Intrinsic ultracontractivity for fractional Schrödinger operators

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Abstract

We establish sharp pointwise estimates for the ground state $\phi_0$ of some singular fractional Schrödinger operators on relatively compact Euclidean subsets. The considered operators are of the type $\left(-\Delta \right)^{\alpha/2}_\Omega - V$, where $V \in L^1_{\text{Loc}}$ and $\left(-\Delta \right)^{\alpha/2}_\Omega$ is the fractional-Laplacian on an open subset $\Omega$ in $\mathbb{R}^d$ with zero exterior condition. The intrinsic ultracontractivity property for such operators is discussed as well and a sharp large time asymptotic for their heat kernels is derived.

Key words: Improved Sobolev inequality, ground state, intrinsic ultracontractivity, Dirichlet form.

1 Introduction

This paper is to study intrinsic ultracontractivity for the Feynman-Kac semigroups generated by Schrödinger operators based on fractional Laplacians and obtain two sharp estimates of the first eigenfunction of these operators, we use potential methods and Sobolev inequalities. Let $\Omega$ be a $C^{1,1}$ bounded domain in $\mathbb{R}^d$ containing the origin. Let $L_0 := \left(-\Delta \right)^{\alpha/2}_\Omega$, $0 < \alpha < \min(2,d)$ be the fractional Laplacian on $\Omega$ with zero exterior condition in $L^2(\Omega, dx)$. It is well known that $L_0$ has purely discrete spectrum $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_k \to \infty$ and that the associated semigroup $T_t := e^{-tL_0}, t > 0$ is irreducible. Hence $L_0$ has a unique strictly positive normalized ground state $\phi_0$. Furthermore Kulczycki proved in [Ku98] that the semigroup $(T_t)_{t>0}$ is intrinsically ultracontractive (IUC for short) regardless the regularity of $\Omega$. The latter property induces among others the large time asymptotic for the heat kernel $p_t$ of $e^{-tL_0}$, $t > 0$:

$$p_t(x, y) \sim e^{-t\lambda_0} \varphi_0(x) \varphi_0(y), \quad \text{on } \Omega \times \Omega. \quad (1.1)$$

Such type of estimates are very important in the sense that they give precise information on the local behavior of the ground state and the heat kernel (for large $t$) as well as on their respective rates of decay at the boundary.

Set $G$ the Green’s Kernel of $L_0$, that since $(T_t)_{t>0}$ is IUC, then there is a finite constant $C_G$, such that
\[ G(x, y) \geq C_G \varphi_0(x) \varphi_0(y), \]  
\text{(1.2)}

yielding,

\[ \xi_0(x) = \int G(x, y) \, dy \geq C_G \varphi_0(x) \int \varphi_0(y) \, dy, \]  
\text{(1.3)}

where \( \xi_0 \) denote the solution of \( L_0 \xi_0 = 1 \).

In this paper we consider the fractional Schrödinger operators

\[ L_V := L_0 - V, \quad V \in L^1_{\text{Loc}}(\Omega). \]

In particular the case

\[ V(x) = \frac{c}{|x|^\alpha}, \quad 0 < c \leq c^* := \frac{2^\alpha \Gamma^2(\frac{d+\alpha}{4})}{\Gamma^2(\frac{d-\alpha}{4})}, \]  
\text{(1.4)}

\[ V(x) = \frac{c}{\delta^\alpha}, \quad 0 < c \leq c^* := \frac{\Gamma^2(\frac{\alpha+1}{2})}{\pi}, \]  
\text{(1.5)}

where \( \delta \) is the Euclidian distance function between \( x \) and \( \Omega \), \( d \geq 3 \) and \( \Omega \) is regular (see [FMT13, FLS08]).

We shall prove that under some realistic assumptions, and especially under the assumptions that some improved Sobolev and Hardy-type inequalities hold true, then the operator \( L_V \) still has discrete spectrum, a unique normalized ground state \( \varphi^V_0 > 0 \ a.e. \) Furthermore \( \varphi^V_0 \) is comparable to the solution, \( \xi^V_0 \) of the equation \( L_V \xi^V_0 = 1 \) (i.e., comparable to \( L^1_V \)). In other words, if we designate by \( G^V \) the Green’s kernel of \( L_V \), then

\[ \varphi^V_0 \sim \xi^V_0 = \int \Omega G^V(x, y) \, dy \quad \text{if} \ \Omega \ \text{is} \ C^{1,1}. \]  
\text{(1.6)}

We shall however, prove that the intrinsic ultracontractivity property is still preserved. Namely, the operator \( e^{-tL_V}, \ t > 0 \) is IUC for domains which are less regular than \( C^{1,1} \) domains.

For \( \alpha = 2 \) (the local case), various types of comparison results as well as pointwise estimates for ground states of the Dirichlet-Schrödinger operator were obtained in [VZ00, DN02, DD03, CG98, Dav89] and in [BBB013] for more general potentials in the framework of (strongly) local Dirichlet. Whereas the preservation of the intrinsic ultracontractivity can be found in [Bañ91] for Kato potentials, in [CG98] and in [BBB013] in the framework of (strongly) local Dirichlet. The potentials satisfying \( (1.4) \) are treated in [BBB13].
Our method relies basically on an improved Sobolev inequality together with a transformation argument (Doob’s transformation) which leads to a generalized ground state representation.

The paper is organized as follows: In section 2 we give the backgrounds together with some preparing results. For the comparability of the ground states we shall consider two situations separately: the subcritical (section 3) and the critical case (section 4).

To get the estimates for the ground states of the approximating operator we shall use on one side the intrinsic ultracontractivity property and on the other side Moser’s iteration technique.

2 Preparing results

We first give some preliminary results that are necessary for the later development of the paper. Some of them are known. However, for the convenience of the reader we shall give new proofs for them.

Let \( 0 < \alpha < \min(2, d) \). Consider the quadratic form \( \mathcal{E}^\alpha \) defined in \( L^2 := L^2(\mathbb{R}^d, dx) \) by

\[
\mathcal{E}^\alpha(f,g) = \frac{1}{2} \mathcal{A}(d, \alpha) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{d+\alpha}} \, dx \, dy,
\]

where

\[
\mathcal{A}(d, \alpha) = \frac{\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{2^{1-\alpha} \pi^{d/2} \Gamma\left(1 - \frac{\alpha}{2}\right)}.
\]  

(2.2)

It is well known that \( \mathcal{E}^\alpha \) is a transient Dirichlet form and is related (via Kato representation theorem) to the selfadjoint operator, commonly named the \( \alpha \)-fractional Laplacian on \( \mathbb{R}^d \) which we shall denote by \( (-\Delta)^{\alpha/2} \).

Alternatively, the expression of the operator \( (-\Delta)^{\alpha/2} \) is given by (see [BBC03, Eq.3.11])

\[
(-\Delta)^{\alpha/2} f(x) = \mathcal{A}(d, \alpha) \lim_{\epsilon \to 0^+} \int_{\{y \in \mathbb{R}^d, |y - x| > \epsilon\}} \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} \, dy,
\]  

(2.3)

provided the limit exists and is finite.

