Enhancement of superconductivity via resonant anti-shielding
with topological plasmon-polarons

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Abstract

Stimulated by papers from the early 1970’s, electromagnetic metamaterial structures have recently been proposed to enhance the critical temperature $T_c$ of BCS superconductivity by modulating the dielectric response. To be successful, such a scheme would require a resonant anti-shielding effect that can lead to a vanishing nonlocal dielectric function of the system. Conventional metamaterials can provide only a local analog of this effect, and so only modest $T_c$ enhancements have been found, to date. Here, we propose a different route to dielectric enhancement, by employing a heterostructure consisting of alternating thin films of a superconductor and a topological crystal. The latter material can host plasmon-polarons, long lived and strongly coupled to the superconductor phonon spectrum, yielding robust, nonlocal resonant anti-shielding. We employ the Eliashberg-Leavens scaling scheme, as well as an ab initio Migdal-Eliashberg approach, to show that resonant anti-shielding can be induced in such a heterostructure, and a 5-fold enhancement of $T_c$ in MgB$_2$ can be reached under realistic conditions.
**Introduction**

Increasing the critical temperature of superconductivity up to and above room temperature has been a type of ‘holy grail’ of condensed matter physics. Discovery of the so-called high-$T_c$ cuprate superconductors in the late 1980s, with $T_c$ up to 92 K, rekindled the field, and raised hopes that room temperature superconductivity could be in sight. Cuprate $T_c$ was increased to 133 K [1] by 1993, but has since stalled there. Despite intense effort, similar sluggish progress has been made on the theoretical front, with the origin of cuprate superconductivity remaining insufficiently clear still today. Even though carrier bosonization remains a key concept, the pairing mechanism seems more subtle than the BCS electron-phonon-electron interaction. True room temperature superconductivity ($T_c = 287$ K) was finally achieved recently in an entirely different class of materials, $\text{H}_2\text{S} + \text{CH}_4$, but the required conditions are currently prohibitive, as samples must be kept under ultrahigh pressure (>250 GPa) [2]. Nevertheless, this achievement demonstrates experimentally that room temperature superconductivity is possible in a steady state supported by solid state interatomic interactions.

In another recent development, following an old reformulation of BCS theory in terms of the effective dielectric function by Kirzhnits, Ginzburg and others [3,4,5], metallic metamaterials have been proposed to increase $T_c$ by controlling the dielectric response [6,7]. Metallic metamaterial structures often rely on plasmonics and nanotechnology, and provide means to effectively customize a metamedium’s dielectric and magnetic response functions to obtain various exotic optical properties, such as negative refraction, superlensing, extraordinary transmission, etc. [8]. In one scenario, such media were engineered to have a vanishingly small effective electronic dielectric function (sometimes called epsilon-near-zero), and this was conjectured as being advantageous for superconductivity enhancement. However, experimentally-observed $T_c$ enhancement has been limited to a few degrees kelvin [6,7].

In this work, we first discuss what we perceive to be the main concern with the above strategy, and propose a manner to address that concern via a resonant anti-shielding (RAS) effect produced by a long-lived topological plasmon-polaron mode, recently discovered on the surface of a topological crystal (TX), $\text{Bi}_2\text{Se}_3$. Strong dielectric coupling between plasmon-polarons and the phonon spectrum of the superconductor can be anticipated in a superlattice of superconductor and TX films (Fig. 1). To calculate the RAS effect on superconductivity, we employ the Eliashberg-Leavens scaling scheme to calculate a $T_{c,max}$, as well as an ab initio Migdal-Eliashberg
approach which directly calculates \( T_c \). When applied to the MgB\(_2\) superconductor, this procedure leads to large enhancement of \( T_c \) after proper adjustment of the system parameters. Even stronger enhancement could occur with the placement of additional ultrathin films (“phonon modifiers”) with engineered phonon bands in the junction regions of the structure.

