Entanglement measure for general pure multipartite quantum states

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We propose an explicit formula for an entanglement measure of pure multipartite quantum states, then study a general pure tripartite state in detail, and at end we give some simple but illustrative examples on four-qubits and m-qubits states.

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I. INTRODUCTION

One of the unsolved problems of modern quantum theory is the quantification of multipartite state entanglement. This is a task that is directly linked to mathematics, such as linear algebra, geometry and functional analysis. The definition of separability and entanglement of a multipartite state was introduced following the definition for bipartite states, given in 1989 by Werner. Eventually, quantitative measures, such as the entanglement of formation and concurrence were formulated for bipartite systems. In recent years, there have been attempts to find an entanglement measure for qubit-qudit states and for multipartite states, i.e., in and for multipartite states, respectively. To exemplify, upper and lower bounds for the quantum relative entropy of entanglement of a multipartite systems in terms of the bipartite entanglements of formation, distillation, and quantum entropy of various subsystems are derived in . Measures based on the Schmidt rank are proposed in and on local unitary and filtering processes in . Furthermore, a very useful tool to detect entanglement, called entanglement witness, is generalized to multipartite states. The tool is the quantification of multipartite state entanglement by the work in , so here we will only give examples for multipartite states.

In this paper, we propose another measure of entanglement for arbitrary, pure multipartite states. Inspired by the work in , we give an explicit expression for such a functional. Our method is based on the joint relative-phase properties of a multipartite quantum system , expressed by a positive operator value measure (POVM) . The POVM is constructed by taking the fold tensor product of the subsystems’ corresponding POVMs. We have already discussed, in detail, our degree of entanglement for a bipartite state in , so here we will only give examples for multipartite states.

II. ENTANGLEMENT FROM A RELATIVE-PHASE POVM

A general and symmetric POVM in a single -dimensional Hilbert space is given by

\[ \hat{\Delta}_Q = \sum_{l_u} \sum_{k_u=1}^{N_u} e^{i\varphi_{k_u,l_u}} |k_u\rangle\langle l_u|, \]

where are the basis vectors in and .

The POVM is a function of the relative phases , for all . It is now possible to form a POVM of a multipartite system by simply forming the tensor product

\[ \hat{\Delta}_Q (\varphi_{Q_1; k_1, l_1}, \ldots, \varphi_{Q_m; k_m, l_m}) = \hat{\Delta}_Q (\varphi_{Q_1; k_1, l_1}) \otimes \cdots \otimes \hat{\Delta}_Q (\varphi_{Q_m; k_m, l_m}), \]

where, e.g., is the set of POVM relative phases associated with subsystems , for all , where we need only to consider when , due to . We can now recast this POVM, expressed in local properties, in terms of the relative-phase sums . Note that if, e.g., , then the term vanishes from the sum due to . From we can form an associated real function expressed in , linearly independent.
relative-phase sums:

\[
\mathcal{P}(\phi_{k_1}^{(M)} j_{i_1}^{(1)} \cdots k_m^{(M)} j_{i_m}^{(M)}) = \text{Tr} \left( \hat{\rho} \Delta(\phi_{k_1}^{(M)} j_{i_1}^{(1)} \cdots k_m^{(M)} j_{i_m}^{(M)}) \right),
\]

where \( \hat{\rho} \) is the state density operator acting on the composite Hilbert space \( \mathcal{H}_2 \). Next, we define to what extent the density operator depends on the particular joint relative-phase sum \( \phi_{k_1}^{(M)} j_{i_1}^{(1)} \cdots k_m^{(M)} j_{i_m}^{(M)} \), e.g.,

\[
\gamma_{k_1}^{(1)} j_{i_1}^{(1)} \cdots k_m^{(M)} j_{i_m}^{(M)} = \left| \int_{2\pi} d\phi \phi_{k_1}^{(1)} j_{i_1}^{(1)} \cdots k_m^{(M)} j_{i_m}^{(M)} e^{-i\phi_{k_1}^{(1)} j_{i_1}^{(1)} \cdots k_m^{(M)} j_{i_m}^{(M)}} \mathcal{P}(\phi_{k_1}^{(1)} j_{i_1}^{(1)} \cdots k_m^{(M)} j_{i_m}^{(M)}) \right|,
\]

