Least Squares Spectral Analysis and Its Application to Superconducting Gravimeter Data Analysis

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Abstract

Detection of a periodic signal hidden in noise is the goal of Superconducting Gravimeter (SG) data analysis. Due to spikes, gaps, datum shifts (offsets) and other disturbances, the traditional FFT method shows inherent limitations. Instead, the least squares spectral analysis (LSSA) has shown itself more suitable than Fourier analysis of gappy, unequally spaced and unequally weighted data series in a variety of applications in geodesy and geophysics. This paper reviews the principle of LSSA and gives a possible strategy for the analysis of time series obtained from the Canadian Superconducting Gravimeter Installation (CSGI), with gaps, offsets, unequal sampling decimation of the data and unequally weighted data points.

Keywords

least squares spectrum; superconducting gravimeter; data analysis

Introduction

The fast Fourier transform (FFT) algorithms are popular spectral estimation techniques for the determination of the power spectrum and show computational efficiency in the analysis of signal process. However, there are certain inherent limitations in the FFT techniques. The most prominent limitations arise from the requirement that the data should be equally spaced and equally weighted with no gaps and datum shifts. Pre-processing of the data is required if there are gaps, spikes, datum shifts and trends in the original data series.

In order to avoid unnecessary data pre-processing that may corrupt or obliterate the useful information hidden in the series (signal), the least squares spectral analysis (LSSA) is used as an alternative to the classical Fourier method. LSSA was first developed by Vanicek. It uses the least squares approximation (LSA) technique which is closely related to the least squares parametric adjustment (LLSPA). The advantages of this technique have already been shown in Reference [4].

The principle of LSSA introduced in this paper is based on the least squares parametric adjustment in vector space. The experimental studies of this method are conducted using the observation series from the Canadian Superconducting Gravimeter Installation (CSGI), which exhibit minor gaps (unequally spaced), offsets, spikes (seismic events) and variable weights due to different noise levels.

1 Principle of LSSA

1.1 Least squares parametric adjustment in vector space

Before describing LSSA, it is expedient to refer to the least squares parametric adjustment in a vector space. As we know, the shortest distance between a point and a plane is the perpendicular from the point to the plane (projection theorem). We can also extend this notion to n-dimensional space, in which the shortest dis-
tance between a point and any sub-space is the perpendicular from the point to the sub-space. Below is the mathematical description of the projection theorem.

1.1.1 Projection theorem

Let $H$ be a vector space with inner product, called Hilbert space. Given $f \in H$ (point) and $M \in H$ (sub-space), then in all the elements $f \in M$, there is one element $m_0 \in M$ so that $\| f - m_0 \| < \| f - m \|$. This element $m_0$ is given by the orthogonal projection of $f$ onto $M$, that is $(f - m_0) \perp M$ (see Fig. 1).

If $M$ is generated by a set of basis vectors $\{\phi_i\}$, then $m$ can be expressed as a linear combination of these basis vectors

$$m_0 = \sum i x_i \phi_i.$$  

(1)

Since $(f - m_0) \perp M$, we have $(f - m_0) \perp \phi_j$, $\forall j$ therefore

$$\langle f - \sum i x_i \phi_i, \phi_j \rangle = 0 , \forall j.$$  

(2)

By use of the inner product, we get

$$\sum i \langle \phi_i, \phi_j \rangle x_i = \langle f, \phi_j \rangle , \forall j.$$  

(3)

In fact, this is the normal equation of parametric adjustment in which $x_i$ is the unknown vector.

1.1.2 Least squares parametric adjustment in vector space

We can now use the concept of Hilbert space to the least squares parametric adjustment. Based upon the Gauss-Markov model $(l, AX, \sigma^2 I)$, $l$ is a vector of observations, $(l \in H)$. The least squares principle aims to find a vector $\hat{l} = AX$ to make the distance between $l$ and $\hat{l}$ shortest or the norm of the residual vector $v$ minimum. Therefore, from the projection theorem, $\hat{l}$ is the orthogonal projection of $l$ onto a sub-space ($M \in H$). So from the above equation, we get

$$\hat{l} = (A^T A)^{-1} A^T l$$  

(4)

$$v = l - \hat{l} = l - A(A^T A)^{-1} A^T l.$$  

(5)

From the projection theorem, we know that $v \perp l$. This means that the projection theorem allows us to decompose $l$ into two orthogonal components, $l$ (the orthogonal projection of $l$ onto $M$) and $v$ (the perpendicular from $l$ to $M$).

