Linear rank-width of distance-hereditary graphs II. Vertex-minor obstructions

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Abstract

In the companion paper [Linear rank-width of distance-hereditary graphs I. A polynomial-time algorithm, Algorithmica 78(1):342–377, 2017], we presented a characterization of the linear rank-width of distance-hereditary graphs, from which we derived an algorithm to compute it in polynomial time. In this paper, we investigate structural properties of distance-hereditary graphs based on this characterization.

First, we prove that for a fixed tree $T$, every distance-hereditary graph of sufficiently large linear rank-width contains a vertex-minor isomorphic to $T$. We extend this property to bigger graph classes, namely, classes of graphs whose prime induced subgraphs have bounded linear rank-width. Here, prime graphs are graphs containing no splits. We conjecture that for every tree $T$, every graph of sufficiently large linear rank-width contains a vertex-minor isomorphic to $T$. Our result implies that it is sufficient to prove this conjecture for prime graphs.

For a class $\Phi$ of graphs closed under taking vertex-minors, a graph $G$ is called a vertex-minor obstruction for $\Phi$ if $G \not\in \Phi$ but all of its proper vertex-minors are contained in $\Phi$. Secondly, we provide, for each $k \geq 2$, a set of distance-hereditary graphs that contains all distance-hereditary vertex-minor obstructions for graphs of linear rank-width at most $k$. Also, we give a simpler way to obtain the known vertex-minor obstructions for graphs of linear rank-width at most 1.

1 Introduction

Linear rank-width is a linear-type width parameter of graphs motivated by the rank-width of graphs [31]. The vertex-minor relation is a graph containment relation which was introduced by Bouchet [7, 8, 10, 9, 11] in his studies of circle graphs and 4-regular Eulerian digraphs. The vertex-minor relation has an important role in the theory of (linear) rank-width [27, 30, 28, 24, 29] as (linear) rank-width does not increase when taking vertex-minors of a graph. We provide concise definitions in Section 2.

The problem of computing linear rank-width has been discussed recently. Kashyap [25] proved that it is NP-hard to compute matroid path-width on binary matroids. Proposition 3.1 in [30] supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC consolidator grant DISTRUCT, agreement No. 648527).

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shows that the problem of determining the linear rank-width of a bipartite graph is equivalent to
the problem of determining the path-width of a binary matroid, and from this relation, we can show
that computing linear rank-width is NP-hard in general. Adler and the authors of this paper [3]
proved that the linear rank-width of distance-hereditary graphs, which are graphs of rank-width 1,
can be computed in time $O(n^2 \log n)$ where $n$ is the number of vertices in an input graph. Jeong,
Kim, and Oum [23] showed that the linear rank-width of distance-hereditary graphs, which are graphs of rank-width 1,
can be computed in time $O(n^2 \log n)$ where $n$ is the number of vertices in an input graph. Jeong,
Kim, and Oum [23] showed that, there is a constructive algorithm to test whether a given graph
has linear rank-width at most $k$ in time $f(k) \cdot n^3$ for some function $f$. Using this, they also proved
that for every fixed integer $w$, there is a polynomial-time algorithm to compute linear rank-width
on graphs of rank-width $w$.

In this paper, we focus on structural aspects of linear rank-width. The first result of the Graph
Minor series papers is that for a fixed tree $T$, every graph of sufficiently large path-width contains
a minor isomorphic to $T$ [32], and this was later used by Blumsath and Courcelle [6] to define a
hierarchy of incidence graphs based on monadic second-order transductions. In order to obtain a
similar hierarchy for graphs, still based on monadic second-order transductions, Courcelle [14] asked
whether for a fixed tree $T$, every bipartite graph of sufficiently large linear rank-width contains a
vertex-minor isomorphic to $T$. We conjecture that it is true for any graph.

Conjecture 1.1. For every fixed tree $T$, there is an integer $f(T)$ such that every graph of linear
rank-width at least $f(T)$ contains a vertex-minor isomorphic to $T$.

We show that Conjecture 1.1 is true if and only if it is true in prime graphs with respect to
split decompositions [16]. A split in a graph is a vertex partition $(A, B)$ such that $|A|, |B| > 2$
and the set of edges joining $A$ and $B$ induces a complete bipartite subgraph. Prime graphs are
graphs without splits and they form, with complete graphs and stars, the basic graphs in the theory
of canonical split decompositions developed by Cunningham [16]. They are also considered when
studying the rank-width of graphs because the rank-width of a graph is the maximum rank-width
over all its prime induced subgraphs.

We prove the following.

Theorem 1.2. Let $p$ be a positive integer and let $T$ be a tree. Let $G$ be a graph such that every
prime induced subgraph of $G$ has linear rank-width at most $p$. If $G$ has linear rank-width at least
$40(p + 2)|V(T)|$, then $G$ contains a vertex-minor isomorphic to $T$.

A graph $G$ is distance-hereditary if for every connected induced subgraph $H$ of $G$ and two
vertices $v$ and $w$ in $H$, the distance between $v$ and $w$ in $H$ is the same as their distance in $G$. It is
known that every prime induced subgraph of a distance-hereditary graph has size at most 3 [10].
Together with this fact, our result implies that Conjecture 1.1 is also true for distance-hereditary
graphs.

To prove Theorem 1.2 we essentially prove that for a fixed tree $T$, every graph admitting a
canonical split decomposition whose decomposition tree has sufficiently large path-width contains a
vertex-minor isomorphic to $T$. Combined with a relation between the linear rank-width of a graph
and the path-width of its canonical split decomposition, we obtain Theorem 1.2. We will obtain such
a relation in Section 4. The vertex-minor relation cannot be replaced with the induced subgraph
relation because there is a cograph admitting a canonical split decomposition whose decomposition
tree has sufficiently large path-width [13, 21], but cographs have no $P_4$ as an induced subgraph.

In the second part, we investigate the set of distance-hereditary vertex-minor obstructions for
graphs of bounded linear rank-width. A graph is a vertex-minor obstruction for graphs of linear
rank-width $k$ if it has linear rank-width $k + 1$ and every proper vertex-minor has linear rank-width $k$. Robertson and Seymour [33] showed that for every infinite sequence $G_1, G_2, \ldots$ of graphs, there exist $G_i$ and $G_j$ with $i < j$ such that $G_i$ is isomorphic to a minor of $G_j$. In other words, graphs are well-quasi-ordered under the minor relation. Interestingly, this property implies that for any proper class $C$ of graphs closed under taking minors, the set of minor obstructions for $C$ is finite.

Oum [27, 29] obtained an analogous result for the vertex-minor relation: for every infinite sequence $G_1, G_2, \ldots$ of graphs of bounded rank-width, there exist $G_i$ and $G_j$ with $i < j$ such that $G_i$ is isomorphic to a vertex-minor of $G_j$. We can obtain the following as a corollary.

**Theorem 1.3** (Oum [27]). For every class $C$ of graphs with bounded rank-width that is closed under taking vertex-minors, there is a finite list of graphs $G_1, G_2, \ldots, G_m$ such that a graph is in $C$ if and only if it has no vertex-minor isomorphic to $G_i$ for some $i \in \{1, 2, \ldots, m\}$.

Theorem 1.3 implies that for every integer $k$, the class of all graphs of (linear) rank-width at most $k$ can be characterized by a finite list of vertex-minor obstructions. However, it does not give any explicit number of necessary vertex-minor obstructions or bound on the size of such graphs. Oum [30] proved that for each $k$, the size of a vertex-minor obstruction for graphs of rank-width at most $k$ is at most $(6^k + 1 - 1)/5$. For linear rank-width, obtaining such an upper bound on the size of vertex-minor obstructions remains an open problem. Jeong, Kwon, and Oum [24] showed that the number of vertex-minor obstructions for linear rank-width at most $k$ is at least $2^{\Omega(3^k)}$.

Adler, Farley, and Proskurowski [1] obtained the set of all three vertex-minor obstructions for graphs of linear rank-width at most 1, depicted in Figure 1, two of which are distance-hereditary. In this paper, we construct a set of graphs containing all vertex-minor obstructions for graphs of linear rank-width at most $k$ that are distance-hereditary. This is an analogous result to the characterization of acyclic minor obstructions for graphs of path-width at most $k$, investigated by Takahashi, Ueno, and Kajitani [34], and Ellis, Sudborough, and Turner [19]. As a similar work, Koutsonas, Thilikos, and Yamazaki [26] characterized matroid obstructions for bounded matroid path-width that are cycle matroids of outerplanar graphs.

Lastly, we obtain simpler proofs of known characterizations of graphs of linear rank-width at most 1 [1, 12].

The paper is organized as follows. Section 2 provides some preliminary concepts, including linear rank-width and vertex-minors. In Section 3, we introduce necessary notions regarding split decompositions, and restate the structural characterization of linear rank-width on distance-hereditary graphs. Section 4 presents a relation between the linear rank-width of a graph whose prime induced subgraphs have bounded linear rank-width and the path-width of its decomposition tree. From this, we prove Theorem 1.2 in Section 5. In Section 6, we provide a way to generate all vertex-minor obstructions for graphs of bounded linear rank-width that are distance-hereditary graphs. Section 7 presents simpler proofs for known characterizations of the graphs of linear rank-width at most 1.
2 Preliminaries

In this paper, graphs are finite, simple and undirected. Our graph terminology is standard, see for instance [13]. Let $G$ be a graph. We denote the vertex set of $G$ by $V(G)$ and the edge set by $E(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of $G$ induced by $X$, and let $G - X := G[V(G) \setminus X]$. For $v \in V(G)$, we write $G - v$ for $G - \{v\}$. For $F \subseteq E(G)$, let $G - F := (V(G), E(G) \setminus F)$. For a vertex $x$ of $G$, let $N_G(x)$ be the set of neighbors of $x$ in $G$ and we call $|N_G(x)|$ the degree of $x$ in $G$. Two vertices $x$ and $y$ are twins if $N_G(x) \setminus \{y\} = N_G(y) \setminus \{x\}$. An edge $e$ of a connected $G$ is a cut-edge if $G - e$ is disconnected. A vertex $v$ in a connected graph $G$ is a cut vertex if $G - v$ is disconnected. A connected graph is 2-connected if it has at least 3 vertices and has no cut vertices.

A tree is a connected graph containing no cycles. A vertex of degree one in a tree is called a leaf. A subcubic tree is a tree with maximum degree at most three, and a path is a tree with maximum degree at most two. The length of a path is the number of its edges. A star is a tree with a distinguished vertex, called its center, adjacent to all other vertices. A complete graph is a graph with all possible edges. A graph $G$ is called distance-hereditary if for every pair of two vertices $x$ and $y$ of $G$ the distance of $x$ and $y$ in $G$ equals the distance of $x$ and $y$ in any connected induced subgraph containing both $x$ and $y$ [1]. It is well-known that a graph is distance-hereditary if and only if it can be obtained from a single vertex by repeated addition of degree one vertices and twins [22]. An induced cycle of length at least 5 is not distance-hereditary.

A subset $F$ of the edge set of $G$ is called a matching if no two edges in $F$ share an end vertex. For an edge $e$ of a graph $G$, we denote by $G/e$ the graph obtained by contracting $e$. A graph $H$ is a minor of a graph $G$ if $H$ is obtained from a subgraph of $G$ by contractions of edges.

2.1 Linear rank-width

For sets $R$ and $C$, an $(R,C)$-matrix is a matrix whose rows and columns are indexed by $R$ and $C$, respectively. For an $(R,C)$-matrix $M$ and subsets $X \subseteq R$ and $Y \subseteq C$, let $M[X,Y]$ be the submatrix of $M$ whose rows and columns are indexed by $X$ and $Y$, respectively.

Let $G$ be a graph. We denote by $A_G$ the adjacency matrix of $G$ over the binary field; that is, for $v, w \in V(G)$, $A_G[v, w] = 1$ if $v$ is adjacent to $w$, and $A_G[v, w] = 0$, otherwise. For a graph $G$, let $\text{cutrk}^*_G : 2^{V(G)} \times 2^{V(G)} \rightarrow \mathbb{Z}$ be a function such that $\text{cutrk}^*_G(X,Y) := \text{rank}(A_G[X,Y])$ for all $X,Y \subseteq V(G)$, where rank is computed over the binary field. The cut-rank function of $G$ is the function $\text{cutrk}_G : 2^{V(G)} \rightarrow \mathbb{Z}$ where for each $X \subseteq V(G)$,

$$\text{cutrk}_G(X) := \text{cutrk}^*_G(X, V(G) \setminus X).$$

An ordering $(x_1, \ldots, x_n)$ of the vertex set $V(G)$ is called a linear layout of $G$. If $|V(G)| \geq 2$, then the width of a linear layout $(x_1, \ldots, x_n)$ of $G$ is defined as

$$\max_{1 \leq i \leq n-1} \{\text{cutrk}_G(\{x_1, \ldots, x_i\})\},$$

and if $|V(G)| = 1$, then the width is defined to be 0. The linear rank-width of $G$, denoted by $\text{lrw}(G)$, is defined as the minimum width over all linear layouts of $G$.

Caterpillars and complete graphs have linear rank-width at most 1. Ganian [20] gave a characterization of graphs of linear rank-width at most 1, and called them thread graphs. Adler and Kanté [2] showed that linear rank-width and path-width coincide on forests, and therefore, there is
a linear-time algorithm to compute the linear rank-width of forests. It is easy to see that the linear rank-width of a graph is the maximum over the linear rank-widths of its connected components.

For a linear layout $L$ of a graph $G$ and $v, w \in V(G)$, we denote $v \leq_L w$ if $v = w$ or $v$ appears before $w$ in the linear layout. For two orderings $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_m)$, we denote $(v_1, v_2, \ldots, v_n) \oplus (w_1, w_2, \ldots, w_m) := (v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m)$.

