Analytic Bertini theorem

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Abstract
We prove an analytic Bertini theorem, generalizing a previous result of Fujino and Matsumura.

Keywords Bertini theorem · Multiplier ideal sheaf · Pluri-subharmonic function · Hodge metric

Mathematics Subject Classification 32U15 · 32Q15

1 Introduction

Let $X$ be a connected complex projective manifold of dimension $n \geq 1$. Given any base-point free linear system $\Lambda$ on $X$, it follows from the classical Bertini theorem [9] that a general hyperplane $H$ of $\Lambda$ is smooth. Let $\varphi$ be a quasi-plurisubharmonic (quasi-psh) function on $X$. For a general member $H \in \Lambda$, the multiplier ideal sheaf $I(\varphi|_H)$ makes sense. It is natural to wonder if

$$I(\varphi|_H) = I(\varphi)|_H$$

holds for general $H$. It is well-known that the $I(\varphi|_H) \subseteq I(\varphi)|_H$ direction always holds for a general $H$, as a consequence of the Ohsawa–Takegoshi $L^2$-extension theorem. Conversely, it is easy to construct examples such that the set $B$ of $H \in \Lambda$ where the equality fails is not contained in any proper Zariski closed subset of $\Lambda$. A natural question arises: is the set $B$ small in a suitable sense? This kind of problem was first studied by Fujino and Matsumura, see [4, 5]. They proved that the complement of $B$ is dense with respect to the complex topology of $\Lambda$ (regarded as a projective space). More recently, Meng and Zhou [11] proved that the complement of $B$ has zero Lebesgue measure. In this paper, we prove the following refinement:

Theorem 1.1 There is a pluripolar set $\Sigma \subseteq \Lambda$ such that for all $H \in \Lambda \setminus \Sigma$, $H$ is smooth and (1.1) holds.
This result affirmatively answers a problem of Boucksom, see [5, Question 1.2]. From the point of view of pluripotential theory, this theorem is quite natural: a small set in pluripotential theory just means a pluripolar set. As shown in [4, Example 3.12], the exceptional set is not contained in a countable union of proper Zariski closed subsets in general, so Theorem 1.1 seems to be the optimal result. We also prove a more general analytic Bertini type result for fibrations Corollary 2.9.

Let us mention a key advantage of Theorem 1.1: our theorem can be applied to a countable family of quasi-psh functions at the same time, see Corollary 2.10. This corollary makes it possible to perform induction on the dimension when studying psh singularities.

2 Analytic Bertini theorem

In this section, varieties or algebraic varieties mean reduced separated schemes of finite type over \( \mathbb{C} \).

**Definition 2.1** Let \( Y \) be a complex projective manifold. A subset \( A \subseteq Y \) is

1. **co-pluripolar** if \( Y \setminus A \) is pluripolar. When \( \dim Y = 1 \), we also say \( A \subseteq Y \) is co-polar.
2. **co-meager** if \( Y \setminus A \) is contained in a countable union of proper Zariski closed sets.

We say a condition in \( y \in Y \) is satisfied *quasi-everywhere* if there is a co-pluripolar subset \( Y_0 \subseteq Y \) such that the condition is satisfied for \( y \in Y_0 \).

Clearly, a co-meager set is co-pluripolar. Both classes are preserved by countable intersections.

**Lemma 2.2** Let \( \pi: Y \to X \) be a smooth morphism of smooth algebraic varieties. Let \( \varphi \) be a quasi-plurisubharmonic function on \( X \), then

\[
\pi^*I(\varphi) = I(\pi^*\varphi).
\] (2.1)

Here \( I(\varphi) \) denotes the multiplier ideal sheaf of \( \varphi \) in the sense of Nadel. Observe that as \( \pi \) is flat, \( \pi^*I(\varphi) \) is a subsheaf of \( \mathcal{O}_Y \), so in (2.1) equality makes sense, the two sheaves are actually equal, not just isomorphic.

