ON THE HOMOLOGY OF THE MODULI SPACE OF PLANE SHEAVES WITH HILBERT POLYNOMIAL $5m + 3$

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ABSTRACT. We compute the Hodge numbers of the moduli space of semi-stable sheaves on the complex projective plane supported on quintic curves and having Euler characteristic 3. For this purpose we study the fixed-point set for a certain torus action on the moduli space.

1. Introduction

Let $M_{P^2}(r,\chi)$ denote the moduli space of semi-stable sheaves on $P^2 = P^2(C)$ having Hilbert polynomial $P(m) = rm + \chi$. Our goal is to determine the additive structure of the singular homology groups with coefficients in $\mathbb{Z}$ for $M = M_{P^2}(5,3)$, which, according to [9], is a smooth projective variety of dimension 26. We refer to the introductory section of [5] for an overview of the present state of research into the geometry of the moduli spaces $M_{P^2}(r,\chi)$. The Poincaré polynomial of $M_{P^2}(5,3)$ has already been computed in [12] by means of a cellular decomposition. We will use, instead, the Bialynicki-Birula method [1, 2, 3], which has the advantage of yielding the Hodge numbers, as well. This method consists of determining the $T$-fixed locus and the $T$-representation of the tangent spaces at the fixed points for the action of a torus $T$ on a smooth projective variety. We refer to [5, Section 2] for a brief outline of the Bialynicki-Birula method. In [5, Sections 5, 6] this method was used to study the homology of $M_{P^2}(4,1)$, relying on a stratification by strata that are easily understood as geometric quotients. We will apply the same technique to $M_{P^2}(5,3)$. We will use the stratification of this moduli space provided in [11], which we recall at the beginning of Section 2. The action of $T$ on each stratum is easy to study because, as mentioned, the strata are geometric quotients. More challenging is the problem of determining the $T$-representation of the normal spaces to the strata, which we solve at Propositions 3.3.1 and 3.4.1. Denote

$$T = (C^*)^3/((c,c,c), c \in C^*).$$

Let $T$ act on $P^2$ by

$$(t_0, t_1, t_2) \cdot (x_0, x_1, x_2) = (t_0^{-1}x_0, t_1^{-1}x_1, t_2^{-1}x_2).$$

Denote by $\mu_t : P^2 \to P^2$ the map of multiplication by $t \in T$. Our main result below concerns the action of $T$ on $M_{P^2}(5,3)$ given by

$$t[F] = [\mu_{t^{-1}}F],$$

where $[F]$ denotes the stable-equivalence class of a sheaf $F$.  

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Theorem. The fixed point locus of $M_{p^2}(5,3)$ consists of 1329 isolated points, 174 projective lines, and three isomorphic surfaces obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at three points on the diagonal, then blowing down the strict transform of the diagonal. The integral homology groups of $M_{p^2}(5,3)$ have no torsion and its Poincaré polynomial is

$$P(x) = x^{32} + 2x^{30} + 6x^{48} + 13x^{46} + 26x^{44} + 45x^{42} + 68x^{40} + 87x^{38} + 100x^{36} + 107x^{34} + 111x^{32} + 112x^{30} + 113x^{28} + 113x^{26} + 113x^{24} + 112x^{22} + 111x^{20} + 107x^{18} + 100x^{16} + 87x^{14} + 68x^{12} + 45x^{10} + 26x^8 + 13x^6 + 6x^4 + 2x^2 + 1.$$  

The Euler characteristic of $M_{p^2}(5,3)$ is 1695 and its Hodge numbers satisfy the relations

$$h^{p,q} = 0 \quad \text{if} \quad p \neq q \quad \text{and} \quad h^{p,p} = b_{2p},$$

where $b_{2p}$ are the Betti numbers from above.

Our calculation of the Poincaré polynomial agrees thus with [12]. Intriguingly, this is the same as the Poincaré polynomial of $M_{p^2}(5,1)$, computed in [12] and [4]. This raises the question whether $M_{p^2}(5,1)$ and $M_{p^2}(5,3)$ are (canonically) isomorphic. Such an isomorphism would imply that there is only one smooth moduli space of semi-stable sheaves supported on plane quintics. Indeed, by duality [10], $M_{p^2}(r,\chi) \simeq M_{p^2}(r,-\chi)$. (Thus, what we say in this paper about $M_{p^2}(5,3)$ is equally valid for $M_{p^2}(5,-3)$, $M_{p^2}(5,2)$, etc.)

2. The torus fixed locus

For the convenience of the reader we recall from [11] the classification of semi-stable sheaves on $\mathbb{P}^2$ having Hilbert polynomial $5m + 3$. In $M_{p^2}(5,3)$ we have an open stratum $M_0$, two locally closed strata $M_1$ and $M_2$ of codimension 2, 3, and a closed stratum $M_3$ of codimension 4. Each $M_{i+1}$ is contained in the closure of $M_i$. The open stratum consists of sheaves $\mathcal{F}$ having a presentation of the form

$$0 \longrightarrow 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\varphi$ is not equivalent to a morphism represented by a matrix having one of the following forms:

$$\begin{bmatrix} \ast & \ast & \ast \\ \ast & 0 & 0 \\ \ast & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & 0 & 0 \end{bmatrix}.$$

We denote by $M_{01}$ the closed subset of $M_0$ given by the condition that the entries of $\varphi_{12}$ span a vector space of dimension 2 in $V^*$. Furthermore, inside $M_{01}$ we distinguish the open subset (in the relative topology) $M_{010}$ of cokernels of morphisms of the form

$$\begin{bmatrix} \ast & \ast & l_1 \\ \ast & \ast & l_2 \\ q_1 & q_2 & 0 \end{bmatrix},$$

where $l_1, l_2 \in V^*$ are linearly independent and $q_1, q_2$ have no common factor. The complement $M_{011} = M_{01} \setminus M_{010}$ is given by the condition that $q_1$ and $q_2$ have a common linear factor.

The sheaves giving points in $M_1$ are precisely those sheaves $\mathcal{F}$ having a resolution of the form

$$0 \longrightarrow 2\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$
where $\varphi_{12} = 0$, $\varphi_{11}$ has linearly independent entries and $\varphi_{22}$ has linearly independent maximal minors. Denote by $M_{10} \subset M_1$ the open subset (in the relative topology) given by the condition that the maximal minors of $\varphi_{22}$ have no common factor. The complement $M_{11} = M_1 \setminus M_{10}$ is given by the condition that the maximal minors of $\varphi_{22}$ have a common linear factor.

The points $[F]$ in $M_2$ are given by exact sequences of the form

$$0 \rightarrow 3\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O}(1) \rightarrow F \rightarrow 0,$$

where $\varphi_{11}$ has linearly independent maximal minors. Denote by $M_{20}$ the open subset (in the relative topology) given by the condition that the maximal minors of $\varphi_{11}$ have no common factor. The sheaves giving points in $M_{20}$ are of the form $\mathcal{O}_Q(-Z)^p(-1)$, where $\mathcal{O}_Q(-Z) \subset \mathcal{O}_Q$ is the ideal sheaf of a zero-dimensional scheme $Z$ of length 3 contained in a quintic curve $Q$, where $Z$ is not contained in a line. We write $\mathcal{O}_Q(Z)(1) = \mathcal{O}_Q(-Z)^p(-1)$. The ideal of $Z$ is generated by the maximal minors of $\varphi_{11}$ and $Q$ is given by the equation $\det(\varphi) = 0$. Denote $M_{21} = M_2 \setminus M_{20}$. The sheaves $F$ giving points in $M_{21}$ are precisely the extension sheaves

$$0 \rightarrow \mathcal{O}_C(1) \rightarrow F \rightarrow \mathcal{O}_L \rightarrow 0$$

satisfying $H^1(F) = 0$. Here $C$ is a quartic curve and $L$ is a line. In fact, $L$ is given by the equation $l = 0$, where $l$ is the common divisor of the maximal minors of $\varphi_{11}$, and $C$ is given by the equation $\det(\varphi)/1 = 0$.

Finally, the sheaves $F$ in the closed stratum are cokernels of the form

$$0 \rightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O} \oplus \mathcal{O}(1) \rightarrow F \rightarrow 0,$$

where $\varphi_{12}$ is non-zero and does not divide $\varphi_{22}$. Equivalently, the sheaves giving points in $M_3$ are of the form $\mathcal{O}_Q(-Y)(2)$, where $\mathcal{O}_Q(-Y) \subset \mathcal{O}_Q$ is the ideal sheaf of a zero-dimensional scheme $Y$ of length 2 contained in a quintic curve $Q$. Here $Y$ is given by the ideal $(\varphi_{12}, \varphi_{22})$ and $Q$ is the zero-set of $\det(\varphi)$.

We denote by $W_i$ the set of morphisms $\varphi$ as above whose cokernels give points in $M_i$. The ambient vector space containing $W_i$ is denoted by $W_i$. For instance

$$W_0 = \text{Hom}(2\mathcal{O}(-2) \oplus \mathcal{O}(-1), 3\mathcal{O}).$$

We denote by $G_0$, $G_1$, $G_2$, $G_3$ the obvious algebraic groups acting by conjugation on $W_i$. For instance

$$G_0 = (\text{Aut}(2\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \text{Aut}(3\mathcal{O}))/\mathbb{C}^*,$$

where $\mathbb{C}^*$ is embedded as the subgroup of homotheties. According to [11], each $W_i$ is a geometric quotient of $W_i$ by $G_i$. In particular, the fibres of the canonical map $W_i \rightarrow M_i$ are precisely the $G_i$-orbits.

2.1. Fixed points in $M_0 \setminus M_{01}$. Consider a $T$-fixed sheaf $F$ that is the cokernel of a morphism

$$\varphi = \begin{bmatrix} q_{11} & q_{12} & X \\ q_{21} & q_{22} & Y \\ q_{31} & q_{32} & Z \end{bmatrix}. \quad \text{For all } t \in T \text{ we have } \varphi = \begin{bmatrix} tq_{11} & tq_{12} & t_0X \\ tq_{21} & tq_{22} & t_1Y \\ tq_{31} & tq_{32} & t_2Z \end{bmatrix}.$$ 

Performing, possibly, column operations on $\varphi$, we may assume that $q_{11}$ and $q_{12}$ are in $\mathbb{C}[Y, Z]$. Since the fibres of the map $W_0 \rightarrow M_0$ are the $G_0$-orbits, we deduce
that there is \((g(t), h(t)) \in G_0\) such that \(t \varphi = h(t) \varphi g(t)\). Clearly, we may assume that

\[
g(t) = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ u_1 & u_2 & 1 \end{bmatrix}, \quad h(t) = \begin{bmatrix} t_0 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_2 \end{bmatrix}.
\]

From the relations

\[
tq_{11} = t_0(q_{11} g_{11} + q_{12} g_{21} + Xu_1),
\]

\[
tq_{12} = t_0(q_{11} g_{12} + q_{12} g_{22} + Xu_2)
\]

we deduce that \(u_1 = 0\) and \(u_2 = 0\). Writing

\[
q_{11} = a_1 Y^2 + b_1 Z^2 + c_1 YZ, \quad q_{12} = a_2 Y^2 + b_2 Z^2 + c_2 YZ,
\]

the above relations are equivalent to the equation

\[
\begin{bmatrix} t^2_1 & 0 & 0 \\ 0 & t^2_2 & 0 \\ 0 & 0 & t_1 t_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} = t_0 \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}.
\]

2.1.1. Case when \(q_{11}, q_{12}\) are linearly independent modulo \(X\). We may assume a priori that one of the matrices

\[
\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}, \quad \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}
\]

is the identity matrix. In the first case we have

\[
\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} t_0^{-1} t_1^j t_2^k & 0 \\ 0 & t_0^{-1} t_1^j t_2^k \end{bmatrix}.
\]

For any \(t \in T\) we have \(tq_{21} = t_0^{-1} t_1^j t_2^k q_{21}\). This shows that \(q_{21} = 0\). Analogously \(q_{22} = 0\), \(q_{31} = 0\), \(q_{32} = 0\), hence \(\det \varphi = 0\), which contradicts our choice of \(\varphi\). In the other two cases we arrive at the same contradiction. We conclude that there are no \(T\)-fixed points in \(M_0 \setminus M_{31}\) for which \(q_{11}\) and \(q_{12}\) are linearly independent modulo \(X\). Analogously, there are no fixed points for which \(q_{21}\), \(q_{22}\) are linearly independent modulo \(Y\) or for which \(q_{31}\), \(q_{32}\) are linearly independent modulo \(Z\).

2.1.2. Case when \(q_{11}, q_{12}\) are linearly dependent but not both zero modulo \(X\). Assume that \(q_{12} = 0\). For each \(t \in T\), \(tq_{11} = t_0 q_{11} g_{11}(t)\), hence \(q_{11}\) is a monomial, say \(Y^l Z^k\), and \(g_{11}(t) = t_0^{-1} t_1^j t_2^k\). Since \(0 = tq_{12} = t_0 q_{11} g_{12}\), we get \(g_{12} = 0\). Thus \(tq_{22} = t_1 q_{22} g_{22}\) and \(tq_{32} = t_2 q_{32} g_{22}\), hence \(q_{22}, q_{32}\) are monomials or zero. We have the relations

\[
tq_{21} = t_0^{-1} t_1^j t_2^k q_{21} + t_1 q_{22} g_{21},
\]

\[
tq_{31} = t_0^{-1} t_1^j t_2^k q_{31} + t_2 q_{32} g_{21}.
\]

Choosing \(t\) such that \(t_0^{-1} t_1^j t_2^k, t_0^{-1} t_1^j t_2^{k+1}\) are different from \(t_0^2, t_1^2, t_2^2, t_0 t_1, t_0 t_2, t_1 t_2\), we see that \(q_{21}\) is a multiple of \(q_{22}\) if \(q_{22} \neq 0\) and \(q_{31}\) is a multiple of \(q_{32}\) if \(q_{32} \neq 0\). In the first case, performing possibly column operations on \(\varphi\), we may assume a priori that \(q_{21} = 0\). From the first relation above we get \(g_{21} = 0\). Likewise, in the second case, we may assume that \(g_{21} = 0\). For all \(t \in T\) we have

\[
tq_{21} = t_0^{-1} t_1^j t_2^k q_{21}, \quad tq_{31} = t_0^{-1} t_1^j t_2^{k+1} q_{31},
\]
forcing $q_{21} = 0$, $q_{31} = 0$. If both $q_{22}$ and $q_{32}$ are non-zero, then
\[
\frac{q_{22}}{q_{32}} = \frac{tq_{22}}{tq_{32}} = \frac{t_1q_{22}}{t_2q_{32}} = t_1t_2^{-1} \frac{q_{22}}{q_{32}}, \text{ so } \frac{q_{22}}{q_{32}} \text{ is a multiple of } YZ^{-1},
\]
i.e. there are $l \in \{X,Y,Z\}$ and $a, b \in \mathbb{C}^*$ such that $q_{22} = a l Y$, $q_{32} = b l Z$. We obtain nine affine lines of $\mathbb{P}^1$ fixed points represented by the matrices
\[
\alpha(q_1, q_2) = \begin{bmatrix} q_1 & 0 & X \\ 0 & q_2 & Y \\ 0 & 0 & Z \end{bmatrix}, \quad q_1 \in \{Y^2, YZ, Z^2\}, \quad q_2 \in \{X^2, XZ, Z^2\},
\]
and eighteen other points if we swap $X$ and $Z$, respectively $Y$ and $Z$. We obtain nine affine lines of $T$-fixed points represented by the matrices
\[
\beta(q_1, l)(a, b) = \begin{bmatrix} q & 0 & X \\ 0 & al Y & Y \\ 0 & b l Z & Z \end{bmatrix}, \quad q \in \{Y^2, YZ, Z^2\}, \quad l \in \{X,Y,Z\}, \quad (a, b) \in \mathbb{P}^1 \setminus \{1\},
\]
and eighteen other lines if we swap $X$ and $Y$, respectively $X$ and $Z$.  

