ON COMPLEMENTABILITY OF $c_0$ IN SPACES $C(K \times L)$

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Abstract. Using elementary probabilistic methods, in particular a variant of the Weak Law of Large Numbers related to the Bernoulli distribution, we prove that for every infinite compact spaces $K$ and $L$ the product $K \times L$ admits a sequence $\langle \mu_n : n \in \mathbb{N} \rangle$ of normalized signed measures with finite supports which converges to 0 with respect to the weak* topology of the dual Banach space $C(K \times L)^*$. Our approach is completely constructive—the measures $\mu_n$ are defined by an explicit simple formula. We also show that this result generalizes the classical theorem of Cembranos and Freniche which states that for every infinite compact spaces $K$ and $L$ the Banach space $C(K \times L)$ contains a complemented copy of the space $c_0$.

1. Introduction

As usual, for a compact (Hausdorff) space $X$ we denote by $C(X)$ the Banach space of continuous real-valued functions on $X$ and by $c_0$ the Banach space of all real-valued sequences which converge to 0, both endowed with the supremum norm.

It is an easy observation that for every infinite compact space $X$ the space $C(X)$ contains a closed linear subspace isomorphic to the space $c_0$. This subspace need not be complemented—e.g., for $X = \beta \mathbb{N}$, the Čech–Stone compactification of the set $\mathbb{N}$ of non-negative integers equipped with the discrete topology, the space $C(X)$ does not contain any complemented copies of $c_0$. However, if $X$ is a product of two infinite compact spaces, then $C(X)$ always contains a complemented copy of $c_0$, as was proved by Cembranos [4] and Freniche [5].

Theorem 1.1 (Cembranos–Freniche). For every infinite compact spaces $K$ and $L$ the Banach space $C(K \times L)$ contains a complemented copy of the space $c_0$.

A crucial step in both of the proofs is an application of the classical Josefson–Nissenzweig theorem which asserts that for every infinite-dimensional Banach space $E$ there exists a sequence $\langle x_n^* : n \in \mathbb{N} \rangle$ in the dual space $E^*$ such that $\|x_n^*\| = 1$ for every $n \in \mathbb{N}$ and $x_n^*(x) \to 0$ for every $x \in E$ (that is, $\langle x_n^* : n \in \mathbb{N} \rangle$ is convergent to 0 with respect to the weak* topology). Since all standard proofs of the Josefson–Nissenzweig theorem are rather intricate and non-constructive, it is hard to deduce from the proofs of Cembranos and Freniche how the constructed complemented copy of $c_0$ in a given space $C(K \times L)$ basically looks like or what properties it has. In this paper we address this issue and provide an elementary and constructive proof of the following strengthening of Theorem 1.1.

Theorem 1.2. For every infinite compact spaces $K$ and $L$ there is a sequence $\langle \mu_n : n \in \mathbb{N} \rangle$ of normalized signed measures on $K \times L$ with finite supports which is weak* convergent to 0.

Note that, by the virtue of the classical Riesz representation theorem, for every infinite compact space $X$, the Josefson–Nissenzweig theorem can be expressed in the following way: there is a...
sequence $\langle \mu_n: n \in \mathbb{N} \rangle$ of normalized signed regular Borel measures on $X$ which converges to 0 with respect to the weak* topology of the dual space $C(X)^*$. Hence Theorem 1.2 might be treated in the first place as a special case of the Josefson–Nissenzweig theorem. However, as said above, in contrast to the latter general result, our proof of Theorem 1.2 is completely constructive—for every compact spaces $K$ and $L$ the measures $\mu_n$ are given by an explicit simple formula. To prove that they weak* converge to 0 we use elementary tools from probability theory, in particular a variant of the Weak Law of Large Numbers related to the Bernoulli distribution.

