Non-linearity analysis for cosmological inflation model with minimal and non-minimal coupling of scalar field from Horndeski theory

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Abstract. In this work, we analyze the non-linearity aspect of perturbations generated in early times for cosmological inflation model with minimal and non-minimal couplings of scalar field from Horndeski theory. We study the spectral index of the perturbations and tensor-to-scalar ratio and can be seen as the evidence for inflation for some responsible coupling constant. We get the sign of non-linearity from the spectral index and tensor-to-scalar ratio for this model, and this result can be analyzed further to find non-Gaussianity.

1. Introduction
Horizon, flatness, and monopole problem are several basic problems in cosmology and inflationary universe is a proposed solution for these problems at the same time \[1,2\]. The searching for cosmological inflation model still becomes one of the big open topics in cosmology. Considering the vacillation of temperature observation data of the Cosmic Microwave Background \[3,4\], there are many models that have been developed to explain this phenomenon \[5,6,7,8,9\], including a model where the curvature tensor is coupled with scalar field \[10,11,12\] gives possibility for an inflationary universe scenario to be right, in spite we do not yet know the origin of the scalar field (“inflaton”). The Cosmological model that couple scalar field with curvature tensor is derived generally by Horndeski \[13\], whereas all models with gravity-coupled scalar fields comprised in \[14,15,16,17,18,19\]. The Horndeski Lagrangian,

\[ L = \sum_{i=2}^{5} L_i, \] (1)
where,
\[ L_2 = K(\phi, X), \]
\[ L_3 = -G_3(\phi, X)\Box \phi, \]
\[ L_4 = G_4(\phi, X)R - 2G_{4X}(\phi, X)[(\Box \phi)^2 - \phi^{ij\nu} \phi_{ij\nu}], \]
\[ L_5 = G_5(\phi, X)G_{\mu\nu} \phi^{ij\nu} + \frac{1}{3} G_{5X}(\phi, X)[(\Box \phi)^3 - 3(\Box \phi)(\phi_{ij\nu} \phi^{ij\nu})]
+ 2(\phi_{ij\nu} \phi^{ij\nu} \phi_{ij\nu}), \]

with \( X = \partial_{\mu} \phi \partial^{\mu} \phi \) while the four coefficient functions \((K(\phi, X), G_i(\phi, X))\) can be chosen particularly for a specific model.

In this work, we consider the coefficient functions as,
\[ K = \omega(\phi)X; \quad G_3 = 0; \quad G_4 = \frac{M^2_{pl}}{2} - \frac{1}{2} \zeta \phi^2; \quad G_5 = \xi \phi, \]

where \( M^2_{pl} = \frac{1}{8\pi G} \) and in this work we take \( M^2_{pl} \approx 1 \). There are two coupling constants, \( \zeta \) and \( \xi \), each for the scalar field coupling and the derivative of scalar field coupling. The background solution of this particular model has been studied in [20], where from the De Sitter expansion and the vanishing scalar field approach, the specific range of coupling constant is obtained, \( 0 < \zeta \leq 0.021 \sim 10^{-2} \). The aim of this work is for obtaining the non-linearity aspect of perturbations generated in early time of this model by deriving the exact power spectrum and show that there exists a little deviation from the invariance scale of the spectral index. From the little deviation obtained we can continue to analyze the non-Gaussianity of the model.

This paper is organized as follows, in Section 2, we derive the solution of the second order Lagrangian equation for both scalar perturbation and tensor perturbation, and consider early regime. In Section 3, we do analysis of non-linearity of the model, where we derive the spectral index of both scalar perturbation and tensor perturbation for two cases, \( \zeta = 10^{-1} \) and \( \zeta = 10^{-2} \) based on the background solution analysis [20]. The last section is for conclusion.

