Minimum Guesswork with an Unreliable Oracle

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Abstract

We study a guessing game where Alice holds a discrete random variable $X$, and Bob tries to sequentially guess its value. Before the game begins, Bob can obtain side-information about $X$ by asking an oracle, Carole, any binary question of his choosing. Carole’s answer is however unreliable, and is incorrect with probability $\epsilon$. We show that Bob should always ask Carole whether the index of $X$ is odd or even with respect to a descending order of probabilities – this question simultaneously minimizes all the guessing moments for any value of $\epsilon$. In particular, this result settles a conjecture of Burin and Shayevitz. We further consider a more general setup where Bob can ask a multiple-choice $M$-ary question, and then observe Carole’s answer through a noisy channel. When the channel is completely symmetric, i.e., when Carole decides whether to lie regardless of Bob’s question and has no preference when she lies, a similar question about the ordered index of $X$ (modulo $M$) is optimal. Interestingly however, the problem of testing whether a given question is optimal appears to be generally difficult in other symmetric channels. We provide supporting evidence for this difficulty, by showing that a core property required in our proofs becomes NP-hard to test in the general $M$-ary case. We establish this hardness result via a reduction from the problem of testing whether a system of modular difference disequations has a solution, which we prove to be NP-hard for $M \geq 3$.

1 Introduction and Main Result

Consider the classical guessing game played by Alice and Bob. Alice holds a discrete random variable (r.v.) $X$ distributed over $[N] \triangleq \{1, 2, \ldots, N\}$ with a probability mass function $p(x)$. Without loss of generality we assume below that the probabilities are in descending order, i.e., $p(1) \geq p(2) \geq \cdots \geq p(N)$. Bob would like to guess $X$ as quickly
as possible. To that end, he is allowed to guess one symbol at a time, namely to ask Alice questions of the form “is $X = x$”. Alice answers truthfully and the game terminates as soon as Bob guesses correctly. It is easy to check that Bob’s optimal strategy attaining the minimal expected guessing time $G(X)$, is to guess the symbols in a decreasing order of probability:

$$G(X) = \sum_{k \in [N]} k \cdot p(k). \quad (1)$$

The guessing game was originally introduced by Massey [1], and studied more specifically for i.i.d sequences by Arikan [2], who drew connections to Rényi entropies and cutoff rates in channel coding. His work has later been generalized by Arikan and Merhav [3] for the case of guessing a possibly continuous r.v. where a guess is considered correct if it satisfies a distortion constraint. Arikan and Boztas considered another variation in [4], where Alice lies with some probability when she rejects Bob’s guesses (but never lies when he guesses correctly). There are many other works tackling related questions, see e.g. [5–10] for a non-exhaustive list.

In this paper, we consider the problem of guessing with a possibly malicious oracle, recently introduced and studied by Burin and Shayevitz [11]. In this setup, before the game begins, Bob can reach out to an Oracle, Carole, and ask her any yes/no question of his choosing. Namely, Bob can choose any subset $A \subseteq [N]$ and ask Carole whether $X$ is in $A$ or in $\bar{A} \triangleq [N] \setminus A$. Below we informally refer to the set $A$ as a partition (of $[N]$). Carole is known to lie with probability $\epsilon$, i.e., that Bob obtains the answer

$$Y_A \triangleq 1(X \in A) \oplus V, \quad (2)$$

where $V \sim \text{Bernoulli}(\epsilon)$ is independent of $X$. What is the best question, namely the best partition $A$, for Bob to choose so that given Carole’s answer his expected guessing time would be minimized? If Carole is always truthful ($\epsilon = 0$) then it is not difficult to check that Bob’s best strategy is asking whether $X$ is even or odd, i.e., using $A_{ZZ} = \{k \in [N] : k \text{ odd}\}$. We will call this choice the zigzag partition. What should the partition $A$ be in the general case? Let $G_A(X)$ be the optimal expected guessing time given Carole’s noisy answer $Y_A$. In [11], the authors reduced the problem of finding the partition that minimizes $G_A(X)$ to a max-cut problem in a certain weighted graph, and then used quadratic relaxation to prove that the zigzag partition is almost optimal, up to a small constant independent of the distribution and the alphabet size.

**Theorem 1 ([11]).** For any r.v. $X$ and lying probability $\epsilon$,

$$G_{A_{ZZ}}(X) \leq \min_{A \subseteq [N]} G_A(X) + \frac{|1 - 2\epsilon|}{4}. \quad (3)$$

In addition, it was conjectured in [11] that the excess term in (3) is an artifact of the proof, and that zigzag is in fact exactly optimal. In this paper, we prove this conjecture in a stronger sense, using an entirely different technique. For any function $f : [N] \to \mathbb{R}$, let $G_A^f(X)$ be the minimal expected value of $f(\text{guessing time})$ given $Y_A$. 

Theorem 2. For any r.v. $X$, lying probability $\epsilon$, and nondecreasing function $f : [N] \to \mathbb{R}$,
\[ G_{Axz}^f(X) = \min_{A \subseteq [N]} G_A^f(X). \] (4)

Thus in particular, for any r.v $X$, zigzag uniformly minimizes all the positive guessing time moments for any $\epsilon$, and more specifically the conjecture in [11] follows by setting $f(k) = k$. It should be noted that the method of [11] cannot be directly extended beyond the expectation of the guessing time, as the max-cut reduction hinges on linearity.

Our approach is based on the observation that the solution to an “unconstrained” version of the minimum guesswork problem is (somewhat surprisingly) achievable. There can generally be many optimal partitions that achieve this unconstrained optimum; in Lemma 11 we characterize and count all the optimal solutions. We then show that the zigzag partition is always a member of the set of optimal solutions. We provide the necessary definitions and basic Lemmas in Section 2, and the proceed to prove Theorem 2 in Section 3. In Section 4, we discuss an extension of our setup to the case where Bob can ask multiple-choice $M$-ary questions, and where Carole’s lies are modeled by a modulo-additive channel. In this case, a (generalized) zigzag is not necessarily optimal, unless the channel is fully symmetric, i.e., Carole has no preference when she lies. Moreover, our proof techniques do not directly extend to this case in a very strong sense: we show that testing for the validity of a core property required by our approach is NP-hard; this is established via a reduction from the problem of testing whether a system of modular difference disequations has a solution, which we prove to be NP-hard for $M \geq 3$. In Section 5, we conclude with a discussion of some interesting directions trying to establish whether testing the achievability of the unconstrained optimum can be done in polynomial-time. We further provide a polynomial-time algorithm for finding the optimal partition in the asymmetric binary case, for sufficiently small lying probabilities.

