ON THE ASYMPTOTICS OF THE AUBIN-YAU FUNCTIONAL

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Abstract. We derive an explicit formula for the asymptotic slope of the Aubin-Yau functional along Bergman geodesics. The result gives a way to check directly whether a projective variety is Chow-Mumford stable.

1. Introduction

The existence of canonical metrics in Kähler geometry is now well understood to be linked to certain notions of stability in the sense of geometric invariant theory. Historically, the first result of this type was the theorem of Donaldson-Uhlenbeck-Yau on the equivalence of the existence of Hermitian-Einstein metrics on holomorphic vector bundles with Mumford-Takemoto stability [6, 18]. Following Yau’s seminal proof of the Calabi conjecture [19], the recent resolution of the Yau-Tian-Donaldson conjecture establishes the equivalence of K-stability and the existence of Kähler-Einstein (KE) metrics on Fano manifolds [20, 3, 4, 5, 9, 17, 1]. This problem of existence of KE metrics and the more general problem of existence of constant scalar curvature Kähler (cscK) metrics may be studied by a variational approach in which the existence of a canonical metric is equivalent to the properness of a certain energy functional (see [8], for example). It is therefore desirable to understand the link between algebraic notions of stability and the behavior of these functionals.

In this article we consider certain special one-parameter degenerations of the Kähler class along which the energy functionals restrict to become convex functions. Along these directions, the relevant energy functionals have an asymptotic slope that is related to algebraic stability invariants, and determines the properness of the energy. For instance, the existence of a degeneration along which the asymptotic slope is negative is an obstruction to the existence of a minimizer for the functional. Asymptotics of energy functionals are also of considerable interest in the study of partition functions over Bergman metrics [10]. The aim of this article is to establish the asymptotics of the Aubin-Yau functional, and express its asymptotic slope as an explicit formula in terms of the data of a test configuration by means of analysis of singular integrals. As a result, we obtain a way of checking the Chow-Mumford stability of a variety.

Let \((X, \omega_0)\) be a Kähler manifold of complex dimension \(n\) with reference Kähler metric \(\omega_0, \omega_\phi = \omega_0 + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi, Ric(\omega_0) = -\frac{\sqrt{-1}}{2\pi} \log \omega_0^n\) the Ricci form of \(\omega_0\), and \(V = \int_X \omega_0^n\).

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The functionals described below are defined on the space of Kähler potentials

\[ \mathcal{K} = \{ \phi \in C^\infty(X), \omega_0 + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi > 0 \}. \]  

**Definition:** The Aubin-Yau functional \( F_{\omega_0}^0(\phi) \) is given by

\[ F_{\omega_0}^0(\phi) = \frac{1}{n+1} \frac{1}{V} \int_X \phi \sum_{i=0}^n \omega_0^i \wedge \omega_{\phi}^{n-i}. \]  

(1.2)

The significance of the Aubin-Yau functional in Kähler geometry is discussed extensively in [16]. Minimizers of \( F_{\omega_0}^0(\phi) \) in the space of Bergman metrics are called balanced metrics. Zhang [21] proved that the existence of a balanced metric is equivalent to Chow-Mumford stability (see also [14]). Donaldson showed that the existence of a cscK metric implies the existence of a balanced metric, hence the Chow-Mumford stability and the existence of a minimizer for \( F_{\omega_0}^0(\phi) \) in the space \( K_k \). In this case, the asymptotic slope of the Aubin-Yau functional is necessarily positive. The asymptotic slope is thus a numerical stability invariant called the Chow weight.

Briefly, let us say how the Aubin-Yau functional is related to other functionals in the literature. It is related to the \( J \)-functional

\[ J_{\omega_0}(\phi) = \frac{\sqrt{-1}}{2\pi V} \int_X \sum_{i=0}^{n-1} \frac{(i+1)}{(n+1)} \partial \phi \wedge \bar{\partial} \phi \wedge \omega_{\phi}^{n-i-1} \wedge \omega_0^i. \]  

(1.3)

by

\[ F_{\omega_0}^0(\phi) = \frac{1}{V} \int_X \phi \omega_0^n - J_{\omega_0}(\phi). \]  

(1.4)

In the special case \([\omega_0] = [K_X^{-1}]\), \( F_{\omega_0}^0 \) is related to the functional \( F_{\omega_0}(\phi) \) by

\[ F_{\omega_0}(\phi) = -F_{\omega_0}^0(\phi) - \log \left( \frac{1}{V} \int_X e^{h_{\omega_0} - \phi} \omega_0^n \right), \quad Ric(\omega_0) - \omega_0 = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} h_{\omega_0}. \]  

(1.5)

Minimizers of \( F_{\omega_0}(\phi) \) are Kähler-Einstein metrics. Its asymptotics are discussed to establish the necessity of K-stability for existence of a KE metric and the issue of uniqueness of KE metrics in [2, 1]. Note the surprising sign on the \( F_{\omega_0}^0 \) term, given that both \( F_{\omega_0}^0 \) and \( F_{\omega_0} \) are convex along the Bergman geodesics we will define below.

Here is the setup for the degenerations we will consider: Let \( L \to X \) be a very ample line bundle, with \( S = \{ S_0, ..., S_N \} \) a basis of sections of \( H^0(X, L) \) furnishing a Kodaira embedding

\[ X \ni z \mapsto \iota_S(z) = [S_0(z), ..., S_N(z)] \in \mathbb{P}^N. \]  

(1.6)

Then the line bundle \( L \) is the pullback of the restriction to \( \iota(X) \) of the hyperplane bundle \( \mathcal{O}_{\mathbb{P}^N}(1) \). We consider the action of one-parameter subgroups \( \sigma_t \in SL(N+1, \mathbb{C}) \) acting diagonally as

\[ \sigma_t \cdot S = (t^{a_0} S_0, ..., t^{a_N} S_N), \quad a_0 + ... + a_N = 0. \]  

(1.7)
Under this action, $X$ acquires a corresponding family of Kähler metrics

$$\omega_t = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \|\sigma_t \cdot S\|_2^2, \quad \|\sigma_t \cdot S\|_2^2 = \sum_{j=0}^{N} |t|^{2a_j} |S_j|^2 \quad (1.8)$$

which are the restrictions to $\sigma_t \cdot \iota(X)$ of the Fubini-Study metric on $\mathbb{P}^N$. Written in terms of potentials, we have $\omega_t = \omega_0 + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi$, where our reference metric is $\omega_0 = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \|S\|_2^2$, and

$$\phi = \log \frac{\|\sigma_t \cdot S\|_2^2}{\|S\|^2} = \log \frac{\sum_{j=0}^{N} |t|^{2a_j} |S_j|^2}{\sum_{j=0}^{N} |S_j|^2}. \quad (1.9)$$

The finite dimensional space of such potentials as the basis of sections varies is called the Bergman space $K_1$. We may also consider larger Bergman spaces $K_k$ as we consider powers of the line bundle $L^k$ with larger bases of sections. Note that if $\phi$ is a potential in $K_1$, then $k \phi$ is a potential in $K_k$, and furthermore,

$$F^0_{k\omega}(k\phi) = kF^0_{\omega}(\phi), \quad (1.10)$$

so for our purposes it suffices to look at a single line bundle $L$.

