A Stochastic Capital-Labour Model with Logistic Growth Function

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Abstract. We propose and study a stochastic capital-labour model with logistic growth function. First, we show that the model has a unique positive global solution. Then, using the Lyapunov analysis method, we obtain conditions for the extinction of the total labour force. Furthermore, we also prove sufficient conditions for their persistence in mean. Finally, we illustrate our theoretical results through numerical simulations.

Keywords: capital-labour mathematical model; stochastic differential equations; Brownian motion; extinction and persistence.

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1 Introduction

Labour supply and demand are essential variables governing the labour market. They are influenced by demographic factors and the gross domestic product, which vary from household to household. In our context, the supply of labour is represented by the number of free jobs and the demand for labour, noting that the workforce or labour force is the total number of people eligible to work \cite{1,3}.

Motivated by the previous information, we propose to model the labour market by ordinary differential equations (ODEs) describing the different interactions between the essential components, that is, the free jobs and the labour force. The suggested model will take the following form:

\begin{align*}
\frac{du}{dt}(t) &= r u(t) \left( 1 - \frac{u(t)}{K} \right) - m u(t) v(t), \\
\frac{dv}{dt}(t) &= m u(t) v(t) - d v(t),
\end{align*}

\textsuperscript{*} This is a preprint whose final form is published by Springer Nature Switzerland AG in the book 'Dynamic Control and Optimization'.
where \( u \) denotes the number of free jobs and \( v \) represents the total unemployed labour force. The positive constant \( r \) is the natural per capita growth of free jobs and \( K \) is the theoretical eventual maximum of the number of free jobs (related to the theoretical maximum of investment capital). The positive parameter \( d \) is the disappearance rate of labour force and \( muv \) is the rate by which the labour force fills in the free jobs. We have adopted the bilinear form to pass from the labour force compartment to the free job one, while the recruitment of people depends progressively and proportionally to the considered employment policy.

It is well known that economies are subject to randomness in terms of natural perturbation processes [10]. Therefore, stochastic models are more suitable than deterministic ones, because they can take into account not only the mean trend but also the variance structure around it. Moreover, deterministic models will always produce the same results for fixed initial conditions, whereas the stochastic ones may give different predicted values. Thus, in order to take into account all the previous arguments, in this paper we propose the following stochastic capital-labour model with a logistic growth function:

\[
\begin{align*}
du(t) &= \left[ ru(t) \left( 1 - \frac{u(t)}{K} \right) - mu(t)v(t) \right] dt - \sigma u(t)v(t)dB, \\
dv(t) &= \left[ mu(t)v(t) - dv(t) \right] dt + \sigma u(t)v(t)dB,
\end{align*}
\]

where \( B(t) \) is a standard Brownian motion with intensity \( \sigma \), defined on a complete filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with the filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions [12]. The motivation to use the logistic growth can be found in [6]; the reader interested in the stochastic techniques is refereed to [7,11] and references therein.

Our work is organized as follows. First, in Section 2, we prove existence and uniqueness of a global positive solution to our stochastic model (2). Then, using the Lyapunov analysis method, we prove in Section 3 the extinction of the total labour force under an appropriate condition. Furthermore, in Section 4 we give sufficient conditions for the persistence in mean of the total labour force. Follows some numerical simulations to illustrate our analytical results (Section 5). We finish with Section 6 of conclusions.

All the equations and inequalities in the paper are understood in the almost surely (a.s.) sense.

2 Existence and uniqueness of global economic solutions

To investigate the dynamical behaviour of a population model, the first concern is whether the solution of the model is positive and global. In order to get a stochastic differential equation for which a unique global solution exists, i.e., there is no explosion within a finite time, for any initial value, standard assumptions for existence and uniqueness of solutions are the linear growth condition and the local Lipschitz condition (cf. Mao [8]). However, the coefficients of system (2) do not satisfy the linear growth condition as the incidence is non-linear.
Therefore, the solution of system (2) may explode at a finite time. In this section, using the Lyapunov analysis method [2,4], we show that the solution of system (2) is positive and global.

**Theorem 1.** For any given initial value \((u(0), v(0)) \in \mathbb{R}_+^2\), there exists a unique positive solution \((u(t), v(t)) \in \mathbb{R}_+^2\) of model (2) for all \(t \geq 0\) a.s. Moreover,

\[
\limsup_{t \to \infty} u(t) \leq \frac{rK}{\mu} \text{ a.s., } \quad \limsup_{t \to \infty} v(t) \leq \frac{rK}{\mu} \text{ a.s.,}
\]

where \(\mu = \min\{r, d\}\).

