NP Stokes fields for radio astronomy

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The spin weighted spherical harmonic (SWSH) description of angular functions typically is associated with the Newman-Penrose (NP) null tetrad formalism. Recently the SWSH description, but not the NP formalism, has been used in the study of the polarization anisotropy of the cosmic microwave background. Here we relate this application of SWSHs to a description of electromagnetic radiation and polarization in the NP formalism. In particular we introduce NP Stokes fields that are the NP equivalent of the Stokes parameters. In addition to giving a more coherent foundation for the recent cosmological SWSH application, the NP formalism aids in the computation of the Lorentz transformation properties of polarization.

I. INTRODUCTION

To describe relativistic fields, the null tetrad, or Newman-Penrose (NP) formalism[1], uses a set of four null spacetime vectors $\vec{\ell}, \vec{n}, \vec{m}, \vec{m}^*$ defined at every point in some region of spacetime. In the NP formalism, rather than deal with the components of a tensor, say the Maxwell tensor $F_{\mu\nu}$ for electromagnetism, one uses the projections of tensors on this tetrad. Because the tetrad field is chosen to satisfy certain properties, it turns out that the NP fields, the projected quantities, are convenient for the mathematics of radiation fields. Since the tetrad legs themselves have angular properties, projection with them adds some angular dependence not inherent in the physical field themselves. This additional angular dependence means that in general NP fields should not be expanded in ordinary spherical harmonics. Rather, all the simplicity of spherical harmonic expansions is regained if NP fields are expanded in sets of angular functions called spin weighted spherical harmonics (SWSHs)[2].

SWSHs have recently played a significant role in research on cosmological anisotropies. The directional nature of linear polarization complicated comparisons of linear polarization in different sky directions, and limited polarization studies to small angular comparisons. In 1996 Seljak[3] found that certain combinations of the Stokes parameters $Q$ and $U$ were particularly well suited to dealing with the angular properties of linear polarization. This insight was still limited, as had been earlier work, to comparisons over small angular regions. That restriction was removed soon after in the breakthrough work by Seljak and Zaldarriaga[4, 5] in which SWSHs were introduced. There soon followed papers[6, 7] with further mathematical details of the use of SWSHs for cosmological anisotropies.

This application of SWSHs to cosmology exploited their connection to the rotation group [8], rather than the role that the SWSHs typically play in physics. To our knowledge, none of the papers on the application of SWSHs to cosmological anisotropies makes reference to the NP formalism itself. One of our purposes here is to show the very natural connection that exists. In this paper we introduce a NP formalism for linear polarization and for Stokes parameters, and we show in this formalism how SWSHs very naturally arise. We argue, furthermore, that the NP formalism has advantages beyond that of giving an explanatory view of a technique already in use.

Since we will only be considering a description of the fields at an observer location, not the propagation of the field, it is appropriate, for simplicity, to limit ourselves to the notation of special, not general relativity. But we make this simplification with some regrets since the NP formalism is extremely useful for dealing with the propagation of fields in curved spacetime. For some problems in curved spacetime, in fact, it is almost indispensable.

We will use the standard special relativistic metric with sign convention $+---$ and with the speed of light $c$ set to unity, so that in Minkowski coordinates

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$  (1.1)

Greek letters $\{\mu, \nu,...\}$ will indicate indices on 4-vectors, and Latin letters $\{i, j, ...\}$ will indicate indices on 3-vectors, also called “spatial vectors.” Arrows over symbols indicate 4-vectors; boldface symbols will be used for 3-vectors. For

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3-vectors, such as the electric and magnetic vectors, subscript and superscript indices are equivalent since the bases used are orthonormal and the 3-metric is $\delta_{ij}$. The 3-vector components are equivalent to the spatial contravariant (superscript) components of a corresponding 4-vector. We use the same root symbol, $\ell$, for the multipole index and for one of the null legs of the NP tetrad. The difference will be clear from context, since in the latter usage it will either appear as a 4-vector $\ell^\mu$ or will be represented by its components $\ell^\mu$.

![Direction of wave propagation and basis vectors.](image)

**FIG. 1:** Direction of wave propagation and basis vectors.

**II. THE NP FORMALISM**

The situation we consider is that of an observer and an antenna at our coordinate origin. Radiation from all directions arrives at the origin, although not necessarily isotropically. This radiation can be considered to be a distribution of plane waves, and we restrict attention to waves propagating in a narrow range of directions around some particular direction $\hat{r}$. As shown in Fig. [1] this direction can also be specified with the angles $\theta$ and $\phi$. The Cartesian components of the vector $\hat{r}$ can be written as

$$\{\hat{r}_x, \hat{r}_y, \hat{r}_z\} = \{\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta\} .$$

(2.1)

It will be very useful to have, in addition to the Cartesian basis, a basis consisting of $\hat{r}$ and the two orthogonal unit vectors tangent to the celestial sphere, $\hat{\theta}$ and $\hat{\phi}$, as shown in Fig. [1].

The direction in which an antenna must be pointed to receive this radiation is, of course, $-\hat{r}$, so that the sky position the observer would assign to the source of this radiation is the antenna direction $\theta_{(A)}$, $\phi_{(A)}$ given by $\theta_{(A)} \equiv \pi - \theta$ and $\phi_{(A)} \equiv \phi + \pi$.