2.2 Vertex-minors

For a graph $G$ and a vertex $x$ of $G$, the local complementation at $x$ in $G$ is an operation to replace the subgraph induced by the set of neighbors of $x$ with its complement. The resulting graph is denoted by $G \� x$. If a graph $H$ can be obtained from $G$ by applying a sequence of local complementations, then $G$ and $H$ are called locally equivalent. A graph $H$ is called a vertex-minor of a graph $G$ if $H$ can be obtained from $G$ by applying a sequence of local complementations and deletions of vertices. Bouchet [11] observed that local complementation does not change the cut-rank function. This directly implies that every vertex-minor $H$ of $G$ satisfies that $lrw_H \leq lrw_G$.

Lemma 2.1 (Bouchet [11]; See Corollary 2). Let $G$ be a graph and let $x$ be a vertex of $G$. Then for every subset $X$ of $V(G)$, we have $\text{cutrk}_G(X) = \text{cutrk}_{G\� x}(X)$.

For an edge $xy$ of $G$, let $W_1 := N_G(x) \cap N_G(y)$, $W_2 := (N_G(x) \setminus N_G(y)) \setminus \{y\}$, and $W_3 := (N_G(y) \setminus N_G(x)) \setminus \{x\}$. The pivoting on $xy$ of $G$, denoted by $G \ast xy$, is the operation to flip the adjacencies between distinct sets $W_i$ and $W_j$, and swap the vertices $x$ and $y$. Flipping the adjacency between two vertices $v$ and $w$ is an operation that add an edge if there was no edge between $v$ and $w$, and remove an edge, otherwise. It is known that $G \ast xy = G \ast x \ast y \ast x = G \ast y \ast x \ast y$ [30, Proposition 2.1]. See Figure 2 for an example.

2.3 Path-width

A path decomposition of a graph $G$ is a pair $(P, B)$, where $P$ is a path and $B = (B_t)_{t \in V(P)}$ is a family of vertex subsets of $G$ such that

1. for every $v \in V(G)$ there exists $t \in V(P)$ such that $v \in B_t$,
2. for every $uv \in E(G)$ there exists $t \in V(P)$ such that $\{u, v\} \subseteq B_t$,
3. for every $v \in V(G)$, the set $\{t \in V(P) : v \in B_t\}$ induces a subpath of $P$. 

The width of a path decomposition \((P, B)\) is defined as \(\max\{|B_t| : t \in V(P)\} - 1\). The path-width of \(G\), denoted by \(pw(G)\), is defined as the minimum width over all path-decompositions of \(G\).

It is well known that if \(H\) is a minor of \(G\), then \(pw(H) \leq pw(G)\). Robertson and Seymour first proved that for a fixed tree \(T\), every graph of sufficiently large path-width contains a minor isomorphic to \(T\). The necessary function was optimized by Bienstock, Robertson, Seymour, and Thomas.

**Theorem 2.2** (Bienstock, Robertson, Seymour, and Thomas). For every forest \(F\), every graph with path-width at least \(|V(F)| - 1\) has a minor isomorphic to \(F\).

We recall the following theorem which characterizes the path-width of trees and is used for computing their path-width in linear time.

**Theorem 2.3** (Ellis, Sudborough, and Turner; Takahashi, Ueno, and Kajitani). Let \(T\) be a tree and let \(k\) be a positive integer. The following are equivalent.

1. \(T\) has path-width at most \(k\).
2. For every node \(x\) of \(T\), at most two of the subtrees of \(T - x\) have path-width \(k\) and all other subtrees of \(T - x\) have path-width at most \(k - 1\).
3. \(T\) has a path \(P\) such that for each node \(v\) of \(P\) and each connected component \(T'\) of \(T - v\) not containing a node of \(P\), \(pw(T') \leq k - 1\).

### 3 Linear rank-width of distance-hereditary graphs

In this section, we recall the characterization of the linear rank-width of distance-hereditary graphs investigated by Adler and the authors of this paper. For this characterization, we need to introduce split decompositions and the new notion of limbs introduced in the previous paper. We will follow the definition for split decompositions used by Bouchet.

A split in a connected graph \(G\) is a vertex partition \((X, Y)\) of \(G\) such that \(|X|, |Y| \geq 2\) and \(cutrk_G(X) = 1\). Prime graphs are connected graphs that do not have a split. Note that every connected graph with at most 3 vertices is a prime graph, by definition. Also, one can observe that every connected graph on 4 vertices admits a split, and it is not a prime graph.

A marked graph is a connected graph \(D\) with a matching \(M(D)\) where every edge in \(M(D)\) is a cut-edge. Every edge in \(M(D)\) is called a marked edge, and the end vertices of marked edges are called marked vertices. The connected components of \(D - M(D)\) are called bags of \(D\). The edges in \(E(D)\setminus M(D)\) are called unmarked edges, and the vertices that are not marked are called unmarked vertices.

If \((X, Y)\) is a split in a marked graph \(G\), then we construct a new marked graph \(D\) such that

- \(V(D) = V(G) \cup \{x', y'\}\) for two distinct new vertices \(x', y' \notin V(G)\),
- \(E(D) = E(G[X]) \cup E(G[Y]) \cup \{x'y'\} \cup E'\) where
  \[
  E' := \{x' : x \in X \text{ and there exists } y \in Y \text{ such that } xy \in E(G)\} \cup \{y'y : y \in Y \text{ and there exists } x \in X \text{ such that } xy \in E(G)\},
  \]
- \(x'y'\) is a marked edge, and all edges in \(E'\) are unmarked edges.
Figure 3: An example of replacing a bag $B$ with its simple decomposition. Circles indicate bags and dotted edges indicate marked edges.

The marked graph $D$ is called a simple decomposition of $G$. See Figure 3 for an example.

A split decomposition of a connected graph $G$ is a marked graph $D$ defined inductively to be either $G$ or a marked graph defined from a split decomposition $D'$ of $G$ by replacing a bag with its simple decomposition. For a marked edge $xy$ of a marked graph $D$, the recomposition of $D$ along $xy$ is the marked graph $(D \land xy) - \{x, y\}$. For a split decomposition $D$, let $G[D]$ denote the graph obtained from $D$ by recomposing all marked edges. Note that if $D$ is a split decomposition of $G$, then $G[D] = G$.

Since each marked edge of a split decomposition $D$ is a cut-edge and all marked edges form a matching, if we contract all unmarked edges in $D$, then we obtain a tree. We call it the decomposition tree of $G$ associated with $D$ and denote it by $T_D$. To distinguish the vertices of $T_D$ from the vertices of $G$ or $D$, the vertices of $T_D$ will be called nodes. For a node $v$ of $T_D$, we write $\text{bag}_D(v)$ to denote the bag of $D$ with which it is in correspondence, and for a bag $B$ of $D$, we write $\text{node}_D(B)$ to denote the node of $T_D$ with which it is in correspondence. Two bags of $D$ are called adjacent bags if their corresponding nodes in $T_D$ are adjacent. A sequence of bags $B_1 - B_2 - \cdots - B_m$ is called a path of bags if for each $i \in \{1, 2, \ldots, m - 1\}$, $B_i$ and $B_{i+1}$ are adjacent bags, and all of $B_1, B_2, \ldots, B_m$ are pairwise distinct. Clearly, for two bags $B$ and $B'$, there is a unique path of bags from $B$ to $B'$, which corresponds to the path from $\text{node}_D(B)$ to $\text{node}_D(B')$ in $T_D$. We denote by $\text{dist}_D(B, B')$ the distance from $\text{node}_D(B)$ to $\text{node}_D(B')$ in $T_D$; in other words, it is one less than the number of bags in the unique path of bags from $B$ to $B'$ in $D$.

### 3.1 Canonical split decompositions and local complementations

A split decomposition is called canonical if each bag is either a prime graph, a star, or a complete graph, and every recomposition of a marked edge in $D$ results in a split decomposition without the same property. The following is due to Cunningham and Edmonds [15], and Dahlhaus [17].

**Theorem 3.1** (Cunningham and Edmonds [15]; Dahlhaus [17]). Every connected graph $G$ has a unique canonical split decomposition, up to isomorphism, and it can be computed in time $O(|V(G)| + |E(G)|)$.

A bag is called a prime bag if it is a prime graph on at least 5 vertices, and a bag is called a complete bag or a star bag if it is a complete graph or a star, respectively.
Let \( D \) be a split decomposition of a connected graph \( G \) with bags that are either a prime graph, a complete graph or a star. The \textit{type of a bag of} \( D \) is either \( P \), \( K \), or \( S \) depending on whether it is a prime graph, a complete graph, or a star, respectively. The \textit{type of a marked edge} \( uv \) is \( AB \) where \( A \) and \( B \) are the types of the bags containing \( u \) and \( v \) respectively. If \( A = S \) or \( B = S \), then we can replace \( S \) by \( S_p \) or \( S_c \) depending on whether the end of the marked edge is a leaf or the center of the star, respectively. Bouchet characterized when it becomes a canonical split decomposition.

**Theorem 3.2** (Bouchet [10]). \textit{Let} \( D \) \textit{be a split decomposition of a connected graph whose bags are either a prime graph, a complete graph, or a star. Then} \( D \) \textit{is a canonical split decomposition if and only if it has no marked edge of type} \( KK \) or \( S_p S_c \).}

We will use the following characterizations of trees and of distance-hereditary graphs.

**Theorem 3.3** (Bouchet [10]).

(1) A connected graph is distance-hereditary if and only if every bag of its canonical split decomposition is of type \( K \) or \( S \).

(2) A connected graph is a tree if and only if every bag of its canonical split decomposition is a star bag whose center is an unmarked vertex.

We now relate the split decompositions of a graph and the ones of its locally equivalent graphs. Let \( D \) be a split decomposition of a connected graph. A vertex \( v \) of \( D \) \textit{represents} an unmarked vertex \( x \) (or is a \textit{representative of} \( x \)) if either \( v = x \) or there is a path of even length from \( v \) to \( x \) in \( D \) starting with a marked edge such that marked edges and unmarked edges appear alternately in the path. Two unmarked vertices \( x \) and \( y \) are \textit{linked} in \( D \) if there is a path from \( x \) to \( y \) in \( D \) such that unmarked edges and marked edges appear alternately in the path. Linkedness of unmarked vertices exactly represents the adjacency relation between those vertices in the original graph.

**Lemma 3.4** (Adler, Kanté, and Kwon [3]). Let \( D \) be a split decomposition of a connected graph \( G \). Let \( v' \) and \( w' \) be two vertices in a same bag of \( D \), and let \( v \) and \( w \) be two unmarked vertices of \( D \) represented by \( v' \) and \( w' \), respectively. The following are equivalent.

1. \( v \) and \( w \) are linked in \( D \).
2. \( vw \in E(G) \).
3. \( v'w' \in E(D) \).

A \textit{local complementation} at an unmarked vertex \( x \) in a split decomposition \( D \), denoted by \( D \ast x \), is the operation to replace each bag \( B \) containing a representative \( w \) of \( x \) with \( B \ast w \). Bouchet observed that \( D \ast x \) is a split decomposition of \( G[D] \ast x \), and \( M(D) = M(D \ast x) \). Two split decompositions \( D \) and \( D' \) are \textit{locally equivalent} if \( D \) can be obtained from \( D' \) by applying a sequence of local complementations at unmarked vertices. As expected, this local complementation also preserves the property that the split decomposition is canonical.

**Lemma 3.5** (Bouchet [10]). Let \( D \) be the canonical split decomposition of a connected graph \( G \). If \( x \) is a vertex of \( G \), then \( D \ast x \) is the canonical split decomposition of \( G \ast x \).
Figure 4: Examples of local complementation and pivoting in a split decomposition.

Let \( x \) and \( y \) be linked unmarked vertices in a split decomposition \( D \), and let \( P \) be the path in \( D \) linking \( x \) and \( y \) such that unmarked edges and marked edges appear alternately in the path. Note that if \( B \) is a bag of type \( S \) containing an unmarked edge of \( P \), then the center of \( B \) is a representative of either \( x \) or \( y \). The pivoting on \( xy \) of \( D \), denoted by \( D \wedge xy \), is the split decomposition obtained as follows: for each bag \( B \) containing an unmarked edge of \( P \), if \( v, w \in V(B) \) represent respectively \( x \) and \( y \) in \( D \), then we replace \( B \) with \( B \wedge vw \). It is worth noticing that by Lemma 3.4, we have \( vw \in E(B) \), hence \( B \wedge vw \) is well-defined.

Lemma 3.6 (Adler, Kanté, and Kwon [3]). Let \( D \) be a split decomposition of a connected graph \( G \). If \( xy \in E(G) \), then \( D \wedge xy = D \ast x \ast y \ast x \).

3.2 Removing vertices

Let \( G \) be a distance-hereditary graph and let \( D \) be its split decomposition. Let \( S \) be a vertex set of \( G \). We explain how we transform \( D \) into a split decomposition of \( G \setminus S \). Note that the split decomposition obtained from \( D \) by removing vertices in \( S \) is not necessarily a split decomposition because the resulting marked graph may have bags of size at most 2. In this case, we need to recompose a marked edge incident with each bag of size at most 2 unless the resulting marked graph has at most two vertices.

Suppose \( D \) is canonical. We frequently consider connected components \( T \) of \( D \setminus V(B) \), for a bag \( B \) of \( D \). This will be used to define limbs in the next subsection. For a bag \( B \) of \( D \) and a connected component \( T \) of \( D \setminus V(B) \), let us denote by \( \zeta_b(D, B, T) \) and \( \zeta_c(D, B, T) \) the end vertices of the marked edge in \( D \) linking \( B \) and \( T \) that are in \( V(B) \) and in \( V(T) \) respectively. Subscripts \( b \) and \( c \) stand for bag and component, respectively. We always treat \( T \) as a canonical split decomposition and regard \( \zeta_c(D, B, T) \) as an unmarked vertex.

