Models of Opinion Dynamics with Random Parametrisation

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Abstract

We analyse a generalisation of the Galam model of binary opinion dynamics in which iterative discussions take place in local groups of individuals and study the effects of random deviations from the group majority. The probability of a deviation or flip depends on the magnitude of the majority. Depending on the values of the flip parameters which give the probability of a deviation, the model shows a wide variety of behaviour. We are interested in the characteristics of the model when the flip parameters are themselves randomly selected, following some probability distribution. Examples of these characteristics are whether large majorities and ties are attractors or repulsors, or the number of fixed points in the dynamics of the model. Which of the features of the model are likely to appear? Which ones are unlikely because they only present as events of low probability with respect to the distribution of the flip parameters? Answers to such questions allow us to distinguish mathematical properties which are stable under a variety of assumptions on the distribution of the flip parameters from features which are very rare and thus more of theoretical than practical interest. In this article, we present both exact numerical results for specific distributions of the flip parameters and small discussion groups and rigorous results in the form of limit theorems for large discussion groups. Small discussion groups model friend or work groups – people that personally know each other and frequently spend time together. Large groups represent scenarios such as social media or political entities such as cities, states, or countries.

Keywords: opinion dynamics, hierarchical voting, contrarianism, random parameters, limit theorems

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1 Introduction

Sociophysics is the study of social phenomena by means of methods developed for the study of the physical world. Among the topics of interest in sociophysics lies the study of opinion dynamics (see e.g. [8, 7, 32]). One of the earliest contributions to the field was the Galam model of opinion dynamics first introduced in [14]. Aside from the Galam model, the situations in which opinion dynamics have been analysed by sociophysics is highly varied: both the time component and the opinion space, i.e. the space of all possible opinions an individual can hold, can be discrete or continuous. For continuous opinion spaces, see the Friedkin-Johnsen model [13] and the bounded confidence models [10]. Beyond sociophysics, opinion dynamics has been studied by scientists from other fields, too: both psychologists and mathematicians have made contributions. Some early contributions of social psychologists are [3] [1] [11]. Recent articles by mathematicians studying opinion dynamics models are [26] [2] [5], which provide mathematically rigorous results about the models involved.

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The Galam model features a binary choice facing each member of a large population. Discussions occur in small groups of size $r \in \mathbb{N}$ which model groups of friends or work colleagues. Once the discussion finishes, all members of the group adopt the majority opinion. Then the population forms new discussion groups, and a new round of discussions takes place with the adoption of the ‘local’ i.e. discussion group majority at the end of the round. We denote by the number $p_0 \in [0, 1]$ the proportion among the population preferring one of the alternatives, say alternative $A$, before any discussions take place, and by $p_t$, $t \in \mathbb{N}$, the same proportion after discussion number $t$ and the adoption of the local majority has taken place. Then the dynamics of the Galam model can be analysed by determining the long term behaviour of the proportion $p_t$, more precisely whether $\lim_{t \to \infty} p_t$ exists and if it does what its value is. The same mathematical formalism can be used to study the different but related scenario of binary voting in a hierarchy of individuals comprising several levels. Each level features groups of $r$ individuals who elect a representative for the next higher level who votes for the majority opinion in the group. Instead of repeated discussions taking place with the population subdivided into several small groups, here we determine which alternative wins, i.e. the winning opinion at the top of the hierarchy. So instead of the time dimension, we have a certain number of levels $n \in \mathbb{N}$ in the hierarchy. Fixing $r$ and $n$ implies the number of voters in the population is $r^n$. Similarly to the dynamics of $p_t$ in the opinion model, here we have $p_k$ for the proportion of voters on level $k \in \{1, \ldots, n\}$ of the hierarchy in favour of $A$, and we are interested in the behaviour of $p_n$.

The main distinction concerning the long term dynamics is whether polarisation takes place, i.e. whether there is a tendency to a tie, leaving the society in a state of conflict, or whether there is a tendency towards a macroscopic majority in favour of one of the alternatives. These long term dynamics are easy to determine for the Galam model: suppose there is an initial majority in favour of one of the alternatives. Then the long term tendency is towards a unanimous majority in favour of the same alternative. Convergence is faster, the larger the size $r$ of the discussion groups is.

To make the model produce qualitatively different behaviour, a number of generalisations have been proposed. Using different versions of this model, attempts have been made to predict the outcome of elections and referenda in the works [28, 18, 19]. One of the generalisations of the basic model is the introduction of a tendency to contrarianism, which manifests as the possibility of rejecting the local majority in favour of the contrary alternative. These models were first studied in [15, 16]. Contrarian tendencies have been studied extensively by other authors using different models, see e.g. [33, 4, 20, 27, 21].

In this tradition of generalisations of the Galam model of opinion dynamics, the so called local flip model with flips against the majority where the likelihood of the flip depends on the magnitude of the majority was introduced in [35]. This model allows us to study scenarios where individuals are guided by the opinions of the others in their local groups – albeit not necessarily in a positive sense. It could be the case that small majorities are not particularly persuasive: individuals may not accept such close outcomes as sufficiently authoritative. Or it could be that particularly large, close to unanimous, majorities are distasteful to some individuals and induce resistance to the majority decision. These two scenarios were called the ‘vertical’ and ‘horizontal frame’, respectively, in [35]. These are aspects which could not be explored using previous models of binary opinion dynamics, and they shed some new light on several social phenomena triggered by one or a few individuals acting against larger local majorities.

Let $r \in \mathbb{N}$ be the size of the local discussion groups. Given a current proportion $p \in [0, 1]$ of the overall population favouring alternative $A$, the proportion in favour after the next round of discussions is given by the update function. The general update function of the local flip model is

$$R_{r, a}(p) = \sum_{i=r+1}^{r} \binom{r}{i} \left[ (1 - a_i) p^i (1 - p)^{r-i} + a_i p^{r-i} (1 - p)^i \right].$$

(1.1)
The vector $\mathbf{a} = (a_{\frac{r+1}{2}}, \ldots, a_r)$ contains all the flip parameters of the model. The parameter $a_i$ gives the probability that the majority is not adopted given that the majority is of size $i \in \left\{ \frac{r+1}{2}, \ldots, r \right\}$. When $i$ voters are in favour of $A$, and we encode each of these votes as a +1 and each vote for $B$ as a −1, the sum of all votes, called the voting margin, is $S = i - (r - i) = 2i - r$. The model is symmetric in that a flip against the local majority is equally likely no matter the alternative the majority favours. Thus, the flip parameters are chosen in such a way that they only depend on the absolute voting margin $|S| = |2i - r|$ but not the sign of $S$. This symmetry implies that the point $p = 1/2$, which signifies a tie between the two alternatives, is a universal fixed point in the dynamics of the model. The local flip model is the most general model of binary opinion dynamics with random deviations from the majority that is symmetric with respect to the two alternatives.

To make the local flip model more accessible than the mere statement of the general update function accomplishes, we describe the simplest case where the local discussion groups are of size 3. Then the model has only two flip parameters: $a_2$ is the probability that a flip against the group majority takes place conditional on there being a 2-1 majority, and $a_3$ is the probability of a flip if there is a unanimous majority. Table 1 describes the process by which the opinions of the members of a local discussion group are updated after discussion ends. The information is read as follows: in the first two rows of the table, we have a unanimous configuration of opinions in favour of $A$ in the local discussion group. The probability that a randomly selected local discussion group of size 3 has the configuration $AAA$, given that there is a proportion $p$ of the overall population in favour of $A$, is $p^3$. In the basic model, the majority would be adopted. However, in the local flip model, there is a probability that this unanimous majority is not adopted given by $a_3 \in [0, 1]$. Thus, we obtain a probability of a group vote in favour of $A$ given by $(1 - a_3)p^3$ and a probability of going for $B$ given by the complementary $a_3p^3$. The updated overall proportion of individuals in favour of $A$, $R_{3,a}(p)$, is obtained by summing the probabilities in the rows where $A$ is adopted.

The aforementioned basic model and the contrarian model introduced in [15, 16] are both special cases of the local flip model. The basic model corresponds to the assumption $\mathbf{a} = 0$, meaning there are no flips against

| Configuration | Group vote | Probability |
|---------------|------------|-------------|
| AAA           | $A$        | $(1 - a_3)p^3$ |
|               | $B$        | $a_3p^3$    |
| $AAB \cdot 3$ | $A$        | $(1 - a_2) \cdot 3p^2 (1 - p)$ |
|               | $B$        | $a_2 \cdot 3p^2 (1 - p)$    |
| $ABB \cdot 3$ | $A$        | $a_2 \cdot 3p (1 - p)^2$    |
|               | $B$        | $(1 - a_2) \cdot 3p (1 - p)^2$ |
| $BBB$         | $A$        | $a_3 (1 - p)^3$             |
|               | $B$        | $(1 - a_3) (1 - p)^3$       |

Table 1: Local Flip Model, $r = 3$
the majority under any circumstances. The contrarian model features flat flip probabilities \( a = (a, \ldots, a) \) for some \( a \in [0,1] \). Both of these models, being special cases of the local flip model, share the universal fixed point \( p = 1/2 \). The basic model also has the fixed points 0 and 1. These features lead to the convergence to unanimous majorities in the basic model described above.

Instead of assuming some fixed vector of flip parameters \( a \) as in [35] and analysing the properties of the model under that assumption, we investigate in this article how likely some of the features of the model are when the parameters are randomly selected. Therefore, in this article, \( a \) is a random vector that follows some probability distribution. As a consequence, features of the model, such as the stability parameter

\[
\lambda_r (a) := R_{r,a} (1/2) = \frac{1}{2^{r-1}} \sum_{i=\lfloor \frac{r}{2} \rfloor}^r \binom{r}{i} (2i - r) (1 - 2a_i)
\]

(1.2)

of the universal fixed point 1/2, will also be random variables. Other features, such as the presence of unanimous attractors will be events of the probability space on which \( a \) is defined. Hence, it makes sense to ask ourselves the question of how likely these events are, or how likely it is that the universal fixed point 1/2 is stable.

There are two possible interpretations of random flip parameters:

1. Different issues to be discussed induce tendencies to adopt the majority opinion or to resist it. In some cases, there may be a strong tendency to align with the majority, and the flip parameters will be close to 0. Other issues may trigger strong resistance against the majority opinion, e.g. if the members of the majority are perceived to be very arrogant or intolerant of divergent opinions. We consider that for each possible issue there is some realisation of the flip parameters \( a \) that affects the dynamics of the model. Thus, if we study the local flip model with random flip parameters, we gain a big picture understanding of how discussions take place and opinions are shaped averaged over a large number of possible discussion topics.

2. We can also consider the mathematical question of what the ‘typical model’ looks like. We know that the model exhibits a variety of features, such as different numbers of fixed points with differing stability properties, depending on the flip parameters, but how likely are these different properties to occur? It may turn out that some features – although possible for special values of the flip parameters – are exceedingly unlikely to occur under random flip parameters. In that case, these features are more of theoretical than practical interest. To make this distinction is especially important given the great generality of the local flip model, which leads to an ‘anything goes’ situation.

This article is organised in four sections and an appendix. After this introduction, Section 2 analyses features of the local flip model with random flip parameters when the local discussion groups are small. This setting allows us to assume a specific distribution of the flip parameters and calculate the probability that the model has certain characteristics explicitly. Afterwards, in Section 3 we analyse properties of the local flip model with random flip parameters when the discussion groups are large. Section 4 presents the conclusions we have reached. Finally, the Appendix contains the proofs of the theorems presented in the article.