The Cembranos–Freniche theorem can be deduced from Theorem 1.2 in the following way, using some $C_p$-theory. Recall that if $X$ is a Tychonoff space, then $C_p(X)$ denotes the space of all continuous real-valued functions on $X$ equipped with the pointwise topology. Similarly, by $(c_0)_p$ we mean the space $c_0$ but endowed with the topology inherited from the product $\mathbb{R}^\mathbb{N}$. The next corollary is an immediate consequence of Theorem 1.2 and the following result due to Banakh, Kąkol, and Śliwa [1] Theorem 1]: a Tychonoff space $X$ admits a sequence $\langle \mu_n: n \in \mathbb{N} \rangle$ of finitely supported signed measures such that $||\mu_n|| = 1$ for every $n \in \mathbb{N}$ and $\int_X f d\mu_n \to 0$ for every $f \in C_p(X)$ if and only if the space $C_p(X)$ contains a complemented copy of the space $(c_0)_p$. Of course, if $X$ is compact, then the condition that $\int_X f d\mu_n \to 0$ for every $f \in C_p(X)$ is equivalent to $\langle \mu_n: n \in \mathbb{N} \rangle$ being convergent to 0 with respect to the weak* topology of $C(X)^*$.

**Corollary 1.3.** For every infinite compact spaces $K$ and $L$ the space $C_p(K \times L)$ contains a complemented copy of the space $(c_0)_p$.

With an aid of the Closed Graph Theorem, the result of Cembranos and Freniche can be easily deduced from the above corollary, see Proposition 2.3 for more details.

Theorem 1.2 gives also applications to Grothendieck $C(X)$-spaces. Recall that a Banach space $E$ is Grothendieck if every sequence $\langle x_n^*: n \in \mathbb{N} \rangle$ in the dual space $E^*$ which is weak* convergent to 0, converges also weakly. It was proved by Cembranos [4] Corollary 2 (cf. also [9] Proposition 5.3)] that, for every infinite compact space $X$, the Banach space $C(X)$ is Grothendieck if and only if $C(X)$ does not contain any complemented copy of $c_0$. The latter result and Theorem 1.2 immediately yield the following theorem of Khurana [7] Theorem 2]: for every infinite compact spaces $K$ and $L$ the space $C(K \times L)$ is never Grothendieck, that is, there exists a sequence $\langle \mu_n: n \in \mathbb{N} \rangle$ of normalized signed regular Borel measures on $K \times L$ which converges weak* to 0 but not weakly. Theorem 1.2 strengthens this observation—there must even exist such $\langle \mu_n: n \in \mathbb{N} \rangle$ which consists only of finitely supported measures (note that, by the Schur property, such $\langle \mu_n: n \in \mathbb{N} \rangle$ cannot be weakly convergent to 0). This yields indeed a strengthening of Khurana’s theorem as there exists a compact space $X$ such that $C(X)$ is not Grothendieck but for every separable closed subspace $Y$ of $X$ the space $C(Y)$ is Grothendieck (see e.g. [6] Example 10)).

Let us finish by mentioning the following problem. It is well-known that the space $C(\beta\mathbb{N})$ is prime, that is, every complemented infinite-dimensional subspace of $C(\beta\mathbb{N})$ is isomorphic to $C(\beta\mathbb{N})$ (see [8]). On the other hand, $C(\beta\mathbb{N} \times \beta\mathbb{N})$ is obviously not prime, we do not know however whether there exist complemented subspaces different than the ones listed below (cf. [3]).

**Question 1.4.** Does there exist a complemented infinite-dimensional subspace of $C(\beta\mathbb{N} \times \beta\mathbb{N})$ which is not isomorphic to any of the following spaces: $C(\beta\mathbb{N} \times \beta\mathbb{N})$, $C(\beta\mathbb{N})$, $c_0$, $c_0 \oplus C(\beta\mathbb{N})$, and $c_0(C(\beta\mathbb{N}))$?

2. Proof of Theorem 1.2

Let us start with some notation and terminology. All topological spaces considered by us are assumed to be Tychonoff. For a Tychonoff space $X$, by $Bor(X)$ we denote the Borel $\sigma$-field of $X$. A function $\mu: Bor(X) \to \mathbb{R}$ is a measure on $X$, if $\mu$ is additive, inner regular with respect
to closed sets and outer regular with respect to open sets, and has bounded variation (that is, \( \|\mu\| = \sup \{ |\mu(A)| + |\mu(B)| : A, B \in \mathcal{B}(X), A \cap B = \emptyset \} < \infty \)). If \( \mu \) attains negative values, then we say that \( \mu \) is signed. If \( \|\mu\| = 1 \), then we say that \( \mu \) is normalized. The support of \( \mu \) is denoted by \( \text{supp}(\mu) \). If there are finite sequences \( x_1, \ldots, x_n \in X \) (with all points pairwise distinct) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) such that \( \mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i} \), where by \( \delta_x \) we denote the one-point measure at a point \( x \in X \), then we say that \( \mu \) is finitely supported. Note that in this case we have \( \text{supp}(\mu) = \{ x_1, \ldots, x_n \} \) and \( \|\mu\| = \sum_{i=1}^{n} |\alpha_i| \). We assume that \( 0 \in \mathbb{N} \) and by \( \mathbb{N}_+ \) we denote the set of positive integers, i.e. \( \mathbb{N}_+ = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \ldots\} \). We identify every \( n \in \mathbb{N} \) with the set \( \{0, \ldots, n-1\} \). If \( A \) is a set, then by \( |A| \) we denote its cardinality.

