On controllability and observability of chains formed by point masses connected with springs and dashpots

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Abstract

We consider a physical system constituted by a finite chain of point masses consecutively linked by linear springs and dashpots. At one of the end points acts an external control force aligned with the chain and the system is observable by the position of the other end point. We show that, whatever is the number of the point masses, if the sequence of the elastic constants is proportional to the sequence of the dashpot constants, then the mechanical system is completely controllable, completely observable, completely reachable and completely re-construcible, in the sense of control theory.

Key words: Control, Dashpots, Springs, Mass chain, Point masses chain, Dashpot-spring-mass chain, Control of physical systems, Viscous damping

1 Introduction

1.1 Premise

In [1] a finite chain of mass points consecutively linked by linear springs is considered; the mechanical system is externally controlled by a control force aligned with the chain, applied at one of the end points, and is observable by the position of the other end point. There it is shown that such a system is completely controllable, completely observable, completely reachable and completely re-constructible, in the sense of control theory, for all choices of the physical parameters.
Here we consider a spring-mass system similar to the one described above but includes viscous damping which is caused by the presence of dashpots connecting consecutive points of the chain. We show that such a system is completely controllable, completely observable, completely reachable and completely re-constructible provided the sequence of the dashpot constants is proportional to the sequence of the elastic constants. This is true for any choice of the number of points, of their masses, and of the elastic constants of the springs.

1.2 On linear systems with control inputs

Following the same notations of [2], we consider the continuous linear first order system

\[ \dot{z} = Fz + gu, \]  
\[ y = Hz + du, \]  

where \( z, u \) and \( y \) are real vectors of respective dimensions \( n, m \) and \( p \), \( \dot{z} = dz/dt \), \( F \) is the \((n \times n)\) state matrix, \( g \) is the \((n \times m)\) input matrix, \( h \) is the \((p \times n)\) output matrix, and \( d \) is the \((p \times m)\) direct transmission matrix. In literature the two equations above are usually referred as state equation and output equation, respectively.

A mechanical system \( \Sigma \) represented by equations (1)-(2) is said to be ([3]–[5])

(A) completely reachable if for each state \( z \) there are an instant \( t > 0 \) and an input function \( u(.) \), defined in \([0, t]\), such that the solution \( z(.) \) to equation (1) satisfies \( z(0) = 0, \ z(t) = z \);

(B) completely controllable if for each state \( z \) there are an instant \( t > 0 \) and an input function \( u(.) \), defined in \([0, t]\), such that the solution \( z(.) \) to equation (1) satisfies \( z(0) = z, \ z(t) = 0 \);

(C) completely observable if every initial state \( z(0) \) can be determined by observing the future outputs \( y(s) \) for \( s \in [0, \tau] \) for some \( \tau > 0 \);

(D) completely re-constructible if every actual state \( z(t) \) can be determined by observing the past outputs \( y(s) \) for \( s \in [t - \tau, t] \) for some \( \tau > 0 \).

By the linearity, for any continuous system (1)-(2),

(E) the controllability space and reachable space coincide and do not depend on the length \( t \) of the time interval; (F) the observability condition and re-constructibility condition are equivalent and do not depend on the length \( \tau \) of the time interval.
Here we consider strict casual systems with one scalar input and one scalar output; hence
\[ d = O, \quad m = 1 = p, \] (3)
\[ g \] is a column vector of dimension \( n \), \( h \) is a row vector of dimension \( n \), and equations (1), (2) become
\[ \dot{z} = F z + g u, \] (4)
\[ y = h \cdot z. \] (5)

We use the following theorem, which holds for system (4)-(5), with one scalar control input and one scalar output (see [2]).

**Theorem 1.1** The mechanical system \( \Sigma \), represented by (4)-(5), is completely reachable and completely observable if and only if the polynomials
\[ \text{det}(zI - F) \quad \text{and} \quad h \text{adj}(zI - F) g \] (6)
have no common root.

1.3 Mechanics of spring-mass chains with viscous damping

We consider the chain of \( N \geq 2 \) mass points \( P_1, P_2, \ldots, P_N \) of respective masses \( m_1, m_2, \ldots, m_N \). Each point interacts only with its nearest neighbours since any two consecutive points \( P_i, P_{i+1} \) are linked by a linear spring having elastic constant \( k_i > 0 \) and natural length \( \ell_i \). Moreover, there is viscous damping since consecutive mass points are connected by dashpots of constants \( c_i \geq 0 \) (see Figure 1 and Figure 2). We denote \( z_i \) the displacement of the point \( P_i \) from its equilibrium position, so that \( z_i = 0 \) for each \( i = 1 \) to \( N \) corresponds to no deformations in the springs. Of course, this system has \( N \) degrees of freedom and the \( z_i \) are free coordinates for it.

The equations of motion of the mass points \( P_i \) have the form
\[ m_i \frac{d^2}{dt^2} z_i = k_i(z_{i+1} - z_i) + c_i(\dot{z}_{i+1} - \dot{z}_i) + f_{ext}^i, \] (7)
\[ m_i \frac{d^2}{dt^2} z_i = k_{i-1}(z_{i-1} - z_i) + k_i(z_{i+1} - z_i) \]
\[ + c_{i-1}(\dot{z}_{i-1} - \dot{z}_i) + c_i(\dot{z}_{i+1} - \dot{z}_i) \quad (i = 2, \ldots, N - 1), \] (8)
Here we study

(i) the controllability of the state system by the input control \( u = f^\text{ext}_1 \), the external force applied to the first mass point \( P_1 \), and

(ii) the observability of each state \( z \) by using as output the position \( y = z_N \) of the last mass point.

