Reduced multibody system transfer matrix method using decoupled hinge equations

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Abstract
In the multibody system transfer matrix method (MSTMM), the transfer matrix of body elements may be directly obtained from kinematic and kinetic equations. However, regarding the transfer matrices of hinge elements, typically information of their outboard body is involved complicating modeling and even resulting in combinatorial problems w.r.t. various types of outboard body’s output links. This problem may be resolved by formulating decoupled hinge equations and introducing the Riccati transformation in the new version of MSTMM called the reduced multibody system transfer matrix method in this paper. Systematic procedures for chain, tree, closed-loop, and arbitrary general systems are defined, respectively, to generate the overall system equations satisfying the boundary conditions of the system during the entire computational process. As a result, accumulation errors are avoided and computational stability is guaranteed even for huge systems with long chains as demonstrated by examples and comparison with commercial software automatic dynamic analysis of the mechanical system.

KEYWORDS
general system topology, multibody system dynamics, reduced transformation, transfer matrix method

1 | INTRODUCTION

Multibody system simulation has been developed and gradually become one of the most important foundations of dynamic system design.1–3 Especially in China, the multibody system transfer matrix method4 (MSTMM) as a rather new modeling method gained great attention and has been applied to many engineering problems. The principle of MSTMM is modular modeling of every element by utilizing linear relationships among kinematic and kinetic quantities acting on it. Any general multibody system can be broken up into two kinds of elements, bodies and hinges, whose dynamics properties can be easily expressed in a transfer matrix form. Both share a parallel status, which avoids the difficulty caused by a special hinge treatment as a constraint as required in ordinary multibody system dynamics. The transfer matrix of each element is regarded as a building block, and a generic library of element transfer matrices5 can be developed in advance, which then can be assembled according to a specific system topology.6

The transfer matrix method was originally developed for linear systems where the relations between modal position coordinates and forces automatically fulfill a linear transfer equation. For application to the simulation of nonlinear multibody systems, the first version of
MSTMM had to linearize the kinematic and kinetic equations and introduce a specific numerical integration scheme to enforce linear transfer equations on position level, which may yield large errors since position kinematics of multibody systems is inherently nonlinear. The new version of MSTMM (NV-MSTMM) avoids this by introducing alternative state vectors on acceleration level and utilizing the strictly linear relationships already existing between accelerations and forces, whereas the nonlinear simulation part on position and velocity level is shifted to the integration scheme. However, there still remain two problems to be resolved in this paper. In principle, the overall transfer matrix of a system is obtained by multiplying its element transfer matrices. For a large system, the accumulative error resulting from successive multiplication of a large number of transfer matrices in a long chain may lead to computational stability problems, which may be resolved by drawing lessons from the Riccati transfer matrix method characterized by the Riccati transformation. A second problem is that hinge transfer equations depend on the information of its outboard body and even the next hinge which hinders its applicability. This can be avoided by a specific description of hinges in the context of Riccati transformation.

In this paper, these two ideas are generalized by defining various reduced transformations to enhance the element library for modeling any type of system, such as chains, trees, closed-loops, and any general system with the features of being time-variant, nonlinear, and involving large motion, Figure 1. After deducing basic element equations in Section 2, systematic procedures for setting up the overall system equations will be discussed in Sections 3–6. Numerical examples in Section 7 will validate the proposed method called the reduced multibody system transfer matrix method (RMSTMM). It should be pointed out that the paper focuses on deriving acceleration equations in a recursive manner similar to classical approaches like Brandl, but requiring only a single sweep. For further numerical integration, well-known schemes are used and not discussed in the paper.

### 2 | BASIC ELEMENT EQUATIONS

An inertial Cartesian coordinate system (Sxyz) is used to describe the motion of elements by position \( \mathbf{r} \) of any connecting point and rotation tensor \( \mathbf{A} \) of body-fixed coordinate systems \( (\mathbf{l},\mathbf{r}) \) and \( (\mathbf{O},\mathbf{r}) \), respectively. The interacting internal forces \( \mathbf{q} \) and torques \( \mathbf{m} \) in Figure 2A are defined such that they fulfill Newton’s third law. Connection of two adjacent elements \( i \) and \( j \) according to \( \mathbf{O}_i \equiv \mathbf{i} \) then yields the identities

\[
\mathbf{r}_O^0 = \mathbf{r}_i^0, \quad \mathbf{A}_O^0 = \mathbf{A}_i^0, \quad \mathbf{q}_O^0 = \mathbf{q}_i^0, \quad \mathbf{m}_O^0 = \mathbf{m}_i^0
\]

between output end \( O_i \) and input end \( i_j \), which are later essential for recursively sweeping through the system and connecting the elements.

In the inertial frame (Sxyz), the position, velocity, and acceleration of an arbitrary point \( P \) on a body in Figure 2B can be denoted as follows:

\[
\mathbf{r}_P = \mathbf{r}_i + \mathbf{r}_{iP} = \mathbf{r}_i + \mathbf{A}_i \mathbf{r}_{iP} \quad \text{→} \quad \dot{\mathbf{r}}_P = \dot{\mathbf{r}}_i + \mathbf{A}_i \dot{\mathbf{r}}_{iP} = \dot{\mathbf{h}} + \mathbf{A}_i \mathbf{r}_{iP} \\
\mathbf{\omega}_P = \mathbf{\omega}_i \quad \text{→} \quad \ddot{\mathbf{r}}_P = \ddot{\mathbf{h}} + \omega_i \mathbf{r}_{iP}
\]

where for rigid bodies \( \mathbf{r}_{iP} = \mathbf{A}_i \mathbf{r}_{iP} \) described in the body-fixed frame is time-independent and \( \mathbf{i} = \mathbf{A}_i \mathbf{A}_i^T = \mathbf{A}_i \mathbf{A}_i^T \) is the skew-symmetric matrix of angular velocity vector \( \mathbf{\omega}_i \). By choosing \( P := O \), we obtain the kinematic input–output relation

\[
\mathbf{r}_O = \mathbf{h} - \mathbf{r}_O \omega \mathbf{O} + \mathbf{\omega}_O \omega \mathbf{O}.
\]

Further, rotation is identical for every point on a rigid body, especially

\[
\mathbf{\omega}_O = \mathbf{\omega}_i \quad \text{→} \quad \mathbf{\omega}_O \equiv \mathbf{\omega}_i.
\]

Newton’s second law \( \mathbf{m}_C = \mathbf{F} \) relates the mass \( m \) of a rigid body, absolute acceleration \( \mathbf{a}_C = \mathbf{\ddot{r}}_O \) of its mass center and resultant force \( \mathbf{F} \). With \( P = C \) in Equation (2) we get from Figure 2B.

