Degree formula for connective $K$-theory

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Abstract

We apply the degree formula for connective $K$-theory to study rational contractions of algebraic varieties. Examples include rationally connected varieties and complete intersections.

1 Introduction

The celebrated Rost degree formula says that given a rational map $f: Y \rightarrow X$ between two smooth projective varieties there is the congruence relation (see [13])

$$\eta_p(Y) \equiv \deg f \cdot \eta_p(X) \mod n_X,$$

(1)

where $p$ is a prime, $\eta_p(X)$ is the Rost number of $X$, $\deg f$ is the degree of $f$ and $n_X$ is the greatest common divisor of degrees of all closed points on $X$ (see [10], [11], [12] and [13]).

It was conjectured by Rost that the degree formula (1) should follow from a generalized degree formula for some universal cohomology theory. This conjecture was proven by Levine and Morel in [9], where they constructed the theory of algebraic cobordism $\Omega$ and provided the respective degree formula (see [9], Theorem 1.2.14).

Unfortunately, the generalized degree formula has one disadvantage: it deals with elements in the cobordism ring which is too big and usually is hard to compute. On the other hand the classical degree formula (1) is easy to apply but it catches only “pro-$p$” effects. The reasonable question would be to find a cohomology theory together with a degree formula which doesn’t loose much information and is still computable.

The natural candidate for such a theory is the connective $K$-theory denoted by $K$. It has two important properties: First, $K$ is the universal oriented cohomology theory for the Chow group $\text{CH}$ and Grothendieck’s $K^0$, meaning the

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following diagram of natural transformations

\[
\begin{align*}
\Omega \\
\downarrow^{\text{pr}_{\mathcal{K}}} \\
\mathcal{K} & \xrightarrow{\beta} K[\beta, \beta^{-1}], \\
\end{align*}
\]

where \(\beta\) denotes the Bott element. Second, it is the universal birational theory in the sense that it preserves the fundamental classes for birational maps, i.e. for any proper birational \(f: Y \to X\) we have \(f_{*}(1_{Y}) = 1_{X}\).

The respective degree formula for \(K\) was predicted by Rost and Merkurjev (see [13, Example 11.4]). In the present notes we deduce this formula from the generalized degree formula of Levine and Morel. Namely, we prove the following

1.1 Lemma. Let \(f: Y \dashrightarrow X\) be a rational map between two smooth projective irreducible varieties of the same dimension over a field \(k\) of characteristic 0. Then there exists a finite family of smooth projective varieties \(\{Z_{i}\}_{i}\) over \(k\) such that each \(Z_{i}\) admits a projective birational map to a proper closed subvariety of \(X\) and

\[
\chi(O_Y) = \deg f \cdot \chi(O_X) + \sum_{i} n_{i} \cdot \chi(O_{Z_{i}}),
\]

where \(n_{i} \in \mathbb{Z}\) and \(\chi(O_X)\) is the Euler characteristic of the structure sheaf of \(X\).

Recall that an algebraic variety \(X\) is called incompressible if any rational map \(X \dashrightarrow X\) is dominant, i.e. has a dense image. The notion of incompressibility appears to be very important in the study of the splitting properties of \(G\)-torsors, where \(G\) is a linear algebraic group, in computations of the essential and the canonical dimension of \(G\) (see [1], [3] and [7]). For instance, using the Rost degree formula (1) Merkurjev provided a uniform and shorten proof of the incompressibility of certain Severi-Brauer varieties, involution varieties and quadrics (see [13, §5 and §7]). Therefore, it is natural to expect that the formula (2) can provide more examples of incompressible varieties. As a demonstration of this philosophy we reduce the formula (2) to the following

1.2 Theorem. Let \(X\) and \(Y\) be smooth projective irreducible varieties over a field of characteristic 0. Let \(n_{X}\) denote the greatest common divisor of degrees of all closed points on \(X\) and let \(\tau_{m}\) denote the \(m\)-th denominator of the Todd genus. Assume there is a rational map \(f: Y \dashrightarrow X\). Then we have the following congruence relation

\[
\chi(O_Y) \cdot \tau_{\dim Y - 1} \equiv \deg f \cdot \chi(O_X) \cdot \tau_{\dim Y - 1} \mod n_{X}.
\]

and as an immediate consequence we obtain

1.3 Corollary. Let \(X\) and \(Y\) be as above. If the image of the rational map \(f: Y \dashrightarrow X\) is of dimension strictly less than the dimension of \(Y\), then \(n_{X}\) divides \(\chi(O_{Y}) \cdot \tau_{\dim Y - 1}\). In particular,
\( i) \) \( n_X \) always divides \( \chi(O_X) \cdot \tau_{\dim X} \);

\( ii) \) if \( n_X \) doesn’t divide \( \chi(O_X) \cdot \tau_{\dim X - 1} \), then \( X \) is incompressible.

The notion of a rationally connected variety \( X \) has been introduced and extensively studied by Campana, Kollár, Miyaoka and Mori (see [8, IV.3.2] for the precise definition). It can be easily seen (see Example 3.6) that for any rationally connected variety \( X \) we have \( \chi(O_X) = 1 \). Then by Corollary 1.3

\( n_X \mid \tau_{\dim X} \), and if \( n_X \nmid \tau_{\dim X - 1} \) then \( X \) is incompressible.