We may assume that $a_0 \geq \ldots \geq a_N$, and we call the sections with weight equal to $a_N$ sections of lowest weight. The path $t \mapsto \phi$ defined above in the space of Kähler potentials is called a Bergman geodesic.

It is known that along such a one-parameter subgroup, $F^0_{\omega}(\phi)$ is convex in $u = \log(1/|t|)$. We aim to describe the asymptotic behavior of $F^0_{\omega}$ as $u \to \infty$, or equivalently, as $|t| \to 0$, and in particular, to determine its asymptotic slope. We employ analysis to establish that the singular behavior of the functional is $O(\log |t|)$, and use some algebra to determine the precise coefficient. The following theorem is in [11]:

**Theorem 1.1.** Let $X$ be a Riemann surface. Then

$$F^0_{\omega}(\phi) = \{-2a_N - \frac{1}{V} \sum_{\text{zeroes of } S_N} \sum_{\alpha=1}^{M} p_{\alpha}^2 (m_{\alpha} - m_{\alpha+1})\} \log \frac{1}{|t|} + O(1) \quad (1.11)$$

as $t \to 0$, where $p_{\alpha}, m_{\alpha}$ refer to the data of the Newton polygon.

Since the expression in braces above is positive, the formula gives another proof of the Chow-Mumford stability of curves. Another proof of Theorem 1.1 is given in [15], along with the slope of the Mabuchi functional. Our approach is inspired by [15], as well as earlier works on asymptotics of oscillatory integrals in [12, 13].

We derive an analogous formula for the Aubin-Yau functional in complex dimension $n = 2$ using a similar asymptotic calculation of singular integrals with certain modifications in order to deal with the new complications in higher dimensions. We expect that the approach is valid in all dimensions with analogous formulas for the slope, but in this paper we stick to dimension 2 for concreteness and ease of notation. In a subsequent note to appear shortly, the author intends to use the same techniques to compute the asymptotic slope of the more complicated Mabuchi K-energy, which is related to K-stability, and whose slopes are related to the Donaldson-Futaki invariant (see [8]).
Here are the results of the paper: First, we observe that for all the integrals that do not involve the highest power of $\omega$, the entire contribution to the slope is from the lowest weight:

**Theorem 1.2.** Assume $X$ has dimension $n = 2$. For $1 \leq i \leq n$,

$$\frac{1}{V} \int_X \phi \omega_0^i \wedge \omega_{\phi}^{n-i} = -2a_N \log \frac{1}{|t|} + O(1)$$  \hspace{1cm} (1.12)

as $t \to 0$.

The main result of this paper is the following formula for the slope:

**Theorem 1.3.** Assume $X$ has dimension $n = 2$. Then we have

$$F_{\omega_0}(\phi t) = \mu \log \frac{1}{|t|} + O(1)$$

as $|t| \to 0$ where the asymptotic slope $\mu$ is given by

$$\mu = (-2a_N) - \frac{1}{3V} \sum_{\text{Sing}(D) \cap Z(S_N)} \sum_{\text{faces } F_c \text{ of } \mathfrak{N}} 16d_c$$

$$\sum_{\{i,j,k,l\}} D_4(i,j,k,l) \int_0^\infty \int_0^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum \alpha x^{2p_\alpha} y^{2r_\alpha})^4} dx dy$$

\hspace{1cm} (1.14)

where $\mathfrak{N}$ is the Newton polytope of the data at a singular point with normal crossings, the exponents $p_i, r_i$, etc. refer to the data of the Newton diagram, and the final sum is over sets of four indices $\{i,j,k,l\}$ corresponding to an unordered selection of four points of the Newton diagram, all lying on the face $F_c$ (not necessarily vertices of $F_c$), at least three of which are distinct, and not all collinear. The sum over $\alpha$ is over all points of the Newton diagram lying on the face $F_i$. The “3” in the denominator comes from $3 = n + 1$. The term $d_c$ is defined by describing the equation of the face $F_i$ in $(p,r,q)$-space as

$$F_c = \{x_c p + b_c r + q = d_c\} \cap \mathfrak{N},$$

\hspace{1cm} (1.15)

and is positive. The positive, symmetric function denoted $D_4(i,j,k,l)$, actually a function of $(p_i, r_i, p_j, r_j, p_k, r_k, p_l, r_l)$, is defined in (2.18) below. The integrals in the formula are all convergent.

**Remark:** We note that as written, the asymptotic slope $\mu$ is a difference of positive terms: a positive trivial contribution $-2a_N$ from the lowest weight, minus the positive nontrivial contribution.

As a corollary, note also that the formula shows that the slope is linear and homogeneous in the weights $a_i$ or $q_i$, at least for fixed geometries of the Newton polytope.

The outline of the paper is as follows: in section two, we describe the proof of Theorem 1.3. We begin by isolating the contribution from the lowest weight. We then calculate the lowest order terms that appear in the volume forms, making use of some algebraic
identities that give us the determinant-like quantities $D_4(i, j, k, l)$. At this point, we introduce and describe the important features of the Newton diagram associated to a one-parameter subgroup, and carry out the computation of the singular part of the integrals. In the final section, we carry through the slope calculation for some simple examples.

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2. Details of the Slope Calculation

It is convenient to utilize the notation from [15] and isolate the lowest power of $|t|$ as follows:

$$
\phi = \log \frac{|\sigma S|^2}{|S|^2} - 2a_N \log \frac{1}{|t|}, \quad |\sigma S|^2 = \sum_{j=0}^{N} |t|^{2q_j}|S_j|^2, \quad |S|^2 = \sum_{j=0}^{N} |S_j|^2,
$$

(2.1)

where the exponents

$$
q_j = a_j - a_N \geq 0
$$

(2.2)

are the nonegative weights. Note that at least one of the non-negative weights is equal to 0. By the assumption that the basis of sections furnishes a smooth Kodaira embedding, there is no point on $X$ where all of the sections vanish. This implies that $\log |S|^2$ is bounded on $X$, and therefore

$$
\int_X \log |S|^2 \omega_0^i \omega_\phi^{-i} \leq CV = O(1),
$$

(2.3)

so we may drop this term from $\phi$ for the calculation of the asymptotic slope. It is then trivial to compute the contribution to the slope from the section of lowest weight, since

$$
\frac{1}{V} \int_X -2a_N \log \frac{1}{|t|} \omega_0^i \omega_\phi^{-i} = -2a_N \log \frac{1}{|t|} \int_X \omega_0^i \omega_\phi^{-i}
$$

(2.4)

for each $0 \leq i \leq n$. These $n + 1$ terms account for the overall contribution of $-2a_N$ (which is non-negative since $a_N \leq 0$) to the asymptotic slope.

