A new non-negative distribution with both finite and infinite support

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Abstract: The Tukey-$\lambda$ distribution has interesting properties including (i) for some parameters values it has finite support, and for others infinite support, and (ii) it can mimic several other distributions such that parameter estimation for the Tukey distribution is a method for identifying an appropriate class of distribution to model a set of data. The Tukey-$\lambda$ is, however, symmetric. Here we define a new class of non-negative distribution with similar properties to the Tukey-$\lambda$ distribution. As with the Tukey-$\lambda$ distribution, our distribution is defined in terms of its quantile function, which in this case is given by the polylogarithm function. We show the support of the distribution to be the Riemann zeta function (when finite), and we provide a closed form for the expectation, provide simple means to calculate the CDF and PDF, and show that it has relationships to the uniform, exponential, inverse beta and extreme-value distributions.

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1. Introduction

The Tukey-$\lambda$ distribution [9,11,17] is a distribution defined in terms of its quantile function. It has interesting properties including that (i) for some parameters values it has finite support, and for others infinite support; and (ii) it can mimic several other distributions. The latter property makes the distribution useful for identifying an appropriate class of distribution to model a set of data. Such properties are rare, as are definitions of distributions in terms of quantiles.

In this short note we present a new distribution defined taking its quantile function to be the polylogarithm function defined by the sum

$$Q(p; s) = \text{Li}_s(p) = \sum_{k=1}^{\infty} \frac{p^k}{k^s}. \tag{1}$$

Although the polylogarithm is defined (either in terms of this series or by analytic continuation) for all $p, s \in \mathbb{C}$, we need only the values $s \in \mathbb{R}$ and $p \in [0, 1]$, noting that the defining sum converges everywhere except for $p = 1$ when $s \leq 1$.
This distribution has similar properties to the Tukey distribution, save that it defines non-negative random variables where the Tukey distribution is symmetric. Like the Tukey distribution it interpolates between several classes of traditional distributions. For

- $s$ large ($\geq 10.0$), it closely approximates the uniform distribution,
- $s \approx 1.6$, it approximates the (non-negative) triangular distribution,
- $s = 1.0$, it is exactly the exponential distribution,
- $s = 0.0$, it is exactly an inverse beta distribution, and
- for large negative $s$ it approximates a generalized extreme value distribution with infinite mean.

This is suggestive that, like the Tukey distribution, this distribution could be used with a probability plot correlation coefficient (PPCC) to determine the best distributional family for a given set of non-negative data.

This highlights that although it is unusual to specify a class of random variates in terms of their quantile function – the inverse cumulative density function (iCDF) – there are advantages to doing so.

2. Background

2.1. The Tukey-$\lambda$ distribution

The Tukey-$\lambda$ distribution [9,11,17] (the original publication of the distribution was in a technical report that no longer appears to be available) is a distribution defined by the quantile function

$$Q(p; \lambda) = \begin{cases} \frac{1}{\lambda} \left[ p^{\lambda} - (1 - p)^{\lambda} \right], & \text{if } \lambda \neq 0, \\ \log \left( \frac{p}{1-p} \right), & \text{if } \lambda = 0, \end{cases}$$

where $\lambda$ is the shape parameter. The distribution is symmetric (about 0).

It is unusual because

- $\lambda > 0$ the distribution has finite support; and when
- $\lambda \leq 0$ the distribution has infinite support.

The Cumulative Distribution Function (CDF) and Probability Density Function (PDF) are not given, in general, by closed forms but its moments are known. Perhaps most interestingly it interpolates between a range of traditional distributions:

- $\lambda = -1$: it is approximately Cauchy;
- $\lambda = 0$: it is the logistic distribution;
- $\lambda = 0.14$: if it approximately normal $N(0, 2.142)$; and
- $\lambda = 1$ and 2: it is uniform, $U(-1, 1)$ and $U(-1/2, 1/2)$, respectively.

A variety of generalization of the distribution exist [11,17] and it has been used in a number of applications (see for example [7,9]). Its main purpose here is to serve as basic for comparison to the new distribution presented in this paper.
2.2. The polylogarithm function

The polylogarithm function is a standard special function, albeit not as commonly used as the gamma or zeta functions. However, it has been much studied, e.g., [1, 2, 4, 5, 8, 20, 22], since the 17th century, and there is at least one book [14] written about its properties. It has many applications; in this context the most obvious are in the moment generating function of the zeta distribution and in the mean of the exponential-logarithmic distribution. There is also a polylogarithmic distribution [12] (not the distribution specified here).

