The quantum query complexity of composition with a relation

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Abstract

The negative weight adversary method, \( \text{ADV}^\pm(g) \), is known to characterize the bounded-error quantum query complexity of any Boolean function \( g \), and also obeys a perfect composition theorem \( \text{ADV}^\pm(f \circ g^n) = \text{ADV}^\pm(f)\text{ADV}^\pm(g) \). Belovs [Bel15b] gave a modified version of the negative weight adversary method, \( \text{ADV}^\pm_{\text{rel}}(f) \), that characterizes the bounded-error quantum query complexity of a relation \( f \subseteq \{0, 1\}^n \times [K] \), provided the relation is efficiently verifiable. A relation is efficiently verifiable if \( \text{ADV}^\pm(f_a) = o(\text{ADV}^\pm_{\text{rel}}(f)) \) for every \( a \in [K] \), where \( f_a \) is the Boolean function defined as \( f_a(x) = 1 \) if and only if \( (x, a) \in f \).

In this note we show a perfect composition theorem for the composition of a relation \( f \) with a Boolean function \( g \)

\[
\text{ADV}^\pm_{\text{rel}}(f \circ g^n) = \text{ADV}^\pm_{\text{rel}}(f)\text{ADV}^\pm(g).
\]

For an efficiently verifiable relation \( f \) this means \( Q(f \circ g^n) = \Theta(\text{ADV}^\pm_{\text{rel}}(f)\text{ADV}^\pm(g)) \).

1 Introduction

Quantum query complexity has been a very successful model for studying quantum algorithms. The most famous quantum algorithms, like Grover’s search algorithm [Gro96] and the period finding routine of Shor’s algorithm [Sho97], can be formulated in this model, and the model has also been fruitful for developing new algorithmic techniques like quantum walks [Amb07, Sze04, MSS07] and learning graphs [Bel12b].

One of the greatest successes of quantum query complexity is the adversary method. The (un-weighted) adversary method began as a lower bound technique developed by Ambainis [Amb02] to show a lower bound on the quantum query complexity of the two-level AND-OR tree, among other problems. It was later generalized to a weighted version in various forms by several authors [Amb06, Zha05, BSS03, LM08]. Špalek and Szegedy [SS06] showed that all of these generalizations were in fact equivalent.

A strictly stronger bound called the negative weights adversary method, \( \text{ADV}^\pm(f) \), was developed by Høyer et al. [HLŠ07]. In a wonderful turn of events, a series of works by Reichardt

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et al. [RS12, Rei14, Rei11] showed that the bounded-error quantum query complexity, $Q(f)$, of a Boolean function $f$ satisfies $Q(f) = O(ADV(f))$, and therefore $ADV(f)$ characterizes bounded-error quantum query complexity up to a constant multiplicative factor.

This characterization has several interesting consequences. For one, it means that upper bounds on quantum query complexity on a function $f$ can now be shown by upper bounding $ADV(f)$, which can be expressed as a relatively simple semidefinite program. This approach has led to improved algorithms for many problems of interest [Bel12a, Bel15a, LMS17], especially via the development of the learning graphs model of Belovs [Bel12b].

Another consequence is that quantum query complexity inherits the nice properties of the adversary bound. One of these nice properties is that the adversary bound behaves perfectly with respect to function composition. For Boolean functions $f : \{0, 1\}^n \rightarrow \{0, 1\}$ and $g : \{0, 1\}^m \rightarrow \{0, 1\}$ define the composition $h = f \circ g^n$ to be the function $h : \{0, 1\}^{nm} \rightarrow \{0, 1\}$ where for an input $x = (x_1, \ldots, x_n) \in \{0, 1\}^n$, with each $x_i \in \{0, 1\}^m$, we have $h(x) = f(g(x_1), \ldots, g(x_n))$. [HLS07] showed that $ADV(h) = ADV(f)ADV(g)$, and [Rei14] showed a matching upper bound.

**Theorem 1 ([HLS07, Rei14]).** Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ and $g : \{0, 1\}^m \rightarrow \{0, 1\}$.

$$ADV(f \circ g^n) = ADV(f)ADV(g).$$

This perfect composition theorem was later extended by [LMR+11] to allow $g$ to be a partial Boolean function and allow the range of $f$ to be non-Boolean. Kimmel [Kim13] showed a perfect composition theorem where both $f$ and $g$ were allowed to be partial Boolean functions. Thus here the domain of $h = f \circ g^n$ is only those $x = (x_1, \ldots, x_n)$ where $(g(x_1), \ldots, g(x_n))$ is in the domain of $f$.

Extensions of the negative weight adversary method have been given for the more general query complexity problem of state generation [Shi02, AMRR11, LMR+11]. In this problem, on input $x$ the algorithm begins in the state $|0\rangle|0\rangle$ as usual but now the goal is to transform this state into the state $|\sigma_x\rangle|0\rangle$ for some target vector $|\sigma_x\rangle$ by making queries to $x$. Lee et al. [LMR+11] have shown that an extension of the negative weight adversary method called the filtered $\gamma_2$ norm gives a semi-tight characterization of the quantum query complexity of the state conversion problem. It is semi-tight because a slightly larger error is needed in the upper bound than in the lower bound.

Belovs [Bel15b] used this characterization to give a modified adversary bound, $ADV_{rel}(f)$, that characterizes the quantum query complexity of a relation $f \subseteq \{0, 1\}^n \times [K]$, provided the relation is efficiently verifiable. Intuitively, a relation is efficiently verifiable if given $a$ the complexity of checking if $(x, a)$ is in the relation is low-order compared to $ADV_{rel}(f)$; the formal definition is given in Definition 15. For an efficiently verifiable relation the success probability of an algorithm can be amplified without increasing the order of the complexity, getting around the semi-tightness of the [LMR+11] characterization.

In this work, we show a perfect composition theorem $ADV_{rel}(f \circ g^n) = ADV_{rel}(f)ADV(g)$ for the composition of a relation $f \subseteq \{0, 1\}^n \times [K]$ with Boolean function $g$. If $f$ is efficiently verifiable this implies that $Q(f \circ g^n) = \Theta(ADV_{rel}(f)ADV(g))$. The lower-bound part of this theorem was required to show a lower bound on the runtime of a quantum algorithm constructing a cut sparsifier of a graph [AdW19]. The perfect adversary composition theorem for functions
has been a very useful tool, both for constructing algorithms and showing lower bounds, and we believe our composition theorem for relations will find additional applications as well.