We set out to determine the nontrivial contribution to the slope, that is, to compute the singular part of

$$
A_i(t) = \int_X \log |\sigma S|^2 \omega_0^i \omega_\phi^{-i}.
$$

(2.5)

Here is the basic idea: The singular part of the global integral $A_i(t)$ may be calculated by integrating only over neighborhoods of isolated points, namely the transverse intersection points of the zero divisor of the section(s) of lowest weight with itself and the zero divisors of the other sections.
Proof. Observe first that the integrand of \( A_i(t) \) is bounded away from the union of the zero sets of the sections of lowest weight. Let \( s_N \) be a section of lowest weight. Suppose that in a neighborhood of a smooth point \( p = 0 \) of \( \{ S_N = 0 \} \), we may take complex coordinates \( z_1, z_2 \) (possibly after a resolution) in which \( \{ S_N = 0 \} = \{ z_1 = 0 \} \), and each of the other sections in this trivialization may be written in the form \( s_i = z_i^p u_i(z_1, z_2) \), where \( u_i \) is a unit. Then it will follow from the calculations below that the volume forms \( \omega_0^2, \omega_0 \wedge \omega_\phi, \) and \( \omega_\phi^2 \) only contain terms of strictly higher order in \( |z_1| \) and \( |z_2| \), and thus there is no contribution to the \( \log |t| \) term in \( F_{\omega_0}^0 \) by Lemma 2.7.

In general, we recall Hironaka’s result on resolution of singularities: There exists a resolution \( \mu : \tilde{X} \to X \) such that \( \mu^* D + \text{Exc}(\mu) = \tilde{D} \) has simple normal crossing support. On \( \tilde{X} \), the nontrivial contributions to the slope come from a finite set of points of intersections with the other divisors with \( \mu^* D_N \).

Assume that we have a set of coordinates in a neighborhood of a point, taken to be the origin, at which the sections vanish with normal crossings. This means that we may take complex holomorphic functions that do not vanish at the origin. For a more detailed account of Hironaka’s theorem and its use in the analysis of integrals see [13].

2.1. Description of the volume form. We must first compute \( \omega_\phi, \omega_0 \wedge \omega_\phi, \) and \( \omega_\phi^2 \). We find

\[
\omega_\phi = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |\sigma S|^2
\]

\[
= \frac{\sqrt{-1}}{2\pi |\sigma S|^4} \sum_{i,j} |t|^{2q_i + 2q_j} (|S_j|^2 \partial S_i \wedge \bar{\partial} S_i - S_i S_j \partial S_j \wedge \bar{\partial} S_j)
\]

\[
= \frac{\sqrt{-1}}{2\pi |\sigma S|^4} \sum_{i,j} |t|^{2(q_i + q_j)} |u_i(0)|^2 |u_j(0)|^2 |x|^{2(p_i + p_j - 1)} |y|^{2(r_i + r_j - 1)}
\]

\[
((p_i^2 - p_i p_j) |y|^2 dx \wedge d\bar{x} + (p_i r_i - p_j r_i) x y dy \wedge d\bar{y} + (p_i r_i - p_j r_j) x \bar{y} dy \wedge d\bar{x} + (r_i^2 - r_j r_i) |x|^2 dy \wedge d\bar{y})
\]

\[+ O(...).
\]  

Here by \( O(\ldots) \) we mean all higher order terms in \( |x| \) and \( |y| \). Taking the wedge product yields

\[
\omega_\phi^2 = \frac{1}{4\pi^2 |\sigma S|^8} \sum_{i,j,k,l} |t|^{2(q_i + q_j + q_k + q_l)} |x|^{2(p_i + p_j + p_k + p_l - 1)} |y|^{2(r_i + r_j + r_k + r_l - 1)} |u_i|^2 |u_j|^2 |u_k|^2 |u_l|^2
\]

\[
2 \left[ (p_i^2 - p_i p_j) (r_k^2 - r_k r_l) - (p_i r_i - p_j r_i) (p_k r_k - p_l r_l) \right] \sqrt{-1} dx \wedge d\bar{x} \wedge \sqrt{-1} dy \wedge d\bar{y}
\]

\[+ O(...),
\]  

where we have used the symmetry in the indices \( (i,j) \leftrightarrow (k,l) \) to obtain twice the quantity in brackets. The quantity in the brackets we will denote by the symbol \( (ijkl) \), and it may be simplified as
\[ (ijkl) = (p_i^2 - p_ip_j)(r_j^2 - r_kr_l) - (p_ir_i - p_jr_j)(p_kr_k - p_kr_l) \\
= (p_ir_k - p_kr_i)(p_i - p_j)(r_k - r_l). \] (2.11)

The other volume forms are the same except for the factors of \(|t|\) that only appear in \(\omega_\phi\):
\[
\omega_0 \wedge \omega_\phi = \frac{1}{4\pi^2|S|^4|\sigma S|^4} \sum_{i,j,k,l} |t|^{2(q_i+q_j)}|x|^{2(p_i+p_j+p_k+p_l-1)}|y|^{2(r_i+r_j+r_k+r_l-1)}|u_i|^2|u_j|^2|u_k|^2|u_l|^2 \\
\left[ (ijkl) + (klji) \right] \sqrt{-1} dx \wedge d\bar{x} \wedge \sqrt{-1} dy \wedge d\bar{y} \\
+ O(...). \] (2.12)

We will see later that it is sufficient to consider only these lowest order terms. From now on, we will also assume that \(u_j(0) = 1\) for \(j = 0, \ldots, N\), which we may do since the asymptotic slope is independent of the basis.

The appearance of the determinant-like quantity \((ijkl)\) is a novel feature in dimension \(n > 1\). In particular, it rules out terms where the same index is taken in each of the four sums. Let us make some simple observations about the symbol \((ijkl)\). First, \((ijkl) = 0\) if \(i = j, k = l,\) or \(i = k\). If \(i\) and \(j\) are distinct indices, the only possibly non-zero symbols involving only \(i\) and \(j\) are \((ijji)\) and \((jiii)\). But
\[
(ijji) = (p_ir_j - p_jr_i)(p_i - p_j)(r_j - r_i) \\
= -(p_jr_i - p_ir_j)(p_j - p_i)(r_i - r_j) \\
= -(jiii), \] (2.13)

so \((ijji) + (jiii) = 0\), and therefore there are no terms in the lowest order part of the volume form with only two distinct indices taken from the sum.

Now consider the case of three distinct indices:

**Lemma 2.1.** The nonzero symbols \((ijkl)\) consisting of a set of three indices with one repeated have sum
\[
(ijki) + (ikji) + (jiik) + (kiii) + (kiji) + (jikii) = (p_jr_i - p_kr_i - p_ir_j + p_kr_k - p_jr_k)^2 \\
= ([ij] - [ik] + [jk])^2, \] (2.14)
where \([ij] = p_ir_j - p_jr_i\). The sum represents the square of the area of any parallelogram with three vertices \(\{(p_i, r_i), (p_j, r_j), (p_k, r_k)\}\), and is nonnegative and symmetric in the indices \(i, j, k\).