Hence we put \( f^\text{ext}_N = 0 \) and by introducing the variables \( z_{N+i} = \frac{d}{dt}z_i \), equations (7)-(9) give rise to the first order system of 2N linear equations

\[
\frac{d}{dt} z_i = z_{N+i} , \quad i = 1, \ldots, N, \tag{10}
\]

\[
m_1 \frac{d}{dt} z_{N+1} = k_1(z_2 - z_1) + c_1(z_{N+2} - z_{N+1}) + u, \tag{11}
\]

\[
m_i \frac{d}{dt} z_{N+i} = k_{i-1}(z_{i-1} - z_i) + k_i(z_{i+1} - z_i) + c_{i-1}(z_{N+i-1} - z_{N+i}) + c_i(z_{N+i+1} - z_{N+i}) \quad (i = 2, \ldots, N - 1) \tag{12}
\]

\[
m_N \frac{d}{dt} z_{2N} = k_{N-1}(z_{N-1} - z_N) + c_{N-1}(z_{2N-1} - z_{2N}). \tag{13}
\]

In order to write equations (10) to (13) in matrix form, we put

\[
X(x_1, x_2, \ldots, x_{N-1}) = \begin{pmatrix} x_1/m_1 & -x_1/m_1 & 0 & \cdots & 0 \\ -x_1/m_2 & x_1/m_2 + x_2/m_2 & x_2/m_2 & \cdots & 0 \\ 0 & -x_2/m_3 & x_2/m_3 + x_3/m_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_{N-2}/m_{N-2} \\ 0 & 0 & 0 & \cdots & -x_{N-1}/m_{N-1} \end{pmatrix} \tag{14}
\]

\[
K_N = X(k_1, k_2, \ldots, k_{N-1}) , \quad C_N = X(c_1, c_2, \ldots, c_{N-1}) ,
\]

and
\[ F_N = \begin{bmatrix} O_N & I_N \\ -K_N & -C_N \end{bmatrix} \]  

(15)

where \( O_N \) is the square zero matrix of order \( N \) and \( I_N \) is the identity matrix of order \( N \); moreover we put

\[
\begin{align*}
& z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \\ z_{N+1} \\ \vdots \\ z_{2N} \end{pmatrix}, \\
& g = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1/m_1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \\
& h = \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{pmatrix},
\end{align*}
\]

(16)

where \( 1/m_1 \) is in \((N + 1)\)th place of column \( g \), of dimension \( 2N \), and \( 1 \) is in \( N \)th place of row \( h \), of dimension \( 2N \).

Hence equations (10)-(13) in matrix form rewrite as:

\[
\begin{align*}
\dot{z} &= Fz + gu, \\
y &= h \cdot z.
\end{align*}
\]

(17)  

(18)

2 Two useful lemmas

For any square matrix \( A \) we use the notation

\[ \|A\| = DetA. \]

(19)

In order to prove the two key theorems of the present paper, \( i.e. \) Theorem 3.1 and Theorem 4.1, we need the two lemmas below.
Lemma 2.1  For any choice of the positive integers $\alpha$, $\beta$ and of the real number $\rho$, let

$O_{\alpha \times \beta}$ be the $\alpha \times \beta$ zero matrix,

$I_\alpha$ be the identity matrix of order $\alpha$,

$B_\beta$ be any matrix of order $\beta$, and

$$
A = \begin{bmatrix}
O_{\alpha \times \beta} & \rho I_\alpha \\
B_\beta & O_{\beta \times \alpha}
\end{bmatrix}.
$$

(20)

Then

$$
\|A\| = \rho^\alpha (-1)^{\alpha(\beta+2)} \|B_\beta\|.
$$

(21)

Proof. By expanding $A$ in the cofactors of the 1st row yields

$$
\|A\| = \rho (-1)^{\beta+2} \left\| \begin{bmatrix}
O_{(\alpha-1) \times \beta} & \rho I_{\alpha-1} \\
B_\beta & O_{\beta \times (\alpha-1)}
\end{bmatrix} \right\|.
$$

(22)

Repeating this step to the matrix

$$
\left\| \begin{bmatrix}
O_{(\alpha-1) \times \beta} & \rho I_{\alpha-1} \\
B_\beta & O_{\beta \times (\alpha-1)}
\end{bmatrix} \right\|,
$$

(23)

consecutively for each $i = 2, \ldots, \alpha$, yields

$$
\|A\| = \rho^i (-1)^{i(\beta+2)} \left\| \begin{bmatrix}
O_{(\alpha-i) \times \beta} & \rho I_{\alpha-i} \\
B_\beta & O_{\beta \times (\alpha-i)}
\end{bmatrix} \right\|.
$$

(24)

and thus for $i = \alpha$ we obtain equality (21).

Q.E.D.
Lemma 2.2 Let $\gamma$ be a positive integer and let $A$ be a square matrix of order $2\gamma - 1$ having the form

$$A = \begin{bmatrix} z I_{\gamma-1} & -I_{\gamma} \\ Row_{\gamma-1}(0) & UpTr(a_1, a_2, \ldots, a_{\gamma-1}) UpTr(b_1, b_2, \ldots, b_{\gamma-1}) Col_{\gamma-1}(\#) \end{bmatrix}, \quad (25)$$

where (i) $Row_{\gamma-1}(0)$ is the row of zeros of dimension $\gamma - 1$, (ii) $UpTr(x_1, x_2, \ldots, x_{\gamma-1})$ denotes any upper triangular matrix having the numbers $x_1, x_2, \ldots, x_{\gamma-1}$ in the main diagonal and (iii) $Col_{\gamma-1}(\#)$ denotes any column of dimension $\gamma - 1$. Then

$$\|A\| = (-1)^{\gamma}(a_1 + zb_1)(a_2 + zb_2) \ldots (a_{\gamma-1} + zb_{\gamma-1}). \quad (26)$$