3.3 Limbs and characterization of linear rank-width

To present the characterization of the linear rank-width of distance-hereditary graphs, we need the new notion called limbs [3]. For an unmarked vertex \( y \) in \( D \) and a bag \( B \) of \( D \) containing a marked vertex representing \( y \), let \( T \) be the connected component of \( D \setminus V(B) \) containing \( y \), and let \( v := \zeta_c(D, B, T) \) and \( w := \zeta_b(D, B, T) \). We define the limb \( \mathcal{L} := L_D[B, y] \) with respect to \( B \) and \( y \) as follows:

1. if \( B \) is of type \( K \), then \( \mathcal{L} := T \ast v - v \),
2. if $B$ is of type $S$ and $w$ is a leaf, then $L := T - v$,

3. if $B$ is of type $S$ and $w$ is the center, then $L := T \setminus vy - v$.

While $T$ is a canonical split decomposition, $L$ may not be a canonical split decomposition, because deleting $v$ may create a bag of size 2. We analyze the cases when such a bag appears, and describe how to transform it into a canonical split decomposition. Suppose that a bag $B'$ of size 2 appears in $L$. If $B'$ has no adjacent bags in $L$, then $B'$ itself is a canonical split decomposition. We may assume there is a bag adjacent to $B'$.

1. ($B'$ has one adjacent bag $B_1$.)
   If $v_1 \in V(B_1)$ is the marked vertex adjacent to a vertex of $B'$ and $r$ is the unmarked vertex of $B'$ in $L$, then we remove the bag $B'$ and replace $v_1$ with $r$. In other words, we recompose along the marked edge connecting $B'$ and $B_1$.

2. ($B'$ has two adjacent bags $B_1$ and $B_2$.)
   If $v_1 \in V(B_1)$ and $v_2 \in V(B_2)$ are the two marked vertices that are adjacent to the two marked vertices of $B'$, then we remove $B'$ and add a marked edge $v_1v_2$. If the new marked edge $v_1v_2$ is of type $KK$ or $S_pS_c$, then by recomposing along $v_1v_2$, we finally transform the limb into a canonical split decomposition.

Let $\mathcal{L}_D[B, y]$ be the canonical split decomposition obtained from $\mathcal{L}_D[B, y]$ and we call it the canonical limb. Let $\mathcal{G}_D[B, y]$ be the graph obtained from $\mathcal{L}_D[B, y]$ by recomposing all marked edges. For a bag $B$ of $D$ and a connected component $T$ of $D - V(B)$, we define $f_D(B, T)$ as the linear rank-width of $\mathcal{G}_D[B, y]$ for some unmarked vertex $y \in V(T)$. It was shown that $f_D(B, T)$ does not depend on the choice of $y$.

**Proposition 3.7** (Adler, Kanté, and Kwon; Proposition 3.4 of [3]). Let $B$ be a bag of $D$ and let $y$ be an unmarked vertex of $D$ represented by a vertex $w$ in $B$. Let $x \in V(G[D])$. If an unmarked vertex $y'$ is represented by $w$ in $D \ast x$, then $\mathcal{G}_D[B, y]$ is locally equivalent to $\mathcal{G}_{D \ast x}[(D \ast x)[V(B)], y']$. Therefore, $f_D(B, T) = f_{D \ast x}((D \ast x)[V(B)], T_x)$ where $T$ and $T_x$ are the components of $D \setminus V(B)$ and $(D \ast x) \setminus V(B)$ containing $y$, respectively.

As a variant of Theorem 2.3, distance-hereditary graphs of bounded linear rank-width can be characterized using limbs.
Theorem 3.8 (Adler, Kanté, and Kwon [3]). Let $k$ be a positive integer and let $D$ be the canonical split decomposition of a connected distance-hereditary graph $G$. Then the following are equivalent.

(1) $G$ has linear rank-width at most $k$.

(2) For each bag $B$ of $D$, $D - V(B)$ has at most two connected components $T$ such that $f_D(B, T) = k$, and every other connected component $T'$ of $D - V(B)$ satisfies that $f_D(B, T') \leq k - 1$.

(3) $T_D$ has a path $P$ such that for each node $v$ of $P$ and each connected component $H$ of $D - V(bag_D(v))$ containing no bags $bag_D(w)$ with $w \in V(P)$, $f_D(bag_D(v), H) \leq k - 1$.

4 Path-width of decomposition trees

To prove Theorem 1.2, we derive a relation between the linear rank-width of a graph whose prime induced subgraphs have bounded linear rank-width and the path-width of its decomposition tree.

Proposition 4.1. Let $p$ be a positive integer. Let $G$ be a connected graph whose prime induced subgraphs have linear rank-width at most $p$, and let $D$ be the canonical split decomposition of $G$, and let $T_D$ be the decomposition tree of $G$ associated with $D$. Then $lrw(G) \leq 2(p + 2)(pw(T_D) + 1)$.

We prove Proposition 4.1 by induction on the path-width of $T_D$. If its path-width is 0, then it consists of one node, and the result directly follows from the given condition that every prime induced subgraph has linear rank-width at most $p$. Note that complete graphs and stars have linear rank-width at most 1. We assume that the path-width of $T_D$ is at least 1. Using Lemma 2.3 $T$ contains a path $P$ such that for each node $v$ of $P$ and each connected component $T'$ of $T - v$ not containing a node of $P$, $pw(T') \leq k - 1$. So, by induction, we can obtain an upper bound of the linear rank-width of split decompositions corresponding to such components $T'$. From this, we will obtain an upper bound of the linear rank-width of the whole graph.

We need the following lemma. We point out that Lemma 4.2 does not require $D$ to be a canonical split decomposition, and this relaxation will be useful for an easier argument in the main proof.

Lemma 4.2. Let $k$ and $p$ be positive integers. Let $B$ be a bag of a split decomposition $D$ with two unmarked vertices $a$ and $b$ such that for every connected component $H$ of $D - V(B)$, $lrw(G[H]) \leq k$. If $B$ has a linear layout of width at most $p$ whose first and last vertices are $a$ and $b$ respectively, then $G[D]$ has a linear layout of width at most $2p + k$ whose first and last vertices are $a$ and $b$ respectively.

Proof. Let $G := G[D]$, and let $L_B := (w_1, w_2, \ldots, w_m)$ be a linear layout of $B$ of width at most $p$ such that $a = w_1$ and $b = w_m$. For each $j \in \{1, 2, \ldots, m\}$,

1. if $w_j$ is an unmarked vertex, then let $L_j := (w_j)$, and

2. if $w_j = \zeta_b(D, B, H)$ for some connected component $H$ of $D - V(B)$, then let $L_j$ be a linear layout of $G[H] - \zeta(D, B, H)$ having width at most $k$.

We define $L := L_1 \oplus L_2 \oplus \cdots \oplus L_m$. We observe that $L$ is a linear layout of $G$. For each $j \in \{1, 2, \ldots, m\}$, we choose an unmarked vertex $y_j$ represented by $w_j$. If $w_j$ is an unmarked vertex, then $y_j = w_j$. 

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We claim that $L$ has width at most $2p+k$. It is sufficient to prove that for every $w \in V(G) \setminus \{a, b\}$, $\text{cutrk}_G(\{v : v \leq_L w\}) \leq 2p + k$. Let $w \in V(G) \setminus \{a, b\}$ and let $S_w := \{v : v \leq_L w\}$ and $T_w := V(G) \setminus S_w$.

Let $H_j$ be a connected component of $D - V(B)$ such that $\zeta_6(D, B, H_j) = w_j$. Observe that if all vertices in $V(H_j) \cap V(G)$ are contained in $S_w$, then all vertices in $V(H_j) \cap V(G)$ that have a neighbor in $T_w$ have exactly the same set of neighbors in $T_w$, which is $N_G(y_j) \cap T_w$. Therefore, when we compute the rank of the matrix $A(G)[S_w, T_w]$, we can replace all vertices in $V(H_j) \cap V(G)$ with $y_j$. The same observation holds for connected components fully contained in $T_w$. Also, for two distinct connected components $H_{j1}, H_{j2}$ of $D - V(B)$ where all vertices of $V(H_{j1}) \cap V(G)$ are contained in $S_w$ and all vertices of $V(H_{j2}) \cap V(G)$ are contained in $T_w$, $y_1$ and $y_2$ are adjacent in $G$ if and only if $\zeta_6(D, B, H_1)$ is adjacent to $\zeta_6(D, B, H_2)$ in $B$. This is an implication of Lemma 3.4.

Having it, we can observe that if $w$ is an unmarked vertex in $B$, then
\[
\text{cutrk}_G(S_w) = \text{cutrk}_B(\{v : v \leq_{L_B} w\}) \leq p.
\]

Thus, we may assume that $w$ is contained in some connected component $H$ of $D - V(B)$. Let $j \in \{1, 2, \ldots, m\}$ such that $\zeta_6(D, B, H) = w_j$.

Note that $H$ is the unique component of $D - V(B)$ possibly intersecting both $S_w$ and $T_w$. Since all vertices of $V(H) \cap V(G)$ having a neighbor in $V(G) \cap V(H)$ have the same neighborhood in $V(G) \cap V(H)$ (that is, $(V(H) \cap V(G), V(G) \cap V(H))$ is a split), we have
\[
\begin{align*}
(1) \quad & \text{cutrk}_G(S_w, T_w \setminus V(H)) \leq \max\{\text{cutrk}_B(\{v : v \leq_{L_B} w_{j_1-1}\}), \text{cutrk}_B(\{v : v \leq_{L_B} w_j\})\} \leq p. \\
(2) \quad & \text{cutrk}_G(S_w \setminus V(H), T_w) \leq \max\{\text{cutrk}_B(\{v : v \leq_{L_B} w_{j_1-1}\}), \text{cutrk}_B(\{v : v \leq_{L_B} w_j\})\} \leq p. \\
(3) \quad & \text{cutrk}_G(S_w \cap V(H), T_w \cap V(H)) \leq k.
\end{align*}
\]

Therefore, we have
\[
\text{cutrk}_G(S_w) \leq \text{cutrk}_G^*(S_w, T_w \setminus V(H)) + \text{cutrk}_G^*(S_w \setminus V(H), T_w) \\
+ \text{cutrk}_G^*(S_w \cap V(H), T_w \cap V(H)) \leq p + p + k \leq 2p + k.
\]

We conclude that $L$ is a linear layout of $G$ of width at most $2p + k$ whose first and last vertices are $a$ and $b$, respectively. 

**Proof of Proposition 4.1.** We prove it by induction on $k := \text{pw}(T_D)$. If $k = 0$, then $T_D$ consists of one node, and $G$ is either a prime graph, a complete graph, or a star. Note that complete graphs and stars have linear rank-width at most 1. Thus, we have $\text{lwr}(G) \leq p \leq 2(p + 2)$. We may assume that $k \geq 1$.

Since $\text{pw}(T_D) = k \geq 1$, by Theorem 2.3, there exists a path $P := v_1v_2 \cdots v_n$ in $T_D$ such that for each node $v$ in $P$ and each connected component $T$ of $T_D - v$ not intersecting $P$, $\text{pw}(T) \leq k - 1$. For each $i \in \{1, 2, \ldots, n\}$, let $B_i := \text{bag}_D(v_i)$. By induction hypothesis, for each $i \in \{1, 2, \ldots, n\}$ and each connected component $H$ of $D - V(B_i)$ not intersecting $\bigcup_{1 \leq j \leq n} V(B_j)$, we have $\text{lwr}(G[H]) \leq 2(p + 2)k$.

Now, let us modify the given canonical split decomposition by two additional unmarked vertices so that we can easily apply Lemma 4.2. For each $i \in \{1, 2, \ldots, n\}$, let $L_{B_i}$ be a linear layout of $B_i$ of width at most $p$. First, we add a twin of the first vertex of $L_{B_1}$ in $B_1$ such that the added vertex
is unmarked. Similarly, we add a twin of the last vertex of $L_{B_n}$ in $B_n$ such that the added vertex is unmarked. Let $a_1$ be the vertex added to $B_1$ and $b_n$ be the vertex added to $B_n$. It is not difficult to see that $B_1$ has a linear layout of width at most $p$ whose first vertex is $a_1$, and $B_n$ has a linear layout of width at most $p$ whose last vertex is $b_n$.

Assume for a moment that $n \geq 2$. For each $i \in \{1, 2, \ldots, n-1\}$, let $b_i$ and $a_{i+1}$ be the marked vertices of $B_i$ and $B_{i+1}$, respectively, such that $b_ia_{i+1}$ is the marked edge connecting $B_i$ and $B_{i+1}$. If $b_i$ is not the end vertex of $L_{B_i}$, then we reorder $L_{B_i}$ so that $b_i$ is the end vertex. Similarly, if $a_{i+1}$ is not the first vertex of $L_{B_{i+1}}$, then we reorder $L_{B_{i+1}}$ so that $a_{i+1}$ is the first vertex. Until now, the width of each $L_{B_i}$ may increase by at most 1. This is because the rank of a matrix increase by at most 1 when we move one element in the column indices (resp. the row indices) to the row indices (resp. the column indices).

Note that the resulting decomposition is not necessarily canonical, as we may add a twin of a vertex in a prime graph. But this is not a problem when we apply Lemma 4.2. By the above modification, we know that for each $i \in \{1, 2, \ldots, n\}$, there is a linear layout of $B_i$ of width at most $p + 2$ whose first and last vertices are $a_i$ and $b_i$, respectively.

We define the following sub-decompositions. See Figure 6 for an illustration. If $n = 1$, then let $D_1 := D$. Otherwise,

1. let $D_1$ be the connected component of $D - V(B_2)$ containing $B_1$,
2. let $D_n$ be the connected component of $D - V(B_{n-1})$ containing $B_n$, and
3. for each $i \in \{2, 3, \ldots, n-1\}$, let $D_i$ be the connected component of $D - (V(B_{i-1}) \cup V(B_{i+1}))$ containing $B_i$.

We regard the vertices $a_i$ and $b_i$ as unmarked vertices of $D_i$.

