The importance of social processes has been the reason of broad interest in their modeling with methods of statistical physics. One of the most interesting problems in that field is the evolution of social networks towards the state known as the Heider (or structural) balance [1–6]. Dynamics of such networks has been studied on various underlying lattices, from chains of actors [7] to complete graphs [8], specifically to find whether vanishing of the ordered and balanced phases is possible and under what conditions. Reviews of the discussed problems and used methods are given in Refs. 9 and 10. Recent examples of the application of the balance theory include usage of the competitive balance model with two different interests [11], the coevolutionary balance model in which actors change both their opinions and relationships [12], the study of social fragmentation and its influence on the dynamics of opinion formation [13], or analysis of indirect reciprocity and its impact on the friendship and enmity relations in the network [14]. In all those cases, the model applied to perform either calculations or numerical simulations is based on Heider’s concept of triads of actors, and friendly or hostile relations between those actors.

The system contains $N$ labeled ($1 \leq i \leq N$) actors (nodes) and $L$ relations (links) among these actors. The positive (negative) link values $x_{ij}$ indicate friendly (hostile) relations among the actors $i$ and $j$. The system dynamics is governed by the changes of links values in actor triangles. There are four available types of triangles in this system, as presented in Figure 1.

Among them, two (those presented on Figures 1a and 1c) are termed as balanced (in Heider’s sense) as the signs of relations obey the following rules:

- a friend of my friend is my friend,
- a friend of my enemy is my enemy,
- an enemy of my friend is my enemy,
- an enemy of my enemy is my friend.

In the imbalanced triangles (Figures 1b and 1d) these rules are violated what leads to the appearance of stress known as cognitive dissonance. Resolving this stress may be achieved by changing some of the links values in the imbalanced triangle, which is the source of the system temporal evolution towards the Heider balance, where the triangles 1b and 1d are absent.

Most of the earlier efforts in Heider balance research were devoted to the study of the structural balance on complete graphs [8] as the assumed topology ensured both:

- presence of triads necessary for introducing cognitive dissonance among actors
and simultaneously allows for some analytical considerations including mean-field calculations [15, 16].

Recently, the systems were enriched with thermal noise modeled either with Glauber dynamics [17–19] or heat-bath [15, 16]. The results of Refs. 15–18 indicate that a phase transition occurs in the system: below the critical noise level \( T < T_C \) the systems orders into the balanced (in Heider’s sense) state, while for the high enough noise level \( T > T_C \) the system is imbalanced. The balance/imbalance phase transition was identified as the first-order phase transition. The critical temperature \( T_C \) increases linearly with the system size \( N \).

On the other hand, the simplest topology (where triangles appear in a natural way) is the triangular lattice. The structural balance on such networks was studied with (deterministic) cellular automata [20] and with (stochastic) heat-bath algorithm [21]. The latter study showed that independently of the assumed noise level, \( T < T_C \) indicated a phase transition occurs in the system: below the critical temperature in such system tends to zero \( (T_C \to 0) \).

Finally, the Heider balance may be observed also in a chain of actors where triangles may be introduced by adding long-range interactions [7]. There, depending on the interaction range, the signatures of absence (for the nearest neighbor interactions assumed), the second-order (for the intermediate range of interaction), and the first-order phase transition (for a very long range of interactions) were observed. The critical temperature \( T_C \) increases non-linearly with the interaction range and the initial fraction of friendly relations.

Intrigued by the above-mentioned (topologically induced) variety of system complex behaviors, here we systematically check how the network topology influences the Heider balance in the presence of (thermal) noise. To that end, we enrich (towards a complete graph) or dilute (towards separated triangles) the triangular lattice. Additionally, for the comprehensiveness of the study, we simulate the evolution towards the Heider balance also on the classical (Erdős–Rényi [22, 23]) random graphs.

We note that more complex system behavior may be observed after further enrichment of the system with additional interactions based not only on link values but also on attributes assigned to the network nodes [12, 13, 24].

