THE BOHR RADIUS FOR AN ELLIPTIC CONDENSER

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Résumé. We compute the exact value of the Bohr radius associated to an elliptic condenser of the complex plane and its Faber polynomial basis.

1. Introduction

Bohr’s classical theorem [1] asserts that if \( f(z) = \sum_{n \geq 0} a_n z^n \) is holomorphic on the unit disc \( D \) and if \( |f(z)| < 1 \), \( \forall z \in D \) then \( \sum_{n \geq 0} |a_n z^n| < 1 \), \( \forall z \in D(0, 1/3) \), and the constant \( 1/3 \) is optimal.

In a previous work [3] we study the Bohr’s phenomenon in the following context : let \( K \subset \mathbb{C} \) be a continuum, \( \Phi : \mathbb{C} \setminus K \to \mathbb{C} \setminus D \) the unique conformal mapping satisfying \( \Phi(\infty) = \infty \), \( \Phi'(\infty) = \gamma > 0 \), and \((F_{K,n})_n\) the sequence of its Faber polynomials [4]. This is a classical fact [4] that \((F_{K,n})_n\) is a Schauder basis for all the spaces \( \mathcal{O}(\Omega_{K,\rho}) \), \( (\rho > 1) \) and also \( \mathcal{O}(K) \). We prove [3, theorem 3.1] that the family \((K, \Omega_{K,\rho}, (F_{K,n})_n)\) satisfies the Bohr phenomenon in the following sense : there exists \( \rho_0 > 1 \) such that for all \( \rho > \rho_0 \), for all \( f = \sum a_n F_{K,n} \in \mathcal{O}(\Omega_{K,\rho}) \), if \( |f(z)| < 1 \) for all \( z \in \Omega_{K,\rho} \), then \( \sum |a_n| \cdot \|F_{K,n}\|_K < 1 \). The infimum \( \rho_K \) of all such \( \rho_0 \) will be called the **Bohr radius** of \( K \).

For example, the Faber polynomial basis for the compact \( \overline{D(0,1)} \) is precisely the Taylor basis i.e. \( F_{K,n}(z) = z^n \) and the levels sets are the discs \( \Omega_{\overline{D(0,1)},\rho} = D(0, \rho) \), \( (\rho > 1) \); then, thanks to the classical Bohr theorem, we have a Bohr phenomenon and the Bohr radius of \( K = \overline{D(0,1)} \) is \( \rho_K = 3 \).

The particular cases \( K := [-1, 1] \subset \mathbb{C} \) is one of the very few more examples (see [4], which is the definite reference on this subject) where

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1. i.e. \( K \) is a compact in \( \mathbb{C} \) including at least two points, and \( \mathbb{C} \setminus K \) is simply connected.

2. this means that for all \( f \in \mathcal{O}(\Omega_{K,\rho}) \) there exists an unique sequence \((a_n)_n\) of complex numbers such that \( f = \sum_{n \geq 0} a_n \phi_n \) for the usual compact convergence of \( \mathcal{O}(\Omega_{K,\rho}) \) and the same is true in \( \mathcal{O}(K) \) equipped with its usual inductive limit topology.
the explicit form of the conformal map $\Phi : \Omega := \mathbb{C} \setminus K \ni z \mapsto w = \Phi(z) \in \mathbb{C} \setminus D(0,1)$ give us more precise estimations. In this simple case $\Phi^{-1}(w) = (w + 1/w)/2$ is the famous Zhukovskii function. Faber polynomials $(F_{K,n})_n$ form a common basis of the spaces $\mathcal{O}(K)$ and $\mathcal{O}(\Omega_{K,\rho})$, $(\rho > 1)$ where the boundary $\partial \Omega_{K,\rho}$ of the level sets are ellipses with focus $1$ and $-1$, and eccentricity $\varepsilon = \frac{2\rho}{1+\rho^2}$. That’s why we will speak of elliptic condenser.

In [2], H.T. Kaptanoglu & N. Sadik study the Bohr phenomenon (with a slightly different approach) in the case of an elliptic condenser and obtain an estimation of its Bohr radius. Their paper inspired our works and in the present one we compute the exact value of this radius. We also compute the exact value of the radius for holomorphic functions with only real coefficients in their Faber expansion. Note that the observation that these two radius can be different (contrary to the classical Bohr’s theorem) seems to be new.

2. THE SKETCH OF THE PROOF AND TECHNICALS NOTATIONS

The Sketch of the Proof: In the proof of the classical Bohr theorem, the main ingredient for an upper estimation of the Bohr radius are Carathéodory inequality:

"let $f(z) = \sum_n a_n z^n \in \mathcal{O}(D(0,1))$. If $\text{re}(f(z)) > 0$ for all $z \in D(0,1)$ then $|a_n| \leq 2\text{re}(a_0)$ for all $n \geq 1$."

In the elliptic case, the procedure is the same. In [3] we already prove the following elliptic-Carathéodory’s inequality:

"([3] prop 2.1) Let $f(z) = a_0 + \sum_1^\infty a_n F_{K,n}(z) \in \mathcal{O}(\Omega_{K,\rho})$ and suppose that $\text{re}(f) > 0$. Then : $|a_n| \leq \frac{2\text{re}(a_0)}{\rho^n - \rho^{-n}}$, $\forall n \in \mathbb{N}^*$. Moreover, if $f(z) = a_0 + \sum_1^\infty a_n F_{K,n}(z) \in \mathcal{O}(\Omega_{K,\rho})$ satisfies $|f| < 1$ with $a_0 > 0$, then we have $|a_n| \leq \frac{2(1-a_0)}{\rho^n - \rho^{-n}}$, for all $n \geq 1$."

to deduce that:

"([3] prop 2.3) The elliptic condenser $(K := [-1,1], \Omega_{K,\rho}, (F_{K,n})_n)$ satisfies Bohr’s phenomenon for all $\rho \geq \rho_0 = 5,1284...""

which gives an eccentricity $\varepsilon_0 = 0.3757...$. This is already better than $\varepsilon_0 = 0.373814... (\rho_0 = 5,1573...)$ find by H.T. Kaptanoglu & N. Sadik in [2].

3. Note that $a_0 < 1$ because $|f| < 1$. 
To find the exact value of $\rho_K$ we will first (in paragraph 3) prove better elliptic Caratheodory’s inequalities. Then (paragraph 4) use these inequalities to get an upper bound for $\rho_K$. Finally in paragraph 5, we prove that the upper bound obtained in the previous paragraph is optimal thanks to explicit test functions.

Our proof is rather technical, so before going into it, let us state clearly the main tools we will use.

**Technical observations and notations:** First, it is fundamental to observe that the expression of $F_{K,n}$ is very more convenient in goal coordinates « $w = \Phi(z)$ » than in source coordinate « $z$ » because in « $w$ » coordinates we have $F_{K,n}(w) = w^n + w^{-n}$, $\forall |w| = |\Phi(z)| > 1$, so $\|F_{K,n}\|_{\Omega_{K,\rho}} = \rho^n + \rho^{-n}$, $\forall \rho > 1$.

To be in the same spirit that the seminal’s work of H.T. Kaptanoglu & N. Sadik, and compute the Bohr radius $\rho_K$ we will proceed as follow:

From now, $\mathcal{E}$ will denote the domain bounded by a non degenerate ellipse with foci $\pm 1$, i.e. a level set $\Omega_{K,\rho}$, ($\rho > 1$) of the biholomorphism $\Phi_K : \overline{C} \setminus K \mapsto \overline{C \setminus D(0,1)}$.

So, the biholomorphism $\Phi_{\mathcal{E}} : \overline{C \setminus \mathcal{E}} \mapsto \overline{C \setminus D(0,1)}$ is $\Phi_{\mathcal{E}} = \Phi_K/\rho$ which extends as a biholomorphism up to $\overline{C \setminus K}$. In another words, the level sets $\Omega_{\mathcal{E},r}$ of $\Phi_{\mathcal{E}}$ are defined not only for $1 < r < +\infty$ but for $1/\rho < r < +\infty$. And we have (\cite{2}) with $R := \rho^{-1}$:

$$F_{\mathcal{E},n}(w) = w^n + \rho^{-2n}w^{-n} = w^n + R^{2n}/w^n, \quad R = \rho^{-1} < |w|.$$  

So, for all $f \in \mathcal{O}(\Omega_{\mathcal{E},r})$ we will have

$$f(w) = \sum_{n \geq 0} a_n \cdot F_{\mathcal{E},n}(w) = \sum_{n \geq 0} a_n \cdot (w^n + \rho^{-2n}w^{-n}), \quad r > |w| > \rho^{-1} = R.$$  

Then, following H.T. Kaptanoglu & N. Sadik, we are going to look for the largest $0 < R < 1$ such that we have a bohr phenomenon for the family $(K, \mathcal{E} = \Omega_{K,R^{-1}}, (F_{K,n})_n)$. If we note $R_B$ this largest $R$, clearly $\rho_K = 1/R_B$.