2.1.3. Case when $q_{11}, q_{12}$ are divisible by $X$, $q_{21}, q_{22}$ are divisible by $Y$, $q_{31}, q_{32}$ are divisible by $Z$. We may write
\[
\varphi = \begin{bmatrix} Xl_{11} & Xl_{12} & X \\ Yl_{21} & Yl_{22} & Y \\ 0 & 0 & Z \end{bmatrix}.
\]
We have the relation
\[
\begin{bmatrix} t_0 X(tl_{11}) & t_0 X(tl_{12}) & t_0 X \\ t_1 Y(tl_{21}) & t_1 Y(tl_{22}) & t_1 Y \\ 0 & 0 & t_2 Z \end{bmatrix} = \begin{bmatrix} t_0 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_2 \end{bmatrix} \varphi \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ u_1 & u_2 & 1 \end{bmatrix}.
\]
From the relations $0 = t_2 Z u_1$ and $0 = t_2 Z u_2$ we see that $u_1 = 0$, $u_2 = 0$. Thus, the above is equivalent to the relation
\[
\begin{bmatrix} tl_{11} & tl_{12} \\ tl_{21} & tl_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}.
\]
Write
\[
l_{11} = a_1 X + b_1 Y + c_1 Z, \quad l_{12} = a_2 X + b_2 Y + c_2 Z,
\]
From the above relation we get the equation
\[
\begin{bmatrix} t_0 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}.
\]
Assume that $l_{11}$ and $l_{12}$ are linearly independent. Performing possibly column operations on $\varphi$, we may assume a priori that one of the matrices
\[
\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix}, \quad \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}
\]
is the identity matrix. In the first case we have
\[
\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} t_0 & 0 \\ 0 & t_1 \end{bmatrix}.
For any \(t_0, t_1, t_2 \in \mathbb{C}^*\) we have \(t_2c_1 = t_0c_1\) and \(t_2c_2 = t_1c_2\). Thus \(c_1 = 0, c_1 = 0, l_{11} = X, l_{12} = Y\). For all \(t \in T\) we have \(t_1l_{21} = t_0l_{21}, t_1l_{22} = t_1l_{22}\), hence \(l_{21} = aX, l_{22} = bY\), where \(a, b \in \mathbb{C}\). We obtain the fixed points represented by the matrices

\[
\gamma(a, b) = \begin{bmatrix} X^2 & XY & X \\ aXY & bY^2 & Y \\ 0 & 0 & Z \end{bmatrix}, \quad a, b \in \mathbb{C}, \quad a \neq b.
\]

For the other two cases we obtain the matrices

\[
\begin{bmatrix} X^2 & XZ & X \\ aXY & bYZ & Y \\ 0 & 0 & Z \end{bmatrix}, \quad \text{respectively} \quad \begin{bmatrix} XY & XZ & X \\ aY^2 & bYZ & Y \\ 0 & 0 & Z \end{bmatrix}.
\]

Note that \(\gamma(a, b)\) is defined for distinct \(a, b \in \mathbb{P}^1\), if we set

\[
\gamma(\infty, b) = \begin{bmatrix} 0 & XY & X \\ XY & bY^2 & Y \\ 0 & 0 & Z \end{bmatrix}, \quad \text{respectively} \quad \gamma(a, \infty) = \begin{bmatrix} X^2 & 0 & X \\ aXY & Y^2 & Y \\ 0 & 0 & Z \end{bmatrix}.
\]

Thus we get a surface \(\gamma(a, b), a, b \in \mathbb{P}^1, a \neq b\), of \(T\)-fixed points in \(M_{5,3}\) isomorphic to the complement of the diagonal in \(\mathbb{P}^1 \times \mathbb{P}^1\). We get two other surfaces of fixed points if we swap \(X\) and \(Z\), respectively if we swap \(Y\) and \(Z\). Note that we have also covered the situation when \(l_{21}\) and \(l_{22}\) are linearly independent. It remains to examine the situation when \(l_{11}, l_{12}\) are linearly dependent and, likewise, \(l_{21}, l_{22}\) are linearly dependent. If we discount \(\gamma(0, \infty), \gamma(\infty, 0)\) and four other points obtained by interchanging \(X\) and \(Z\), respectively by interchanging \(Y\) and \(Z\), leaves us with three isolated \(T\)-fixed points represented by the matrices

\[
\alpha(q_1, q_2) = \begin{bmatrix} q_1 & 0 & X \\ 0 & q_2 & Y \\ 0 & 0 & Z \end{bmatrix}, \quad (q_1, q_2) \in \{(X^2, XY), (XY, Y^2), (XZ, YZ)\}.
\]

2.2. Fixed points in \(M_{0,1}\). Assume that the point in \(M_{5,3}\) represented by the morphism

\[
\varphi = \begin{bmatrix} q_{11} & q_{12} & l_1 \\ q_{21} & q_{22} & l_2 \\ q_1 & q_2 & 0 \end{bmatrix}
\]

is fixed by \(T\). Then, as noted before, for each \(t \in T\), there is \((g(t), h(t)) \in G_0\) such that \(t\varphi = h(t)\varphi g(t)\). We may write

\[
g(t) = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ u_1 & u_2 & 1 \end{bmatrix}, \quad h(t) = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}.
\]

From the relation \(0 = h_{31}l_1 + h_{32}l_2\) we get \(h_{31} = 0, h_{32} = 0\). We have the relation

\[
\begin{bmatrix} tl_1 \\ tl_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}.
\]

We can argue as at \([2.1.9]\) to deduce that \(l_1\) and \(l_2\) are distinct elements in the set \(\{X, Y, Z\}\) and that

\[
\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} (tl_1)/l_1 & 0 \\ 0 & (tl_2)/l_2 \end{bmatrix}.
\]
We have the relation
\[
\begin{bmatrix}
t_1q_1 & t_2q_2
\end{bmatrix} = h_{33} \begin{bmatrix}
q_1 & q_2
\end{bmatrix} \begin{bmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{bmatrix}.
\]

Arguing as at 2.1.1, we can show that \(q_1\) and \(q_2\) are distinct elements in the set \(\{X^2, Y^2, Z^2, XY, XZ, YZ\}\) and that
\[
\begin{bmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{bmatrix} = h_{33}^{-1} \begin{bmatrix}
(t_1q_1)/q_1 & 0 \\
0 & (t_2q_2)/q_2
\end{bmatrix}.
\]

One linear form among \(l_1, l_2\) does not divide \(q_1\) or \(q_2\), otherwise \(q_1\) and \(q_2\) would be both equal to \(l_1l_2\). Performing, possibly, elementary operations on \(\varphi\), we may assume that \(l_1\) does not divide \(q_1\), that \(q_{11}\) and \(q_{12}\) do not contain any monomial divisible by \(l_1\), and that \(q_{11}\) and \(q_{21}\) do not contain the monomial \(q_1\). From the relation
\[
t_{q_{11}} = \left(\frac{t_1}{l_1}q_{11} + h_{13}q_1\right) h_{33}^{-1} \frac{t_q}{q_1} + (t_1)u_1
\]
we get \(h_{13} = 0\), \(u_1 = 0\). From the relation
\[
t_{q_{12}} = \left(\frac{t_1}{l_1}q_{12} + h_{13}q_1\right) h_{33}^{-1} \frac{t_q}{q_2} + (t_1)u_2 = \frac{t_1}{l_1}q_{12}h_{33}^{-1} \frac{t_q}{q_2} + (t_1)u_2
\]
we get \(u_2 = 0\). From the relation
\[
t_{q_{21}} = \left(\frac{t_1}{l_2}q_{21} + h_{23}q_1\right) h_{33}^{-1} \frac{t_q}{q_1} + (t_2)u_1 = \left(\frac{t_1}{l_2}q_{21} + h_{23}q_1\right) h_{33}^{-1} \frac{t_q}{q_1}
\]
we get \(h_{23} = 0\). We deduce that the equation \(t\varphi = h(t)\varphi g(t)\) is equivalent to the relation
\[
\begin{bmatrix}
t_q_{11} & t_{q_{12}} \\
t_{q_{21}} & t_{q_{22}}
\end{bmatrix} = h_{33}^{-1} \begin{bmatrix}
(t_1)/l_1 & 0 \\
0 & (t_2)/l_2
\end{bmatrix} \begin{bmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{bmatrix} \begin{bmatrix}
(t_1)/q_1 & 0 \\
0 & (t_2)/q_2
\end{bmatrix}.
\]

It becomes clear now that \(q_{11}, q_{12}, q_{21}, q_{22}\) are monomials or zero and that
\[
h_{33}(t) = t_i^k t_j^{{-k}} t_k^l
\]
for some integers \(i, j, k\) satisfying \(i + j + k = 1\). In fact,
\[
\begin{bmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{bmatrix} = \begin{bmatrix}
c_{11}X^{-i}Y^{-j}Z^{-k} & c_{12}X^{-i}Y^{-j}Z^{-k} \\
c_{21}X^{-i}Y^{-j}Z^{-k} & c_{22}X^{-i}Y^{-j}Z^{-k}
\end{bmatrix}
\]
where \(c_{rs} = 0\) if the corresponding monomial has negative exponents.

Given \(l_1, l_2, q_1, q_2\) as above, and a monomial \(d\) of degree 5, we denote by \(M(l_1, l_2, q_1, q_2)\) the image in \(M_{p^2}(5, 3)\) of the set of morphisms of the form
\[
\varphi = \begin{bmatrix}
* & * & l_1 \\
* & * & l_2 \\
q_1 & q_2 & 0
\end{bmatrix},
\]
and by \(M(l_1, l_2, q_1, q_2, d)\) the subset given by the additional condition that \(\det(\varphi)\) be a multiple of \(d\). Clearly, this sets are \(T\)-invariant.

**Proposition 2.2.1.** Assume that \(l_1, l_2 \in \{X, Y, Z\}\) are distinct one-forms and \(q_1, q_2 \in \{X^2, Y^2, Z^2, XY, XZ, YZ\}\) are distinct two-forms. Then, for any monomial \(d\) of degree 5 belonging to the ideal \((l_1q_1, l_1q_2, l_2q_1, l_2q_2)\), the set of fixed points for the action of \(T\) on \(M(l_1, l_2, q_1, q_2, d)\) has precisely one irreducible component, which is either a point or an affine line.
Proof. We will only examine the case when \( l_1 = X, l_2 = Y, q_1 = Z^2, q_2 = XZ \), all other cases being analogous. Consider, therefore, a morphism of the form

\[
\varphi = \begin{bmatrix}
c_{11} X^{1-i} Y^{-i} Z^{2-k} & c_{12} X^{2-i} Y^{-i} Z^{1-k} \\
c_{21} X^{-i} Y^{-1} Z^{2-k} & c_{22} X^{1-i} Y^{-1} Z^{1-k}
\end{bmatrix}
\]

where \( i + j + k = 1 \). Assume that \( c_{12} \neq 0 \). Then \( i = 2, c_{11} = 0, c_{21} = 0, c_{22} = 0 \), so we get three isolated fixed points:

\[
\begin{bmatrix}
0 & Y^2 & X \\
0 & 0 & Y \\
Z^2 & XZ & 0
\end{bmatrix}, \quad \begin{bmatrix}
0 & YZ & X \\
0 & 0 & Y \\
Z^2 & XZ & 0
\end{bmatrix}, \quad \begin{bmatrix}
0 & Z^2 & X \\
0 & 0 & Y \\
Z^2 & XZ & 0
\end{bmatrix}.
\]

Assume now that \( c_{12} = 0 \) and \( c_{11} \neq 0 \). Then \( i = 1, c_{21} = 0, j + k = 0, j \leq 0, k \leq 2 \). Thus \( (j, k) \) is one of the following pairs: \( (0,0), (-1,1), (-2,2) \). The first case is not feasible because of our convention that \( q_{11} \) do not contain the monomial \( q_1 \). We obtain the fixed points

\[
\varphi_1(c) = \begin{bmatrix}
YZ & 0 & X \\
0 & cY^2 & Y \\
Z^2 & XZ & 0
\end{bmatrix}, \quad c \in \mathbb{C} \setminus \{-1\}, \quad \begin{bmatrix}
Y^2 & 0 & X \\
0 & 0 & Y \\
Z^2 & XZ & 0
\end{bmatrix}.
\]

Assume next that \( c_{11} = 0, c_{12} = 0, c_{21} \neq 0 \). Then \( i \leq 0, j \leq 1, k \leq 2 \), hence \( (i, j, k) \) is one of the following triples: \( (0,1,0), (0,0,1), (-1,1,1), (0,-1,2), (-1,0,2), (-2,1,2) \). The first case is not feasible because of our convention that \( q_{21} \) do not contain the monomial \( q_1 \). We obtain the fixed points

\[
\varphi_2(c) = \begin{bmatrix}
0 & 0 & X \\
YZ & cX^2 & Y \\
Z^2 & XZ & 0
\end{bmatrix}, \quad \varphi_3(c) = \begin{bmatrix}
0 & 0 & X \\
XY & 0 & Y \\
Z^2 & XZ & 0
\end{bmatrix}, \quad c \in \mathbb{C} \setminus \{1\},
\]

The final case to examine is when \( c_{11} = 0, c_{12} = 0, c_{21} = 0, c_{22} \neq 0 \). We have \( i \leq 1, j \leq 1, k \leq 1 \), hence \( (i, j, k) \) belongs to the list \( \{1,1,-1\}, \{1,0,0\}, \{1,-1,1\}, \{0,1,0\}, \{0,0,1\}, \{-1,1,1\} \). We get the fixed points

\[
\varphi_1(\infty) = \begin{bmatrix}
0 & 0 & X \\
0 & YZ & Y \\
Z^2 & XZ & 0
\end{bmatrix}, \quad \varphi_2(\infty) = \begin{bmatrix}
0 & 0 & X \\
0 & XY & Y \\
Z^2 & XZ & 0
\end{bmatrix}, \quad \varphi_3(\infty) = \begin{bmatrix}
0 & 0 & X \\
0 & X^2 & Y \\
Z^2 & XZ & 0
\end{bmatrix}.
\]

In conclusion, for the action of \( T \) on \( M(X,Y,Z^2,XZ) \), we have nine fixed isolated points and three affine lines, namely

\[
\{\varphi_1(c), \ c \in \mathbb{P}^1 \setminus \{-1\}\}, \ \{\varphi_2(c), \ c \in \mathbb{P}^1 \setminus \{1\}\}, \ \{\varphi_3(c), \ c \in \mathbb{P}^1 \setminus \{1\}\}. \quad \square
\]

We denote by \( \delta(1_1,1_2,q_1,q_2,d) \) the set of \( T \)-fixed points in \( M(1_1,1_2,q_1,q_2,d) \). The list of fixed affine lines can be found in Table 1 at the end of this section.
2.2.2. **Limits of sequences of points in** $M_0 \setminus M_{01}$. By an abuse of notation, in the sequel a matrix will denote its image in the moduli space. We have

$$\lim_{a \to 1} \beta(q, l)(a, 1) = \lim_{a \to 1} \begin{bmatrix} q & 0 & X \\ 0 & aY & Y \\ 0 & lZ & Z \end{bmatrix} = \lim_{a \to 1} \begin{bmatrix} q & -lX & X \\ 0 & (a - 1)lY & Y \\ 0 & 0 & Z \end{bmatrix}$$

$$= \lim_{c \to 0} \begin{bmatrix} q & -lX & cX \\ 0 & lY & Y \\ 0 & 0 & Z \end{bmatrix} = \delta(Y, Z, q, lX, qlYZ).$$

Thus, each of the 27 affine fixed lines in $M_0 \setminus M_{01}$ contains a point from $M_{01}$ in its closure. We obtain 27 irreducible components of the $T$-fixed locus isomorphic to $\mathbb{P}^1$. Denote by $S$ the closure of the set $\{\gamma(a, b) \mid a, b \in \mathbb{P}^1, a \neq b\}$. Assume that $a, b \in \mathbb{C}$ are distinct. We have

$$\lim_{c \to 0} \gamma(ca, cb) = \lim_{c \to 0} \begin{bmatrix} X^2 & XY & cX \\ acXY & bcY^2 & Y \\ 0 & 0 & Z \end{bmatrix} = \lim_{c \to 0} \begin{bmatrix} X^2 & XY & cX \\ cXY & bY^2 & Y \\ 0 & 0 & Z \end{bmatrix}$$

$$= \delta(Y, Z, X, XY, X^2Y^2Z)(a, b).$$

$$\lim_{c \to 0} \gamma(ca + 1, cb + 1) = \lim_{c \to 0} \begin{bmatrix} X^2 & (ca + 1)XY & (cb + 1)Y^2 & X \\ acXY & cbY^2 & Y \\ -XZ & -YZ & Z \end{bmatrix}$$

$$= \lim_{c \to 0} \begin{bmatrix} 0 & 0 & X \\ acXY & cbY^2 & Y \\ -XZ & -YZ & Z \end{bmatrix} = \delta(X, Y, XZ, YZ, X^2Y^2Z)(a, b).$$