\section{Model}

The system Hamiltonian for a fully connected graph [8, 15],

\[
\mathcal{H} = - \sum_{i<j} \sum_{k>j} x_{ij} \cdot x_{jk} \cdot x_{ki},
\]

may be easily adopted for any network with \( N \) nodes and \( L \) links

\[
\mathcal{H} = - \sum_{i,j>i,k>j} a_{ij} x_{ij} \cdot x_{jk} \cdot x_{ki},
\]

where (binary and symmetric) adjacency matrix \( A = [a_{ij}] \) elements

\[
a_{ij} = \begin{cases} 
1 & \iff i \text{ and } j \text{ are connected}, \\
0 & \iff \text{otherwise},
\end{cases}
\]

define topology of the network. For a complete graph, \( a_{ij} = 0 \) and \( a_{ij} \neq 1 \).

A triangular lattice may be constructed based on a square grid with \( N = W^2 \) sites, where \( W \) is the linear size of the lattice. Then the neighbors of site \((n, m)\), \( 1 \leq n, m \leq W \), are located at \((n - 1, m), (n + 1, m), (n, m - 1), (n, m + 1), (n + 1, m - 1), (n - 1, m + 1)\). We assume periodic boundary conditions. The adjacency matrix \( A \) for the triangular lattice with periodic boundary conditions is schematically sketched in Figure 2.

To construct the diluted triangular lattice, a fraction of the unit elements \((a_{ij} = a_{ji} = 1)\) must be replaced with zeros \((a_{i,j} = a_{j,i} = 0)\). We implement this dilution by removing exactly \( q_- \) randomly chosen links. Similarly, adding extra links between the nodes of the triangular lattice requires substituting \( 0 \to 1 \) in the adjacency matrix and keeping it symmetric. It means that adding \( q_+ \) links is performed in the opposite way, by replacing \( q_+ \) randomly chosen pairs of zero elements \((a_{ij} = a_{ji} = 0)\) by unit values \((a_{ij} = a_{ji} = 1)\).

In the case of the other considered topology, i.e., the classical random graph, the procedure comes down to adding the required number of links between randomly chosen nodes \( i \) and \( j \), where \( i \neq j \). For that purpose, we simply start with a matrix filled with zeros, and then continue to substitute \( a_{ij} = 1 \) at random positions \((i, j)\) up to the limit of the needed graph density. Since for all considered topologies the adjacency matrix must be symmetric, therefore also for the classical random graph we each time substitute two matrix elements with the unit value \((a_{ij} = a_{ji} = 1)\).

Having the adjacency matrix \( A \), the number of links is

\[
L = \sum_{i,j,i<j} a_{ij}
\]

and the number of triads may be calculated as

\[
\Delta = \sum_{i,j,k} a_{ij} a_{jk} a_{ki}.
\]

In the limiting cases, these values are known analytically, as for a complete graph \( L_{\text{CC}} = \binom{N}{2} \) and \( \Delta_{\text{CC}} = \binom{N}{3} \) while for a triangular lattice with periodic boundary conditions \( L_{\text{TR}} = 3N = 3W^2 \) and \( \Delta_{\text{TR}} = 2N = 2W^2 \).

For the diluted systems, removing \( q_- \) links yields the bond occupation probability \( p_- = 1 - q_- / L_{\text{TR}} \) which
It can be related to the bond occupation probability if we note that the number of links removed from the triangular lattice to create its diluted version is \( q_+ = L_{\text{TR}} - L \), or \( q_+ = L - L_{\text{TR}} \) when we enhance the triangular lattice with \( q_+ \) more links, where \( L \) is the number of links in the resulting diluted or enriched lattice. For both of them, it leads to \( p_{\pm} = L / L_{\text{TR}} = D / D_{\text{TR}} \). Because \( D_{\text{TR}} = L_{\text{TR}} / L_c = 6 / (N - 1) \), it follows that the graph density depends on the bond occupation probability as \( D = 6 p_{\pm} / (N - 1) = \bar{k} / (W^2 - 1) \). For example, the density of the triangular lattice with linear size \( W = 10 \) is \( D_{\text{TR}} \approx 0.061 \), while for \( W = 20 \) it is \( D_{\text{TR}} \approx 0.015 \).