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**FIGURE 1** Dynamics models of chain (A), tree (B), closed-loop (C), and general system (D)
\[ m_\dot{\omega} - m_\ddot{\omega} \omega + \omega \ddot{\omega} + \omega \dot{\omega} = q_0 + \sum_{k=2}^{N} q_k + f_c \]  

(5)

or, after reordering, the kinetic input–output relation

\[ q_0 = m_\dot{\omega} - m_\ddot{\omega} \omega + f_c - \omega \dot{\omega} + \sum_{k=2}^{N} q_k. \]  

(6)

By applying Euler's theorem of momentum \( dG/dt + \ddot{\omega}m_\eta = M_I \) to Figure 2B w.r.t. input end \( \eta \) we obtain

\[ \frac{dG}{dt} + m_\ddot{\eta} \dot{\omega} = m_\dot{\eta} q_0 + m_c \ddot{\eta} + m_\ddot{\omega} \omega + \sum_{k=2}^{N} m_k \]  

(7)

The moment of momentum is given as follows:

\[ G_I = A_I G'_I = A_I^T \dot{\omega} = A_I^T A_I^T \omega. \]  

(8)

Based on time-invariant inertia tensor \( A_I \) w.r.t. to body-fixed frame \([I \eta \zeta]\). With \( \dot{A}_I = \ddot{G}_I \) and \( \dot{A}_I^T = A_I^T G_I \), time differentiating w.r.t. the inertial frame yields

\[ \frac{dG}{dt} = \dot{A}_I^T A_I^T \omega + A_I^T \dot{A}_I^T \omega + A_I^T \dot{A}_I^T \omega \]  

(9)

due to \( \dot{G}_I \omega = \ddot{G}_I \omega = O_{3 \times 1} \). Substitution of Equations (6) and (9) into Equation (7) and use of \( f_{_c} \equiv f_0 - f_c \), \( n_{_c} \equiv n_0 - n_c \), \( \tau_{_c} \equiv \tau_0 - \tau_c \) yields the rotational dynamics equation as second kinetic input–output relation:

\[ m_\dot{\omega} = m_\dot{\omega} \omega + \sum_{k=2}^{N} m_k \]  

(10)

\[ f_c = [\ddot{G}_I \omega, \tau_0 - \tau_c, -f_0 + \tau_0 \omega, \tau_0 \omega]. \]  

(11)

as state vectors of the new version of MSTMM (NV-MSTMM), kinetics Equations (10) and (6) can be summarized as transfer equation

\[ z_{\eta,0} = T_{0 \eta} z_{0,0} + T_{0 \eta} f_0 + \sum_{k=2}^{N} T_{0 \eta,k} z_{0,k}. \]  

(12)

where \( z_{\eta,k} \) refers to forces and torques acting on input end \( k \) and

\[ T_{0 \eta} = \begin{bmatrix} I_3 & \tau_0 \\ O_{3 \times 3} & I_3 \end{bmatrix}, \]  

(13)

\[ T_{0 \eta,k} = \begin{bmatrix} I_3 & \tau_0 \\ O_{3 \times 3} & I_3 \end{bmatrix}, \]  

(14)

Further, kinematics Equations (3) and (4) may be summarized as follows:

\[ z_{0,0} = T_{0 \eta} z_{0,0} + f_0. \]  

(15)

For multi-input bodies, rigid body kinematics (3), (4) is also valid between the first input end \( \eta = \eta_1 \) and remaining input ends \( k \), which may be written as consistency equations

\[ z_{0,k} = H_k z_{0,1} + h_k (k = 2, 3, ..., N), \]  

(16)

where

\[ z_{0,k} = \begin{bmatrix} \ddot{\eta}_k \\ \dot{\omega}_k \\ \omega_k \end{bmatrix}, \]  

(17)
For a massless and dimensionless smooth ball-and-socket hinge in Figure 2C, we find

$$m_0 = m_i = 0, q_i = q, \ddot{r}_0 = \ddot{r}_i.$$  
(18)

According to the definitions (11), these equations may be summarized as follows:

$$z_{0,i} = E_e z_{a,i}, E_i z_{a,i} = 0, E_i E_{0,i} + E_{i,0} + E_f = 0,$$  
(19)

where

$$E_e = [I_0 O_{3x3}], E_i = [I_3 O_{3x3}], E_0 = [-I_3 O_{3x3}],$$

$$E_f = O_{3x1}.$$  
(20)

Such sets of three hinge equations may be also found for pin hinges and translational hinges, where the first describes the force/torque equilibrium, the second vanishing forces/torques in free moving directions due to smoothness, and the third motion consistency in constrained directions of the hinge. In the following, these basic body and hinge transfer relations will be utilized as building blocks for obtaining the modeling procedure for the multibody system with various topologies.

3 | RECURSIVE EQUATIONS FOR CHAIN SYSTEMS

A chain system as shown in Figure 1A with its corresponding topology graph in Figure 3A always has at least one free boundary end, which is regarded as the tip corresponding to the first element of the system. In such a case, the Riccati transformation assumes that one part of state vectors (11) can be expressed as a function of the other, for example, as follows:

$$z_a = z_{a,i} + e.$$  
(21)

Due to vanishing forces and torques at a free boundary, this is obvious for the tip body 1 reading as follows:

$$z_{1,i}^{(1)} = z_{1,i}^{(1)} + e^{(1)}$$ where $$S_1 = O_{6x6}, e^{(1)} = O_{6x1}.$$  
(22)

Applied to connection points between the elements of the chain, the reduced transformation (21)

$$z_{a,i}^{(i)} = S_{0,i}^{(i)} z_{a,i}^{(i)} + e_{i}^{(i)}, z_{a,0}^{(i)} = S_{0,i}^{(i)} z_{a,0}^{(i)} + e_{O}^{(i)}$$  
(23)

and identities $f^{(i+1)} \equiv O^{(i)}$ result in transition conditions

$$z_{a,i}^{(i+1)} = S_{0,i}^{(i+1)} z_{a,i}^{(i+1)} + e^{(i+1)} = z_{a,0}^{(i)} + e^{(i)}, z_{a,0}^{(i)} = z_{a,0}^{(i)},$$  
(24)

which allows to transfer from tip to root with identities

$$S_i^{(i+1)} \equiv S_{0,i}^{(i)}, e^{(i+1)} \equiv e_{O}^{(i)}.$$  
(25)

By sweeping through the chain along the transfer path in Figure 3A, we finally end up with $z_{n,i}^{(n)} = S_{0,i}^{(n)} z_{n,i}^{(n)} + e_{n}^{(n)}$. Since the output end of body $n$ is a boundary end with the property that always half of state variables at $z_{a,i}^{(n)}$ and $z_{a,0}^{(n)}$ are known, the equation can always be solved for the unknown boundary state variables. For a free boundary, the condition is $z_{a,0}^{(n)} = 0$ or, for example, in the case of a pinned support, translational accelerations and torques are zero. Thus, the missing link of the procedure is to deduce transfer equations for $S$ and $e$ from input to output for bodies and hinges from their basic equations.

