On the Convergence of Network Systems

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Abstract
The apparent disconnection between the microscopic and the macroscopic is a major issue in the understanding of complex systems. To this extent, we study the convergence of repeatedly applying local rules on a network, and touch on the expressive power of this model. We look at network systems and study their behavior when different types of local rules are applied on them. For a very general class of local rules, we prove convergence and provide a certain member of this class that, when applied on a graph, efficiently computes its $k$-core and its $(k-1)$-crust giving hints on the expressive power of such a model. Furthermore, we provide guarantees on the speed of convergence for an important subclass of the aforementioned class. We also study more general rules, and show that they do not converge. Our counterexamples resolve an open question of (Zhang, Wang, Wang, Zhou, KDD - 2009) as well, concerning whether a certain process converges. Finally, we show the universality of our network system, by providing a local rule under which it is Turing-Complete.

2012 ACM Subject Classification
Networks \rightarrow Network dynamics

Keywords and phrases network systems, network dynamics, convergence

1 Introduction
There is an increasing interest on the interplay between the microscopic and the macroscopic in terms of emergent behavior. This is a crucial point for a better understanding of complex systems. Biological systems seem to form macroscopic structures out of local interactions between simpler structures, on all levels of organization. Physarum polycephalum (a slime mold) has been shown to be able to solve computational problems such as the shortest path. The underline common characteristic of these systems is the emergent behavior in the macroscopic level out of simple local interactions at the microscopic level. Motivated by the plethora of such examples, we initiate (to the best of our knowledge) research from a theoretical perspective related to the study of repeatedly applying local rules on a network, and look at the expressive power of this model from a computer science perspective.

1.1 Related Work
Efforts on enriching our understanding on how the microscopic gives rise to the macroscopic has produced several interesting results. One of the most well-known such results is the experiments of Nakagaki et al. [17], who presented the ability of a slime-mold (Physarum polycephalum) to solve mazes. Later on, researchers have established the validity of the aforementioned claim, and provided more functions that Physarum can compute, from a theoretical point of view [4, 13, 19]. Bird-flocking is also an intriguing such system, where Chazelle managed to prove convergence [7]. Based on these results Chazelle coined the term Natural Algorithms [6] and argued that traditional mathematics seems to fail to attack such problems in an efficient manner (efficiency refers to expressive power) especially due to the existence of memory within these systems that seems to break any symmetry on which traditional mathematics can be based on. There are also many active models,
Programmable Matter [12, 15, 16], which models any type of matter that can algorithmically change its physical properties. The goal is to check whether a desired configuration can be reached, and if so, to provide a method to achieve it as fast as possible. What distinguishes such active models from passive ones is the fact that movements are in complete control of the algorithm. On the other hand, on passive models the movements occur by the environment, and the algorithm can only decide whether to accept them or not.

On more theoretical constructions, important progress has been made on Population Protocols [1, 5, 10], where anonymous agents with only a constant amount of memory available randomly interact with each other and are able to compute functions that do not seem possible at first, like electing leaders. Population protocols are a typical example of a passive model, and are related to chemical reaction networks [9]. Additionally, cellular automata use very simple update rules that give rise to interesting patterns [2, 11]. A great example is one of the most simple cellular automata, Rule 110, which has been shown by Cook to be Turing-Complete [8].

Except for the analysis of such complex systems, there are some important results on the design of local rules that give rise to a desired global behaviour. Silk et al. [18] designed local rules that allow a network to reach a desired steady-state degree distribution, while Valentini et al. proposed a global-to-local design methodology to compose heterogeneous swarms for self-organized task allocation [20].

1.2 Our results

As stated by Wolfram [21] for the majority of the complex systems: 'there is in effect always a fixed underlying geometrical structure which remains unchanged throughout the evolution of the system... it is possible to construct systems in which there is no such invariance... network systems'.

We provide a definition of a network system and study its behavior when different types of local rules are applied on it. We prove convergence for a very general class of local rules. Then, to exhibit the expressiveness in terms of emergent behavior we provide a certain member of this class (very simple rule) that, when applied on a graph, efficiently computes its $k$-core and its $(k - 1)$-crust [3]. Furthermore, we provide guarantees on the speed of convergence for an important subclass of the aforementioned class. We also study more general rules and show that they do not converge. Our counterexamples resolve an open question of Zhang et al. [22] as well, concerning whether a certain dynamic process to enhance network communities converges. Finally, we prove that the network system with a particular local rule is Turing-Complete.

2 Preliminaries

Let $G = (V, E)$ be an undirected simple graph. The edges of this graph evolve over time (discrete time) based on a set of rules. We represent the graph at time $t$ by $G^{(t)} = (V, E^{(t)})$, and $G^{(0)} = G$. At time $t$, a certain set of pairs of distinct nodes (namely $C^{(t)} \subseteq V^2$) determines whether our rules will be applied on some pair or not. We will slightly abuse notation and say that $C^{(t)} = V^2$ to denote that the rules are applied on all edges whose endpoints are distinct, even though $V^2$ contains pairs of non-distinct nodes. Let the energy of the edge $e = (u, v)$ at time $t$ be the score of this edge which is represented by $E^{(t)}_{G^{(t)}}(e)$ or $E^{(t)}_{G^{(t)}}(u, v)$. We write $E^{(t)}(u, v)$ or $E(u, v)$ when the graph and the time we are referring to are clear from the context. Finally, assume that $n = |V|$ and $m^{(t)} = |E^{(t)}|$. Notice that the set of nodes is static and never changes, while the edges are affected by the rules.
Let \( N_G(u) \) be the set of all neighbors of node \( u \) and let \( d_G(u) \) be the degree of node \( u \) in graph \( G \). We define \( |E^{(t)}(u, v)| \) to be the number of edges between \( u \) and \( v \) at time \( t \) (either 0 or 1), and \( |E(G[N_G^{(0)}(u) \cap N_G^{(0)}(v)])) \) to be the number of edges between common neighbors of \( u \) and \( v \) at time \( t \).

Let \( f : \mathbb{N}^2 \rightarrow \mathbb{R} \) be a continuous function having the following two properties: i) Non-decreasing, that is \( f(x, y + \epsilon) \geq f(x, y) \) for \( \epsilon > 0 \) (similarly \( f(x + \epsilon, y) \geq f(x, y) \)) and ii) Symmetric, \( f(x, y) = f(y, x) \). The second property is related to the fact that we consider undirected graphs. We call these functions proper. We call a function \( g_G(u) \) degree-like if it only depends on the neighborhood \( N_G(u) \) of its argument. Furthermore, if for the neighborhoods of two nodes \( u \) and \( v \) it holds that \( N_G(u) \supseteq N_G(v) \), then \( g_G(u) \geq g_G(v) \).

Our goal is to find whether the following dynamical process converges or alternatively whether the following algorithm terminates. In the following, the edge \( e^{(t)} \) is also used as a boolean variable. In particular, when \( e^{(t)} = 0 \) it means that \( e^{(t)} \notin E^{(t)} \) (the edge is non-existent) while \( e^{(t)} = 1 \) means that \( e^{(t)} \in E^{(t)} \) (the edge is existent). Let \( \alpha \) and \( \beta \) be two parameters that correspond to two thresholds, the lower and the upper threshold respectively.

We iterate the following (\( t \) corresponds to iterations):

For each edge \( e^{(t)} \notin C^{(t)} \) in the graph, edge \( e^{(t+1)} = e^{(t)} \).

For each edge \( e^{(t)} \in C^{(t)} \) in the graph we compute the energy \( \mathcal{E}(e^{(t)}) \).

Three cases for \( \mathcal{E}(e^{(t)}) \):
1. \( \mathcal{E}(e^{(t)}) < \alpha \): Then edge \( e^{(t+1)} = 0 \) (non-existent).
2. \( \alpha \leq \mathcal{E}(e^{(t)}) < \beta \): Then edge \( e^{(t+1)} = e^{(t)} \).
3. \( \mathcal{E}(e^{(t)}) \geq \beta \): Then edge \( e^{(t+1)} = 1 \) (existent).

Until the graph does not change, that is \( G^{(t)} = G^{(t+1)} \).

A discussion is in order with respect to the terminating condition of this algorithm. The user, apart from defining the local rule, must also define \( C^{(t)} \). This terminating condition makes the silent and plausible assumption that the user does not behave as an adversary to the algorithm but in fact it tries to improve on the algorithm by trying to guarantee convergence, among others. In this sense, there is no meaning from the side of the algorithm to make an iteration without changing the graph. For example, \( C^{(t)} = \emptyset \) always causes the algorithm to terminate. This is why this terminating condition for convergence checks whether the graphs changes between successive iterations. Notice that the terminating condition can be also related to other conditions based on the specified goal of the dynamic process; for example, the process may terminate as soon as a clique of a particular size occurs in the graph.

