SPLITTINGS OF RIGHT-ANGLED ARTIN GROUPS

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Abstract. We show that if a right-angled Artin group $A(\Gamma)$ has a non-trivial, minimal action on a tree $T$ which is not a line, then $\Gamma$ contains a separating subgraph $\Lambda$ such that $A(\Lambda)$ stabilizes an edge in $T$.

1. Introduction

Given a graph $\Gamma$, let $A(\Gamma)$ denote the associated right-angled Artin group. That is, $A(\Gamma)$ is the group given by the presentation

$$\langle V(\Gamma) \mid [u, v] = 1 \text{ whenever } (u, v) \in E(\Gamma) \rangle.$$ 

A natural question in the study of right-angled Artin groups is how are the graph-theoretic properties of $\Gamma$ related to group-theoretic properties of $A(\Gamma)$? We consider this question in the context of splittings of the group $A(\Gamma)$. Here by a splitting of a group $G$ we mean a graph of groups decomposition of $G$ and by a splitting over a subgroup $H$ we mean a graph of groups decomposition where $H$ is an edge group. By Bass-Serre theory, for any such splitting there is an action of $G$ on a tree (called the Bass-Serre tree of the splitting) and conversely any action of $G$ on a tree has a corresponding splitting where the edge groups are stabilizers of edges in the tree.

It is well-known that $A(\Gamma)$ splits as a non-trivial free product if and only if $\Gamma$ is disconnected. Clay showed that $A(\Gamma)$ splits over $\mathbb{Z}$ if and only if $\Gamma$ contains a cut-vertex [1]. Groves and the author generalized Clay’s result to show that unless $\Gamma$ is a complete graph, $A(\Gamma)$ splits over an abelian subgroup if and only if $\Gamma$ contains a cut-clique, that is a complete subgraph $\Lambda$ such that $\Gamma \setminus \Lambda$ is disconnected [2].

One direction of these implications is straightforward and follows from the general observation that if $\Lambda$ is a separating subgraph of $\Gamma$, that is a subgraph such that $\Gamma \setminus \Lambda$ is disconnected, then $A(\Gamma)$ splits over $A(\Lambda)$ as an amalgamated product

$$A(\Gamma) \cong A(\Gamma_1 \cup \Lambda) *_{A(\Lambda)} A(\Gamma_2 \cup \Lambda)$$

where $\Gamma_1$ is a connected component of $\Gamma \setminus \Lambda$ and $\Gamma_2 = \Gamma \setminus (\Gamma_1 \cup \Lambda)$.

Another way to construct splittings of $A(\Gamma)$ is to consider actions of $A(\Gamma)$ on a line. Any such action will produce a splitting of $A(\Gamma)$ as an HNN-extension over the kernel of the action. When $\Gamma$ is connected the set of such actions is equivalent to the set homomorphisms from $A(\Gamma) \rightarrow \mathbb{Z}$. For example, $A(\Gamma)$ splits as an HNN-extension over the Bestvina-Brady subgroup which is the kernel of the homomorphism $A(\Gamma) \rightarrow \mathbb{Z}$ defined by sending each generator of $A(\Gamma)$ to 1. Homomorphisms $A(\Gamma) \rightarrow \mathbb{Z}$ all factor through the abelianization of $A(\Gamma)$ which is isomorphic to $\mathbb{Z}^n$ (where $n$ is the number of vertices of $\Gamma$). Hence the set of homomorphisms $\mathbb{Z}^n \rightarrow \mathbb{Z}$ gives a parameterization of these types of splittings of $A(\Gamma)$.

\[1\] All trees in this paper are assumed to be simplicial trees.
We now consider splittings of $A(\Gamma)$ where the corresponding Bass-Serre tree is not a line. An action of a group $G$ on a tree $T$ is called minimal if $T$ has no proper $G$-invariant subtrees and non-trivial if there is no point of $T$ which is fixed by all elements of $G$.

**Theorem 1.** Suppose $A(\Gamma)$ has a non-trivial minimal action on a tree $T$ which is not a line. Then $\Gamma$ has an induced subgraph $\Lambda$ such that $\Gamma \setminus \Lambda$ is disconnected and $A(\Lambda)$ stabilizes an edge in $T$.

We note that the edge group in a given splitting of $A(\Gamma)$ may be strictly larger than the subgroup $A(\Lambda)$ produced in the above theorem. For example, if there is an epimorphism $f: A(\Gamma_1) \to A(\Gamma_2)$ and $\Lambda_2$ is a separating subgraph of $\Gamma_2$, then the splitting of $A(\Gamma_2)$ over $A(\Lambda_2)$ induces a splitting of $A(\Gamma_1)$ over $f^{-1}(A(\Lambda_2))$. The subgraph $\Lambda_1$ constructed in Theorem 1 will consist of those vertices of $\Gamma_1$ which $f$ maps into $A(\Lambda_2)$. As long as some element of $\ker(f)$ does not belong to $A(\Lambda_1)$, $A(\Lambda_1)$ will be a proper subgroup of $f^{-1}(A(\Lambda_2))$.

Nevertheless, it is common to consider all possible splittings over a particular family of subgroups, for example the family of all abelian subgroups. If $\mathcal{A}$ is any family of subgroups of a group $G$, then a splitting of $G$ is called an $\mathcal{A}$-splitting if all edge groups of the splitting belong to $\mathcal{A}$.

