A NOTE ON THE GUTMAN INDEX OF JACO GRAPHS

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Abstract: The concept of the Gutman index, denoted $Gut(G)$ was introduced for a connected undirected graph $G$. In this note we apply the concept to the underlying graphs of the family of Jaco graphs, (directed graphs by definition), and describe a recursive formula for the Gutman index $Gut(J_{n+1}^+(x))$. We also determine the Gutman index for the trivial edge-joint between Jaco graphs.

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1. Introduction

For general reference to notation and concepts of graph theory see [2]. Unless mentioned otherwise, a graph $G = G(V,E)$ on $\nu(G)$ vertices (order of $G$) with $\epsilon(G)$ edges (size of $G$) will be a finite undirected and connected simple graph. The degree of a vertex in $G$ is denoted $d_G(v)$ and if the context of $G$ is clear the degree is denoted $d(v)$ for brevity. Also in a directed graph $G\rightarrow$ the degree is $d_{G\rightarrow}(v) = d_{G\rightarrow}^+(v) + d_{G\rightarrow}^-(v)$ or for brevity, $d(v) = d^+(v) + d^-(v)$ if $G$ is clear.
The concept of the Gutman index $Gut(G)$ of a connected undirected graph $G$ was introduced in 1994 by Gutman [4]. It is defined to be $Gut(G) = \sum_{\{v,u\} \subseteq V(G)} d_G(v)d_G(u)d_G(v,u)$, where $d_G(v)$ and $d_G(u)$ are the degree of $v$ and $u$ in $G$ respectively, and $d_G(v,u)$ is the distance between $v$ and $u$ in $G$. Clearly, if the vertices of $G$ of order $n$ are randomly labeled $v_1, v_2, v_3, \ldots, v_n$ the definition states that $Gut(G) = \sum_{\ell=1}^{n-1} \sum_{j=\ell+1}^{n} d_G(v_\ell)d_G(v_j)d_G(v_\ell, v_j)$. Worthy results are reported in Andova et al. [1] and Dankelmann et al. [3].

2. The Gutman Index of the Underlying Graph of a Jaco Graph

Despite earlier definitions in respect of the family of Jaco graphs [5, 6], the definitions found in [7] serve as the unifying definitions. For ease of reference some of the important definitions are repeated here.

Definition 2.1. [7] Let $f(x) = mx + c; x \in \mathbb{N}$, $m, c \in \mathbb{N}_0$. The family of infinite linear Jaco graphs denoted by $\{J_\infty(f(x)) : f(x) = mx + c; x \in \mathbb{N} \text{ and } m, c \in \mathbb{N}_0\}$ is defined by $V(J_\infty(f(x))) = \{v_i : i \in \mathbb{N}\}, A(J_\infty(f(x))) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in A(J_\infty(f(x)))$ if and only if $(f(i) + i) - d^-(v_i) \geq j$.

Definition 2.2. [7] The family of finite linear Jaco graphs denoted by $\{J_n(f(x)) : f(x) = mx + c; x \in \mathbb{N} \text{ and } m, c \in \mathbb{N}_0\}$ is defined by $V(J_n(f(x))) = \{v_i : i \in \mathbb{N}, i \leq n\}, A(J_n(f(x))) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j \leq n\}$ and $(v_i, v_j) \in A(J_n(f(x)))$ if and only if $(f(i) + i) - d^-(v_i) \geq j$.

The reader is referred to [7] for the definition of the prime Jacoian vertex and the Hope graph. The graph has four fundamental properties which are:

(i) $V(J_\infty(f(x))) = \{v_i : i \in \mathbb{N}\}$ and,

(ii) if $v_j$ is the head of an arc then the tail is always a vertex $v_i$, $i < j$ and,

(iii) if $v_k$, for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices $v_\ell$, $k < \ell < j$ are tails of arcs to $v_j$ and finally,

(iv) the degree of vertex $k$ is $d(v_k) = f(k)$.