The Heider balance may be easily identified by checking the system work function \([8, 25]\)

\[
U = -\frac{1}{\Delta} \sum_i \sum_{j>i} \sum_{k>j} a_{ij} x_{ij} \cdot a_{jk} x_{jk} \cdot a_{ki} x_{ki},
\]

(7)

which is equal to \(-1\) if and only if all the triangles in the system are balanced.

We assume that the system evolution is governed by the heat-bath algorithm. If \( a_{ij} \neq 0 \) then evolution of the link between nodes \( i \) and \( j \) of value \( x_{ij} \) is given by

\[
x_{ij}(t + 1) = \begin{cases} 
1 & \text{with probability } p_{ij}(t), \\
-1 & \text{with probability } [1 - p_{ij}(t)],
\end{cases}
\]

(8a)

where

\[
p_{ij}(t) = \frac{\exp[\xi_{ij}(t)/T]}{\exp[\xi_{ij}(t)/T] + \exp[-\xi_{ij}(t)/T]}.
\]

(8b)

\( T \) is the temperature (noise level) at which the evolution occurs, and

\[
\xi_{ij}(t) = \sum_k a_{ik} x_{ik}(t) \cdot a_{kj} x_{kj}(t).
\]

(8c)
This procedure allows us to carry out the stochastic evolution of the system. It means that in addition to the tendency to minimize the work function $U$ by changing the state of triads from imbalanced to balanced, which modifies the state of the system towards the limit of $U \to -1$, also the changes increasing $U$ are possible with non-zero probability given by Equation (8). As a result, the system evolves towards the thermal equilibrium realized at some $U > -1$. In each time step of the evolution, all links are updated synchronously.

### III. RESULTS

Simulations based on the models described in the previous section were performed to find under what conditions the phase transition between the Heider balance and the imbalanced state occurs. Since the main objective was to identify the values of the critical temperature characterizing such transitions, we (quite arbitrarily) assumed that $T = T_C$ when $U(T) = -0.5$ to be consistent with Refs. 7 and 18. Below we discuss our results obtained for systems of three different types: when links are gradually randomly removed from a triangular lattice, thus reducing the graph density (Section III A); when the underlying lattice is created by adding links to a triangular lattice up to the point when it becomes a complete graph (Section III B); and finally when the topology of the simulated system follows the concept of classical random graph (Section III C). In each of those cases, we examine the full range of the possible densities of the considered graphs.

We start simulation with the system in the imbalanced ($U = 0$) initial state achieved by setting at $t = 0$ the same number of positive and negative link values. The simulations are carried out for systems with $N = W^2$ sites and $W = 10$ or $W = 20$.

#### A. Diluted triangular lattice

In Figure 3 we show a single realization ($R = 1$) of the time evolution of the work function $U(t)$ for various values of the thermal noise $T$ and various numbers of removed links $q_-$ on the lattice with $W^2 = 20^2$ sites. As it was shown in Ref. 21, for the triangular lattice ($q_- = 0$, $p_- = 1$) the system reaches the imbalanced state independently on the assumed temperature $T$.

We note that the system work function $U$ is multiply degenerated as dozens among $2^L$ available positive/negative link distributions may lead to the same value of $U$. The triangular lattice dilution ($q_- > 0$, $p_- < 1$) introduces defects in the system which reduce the number of available work function $U$ values, which is clearly visible at high level of dilution ($q_- \to N, p_- \to 0$) in Figures 3c and 3f.

For high temperature $T$, independently on the system dilution $q_-$, the work function $U$ fluctuates around $U = 0$ (see Figures 3d to 3f). In case of the example presented in Figure 3b (for $T = 0.5$ and $p_- = 1/2$) the work function fluctuates around $-0.7$ and for further lattice dilution ($p_- = 1/5$) it switches mainly between the four discrete, well separated but negative values of $-1 \leq U < 0$ ($U \in \{-1, -0.714, -0.428, -0.143\}$, see Figure 3c).

