A NOTE ON A CLASS OF \(p\)-VALENT STARLIKE FUNCTIONS OF ORDER BETA

SWADESH SAHOO* AND NAVNEET LAL SHARMA

Abstract. In this paper we obtain sharp coefficient bounds for certain \(p\)-valent starlike functions of order \(\beta\), \(0 \leq \beta < 1\). Initially this problem was handled by Aouf in M.K. Aouf, On a class of \(p\)-valent starlike functions of order \(\alpha\), Internat. J. Math. & Math. Sci. 1987;10:733–744. We pointed out that the proof given by Aouf was incorrect and a correct proof is presented in this paper.

2010 Mathematics Subject Classification. Primary 30C45; Secondary 30C55
Key words. \(p\)-valent analytic functions, starlike functions, differential subordination

1. Introduction

It is well-known that each univalent functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]

in the open unit disk \(\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}\) has the property \(|a_2| \leq 2\), with equality occurring only for rotations of the Koebe function

\[ k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n. \]

This suggests the famous conjecture of Bieberbach [2], first proposed in 1916. This states that if \(f\) in the above form is univalent in \(\mathbb{D}\) then \(|a_n| \leq n\) for all \(n \geq 2\). Initially this conjecture was proved in many special cases and has a long history. It was finally settled after several years by De Branges [4] in 1985. For basic theory of Bieberbach conjecture problem for number of classes of univalent functions we refer to [5, 9]. Part of this development, it was not generalized to the class of \(p\)-valent functions until 1948. The initiative was first taken by Goodman, see [8]. Similar problem for many other classes of \(p\)-valent functions can be found, for instance in [1, 7, 12]. In this paper we consider certain classes of \(p\)-valent functions in the unit disk and prove Bieberbach’s conjecture for these functions.

For a natural number \(p\), let \(A_p\) denote the class of functions of the form

\[ f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \]

which are analytic and \(p\)-valent in the open unit disk.

Let \(g(z)\) and \(f(z)\) be analytic in \(\mathbb{D}\). A function \(g(z)\) is called to be subordinate to \(f(z)\) if there exists an analytic function \(\phi(z)\) in \(\mathbb{D}\) with \(\phi(0) = 0\) and \(|\phi(z)| < 1\) \((z \in \mathbb{D})\) such that \(g(z) = f(\phi(z))\). We denote this subordination by \(g(z) \prec f(z)\) (see [11]).

* The corresponding author.
Let $\mathcal{S}_p(A, B, \beta)$ denote the class of functions $f(z) \in \mathcal{A}_p$ satisfying
\begin{equation}
\frac{zf'(z)}{f(z)} \prec p + \frac{[pB + (A - B)(p - \beta)]z}{1 + Bz}, \quad z \in \mathbb{D}, \quad 0 \leq \beta < 1,
\end{equation}
where $A$ and $B$ have the restriction $-1 \leq B < A \leq 1$. The class $\mathcal{S}_p(A, B, \beta)$ was considered by Aouf in [1]. As a special case, we see that
\begin{equation}
\mathcal{S}_p(1, -1, \beta) = \mathcal{S}_p(\beta), \quad \mathcal{S}_1(\beta) = \mathcal{S}_p(\beta), \quad \mathcal{S}_p(0) = \mathcal{S}_p \quad \text{and} \quad \mathcal{S}_1(A, B, 0) = \mathcal{S}_*(A, B).
\end{equation}
Note that $\mathcal{S}_p(\beta)$, the class of $p$-valent starlike functions of order $\beta$, was studied by Goluzina in [7]; $\mathcal{S}_*(\beta)$, the class of starlike functions of order $\beta$ was introduced by Robertson in [13]; $\mathcal{S}_p$, the usual class of $p$-valent starlike functions; and $\mathcal{S}_*(A, B)$ was introduced by Janowski in [10].

Aouf estimated the coefficient bounds for the functions from the class $\mathcal{S}_p(A, B, \beta)$ in [1] in which the proof is found to be incorrect. In this paper, we provide a correct proof.

