ON UENO’S CONJECTURE K

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Abstract. We show that if $X$ is a smooth complex projective variety with Kodaira dimension $0$ then the Kodaira dimension of a general fiber of its Albanese map is at most $h^0(\Omega^1_X)$.

1. Introduction

Let $X$ be a smooth projective variety with $\kappa(X) = 0$. Let $a : X \to A$ be its Albanese map with general fiber $F$. Then Ueno’s Conjecture K states that:

Conjecture 1.1. 1) $a$ is an algebraic fiber space (i.e. it is surjective with connected fibers),
2) $\kappa(F) = 0$ and
3) there is an étale cover $B \to A$ such that $X \times_A B$ is birationally equivalent to $F \times B$ over $A$.

This conjecture is an important test case of the more general $C_{n,m}$ conjecture of Iitaka which states that: Given a surjective morphism of smooth complex projective varieties $f : X \to Y$, $n = \dim X$ and $m = \dim Y$, with connected general fiber $F$, then

$$\kappa(X) \geq \kappa(F) + \kappa(Y).$$

Kawamata has shown (cf. [Kawamata85]) that these conjectures follow from the conjectures of the Minimal Model Program (including abundance). He has also shown:

Theorem 1.2. Conjecture K 1) is true (see Theorem 1 of [Kawamata81]).

Theorem 1.3. [Kawamata82] Let $X$ be a smooth projective variety with $\kappa(X) = 0$ and $a : X \to A$ its Albanese map with general fiber $F$. If $\dim A = 1$, then $\kappa(F) = 0$.

$C_{n,m}$ then follows easily for any fiber space $f : X \to Y$ where $Y$ is an elliptic curve. Since it is known that the $C_{n,m}$ conjecture holds when the base is of general type (cf. [Kawamata81] and [Viehweg82]), then

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$C_{n,1}$ holds. It is also worth noting that by [Kollár87], conjecture $C_{n,m}$ is known to hold when the general fiber $F$ is of general type.

In this paper we prove the following:

**Theorem 1.4.** Let $X$ be a smooth projective variety with $\kappa(X) = 0$ and $a : X \to A$ be its Albanese map with general fiber $F$. Then $\kappa(F) \leq \dim A$.

We now proceed to briefly sketch the proof of (1.4). This loosely follows the main ideas that Kawamata uses in the proof of (1.3).

In [Hacon04], using the theory of Fourier-Mukai transforms, it is shown that if $N \geq 2$ and $P_N(X) = 1$, then there exists an ideal sheaf $\mathcal{I}_{N-1} \subset \mathcal{O}_X$ such that $V_N := a_*(\omega_X^N \otimes \mathcal{I}_{N-1})$ is a unipotent vector bundle (i.e. given by successive extensions by $\mathcal{O}_A$) with $h^0(V_N) = 1$ and rank$(V_N) = P_N(F)$.

We fix an integer $N_1$ such that $|N_1K_F|$ induces the Iitaka fibration of $F$. Let $U_t$ be the image of $V_{N_1}^{\otimes t}$ in $V_{N_1}$ (under the natural multiplication map). Then $U_t$ is also a unipotent vector bundle and its rank (as a function of $t$) grows like $t^{\kappa(F)}$. In order to bound the rate of growth of the ranks of $U_t$, using the theory of Fourier-Mukai transforms, we consider an equivalent problem concerning modules $M_t := R\mathcal{S}(U_t)$ over the regular local ring $\mathcal{O}_{A,0}$ of length equal to the rank of $U_t$. Since $h^0(U_N) = 1$ and the multiplication maps above are non-zero on simple tensors, it turns out that the modules $M_t$ have no decomposable submodules, and for any submodules $L \subset M_t$ and $L' \subset M_{t+s}$, the dimension of the image of the Pontryagin product $L \ast L'$ under the natural map from $M_t \ast M_{t+s}$ to $M_{t+s}$, always has length at least $\dim_k(L) + \dim_k(L') - 1$. This allows us to define natural extensions $M_t \leftarrow \tilde{M}_t$ and multiplication maps $\tilde{M}_t \ast \tilde{M}_s \to \tilde{M}_{t+s}$ which behave analogously to the case in which $A$ is a $g$-fold product of elliptic curves. We are hence able to show that the rate of growth of $\dim_k(\tilde{M}_t)$ (and hence of $\dim_k(M_t)$) is bounded by $O(t^g)$. Therefore $\kappa(F) \leq g$ where $g = \dim A$.

