Spectrum of a stochastic diffusion in an expanding universe

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Abstract

We discuss the diffusion equation resulting from a strong dissipation limit of the random wave equation arising in the models of warm inflation. We show that the long wave power spectrum of scalar perturbations in the model of an exponential expansion coincides with the spectrum of quantum fluctuations on this space-time.

1 Introduction

The standard consequence of the inflationary paradigm is an almost scale invariant spectrum which is confirmed by WMAP observations [1]. The power spectrum is obtained by a quantization of the quadratic fluctuations around the homogeneous solution (derived first in [2], see also the review [3]). It has been pointed out [4] that the almost scale invariant spectrum may arise also in some versions of the Pre-Big-Bang and Ekpyrotic cosmology [5][6] as well as in the bounce models [7]. The power spectrum close to the scale invariant one has also been obtained in warm inflation models [8][9]. It is usually believed that in the standard models of inflation the thermal fluctuations are negligible [10]. However, such fluctuations may be important in other cosmological theories [11]. In most of these models the fluctuations satisfy a wave equation. In this paper we study the stochastic wave equation in the form derived from an interaction of scalar fields with an environment [12][13]. We approximate solutions of the dissipative random wave equation by a solution of a diffusion equation. In the case of a random diffusion the calculation of the power spectrum is much simpler. Its dependence on the evolution law can be seen in a more transparent way. We show that if the evolution of the scale $a(t)$ is close to exponential and the initial fluctuations are normalized to zero at $a = 0$ then
we obtain the same power spectrum as in the quantum case when the classical modes are normalized asymptotically to the Minkowski plane waves [3]. This raises the question whether the observed fluctuations [1] are really of quantum origin or perhaps they are thermal and classical. Beyond the inflationary models the behaviour of quantum and thermal fluctuations may be different. The plan of the paper is the following. In sec.2 we discuss a dissipative stochastic wave equation. In sec.3 we consider a diffusion approximation. In sec.4 we calculate the power spectrum under the assumption that the evolution of the scale factor is almost exponential. In the summary and outlook we discuss the differences between stochastic and quantum fluctuations if the evolution is power-law. In the Appendix we outline a method which allows to derive exact formulas for the diffusion as well as wave equations if the evolution is precisely exponential.

2 Random wave equation

In a flat FLWR expanding metric
\[ ds^2 = dt^2 - a^2 dx^2, \]
we consider the wave equation
\[
\partial_t^2 \phi - a^{-2} \Delta \phi + (3H + \gamma^2) \partial_t \phi + m^2 \phi + V' \phi + \frac{3}{2} \gamma^2 H \phi = \gamma a^{-\frac{3}{2}} \eta, \tag{1}
\]
where \( H = a^{-1} \partial_t a \) and
\[
\langle \eta_s(x) \eta_t(y) \rangle = \delta(t - s) \delta(x - y). \tag{2}
\]
This equation has been derived from an interaction of the inflaton with an environment in [12],[13] (with the \( \frac{3}{2} \gamma^2 H \phi \) term) and is the basis of the warm inflation approach to cosmology [14].

We can transform eq.(1) to another form. Let
\[
\phi = a^{-\frac{3}{2}} \exp(-\frac{1}{2} \gamma^2 t) \Phi \tag{3}
\]
Then
\[
\partial_t^2 \Phi - a^{-2} \Delta \Phi - \Omega^2 \Phi + a^4 \exp(\frac{1}{2} \gamma^2 t) V'(\Phi) \exp(-\frac{1}{2} \gamma^2 t) \Phi = \exp(\frac{1}{2} \gamma^2 t) \gamma \eta, \tag{4}
\]
where
\[
\Omega^2 = -m^2 + \frac{9}{4} H^2 + \frac{3}{2} \partial_t H + \frac{1}{4} \gamma^4.
\]
Hence, the wave equation with friction is transformed into a wave equation with a complex mass \( \Omega \). Note that large \( 3H + \gamma^2 \) means large \( \Omega \).
3 Diffusion approximation

