Abstract—A general multi-terminal source code and a general multi-terminal channel code are presented. Constrained-random-number generators with sparse matrices, which are building blocks for the code construction, are used in the construction of both encoders and decoders. Achievable regions for source coding and channel coding are derived in terms of entropy functions, where the capacity region for channel coding provides an alternative to the region of [Somekh-Baruch and Verdú, ISIT2006].

I. INTRODUCTION
In this paper, we consider the problems of multi-terminal source coding (Fig. 1) and channel coding (Fig. 2). First, we construct a code for correlated sources and derive an achievable region. Our setting extends separate coding for correlated sources [8] [13] [15]. We derive multi-letter characterized capacity regions for these problems by showing that they are achievable with the constructed codes. Our capacity region for the channel coding is specified in terms of entropy functions and provides an alternative to the region derived in [28]. It should be noted that, when auxiliary random variables are assumed to be independent and identically distributed (i.i.d.), the channel coding is specified in terms of entropy functions, where the capacity region for channel coding provides an alternative to the region derived in [28].

Throughout this paper, we use the information spectrum method introduced in [11], and we do not assume such conditions as consistency, stationarity and ergodicity. Let \( P(\cdot) \) denote the probability of an event. For a sequence \( \{\mu_{n}\}_{n=1}^{\infty} \) of probability distributions corresponding to \( U \equiv \{U_{n}\}_{n=1}^{\infty} \), \( \mu(U) \) denotes the spectral inf-entropy rate. For a sequence \( \{\mu_{n}V_{n}\}_{n=1}^{\infty} \) of joint probability distributions corresponding to \( (U,V) \equiv \{(U_{n},V_{n})\}_{n=1}^{\infty} \), \( \mu(U|V) \) denotes the spectral conditional sup-entropy rate. Formal definitions are given in Appendix A.

We define \( \chi(S) \equiv 1 \) if \( S \) is true. Otherwise, we define \( \chi(S) \equiv 0 \). The set \( U \setminus V \) denotes the set difference and \( U^{c} \equiv U \setminus U' \). For a set \( f_{s} \equiv \{f_{s}\}_{s \in S} \) of functions and a set \( c_{s} \equiv \{c_{s}\}_{s \in S} \) of vectors, let \( \epsilon_{f_{s}}(c_{s}) \equiv \{z_{s} : f_{s}(z_{s}) = c_{s} \} \) for all \( s \in S \), where \( z_{s} \equiv \{z_{s}\}_{s \in S} \).

II. CONSTRUCTION OF SOURCE CODE
We introduce the single-hop multi-terminal source coding problem illustrated in Fig. 1. This code is used to construct a channel code.

For an index set \( S \) of messages and an index set \( J \) of decoders, let \( (Z_{S},Y_{J}) \equiv \{\{Z_{n}\}_{s \in S},\{Y_{j}\}_{j \in J}\}_{n=1}^{\infty} \) be a general correlated source, which is characterized by joint distributions \( \mu_{Z_{S}Y_{J}} \) of \( Z_{S,n} \equiv \{Z_{n}\}_{s \in S} \) and \( Y_{J,n} \equiv \{Y_{j}\}_{j \in J} \). For each \( s \in S \) and \( n \in \mathbb{N} \), let \( Z_{n}^{s} \) be the alphabet of a message \( Z_{n}^{s} \). For each \( j \in J \) and \( n \in \mathbb{N} \), let \( Y_{j}^{n} \) be the alphabet of side information \( Y_{j}^{n} \) available for the \( j \)-th decoder. It should be noted that we assume that \( Z_{n}^{s} \) is a finite set but \( Y_{j}^{n} \) is allowed to be an arbitrary (infinite, continuous) set.

For each \( s \in S \) and \( n \in \mathbb{N} \), let \( F_{s,n} : Z_{n}^{s} \to C_{s,n} \) be the \( s \)-th (possibly stochastic) encoding function, where \( C_{s,n} \) is the set of all codewords. Let \( C_{s,n} \equiv F_{s,n}(Z_{n}^{s}) \) be the codeword of the \( s \)-th encoder. For each \( j \in J \), let \( D_{j} \) be the index set of codewords available for the \( j \)-th decoder, which is also
the index set of messages reproduced by the decoder, where \( \mathcal{D}_j \subset S \). Let \( C_{D_{j,n}} \equiv \{ C_{s,n} \}_{s \in \mathcal{D}_j} \) be the set of codewords available for the \( j \)-th decoder, \( \Psi_{j,s,n} : [X_{s \in \mathcal{D}_j}, C_{s,n}] \times Y_{D_{j,n}} \rightarrow X_{s \in \mathcal{D}_j}, Z^n \) be the \( j \)-th (possibly stochastic) decoding function, and \( \tilde{Z}^n_{D_{j,n}} \equiv \Psi_{j,s,n}(C_{D_{j,n}}, n) \) be the reproduction of messages by the \( j \)-th decoder. For each \( j \in J \) and \( s \in \mathcal{D}_j \), let \( \tilde{Z}^n_{j,s} \) be the reproduction of the \( s \)-th message by the \( j \)-th decoder. We expect \( \tilde{Z}^n_{j,s} = Z^n_s \) for all \( j \in J \) and \( s \in \mathcal{D}_j \) with a small error probability by letting \( n \) be sufficiently large. Let \( \mathcal{Z}^n_{j,s,n} \equiv \{ \tilde{Z}^n_{j,s} \}_{s \in \mathcal{D}_j, s \in \mathcal{D}_n} \) be the random variable of all reproductions.

We call a rate vector \( \{ r_s \}_{s \in S} \) achievable if there is a (possibly stochastic) code \( \{ (F_{s,n}, \Psi_{j,s,n}) \}_{s \in S, j \in J} \) with a small error probability for all \( (j, \mathcal{D}_j) \) satisfying \( j \in J \) and \( \emptyset \not\subset \mathcal{D}_j \subset \mathcal{D}_J \). Then we have the following theorem, which is a generalization of (5).

**Theorem 1:** Let \( \mathcal{Z}_{s,n} \) be a set of random variables. Then we have the following theorem, which is a generalization of (5).

The converse part \( R_{c} \subset R_{c} \) is shown in Appendix B. To prove the achievability part \( R_{a} \subset R_{a} \), we construct a code by assuming that \( s \in \mathcal{Z}_{c} \). For each \( s \in S \), a source \( Z^n_s \) is encoded by using a deterministic function \( f_s : Z^n_s \rightarrow C_{s,n} \), where we omit the dependence of \( f_s \) on \( n \) and \( r_s = \log(\|C_{s,n}\|)/n \) represents the encoding rate of the \( s \)-th message. We can use a sparse matrix as a function \( f_s \) by assuming that \( Z^n_s \) is an \( n \)-dimensional linear space on a finite field.

Let \( c_s \in C_{s,n} \) be the \( s \)-th codeword. For each \( j \in J \), the decoder generates \( \tilde{Z}^n_{j,s} \) by using a constrained-random-number generator with a distribution given as

\[
\mu_{Z^n_{j,s}, Y^n_j} \left( \tilde{Z}^n_{j,s}, (C_{j,n}, y_j) \right) = \frac{\mu_{Z^n_{j,s}, Y^n_j} (\tilde{Z}^n_{j,s}, y_j)}{\mu_{Z^n_{j,s}, Y^n_j} (C_{j,n}, (C_{j,n}, y_j))} \tag{3}
\]

for a given codeword \( C_{j,n} \equiv \{ c_s \}_{s \in \mathcal{D}_j} \) and side information \( y_j \in Y^n_j \). where \( f_{j,n}(\tilde{Z}^n_{j,s}) \equiv \{ f_s(\tilde{Z}^n_{j,s}) \}_{s \in \mathcal{D}_j} \). It should be noted that the constrained-random-number generator is sufficient to achieve the fundamental limit. When sources are i.i.d., tractable approximation algorithms for a constrained-random-number generator summarized in [24] are available. While the maximum a posteriori probability decoder is optimal, it may not be tractable.

