On non-geometric augmentations in high dimensions

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Abstract
In this note we construct augmentations of Chekanov–Eliashberg algebras of certain high dimensional Legendrian submanifolds that are not induced by exact Lagrangian fillings. The obstructions to the existence of exact Lagrangian fillings that we use are Seidel’s isomorphism and the injectivity of a certain algebraic map between the corresponding augmentation varieties proven by Gao and Rutherford. In addition, along the way we discuss the relation between augmentation varieties of Legendrian submanifolds and their spherical spuns (Proposition 4.3).

Keywords Legendrian submanifold · Chekanov–Eliashberg algebra · Augmentation · Exact Lagrangian filling

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1 Introduction and main results

It is natural to study symplectic manifolds with contact boundary by studying Lagrangian submanifolds with Legendrian boundary; in particular one can study exact Lagrangian fillings of Legendrian submanifolds. In this paper we consider closed Legendrian submanifolds $\Lambda$ in the standard contact vector space $\mathbb{R}^{2n+1}_{st} := (\mathbb{R}^{2n+1}, \alpha_{st} := dz - \Sigma_i y_i dx_i)$ and their exact Lagrangian fillings, i.e. smooth cobordisms $(L; \emptyset, \Lambda)$ and Lagrangian embeddings $L \hookrightarrow (\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t\alpha_{st}))$ satisfying $L|_{[T_L, \infty) \times \mathbb{R}^{2n+1}} = [T_L, \infty) \times \Lambda$ for some $T_L \gg 0$, $L^c := L|_{(-\infty, T_L] \times \mathbb{R}^{2n+1}}$ is compact, and there is $f \in C^\infty(L)$ which is constant on $[T_L, \infty) \times \Lambda$ satisfying $df = e^t\alpha_{st}$.

Legendrian contact homology is a modern invariant of Legendrian submanifolds in $\mathbb{R}^{2n+1}_{st}$ which is a variant of the symplectic field theory (SFT) introduced by Eliashberg, Givental, and Hofer in [13]. For Legendrian submanifolds in $\mathbb{R}^3_{st}$, it was defined by Chekanov in [3] and the version of Legendrian contact homology for Legendrian submanifolds of $\mathbb{R}^{2n+1}_{st}$ was developed by Ekholm–Etnyre–Sullivan in [9, 10].

Legendrian contact homology is a homology of the differential graded algebra (DGA) $\mathcal{A}(\Lambda)$, which is called the Chekanov–Eliashberg DGA of $\Lambda$ or Legendrian contact homology.

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DGA of \( \Lambda \). Chekanov–Eliashberg DGA \( \mathcal{A}(\Lambda) \) is defined to be the non-commutative unital differential graded algebra freely generated by the set of Reeb chords of \( \Lambda \), denoted by \( \mathcal{Q}(\Lambda) \), it is defined over a unital ring \( R \). The differential \( \partial(a) \) on \( a \in \mathcal{Q}(\Lambda) \) is defined by a count of rigid pseudo-holomorphic disks for some choice of compatible almost complex structure, and is then extended using the Leibniz rule. The homology of the Chekanov–Eliashberg algebra is called the Legendrian contact homology of \( \Lambda \). Following the result of Ekholm–Etnyre–Sullivan in [9], Legendrian contact homology is independent of the choice of an almost complex structure and is invariant under Legendrian isotopy. When \( \mathcal{A}(\Lambda) \) is defined over a unital ring \( R \), we will sometimes write it as \( \mathcal{A}_R(\Lambda) \).

An augmentation is a unital DGA-morphism \( \varepsilon : \mathcal{A}(\Lambda) \to (\mathbb{F}, 0) \) which thus satisfies \( \varepsilon \circ \partial = 0 \). In our constructions we will use only graded augmentations which by definition vanish on all generators in nonzero degrees.

The following fact can be seen as a consequence of the discussion in [11]: An exact Lagrangian filling \( L \) of \( \Lambda \) gives rise to a unital DGA morphism \( \varepsilon : \mathcal{A}(\Lambda) \to \mathbb{F}_2 \) defined by an appropriate count of rigid pseudoholomorphic discs with boundary on the filling. Even though the result in [11] is written for \( \mathbb{F}_2 \)-coefficients only, it admits a natural extension to more general coefficients. In this work we will only consider spin Maslov number 0 exact Lagrangian cobordisms of spin Maslov number 0 Legendrian submanifolds of the standard contact vector space, where we will always choose the spin structure on a Legendrian which is a restriction of the spin structure of the exact Lagrangian filling. In this case there is a map \( \varepsilon : \mathcal{A}(\Lambda) \to \mathbb{F} \) for an arbitrary field \( \mathbb{F} \), and all the homology groups that we will consider will have a \( \mathbb{Z} \)-grading.