We may assume that the second order time derivative in eq.(1) is negligible
\[
(3H + \gamma^2)\partial_t \phi - a^{-2}\triangle \phi + m^2 \phi + V'(\phi) + \frac{3}{2}\gamma^2 H \phi = \gamma a^{-\frac{3}{2}} \eta. \tag{5}
\]
Let in eq.(4)(momentum space)
\[
\omega^2 = \Omega^2 - a^{-2}k^2 = -m^2 - a^{-2}k^2 + \frac{3}{4}H^2 + \frac{3}{2}\partial_k H + \frac{1}{4}\gamma^4 - \frac{3}{2}\gamma^2 H. \tag{6}
\]
Then, the neglect of the second order time derivative in eq.(1) is equivalent to an expansion in powers of \((3H + \gamma^2)^{-1}\)
\[
\omega = \frac{3}{2}H + \frac{1}{2}\gamma^2 - (m^2 + a^{-2}k^2 - \frac{3}{2}\partial_k H + \frac{3}{2}\gamma^2 H)(3H + \gamma^2)^{-1}. \tag{7}
\]
We compare solutions of the wave equation (1) with solutions of the diffusion equation (5). First, let \(V = 0\) then the diffusion equation (in momentum space) is
\[
\partial_t \phi = - (3H + \gamma^2)^{-1} \left( a^{-2}k^2 + m^2 + \frac{3}{2}\gamma^2 H \right) \phi + \gamma(3H + \gamma^2)^{-1} a^{-\frac{3}{2}} \eta. \tag{8}
\]
On the other hand we may write the solution \(\Phi_\eta\) of the linear wave equation (4) \((V = 0)\)with the source \(\eta\) and prescribed initial conditions as a sum of the solution \(\Phi_q\) of the homogeneous equation with these initial conditions and the solution \(\Phi\) of eq.(4) with zero boundary conditions. In the linear case the solution \(\Phi_q\) could be directly quantized. In the non-linear theory with the gravitational fluctuations we quantize fluctuations around \(\Phi_q\). There still remains the classical term \(\Phi\) resulting from the stochastic source \(\eta\). In the lowest order quadratic fluctuations of \(\Phi_\eta\) are a sum of quantum fluctuations \(\langle \Phi_q^2 \rangle\) and thermal fluctuations \(\langle \Phi^2 \rangle\). We can express \(\Phi\) by means of the Green function \(G\)
\[
\Phi(t) = \gamma \int ds G(t,s) \exp \left( \frac{1}{2}\gamma^2 s \right) \eta_s ds \tag{9}
\]
We apply the approximate Green function
\[
G(t,s) = \omega(s)^{-\frac{1}{2}} \omega(t)^{-\frac{1}{2}} \sinh \left( \int_s^t d\tau \omega(\tau) \right). \tag{10}
\]
It can be seen that neglecting the fast decaying mode the Green function (10) approximates for a small \(k\) and slowly varying \(H\) the Green function of the diffusion equation. Using the Green function (10) we can extend the approximation of the integral form of the wave equation to the integral form of the diffusion equation with \(V \neq 0\). We skip the study of the exact relation between these equations concentrating on the linearization of both equations which involves a time dependent mass term \(m^2 \to m^2(t)\) (resulting from an expansion of \(V'\) around the \(x\)-independent solution \(\Phi_q\) of the non-linear equation, see e.g.,[3]).
4 Spectrum of diffusion

Let us assume that a random field \( \phi_t(x) \) is invariant under space translations and define

\[
\langle \phi_t(x) \phi_t(y) \rangle = \int dk \rho_t(k) \exp(i k (x - y))
\]  

(11)

or in Fourier transform

\[
\langle \phi_t(k) \phi_t(k') \rangle = (2\pi)^3 \delta(k + k') \rho_t(k).
\]  

(12)

The spectral index \( 2\sigma \) is defined by the low \( k = |k| \) behaviour \( \rho_t(k) \propto k^{-2\sigma} \).