For every open subset \( \Omega \subset \mathbb{R}^d \), we denote by \( L_0 := (-\Delta)^{\alpha/2}|_\Omega \) the localization of \( (-\Delta)^{\alpha/2} \).
on $\Omega$, i.e., the operator which Dirichlet form in $L^2(\overline{\Omega}, dx)$ is given by

$$D(\mathcal{E}) = W_0^{\alpha/2}(\Omega) := \{ f \in W^{\alpha/2,2}(\mathbb{R}^d): f = 0 \text{ } \mathcal{E} \text{- q.e. on } \Omega^c \}$$

$$\mathcal{E}(f, g) = \frac{1}{2} \mathcal{A}(d, \alpha) \int \int \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{d+\alpha}} dxdy$$

$$= \frac{1}{2} \mathcal{A}(d, \alpha) \left( \int \int \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{d+\alpha}} dxdy \right)$$

$$+ \int f(x)g(x)\kappa_\Omega^{(\alpha)}(x) dx, \forall f, g \in W_0^{\alpha/2}(\Omega).$$

where

$$\kappa_\Omega^{(\alpha)}(x) := \mathcal{A}(d, \alpha) \int_{\Omega^c} \frac{1}{|x - y|^{d+\alpha}} dy. \quad (2.4)$$

The Dirichlet form $\mathcal{E}$ coincides with the closure of $\mathcal{E}^\alpha$ restricted to $C_0^\infty(\Omega)$, and is therefore regular and furthermore transient.

We also recall the known fact that $L_0$ is irreducible even when $\Omega$ is disconnected [BBC03, p.93].

If moreover $\Omega$ is bounded, thanks to the well known Sobolev embedding,

$$\left( \int_\Omega |f|^{\frac{2d}{d-\alpha}} dx \right)^{\frac{d-\alpha}{2d}} \leq C(\Omega, d, \alpha) \mathcal{E}[f], \forall f \in W_0^{\alpha/2,2}(\Omega), \quad (2.5)$$

the operator $L_0$ has compact resolvent (that we shall denote by $K := L_0^{-1}$) which together with the irreducibility property imply that there is a unique continuous bounded, $L^2(\Omega, dx)$ normalized function $\varphi_0 > 0$ and $\lambda_0 > 0$ such that

$$L_0\varphi_0 = \lambda_0 \varphi_0 \text{ on } \Omega. \quad (2.6)$$

We shall prove that this property of $L_0$ is still preserved by perturbations of the form $V \in L^1_{loc}$. However, singularities will appear for the ground state of the perturbed operator provided $\Omega$ contains the origin.

Let $V_\ast$ be a fixed positive potentials such that $V_\ast \in L^1_{loc}(\Omega)$, we shall also adopt some assumptions along the paper.

The first assumption is the following Hardy-type inequality : There is a finite constant $C_H > 0$ such that

$$\int \frac{f^2(x)}{\varphi_0^2(x)} dx \leq C_H \mathcal{E}[f], \forall f \in W_0^{\alpha/2,2}(\Omega). \quad (2.7)$$

**Remark 2.1.** The latter inequality holds true for bounded domains satisfying the uniform interior ball condition and $d \geq 2, \alpha \neq 1$. Indeed for this class of domains we already observed that

$$\varphi_0 \geq C \delta^{\alpha/2}, \quad (2.8)$$
whereas Corollary 2.4 asserts that if $\Omega$ is a Lipschitz domain then for every $\alpha \neq 1$ and $d \geq 2$ we have
\[
\int \frac{f^2(x)}{\delta^\alpha(x)} \, dx \leq C_H \mathcal{E}[f], \quad \forall f \in W_0^{\alpha/2,2}(\Omega), \tag{2.9}
\]
Combining the two inequalities yields (2.7).

3 The subcritical case

In this section we fix:
A positive potentials $V \in L^1_{loc}(\Omega)$, such that there is $\kappa \in (0,1)$, with
\[
\int f^2(x)V(x) \, dx \leq \kappa \mathcal{E}[f], \quad \forall f \in W_0^{\alpha/2,2}(\Omega). \tag{3.1}
\]
Having in mind that $0 < \kappa < 1$, we conclude that the quadratic form which we denote by $\mathcal{E}_V$ and which is defined by
\[
D(\mathcal{E}_V) = W_0^{\alpha/2,2}(\Omega), \quad \mathcal{E}_V[f] = \mathcal{E}[f] - \int f^2 V \, dx, \quad \forall f \in W_0^{\alpha/2,2}(\Omega), \tag{3.2}
\]
is closed in $L^2(\Omega, dx)$ and is even comparable to $\mathcal{E}$. Hence setting $L_V$ the positive selfadjoint operator associated to $\mathcal{E}_V$, we conclude that $L_V$ has purely discrete spectrum $0 < \lambda_0^V < \lambda_1^V < \cdots < \lambda_k^V \to \infty$, as well.
Furthermore the associated semigroup $e^{-tL_V}$, $t > 0$ is irreducible (it has a kernel which dominates the heat kernel of the free operator $L_0$). Thereby there is a unique $\varphi_0^V \in W_0^{\alpha/2,2}(\Omega)$ such that
\[
\|\varphi_0^V\|_{L^2} = 1, \quad \varphi_0^V > 0 \text{ q.e. and } L_V\varphi_0^V = \lambda_0^V \varphi_0^V. \tag{3.3}
\]
Two real-valued, measurable a.e. positive and essentially bounded functions $S$ and $F$ on $\mathbb{R}^d$ such that either $S \neq 0$ or $F \neq 0$. Let $w \in W_0^{\alpha/2,2}(\Omega)$, we say that $w$ is a solution of the equation
\[
L_V w = Sw + F, \tag{3.4}
\]
if
\[
\mathcal{E}_V(w, f) = \int fSw \, dx + \int fF, \quad \forall f \in W_0^{\alpha/2,2}(\Omega). \tag{3.5}
\]
In the goal of obtaining the precise behavior of the ground state, we proceed to transform the form $\mathcal{E}_V$ into a Dirichlet form on $L^2(\Omega, w^2 \, dx)$, where $w > 0$ q.e. is a solution of
the equation $L_V w = Sw + F$.
Let $Q^w$ be the $w$-transform of $L_V - S$, i.e., the quadratic form defined in $L^2(\Omega, w^2 dx)$ by

$$D(Q^w) := \{ f : w f \in W_0^{\alpha/2,2}(\Omega) \} \subset L^2(\Omega, w^2 dx),$$

$$Q^w[f] = \mathcal{E}_V^w[f] - \int w^2 f^2 S \, dx, \quad \forall f \in D(Q^w) \quad \text{where} \quad \mathcal{E}_V^w[f] = \mathcal{E}_V[w f].$$

**Lemma 3.1.** The form $Q^w$ is a regular Dirichlet form and

$$Q^w[f] = \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} w(x)w(y) \, dx \, dy + \int f^2 F w, \quad \forall f \in D(Q^w).$$

**Proof.** Obviously $Q^w$ is closed and densely defined as it is unitary equivalent to the closed densely defined form $\mathcal{E}_V^w$. Let us prove (3.8).

Writing

$$w(x)w(y)\left(\frac{g(x)}{w(x)} - \frac{g(y)}{w(y)}\right)^2 = (g(x) - g(y))^2 + g^2(x)\left(\frac{w(y) - w(x)}{w(x)}\right) + g^2(y)\left(\frac{w(x) - w(y)}{w(y)}\right),$$

(3.9)

and setting $g = w f$, we get

$$Q^w[f] = \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} w(x)w(y) \, dx \, dy + A(d, \alpha) \int \int \frac{w(x) - w(y)}{|x - y|^{d+\alpha}} f^2(x)w(x) \, dx \, dy$$

$$- \int f^2(x)w^2(x)V(x) \, dx$$

$$- \int f^2(x)w^2(x)S(x), \quad \forall f \in D(Q^w),$$

$$\geq A(d, \alpha) \int \int \frac{w(x) - w(y)}{|x - y|^{d+\alpha}} f^2(x)w(x) \, dx \, dy$$

$$- \int f^2(x)w^2(x)V(x) \, dx$$

$$- \int f^2(x)w^2(x)S(x) \, dx, \quad \forall f \in D(Q^w),$$

(3.10)

we derive in particular that the integral

$$\int \int \frac{w(x) - w(y)}{|x - y|^{d+\alpha}} f^2(x)w(x) \, dx \, dy$$

is finite.