3.1 | Body treatment

For the single-input–single-output rigid body $i$ of a chain, transfer Equations (12), (14) reduce to

$$S_i = S_{0,i} z_{i}^{(i)} + e_{i}^{(i)}.$$  
(26)

![Figure 3](image-url) Topology figures corresponding to Figure 1 for chain (A), tree (B), closed-loop (C), and general system (D) showing bodies as circles and hinges as arrows along transfer path.
The second equation yields

\[ z_{6,1}^{(l)} = p^{(l)} z_{6,0}^{(l)} + q^{(l)} \]  

(27)

with abbreviations

\[ p^{(l)} = \left( T_{0,1}^{(l)} \right)^{-1}, \quad q^{(l)} = -p^{(l)} f_{6,0}^{(l)}. \]  

(28)

Substituting the first equation of (23) and Equation (27) into the first equation of (26) yields

\[
\begin{align*}
 z_{6,0}^{(l)} &= T_{0,1}^{(l)} \left( S_{0,0}^{(l)} \left( p^{(l)} z_{6,0}^{(l)} + q^{(l)} \right) + e_{0}^{(l)} \right) + T_{0,1}^{(l)} \left( p^{(l)} z_{6,0}^{(l)} + q^{(l)} \right) + f_{6,0}^{(l)} \\
 &= \left( T_{0,1}^{(l)} S_{0,0}^{(l)} + T_{0,1}^{(l)} p^{(l)} \right) z_{6,0}^{(l)} + T_{0,1}^{(l)} e_{0}^{(l)} + f_{6,0}^{(l)} + \left( T_{0,1}^{(l)} e_{0}^{(l)} + T_{0,1}^{(l)} q^{(l)} \right) \\
 &= : S_{0,0}^{(l)} z_{6,0}^{(l)} + e_{0}^{(l)}
\end{align*}
\]

(29)

relating \( z_{6,0}^{(l)} \) and \( z_{6,1}^{(l)} \) as desired in Equation (23). This results in body transfer equations \( (S_{0,0}^{(l)}, e_{0}^{(l)}) \rightarrow (S_{0,0}^{(l)}, e_{0}^{(l)}) \) reading as follows:

\[
S_{0}^{(l)} = \left( T_{0,1}^{(l)} S_{0,0}^{(l)} + T_{0,1}^{(l)} p^{(l)} \right) p^{(l)}, \quad e_{0}^{(l)} = T_{0,1}^{(l)} e_{0}^{(l)} + f_{6,0}^{(l)} + \left( T_{0,1}^{(l)} S_{0,0}^{(l)} + T_{0,1}^{(l)} q^{(l)} \right) Q^{(l)}.
\]

(30)

### 3.2 | Hinge treatment

The reduced transfer equation of a hinge can be obtained by substituting the reduced transformation (23) into the set of basic Equation (19):

\[
\begin{align*}
 z_{6,0}^{(l)} &= E_{6}^{(l)} z_{6,0}^{(l)} + e_{6}^{(l)}, \quad E_{6}^{(l)} = \left( S_{0,0}^{(l)} e_{0}^{(l)} + e_{0}^{(l)} \right) = 0, \\
 E_{6}^{(l)} z_{6,0}^{(l)} + E_{6}^{(l)} q^{(l)} &= 0.
\end{align*}
\]

(31)

By assembling the last two equations in matrix form and solving it for \( z_{6,0}^{(l)} \), in terms of \( z_{6,0}^{(l)} \), we also obtain relation (27) where now

\[
\begin{align*}
p^{(l)} &= \begin{bmatrix} E_{6}^{(l)} \end{bmatrix}^{-1} \begin{bmatrix} E_{6}^{(l)} \\ E_{6}^{(l)} S_{0,0}^{(l)} \end{bmatrix} Q^{(l)} = \begin{bmatrix} E_{6}^{(l)} S_{0,0}^{(l)} \\ E_{6}^{(l) e_{0}^{(l)}} \end{bmatrix}^{-1} E_{6}^{(l) e_{0}^{(l)}}.
\end{align*}
\]

(32)

Finally, substituting Equation (27) into the first equation of (31) yields the Riccati equation of the hinge output end as follows:

\[
\begin{align*}
z_{6,0}^{(l)} &= E_{6}^{(l)} S_{0,0}^{(l)} p^{(l)} z_{6,0}^{(l)} + \left( E_{6}^{(l)} e_{0}^{(l)} + E_{6}^{(l) S_{0,0}^{(l)} Q^{(l)}} \right) \\
&=: S_{0,0}^{(l)} z_{6,0}^{(l)} + e_{0}^{(l)}
\end{align*}
\]

(33)

with desired hinge transfer equations \( (S_{0,0}^{(l)}, e_{0}^{(l)}) \rightarrow (S_{0,0}^{(l)}, e_{0}^{(l)}) \) reading as follows:

\[
S_{0}^{(l)} = E_{6}^{(l)} S_{0,0}^{(l)} p^{(l)} + E_{6}^{(l)} e_{0}^{(l)} + E_{6}^{(l) S_{0,0}^{(l)} Q^{(l)}}.
\]

(34)

### 3.3 | Recursive procedure for chain systems

With the above relations we may now summarize the complete recursive procedure for a chain system as follows:

(i) Start at the tip with \( S_{1}^{(l)} = O_{n+6}, e_{1}^{(l)} = O_{n+1} \) according to Equation (22).

(ii) Sweep through the chain along the transfer path to compute \( (S_{0}^{(l)}, e_{0}^{(l)}) \) based on body Equation (30) and hinge Equation (34) with auxiliary quantities (28) for bodies and (32) for hinges, respectively, as well as Equation (25).

(iii) Solve final equation \( z_{6,0}^{(l)} = S_{0}^{(l)} z_{6,0}^{(l)} + e_{0}^{(l)} \) for unknown state variables of \( z_{6,0}^{(l)} \) and \( z_{6,0}^{(l)} \) after substituting known boundary states.

(iv) Apply Equation (27) and the first equation of (23) along the inverse transfer path to find all intermediate state vectors.

(v) Extract unknown accelerations for feeding a numerical integration scheme.

(vi) For the chain system, it is obvious that the number of multiplications of matrices scales linearly with the number of elements where the transfer matrices have a rather low order independent of the DOF of the multibody system.