Proof. For each $i = 1, \ldots, N - 1$ multiply the $(N - 1 + i)th$ column of $A$ by $z$ and add the result to the $ith$ column; we obtain a matrix $B$ having the form

$$B = \begin{bmatrix} O_{\gamma \times (\gamma - 1)} & -I_{\gamma} \\ UpTr(a_j + zb_j : 1 \leq j \leq \gamma - 1) & UpTr(b_j : 1 \leq j \leq \gamma - 1) \ Col_{\gamma - 1}(\#) \end{bmatrix}. \quad (27)$$

Now for $i = 1, \ldots, \gamma$, multiplying the $ith$ row of $B$ by suitable quantities and adding the results to the $(N + 1 + i)th$ row, one obtains a matrix $B'$ having the form

$$B' = \begin{bmatrix} O_{\gamma \times (\gamma - 1)} & -I_{\gamma} \\ UpTr(a_j + zb_j : 1 \leq j \leq \gamma - 1) & O_{(\gamma - 1) \times \gamma} \end{bmatrix}. \quad (28)$$

By the properties of determinants we have $\|A\| = \|B\| = \|B'\|$. By Lemma 2.1 with $\rho = -1$ we have

$$\|B'\| = (-1)^{\gamma}(-1)^{\gamma + 1} \|UpTr(a_j + zb_j : 1 \leq j \leq \gamma - 1)\| ;$$

thus, since $\gamma(\gamma + 1)$ is even,

$$\|A\| = \|B'\| = (-1)^{\gamma} \|UpTr(a_j + zb_j : 1 \leq j \leq \gamma - 1)\|. \quad$$

Now equality (26) holds since the determinant of a upper triangular matrix equals the product of its diagonal elements.
3 Form of the adjoint polynomial

**Theorem 3.1** In connection with any positive integer \( N \geq 2 \) consider the square matrix \( F_N \) of order \( 2N \) defined by (14)-(15); moreover

(i) let \( a_N = zI_{2N} - F_N \), thus
\[
a_N = \begin{bmatrix} zI_N & -I_N \\ K_N & C_N + zI_N \end{bmatrix};
\]  \hspace{1cm} (29)

(ii) let
\[
h_N = \begin{bmatrix} 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \end{bmatrix}
\]  \hspace{1cm} (30)

be the row of dimension \( 2N \), with components \( \delta^{Ni} \), \( i = 1, 2, \ldots, 2N \) (\( \delta \) = Kronecker’s delta);

(iii) let
\[
g_N = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix}^T
\]  \hspace{1cm} (31)

be the column of dimension \( 2N \), with components \( \frac{1}{m_i} \delta^{(N+1)i} \), \( i = 1, 2, \ldots, 2N \).

Then
\[
h_N \text{adj}(a_N) g_N = \frac{(k_1 + zc_1)(k_2 + zc_2) \ldots (k_{N-1} + zc_{N-1})}{m_1 m_2 \ldots m_N}.
\]  \hspace{1cm} (32)

**Proof.** The polynomial \( h_N \text{adj}(a_N) g_N \) equals \( c_N^{N+1}/m_1 \), where \( c_N^{N+1} \) is the \((N, N+1)\) component of the matrix \( c = \text{adj}(a_N) \); hence we have
\[
m_1 h_N \text{adj}(a_N) g_N = c_N^{N+1} = (-1)^{N+(N+1)} \|A_{N+1N}\| = -\|A_{N+1N}\|,
\]  \hspace{1cm} (33)

where \( A_{N+1N} \) is the submatrix of \( a_N \) obtained by deleting the \((N+1)\)th row and the \( N \)th column. Now we observe that \( A_{N+1N} \) has the form (25), that
is,

\[
A_{N+1} N = \begin{bmatrix}
  zI_{N-1} & -I_N \\
  \\
  \text{Row}_{N-1}(0) \\
  \\
  \text{Up}^T(a_1, a_2, \ldots, a_{N-1}) \text{Up}^T(b_1, b_2, \ldots, b_{N-1}) \text{Col}_{N-1}(\#)
\end{bmatrix},
\tag{34}
\]

with

\[
a_1 = -\frac{k_1}{m_2}, \quad a_2 = -\frac{k_2}{m_3}, \quad \ldots, \quad a_{N-1} = -\frac{k_{N-1}}{m_N},
\tag{35}
\]

\[
b_1 = -\frac{c_1}{m_2}, \quad b_2 = -\frac{c_2}{m_3}, \quad \ldots, \quad b_{N-1} = -\frac{c_{N-1}}{m_N},
\tag{36}
\]

and \(\text{Col}_{N-1}(\#)\) is any column of dimension \(N-1\). Now Lemma 2.2 yields

\[
\|A_{N+1} N\| = (-1)^N(a_1 + zb_1)(a_2 + zb_2) \ldots (a_{N-1} + zb_{N-1}),
\tag{37}
\]

and thus equalities (33) to (37) yield

\[
h_N \text{adj}(a_N) g_N = \frac{c_{N+1}^N}{m_1} = (-1)^{N+1} \frac{(a_1 + zb_1)(a_2 + zb_2) \ldots (a_{N-1} + zb_{N-1})}{m_1}
\]

\[
= (-1)^{N+1+N-1} \frac{(k_1 + zc_1)(k_2 + zc_2) \ldots (k_{N-1} + zc_{N-1})}{m_1 m_2 \ldots m_N}.
\]

Q.E.D.

\[ \text{4 Form of the characteristic polynomial} \]

Let us remind that the characteristic polynomial \(P_N(z)\) of \(F_N\) is defined by

\[
P_N(z) = \|a_N\|, \quad \text{with} \quad a_N = zI_{2N} - F_N.
\tag{38}
\]

The next theorem characterizes the characteristic polynomials of the mechanical systems considered here.