Recall that $\text{pw}(T) \leq k - 1$ for every node $v$ of $P$ and every connected component $T$ of $T_D - v$ not intersecting $P$. Therefore, $\text{lrw}(G[H]) \leq 2(p + 2)k$, for each connected component $H$ of $D_i - V(B_i)$, by induction hypothesis. Thus, by Lemma 4.2, $G[D_i]$ has a linear layout $L_i$ of width at most $2(p + 2) + 2(p + 2)k = 2(p + 2)(k + 1)$ whose first and last vertices are $a_i$ and $b_i$, respectively. For each $i \in \{1, 2, \ldots, n\}$, let $L_i'$ be the linear layout obtained from $L_i$ by removing $a_i$ and $b_i$. Then it is not hard to check that

$$L_1' \oplus L_2' \oplus \cdots \oplus L_n'$$

is a linear layout of $G$ having width at most $2(p + 2)(k + 1)$. We conclude that $\text{lrw}(G) \leq 2(p + 2)(\text{pw}(T_D) + 1)$.

For distance-hereditary graphs, the following establishes a lower bound and the tight upper bound of linear rank-width with respect to the path-width of their canonical split decompositions.
Proposition 4.3. Let \( D \) be the canonical split decomposition of a connected distance-hereditary graph \( G \). Then \( \frac{1}{2} \operatorname{pw}(T_D) \leq \operatorname{lrw}(G) \leq \operatorname{pw}(T_D) + 1 \).

The upper bound part is tight. For instance, every complete graph with at least two vertices has linear rank-width 1 and the path-width of its decomposition tree has path-width 0. Also, for each odd integer \( k = 2n + 1 \) with \( n \geq 1 \), every complete binary tree of height \( k \) (each path from a leaf to the root has distance \( k \)) has linear rank-width \( \lceil k/2 \rceil = n + 1 \), and its decomposition tree has path-width \( \lceil (k-1)/2 \rceil = n \). (Note that the linear rank-width and the path-width of a tree are the same \[2\].) We will need the following lemmas.

Lemma 4.4. Let \( G \) be a graph and let \( uv \in E(G) \). Then \( \operatorname{pw}(G) \leq \operatorname{pw}(G/uv) + 1 \).

Proof. Let \((P, \mathcal{B})\) be an optimal path-decomposition of \( G/uv \), and let \( z \) be the contracted vertex in \( G/uv \). It is not hard to check that a new path-decomposition obtained by removing \( z \) and adding \( u \) and \( v \) in each bag containing \( z \) is a path-decomposition of \( G \). We conclude that \( \operatorname{pw}(G) \leq \operatorname{pw}(G/uv) + 1 \).

Lemma 4.5. Let \( G \) be a graph. Let \( u \) be a vertex of degree 2 in \( G \) such that \( v_1, v_2 \) are the neighbors of \( u \) in \( G \) and \( v_1v_2 \notin E(G) \). Then \( \operatorname{pw}(G) \leq \operatorname{pw}(G/uv_1/uv_2) + 1 \).

Proof. Let \( w \) be the contracted vertex in \( G/uv_1/uv_2 \), and let \((P, \mathcal{B})\) be an optimal path-decomposition of \( G/uv_1/uv_2 \) of width \( t := \operatorname{pw}(G/uv_1/uv_2) \). We may assume that no two adjacent bags in \((P, \mathcal{B})\) are equal.

We obtain a path-decomposition \((P, \mathcal{B}')\) from \((P, \mathcal{B})\) by replacing \( w \) with \( v_1 \) and \( v_2 \) in all bags containing \( w \). Since no two adjacent bags in \((P, \mathcal{B})\) are equal, no two adjacent bags in \((P, \mathcal{B}')\) are equal.

We first assume that there are two adjacent bags \( B_1 \) and \( B_2 \) in \((P, \mathcal{B}')\) containing both \( v_1 \) and \( v_2 \), respectively. We obtain a path-decomposition \((P', \mathcal{B}'')\) from \((P, \mathcal{B}')\) by subdividing the edge between \( B_1 \) and \( B_2 \), and adding a new bag \( B' = (B_1 \cap B_2) \cup \{u\} \). Since \( B_1 \) and \( B_2 \) are not the same, \( |B_1 \cap B_2| \leq t + 1 \) and therefore, \( |B'| \leq t + 2 \). Thus, \((P', \mathcal{B}'')\) is a path-decomposition of \( G \) of width at most \( t + 1 \), and \( \operatorname{pw}(G) \leq \operatorname{pw}(G/uv_1/uv_2) + 1 \).

Now we may assume that there is only one bag \( B \) in \((P, \mathcal{B}')\) containing both \( v_1 \) and \( v_2 \). In this case, since \( v_1v_2 \notin E(G) \), we can obtain a path decomposition of \( G \) by replacing this bag \( B \) with a sequence of two bags \( B_1 \) and \( B_2 \), where \( B_1 := B\setminus\{v_2\} \cup \{u\} \) and \( B_2 := B\setminus\{v_1\} \cup \{u\} \). This implies that \( \operatorname{pw}(G) \leq \operatorname{pw}(G/uv_1/uv_2) + 1 \).

We are now ready to prove Proposition 4.3. We need the split decomposition characterization of graphs of linear rank-width at most 1 proved by Bui-Xuan, Kanté, and Limouzy \[12\] for the base case, which can be easily obtained by Theorem 3.8. We give a proof of this characterization in Theorem 7.1.

Proof of Proposition 4.3. Let us first prove that \( \operatorname{pw}(T_D) \leq 2 \operatorname{lrw}(G) \) by induction on the linear rank-width of \( G \). Let \( k := \operatorname{lrw}(G) \). If \( k = 0 \), then \( G \) consists of a vertex, and \( \operatorname{pw}(T_D) = 0 \). If \( k = 1 \), then by Theorem 7.1, \( T_D \) is a path and we have \( \operatorname{pw}(T_D) \leq 1 \leq 2k \). Thus, we may assume that \( k \geq 2 \). By Theorem 3.8, there exists a path \( P \) in \( T_D \) such that

- for every node \( v \) in \( P \) and every connected component \( H \) of \( D - V(\operatorname{bag}_D(v)) \) containing no bag in \( \{\operatorname{bag}_D(w) \mid w \in V(P)\} \), \( f_D(\operatorname{bag}_D(v), H) \leq k - 1 \).

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Let $v$ be a node of $P$ and $C$ be a connected component of $D - V(\text{bag}_D(v))$ containing no bag $\text{bag}_D(w)$ with $w \in V(P)$. Let $y$ be an unmarked vertex of $C$ represented by $\zeta_*(D, \text{bag}_D(v), C)$, and let $L := \mathcal{L}(D[V(\text{bag}_D(v)), y]$. By induction hypothesis, the decomposition tree $T_L$ of $L$ has path-width at most $2k - 2$. We claim that $\text{pw}(T_C) \leq 2k - 1$, where $T_C$ is the decomposition tree of $C$. By the definition of canonical limbs, either $T_L = T_C$ or $T_L$ is obtained from $T_C$ using one of the following operations:

1. Removing a node of degree 1.
2. Removing a node of degree 2 with its neighbors $v_1, v_2$ and adding an edge $v_1v_2$.
3. Removing a node of degree 2 with its neighbors $v_1, v_2$ and identifying $v_1$ and $v_2$.

The first two cases can be regarded as contracting one edge. So, $\text{pw}(T_C) \leq \text{pw}(T_L) + 1 \leq (2k - 2) + 1 = 2k - 1$ by Lemma \ref{lem:pw_contraction}. The last case corresponds to contracting two edges incident with a vertex of degree 2. By Lemma \ref{lem:pw_degree2}, $\text{pw}(T_C) \leq \text{pw}(T_L) + 1 \leq 2k - 1$.

Therefore, for each node $v$ of $P$ and each connected component $T'$ of $T_D - v$ not containing a node of $P$ we have that $\text{pw}(T') \leq 2k - 1$. By Theorem \ref{thm:lw_prime}, $T_D$ has path-width at most $2k$, as required.

(2) We prove that $\text{lrw}(G) \leq \text{pw}(T_D) + 1$ by induction on the path-width of $T_D$. Let $k := \text{pw}(T_D)$.

If $k = 0$, then $T_D$ consists of one node. Since $G$ is distance-hereditary, $G$ should be a star or a complete graph, and therefore, we have $\text{lrw}(G) \leq 1 = \text{pw}(T_D) + 1$. We may assume that $k \geq 1$.

By Theorem \ref{thm:lw_prime}, there exists a path $P = v_0v_1 \cdots v_nv_{n+1}$ in $T_D$ such that for every node $v$ in $P$ and every connected component $F$ of $T_D - v$ containing no nodes of $P$, $\text{pw}(F) \leq k - 1$. Let $v$ be a node of $P$ and let $C$ be a connected component of $D - V(\text{bag}_D(v))$ containing no bags $\text{bag}_D(w)$ with $w \in V(P)$. By induction hypothesis, $G[C]$ has linear rank-width at most $(k - 1) + 1 = k$. By the definition of limbs, we conclude that $f_D(\text{bag}_D(v), C) \leq k$. Thus, by Theorem \ref{thm:lw_containment}, we conclude that $\text{lrw}(G) \leq k + 1$.

We could not confirm that the lower bound in Proposition \ref{prop:lw_bounds} is tight. We leave the following as an open question.

**Question 1.** Let $D$ be the canonical split decomposition of a connected distance-hereditary graph $G$. Is it true that $\text{pw}(T_D) \leq \text{lrw}(G)$?

## 5 Containing a tree as a vertex-minor

In this section, we prove our first main result.

**Theorem 1.2.** Let $p$ be a positive integer and let $T$ be a tree. Let $G$ be a graph such that every prime induced subgraph of $G$ has linear rank-width at most $p$. If $\text{lrw}(G) \geq 40(p + 2)|V(T)|$, then $G$ contains a vertex-minor isomorphic to $T$.

To prove it, we observe that the decomposition tree of the canonical split decomposition of $G$ has large path-width using Theorem \ref{thm:lw_containment}. The main argument of this section is that if $G$ admits a canonical split decomposition whose decomposition tree has sufficiently large path-width, then $G$ contains a vertex-minor isomorphic to $T$. 

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Therefore, we have $\phi(T)$ vertices of degree at most 3 in $p$, at least 4, and let $v$.

Since $T$ is a tree, we have $\phi(T) = 0$.

**Lemma 5.1.** Let $k$ be a positive integer and let $T$ be a tree with $\phi(T) = k$. Then $T$ is a vertex-minor of a tree $T'$ with $\phi(T') = k - 1$ and $|V(T')| = |V(T)| + 2$.

**Proof.** Since $\phi(T) \geq 1$, $T$ has a vertex of degree at least 4. Let $v \in V(T)$ be a vertex of degree at least 4, and let $v_1, v_2, \ldots, v_m$ be its neighbors. We obtain $T'$ from $T$ by replacing the edge $vv_1$ with the path $v_2p_1v_1$, removing $vv_2$ and adding an edge between $p_1$ and $v_2$. It is easy to verify that $(T' \setminus p_1p_2) \setminus \{p_1, p_2\} = T$. We depict this procedure in Figure 7. We observe that $p_1$ and $p_2$ are vertices of degree at most 3 in $T'$, and the degree of $v$ in $T'$ is one less than the degree of $v$ in $T$. Therefore, we have $\phi(T') = k - 1$.

**Lemma 5.2.** Every tree $T$ is a vertex-minor of a subcubic tree $T'$ such that $|V(T')| \leq 5|V(T)|$.

**Proof.** By Lemma 5.1, $T$ is a vertex-minor of a subcubic tree $T'$ with $|V(T')| \leq |V(T)| + 2\phi(T)$. Since $\phi(T) \leq 2|E(T)| \leq 2|V(T)|$, we conclude that $|V(T')| \leq |V(T)| + 2\phi(T) \leq 5|V(T)|$.

We recall that by (2) of Theorem 3.3, a connected graph is a tree if and only if every bag of its canonical split decomposition is a star bag whose center is an unmarked vertex. The basic strategy is to extract the canonical split decomposition of a subcubic tree from the canonical split decomposition of $G$. To do this, we will obtain a star from each prime bag, without changing too much the shape of the obtained canonical split decomposition. Lemma 5.4 describes how to obtain a star from a prime graph as a vertex-minor, without applying local complementations at some special vertices, which will correspond to marked vertices.

We observe that every prime graph on at least 5 vertices is 2-connected. This is because if a connected graph $G$ contains a cut vertex $v$ and $T_1, T_2, \ldots, T_m$ are connected components of $G - v$, then $\left( V(T_1) \cup \{v\}, \bigcup_{j \in \{2, \ldots, m\}} V(T_j) \right)$ is a split of $G$. We use this observation in Lemma 5.4.

**Lemma 5.3.** Let $abc$ be an induced path in a 2-connected graph $G$. By applying local complementations at vertices in $V(G) \setminus \{a, b\}$, we can obtain $G'$ locally equivalent to $G$ such that $G'[\{a, b, c\}]$ is a triangle.

**Proof.** As $b$ is not a cut vertex of $G$, there is a path from $a$ to $c$ in $G - b$. Let $r_1r_2 \cdots r_s$ be the shortest path from $c = r_1$ to $a = r_s$ in $G - b$. Note that $s \geq 3$ as $a$ is not adjacent to $c$. See Figure 8 for an illustration.

We prove by induction on $s$ that $G[\{b, r_1, r_2, \ldots, r_s\}]$ can be transformed into an induced path $acb$ by applying local complementations only at vertices in $\{r_1, r_2, \ldots, r_{s-1}\}$. We illustrate this
procedure in Figure 8. Assume \( s = 3 \). If \( b \) is adjacent to \( r_2 \), then we remove this edge by applying a local complementation at \( c = r_1 \). And then we apply a local complementation at \( r_2 \) to create an edge between \( a \) and \( c \). Then \( abc \) becomes a triangle.