2. Main result

The following Lemma is obtained by Goel and Mehrok:

**Lemma 2.1.** [6, Theorem 1] Let $-1 \leq B < A \leq 1$ and $f \in \mathcal{S}_*(A, B)$. Then
\begin{equation}
|a_2| \leq A - B;
\end{equation}
for $A - 2B \leq 1$, $n \geq 3$,
\begin{equation}
|a_n| \leq \frac{A - B}{n - 1};
\end{equation}
and for $A - (n - 1)B > (n - 2)$, $n \geq 3$,
\begin{equation}
|a_n| \leq \frac{1}{(n - 1)!} \prod_{j=2}^{n} (A - (j - 1)B).
\end{equation}
The equality signs in (2.1) and (2.2) are attained for the functions
\begin{equation}
k_{n,A,B}(z) = \begin{cases}
z(1 + B\delta z^{n-1})^{(A-B)/(n-1)B}, & \text{if } B \neq 0; \\
z \exp \left(\frac{A\delta z^{n-1}}{n-1}\right), & \text{if } B = 0,
\end{cases}
\end{equation}
and in (2.3) equality is attained for the functions
\begin{equation}
k_{A,B}(z) = \begin{cases}
z(1 + B\delta z)^{(A-B)/B}, & \text{if } B \neq 0; \\
z e^{A\delta}, & \text{if } B = 0,
\end{cases}\quad |\delta| = 1.
\end{equation}

However, a $p$-valent analog of Lemma 2.1 was wrongly proven by Aouf in the following form:

**Theorem A.** [1, Theorem 3] Let $-1 \leq B < A \leq 1$ and $p \in \mathbb{N}$. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{S}_p(A, B, \beta)$, then
\begin{equation}
|a_n| \leq \prod_{j=0}^{n-p-1} \frac{|(B - A)(p - \beta) + Bj|}{j + 1}
\end{equation}
for $n \geq p + 1$, and these bounds are sharp for all admissible $A, B, \beta$ and for each $n$.

We now give the correct form of the statement stated in Theorem A and it’s proof.
Theorem 2.2. Let \(-1 \leq B < A \leq 1\) and \(p \in \mathbb{N}\). If \(f(z) \in S_p(A, B, \beta)\) is in the form (1.1), then we have
\[
|a_{p+1}| \leq (A - B)(p - \beta);
\]
for \(A(p - \beta) - B(p - \beta - 1) \leq 1\) (or \(A(p - \beta) - B(n - \beta - 1) \leq (n - p - 1)\), \(n \geq p + 2\),
\[
|a_n| \leq \frac{(A - B)(p - \beta)}{n - p};
\]
and for \(A(p - \beta) - B(n - \beta - 1) > (n - p - 1)\), \(n \geq p + 2\),
\[
|a_n| \leq \frac{1}{n-p} \left( A(p - \beta) - B(p - \beta + j - 1) \right).
\]
The inequalities (2.6), (2.7) and (2.8) are sharp.

Proof. Let \(f(z) \in S_p(A, B, \beta)\). By the relation (1.2) we can guarantee an analytic function \(\phi : \mathbb{D} \to \overline{\mathbb{D}}\) with \(\phi(0) = 0\) such that
\[
\frac{zf'(z)}{f(z)} = p + \frac{[pB + (A - B)(p - \beta)]\phi(z)}{1 + B\phi(z)},
\]
i.e.
\[
zf'(z) - pf(z) = [(pB + (A - B)(p - \beta))f(z) - Bzf'(z)]\phi(z).
\]
Substituting the series expansion (1.1), of \(f(z)\), and canceling the factor \(z^p\) on both sides, we obtain
\[
\sum_{k=1}^{\infty} ka_{p+k}z^k = \left( (A - B)(p - \beta) - \sum_{k=1}^{\infty} [B(p + k) + (-pB + (B - A)(p - \beta))]a_{p+k}z^k \right) \phi(z).
\]
Rewriting it, we get
\[
\sum_{k=1}^{\infty} ka_{p+k}z^k = \left( (A - B)(p - \beta) + \sum_{k=1}^{\infty} [A(p - \beta) - B(k + p - \beta)]a_{p+k}z^k \right) \phi(z).
\]
By Clunie’s method [3] (for instance see [15, 14]) for \(n \in \mathbb{N}\), we observe that
\[
\sum_{k=1}^{n} k^2|a_{p+k}|^2 \leq (A - B)^2(p - \beta)^2 + \sum_{k=1}^{n-1} [A(p - \beta) - B(k + p - \beta)]^2|a_{p+k}|^2.
\]
Simplification of the above inequality leads to
\[
|a_{p+n}|^2 \leq \frac{1}{n^2} \left( (A - B)^2(p - \beta)^2 + \sum_{k=1}^{n-1} \left( [A(p - \beta) - B(k + p - \beta)]^2 - k^2 \right)|a_{p+k}|^2 \right)
\]
or
\[
|a_{p+n}|^2 \leq \frac{1}{n^2} \left( (A - B)^2(p - \beta)^2 + \sum_{k=2}^{n} \left( [A(p - \beta) - B(k + p - \beta - 1)]^2 - (k - 1)^2 \right)|a_{p+k-1}|^2 \right).
\]
Above inequality can be rewritten by replacing $p + n$ by $n$ as

$$
|a_n|^2 \leq \frac{1}{(n-p)^2} \left( (A-B)^2(p-\beta)^2 + \sum_{k=2}^{n-p} \left( [A(p-\beta) - B(k+p-\beta-1)]^2 - (k-1)^2 \right) |a_{p+k-1}|^2 \right)
$$

for $n \geq p + 1$.