Note that the chain of mass points which is considered here has no springs connecting the first point and the latter to a fixed external wall.

For this reason the characteristic polynomial has the twice root zero.
Theorem 4.1  In connection with any positive integer $N$ consider the square matrix $F_N$ of order $2N$ defined by (14)-(15), and put

$$b_N = zc_N + k_N.$$  

(39)

Then the equalities

$$P_1(z) = z^2, \quad D_0 = 1,$$  

(40)

$$D_N(z) = P_N(z) + \frac{b_N}{m_N} D_{N-1}(z), \quad N = 1, 2, \ldots,$$  

(41)

$$P_N(z) = \left(z^2 + \frac{b_{N-1}}{m_N}\right) P_{N-1}(z) + z^2 \frac{b_{N-1}}{m_{N-1}} D_{N-2}(z), \quad N = 2, 3, \ldots,$$  

(42)

inductively define the sequence $\{P_N(z)\}_{N \geq 1}$ of the characteristic polynomials of $F_N$.

Moreover, let

$$\{1\}_N = z^2 I_N + zC_N + K_N + W_N,$$  

(43)

with

$$W_N = \begin{bmatrix} O_{(N-1)} & Col_{(N-1)}(0) \\ Row_{(N-1)}(0) \left( z\frac{c_N}{m_N} + \frac{k_N}{m_N} \right) \end{bmatrix}.$$  

(44)

Then the quantities $D_k$ in (40)-(42) are the determinants of matrices $\{1\}_k$:

$$D_k = \|\{1\}_k\|.$$  

(45)

Proof.  We prove the theorem by induction. Hence we firstly note that the inductive formulae (40)-(42) hold for $N = 2$; in fact, see (15), for

$$F_2 = \begin{bmatrix} O_2 & I_2 \\ -K_2 & -C_2 \end{bmatrix},$$  

(46)

by (38) we have

$$a_2 = \begin{bmatrix} zI_2 & -I_2 \\ K_2 (zI_2 + C_2) \end{bmatrix}.$$  

(47)
and

\[ P_2(z) = \|a_2\| = \left\| \begin{pmatrix} z^2 + z \frac{c_{m_1}}{m_1} + \frac{k_{m_1}}{m_1} \\ -z \frac{c_{m_2}}{m_2} - \frac{k_{m_2}}{m_2} \end{pmatrix} \begin{pmatrix} \left(-z \frac{c_{m_1}}{m_1} - \frac{k_{m_1}}{m_1}\right) \\ \left(z^2 + z \frac{c_{m_2}}{m_2} + \frac{k_{m_2}}{m_2}\right) \end{pmatrix} \right\| \]

\[ = \left( \frac{k_{m_1}}{m_1} + \frac{k_{m_2}}{m_2} \right) z^2 + \left( \frac{c_{m_1}}{m_1} + \frac{c_{m_2}}{m_2} \right) z^3 + z^4 . \] (48)

It is easy to verify that using (40)-(42) with \( N = 2 \) one arrives at the same result. That is, the expression of \( P_2(z) \) given by its definition (38) coincides with the expression given by equalities (40)-(42).

Now, in order to use the induction principle, assume the inductive hypothesis: let (40)-(42) hold for an arbitrarily given integer \( N \); then consider

\[ F_{N+1} = \begin{bmatrix} O_{N+1} & I_{N+1} \\ -K_{N+1} & -C_{N+1} \end{bmatrix}, \] (49)

with \( K_{N+1} \) and \( C_{N+1} \) defined accordingly with (14), (14).

By (38) we have

\[ P_{N+1}(z) = \|a_{N+1}\|, \quad a_{N+1} = \begin{bmatrix} zI_{N+1} & -I_{N+1} \\ K_{N+1} (zI_{N+1} + C_{N+1}) \end{bmatrix} \] (50)

Now, for each \( i = 1, \ldots, N \), we multiply the \((N + 1 + i)\)th column of the matrix \( a_{N+1} \) by \( z \) and add the result to the \( i \)th column; we obtain the matrix

\[ \begin{bmatrix} O_{N+1} & I_{N+1} \\ (z^2 I_{N+1} + z C_{N+1} + K_{N+1}) (z I_{N+1} + C_{N+1}) \end{bmatrix} . \] (51)

Hence for each \( i = 1, \ldots, N + 1 \) multiplying the \( i \)th row of the latter matrix by suitable quantities and adding the results to the \((N + 1 + i)\)th row, we obtain the matrix

\[ a'_{N+1} = \begin{bmatrix} O_{N+1} & -I_{N+1} \\ (z^2 I_{N+1} + z C_{N+1} + K_{N+1}) O_{N+1} \end{bmatrix} . \] (52)
Of course, by the determinant properties,
\[ \| a'_{N+1} \| = \| a_{N+1} \| = P_{N+1}(z). \tag{53} \]

By (52)-(53) Lemma 2.1, for \( \alpha = \beta = N + 1 \) and \( A = a_{N+1} \), yields
\[ \| a'_{N+1} \| = (-1)^{N+1} (N+1)(N+3) \| z^2 I_{N+1} + z C_{N+1} + K_{N+1} \| = P_{N+1}(z). \tag{54} \]

Since \( N + 1 + (N + 1)(N + 3) = (N + 1)(N + 4) \) is even for each \( N \), we have
\[ (-1)^{N+1} (N+1)(N+3) = 1, \]
and thus
\[ \| z^2 I_{N+1} + z C_{N+1} + K_{N+1} \| = P_{N+1}(z) \tag{55} \]