The paper is organized as follow. First, we provide some preliminaries on algebraic cobordism and connective \( K \)-theory. Then we prove Lemma 1.1 and Theorem 1.2 and discuss its applications (Corollary 1.3) to the question of incompressibility of algebraic varieties. In the last section we relate the degree formula (3) with the classical Rost degree formula (1).

**Notation and conventions** By \( k \) we denote a field of characteristic 0. A variety will be a reduced and separated scheme of finite type over \( k \). By \( pt \) we denote \( \text{Spec} \ k \). Given a cycle \( \alpha \in \text{CH}(X) \) by \( \deg \alpha \) we denote the push-forward \( p^*(\alpha) \in Z \), where \( p: X \to pt \) is the structure map.

2 **Algebraic cobordism and connective \( K \)-theory**

2.1. In [9] M. Levine and F. Morel introduced the theory of algebraic cobordism \( \Omega \) that is a contravariant functor from the category of smooth quasi-projective varieties over a field \( k \) of characteristic 0 to the category of graded commutative rings. An element of codimension \( i \) in \( \Omega(X) \) is the class \([f: Y \to X]\) of a projective map of pure codimension \( i \) between smooth quasi-projective varieties \( X \) and \( Y \). Given \( f \) we denote by \( f_*: \Omega_i(Y) \to \Omega_i(X) \) the induced push-forward and by \( f^*: \Omega_i(X) \to \Omega_i(Y) \) the induced pull-back.

2.2. Let \( h \) be an oriented cohomology theory as defined in [9]. Roughly speaking, \( h \) is a cohomological functor endowed with characteristic classes \( c^h \). The main result of [9] says that \( \Omega \) is a universal oriented cohomology theory, i.e. any such \( h \) admits a natural transformation of functors \( pr_h: \Omega \to h \) preserving the characteristic classes.

To any oriented cohomology theory \( h \) one assigns a one-dimensional commutative formal group law \( F_h \) over the coefficient ring \( h(pt) \) via

\[
c^h_1(L_1 \otimes L_2) = F_h(c^h_1(L_1), c^h_1(L_2)),
\]

where \( L_1 \) and \( L_2 \) are lines bundles on \( X \) and \( c^h_1 \) is the first Chern class. For \( \Omega \) the respective formal group law turns to be a universal formal group law

\[
F(x, y) = x + y + \sum_{i,j \geq 1} a_{ij} x^i y^j.
\]

The ring of coefficients \( \Omega(pt) \) is generated by the coefficients \( a_{ij} \) of \( F \) and coincides with the Lazard ring \( \mathbb{L} \).
2.3. We recall several auxiliary facts about Chow groups CH and graded $K_0$:

Consider the augmentation map $\varepsilon_a: \mathbb{L} \to \mathbb{Z}$ defined by $a_{ij} \mapsto 0$. Define a cohomology theory $\Omega_a$ by $\Omega_a(X) = \Omega(X) \otimes_{\varepsilon_a} \mathbb{Z}$. According to [9]

- $\Omega_a(X)$ coincides with the Chow group $\text{CH}(X)$ of algebraic cycles on $X$ modulo rational equivalence;
- the natural transformation $\text{pr}_a: \Omega(X) \to \Omega_a(X)$ is surjective and its kernel is generated by elements of positive dimensions $L_{>0}$ of the Lazard ring;
- $\Omega_a$ is a universal theory for the additive formal group law $F_a(x, y) = x + y$.

Consider the map $\varepsilon_m: \mathbb{L} \to \mathbb{Z}[\beta, \beta^{-1}]$ defined by $a_{11} \mapsto -\beta$ and $a_{ij} \mapsto 0$ for $(i, j) \neq (1, 1)$. Define a cohomology theory $\Omega_m$ by $\Omega_m(X) = \Omega(X) \otimes_{\varepsilon_m} \mathbb{Z}[\beta, \beta^{-1}]$. According to [9]

- $\Omega_m(X)$ coincides with $K^0(X)[\beta, \beta^{-1}]$, where $K^0(X)$ is Grothendieck’s $K^0$ of $X$;
- $\Omega_m$ is a universal theory for the multiplicative periodic formal group law $F_m(x, y) = x + y - \beta xy$.

2.4. The cohomology theory $K$ which will be the central object of our discussion is a universal theory for both additive and multiplicative periodic formal group laws. It is called the connective $K$-theory and is defined as $K(X) = \Omega(X) \otimes_{\varepsilon} \mathbb{Z}[v]$, where $\varepsilon: \mathbb{L} \to \mathbb{Z}[v]$ is given by $a_{11} \mapsto -v$ and $a_{ij} \mapsto 0$ for $(i, j) \neq (1, 1)$. It has the following properties (see [9] §4.3.3):

- The natural transformations $K \to \Omega_a$ and $K \to \Omega_m$ are given by the evaluations $v \mapsto 0$ and $v \mapsto \beta$ respectively. Roughly speaking, $K$ can be viewed as a homotopy deformation between $\Omega_a$ and $\Omega_m$.
- The natural transformations $\Omega \to K$ and $K \to \Omega_a = \text{CH}$ are surjective.
- The respective formal group law $F$ is the multiplicative non-periodic formal group law $F_K(x, y) = x + y - vxy$, where $v$ is non-invertible.