The following lemma justifies the terminating condition for the case where \( C^{(t)} = V^2 \). The choice of \( C^{(t)} \) is imposed by the fact that careful choices of \( C^{(t)} \) can make the algorithm loop for ever, while for this case \( (C^{(t)} = V^2) \) it terminates (an example is shown in 6.1).

\textbf{Lemma 1.} If for some \( t' \) it holds that \( G^{(t'-1)} = G^{(t')} \) and it holds that \( \forall t : C^{(t)} = V^2 \), then for any \( t > t' \) it holds that \( G^{(t)} = G^{(t')} \).

\textbf{Proof.} Since \( C^{(t)} = V^2 \), the update rule applies on all possible pairs of nodes. Since the process is deterministic and from the hypothesis that \( G^{(t'-1)} = G^{(t')} \) the lemma follows. \( \square \)

In general, we can iterate the procedure until \( |G^{(t+1)} - G^{(t)}| < \epsilon \) for some user-defined parameter \( \epsilon \). Note that in the description of the algorithm we set \( \epsilon = 0 \). Whenever we prove convergence, it holds for all values of \( \epsilon \) since it holds for \( \epsilon = 0 \), while when we disprove convergence, we present counter-examples where consecutive graphs do not share any edges at all (and thus the process wouldn’t stop even for high values of \( \epsilon \)).
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Suppose that for all \( t \) it holds that \( C^{(t)} \) only depends on \( G^{(t)} \), and not on \( t \) itself. Since \( |V| \) stays the same throughout the process, there are finitely many graphs with that many nodes. Thus, if the process doesn’t converge, at some point \( t \) we reached a graph from a previous step (say \( t' < t - 1 \)). That is \( G^{(t')} = G^{(t)} \). We say that the cycle size of this process is \( t - t' \), since the graphs \( G^{(t')}, G^{(t'+1)}, ..., G^{(t-1)} \) will periodically repeat.

We study the following update rules:

1. \( \mathcal{E}(u, v) = \min\{d_{G^{(t)}}(u), d_{G^{(t)}}(v)\} \) and the rule is applied on all edges (\( \forall t : C^{(t)} = V^2 \)). We prove convergence, as well as matching upper and lower bounds on the number of steps required. This update rule is related to the discovery of the \( \alpha \)-core of a graph [3].

2. \( \mathcal{E}(u, v) = f(d_{G^{(t)}}(u), d_{G^{(t)}}(v)) \), where \( f \) is any proper function. The rule is always applied on all edges (\( \forall t : C^{(t)} = V^2 \)). We prove convergence, as well as an upper bound on the number of steps required when \( \alpha = \beta \).

3. \( \mathcal{E}(u, v) = f(g_{G^{(t)}}(u), g_{G^{(t)}}(v)) \). This is a continuation of the previous model, where at any time and for any edge we can arbitrarily decide whether the rule will be applied (\( C^{(t)} \) can change for different values of \( t \), without any restriction). The function \( f \) is proper, and \( g_{G^{(t)}}(u) \) is any degree-like function. We prove that the above process always converges.

4. \( \mathcal{E}(u, v) = |N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)| \). The rule is only applied on pairs of nodes whose distance is at most 2 (they are either directly connected or they share a common neighbor). We prove that there are cases where this process doesn’t converge.

5. The update rule is described in [22] as a preprocessing algorithm to enhance communities within a given network. The energy definition is \( \mathcal{E}(u, v) = |N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)| + |E^{(t)}(u, v)| + |E(G[N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)])| \). The rule is only applied on pairs of nodes whose distance is at most 2. We disprove their conjecture that their algorithm always converges, by providing counterexamples.

Note that when \( \alpha = \beta \), then the system has in effect no memory and it is in a sense an initial value problem. When \( \alpha < \beta \), then the system is equipped with memory since the status of edges in maintained from the previous time point in case the energy of the edge falls in the range \( [\alpha, \beta] \). This extra memory of the network system renders its analysis more complicated.

## 3 Taking the Minimum of Degrees

In this section we define \( \mathcal{E}'(u, v) = \min\{d_{G^{(t)}}(u), d_{G^{(t)}}(v)\} \), that is we choose the minimum degree of its two nodes as the new energy of the edge. The rule is always applied on all edges (\( \forall t : C^{(t)} = V^2 \)). To investigate convergence we take cases for the nodes in \( G^{(t)} \) for arbitrary \( t \). We prove that with this definition of energy the process converges in \( O(n) \) steps, and provide a matching lower bound on the number of steps needed.

**Theorem 2.** The process always converges in \( O(n) \) steps, and there are cases where \( \Omega(n) \) steps are required.

**Proof.** For all nodes \( v \in V, \) for any \( t, \) such that \( d(v) < \alpha \) it holds that they become isolated (their degree is zero) in the next step. This is because \( \min\{d(u), d(v)\} \leq d(v) < \alpha \), for any \( d(u) \). As soon as a node becomes isolated, it will be isolated forever since again \( d(v) = 0 < \alpha \).

There can be \( O(n) \) consecutive rounds where at least one node becomes isolated; after that we end up with a graph \( G^{(t)} \) for which there are two sets of nodes: the set \( V_0^{(t)} \) which contains all nodes with degree 0 as well as the set \( V_\alpha^{(t)} \) that contains all nodes with degree at least \( \alpha \).
The question is how many rounds are needed for \( V_\alpha^{(t)} \) to settle down. No edge with both endpoints in \( V_\alpha^{(t)} \) will cease to exist at time \( t+1 \) since this would mean that some node in \( V_\alpha^{(t)} \) has degree less than \( \alpha \) which is a contradiction. Thus, the degrees do not decrease.

We notice that all nodes with degree at least \( \beta \) will form a clique at time \( t + 1 \). On the other hand, nodes of \( V_\alpha^{(t)} \) with degree less that \( \beta \) will not form any new edge, effectively having the same neighbors at time \( t + 1 \). The exact same reasoning gives \( G^{(t+1)} = G^{(t+2)} \) and thus we have convergence in at most 1 step, due to Lemma 1.

The above discussion gives an upper bound of \( O(n) \) time steps for convergence. Let us give a simple matching lower bound. If \( G^{(0)} \) is a path, \( \alpha = 2 \) and \( \beta > 2 \), then the graph converges in \( \lfloor \frac{n}{2} \rfloor \) time steps, since no edge will ever be created, and at every time step, the two ends of the remaining path become isolated.

It is interesting to notice the similarity of our process, and the process of acquiring the \( \alpha \)-core (or complementary the \( \alpha \)-crust) of a simple undirected graph [3].

**Definition 3.** The \( \alpha \)-core \( H \) of a graph \( G \) is the unique maximal subgraph of \( G \) such that for the degree of every node \( u \in H \) it holds that \( \deg_H(u) \geq \alpha \). All nodes not in \( H \) form the \( \alpha - 1 \) crust of graph \( G \).

The \( \alpha \)-core is a notion that plays an important role in studying the clustering structure of social networks. Batagelj et al. [3] proved that the following process efficiently computes the \( \alpha \)-core of a graph:

**Lemma 4.** Given a graph \( G \) and a number \( \alpha \), one can compute \( G \)'s \( \alpha \)-core by repeatedly deleting all nodes whose degree is less than \( \alpha \).

What makes our process and the process of detecting the \( \alpha \)-core different is the fact that new edges can emerge in our process. However, we can disallow this by setting \( \beta \geq n \).

**Lemma 5.** When \( E(u,v) = \min\{d_{G^{(0)}}(u), d_{G^{(0)}}(v)\} \) and \( \forall t : C^{(t)} = V^2 \) the dynamic process for any value of \( \alpha \) and \( \beta \geq n \) is essentially the same process with the one for detecting the \( \alpha \)-core. Furthermore, all isolated nodes in our process form the \( (\alpha - 1) \)-crust of \( G^{(0)} \), while the remaining graph forms the \( \alpha \)-core.

**Proof.** First of all, even if a node connects with any other node, its degree will be \( n - 1 \). Thus, it holds that \( \min\{d(u), d(v)\} \leq n - 1 < \beta \). This ensures that no edge will ever be formed by the dynamic process.