**Proof.** Since the drift and the diffusion of (2) are locally Lipschitz, then for any given initial value \((u(0), v(0)) \in \mathbb{R}_+^2\), there exists a unique local solution for \(t \in [0, \tau_e)\), where \(\tau_e\) is the explosion time. To show that this solution is global, we need to show that \(\tau_e = +\infty\). Define the stopping time \(\tau^+ = \inf\{t \in [0, \tau_e) : u(t) \leq 0 \text{ or } v(t) \leq 0\}\).

We suppose that \(\tau^+ < +\infty\). For any \(t \leq \tau^+\), we define the following function:

\[
F(t) := \ln(u(t)v(t)).
\]

By using Itô’s formula and system (2), we obtain that

\[
dF = r \left(1 - \frac{u}{K}\right) - mv + mu - d - \frac{\sigma^2}{2}(u^2 + v^2) + \sigma(u - v)dB \\
\geq -\frac{r}{K}u - mv - d - \frac{\sigma^2}{2}(u^2 + v^2) + \sigma(u - v)dB.
\]

Integrating both sides between 0 and \(t\), we get that

\[
F(t) \geq F(0) + \int_0^t H(s)ds + \sigma \int_0^t (u(s) - v(s))dB(s),
\]

where \(H(s) = -\frac{r}{K}u(s) - mv(s) - d - \frac{\sigma^2}{2}(u^2(s) + v^2(s))\). At least one among \(u(\tau^+)\) and \(v(\tau^+)\) is equal to 0. Then, we get

\[
\lim_{t \to \tau^+} F(t) = -\infty.
\]

Letting \(t \to \tau^+\) in (3) we obtain

\[
-\infty \geq F(t) \geq F(0) + \int_0^{\tau^+} H(s)ds + \sigma \int_0^{\tau^+} (u(s) - v(s))dB(s) > -\infty,
\]

which is a contradiction. Thereby, \(\tau^+ = +\infty\), which means that the model has a unique global solution \((u(t), v(t)) \in \mathbb{R}_+^2\) a.s. We now prove the boundedness. If we sum the equations from system (2), then

\[
dN(t) = \left(r u(t) \left(1 - \frac{u(t)}{K}\right) - dv(t)\right) dt,
\]
where $N(t) = u(t) + v(t)$. Thus,
\[
\frac{dN}{dt} = ru \left(1 - \frac{u}{K}\right) - dv
\]
\[
= ru - dv - \frac{r}{K}(u - K)^2 - 2ur + rK
\]
\[
= -ru - dv - \frac{r}{K}(u - K)^2 + rK
\]
\[
\frac{dN}{dt} \leq -\mu N + rk,
\]
where $\mu = \min\{r, d\}$, and so
\[
e^{\mu t} \frac{dN}{dt} \leq e^{\mu t} (-\mu N + rk),
\]
\[
\int_0^t e^{\mu s} \frac{dN}{ds} ds \leq \int_0^t e^{\mu s} (-\mu N(s) + rK) ds,
\]
\[
e^{\mu t} N(t) \leq \frac{rK}{\mu} (e^{\mu t} - 1) + N(0),
\]
\[
N(t) \leq \frac{rK}{\mu} (1 - e^{-\mu t}) + N(0) e^{-\mu t},
\]
\[
\limsup_{t \to \infty} N(t) \leq \frac{rK}{\mu} \text{ a.s.,}
\]
This fact implies that $\limsup_{t \to \infty} u(t) \leq \frac{rK}{\mu} \text{ a.s.}$ and $\limsup_{t \to \infty} v(t) \leq \frac{rK}{\mu} \text{ a.s.}$, which completes the proof. \(\square\)

3 Extinction of total labour force

When studying dynamical systems, it is important to discuss the possibility of extinction or persistence of a population. Here we investigate extinction of the capital-labour. The question of persistence will be addressed in Section 4.

**Theorem 2.** For any initial data $(u(0), v(0)) \in \mathbb{R}_+^2$, if $\frac{m^2}{2\sigma^2} - d < 0$, then one has $v(t) \to 0 \text{ a.s.}$ when $t \to +\infty$.

