LAX EQUATIONS AND KNIZHNIK-ZAMOLODCHIKOV CONNECTION

OLEG K. SHEINMAN

Abstract. Given a Lax system of equations with the spectral parameter on a Riemann surface we construct a projective unitary representation of the Lie algebra of Hamiltonian vector fields by Knizhnik-Zamolodchikov operators. This provides a prequantization of the Lax system. The representation operators of Poisson commuting Hamiltonians of the Lax system projectively commute. If Hamiltonians depend only on action variables then the corresponding operators commute.

1. INTRODUCTION

In [7] I.Krichever proposed a new notion of Lax operator with a spectral parameter on a Riemann surface. He has given a general and transparent treatment of Hamiltonian theory of the corresponding Lax equations. This work has called into being the notion of Lax operator algebras [13] and consequent generalization of the Krichever’s approach on Lax operators taking values in the classical Lie algebras over $\mathbb{C}$ [20, 21]. The corresponding class of Lax integrable systems contains Hitchin systems and their analog for pointed Riemann surfaces, integrable gyroscopes and similar examples.

In the present paper we address the following problem: given a Lax integrable system of the just mentioned type, to construct a unitary projective representation of the corresponding Lie algebra of Hamiltonian vector fields. For the Lax equations in question, we propose a way to represent Hamiltonian vector fields by covariant derivatives with respect to the Knizhnik-Zamolodchikov connection. It is conventional that Knizhnik-Zamolodchikov-Bernard operators provide a

1991 Mathematics Subject Classification. 17B66, 17B67, 14H10, 14H15, 14H55, 30F30, 81R10, 81T40.

Key words and phrases. Current algebra, Lax operator algebra, Lax integrable system, Knizhnik-Zamolodchikov connection.

Supported in part by the program "Fundamental Problems of Nonlinear Dynamics" of the Russian Academy of Sciences.
quantization of Calogero-Moser and Hitchin second order Hamiltonians [3, 5]. Unexpectedly, we observed such relation for all Hamiltonians, and, moreover, for all observables of the Hamiltonian system given by the Lax equations in question.

Consider the phase space $\mathcal{P}^D$ of the Lax system in the Krichever formalism. Every element of $\mathcal{P}^D$ is a meromorphic $g$-valued function on a Riemann surface $\Sigma$ satisfying certain constrains where $g$ is a classical Lie algebra over $\mathbb{C}$. Assign every $L \in \mathcal{P}^D$ with its spectral curve $\det(L(z) - \lambda) = 0$ which is a finite branch covering of $\Sigma$. Thus we have a family of Riemann surfaces over $\mathcal{P}^D$. Following the lines of [15] (see also [18]) we construct an analog of the Knizhnik-Zamolodchikov connection for this family. For this end, we construct a sheaf of admissible representations of the Lax operator algebras canonically related to the integrable system. For any tangent vector $X$ to $\mathcal{P}^D$ denote by $\rho(X)$ its image under the Kodaira-Spencer mapping. In general $\rho(X) \in H^1(\Sigma_L, T\Sigma_L)$ where $\Sigma_L$ is the spectral curve at the corresponding point $L \in \mathcal{P}^D$ and $T\Sigma_L$ is its tangent sheaf. It is shown in [15] that $\rho(X)$ can be considered as a Krichever-Novikov vector field, and the arising ambiguity is compensated by passing to a certain quotient sheaf called a sheaf of coinvariants. Hence we may consider $T(\rho(X))$ where $T$ is the Sugawara representation, and further on define the high genus Knizhnik-Zamolodchikov connection

$$\nabla_X = \partial_X + T(\rho(X)).$$

By projective flatness of $\nabla$ [15] we have

$$[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + \lambda(X,Y) \cdot id,$$

i.e. $X \to \nabla_X$ is a projective representation of the Lie algebra of vector fields on $\mathcal{P}^D$.

Now observe that $\mathcal{P}^D$ is a symplectic manifold with respect to the Krichever-Phong symplectic structure $\omega$ [7, 12]. For any $f \in C^\infty(\mathcal{P}^D)$ we can consider the corresponding Hamiltonian vector field $X_f$. The correspondence $f \to \nabla_{X_f}$ is a representation of the Poisson algebra of classical observables of the Lax integrable system. An easy corollary is as follows. If $\{H_a\}$ is a Poisson commuting family of Hamiltonians labelled by an index $a$ then the corresponding Knizhnik-Zamolodchikov operators projectively commute.
The just introduced representation of the Lie algebra of Hamiltonian vector fields is unitary. This claim relies on two facts. First, by Poincaré theorem the symplectic form and its degrees are absolute integral invariants of Hamiltonian phase flows. Hence the measure on $\mathcal{P}^D$ defined by the volume form $\omega^p/p!$ (where $p = (\dim \mathcal{P}^D)/2$) is invariant with respect to the Hamiltonian flows. For this reason the $\partial_X$ operator is skew-symmetric for any real Hamiltonian $X$ in a certain subspace in $L^2(\mathcal{P}^D, \omega^p/p!)$. Second, the Sugawara representation is unitary as soon as the underlying representation of the current algebra possesses this property. For the Virasoro and loop algebras it is proved in [6], and for Krichever-Novikov vector field and commutative current algebras in [11].

The subject of the paper certainly must be discussed in the context of quantization of Lax integrable systems. Our work is very similar to Hitchin [4]. Both are devoted to the problem of a correspondence between an integrable system and a connection on a certain moduli space, in a different set-up. In [4] N.J.Hitchin notes two problems of his approach: taking into account the marked points on Riemann surfaces and unitarity of the connection. He points out that the Knizhnik-Zamolodchikov connection could be a solution of the first problem. As it is shown below, it resolves also the second one.

A large number of works is devoted to quantum integrable systems. Let us give here a brief outline of the construction due to B.Feigin and E.Frenkel [1], A.Beilinson and V.Drinfeld [2]. Denote by $U(\mathfrak{g})_\kappa$ the universal enveloping algebra of $\mathfrak{g}$ on the level $\kappa$. There is the only case when $U(\mathfrak{g})_\kappa$ has an infinite-dimensional center, namely if $\kappa$ is equal to the dual Coxeter number for $\mathfrak{g}$. By means of the double-coset construction one can obtain an action of the finite-dimensional quotient space of the center by differential operators in the smooth sections of some vector bundle on the moduli space of holomorphic vector bundles on a Riemann surface. Indeed, $\mathcal{M} = G_{left} \setminus G/G_+$ where $\mathcal{M}$ is the moduli space of holomorphic vector bundles, $G$ is the loop group corresponding to $\mathfrak{g}$, $G_+$ is its subgroup corresponding to $\mathfrak{g}_+$ and $G_{left}$ is a certain subgroup of $G_-$. The center of $U(\mathfrak{g})_\kappa$ acts on $G$ by casimirs. This action pushes down to $\mathcal{M}$. Not every vector bundle on $\mathcal{M}$ admits a nontrivial space of differential operators. It exists in a more-less unique case of $\sqrt{\kappa}$ — the bundle of half-forms.
on $\mathcal{M}$ (where $\mathcal{K}$ is the canonical bundle on $\mathcal{M}$). This construction results in the quantum Hitchin system in a sense that the symbols of the obtained operators are equal to the corresponding classical Hitchin Hamiltonians.