where \( \mathcal{P} \) must be expressed in the relative-phase sum parameter \( \phi_{k_1}^{(1)} j_{i_1}^{(1)} \cdots k_m^{(M)} j_{i_m}^{(M)} \), but the particular choice of the remaining \( M-1 \) linearly independent relative-phase sum parameters is inconsequential for the absolute value of the integral. The coefficients \( \gamma_{k_1, l_1 \ldots l_m, a} \) (where, here, and in the following, we will omit the superscript on the indices) are proportional to the Fourier components of the joint relative-phase distribution. Now, let us introduce the following index operator to connect the notation using the subsystem indices, and the notation using the joint-system index running from 1 to \( N_1 N_2 \cdots N_m \):

\[
\Pi(k_1, l_1, k_2, l_2, \ldots, k_m, l_m) = (k_1 - 1) N_2 \cdots N_m + (k_2 - 1) N_3 \cdots N_m + \cdots + (k_m - 1) N_{m+1} + k_m,
\]

\[
+ (l_1 - 1) N_2 \cdots N_m + (l_2 - 1) N_3 \cdots N_m + \cdots + (l_m - 1) N_m + l_m.
\]

Note that the index operator generates two indices based on the set \( \{k_u\} \) and \( \{l_u\} \), respectively. Evaluating the Fourier components, one finds, not surprisingly, that \( \gamma_{k_1, l_1 \ldots k_m, l_m} = 2\pi |\rho_{\Pi(k_1, l_1 \ldots k_m, l_m)}| \). That is, to each relative-phase sum there is an associated joint-system density matrix coefficient. We now define an index permutation operator \( P_j \) operating on any function \( f(k_1, l_1, \ldots, k_m, l_m) \) by

\[
P_j f(k_1, l_1, \ldots, k_j, l_j, \ldots, k_m, l_m) = f(k_1, l_1, \ldots, k_j, l_j, \ldots, k_m, l_m) - f(k_1, l_1, \ldots, k_j, l_j, \ldots, k_m, l_m).
\]

Using this operator we can generalize our earlier results for bipartite systems. We form an entanglement function by summing the absolute difference between pairwise relative-phase sums. The function is given by

\[
\Gamma(\hat{\rho}) = |N_2| \sum_{l_1 > k_1 = 1} N_1 \sum_{l_2 > k_2 = 1} N_2 \cdots \sum_{l_m > k_m = 1} N_m \sum_{k_m = l_m = 1} N_m \cdots \sum_{k_1 = 1} N_1 \sum_{l_1 > k_1 = 1} |P_2| \rho_{\Pi(k_1, l_1, k_2, l_2, \ldots, k_{m-1}, l_{m-1}, k_m, l_m)}|^2 + \cdots
\]

This is our central equation. It looks messy, but has a rather logical inner structure. The factors \( N_u \)
are normalization factors, and they should not be confused with the space dimensions \(N_u\). The first sums, where two of them are written explicitly (with normalization factors \(N_Q\)) on the right hand side of (8), only contributes the respective subsystem’s bipartite entanglement. There are \(\binom{m}{3}\) triparticle combinations, and for every choice \(\{Q_u, Q_v, Q_w\}\), where \(u < v < w\), there are \(N_u(N_u-1)N_v(N_v-1)N_w(N_w-1)/8\) combination of system relative phases. For each combination, we can sum all three relative phases, sum the first two and subtract the third, etc. To form differences of all combinations, we use both the permutation operators \(P_v\) and \(P_w\). Hence, we get \(4 = 2^{3-1}\) contributions within the first curly bracket in (8) above. For each choice, the other systems indices can be chosen in \(\prod_{j=1}^m N_j/(N_uN_vN_w)\) different ways. For the quadraparticle contribution we proceed in the same way. For every choice \(\{Q_u, Q_v, Q_w, Q_z\}\), where \(u < v < w < z\), we use the permutation operators \(P_v, P_w, P_z\). We get \(8 = 2^{4-1}\) contributions inside the corresponding curly brackets, each being a sum of \(\prod_{j=1}^m N_j/(N_uN_vN_wN_z)\) terms. The sum proceed in this fashion until the \(m\)-partite entanglement contributions are to be added. There is only one way \(\binom{m}{m} = 1\) to chose all subsystems, and we insert \(m\) as index in our permutation operator, we use the permutation operators \(P_2, P_3, \ldots, P_m\). (We do not permute \(k_1\) and \(l_1\).) In all, we get \(2^m - 1\) terms inside the curly brackets of the last sum in (8), above. These terms represent all the possible relative-phase sums and differences between all the \(m\)-systems, so there are no further terms.