There are a few obstructions to the existence of an exact Lagrangian filling which induces a given augmentation, see [2, 5, 8, 12, 14–16], and quite a few examples of augmentations of Legendrian knots that are not induced by exact Lagrangian fillings (i.e., are non-geometric). We would like to construct non-geometric augmentations for high dimensional Legendrian submanifolds.

### 1.1 Seidel’s isomorphism

One of the most well-known obstruction comes from the so-called Seidel’s isomorphism. In [8], Ekholm outlined an isomorphism, first conjectured by Seidel, which relates the linearised Legendrian contact cohomology of a Legendrian and the singular homology of its embedded exact Lagrangian filling. The details of this isomorphism were later completed in the work of Dimitroglou Rizell, see [5]:

**Theorem 1.1** ([5, 8]) Given a Legendrian submanifold \( \Lambda \subset \mathbb{R}^{2n+1} \) and its exact Lagrangian filling \( L \) of Maslov number 0. For the augmentation \( \varepsilon_L : \mathcal{A}_{\mathbb{F}_2}(\Lambda) \to (\mathbb{F}_2, 0) \) induced by \( L \), there is an isomorphism

\[
LCH^{n-i}_{\varepsilon_L}(\Lambda; \mathbb{F}_2) \simeq H_i(L; \mathbb{F}_2).
\]

**Remark 1.2** Note that Seidel’s isomorphism has been proven by Dimitroglou Rizell in [5] only with \( \mathbb{F}_2 \)-coefficients. On the other hand, signs of the cobordism maps have been worked out by Karlsson in [20]. Besides that the stronger result of Ekholm–Lekili [12] which compares the corresponding \( A_\infty \)-structures has been proven with signs, in particular it works over an arbitrary field. From this perspective we can say that Seidel’s isomorphism holds with \( \mathbb{Q} \)-coefficients and with \( \mathbb{Z} \)-coefficients. In addition, observe that there is a version of Seidel’s isomorphism with local coefficients recently proven by Gao–Rutherford, see [15].
1.2 Obstruction of fillability from augmentation varieties

In [15] Gao and Rutherford use local structure of the augmentation variety to obstruct Lagrangian fillings, and to provide the examples of Legendrian twist knots with augmentations for which the other known obstructions to the existence of an exact Lagrangian filling inducing it such as the one coming from Thurston-Bennequin number [2], Seidel’s isomorphism [5, 8], the extension of Seidel’s isomorphism of Ekholm-Lekili [12] and the examples of Etgü [14] based on the result of Ekholm-Lekili do not apply.

Now we recall some details of the obstruction of Gao and Rutherford from [15]. Let $L$ be an exact Lagrangian filling of a Legendrian submanifold $\Lambda_1 \subset \mathbb{R}^{2n+1}_{st}$, then

$$H_1(L; \mathbb{Z}) \simeq \mathbb{Z}^k \oplus \mathbb{Z}/k_1 \oplus \cdots \oplus \mathbb{Z}/k_s$$

for some $k, k_1, \ldots, k_s$. We define

$$\text{Aug}(L; F) \simeq (F^*)^k \oplus C_{k_1}(F) \oplus \cdots \oplus C_{k_s}(F),$$

where $C_{k_i}(F)$ is the group of $k_i$-th roots of unity in $F$, i.e.

$$C_{k_i}(F) = \{ x \mid x^{k_i} = 1, x \in F \}$$

and $i = 1, \ldots, k_s$.

**Proposition 1.3** (Proposition 2.6 in [15]) Let $L$ be an exact Lagrangian filling of a Legendrian submanifold $\Lambda \subset (\mathbb{R}^{2n+1}_{st}, \alpha_{st})$ such that the Maslov number of $L$ vanishes. If $F$ has a characteristic different from 2, assume that $L$ is equipped with a choice of spin structure. Then, the map $f^*: \text{Aug}(L; F) \to \text{Aug}(\Lambda, F)$ is an injective, algebraic map.

