Finding and proving supremum and infimum: students’ misconceptions

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Abstract. The real numbers system is one of the topics that pre-service mathematics teachers have to master. There is no exception to supremum and infimum, the basic concept of completeness properties of the real numbers system. They face with these two concepts in the second year and they require these two concepts all their undergraduate-study long. This paper focuses on analyzing pre-service mathematics teachers’ misconceptions on finding and proving supremum and infimum of a set of the real number system. The study reported in this paper was done by qualitative research. A test was given to 62 students who took Introduction of Real Analysis. Numerous misconceptions found between supremum and upper bound, not to mention infimum and lower bound. The misconception about the definition of supremum and infimum, the theorem of supremum and infimum, and the basic concept of mathematics were described as well.

1. Introduction
On the topic of the real number system ($\mathbb{R}$), the supremum and infimum of a set are the new abstract concept that students of the Department of Mathematics Education, Universitas Muhammadiyah Surakarta face with. Though upper bound and lower bound is not new not concepts for them, least upper bound and great lower bound is new things they should master to study other advanced mathematics concepts, more precisely from calculus to real analysis [1]. These two concepts are included in the completeness property on the real number system, that belong to crucial concepts on $\mathbb{R}$ [2]. These properties taught in Introduction to Real Analysis course on the second year, ask students not only to find but also to prove. What students do on proving at this course surely different with their demands on proving at high school [3]. It is no secret that proving require students’ struggle [4][5][6]. Nonetheless, students, as prospective mathematics teachers, have to master how to find and how to prove these properties. Based on the experience of teaching Introduction to Real Analysis course for two years, it is noticeable that many students often run into failure on mid-semester test, especially in the part of supremum and infimum. Some of them could guess the value of supremum and infimum of a set but left a blank answer on proving it. Besides, they looked reckless on writing a definition or some part of a definition, such as the upper bound or the warranty of the word “least”. In this paper, students’ misconceptions on finding and proving supremum and infimum of a set on $\mathbb{R}$ are discussed. Besides, some errors on students’ answer described as well.

In his paper, Weber [7] stated that there were four categories of students’ proofs, they are 1) valid proof, 2) failure on assembling the syntactic knowledge, 3) lack adequate syntactic knowledge needed
for proving, or 4) logical errors. A student’s proof is valid when it is correct. If students blunder on congregate their syntactic knowledge, they belong to the second category. Knowing a theorem or definition does not ensure students can apply it [7]. If they do not know definitions, theorems or other mathematical facts those underlying the proof, then their proofs are in the third category. The latter happens if students feel their proof are valid, but they are incorrect. This could be caused by misconception or belief that proving a general theorem by confirming a specific example. Student’s understanding can be merged into either correct conception or misconception [8]. Misconception is not similar to error [9], students’ misconceptions on understanding the concepts of mathematics cause students’ mathematical errors [10].

2. Methods
This presented paper was the result of a qualitative research to analyzing pre-service teachers’ misconceptions on finding and proving supremum and infimum of a set of the real number system. The subject consists of 62 pre-service mathematics teacher who took Introduction of Real Analysis course in the academic year 2017/2018. The main data were students’ answer to writing test, while the supporting data were gotten by observation of teaching and learning processes and the result of structured tasks. The validity of data was done by investigator triangulation [11]. For ease, this reported research uses pseudonyms to mention the subjects, such as Ari, Budi, Candra, etcetera.

3. Results and Discussion
3.1. Proving the theorem of supremum
A case of proving a theorem about supremum and a case of finding and proving supremum and infimum were given to 62 preservice mathematics teachers, these were:

a. Let $S \neq \emptyset$, $A \subset \mathbb{R}$, and $u$ is the upper bound of $A$. Proof that:
   If for every $\varepsilon > 0$, there exists an $s_{\varepsilon} \in S$ such that $u - \varepsilon < s_{\varepsilon}$, then $u = \sup(S)$.

b. Let $S = [1,3] \cup \{0\}$. Find $\sup(S)$ and $\inf(S)$. Give your explanation.

From 62 students, only 24.19% who able to answer the first case inappropriate concept of supremum. The rest proved in errors, misconceptions, or left it blank. Indeed, from the 24.19%, only three students who gain perfect score, the rest did some failure on their processes so that their proof was invalid. The whole students’ answer is depicted in the Figure 1 as follow.