\footnote{4. Of course we could have done the same by looking for $\rho > 1$ such that all $f = \sum_n a_n F_{n,K} \in \mathcal{O}(\Omega_K, \rho)$ satisfies $\sum_n |a_n| \cdot \|F_{n,K}\|_K < 1$.}
3. « Elliptic » Carathéodory Inequalities.

Let \( f = \sum_{n \geq 0} a_n F_{\varphi,n} \in \mathcal{O}(\mathcal{E}) \), up to a rotation, we always suppose in this paragraph that \( a_0 > 0 \). Then elementary computations gives for all \( n \geq 1 \):

1. \[
\int_{0}^{2\pi} f \left( \Phi^{-1}(e^{i\theta}) \right) e^{in\theta} d\theta = \int_{0}^{2\pi} \left( a_0 + \sum_{k \geq 1} a_k \left( e^{ik\theta} + R^{2k} e^{-ik\theta} \right) \right) e^{in\theta} d\theta = R^{2n} a_n
\]
2. \[
\int_{0}^{2\pi} \frac{f(\Phi^{-1}(e^{i\theta}))}{e^{i\theta}} e^{in\theta} d\theta = \int_{0}^{2\pi} \left( \frac{1}{2\pi} + \sum_{k \geq 1} \frac{1}{2\pi} \left( e^{-ik\theta} + R^{2k} e^{ik\theta} \right) \right) e^{in\theta} d\theta = \frac{a_n}{2n}
\]

specially :
3. \[
R^{4n} a_{2n} = \int_{0}^{2\pi} f \left( \Phi^{-1}(e^{i\theta}) \right) e^{2in\theta} d\theta, \quad a_{2n} = \int_{0}^{2\pi} f \left( \Phi^{-1}(e^{i\theta}) \right) e^{-2in\theta} d\theta,
\]
4. \[
R^{4n} \overline{a}_{2n} = \int_{0}^{2\pi} \frac{f(\Phi^{-1}(e^{i\theta}))}{e^{-i\theta}} e^{-2in\theta} d\theta, \quad \overline{a}_{2n} = \int_{0}^{2\pi} \frac{f(\Phi^{-1}(e^{i\theta}))}{e^{i\theta}} e^{2in\theta} d\theta.
\]

Our goal in this paragraph is to prove the following "elliptic Carathéodory's type inequality" :

**Proposition 3.1.** Let \( f = \sum_{n \geq 0} a_n F_{\varphi,n} \in \mathcal{O}(\mathcal{E}) \). If \( \text{re}(f) \geq 0 \) on \( \mathcal{E} \) and \( R \leq 0.2053... \), then for all \( n \in \mathbb{N}^* \) :

\[
|a_n| R^n + |a_{2n}| R^{2n} \leq \frac{2 \text{re}(a_0) R^n}{1 - R^{2n}} + \frac{2 \text{re}(a_0) R^{2n}}{1 + R^{4n}}.
\]

First we need two lemmas :

**Lemma 3.2. ("Classical" Carathéodory’s inequality)**

Let \( f(z) = \sum_{n \geq 0} a_n F_{\varphi,n}(z) \in \mathcal{O}(\mathcal{E}) \). If \( \text{re}(f) \geq 0 \) on \( \mathcal{E} \), then for all \( n \in \mathbb{N}^* \):

5. \[
|R^{2n} a_n + \overline{a}_n| = (1 + R^{2n})^2 \text{re}^2(a_n) + (1 - R^{2n})^2 \text{im}^2(a_n) \leq 4 \text{re}^2(a_0).
\]

Particulary :
6. \[
|\text{re}(a_n)| \leq \frac{2 \text{re}(a_0)}{1 + R^{2n}}, \quad \forall n \in \mathbb{N}^*.
\]

**Proof:** For all \( n \in \mathbb{N}^* \) with (1) and (2) we have :

\[
|R^{2n} a_n + \overline{a}_n| = \left| \int_{0}^{2\pi} 2\text{re} \left( \Phi^{-1}(e^{i\theta}) \right) e^{in\theta} d\theta \right| \leq 2 \int_{0}^{2\pi} 2\text{re} \left( \Phi^{-1}(e^{i\theta}) \right) d\theta = 2 \text{re}(a_0).
\]
So for all \( n \in \mathbb{N}^* \):
\[
|R^{2n}a_n + \overline{a_n}| = (1 + R^{2n})^2 \text{re}^2(a_n) + (1 - R^{2n})^2 \text{im}^2(a_n) \leq 4\text{re}^2(a_0),
\]
i.e.
\[
\text{re}(a_n) \leq \frac{\sqrt{4\text{re}^2(a_0) - (1 - R^{2n})^2 \text{im}^2(a_n)}}{1 + R^{2n}} \leq \frac{2\text{re}(a_0)}{1 + R^{2n}}.
\]
QED.

\[\square\]

**Lemma 3.3.** Under the latest assumptions we have for all \( n \in \mathbb{N}^* \):
\[
|a_n| \leq 2 \cdot \frac{\sqrt{1 + R^{4n}}}{1 - R^{4n}} \sqrt{\text{re}(a_0) - R^{2n}\text{re}(a_{2n})} \cdot \sqrt{\text{re}(a_0)}.
\]

**Proof:** It will be more convenient in the sequel to state
\[
\Theta_n(e^{i\theta}) := e^{in\theta} - R^{2n}e^{-in\theta}.
\]
Then \( |\Theta_n(e^{i\theta})| = 1 + R^{4n} - R^{2n}(e^{-2in\theta} + e^{2in\theta}) \). So, with (3) and (4), we can write:
\[
2 \int_0^{2\pi} \text{re} (f(\Phi^{-1}(e^{i\theta}))) \cdot |\Theta_n(e^{i\theta})| d\theta = 2(1 + R^{4n}) \int_0^{2\pi} \text{re} (f(\Phi^{-1}(e^{i\theta}))) d\theta
\]
\[
- R^{2n} \int_0^{2\pi} 2\text{re} (f(\Phi^{-1}(e^{i\theta}))) (e^{-2in\theta} + e^{2in\theta}) d\theta
\]
\[
= 2(1 + R^{4n}) \text{re}(a_0) - R^{2n} [a_{2n} + \overline{a_{2n}} + R^{4n}(a_{2n} + \overline{a_{2n}})]
\]
\[
= 2(1 + R^{4n}) [\text{re}(a_0) - R^{2n}\text{re}(a_{2n})].
\]

And also:
\[
\int_0^{2\pi} f(\Phi^{-1}(e^{i\theta})) \cdot \Theta_n(e^{i\theta}) d\theta = 0,
\]
\[
\int_0^{2\pi} \overline{f(\Phi^{-1}(e^{i\theta}))} \cdot \Theta_n(e^{i\theta}) d\theta = \overline{a_n}(1 - R^{4n}).
\]

Consequently
\[
\left| \int_0^{2\pi} (f(\Phi^{-1}(e^{i\theta}))) \cdot |\Theta_n(e^{i\theta})| d\theta \right|
\]
\[
\leq 2 \int_0^{2\pi} \text{re} (f(\Phi^{-1}(e^{i\theta}))) |\Theta_n(e^{i\theta})| d\theta
\]
\[
\leq 2 \left( \int_0^{2\pi} \text{re} (f(\Phi^{-1}(e^{i\theta}))) |\Theta_n(e^{i\theta})|^2 d\theta \right)^{1/2} \cdot \left( \int_0^{2\pi} \text{re} (f(\Phi^{-1}(e^{i\theta}))) d\theta \right)^{1/2}
\]

5. Observe that, if all the \( a_n \in \mathbb{R} \) then we have the stronger inequalities \( |a_n| \leq \frac{2\text{re}(a_0)}{1 - R^{2n}}, \ n \geq 1 \) which will be fundamental for the "real" case.
thanks to Cauchy-Schwarz. With the three previous identities we get:

\[ |a_n| \cdot (1 - R^{4n}) \leq 2\sqrt{\text{re}(a_0)} \cdot (1 + R^{4n}) \cdot (\text{re}(a_0) - R^{2n}\text{re}(a_{2n})) \]

for all \( n \geq 1 \), the desired inequality. QED.