We assume \( s \geq 4 \). Similarly, if \( b \) is adjacent to \( r_2 \), then we remove this edge by applying a local complementation at \( c = r_1 \), and then we apply a local complementation at \( r_2 \) to create an edge between \( c \) and \( r_3 \). Let \( G_1 \) be the resulting graph. Then \( r_1r_3r_4\cdots r_s \) is an induced path in \( G_1 - b \). Thus, by induction hypothesis, we can obtain \( G_2 \) locally equivalent to \( G_1[{\{b, r_1, r_3, r_4, \ldots, r_s}\}] \) by applying local complementations only at vertices in \( \{r_1, r_3, \ldots, r_{s-1}\} \) such that \( G_2[{\{a, b, c\}}] \) is a triangle.

\[ G[{\{b, r_1, r_2, \ldots, r_5\}}] \quad G[{\{b, r_1, r_2, \ldots, r_5\}}] * r_1 * r_2 \]

Figure 8: Reducing from \( G[{\{b, r_1, r_2, \ldots, r_s\}}] \) in Lemma 5.3.

Lemma 5.4. Let \( G \) be a prime graph on at least 5 vertices, and let \( a, b, c \in V(G) \). Then there exists a sequence \( x_1, x_2, \ldots, x_t \) of vertices in \( V(G) \setminus \{a, b\} \) (not necessarily all distinct) such that \( abc \) is an induced path of \( G * x_1 * x_2 * \cdots * x_t \).

Proof. We first create a triangle or an induced path of length 2 on \( \{a, b, c\} \) by applying local complementations at vertices in \( V(G) \setminus \{a, b, c\} \). For this argument, \( a, b, c \) are symmetric. Without loss of generality, we assume the distance between \( a \) and \( b \) is at most the distance between \( a \) and \( c \) or between \( b \) and \( c \). Let \( P = p_1p_2\cdots p_m \) be a shortest path from \( a = p_1 \) to \( b = p_m \) in \( G \). By the distance property, \( c \notin V(P) \). We define

\[
G_1 := \begin{cases} 
G * p_2 * p_3 * \cdots * p_{m-1} & \text{if } m \geq 3, \\
G & \text{otherwise}.
\end{cases}
\]

It is not difficult to observe that \( a \) and \( b \) are adjacent in \( G_1 \). Now, we take a shortest path \( Q = q_1q_2\cdots q_n \) from \( c = q_1 \) to \( q_n \in \{a, b\} \) in \( G_1 \). We define

\[
G_2 := \begin{cases} 
G_1 * q_2 * q_3 * \cdots * q_{n-1} & \text{if } n \geq 3, \\
G_1 & \text{otherwise}.
\end{cases}
\]

We observe that \( c \) has a neighbor on \( \{a, b\} \) in \( G_2 \). Furthermore, if \( a \) and \( b \) are not adjacent in \( G_2 \), it means that the last local complementation removed this edge, and it implies that \( c \) should be adjacent to both \( a \) and \( b \) in \( G_2 \). Therefore, either \( G_2[{\{a, b, c\}}] \) is a triangle or an induced path of length 2.
We do not want to apply local complementation at \(a, b\) to create a required induced path. If \(acb\) is already an induced path, then we are done. If \(G_2[\{a, b, c\}]\) is a triangle, then we apply local complementation at \(c\). Therefore, we may assume that \(abc\) or \(bac\) is an induced path. Note that \(G_2\) is 2-connected.

**Case 1.** \(abc\) is an induced path in \(G_2\).

We apply Lemma 5.3. Then by applying local complementations at vertices in \(V(G) \setminus \{a, b\}\), we can obtain \(G_3\) locally equivalent to \(G_2\) such that \(G_3[\{a, b, c\}]\) is a triangle. By applying a local complementation at \(c\), we obtain the required path.

**Case 2.** \(bac\) is an induced path in \(G_2\).

We apply Lemma 5.3. Then by applying local complementations at vertices in \(V(G) \setminus \{a, b\}\), we can obtain \(G_3\) locally equivalent to \(G_2\) such that \(G_3[\{a, b, c\}]\) is a triangle. By applying a local complementation at \(c\), we obtain the required path.

We conclude the lemma.

Starting from a split decomposition whose decomposition tree is a subdivision of a huge binary tree, we will extract a split decomposition of some fixed binary tree. To do this, we need to explain how we sequentially transform each bag into a star whose center is unmarked. Lemma 5.5 deal with the case when a bag has two neighbor bags, and Lemma 5.6 deal with the case when a bag has three neighbor bags.

A canonical split decomposition \(D\) is *rooted* if we distinguish a leaf bag and call it the *root* of \(D\). Let \(D\) be a rooted canonical split decomposition with root bag \(R\). A bag \(B\) is a *descendant* of a bag \(B'\) if \(B'\) is on the path of bags from \(R\) to \(B\) in \(D\). If \(B\) is a descendant of \(B'\) and \(B\) and \(B'\) are adjacent bags, then we call \(B\) a *child* of \(B'\) and \(B'\) the *parent* of \(B\). A bag in \(D\) is called a *non-root bag* if it is not the root bag.

**Lemma 5.5.** Let \(D\) be a rooted canonical split decomposition of a connected graph with root bag \(R\) and let \(B\) be a non-root bag of \(D\) such that

- \(D - V(B)\) has exactly two connected components \(T_1\) and \(T_R\) where \(T_R\) contains \(R\),
• the parent of $B$ is a star and $\zeta_c(D, B, T_R)$ is a leaf.

Then by possibly applying local complementations at unmarked vertices of $D$ contained in $V(T_1) \cup V(B)$ and deleting some unmarked vertices in $B$, we can transform $D$ into a canonical split decomposition $D'$ containing a bag $P$ such that

1. $D' - V(P)$ consists of exactly two connected components $F_R$ and $F_1$,
2. $F_R = T_R$ or $F_R = T_R \ast \zeta_c(D, B, T_R)$,
3. $F_1$ is locally equivalent to $T_1$, and
4. $P$ is a star bag whose center is unmarked.

Proof. Let $v := \zeta_b(D, B, T_R)$ and $w := \zeta_b(D, B, T_1)$. Let $y$ be an unmarked vertex in $D$ represented by $w$. See Figure 9 for the setting.

First assume that $B$ is a star bag. Since $\zeta_b(D, B, T_R)$ is a leaf, $v$ is not the center of $B$. If its center is unmarked, then we are done. We may assume the center of $B$ is $w$. Since $|V(B)| \geq 3$, $B$ contains at least one unmarked vertex, which is adjacent to $w$. We choose an unmarked leaf vertex $z$ in $B$. We observe that $y$ is linked to $z$, that is, $yz \in E(G)$. Then in $D \ast yz$, $z$ becomes the center of a star, and $T_R$ does not change. Also, $T_1$ is changed to the decomposition obtained from $T_1$ by pivoting $yz'$ where $z' = \zeta_c(D, B, T_1)$. Thus, the resulting decomposition satisfies the required property. If $B$ is a complete bag, then we choose an unmarked vertex in $B$, and apply a local complementation at this vertex. Then the resulting decomposition satisfies the required property.

Now, suppose $B$ is a prime bag. Choose an unmarked vertex $z$ of $B$ that is adjacent to $w$. Since a prime graph with at least 5 vertices is 2-connected, there is always an unmarked vertex adjacent to $w$. Note that $y$ and $z$ are linked.

Let $B_1$ be the child of $B$. If $B_1$ is a star bag whose center is adjacent to $B$, then by pivoting $yz$ we transform $B_1$ into a star bag having $\zeta_c(D, B, T_1)$ as a leaf. If $B_1$ is a complete bag, then we apply a local complementation at $y$. In the resulting decomposition, either $B_1$ is a prime bag or $\zeta_c(D, B, T_1)$ is a leaf of a star bag. Let $B'$ be the bag modified from $B$ in the resulting decomposition. Note that $B'$ is still a prime graph by Lemma 2.1.

We apply Lemma 5.4 with $(a, b, c) = (v, w, z)$. By Lemma 5.4, we can modify $B'$ into an induced path $vwz$ by only applying local complementations at unmarked vertices in $B'$ and removing all unmarked vertices in $B'$ except $z$. Note that the marked edges incident with $B'$ are still marked edges that cannot be recomposed, as both have types $S_pS_p$ or $S_pP$. Let $D'$ be the modified decomposition and let $P$ be the new bag in $D'$ modified from $B'$. Then $D' - V(P)$ has two connected components $F_R$ and $F_1$ where

- $F_R = T_R$ or $F_R = T_R \ast \zeta_c(D, B, T_R)$,
- $F_1$ is locally equivalent to $T_1$, and
- $P$ is a star whose center is unmarked,

as required.

Lemma 5.6. Let $D$ be a rooted canonical split decomposition of a connected graph with root bag $R$ and let $B$ be a non-root bag of $D$ such that
• $D - V(B)$ has exactly three connected components $T_1, T_2,$ and $T_R$ where $T_R$ contains $R$.

• the distance from node$_D(B)$ to node$_D(R)$ is at least 3 in $T_D$.

• the parent $P_1$ of $B$ and its parent $P_2$ satisfy that node$_D(P_1)$ and node$_D(P_2)$ have degree 2 in $T_D$.

• $P_1$ and $P_2$ are stars whose centers are unmarked, and

• for each $i \in \{1, 2\}$, the child $B_i$ of $B$ in $T_i$ satisfies that node$_D(B_i)$ has degree 2 in $T_D$.

Then, by possibly applying local complementations at unmarked vertices of $D$ contained in $V(T_1) \cup V(T_2) \cup V(B) \cup V(P_1) \cup V(P_2)$ and deleting some unmarked vertices in $V(T_1) \cup V(T_2) \cup V(B) \cup V(P_1) \cup V(P_2)$ and recomposing some marked edges, we can transform $D$ into a canonical split decomposition $D'$ containing a bag $P$ such that

1. $D' - V(P)$ consists of exactly three connected components $F_1, F_2,$ and $F_R$,

2. $F_R = T_R - (V(P_1) \cup V(P_2))$,

3. for each $i \in \{1, 2\}$, $F_i$ is locally equivalent to $T_i$ or $T_i - V(B_i)$, and

4. $P$ is a star bag whose center is unmarked.

Proof. For each $i \in \{1, 2\}$, let $x_i$ be the center of $P_i$, and let $v := \zeta_6(D, B, T_R)$, and for each $i \in \{1, 2\}$, let $v_i := \zeta_6(D, B, T_i)$, and $y_i$ be an unmarked vertex represented by $v_i$.

We first deal with an easier case.

Case 1. $B$ is either a star or a complete graph, and has an unmarked vertex.

The case when $B$ is a complete graph is depicted in Figure 10. We first transform $B$ into a star whose center is unmarked. Let $z$ be an unmarked vertex in $B$.

Assume $B$ is a star. Since $\zeta_6(D, B, T_R)$ is a leaf of a star, $v$ is not the center of $B$. We may assume that the center of $B$ is either $v_1$ or $v_2$. By symmetry, we may assume it is $v_1$. In this case, $y_1$ and $z$ are linked in $D$. Thus, $B$ becomes a star whose center is $z$ in $D \setminus y_1z$. If $B$ is a complete bag, then we apply a local complementation at $z$. Then $B$ becomes a star whose center is $z$. Note that in any case, $T_R$ does not change by this local complementation as $\zeta_6(D, B, T_R)$ is a leaf of a star, and $T_i$ becomes a split decomposition locally equivalent to $T_i$.

Let $D_1$ be the resulting decomposition. Lastly, we transform $D_1$ into a split decomposition $D_2$ as follows:

1. We pivot $x_1x_2$ and then remove all unmarked vertices contained in $P_1$ and $P_2$.

2. We recompose marked edges incident with $P_1$ and $P_2$. Equivalently, we remove all vertices in $P_1$ and $P_2$ in the decomposition, and add a new marked edge between $v$ and the marked vertex in the parent of $P_2$ that is adjacent to $P_2$.

Note that $D_2$ is canonical, as the new marked edge has the same type as before. Thus, we obtained a required decomposition.

Now, we may assume that either $B$ is a prime bag, or $|V(B)| = 3$.

Case 2. $|V(B)| = 3$. 

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Figure 10: When $B$ is a complete bag and has an unmarked vertex in Lemma 5.6.
Figure 11: When $B$ is a complete bag and has no unmarked vertices in Lemma 5.6.
An example case is depicted in Figure \[11\]

Since $|V(B)| = 3$, $B$ is either a star or a complete graph. We first modify $B$ into a star whose center is $v_1$. First assume that $B$ is a star. Since $\zeta_c(D, B, T_R)$ is a leaf of a star, the center of $B$ is either $v_1$ or $v_2$. We may assume the center of $B$ is $v_2$. Since $v_1$ is adjacent to $v_2$, $y_1$ and $y_2$ are linked in $D$. Then $B$ becomes a star whose center is $v_1$ in $D \setminus y_1y_2$. If $B$ is a complete bag, then we apply local complementation at $y_1$. Then $B$ becomes a star whose center is $v_1$. Note that $T_R$ does not change by this local complementation as $\zeta_c(D, B, T_R)$ is a leaf of a star and the center of the parent of $B$ is unmarked. Let $D_1$ be the resulting decomposition.

Let $w$ be the marked vertex in $P_2$ that is adjacent to $P_1$. We transform $D_1$ into a split decomposition $D_2$ as follows:

1. We pivot $x_1x_2$.
2. We delete the vertices of $V(P_1)$, and add a marked edge between $v$ and $w$.
3. We recompose the new marked edge $vw$ (it is of type $S_pS_c$).

Observe that the bag $B'$ in $D_2$ obtained by merging $B$ and $P_2$ is a star whose center is $v_1$, and it contains an unmarked vertex $x_2$. Moreover, $D_2$ is canonical. Lastly, we pivot $y_1x_2$. Then $B'$ becomes a star whose center is $x_2$. Note that the connected components of $D_2 - V(B')$ are respectively $T_R - (V(P_1) \cup V(P_2))$ and $F_1$ and $F_2$ such that $F_i$ is locally equivalent to $T_i$ for $i \in \{1, 2\}$.

Now, it remains to show when $B$ is a prime bag. We reduce this case to Case 1 or Case 2 applying Lemma \[5.4\]. Note that in the previous cases, we deduce that $F_i$ is locally equivalent to $T_i$ for each $i \in \{1, 2\}$. But when we transform $B$ into a star bag, we may merge $B$ with one of its child bags.