Resonant anti-shielding effect

First, to illustrate the basic physics of the anti-shielding effect, we employ the well-known formula for the dressed combined electron-electron interaction in a jellium metal [9],

\[
\frac{V_q}{\varepsilon} + \frac{|g_q|}{\varepsilon} \left( \frac{2\omega_q}{(\omega^2 - \omega^2_q) + i\delta} \right) = \frac{V_q}{\varepsilon_{\text{eff}}}
\]

where \( V_q = \frac{4\pi e^2}{q^2} \) is the bare electrostatic potential, \( \varepsilon \) the electronic dielectric function of the environment, \( g_q \) the matrix element for electron-phonon scattering, averaged over all electronic states, \( \omega_q \) the phonon dispersion, and \( i\delta \) a small imaginary constant loss factor. The first term in Eq. (1) is the electron-electron interaction, mediated (screened) by other electrons, and the second, Fröhlich, term is the electron-electron interaction mediated by phonons with frequency \( \omega_q \). This term, when negative, can lead to electron (Cooper) pairing, and to BCS-type superconductivity. This typically occurs at frequency \( \omega \approx \omega_q \), with the wavevector \( q \) of the order of \( k_F \) (Fermi wave vector). Eq. (1) also shows that this Cooper pairing interaction can be enhanced by making \( |\varepsilon| < 1 \), i.e. anti-shielding, which represents enhancement, rather than suppression (shielding) of the interactions. A resonant anti-shielding effect (RAS) can thus occur if \( |\varepsilon| \ll 1 \). RAS is much stronger for the Fröhlich term in Eq. (1). This is expected, because any more realistic treatment would require a spectral averaging which, as a result of \( \varepsilon \) changing sign about the vanishing point, would lead to cancellations in the first term \( \sim 1/\varepsilon \), and accumulations for the second, which goes as \( \sim 1/|\varepsilon|^2 \).

For a typical superconductor, \( \varepsilon > 1 \) (and is of order 1) and thus, anti-shielding is impossible without some additional mechanism. Similarly, conventional metamaterial structures cannot provide a RAS effect since, while a vanishing of \( \varepsilon \) at \( \omega \approx \omega_q \) is relatively easy to accomplish, achieving this simultaneously at very large \( q \approx 2k_F \) is an exceedingly difficult task. To show this, consider first a superconducting region in contact with a structured, normal metal, with the smallest structured feature size \( a_{\text{min}} \) (e.g., grating period, edge sharpness, surface
roughness, etc.) much larger than the thickness of the shielding field penetration layer $\zeta \approx \frac{1}{q}$. In this case, the majority of the superconducting electrons within this layer effectively “see” a flat metallic surface. The effective dielectric function of a metal, which properly describes the nonlocal, flat surface response, has been derived [10], and for $\omega = \omega_q$ and $q = 2k_F$, it is greater than 1. Thus, it represents shielding and is incapable of enhancing Cooper pairing in any structural combination with a superconductor.

Consider now a metamaterial scheme in another limit of $a_{min} \approx \zeta$. In this case, in general, the Fourier spectrum of the response (e.g. fields) could contain components with maximum momentum magnitude $q_{max}$ of order $1/a_{min}$. For example, in the simplest case of a metamaterial surface structured in the form of a 1D grating (say, in the $x$-direction) with period $a_{min}$, the reciprocal lattice vector component $G_x \approx q_{max} \approx 2\pi/a_{min}$ produces an Umklapp response $q_x \rightarrow q_x - G_x = 2k_F - \frac{2\pi}{a_{min}} \approx 0$. Thus, the issue with a large momentum required in the $x$-direction can be resolved, as long as $a_{min} \approx \pi/k_F$. However, several additional facts must be kept in mind at this point. First, simple 1D texturing is insufficient for significant enhancement of superconductivity, since this can boost the pairing interaction only in a narrow cone of momentum space, along the chosen direction ($x$, above). 2D structuring of a metamaterial, with the 2D vector version of the Umklapp condition $q \rightarrow q - G$, improves the situation by providing multiple directional alternatives at the circumference of the Brillouin zone (BZ). One route to this could be to employ surface roughness, with a 2D Fourier spectrum peaked at $|q_{max}| \approx 2\pi/a_{min}$. Second, $a_{min}$ needs to be very small. While for cuprates, with $k_F \approx 2$ nm$^{-1}$, $a_{min} \approx \pi/k_F \approx 1.5$ nm, which is experimentally difficult though possible, while for good metals, with $k_F \approx 20$ nm$^{-1}$, $a_{min} \approx \pi/k_F \approx 0.2$ nm, i.e. atomically small, and thus very challenging to control with current nanotechnology. Of course, these conclusions are limited to $\zeta \approx a_{min}$, i.e. the thickness of the superconductor film must be of the same order. Thus, while achieving significant superconductivity enhancement with metamaterials is very difficult, it is in principle possible, and will be discussed elsewhere [11].