The solution \( \phi_t = \phi_q + \phi \) of eq.(8) with a given initial condition is a sum of the solution \( \phi_q \) of the homogeneous equation with this initial condition and \( \phi \) with 0 as an initial condition at \( t_0 \). Explicitly

\[
\phi_t = \gamma \int_{t_0}^t \exp \left( - \int_s^t (3H + \gamma^2)^{-1} (k^2 a^{-2} + m^2(s') + \frac{3}{2} \gamma^2 H) ds' \right) a(s)^{-\frac{3}{2}} (3H + \gamma^2)^{-1} ds.
\]  

(13)

Hence,

\[
\rho_t(k) = \gamma^2 \int_{t_0}^t \exp \left( - 2 \int_s^t (3H + \gamma^2)^{-1} (k^2 a^{-2} + m^2(s') + \frac{3}{2} \gamma^2 H) ds' \right) a(s)^{-3} (3H + \gamma^2)^{-2} ds.
\]  

(14)

If we introduce the e-fold time

\[
d\nu = H dt,
\]  

(15)

then

\[
\rho_t(k) = \gamma^2 \int_{\nu(t_0)}^{\nu(t)} (3H + \gamma^2)^{-2} \exp \left( - 2 \int_{\nu(t_0)}^{\nu(t)} (3H^2)^{-1} (1 + \Gamma)^{-1} \left( k^2 \exp(-2\tau') + m^2(\tau') + \frac{3}{2} \gamma^2 H \right) d\tau' \right) H^{-1} \exp(-3\tau) d\tau,
\]  

(16)

where

\[
\Gamma = (3H)^{-1} \gamma^2.
\]  

(17)

We introduce the variable

\[
u = \exp(-2\tau)
\]  

(18)

and assume that \( H(\nu) \propto const \) and \( m^2 H^{-2} \propto const \) then

\[
\rho_t(k) = \left( \frac{1}{2H} \right)^2 \gamma^2 \exp \left( (3H^2)^{-1} (1 + \Gamma)^{-1} k^2 \exp(-2\nu) \right) \int_{\nu(t_0)}^{\nu(t)} \exp \left( -(3H^2)^{-1} (1 + \Gamma)^{-1} k^2 u - 2q \nu \right) u^{\frac{3}{2} - q} du,
\]  