2.5. We will extensively use the following fact (see [9] Cor.4.2.5 and 4.2.7]):

The theory $K$ is universal among all oriented theories for which the birational invariance holds, i.e. $F_*(1_Y) = 1_X$ for any birational projective map $f: Y \to X$. As a consequence, the kernel of the map $\varepsilon: \mathbb{L} \to \mathbb{Z}[v]$ is the ideal generated by elements $[W] - [W']$, where $W$ and $W'$ are birationally equivalent.

3 Degree formula and Todd genus

In the present section we prove Lemma 1.1 and Theorem 1.2 of the introduction. By $X$ and $Y$ we denote smooth projective irreducible varieties of the same dimension $d$ over a field $k$ of characteristic 0.
3.1. Consider a rational morphism $f: Y \to X$. Let $\overline{\Gamma}_f$ be the closure of its graph in $Y \times X$ and $\overline{\Gamma}_f' \to \overline{\Gamma}_f$ be its resolution of singularities. By the generalized degree formula for the composite $\overline{\Gamma}_f' \to Y \times X \xrightarrow{pr_2} X$ (see [9, Thm. 1.2.14]) there exists a finite family of smooth projective varieties $\{Z_i\}$ such that each $Z_i$ admits a projective birational map $f_i$ on the proper closed subvariety of $X$ and

$$[\overline{\Gamma}_f' \to X] = \deg f \cdot [X \xrightarrow{id} X] + \sum_i u_i \cdot [Z_i \xrightarrow{f_i} X],$$

where $\deg f = [k(Y) : k(X)]$ if $f$ is dominant, and $\deg f = 0$ otherwise. Observe that by definition $\dim Z_i < d$ and $u_i \in \mathbb{L}_{>0}$.

Applying the push-forward $p_*: \Omega(X) \to \mathbb{L}$ induced by the structure map $p: X \to pt$ we obtain

$$[\overline{\Gamma}_f'] = \deg f \cdot [X] + \sum_i u_i \cdot [Z_i].$$

Then projecting on $K(pt)$ we obtain the following equality:

$$[Y]_K = \deg f \cdot [X]_K + \sum_i (u_i)_K \cdot [Z_i]_K. \quad (4)$$

Observe that $[Y]_K = [\overline{\Gamma}_f']_K$, since $Y$ and $\overline{\Gamma}_f'$ are birationally isomorphic and we have property 2.5.

The Euler characteristic. It turns out that the class $[X]_K$ can be expressed in terms of the Euler characteristic $\chi(\mathcal{O}_X)$ of the structure sheaf of $X$. Namely,

3.2 Lemma. We have $[X]_K = \chi(\mathcal{O}_X) \cdot v^d$.

Proof. Consider the map $\varepsilon_m: \mathbb{L} \to \mathbb{Z}[\beta, \beta^{-1}]$. The image of the class $[X]$ is equal to $[X]_m = p_*([\mathcal{O}_X]) \cdot \beta^d$, where $p_*$ is the push-forward induced by the structure map $p: X \to pt$ and the number $p_*([\mathcal{O}_X])$ coincides with the Euler characteristic $\chi(\mathcal{O}_X)$ of the structure sheaf of $X$ (see [13, §10]). Since $\varepsilon_m$ factors through $\mathbb{Z}[v]$ and the map $\mathbb{Z}[v] \to \mathbb{Z}[\beta, \beta^{-1}]$, $v \mapsto \beta$, is injective, we obtain the desired formula.

3.3 Corollary. If $X$ is birationally isomorphic to $Y$, then $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$.

Proof. By property 2.5 of $K$ we have the equality $[X]_K = [Y]_K$. $\square$

3.4. Observe that by the very definition

$$\chi(\mathcal{O}_X) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{O}_X)$$

and the number $\dim_k H^i(X, \mathcal{O}_X)$ is known to be a birational invariant for any smooth projective geometrically irreducible variety $X$ over a field $k$ of characteristic 0.
3.5. Since $\dim_k H^i(X, \mathcal{O}_X)$ doesn’t depend on a base change, so is $\chi(\mathcal{O}_X)$. Namely, if $X_l = X \times_k l$ is a base change by means of a field extension $l/k$, then $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_X)$.

3.6 Example. Let $X$ be a rationally connected smooth projective variety over a field $k$ of characteristic $0$ (see [8, IV.3.2]). Observe that any geometrically rational or unirational variety (see [8, IV.1]) provides an example of a rationally connected variety.

By [8, IV.3.3 and IV.3.8] we have $H^0(X, (\Omega^1_X)^{\otimes m}) = 0$ for every $m > 0$. By [6, Example 15.2.14] $\dim_k H^0(X, (\Omega^1_X)^{\otimes m}) = \dim_k H^m(X, \mathcal{O}_X)$ for every $m$ and, therefore, $\chi(\mathcal{O}_X) = 1$.

