Multisymplectic 3-forms on 7-dimensional manifolds.

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Abstract: The goal of the paper is to give characterization of closed connected manifolds which admit a global multisymplectic 3-form of some algebraic type. A generic type of such 3-form is equivalent to a $G_2$-structure. This is the most interesting case and was solved in [Gr]. Some other algebraic types were solved quite recently. In this paper we give characterization in the remaining cases.

0.1 Introduction

We call a 3-form $\omega \in \Lambda^3((\mathbb{R}^7)^*)$ multisymplectic iff the map

$$i_\omega : \mathbb{R}^7 \to \Lambda^2((\mathbb{R}^7)^*)$$

is injective. There are eight orbits of multisympletic forms under the natural action of $GL(7, \mathbb{R})$ on $\Lambda^3((\mathbb{R}^7)^*)$. We will denote preferred representatives of the orbits by $\omega_i$, $i = 1, \ldots, 8$. Let us denote the stabilizer of $\omega_i$ by $O_i$ and a maximal compact subgroup of $O_i$ by $K_i$. The connected component of identity of $K_i$ is denoted by $K_0^i$.

Let $M$ be a 7-dimensional closed connected manifold and let $\rho \in \Omega^3(M)$ be a global 3-form. We say that $\rho$ is of algebraic type $i = 1, \ldots, 8$ if $\forall x \in M$ there exists a basis of $T^*_x M$ at the point $x$ such that the 3-form $\rho(x) \in \Lambda^3(T^*_x M)$ belongs to the $i$-th orbit of the multisymplectic forms. This notion clearly does not depend on a choice of frame. The main goal of this paper is to give topological restrictions on $M$ to admit a global 3-form of a given algebraic type. We will need the following observations.

A global 3-form of the $i$-th algebraic type on the manifold $M$ is equivalent to a reduction of structure group of the tangent bundle $TM$ of $M$ to $K_i$. The first goal is to find maximal compact subgroups $K_i$. The groups $O_i$ were given in [BV]. Without loss of generality we may take $K_i := O_i \cap O(7)$. This is the content of the first part of this paper.

Topological conditions on manifolds which admit a global 3-form of a given algebraic type is given in the second part. Solved cases include the algebraic types 3, 5 and 8. The type 8 is the first solved case and is equivalent to a $G_2$-structure. The manifold $M$ admits such structure iff $M$ is orientable and admits a spin structure. This is originally a result of [Gr]. The type 5 was solved in [Le]. It turns out that the case 5 is equivalent to the case 8. We show that the cases 5, 8 are equivalent to the cases 6, 7. We use similar ideas as in the paper [Le]. The type 3 was solved in [D] using techniques introduced in [Th]. We use the same machinery to handle remaining cases with additional assumption on orientability or simple connectedness of manifolds. These are the theorems 9-11. The last section consists of lemma needed in the two first chapters.

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0.1.1 Notation

Notation for the first chapter. We denote the space $\mathbb{R}^7$ by $V$ and the standard basis by $\{e_1, \ldots, e_7\}$. We will denote the dual basis by $\{\alpha_1, \ldots, \alpha_7\}$. Let us denote by $(v_1, \ldots, v_i)$ the linear span of vectors $v_1, \ldots, v_i$. We will denote the stabilizer of the preferred multisymplectic 3-form $\omega_i$ by $O_i$, its maximal compact subgroup by $K_i$ and the connected component of $K_i$ is denoted by $K_0^i$.

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A representative of the orbit is

\[ \omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 \wedge \alpha_6. \]  

(2)

Let us denote by \( V_i = \langle e_7 \rangle, V_1 = \langle e_3, e_4 \rangle, V_2 = \langle e_5, e_6 \rangle, V_3 = \langle e_1, e_2 \rangle, W_1 = (V_1^2 \oplus V_1)/V_1, W_2 = (V_2^2 \oplus V_1)/V_1 \). Let \( \varphi \) be any element of \( O_1 \). The following statements were proved in [BY].

- \( \varphi \) preserves the subspace \( V_1 \).
- \( \varphi \) induces an automorphism of \( W_1^2 \oplus W_2^2 \) such that:
  1. \( \varphi(W_1^2) = W_1^2, \varphi(W_2^2) = W_2^2 \) or
  2. \( \varphi(W_1^2) = W_2^2, \varphi(W_2^2) = W_1^2 \).

We define a map \( sgn: O_1 \to \mathbb{Z}_2 \) by \( sgn(\varphi) = 1 \) if the first possibility holds and \( sgn(\varphi) = -1 \) if the latter condition holds. We call \( sgn(\varphi) \) the sign of \( \varphi \). Clearly the map \( sgn \) is a group homomorphism.

- The stabilizer \( O_1 \) of \( \omega_1 \) is isomorphic to a semi-direct product

\[ (N \rtimes (GL(W_1^2) \times GL(W_2^2))) \rtimes \mathbb{Z}_2 \]  

(3)

such that:

- The first semi-direct product is given by the homomorphism \( sgn \).
- The map \( \varphi \in Ker(sgn) \mapsto \varphi|_{W_1^2 \oplus W_2^2} \in GL(W_1^2) \times GL(W_2^2) \) is surjective.
- The group \( N \) consists of transformations of the form \( Id_V + \varphi_1 + \varphi_2 \) where

\[ \varphi_1: V_2^3 \to V_1^1 \oplus V_2^2 \oplus V_1, \varphi_2: V_2^1 \oplus V_2^2 \to V_1. \]

In particular, with respect to usual convention, any element of \( N \) is an upper triangular matrix.

Let us define an embedding

\[ \rho: GL(2, \mathbb{R}) \times GL(2, \mathbb{R}) \to GL(\langle e_1 \rangle) \times GL(\langle e_2 \rangle) \times GL(V_1^2) \times GL(V_2^2) \times GL(V_1) \]  

(4)

\[ (\rho(a, b))(v_1, v_2, v_3, v_4, v_5) = (det(a^{-1})v_1, det(b^{-1})v_2, av_3, bv_4, det(ab)v_5), \]

where \( GL(2, \mathbb{R}) \) acts naturally on \( V_1^2 \cong V_2^2 \cong \mathbb{R}^2 \). Let us denote the image of \( GL(2, \mathbb{R}) \times GL(2, \mathbb{R}) \) under \( \rho \) by \( GL(2, \mathbb{R})_{2,2}^2 \).

**Lemma 1.** The group \( GL(2, \mathbb{R})_{2,2}^2 \) gives a splitting of \( GL(W_1^2) \times GL(W_2^2) \) in the formula (3).  

