:
\[ A_m(W_1, W_2) = \begin{pmatrix} O_m & W_1' & W_2' \\ W_1'^T & O_n & \frac{1}{\sqrt{\alpha}} W_1^T W_2 \\ W_2'^T & \frac{1}{\sqrt{\alpha}} W_2^T W_1 & O_n \end{pmatrix}. \]

**Proof.** Let \( B \) be the matrix \((W_1' | W_2')\), and let

\[ C = \begin{pmatrix} O_n & \frac{1}{\sqrt{\alpha}} W_1^T W_2 \\ \frac{1}{\sqrt{\alpha}} W_2^T W_1 & O_n \end{pmatrix}. \]

By a straightforward computation we arrive at the following equality (see, for example, \[10\]):

\[ (2\sqrt{\alpha}I_{2n} - C)^{-1} = \begin{pmatrix} \frac{2}{3\sqrt{\alpha}} I_n & \frac{1}{3\sqrt{\alpha}} W_1^T W_2 \\ \frac{1}{3\sqrt{\alpha}} W_2^T W_1 & \frac{3}{3\sqrt{\alpha}} I_n \end{pmatrix}. \]

Therefore \( B(2\sqrt{\alpha}I_{2n} - C)^{-1} B^T = 2\sqrt{\alpha}I_m \), and it follows that the vectors

\[ \left( \mu I - C \right)^{-1} B^T x \]

for \( x \in \mathbb{R}^m \), lie in the eigenspace for the eigenvalue \( 2\sqrt{\alpha} \) of \( A_m(W_1, W_2) \). Thus \( 2\sqrt{\alpha} \) is an eigenvalue of \( A_m(W_1, W_2) \) with multiplicity at least \( m \). A similar argument applied for \( C = O_n \) shows that \( -\sqrt{\alpha} \) is an eigenvalue with multiplicity at least \( n + m \). In addition, by eigenvalue interlacing, \( \sqrt{\alpha} \) is an eigenvalue of \( A_m(W_1, W_2) \) with multiplicity at least \( n - m \) since the submatrix

\[ \begin{pmatrix} O_n & \frac{1}{\sqrt{\alpha}} W_1^T W_2 \\ \frac{1}{\sqrt{\alpha}} W_2^T W_1 & O_n \end{pmatrix} \]

has the spectrum \([-\sqrt{\alpha}^n, \sqrt{\alpha}^n]\).

This theorem enables us to construct an infinite family of signed graphs with spectrum \([-\sqrt{\alpha}^{n+m}, \sqrt{\alpha}^{n-m}, 2\sqrt{\alpha}^m]\) for some appropriate \( \alpha \). We remark that the matrix \((11)\) does not always correspond to a signed graph. For a signed graph, the inner product of any row of \( W_1 \) and any row of \( W_2 \) has to be 0 or \( \pm \sqrt{\alpha} \), and in \[11\] one can find a method for constructing a family of weighing matrices of weight 4 with this property. The method can be used to construct signed graphs with spectrum \([-2^{n+m}, 2^{n-m}, 4^m]\), as in the following example.

**Example 16.** Let \( W_1 \) and \( W_2 \) be as follows:
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\[
W_1 = \begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & -1 \\
1 & -1 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 & -1 & -1 \\
1 & 1 & 0 & 0 & -1 & 1 \\
1 & -1 & -1 & 1 & 0 & 0
\end{pmatrix}, \quad
W_2 = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 1 \\
1 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 & -1 \\
1 & 1 & 0 & 0 & -1 & -1 \\
1 & -1 & -1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 1
\end{pmatrix}.
\]

Considering the first two rows of \(W_1\) and \(W_2\), we see from Theorem 15 that the signed graph \(\hat{G}\) with adjacency matrix \(A_2(W_1, W_2)\) has spectrum \([-2^8, 2^4, 4^2]\). Since \(\hat{G}\) has no vertices of degree 4, Theorem 4.3 shows that if a vertex-deleted subgraph of \(\hat{G}\) has 3 eigenvalues, then it is obtained by deleting a vertex of degree 8. There are exactly two such vertices, labelled by 1 and 2 in (11), and the deletion of either leads to a subgraph with spectrum \([-2^7, 2^5, 4]\). This is because, in the notation of Theorem 4.3, \(r\) is an eigenvector of the vertex-deleted subgraph corresponding to the eigenvalue 2.

5.2. Symmetric 3-class association schemes

A symmetric 3-class association scheme \(\mathcal{R}\) consists of a set \(X\) and a partition of \(X \times X\) into 4 non-empty binary relations \(R_0, R_1, R_2, R_3\) satisfying the following constraints:

- \(R_0 = \{(x, x) \mid x \in X\}\);
- If \((x, y) \in R_i\), then \((y, x) \in R_i\) and if \((x, y) \in R_k\), then the number of \(z \in X\) such that \((x, z) \in R_i\) and \((z, y) \in R_j\) is a constant \(p_{ij}^k\) depending on \(i, j, k\), but not on a particular choice of \(x, y\).