We believe that a more detailed analysis of the modules $M_t$ would show that the image of the relative Iitaka fibration, denoted by $f : X \to W$, of the Albanese map $X \to A$ is a $\mathbb{P}^{\kappa(F)}$-bundle over $A$. We do not pursue this here.

1.1. **Notation and conventions.** Throughout this paper, we work over $k = \mathbb{C}$. If $X$ is a smooth projective variety, then $K_X$ will denote a canonical divisor and we let $\omega_X = \mathcal{O}_X(K_X)$. For any integer $N > 0$, we let $P_N(X) = h^0(X, \omega_X^N)$ so that $P_0(X) = P_1(X)$. The Kodaira dimension of $\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim X\}$ is defined as follows: if $P_N(X) = 0$ for all $N > 0$, then $\kappa(X) = -\infty$, otherwise we let $\kappa(X) = m$, where $0 \leq m \leq \dim X$ is an integer such that $P_N(X) = O(N^m)$ (that is, there are constants $\alpha, \beta > 0$ such that $\alpha N^m \leq P_N(X) \leq \beta N^m$ for all $N \gg 0$ sufficiently divisible).
1.2. The Fourier-Mukai Transform. For any abelian variety $A$ of
dimension $g$, we let $\hat{A} = \text{Pic}^0(A)$ be the dual abelian variety and $\mathcal{P}$
the (normalized) Poincaré line bundle on $A \times A$. By [Mukai81], there
is a functor $\hat{\mathcal{S}}$ from the category of $\mathcal{O}_A$-modules to the category of
$\mathcal{O}_{\hat{A}}$-modules defined by $\hat{\mathcal{S}}(M) = (p_{\hat{A}})_*(p_\hat{A}^*M \otimes \mathcal{P})$ where $p_A, p_{\hat{A}}$
denote the projections from $A \times A$ to $A, \hat{A}$. Similarly, for any $\mathcal{O}_{\hat{A}}$-module $N$,
one defines $\mathcal{S}(N) = (p_A)_*(p_\hat{A}^*N \otimes \mathcal{P})$. By (2.2) of [Mukai81], there
are isomorphisms of derived functors
\[ R\hat{\mathcal{S}} \circ RS \cong (-1)\hat{\mathcal{A}}^*[-g], \quad RS \circ R\hat{\mathcal{S}} \cong (-1)\mathcal{A}^*[g], \]
where $R\hat{\mathcal{S}}, RS$ are the the derived functors of $\hat{\mathcal{S}}, \mathcal{S}$ and $[-g]$ denotes
shifting a complex $g$ spaces to the right.

We will consider the dualizing functor on $A$ given by
\[ \Delta_A(?) = R\text{Hom}(?, \mathcal{O}_A)[g]. \]
By 3.8 of [Mukai81], we have
\[ \Delta_A \circ RS = ((-1)\mathcal{A}^* \circ RS \circ \Delta_A)[g]. \]
A vector bundle $U$ on $A$ is unipotent if there is a sequence of sub-bundles
\[ 0 = U_0 \subset U_1 \subset \ldots \subset U_r = U \]
such that $U_i/U_{i-1} \cong \mathcal{O}_A$ for all $1 \leq i \leq r = \text{rank}(U)$. We have
that $R^g\hat{\mathcal{S}}$ gives an equivalence in categories between unipotent vector
bundles on $A$ and the category of coherent sheaves supported on $0$, i.e.
the category of Artinian $\mathcal{O}_{\hat{A}, \hat{0}}$ modules of finite length.

Recall that we have (cf. §3 of [Mukai81])

**Proposition 1.5.** Let $U, V$ be unipotent vector bundles on $A$, then
$U \otimes V$ and $U^*$ are unipotent vector bundles and we have
$R^g\hat{\mathcal{S}}(U \otimes V) \cong R^g\hat{\mathcal{S}}(U) \ast R^g\hat{\mathcal{S}}(V)$ and
$R^g\hat{\mathcal{S}}(U^*) \cong (-1)\mathcal{A}^*(R^g\hat{\mathcal{S}}(U))$.