## 2 Preliminaries

For a positive integer $m$ we let $[m] = \{1, \ldots, m\}$. For two matrices $A, B$ of the same size, $A \circ B$ denotes the Hadamard or entrywise product: $(A \circ B)(x, y) = A(x, y)B(x, y)$. We use $\|A\|$ to denote the spectral norm of a matrix $A$ and $\|A\|_r$ to denote the trace norm. For a symmetric matrix $A$, we use $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ to denote the largest and smallest eigenvalues of $A$, respectively. We use $I_n$ for the $n$-by-$n$ identity matrix, and drop the subscript when the dimension is implied from context. For a vector $v \in \mathbb{R}^n$ and natural numbers $a \leq b$ we let $v(a : b) \in \mathbb{R}^{b-a+1}$ be the vector $(v(a), v(a+1), \ldots, v(b))$.

We will need a few simple facts about positive semidefinite matrices. First, if $A, B \succeq 0$, then $A \otimes B \succeq 0$. Also, a principal submatrix of a positive semidefinite matrix is positive semidefinite. A matrix $A$ obtained from $A$ by duplicating rows and columns is again positive semidefinite. The last two facts follow from the following more general observations:

**Fact 2.** Let $A \succeq 0$ be an $M$-by-$M$ matrix, and $h : [N] \to [M]$ a function. Then the $N$-by-$N$ matrix $\hat{A}$ defined by $\hat{A}(x, y) = A(h(x), h(y))$ is also positive semidefinite.

**Fact 3.** If $A, B$ are the same size and $A \succeq 0, B \succeq 0$ then $A \circ B \succeq 0$. Similarly, if $A \succeq 0, B \preceq 0$ then $A \circ B \preceq 0$.

**Fact 4.** Let $A$ be a symmetric matrix. Then $\lambda_{\max}(A) \cdot I \succeq A \succeq \lambda_{\min}(A) \cdot I$. In particular, $\lambda_{\max}(A) \geq v^T Av \geq \lambda_{\min}(A)$ for any unit vector $v$.

### 2.1 Quantum query complexity

The bounded-error quantum query complexity of the function $g : \{0, 1\}^n \to \{0, 1\}$, denoted $Q(g)$, is the minimum number of queries needed by a quantum algorithm that outputs $g(x)$ with probability at least $2/3$ for every input $x \in \{0, 1\}^n$. Let $K$ be a positive integer and $f \subseteq \{0, 1\}^n \times [K]$ be a relation. We say that a quantum algorithm computes $f$ if for every $x \in \{0, 1\}^n$ the algorithm outputs an $a \in [K]$ such that $(x, a) \in f$ with probability at least $2/3$. We let $Q(f)$ denote the minimum cost of a quantum query algorithm that computes $f$.

We will always assume that for every $x \in \{0, 1\}^n$ there exists $a \in [K]$ such that $(x, a) \in f$. This assumption is without loss of generality as if $f$ defined on a strict subset $S$ of $\{0, 1\}^n$ we can consider instead $f'$ which is defined as $f$ together with every pair $(x, a) \in (\{0, 1\}^n \setminus S) \times [K]$. In this way any algorithm that computes $f$ on $S$ also computes $f'$ on $\{0, 1\}^n$, as any output is accepted for inputs not in $S$.

### 2.2 Adversary bound

**Definition 5** (Adversary matrix for a function). Let $g : \{0, 1\}^n \to \{0, 1\}$ be a Boolean function. Let $\Gamma_g \in \mathbb{R}^{2n \times 2n}$, with rows and columns labeled by elements of $\{0, 1\}^n$. We say that $\Gamma_g$ is
a functional adversary matrix for \( g \) iff \( \Gamma_g \) is symmetric and \( \Gamma_g(x, y) = 0 \) for all \( x, y \) such that \( g(x) = g(y) \).

Up to a permutation of rows and columns, we may always assume that the first \( |g^{-1}(0)| \) rows and columns of a functional adversary matrix \( \Gamma_g \) are labeled by elements of \( g^{-1}(0) \). Then there is a \( |g^{-1}(0)| \times |g^{-1}(1)| \) matrix \( Z \) such that

\[
\Gamma_g = \begin{bmatrix} 0 & Z \\ Z^T & 0 \end{bmatrix}.
\]

In this paper we always assume that adversary matrices for functions are presented in this form.

**Definition 6** (Functional adversary bound \cite{HLS07}). Let \( g : \{0, 1\}^n \rightarrow \{0, 1\} \) be a Boolean function. For \( i \in [n] \), let \( D_i \) be a \( 2^n \times 2^n \) Boolean matrix where \( D_i(x, y) = 1 \) iff \( x_i \neq y_i \). The adversary bound \( \text{ADV}^\pm(g) \) for \( g \) is defined as

\[
\text{ADV}^\pm(g) = \max_{\Gamma} \|\Gamma\| \text{ subject to } \|\Gamma \circ D_i\| \leq 1 \text{ for all } i = 1, \ldots, n, \\
\Gamma(x, y) = 0 \text{ for all } x, y \in \{0, 1\}^n \text{ with } g(x) = g(y).
\]

We will also need the dual formulation of the adversary bound.

**Theorem 7** (\cite{LMR11} Theorem 3.4). Let \( g : \{0, 1\}^n \rightarrow \{0, 1\} \) be a Boolean function. Let \( \{u_{x,i}\}, \{v_{x,i}\} \) be two families of vectors of arbitrary finite dimension indexed by \( x \in \{0, 1\}^n \) and \( i \in [n] \).

\[
\text{ADV}^\pm(g) = \min_{\{u_{x,i}\}, \{v_{x,i}\}} \max \left\{ \max_{x \in \{0, 1\}^n} \sum_{i \in [n]} \|u_{x,i}\|^2, \max_{x \in \{0, 1\}^n} \sum_{i \in [n]} \|v_{x,i}\|^2 \right\} \\
\text{subject to } \sum_{i : x_i \neq y_i} \langle u_{x,i}, v_{y,i} \rangle = \begin{cases} 1 & \text{ if } g(x) \neq g(y) \\ 0 & \text{ if } g(x) = g(y) \end{cases} \text{ for all } x, y \in \{0, 1\}^n
\]

If one simply takes the dual of **Definition 6** one gets the above optimization problem without the constraint on \( x, y \) pairs where \( g(x) = g(y) \). These additional constraints are needed to show the upper bound in the composition theorem, and \cite{LMR11} show they can be added without increasing the objective value of the program.