(19)
where
\[ q = (\delta + \frac{3}{2}\Gamma)(1 + \Gamma)^{-1} \]  
and
\[ \delta = m^2(3H^2)^{-1}. \]
The result of integration in eq.(19) can be expressed by the incomplete Γ function
\[ \rho_t(k) = \left( \frac{1}{2\pi} \right)^2 (3H + \gamma^2)^{-2} \exp(-2q\nu)\gamma^2 \exp \left( (3H^2)^{-1}(1 + \Gamma)^{-1}k^2 \exp(-2\nu) \right) \]
\[ \left( (3H^2)^{-1}k^2(1 + \Gamma)^{-1} \right)^{-\sigma} \Gamma(\sigma, (3H^2)^{-1}(1 + \Gamma)^{-1}k^2 \exp(-2\nu)) \]
\[ - \left( (3H^2)^{-1}(1 + \Gamma)^{-1}k^2 \right)^{-\sigma} \Gamma(\sigma, (3H^2)^{-1}(1 + \Gamma)^{-1}k^2 \exp(-2\nu_0)) \],
where
\[ \sigma = \frac{3}{2} - (\delta + \frac{3}{2}\Gamma)(1 + \Gamma)^{-1}. \]  
We have for \( x << 1 \)
\[ \Gamma(\alpha, x) = \Gamma(\alpha) - x^\alpha \left( n!(\alpha + n) \right)^{-1}, \]
and for \( x >> 1 \)
\[ \Gamma(\alpha, x) = x^{\alpha-1} \exp(-x). \]
If \( \nu_0 \rightarrow -\infty \) then the second term in eq.(22) is vanishing and the first factor is dominating. In such a case for a small \( k \)
\[ \rho_t(k) \approx k^{-2\sigma}. \]
More precisely the behaviour (25) takes place if \( k \) is small and
\[ k(a(\nu_0)H)^{-1} >> 1. \]
If
\[ k(a(\nu_0)H)^{-1} << 1, \]
(and \( a(\nu_0) \leq a(\nu) \)) then
\[ \rho_t(k) \approx \text{const}. \]
The spectral index of cosmological perturbations \( n_S \) is related to the fluctuations of \( \phi \) as
\[ n_S - 1 = 2\sigma - 3. \]
At \( \gamma = 0 \) the result (25) coincides with the power spectrum of quantum fluctuations which are derived by a calculation of \( \langle \Phi^2 \rangle \) in the Bunch-Davis vacuum [2][15][16](sec.24.3) (normalized so that the scalar modes behave as plane waves at large \( k(aH)^{-1} \)). It follows from eq.(22) that the amplitude of thermal fluctuations is determined by \( H, \sigma \) (known from CMB measurements), and \( \gamma \) (which
this way would be fixed by \( \rho(k) \). On the other hand the friction \( \gamma \) is related (depending on the model) to other measurable quantities as, e.g., the diffusion constant [13] [17] or the density of radiation at the end of inflation in the warm inflation scenario [18]. The theory shows that under the assumption of almost exponential expansion both the quantum fluctuations and the thermal fluctuations lead to almost the same spectral index. It can depend on the potential \( V \) (which has not been taken into account here; see [19] [20] on a reconstruction of the potential from observational data). However, with the present observational sensitivity the contributions of the thermal and quantum fluctuations may be indistinguishable if the expansion is almost exponential.

5 Summary and outlook

The \( \Lambda \)CDM model with the inflationary scenario is the basic paradigm of the contemporary cosmology. The measured power spectrum is believed to result from an almost exponential expansion. The results of this paper show that it may be difficult on the basis of the present astronomical observations to distinguish the contribution of quantum noise from the thermal noise if the expansion is exponential. If \( H \) is varying (as is the case in the alternative cosmological models [4] [5] [6]) then some more precise estimates in both the quantum calculations of the spectrum as well as in the calculations in the warm inflation models can be compared with observational data to distinguish the contribution of quantum and thermal fluctuations. In the power-law expansion \( a \simeq t^\alpha \) (with \( H = \frac{\alpha}{t} \)) the diffusion equation (8) reads

\[
\frac{\partial}{\partial t} \phi = - \left( t^{-2\alpha} \left( \frac{3\alpha}{2} + \gamma^2 \right)^{-1} k^2 + \left( m^2 + \frac{3\alpha\gamma^2}{2t} \right) \left( \frac{3\alpha}{2} + \gamma^2 \right)^{-1} \right) \phi \\
+ \gamma \left( \frac{3\alpha}{2} + \gamma^2 \right)^{-1} t^{-\frac{3\alpha}{2}} \eta.
\]

(30)

The exploration of the solution of eq. (30) is more involved. However, direct conclusions are possible in some special cases. If in eq.(30) we neglect \( m^2 \) and \( \gamma^2 \) in all factors \( \left( \frac{3\alpha}{2} + \gamma^2 \right)^{-1} \) (as well as the term \( \frac{3\alpha}{2} \gamma^2 H \) ) then we get the scale invariant power spectrum \( \rho \simeq k^{-3} \). If the noise \( \eta \) on the rhs of eq.(1) is multiplied by \( \sqrt{3H + \gamma^2} \) (as in [14]) and we neglect \( m^2 + \frac{3H\gamma^2}{2} \) then the power index will be \( \sigma = \frac{3}{2} + \frac{1}{\alpha-1} \) exactly as in the case of quantum fluctuations on a power-law expanding universe [21].