We now use the null tetrad to help in the discussion of radio waves propagating in the $\hat{r}$ direction. The NP null tetrad consists of two real null 4-vectors $\ell$ and $\bar{n}$, and the complex null vector $\bar{m}$, which can usefully be considered to be equivalent to two null 4-vectors $\bar{m}$ and its complex conjugate $\bar{m}^\ast$. The four null vectors must have all dot products to be zero except for the dot products in the following normalization conditions

$$\ell \cdot \bar{n} = 1 \quad \bar{m} \cdot \bar{m}^\ast = -1 .$$

(2.2)

In terms of the notation just defined, we choose our NP tetrad to be the null vectors with the following contravariant components

$$\ell^\mu = \{\ell^0, \ell^j\} = 2^{-1/2}\{1, \hat{r}_j\}$$

(2.3)

$$n^\mu = 2^{-1/2}\{1, -\hat{r}_j\}$$

(2.4)

$$m^\mu = 2^{-1/2}\{0, -\hat{\theta}_j - i\hat{\phi}_j\} .$$

(2.5)

The Minkowski components of these tetrad legs are

$$\ell^\mu = \{\ell^0, \ell^x, \ell^y, \ell^z\} = 2^{-1/2}\{1, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta\}$$

(2.6)

$$n^\mu = 2^{-1/2}\{1, -\sin \theta \cos \phi, -\sin \theta \sin \phi, -\cos \theta\}$$

(2.7)

$$m^\mu = 2^{-1/2}\{0, -\cos \theta \cos \phi + i \sin \phi, -\cos \theta \sin \phi - i \cos \phi, \sin \theta\} .$$

(2.8)
III. STOKES PARAMETERS IN THE NP FORMALISM

We note that the plane waves traveling in the direction shown in Fig. 1 must have their Poynting vector directed in the \( \hat{r} \) direction, and hence must have \( E^\phi = B^r = 0 \), \( E^\theta = B^\phi \) and \( E^\phi = -B^\theta \). In this case, \( \Phi_0 \) and \( \Phi_1 \) vanish, and all the electromagnetic information is carried in \( \Phi_2 \) which reduces to

\[
\Phi_2 = E^\phi - i E^\theta . \tag{3.1}
\]

In the description of Stokes parameters a complex representation of the electric field is typically used. Because the NP quantities have their own complex nature, we avoid the more typical approach and describe a frequency component of the electric field as

\[
\tilde{E}^\phi = \mathcal{E}^\phi \cos (\omega t + \delta_\phi (\omega)) \quad \tilde{E}^\theta = \mathcal{E}^\theta \cos (\omega t + \delta_\theta (\omega)) . \tag{3.2}
\]

We next define a field that is phase shifted by \( \pi/2 \):

\[
\tilde{\Phi}_2 = \mathcal{E}^\phi \cos (\omega t + \delta_\phi (\omega) + \pi/2) - i \mathcal{E}^\phi \cos (\omega t + \delta_\theta (\omega) + \pi/2) = -\mathcal{E}^\phi \sin (\omega t + \delta_\phi (\omega)) + i \mathcal{E}^\theta \sin (\omega t + \delta_\theta (\omega)) . \tag{3.3}
\]

To construct the NP quantities for radio astronomy we average over the range of frequencies of interest to form the following “NP Stokes fields”

\[
\mathcal{I} = 2 \left\langle \Phi_2 \Phi_2^* \right\rangle \quad \mathcal{S} = 2 \left\langle \Phi_2 \Phi_2 \right\rangle \quad \mathcal{V} = 2 i \left\langle \Phi_2 \tilde{\Phi}_2 \right\rangle . \tag{3.4}
\]

The angle brackets \( \langle \rangle \) denote the average over some frequency interval. Unless the radiation of interest is totally unpolarized we assume that \( \delta_\theta \equiv \langle \delta_\theta (\omega) \rangle \) and \( \delta_\phi \equiv \langle \delta_\phi (\omega) \rangle \) are not both zero.
The first of these quantities is straightforward to evaluate with Eq. (3.1) to show that
\[ I = 2 \left( \langle \mathbf{E}^\theta \rangle^2 + \langle \mathbf{E}^\phi \rangle^2 \right) = \langle \mathbf{E}^\theta \rangle^2 + \langle \mathbf{E}^\phi \rangle^2 \equiv I. \] (3.5)

The last symbol in this equation is the usual Stokes parameter \( I \). (See, e.g., Sec. 7.2 of Ref. [11].) Up to multiplicative factors this quantity is clearly just the intensity of radiative energy, i.e., the magnitude of the Poynting vector. The NP quantity \( I \) has spin weight zero, in complete accord with what one would expect on physical grounds, since the intensity, or temperature, is a scalar under rotations.

In terms of the electric field, the NP quantity \( S \) has the form
\[ S = 2 \left( \langle \mathbf{E}^\theta \rangle^2 \right) - 2i\mathbf{E}^\theta \cdot \mathbf{E}^\phi \cos (\delta_\theta - \delta_\phi) = Q - iU, \] (3.6)
where \( Q \) and \( U \) are the usual Stokes parameters of the radiation. (See, e.g., Sec. 7.2 of Ref. [11].) Unlike \( I \), the \( S \) field has spin weight -2, which means that an expansion of \( S \) in functions on the sphere is best carried out with spin weight -2 harmonics[12]. The fact that this spin weight -2 NP Stokes field (or its spin weight +2 complex conjugate) carries the information about linear polarization explains the applicability of spin weight \pm 2 SWSHs to the studies of the linear polarization of the CMB.