Note that the terms under the summation in the right hand side of (2.9) may be positive as well as negative. We investigate it by including here a table (see Table 1) for values of $W := (A(p-\beta) - B(k+p-\beta-1))^2 - (k-1)^2$ for various choices of $A, B, k, \beta$ and $p$.

| $k$ | $p$ | $A$ | $B$ | $\beta$ | $W$  |
|-----|-----|-----|-----|--------|-----|
| 2   | 1   | 0.8 | 0.5 | 0      | -0.96 |
| 2   | 1   | -0.5| -0.8| 0      | 0.21  |
| 3   | 2   | 0.5 | 0.4 | 0.5    | -3.5775 |
| 3   | 2   | -0.1| -0.7| 0.5    | 1.29  |

Table 1

(This the place where the incorrectness of Aouf’s proof is found!)

So, we can not apply direct mathematical induction in (2.9) to establish the required bounds for $|a_n|$. Therefore, we are considering different cases for this.

First, for $n = p + 1$, we easily see that (2.9) reduces to

$$
|a_{p+1}| \leq (A-B)(p-\beta)
$$

which establishes (2.6).

Secondly, $A(p-\beta) - B(p-\beta-1) \leq 1$ if and only if $A(p-\beta) - B(n-\beta-1) \leq (n-p-1)$ for $n \geq p + 2$. Since all the terms under the summation in (2.9) are non-positive, we reduce to

$$
|a_n| \leq \frac{(A-B)(p-\beta)}{n-p}
$$

for $A(p-\beta) - B(p-\beta+1) \leq 1, n \geq p + 2$. This proves (2.7). The equality holds in (2.6) and (2.7) for the functions

$$
k_{n,A,B,p}(z) = \begin{cases} 
  z^p \left( 1 + B\delta z^{n-1} \right) (A-B)(p-\beta)/(n-1)B, & B \neq 0; \\
  z^p \exp \left( \frac{A(p-\beta)\delta z^{n-1}}{n-1} \right), & B = 0, \quad |\delta| = 1.
\end{cases}
$$

Finally let us prove (2.8) when $A(p-\beta) - B(n-\beta-1) > (n-p-1), n \geq p + 2$. We see that all the terms under the summation in (2.9) are positive. We prove the inequality by the usual mathematical induction. Fix $n, n \geq p + 2$ and suppose that (2.8) holds for
k = 3, 4, \ldots, n - p. Then from (2.9), we find

\begin{equation}
|a_n|^2 \leq \frac{1}{(n-p)^2} \left( (A-B)^2(p-\beta)^2 + \sum_{k=2}^{n-p} \left( [A(p-\beta) - B(k + p - \beta - 1)]^2 - (k-1)^2 \right) \prod_{j=1}^{k-1} \frac{[A(p-\beta) - B(p-\beta + j - 1)]^2}{j^2} \right).
\end{equation}

It is now enough to show that the square of the right hand side of (2.8) is equal to the right hand side of (2.10), that is

\begin{equation}
\prod_{j=1}^{m-p} \frac{[A(p-\beta) - B(p-\beta + j - 1)]^2}{j^2} = \frac{1}{(m-p)^2} \left( (A-B)^2(p-\beta)^2 + \sum_{k=2}^{m-p} \left( [A(p-\beta) - B(k + p - \beta - 1)]^2 - (k-1)^2 \right) \prod_{j=1}^{k-1} \frac{[A(p-\beta) - B(p-\beta + j - 1)]^2}{j^2} \right)
\end{equation}

for \( A(p-\beta) - B(m-\beta - 1) > (m-p-1) \), \( m \geq p+2 \). We also use the induction principle to prove (2.11).