And the remaining case of four distinct indices:

**Lemma 2.2.** The summation of symbols \((ijkl)\) over all permutations of 4 distinct indices yields
\[
\sum_{\sigma \in S_4} (\sigma(i)\sigma(j)\sigma(k)\sigma(l)) = ([ij] - [ik] + [jk])^2 + ([jk] - [jl] + [kl])^2 + ([kl] - [ki] + [li])^2 + ([li] - [lj] + [ij])^2 \\
+ O(...). \] (2.16)
These two algebraic identities may be verified by a computer algebra system.

As a corollary, it is clear from these formulas that the lowest order terms in the volume form are non-negative, and are equal to 0 if the four indices correspond to collinear points. We set

$$D_3(i, j, k) = (ij - [ik] + [jk])^2$$  \hspace{1cm} (2.17)

and

$$D_4(i, j, k, l) = \begin{cases} D_3(i, j, k) + D_3(j, k, l) + D_3(k, l, i) + D_3(l, i, j) & \text{if all indices distinct} \\ \frac{1}{2} (D_3(i, j, k) + D_3(j, k, l) + D_3(k, l, i) + D_3(l, i, j)) & \text{if any two indices are the same} \end{cases}$$ \hspace{1cm} (2.18)

The factor of 1/2 in the case of a repeated index compensates for the overcounting by transposing the slots of the repeated index, and so $D_4(i, j, k, k) = D_3(i, j, k)$.

We may thus rewrite the lowest-order part of $\omega^2$ as a sum of positive terms as

$$\omega^2 = \frac{1}{|\sigma S|^8} \sum_{i,j,k,l} \left[ 2D_4(i, j, k, l)|t|^{2(q_i + q_j + q_k + q_l)}|x|^{2(p_i + p_j + p_k + p_l)} - 2|y|^{2(r_i + r_j + r_k + r_l)} - 2 \sqrt{\frac{-1}{2\pi}} \frac{d\xi \wedge d\xi}{2\pi} \frac{d\eta \wedge d\eta} + O(...) \right]$$ \hspace{1cm} (2.19)

and similarly for $\omega^2_0$ and $\omega_0 \wedge \omega_\phi$.

2.2. Newton diagram. The analysis of the singular integrals appearing in the Aubin-Yau functional is well facilitated by appealing to the geometry of the Newton polytope.

**Definition:** We call the set of points $\{(p_j, r_j, q_j) \in \mathbb{R}_+^3 \}_{j=0}^N$ the **Newton diagram** of the data. The **Newton polytope** $\mathfrak{N}$ is the region given by

$$\mathfrak{N} = \text{ConvexHull} \left( \bigcup_{j=0}^N \{(p_j, r_j, q_j) + \mathbb{R}_+^3 \} \right),$$ \hspace{1cm} (2.20)

that is, the unbounded convex polytope which is the convex hull of the union of the positive orthant $\mathbb{R}_+^3 = \{(p, r, q) \in \mathbb{R}^3 | p, r, q \geq 0 \}$ translated to each point in the Newton diagram.

The vertices of $\mathfrak{N}$ are a subset of the data $\{p_v, r_v, q_v\}_{v=1}^L$.

Fix $t$ with $0 < |t| < 1$. We examine the sum $|\sigma S|^2 = \sum_{i=0}^N |t|^{2q_i} |x|^{2p_i} |y|^{2r_i}$ that appears in the denominator and as a factor of $\log |\sigma S|^2$ in the integrals under consideration. It is convenient to describe the regions in the $x, y$ variables where each particular term is dominant; for fixed $x, y, t$, we say that the term $|t|^{2q_i} |x|^{2p_i} |y|^{2r_i}$ dominates the other terms in the sum if

$$|t|^{2q_i} |x|^{2p_i} |y|^{2r_i} \geq |t|^{2q_j} |x|^{2p_j} |y|^{2r_j} \text{ for all } j \neq i. \hspace{1cm} (2.21)$$

It is simple to describe the regions where the various terms dominate in terms of variables $\alpha, \beta$ where we set $|x| = |t|^{\alpha}, |y| = |t|^{\beta}$, where $\alpha, \beta$ are real and non-negative. The term $|t|^{2q_i} |x|^{2p_i} |y|^{2r_i}$ dominates when

$$|t|^{2(q_i + p_i \alpha + r_i \beta)} \geq |t|^{2(q_j + p_j \alpha + r_j \beta)}$$ \hspace{1cm} (2.22)
for all other points \((p_j, r_j, q_j)\) in the diagram, or
\[
q_j + p_j \alpha + r_j \beta \geq q_i + p_i \alpha + r_i \beta, \quad j \neq i.
\]
(2.23)
Together with the constraints \(\alpha \geq 0, \beta \geq 0\), these inequalities describe the region \(M_i\) where \(|t|^{2q_i}|x|^{2p_i}|y|^{2r_i}\) dominates as the intersection of a set of half-planes. \(M_i\) describes the set of planes passing through the point lying below all the other points of the diagram, so it has empty interior unless the point \((p_i, r_i, q_i)\) is a vertex of the Newton polytope.

**Lemma 2.3.** The positive quadrant \(\alpha, \beta \geq 0\) is partitioned into polygonal regions \(M_v, v = 1, \ldots, L\), on which the term \(|t|^{2q_v}|x|^{2p_v}|y|^{2r_v}\) dominates. For each vertex with \((p_v, r_v) \neq (0, 0)\), the region \(M_v\) is the convex hull of the points \((m^x_{v,i}, m^y_{v,i})\), where \(i\) ranges over the number of faces of \(\mathcal{R}\) incident to the vertex, with normal vector \((m^x_{v,i}, m^y_{v,i}, 1)\).

Note that the set of points \((m^x_{v,i}, m^y_{v,i})\) forms a natural dual of the Newton polyhedron. Every Newton polytope for test configurations of this form also contains vertical faces \(x = 0\) and \(y = 0\), which do not play a role in the analysis.

In the interior of the region where \(|t|^{2q_v}|x|^{2p_v}|y|^{2r_v}\) dominates, we may Taylor expand to obtain
\[
\log |\sigma S|^2 = \log(|t|^{2q_v}|x|^{2p_v}|y|^{2r_v}) + O\left(\frac{\sum_{j \neq v} |t|^{2q_j}|S_j|^2}{|t|^{2q_v}|x|^{2p_v}|y|^{2r_v}}\right),
\]
(2.24)
\[
\frac{1}{|\sigma S|^2} = \frac{1}{|t|^{2q_v}|x|^{2p_v}|y|^{2r_v}} + O\left(\frac{\sum_{j \neq v} |t|^{2q_j}|S_j|^2}{|t|^{2q_v}|x|^{2p_v}|y|^{2r_v}}\right).
\]
(2.25)
At the boundary of the region \(M_v\), the term \(|t|^{2q_v}|x|^{2p_v}|y|^{2r_v}\) shares the same order as other terms.