Now we are able to give the

**Proof of the proposition 3.1**: The "classical Caratheodory inequality" (lemma 3.1) gives for \( a_{2n} \)

\[ (1 + R^{4n})^2\text{re}^2(a_{2n}) + (1 - R^{4n})^2\text{im}^2(a_{2n}) \leq 4\text{re}^2(a_0), \]

which implies

\[ \text{im}^2(a_{2n}) \leq \frac{4\text{re}^2(a_0) - (1 + R^{4n})^2\text{re}^2(a_{2n})}{(1 - R^{4n})^2} \]

so

\[ |a_{2n}|^2 \leq \frac{4\text{re}^2(a_0)}{(1 - R^{4n})^2} + \text{re}^2(a_{2n}) \left[ 1 - \frac{(1 + R^{4n})^2}{(1 - R^{4n})^2} \right] \]

\[ \leq \frac{4}{(1 - R^{4n})^2} \cdot [\text{re}^2(a_0) - R^{4n}\text{re}^2(a_{2n})] \]

or:

\[ |a_{2n}| \leq \frac{2}{1 - R^{4n}} \cdot \sqrt{\text{re}^2(a_0) - R^{4n}\text{re}^2(a_{2n})}, \]

for all \( n \in \mathbb{N}^* \). This last inequality associated with lemma 3.2 lead us to the main estimation:

\[ |a_n| R^n + |a_{2n}| R^{2n} \leq \]

\[ \leq \frac{2R^n}{1 - R^{4n}} \left[ \sqrt{\text{re}(a_0)(1 + R^{4n})\text{re}(a_0) - R^{2n}\text{re}(a_{2n})} + R^n\sqrt{\text{re}^2(a_0) - R^{4n}\text{re}^2(a_{2n})} \right] \]

\[ \leq \frac{2R^n}{1 - R^{4n}} \left[ \sqrt{\text{re}(a_0)(1 + R^{4n})\text{re}(a_0) + R^{2n}\text{re}(a_{2n})} + R^n\sqrt{\text{re}^2(a_0) - R^{4n}\text{re}^2(a_{2n})} \right] \]

\[ = \frac{2R^n}{1 - R^{4n}} \left[ \sqrt{\text{re}(a_0)(1 + R^{4n})\text{re}(a_0) + R^{2n}\text{re}(a_{2n})} + R^n\sqrt{\text{re}^2(a_0) - R^{4n}\text{re}^2(a_{2n})} \right] \]

\[ := \frac{2R^n}{1 - R^{4n}} G(x) \]

where \( x = |\text{re}(a_{2n})| \in [0, \frac{2\text{re}(a_0)}{1 + R^{4n}}] \). Now, let us maximize \( G(x) \) on \([0, \frac{2\text{re}(a_0)}{1 + R^{4n}}]\):

\[ G(x) = \sqrt{\text{re}(a_0)(1 + R^{4n})\text{re}(a_0) + R^{2n}\text{re}(a_{2n})} + R^n\sqrt{\text{re}^2(a_0) - R^{4n}\text{re}^2(a_{2n})} \]

\[ G'(x) = \frac{R^{2n}}{2\sqrt{\text{re}(a_0) + R^{2n}\text{re}(a_{2n})}} \left( \sqrt{\text{re}(a_0)(1 + R^{4n})} - \frac{2R^{3n}x}{\sqrt{\text{re}(a_0) - R^{2n}\text{re}(a_{2n})}} \right). \]

So

\[ G'(x) = 0 \iff 4R^{6n}x^2 + \text{re}(a_0)R^{2n}(1 + R^{4n})x - \text{re}^2(a_0)(1 + R^{4n}) = 0 \]
whose roots are
\[
\begin{align*}
x_1 &= \frac{\text{re}(a_0)\sqrt{1 + R^{4n}}}{8R^{4n}} \left[ \sqrt{1 + R^{4n} + 16R^{2n}} + \sqrt{1 + R^{4n}} \right] < 0, \\
x_2 &= \frac{\text{re}(a_0)\sqrt{1 + R^{4n}}}{8R^{4n}} \left[ \sqrt{1 + R^{4n} + 16R^{2n}} - \sqrt{1 + R^{4n}} \right] > 0.
\end{align*}
\]
Because \( G'(x) \geq 0 \) for \( x \in [0, x_2] \) and \( G'(x) < 0 \) for \( x > x_2 \), we have to study the sign of \( x_2 - \frac{2\text{re}(a_0)}{1 + R^{4n}} \). First, observe that
\[
\sqrt{1 + R^{4n} + 16R^{2n}} - \sqrt{1 + R^{4n}} \geq \frac{1}{2} \cdot \frac{16R^{2n}}{\sqrt{1 + R^{4n} + 16R^{2n}}},
\]
so
\[
x_2 - \frac{2\text{re}(a_0)}{1 + R^{4n}} = \frac{\text{re}(a_0)\sqrt{1 + R^{4n}}}{8R^{4n}} \left[ \sqrt{1 + R^{4n} + 16R^{2n}} - \sqrt{1 + R^{4n}} - \frac{2\text{re}(a_0)}{1 + R^{4n}} \right]
\geq \frac{\text{re}(a_0)\sqrt{1 + R^{4n}}}{8R^{4n}} \cdot \frac{1}{2} \cdot \frac{16R^{2n}}{\sqrt{1 + R^{4n} + 16R^{2n}}} - \frac{2\text{re}(a_0)}{1 + R^{4n}}
\geq \frac{\text{re}(a_0)\sqrt{1 + R^{4n}}}{2R^{2n} \left( \sqrt{1 + R^{4n} + 16R^{2n}} \right)} - \frac{2\text{re}(a_0)}{1 + R^{4n}}
\geq \frac{\text{re}(a_0)\left[ (1 + R^{4n})\sqrt{1 + R^{4n} + 16R^{2n}} - 2R^{2n}\sqrt{1 + R^{4n} + 16R^{2n}} \right]}{2R^{2n}(1 + R^{4n})\sqrt{1 + R^{4n} + 16R^{2n}}}
\]
Now let us study the sign of the numerator \( (1 + R^{4n})^{3/2} - 2R^{2n}\sqrt{1 + R^{4n} + 16R^{2n}} \): as we saw it just few lines above the inequality \( \frac{1}{2\sqrt{a}}(b - a) \geq \sqrt{a} - \sqrt{b} \) gives
\[
\sqrt{1 + R^{4n} + 16R^{2n}} \leq \sqrt{1 + R^{4n}} + \frac{8R^{2n}}{\sqrt{1 + R^{4n}}},
\]
which implies
\[
(1 + R^{4n})^{3/2} - 2R^{2n}\sqrt{1 + R^{4n} + 16R^{2n}} \geq
\geq (1 + R^{4n})^{3/2} - 2R^{2n}\sqrt{1 + R^{4n}} - \frac{16R^{4n}}{\sqrt{1 + R^{4n}}}
\geq \sqrt{1 + R^{4n}(1 + R^{4n} - 2R^{2n})} - \frac{16R^{4n}}{\sqrt{1 + R^{4n}}}
\geq \sqrt{1 + R^{4n}(1 - R^{2n})^2} - 16R^{4n}
\geq (1 - R^{2n})^2 - 16R^{4n} = -15R^{4n} - 2R^{2n} + 1.
\]
But, \(-15R^{4n} - 2R^{2n} + 1 \geq 0\) if \( R^{2n} \in [-1/3, 1/5] \) which is the case if \( R^2 \leq 1/5 \) i.e. \( 0 \leq R \leq 1/\sqrt{5} \approx 0.447... \) which is more than confortable because\( R \leq 0.2053...\)

6. because \( \frac{1}{2\sqrt{a}}(b - a) \geq \sqrt{b} - \sqrt{a} = \int_a^b \frac{dt}{2\sqrt{t}} \geq \frac{1}{2\sqrt{a}}(b - a), \quad \forall a < b \in \mathbb{R}^+ \).
7. Remember that if \( R \geq 0, 2053... \) we have no Bohr’s phenomenon as we saw it in the last paragraph.
So, for $R \leq 0.2053\ldots$ we have for all $n \in \mathbb{N}^*$:

$$|a_n|R^n + |a_{2n}|R^{2n} \leq \frac{2R^n}{1 - R^{4n}}G\left(\frac{2\text{re}(a_0)}{1 + R^{4n}}\right)$$

$$\leq \frac{2\text{re}(a_0)R^n}{1 - R^{2n}} + \frac{2\text{re}(a_0)R^{2n}}{1 + R^{4n}}$$