Observe that the authors do not quantize the full algebra of observables but only its commutative subalgebra (see [2, 2.2.5] for example).

Especially detailed information is obtained on quantum Calogero-Moser systems due to the works of A.P.Veselov, A.N.Sergeev, G.Felder, M.V.Feigin.

The idea of quantization of Hitchin systems by means Knizhnik-Zamolodchikov connection was also addressed, or at least mentioned, many times in the theoretical physics literature (D.Ivanov [5], G.Felder and Ch.Wieczorkowski [3], M.A.Olshanetsky and A.M.Levin) but only the second order Hamiltonians have been involved.

To conclude with, let us note that the results of the present paper provide a prequantization of the Lax integrable systems with the spectral parameter on a Riemann surface. We prequantize the whole algebra of observables rather than any commutative subalgebra of it.

The paper is organized as follows. In Section 2 we give a description of Lax integrable systems with the spectral parameter on a Riemann surface. This section is a survey of the results of [7, 13, 20, 21]. We introduce the phase parameters and explain their relation to the Tjurin parametrization of holomorphic vector bundles on Riemann surfaces. Then we define the notion of a Lax operator with the spectral parameter on a Riemann surface, and values in a classical complex Lie algebra. We introduce the corresponding Lax equations, construct their hierarchy of commuting flows and present their Hamiltonian theory including the Krichever-Phong symplectic structure.

In Section 3 we introduce a conformal field theory related to an integrable system of the above described type. By conformal field theory we mean a family of Riemann surfaces, a finite rank bundle (of coinvariants) on this family, and a flat connection (the Knizhnik-Zamolodchikov connection) on this bundle.

As a family of Riemann surfaces we take the family of spectral curves corresponding to the Lax integrable system in question. We recall from [15] [18] the above mentioned version of the Kodaira-Spencer mapping.
Further on we introduce a certain commutative current Krichever-Novikov algebra which is required by the Sugawara construction. For a Lax operator algebra this commutative algebra plays a role similar to Cartan subalgebra in the Kac-Moody and Cartan-Weil theories. At last, we recall from [22] the construction of the fermionic representation of Krichever-Novikov current algebras and carry out the Sugawara construction.

In Section 4 we formulate and prove main results of the paper. We construct the Knizhnik-Zamolodchikov connection on the family of spectral curves and prove that the Knizhnik-Zamolodchikov operators give a projective unitary representation of the Lie algebra of Hamiltonian vector fields. As a corollary we obtain that the operators corresponding to the family of commuting Hamiltonians commute up to scalar operators.

The author is grateful to I.M.Krichever and to M.Schlichenmaier for many fruitful discussions. The majority of authors results the present work relies on are obtained in collaboration with them. I am also grateful to D.Talalaev. Our discussions were helpful in order to realize a role of the spectral curve in quantization. I am thankful to A.P.Veselov and M.A.Olshanetsky for useful discussions.

2. Phase space and Hamiltonians of a Lax integrable system

In the present section following the lines of [7, 21] we consider a certain class of integrable systems given by $\mathfrak{g}$-valued (in particular matrix-valued) Lax operators of zero order with a spectral parameter on a Riemann surface. The examples include Calogero-Moser systems, Hitchin systems and their generalizations, gyroscopes etc.

2.1. Geometric data. Every integrable system in question is given by the following geometrical data: a Riemann surface $\Sigma$ with a given complex structure, a classical Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, fixed points $P_1, \ldots, P_N \in \Sigma \ (N \in \mathbb{Z}_+)$, a positive divisor $D = \sum_{i=1}^{N} m_i P_i$, points $\gamma_1, \ldots, \gamma_K \in \Sigma \ (K \in \mathbb{Z}_+)$, vectors $\alpha_1, \ldots, \alpha_K \in \mathbb{C}^n$ associated with $\gamma$’s and given up to the right action of a classical group $G$ corresponding to $\mathfrak{g}$. The
last two items (γ’s and α’s) are joined under the name Tyurin data, because of the following

**Theorem 2.1** (A.N.Tyurin). Let \( g = \text{genus } \Sigma, n \in \mathbb{Z}_+ \). Then there is a 1-to-1 correspondence between the following data:

1) points \( \gamma_1, \ldots, \gamma_{ng} \) of \( \Sigma \);
2) \( \alpha_1, \ldots, \alpha_{ng} \in \mathbb{C}P^{n-1} \)

and the equivalence classes of the equipped semi-stable holomorphic rank \( n \) vector bundles on \( \Sigma \)

where equipment means fixing \( n \) holomorphic sections linear independent except at \( ng \) points.

2.2. Lax operators on Riemann surfaces. Let \( \{\alpha\} = \{\alpha_1, \ldots, \alpha_K\} \), \( \{\gamma\} = \{\gamma_1, \ldots, \gamma_K\} \), \( \{\kappa\} = \{\kappa_i \in \mathbb{C} | i = 1, \ldots, K\} \). Below, we will avoid the indices using \( \alpha \) instead \( \alpha_i \) etc. Consider a set \( \{\beta\} = \{\beta_1, \ldots, \beta_K\} \) dual to \( \{\alpha\} \) with respect to the symplectic structure to appear below.

Consider a function \( L(P, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\kappa\}) \) \( (P \in \Sigma) \) obeying certain requirements which will follow immediately. In the local coordinate \( z \) on \( \Sigma \) we refer to this function as to \( L(z) \) omitting the indication to other arguments. We require that \( L \) is meromorphic as a function of \( P \), has arbitrary poles at \( P_i \)'s, simple or double poles at \( \gamma \)'s (depending on \( g \)), is holomorphic elsewhere, and at every \( \gamma \) is of the form

\[
L(z) = \frac{L_{-2}}{(z-z_\gamma)^2} + \frac{L_{-1}}{(z-z_\gamma)} + L_0 + L_1(z-z_\gamma) + O((z-z_\gamma)^2)
\]

where \( z \) is a local coordinate at \( \gamma, z_\gamma = z(\gamma) \) and the following relations hold:

\[
(2.1) \quad L_{-2} = \nu \alpha \alpha^t \sigma, \quad L_{-1} = (\alpha \beta^t + \varepsilon \beta \alpha^t) \sigma, \quad \beta^t \sigma \alpha = 0, \quad L_0 \alpha = \kappa \alpha
\]

where \( \alpha, \beta \in \mathbb{C}^n \) (\( \alpha \) is associated with \( \gamma \), \( \beta \) is arbitrary), \( \nu \in \mathbb{C}, \sigma \) is a \( n \times n \) matrix. \( L \) is called a Lax operator with a spectral parameter on the Riemann surface \( \Sigma \). The \( \nu, \varepsilon, \sigma \) in (2.1) depend on \( g \) as follows:

\[
(2.2) \quad \nu \equiv 0, \varepsilon = 0, \quad \sigma = id \quad \text{for } g = \mathfrak{gl}(n), \mathfrak{sl}(n),
\]

\[
\varepsilon = 1 \quad \text{for } g = \mathfrak{sp}(2n),
\]

and \( \sigma \) is a matrix of the symplectic form for \( g = \mathfrak{sp}(2n) \).
In addition we assume that
\[(2.3) \quad \alpha^t \alpha = 0 \quad \text{for} \quad g = \mathfrak{so}(n)\]
and
\[(2.4) \quad \alpha^t \sigma L_1 \alpha = 0 \quad \text{for} \quad g = \mathfrak{sp}(2n).\]

2.3. Lax operator algebras.

**Theorem 2.2** (Lie algebra structure, [13]). For fixed Tyurin data the space of Lax operators is closed with respect to the point-wise commutator \([L, L'](P) = [L(P), L'(P)]\) \((P \in \Sigma)\) (in the case \(g = \mathfrak{gl}(n)\) also with respect to the point-wise multiplication).