2. Second Order Lagrangian Equation

In this work, we consider the flat homogenous and isotropic (FLRW) metric to be our background framework,
\[ ds^2 = -dt^2 + a^2 \delta_{ij} dx^i dx^j, \]
with \( a \) is a scale factor. From model [6] we can get the second order Lagrangian density for scalar perturbation \( \Theta \),
\[ L^{(s)}_2 = a^3 Q_s [(\dot{\Theta})^2 - \frac{c_s^2}{a^2} (\partial \Theta)^2]; \]
\[ Q_s = \frac{2L_S(9\mathcal{W}^2 + 8L_S\mathcal{W})}{\mathcal{W}^2}; \]
\[ c_s^2 = \frac{2}{Q_s}(\dot{\mathcal{M}} + H\mathcal{M} - \mathcal{E}). \]
where,

\[
L_S = \frac{1}{2} [1 - \zeta \phi^2 - \xi \dot{\phi}^2] \tag{11}
\]

\[
\mathcal{W} = 2 [H - \zeta (H \phi^2 + \phi \dot{\phi}) - \frac{3 \xi H \dot{\phi}^2}{2}] \tag{12}
\]

\[
w = 3 [-9 H^2 + 9 \zeta (H^2 \phi^2 + 2 H \dot{\phi}) + 27 \xi H^2 \dot{\phi}^2] \tag{13}
\]

\[
\mathcal{M} = \frac{4 \zeta^2 \dot{\phi}^4 + 4 \zeta \xi \phi^2 \dot{\phi}^2 + \xi^2 \dot{\phi}^4 + 16 L_S - 4}{8 [H(2 L_S - \xi \phi^2) - \xi \dot{\phi}^2]}, \tag{14}
\]

\[
\mathcal{E} = \frac{1}{2} [1 - \zeta \phi^2 + \xi \dot{\phi}^2] \tag{15}
\]

with dot represents a derivative with respect to \( t \) and \( H = \frac{\dot{a}}{a} \) is defined as the Hubble parameter.

Using Euler-Lagrange equation, the equation of motion for scalar perturbation for each Fourier mode is obtained,

\[
\ddot{\theta} + (3 H + \frac{\dot{Q}_s}{Q_s}) \dot{\theta} + c_s^2 k^2 a^2 \theta = 0. \tag{16}
\]

The solution to equation \( \text{(16)} \) is given by \([21, 22, 23]\),

\[
\theta(\tau, k) = \frac{i H e^{-ic_k \tau}}{2(c_s k)^{3/2} \sqrt{Q_s}} (1 + ic_s k \tau). \tag{17}
\]

In \( \tau \to 0 \) limit, it follows that,

\[
\theta(0, k) = \frac{-i H}{2(c_s k_1)^{3/2} \sqrt{Q_s}}. \tag{18}
\]

For tensor perturbation \((\gamma_{ij})\), the second order Lagrangian equation have form as,

\[
\mathcal{L}_2^{(h)} = \frac{a^3}{4} Q_t [\gamma_{ij}^2 - \frac{c_t^2}{a^2} (\partial_k \gamma_{ij})^2]; \tag{19}
\]

\[
Q_t = \frac{1}{2} [1 - \zeta \phi^2 - \xi \dot{\phi}^2]; \tag{20}
\]

\[
c_t^2 = \frac{1 - \zeta \phi^2 + \xi \dot{\phi}^2}{1 - \zeta \phi^2 - \xi \dot{\phi}^2}. \tag{21}
\]

Following same procedure as scalar perturbation, the solution for tensor perturbation,

\[
\gamma(t, k) = \frac{i H e^{-ic_k \tau}}{2(c_t k)^{3/2} \sqrt{Q_t}} (1 + ic_t k \tau), \tag{22}
\]

which is for \( \tau \to 0 \) limit,

\[
\gamma(0, k) = \frac{-i H}{2(c_t k_1)^{3/2} \sqrt{Q_t}}. \tag{23}
\]

3. Non-linearity Aspect of The Perturbations

We can compute the power spectrum of the field perturbation as a two-point function,

\[
\langle R_{k_1} R_{k_2} \rangle = (2\pi)^3 \delta(k_1 + k_2) P_R(k_1), \tag{24}
\]
with $R = \theta$ for scalar perturbation, and $R = \gamma$ for tensor perturbation. We define the dimensionless power spectrum $\Delta^2_R = \frac{k^3}{2\pi^2} P_R(k)$. Using the solution of the second order Lagrangian equation from previous section, equation (18), we can get the expression of the power spectrum.