For every \( n \in \mathbb{N}_+ \) put \( \Omega_n = \{-1, 1\}^n \) and \( \Sigma_n = n \times \{n\} \) (so \( |\Omega_n| = 2^n \) and \( |\Sigma_n| = n \)). To simplify the notation, we will usually write \( i \in \Sigma_n \) instead of \( (i, n) \in \Sigma_n \)—this should cause no confusion. Put also \( \Omega = \bigcup_{n \in \mathbb{N}_+} \Omega_n \) and \( \Sigma = \bigcup_{n \in \mathbb{N}_+} \Sigma_n \), and endow these two sets with the discrete topology. This way, we can think of the product space \( \Omega \times \Sigma \) as a countable union of pairwise disjoint discrete rectangles \( \Omega_k \times \Sigma_m \) of size \( n2^k \)—the rectangles \( \Omega_n \times \Sigma_n \), lying along the diagonal, will bear a special meaning, namely, they will be the supports of measures from the special sequence \( \{\mu_n : n \in \mathbb{N}_+\} \) on the space \( \beta\Omega \times \beta\Sigma \) (that is, on the product of the Čech–Stone compactifications of \( \Omega \) and \( \Sigma \)) defined as follows \( (n \in \mathbb{N}_+) \):

\[
\mu_n = \sum_{s \in \Omega_n, \ i \in \Sigma_n} \frac{s(i)}{n2^n} \delta_{(s, i)}.
\]

Then, \( \text{supp}(\mu_n) = \Omega_n \times \Sigma_n \), so \( |\text{supp}(\mu_n)| = n2^n \), \( \|\mu_n\| = 1 \), and

\[
\pi_i[\text{supp}(\mu_n)] \cap \pi_i[\text{supp}(\mu_{n'})] = \emptyset
\]

for every \( n \neq n' \) and \( i \in \{0, 1\} \) (here \( \pi_i \) denotes the projection on the \( i \)-th coordinate). Note that for each \( n \in \mathbb{N}_+ \) and any two sets \( A \in \wp(\Omega) \) and \( B \in \wp(\Sigma) \) we have:

\[
\mu_n([A] \times [B]) \leq \frac{|A \cap \Omega_n|}{2^n} \cdot \frac{|B \cap \Sigma_n|}{n},
\]

where \([A] \) and \([B] \) always denote the clopen subsets of \( \beta\Omega \) and \( \beta\Sigma \) corresponding in the sense of the Stone duality to \( A \) and \( B \), respectively—since \( \beta\Omega \) and \( \beta\Sigma \) are extremely disconnected, we have \([A] = \overline{A}^{\beta\Omega} \) and \([B] = \overline{B}^{\beta\Sigma} \).

In the next proposition we will prove that the sequence \( \{\mu_n : n \in \mathbb{N}\} \), as defined above, is weak* convergent to 0 on the space \( \beta\Omega \times \beta\Sigma \). However, before we do that, we need to provide a bit of explanation of probability tools we use in the proof. For every \( n \in \mathbb{N}_+ \) and \( i \in n \) define the function \( X_i : \Omega_n \to \{0, 1\} \) as follows: \( X_i(r) = 1 \) if and only if \( r(i) = 1 \), where \( r \in \Omega_n \). Put \( S_n = \sum_{i=0}^{n-1} X_i \), so \( S_n : \Omega_n \to n \) is the function computing the number of 1's in the argument sequence \( r \in \Omega_n \). For a finite set \( A \in [\omega]^{<\omega} \), let \( P_A \) denotes the standard product probability on \( \{-1, 1\}^A \) (assigning \( 1/2^{|A|} \) to each elementary event, i.e. \( P_A(\{r\}) = 1/2^{|A|} \) for each \( r \in \{-1, 1\}^{|A|} \)). Recall that for every \( k \leq n \) it holds:

\[
P_n(S_n = k) = P_n(\{r \in \Omega_n : S_n(r) = k\}) = \binom{n}{k}1/2^n.
\]

We will need the following fact, being a variant of the Weak Law of Large Numbers, which estimates the probability that \( S_n(r) \) has value “far” (with respect to \( \varepsilon \)) from \( n/2 \), i.e. that “r contains significantly more (with respect to \( \varepsilon \)) 1’s than -1’s, or vice versa”.