It is called Lax operator algebra and denoted by \(\mathfrak{g}\).

**Theorem 2.3** (almost graded structure, [13]). There exist such finite-dimensional subspaces \(g_m \subset \mathfrak{g}\) that
\[
\begin{align*}
&\text{(1)} \quad \mathfrak{g} = \bigoplus_{m=-\infty}^{\infty} g_m; \quad \text{(2)} \quad \dim g_m = \dim \mathfrak{g}; \quad \text{(3)} \quad [g_k, g_l] \subseteq \bigoplus_{m=k+l}^{k+l+g} g_m.
\end{align*}
\]

**Theorem 2.4.** If \(g\) is simple then \(\mathfrak{g}\) has only one almost graded central extension, up to equivalence [16]. It is given by a cocycle \(\gamma(L, L') = -\text{res}_{P_\infty} \text{tr}(LdL' - [L, L']\theta)\) where \(\theta\) is a certain 1-form [13].

2.4. M-operators. \(M = M(z, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\kappa\})\) is defined by the same constrains as \(L\), excluding \(\beta^t \sigma \alpha = 0\) and \(L_0 \alpha = \kappa \alpha\), namely
\[
M = \frac{M_{-2}}{(z - z_\gamma)^2} + \frac{M_{-1}}{z - z_\gamma} + M_0 + M_1(z - z_\gamma) + O((z - z_\gamma)^2)
\]
where
\[(2.5) \quad M_{-2} = \lambda \alpha^t \sigma, \quad M_{-1} = (\alpha \mu^t + \varepsilon \mu \alpha^t) \sigma
\]
\(M\) also takes values in \(\mathfrak{g}\), \(\lambda \in \mathbb{C}, \mu \in \mathbb{C}^n\).

2.5. Lax equations. For variative Tyurin data, the collection of equations on \(\{\alpha\}, \{\beta\}, \{\gamma\}, \{\kappa\}\) and main parts of \(L\) at \(\{P_i\}_{i=1, \ldots, N}\) equivalent to the relation
\[(2.6) \quad \dot{L} = [L, M]
\]
is called a Lax equation.
Motion equations of Tyurin data assigned to a point $\gamma$:

\[
\dot{z}_\gamma = -\mu^t \sigma \alpha, \quad \dot{\alpha} = -M_0 \alpha + k_a.
\]

Besides, there are motion equations of main parts of the function $L$ at $P_i$'s.

Let $D = \sum m_i P_i$ ($i = 1, \ldots, N, \infty$) be a divisor such that $\text{supp} D \cap \{\gamma\} = \emptyset$, $\mathcal{L}^D := \{L | (L) + D \geq 0 \text{ outside } \gamma \text{'s}\}$. Stress again that the elements of $\mathcal{L}^D$ have a two-fold interpretation: as Lax operators and as sets of Tyurin data and main parts of $L$-operators.

Under a certain (effective) condition [13, 16] the Lax equation defines a flow on $\mathcal{L}^D$.

2.6. Examples. 1) $g = 0$, $\alpha = 0$ (i.e. $\Sigma = \mathbb{C}P^1$, the bundle is trivial), $P_1 = 0$, $P_2 = \infty$. Then $\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ is a loop algebra, (2.6) is a conventional Lax equation with rational spectral parameter:

\[
L_t = [L, M], \quad L, M \in \mathfrak{g} \otimes \mathbb{C} [\lambda^{-1}, \lambda], \quad \lambda \in \mathcal{D}^1.
\]

The Lax equations of this type are considered by I.Gelfand, L.Dikii, I.Dorfman, A.Reyman, M.Semenov-Tian-Shanskii, V.Drinfeld, V.Sokolov, V.Kac, P. van Moerbeke. Many known integrable cases of motion and hydrodynamics of a solid body belong to this class.

2) Elliptic curves: Calogero-Moser systems [21].

3) Arbitrary genus: Hitchin systems

2.7. Hierarchy of commuting flows.

Theorem 2.5 ([7, 20, 21]). Given a generic $L$ and effective divisor $D = \sum m_i P_i$ ($i = 1, \ldots, N, \infty$), there is a family of $M$-operators $M_a = M_a(L)$ ($a = (P_i, n, m), n > 0, m > -m_i$) uniquely defined up to normalization, such that outside the $\gamma$-points $M_a$ has pole at the point $P_i$ only, and in the neighborhood of $P_i$

\[
M_a(w_i) = w_i^{-m} L^n (w_i) + O(1),
\]

The equations

\[
(2.7) \quad \partial_a L = [L, M_a], \quad \partial_a = \partial/\partial t_a
\]

define a family of commuting flows on an open set of $\mathcal{L}^D$. 

2.8. Krichever-Phong symplectic structure. We define an external 2-form on $L^D$. For $L \in L^D$ let $\Psi$ be a matrix-valued function formed by the eigenvectors of $L$: $\Psi L = K \Psi$ ($K$ — diagonal).

\[ \Omega := \text{tr}(\delta\Psi \wedge \delta L \cdot \Psi^{-1} - \delta K \wedge \delta \Psi \cdot \Psi^{-1}) = 2\delta \text{tr}(\delta\Psi \cdot \Psi^{-1}K) \]

where $\delta\Psi$ is the differential of $\Psi$ in $\alpha, \beta, \ldots$.

Let $dz$ be a holomorphic 1-form on $\Sigma$ and

\[ \omega := -\frac{1}{2} \left( \sum \text{res}_{\gamma} \Omega dz + \sum \text{res}_{P} \Omega dz \right). \]

Theorem 2.6 ([7, 21]). $\omega$ is a symplectic form on a certain closed invariant submanifold $P^D \subset L^D$.

2.9. Hamiltonians.

Theorem 2.7 ([7, 21]). The equations of the above commutative family are Hamiltonian with respect to the Krichever-Phong symplectic structure on $P^D$, with the Hamiltonians given by

\[ H_a = -\frac{1}{n+1} \text{res}_{P_i} \text{tr}(w^{-m}i L^{n+1})dw_i \]

Example. Let $g = \mathfrak{gl}(n), D$ be a divisor of a holomorphic 1-form. Then $L^D \simeq T^*(M_0)$ where $M_0$ is an open subset of the moduli space of holomorphic vector bundles on $\Sigma$, $H_a$ are Hitchin Hamiltonians.