### 4 | TREE STRUCTURES

For a tree system as shown in Figure 1B, one may start from all tip boundary ends in Figure 3B in parallel and proceed forward recursively through the chain subsystems following the procedure in Section 3 to finally arrive at the input ends \( P_{t}^{(m1)} \) and \( P_{t}^{(m1)} \) of the N-input body, respectively:

\[
\begin{align*}
 S_{1}^{(l)} &= O_{n+6}, \quad e_{1}^{(l)} = O_{n+1} \rightarrow S_{O}^{(l)} = S_{2}^{(l)}, \\
 e_{O}^{(l)} &= e_{1}^{(l)} \rightarrow \cdots \rightarrow S_{N}^{(l)} = S_{1}^{(l)}, \quad e_{N}^{(l)} = e_{1}^{(l)}, \\
 S_{O}^{(l)} &= O_{n+6}, \quad e_{O}^{(l)} = O_{n+1} \rightarrow S_{I}^{(l)} = S_{1}^{(l)}, \quad e_{I}^{(l)} = e_{1}^{(l)}.
\end{align*}
\]

(35)

Next, we require a procedure to obtain output quantities \( S_{O}^{(l)}, e_{O}^{(l)} \) from input information \( S_{I}^{(l)}, e_{I}^{(l)} \) for all input ends \( l \) of a general N-input rigid body i (here \( i = 11, N = 2 \)) satisfying Equation (21).

By substituting

\[
z_{6,k}^{(l)} = S_{6,k}^{(l)} z_{6,k}^{(l)} + e_{6,k}^{(l)} \quad (k = 1, 2, ..., N)
\]

(36)

into transfer Equation (12) yields

\[
\begin{align*}
z_{6,1}^{(l)} &= \left( T_{0,1}^{(l)} S_{0,0}^{(l)} + e_{0}^{(l)} \right) + \left( T_{0,1}^{(l)} e_{0}^{(l)} + T_{0,1}^{(l)} q^{(l)} \right) \\
&\quad + \sum_{k=2}^{N} \sum_{k=1}^{N} T_{0,1}^{(l)} \left( z_{6,k}^{(l)} + e_{6,k}^{(l)} \right) = \sum_{k=1}^{N} T_{0,1}^{(l)} z_{6,k}^{(l)} + e_{6,k}^{(l)}
\end{align*}
\]

(37)
where
\[
T_{d,k}^{0} = \begin{cases} T_{d,k}^{0} z_{o}^{0} + T_{d,k}^{0} & \text{for } k = 1, \\
T_{d,k}^{0} z_{x}^{k} + T_{d,k}^{0} & \text{for } k \geq 2, \\
T_{d,k}^{0} & \text{for } k \geq 2.
\end{cases}
\] (38)

Substituting consistency Equation (16), that is,
\[
z_{o,k}^{0} = H_{k}^{0} z_{x}^{k} + h_{k}^{0} \quad (k = 2, 3, \ldots, N)
\] (39)
into Equation (37) yields
\[
z_{o,k}^{0} = T_{d,k}^{0} z_{x}^{k} + \sum_{k=2}^{N} T_{d,k}^{0} H_{k}^{0} z_{x}^{k,1} + h_{k}^{0} + t_{k}^{0}
\] (40)
being identical to Equations (27) and (28). Finally, substituting Equation (41) into Equation (40) yields the reduced transfer equation of a multi-input body as follows:
\[
z_{o,k}^{0} = T_{d,k}^{0} z_{x}^{k} + \sum_{k=2}^{N} T_{d,k}^{0} H_{k}^{0} z_{x}^{k} + \sum_{k=2}^{N} T_{d,k}^{0} H_{k}^{0} + t_{k}^{0}
\] (42)
where
\[
S_{0}^{0} = \left( T_{d,1}^{0} + \sum_{k=2}^{N} T_{d,k}^{0} H_{k}^{0} \right) \phi_{k,1}^{0},
\]
\[
e_{0}^{0} = \left( T_{d,1}^{0} + \sum_{k=2}^{N} T_{d,k}^{0} H_{k}^{0} \right) \phi_{k,1}^{0} + \sum_{k=2}^{N} T_{d,k}^{0} H_{k}^{0} + t_{k}^{0}
\] (43)
are reduced transfer matrices of the output end. The influence of input information \((S_{0}^{0}, e_{0}^{0})\) results from auxiliary matrices (38). For completeness, remaining input states \(z_{d,k}^{0}\) can be determined from the output state by Equations (39) and (41) as follows:
\[
z_{d,k}^{0} = P_{d,k}^{0} z_{o,k}^{0} + Q_{d,k}^{0},
\] (44)
where \(P_{d,k}^{0} = H_{k}^{0} P_{d,k}^{0}, \ Q_{d,k}^{0} = H_{k}^{0} Q_{d,k}^{0} + h_{k}^{0}\) \((k = 2, 3, \ldots, N)\).

After obtaining matrices for output end of element 11 from Equation (43), we may proceed along the output chain in Figure 3B from element 12 to the output end of the element \((2n-1)\) with the procedure in Section 3:
\[
S_{o}^{12} = S_{o}^{11}, \ e_{o}^{12} = e_{o}^{11} \rightarrow S_{o}^{13} = S_{o}^{12}, \ e_{o}^{13} = e_{o}^{12} \rightarrow \ldots \rightarrow S_{o}^{2n-1}, \ e_{o}^{2n-1}
\] (45)

Finally, the boundary condition can be applied to the final relation \(z_{o}^{0,(2n-1)} = S_{o}^{2n-1} z_{o}^{0,(2n-1)} + e_{o}^{0,(2n-1)}\). This demonstrates that tree structures can be treated almost in the same way as chains, where tree nodes need to be treated according to Equation (43). It is worth mentioning that the final output relation (42) of a multi-input body is the same as Equation (23) for single-input bodies.