Which is better than the expected estimation:

$$|a_n| \leq \frac{2\text{re}(a_0)}{1 - R^{2n}}, \quad \forall n \in \mathbb{N}^*, \ a_n \not\in \mathbb{R}.$$
4. Minoration for the Bohr radius

Remember the notations: \( \Theta_n(e^{i\theta}) = e^{in\theta} - R^{2n}e^{-in\theta} \), \( F_{\varphi,n}(\Phi^{-1}(e^{i\theta})) = e^{in\theta} + R^{2n}e^{-in\theta} \) is \( n \)-th Faber’s polynomial for the ellipse \( \varphi \). Let \( f = \sum a_n F_{\varphi,n} \in \mathcal{O}(\varphi) \) with (without losing any generality) a positive real part and \( a_0 > 0 \). Then we have for all \( n \geq 1 \):

\[
\int_0^{2\pi} f(\Phi^{-1}(e^{i\theta})) \Theta_n^2(e^{i\theta}) d\theta = -2a_0 R^{2n} + 2a_{2n} R^{4n},
\]

\[
\int_0^{2\pi} \overline{f(\Phi^{-1}(e^{i\theta}))} \Theta_n^2(e^{i\theta}) d\theta = -2a_0 R^{2n} + \overline{a}_{2n} (1 + R^{8n}).
\]

So:

\[
\int_0^{2\pi} \left( f(\Phi^{-1}(e^{i\theta})) + \overline{f(\Phi^{-1}(e^{i\theta}))} \right) \Theta_n^2(e^{i\theta}) d\theta = -4a_0 R^{2n} + 2a_{2n} R^{4n} + \overline{a}_{2n} (1 + R^{8n})
\]

which implies:

\[
\left| \int_0^{2\pi} \left( f(\Phi^{-1}(e^{i\theta})) + \overline{f(\Phi^{-1}(e^{i\theta}))} \right) \Theta_n^2(e^{i\theta}) d\theta \right|^2 \geq (-4a_0 R^{2n} + \text{re}(a_{2n})(1 + R^{4n})^2).
\]

On the other side, using Cauchy-Schwarz’s inequality (remember that \( \text{re}(f) \geq 0 \)), we have

\[
\left| \int_0^{2\pi} \left( f(\Phi^{-1}(e^{i\theta})) + \overline{f(\Phi^{-1}(e^{i\theta}))} \right) \Theta_n^2(e^{i\theta}) d\theta \right|
\]

\[
\leq \int_0^{2\pi} \left| f(\Phi^{-1}(e^{i\theta})) + \overline{f(\Phi^{-1}(e^{i\theta}))} \right| \cdot |\Theta_n(e^{i\theta})|^2 d\theta
\]

\[
\leq \left( \int_0^{2\pi} 2\text{re} \left( f(\Phi^{-1}(e^{i\theta})) \right) |\Theta_n(e^{i\theta})|^4 d\theta \right)^{1/2} \left( \int_0^{2\pi} 2\text{re} \left( \overline{f(\Phi^{-1}(e^{i\theta}))} \right) d\theta \right)^{1/2}.
\]

Easy computation gives

\[
|\Theta_n(e^{i\theta})|^4 = (1 + 4R^{4n} + R^{8n}) - 2R^{2n}(1 + R^{4n})(e^{2in\theta} + e^{-2in\theta}) + R^{4n}(e^{4in\theta} + e^{-4in\theta}),
\]

so

\[
\int_0^{2\pi} 2\text{re} \left( f(\Phi^{-1}(e^{i\theta})) \right) |\Theta_n(e^{i\theta})|^4 d\theta
\]

\[
= 2(1 + R^{4n} + R^{8n})a_0 - 4R^{2n}(1 + R^{4n})^2 \text{re}(a_{2n}) + 2R^{4n}(1 + R^{8n}) \text{re}(a_{4n}).
\]

Then, we can deduce the main inequality

\[
\text{re}^2(a_{2n}) \leq \frac{4a_0 (1 + R^{8n})}{(1 + R^{4n})^4} (a_0 + R^{4n} \text{re}(a_{4n})).
\]
This implies
\[ \text{re}(a_{4n}) \geq \frac{1}{R^{4n}} \left( \frac{\text{re}(a_{2n})^2(1 + R^{4n})^4}{4a_0(1 + R^{8n})} - a_0 \right) := h_n(\text{re}(a_{2n})). \]

Using Carathéodory’s inequality (7) we have:
\[ |a_{4n}| \leq \frac{2}{(1 - R^{8n})} \sqrt{a_0^2 - R^{8n} \text{re}(a_{4n})^2} := g_n(\text{re}(a_{4n})), \]

Because of these two inequalities, define
\[ g_n(u) = \frac{2}{(1 - R^{8n})} \sqrt{a_0^2 - R^{8n}u^2}, \quad 0 \leq u = \text{re}(a_{4n}) \leq \frac{2a_0}{1 + R^{8n}} \]
\[ h_n(v) = \frac{1}{R^{4n}} \left( \frac{v^2(1 + R^{4n})^4}{4a_0(1 + R^{8n})} - a_0 \right), \quad 0 \leq v = \text{re}(a_{2n}) \leq \frac{2a_0}{1 + R^{4n}}. \]

Elementary computation assures that \( g_n \) decrease on \([0, 2a_0/(1 + R^{8n})]\), \( h_n \) increase on \([0, 2a_0/(1 + R^{4n})]\), with
\[ g_n(\frac{2a_0}{1 + R^{8n}}) = \frac{2a_0}{1 + R^{8n}}, \quad g_n'(\frac{2a_0}{1 + R^{8n}}) = -\frac{4R^{8n}}{(1 - R^{8n})^2} \]
\[ h_n(\frac{2a_0}{1 + R^{4n}}) = \frac{2a_0}{1 + R^{8n}}, \quad h_n'(\frac{2a_0}{1 + R^{4n}}) = \frac{(1 + R^{4n})^3}{R^{4n}(1 + R^{8n})} \]

And (remember that we have already \( R < 0.2053 \))
\[ h_n'(\frac{2a_0}{1 + R^{4n}}) \leq \frac{2}{R^{4n}}, \quad \forall 0 \leq R \leq \frac{1}{2}. \]

From now on, to simplify, we will note \( x_0^n := \frac{2a_0}{1 + R^{4n}} \).

**Lemma 4.1.** Let \( x_1 \) be the unique value in \([0, x_0^n]\) such that \( h_n(x_1) = 0 \). Define \( \phi_n \) on \([0, x_0^n]\) by
\[ \phi_n(t) = \begin{cases} \frac{2a_0}{1 - R^{8n}}, & \text{if } t \in [0, x_1], \\ g_n \circ h_n(t), & \text{if } t \in [x_1, x_0^n]. \end{cases} \]

Then, \( \phi_n \in C^1([0, x_0^n]) \) and we have the following estimation:
\[ R^{4n} \phi_n'(t) \geq -4 \frac{R^{8n}(1 + R^{4n})}{(1 - R^{4n})^2(1 + R^{8n})}, \quad \forall t \in [x_1, x_0^n]. \]

Moreover, if \( R \leq \frac{1}{2} \) then:
\[ R^{4n} \phi_n'(t) \geq -8R^{8n}. \]

---

8. This is inequality (7) in the proof of proposition 3.1 page 6.
Lemma 4.2. Computation gives the required inequality.

First part is trivial. For the last one, we have

Proof: First part is trivial. For the last one, we have \( R^{4n}(g_n \circ h_n)'(t) = R^{4n} h_n'(t) g_n'(h_n(t)) \) and we now that:

- \( h_n' \) increase on \([0, \frac{2a_k}{1+R^{4n}}]\) and take positive values.
- \( g_n' \) decrease
- \( h_n' \geq 0 \) on \( x \geq x_1 \)

so, \( g_n(h_n(x)) \) is negative and reach its minimum at \( \frac{2a_k}{1+R^{4n}} \). The easy computation gives the required inequality.