2 Definitions and Basic Lemmas

Let us proceed more rigorously. We assume throughout without loss of generality that $0 < \epsilon < 1/2$. First, observe the following simple rearrangement lemma.

Lemma 3. Let $a_1, \ldots, a_N$ be a sequence of real numbers, and let $a_1^\downarrow, \ldots, a_N^\downarrow$ be the same sequence ordered in descending order. Then
\[ \sum_{k \in [N]} f(k) \cdot a_k^\downarrow \leq \sum_{k \in [N]} f(k) \cdot a_k, \] (5)
for any nondecreasing function $f : [N] \to \mathbb{R}$.

Proof. If $\{a_k\}$ is in descending order, we are done. Otherwise, there must exist a pair $a_i > a_j$ for $i > j$. Switching between them clearly reduces the sum. Iterating this procedure, we terminate at a descending order after a finite number of iterations. 

$\blacksquare$
In light of Lemma 3, it should be clear that for any partition $A$ and any nondecreasing function $f$, Bob’s optimal guessing strategy in terms of minimizing the expectation of $f$ applied to his guessing time given $Y_A$, is to guess in decreasing order of posterior probabilities. Namely,

$$G_A^f(X) = \mathbb{E} \left( \sum_{k \in [N]} f(k) \cdot P_{X|Y_A}(x_k^y | Y_A) \right),$$

(6)

where $\{x_k^y\}_{k \in [N]}$ is a permutation of $[N]$ that pertains to the posterior order given $Y_A = y$, i.e., such that

$$P_{X|Y_A}(x_k^y | y) \geq P_{X|Y_A}(x_{k+1}^y | y).$$

(7)

We are interested in studying the optimal partition, i.e., one that minimizes $G_A^f(X)$ over $A \subseteq [N]$. Writing $\bar{\epsilon} = 1 - \epsilon$, let us expand the expression for $G_A^f(X)$:

$$G_A^f(X) = \sum_{y \in \{0,1\}} \sum_{k \in [N]} f(k) \cdot P_{X,Y_A}(x_k^y, y) \cdot P_Y(y)$$

(8)

$$= \sum_{y \in \{0,1\}} \sum_{k \in [N]} f(k) \cdot P_{X,Y_A}(x_k^y, y) \cdot \left\{ \begin{array}{ll}
\epsilon p(x_k^0) + \bar{\epsilon} p(x_k^1) & x_k^0 \in A, x_k^1 \in A \\
\epsilon p(x_k^0) + \epsilon p(x_k^1) & x_k^0 \in A, x_k^1 \in \bar{A} \\
\bar{\epsilon} p(x_k^0) + \bar{\epsilon} p(x_k^1) & x_k^0 \in \bar{A}, x_k^1 \in A \\
\bar{\epsilon} p(x_k^0) + \epsilon p(x_k^1) & x_k^0 \in \bar{A}, x_k^1 \in \bar{A} \\
\end{array} \right\}$$

(10)

In what follows, we refer to any pair of the form $(\epsilon p(k), \bar{\epsilon} p(k))$ as posterior-siblings. The set of all posterior-siblings is defined to be

$$\Pi \triangleq \{\epsilon p(k), \bar{\epsilon} p(k)\}_{k \in [N]},$$

(11)

and is of cardinality $2N$. Note that in general this is a multiset; however, for brevity of exposition we will assume that all the elements in $\Pi$ are distinct. This incurs no loss of generality, since if this is not the case then we can always consider an arbitrarily small perturbation of the distribution that satisfies this. The set $\Pi$ can be naturally written as a disjoint union of two posterior sets $\Pi = \Pi_0^y \cup \Pi_1^y$, where

$$\Pi_0^y \triangleq \{P_{X,Y_A}(x_k^y, y)\}_{k \in [N]},$$

(12)

collects the posterior terms corresponding to answer $y$ by Carole. We note the following simple fact.

**Lemma 4.** The posterior sets separate all the posterior-siblings, i.e., they never both belong to the same posterior set $\Pi_0^y$. Conversely, for any partition of $\Pi = \Pi_0^y \cup \Pi_1^y$ that separates all the posterior-siblings, there exists a unique partition $A$ such that $\Pi_0^y = \Pi^0$ and $\Pi_1^y = \Pi^1$. 
Proof. The first direction follows immediately from the definition. For the converse, write

\[ \Pi_1^1 = \{ \bar{\epsilon}p(k) : k \in A \} \cup \{ \epsilon p(k) : k \in \bar{A} \}, \] (13)

Hence \( \Pi_A^1 = \Pi^1 \) implies that

\[ A \triangleq \{ k \in [N] : \bar{\epsilon}p(k) \in \Pi^1 \}. \] (14)

Since \( \Pi^0, \Pi^1 \) separate the posterior-siblings, we also have

\[ \bar{A} = \{ k \in [N] : \epsilon p(k) \in \Pi^1 \}, \] (15)

and using (13) again we have that \( \Pi_A^1 = \Pi^1 \). It is easy to check that \( \Pi_A^0 = \Pi^0 \) as well. □

Let \( \pi_A^y : [N] \to \Pi_A^y \) be the bijection recording the descending order on \( \Pi_A^y \), i.e.:

\[ \pi_A^y(k) \geq \pi_A^y(k + 1). \] (16)

This bijection is unique by our assumption that all the terms in \( \Pi \) are distinct. With this notation at hand, we can write

\[ G_f^f(X) = \sum_{k \in [N]} f(k) \cdot \left[ \pi_A^0(k) + \pi_A^1(k) \right]. \] (17)