**Plasmon-polaron at the surface of a topological crystal**

An alternative strategy to achieving RAS suggests itself from the fact that Maxwell’s equations allow for the existence of longitudinal plasmon modes in various uniform and non-
uniform conductors, for which \( \varepsilon(q, \omega) = 0 \). However, conventional plasmon modes occur in the sector of phase space far from the required condition \( \omega \approx \omega_q \), with \( q \) of the order of \( k_F \).

Recently, an unusual plasmonic mode (called an \( \alpha \)-mode) was observed in the topological crystal Bi\(_2\)Se\(_3\) [12], in that phase space sector. This mode exists only if Bi\(_2\)Se\(_3\) remains topological, i.e. Dirac electrons exist on its surface. The dispersion curve for this mode is close to linear, \( \omega \propto q \) (see Fig. 2a) and it is clearly not a pure phonon mode, since it crosses the BZ edge without any momentum Umklapp (LEED showed no surface reconstruction [12]). The most striking observation was that this mode, even at room temperature and very large momentum, remains strong and extremely weakly damped, with the damping rate and intensity almost constant for \( 2k_F < q < 6k_F \) [12]. All other known plasmon modes are unobservable in that range. An interesting observation was that in the non-topological form of Bi\(_2\)Se\(_3\) (induced \textit{in situ} by Mn doping), this \( \alpha \)-mode disappears and is replaced by a conventional, transverse acoustic phonon mode (Fig. 2a). The new acoustic phonon mode has a standard dispersion, close to that of the \( \alpha \)-mode in the first BZ [12].

A recent theoretical study [13] is consistent with these discoveries. In particular, it shows that the \( \alpha \)-mode is a plasmon-polaron, a hybrid of plasmon excitations of the Dirac surface electrons, and the transverse acoustic phonon mode. This \( \alpha \)-mode thus has topological character, as it involves collective electron spin fluctuations of the topological 2D Dirac band electron states at the surface. Ref. [13] demonstrated that the \( \alpha \)-mode has near perfect suppression of forward and backward scattering, resulting in the experimentally-observed ultralow damping, and the absence of Umklapp scattering at the first BZ boundary.

The \( \alpha \)-mode is similar to the phonon-polariton mode, the well-known hybrid of photon and phonon excitations. The standard way to obtain the dispersion relation for the polariton [14] is to start with the dispersion relation for photons (light line) \( \omega = q c / \sqrt{\varepsilon} \), and to replace \( \varepsilon \) with the usual Lyddane–Sachs–Teller-like phonon formula [15], \( \varepsilon = \varepsilon_{eff} = \frac{\omega_{T0}^2 - \omega^2}{\omega_{LO}^2 - \omega^2} \) (TO, LO = transverse, longitudinal optical). By analogy, one can derive the dispersion relation for the plasmon-polaron by starting with the dispersion relation for the topological 2D Dirac plasmon [13]:

\[
\varepsilon_{TI}(q, \omega) = 1 - \tilde{V}_q \Pi(q, \omega) = 0
\] (2)
where $\Pi(q, \omega)$ is the RPA susceptibility of the electrons [13], and $\tilde{V}_q = V_q / 2\varepsilon$. As in the case of a polariton, we assume that $\varepsilon = \varepsilon_{eff}$, except now $\varepsilon_{eff}$ is given by Eq. (1), with $\varepsilon = \tilde{\varepsilon}$ (a background dielectric constant). This transforms Eq. (2) into the dispersion relation for a plasmon-polaron:

$$\varepsilon_T(q, \omega) = 1 - \frac{\Pi(q, \omega)}{2} \left[ \frac{V_q}{\varepsilon} + \left| \frac{g_q}{\varepsilon} \right|^2 \frac{2\omega_q}{(\omega^2 - \omega_q^2 + i\delta)} \right] = 0$$  \hspace{1cm} (3)

Eq. (3) demonstrates that the plasmonic response in this case becomes strongly synchronized / correlated with the phonon oscillations in the vicinity of the interface, on which the topological Dirac electron states reside. In the limit of interest in this work ($q$ of order $k_F$ and $\omega$ of order $\omega_q$), Eq. (3) can be simplified to

$$\varepsilon_T(q, \omega) \approx 1 + A\left| g_q \right|^2 \frac{2\omega_q}{(\omega^2 - \omega_q^2 + i\delta)} \approx \frac{(\omega^2 - \omega_q^2)}{(\omega^2 - \omega_q^2 + i\delta^2)}$$  \hspace{1cm} (4)

with parameter $A = \frac{1}{4\varepsilon^2(\alpha_v v_F)} > 0$, where $\alpha_v$ is a Bohr radius, $v_F$ is the Fermi velocity and the plasmon-polaron frequency is given by

$$\tilde{\omega}_q \approx \omega_q - A\left| g_q \right|^2 < \omega_q .$$  \hspace{1cm} (5)

Eq. (5) is confirmed by experiment [12] and theory [13]: the plasmon-polaron mode is negatively depolarization-shifted, i.e. it follows roughly the phonon mode in the first BZ, but always at frequencies lower than the phonon mode (see Fig. 2a).