The family of finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and arcs to vertices) $v_1, t > n$. Hence, trivially $d(v_i) \leq i$ for $i \in \mathbb{N}$. For $m = 0$ and $c \geq 0$ two special classes of disconnected linear Jaco graphs exist. For $c = 0$ the Jaco graph $J_n(0)$ is a null graph (edgeless graph) on $n$ vertices. For $c > 0$, the Jaco graph $J_n(c) = \bigcup_{[n/c+1]} K_{c+1}^{\rightarrow} \bigcup_{[n-(c+1)/c+1]} K_{c+1}^{\rightarrow}$. 

since the Gutman index is defined for connected graphs the bound \( m \geq 1 \) will apply.

In this note we only consider the case \( m = 1 \), \( c = 0 \). The generalisation for \( f(x) = mx + c \) in general remains open. Denote the underlying Jaco graph by \( J^*_n(f(x)) \). A recursive formula of the Gutman index \( \text{Gut}(J^*_n(x)) \) in terms of \( \text{Gut}(J^*_n(x)) \) is given in the next theorem.

**Theorem 2.1.** For the underlying graph \( J^*_n(x) \) of a finite Jaco Graph \( J_n(x), n \in \mathbb{N}, n \geq 2 \) with Jacoian vertex \( v_i \) we have that recursively:

\[
\text{Gut}(J^*_n(x)) = \text{Gut}(J^*_n(x)) + \sum_{k=1}^{i} \sum_{t=i+1}^{n} d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_k, v_t)
+ \sum_{t=i+1}^{n-1} \sum_{q=t+1}^{n} (d_{J^*_n(x)}(v_t) + d_{J^*_n(x)}(v_q)) + (n-i) \sum_{k=1}^{i} d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_k, v_n)
+ \sum_{t=i+1}^{n} d_{J^*_n(x)}(v_t)) + (n-i-1) + i(n-i).
\]

**Proof.** Consider the underlying Jaco graph, \( J^*_n(x), n \in \mathbb{N}, n \geq 2 \) with prime Jacoian vertex \( v_i \). Now consider \( J^*_n+1(x) \). From the definition of a Jaco graph the extension from \( J^*_n(x) \) to \( J^*_n+1(x) \) adds the vertex \( v_{n+1} \) and the edges \( v_{i+1}v_{n+1}, v_{i+2}v_{n+1}, \ldots, v_{n}v_{n+1} \).

Step 1: Consider any ordered pair of vertices \((v_k, v_q)_{k<q}, 1 \leq k \leq i-1, k + 1 \leq q \leq i\). By applying the definition of the Gutman index to this pair of vertices we have the term:

\[
d_{J^*_n+1(x)}(v_k)d_{J^*_n+1(x)}(v_q) = d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_q) + d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_k, v_q).
\]

By applying this step \( \forall v_k, 1 \leq k \leq i-1 \), and \( \forall v_q, k+1 \leq q \leq i \) with \( k < q \) we obtain:

\[
\sum_{k=1}^{i-1} \sum_{q=k+1}^{i} d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_q)d_{J^*_n(x)}(v_k, v_q).
\]

Step 2: Consider any vertex \( v_k, 1 \leq k \leq i \) and any other vertex \( v_t, i+1 \leq t \leq n \). By applying the definition of the Gutman index to this pair of vertices we have the term:

\[
d_{J^*_n+1(x)}(v_k)d_{J^*_n+1(x)}(v_t) = d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_t) + d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_k, v_t)
= d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_t) + d_{J^*_n(x)}(v_k)d_{J^*_n(x)}(v_k, v_t).
\]
By applying this step $\forall v_k, 1 \leq k \leq i$ and $\forall v_t, i + 1 \leq t \leq n$, we obtain:

$$\sum_{k=1}^{i} \sum_{t=i+1}^{n} d_{J_n^*}(v_k) d_{J_n^*}(v_t) d_{J_n^*}(v_k, v_t) + \sum_{k=1}^{i} \sum_{t=i+1}^{n} d_{J_n^*}(v_k) d_{J_n^*}(v_k, v_t).$$

Step 3: Consider any two distinct vertices $v_t, v_q, i + 1 \leq t \leq n - 1$, and $t + 1 \leq q \leq n$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n+1}^*}(v_t) d_{J_{n+1}^*}(v_q) d_{J_{n+1}^*}(v_t, v_q) = (d_{J_n^*}(v_t) + 1)(d_{J_n^*}(v_q) + 1)d_{J_n^*}(v_t, v_q) = d_{J_n^*}(v_t) d_{J_n^*}(v_q) + d_{J_n^*}(v_t) + d_{J_n^*}(v_q) + 1.$$