Proof: Clearly \( GL(2, \mathbb{R})_{2,2}^2 \subset O_1 \) and \( GL(2, \mathbb{R})_{2,2}^2 \) satisfies all conditions to be a splitting of \( GL(W_1^2) \times GL(W_2^2) \). \( \square \)

Now it is easy to find a maximal compact subgroup \( K_1 \). From the lemmas [8], [9] and the description of \( N \) given above follows that \( K_1 \) is a semi-direct product of a maximal compact subgroup of \( GL(2, \mathbb{R})_{2,2}^2 \) and \( \mathbb{Z}_2 \). A maximal compact subgroup of \( GL(2, \mathbb{R})_{2,2}^2 \) is the group \( O(2) \times O(2) \). A splitting of the group \( \mathbb{Z}_2 \) of the homomorphism \( sgn \) is given for example by the transformation

\[ e_1 \mapsto e_2, e_2 \mapsto e_1, e_3 \mapsto e_5, e_4 \mapsto e_6, e_5 \mapsto e_3, e_6 \mapsto e_4, e_7 \mapsto -e_7 \]  

(5)
Theorem 1. A maximal compact subgroup $K_1$ of $O_1$ is generated by the subgroup $O(2) \times O(2)$ of the group defined [4] and the transformation given in [3]. In particular $K_1$ is a subgroup of $SO(7)$.

0.2.2 The 3-form $\omega_2$.

A representative chosen with respect to a basis $\{e_1, \ldots, e_7\}$ in [BV] is

$$\omega_2 = \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 - \alpha_2 \wedge \alpha_3 \wedge \alpha_7$$

(6)

Let us denote by $\{f_1, \ldots, f_7\}$ which is given by

$$f_1 = e_5 + e_6$$

$$f_2 = -e_5 + e_6$$

$$f_3 = -e_5 - e_6 + e_7$$

$$\sqrt{2} f_4 = -e_1 - e_4$$

$$\sqrt{2} f_5 = -e_2 + e_3$$

$$\sqrt{2} f_6 = -e_2 - e_3$$

$$\sqrt{2} f_7 = -e_1 + e_4$$

Let us denote by $\{\beta_1, \ldots, \beta_7\}$ the dual basis to $\{f_1, \ldots, f_7\}$. A new 3-form $\omega'_2$ in the orbit of $\omega_2$ is

$$\omega'_2 = \beta_1 \wedge \beta_4 \wedge \beta_5 - \beta_1 \wedge \beta_6 \wedge \beta_7 + \beta_2 \wedge \beta_5 \wedge \beta_7$$

$$- \beta_2 \wedge \beta_4 \wedge \beta_6 - \beta_3 \wedge \beta_4 \wedge \beta_7 - \beta_3 \wedge \beta_5 \wedge \beta_6.$$  

(7)

We will give a description of a maximal compact subgroup of the stabilizer $O_2 := Stab(\omega'_2)$. Let us denote by $V_3 = \langle f_1, f_2, f_3 \rangle, V_4 = \langle f_4, \ldots, f_7 \rangle, W_4 = V/V_3$. We notice that

$$\omega'_2 + \beta_1 \wedge \beta_2 \wedge \beta_3 = \omega'_5.$$  

(8)

3-form $\omega'_5$ is the 3-form in the orbit of $\omega_5$ given by the formula (41) with respect to the basis $\{i, j, k, e_4, e_4i, e_4j, e_4k| i, j, k \in \mathbb{H}\}$ of $\mathcal{O}$. This leads to the following observation.

**Lemma 2.** The group $\text{SL}(2, \mathbb{R})^3_4$ given in the formula (44) is a subgroup of $K_2$.

Proof: Let $g \in \text{SL}(2, \mathbb{R})^3_4$. From the formula (44) follows that $g^*(\beta_1 \wedge \beta_2 \wedge \beta_3) = \beta_1 \wedge \beta_2 \wedge \beta_3$. Since $\text{SL}(2, \mathbb{R})^{3, 4}_4 \subset \mathcal{G}_2$, then $g^*\omega'_5 = \omega'_5$. Thus $g^*\omega'_2 = \omega'_2$. □

**Lemma 3.** There is an isomorphism

$$O_2 \cong (L \rtimes \text{SL}(2, \mathbb{R})^{3, 4}_4) \rtimes \mathbb{R}^*$$

(9)

where:

- The group $\mathbb{R}^*$ is the cokernel of $O_2$ under the homomorphism $g \in O_2 \mapsto \det(g|_{V_3})$.
- The group $L$ consists of endomorphisms of the form $Id_V + \varphi$ where $\varphi : V_4 \to V_3$.

Proof: In [BV] is given an isomorphism

$$O_2 \cong (L \rtimes \text{Spin}(1,2)) \rtimes \text{SO}(1,2)$$

where $V_3 \cong \text{Im} \mathbb{H}$ by

$$f_1 \mapsto i, f_2 \mapsto j, f_3 \mapsto k,$$

and $V_4 \cong \text{Im} \mathcal{O}$ by

$$f_4 \mapsto 1, f_5 \mapsto i, f_6 \mapsto j, f_7 \mapsto k$$

Thus we have to verify isomorphism between the semi-direct product $\text{Spin}(1,2) \rtimes \text{SO}(1,2)$ given in [BV] and the group $\text{SL}(2, \mathbb{R})^{3, 4}_4$. We identify $V_4 \cong \mathbb{H}$ by a map

$$f_4 \mapsto 1, f_5 \mapsto i, f_6 \mapsto j, f_7 \mapsto k$$

where $i, j, k$ are given in (34) and $I_2 \in \text{GL}(2, \mathbb{R})$ is the identity matrix. This also identifies $V \cong \text{Im} \mathcal{O}$. Let $g \in \text{SL}(2, \mathbb{R})^{3, 4}_4$, then:
A standard quadratic form\textsuperscript{2} of signature \{+,-,-\} on \(V_3\) is invariant under \(g\).

A standard quadratic form\textsuperscript{3} of signature \{++,+-,-\} on \(V_4\) is invariant under \(g\).

- If \(g = (1,2,a)\), compare to the formula (14), then \(g\) commutes with the image of the map \(V_3 \to \text{End}(W_4)\) induced by \(\omega_2\).