In Figure 4 the thermal evolution of the average work function $\langle U \rangle$ for various values of the system dilution (expressed in terms of the bond occupation probability $p_-$) and various lattice sizes ($W = 10$ and 20) is presented. A single point in this chart comes from averaging over the last $\tau = 10^4$ among $t_{\max} = 10^4$ time steps of the evolution. The points uncertainty comes from averaging over $R = 100$ simulations. The averaging procedure over both last time steps and various independent simulations is denoted as $\langle \cdots \rangle$. The system size $N = W^2$ does not influence the results significantly. In this aspect (i.e., independence of the results on the system size $N$) the diluted triangular lattice is similar to the one-dimensional chain of nodes/links with long-range interactions [7]. However, this is quite different than, for instance, for a complete graph, where $T_C$ increases with system size as $T_C \propto (N - 2)^2$ [16].

Independently on the lattice dilution—in the low temperature limit—we have $U_0 \equiv U(T \to 0) > -1$ (see Figure 4). This limiting value $U_0$ decreases with decreasing bond occupation probability $p_-$. However, for $p_- > 0.83$ we have $U_0 > -0.5$ and thus the critical temperature $T_C$ is undefined, as the curve $U(T)$ does not intersect the line $U = -0.5$. As for the dilution level in the range $0.83 < p_- < 1$ where the system stays closer to the imbalanced state than to the balanced one, we may assume that in this dilution region the critical temperature tends to zero ($T_C \to 0$) what is consistent with our earlier studies [21]. Finally, independently on the lattice dilution $p$, the system reaches the imbalanced ($U = 0$) for high enough noise level $T > 5$.

In Figure 5 the dependencies of $\langle U \rangle$ vs. the occupation probability $p_-$ and thermal noise $T$ for $W = 10$ and $W = 20$ are presented. The results are averaged over $R = 10$ simulations. The phase border corresponding to $U = -0.5$ (i.e. the critical temperature $T_C(p_-)$ dependence on the occupation probability $p_-$) is marked with green points.

To sum up the results obtained for the diluted triangular lattice:

- the triangular lattice dilution highlights the degeneration of the system work function $U$ (see Figure 3);
- in topological neighborhood of the triangular lattice ($0.8 < p_- \leq 1$) the system never reaches $U < -0.5$ (see Figures 4 and 5), which may be considered as $T_C \to 0$;
- the average work function $\langle U \rangle$ depends on the triangular network dilution $q_-$ and the assumed noise level $T$ (see Figure 5).
reaching in dilution the bond percolation threshold and further \((p_- < p_c^{\text{bond}} = 2\sin(\pi/18) \approx 0.347296355 \cdots [26])\) does not change the system behavior qualitatively, and for \(p_- \to 0\) we have \(T_C \to 1\);

- and the system size \(N = W^2\) does not influence the results (compare Figure 4a with Figure 4b and Figure 5a with Figure 5b).

B. Enhanced triangular lattice

Simulations on the lattices created by adding links to the initial triangular topology were performed as in the previous case, starting from a random imbalanced \((U = 0)\) initial state with fifty-fifty distribution of friendly and hostile relations. Examples of the time evolution of the work function for \(p_+ > 1\), i.e., when links are added to the triangular lattice instead of being removed from it, are presented in Figure 6. Same as for the diluted triangular lattice \((p_- < 1)\), the imbalanced state is retained at higher temperatures for all considered graph densities, however the required temperatures tend to be larger as \(p_+\) increases. On the other hand, no quantization of the work function values is visible, because adding links does not lead to reduction of the available values of \(U\); on the contrary, the number of possible work function values increases and the spectrum remains quasi-continuous.