Assume that $a, b \in \mathbb{C}$ are distinct and non-zero. We have

$$\lim_{c \to \infty} \gamma(ca, cb) = \lim_{c \to \infty} \begin{bmatrix} X^2 & XY & X \\ caXY & cbY^2 & Y \\ 0 & 0 & Z \end{bmatrix} = \lim_{c \to \infty} \begin{bmatrix} a^{-1}X^2 & b^{-1}XY & X \\ a^{-1}caXY & a^{-1}cbY^2 & Y \\ 0 & 0 & Z \end{bmatrix}$$

$$= \delta(X, Z, X, XY, X^2Y^2Z)(a^{-1}, b^{-1}).$$

Thus, $S$ contains three affine lines of fixed points for the action of $T$ on $M_{01}$ denoted $\delta_1, \delta_2, \delta_3$. Denote $S_0 = \{\gamma(a, b) \mid a, b \in \mathbb{P}^1, a \neq b\} \cup \delta_1 \cup \delta_2 \cup \delta_3$. Denote by $\Delta$ the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. From the above calculations it is clear that $S_0$ is isomorphic to an open subset of the blow-up $B$ of $\mathbb{P}^1 \times \mathbb{P}^1$ at $\{(0, 0), (1, 1), (\infty, \infty)\}$. In fact, $S_0 \simeq B \setminus \Delta$, where $\Delta$ is the strict transform of $\Delta$. In Section 3 we will show that $S \setminus S_0$ consists of a single point (in $M_1$). It follows that $S$ is isomorphic to the blow-down of $B$ along $\Delta$. 
From what was said above, we have a complete picture of the fixed locus for the action of T on $M_0$. There are thirty isolated points of the form $\alpha(q_1, q_2)$, twenty-seven projective lines that are the closure of the affine lines $\beta(q_1, l_1)$, three surfaces isomorphic to $B \setminus \Delta$ and a number of isolated points and affine lines of the form $\delta$. The information about the latter is summarised in the Table 1 below. We assume that $l_1 = X$, $l_2 = Y$, the other cases being obtained by a permutations of variables. The first column contains the pair $(q_1, q_2)$ (again modulo permutations of variables), the second column contains the monomials $d$ of degree 5 that are in the ideal $(l_1q_1, l_1q_2, l_2q_1, l_2q_2)$, the third column contains those $d$ for which $\delta(X, Y, q_1, q_2, d)$ is a line, and in the fourth column are listed those $d$ for which $\delta(X, Y, q_1, q_2, d)$ is contained in the closure of a line or surface of fixed points in $M_0 \setminus M_0$. We will use the abbreviation $\Sigma^5 = \{XY^iZ^k, i, j, k \geq 0, i + j + k = 5\}$.

| $(q_1, q_2)$ | equation of support | affine lines | limit sheaves |
|---------------|---------------------|--------------|--------------|
| $(X^2, Y^2)$  | $\Sigma^5 \setminus \{Z^5, Z^2X, Z^2Y, Z^2X^2, Z^2XY, Z^2Y^2\}$ | $X^2Y^2Z$   |              |
| $(X^2, Z^2)$  | $\Sigma^5 \setminus \{Y^5, Z^3, XY^4, Y^3Z, XV^4\}$ | $X^3YZ$    |              |
| $(Z^2, XY)$   | $\Sigma^5 \setminus \{X^5, Z^3, XY^4, Z^4, YZ^3, Y^2Z^3, Y^3Z^2, Y^4Z, XYZ^3\}$ | $X^2Y^2Z$   | $X^3Y^2$     |
| $(X^2, YZ)$   | $\Sigma^5 \setminus \{X^5, Y^5, Z^3, XZ^4, YZ^3, XY^4, X^2Y^3, X^3Y^2, X^4Y\}$ | $X^2Y^2Z$   |              |
| $(XZ, YZ)$    | $\Sigma^5 \setminus \{X^5, Y^5, Z^3, XZ^4, YZ^3, XY^4, X^2Y^3, Y^3Z^2, Y^4Z\}$ | $X^2Y^2Z$   |              |
| $(X^2, XZ)$   | $\Sigma^5 \setminus \{Y^5, Z^5, X^4, XZ^4, YZ^4, Y^2Z^3, Y^3Z^2, Y^4Z\}$ | $X^2Y^2Z$   | $X^3Y^3$     |
| $(XY, YZ)$    | $\Sigma^5 \setminus \{X^5, Y^5, Z^3, Y^4, Z^4, XZ^3, X^2Z^3, X^3Z^2, X^4Z\}$ | $X^2Y^2Z$   |              |
| $(XZ, Z^2)$   | $\Sigma^5 \setminus \{X^5, Y^5, Z^3, X^4Z, X^2Y^3, X^3Y^2, X^4Y\}$ | $X^2Y^2Z$   | $X^3Z^2$     |

2.3. Fixed points in $M_1$. Assume that the point in $M_{d2}(5, 3)$ represented by the morphism

$$\varphi = \begin{bmatrix} \varphi_{11} & 0 \\ \varphi_{21} & \varphi_{22} \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & 0 & 0 \\ q_{11} & q_{12} & l_{11} & l_{12} \\ q_{21} & q_{22} & l_{21} & l_{22} \\ q_{31} & q_{32} & l_{31} & l_{32} \end{bmatrix}$$

is fixed by T. Then, for each $t \in T$, there is $(g(t), h(t)) \in G_1$ such that $t \varphi = h(t)\varphi g(t)$. We write

$$g(t) = \begin{bmatrix} g_1 & 0 \\ u & g_2 \end{bmatrix}, \quad h(t) = \begin{bmatrix} h_{11} & 0 \\ v & h_2 \end{bmatrix}.$$
From the relations $t\varphi_{11} = h_{11}(t)\varphi_{11}g_1(t)$ and $t\varphi_{22} = h_2(t)\varphi_{22}g_2(t)$ we see that $\varphi_{11}$ gives a $T$ fixed point in $N(3, 2, 1)$ while $\varphi_{22}$ gives a $T$-fixed point in $N(3, 2, 3)$. The fixed points in $N(3, 2, 1)$ have been described at 2.1.3. They are completely determined by the $T$-fixed point $x \in \mathbb{P}^2$ given by the ideal $(l_1, l_2)$. The fixed points in $N(3, 2, 3)$ have been described in Chapter VI]. They are completely determined by the ideal generated by the maximal minors $q_1, q_2, q_3$ of $\varphi_{22}$. We have ten fixed points corresponding to the ideals of $T$-fixed subschemes $Z$ of $\mathbb{P}^2$ of length 3, where $Z$ is not contained in a line, and three more points corresponding to the ideals $(X^2, XY, XZ), (XY, Y^2, YZ), (XZ, YZ, Z^2)$. Let $M(l_1, l_2, q_1, q_2, q_3)$ be the image in $M_{P, 5, 3}$ of the set of morphisms $\varphi$ as above with fixed $\varphi_{11}$ and $\varphi_{22}$. Given a monomial $d$ of degree 5, we denote by $M(l_1, l_2, q_1, q_2, q_3, d)$ the subset given by the additional condition that $\det(\varphi)$ be a multiple of $d$.

**Proposition 2.3.1.** Assume that $\varphi_{11}$ and $\varphi_{22}$ give $T$-fixed points in $N(3, 2, 1)$, respectively $N(3, 2, 3)$. Let $l_1, l_2$ be the entries of $\varphi_{11}$, and let $q_1, q_2, q_3$ be the maximal minors of $\varphi_{22}$. Then, for any monomial $d$ of degree 5 belonging to the ideal $(l_1q_1, l_1q_2, l_1q_3, l_2q_1, l_2q_2, l_2q_3)$, the set of fixed points for the action of $T$ on $M(l_1, l_2, q_1, q_2, q_3, d)$ has precisely one irreducible component, which is either a point or an affine line.

**Proof.** We will only examine the case when $Z$ is a triple point supported on $\{x\}$, the other cases being analogous. Consider morphisms of the form

$$\varphi = \begin{bmatrix}
X & Y & 0 & 0 \\
q_{11} & q_{12} & X & 0 \\
q_{21} & q_{22} & 0 & Y \\
q_{31} & q_{32} & Y & X
\end{bmatrix}.$$ 

Performing elementary operations on $\varphi$ we may assume that

$$q_{11} \in \mathbb{C}[Y, Z], \quad q_{12} \in \mathbb{C}[Z], \quad q_{21} \in \mathbb{C}[Z], \quad q_{22} \in \mathbb{C}[X, Z], \quad q_{32} \in \mathbb{C}[X, Z].$$

We have the relations

$$t\varphi_{11} = \varphi_{11} \begin{bmatrix}
t_0 & 0 & 0 \\
0 & t_1 & 0
\end{bmatrix}, \quad t\varphi_{22} = \begin{bmatrix}
t_0t_1^{-1} & 0 & 0 & 0 \\
0 & t_1^{-1}t_0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \varphi_{22} \begin{bmatrix}
t_1 & 0 & 0 \\
0 & t_0 & 0
\end{bmatrix}.$$ 

The morphisms $\varphi_{11}$ and $\varphi_{22}$ are stable as Kronecker modules, hence they have trivial stabilisers in $\text{GL}(2, \mathbb{C})$, respectively in $(\text{GL}(2, \mathbb{C}) \times \text{GL}(3, \mathbb{C}))^*/\mathbb{C}^*$. Thus, we may assume that

$$g_1(t) = h_{11}^{-1} \begin{bmatrix}
t_0 & 0 & 0 \\
0 & t_1 & 0
\end{bmatrix}, \quad g_2 = \begin{bmatrix}
t_1 & 0 & 0 \\
0 & t_0 & 0
\end{bmatrix}, \quad h_2 = \begin{bmatrix}
t_0t_1^{-1} & 0 & 0 & 0 \\
0 & t_1^{-1}t_0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$
The relation \( t\varphi = h(t)\varphi g(t) \) is equivalent to the relation \( t\varphi_{21} = v\varphi_{11}g_1 + h_2\varphi_{21}g_1 + h_2\varphi_{22}u \), or

\[
\begin{bmatrix}
tq_{11} & tq_{12} \\
tq_{21} & tq_{22} \\
tq_{31} & tq_{32}
\end{bmatrix} =
\begin{bmatrix}
v_1 \\ v_2 \\ v_3
\end{bmatrix}
\begin{bmatrix}
X & Y
\end{bmatrix}
\begin{bmatrix}
h_{11} \\
h_{12}
\end{bmatrix}
\begin{bmatrix}
t_0 & 0 \\
0 & t_1
\end{bmatrix} +
\begin{bmatrix}
t_0t_1^{-1} & 0 & 0 \\
0 & t_0^{-1}t_1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22} \\
q_{31} & q_{32}
\end{bmatrix}
\begin{bmatrix}
h_{11} \\
h_{12}
\end{bmatrix}
\begin{bmatrix}
t_0 & 0 \\
0 & t_1
\end{bmatrix} +
\begin{bmatrix}
t_0t_1^{-1} & 0 & 0 \\
0 & t_0^{-1}t_1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
X & 0 & 0 \\
0 & Y & X
\end{bmatrix}
\begin{bmatrix}
u_{11} & u_{12} \\
u_{21} & u_{22}
\end{bmatrix}.
\]

From the relation

\[ tq_{12} = h_{11}^{-1}t_1v_1Y + h_{11}^{-1}t_0q_{12} + t_0t_1^{-1}u_{12}X \]

we get the relations

(1) \( tq_{12} = h_{11}^{-1}t_0q_{12}, \quad u_{12} = aY, \quad v_1 = -h_{11}t_0t_1^{-2}aX \) for some \( a \in \mathbb{C} \).

From the relation

\[ tq_{11} = h_{11}^{-1}t_0v_1X + h_{11}^{-1}t_0^{-2}t_1^{-1}q_{11} + t_0t_1^{-1}u_{11}X \]

we get the relations

(2) \( tq_{11} = h_{11}^{-1}t_0^{-2}t_1^{-1}q_{11}, \quad u_{11} = t_0t_1^{-1}aX. \)

From the relation

\[ tq_{21} = h_{11}^{-1}t_0v_2X + h_{11}^{-1}t_1q_{21} + t_0^{-1}t_1u_{21}Y \]

we get the relations

(3) \( tq_{21} = h_{11}^{-1}t_1q_{21}, \quad u_{21} = bX, \quad v_2 = -h_{11}t_0^{-2}t_1bY \) for some \( b \in \mathbb{C} \).

From the relation

\[ tq_{22} = h_{11}^{-1}t_1v_2Y + h_{11}^{-1}t_0^{-1}t_1^{-1}q_{22} + t_0^{-1}t_1u_{22}Y \]

we get the relations

(4) \( tq_{22} = h_{11}^{-1}t_0^{-1}t_1^{-2}q_{22}, \quad u_{22} = t_0^{-1}t_1bY. \)

From the relation

\[ tq_{32} = h_{11}^{-1}t_1v_3Y + h_{11}^{-1}t_1q_{32} + u_{12}Y + u_{22}X \]

we get the relation

(5) \( tq_{32} = h_{11}^{-1}t_1q_{32}, \quad v_3 = -h_{11}(t_1^{-1}aY + t_0^{-1}bX). \)

Finally, we have the relations

\[
\begin{align*}
tq_{31} &= h_{11}^{-1}t_0v_3X + h_{11}^{-1}t_0q_{31} + u_{11}Y + u_{21}X \\
&= h_{11}^{-1}t_0q_{31} - t_0(t_1^{-1}aY + t_0^{-1}bX)X + t_0t_1^{-1}aXY + bX^2,
\end{align*}
\]

(6) \( tq_{31} = h_{11}^{-1}t_0q_{31}, \)
Combining relations (1)–(6), yields the equation \(t \varphi_{21} = h_2(t) \varphi_{21} \varphi_1(t)\). It becomes clear now that \(q_{rs}\) are monomials and \(h_{11}(t) = t_0^{-i} t_1^{-j} t_2^{-k}\), where \(i, j, k\) are integers satisfying \(i + j + k = 1\). In fact,

\[
\varphi_{21} = \begin{bmatrix}
    c_{11}X^{2+i}Y^{-1+i}Z^k & c_{12}X^{1+i}Y^jZ^k \\
    c_{21}X^iY^{1+j}Z^k & c_{22}X^{i-1}Y^2Z^k \\
    c_{31}X^iY^{1+j}Z^k & c_{32}X^iY^jZ^k 
\end{bmatrix}
\]

Assume that \(c_{12} \neq 0\). Then \(i = -1, j = 0, c_{11} = 0, c_{21} = 0, c_{22} = 0, c_{31} = 0, c_{32} = 0\). We get the \(T\)-fixed points represented by the matrices

\[
\varphi_1(c) = \begin{bmatrix}
    X & Y & 0 & 0 \\
    0 & Z^2 & X & 0 \\
    0 & 0 & Y & 0 \\
    cZ^2 & 0 & Y & X
\end{bmatrix}, \quad c \in \mathbb{C} \setminus \{-1\}.
\]

Assume that \(c_{12} = 0, c_{21} \neq 0\). Then \(i = 0, j = -1, c_{11} = 0, c_{22} = 0, c_{31} = 0, c_{32} = 0\). We obtain the fixed points

\[
\varphi_2(c) = \begin{bmatrix}
    X & Y & 0 & 0 \\
    0 & 0 & X & 0 \\
    Z^2 & 0 & Y & 0 \\
    0 & cZ^2 & Y & X
\end{bmatrix}, \quad c \in \mathbb{C} \setminus \{-1\}.
\]

Assume that \(c_{12} = 0, c_{21} = 0, c_{11} \neq 0\). Then \(i = -2, c_{22} = 0, c_{31} = 0, c_{32} = 0,\) and we get the fixed points

\[
\begin{bmatrix}
    X & Y & 0 & 0 \\
    q & 0 & X & 0 \\
    0 & 0 & Y & 0 \\
    0 & 0 & Y & X
\end{bmatrix}, \quad q \in \{Y^2, YZ, Z^2\}.
\]

Assume that \(c_{11} = 0, c_{12} = 0, c_{21} = 0, c_{22} \neq 0\). Then \(j = -2, c_{31} = 0, c_{32} = 0,\) and we get the fixed points

\[
\begin{bmatrix}
    X & Y & 0 & 0 \\
    0 & 0 & X & 0 \\
    0 & q & 0 & Y \\
    0 & 0 & Y & X
\end{bmatrix}, \quad q \in \{X^2, XZ, Z^2\}.
\]

Assume that \(c_{11} = 0, c_{12} = 0, c_{21} = 0, c_{22} = 0, c_{32} \neq 0\). Then \(j = -1, c_{31} = 0,\) and we obtain the fixed points

\[
\begin{bmatrix}
    X & Y & 0 & 0 \\
    0 & 0 & X & 0 \\
    0 & 0 & 0 & Y \\
    0 & q & Y & X
\end{bmatrix}, \quad q \in \{X^2, XZ, Z^2\}.
\]

Note that for \(q = Z^2\) we have the point \(\varphi_2(\infty)\). Assume, finally, that \(c_{11} = 0, c_{12} = 0, c_{21} = 0, c_{22} = 0, c_{32} = 0, c_{31} \neq 0\). We have the fixed points

\[
\begin{bmatrix}
    X & Y & 0 & 0 \\
    0 & 0 & X & 0 \\
    0 & 0 & 0 & Y \\
    q & 0 & Y & X
\end{bmatrix}, \quad q \in \{X^2, Y^2, Z^2, XY, XZ, YZ\}.
Note that for \( q = Z^2 \) we have the point \( \varphi_1(\infty) \). In conclusion, the set of fixed points for the action of \( T \) on \( M(X,Y,X^2,XY,Y^2) \) consists of thirteen isolated points and two affine lines, namely

\[
\{([\varphi_1(c)], c \in \mathbb{P}^1 \setminus \{-1\}) \cup ([\varphi_2(c)], c \in \mathbb{P}^1 \setminus \{-1\}) \}.
\]

2.4. Fixed points in \( M_2 \). Assume that \( \varphi \in W_2 \) gives a \( T \)-fixed point \( \mathcal{F} \) in \( M_{p_2}(5,3) \). Then it is easy to see that \( \varphi_{11} \) gives a fixed point in the Kronecker moduli space \( N(3,3,2) \). Given quadratic forms \( q_1, q_2, q_3 \), we denote by \( M(q_1, q_2, q_3) \) be the ideal generated by the maximal minors of \( \varphi \). Let \( \mathcal{F} \) be the image of \( M_{p_2}(5,3) \) of the set of morphisms \( \varphi \in W_2 \) such that \( (q_1, q_2, q_3) \) is the ideal generated by the maximal minors of \( \varphi_{11} \). Given a monomial \( d \) of degree 5, we denote by \( M(q_1, q_2, q_3, d) \) the subset given by the additional condition that \( \det(\varphi) \) is a multiple of \( d \).