Here the Pontryagin product $R^g\hat{\mathcal{S}}(U) \ast R^g\hat{\mathcal{S}}(V)$ is just the push-forward
$\mu_*(R^g\hat{\mathcal{S}}(U) \boxtimes R^g\hat{\mathcal{S}}(V))$ via the multiplication map $\mu : \hat{A} \times \hat{A} \to \hat{A}$
given by $\mu(x, y) = x + y$. Equivalently, we regard the vector space
$R^g\hat{\mathcal{S}}(U) \otimes_k R^g\hat{\mathcal{S}}(V)$ as a $\mathcal{O}_{\hat{A}, \hat{0}}$-module via the corresponding comulti-
plication map $\mu^* : \mathcal{O}_{\hat{A}, \hat{0}} \to \mathcal{O}_{\hat{A}, \hat{0}} \otimes \mathcal{O}_{\hat{A}, \hat{0}}$ given by $\mu^*(x) = x \otimes 1 + 1 \otimes x$.

**Proposition 1.6.** Let $\phi : U \to V$ be a homomorphism between unipo-
tent vector bundles. Then $\text{im}(\phi), \ker(\phi), \text{coker}(\phi)$ are also unipotent.

**Proof.** We proceed by induction on $r := \text{rank}(U)$. If $r = 1$ the assertion
is clear. If $r \geq 2$, we consider the composition $\mathcal{O}_A \xrightarrow{\iota} U \xrightarrow{\phi} V$.

If $\phi \circ \iota = 0$, then we have an induced homomorphism $\phi' : U/\mathcal{O}_A \to V$.
Therefore, by induction, $\text{im}(\phi) = \text{im}(\phi')$ and $\ker(\phi) = \ker(\phi')$ are unipotent. Moreover we have a short exact sequence
$0 \to \mathcal{O}_A \to \ker(\phi) \to \ker(\phi') \to 0$. Since $\ker(\phi)$ is an extension of unipotent
vector bundles, it is also unipotent.
If \( \phi \circ \iota \neq 0 \), then we have a homomorphism \( \phi' : U/O_A \rightarrow V/O_A \) such that \( \ker(\phi) \cong \ker(\phi') \), \( \operatorname{coker}(\phi) \cong \operatorname{coker}(\phi') \). Finally, since \( \operatorname{im}(\phi) \) is an extension of \( \operatorname{im}(\phi') \) by \( O_A \), it is also unipotent.

2. Artinian modules without decomposable submodules

Let \( \hat{\mathcal{A}} \) be an abelian variety of dimension \( g \) with origin \( \hat{0} \) and \( B \) be the regular local ring \( O_{A,0} \cong O_{A,\hat{0}} \) with maximal ideal \( m \).

Lemma 2.1. The set of all unipotent vector bundles \( U \) on \( A \) with \( h^0(A,U) = 1 \) is in one to one correspondence with the set of all Artinian \( B \)-modules of finite length without decomposable submodules.

Proof. By [Mukai81, Example 2.9], the Fourier-Mukai transform gives a bijection between the category of unipotent vector bundles \( U \) on \( A \) and the category of Artinian \( B \)-modules of finite length \( M \), given by \( U \rightarrow M = \mathcal{R}_{\phi}^{\hat{S}}(U) \) and \( M \rightarrow U = \mathcal{R}_{\phi}^{0}\hat{S}(M) \). Suppose that \( M \) has a decomposable submodule, then there is a injective homomorphism of \( B \)-modules \( k \oplus k \rightarrow M \). Taking the Fourier-Mukai transform we get an injective homomorphism \( O_A \oplus O_A \rightarrow \mathcal{R}_{\phi}^{0}\hat{S}(M) \) and hence \( h^0(\mathcal{R}_{\phi}^{0}\hat{S}(M)) \geq 2 \). Suppose on the other hand that \( h^0(U) \geq 2 \), then there is a homomorphism \( \phi : O_A \oplus O_A \rightarrow U \) which is injective on global sections. Let \( V \) be the image of \( \phi \), then by \((1.6)\), \( V \) is a unipotent vector bundle of rank at most 2. If \( \operatorname{rank}(V) = 1 \) then \( U \) is a unipotent line bundle with \( h^0(V) = 2 \) which is impossible. Therefore \( \operatorname{rank}(V) = 2 \) and so \( \phi \) is an injection. By \((1.6)\), \( \operatorname{coker}(\phi) \) is also unipotent. In particular \( \mathcal{R}_{\phi}(\operatorname{coker}(\phi)) = \mathcal{R}_{\phi}^{0}\hat{S}(\operatorname{coker}(\phi)) \). Therefore, taking the Fourier-Mukai transform we get an injective homomorphism \( \psi : k \oplus k \rightarrow \mathcal{R}_{\phi}^{0}\hat{S}(U) \) and so \( \mathcal{R}_{\phi}^{0}\hat{S}(U) \) has a decomposable submodule.