**Proof** As pointed out by the referee, \( \pi^*I(\varphi) \supseteq I(\pi^*\varphi) \) is proved in [1, Proposition 14.3]. So it suffices to prove the reverse inclusion.

By decomposing \( \pi \) into the composition of an étale morphism and a projection locally, it suffices to deal with the two cases separately. Fix a local section \( f \) of \( I(\varphi) \).

Assume that \( \pi: X \times \mathbb{C}^n \to X \) is the natural projection. Fix a volume form \( dV \) on \( X \). Take the product volume form \( dV \otimes d\lambda \) on \( X \times \mathbb{C}^n \), where \( d\lambda \) denotes the Lebesgue measure. It follows from Fubini theorem that \( |\pi^* f|^2 e^{-\pi^* \varphi} \) is locally integrable with respect to \( dV \otimes d\lambda \).

Now assume that \( \pi \) is étale. The change of variable formula shows that \( |\pi^* f|^2 e^{-\pi^* \varphi} \) is locally integrable. \( \square \)

In Lemma 2.2, we do not really need the algebraic structures on \( X \) and \( Y \). For general complex manifolds, it suffices to apply the co-area formula.

We recall the notion of positive metrics on a torsion-free coherent sheaf.

**Definition 2.3** Let \( X \) be a smooth complex algebraic variety. Let \( \mathcal{F} \) be a torsion-free (algebraic) coherent sheaf on \( X \). Let \( Z \subseteq X \) be the smallest Zariski closed set such that \( \mathcal{F}|_{X \setminus Z} = \mathcal{O}_{X \setminus Z}(F) \) for some vector bundle \( F \) on \( X \setminus Z \). A **singular Hermitian metric** (resp. **positive singular Hermitian metric**) on \( \mathcal{F} \) is a singular Hermitian metric (resp. Griffiths positively curved singular Hermitian metric) on \( F \) in the sense of [13].

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Theorem 2.4 Let $X$ be a connected projective manifold of dimension $n \geq 1$. Let $\varphi$ be a quasi-plurisubharmonic function on $X$. Let $p: X \to \mathbb{P}^N$ be a morphism ($N \geq 1$). Define

$$G := \{ H \in |O_{\mathbb{P}^N}(1)|: H' := H \cap X is smooth and \mathcal{I}(\varphi|_{H'}) = \mathcal{I}(\varphi)|_{H'} \}.$$ 

Then $G \subseteq |O_{\mathbb{P}^N}(1)|$ is co-pluripolar.

Remark 2.5 Here and in the sequel, we slightly abuse the notation by writing $H \cap X$ for $p^{-1}H$, the scheme-theoretic inverse image of $H$. In other words, $H \cap X := H \times_{\mathbb{P}^N} X$.

By definition, any $H \in |O_{\mathbb{P}^N}(1)|$ such that $p^{-1}H = \emptyset$ lies in $G$.

Proof Take an ample line bundle $L$ with a smooth Hermitian metric $h$ such that $c_1(L, h) + dd^c\varphi \geq 0$, where $c_1(L, h)$ is the first Chern form of $(L, h)$, namely the curvature form of $h$. Let $\mathcal{L}$ be the invertible sheaf corresponding to $L$. We introduce $\Lambda := |O_{\mathbb{P}^N}(1)|$ to simplify our notations.

Step 2.6 We prove that the following set is co-pluripolar:

$$G_{\mathcal{L}} := \{ H \in \Lambda : H \cap X is smooth and H^0(H \cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})) = H^0(H \cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X} \otimes \mathcal{I}(\varphi)|_{H\cap X}) \}.$$ 

Here $\omega_{H\cap X}$ denotes the dualizing sheaf of $H \cap X$.

Let $U \subseteq \Lambda \times X$ be the closed subvariety whose $\mathbb{C}$-points correspond to pairs $(H, x) \in \Lambda \times X$ with $p(x) \in H$. Let $\pi_1: U \to \Lambda$ be the natural projection. We may assume that $\pi_1$ is surjective, as otherwise there is nothing to prove.