In terms of the electric field the third of our NP Stokes fields is
\[ V = 2\mathbf{E}^\theta \mathbf{E}^\phi \sin (\delta_\phi - \delta_\theta) = V, \] (3.7)
where \( V \) is the last of the usual Stokes parameters. Like \( I \), and unlike \( S \), the field \( V \) has spin weight 0. The mathematical reason for this is that the Stokes parameter \( V \) is invariant with respect to rotations about the direction of propagation. Like \( I \), the circular polarization measure \( V \) should be expanded in ordinary spherical harmonics.

All three NP Stokes fields have boost weight -2. We will see below that the boost weight of an NP quantity indicates the quantity’s transformation property under a Lorentz boost. In particular a boost weight \( w \) quantity is multiplied by the Doppler shift (as defined in Sec. IV) raised to the power \( w \). Thus, all the Stokes fields must be multiplied by the square of the Doppler shift under a boost. This is in accord with the fact that all the Stokes fields are quadratic in the transverse electric fields, which themselves are multiplied by a Doppler shift factor under a boost. (An alternate viewpoint is that every photon is Doppler shifted, and the rate of photon arrival is Doppler shifted, hence all radiative intensities gets doubly Doppler shifted.)

### IV. LORENTZ TRANSFORMATIONS

Here we consider the behavior of the NP radiative quantities under a Lorentz transformation, the transformation of measured quantities associated with a change of observer rest frame, in particular for a “boost,” a transformation from the reference frame of an observer to the frame of a relatively moving observer. Transformations of Stokes parameters have been considered by Challinor and van Leeuwen [13], who did not use SWSHs or the NP formalism; rather they confined their study to small angular scales. We show here that the NP formalism removes the restriction of small angular scales for Lorentz transformations, just as it does for angular correlations.

The complete set of Lorentz transformations of reference frame can be considered to be equivalent to the transformations of the null tetrad that maintain the conditions in Eq. (2.2). The group of transformations of the tetrad is usefully divided into several separate subclasses[14, 15], as follows.

**Class I:**

\[ \mathbf{n}' = \mathbf{n} \] (4.1)
\[ \mathbf{m}' = \mathbf{m} + b \mathbf{n} \] (4.2)
\[ \mathbf{\ell}' = \mathbf{\ell} + b \mathbf{m}^* + b^* \mathbf{m} + bb^* \mathbf{n}, \] (4.3)

**Class II:**

\[ \mathbf{\ell}' = \mathbf{\ell} \] (4.4)
\[ \mathbf{m}' = \mathbf{m} + a \mathbf{\ell} \] (4.5)
\[ \mathbf{n}' = \mathbf{n} + a^* \mathbf{m}^* + a \mathbf{m}^* + aa^* \mathbf{\ell}, \] (4.6)
Class III: 
\[ \vec{m}' = e^{i\lambda} \vec{m}, \quad \vec{\ell}' = \vec{\ell}, \quad \vec{n}' = \vec{n}, \quad (4.7) \]

Class IV: 
\[ \vec{m}' = \vec{m}, \quad \vec{\ell}' = K\vec{\ell}, \quad \vec{n}' = K^{-1}\vec{n}. \quad (4.8) \]

The two complex parameters \( a, b \) and the two real parameters \( \lambda \) and \( K \) contain the six degrees of freedom of the Lorentz group. They can be related to the usual boost and rotation parameters of the coordinate basis vectors of observers’ reference frames if a specific association is made between the null tetrad \( \{ \vec{\ell}, \vec{n}, \vec{m}, \vec{m}^* \} \) and the reference frame of an observer.

Since the radiative quantities defined in Eqs. (2.9) are constructed from the tetrad legs, the transformations in Eqs. (4.1)–(4.8) induce transformations to new radiative quantities \( \Phi_0', \Phi_1', \Phi_2' \), as follows.

Class I: 
\[ \begin{align*} 
\Phi_0' &= \Phi_0 + 2b\Phi_1 + b^2\Phi_2 \\
\Phi_1' &= \Phi_1 + b\Phi_2 \\
\Phi_2' &= \Phi_2 
\end{align*} \quad (4.9) \]

Class II: 
\[ \begin{align*} 
\Phi_0' &= \Phi_0 \\
\Phi_1' &= \Phi_1 + a^*\Phi_0 \\
\Phi_2' &= \Phi_2 + 2a^*\Phi_1 + a^{*2}\Phi_0 
\end{align*} \quad (4.10) \]

Class III: 
\[ \Phi_0' = e^{i\lambda}\Phi_0, \quad \Phi_1' = \Phi_1, \quad \Phi_2' = e^{-i\lambda}\Phi_2 \quad (4.11) \]

Class IV: 
\[ \Phi_0' = K\Phi_0, \quad \Phi_1' = \Phi_1, \quad \Phi_2' = K^{-1}\Phi_2. \quad (4.12) \]

We are now in a position to apply this mathematical infrastructure to the radiation problem. We consider two relatively moving observers \( \mathcal{O} \) and \( \mathcal{O}' \), with 4-velocities \( \vec{u} \) and \( \vec{u}' \), and we focus our attention on a particular beam of radiation.