**RAS at interface with topological crystals**

Consider now a thin superconductor film sandwiched between two thick TX = Bi$_2$Se$_3$ slabs, as schematically shown in Fig. 1. We can assume initially that the superconducting film is sufficiently thin (thickness $< \zeta \leq 1/q$), so that RAS is uniformly extended throughout the superconductor. The topological proximity effect, discussed below, can significantly relax this requirement [16]. Then, the effective dielectric function experienced by electrons in the superconductor is given by

$$\tilde{\varepsilon}_{sup}(q, \omega) \approx \varepsilon_{sup} + [\varepsilon_T(q, \omega) - 1],$$  \hspace{1cm} (6)

where $\varepsilon_{sup}$ is the dielectric constant of the bulk superconductor, of order 1 in the required domain of phase space, and the term in the square parentheses is the polarizability of the Dirac surface electrons of the topological Bi$_2$Se$_3$. 

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Let us assume now that phonons of the superconductor control the behavior of the plasmon-polaron, and generalize the formula in Eq. (3) by relaxing the jellium assumption and by including all relevant phonon bands. Then, with \( \epsilon_k \) the electron energy, \( g_{kk'} \) the generalized matrix element for scattering between electronic states \( k' \) and \( k \) through a phonon \( q = (k' - k , \omega_{qp}) \) in the phonon branch \( v \), and \( \delta \to 0^+ \), Eq. (6) with the modified Eq. (3) becomes

\[
\bar{\epsilon}_{sup}(\omega) = \epsilon_{sup} + \bar{A} \frac{1}{N} \sum_{kk'v} |g_{kk'}|^2 \frac{2\omega_{qp}}{(\omega^2-\omega_{qp}^2)} - i\pi \bar{A} \frac{1}{N} \sum_{kk'v} |g_{kk'}|^2 \Delta(\omega - \omega_{qp}) \delta(\epsilon_k) \delta(\epsilon_{k'})
\]

where \( N \) is the overall normalization factor. We recognize the last term in Eq. (7) as \(-i\pi\bar{A}\alpha^2F(\omega)\), where \( \alpha^2F(\omega) \) is the Eliashberg function of the superconductor [17, 18], defined explicitly in Eq. 39 of [19]. This allows us to rewrite Eq. (7) as follows

\[
\bar{\epsilon}_{sup}(\omega) = \frac{\bar{\epsilon}_{sup}(\omega)}{\epsilon_{sup}} = 1 - \kappa \left[ \int_0^\infty \frac{\alpha^2F(\omega)}{\omega^2-\omega^2} d\omega + i\pi\alpha^2F(\omega) \right] = \bar{\epsilon}' + i\bar{\epsilon}''
\]

(8)

where \( \kappa = \bar{A}/\epsilon_{sup} \). Since \( \epsilon_{sup} \) is already included in the unshielded \( \alpha^2F(\omega) \), this renormalization reduces the number of adjustable parameters. The integral in Eq. (8) is the Kramers-Kronig transform of \(-\pi\alpha^2F(\omega)\), and the singular character of the integrand can be eliminated by using the familiar procedure

\[
\int_{\omega_{min}}^{\omega_{max}} \frac{f(\omega)d\omega}{\omega-\omega} = \int_{\omega_{min}}^{\omega_{max}} \frac{f(\omega)-f(\omega)}{\omega-\omega} d\omega + f(\omega) \ln \left| \frac{\omega_{max}-\omega}{\omega_{min}-\omega} \right|
\]