Let \( \text{Error}(s) \) be the decoding error probability of a set of functions \( f_s \equiv \{ f_s \}_{s \in S} \). Then we have the following theorem, which concludes the achievability part \( R_{a} \supset R_{c} \). The proof is given in Appendix B.

**Theorem 2:** Let \( (Z^n_{s,D}, Y^n_j) \) be a pair of general correlated sources. Let us assume that \( \{ r_s \}_{s \in S} \) satisfies

\[
\sum_{s \in \mathcal{D}_j} r_s \geq \mathcal{W} (Z^n_{D}, Y^n_j, Z^n_{D'}) \tag{4}
\]

for every \( (j, \mathcal{D}_j') \) satisfying \( j \in J \) and \( \emptyset \not\subset \mathcal{D}_j' \subset \mathcal{D}_j \). Then there is a set of functions (sparse matrices) \( f_s \) such that \( \text{Error}(s) \leq \delta \) for all \( \delta > 0 \) and all sufficiently large \( n \).

### III. CONSTRUCTION OF CHANNEL CODE

#### A. General Formulas for Capacity Region

Let \( S \) be the index set of multiple messages, \( I \) be the index set of channel inputs, and \( J \) be the index set of channel outputs. A general channel is characterized by a sequence \( \{ \mu_{y|z} \}_{z \in S} \) of conditional distributions, where \( Z^n_j \equiv \{ Z^n_j \}_{j \in J} \) is a set of random variables of multiple channel inputs, and \( Y^n_j \equiv \{ Y^n_j \}_{j \in J} \) is a set of random variables of multiple channel outputs. For each \( i \in I \) and \( n \in \mathbb{N} \), let \( X^n_i \) be the alphabet of random variable \( X^n_i \).

For each \( j \in J \) and \( n \in \mathbb{N} \), let \( Y^n_j \) be the alphabet of random variable \( Y^n_j \).

For each \( s \in S \) and \( n \in \mathbb{N} \), let \( M_{s,n} \) be a random variable of the \( s \)-th message subject to the uniform distribution on an alphabet \( M_{s,n} \). We assume that \( \{ M_{s,n} \}_{s \in S} \) are mutually independent. For each \( i \in I \), let \( S_i \) be the index set of sources available for the \( i \)-th encoder, where \( S_i \subset S \). The \( i \)-th encoder generates the channel input \( X^n_i \) from the set of messages \( M_{S_i,n} \equiv \{ M_{i,n} \}_{s \in S_i} \). For each \( j \in J \), let \( D_j \) be the index set of messages reproduced by the \( j \)-th decoder, where \( D_j \subset S \). The \( j \)-th decoder receives the channel output \( Y^n_j \) and reproduces a set of messages \( \tilde{M}_{D_j,n} \equiv \{ \tilde{M}_{j,n} \}_{s \in D_j} \), where \( \tilde{M}_{j,n} \) is the \( s \)-th message reproduced by the \( j \)-th decoder. We expect \( \tilde{M}_{j,n} = M_{s,n} \) with a small error probability for all \( j \in J \) and \( s \in D_j \) by letting \( n \) be sufficiently large.

We call a rate vector \( \{ R_s \}_{s \in S} \) achievable if there is a (possibly stochastic) code \( \{ (\Phi_{s,n}, \Psi_{j,s,n}) \}_{s \in S, j \in J} \) consisting of encoders \( \Phi_{s,n} : X_{s \in \mathcal{D}_j}, M_{s,n} \rightarrow X^n_j \) and decoders \( \Psi_{j,s,n} : Y^n_j \rightarrow X_{s \in \mathcal{D}_j}, M_{s,n} \) such that

\[
\liminf_{n \to \infty} \frac{\log |M_{s,n}|}{n} \geq R_s \quad \text{for all } s \in S \tag{5}
\]

\[
\lim_{n \to \infty} P \left( \tilde{M}_{j,s,n} \neq M_{s,n} \text{ for some } j \in J \text{ and } s \in D_j \right) = 0 \tag{6}
\]
where $X_i^n \equiv \Phi_{i,n}(S_i)$, $\tilde{M}_{D,j,n} \equiv \Psi_{j,n}(Y^n_j)$, and the joint distribution of $(M_{S,n}, X^n_2, Y^n_2, \tilde{M}_{D,j,n})$ is given as

$$H_{M_{S,n}, X^n_2, Y^n_2, \tilde{M}_{D,j,n}}(m_{S,n}, x_2, y_2, \tilde{m}_{D,j}) = \prod_{j \in J} \mu_{\tilde{M}_{D,j,n}}(\tilde{m}_{D,j}|y_2) \cdot \prod_{j \in J} \mu_{\tilde{M}_{D,j,n}}(x_2|m_{S,n}) \cdot \prod_{s \in S} \left[1 - M_{S,n}(m_{S,n}) \right]$$

by letting $\tilde{M}_{D,j,n} \equiv \{\tilde{m}_{D,j}\}_{j \in J, s \in D_j}$. Let $R_{\text{OP}}$ be the set of all achievable rate vectors.

Let $R_{\text{IT}}$ be defined as the set of all $\{R_s\}_{s \in S}$ satisfying the condition that there are random variables $\{Z_s\}_{s \in S}$ and positive numbers $\{r_s\}_{s \in S}$ such that

$$R_s \geq 0 \quad \text{for all } s \in S$$

$$\sum_{s \in D_j} r_s \geq \mathcal{H}(Z_{D_j}|Y_j, Z_{D_j}) \quad \text{for all } j \in J$$

$$R_s + r_s \leq H(Z_s) \quad \text{for all } s \in S$$

It should be noted that we can eliminate $\{r_s\}_{s \in S}$ from the above conditions by employing the Fourier-Motzkin method [27 Appendix D].

We show the following theorem, which is a generalization of the result presented in [18].

**Theorem 3:** $R_{\text{OP}} \supseteq R_{\text{IT}}$.

**Remark 1:** The capacity region of this type of channel is derived in [28] as the set of all $\{R_s\}_{s \in S}$ that satisfy

$$0 \leq R_s \leq \min_{x_2 \in X_2} I(Z_s; Y_j) \quad \text{for all } s \in S,$$

where $I(U; V)$ denotes the spectral inf-mutual information rate. It should be noted that Theorem 3 provides an alternative capacity region to that derived in [28].

The converse part $R_{\text{OP}} \supseteq R_{\text{IT}}$ will be shown in Appendix C. To prove the achievability part $R_{\text{OP}} \supseteq R_{\text{IT}}$, we first consider a special case where $S$ is the disjoint union of $\{S_i\}_{i \in I}$. We have the following theorem, where the code construction is given in Section IIB.

**Theorem 4:** Assume that $S$ is the disjoint union of $\{S_i\}_{i \in I}$, that is, $S = \bigcup_{i \in I} S_i$ and $S_i \cap S_j = \emptyset$ for all $i \neq j$. Let $R_{\text{IT}}$ be defined as the set of all $\{R_s\}_{s \in S}$ satisfying the condition that there are the random variables $\{Z_s\}_{s \in S}$ and positive numbers $\{r_s\}_{s \in S}$ satisfying (7), (8), and

$$\sum_{s \in S} [R_s + r_s] \leq H(Z_s) \quad \text{for all } s \in S$$

for all $(s, i, S'_i, j, D'_j)$ satisfying $s \in S_i$, $i \in I$, $j \in J$, and $\emptyset \neq D'_j \subset D_j$.