**Remark 1.4** Note that when $\Lambda$ is a Legendrian knot in $(\mathbb{R}^3, \alpha_{st})$ and $L$ is an orientable exact Lagrangian filling of $\Lambda$, then $H_1(L; \mathbb{Z})$ is a free abelian group. Hence $\text{Aug}(L, F) \simeq (F^*)^k$ for some $k$.

1.3 Main result

Our main result is the extension of the examples in $\mathbb{R}^3_{st}$ of augmentations whose geometricity is obstructed by Seidel’s isomorphism (see Sect. 2.1) to high dimensions (see Sect. 3) and the extension of the examples in $\mathbb{R}^3_{st}$ of augmentations whose geometricity is not obstructed by Seidel’s isomorphism, but is obstructed by Proposition 1.3 (see Sect. 2.2) to high dimensions (see Sect. 4).

The examples we get will be obtained using the spherical spinning construction.

1.3.1 Spherical spinning

Recall that the front $S^k$-spinning construction is a Legendrian version of suspension. The front $S^k$-spinning construction from a Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}_{st}$ produces a Legendrian embedding of $\Lambda \times S^k$ inside $\mathbb{R}^{2(n+k)+1}_{st}$ whose image is denoted by $\Sigma_{S^k} \Lambda$. When $k = 1$ it has appeared in the work of Ekholm–Etnyre–Sullivan [10] and in the case when $k > 1$ it has been discussed by the author in [17]. One of the important properties of the spherical front spinning that we will need is that as shown in [17] it can be extended to exact Lagrangian cobordisms. For other details of the construction and its properties we refer the reader to [6, 10, 17–19].

Both types of examples (from Sections 3, 4) are joined in the following theorem.
Theorem 1.5 There is a Legendrian submanifold $\Lambda$ in $\mathbb{R}^{2n+1}$ of Maslov number 0 such that the Chekanov-Eliashberg algebra of $\Lambda$ admits an augmentation $\varepsilon: A(\Lambda) \to (\mathbb{F}_2, 0)$ which is not induced by a spin exact Lagrangian filling of Maslov number 0.

In addition, along the way we discuss the relation between augmentation varieties of Legendrian submanifolds and their spherical spuns (Proposition 4.3).

2 Examples in low dimensions

2.1 Class A

In this class we consider Legendrian knots $\Lambda$ in $\mathbb{R}^3$ such that the Chekanov–Eliashberg algebra admits an augmentation

$$\varepsilon: A(\Lambda) \to (\mathbb{F}_2, 0)$$

satisfying that for some $i > 1$ or $i < 0$, $LCH^i(\Lambda; \mathbb{F}_2)$ is not isomorphic to $H_{1-i}(L\Lambda; \mathbb{F}_2)$ for all exact Lagrangian fillings $L\Lambda$ of Maslov number 0. In other words, in this class we consider Legendrian knots with augmentations that are not geometric because they violate the obstruction coming from Seidel’s isomorphism.

Remark 2.1 From Remark 1.2 it follows that in the description of Class A we can rely on Seidel’s isomorphism not only with $\mathbb{F}_2$-coefficients, but with more general field coefficients such as $\mathbb{Q}$ and $\mathbb{R}$, and also with $\mathbb{Z}$-coefficients.

We rely on the work of Chongchitmate–Ng [4]:

(i) Legendrian representative of $m(8_{21})$ from [4], see Fig. 1, has a vanishing rotation number, and hence Maslov number 0, and two Poincaré polynomials, one of which is of the form

$$P_{m(8_{21})}(t) = t^{-1} + 4 + 2t.$$

The augmentation $\varepsilon_{m(8_{21})}$ which corresponds to this polynomial has the property that

$$LCH^1_{-1}(\Lambda; \mathbb{F}_2) \simeq LCH^{-1}_{m(8_{21})}(\Lambda; \mathbb{F}_2) \simeq \mathbb{F}_2.$$
From Theorem 1.1 it follows that $\varepsilon_m(8_{21})$ is not induced by a Maslov number 0 exact Lagrangian filling $L$, since otherwise $H_2(L; \mathbb{F}_2) \cong \mathbb{F}_2$, which is impossible from the topological point of view.

(ii) We can argue the same way for many other Legendrian knots $\Lambda$ (in particular from the atlas in [4]), whose Poincaré polynomials are computed with respect to certain augmentations that are by topological reasons in conflict with Seidel’s isomorphism. For example, one can take Legendrian representatives of $9_{46}, 9_{48}$ and so on from [4].