By applying this step $\forall v_t, i + 1 \leq t \leq n - 1$ and $\forall v_q, t + 1 \leq q \leq n$, we obtain:

$$\sum_{t=i+1}^{n-1} \sum_{q=t+1}^{n} d_{J_n^*}(v_t) d_{J_n^*}(v_q) + \sum_{t=i+1}^{n-1} \sum_{q=t+1}^{n} (d_{J_n^*}(v_t) + d_{J_n^*}(v_q)) + (n - i - 1).$$

Step 4: Consider any vertex $v_k, 1 \leq k \leq i$ and the vertex $v_{n+1}$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n+1}^*}(v_k) d_{J_{n+1}^*}(v_{n+1}) d_{J_{n+1}^*}(v_k, v_{n+1}) = d_{J_n^*}(v_k)(n-i)(d_{J_n^*}(v_k, v_{n+1}).$$

By applying this step $\forall v_k, 1 \leq k \leq i$ we obtain:

$$\sum_{k=1}^{i} d_{J_n^*}(v_k)(n-i)(d_{J_n^*}(v_k, v_{n+1}) + 1) = (n-i) \sum_{k=1}^{i} d_{J_n^*}(v_k)d_{J_n^*}(v_k, v_{n}) + i(n-i).$$

Step 5: Consider any vertex $v_t, i + 1 \leq t \leq n$ and the vertex $v_{n+1}$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n+1}^*}(v_t) d_{J_{n+1}^*}(v_{n+1}) d_{J_{n+1}^*}(v_t, v_{n+1}) = d_{J_n^*}(v_t)(n-i)d_{J_n^*}(v_t, v_{n}).$$
By applying this step \( \forall v_t, i+1 \leq t \leq n \) we obtain:

\[
\sum_{t=i+1}^{n} d_{J_n^*(x)}(v_t)(n-i) = (n-i) \sum_{t=i+1}^{n} d_{J_n^*(x)}(v_t).
\]

**Final Summation Step:** Adding Steps 1 to 5 and noting that:

\[
Gut(J_n^*(x)) = \sum_{k=1}^{i-1} \sum_{q=k+1}^{i} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_q)d_{J_n^*(x)}(v_k, v_q)
\]

\[
+ \sum_{k=1}^{i} \sum_{t=i+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_t)d_{J_n^*(x)}(v_k, v_t)
\]

\[
+ \sum_{t=i+1}^{n-1} \sum_{q=t+1}^{n} d_{J_n^*(x)}(v_t)d_{J_n^*(x)}(v_q),
\]

provides the result:

\[
Gut(J_{n+1}^*(x)) = Gut(J_n^*(x)) + \sum_{k=1}^{i} \sum_{t=i+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_k, v_t)
\]

\[
+ \sum_{t=i+1}^{n-1} \sum_{q=t+1}^{n} (d_{J_n^*(x)}(v_t) + d_{J_n^*(x)}(v_q)) + (n-i)(\sum_{k=1}^{i} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_k, v_n)
\]

\[
+ \sum_{t=i+1}^{n} d_{J_n^*(x)}(v_t)) + (n-i-1) + i(n-i).
\]

3. The Gutman Index of the Edge-Joint between

\( J_n^*(x), n \in \mathbb{N} \) and \( J_m^*(x), m \in \mathbb{N} \)

The concept of an *edge-joint* between two simple undirected graphs \( G \) and \( H \) is defined below.

**Definition 3.1.** The edge-joint of two simple undirected graphs \( G \) and \( H \) is the graph obtained by linking the edge \( vu, v \in V(G), u \in V(H) \) and denoted, \( G \rightsquigarrow vu H \).
Note. $G \sim_{vu} H = G \cup H + vu, v \in V(G), u \in V(H)$.