**Remark 2:** Theorem 3 provides an alternative rate. It should be noted that Theorem 3 provides an alternative rate. Let $R_{\text{OP}}$ be the set of all achievable rate vectors.

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$$R_s \geq 0 \quad \text{for all } s \in S$$

$$\sum_{s \in D_j} r_s \geq \mathcal{H}(Z_{D_j}|Y_j, Z_{D_j}) \quad \text{for all } j \in J$$

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$$\sum_{s \in S} [R_s + r_s] \leq H(Z_s) \quad \text{for all } s \in S$$

for all $(s, i, S'_i, j, D'_j)$ satisfying $s \in S_i$, $i \in I$, $j \in J$, and $\emptyset \neq D'_j \subset D_j$.

**Remark 2:** Theorem 3 provides an alternative rate.
Here, we define a constrained-random-number generator, which is used by the i-th encoder. Let $Z_i^*$ be a random variable corresponding to a distribution:

$$\mu_{Z_i^*|c_{s,i,n} M_{s,i,n}}(z_{s,i}|c_{s,i}, m_{s,i}) = \frac{\mu_{Z_{i,n}^*}(s_i(ze_{s,i}) = c_{s,i}, g_{s,i}(z_{s,i}) = m_{s,i})}{\mu_{Z_{i,n}^*}(c_{s,i}, j s_{i} g_{s,i}(m_{s,i}))}.$$ 

We define the i-th encoder $\Phi_{i,n} : M_{s,i,n} \rightarrow X_i^n$ and the j-th decoder $\Psi_{j,n} : Y_j^n \rightarrow M_{D,j,n}$ as:

$$\Phi_{i,n}(m_{s,i}) \equiv W_i(Z_{i}^*) \quad \Psi_{j,n}(y_j) \equiv g_j(\tilde{Z}_j^n) \in D_j,$$

where the decoder claims an error when $c_{s,i} \cap e_{s,i} (m_{s,i}) = \emptyset$, $W_i$ is the channel subject to the conditional probability distribution $\mu_{X_i|Z_i^*}$, and $\tilde{Z}_j^n$ is the projection of $\tilde{Z}_j^n$ on $M_{D,j,n}$. The flow of vectors is illustrated in Fig. 5 in Appendix C.

Let $M_{D,j,n} \equiv \Psi_{j}(Y_j^n)$ and $\text{Error}(f_S, g_S, c_S)$ be the error probability. We have the following lemma, where the proof is given in Appendix C.

Theorem 6: Let us assume that $S$ is the disjoint union of $\{S_i\}_{i \in I}$, that is, $S = \bigcup_{i \in I} S_i$ and $S_i \cap S_j = \emptyset$ for all $i \neq j$.

Let us assume that $(r_s, R_s)$ $s \in S$ satisfies

$$\sum_{s \in S} [R_s + r_s] < H(Z_S) \quad \text{(14)}$$

for all $(i, S'_i)$ satisfying $i \in I$ and $0 \neq S'_i \subset S_i$. Then for any $\delta > 0$ and all sufficiently large $n$ there are functions (sparse matrices) $g_S$ and vectors $c_S$ such that $\text{Error}(f_S, g_S, c_S) \leq \text{Error}(f_S) + \delta$.

Immediately from Theorems 2 and 6 we have Theorem 4.

IV. APPLICATION TO MULTIPLE ACCESS CHANNEL

In this section, we apply code construction to a multiple access channel, where $S$ $\equiv \{0, 1, 2\}$, $I$ $\equiv \{1, 2\}$, $S_i \equiv \{0, i\}$ for each $i \in I$, $J$ $\equiv \{0\}$, and $D_0$ $\equiv \{1, 2\}$. This setting corresponds to the situation that the first and the second encoder have access to the i-th message. In the following, we denote $D \equiv D_0$, $Y \equiv Y_0$, $Y^n \equiv Y_0^n$, and $y \equiv y_0$.

From Theorem 5 we have an achievable region as the set of all $(R_s)_{s \in S}$ such that there are $Z_S$ and positive numbers $(r_s)_{s \in S}$ satisfying

$$R_s \geq 0$$

$$\sum_{s \in S'} r_s \geq H(Z_S)$$

$$R_s + r_s \leq H(Z_s)$$

for all $(s, S')$ satisfying $s \in S$ and $0 \neq S' \subset D$, where the joint distribution of $(Z_S, X_{S'}, Y_S)$ is given by (10).

Here, let us assume that a multiple access channel is i.i.d. given by a conditional distribution $\mu_{Y|X_S, X_{S'}}$. In addition, let us assume that the $(Z_S, Z_0, Z_1, Z_2, X_{S'}, X_2)$ are i.i.d. sources given by the distributions $\mu_{Z_S} = \mu_{Z_0} = \mu_{Z_1} = \mu_{Z_2} = 0$ and $\mu_{X_{S'}} = 0$. By employing the Fourier-Motzkin method [2, Appendix D] to eliminate $(r_s)_{s \in S}$, we have following equivalent conditions for $(R_0, R_1, R_2)$ as

$$R_0 \leq \mu_{Y|X_S} = 0$$

$$R_1 \leq \mu_{Y|Z_0} + \mu_{Y|Z_1} + \mu_{Y|Z_2} = 0$$

$$R_0 + R_2 \leq \mu_{Y|Z_0} + \mu_{Y|Z_1} + \mu_{Y|Z_2} = 0$$

$$R_0 + R_2 \leq \mu_{Y|Z_0} + \mu_{Y|Z_1} + \mu_{Y|Z_2} = 0$$
By making the convex closure after the union over all i.i.d.
distributions \( \mu_{X_1,Z_0}, \mu_{Z_0}, \mu_{X_1|Z_0,Z_1}, \) and
\( \mu_{X_2|Z_0}, \) we have the region equivalent to that derived in
[9]. It is shown in [9] that this region is equivalent to the region derived in
[17] specified by the conditions
\[
0 \leq R_0 \leq I(Z_1;Y_1|Z_0) + I(Z_2;Y_2|Z_0) - I(Z_1;Z_2|Z_0) + \min\{I(Z_0;Y_1), I(Z_0;Y_2)\},
\]
where the equivalence comes from the result presented in [17].
It should be noted that this region is also equivalent to the Gel’fand-Pinsker region
[8] specified by inequalities [16], [17], and
\[
0 \leq R_0 + R_1 + R_2 \leq I(Z_1;Y_1|Z_0) + I(Z_2;Y_2|Z_0) - I(Z_1;Z_2|Z_0) + \min\{I(Z_0;Y_1), I(Z_0;Y_2)\},
\]
and the Liang-Kramer-Poor region [14] specified by inequalities
[15], [16], and [18]–[20].