(9)

where the integrand on the right is no longer singular, as long as \( \frac{df}{d\omega} \) is nonsingular at any \( \omega \) in the integration range. Now, Eq. (8) takes the final form

\[
\bar{\epsilon}_{sup}(\omega) = 1 - \kappa \left\{ \alpha^2F(\omega) \ln \left| \frac{\omega_{max}-\omega}{\omega_{min}-\omega} \right| + \int_{\omega_{min}}^{\omega_{max}} \frac{\alpha^2F(\omega) - \alpha^2F(\omega)}{\omega-\omega} + \frac{\alpha^2F(\omega)}{\omega+\omega} d\omega + i\pi\alpha^2F(\omega) \right\}
\]

(10)

We can employ this expression to estimate the critical temperature of the superconductor sandwiched between Bi₂Se₃, but we first need to estimate a renormalized value for \( \kappa \). To do that, we consider Bi₂Se₃ interfacing vacuum. In that case, the phonon spectrum of Bi₂Se₃ controls the physics of the plasmon-polaron. Since this is only an order of magnitude estimate of \( \kappa \), we model the Eliashberg function of Bi₂Se₃ as a step function defined as \( \alpha^2F(\omega) = 1 \) (typical average value), in the range \( \omega_{min} < \omega < \omega_{max} \), and \( \alpha^2F(\omega) = 0 \) otherwise. Then, using Eq. (10) and choosing \( \kappa = 1 \), we obtain the result shown in Fig. 2(b), with \( \omega_q = 5.5 \text{ meV} \), and \( \bar{\omega}_q = 4.3 \text{ meV} \). This is
in quantitative agreement with the experimental result shown in Fig. 2(a) at \( q = 0.53 \, \text{Å}^{-1} \), and overall (on average) agreement with experiment in the entire \( q \) range in the first BZ, where \( \Delta = \frac{\omega_q - \bar{\omega}_q}{\omega_q} \approx 20\% \) (see Fig. 2b). We use in our calculations \( \kappa \) as a variable, but this estimate limits its variation range to order 1.

**\( T_c \) enhancement with RAS**

There is a vast literature devoted to estimating/calculating \( T_c \) of superconductors [17], and the retarded Migdal-Eliashberg theory [18] seems to have survived the test of time, with suggested possible applications to even non-phonon mediated superconductors. We calculate the key Eliashberg function of this theory *ab initio* by employing the method described in [19]. Specifically, we compute the electronic Hamiltonian, phonon dynamical matrix, and electron-phonon matrix elements on a coarse (6 x 6 x 6) wave vector mesh using the Quantum Espresso [20] and EPW [21] codes. The latter generates these quantities in a real space, Wannier representation using the maximally-localized Wannier functions [22]. These data are read into the *elphbolt* program [23], which has been extended to provide a parallel solver for the Migdal-Eliashberg equations. Within *elphbolt*, the electron-phonon matrix elements are interpolated onto a fine (40 x 40 x 40) wave vector mesh. The fully anisotropic and the standard, isotropic versions of the \( \alpha^2 F \) function, as given respectively by Eqs. (19) and (39) in Ref. [19], are then computed. All energy-conserving delta functions are calculated using the analytic tetrahedron method [24].

It is clear from Eq. (7) that \( \alpha^2 F(\omega) \) is screened as is the generalized matrix element \( |g_{kk'}| \), i.e.

\[
\overline{\alpha^2 F(\omega)} = \frac{\alpha^2 F(\omega)}{|\bar{\varepsilon}_{\text{sup}}(\omega)|^2}
\]  

Clearly, RAS occurs for \( |\bar{\varepsilon}_{\text{sup}}(\omega)| < 1 \) and, for \( |\bar{\varepsilon}_{\text{sup}}(\omega)| \ll 1 \), it strongly enhances the screened Eliashberg function. This is the main effect of RAS, and the next step is to calculate the critical temperature \( T_c \) from the screening-renormalized Eliashberg function. We consider here two methods: the simple, yet general scaling method (originally due to Leavens [25]), and the method based on a direct solution of the coupled Eliashberg equations [17,18]. Both methods, in contrast to many others (e.g. [17]), are valid for an arbitrary strength of the electron-phonon mass enhancement parameter \( \lambda = \int_0^{\infty} \frac{\alpha^2 F(\omega)}{\omega} \, d\omega \), required while dealing with RAS. We begin with the
scaling method. This powerful method estimates not $T_c$, but its upper limit $T_{c,max}$. It is based on a proper scaling of the linearized Eliashberg equations which gives the following simple formula

$$T_{c,max} = c(\mu^*) \int_0^\infty \alpha^2 F(\omega) d\omega$$  

(12)