# 2 Small Local Discussion Groups

In this section, we analyse properties of the versions of the local flip model discussed in [35] under the assumption that the flip parameters of the model are independent and uniformly distributed on the interval
dynamics is delimited by the inequalities 1 − / 3. The dynamics are given by the inequalities 1 − / 3 < a < b < 1. The other flip parameter, a3, describes the probability of disregarding an unanimous 3-0 majority. We will use the notation a2 = a and a3 = b as in [35] and assume a uniform distribution, i.e. the random vector (a, b) is uniformly distributed on the set [0, 1]2, which we write as (a, b) ∼ U [0, 1]2.

In order to determine the behaviour of the model for different values of the flip parameters a and b, the stability of the universal fixed point 1/2 is calculated. There are four different regions in the parameter space of this model [0, 1]2 with differing stability properties: in the region L given by the inequality b < 1/3 − a, 1/2 is unstable and the dynamics are monotonically away from 1/2. The stable region M1 with monotonic dynamics is given by the inequalities 1/3 − a < b < 1 − a. The other stable region M2 with alternating dynamics is delimited by the inequalities 1 − a < b < 5/3 − a. Finally, there is an unstable region H with alternating dynamics which lies above b = 5/3 − a. A graphical representation of these results can be found in Figure 1.

Having identified these regions, we can calculate the probability that the fixed point 1/2 is stable or unstable: P {1/2 is stable} = 8/9, P {1/2 is unstable} = 1/9. Under the uniform distribution U [0, 1]2, these probabilities are simply the areas of the respective regions in the parameter space [0, 1]2.

The other aspect analysed in [35] was the number of fixed points. There are three regions: the region in which every value p ∈ [0, 1] is a fixed point consists of a single point, F∞ := {(1/3, 0)}. The region F1 with only a single fixed point which is 1/2 is given by the inequalities 1/3 − a ≤ b and b > 0. The region with three different fixed points is the complement F3 := [0, 1]2 \ (F∞ ∪ F1), i.e. the corner region around the origin and the a-axis excluding the point: (1/3, 0).

Thus, the probabilities of the number of fixed points taking the values 1, 3, and ∞ are, respectively, P {there is 1 fixed point} = 17/18, P {there are 3 fixed points} = 1/18, and P {there are infinitely many fixed points} = 0.

As shown in the appendix A.2 of [35], for any group size r the unanimous majority points 0 and 1 are fixed points if and only if ar = 0. Furthermore, in case 0 and 1 are fixed points, they are stable if and only if a_r−1 ≤ 1/r. The event of having unanimous attractors for group size r = 3, i.e. the points 0 and 1 being fixed points and attractive, has probability 0: P {unanimous attractors} ≤ P {a_r = 0} = 0.

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1For any x ∈ R, the Dirac distribution or point mass at x is defined as δ_x A := \begin{cases} 1, & \text{if } x ∈ A \\ 0, & \text{if } x \notin A \end{cases} for all subsets A of R.
2.2 Group Size $r = 5$

We consider the same models as in Section 4.2 of [35] with the same notation: $a_3 = a$, $a_4 = b$, and $a_5 = c$. Each of these models has some constraint on the possible values of the flip parameters. Similarly to the discussion group size $r = 3$, we can conduct an analysis of the model’s behaviour. See Section 4.2 of [35] for the precise results, which, due to length constraints on this article, we choose to omit. We present a graphical representation of the version of the model with $c = 0$, i.e. there are no flips against unanimous majorities, in Figure 2.

The probabilities of a number of properties of four versions of local flip models analysed in Section 4.2 of [35] can be found in Table 2. These are exact figures with no rounding applied. Due to the uniform distribution we have assumed for the flip parameters, the probabilities presented in Table 2 are the areas of the corresponding regions in the parameter space of the respective version of the model. E.g., for the model with $(a, b) \sim \mathcal{U} [0, 1]^2$, $c = 0$ presented in the last column in Table 2, the probabilities can be calculated directly from the diagrams in Figure 2.

As we see, under uniform distribution of the flip parameters, there generally is a small number of fixed points.
Specifically, the probability that a full complement of \( r \) fixed points exists is small for both \( r = 3 \) and \( r = 5 \), with the one exception of the distribution \((a, b) \sim \mathcal{U}[0, 1]^2, c = 0\). This exception is reached by setting the flip parameter for unanimous majorities to the only value that allows for unanimous attractors. This is thus not a typical result. In line with the generally small number of fixed points, the unanimous majorities are rarely attractive fixed points. This is thus a stable result even under randomly selected flip parameters which clearly distinguishes the local flip model from the basic Galam model of opinion dynamics, which features unanimous attractors for any local discussion group size.

As for the universal fixed point \( 1/2 \), we observe mostly stable fixed points. However, there are some exceptions when the right set of constraints is imposed on the flip parameters. For \( r = 3 \) and unrestricted flip parameters, we have a high probability of a stable fixed point \( 1/2 \). Similarly, for \( r = 5 \) and \((a, b) \sim \mathcal{U}[0, 1]^2, c = 0\), we have a stable fixed point. On the other hand, if we set both \( b \) and \( c \) equal to 0, i.e. we only allow flips when the majority is paper thin, an unstable fixed point \( 1/2 \) is more likely than a stable one.

We could try using a different distribution for the flip parameters to illustrate that these conclusions depend on the particular distribution chosen. Due to the somewhat arbitrary nature of such an undertaking, we choose instead to investigate the case of large update group sizes, which allows more robust conclusions which do not require assuming specific distributions to obtain results.

### 3 Large Local Discussion Groups

In this section, we present results concerning mainly large local discussion groups, which is understood to be the limit as \( r \) goes to infinity, although some of the following results hold for all values of \( r \). Contrary to the last section, which was about small discussion groups, such as friend or work groups which are typically in the single digits, here we are concerned with structures such as social media, where large groups of individuals discuss topics – frequently in a very controversial manner. Another scenario would be identifying the discussion groups with the states of a federal republic. As the overall population becomes very large, we have either a bounded number of local discussion groups or else the number of discussion groups also grows without bound but more slowly than the overall population.
3.1 Unanimous Attractors

For any \( r, 0 \) and 1 are fixed points if and only if \( a_r = 0 \) (see appendix A2 of [35]). So \( \mathbb{P}\{0, 1 \text{ fixed}\} = \mathbb{P}\{a_r = 0\} \) depends only on the marginal distribution of the single parameter \( a_r \). How likely is it that 0 and 1 are fixed points and they are attractors? This probability is given by

\[
\mathbb{P}\{0, 1 \text{ stable and } 0, 1 \text{ fixed}\} = \mathbb{P}\{a_r - 1 \leq 1/r \text{ and } a_r = 0\}.
\]

We consider two cases which represent the extremes among the possible distributions of \( a \):

1. \( a_i \) are stochastically independent. Then

\[
\mathbb{P}\{0, 1 \text{ stable and } 0, 1 \text{ fixed}\} = \mathbb{P}\{a_{r-1} \leq 1/r\} \mathbb{P}\{a_r = 0\}.
\]

2. All \( a_i \) are equal almost surely. This is the case of the contrarian model with a flat flip probability \( a \). Then

\[
\mathbb{P}\{0, 1 \text{ stable and } 0, 1 \text{ fixed}\} = \mathbb{P}\{a = 0\}.
\]

We conclude that unanimous attractors, which is a feature of the basic model of opinion dynamics with no local flips, is relatively unlikely to occur in both of these extreme scenarios. To find a stochastic model for the flip parameters which makes unanimous attractors more likely, we would have to look for some joint distribution that lies somewhere in between the two extremes, tailoring it specifically to achieve the desired outcome.

3.2 Stability of the Fixed Point \( p = 1/2 \)

In Figure 3, we see the update function (1.1) of the basic model, i.e. when \( a = 0 \). The stability parameter \( \lambda_r(0) \) of the universal fixed point defined in (1.2) is the slope of each curve at \( p = 1/2 \). The diagram illustrates that as the size of the local discussion group increases, so does \( \lambda_r(0) \). It was proved in [17] (see p. 27), that
as \( r \) goes to infinity, \( \lambda_r (0) \) behaves asymptotically like the expression \( \sqrt{\frac{2r}{\pi}} \). So the fixed point 1/2 becomes more and more unstable, with the dynamics being monotonic for any value of \( r \), meaning that if the initial proportion \( p_0 \) of the overall population in favour of \( A \) starts close by 1/2 – but not exactly at a tie – then for large local discussion groups, the dynamics tend away from a tie to a unanimous majority in favour of the initially favoured alternative, and convergence is faster the larger \( r \) is.

We start our investigation of the stability of the fixed point 1/2 by considering the two cases treated in the section about unanimous attractors: independent flip parameters and flat flip parameters. Afterwards, we investigate what happens in between these two extremes using a model of ferromagnetism that features different degrees of correlation between binary random variables.

### 3.2.1 Independent \( a_i \)

Let \( (a_i)_{i \in \mathbb{N}} \) be an infinite sequence of stochastically independent and identically distributed random variables. To avoid dealing with the trivial case that there is no randomness in the model, which is the case if the variance \( V \) and \( \lambda_r \) are equal to 1/2, then the sign of \( \mathbb{E} \lambda_r (a) \) is given by the sign of \( 1 - 2 \mathbb{E} a_i \). However, this in of itself is of limited usefulness. The entire distribution of the stability parameter matters. Let us start with \( \mathbb{V} \lambda_r (a) \), the variance of \( \lambda_r (a) \).

Its typical magnitude will turn out to be smaller than \( r \). This implies that the stability parameter under the present assumptions scales more slowly as \( r \) goes to infinity compared to the deterministic basic model. We have the following limit theorem from which the previous claim follows. In the statement of the limit theorems in this article, the symbol \( \frac{d}{r \to \infty} \) stands for convergence in distribution as \( r \) goes to infinity. For the remainder of this article, we will be using the usual notation for asymptotic expressions.

**Theorem 1.** Let \( (a_i)_{i \in \mathbb{N}} \) be a sequence of independent and identically distributed random variables with support in \([0, 1]\) and \( \mathbb{E} a_i \neq 1/2 \) for each \( i \in \mathbb{N} \). Then the sequence of random variables \( (\lambda_r (a)) \) defined in \((1.2)\) and normalised by \( \sqrt{r} \) converges in distribution. More specifically,

\[
\frac{\lambda_r (a)}{\sqrt{r}} \xrightarrow{d} r \to \infty \delta \sqrt{\frac{1}{\pi} (1 - 2 \mathbb{E} a_1)}
\]

and \( \mathbb{V} \lambda_r (a) = \Theta (\sqrt{r}) \).

On the other hand, if the expectations \( \mathbb{E} a_i \) are equal to 1/2, then \( \lambda_r (a) \) is a centred random variable. The fixed point 1/2 is therefore ‘on average stable’. However, this does not imply that a stable fixed point is

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2Let \( (a_n)_{n \in \mathbb{N}} \) and \( (b_n)_{n \in \mathbb{N}} \) be sequences in the real numbers. Then we will write

\[
\begin{align*}
  a_n &= o (b_n) \quad \text{if} \quad \lim_{n \to \infty} \frac{a_n}{b_n} &= 0, \\
  a_n &= \Theta (b_n) \quad \text{if} \quad \liminf_{n \to \infty} \frac{a_n}{b_n} > 0 \quad \text{and} \quad \limsup_{n \to \infty} \left| \frac{a_n}{b_n} \right| < \infty, \\
  a_n &\approx b_n \quad \text{if} \quad \lim_{n \to \infty} \frac{a_n}{b_n} = 1.
\end{align*}
\]

Informally, \( a_n = o (b_n) \) means \( a_n \) is ‘asymptotically smaller’ than \( b_n \), and \( a_n = \Theta (b_n) \) means \( a_n \) and \( b_n \) are ‘asymptotically of the same order’. \( a_n \approx b_n \) is a stronger condition than being of the same order. It means we can asymptotically substitute \( b_n \) for \( a_n \).
typical behaviour for this model under these assumptions. A better description of the typical behaviour is
given by its variance, which is of order $\Theta (\sqrt{r})$. The formal result is the following limit theorem. Below and
in the rest of this article, $\mathcal{N}(\mu, \sigma^2)$ stands for the normal distribution with expectation $\mu \in \mathbb{R}$ and variance
$\sigma^2 \geq 0$. The special case $\sigma = 0$ can be identified with $\delta_\mu$.