The average value of the work function \((\langle U \rangle)\) is presented as a function of temperature in Figure 7 for several values of \(p_+\), starting from \(p_+ = 1\) which corresponds to the triangular lattice. All points were obtained from the averaging procedure over the last \(\tau = 10^3\) steps of \(R = 10\) simulations which took \(t_{\text{max}} = 10^4\) time steps each. Comparison of Figures 7a and 7b reveals that those characteristics depend on the system size \(N = W^2\), unlike for \(p_- < 1\). For the larger size \(W\), the balanced phase can exist at temperatures which are higher than for smaller systems with the same \(p_+\).

The dependence of the average value of the work function \((\langle U \rangle)\) on the occupation parameter \(p_+\) and the temperature \(T\) is presented in Figure 8 for \(W = 10\) and \(W = 20\). It shows that the boundary between the regions of the \((p_+, T)\)-space with balanced and imbalanced phases is very sharp, except for the smallest values of \(p_+\) (see the insets of Figure 8). This feature is also visible in the \(U(T)\) dependencies presented in Figure 7a (for \(p_+ \geq 3\)) and Figure 7b (for \(p_+ \geq 5\)).

In contrast to the diluted triangular lattice, in the case of the enhanced triangular lattice the critical temperatures are not size-independent. Their values are indicated by the boundary of the phases (and green dots) in Figure 8, following the condition \(U = -0.5\). The maximum value of the bond occupation parameter, \(p_{\text{max}}\), for which the results are presented is different in Figure 8a and in Figure 8b which stems from the fact that the limiting case is always the complete graph. In that case, the density is \(D = 1\), and the corresponding \(p_{\text{max}} = (W^2 - 1)/6\) depends on the size of the system: for \(W = 10\) we have \(p_{\text{max}} = 16.5\) while for \(W = 20\) it is equal \(p_{\text{max}} = 66.5\). At those values of the bond occupation parameter, the lat-
practice reaches the limit of the complete graph for which the critical temperatures are known to be size-dependent \[18\] and are in agreement with our results for \( p_+ = p_{\text{max}} \).

Transition from an imbalanced state to a balanced state is observed at a positive critical temperature only if the number of links added to the triangular lattice is above some minimum value, and hence the bond occupation parameter is above the corresponding minimum value \( p^* > 1 \). In the case of the diluted lattices discussed in Section IIIA, \( T_C > 0 \) was found only if the number of removed links was large enough, specifically when \( p_- < 0.83 \). It means that for a certain range of parameter values around \( p_+ = 1 \) (which corresponds to the triangular lattice) \( T_C \rightarrow 0 \), which proves that the topology has to be considerably different from the triangular lattice in order to produce \( T_C > 0 \) and allow for a balanced state. However, there is an important difference between the situations for the enhanced (\( p_+ > 1 \)) and the diluted (\( p_- < 1 \)) lattices. Unlike in the latter case, the system size affects the value of \( p_+ = p^* \) which separates the cases where the finite and positive critical temperatures can be found from those with \( T_C \rightarrow 0 \), e.g. for \( W = 10 \) it is \( p^* \approx 1.7 \) (see the inset in Figure 8a) and increases to \( p^* \approx 2.9 \) for \( W = 20 \) (see the inset in Figure 8b).

C. Classical random graphs

In addition to simulations on lattices created when links are either gradually removed from a triangular lattice or introduced to this kind of lattice, we also performed simulations on classical random graphs. In that case, the whole range of possible graph densities was cov-
FIG. 6: (Color online). Time evolution of $U(t)$ for enhanced triangular lattices, $W = 10$ and 20, $R = 1$. (a) $T = 0.5$, $p_+ = 1.5$, (b) $T = 0.5$, $p_+ = 3$, (c) $T = 0.5$, $p_+ = 9$, (d) $T = 5$, $p_+ = 1.5$, (e) $T = 5$, $p_+ = 3$, (f) $T = 5$, $p_+ = 9$.