Thus \(\text{SL}(2,\mathbb{R})_{3,4}^2\) satisfies all the conditions which define the semi-direct product \(\text{Spin}(1,2) \times \text{SO}(1,2)\) inside \(O_2\). □

From the lemmas \[9\], \[8\] and the description of \(L\) given above follows that \(K_1\) is a semi-direct product of a maximal compact subgroup of \(\text{SL}(2,\mathbb{R})_{3,4}^2\) and a maximal compact subgroup of \(\mathbb{R}^*\). A maximal compact subgroup \(T_{2,2,2}^2\) of \(\text{SL}(2,\mathbb{R})_{3,4}^2\) is given \[45\] with explicit matrix realizations of the connected component of the identity given in \[46\]. The other component of \(T_{2,2,2}^2\) contains for example a transformation

\[
f_1 \mapsto -f_1, f_2 \mapsto f_2, f_3 \mapsto -f_3, f_4 \mapsto -f_4, f_5 \mapsto -f_5, f_6 \mapsto f_7, f_7 \mapsto f_6
\]

(10)

One can verify that a transformation

\[
f_1 \mapsto f_1, f_2 \mapsto f_2, f_3 \mapsto -f_3, f_4 \mapsto f_4, f_5 \mapsto f_5, f_6 \mapsto f_6, f_7 \mapsto f_7
\]

(11)

gives a splitting of a maximal compact subgroup \(Z_2\) of \(\mathbb{R}^*\) of the map \(g \in O_2 \mapsto \det(g|_{V_3})\). Thus we have the following characterization of \(K_2\).

**Theorem 2.** The group \(K_2\) has four components and is generated by the connected component of the identity \(K_0^2\) given in the formula \[46\] and the transformations given in \[10\] and \[11\].

**0.2.3 The 3-form \(\omega_3\).**

A representative is

\[
\omega_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3 - \alpha_6 \wedge \alpha_7 + \alpha_4 \wedge \alpha_5).
\]

(12)

Let us denote by \(V_1 = \langle e_1 \rangle, V_6 = \langle e_2, \ldots, e_7 \rangle\). Then from \[BV\] we have that:

- Any element \(\varphi \in O_3\) preserves the subspace \(V_6\).
- The 3-form \(\omega_3\) induces a symplectic structure on \(V_6\).
- The stabilizer \(O_3\) is isomorphic to a semi-direct product

\[
O_3 \cong N \ltimes \text{CSp}(3,\mathbb{R}),
\]

(13)

where

- The group \(\text{CSp}(3,\mathbb{R})\) is isomorphic to the cokernel of \(O_4\) under the homomorphism \(g \in O_4 \mapsto g|_{V_5} \in \text{End}(V_6)\).
- The group \(N\) consists of the endomorphisms of the form \(Id_V + \varphi\) where \(\varphi : V_1 \to V_6\).

The connected component of the identity \(K_0^2\) is isomorphic to a maximal compact subgroup \(U(3)\) of \(\text{Sp}(3,\mathbb{R})\). The other component contains for example the transformation

\[
e_1 \mapsto -e_1, e_2 \mapsto e_2, e_3 \mapsto -e_3, e_4 \mapsto e_4, e_5 \mapsto -e_5, e_6 \mapsto e_6, e_7 \mapsto -e_7.
\]

(14)

**Theorem 3.** The group \(K_2\) has two components and is generated by the connected component of the identity of the identity which is isomorphic to a maximal compact subgroup \(U(3)\) of \(\text{Sp}(3,\mathbb{R})\) and the transformation given in \[74\].

\textsuperscript{2}With respect to the basis \(\{f_1, f_2, f_3\}\).

\textsuperscript{3}With respect to the basis \(\{f_4, \ldots, f_7\}\).
0.2.4 The 3-form $\omega_4$.

A representative of the orbit is

$$\omega_4 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_7) + \alpha_2 \wedge \alpha_3 \wedge \alpha_4. \quad (15)$$

Let us denote by $V_1 = \langle e_1 \rangle$, $V_3 = \langle e_5, e_6, e_7 \rangle$, $U_3 = \langle e_2, e_3, e_4 \rangle$, $V_6 = \langle e_2, \ldots, e_7 \rangle$, $W_3 = V_6/V_3$, $W_1 = V/V_6$. Then from [BV] we have that:

- Any element $\varphi \in O_3$ preserves the subspaces $V_3, V_6$.
- The 3-form $\omega_3$ induces a symplectic structures on $V_6$ and a volume form on $W_3$.
- The stabilizer is isomorphic to a semi-direct product

$$O_3 \cong (N \ltimes SL(W_3)) \ltimes \mathbb{R}^*, \quad (16)$$

where

- The group $\mathbb{R}^*$ is isomorphic to the cokernel of $O_4$ under the homomorphism $\varphi \in O_6 \mapsto (\varphi|_{V_1}) \in \text{End}(W_1)$.
- The restriction to $W_3$ gives epimorphism

$$\{ \varphi \in O_4 \mid \varphi|_{V_1} = Id_{V_1} \} \rightarrow SL(W_3). \quad (17)$$

- The group $N$ consists of endomorphisms of the form $Id_V + \varphi_1 + \varphi_2$ where

$$\varphi_1 : V_1 \rightarrow V_6, \varphi_2 : U_3 \rightarrow V_3$$

Let us define

$$\phi : SO(3) \rightarrow \text{End}(V_1) \oplus \text{End}(U_3) \oplus \text{End}(V_3) \quad (18)$$

$$(\phi(A))(v_1, v_2, v_3) = (v_1, Av_2, Av_3),$$

where $SO(3)$ acts naturally on $V_3, U_3 \cong \mathbb{R}^3$ with respect to the preferred basis given above. Let us denote the group $\phi(SO(3))$ by $SO(3,3,3)$.

**Lemma 4.** The group $SO(3,3,3)$ given in (18) is a subgroup of $O_4$.

**Proof:** Let $g \in SO(3,3,3)$. We have to check that:

- $g^*(\alpha_2 \wedge \alpha_3 \wedge \alpha_4) = \alpha_2 \wedge \alpha_3 \wedge \alpha_4$.
- $g^*(\alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_7) = \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_7$.

