Cartan Calculus: Differential Geometry for Quantum Groups

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1 Introduction

The topic of this lecture is differential geometry on quantum groups. Several lecturers at this conference have talked about this subject. For a review and a fairly extensive list of references I would like to point the reader to the contributions of B. Jurco and S. L. Woronowicz in this proceedings. Here we shall propose a new rigid framework for the so-called Cartan calculus of Lie derivatives, inner derivations, functions, and forms. The construction employs a semi-direct product of two graded Hopf algebras, the respective super-extensions of the deformed universal enveloping algebra and the algebra of functions on a quantum group. All additional relations in the Cartan calculus follow as consistency conditions. The approach is not based on the Leibniz rule for the exterior derivative and might hence also be of interest in the recent work on its deformations. However, given a $d$ that satisfies the Leibniz rule, the Cartan identity (77) follows as a theorem.

Rigorous proofs for many statements in this presentation can be found in some form or other in [1]. For a nice review see e.g. [2]. For Quantum Groups and (quasitriangular) Hopf algebras one could consult [3, 4].

In the next section we would like to motivate the semi-direct product construction by some geometrical considerations.

1.1 Classical Left Invariant Vector Fields

Let’s recall the left-invariant classical case: The Lie algebra is spanned by left-invariant vector fields on the group manifold of a Lie group $G$. These are uniquely

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determined by the tangent space at 1 (the identity of $G$). Let $g, h \in G$. Curves on $G$ can be naturally transported by left (or right) translation i.e. $h \mapsto gh$ ($h \mapsto hg$).

This defines a left transport $L_g^{-1}$ of the tangent vectors: $L_g^{-1}(\chi_1) = \tilde{\chi}_g$. $\chi_1$ is the vector field $\chi$ at the identity of the group and $\tilde{\chi}_g$ is the new vector field $\tilde{\chi}$ evaluated at the point of the group manifold corresponding to the group element $g$; if $\chi$ is left invariant then $\chi = \tilde{\chi}$ and in particular $L_g^{-1}(\chi_1) = \tilde{\chi}_g = \chi_g$. An inner product for a vector field $\chi$ with a function $f$ can be defined by acting with the vector field on the function and evaluating the resulting function at the identity of the group:

$$\langle \chi, f \rangle := \chi_1 \triangleright f|_1 \in k.$$  

If we know these values for all functions, we can reconstruct the action of $\chi$ on a function $f$, $\chi_g \triangleright f|_g$, at any (connected) point of the group manifold. The construction goes as follows (see figure):

We start at the point $g$, transport $f$ and $\chi$ back to the identity by left translation and then evaluate them on each other. The result, being a number, is invariant under translations and hence gives the desired quantity. The left translation $L_g(f)$ of a function, implicitly defined through $L_g(f)(h) = f(gh)$, finds an explicit expression in Hopf algebra language (here and in what follows we use the formal notation $\Delta f \equiv f^{(1)} \otimes f^{(2)}$ for the coproduct)

$$L_g(f) = f^{(1)}(g)f^{(2)},$$

that we now use to express

$$\chi_g \triangleright f|_g = L_g(\chi_1) \triangleright f^{(1)}(g)f^{(2)}|_1$$

$$= \chi_1 \triangleright f^{(1)}(g)f^{(2)}|_1$$

$$= f^{(1)}(g)\langle \chi, f^{(2)} \rangle,$$

for a left-invariant vector field $\chi$. If the drop $g$, we obtain the familiar expression for the action of a vector field on a function valid on the whole group manifold

$$\chi \triangleright f = f^{(1)}(\chi, f^{(2)}). \quad (1)$$

The left and right vacua find the following geometric interpretation:

left vacuum $\langle \cdot \rangle$: “Evaluate at the identity (of the group).”
right vacuum $\triangleright f$: “Evaluate on the unit function.”
1.2 “Quantum Geometry”

Group elements \( (g) \) “an sich” do not exist for quantum groups, everything has to be formulated in terms of a Hopf algebra of functions. The group operation is replaced by the coproduct of functions. If we take care only to speak about functions in \( \mathcal{A} \) and its dual Hopf algebra \( \mathcal{U} \), we can, however, still develop a geometric picture for vector fields on quantum groups. “Points” will be labeled by elements of \( \hat{\mathcal{U}} \), which is the same as \( \mathcal{U} \) but has the opposite multiplication; if elements of \( \mathcal{U} \) are left-invariant then elements of \( \hat{\mathcal{U}} \) are right-invariant. Lie derivatives along elements of \( \hat{\mathcal{U}} \) take the place of left translations, while Lie derivatives along elements of \( \mathcal{U} \) correspond to right translations. Here is the quantum picture of the classical construction given above:

\[
\mathcal{L}_\hat{y}(f) = \langle y, f(2) \rangle f(1)
\]

\[
\mathcal{L}_\hat{y}(\chi) = \chi \epsilon(y)
\]

\[
\mathcal{L}_\hat{y}(x) = x \epsilon(\hat{y}) \text{ because } x \text{ is left-invariant. Note that multiple occurrences of the same Hopf algebra element in a single term are unnatural. Instead one should use the parts of the coproduct of this element. We now compute } x \triangleright f \text{ in complete analogy to the classical case}
\]

\[
x \triangleright f|_{\hat{y}} = \mathcal{L}_{\hat{y}(1)}(x) \triangleright \mathcal{L}_{\hat{y}(2)}(f) \bigg|_1 \quad (\equiv \mathcal{L}_\hat{y}(x \triangleright f) \bigg|_1)
\]

\[
= \epsilon(\hat{y}(1)) x \triangleright \mathcal{L}_{\hat{y}(2)}(f) \bigg|_1
\]

\[
= x \triangleright \mathcal{L}_\hat{y}(f) \bigg|_1
\]

\[
= x \triangleright \langle y, f(1) \rangle f(2) \bigg|_1
\]

\[
= \langle y, f(1) \rangle \langle x, f(2) \rangle
\]

or, for arbitrary \( \hat{y} \):

\[
x \triangleright f = f(1) \langle x, f(2) \rangle,
\]

giving a geometric justification for the familiar left action of \( \mathcal{U} \) on \( \mathcal{A} \).