As far as existing edges are concerned, both processes delete edges where at least one endpoint \( v \) has degree less than \( \alpha \), since \( \min\{d(u), d(v)\} \leq d(v) < \alpha \), for any \( d(u) \). By the same reasoning, these nodes (which, by definition, belong to the \( (\alpha - 1) \)-crust) remain isolated forever. Furthermore, edges with both endpoints having degree at least \( \alpha \) will be preserved as the minimum of their degrees will still be at least \( \alpha \).

### 4 Symmetric Non-Decreasing Function on the Degrees

We don’t study the case where \( \alpha < \beta \), as it is just a special case of Section 5. However, we study the case \( \alpha = \beta \), as we are not only able to prove convergence of the process, but we also prove an upper bound on the number of steps needed for convergence.

We define the energy of an edge \( (u,v) \) to be \( E(u,v) = f(d_{G^{(0)}}(u), d_{G^{(0)}}(v)) \), where \( f \) is a proper (symmetric and non-decreasing in both variables) function. The rule is always applied on all edges \( \forall t : C^{(t)} = V^2 \).
For the graph $G^{(t)}$, let $R^{(t)}(u, v)$ be an equivalence relation defined on the set of nodes $V$ for time $t$, such that $(u, v) \in R^{(t)}$ if and only if $d_{G^{(t)}}(u) = d_{G^{(t)}}(v)$. The equivalence class $R_i^{(t)}$ corresponds to all nodes with degree $d(R_i^{(t)})$, where $i$ is the rank of the degree in decreasing order. This means that the equivalence class $R_1^{(t)}$ contains all nodes with maximum degree in $G^{(t)}$. Apparently, the maximum number of equivalence classes is $n = |V|$, since the degree can be in the range $[0, n-1]$. Let $|G^{(t)}|$ be the number of equivalence classes in graph $G^{(t)}$.

Before moving to the proof, we notice certain properties of the dynamic process that hold for all $t \geq 1$, that is they hold after at least one round of the process (initialization). First, we show that nodes have an implicit hierarchy with respect to degrees.

\> **Property 1.** If $d_{G^{(t)}}(u) \geq d_{G^{(t)}}(w)$, then $d_{G^{(t+1)}}(u) \geq d_{G^{(t+1)}}(w)$, for all $t \geq 1$.

\> **Proof.** For any neighbor $v$ of $w$ in $G^{(t+1)}$ it holds that $E^{(t)}(v, w) \geq \beta$. Then it also holds that $E^{(t)}(v, u) \geq \beta$, since $f$ is non-decreasing, which means $v$ is also a neighbor of $u$ in $G^{(t+1)}$. ◀

Nodes that have the same degree at time $t$, share the same neighbors at time $t+1$.

\> **Property 2.** If $d_{G^{(t)}}(u) = d_{G^{(t)}}(w)$, then $N_{G^{(t+1)}}(u) = N_{G^{(t+1)}}(w)$.

\> **Proof.** As in the proof of Property 1 due to the equality of the degrees, it also holds that any neighbor $v$ of $u$ is a neighbor of $w$ and respectively any neighbor $v$ of $w$ is a neighbor of $u$. ◀

In the following, we discuss properties related to equivalence classes.

\> **Property 3.** The number of equivalence classes in $G^{(t+1)}$ is less than or equal to the number of equivalence classes in $G^{(t)}$.

\> **Proof.** By Property 2, nodes that belong to the same equivalence class at time $t > 0$ will always belong to the same equivalence class for all $t' > t$. ◀

\> **Property 4.** If $G^{(t+1)}$ has the same number of equivalence classes as $G^{(t)}$, then $\forall i$, $|R_i^{(t)}| = |R_i^{(t+1)}|$, where $|R_i^{(t)}|$ is the number of nodes in the equivalence class $R_i^{(t)}$.

\> **Proof.** Suppose that the above doesn’t hold. Then, there is some $i$ for which $|R_i^{(t)}| \neq |R_i^{(t+1)}|$. This means that there must be two nodes in some equivalence class $R_j^{(t)}$ that landed to different classes in $G^{(t+1)}$. However, Property 2 implies that this is impossible. ◀

The following lemma describes how equivalence classes behave with respect to edge distribution.

\> **Lemma 6.** If an arbitrary node $u$ in $R_i^{(t)}$ is connected with some node $w$ in $R_j^{(t)}$, then $u$ is connected with every node $x$ in every equivalence class $R_k^{(t)}$, such that $k \leq j$ and $t > 0$.

\> **Proof.** Due to Property 1, for all nodes $x \in R_k^{(t)}$ it holds that $d_{G^{(t)}}(x) \geq d_{G^{(t)}}(w)$ and so they are also neighbors of $u$. ◀

We prove by induction that this process always converges in at most $2|G^{(0)}|$ steps. To begin with, it is obvious that the clique $K_n$ as well as the null graph $K_n$ both converge in at most one step, for any value of $\beta$. The following renormalization lemma describes how the number of equivalence classes is reduced and is crucial to the induction proof.
Lemma 7. If \( d(R_1(t)) = n - 1 \) for every \( t \geq c \), for some \( c \in \mathbb{N} \cup \{0\} \), and the subgraph \( G^{(c)} \setminus R_1^{(c)} \) converges for any value of \( \beta \) and proper function \( f \), then \( G^{(c)} \) converges as well. Similarly, if \( d(R_1(t)) = 0 \) for every \( t \geq c \), for some \( c \in \mathbb{N} \cup \{0\} \), and the subgraph \( G^{(c)} \setminus R_{G^{(c)}}^{(c)} \) converges for any value of \( \beta \) and proper function \( f \), then \( G^{(c)} \) converges as well. The time it takes for \( G^{(c)} \) to converge is the same as the time it takes for the induced subgraph to converge, for both cases.

Proof. The main idea is that we consider two different sets of nodes: \( R_1^{(c)} \) and \( V \setminus R_1^{(c)} \). Due to our hypothesis, at all future time-steps the edges between these two groups, and the edges with both endpoints in \( V \setminus R_1^{(c)} \) are fixed. Concerning the edges with both endpoints in \( V \setminus R_1^{(c)} \), we can almost study this subgraph independently. That’s because the effect of \( R_1^{(c)} \) on \( V \setminus R_1^{(c)} \) is completely predictable: it always increases the degree of all nodes by the exact same amount. The same reasoning applies for \( R_{G^{(c)}}^{(c)} \).

More formally, by Property 1 for all \( t \geq c \) it holds that \( R_1^{(c)} \subseteq R_1^{(c+1)} \). This means that the nodes in \( R_1^{(c)} \) are always connected to every node after time \( c \). As a result, for all \( u \in V \setminus R_1^{(c)} \) it holds that their degree in the induced subgraph \( G^{(c)} \setminus R_1^{(c)} \) is \( d_{G^{(c)}}(u) - |R_1^{(c)}| \).

Thus, the decision for the existence of an edge \((u, v)\), where \( u, v \in G^{(c)} \setminus R_1^{(c)} \) is the following:

\[
E^{(c)}(u, v) = f(d_{G^{(c)}}(u) + |R_1^{(c)}|, d_{G^{(c)}}(v) + |R_1^{(c)}|) \geq \beta
\]

which can be written as:

\[
E^{(c)}(u, v) = g(d_{G^{(c)}}(u), d_{G^{(c)}}(v)) \geq \beta
\]

where

\[
g(x, y) = f(x + |R_1^{(c)}|, y + |R_1^{(c)}|)
\]

Clearly, \( g \) is a proper function assuming that \( f \) is a proper function. Thus, the choice of whether the edge exists between \( u \) and \( v \) is equivalent between \( G^{(c)} \) and \( G^{(c)} \setminus R_1^{(c)} \) by appropriately changing \( f \) to \( g \). But due to our hypothesis \( G^{(c)} \setminus R_1^{(c)} \) converges, and thus \( G^{(c)} \) also converges in the same number of steps. Note that we need not compute \( g \) since this is only an analytical construction; the dynamic process continues as defined. The proof of the second part of the lemma is similar in idea but much simpler since function \( f \) does not change due to the fact that the removed nodes have degree 0.

The following theorem establishes that the dynamic process converges in linear time.

Theorem 8. When \( \alpha = \beta \), \( E(u, v) = f(d_{G^{(c)}}(u), d_{G^{(c)}}(v)) \), and \((\forall t : C^{(t)} = V^2)\), the dynamic process on an undirected simple graph \( G \) converges in at most \( 2|G^{(0)}| \) steps.