Now, by expanding the matrix \( z^2 I_{N+1} + z C_{N+1} + K_{N+1} \) in the cofactors of its last row, we obtain
\[ P_{N+1}(z) = \| z^2 I_{N+1} + z C_{N+1} + K_{N+1} \| \]
\[ = (-1)^{2N+2} \left( z^2 + z \frac{c_N}{m_{N+1}} + \frac{k_N}{m_{N+1}} \right) \| \{1\}_N \| + (-1)^{2N+1} \left( -z \frac{c_N}{m_{N+1}} - \frac{k_N}{m_{N+1}} \right) \| \{2\}_N \| \tag{56} \]

and hence,
\[ P_{N+1}(z) = (z^2 + z \frac{c_N}{m_{N+1}} + \frac{k_N}{m_{N+1}}) \| \{1\}_N \| + (z \frac{c_N}{m_{N+1}} + \frac{k_N}{m_{N+1}}) \| \{2\}_N \| \tag{57} \]

where, for
\[ \alpha_j = z^2 + z \frac{c_j + c_{j+1}}{m_j} + \frac{k_j + k_{j+1}}{m_j}, \tag{58} \]

the matrices \( \{1\}_N, \{2\}_N \) are defined by
\[ \{1\}_N = \begin{bmatrix} Z_1 Z_2 \end{bmatrix}, \quad \{2\}_N = \begin{bmatrix} Z_1 Z_3 \end{bmatrix} \tag{59} \]
with $Z_1$, $Z_2$ and $Z_3$ respectively given by

\[
\begin{bmatrix}
(z^2 + z \frac{c_1}{m_1} + \frac{k_1}{m_1}) & (-z \frac{c_1}{m_1} - \frac{k_1}{m_1}) & 0 & \ldots \\
(-z \frac{c_1}{m_2} - \frac{k_1}{m_2}) & \alpha_1 & (-z \frac{c_2}{m_2} - \frac{k_2}{m_2}) & \ldots \\
0 & (-z \frac{c_2}{m_3} - \frac{k_2}{m_3}) & \alpha_2 & \ldots \\
0 & 0 & (-z \frac{c_3}{m_4} - \frac{k_3}{m_4}) & \ldots \\
0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\ldots & \ldots & \ldots & \ldots \\
\ldots & 0 & 0 & 0 \\
\ldots & 0 & 0 & 0 \\
\ldots & (-z \frac{c_{N-3}}{m_{N-3}} - \frac{k_{N-3}}{m_{N-3}}) & 0 & 0 \\
\ldots & \alpha_{N-3} & (-z \frac{c_{N-2}}{m_{N-2}} - \frac{k_{N-2}}{m_{N-2}}) & 0 \\
\ldots & (-z \frac{c_{N-2}}{m_{N-1}} - \frac{k_{N-2}}{m_{N-1}}) & \alpha_{N-2} & (-z \frac{c_{N-1}}{m_{N-1}} - \frac{k_{N-1}}{m_{N-1}}) \\
\ldots & 0 & (-z \frac{c_{N-1}}{m_N} - \frac{k_{N-1}}{m_N}) & \alpha_{N-1}
\end{bmatrix}
\]

For instance, for $N = 4$ we have

\[
Z_1 = \begin{bmatrix}
(z^2 + z \frac{c_1}{m_1} + \frac{k_1}{m_1}) & (-z \frac{c_1}{m_1} - \frac{k_1}{m_1}) \\
(-z \frac{c_1}{m_2} - \frac{k_1}{m_2}) & (z^2 + z \frac{c_1 + c_2}{m_2} + \frac{k_1 + k_2}{m_2}) \\
0 & (-z \frac{c_2}{m_3} - \frac{k_2}{m_3}) \\
0 & 0
\end{bmatrix}
\]
\[ Z_2 = \begin{bmatrix} 0 & 0 \\ (-z \frac{c_1}{m_2} - \frac{k_1}{m_2}) & 0 \\ (z^2 + z \frac{c_1 + c_3}{m_3} + \frac{k_3 + k_4}{m_3}) & (-z \frac{c_1}{m_3} - \frac{k_3}{m_3}) \\ (-z \frac{c_1}{m_4} - \frac{k_3}{m_4}) & (z^2 + z \frac{c_1 + c_4}{m_4} + \frac{k_3 + k_4}{m_4}) \end{bmatrix} \] (64)

\[ Z_3 = \begin{bmatrix} 0 & 0 \\ (-z \frac{c_1}{m_2} - \frac{k_1}{m_2}) & 0 \\ (z^2 + z \frac{c_1 + c_3}{m_3} + \frac{k_3 + k_4}{m_3}) & 0 \\ (-z \frac{c_1}{m_4} - \frac{k_3}{m_4}) & (-z \frac{c_1}{m_4} - \frac{k_4}{m_4}) \end{bmatrix} \] (65)

\[ \{1\}_4 = \begin{bmatrix} (z^2 + z \frac{c_1}{m_1} + \frac{k_1}{m_1}) & (-z \frac{c_1}{m_1} - \frac{k_1}{m_1}) & 0 & 0 \\ (-z \frac{c_1}{m_2} - \frac{k_1}{m_2}) & \alpha_1 & (-z \frac{c_1}{m_2} - \frac{k_3}{m_2}) & 0 \\ 0 & (-z \frac{c_1}{m_3} - \frac{k_4}{m_3}) & \alpha_2 & (-z \frac{c_1}{m_3} - \frac{k_1}{m_3}) \\ 0 & 0 & (-z \frac{c_1}{m_4} - \frac{k_3}{m_4}) & \alpha_3 \end{bmatrix} \] (66)

and

\[ \{2\}_4 = \begin{bmatrix} (z^2 + z \frac{c_1}{m_1} + \frac{k_1}{m_1}) & (-z \frac{c_1}{m_1} - \frac{k_1}{m_1}) & 0 & 0 \\ (-z \frac{c_1}{m_2} - \frac{k_1}{m_2}) & \alpha_1 & (-z \frac{c_1}{m_2} - \frac{k_3}{m_2}) & 0 \\ 0 & -z \frac{c_1}{m_3} - \frac{k_4}{m_3} & \alpha_2 & 0 \\ 0 & 0 & (-z \frac{c_1}{m_4} - \frac{k_3}{m_4}) & (-z \frac{c_1}{m_4} - \frac{k_1}{m_4}) \end{bmatrix} \] (67)