Example 5. Let \( N = 2 \) and \( p(1) = 0.8 > p(2) = 0.2 \). Carole lies with probability \( \epsilon = 0.1 \), and Bob chooses \( A = \{1\} \). This choice generates the following \( \Pi_A^y \) sets:

\[ \Pi_A^0 = \{ \epsilon p(1), \bar{\epsilon}p(2) \} \] (18)
\[ \Pi_A^1 = \{ \bar{\epsilon}p(1), \epsilon p(2) \}, \] (19)

and \( \pi_A^y \) in this case is:

\[ \pi_A^0(1) = \bar{\epsilon}p(2) = (1 - 0.1) \cdot 0.2 = 0.18 \] (20)
\[ \pi_A^0(2) = \epsilon p(1) = 0.1 \cdot 0.8 = 0.08 \] (21)
\[ \pi_A^1(1) = \bar{\epsilon}p(1) = (1 - 0.1) \cdot 0.8 = 0.72 \] (22)
\[ \pi_A^1(2) = \epsilon p(2) = 0.1 \cdot 0.2 = 0.02. \] (23)

A bijection \( \sigma : [2N] \to \Pi \) is *induced by* \( A \subseteq [N] \), if for all \( k \in [N] \)

\[ \{ \sigma(2k - 1), \sigma(2k) \} = \{ \pi_A^0(k), \pi_A^1(k) \}. \] (24)

Below we refer to \( \sigma(2k - 1) \) and \( \sigma(2k) \) as \( \sigma \)-siblings. Note that for this \( \sigma \),

\[ G_f^f(X) = \sum_{k \in [N]} f(k) \cdot \left[ \sigma(2k - 1) + \sigma(2k) \right]. \] (25)
We now characterize the bijections that are induced by some $A$. We say that $A$ and $\sigma$ are posterior-respecting if $\Pi_A^0$ and $\Pi_A^1$ separate all the $\sigma$-siblings, i.e.,

$$\{\sigma(2k-1), \sigma(2k)\} \not\subseteq \Pi_A^y.$$  

(26)

for any $k \in [N], y \in \{0,1\}$. We further say that a set $A$ and $\sigma$ are order-preserving if the elements of both $\Pi_A^0$ and $\Pi_A^1$ are ordered within the bijection, i.e., $\sigma(i) > \sigma(j)$ whenever $\{\sigma(i), \sigma(j)\} \subseteq \Pi_A^y$ for $i < j$ and some $y$.

**Lemma 6.** $\sigma$ is induced by $A$ if and only if they are posterior-respecting and order-preserving.

**Proof.** If $\sigma$ is induced by $A$, the claim follows trivially from definition. Suppose $A$ and $\sigma$ are posterior-respecting. Then for any $k \in [N]$ there exists $y_k \in \{0,1\}$ such that the $k$th $\sigma$-siblings are separated:

$$\sigma(2k-1) \in \Pi_A^{y_k}, \sigma(2k) \in \Pi_A^{1-y_k}.$$  

(27)

This enables us to define the bijections $\sigma^0 : [N] \to \Pi_A^0$ and $\sigma^1 : [N] \to \Pi_A^1$ by

$$\sigma^{y_k}(k) \triangleq \sigma(2k-1)$$  

(28)

$$\sigma^{1-y_k}(k) \triangleq \sigma(2k).$$  

(29)

If $A$ and $\sigma$ are also order-preserving then it must be that $\sigma^0(1) > \sigma^0(2) > \ldots > \sigma^0(N)$, which means that $\sigma^0$ is a bijection from $[N]$ to $\Pi_A^0$ that agrees with the posterior order bijection $\pi_A^0$. Since $\pi_A^0$ is unique, we conclude that $\sigma^0 = \pi_A^0$. Similarly, $\sigma^1 = \pi_A^1$. We have thus obtained the following set equalities:

$$\{\sigma(2k-1), \sigma(2k)\} = \{\sigma^{y_k}(k), \sigma^{1-y_k}(k)\} = \{\sigma^0(k), \sigma^1(k)\} = \{\pi_A^0(k), \pi_A^1(k)\},$$

(30)

(31)

(32)

and hence $\sigma$ is induced by $A$.  

$\blacksquare$

It is not difficult to check (e.g., by counting) that not all bijections $\sigma$ are induced by some partition $A$. As it turns out, the obstacle is being order preserving; we now show that the posterior-respecting property can always be satisfied. To that end, we introduce the graph $G_\sigma$ induced by a bijection $\sigma : [2N] \to \Pi$. The vertex set of $G_\sigma$ is the set $\Pi$ of posterior terms, and we draw an edge between any two vertices that are either posterior-siblings or $\sigma$-siblings.

**Lemma 7.** $G_\sigma$ is a disjoint union of even cycles and isolated edges.

**Proof.** By definition, the degree of each vertex $v$ is $\deg(v) \in \{1,2\}$. If $\deg(v) = 1$, and denoting its single adjacent vertex by $v'$, then $(v,v')$ are both posterior-siblings and $\sigma$-siblings, and hence $\deg(v') = 1$. Thus the graph is a disjoint union of degree-1 vertices (i.e., isolated edges) and degree-2 vertices. The component of degree-2 vertices must be a disjoint union of cycles. Because both posterior-siblings must be in the same cycle, each cycle is of even length.  

$\blacksquare$
The following corollary is immediate.

**Corollary 8.** \( \mathcal{G}_\sigma \) is 2-colorable, and the number of distinct colorings is \( 2^c \), where \( c \) is the number of connected components of \( \mathcal{G}_\sigma \).

With this in hand, we can prove the following.

**Lemma 9.** For any bijection \( \sigma : [2N] \to \Pi \) there exists a partition \( A \) such that \( A \) and \( \sigma \) are posterior-respecting. Moreover, the number of such partitions \( A \) is \( 2^c \), where \( c \) is the number of connected components of \( \mathcal{G}_\sigma \).