**Theorem 2.** Let $(a_i)_{i \in \mathbb{N}}$ be a sequence of independent and identically distributed random variables with support
in $[0, 1]$ and $Ea_i = 1/2$ for each $i \in \mathbb{N}$. We define the constants

$$b_{ri} := \binom{r}{i} \frac{2i - r}{2^{r-1}},$$

for all $r$ and all $i \in \{\frac{r+1}{2}, \ldots, r\}$ and the sequence $s_r := \sqrt{\sum_{i=\frac{r+1}{2}}^{r} b_{ri}^2}$. Asymptotically, $s_r$ is of order $\Theta (r^{1/4})$.

The sequence of random variables $(\lambda_r (a))$ defined in (1.2) and normalised by $s_r$ converges in distribution to
a centred normal distribution. Specifically,

$$\frac{\lambda_r (a)}{s_r} \xrightarrow{d} \mathcal{N}(0, 4(1 - 2Ea_1)).$$

Also, $\forall \lambda_r (a) = \Theta (\sqrt{r})$.

We can summarise our findings regarding independent flip parameters as follows:

1. If for each $i \ Ea_i \neq 1/2$, then we have a bias towards either going along with the majority or rejecting
   it. The high flip probability regime, given by $Ea_i > 1/2$, leads to a negative stability parameter. This
   means there are alternating dynamics of the majorities around the fixed point 1/2, repeatedly switching
   between a majority in favour of alternative $A$ and a majority in favour of $B$. If $Ea_i < 1/2$ for each $i$,
   then the dynamics around the fixed point 1/2 are monotonic. The typical magnitude of the stability
   parameter scales with $r$ and is of order $\Theta (\sqrt{r})$. This is the same magnitude as in the basic model
   with all flip parameters equal to 0, but keep in mind that in the basic model, the sign of the stability
   parameter is always positive, whereas here the sign is determined by $Ea_i$.

   Informally, we can say that the typical realisation of the stability parameter can be expressed as
   $\lambda_r (a) = \sqrt{\frac{2r}{\pi}} (1 - 2Ea_1) \pm \Theta (r^{1/4})$.

2. If $Ea_i = 1/2$ for each $i$, the stability parameter is a centred random variable with equal probabilities
   of being positive or negative. Its fluctuations are described by a normal distribution whose variance is
   proportional to the variance of each flip parameter, and their magnitude is of order $\Theta (\sqrt{r})$.

   Informally, the typical magnitude of the stability parameter is given by the expression $\lambda_r (a) =
   \pm \Theta (r^{1/4})$.

In both cases, the fixed point 1/2 becomes unstable as the discussion groups grow large, meaning there is
a tendency towards majorities in favour of one of the alternatives, albeit the dynamics can be alternating
which implies there is no convergence of the distribution of opinions towards a limit. Therefore, independent
and identically distributed flip parameters do not give rise to a tendency towards ties in the dynamics of
the model. Under independence of the flip parameters, we would have to relax the identical distribution
assumption and tailor the distributions of the $a_i$ to obtain a stable fixed point 1/2 and thus a tendency
towards ties.
3.2.2 Flat Flip Probabilities

The other extreme is when there is perfect positive correlation between all flip parameters as in the contrarian model. Let \( a \) be the random variable which gives the value for each flip parameter. Then, as \( r \) grows large, the probability that we observe a model with a stable universal fixed point \( 1/2 \) is given by \( \mathbb{P}\{ a = 1/2 \} \), as only a realisation of \( a = 1/2 \) will make \( 1/2 \) stable. All other values of the flip parameters will lead to a highly unstable fixed point, either with alternating dynamics if \( a > 1/2 \) or monotonic dynamics if \( a < 1/2 \). So we can explicitly calculate the probabilities of certain properties of the contrarian model, similarly to the case of small \( r \) treated above.

3.2.3 Correlated Flip Parameters

In between the two extremes of stochastic independence (i.e. all flip parameters vary in value without any correlation to each other) and the case of perfect correlation in which all flip parameters assume the same value, we want to analyse a range of distributions for positively correlated flip parameters. We cover the range of very weakly correlated to strongly correlated flip parameters. The latter we understand as the situation where the typical sum \( \sum_{i=1}^{n} a_i \) is of order \( r \), or ‘macroscopic’, whereas in the former case that same sum would be \( o(r) \) with high probability.

To obtain varying degrees of positive correlation between the flip parameters, we employ the Curie-Weiss model (CWM) of ferromagnetism, as it allows us to observe three different regimes of correlations while also being amenable to analytic solutions at least asymptotically. The CWM describes a set of elementary magnets (or ‘spins’) that tend to align with each other. The two possible states for each magnet are encoded in the random variables as the values \(-1, 1\). The CWM was first applied to the study of problems in the social sciences in [6]. Since then, there have been several articles using the CWM to explore some social or economic phenomenon (see e.g. [24, 9, 30, 29, 34, 25]).

Let \( n \in \mathbb{N} \) be the number of random variables \((X_1^{(n)}, \ldots, X_n^{(n)})\). The CWM is defined by the so called ‘canonical ensemble’, the probability of each configuration \( x \in \{-1, 1\}^n \):

\[
\mathbb{P}\left(X_1^{(n)} = x_1, \ldots, X_n^{(n)} = x_n\right) = Z^{-1} \exp\left(\frac{\beta}{2n} \sum_{i=1}^{n} x_i\right),
\]

where \( \beta \geq 0 \) is the inverse temperature parameter and \( Z \) is a normalisation constant. \( \beta \) regulates the interaction strength between the random variables. For positive \( \beta \), the configurations with the highest probability are \((1, \ldots, 1)\) and \((-1, \ldots, -1)\). On a technical note, contrary to the case of independent random variables where we can assume that there is a single infinite sequence of independent random variables, of which we are free to take the first \( n \) variables, there is no such infinite sequence of Curie-Weiss random variables. Instead, we have a separate model with \( n \) variables for each \( n \in \mathbb{N} \). In order to emphasize this difference, we included the superindex \((n)\) in the notation.

The CWM has three regimes of distinct behaviour:

1. For \( \beta < 1 \), the interaction between the \( X_i^{(n)} \) is weak.
2. \( \beta = 1 \) is a critical point with its own distinct behaviour.
3. For \( \beta > 1 \), the interaction is strong.
Each of these descriptions can be made precise in terms of the limiting distribution of the so called magnetisation $\sum_{i=1}^{n} X_{i}^{(s)}/n$ as $n$ goes to infinity. We note that for $\beta = 0$ the variables $X_{i}^{(s)}$ are independent. Thus, we have a special case of our assumptions in Section 3.2.1.

Now re-index the random variables $X_{i}^{(s)}$ such that the index $i$ runs from $\frac{1}{2} + 1$ to $r$ instead of 1 to $n$. We will write $X_{i}^{(r)}$ for each $r$ and each $i$. As $1 - 2a_{i}$ takes values in the interval $[-1, 1]$, we can identify $1 - 2a_{i}$ with $X_{i}^{(r)}$ and obtain a stochastic model of the flip parameters with

$$\lambda_{r} (a) = \frac{1}{2^{r-1}} \sum_{i=\frac{1}{2}+1}^{r} \binom{r}{i} (2i - r) X_{i}^{(r)}.$$

However, due to the binary nature of the random variables $X_{i}^{(r)}$ we only get the two levels of flip probabilities $X_{i}^{(r)} = 1 = 1 - 2a_{i}$, which is equivalent to $a_{i} = 0$, and $X_{i}^{(r)} = -1$, which is equivalent to $a_{i} = 1$. This all-or-nothing setup is somewhat extreme. It only considers the possibilities of either respecting the majority when $a_{i} = 0$ or flipping with probability 1. Hence, we introduce a scale $s \in (0, 1)$, which is constant, and we identify $1 - 2a_{i}$ with $sX_{i}$, thus obtaining two possible levels of flip probabilities $\frac{1+s}{2}$ and $\frac{1-s}{2}$. So there is the possibility of a low flip probability and a high flip probability, both of which lie in $(0, 1)$. The stability parameter is therefore

$$\lambda_{r} (a) = \frac{s}{2^{r-1}} \sum_{i=\frac{1}{2}+1}^{r} \binom{r}{i} (2i - r) X_{i}^{(r)}.$$

Note that $Ea_{i} = 1/2$ holds for all $i$, and hence $E\lambda_{r} (a) = 0$.

The results for this setup are as follows:

1. For $\beta = 0$, the model behaves just as outlined in Section 3.2.1. The stability parameter $\lambda_{r} (a)$ normalised by a term of order $r^{1/4}$ tends to a normal distribution.

   For $0 < \beta < 1$, there is positive correlation between the flip parameters $a_{i} = \frac{1-sX_{i}^{(r)}}{2}$. However, this correlation is relatively weak, and the stability parameter behaves as for independent flip parameters.

2. If $\beta = 1$, then there is stronger correlation between the flip parameters than for $\beta < 1$. However, this increase in correlation is not reflected in the typical magnitude of $\lambda_{r} (a)$. We once again find a typical magnitude of order $\Theta (r^{1/4})$ with equal probabilities of $\lambda_{r} (a)$ being positive or negative. It is only the multiplicative constant in the term $\Theta (r^{1/4})$ which differs. See Theorem 3 for more details.

3. For $\beta > 1$, the correlation between the $a_{i}$ is stronger than for $\beta = 1$. Now we see this reflected in the typical magnitude of $\lambda_{r} (a)$ which is of order $\Theta (\sqrt{r})$. Thus, we conclude that the strong correlation between the flip parameters induces the same order of instability of the fixed point as we observe in the basic model. However, due to the expectation of each flip parameter being 1/2, we once again have equal probabilities of observing monotonic dynamics around 1/2 or alternating dynamics.

As we see, using the CWM for the flip parameters, we cover the cases of $\lambda_{r} (a)$ being typically of order $\Theta (r^{1/4})$ similarly to the case of independent flip parameters and $\lambda_{r} (a)$ being typically of order $\Theta (\sqrt{r})$ which is the same magnitude as in the basic model. The stronger the correlation between the flip parameters, the more unstable the fixed point 1/2 becomes. This means for strongly correlated flip parameters, there is a pronounced tendency to a large majority in favour of one of the alternatives, or else there is oscillating

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behaviour that alternates between majorities in favour of each of the alternatives in turn. For weak correlation, signified by $\beta \leq 1$, we obtain an unstable fixed point $1/2$, where the magnitude scales more slowly than for the strong correlation regime $\beta > 1$. Note that similarly to the case of independent flip parameters with $\mathbb{E}a_i = 1/2$, the CWM also yields a random sign of $\lambda_r(\mathbf{a})$ with each sign having probability $1/2$.

Formally, we have the following limit theorem:

**Theorem 3.** Let for each $r$, $(X_{i}^{(r)})_{\frac{r+1}{2} \leq i \leq r}$ be $\{-1, 1\}$-valued random variables with joint distribution given by (3.1), $s \in (0, 1)$, and let $a_i := \frac{1-s X_{i}^{(r)}}{2}$ for each $i \in \{ \frac{r+1}{2}, \ldots, r \}$. We define the constants

$$b_{ri} := s \left( \begin{array}{c} r \\ i \\ \end{array} \right) \frac{2i - r}{2^{r-1}}$$

for all $r$ and all $i \in \{ \frac{r+1}{2}, \ldots, r \}$ and the sequence $s_r := \sqrt{\sum_{i=\frac{r+1}{2}}^{r} b_{ri}^2}$. Asymptotically, $s_r$ is of order $\Theta(r^{1/4})$. We have the following limiting distributions:

1. If $\beta < 1$, then
   $$\frac{\lambda_r(\mathbf{a})}{s_r} \xrightarrow{d} \mathcal{N}(0, 1).$$

2. If $\beta = 1$, then $\frac{\lambda_r(\mathbf{a})}{s_r}$ converges in distribution to a symmetric non-normal distribution.