The first point is straightforward since $g$ restricted to $U_3$ is a volume preserving transformation. The second point follows from the following computation. Let us denote by $\omega = \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_7$. Then we can write

$$\omega(-,-) = g(J-, -),$$

where $g$ is the standard scalar product on $V_6$ with respect to the basis $\{e_2, \ldots, e_7\}$ and $J$ is the complex structure on $V_6$ with the block matrix representation

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where all the block matrices are of rank $3 \times 3$. The lemma follows from the matrix identity $q^T J^T q = J^T$ for any $q \in SO(3,3,3). \square$

The group $SO(3,3,3)$ gives a splitting of a maximal compact subgroup of (17). From the lemmas [9] [8] and the description of $N$ follows that a maximal compact subgroup $K_4$ is isomorphic to a semi-direct product $SO(3,3,3) \ltimes \mathbb{Z}_2$. The connected component of the identity is isomorphic to $SO(3)_{3,3}$. The other component of $K_4$ contains for example the transformation

$$e_1 \mapsto -e_1, e_2 \mapsto e_2, e_3 \mapsto e_3, e_4 \mapsto e_4, e_5 \mapsto -e_5, e_6 \mapsto -e_6, e_7 \mapsto -e_7. \quad (19)$$

**Theorem 4.** The group $K_4$ has two components and is generated by its connected component of the identity $SO(3)_{3,3}$ and the transformations given in (19).
0.2.5 The 3-form $\omega_5$.
A preferred representative is
\begin{align}
\omega_5 &= \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7 \\
&\quad + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6.
\end{align}

Maximal compact is $SO(4)_{3,4}$ in $\tilde{G}_2$, see [Le]. The 3-form $\omega_5$ is given by the multiplication of $\mathcal{O} \cong \tilde{H} \oplus \tilde{H}$ with respect to the basis
\[ e_1 \sim (i,0), e_2 \sim (0,1), e_3 \sim (0,i), e_4 \sim (j,0), e_5 \sim (k,0), e_6 \sim (0,j), e_7 \sim (0,k). \]

0.2.6 The 3-form $\omega_6$.
Let us choose a representative
\[ \omega_6 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7 - \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 \wedge \alpha_7. \]

A 3-form $\omega_6^*$ used in [BV] is related to $\omega_6$ by the formula $\omega_6^* = (g^{-1})^* \omega_6$ where $g : V \to V$ is given by
\[ g(e_1) = -e_7, g(e_7) = e_3, g(e_3) = -e_5, g(e_6) = -e_6, g(e_i) = e_i, i = 1,2,4. \]

Let us denote by $V_1 = (e_3), V_2 = (e_1, e_2), V_3 = V_1 \oplus V_2, V_4 = (e_4,\ldots,e_7), W_4 = V/V_3$. Let $U(1)_3$ be the subgroup of diagonal matrices of the unitary group $SU(2)$ and let $U(1)_3 \times_{\mathbb{Z}_2} SU(2)$ be the natural subgroup of $SO(4)_{3,4} \subset \tilde{G}_2$ below the formula (42).

**Lemma 5.** The group $U(1)_3 \times_{\mathbb{Z}_2} SU(2) = \{g \in SO(4)_{3,4} | g^*(\alpha_3) = \alpha_3\}$ is a subgroup of $K_6$.

**Proof:** We have that $\omega_6 - \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_5 \wedge \alpha_6 = \tilde{\omega}_5$, where $\tilde{\omega}_5$ is the 3-form in the orbit of $\omega_5$ given by the formula (11) with respect to the basis \{i,j,k,e_4,i_4,j_4,k_4\} of $\mathcal{O}$ as in the formula (37). Let $g \in SO(4)_{3,4}$, then $g^*(\tilde{\omega}_5) = \tilde{\omega}_6$ and $g^*(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) = \alpha_1 \wedge \alpha_2 \wedge \alpha_3$. Since $g^*(\alpha_3) = \alpha_3$, we have that $g(V_2) = V_2$ and that
\[ g^*(\tilde{\omega}_6) = g^*(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7) \]
\[ - \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 \wedge \alpha_7 - \alpha_3 \wedge \alpha_4 \wedge \alpha_7 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6) \]
\[ = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + g^*(-\alpha_1 \alpha_2 \wedge \alpha_5 + \alpha_1 \alpha_5 \wedge \alpha_7) \]
\[ - \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 \wedge \alpha_7 - \alpha_3 \wedge \alpha_4 \wedge \alpha_7 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6) \]
\[ = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 - \alpha_3 \wedge g^*(\alpha_4 \wedge \alpha_7 - \alpha_5 \wedge \alpha_6) \]
for some two forms $\beta_1, \beta_2 \in \Lambda^2 V_4$. Comparing this to $g^*(\tilde{\omega}_6) = \tilde{\omega}_6$ we obtain that
\[ \beta_1 = -\alpha_4 \wedge \alpha_5 + \alpha_6 \wedge \alpha_7 \]
\[ \beta_2 = -\alpha_4 \wedge \alpha_6 - \alpha_5 \wedge \alpha_7 \]
for some $\beta_1, \beta_2 \in \Lambda^2 V_4$. Comparing this to $g^*(\tilde{\omega}_6) = \tilde{\omega}_6$ we obtain that
\[ g^*(\alpha_4 \wedge \alpha_7 - \alpha_5 \wedge \alpha_6) = \alpha_4 \wedge \alpha_7 - \alpha_5 \wedge \alpha_6. \]

This proves that $g^*(\omega_6) = \omega_6$. $\Box$

**Lemma 6.** The connected component of $K_6$ is isomorphic to $U(1)_3 \times_{\mathbb{Z}_2} SU(2)$.

**Proof:** From [BV] we have an isomorphism
\[ O_6 \cong ((\Lambda \times \operatorname{sl}(2,\mathbb{C})) \times SO(2)) \rtimes \mathbb{R}^*\]
where:
- the group $\mathbb{R}^*$ is the cokernel of $O_6$ under the homomorphism $g \in O_6 \mapsto \det(g|_{V_2})$. 

• the group L consists of transformations of the form $Id + \varphi_1 + \varphi_2$ where
\[ \varphi_1 : V_2 \to (V_4 \oplus V_1), \varphi_2 : V_4 \to V_1. \] (24)

We have that the group $U(1)_3 \times \mathbb{Z}_2$ $SU(2)$ satisfy the following conditions:

• The subgroup $U(1)_3$ operates with respect to the standard scalar product on $V_2$ as orthogonal transformations.

• The subgroup $SU(2)$ commutes with the image of the map $\lambda : V_3 \to \text{End}(W_4)$ given in [BV].

Comparing this to [BV], we deduce that $U(1)_3$ is a splitting of a maximal compact subgroup of $SL(2, \mathbb{C}) \times SO(2)$. The rest is a consequence of the lemma 9 description of L and the formula (23). □

From the lemma 9 follows that $K_6 \cong (U(1)_3 \times \mathbb{Z}_2) SU(2)$. Thus to complete the description of $K_6$, we have to find an element $g \in K_6$ which does not belong to the connected component of the identity. Such element is for example the transformation
\[ e_1 \mapsto e_2, e_2 \mapsto e_1, e_3 \mapsto -e_3, e_4 \mapsto e_4, e_5 \mapsto e_6, e_6 \mapsto e_7, e_7 \mapsto -e_7. \] (25)

Thus we can formulate the following theorem.