(3.11)
Thus using Fubini’s together with dominated convergence theorem, we achieve

\[
Q^w[f] = \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} w(x)w(y) \, dx \, dy
+ A(d, \alpha) \int f^2(x)w(x) \left( \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{w(x) - w(y)}{|x-y|^{d+\alpha}} \, dy \right) \, dx
- \int f^2(x)w^2(x)V(x) \, dx
- \int f^2(x)w^2(x)S(x) \, dx \quad \forall \ f \in D(Q^w).
\]

(3.12)

Now, owing to the fact that \( w \) is a solution of the equation

\[
L_Vw = Sw + F,
\]

having (2.3) in hands and substituting in (3.12) we get formula (3.8) from which we read that \( Q^w \) is Markovian and hence a Dirichlet form.

**Regularity:** Relying on the expression (3.8) of \( Q^w \), we learn from [FOT94, Example 1.2.1.], that \( C^\infty_c(\Omega) \subset D(Q^w) \) if and only if

\[
J := \int \int_\Omega \frac{|x-y|^2}{|x-y|^{d-r}} w(x)w(y) \, dx \, dy < \infty.
\]

(3.14)

Set \( r' = 2 - \alpha \). Then \( 0 < r' < d \). We rewrite \( J \) as

\[
J := \int \int_\Omega \frac{w(x)w(y)}{|x-y|^{d-r}} \, dx \, dy \\
\leq \frac{1}{2} \int \int_\Omega \frac{w(x)^2 + w(y)^2}{|x-y|^{d-r}} \, dx \, dy \\
= \int \int_\Omega \frac{w(x)^2}{|x-y|^{d-r}} \, dx \, dy \\
= \int w(x)^2 \left( \int_\Omega \frac{dy}{|x-y|^{d-r}} \right) \, dx < \infty,
\]

with

\[
\sup_{x \in \Omega} \left( \int_\Omega \frac{dy}{|x-y|^{d-r}} \right) < \infty.
\]

Hence \( J \) is finite.

Hence from the Beurling–Deny–LeJan formula (see [FOT94, Theorem 3.2.1, p.108]) together with the identity (3.8), we learn that \( Q^w \) is regular, which completes the proof.

We designate by \( L^w \) the operator associated to \( Q^w \) in the weighted Lebesgue space \( L^2(\Omega, w^2 \, dx) \) and \( T^w_t, \ t > 0 \) its semigroup. Then

\[
L^w = w^{-1}(L_V - S)w \quad \text{and} \quad T^w_t = w^{-1}e^{-t(L_V - S)}w, \ t > 0.
\]

(3.15)
In the sequel set:

\[ C_0 = C_G \int \varphi_0(y)S(y)w(y) + C_G \int \varphi_0(y)F(y), \]

\[ r := \frac{d}{d - \alpha}, \ A := (C_0C_H + C_S)(1 + \lambda_0C_S|\Omega|^{1-1/r}) \quad \text{and} \quad q := \frac{2r - 1}{r}. \quad (3.16) \]

**Theorem 3.1.** For every \( f \in D(Q^w) \), we have

\[ (IS1) \quad \| f^2 \|_{L^q(w^2dx)} \leq A(Q^w[f] + \int Sf^2w^2). \]

The proof of Theorem 3.1 relies upon auxiliary results which we shall state in three lemmata.

**Lemma 3.2.** The following identity holds true

\[ \varphi_0^V = K(V\varphi_0^V) + \lambda_0^V K\varphi_0^V \ a.e., \quad (3.17) \]

where

\[ K\varphi := \int G(., y)\varphi(y) \, dy. \]

**Proof.** Set

\[ u = \varphi_0^V - K(V\varphi_0^V) - \lambda_0^V K\varphi_0^V. \quad (3.18) \]

Owing to the fact that \( \varphi_0^V \) lies in \( W_0^{\alpha/2,2}(\Omega) \) and hence lies in \( L^2(Vdx) \), we obtain that the measure \( \varphi_0^V V \) has finite energy integral with respect to the Dirichlet form \( E_\Omega \), i.e.,

\[ \int |f\varphi_0^V V| \, dx \leq \gamma(E[f])^{1/2}, \ \forall \ f \in C^\infty_c(\Omega), \quad (3.19) \]

and therefore \( K(V\varphi_0^V) \in W_0^{\alpha/2,2}(\Omega) \). Thus \( u \in W_0^{\alpha/2,2}(\Omega) \) and satisfies the identity

\[ E(u, g) = E(\varphi_0^V, g) - \int \varphi_0^V gV \, dx - \lambda_0^V \int \varphi_0^V g \, dx = E_V(\varphi_0^V, g) - \lambda_0^V \int \varphi_0^V g \, dx = 0, \ \forall \ g \in W_0^{\alpha/2,2}(\Omega). \quad (3.20) \]

Since \( E \) is positive definite we conclude that \( u = 0 \ a.e. \), which yields the result. \( \square \)
Lemma 3.3. Let $w$ be as in Theorem 3.1. Then the following inequality holds true

$$w \geq C_0 \varphi_0 \text{ q.e.} \quad (3.21)$$

where

$$C_0 = C_G \int \varphi_0(y)S(y)w(y) + C_G \int \varphi_0(y)F(y).$$

Proof. As in the proof of Lemma 3.2 we show that $w$ satisfies

$$w - KVw = KSw + KF,$$

We also recall the known fact that since $T_t = e^{-L_0t}$ is IUC (see [Kul98]), then there is a finite constant $C_G$, such that

$$G(x, y) \geq C_G \varphi_0(x)\varphi_0(y), \quad (3.22)$$

yielding,

$$w \geq KSw + KF \geq C_G \varphi_0 \int \varphi_0(y)S(y)w(y) + C_G \varphi_0 \int \varphi_0(y)F(y) \text{ q.e.}.$$

and

$$C_0 = C_G \int \varphi_0(y)S(y)w(y) + C_G \int \varphi_0(y)F(y)$$

Lemma 3.4. We have,

$$\int f^2 \leq C_0 C_H \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}} w(x)w(y) dxdy + C_0 C_H \lambda_0 \int w^2 f^2, \forall f \in D(Q^w). \quad (3.23)$$

Proof. At this stage we use Hardy’s inequality (2.7), which states that there is a constant $C_H > 0$ such that

$$\int \frac{u^2}{\varphi_0^2} dx \leq C_H \frac{A(d, \alpha)}{2} \int \int \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} dxdy, \forall u \in W^{n/2,2}_0(\Omega). \quad (3.24)$$

Let $f \in D(Q^w) \subset D(Q_0^w)$. Taking $u = f \varphi_0$ in inequality (3.24) yields

$$\int f^2 = \int \frac{f^2 \varphi_0^2}{\varphi_0^2} \leq C_H \frac{A(d, \alpha)}{2} \int \int \frac{(f \varphi_0(x) - f \varphi_0(y))^2}{|x - y|^{n+\alpha}} dxdy$$

$$= C_H \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}} \varphi_0(x)\varphi_0(y) dxdy$$

$$+ C_H \frac{A(d, \alpha)}{2} \int \int \frac{(\varphi_0(x) - \varphi_0(y))(f^2 \varphi_0(x) - f^2 \varphi_0(y))}{|x - y|^{n+\alpha}} dxdy. \quad (3.25)$$
Thanks to the fact that $\varphi_0$ is an eigenfunction associated to $\lambda_0$, we achieve
\[
\frac{A(d, \alpha)}{2} \int \int \frac{(\varphi_0(x) - \varphi_0(y))(f^2 \varphi_0(x) - f^2 \varphi_0(y))}{|x - y|^{n+\alpha}} dxdy = \lambda_0 \int f^2 \varphi_0^2.
\] (3.26)
Combining (3.26) with (3.25) we obtain
\[
\int f^2 \leq C_H \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}} \varphi_0(x) \varphi_0(y) dxdy
+ C_H \lambda_0 \int (\varphi_0)^2 f^2, \ \forall \ f \in D(Q^w).
\] (3.27)
Having the lower bound for $w$ given by Lemma 3.3 in hand, we establish
\[
\int f^2 \leq C_H C_0 \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}} w(x)w(y) dxdy
+ C_H C_0 \lambda_0 \int w^2 f^2, \ \forall \ f \in D(Q^w).
\] (3.28)