*Proof.* Fix some 2-coloring of \( \mathcal{G}_\sigma \), which must exist by Corollary 8. Let \( \Pi^0_1 \) and \( \Pi^1_1 \) be the color classes associated with this coloring, which form a partition of the vertex set \( \Pi \) into two independent sets. Since posterior-siblings are connected by an edge, it follows that \( \Pi^0_1 \) and \( \Pi^1_1 \) separate all the posterior siblings. Thus according to Lemma 4 there exists a set \( A \) such that \( \Pi^0_1 = \Pi^0_A \) and \( \Pi^1_1 = \Pi^1_A \). Since \( \sigma \)-siblings are also connected by an edge, it follows that \( \Pi^0_A \) and \( \Pi^1_A \) separate all the \( \sigma \)-siblings. Hence, \( A \) and \( \sigma \) are posterior-respecting. Finally, any 2-coloring clearly results in a distinct and unique \( A \) satisfying the condition, hence in light of Corollary 8 there are \( 2^c \) such partitions. \( \blacksquare \)

**Example 10.** Let \( N = 4 \) and define the following bijection \( \sigma \):

\[
\begin{align*}
\sigma(2 \cdot 1 - 1) &= \bar{\varepsilon}p(1) & \sigma(2 \cdot 1) &= \varepsilon p(2) \\
\sigma(2 \cdot 2 - 1) &= \varepsilon p(1) & \sigma(2 \cdot 2) &= \bar{\varepsilon}p(3) \\
\sigma(2 \cdot 3 - 1) &= \varepsilon p(2) & \sigma(2 \cdot 3) &= \varepsilon p(3) \\
\sigma(2 \cdot 4 - 1) &= \bar{\varepsilon}p(4) & \sigma(2 \cdot 4) &= \varepsilon p(4).
\end{align*}
\]

The corresponding graph \( \mathcal{G}_\sigma \) and a legal 2-coloring appears in Figure 1. If \( \Pi^1 \) is the set of the red nodes, then \( \Pi^1 = \Pi^1_A \) for \( A = \{2\} \). Otherwise, If \( \Pi^1 \) is the set of the yellow nodes, then \( \Pi^1 = \Pi^1_A \) for \( A = \{1, 3, 4\} \). The number of legal 2-colorings is 2 for each one of the connected components, in total \( \mathcal{G}_\sigma \) has \( 2 \cdot 2 = 4 \) legal 2-colorings, and each 2-coloring corresponds to different partition \( A \).
3 Proof of Main Result (Theorem 2)

Define $\sigma \downarrow : [2N] \to \Pi$ to be the unique bijection corresponding to the natural descending order on $\Pi$, i.e., such that

$$\sigma \downarrow (1) > \sigma \downarrow (2) > \cdots > \sigma \downarrow (2N).$$

(33)

In light of Lemma 3 and since (25) holds for any permutation $\sigma$ induced by some partition $A$, we clearly have that

$$\min_{A \subseteq [N]} G_A^f(X) \geq \sum_{k \in [N]} f(k) \cdot [\sigma \downarrow (2k - 1) + \sigma \downarrow (2k)].$$

(34)

The right-hand-side of (34) is an unconstrained minimum, since not all permutations are induced by a partition. Somewhat surprisingly, the permutation $\sigma \downarrow$ that achieves the unconstrained minimum, is in fact always induced by some partition.

Lemma 11. For an optimal partition, it holds that

$$\min_{A \subseteq [N]} G_A^f(X) = \sum_{k \in [N]} f(k) \cdot [\sigma \downarrow (2k - 1) + \sigma \downarrow (2k)].$$

(35)

Moreover, the number of optimal partitions is $2^c$, where $c$ is the number of connected components of $G_{\sigma \downarrow}$.

Proof. By Lemma 9, there exists some partition $A \downarrow$ such that $A \downarrow$ and $\sigma \downarrow$ are posterior-respecting. It is easy to see that $A \downarrow$ and $\sigma \downarrow$ are order-preserving; this in fact holds for any partition $A$ simply since $\sigma \downarrow$ is ordered. Invoking Lemma 6, $\sigma \downarrow$ is induced by $A \downarrow$ and the claim follows from (25). The number of optimal partitions now follows from Corollary 8. ■

Remark 12. Note that when counting the number of optimal partitions, we are counting partitions and their complements, which essentially corresponds to the same solution. The number of truly distinct solutions is therefore $2^{c-1}$.

We have seen that the unconstrained minimum can be attained, and that in general, there may be many partitions that attain it. But it is still unclear what these optimal partitions look like. Interestingly, we now show that the zigzag partition is always a member of the set of optimal partitions, which concludes the proof of Theorem 2. To that end, it suffices to show the following:

Lemma 13. $A_{ZZ}$ and $\sigma \downarrow$ are posterior-respecting.

Proof. $A_{ZZ}$ partitions $\Pi$ into the following two posterior sets:

$$\Pi^0_{A_{ZZ}} = \{\bar{\epsilon}p(k) : k \text{ odd}\} \cup \{\epsilon p(k) : k \text{ even}\}$$

(36)

$$\Pi^1_{A_{ZZ}} = \{\epsilon p(k) : k \text{ odd}\} \cup \{\bar{\epsilon}p(k) : k \text{ even}\}.$$  

(37)

To prove our claim, we need to show that this partition separates all the $\sigma \downarrow$-siblings. To that end, we make a distinction between different types of $\sigma \downarrow$-siblings:
(i) The $\sigma^\downarrow$-siblings are of the form $\{ep(i), ep(j)\}$: Since the probabilities are descending order ($p(k) > p(k+1)$) and $\sigma^\downarrow$ also orders the posterior terms in descending order (cf. (33)), then it must be that $|j-k|=1$. Hence $j$ and $k$ have different parities. In light of (36)-(37), it is clear that these $\sigma^\downarrow$-siblings cannot belong to the same posterior set.

(ii) The $\sigma^\downarrow$-siblings are of the form $\{\bar{ep}(i), \bar{ep}(j)\}$: This follows similarly to the previous case.

(iii) The $\sigma^\downarrow$-siblings are of the form $\{ep(i), \bar{ep}(j)\}$: Since the probabilities are descending order ($p(k) > p(k+1)$) and $\sigma^\downarrow$ also orders the posterior terms in descending order (cf. (33)), and $\epsilon < 1/2$, it must hold that $i \leq j$. Let us count how many posterior terms are greater than both $\{ep(i), \bar{ep}(j)\}$. These terms are exactly all the terms of the form $\{ep(k)\}_{k=1}^{j-1}$, $\{ep(k)\}_{k=1}^{i-1}$, a total of exactly $(j-1) + (i-1) = i + j - 2$ terms. This number must be even, since the $\sigma^\downarrow$-siblings come in pairs. Therefore, $i$ and $j$ must have the same parity and again, in light of (36)-(37), it is clear that these $\sigma^\downarrow$-siblings cannot belong to the same posterior set.