5 | TREATMENT OF ISOLATED CLOSED-LOOPS

The treatment of a closed-loop system as shown in Figure 1C with \(n\) single-input elements is different from an open-loop chain because no boundary conditions can be used to determine initial values for \(z\) such as (22) in reduced transformation (21). Let us assume a cutoff between elements \(n\) and 1 (see Figure 3C) accompanied by the closing conditions
\[
z_{o}^{(n)} = z_{o}^{n}, \ z_{o}^{1} = z_{o}^{1},
\] (46)
Then, solution of the second equation of body transfer Equation (26) for \(z_{o}^{(1)}\) yielding
\[
z_{o}^{(1)} = T_{ab}^{(1)} z_{o}^{(1)} - T_{ab}^{(1)} f_{c}^{(1)}
\] (47)
and substitution into the first equation results in
\[
z_{o}^{(1)} = S_{0}^{(1)} z_{o}^{(1)} + D_{0}^{(1)} z_{o}^{c} + e_{o}^{(1)} \quad \text{where } S_{0}^{(1)} = T_{ab}^{(1)} T_{ab}^{(1)},
\]
\[
D_{0}^{(1)} = T_{ab}^{(1)} f_{c}^{(1)} - T_{ab}^{(1)} f_{c}^{(1)}
\] (48)
This indicates that in a closed-loop the reduced transformation (21) should be generally expanded by the unknown cut-state \(z_{o,c}\) as follows:
\[
z_{o} = S_{0}^{(1)} z_{o}^{c} + D_{0}^{(1)} z_{o}^{c} + e_{o}^{(1)}
\] (49)
Especially for the input end of the element \(i = 1\) of the spanning chain in Figure 1C, the first equation of identity (46) may be then written as follows:
\[
z_{o}^{(1)} = S_{0}^{(1)} z_{o}^{(1)} + D_{0}^{(1)} z_{o}^{c} + e_{o}^{(1)}
\] (50)
where \(S_{0}^{(1)} = O_{b,6}, \ D_{0}^{(1)} = I_{b}, \ e_{o}^{(1)} = O_{b,1}.\)

As will be shown later, a recursive procedure similar to that for chains can be established for the spanning chain finally starting from Equation (50) and ending in \(z_{o}^{(n)} = S_{0}^{(n)} z_{o}^{(n)} + D_{0}^{(n)} z_{o}^{c} + e_{o}^{(n)}\) for the output end of element \(n\). The closing condition (46) then reads as follows:
\[
\left( I_{b} - D_{0}^{(b)} \right) z_{o}^{(c)} = S_{0}^{(n)} z_{o,c} + e_{o}^{(n)}
\] (51)
where both \(z_{o,c}\) and \(z_{o,c}\) are unknown and are not fully determined by this equation because there are only six equations for 12 unknowns.
Noting that the second part of the closed-loop condition (46) has not been taken into account yet, substituting Equation (47) into the second equation of (46) yields

\[
z_{0,C} = B_{0,C}^{(1)} z_{0,0}^{(1)} + B_{0,C}^{(2)} z_{0,C} + b_{0}^{(3)} ,
\]
where

\[
B_{0,C}^{(3)} = \left( T_{0,C}^{(1)} \right)^{-1} . \quad B_{0,C}^{(4)} = O_{0,4} , \quad b_{0}^{(5)} = - \left( T_{0,C}^{(1)} \right)^{-1} f_{0}^{(1)} .
\] (52)

We may now assume that state vectors of all connecting points in a closed-loop satisfy such a relation between actual state \( z \) and cut state \( z_{0,C} \). \( z_{0,C} \) of the spanning chain, that is,

\[
z_{0,C} = B_{0,C} z_{0} + B_{0,C} z_{0,C} + b,
\] (53)

which may be interpreted as a complementary equation for a closed-loop system. Especially for the input end of element \( i = 1 \) of the spanning chain in Figure 3C, this reads as follows:

\[
z_{0,C} = B_{1,C}^{(1)} z_{0,1}^{(1)} + B_{1,C}^{(2)} z_{0,C} + b_{1}^{(3)} ,
\]
with

\[
B_{1,C}^{(3)} = I_{1} , \quad B_{1,C}^{(4)} = O_{0,4} , \quad b_{1}^{(5)} = O_{0,3} ,
\] (54)
due to closing condition (46) and for the output end of element \( n \) as

\[
z_{0,C} = B_{n,C}^{(1)} z_{0,n}^{(1)} + B_{n,C}^{(2)} z_{0,C} + b_{n}^{(3)} ,
\]
with

\[
B_{n,C}^{(3)} = I_{n} , \quad B_{n,C}^{(4)} = O_{0,4} , \quad b_{n}^{(5)} = O_{0,3} .
\] (55)

Combining it with Equation (51) yields

\[
\begin{bmatrix}
I - D_{0}^{(1)} & - S_{0}^{(3)} \\
-B_{0,C}^{(3)} & I - B_{0,C}^{(3)}
\end{bmatrix}
\begin{bmatrix}
z_{0,C}^{(2)} \\
b_{0}^{(5)}
\end{bmatrix}
= \begin{bmatrix}
e_{0}^{(5)} \\
b_{0}^{(5)}
\end{bmatrix}
\] (56)

where now the number of equations equals that of unknowns and allows to compute the unknown cut state \( z_{0,C} \) and \( z_{0,0} \). Next, the reduced transfer matrices of Equations (49) and (53) for bodies and hinges need to be determined to complete the concept.

### 5.1 Body treatment

Equations (49) and (53) for the input of any single-input body \( i \) in a closed-loop read as

\[
z_{0,i}^{(2)} = S_{0}^{(1)} z_{0,i}^{(1)} + D_{0}^{(1)} z_{0,0}^{(1)} + e_{0}^{(1)} , \quad z_{0,C} = B_{0,C}^{(1)} z_{0,0}^{(1)} + B_{0,C}^{(2)} z_{0,C} + b_{0}^{(3)} .
\] (57)

Substituting the first equation into body transfer Equation (26) yields

\[
z_{0,i}^{(2)} = \left( T_{0,i}^{(1)} s_{0}^{(1)} + T_{0,i}^{(1)} b_{0}^{(1)} \right) z_{0,i}^{(1)} + T_{0,i}^{(1)} D_{0}^{(1)} z_{0,0}^{(1)} + \left( T_{0,i}^{(1)} b_{0}^{(1)} + f_{0}^{(1)} \right) ,
\] (58)

The second body transfer equation still results in Equation (27) and (28) which, however, may be extended artificially as follows:

\[
z_{0,i}^{(2)} = P_{0}^{(1)} z_{0,0}^{(1)} + W_{0}^{(1)} z_{0,C} + Q_{0}^{(1)} .
\] (59)

with

\[
P_{0}^{(1)} = \left( T_{0,i}^{(1)} \right)^{-1} , \quad W_{0}^{(1)} = O_{0,6} , \quad Q_{0}^{(1)} = - P_{0}^{(1)} f_{0}^{(1)} .
\] (60)