The next theorem provides $Gut(J^*_n(x) \sim_{v_1u_1} J^*_m(x))$ in terms of $Gut(J^*_n(x))$ and $Gut(J^*_m(x))$. The edge-joint $J^*_n(x) \sim_{v_1u_1} J^*_m(x)$ is called trivial. Edge-joints $J^*_n(x) \sim_{v_1u_j} J^*_m(x), i \neq j$ or $j \neq i$ are called non-trivial. For families (classes) of graphs such as paths $P_n$, cycles $C_n$, complete graphs $K_n$, Jaco graphs $J_n(f(x))$, etc., the notation is abbreviated as $P_n \sim_{vu} P_m = P_{n,m}^\sim$ and $J_n^*(f(x)) \sim_{v_1u_j} J_m^*(f(x)) = J_{n,m}^{\sim,v_1u_j}$, etc.

**Theorem 3.1.** For the underlying graphs $J^*_n(x)$ and $J^*_m(x)$ of the finite Jaco Graphs $J_n(x), J_m(x), n, m \in \mathbb{N}$ and $n \geq m \geq 2$:

$Gut(J^*_n(x) \sim_{v_1u_1} J^*_m(x)) = Gut(J^*_{n,m}^{v_1u_1}) = Gut(J^*_n(x)) + Gut(J^*_m(x))$

$$+ \sum_{t=2}^n d_{J^*_n(x)}(v_t) d_{J^*_n(x)}(v_1, v_t) + \sum_{s=2}^m d_{J^*_m(x)}(u_s) d_{J^*_m(x)}(u_1, u_s)$$

$$+ \sum_{t=2}^m (d_{J^*_m(x)}(v_1) + 1) d_{J^*_m(x)}(u_t)(d_{J^*_m(x)}(u_1, u_t) + 1)$$

$$+ \sum_{k=2}^n \sum_{t=2}^m d_{J^*_n(x)}(v_k) d_{J^*_m(x)}(u_t)(d_{J^*_n(x)}(v_1, v_k) + d_{J^*_m(x)}(u_1, u_t) + 1) + 4.$$

**Proof.** Consider the underlying Jaco graphs, $J^*_n(x), J^*_m(x)$, with $n, m \in \mathbb{N}$ and $n \geq m \geq 2$ with $J_m(x)$ having prime Jacoian vertex $u_i$. Also label the vertices of $J^*_n(x)$ and $J^*_m(x)$; $v_1, v_2, v_3, ..., v_n$ and $u_1, u_2, u_3, ..., u_m$, respectively. Consider $J^*_{n,m}^{v_1u_1} = J^*_n(x) \cup J^*_m(x) + v_1u_1$. Without loss of generality apply the piecewise definition:

$$Gut(J^*_{n,m}^{v_1u_1}) = \sum_{k=1}^{n-1} \sum_{\ell=k+1}^n d_{J^*_{n,m}^{v_1u_1}}(v_k) d_{J^*_{n,m}^{v_1u_1}}(v_\ell) d_{J^*_{n,m}^{v_1u_1}}(v_k, v_\ell)$$

$$+ \sum_{t=1}^{m-1} \sum_{s=t+1}^m d_{J^*_{n,m}^{v_1u_1}}(u_t) d_{J^*_{n,m}^{v_1u_1}}(u_s) d_{J^*_{n,m}^{v_1u_1}}(u_t, u_s)$$

$$+ \sum_{k=1}^n \sum_{t=2}^m d_{J^*_{n,m}^{v_1u_1}}(v_k) d_{J^*_{n,m}^{v_1u_1}}(u_t) d_{J^*_{n,m}^{v_1u_1}}(v_k, u_t) + d_{J^*_{n,m}^{v_1u_1}}(v_1) d_{J^*_{n,m}^{v_1u_1}}(u_1, v_1).$$

Step 1(a): Consider vertex $v_1$ and vertex $v_\ell, 2 \leq \ell \leq n$. By applying the definition of the Gutman index to this pair of vertices we have the term:
\[ d_{J_{n,m}^{-v_1u_1}}(v_1)d_{J_{n,m}^{-v_1u_1}}(v_\ell)d_{J_{n,m}^{-v_1u_1}}(v_1, v_\ell) = (d_{J_n^+(x)}(v_1)+1)d_{J_n^+(x)}(v_\ell)d_{J_n^+(x)}(v_1, v_\ell) \]
\[ = d_{J_n^+(x)}(v_1)d_{J_n^+(x)}(v_\ell)d_{J_n^+(x)}(v_1, v_\ell) + d_{J_n^+(x)}(v_\ell)d_{J_n^+(x)}(v_1, v_\ell). \]