From our definitions, it is clear that for any product state

\[
\rho_{\Pi(k_1,l_1,k_2,l_2,\ldots,k_m,l_m)} = \rho_{k_1,l_1}\rho_{k_2,l_2}\cdots\rho_{k_m,l_m},
\]

where \(\rho_{k_u,l_u}\) is the indicated density matrix coefficient of system \(u\). In this case, one gets \(P_u|\rho_{\Pi(k_1,l_1,k_2,l_2,\ldots,k_m,l_m)}\rangle = 0\) for any \(u\) and any set of indices \(k_1, l_1, \ldots, k_m, l_m\). Hence, our entanglement function \(\Gamma(\bar{\rho}) = 0\) for any tensor product of \(m\) density operators. For entangled states, the function is not invariant to local unitary transformations. In analogy with our definitions for bipartite states, we define our measure of entanglement \(\Gamma_{\text{sup}}\), where \(\text{sup}\) refers to the supremum of \(\Gamma\) under all possible local unitary transformations.

Let us now write out and use (8) in a few explicit cases. The degree of entanglement for a \(\mathcal{H}_{Q_1} \otimes \mathcal{H}_{Q_2}\) bipartite states is given by

\[
\Gamma(\bar{\rho}) = \left(\mathcal{N}_2 \sum_{l_1 > k_1} \sum_{l_2 > k_2} \sum_{k_2 = 1}^2 \sum_{k_1 = 1}^2 |\rho(k_1-1)N_2 + k_2,(l_1-1)N_2 + l_2| - |\rho(k_1-1)N_2 + l_2,(l_1-1)N_2 + k_2|^{2} \right)^{\frac{1}{2}}.
\]

This special case has already been discussed in detail in \(21, 22\), and we have shown that the equation coincides with the concurrence \(R\) for pure bipartite states in \(2 \otimes 2\) (provided that one sets \(N_2 = 2\)) and with generalized concurrence measures in \(2 \otimes 3\) dimensions \(7, 8, 9\).

Note that our measure sums all the state’s entanglement. That is, although, e.g., a state’s bipartite entanglement between subsystems \(Q_1\) and \(Q_2\) cannot be used simultaneously neither with its bipartite entanglement between subsystems \(Q_1\) and \(Q_3\), nor, e.g., with its tripartite entanglement between subsystems \(Q_1, Q_2,\) and \(Q_3\), all contributions are added in our measure. That is, our measure characterizes the entanglement contained in a state, but in general the measure exceeds the usable entanglement. However, by looking at the various terms in the sum, the usable entanglement can be extracted as the measure is composed of sub-sums containing the bipartite \(Q_1\) and \(Q_2\) entanglement, the bipartite \(Q_1\) and \(Q_3\) entanglement, the tripartite \(Q_1, Q_2,\) and \(Q_3\) entanglement, etc., as can explicitly be seen in (10), below. Also note that our measure sums the possible cooperative entanglement. That is, if some subsystems are ignored, or the information contained in a subsystem is lost, then the ensuing state’s entanglement is in general lower than what our measure predicts. We shall give a concrete example of this in Sec. [IV] below.