3. If $\beta > 1$, then
   $$\frac{\lambda_r(\mathbf{a})}{\sqrt{r}} \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{2}(\delta_m + \delta_m)\right),$$
   where $m > 0$ is a constant independent of $r$ and strictly increasing in $\beta$.

Informally, if $\beta \leq 1$, the typical stability parameter value is $\lambda_r(\mathbf{a}) = \pm \Theta(r^{1/4})$. For $\beta > 1$, the typical value is $\lambda_r(\mathbf{a}) = \pm m \sqrt{r} \pm \Theta(r^{1/4})$.

The limiting distributions of (a suitably normalised) $\lambda_r(\mathbf{a})$ broadly follow the same pattern as for the magnetisation

$$S_r := \sum_{i=\frac{r+1}{2}}^{r} X_{i}^{(r)}.$$ 

There are, however, some interesting differences:

1. The limiting distribution of the normalised magnetisation $\sqrt{\frac{2}{r}} S_r$ when $\beta < 1$ is also normal, but the variance depends on the parameter $\beta$ (see Sections IV.4 and V.9 of [12] for a detailed analysis of the CWM):
   $$\sqrt{\frac{2}{r}} S_r \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{1-\beta}\right).$$

The variance is $1$ for $\beta = 0$, i.e. when the $X_{i}^{(r)}$ are independent, and then it increases as $\beta$ grows, diverging to infinity as $\beta$ approaches $1$ from below. This is not the case for $\frac{\lambda_r(\mathbf{a})}{s_r}$, which converges in distribution to a standard normal, independently of the value of $\beta < 1$. Note that $s_r$ does not depend on $\beta$. 

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2. The normalisation required in each regime of the model in order to obtain convergence in distribution is different. For the stability parameter \( \lambda_r(\mathbf{a}) \), we normalise by \( s_r = \Theta\left(\frac{1}{\sqrt{r}}\right) \) in both the regimes \( \beta < 1 \) and \( \beta = 1 \), while \( \beta > 1 \) requires normalisation by \( \sqrt{r} \). The magnetisation \( S_r \), on the other hand, requires normalisation by \( \sqrt{\frac{r}{2}} \), \( \left(\frac{r}{2}\right)^{3/4} \), and \( \frac{r}{2} \), respectively, in each regime. This difference is crucial since it directly affects the typical magnitude of the stability parameter which is different than that of the magnetisation.

For independent flip parameters, we analysed the case that \( \mathbb{E} a_i = 1/2 \), which makes \( \lambda_r(\mathbf{a}) \) a centred random variable, and the case \( \mathbb{E} a_i \neq 1/2 \). We can do the same for flip parameters with a joint Curie-Weiss distribution. The first case was treated above. If we allow an external magnetic field in the model, the random variables \( X_i^{(r)} \) no longer have expectation 0, and thus the flip parameters \( a_i \) will have expectation different than 1/2. The joint distribution is given by

\[
P \left( X_1^{(n)} = x_1, \ldots, X_n^{(n)} = x_n \right) = Z^{-1} \exp \left( \frac{\beta}{2n} \sum_{i=1}^{n} x_i \right)^2 + h \sum_{i=1}^{n} x_i \),
\]

for each configuration \( x \in \{-1, 1\}^n \). Above, the parameter \( h \in \mathbb{R} \) gives the strength of the external magnetic field. We once again re-index the variables such that for each \( r \) the index \( i \) of \( X_i^{(r)} \) runs from \( \frac{r+1}{2} \) to \( r \). In the context of the flip parameters \( a_i = \frac{1-sX_i^{(r)}}{2} \), a positive \( h \) implies a bias towards lower flip probabilities, whereas a negative \( h \) means there is a bias towards higher flip probabilities. Hence, we can regard \( h \) as a parameter that regulates how contrarian the people are.

Under the presence of such a bias, the stability parameter will be of order \( \Theta(\sqrt{r}) \), regardless what the value of the parameter \( \beta \geq 0 \) is. The sign of \( \lambda_r(\mathbf{a}) \) will be the same as that of \( h \). Thus, for \( h > 0 \) which is low contrarianism, we obtain a model which is qualitatively similar to the basic model, featuring an unstable fixed point 1/2 with monotonic dynamics. A value \( h < 0 \) which indicates high contrarianism, on the other hand, yields an unstable fixed point with alternating dynamics. The magnitude of the stability parameter will be \( \Theta(\sqrt{r}) \) for all \( h \neq 0 \) and all \( \beta \geq 0 \).

The formal result is the following theorem:

**Theorem 4.** Let for each \( r \), \( \left(X_i^{(r)}\right)_{\frac{r+1}{2} \leq i \leq r} \) be \( \{-1, 1\}\)-valued random variables with joint distribution given by \( (3.4) \), \( h \neq 0 \), \( s \in (0, 1) \), and let \( a_i := \frac{1-sX_i^{(r)}}{2} \) for each \( i \in \left\{ \frac{r+1}{2}, \ldots, r \right\} \). Let \( b_{ri} \) and \( s_r \) be defined as in Theorem 3 for all \( r \) and all \( i \in \left\{ \frac{r+1}{2}, \ldots, r \right\} \). Then we have for all \( \beta \geq 0 \)

\[
\frac{\lambda_r(\mathbf{a}) - x(\beta, h) \sum_{i=\frac{r+1}{2}}^{r} b_{ri}}{s_r} \quad \xrightarrow{\quad r \to \infty \quad} \quad \mathcal{N}(0, 1).
\]

Above, \( x(\beta, h) \) is a constant independent of \( r \) that has the same sign as \( h \). \( |x(\beta, h)| \) is increasing in \( \beta \).

The proofs of this theorem and all previous ones can be found in the Appendix.

Informally, for \( h \neq 0 \), the typical value of the stability parameter is \( \lambda_r(\mathbf{a}) = x(\beta, h) \sqrt{\frac{2r}{\pi}} + \Theta(\frac{1}{\sqrt{r}}) \).

Contrary to the absence of an external magnetic field, the CWM behaves in a qualitatively similar fashion for all values of \( \beta \geq 0 \) when there is an external magnetic field \( h \neq 0 \). This is similar to the behaviour of the magnetisation \( S_r \). A central limit theorem similar to the statement \( (3.3) \) holds for any value \( \beta \geq 0 \) provided
that $h \neq 0$. (3.5) is a central limit theorem that holds for all values of $\beta$ and $h \neq 0$. However, just as without the external magnetic field, the stability parameter and the magnetisation have different typical magnitudes: the typical magnitude of $\lambda_r(a)$ is of order $\Theta(\sqrt{r})$, whereas the typical magnitude of $S_r$ is of order $\Theta(r)$ under the assumption of an external magnetic field.

Even the strong correlation regime of the CWM does not give a stable fixed point $1/2$ with positive probability. One reason for this is that there are no realisations of the vector of flip parameters $a$ where each entry is equal to $1/2$. Also, the typical realisations of $a$ feature a sizeable majority of entries $a_i$ pointing in the same direction. In conclusion, strong correlation on its own is not necessarily enough to reproduce the result we saw in Section 3.2.2 of a stable fixed point under perfect positive correlation of the flip parameters and a positive probability of all of them being equal to $1/2$.

4 Conclusion

We investigated the behaviour of the local flip model when the flip parameters, which are ordinarily fixed constants, are randomly selected. First we studied the case of small local discussion groups such as real-life friend or work groups in Section 2. Under independent uniformly distributed flip parameters, we found that

- The number of fixed points in the dynamics of the model is generally small (meaning smaller than the possible maximum of $r$ fixed points). Thus, the dynamics of the model are fairly simple, with large basins of attraction to the few attractors that mark the limits public discourse tends towards as time passes.

- Unless we exclude the possibility of flips against unanimous majorities, the local flip model typically does not present unanimous attractors. This is a major difference compared to the basic model. As unanimous majorities are extremely rare in real life, this is a point in favour of the local flip model.

- The universal fixed point $1/2$ is mostly stable. There is a tendency towards ties in most discussions.

Then, in Section 3 we studied large discussion groups. Unanimous majorities are very unlikely to occur for both large and small discussion groups. On the other hand, we found that the picture differs considerably between small and large groups as far as the stability of the fixed point $1/2$ is concerned. Whereas the former feature mostly stable fixed points at $1/2$, the latter do not. In fact, we found that the larger the local discussion groups, generally, the more unstable the fixed point $1/2$ becomes. The many degrees of freedom inherent in large discussion groups do not lead to the coexistence of stable and unstable fixed point at $1/2$. Contrary to the basic model, the sign of the stability parameter, which determines whether the dynamics close to a tie are monotonic or alternating, can be positive or negative. We showed that for unbiased flip parameters, i.e. flip parameters with expectation $1/2$, monotonic and alternating dynamics around a tie are equally likely. If the flip parameters are biased, then the sign of the bias determines the sign of the stability parameter and thus whether we only see monotonic or alternating dynamics but never both with positive probability. If the flip parameters are biased towards accepting the majority, then the local flip model behaves similarly to the basic model with monotonic dynamics. These findings are stable across a range of degrees of dependency of the flip parameters, all the way from independence to strong correlation in the low temperature regime of the CWM. Thus, the picture the local flip model with random flip parameters paints is that for small discussion groups there is a tendency towards ties, whereas for large discussion groups there is a tendency to move away from ties. We either see convergence to a majority in favour of one of the alternatives, or we observe perpetual swings from majorities in favour of one alternative to majorities for the other one.
Appendix

We prove the results presented in Theorems 1, 2, 3, and 4. We will use the notation \( a \land b = \min\{a, b\} \) and \( a \lor b = \max\{a, b\} \) for all real numbers \( a, b \) throughout the rest of this article.

A Proof of Theorem 1

The first two of these theorems pertain to sums of independent random variables, and therefore we can apply the central limit theorem by Feller-Li̇šev (see e.g. Theorem 6.16 in [22]). The sequence of random variables for which we want to determine a limiting distribution is

\[
\lambda_r(a) \Bigg/ \sqrt{r} = \frac{1}{\sqrt{r}} \sum_{i=\lceil \frac{r}{2} \rceil}^{r} s \left( \frac{r}{i} \right) \frac{2i-r}{2r-1} (1-2a_i) = \frac{1}{\sqrt{r}} \sum_{i=\lceil \frac{r}{2} \rceil}^{r} b_{ri} (1-2a_i),
\]

where the coefficients \( b_{ri} \) equal \( \left( \frac{r}{i} \right) \frac{2i-r}{2r-1} \) as defined in the statement of Theorem 2. We define for each \( r \) and each \( i \in \{ \frac{r+1}{2}, \ldots, r \} \) the random variable \( Y_{ri} := \frac{b_{ri}(1-2a_i)}{\sqrt{r}} \). In order to show Theorem 1 for this triangular array of random variables \( (Y_{ri})_{r \in \mathbb{N}, i \in \{ \frac{r+1}{2}, \ldots, r \}} \), we need to verify four statements:

**Condition 5.**
1. \( (Y_{ri})_{r \in \mathbb{N}, i \in \{ \frac{r+1}{2}, \ldots, r \}} \) is a null array, i.e. for each \( r \) the random variables \( Y_{ri} \) are independent and \( \lim_{r \to \infty} \sup_i \mathbb{E} (|Y_{ri}| \land 1) = 0 \).
2. \( \sum_{i=\lceil \frac{r}{2} \rceil}^{r} \mathbb{P} \{ |Y_{ri}| > \varepsilon \} \xrightarrow{r \to \infty} 0, \varepsilon > 0 \).
3. \( \sum_{i=\lceil \frac{r}{2} \rceil}^{r} \mathbb{E} (Y_{ri} \mathbb{1}_{|Y_{ri}| \leq 1}) \xrightarrow{r \to \infty} b \in \mathbb{R} \), where for any measurable set \( A \) \( \mathbb{1}_A \) refers to the indicator function.
4. \( \sum_{i=\lceil \frac{r}{2} \rceil}^{r} \mathbb{V} (Y_{ri} \mathbb{1}_{|Y_{ri}| \leq 1}) \xrightarrow{r \to \infty} c \geq 0 \).