Proof. We use induction on the number of equivalence classes. For the base case, the graph \( G^{(0)} \) has only one equivalence class \( R_1^{(0)} \) (the graph is regular). There are two cases: either \( f(d(R_1^{(0)}), d(R_1^{(0)})) \geq \beta \) and all edges are created \((G^{(1)} = K_n)\), or \( f(d(R_1^{(0)}), d(R_1^{(0)})) < \beta \) and no edge is created \((G^{(1)} = \overline{K_n})\). Either way, \( G^{(1)} \) converges in at most 1 step, and thus \( G^{(0)} \) converges in at most 2 steps.

Suppose the theorem holds for \( i - 1 \) equivalence classes and let \( G^{(i)} \) be a graph with \( |G^{(i)}| = i > 1 \). If the number of equivalence classes decreases within the first two steps, then the process converges in at most \( 2i \) steps, by the inductive hypothesis. Thus, we only look at
Thus, we assume while the maximum degree will be at most we always get Cases (1) or (2) because, inductively, the minimum degree will always be

other hand, if then from this point on we always get Cases (3) or (4) since, inductively, the maximum degree will always be and the minimum degree will always be at least . On the other hand, if

then we always get Cases (3) or (4) respectively, and thus would converge in at most steps. Else it holds that:

because otherwise and would unite in , effectively reducing the number of equivalence classes. Thus, , due to Lemma Property and the fact that . In the case where

we always get Cases (1) or (2) because, inductively, the minimum degree will always be 0, while the maximum degree will be at most . In this case, the theorem is proved due to Lemma and the inductive hypothesis.

On the other hand, if

then we always get Cases (3) or (4) since, the maximum degree will always be as we prove below. In this case, the theorem is also proved due to Lemma and the inductive hypothesis.

The same reasoning works for Case (2), which results in . Like before, we assume (otherwise the equivalence classes are reduced) and so

Thus, , due to Lemma and Property In the case where

then from this point on we always get Cases (3) or (4) since, inductively, the maximum degree will always be and the minimum degree will always be at least . On the other hand, if

then we always get Cases (1) or (2) because, inductively, the minimum degree will always be 0.

In all possible cases, after at most 2 rounds it suffices to examine graphs with reduced number of equivalence classes. This proves the upper bound for the convergence.
5 Local Rules

In this section we extend the update rule given in Section 4 More specifically, we change the definition of energy, from $E(u, v) = f(d_{G^{(t)}(u)}, d_{G^{(t)}(v)})$ to $E(u, v) = f(g_{G^{(t)}(u)}, g_{G^{(t)}(v)})$.

The function $g$ can be any degree-like function. That is, $g$ must only depend on the neighborhood of the given node. Furthermore, $g$ must capture the property that the bigger the neighborhood of some node, the higher the assigned value. Formally, assuming that the neighborhood of node $u$ at time $t$ is $N_{G^{(t)}}(u)$, the neighborhood of node $v$ at time $t'$ is $N_{G^{(t')}}(v)$, and $N_{G^{(t)}}(u) \supseteq N_{G^{(t')}}(v)$, then $g_{G^{(t)}}(u) \geq g_{G^{(t')}}(v)$. Notice that, generally, the values $t$ and $t'$ may differ. The reason we extend the notion of degree is so that $g$ can represent more interesting rules. For example, we are no longer obligated to handle all nodes in the same manner; nodes can be assigned an importance factor (e.g., a known centrality measure such as their betweenness centrality in $G^{(0)}$), and let $g(u)$ be the sum of these factors of nodes in the neighborhood of $u$.

Additionally, at any time and for any edge we can arbitrarily decide whether the rule will be applied. This means that $C^{(t)}$ can change for different values of $t$, with no restrictions posed. For example, allowing only preserving of edges from time $t_0$ to time $t_0 + 1$ would be achieved by setting $C^{(t_0)} = E^{(t_0)}$, and applying the rules only on pairs of nodes whose distance is at most 2 would be $C^{(t_0)} = \{(u, v) \text{ s.t. } (u, v) \in E^{(t_0)}\} \cup \{3 w \text{ s.t. } ((u, w), (w, v)) \in E^{(t_0)} \times E^{(t_0)}\}$. We also assume that the function $f$ is proper (symmetric and non-decreasing in both variables).

It is easy to see that the update rule in Section 4 is a special case of the current update rule, where the function $g$ is the degree of the node, and $\forall t : C^{(t)} = V^2$.

Notice that the introduction of $C^{(t)}$ allows us to define local update rules. For example, $C^{(t)}$ could be defined in a way that allows an edge to be formed if and only if the previous distance between the two nodes is bounded by some constant $K$.

To show that any such process converges, we define the following:

Definition 9. A pair $(t, D)$ is said to be $|D| - Done$ if $t$ is a natural number, $D \subseteq V$ and it holds that the neighborhood of all nodes $u \in D$ doesn’t change after time $t$. That is, $N_{G^{(t')}}(u) = N_{G^{(t)}}(u)$, for $t' \geq t$.

Our convergence proof repeatedly detects $|D| - Done$ pairs with increasing $|D|$. When $D = V$, all neighborhoods do not change, and thus the process converges.

Lemma 10. Given a $|D| - Done$ pair $(t, D)$, we can find a $(|D| + 1) - Done$ pair $(t', D')$.

Proof. Let $t_1 \geq t$ be a time-step where some node $u \notin D$ maximizes the function $g$ over all future time-steps and nodes not in $D$. More formally, we define $t_1 \geq t$ as the time-step where there is some node $u \notin D$ such that $g_{G^{(t_1)}}(u) \geq g_{G^{(t_1)}}(v)$, for all $t' \geq t_1$ and $v \in V \setminus D$. If there are many such choices, we arbitrarily pick one where the degree of $u$ is the highest. Let us note that, later in time (say at $t_1'$), it is entirely possible that $u$’s neighborhood shrinks and thus its $g$ value drops ($g_{G^{(t_1')}}(u) < g_{G^{(t_1)}}(u)$).

It is guaranteed that $t_1$ exists, as there are finitely many graphs with $|V|$ nodes, and finitely many nodes. Thus, there are finitely many values of $g_C(u)$ to appear after time $t$.

Our core idea is that either $u$’s neighborhood stays the same in all subsequent time-steps (and thus $D$ is extended by $u$), or some edge is lost along the way. But if the other endpoint $w$ of the edge can’t preserve an edge with $u$, which maximizes $g$, then it doesn’t preserve any other edge. Inductively, it will never form any new edge, and thus $D$ can be extended by $w$.

More formally, if neighbors of $u$ in $G^{(t_1)}$ remain neighbors of $u$ in all subsequent time-steps, then, in future time-steps, its neighborhood can only grow from $N_{G^{(t_1)}}(u)$, or stay the same.
10 On the Convergence of Network Systems

But if its neighborhood grows, due to the properties of function $g$, its value will not drop and the degree of $u$ will increase. However, the way we picked $u$ doesn’t allow this. We conclude that the neighborhood of $u$ doesn’t change after time $t_1$, and thus we can extend $D$ by $\{u\}$, that is, $(t_1, D \cup \{u\}) = (|D| + 1) - Done$.

Else, let $t_2 > t_1$ be the first time-step that a neighbor $w$ of $u$ in $G(t_2)$ is not a neighbor of $u$ in $G(t_2)$. It follows directly from the fact that $u$’s neighborhood stays the same until $t_2 - 1$ that $g_{G(t_2-1)}(u) = g_{G(t_2-1)}(u)$. Then $w$ has no neighbor $v \in V \setminus D$ in $G(t_2)$, as it holds that $\alpha > f(g_{G(t_2-1)}(u), g_{G(t_2-1)}(w)) = f(g_{G(t_2-1)}(u), g_{G(t_2-1)}(w)) \geq f(g_{G(t_2-1)}(v), g_{G(t_2-1)}(w))$. The latter inequality follows from the way we picked $t_1$ and $u$. Of course, due to the definition of $D$, no new edge is formed between $w$ and a node in $D$. Thus, the neighborhood of $w$ shrinks, and due to $g$’s properties $g_{G(t_2-1)}(w) \geq g_{G(t_2)}(w)$.

We argue that the neighborhood of $w$ at all subsequent time steps will stay the same, that is $N_{G(t_2)}(w) = N_{G(t_2)}(w), t_2 \geq t_2$. We prove this inductively. It trivially holds for $t_2 = t_2$. Supposing it holds for some $t_2$, we prove that it also holds for $t_2 + 1$. If it doesn’t, then $w$ forms an edge with some node $v \in V \setminus D$, due to the definition of $D$. But we know that $\beta \geq \alpha > f(g_{G(t_2-1)}(u), g_{G(t_2-1)}(w)) = f(g_{G(t_2-1)}(u), g_{G(t_2-1)}(w)) \geq f(g_{G(t_2)}(v), g_{G(t_2)}(w))$ due to $f$ being non-decreasing. We conclude that the neighborhood of $w$ doesn’t change after time $t_2$, and thus we can extend $D$ by $\{w\}$, that is, $(t_2, D \cup \{w\}) = (|D| + 1) - Done$.