Observe that $U$ is a local complete intersection scheme by Krull’s Hauptidealsatz and a fortiori a Cohen–Macaulay scheme. It follows from miracle flatness [10, Theorem 23.1] that the natural projection $\pi_2: U \to X$ is flat. As the fibers of $\pi_2$ over closed points of $X$ are isomorphic to $\mathbb{P}^{N-1}$, it follows that $\pi_2$ is smooth. Thus, $U$ is smooth as well.

In the following, we will construct pluripolar sets $\Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \Sigma_4 \subseteq \Lambda$ such that the behaviour of $\pi_1$ is improved successively on the complement of $\Sigma_i$.

Step 2.6.1 The usual Bertini theorem shows that there is a proper Zariski closed set $\Sigma_1 \subseteq \Lambda$ such that $\pi_1$ has smooth fibres outside $\Sigma_1$. This is slightly more general than the version that one finds in [7], see [9, Théorème 6.3] for a proof.

Step 2.6.2 By Kollár’s torsion-free theorem [4, Theorem C],

$$\mathcal{F}^i := R^i\pi_1_{\ast}(\omega_{U/\Lambda} \otimes \pi_2^*\mathcal{L} \otimes \mathcal{I}(\pi_2^*\varphi))$$

is torsion-free for all $i$. Here $\omega_{U/\Lambda}$ denotes the relative dualizing sheaf of the morphism $U \to \Lambda$. Thus, there is a proper Zariski closed set $\Sigma_2 \subseteq \Lambda$ such that

(1) $\Sigma_2 \supseteq \Sigma_1$.

(2) The $\mathcal{F}^i$’s are locally free outside $\Sigma_2$.

(3) $\omega_{U/\Lambda} \otimes \pi_2^*\mathcal{L} \otimes \mathcal{I}(\pi_2^*\varphi)$ is $\pi_1$-flat on $\pi_1^{-1}(\Lambda \setminus \Sigma_2)$ [3, Théorème 6.9.1].
We write $\mathcal{F} = \mathcal{F}^0$. By cohomology and base change [7, Theorem III.12.11], for any $H \in \Lambda \setminus \Sigma_2$, the fibre $\mathcal{F}|_H$ of $\mathcal{F}$ is given by

$$\mathcal{F}|_H = H^0(\pi_{1,H}, \omega_{U/\Lambda}|_{\pi_{1,H}} \otimes \tau_2^* \mathcal{L}|_{\pi_{1,H}} \otimes \mathcal{I}(\tau_2^* \varphi)|_{\pi_{1,H}}).$$

Here $\pi_{1,H}$ denotes the fibre of $\pi_1$ at $H$.

**Step 2.6.3** In order to proceed, we need to make use of the Hodge metric $h_{\mathcal{H}}$ on $\mathcal{F}$ defined in [8]. We briefly recall its definition in our setting. By [8, Section 22], we can find a proper Zariski closed set $\Sigma_3 \subseteq \Lambda$ such that

1. $\Sigma_3 \supseteq \Sigma_2$.
2. $\pi_1$ is submersive outside $\Sigma_3$.
3. Both $\mathcal{F}$ and $\pi_1^*(\omega_{U/\Lambda} \otimes \tau_2^* \mathcal{L}) / \mathcal{F}$ are locally free outside $\Sigma_3$.
4. For each $i$,

$$R^i\pi_1^*(\omega_{U/\Lambda} \otimes \tau_2^* \mathcal{L})$$

is locally free outside $\Sigma_3$.

Then for any $H \in \Lambda \setminus \Sigma_3$,

$$H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi|_{H \cap X})) \subseteq \mathcal{F}|_H \subseteq H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X}).$$

See [8, Lemma 22.1].