As mentioned at the beginning of this section, the points in \( M_{20} \) are of the form \( \mathcal{O}_Q(Z)/(1) \). Clearly, \( \mathcal{O}_Q(Z)/(1) \) is fixed by \( T \) if and only if \( Q \) and \( Z \) are \( T \)-invariant, so we will concentrate on finding the fixed points \( \mathcal{F} \) in \( M_{21} \). As mentioned at the beginning of this section, \( \mathcal{F} \) is an extension of \( \mathcal{O}_L \) by \( \mathcal{O}_C/(1) \). Denote

\[
M_{L,C} = \mathbb{P}(\text{Ext}^1(\mathcal{O}_L, \mathcal{O}_C/(1)) \cap M_{21}.
\]

The common factor of the maximal minors of \( \varphi_{11} \) is a monomial because \( \varphi_{11} \) gives a fixed point in the Kronecker moduli space. Thus \( L \) is \( T \)-invariant, and the same is true of \( C \). We will assume that \( L \) is given by the equation \( X = 0 \), the other cases being analogous. Write \( d = \det(\varphi) = X^iY^jZ^k \) and fix \( f_1, f_2, f_3 \in S^3V^* \) satisfying the equation

\[
-f_1Y - f_2Z + f_3X = d/l = X^{i-1}Y^jZ^k.
\]

Consider the set \( U \) of morphisms of the form

\[
\begin{bmatrix}
X & 0 & Y \\
0 & X & Z \\
f_1 - qZ & f_2 + qY & f_3
\end{bmatrix}, \quad q = aY^2 + bYZ + cZ^2, \quad a, b, c \in \mathbb{C}.
\]

By the argument at \([5, Proposition 5.1]\), we can show that the map \( U \to M_{L,C} \) sending \( \varphi \) to \([\text{Coker}(\varphi)]\) is an isomorphism and the induced action of \( T \) on \( U \) via this isomorphism is given by \( (t, q) \mapsto t^{i-1}q^{i-1}t_2^{l-k}(tq) \). Choosing coordinates \((a, b, c)\) we identify \( U \) with \( \mathbb{A}^3 \). The induced action of \( T \) on \( \mathbb{A}^3 \) is given by

\[
t(a, b, c) = t^{i-1}(t_1^{i-1}t_2^{l-k}a, \ t_2^{i-1}t_2^{l-k}b, \ t_1^{i-1}t_2^{l-k}c).
\]

If \( i \neq 1 \), or if \((i,j,k) \notin \{(1,4,0), (1,0,4)\}\), we get a single isolated fixed point, namely \((0,0,0)\). If \((i,j,k) \in \{(1,3,1), (1,2,2), (1,1,3)\}\), then we get an affine line of fixed points. Summarising we obtain the following proposition.

**Proposition 2.4.1.** Assume that \( \varphi_{11} \) gives a \( T \)-fixed point in \( N(3,3,2) \). Let \( q_1, q_2, q_3 \) be the maximal minors of \( \varphi_{11} \). Then, for any monomial \( d \) of degree 5 belonging to the ideal \( (q_1, q_2, q_3) \), the set of fixed points for the action of \( T \) on \( M(q_1, q_2, q_3, d) \) has precisely one irreducible component, which is either a point or an affine line. We have a line in the following cases:

\[
\begin{array}{c|c}
(q_1, q_2, q_3) & d \\
\hline
(X^2, XY, XZ) & XY^2Z, XY^2Z^2, XYZ^2 \\
(XY, Y^2, YZ) & X^3YZ, X^2YZ^2, XYZ^3 \\
(XZ, YZ, Z^2) & X^3YZ, X^2YZ^2, XYZ^3
\end{array}
\]
2.5. **Fixed points in** \( M_3 \). Clearly, \( O_Q(-Y)(2) \) gives a T-fixed point in \( M_2: (5,3) \) if and only if \( Q \) and \( Y \) are T-invariant. Equivalently, the fixed points in \( M_3 \) are given by matrices \( \varphi \in W_3 \) that have monomial entries.

3. **The torus representation of the tangent spaces at the fixed points**

Let \( F = \text{Coker}(\varphi) \) give a T-fixed point in \( M_k \), where \( \varphi \in W_k \). Since \( W_k \to M_k \) is a geometric quotient map, the tangent space \( T_{[F]} M_k \) is the quotient of \( T_{\varphi} W_k \) by the tangent space \( T_{\varphi}(G_k \varphi) \) to the orbit of \( \varphi \). The latter is a T-invariant subspace because \( [F] \) is T-fixed. Thus, the list of weights for the action of \( T \) on \( T_{[F]} M_k \) is obtained by subtracting the list of weights for \( T_{\varphi}(G_k \varphi) \) from the list of weights for \( T_{\varphi} W_k \). These two lists can be determined as in [5 Section 6] provided that there is a morphism of groups

\[
(C^*)^3 \to G_k, \quad t \mapsto ((u(t), v(t)), \quad \text{such that} \quad t\varphi = v(t)u(t) \quad \text{for all} \quad t \in (C^*)^3
\]

and such that \( u(t) \) and \( v(t) \) are diagonal matrices. The diagonal entries \( u_i, v_i \) of \( u, v \) are characters of \((C^*)^3\). The existence of \( u \) and \( v \) is obvious when \( \varphi \) is \( \alpha, \beta \) or \( \gamma \) from [2.1]. In the sequel, for all other fixed points \([F]\) in \( M_2: (5,3) \) we will give \( \varphi \) for which \( u \) and \( v \) exist.

For convenience, we will use additive notation when we deal with characters of \((C^*)^3\) or of \( T \). Denote by \( x, y, z \) the standard basis for the lattice of characters of \((C^*)^3\). The characters of \( T \) are of the form \( ix + jy + kz, \ i, j, k \in \mathbb{Z}, \ i + j + k = 0 \).

We will use the following abbreviations:

\[
\Sigma^3 = \left\{ X^iY^jZ^k \mid i, j, k \geq 0, \ i + j + k = 5 \right\},
\]

\[
\chi_0 = \text{the trivial character of } (C^*)^3 \text{ or of } T,
\]

\[
\sigma^0 = \{x - y, \ y - x, \ x - z, \ z - x, \ y - z, \ z - y\},
\]

\[
\sigma^1 = \{ix + jy + kz, \ i, j, k \in \mathbb{Z}, \ i, j, k \geq 0, \ i + j + k = l\}, \quad l \geq 1,
\]

\[
\sigma^1_x = \sigma^1 \setminus (x + \sigma^{1-1}),
\]

\[
\sigma^1_y = \sigma^1 \setminus (y + \sigma^{1-1}),
\]

\[
\sigma^1_z = \sigma^1 \setminus (z + \sigma^{1-1}).
\]

We also adopt the following convention: whenever a monomial \( f = X^iY^jZ^k \) appears in a list of characters, it stands for the expression \( ix + jy + kz \).

3.1. **Fixed points in** \( M_0 \). According to [5 6.1.1], the action of \( T \) on \( T_{\varphi} W_0 \) is given by the formula

\[
(t, w) \mapsto v(t)^{-1}(tw)u(t)^{-1},
\]

where \( tw \) refers to the canonical action of \((C^*)^3\) on the symmetric powers of \( V^* \). Thus, the list of weights for the action of \( T \) on \( T_{\varphi} W_0 \) is represented by the array

\[
\begin{align*}
-v_1 - u_1 + \sigma^2 & \quad -v_1 - u_2 + \sigma^2 & \quad -v_1 - u_3 + \sigma^1 \\
-v_2 - u_1 + \sigma^2 & \quad -v_2 - u_2 + \sigma^2 & \quad -v_2 - u_3 + \sigma^1 \\
-v_3 - u_1 + \sigma^2 & \quad -v_3 - u_2 + \sigma^2 & \quad -v_3 - u_3 + \sigma^1
\end{align*}
\]

According to [5 6.1.3], the action of \( T \) on \( T_{\varphi}(G_0 \varphi) \), which is identified with the tangent space of \( G_0 \) at the neutral element, is given by the formula

\[
(t(A, B) = (u(t)(tA)u(t)^{-1}, \ v(t)^{-1}(tB)v(t)).
\]
It follows that the list of weights for the action of $T$ on $T_{\varphi}(G_0 \varphi)$ is represented by the array

$$\begin{align*}
\chi_0 & \quad u_1 - u_2 & \quad \chi_0 & \quad -v_1 + v_2 & \quad -v_1 + v_3 \\
\chi_0 & \quad u_2 - u_1 & \quad -v_2 + v_1 & \quad \chi_0 & \quad -v_2 + v_3 \\
\chi_0 & \quad u_3 - u_1 + \sigma^1 & \quad u_3 - u_2 + \sigma^1 & \quad -v_3 + v_1 & \quad -v_3 + v_2 & \quad \chi_0 \\
\end{align*}$$

Thus, the list of weights for the action of $T$ on $T_{\varphi}(G_0 \varphi)$ is represented by the array

$$\begin{bmatrix}
q_1 & 0 & X \\
0 & q_2 & Y \\
0 & 0 & Z \\
\end{bmatrix}$$

Assume now that

$$\varphi = \alpha(q_1, q_2)$$

We have

$$\begin{align*}
u_1 &= q_1 - x & v_1 &= x \\
u_2 &= q_2 - y & v_2 &= y \\
u_3 &= 0 & v_3 &= z \\
\end{align*}$$

Thus, the list of weights for the action of $T$ on $T_{\varphi}(G_0 \varphi)$ is represented by the following table:

$$\begin{align*}
-q_1 + (\sigma^2 \setminus \{q_1\}) & \quad -q_2 + (\sigma^2 \setminus \{q_2\}) \\
x - y - q_1 + (\sigma^2 \setminus \{q_2\}) & \quad y - x - q_2 + (\sigma^2 \setminus \{q_1\}) \\
x - z - q_1 + \sigma^2_x & \quad y - z - q_2 + \sigma^2_x \\
\end{align*}$$

Here $(q_1, q_2) \in (\sigma^2_x \times \sigma^2_y) \cup \{(2x, x + y), (x + y, 2y), (x + z, y + z)\}$. Assume next that

$$\varphi = \beta(q, l)$$

We have

$$\begin{align*}
u_1 &= q - x & v_1 &= x \\
u_2 &= l & v_2 &= y \\
u_3 &= 0 & v_3 &= z \\
\end{align*}$$

Thus, the list of weights for the action of $T$ on $T_{\varphi}(G_0 \varphi)$ is represented by the following array:

$$\begin{align*}
-q + (\sigma^2 \setminus \{q\}) & \quad -z - l + (\sigma^2 \setminus \{q\}) \\
-y + z - q + (\sigma^2 \setminus \{y + l\}) & \quad -x - l + (\sigma^2 \setminus \{x + l\}) \\
-x + z - q + \sigma^2_x & \quad -y - l + \sigma^2_y \\
\end{align*}$$

Here $q \in \sigma^2_x, l \in \sigma^4$. We next examine the case of the fixed surfaces. Assume that

$$\varphi = \gamma$$

Clearly, we have

$$\begin{align*}
u_1 &= x & v_1 &= x \\
u_2 &= y & v_2 &= y \\
u_3 &= 0 & v_3 &= z \\
\end{align*}$$

The list of weights for the action of $T$ on $T_{\varphi}(G_0 \varphi)$ is represented by the following array:

$$\begin{align*}
-2x + (\sigma^2 \setminus \{2x\}) & \quad -x - y + (\sigma^2 \setminus \{2x\}) & \quad -x - z + \sigma^2_x \\
-2y + (\sigma^2 \setminus \{2y\}) & \quad -x - y + (\sigma^2 \setminus \{2y\}) & \quad -y - z + \sigma^2_y \\
\end{align*}$$
Given $q_1$, $q_2$ and $d$ as in Table 1, Section 2.2 we consider morphisms of the form

$$\varphi = \delta(X,Y,q_1,q_2,d) = \begin{bmatrix} c_{11}d/q_2Y & c_{12}d/q_1Y & X \\ c_{21}d/q_2X & c_{22}d/q_1X & Y \\ q_1 & q_2 & 0 \end{bmatrix}. $$

We have

$$u_1 = d - x - y - q_2 \quad v_1 = x$$

$$u_2 = d - x - y - q_1 \quad v_2 = y$$

$$u_3 = 0 \quad v_3 = -d + q_1 + q_2 + x + y$$

The list of weights for the action of $T$ on $T_{\delta}M$ is obtained by removing an extra copy of $\chi_0$ from the following table:

$$\begin{array}{c c c c c}
-d + q_2 + y + \sigma_y^2 & -d + q_2 + x + (\sigma_y^2 \setminus \{q_1\}) \\
-d + q_1 + x + \sigma_y^2 & -d + q_1 + y + (\sigma_y^2 \setminus \{q_2\}) \\
d - q_1 - q_2 - x - y + z & -q_1 + (\sigma_y^2 \setminus \{q_1,q_2\}) \\
-x + z & -q_2 + (\sigma_y^2 \setminus \{q_1,q_2\}) \\
-y + z & \end{array}$$

It is known that $\dim(T_{\{\delta\}}M)_{\chi_0}$ equals the dimension of the irreducible component of $M^T$ that contains $[\mathcal{F}]$. Thus, counting how many times $\chi_0$ appears in the above list, gives us another approach for determining column four of Table 1 in Section 2.2. For instance, $\dim(T_{\{\delta\}}M)_{\chi_0} = 2$ for $\varphi = \delta(X,Y,XZ,YZ,X^2Y^2Z)$ or $\delta(X,Y,XZ,Z^2,X^2YZ^2)$ even though these points belong to irreducible components of $M^T_{\chi_0}$ of dimension 1. This shows that these points belong to the closure of irreducible components of $$(M \setminus M_{\chi_0})^T = (M_0 \setminus M_{\chi_0})^T$$

of dimension 2. For the other points in column four of Table 1 $\dim(T_{\{\delta\}}M)_{\chi_0} = 1$ even though they belong to isolated points of $M^T_{\chi_0}$; thus they belong to the closure of lines in $(M_0 \setminus M_{\chi_0})^T$.

All fixed points in $M_0$ can be obtained from the points $\alpha$, $\beta$, $\gamma$, $\delta$ above by a permutation of variables. We have thus found the $T$-representation for all tangent spaces at fixed points in $M_0$.