**Case 3.** $B$ is a prime bag.

Note that applying a local complementation at an unmarked vertex in $B$ does not change the fact that $y_1$ is represented by $v_1$. This is because the alternating path from $y_1$ to $v_1$ does not change when we apply a local complementation at an unmarked vertex in $B$.

We apply Lemma \[5.4\] with $(a, b, c) = (v, v_2, v_1)$ so that $B$ is transformed into an induced path $vv_1v_2$. Note that applying a local complementation at $v_1$ can be simulated by applying a local complementation at $y_1$. Since $B$ is a prime graph on at least 5 vertices, by Lemma \[5.4\], we can modify $B$ into an induced path $vv_1v_2$ by only applying local complementations at unmarked vertices in $B$ and $y_1$. Then we remove all the other vertices of $B$.

Note that the marked edge connecting $B$ and $P_1$ is still a valid marked edge as $\zeta_c(D, B, T_R)$ is a leaf of a star. However, for $i \in \{1, 2\}$, the marked edge incident with $v_i$ and $\zeta_c(D, B, T_i)$ may have type $S_pS_c$. In this case, we recompose this marked edge so that the resulting decomposition is canonical.

Let $D_1$ be the modified decomposition. Since both $\text{node}_D(P_1)$ and $\text{node}_D(P_2)$ have degree 2 in $T_D$, the bag $B'$ of $D_1$ modified from $B$ still has 3 adjacent bags in $D_1$. As $B'$ is star bag of $D_1$, we can reduce the remaining steps to Case 1 or Case 2 depending on the size of $B'$, from which we can construct the required canonical split decomposition.

We are ready to prove the main result of the section. We note that for a graph $H$, any subdivision of $H$ contains a vertex-minor isomorphic to $H$. We will use this fact. For a tree $T$, let $\eta(T)$ be the tree obtained from $T$ by replacing each edge with a path of length 4.
Proof of Theorem 2.2. Let \( t := |V(T)| \) and suppose that \( \text{lrw}(G) \geq 40(p + 2)t \). By Lemma 5.2, there exists a subcubic tree \( T' \) such that \( T \) is a vertex-minor of \( T' \) and \( |V(T')| \leq 5t \). We consider the tree \( \eta(T') \) which is the tree obtained from \( T' \) by replacing each edge with a path of length 4. Observe that \( |V(\eta(T'))| \leq 20t \).

Let \( D \) be the canonical split decomposition of \( G \) and let \( T_D \) be the decomposition tree of \( D \). Since \( \text{lrw}(G) \geq 40(p + 2)t \), by Proposition 4.1, \( \text{pw}(T_D) \geq 20t - 1 \). Since \( |V(\eta(T'))| \leq 20t \), from Theorem 2.2, \( T_D \) contains a minor isomorphic to \( \eta(T') \). Since the maximum degree of \( \eta(T') \) is at most 3, \( T_D \) contains a subgraph \( T_1 \) that is isomorphic to a subdivision of \( \eta(T') \). Let \( D_1 := D[\bigcup_{v \in V(T_1)} V(\text{bag}_D(v))] \). Observe that \( D_1 \) is not necessarily a decomposition of an induced subgraph of \( G \), as the unmarked vertex which was a marked vertex before does not correspond to a real vertex of \( G \). To make it as a decomposition of an induced subgraph of \( G \), we obtain a new decomposition \( D_2 \) from \( D_1 \) as follows:

- For every unmarked vertex \( x \) of \( D_1 \) that was a marked vertex in \( D \), there is a vertex \( y \in V(G) \) represented by \( x \) in \( D \). We choose such a vertex and replace \( x \) with \( y \).

We can observe that \( D_2 \) is a canonical split decomposition of an induced subgraph of \( G \), and \( T_{D_2} \) is isomorphic to \( T_{D_1} \).

We choose a leaf bag \( R_2 \) of \( D_2 \) and regard it as the root of \( D_2 \). We first transform \( R_2 \) into a star where the marked vertex in \( R_2 \) is a leaf by applying local complementations at unmarked vertices of \( D_2 \).

(*) Let \( v \) be the marked vertex of \( R_2 \), and \( v' \) be a neighbor of \( v \) in \( R_2 \), and \( w \) be an unmarked vertex of \( D_2 \) represented by \( v \). If \( R_2 \) is a star whose center is unmarked, then we do nothing. If \( R_2 \) is a star whose center is \( v \), then we pivot \( v'w \). If \( R_2 \) is a complete bag, then we apply local complementation at \( v' \). Then \( R_2 \) becomes a star whose center is unmarked.

Assume \( R_2 \) is a prime bag and let \( C \) be the child of \( R_2 \). If \( C \) is a star whose center is \( c \) is adjacent to \( v \), then we do a pivot at \( v'w \) to turn \( C \) into a star with \( c \) as a leaf. If \( C \) is a complete graph, then we apply a local complementation at \( w \). The bag modified from \( C \) is either a prime graph or a star whose leaf is adjacent to \( v \). Let \( R'_2 \) be the resulting bag from \( R_2 \).

Now, we choose one more unmarked vertex \( v'' \) in \( R'_2 \) adjacent to \( v \). Such a vertex exists as \( R'_2 \) is 2-connected. Applying Lemma 5.4 to \( R'_2 \) with \( (a, b, c) = (v, v', v'') \), there exists a sequence \( x_1, x_2, \ldots, x_{\ell} \) of vertices in \( V(R'_2) \setminus \{v, v'\} \) such that \( vv''v' \) is an induced path of \( R'_2 * x_1 * x_2 * \cdots * x_{\ell} \). We apply this sequence of local complementations and then remove all vertices in \( R'_2 \) except \( v, v' \), and \( v'' \). By the previous procedure, the resulting decomposition is canonical and the bag modified from \( R'_2 \) is a star whose center is unmarked.

Let \( D_3 \) be the resulting decomposition, and \( R_3 \) be the root bag that is modified from \( R_2 \). Note that \( T_{D_3} \) is isomorphic to \( T_{D_2} \).

As \( T_{D_3} \) is isomorphic to a subdivision of \( \eta(T') \), there is a subdivision mapping \( g \) from \( T' \) to \( T_{D_3} \) such that for each edge \( e \) of \( T' \), \( g(e) \) is a path of length at least 4. Note that \( g(V(T')) \) is exactly the set of all leaves and all vertices of degree at least 3 in \( T_{D_3} \).

A bag \( B \) is processed if every bag on the path from \( B \) to the root bag is a star whose center is unmarked. Let \( B_1, B_2, \ldots, B_m \) be an ordering of bags in \( \{ \text{bag}_{D_3}(v) : v \in g(V(T')) \} \) such that

- for each \( i \in \{2, 3, \ldots, m\} \), every ascendant bag of \( B_i \) in the set \( \{ \text{bag}_{D_3}(v) : v \in g(V(T')) \} \) is contained in \( \{ B_1, B_2, \ldots, B_{i-1} \} \).
Such an ordering can be found using a BFS. For each \( i \in \{2, 3, \ldots, m\} \), let \( F(B_i) \) be the bag \( B \) in \( \{B_1, B_2, \ldots, B_{i-1}\} \) such that \( B \) is an ascendant bag of \( B_i \), and \( B \) is closest to \( B_i \). We will define below a sequence \( F_1, F_2, \ldots, F_m \) of rooted canonical split decompositions such that \( \text{node}_{D_3}(B_j) \in V(T_{F_i}) \) for \( 1 \leq i, j \leq m \), and for convenience we keep continuing calling \( B_j \) the bag \( \text{bag}_{F_i}(\text{node}_{D_3}(B_j)) \).

For \( j \in \{1, 2, \ldots, m\} \), let \( F_1, F_2, \ldots, F_j \) be a maximal sequence of rooted canonical split decompositions such that

- \( D_3 = F_1 \),
- \( \text{for each } i \in \{1, 2, \ldots, j-1\}, G[F_{i+1}] \) is a vertex-minor of \( G[F_i] \),
- \( \text{in } F_i \) with \( i \in \{1, 2, \ldots, j\} \),
  - \( B_1, B_2, \ldots, B_i \) are processed,
  - \( \text{for } \ell \in \{2, 3, \ldots, i\}, \text{dist}_{F_i}(B_{\ell}, F(B_{\ell})) \geq 1 \),
  - \( \text{if } B \in \{B_{i+1}, B_{i+2}, \ldots, B_m\} \) is a bag where \( F(B) \) is processed, then \( \text{dist}_{F_i}(B, F(B)) \geq 3 \),
  - \( \text{if } B \in \{B_{i+1}, B_{i+2}, \ldots, B_m\} \) is a bag where \( F(B) \) is not processed, then \( \text{dist}_{F_i}(B, F(B)) \geq 4 \),
  - \( \text{node}_{D_3}(R_3) \in V(T_{F_i}) \) and \( F_i \) is rooted at \( \text{bag}_{F_i}(\text{node}_{D_3}(R_3)) \).

By (*), \( B_1 = R_3 \) is processed. Thus, \( F_1 \) is indeed a sequence satisfying those conditions when \( j = 1 \). We claim that \( j = m \). In other words, all bags in \( \{\text{bag}_{D_3}(v) : v \in g(V(T'))\} \) can be sequentially processed.

**Claim 1.** \( j = m \).

**Proof.** Suppose for contradiction that \( j < m \). We may assume that \( B_{j+1} \) is not processed in \( F_j \), otherwise, \( F_1, F_2, \ldots, F_j, F_{j+1} \) is a longer sequence satisfying the required conditions. Clearly, \( F(B_{j+1}) \) is processed. The induction hypothesis for \( j \) implies that \( \text{dist}_{F_j}(B_{j+1}, F(B_{j+1})) \geq 3 \).

Let \( F(B_{j+1}) = U_1 - U_2 - \cdots - U_y = B_{j+1} \) be the path of bags in \( F_j \) from \( F(B_{j+1}) \) to \( B_{j+1} \).

We recursively apply Lemma 5.5 to \( U_2, U_3, \ldots, U_{y-1} \) so that the bag modified from each of \( U_2, U_3, \ldots, U_{y-1} \) is a star whose center is unmarked. Note that when we apply Lemma 5.5 to \( U_2, U_3, \ldots, U_{y-1} \), the decomposition tree does not change.

After that, we apply Lemma 5.6 to \( B_{j+1} \) so that the bag modified from \( B_{j+1} \) is a star whose center is unmarked. When we apply Lemma 5.6 to \( B_{j+1} \), some child bags of \( B_{j+1} \) may be merged with \( B_{j+1} \). Thus if \( U \) is a bag with \( F(U) = B_{j+1} \), then the value \( \text{dist}_{F_j}(U, B_{j+1}) \) may decrease by at most 1.

Let \( F_{j+1} \) be the resulting decomposition. We can verify that in \( F_{j+1} \),

- \( B_1, B_2, \ldots, B_{j}, B_{j+1} \) are processed,
- \( \text{for } \ell \in \{2, 3, \ldots, j+1\}, \text{dist}_{F_{j+1}}(B_{\ell}, F(B_{\ell})) \geq 1 \),
- \( \text{if } B \in \{B_{j+2}, B_{j+3}, \ldots, B_m\} \) is a bag where \( F(B) \) is processed, then \( \text{dist}_{F_{j+1}}(B, F(B)) \geq 3 \),
- \( \text{if } B \in \{B_{j+2}, B_{j+3}, \ldots, B_m\} \) is a bag where \( F(B) \) is not processed, then \( \text{dist}_{F_{j+1}}(B, F(B)) \geq 4 \).
\begin{itemize}
\item $\text{node}_{D_3}(R_3) \in V(T_{F_3})$ and $F_i$ is rooted at $\text{bag}_{F_i}(\text{node}_{D_3}(R_3))$
\end{itemize}

This contradicts the maximality of the sequence. We conclude that $j = m$. \hfill \Box

Let $D_4 := F_m$. Note that $T_{D_4}$ is isomorphic to a subdivision of $T'$, and every bag of $D_4$ is a star whose center is unmarked. Therefore, $G[D_4]$ is isomorphic to a tree that can be obtained from a subdivision of $T'$ by adding some leaves, and in particular, $G[D_4]$ contains an induced subgraph isomorphic to a subdivision of $T'$. Thus, $G$ contains a vertex-minor isomorphic to $T'$, and also contains a vertex-minor isomorphic to $T$, as required. \hfill \Box

6 Distance-hereditary vertex-minor obstructions for graphs of bounded linear rank-width

In this section, we describe a way to generate all vertex-minor obstructions for graphs of bounded linear rank-width that are distance-hereditary graphs. It generalizes the constructions developed by Jeong, Kwon, and Oum [24].

For a distance-hereditary graph $G$, a connected distance-hereditary graph $G'$ is a one-vertex DH-extension of $G$ if $G = G' - v$ for some vertex $v \in V(G')$. For convenience, if $G'$ is a one-vertex DH-extension of $G$, and $D$ and $D'$ are canonical split decompositions of $G$ and $G'$ respectively, then $D'$ is also called a one-vertex DH-extension of $D$.

Let $D_1, D_2$ and $D_3$ be three canonical split decompositions. For each $i \in \{1, 2, 3\}$, let $D'_i$ be a one-vertex DH extension of $D_i$ with a new unmarked vertex $w_i$ and such that $w_i$ is not contained in a star bag centered at $w_i$. Furthermore, we choose an unmarked vertex $z_i$ linked to $w_i$. Let $B$ be a complete graph or a star, on three vertices $v_1, v_2, v_3$. For each $i \in \{1, 2, 3\}$, let $D''_i$ be a split decomposition such that

1. if $B$ is a complete graph, then $D''_i := D'_i \ast w_i$;
2. if $B$ is a star with center $v_i$, then $D''_i := D'_i \wedge w_i z_i$;
3. if $B$ is a star with $v_i$ a leaf, then $D''_i := D'_i$.