▌Theorem 11. When $E(u, v) = f(g_{G(t)}(u), g_{G(t)}(v))$, the dynamic process on an undirected simple graph $G$ converges for any $\alpha, \beta$, proper function $f$, degree-like function $g$ and sequence of sets $C(t)$.

▌Proof. It trivially holds that $(0, \emptyset) = 0 - Done$. By applying Lemma 10 once, we increase the size of $D$ by 1. Thus, by applying it $|V|$ times, we end up with a $|V| - Done$ pair $(t, V)$. Since all neighborhoods stay the same for all future steps, $G(t) = G(t)$ for all $t' \geq t$.

6 Moving Beyond Degree

We define the energy of an edge $(u, v)$ to be $E(u, v) = N_{G(t)}(u) \cap N_{G(t)}(v)$. The rule is only applied on pairs of nodes whose distance is at most 2 (they are either directly connected or they share a common neighbor).

In the following, we prove that the process may not converge. To prove this, we provide a certain family of graphs, such that when $G(t)$ is any member of this family and the parameters are $\alpha = \beta = 2$, the process doesn’t converge. Furthermore, this family has the property that for any positive number $c$, there exists a member of it such that the cycle size of the process with $\alpha = \beta = 2$ is at least $c$. Finally, we provide examples where consecutive graphs do not share any edges at all; thus, even if we stop when consecutive graphs are ‘close enough’, and not necessarily the same, the process still doesn’t terminate. We even give an example where we infinitely swap between a graph and its complement, which is the farthest we could get from ‘close enough’.

▌Definition 12. Let $[S]$ denote the set $\{0, 1, ..., S - 1\}$. For each $S$, we define the graph $G_S = (V_S, E_S)$ to be the undirected graph with $S^2$ nodes, where each node is identified by a pair of integers (that is $V_S = [S] \times [S]$) and for each node $(x, y)$ its neighbors are the nodes $(x, y \pm 1 \mod S)$ and $(x \pm 1 \mod S, y)$.

Notice that applying the dynamic process to $G(0) = G_3$, we infinitely swap between $G_3$ and its complement.
Except for the case where \( S = 1 \), which trivially converges, we prove that the process with \( G^{(0)} = G_S \) and parameters \( \alpha = \beta = 2 \) doesn’t converge, for all other odd values of \( S \). To prove this, we need the following lemma.

**Lemma 13.** Let \( i, j \) be natural numbers (including zero), and \( S > 1 \) be an odd number. Then it holds that \( i \not\equiv i + 2^j \pmod{S} \).

**Proof.** Suppose that \( i \equiv i + 2^j \pmod{S} \). Then \( i + 2^j = i + kS \implies 2^j = kS \) for some integer \( k \). But this is impossible, since \( 2^j \) only contains the number 2 in its (unique) prime factorization, while \( kS \) contains at least one odd prime, due to \( S \) being odd. ▶

Using the above, we are ready to describe all \( G^{(t)} \) in the process.

**Lemma 14.** Let \( S \geq 3 \) be an odd integer, and the process have parameters \( \alpha = \beta = 2 \) and \( G^{(0)} = G_S \). Then any node \((x, y)\) in \( G^{(t)} \) has the following 4 neighbors:

- \((x \pm 2^{t+1} \pmod{S}, y)\) and \((x, y \pm 2^t \pmod{S})\), if \( t = 2l \) is even.
- \((x \pm 2^t \pmod{S}, y \pm 2^t \pmod{S})\), if \( t = 2l + 1 \) is odd.

**Proof.** We use induction. The lemma holds for \( t = 0 \) due to the definition of \( G^{(0)} \). Suppose it holds for \( t \), we show that it also holds for \( t + 1 \). If \( t \) is even, \( t = 2l \), then due to our inductive hypothesis, the neighbors of \((x, y)\) are \((x \pm 2^l \pmod{S}, y)\) and \((x, y \pm 2^l \pmod{S})\).

These 4 nodes are all distinct with each other and distinct from \((x, y)\). We show this for just one pair, namely \((x + 2^l \pmod{S}, y)\) and \((x - 2^l \pmod{S}, y)\), as all others follow the same reasoning. Suppose they coincided; then \( x + 2^l \equiv x - 2^l \pmod{S} \implies x + 2^{l+1} \equiv x \pmod{S} \), which is not allowed by Lemma 13.

To find nodes sharing common neighbors with \((x, y)\), it suffices to check at neighbors of \((x, y)\)’s neighbors. Each of these 4 nodes only has 4 neighbors. There are 16 such nodes, but since \((x, y)\) obviously appears 4 times, only 12 nodes are of interest. We see that the nodes \((x \pm 2^l \pmod{S}, y \pm 2^l \pmod{S})\) appear in the neighborhood of \( \beta = 2 \) of \((x, y)\)’s neighbors (and thus form an edge with \((x, y)\) at time \( t + 1 \), since they share \( \beta \) common neighbors with it).

The nodes \((x \pm 2^{l+1} \pmod{S}, y)\) and \((x, y \pm 2^{l+1} \pmod{S})\) appear in the neighborhood of only 1 of \((x, y)\)’s neighbors, and thus do not form an edge with \((x, y)\) at time \( t + 1 \). To complete the proof, we use the technique of the previous paragraph to show that all aforementioned nodes are distinct.

The case where \( t = 2l + 1 \) is completely analogous. ▶

Now that we have a description of all \( G^{(t)} \), it is easy to see that the process doesn’t converge.

**Lemma 15.** Let \( S \geq 3 \) be an odd integer, and the process have parameters \( \alpha = \beta = 2 \) and \( G^{(0)} = G_S \). Then the process doesn’t converge, and the cycle size is \( 2k \), where \( k > 0 \) is the smallest integer such that \( 2^k \equiv \pm 1 \pmod{S} \).

**Proof.** First of all we prove that such a number \( k \) exists. Due to the pigeonhole principle, there is some pair of integers \( i, j \), where \( 0 \leq i < j \leq S \) such that \( 2^i \equiv 2^j \pmod{S} \), which means that \( 2^i \equiv 2^i \times 2^{-i} \pmod{S} \). It follows that \( 2^i(1 - 2^{-i}) = z_1S \) for some integer \( z_1 \), and since \( 2^i \) and \( S \) do not share any common prime factors (due to \( S \) being odd), then \( z_1 = 2^i z_2 \) for some integer \( z_2 \). Thus \( 1 - 2^{-i} = z_2N \implies 2^{-i} \equiv 1 \pmod{S} \), which proves our point.

We notice that \( G^{(0)} \) is different from every \( G^{(t)} \) for odd \( t = 2l + 1 \). To show this, we note that \((0, 0)\) and \((0, 1)\) are neighbors at \( G^{(0)} \), but all neighbors of \((0, 0)\) at \( G^{(t)} \) are distinct from \((0, 1)\). That’s because if any of \((\pm 2^t \pmod{S}, \pm 2^t \pmod{S})\) coincides with \((0, 1)\), then it holds that \( 2^t \equiv 0 \pmod{S} \), which is not allowed by Lemma 13.
It is also straightforward to verify that if $2^t \equiv \pm 1 \pmod S$, then $G^{(0)} = G^{(2t)}$, using Lemma 14. On the other hand, if $2^t \not\equiv \pm 1 \pmod S$, then $G^{(0)} \neq G^{(2t)}$, since the edge connecting $(0,0)$ and $(0,1)$ in $G^{(0)}$ doesn’t correspond to any edge in $G^{(2t)}$. The latter follows from the fact that $(0,0)$, in $G^{(2t)}$, is connected to $(\pm 2^t \pmod S, 0)$, which both differ from $(0,1)$, and to $(0,\pm 2^t \pmod S)$, which also differ because we assumed $2^t \not\equiv \pm 1 \pmod S$. ◀

The above discussion naturally leads us to our main theorem concerning the convergence of this process.

**Theorem 16.** Let $c$ be any natural number. When $E(u,v) = |N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)|$ and the rule is only applied on pairs of nodes whose distance is at most $c$, it is feasible to find a value $S(c)$ such that the process with parameters $\alpha = \beta = 2$ and $G^{(0)} = G_{S(c)}$ has a cycle size of at least $c$.