We will now define a natural extension of Artinian \( B \)-modules of finite length with no decomposable submodules. We begin by considering the corresponding dual objects.

Lemma 2.2. Let \( M \) be an Artinian \( B \)-module of finite length with no decomposable submodules, then \( \Delta(M) \) is an Artinian \( B \)-module of finite length with no decomposable quotients.

Proof. Clear.

Lemma 2.3. Let \( (B,m,k) \) be a local ring with an inclusion \( k \hookrightarrow B \) and \( M \) be an Artinian \( B \)-module of finite length without decomposable quotient modules. Then \( M = Ba \) for some \( a \in M \). In particular, \( M \cong B/\operatorname{Ann}(a) \).

Proof. \( M/mM \) is a vector space over \( k \). If \( \dim_k M/mM = 0 \), then by Nakayama’s Lemma we have \( M = 0 \). If \( \dim_k M/mM = 1 \), pick any element \( a \in M - mM \), then \( Ba + mM = M \). By Nakayama’s Lemma again, \( Ba = M \), so that \( M \cong B/\operatorname{Ann}(a) \).
If \( \dim_k M/\mathfrak{m}M \geq 2 \), we pick \( a, b \in M - \mathfrak{m}M \) such that their image \( \overline{a}, \overline{b} \in M/\mathfrak{m}M \) are linearly independent. Then \( B\overline{a} + B\overline{b} \) is a decomposable quotient module of \( M \), which is the required contradiction. \( \square \)

Consider now an Artinian \( B \)-module of finite length \( M \) without decomposable submodules, where \( B \) is the local ring \( \mathcal{O}_{\overline{A}, \overline{b}} \). Then \( N = \Delta(M) \) is an Artinian \( B \)-module of finite length without decomposable quotient modules. Therefore \( N \cong B/I \) for some \( I \) with \( \sqrt{I} = \mathfrak{m} \).

For any \( x \in \mathfrak{m} - \mathfrak{m}^2 \), we let \( e(x) := \min\{k | x^k \in I \} \). We pick successive elements

\[
x_i \in \mathfrak{m} - \text{Span}(x_1, \ldots, x_{i-1}) + \mathfrak{m}^2
\]

with minimal \( e_i = e(x_i) \). We then let \( J = J(I) := (x_1^{e(x_1)}, \ldots, x_g^{e(x_g)}) \subset I \). The elements \( (x_1, \ldots, x_g) \) form a system of parameters and generate \( \mathfrak{m} \). We let \( \overline{N} := B/J \) and \( \overline{M} = \Delta(\overline{N}) \). From the surjection \( \overline{N} \to N \), we obtain an injection \( M \to \overline{M} \).

We say that the module \( \overline{M} \) (or more precisely the injection \( M \to \overline{M} \)) is an algebraically splitting extension (or \( AS \)-extension), of \( M \).

We will say that a \( B \)-module of the form \( B/(x_1^{e_1}, \ldots, x_g^{e_g}) \) is an \( AS \)-module. Note that by Lemma 2.4 below, the \( AS \)-extensions defined above are \( AS \)-modules.

**Lemma 2.4.** Let \( M \) be any \( AS \)-\( B \)-module then \( M \cong \Delta(M) \).

**Proof.** The proof follows easily by considering the Koszul complex given by the regular sequence \( x_1^{e_1}, \ldots, x_g^{e_g} \). \( \square \)

In summary we have:

**Proposition 2.5.** Let \( M \) be a Artinian \( B \)-module without decomposable submodules, then \( M \cong I/J \) for some ideals \( I \) and \( J = (x_1^{e_1}, \ldots, x_g^{e_g}) \).

**Example 2.6.** Let \( M = B/(x_1^{e_1}, \ldots, x_g^{e_g}) \) and \( M' = B/(x_1^{f_1}, \ldots, x_g^{f_g}) \). We wish to compute \( M \ast M' \).