**Theorem 5.** The group $K_6$ is generated by the connected component of the identity which is equal to $U(1)_3 \times \mathbb{Z}_2$ $SU(2)$ defined in the lemma 9 and the transformation (25). In particular $K_6 \subset SO(4)_{3,4}$.

**0.2.7 The 3-form $\omega_7$.**

Let us choose a 3-form
\[ \omega_7 = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 - \alpha_1 \wedge \alpha_6 \wedge \alpha_7 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 - \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6. \] (26)

The 3-form $\omega'_7$ used in [BV] is related to $\omega_7$ by the formula $\omega'_7 = (g^{-1})^*\omega_7$ where $g : V \to V$ is given by
\[ g(e_1) = -e_4, g(e_2) = -e_7, g(e_3) = e_5, g(e_4) = e_6, g(e_5) = e_3, g(e_6) = -e_1, g(e_7) = e_2. \]

Let us denote by $V_3$ the span $\langle e_1, e_2, e_3 \rangle$ and by $V_4$ the span $\langle e_4, ..., e_7 \rangle$. Let us denote by $W_4 = V/V_3$.

**Theorem 6.** $SO(4)_{3,4} = \{g \in G_2 | g^*(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) = \alpha_1 \wedge \alpha_2 \wedge \alpha_3\}$ is a maximal compact subgroup of $K_7$.

Proof: We notice that
\[ \omega_8 = \omega_7 + \alpha_1 \wedge \alpha_2 \wedge \alpha_3, \] (27)

where $\omega_8$ is the 3-form (29). This implies that $SO(4)_{3,4} \subset K_7$. We now show also the opposite inclusion. The equation (27) naturally leads to an isomorphism $V \cong Im H \oplus H \cong I, \bar{O}$ given by
\[ e_1 \mapsto (i, 0), e_2 \mapsto (0, 1), e_3 \mapsto (0, i), e_4 \mapsto (j, 0), e_5 \mapsto (k, 0), e_6 \mapsto (0, j), e_7 \mapsto (0, k). \] (28)

This implies that

• The action of $SO(4)_{3,4}$ restricted to $V_3$ is orthogonal with respect to the standard scalar product.

• The action of the subgroup $1 \times SU(2) \subset SO(4)_{3,4}$ on $V_4$ commutes with the image of $\lambda : V_3 \to \text{End}(W_4)$ introduced in [BV].

This observation together with the lemma 9 and [BV] gives that $SO(4)_{3,4}$ is a splitting of a maximal compact subgroup of $O_7$. □
0.2.8 The 3-form $\omega_8$.

This is a well known case. A representative is

$$
\omega_8 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 - \alpha_1 \wedge \alpha_6 \wedge \alpha_7
+ \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6.
$$

The stabilizer is the exceptional compact 14 dimensional Lie group $O_8 \cong G_2$. The 3-form $\omega_8$ is given by the standard multiplication table given by the Cayley-Dickinson construction.

0.3 Manifolds admitting a global multisymplectic 3-form of one type.

We will first consider the global 3-forms of type $5, 6, 7, 8$. The known implications in the following theorem are $1 \leftrightarrow 2$, which is result of [Gr], and $2 \leftrightarrow 3$ which was proved in [Le]. We use similar arguments as in [Le] to prove also the remaining implications.

The case 3 was solved in [D]. The method used in [D] is based on computing Postnikov invariants. The method is explained in [Th]. We use the same approach to handle the remaining cases.

0.3.1 Global 3-forms of type $\omega_5, \omega_6, \omega_7, \omega_8$.

**Theorem 7.** Let $M$ be a closed connected 7-dimensional manifold. Then the following are equivalent:

1. $M$ is orientable and spinnable.
2. $M$ admits a global 3-form of algebraic type 8.
3. $M$ admits a global 3-form of algebraic type 5.
4. $M$ admits a global 3-form of algebraic type 6.
5. $M$ admits a global 3-form of algebraic type 7.

Proof: (1)$\leftrightarrow$(2) is given in [Gr]. (2)$\leftrightarrow$(3) is given in [Le]. (2)$\leftrightarrow$(4)$\leftrightarrow$(5). From the theorem 12 and the remark given below the theorem follows that a $G_2$-structure implies $SU(2)_{0,4}$-structure where $SU(2)_{0,4}$ is the subgroup $SU(2) \cong 1 \times SU(2) \subset SO(4)_{3,4}$ where $SO(4)_{3,4}$ is given by the formula (42). Thus $G_2$-structure implies also reduction to all subgroups which contain $SU(2)_{0,4}$. We have proved that $SU(2)_{0,4} \leq K_7, K_6 \leq G_2$. The rest follows from the theorem 13.

0.3.2 A global 3-form of type $\omega_4$.

We will consider only the orientable case. We first prove the following observation.

**Lemma 7.** Let $M$ be an orientable 7-dimensional closed connected manifold such that $M$ admits a global 3-form of type 4. Then $M$ admits a reduction to $U(1)$.

Proof: By assumption $M$ admits a reduction to $K_4 \cap SO(7) = SO(3)_{3,3}$. The formula (18) implies that the tangent bundle $TM$ decomposes as $TM \cong \xi^1 \oplus \rho_1^3 \oplus \rho_2^3$, where $\rho_1^3$ and $\rho_2^3$ are isomorphic real orientable 3-dimensional vector bundles. Moreover we can choose a complex structure $J$ on $\rho_1^3 \oplus \rho_2^3$ such that $J(\rho_1^3) = \rho_2^3$ and $J(\rho_2^3) = \rho_1^3$. This implies that $w_2(M) = w_2(\rho_1^3) + w_2(\rho_2^3)$ is zero and thus the manifold $M$ admits a $SU(2)$-structure, see the theorem 7 and the lemma 12. Since the Chern classes of a complex vector bundle do not depend on a choice of
a complex structure, see [MS], the first Chern class of the complex bundle $(\rho_1^3 \oplus \rho_2^3)$ is trivial and thus the bundle admits a SU(3)-structure. By a result of Thomas, see [T], $M$ has two everywhere linearly independent sections and thus the bundle $(\rho_1^3 \oplus \rho_2^3)$ has a nowhere zero section $\zeta$. Let us denote by $\xi^i = \langle \zeta, J\zeta \rangle$. Then $\rho_1^i \oplus \rho_2^3 \cong \xi^i \oplus \eta^i$ where $\eta^i$ is a complex 2-dimensional bundle with SU(2)-structure and the with complex structure $J|_{\eta^i}$. Let us denote $\eta^i := \eta^i \cap \rho_i^j$ for $i = 1, 2$. Then $\eta_1^1 \oplus \eta_2^3 = \eta^i$ is a splitting of $\eta^i$ into two complex line bundles. Then $c_1(\eta_1^1) = -c_1(\eta_2^3)$ and in particular $J$ is a complex anti-linear isomorphism of these two line bundles. □