There is also a geometric picture for the adjoint action in \( \mathcal{U} \), which can be interpreted as a quantum Lie bracket. Recall the classical construction: Functions and hence curves on a group manifold can be transported along a vector field. With the curves we implicitly also transport their tangent vectors. This transport is called the Lie derivative of a (tangent) vector along a vector field. Classically we
find it to be equal to the commutator (Lie bracket) of the two vector fields. In the quantum case we have

$$\mathcal{L}_y(x) = y_{(1)} x S(y_{(2)}) = y \overset{\text{ad}}{\triangleright} x.$$  \hspace{1cm} (3)

### 1.3 Action of General Vector Fields

Our derivation of the action of a vector field on a function in the previous section relied on the use of left translations in conjunction with left-invariant vector fields. For completeness we will now consider the action of a general vector field — neither necessarily left or right invariant — on a function using alternatively left or right translations.

Left and right coactions $\Delta_A$, $\Delta_A$ contain the information about transformation properties of vector fields. Here is how a vector field transforms (classically) if we left-transport it from a point $g$ on the group manifold back to the identity

$$\chi|_g \mapsto \chi^{(1)'}(g) \cdot \chi^{(2)}|_1, \quad \Delta_A(\chi) \equiv \chi^{(1)'} \otimes \chi^{(2)};$$

here is the behavior under a right translation:

$$\chi|_g \mapsto \chi^{(1)} \cdot \chi^{(2)'}(g)|_1, \quad \Delta_A(\chi) \equiv \chi^{(1)} \otimes \chi^{(2)'}.$$

If we now redo the construction of the previous section for general vector fields $\chi$, both for left and right translations, we get the following two equivalent results for actions on functions:

$$\chi(f) = \left\langle \chi^{(1)}, f^{(1)} \right\rangle \chi^{(2)'} f^{(2)} = \chi^{(1)'} f^{(1)} \left\langle \chi^{(2)}, f^{(2)} \right\rangle .$$  \hspace{1cm} (4)

Technically there is an ordering ambiguity for $f$ and the primed parts of $\chi$, but this can be easily resolved by requiring $a(f) = af$ for $a \in \mathcal{A}$ in both cases; both expressions are written as left actions. Luckily we do not have to work with these complicated formulas — it is sufficient to consider left-invariant vector fields as we will see.

### 1.4 Algebra of Differential Operators

In what follows we will fix the convention that elements in $\mathcal{U} \cong \mathcal{A}^*$ are left-invariant. Vector fields with different behavior under transformations can be realized as left-invariant vector fields with functional coefficients, they hence live in $\mathcal{A} \otimes \mathcal{U}$. The left action of $x \in \mathcal{U}$ on products in $\mathcal{A}$, say $bf$, is given via the coproduct in $\mathcal{U}$,

$$x \triangleright bf = (bf)_{(1)} \langle x, (bf)_{(2)} \rangle = b_{(1)} \langle x_{(1)}, b_{(2)} \rangle x_{(2)} \triangleright f.$$  \hspace{1cm} (5)

Dropping the “$\triangleright$” we can write this for arbitrary functions $f$ in the form of commutation relations

$$xb = b_{(1)} \langle x_{(1)}, b_{(2)} \rangle x_{(2)}. \hspace{1cm} (6)$$
This commutation relation provides \( A \otimes U \) with an algebra structure
\[
\cdot : (A \otimes U) \otimes (A \otimes U) \to A \otimes U : \\
(a \otimes x) \cdot (b \otimes y) = a b \langle x(1), b(2) \rangle \otimes x(2) y. 
\] (7)

The resulting associative algebra is the generalized semi-direct product of \( A \) and \( U \); it is denoted \( A \ltimes U \) and we call it the quantized algebra of differential operators. The commutation relation (6) should be interpreted as a product in \( A \ltimes U \). (From now on we will omit \( \otimes \)-signs in \( A \ltimes U \).) This algebra contains all information about general vector fields, their transformation properties and actions on functions. It is bicovariant in the sense that it is a bi-\( A \)-comodule and a bi-\( U \)-module (where the elements of \( U \) should be interpreted as Lie derivatives.)