...ered, from values close to zero (as at least one triad is needed to proceed) up to $D = 1$ (complete graph). Figure 9 displays the dependence of the average work function $\langle U \rangle$ on temperature $T$ for various graph densities $D$ and two different system sizes, $W = 10$ (Figure 9a) and $W = 20$ (Figure 9b). In all those examples $\langle U \rangle < -0.5$ at low temperatures and $\langle U \rangle \to 0$ at high $T$, which means that for each of them a transition between the balanced (at least partially, as not always $\langle U \rangle = -1$) and imbalanced state occurs and it is possible to find the critical temperature $T_C$. Comparison of the results obtained for $W = 10$ and $W = 20$ shows that except for the smallest densities, those critical temperatures depend on the system size. We can also see that at low densities the increase of the average work function $\langle U \rangle$ with temperature is much smoother than at higher $D$.

The details of the critical temperature dependence on the density $D$ of classical random graphs is shown in Figure 10. In the two limiting cases of $D \to 0$ and $D = 1$, the results are obviously the same as those obtained when topologies based on the diluted or enhanced triangular lattices were used ($D \to 0$ which corresponds to $p_- \to 0$, and $D = 1$ which corresponds to $p_+ = p_{\text{max}}$). For the classical random graphs with $D \to 0$, the lattices also reduce to single triads, and independent of the system size $T_C \to 1$. When the density $D$ is increased, the critical temperature initially decreases and reaches a minimum of $T_C \approx 0.4$ for values of $D$ which depend on the system size (see the inset of Figure 10). If the density is further increased, $T_C$ grows monotonically up to the maximum value at $D = 1$, which means that we are dealing with a complete graph for which $T_C$ also depends on the system size. Ultimately, it means that for a classical random graph of any density, both imbalanced and balanced states may exist depending on the temperature, and the critical temperature separating those two phases depends on the system size.

IV. DISCUSSION

In Figure 11 the dependencies $U(T)$ for various lattice sizes $N = W^2$ and various classical random graphs are presented. The classical random graphs utilized for these simulations have either the number of links $L_{\text{CRG}}$ or number of triads $\Delta_{\text{CRG}}$ identical with those numbers on the regular triangular lattice, i.e. with fully occupied ($p = 1$) $L_{\text{TR}} = 3W^2$ links, what corresponds to $\Delta_{\text{TR}} = 2W^2$ triangles. The results for the triangular lattice are also included. In both cases, the results are averaged over one hundred ($R = 100$) independent simulations.

The expected number of triads in a random graph is $\Delta = \Delta_{\text{CG}}D^3$, where $\Delta_{\text{CG}} = \binom{N}{3}$ is the number of triads in a complete graph with $N$ nodes. Since for the triangular lattices $D_{\text{TR}} = 6/(N - 1)$, it follows that for any random graph with the number of links (and therefore also density) equal to the number of links in a triangular lattice of the same size $N$, the number of triads is $\Delta = 36N(N - 2)/(N - 1)^2$ which gives $\Delta \approx 36$ in the limit of large $N$.

Independently on the assumed $W$ the results for the triangular lattice are qualitatively and quantitatively dif-
different from those obtained on random topology with the same number of links $L_{\text{Crg}} = L_{\text{TR}}$ and the same number of triangles $\Delta_{\text{Crg}} = \Delta_{\text{TR}}$ as for the triangular lattice. We note that the triangular lattice topology may lead to the complete absence of the balanced states, as we have shown in Ref. 21.

To compare the critical temperatures found for the classical random graphs (Section III C) with those for both the diluted (Section III A) and enhanced (Section III B) triangular lattices, it is necessary to present all of them as functions of the same parameter. Hence, for each density $D$ characterizing a classical random graph, a corresponding value of the average node degree $\bar{k} = 2L/N = D(N - 1) = D(W^2 - 1)$ is used.