**Proof.** Let us define $G_1(t) := \log(v(t))$. Applying Itô’s formula to $G_1$ leads to
\[
dG_1(t) = \left(\frac{mu(t) - d - \frac{\sigma^2}{2} u^2(t)}{2} dt + \sigma u(t) dB(t),
\right.
\[
\frac{dG_1}{dt} = \left(-\frac{\sigma^2}{2} u(t) - \frac{m^2}{\sigma^2} \right) + \frac{m^2}{2\sigma^2} - d
\right) dt + \sigma u(t) dB(t),
\]
\[
\frac{dG_1}{dt} \leq \left(\frac{m^2}{2\sigma^2} - d\right) dt + \sigma u(t) dB(t).
\]
Integrating from 0 to \( t \) and dividing both sides by \( t \), we have
\[
\frac{\log(v(t))}{t} \leq \frac{\log(v_0)}{t} + \frac{1}{t} \int_0^t \left( \frac{m^2}{2\sigma^2} - d \right) ds + \frac{\sigma}{t} \int_0^t u(s) dB(s)
\]
\[
\leq \frac{\log(v_0)}{t} + \frac{1}{t} \int_0^t \left( \frac{m^2}{2\sigma^2} - d \right) ds + \frac{\sigma}{t} \int_0^t u(s) dB(s)
\]
\[
\leq \frac{\log(v_0)}{t} + \frac{m^2}{2\sigma^2} - d + \frac{\sigma}{t} \int_0^t u(s) dB(s).
\]

Let \( M_t := \int_0^t \sigma u(s) dB_s \). Then,
\[
\limsup_{t \to +\infty} \frac{M_t}{t} = \limsup_{t \to +\infty} \frac{\sigma}{t} \int_0^t u^2(s) ds \leq \sigma^2 \left( \frac{rK}{\mu} \right)^2 < +\infty
\]
and, by using the strong law of large numbers for martingales (see, e.g., [8]),
\[
\limsup_{t \to +\infty} \frac{M_t}{t} = 0.
\]

Therefore,
\[
\limsup_{t \to +\infty} \frac{\log(v(t))}{t} \leq \frac{\log(v_0)}{t} + \frac{m^2}{2\sigma^2} - d.
\]

Thus, if \( \frac{m^2}{2\sigma^2} - d < 0 \), then \( v(t) \to 0 \) when \( t \to +\infty \) a.s. \( \square \)

4 Persistence in the mean of total labour force

Now, we investigate the persistence property of \( v(t) \) in the mean, that is, we give conditions for which
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t v(s) ds > 0.
\]

For convenience, we introduce the following notation:
\[
\langle x(t) \rangle := \frac{1}{t} \int_0^t x(s) ds.
\]

**Theorem 3.** Let \((u(t), v(t))\) be a solution of system (2) with initial value \((u(0), v(0)) \in \mathbb{R}_+^2\).

If
\[
R_0^* := \frac{r}{d} - \frac{\sigma^2 K^2}{2d} > 1 \quad \text{and} \quad m > rK,
\]
then the variable \( v(t) \) satisfies the following expression:
\[
\liminf_{t \to \infty} \langle v \rangle \geq \frac{d(R_0^* - 1)}{m - d} > 0.
\]
Proof. Using the second equation of system (2), we have
\[ \frac{v(t) - v(0)}{t} = m\langle uv \rangle - d\langle v \rangle + \frac{\sigma}{\sqrt{t}} \int_0^t u(s)v(s)dB. \] (6)

Applying Itô’s formula on model (2) leads to
\[ d\ln(u(t)) = \left[ r \left( 1 - \frac{u}{K} \right) - m\langle u \rangle - \frac{1}{2} \sigma^2 v^2 \right] dt - \sigma v dB. \] (7)

and
\[ d\ln(v(t)) = \left[ m\langle u \rangle - d - \frac{1}{2} \sigma^2 u^2 \right] dt + \sigma u dB. \] (8)

Integrating both sides of (7) and (8) from 0 to \( t \), and dividing by \( t \), leads to
\[ \frac{\ln(u(t)) - \ln(u(0))}{t} = r - \frac{r}{K} \langle u \rangle - m\langle v \rangle - \frac{\sigma^2}{2} \langle v^2 \rangle - \frac{\sigma}{\sqrt{t}} \int_0^t v(s)dB \] (9)

and
\[ \frac{\ln(v(t)) - \ln(v(0))}{t} = m\langle u \rangle - d - \frac{\sigma^2}{2} \langle u^2 \rangle + \frac{\sigma}{\sqrt{t}} \int_0^t u(s)dB. \] (10)

Combining (6), (9), and (10), we derive that
\[ \frac{v(t) - v(0)}{t} + \frac{\ln(u(t)) - \ln(u(0))}{t} + \frac{\ln(v(t)) - \ln(v(0))}{t} \]
\[ = r - \frac{r}{K} \langle u \rangle + m\langle u \rangle + (m - \sigma^2)\langle uv \rangle - (d + m)\langle v \rangle - d \]
\[ - \frac{\sigma^2}{2} \langle (u + v)^2 \rangle + \frac{\sigma}{\sqrt{t}} \int_0^t \langle u(s) - v(s) + u(s)v(s) \rangle dB_s \]
\[ \geq r - d - \frac{\sigma^2}{2} K^2 - (d + m)\langle v \rangle + \left( m - \frac{r}{K} \right) \langle u \rangle \]
\[ + \frac{\sigma}{\sqrt{t}} \int_0^t \langle u(s) - v(s) + u(s)v(s) \rangle dB_s. \]