Equation (6) can be used to calculate arbitrary inner products of \( U \) with \( A \), if we define a right vacuum \( \langle \cdot \rangle \) to act like the counit in \( U \) and a left vacuum \( | \cdot \rangle \) to act like the counit in \( A \)
\[
\langle x b \rangle = \langle b(1) \langle x(1), b(2) \rangle x(2) \rangle \\
= \epsilon(b(1)) \langle x(1), b(2) \rangle \epsilon(x(2)) \\
= \langle x, b \rangle, \quad \text{for } \forall \ x \in U, \ b \in A. 
\] (8)

Using only the right vacuum we recover the formula for left actions
\[
x b \rangle = b(1) \langle x(1), b(2) \rangle x(2) \\
= b(1) \langle x(1), b(2) \rangle \epsilon(x(2)) \\
= b(1) \langle x, b(2) \rangle \\
= x \triangleright b, \quad \text{for } \forall \ x \in U, \ b \in A. 
\] (9)

It can be shown that the right coaction of \( A \) on \( A \ltimes U \) with the correct geometrical meaning is obtained through conjugation by the canonical element \( C \) of \( A \otimes U \)
\[
\Delta_A(\alpha) \equiv \alpha^{(1)} \otimes \alpha^{(2)'} = C(\alpha \otimes 1)C^{-1} 
\] (10)
for any \( \alpha \in A \ltimes U \). This expression shows explicitly that \( \Delta_A \) is an algebra homomorphism and that it is consistent with the algebra structure of \( A \ltimes U \). Given linear dual bases \( \{e_\alpha\} \) of \( U \) and \( \{f^\alpha\} \) of \( A \) the canonical element is \( C = e_\alpha \otimes f^\alpha \). We can also define a left \( U \)-coaction for elements of \( A \ltimes U \)
\[
\Delta_U(\alpha) \equiv \alpha_1' \otimes \alpha_2 = C^{-1}(1 \otimes \alpha)C, 
\] (11)
that appears in the general commutation relation
\[
\alpha \beta = \beta^{(1)} \langle \alpha_1', \beta^{(2)'} \rangle \alpha_2 
\] (12)
for arbitrary elements \( \alpha, \beta \in A \ltimes U \).
2 Cartan Calculus

So far we have considered the dually paired Hopf algebras $\mathcal{U}$ of left-invariant operators (vector fields) and $\mathcal{A}$ of functions on a quantum group. All information about a Quantum Group is contained in $\mathcal{A} \rtimes \mathcal{U}$, but to obtain more tools for applications and in particular to introduce the concept of "infinitesimality", we still have to consider additional structure. We will extend the Hopf algebras $\mathcal{U}$ and $\mathcal{A}$ by formally adjoining spaces $\mathcal{I}_{\text{left}}$ and $\Gamma_{\text{left}}$ respectively. $\mathcal{I}_{\text{left}}$ is spanned by a basis of symbols $\{i_i\}$ of degree $-1$, called inner derivations, similarly $\Gamma_{\text{left}}$ is spanned by a basis $\{\omega^i\}$ of left-invariant 1-forms with degree +1 and $\langle i_i, \omega^j \rangle = \delta^j_i$. In the following we shall derive consistency conditions under which the spaces

$$\Omega = \mathcal{A} \otimes \left( \bigoplus_{i=1}^{\infty} \Gamma_{\text{left}}^{\otimes i} \right)$$

and

$$\mathcal{D} = \mathcal{U} \otimes \left( \bigoplus_{i=1}^{\infty} \mathcal{I}_{\text{left}}^{\otimes i} \right)$$

are again (graded) Hopf algebras. The semi-direct product of these Hopf algebras is an bicovariant associative graded algebra of inner derivations, Lie derivatives and forms, namely the Cartan Calculus\cite{5, 2, 6}. This construction gives an economic way to derive all commutation relations in the Cartan Calculus. The possible commutation relations within $\Omega$ and $\mathcal{D}$ are for instance restricted by aforementioned consistency conditions.

2.1 Derivatives and Differential Forms

2.1.1 Graded Hopf Algebra of Derivatives

Let us start by introducing an operation $\iota_\mathcal{U} \Delta : \mathcal{I}_{\text{left}} \to \mathcal{U} \otimes \mathcal{I}_{\text{left}}$ on the space of inner derivations. We would like to investigate under what conditions

$$\Delta(i_i) = i_i \otimes 1 + \iota_\mathcal{U} \Delta(i_i) = i_i \otimes 1 + L_i^j \otimes i_j$$

is a coproduct in $\mathcal{D}$ satisfying the Hopf algebra axioms:

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta, \quad (\epsilon \otimes \text{id}) = \text{id}, \quad \text{and} \quad (\text{id} \otimes \epsilon) = \text{id}$$

require

$$\Delta L = L \otimes L, \quad \epsilon L = 1, \quad SL = L^{-1}, \quad \text{and} \quad \epsilon(i_i) = 0.$$  \hspace{1cm} (16)

$$(S \otimes \text{id})\Delta = \epsilon$$ implies

$$S i_i = -SL_i^j i_j.$$  \hspace{1cm} (17)

$$(\text{id} \otimes S)\Delta = \epsilon$$ is then automatically satisfied. Next we will consider commutation relations between elements of $\mathcal{U}$ and $i$. As we have in mind to let the elements of $\mathcal{U}$

\footnote{We shall also require the existence of a biinvariant operator $d$ of degree +1}
act as Lie derivatives or Lie transports, the action of these left-invariant differential operators on those inner derivations should be given via a right \( \mathcal{A} \)-coaction \[ \mathcal{L}_x \hat{i}_i = i_j \langle x(1), T^j_i \rangle \mathcal{L}_x(x(2)) \] (18)

where \( \Delta_A(i_i) = i_j \otimes T^j_i \) and \( x \in \mathcal{U} \).