\textbf{Lemma 3.5.} Set
\[
\Lambda_1 = 1 + \frac{C_H C_0}{2}, \ \Lambda_2 = \frac{\|F\|_\infty^2}{2} + \frac{C_H C_0 \lambda_0}{2},
\] (3.29)
$C_0$ being the constant appearing in Lemma 3.3. Then
\[
Q^w[f] \leq \Lambda_1 \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}} w(x)w(y) dxdy + \Lambda_2 \int f^2 w^2, \ \forall \ f \in D(Q^w).
\] (3.30)
\textbf{Proof.} We have already established that
\[
Q^w[f] = \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}} w(x)w(y) dxdy + \int f^2 Fw, \ \forall \ f \in D(Q^w).
\] (3.31)
Making use of Hölder’s and Young’s inequality together with inequality (3.23) we obtain
\[
Q^w[f] \leq \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}} w(x)w(y) dxdy + \left( \int f^2 \right)^\frac{1}{2} \left( \int f^2 F^2 w^2 \right)^\frac{1}{2}
\leq \Lambda_1 \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}} w(x)w(y) dxdy + \Lambda_2 \int f^2 w^2, \ \forall \ f \in D(Q^w),
\]
which finishes the proof.

\[\square\]
Proof. of Theorem 3.1. We observe first that

\[ Q^w[f] + \int S f^2 w^2 = \mathcal{E}^w_V[f] := \mathcal{E}_V[w f], \quad \forall f \in D(Q^w). \quad (3.32) \]

By Hölder’s inequality, we get for every \( f \in D(Q^w) \),

\[ \int w^2 f^{2(2-1/r)} \leq \left( \int w^{2r} f^{2r} \right)^{1/r} \left( \int f^2 \right)^{1-1/r} \quad (3.33) \]

Using that \( \mathcal{E} \) and \( \mathcal{E}_V \) are equivalent, and by the Sobolev inequality 2.5, then there exists a finite constant positive \( C_S \) and \( r := \frac{d}{d-\alpha} > 1 \) such that

\[ \left( \int g^{2r} \right)^{1/r} \leq C_S \mathcal{E}_V[g], \quad \text{for all } g \in W^{\alpha/2,2}(\Omega). \quad (3.34) \]

Taking \( g = w f \), we have

\[ \left( \int w^{2r} f^{2r} \right)^{1/r} \leq C_S \mathcal{E}^w_V[f]. \quad (3.35) \]

On the other hand we have, according to Lemma 3.4

\[ \int f^2 \leq C' \left( \frac{A(d, \alpha)}{2} \right) \int \int \frac{(f(x) - f(y))^2}{|x-y|^{n+\alpha}} w(x) w(y) dxdy + \lambda_0 \int w^2 f^2. \quad (3.36) \]

Applying another time Hölder’s inequality we get

\[ \int (f w)^2 \leq |\Omega|^{1-1/r} ||(f w)^2||_{L^r} \leq C_S |\Omega|^{1-1/r} \mathcal{E}^w_V[f], \quad \forall f \in D(Q^w). \quad (3.37) \]

Recalling that \( \mathcal{E}^w_V[f] \geq \frac{A(d, \alpha)}{2} \int \int \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}} w(x) w(y) dxdy \), we achieve

\[ \int f^2 \leq C_H C_0 (1 + \lambda_0 C_S |\Omega|^{1-1/r}) \mathcal{E}^w_V[f], \quad \forall f \in D(Q^w). \quad (3.38) \]

Combining (3.33), (3.34) and (3.38), we get (IS1).

For every \( t > 0 \) we designate by \( T^w_t \) the semigroup associated to the form \( Q^w \) in the space \( L^2(w^2 dx) \). We are yet ready to prove the ultracontractivity of \( T^w_t \).

Set

\[ s := \frac{2}{q-1} := \frac{2r}{r-1}. \quad (3.39) \]

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Theorem 3.2. Then $T^w_t$ is ultracontractive for every $t > 0$ and there exists $C_1 > 0$ depends only on $A$ and $s$ such that

$$
\|T^w_t\|_{L^1(w^2dx),L^\infty} \leq C_1 t^{-s/2} e^{t\|S\|_\infty}, \forall t > 0.
$$

(3.40)

Proof. From Theorem 3.1, we derive

$$
\| f^2 \|_{L^r(w^2dx)} \leq A(Q^w[f] + \|S\|_\infty \int_\Omega f^2 w^2 dx), \forall f \in D(Q^w).
$$

(3.41)

Since $Q^w$ is a Dirichlet form, it is known that a Sobolev embedding for the domain of a Dirichlet form yields the ultracontractivity of the related semigroup (see [SC02, Theorems 4.1.2,4.1.3]), which ends the proof.

We shall apply Theorem 3.1 to the special cases $V = 0, F = 1$

Theorem 3.3. Let $\xi^V := L^{-1}_t 1$, then

$$
\varphi^V_0 \leq C(V,t)\xi^V, \text{ a.e. } \forall t > 0,
$$

where

$$
C(V,t) := C_1 t^{-s/2} e^{t\lambda_0} \forall t > 0.
$$

(3.43)

Proof. Applying Theorem 3.1 to the case $V = 0, F = 1$, we get $w = \xi^V$, and it yields that the semi-group $T^\xi^V_t$ is ultracontractive and $\frac{\varphi^V}{\xi^V}$ is an eigenfunction for $T^\xi^V_t$ associated to the eigenvalue $e^{-t\lambda_0}$, \forall t > 0. Thus

$$
\frac{\|\varphi^V_0\|_{\xi^V}}{\|\varphi^V_0\|_{\xi^V}} \leq e^{t\lambda_0} \|T^\xi^V_t\|_{L^2((\xi^V)^2dm),L^\infty} \leq C_1 t^{-s/2} e^{t\lambda_0}, \forall t > 0,
$$

(3.44)

and

$$
\varphi^V_0 \leq C_1 t^{-s/2} e^{t\lambda_0} \xi^V, \text{ a.e. } \forall t > 0,
$$

(3.45)

which was to be proved.

While for the upper pointwise estimate we exploited the idea of intrinsic ultracontractivity, for the reversed estimate we shall however, make use of Moser’s iteration technique.
Theorem 3.4. For every $t > 0$, the following estimate holds true

$$\xi^V \leq (AC_0C_HC^2(V, t) + 2\lambda_0^V)\varphi_0^V, \ a.e., \ (3.46)$$

where

$$C(V, t) := C_1t^{-s/2}e^{t\lambda_0}\forall \ t > 0. \ (3.47)$$

For the proof we establish the following lemma:

Lemma 3.6. Assume that $V \in L^\infty(\Omega)$, then (3.46) holds true.