Since \( m - \frac{r}{K} > 0 \), then
\[ \frac{v(t) - v(0)}{t} + \frac{\ln(u(t)) - \ln(u(0))}{t} + \frac{\ln(v(t)) - \ln(v(0))}{t} \]
\[ \geq r - d - \frac{\sigma^2}{2} K^2 - (d + m)\langle v \rangle + \frac{\sigma}{\sqrt{t}} \int_0^t \langle u(s) - v(s) + u(s)v(s) \rangle dB_s. \]

Therefore,
\[ \langle d + m \rangle \langle v \rangle \geq r - d - \frac{\sigma^2}{2} K^2 + \frac{\sigma}{\sqrt{t}} \int_0^t \langle u(s) - v(s) + u(s)v(s) \rangle dB(s) \]
\[ - \frac{v(t) - v(0)}{t} - \frac{\ln(u(t)) - \ln(u(0))}{t} - \frac{\ln(v(t)) - \ln(v(0))}{t}. \]
Let us denote
\[ M_1(t) := \sigma \int_0^t (u(s) - v(s) + u(s)v(s)) dB(s). \]

Using the strong law of large numbers for martingales, together with the fact that almost surely for every \( \varepsilon \), there exists \( T \) such that \( 0 < u(t), v(t) < \frac{vK}{\mu} + \varepsilon \) for every \( t > T \), we can say that
\[
\lim_{t \to \infty} \frac{v(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{u(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{M_1(t)}{t} = 0 \quad a.s.
\]
Thus,
\[
\liminf_{t \to \infty} \langle v \rangle \geq \frac{r - d - \frac{\sigma^2 K^2}{2}}{(d + m)} = \frac{d(R_0^* - 1)}{m + d} > 0,
\]
where \( R_0^* = \frac{r}{d} - \frac{\sigma^2 K^2}{2d} \). The proof is complete. \( \square \)

5 Numerical simulations

In this section, we illustrate our mathematical results through numerical simulations. In the two examples considered, we apply the algorithm presented in [5] to solve system (2) and we use the parameter values from Table 1, inspired from [9].

Table 1: Parameter values used in the numerical simulations.

| Parameters | Fig. 1 | Fig. 2 |
|------------|--------|--------|
| \( r \)    | 1      | 1      |
| \( d \)    | 0.2    | 0.2    |
| \( m \)    | 0.001  | 0.1    |
| \( K \)    | 100    | 100    |
| \( \sigma \)| 0.09  | 0.001  |

Figure 1 shows the evolution of the free jobs and the total labour force during the period of observation. It can be seen that both curves of the total labour force, corresponding to the deterministic and to the stochastic models, converge toward zero. This indicates the extinction of the total labour force, which is consistent with our theoretical results. Indeed, for the used parameters (see Table 1), one has \( \frac{m^2}{2\sigma^2} - d = -0.19 < 0 \) and it follows, from Theorem 2, that \( v(t) \to 0 \) with probability one when \( t \to +\infty \).

The evolution of the free jobs and the total labour force, for both deterministic and stochastic models, is also illustrated in Fig. 2. In this case, the key
conditions (4) of our Theorem 3 are satisfied: \( R_0^\mu = \frac{r}{d} - \frac{\sigma^2 K^2}{2d} = 4.99 > 1 \) and \( m - \frac{r}{K} = 0.09 > 0 \). As predicted by Theorem 3, one can clearly see in Fig. 2 the persistence of the total labour force.

6 Conclusions

The labour force (workforce), which can be defined as the total number of people who are eligible to work, is a centred component of each modern economy, whereas free jobs are systematically supplied by companies. In this work, we have proposed and analysed a capital-labour model by means of an economic dynamical system describing the interaction between free jobs and labour force. Mathematically, our model is governed by stochastic differential equations, where the component of stochastic noise is considered for an additional degree of realism, intended to describe well reality. Furthermore, the transmission rate by which the labour force individuals are moving to the free jobs compartment is modelled by the logistic growth function with an appropriate carrying capacity \( K \). Some relevant results were obtained. First of all, by proving existence and uniqueness of a global positive solution, as well as its boundedness, we have shown that the proposed model is mathematically and economically well-posed.
Moreover, a sufficient condition for the extinction of labour force is obtained, via the strong law of large numbers for martingales, in addition to adequate sufficient conditions for the persistence in mean. In order to illustrate our theoretical results, we have implemented some numerical simulations where, for a good accuracy of the approximate numerical solutions, the Milstein scheme has been used.

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