Let us denote the subgroup in the lemma 7. isomorphic to U(1) by $L_4$. An inclusion $L_4 \hookrightarrow SO(7)$ factors through $L_4 \hookrightarrow SU(2) \hookrightarrow SO(7)$. The long exact sequence for the homotopy groups of the principal fibration $L_4 \rightarrow SO(7) \rightarrow Q_4$ yields that the homotopy groups of $Q_4 = SO(7)/L_4$ are $\pi_1(Q_4) \cong \mathbb{Z}_2$, $\pi_2(Q_4) \cong \mathbb{Z} \cong \pi_3(Q_4)$ and are trivial for $7 > i > 3$ and $i = 0$. Thus from the lemma [11] follows that it is sufficient to find a lift of $f : M \rightarrow SO(7)$ to $BL_4$ over a 4-skeleton $M^4$ of $M$ where $f$ is the tautological map.

Let us factor the fibration $p : BL_4 \rightarrow BSO(7)$ into fibrations $BL_4 \rightarrow BSU(2) \rightarrow BG_2 \rightarrow BSO(7)$. Recall that there exist a lift of $f$ to $BSU(2)$ iff $w_3(M) = 0$. Let us build a Postnikov tower for a fibration $q : BL_4 \rightarrow BSU(2)$. We have that $SU(2)/L_4 \cong S^2$ and thus we have

$$
\begin{array}{ccc}
S^2 & \xrightarrow{\alpha} & K(\mathbb{Z}, 2) \\
\downarrow & & \downarrow \\
\text{BL}_4 & \xrightarrow{\alpha} & BSU(2) \times K(\mathbb{Z}, 2) \\
\downarrow & & \downarrow \chi \times \pi \\
\text{BSU}(2) & \xrightarrow{\ast} & K(\mathbb{Z}, 3) \\
\end{array}
$$

(30)

Here the map $h$ has to be a homotopy equivalence $BL_4 \cong K(\mathbb{Z}, 2)$ such that $h^*(\alpha) = c_1$ where $\alpha \in H^2(K(\mathbb{Z}, 2), \mathbb{Z})$ and $c_1 \in H^2(L_4, \mathbb{Z})$ are generators. We may choose $c_1$ such that $c_2(\eta_2^3) = -c_1^2$ where $c_2 \in H^4(BSU(2), \mathbb{Z})$ is the second Chern class of the tautological bundle. Thus the second Postnikov invariant is $c_2 \otimes 1 - 1 \otimes c_1^2$.

**Theorem 8.** Let $M$ be a closed orientable manifold. Then $M$ admits a global 3-form of algebraic type 4 iff $w_3(M) = 0$ and there exist a class $e \in H^2(M, \mathbb{Z})$ such that $e^2 = 2p_1(M)$.

Proof: Let us assume that $w_3(M) = 0$ and that $M$ is orientable. Then the tangent bundle $TM$ of $M$ is isomorphic to $\eta^4 \oplus \xi^3$ where $\xi^3$ is trivial 3-dimensional bundle and $\eta^4$ has a SU(2)-structure. Then $-2c_2(\eta^4) = p_1(M)$. As we have argued above, the bundle $\eta^4$ decomposes as a sum of two complex line bundles iff $\exists e \in H^2(M, \mathbb{Z})$ such that $-e^2 = c_2(\eta^4)$. □

0.3.3 A global 3-form of type $\omega_3$.

This was solved in [D].

**Theorem 9.** Let $M$ be an orientable closed connected 7-dimensional manifold. Then $M$ admits a global 3-form of type 3 iff $\beta(w_3(M)) = 0$.

0.3.4 A global 3-form of type $\omega_2$.

We will consider reduction to the connected component of the identity of $K_2$. Explicit realization of $K_2^0$ is given in the formula [46]. In particular we see that $BK_2^0$ is an Eilenberg-MacLane space $K(\mathbb{Z} \times \mathbb{Z}, 2)$. In particular $H^2(BK_2^0, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$ and is generated by the Euler classes of two 2-dimensional bundles of the tautological bundle corresponding to the subgroups $((\eta)_2, (\rho)_2)$ in the formula [46]. Let us denote these generators by $\alpha, \beta$.

\[\text{Recall that SO(3) sits in U(3)}.\]
The homotopy groups $\pi_i(\text{SO}(7)/K_2^1)$ are zero for $4 < i < 7$. From the lemma \[\square\], as in the case of $\omega_3$, follows that it suffices to consider Postnikov invariants up to dimension four. The group $K_2^1$ is a subgroup of $G_2$ and thus the necessary condition is $w_2(M) = 0$. The group $K_2^0$ is also a subgroup of SU(3). Let us denote by $\omega_2$ the homogeneous space $Q_2 := \text{SU}(3)/K_2^0$. Then $\pi_1(Q_2) \cong \pi_0(Q_2) = 0, \pi_2(Q_2) = \mathbb{Z} \times \mathbb{Z}, \pi_3(Q_2) \cong \mathbb{Z}$. We consider a Postnikov tower

\[
Q_2 \xrightarrow{q} K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \xrightarrow{k_2} K(\mathbb{Z}, 4)
\]

for the fibration $Q_2 \to \text{BK}_2^0 \to \text{BSU}(3)$.

**Theorem 10.** Let $M$ be a 7-dimensional closed simply connected and connected manifold. Then $M$ admits a global 3-form of type $\omega_2$ iff $w_2(M) = 0$ and there exist classes $e, f \in H^2(M, \mathbb{Z})$ such that $e^2 + f^2 + ef = \frac{1}{2}p_1(M)$.