The term $c(x)$ is a slowly, monotonically decreasing function of $x$ (see Ref. 17) and the Coulomb pseudopotential is

$$\mu^* \approx \frac{N(\mu)U}{1+N(\mu)U \ln\left(\frac{\epsilon_F}{\hbar \omega_0}\right)}$$  

(13)

where $\epsilon_F$ is the Fermi energy, and $U$ is the double Fermi surface average of the screened Coulomb potential. Here we notice, that typically $\ln\left(\frac{\epsilon_F}{\hbar \omega_0}\right)$ ranges from 5 to 10, and $N(\mu)U \gg \mu^*$. Thus, one can approximate Eq. (13) with $\mu^* \approx \frac{1}{\ln\left(\frac{\epsilon_F}{\hbar \omega_0}\right)}$, i.e. independent of $\tilde{\varepsilon}_{sup}(\omega)$. The slow effect of the repulsive Coulomb interaction on shielding is reflecting the cancellations upon spectral averaging of the electron-electron interaction, as mentioned in section 2 above. The general formula for $T_{c,max}$, including anti-shielding effect is given in this method by

$$T_{c,max} = c(\mu^*) \int_0^\infty \frac{\alpha^2 F(\omega)}{\tilde{\varepsilon}_{sup}(\omega)} d\omega = c(\mu^*) \int_0^\infty \frac{\alpha^2 F(\omega)}{\epsilon_{sup}(\omega)} d\omega = c(\mu^*) \frac{\alpha^2 F(\omega)}{\epsilon_{sup}(\omega)} d\omega$$  

(14)

where the last integral form results from Eq. (8), where it was identified that $\tilde{\varepsilon}'' = -\kappa \pi \alpha^2 F(\omega)$.

Next, we consider the second, direct method. Here, to calculate $T_c$, we solve directly the Eliashberg equations given by Eqs. 21 and 22 (anisotropic theory), as well as Eqs. 35 and 36 (isotropic theory), of Ref. [19] that relate the mass renormalization ($Z$) and gap ($\Delta$) functions on a Matsubara grid. The electron-phonon coupling function $\lambda$ entering these equations is computed from the $\alpha^2 F$ function. The Eliashberg equations are self-consistently solved at different temperatures. Above the transition temperature, $\Delta$ vanishes. For the isotropic solver, we may choose to (anti)shield the $\alpha^2 F$ function, as shown in Eq. (11). In principle, the same can be done for the anisotropic case. However, that would require calculating the anisotropic dielectric function, which is currently beyond the scope of this work. In this work, we use the fully anisotropic solver to validate our code, while using the isotropic solver for studying the effect of anti-shielding.
Discussion

As examples, we consider the interfaces of Pb, a classic low $T_c$ superconductor, and MgB$_2$, the acknowledged highest $T_c$ BCS-type superconductor, with Bi$_2$Se$_3$. We employ the \textit{ab initio} calculated Eliashberg function $\alpha^2 F(\omega)$, with $\mu^* = 0.1$ (0.16) for Pb (MgB$_2$) [25]. First, by following the procedure to estimate $T_{c,max}$, we obtain the results summarized in Fig. 3(a). This figure shows $T_{c,max}$ versus $\kappa$ for Pb and for MgB$_2$, each interfaced with Bi$_2$Se$_3$. Each curve starts (at $\kappa = 0$) at the corresponding maximum critical temperatures without the RAS effect: $T_{c,max} = 7$ K (Pb), and $T_{c,max} = 43$ K (MgB$_2$). These are close to the experimentally observed values of 7.2 K and 39 K, respectively. The overall shape of these curves seems universal, with $T_{c,max}$ values first steadily increasing, and then rapidly peaking at strongly enhanced levels, which for Pb is about 50 K, and for MgB$_2$ exceeds 260 K (approaching room temperature). At even higher values of $\kappa$, $T_{c,max}$ collapses to very small values. Next, we find the directly-calculated $T_c$ for MgB$_2$ interfaced with Bi$_2$Se$_3$ by solving the Eliashberg equations. The results appear in Fig. 3(a) as solid circles, and are located below the corresponding $T_{c,max}$ line, as required. They also follow qualitatively the behavior of the $T_{c,max}$ line, with $T_c$ increasing with $\kappa$, for $\kappa < 1.3$. For $\kappa = 1.3$ the maximum $T_c$ is $\approx 100$K, which is smaller than the $T_{c,max}$ of $\approx 260$ K. In our calculations, we employ the isotropic (momentum-averaged) version of the Migdal-Eliashberg theory, which is known to underestimate $T_c$, and so for $\kappa = 0$ (absence of RAS), the $T_c$ of 23 K is lower than the experimental result of $T_c = 39$ K. In turn, the corresponding anisotropic calculation yields $T_c = 54$ K, \emph{i.e.} much larger than the experiment. Instead of absolute values of $T_c$, one could use the $T_c$ ratio, relative to the no-screening case. Then, these calculations predict greater than a 4-fold increase of $T_c$ at $\kappa = 1.3$. The corresponding $T_{c,max}$ ratio from the scaling method is greater than 5.