The interpretation of the NP Stokes fields given in Secs. III requires that \( \vec{\ell} \) be in the null direction of propagation of the radiation and that \( \vec{\ell} \cdot \vec{u} = 1/\sqrt{2} \). Since we want both observers to be making the same physical measurements, only in different frames, we must have that \( \vec{\ell}' \) is also in the null direction of propagation (the same spacetime direction as \( \vec{\ell} \)) and that \( \vec{\ell}' \cdot \vec{u}' = 1/\sqrt{2} \). From the requirement that \( \vec{\ell} \) and \( \vec{\ell}' \) be in the same null direction, we infer that the tetrads of the two observers must be related only by transformations of Classes II, III, and IV, since the transformations of Class I in Eqs. (4.1)–(4.3), change the direction of \( \vec{\ell} \). Equivalently, \( b \) must vanish and the relationship of the measurements of the two observers must depend only on the real parameters \( K, \lambda \) and on the complex parameter \( a \).

An interesting conclusion now follows. As pointed out at the start of Sec. III, \( \Phi_0 = \Phi_1 = 0 \). The transformations in Eqs. (4.9)–(4.11), with \( b = 0 \), then tell us that \( \Phi_0' = \Phi_1' = 0 \). This, of course, must be the case on physical grounds. Furthermore, these relations tell us that the most general nontrivial transformation of \( \Phi_2 \) is \( \Phi_2' = e^{-i\lambda}K^{-1}\Phi_2 \). For the NP Stokes fields this means that
\[ \begin{align*} 
I' &= K^{-2}I, \\
S' &= e^{-2i\lambda}K^{-2}S, \\
V' &= K^{-2}V. \quad (4.13) \]

In a Lorentz transformation, then, the polarization properties in \( S \) are affected by both \( K \) and by the \( \lambda \) parameter of a Class III transformation, while the other two NP Stokes fields, as well as the magnitude of \( S \), are affected only by \( K \). The detailed relationship of the parameters \( K \) and \( \lambda \), and the frames of observers, are given in Appendix A. Here we only give the result for \( K \). If \( \beta \) is the 3-velocity of a frame \( \mathcal{O}' \), as observed by \( \mathcal{O} \), then
\[ K = \frac{\sqrt{1 - \beta^2}}{1 - \hat{\beta} \cdot \beta} \quad (4.14) \]
This is the well known Doppler factor that relates observations of radiation by the two observers, and is the ratio $E/E'$ of the observed energies.

Although the formal transformation properties in Eq. (4.17) are correct, they cannot directly be related to observations of the CMB. The Stokes parameters, and the NP Stokes fields, measure radiative power per unit area. Radio telescopes, however, measure radiative power per unit area per unit frequency interval $d
u$, per unit solid angle $d\Omega$. In considering transformation properties we must take into account that

$$\nu' = K^{-1}\nu \quad d\Omega' = K^2 d\Omega.$$  \hspace{1cm} (4.19)

Thus, a radio telescope does not measure e.g., the Stokes parameter $I$, but rather the specific intensity

$$I_\nu \equiv \frac{dI}{d\nu d\Omega}.$$  \hspace{1cm} (4.20)

From the transformation properties in Eqs. (4.17) and (4.19) it follows that the specific intensity $I_\nu$ transforms according to

$$I_\nu' = K^{-3}I_\nu.$$  \hspace{1cm} (4.21)

We can apply this to the NP Stokes fields and can introduce measureable quantities $I_\nu$, $S_\nu$, $V_\nu$, defined per unit frequency per unit solid angle. The transformation laws for these quantities are those of Eq. (4.17), with $K^{-2}$ replaced by $K^{-3}$.

These specific intensities are directly related to what is measured by radiotelescopes, but it is simple, and practical, to construct other quantities directly related to telescope observables. The specific intensities can be combined with the frequency at which the specific intensity is measured to form quantities $I_\nu/\nu^N$, and the equivalent for NP Stokes fields. For these fields the transformation properties are

$$\left(I_\nu/\nu^N\right)' = K^{N-3} \left(I_\nu/\nu^N\right) \quad \left(S_\nu/\nu^N\right)' = e^{-2i\lambda}K^{N-3} \left(S_\nu/\nu^N\right) \quad \left(V_\nu/\nu^N\right)' = K^{N-3} \left(V_\nu/\nu^N\right).$$  \hspace{1cm} (4.22)

In cosmology, the Lorentz transformation properties of radiation quantities are important for transforming observations to a “rest frame” of the CMB. In principle this is the frame that must be used for comparisons of observations and theories of early universe processes. The general prescription for finding this frame is to identify a dipole in the radiation. A transformation can then be made to a frame in which this dipole vanishes. The process, however, is not unique. One can, for example, choose to eliminate the dipole in circular polarization rather than intensity.

The problem of finding a rest frame is essentially the problem of understanding how the radiation gives a representation of the Lorentz group. The general theory of such representations can be based on the work of Gel’fand, Graev and Vilenkin[16] and is sketched in Appendix B. With this theory, Lorentzian vectors and tensors are constructed from the multipoles, the 2-sphere integrals of radiation quantities. This issue of a choice of rest frames is taken up in greater detail in Appendix B.