3. Conformal field theory related to a Lax integrable system

By conformal field theory we mean a family of Riemann surfaces, a finite rank bundle (of conformal blocks) on this family, and a flat connection (Knizhnik-Zamolodchikov connection) on this bundle.

We consider the universal bundle of spectral curves over the phase space $P^D$ of the integrable system as a family of Riemann surfaces. For this family we introduce the Kodaira-Spencer mapping sending tangent vectors on the base ($P^D$ in our case) to the vector fields on the corresponding Riemann surfaces of the family. We use the Sugawara construction to obtain the analog of Knizhnik-Zamolodchikov connection on this family. The Knizhnik-Zamolodchikov operators give a projective representation of the Lie algebra of Hamiltonian vector fields. We prove that operators corresponding to the family of commuting Hamiltonians commute up to scalar operators.
In this section we realize the first part of the programme. The second part starting from construction of the Knizhnik-Zamolodchikov connection is delayed to the next section.

3.1. Spectral curves. For every $L \in \mathcal{P}^D$ (i.e. all arguments of $L$ are fixed except for $z$) let $\Sigma_L$ be a curve given by the equation $\det(L(z) - \lambda) = 0$. $\Sigma_L$ is called a spectral curve of $L$. It is a $n$-fold branch covering of $\Sigma$.

Thus a family of curves parameterized by points of $\mathcal{P}^D$ is obtained. We apply a conformal field theory technique (see [14, 15, 17, 18] and references there) to construct a bundle of conformal blocks with a projective flat connection on $\mathcal{P}^D$. We will represent a hamiltonian vector field on $\mathcal{P}^D$ by the operator of covariant derivative along that.

3.2. Krichever-Novikov vector fields. Consider an arbitrary Riemann surface with marked points and the Lie algebra of meromorphic vector fields on it holomorphic outside the marked points. It is called Krichever-Novikov vector field algebra and denoted by $\mathcal{V}$. Below we consider Krichever-Novikov vector field algebra on $\Sigma_L$ with the preimages of $P_1, \ldots, P_N$ as marked points. With every $P_i$ and an arbitrary $n \in \mathbb{N}$ we associate an element $e_{i,n}$ such that the collection of those form a base in $\mathcal{V}$ as of a vector space, and

$$[e_{n,p}, e_{m,r}] = \delta^r_p (m - n) e_{n+m,p} + \sum_{h=n+m+1}^{n+m+l} \sum_{s=1}^{N} \gamma^{(h,s)}_{(n,p),(m,r)} e_{h,s}$$

with some $\gamma^{(h,s)}_{(n,p),(m,r)} \in \mathbb{C}$, $l \in \mathbb{N}$ (we refer to [14, 15, 17, 18] for details). Hence the subspaces $\mathcal{V}_n = \bigoplus_{i=1}^{N} \mathbb{C} e_{i,n}$ give an almost graded structure on $\mathcal{V}$ in the same sense as in the Theorem 2.3.

3.3. Kodaira-Spencer cocycle. Denote the spectral curve over $L$ by $\Sigma_L$ and the Lie algebra of Krichever-Novikov vector fields on $\Sigma_L$ by $\mathcal{V}_L$. Our next goal is to define a map $\rho : T_L \mathcal{P}^D \to \mathcal{V}_L$. Fix a certain point, say $P_\infty \in \Sigma_L$. We may think of $P_\infty$ as of analitically depending on $L$. Choose a local family of transition functions $d_L$ giving the complex structure on $\Sigma_L$ and analitically depending on $L$. Let us take $X \in T_L \mathcal{P}^D$ and a curve $L_X(t)$ in $\mathcal{P}^D$ with the initial point $L$ and the
tangent vector $X$ at $L$. By definition
\begin{equation}
\rho(X) = d_L^{-1} \cdot \partial_X d_L.
\end{equation}

We consider $\rho(X)$ as a local vector field on the Riemann surface $\Sigma_L$. It can be explained in two ways. First, every transition function $d_L$ can be considered as a local analytic diffeomorphism with annulus domain of definition. Such diffeomorphisms form a group provided two diffeomorphisms coinciding in a domain are identified. Hence $\rho(X)$ is a tangent vector at the unit of the group, i.e. a local vector field, in a standard way. The relation (3.1) is an invariant definition of that vector field. Now let us give a definition using local coordinates. Let $\tau$ be a set of local coordinates on $L^D$. A family of local diffeomorphisms $d_\tau$ can be considered as an analytic function $d(z, \tau)$ where $z$ is a local coordinate in the neighborhood of $P_\infty$ on $\Sigma_L$. For $X = \sum_i X_i \partial_{\tau_i} \in T_\tau P^D$
\begin{equation}
(\partial_X d_\tau)(z) = \sum_i X_i \frac{\partial d(z, \tau)}{\partial \tau_i}
\end{equation}
is a local function. It gives a tangent vector to the group of local diffeomorphisms at the point $d_\tau$. Now set
\begin{equation}
\rho_z(X) = d_\tau^{-1}(\partial_X d_\tau(z)) \quad \text{and} \quad \rho(X) = \rho_z(X) \frac{\partial}{\partial z}
\end{equation}
where $\rho_z(X)$ is a local function, and $\rho(X)$ is a local vector field expressed in the coordinate $z$.

Summarizing the results of [18, Sect. 5.1] we obtain

**Proposition 3.1.** There exist $e \in \mathcal{V}$ such that locally (in the neighborhood of $P_\infty$) $\rho(X) = e$. The vector field $e$ is defined up to adding elements of $\mathcal{V}^{(1)} \oplus \mathcal{V}^{\text{reg}}$ where $\mathcal{V}^{(1)}$ is the direct sum of homogeneous subspaces of degree $\geq 0$ in $\mathcal{V}$, and $\mathcal{V}^{\text{reg}} \subset \mathcal{V}$ is the subspace of vector fields having zero at $P_\infty$. Both $\mathcal{V}^{(1)}$ and $\mathcal{V}^{\text{reg}}$ are Lie subalgebras.

Below, we always regard to $\rho(X)$ as to an element of $\mathcal{V}^{\text{reg}} \setminus \mathcal{V} / \mathcal{V}^{(1)}$.

A local vector field in the annulus centered at $P_\infty$ gives a class of Chech 1-cohomologies of the Riemann surface $\Sigma$ with coefficients in the tangent sheaf. The cohomology class represented by the vector field $\rho(X)$ is called the *Kodaira-Spencer class* of $X$. It is responsible for the deformation of moduli of the pointed surface along $X$. In this
form the Kodaira-Spencer class was used for example in [24, Lemma 1.3.8], and also in [18].

3.4. From Lax operator algebra to commutative Krichever-Novikov algebra. In this section, we canonically associate a commutative Krichever-Novikov algebra to a generic element $L \in \mathfrak{g}$. We need that for the Sugawara construction below. Indeed, the Sugawara construction [6, 10, 15] requires that the current algebra splitted to the tensor product of a functional algebra and a finite dimensional Lie algebra. Krichever-Novikov algebras are of this type, and Lax operator algebras are not.