2.2 Class B

Consider the following Legendrian representatives of twist knots $\Lambda_n$, $n$ is odd and $n > 3$, investigated by Rutherford and Gao in [15]. The set of Reeb chords $Q(\Lambda_n)$ consists of chords $a, b, c_1, \ldots, c_n$ and $e_0, e_1, \ldots, e_n$, see Fig. 2. Reeb chords $a, b$ and $c_1, \ldots, c_k$ appear on the right, and crossings $c_{k+1}, \ldots, c_n$ appear on the left. The right cusps are labeled in clockwise order starting at the upper right as $e_0, e_1, \ldots, e_{k+1}$ (appearing on the right) and $e_{k+2}, \ldots, e_n$ (appearing on the left).

As proven in [15], there is an augmentation $\varepsilon_{\Lambda_n}$ to $\mathbb{F}_2$ defined by

$$\varepsilon_{\Lambda_n}(a) = 0, \varepsilon_{\Lambda_n}(b) = 0, \varepsilon_{\Lambda_n}(c_i) = 1, \varepsilon_{\Lambda_n}(e_j) = 0$$

for all $i = 1, \ldots, n$ and $j = 0, \ldots, n$, that is not induced by any Maslov 0 exact Lagrangian filling.

3 High dimensional analogue of Class A

We now consider the collection of inductive spuns of the Legendrian representative $\Lambda$ of $m(8_{21})$ that we discussed in Sect. 2.1. Following the discussion in [6] observe that there is an inclusion of DGAs $i : \mathcal{A}(\Lambda) \hookrightarrow \mathcal{A}(\Sigma S^1 \Lambda)$, which can be left inverted by a surjective DGA map $\pi : \mathcal{A}(\Sigma S^1 \Lambda) \twoheadrightarrow \mathcal{A}(\Lambda)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig. 2 The front projection of $\Lambda_7$}
\caption{The front projection of $\Lambda_7$}
\end{figure}
Theorem 3.1 Given $\Sigma_{S^1} \ldots \Sigma_{S^1} \Lambda$ and an augmentation $\epsilon_{\Sigma_{S^1} \ldots \Sigma_{S^1} \Lambda}$ of $A_{F_2}^{\Sigma_{S^1} \ldots \Sigma_{S^1} \Lambda}$ which is given by the inductive application of $\pi^*$ to $\epsilon_\Lambda$. Then $\epsilon_{\Sigma_{S^1} \ldots \Sigma_{S^1} \Lambda}$ is not induced by a Maslov number 0 exact Lagrangian cobordism.

Proof First we consider the case of $S^1$-spun of $\Lambda$. From the Künneth-type formula described in [7, Theorem 4.1, Remark 4.2], it follows that

$$LCH_i^{\epsilon_{\Sigma_{S^1} \Lambda}}(\Sigma_{S^1} \Lambda; F_2) \simeq LCH_i^{\epsilon_\Lambda}(\Lambda; F_2) \oplus LCH_{i-1}^{\epsilon_\Lambda}(\Lambda; F_2).$$

Then using it and the fact that $LCH_i^{\epsilon_\Lambda}(\Lambda; F_2) \simeq F$, $LCH_{i-1}^{\epsilon_\Lambda}(\Lambda; F_2) \simeq 0$, we obtain

$$\dim(LCH_i^{\epsilon_{\Sigma_{S^1} \Lambda}}(\Sigma_{S^1} \Lambda; F_2)) = 1.$$ (3.1)

Assume that $\epsilon_{\Sigma_{S^1} \Lambda}$ is induced by the Maslov number 0 spin exact Lagrangian cobordism $L$. Then from Theorem 1.1 we see that

$$LCH_{2-i}^{\epsilon_{\Sigma_{S^1} \Lambda}}(\Sigma_{S^1} \Lambda; F_2) \simeq LCH_{2-i}^{\epsilon_{\Sigma_{S^1} \Lambda}}(\Sigma_{S^1} \Lambda; F_2) \simeq H_i(L; F_2),$$

and hence from Eq. 3.1 it follows that $\dim H_3(L; F_2) = 1$, which is impossible since $L$ is a spin 3-dimensional filling of $\Sigma_{S^1} \Lambda$.