Combining (4) and Lemma 3.2 we prove the Lemma 1.1 of the introduction:

3.7 Lemma. Let $f : Y \dashrightarrow X$ be a rational map between two smooth projective irreducible varieties of the same dimension $d$ over a field of characteristic $0$. Then there exists a finite family of smooth projective varieties $\{Z_i\}_i$ such that each $Z_i$ admits a projective birational map $f_i$ on a proper closed subvariety of $X$ and

$$\chi(\mathcal{O}_Y) = \deg f \cdot \chi(\mathcal{O}_X) + \sum_i n_i \cdot \chi(\mathcal{O}_{Z_i}), \text{ where } n_i \in \mathbb{Z}. \quad (5)$$

The Todd genus. It is well-know that the Euler characteristic $\chi(\mathcal{O}_X)$ is related with the Todd genus of $X$.

3.8. According to [6, Example 3.2.4] the Todd class of the tangent bundle of a smooth projective variety $X$ is the following polynomial in Chern classes

$$\text{Td}(X) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \frac{-c_1^4 + 4c_1^2 c_2 + 3c_2^2 + c_1 c_3 - c_4}{720} + \ldots, \quad (6)$$

where $c_i \in \text{CH}^i(X)$ denotes the $i$-th Chern class of the tangent bundle of $X$. The denominators $\tau_0 = 1, \tau_1 = 2, \tau_2 = 12, \tau_3 = 24, \tau_4 = 720 \ldots$ are called Todd numbers. We have the following explicit formula for $\tau_d$ (see [13, Example 9.9]):

$$\tau_d = \prod_{p \text{ prime}} \left[ \frac{d}{p^{d-1}} \right]. \quad (7)$$

In particular, $\tau_{d-1} | \tau_d$ for any $d$.

3.9. To compute the Euler characteristic $\chi(\mathcal{O}_X)$ we may use the following equality (see [6, Corollary 18.3.1]):

$$\chi(\mathcal{O}_X) = \deg \text{Td}(X), \quad (8)$$

where $\deg \text{Td}(X)$ is the degree of the $d$-th homogeneous component of the Todd class $\text{Td}(X)$ and is called the Todd genus of $X$.

Observe that the Euler characteristic and the Todd genus are multiplicative, i.e. $\chi(\mathcal{O}_{X \times Y}) = \chi(\mathcal{O}_X) \cdot \chi(\mathcal{O}_Y)$. 
3.10 Example. Let $X$ be a complete intersection of $m$ smooth hypersurfaces of degrees $d_1, \ldots, d_m$ in $\mathbb{P}^n_k$. Then

$$
\chi(\mathcal{O}_X) = \text{Res}_{z=0} \prod_{i=1}^m (1 - e^{-d_i z})/(1 - e^{-z})^{n+1}.
$$

Indeed, by [6, Example 15.2.12.(iii)] we have

$$
\chi(\mathcal{O}_X) = \deg (\text{Td}(\mathbb{P}^n) \cdot \prod_{i=1}^m (1 - e^{-d_i z})) = \deg \left( \left( \frac{z}{1 - e^{-z}} \right)^{n+1} \cdot \prod_{i=1}^m (1 - e^{-d_i z}) \right),
$$

where $z = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.

We are ready now to prove the main result of this paper (Theorem 1.2).

3.11 Theorem. Let $X$ and $Y$ be smooth projective irreducible varieties over a field $k$ of characteristic 0. Let $n_X$ denote the greatest common divisor of degrees of all closed points on $X$ and let $\tau_m$ denote the $m$-th Todd number. Assume there is a rational map $f : Y \dashrightarrow X$. Then

$$
\chi(\mathcal{O}_Y) \cdot \tau_{\dim Y - 1} \equiv \deg f \cdot \chi(\mathcal{O}_X) \cdot \tau_{\dim Y - 1} \mod n_X.  \quad (9)
$$

Proof. Taking the product with a projective space we may assume that $X$ and $Y$ have the same dimension $d$. By the formula 5 we have

$$
\chi(\mathcal{O}_Y) = \deg f \cdot \chi(\mathcal{O}_X) + \sum_i n_i \chi(\mathcal{O}_{Z_i}) = \deg f \cdot \chi(\mathcal{O}_X) + \sum_{j=0}^{d-1} \left( \sum_{i, \dim Z_i = j} n_i \chi(\mathcal{O}_{Z_i}) \right),
$$

where each $Z_i$ admits a projective morphism $Z_i \rightarrow X$. Identifying the Euler characteristic with the Todd number using (8) we see that all the characteristic numbers in the numerator of the $(\dim Z_i)$-th homogeneous component of the polynomial (6) for $\text{Td}(Z_i)$ are divisible by $n_X$. Therefore, we have

$$
\sum_{j=0}^{d-1} \left( \sum_{i, \dim Z_i = j} n_i \cdot \frac{n_X \cdot m_i}{\tau_j} \right) = \sum_{j=0}^{d-1} \frac{n_X}{\tau_j} \left( \sum_{i, \dim Z_i = j} n_i m_i \right) =
$$

$$
= \frac{n_X}{\tau_{d-1}} \sum_{j=0}^{d-1} \frac{\tau_{d-1}}{\tau_j} \left( \sum_{i, \dim Z_i = j} n_i m_i \right) \equiv \frac{n_X \cdot m}{\tau_{d-1}},
$$

where $m = \sum_{j=0}^{d-1} \frac{\tau_{d-1}}{\tau_j} \left( \sum_{i, \dim Z_i = j} n_i m_i \right) \in \mathbb{Z}$ according to 7. This completes the proof of the theorem. \hfill \Box