The outcome of this procedure is used to compare the critical temperature characteristics of both topologies in Figure 12. In the case of lattices created from the triangular lattice ($\bar{k} = 6$) by removing some links ($\bar{k} < 6$), or by adding more links ($\bar{k} > 6$), the critical temperature is not observed when $5 < \bar{k} < \bar{k}^*$. No such range exists for classical random graphs, for which the transition between the balanced and imbalanced states is always possible, even at $\bar{k} = 6$ when the graph of the same density as the regular triangular lattice of identical size is considered.

Analysis of the lattices with $\bar{k} < 6$ ($p_+ < 1$) reveals that the critical temperatures found for the diluted triangular lattices and for the random graphs approach the common limiting value $T_C = 1$ when $\bar{k} \to 0$. It is not surprising since for both types of lattices the reduction of the graph density inevitably leads towards systems consisting of a small number of disconnected triads.

When $\bar{k}$ increases, initially for both types of lattices a

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**Figure 7**: (Color online). Thermal evolution of average work function $\langle U \rangle$ for enhanced triangular lattices, $t_{\text{max}} = 10^4$, $\tau = 10^3$, $R = 10$, various $p_+$, (a) $W = 10$, (b) $W = 20$.

**Figure 8**: (Color online). Average work function $\langle U \rangle$ vs. occupation parameter $p_+$ and temperature $T$ for enhanced triangular lattices, $R = 10$, $t_{\text{max}} = 10^4$, $\tau = 10^3$, (a) $W = 10$, (b) $W = 20$. Green dots indicate the phase border, i.e. pairs $(p_+, T)$, where $U = -0.5$ and thus at this border we have $T = T_C$. 
FIG. 9: (Color online). Thermal evolution of work function $\langle U \rangle$ for classical random graphs, $t_{\text{max}} = 10^4$, $\tau = 10^3$, $R = 10$, various $D$, (a) $W = 10$, (b) $W = 20$.

FIG. 10: (Color online). Critical temperature $T_C$ vs. density $D$ of classical random graphs, $R = 10$, $t_{\text{max}} = 10^4$, $\tau = 10^3$, $W = 10$ and 20.

FIG. 11: (Color online). Work function $U(T)$ for classical random graphs (CRG) and triangular lattice (TR) for $t_{\text{max}} = 10^4$, $\tau = 10^3$ and various $W$. The results are averaged $\langle \cdots \rangle$ over $R = 100$ simulations.

FIG. 12: (Color online). Critical temperature $T_C$ vs. average node degree $\bar{k}$ for triangular lattices (TR), diluted when $\bar{k} < 6$ or enhanced when $\bar{k} > 6$, and for classical random graphs (CRG). Inset: the minimum average node degree $\bar{k}^*$ (also indicated with the dashed lines in the main part of this figure) required for $T_C > 0$ in enhanced triangular lattices depending on their size $W$.

On the other hand, for the classical random graph, the critical temperature remains positive for all values of the average node degree $\bar{k}$. After the initial decrease, $T_C$ has a minimum, whose position in terms of $\bar{k}$ depends on the system size. For triangular lattices, $T_C \to 0$ at $\bar{k} = 5$ and the critical temperature remains undefined until the lattice becomes an enhanced triangular lattice and $\bar{k}$ reaches the size-dependent $\bar{k}^*$ (corresponding to $p^* > 1$) and its value increases with the size of the system (see the inset in Figure 12 which presents $\bar{k}^*$ for $W = 10, 12, \ldots, 20$).
on the system size. Figure 12 shows that this minimum occurs at the value of the average node degree \( \bar{k} \) which increases with the size of the system and is larger than \( \bar{k}^* \) characterizing the enhanced triangular lattices.

For large values of \( \bar{k} \), when huge numbers of links make it hardly possible to distinguish between graphs created randomly and by enhancing a regular triangular lattice, the critical temperatures assume the same values for both topologies provided that the system sizes are equal. Additionally, in the limit of large \( \bar{k} \) those common values of the critical temperature depend on \( \bar{k} \) as \( T_C \propto \bar{k}^\gamma \). To find the exponent \( \gamma \), least-squares fitting procedure was performed for those points in Figure 12 which represent graphs with \( D > 0.5 \), i.e., when approximately \( \bar{k} > 50 \) for \( W = 10 \) or \( \bar{k} > 200 \) for \( W = 20 \). It reveals that \( \gamma \approx 1.6 \pm 0.7 \)—see Table I for the complete list of the values of \( \gamma \) and their uncertainties.