By the description following Proposition 1.5, it can be realized as

\[
M \otimes_k M' = k[x_1, \ldots, x_g]/(x_1^{e_1}, \ldots, x_g^{e_g}) \otimes k[y_1, \ldots, y_g]/(y_1^{f_1}, \ldots, y_g^{f_g}),
\]

regarded as \( k[t_1, \ldots, t_g] \)-module by letting \( t_i := x_i + y_i \). We first treat the case that \( g = 1 \). The general case is obtained by taking the tensor product.

Let \( e = e_1, f = f_1, x = x_1, y = y_1 \) and \( t = t_1 \). Assume that \( e \leq f \). We will need the following two facts:

(1) Let \( 0 \leq d \leq e - 1 \). Then \( t^{e+f-2d-2}v_d \neq 0 \) for any \( v_d \neq 0 \) homogeneous of degree \( d \).

(2) Let \( 0 \leq d \leq e - 1 \). Then there exists an homogeneous element \( t^d \) of degree \( d \), which is unique up to multiplication by a scalar and such that \( t^{e+f-2d-1}t^d = 0 \).

To see these two facts, we write

\[
v_d = \sum_{i=0}^{d} a_i x^i y^{d-i} \quad \text{and} \quad t^{e+f-2d-2}v_d = \sum_{i=1}^{d+1} c_i x^{e-i} y^{f-d+i-2}.
\]

Then we have
where \( C_i^j = j \cdots (j-i+1)/i! \). An explicit computation (cf. [Roberts94]) shows that the \((d+1) \times (d+1)\) matrix is non-singular. This proves (1).

Similarly, if we now write \( t^{(d)} = \sum_{i=0}^{d} a_i x^i y^{d-i} \) and \( t^{e+f-2d-1} t^{(d)} = \sum_{i=1}^{d} c_i x^i y^{d-i} \). Then we have

\[
\begin{pmatrix}
C_1 \\
C_2 \\
\vdots \\
C_{d+1}
\end{pmatrix}
= \begin{pmatrix}
C_{e+d-1}^{e+f-2d-2} & \cdots & C_{e-1}^{e+f-2d-2} \\
\vdots & \ddots & \vdots \\
C_{e+d}^{e+f-2d-2} & \cdots & C_{e-1}^{e+f-2d-2}
\end{pmatrix}
\begin{pmatrix}
a_d \\
a_{d-1} \\
\vdots \\
a_0
\end{pmatrix}
\]

This \((d \times (d+1))\) matrix has rank \( d \) by an analogous computation. Thus (2) follows.

We next claim that, as a \( k[t] \)-module

\[
k[x, y]/(x^e, y^f) = \bigoplus_{d=0}^{e-1} t^{(d)} k[t]/(t^{e+f-2d-1}).
\]

To see this, it suffices to verify the equality as \( k \)-vector spaces. Let \( F_d \) be the subspace of homogeneous polynomial of degree \( d \). Clearly \( F_d \) contains \( \{t^{(0)} t^d, t^{(1)} t^{d-1}, \ldots, t^{(d-1)} t, t^{(d)}\} \). It suffices to show that these are linearly independent. We proceed by induction on \( d \in \{0, 1, \ldots, e-1\} \). Hence we assume that \( \{t^{(0)} t^{d-1}, t^{(1)} t^{d-2}, \ldots, t^{(d-2)} t, t^{(d-1)} t\} \) are independent, therefore so are \( \{t^{(0)} t^d, t^{(1)} t^{d-1}, \ldots, t^{(d-1)} t\} \). If \( t^{(d)} = t v_{d-1} \) for some \( v_{d-1} \), then \( t^{e+f-2d-1} t^{(d)} = t^{e+f-2d} v_{d-1} = 0 \) which contradicts (1). Therefore, \( t^{(d)} \neq t v_{d-1} \) and so \( t^{(d)} \) is not contained in the subspace generated by \( \{t^{(0)} t^d, t^{(1)} t^{d-1}, \ldots, t^{(d-1)} t\} \). For \( d \in \{e, e+f-2\} \) the proof follows by a similar argument. This completes the proof of the claim.