### 2.3. Singular integral analysis

We aim to calculate the divergent term in the Aubin-Yau functional as \(|t| \to 0\), which is of the form \(\log(1/|t|)\). Let \(U\) be a small polydisk around the origin, which we may take to have radius 1. Subtracting off contributions of order \(O(1)\), we calculate the contribution from the term \(\int_U \phi \omega_\phi^2\) as follows:

\[
\int_U \phi \omega_\phi^2 = \int_U \frac{\log |\sigma S|^2}{4\pi^2 |\sigma S|^8} \sum_{\{i,j,k,l\}} \left|t\right|^{2(q_i+q_j+q_k+q_l)}|x|^{2(p_i+p_j+p_k+p_l-1)}|y|^{2(r_i+r_j+r_k+r_l-1)} \\
2D_4(i,j,k,l) \sqrt{-1} dx \wedge d\bar{x} \wedge \sqrt{-1} dy \wedge d\bar{y} + O(1)
\]
(2.26)
\[
= 8 \int_0^1 \int_0^1 \sum_{\{i,j,k,l\}} D_4(i,j,k,l) \left|t\right|^{2(q_i+q_j+q_k+q_l)}|x|^{2(p_i+p_j+p_k+p_l-1)}|y|^{2(r_i+r_j+r_k+r_l-1)} \frac{\log |\sigma S|^2}{|\sigma S|^8} dx\, dy + O(1).
\]
(2.27)
where we integrate out the angular variables. In the last line, at the risk of some confusion, we have allowed \(x\) and \(y\) to stand for the real absolute values of the complex numbers in the first line. We must be careful with all factors of 2 and \(\pi\) in this calculation, since we must compare these localized integrals to the global contribution from the lowest weight.
The factor of 4 comes from the change to polar coordinates in both $x$ and $y$ (as complex variables):

$$\frac{\sqrt{-1}}{2\pi} dx \wedge d\bar{x} = \frac{\sqrt{-1}}{2\pi} (du + \sqrt{-1}dv) \wedge (du - \sqrt{-1}dv)$$  \hspace{1cm} (2.28)

$$= \frac{2}{2\pi} du \wedge dv$$  \hspace{1cm} (2.29)

$$= \frac{2}{2\pi} r dr \wedge d\theta$$  \hspace{1cm} (2.30)

Integrating each of the $\theta$ variables cancels a factor of $2\pi$ in denominator.

We set

$$A(t) = 8 \int_0^1 \int_0^1 \sum_{\{i,j,k,l\}} D_4(i, j, k, l) |t|^{2(q_i+q_j+q_k+q_l)x^2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1} \log |\sigma S|^2 dxdy$$

and set to determine the asymptotic behavior of $A(t)$. We begin with two very simple lemmas.

**Lemma 2.4.** Let $A, B, \{p_i\}_{i=0}^N, \{r_i\}_{i=0}^N$ be non-negative integers such that $A + B \geq \min_i\{p_i + r_i\}$ and $A + B \leq \max_i\{p_i + r_i\} - 3$. Then the integral

$$I = \int_0^{\infty} \int_0^{\infty} \frac{x^A y^B}{\sum_{i=0}^n x^{p_i} y^{r_i}} dxdy < +\infty.$$  \hspace{1cm} (2.31)

*Proof.* We set $r^2 = x^2 + y^2$ and let $I = I_1 + I_2$ where the domain of integration of $I_1$ is $r < 1$ and that of $I_2$ is $r > 1$. We have

$$I_1 \leq \int_0^1 \int_0^1 \left| \frac{x^A y^B}{\sum_{i=0}^n x^{p_i y^{2r_i}}} \right| dxdy$$  \hspace{1cm} (2.32)

$$\leq c \int_0^1 r^{A+B+1} \frac{d}{r^{\min\{p_i+r_i\}}} dr$$  \hspace{1cm} (2.33)

$$\leq c \int_0^1 r dr$$  \hspace{1cm} (2.34)

$$< \infty.$$  \hspace{1cm} (2.35)

Similarly,

$$I_2 \leq c \int_1^{\infty} \frac{r^{A+B+1}}{r^{\max\{p_i+r_i\}}} dr$$  \hspace{1cm} (2.36)

$$\leq c \int_1^{\infty} r^{-2}$$  \hspace{1cm} (2.37)

$$< \infty.$$  \hspace{1cm} (2.38)

$\square$

Note that this argument also shows...
Lemma 2.5.\[I = \int_0^\infty \int_0^\infty x^A y^B \log P(x, y) dx dy < +\infty,\] (2.39)

where \(P(x, y)\) is a positive polynomial.

Now we return to the setting of our Bergman geodesic and the integral \(A(t)\). First we prove that we may take the upper limits of integration to \(+\infty\) without changing the singular part of the integral. This will facilitate the scaling we wish to make later on.

Lemma 2.6. Suppose \(\{i, j, k, l\}\) are a set of indices so that

\[2(p_i + p_j + p_k + p_l + r_i + r_j + r_k + r_l) - 1 \leq 4 \max_{\alpha; q_\alpha = 0} \{2(p_\alpha + r_\alpha)\} - 2.\] (2.40)

Then

\[
\begin{align*}
\int_0^1 \int_0^1 \frac{t|t|^{2(q_i + q_j + q_k + q_l)} x^{2(p_i + p_j + p_k + p_l) - 1} y^{2(r_i + r_j + r_k + r_l) - 1}}{|\sigma S|^8} \log |\sigma S|^2 dx dy &= \\
\int_0^\infty \int_0^\infty \frac{t|t|^{2(q_i + q_j + q_k + q_l)} x^{2(p_i + p_j + p_k + p_l) - 1} y^{2(r_i + r_j + r_k + r_l) - 1}}{|\sigma S|^8} \log |\sigma S|^2 dx dy + O(1)
\end{align*}
\] (2.41)

Proof. We have

\[
\begin{align*}
\int_1^\infty \int_1^\infty &\frac{t|t|^{2(q_i + q_j + q_k + q_l)} x^{2(p_i + p_j + p_k + p_l) - 1} y^{2(r_i + r_j + r_k + r_l) - 1}}{|\sigma S|^8} \log |\sigma S|^2 dx dy \\
&\leq \int_1^\infty \int_1^\infty \frac{t|t|^{2(q_i + q_j + q_k + q_l)} x^{2(p_i + p_j + p_k + p_l) - 1} y^{2(r_i + r_j + r_k + r_l) - 1}}{(\sum_{\alpha; q_\alpha = 0} x^{2p_\alpha y^{2r_\alpha}})^4} \log |\sigma S|^2 dx dy.
\end{align*}
\] (2.42)