**Remark:** The form of (18) can also be derived as follows: Introduce a right \( \mathcal{U} \)-coaction \( \Delta_u \), such that \( \Delta_u \equiv \Delta \) on \( \mathcal{U} \) and \( \Delta_u(i_i) = i_i \otimes 1 \), then make an ansatz for the \( \mathcal{L} - \hat{i} \) commutation relations that is linear and homogeneous in \( \hat{i} \): \( \mathcal{L}_x \hat{i}_i = i_j \mathcal{L}_x T^j_i(x) \), where \( T^j_i : \mathcal{U} \rightarrow \mathcal{U} \) are linear maps. Requiring covariance under \( \Delta_u \) gives

\[ T^j_i(x) = \epsilon(T^j_i(x(1)))x(2) \] or \( T^j_i(x) = \langle x(1), T^j_i(x(2)) \rangle \) with \( T^j_i := f^\alpha \epsilon(T^j_i(e_\alpha)) \in \mathcal{A} \).

The requirement that \( \Delta_A \) be a coaction and in particular an algebra homomorphism immediately gives \( \Delta T = T \otimes T \), \( \epsilon T = I \), and \( ST = T^{-1} \). It also put constraints on the \( T - T \) commutation relations. We will come back to that (19). The coproduct \( \Delta \) must be a homomorphism in the extended algebra \( \mathcal{D} \), i.e. it should preserve all its commutation relations. The \( \mathcal{L} - \hat{i} \) commutation relations give

\[ \Delta(\mathcal{L}_x)_{\mathcal{U}} \Delta(i_i) = \mathcal{U} \Delta(i_j) \langle x(1), T^j_i \rangle \mathcal{U} \Delta(\mathcal{L}_x)(2) \] (19)
i.e. that \( \mathcal{U} \Delta \) must be a coaction and in particular a homomorphism of the \( \mathcal{L} - \hat{i} \) commutation relations. For the specific form of the coaction on the \( \hat{i} \)’s we find:

\[ x(1)L^j_i(x(2), T^k_i) = \langle x(1), T^j_i \rangle L^k_j x(2) \] (20)
or, with \( x = L \) and \( \hat{R}^{ab}_{cd} := \langle L^c_b, T^a_d \rangle \),

\[ \hat{R}_{12} \hat{L}_{12} = L_2 \hat{L}_1 \hat{R}_{12} \] (21)

The commutation relations between inner derivations should be homogeneous to preserve the grading. They can therefore be written in terms of projection operators on quadratic and perhaps higher order combinations of inner derivations. Here we would like to consider quadratic relations with projection operators \( P^{(a)} \). \n
\[ \Delta(i_i i_k) = i_i i_k \otimes 1 + \mathcal{U} \Delta(i_i)(i_k \otimes 1) + (i_i \otimes 1) \mathcal{U} \Delta(i_k) + \mathcal{U} \Delta(i_i) \mathcal{U} \Delta(i_k) \] (22)
must be consistent with relations of the form \( i_i i_k P^{(a)}_{ik} = 0 \).

The first term of (22) is trivially consistent. The fourth term requires \( \mathcal{U} \Delta \) to be a homomorphism of the \( \hat{i} - \hat{i} \) relations and hence (recalling our previous results and the fact that \( \Delta \equiv \mathcal{U} \Delta \) on \( \mathcal{U} \)) a homomorphism of all of \( \mathcal{D} \). We will see that this is indeed satisfied after considering the second and third terms in (22):

\[ 0 = (\mathcal{U} \Delta(i_i)(i_k \otimes 1) + (i_i \otimes 1) \mathcal{U} \Delta(i_k)) P^{(a)}_{ik} \]
\[ = (-L^j_i i_k \otimes i_j + i_i L^k_j \otimes i_j) P^{(a)}_{ik} \]
\[ = i_j L^k_j \otimes i_l (\delta^i_l \delta^j_k - \langle L^i_j, T^j_k \rangle) P^{(a)}_{ik} \]
\[ = i_j L^k_j \otimes i_l (I - \hat{R})^{r,j} P^{(a)}_{ik} \] (24)

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\( ^3 \) \( \mathcal{L}_x \) reads: “Lie derivative along the vector field \( x \).”

\( ^4 \) \( GL_q(n), SL_q(n) \) and \( SU_q(n) \) only have quadratic relations. For \( SO_q(n) \) see [1].
\[(I - \hat{R})P^{(\alpha)} = 0 \text{ for all } \alpha. \] (25)

Conditions on the projectors of higher order relations of the \(i\)'s are obtained analogously. However, do to the characteristic equation for \(\hat{R}\) these relations may already be implied by the quadratic relations. The “−” sign in the second line of (24) comes from the graded tensor product structure of \(D\):

\[(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\cdot\deg(c)}ac \otimes bd. \] (26)

Note that all \(P^{(\alpha)}\) have to satisfy (24). Adding additional relations “by hand”, as it was done in [7], breaks the structure under consideration and also leads to problems for higher order relations [8]. The homomorphism of \(\Delta\) with respect to the last term in (22) can be seen in two ways: In view of equation (24) we can represent \(i_\ell i_j\) as \(i_k \otimes i_l(I - \hat{R})^{kl}ij\) and then we get \(L_m^iL_n^j \otimes i_\ell i_j P^{(\alpha)mn}_{\text{op}} = L_m^iL_n^j \otimes (i_k \otimes i_l)(I - \hat{R})^{kl}ijmnP^{(\alpha)mn}_{\text{op}} = 0\), where we have used equation (21). Alternatively we could write \(P^{(\alpha)}\) in terms of \(\hat{R}\) employing its characteristic equation; equation (21) will then give the desired result again.