The overall behavior of the curves in Fig. 3(a) can be understood analytically by using the last integral form in Eq. (14), and for simplicity employing Eq. (4), which amounts to a single peak model of the unshielded Eliashberg function. Then, the resulting simple formula for $T_{c,max}$ is

$$T_{c,max} \approx \frac{c(\mu^*)\delta^2}{\pi} \int_0^\infty \frac{d\omega}{(\omega^2-\tilde{\omega}_q^2)^2+\delta^4} = \frac{c(\mu^*)}{2} \frac{1}{(\omega_q^4+\delta^4)^{1/4}} \frac{\cos \left[ \frac{1}{2} \arctan \left( \frac{\delta^2}{\omega_q^2} \right) \right]}{2}$$

where $\omega_q^2 = \omega_q^2 - \kappa$. A normalized plot of Eq. (15) is shown in Fig. 3(b), for $\omega_q^2 = 1$, $c(\mu^*) = 2$, $\delta = 0.17\omega_q$ (red curve), $\delta = 0.3\omega_q$ (green) and $\delta = 0.56\omega_q$ (black). There is a pronounced maximum at $\kappa = \omega_q^2$, the condition for vanishing plasmon-polaron frequency. Also, analogous to
the result shown in Fig. 3(a), the $T_{c,\text{max}}$ versus $\kappa$ dependence in Eq. (15) is asymmetric about this maximum, and $T_{c,\text{max}}$ vanishes for $\kappa \to \infty$.

Figure 4 further explains/confirms this behavior. Fig. 4(a) shows the ab initio-calculated $|\tilde{\varepsilon}_{\text{sup}}(\omega)|^2$ for $\kappa = 1$ and 1.3, as well as $\alpha_2 F(\omega)$ without RAS. Fig. 4(b) shows the corresponding $\alpha_2 F(\omega)$ curves, as well as curves representing $\lambda(\omega) = \int_0^\omega \frac{\alpha_2 F(\omega')}{\omega'} d\omega'$, which for large $\omega$ approaches the electron-phonon mass enhancement parameter $\lambda$ of the Eliashberg theory. There is a strong enhancement of $\alpha_2 F(\omega)$ and $\lambda(\omega)$ with increasing values of $\kappa$. The inset in Fig. 4(a) explains this, by showing an expanded section of the main plot in the regions of RAS, marked yellow. The dramatic peak locations of $\alpha_2 F(\omega)$ for $\kappa = 1$ and 1.3 correlate with the locations of the nearly vanishing $|\tilde{\varepsilon}_{\text{sup}}(\omega)|^2$. For $\kappa > 1.3$, the plasmon-polaron mode disappears, and anti-shielding turns into shielding. This weakens the pairing interaction, as shown also in the context of the model calculation of Eq. (15) above. The above analysis suggests that further dramatic enhancement could be possible if the RAS condition were to occur at higher frequencies, closer to the main Eliashberg peak of the superconductor, located at 55 - 60 meV. This might be facilitated by additional “phonon modifier” layers, as sketched in Fig. 1(c), placed between the topological crystal and the superconductor, and if its phonon spectrum could adjust the plasmon-polaron response so as to achieve this spectral alignment.

As discussed in Section 3 above, the key advantage of the topological plasmon-polaron is that it remains weakly damped at large momentum and temperature. This is a unique feature, unmatched by any conventional plasmonic mode. In addition, these modes benefit greatly from the recently-discovered topological proximity effect [16], observed at the surface of the topological crystal TlBiSe$_3$, coated with superconducting Pb. It was shown there that the topological state of the crystal extends through up to 20 monolayers of the superconductor, without any admixing with this superconductor. This effect is expected to improve the plasmon-polaron penetration into the superconductor films, as well as it could increase the efficiency of the “phonon modifier” layers, discussed in the previous paragraph.