The functional adversary bound characterizes the bounded-error quantum query complexity of any function \( g \). The lower bound is due to Høyer et al. \cite{HLS07} and the upper bound due to Reichardt \cite{Rei11}.

**Theorem 8** (\cite{HLS07}, \cite{Rei11}). Let \( g : \{0, 1\}^n \rightarrow \{0, 1\} \) be a Boolean function. Then \( Q(g) = \Theta(\text{ADV}^\pm(g)) \)
3 Adversary bound for relations

Belovs [Bel15b] developed a modification, \( \text{ADV}_{rel}^\pm(f) \), of the adversary bound that relates to the bounded-error quantum query complexity of a relation \( f \subseteq \{0, 1\}^n \times [K] \). To motivate this bound, we first review the state generation problem [Shi02, AMRR11, LMR+11]. A state generation problem is specified by a family of vectors \( |\sigma_x\rangle \in \mathbb{R}^M \) for each \( x \in \{0, 1\}^n \). On input \( x \) the algorithm begins in the state \( |0\rangle|0\rangle \) and the goal is for the algorithm to finish in the target state \( |\sigma_x\rangle|0\rangle \) after making as few queries to \( x \) as possible.

Lee et al. [LMR+11] give the following definition and theorem.

**Definition 9** (Filtered \( \gamma_2 \) norm). Let \( A \) be a \( 2^n \)-by-\( 2^n \) matrix and \( D = \{D_1, \ldots, D_n\} \) where each \( D_i \) is a \( 2^n \)-by-\( 2^n \) Boolean matrix defined as \( D_i(x, y) = 1 \) iff \( x_i \neq y_i \). Define

\[
\gamma_2(A|D) = \min_{\{u_{x,i}, v_{x,i}\}} \max \left\{ \frac{1}{\max_{x \in \{0,1\}^n}} \sum_{i \in [n]} \|u_{x,i}\|^2, \frac{1}{\max_{x \in \{0,1\}^n}} \sum_{i \in [n]} \|v_{x,i}\|^2 \right\} \\
\text{subject to } \sum_{i \in [n]} \langle u_{x,i}, v_{y,i} \rangle \cdot D_i(x, y) = A(x, y) \text{ for all } x, y \in \{0, 1\}^n
\]

**Theorem 10** ([LMR+11]). Let \( M, n \) be positive integers and \( \{|\sigma_x\rangle\}_{x \in \{0,1\}^n} \) be a family of vectors with each \( |\sigma_x\rangle \in \mathbb{R}^M \). Let \( A \) be a \( 2^n \)-by-\( 2^n \) matrix where \( A(x, y) = 1 - \langle \sigma_x | \sigma_y \rangle \) for all \( x, y \in \{0, 1\}^n \). Let \( D = \{D_1, \ldots, D_n\} \) where each \( D_i \) is a \( 2^n \)-by-\( 2^n \) Boolean matrix defined as \( D_i(x, y) = 1 \) iff \( x_i \neq y_i \). For any \( 0 < \epsilon < \gamma_2(A|D) \) there is a quantum algorithm that for every \( x \in \{0, 1\}^n \) terminates in a state \( |\sigma_x^\epsilon\rangle \) satisfying \( \langle \sigma_x^\epsilon | (|\sigma_x\rangle \otimes |0\rangle) \geq \sqrt{1 - \epsilon} \) after making \( O(\gamma_2(A|D) \frac{\log(1/\epsilon)}{\epsilon^2}) \) many queries to \( x \).

In computing a relation \( f \subseteq \{0, 1\}^n \times [K] \), there is not a fixed ideal target state; rather the algorithm has the freedom to optimize over a set of target states which we call perfect target states for \( f \).

**Definition 11** (Perfect target states). Let \( f \subseteq \{0, 1\}^n \times [K] \) and \( \{|\psi_x\rangle\}_{x \in \{0,1\}^n} \) a family of unit vectors. We say that \( \{|\psi_x\rangle\}_{x \in \{0,1\}^n} \) are perfect target states for \( f \) if there exists a complete family of orthogonal projectors \( \{\Pi_a\}_{a \in [K]} \) such that each \( |\psi_x\rangle \) can be decomposed as \( |\psi_x\rangle = \sum_{a: (x,a) \in f} |\sigma_{x,a}\rangle \) for some (un-normalized) vectors \( |\sigma_{x,a}\rangle \) satisfying \( \Pi_a |\sigma_{x,a}\rangle = |\sigma_{x,a}\rangle \) for all \( x, a \).

Let \( \{|\psi_x\rangle\}_{x \in \{0,1\}^n} \) be a family of perfect target states for \( f \) and suppose there is a state generation algorithm that on input \( x \) terminates in the state \( |\psi_x\rangle|0\rangle \) for every \( x \in \{0, 1\}^n \). If the algorithm measures according to the projectors \( \{\Pi_a \otimes I\}_{a \in [K]} \), for the projectors \( \{\Pi_a\}_{a \in [K]} \) witnessing the perfection of \( \{|\psi_x\rangle\}_{x \in \{0,1\}^n} \), then for every \( x \) it will output an \( a \) with \( (x, a) \in f \) with certainty. This motivates studying the optimization problem of minimizing the filtered \( \gamma_2 \) norm \( \gamma_2(1 - [\langle |\psi_x| \psi_y\rangle]_{x,y \in \{0,1\}^n}|D) \) over all families of perfect target states \( \{|\psi_x\rangle\}_{x \in \{0,1\}^n} \) for \( f \). This optimization problem gives Belovs’ definition of the relational adversary bound.
Definition 12 ([Bell15b] Equation 20). Let \( f \subseteq \{0, 1\}^n \times [K] \).

\[
\text{ADV}_{\text{rel}}^\pm(f) = \min_{\{u_{x,i}, v_{y,i}, \sigma_{x,a}\}} \max \left\{ \max_{x \in \{0, 1\}^n} \sum_{i \in [n]} \|u_{x,i}\|^2, \max_{x \in \{0, 1\}^n} \sum_{i \in [n]} \|v_{y,i}\|^2 \right\}
\]
subject to
\[
\sum_{i:x \neq y} \langle u_{x,i}, v_{y,i} \rangle = 1 - \sum_{a \in [K]} \langle \sigma_{x,a}, \sigma_{y,a} \rangle \quad \text{for all } x, y \in \{0, 1\}^n
\]
\[
\|\sigma_{x,a}\|^2 = 0 \quad \text{for all } x, a \text{ with } (x, a) \notin f
\]

Theorem 13. Let \( f \subseteq \{0, 1\}^n \times [K] \). Then \( Q(f) = O(\text{ADV}_{\text{rel}}^\pm(f)) \).