The polylogarithm is a generalization of the log function in the sense that the Taylor series of the ordinary logarithm

\[-\ln(1 - z) = \sum_{n=1}^{\infty} \frac{z^n}{n},\]

which matches the definition (1) for \(s = 1\), but also note that this is the quantile function of the exponential distribution.

The polylogarithm function is defined for \(s \in \mathbb{C}\); however, we only consider real parameters here in order to generate real random variables.

The polylogarithm function is well-behaved in the region of interest. It is monotonically increasing from 0 at \(z = 0\) to

\[\lim_{z \to 1} \operatorname{Li}_s(z) = \begin{cases} \zeta(s), & \text{for } s > 1, \\ \infty, & \text{otherwise}, \end{cases}\]

(2)

where \(\zeta(s)\) is the well-known Riemann zeta function. Thus the function forms a valid quantile or inverse CDF function.

There are a number of computer packages for numerical calculation of polylogarithms e.g., [15, 16, 21]. We use the package described in [18], written in Julia [3].

3. Properties of the Random Variate

Note that throughout this letter we focus on the single shape parameter \(s\) and neglect separate scale and location parameters, but these can easily be incorporated by addition and multiplication given the underlying properties given here.

3.1. Support

From (2) we can immediately identify that the support of the distribution is

\[\text{supp}(X) = \begin{cases} [0, \zeta(s)], & \text{for } s > 1, \\ [0, \infty), & \text{otherwise}. \end{cases}\]
3.2. Cumulative distribution and density functions

We know of no closed form for the inverse of the polylogarithm and so the CDF must be calculated numerically. However, \( Q = F^{-1} \) is monotonically increasing so this is an easy computation, involving a one-dimensional search to find the point \( p \) such that \( F^{-1}(p) = x \) (or equivalently \( F(x) = p \)).

The density function \( f(x) \) can be computed by noting
\[
\frac{dF^{-1}}{dp} \bigg|_{F^{-1}(p)=x} = 1/f(x).
\]

As before a search is needed to find the point where \( F^{-1}(p) = x \); however the derivative of the polylogarithm function can be found directly from the definition
\[
\frac{d}{dx} \text{Li}_s(x) = \frac{\text{Li}_{s-1}(x)}{x}.
\]

Staudte [19] argues that this function (scaled by the mean where this is finite) should have its own place in the important descriptions of a probability distribution and calls it the probability density quantile (pdQ) function. He proposes and later authors confirm [6] that many applications can use this function, for instance for parameter estimation, more effectively than traditional characterizations. From it we obtain the density
\[
f(x) = \frac{p}{\text{Li}_{s-1}(p)} \bigg|_{\text{Li}_s(p)=x}.
\]

Figure 1 shows the CDF and density for a range of parameters \( s \). We see several features; notably all of the densities are 1 at the origin, which fact can be derived from the limit
\[
\frac{\text{Li}_{s-1}(x)}{x} \rightarrow 1 \text{ as } x \rightarrow 0.
\]

3.3. Simulation

One advantage of defining a random variable in terms of its quantile function is that simulation is easy. Given a set of random variables \( X_i \sim U(0, 1) \), i.e., that are distributed as uniform random variables on the interval \([0, 1]\), then a random variable with quantile function \( Q(p) = F^{-1}(p) \) can be generated by taking
\[
Y_s = Q(X).
\]

3.4. Expectation

The following result appears almost trivial, but we have not seen it stated explicitly anywhere.
Lemma 3.1. The expectation of a random variable $Y$ with quantile function $Q(p)$ is given by

$$E[Y] = \int_0^1 Q(p) \, dp,$$

when this integral exists.