This concludes the proof of our main result. A simple consequence is the following:

Corollary 14. The zigzag partition is the unique optimal partition (up to complements) if and only if $G_{\sigma^\downarrow}$ is a cycle on $2N$ vertices.

4 Multiple-choice questions

A natural extension of the problem above is considering a setup in which Bob communicates with Carole over a $M$-ary symmetric modulo-additive channel ($M > 2$). In this setup, Bob can ask multiple-choice $M$-ary question, i.e., to partition $[N]$ into $M$ sets $\{A^i\}_{i=0}^{M-1}$ and ask Carole which one contains $X$. Denoting by $A \triangleq \{A^i\}_{i=0}^{M-1}$ for short and given that $X \in A^j$, Bob receives a noisy answer

$$Y_A \triangleq j + V \mod M,$$

(38)

where $P_V(v) = \epsilon_v$ for $v \in \{0,1,\ldots,M-1\}$. In what follows, we generalize our previous definitions and show that it is computationally hard to test whether, for a general bijection $\sigma$, there exists a partition $A$ such that $A$ and $\sigma$ are posterior respecting; namely, we show that Lemma 9 no longer holds for $M > 2$. This observation seems to suggest that it is perhaps also hard to test whether a partition $A$ is optimal. We conclude this section with an example showing that unlike in the binary case, for $M > 2$ the unconstrained optimum cannot be always achieved. It is nevertheless worth noting that in the special case where the channel is fully symmetric, i.e., where Carole has no preference when she
lies, a generalized modulo-$M$ zigzag question achieves the unconstrained optimum; see a brief discussion in Section 5.

We refer to \( \{ \epsilon_0 p(k), \epsilon_1 p(k), \ldots, \epsilon_{M-1} p(k) \} \) as posterior-siblings and
\[
\Pi \triangleq \bigcup_{k \in [N]} \{ \epsilon_0 p(k), \epsilon_1 p(k), \ldots, \epsilon_{M-1} p(k) \},
\]
is the set of all posterior-siblings. Following the same definition for posterior sets \( \Pi^y_A \),
\[
\Pi^y_A \triangleq \{ P_{X,Y_A}(x^y_k, y) \}_{k \in [N]},
\]
and the corresponding bijections \( \pi^y_A \), we can write Bob’s expected guessing time as
\[
G^f_A(X) = \sum_{k \in [N]} f(k) \cdot \sum_{y=0}^{M-1} \pi^y_A(k).
\]

Given a partition \( \{ \Pi^i \}_{i=0}^{M-1} \) of \( \Pi \), we say that it cyclically-separates the posterior-siblings \( \{ \epsilon_0 p(k), \epsilon_1 p(k), \ldots, \epsilon_{M-1} p(k) \} \) if
\[
\epsilon_i p(k) \in \Pi^j \iff \epsilon_{i+1} p(k) \in \Pi^{j+1},
\]
where indices are calculated modulo \( M \).

**Lemma 15. (Generalization of Lemma 4)** For any \( A \), the posterior sets cyclically-separate all the posterior-siblings. Conversely, for any partition \( \{ \Pi^i \}_{i=0}^{M-1} \) of \( \Pi \) that cyclically-separates all the posterior-siblings, there exists a unique partition \( A \) such that \( \Pi^y_A = \Pi^y \).

**Proof.** The first direction follows immediately from the definition. The converse is a trivial generalization of the converse in Lemma 4. \( \square \)

Continuing generalization of previous definitions, a bijection \( \sigma : [MN] \rightarrow \Pi \) is induced by \( A \), if for all \( k \in [N] \)
\[
\{ \sigma(Mk), \sigma(Mk-1), \ldots, \sigma(M(k-1)+1) \} = \{ \pi^0_A(k), \pi^1_A(k), \ldots, \pi^{M-1}_A(k) \},
\]
and for such \( \sigma \)
\[
G^f_A(X) = \sum_{k \in [N]} f(k) \cdot \sum_{i=0}^{M-1} \sigma(Mk - i).
\]

where \( \{ \sigma(Mk-i) \}_{i=0}^{M-1} \) are \( \sigma \)-siblings in the general case. \( A \) and \( \sigma \) are posterior-respecting if \( \{ \Pi^y_A \}_{y=0}^{M-1} \) separate (not necessary cyclically-separate) all the \( \sigma \)-siblings, i.e.,
\[
\{ \sigma(Mk-i), \sigma(Mk-j) \} \not\subseteq \Pi^y_A,
\]
for any \( k \in [N], y, j, i \in \{0, 1, \ldots, M-1\} \) and \( i \neq j \). Completing the generalization, \( A \) and \( \sigma \) are order-preserving if the elements of \( \Pi^y_A \) are ordered within the bijection, i.e., \( \sigma(i) > \sigma(j) \) whenever \( \{ \sigma(i), \sigma(j) \} \subseteq \Pi^y_A \) for \( i < j \) and some \( y \).
Lemma 16. (Generalization of Lemma 6) $\sigma$ is induced by $A$ if and only if they are posterior-respecting and order-preserving.

Proof. If $\sigma$ is induced by $A$, the claim follows trivially from definition. Suppose $A$ and $\sigma$ are posterior-respecting. Then for any $k \in [N]$ there exists a permutation $\gamma_k$ of $\{0, 1, \ldots, M-1\}$ such that the $k$th $\sigma$-siblings separated as follows

$$\sigma(Mk - i) \in \Pi^{\gamma_k(i)}_A.$$  

(46)

This enables us to define $M$ bijections $\sigma^i : [N] \to \Pi^i_A$ as follows

$$\sigma^{\gamma_k(i)}(k) \triangleq \sigma(Mk - i).$$  

(47)

If $A$ and $\sigma$ are also order-preserving then it must be that $\sigma^i(1) > \sigma^i(2) > \ldots > \sigma^i(N)$, which means that $\sigma^i$ is a bijection from $[N]$ to $\Pi^i_A$ that agrees with the posterior order bijection $\pi^i_A$. Since $\pi^i_A$ is unique, we conclude that $\sigma^i = \pi^i_A$. We have thus obtained the following set equalities:

$$\{\sigma(Mk - i)\}_{i=0}^{M-1} = \{\sigma^{\gamma_k(i)}(k)\}_{i=0}^{M-1}$$  

(48)

$$= \{\sigma^i(k)\}_{i=0}^{M-1}$$  

(49)

$$= \{\pi^i_A(k)\}_{i=0}^{M-1},$$  

(50)

and hence $\sigma$ is induced by $A$. \[\blacksquare\]

Theorem 17. Deciding for general $\sigma$ if it is induced by some partition $A$ is NP-hard

To prove this, we will show that testing whether $\sigma$ has a posterior respecting partition, is NP-hard. In light of Lemma 16 it is in fact a stronger statement.