Proof. Step 1: Iteration formula

We claim that, there exists $q > 1$ such that for all $j \geq 1$, we have

$$\left(\int \rho^{2j}(\varphi_0^V)^2 \ dx\right)^{\frac{1}{j}} \leq (ACC^2(V, t) + 2\lambda_0^V)j^2 \int \rho^{2j}(\varphi_0^V)^2 \ dx. \ (3.48)$$

Consider the family of smooth domains

$$\Omega_\epsilon = \{x \in \Omega / \text{dist}(x, \partial \Omega) > \epsilon\}.$$ 

Let $\xi_\epsilon^V \in W^{\alpha/2, 2}((\Omega_\epsilon))$ be the solution of $L_\epsilon^V \xi_\epsilon^V = 1$ in $\Omega_\epsilon$. Since $V \in L^\infty$, then by [BBB013, Lemma 4.4], the function $\xi_\epsilon^V \in W^{\alpha/2, 2}((\Omega_\epsilon) \cap L^\infty(\Omega_\epsilon))$, increasing and convergent. We assume that $\xi_\epsilon^V \rightharpoonup u$ as $\epsilon \to 0$ (uniformly, $\xi_\epsilon^V \in C^{\alpha/2}(\Omega_\epsilon)$). On the other hand, we have $\xi_\epsilon^V = K(V \xi_\epsilon^V) + K(1_{\Omega_\epsilon})$ converge to $u = K(Vu) + K(1)$, thus $L_\epsilon^Vu = 1$ and by unicity of solution we have $u = \xi^V$.

Letting

$$\rho_\epsilon := \frac{\xi_\epsilon^V}{\varphi_0^V}, \ (3.49)$$

Since $\varphi_0^V > 0$ in $\Omega$ there exists $C_\epsilon > 0$ such that $\varphi_0^V > C_\epsilon$ in $\Omega_\epsilon$, it follows that $\rho_\epsilon \in W^{\alpha/2, 2}(\Omega_\epsilon) \cap L^\infty(\Omega_\epsilon)$ and

$$\rho_\epsilon \rightharpoonup \rho := \frac{\xi_\epsilon^V}{\varphi_0^V}.$$ 

Now using

$$w(x)w(y)\left(\frac{g_1(x)}{w(x)} - \frac{g_1(y)}{w(y)}\right)\left(\frac{g_2(x)}{w(x)} - \frac{g_2(y)}{w(y)}\right) = (g_1(x) - g_1(y))(g_2(x) - g_2(y))$$

$$-(w(x) - w(y))\left[\frac{g_1(x)g_2(x)}{w(x)} - \frac{g_1(x)g_2(x)}{w(x)}\right],$$

with the equations satisfied by the ground state $\varphi_0^V$ and the function $\xi_\epsilon^V$, setting $g_1 = \varphi_0^V f$, $g_2 = \xi_\epsilon^V$ and $w = \varphi_0^V$, we find, for every $f \in W^{\alpha/2, 2}(\Omega_\epsilon) \cap L^\infty(\Omega_\epsilon)$,
According to Theorem 3.3, we obtain

\[
\frac{A(d, \alpha)}{2} \int \frac{(f(x) - f(y))(\rho_\epsilon(x) - \rho_\epsilon(y))}{|x - y|^{d+\alpha}} \varphi_0^V(x)\varphi_0^V(y) \, dx \, dy =
\]

\[
\frac{A(d, \alpha)}{2} \int \frac{(\varphi_0^V f(x) - \varphi_0^V f(y))(\xi_\epsilon^V(x) - \xi_\epsilon^V(y))}{|x - y|^{d+\alpha}} \, dx \, dy
\]

\[
- \frac{A(d, \alpha)}{2} \int \frac{(\xi_\epsilon^V f(x) - \xi_\epsilon^V f(y))(\varphi_0^V(x) - \varphi_0^V(y))}{|x - y|^{d+\alpha}} \, dx \, dy
\]

\[
= \frac{A(d, \alpha)}{2} \int \varphi_0^V f(x) \left( \lim_{\epsilon \to 0} \int_{|x - y| > \epsilon} \frac{\xi_\epsilon^V(x) - \xi_\epsilon^V(y)}{|x - y|^{d+\alpha}} \, dy \right) \, dx - \int f \xi_\epsilon^V \varphi_0^V V(x) \, dx
\]

\[
- \left( \frac{A(d, \alpha)}{2} \int \xi_\epsilon^V f(x) \left( \lim_{\epsilon \to 0} \int_{|x - y| > \epsilon} \frac{\varphi_0^V(x) - \varphi_0^V(y)}{|x - y|^{d+\alpha}} \, dy \right) \, dx - \int f \xi_\epsilon^V \varphi_0^V V(x) \, dx \right)
\]

\[
= \int \varphi_0^V f(x) \, dx - \lambda_\epsilon^V \int \xi_\epsilon^V \varphi_0^V f(x) \, dx. \tag{3.50}
\]

Testing the latter equation with \( f = \rho_\epsilon^{2j-1}, \ j \geq 1, \ (f \in W^{\alpha/2,2}(\Omega_\epsilon) \cap L^\infty(\Omega_\epsilon)) \), we deduce

\[
\frac{A(d, \alpha)}{2} \int \frac{(\rho_\epsilon^{2j-1}(x) - \rho_\epsilon^{2j-1}(y))(\rho_\epsilon(x) - \rho_\epsilon(y))}{|x - y|^{d+\alpha}} \varphi_0^V(x)\varphi_0^V(y) \, dx \, dy =
\]

\[
\int \rho_\epsilon^{2j-1}(x)\varphi_0^V(x) \, dx - \lambda_\epsilon^V \int \xi_\epsilon^V \varphi_0^V(x) \, dx. \tag{3.51}
\]

Using that for all \( a, b \geq 0 \) and \( j \geq 1 \) we have

\[
(a^j - b^j)^2 = (a - b)^2(a^{j-1} + a^{j-2}b + a^{j-3}b^2 + \ldots + b^{j-1})^2
\]

\[
\leq j(a - b)^2(a^{2j-2} + a^{2j-3}b + a^{2j-4}b^2 + \ldots + b^{2j-2})
\]

\[
\leq j(a - b)^2(a^{2j-2} + a^{2j-3}b + a^{2j-4}b^2 + \ldots + b^{2j-2})
\]

\[
= j(a - b)(a^{2j-1} - b^{2j-1}) \tag{3.52}
\]

which yields, using (3.51) and (3.52)

\[
\frac{A(d, \alpha)}{2} \int \frac{(\rho_\epsilon^j(x) - \rho_\epsilon^j(y))^2}{|x - y|^{d+\alpha}} \varphi_0^V(x)\varphi_0^V(y) \, dx \, dy \leq
\]

\[
j \int \rho_\epsilon^{2j-1}(x)\varphi_0^V(x) \, dx - \lambda_\epsilon^V \int \xi_\epsilon^V \varphi_0^V(x) \, dx \leq \int j \rho_\epsilon^{2j-1}(x)\varphi_0^V(x) \, dx. \tag{3.53}
\]

According to Theorem 3.3, we obtain

\[
\frac{A(d, \alpha)}{2} \int \frac{(\rho_\epsilon^j(x) - \rho_\epsilon^j(y))^2}{|x - y|^{d+\alpha}} \varphi_0^V(x)\varphi_0^V(y) \, dx \, dy \leq C(V, t) \int j \rho_\epsilon^{2j-1}(x)\varphi_0^V(x) \, dx. \tag{3.54}
\]
Using Hölder inequality and Lemma 3.4 (with $S = \lambda_0^V$, $F = 0$, $w = \varphi_0^V$ and $f = \rho_t^j$), it follows from (3.54) that

$$\frac{A(d, \alpha)}{2} \int \int \frac{(\rho_t^j(x) - \rho_t^j(y))^2}{|x - y|^{d+\alpha}} \varphi_0^V(x) \varphi_0^V(y) \, dx \, dy \leq$$

$$C(V, t) j \left( \int (\varphi_0^V)^2 \rho_t^{2j-2} \varphi_0^V \, dx \right)^{\frac{1}{2}} \left( \int \rho_t^{2j} \, dx \right)^{\frac{1}{2}} \leq$$

$$C^{1/2} C(V, t) j \left( \int (\varphi_0^V)^2 \rho_t^{2j-2} \, dx \right)^{\frac{1}{2}}$$

$$\times \left( \frac{A(d, \alpha)}{2} \int \int \frac{(\rho_t^j(x) - \rho_t^j(y))^2}{|x - y|^{d+\alpha}} \varphi_0^V(x) \varphi_0^V(y) \, dx \, dy + \lambda_0^V \int (\varphi_0^V)^2 \rho_t^{2j} \, dx \right)^{\frac{1}{2}}. \quad (3.55)$$