1.2 Least squares spectrum

Given a vector of observations $f = f(t) = \{ f_i \}_{i=1,2, \ldots, n}$, we can set up a model $p$ that can be expressed as follows.

$$p = \sum i \phi_i x_i = \phi \hat{x}.$$  

(6)

where $\Phi$ is a matrix of known base functions and $\hat{x}$ is the vector of unknown parameters. Here we do not assume $t_i$ to be equally spaced. But we assume that the observations $f_i, i = 1,2, \ldots, n$ possess a fully populated covariance matrix $C_f$. For simplicity here we assume $C_f = I$.

To estimate the model parameters $\hat{x}$, the projection theorem is used and the difference between $p$ and $f$ becomes minimum in the least squares sense. The estimation of the model parameters can be obtained as follows:

$$\hat{x} = \Phi(\Phi^T \Phi)^{-1} \Phi^T f.$$  

(7)

$$p = \hat{\Phi} \hat{x} = \Phi(\Phi^T \Phi)^{-1} \Phi^T f.$$  

(8)

The residual vector can be written as

$$v = f - p = f - \Phi(\Phi^T \Phi)^{-1} \Phi^T f.$$  

(9)

From the projection theorem, we know that $v \perp p$. This means that $f$ has been decomposed into a signal $p$ and noise $v$ (residual series). If we project $p$ back onto $f$, we can get the length of this orthogonal projection as follows (see Fig. 2)

$$\frac{\langle f, p \rangle}{\| f \|}.$$  

(10)

Fig. 2 Second projection

To describe how closely $p$ approaches $f$, we use a fractional measure $S$ that is the ratio of the length of
In spectral analysis we usually try to search for the hidden periodic signals that are expressed in terms of sine and cosine base functions. So, if we specify the form of the base functions to be trigonometric based on a set of spectral frequencies \( \omega_i, i = 1, 2, \ldots, m \), we have

\[
p(\omega_i) = X_1 \cos \omega_i t + X_2 \sin \omega_i t.
\]

(12)

Let \( \hat{X} = [X_1, X_2]^T \) and \( \Phi = [\cos \omega_i t, \sin \omega_i t] \), then \( \hat{X} \) can be determined from Eq. (7). So for different frequencies \( \omega_i, i = 1, 2, \ldots, m \), we can get different spectral values.

\[
s(\omega_i) = \frac{f^T p(\omega_i)}{f^T f}, \quad i = 1, 2, \ldots, m.
\]

(13)

Eq. (13) describes the least squares spectrum. Obviously the least squares spectrum of \( f \) is the collection of the spectral values for all desired frequencies \( \omega_i, i = 1, 2, \ldots, m \). The bigger the spectral value at a frequency \( \omega_i \), the more powerful \( f \) is at this frequency.

A least squares spectrum with a fully populated covariance matrix \( C_f \) can be expressed directly\(^4\) as follows.

\[
s = \frac{f^T C_f^{-1} \Phi (\Phi^T C_f^{-1} \Phi)^{-1} \Phi^T C_f^{-1} f}{f^T C_f^{-1} f}
\]

(14)

### 2.1 Data acquisition

A six-year-long data series from the Canadian superconducting gravimeter installation (CSGI) have been assembled for analysis. All raw data were digitally recorded at the sampling rate of 1/sec in units of Volt. Volts can be changed into units of gravity (\( \mu \text{Gal} \)) by using a calibration factor usually obtained from parallel absolute gravity measurements. This calibration factor for CSGI is \(-78.48 \mu \text{Gal/V} \). Due to environmental disturbances and other processes (earthquakes and hydro-geological process) and instrumental factors (helium refills, failures, maintenance, calibration, system upgrades etc), the SG data are not always continuous and quiet. Gaps, spikes and datum shifts as well as occasional high-level noise exist in the raw data.

### 2.2 Earth tide removal

A synthetic tide by the gravity tide prediction program G-WAVE was calculated at 1s intervals and removed from the superconducting gravimeter record. Program G-WAVE was written by J. B. Merriam\(^7\) at the University of Saskatchewan, Canada with predication accuracy of about 40 nGal, when over 3000 tidal waves are used. The colatitude and longitude of the station are required to compute the gravity tide at different levels of accuracy depending on the number of waves used.