V. SUMMARY AND CONCLUSIONS

The introduction of SWSHs to the analysis of polarization anisotropy followed from a consideration of the explicit rotation properties of the Stokes parameters $Q, U$ and to the subsequent realization that a combination of the two had convenient transformation properties under rotation. We have shown here that the NP formalism leads very naturally to an NP formulation of the Stokes parameters, and that in this formulation the use of spin weight $\pm 2$ SWSHs is immediate. We have also shown the convenience of the NP formalism for “boosts,” the transformation of radiation properties from an observer’s frame to the frame of a relatively moving observer. More specifically, we have shown that the NP formalism allows for the computation of boosts of any radiation quantity without the limitation of small angular scale.

The convenience of the NP formalism is not a coincidence. Its mathematical underpinnings are deeply rooted in the structure of spacetime, so the formalism is very naturally suited to the description of fields that propagate at the speed of light. This can be taken as the reason that Stokes parameters arise simply as fields quadratic in the NP field describing electromagnetic radiation. This even more certainly explains the relative simplicity of boost transformations. The brief discussion given of the nonuniqueness of the “rest frame” of the CMB is closely associated with the convenience of the NP formalism for boosts.

Since the advantages, or conveniences, follow from the appropriateness of the NP formalism for fields propagating along null directions in spacetime, they apply also to non-electromagnetic fields. In particular, gravitational waves,
whether the standard transverse-traceless waves of Einstein’s theory, or the full set of six possible polarization states, all have gravitational Stokes parameters that follow from fields quadratic in the NP projections of the Riemann tensor.

Lastly, we mention that the NP formalism is very naturally suited to calculations of the propagation of radiation in curved spacetime. In the cosmological context this means that the NP formalism should greatly simplify, for example, the calculation of the effect on polarization of gravitational lensing or inhomogeneous cosmological expansion.

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Appendix A: The general Lorentz transformation

To relate the value of $K$ to the usual parameterization of a Lorentz transformation we consider, as in Sec. IV, a frame $\mathcal{O}'$ observed in frame $\mathcal{O}$ to be moving with 3-velocity $\beta$. The propagation 4-vector $\tilde{k}$ of a plane electromagnetic wave (equivalently, the 4-momentum of a photon) must be parallel to $\tilde{\ell}$, so we can write $\tilde{k} = \kappa \tilde{\ell}$, where $\kappa$ is proportional to the energy $E$ of a “photon” in frame $\mathcal{O}$. With $\ell' = K\ell$, we have from this

$$\frac{E'}{E} = \frac{\tilde{k} \cdot \tilde{u}'}{\kappa \cdot \tilde{u}} = \frac{\tilde{\ell} \cdot \tilde{u}'}{\kappa \cdot \tilde{u}} = K^{-1} \frac{\tilde{\ell}' \cdot \tilde{u}'}{\ell \cdot \tilde{u}} = K^{-1}. \quad (A1)$$

To get the specific dependence of $K$ on the relative direction of motion, and the direction of observation, we can use a straightforward evaluation of $\tilde{\ell} \cdot \tilde{u}'$ in the $\mathcal{O}$ frame, in which $\tilde{\ell}$ has the contravariant components in Eq. (2.3), and $\tilde{u}'$ has the contravariant components

$$u'^\mu = \gamma \{1, \beta^j\}, \quad (A2)$$

where $\gamma$ is the usual Lorentz factor $1/\sqrt{1 - \beta^2}$. From these components we get

$$\tilde{\ell} \cdot \tilde{u}' = \gamma(1 - \hat{r} \cdot \beta)/\sqrt{2}. \quad (A3)$$

But we have $\tilde{\ell} = K^{-1}\tilde{\ell}'$ so that $\tilde{\ell} \cdot \tilde{u}' = K^{-1}\tilde{\ell}' \cdot \tilde{u}' = K^{-1}/\sqrt{2}$. From this we have $K^{-1} = \sqrt{2} \tilde{\ell} \cdot \tilde{u}'$ and

$$K = \frac{1}{\sqrt{2} \tilde{\ell} \cdot \tilde{u}'} = \frac{1}{\gamma(1 - \hat{r} \cdot \beta)}. \quad (A4)$$

It should be noticed that we have not completely specified the Lorentz transformation between the reference frames of observers $\mathcal{O}$ and $\mathcal{O}'$; we have specified only the relative velocity, not the rotations that relate them. From the point of view of the transformations in Eqs. (4.1)–(4.8), we have fixed the complex parameter $\hat{b}$ to be zero (since the direction of $\tilde{\ell}$ is fixed) and we have fixed the real parameter $K$ with Eq. (A4). What remains is the three degrees of freedom in the parameters $\lambda$ and $a$. These three degrees of freedom can be fixed independently for each direction $\theta, \phi$. That is, $\lambda$ and $a$ can, like $K$ in Eq. (A4), be functions of $\hat{r}$. These three degrees of freedom are equivalent to the freedom to make spatial rotations connecting frames $\mathcal{O}$ and $\mathcal{O}'$.