For brevity, before proceeding to prove the theorem, we introduce a more accessible equivalent formulation of the problem. From this point on, we redefine $\Pi$ as $\Pi \triangleq [N] \times [M]$ and each term $\epsilon_v p(k)$ is replaced with a tuple $(k, v + 1)$ (notice that the $\pm 1$ correction is due to a change from zero-based to one-based indexing). Respectively, we will change the definition of $\sigma$ to an equivalent definition $\sigma : [N] \times [M] \to [N] \times [M]$, which follows the following equivalence relation with the previous definition

$$\sigma(Mj - i + 1) = \epsilon_{v-1} p(k) \iff \sigma(j, i) = (k, v),$$  

(51)

for all $k, j \in [N]$ and $i, v \in [M]$. Let us define a mapping $\zeta : [N] \times [M] \to \mathbb{Z}_M$, that collects for a given partition $A$, the index of the posterior set that each $(n, m)$ belongs to. i.e., if $(n, m) \in \Pi^i_A$ then $\zeta(n, m) = y$. Due to Lemma 15

$$\zeta(n, m) = \zeta(n, 1) + m - 1,$$  

(52)
where additions from this point on are done modulo $M$, and we define $z_n \triangleq \zeta(n, 1)$. Now, given that $\sigma(k, i) = (n_1, m_1)$ and $\sigma(k, j) = (n_2, m_2)$, (15) becomes

$$\zeta(\sigma(k, i)) \neq \zeta(\sigma(k, j))$$

$$\implies \zeta(n_1, m_1) \neq \zeta(n_2, m_2)$$

$$\implies z_{n_1} + m_1 \neq z_{n_2} + m_2$$

$$\implies z_{n_1} - z_{n_2} \neq m_2 - m_1.$$  

Any partition $A$ which is posterior respecting with respect to $\sigma$ must obey the above condition for all $k \in [N]$ and $i \neq j \in [M]$. Conversely, any assignment to $\{z_n\}_{n=1}^N$ that satisfies the above condition for all $k \in [N]$ and $i \neq j \in [M]$, uniquely defines a posterior respecting partition. We will show, that testing whether such an assignment exists, is NP-hard, and this is why also asking if there exists a posterior respecting partition is NP-hard. We call (56) a Difference Modular Disequation (DMD). In [13], Himanshu et al. showed that classifying whether a system of linear modular disequations is satisfiable is NP-hard. They do this, by reducing a 3-SAT problem to a system of linear modular disequations. We will prove that also answering whether a system of DMDs, i.e set of equations of the form $w_i - w_j \not\equiv c_k \pmod{M}$ (for some constants $c_k$) is NP-hard, which is a strengthening of [13]. To prove this we will reduce another well known NP-hard problem, the Not All Equal-3SAT (NAE-3SAT) [14] to a system of DMDs.

**Lemma 18.** The problem of deciding whether a system of DMDs can be satisfied is NP-hard.

**Proof.** We prove the lemma for $M = 3$ and discuss the (trivial) extension to $M > 3$ at the end. Let the variables of the NAE-3SAT problem be $x_1, x_2, \ldots, x_n$. For each variable $x_i$ in NAE-3SAT, we introduce two integer variables $w_i$ and $\hat{w}_i$ and we add the following equations to the system:

$$w_i - s \neq 0$$

$$\hat{w}_i - s \neq 0$$

$$w_i - \hat{w}_i \neq 0,$$

where $s$ is a common variable. We now may define a mapping between the value of $w_i - s \pmod{3}$ and the value of $x_i$. It is not very important, so we will choose that if the difference is 1, $x_i$ is false, and if the difference is 2, $x_i$ is true. Then the value of $\hat{w}_i - s \pmod{3}$ represents $\neg x_i$. Let $\delta$ be a mapping between literals and their corresponding variable, i.e. $\delta(x_i) = w_i$ and $\delta(\neg x_i) = \hat{w}_i$. For each clause $u \lor v \lor w$, we introduce an integer variable $c_i$ and add the following equations to the system:

$$\delta(u) - c_i \neq 0$$

$$\delta(v) - c_i \neq 1$$

$$\delta(w) - c_i \neq 2.$$
Given a solution to the system, we use the mapping that was defined above to find the corresponding assignment to \( \{x_i\}_{i=1}^{n} \). Notice that for some clause \( i \), \( c_i \) has no legal value if and only if \( \delta(u) = \delta(v) = \delta(w) \), so if there is a solution, it cannot be that all the literals of some clause are equal. The reduction from the given NAE-3SAT to the system of equations is polynomial time. To extend this to \( M > 3 \), we reduce a NAE-\( M \)-SAT to a system of DMDs modulo \( M \) the same way. Obviously NAE-\( M \)-SAT for \( M > 3 \) is still NP-hard. It can be proven for example recursively, by reducing a NAE-(\( M - 1 \))-SAT to NAE-\( M \)-SAT. Given that \( c \) is a clause from a NAE-(\( M - 1 \))-SAT instance, we replace it with \((c \lor x) \land (c \lor y)\), where \( x \) and \( y \) are dummy variables, and the extra clause \( x \lor x \ldots \lor x \lor y \) (\( x \) appears \( M - 1 \) times) forces \( x = \neg y \). Then, for example if \( x = 0 \), the clause \( c \lor x \), forces the literals in \( c \) to be not all equal to 0, and the clause \( c \lor y \) forces the literals in \( c \) to be not all equal to 1. ■

Next, we will show that it is possible to reduce any system of DMDs to a problem of deciding whether for some partition \( \sigma \), there is an assignment to \( \{z_n\}_{n=1}^{N} \) such that the corresponding partition is posterior respecting with respect to \( \sigma \).