We will also need the following estimations:

Lemma 4.2. Fix \( n_0 \geq 1 \), then:

1. Fix \( k \in \mathbb{N} \) and let \( x_1 := h_{n_0}^{-1} \circ \cdots \circ h_{2k n_0}^{-1}(0) \). Then the function defined on \([0, x_0]\) by

\[
\phi_{2k n_0}(t) = \begin{cases} 
\frac{2a_k}{1-R^{2k+1}}, & \text{if } t \in [0, x_1], \\
g_{2k n_0} \circ h^{-1}_{2k n_0} \circ h_{2k+1} \cdots \circ h_{n_0}(t), & \text{if } t \in [x_1, x_0] . 
\end{cases}
\]

satisfies

\[
\forall t \in [0, x_0], R \in [0, 1/2] : R^{2k+n_0} \phi_{2k n_0}'(x) \geq -2^{k+3} R^{n_0 2^{k+3}} .
\]

2. \(-\sum_{k=0}^{\infty} 2^{k+3} R^{n_0 2^{k+3}} \geq -\frac{8 R^{8 n_0}}{1 - 2 R^{8 n_0 / 3}} \geq -16 R^{8 n_0}, \quad \forall R \leq 1/2.\)

Proof: 1) We have:

\[
R^{2k+n_0}(\phi_{2k n_0}(x))' = R^{2k+n_0} g_{2k n_0}'(x_0^{2k n_0}) \cdots h_{n_0}'(x_0^{2k n_0}) .
\]

By the lemma 4.1 and the remarks before, we have the minoration

\[
R^{2k+n_0}(\phi_{2k n_0}(x))' \geq -8 R^{n_0 2^{k+3}} \times \cdots \times \frac{2}{R^{n_0 2^{k+1}}} .
\]

that is \( R^{2k+n_0}(\phi_{2k n_0}(x))' \geq -2^{k+3} R^{n_0 2^{k+3}} .\)

2) \( n_0 \geq 1 \) being fixed

\[
-\sum_{k=0}^{\infty} 2^{k+3} R^{n_0 2^{k+3}} \geq -\sum_{k=0}^{\infty} \left( 2^{(k+3)2-3/n_0 R} \right)^{n_0 2^{k+3}} \geq -\sum_{k=0}^{\infty} \left( 2^{8 n_0 / 3 R} \right)^{n_0 2^{k+3}} .
\]

Now, \( R \leq 1/2 \) implies \( 2^{3/8 n_0 R} \leq 1 \) so

\[
-\sum_{k=0}^{\infty} \left( 2^{3/8 n_0 R} \right)^{n_0 2^{k+3}} \geq -\sum_{k=0}^{\infty} \left( 2 R^{8 n_0 / 3} \right)^{k+3} \geq -\frac{8 R^{8 n_0}}{1 - 2 R^{8 n_0 / 3}} .
\]
because $n_0^{2^{k+3}} \geq 8(k + 3)n_0/3$. Finally
\[
- \sum_{k=0}^{\infty} 2^{k+3} R^{n_0^{2^{k+3}}} \geq -\frac{8R^{8n_0}}{1 - 2R^{8n_0}/3} \geq -16R^{8n_0}, \quad \forall R \leq 1/2.
\]

Because one more time of inequality (7) we have $n \geq 1$:
\[
|a_n| R^n + |a_{2n}| R^{2n} \leq f_n(x), \quad \forall n \geq 1.
\]
where
\[
f_n(x) := \frac{2R^n}{1 - R^{4n}} \left( \sqrt{a_0(1 + R^{4n})(a_0 - R^{2n}x)} + R^n \sqrt{a_0^2 - R^{4n}x^2} \right).
\]

**Lemma 4.3.** For all $R \leq \frac{1}{2}$ and $n \geq 1$, we have:
\[
\forall x \in \left[0, \frac{2a_0}{1 + R^{4n}}\right] : f_n'(x) \geq \frac{R^{3n}}{4}.
\]

**Proof:** Write $f_n(x) = \frac{R^n}{1 - R^{4n}} \theta_n(x)$. Then, we have
\[
\theta_n'(x) = \frac{1}{a_0^2 - R^{4n}x^2} \left( \sqrt{a_0(1 + R^{4n})(a_0 - R^{2n}x)} - 2R^{3n}x \right);
\]
and after some computations:
\[
\theta_n''(x) = -\frac{R^{2n}}{(a_0^2 - R^{4n}x^2)^2} \left( 2R^{2n}x \sqrt{a_0(1 + R^{4n})(a_0 - R^{2n}x)} - 4R^{5n}x^2 \\
+ \frac{1}{2} \sqrt{a_0(1 + R^{4n})(a_0 - R^{2n}x)}(a_0 + R^{2n}x) + 2R^n(a_0^2 - R^{4n}x^2) \right)
\]
\[
= -\frac{R^{2n}}{(a_0^2 - R^{4n}x^2)^2} \left( \frac{1}{2} \sqrt{a_0(1 + R^{4n})(a_0 - R^{2n}x)}(5R^{2n}x + a_0) \\
+ 2R^n(a_0^2 - 3R^{4n}x^2) \right).
\]
If $x_0^n = \frac{2a_0}{1 + R^{4n}}$, we have $a_0^2 - 3R^{4n}x^2 = \frac{a_0^2}{(1 + R^{4n})^2}(1 - 10R^{4n} + R^{8n}) \geq a_0^2(1 - 10R^4)$ (for all $n \geq 1$) and so is positive if $R \leq 1/2$. So $\theta_n''$ is negative on $[0, x_0^n]$ and the infimum of $f_n'$ on $[0, x_0^n]$ is $\frac{R^n}{1 - R^{4n}} \theta_n'(x_0^n)$. After some simple computations we get $\theta_n'(x_0^n) = \frac{R^{2n}}{1 - R^{4n}}(1 - R^{2n} - 4R^{3n} + R^{4n} - R^{6n})$, so, $R \leq 1/2$ implies $\theta_n'(x_0^n) \geq R^{2n}/4$. Consequently $f_n'(x) \geq R^{3n}/4$ if $R \leq 1/2$, on $[0, x_0^n]$. Q.E.D.
Lemma 4.4. We have the following elliptic version of Carthéodory’s inequality: let \( f = a_0 + \sum_{n=1}^{\infty} a_n F_{n,\mathcal{E}} \) be holomorphic on the ellipse \( \mathcal{E} \). If \( \text{re}(f) > 0 \) then:

\[
\sum_{n=1}^{\infty} R^n |a_n| \leq \sum_{n=0[2], n \geq 1} \frac{2R^n a_0}{1 + R^{2n}} + \sum_{n=1[2]} \frac{2R^n a_0}{1 - R^{2n}}.
\]

Proof: Suppose \( n_0 \) odd and let \( x = \text{re}(a_{2n_0}) \in [0, x_0] \). Then because of the preceding lemmas

\[
R^{n_0} |a_{n_0}| + R^{2n_0} |a_{2n_0}| + \sum_{k=0}^{\infty} R^{k+2n_0} |a_{2k+2n_0}| \leq f_{n_0}(x) + \sum_{k=0}^{\infty} \phi_{2k,n_0}(x),
\]

The derivative of the function on the right side is greater than \( \frac{R^{3n_0}}{4} - 16R^{8n_0} \) on \([0, x_0]\); so it is positive if \( R \leq 0.4 \). This implies that the right side of the inequality is an increasing function and so:

\[
R^{n_0} |a_{n_0}| + R^{2n_0} |a_{2n_0}| + \sum_{k=0}^{\infty} R^{k+2n_0} |a_{2k+2n_0}| \leq \frac{2R^n a_0}{1 - R^{2n_0}} + \sum_{k=1}^{\infty} \frac{2R^{2k+n_0} a_0}{1 + R^{2k+1+n_0}},
\]

summing these inequalities for all odd \( n_0 \) we get the desired conclusion.

\[\square\]

Proposition 4.5. 1) Let \( R_0 \) the unique solution in \([0, 1]\) of the following equation:

\[
\sum_{n=1}^{\infty} R^n |a_n| \leq \sum_{n=0[2], n \geq 1} \frac{4R^n}{1 + R^{2n}} + \sum_{n=1[2]} \frac{4R^n}{1 - R^{2n}} = 1.
\]

Then, we will have Bohr’s phenomenon if \( R \leq R_0 \), for all \( f \in \mathcal{O}(\mathcal{E}, \mathbb{D}) \).

2) Let \( R_1 \) be the unique solution in \([0, 1]\) of

\[
\sum_{n=1}^{\infty} \frac{4R^n}{(1 + R^{2n})} = 1
\]

Then, we will have Bohr’s phenomenon if \( R \leq R_1 \), for all holomorphic functions \( f \in \mathcal{O}(\mathcal{E}, \mathbb{D}) \) with real coefficients.