All that is left to check for the Hopf algebra structure of \(D\) is the antipode and the counit. The counit of \(U\) extends trivially to a homomorphism of all of \(D\) because of \(\epsilon(\hat{i}_i) = 0\). The antipode should be a graded anti-homomorphism of \(D\):

\[S(ab) = (-1)^{\deg(a)\cdot\deg(b)}S(b)S(a). \] (27)

Consistency of the \(\mathcal{L} - i\) commutation relation:

\[S(\mathcal{L}_x i_\ell) = S(i_\ell(x_{(1)}, T_i)(\mathcal{L}_x x_{(2)}) = -S(\mathcal{L}_x x_{(3)})(x_{(1)}, T_i)SL_i^k i_k = -SL_i^k i_k S(\mathcal{L}_x x_{(3)})(x_{(2)}, T_i^k) i_k = -SL_i^k i_k S(\mathcal{L}_x x_{(3)})(x_{(2)}, T_i^k) S(\mathcal{L}_x x_{(1)}) \] (28)

where equation (21) was used in the second line. The consistency of \(i - i\) type relations can be reduced to the requirement that \(\Delta\) be a homomorphism:

\[S(i_\ell i_j)P^{(\alpha)ij}_{\text{op}} = -SL_j^k i_k SL_i^l i_l P^{(\alpha)ij}_{\text{op}} = -SL_j^k SL_i^l i_m i_m \hat{R}_{nk}^{mn} P^{(\alpha)ij}_{\text{op}} = -SL_j^k SL_i^l i_m i_m \hat{R}_{nk}^{mn} P^{(\alpha)ij}_{\text{op}} \] (29)

Given an \(L \in M_n(U)\), \(n = \dim(\mathcal{L}_{\text{left}})\) that forms a representation of \(A\) and satisfies equation (21) and a \(T \in M_n(A)\) (see below) we have hence succeeded in extending the Hopf algebra structure of \(U\) to \(D\) while obtaining natural commutation relations among the elements of \(\mathcal{L}_{\text{left}}\) on the way.
2.1.2 Graded Hopf Algebra of Differential Forms

In pretty much the same way as for $U$ we can extend the Hopf algebra structure of $A$ to $Ω$ if we introduce a $Δ_A$ and a $Δ$ on $Γ_{\text{left}}$:

$$Δ_A(ω^i) = ω^j \otimes A^i_j,$$

$$Δ(ω^i) = Δ_A(ω^i) + 1 \otimes ω^i,$$

$$S(ω^i) = -ω^j SA^i_j,$$

$$ε(ω^i) = 0.$$

(30)  

(31)  

(32)  

(33)

The multiplication between elements of $A$ and $Γ_{\text{left}}$ should be covariant under a left $A$-coaction $AΔ_A$ and is hence given via a right $U$-coaction $Δ_U(ω^i) = ω^j \otimes B^i_j$ (see the remark in section 2.1.1):

$$aω^i = ω^j B^i_j(a).$$

(34)

(In [10] equation (30) is analyzed from the point of view of inhomogeneous quantum groups and (34) is interpreted as a braiding arising from an universal $R$ in the quantum double of $U$.) Having in mind to combine $Ω$ and $D$ into a semi-direct product algebra $Ω \rtimes D$ we would like them to be dually paired Hopf algebras with respect to an inner product $< , > : D \otimes Ω \rightarrow k$ in the sense of

$$< xy, a > = (-1)^{\text{deg}(x) \cdot \text{deg}(y)} < x, a(1) > < y, a(2) >,$$

$$< x, ab > = (-1)^{\text{deg}(a) \cdot \text{deg}(b)} < x(1), a > < x(2), b >,$$

$$< Sx, a > = < x, Sa >.$$

(35)  

(36)  

(37)

On $U \otimes A$ this new inner product reduces to the old one ($\langle , \rangle$) with $L_x$ interpreted as $x \in U$, furthermore it is zero unless the overall degree of the elements in $< , >$ is zero. The only thing left to fix is: $< i_i, ω^j > = δ^i_j$.

$A$ and $B$ are no longer independent from $L$ and $T$:

$$ε(x)δ^i_j = \langle L_x i_k, ω^i >$$

$$= \langle i_t \langle x(1), T^t_k \rangle L_{x(2)}, ω^i >$$

$$= δ^i_j \langle x(1), T^t_k \rangle \langle x(2), A^t_j \rangle$$

$$= \langle x, T^j_k A^t_j \rangle,$$

i.e. $A = S^{-1} T$.

$$\langle L^i_k, a > = \langle i_k, aω^i >$$

$$= \langle i_k, ω^j B^i_j(a) >$$

$$= \langle B^i_k, a >,$$

i.e. $B = L$.

It has been known for some time [9] that the Woronowicz type calculus of forms and functions on $A_n$ quantum groups forms a Hopf algebra. Here we are just reversing the logic: We require the calculus to be a graded Hopf algebra and then derive consistency relations from this. Following is a summary of definitions and relations that are needed to turn both $D$ and $Ω$ into Hopf algebras:
Dually Paired $\mathcal{D}$- and $\omega$-Hopf Algebras (Summary)

$$\Delta(i_i) = L^j_i \otimes i_j \quad (40)$$
$$\Delta_\mathcal{U}(\omega^i) = \omega^j \otimes L^j_i \quad (41)$$
$$\Delta_A(i_i) = i_j \otimes T^j_i \quad (42)$$
$$\Delta_A(\omega^i) = \omega^j \otimes S^{-1}T^j_i \quad (43)$$
$$\Delta(L) = L \hat{\otimes} L \quad (44)$$
$$\epsilon(L) = I \quad (45)$$
$$S(L) = L^{-1} \quad (46)$$
$$\Delta(T) = T \hat{\otimes} T \quad (47)$$
$$\epsilon(T) = I \quad (48)$$
$$S(T) = T^{-1} \quad (49)$$