Then we inductively apply the same strategy and get that $\Sigma_{S^1} \ldots \Sigma_{S^1} \Lambda$ admits an augmentation, $\epsilon_{\Sigma_{S^1} \ldots \Sigma_{S^1} \Lambda}$, which is given by the inductive application of $\pi^*$ on $\epsilon_\Lambda$, and which is not induced by any spin, Maslov number 0 exact Lagrangian cobordism.

Remark 3.2 Other Legendrian knots from Class A can be treated in the absolutely analogous way, i.e. the result of Theorem 3.1 will work for other Legendrian knots from Class A.

Remark 3.3 Combining the geography/realization result for Poincaré polynomials by Bourgeois–Galant [1] with Seidel’s isomorphism, one can find more non-geometric augmentations in the high dimensional analogue of Class A.

4 High dimensional analogue of Class B

In this section we consider the collection of inductive spuns of the Legendrian twist knots $\Lambda_n$ from Sect. 2.2. Again, following the discussion in [6] observe that there is an inclusion of DGAs $i : A(\Lambda_n) \hookrightarrow A(\Sigma_{S^1} \Lambda_n)$, which can be left inverted by a surjective DGA map $\pi : A(\Sigma_{S^1} \Lambda_n) \twoheadrightarrow A(\Lambda_n)$ for $l \geq 1$.

Theorem 4.1 For a Legendrian representative $\Lambda_n$, $\Sigma_{S^1} \ldots \Sigma_{S^1} \Lambda_n$ admits a graded augmentation $\epsilon_{\Sigma_{S^1} \ldots \Sigma_{S^1} \Lambda_n}$ of $A_{F_2}^{(\Sigma_{S^1} \ldots \Sigma_{S^1} \Lambda_n)}$ which is defined by the inductive application of $\pi^*$ to $\epsilon_\Lambda_n$, and which is not induced by a Maslov number 0 spin exact Lagrangian cobordism.

Proof First we consider the case of $S^1$-spun of $\Lambda_n$. We take the graded augmentation $\epsilon_{\Sigma_{S^1} \Lambda_n}$ of $A_{F_2}^{(\Sigma_{S^1} \ldots \Sigma_{S^1} \Lambda_n)}$ defined by $\epsilon_{\Sigma_{S^1} \Lambda_n} := \pi^*(\epsilon_\Lambda_n)$, for simplicity we will denote $\epsilon_\Lambda_n$ by $\epsilon$ and $\epsilon_{\Sigma_{S^1} \Lambda_n}$ by $\tilde{\epsilon}$. Now we prove that $\tilde{\epsilon}$ is not induced by a Maslov 0, spin exact Lagrangian cobordism $\tilde{L}$ of $\Sigma_{S^1} \Lambda_n$.

From the proof of [15, Proposition 4.1] we know that

$$LCH_i^0(\Lambda_n; F) \simeq F^2 \text{ and } LCH_i^1(\Lambda_n; F) \simeq F$$ (4.1)

for a field $F$. Now we claim that $LCH_i^0(\Lambda_n; Z) \simeq Z^2$, $LCH_i^1(\Lambda_n; Z) \simeq Z$. One way to see that is to perform a direct computation of $LCH_i^0(\Lambda_n; Z)$, which is similar to the discussion in [15]. We would like to thank Dan Rutherford for discussing the following computation.

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We consider the augmentation $\varepsilon$, with $\varepsilon(a) = \varepsilon(b) = 0$. We will base our consideration on [15, Formulas (4.1) – (4.4)].

With $\varepsilon(a) = \varepsilon(b) = 0$, the value of $\varepsilon$ on the other degree 0 generators must be

$$\varepsilon(c_i) = \begin{cases} 1, & \text{for } i \text{ is odd;} \\ -1, & \text{for } i \text{ is even.} \end{cases}$$

Then we specialize $t = \varepsilon(t) = -1$, and then take the word length 1 part of the conjugated differential $\Phi_\varepsilon \circ \partial \circ \Phi_\varepsilon^{-1}$, where $\Phi_\varepsilon(c) = c + \varepsilon(c)$ for any Reeb chord $c$.