**Lemma 19.** For a general \( \sigma \), testing whether there is a posterior respecting partition is NP-hard.

**Proof.** Throughout the proof, we show how to construct \( \sigma \), that generates an equivalent system of DMDs for any given system of DMDs, thus showing that finding a posterior respecting partition is generally at least as hard as solving a system of DMDs. We will prove it for \( M = 3 \), but the same technique also extends to \( M > 3 \). Given the following equation \( i \)

\[
\text{we may try a straightforward mapping, and have } w_k \text{ and } w_l \text{ to be mapped to } z_k \text{ and } z_l \text{ respectively. By having } \sigma(1, 1) = (k, 1) \text{ and } \sigma(1, 2) = (l, c_i + 1), \text{ according to } (56), \text{ this mapping generates an equivalent equation to } (63). \text{ However, this naive approach does not scale, since we run into trouble in case } w_k \text{ appears in another disequation. Recall that because } \sigma \text{ is a bijection, we cannot use } (k, 1) \text{ for a different input to } \sigma. \text{ To be able to scale, we introduce a duplication gadget that duplicates the variable } z_k, \text{ i.e., we will add a row to } \zeta \text{ such that } \zeta(k, *) = \zeta(k', *). \text{ We add the following to } \sigma \]

\[
\sigma(1, *) = \{(k, 1), (j, 1), (j', 1)\} \quad (64) \\
\sigma(2, *) = \{(k', 2), (j, 2), (j', 2)\} \quad (65) \\
\sigma(3, *) = \{(k'', 3), (j, 3), (j', 3)\}, \quad (66)
\]

where \( k', k'' \) are duplication rows, and \( j, j' \) are some helper rows that do not correspond to any variable in the original system of equations (we use * because the order between the \( \sigma \)-siblings does not matter). These rows in \( \sigma \) generate the following DMDs:

\[
z_k - z_j \neq 0 \quad (67)
\]

\( ^* \) stands for any index
A solution to this system must have $z_j \neq z'_j$ and $z_k, z_{k'}$ and $z_{k''}$ must be different from both $z_j$ and $z'_j$. So it must be that

$$z_k = z_{k'} = z_{k''},$$

(74)

thus we can use $(k, *), (k', *)$ or $(k'', *)$ interchangeably when constructing $\sigma$. In order to maintain clearer presentation, we will not carry the $k'$ and $k''$, and just assume that we have three copies of $(k, 1), (k, 2)$ and $(k, 3)$, where one of each was already been used for duplication. Now, to represent the disequation we add the following to $\sigma$

$\sigma(4, *) = \{(k, 1), (l, c_i + 1), (i, 1)\}$

(75)

$\sigma(5, *) = \{(k, 2), (l, c_i + 2), (i, 2)\}$

(76)

$\sigma(6, *) = \{(k, 3), (l, c_i + 3), (i, 3)\}$

(77)

where $i$ is a helper row that correspond to equation $i$. Indeed all of the above rows in $\sigma$ generate the same DMDs, but we need to place all the $(k, *), (l, *)$ and $(i, *)$ somewhere in $\sigma$. We use the last copy of $(k, *)$ to duplicate and/or represent another disequation. For each disequation, we generate at most a constant number of corresponding duplications, hence showing that there is an assignment to $\{z_n\}^N_{n=1}$ that corresponds to a posterior respecting partition with respect to the $\sigma$ we have constructed, is at least as hard as deciding whether a system of DMDs can be satisfied, and using Lemma 18 it is NP-hard.

Theorem 17 is a direct corollary of Lemma 16 and Lemma 19. We will conclude with an example of a case where the unconstrained optimum is not achievable and propose a non hermetic method of testing whether it is achievable.

Example 20. For $M = 3$ and $V \sim [\epsilon_0 = 0.5, \epsilon_1 = 0.3, \epsilon_2 = 0.2]$, then for $X \sim [0.35 0.26 0.24 0.15]$, there is no partition $\{A^i\}_{i=0}^2$ that achieves the unconstrained optimum. A necessary condition for $\sigma^i$ to be induced by some partition is that there exists a posterior-respecting partition. We will try to construct such a partition. Without loss of generality, we start by assigning $x_4$ to $A^0$, therefore $\epsilon_0 p(4) \in \Pi^0_\Delta$, $\epsilon_1 p(4) \in \Pi^1_\Delta$ and $\epsilon_2 p(4) \in \Pi^2_\Delta$. Table 7 shows the ternary $\sigma^i$-siblings. Then, in order to split $\{\sigma^1(4, 1), \sigma^1(4, 2), \sigma^1(4, 3)\}$ between the posterior sets, we must assign $x_3$ to $A^1$. It is left to the reader to verify that any assignment to $x_1$ and $x_2$ does not end up with a posterior-respecting partition.
Table 1: Table of $\sigma^i$-siblings generated by Example 20. Each cell contains posterior probability expression/value pair. The coloring refers to the posterior sets.

| $\sigma$ | 1     | 2     | 3     |
|----------|-------|-------|-------|
| 1        | $\epsilon_0 p(1)/.175$ | $\epsilon_0 p(2)/.13$ | $\epsilon_0 p(3)/.12 \in \Pi_A^1$ |
| 2        | $\epsilon_1 p(1)/.105$ | $\epsilon_1 p(2)/.078$ | $\epsilon_0 p(4)/.075 \in \Pi_A^1$ |
| 3        | $\epsilon_1 p(3)/.072 \in \Pi_A^2$ | $\epsilon_2 p(1)/.07$ | $\epsilon_2 p(2)/.052$ |
| 4        | $\epsilon_2 p(3)/.048 \in \Pi_A^2$ | $\epsilon_1 p(4)/.045 \in \Pi_A^1$ | $\epsilon_2 p(4)/.03 \in \Pi_A^2$ |

5 Discussion

We have shown that the zigzag partition, which amounts to querying whether $X$ has an odd or even index when ordered in descending order of probabilities, is the best question Bob can ask Carole in order to uniformly minimize the expectation of any nondecreasing function of his guessing time, regardless of Carole’s lying probability. This result is limited to the case of yes/no questions and a binary symmetric channel from Carole to Bob. Natural extensions of this problem are therefore 1) let Bob ask multiple-choice $M$-ary questions, i.e., to partition $[N]$ into $M$ sets $\{A_i\}_{i=0}^{M-1}$ and ask Carole which one contains $X$, and 2) consider more general channel models for Carole’s noisy reply.