### 2.3 Data filtering

The 1s residual data series produce a very large data set and perhaps with a sampling interval that is unnecessarily short for our scope of analysis. Also, due to the limitation of the computer memory (desktop), the data series must be filtered in the time domain. A Parzen window is used as a weighting function throughout the whole series, purposely centered at random times with no overlap between successive windows to avoid correlation. The series produced with this approach is unequally spaced with variances derived from rigorous error propagation at the noise levels within the window to the filtered data value. To avoid any possibility of aliasing, about 24 points per
hour with a window lag of 50 s and randomly sampled with maximum step 100 s were obtained. Fig. 3 shows the filtered data series and some of the data sets (Table 1). As an example, Table 1 shows a short segment of the series with unequally spaced and weighted values.

| Times/h | Gravity value/μGal | Standard deviation/μGal |
|---------|---------------------|-------------------------|
| 0.013   | 113,192 6           | 2.234 2                 |
| 0.047   | 113,555 5           | 3.217 6                 |
| 0.101   | 113,576 1           | 3.079 8                 |
| 0.145   | 113,317 3           | 2.058 3                 |
| 0.193   | 113,271 6           | 2.347 3                 |
| 0.226   | 113,196 8           | 3.993 5                 |
| 0.265   | 113,132 8           | 2.155 2                 |
| 0.320   | 113,464 4           | 1.502 7                 |
| 0.376   | 113,190 3           | 2.051 3                 |
| 0.426   | 113,159 1           | 2.383 8                 |
| 0.454   | 113,341 0           | 2.187 9                 |
| 0.502   | 113,052 3           | 3.090 6                 |
| 0.557   | 113,103 7           | 2.444 3                 |
| 0.612   | 113,029 2           | 2.404 9                 |

Table 1 SG filtered data series
(only short segment shown here)

Fig. 3 Filtered data series in 2001

2.4 Least squares spectral analysis

After the initial processing of the data (unequally spaced decimation), we obtained filtered data sets with unequally spaced and unequally weighted values (see Table 1). From Fig. 3, we can see clearly that there exists a sizeable noise in April 2001. The reason for this is a serious disturbance to the instrument. This high level noise will certainly distort the spectrum of the data series. Therefore, we eliminated these data segments leaving a gap. Fig. 4 (a) shows the segment in April 2001 with the said gap and Fig. 4(b) shows a data segment in June 2001 with datum shifts and spikes. Based on the technique of LSSA, we did not need to take any corrective action (e.g., interpolation between gaps, smooth of spikes and datum shifts). The spectra were obtained by the weighted least squares spectral analysis method. Each spectrum corresponds to a time series of one month. For example, the least squares spectra of April 2001 and June 2001 are shown in Fig. 5.

3 Discussion and Conclusions

The superconducting gravimeter time series contain a wealth of information about the internal process of the Earth that may be covered by high frequency noise or may be destroyed by careless and unnecessary preprocessing of the original data. The experimental studies with LSSA not only show a great advantage when analyzing gappy, unequally spaced and unequally weighted data, but also allow us to draw a few useful conclusions as follows:
1) Data filtering can be done at unequal space intervals prior to the analysis by the LSSA. The so produced unequally spaced value series with variances derived from rigorous error propagation can eliminate possible aliasing that might have occurred during sampling in the data acquisition stage (hardware filters). In addition, the filtered series contain much less data points thus making their analysis possible with small computer systems (e.g. a desktop computer).

2) The interpolation of gaps in the data series is unnecessary when using LSSA. This is efficient for the data analysis especially in the case when high level noise (e.g. from earthquakes) exists in the series. A very simple way to avoid saturated data segments is to eliminate the said segment altogether and proceed with the analysis with no need to interpolate the data.

3) As shown, one of the advantages of the LSSA is that there is no need to smooth any spikes before analysis. However, the distinction between spikes and extended high level noise should be clearly identified because the latter will certainly distort the spectra and the information we are interested in.

4) A Parzen window is suggested for filtering (or decimating) the data series, but care must be exercised when selecting its bandwidth so as not to over-smooth the final spectrum (loss of essential information) or, at the other extreme, produce a very noisy spectrum in which the signal will not be detectable. For detecting the minute translational modes in the Earth's inner core from superconducting gravimeter records, further research is needed for the estimation of the optimum bandwidth of the data decimation filter. This is the subject of another investigation.

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