Proof: First we easily find that $q^* : H^4(\text{BSU}(3), \mathbb{Z}) \to H^4(\text{BK}_2^0, \mathbb{Z})$ is given by $q^*(c_2) = -(\alpha^2 + \beta^2 + \alpha \beta)$. This implies that Postnikov invariant $k_2$ is equal to $c_2 + \alpha^2 + \beta^2 + \alpha \beta$. The map $H^4(\text{BSO}(7), \mathbb{Z}) \to H^4(\text{BSU}(3), \mathbb{Z})$ is given $p_1 \mapsto -2c_2$. This gives the theorem. \[\square\]

### 0.3.5 A global 3-form of type $\omega_1$.

We will consider reduction to the connected component of the identity of $K_1$. The connected component of the identity $K_1^0$ is isomorphic to the two dimensional torus $\mathbb{T}^2$. In particular $\text{BK}_1^0$ is an Eilenberg-MacLane space $K(\mathbb{Z} \times \mathbb{Z}, 2)$. Let us denote by $\alpha, \beta$ the Euler classes of the two 2-dimensional vector bundles of the tautological bundle. The non-trivial $i$-th homotopy groups for $i < 7$ of the quotient $Q_1 := \text{SO}(7)/K_1^0$ are $i = 3, 4, \text{i.e. } \pi_3(Q_1) \cong \mathbb{Z} \times \mathbb{Z}, \pi_4(Q_1) \cong \mathbb{Z}$. The lemma \[\square\] implies that we may consider Postnikov invariants of dimension equal or smaller to four. We consider a Postnikov tower

\[
Q_1^1 
\xrightarrow{q_1} K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) 
\xrightarrow{i} E_0 
\xrightarrow{k_2} K(\mathbb{Z}, 4) 
\xrightarrow{p} K(\mathbb{Z}, 3) 
\xrightarrow{1 \times w_3 + w_3 \times 1} K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 3)
\]

for the fibration $Q_1 \to \text{BK}_1^0 \to \text{BSO}(7)$, where $W_3$ is the generator of $H^3(\text{BSO}(7), \mathbb{Z}) \cong \mathbb{Z}_2$.

**Theorem 11.** Let $M$ be a 7-dimensional closed simply connected and connected manifold. Then $M$ admits a global 3-form of type 1 iff there exist classes $e, f \in H^2(M, \mathbb{Z})$ such that $\rho_2(e+f) = w_2(M)$ and $e^2 + f^2 = p_1(M)$.

Proof: The Serre sequence gives exact sequences

\[
0 \to H^2(E_0, \mathbb{Z}) \to H^2(\text{BK}_1^0, \mathbb{Z}) \to 0 \\
0 \to H^2(E_0, \mathbb{Z}) \to H^2(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2), \mathbb{Z}) \to H^3(\text{BSO}(7)) \to 0
\]
In particular we may view $\alpha, \beta$ as generators of $H^2(E_0, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$ and up to a change of basis \{e_1, e_2\} of $H^2(K(2, 2) \times K(2, 2), \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$ we may also assume that $i^*\alpha = 2e_1, i^*\beta = e_2$. We also get a diagram

$$
\begin{array}{c}
H^4(BSO(7), \mathbb{Z}) \xrightarrow{p^*} H^4(E_0, \mathbb{Z}) \xrightarrow{c'} H^4(K(2, 2) \times K(2, 2), \mathbb{Z}) \\
0 \to H^3(Q^1_4, \mathbb{Z}) \xrightarrow{\tau} H^4(E_0, \mathbb{Z}) \xrightarrow{q^*} H^4(BK^0_1, \mathbb{Z}) \to 0,
\end{array}
$$

where the bottom row is exact and the upper row is a complex. In particular we obtain that $H^4(E_0, \mathbb{Z})$ is a free abelian group isomorphic to $\mathbb{Z}^4$ which is generated by $k_2, \alpha^2, \alpha\beta, \beta^2$. Now $q^*(p_1) = \alpha^2 + \beta^2 \in H^4(BK^0_1, \mathbb{Z})$ where $p_1 \in H^4(BSO(7), \mathbb{Z})$ is the first Pontryagin class which is a generator of $H^4(BSO(7), \mathbb{Z})$. Thus there exists an integer $a \in \mathbb{Z}$ such that $p_1 = ak_2 - \alpha^2 - \beta^2$. This implies that $q^*(ak_2) = 4e_1^2 + e_2^2$. But this implies that $a = \pm 1$. $\square$

### 0.4 Appendix.

#### 0.4.1 Algebras.

Let us recall the Cayley-Dickinson construction of $*$-algebras. Let $(A, *)$ be a $*$-algebra. We write the conjugation as $a \mapsto *a = \bar{a}$ for $a \in A$. If $A$ is an $*$-algebra, we define a new $*$-algebra $CD(A)$ such that $CD(A) \cong A \oplus A$ as a vector space with the multiplication and conjugation given by

$$
(a, b). (c, d) = (ac - \bar{d}b, bc + ad) \quad (a, b) = (\bar{a}, -b).
$$

For example $CD(\mathbb{R}) = \mathbb{C}, CD(\mathbb{C}) = \mathbb{H}, CD(\mathbb{H}) = \mathbb{O}, CD(\mathbb{H}) = \O$ where $\mathbb{R}$, resp. $\mathbb{C}$, resp. $\mathbb{H}$, resp. $\mathbb{O}$, resp. $\O$ denote the real numbers, resp. the complex numbers, resp. quaternions, resp. pseudo-quaternions, resp. octonions, resp. pseudo-octonions.

**Pseudo-quaternions.** The algebra $\mathbb{H}$ of pseudo-quaternions is isomorphic to the algebra $M(2, \mathbb{R})$ of real $2 \times 2$ matrices. Let us use the following notation

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad ij = k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Let us denote by $1_2$ the identity $2 \times 2$-matrix. Let us denote by $\SL(2, \mathbb{R}) = \{g \in \GL(2, \mathbb{R}) | \det(g) = \pm 1\}$. If $g \in \SL(2, \mathbb{R})$, then the conjugation by $g$ is an automorphism of the algebra $\mathbb{H}$.

**Octonions.** A choice of a subalgebra $A$ of $\mathbb{O}$ isomorphic to $\mathbb{H}$ and a vector $e$ in the orthogonal complement $A^\perp$ with respect to the standard scalar product on $\mathbb{O}$ gives isomorphism of $\mathbb{O}$ with the algebra $\mathbb{H} \oplus \mathbb{H}$ via

$$
(a + be) \in \mathbb{O} \leftrightarrow (a, b) \in \mathbb{H} \oplus \mathbb{H},
$$

with the multiplication and the conjugation given by the same formulas as in (33). Alternatively we may write octonions as pairs of quaternions $(p, q) \leftrightarrow p + eq$. In this case the multiplication is given by

$$
(a, b)(c, d) = (ac - \bar{d}b, cb + \bar{a}d).
$$
**Pseudo-octonions.** A choice of a subalgebra $\widetilde{A}$ of $\widetilde{\mathbb{O}}$ isomorphic to $\mathbb{H}$ and a vector $e$ in the orthogonal complement $\widetilde{A}^\perp$ with respect to the standard scalar product on $\widetilde{\mathbb{O}}$ gives isomorphism of $\widetilde{\mathbb{O}}$ with the algebra $\mathbb{H} \oplus \mathbb{H}$ via