By taking \(|t|\) sufficiently small, we may write the term \(\log |\sigma S|^2\) as

\[
\log |\sigma S|^2 \leq \log(2 \sum_{\alpha; q_\alpha = 0} x^{2p_\alpha y^{2r_\alpha}}) = \log(\sum_{\alpha; q_\alpha = 0} x^{2p_\alpha y^{2r_\alpha}}) + \log 2.
\] (2.43)

(2.44)

By (2.40), the integral

\[
\int_1^\infty \int_1^\infty \frac{x^{2(p_i + p_j + p_k + p_l) - 1} y^{2(r_i + r_j + r_k + r_l) - 1}}{(\sum_{\alpha; q_\alpha = 0} x^{2p_\alpha y^{2r_\alpha}})^4} \log |\sigma S|^2 = O(1)
\] by Lemma 2.5, and the same holds for the same integral multiplied by \(|t|^{2(q_i + q_j + q_k + q_l)}\), a non-negative power of \(|t|\). \(\square\)

Let us remark that in the case when the inequality (2.40) is not satisfied, then

\[
\begin{align*}
\int_0^1 \int_0^1 &\frac{t|t|^{2(q_i + q_j + q_k + q_l)} x^{2(p_i + p_j + p_k + p_l) - 1} y^{2(r_i + r_j + r_k + r_l) - 1}}{|\sigma S|^8} \log |\sigma S|^2 dx dy \\
&\leq \int_0^\infty \int_0^\infty \frac{t|t|^{2(q_i + q_j + q_k + q_l)} x^{2(p_i + p_j + p_k + p_l) - 1} y^{2(r_i + r_j + r_k + r_l) - 1}}{(\sum_{\alpha; q_\alpha = 0} x^{2p_\alpha y^{2r_\alpha}})^4} \log |\sigma S|^2 dx dy \\
&= O(1),
\end{align*}
\] (2.45)
Lemma 2.7. Let \( \{i, j, k, l\} \) be a set of indices such that \( D_4(i, j, k, l) \neq 0 \) and such that (2.40) holds. Then

\[
I(t) = \int_0^\infty \int_0^\infty \frac{|t|^{q_i + q_j + q_k + q_l} x^{2(p_i + p_j + p_k + p_l) - 1} y^{2(r_i + r_j + r_k + r_l) - 1}}{|\sigma S|^8} \log |\sigma S|^2 dx dy
= 2d_F \log |t| \int_0^\infty \int_0^\infty \frac{x^{2(p_i + p_j + p_k + p_l) - 1} y^{2(r_i + r_j + r_k + r_l) - 1}}{\sum_\alpha x^{2\alpha} y^{2r_\alpha}} dx dy + O(1) \tag{2.46}
\]

where the sum is over indices \( \alpha \) where the points \( (p_\alpha, r_\alpha, q_\alpha) \) lie on the face \( F \) of \( \mathfrak{N} \), where \( F \) is a subset of the plane given by the equation \( m^x_F x + m^y_F y + z = d_F \).

Proof. We compute the integral by rescaling. Let \( x \to |t|^{m^x} x, y \to |t|^{m^y} y \), where \( m^x, m^y \geq 0 \). The integral becomes

\[
I(t) = \int_0^\infty \int_0^\infty \frac{|t|^{q_i + q_j + q_k + q_l} x^{2(p_i + p_j + p_k + p_l) - 1} y^{2(r_i + r_j + r_k + r_l) - 1}}{|\sigma S|^8} \log |\sigma S|^2 dx dy
= \int_0^\infty \int_0^\infty \frac{x^{2(p_i + p_j + p_k + p_l) - 1} y^{2(r_i + r_j + r_k + r_l) - 1}}{(\sum_\alpha x^{2\alpha} y^{2r_\alpha})^4} \log(\sum_{u=0}^N |t|^{2(q_u + m^x p_u + m^y r_u)} x^{2p_u} y^{2r_u}) dx dy. \tag{2.47}
\]

Setting \( \gamma = q_v + m^x p_v + m^y r_v = \min_u \{q_u + m^x p_u + m^y r_u\} \), we may factor this term out of the denominator. The integrand thus acquires an overall factor of

\[
|t|^{2((q_i + q_j + q_k + q_l) + m^x (p_i + p_j + p_k + p_l) + m^y (r_i + r_j + r_k + r_l) - 4\gamma)}.
\]

Now the minimum \( \gamma \) is realized at \( q_v + m^x p_v + m^y r_v \) if \( (p_v, r_v, q_v) \) is a vertex of \( \mathfrak{N} \) and \( (m^x, m^y) \in M_v \). We observe

\[
[(q_i - q_v) + m^x (p_i - p_v) + m^y (r_i - r_v)] + \cdots + [(q_l - q_v) + m^x (p_l - p_v) + m^y (r_l - r_v)] \geq 0. \tag{2.48}
\]

If this exponent is strictly greater than 0, then by Lemma 2.5, the integral is \( O(1) \) as \( t \to 0 \). Equality is obtained only if each term in brackets is 0. In this case, the point \( (m^x, m^y) \) must be contained in \( M_i \cap M_j \cap M_k \cap M_l \), and since at least three of \( i, j, k, l \) must be distinct, \( (m^x, m^y) \) must lie at a corner of \( M_v \), or in other words, each of the points \( (p_i, r_i, q_i), \ldots, (p_l, r_l, q_l) \) lies on a common face \( F \) of \( \mathfrak{N} \) with normal vector \( (m^x, m^y, 1) \), and \( \gamma = d_F \). This shows that only sets of indices corresponding to four points lying on a single face of the Newton polytope contribute to the asymptotic slope, and moreover, that all higher-order terms in \( \omega^0_i \land \omega^{m^x-1}_i \) do not contribute to the slope.
For a given term with all four indices corresponding to points on a face \( F \) of the Newton diagram, after scaling by the coordinates of the normal vector of the face, we have

\[
I(t) = \int_0^\infty \int_0^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum_\alpha x^{2\rho_\alpha}y^{2\alpha} + |t|P(x,y,|t|))^{\frac{1}{4}}} (\log |t|^{2\gamma} + \log(\sum_\alpha x^{2\rho_\alpha}y^{2\alpha} + |t|P(x,y,|t|))) dxdy
\]

(2.49)

\[
= 2\gamma \log |t| \int_0^\infty \int_0^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum_\alpha x^{2\rho_\alpha}y^{2\alpha} + |t|P(x,y,|t|))^{\frac{1}{4}}} dxdy + O(1)
\]

(2.50)

where the sum is over all the indices \( \alpha \) of points on the face \( F \), and \( P(x,y,|t|) \) is a polynomial. The last line is obtained by Taylor expanding the denominator and \( \log(\sum_\alpha x^{2\rho_\alpha}y^{2\alpha} + |t|P(x,y,|t|)) \) term. We may use Lemma 2.5 since if all the points \((p_i,r_i,q_i),..., (p_i,r_i,q_i)\) lie on the face \( F \) and \( D(x,j,k,l) \neq 0 \), then

\[
2(p_i + p_j + p_k + p_l + r_i + r_j + r_k + r_l) - 1 \leq 4 \max_\alpha \{2(p_\alpha + r_\alpha)\} - 2
\]

(2.51)

and

\[
2(p_i + p_j + p_k + p_l + r_i + r_j + r_k + r_l) - 1 \geq 4 \min_\alpha \{2(p_\alpha + r_\alpha)\} + 1.
\]

(2.52)

\[
\square
\]

Now let us show there is no non-trivial contribution to the slope from the terms in the Aubin-Yau functional involving \( \omega^2 \) and \( \omega \wedge \omega_\phi \); that is, their only contribution is from the highest weight. For the integral \( \int_X \log |\sigma S|^2 \omega_0^n \) this is easy to see, since \( \omega_0^n \) is bounded independent of \( t \) and \( \log |\sigma S|^2 \leq \log |S_N|^2 + c \) is integrable on \( X \), therefore \( \int_X \log |\sigma S|^2 \omega_0^n \leq C = O(1) \).