Proof: 1) Let \( f = a_0 + \sum_{n=1}^{\infty} a_n F_{n,\mathcal{E}} \in \mathcal{O}(\mathcal{E}, \mathbb{D}) \). Up to a rotation we have \( a_0 \geq 0 \). Consider \( g = 1 - f \), she satisfies \( \text{re}(g) > 0 \) and we can
applies to all the proceedings results. We will have Bohr’s phenomena if we can find $R \leq r \leq 1$ such that

$$|a_0| + \sum_{1}^{\infty} |a_n| \left( r^n + \frac{R^{2n}}{r^n} \right) \leq 1,$$

The left side of this inequality is an increasing function of $r$, so such an inequality will be possible if

$$|a_0| + \sum_{1}^{\infty} 2|a_n|R^n \leq 1.$$

But, because the lemma 4.4:

$$|a_0| + \sum_{1}^{\infty} 2|a_n|R^n \leq |a_0| + \sum_{n=0[2],n\geq1}^{\infty} \frac{4R^n(1-a_0)}{1+R^{2n}} + \sum_{n=1[2]}^{\infty} \frac{4R^n(1-a_0)}{1-R^{2n}},$$

and so, if

$$\sum_{n=0[2],n\geq1} \frac{4R^n}{1+R^{2n}} + \sum_{n=1[2]} \frac{4R^n}{1-R^{2n}} \leq 1,$$

we will assure the existence of Bohr’s phenomena.

2) If the coefficients $a_n$ are reals we then can use the inequality $|a_n| \leq \frac{2\text{re}(a_0)}{1+R^{2n}}, \ n \geq 1$ (observed in (footnote 5) the proof of the lemma 3.2). The result follow immediately. ■
5. Optimality

5.1. Strategy. In this paragraph, we construct families of holomorphic functions \( \phi_1(r, z) \) et \( \phi_2(r, z) \) which gives optimality for the Bohr radius of the ellipse in the category of holomorphic functions with arbitrary coefficients and also in the category of holomorphic functions with real coefficients. One more time let \( F_{n,E} \) be the Faber polynomials of the ellipse [9], this is an orthogonal (not orthonormal) family of polynomials for the image measure on the boundary of the ellipse of the Lesbesgue measure on the unit circle via \( \phi^{-1} \). Let us now consider the Bergman function associated:

\[ \sum_n F_{n,E}(w_0) F_{n,E}(z) \]

where \( w_0 \in \partial E \) is fixed.

To define extremal functions for Bohr’s problem on the ellipse, the idea is to take sequences of points \((w_k^0)_k\) inside the ellipse which tends to the boundary point \( w_0 \) (observe that this is the same in classical cases of the unit disc) and to perturb the family of Bergman function associated \( \sum_n F_{n,E}(w_k^0) F_{n,E}(z) \). Because of the geometry of the ellipse, it seems reasonable to expect that we should choose the boundary points \( w_0 \) also on the axes of the ellipse and choosing the sequences \((w_k^0)_k\) associated tending on the semi-axes to the boundary points. And that’s really what occurs as we soon shall see.

Clearly, such an asymmetry doesn’t occur for the disc. Observe also that in the cases of the disc (i.e. \( R = 0 \)) we fall down on the classical functions giving optimality. That’s what we get when choosing the sequences \((w_k^0)_k\)

5.2. Somme technical lemmas. Fix \( 0 < R < 1 \) and consider for \( R < r < 1 \) the function

\[ \phi_1(r, z) = -r + \frac{1 + r}{\gamma(r)} \sum_{n=1}^{\infty} \frac{r^n + R^{2n} r^{-n}}{(1 + R^{2n})^2} (z^n + R^{2n} z^{-n}), \]

where

\[ \gamma(r) = \sum_{n=1}^{\infty} \frac{r^n + R^{2n} r^{-n}}{1 + R^{2n}}, \]

Let \((r_k)_k\) a real sequence converging to 1 and consider the complex sequence \((z_k)_k\) defined by

\[ |\phi_1(r_k, z_k)| := \sup_{|z|=1} |\phi_1(r_k, z)|. \]

---

9. Note that these two radius are equal for the disc.
10. Remember that we have \( F_{n,E}(\Phi^{-1}(z)) = z^n + \frac{R^{2n}}{n} \) et \( F_{0,E} = 1 \)
Up to replace \((r_k)_k\) by a a subsequence, we can suppose \((z_k)_k\) converge, say to \(z_0 \in \partial \mathbb{D}\).

In a same spirit, define for all \(R < r < 1\):

\[
\phi_2(r, z) = -r + \frac{1 + r}{\theta(r)} \left( \sum_{n=0[2], n \geq 1} i^n \frac{(r^n + R^{2n}r^{-n})(z^n + R^{2n}z^{-n})}{(1 + R^{2n})^2} \right) - \sum_{n=1[2], n \geq 1} \frac{i^n (r^n - R^{2n}r^{-n})(z^n + R^{2n}z^{-n})}{(1 - R^{2n})^2},
\]

where

\[
\theta(r) = \sum_{n=0[2], n \geq 1} \frac{(r^n + R^{2n}r^{-n})}{(1 + R^{2n})} + \sum_{n=1[2], n \geq 1} \frac{(r^n - R^{2n}r^{-n})}{(1 - R^{2n})}.
\]

The sequence \((z_k)_k\) being defined as for the \((\phi_1)_r\). We have:

**Proposition 5.1.**

\[
\lim_{k \to \infty} \frac{|\phi_1(r_k, z_k)|^2 - 1}{1 - r_k} = 0, \quad \text{and} \quad \lim_{k \to \infty} \frac{|\phi_2(r_k, z_k)|^2 - 1}{1 - r_k} = 0.
\]

This clearly implies that

\[
\lim_{k \to \infty} \frac{|\phi_1(r_k, z_k)| - 1}{1 - r_k} = 0, \quad \text{and} \quad \lim_{k \to \infty} \frac{|\phi_2(r_k, z_k)| - 1}{1 - r_k} = 0.
\]

We will prove the proposition 5.1 in the next paragraph. Before, we need some technical lemmas.

**Lemma 5.2.** We have the following estimations:

\[
(1 - r) \gamma(r) = (1 - r) \sum r^n + (1 - r) \epsilon_1(r) = r + (1 - r) \epsilon_1(r),
\]

\[
(1 - r) \theta(r) = (1 - r) \sum r^n + (1 - r) \epsilon_2(r) = r + (1 - r) \epsilon_2(r),
\]

where \(\lim_{r \to 1} \epsilon_1(r) = \lim_{r \to 1} \epsilon_2(r) = 0\).

Proof:
(1) Straight computation gives
\[
\gamma(r) - \sum_{n=1}^{\infty} r^n = \sum_{n=1}^{\infty} \frac{R^{2n}r^n - r^n R^{2n}}{1 + R^{2n}},
\]
and the left side of the equality is real analytic on a neighborhood of \( r = 1 \) because \( R < 1 \) and takes value 0 if \( r = 1 \). The result follows.

(2) Similarly :
\[
\theta(r) - \sum_{n=1}^{\infty} r^n = \sum_{n=0[2]} R^{2n}r^n - r^n R^{2n} + \sum_{n=1[2]} r^n R^{2n} - R^{2n}r^n,
\]
and as in the first cases, the right part of the equality is real analytic on a neighborhood of \( r = 1 \) because \( R < 1 \) and takes value 0 if \( r = 1 \); this gives the result.