Proof. Let \( \{ |\sigma_{x,a}\rangle \}_{x \in \{0, 1\}^n, a \in [K]} \) be part of an optimal solution to the program for \( \text{ADV}_{\text{rel}}^\pm(f) \). Let \( m_a \) be the dimension of \( |\sigma_{x,a}\rangle \) and define \( |\psi_x\rangle = |\sigma_{x,1}\rangle \otimes \cdots \otimes |\sigma_{x,K}\rangle \in \mathbb{R}^M \) where \( M = \sum_{a \in [K]} m_a \). For \( a \in [K] \), let \( s_a = \sum_{i < a} m_i \) and define \( \Pi_a = \sum_{i = s_i + 1}^{s_a + m_a} |e_i\rangle \langle e_i| \), where \( |e_i\rangle \) is the \( i^{th} \) standard basis vector. For \( A(x, y) = 1 - \langle \psi_x | \psi_y \rangle \) we have \( \gamma_2(A|D) = \text{ADV}_{\text{rel}}^\pm(f) \). Therefore by applying Theorem 10 with \( \epsilon = 1/3 \) there is a quantum algorithm that on input \( x \) terminates in a state \( |\psi_x'\rangle \) satisfying \( \langle \psi_x' | (|\psi_x\rangle \otimes |0\rangle) \geq \sqrt{1 - \epsilon} \) after making \( O(\text{ADV}_{\text{rel}}^\pm(f)) \) many queries. This implies \( ||\psi_x'\rangle - |\psi_x\rangle|0\rangle||^2 \leq \epsilon \) and so
\[
\epsilon \geq ||\psi_x'\rangle - |\psi_x\rangle|0\rangle||^2
\]
\[
= \sum_{a \in [K]} \|(\Pi_a \otimes I)|\psi_x'\rangle - |\sigma_{x,a}\rangle \otimes |0\rangle||^2
\]
\[
\geq \sum_{a : (x,a) \notin f} ||(\Pi_a \otimes I)|\psi_x'\rangle||^2
\]

Thus running the state generation algorithm for the states \( \{|\psi_x\rangle\} \) and measuring according to the projectors \( \{\Pi_a \otimes I\}_{a \in [K]} \) gives an algorithm with error probability at most \( \epsilon = 1/3 \) as desired. \( \square \)

For the composition theorem we will also need the dual formulation of the relational adversary bound from [Definition 6]

Theorem 14 ([Bell15b] Equation 22). Let \( f \subseteq \{0, 1\}^n \times [K] \) be a relation. For \( i \in [n] \), let \( D_i \) be a \( 2^n \)-by-\( 2^n \) Boolean matrix where \( D_i(x, y) = 1 \) iff \( x_i \neq y_i \). For \( a \in [K] \) let
\[
\chi_a(x) = \begin{cases} 
1 & \text{if } (x, a) \in f \\
0 & \text{otherwise.}
\end{cases}
\]

Then
\[
\text{ADV}_{\text{rel}}^\pm(f) = \max_{\Gamma} \lambda_{\text{max}}(\Gamma)
\]
subject to
\[
\|\Gamma \circ D_i\| \leq 1 \quad \text{for all } i = 1, \ldots, n
\]
\[
\Gamma \circ \chi_a \chi_a^T \preceq 0 \quad \text{for all } a \in [K].
\]

We will call a matrix \( \Gamma \) satisfying \( \Gamma \circ \chi_a \chi_a^T \preceq 0 \) for all \( a \in [K] \) a relational adversary matrix.
Definition 15 (Efficiently verifiable). Let $K$ be a positive integer and $f \subseteq \{0,1\}^n \times [K]$ be a relation. For each $a \in [K]$ define a Boolean function $f_a$ by $f_a(x) = 1$ iff $(x,a) \in f$. We say that $f$ is efficiently verifiable if $\text{ADV}^\pm(f_a) = o(\text{ADV}^\pm_{rel}(f))$ for all $a \in [K]$.

Belovs [Bel15b] has shown that the relational adversary bound is also a lower bound on the bounded-error quantum query complexity of a relation $f$ that is efficiently verifiable.

Theorem 16 (Belovs [Bel15b] Theorem 40). Let $f \subseteq \{0,1\}^n \times [K]$ be an efficiently verifiable relation. Then $Q(f) = \Omega(\text{ADV}^\pm_{rel}(f))$.

4 Composition Theorem

Definition 17. Let $A$ be an $m$-by-$n$ matrix. Define

$$\hat{A} = \begin{bmatrix} \|A\|I_m & A \\ A^T & \|A\|I_n \end{bmatrix}.$$  \hspace{1cm} (2)

Lemma 18. $\hat{A}$ is positive semidefinite for any $A$.

Proof. The minimum eigenvalue of $A' = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$ is $-\|A\|$ thus $\hat{A} = \|A\|I + A' \succeq 0$ by Fact 4. \hfill $\Box$

Definition 19 (Matrix Composition). Let $N$ be a positive integer and let $B$ be a symmetric $2^N$-by-$2^N$ matrix. Let $A_1, \ldots, A_N$ be matrices where $A_i$ is of size $m_i$-by-$n_i$. Define the matrix composition of $B$ with $A_1, \ldots, A_N$ to be a matrix $C$ of size $\prod_{i=1}^N (m_i + n_i)$ with rows and columns labeled by elements of $[m_1 + n_1] \times \cdots \times [m_N + n_N]$. For $a = (a_1, \ldots, a_N) \in [m_1 + n_1] \times \cdots \times [m_N + n_N]$ define $\tilde{a} \in \{0,1\}^N$ to be the string where $\tilde{a}_i = 1$ if $a_i > m_i$ and $\tilde{a}_i = 0$ otherwise. Let $\tilde{B}$ be the matrix of the same size as $C$ where $\tilde{B}(a,b) = B(\tilde{a}, \tilde{b})$. Then

$$C = \tilde{B} \circ (\otimes_{i=1}^N \hat{A}_i).$$

Remark 20. Each $A_i$ implicitly defines a function $g_i : [m_i + n_i] \to \{0,1\}$ where $g_i(a) = 1$ iff $a > m_i$. Under this interpretation, the definition of $\tilde{a}$ arises naturally as $\tilde{a} = (g_1(a_1), \ldots, g_N(a_N))$ for $a = (a_1, \ldots, a_N) \in [m_1 + n_1] \times \cdots \times [m_N + n_N]$.