3.2. Fixed points in $M_1$. The list of weights for the action of $T$ on $T_{\varphi}W_1$ given by formula (3.1.1) is represented by the array

$$\begin{array}{c c c c c c c}
-v_1 - u_1 + \sigma^1 & -v_1 - u_2 + \sigma^1 \\
-v_2 - u_1 + \sigma^2 & -v_2 - u_2 + \sigma^2 & -v_2 - u_3 + \sigma^1 \\
-v_3 - u_1 + \sigma^2 & -v_3 - u_2 + \sigma^2 & -v_3 - u_3 + \sigma^1 & -v_3 - u_4 + \sigma^1 \\
-v_4 - u_1 + \sigma^2 & -v_4 - u_2 + \sigma^2 & -v_4 - u_3 + \sigma^1 & -v_4 - u_4 + \sigma^1 & \end{array}$$

The list of weights for the action of $T$ on $T_{\chi}G_1$ given by formula (3.1.2) is represented by the table

$$\begin{array}{c c c c c c c}
-v_2 + v_1 + \sigma^1 & \chi_0 & -v_2 + v_3 & -v_2 + v_4 \\
-v_3 + v_1 + \sigma^1 & -v_3 + v_2 & \chi_0 & -v_3 + v_4 \\
-v_4 + v_1 + \sigma^1 & -v_4 + v_2 & -v_4 + v_3 & \chi_0 \end{array}$$
weights for the action of $T \times N$ on the torus $T^3$.

Proposition 3.2.1. Let $\mathcal{N}_1$ be the normal space to $M_1$ at $[\mathcal{F}]$. The torus $T$ acts on $\mathcal{N}_1$ with weights $-v_1 - u_3, -v_1 - u_4$.

Proof. By an argument analogous to the argument at [8, Theorem 4.3.3], we can show that $M_0 \cup M_{01}$ is the good quotient of an open subset $W \subset W_1$ modulo $G_1$. Here $W$ is simply the set of injective morphisms that have semi-stable cokernel. The list of weights for the action of $T$ on $W_1$ given by formula (3.1.1) is the same as the list for $T \times G_1$, except that it contains $-v_1 - u_3$ and $-v_1 - u_4$ in the upper-right corner, accounting for the two normal directions to $W_1$ at $\varphi$. □

Recall from Proposition 2.3.1 that an irreducible component of $M_1$ is uniquely determined by $l_1, l_2, q_1, q_2, q_3$ and $d$. The ideal $(q_1, q_2, q_3)$ defines a $T$-invariant zero-dimensional scheme $Z$ of length 3 that is not contained in a line or is of the form $(lX, lY, lZ)$, $l \in \{X, Y, Z\}$. Assume firstly that $\varphi$ has the form

$$\epsilon(d) = \begin{bmatrix} X & 0 & 0 \\ c_{11}d/Y^2Z & c_{12}d/XYZ & X \\ c_{21}d/XYZ & c_{22}d/X^2Y & Y \\ c_{31}d/XYZ & c_{32}d/X^2Y & Z \end{bmatrix}$$

corresponding to the case when $Z$ consists of three distinct points, namely $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. We have

$$u_1 = d - y \quad v_1 = x + y - d$$
$$u_2 = d - x \quad v_2 = -y - z$$
$$u_3 = x + y + z \quad v_3 = -x - z$$
$$u_4 = x + y + z \quad v_4 = -x - y$$

The torus $T$ acts on both $C_{S_1}$ and $C_{S_2}$ with weight $-d + 2x + 2y + z$. The list of weights for the action of $T$ on $T_{[\epsilon(d)\Lambda]} M$ is obtained by removing an extra copy of
χ₀ from the following table:

|             | -d + 2x + z + σ₂ₓ | -d + x + y + z + σ₂ᵧ | -d + x + y + z + σ₂ᵧ |
|-------------|-------------------|----------------------|----------------------|
| d - 2x - 2y - z | -d + 2y + σ₂ₓ     | -d + x + y + z + σ₂ᵧ | -d + x + y + z + σ₂ᵧ |
| d - 2x - 2y - z | -d + x + y + z + σ₂ᵧ | -d + x + y + z + σ₂ᵧ | -d + x + y + z + σ₂ᵧ |
| -d + 2x + 2y + z | -d + x + y + z + σ₂ᵧ | -d + x + y + z + σ₂ᵧ | -d + x + y + z + σ₂ᵧ |
| -d + 4x + y    | -d + x + 2y + σ₂ᵧ | -d + x + 2y + σ₂ᵧ   | -d + x + 2y + σ₂ᵧ   |
| z - x          | -d + 2x + z + σ₂ᵧ | -d + 2x + z + σ₂ᵧ   | -d + 2x + z + σ₂ᵧ   |
| z - y          | σ₀                | σ₀                  | σ₀                  |

Consider next morphisms of the form

\[ ζ(l₁, l₂, d) = \begin{bmatrix}
   l₁ & l₂ & 0 & 0 \\
   c₁₁ d/l₂ X² & c₁₂ d/l₁ X² & Y & 0 \\
   c₂₁ d/l₂ YZ & c₂₂ d/l₁ YZ & 0 & X \\
   c₃₁ d/l₂ XY & c₃₂ d/l₁ XY & X & Z
\end{bmatrix} \]

corresponding to the case when \( Z \) is the union of a double point and a simple point. We have

\[
\begin{align*}
  u₁ &= d - l₂ \\
  u₂ &= d - l₁ \\
  u₃ &= 2x + y \\
  u₄ &= x + y + z
\end{align*}
\]

The torus \( T \) acts on \( \mathbb{C}s₁ \) with weight \( 2x + y - d + l₁ + l₂ \) and on \( \mathbb{C}s₂ \) with weight \( x + y + z - d + l₁ + l₂ \). The list of weights for the action of \( T \) on \( T_{[ζ]} M \) is obtained by subtracting the list \( \{χ₀, -d + l₁ + l₂ + x + 2y\} \) from the list

\[
\begin{align*}
  &d - l₁ - l₂ - 2x - y & -d + l₂ + 2x + σ₁₁ & x - y \\
  &d - l₁ - l₂ - x - y - z & -d + l₁ + y + z + σ₁₂ & y - x \\
  &l₃ - l₁ & -d + l₂ + y + z + σ₂ₓ & z - x \\
  &l₃ - l₂ & -d + l₁ + x + y + σ₂ᵧ & 2z - 2x \\
  &     & -d + l₁ + 2x + σ₂ᵧ & y - z \\
  &     & -d + l₁ + l₂ + x + y + σ₂ᵧ & z - y
\end{align*}
\]

We next examine the case when \( Z \) is a triple point. We may assume that \( Z \) is supported on \((0,0,1)\), the other cases being obtained by a permutation of variables. Consider thus morphisms of the form

\[ η(l₁, l₂, d) = \begin{bmatrix}
   l₁ & l₂ & 0 & 0 \\
   c₁₁ d/l₂ Y² & c₁₂ d/l₁ Y² & X & 0 \\
   c₂₁ d/l₂ X² & c₂₂ d/l₁ X² & 0 & Y \\
   c₃₁ d/l₂ XY & c₃₂ d/l₁ XY & Y & X
\end{bmatrix} \]

We have

\[
\begin{align*}
  u₁ &= d - l₂ \\
  u₂ &= d - l₁ \\
  u₃ &= x + 2y \\
  u₄ &= 2x + y
\end{align*}
\]

The torus \( T \) acts on \( \mathbb{C}s₁ \) with weight \( x + 2y - d + l₁ + l₂ \) and on \( \mathbb{C}s₂ \) with weight \( 2x + y - d + l₁ + l₂ \). The list of weights for the action of \( T \) on \( T_{[η]} M \) is obtained
by subtracting the list \(\{\chi_0, -d + l_1 + l_2 + x + y + z\}\) from the list
\[
\begin{align*}
&d - l_1 - l_2 - x - 2y & -d + l_2 + 2y + \sigma_y^2 & z - x \\
&d - l_1 - l_2 - 2x - y & -d + l_1 + 2x + \sigma_z^2 & z - x \\
l_3 - l_1 & -d + l_2 + x + y + \sigma_y^2 & z - y \\
l_3 - l_2 & -d + l_1 + x + y + \sigma_z^2 & x - 2y + z \\
& & -d + l_1 + 2y + \sigma_x^2 & y - 2x + z
\end{align*}
\]

Finally, we assume that \((q_1, q_2, q_3) = (X^2, XY, XZ)\), that is, we consider morphisms of the form
\[
\theta(l_1, l_2, d) = \begin{bmatrix}
l_1 & l_2 & 0 & 0 \\
c_{11}d/l_2XY & c_{12}d/l_1XY & X & 0 \\
c_{21}d/l_2XZ & c_{22}d/l_1XZ & 0 & X \\
c_{31}d/l_2X^2 & c_{32}d/l_1X^2 & Y & Z
\end{bmatrix}.
\]

We have
\[
\begin{align*}
u_1 &= d - l_2 & v_1 &= -d + l_1 + l_2 \\
u_2 &= d - l_1 & v_2 &= -x - y \\
u_3 &= 2x + y & v_3 &= -x - z \\
u_4 &= 2x + z & v_4 &= -2x
\end{align*}
\]

The torus \(T\) acts on \(\mathbb{C}s_1\) with weight \(2x + y - d + l_1 + l_2\) and on \(\mathbb{C}s_2\) with weight \(2x + z - d + l_1 + l_2\). The list of weights for the action of \(T\) on \(T_{[0]} M\) is obtained by subtracting the list \(\{\chi_0, -d + l_1 + l_2 + 3x\}\) from the list
\[
\begin{align*}
&d - l_1 - l_2 - 2x - y & -d + l_2 + x + y + \sigma_z^2 & y - x \\
&d - l_1 - l_2 - 2x - z & -d + l_1 + x + y + \sigma_z^2 & z - x \\
l_3 - l_1 & -d + l_2 + x + z + \sigma_{l_1}^2 & z - x \\
l_3 - l_2 & -d + l_1 + 2x + \sigma_z^2 & 2z - x - y \\
& & -d + l_1 + x + z + \sigma_x^2 & 2y - x - z
\end{align*}
\]

All fixed points in \(M_1\) can be obtained from the points \(\varepsilon, \zeta, \eta, \theta\) above by a permutation of variables. We have thus found the \(T\)-representation for all tangent spaces at fixed points in \(M_1\). To complete the picture, we need to determine which points of \(M_1^\perp\) lie in the closure of
\[
(M \setminus \overline{M_1})^T = M_1^{\perp}.
\]

This can be done using Proposition \(3.2\). For instance, \(T\) acts trivially on \(N_{[\varphi]}\) for \(\varphi = \varepsilon(X^2Y^2Z)\). This shows that \(\{\varepsilon(X^2Y^2Z)\}\) belongs to the closure of a surface in \(M_1^\perp\). There are only three such surfaces: \(S\) and two others obtained by interchanging \(X\) and \(Z\), respectively, \(Y\) and \(Z\). The sheaves giving points in \(S\) have support given by the equation \((X^2Y^2Z = 0)\). We deduce that \(\{\varepsilon(X^2Y^2Z)\}\) belongs to \(S\) and that \(\{\varepsilon(X^2Y^2Z)\}, \{\varepsilon(X^2Y^2Z)\}\) belong to the other two surfaces. For the remaining points in column five of Table 2 below, we have \(\dim N_{\chi_0} = 1\), which shows that they belong to the closure of affine lines in \(M_1^\perp\).
Table 2. Fixed points in $M_1$.

| $(q_1, q_2, q_3)$ | $(l_1, l_2)$ | equation of support | affine lines | limit sheaves |
|------------------|----------------|---------------------|-------------|---------------|
| $(XY, XZ, YZ)$   | $(X, Y)$       | $\Sigma^5 \{X^5, Y^5, Z^5, XZ^4, YZ^4\}$ | $X^2Y^2Z$   | $X^2Y^2Z$     |
|                  | $(X, Z)$       | $\Sigma^5 \{X^5, Y^5, Z^5, XZ, YZ\}$   | $X^3Y^2Z$   | $X^3Y^2Z$     |
|                  | $(Y, Z)$       | $\Sigma^5 \{X^5, Y^5, Z^5, X^2Y, X^2Z\}$ | $X^3Y^2Z$   | $X^3Y^2Z$     |
| $(X^2, XY, YZ)$  | $(X, Y)$       | $\Sigma^5 \{X^5, Y^5, Z^5, X^2Z^3, Y^2X, Z^3XY, Z^3Y^2\}$ | $X^2Y^2Z$   | $X^2Y^2Z$     |
|                  | $(X, Z)$       | $\Sigma^5 \{X^5, Y^5, Z^5, XZ, YZ\}$   | $X^2Y^2Z$   | $X^2Y^2Z$     |
|                  | $(Y, Z)$       | $\Sigma^5 \{X^5, Y^5, Z^5, XZ, YZ\}$   | $X^2Y^2Z$   | $X^2Y^2Z$     |
| $(X^2, XY, XZ)$  | $(X, Y)$       | $\Sigma^5 \{X^5, Y^5, Z^5, XZ, YZ\}$   | $X^3Y^2Z$   | $X^3Y^2Z$     |
|                  | $(X, Z)$       | $\Sigma^5 \{X^5, Y^5, Z^5, XZ, YZ\}$   | $X^3Y^2Z$   | $X^3Y^2Z$     |
|                  | $(Y, Z)$       | $\Sigma^5 \{X^5, Y^5, Z^5, XZ, YZ\}$   | $X^3Y^2Z$   | $X^3Y^2Z$     |

3.3. Fixed points in $M_2$. The list of weights for the action of $T$ on $T_{\varphi} W_2$ given by formula (3.1.1) is represented by the array

$-v_1 - u_1 + \sigma^1$ $-v_1 - u_2 + \sigma^1$ $-v_1 - u_3 + \sigma^1$
$-v_2 - u_1 + \sigma^1$ $-v_2 - u_2 + \sigma^1$ $-v_2 - u_3 + \sigma^1$
$-v_3 - u_1 + \sigma^3$ $-v_3 - u_2 + \sigma^3$ $-v_3 - u_3 + \sigma^3$

The list of weights for the action of $T$ on $T_{\varphi}(G_{2q})$ given by formula (3.1.2) is represented by the array

$\chi_0$ $u_1 - u_2$ $u_1 - u_3$ $\chi_0$ $-v_1 + v_2$
$u_2 - u_1$ $\chi_0$ $u_2 - u_3$ $-v_2 + v_1$ $\chi_0$
$u_3 - u_1$ $u_3 - u_2$ $\chi_0$ $-v_3 + v_1 + \sigma^2$ $-v_3 + v_2 + \sigma^2$

Proposition 3.3.1. The normal space $N_{[F]}$ to $M_2$ at $[F]$ can be identified with $H^0(F(-1))^* \otimes H^1(F(-1))$.

The torus $T$ acts on $N_{[F]}$ with weights $u_i + v_3 - x - y - z$, $i = 1, 2, 3$.