Fact 2.1. If \( n \in \mathbb{N}_+ \) and \( \varepsilon \in (0, 1/12] \) are such numbers that \( n \geq 48/\varepsilon \), then:

\[
P_n(\{|S_n - n/2| \geq \varepsilon n/2\}) \leq \frac{\sqrt{2}}{\varepsilon \sqrt{n}}
\]

Proof. See Bollobás [2, Theorem 1.7.(i)]. \( \square \)

We are ready to prove the aforementioned auxiliary proposition.

Proposition 2.2. The sequence \( \langle \mu_n : n \in \mathbb{N}_+ \rangle \) defined above is convergent to 0 with respect to the weak* topology of the dual space \( C(\beta \Omega \times \beta \Sigma)^* \).

Proof. Since \( \beta \Omega \times \beta \Sigma \) is a totally disconnected compact space, to prove that \( \langle \mu_n : n \in \mathbb{N}_+ \rangle \) is weak* convergent to 0 it is enough to show that it converges to 0 on every clopen subset of the form \([A] \times [B]\), where \( A \in \varphi(\Omega) \) and \( B \in \varphi(\Sigma) \) (to see this, use e.g. the Stone–Weierstrass theorem applied to the linear subspace of simple functions defined on clopen subsets of \( \beta \Omega \times \beta \Sigma \)). So let us fix two such sets \( A \) and \( B \).

Fix \( \varepsilon \in (0, 1/12] \) and put:

\[
I_0 = \{ n \in \mathbb{N}_+ : |B \cap \Sigma_n| < 2/\varepsilon^4 \}
\]

and

\[
I_1 = \mathbb{N}_+ \setminus I_0 = \{ n \in \mathbb{N}_+ : |B \cap \Sigma_n| \geq 2/\varepsilon^4 \}.
\]

For each \( i \in \{0, 1\} \) we will find \( N_i \in \mathbb{N}_+ \) such that for every \( n \geq N_i, n \in I_i \), it holds

\[
|\mu_n([A] \times [B])| = |\mu_n([A \cap \Omega_n] \times [B \cap \Sigma_n])| < 2\varepsilon.
\]

We first look for \( N_0 \). By (†) for every \( n \in I_0 \) we have:

\[
|\mu_n([A] \times [B])| \leq \frac{|A \cap \Omega_n|}{2^n} \cdot \frac{|B \cap \Sigma_n|}{n} \leq \frac{|B \cap \Sigma_n|}{n} < \frac{2}{n \varepsilon^4},
\]

so if \( I_0 \) is infinite, then there exists \( N_0 \in \mathbb{N}_+ \) such that for every \( n \geq N_0, n \in I_0 \), we have:

\[
|\mu_n([A] \times [B])| < 2\varepsilon.
\]

If, on the other hand, \( I_0 \) is finite, then simply set \( N_0 = 1 + \max I_0 \).

Let us now look for \( N_1 \). We assume that \( I_1 \) is non-empty—otherwise simply set \( N_1 = 0 \). For every \( n \in I_1 \) define the set \( \Delta_{n, \varepsilon} \) as follows:

\[
\Delta_{n, \varepsilon} = \{ s \in \Omega_n : \left| \left\{ i \in B \cap \Sigma_n : s(i) = 1 \right\} - \frac{|B \cap \Sigma_n|}{2} \right| \geq \frac{|B \cap \Sigma_n|}{2} \varepsilon \}.
\]