Proof. Take random variable $X \sim U(0, 1)$, and note that the expectation of a $L_1$ function $g(x)$ of a random variable is given by [10, Thm 4.26]

$$E[g(X)] = \int g(x) \, dF_X.$$

Then note that we can create a random variable $Y$ with quantile function $Q(p)$ by taking $Y = Q(X)$ and hence when the integral exists

$$E[Y] = \int Q(x) \, dF_X = \int_0^1 Q(p) \, dp.$$

The lemma leads directly to the expectation for our random variable $Y \sim \text{Li}_s(X)$ as being

$$E[Y_s] = \int_0^1 \text{Li}_s(x) \, dx.$$

Substituting the definition (1) and taking care that we integrate over a finite series we get

$$E[Y_s] = \lim_{T \to 1} \sum_{k=1}^{\infty} \frac{1}{k^s} \int_0^T x^k \, dx$$

$$= \lim_{T \to 1} \sum_{k=1}^{\infty} \frac{T^{k+1}}{k^s(k + 1)}$$
Figure 2 shows the expectation as a function of $s$. Interestingly, there is a range of values $0 < s \leq 1$ where the distribution has infinite support but finite expectation, and a range where the expectation is infinite $s \leq 0$ where the distribution resembles the Pareto or power-law distribution. However, the Pareto distribution does not include parameters with finite support as this distribution does, nor does it interpolate from the exponential distribution (at $s = 1$).

There are many identities known for polylogarithms. One of use here is that

$$\text{Li}_{s+1}(z) = \int_0^z \frac{\text{Li}_s(t)}{t} dt,$$

which leads via integration by parts to

$$E[Y_s] = \zeta(s + 1) - E[Y_{s+1}],$$

for random variable $Y_s$ which has quantile function $\text{Li}_s(z)$. This has advantages for calculating the expectation particularly for $s \downarrow 0$ where the series converges only very slowly and direct integration is more problematic because of the pole.
3.5. Higher moments

We do not have a closed form for the higher moments, but we can estimate them through integration, using the above result to note that

$$E[Y^m] = \int_0^1 Li_s(x)^m dx.$$  

Figure 2 also shows the resulting variance computed by numerical integration. It is noteworthy that the asymptote for variance appears to be \( s_2 = 1/2 \), which we show in the following result.

**Theorem 3.1.** The random variable \( Y_s \) with quantile function \( Q(p; s) = Li_s(p) \) has infinite \( m \)th moment for \( s < s_m = 1 - 1/m \).

**Proof.** Chebyshev’s inequality [10, Thm 4.40] states that for a non-negative random variable \( X \)

$$P(X \geq a) \leq \frac{E[\varphi(X)]}{\varphi(a)},$$

for \( a \geq 0 \) and \( \varphi \) a positive, monotonically increasing function on \( \mathbb{R}^+ \). Taking \( \varphi(x) = x^m \) we get

$$P(Y_s \geq a) \leq \frac{E[Y^m_s]}{a^m}.$$  

Now take \( a = Q_s(1-p) \) and note that \( p = P(Y_s \geq Q_s(1-p)) \) to get

$$Q_s(1-p)^m p \leq E[Y^m_s].$$

Now the limit of \( Li_s(z) \) near \( z = 1 \) is (for non-integer \( s \)) given by [5, 13, 22] to be

$$Li_s(z) \to \Gamma(1-s)(-\ln z)^{s-1}.$$  

If we take the first order term in the Taylor series of \( \ln z \) we get

$$Li_s(1-p)^p \sim p^{s-1}.$$  

Therefore

$$Q_s(1-p)^m p \sim p^{m(s-1)+1},$$

and hence for \( m = 1 \) if \( s < 0 \), the bound pushes the expected value to \( \infty \) as \( p \) goes to 0. In general this pushes the \( m \)th moment to be infinite when \( m(s-1)+1 < 0 \) or for

$$s < s_m = 1 - 1/m.$$  

\[ \Box \]

**Remark:** The proof above does not show that the moments are finite for \( s > 1 - 1/m \); however, numerical experiments suggest that the bound is tight. The consequence is that we see the asymptote for the expectation at \( s_1 = 0 \), the variance at \( s_2 = 1/2 \), and if we plot it we see the asymptote for the third moment at \( s_3 = 2/3 \). The increasing nature of the sequence means that there are intervals \([1 - 1/n, 1 - 1/(n+1)]\) where all moments up to \( n \) are finite, and the \((n+1)\)th moment is infinite.
4. Relationships

The polylogarithm function has a range of relationships to other functions most notably the Riemann zeta function \([13, (a)]\)

\[ \text{Li}_s(1) = \zeta(s), \text{ for } \Re(s) > 1. \]

There are many other such relationships known, but few are relevant as they often involve complex parameters or arguments.