![Figure 1. Students’ results on proving the first case](image-url)
As shown in Figure 1, about 96.77% students who failed on proving the given theorem. Surely this was an awful condition. They wrote an incorrect definition of the supremum of a set of $\mathbb{R}$, wrote carelessly without meaning, or stated directly proven without explanation. The incorrect definition appeared at students’ error on took the symbols. In agreement with Stravou [12], when students faced with problem-solving, they used incorrect definition. As reported by Alcock and Simpson [13], only 29% students who could implement a definition correctly. There were students who assumed the statement that would prove to generate its proof. Figure 2 showed how Ari proved the given theorem in the first case.

**Translation:**

\[
S \subseteq \mathbb{R} \text{ and } u \text{ is an upper bound of } S \\
\text{Take any } \varepsilon > 0 \\
\text{We get } u - \varepsilon > s_\varepsilon \\
\text{Because } u = \sup S \text{ then } s_\varepsilon < s. \text{ Therefore } u - \varepsilon < s_\varepsilon
\]

**Figure 2.** Ari’s answer in the first case

Ari clearly stated that “because of $u = \sup S$ then ….”. His statement was what should be proved. This result is in line with Stravou [12].

Of interest to note there were 19% students who only restated the given hypothesis and wrote that “…so $u = \sup S$”. Some of them only changed “for every $\varepsilon > 0$” with “Take any $\varepsilon > 0$”. They added nothing or just manipulated simple algebra as shown in the following figure.

**Translation:**

\[
\text{Proof:} \\
\text{Take any } \varepsilon > 0, \text{ it has characteristic } u - \varepsilon < s_\varepsilon \\
\text{ } u < s_\varepsilon + \varepsilon \\
\text{Then } u = \sup S \\
\text{And } u \text{ is upper bound of } S
\]

**Figure 3.** Budi’s answer in the first case

Figure 3 depicts how Budi wrote his proof. He did not address his proof to the definition of supremum. Because $u$ is the upper bound of $S$, Budi should only prove that $u$ is the least upper bound. All Budi did was only transformed “$u - \varepsilon < s_\varepsilon$” to “$u < s_\varepsilon + \varepsilon$”. 
It is also noticeable that there were two students who constructed a proof by using specific example. This is in accordance with Sari et al [14], student used a specific example to generate the proof of general implication statement.

3.2. Students’ misconceptions on finding and proving supremum and infimum

There were 8 students who not understood the concept of supremum and infimum of a set correctly. Founded on the data analysis, they answered the second case by invalid proof. Four of them were seen blurring between supremum and upper bound. The example of these proofs could be seen at Candra’s proof and Dedi’s proof on Figure 4 as follows.

**Translation:**

**Solution:**
- \( \sup S \) is 3 because \( x \in S \) satisfies \( x \leq 3 \)
- \( \inf S \) is 1 because \( x \in S \) satisfies \( x > 1 \)

**Figure 4.** Candra’s answer in the second case

It can be seen that Candra stated \( \sup S = 3 \) because \( x \in S \) satisfies \( x \leq 3 \), and \( \inf S = 1 \) because \( x \in S \) satisfies \( x > 1 \). Based on student’s note when they took the course (see Figure 5).

**Translation:**

The Completeness Property on \( \mathbb{R} \)

Given a set \( S \subseteq \mathbb{R} \).
- \( u \in \mathbb{R} \) is called upper bound of \( S \) if \( \forall s \in S \) satisfies \( s \leq u \)
- \( v \in \mathbb{R} \) is called lower bound of \( S \) if \( \forall s \in S \) satisfies \( v \leq s \)

**Figure 5.** A part of student’s note in the Introduction to Real Analysis course

What Candra did was only proved that 3 is an upper bound of \( S \) and 1 is a lower bound of \( S \). Moreover, what Candra did was same with what Dedi’s answer on Figure 6 below.
sup$S = 0$ because

1. 0 is upper bound on $S$, $0 > x$

inf$S = 1$ because:

1. 1 is lower bound on $S$, $1 < x$

Figure 6. Dedi’s answer in the second case

The difference between Candra’s proof and Dedi’s proof was the value of sup $S$. Though Chandra guessed right and Dedi mentioned wrong on sup $S$, they got the same poor score. Both of them took wrong value on inf$S$ and only focused on upper and lower bound of $S$.