**Proof.** Picking $S(c) = 2^\lceil\frac{c}{2}\rceil + 1$, we have that the process has a cycle size of $2k$, where $k > 0$ is the smallest integer such that $2^k \equiv \pm 1 \pmod S(c)$, due to Lemma 15. For $k = \lceil\frac{c}{2}\rceil$ we have that $2^k \equiv -1 \pmod S(c)$. All values $t$, where $0 < t < k$ have $1 < 2^t < S(c) - 1$, and thus $2^t \not\equiv \pm 1 \pmod S(c)$. The cycle size is therefore $2\lceil\frac{c}{2}\rceil \geq c$. ◀

### 6.1 Dependence of Convergence on $C^{(t)}$

In Lemma 14, we assumed that $C^{(t)} = V^2$ to prove that the terminating condition is $G^{(t' - 1)} = G^{(t')}$ and convergence is guaranteed while other choices of $C^{(t)}$ for the same initial graph lead to infinite loops. To demonstrate this, we use the machinery developed to prove Theorem 14. Consider the two following instances of the problem. Both instances have $E^{(t)}(u,v) = |N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)|$ if $d_{G^{(t)}}(u) = d_{G^{(t)}}(v) = 4$, and $E^{(t)}(u,v) = 0$ otherwise, $G^{(0)} = K_9$ (clique with 9 nodes) and $\alpha = \beta = 2$. On the first instance, it holds that $\forall t : C^{(t)} = V^2$. On the second one, it holds that $C^{(0)}$ contains the edges of $G_9$ (which is defined in Definition 12), and for $t > 0 : C^{(t)} = V^2$.

Since $d_{G^{(0)}}(u) = d_{G^{(0)}}(v) = 8 \not= 4$, the energy of all pairs of nodes is zero. Thus, on the first instance we get $G^{(1)} = K_9$ (the null graph with 9 nodes). But since all nodes are isolated on $G^{(1)}$, we get that $d_{G^{(1)}}(u) = d_{G^{(1)}}(v) = 0 \not= 4$ for all pairs of nodes $(u,v)$, which implies convergence, due to the fact that $G^{(1)} = G^{(2)} = K_9$. On the second instance, the same reasoning for $G^{(0)}$ gives $G^{(1)} = G_9$, due to $C^{(0)}$, which preserves edges. But then it is trivial to use Lemma 15 to prove that this process doesn’t converge.

### 6.2 Disproving a Convergence Conjecture

Zhang et al. defined in [22], the energy of an edge to be $E(u,v) = |N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)| + |E^{(t)}(u,v)| + |E(G[N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)])|$. This is an extension of our model described above, where we also add the number of edges between common neighbors of $u$ and $v$, denoted by $|E(G[N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)])|$, and the binary term $|E^{(t)}(u,v)|$, which is the number of edges between $u$ and $v$. The rule is only applied on pairs of nodes whose distance is at most 2.

Zhang et al. proposed the above process as an enhancement of the contrast between communities so that a community detection algorithm can discover them more easily. They conjectured that this process always converges. However, we disprove their conjecture by providing a counterexample. As in Section 4, we provide examples where consecutive graphs do not share any edges at all; thus, the above holds even if we stop when consecutive graphs are "close enough", and not necessarily the same.
Our proof is heavily based on the counterexamples given above. We first prove that these counterexamples do not contain any triangles for certain values of $S$.

**Lemma 17.** Let $S \geq 5$ be any odd integer not divisible by 3, $\mathcal{E}(u, v) = |N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)|$, $\alpha = \beta = 2$ and $G^{(0)} = G_S$. Then $G^{(t)}$ doesn’t contain any triangle, for any $t$.

**Proof.** For simplicity, suppose that $t$ is even, $t = 2l$, as the other case is analogous. Suppose that a triangle exists; then, due to the symmetry of $G^{(t)}$ we can assume that 2 of the 3 nodes are the neighboring nodes $(i, j)$ and $(i, j + 2^l \mod S)$ (Lemma 14). Since the third node is a neighbor of $(i, j + 2^l \mod S)$, then it is either one of $(i \pm 2^l \mod S, j + 2^l \mod S)$ or $(i, j + 2^{l+1} \mod S)$. But if $(i \pm 2^l \mod S, j + 2^l \mod S)$ were neighbors with $(i, j)$, then either $i \equiv i + 2^l \mod S$ or $j = j + 2^l \mod S$, which doesn’t hold, due to Lemma 13.

Thus, the third node must be $(i, j + 2^{l+1} \mod S)$. For $(i, j)$ to be neighbors with $(i, j + 2^{l+1} \mod S)$, it holds that $2^{l+1} \equiv -2^l \mod S \implies 3 \times 2^l = z_1S$, where $z_1$ is integer. However, $S$ is odd and not divisible by 3. Thus $z_1 = 3 \times 2^l \times z_2$ for some integer $z_2$, which means $z_2S = 1 \implies S = 1$. But this is a contradiction, as $S \geq 5$.

We are now ready to prove that for certain values of $S$, the counterexamples of Section 6 produce the same sequence of graphs for both processes, and thus the current process doesn’t always converge.

**Lemma 18.** Let $S \geq 5$ be any odd integer not divisible by 3. Then, when $\alpha = \beta = 2$ and $G^{(0)} = G_S$, both the process with $\mathcal{E}(u, v) = |N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)|$ and the process with $\mathcal{E}(u, v) = |N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)| + |\mathcal{E}(t)(u, v)| + |E(G[N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)])|$ produce the same sequence of graphs.

**Proof.** Let $G^{(t)}$ be the graphs of the process with $\mathcal{E}(u, v) = |N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)|$ and $G^{(t)}$ be the graphs of the other process. It trivially holds that $G^{(t)} = G^{(0)}$. Suppose that it holds that $G^{(t)} = G^{(t)}$. We prove that $G^{(t+1)} = G^{(t+1)}$. First of all, the number of edges between common neighbors at time $t$ is the same, since both graphs are equal. That is $|E(G[N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)])| = |E(G[N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)])|$, which, due to Lemma 17 is equal to 0. If there is no edge between $u$ and $v$ in $G^{(t)}$, the energy of $(u, v)$ is the same in both processes, while, if an edge exists, the energy in $G^{(t)}$ is equal to the energy in $G^{(t)}$ plus 1, due to the term $|\mathcal{E}(t)(u, v)|$. But, even though it is incremented by 1, it still holds that $\mathcal{E}(t)(u, v) < \alpha = 2$. That is because $(u, v)$ do not have any common neighbor, for, if they had, these three nodes would form a triangle at time $t$ and this would contradict Lemma 17.

Thus, $G^{(t+1)} = G^{(t+1)}$.

We are now ready to prove our main theorem for this process.

**Theorem 19.** Let $c > 4$ be any natural number. When $\mathcal{E}(u, v) = |N_{G^{(0)}}(u) \cap N_{G^{(0)}}(v)| + |\mathcal{E}(t)(u, v)| + |E(G[N_{G^{(t)}}(u) \cap N_{G^{(t)}}(v)])|$ and the rule is only applied on pairs of nodes whose distance is at most 2, it is feasible to find a value $S(c)$ such that the process with parameters $\alpha = \beta = 2$ and $G^{(0)} = G_S(c)$ has a cycle size of at least $c$.

**Proof.** Due to Lemma 18 it suffices to prove this for the process with energy definition $\mathcal{E}(u, v) = |N_{G^{(0)}}(u) \cap N_{G^{(0)}}(v)|$, as long as $S$ is odd and not divisible by 3. If $2^{[\frac{c}{2}]} + 1$ is not divisible by 3, then our theorem holds for $S(c) = 2^{[\frac{c}{2}]} + 1$, as in Theorem 16. Else, $2^{[\frac{c}{2}]} + 1 = 3z_1$ for some integer $z_1$. We set $S(c) = 2^{[\frac{c}{2}]} + 1 = 3z_1 - 2$, which is not divisible by 3. We know that the process has a cycle size of 2$k$, where $k > 0$ is the smallest integer such that $2^k \equiv 1 \mod S(c)$, due to Lemma 15. For $k = [\frac{c}{2}]$ we have that $2^k \equiv 1 \mod S(c)$. All values $t$, where $0 < t < k - 1$ have $1 < 2^t \leq \frac{c}{2} = \frac{2c + 1}{2} < S(c) - 1$, and thus it holds that $2^t \neq \pm 1 \mod S(c)$. The cycle size is $2[\frac{c}{2}] \geq c$. 