There are several forms of analytic continuation for the polylogarithm function. For instance, when \(\Re(s) > 0\) we can define it using the integral

\[ \text{Li}_s(z) = \frac{1}{z} \int_0^\infty \frac{t^{s-1}}{e^{t/z} - 1} dt, \tag{3} \]

except for a pole at \(z = 1\) for \(\Re(s) < 2\), which relates the polylogarithm to the Bose-Einstein distribution, and a similar relation connects it to the Fermi-Dirac and Maxwell–Boltzmann distributions.

More directly useful is that fact that there are several parameter values for which the polylogarithm function simplifies dramatically, \(i.e., [13, (3) and (4)]\)

\[ \begin{align*}
    \text{Li}_1(z) &= -\ln(1-z), \\
    \text{Li}_0(z) &= z/(1-z), \\
    \text{Li}_{-1}(z) &= z/(1-z)^2, \\
    \text{Li}_{-2}(z) &= z(1+z)/(1-z)^3, \\
    \text{Li}_{-3}(z) &= z(1+z^2)/(1-z)^4,
\end{align*} \]

and so on. In particular the case \(s = 1\) leads to the exponential distribution and \(s = 0\) is interesting because \(\text{Li}_0(z)\) is the quantile function of the inverse beta distribution (or beta prime distribution). The PDF of the inverse beta is

\[ iB(\alpha, \beta) = \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha, \beta)}, \]

where \(B(\alpha, \beta)\) is the beta function. When \(\alpha = 1\) and \(\beta = 1\) we get

\[ iB(1, 1) = \frac{1}{(1+x)^2}, \]

which has CDF \(x/(1+x)\), and inverse CDF \(\text{Li}_0(z)\). Note that the mean of the inverse beta is only finite for \(\beta > 1\), and that its mode is at \((\alpha - 1)/(\beta + 1) = 0\) as we expect. Also the inverse beta with these parameters is a power-law type of heavy-tailed distribution.

There are a number of useful limits as well \([5,13,22]\). For instance,

\[ \lim_{s \to \infty} \text{Li}_s(z) = z, \]
which implies that for large $s$ the distribution tends towards the uniform distribution. We see this clearly in the above results for values of $s$ as small as 10.

The limit for negative $s$ is also useful:

$$\lim_{s \to -\infty} \text{Li}_s(z) = \Gamma(1-s)(-\ln(z))^{s-1},$$

which leads to a closed form to calculate the CDF for large negative $s$. It is also closely linked to the quantile function for the generalized extreme value distribution for $\xi > 0$

$$Q_{GEV}(p; \mu, \sigma, \xi) = \mu + \frac{\sigma}{\xi} \left[ (-\ln(p))^{-\xi} - 1 \right],$$

if we identify

$$\xi = 1-s,$$
$$\sigma = \xi \Gamma(1-s) = \Gamma(2-s),$$
$$\mu = \frac{\sigma}{\xi} = \Gamma(1-s).$$

Another advantage of defining random variates in terms of their quantile function is that it is then tautologically trivial to calculate the quantiles. Notably in this case, the median, i.e., $Q(0.5)$, is easy because many values of $\text{Li}_s(0.5)$ are known, e.g.,

$$\text{Li}_1(0.5) = \ln 2,$$
$$\text{Li}_2(0.5) = \frac{1}{12} \pi^2 - \frac{1}{2} (\ln 2)^2.$$

In general $\text{Li}_n(0.5)$ takes values from the multiple zeta function.

Finally, note that there are a small selection of other random variates that have the property that their support is both finite or infinite (apart from the the Tukey lambda distribution mentioned earlier and its generalizations [11,17]), e.g.,

- the generalized Pareto distribution; and
- the Wakeby distribution, which can also be defined in terms of quantiles, and whose separation into different classes has been used to explain phenomena in hydrology [7].

However, neither of these have the property that they interpolate between such a range of standard distributions.

5. Conclusion

This paper has presented a new random variate that has properties similar to the Tukey-$\lambda$ distribution except that it is non-negative.
The distribution is defined in terms of its quantile function, which is taken to be a polylogarithm function, and from which we can derive many properties such as the PDF and expectation. It is noteworthy that this distribution interpolates all the way from the uniform distribution to heavy-tailed distributions with some set of infinite moments.

There are no doubt many generalizations possible — there are certainly many generalizations of the polylogarithm function. At a deeper level this distribution highlights some advantages of defining a random variate by its quantile.

Finally, the natural name for this distribution might be thought to be the polylogarithm distribution, however, that name is unfortunately already taken [12], as is the zeta distribution, and so as yet it remains nameless.

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