3.12 Example. If $X$ and $Y$ are curves, then the degree formula (9) turns into

$$
\chi(\mathcal{O}_Y) \equiv \deg f \cdot \chi(\mathcal{O}_X) \mod n_X,  \quad (10)
$$
where \( n_X \) denotes the g.c.d. of degrees of all closed points on \( X \). Observe that 
\[
\chi(O_X) = 1 - p_g, \quad \text{where} \quad p_g \text{ is the geometric genus of a geometrically irreducible curve } X.
\]
For surfaces \( X \) and \( Y \) it can be stated as
\[
2\chi(O_Y) \equiv \deg f \cdot 2\chi(O_X) \mod n_X. \tag{11}
\]
Observe that \( \chi(O_X) = 1 - q + p_g \), where \( q \) is the irregularity and \( p_g \) is the geometric genus of a geometrically irreducible surface \( X \).

4 Incompressibility

4.1. The notion of (in-)compressibility of algebraic varieties appears naturally in the study of the splitting properties of \( G \)-torsors and their canonical dimensions. Recall that (see [7, §4] and [3, §1]) a canonical dimension \( \text{cdim} X \) of a smooth projective irreducible variety \( X \) over \( k \) is defined to be the minimal dimension of a closed irreducible subvariety \( Z \) of \( X \) such that \( Z_k(X) \) has a rational point. By definition we have \( \text{cdim} X \leq \dim X \). If \( \text{cdim} X = \dim X \), then \( X \) is called incompressible.

To say that “\( Z_k(X) \) has a rational point” is the same as to say that there is a rational dominant map \( X \rightarrow Z \). Therefore, a variety \( X \) is incompressible if and only if any rational map \( X \rightarrow X \) is dominant.

As an immediate consequence of Theorem 3.11 we obtain

4.2 Corollary. Let \( X \) and \( Y \) be smooth projective irreducible varieties over a field \( k \) of characteristic 0. Let \( n_X \) denote the greatest common divisor of degrees of all closed points on \( X \) and let \( \tau_m \) denote the \( m \)-th Todd number. Assume there is a rational map \( f : Y \rightarrow X \).

If the image of \( f \) is of dimension strictly less than the dimension of \( Y \), then \( n_X \) divides \( \chi(O_Y) \cdot \tau_{\dim Y-1} \). In particular, i) \( n_X \mid \chi(O_X) \cdot \tau_{\dim X} \), and ii) if \( n_X \mid \chi(O_X) \cdot \tau_{\dim X-1} \), then \( X \) is incompressible.

Proof. To prove i) we apply Theorem 3.11 to the projection \( X \times \mathbb{P}^1 \rightarrow X \), and to prove ii) we apply 3.11 to the identity \( X \rightarrow X \).

4.3 Example. Let \( X \) be a rationally connected smooth projective variety over a field \( k \) of characteristic 0. Then according to Example 3.6 we have \( \chi(O_X) = 1 \). Therefore, \( n_X \mid \tau_{\dim X} \) and
\[
n_X \nmid \tau_{\dim X-1} \implies X \text{ is incompressible}. \tag{12}
\]
Observe that if \( X \) is a curve or a surface, the implication (12) can be proven directly using the geometry (see [11, §8] and [3, §2, §3]).

4.4 Example. (cf. [11, Example 8.2] and [10, §7.3]) Let \( X \) be a complete intersection of \( m \) smooth hypersurfaces of degrees \( d_1, \ldots, d_m \) in \( \mathbb{P}^n_k \).
Assume that \( \dim X = p - 1 \) for some prime \( p \). Let \( m_p \) denote the number of degrees \( d \) which are divisible by \( p \). We claim that
\[
p \nmid m_p \quad \text{and} \quad p \nmid \frac{d_1d_2 \ldots d_m}{n_X} \implies X \text{ is incompressible.} \tag{13}
\]

Indeed, by the formula from Example 3.10 the Euler characteristic \( \chi(\mathcal{O}_X) \) is equal to the coefficient at \( z^n \) in the expansion of
\[
d_1d_2 \ldots d_m z^m \prod_{i=1}^m \left( \sum_{r=0}^{\dim X} \frac{(-d_i)^r}{(r+1)!} z^r \right) \left( \sum_{r=0}^{\dim X} \frac{B_r}{r!} z^r \right)^{p+m},
\]
where \( B_r \) denotes the \( r \)-th Bernoulli number. Since the denominator of \( \frac{B_r}{r!} \) is not divisible by \( p \) for any \( r < p - 1 \) and is divisible by \( p \) for \( r = p - 1 \), we obtain
\[
\frac{\chi(\mathcal{O}_X)\tau_{p-2}}{n_X} = \tau_{p-2} \cdot \frac{d_1d_2 \ldots d_m}{n_X} \cdot \left( \frac{a}{b} - \frac{m_p}{p!} \right) \notin \mathbb{Z}, \text{ where } p \nmid ab.
\]

4.5 Example. Let \( Y \) be a smooth hypersurface of degree \( p^r \), \( r > 0 \), in \( \mathbb{P}_k^n \) with \( n_Y = p^r \). Assume that there is a rational map \( Y \dashrightarrow X \) to a smooth projective variety \( X \) with \( \dim X < \dim Y \). Then \( n_X \mid p^{r-1} \).