By applying this step \( \forall v_\ell, 2 \leq \ell \leq n \) we obtain:
\[
\sum_{\ell=2}^{n} d_{J_n^+(x)}(v_1)d_{J_n^+(x)}(v_\ell)d_{J_n^+(x)}(v_1, v_\ell) + \sum_{\ell=2}^{n} d_{J_n^+(x)}(v_\ell)d_{J_n^+(x)}(v_1, v_\ell).
\]

Step 1(b): For all ordered pairs of vertices \((v_k, v_\ell)_{k<\ell}\) with \(2 \leq k \leq n-1\) and \(3 \leq \ell \leq n\) we have that:
\[
\sum_{k=2}^{n-1} \sum_{\ell=k+1}^{n} d_{J_{n,m}^{-v_1u_1}}(v_k)d_{J_{n,m}^{-v_1u_1}}(v_\ell)d_{J_{n,m}^{-v_1u_1}}(v_k, v_\ell)\]
\[ = \sum_{k=2}^{n-1} \sum_{\ell=k+1}^{n} d_{J_n^+(x)}(v_k)d_{J_n^+(x)}(v_\ell)d_{J_n^+(x)}(v_k, v_\ell). \]

By applying this step \( \forall (v_k, v_\ell)_{k<\ell}, 1 \leq k \leq n-1 \) and \( 2 \leq \ell \leq n \), we obtain:
\[
\sum_{k=1}^{n-1} \sum_{\ell=k+1}^{n} d_{J_n^+(x)}(v_k)d_{J_n^+(x)}(v_\ell)d_{J_n^+(x)}(v_k, v_\ell) + \sum_{\ell=2}^{n} d_{J_n^+(x)}(v_\ell)d_{J_n^+(x)}(v_1, v_\ell)\]
\[ = \text{Gut}(J_n^+(x)) + \sum_{\ell=2}^{n} d_{J_n^+(x)}(v_\ell)d_{J_n^+(x)}(v_1, v_\ell). \]

Step 2: Similar to Step 1 we have that:
\[
\sum_{t=1}^{m-1} \sum_{s=t+1}^{m} d_{J_{n,m}^{-v_1u_1}}(u_t)d_{J_{n,m}^{-v_1u_1}}(u_s)d_{J_{n,m}^{-v_1u_1}}(u_t, u_s)\]
\[ = \sum_{t=1}^{m-1} \sum_{s=t+1}^{m} d_{J_m^+(x)}(u_t)d_{J_m^+(x)}(u_s)d_{J_m^+(x)}(u_t, u_s) + \sum_{s=2}^{m} d_{J_m^+(x)}(u_s)d_{J_m^+(x)}(u_1, u_s)\]
\[ = \text{Gut}(J_m^+(x)) + \sum_{s=2}^{m} d_{J_m^+(x)}(u_s)d_{J_m^+(x)}(u_1, u_s). \]

Step 3: To conclude this step we will provide the next partial summation as a piecewise summation, to be:
By applying the step \( \forall \) term:

By applying this step \( \forall \) term:

\[
\sum_{k=1}^{n} \sum_{t=2}^{m} d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_k) d_{J_{n,m}^{\rightarrow u_1 v_1}}(u_t) d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_k, u_t)
\]

\[
= \sum_{t=2}^{m} d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_1) d_{J_{n,m}^{\rightarrow u_1 v_1}}(u_t) d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_1, u_t)
\]

\[
+ \sum_{k=2}^{n} \sum_{t=2}^{m} d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_k) d_{J_{n,m}^{\rightarrow u_1 v_1}}(u_t) d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_k, u_t).
\]

Step 3(a): Consider vertex \( v_1 \) and vertex \( u_t, 2 \leq t \leq m \). By applying the definition of the Gutman index to this pair of vertices we have the term:

\[
d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_1) d_{J_{n,m}^{\rightarrow u_1 v_1}}(u_t) d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_1, u_t)
\]

\[
= (d_{J_{n}^{*}(x)}(v_1) + 1) d_{J_{m}^{*}(x)}(u_t) (d_{J_{n}^{*}(x)}(u_1, u_t) + 1).
\]