We note that our proof of Theorem 2 is almost trivially extended to the $M$-ary case when the channel from Carole to Bob is modulo-additive with a uniform crossover probability, i.e., where Carole answer truthfully with probability $1 - \epsilon$, and gives any one of the other $M - 1$ incorrect answers with probability $\frac{\epsilon}{M-1}$. This setup reduces to the one discussed on this paper when $M = 2$. For arbitrary $M$, the corresponding zigzag partition is the collection of disjoint subsets $\{A_{ZZ}^i\}_{i=0}^{M-1}$ given by

$$A_{ZZ}^i \triangleq \{k \in [N] : k \equiv i \pmod{M}\}.$$  (78)

This choice is optimal and achieves the corresponding unconstrained optimum (just as (34) is achieved in the $M = 2$ case).

Interestingly, our approach does not extend when replacing the special symmetric channel above with a general (symmetric!) modulo-additive channel; in fact, for such channels the unconstrained optimum cannot always be achieved (by any partition), and zigzag is not always optimal. The problem of exactly characterizing the optimal partition or even its performance in this setup appears to be hard. It is thus interesting to examine the applicability of the max-cut / quadratic relaxation approach of [11] to possibly obtain bounds. It is possible to transform a problem of testing whether there exists a solution to system of DMDs, to a problem of testing if a maximum independent set of a certain graph $G$ is of size $N$, where $N$ is the number of variables in the system. This alternative formulation may hopefully allow the use of graph-theoretic techniques to show interesting properties of specific bijections, such as the unconstrained optimum
bijection $\sigma^\dagger$. We construct $G$ in the following way. For each variable $w_k$, we add a clique of $M$ vertices, indexed $0, 1, \ldots, M - 1$. For each equation $w_k - w_l \not\equiv c_i$, we add an edge between the vertex $c_i + m$ of $w_k$ and the vertex $m$ of $w_l$, for all $m \in [M]$. This ensures that if we take two nonadjacent (independent) vertices from these two cliques, then their indices will satisfy the DMD. Thus, if there is an independent set of size $N$ in $G$, then there is a solution to the system of DMDs. Any graph $G$ has the following property [15]

$$\alpha(G) \leq \vartheta(G) \leq \bar{\chi}(G),$$

(79)

where $\alpha(G)$ and $\bar{\chi}(G)$ are the independence number and the clique partition number, both of which are NP-hard to compute, and $\vartheta(G)$ is the Lovász number which can be computed in polynomial time. For graph instances that represent a system of DMDs, $\alpha(G)$ and $\bar{\chi}(G)$ are bounded

$$\alpha(G) \leq \bar{\chi}(G) \leq N,$$

(80)

so, if $\vartheta(G) < N$, then there is no solution to the system of DMDs. Otherwise, as a result of what we have proven before, it is computationally NP-hard to test whether $\alpha(G) < N$ (this was also proved in [16]). This is not necessarily the case for the system of DMDs that is generated by $\sigma^\dagger$, and it is an open question whether it remains NP-hard to test if the unconstrained optimum is achievable. For example, if the following property is true

$$\alpha(G^\dagger) < N \Rightarrow \vartheta(G^\dagger) < N,$$

(81)

it the would make the problem of testing whether the unconstrained optimum can be achieved solvable in polynomial time.

It is also interesting to go back to the binary case but consider an asymmetric channel model, i.e., where the crossover probability depends on the input. For this channel, we have derived a quadratic time algorithm for finding the optimal partition for the expected guessing time (not for a general function), in the case of “sufficiently small” crossover probabilities, satisfying for all $i, j \in [N]$

$$\epsilon p(i) < \delta p(j)$$

(82)

$$\delta p(i) < \epsilon p(j),$$

(83)

where $\epsilon$ (resp. $\delta$) is the probability of crossing $0 \rightarrow 1$ (resp. $1 \rightarrow 0$). In this case, Bob’s optimal strategy regardless of the partition he has used, is to first guess the values of $X$ from the set pointed out by Carole, and only then go over the values in the complement set (according to the posterior order). Given a partition $A$, let $x_k^A$ (resp. $x_k^{\bar{A}}$) be the posterior order within $A$ (resp. $\bar{A}$), i.e. $p(x_k^A) \geq p(x_{k+1}^A)$ (resp. $p(x_k^{\bar{A}}) \geq p(x_{k+1}^{\bar{A}})$). Then Bob’s expected guessing time ($f(k) = k$) is given by

$$G_A(X) = \sum_{k=1}^{\lfloor |A| \rfloor} k \cdot \bar{\epsilon} p(x_k^A) + \sum_{k=1}^{\lfloor |A| \rfloor} (k + \lfloor |\bar{A}| \rfloor) \cdot \delta p(x_k^{\bar{A}})$$

(84)
\[ + \sum_{k=1}^{|A|} k \cdot \tilde{\delta}p(x_k^A) + \sum_{k=1}^{|A|} (k + |A|) \cdot \epsilon p(x_k^A) \quad (85) \]

\[ = \sum_{k=1}^{|\tilde{A}|} (k + \epsilon |\tilde{A}|) \cdot p(x_k^{\tilde{A}}) + \sum_{k=1}^{|A|} (k + \delta |\tilde{A}|) \cdot p(x_k^A) \quad (86) \]

\[ = \sum_{k=1}^{|\tilde{A}|} \bar{c}_k \cdot p(x_k^{\tilde{A}}) + \sum_{k=1}^{|A|} c_k \cdot p(x_k^A). \quad (87) \]

Fixing the size of \( A \), the coefficients \( c_k \) and \( \bar{c}_k \) in (87) are known. Ordering these \( N \) coefficients in descending order, denoted by \( d_k \), and noting that \( p(x_k) \) is a nondecreasing function, we can appeal to Lemma 3 and obtain

\[ G_A(X) \geq \sum_{k \in [N]} d_k \cdot p(x_k). \quad (88) \]

We can easily achieve this bound by assigning \( x_k \) to \( A \) if and only if \( d_k \) is in \( \{c_k\}_{k=1}^{|A|} \). By iterating over the size of \( A \), it is possible to find the optimal partition in \( O(N^2) \) steps (\( N \) evaluations of \( G_A(X) \)).

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