**APPENDIX**

**A. Entropy and Mutual Information for General Sources**

First, we review the definition of the limit super-
ior/inferior probability introduced in [11]. For a sequence
\( \{ U_n \}_{n=1}^{\infty} \) of random variables, the limit superior in probability
p-limsup
\( n \to \infty \) and the limit inferior in probability
p-liminf
\( n \to \infty \) are defined as
\[
\begin{align*}
p\text{-limsup } U_n &= \inf \left\{ \theta : \lim_{n \to \infty} P(U_n > \theta) = 0 \right\} \\
p\text{-liminf } U_n &= \sup \left\{ \theta : \lim_{n \to \infty} P(U_n < \theta) = 0 \right\}.
\end{align*}
\]
We have the following relations [11] Section 1.3:
\[
\begin{align*}
p\text{-limsup } [U_n + V_n] &\leq p\text{-limsup } U_n + p\text{-limsup } V_n \quad (21) \\
p\text{-liminf } [U_n + V_n] &\geq p\text{-liminf } U_n + p\text{-liminf } V_n \quad (22) \\
p\text{-limsup } [U_n + V_n] &\leq p\text{-limsup } U_n + p\text{-liminf } V_n \quad (23) \\
p\text{-liminf } [U_n + V_n] &\geq p\text{-liminf } U_n + p\text{-liminf } V_n \quad (24) \\
p\text{-limsup } [-U_n] &= p\text{-liminf } U_n. \quad (25)
\end{align*}
\]
For a sequence \( \{ \mu_n \}_{n=1}^{\infty} \) of probability distributions corresponding to
\( U \), we define the spectral inf-entropy rate \( H(U) \) as
\[
H(U) = \liminf_{n \to \infty} \frac{1}{n} \log_2 \frac{1}{\mu_n(U^n)}.
\]
For a general sequence \( \{ U_n|V_n \}_{n=1}^{\infty} \) of joint probability distributions corresponding to
\( (U,V) = \{(U^n,V^n)\}_{n=1}^{\infty} \), we define the spectral conditional sup-entropy rate \( \overline{H}(U|V) \), the
spectral conditional inf-entropy rate \( \underline{H}(U|V) \), and the spectral
inf-information rate \( I(U;V) \) as
\[
\begin{align*}
\overline{H}(U|V) &= p\text{-limsup } \frac{1}{n} \log_2 \frac{1}{\mu_{U^n|V^n}(U^n|V^n)} \\
\underline{H}(U|V) &= p\text{-liminf } \frac{1}{n} \log_2 \frac{1}{\mu_{U^n|V^n}(U^n|V^n)} \\
I(U;V) &= p\text{-liminf } \frac{1}{n} \log_2 \frac{1}{\mu_{U^n|V^n}(U^n|V^n)}
\end{align*}
\]
In the following, we introduce some inequalities that we use in the
proof of the converse part. Trivially, we have
\[
\bar{H}(U|V) \geq H(U|V) \geq 0.
\]
From [11] Lemma 3.2.1, Definition 4.1.3, we have
\[ p\text{-liminf}_{n \to \infty} \frac{1}{n} \log_2 \frac{\mu_{U_n}(U_n)}{\mu_{V_n}(U_n)} \geq 0, \]
which implies that
\[ I(U; V) = p\text{-liminf}_{n \to \infty} \frac{1}{n} \log_2 \frac{\mu_{U_n|V_n}(U_n|V_n)}{\mu_{V_n}(U_n)} \]
\[ = p\text{-liminf}_{n \to \infty} \frac{1}{n} \log_2 \frac{\mu_{U_n|V_n}(U_n, V_n)}{\mu_{V_n}(U_n)\mu_{V_n}(V_n)} \]
\[ \geq 0. \]

We show the following lemmas.

**Lemma 1:** Let \( U_n \) be the alphabet of \( U^n \). Then
\[ \overline{H}(U) \leq \limsup_{n \to \infty} \frac{\log_2 |U_n|}{n}. \]

**Proof:** We have
\[ \limsup_{n \to \infty} \frac{\log_2 |U_n|}{n} - \overline{H}(U) \]
\[ = p\text{-limsup}_{n \to \infty} \frac{1}{n} \log_2 \frac{\mu_{U_n}(U_n)}{\mu_{U_n}(U_n)} \]
\[ = p\text{-limsup}_{n \to \infty} \frac{1}{n} \log_2 \frac{1}{\mu_{V_n}(U_n)} \]
\[ \geq \frac{1}{n} \log_2 \frac{1}{\mu_{U_n}(U_n)} \]
\[ \geq 0, \]
where the second equality comes from (25), the first inequality comes from (24), and the second inequality comes from (26). \[ \square \]

**Lemma 2:** \( \overline{H}(U|V) \geq \overline{H}(U|V, W) \).

**Proof:** We have
\[ \overline{H}(U|V) = \overline{H}(U|V, W) \]
\[ = \text{p-limsup}_{n \to \infty} \frac{1}{n} \log_2 \frac{1}{\mu_{U_n|V}(U_n|V_n)} \]
\[ = \text{p-limsup}_{n \to \infty} \frac{1}{n} \log_2 \frac{1}{\mu_{U_n|V, W}(U_n|V_n, W_n)} \]
\[ = \text{p-limsup}_{n \to \infty} \frac{1}{n} \log_2 \frac{\mu_{U_n|V, W}(U_n|V_n, W_n)}{\mu_{U_n|V}(U_n|V_n)} \]
\[ + \text{p-limsup}_{n \to \infty} \frac{1}{n} \log_2 \frac{\mu_{U_n|V, W}(U_n|V_n, W_n)}{\mu_{U_n|V}(U_n|V_n)} \]
\[ \geq \frac{1}{n} \log_2 \frac{\mu_{U_n|V, W}(U_n|V_n, W_n)}{\mu_{U_n|V}(U_n|V_n)} \]
\[ \geq 0, \]
where the second equality comes from (25), the first inequality comes from (24), and the second inequality comes from (26). \[ \square \]

**B. Proof of \( R_{\text{source}}^{\text{DP}} \subset R_{\text{source}}^{\text{IT}} \)**

We use the following lemma, which is analogous to the Fano inequality.

**Lemma 3** ([18] Lemma 7): Let \( (U, V) \equiv \{(U_n, V_n)\}_{n=1}^{\infty} \) be a sequence of two random variables. If there is a sequence \( \{\Psi_n\}_{n=1}^{\infty} \) of (possibly stochastic) functions independent of \( (U, V) \) satisfying the condition
\[ \lim_{n \to \infty} P(\Psi_n(V_n) \neq U_n) = 0, \]
then
\[ \overline{I}(U|V) = 0. \]

**Proof:** When \( \{\Psi_n\}_{n=1}^{\infty} \) is a sequence of deterministic functions, the lemma is the same as [18, Lemma 7]. When \( \{\Psi_n\}_{n=1}^{\infty} \) is a sequence of stochastic functions, we can obtain a sequence \( \{\psi_n\}_{n=1}^{\infty} \) of deterministic functions such that
\[ P(\psi_n(V_n) \neq U_n) \leq \sum_{\psi \in \Psi} P(\psi = \psi_n)P(\psi_n(V_n) \neq U_n) \]
\[ = P(\Psi_n(V_n) \neq U_n) \]
for all \( n \) from the random coding argument and the fact that \( \Psi_n \) is independent of \( (U_n, V_n) \). Then we have the lemma by using [18, Lemma 7]. \[ \square \]

In the following, we show \( R_{\text{source}}^{\text{DP}} \subset R_{\text{source}}^{\text{IT}} \) by using the above lemma.

Assume that \( \{r_s\}_{s \in S} \in R_{\text{source}}^{\text{DP}} \). Then there is a code \( \{(F_s, n)\}_{s \in S}, \{\{\psi_{j, n}\}_{j \in J}\}_{n=1}^{\infty} \) satisfying (3) and (3).