Once the four conditions are verified, we have obtained convergence in distribution of \( \sum_{i=\lceil \frac{r+1}{2} \rceil}^{r} Y_{ri} \) to \( \mathcal{N}(b, c) \). This includes the degenerate normal distribution \( \mathcal{N}(b, 0) = \delta_b \) as a special case.

In order to show the first statement, we need to determine the asymptotic order of each \( Y_{ri} \). That means we need to evaluate each \( b_{ri} \) asymptotically. We will next present and prove several statements regarding these coefficients before proceeding to the proof of the four statements in Condition 5. Since we will later also need powers \( b_{ri}^k \) for all \( k \in \mathbb{N} \), we will take care of this in the following lemma:

**Lemma 6.** Let \( r, k \in \mathbb{N} \) and \( i = \frac{r+1}{2} + \eta_r \in \{ \frac{r+1}{2}, \ldots, r \} \). We distinguish four classes of \( \eta_r \):

1. If \( \eta_r = o(\sqrt{r}) \), then \( b_{ri}^k = \Theta \left( \frac{\eta_r^k}{r^{k/2}} \right) \).
2. If \( \lim_{r \to \infty} \frac{\eta_r}{\sqrt{r}} = h > 0 \), then \( b_{ri}^k \xrightarrow{r \to \infty} (\frac{2h}{\pi})^{k/2} \exp(-2kh^2) h^k > 0 \).
3. If \( \sqrt{r} = O(\eta_r) \) and \( \eta_r = o(r) \), then \( b_{ri}^k = \Theta \left( \exp(-2kh^2) \frac{\eta_r^k}{r^{k/2}} \right) \).

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4. If \( \lim_{r \to \infty} \frac{\eta_r}{\sqrt{r}} = h \in (1/2, 1] \), then \( b_{r_i} = \Theta \left( \exp \left( -2kh \right) r^{k/2} \right) \).

Proof. The proof of this lemma uses the local limit theorem for the binomial distribution by de Moivre-Laplace:

**Theorem 7.** Let \( P_n \) be the binomial distribution with \( n \in \mathbb{N} \{0, 1\} \)-valued variables and parameter \( 1/2 \) and let \( \phi \) be the Lebesgue density function of the standard normal distribution. Then we have

\[
\sup_{i \in \{0, \ldots, n\}} \left| \sqrt{\frac{n}{2}} P_n \{ i \} - \phi \left( \frac{i - n/2}{\sqrt{n/2}} \right) \right| \to 0. \quad r \to \infty
\]

The theorem states that the scaled point probabilities \( \sqrt{\frac{n}{2}} P_n \{ i \} \) of the binomial distribution are well approximated for large values of \( n \) by the corresponding values of the density function of the standard normal distribution. Note that the convergence is uniform over all values of \( i \in \{0, \ldots, n\} \).

An elementary calculation shows that for all \( r \) and all \( i \)

\[
b_{r_i} = 4 P_r \{ i \} \left( i - r/2 \right). \]

Translating Theorem 7 to our setting, we have

\[
\sup_{i \in \left\{ \frac{r+1}{2}, \ldots, r \right\}} \left| b_{r_i} - 4 \phi \left( \frac{i - r/2}{\sqrt{r}/2} \right) \frac{i - r/2}{\sqrt{r}/2} \right| \to 0. \quad r \to \infty \quad (A.1)
\]

Due to the exponential nature of \( \phi \), the set

\[
A := \left\{ 4 \phi \left( \frac{i - r/2}{\sqrt{r}/2} \right) \frac{i - r/2}{\sqrt{r}/2} \bigg| r \in \mathbb{N}, i \in \left\{ \frac{r+1}{2}, \ldots, r \right\} \right\} \quad (A.2)
\]

is bounded. Hence, the uniform convergence stated in (A.1) follows from Theorem 7.

Now we evaluate the expression \( 4 \phi \left( \frac{i - n/2}{\sqrt{n/2}} \right) \frac{i - n/2}{\sqrt{n/2}} \) to calculate the asymptotic expressions in the lemma. Let \( r \in \mathbb{N} \) and \( i = \frac{r+1}{2} + \eta_r \in \left\{ \frac{r+1}{2}, \ldots, r \right\} \). We calculate

\[
b_{r_i} \approx 4 \phi \left( \frac{i - r/2}{\sqrt{r}/2} \right) \frac{i - r/2}{\sqrt{r}/2} = 2 \sqrt{\frac{2}{\pi}} \exp \left( - \left( \frac{r+1}{2} + \eta_r - \frac{\eta_r}{\sqrt{r}/2} \right)^2 \right) \frac{r+1}{2} + \eta_r - \frac{\eta_r}{\sqrt{r}}.
\]

If \( \eta_r = o \left( \sqrt{r} \right) \), then

\[
b_{r_i} \approx 2 \sqrt{\frac{2}{\pi}} \exp \left( - \frac{4\eta_r^2}{2r} \right) \frac{2\eta_r}{\sqrt{r}} = \Theta \left( \frac{\eta_r}{\sqrt{r}} \right).
\]

If \( \lim_{r \to \infty} \frac{\eta_r}{\sqrt{r}} = h > 0 \), then

\[
b_{r_i} \approx 2 \sqrt{\frac{2}{\pi}} \exp \left( - \frac{4\eta_r^2}{2r} \right) \frac{2\eta_r}{\sqrt{r}} \to \frac{32}{\pi} \frac{1}{h} \exp (-2h^2) h, \quad r \to \infty.
\]
which is clearly positive. If $\sqrt{r} = o(\eta_r)$ and $\eta_r = o(r)$, then

$$b_{ri} \approx 2\sqrt{\frac{2}{\pi}} \exp \left( -\frac{4\eta_r^2}{2r} \right) \frac{2\eta_r}{\sqrt{r}} = \Theta \left( \exp \left( -\frac{2\eta_r^2}{r} \right) \frac{\eta_r}{\sqrt{r}} \right).$$

If $\lim_{r \to \infty} \frac{\eta_r}{r} = h \in (1/2, 1]$, then

$$b_{ri} \approx 2\sqrt{\frac{2}{\pi}} \exp \left( -2h - 2hr \right) \cdot 2h\sqrt{r} = \Theta \left( \exp \left( -2hr \right) \sqrt{r} \right).$$

Thus, the statements of Lemma 6 hold for $k = 1$. The statements for general $k \in \mathbb{N}$ follow from the observation that the set $A$ in (A.2) is bounded, and the function $x \mapsto x^k$ restricted to any bounded subset of the reals is uniformly continuous.

Lemma 6 states that the only coefficients $b_{ri}$ which do not decay as $r$ goes to infinity are those for which $i - \frac{r + 1}{2}$ is of order $\sqrt{r}$, in which case $b_{ri}$ converges to some positive constant. Therefore, if we are looking for the maximum of $b_{ri}$ for a fixed $r$ over all $i$, we only need to look at those $i$ which are of order $\sqrt{r}$, provided $r$ is large enough.

Lemma 8. Let $k \in \mathbb{N}$. For large enough $r$, the maximum of $\{b_{ri}^k | i \in \{\frac{r + 1}{2}, \ldots, r\}\}$ is located at $i \approx \frac{r + 1}{2} + \frac{\sqrt{r}}{2}$ and the value of the corresponding $b_{ri}^k$ is asymptotically given by

$$0 < \left( \frac{2\sqrt{2}}{\sqrt{e\pi}} \right)^k < 1.$$

Also, for all $r$ large enough, the set of coefficients $b_{ri}^k$ for all $i \in \{\frac{r + 1}{2}, \ldots, r\}$ is bounded above by some constant strictly smaller than 1.

Proof. Let $k \in \mathbb{N}$. Let $h > 0$ and $i = \frac{r + 1}{2} + \lfloor h\sqrt{r} \rfloor$. The brackets $\lfloor \cdot \rfloor$ denote the floor function which rounds down the expression on the inside to the nearest integer. Then by Lemma 6

$$b_{ri}^k \to \left( \frac{32}{\pi} \right)^{k/2} \exp \left( -2kh^2 \right) h^k.$$

We identify the value of the constant $h$ which maximises the limit on the right hand side above. This expression can be understood as a function $f : (0, \infty) \to \mathbb{R}$ of $h$. Calculating the first derivative, we obtain

$$f'(h) = \exp \left( -2kh^2 \right) k \left( -4h^{k+1} + h^{k-1} \right).$$

Equating the first derivative to 0, we identify the critical point $h_0 = 1/2$. We calculate the second derivative

$$f''(h) = \exp \left( -2kh^2 \right) k \left( -4kh^k \left( -4h^2 + 1 \right) + h^{k-2} \left( -4 \left( k + 1 \right) h^2 + k - 1 \right) \right).$$

The value of the second derivative at the critical point $h_0$ is negative, hence $h_0$ is indeed the unique global maximum of $f$ on $(0, \infty)$. By substituting $h_0$ into $f$, we obtain the asymptotic maximum of $b_{ri}^k$. It is easily calculated that

$$f(h_0) = \left( \frac{2\sqrt{2}}{\sqrt{e\pi}} \right)^k.$$
and this value lies in \((0, 1)\). As for any \(h > 0\) and \(i = \frac{i+1}{2} + |h\sqrt{r}|\) the coefficients converge uniformly over all values of \(h\) to \(f(h)\), we can choose \(r_0 \in \mathbb{N}\) large enough that for all \(r \geq r_0\) and all \(i \in \left\{\frac{i+1}{2}, \ldots, r\right\}\) we have

\[
0 < b_{ri}^k < 1 \quad \text{and} \quad \sup_{r \geq r_0} \sup_{i \in \left\{\frac{i+1}{2}, \ldots, r\right\}} b_{ri}^k < 1.
\]

So starting at \(r_0\), all coefficients \(b_{ri}^k\) are uniformly bounded away from 1.

We next determine the asymptotic order of the sums \(\sum_{i=\frac{i+1}{2}}^{r} b_{ri}^k\).

**Lemma 9.** Let \(k \in \mathbb{N}\). Then we have

\[
\sum_{i=\frac{i+1}{2}}^{r} b_{ri}^k = \Theta(\sqrt{r}).
\]

The lemma states that for all \(k \in \mathbb{N}\) the asymptotic order of the sum \(\sum_{i=\frac{i+1}{2}}^{r} b_{ri}^k\) is the same, namely \(\Theta(\sqrt{r})\). This has important implications for the remaining proofs.

**Proof.** We first show an asymptotic upper bound for \(\sum_{i=\frac{i+1}{2}}^{r} b_{ri}^k\). Lemma\footnote{8} says that, for all \(r\) large enough, all summands \(b_{ri}\) are smaller than 1. Hence, \(b_{ri}^k \leq b_{ri}\) for all \(i\). Therefore, we have the upper bound

\[
\sum_{i=\frac{i+1}{2}}^{r} b_{ri}^k \leq \sum_{i=\frac{i+1}{2}}^{r} b_{ri} \approx \sqrt{\frac{2r}{\pi}} = \Theta(\sqrt{r}),
\]

using the asymptotic expression for \(\sum_{i=\frac{i+1}{2}}^{r} b_{ri}\) from p. 27 of \cite{17}.