Substitution into Equation (58) leads to the reduced transfer Equation (49) of the output end

\[
z_{0,i}^{(2)} = S_{0}^{(1)} z_{0,i}^{(1)} + D_{0}^{(1)} z_{0,C} + e_{0}^{(1)} ,
\] (61)

where

\[
S_{0}^{(1)} = \left( T_{0,i}^{(1)} s_{0}^{(1)} + T_{0,i}^{(1)} b_{0}^{(1)} \right) , \quad D_{0}^{(1)} = T_{0,i}^{(1)} D_{0}^{(1)} ,
\]
\[
e_{0}^{(1)} = \left( T_{0,i}^{(1)} s_{0}^{(1)} + T_{0,i}^{(1)} b_{0}^{(1)} \right) Q_{0}^{(1)} + T_{0,i}^{(1)} e_{0}^{(1)} + f_{0}^{(1)} .
\] (62)

can be computed from the element transfer matrices and the input information \( S_{0}^{(1)} , D_{0}^{(1)} \), and \( e_{0}^{(1)} \) as desired. Similarly, substituting (59), (60) into the second equation of (57) yields the complementary output relation

\[
z_{0,C} = B_{0,C}^{(1)} z_{0,0}^{(1)} + B_{0,C}^{(2)} z_{0,C} + b_{0}^{(3)}
\] (63)

with recursive formulas

\[
B_{0,C}^{(1)} = B_{0,C}^{(1)} z_{0,0}^{(1)} + B_{0,C}^{(2)} z_{0,C} + b_{0}^{(3)}
\] (64)

### 5.2 Hinge treatment

The relations (57) are also applicable here. Substitution into hinge Equation (19) yields

\[
z_{0,i}^{(2)} = E_{0}^{(1)} s_{0}^{(1)} z_{0,i}^{(1)} + z_{0,C}^{(2)} + E_{0}^{(1)} e_{0}^{(1)} ,
\]
\[
E_{0}^{(1)} s_{0}^{(1)} z_{0,i}^{(1)} + E_{0}^{(1)} D_{0}^{(1)} z_{0,0}^{(1)} + E_{0}^{(1)} e_{0}^{(1)} = 0 , \quad E_{0}^{(1)} z_{0,i}^{(1)} + E_{0}^{(1)} z_{0,C}^{(2)} + E_{0}^{(1)} e_{0}^{(1)} = 0 .
\] (65)

Combining the last two equations and solving them for \( z_{0,i}^{(2)} \) in terms of \( z_{0,0}^{(1)} \) and \( z_{0,C}^{(2)} \) yields Equation (59) with

\[
p_{0}^{(1)} = \begin{bmatrix}
E_{0}^{(1)} \\
E_{0}^{(1)} s_{0}^{(1)}
\end{bmatrix}^{-1} \begin{bmatrix}
E_{0}^{(1)} \\
O
\end{bmatrix} , \quad W_{0}^{(1)} = \begin{bmatrix}
E_{0}^{(1)} \\
E_{0}^{(1)} s_{0}^{(1)}
\end{bmatrix}^{-1} \begin{bmatrix}
O \\
E_{0}^{(1)} D_{0}^{(1)}
\end{bmatrix} ,
\] (66)
\[
Q_{0}^{(1)} = \begin{bmatrix}
E_{0}^{(1)} \\
E_{0}^{(1)} s_{0}^{(1)}
\end{bmatrix}^{-1} \begin{bmatrix}
E_{0}^{(1)} \\
O
\end{bmatrix} , \quad E_{0}^{(1)} = \begin{bmatrix}
E_{0}^{(1)} \\
E_{0}^{(1)} s_{0}^{(1)}
\end{bmatrix} .
\]

Then, substituting Equation (59) into the first equation of (65) yields Equation (61) with hinge transfer equations

\[
S_{0}^{(1)} = E_{0}^{(1)} s_{0}^{(1)} p_{0}^{(1)} , \quad D_{0}^{(1)} = E_{0}^{(1)} s_{0}^{(1)} w_{0}^{(1)} + E_{0}^{(1)} d_{0}^{(1)} ,
\]
\[
e_{0}^{(1)} = E_{0}^{(1)} e_{0}^{(1)} .
\] (67)

Finally, substituting Equation (59) into the second equation of (57) yields
\[
\begin{align*}
  z_{a,C} &= B_{a,C}^{[i]} \rho_{a,C} + \left( B_{a,C}^{[i]} + B_{a,C}^{[0]} W_{a,C} \right) z_{a,C} + \left( b_a^{[i]} + B_{b_a,C}^{[0]} Q_{b_a,C} \right) \quad (68)
\end{align*}
\]

identical with Equation (63) with abbreviations (64).

5.3 Recursive procedure for closed loops

The whole loop procedure may be summarized as follows:

(i) Cut off the loop and set the initial values for the reduced transfer matrices according to Equations (50) and (54) for the cutting point.

(ii) Sweep through the spanning chain using Equations (62), (64) for bodies, and (67), (66) for hinges based on auxiliary quantities (60) for bodies and (66) for hinges.

(iii) Once the cutting point is reached from the other side, closing condition (56) can be established and solved for the cutting state \( z_{a,C} \) and \( z_{b,C} \).

(iv) Finally, all intermediate state vectors can be calculated recursively by reduced transfer Equations (59) and (57) in inverse transfer path direction.

In essence, there are totally \( 4 \times 6 = 24 \) equations including the reduced transformation (49), complementary Equation (53) and closed-loop conditions (46). The number of equations equals that of the involved 24 state variables associated with a couple of cutting points, which are considered as root and tip of the spanning chain, respectively. Compared to classical multibody system formulations, the additional coordinates (46) introduced by loop-cutting are rather similar, where the \( z_{a,C} \) part is somehow related to Lagrange multipliers of a classical differential algebraic equation (DAE) approach.

6 ATTACHMENT OF A LOOP TO A GENERAL SYSTEM

The simplest general system may be composed of a tree subsystem and a closed-loop subsystem as shown in Figure 1D, where the two subsystems are connected in point \( p_{1}^{[18]} = p_{2}^{[1]} \) of Figure 3D. The closed-loop is cut off between elements 1 and 10 leading to loop-closing conditions

\[
\begin{align*}
  z_{a,i}^{[1]} &= z_{a,0}^{[1]}, & z_{b,i}^{[1]} &= z_{b,0}^{[1]} \quad (69)
\end{align*}
\]

where the new terms \( D z_{a,A} \) and \( B z_{a,A} \) are related to the attachment forces and torques. Especially for the input end of body 1 we get

\[
\begin{align*}
  z_{a,1}^{[1]} &= S_{a,1}^{[0]} z_{a,0}^{[1]} + D_{a,1}^{[0]} z_{a,0}^{[1]} + D_{a,1}^{[1]} z_{a,A} + e_{1}^{[1]}, \\
  z_{b,0}^{[1]} &= B_{b,1}^{[1]} z_{b,0}^{[1]} + B_{b,1}^{[0]} z_{b,A} + b_{1}^{[1]}.
\end{align*}
\]