Now let

\[ E_N = \begin{bmatrix} (z^2 + z \frac{c_1}{m_1} + \frac{k_1}{m_1}) & \ldots & \ldots & 0 \\ \cdot & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \alpha_{N-2} \end{bmatrix} \] (68)

Since

\[ z^2 I_N + zC_N + K_N \]
by the linearity property of determinants with respect to columns, we can express \(\|\{1\}_N\|\) within (57) as

\[
\|\{1\}_N\| = \left\| z^2 I_N + z C_N + K_N \right\| + \| E_N \| = P_N(z) + \| E_N \| ,
\]

(70)

Again by the linearity property of determinants we have \(\| E_N \| + \|\{2\}_N\| = 0\) since the last column of the matrix \(E_N + \{2\}_N\) vanishes; hence, taking into account equalities (56) through (68), equality (57) yields

\[
P_{N+1}(z) = \left( z^2 + z \frac{c_N}{m_{N+1}} + \frac{k_N}{m_{N+1}} \right) P_N(z) + z^2 \| E_N \| .
\]

(71)

Replacing determinant \(\| E_N \|\) by its expansion in the cofactors of the \(n\)th column in equalities (71) and (70), respectively yields

\[
P_{N+1}(z) = \left( z^2 + z \frac{c_N}{m_{N+1}} + \frac{k_N}{m_{N+1}} \right) P_N(z) + z^2 \left( z \frac{c_N}{m_N} + \frac{k_N}{m_N} \right) \|\{1\}_{N-1}\|
\]

(72)

and

\[
\|\{1\}_N\| = P_N(z) + z^2 \left( z \frac{c_N}{m_N} + \frac{k_N}{m_N} \right) \|\{1\}_{N-1}\| ,
\]

(73)

where the matrix \(\{1\}_{N-1}\) is defined by (43) for \(i = N - 1\).

Thus we have proved (40) through (42).

Lastly note that by (59)- (61) equalities (43), (44) yield (45).

Q.E.D.
Remark 4.1 By the above inductive definition, the polynomial \(P_N(z)\) can be written in the form

\[
P_N(z) = \left( z^2 + \frac{zc_{N-1} + k_{N-1}}{m_N} \right) P_{N-1}(z) + z^2 \frac{zc_{N-1} + k_{N-1}}{m_{N-1}} (P_{N-2}(z) + \ldots
\]

\[
\ldots + \frac{zc_3 + k_3}{m_3} \left( P_2(z) + \frac{zc_2 + k_2}{m_2} (z^2 + \frac{zc_1 + k_1}{m_1}) \ldots \right). \tag{74}
\]

Indeed, replacing back with equalities (40)-(41) in (42) yields (74).

Moreover, the above inductive definition implies the following expression for \(P_N(z)\):

\[
P_N(z) = \sum_{\rho=0}^{2N-2} \sum_{0 \leq M; R \leq \rho} \left( \sum_{1 \leq r_1 < r_2 < \ldots < r_M < N} \frac{k_{r_1} \ldots k_{r_M} c_{s_1} c_{s_2} \ldots c_{s_R}}{m_{t_1} m_{t_2} \ldots m_{t_{M+R}}} \right) z^{2N-\rho}, \tag{75}
\]

For example, for \(N = 2\) we have

\[
P_2(z) = \left( z^2 + \frac{zc_1 + k_1}{m_2} \right) P_1(z) + z^2 \frac{zc_1 + k_1}{m_1} D_0,
\]

thus

\[
P_2(z) = z^4 + z^3 \left( \frac{c_1}{m_1} + \frac{c_1}{m_2} \right) + z^2 \left( \frac{k_1}{m_1} + \frac{k_1}{m_2} \right).
\]

For \(N = 3\) we have

\[
P_3(z) = \left( z^2 + \frac{zc_2 + k_2}{m_3} \right) P_2(z) + z^2 \frac{zc_2 + k_2}{m_2} \left( P_1(z) + \frac{zc_1 + k_1}{m_1} D_0 \right),
\]

thus

\[
P_3(z) = \left( z^2 + \frac{zc_2 + k_2}{m_3} \right) \left[ z^4 + z^3 \left( \frac{c_1}{m_1} + \frac{c_1}{m_2} \right) + z^2 \left( \frac{k_1}{m_1} + \frac{k_1}{m_2} \right) \right] + z^2 \frac{zc_2 + k_2}{m_2} \left( z^2 + \frac{zc_1 + k_1}{m_1} \right),
\]

and

\[
P_3(z) = z^6 + z^5 \left( \frac{c_1}{m_1} + \frac{c_1}{m_2} + \frac{c_2}{m_2} + \frac{c_2}{m_3} \right) + z^4 \left( \frac{k_1}{m_1} + \frac{k_1}{m_2} + \frac{k_2}{m_2} + \frac{k_2}{m_3} + \frac{c_1 c_2}{m_1 m_2} + \frac{c_1 c_2}{m_1 m_3} + \frac{c_1 c_2}{m_2 m_3} \right).
\]
\[ +z^3 \left( \frac{k_1 c_2}{m_1 m_2} + \frac{k_1 c_2}{m_2 m_3} + \frac{k_2 c_1}{m_1 m_2} + \frac{k_1 c_2}{m_1 m_2} + \frac{k_2 c_1}{m_1 m_2} + \frac{k_2 c_1}{m_1 m_2} + \frac{k_1 k_2}{m_1 m_2} + \frac{k_1 k_2}{m_1 m_2} + \frac{k_1 k_2}{m_1 m_2} \right). \]

One may verify that the same expressions can be obtained by using (75).