For \( j \in J \) and \( D_j' \subset D_j \), let \( \psi_{j, n}(C_{D_j, n}) \) be the projection of \( \psi_{j, n}(C_{D_j, n}) \) on \( \times_{s \in D_j'} Z_s^n \). Then we have
\[ \lim_{n \to \infty} P(\psi_{j, n}(C_{D_j, n}) \neq Z_{D_j'}^n) = 0 \]
from (2). From Lemma 3 we have
\[ \overline{H}(Z_{D_j'}|C_{D_j'}, C_{D_j}, Y_j, Z_{D_j''}) \leq \overline{H}(Z_{D_j'}|C_{D_j}, Y_j) = 0. \]

From (30) and \( \overline{H}(Z_{D_j'}|C_{D_j'}, Y_j, Z_{D_j''}) \geq 0 \), we have
\[ \overline{H}(Z_{D_j'}|C_{D_j'}, Y_j, Z_{D_j''}) = 0 \]
for any \( (j, D_j') \) satisfying \( j \in J \) and \( \emptyset \neq D_j' \subset D_j \).

Next, we show the relation (32), which appears on the top of the next page, where the second equality comes from the fact that \( C_{D_j, n} \leftrightarrow Z_{D_j}^n \leftrightarrow (C_{D_j', n}, Y_j, Z_{D_j''}) \) and \( C_{D_j', n} \leftrightarrow Z_{D_j''} \leftrightarrow (C_{D_j', n}, Y_j, Z_{D_j''}) \) form Markov chains, and inequalities comes from (21), (22), and (24).

Finally, we have
\[ \sum_{s \in D_j'} r_s \]
\[ \geq \sum_{s \in D_j'} \limsup_{n \to \infty} \frac{\log_2 |C_{s, n}|}{n} \]
\[ \geq \limsup_{n \to \infty} \frac{\log_2 |C_{s, n}|}{n} \]
\[ \geq \overline{H}(C_{D_j'}) \]
second inequality comes from the fact that \( R_j \geq \log Z_{D_j} \) and \( \mu_{C_j,n}(Y_j^n, Z_{D_j}^n) \). Assume that \( i, n \in \Phi_j \). Then it is clear that \( \log Z_{D_j} \) satisfies (33) and (38), which we use the fact that \( \log Z_{D_j} \) is given as a general source without loss of generality.

\[
\sum_{s \in D_j'} r_s \geq \overline{H}(M_{D_j'}|Y_j, M_{D_j'}) \geq 0
\]

for all \( (j, D_j') \) satisfying \( j \in J \) and \( \emptyset \neq D_j' \subset D_j \). Then we have

\[
\text{if } n \to \infty, \quad \sum_{s \in D_j'} r_s \geq \overline{H}(M_{D_j'}|Y_j, M_{D_j'}) = 0
\]

Assume that \( s \in S \). Since the distribution \( \mu_{M_s,n} \) of \( M_s,n \) is uniform on \( M_s,n \), we have the fact that

\[
\frac{1}{n} \log \frac{1}{\mu_{M_s,n}(m_s)} = \frac{1}{n} \log |M_s,n| \\
\geq \liminf_{n \to \infty} \frac{1}{n} \log |M_s,n| - \delta
\]

for all \( m_s \in M_s,n \), \( \delta > 0 \), and sufficiently large \( n \). This implies that

\[
\lim_{n \to \infty} P\left( \frac{1}{n} \log \frac{1}{\mu_{M_s,n}(M_s,n)} < \liminf_{n \to \infty} \frac{1}{n} \log |M_s,n| - \delta \right) = 0
\]

from (37) and the definition of \( \overline{H}(M_s) \). We have

\[
R_s + r_s = R_s
\]

\[
\leq \liminf_{n \to \infty} \frac{\log |M_s,n|}{n} \leq \overline{H}(M_s) + \delta
\]

for all \( s \in S \), where the equality comes from the fact that \( r_s = 0 \), the first inequality comes from (5), and the second inequality comes from (38). By letting \( \delta \to 0 \), we have

\[
R_s + r_s \leq \overline{H}(M_s)
\]

Let \( Z_s \equiv M_s \) for each \( s \in S \) and \( X_i \equiv \Phi_{i,n}(M_{S_i,n}) \) \( \forall n \) for each \( i \in I \). Since messages \( \{M_s\}_{s \in S} \) are mutually independent, the joint distribution of \( \{Z_s, X_i^n, Y_i^n\} \) is given as (10). Then, from (33) and (39), we have \( \{R_s\}_{s \in S} \in R_{\text{op}} \) which implies \( R_{\text{op}} \subset R_{\text{IT}} \).

\[\text{We can assume that } M_{s,n} \in Z_s^n \text{ without loss of generality.}\]
D. \((\alpha, \beta)-\text{hash property}\)

In this section, we review the hash property introduced in [18, 22] and show two basic lemmas. For the set of functions, let \(\text{Im}_F = \bigcup_{F \in F_{\mathcal{Z}}} \{Fz : z \in \mathbb{Z}^n\}\).

Definition 1 ([18 Definition 3]): Let \(F_s\) be a set of functions on \(\mathbb{U}_n\). For a probability distribution \(p_{F_s}\) on \(F_s\), we call a pair \((F_s, p_{F_s})\) an ensemble. Then, \((F_s, p_{F_s})\) has an \((\alpha_{F_s}, \beta_{F_s})\)-hash property if there is a pair \((\alpha_{F_s}, \beta_{F_s})\) depending on \(p_{F_s}\) such that

\[
\sum_{\tilde{z} \in \mathbb{U}_n \setminus \{z\}} p_{F_s}(\{f : f(z) = f(\tilde{z})\}) \leq \beta_{F_s},
\]

for any \(z \in \mathbb{Z}^n\). Consider the following conditions for two sequences \(\alpha_F \equiv \{\alpha_{F_n}\}_{n=1}^{\infty}\) and \(\beta_F \equiv \{\beta_{F_n}\}_{n=1}^{\infty}\)

\[
\lim_{n \to \infty} \frac{1}{n} \log(1 + \beta_{F_n}) = 0,
\]

\[
\lim_{n \to \infty} \frac{1}{n} \log \alpha_{F_n} = 0,
\]

\[
\lim_{n \to \infty} \beta_{F_n} = 0.
\]

(41)

Then, we say that \((F_s, p_{F_s})\) has an \((\alpha_{F_s}, \beta_{F_s})\)-balanced-coloring property if \(\alpha_F\) and \(\beta_F\) satisfy (41), (42), and (43). We say that \((F_s, p_{F_s})\) has an \((\alpha_F, \beta_F)\)-collision-resistant property if \(\alpha_F\) and \(\beta_F\) satisfy (41), (44), and (45). We say that \((F_s, p_{F_s})\) has an \((\alpha_{F_s}, \beta_{F_s})\)-hash property if \(\alpha_F\) and \(\beta_F\) satisfy (41), (42), and (45). Throughout this paper, we omit the dependence of \(F\) and \(p\) on \(n\).

It should be noted that when \(F\) is a two-universal class of hash functions \([6]\) and \(p_F\) is the uniform distribution on \(F\), then \((F, p_F)\) has a \((1, 0)\)-hash property. Random binning \([2]\) and the set of all linear functions \([3]\) are two-universal classes of hash functions. It is proved in [21 Section III-B] that an ensemble of sparse matrices has a hash property. It is proved in [19 Section IV-B] that an ensemble of systematic sparse matrices has a balanced-coloring property.