Let $C$ be the matrix composition of $B$ with $A_1, \ldots, A_N$. The key to adversary composition theorems is the fact that $\|C\| = \|B\| \cdot \prod_{i=1}^N \|A_i\|$, as originally shown in [HLS07] Lemma 16. The statement we give here differs in two respects from the statement given in [HLS07]. First, in [HLS07] the lemma was only stated for $B$ restricted to be of the form

$$B = \begin{bmatrix} 0 & Z \\ Z^T & 0 \end{bmatrix},$$

for some matrix $Z$. To show the relational composition theorem we need to allow an arbitrary symmetric matrix $B$. Second, as the objective function of the relational adversary bound in [Theorem 14]
is in terms of $\lambda_{\max}$ we need a statement both about the spectral norm of $C$ and about $\lambda_{\max}(C)$. The proof given in [HLS07] can handle both of these generalizations; the assumption about the structure of $B$ is not used and the proof already determines the entire spectrum of $C$. Instead, however, we give a new proof which is substantially shorter than the one from [HLS07].

**Lemma 21.** Let $N$ be a positive integer. Let $B \in \mathbb{R}^{2^N \times 2^N}$ be a symmetric matrix and $A_1, \ldots, A_N$ be arbitrary matrices. If $C$ is the matrix composition of $B$ with $A_1, \ldots, A_N$ then

$$
\|C\| = \|B\| \cdot \prod_{i=1}^{N} \|A_i\| \quad \text{and} \quad \lambda_{\max}(C) \geq \lambda_{\max}(B) \cdot \prod_{i=1}^{N} \|A_i\| .
$$

**Proof.** First we set up some notation which will be used for both the upper and lower bounds of the proof. For $i = 1, \ldots, N$ let $A_i$ be a $m_i$-by-$n_i$ matrix. Let $\|A_i\| = \lambda_1^{(i)} \geq \cdots \geq \lambda_{m_i+n_i}^{(i)}$ be the eigenvalues of the matrix $\begin{bmatrix} A_i & 0 \\ A_i^T & 0 \end{bmatrix}$ and let $z_1^{(i)}, \ldots, z_{m_i+n_i}^{(i)}$ be the corresponding eigenvectors.

Note that $z_1^{(i)}, \ldots, z_{m_i+n_i}^{(i)}$ form an orthonormal basis for $\mathbb{R}^{m_i+n_i}$. Finally, let $z_1^{(i)} = z_1^{(i)}(1 : m_i)$ and $z_2^{(i)} = z_1^{(i)}(m_i + 1 : m_i + n_i)$, and for $b_1, b_2 \in \{0, 1\}$ define $\hat{A}_i^{(b_1, b_2)}$ by

$$
\begin{bmatrix}
\hat{A}_i^{(0,0)} & \hat{A}_i^{(0,1)} \\
\hat{A}_i^{(1,0)} & \hat{A}_i^{(1,1)}
\end{bmatrix} = \begin{bmatrix} \|A_i\| I_{m_i} & A_i \\ A_i^T & \|A_i\| I_{n_i} \end{bmatrix} .
$$

From the eigenvalue equation we see that $A_i z_2^{(i)} = \lambda_1^{(i)} z_2^{(i)}$ and $A_i^T z_1^{(i)} = \lambda_1^{(i)} z_1^{(i)}$. This means

$$
\lambda_1^{(i)} \|z_2^{(i)}\|^2 = (z_1^{(i)})^T A_i z_2^{(i)} = \lambda_1^{(i)} (z_2^{(i)})^T A_i^T z_2^{(i)} = \lambda_1^{(i)} \|z_1^{(i)}\|^2
$$

and so $\|z_2^{(i)}\|^2 = \|z_1^{(i)}\|^2 = 1/2$ whenever $\lambda_1^{(i)} \neq 0$. Thus for $b_1 \in \{0, 1\}$ we can succinctly write

$$
(z_2^{(i)} b_1)^T \hat{A}_i^{(b_1, 1-b_1)} z_2^{(i)1-b_1} = \lambda_1^{(i)} \|z^{(i)}\|^2 \|z_2^{(i)}\|^2 .
$$

These observations lead to the crucial fact we need, for $b_1, b_2 \in \{0, 1\}$

$$
z_1^{(i)} b_1 \hat{A}_i^{(b_1, b_2)} z_2^{(i)} b_2 = \begin{cases} 0 & \text{if } j \neq k \\
\|A_i\| \|z_1^{(i)}\|^2 & \text{if } j = k, b_1 = b_2 \\
\lambda_1 \|z_2^{(i)}\|^2 \|z_1^{(i)}\|^2 & \text{if } j = k, b_1 \neq b_2 .
\end{cases}
$$

We now turn to showing the most difficult part of the lemma, that $\|C\| \leq \|B\| \cdot \Pi_{i=1}^{N} \|A_i\|$. Let $\psi_{j_1, \ldots, j_N} = \psi_{j_1}^{(1)} \otimes \cdots \otimes \psi_{j_N}^{(N)}$ where each $j_i \in [m_i + n_i]$. We compute

$$
\psi_{j_1, \ldots, j_N}^T C \psi_{k_1, \ldots, k_N} = \sum_{\alpha, \beta \in \{0, 1\}^N} B(\alpha, \beta) \prod_{i \in [N]} z_1^{(i)} \alpha_i \hat{A}_i^{(\alpha_i, 1-\beta_i)} \alpha_i^{(i)} z_2^{(i)} \beta_i .
$$

Suppose that $j_i \neq k_i$ for some $i$. Then by Eq. (4) $z_1^{(i)} \alpha_i \hat{A}_i^{(\alpha_i, 1-\beta_i)} z_2^{(i)} \beta_i = 0$ for all values of $\alpha_i, \beta_i$. Thus $\psi_{j_1, \ldots, j_N}^T C \psi_{k_1, \ldots, k_N} = 0$ unless $j_i = k_i$ for all $i = 1, \ldots, N$. 8
Let us now consider the value in this case. For \( i = 1, \ldots, N \) let

\[
D_i = \begin{bmatrix} \|A_i\| & \lambda_{j_i}^{(i)} \\ \lambda_{j_i}^{(i)} & \|A_i\| \end{bmatrix} \odot \left( \begin{bmatrix} \|z_j^{(i),0}\| \\ \|z_j^{(i),1}\| \end{bmatrix} \begin{bmatrix} \|z_j^{(i),0}\| & \|z_j^{(i),1}\| \end{bmatrix} \right) .
\]