Next, we show a lower bound for \(\sum_{i=\frac{i+1}{2}}^{r} b_{ri}^k\). By Statement 2 of Lemma\footnote{6} we have for any constants 0 < \(c < C < \infty\), \(r\) large enough, and \(i \in B := \{i \in \mathbb{N} | \frac{i+1}{2} + c\sqrt{r} < i < \frac{i+1}{2} + C\sqrt{r}\}\)

\[
b_{ri}^k \geq \frac{1}{2} \min \left\{ \left(\frac{32}{\pi}\right)^{k/2} \exp\left(-2kh^2\right) h^k \left| h \in [c, C]\right. \right\} =: \tau.
\]

Then \(\tau > 0\) holds. We note that the cardinality of index set \(B\) is at least \(|(C - c)\sqrt{r}|\), so a lower bound for the sum \(\sum_{i=\frac{i+1}{2}}^{r} b_{ri}^k\) is given by

\[
\tau \left[(C - c)\sqrt{r}\right] = \Theta(\sqrt{r}).
\]

**Statement 1**

After these preparatory lemmas which we will also use for later proofs, we are ready to show Statement 1 concerning the \((Y_{ri})_{r \in \mathbb{N}, i \in \left\{\frac{i+1}{2}, \ldots, r\right\}}\) being a null array.

By Lemma\footnote{8} there is a constant \(K > 0\) such that

\[
|Y_{ri}| = \frac{b_{ri} |1 - 2a_i|}{\sqrt{r}} \leq \frac{K}{\sqrt{r}} \quad \text{(A.3)}
\]
holds for all $r$ and all $i \in \{ \frac{r+1}{2}, \ldots, r \}$. Therefore, we have
\[ \mathbb{E}(\lvert Y_{ri} \rvert \land 1) = \mathbb{E}\left(\frac{b_{ri} \lvert 1 - 2a_i \rvert}{\sqrt{r}} \land 1 \right) \leq \frac{K}{\sqrt{r}} \rightarrow r \rightarrow \infty \rightarrow 0 \]
uniformly over all $r \in \mathbb{N}, i \in \{ \frac{r+1}{2}, \ldots, r \}$. This proves Statement 1 in Condition 5.

**Statement 2**

Let $\varepsilon > 0$. By (A.3), we have for all $r > K^2 / \varepsilon^2$
\[ \lvert Y_{ri} \rvert < \varepsilon \]
for all $i \in \{ \frac{r+1}{2}, \ldots, r \}$. Therefore,
\[ \sum_{i = \frac{r+1}{2}}^{r} \mathbb{P}\{ \lvert Y_{ri} \rvert > \varepsilon \} = 0 \]
for all $r > K^2 / \varepsilon^2$. This proves Statement 2 in Condition 5.

**Statement 3**

By (A.3), for all $r \geq K^2$,
\[ \sum_{i = \frac{r+1}{2}}^{r} \mathbb{E}(Y_{ri} \mathbb{1}_{\{\lvert Y_{ri} \rvert \leq 1\}}) = \sum_{i = \frac{r+1}{2}}^{r} \mathbb{E}(Y_{ri}) = \frac{1}{\sqrt{r}} \sum_{i = \frac{r+1}{2}}^{r} b_{ri} \mathbb{E}(1 - 2a_i) \]
\[ = \mathbb{E}(1 - 2a_1) \frac{1}{\sqrt{r}} \sum_{i = \frac{r+1}{2}}^{r} b_{ri} \rightarrow r \rightarrow \infty \sqrt{\frac{2}{\pi}} (1 - 2\mathbb{E}a_1) =: b \in \mathbb{R}. \]

**Statement 4**

We again employ (A.3) to calculate that for all $r \geq K^2$
\[ \sum_{i = \frac{r+1}{2}}^{r} \mathbb{V}(Y_{ri} \mathbb{1}_{\{\lvert Y_{ri} \rvert \leq 1\}}) = \sum_{i = \frac{r+1}{2}}^{r} \mathbb{V}Y_{ri} = \sum_{i = \frac{r+1}{2}}^{r} \left( \mathbb{E}(Y_{ri}^2) - (\mathbb{E}Y_{ri})^2 \right) \]
\[ \approx \frac{1}{r} \sum_{i = \frac{r+1}{2}}^{r} b_{ri}^2 \left( \mathbb{E}(1 - 2a_i)^2 - (1 - 2\mathbb{E}a_i)^2 \right) \]
\[ = \frac{1}{r} \sum_{i = \frac{r+1}{2}}^{r} b_{ri}^2 \left( 4\mathbb{E}a_i^2 - 4(\mathbb{E}a_i)^2 \right) \]
\[ = \frac{4\mathbb{V}a_1}{r} \sum_{i = \frac{r+1}{2}}^{r} b_{ri}^2 \rightarrow r \rightarrow \infty 0. \quad (A.4) \]
In the last step, we used Lemma 9, by which \( r_{\lambda_r} (a) = \Theta (\sqrt{r}) \).

In virtue of the verification of Condition 5, we have thus shown the limit theorem
\[
\lambda_r (a) \overset{d}{\to} \mathcal{N} \left( \sqrt{\frac{r}{\pi}} (1 - 2Ea_1), 0 \right) = \delta \sqrt{\frac{1}{\pi} (1 - 2Ea_1)}.
\]

The last statement in Theorem 1, \( \forall \lambda_r (a) = \Theta (\sqrt{r}) \), follows from our calculation (A.4), Lemma 9, and
\[
\forall \lambda_r (a) = r \sum_{i = \frac{r+1}{2}}^{r} \forall y_{ri} = 4Va_1 \sum_{i = \frac{r+1}{2}}^{r} b_{ri}^2 = \Theta (\sqrt{r}).
\]

This concludes the proof of Theorem 1.

**B Proof of Theorem 2**

Using the lemmas from the last section, we now show Theorem 2 by verifying Condition 5 for the triangular array \( (Y_{ri})_{r \in \mathbb{N}, i \in \{ \frac{r+1}{2}, \ldots, r \}} \) defined by \( Y_{ri} := \frac{b_{ri} (1 - 2a_i)}{s_r} \) under the assumption that for all \( i \) \( Ea_i = 1/2 \) holds.

**Statement 1**

We have to show that \( (Y_{ri})_{r \in \mathbb{N}, i \in \{ \frac{r+1}{2}, \ldots, r \}} \) is a null array. By Lemma 8, there is a constant \( K > 0 \) such that
\[
|Y_{ri}| = \frac{b_{ri} |1 - 2a_i|}{s_r} \leq \frac{K}{s_r}
\]
holds for all \( r \) and all \( i = \frac{r+1}{2} \). We recall the definition of \( s_r := \sum_{i = \frac{r+1}{2}}^{r} b_{ri}^2 \). Using Lemma 9, we obtain \( s_r = \Theta (r^{1/4}) \), and by Lemma 8 we have
\[
E (|Y_{ri}| \wedge 1) = E \left( \frac{b_{ri} |1 - 2a_i|}{s_r} \wedge 1 \right) = \Theta \left( \frac{1}{r^{1/4}} \right) \overset{r \to \infty}{\longrightarrow} 0
\]
uniformly over all \( r \in \mathbb{N}, i \in \{ \frac{r+1}{2}, \ldots, r \} \). This proves Statement 1 in Condition 5.

**Statement 2**

Let \( \varepsilon > 0 \). \( s_r = \Theta (r^{1/4}) \) implies \( \tau := \liminf_{r \to \infty} \frac{r^{1/4}}{s_r} > 0 \). Hence there is an \( r_0 \in \mathbb{N} \) such that for all \( r \geq r_0 \) the inequality \( \tau/2 < s_r/r^{1/4} \) holds. It follows from (B.1) that, for all \( r > r_0 \) \( \forall \left( \frac{2K}{\tau^2} \right)^{4} \) and all \( i \in \{ \frac{r+1}{2}, \ldots, r \} \), we have
\[
|Y_{ri}| \leq \frac{K}{s_r} < \frac{2K}{\tau r^{1/4}} < \varepsilon.
\]
As a consequence,
\[
\sum_{i = \frac{r+1}{2}}^{r} \mathbb{P} (|Y_{ri}| > \varepsilon) = 0
\]
for all \( r > r_0 \) \( \forall \left( \frac{2K}{\tau^2} \right)^{4} \). This proves Statement 2 in Condition 5.
Statement 3

Using (B.2) with $\varepsilon = 1$, we have for all $r > r_0 \lor \left( \frac{2K}{\tau} \right)^4$

$$\sum_{i=\frac{r+1}{2}}^{r} \mathbb{E} \left( Y_{ri} I_{\{|Y_{ri}| \leq 1\}} \right) = \sum_{i=\frac{r+1}{2}}^{r} \mathbb{E} (Y_{ri}) = \frac{1}{s_r} \sum_{i=\frac{r+1}{2}}^{r} b_{ri} \mathbb{E} (1 - 2a_i)$$

$$= \mathbb{E} (1 - 2a_1) \frac{1}{s_r} \sum_{i=\frac{r+1}{2}}^{r} b_{ri} = 0 =: b \in \mathbb{R}.$$

Statement 4

We again employ (B.2) to calculate that for all $r > r_0 \lor \left( \frac{2K}{\tau} \right)^4$

$$\sum_{i=\frac{r+1}{2}}^{r} \mathbb{V} \left( Y_{ri} I_{\{|Y_{ri}| \leq 1\}} \right) = \sum_{i=\frac{r+1}{2}}^{r} \mathbb{V} Y_{ri} = \sum_{i=\frac{r+1}{2}}^{r} \left( \mathbb{E} Y_{ri}^2 - (\mathbb{E} Y_{ri})^2 \right)$$

$$\approx \frac{1}{s_r^2} \sum_{i=\frac{r+1}{2}}^{r} b_{ri}^2 \left( \mathbb{E} (1 - 2a_i)^2 - (1 - 2\mathbb{E} a_i)^2 \right)$$

$$= \frac{1}{s_r^2} \sum_{i=\frac{r+1}{2}}^{r} b_{ri}^2 \left( 4\mathbb{E} a_i^2 - 4 (\mathbb{E} a_i)^2 \right)$$

$$= \frac{4V_{a_1}}{s_r^2} \sum_{i=\frac{r+1}{2}}^{r} b_{ri}^2 = 4V_{a_1}. \quad (B.3)$$

In the last step, we used the definition of $s_r$.

We have verified Condition 5 and thus shown the limit theorem

$$\frac{\lambda_r (a)}{s_r} \xrightarrow{d} r \rightarrow \infty \mathcal{N} (0, 4V_{a_1}).$$

The last statement in Theorem 2, $\mathbb{V} \lambda_r (a) = \Theta (\sqrt{r})$, follows from our calculation (B.3), Lemma 9 and

$$\mathbb{V} \lambda_r (a) = s_r^2 \sum_{i=\frac{r+1}{2}}^{r} \mathbb{V} Y_{ri} = 4V_{a_1} \sum_{i=\frac{r+1}{2}}^{r} b_{ri}^2 = \Theta (\sqrt{r}).$$

This concludes the proof of Theorem 2.