5 On controllability and observability of the physical systems considered

**Theorem 5.1** For \( N = 2 \) the mechanical system \( \Sigma \) is completely reachable and completely observable for any choice of \( m_1, m_2, c_1 \) and \( k_1 \).

**Proof.** In connection with the square matrix \( F_2 \) of order 4 defined in (15), consider the characteristic polynomial \( P_2(z) \) and the adjoint polynomial \( h_2 \text{adj}(a_2) g_2 \), respectively defined by equalities (38) and (29) through (32).

By Theorem 4.1 and Theorem 3.1, we respectively have

\[ P_2(z) = z^4 + (zc_1 + k_1) \left( \frac{1}{m_1} + \frac{1}{m_2} \right) z^2, \quad h_2 \text{adj}(a_2) g_2(z) = \frac{zc_1 + k_1}{m_1 m_2}. \]

Hence by replacing in \( P_2(z) \) the unique root \( z_1 = -k_1/c_1 \) of \( h_2 \text{adj}(a_2) g_2(z) \), we have \( P_2(z_1) = z_1^4 = \left( \frac{k_1}{c_1} \right)^4 \neq 0 \). Thus the characteristic polynomial and the adjoint polynomial have no common root and Theorem 1.1 implies that \( \Sigma \) is completely reachable and completely observable; moreover, by assertions \((E), (F)\) of Section 1.3, \( \Sigma \) is completely controllable and completely reconstrucitble too.

Q.E.D.

**Theorem 5.2** Let \( N \) be any integer \( > 2 \). If

\[ c_1/k_1 = c_2/k_2 = \ldots = c_{N-1}/k_{N-1}, \] (76)

then

(A) the mechanical system \( \Sigma \) is completely reachable and completely observable for any choice of \( m_1, m_2, \ldots, m_N \).

**Proof.** By Theorem 3.1 the roots of the adjoint polynomial (32) are \( z_i = -k_i/c_i, \quad i = 1, \ldots, N - 1 \). Let

\[ b_i = b_i(z) = zc_i + k_i \quad \beta_{ij} = -k_j \frac{c_i}{c_j} + k_i, \] (77)
hence \( b_i = 0 \) if and only if \( z = z_i \). By hypothesis (76) we have
\[
\beta_{ij} = 0, \quad i, j = 1, 2, \ldots, N - 1.
\] (78)

Hence the inductive definition (40)-(42) of \( P_N(z) \) yields
\[
P_N(z_j) = z_j^2 P_{N-1}(z_j) = z_j^4 P_{N-2}(z_j) = \ldots = (k_j/c_j)^{2N} \neq 0.
\] (79)

Thus any root \( z_j \) of the adjoint polynomial cannot be a root for the characteristic polynomial too.

Now, Theorem 1.1 implies that \( \Sigma \) is completely reachable and completely observable; and by assertions (E), (F) of Section 1.3, \( \Sigma \) is completely controllable and completely re-constructible too.

Q.E.D.

In particular, by the above theorem, for \( N = 3 \) the mechanical system \( \Sigma \) is completely reachable and completely observable for any choice of \( m_1, m_2, m_3, c_1, k_1, c_2 \) and \( k_2 \) provided that \( c_1/k_1 = c_2/k_2 \).

The next two theorems respectively show that when the sequences \( \{c_i\} \) and \( \{k_i\} \) are not proportional proposition (A) can be either true or not true. Hence, in particular, condition (76) is sufficient but not necessary for the validity of (A).

**Theorem 5.3** Proposition (A) is not true for some mechanical system \( \Sigma \) such that (76) does not hold.

**Proof.** Let \( N = 3 \). We have
\[
P_3(z) = (z^2 + \frac{b_2}{m_3}) P_2(z) + z^2 \frac{b_2}{m_2} (P_1(z) + \frac{b_1}{m_1}),
\]
i.e.,
\[
P_3(z) = \left(z^2 + \frac{b_2}{m_3}\right) \left(z^4 + b_1 \frac{1}{m_2} z^2 + z^2 \frac{b_1}{m_1}\right) + z^2 \frac{b_2}{m_2} \left(z^2 + \frac{b_1}{m_1}\right),
\] (80)

Thus, if \( z_1 = 0 \) is root of the adjoint polynomial, i.e.

if \( b_1 = 0 \), then \( P_3(z_1) = z_1^6 + \beta_{12} H z_1^4 \), with \( H = \frac{1}{m_2} + \frac{1}{m_3} \).

Hence \( P_3(z_1) = 0 \) if and only if \( z_1^2 + \beta_{21} H = 0 \), i.e.,
\[ \frac{k_2^2}{c_1^2} + (k_2 - \frac{k_1}{c_1}c_2)H = 0 \]

This relation is linear in \( k_2 \); hence, if \( H, k_1, c_1, c_2 \) are given, then one determines the unique \( k_2 \) such that the last equality holds. With regard to these values of the material parameters it follows that \( z_1 \) is a root for both the characteristic polynomial and adjoint polynomial.

Q.E.D.