We introduce lemmas that are multiple extensions of the balanced-coloring property and the collision-resistant property. We use the following notations. For each \(s \in S\), let \(F_s\) be a set of functions on \(\mathbb{Z}^n_s\) and \(c_s \in \text{Im}_F\). Let \(\mathbb{Z}_S \equiv \times_{s \in S} \mathbb{Z}_s^n\) and

\[
\alpha_{F_s} \equiv \prod_{s \in S} \alpha_F, \\
\beta_{F_s} \equiv \prod_{s \in S} \beta_F + 1 - 1,
\]

where \(\prod_{s \in \emptyset} \equiv 1\). It should be noted that

\[
\lim_{n \to \infty} \frac{1}{n} \log(1 + \alpha_{F_s}) = 0,
\]

\[
\lim_{n \to \infty} \frac{1}{n} \log \alpha_{F_s} = 0,
\]

\[
\lim_{n \to \infty} \beta_{F_s} = 0.
\]

(45)

3The square part of the matrix is identity.
for all \((z_S, S')\) satisfying \(z_S \in T\) and \(\emptyset \neq S' \subseteq S\), where the second inequality comes from (46) and the third inequality comes from (47). It should be noted that (48) is valid for the cases \(S_c = \emptyset\) and \(S_c = S\) by letting \(Q_0 = Q(T)\) because

\[
\sum_{p_z, z'_z \leq \frac{\alpha_{p_z}}{\sum_{p_z, z'_z} \text{Im} F_z}} Q(z_S) \prod_{s \in S} p_{z, z'_s} \leq \frac{1}{\sum_{p_z, z'_z} \text{Im} F_z} \sum_{p_z, z'_z \leq \frac{\alpha_{p_z}}{\sum_{p_z, z'_z} \text{Im} F_z}} Q(z_S) \prod_{s \in S} p_{z, z'_s}
\]

Then we have

\[
E_{F_S C_S} \left[ \frac{\sum_{z_S} Q(z_S) \chi(F_S(z_S) = C_S)}{Q(T)} \right]^2 = \sum_{z_S \in T} Q(z_S) \prod_{s \in S} p_{z, z'_s} \leq \frac{1}{\sum_{p_z, z'_z} \text{Im} F_z} \sum_{p_z, z'_z \leq \frac{\alpha_{p_z}}{\sum_{p_z, z'_z} \text{Im} F_z}} Q(z_S) \prod_{s \in S} p_{z, z'_s}
\]

Next, let \(C_S\) be the random variable subject to the uniform distribution on \(\times_{s \in S} \text{Im} F_s\). From (51), we have

\[
E_{F_S C_S} \left[ \frac{\sum_{z_S} Q(z_S) \chi(F_S(z_S) = C_S)}{Q(T)} \right]^2 = \sum_{z_S \in T} Q(z_S) \prod_{s \in S} p_{z, z'_s} \leq \frac{1}{\sum_{p_z, z'_z} \text{Im} F_z} \sum_{p_z, z'_z \leq \frac{\alpha_{p_z}}{\sum_{p_z, z'_z} \text{Im} F_z}} Q(z_S) \prod_{s \in S} p_{z, z'_s}
\]

Then we have

\[
E_{F_S C_S} \left[ \frac{\sum_{z_S} Q(z_S) \chi(F_S(z_S) = C_S)}{Q(T)} \right]^2 = \sum_{z_S \in T} Q(z_S) \prod_{s \in S} p_{z, z'_s} \leq \frac{1}{\sum_{p_z, z'_z} \text{Im} F_z} \sum_{p_z, z'_z \leq \frac{\alpha_{p_z}}{\sum_{p_z, z'_z} \text{Im} F_z}} Q(z_S) \prod_{s \in S} p_{z, z'_s}
\]

Next, let \(C_S\) be the random variable subject to the uniform distribution on \(\times_{s \in S} \text{Im} F_s\). From (51), we have

\[
E_{F_S C_S} \left[ \frac{\sum_{z_S} Q(z_S) \chi(F_S(z_S) = C_S)}{Q(T)} \right]^2 = \sum_{z_S \in T} Q(z_S) \prod_{s \in S} p_{z, z'_s} \leq \frac{1}{\sum_{p_z, z'_z} \text{Im} F_z} \sum_{p_z, z'_z \leq \frac{\alpha_{p_z}}{\sum_{p_z, z'_z} \text{Im} F_z}} Q(z_S) \prod_{s \in S} p_{z, z'_s}
\]

Finally, the lemma is shown as

\[
E_{F_S} \left[ \frac{\sum_{z_S} Q(T \cap C_S(z_S))}{Q(T)} \right] = \sum_{z_S \in T} Q(z_S) \prod_{s \in S} p_{z, z'_s} \leq \frac{1}{\sum_{p_z, z'_z} \text{Im} F_z} \sum_{p_z, z'_z \leq \frac{\alpha_{p_z}}{\sum_{p_z, z'_z} \text{Im} F_z}} Q(z_S) \prod_{s \in S} p_{z, z'_s}
\]

for all \(z_S' \in T\), where the equality comes from the fact that \(Q_0 = Q(T)\) and \(\beta_{F_S} = 0\).
for all $z \in Z^n$. and $s', s'' \in S$ from (48). Then we have

$$p_{F_S} \left( \{ f : s \in T \setminus \{ z_s \} \setminus C_{F_S}(F_S(z_S)) \neq 0 \} \right) \leq \sum_{z_s \in T \setminus \{ z_s \}} \prod_{s \in S} p_{z_s, z_s'} = \prod_{s \in S} p_{z_s, z_s'} - 1 \left( z_s \in S \right), \sum_{S' \subseteq S} \prod_{s \in S'} p_{z_s, z_s'} - 1 \leq \sum_{S' \subseteq S} \frac{\alpha_{F_S} [\beta_{F_S} + 1] |S'_c|}{\prod_{s \in S'} |\text{Im} F_s|} + \beta_{F_S}$$

(56)

for all $T \subseteq Z^n$ and $S', S'' \subseteq S$, where the third equality comes from the fact that $p_{z_s, z_s'} = 1$, the second inequality comes from (55), and the last equality comes from the fact that $\alpha_{F_S} = 1, \beta_{F_{S', c}} = \beta_{F_S}, \prod_{s \in \emptyset} |\text{Im} F_s| = 1$, and $\mathcal{O}_S = 1$. 

E. Proof of Theorem 3

In the following, we omit the dependence of $Z, C, Y$, and $\hat{Z}$ on $n$ when they appear in the subscript of $\mu$.

First, we prove the following lemma, where we omit the dependence of $Y, D, \delta$ on $j$.

**Lemma 6:** Let $(Z_D, Y)$ be a pair of correlated sources and

$$T_{Z_D} \equiv \left\{ (z_D, y) : \frac{1}{n} \log \frac{\mu_{Z_D'}|Z_{D'}^c}(z_D', z_{D'}^c, y) - \frac{1}{|Z_D|} \mu_{Z_D' | Y, Z_{D'}^c}(y, z_{D'}^c) + \varepsilon \right\}$$

where $\mu_{Z_D'}|Z_{D'}^c}(y, z_{D'}^c)$ outputs one of the elements in $T_{Z_D} \cap C_{F_{D'}}(c_D)$ and declares an error when $T_{Z_D} \cap C_{F_{D'}}(c_D) = \emptyset$.