According to Equation (69), the reduced transfer matrices can be initialized as follows:

\[
\begin{align*}
  S_{a,1}^{[0]} &= O_{a,6}, & D_{a,1}^{[0]} &= I_b, & D_{a,1}^{[1]} &= O_{a,6}, & e_{1}^{[1]} &= O_{b,1}. \\
  B_{b,1}^{[0]} &= I_b, & B_{b,1}^{[1]} &= O_{a,6}, & b_{1}^{[1]} &= O_{b,1}. \quad (72)
\end{align*}
\]

Sweeping through the closed-loop subsystem along the transfer path from the input end of body 1 to the output end of hinge 10 finally yields

\[
\begin{align*}
  z_{a,10}^{[10]} &= S_{a,10}^{[10]} z_{a,0}^{[10]} + D_{a,10}^{[10]} z_{a,0}^{[10]} + D_{a,10}^{[1]} z_{a,A} + e_{10}^{[10]}, \\
  z_{b,0}^{[10]} &= B_{b,10}^{[1]} z_{b,0}^{[10]} + B_{b,10}^{[0]} z_{b,A} + b_{10}^{[10]}.
\end{align*}
\]

The closing conditions (69) then result in

\[
\begin{align*}
  \left( D_{a,10}^{[10]} - I_b \right) z_{a,10}^{[10]} + S_{a,10}^{[10]} z_{a,0}^{[10]} + D_{a,10}^{[1]} z_{a,A} + e_{10}^{[10]} &= 0, \\
  B_{b,10}^{[0]} z_{b,0}^{[10]} + b_{10}^{[10]} &= 0. \quad (74)
\end{align*}
\]

Combination of the two equations in matrix form and solving them for \( [z_{a,C}^{[1]} z_{b,C}^{[1]}]^T \) in terms of \( z_{a,A} \) yields

\[
\begin{align*}
  [z_{a,C}^{[1]} z_{b,C}^{[1]}] &= [\Phi_1 \Phi_2] z_{a,A} + [\Phi_1 \Phi_2]. \quad (76)
\end{align*}
\]

where

\[
\begin{align*}
  \Phi_1 &= \left( D_{a,10}^{[10]} - I_b \right) S_{a,10}^{[10]} \left( B_{b,10}^{[0]} - I_b \right)^{-1} D_{a,10}^{[10]} B_{b,10}^{[0]} , \\
  \Phi_2 &= \left( D_{a,10}^{[10]} - I_b \right) S_{a,10}^{[10]} \left( B_{b,10}^{[0]} - I_b \right)^{-1} e_{10}^{[10]} B_{b,10}^{[0]} . \quad (77)
\end{align*}
\]

The kinematics relationships (2) and (4) of rigid body 1 between cutoff point \( p_{1}^{[18]} \) and attachment point \( p_{0}^{[1]} \) result in

\[
\begin{align*}
  \omega_{a,1}^{[1]} &= \omega_{a,1}^{[18]} \quad (78)
\end{align*}
\]

where \( \omega_{a,1}^{[18]} = p_{1}^{[1]} p_{0}^{[1]} \). With \( z_{b,c} = \Phi_2 z_{a,A} + \Phi_2 \) from Equation (76) we get

\[
\begin{align*}
  z_{b,c} = E_1 \Phi_2 z_{a,A} + E_1 \Phi_2 + e_2 \quad (79)
\end{align*}
\]
which may be solved for $z_{dA}$ as $z_{dA} = (E_T \Phi_T)^{-1} z_{dA} - (E_T \Phi_T)^{-1} (E_T \varphi_T + e_2)$. Due to $z_{dA} = z_{d1}^{(18)}$ and $z_{dA} = z_{d1}^{(18)}$, this serves as input to hinge 18 as

$$z_{d1}^{(18)} = S_{d1}^{(18)} z_{d1}^{(18)} + e_1^{(18)}$$

where $S_{d1}^{(18)} = (E_T \Phi_T)^{-1}, e_1^{(18)} = -S_{d1}^{(18)} (E_T \varphi_T + e_2)$. (80)

which means that the whole loop subsystem may be treated like a single-input element with reduced transformation (21). Thus, the loop-cutting state is totally eliminated.

The remaining part of the system in Figure 3D is a tree subsystem with initial values $S_{h1}^{(11)} = O_{h1} = e_{h1}^{(11)}$ for the free tip $P_1^{(11)}$ and $S_{h1}^{(18)}, e_{h1}^{(18)}$ for the attachment point as determined by Equation (80), which can be treated according to Section 4:

$$
\begin{align*}
S_{h1}^{(11)} & \rightarrow \ldots \rightarrow S_{d0}^{(20)} = S_{d0}^{(17)} e_{d0}^{(17)} = e_{d1}^{(17)} \rightarrow \ldots \rightarrow S_{d0}^{(23)} e_{d0}^{(23)} \\
S_{h1}^{(18)} & \rightarrow \ldots \rightarrow S_{d0}^{(20)} = S_{d0}^{(17)} e_{d0}^{(17)} = e_{d1}^{(17)} \rightarrow \ldots \rightarrow S_{d0}^{(23)} e_{d0}^{(23)}.
\end{align*}
$$

(81)

The remaining question is how to transfer the matrices in Equation (70) from element input to output within the spanning chain of the subloop. Comparing Equation (70) with (49) and (53), respectively, shows that the new terms $D z_{dA}$ and $B z_{dA}$ added look very similar to the terms $D z_{dC}$ and $B z_{dC}$, respectively. Therefore, a similar way as in Section 5 may be used to deduce reduced transfer relations in Equation (70). For bodies, we may find that Equations (60), (62), and (64) have to be accompanied by

$$
W^{(0)} = O_{bAc}, \quad D^{(0)} = T^{(0)} D^{(0)}, \quad B^{(0)} = B^{(0)} + B^{(0)} W^{(0)}.
$$

(82)

For hinges, Equations (66), (67), and (64) have to be supplemented by

$$
W^{(0)} = -\left[ \begin{bmatrix} E^{(0)} \\ E^{(0)} S^{(0)} \end{bmatrix} \right]^{-1} O, \quad D^{(0)} = E^{(0)} S^{(0)} W^{(0)} + E^{(0)} D^{(0)}, \quad B^{(0)} = B^{(0)} + B^{(0)} W^{(0)}.
$$

(83)

The whole procedure for a general system may be summarized as follows:

(i) Cut off loops at the connecting body, respectively, and set the initial values for the reduced transfer matrices of the cutting point according to Equation (72).

(ii) Sweep through the spanning chain of the closed-loop subsystem to update the reduced transfer matrices by applying Equations (60), (62), (64), and (82) for bodies, and (66), (67), (64), and (83) for hinges.