So \( \Delta_{n, \varepsilon} \) denotes the event that \( s \in \Omega_n \) is “far” (with respect to \( \varepsilon \)) from having the same numbers of 1’s and \(-1\)’s when restricted to the set \( B \). If we put similarly:

\[
\Gamma_{n, \varepsilon} = \{ s \in \{-1, 1\}^{B \cap \Sigma_n} : \left| \left\{ i \in B \cap \Sigma_n : s(i) = 1 \right\} - \frac{|B \cap \Sigma_n|}{2} \right| \geq \frac{|B \cap \Sigma_n|}{2} \varepsilon \},
\]

then we trivially have:

(\( \times \))

\[
\Delta_{n, \varepsilon} = \Gamma_{n, \varepsilon} \times \{-1, 1\}^{\Sigma_n \setminus B}.
\]

Using this, for every \( n \in I_1 \) we will estimate the following values (see (1) and (3)):

(\( \ast \))

\[
|\mu_n([A \cap \Delta_{n, \varepsilon}] \times [B \cap \Sigma_n])| = \left| \sum_{s \in A \cap \Delta_{n, \varepsilon}, i \in B \cap \Sigma_n} s(i) \frac{n^2}{n^2} \right|
\]
and

\[ (*) \quad \left| \mu_n \left( [A \cap (\Omega_n \setminus \Delta_{n,\varepsilon})] \times [B \cap \Sigma_n] \right) \right| = \left| \sum_{s \in A \cap (\Omega_n \setminus \Delta_{n,\varepsilon})} \sum_{i \in B \cap \Sigma_n} \frac{s(i)}{n^{2n}} \right|. \]

Note that:

\[
\left| \mu_n ([A] \times [B]) \right| \leq \left| \mu_n ([A \cap \Delta_{n,\varepsilon}] \times [B \cap \Sigma_n]) \right| + \left| \mu_n ([A \cap (\Omega_n \setminus \Delta_{n,\varepsilon})] \times [B \cap \Sigma_n]) \right|,
\]

so obtaining “good” estimations of (*) and (**) will finish the proof.

Fix \( n \in I_1 \) and let us start with the estimation of (*). Note that \( |B \cap \Sigma_n| \geq 48/\varepsilon \), so recall (x) and apply Fact 21 with the set \( B \cap \Sigma_n \) instead of the set \( n = \{0, \ldots, n-1\} \) to get that:

\[ (0) \quad P_n (\Delta_{n,\varepsilon}) = P_{B \cap \Sigma_n} (\Gamma_{n,\varepsilon}) \cdot P_{\Sigma_n \setminus B} (\{1, 1\} \Sigma_n \setminus B) = P_{B \cap \Sigma_n} (\Gamma_{n,\varepsilon}) \leq \frac{\sqrt{2}}{\varepsilon \sqrt{|B \cap \Sigma_n|}}. \]

It holds that:

\[
\left| \sum_{s \in A \cap \Delta_{n,\varepsilon}} \sum_{i \in B \cap \Sigma_n} \frac{s(i)}{n^{2n}} \right| \leq \sum_{s \in A \cap \Delta_{n,\varepsilon}} \sum_{i \in B \cap \Sigma_n} \frac{1}{n^{2n}} = \left| A \cap \Delta_{n,\varepsilon} \right| \cdot \left| B \cap \Sigma_n \right| \leq \frac{|\Delta_{n,\varepsilon}| \cdot |\Sigma_n|}{n^{2n}} = P_n (\Delta_{n,\varepsilon}),
\]

so by (0) we get the following estimation of (*):

\[ (1) \quad \left| \sum_{s \in A \cap \Delta_{n,\varepsilon}} \sum_{i \in B \cap \Sigma_n} \frac{s(i)}{n^{2n}} \right| \leq \frac{\sqrt{2}}{\varepsilon \sqrt{|B \cap \Sigma_n|}}. \]

We now estimate (**) For every \( s \in \Omega_n \setminus \Delta_{n,\varepsilon} \) we have:

\[
\left| \sum_{i \in B \cap \Sigma_n} s(i) \right| = \left| \{ i \in B \cap \Sigma_n : s(i) = 1 \} \right| - \left| \{ i \in B \cap \Sigma_n : s(i) = -1 \} \right| \leq \left| \{ i \in B \cap \Sigma_n : s(i) = 1 \} \right| - \left| \{ i \in B \cap \Sigma_n : s(i) = -1 \} \right| \leq 2 \cdot \frac{|B \cap \Sigma_n|}{2} = \varepsilon |B \cap \Sigma_n|,
\]

so:

\[ (2) \quad \left| \sum_{i \in B \cap \Sigma_n} s(i) \right| < \varepsilon |B \cap \Sigma_n|, \]