By Young’s inequality, we obtain

$$\frac{A(d, \alpha)}{2} \int \int \frac{(\rho_t^j(x) - \rho_t^j(y))^2}{|x - y|^{d+\alpha}} \varphi_0^V(x) \varphi_0^V(y) \, dx \, dy \leq$$

$$\frac{1}{2} C C^2 (V, t) j^2 \int (\varphi_0^V)^2 \rho_t^{2j-2} \, dx + \frac{\lambda_0(V)}{2} \int (\varphi_0^V)^2 \rho_t^{2j}$$

$$= \frac{1}{2} \frac{A(d, \alpha)}{2} \int \int \frac{(\rho_t^j(x) - \rho_t^j(y))^2}{|x - y|^{d+\alpha}} \varphi_0^V(x) \varphi_0^V(y) \, dx \, dy, \quad (3.56)$$

so that

$$\frac{A(d, \alpha)}{2} \int \int \frac{(\rho_t^j(x) - \rho_t^j(y))^2}{|x - y|^{d+\alpha}} \varphi_0^V(x) \varphi_0^V(y) \, dx \, dy \leq$$

$$C C^2 (V, t) j^2 \int (\varphi_0^V)^2 \rho_t^{2j-2} \, dx + \lambda_0(V) \int (\varphi_0^V)^2 \rho_t^{2j}. \quad (3.57)$$

By Theorem (3.1), with $S = \lambda_0^V$, $F = 0$, $w = \varphi_0^V$ and $f = \rho_t^j$, we get from (IS1)

$$\| \rho_t^{2j} \|_{L^2((\varphi_0^V)^2 \, dx)} \leq A \frac{(A(d, \alpha))}{2} \int \int \frac{(\rho_t^j(x) - \rho_t^j(y))^2}{|x - y|^{d+\alpha}} \varphi_0^V(x) \varphi_0^V(y) \, dx \, dy$$

$$+ \lambda_0(V) \int \rho_t^{2j} (\varphi_0^V)^2 \, dx, \quad (3.58)$$

using (3.57),

$$\| \rho_t^{2j} \|_{L^2((\varphi_0^V)^2 \, dx)} \leq A C C^2 (V, t) j^2 \int (\varphi_0^V)^2 \rho_t^{2j-2} \, dx$$

$$+ 2 \lambda_0^V j^2 \int (\varphi_0^V)^2 \rho_t^{2j} \, dx. \quad (3.59)$$
Thus

\[
\left( \int \rho_{\epsilon}^{2j}(\varphi_0^V)^2 \, dx \right)^{\frac{1}{2}} \leq \text{ACC}^2(V, t)j^2 \int (\varphi_0^V)^2 \rho_{\epsilon}^{2j-2} \rho^2 \, dx + 2\lambda_0^V j^2 \int (\varphi_0^V)^2 \rho_{\epsilon}^{2j} \, dx. \tag{3.60}
\]

It is then easy to pass to the limit as \( \epsilon \to 0 \), using e.g. monotone convergence to obtain (3.48).

**Step 2** we show when \( V \) is bounded that

\[
\xi^V \leq M(V, t)\varphi_0^V, \quad \forall \, t > 0, \tag{3.61}
\]

iterate (3.48), define \( j_k = 2^{q_k} \) for \( k = 0, 1, \ldots \) and

\[
\Theta_k = \left( \int \rho_{\epsilon}^{2j_k}(\varphi_0^V)^2 \, dx \right)^{\frac{1}{2}} \quad \text{and} \quad M(V, t) := (\text{ACC}^2(V, t) + 2\lambda_0^V).
\]

Then (3.48) can be written as

\[
\Theta_{k+1} \leq \left( M(V, t)(q^{2k})^2 \right)^{\frac{1}{2}} \Theta_k. \tag{3.63}
\]

Using this recursively yields

\[
\Theta_k \leq M(V, t)\Theta_0 = M(V, t)\left( \int \xi_0^V \right)^{\frac{1}{2}} \leq M(V, t), \tag{3.64}
\]

for all \( k = 0, 1, \ldots \). Since the right-hand-side of the latter inequality is independent from \( k \), we deduce

\[
\lim_{k \to \infty} \Theta_k = \sup \Omega \rho \leq M(V, t), \tag{3.65}
\]

and this shows (3.61).

**Proof of Theorem 4.3.4.** Let

\[
V_k(x) := \min(V, k), \quad k > 0.
\]

Then \( L_k := L_0 - V_k \) increases in the strong resolvent sense to \( L_V \). Since \( L_V \) has compact resolvent, the latter convergence is even uniform (see [BAB11] Lemma 2.5]). Thus setting \( \lambda_0^{(k)} \)'s the ground state energy of the \( L_k \)'s, \( \varphi_0^{(k)} \) its associated ground state and \( \xi_0^{(k)} := L_k^{-1} \)

we obtain

\[
\lambda_0^{(k)} \to \lambda_0, \quad \varphi_0^{(k)} \to \varphi_0^V \quad \text{and} \quad \xi_0^{(k)} \to \xi^V \quad \text{in} \quad L^2(\Omega, dx). \tag{3.66}
\]

Using Lemma 4.3.6, 4.3.3, 4.3.4 and Theorem 4.3.2, it easy to be prove that

\[
\lim_{k \to \infty} M(V, t) = M(V, t) \in (0, \infty) \quad \text{and} \quad \xi^{(k)} \leq M(V, t)\varphi_0^{(k)}, \quad \text{a.e.} \quad \forall \, k \, \text{large}. \tag{3.67}
\]

Then \( \xi^V \leq M(V, t)\varphi_0^V \), a.e., which was to be proved. \( \square \)
4 The critical case

The critical case differs in some respects from the subcritical one. The most apparent difference is that the critical quadratic form is no longer closed on the starting fractional Sobolev space $W_0^{\alpha/2,2}(\Omega)$. Consequently the proof of Lemma 3.2 is no more valid to express the ground state for the simple reason that it may not belong to $W_0^{\alpha/2,2}(\Omega)$. (See [BBB13]) We shall however prove that the critical form is closable and has compact resolvent by mean of a Doob’s transformation. An approximation process will then lead to extend the identity of Lemma 3.2 helping therefore to get the sharp estimate of the ground state.

The development of this section depends heavily on the following improved Sobolev inequality holds true: there is a finite constant $C > 0$ and $r > 1$ such that

\[
(IS) : \quad \| f^2 \|_{L^r} \leq C S (\mathcal{E}[f] - \int V_* f^2(x) \, dx), \quad \forall f \in C^1_c(\Omega). \tag{4.1}
\]

**Remark 4.1.** We observe that if $d \geq 3$ the potentials (1.4) and (1.3) satisfy (4.1) with $r := \frac{d}{d-\alpha}$ if $0 < c < c^*$ and with any $1 < r < \frac{d}{d-\alpha}$ for $c = c^*$. (See [FLS08, FMT13])

Hence we only consider solutions that belong to the hilbert space $H$, defined as the completion of $C^\infty_c(\Omega)$ with respect to the norm

\[
\| f \|_H^2 = \mathcal{E}_\Omega[f] - \int V_* f^2(x) \, dx.
\]

We denote by $H'$ the dual of $H$. Observe that $W_0^{\alpha/2,2}(\Omega) \subset H \subset L^2(\Omega)$.