After that we see that the linearized differential is defined by $\partial_{\varepsilon}(e_0) = c_n$, $\partial_{\varepsilon}(e_1) = -c_1$ and for $i > 1$

$$\partial_{\varepsilon}(e_i) = \begin{cases} c_{i-1} - c_i, & \text{for } i \text{ is odd;} \\ -c_{i-1} + c_i, & \text{for } i \text{ is even.} \end{cases}$$

Then we see that there is an isomorphism of free $\mathbb{Z}$-modules

$$\partial_{\varepsilon}|_{\mathbb{Z}\langle e_1, e_2, \ldots, e_n \rangle} : \mathbb{Z}\langle e_1, e_2, \ldots, e_n \rangle \to \mathbb{Z}\langle c_1, c_2, \ldots, c_n \rangle,$$

and hence we can see that there is an acyclic subcomplex $C$

$$0 \to \mathbb{Z}\langle e_1, e_2, \ldots, e_n \rangle \to \mathbb{Z}\langle c_1, c_2, \ldots, c_n \rangle \to 0,$$

and we can take a quotient of $LCC^e / \mathcal{C}$, where the quotient map $LCC^e \to LCC^e / \mathcal{C}$ is a quasi-isomorphism, which leads to the complex $0 \to \mathbb{Z}\langle e_0 \rangle \to \mathbb{Z}\langle a, b \rangle \to 0$ with the vanishing differential which leads to $LCH^0_\varepsilon = \mathbb{Z}^2$ and $LCH^1_\varepsilon = \mathbb{Z}$, which using the universal coefficient theorem implies that $LCH^0_\varepsilon = \mathbb{Z}^2$ and $LCH^1_\varepsilon = \mathbb{Z}$.

The alternative way to get the same result is to observe that from Isomorphisms 4.1 it follows that

$$LCH^0_\varepsilon(\Lambda_n; \mathbb{C}) \simeq \mathbb{C}^2 \text{ and } LCH^1_\varepsilon(\Lambda_n; \mathbb{C}) \simeq \mathbb{C},$$

and therefore the rank of $LCH^0_\varepsilon(\Lambda_n; \mathbb{Z})$ equals 2 and the rank of $LCH^1_\varepsilon(\Lambda_n; \mathbb{Z})$ equals 1. In order to avoid $p$-torsion, one can use computations with a field of characteristic $p$.

From the Künneth-type formula described in [7, Theorem 4.1, Remark 4.2], it follows that

$$LCH^1_\varepsilon(\Sigma S^1 \Lambda_n; \mathbb{Z}) \simeq LCH^i_\varepsilon(\Lambda_n; \mathbb{Z}) \oplus LCH^{i-1}_\varepsilon(\Lambda_n; \mathbb{Z}).$$

Hence $LCH^1_\varepsilon(\Sigma S^1 \Lambda_n) \simeq \mathbb{Z}^3$. Then we use the Seidel’s isomorphism over $\mathbb{Z}$ and see that $H_1(\tilde{L}) \simeq \mathbb{Z}^3$, and hence from the discussion in Sect. 1.2 we get that

$$\text{Aug}(\tilde{L}, \mathbb{F}) \simeq (\mathbb{F}^*)^3. \tag{4.2}$$

\[\square\]

**Remark 4.2** Note that if we apply the argument described above to the $m$-th iterated spun of $\Lambda_n$, i.e. $\Sigma S^1 \ldots \Sigma S^1 \Lambda_n$, then $H_1(\tilde{L}) \simeq \mathbb{Z}^{2+m}$ and $\text{Aug}(\tilde{L}, \mathbb{F}) \simeq (\mathbb{F}^*)^{2+m}$.

Then we need the following proposition:

**Proposition 4.3** Let $\Lambda$ be a Maslov number 0 spin Legendrian submanifold such that $Q_i(\Lambda) = \emptyset$ for all $i < 0$. Then there is an isomorphism of (graded) augmentation varieties $\text{Aug}(\Lambda; \mathbb{F}) \simeq \text{Aug}(\Sigma S^m \Lambda; \mathbb{F})$ for all $m \geq 2$, and $\text{Aug}(\Sigma S^1 \Lambda; \mathbb{F}) \simeq \text{Aug}(\Lambda; \mathbb{F}) \times \mathbb{F}^*$. 