It remains to show \( \int_X \log |\sigma S|^2 \omega_0 \wedge \omega_\phi \) is bounded as \( |t| \to 0 \). This can be seen by computing as before:

\[
\int_U \log |\sigma S|^2 \omega \wedge \omega_\phi = \int_U \frac{\log |\sigma S|^2}{4\pi^2 |S|^4 |\sigma S|^4} \sum_{i,j,k,l} |t|^{2(q_i+q_j)} |x|^{2(p_i+p_j+p_k+p_l)-1} |y|^{2(r_i+r_j+r_k+r_l)-1} ((ijkl) + (klij)) \sqrt{-dx \wedge d\bar{x} \wedge \sqrt{-dy \wedge d\bar{y}}} + O(1)
\]

(2.53)

\[
= 4 \int_0^\infty \int_0^\infty \sum_{i,j,k,l} ((ijkl) + (klij)) \log |\sigma S|^2
\]

\[
\frac{|t|^{2(q_i+q_j)} x^{2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1}}{|S|^4 |\sigma S|^4} dxdy + O(1)
\]

(2.54)

Under the scaling \( x \to |t|^m x \), \( y \to |t|^m y \), we may pull out a factor of

\[
|t|^{2(q_i+q_j)+m^x(p_i+p_j+p_k+p_l)+m^y(r_i+r_j+r_k+r_l)-2(q_\alpha+m^xp_\alpha+m^yr_\alpha)},
\]

where \( q_\alpha + m^x p_\alpha + m^y r_\alpha = \min_i \{q_i + m^x p_i + m^y r_i\} \). We find that the exponent

\[
[(q_i - q_\alpha) + m^x(p_i - p_\alpha) + m^y(r_i - r_\alpha)] + [(q_j - q_\alpha) + m^x(p_j - p_\alpha) + m^y(r_j - r_\alpha)]
\]

\[
+ [m^x p_k + m^y r_k] + [m^x p_l + m^y r_l] \geq 0
\]

(2.55)
with equality only if each term in brackets is 0. But this can only happen when \((p_k, r_k) = (p_l, r_l) = (0, 0)\), in which case \((ijkl) = (kl ij) = 0\). It follows that the lowest order integral is a convergent integral multiplied by a positive power of \(|t|\), which is \(O(1)\) as \(|t| \to 0\).

Thus we have proven Theorem 1.2, and when combined with Lemma 2.7, we obtain Theorem 1.3.

3. Examples

The formula in (1.3) is most easily applicable in the case of toric surfaces. A polarized toric surface \((X, L)\) is associated to a polygon \(P\) in the first quadrant of \(\mathbb{R}^2\) with integral vertices including the point \((0, 0)\). The lattice points \((p_i, r_i)\) of \(P\) are in 1-1 correspondence with a basis of sections \(S_i = x^{p_i} y^{r_i}\) of \(L\) in the coordinates of an open, dense subset of \(X\). We must also be careful with the computation of our normalized volume. The volume of \(X, V = \int_X \omega_0^2\) is related to the Euclidean volume of the polygon \(P\) by

\[
Vol_{Euc}(P) = \int_X \frac{\omega_0^2}{2} = \frac{V}{2}.
\]

3.1. Projective space \(\mathbb{P}^2\). The bundle \(O(1)\) over \(\mathbb{P}^2\) is represented by a triangle with vertices \((0, 0), (1, 0), (0, 1)\). The area of the triangle is 1/2, so \(V = \int_X \omega_0^2 = 1\). We may specify a test-configuration or Bergman geodesic by assigning a non-negative weight over each point. The lowest weight contribution is twice the average of the non-negative weights. The setup is symmetric with respect to the points \((1, 0)\) and \((0, 1)\), so there are not many essentially different configurations. Here are the possibilities:

1. The weight at \((0, 0)\) is 0. In this case, the Newton polytope is trivial, consisting of the entire positive orthant, and there is no non-trivial contribution to the slope. The slope is positive and comes entirely from the lowest weight.
2. The weight at \((0, 0)\) is greater than zero, which we may take to be 1 by the linear homogeneity of the slope in the weights. At least one of the remaining weights must be zero. We take our Newton diagram to be \[\{(0, 0, 1), (0, 1, q), (1, 0, 0)\}\]. There are two possibilities for \(q\):
   a. \(q > 1\): In this case, the Newton polytope has only one non-trivial face, given by the equation \(x + z = 1\), and only the points \((0, 0, 1)\) and \((1, 0, 0)\) lie on it. There is no non-trivial contribution to the slope. The slope is equal to \(\mu = 2(1 + q)/3\).
   b. \(0 \leq q \leq 1\): Again the Newton polytope consists of just the face \(F : x + (1 - q)y + z = 1 = d_F\). Now all three points lie on the face. We have \(D_4(1, 2, 3, 3) = 1\). The slope is given by

\[
\mu = \frac{2(1 + q)}{3} - \frac{1}{3} 16 \cdot 1 \cdot 1 \cdot (I_1 + I_2 + I_3)
\]
where the three integrals come from the three choices of repeated index. We may compute

\[ I_1 = \int_0^\infty \int_0^\infty \frac{xy}{(1 + x^2 + y^2)^4} \, dx \, dy = \frac{1}{24}, \quad (3.3) \]

\[ I_2 = \int_0^\infty \int_0^\infty \frac{x^3y}{(1 + x^2 + y^2)^4} \, dx \, dy = \frac{1}{24}, \quad (3.4) \]

\[ I_3 = \int_0^\infty \int_0^\infty \frac{xy^3}{(1 + x^2 + y^2)^4} \, dx \, dy = \frac{1}{24}, \quad (3.5) \]

so the slope comes to \( \mu = 2(1 + q)/3 - 2/3 = q/3 \), which is 0 if \( q = 0 \). In particular, \( \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\} \) is the configuration with the smallest slope, and it is non-negative.