**Corollary 2.** Let $\Gamma$ be a connected graph and let $\mathcal{A}$ be a family of subgroups of $A(\Gamma)$ such that $\mathcal{A}$ is closed under taking subgroups. If $A(\Gamma)$ has a non-trivial $\mathcal{A}$-splitting then either there exists a homomorphism $\varphi: A(\Gamma) \to \mathbb{Z}$ with $\ker(\varphi) \in \mathcal{A}$ or $\Gamma$ has a separating subgraph $\Lambda$ such that $A(\Lambda) \in \mathcal{A}$.

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### 2. Proof

The proof of Theorem 1 only uses elementary properties of group actions on trees and right-angled Artin groups which we now describe.

Let $G$ be a group acting on a tree $T$. For an element $g \in G$, let $\text{Fix}(g)$ denote the set $\{x \in T \mid gx = x\}$. If $\text{Fix}(g) \neq \emptyset$, the $g$ is called elliptic. In this case $\text{Fix}(g)$ is connected and hence a subtree of $T$. If $g \in G$ is not elliptic, then $g$ is hyperbolic which means that $T$ contains a unique line which is fixed by $g$ set-wise and on which $g$ acts as a non-trivial translation. In this case the corresponding line is called the axis of $g$ which we denote by $\text{Axis}(g)$. If $g$ is hyperbolic and $h$ is any element of $G$, then $h^{-1}gh$ is hyperbolic with axis $h(\text{Axis}(g))$. In particular, if $g$ and $h$ commute, then $h$ fixes the axis of $g$ set-wise. The next lemma follows easily from this observation.

**Lemma 3.** Suppose that a group $G$ acts on a tree $T$ and that $g, h \in G$ are commuting elements with $g$ acting hyperbolically on $T$.

1. If $h$ is elliptic, then $\text{Axis}(g) \subseteq \text{Fix}(h)$.
2. If $h$ is hyperbolic, then $\text{Axis}(g) = \text{Axis}(h)$.

For commuting elliptic elements we use the following.

**Lemma 4.** [Lemma 1.1] Suppose that a group $G$ acts on a tree $T$ and that $g, h \in G$ are commuting elements which both act elliptically on $T$. Then $\text{Fix}(g) \cap \text{Fix}(h) \neq \emptyset$.

**Proof of Theorem 2.** We identify vertices of $\Gamma$ with generators of $A(\Gamma)$ in the natural way. First we suppose that each vertex $v$ acts elliptically on $T$; in this case the proof is similar to the corresponding
case in the proof of [2, Theorem A], but we include the details for the sake of completeness. Since the action is non-trivial, there must be some vertices \(v\) and \(u\) in \(\Gamma\) such that \(\text{Fix}(v) \cap \text{Fix}(u) = \emptyset\).

Let \(e\) be an edge in the tree \(T\) on the path connecting the subtrees \(\text{Fix}(v)\) and \(\text{Fix}(u)\). Let \(\Lambda\) be the induced subgraph of \(\Gamma\) on the set of vertices which fix \(e\). Clearly \(A(\Lambda) \leq \text{Stab}(e)\). Now we will show that the path in \(\Gamma\) from \(u\) to \(v\) contains a vertex in \(\Lambda\). Note that \(\Lambda\) may be empty, in which case the proof will show that there is no path from \(u\) to \(v\), i.e. \(\Gamma\) is disconnected. To that end, let \(v = v_0, v_1, ..., v_n = u\) be the vertices of a path from \(v\) to \(u\) in \(\Gamma\). Note that \(n \geq 2\) since Lemma 4 implies that \(u\) is not adjacent to \(v\) in \(\Gamma\) since each \(v_i\) is adjacent to \(v_{i+1}\) in \(\Gamma\) they must commute as elements of \(A(\Gamma)\). Hence by Lemma 4 \(\text{Fix}(v_i) \cap \text{Fix}(v_{i+1}) \neq \emptyset\) for \(0 \leq i \leq n - 1\). It follows that there is a path in the tree \(T\) from \(\text{Fix}(v)\) to \(\text{Fix}(u)\) which is contained in \(\bigcup_{i=1}^{n-1} \text{Fix}(v_i)\). Since the path in \(T\) from \(\text{Fix}(v)\) to \(\text{Fix}(u)\) is unique, the edge \(e\) must belong to \(\text{Fix}(v_i)\) for some \(i\), and hence \(v_i \in \Lambda\). Therefore, \(v\) and \(u\) are in different connected components of \(\Gamma \setminus \Lambda\).

Now suppose that some vertex \(v\) acts hyperbolically on \(T\). Let \(\Lambda\) be the subgraph induced by the set of vertices which act elliptically on \(T\) and which fix the axis of \(v\) point-wise. Then for any edge \(e\) on this axis, \(A(\Lambda) \leq \text{Stab}(e)\).

Now consider the connected component of \(v\) in \(\Gamma \setminus \Lambda\). Let \(u\) be a vertex in this component, and let \(v = v_0, ..., v_n = u\) be a path from \(v\) to \(u\). Notice that \(v_1\) must be hyperbolic, because elliptic elements which commute with \(v\) will belong to \(\Lambda\) by Lemma 3. But then \(v\) and \(v_1\) have the same axis byLemma 3. Repeating this argument, we get that \(v_2, ..., u\) are all hyperbolic with the same axis as \(v\). But since the action of \(A(\Gamma)\) is minimal and \(T\) is not a line, there must exist some vertex \(w\) which does not set-wise fix the axis of \(v\). Hence \(v\) and \(w\) are in different connected components of \(\Gamma \setminus \Lambda\).

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References

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