$$x(1)L^j_i \langle x(2), T^k_j \rangle = \langle x(1), T^j_i \rangle L^k_j x(2) \quad (50)$$
$$a(1)T^k_j \langle L^j_i, a(2) \rangle = \langle L^k_i, a(1) \rangle T^j_i a(2) \quad (51)$$
$$\hat{R}^{ij}_{kl} : = \langle L^j_i, T^k_l \rangle \quad (52)$$
$$\hat{\sigma}^{ij}_{kl} : = \langle L^j_i, S^{-1}T^k_l \rangle = \hat{R}^{-1} \quad (53)$$
$$i_i j_i P^{(a)ij}_{kl} = 0 \Rightarrow (I - \hat{R})P^{(a)} = 0 \quad (54)$$
$$P^{(a)ij}_{kl} \omega^k \omega^l = 0 \Rightarrow P^{(a)}(I - \hat{\sigma}) = 0 \quad (55)$$

Equations similar two the last two relations are also true for higher order relations in the $i$'s and $\omega$'s.

### 2.2 Semidirect Product of $\omega$ and $\mathcal{D}$

Having obtained dually paired graded Hopf algebras $\Omega$ and $\mathcal{D}$ we will now construct a bi-$\mathcal{A}$-covariant (and bi-$\mathcal{U}$-covariant) algebra of Lie derivatives, inner derivations, functions, and forms from their semi-direct product. The multiplication in $\Omega \ltimes \mathcal{D}$ is given by

$$xa = (-1)^{\text{deg}(x) \cdot \text{deg}(a)} a(1)(a(2), x(1))x(2) \quad (56)$$

where $x \in \mathcal{D}$, $a \in \Omega$ and

$$(b, y) \equiv (-1)^{\text{deg}(b) \cdot \text{deg}(y)} < y, b >, \quad \text{forally} \in \mathcal{D}, \ b \in \Omega. \quad (57)$$

In the calculation of the grading that we use one acquires a minus sign whenever an odd quantity moves past another odd quantity and $< y, b > = 0$ unless $\text{deg}(y) + \text{deg}(b) = 0$. (A graphical presentation of the graded semi-direct product or rather its braided generalization is given below.) The grading was chosen such that $\Omega \ltimes \mathcal{D}$ is an associative algebra. Actions of $\mathcal{D}$ on $\Omega$ can be recovered either via the adjoint action

$$x(a) \equiv x \triangleright a = (-1)^{\text{deg}(a) \cdot \text{deg}(x(2))} x(1)aS(x(2)) \quad (58)$$
or through a right \( U \)-vacuum that acts like the counit on \( D \):

\[
xa > = (-1)^{\deg(x) \cdot \deg(a)} (a(1), x(1)) x(2) > \\
= (-1)^{\deg(x) \cdot \deg(a)} a(1) (a(2), x(1)) \epsilon(x(2)) \\
= (-1)^{\deg(a) + \deg(x) \cdot \deg(x)} a(1) < x, a(2) > .
\] (59)

Inner products can be calculated from the commutation relations with the additional help of a left \( A \)-vacuum that acts like the counit on \( \Omega \):

\[
< xa > = (-1)^{\deg(x) \cdot \deg(a)} < a(1) (a(2), x(1)) x(2) > \\
= (-1)^{\deg(x) \cdot \deg(a)} \epsilon(a(1)) (a(2), x(1)) \epsilon(x(2)) \\
= + < x, a > .
\] (60)

The proof of the bicovariance of the semidirect product algebra is virtually identical to the ungraded case \( [] \) and therefore left as an exercise. Following is the graphical picture of the semi-direct product construction and an example of the computation of a commutation relation:

\[
\begin{align*}
&\begin{array}{c}\phantom{+}~\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}~\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phantom{+}\phan{
2.3 Some Remarks on the Exterior Derivative

So far we have constructed a consistent bicovariant algebra of derivatives and forms. This is the rigid framework for a differential geometry of quantum groups. It is now of interest to investigate the possibility of a biinvariant map \( d : \mathcal{A} \to \Gamma_{\text{left}} \), called the exterior derivative (on functions). Biinvariance means

\[
d \circ \Delta = (\text{id} \otimes d) \Delta, \quad \Delta \circ d = (d \otimes \text{id}) \Delta.
\]

(61)

\( d(a) \) should be of first order in the \( \omega^i \) and linear in \( a \), \( i.e. \ d(a) = \omega^i \tilde{\chi}_i(a) \) with linear maps \( \tilde{\chi}_i : \mathcal{A} \to \mathcal{A} \). The requirement of left-invariance leads to

\[
\tilde{\chi}_i(a) = a_{(1)} \epsilon(\tilde{\chi}_i(a_{(2)})) = a_{(1)}(\chi_i, a_{(2)})
\]

(62)

where \( \chi_i = e_{\alpha} \tilde{\chi}_i(f^\alpha) \). (We will drop the “\( \tilde{\} \)” from here on.) We can learn a couple of things from this equation:

i) \( d \) satisfies the Leibniz rule on \( \mathcal{A} \) iff \( \Delta \chi_i = \chi_i \otimes 1 + L^j_i \otimes \chi_j \).

ii) \( i_i(d(a)) = i_i(\omega^j \chi_j(a)) = \chi_i(a) \), \( i.e. \ i_i \equiv i_{\chi_i} \).

iii) \( d \) can be written as an operator \( \omega^i \chi_i \) on \( \mathcal{A} \).