Let $\Psi$ be the matrix formed by the canonically normalized left eigenvectors of $L$. In the case $\mathfrak{g} = \mathfrak{gl}(n)$ we consider a vector $\psi$ to be canonically normalized if $\sum \psi_i = 1$ [7, 21]. In the other cases we require that $\Psi \in G$ point-wise, i.e. $\Psi^t = -\varepsilon \sigma \Psi^{-1} \sigma^{-1}$ where $G = SO(n), Sp(2n)$ depending on $\mathfrak{g}$, and $\varepsilon$ satisfies to $\sigma^t = -\varepsilon \sigma$, i.e. $\varepsilon = -1$ if $\sigma$ is symmetric (the case $\mathfrak{g} = \mathfrak{so}(n)$), $\varepsilon = 1$ if $\sigma$ is skew-symmetric (the case $\mathfrak{g} = \mathfrak{sp}(2n)$). As usual, $\varepsilon = 0$ for $\mathfrak{g} = \mathfrak{gl}(n)$. $\Psi$ is defined modulo normalization and permutations of its rows (such normalization descends to the left multiplication $\Psi$ by a diagonal matrix). We also consider the diagonal matrix $K$ defined by

$$\Psi L = K \Psi,$$

i.e. formed by the eigenvalues of $L$. Similar to [7, 21]

$$\Psi(z) = \varepsilon \tilde{\beta} \alpha^t \sigma + \Psi_0 + \Psi_1(z - z_\gamma) + \ldots,$$

$$(3.5)$$

$$\Psi^{-1}(z) = \alpha \tilde{\beta}^t \sigma + \tilde{\Psi}_0 + \tilde{\Psi}_1(z - z_\gamma) + \ldots$$

in the neighborhood of a $\gamma$. The residue of $\Psi$ which is absent in [7, 21] appears here for $\mathfrak{g} = \mathfrak{so}(n)$ and $\mathfrak{g} = \mathfrak{sp}(2n)$ due to the requirement $\Psi^t = -\sigma \Psi^{-1} \sigma^{-1}$. As it is shown in [21] the following relations hold which are essentially equivalent to holomorphy of $\Psi L$ and the relation $\Psi \Psi^{-1} = id$:

$$\Psi_0 \alpha = 0, \varepsilon \alpha^t \sigma \tilde{\Psi}_0 = 0.$$
Observe that if \( \varepsilon = 0 \) (i.e. \( g = gl(n) \)) then \( \nu = \lambda = 0 \). Hence \( \nu \alpha^{t} \sigma \Psi_{0} = \lambda \alpha^{t} \sigma \Psi_{0} = 0 \) as well. For the same reason \( \varepsilon \alpha^{t} \sigma \alpha = \nu \alpha^{t} \sigma \alpha = \lambda \alpha^{t} \sigma \alpha = 0 \). The following Lemma is important only as a motivation for the next Lemma 3.3, no proof below relies on it.

**Lemma 3.2 ([21], Lemma 7.2).** The matrix-valued function \( K \) is holomorphic at all \( \gamma \)-points provided (3.5), (3.6) hold there.

Hence \( K \) is a meromorphic diagonal matrix-valued function on \( \Sigma \) holomorphic outside \( P_{i} \)'s. If we denote the algebra of scalar-valued functions possessing this property by \( A \) then \( K \in \mathfrak{h} \otimes A \) where \( \mathfrak{h} \subset g \) is a diagonal (Cartan) subalgebra. \( A \) is called Krichever-Novikov function algebra on \( \Sigma \), and \( \mathfrak{h} \otimes A \) the corresponding Krichever-Novikov current algebra. We obtain only commutative current algebras here. Let be \( \overline{\mathfrak{h}} = \mathfrak{h} \otimes A \).

In what follows we need a slightly different set-up. We consider the algebra \( A_{L} \) similar to \( A \) but defined on \( \Sigma_{L} \), and having pre-images of the points \( P_{i} \) as the collection of poles. Let us take an arbitrary element of \( A_{L} \) and push it down to \( \Sigma \) as a diagonal matrix \( h \). Every sheet is assigned with a certain row of \( h \). At the branching points we may obtain a coincidence of eigenvalues of \( h \). The order of diagonal elements of the matrix \( h \) will depend on the order of the sheets. This ambiguity is the same as for \( \Psi \). The permutation of rows corresponding to an element \( w \) of the Weil group descends to \( \Psi \rightarrow w \Psi \) (which is easy verified for \( w \) to be a transposition), and \( h \) transforms as \( h \rightarrow whw^{-1} \). Thus \( L = \Psi^{-1} h \Psi \rightarrow L \). Thus the mapping \( A_{L} \rightarrow \overline{\mathfrak{h}} \) is just the direct image of functions from \( A_{L} \).

### 3.5. The representation of \( \overline{\mathfrak{h}} \).

We need a representation of \( \overline{\mathfrak{h}} \) in order to construct the bundle of conformal blocks and Knizhnik-Zamolodchikov connection in the next section. Recall that \( \overline{g} \) stays for the Lax operator algebra in question.

**Lemma 3.3.** For any \( h \in \overline{\mathfrak{h}} \) we have \( \Psi^{-1} h \Psi \in \overline{g} \).

**Proof.** Let us take \( h \) in the form

\[
h = h_{0} + h_{1} z + \ldots
\]
in a neighborhood of a $\gamma$ where we set $z_\gamma = 0$ for simplicity. By (3.5) we obtain

$$L = \Psi^{-1} h \Psi = \frac{\varepsilon \alpha \tilde{\beta}^t \sigma h_0 \tilde{\beta}^t \sigma}{z^2} + \frac{\alpha \tilde{\beta}^t \sigma h_0 \Psi_0 + \varepsilon \tilde{\Psi}_0 h_0 \tilde{\beta}^t \sigma}{z}$$

(3.7)

$$+ (\alpha \tilde{\beta}^t \sigma h_0 \Psi_1 + \alpha \tilde{\beta}^t \sigma h_1 \Psi_0 + \varepsilon \alpha \tilde{\beta}^t \sigma h_2 \tilde{\beta}^t \sigma + \tilde{\Psi}_0 h_0 \Psi_0$$

$$+ \varepsilon \tilde{\Psi}_0 h_1 \tilde{\beta}^t \sigma + \varepsilon \tilde{\Psi}_1 h_0 \tilde{\beta}^t \sigma)$$

$$+ \ldots .$$

If $g = \mathfrak{so}(n)$ then $\sigma^t = \sigma$. We have $(\sigma h_0)^t = h_0^t \sigma^t = -\sigma h_0 \sigma^{-1} \cdot \sigma = -\sigma h_0$, i.e. $\sigma h_0$ is skew-symmetric. Hence $\tilde{\beta}^t (\sigma h_0) \tilde{\beta} = 0$ and the term with $z^{-2}$ in (3.7) vanishes. Observe that for $g = \mathfrak{sp}(2n)$ it does not vanish because $\sigma h_0$ is symmetric.