### III. TRIPARTITE ENTANGLEMENT

The degree of entanglement for a \(\mathcal{H}_{Q_1} \otimes \mathcal{H}_{Q_2} \otimes \mathcal{H}_{Q_3}\) tripartite state is given by
Our measure of entanglement of this state is

\[ \Gamma(\rho) = (N_2| \sum_{l_1 \geq k_1 = 1} N_1 \sum_{l_2 \geq k_2 = 1} N_2 \sum_{l_3 \geq k_3 = 1} N_3 \sum_{l_4 \geq k_4 = 1} N_4 \left| \rho\Pi(k_1,l_1,k_2,l_2,k_3,l_3) \right| - |\rho\Pi(k_1,l_1,k_2,l_2,k_3,l_3)|^2_{Q_1,Q_2} \]

\[ + \sum_{l_1 \geq k_1 = 1} N_1 \sum_{l_2 \geq k_2 = 1} N_2 \sum_{l_3 \geq k_3 = 1} N_3 \sum_{l_4 \geq k_4 = 1} N_4 \left| \rho\Pi(k_1,l_1,k_2,l_2,k_3,l_3) \right| - |\rho\Pi(k_1,l_1,k_2,l_2,k_3,l_3)|^2_{Q_2,Q_3} \]

\[ + \sum_{l_2 \geq k_2 = 1} N_2 \sum_{l_3 \geq k_3 = 1} N_3 \sum_{l_4 \geq k_4 = 1} N_4 \sum_{l_1 \geq k_1 = 1} N_1 \left| \rho\Pi(k_1,l_1,k_2,l_2,k_3,l_3) \right| - |\rho\Pi(k_1,l_1,k_2,l_2,k_3,l_3)|^2_{Q_1,Q_3} \]

\[ + N_3 \sum_{l_1 \geq k_1 = 1} N_1 \sum_{l_2 \geq k_2 = 1} N_2 \sum_{l_3 \geq k_3 = 1} N_3 \sum_{l_4 \geq k_4 = 1} N_4 \left| \rho\Pi(k_1,l_1,k_2,l_2,k_3,l_3) \right| - |\rho\Pi(k_1,l_1,k_2,l_2,k_3,l_3)|^2_{Q_1,Q_2,Q_3} \]

\[ - (|\rho\Pi(k_1,l_1,k_2,l_2,k_3,l_3)|^2_{Q_1,Q_2,Q_3}) \]}

(11)

Let us now give two concrete examples of this measure for some three-qubit states. In the three-qubit space there exist two classes of states, inequivalent under local operations and classical communication (LOCC), called $\Psi_{\text{GHZ}}$ and $\Psi_{\text{W}}$ states. They are, e.g., $\Psi_{\text{GHZ}} = (|000\rangle + |111\rangle)/\sqrt{2}$ and $\Psi_{\text{W}} = (|001\rangle + |101\rangle + |110\rangle)/\sqrt{3}$. For these states, we have

\[ \Gamma(\hat{\rho}_{\text{GHZ}}) = \left( N_3 |\rho_{1,3}|^2_{Q_1,Q_2,Q_3} \right)^\frac{1}{2} = \sqrt{\frac{N_3}{4}}, \]

and

\[ \Gamma(\hat{\rho}_{\text{W}}) = \left( N_2 (|\rho_{2,3}|^2_{Q_1,Q_2} + |\rho_{2,3}|^2_{Q_2,Q_3} + |\rho_{2,3}|^2_{Q_1,Q_3}) \right)^\frac{1}{2} = \sqrt{\frac{N_2}{3}}. \]

Here, we see that the normalization factors must be retained (or, possibly be chosen with particular relative weights) in order for the entanglement measure to make sense for general states. Because the GHZ and the W states belong to different equivalence classes, their relative entanglement weights are not obvious. This issue is tied to the, still open, question about minimum reversible entanglement generating sets $2\mid 12 \mid 13 \mid 18 \mid 23$.