Further possible architectures include natural or engineered bulk SC-TX layered materials, wherein the properties of the superconductor are modulated by the properties of proximate TX. For example, the cuprate superconductors generally consist of hole- or electron-doped conducting CuO$_2$ layers sandwiched by nonconducting layers (e.g. yttrium- or bismuth- oxide). One should
consider the possibility of synthesizing cuprate systems modified to incorporate known TX layers (e.g. metal chalcogenides). Similarly, many molecular organic superconductors are comprised of 2D superconducting layers sandwiched by nonconducting layers, the latter of which might be engineered to have TX character. The same in situ strategy could be applied to MgB$_2$, the known BCS superconductor with very large, relevant phonon frequencies. Such incorporated topological modifications could produce atomic/molecular layers functioning as charge reservoirs as well as providing the $T_c$-enhancing RAS effect. These kinds of systems could facilitate high temperature superconductivity in multiple physical forms, from single crystalline to nanocrystalline / ceramic, so long as the core TX–superconductor–TX character was preserved.

**Conclusions**

In this work, we investigated resonant anti-screening induced in a superconductor film sandwiched between topological crystals which support, at their surfaces, plasmon-polaron collective modes characterized by long life times at very large momentum transfer, facilitating robust nonlocality. These plasmon-polarons (hybrids of plasmon and phonon excitations) can strongly modify the dynamics of the electron-phonon interaction in the superconductor, and can be adjusted for maximum effect via the topological proximity effect, leading to a prediction of significantly enhanced superconductivity. We employed the Eliashberg-Leavens scaling scheme, as well as an ab initio Migdal-Eliashberg approach, to show that resonant anti-shielding can be induced in such a heterostructure, and demonstrate that more than a 5-fold enhancement of $T_c$ can occur in MgB$_2$ under proper choice of coupling parameter $\kappa$, which is related to $\lambda$. Successfully implemented experimentally, this can represent a new path to ultrahigh transition temperature superconductivity at ambient pressure.

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Fig. 1. (a) Superconductor (SC) -topological crystal (TX) superlattice structure designed to exploit the proposed resonant anti-shielding (RAS) effect produced by a surface plasmon-polaron. (b) Expanded view of the superlattice, also indicating the decaying amplitudes of the electric field (dashed lines) produced by the plasmon-polaron mode propagating (yellow arrows) along each interface. (c) Alternate structure containing additional phonon-modifier films. Note that, due to the topological proximity effect, plasmon-polaron modes occur at the interfaces of the modifiers with the superconductor.
Fig. 2. (a) Collective modes of the 2D Dirac electron gas on the surface of a topological crystal Be$_2$Se$_3$ (interpolated from experimental data of Ref. [12]): $\alpha-$mode (blue line), acoustic phonon mode (red line). BZ - Brillouin zone. (b) Calculated dielectric function of the 2D Dirac electron gas using Eq. (10) with a step model of $\alpha^2F$, and assuming $\kappa = 1$. solid line: real part, dashed line: imaginary part.
Fig. 3 (a) Calculated superconducting critical temperatures vs. coupling parameter $\kappa$, using the \textit{ab initio} method by directly solving the Eliashberg equations ($T_c$ for MgB$_2$, circles) and the scaling method ($T_{c,max}$ for Pb, solid line and MgB$_2$, dashed) – all interfaced with Be$_2$Se$_3$. Arrow points to $T_c$ accounting for anisotropic (not averaged-out $q$-dependence) Eliashberg function. (b) $T_{c,max}$ calculated using Eq. (15), representing a simple single peak Eliashberg function model. Parameters chosen are $\tilde{\omega}_q = 1$, $c(\mu^*) = 2$, $\delta = 0.17 \tilde{\omega}_q$ (red curve), $\delta = 0.3 \tilde{\omega}_q$ (green) and $\delta = 0.56 \tilde{\omega}_q$ (black).
Fig. 4  (a) *ab initio* calculated $|\tilde{\varepsilon}_{\text{sup}}(\omega)|^2$ for $\kappa = 1$ (blue line) and 1.3 (purple), as well as $\alpha^2 F(\omega)$ without RAS (red-bold line). (b) *ab initio* calculated $\lambda(\omega)$ (dashed lines) and $\alpha^2 F(\omega)$ (solid) for $\kappa = 0$ (red lines), $\kappa = 1$ (blue), 1.3 (purple).
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