C Proof of Theorem 3

Theorems 3 and 4 are limit theorems for sums of triangular arrays of dependent random variables. Therefore, we cannot employ the classic central limit theorem by Feller-Liouville. Instead, we will use the method of
moments. Let \((Y_{ni})_{n \in \mathbb{N}, i \in \{1, \ldots, n\}}\) be a triangular array of random variables. The method of moments consists of showing the convergence of moments

\[
E \left( \sum_{i=1}^{n} Y_{ni} \right)^k \xrightarrow{n \to \infty} m_k \in \mathbb{R}
\]

for each \(k \in \mathbb{N}\). Provided the constants \(m_k\) satisfy appropriate upper bounds, e.g.

\[
m_k \leq AC^k k! \quad k \in \mathbb{N},
\]

(C.1)

for fixed constants \(A, C \in \mathbb{R}\), the convergence of the moments implies convergence in distribution of the sequence \((\sum_{i=1}^{n} Y_{ni})_{n \in \mathbb{N}}\) to a limiting distribution \(\mu\) with moments of all orders \(k\) given by the limits \(m_k\) above. Provided the constants \(m_k\) satisfy appropriate upper bounds, e.g.

\[
m_k \leq AC^k k! \quad k \in \mathbb{N},
\]

(C.1)

for fixed constants \(A, C \in \mathbb{R}\), the convergence of the moments implies convergence in distribution of the sequence \((\sum_{i=1}^{n} Y_{ni})_{n \in \mathbb{N}}\) to a limiting distribution \(\mu\) with moments of all orders \(k\) given by the limits \(m_k\) above. Moreover, the limiting distribution \(\mu\) is uniquely determined by its moments \(m_k\). Consult e.g. Example 4 on p. 205 of [31] for a proof of these statements. We note that all limiting distributions demonstrated in this article satisfy the condition (C.1), so the method of moments is applicable.

Let \(\beta \leq 1\). We define \((Y_{ri})_{r \in \mathbb{N}, i \in \{\frac{r+1}{2}, \ldots, r\}}\) by

\[
Y_{ri} := \frac{b_{ri}(1-2a_i)}{s_r} X_i s_r
\]

for all \(r \in \mathbb{N}\) and all \(i \in \{\frac{r+1}{2}, \ldots, r\}\).

To calculate the limits of the moments \(E \left( \sum_{i=\frac{r+1}{2}}^{r} Y_{ri} \right)^k\), we have to evaluate asymptotically sums of the type

\[
\sum_{i_1, \ldots, i_k=\frac{r+1}{2}}^{r} \mathbb{E}Y_{ri_1} \cdots Y_{ri_k}.
\]

Therefore, we will need to know the asymptotic behaviour of correlations such as

\[
\mathbb{E}Y_{ri_1} \cdots Y_{ri_k} = \frac{b_{ri_1} \cdots b_{ri_k}}{s_r^k} \mathbb{E}X_{ri_1} \cdots X_{ri_k}.
\]

We thus start the proof of Theorem [3] by presenting asymptotic expressions for correlations of the type above in the CWM. These facts are well known and presented here for the convenience of the reader. The Curie-Weiss random variables \((X_{\frac{r+1}{2}}, \ldots, X_r)\) are exchangeable. It follows that for any set of indices \(\{i_1, \ldots, i_k\}\) with cardinality \(k\) we have \(\mathbb{E}X_{ri_1} \cdots X_{ri_k} = \mathbb{E}X_{r \frac{k+1}{2}} \cdots X_{r \frac{k+1}{2}+k}\), i.e. the correlation \(\mathbb{E}X_{ri_1} \cdots X_{ri_k}\) depends only on the number of different random variables included and not their specific identities. We are thus free to look only at the first \(k\) random variables. The value of these correlations depends on the regime of the model, and we have the following proposition. Below \((k-1)!!\) stands for \((k-1)(k-3) \cdots 5 \cdot 3 \cdot 1\), and we set \(\tilde{\beta} := \frac{\beta}{1-\beta}\) for all \(\beta < 1\).

**Proposition 10.** Let \(k \in \mathbb{N}\). If \(k\) is odd, then \(\mathbb{E}X_{r \frac{k+1}{2}} \cdots X_{r \frac{k+1}{2}+k}\) equals 0 for all values of \(\beta \geq 0\). If \(k\) is even, then we have:

1. If \(\beta < 1\), then

\[
\mathbb{E}X_{r \frac{k+1}{2}} \cdots X_{r \frac{k+1}{2}+k} \approx (k-1)!! \tilde{\beta}^{k/2} \left(\frac{2}{r}\right)^{k/2}.
\]

2. If \(\beta = 1\), then

\[
\mathbb{E}X_{r \frac{k+1}{2}} \cdots X_{r \frac{k+1}{2}+k} \approx c_k \left(\frac{2}{r}\right)^{k/4}.
\]

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3. If $\beta > 1$, then
\[ \mathbb{E} X_{\frac{r+1}{2}} \cdots X_{\frac{r+1}{2}+k} \xrightarrow{r \to \infty} x(\beta)^k. \]

The constant $c_k$ is given by $12^{k/4} \frac{\Gamma\left(\frac{k+1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$, where $\Gamma$ stands for the gamma function, and $x(\beta)$ is the positive solution of the equation $\tanh \beta x = x$.

**Proof.** This is Theorem 5.17 in [23] and its proof can be found there. \qed

The second ingredient we need in order to calculate the moments
\[ M_k := \mathbb{E} \left( \sum_{i=1}^{r} Y_{ri} \right)^k = \frac{1}{s_k^{\bar{r}}} \sum_{i_1, \ldots, i_k = \frac{r+1}{2}} b_{ri_1} \cdots b_{ri_k} \mathbb{E} X_{ri_1} \cdots X_{ri_k} \]

is a count of the number of what we will call profiles. If we have an index vector $I = (i_1, \ldots, i_k)$, where all indices belong to $\{1, \ldots, k\}$ (possibly including repeated indices), there is a profile $\underline{r} = (r_1, \ldots, r_k) \in \{0, 1, \ldots, k\}^k$ which registers the number of indices in $I$ according to their multiplicity: $r_j$ stands for the number of indices in $I$ that occur exactly $j$ times for $j \in \{1, \ldots, k\}$. Let $\Pi^k$ be the set of all profiles of length $k$. All profiles $\underline{r} \in \Pi^k$ satisfy the identity $\sum_{j=1}^{k}jr_j = k$. Also, the number of profiles is finite. A simple upper bound on $|\Pi^k|$ is given by $(k+1)^k$.

We will need to know how many index vectors there are for each profile. The following lemma provides the answer:

**Lemma 11.** Let $k \in \mathbb{N}$ and $\underline{r} \in \Pi^k$. The number of index vectors $(i_1, \ldots, i_k)$ with entries in $\{1, \ldots, k\}$ and profile $\underline{r}$ is given by
\[ \frac{k!}{r_1! \cdots r_k! 1^{r_1} \cdots k^{r_k}}. \]

**Proof.** Let $k \in \mathbb{N}$ and $\underline{r} \in \Pi^k$. We construct an index vector $I = (i_1, \ldots, i_k)$ with entries in $\{1, \ldots, k\}$ and profile $\underline{r}$ in two steps:

1. We first partition the set $\{1, \ldots, k\}$ into $k$ sets $A_j$, indexed by $j = 1, \ldots, k$, with cardinality $|A_j| = jr_j$ in accordance with the profile $\underline{r}$. Each set $A_j$ indicates the positions $\ell \in \{1, \ldots, k\}$ such that the index $i_\ell$ occurs exactly $j$ times in $I$. There are
\[ \begin{pmatrix} k! \\ r_1! \ (2r_2)! \cdots \ (kr_k)! \end{pmatrix} \]  
(C.2)
n ways to partition $\{1, \ldots, k\}$ as described.

2. We join the selected the positions $\ell \in A_j$ such that the $i_\ell$ have the same value, i.e. we have to form partitions of $A_j$ into subsets of $j$ elements each for all $j = 1, \ldots, k$. Let the elements of $A_j$ be arranged in ascending order. We form the first set of the partition of $A_j$, which contains the smallest element of $A_j$, by selecting $j - 1$ from all remaining $jr_j - 1$ elements of $A_j$. There are
\[ \frac{(jr_j - 1) \cdots (j (r_j - 1) + 1)}{(j - 1)!} \]
ways to make this selection. Once the first \( n \in \{1, \ldots, r_j - 1\} \) of the subsets of \( A_j \) have been selected in this fashion, we generate subset \( n + 1 \) by choosing the smallest of the elements not yet assigned to any of the previous subsets and choosing \( j - 1 \) from all remaining \( j (r_j - n - 1) \) elements of \( A_j \). There are

\[
\frac{(j (r_j - n) - 1) \cdots (j (r_j - n - 1) + 1)}{(j - 1)!}
\]

ways to make this selection. Thus, we have defined the algorithm to determine the partition of \( A_j \) into subsets of cardinality \( j \). We observe that there are

\[
\left( \frac{j (r_j - n) - 1) \cdots (j (r_j - n - 1) + 1)}{(j - 1)!} \right) \cdots \left( \frac{(j - 1)\cdots 1}{(j - 1)!} \right) = \frac{(jr_j)!}{r_1!1\cdots r_k!}
\]

such partitions.

We multiply (C.2) and (C.3) for each \( j \) and obtain

\[
\frac{k!}{r_1! \cdots r_k!} \frac{r_1!(2r_2)! \cdots (kr_k)!}{r_1! \cdots r_k!} = \frac{k!}{r_1! \cdots r_k!} \cdots \frac{r_1! \cdots r_k!}{k!}.
\]

We define the correlation \( \mathbb{E}X(r) \) corresponding to a profile \( r \in \Pi^k \) by choosing any \( \sum_{j=1}^k r_j \) different \( X_{ri} \). Then raise the first \( r_1 \) of them to the power 1, the next \( r_2 \) to the power 2, etc. Take the product of all these powers and take their expectation. This expectation is what we will refer to as \( \mathbb{E}X (r) \). Due to the exchangeability of the Curie-Weiss random variables, the identity of the selected \( X_{ri} \) in this definition is inconsequential.

Using Lemma 11, we can express \( M_k \) as

\[
M_k = \frac{1}{s_k^k} \sum_{r_1, \ldots, r_k} \frac{k!}{r_1! \cdots r_k!} \mathbb{E}X_{r_1} \cdots X_{r_k}
\]

\[
\approx \frac{1}{s_k^k} \sum_{r \in \Pi^k} \frac{k!}{r_1! \cdots r_k!} \left( \sum_{i=1}^{r_1} b_{r_i} \right)^{r_1} \cdots \left( \sum_{i=1}^{r_k} b_{r_i}^k \right)^{r_k} \mathbb{E}X (r) \cdot (C.4)
\]

Each summand in (C.4) is indexed by a profile \( r \in \Pi^k \). Thus there are only finitely many of them (as noted above, at most \((k + 1)^k\), which is independent of \( r \)), and each summand depends on \( r \). Most of these summands do not contribute to the moment \( M_k \) in the sense that the corresponding summand goes to 0 as \( r \to \infty \). Our first task is to determine which of the summands contribute and discard the others. Secondly, we calculate the limit of the contributing summands and show they converge to the moment of order \( k \) of the claimed limiting distribution.