The four other students who misunderstood the concept of supremum and infimum of a set correctly seen on their answers that did not lead to the uniqueness of supremum or infimum of a set of $\mathbb{R}$. The first, Figure 7 depicts how Eko answered the second case, finding and proving sup$S$ and inf$S$.

sup$S$ is $(0,3)$ because supremum $S$ is the least of upper part of . . . .

inf$S$ is $(1,0)$ because infimum $S$ is the least of lower part of . . . .

Figure 7. Eko’s answer in the second case

Eko understood supremum of a set is a set too. This means Eko’s beliefs on the concept of supremum and infimum of a set was faulty. Moreover, it seems that Eko had a misconception on upper bound and lower bound because he wrote “upper part” instead of “upper bound”. These two terms are different. What is the upper part of a set? His beliefs on upper bound and lower bound led to misconceptions on supremum and infimum. Misconceptions could occur because of students’ lack of the concepts [15].

In the following figure, Figure 8, what Fandi did was apparently analog with Eko’s answer.

sup $S = \{1,3\} \cup \{0\}$

Supremum $S = \{1,3\} \cup \{0\}$

inf $S = \{0\} \cup \{1,3\}$

Figure 8. Fandi’s answer in the second case

Though Fandi left his answer without explanation, it is interesting to note that his belief in sup$(S)$ and inf$(S)$ was totally wrong. Fandi only wrote sup$(S)$ and inf$(S)$ as a union of two set and transposed the
order. In the Introduction of Real Analysis course, students work in the real number system [2], so that the value of supremum and infimum of a set, if they exist, must be a real number. Now in Figure 9, Galuh, one who stated sup $S$ and inf $S$ are set as well.

**Translation:**

$S = [1,3] \cup \{0\}$

$1 > 0$

$3 > 0$

$[1,3] \in \mathbb{R}$. Therefore sup $S = [1,3]$

inf $S = [1,3]$

**Figure 9.** Galuh’s answer in the second case

Just a single flash, Galuh’s answer was entirely wrong. Though her writing that $1 > 0$ and $3 > 0$ were true, her statement that $[1,3] \in \mathbb{R}$, sup $S = [1,3]$, and inf $S = [1,3]$ ruined her work. This is proof that Galuh did not understand the unique concept of supremum and infimum. The other student who believed that sup $S$ was not a unique real number was Hesti. Her work on finding and proving supremum could be identified on the following picture.

**Translation:**

$S = [1,3] \cup \{0\}$

$sup S = \{0,1,2,3\}$ because numbers 1 to 3 joined with empty set

**Figure 10.** Hesti’s answer in the second case

Based on Figure 10, it is obvious that Hesti stated there were four elements of $\mathbb{R}$ such that all of them was sup $S$. However lecturer emphasized the uniqueness of supremum and infimum when they encountered the transition from upper bound to supremum, it did not work to students’ understanding. Besides Hesti, Intan also did the same work.

**Translation:**

$sup S \rightarrow S = [1,3] \cup \{0\}$

$1,3 \leq 0$

$0 \geq 1,3$

$Then sup S = S = 1,3$

$inf S \rightarrow S = [1,3] \cup \{0\}$

$\forall \epsilon > 0. \exists s_{\epsilon} \in \mathbb{N}$

$s_{\epsilon} \in S \rightarrow [1,3] \cup \{0\}$

$0 < 1,3$

$Then inf S = -1, -3$

**Figure 11.** Intan’s answer in the second case

According to Figure 11, Intan ended up her answer by stating that sup $(S) = 1,3$ and inf $(S) = -1, -3$. Based on the definition of supremum, it contains the term “least” so that it should be unique.
At a glance, Intan’s answer on infimum case was addressed to the theorem of infimum, that is “If for every $\varepsilon > 0$, there exists an $s_\varepsilon \in S$ such that $s_\varepsilon < u + \varepsilon$, then $u = \inf(S)$.” It is clear from the given hypothesis that $S = [1, 3] \cup \{0\}$, but Intan wrote that $s_\varepsilon \in \mathbb{N}$. Besides, Intan did not choose the suitable value of $s_\varepsilon$. Intan’s faulty understanding of the theorem of infimum trapped her on error.