**Theorem 5.4** Proposition (A) is true for some mechanical system \( \Sigma \) such that (76) does not hold.

**Proof.** Let \( N = 3 \). By (80) we have

\[
P_3(z_1) = \left( z_1^2 + \frac{b_2(z_1)}{m_3} \right) z_1^4 + z_1^2 \frac{b_2(z_1)}{m_2} z_1^2 = \frac{k_1^4}{c_1^2} \left[ \frac{k_2^2}{c_2^2} + (k_2 - \frac{k_1}{c_1}c_2)H_{23} \right], \tag{81}
\]

\[
P_3(z_2) = z_2^4 \left( \frac{b_1(z_2)}{m_3} + \frac{b_2(z_1)}{m_2} \right) z_1^4 + z_1^2 \frac{b_2(z_1)}{m_2} z_1^2 = \frac{k_2^4}{c_2^2} \left[ \frac{k_1^2}{c_1^2} + (k_1 - \frac{k_2}{c_2}c_1)H_{12} \right]. \tag{82}
\]

where

\[
H_{ij} = \frac{1}{m_i} + \frac{1}{m_j} \tag{83}
\]

and

\[
z_i = -\frac{k_1}{c_i} \tag{84}
\]

are the roots of the adjoint polynomial. Hence

\[
P_3(z_1) = 0 \iff \frac{k_1^2}{c_1^2} + (k_2 - \frac{k_1}{c_1}c_2)H_{23} = 0 \tag{85}
\]

and

\[
P_3(z_2) = 0 \iff \frac{k_2^2}{c_2^2} + (k_1 - \frac{k_2}{c_2}c_1)H_{12} = 0. \tag{86}
\]

Thus, given \( m_1, m_2, m_3, c_1 \) and \( c_2 \) one can choose \((k_1, k_2)\) such that both \( P_3(z_1) \neq 0 \) and \( P_3(z_2) \neq 0 \). Hence the adjoint polynomial and the characteristic polynomial have no common root. In connection with such a choice for the mechanical system \( \Sigma \) proposition (A) is true.
6 On applications of the above results

The system of equations $\Sigma_N$ studied here is a mathematical model suitable to describe various rectilinear physical systems as chains of three-dimensional bodies undergoing translational motions.

By the results of Theorems 5.1-5.2 above, people involved in the design of such physical systems with $N > 2$ can choose the material constants in order to render the system completely controllable, observable, reachable and reconstructible; instead for $N = 2$ any such a system has such properties for any choice of the material constants.

The key advantage of the theorems proved here for engineering applications is that, when control is a property useful for the system under design, one can design the system by choosing the material parameters in such a way as to have controllability.

A further theoretical task will be to construct or to find the controlling input functions for the system. The present paper clarifies when such a problem has a solution.

An example of a physical system, which can be modelled by $\Sigma_N$, is given by a chain of $N$ pistons consecutively connected by springs and constrained inside a cylindrical cavity containing a fluid. Assuming that an external input force $u(\tau)$ can act normally on the first piston, and taking as output the position of the latter piston, the system will be controllable and observable by the history of the latter provided the elastic springs be chosen proportional to the constants of viscous damping between consecutive pistons. Hence, (i) the system can reach any prescribed state by a suitable input force $u(\tau)$; and (ii) every state of the system (that is, position and velocity of each piston) can be determined by observing the excursions of the latter piston.

In particular, system $\Sigma_2$ can also be used to model the behaviour of car-wheel suspensions. In more detail, when a car travels along a bumpy road the wheel tyre copies roughness of the road surface. The wheel is thus driven up or down in the vertical direction along the $z$ axis. In this case, the rectilinear system under consideration consists of four bodies: the spring, the shock absorber, the wheel and the quarter of the car body. The system excitation by the road surface and the gravitational attraction of the wheel and the quarter-car body form the system surroundings. The model is excited by the source of the velocity $\dot{z}_o$ resulting from the tyre copying the road bumps.
Figure 3 on the left shows a detail of the suspension of one of its wheels, and on the right shows a scheme of it. The point $R_z$ represents the $z$-position of the contact point between the car tyre and the road surface $R$ when the car moves in the $x$-direction. The points $W_z$ and $B_z$ represent the $z$-motion of the wheel axis $W$ and the body-suspension interaction point $B$, respectively. The constants $k$ and $c$ are the compliance and damping of the wheel tyre, $m_1$ is the mass of the complete wheel and $m_2$ is the quarter of the mass of the car body. The model is excited by the source of the velocity $\dot{z}_o$ resulting from the tyre copying the road bumps. Then by applying Newton’s second law we obtain the differential equations of this physical model:

$$m_1 \frac{d^2}{dt^2} z_1 = k_1(z_2 - z_1) + c_1(\dot{z}_2 - \dot{z}_1) + f_{ext}^1,$$

(87)

$$m_2 \frac{d^2}{dt^2} z_2 = k_1(z_1 - z_2) + c_1(\dot{z}_1 - \dot{z}_2),$$

(88)

with

$$f_{ext}^1 = k(z_o - z_1) + c(\dot{z}_o - \dot{z}_1),$$

(89)

We see that the system considered is a particular $\Sigma_2$ system (see equations (7)-(9)). Hence by Theorem 5.1 it is observable and reconstrucible for arbitrary choices of all the material parameters.

If one wants to know how e.g. the wheel center $W$ behaves in response to the external force road input, without a priori knowing the latter, he may observe the output past excursions $y(s) := z_2(s)$ of the body-suspension interaction point: they determine the actual state of the system, thus the actual position of $W$ too. Thus he can go back to the external force road input which generates the vertical motion of the suspension.

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Fig. 1. Physical model of system $\Sigma_2$.

Fig. 2. Physical model of system $\Sigma_N$, $N > 2$.

Fig. 3. Quarter car model