Therefore, for any \( g > 0 \), we have a decomposition

\[
M \otimes M' = \bigoplus_{I \in [0, e_1-1] \times \cdots \times [0, e_g-1]} V_I
\]

where \( e_i = \min\{e_i, f_i\} \), \( I = (i_1, \ldots, i_g) \) and \( V_I \) is generated by \( t^{(i_1)}_1 \cdots t^{(i_g)}_g \) and \( V_I \cong B/(t^{(i_1)}_1 + f_1 - 1 - 2h_1, \ldots, t^{(i_g)}_g + f_0 - 1 - 2i_0) \).

We let \( V_{\text{max}} := V_{(0, \ldots, 0)} \) be the maximal component of \( M \otimes M' \).

**Definition 2.7.** Given a multiplication map \( \varphi : M \otimes M' \to M_2 \) of Artinian \( B \)-modules of finite length. We say that it is geometric if \( M_2 \) has no decomposable submodules and for every submodules \( L < M \)
and $L' < M'$, the image of $\varphi(L \ast L')$ has dimension at least $\dim_k(L) + \dim_k(L') - 1$.

**Proposition 2.8.** Given a geometric multiplication map $\varphi : M \ast M' \to M_2$. If $M$ and $M'$ are AS-modules, then restriction of $\varphi$ to the maximal component $V_{max}$ of $M \ast M'$ is injective. Moreover, $\varphi(M \ast M') = \varphi(V_{max})$.

**Proof.** Let $M = B/(x_1^a, \ldots, x_g^a)$ and $M' = B/(x_1^{f_1}, \ldots, x_g^{f_g})$. Then $M \ast M' = \bigoplus V_i$ as above.

We first claim that $\varphi : V_{max} \to M_2$ is injective. To see this, let $N = (x_1^{e_1-1} \ldots x_g^{e_g-1})/(x_1^a, \ldots, x_g^a) = \operatorname{Soc}(M)$ and $N' = (x_1^{f_1-1} \ldots x_g^{f_g-1})/(x_1^a, \ldots, x_g^a) = \operatorname{Soc}(M')$ be the unique rank 1 submodule given by the annihilator of the maximal ideal. Then $N \ast N'$ is the unique rank 1 submodule in $V_{max}$. Since $\varphi$ is geometric, $N \ast N'$ is not in the kernel of $\varphi$. Suppose now that $0 \neq f \in V_{max}$ is in the kernel of $\varphi$, then there exist integers $a_1, \ldots, a_g \geq 0$ such that $t_1^{a_1} \ldots t_g^{a_g} f$ generates $N \ast N'$ and this leads to an easy contradiction.

We will now show that $\varphi(M \ast M') = \varphi(V_{max})$. Let $M'_2$ be the image of $\varphi$, then clearly the restriction $\varphi' : M \ast M' \to M'_2$ is also geometric. By Proposition 2.9 below, $M'_2$ admits an extension to $V_{max}$ and hence they are isomorphic and the claim follows. \qed

**Proposition 2.9.** Given a surjective geometric multiplication map $\varphi : M \ast M' \to M_2$. Let $\bar{M}$ and $M'$ be a AS-extensions of $M$ and $M'$. Then $M_2$ admits an extension to the maximal component of $M \ast M'$.

**Proof.** We write $\bar{M} = B/(x_1^a, \ldots, x_g^a)$ and $M' = B/(x_1^{f_1}, \ldots, x_g^{f_g})$. We keep the notation as in Example 2.6.

Let $J = (t_1^{e_1+f_1-1}, \ldots, t_g^{e_g+f_g-1})$. We note that $J$ annihilates $\bar{M} \ast M'$, and hence it also annihilates $M \ast M'$ and $M_2$. We claim that $J$ annihilates $\Delta(M_2)$. To see this, we regard $t_1, \ldots, t_g$ as elements of $\operatorname{End}_k(M_2)$. Thus $t_1^{e_1+f_1-1} = 0 \in \operatorname{End}_k(M_2)$ implies that $t_1^{e_1+f_1-1} = 0 \in \operatorname{End}_k(\Delta(M_2))$.

Note that $\Delta(M_2)$ is principal and $\Delta(M_2) \cong B/\operatorname{Ann}(\Delta(M_2))$ by Lemma 2.3. Therefore, we have surjective homomorphism $B/J \to \Delta(M_2)$. Dualizing it, we get an injective homomorphism $M_2 \hookrightarrow \Delta(B/J) \cong B/J$. Where $B/J$ is isomorphic to the maximal component of $\bar{M} \ast M'$.