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Proof We will again rely on the analysis from [6]. Since $A(\Lambda)$ is supported in non-negative degrees, we see that $A(\Sigma^m \Lambda)$ is supported in non-negative degrees. There is an inclusion of DGAs $i : A(\Lambda) \to A(\Sigma^m \Lambda)$ which admits a left-inverse $\pi : A(\Lambda) \to A(\Sigma^m \Lambda)$ that has been proven with $\mathbb{F}_2$-coefficients, but admits a natural extension to group ring coefficients. We now assume that $m \geq 2$. Then $\mathbb{Z}[\pi_1(\Lambda)] \cong \mathbb{Z}[\pi_1(S^m \times \Lambda)] \cong \mathbb{Z}[\pi_1(\Sigma^m \Lambda)]$, and hence the coefficients of the corresponding DGAs are equivalent. Since both $A(\Lambda)$ and $A(\Sigma^m \Lambda)$ are supported in non-negative degrees, the maps $i^*, \pi^*$ between graded augmentations of $A(\Lambda)$ and of $A(\Sigma^m \Lambda)$ induced by $i$ and $\pi$ provide a one-to-one correspondence, which leads to the isomorphism of (graded) augmentation varieties $\text{Aug}(\Lambda, \mathbb{F}) \cong \text{Aug}(\Sigma^m \Lambda, \mathbb{F})$ for $m \geq 2$.

Then we consider the case when $m = 1$. From the existence of $i$ and its left inverse $\pi$ and the fact that $\mathcal{Q}_i(\Lambda) = \emptyset$ for $i < 0$ it follows that for every graded augmentation $\varepsilon$ of $\Lambda$ and every generator $c$ of $A(\Lambda)$ with grading 0, $\pi^*(\varepsilon)(c_S) = \varepsilon(c)$. This, together with the fact that $\pi_1(\Sigma^1 \Lambda) \simeq \pi_1(\Lambda) \times \mathbb{Z}$ (and hence $\pi_1(\Sigma^1 \Lambda)$ has an extra generator compared to $\pi_1(\Lambda)$), implies that there is an identification $\text{Aug}(\Sigma^1 \Lambda; \mathbb{F}) \cong \text{Aug}(\Lambda; \mathbb{F}) \times \mathbb{F}^*$. \qed

Now we assume that $\mathbb{F} = \mathbb{F}_2$, and $\varepsilon, \tilde{\varepsilon}$ are augmentations to $\mathbb{F}_2$. Note that in this case we can naturally extend $\tilde{\varepsilon}$ to be an augmentation to $\mathbb{F}_2$ by applying the inclusion $\mathbb{F}_2 \subset \tilde{\mathbb{F}}_2$. Here $\tilde{\mathbb{F}}_2$ denotes the algebraic closure of $\mathbb{F}_2$. Now we recall that according to the computation in [15, Proposition 4.1],

$$\text{Aug}(\Lambda_n; \mathbb{F}_2) \cong V = \{(a, b) \in \mathbb{F}_2^2 \mid ab \neq -1\},$$

and hence by Proposition 4.3 $\text{Aug}(\Sigma^1 \Lambda_n; \mathbb{F}_2) \cong V \times \mathbb{F}_2^*$. From Proposition 1.3 combined with Formula 4.2 and Proposition 4.3 we know that there is an injective algebraic map

$$f^*_L : \text{Aug}(\bar{L}, \mathbb{F}_2) \cong (\mathbb{F}_2^*)^3 \to \text{Aug}(\Sigma^1 \Lambda_n; \mathbb{F}_2) \cong V \times \mathbb{F}_2^*.$$ 

Note that graded augmentation $\tilde{\varepsilon}$ determines a point

$$(\tilde{\varepsilon}(a^S), \tilde{\varepsilon}(b^S), \ldots) = (0, 0, \ldots) \in \text{Aug}(\Sigma^1 \Lambda_n; \mathbb{F}_2) \cong V \times \mathbb{F}_2^*.$$ 

Then we prove the “stabilized” version of [15, Proposition 4.3]

**Proposition 4.4** Let $k$ be an algebraically closed field, and $V = \{(a, b) \in k^2 \mid ab \neq -1\}$. There is no injective algebraic map $\varphi : (k^*)^2 \times (k^*)^m \to V \times (k^*)^m$ having $(0, 0, x)$ in its image, where $x \in (k^*)^m$.