By [3] Prop. 6.2 we have \( n_X \mid n_Y \). Therefore, \( n_X \mid p^r \). By Corollary 4.2 \( n_X \mid \chi(\mathcal{O}_Y) \cdot \tau_{p-2} \) and by the previous example the right hand side is not divisible by \( n_Y = p^r \).

The next proposition provides a version of the index reduction formula for varieties with different Euler characteristics.

4.6 Proposition. Let \( X \) and \( Y \) be smooth projective geometrically irreducible varieties over a field \( k \) of characteristic 0. Let \( p \) be a prime. Assume that
\[
\begin{align*}
&\bullet \ \dim X, \dim Y < p; \\
&\bullet \ n_X = n_Y = p; \\
&\bullet \ p \mid \chi(\mathcal{O}_X) \text{ and } p \nmid \chi(\mathcal{O}_Y).
\end{align*}
\]
Then \( n_{X,k(Y)} = p \), where \( k(Y) \) is the function field of \( Y \).

Proof. Taking the product with a projective space we may assume that \( \dim X = \dim Y = p - 1 \). Obviously, \( n_{X,k(Y)} \mid n_X \). Assume that \( n_{X,k(Y)} = 1 \). Then \( X_{k(Y)} \) has a closed point \( P \) of degree \( m \) coprime to \( p \).

We follow the proof of [4] Thm. 3.3 (see also [2] Thm. 5.1). Let \( K \) denote the residue field of \( P \) and let \( Y' \) be a smooth projective variety over \( k \) such that \( K = k(Y') \). Such \( Y' \) can be always obtained from \( Y \) by taking the normalization and then resolving the singularities. The condition that \( X_{k(Y')} \) has the point \( P \) means that there is a rational map \( f_P: Y' \rightarrow X \).

Assume that \( f_P \) is not dominant. Then by Theorem 3.11 applied to the maps \( f_P: Y' \rightarrow X \) and \( Y' \rightarrow Y \) we obtain that
\[
n_X \mid \chi(\mathcal{O}_{Y'}) \cdot \tau_{p-2} \quad \text{and} \quad \chi(\mathcal{O}_{Y'}) \cdot \tau_{p-2} \equiv m \cdot \chi(\mathcal{O}_Y) \cdot \tau_{p-2} \pmod{n_Y}.
\]

9
Therefore, $p \mid \chi(\mathcal{O}_Y)$, a contradiction. Hence, $f_P$ has to be dominant.

If $f_P$ is dominant, then by (9) we have

$$
\chi(\mathcal{O}_Y) \cdot \tau_{p-2} \equiv \deg(f_P) \cdot \chi(\mathcal{O}_X) \cdot \tau_{p-2} \mod n_X.
$$

Since $p \mid \chi(\mathcal{O}_X)$, the left hand side has to be divisible by $p$, a contradiction.

Therefore, $n_{X_{k(Y)}} \neq 1$, and the proposition is proven. 

4.7 Corollary. (cf. [2, Thm. 1.1]) Let $X/k$ be as above, i.e. $\dim X < p$, $n_X = p$ and $p \mid \chi(\mathcal{O}_X)$ for some prime $p$. Then there exists a field extension $K/k$ of cohomological dimension one such that $n_{X_K} = p$.

Proof. Take $Y$ to be a Severi-Brauer variety of a division algebra of degree $p$ and apply Proposition 4.6. We obtain that $n_{X_{k(Y)}} = n_X = p$. Repeating the arguments of [2, §2] for the given $X$ and $Y$ we finish the proof.

4.8 Example. If $X$ is a curve of genus one (here $\chi(\mathcal{O}_X) = 0$) the existence of such a field $K$ was proven in [5].

Let $p$ be an odd prime and let $X$ be a smooth hypersurface of degree $p$ in $\mathbb{P}_k^{p-1}$ with $n_X = p$. Observe that $X$ is an anisotropic Calabi-Yau variety with $\chi(\mathcal{O}_X) = 0$. By Corollary 4.7 we obtain that there exists a field $K/k$ of cohomological dimension one such that $n_{X_K} = p$. In particular, $X$ has no zero-cycles of degree one over $K$.

5 Comparison with the classical degree formulas

5.1. We follow the notation of [12]. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ be a partition, i.e. a sequence of integers (possibly empty) $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_r$, and let $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_r$ denote its degree. For any $\alpha$ we define the smallest symmetric polynomial

$$
P_\alpha(x_1, x_2, \ldots) = \sum_{(i_1, i_2, \ldots, i_r)} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_r^{\alpha_r} = Q_\alpha(\sigma_1, \sigma_2, \ldots)
$$

containing the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_r^{\alpha_r}$ where the $\sigma_i$ are the elementary symmetric functions.