Now we can give the definition of the Hodge metric on $\Lambda \setminus \Sigma_3$. Given any $H \in \Lambda \setminus \Sigma_3$, any $\alpha \in \mathcal{F}|_H$, the Hodge metric is defined as

$$h_{\mathcal{H}}(\alpha, \alpha) := \int_{X \cap H} |\alpha|^2_{he^{-\varphi}|_{X \cap H}} \in [0, \infty].$$

Observe that $h_{\mathcal{H}}(\alpha, \alpha) < \infty$ if and only if $\alpha \in H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi|_{H \cap X}))$. Moreover, $h_{\mathcal{H}}(\alpha, \alpha) > 0$ if $\alpha \neq 0$. It is shown in [8] (c.f. [12, Theorem 3.3.5]) that $h_{\mathcal{H}}$ is indeed a singular Hermitian metric and it extends to a positive metric on $\mathcal{F}$.

**Step 2.6.4.** The determinant $\det h_{\mathcal{H}}$ is singular at all $H \in \Lambda \setminus \Sigma_3$ such that

$$H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi|_{H \cap X})) \neq \mathcal{F}|_H.$$

As the map $\pi_2$ is smooth, we have $\pi_2^* \mathcal{I}(\varphi) = \mathcal{I}(\pi_2^* \varphi)$ by Lemma 2.2. Under the identification $\pi_{1,H} \cong H \cap X$, we have

$$\pi_2^* \mathcal{I}(\varphi)|_{\pi_{1,H}} \cong \mathcal{I}(\varphi)|_{H \cap X}.$$

Thus we have the following inclusions:

$$H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi|_{H \cap X})) \subseteq H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi)|_{H \cap X}).$$

Recall that the first inclusion follows from the Ohsawa–Takegoshi $L^2$-extension theorem. Hence $\det h_{\mathcal{H}}$ is singular at all $H \in \mathcal{O}_{\mathcal{F}}(1)|\setminus \Sigma_3$ such that

$$H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi|_{H \cap X})) \neq H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi)|_{H \cap X}).$$

Let $\Sigma_4$ be the union of $\Sigma_3$ and the set of all such $H$. Since the Hodge metric $h_{\mathcal{H}}$ is positive ([12, Theorem 3.3.5] and [8, Theorem 21.1]), its determinant $\det h_{\mathcal{H}}$ is also positive ([13, Proposition 1.3] and [8, Proposition 25.1]), it follows that $\Sigma_4$ is pluripolar. As a consequence, $G_\mathcal{E}$ is co-pluripolar.
Step 2.7 Fix an ample invertible sheaf $S$ on $X$. The same result holds with $\mathcal{L} \otimes S^{\otimes a}$ in place of $\mathcal{L}$. Thus the set

$$A := \bigcap_{a=0}^{\infty} \mathcal{G}_{\mathcal{L} \otimes S^{\otimes a}}$$

is co-pluripolar. For each $H \in W$ such that $X \cap H$ is smooth and $\mathcal{I}(\varphi|_{X \cap H}) \neq \mathcal{I}(\varphi)|_{X \cap H}$, let $K$ be the following cokernel:

$$0 \rightarrow \mathcal{I}(\varphi|_{X \cap H}) \rightarrow \mathcal{I}(\varphi)|_{X \cap H} \rightarrow K \rightarrow 0.$$  

By Serre vanishing theorem, taking $a$ large enough, we may guarantee that

$$H^1(X \cap H, \omega_{X \cap H} \otimes (\mathcal{L} \otimes S^{\otimes a})|_{X \cap H} \otimes \mathcal{I}(\varphi|_{X \cap H})) = 0$$

and

$$H^0(X \cap H, \omega_{X \cap H} \otimes (\mathcal{L} \otimes S^{\otimes a})|_{X \cap H} \otimes K) \neq 0.$$  