For scalar perturbation,

$$\langle \theta_1 \theta_2 \rangle = \langle \{ -i\frac{H}{2(c_s k_1)^{3/2} \sqrt{Q_s}} \} \{ i\frac{H}{2(c_s k_2)^{3/2} \sqrt{Q_s}} \} \rangle$$

$$= (2\pi)^3 \delta(k_1 + k_2) \left\{ \frac{H^2}{4(c_s k)^3 Q_s} \right\}$$

$$= \frac{2\pi^2}{k^3} (2\pi)^3 \delta(k_1 + k_2) \left\{ \frac{H^2}{8\pi^2 c_s^2 Q_s} \right\},$$

so we get the dimensionless power spectrum for scalar perturbation of this model,

$$\Delta^2_\theta = \frac{H^2}{8\pi^2 c_s^2 Q_s}.$$  \hspace{1cm} (26)

Therefore we can get the spectral index of scalar perturbation,

$$n_s - 1 = \frac{2\dot{H}}{H^2} - \frac{1}{H} \frac{\dot{Q}_s}{Q_s} - \frac{3}{H} \frac{\dot{c}_s}{c_s}.$$  \hspace{1cm} (27)

Analogously for tensor perturbation, we get the dimensionless power spectrum of this model using (23),

$$\langle \gamma_1 \gamma_2 \rangle = \langle \{ -i\frac{H}{2(c_t k_1)^{3/2} \sqrt{Q_t}} \} \{ i\frac{H}{2(c_t k_2)^{3/2} \sqrt{Q_t}} \} \rangle$$

$$= (2\pi)^3 \delta(k_1 + k_2) \left\{ \frac{H^2}{4(c_t k)^3 Q_t} \right\}$$

$$= \frac{2\pi^2}{k^3} (2\pi)^3 \delta(k_1 + k_2) \left\{ \frac{H^2}{8\pi^2 c_t^2 Q_t} \right\},$$

Because of the tensor perturbation polarization, the dimensionless power spectrum is multiplied by 4 (2 for each $k$),

$$\Delta^2_\gamma = 4 \times \frac{H^2}{8\pi^2 c_t^2 Q_t} = \frac{H^2}{2\pi^2 c_t^2 Q_t}.$$  \hspace{1cm} (29)

Therefore we can get the spectral index of scalar perturbation,

$$n_t = \frac{2\dot{H}}{H^2} - \frac{1}{H} \frac{\dot{Q}_t}{Q_t} - \frac{3}{H} \frac{\dot{c}_t}{c_t}.$$  \hspace{1cm} (30)

To get the the indirect probe of the inflationary dynamics, we plot the evolution of time from the spectral index of both perturbations for some cases ($\zeta = 10^{-1}$ and $\zeta = 10^{-2}$) as shown in Fig. 4. We can see that for each case considered, there are little deviations from scale-invariance for $n_s = 1$ and $n_t = 0$. 
4. Conclusion
We have studied and analyzed the spectral index for scalar and tensor perturbations of a specific inflation model derived from the Horndeski theory. From Fig. 1 we can show the sign of non-linearity of this model for two cases ($\zeta = 10^{-1}$ and $\zeta = 10^{-2}$) as seen from a little deviation from scale-invariance for $n_s$ and $n_t$. For the next step, we will analyze the non-gaussianity of the model.

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