5.2. Let $X$ be a smooth projective variety of dimension $d$. Let $c_i = c_i(-T_X)$ denote the $i$-th Chern class of the inverse of the tangent bundle. Let $\alpha$ be a partition of $d$. We define the $\alpha$-characteristic number of $X$ by

$$
c_\alpha = \deg Q_\alpha(c_1, c_2, \ldots).
$$

Observe that $c_{(1,1,\ldots,1)} = \deg c_{d}(-T_X)$ and $c_{(d)}$ defines the so called additive characteristic number of $X$. 

10
5.3. We fix a prime $p$. Consider a partition

$$\alpha = (p-1, p-1, p-1, p^2-1, p^2-1, \ldots),$$  \hspace{1cm} (14)

where $p^i - 1$ is repeated $r_i$ times (in [10] it was denoted by the sequence $R = (r_1, r_2, \ldots)$). The set of all such partitions $\alpha$ will be denoted by $\Lambda_p$. According to [10, §6] for any $\alpha \in \Lambda_p$ the characteristic number $c_\alpha$ is divisible by $p$. The integer $\frac{1}{p}c_{p-1, \ldots, p-1}$ is called the Rost number and is denoted by $\eta_p$.

5.4. By [10, Theorem 6.4] for any prime $p$ and any partition $\alpha \in \Lambda_p$ of $d$ we have the degree formula:

$$\frac{c_\alpha(Y)}{p} \equiv \deg f \frac{c_\alpha(X)}{p} \mod n_X, \hspace{1cm} (15)$$

where $\deg f$ is the degree of a rational map $f: Y \to X$ and $d = \dim Y = \dim X$. In particular, if $n_X \nmid \frac{1}{p}c_\alpha(X)$, then $X$ is incompressible.

In the present section we discuss the relations between the classical degree formulas (15) and the degree formula (9). The following lemma provides an explicit formula for $\chi(\mathcal{O}_X)$ in terms of characteristic numbers $c_\alpha(X)$

5.5 Lemma. Let $X$ be a smooth projective variety of dimension $d$ over $k$. Then

$$\chi(\mathcal{O}_X) = (-1)^d \sum_{\alpha, |\alpha| = d} \frac{c_\alpha(X)}{(\alpha_1 + 1)!\alpha_2 + 1)! \ldots \alpha_r + 1)!}.$$  

Proof. By the very definition $\text{Td}(X) = \text{Td}(T_X) = \prod_{i=1}^d Q(x_i)$, where $Q(x_i) = \frac{x_i^{1-e^{-x_i}}}{1-e^{-x_i}}$ and $x_i$ are the roots of the tangent bundle $T_X$. Since $\text{Td}(T_X) = \text{Td}(-T_X)^{-1} = \prod_{i=1}^d Q(-x_i)^{-1}$, its component of degree $d$ is equal to the coefficient at $z^d$ in the expansion of the product

$$\prod_{i=1}^d (1 - \frac{x_i}{p}z + \frac{x_i^2}{p^2}z^2 - \frac{x_i^3}{p^3}z^3 + \ldots).$$

Analyzing the product we see that this coefficient is, indeed, given by

$$(-1)^d \sum_{\alpha, |\alpha| = d} \frac{P_\alpha(x_1, x_2, \ldots)}{(\alpha_1 + 1)!\alpha_2 + 1)! \ldots \alpha_r + 1)!},$$

where $P_\alpha$ is the minimal symmetric polynomial from 5.1.  \hspace{1cm} \Box

5.6 Lemma. Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a partition of $d$ and let $v_p(m)$ denote the $p$-adic valuation of an integer $m$. Then

$$v_p(\tau_{d-1}) + 1 \geq v_p\left(\prod_{i=1}^r (\alpha_i + 1)\right),$$

where the equality holds if and only if $\alpha \in \Lambda_p$. 11
Proof. Follows from the formulas $v_p(\tau_{d-1}) = \left[ \frac{p-1}{p} \right]$ and $v_p(m!) = \sum_{j=1}^{\infty} \left[ \frac{m}{p^j} \right]$. □

5.7 Definition. Let $p$ be a prime $p$ and let $d$ be an integer. We define a linear combination $u_p$ of characteristic numbers $c_\alpha$, $\alpha \in \Lambda_p$, as

$$u_p = \sum_{\alpha \in \Lambda_p, |\alpha| = d} \frac{\tau_{d-1}}{\prod_{i=1}^{\tau_{d-1}} (\alpha_i + 1)!} c_\alpha.$$ 

According to Lemma 5.6 we have

$$u_p = \sum_{\alpha \in \Lambda_p, |\alpha| = d} \frac{n_\alpha c_\alpha}{p},$$

where $n_\alpha = \frac{p \cdot \tau_{d-1}}{\prod_{i=1}^{\tau_{d-1}} (\alpha_i + 1)!} \in \mathbb{Z}$ and $p \nmid n_\alpha$.