\qed

3. **Proof of the main theorem**

In this section we will prove the main theorem.

We will make use of various multiplier ideal sheaves. We refer the reader to [Lazarsfeld04] for their definitions and main properties.
**Theorem 3.1.** Let \( a : X \to A \) be a surjective morphism with general fiber \( F \) from a smooth projective variety \( X \) with \( \kappa(X) = 0 \) to an abelian variety with \( \dim A = g \). Then \( \kappa(F) \leq g \).

**Proof.** By [Hacon04, §5], for all \( N \geq 2 \) such that \( P_N(X) = 1 \), there exists an ideal sheaf \( \mathcal{I}_{N-1} \subset \mathcal{O}_X \) such that \( V_N = a_*(\omega_X^N \otimes \mathcal{I}_{N-1}) \) is a unipotent vector bundle of rank \( P_N(F) \) and such that \( H^0(A, V_N) = P_N(X) = 1 \). (If \( P_N(X) \neq 1 \), then \( a_*(\omega_X^N \otimes \mathcal{I}_{N-1}) \) is given by successive extensions by some \( P \in \text{Pic}_{\text{tors}}^0(A) \).) We let \( V_0 = \mathcal{O}_A \). (Note that by Theorem 4 of [CH02], if \( P_1(X) = 1 \), then \( a_*(\omega_X^N) = \mathcal{O}_A \).) We fix an integer \( e \geq 2 \) such that \( P_N(X) = 1 \) for any integer \( N > 0 \) divisible by \( e \).

**Lemma 3.2.** If \( N \) and \( M \) are positive integers divisible by \( e \) then the homomorphisms \( V_N \otimes V_M \to a_!\omega_X^{N+M} \) factor through \( V_{N+M} \).

**Proof.** Let \( H \) be an ample line bundle on \( A \). Recall that by definition (cf. §5 of [Hacon04]), there exists a number \( e_0 > 0 \) such that \( \mathcal{I}_N = \mathcal{I}(\lceil NK_X + e_0^*H \rceil) \) for any rational number \( 0 < e \leq e_0 \). Therefore, we may assume a rational number \( 0 < e \leq e_0 \). Then \( \mathcal{I}_N = \mathcal{I}(\lceil t(N-1)K_X + e_0^*H \rceil) \) for any rational number \( 0 < e \leq e_0 \). Therefore, we may assume a rational number \( 0 < e \leq e_0 \). Then \( \mathcal{I}_N = \mathcal{I}(\lceil t(N-1)K_X + e_0^*H \rceil) \) for any integer \( N > 0 \) divisible by \( e \).

**Claim 3.3.** We have \( V'_N = (a')_!(\omega_X^N \otimes \mathcal{I}'_{N-1}) \) where \( \mathcal{I}'_{N-1} = \mathcal{I}(\lceil t(N-1)K_X + e_0^*H \rceil) \).

**Proof.** This follows easily by flat base change and the fact that, by Theorem 11.2.16 of [Lazarsfeld04], we have \( (m')^*\mathcal{I}(\lceil (N-1)K_X + e_0^*H \rceil) = \mathcal{I}(\lceil (m')^*\lceil (N-1)K_X + e_0^*H \rceil \rceil) \).

We now claim that there is a homomorphism
\[
(\omega_X^N \otimes \mathcal{I}'_{N-1}) \otimes H^0(X', \mathcal{O}_{X'}(\omega_{X'}^M \otimes \mathcal{I}'_{M-1} \otimes (a')^*H')) \to \omega_{X'}^{N+M} \otimes \mathcal{I}'_{N+M-1} \otimes (a')^*H'.
\]
To check this, it suffices to verify that for any section \( s \in H^0(X', \mathcal{O}_{X'}(\omega_{X'}^M \otimes \mathcal{I}'_{M-1} \otimes (a')^*H')) \), one has that \( \mathcal{I}'_{N-1} \cdot s \subset \omega_{X'}^M \otimes (a')^*H' \otimes \mathcal{I}'_{N+M-1} \). This in turn follows from the inclusion of linear series
\[
|t((N-1)K_X + (a')^*H')| \times |MK_{X'} + (a')^*H'| \to |t((N+M-1)K_X + 2(a')^*H')|.
\]

Pushing forward, we obtain a homomorphism
\[
V'_N \otimes H^0(A', \mathcal{O}_{X'} \otimes (a')^*H') \to V'_{N+M} \otimes H' \subset (a')_!(\omega_{X'}^{N+M}) \otimes H'.
\]