The choice of the orientations of these frames is equivalent to a specification of how the null tetrad legs are chosen. We start by considering observer $\mathcal{O}$ with 4-velocity $\tilde{u}$ and radiation in a particular direction. In Eq. (2.3) we chose $\tilde{\ell}$ in the direction of propagation, and normalized $\tilde{\ell}$ so that $\tilde{u} \cdot \tilde{\ell} = 1/\sqrt{2}$, for reasons already discussed. We next, in Eq. (2.4), defined $\tilde{n}$ by

$$\tilde{n} \equiv \sqrt{2} \tilde{u} - \tilde{\ell}. \quad (A5)$$

This choice guarantees that $\tilde{n}$ is null and satisfies the normalization condition in Eq. (2.2). It also means that for any direction of incoming radiation we have a unit “outward” spatial vector $\tilde{r} \equiv (\tilde{n} - \tilde{\ell})/\sqrt{2}$ that is orthogonal to $\tilde{u}$, and that is parallel to the spatial direction of propagation. (The vector $\tilde{r}$ is just $\sqrt{2}$ times the projection of $\tilde{\ell}$ orthogonal to $\tilde{u}$.) In addition, the form of $\tilde{n}$ in Eq. (A5) provides a unit timelike vector $\tilde{t} \equiv (\tilde{n} + \tilde{\ell})/\sqrt{2} = \tilde{u}$ that is
independent of the direction of propagation of the radiation. In short, we have a tetrad compatible with the $\vec{\ell}$ and $\vec{n}$ legs of Eqs. (2.3) and (2.4). We also make the choice

$$\vec{n}' = \sqrt{2} \vec{u}' - \vec{\ell}$$

(A6)

so that the tetrad for $\mathcal{O}'$ is also compatible with the conditions in Eqs. (2.3) and (2.4).

From Eqs. (4.1)–(4.8), the set [17] of allowed transformations (with $b = 0$) of the tetrad for $\mathcal{O}'$ in terms of the tetrad for $\mathcal{O}$ is

$$\vec{\ell}' = K \vec{\ell}$$

(A7)

$$\vec{m}' = e^{i\lambda} \left( \vec{m} + a \vec{\ell} \right)$$

(A8)

$$\vec{n}' = K^{-1} \left( \vec{n} + a \vec{m} + a^* \vec{m}^* + aa^* K \vec{\ell} \right).$$

(A9)

For simplicity, we first make the restriction that the $z$ axis is chosen in the direction of $\beta$. (The $\beta$ direction can always be put in the $z$ direction by using pure rotations.) We then set equal the right-hand sides of the expressions for $\vec{n}'$ given in Eqs. (A6) and (A9)

$$\sqrt{2} \gamma \{1, \beta\} - K \{1, \vec{r}\} = K^{-1} \left[ \{1, -\vec{r}\} + a\{0, -\theta - i\phi\} + a^*\{0, -\theta + i\phi\} + aa^* K \{1, \vec{r}\} \right].$$

(A10)

Taking the dot product of this equation, first with $\vec{\theta}$ and then with $\vec{\phi}$, for $\beta$ in the $z$ direction, gives us

$$\sqrt{2} \gamma \alpha \sin \omega = K^{-1} \left[ \left( a / \sqrt{2} \right) + \left( a^* / \sqrt{2} \right) \right]$$

(A11)

$$0 = K^{-1} \left[ \left( a / \sqrt{2} \right) i + \left( a^* / \sqrt{2} \right) (-i) \right],$$

(A12)

where $\alpha$ is the angle between the directions of $\beta$ and $\vec{r}$. (This is also $\theta$ for $\beta$ in the $z$ direction.) From this we conclude

$$a = \beta \gamma K \sin \alpha = \frac{\beta \sin \alpha}{1 - \beta \cos \alpha}.$$  

(A13)

Finally, we note that the direction of $\vec{\phi}$ is invariant under a Lorentz boost, since $\vec{\phi}$ is orthogonal to $\beta$. Since $\vec{m} \cdot \vec{\phi} = \vec{m}' \cdot \vec{\phi}'$, this means that $\vec{m} \cdot \vec{\phi} = \vec{m}' \cdot \vec{\phi}'$. In addition, $\vec{m} \cdot \vec{\ell} = 0$, so that the dot product of $\vec{\phi}$ with Eq. (A8) gives us $\lambda = 0$.

In summary, for a pure boost in the $z$ direction the tetrad transformation parameters are

$$\lambda = b = 0 \quad K = \frac{1}{\gamma (1 - \beta \cos \alpha)} \quad a = \frac{\beta \sin \alpha}{(1 - \beta \cos \alpha)}.$$  

(A14)

It is interesting that $\lambda = 0$ for this pure boost in the $z$ direction. According to Eq. (4.17) this means that $\mathcal{S}/\mathcal{I}$ is invariant for such a transformation, and therefore that both observers will agree on the linear polarization of the radiation.