Proof. We apply [5, Proposition 6.2] to the sheaf $F^0(1)$, which gives a point in $M_{p_2}(5, 2)$. We have a resolution

$0 \rightarrow O(-3) \oplus 2O(-1) \xrightarrow{\varphi} 3O \rightarrow F^0(1) \rightarrow 0$. 
Let $N^D$ denote the normal space to $M_2^T \subset M_2(5,2)$ at $[\mathcal{F}_2^0(1)]$. As $\mathbb{T}$-modules, $N$ and $N^D$ are isomorphic. This proves the second part of the proposition. Moreover,

$$N \simeq N^D \simeq H^0(\mathcal{F}_2^0(1))^* \otimes H^1(\mathcal{F}_2^0(1)) \quad \text{by \cite{5} Proposition 6.2}$$

$$\simeq H^1(\mathcal{F}(-1)) \otimes H^2(\omega_{P^2})^* \otimes H^0(\mathcal{F}(-1))^* \otimes H^2(\omega_{P^2}) \quad \text{by Serre Duality}$$

$$\simeq H^1(\mathcal{F}(-1)) \otimes H^0(\mathcal{F}(-1))^*. \quad \square$$

Recall from Proposition 2.4.1 that an irreducible component of $M_2^T$ is uniquely determined by $q_1, q_2, q_3$ and $d$. The ideal $(q_1, q_2, q_3)$ defines a $\mathbb{T}$-invariant zero-dimensional scheme $Z$ of length 3 that is not contained in a line or is of the form $(lX, lY, lZ), l \in \{X, Y, Z\}$. Assume firstly that $\varphi$ has the form

$$\iota(d) = \begin{bmatrix} X & Y & 0 \\ X & 0 & Z \\ c_1d/YZ & c_2d/XZ & c_3d/XY \end{bmatrix}$$

corresponding to the case when $Z$ consists of three distinct points. Clearly, we have

$$u_1 = x \quad v_1 = 0$$
$$u_2 = y \quad v_2 = 0$$
$$u_3 = z \quad v_3 = d - x - y - z.$$ 

The list of weights for the action of $\mathbb{T}$ on $T_{\iota(d)} M$ is obtained by removing an extra copy of $\chi_0$ from the table

$$\sigma^0 \quad d - x - 2y - 2z \quad -d + y + z + \sigma^3$$
$$d - 2x - y - 2z \quad -d + x + z + \sigma^3$$
$$d - 2x - 2y - z \quad -d + x + y + \sigma^3.$$ 

We next examine the case when $Z$ is the union of a double point and a simple point. Consider thus morphisms of the form

$$\kappa(d) = \begin{bmatrix} Y & 0 & X \\ 0 & X & Z \\ c_1d/X^2 & c_2d/YZ & c_3d/XY \end{bmatrix}.$$ 

We have

$$u_1 = y - x \quad v_1 = x$$
$$u_2 = x - z \quad v_2 = z$$
$$u_3 = 0 \quad v_3 = d - x - y.$$ 

The list of weights for the action of $\mathbb{T}$ on $T_{\kappa(d)} M$ is obtained by removing an extra copy of $\chi_0$ from the array

$$x - y \quad y - z \quad d - 3x - y - z \quad -d + y + z + \sigma^3$$
$$y - x \quad z - y \quad d - x - 2y - 2z \quad -d + 2x + \sigma^3$$
$$z - x \quad 2z - 2x \quad d - 2x - 2y - z \quad -d + x + y + \sigma^3.$$ 

We assume now that $Z$ is a triple point supported at $(0, 0, 1)$, that is, we consider morphisms of the form

$$\lambda(d) = \begin{bmatrix} X & 0 & Y \\ 0 & Y & X \\ c_1d/Y^2 & c_2d/X^2 & c_3d/XY \end{bmatrix}.$$
We have
\[
\begin{align*}
  u_1 &= x - y & v_1 &= y \\
  u_2 &= y - x & v_2 &= x \\
  u_3 &= 0 & v_3 &= d - x - y
\end{align*}
\]

The list of weights for the action of \(T\) on \(T_{[\lambda(d)]} M\) is obtained by removing an extra copy of \(\chi_0\) from the array
\[
\begin{align*}
  z - x & & d - x - y - z & & -d + 2x + \sigma^3_y \\
  z - x & & d - 3x - y - z & & -d + 2y + \sigma^3_x \\
  z - 2x + y & & d - 2x - 2y - z & & -d + 2y + \sigma^3_x \\
  z + x - 2y & & d - 2x - 2y - z & & -d + 2y + \sigma^3_x
\end{align*}
\]

Finally, we assume that \((q_1, q_2, q_3) = (X^2, XY, XZ)\), that is, we consider morphisms of the form
\[
\mu(d) = \begin{bmatrix}
  X & 0 & Y \\
  0 & X & Z \\
  c_1 d/XY & c_2 d/XZ & c_3 d/X^2
\end{bmatrix}.
\]

We have
\[
\begin{align*}
  u_1 &= x - y & v_1 &= y \\
  u_2 &= x - z & v_2 &= z \\
  u_3 &= 0 & v_3 &= d - 2x
\end{align*}
\]

The list of weights for the action of \(T\) on \(T_{[\mu(d)]} M\) is obtained by removing an extra copy of \(\chi_0\) from the table
\[
\begin{align*}
  z - x & & y - x & & d - 2x - 2y - z & & -d + x + y + \sigma^3_x \\
  z - x & & y - x & & d - 2x - 2y - z & & -d + x + y + \sigma^3_x \\
  2z - x - y & & 2y - x - z & & d - 3x - y - z & & -d + x + z + \sigma^3_y
\end{align*}
\]

All fixed points in \(M_2\) can be obtained from the points \(\iota, \kappa, \lambda, \mu\) above by a permutation of variables. We have thus found the \(T\)-representation for all tangent spaces at fixed points in \(M_2\). To complete the picture, we need to determine which points of \(M_2^T\) lie in the closure of \((M \setminus M_2)^T = (M_0 \cup M_1)^T\).

This can be done using Proposition 3.3.1. For all points in column four of Table 3 below, we have \(\dim N_{X_0} = 1\). This shows that these points lie in the closure of affine lines in \((M_0 \cup M_1)^T\).
3.4. Fixed points in $M_3$. The list of weights for the action of $T$ on $T_\phi W_3$ given by formula (3.1.1) is represented by the array

$$-v_1 - u_1 + \sigma^3 \quad -v_1 - u_2 + \sigma^1 \quad -v_2 - u_1 + \sigma^4 \quad -v_2 - u_2 + \sigma^2.$$ 

The list of weights for the action of $T$ on $T_\phi (G_3 \phi)$ given by formula (3.1.2) is represented by the array

$$\chi_0 \quad u_2 - u_1 + \sigma^2 \quad -v_2 + v_1 \quad \chi_0.$$ 

**Proposition 3.4.1.** The normal space $N_{[\mathcal{F}]}$ to $M_3$ at $[\mathcal{F}]$ can be identified with

$$H^0(\mathcal{F})^* \otimes H^1(\mathcal{F}).$$

The torus $T$ acts on $N_{[\mathcal{F}]}$ with weights

$$u_1 + v_1 - x - y - z, \quad u_1 + v_2 - 2x - y - z, \quad u_1 + v_2 - x - 2y - z, \quad u_1 + v_2 - x - y - 2z.$$ 

**Proof.** The argument is analogous to the argument at [5, Proposition 6.2]. Let $\kappa : H^0(\mathcal{F}) \otimes \mathcal{O} \to \mathcal{F}$ be the canonical morphism and let $\mathcal{K} = \text{Ker}(\kappa)$. Applying the snake lemma to the diagram

$$\begin{array}{ccc}
0 & \to & \Omega^1(1) \\
\downarrow & & \downarrow \\
\mathcal{K} & \to & H^0(\mathcal{F}) \otimes \mathcal{O} \\
\downarrow & & \downarrow \kappa \\
0 & \to & \mathcal{O}(-3) \oplus \mathcal{O}(-1) \\
\downarrow & \phi & \downarrow \\
0 & \to & \mathcal{O} \oplus \mathcal{O}(1) \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{F} \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}$$

we get the exact sequence

$$0 \to \Omega^1(1) \to \mathcal{K} \to \mathcal{O}(-3) \oplus \mathcal{O}(-1) \to 0.$$ 

We claim that $\text{Ext}^1(\mathcal{K}, \mathcal{F}) = 0$. This follows from the long $\text{Ext}(\_, \mathcal{F})$-sequence associated to the above short exact sequence and from the vanishing of

$$\text{Ext}^1(\mathcal{O}(-3), \mathcal{F}), \quad \text{Ext}^1(\mathcal{O}(-1), \mathcal{F}), \quad \text{Ext}^1(\Omega^1(1), \mathcal{F}).$$

To see that the last group vanishes we apply the long $\text{Ext}(\Omega^1(1), \_)$-sequence to the short exact sequence expressing $\mathcal{F}$ as the cokernel of $\phi$ and we use the vanishing of

$$\text{Ext}^1(\Omega^1(1), \mathcal{O}), \quad \text{Ext}^1(\Omega^1(1), \mathcal{O}(1)), \quad \text{Ext}^2(\Omega^1(1), \mathcal{O}(-3)), \quad \text{Ext}^2(\Omega^1(1), \mathcal{O}(-1)).$$

Applying the long $\text{Ext}(\_, \mathcal{F})$-sequence to the exact sequence

$$0 \to \mathcal{K} \to H^0(\mathcal{F}) \otimes \mathcal{O} \to \mathcal{F} \to 0$$

we get the exact sequence

$$0 \to \Omega^1(1) \to \mathcal{K} \to \mathcal{O}(-3) \oplus \mathcal{O}(-1) \to 0.$$
we obtain a surjective map $\epsilon$: $\text{Ext}^1(\mathcal{F}, \mathcal{F}) \to H^0(\mathcal{F})^* \otimes H^1(\mathcal{F})$. Consider an extension sheaf

$$0 \to \mathcal{F} \to \mathcal{E} \to \pi \to 0.$$  

Assume that the extension class of $\mathcal{E}$ belongs to $\text{Ker}(\epsilon)$. Then, as in [5 Proposition 6.2], we deduce that $H^0(\pi)$ has a splitting. Thus, $h^0(\mathcal{E}) = 8$ and, from Lemma 3.4.2 below, we have $h^0(\mathcal{E}(-1)) = 2$. It becomes clear now that we can apply the horseshoe lemma to the above extension in order to obtain a resolution of the form

$$0 \to (\mathcal{O}(-3) \oplus \mathcal{O}(-1)) \oplus (\mathcal{O}(-3) \oplus \mathcal{O}(-1)) \oplus (\mathcal{O} \oplus \mathcal{O}(1)) \to \mathcal{E} \to 0,$$

$$\psi = \left[ \begin{array}{cc} \varphi & w \\ 0 & \varphi \end{array} \right].$$

Thus, $\mathcal{E}$ gives a tangent vector to $M_3$, namely the image of $w$ in $W_3/T_eG_3 \simeq T_{[\mathcal{F}]} M_3$. Since $\text{Ker}(\epsilon)$ is contained in $T_{[\mathcal{F}]} M_3$ and both spaces have dimension $22$, we conclude that $\text{Ker}(\epsilon) = T_{[\mathcal{F}]} M_3$. This proves the first part of the proposition.

To determine the action of $\mathcal{T}$ on $N$ consider the commutative diagram

$$\begin{array}{cccccc}
0 & \to & \mathcal{O}(-3) \oplus \mathcal{O}(-1) & \xrightarrow{\psi} & \mathcal{O} \oplus \mathcal{O}(1) & \to & \mathcal{F} & \to & 0 \\
\downarrow{u(t)^{-1}} & & \downarrow{v(t)} & & \downarrow{u(t)} & & & & \\
0 & \to & \mathcal{O}(-3) \oplus \mathcal{O}(-1) & \xrightarrow{t\psi} & \mathcal{O} \oplus \mathcal{O}(1) & \to & t\mathcal{F} & \to & 0
\end{array}$$

The action of $\mathcal{T}$ on $H^0(\mathcal{F})$ is given by $ts = u(t)^{-1}\mu^*_t(s)$. Under the identification $\mathbb{C} \oplus V^* \simeq H^0(\mathcal{F})$, we have $ts = v(t)^{-1}\mu^*_t(s)$. Thus, $\mathcal{T}$ acts on $H^0(\mathcal{F})$ with weights $-v_1, -v_2 + x, -v_2 + y, -v_2 + z$. As in [5 Proposition 6.2], $\mathcal{T}$ acts on $H^1(\mathcal{F})$ with weight $u_1 - x - y - z$. $\square$

**Lemma 3.4.2.** Assume that $\mathcal{F}$ gives a point in $M_3$. Let $\mathcal{E}$ be an extension of $\mathcal{F}$ by $\mathcal{F}$. If $h^0(\mathcal{E}) = 8$, then $h^0(\mathcal{E}(-1)) = 2$.

**Proof.** Assume, instead, that $h^0(\mathcal{E}(-1)) = 1$. Denote $m = h^1(\mathcal{E} \otimes \Omega^1(1))$. The $E^1$-level of the Beilinson spectral sequence [8 (2.2.3)] converging to $\mathcal{E}$ reads

$$\begin{array}{cccc}
5\mathcal{O}(-2) & \xrightarrow{\varphi_1} & m\mathcal{O}(-1) & \xrightarrow{\varphi_2} & 2\mathcal{O}. \\
\mathcal{O}(-2) & \xrightarrow{\varphi_3} & (m + 2)\mathcal{O}(-1) & \xrightarrow{\varphi_4} & 8\mathcal{O}
\end{array}$$

Note that $m \geq 5$ because $\varphi_2$ is surjective. Note that $m \leq 7$ because $\mathcal{E}$ maps surjectively to $\text{Ker}(\varphi_2)/\text{Im}(\varphi_1)$. If $m = 7$, then the first row above is a monad, so its cohomology has slope $-2/3$, destabilising $\mathcal{E}$. Thus, $m = 5$ or 6.

Assume, firstly, that $m = 5$. Denote $\mathcal{G} = \mathcal{E}^0(1)$. The Beilinson free monad [8 (2.2.1)] yields a resolution

$$0 \to 2\mathcal{O}(-2) \xrightarrow{\psi} 8\mathcal{O}(-2) \oplus 5\mathcal{O}(-1) \xrightarrow{\varphi} \Omega^1 \oplus 4\mathcal{O}(-1) \oplus 5\mathcal{O} \to \mathcal{G} \to 0$$

in which $\varphi_{12} = 0$, $\varphi_{22} = 0$. The map $5\mathcal{O} \to \mathcal{G}$ is an isomorphism on global sections and $\mathcal{G}$ has no zero-dimensional torsion. This shows, as in the proof of [11]
Proposition 3.1.3], that \( \psi_{21} \) has one of the following canonical forms:

\[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
x & 0 \\
y & s \\
z & t \\
\end{bmatrix},
\begin{bmatrix}
0 & 0 \\
x & 0 \\
y & 0 \\
z & r \\
0 & t \\
\end{bmatrix},
\begin{bmatrix}
x & 0 \\
0 & 0 \\
y & 0 \\
z & r \\
0 & t \\
\end{bmatrix}.
\]

Here \( \{R, S, T\} \) is a basis of \( V^* \). At [11 Proposition 3.1.3] it is shown how each of these forms leads to a contradiction.

Assume next that \( m = 6 \). The Beilinson free monad [8 (2.2.1)] yields the resolution

\[
0 \rightarrow O(-2) \rightarrow 5O(-2) \oplus 8O(-1) \rightarrow 2O^1 \oplus 8O \rightarrow E \rightarrow 0,
\]

hence the resolution

\[
0 \rightarrow O(-2) \rightarrow 2O(-3) \oplus 5O(-2) \oplus 8O(-1) \xrightarrow{\psi} 6O(-2) \oplus 8O \rightarrow E \rightarrow 0.
\]

As in the argument at [11 Proposition 2.1.4], the rank of \( \varphi_{12} \) is maximal, otherwise \( E \) would map surjectively to the cokernel of a morphism \( 2O(-3) \rightarrow 2O(-2) \), violating semi-stability. We arrive at the exact sequence

\[
0 \rightarrow O(-2) \xrightarrow{\psi} 2O(-3) \oplus 8O(-1) \rightarrow O(-2) \oplus 8O \rightarrow E \rightarrow 0.
\]

The map \( 8O \rightarrow E \) is an isomorphism on global sections. It follows, as in the argument at [11 Proposition 2.1.4], that \( \text{Coker}(\varphi_{12}) \simeq 5O(-1) \oplus O^1(1) \). Combining the resolution

\[
0 \rightarrow 2O(-3) \oplus 5O(-1) \oplus O^1(1) \rightarrow O(-2) \oplus 8O \rightarrow E \rightarrow 0
\]

with the Euler sequence we obtain the resolution

\[
0 \rightarrow 2O(-3) \oplus 5O(-1) \oplus 3O \xrightarrow{\varphi} O(-2) \oplus 8O \oplus O(1) \rightarrow E \rightarrow 0.
\]

As before, \( \varphi_{23} \) has maximal rank, otherwise \( E \) would map surjectively to the cokernel of a morphism \( 2O(-3) \oplus 5O(-1) \rightarrow O(-2) \oplus 6O \), contradicting semi-stability. Canceling \( 3O \) we obtain a resolution of \( E \) that fits into a commutative diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & O(-3) & \oplus & O(-1) & \rightarrow & O & \oplus & O(1) & \rightarrow & F & \rightarrow & 0 \\
& & \downarrow{\beta} & & \downarrow{\alpha} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & 2O(-3) & \oplus & 5O(-1) & \rightarrow & O(-2) & \oplus & 5O & \oplus & O(1) & \rightarrow & E & \rightarrow & 0
\end{array}
\]

The map \( F \rightarrow E \) above is the inclusion given in the hypothesis of the lemma, so its cokernel is \( F \). From the snake lemma we obtain the exact sequence

\[
0 \rightarrow \text{Coker}(\beta) \rightarrow \text{Coker}(\alpha) \rightarrow F \rightarrow 0.
\]