Each \( D_i \succeq 0 \) because it is the Hadamard product of two positive semidefinite matrices (the first matrix is diagonally dominant as \( \|A_i\| \geq |\lambda_{j_i}^{(i)}| \) and so is psd). Thus \( \|D_i\|_{tr} = \text{Tr}(D_i) = \|A_i\| \). Let \( D = \bigotimes_{i=1}^N D_i \) and note that \( \|D\|_{tr} = \prod_{i=1}^N \|A_i\| \). Then from Eq. (4) and Eq. (5) we see that

\[
\psi^T_{j_1,\ldots,j_N} C \psi_{j_1,\ldots,j_N} = \text{Tr}(BD) \leq \|B\|\|D\|_{tr} = \|B\| \prod_{i=1}^N \|A_i\| .
\]

Let \( M = \prod_{i=1}^N (m_i + n_i) \) and let \( \psi = \sum_{j_1,\ldots,j_N} \alpha_{j_1,\ldots,j_N} \psi_{j_1,\ldots,j_N} \in \mathbb{R}^M \) be an arbitrary unit vector. Then

\[
\psi^T C \psi = \sum_{j_1,\ldots,j_N,k_1,\ldots,k_N} \alpha_{j_1,\ldots,j_N,k_1,\ldots,k_N} (\psi^T_{j_1,\ldots,j_N} C \psi_{k_1,\ldots,k_N})
\]

\[
= \sum_{j_1,\ldots,j_N} \alpha_{j_1,\ldots,j_N}^2 (\psi^T_{j_1,\ldots,j_N} C \psi_{j_1,\ldots,j_N})
\]

\[
\leq \|B\| \prod_{i=1}^N \|A_i\| .
\]

This shows \( \|C\| \leq \|B\| \cdot \prod_{i=1}^N \|A_i\| . \)

We now turn to showing the lower bounds

\[
\|C\| \geq \|B\| \cdot \prod_{i=1}^N \|A_i\| \quad (6)
\]

\[
\lambda_{\max}(C) \geq \lambda_{\max}(B) \cdot \prod_{i=1}^N \|A_i\| \quad (7)
\]

Recall that \( z_1^{(i)} \) is an eigenvector of \( \begin{bmatrix} 0 & A_i \\ A_i^T & 0 \end{bmatrix} \) corresponding to eigenvalue \( \|A_i\| \). Also \( \|z_1^{(i),0}\|^2 = \|z_1^{(i),1}\|^2 = 1/2 \) by Eq. (3). Let \( v \) be a unit norm eigenvector of \( B \) corresponding to eigenvalue \( \lambda \) (which later will either be set to \( \lambda_{\max}(B) \) or \( \lambda_{\min}(B) \)). For \( x = (x_1, \ldots, x_N) \in \{0,1\}^N \) let \( \tilde{x} = (g(x_1), \ldots, g(x_N)) \). We now define our witness \( \psi \) which we will use to show Eqs. (6) and (7) via Fact 4:

\[
\psi((x_1, \ldots, x_N)) = v(\tilde{x}) \prod_{i=1}^N \sqrt{2} z_1^{(i),\tilde{x}_i}(x_i) .
\]
We have
\[
\sum_{x_1, \ldots, x_N} \psi((x_1, \ldots, x_N))^2 = \sum_{\alpha \in \{0, 1\}^N} v(\alpha)^2 \prod_{i=1}^N \left( \sum_{x_i \colon z_i = \alpha_i} 2z_1^{(i), \alpha_i} (x_i)^2 \right)
\]
\[
= \sum_{\alpha \in \{0, 1\}^N} v(\alpha)^2 \prod_{i=1}^N \left( 2\|z_1^{(i), \alpha_i}\|^2 \right)
\]
\[
= \sum_{\alpha \in \{0, 1\}^N} v(\alpha)^2
\]
\[
= 1.
\]

Thus \(\psi\) is a unit vector. Now
\[
\psi^T C \psi = \sum_{\alpha, \beta \in \{0, 1\}^N} B(a, b) v(\alpha) v(\beta) \prod_{i=1}^N \|A_i\| \cdot \left( z_1^{(i), \alpha_i} A^{(\alpha_i, \beta_i)} z_1^{(i), \beta_i} \right)
\]
\[
= \sum_{\alpha, \beta \in \{0, 1\}^N} B(a, b) v(\alpha) v(\beta) \prod_{i=1}^N \|A_i\|
\]
\[
= \lambda \prod_{i=1}^N \|A_i\|.
\]

Taking \(\lambda = \lambda_{\text{max}}(B)\) shows \text{Eq. (7)} by \text{Fact 4}. If \(\lambda_{\text{max}}(B) = \|B\|\) this also shows \text{Eq. (6)}. Otherwise, if \(\|B\| = -\lambda_{\text{min}}(B)\) then taking \(\lambda = \lambda_{\text{min}}(B)\) we have \(\psi^T C \psi = -\|B\| \prod_{i=1}^N \|A_i\|\), implying \text{Eq. (6)} again by \text{Fact 4}.

We now turn to showing the main result of this note, that \(\text{ADV}_{\text{rel}}^\pm(f \circ g^N) = \text{ADV}_{\text{rel}}^\pm(f) \text{ADV}_{\text{rel}}^\pm(g)\) for a relation \(f\) and Boolean function \(g\). We start with the more difficult direction which is showing the lower bound.

**Theorem 22.** Let \(f \subseteq \{0, 1\}^N \times [K]\) be a relation and \(g : \{0, 1\}^m \rightarrow \{0, 1\}\) be a Boolean function. Then \(\text{ADV}_{\text{rel}}^\pm(f \circ g^N) \geq \text{ADV}_{\text{rel}}^\pm(f) \text{ADV}_{\text{rel}}^\pm(g)\).

**Proof.** Let \(\Gamma_f\) be a relational adversary matrix achieving the optimal bound for the relation \(f\). Let
\[
\Gamma_g = \begin{bmatrix} 0 & Z^T \\ Z & 0 \end{bmatrix}
\]
be an optimal functional adversary matrix for \(g\). Define \(\Gamma_h\) as the matrix composition of \(\Gamma_f\) with \(N\) copies of \(Z\).