For \(0 \leq i \leq 3\), we define the \((0, 1)\)-matrix \(A_i\) with rows and columns indexed by the elements of \(X\), and \((x, y)\)-entry 1 if and only if \((x, y) \in R_i\). It follows that \(A_0 = I\) and \(A_iA_j = \Sigma_k p_{ij}^k A_k\). For \(i \in \{1, 2, 3\}\) let \(G_i\) be the graph with adjacency matrix \(A_i\), and for distinct \(i, j \in \{1, 2, 3\}\) let \(\hat{G}_{i,j}\) be the signed graph with adjacency matrix \(A_i - A_j\). The matrices \(A_i\) span a 4-dimensional commutative \(\mathbb{R}\)-algebra (called the Bose-Mesner algebra, cf. [3, Chapter 17]). It follows that the signed graphs \(\hat{G}_{i,j}\) have at most 4 eigenvalues.

**Theorem 17.** Let \(\hat{G}_{i,j}\) be a signed graph arising from a 3-class association scheme. Then \(\hat{G}_{i,j}\) is strongly regular with parameters

\[
r = p_{ij}^0 + p_{ji}^0, \quad a = p_{ij}^i + p_{ji}^j - 2p_{ij}^i, \quad b = -2p_{ij}^j + p_{ji}^i + p_{ij}^j, \quad c = p_{ij}^k + p_{ji}^k - 2p_{ij}^k
\]

where \(\{i, j, k\} = \{1, 2, 3\}\). Moreover, \(\hat{G}_{i,j}\) has 3 eigenvalues if and only if \(a + b = 2c \neq 0\), and 2 eigenvalues if and only if \(a + b = 2c = 0\).
Proof. The first assertion follows from [12, Theorem 2.2], and the second from Lemma 7. Lastly, $\hat{G}_{ij}$ is not a complete graph since $A_k \neq 0$, and so the third assertion follows from [18, Theorem 4.2].

We note that $b$ is replaced by $-b$ in [12, Definition 1.4]. Now we are ready to provide some examples of signed graphs with at most 3 distinct eigenvalues.

Example 18. The 3-class Johnson scheme $J(n,3)$ ($n \geq 6$), also known as the tetrahedral scheme, is defined on the 3-subsets of an $n$-set, with two subsets in the relation $R_i$ if they intersect in $3 - i$ elements. The following scheme provides the numbers $p_{ij}^3$ relevant to $\hat{G}_{1,3}$: they are obtained by a simple computation.

\[
\begin{align*}
p_{11}^3 & \quad p_{13}^3 & \quad p_{11}^3 & \quad p_{13}^3 & \quad p_{33}^3 & \quad p_{11}^2 & \quad p_{33}^2 & \quad p_{13}^2 \\
n - 2 & \quad \binom{n-4}{3} & \quad 0 & \quad 0 & \quad 3(n-6) & \quad \binom{n-6}{3} & \quad 4 & \quad \binom{n-5}{3} & \quad n - 5
\end{align*}
\]

Therefore,

\[
p_{11}^3 + p_{33}^3 - 2p_{13}^3 - 2p_{11}^3 + p_{33}^3 = n - 2 + \binom{n-4}{3} - 6(n-6) + \binom{n-6}{3},
\]

\[
2(p_{11}^2 + p_{33}^2 - 2p_{13}^2) = 8 + 2\binom{n-5}{3} - 4n + 20.
\]

Both of the above expressions are equal to $\frac{1}{2}(n-2)(n-7)(n-9)$. By Theorem 17, $\hat{G}_{1,3}$ has 2 eigenvalues when $n = 7$ or $n = 9$, and 3 eigenvalues otherwise.

Alternatively we can find the spectrum of $\hat{G}_{i,j}$ by using the following information from [2, 9]:

\[
\text{Spec}(A_1) = \left[3(n-3), (2n-9)^{n-1}, (n-7)^{\binom{n}{2}-n}, -3^{\binom{n}{2}}-\binom{n}{2}\right],
\]

\[
\text{Spec}(A_3) = \left[\binom{n-3}{3}, (-n^2 + 9n)/2 - 10^{n-1}, (n-5)^{\binom{n}{2}-n}, -1^{\binom{n}{2}}-\binom{n}{2}\right].
\]

Here the eigenvalues are ordered by common eigenvectors and so the eigenvalues of $A_1 - A_3$ are $\rho$, $\mu^{n-1}$ and $\lambda^{\binom{n}{2}-n}$, where $\rho, \mu, \lambda$ are not necessarily distinct, and

\[
\rho = 3(n-3) - \binom{n-3}{3}, \quad \mu = \frac{1}{2}(n^2 - 5n + 2), \quad \lambda = -2.
\]

Note that $G_1$ is regular of degree $r_1 = 3(n-3)$, $G_3$ is regular of degree $r_2 = \binom{n-3}{3}$ and the graph underlying $\hat{G}_{1,3}$ is regular of degree $r_1 + r_2$.