7 Turing-Completeness

In this section we describe a local rule under which our Network System is able to simulate Rule 110, an one-dimensional cellular automaton that Cook proved to be Turing-Complete [8]. Thus, we prove that there exist local rules for which our Network System is Turing-Complete.

Definition 20. Rule 110 is an one-dimensional cellular automaton. Let cell\(t(i)\) be the binary value of the \(i\)-th cell at time \(t\). If cell\(t(i) = 0\), then cell\(t(i+1) = cell^{t(i)(i+1)} = 1\). Else, cell\(t(i+1) = 0\) if cell\(t(i-1) = cell^{t(i)(i+1)} = 1\), and 1 otherwise.

Let \(CN(t(u,v) = |N_{G^{(i)}}(u) \cap N_{G^{(i)}}(v)|\) be the number of common neighbors of \(u\) and \(v\) at time \(t\), and \(CE(t(u,v) = |E(G|CN(t)|)|\) be the number of edges between the common neighbors of \(u\) and \(v\) at time \(t\). We pick an arbitrary value for \(\beta\) and then set \(\alpha = \beta\). The energy between \(u\) and \(v\) is defined as follows:

\[
E(t(u,v) = \begin{cases} 
CE(t(u,v) + \beta - 10 & \text{if } CN(t(u,v) = 10 \text{ and } |E(t(u,v)| = 0 \\
\beta + 12 - CE(t(u,v) & \text{if } CN(t(u,v) = 10 \text{ and } |E(t(u,v)| = 1 \\
CE(t(u,v) + \beta - 6 & \text{if } CN(t(u,v) = 6 \\
\beta - 1 & \text{otherwise}
\end{cases}
\]

Informally, our simulation of Rule 110 follows these steps. First, we design a primitive cell-gadget (henceforth PCG) that stores binary values, but fails to capture Rule 110 since it doesn’t distinguish between the left and the right cell. Then, by making use of the PCG as a building block, we build the main cell-gadget (henceforth CG) that is used to simulate a single cell of the cellular automaton. Finally, each time-step from rule 110 is simulated using 2 time-steps of our process; on the first one, some PCGs acquire their proper value. On the second step, the rest of the PCGs copy the correct value from the ones that already acquired it. Our construction, along with the piecewise energy function allow us to make these two sets of PCGs behave differently (some PCGs compute the correct value, while the others copy). For clarity purposes, we slightly abuse notation, and consider the time steps of our process to differ by 0.5 instead of 1. Thus, we write that the sequence of graphs is \(G(0), G(0.5), G(1), \ldots\), where graphs \(G(t+0.5),\) for \(t \in \mathbb{N}\), are transitional states of the graph and have no correspondence with cell states of the cellular automaton.

More formally, a PCG is a pair of nodes \((h_i,l_i)\), such that the existence of an edge between them corresponds to value 1 and otherwise it corresponds to value 0. This PCG is connected to another PCG \((h_{i+1},l_{i+1})\) by adding all possible edges between these nodes as shown in Figure 1. In this way, \(CE(t(h_i,l_i)\) would be the sum of values of the two adjacents cell gadgets.

The \(i\)-th CG that corresponds to the \(i\)-th cell (we write CG\((i)\)) consists of 4 PCGs, which we identify as \(A_1(i), A_2(i), B_1(i)\) and \(B_2(i)\). We connect each \(A_j(i)\) with each \(B_k(i)\) (4 connections in total, where each connection uses 4 edges, as depicted in Figure 1). In order to connect CG\((i)\) (cell \(i\)) with CG\((i+1)\) (cell \(i+1\)) we connect \(A_j(i)\) with \(A_j(i+1)\), and \(A_j(i)\) with \(B_j(i+1)\), as shown in Figure 2. A CG is said to have value 0 if all 4 of its PCGs are set to 0 and 1 if all PCGs are set to 1. We guarantee that no other case can occur in \(G(t), t \in \mathbb{N}\), although certain cases can occur in the intermediate graphs \(G(t+0.5), t \in \mathbb{N}\).

Each cell from Rule 110 is represented by a CG, and they are connected by the aforementioned method. At time \(t\), where \(t \geq 0\) is an integer, we have that \(C^{t}\) contains all pairs of nodes both belonging in the same \(A_j(i)\). In other words, only the edges that define the value of an \(A_j(i)\) are allowed to change from time \(t\) to time \(t+0.5\). Similarly, at time \(t+0.5\)
we have that $C^{(t+0.5)}$ contains all pairs of nodes both belonging in the same $B_j(i)$, for any valid $i,j$.

We notice that due to the definition of $C^{(t)}$ for any $t$, only edges inside PCGs may be allowed to change, meaning that all connections between PCGs will remain as is forever. In addition, the number of common neighbors of the pair of nodes $A_j(i)$ is always $CN^{(t)}(A_j(i)) = 10$, for all valid $t,i,j$, as it has 5 neighboring PCGs, and each PCG has two nodes. Furthermore, it holds that $CE^{(t)}(A_j(i)) = 8 + A_j^{(t)}(i-1) + B_1^{(t)}(i) + B_2^{(t)}(i) + A_j^{(t)}(i+1) + B_j^{(t)}(i+1)$, as the edges between common neighbors are the internal edges of neighboring PCGs, plus the connection between $A_j^{(t)}(i-1)$ and $B_j^{(t)}(i)$ (4 edges), plus the connection between $A_j^{(t)}(i+1)$ and $B_j^{(t)}(i+1)$ (4 edges). Similarly, for a $B_j(i)$ we have that $CN^{(t)}(B_j(i)) = 6$, and $CE^{(t)}(B_j(i)) = 4 + A_j^{(t)}(i-1) + A_j^{(t)}(i) + A_j^{(t)}(i+1)$.

Lemma 21. It holds that $A_j^{(t)}(i) = B_j^{(t)}(i) = cell^{(t)}(i)$ for $j \in \{1,2\}$ and all $i,t \in \mathbb{N}$.

Proof. It holds that $A_j^{(0)}(i) = B_j^{(0)}(i) = cell^{(0)}(i)$ by the initialization of our construction. Suppose that $A_j^{(t)}(i) = B_j^{(t)}(i) = cell^{(t)}(i)$ for integer $t \geq 0$. By using induction we show that the lemma holds for time $t+1$.

First of all, we prove that $A_j^{(t+0.5)}(i) = cell^{(t+1)}(i)$. If $cell^{(t)}(i) = 0$, then it holds that $cell^{(t+1)}(i) = cell^{(t)}(i+1) = A_j^{(t)}(i+1) = B_j^{(t)}(i+1)$, due to our inductive hypothesis. Furthermore, due to our inductive hypothesis it holds that $A_j^{(t)}(i) = B_1^{(t)}(i) = B_2^{(t)}(i) = 0$. Thus, since $CN^{(t)}(A_j(i)) = 10$ and $|E^{(t)}(A_j(i))| = 0$ (there is no edge between the two nodes in $A_j(i)$) the energy between the pair of nodes is $E^{(t)}(A_j(i)) = CE^{(t)}(A_j(i)) + \beta - 10$. To find the energy of the pair of nodes $A_j(i)$ we compute:

$$CE^{(t)}(A_j(i)) = 8 + A_j^{(t)}(i-1) + B_1^{(t)}(i) + B_2^{(t)}(i) + A_j^{(t)}(i+1) + B_j^{(t)}(i+1) =$$

$$8 + cell^{(t)}(i-1) + 2cell^{(t)}(i+1)$$

Thus, it follows that the energy of $A_j(i)$ is $E^{(t)}(A_j(i)) = \beta + cell^{(t)}(i-1) + 2cell^{(t)}(i+1) - 2$, which is at least $\beta$ if and only if $cell^{(t)}(i+1) = 1$. Thus, in the case where $cell^{(t)}(i) = 0$ we proved that indeed it holds that $A_j^{(t+0.5)}(i) = cell^{(t+1)}(i)$.