For \(a \in [K]\) let \(\chi_a \in \mathbb{R}^{2N}\) be defined as in \text{Eq. (1)} and define \(\phi_a \in \mathbb{R}^{2mN}\) similarly but for the composed function:
\[
\phi_a((x_1, \ldots, x_N)) = \begin{cases} 1, & ((g(x_1), \ldots, g(x_N)), a) \in f; \\
0, & \text{otherwise.} \end{cases}
\]

We will show three things
(1) \(\lambda_{\max}(\Gamma_h) = \lambda_{\max}(\Gamma_f) \cdot \|\Gamma_g\|^N\);

(2) \(\Gamma_h \circ \phi_a \phi_a^T \preceq 0\) for all \(a \in [K]\); and

(3) \(\|\Gamma_h \circ D_{\ell}\| \leq \|\Gamma_f \circ D_p\| \cdot \|\Gamma_g \circ D_q\| \cdot \|\Gamma_g\|^{N-1}\), for any \(p \in [N], q \in [m]\) where \(\ell = (p-1)m + q\) is the \(q^{\text{th}}\) bit in the \(p^{\text{th}}\) block.

These three items give the theorem since item (2) implies \(\Gamma_h\) is a relational adversary matrix and

\[
\frac{\lambda_{\max}(\Gamma_h)}{\|\Gamma_h \circ D_{\ell}\|} \geq \frac{\lambda_{\max}(\Gamma_f) \cdot \|\Gamma_g\|^N}{\|\Gamma_f \circ D_p\| \cdot \|\Gamma_g \circ D_q\| \cdot \|\Gamma_g\|^{N-1}} \\
\geq \left(\frac{\lambda_{\max}(\Gamma_f)}{\|\Gamma_g \circ D_q\|}\right) \left(\frac{\|\Gamma_g\|^N}{\|\Gamma_g\|^{N-1}}\right) \\
\geq \text{ADV}_{rel}(f) \text{ADV}(g).
\]

We now show the three items. Item (1) follows immediately from the second equation of Lemma 21 as \(\Gamma_h\) is defined to be the matrix composition of \(\Gamma_f\) with \(N\) copies of \(Z\), and \(\|\Gamma_g\| = \|Z\|\).

Item (2) is the main novelty in the relational case. From Definition 19 we see that

\(\Gamma_h \circ \phi_a \phi_a^T = (\hat{\Gamma}_f \circ \phi_a \phi_a^T) \circ (\otimes^N \hat{\Gamma}_g)\),

where \(\hat{\Gamma}_g\) is defined via Eq. (2) and \(\hat{\Gamma}_f\) is a \(2^{mN}\)-by-\(2^{mN}\) matrix defined as

\(\hat{\Gamma}_f((x_1, \ldots, x_N), (y_1, \ldots, y_N)) = \Gamma_f((g(x_1), \ldots, g(x_N)), (g(y_1), \ldots, g(y_N)))\).

By Lemma 18, \(\hat{\Gamma}_g \succeq 0\), which gives us \(\otimes^N \hat{\Gamma}_g \succeq 0\). Also, \(\Gamma_f \circ \chi_a \chi_a^T \preceq 0\) by the definition of the relational adversary matrix, thus by Fact 2 we get that \(\hat{\Gamma}_f \circ \phi_a \phi_a^T \preceq 0\). Combining the last two observations with Fact 3 we get Item (2).

Item (3) follows as in HLS07, but for completeness we give the details here. Write \(\ell \in [mN]\) as \(\ell = (p-1)m + q\) with \(p \in [N]\) and \(q \in [m]\), i.e. \(\ell\) refers to \(x_p(q)\), the \(q^{\text{th}}\) bit in the \(p^{\text{th}}\) block. It is shown in HLS07 that

\(\Gamma_h \circ D_{\ell} = (\Gamma_f \circ D_p) \circ ((\otimes^{p-1} \hat{\Gamma}_g) \otimes (\hat{\Gamma}_g \circ D_q) \otimes (\otimes^{N-p} \hat{\Gamma}_g))\),

where

\((\Gamma_f \circ D_p)((x_1, \ldots, x_N), (y_1, \ldots, y_N)) = (\Gamma_f \circ D_p)((g(x_1), \ldots, g(x_N)), (g(y_1), \ldots, g(y_N)))\).

The right hand side is not obviously in the proper form to apply Lemma 21 because Lemma 21 requires \(\hat{\Gamma}_g \circ D_q\) in place of the term \(\hat{\Gamma}_g \circ D_q\) in Eq. (8). We can show, however, that the right hand side of Eq. (8) does not change if we replace \(\hat{\Gamma}_g \circ D_q\) with \(\hat{\Gamma}_g \circ D_q\).

Claim 23.

\((\Gamma_f \circ D_p) \circ ((\otimes^{p-1} \hat{\Gamma}_g) \otimes (\hat{\Gamma}_g \circ D_q) \otimes (\otimes^{N-p} \hat{\Gamma}_g)) = (\Gamma_f \circ D_p) \circ ((\otimes^{p-1} \hat{\Gamma}_g) \otimes (\hat{\Gamma}_g \circ D_q) \otimes (\otimes^{N-p} \hat{\Gamma}_g))\)
Proof. \( \tilde{\Gamma}_g \circ D_q \) and \( \Gamma_g \circ D_q \) differ only on the diagonal, therefore it suffices to show that for all entries \((x_1, \ldots, x_N), (y_1, \ldots, y_N)\) where \(x_p = y_p\) the left and right hand sides agree. When \(x_p = y_p\) the left hand side will be 0 because \( \tilde{\Gamma}_g \circ D_q \) is zero on the diagonal. When \(x_p = y_p\) the right hand side will also be zero because then \( \tilde{x}_p = y_p \) and so \( (\Gamma_f \circ D_p)(x, y) = 0 \).

\( \square \)

The right hand side of the expression in Claim 23 is now in the right form to apply the first equation of Lemma 21, and we can conclude \( \|\Gamma_h \circ D_t\| \leq \|\Gamma_f \circ D_p\| \|\Gamma_g \circ D_q\| \|\Gamma_g\|^N \) \(\rho\) as desired.

Next we show the upper bound.