**TABLE I: Exponent \( \gamma \) and its uncertainty \( u(\gamma) \) found for triangular lattices (TR) and classical random graphs (CRG), for \( W = 10 \) and 20.**

| \( W \) | TR  | CRG |
|------|-----|-----|
| 10   | 1.649 | 1.616 | 1.700 |
| 20   | 1.689 | 0.007 | 0.004 |

V. CONCLUSIONS

The underlying network topology has a decisive impact on the thermally driven phase transitions between the balanced and imbalanced states in systems used to model the dynamics of hostile and friendly attitudes. We show that on the example of two different types of lattices used for that purpose, both of them discussed in the full range of possible graph densities, from a very low number of links (and only one or several triads) to complete graphs. The first is based on the regular triangular lattice modified by removing or adding some links, while the second is the classical random graph.

In either of them, reaching the Heider balance is possible under certain conditions. For systems in which the lattice is a classical random graph, it is enough to keep the thermal noise below the value of the critical temperature which depends on the number of nodes in the lattice and thus the density of the graph. In the case of systems based on triangular lattices, an additional condition is that the lattice differs significantly from the triangular lattice, i.e., enough links were removed to create a diluted triangular lattice or added to produce an enhanced triangular lattice, and our results provide quantitative details of the required modification. It is related to the fact that for the ideal triangular lattice the thermal noise completely prevents even partial Heider balance [21], which may have its origin in a large density of triads (compared to the random lattices with the same number of links), responsible for frustration similar to that observed in spin glasses. In the case of the diluted triangular lattice the average node degree must be reduced at least by one. In the enhanced triangular lattices, two classes of links are present: the nearest-neighbor links preserved from the triangular lattice, and the additional random links, statistically mostly long-range (which are the only kind existing in the random links). The influence of those two classes of links on the existence of the balanced and imbalanced phases seems to be opposite, with the first acting against the balanced state and the second helping to achieve it. In the end, a sufficient number of the long-range links needs to be added to overcome the effect of the triangular lattice. The required minimum of the average node degree depends on the size of the system which may suggest that emergence of the balanced phase demands more relatively short-ranged (but not nearest-neighbor) links, which are less likely in the larger lattices when created randomly.

The common property of the critical temperature in both topologies is that it initially decreases from \( T_C = 1 \) with increasing graph density, while for large densities it increases and assumes values depending only on the system size following a power law with the same exponent for both types of lattices, to reach the limiting values known for complete graphs [18]. The difference is that for the diluted triangular lattices the critical temperature approaches zero after the initial decrease and is again observed only when enough links are added, while for random graphs it slowly decreases to a minimum value but stays positive, and then increases. It should be noted, though, that the values of the critical temperature presented in this work were obtained with the arbitrary criterion of association of critical temperature \( T_C \) with temperature \( T \) for which \( \langle U(T) \rangle = -0.5 \). Moreover, caution must be taken even if the graph is connected but contains only a small number of noninteracting triads, which makes it difficult to define the phase transition.

Another interesting observation can be made concerning the dependence of the work function on temperature, which reveals the transition between balanced and imbalanced states. That transition is significantly sharper at higher graph density, which resembles the situation in the case of chains of actors when the range of coupling between the nodes is larger [7]. However, this intriguing property (discussed also in Ref. [12]) still requires a detailed analysis, which is far beyond the scope of this work.

In summary, our results demonstrate how the topology of two types of lattices affect the process of reaching the structural balance in the presence of thermal noise simulated using the heat-bath dynamics. It is shown where such a balanced state is possible, and in that case the critical temperatures are provided as functions of the bond occupation parameter or the graph density.
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