**Proof:** Let $T_{Z_D}(y) \equiv \left\{ z_D : (z_D, y) \in T_{Z_D} \right\}$ and assume that $(z_D, y) \in T_{Z_D}$ and $\zeta D'(f_{D'}(z_D)) \neq Z_D$. Then $T_{Z_D}(y) \setminus \{ z_D \} \subseteq C_{F_{D'}}(f_{D'}(z_D)) = 0$. We have

$$E_{F_D} \left[ \log \frac{\mu_{Z_D'}|Z_{D'}^c}(y) \right] \leq \sum_{D' \subseteq D} \frac{\alpha_{F_{D'}} [\beta_{F_{D'}} + 1] |S'_c|}{\prod_{s \in S'} |\text{Im} F_s|} + \beta_{F_{D'}}$$

(57)

where the second inequality comes from Lemma 8 and the third inequality comes from $\mathcal{O}_D = 2^{-n |Z_{D'} \setminus Y, Z_{D'}^c|} + \varepsilon$. We have

$$E_{F_D} \left[ \log \frac{\mu_{Z_D'}|Z_{D'}^c}(y) \right] = \sum_{(z_D, y) \in T_{Z_D}} \log \frac{\mu_{Z_D'}|Z_{D'}^c}(y)$$

(58)
From this inequality and the fact that \(\log(\alpha_{FG})/n \to 0\), \(\beta_{FG} \to 0\), \(\mu_{ZD} s r \to 0\), we have the fact that for all \(\delta > 0\) and sufficiently large \(n\) there are \(\{f_s\}_{s \in D}\) such that the error probability is less than \(\delta\) for all sufficiently large \(n\) when \(\{r_s\}_{s \in D}\) satisfies

\[
\sum_{s \in D'} r_s > \mathbb{P}(Z_{D'}, Z_{D''}, Y) + \varepsilon
\]

for all \(D'\) satisfying \(\emptyset \neq D' \subset D\) by letting sufficiently small \(\varepsilon > 0\).

Next, we introduce the following lemma.

**Lemma 7 (Corollary 2):** Let \((U, V)\) be a pair consisting of a state \(U\) and an observation \(V\) and \(\mu_{UV}\) be the joint distribution of \((U, V)\). We make a stochastic decision with \(\mu_{UV}\), that is, the joint distribution of \((U, V)\) and a guess \(\hat{U}\) of the state \(U\) is given as

\[
\mu_{UV, \hat{U}}(u, v, \hat{u}) = \mu_{UV}(u, v)\mu_{U|V}(\hat{u}|v).
\]

Then the decision error probability of this rule is at most twice the decision error probability of any (possibly stochastic) decision, that is,

\[
P(\hat{U} \neq U) \leq 2P(\hat{U}' \neq U),
\]

where

\[
\mu_{UV, \hat{U}'}(u, v, \hat{u}) = \mu_{UV}(u, v)\mu_{U|V}(\hat{u}'|v)
\]

for any probability distribution \(\mu_{U|V}\).

Finally, we prove Theorem 2. The joint distribution of \((Z_{D'}, C_{D'}, Y_{D'})\) is given as

\[
\mu_{Z_{D'}, C_{D'}, Y_{D'}}(z_{D'}, c_{D'}, y_{D'}) = \mu_{Z_{D'}, Y_{D'}}(z_{D'}, y_{D'}) = \mu_{F_{D'}(z_{D'}, c_{D'})}.
\]

Then we have

\[
\mu_{Z_{D'}, C_{D'}, Y_{D'}}(z_{D'}, c_{D'}, y_{D'}) = \sum_{s \in J} \mu_{Z_{D'}, C_{D'}, Y_{D'}}(z_{D'}, c_{D'}, y_{D'})
\]

\[
= \sum_{s \in J} \mu_{Z_{D'}, Y_{D'}}(z_{D'}, y_{D'}) \chi(F_{D'}(z_{D'}, c_{D'}))
\]

\[
= \sum_{s \in J} \mu_{Z_{D'}, Y_{D'}}(z_{D'}, y_{D'}) \chi(F_{D'}(z_{D'}, c_{D'}))
\]

\[
= \sum_{s \in J} \mu_{Z_{D'}, Y_{D'}}(z_{D'}, y_{D'}) \chi(F_{D'}(z_{D'}, c_{D'})),
\]

for given positive numbers \(\{\delta_j\}_{j \in J}\) and all sufficiently large \(n\). When \(Z^n_{D'} \neq Z^n_{D''}\) for some \(j \in J\) and \(s \in D_j\), we have \(Z^n_{D_j} \neq Z^n_{D_j}\) for some \(j \in J\). From this, we have

\[
E_{F_{D'}} [\text{Error}(F_{S'})] \leq E_{F_{S'}} \sum_{j \in J} \mathbb{P}(Z^n_{D_j} \neq Z^n_{D_j})
\]

\[
\leq 2 \sum_{j \in J} \delta_j
\]

for all positive values \(\{\delta_j\}_{j \in J}\) and all sufficiently large \(n\). We obtain the theorem by letting \(2 \sum_{j \in J} \delta_j < \delta\).

**F. Proof of Theorem 6**

Before the proof of the theorem, in Fig. 5 we illustrate the code construction for a simple case.

Here, we prove the theorem. For a given \((r_s, R_s)\) satisfying \(\{i, S_i\}\) for all \(i \in I\) that satisfies \(\emptyset \neq S_i \subset S\), assume that \((G_s, p_s)\) have a balanced-coloring property for every \(s \in S\), where \(r_s = \log(|\text{Im}G_s|)/n\) and \(R_s = \log(|\text{Im}G_s|)/n\).

In the following, we omit the dependence of \(Z, C, Y\), and \(\hat{Z}\) on \(n\). Let

\[
\mathcal{E}_{f_S, g_S}(c_S, m_S) = \mathcal{E}_{f_S}(c_S) \cap \mathcal{E}_{g_S}(m_S).
\]

We use the fact without notice that \(\{\mathcal{E}_{f_S, g_S}(c_S, m_S)\}\) is a partition of \(Z^n_{D'}\) and \(\{S_i\}_{i \in I}\) is a partition of \(S\).

The error probability \(\text{Error}(f_S, g_S, c_S)\) is evaluated as \(\mathcal{E}_{f_S, g_S}(c_S, m_S)\), which appears on the top of the next page, where the first and the second terms on the right hand side of \(\mathcal{E}_{f_S, g_S}(c_S, m_S)\) correspond to the encoding error and the decoding error probabilities, respectively.

The expectation of the first term on the right hand side of \(\mathcal{E}_{f_S, g_S}(c_S, m_S)\) is evaluated as

\[
E_{G_{D'}} \left[ \sum_{m_S, z_S, \hat{z}_{D'}: \{z_S, \hat{z}_{D'}\}} \mu_{Z_s} \hat{Z}_{D'}(z_S, \hat{z}_{D'}) \mu_{Z_s} \mathcal{E}_{f_S}(c_S) \right]
\]

\[
= E_{G_{D'}} \left[ \sum_{m_S, z_S, \hat{z}_{D'}: \{z_S, \hat{z}_{D'}\}} \mu_{Z_s} \hat{Z}_{D'}(z_S, \hat{z}_{D'}) \mathcal{E}_{f_S}(c_S) \right]
\]

\[
= E_{G_{D'}} \left[ \sum_{m_S, z_S, \hat{z}_{D'}: \{z_S, \hat{z}_{D'}\}} \mu_{Z_s} \hat{Z}_{D'}(z_S, \hat{z}_{D'}) \mathcal{E}_{f_S}(c_S) \right]
\]

\[
= E_{G_{D'}} \left[ \sum_{m_S, z_S, \hat{z}_{D'}: \{z_S, \hat{z}_{D'}\}} \mu_{Z_s} \hat{Z}_{D'}(z_S, \hat{z}_{D'}) \mathcal{E}_{f_S}(c_S) \right]
\]

where the first equality comes from the fact that \(C_S\) is generated at random subject to the distribution \(\{\mu_{Z_S}(\mathcal{E}_{f_S}(c_S))\}\), and the last equality comes from the definition of \(\text{Error}(f_S)\).
Let $\mathcal{T}_S$ be defined as

\[ \mathcal{T}_S \equiv \left\{ z_S : \frac{1}{n} \log \frac{1}{\mu_{Z_{S'}}(z_{S'})} \geq H(Z_{S'}) - \epsilon \right\} . \]