3.2. First Hirzebruch surface. We may represent this toric surface as the convex polygon with vertices \( \{(0, 0), (2, 0), (1, 1), (0, 1)\} \) with volume \( 3/2 \), and a Bergman geodesic is specified by a choice of weights for a Newton diagram \( \{P_0 : (0, 0, q_{00}), P_1 : (1, 0, q_{10}), P_2 : (2, 0, q_{20}), P_3 : (0, 1, q_{01}), P_4 : (1, 1, q_{11})\} \). Now there are many more possibilities for the shape of the Newton polytope. Note that if, for example, \( q_{00} = 0 \), \( q_{10} = 0 \), or \( q_{01} = 0 \), then the Newton polytope will be of the same types as for \( \mathbb{P}^2 \).

Let us give two examples of test configurations. If we seek slopes that are as small as possible, we want the Newton diagrams to consist entirely of points on the boundary of the Newton polytope. In particular, if \( q_{20} = 0 \), we must have

\[ q_{10} \leq \frac{1}{2} q_{00}. \]

The maximal number of faces each containing at least three points of the Newton polytope is three, and these may occur in two shapes: \( F_1 = \{P_0, P_1, P_4\} \), \( F_2 = \{P_0, P_3, P_4\} \), \( F_3 = \{P_1, P_2, P_3\} \) or \( \tilde{F}_1 = \{P_0, P_1, P_3\} \), \( \tilde{F}_2 = \{P_1, P_3, P_4\} \), \( \tilde{F}_3 = \{P_1, P_2, P_4\} \). The first case occurs if

\[ q_{10} + q_{01} - q_{00} - q_{11} > 0, \quad (3.6) \]

and the second case if the inequality is reversed.

Let us take as an example the first case. By solving the inequalities (2.23), we obtain the equations of the faces of the polytope:

\[ F_1 = \{(q_{00} - q_{10})p + (q_{10} - q_{11})r + q = q_{00}\} \quad (3.7) \]

\[ F_2 = \{(q_{01} - q_{11})p + (q_{00} - q_{11})r + q = q_{00}\} \quad (3.8) \]

\[ F_3 = \{q_{10}p + (q_{10} - q_{01})r + q = 2q_{10}\} \quad (3.9) \]

The contribution from each face requires the evaluation of three integrals of the form

\[ I_{ijkl} = \int_0^\infty \int_0^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum_{\alpha \in F} x^{2p_\alpha} y^{2r_\alpha})^4} \, dx \, dy. \]
For example, on face $F_3$,

$$I_{1241} = \int_0^\infty \int_0^\infty \frac{x^9 y}{(x^2 + x^2 y^2 + x^4)^4} \, dx \, dy = \frac{1}{24}$$

(3.10)

and in fact all of the integrals are the same as the integrals appearing in the $\mathbb{P}^2$ calculation, and are all equal to $1/24$. Also, all the relevant factors $D_4(i, j, k, l)$ are equal to 1. For the total slope we have

$$\mu = \frac{2(q_{00} + q_{10} + q_{01} + q_{11} + q_{20})}{5} - \frac{1}{32} \cdot \frac{1}{3/2} \cdot 16 \left( \frac{q_{00}}{8} + \frac{q_{00}}{8} + \frac{2q_{10}}{8} \right)$$

(3.11)

$$= \frac{-2(q_{00} + q_{10}) + 18(q_{01} + q_{11})}{45}$$

(3.12)

Combining the inequality (3.6) with $q_{10} < q_{00}/2$, we have $q_{01} > q_{00}/2 + q_{11}$, so

$$\mu > \frac{-5q_{00}/2 + 18(q_{01} + q_{11})}{45} > \frac{13q_{00}/2 + 36q_{11}}{45}.$$  

(3.13)

For the final example, suppose all of the points of the Newton diagram lie on a single face and $q_{20} = 0$. Setting $q_{00} = 1$, we have that the equation of the face must be $F : 1/2x + cy + z = 1$, where $0 \leq c \leq 1/2$. The sum

$$m = \sum_{\{i, j, k, l\}^*} D_4(i, j, k, l) \int_0^\infty \int_0^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{\sum_{\alpha} x^{2p_{\alpha}} y^{2r_{\alpha}}} \, dx \, dy$$

in (1.14) for the non-trivial part of the slope contains 32 terms: 5 choices for sets of 4 distinct indices, and 9 choices for sets of 3 indices with one repeated (the set \{0, 1, 2\} excluded for being collinear), each of which gives three terms by the choice of the repeated
index. The integrals in $m$ can be computed by Mathematica, for example:

$$I_{0013} = \int_0^\infty \int_0^\infty \frac{xydx dy}{(1 + x^2 + y^2 + x^2y^2 + x^4)^4} = \frac{7(-9 + 2\sqrt{3}\pi)}{648}$$

(3.14)

$$I_{0113} = \int_0^\infty \int_0^\infty \frac{x^3ydx dy}{(1 + x^2 + y^2 + x^2y^2 + x^4)^4} = \frac{6 - \sqrt{3}\pi}{108}$$

(3.15)

$$I_{0133} = \int_0^\infty \int_0^\infty \frac{xydxdy}{(1 + x^2 + y^2 + x^2y^2 + x^4)^4} = \frac{9 - \sqrt{3}\pi}{324}$$

(3.16)

$$I_{0014} = \int_0^\infty \int_0^\infty \frac{x^3y dx dy}{(1 + x^2 + y^2 + x^2y^2 + x^4)^4} = \frac{6 - \sqrt{3}\pi}{108}$$

(3.17)

$$I_{0114} = \int_0^\infty \int_0^\infty \frac{x^5ydxdy}{(1 + x^2 + y^2 + x^2y^2 + x^4)^4} = \frac{-9 + 2\sqrt{3}\pi}{648}$$

(3.18)

$$I_{0144} = \int_0^\infty \int_0^\infty \frac{x^5y^3 dx dy}{(1 + x^2 + y^2 + x^2y^2 + x^4)^4} = \frac{45 - 8\sqrt{3}\pi}{648}$$

(3.19)

$$I_{0124} = \int_0^\infty \int_0^\infty \frac{x^7ydxdy}{(1 + x^2 + y^2 + x^2y^2 + x^4)^4} = \frac{-9 + 2\sqrt{3}\pi}{648}$$

(3.20)

and so on. Incredibly, the overall sum $m$ of these integrals weighted by the numbers $D_4(i, j, k, l)$ is rational: $m = 3/8$. The total asymptotic slope is

$$\mu = \frac{2(q_{00} + q_{10} + q_{01} + q_{11} + q_{20})}{5} - \frac{1}{\frac{3}{2} \cdot \frac{3}{2}} \frac{1}{16q_{00}} \frac{3}{8}$$

(3.21)

$$\mu = \frac{2(q_{10} + q_{01} + q_{11} + q_{20})}{5} - \frac{4q_{00}}{15}$$

(3.22)

Setting $q_{00} = 1$, $\mu$ attains its smallest value when $q_{10} = q_{01} = 1/2, q_{20} = q_{11} = 0$, in which case $\mu = 2/15$.

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