For all \( k \geq 1 \), let us fix the following notations :
\[
A_k + iB_k = \sum_{n=1}^{\infty} \frac{r^n_k + R^{2n}r^{-n}_k}{(1 + R^{2n})^2} (z^n_k + R^{2n}z^{-n}_k),
\]
\[
C_k + iD_k = \sum_{n=1}^{\infty} \frac{r^n_k + R^{2n}r^{-n}_k}{(1 + R^{2n})^2} (z^n_k + R^{2n}z^{-n}_k)
- \sum_{n=1}^{\infty} \frac{i^n(r^n_k - R^{2n}r^{-n}_k)(z^n_k + R^{2n}z^{-n}_k)}{(1 - R^{2n})^2}.
\]

**Lemma 5.3.** Write :
\[
A_k = \text{re} \left( \sum_{n \geq 1} z^n_k r^n_k \right) + \alpha_k,
\]
\[
C_k = \text{re} \left( \sum_{n \geq 1} (ir_k z_k)^n - \sum_{n \geq 1, n=0[2]} (ir_k z_k)^n \right) + \beta_k.
\]
Then \( \lim_{k \to \infty} \alpha_k = 0 = \lim_{k \to \infty} \beta_k. \)

Moreover
- If \( \lim_{k \to \infty} z_k = z_0 \neq 1 \), then, there exists a constant \( M_1 > 0 \) such that \( |A_k + iB_k| \leq M_1 \) for all \( k \) large enough.
- If \( \lim_{k \to \infty} z_k = z_0 \neq i \), then, there exists a constant \( M_2 > 0 \) such that \( |C_k + iD_k| \leq M_2 \) for all \( k \) large enough.
**Proof:** We have:

\[ A_k + iB_k - \sum_{n=1}^{\infty} z_k^n r_k^n = \sum_{n=1}^{\infty} \frac{r_k^n R^{2n} z_k^{-n} + z_k^n R^{2n} r_k^{-n} + R^{4n} r_k^{-n} z_k^{-n} - 2R^{2n} r_k^{-n} z_k^n - R^{4n} r_k^{-n} z_k^n}{(1 + R^{2n})^2}. \]

One more time, because \( R < 1 \), the function on the right side is real analytic on a neighborhood of \( r = 1 \) and \( z = z_0 \) with \( |z_0| = 1 \), so is bounded for \( k \) large enough. Function \( \sum_{n=1}^{\infty} z_k^n r_k^n \) is also (for \( k \) large enough) bounded if \( z_0 \neq 1 \). These two observations assure the second part of the lemma for \( A_k + iB_k \).

Moreover, observe that \( |z_0| = 1 \) implies that the real part of the same function on the right side of the equality tends to 0 as \( k \to +\infty \). This is the first part of the lemma for \( A_k + iB_k \). Moreover, for \( z_0 = 1 \) the function itself tends to 0 that gives the lemma for \( A_k + iB_k \).

We have the identity:

\[ C_k + iD_k - \sum_{n \geq 1, n=0\{2\}} (r_k i z_k)^n + \sum_{n \geq 1, n=1\{2\}} (r_k i z_k)^n = \sum_{n=0\{2\}} \frac{r_k^n R^{2n} (i z_k)^{-n} + (i z_k)^n R^{2n} r_k^{-n} + R^{4n} (r_k i z_k)^{-n} - 2R^{2n} (r_k i z_k)^n - R^{4n} (r_k i z_k)^n}{(1 + R^{2n})^2} \]

\[ + \sum_{n=1\{2\}} \frac{r_k^n R^{2n} (i z_k)^{-n} + (i z_k)^n \frac{R^{2n} r_k^{-n}}{r_k} - R^{4n} (r_k i z_k)^{-n} - 2R^{2n} (r_k i z_k)^n + R^{4n} (r_k i z_k)^n}{(1 - R^{2n})^2}. \]

The function on the right side is real analytic on a neighborhood of \( r = 1 \) and \( z = z_0 \) with \( |z_0| = 1 \), (because \( R < 1 \) so is bounded for \( k \) large enough. The function \( \sum_{n \geq 1, n=0\{2\}} (r_k i z_k)^n - \sum_{n \geq 1, n=1\{2\}} (r_k i z_k)^n \) is also bounded (for \( k \) large enough) if \( z_0 \neq i \). These two observations imply the second part of the lemma for \( C_k + iD_k \).

Note also that \( |z_0| = 1 \) implies that the real part of the same function on the right side of the equality tends to 0 as \( k \to +\infty \). This is the first part of the lemma for \( C_k + iD_k \). Moreover, if \( z_0 = i \), the function itself tends to 0 that gives the lemma for \( C_k + iD_k \). \( \blacksquare \)

The two properties in the preceding lemma means for \( A_K + iB_k \) and \( C_k + iD_k \):

- If \( \lim_{k \to \infty} z_k = z_0 = 1 \), then:
  \[ A_k + iB_k = \sum_{n \geq 1} r_k^n z_k^n + \lambda_k, \]
with \( \lim_{k \to \infty} \lambda_k = 0 \).

- If \( \lim_{k \to \infty} z_k = z_0 = i \), then

\[
C_k + iD_k = \sum_{n \geq 1, n=0[2]} (r_k i z_k)^n - \sum_{n \geq 1, n=1[2]} (r_k i z_k)^n + \nu_k,
\]

with \( \lim_{k \to \infty} \nu_k = 0 \).

### 5.3. The proof of proposition 5.1.

Now, we can write:

\[
\frac{\phi_1(r_k, z_k)^2 - 1}{1 - r_k} = -(1+r_k) - 2 \frac{r_k (1 + r_k)}{(1 - r_k) \gamma(r_k)} A_k + \frac{(1 + r_k)^2 (1 - r_k)}{(1 - r_k)^2 \gamma^2(r_k)} (A_k^2 + B_k^2),
\]

\[
\frac{\phi_2(r_k, z_k)^2 - 1}{1 - r_k} = -(1+r_k) - 2 \frac{r_k (1 + r_k)}{(1 - r_k) \theta(r_k)} C_k + \frac{(1 + r_k)^2 (1 - r_k)}{(1 - r_k)^2 \theta^2(r_k)} (C_k^2 + D_k^2),
\]

So, to prove proposition 5.1, it is sufficient to show

\[
\text{(8)} \quad \lim_{k \to \infty} -2 \frac{r_k (1 + r_k)}{(1 - r_k) \gamma(r_k)} A_k + \frac{(1 + r_k)^2 (1 - r_k)}{(1 - r_k)^2 \gamma^2(r_k)} (A_k^2 + B_k^2) = 2,
\]

\[
\text{(9)} \quad \lim_{k \to \infty} -2 \frac{r_k (1 + r_k)}{(1 - r_k) \theta(r_k)} C_k + \frac{(1 + r_k)^2 (1 - r_k)}{(1 - r_k)^2 \theta^2(r_k)} (C_k^2 + D_k^2) = 2.
\]

- First let us prove (8) with \( \lim_{k \to \infty} z_k = z_0 \neq 1 \). Because of lemma 5.3:

\[
\lim_{k \to \infty} \left( -2 \frac{r_k (1 + r_k)}{(1 - r_k) \gamma(r_k)} A_k + \frac{(1 + r_k)^2 (1 - r_k)}{(1 - r_k)^2 \gamma^2(r_k)} (A_k^2 + B_k^2) \right)
\]

\[
= \lim_{k \to \infty} -2 \frac{r_k (1 + r_k)}{(1 - r_k) \gamma(r_k)} A_k = -4 \lim_{k \to \infty} A_k.
\]

But we have

\[
\lim_{k \to \infty} A_k = \lim_{k \to \infty} \text{re} \left( \sum_{n=1}^{\infty} r_k^n z_k^n \right) = \lim_{k \to \infty} \text{re} \left( \frac{r_k z_k}{1 - r_k z_k} \right) = \frac{\text{re}(z_0) - 1}{2 - 2 \text{re}(z_0)} = -1/2,
\]

what we had to prove.

- For (9) with \( \lim_{k \to \infty} z_k = z_0 \neq i \).

Again because of lemma 5.3, it is sufficient to prove that

\[
\lim_{k \to \infty} -2 \frac{r_k (1 + r_k)}{(1 - r_k) \theta(r_k)} C_k = \lim_{k \to \infty} -4C_k = 2.
\]

This is the case because (see below)

\[
\lim_k C_k = - \lim_{k \to \infty} \text{re} \left( \frac{r_k z_k}{r_k z_k - i} \right) = -1/2,
\]
• Now let us look at (9) with \( \lim_{k \to \infty} z_k = z_0 = i \).