Then

$$H^0(X \cap H, \omega_{X \cap H} \otimes (\mathcal{L} \otimes S^{\otimes a})|_{X \cap H} \otimes \mathcal{I}(\varphi|_{X \cap H})) \neq \mathcal{H}^0(X \cap H, \omega_{X \cap H} \otimes (\mathcal{L} \otimes S^{\otimes a})|_{X \cap H} \otimes \mathcal{I}(\varphi)|_{X \cap H}).$$

Thus $H \notin A$. We conclude that $\mathcal{G}$ is co-pluripolar. □

Remark 2.8 As pointed out by the referee, in [4], Fujino and Matsumura also treated the case when $X$ is not projective. It is of interest to understand if Theorem 2.4 can be extended to non-projective complex manifolds as well.

Note that the argument for $\pi: U \rightarrow \Lambda$ in the proof of Theorem 2.4 works for more general fibrations. With essentially the same proof, we can similarly prove an analytic Bertini type theorem for fibrations.

Corollary 2.9 Let $\pi: U \rightarrow W$ be a surjective morphism of projective varieties. Let $(L, \phi)$ be a Hermitian pseudo-effective line bundle on $U$, namely $L$ is a holomorphic line bundle on $U$ and $\phi$ is a plurisubharmonic metric on $L$. Then there is a pluripolar subset $\Sigma \subseteq W$ such that for all $w \in W \setminus \Sigma$, $U_w := \pi^{-1}(w)$ is smooth and we have $\mathcal{I}(\phi|_{U_w}) = \mathcal{I}(\phi)|_{U_w}$.

Corollary 2.10 Let $X$ be a projective manifold of pure dimension $n \geq 1$. Let $\Lambda$ be a base-point free linear system. Let $\varphi$ be a quasi-psh function on $X$. Then there is a pluripolar subset $\Sigma \subseteq \Lambda$ such that for any $H \in \Lambda \setminus \Sigma$ and any real number $k > 0$,

$$\mathcal{I}(k\varphi|_{H}) = \mathcal{I}(k\varphi)|_{H} \quad (2.2)$$

and we have a short exact sequence for all $k > 0$,

$$0 \rightarrow \mathcal{I}(k\varphi) \otimes O_X(-H) \rightarrow \mathcal{I}(k\varphi) \rightarrow \mathcal{I}(k\varphi|_{H}) \rightarrow 0. \quad (2.3)$$

Proof First observe that by the strong openness theorem [6] in order to verify (2.2) for all real $k > 0$, it suffices to verify it for $k$ lying in a countable subset $K \subseteq \mathbb{R}_{>0}$.

Applying Theorem 2.4 to each $k\varphi$ with $k \in K$ and each connected component of $X$, we find that there is a pluripolar set $\Sigma_1 \subseteq \Lambda$ such that for any $H \in \Lambda \setminus \Sigma_1$ and any $k \in K$, (2.2) holds. On the other hand, the union of the sets of associated primes of $\mathcal{I}(k\varphi)$ for $k > 0$ is a countable set, hence the set $A$ of $H \in \Lambda$ that avoids them is co-meager. It suffices to take $\Sigma = \Sigma_1 \cup (\Lambda \setminus A)$.
Following the terminology of [2], given quasi-psh functions $\varphi$ and $\psi$ on $X$, we say $\varphi \sim_I \psi$ if for all real $k > 0$, $I(k\varphi) = I(k\psi)$.

**Corollary 2.11**  
Let $X$ be a projective manifold of pure dimension $n \geq 1$. Let $\varphi, \psi$ be quasi-psh functions on $X$ such that $\varphi \sim_I \psi$. Let $\Lambda$ be a base-point free linear system. Then there is a pluripolar subset $\Sigma \subseteq \Lambda$ such that for any $H \in \Lambda \setminus \Sigma$, $\varphi|_H$ and $\psi|_H$ are both quasi-psh functions on $H$ and we have $\varphi|_H \sim_I \psi|_H$.

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**Declaration**

**Competing interests**  
I declare that I have no competing interests related to this paper.

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