(iii) Compute matrices of Equation (21) at the output end of attachment body (here body 1 identical with the input of hinge 18 delivering $S_{d1}^{(18)} e_{d1}^{(18)}$ according to Equation (80).

(iv) Sweep through the remaining tree subsystem toward the root of the system.

(v) Solve the boundary conditions for the unknowns of the state vector at the root of the system.

(vi) Finally, state vectors of all connecting points in the system can be calculated one by one against transfer paths.

7 | VALIDATION BY NUMERICAL EXAMPLES

To validate the proposed strategies and demonstrate their efficiency, three types of systems will be simulated: a tree system, an isolated closed loop, and a combination of loop and tree as shown in Figure 1.

7.1 | Huge spatial tree systems

To see the computational efficiency of the proposed approach called RMSTMM, numerical simulations of the spatial tree system in Figure 1B with a wide range of DOFs are carried out and compared with commercial software automatic dynamic analysis of mechanical system (ADAMS). The mass, moment of inertia w.r.t. the mass center and length of all bodies are $m = 1$ kg, $J_1 = 1/12$ kg m$^2$ and $l = 1$ m, respectively. The bodies are connected with ball-and-socket hinges and the initial condition is according to Figure 1B.

Both simulations are performed on a Xeon E5-2640 CPU with 2.00 GHz. Regarding the simulation of RMSTMM, the second time-derivatives of generalized coordinates $\dot{\ddot{q}} = g(t, q, \dot{q})$ are deduced from the state vectors of the connection points between the elements using kinematic hinge relations. For numerical integration, these second-order equations are reduced to first-order ordinary differential equations $\ddot{y} = f(t, y)$ where $y = [\dot{q}^T \dot{\dot{q}}^T]^T$, $f(t, y) = [\dot{q}^T, \dot{\dot{q}}^T(t, q, \dot{q})]^T$, and integrated with a fourth-order Runge–Kutta scheme with constant step size $\Delta t = 0.001$ s. The simulation results of the rotation about the z-axis of two boundary bodies 1 and 2$n - 1$ and the 2-input body 11 are shown in Figure 4, which agree very well for both procedures for 500 bodies associated with 2$n - 1 = 999$. However, as shown in Table 1, the time cost for RMSTMM is by a factor of 36 lower than that for ADAMS. Even more important, ADAMS fails for a higher number of bodies due to running out of computer memory, whereas the run time of RMSTMM scales only linearly with the number of bodies because it avoids the mass matrix as mentioned in Section 3.

7.2 | Some closed-loop simulations

Next, the planar closed-loop system in Figure 1C with a circular initial configuration is taken as an example. The parameters of all bodies are the same as in Section 7.1. Besides gravity, an external force equaling the total weight of all bodies with opposite gravity direction is applied to the mass center of body 1 pulling it up. For planar simulation, state
vectors \( \mathbf{z}_a = \begin{bmatrix} m_2 & \mathbf{q}_\mathbf{k} & \mathbf{q}_r \end{bmatrix}^T \) and \( \mathbf{z}_b = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{v}_2 \end{bmatrix}^T \) and hinge matrices \( \mathbf{H}_j \) reduce to

\[
E_c = I_3, \quad E_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\] (84)

Simulation results are compared to those of NV-MSTMM, where the overall transfer equation of the system is computed from \( 7 \times 7 \) transfer matrices \( \mathbf{U}_i \) as follows:

\[
(l - \mathbf{U}_{i_{-1}})x_{(i)} = 0, \quad \text{where} \quad \mathbf{U}_{i_{-1}} = \mathbf{U}_i \mathbf{U}_{i_{-2}} \ldots \mathbf{U}_2 \mathbf{U}_1.
\] (85)

as well as DAE simulations obtained by a Lagrangian approach. For numerical integration, the same scheme as above with \( \Delta t = 0.001 \text{s} \) is used. However, to guarantee the exact fulfilling of loop-closing condition

\[
\mathbf{C}(\mathbf{q}) = r_{c,0} - r_{c,1} = \begin{bmatrix} x_{c,0} - x_{c,1} \\ y_{c,0} - y_{c,1} \end{bmatrix} = 0,
\] (86)

after each time-step the positions are corrected with a Newton-Raphson iteration solving Equation (86) according to Zhang.15

Results of all three methods agree well up to \( n/2 = 24 \) bodies, see Figure 5. However, for \( n/2 = 25 \) bodies, computational instability occurs.
occurs with the original NV-MSTMM, whereas the computation by RMSTMM is still stable and the results have good agreement with the Lagrangian simulation results for $n/2 = 100$ in Figure 6.

For computational speed assessment, RMSTMM and Lagrangian equation are compared for a different total number of bodies in Figure 7, where again a Xeon E5-2640 CPU with 2.00 GHz is used. As before, the proposed strategy shows only a linear increase of CPU time, and thus is far more efficient than ordinary multibody system formulations not involving recursive schemes like the one described in Brandl et al.\textsuperscript{12}

### 7.3 Simulation of a general system with a closed-loop subsystem

Finally, the planar general system composed of a closed-loop and a tree as shown in Figure 1D is computed with RMSTMM, NV-MSTMM,\textsuperscript{13} and ADAMS. All hinges are smooth pin hinges. The body parameters are the same as in Section 7.1 except for two input body 17 with $m^{(17)} = 2\text{kg}$, $J_x^{(17)} = 2/3\text{kg}\cdot\text{m}^2$, $r_{17}^{(17)} = [1\ 0]^T\text{m}$, $r_{27}^{(17)} = [1\ 0]^T\text{m}$, $r_{37}^{(17)} = [2\ 0]^T\text{m}$. The initial position is shown in Figure 1D, all initial speeds are zero. The simulation results for some angular speeds shown in Figure 8 have good agreement validating the concept.

It may be pointed out that the same scheme as in Section 7.2 was used for RMSTMM. Obviously, the position correction of (86) works well as the total energy stays constant in Figure 9A and position violations are small in Figure 9B (green and black curves). Without this correction, position errors are higher and may grow (blue and red curves in Figure 9B).

### 8 CONCLUSIONS

The paper systematically develops reduced transformations and recursive schemes, which make RMSTMM generally and easily applicable to multibody systems with chain, closed-loop, tree, and general topology. Compared with NV-MSTMM, the method not only benefits from high algorithmic stability but is much easier to use due to fully independent hinge equations and ensures strict satisfaction of system boundary conditions, kinematics, and kinetics, which makes it an accurate analysis method. It has the same high computational speed as NV-MSTMM and scales only linearly with the number of bodies. Applications to various kinds of examples demonstrate the effectiveness and applicability of RMSTMM even to huge systems with long chains and more than 100 000 DOFs.

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### CONFLICT OF INTEREST

The authors declare that there are no conflict of interest.
DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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