Let \( c_k + id_k = C_k + iD_k - \nu_k \). Remember that:

\[
C_k + iD_k = \sum_{n \geq 1, n=0[2]} (r_k iz_k)^n - \sum_{n \geq 1, n=1[2]} (r_k iz_k)^n + \nu_k,
\]

so

\[
c_k + id_k := \sum_{n \geq 1, n=0[2]} (ir_k z_k)^n - \sum_{n \geq 1, n=1[2]} (ir_k z_k)^n.
\]

After elementary computations we have the following equalities (where \( w_k := \text{im}(z_k) \)):

\[
c_k = -\text{re}\left(\frac{r_k z_k}{r_k z_k - i}\right),
\]

\[
|c_k + id_k|^2 = \frac{r_k^2}{1 - 2r_k w_k + r_k^2},
\]

so, the key equalities:

\[
c_k = \frac{r_k(r_k - w_k)}{1 - 2r_k w_k + r_k^2}, \quad |c_k + id_k|^2 = \frac{r_k^2}{1 - 2r_k w_k + r_k^2}.
\]

Write in polar coordinates:

\[
w_k = 1 + \rho_k \cos(\Lambda_k), \quad r_k = 1 + \rho_k \sin(\Lambda_k),
\]

where \( \rho_k \geq 0 \) and \( \Lambda_k \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \) because \( r_k \leq 1 \) (the same for \( w_k \)). We get the following

\[
1 - 2r_k w_k + r_k^2 = -2\rho_k \cos(\Lambda_k) - 2\rho_k^2 \cos(\Lambda_k) \sin(\Lambda_k) + \rho_k^2 \sin^2(\Lambda_k),
\]

\[
2(r_k - w_k) + (1 - r_k)^2(2 - r_k) = -2\rho_k \cos(\Lambda_k) + \rho_k^2 \sin^2(\Lambda_k) + o(\rho_k^2).
\]

From these, we can deduce that for \( k \) large enough:

\[
1 - 2r_k w_k + r_k^2 \geq a\rho_k^2,
\]

where \( a > 0 \) is a constant. And

\[
-2\rho_k^2 \cos(\Lambda_k) \sin(\Lambda_k) = ( -2\rho_k \cos(\Lambda_k) + \rho_k^2 \sin^2(\Lambda_k) )\mu_k,
\]

where \( \lim_{k \to \infty} \mu_k = 0 \). Using (12), lemma 5.4 and 5.2, we can replace in (9), \( C_k \) and \( D_k \) by \( c_k \) and \( d_k \).

Now, because of (9) and lemma 5.2, we have to prove that

\[
\lim_{k \to \infty} \left(-2c_k r_k + \frac{(1 - R_k^2)}{(1 - r_k^2)\theta(r_k)}|c_k + id_k|^2\right) = 1
\]


or, using the last expressions for $c_k$ and $d_k$ and always lemma 5.2:
\[
\lim_{k \to \infty} \frac{2(r_k - w_k) + (1 - r_k^2)(1 + (1 - r_k) + (1 - r_k)c_2(r_k))}{1 - 2r_k w_k + r_k^2} = 1.
\]
Because of (12), this limit is the same as
\[
\lim_{k \to \infty} \frac{2(r_k - w_k) + (1 - r_k^2)(2 - r_k)}{1 - 2r_k w_k + r_k^2},
\]
and because of (11) this last one is equal to
\[
\lim_{k \to \infty} \frac{-2\rho_k \cos(\Lambda_k) + \rho_k^2 \sin^2(\Lambda_k) + o(\rho_k^2)}{1 - 2r_k \rho_k \cos(\Lambda_k) + r_k^2}.
\]
Then, we have the required conclusion by (12) and (13).

\* Conclude with (8) with $\lim_{k \to \infty} z_k = 1$.

Write $a_k + ib_k = \sum r_k^n z_k$. After elementary computations, we have the following (with $h_k = \Re(z_k)$):
\[
a_k = -\frac{r_k(r_k - \Re(z_k))}{1 - 2r_k \Re(z_k) + r_k^2}, \quad |a_k + ib_k|^2 = \frac{r_k^2}{1 - 2r_k \Re(z_k) + r_k^2},
\]
which assure that this case goes mutatis-mutandis as the last one, replacing $w_k$ by $\Re(z_k)$.

5.4. Optimality : Functions with reals coefficients : Let us consider the family $\phi_1(r_k, z)$ on the unit disc. Their modulus less than 1. Bohr’s phenomenon on the ellipse will occurs only of there exists $1 > r_1 > R$ such that
\[
\frac{1}{\phi_1(r_k, z_k)} \left( r_k + \frac{1 + r_k}{\gamma(r_k)} \sum_{n=1}^{\infty} \frac{r_k^n + R^{2n}r_k^{-n}}{1 + R^{2n}^2} \sup_{|z_1| = r_1} (z^n + R^{2n}z^{-n}) \right) \leq 1,
\]
for all $k \in \mathbb{N}$, i.e.
\[
(1 + r_k) \sum_{n=1}^{\infty} \frac{r_k^n + R^{2n}r_k^{-n}}{1 + R^{2n}^2} \sup_{|z_1| = r_1} (z^n + R^{2n}z^{-n}) \leq (\phi_1(r_k, z_k) - r_k)\gamma(r_k);
\]
which leads to the existence of $R < r_1 \leq 1$, such that:
\[
(1 + r_k) \sum_{n=1}^{\infty} \frac{r_k^n + R^{2n}r_k^{-n}}{1 + R^{2n}^2} (r_k^n + R^{2n}r_k^{-n}) \leq (\phi_1(r_k, z_k) - r_k)\gamma(r_k),
\]
for all $k \in \mathbb{N}$. 
Because of proposition 1.1, \( \phi_1(r_k, z_k) = (r_k - 1)\varepsilon(r_k) + 1 \) with \( \lim_k \varepsilon(r_k) = 0 \). From this equality, the lemma 5.2, letting \( k \) goes to infinity in the last inequality leads to:

\[
\sum_{n=0}^{\infty} \frac{2}{(1 + R^{2n})} (r_1^n + R^{2n}r_1^{-n}) \leq 1.
\]

This inequality is possible only if

\[
\sum_{n=1}^{\infty} \frac{4R^n}{(1 + R^{2n})} \leq 1,
\]

because \( R < r_1 \leq 1 \). This implies \( R \leq R_1 \) with \( R_1 \approx 0.205328678165046 \).

5.5. **Optimality : The general case.** We follow steep by steep the « real coefficients cases » replacing \( \phi_1 \) by \( \phi_2 \).

Let us consider the family \( \left( \frac{\phi_2(r_k, z_k)}{\phi_2(r_k, z_k)} \right)_k \) of holomorphic functions on the unit disc, their modulus is less than 1, so Bohr’s phenomenon on the ellipse will occurs only of there exists \( R < r_1 \leq 1 \) such that for all \( k \in \mathbb{N} \).

\[
\frac{1}{\phi_2(r_k, z_k)} \left[ r_k + \frac{1 + r_k}{\theta(r_k)} \sum_{n=0}^{[2]} \frac{r^n_k + R^{2n}r_k^{-n}}{(1 + R^{2n})^2} \sup_{|z| = r_1} (z^n + R^{2n}z^{-n}) \right] \leq 1
\]

This leads to the existence (en exprimant le Sup) of \( R < r_1 \leq 1 \), such that

\[
(1 + r_k) \sum_{n=0}^{[2]} \frac{r^n_k + R^{2n}r_k^{-n}}{(1 + R^{2n})^2} (r_1^n + R^{2n}r_1^{-n})
\]

\[
+(1 + r_k) \sum_{n=1}^{[2]} \frac{r^n_k - R^{2n}r_k^{-n}}{(1 - R^{2n})^2} (r_1^n + R^{2n}r_1^{-n}) \leq (\phi_2(r_k, z_k) - r_k)\theta(r_k),
\]

for all \( k \in \mathbb{N} \).

Because proposition 5.1, \( \phi_2(r_k, z_k) = (r_k - 1)\varepsilon(r_k) + 1 \) with \( \lim_k \varepsilon(r_k) = 0 \). One more time, this equality, the lemma 5.2, and letting \( k \) goes to infinity in the last inequality leads to:

\[
\sum_{n=0}^{[2]} \frac{2}{(1 + R^{2n})} (r_1^n + R^{2n}r_1^{-n}) + \sum_{n=1}^{[2]} \frac{2}{(1 - R^{2n})} (r_1^n + R^{2n}r_1^{-n}) \leq 1.
\]
But this last inequality is possible only if
\[
\sum_{n=0}^{\infty} \frac{4R^n}{(1 + R^{2n})} + \sum_{n=1}^{\infty} \frac{4R^n}{(1 - R^{2n})} \leq 1,
\]
because \( R < r_1 \leq 1 \). This implies \( R \leq R_0 \).

We have proved that

**Theorem 5.4.** Let \( R_1 \) be the unique solution in \([0, 1]\) of
\[
\sum_{n=0}^{\infty} \frac{4R^n}{(1 + R^{2n})} = 1
\]
and \( R_0 \) the unique solution in \([0, 1]\) of
\[
\sum_{n=0}^{\infty} \frac{4R^n}{(1 + R^{2n})} + \sum_{n=1}^{\infty} \frac{4R^n}{(1 - R^{2n})} = 1.
\]

1) If \( R > R_1 \) then, there are no Bohr’s phenomenon for the ellipse in the category of holomorphic functions with reals coefficients.

2) If \( R > R_0 \) then, there are no Bohr’s phenomenon for the ellipse in the category of holomorphic functions.

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