If $F \in H'$ we say that $f \in H$ is solution of

\[
L_{V_*} f = (L_0 - V_*) f = F \tag{4.2}
\]

if

\[
\mathcal{E}_{V_*}(f, g) = \int_\Omega F g \, dx, \quad \text{for all } g \in H.
\]

**Lemma 4.1.** Suppose (4.1) and let $F \in H'$. Then there exists a unique solution $f \in H$ which is a solution of (4.3), and if $F \geq 0$ in the sense of distributions then $f \geq 0$ a.e.

**Proof.** We can assume that $F > 0$. It follows from Lax-Milgram lemma that there exists a unique $f \in H$ such that

\[
\mathcal{E}_{V_*}(f, g) = \int_\Omega F g \, dx, \quad \forall g \in H.
\]

We now show that $f \geq 0$. By definition of $H$, there exists $f_k$ in $C^\infty_c(\Omega)$ converging to $f$ in $H$. Letting $F_k = (-\Delta)^{\frac{\alpha}{2}} f_k - V_* f_k$, it follows that $F_k \in H'$ and $F_k \to F$ in $H'$. 

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Then by [FLS08, Lemma 3.3] \( f_k \in W^{\alpha/2,2}_0(\Omega) \), yielding that \( f_k^- \in W^{\alpha/2,2}_0(\Omega) \).
Activating Sobolev inequality (4.1) together with identity (2.3) and utilizing the fact that \( F_k = (-\Delta)^{\frac{\alpha}{2}} f_k - V_* f_k \), we obtain:

\[
\| (f_k^-)^2 \|_H = \left( \frac{1}{2} A(d, \alpha) \right) \int \int \frac{(f_k^-(x) - f_k^-(y))^2}{|x - y|^{d+\alpha}} \, dx \, dy - \int V_*(x)(f_k^-)^2(x) \, dx \\
\leq - \left( \frac{1}{2} A(d, \alpha) \right) \int \int \frac{(f_k^-(x) - f_k(0))(f_k^-)(x) - f_k^-(y))}{|x - y|^{d+\alpha}} \, dx \, dy \\
- \int V_*(x)f_k(x)(f_k^-)(x) \, dx \\
= - \mathcal{E}(f, f^-) \\
= - \int F_k f_k^-(x) \, dx \leq 0. \tag{4.3}
\]

In the ’passage’ from the first to the second inequality, we used the fact that for any Dirichlet form \( D \) one has \( D(f^+, f^-) \leq 0 \) (See [MR92, Theorem 4.4-i]), whereas the equality before the last one is obtained with the help of the identity (2.3).
To pass to the limit in the last equation, we just need to prove that \( f_k^- \) remains bounded in \( H \).

\[
\| (f_k^-)^2 \|_H^2 = \frac{1}{2} A(d, \alpha) \int \int \frac{(f_k^-(x) - f_k^-(y))^2}{|x - y|^{d+\alpha}} \, dx \, dy - \int V_*(x)(f_k^-)^2(x) \, dx \\
= \frac{1}{2} A(d, \alpha) \int \int \frac{(f_k^-(x) - f_k^-(y))^2}{|x - y|^{d+\alpha}} \, dx \, dy - \int V_*(x)(f_k^2(x) \, dx \\
+ \int V(x)(f_k^2)^2(x) \, dx \\
\geq \frac{1}{2} A(d, \alpha) \int \int \frac{(f_k^-(x) - f_k^-(y))^2}{|x - y|^{d+\alpha}} \, dx \, dy - \int V_*(x)(f_k^2(x) \, dx \\
+ \frac{1}{2} A(d, \alpha) \int \int \frac{(f_k^+(x) - f_k^+(y))^2}{|x - y|^{d+\alpha}} \, dx \, dy \\
= \frac{1}{2} A(d, \alpha) \int \int \frac{(f_k(x) - f_k(y))^2}{|x - y|^{d+\alpha}} \, dx \, dy - \int V_*(x)(f_k^2(x) \, dx \\
+ 2\mathcal{E}(f_k^-, f_k^+) \\
\leq \| (f_k)^2 \|_H^2. \tag{4.4}
\]

Letting \( k \to \infty \) in (4.3), we get \( f^- \equiv 0 \) in \( \Omega \) yielding \( f \geq 0 \).
Let $\dot{E}_*$ be the quadratic form defined by
\begin{align*}
F := D(\dot{E}_*) &= \{ f \in H, L_V f \in L^2(\Omega) \}, \\
\dot{E}_*[f] &= \mathcal{E}[f] - \int V_* f^2 \, dx, \quad \forall f \in \mathcal{F}. 
\end{align*}
(4.5)
(4.6)

Let $S$ and $F$ be two real-valued, measurable a.e. positive and essentially bounded functions on $\mathbb{R}^d$. Let $w_* \in H$ be solution of
\begin{equation}
L_* w_* = Sw_* + F. 
\end{equation}
(4.7)

**Lemma 4.2.** There is a finite constant $\tilde{C}_0$ such that
\begin{equation}
w_* \geq \tilde{C}_0 \varphi_0 \text{ a.e.} 
\end{equation}
(4.8)

**Proof.** Let $V_k(x) = \min(V_*(x), k), k > 0,$

$w_k$ is a solution of (4.7) with the potential $V_*(x)$ replaced by the potential $V_k(x)$, we obtain
\[(\Delta)^{\alpha/2} w_k, w_k)_{L^2(\Omega)} \leq \|w_k\|_H + \|V_k\|_{L^\infty(\Omega)} (w_k, w_k)_{L^2(\Omega)} < \infty.
\]

Since $(w_k)_k$ is bounded in $W_0^{\alpha/2, 2}(\Omega)$ and nondecreasing in $k$, it converges to $w$ in $L^2(\Omega)$ and that $w_k$ remains bounded in $H$ so that $w \in H$ and $w = w_*$. On the other hand by Lemma 3.3 it yields
\[w_k \geq C_G \varphi_0 \int \varphi_0(y) S(y) w_k(y) + C_G \varphi_0 \int \varphi_0(y) F(y) \text{ q.e.}
\]

We note that here all the integrals are finite, and that we can pass to the limit in the equation satisfied by $w_k$ and conclude that
\[w_* \geq C_G \varphi_0 \int \varphi_0(y) S(y) w_*(y) + C_G \varphi_0 \int \varphi_0(y) F(y) \text{ q.e.},
\]

\[\square\]

By analogy to the subcritical case we define the $w_*$-transform of $\dot{E}_*$ which we denote by $\dot{Q}_*$ and is defined by
\begin{align*}
D(\dot{Q}_*) &:= \{ f : w_* f \in \mathcal{F} \} \subset L^2(\Omega, w_*^2 \, dx), \\
\dot{Q}_*[f] &= \dot{E}_*[w_* f] - \int w_*^2 f^2 S \, dx, \quad \forall f \in D(\dot{Q}_*). 
\end{align*}
(4.9)

Following the computations made in the proof of Lemma 3.3 we realize that $\dot{Q}_*$ has the following representation
\begin{align*}
\dot{Q}_*[f] &= \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} w_*(x) w_*(y) \, dx \, dy \\
&\quad + \int f^2 F \, w_* \, dx, \quad \forall f \in D(\dot{Q}_*). 
\end{align*}
(4.10)
Lemma 4.3. The form $\dot{\mathcal{Q}}_*$ is closable in $L^2(\Omega, w_2^*dx)$. Furthermore its closure is a Dirichlet form in $L^2(\Omega, w_2^*dm)$. It follows, in particular that $\dot{\mathcal{E}}_*$ is closable.