Since \( \alpha(-1) \) and \( \alpha \) are injective on global sections, we see that \( \alpha \) is injective, hence \( \text{Coker}(\alpha) \simeq O(-2) \oplus 4O \). From the above exact sequence we see that \( h^1(\text{Coker}(\beta)(-1)) = 1 \). On the other hand, if \( \beta_{11} \neq 0 \), then \( h^1(\text{Coker}(\beta)(-1)) = 0 \). If \( \beta_{11} = 0 \), then \( h^1(\text{Coker}(\beta)(-1)) = 3 \). We have arrived at a contradiction. Our original assumption that \( h^0(E(-1)) = 1 \) must be wrong. This proves the lemma. \( \Box \)
Recall from Section 2.5 that $M^T_d$ consists of isolated points of the form $O_Q(-Y)^{(2)}$. They are cokernels of morphisms in $W_3$ of the form

$$v(l, q, d) = \begin{bmatrix} c_1 d/q & l \\ c_2 d/l & q \end{bmatrix},$$

where $l \in \{X, Y, Z\}$, $q$ is a quadratic monomial, $l$ does not divide $q$, $d$ is a monomial of degree 5 in the ideal $(l, q)$, and $c_1, c_2 \in \mathbb{C}$. We have

$$u_1 = d - q - l \quad \quad v_1 = l$$
$$u_2 = 0 \quad \quad v_2 = q.$$

All fixed points in $M_3$ can be obtained from the points $v(X, q, d)$ by swapping $X$ and $Y$, or by swapping $X$ and $Z$. Thus, in order to describe the $T$-representation for all tangent spaces at fixed points in $M_3$, it is enough to give the list of weights for the action of $T$ on $T_{v(X,q,d)} M$. This is

$$d - q - x - y - z \quad -d + x + (\sigma^4 \setminus \{d - x\}) \quad -x + y$$
$$d - 3x - y - z \quad -d + q + \sigma^4_x \quad -x + z$$
$$d - 2x - 2y - z \quad -q + (\sigma^4_x \setminus \{q\}) \quad .$$

To give a complete picture for the torus fixed locus in $M_3$ we need to determine which points of $M^T_d$ lie in the closure of

$$(M \setminus M_3)^T = (M_0 \cup M_1 \cup M_2)^T.$$

This can be done using Proposition 3.4.1. For all points represented in column 4 of Table 4 below we have $\dim N_{X_0} = 1$. This shows that these points lie in the closure of affine lines in $(M_0 \cup M_1 \cup M_2)^T$.

| $l$ | $q$ | $d$ | limit sheaves |
|-----|-----|-----|--------------|
| $X$ | $Y^2$ | $\Sigma \setminus \{Z^2, YZ\}$ | $Y^2Z, X^2YZ, X^2Z, XYZ^2$ |
| $X$ | $Y^2$ | $\Sigma \setminus \{Y^2, YZ\}$ | $Y^2Z, X^2YZ, X^2Z, XYZ^2$ |
| $X$ | $Y^2$ | $\Sigma \setminus \{Y^2, Z^2\}$ | $Y^2Z, X^2YZ, X^2Z, XYZ^2$ |
| $Y$ | $X^2$ | $\Sigma \setminus \{Z^2, XZ\}$ | $X^2Z, X^2YZ, X^2Z, XYZ^2$ |
| $Z$ | $X^2$ | $\Sigma \setminus \{X^2, XZ\}$ | $X^2Z, X^2YZ, X^2Z, XYZ^2$ |
| $Z$ | $X^2$ | $\Sigma \setminus \{X^2, Z^2\}$ | $X^2Z, X^2YZ, X^2Z, XYZ^2$ |
| $Z$ | $X^2$ | $\Sigma \setminus \{X^2, Y^2\}$ | $X^2Y, X^2YZ, X^2Z, XYZ^2$ |
| $Z$ | $X^2$ | $\Sigma \setminus \{X^2, Y^2\}$ | $X^2Y, X^2YZ, X^2Z, XYZ^2$ |

3.5. Proof of the main theorem. We are now ready to prove the theorem announced in the introduction. We first recall some general facts. It is known that every irreducible component $X$ of $M^T_d$ is a smooth subvariety of $M$. The irreducible components of $M^T_d$ are disjoint. For every $x \in X$ we have $\dim(T_x M)|_{X_0} = \dim X$. Given points $x, y \in X$, the tangent spaces $T_x M$ and $T_y M$ are isomorphic as $T$-modules. Inspecting Tables 2, 3, and 4, we see that the only fixed points $[q]$ outside $M_0$ for which $\dim(T_{[q]} M)|_{X_0} = 2$ are $[\epsilon(X^2Y^2Z)], [\epsilon(XY^2Z^2)]$, and $[\epsilon(X^2YZ)^2)]$. Therefore, these are the only points of $M \setminus M_0$ that belong to an irreducible component of $M^T_d$ of dimension 2. All other points in $(M \setminus M_0)^T$ belong to an irreducible component of the fixed locus of dimension 0 or 1. In view of Section 2.2.2 and
of the comments at the end of Section 3.2 we conclude that there are only three irreducible components of \( M^1 \) of dimension greater than 1, namely \( S \) and two other surfaces obtained from \( S \) by interchanging \( X \) and \( Z \), respectively, by interchanging \( Y \) and \( Z \). As mentioned at Section 3.2 \( S \) can be obtained from \( \mathbb{P}^1 \times \mathbb{P}^1 \) by blowing up three points on the diagonal and then blowing down the strict transform of the diagonal. The Poincaré polynomial of \( S \) is \( P(x) = x^4 + 4x^2 + 1 \) and its Hodge numbers satisfy the relation \( h^{p,q} = 0 \) if \( p \neq q \). A further examination of Tables 1, 2, 3, and 4, proves the first part of the theorem concerning the structure of \( M^1 \). Beside the three surfaces, there are 1329 isolated points and 174 irreducible components isomorphic to \( \mathbb{P}^1 \). Let \( \Pi \) be the set of points, let \( \Lambda \) be the set of lines, let \( \Xi \) be the set of surfaces in \( M^1 \). Let \( \Sigma \) be the set of irreducible components of \( M^1 \).

Let \( \lambda(t) = (t^{n_0}, t^{n_1}, t^{n_2}) \) be a one-parameter subgroup of \( T \) that is orthogonal to any non-zero character \( \chi \in \chi^*(T) \) appearing in the weight decomposition of some \( T_x M, x \in M^1 \). Inspecting the tables of characters from Section 3, we see that the set of characters \( \chi \neq \chi_0 \) for which \( \dim(T_x M)_x \neq 0 \) for some \( x \in M^1 \) is contained in the set

\[
\{ix + jy + kz, \quad -6 \leq i, j, k \leq 6\}.
\]

We can thus choose \( \lambda(t) = (1, t, t^7) \). We consider the \( \mathbb{C}^* \) action on \( M \) induced by \( \lambda \). Given \( X \in \Xi \) and \( x \in \Xi \), we denote by \( p(X) \) the dimension of the subspace of \( T_x M \) on which \( \mathbb{C}^* \) acts with positive weights. In other words, \( p(X) \) is the sum of the dimensions of the non-zero spaces \( (T_x M)_x \) for which \( \langle \lambda, T \chi \rangle > 0 \). According to the comments at the beginning of this proof, \( p(X) \) does not depend on the choice of \( x \in X \). These numbers are computed in Appendix A using Singular programs.

The Homology Basis Formula [3, Theorem 4.4]

\[
H_m(M, \mathbb{Z}) \cong \bigoplus_{X \in \Xi} H_{m-2p(X)}(X, \mathbb{Z})
\]

implies that the homology groups \( H_m(M, \mathbb{Z}) \), \( 0 \leq m \leq 52 \), are torsion-free. This also yields a formula for the Poincaré polynomial:

\[
P_M(x) = \sum_{X \in \Xi} x^{2p(X)} + \sum_{X \in \Lambda} (x^2 + 1)x^{2p(X)} + \sum_{X \in \Sigma} (x^4 + 4x^2 + 1)x^{2p(X)}.
\]

Starting from this, the Singular [6] program in Appendix A computes the expression for \( P_M \) given in the introduction. Owing to the fact [3, (4.7)] that the Homology Basis Formula respects the Hodge decomposition, we have the isomorphism

\[
H^p(M, \Omega_M^q) \cong \bigoplus_{X \in \Xi} H^{p-p(X)}(X, \Omega_X^{q-p(X)}).
\]

This shows that \( h^{p,q}(M) = 0 \) for \( p \neq q \), the same property being true for the Hodge numbers of all \( X \in \Xi \).

APPENDIX A. SINGULAR PROGRAMS

```plaintext
ring r=0,(x,y,z),dp;

int i,j; poly P, q, q1, q2, l1, l2, l3; P=0;
int points, lines; points = 0; lines = 0;

list s1, s2, s2_0, s2_1, s2_2, s3, s4, s5, d;
```
s1 = list(x, y, z);
s2 = list(2x, 2y, 2z, x+y, x+z, y+z);
s2_0 = list(2y, 2z, y+z);
s2_1 = list(2x, 2z, x+y);
s2_2 = list(2x, 2y, x+y);
s3 = list(3x, 3y, 3z, 2x+y, 2x+z, x+2y, 2y+z, x+2z, y+2z, x+y+z);
s4 = list(4x, 4y, 4z, 3x+y, 2x+2y, x+3y, 3x+z, 2x+2z, x+3z, 3y+z, 2y+2z, y+3z, 2x+y+z, x+2y+z, x+y+2z);
s5 = list(5x, 5y, 5z, 4x+y, 3x+2y, 2x+3y, x+4y, 4x+z, 3x+2z, 2x+3z, x+4z, 4y+z, 3y+2z, 2y+3z, y+4z, 3x+y+z, x+3y+z, x+y+3z, 2x+2y+z, 2x+y+2z, x+2y+2z);

proc add(poly p, list l)
{int i; list ll; ll=list();
  for (i=1; i<=size(l); i=i+1) {ll=ll+list(p+l[i]);};
  return(ll);};

proc positive_part(list l)
{int i; int p; p=0;
  for (i=1; i<=size(l); i=i+1) {if (l[i]>0) {p=p+1;};
  return(p);};

proc values(list w, list l)
{int i; list v; v=list();
  for (i=1; i<=size(w); i=i+1)
    {v=v+list((w[i]/x)*l[1]+(w[i]/y)*l[2]+(w[i]/z)*l[3]);};
  return(v);};

proc sub(list l, list ll)
{list lll; int i, j, e; lll=list(); for (j=1; j<=size(ll); j=j+1)
  {e=1; for (i=1; i<=size(lll); i=i+1)
    {if (lll[i]==ll[j] and e==1) {lll=delete(lll, i); e=0;};};
  return(lll);};

proc id2(list l)
{list ll; ll = list(); int i; for (i=1; i<=size(l); i=i+1)
  {ll = ll+ add(l[i], s3);};
  return(sub(s5, (sub(s5, ll))));};

proc id3(list l)
{list ll; ll = list(); int i; for (i=1; i<=size(l); i=i+1)
  {ll = ll+ add(l[i], s2);};
  return(sub(s5, (sub(s5, ll))));};

proc point_3(list l)
{points=list(points+3;
  return(x^2*positive_part(values(l, list(0, 1, 7)))+x^2*positive_part(values(l, list(7, 1, 0)))
  +x^2*positive_part(values(l, list(0, 7, 1))));};
proc point_3_1(list l)
{points=points+3;
 return(x^{2*positive_part(values(1, list(0,1,7)))}
 +x^{2*positive_part(values(1, list(1,0,7)))}
 +x^{2*positive_part(values(1, list(7,1,0))))};
}

proc point_6(list l)
{points=points+6;
 return(x^{2*positive_part(values(1, list(0,1,7)))}
 +x^{2*positive_part(values(1, list(1,0,7)))}
 +x^{2*positive_part(values(1, list(7,1,0)))}
 +x^{2*positive_part(values(1, list(1,7,0)))}
 +x^{2*positive_part(values(1, list(0,7,1)))}
 +x^{2*positive_part(values(1, list(7,0,1))))};
}

proc line_3(list l)
{lines=lines+3;
 return((1+x^2)*x^{2*positive_part(values(1, list(0,1,7)))}
 +(1+x^2)*x^{2*positive_part(values(1, list(7,1,0)))}
 +(1+x^2)*x^{2*positive_part(values(1, list(0,7,1))))};
}

proc line_3_1(list l)
{lines=lines+3;
 return((1+x^2)*x^{2*positive_part(values(1, list(0,1,7)))}
 +(1+x^2)*x^{2*positive_part(values(1, list(1,0,7)))}
 +(1+x^2)*x^{2*positive_part(values(1, list(7,1,0))))};
}

proc line_6(list l)
{lines=lines+6;
 return((1+x^2)*x^{2*positive_part(values(1, list(0,1,7)))}
 +(1+x^2)*x^{2*positive_part(values(1, list(1,0,7)))}
 +(1+x^2)*x^{2*positive_part(values(1, list(7,1,0)))}
 +(1+x^2)*x^{2*positive_part(values(1, list(0,7,1)))}
 +(1+x^2)*x^{2*positive_part(values(1, list(7,0,1))))};
}

proc surface_3(list l)
{return((1+4*(x^2)+x^4)*x^{2*positive_part(values(1, list(0,1,7)))}
 +(1+4*(x^2)+x^4)*x^{2*positive_part(values(1, list(7,1,0)))}
 +(1+4*(x^2)+x^4)*x^{2*positive_part(values(1, list(0,7,1))))};
}

proc w0(list u, list v)
{return(add(-v[1]-u[1], s2) + add(-v[1]-u[2], s2) + add(-v[1]-u[3], s1)
 +add(-v[2]-u[1], s2) + add(-v[2]-u[2], s2) + add(-v[2]-u[3], s1)
 +add(-v[3]-u[1], s2) + add(-v[3]-u[2], s2) + add(-v[3]-u[3], s1));
}

proc g0(list u, list v)
{return(list(u[1]-u[1], u[1]-u[2], -v[1]+v[1], -v[1]+v[2], -v[1]+v[3])}
HOMOLOGY OF THE SPACE OF SHEAVES OF WITH HILBERT POLYNOMIAL $5m + 3$ 31

+ list(u[2]-u[1], u[2]-u[2], -v[2]+v[1], -v[2]+v[2], -v[2]+v[3])
+ add(u[3]-u[1], s1) +
add(u[3]-u[2], s1) + list(-v[3]+v[1], -v[3]+v[2], -v[3]+v[3]));}

proc m0(list u, list v)
{return(sub(w0(u,v), g0(u,v)));}

list alpha;
for (i=1; i<=3; i=i+1) {for (j=1; j<=3; j=j+1)
{alpha = m0(list(s2_0[i]-x, s2_1[j]-y, 0), list(x, y, z));
P=P+point_3(alpha);};
alpha = m0(list(x, x, 0), list(x, y, z));
P=P+point_3_1(alpha);
}

list beta;
for (i=1; i<=3; i=i+1) {for (j=1; j<=3; j=j+1)
{beta = m0(list(s2_0[i]-x, s1[j], 0), list(x, y, z));
P=P+line_3_1(beta);};
list gamma;
gamma = m0(list(x,y,0), list(x,y,z));
P = P + surface_3(gamma);

list delta;
q1=2x; q2=2y;
d=sub(id3(list(3x, x+2y, 2x+y, 3y)), list(2x+2y+z));
for (i=1; i<=size(d); i=i+1)
{delta= m0(list(d[i]-x-y-q2, d[i]-x-y-q1, 0),
list(x, y, -d[i]+q1+q2+x+y));
P = P + point_3(delta);};
d=list(2x+2y+z);
for (i=1; i<=size(d); i=i+1)
{delta= m0(list(d[i]-x-y-q2, d[i]-x-y-q1, 0),
list(x, y, -d[i]+q1+q2+x+y));
P = P + line_3_1(delta);};
q1=2x; q2=2z;
d=sub(id3(list(3x, x+2z, 2x+y, y+2z)), list(3x+y+z));
for (i=1; i<=size(d); i=i+1)
{delta= m0(list(d[i]-x-y-q2, d[i]-x-y-q1, 0),
list(x, y, -d[i]+q1+q2+x+y));
P = P + point_6(delta);};
q1=2z; q2=x+y;
d=sub(id3(list(x+2z, 2x+y, y+2z, x+2y)), list(2x+2y+z));
for (i=1; i<=size(d); i=i+1)
\{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{point}_3(\text{delta});\}\}

q1 = 2x; q2 = y+z;
d = \text{sub}(\text{id3}(\text{list}(3x, x+y+z, 2x+y, 2y+z)), \text{list}(2x+y+2z, 3x+2y));
for (i=1; i<=\text{size}(d); i=i+1)
{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{point}_6(\text{delta});}\}

\{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{line}_6(\text{delta});\}\}

\{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{point}_6(\text{delta});\}\}

q1 = 2x; q2 = x+y;
d = \text{sub}(\text{id3}(\text{list}(3x, 2x+y, x+2y)), \text{list}(3x+y+z, 2x+2y+z, 2x+3y));
for (i=1; i<=\text{size}(d); i=i+1)
{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{point}_6(\text{delta});}\}

\{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{line}_6(\text{delta});\}\}

q1 = x+z; q2 = y+z;
d = \text{sub}(\text{id3}(\text{list}(2x+z, x+y+z, 2y+z)), \text{list}(x+y+3z, x+3y+z, 2x+2y+z, 3x+y+z));
for (i=1; i<=\text{size}(d); i=i+1)
{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{point}_3(\text{delta});}\}