Besides misconceptions on the concepts of supremum and infimum, it is reported that students also did misconceptions on the fundamental knowledge of mathematics. According to Muzangwa and Chifamba [15], the main cause of students’ errors on calculus (the preliminary course of real analysis) is closely related with basic algebra. This happened on two matters. The first, 29.03% of 62 students found not having a correct understanding on upper bound or lower bound. The definition of supremum can be divided into two part, upper bound and the guarantee of the word “least”. As shown in the student’s note on Figure 5 and in the students’ handout, lecturer transferred knowledge that upper bound of $S$ could be equals to an element of $S$. Moreover, to emphasize this concept, the lecturer made a game in the teaching and learning activity. In fact, in order to prove supremum, students still wrote: “$s < u$”, instead of writing “$s \leq u$” when they proved $u$ is an upper bound of $S$. For instance, Janu’s answer on Figure 12 could illustrate this kind of misconception.

### Translation:

$S = [1, 3] \cup \{0\}$. Find $\sup S$ and $\inf S$

- $S = \{0,1,3\}$, $\sup S = 3$ because:
  
  i). $\forall x \in S$ then it satisfies $x \leq 3$
  
  ii). $\forall \varepsilon > 0$, there exists $s_\varepsilon(3 - \varepsilon) \in S$ so that $3 - \varepsilon < 3$

- $S = \{0,1,3\}$, $\inf S = 0$ because:
  
  i). $\forall x \in S$ then it satisfies $x \geq 0$

  ii). $\exists w$

**Figure 12.** Janu’s answer in the second case

It is also noticeable that Janu clearly stated that the set $S = \{0, 1, 3\}$. Though his answer on finding $\sup(S)$ and $\inf(S)$ was correct, his belief about $S = [1, 3] \cup \{0\}$ is similar to $S = \{0, 1, 3\}$ could cause his to be at risk as a preservice mathematics teacher. However, his later answer on proving $\sup(S)$ by using theorem on the first case was correct.

The similar idea of a set was had by Karim. His work on finding and proving $\sup S$ and $\inf S$ could be seen as follows.
Translation:

Proof:

Because the union then 0,1,3 so for sup S

\( \sup S \): (i) 3 is upper bound of S (it means that \( \forall s \in S, s \leq u \))

(ii) \( \forall t < u, then \exists s_0 \in S \) so that \( t \leq s_0 \leq t \)

\( \forall t < u, then \exists s_0 = \frac{1 + u}{2} < u \)

\( \inf S \): (i) 0 is lower bound of S (it means that \( \forall s \in S, s \geq v \))

(ii) \( \forall w > v, then \exists s_0 \in S \) so that \( t \leq w \)

\( \forall w > v, then \exists s_0 = \frac{1 + v}{w} \geq v \)

Figure 13. Karim’s answer in the second case

It is obvious that the first Karim’s statement was about the union of the set because \( S = [1, 3] \cup \{0\} \) then he had 0,1,3. Though he did not write \( S = \{0, 1, 3\} \), based on his statement about the union, he possible concluded that \( S = [1, 3] \cup \{0\} \) as same as \( S = \{0, 1, 3\} \). Furthermore, he found that \( \sup(S) = 3 \) and \( \inf(S) = 0 \). However the way he proved \( \sup(S) \) and \( \inf(S) \) by the definition supremum and infimum was correct, he ended up with incorrect choices of \( s_0 \). This means that he failed on assembling his syntactic knowledge [7]. This report is in line with the result of Luneta and Makonye [9], powerless competencies on pre-calculus, such as factorization and solving equation, led to negative impact on studying calculus.

4. Conclusion

According to results and discussions, pre-service mathematics teachers had misconceptions on finding and proving supremum and infimum on two matters. The first, they encountered misconceptions on the concept of supremum and infimum, specifically the differences between supremum and upper bound; and the uniqueness of supremum and infimum of a set of \( \mathbb{R} \). The second, they got misconception about the basic concepts of mathematics that needed in studying supremum and infimum, such as the dissimilarity between the sign of “less than” and “less than or equal”; and belief about an interval \( S = [a, b] \) equal to a set \( S = \{a, b\} \). These misconceptions led to errors when students found and proved the supremum and infimum of a set.

5. Suggestion

Based on this reported research, some suggestion are given to the lecturers. They have to emphasize the new concepts clearly and correctly. As stated by Dumitrașcu [16], the lecturer can implement both lecturing and the small-group discovery method to make effective and enjoyable real analysis class. This
method should be supported by well-prepared handouts for working on the group. The handout should encourage reasoning or checking students’ error [10]. Besides, the fundamental concept of mathematics, such as “greater than”, “less than”, or set were studied since students were in elementary school. This is crucial to improve pre-service mathematics teacher understanding on the fundamental concept of mathematics.

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