Let us consider the term with $z^{-1}$. We must represent the nominator in the form $(\alpha \tilde{\beta}^t + \varepsilon \tilde{\beta}^t \alpha^t) \sigma$, i.e. prove that $\alpha \tilde{\beta}^t \sigma h_0 \Psi_0 \sigma^{-1} = (\tilde{\Psi}_0 h_0 \tilde{\beta}^t \sigma)^t$. By $h_0^t = -\sigma h_0 \sigma^{-1}$ it would be sufficient that $\tilde{\Psi}_0^t = -\sigma \tilde{\Psi}_0 \sigma^{-1}$. The latter follows from the requirement $(\Psi^{-1})^t = -\sigma \Psi^t \sigma^{-1}$, i.e. $\Psi \in G$ where $G$ is a classical group corresponding to $g$, and which is included into the definition of $\Psi$ above.

Next, let us check the eigenvalue condition on $L_0$. To this end, let us operate on $\alpha$ by the six-term expression in bracket in (3.7). Three summands containing the combination $\varepsilon \alpha^t \sigma$ will vanish by $\varepsilon \alpha^t \sigma \alpha = 0$. Two more summands having $\Psi_0$ on the right hand side will vanish by $\Psi_0 \alpha = 0$. The remaining summand will give $\alpha (\tilde{\beta}^t \sigma h_0 \Psi_1 \alpha)$. Observe that the expression in brackets is a scalar.

To complete the proof of the theorem we must check that $\Psi^{-1} h \Psi$ is holomorphic except at $P_i$’s and $\gamma$’s. $\Psi$ possesses certain singular points due to the normalization ($\sum \psi_i = 1$ where $\psi$ is a row of $\Psi$). Let us prove that $\Psi^{-1} h \Psi$ is holomorphic at those points for any $h \in \mathfrak{h}$. By (3.4) $\Psi$ is defined up to left multiplication by a diagonal matrix $d$ (corresponding to normalization of the rows of $\Psi$). But since $h$ and $d$ are both diagonal, $(d \Psi)^{-1} h d \Psi = \Psi^{-1} h \Psi$, hence $\Psi^{-1} h \Psi$ does not depend on the normalization and can be defined without any normalization. But then $\Psi$ has no singularities which are due to normalization. □

By Lemma 3.3 any representation of $\overline{\mathfrak{g}}$ induces the corresponding representation of $\overline{\mathfrak{h}}$. Since $\Psi$ is meromorphic at $P_i$’s, the just constructed mapping $\overline{\mathfrak{h}} \to \overline{\mathfrak{g}}$ preserves degree. Hence an almost graded
\( \mathfrak{g} \)-module induces the almost-graded \( \mathfrak{h} \)-module. Let us remind that \( \Psi \) is defined up to left multiplication by a diagonal matrix. For this reason \( \mathfrak{h} \) needs to be diagonal, otherwise the mapping \( \mathfrak{h} \rightarrow \mathfrak{g} \) is not well-defined.

Consider the following natural representation of \( \mathfrak{g} \). Let \( \mathcal{F} \) be the space of meromorphic vector-valued functions \( \psi \) holomorphic except at \( P_1, \ldots, P_N \) and \( \gamma \)'s, such that

\[
\psi(z) = \nu \frac{\alpha}{z} + \psi_0 + \ldots
\]

at any point \( \gamma \). \( \mathcal{F} \) is an almost graded \( \mathfrak{g} \)-module with respect to the Krichever-Novikov base introduced in [22]. Consider the semi-infinite degree of this module which is also constructed in [22]. Denote it by \( \mathcal{F}^{\infty/2} \). The induced \( \mathfrak{h} \)-module is what we will consider below. This is an admissible module in the sense that every its element annihilates having been multiply operated by an element of \( \mathfrak{h} \) of a positive degree. Moreover, it is generated by a vacuum vector. By the above constructed mapping \( A_L \rightarrow \mathfrak{h} \) we also consider \( \mathcal{F}^{\infty/2} \) as an \( A_L \)-module.

3.6. **Sugawara representation.** In this paper we need a ”commutative” version of Sugawara construction developed in [11] to be applied to \( \mathfrak{h} \). See [15, 18, 17] for more details.

Consider an admissible \( \mathfrak{h} \)-module \( V \). We are mainly interested in the case \( V = \mathcal{F}^{\infty/2} \) here.

For any admissible representation of an affine Krichever-Novikov algebra (say \( \mathfrak{h} \)) there exist a projective representation \( T \) of the Krichever-Novikov vector field algebra canonically defined by the relation

\[
[T(e), h(A)] = -c \cdot h(eA)
\]

where \( h \in \mathfrak{h} \), \( A \in A_L \), \( e \in \mathcal{V}_L \), \( h(A) \) denotes the representation operator of \( h \otimes A \in \mathfrak{h} \), \( eA \) denotes the natural action of a vector field on a function, \( c \) is a level of the \( \mathfrak{h} \)-module (in the non-commutative case we would have \( c + \kappa \) instead \( c \) where \( \kappa \) is the dual Coxeter number).

The representation \( T \) has the following effective definition. Let \( \{A_j\}, \{\omega^k\}, \{e_m\} \) be the Krichever-Novikov bases in \( A_L \), the space of Krichever-Novikov 1-forms and Krichever-Novikov vector fields on \( \Sigma_L \) (the first two are dual), \( \{h_i\} \) and \( \{h^i\} \) is a pair of dual bases in \( \mathfrak{h} \) and
Then $T(e_m) = \sum_{i,j,k} c_{jk}^m h_i(A_j) h^j(A_k)$: where $c_{jk}^m = \text{res}_{\infty} \omega_j \omega^k e_m$,
:: is a normal ordering.

Remark. There would be two problems if we wanted to repeat the Sugawara construction for Lax operator algebras. The first is related to generalization of the relation $[T(e), u(L)] = u(e.L)$. The right hand side must contain a connection $\nabla = d + \omega$ and have the form $[T(e), u(L)] = u(\nabla e L)$ but it is not evident how $\nabla$ enters the left hand side. Another problem is related to the definition of $c_{jk}^m$, namely how make the corresponding 1-form $\omega^j \omega^k e_m$ to be regular at the $\gamma$-points. Another face of the problem is that we need a current algebra to be splitted to tensor product of a finite-dimensional and a function algebra to carry out the Sugawara construction.

By introducing $\mathfrak{h}$ and $A_L$ we manage with the known version of Sugawara construction for Krichever-Novikov algebras, and even for commutative Krichever-Novikov algebras.

The role of $L$ is to define an admissible representation of $\mathfrak{h}$, $A_L$. Observe that the Lax operator $L$ is fixed as a function on $L^D$ and of the spectral parameter, hence $\Psi$ is uniquely defined up to the above discussed equivalence, and the representation is well-defined.

4. Representation of the algebra of Hamiltonian vector fields

In this section we construct the Knizhnik-Zamolodchikov connection on the family of spectral curves. The Knizhnik-Zamolodchikov operators give a projective representation of the Lie algebra of Hamiltonian vector fields. We prove that operators corresponding to the family of commuting Hamiltonians commute up to scalar operators. We prove unitarity of that representation.