We use a similar reasoning for the case where $cell^{(t)}(i) = 1$. In particular, since $CN^{(t)}(A_j(i)) = 10$ and $|E^{(t)}(A_j(i))| = 1$ (there is an edge between the two nodes in $A_j(i)$) the energy between the pair of nodes is $E^{(t)}(A_j(i)) = \beta + 12 - CE^{(t)}(A_j(i))$. We compute:

$$CE^{(t)}(A_j(i)) = 8 + A_j^{(t)}(i-1) + B_1^{(t)}(i) + B_2^{(t)}(i) + A_j^{(t)}(i+1) + B_j^{(t)}(i+1) =$$
Figure 2 Each circle represents a PCG (2 nodes) and each line represents a connection between PCGs (4 edges) as in Figure 1. Only connections relevant to $A_1(i)$, $A_2(i)$, $B_1(i)$, and $B_2(i)$ are shown. The 4 dotted connections in the second column are internal connections of CG(i). All other continuous connections correspond to how CG(i−1) is connected with CG(i) and CG(i) is connected with CG(i+1). None of the edges of these connections is ever included in any C(t), and thus they are always preserved.

$$= 10 + \text{cell}^{(t)}(i-1) + 2\text{cell}^{(t)}(i+1)$$

Thus, it follows that the energy of $A_j(i)$ is $E^{(t)}(A_j(i)) = \beta + 2 - \text{cell}^{(t)}(i-1) - 2\text{cell}^{(t)}(i+1)$, which is less than $\beta$ if and only if $\text{cell}^{(t)}(i-1) = \text{cell}^{(t)}(i+1) = 1$. This proves that $A_j^{(t+0.5)}(i) = \text{cell}^{(t+1)}(i)$.

Furthermore, it trivially holds that $A_j^{(t+1)}(i) = \text{cell}^{(t+1)}(i)$. That’s because, by definition, $A_j(i) \notin C^{(t+0.5)}$, and thus $A_j^{(t+0.5)}(i) = A_j^{(t+1)}(i)$. The energy of $B_j(i)$ at time $t + 0.5$ is (recall that $CN^{(t)}(B_j(i)) = 6$):

$$E^{(t+0.5)}(B_j(i)) = CE^{(t+0.5)}(B_j(i)) + \beta - 6 = \beta + 2A_j^{(t+0.5)}(i) + A_j^{(t+0.5)}(i-1) - 2$$

This is at least $\beta$ if and only if $A_j^{(t+0.5)}(i) = 1$, which proves that $B_j^{(t+1)}(i) = \text{cell}^{(t+1)}(i)$.

The following corollary is a straightforward consequence of this lemma.

\textbf{Corollary 22.} It holds that $\text{cell}^{(t)}(i) = CG^{(t)}(i)$.

We are now ready to prove our main theorem.

\textbf{Theorem 23.} The Network System we are studying is Turing-Complete.

\textbf{Proof.} By Corollary 22 it follows that Rule 110 is simulated by the particular network system constructed above. If Rule 110 converges at step $t$ (meaning that no cell changes state for
\( t' > t \), then our simulation terminates also at time \( t + 1 \) since no change will have taken place in the graph from time \( t \) to time \( t + 1 \). Since Rule 110 is Turing-Complete it follows that the particular Network System is also Turing-Complete.

As a final note, there are local rules that make the Network System Turing-Complete even if we are not allowed to use \( C(t) \), that is if all pairs of nodes are always allowed to create an edge (\( \forall t : C(t) = V^2 \)). The construction uses properties of cliques to create gadgets that work like always-on edges, as well as gadgets that work like always-off edges. Using the always-on gadgets, we can create new gadgets that work like edges that always flip their status. Of course the energy definition should change accordingly. These are enough to replace \( C(t) \) in the above simulation. Although feasible, the construction is too technical and we decided not to include it.

8 Conclusions

In this paper, we try to pinpoint properties of network systems that, to our opinion, can provide a framework for better understanding the emergent behavior of a complex system based on local interactions. We proved convergence depending on the type of rules and also provide some hints as to the power of this model (the computation of the \( \alpha \)-core of a graph). Furthermore, in the extended version [14] we prove that there exist local rules under which our network system is Turing-Complete. In fact, it seems that as soon as local rules look at the neighbor of a node convergence is not guaranteed and depends on the parameters as well as on the input graph. Notice, that we do not envision the network system as an alternative to a Turing machine but as a framework to study emergent behavior in a mesoscopic scale where other approaches like statistical physics (macroscopic scale) and dynamical systems theory (microscopic scale) seems not to be able to reach easily meaningful results.

In the future, we will try to capture conditions related to the input graph, values \( \alpha \) and \( \beta \) as well as to the definition of the energy under which convergence is guaranteed. In addition, we are really interested in finding a set of simple local rules such that when combined, we can make meaningful local programs which could be analyzed with respect to properties like convergence as well as with respect to the emergent behavior of the network. This is no small task and may have a great impact to various scientific fields. Finally, note that the backbone of the provided proofs is induction, which is closely related to algorithms. Maybe an algorithmic approach to analyzing such local rules would have success since it would be more natural to reason about - for example the "there exists some time \( t \) where a property holds" would become the assumption of an if statement and the result would more naturally tumble out.

References

1 Dana Angluin, James Aspnes, David Eisenstat, and Eric Ruppert. The computational power of population protocols. Distributed Computing, 20(4):279–304, 2007.
2 Pablo Arrighi and Gilles Dowek. Free fall and cellular automata. In DCM, volume 204 of EPTCS, pages 1–10, 2015.
3 Vladimir Batagelj, Andrej Mrvar, and Matjaz Zaversnik. Partitioning approach to visualization of large graphs. In Graph Drawing, volume 1731 of Lecture Notes in Computer Science, pages 90–97. Springer, 1999.
4 Ruben Becker, Vincenzo Bonifaci, Andreas Karrenbauer, Pavel Kolev, and Kurt Mehlhorn. Two results on slime mold computations. CoRR, abs/1707.06631, 2017.
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5 Ioannis Chatzigiannakis, Othon Michail, Stavros Nikolaou, Andreas Pavlogiannis, and Paul G. Spirakis. Passively mobile communicating machines that use restricted space. *Theor. Comput. Sci.*, 412(46):6469–6483, 2011.

6 Bernard Chazelle. Natural algorithms and influence systems. *Commun. ACM*, 55(12):101–110, December 2012. URL: [http://doi.acm.org/10.1145/2380656.2380679](http://doi.acm.org/10.1145/2380656.2380679)

7 Bernard Chazelle. The convergence of bird flocking. *J. ACM*, 61(4):21:1–21:35, 2014.

8 Matthew Cook. Universality in elementary cellular automata. *Complex Systems*, 15(1), 2004.

9 David Doty and Shaopeng Zhu. Computational complexity of atomic chemical reaction networks. *Natural Computing*, 17(4):677–691, 2018.

10 Javier Esparza, Pierre Ganty, Jérôme Leroux, and Rupak Majumdar. Verification of population protocols. *Acta Inf.*, 54(2):191–215, 2017.

11 Bernd Gärtner and Ahad N. Zehmakan. (biased) majority rule cellular automata. *CoRR*, abs/1711.10920, 2017.

12 Nicolas Gastineau, Wahabou Abdou, Nader Mbarek, and Olivier Togni. Distributed leader election and computation of local identifiers for programmable matter. *CoRR*, abs/1807.10461, 2018.

13 Andreas Karrenbauer, Pavel Kolev, and Kurt Mehlhorn. Convergence of the non-uniform physarum dynamics. *CoRR*, abs/1901.07231, 2019.

14 Evangelos Kipouridis and Kostas Tsichlas. On the convergence of network systems. *CoRR*, abs/1902.04121, 2019.

15 Othon Michail, George Skretas, and Paul G. Spirakis. On the transformation capability of feasible mechanisms for programmable matter. In *ICALP*, volume 80 of *LIPIcs*, pages 136:1–136:15. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017.

16 Othon Michail and Paul G. Spirakis. Elements of the theory of dynamic networks. *Commun. ACM*, 61(2):72, 2018.

17 Toshiyuki Nakagaki, Hiroyasu Yamada, and Ágota Tóth. Maze-solving by an amoeboid organism. *Nature*, 407:470, 09 2000. doi:10.1038/35035159

18 H. Silk, M. Homer, and T. Gross. Design of self-organizing networks: Creating specified degree distributions. *IEEE Transactions on Network Science and Engineering*, 3(3):147–158, 07 2016. doi:10.1109/TNSE.2016.2586762

19 Damian Straszak and Nisheeth K. Vishnoi. On a natural dynamics for linear programming. In *ITCS*, page 291. ACM, 2016.

20 G. Valentini, H. Hamann, and M. Dorigo. Global-to-local design for self-organized task allocation in swarms. Technical Report TR/IRIDIA/2016-002, IRIDIA, Université Libre de Bruxelles, Brussels, Belgium, 03 2016.

21 Stephen Wolfram. *A New Kind of Science*. Wolfram Media Inc., Champaign, Illinois, US, United States, 2002.

22 Yuzhou Zhang, Jianyong Wang, Yi Wang, and Lizhu Zhou. Parallel community detection on large networks with propinquity dynamics. In *KDD*, pages 997–1006. ACM, 2009.