We let $\mathcal{N}(D_1, D_2, D_3, K)$ be the set of all possible canonical split decompositions obtained from the disjoint union of such $D''_1, D''_2, D''_3$ and a complete bag $B$ on three vertices $v_1, v_2, v_3$, by adding the marked edges $v_1 w_1, v_2 w_2$, and $v_3 w_3$. For $i \in \{1, 2, 3\}$, we let $\mathcal{N}(D_1, D_2, D_3, (S, i))$ be the set of all possible canonical split decompositions obtained from the disjoint union of such $D''_1, D''_2, D''_3$ and a star bag $B$ on three vertices $v_1, v_2, v_3$ whose center is $v_i$, by adding the marked edges $v_1 w_1, v_2 w_2$, and $v_3 w_3$.

For a set $\mathcal{D}$ of canonical split decompositions, we let

$$
\Delta(\mathcal{D}) := \{ \mathcal{N}(D_1, D_2, D_3, K) \mid D_1, D_2, D_3 \in \mathcal{D} \}
\cup \{ \mathcal{N}(D_1, D_2, D_3, (S, i)) \mid D_1, D_2, D_3 \in \mathcal{D}, i \in \{1, 2, 3\} \},
\mathcal{D}^+ := \mathcal{D} \cup \{ D' : D'$ is a one vertex DH-extension of $D \in \mathcal{D} \}.
$$

For each non-negative integer $k$, we recursively construct the set $\Psi_k$ of canonical split decompositions as follows.

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1. Ψ₀ := {K₂} (K₂ is the canonical split decomposition of itself.)

2. For κ ≥ 0, let Ψₖ₊₁ := Δ(Ψₖ⁺).

We prove the following.

**Theorem 6.1.** Let κ be a non-negative integer. Every distance-hereditary graph of linear rank-width at least κ + 1 contains a vertex-minor isomorphic to a graph whose canonical split decomposition is isomorphic to a decomposition in Ψₖ.

We prove some intermediate lemma.

**Lemma 6.2.** Let D be the canonical split decomposition of a connected distance-hereditary graph containing two distinct bags B₁ and B₂, and for each i ∈ {1, 2}, let Tᵢ be the connected component of D − V(Bᵢ) such that Tᵢ contains B₃₋ᵢ. If

- ζₒ(D, B₁, T₁) is not the center of a star and
- B₂ is a star bag and ζₒ(D, B₂, T₂) is a leaf of B₂,

then there exists a canonical split decomposition D′ such that

1. G[D] has G[D′] as a vertex-minor,
2. D[V(T₂)\V(T₁)] = D′[V(T₂)\V(T₁)],
3. D[V(T₁)\V(T₂)] = D′[V(T₁)\V(T₂)], and
4. either B₁ and B₂ are adjacent in D′, or there is a path of bags B₁ − B₁ − B₂ in D′ such that |V(B)| = 3 and B is a star bag whose center is unmarked.
Proof. If $B_1$ and $B_2$ are adjacent bags in $D$, then we are done. We assume that $B_1$ and $B_2$ are not adjacent. Let $B_1 = U_1 - U_2 - \cdots - U_m = B_2$ be the path of bags in $D$. Also, let $P = p_1p_2\ldots p_\ell$ be the shortest path from $\zeta_0(D, B_1, T_1) = p_1$ to $\zeta_0(D, B_2, T_2) = p_\ell$ in $D$. Note that $\ell \geq 4$ as $m \geq 3$.

Suppose there exists a bag $U_i$ containing exactly two consecutive vertices $p_j$, $p_{j+1}$ of $P$. In this case, we remove $U_i$ and remove all the connected components of $D - V(U_i)$ that contain neither $B_1$ nor $B_2$, and add a marked edge $p_{j-1}p_{j+2}$. This procedure corresponding to removing all unmarked vertices in the removed sub-decomposition. Since this operation does not change the parts $D[V(T_2)\backslash V(T_1)]$ and $D[V(T_1)\backslash V(T_2)]$, applying this operation consecutively, we may assume that for $i \in \{2, 3, \ldots, m - 1\}$, $U_i$ contains three consecutive vertices of $P$. In other words, $U_i$ is a star whose center is adjacent to neither a vertex of $U_{i-1}$ nor to a vertex of $U_{i+1}$. See 2) of Figure 12.

Suppose $m \geq 4$. Note that $U_2$ contains $p_2, p_3, p_4$ and $U_3$ contains $p_5, p_6, p_7$. Take two unmarked vertices $x_3$ and $x_6$ of $D$ that are represented by $p_3$ and $p_6$, respectively. Observe that $x_3$ and $x_6$ are linked in $D$. Let $D' := D \cup x_3x_6$. Notice that $D'[V(U_2)]$ and $D'[V(U_3)]$ are stars whose centers are adjacent to each other. Moreover, $D'[V(T_2)\backslash V(T_1)] = D[V(T_2)\backslash V(T_1)]$ and similarly, $D'[V(T_1)\backslash V(T_2)] = D[V(T_1)\backslash V(T_2)]$. For each $i \in \{2, 3\}$, we delete from $D'$, $U_i$ and all the connected components of $D' - V(U_i)$, except two connected components containing $B_1$ and $B_2$ respectively, and add the marked edge $p_1p_8$. See 3) and 4) of Figure 12. By the assumption that $p_1$ is not the center of $B_1$, the marked edge incident with $B_1$ is of type $S_pS_p$ or $KS_p$.

Therefore, the resulting decomposition is a canonical split decomposition satisfying the conditions (1), (2), (3), and the number of bags containing $P$ is decreased by two.

Applying this procedure recursively, at the end, we obtain a canonical split decomposition such that either $B_1$ and $B_2$ are adjacent, or there is a path of bags $B_1 - B - B_2$ such that $B$ is a star bag whose center is adjacent to neither $B_1$ nor $B_2$. In the latter case, we remove all unmarked leaves of $B$, and remove all connected components of $D - V(B)$ containing neither $B_1$ nor $B_2$, and replace the center of $B$ with an unmarked vertex represented by it. Then we obtain the required decomposition.

The next proposition says how we can replace limbs having linear rank-width $k \geq 1$ into a canonical split decomposition in $\Psi_{k-1}^L$ using Lemma 6.2. In this proposition, we sometimes remove unmarked vertices from a given split decomposition, to take a split decomposition of the graph obtained by removing the corresponding vertices. We described this operation in Section 3.2.

Proposition 6.3. Let $D$ and $A$ be the canonical split decompositions of some connected distance-hereditary graphs. Let $B$ be a star bag of $D$ and $v$ be a leaf of $B$, and $T$ be a connected component of $D - V(B)$ such that $\zeta_5(D, B, T) = v$, and let $w$ be an unmarked vertex of $D$ represented by $v$. If $L_D[B, w]$ has a vertex-minor that is either $G[A]$ or a one-vertex DH extension of $G[A]$, then there exists a canonical split decomposition $D'$, a vertex-minor of $D$, such that

1. either $D' - V(T) = D - V(T)$ or $D' - V(T) = (D - V(T)) * v$, and

2. for some unmarked vertex $w'$ of $D'$ represented by $v$, $L_{D'}[B, w']$ is either $A$ or a one-vertex DH-extension of $A$.

Proof. Suppose $L_D[B, w]$ has a vertex-minor that is either $G[A]$ or a one-vertex DH extension of $G[A]$. It means that there exist a sequence $x_1, x_2, \ldots, x_m$ of vertices of $L_D[B, w]$ and $S \subseteq V(L_D[B, w])$ such that $(L_D[B, w] * x_1 * x_2 * \ldots * x_m) - S$ is either $G[A]$ or a one-vertex DH-extension of $G[A]$. So, there exists $Q \subseteq V(L_D[B, w])$ such that the graph obtained from $(L_D[B, w]*
Let $B_2$ be the bag of $D_3$ containing $y$. We divide into cases depending on whether $B$ and $B_2$ are adjacent or not.

**Case 1.** $B$ and $B_2$ are adjacent in $D_3$.

In this case, $D_3$ itself is a required decomposition. Choose an unmarked vertex $z$ in $D_3$ represented by $v$. Then $\mathcal{L}_{D_3}[B, z]$ is the same as the split decomposition obtained from $(\mathcal{L}_D[B, w]$ *
Therefore, the construction, \(D_H\)-extension of \(D\), then let \(y\) from the disjoint union of the two connected components of \(B\) recomposing a marked edge incident with \(z\), special case.

operation of removing a new one-vertex extension of \(c\) vertex added vertex \(a\) in \(H\).

We would like to remove the extended vertex \(a\) from \(H\), and then add \(c\) to \(H\) so that we obtain a new one-vertex extension of \(A\) which contains \(c\). But this is not always possible because the operation of removing \(a\) may disconnect the remaining part of \(H\) from \(c\). We first deal with this special case.

Assume \(B_2\) is a star whose center is an unmarked vertex in \(D_3\). In this case this center should be \(z\). We obtain a new decomposition \(D_4\) by applying a local complementation at \(c\), removing \(c\) and recomposing a marked edge incident with \(B_s\). Note that \(D_4\) is exactly the decomposition obtained from the disjoint union of the two connected components of \(D_3 - V(B_s)\) by adding a marked edge \(yv\), and thus it is canonical. Also, \(z\) is represented by \(v\) in \(D_4\), and we have \(LC_{D_4}(B, z) = H\). Thus, \(D_4\) is a required decomposition.

Now we assume that \(c\) is linked to at least two vertices of \(H\) in \(D_3\). Since \(H\) is a one vertex \(D_H\)-extension of \(A\) and \(A\) was chosen as a canonical split decomposition of a connected graph, \(G[H]\) - \(a\) is connected. So, if we define \(D_4\) as the canonical split decomposition obtained from \(D_3 - a\), then \(D_4\) is connected and \(LC_{D_4}(B, c)\) can be regarded as a one vertex \(D_H\)-extension of \(A\). Therefore, \(D_4\) is a required decomposition.

Proof of Theorem 6.1. We prove it by induction on \(k\). If \(k = 0\), then \(\text{lrw}(G) \geq 1\) and \(G\) has an edge. We may assume \(k \geq 1\).

Let \(D\) be the canonical split decomposition of \(G\). Since \(G\) has linear rank-width at least \(k + 1\), by Theorem 3.3, there exists a bag \(B\) in \(D\) with three connected components \(T_1, T_2, T_3\) of \(D - V(B)\) such that \(f_{D}(B, T_i) \geq k\) for each \(i \in \{1, 2, 3\}\).

We remove all connected components of \(D - V(B)\) other than \(T_1, T_2, T_3\), and for each marked vertex \(w\) in \(B\) that was adjacent to some removed component, we choose a vertex \(w'\) in \(D\) represented by \(B\) and replace \(w\) with \(w'\). Note that the resulting decomposition is a canonical split decomposition of an induced subgraph of \(G\).

Now, if \(B\) is a star whose center is unmarked, then we apply a local complementation at this vertex, and otherwise, we change nothing. Then we obtain a new decomposition by removing all unmarked vertices in \(B\). Let us denote by \(D'\) this canonical split decomposition and denote by \(B'\) the bag modified from \(B\).

For each \(i \in \{1, 2, 3\}\), let \(v_i := \zeta_b(D', B', T_i)\) and \(w_i := \zeta_c(D', B', T_i)\), and \(z_i\) be an unmarked vertex of \(D'\) represented by \(v_i\) in \(D'\).

We define a new decomposition \(D_1\) as follows. If \(B'\) is a star bag centered at \(v_3\), then let \(D_1 := D'\). If \(B\) is a complete bag, then let \(D_1 := D' \ast z_3\). If \(B\) is a star bag centered at \(v_i \in \{v_1, v_2\}\), then let \(D_1 := D \ast z_i \ast z_3\). One easily checks that \(D_1[\{v_1, v_2, v_3\}]\) is a star centered at \(v_3\). Let \(B_1 := D_1[\{v_1, v_2, v_3\}]\) and, for \(j \in \{1, 2, 3\}\), let \(T_j := D_1[V(T_j)]\). Note that \(z_i\) is still represented
We apply Proposition 6.3 to $z$ hypothesis, there exists a canonical split decomposition $w$ of $v_i$. Let $D_i = \text{LC}_{D_i}[B^1, z_i] = T_i^1 - w_i$ and by the induction hypothesis, there exists a canonical split decomposition $F_i$ in $\Psi_{k-1}$ such that $\text{LC}_{G_i}[B^1, z_i]$ has a vertex-minor isomorphic to $G[F_i]$. By applying Proposition 6.3 to $T_1^1$ and $T_2^1$, we can obtain a canonical split decomposition $D_2$ satisfying that

1. $D_2[V(B^1)] = D_1[V(B^1)]$,
2. $D_2[V(T_3)]$ is either $T_3$ or $T_3 \ast w_3$, and
3. for each $i \in \{1, 2\}$, $\text{LC}_{D_2}[D_2[V(B^1)], z_i^2]$ is isomorphic to a canonical split decomposition in $\Psi_{k-1}$ for some unmarked vertex $z_i^2$ of $D_2$ represented by $v_i$.

Let $B^2 := D_2[V(B^1)]$. For each $i \in \{1, 2\}$, let $T_i^2$ be the connected component of $D_2 - V(B^2)$ containing $z_i^2$, and $w_i^2 := z_i^2$. Let $w_3^2 := w_3$, $z_3^2 := z_3$, and $T_3^2 := D_2[V(T_3)]$.

Now, we want to transform $B^2$ into a star whose center is $v_2$ by applying local complementations at $z_3^2$ and $z_2^2$. We can verify that

1. $(D_2 \ast z_3^2 \ast z_2^2)[V(B^2)]$ is a star whose center is $v_2$,
2. $(D_2 \ast z_3^2 \ast a_2)[V(T_1^2)] = T_1^2$,
3. $(D_2 \ast z_3^2 \ast a_2)[V(T_2^2)] = T_2^2 \ast w_3^2 \ast z_2^2$,
4. $(D_2 \ast z_3^2 \ast a_2)[V(T_3^2)] = T_3^2 \ast z_3^2 \ast w_3^2$.