4.1. Conformal blocks and Knizhnik-Zamolodchikov connection. Let us consider the sheaf of $A_L$-modules $\mathcal{F}^{\infty/2}$ on $\mathcal{P}^D$. Let $\mathfrak{h}^{reg} \subset \mathfrak{h}$ be a subalgebra consisting of the functions regular at $P_{\infty}$. The sheaf of quotients $\mathcal{F}^{\infty/2}/\mathfrak{h}^{reg}$ on $\mathcal{P}^D$ is called the sheaf of covariants (over a different base it was defined in [15] in this way).

Let $X$ be a vector field on $\mathcal{P}^D$. By definition

$$\nabla_X = \partial_X + T(\rho(X))$$
where $\rho$ is the Kodaira-Spencer mapping, $T$ is the Sugawara representation in $\mathcal{F}^{\infty/2}/\mathfrak{h}^{\text{reg}}$.

**Theorem 4.1** ([15, 18]). $\nabla$ is a projective flat connection on the sheaf of coinvariants:

$$[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + \lambda(X,Y) \cdot \text{id}$$

where $\lambda$ is a certain cocycle, $\text{id}$ is the identity operator.

In [15], Theorem 4.1 has been formulated and proven in the Conformal Field Theory setup, i.e. for a certain moduli space of Riemann surfaces with marked points and fixed jets of local coordinates at those points. We assert that the situation here is quite similar and the proof is the same. In analogy with CFT we refer to the projective flat connection defined by Theorem 4.1 as Knizhnik-Zamolodchikov connection.

The horizontal sections of the sheaf of covariants are called conformal blocks.

### 4.2. Representation of Hamiltonian vector fields and commuting Hamiltonians

By Theorem 4.1 $X \rightarrow \nabla_X$ is a projective representation of the Lie algebra of vector fields on $P^D$ in the space of sections of the sheaf of covariants. Denote this representation by $\nabla$. The restriction of $\nabla$ to the subalgebra of Hamiltonian vector fields gives the projective representation of that.

**Theorem 4.2.** If $X$, $Y$ are Hamiltonian vector fields such that their Hamiltonians Poisson commute then $[\nabla_X, \nabla_Y] = \lambda(X,Y) \cdot \text{id}$. If the Hamiltonians depend only on action variables, then $[\nabla_X, \nabla_Y] = 0$.

**Proof.** The projective commutativity immediately follows from Theorem 4.1 since $[X, Y] = 0$.

For a Lax equation the spectral curve is an integral of motion. This means that the complex structure, hence the transition functions, are invariant along the phase trajectories. Hence $\rho(X) = 0$ and $\nabla_X = \partial_X$. This implies the commutativity of Hamiltonians depending only on the action variables. \hfill $\square$

### 4.3. Unitarity

The goal of the section is to specify a subspace of the representation space and introduce a scalar product there such that the above representation $\nabla$ becomes unitary.
Recall that in general only real vector field Lie algebras admit unitary representations in the classical sense, i.e. such that the representation operators are skew-Hermitian. For complexifications, a convenient way to define unitarity is as follows [6]. Let $G$ be a Lie algebra and $T$ its representation in the space $V$. Consider an antilinear antinvolution $\dagger$ on $G$ ($X \rightarrow X^{\dagger}$, $X \in G$). An Hermitian scalar product in $V$ is called contravariant if $T(X)^{\dagger} = T(X^{\dagger})$ where the $\dagger$ on the left hand side means the Hermitian conjugation. A pair consisting of $T$ and a contravariant scalar product is called a unitary representation of $G$. The restriction of $T$ to the Lie subalgebra of the elements such that $X^{\dagger} = -X$ is unitary in the classical sense.

To construct a contravariant Hermitian scalar product in the space of the representation $\nabla$ we first introduce a point-wise scalar product in the sheaf of covariants, and then integrate it over the phase space $\mathcal{P}^D$ by an invariant volume form.

Over every point in $\mathcal{P}^D$ we introduce an Hermitian scalar product in $\mathcal{F}^{\infty}/2$ declaring semi-infinite monomials with basis entries to be orthonormal ([6, p.39], [11]). Below (Theorem 4.3) we prove that it gives a well-defined point-wise scalar product on coinvariants.

By Poincaré theorem the symplectic form and its degrees are absolute integral invariants of Hamiltonian phase flows. Hence $\omega^p/p!$ defines an invariant measure with respect to Hamiltonian phase flows. Let $L^2(C, \omega^p/p!)$ be the space of quadratically integrable sections of the sheaf $C$ with respect to that measure where the square over a point is given by the above point-wise scalar product.

**Theorem 4.3.** The representation $\nabla : X \rightarrow \nabla_X$ of the Lie algebra of Hamiltonian vector fields on $\mathcal{P}^D$ in the subspace of smooth sections in $L^2(C, \omega^p/p!)$ is unitary.

**Proof.** First let us prove that the point-wise scalar product on $\mathcal{F}^{\infty}/2$ is well-defined on coinvariants. The different quasihomogeneous almost-graded components of the module $\mathcal{F}^{\infty}/2$ are orthogonal. The point-wise coinvariants are defined as $\mathcal{V}/g^{\text{reg}}\mathcal{V}$. The degrees of monomials occurring in the subspace $g^{\text{reg}}\mathcal{V}$ (including summands in linear combinations) are obviously less than those of monomials which form coinvariants. Hence these spaces of monomials are orthogonal and the induced scalar product on the quotient does not depend on the choice
of representatives in the equivalence classes, and we get a well-defined point-wise scalar product on coinvariants.

Next let us construct a certain antilinear antiinvolution on the tangent vector fields on $\mathcal{P}^D$. We will push it down from $\mathcal{V}$ by the inverse to the Kodaira-Spencer mapping making use of the double coset description of the tangent space to moduli space of curves ($T_L\mathcal{M} = \mathcal{V}_-^{(1)} \setminus \mathcal{V} / \mathcal{V}_+^{(1)}$). The Lie subalgebras $\mathcal{V}_±^{(1)}$ will be defined in the end of the paragraph. Recall from [6, p.39] that the convenient antiinvolution in $\mathcal{V}$ is induced by its embedding into the Lie algebra $\mathfrak{a}_\infty$ of infinite matrices with finite number of diagonals. Following [6] we denote this embedding by $r$. The antilinear antiinvolution in $\mathfrak{a}_\infty$ amounts in $r(e_i) \rightarrow -r(e_i)^\dagger$ for the elements of the Krichever-Novikov base, with the consequent antilinear continuation to the complexification. We go on denoting this antiinvolution by $\dagger$. For example in the case of two marked points $\{P_1, P_\infty\} = \{P_+, P_-\}$ the $\dagger$ sends the subspace $\mathcal{V}_+^{(p)}$ of vector fields regular at $P_+$ and vanishing there with the order at least $p$ to the similar subspace $\mathcal{V}_-^{(p)}$ at $P_-$ (for an arbitrary $p \in \mathbb{Z}$). Hence $\mathcal{V}_+^{(p)} \oplus \mathcal{V}_-^{(p)}$ is invariant under the antiinvolution. By Proposition 3.1 $\mathcal{V}_+^{(1)} \oplus \mathcal{V}_-^{(1)} = \ker \rho^{-1}$ in this case. Hence $\dagger$ is well defined on the tangent space to $\mathcal{P}^D$ at the corresponding point.