We now turn to the general transformation, with $\beta$ not necessarily in the $z$ direction. We have already constructed $\vec{\ell}'$ and $\vec{n}'$ from the propagation vector and the 4-velocity $\vec{u}'$. What remains is to find $\vec{m}'$. To do this we note that from the forms of $\vec{m}$ and $\vec{\ell}$ given in Sec. II we have

$$\vec{m} (\theta, \phi) = - (\partial_\theta + (i / \sin \theta) \partial_\phi) \vec{\ell} (\theta, \phi).$$

(A15)

An analogous relationship must obtain in the primed frame, so we can write

$$\vec{m}' (\theta', \phi') = - (\partial_{\theta'} + (i / \sin \theta') \partial_{\phi'}) \vec{\ell}' (\theta', \phi') = - (\partial_{\theta'} + (i / \sin \theta') \partial_{\phi'}) K (\theta, \phi) \vec{\ell} (\theta, \phi).$$

(A16)

In order to compute the right hand side of this equation, we must know $\theta, \phi$ as functions of $\theta', \phi'$ for an arbitrary Lorentz transformation.

The general Lorentz transformation, for an arbitrary boost combined with an arbitrary rotation, requires the specification of six parameters. It is convenient here to utilize the description and parameterization related to the “aberration transform,” as described e.g., in Ref. [18], with a slightly different notation [19]. In this description the direction of photon propagation $\theta', \phi'$ in frame $\mathcal{O}'$ is related to the direction $\theta, \phi$ for the same photon in frame $\mathcal{O}$ by

$$e^{i\phi'} \cot (\theta'/2) = A e^{i\phi} \cot (\theta/2) + B \quad C e^{i\phi} \cot (\theta/2) + D.$$  

(A17)
Here $A, B, C, D$ are complex numbers. The eight degrees of freedom in these complex numbers are constrained by the two conditions in $AD - BC = 1$, leaving six unconstrained degrees of freedom in the transformation (A17). Any proper (non-time reversing, non-parity reversing) Lorentz transformation can be represented with these six degrees of freedom.

Carrying out the differentiations indicated gives

$$\vec{m}'(\theta', \phi') = e^{i(\sigma - \phi + \phi')} \left( \vec{m}(\theta, \phi) + J(\theta, \phi) \vec{\ell}(\theta, \phi) \right),$$

(A18)

where

$$J(\theta, \phi) = - (\partial_\theta + (i/\sin \theta) \partial_\phi) [\log K(\theta, \phi)],$$

(A19)

and

$$e^{i\sigma} = \sqrt{\frac{C^* e^{-i\phi} \cot(\theta/2) + D^*}{C e^{i\phi} \cot(\theta/2) + D}}.$$  

(A20)

The values of $a$ and $\lambda$ are inferred from a comparison of the result in Eq. (A18) and Eq. (A8).

The special case of the pure boost in the $z$ direction is given by $B = C = 0$ and $D = 1/A$, with $A$ a real number. Equation (A20) gives $\sigma = 0$. Since $K$, from Eq. (A14), in a slightly different notation, is $K = [\gamma(1 - \beta \cos \theta)]^{-1}$ we have, from Eq. (A19),

$$J = \partial_\theta [\log \gamma(1 - \beta \cos \theta)] = \frac{\beta \sin \theta}{1 - \beta \cos \theta},$$

(A21)

which means, according to Eq. (A18) that

$$\vec{m}' = \vec{m} + \frac{\beta \sin \theta}{1 - \beta \cos \theta} \vec{\ell},$$

(A22)

in agreement with Eqs. (A8), (A14) for the case of a pure boost in the $z$ direction.

Appendix B: Representations of the Lorentz group and the relevance to the CMB

Following the beautiful theory of representations of the Lorentz group developed by Gel’fand, Graev and Velinkin (GGV)[16], one finds that the different Stokes fields lying on the celestial sphere, lie in different vector spaces of infinite dimensional representations of the Lorentz group. In the notation of GGV, representations are given by homogeneous functions of two complex variables $(z_1, z_2, \bar{z}_1, \bar{z}_2)$, each representation being labeled by the homogeneity degree, $(n_1 - 1, n_2 - 1)$ of each pair of the variables. We consider only the so-called integer representations, for which $n_1$ and $n_2$ are either both positive or both negative integers. These representations, via homogeneous functions, can be mapped into functions on the sphere with well defined spin and conformal weights, $s$ and $w$. The $s$ and $w$ are related to $n_1$ and $n_2$ by

$$(n_1, n_2) = (w - s + 1, w + s + 1).$$

(B1)

A variety of results can be extracted from the GGV work. If $n_1$ and $n_2$ are either both positive or both negative, one can find (or construct), from the infinite dimensional representation spaces, specific finite dimensional vector spaces that are equivalent to the standard tensor representations. (A simple example is that the $\ell = 0$ harmonic coefficient of a $w = -2, s = 0$ function is a Lorentz scalar. See below.)

We review the basic argument and results for negative integer representations, taking for simplicity the special case of $s = 0$. First, we note that the area element on the unit sphere (or equivalently, the solid angle as seen on the celestial sphere) [20]

$$d\Omega = \sin \theta d\theta d\varphi,$$

(B2)

transforms (under the aberration transformation of Eq. (A17)) as

$$d\Omega' = K^2 d\Omega.$$

(B3)
Second, we consider functions \( F_{(-n-2)} \) and \( G_{(n)} \) on the sphere, respectively with boost weight \(-n - 2\) and \( n \), that is with Lorentz transformation properties
\[
F'_{(-n-2)} = K^{-n-2}F_{(-n-2)} \quad G'_{(n)} = K^nG_{(n)}.
\]

The integral
\[
\int F_{(-n-2)}G_{(n)} \, d\Omega
\]
over the 2-sphere is therefore a Lorentz invariant:
\[
\int F_{(-n-2)}G_{(n)} \, d\Omega = \int F'_{(-n-2)}G'_{(n)} \, d\Omega'.
\]

In a similar manner, we can create Lorentz vectors and tensors.