We also find

\[
\Delta(d(a)) = d(a_{(1)}) \otimes a_{(2)} + a_{(1)} \otimes d(a_{(2)})
\]

(63)

essentially because \( d \) is biinvariant. The first term of this equation implies

\[
\mathcal{L}_x \circ d = d \circ \mathcal{L}_x
\]

(64)

\[ — \text{ and in particular } L^j_i \circ d = d \circ L^j_i, \] \[ — \]

the second term implies

\[
\mathcal{L}_{\chi_i}(a) = i_i(da).
\]

(65)

2.4 Explicit Realizations of all this

Right invariance of \( d \) imposes another condition on \( \chi_i \):

\[
\Delta_{\mathcal{A}}(\chi_i) = \chi_j \otimes T^i_j,
\]

(66)

where \( \Delta_{\mathcal{A}}(\chi_i) \equiv e_{\alpha(1)} \chi_i S(e_{\alpha(2)}) \otimes f^\alpha \) from (\[\[\]). All together there are quite a few conditions on the objects under consideration and one may wonder whether they can all be satisfied. Luckily that is so and there are even (at least) two possibilities:

\[ \text{That is why we chose the letter “} L \text{” in the first place.} \]
Extrinsic Braiding — Plane Type Calculi  For quasitriangular Hopf algebras one may choose \( L \) proportional to either \( \langle R, T_{ij} \otimes \text{id} \rangle \) or \( \langle R, \text{id} \otimes S^{-1}T_{ij} \rangle \), where \( T \) is the adjoint representation corresponding to a set of bicovariant generators \( \{ \chi_i \} \). The corresponding \( f - \omega \) relations are used in [11] to reduce the number of 1-forms. That is possible because \( L \) and \( T \) are simultaneously reducible in this formulation. The price one has to pay is that the Leibniz rule for \( d \) will no longer be satisfied.

The commutation relations of quantum planes are incidentally derived from an \( L \) that is of the above form [12].

Intrinsic Braiding — Woronowicz [1] Type Calculi  Here one assumes again the existence of a set of bicovariant generators \( \{ \chi_i \} \) but further requires a coproduct of the form \( \Delta \chi_i = \chi_i \otimes 1 + L_i^j \otimes \chi_j \), so that the Leibniz rule for \( d \) is satisfied. Given Hopf algebras \( A \) and \( U \) it is always possible to find such generators. The trivial choice being the elements of a (formal) linear basis \( \{ e_\alpha \} \) of \( U \) — except for the element “1” of course. This choice will then lead to a universal Cartan calculus [13]. This universal Calculus can be reduced in dimensionality by dividing by certain ideals. One typically ends up with more generators than in the classical case though. In the following we will assume that we are dealing with this type of calculus. An explicit realization for quantum groups of the type described in [4] was given by B. Jurco [14].

Now we have all the tools necessary to investigate the meaning of \( S(i_i) \):

## 2.5 Antipode of Inner Derivations

When we calculate the action of \( S(i_i) \) on functions and differentials we find

\[
S_i(f) = 0, \quad S_i(df) = S\chi_i(f),
\]  

(67)

as expected. If we however calculate commutation relations using \( S_i = -SL_i^j i_j \), we are in for a surprise:

\[
S_i f = f S_i,
\]  

(68)

i.e. \( S_i \) commutes with functions — there is no braiding here —, but

\[
S_i df = -df S_i + S\chi_j(f)SL_i^j,
\]  

(69)

so \( S_i \) braids as it acts on something, not when it moves through a term. This is the post-grading calculus as opposed to the standard pre-grading calculus of the \( i \)’s. In effect (69) should be read backwards, i.e. from right to left. To find an interpretation of \( S_i \) we have to write the equations in a more standard way. As all what really matters are inner products (“matrix elements”) we can achieve this goal by introducing a left \( U \)-vacuum and a right \( A \)-vacuum together with a left-acting \( S_i \) and a new commutation relation

\[
d(f) \ S_i = S\chi_i(f) - S_i SL_i^j(df).
\]  

(70)
A real nice feature is that $\tilde{S}i_i$ is already realized in $\Omega \rtimes D$: Multiply equation (69) by $S^2 L_j^i$ from the right to find after some manipulations using $S^2 i_i = S i_j S^2 L_j^i$ that

$$d(f)(-S^2 i_i) = S\chi_i(f) - (-S^2 i_j) S L_j^i (df),$$

i.e.

$$\tilde{S}i_i \equiv -S^2 i_i.$$

The way it acts, $S\tilde{i}$ should be interpreted as a right-invariant object. From its realization in $\Omega \rtimes D$ we can compute its left transformation property:

$$\mathcal{A} \Delta (S i_i) = S^{-2} T^j_i \otimes S i_j.$$  \hspace{1cm} (73)

Remark: This is also a solution to the problem that the antipodes of a set of $\{\chi_i\}$ that closes under right coactions does not in general close under $\Delta_{\mathcal{A}}$ again. If interpreted as right-invariant and left-acting objects, the $\{S\chi_i\}$ will however close under $\mathcal{A}\Delta$.