6 Appendix: Exact formula for the exponential expansion

When \( a(t) = \exp(\lambda t) \) then the solutions of the diffusion as well as the wave equation can be obtained with precisely controlled approximations. Let us de-
\[ M^2 = m^2 + \frac{3}{2} \gamma^2 H. \]  \quad (31)

The solution of the linear diffusion equation with zero initial condition is
\[ \phi_t = \gamma \int_{t_0}^t ds \frac{1}{3H + \gamma^2} \eta_s \exp\left(-\frac{3}{2} H(t-s)\right) \exp\left(-\frac{k^2}{3H + \gamma^2}\left(\exp(-2hs) - \exp(-2Ht) - \frac{M^2}{3H + \gamma^2} (t-s)\right)\right) \]  \quad (32)

Let
\[ u(s) = \exp(-2Hs) \]
\[ r = 3H^2(1 + \frac{1}{3} \gamma^2 H^{-1}) = 3H^2(1 + \Gamma) \]  \quad (33)

Then
\[ \rho_t(k) = \frac{\gamma^2}{2H} \left(\frac{1}{3H + \gamma^2}\right)^2 \exp\left(\frac{k^2}{r}\right) \int_{u(t_0)}^{u(t)} \exp\left(-\frac{k^2}{r} u^{\sigma-1} \right) du \]  \quad (34)

where
\[ \sigma = \frac{3}{2} - \frac{M^2}{r} \]  \quad (35)

We have in eq.(34) the same integral as in eq.(22)
\[ \rho_t(k) = \frac{\gamma^2}{2H} \left(\frac{1}{3H + \gamma^2}\right)^2 \exp\left(\frac{k^2}{r}\right) \left(\frac{k^2}{r}\right)^{-\sigma} \left(\Gamma(\sigma, \frac{k^2}{r} u(t)) - \Gamma(\sigma, \frac{k^2}{r} u(t_0))\right) \]  \quad (36)

Hence, if \( t_0 = -\infty \), i.e., \( a(t_0) = 0 \), then
\[ \rho_t(k) \simeq k^{-2\sigma} \]  \quad (37)

In general, if \( k \to 0 \) with \( \frac{k^2}{r} = k \exp(-Ht_0) \) bounded then still we have \( \rho_t(k) \simeq k^{-2\nu} \) without referring to initial conditions, i.e., we obtain the behaviour discussed at eqs.(26)-(27). If \( \gamma = 0 \) then
\[ \sigma = \frac{3}{2} - \delta \]  \quad (38)

This is exactly the index resulting from a quantization of the scalar field in an exponentially expanding universe \[2][15][16]. Such a conclusion has also been derived in warm inflation for \( M = 0 \) in \[8\].

We can estimate the solution of the stochastic wave equation in the WKB approximation (10). Let
\[ q = \frac{9}{4} H^2 + \frac{1}{4} \gamma^4 + \frac{3}{2} \gamma^2 H - m^2. \]  \quad (39)
Now
\[ \int_s^t \omega = -\frac{1}{2H} \int (q - k^2 u)^{\frac{3}{2}} u^{-1} du \] (40)

or if we introduce
\[ v = q - k^2 u \] (41)

\[ \int_s^t \omega = H^{-1} \sqrt{v(t)} - H^{-1} \sqrt{v(s)} - \frac{\sqrt{q}}{2H} \left( 2Ht - 2Hs + \ln \left( 2q - k^2 u(t) + 2\sqrt{q} \sqrt{v(t)} \right) \right) 
- \ln \left( 2q - k^2 u(s) + 2\sqrt{q} \sqrt{v(s)} \right) . \] (42)

Then, if we first expand \( \int_s^t \omega \) in \( \frac{1}{2H} \) and subsequently calculate \( \Phi \) according to eq. (9) then from the solution (9) of the wave equation we get the same result (38) for the power spectrum.

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