The following are several simple specific examples:

(a) If we choose \( G_{(0)} = 1 \) and choose \( F_{(-2)} \) arbitrarily, we have that
\[
\text{Harmonic}_{\ell=0} = \int F_{(-2)} \, d\Omega = \int F'_{(-2)} \, d\Omega',
\]
is a Lorentz invariant. That is, we have that the monopole, the \( \ell = 0 \) harmonic coefficient, of \( F_{(-2)} \) is Lorentz invariant.

(b) If we choose \( G_{(1)} = \ell^a \) and choose \( F_{(-3)} \) arbitrarily we have that
\[
w^a = \int F_{(-3)} \ell^a \, d\Omega,
\]
is a Lorentzian 4-vector extracted from \( F_{(-3)} \).

(c) If we choose \( G_{(2)} = \ell^a \ell^b \) and choose \( F_{(-4)} \) arbitrarily we have that
\[
w^{ab} = \int F_{(-4)} \ell^a \ell^b \, d\Omega,
\]
a trace-free symmetric tensor extracted from \( F_{(-4)} \).

A subtle issue arises in the above constructions. In the body of the paper we took the components \( \ell^a \) in one particular Minkowski coordinate system to have the canonical form
\[
\ell^a = \frac{\sqrt{2}}{2}(1, \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta).
\]
For \( \tilde{\ell} \) in the primed frame, we must have \( \tilde{\ell}' = K\ell \), and hence the components of \( \tilde{\ell}' \) must be (\( \theta, \phi \)-dependent) linear combinations of the same harmonics, i.e., of \( (1, \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \). But the components of \( \tilde{\ell}' \), must have the canonical form \( 2^{-1/2}(1, \sin \theta' \cos \varphi', \sin \theta' \sin \varphi', \cos \theta') \), in the primed Minkowski frame. This means that a Lorentz transformation of coordinates
\[
\ell'^{a'} = L^a_a \ell^a = \frac{\sqrt{2}}{2}(1, \sin \theta' \cos \varphi', \sin \theta' \sin \varphi', \cos \theta')
\]
is necessary to give the canonical form for the components \( \tilde{\ell}' \). This in turn implies, via \( \ell'^{a'}w^{a'} = \ell^aw^a \), a (coordinate) Lorentz transformation on the \( w^a \),
\[
w'^{a'} = L^a_a w^a.
\]

These relations are easily translated to the Stokes fields, \( I(\theta, \varphi), S(\theta, \varphi) \) and \( V(\theta, \varphi) \) discussed in the text and, as well, to the closely associated quantity, the intensity per unit solid angle per unit frequency interval \( I_\nu(\theta, \varphi) \). The \( I_\nu \) transforms as with \( s = 0, \) and \( w = -3 \).
Since $I(\theta, \varphi)$ has boost weight $w = -2$ and spin weight $s = 0$, a formal set of relations arise from the GGV work, and a set of Lorentzian tensors can be directly extracted as follows from $I$, $I_\nu$, or more generally from $I/\nu^N$.

(1) From Eq. (B7), it follows that the coefficient of the $\ell = 0$ harmonic, the monopole of $I$, is a Lorentz invariant, $I_0$.

(2) From Eq. (B8), in the spherical harmonic expansion of the $w = -3$ quantity $I^2$, the coefficients of the $(\ell = 0, 1)$ harmonics form and transform as a Lorentzian four-vector, say $T^a$. In the same manner, the coefficients of the $(\ell = 0, 1)$ harmonics in $I_\nu$ determine a four-vector, $v^a$ that can be considered as defining the rest-frame for the radiation bath.

(3) From Eq. (B9), the spherical harmonic expansion of either of the $w = -4$ quantities $I^2$, $\nu I$, the coefficients of the $(\ell = 0, 1, 2)$ harmonics form and transform as a Lorentzian trace-free, symmetric two-index tensor, $T^{ab}$ and $V^{ab}$ respectively.

(4) These results generalize to all higher powers of $I$, i.e. to all cases of integer $w < -4$, and $s = 0$.

These results allow us to assign a physical meaning for the vector defined in (2). The quantity $I_\nu$, \textit{the intensity per unit solid angle per unit frequency interval}, is the physical quantity that is usually measured in CMB observations. Since it is a quantity of weight $w = -3$ it has, as was mentioned above, an invariantly defined Lorentzian four-vector, $v^a$, the $\ell = 0, 1$ harmonics. Since the monopole is far larger than the dipole term, this vector is time-like and can be identified with the observers velocity with respect to the CMB background rest-frame. To use the Lorentzian vector $v^a$ as a ‘rest-frame’ for the background radiation field one finds a moving frame in which $I_\nu$ has no dipole component. More explicitly, the velocity of the rest frame, thus defined, is

$$v^a \equiv \int I_\nu I^a d\Omega$$

This in turn could be used to find the $K$ and then determine the rest-frame distribution

$$I_\nu' = K^{-3}I_\nu.$$

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