Let \( \beta < 1 \). The limit given in Theorem 3 is standard normal, and we have to show that \( M_k \) converges to the moment of order \( k \) of a standard normal distribution. The moments of the standard normal are given by

\[
\begin{cases} 
(k - 1)!!, & k \text{ even}, \\
0, & k \text{ odd}.
\end{cases}
\]

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First we note that due to the symmetric nature of the Curie-Weiss distribution defined in (3.1), all odd moments \( M_k \) are 0. Let \( k \) be even. Employing Proposition 10 and noting that for all \( i \in \left\{ \frac{r+1}{2}, \ldots, r \right\} \) and all \( j \in \mathbb{N}_0 \)

\[
X_{ri}^j = \begin{cases} 
1 & \text{if } j \text{ is even,} \\
X_{ri} & \text{otherwise,}
\end{cases}
\]

the correlation \( \mathbb{E}X(\tau) \) can be expressed for each profile \( \tau \in \Pi^k \) as

\[
(o - 1)!! \bar{\beta}^{o/2} \left( \frac{2}{r} \right)^{o/2},
\]

where \( o \) stands for the sum of \( r_j \) over all odd \( j \). By (C.4), we have

\[
M_k \approx \frac{1}{s_r^k} \sum_{\tau \in \Pi^k} \frac{k!}{r_1! \cdots r_k!} \left( \sum_{i=\frac{r+1}{2}}^r b_{ri} \right)^{r_1} \cdots \left( \sum_{i=\frac{r+1}{2}}^r b_{ki} \right)^{r_k} (o - 1)!! \bar{\beta}^{o/2} \left( \frac{2}{r} \right)^{o/2}.
\]

We inspect the factors in each summand which depend on \( r \). These are:

\[
\frac{1}{s_r^k} \left( \sum_{i=\frac{r+1}{2}}^r b_{ri} \right)^{r_1} \cdots \left( \sum_{i=\frac{r+1}{2}}^r b_{ki} \right)^{r_k} \left( \frac{1}{r} \right)^{o/2} = \Theta \left( \frac{1}{r^{k/4}} \right) \Theta \left( r^{\frac{k}{2} \sum_{j=1}^k r_j} \right) \frac{1}{r^{o/2}}.
\]

Above we used the definition of \( s_r \) and the asymptotic expression provided by Lemma 9. The powers of \( r \) are

\[
-k/4 + 1/2 \sum_{j=1}^k r_j - o/2 = -k/4 + 1/2 \sum_{j=1}^{k/2} r_{2j}.
\]

The summand converges (to a positive constant or to 0) if and only if (C.5) is smaller or equal to 0. Since we have

\[
k = \sum_{j=1}^k j r_j \geq 2 \sum_{j=1}^{k/2} r_{2j},
\]

all summands in \( M_k \) converge. Moreover, as (C.5) being smaller or equal to 0 is equivalent to the last inequality presented, (C.5) equals 0 if and only if \( 2r_2 = k \). Hence, the only summand in (C.4) which contributes is the one corresponding to the profile \((0, k/2, 0, \ldots, 0)\). For all other profiles, (C.5) is negative, and thus the summand converges to 0. We have thus shown

\[
M_k \approx \frac{1}{s_r^k} \left( \frac{k!}{(k/2)! 2^{k/2}} \right)^{k/2} \left( \sum_{i=\frac{r+1}{2}}^r b_{ri}^2 \right)^{k/2} = \frac{k!}{(k/2)! 2^{k/2}} = (k - 1)!!.
\]

This proves the moments \( M_k \) converge to the moments of the standard normal distribution and concludes the proof of the statement for \( \beta < 1 \) in Theorem 3.
Now let $\beta = 1$. We analyse the moments $M_k$ given in (C.4). Odd moments are 0, so let $k$ be even. According to Proposition 10, the correlations $E X(y)$ can be expressed as

$$c_0 \left( \frac{2}{r} \right)^{\alpha/4},$$

where $o$ again stands for the sum of $r_j$ over all odd $j$. We inspect the factors in each summand in (C.4) which depend on $r$. These are:

$$\frac{1}{s_r} \left( \sum_{i=\frac{r}{2}+1}^{r} b_{ri} \right)^{r_1} \cdots \left( \sum_{i=\frac{r}{2}+1}^{r} b_{ri} \right)^{r_k} \left( \frac{1}{r} \right)^{\alpha/4} = \Theta \left( \frac{1}{r^{k/4}} \right) \Theta \left( r^{\frac{1}{2} \sum_{j=1}^{k} r_j} \right) 1 \frac{r^{-\alpha/4}}{r^{\alpha/4}}.$$

The powers of $r$ are

$$-\frac{k}{4} + \frac{1}{2} \sum_{j=1}^{k} r_j - \frac{o}{4} = -\frac{k}{4} + \frac{1}{2} \sum_{j=1}^{k/2} r_{2j} + \frac{1}{4} o. \quad (C.6)$$

The above expression is smaller or equal to 0 if and only if

$$k = \sum_{j=1}^{k} j r_j \geq o + 2 \sum_{j=1}^{k/2} r_{2j}.$$

Since this equality holds for all profiles, none of the summands diverges. The contributing summands are those for which equality holds in the inequality above. The profiles for which equality holds are $\{(2j; \frac{k}{2} - j, 0, \ldots, 0) \mid j = 0, \ldots, \frac{k}{2}\}$, i.e. each index occurs either once or twice. Therefore,

$$M_k \approx \frac{1}{s_r} \frac{k!}{(2j)!(\frac{k}{2} - j)!2^{k/2-j}} \left( \sum_{i=\frac{r}{2}+1}^{r} b_{ri} \right)^{2j} \left( \sum_{i=\frac{r}{2}+1}^{r} b_{ri}^2 \right)^{k/2-j} c_{2j} \left( \frac{2}{r} \right)^{j/2}.$$

So we have convergence of the moments to fixed limits and as a consequence also convergence in distribution. The moments given above are not those of a normal distribution, which can be verified by calculating $M_2$ and $M_4$ and noting that $\lim_{r \to \infty} M_4 \neq 3 (\lim_{r \to \infty} M_2)^2$, whereas the moments of any centred normal distribution satisfy this equality.

Finally, we treat the case $\beta > 1$. Since the normalisation of $\lambda_r(a)$ is now $\sqrt{r}$, the triangular array $(Y_{ri})_{r \in \mathbb{N}, i \in \left\{ \frac{r}{2} + \cdots + r \right\}}$ is defined by $Y_{ri} := \frac{b_r(1-2a_i)}{\sqrt{r}} = \frac{b_r X_{ri}}{\sqrt{r}}$ and the moments $M_k$ by

$$M_k := E \left( \sum_{i=1}^{n} Y_{ni} \right)^k = \frac{1}{r^{k/2}} \sum_{i_1, \ldots, i_k = \frac{r}{2} + 1}^{r} b_{ri_1} \cdots b_{ri_k} E X_{ri_1} \cdots X_{ri_k}$$

$$\approx \frac{1}{r^{k/2}} \sum_{i \in \Pi^k} r_1! \cdots r_k!1^{r_1} \cdots k^{r_k} \left( \sum_{i=\frac{r}{2}+1}^{r} b_{ri} \right)^{r_1} \cdots \left( \sum_{i=\frac{r}{2}+1}^{r} b_{ri}^2 \right)^{r_k} E X(y).$$

By Proposition 10, the correlations $E X(y)$ converge to

$$x(\beta)^\alpha$$

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in the limit \( r \to \infty \), where \( o \) again stands for the sum of \( r_j \) over all odd \( j \).

The odd moments \( M_k \) are 0. For the even moments, each summand of \( M_k \) has the following factors which depend on \( r \):

\[
\frac{1}{r^{k/2}} \left( \sum_{i=\frac{r+1}{2}}^r b_{ri} \right) \cdots \left( \sum_{i=\frac{r+1}{2}}^r b_{ri}^k \right) = \frac{1}{r^{k/2}} \Theta \left( r^{\frac{k}{2}} \sum_{j=1}^k r_j \right).
\]

Thus, the powers of \( r \) in each summand are

\[
-\frac{k}{2} + \frac{1}{2} \sum_{j=1}^k r_j.
\]

This expression is non-positive for all profiles, so none of the summands diverges. The inequality holds with equality for the profile \((k,0,\ldots,0)\), and it is strict for all other profiles. Hence, the moment \( M_k \) is asymptotically equal to

\[
M_k \approx \frac{1}{r^{k/2}} \left( \sum_{i=\frac{r+1}{2}}^r b_{ri} \right)^k x(\beta)^k \approx \frac{1}{r^{k/2}} \left( \frac{2r}{\pi} \right)^{k/2} x(\beta)^k = \left( \frac{2}{\pi} \right)^{1/2} x(\beta)^k =: m^k.
\]

Since all odd moments are 0 and all even moments are \( m^k \), we conclude that \( \frac{\lambda_r(a)}{\sqrt{r}} \) converges in distribution to \( \frac{1}{2}(\delta_m + \delta_m) \).

\[\text{D Proof of Theorem 4}\]

Contrary to the CWM defined in (3.1), the model with an external magnetic field defined in (3.4) features flip parameters with expectations different than \( 1/2 \), analogous to the case of independent flip parameters treated in Theorem 1. The CWM with an external magnetic field shows a different behaviour and we cannot employ Proposition 10 for the correlations.

Instead of the equation \( \tanh \beta x = x \), we have to analyse the more general equation

\[
\tanh (\beta (x + h)) = x.
\]

(D.1)

For any value of \( \beta \geq 0 \), \( h \neq 0 \), there is a solution \( x(\beta, h) \) of (D.1) with the same sign as \( h \). (We remark that if \( \beta > 1 \) and \( |h| \) is small enough, then there are one or two solutions of the opposite signs as well. These are of no consequence for our analysis. See Sections IV.4 and V.9 of [12] for a detailed analysis of this model.)

We have the following limit result for the correlations:

**Proposition 12.** Let \( \beta \geq 0 \), \( h \neq 0 \), and \( x(\beta, h) \) the solution of (D.1) defined above. Then we have for all \( k \in \mathbb{N} \)

\[
\mathbb{E} X_{r+1} \cdots X_{r+k} \xrightarrow{r \to \infty} x(\beta, h)^k.
\]

As this proposition suggests, contrary to the model without an external magnetic field treated previously, we do not have distinct behaviour in the form of three different regimes as a function of \( \beta \).

We define the triangular array \((Y_{ri})_{r \in \mathbb{N}, i \in \{\frac{r+1}{2}, \ldots, r\}}\) by

\[
Y_{ri} := \frac{b_{ri}(1-2X_i-x(\beta, h))}{s_r} = \frac{b_{ri}(X_i-x(\beta, h))}{s_r}
\]

and the moments \( M_k \) for all \( k \in \mathbb{N} \) by
\[ M_k := E \left( \sum_{i=\frac{r+1}{2}}^{r} Y_{ri} \right)^k = \frac{1}{s_r^k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} x (\beta, h)^{k-j} E \lambda_r (a)^j. \]

The last factor in each summand above, \( E \lambda_r (a)^j \), can be expanded as we did in the proof of Theorem 3:

\[ E \lambda_r (a)^j = \sum_{i_1, \ldots, i_j = \frac{r+1}{2}} b_{ri_1} \cdots b_{ri_j} E X_{ri_1} \cdots X_{ri_j} \]
\[ \approx \sum_{\mathfrak{r} \in \Pi^j} \frac{j!}{r_1! \cdots r_j! \Pi r_1^{r_1} \cdots j! r_j^{r_j}} \left( \sum_{i_{t=\frac{r+1}{2}}}^{r} b_{ri_t} \right)^{r_1} \cdots \left( \sum_{i_{t=\frac{r+1}{2}}}^{r} b_{ri_t}^{r_j} \right)^{r_j} E X (\mathfrak{r}). \]

As a consequence of Proposition 12, the correlations \( E X (\mathfrak{r}) \) for all \( \mathfrak{r} \in \Pi^j \) converge to

\[ E X (\mathfrak{r}) \xrightarrow{r \to \infty} x (\beta, h)^o, \]

where \( o \) is the sum of the \( r_\ell \) over all odd natural numbers \( \ell \) in the range \([1, j]\).

Thus, \( M_k \) is a double sum over \( j = 0, \ldots, k \) and \( \mathfrak{r} \in \Pi^j \). The remainder of the proof consists of determining the factors which depend on \( r \) in each summand,

\[ \Theta \left( \frac{1}{s_r} \right) \Theta \left( r \frac{j}{2} \sum_{\ell=1}^{r_j} r_\ell \right), \]

analysing which of the summands contribute asymptotically to \( M_k \), calculating their sum, and showing that it converges to the corresponding moment of order \( k \) of a centred normal distribution. Since this procedure is very similar to the three cases we have already treated in such fashion in the proof of Theorem 3, we will omit this part and conclude here.

**Conflicts of Interest**

The author declares that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Data Availability**

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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