**Proof** Here we follow the proof of [15, Proposition 4.3]. Let $(s_1, s_2, t_1, \ldots, t_m)$ be the coordinates on $(k^*)^2 \times (k^*)^m$, and let $(a, b, c, x_1, \ldots, x_m)$ be the coordinates on $k^3 \times (k^*)^m$, where $V \times (k^*)^m$ is a closed subvariety cut of by the equation $(ab + 1)c = 1$. Let $A = A(s_1, s_2, t_1, \ldots, t_m)$, $B = B(s_1, s_2, t_1, \ldots, t_m)$, $C = C(s_1, s_2, t_1, \ldots, t_m)$, $X_i = X_i(s_1, s_2, t_1, \ldots, t_m)$, $i = 1, \ldots, m$, be the functions defined by the injective map $\varphi$. Then $A, B, C, X_1, \ldots, X_m$ satisfy

$$(AB + 1)C = 1.$$ 

Following the same argument as in the proof of [15, Proposition 4.3] we can assume that

$$AB = \alpha s_1^{l_1} s_2^{l_2} t_1^{n_1} \cdots t_m^{n_m} - 1,$$

where $\alpha \in k^*$, $A$ and $B$ are polynomials in $k[s_1, s_2, t_1, \ldots, t_m]$, $l_i, n_j \geq 0$, where $i = 1, 2$, $j = 1, \ldots, m$. Now we use the fact that there exists certain $p = (s_1', s_2', t_1', \ldots, t_m')$ such that $A(p)B(p) = 0$. Then, as in the proof of [15, Proposition 4.3], there are two cases.
The first case is when $l_1, l_2, n_1, \ldots, n_m = 0$. In this case we see that $\alpha = 1$, since otherwise $AB$ would not have any zero. Therefore, $AB = 0$, which implies that one of $A$ or $B$ is 0. If $A = 0$, then

$$\varphi(s_1, s_2, t_1, \ldots, t_m) = (0, B(s_1, s_2, t_1, \ldots, t_m), \ldots),$$

which contradicts injectivity of $\varphi$. We get to the same contradiction when $B = 0$.

The second case concerns the situation when there is at least one non-zero number in $\{l_1, l_2, n_1, \ldots, n_m\}$. Now assume that $\text{char}(k) = 0$ and, for example, $l_1 \neq 0$. Then we observe that

$$(AB)(s_1, s_2', t_1', \ldots, t_m') = (\alpha(s_2')^{l_2} (t_1')^{n_1} \ldots (t_m')^{n_m}) s_1^{l_1} - 1.$$

This leads to contradiction since $s_1'$ would be a multiple root of $(AB)(s_1, s_2', t_1', \ldots, t_m')$ and $(\alpha(s_2')^{l_2} (t_1')^{n_1} \ldots (t_m')^{n_m}) s_1^{l_1} - 1 \in k[s_1]$ is separable, in other words $(AB)(s_1, s_2', t_1', \ldots, t_m')$ has no multiple roots, since it is relatively prime to its formal derivative:

$$(-1)((\alpha(s_2')^{l_2} (t_1')^{n_1} \ldots (t_m')^{n_m}) s_1^{l_1} - 1) + \frac{s_1}{l_1} ((l_1 \alpha(s_2')^{l_2} (t_1')^{n_1} \ldots (t_m')^{n_m}) s_1^{l_1-1}) = 1.$$

We can use the same argument in the situation when other numbers from $\{l_1, l_2, n_1, \ldots, n_m\}$ are greater than 0. The case when $\text{char}(k) \neq 0$ completely mimics the corresponding part in the proof of [15, Proposition 4.3], and is based on the same formal derivative computation we did in the case when $\text{char}(k) = 0$.

Proposition 4.4 implies that $\bar{\varepsilon}$ is not induced by an embedded Maslov number 0 spin exact Lagrangian filling $\bar{L}$ of $\Sigma S^1 \Lambda_n$. The same way Proposition 4.3, Proposition 4.4 and Remark 4.2 imply that $\bar{\varepsilon}$, which is an augmentation of $\Sigma S^1 \ldots S^1 \Lambda_n$ defined by the inductive application of $\pi^*$ to $\varepsilon \Lambda_n$, is not induced by an embedded Maslov number 0 spin exact Lagrangian filling $\bar{L}$ of $\Sigma S^1 \ldots S^1 \Lambda_n$.

**Remark 4.5** Note that instead of the $S^1$-spinning construction, one can apply the $S^k$-spinning construction, $k \geq 2$, to Classes A and B. In order for the arguments from Sects. 3 and 4 to work, one needs to know the analogue of Künneth formula for high dimensional spuns, i.e. analogue of [7, Theorem 4.1, Remark 4.2]. This analogue of Künneth formula is expected to appear in the forthcoming work of Strakoš [21].

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**Declarations**

**Conflict of interest** The authors declare no competing interests.
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