**IV. BEYOND TREE-PARTITE QUBIT ENTANGLEMENT**

Next, let us look at an interesting four-qubit state $|\Psi_1\rangle = (|0,0,0,1\rangle + |0,1,0,0\rangle + |1,0,1,0\rangle + |1,1,1,1\rangle)/2$. Our measure of entanglement of this state is

\[ \Gamma(\Psi_1) = (N_2(|\rho_{2,5}|^2_{Q_2,Q_4} + |\rho_{11,16}|^2_{Q_2,Q_4}) \]

\[ + N_3(|\rho_{2,3}|^2_{Q_1,Q_2,Q_3} + |\rho_{5,16}|^2_{Q_1,Q_2,Q_3} + |\rho_{2,11}|^2_{Q_1,Q_3,Q_4} + |\rho_{5,16}|^2_{Q_1,Q_3,Q_4})) \]

\[ = \left( \frac{N_2}{8} + \frac{N_3}{4} \right)^\frac{1}{2}. \]

The state has both bipartite and tripartite entanglement. In order to use the bipartite entanglement the parties possessing the different qubit subsystems must cooperate. If, e.g., qubit 1 and 3 are measured in the standard basis, the result is either two zeros or two ones. If this result is communicated to the parties holding qubit 2 and 4, (that is, we perform a LOCC, optimal for bringing out the bipartite entanglement) the remaining two-qubit state will be in a known pure EPR-state. If, on the other hand, if we simply trace out subsystems $Q_1$ and $Q_3$, (or measure qubit 1 and 3 but keep the result secret), then the remaining state is in an an equal mixture of the EPR-states, and this state is directly separable. This means that if the different parties do not cooperate, the state’s bipartite entanglement in subspace 1 and 3 vanishes.

The entanglement of the state $|\Psi_2\rangle = (|0,1,1,0\rangle + |1,0,0,1\rangle + |0,1,1,1\rangle + |1,0,0,0\rangle)/2$, on the other hand, is given by

\[ \Gamma(\Psi_2) = (N_3 |\rho_{7,9}|^2_{Q_1,Q_2,Q_3} \right)^\frac{1}{2} = \sqrt{\frac{N_3}{4}}. \]

That is, the state has only tripartite entanglement and no bipartite entanglement. To arrive at the result, we note that a unitary transformation $U_4$ local to $Q_4$ can transform the state into, e.g., $U_4|\Psi_2\rangle = (|0,1,1\rangle + |1,0,0\rangle \otimes |0\rangle)/\sqrt{2}$ for which one finds the supremum of $\Gamma$. In this case, the state’s entanglement is the same whether or not the person in possession of qubit 4 cooperates or not.

As a last example, consider a a $m$-qubit density operator given by a mixture of the two orthogonal, so called, $m$-Cat states

\[ |\Psi_{\text{Cat}}\rangle = \frac{1}{\sqrt{2}}(|0_1,0_2,\ldots,0_m\rangle + |1_1,1_2,\ldots,1_m\rangle) \]
Then, our degree of entanglement gives

$$\Gamma(\Psi_{\text{Cat}}) = \left( N_m |p_{1,2,m}|_{Q_1, Q_2,..., Q_m}^2 \right)^{\frac{1}{2}}$$

$$= \left( \frac{N_m}{4} \right)^{\frac{1}{2}}.$$ 

V. CONCLUSIONS

In conclusion, we have proposed an entanglement measure for pure multipartite quantum states. The measure directly detects product states (it is zero for such states), and quantifies the entanglement of any pure state up to the bipartite, tripartite, ..., m-partite normalization coefficients. Since it is not possible to use, nor convert, the entanglement in states with incompatible entanglement classes such as GHZ- and W-states into each other, it may not be meaningful to specify the coefficients relative to each other. Rather, from an operational point of view, it seems more meaningful to specify each type of entanglement separately, e.g. in a system composed of four subsystems $Q_1$, $Q_2$, $Q_3$, and $Q_4$, it is meaningful to discuss, separately, the bipartite entanglement between, e.g., systems $Q_1$ and $Q_2$, and $Q_1$ and $Q_4$. We do not see how the bipartite entanglement, in an operational sense, could (or should) be compared to, e.g., the tripartite entanglement between $Q_1$, $Q_2$, and $Q_4$. Our measure sums all contributions to quantify the state’s entire entanglement, but, as just indicated, from an operational viewpoint, it is rather the the sum’s various contributions that have a well defined operational meaning. This is in contrast to, e.g., Partovi’s measure [19], that is a minimum entropic distance measure between the state and a separable state with the same statistical marginal distributions. In this sense, Hossein Partovi’s measure is a better measure of the “quantunness” of the state, while our measure emphasize the state’s usefulness as a quantum information carrier.

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