Proof. We first mention that since $\dot{\mathcal{E}}_*$ is densely defined then $\dot{\mathcal{Q}}_*$ is densely defined as well. Now we proceed to show that $\dot{\mathcal{Q}}_*$ possesses a closed extension. To that end we introduce the form $\tilde{\mathcal{Q}}$ defined by

$$D(\tilde{\mathcal{Q}}) := \{ f : f \in L^2(\Omega, w_2^*dx), \tilde{\mathcal{Q}}[f] < \infty \}$$

$$\tilde{\mathcal{Q}}[f] = \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} w_*(x)w_*(y) \, dx \, dy$$

$$+ \int f^2 Fw_* \, dx, \quad \forall f \in D(\tilde{\mathcal{Q}}). \quad (4.11)$$

Arguing as in the proof of Lemma 3.1 we obtain that $C^\infty_c(\Omega) \subset D(\tilde{\mathcal{Q}})$. Hence from the Beurling–Deny–LeJan formula (see [FOT94, Theorem 3.2.1, p.108]), the form $\tilde{\mathcal{Q}}$ is the restriction to $C^\infty_c(\Omega)$ of a Dirichlet form and therefore closable and Markovian. Since $D(\dot{\mathcal{Q}}_*) \subset D(\tilde{\mathcal{Q}})$ we conclude that $\dot{\mathcal{Q}}_*$ is closable and Markovian as well, yielding that its closure is a Dirichlet form. Now the closability of $\dot{\mathcal{E}}_*$ is an immediate consequence of the closability of $\dot{\mathcal{Q}}_*$ which finishes the proof.

From now on we set $\mathcal{E}_*$ the closure of $\dot{\mathcal{E}}_*$ and $L_*$ the selfadjoint operator related to $\mathcal{E}_*$, respectively $Q_*$ the closure of $\dot{\mathcal{Q}}_*$ and $H_*$ its related selfadjoint operator. Finally $T^*_t := e^{-tL_*}, \ t > 0$ and $S_t := e^{-tH_*}, \ t > 0$. Obviously $H_* = w_*^{-1}L_*w_*$.

Of course the inequality (4.1) extends to the elements of $D(\mathcal{E}_*)$ with $\dot{\mathcal{E}}_*$ replaced by $\mathcal{E}_*$. The idea of using improved Sobolev type inequality to get estimates for the ground state was already used in [BBB013, DD03].

Theorem 4.1. For every $t > 0$, the operator $S_t$ is ultracontractive. It follows that

i) The operators $S_t, \ t > 0$ and hence $T^*_t, \ t > 0$ are Hilbert-Schmidt operators and the operator $L_*$ has a compact resolvent.

ii) $\ker(L_* - \lambda_0^*) = \mathbb{R}\varphi_0^*$ with $\varphi_0^* > 0 \ a.e.$

iii) If $\Omega$ satisfies the uniform interior ball condition then

$$\varphi_0^*(x) \geq (C_G\lambda_0^* \int \varphi_0(y)^{\alpha/2} \varphi_0^*(y) \, dy) \varphi_0(x)^{\alpha/2}, \ a.e. \quad (4.12)$$

Proof. The proof that $S_t, \ t > 0$ is ultracontractive runs as the one corresponding to the subcritical case with the help of Lemma 4.3 and inequality (4.1) as main ingredient.

i) Every ultracontractive operator has an almost everywhere bounded kernel and since $w_* \in L^2(\Omega)$ one get that $S_t, \ t > 0$ is a Hilbert-Schmidt operator as well as $T^*_t$ and hence $L_*$ has compact resolvent.
ii) Since $T^*_t$, $t > 0$ has a nonnegative kernel it is irreducible and the claim follows from the well known fact that the generator of every irreducible semigroup has a nondegenerate ground state energy with a.e. nonnegative ground state.

iii) The fact that $T^*_t$ is a Hilbert-Schmidt operator yields that $L_*$ possesses a Green kernel, $G_*$ and that $G_* \geq G$. Writing

$$\varphi^*_0 = \lambda^*_0 \int G_*(\cdot,y) \varphi^*_0(y) \, dy \geq \int G(\cdot,y) \varphi^*_0(y) \, dy$$

(4.13)

and using the lower bound \text{(3.22)} yields the result.

Let $(V_k)$ be an increasing sequence of positive potentials such that $V_k \uparrow V_*$ and there is a constant $0 < \kappa_k < 1$ such that for every $k \in \mathbb{N}$ we have

$$\int f^2 V_k(x) \, dx \leq \kappa_k E[f], \quad \forall f \in F.$$  \hspace{1cm} (4.14)

For example the sequence $V_k = (1 - \frac{1}{k}) V_*$ satisfies the above conditions.

By the assumption $0 < \kappa_k < 1$, we conclude that the following forms

$$D(\mathcal{E}_{V_k}) = \mathcal{F}, \quad \mathcal{E}_{V_k}[f] = \mathcal{E}[f] - \int \Omega f^2 V_k(x) \, dx, \quad \forall f \in F,$$

are closed in $L^2$. For every integer $k$, we shall designate by $L_k$ the self-adjoint operator related to $\mathcal{E}_{V_k}$.

Let $0 < V_k \uparrow V_*$, then $L_k := L - V_k$, increases in the strong resolvent sense to $L_*$. Since $L_*$ has compact resolvent, the latter convergence is even uniform (see \text{[BABP]} Lemma 2.5). Thus setting $\lambda^{(k)}_0$’s the ground state energy of the $L_k$’s , $\varphi^{(k)}_0$ its associated ground state and $\xi^{(k)}_0 := L^{-1} k$ we obtain

$$\lambda^{(k)}_0 \to \lambda^*_0, \quad \varphi^{(k)}_0 \to \varphi^*_0 \quad \text{and} \quad \xi^{(k)}_0 \to \xi^* \text{ in } L^2(\Omega, dx).$$ \hspace{1cm} (4.15)

For an accurate description of the behavior of the ground state, we shall extend formula \text{(3.2)} to $\varphi^*_0$.

Finally we resume.

**Theorem 4.2.** Let $V \in L^1_{\text{loc}}$ be a positive potential. Then under assumptions , (IS)\text{and} (HI) the following sharp estimate for the ground state $\varphi^*_0$ holds true

$$((ACC_1^2 \inf_{t > 0} t^{-s} e^{2t \lambda^*_0} + 1))^{-1} \xi^* \leq \varphi^*_0 \leq \xi^*(C_1 \inf_{t > 0} t^{-s/2} e^{t \lambda^*_0}), \quad \text{a.e.}$$

**Corollary 4.3.** We have

$$\varphi^*_0 \sim \int G^*(\cdot,y) \, dy, \quad \text{a.e.}$$ \hspace{1cm} (4.16)
We also derive by standard way the following large time asymptotics for the heat kernel.

**Corollary 4.4.** There is \( T > 0 \) such that for every \( t > T \),

\[
p_t^*(x, y) \sim e^{-\lambda_0^* t} \varphi_0^*(x) \varphi_0^*(y) \sim e^{-\lambda_0^* t} \xi^*(x) \xi^*(y), \text{ a.e..}
\]

(4.17)

It follows, in particular that

\[
-\lambda_0^* = \lim_{t \to \infty} \frac{1}{t} \ln \left( \frac{p_t^*(x, y)}{\xi^*(x) \xi^*(y)} \right).
\]

(4.18)

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