We apply Proposition 6.3 to $D_2 \ast z_3^2 \ast z_2^2$ and obtain a canonical split decomposition $D_3$ so that

1. $D_3[V(B^2)] = (D_2 \ast z_3^2 \ast z_2^2)[V(B^2)]$ and $D_3[V(T_1^2)] = (D_2 \ast z_3^2 \ast z_2^2)[V(T_1^2)]$,
2. $D_3[V(T_2^2)]$ is either $(D_2 \ast z_3^2 \ast z_2^2)[V(T_2^2)]$ or $(D_2 \ast z_3^2 \ast z_2^2)[V(T_2^2)] \ast w_3^2$, and
3. $\text{LC}_{D_3}[D_3[V(B^2)], z_3^2]$ is isomorphic to a canonical split decomposition in $\Psi_{k-1}$ for some unmarked vertex $z_3^2$ of $D_3$ represented by $v_3$.

Let $B^3 := D_3[V(B^2)]$. Let $T_3^3$ be the connected component of $D_3 - V(B^3)$ containing $z_3^3$, and $w_3^3 := \zeta(h, T_3^3)$. Note that $T_3^3 \ast w_3^3 \ast z_3^3 \in \Psi_{k-1}$ and for $i \in \{1, 2\}$, $z_i^3$ is still represented by $v_i$ in $D_3$. We define $T_1^3 := D_3[V(T_1^2)]$, $T_2^3 := D_3[V(T_2^2)]$ and define $w_1^3, w_2^3, z_1^3, z_3^3$ as the same as $w_1^2, w_2^2, z_1^2, z_2^2$, respectively.

Now we claim that $D_3 \in H_k$ or $D_3 \ast z_3^3 \in H_k$. We observe two cases depending on whether $T_3^3$ is equal to $(D_2 \ast z_3^2 \ast z_2^2)[V(T_2^2)]$ or to $(D_2 \ast z_3^2 \ast z_2^2)[V(T_2^2)] \ast w_3^2$.

**Case 1.** $T_3^3 = (D_2 \ast z_3^2 \ast z_2^2)[V(T_2^2)]$.

We observe that $B^3$ is a star whose center is $v_2$, and the three connected components of $D_3 - V(B^3)$ are $T_1^3, T_2^3 \ast w_3^2 \ast z_2^2$, and $T_3^3$. In this case, $D_3 \ast z_3^2 \in H_k$ because

1. $(D_3 \ast z_3^2)[V(B^3)]$ is a complete bag, and
2. the three components of $D_3 - V(B^3)$ are $T_1^3 \ast w_1^3, T_2^3 \ast w_2^3, T_3^3 \ast w_3^3$.  

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and the limbs of $D_3 \ast z_2^2$ with respect to $B^3$ are $T_1^3 - w_1^2$, $T_2^3 - w_2^2$, and $T_3^3 - w_3^2$, which are contained in $\Psi_{k-1}$.

**Case 2.** $T_3^3 = (D_2 \ast z_2^2 \ast z_2^2)[V(T_2^3)] \ast w_2^2$.

We observe that $B^3$ is a star centered at $v_2$, and the three components of $D_3 - V(B^3)$ are $T_1^3$, $T_2^3 \ast w_2^2 \ast z_2^2 \ast w_2^2 = T_2^3 \ast w_2^2 \ast z_2^2$, and $T_3^3$. We can see that $D_3 \in \Psi_k$ because the limbs with respect to $B^3$ are $T_1^3 - w_1^2$, $T_2^3 - w_2^2$, and $T_3^3 - w_3^2$, which are contained in $\Psi_{k-1}$.

We conclude that $G$ has a vertex-minor isomorphic to $G[D_3]$ where $D_3 \in \Psi_k$, as required. □

In order to prove that $\Psi_k$ is a minimal set of canonical split decompositions of distance-hereditary vertex-minor obstructions for linear rank-width at most $k$, we need to prove that for every $D \in \Psi_k$, $G[D]$ has linear rank-width $k + 1$ and all its proper vertex-minors have linear rank-width at most $k$. However, while $lrw(G[D]) = k + 1$ for all $D \notin \Psi_k$, they are not minimal with respect to having linear rank-width $k + 1$. For instance for many canonical split decompositions $D$ in $\Psi_1$, $G[D]$ is not a vertex-minor obstruction for linear rank-width $k + 1$ as it contains either $\alpha_1$ or $\gamma_1$ as a vertex-minor (see Section 7). We guess that the following set $\Phi_k$ would form a minimal set of distance-hereditary vertex-minor obstructions, but we leave it as an open problem.

1. $\Phi_0 := \{K_2\}$.

2. For $k \geq 0$, let $\Phi_{k+1} := \Delta(\Phi_k)$.

Our intuition is supported by the following.

**Proposition 6.4.** Let $k$ be a non-negative integer and let $D \in \Phi_k$. Then $lrw(G[D]) = k + 1$ and every proper vertex-minor of $G[D]$ has linear rank-width at most $k$.

We need the following two lemmas.

**Lemma 6.5.** Let $D \in \Phi_k$ and $v$ be an unmarked vertex in $D$. Then $D \ast v \in \Phi_k$.

**Proof.** We proceed by induction on $k$. We may assume that $k \geq 1$. By the construction, there exists a bag $B$ of $D$ such that the three limbs $D_1$, $D_2$, $D_3$ in $D$ corresponding to the bag $B$ are contained in $\Phi_{k-1}$.

Let $B' := B$ or $B' := B \ast v'$ be a bag of $D \ast v$ depending on whether $v$ has a representative $v'$ in $B$. Let $D_1'$, $D_2'$ and $D_3'$ be the three limbs of $D \ast v$ corresponding to the bag $B'$ such that $D_1'$ and $D_i$ came from the same component of $D - V(B)$. One checks by Proposition 3.7 that $D_i'$ is locally equivalent to $D_i$. So by the induction hypothesis, $D_i' \in \Phi_{k-1}$. And $D \ast v$ is the canonical split decomposition obtained from $D_i'$ following the construction of $\Phi_k$. Therefore, $D \ast v \in \Phi_k$. □

**Lemma 6.6** (Bouchet [9]). Let $G$ be a graph, $v$ be a vertex of $G$ and $w$ be an arbitrary neighbor of $v$. Then every elementary vertex-minor obtained from $G$ by deleting $v$ is locally equivalent to either $G - v$, $G \ast v - v$, or $G \ast vw - v$.

**Proof of Proposition 6.4.** By construction, it is not hard to prove by induction with the help of Theorem 3.8 that $lrw(G[D]) = k + 1$ for every split decomposition $D \in \Phi_k$. For the second statement, by Lemmas 6.5 and 6.6 it is sufficient to show that if $D \in \Phi_k$ and $v$ is an unmarked vertex of $D$, then $G[D] - v$ has linear rank-width at most $k$. We use induction on $k$ to prove it. We may assume that $k \geq 1$. Let $B$ be the bag of $D$ such that $D - V(B)$ has exactly three limbs.
that are contained in $\Phi_{k-1}$. Clearly there is no other bag having the same property. Since $B$ has no unmarked vertices, $v$ is contained in one of the limbs $D'$, and by induction hypothesis, $G[D'] - v$ has linear rank-width at most $k - 1$. Therefore, by Theorem 3.8, $G[D] - v$ has linear rank-width at most $k$.

We finish by pointing out that it is proved in [21] that the number of distance-hereditary vertex-minor obstructions for linear rank-width $k$ is at least $2^{\Omega(3^k)}$. One can easily check by induction that the number of graphs in $\Phi_k$ is bounded by $2^{O(3^k)}$. Therefore, we can conclude that the number of distance-hereditary vertex-minor obstructions for linear rank-width $k$ is equal to $2^{\Theta(3^k)}$.

7 Simpler proofs for the characterizations of graphs of linear rank-width at most 1

In this section, we obtain simpler proofs for known characterizations of the graphs of linear rank-width at most 1 using Theorem 3.8. Theorem 7.1 was originally proved by Bui-Xuan, Kanté, and Limouzy [12].

**Theorem 7.1** (Bui-Xuan, Kanté, and Limouzy [12]). Let $G$ be a connected graph and let $D$ be the canonical split decomposition of $G$. The following two are equivalent.

1. $G$ has linear rank-width at most 1.
2. $G$ is distance-hereditary and $T_D$ is a path.

**Proof.** We first prove that (2) implies (1). Let $T_D := u_1 u_2 \cdots u_m$. For each $1 \leq i \leq m$, we take any ordering $L_i$ of unmarked vertices in bag $D(u_i)$. Since $G$ is distance-hereditary, by Theorem 3.3, each bag of $D$ is a complete graph or a star. Thus, we can easily check that $L_1 \oplus L_2 \oplus \cdots \oplus L_m$ is a linear layout of $G$ having width at most 1.

We prove that (1) implies (2). Suppose $G$ has linear rank-width at most 1. From the known fact that a connected graph has rank-width at most 1 if and only if it is distance-hereditary [30], $G$ is distance-hereditary. Suppose $T_D$ is not a path. Then there exists a bag $B$ of $D$ such that $B$ has at least three neighbor bags in $D$. Thus, $D - V(B)$ has at least three components $T$ where $f_D(B,T) \geq 1$. By Theorem 3.8, $G$ has linear rank-width at least 2, which is a contradiction.

From Theorem 7.1, we have a linear-time algorithm to recognize the graphs of linear rank-width at most 1.

**Theorem 7.2.** For a given graph $G$, we can test whether $G$ has linear rank-width at most 1 or not in time $O(|V(G)| + |E(G)|)$.

**Proof.** We first compute the canonical split decomposition $D$ of each connected component of $G$ using the algorithm from Theorem 3.1. It takes $O(|V(G)| + |E(G)|)$ time. Furthermore, this algorithm outputs the type of each bag together. Note that each bag of a canonical split decomposition of a connected distance-hereditary graph is either a complete graph or a star by Theorem 3.3. Thus, if there is a prime bag, then we answer that $G$ has linear rank-width more than 1.

Additionally, we check whether $T_D$ is a path or not. By Theorem 7.1, if $T_D$ is a path and each bag is not prime, then we conclude that $G$ has linear rank-width at most 1, and otherwise, $G$ has linear rank-width at least 2.
Figure 13: The induced subgraph obstructions for graphs of linear rank-width at most 1 that are distance-hereditary.

The list of induced subgraph obstructions for graphs of linear rank-width at most 1 was characterized by Adler, Farley, and Proskurowski [1]. The obstructions consist of the known obstructions for distance-hereditary graphs [4], and the set $\Omega_T$ of the induced subgraph obstructions for graphs of linear rank-width at most 1 that are distance-hereditary. See Figure 13 for the list of obstructions $\alpha_i, \beta_j, \gamma_k$ in $\Omega_T$ where $1 \leq i \leq 4$, $1 \leq j \leq 6$, $1 \leq k \leq 4$. This set $\Omega_T$ can be obtained from Theorem 7.1 in a much easier way than the previous result.

A graph $H$ is called a pivot-minor of a graph $G$ if $H$ can be obtained from $G$ by applying a sequence of pivoting on edges and deletions of vertices.

**Theorem 7.3** (Adler, Farley, and Proskurowski [1]). Let $G$ be a connected graph. The following are equivalent.

1. $G$ has linear rank-width at most 1.

2. $G$ is distance-hereditary and $G$ has no induced subgraph isomorphic to a graph in
   \[ \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}. \]

3. $G$ has no pivot-minor isomorphic to a graph in $\{C_5, C_6, \alpha_1, \alpha_2, \beta_1, \beta_3, \beta_4, \beta_6\}$.

4. $G$ has no vertex-minor isomorphic to a graph in $\{C_5, \alpha_1, \beta_1\}$.

**Proof.** By Lemma 2.1, $(1) \to (4)$ is clear as $C_5$, $\alpha_1$ and $\beta_1$ have linear rank-width 2. We can easily confirm the directions $((4) \to (3) \to (2))$; see [1]. We add a proof for $((2) \to (1))$.

Suppose that $G$ has linear rank-width at least 2 and it is distance-hereditary. Let $D$ be the canonical split decomposition of $G$. By Theorem 7.1, $T_D$ is not a path. Thus there exists a bag $B$
of $D$ such that $D - V(B)$ has at least three connected components $T_1, T_2, T_3$. For each $i \in \{1, 2, 3\}$, let $v_i := \zeta_b(D, B, T_i)$ and $w_i := \zeta_c(D, B, T_i)$. We have three cases; $B$ is a complete bag, or $B$ is a star bag with the center at one of $v_1, v_2, v_3$, or $B$ is a star bag with the center at a vertex of $V(B) \setminus \{v_1, v_2, v_3\}$.

If $B$ is a complete bag, then $G$ has an induced subgraph isomorphic to one of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ depending on the types of the marked edges $v_iw_i$. If $B$ is a star bag with the center at one of $v_1, v_2, v_3$, then $G$ has an induced subgraph isomorphic to one of $\beta_1, \beta_2, \ldots, \beta_6$. Finally, if $B$ is a star bag with the center at a vertex of $V(B) \setminus \{v_1, v_2, v_3\}$, then $G$ has an induced subgraph isomorphic to one of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. We summarize all the cases in Table 1.

8 Conclusion

In this paper we used the characterization of the linear rank-width of distance-hereditary graphs given in [3] to prove that Question 1.1 is true if and only if it is true in prime graphs. Also, for each non-negative integer $k$, we compute a set of distance-hereditary graphs such that every distance-hereditary graph of linear rank-width at least $k+1$ contains a vertex-minor isomorphic to one of the graphs in the set.

Computing an upper bound on the size of vertex-minor obstructions for graphs of bounded linear rank-width is a challenging open question. Until now only a bound on obstructions for graphs of bounded rank-width is known [30]. Secondly, resolving Question 1.1 in all graphs seems to require new techniques. We currently do not have any idea on how to reduce any graph of small rank-width but large linear rank-width into a distance-hereditary graph whose decomposition tree has large path-width. One might start with graphs of rank-width 2.
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