For a local vector field $X$ on $\mathcal{P}^D$ by $\partial_X$ we mean the corresponding derivative of local (for example finitary) smooth sections of the sheaf $C$. For a Hamiltonian $X$ the $\partial_X$ is skew-Hermitian by $X$-invariance of the measure, hence $\partial_X^\dagger = -\partial_X$. Further on, for a real Hamiltonian vector field $X$ (i.e. $X^\dagger = -X$) we have $-\partial_X = \partial_X^\dagger$, hence $\partial_X^\dagger = \partial_X^\dagger$. By complex antilinearity we obtain the same relation for all Hamiltonian vector fields.

Let $\langle \cdot | \cdot \rangle$ be the above introduced point-wise scalar product, and $\langle \cdot | \cdot \rangle = \frac{1}{p!} \int_{\mathcal{P}^D} \langle \cdot | \cdot \rangle \omega^p$. By [15] $\langle \cdot | \cdot \rangle$ is a contravariant form with respect to the Sugawara representation (in the abelian case). This implies that the $\langle \cdot | \cdot \rangle$ possesses the same property, hence $T(\rho(X))^\dagger = T(\rho(X)^\dagger)$. By definition $\rho(X)^\dagger = \rho(X^\dagger)$. Hence $T(\rho(X))^\dagger = T(\rho(X^\dagger))$.

By the last two paragraphs we have for a Hamiltonian vector field $X$

$$\partial_X^\dagger = \partial_X^\dagger \text{ and } T(\rho(X))^\dagger = T(\rho(X^\dagger)).$$
Hence
\[(\partial X + T(\rho(X))) = \partial X + T(\rho(X^\dagger))\],
i.e.
\[(\nabla X)^\dagger = \nabla X^\dagger.\]

References

[1] Feigin, B., Frenkel, E. Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras. In: Infinite analysis, A.Tsuchiya, T.Eguchi, M.Jimbo eds., Adv. Ser. in Math. Phys., vol. 16, 197-215. World Scientific, Singapore, 1992.
[2] Beilinson, A., Drinfeld, V. Quantization of Hitchin’s integrable system and Hecke eigensheaves. Preprint.
[3] Felder, G., Wieczerkowski, Ch. Conformal blocks on elliptic curves and Knizhnik-Zamolodchikov-Bernard equations. Comm.Math.Phys. 176, 133-161 (1996).
[4] Hitchin, N.J. Flat connections and geometric quantization. Comm. Math. Phys. 131 (1990), 347-380.
[5] Ivanov, D. Knizhnik-Zamolodchikov-Bernard equations as a quantization of nostationary Hitchin systems. [hep-th/9610207]
[6] Kac V.G., Raina A.K. Highest Weight Representations of Infinite Dimensional Lie Algebras. Adv. Ser. in Math. Physics Vol.2, World Scientific, 1987.
[7] Krichever, I.M. Vector bundles and Lax equations on algebraic curves. Comm. Math. Phys. 229, 229–269 (2002).
[8] Krichever I.M., Novikov S.P. Holomorphic bundles on algebraic curves and nonlinear equations. Uspekhi Math. Nauk (Russ. Math.Surv), 35 (1980), 6, 47–68.
[9] Krichever I.M., Novikov S.P. Holomorphic bundles on Riemann surfaces and Kadomtsev-Petviashvili equation. I. Funct. Anal. and Appl., 12 (1978), 4, 41–52.
[10] Krichever, I.M. Novikov, S.P. Algebras of Virasoro type, Riemann surfaces and structures of the theory of solitons. Funct. Anal. Appl. 21 (1987), 2, 126-141.
[11] Krichever, I.M. Novikov, S.P. Algebras of Virasoro type, Riemann surfaces and strings in Minkowski space. Funct. Anal. Appl. 21 (1987), 4, 294-307.
[12] Krichever, I.M., Phong, D.H. Symplectic forms in the theory of solitons. Surveys in Diffferential Geometry IV (1998),239-313, International press, C.L.Terng and K.Ulenabeck eds.
[13] Krichever, I.M., Sheinman, O.K. Lax operator algebras. Funct. Anal. i Prilozhen., 41 (2007), no. 4, p. 46-59, [math.RT/0701648]
[14] M.Schlichenmaier, O.Sheinman. Wess-Zumino-Witten-Novikov Theory, Knizhnik-Zamolodchikov equations and Krichever-Novikov algebras. Russ.Math.Surv., 1999, v. 54, no. 1, p. 213–249.
[15] M.Schlichenmaier, O.Sheinman. Knizhnik-Zamolodchikov equations for positive genus and Krichever-Novikov algebras. Russ.Math.Surv., 2004, v. 59, no. 4, p. 737–770.
[16] Schlichenmaier, M., Sheinman, O.K. Central extensions of Lax operator algebras. Russ.Math.Surv., 63, no.4, p. 131-172, arXiv:0711.4688
[17] Sheinman, O.K. Krichever-Novikov algebras, their representations and applications. In: Geometry, Topology and Mathematical Physics. S.P.Novikov’s Seminar 2002-2003, V.M.Buchstaber, I.M.Krichever, eds., AMS Translations, Ser.2, v. 212 (2004), 297–316, [math.RT/0304020]
[18] Sheinman, O.K. Krichever-Novikov algebras, their representations and applications in geometry and mathematical physics. In: Contemporary mathematical problems, Steklov Mathematical Institute publications, v. 10 (2007), 142 p. (in Russian)
[19] Sheinman, O.K. *On certain current algebras related to finite-zone integration*. In: Geometry, Topology and Mathematical Physics. S.P.Novikov’s Seminar 2004-2008, V.M.Buchstaber, I.M.Krichever, eds., AMS Translations, Ser.2, v.224 (2008).

[20] Sheinman, O.K. *Lax operator algebras and integrable hierarchies*. In: Proc. of the Steklov Mathematical Institute, 2008, v.263.

[21] Sheinman, O.K. *Lax operator algebras and Hamiltonian integrable hierarchies*. Arxiv.math.0910.4173. To appear in Russ. Math.Surv., 2011, no.1.

[22] Sheinman, O.K. *The fermion model of representations of affine Krichever-Novikov algebras*. Func. Anal. Appl. 35 (2001), 3, p. 209-219.

[23] Tyurin, A.N. *Classification of vector bundles on an algebraic curve of an arbitrary genus*. Soviet Izvestia, ser. Math., 29, 657–688.

[24] Ueno K. *Introduction to conformal field theory with gauge symmetries*. Geometry and Physics, Proceed. Aarhus conference 1995 (Andersen J.E. et. al., ed.), Marcel Dekker, 1997, pp. 603–745.