Then the expectation of the second term on the right hand side of (53) is evaluated as (54), which appears in the top of the next page. The third inequality comes from Lemma 4 by letting

\[ \mathcal{T} \equiv \mathcal{T}_S \cap \mathcal{C}_f(c_S) \]

\[ Q \equiv \mu_{Z_S} \]

and using the relations

\[ \mathcal{T}_{S'} \subset \left\{ z_{S'} : \frac{1}{n} \log \frac{1}{\mu_{Z_{S'}}(z_{S'})} \geq H(Z_{S'}) - \epsilon \right\} \]

\[ \mathcal{T}_{S'} \cap \mathcal{C}_f(c_{S''}) \subset \mathcal{C}_{f_{S''}}(c_{S''}) \]

\[ \sqrt{n} \frac{1}{\mu_{Z_S}(z_S)} \geq H(Z_{S'}) - \epsilon \]

\[ \leq \frac{1}{n} \log \frac{1}{\mu_{Z_{S'}}(z_{S'})} \geq H(Z_{S'}) - \epsilon \]

\[ \mathcal{Q}_{S''} \leq \mu_{Z_{S''}}(z_{S''}) \]

\[ \leq 2^{-n[H(Z_{S'})-\epsilon]} \mu_{Z_{S''}}(\mathcal{C}_{f_{S''}}(c_{S''})) \]

The fourth inequality comes from the Jensen inequality. The last equality comes from the fact that

\[ \sum_{x_{S''}} \mu_{Z_{S''}}(\mathcal{C}_{f_{S''}}(c_{S''})) = \prod_{x_{S'} \in S'} \left[ \prod_{x_{S'} \in S'} \mathcal{L} \right] \]

and $\mu_{Z_S}(\mathcal{T}_S) \leq 1$. 

\[ \prod_{x_{S'} \in S'} \mathcal{L} \]

\[ \prod_{x_{S'} \in S'} \mathcal{L} \]

\[ \prod_{x_{S'} \in S'} \mathcal{L} \]

\[ \prod_{x_{S'} \in S'} \mathcal{L} \]
\[ E_{\mathcal{G},\gamma}[\text{Error}(f_S, G_S, C_S)] \]
\[ \leq \text{Error}(f_S) + \sqrt{\alpha G_S - 1 + \sum_{S' \subset S, S' \neq \emptyset} \alpha G_{S'} \left( \beta G_{S'} + 1 \right) 2^{-n \gamma}} + 2 \mu \gamma \left( T_\gamma \right), \]
\[ \gamma \equiv H(Z_{S'}) - \sum_{s \in S'} [R_s + r_s] - \epsilon. \]

Finally, let us assume that \( \{(r_s, R_s)\}_{s \in S} \) satisfies (11) for all \((i, S')\) satisfying \( i \in I \) and \( \emptyset \neq S' \subset S_i \). Then we have
\[ \sum_{s \in S'} [R_s + r_s] = \sum_{i \in I} \sum_{S \cap S_i} [R_s + r_s] < \sum_{i \in I} \sum_{S \cap S_i} H(Z_{S \cap S_i}) \leq H(Z_{S'}), \]
where the last inequality comes from (24) and the fact that \( \{Z_s\}_{s \in S} \) and \( \{Z_s\}_{s \in S_i} \) are mutually independent when \( i \neq i' \). Then, by letting \( \epsilon \to 0 \), \( \alpha G_s \to 1 \), \( \log(1 + \beta G_s) / n \to 0 \) and \( \mu \gamma \left( T_\gamma \right) \to 0 \), we have the fact that for all \( \delta > 0 \) and sufficiently large \( n \) there are \( \{g_s\}_{s \in S} \) and \( \{c_s\}_{s \in S} \) such that
\[ \text{Error}(f_S, G_S, C_S) \leq \text{Error}(f_S) + \delta. \]

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\[E_{G_{s}C_{s}} \left( \sum_{m_{s}} \frac{\mu_{Z_{s}}(e_{f_{s}G_{s}}(C_{s}, m_{s}))}{\mu_{Z_{s}}(e_{f_{s}}(C_{s}))} \right) = \frac{1}{\prod_{s \in S} \text{Im} \hat{g}_{s}} \]

\[\leq E_{G_{s}} \left[ \sum_{m_{s}, e_{s}} \mu_{Z_{s}}(e_{f_{s}}(C_{s}) \cap e_{G_{s}}(m_{s})) - \frac{\mu_{Z_{s}}(T_{Z_{s}} \cap e_{f_{s}}(C_{s}))}{\prod_{s \in S} \text{Im} \hat{g}_{s}} \right] \]

\[\leq E_{G_{s}} \left[ \sum_{m_{s}, e_{s}} \mu_{Z_{s}}(T_{Z_{s}} \cap e_{f_{s}}(C_{s}) \cap e_{G_{s}}(m_{s})) - \frac{\mu_{Z_{s}}(T_{Z_{s}} \cap e_{f_{s}}(C_{s}))}{\prod_{s \in S} \text{Im} \hat{g}_{s}} \right] \]

\[\leq E_{G_{s}} \left[ \sum_{m_{s}, e_{s}} \mu_{Z_{s}}(T_{Z_{s}} \cap e_{f_{s}}(C_{s}) \cap e_{G_{s}}(m_{s})) - \frac{\mu_{Z_{s}}(T_{Z_{s}} \cap e_{f_{s}}(C_{s}))}{\prod_{s \in S} \text{Im} \hat{g}_{s}} \right] + E_{G_{s}} \left[ \sum_{m_{s}, e_{s}} \mu_{Z_{s}}(T_{Z_{s}} \cap e_{f_{s}}(C_{s})) \right] \]

\[= \sum_{e_{s}} \mu_{Z_{s}}(T_{Z_{s}} \cap e_{f_{s}}(C_{s})) E_{G_{s}} \left[ \sum_{m_{s}} \frac{\mu_{Z_{s}}(T_{Z_{s}} \cap e_{f_{s}}(C_{s}) \cap e_{G_{s}}(m_{s}))}{\mu_{Z_{s}}(T_{Z_{s}} \cap e_{f_{s}}(C_{s}))} - \frac{1}{\prod_{s \in S} \text{Im} \hat{g}_{s}} \right] + 2\mu_{Z_{s}}(T_{Z_{s}}) \]

\[\leq \sum_{e_{s}} \mu_{Z_{s}}(T_{Z_{s}} \cap e_{f_{s}}(C_{s})) \sqrt{\alpha_{G_{s}} - 1 + \frac{\sum_{S' \subset S \cup S'} \alpha_{G_{s}'}[\beta_{G_{s}'} + 1] \prod_{s' \in S'} \text{Im} \hat{g}_{s'} ||_{\text{Im} \hat{g}_{s'}}^{2} - n ||_{\text{Im} \hat{g}_{s'}} - \varepsilon}{\mu_{Z_{s}}(T_{Z_{s}} \cap e_{f_{s}}(C_{s}))}} \]

\[+ 2\mu_{Z_{s}}(T_{Z_{s}}) \]

\[\leq \sqrt{\alpha_{G_{s}} - 1 + \frac{\sum_{S' \subset S \cup S'} \alpha_{G_{s}'}[\beta_{G_{s}'} + 1] \prod_{s' \in S'} \text{Im} \hat{g}_{s'} ||_{\text{Im} \hat{g}_{s'}}^{2} - n ||_{\text{Im} \hat{g}_{s'}} - \varepsilon}{\mu_{Z_{s}}(T_{Z_{s}})}} \]

\[+ 2\mu_{Z_{s}}(T_{Z_{s}}) \]