\{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{line}_3(\text{delta});\}\}

q1 = 2x; q2 = x+z;
d = \text{sub}(\text{id3}(\text{list}(3x, 2x+z, 2x+y, x+y+z)), \text{list}(3x+2z, 2x+y+2z, 2x+2y+z, 4x+y));
for (i=1; i<=\text{size}(d); i=i+1)
{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{point}_6(\text{delta});}\}

\{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{line}_6(\text{delta});\}\}

\{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{point}_3(\text{delta});\}\}

q1 = 2x; q2 = 2x+y+2z;
d = \text{sub}(\text{id3}(\text{list}(3x, 2x+y, 2x+2y+z, 2x+3y));
for (i=1; i<=\text{size}(d); i=i+1)
{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{point}_6(\text{delta});}\}

\{\text{delta} = m0(\text{list}(d[i]-x-y-q2, d[i]-x-y-q1, 0), \text{list}(x, y, -d[i]+q1+q2+x+y));
P = P + \text{line}_6(\text{delta});\}\}
\[ P = P + \text{line}_6(\text{delta}); \]

\[ q_1 = x+y; \]
\[ q_2 = y+z; \]
\[ d = \text{sub}(\text{id}_3(\text{list}(2x+y, x+y+z, x+2y, 2y+z)), \]
\[ \text{list}(x+2y+2z, 2x+y+2z, 2x+3y, 3x+y+z)); \]
\[ \text{for} \ (i=1; \ i<\text{size}(d); \ i=i+1) \]
\[ \{ \text{delta} = \text{m}_0(\text{list}(d[i]-x-y-q_2, d[i]-x-y-q_1, 0), \]
\[ \text{list}(x, y, -d[i]+q_1+q_2+x+y)); \]
\[ P = P + \text{point}_6(\text{delta}); \}; \]

\[ d = \text{list}(x+2y+2z, 2x+y+2z, 3x+y+z); \]
\[ \text{for} \ (i=1; \ i<\text{size}(d); \ i=i+1) \]
\[ \{ \text{delta} = \text{m}_0(\text{list}(d[i]-x-y-q_2, d[i]-x-y-q_1, 0), \]
\[ \text{list}(x, y, -d[i]+q_1+q_2+x+y)); \]
\[ P = P + \text{line}_6(\text{delta}); \}; \]

\[ q_1 = x+z; \]
\[ q_2 = 2z; \]
\[ d = \text{sub}(\text{id}_3(\text{list}(2x+z, x+2z, x+y+z, y+2z)), \]
\[ \text{list}(x+2y+2z, 2x+y+2z, 3x+2z)); \]
\[ \text{for} \ (i=1; \ i<\text{size}(d); \ i=i+1) \]
\[ \{ \text{delta} = \text{m}_0(\text{list}(d[i]-x-y-q_2, d[i]-x-y-q_1, 0), \]
\[ \text{list}(x, y, -d[i]+q_1+q_2+x+y)); \]
\[ P = P + \text{point}_6(\text{delta}); \}; \]

\[ d = \text{list}(x+2y+2z, 3x+2z); \]
\[ \text{for} \ (i=1; \ i<\text{size}(d); \ i=i+1) \]
\[ \{ \text{delta} = \text{m}_0(\text{list}(d[i]-x-y-q_2, d[i]-x-y-q_1, 0), \]
\[ \text{list}(x, y, -d[i]+q_1+q_2+x+y)); \]
\[ P = P + \text{line}_6(\text{delta}); \}; \]

\[ \text{proc w1(list u, list v) \{return(\text{add}(-v[1]-u[1], s1)+\text{add}(-v[2]-u[2], s1)+\text{list}(-v[1]-u[3], -v[1]-u[4]) +\text{add}(-v[2]-u[1], s2)+\text{add}(-v[2]-u[2], s2)+\text{add}(-v[2]-u[3], s1)+\text{add}(-v[2]-u[4], s1) +\text{add}(-v[3]-u[1], s2)+\text{add}(-v[3]-u[2], s2)+\text{add}(-v[3]-u[3], s1)+\text{add}(-v[3]-u[4], s1) +\text{add}(-v[4]-u[1], s2)+\text{add}(-v[4]-u[2], s2)+\text{add}(-v[4]-u[3], s1)+\text{add}(-v[4]-u[4], s1)); \}; \]

\[ \text{proc g1(list u, list v) \{return(\text{add}(-v[2]+v[1], s1) + \text{list}(-v[2]+v[2], -v[2]+v[3], -v[2]+v[4]) + \text{add}(-v[3]+v[1], s1) + \text{list}(-v[3]+v[2], -v[3]+v[3], -v[3]+v[4]) + \text{add}(-v[4]+v[1], s1) + \text{list}(-v[4]+v[2], -v[4]+v[3], -v[4]+v[4]) + \text{list}(0, 0, 0, 0, u[1]-u[2], u[2]-u[1], u[3]-u[4], u[4]-u[3]) + \text{add}(u[3]-u[1], s1) + \text{add}(u[3]-u[2], s1) + \text{add}(u[4]-u[2], s1)); \}; \]

\[ \text{proc m1(list u, list v, list s) \{return(\text{sub}(w1(u,v), \text{sub}(g1(u,v), s))); \}; \]

\[ \text{list epsilon; } \]
d = sub(id3(add(x, list(x+y, x+z, y+z)) + add(y, list(x+y, x+z, y+z))), list(2x+2y+z, x+y+3z));
for (i=1; i<size(d); i=i+1)
{epsilon = m1(list(d[i]-y, d[i]-x, x+y+z, x+y+z),
list(x+y-d[i], -y-z, -x-z, -x-y),
list(-d[i]+2x+2y+z, -d[i]+2x+2y+z));
P = P + point_3(epsilon);};

for (i=1; i<size(d); i=i+1)
{epsilon = m1(list(d[i]-y, d[i]-x, x+y+z, x+y+z),
list(x+y-d[i], -y-z, -x-z, -x-y),
list(-d[i]+2x+2y+z, -d[i]+2x+2y+z));
P = P + point_3(epsilon);};

d = list(x+y+3z);
list zeta;
l1 = x; l2 = y;
d = sub(id3(add(x, list(2x, x+y, y+z)) + add(y, list(2x, x+y, y+z))), list(3x+2y, 2x+2y+z, 2x+y+2z));
for (i=1; i<size(d); i=i+1)
{zeta = m1(list(d[i]-l2, d[i]-l1, 2x+y, x+y+z),
list(-d[i]+l1+l2, -2x, -y-z, -x-y),
list(2x+y-d[i]+l1+l2, x+y+z-d[i]+l1+l2));
P = P + point_6(zeta);};

d = list(2x+y+2z);
for (i=1; i<size(d); i=i+1)
{zeta = m1(list(d[i]-l2, d[i]-l1, 2x+y, x+y+z),
list(-d[i]+l1+l2, -2x, -y-z, -x-y),
list(2x+y-d[i]+l1+l2, x+y+z-d[i]+l1+l2));
P = P + point_6(zeta);};

l1 = x; l2 = z;
d = sub(id3(add(x, list(2x, x+y, y+z)) + add(z, list(2x, x+y, y+z))), list(3x+y+z, 2x+y+2z, x+3y+z));
for (i=1; i<size(d); i=i+1)
{zeta = m1(list(d[i]-l2, d[i]-l1, 2x+y, x+y+z),
list(-d[i]+l1+l2, -2x, -y-z, -x-y),
list(2x+y-d[i]+l1+l2, x+y+z-d[i]+l1+l2));
P = P + point_6(zeta);};

d = list(x+3y+z);
for (i=1; i<size(d); i=i+1)
{zeta = m1(list(d[i]-l2, d[i]-l1, 2x+y, x+y+z),
list(-d[i]+l1+l2, -2x, -y-z, -x-y),
list(2x+y-d[i]+l1+l2, x+y+z-d[i]+l1+l2));
P = P + line_6(zeta);};
\[ HOMOLOGY \ OF \ THE \ SPACE \ OF \ SHEAVES \ OF \ SHEAVES \ WITH \ HILBERT \ POLYNOMIAL \ 5m + 3 35 \]

\[ l1 = y; \ l2 = z; \]
\[ d = sub(id3(add(y, list(2x, x+y, y+z)) + add(z, list(2x, x+y, y+z))), list(2x+2y+z, x+2y+2z)); \]
\[ list \ s_z; \ s_z = list(); \]
\[ for \ (i=1; \ i <= size(d); \ i = i + 1) \]
\[ \{ zeta = m1(list(d[i] - l2, d[i] - l1, 2x+y, x+y+z), list(-d[i] + l1 + l2, -2x, -y-z, -x-y), list(2x+y-d[i] + l1 + l2, x+y+z-d[i] + l1 + l2)); \]
\[ \ s_z = s_z + list(size(zeta)); \]
\[ P = P + point_6(zeta); \}; \]

\[ list \ eta; \]
\[ l1 = x; \ l2 = y; \]
\[ d = sub(id3(add(x, list(2x, x+y, 2y)) + add(y, list(2x, x+y, 2y))), list(3x+2y, 2x+3y, x+2y+2z, 2x+y+2z)); \]
\[ for \ (i=1; \ i <= size(d); \ i = i + 1) \]
\[ \{ eta = m1(list(d[i] - l2, d[i] - l1, x+2y, 2x+y), list(-d[i] + l1 + l2, -2y, -2x, -x-y), list(x+2y-d[i] + l1 + l2, 2x+y-d[i] + l1 + l2)); \]
\[ P = P + point_3(eta); \}; \]

\[ d = list(x+2y+2z, 2x+y+2z); \]
\[ for \ (i=1; \ i <= size(d); \ i = i + 1) \]
\[ \{ eta = m1(list(d[i] - l2, d[i] - l1, x+2y, 2x+y), list(-d[i] + l1 + l2, -2y, -2x, -x-y), list(x+2y-d[i] + l1 + l2, 2x+y-d[i] + l1 + l2)); \]
\[ P = P + line_3(eta); \}; \]

\[ l1 = x; \ l2 = z; \]
\[ d = sub(id3(add(x, list(2x, x+y, x+z)) + add(z, list(2x, x+y, x+z))), list(2x+2y+z, 3x+y+z)); \]
\[ for \ (i=1; \ i <= size(d); \ i = i + 1) \]
\[ \{ eta = m1(list(d[i] - l2, d[i] - l1, 2x+y, 2x+z), list(-d[i] + l1 + l2, -x-y, -x-z, -2x), list(2x+y-d[i] + l1 + l2, 2x+z-d[i] + l1 + l2)); \]
\[ P = P + point_6(eta); \}; \]

\[ list \ theta; \]
\[ l1 = x; \ l2 = y; \]
\[ d = sub(id3(add(x, list(2x, x+y, x+z)) + add(y, list(2x, x+y, x+z))), list(3x+y+z, 3x+2y, 2x+y+2z, x+3y+z, x+2y+2z)); \]
\[ for \ (i=1; \ i <= size(d); \ i = i + 1) \]
\[ \{ theta = m1(list(d[i] - l2, d[i] - l1, 2x+y, 2x+z), list(-d[i] + l1 + l2, -x-y, -x-z, -2x), list(2x+y-d[i] + l1 + l2, 2x+z-d[i] + l1 + l2)); \]
\[ P = P + point_6(theta); \}; \]

\[ d = list(2x+y+2z, x+3y+z, x+2y+2z); \]
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for (i=1; i<=size(d); i=i+1)
    \{theta = m1(list(d[i]-l2, d[i]-l1, 2x+y, 2x+z),
           list(-d[i]+l1+l2, -x-y, -x-z, -2x),
           list(2x+y-d[i]+l1+l2, 2x+z-d[i]+l1+l2));
    P = P + line_6(theta);\};

l1=y; l2=z;
d=sub(id3(add(y, list(2x, x+y, x+z)) + add(z, list(2x, x+y, x+z))),
    list(2x+2y+z, 2x+y+2z, x+y+3z, x+2y+2z, x+3y+z));
for (i=1; i<=size(d); i=i+1)
    \{theta = m1(list(d[i]-l2, d[i]-l1, 2x+y, 2x+z),
           list(-d[i]+l1+l2, -x-y, -x-z, -2x),
           list(2x+y-d[i]+l1+l2, 2x+z-d[i]+l1+l2));
    P = P + point_3_1(theta);\};

d=list(x+y+3z, x+2y+2z, x+3y+z);
l1=y; l2=z;
for (i=1; i<=size(d); i=i+1)
    \{theta = m1(list(d[i]-l2, d[i]-l1, 2x+y, 2x+z),
           list(-d[i]+l1+l2, -x-y, -x-z, -2x),
           list(2x+y-d[i]+l1+l2, 2x+z-d[i]+l1+l2));
    s_t = s_t + list(size(theta));
    P = P + line_3_1(theta);\};

proc w2(list u, list v)
    \{return(add(-v[1]-u[1], s1) + add(-v[1]-u[2], s1) + add(-v[1]-u[3], s1)
           +add(-v[2]-u[1], s1) + add(-v[2]-u[2], s1) + add(-v[2]-u[3], s1)
           +add(-v[3]-u[1], s3) + add(-v[3]-u[2], s3) + add(-v[3]-u[3], s3));\};

proc g2(list u, list v)
    \{return(list(u[1]-u[1], u[1]-u[2], u[1]-u[3])
           + list(u[2]-u[1], u[2]-u[2], u[2]-u[3])
           + list(u[3]-u[1], u[3]-u[2], u[3]-u[3])
           + list(0, 0, -v[1]+v[2], -v[2]+v[1])
           + add(-v[3]+v[1], s2) + add(-v[3]+v[2], s2));\};

proc m2(list u, list v)
    \{return(sub(w2(u,v), g2(u,v))
           + list(u[1]+v[3]-x-y-z, u[2]+v[3]-x-y-z, u[3]+v[3]-x-y-z));\};

list iota;
d=sub(id2(list(x+y, x+z, y+z)), list(2x+2y+z, x+2y+2z, 2x+y+2z));
for (i=1; i<=size(d); i=i+1)
    \{iota = m2(list(x, y, z), list(0, 0, d[i]-x-y-z));
    P = P + x^(2*positive_part(values(iota, list(0,1,7))));\};

points=points+15;
list kappa;
d= sub(id2(list(2x, x+y, y+z)), list(x+2y+2z, 3x+y+z, 2x+2y+z));
for (i=1; i <=size(d); i=i+1)
{kappa = m2(list(y-x, x-z, 0), list(x, z, d[i]-x-y));
P = P + point_6(kappa);};

list lambda;
d= sub(id2(list(2x, x+y, 2y)), list(x+3y+z, 3x+y+z, 2x+2y+z));
for (i=1; i <=size(d); i=i+1)
{lambda = m2(list(x-y, y-x, 0), list(y, x, d[i]-x-y));
P = P + point_3(lambda);};

list mu;
d=sub(id2(list(2x, x+y, x+z)),
list(x+y+3z, x+2y+2z, x+3y+z, 2x+y+2z, 2x+2y+z, 3x+y+z));
for (i=1; i <=size(d); i=i+1)
{mu = m2(list(x-y, x-z, 0), list(y, z, d[i]-2x));
P = P + point_3_1(mu);};

d = list(x+y+3z, x+2y+2z, x+3y+z);
for (i=1; i <=size(d); i=i+1)
{mu = m2(list(x-y, x-z, 0), list(y, z, d[i]-2x));
P = P + line_3_1(mu);};

proc w3(list u, list v)
{return(add(-v[1]-u[1], s3) + add(-v[1]-u[2], s1)
+ add(-v[2]-u[1], s2) + add(-v[2]-u[2], s2));};

proc g3(list u, list v)
{return(list(0, 0, 0)
+ add(u[2]-u[1], s2) + add(-v[2]+v[1], s1));};

proc m3(list u, list v)
{return(sub(w3(u,v), g3(u,v))
+ list(u[1]+v[1]-x-y-z, u[1]+v[2]-2x-y-z, u[1]+v[2]-x-2y-z,
 u[1]+v[2]-x-y-2z));};

list nu;
q=2y;
d= sub(s5, list(x+3y+z, 2x+y+2z, 3x+y+z, 2x+2y+z, 5z, y+4z));
for (i=1; i <=size(d); i=i+1)
{nu = m3(list(d[i]-q-x, 0), list(x, q));
P = P + point_6(nu);};

q=y+z;
d=sub(s5, list(x+2y+2z, 2x+y+2z, 3x+y+z, 2x+2y+z, 5y, 5z));
for (i=1; i <=size(d); i=i+1)
{nu = m3(list(d[i]-q-x, 0), list(x, q));}
P = P + point_3_1(nu);};

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