We find that $\rho \geq \mu$ if and only if $n(n-2)(n-7) \leq 0$. When $n = 7$, $\hat{G}_{1,3}$ has spectrum $[8^7, -2^{28}]$ and by [14, Corollary 4.4] each vertex-deleted subgraph of $\hat{G}_{1,3}$ has 3 eigenvalues. Such a subgraph necessarily has spectrum $[6, 8^6, -2^{27}]$.

When $n = 6$ the spectrum of $\hat{G}_{1,3}$ is $[8, 4^3, -2^{14}]$ and each vertex of $\hat{G}_{1,3}$ has degree 10. Since $-\rho \lambda \neq 10$, Theorem 10(i) shows that no vertex-deleted subgraph of $\hat{G}_{1,3}$ has 3 eigenvalues.
Secondly, we find that \( \rho \leq \lambda \) if and only if \( (n - 1)(n - 2)(n - 9) \geq 0 \). When \( n = 9 \), the spectrum of \( \hat{G}_{1,3} \) is \([19^8, -276]\) and by [14, Corollary 4.4] each vertex-deleted subgraph has spectrum \([17, 19^7, -275]\). When \( n > 9 \), no vertex-deleted subgraph of \( \hat{G}_{1,3} \) has 3 eigenvalues for otherwise, in the notation of Theorem 10(ii), we have \( ||r||^2 = -\rho \mu \); but \( r_1 + r_2 = -\rho \mu \) if and only if \( (n - 1)(n - 2)(n - 9) = 0 \), equivalently \( n = 9 \).

The remaining case is \( n = 8 \), when Theorem 10(iii) shows in similar fashion that a vertex-deleted subgraph of \( \hat{G}_{1,3} \) does not have 3 distinct eigenvalues. In summary, a vertex-deleted subgraph of the signed graph \( \hat{G}_{1,3} \) derived from \( J(n, 3) \) has 3 eigenvalues if and only if \( n = 7 \) or 9.

The definition of \( J(n, 3) \) may be extended to \( J(5, 3) \), but this scheme is degenerate in our context because then \( A_3 = 0 \). In fact, \( J(5, 3) \) is a 2-class association scheme, with \( G_1 \cong L(K_5) \) and \( G_2 \cong \overline{G}_1 \) (the Petersen graph). Here \( \hat{G}_{1,2} \) has spectrum \([-3^5, 3^3]\), while any vertex-deleted subgraph has spectrum \([-3^4, 0, 3^4]\).

**Example 19.** The 3-class Hamming scheme \( H(3, q) \) is defined on the triples of \( q \) symbols (words of length 3 over an alphabet with \( q \) letters), where two triples are in the relation \( R_i \) if they differ in \( i \) coordinates \((i = 0, 1, 2, 3)\). In this case we have the following:

\[
\begin{align*}
p_{11} & \quad p_{13} & \quad p_{11} & \quad p_{13} & \quad p_{31} & \quad p_{33} & \quad p_{31} & \quad p_{33} & \quad p_{13} \\
q - 2 & \quad (q - 2)(q - 1)^2 & \quad 0 & \quad 0 & \quad 3(q - 2) & \quad (q - 2)^3 & \quad 2 & \quad (q - 2)^2(q - 1) & \quad q - 1
\end{align*}
\]

Therefore,

\[
p_{11}^4 + p_{33}^4 - 2p_{13}^4 - 2p_{13}^3 + p_{11}^3 + p_{33}^3 = q - 2 + (q - 2)(q - 1)^2 - 6(q - 2) + (q - 2)^3,
\]

\[
2(p_{11}^2 + p_{33}^2 - 2p_{13}^2) = 4 + 2(q - 2)^2(q - 1) - 4q + 4.
\]

Note that both of the above expressions are equal to \( 2q(q - 2)(q - 3) \). By Theorem 17, the signed graph \( \hat{G}_{1,3} \) has at most 3 distinct eigenvalues. Moreover, \( c = 0 \) if and only if \( q = 2 \) or 3, and these are the cases in which \( \hat{G}_{1,3} \) has only 2 distinct eigenvalues. Thus, by [14, Corollary 4.4] any vertex deleted subgraph of \( \hat{G}_{1,3} \) has 3 distinct eigenvalues when \( q \in \{2, 3\} \). On the other hand, by the proof of [8, Theorem 4.2] the eigenvalues of \( \hat{G}_{1,3} \) other than its net-degree are the roots of the quadratic

\[
x^2 + \frac{b - a}{2}x + \frac{a + b}{2} - r.
\]

Now, by Theorem 17, we conclude that the eigenvalues of \( \hat{G}_{1,3} \) are

\[
\rho = 3(q - 1) - (q - 1)^3, \quad \mu = q^2 - 2, \quad \lambda = -2.
\]

For \( q > 3 \), we find \( \rho < \lambda \), and so by Theorem 10(ii), if a vertex-deleted subgraph of \( \hat{G}_{1,3} \) has only 3 distinct eigenvalues then \( ||r||^2 = -\rho \mu \). This equality holds if
and only if
\[ q(q-1)^2(q+2)(q-3) = 0. \]
Accordingly, no vertex-deleted subgraph of $\hat{G}_{1,3}$ has 3 eigenvalues when $q > 3$.

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