Next, it holds:

\[
\left| \sum_{s \in A \cap (\Omega_n \setminus \Delta_{n,\varepsilon})} \sum_{i \in B \cap \Sigma_n} \frac{s(i)}{n^{2n}} \right| \leq \frac{1}{n^{2n}} \sum_{s \in A \cap (\Omega_n \setminus \Delta_{n,\varepsilon})} \sum_{i \in B \cap \Sigma_n} s(i) = \frac{1}{n^{2n}} \sum_{s \in A \cap (\Omega_n \setminus \Delta_{n,\varepsilon})} \left| \sum_{i \in B \cap \Sigma_n} s(i) \right| \leq \frac{1}{n^{2n}} \sum_{s \in A \cap (\Omega_n \setminus \Delta_{n,\varepsilon})} \sum_{i \in B \cap \Sigma_n} \left| s(i) \right| \leq \frac{1}{n} \max \left\{ \left| \sum_{i \in B \cap \Sigma_n} s(i) \right| : s \in \Omega_n \setminus \Delta_{n,\varepsilon} \right\},
\]

where
so by (2):

\[
\left| \sum_{s \in A \cap (\Omega_n \setminus \Delta_n, \varepsilon)} \frac{s(i)}{n2^n} \right| < \frac{\varepsilon |B \cap \Sigma_n|}{n} \leq \frac{\varepsilon n}{n} = \varepsilon.
\]

Using (1), (3), and the fact that \(|B \cap \Sigma_n| \geq 2/\varepsilon^4\), we conclude that for every \(n \in I_1\) we have:

\[
|\mu_n([A] \times [B])| = \left| \sum_{s \in A \cap \Omega_n} \frac{s(i)}{n2^n} \right| \leq \left| \sum_{s \in A \cap \Delta_n, \varepsilon} \frac{s(i)}{n2^n} \right| + \left| \sum_{s \in A \cap (\Omega_n \setminus \Delta_n, \varepsilon)} \frac{s(i)}{n2^n} \right| < \frac{\sqrt{2}}{\varepsilon \sqrt{|B \cap \Sigma_n|}} + \varepsilon \leq \varepsilon + \varepsilon = 2\varepsilon.
\]

It follows that if for \(N_1\) we take any number from \(I_1\), e.g. we set \(N_1 = \min I_1\), then for every \(n \in I_1\), \(n \geq N_1\), we have:

\[
|\mu_n([A] \times [B])| < 2\varepsilon.
\]

We finish the proof by denoting \(N = \max (N_0, N_1)\) and seeing that for every \(n \geq N\) we obviously have the same inequality, i.e.:

\[
|\mu_n([A] \times [B])| < 2\varepsilon.
\]

Since \(\varepsilon \in (0, 1/12]\) is arbitrary, it holds that \(\lim_{n \to \infty} \mu_n([A] \times [B]) = 0\) and hence \(\langle \mu_n : n \in \mathbb{N}^+ \rangle\) is weak* convergent to 0.

**Remark 2.3.** Let us note that from the above proof we can deduce for every \(n \in \mathbb{N}^+\) the following estimations of the value \(|\mu_n([A] \times [B])|\), depending only on the size of the intersection \(B \cap \Sigma_n:\)

\[
|\mu_n([A] \times [B])| \leq \begin{cases} \frac{|B \cap \Sigma_n|}{n} \frac{\sqrt{2}}{\varepsilon \sqrt{|B \cap \Sigma_n|}} & \text{if } |B \cap \Sigma_n| < \frac{2}{\varepsilon^4}, \\ \frac{|B \cap \Sigma_n|}{n} \varepsilon \sqrt{|B \cap \Sigma_n|} & \text{if } |B \cap \Sigma_n| \geq \frac{2}{\varepsilon^4}. \end{cases}
\]

We are in the position to prove the main result of this paper, Theorem 1.2.