By applying this step \( \forall u_t, 2 \leq t \leq m \) we obtain:

\[
\sum_{t=2}^{m} (d_{J_{n}^{*}(x)}(v_1) + 1) d_{J_{m}^{*}(x)}(u_t) (d_{J_{n}^{*}(x)}(u_1, u_t) + 1).
\]

Step 3(b): Consider vertex \( v_k, 2 \leq k \leq n \) and vertex \( u_t, 2 \leq t \leq m \). By applying the definition of the Gutman index to this pair of vertices we have the term:

\[
d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_k) d_{J_{n,m}^{\rightarrow u_1 v_1}}(u_t) d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_k, u_t)
\]

\[
= d_{J_{n}^{*}(x)}(v_k) d_{J_{m}^{*}(x)}(u_t) (d_{J_{n}^{*}(x)}(v_1, v_k) + d_{J_{m}^{*}(x)}(u_1, u_t) + 1).
\]

By applying the step \( \forall v_k, 2 \leq k \leq n \) and \( \forall u_t, 2 \leq t \leq m \), we obtain:

\[
\sum_{k=2}^{n} \sum_{t=2}^{m} d_{J_{n}^{*}(x)}(v_k) d_{J_{m}^{*}(x)}(u_t) (d_{J_{n}^{*}(x)}(v_1, v_k) + d_{J_{m}^{*}(x)}(u_1, u_t) + 1).
\]

Step 4: It is easy to see that:

\[
d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_1) d_{J_{n,m}^{\rightarrow u_1 v_1}}(u_1) d_{J_{n,m}^{\rightarrow u_1 v_1}}(v_1, u_1) = 4.
\]

**Final Summation Step**: Adding Steps 1 to 4 provides the result:

\[
Gut(J_{n,m}^{\rightarrow u_1 v_1}) = Gut(J_{n}^{*}(x)) + Gut(J_{m}^{*}(x)) + \sum_{\ell=2}^{n} d_{J_{n}^{*}(x)}(v_\ell) d_{J_{m}^{*}(x)}(v_1, v_\ell)
\]
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\[ + \sum_{s=2}^{m} d_{J_{s}^{*}}(x)(u_{s})d_{J_{s}^{*}}(x)(u_{1}, u_{s}) + \sum_{t=2}^{m} (d_{J_{t}^{*}}(x)(v_{1}) + 1)d_{J_{t}^{*}}(x)(u_{t})(d_{J_{m}^{*}}(x)(u_{1}, u_{t}) + 1) \]

\[ + \sum_{k=2}^{n} \sum_{t=2}^{m} d_{J_{k}^{*}}(x)(v_{k})d_{J_{m}^{*}}(x)(u_{t})(d_{J_{k}^{*}}(x)(v_{1}, v_{k}) + d_{J_{m}^{*}}(x)(u_{1}, u_{t}) + 1) + 4. \]

4. Conclusion

For the simple case \( f(x) = x \) the calculation of the Gutman index for Jaco graph and the edge-joint between them is immediately complicated. Finding a result similar to Theorem 3.1 for \( J_{n}^{*}(x) \sim_{v_{i}, u_{j}} J_{m}^{*}(x), i \neq 1 \) or \( j \neq 1 \) (non-trivial edge-joints) remains open. The single most important challenge is to find a closed formula for the number of edges in \( J_{n}(x) \). Such closed formula will enable finding a closed formula for distances between given vertices and a simplified formula for many invariants of Jaco graphs might result from such finding. Hence, important open questions remain such as: Is there a closed formula for the number of edges of \( J_{n}(x), n \in \mathbb{N} \)? Is there a closed formula for the cardinality of the Jaconian set \( \mathcal{J}(J_{n}(x)) \) of \( J_{n}(x), n \in \mathbb{N} \)? Is there a closed formula for \( d_{J_{n}^{*}}(x)(v_{1}, v_{n}) \) in \( J_{n}^{*}(x), n \in \mathbb{N} \)? Refer to [7] for further reading.

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