### 2.6 Cartan Identity

The graded Hopf algebra of $\Omega$ can be very easily expressed if one treats $d$ as an independent element with

$$\Delta(d) = d \otimes 1 + 1 \otimes d,$$

$$S(d) = -d,$$

$$\epsilon(d) = 0.$$  \hspace{1cm} (74, 75, 76)

The exterior derivative $d$ can now be viewed as an additional operator in $D$; equations (30, 63) follow then automatically. We would now like to proof the following interesting relation among Lie derivatives, inner derivations, and the exterior derivative

$$L_{\chi_k} = di_k + i_k d,$$  \hspace{1cm} (77)

due to Cartan. We start by checking the coproduct:

$$\Delta(d i_k + i_k d) = (i_k \otimes 1 + L_k^l \otimes i_l)(d \otimes 1 + 1 \otimes d)$$

$$+ (d \otimes 1 + 1 \otimes d)(i_k \otimes 1 + L_k^l \otimes i_l)$$

$$= (d i_k + i_k d) \otimes 1 + L_k^l \otimes (d i_l + i_l d).$$  \hspace{1cm} (78)

Similar one checks $S$ and $\epsilon$. Using relations that we have previously derived we see that

$$(d i_i + i_i d)(f) = 0 + i_i (df) = \chi_i(f)$$

and

$$(d i_i + i_i d)(df) = d(\chi_i(f))$$

thus verifying (77) on $f$ and $df$. Using the coproduct (78) in $D$ this immediately extends to all of $\Omega$. 

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2.7 Summary of Relations in the Cartan Calculus

**Commutation Relations** For any $p$-form $\alpha$:

\[
\begin{align*}
\text{d} \alpha &= \text{d}(\alpha) + (-1)^p \alpha \text{d} \\
i_{\chi_i} \alpha &= i_{\chi_i}(\alpha) + (-1)^p L_{i}^{j}(\alpha) i_{\chi_j} \\
\mathcal{L}_{\chi_i} \alpha &= \mathcal{L}_{\chi_i}(\alpha) + L_{i}^{j}(\alpha) \mathcal{L}_{\chi_j}
\end{align*}
\]

**Actions** For any function $f \in \mathcal{A}$, 1-form $\omega_f \equiv S_f(1) \text{d} f(2)$ and vector field $\phi \in \mathcal{A} \times \mathcal{U}$:

\[
\begin{align*}
i_{\chi_i}(f) &= 0 \\
i_{\chi_i}(\text{d} f) &= \text{d} f(1) < \chi_i, f(2)> \\
i_{\chi_i}(\omega_f) &= -<\chi_i, S_f> \\
\mathcal{L}_{\chi}(f) &= \chi(f) = f(1) < \chi, f(2)> \\
\mathcal{L}_{\chi}(\omega_f) &= \omega_f(2) <\chi, S(f(1)) f(3)> \\
\mathcal{L}_{\chi}(\phi) &= \chi(1) \phi S(\chi(2))
\end{align*}
\]

**Graded Quantum Lie Algebra of the Cartan Generators**

\[
\begin{align*}
\text{d} \text{d} &= 0 \\
\text{d} \mathcal{L}_{\chi} &= \mathcal{L}_{\chi} \text{d} \\
\mathcal{L}_{\chi_i} &= \text{d} i_{\chi_i} + i_{\chi_i} \text{d} \\
[\mathcal{L}_{\chi_i}, \mathcal{L}_{\chi_k}]_q &= \mathcal{L}_{\chi_i} f_{i}^{l} f_{l}^{k} \\
[\mathcal{L}_{\chi_i}, i_{\chi_k}]_q &= i_{\chi_l} f_{i}^{l} f_{l}^{k}
\end{align*}
\]

Where $f_{i}^{l} f_{l}^{k} = \langle \chi_i, T_{i}^{k} \rangle$ and $[,]_q$ is defined as follows

\[
[\mathcal{L}_{\chi_i}, \Box]_q := \mathcal{L}_{\chi_i} \Box - L_{i}^{j}(\Box) \mathcal{L}_{\chi_j}.
\]

This quantum Lie algebra becomes infinite dimensional as soon as we introduce derivatives along general vector fields (see below).

2.8 Lie Derivatives Along General Vector Fields

We have seen in the introduction that general vector fields are of the form $a^i \chi_i$, where $a^i \in \mathcal{A}$ are functional coefficients. Inner derivations contract forms with vectors “at each point”. Classically the $a^i$ are just numerical coefficients at each point of the group manifold. Here this property is expressible as

\[
i_{a^i \chi_i} = a^i i_{\chi_i}.
\]
Through the Cartan identity
\[ \mathcal{L}_{a^i \chi_i} = di_{a^i \chi_i} + i_{a^i \chi_i} d \]  
we can also consistently\(^6\) introduce Lie derivatives along general vector fields. Combining these two equations we find
\[ \mathcal{L}_{a^i \chi_i} = a^i \mathcal{L}_{\chi_i} + d(a^i) \chi_i \]  
as in the classical case.

2.9 A glimpse at extended calculi on the plane

In the case of GL\(_q(n)\) the (bicovariant) vector fields \(\chi_i\) find a simple realization in the differential operators of the linear quantum plane that can be used to induce a Cartan calculus on the plane from the one of the group\(^{[15]}\). This extended calculus on the plane has some unexpected features like for instance the appearance of differentials and inner derivations in the commutation relations of Lie derivatives with functions. For other quantum groups the \(\chi_i\)'s have non-linear realizations on the plane and an extended calculus has not yet been successfully induced.

It would be nice if the formalism presented in this lecture could be generalized to the case of quantum planes (\(i.e.,\) pseudo-Hopf algebras with non-trivial braiding) to give an independent approach to the construction of extended calculi.

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\(^{6}\)On all forms.
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