**Proof of Theorem 1.2.** First, notice that the space \(\mathbb{N}\) of all non-negative integers, endowed with the discrete topology, is homeomorphic to both \(\Omega\) and \(\Sigma\), so the Čech–Stone compactifications \(\beta\mathbb{N}\), \(\beta\Omega\), and \(\beta\Sigma\) are mutually homeomorphic. Consequently, by Proposition 2.2, \(\beta\mathbb{N} \times \beta\mathbb{N}\) admits a weak* convergent to 0 sequence \(\langle \nu_n : n \in \mathbb{N} \rangle\) of finitely supported normalized signed measures with pairwise disjoint supports, contained completely in \(\mathbb{N} \times \mathbb{N}\) (as measures \(\mu_n\)’s defined above on \(\beta\Omega \times \beta\Sigma\) have pairwise disjoint supports contained in \(\Omega \times \Sigma\)).

Let \(D\) and \(E\) be discrete countable subsets of \(K\) and \(L\), respectively. Let \(\varphi : \mathbb{N} \to D\) and \(\psi : \mathbb{N} \to E\) be bijections. By the Stone Extension Property of \(\beta\mathbb{N}\), there are continuous maps \(\Phi : \beta\mathbb{N} \to K\) and \(\Psi : \beta\mathbb{N} \to L\) such that \(\Phi \upharpoonright \mathbb{N} = \varphi\) and \(\Psi \upharpoonright \mathbb{N} = \psi\). For each \(n \in \mathbb{N}\) define a measure \(\rho_n\) on \(K \times L\) as follows:

\[
\rho_n = \sum_{(x,y) \in \text{supp}(\nu_n)} \nu_n([\{(x,y)\}]) \cdot \delta_{\varphi(x), \psi(y)},
\]

it follows that \(\|\rho_n\| = 1\) and \(\text{supp} (\rho_n)\) is finite. Since \(\langle \nu_n : n \in \mathbb{N} \rangle\) weak* converges to 0, for every \(f \in C(K \times L)\) we have:

\[
\lim_{n \to \infty} \int_{K \times L} f d\rho_n = \lim_{n \to \infty} \int_{\beta\mathbb{N} \times \beta\mathbb{N}} f(\Phi, \Psi) d\nu_n = 0,
\]
where \( f(\Phi, \Psi)(x, y) = f(\Phi(x), \Psi(y)) \in C(\beta\mathbb{N} \times \beta\mathbb{N}) \), so \( \langle \rho_n : n \in \mathbb{N} \rangle \) is also weak* convergent to 0.

Theorem 1.2 and the aforementioned characterization from II of those spaces \( C_P(X) \) which contain a complemented copy of the space \( (c_0)_p \) imply immediately Corollary 1.3. It appears that using this corollary and the Closed Graph Theorem one can easily get Cembranos’ and Freniche’s Theorem 1.1 as shown in the next proposition.

**Proposition 2.4.** Let \( X \) be a compact space such that \( C_P(X) \) contains a complemented closed linear subspace \( E \) isomorphic to \( (c_0)_p \). Then, \( E \) with the norm topology of \( C(X) \) is complemented in \( C(X) \) and isomorphic to the Banach space \( c_0 \).

**Proof.** Let \( F \) be a closed linear subspace of \( C_P(X) \) such that \( C_P(X) = E \oplus F \). Then, since the norm topology of \( C(X) \) is finer than the product topology of \( C_P(X) \), the spaces \( (E, \| \cdot \|) \) and \( (F, \| \cdot \|) \) (i.e. endowed with the inherited norm topology of \( C(X) \)) are still closed in \( C(X) \) and hence \( C(X) = E \oplus F \). It is enough now to show that \( (E, \| \cdot \|) \) is isomorphic to the Banach space \( c_0 \).

Since \( (E, \tau_p) \) (i.e. with the inherited product topology of \( C_P(X) \)) is isomorphic to \( (c_0)_p \), there is a topology \( \tau \) on \( E \) stronger than \( \tau_p \) and such that \( (E, \tau) \) is isomorphic to \( c_0 \). The identity operator \( T : (E, \| \cdot \|) \rightarrow (E, \tau) \) has closed graph, so it is continuous, and hence \( \tau \) is a Banach space topology on \( E \) smaller than the norm topology of \( E \). On the other hand, the identity operator \( S : (E, \tau) \rightarrow (E, \| \cdot \|) \) has closed graph, too, so it is also continuous, and hence the topology \( \tau \) on \( E \) is greater than the norm topology of \( E \). It follows that both topologies are equal, and hence \( (E, \| \cdot \|) \) is isomorphic to the Banach space \( c_0 \). \( \square \)

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