**Theorem 24.** Let \( f \subseteq \{0, 1\}^N \times [K] \) be a relation and \( g : \{0, 1\}^m \to \{0, 1\} \) be a Boolean function. Then \( \text{ADV}_{rel}^\pm(f \circ g^N) \leq \text{ADV}_{rel}^\pm(f) \text{ADV}^\pm(g) \).

**Proof.** Let \( \{u_{y,i}\}_{y \in \{0,1\}^m, i \in [m]}, \{u_{x,i}\}_{x \in \{0,1\}^m, i \in [m]} \) be an optimal solution to the \( \text{ADV}^\pm(g) \) program from Theorem 7, and let \( \{\psi_{x,i}\}_{x \in \{0,1\}^N, i \in [N]}, \{\phi_{x,i}\}_{x \in \{0,1\}^N, i \in [N]}, \{\sigma_{x,a}\}_{x \in \{0,1\}^N, a \in [K]} \) be an optimal solution to the \( \text{ADV}_{rel}^\pm(f) \) program from Definition 12. We will construct a solution to the program from Definition 12 for \( h = f \circ g^N \) of cost \( \text{ADV}_{rel}^\pm(f) \text{ADV}^\pm(g) \). For \( x = (x_1, \ldots, x_N) \in \{0, 1\}^N \) let \( \tilde{x} = (g(x_1), \ldots, g(x_N)) \).

Define

- \( \alpha_{x,\ell} = \psi_{x,p} \otimes u_{x,p,q} \) for \( \ell = (m-1)p + q \) where \( p \in [N], q \in [m] \).
- \( \beta_{x,\ell} = \phi_{x,p} \otimes v_{x,p,q} \) for \( \ell = (m-1)p + q \) where \( p \in [N], q \in [m] \).
- \( \rho_{x,a} = \sigma_{x,a} \).

Then for any \( x = (x_1, \ldots, x_N) \in \{0, 1\}^N \)

\[
\sum_\ell \|\alpha_{x,\ell}\|^2 = \sum_p \|\psi_{x,p}\|^2 \sum_{i \in [m]} \|u_{x,p,i}\|^2 
\leq \text{ADV}^\pm(g) \sum_p \|\psi_{x,p}\|^2 
\leq \text{ADV}_{rel}^\pm(f) \text{ADV}^\pm(g).
\]

A similar calculation shows \( \sum_\ell \|\beta_{x,\ell}\|^2 \leq \text{ADV}_{rel}^\pm(f) \text{ADV}^\pm(g) \) for any \( x \in \{0, 1\}^N \).

Having established the objective value, we move on to the constraints.

\[
\sum_{\ell, x_\ell \neq y_\ell} \langle \alpha_{x,\ell}, \beta_{x,\ell} \rangle = \sum_{p \in [N]} \langle \psi_{x,p}, \phi_{y,p} \rangle \sum_{i \in [m]} \langle u_{x,p,i}, v_{y,p,i} \rangle 
= \sum_{p \in [N]} \langle \psi_{x,p}, \phi_{y,p} \rangle 
= 1 - \sum_{a: (\tilde{x}, a) \notin f, (\tilde{y}, a) \in f} \langle \sigma_{\tilde{x},a}, \sigma_{\tilde{y},a} \rangle 
= 1 - \sum_{a: (x,a) \notin h, (y,a) \in h} \langle \rho_{x,a}, \rho_{y,a} \rangle,
\]

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as desired.

Finally, \(\|\rho_{x,a}\|^2 = \|\sigma_{x,a}\|^2 = 0\) if \((\bar{x},a) \not\in f\). As \((x,a) \in h\) iff \((\bar{x},a) \in f\) this shows \(\|\rho_{x,a}\|^2 = 0\) for all \((x,a) \not\in h\).

Corollary 25. Let \(f \subseteq \{0,1\}^N \times [K]\) be a relation and \(g : \{0,1\}^m \to \{0,1\}\) be a Boolean function. Then \(ADV_{rel}^\pm(f \circ g^N) = ADV_{rel}^\pm(f)ADV^\pm(g)\).

Proof. This follows from Theorem 22 and Theorem 24.

Corollary 26. Let \(K\) be a positive integer, \(f \subseteq \{0,1\}^N \times [K]\) be an efficiently verifiable relation, and \(g : \{0,1\}^m \to \{0,1\}\) be a Boolean function. Then \(Q(f \circ g^N) = \Theta(ADV_{rel}^\pm(f)ADV^\pm(g))\).

Proof. Let \(h = f \circ g^N\). We start by showing \(Q(h) = \Omega(ADV_{rel}^\pm(f)ADV^\pm(g))\). By Theorem 22 \(ADV_{rel}^\pm(h) \geq ADV_{rel}^\pm(f)ADV^\pm(g)\). For \(a \in [K]\) let \(h_a : \{0,1\}^{mN} \to \{0,1\}\) be defined as \(h_a((x_1,\ldots,x_N)) = 1\) iff \((g(x_1),\ldots,g(x_N)),a) \in f\). Letting \(f_a : \{0,1\}^N \to \{0,1\}\) be defined as \(f_a(x) = 1\) iff \((x,a) \in f\), we see that \(h_a = f_a \circ g^N\). By Theorem 1 we have that \(ADV^\pm(h_a) = ADV^\pm(f_a)ADV^\pm(g)\) and therefore, as \(f\) is efficiently verifiable, \(ADV^\pm(h_a) = o(ADV_{rel}^\pm(h))\) for every \(a \in [K]\). Thus \(h\) is also efficiently verifiable and the corollary follows by Theorem 16.

For the other direction, we use Theorem 24 to obtain \(ADV_{rel}^\pm(h) \leq ADV_{rel}^\pm(f)ADV^\pm(g)\) and then apply Theorem 13.

Corollary 27. Let \(K\) be a positive integer, \(f \subseteq \{0,1\}^N \times [K]\) be an efficiently verifiable relation, and \(g : \{0,1\}^m \to \{0,1\}\) be a Boolean function. Then \(Q(f \circ g^N) = \Theta(Q(f)Q(g))\).

Proof. By Corollary 26 \(Q(f \circ g^N) = \Theta(ADV_{rel}^\pm(f)ADV^\pm(g))\). We have \(Q(g) = \Theta(ADV^\pm(g))\) by Theorem 8. Also, \(Q(f) = O(ADV_{rel}^\pm(f))\) by Theorem 13 and \(Q(f) = \Omega(ADV_{rel}^\pm(f))\) by Theorem 16 as \(f\) is efficiently verifiable. The corollary follows.

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