The equation (2.11) is recognized for \( m = p+2 \). Suppose that (2.11) is true for all \( m, p+2 < m \leq n-p \). Then from (2.10), we obtain

\begin{align*}
|a_n|^2 &\leq \frac{1}{(n-p)^2} \left( (A-B)^2(p-\beta)^2 + \sum_{k=2}^{n-p-1} \left( [A(p-\beta) - B(k + p - \beta - 1)]^2 - (k-1)^2 \right) \right) \times \prod_{j=1}^{k-1} \frac{[A(p-\beta) - B(p-\beta + j - 1)]^2}{j^2} + \left( [A(p-\beta) - B(n-p - 1)]^2 - (n-p-1)^2 \right) \times \prod_{j=1}^{n-p-1} \frac{[A(p-\beta) - B(p-\beta + j - 1)]^2}{j^2}.
\end{align*}

Using the induction hypothesis, for \( m = n-1 \), we get

\begin{align*}
|a_n|^2 &\leq \frac{1}{(n-p)^2} \left( (n-p-1)^2 \prod_{j=1}^{n-p-1} \frac{[A(p-\beta) - B(p-\beta + j - 1)]^2}{j^2} \right) + \left( [A(p-\beta) - B(n-\beta - 1)]^2 - (n-p-1)^2 \right) \prod_{j=1}^{n-p-1} \frac{[A(p-\beta) - B(p-\beta + j - 1)]^2}{j^2}.
\end{align*}

Hence

\begin{equation}
|a_n| \leq \prod_{j=1}^{n-p} \frac{[A(p-\beta) - B(p-\beta + j - 1)]}{j}.
\end{equation}
It is easy to prove that the bounds are sharp for the function

\[ k_{A,B,p}(z) = \begin{cases} 
  z^p \left( 1 + B \delta z \right)^{(A-B)(p-\beta)/B}, & B \neq 0; \\
  z^p e^{A(p-\beta)z\delta}, & B = 0,
\end{cases} \quad |\delta| = 1. \]

This completes the proof of Theorem 2.2. \( \square \)

We remark that, choosing \( p = 1 \) and \( \beta = 0 \) in Theorem 2.2 we turned into Lemma 2.1.

**Acknowledgements.** The second author acknowledges the support of National Board for Higher Mathematics, Department of Atomic Energy, India (grant no. 2/39(20)/2010-R&D-II).

**References**

[1] M. K. Aouf, On a class of \( p \)-valent starlike functions of order \( \alpha \), *Internat. J. Math. & Math. Sci.*, 10 (1987), 733–744.

[2] L. Bieberbach, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. *S.-B. Preuss. Akad. Wiss.*, 1916, 940–955.

[3] J. Clunie, On meromorphic schlicht functions, *J. London Math. Soc.*, 34 (1959), 215–216.

[4] L. de Branges, A proof of the Bieberbach conjecture, *Acta Math.*, 154 (1985), 137152.

[5] P. L. Duren, *Univalent Functions*, Springer-Verlag, 1983.

[6] R. M. Goel and B. S. Mehrok, On the coefficients of a subclass of starlike functions, *Indian J. Pure Appl. Math.*, 12(5) (1981), 634–647.

[7] E. G. Goluzina, On the coefficients of a class of functions, regular in a disk and having an integral representation in it, *J. of Soviet Math.*, 6 (1974), 606–617.

[8] A. W. Goodman, On some determinants related to \( p \)-valent functions, *Trans. Amer. Math. Soc.*, 63 (1948), 175–192.

[9] A. W. Goodman, *Univalent Functions*, Vol. 12, Mariner, Tampa, Florida, 1983.

[10] W. Janowski, Some extremal problem for certain families of analytic functions, *Ann. Polon. Math.*, 28 (1973), 297–326.

[11] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, New York and Basel, Marcel Dekker, 2000.

[12] D. A. Patil and N. K. Thakare, On convex hulls and extreme points of \( p \)-valent starlike and convex classes with applications, *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.)* 27(75) (1983), 145–160.

[13] M. S. Robertson, On the theory of univalent functions, *Annals of Mathematics*, 37 (1936), 374–408.

[14] M. S. Robertson, Quasi-subordination and coefficient conjectures, *J. Bull. Amer. Math. Soc.*, 76 (1970), 1–9.

[15] W. Rogosinski, On the coefficients of subordinate functions, *Proc. London Math. Soc.*, 48(2) (1943), 48–82.

Swadesh Sahoo, Discipline of Mathematics, Indian Institute of Technology Indore, Indore 452 017, India

E-mail address: swadesh@iiti.ac.in

Navneet Lal Sharma, Discipline of Mathematics, Indian Institute of Technology Indore, Indore 452 017, India

E-mail address: sharma.navneet23@gmail.com