Since \( V'_M \otimes H' \) is generated, the map \( V'_N \otimes V'_M \otimes H' \to (a')_!(\omega_{X'}^{N+M}) \otimes H' \) factors through \( V'_{N+M} \otimes H' \). Therefore, we have a homomorphism \( V'_N \otimes
V′_{M} \to V′_{N+M} \text{ i.e. an element of}
H^0(A', (V'_{N} \otimes V'_{M})^* \otimes (V'_{N+M})^* \otimes P) 
\approx \bigoplus_{P \in \text{Ker}(\hat{\cdot})} H^0(A, (V_N \otimes V_M)^* \otimes V_{N+M} \otimes P)
\approx H^0(A, (V_N \otimes V_M)^* \otimes V_{N+M}).
This is the required homomorphism $V_N \otimes V_M \to V_{N+M}$. □

We now fix a positive integer $N_1 > 0$ such that $|N_1K_F|$ defines a rational map which is birational to the Iitaka fibration of $F$. Let $U_t$ be the image of the homomorphisms $V^\otimes_{N_1} \to V_{tN_1}$. By Proposition 1.6 the $U_t$ are unipotent vector bundles of rank $r_t$ where $r_t = O(t^{\kappa(F)})$.
Let $M_t$ be the Artinian $O_{A,0}$ module given by $M_t = R^q\tilde{S}(U_t)$. Then
\[ \dim_k M_t = r_t. \]
We have surjective homomorphisms
\[ \varphi_{t,s} : M_t \otimes M_s \to M_{t+s}, \]

**Lemma 3.4.** For any submodules $N \subset M_t$ and $N' \subset M_s$ one has that
\[ \dim \varphi_{t,s}(N \otimes N') \geq \dim N + \dim N' - 1. \]

**Proof.** It suffices to show that given the sub-bundles $W = R^0\tilde{S}(N) \subset U_t$ and $W' = R^0\tilde{S}(N') \subset U_s$, one has that the rank of the image of $W \otimes W'$ in $U_{t+s}$ is at least $\dim V + \dim V' - 1$. This follows easily from the fact that the map
\[ \psi : H^0(F, \omega_F^{tN_1}) \otimes H^0(F, \omega_F^{sN_1}) \to H^0(F, \omega_F^{(t+s)N_1}) \]
is non-zero on tensors of the form $0 \neq v \otimes w$ and therefore (by a result of H. Hopf) for any sub-spaces $V$ of $H^0(F, \omega_F^{tN_1})$ and $V'$ of $H^0(F, \omega_F^{sN_1})$, one has
\[ \dim \psi(V \otimes V') \geq \dim V + \dim V' - 1. \]

We now define a sequence of finite length modules
\[ \tilde{M}_t \cong B/(x_1^{te_1-t+1}, \ldots, x_g^{te_g-t+1}) \]
and injective homomorphisms $M_t \hookrightarrow \tilde{M}_t$ as follows:

For $t = 1$, $\tilde{M}_1$ is the AS-extension of $M_1$ defined as in §2. In particular there is an injective homomorphism $M_1 \hookrightarrow \tilde{M}_1$. Assume now that the inclusion $M_{t-1} \hookrightarrow \tilde{M}_{t-1}$ has been defined. We consider the geometric multiplication map (in the sense of §2)
\[ \varphi_{t-1,1} : M_{t-1} \otimes M_1 \to M_t. \]
Since $\varphi_{t-1,1}$ is surjective, by Proposition 2.9 as $\tilde{M}_t$ is the maximal component of $M_{t-1} \otimes M_1$, then we have an injective homomorphism $M_t \hookrightarrow \tilde{M}_t$ and a surjection $\tilde{M}_{t-1} \otimes M_1 \to \tilde{M}_t$.

Let $e = e_1 \cdots e_g$ be the length of $\tilde{M}_1$, then by Example 2.6 one easily sees that $\tilde{M}_t \cong B/(x_1^{te_1-t+1}, \ldots, x_g^{te_g-t+1})$ as claimed above. It follows that
\[ r_t \leq \dim_k \tilde{M}_t = \prod (t(e_i - 1) + 1) < e \cdot t^g, \]
and therefore $\kappa(F) \leq g$. 

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