5.8 Proposition. Let $X$ be a smooth projective variety of dimension $d$ over a field of characteristic 0. Then

$$n_X \nmid \chi(O_X) \tau_{d-1} \iff \exists p \text{ such that } n_X \nmid u_p(X).$$

Proof. Since $n_X \mid c_\alpha$ for any $\alpha$, $u_p = \frac{n_\alpha c_\alpha}{p} \in \mathbb{Z}$. We have the following chain of equivalences

$$n_X \mid \chi(O_X) \tau_{d-1} \iff n_X \mid \sum_{p} \frac{n_\alpha}{p} u_p' \iff \sum_{p} \frac{u_p'}{p} \in \mathbb{Z} \iff \forall p \ n \mid u_p' \iff \forall p \ n \mid \frac{u_p'}{n} \in \mathbb{Z}. \quad \square$$

5.9 Example. Let $X$ be a smooth projective curve. In this case we have only one non-trivial partition $\alpha = (1) \in \Lambda_2$ and

$$\chi(O_X) = -\frac{1}{2} c_{(1)}(X) = -\eta_2(X) = -u_2(X).$$

Therefore, for curves the degree formula (9) coincides with the classical one.

5.10 Example. For a smooth projective surface $X$ we have two partitions $\alpha = (1,1)$ and $(2)$, where the first one belongs to $\Lambda_2$ and the second one to $\Lambda_3$. We have

$$\chi(O_X) = \frac{1}{4} c_{(1,1)}(X) + \frac{1}{4} c_{(2)}(X)$$

and $u_2 = \frac{1}{2} c_{(1,1)} = \eta_2$, $u_3 = \frac{1}{4} c_{(2)} = \eta_3$.

The degree formula (9) turns into a sum of the classical degree formulas

$$(\eta_2 + \eta_3)(Y) \equiv \deg f \cdot (\eta_2 + \eta_3)(X) \mod n_X$$

and

$$n_X \mid \tau_{d-1} \chi(O_X) \iff n_X \mid \eta_2(X) \text{ or } n_X \mid \eta_3(X).$$

So from the point of view of incompressibility the degree formula (9) provides the same answer as the classical degree formulas.

5.11 Example. Let $X$ be a smooth projective 3-fold. In this case we have three partitions $(1,1,1)$, $(1,2)$ and $(3)$, where the first and the last one belong to $\Lambda_2$. We have

$$\chi(O_X) = -\frac{1}{8} c_{(1,1,1)} - \frac{1}{12} c_{(1,2)} - \frac{1}{24} c_{(3)}$$

and $u_2 = \frac{3}{2} c_{(1,1,1)} + \frac{1}{2} c_{(2)}$. 

12
Therefore,
\[ n_X \nmid \chi(O_X) \tau_2 \iff n_X \nmid u_2 \stackrel{(\ast)}{\iff} n_X \nmid \frac{1}{2}c_{(1,1,1)} \text{ or } n_X \nmid \frac{1}{2}c_{(3)} \]
which means that for 3-folds the classical degree formulas \([15]\) detect the incompressibility better than \([5]\).

5.12. Since each characteristic number \(c_\alpha\) is divisible by \(n_X\), to say that \(n_X \nmid q \cdot c_\alpha, q \in \mathbb{Q}\), is equivalent to say that \(q \cdot C_\alpha \in \mathbb{Z}\), where \(C_\alpha := c_\alpha/n_X \in \mathbb{Z}\). Hence, the implication \((\ast)\) can be rewritten as
\[ \frac{1}{2}C_{(1,1,1)} + \frac{1}{2}C_{(3)} \not\in \mathbb{Z} \iff \frac{1}{2}C_{(1,1,1)} \not\in \mathbb{Z} \text{ or } \frac{1}{2}C_{(3)} \not\in \mathbb{Z} \]
In particular, the implication \((\ast)\) becomes an equivalence if and only if a 3-fold \(X\) satisfies the following condition
\[ C_{(1,1,1)} = \frac{c_{(1,1,1)}(X)}{n_X} \text{ is even or } C_{(3)} = \frac{c_{(3)}(X)}{n_X} \text{ is even.} \] (16)

5.13 Example. Let \(X\) be a complete intersection of \(m\) hypersurfaces of degrees \(d_1, \ldots, d_m\) in \(\mathbb{P}^{m+3}\). Using the formula from [11, \S 8] we obtain:
\[ C_{(3)} = \frac{\partial}{\partial z} \left( \prod_{i=1}^{m} \frac{(1 + d_i z)}{(1 + z)^{m+4}} \right)_{z=0} = \left( \sum_{i=1}^{m} d_i^3 \right) - m - 4 \equiv \sigma_1 + m \mod 2, \]
where \(\sigma_1\) denotes the sum of all degrees. And
\[ C_{(1,1,1)} = \frac{1}{6} \frac{\partial^3}{\partial z^3} \left( \prod_{i=1}^{m} \frac{(1 + d_i z)}{(1 + z)^{m+4}} \right)_{z=0} \equiv \left( \binom{m+2}{3} + \binom{m+1}{2} \right) \sigma_1 + m \sigma_2 \mod 2, \]
where \(\sigma_2\) is the second elementary symmetric function in \(d_i\)-s.

Hence, a complete intersection \(X\) satisfies (16) if and only if it satisfies one of the following conditions
- \(m = 4k\);
- \(m = 4k + 1, \sigma_1 \text{ or } \sigma_2 \text{ is odd; }\)
- \(m = 4k - 1, \sigma_1 \text{ is odd or } \sigma_2 \text{ is even.}\)

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