$$(a + be) \in \widetilde{\mathbb{O}} \leftrightarrow (a, b) \in \mathbb{H} \oplus \mathbb{H},$$

with the multiplication

$$(a, b)(c, d) = (ac + db, cb + ad).$$

If we choose a subalgebra $A$ of $\widetilde{\mathbb{O}}$ isomorphic to $\tilde{\mathbb{H}}$ and $e$ as above, then we can identify

$$\tilde{\mathbb{H}} \oplus \tilde{\mathbb{H}} \cong \widetilde{\mathbb{O}},$$

$$(p, q) \mapsto p + eq.$$

### 0.4.2 3-forms from algebras.

Let us denote the group of automorphisms of the algebra $\widetilde{\mathbb{O}}$ associated to the standard scalar product is positive definite in the case of octonions and of signature $(3, 4)$ in the case of pseudo-octonions. Then the formula

$$\omega(a, b, c) = g(ab, c)$$

and

$$\tilde{\omega}(e, f, g) = \tilde{g}(ef, g)$$

for $a, b, c \in \mathbb{O}$ or $e, f, g \in \tilde{\mathbb{O}}$ defines a 3-form. In particular $\omega$ belongs to the orbit $[\omega_8]$ while $\tilde{\omega}$ belongs to the orbit $[\omega_5]$.

### 0.4.3 Subgroups of the exceptional groups $G_2$ and $\tilde{G}_2$.

Let us recall that the exceptional Lie group $G_2$ is the group of automorphisms of the algebra $\mathbb{O}$. Let us denote the group of automorphisms of the algebra $\widetilde{\mathbb{O}}$ by $\tilde{G}_2$.

**Subgroup $SO(4)_{3,4}$ in $G_2$ and in $\tilde{G}_2$.** Let us define an embedding of $SO(4)$ in $\text{End}(\mathbb{H}) \times \text{End}(\tilde{\mathbb{H}})$ by the formula

$$\phi : SU(2) \times \mathbb{Z}_2 \rightarrow SU(2) \rightarrow \text{End}(\mathbb{H}) \times \text{End}(\tilde{\mathbb{H}})$$

$$(\phi(a, b))(p, q) = (apa^{-1}, bqa^{-1}),$$

where $(p, q) \in \mathbb{H} \oplus \tilde{\mathbb{H}}$. Let us denote the image of $\phi$ by $SO(4)_{3,4}$. It is shown in [Y] that $SO(4)_{3,4} \subset G_2$ where we identify $\mathbb{O}$ with $\mathbb{H} \oplus \tilde{\mathbb{H}}$ as in [35].

If we identify $\widetilde{\mathbb{O}}$ with $\mathbb{H} \oplus \tilde{\mathbb{H}}$ as in the formula [37], then the formula [42] gives an embedding of $SO(4)$ in $G_2$. We denote the image of this embedding also as $SO(4)_{3,4}$.

**Subgroup $\widetilde{SL}(2, \mathbb{R})_{3,4}^2$ of $G_2$.** Let us denote by $\widetilde{SL}(2, \mathbb{R}) = \{ a \in \text{GL}(2, \mathbb{R}) | \text{det}(a) = \pm 1 \}$. Let us denote by $\widetilde{SL}(2, \mathbb{R})_{3,4}^2 := \{(a, b)|a, b \in \widetilde{SL}(2, \mathbb{R}), \text{det}(ab) = 1\}/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{ \pm(1_2, 1_2) \}$ and $1_2$ is the identity $2 \times 2$-matrix. We embed $\widetilde{SL}(2, \mathbb{R})_{3,4}^2$ in $\text{End}(\mathbb{H}) \times \text{End}(\tilde{\mathbb{H}})$ by the formula

$$(a, b)(p, q) := (apa^{-1}, aqb^{-1}),$$

with $a, b \in \widetilde{SL}(2, \mathbb{R})$ and $(p, q) \in \mathbb{H} \oplus \tilde{\mathbb{H}}$. As in [42], one can easily verify that $\widetilde{SL}(2, \mathbb{R})_{3,4}^2$ is a subgroup of $G_2$ where we identify $\widetilde{\mathbb{O}}$ with $\mathbb{H} \oplus \tilde{\mathbb{H}}$ as in [39].

From the lemma [42] follows that the group

$$S(O(2) \times \mathbb{Z}_2 O(2)) := \{(a, b)|a, b \in O(2), \text{det}(ab) = 1\} \subset \widetilde{SL}(2, \mathbb{R})_{3,4}^2$$

(45)
is a maximal compact subgroup of $\text{SL}(2, \mathbb{R})_3^2$. We denote this group by $T^2_2,2$. The connected component of the identity of $T^2_2,2$ is realized with respect to the standard basis of $Im(\tilde{\mathcal{O}})$ by matrices

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & (-2\alpha)_2 & 0 & 0 \\
0 & 0 & (\alpha + \beta)_2 & 0 \\
0 & 0 & 0 & (\alpha - \beta)_2
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & (-\theta - \rho)_2 & 0 & 0 \\
0 & 0 & (\theta)_2 & 0 \\
0 & 0 & 0 & (\rho)_2
\end{pmatrix}, \quad (46)
$$

where $(\gamma)_2 \in \text{SO}(2)$ denotes rotation by the angle $\gamma \in \mathbb{R}$ and $\alpha + \beta = \theta, \alpha - \beta = \rho$. In particular the connected component of $T^2_2,2$ is isomorphic to $\text{SO}(2) \times \text{SO}(2)$.

0.4.4 Semi-direct product of Groups.

**Lemma 8.** Let $N \subset \text{GL}(n, \mathbb{R})$ be a closed subgroup. Suppose that $N$ consists of matrices of the form $1_n + A$ where $1_n$ is the identity matrix and $A$ is a strictly upper (lower) triangular matrix. Then the only subgroup $N'$ of $N$ for which there exists an $N'$-invariant scalar product on $\mathbb{R}^n$ is the trivial subgroup. Thus the maximal compact subgroup of $N$ is the trivial subgroup.

Proof: Easy exercise.$\square$.

**Lemma 9.** Let $G_0 = G_1 \rtimes G_2$ be a semi-direct product of Lie groups. Let $K_0 \subset G_0$ be a maximal compact subgroup. Then $K_1 \rtimes K_2$ is a maximal compact subgroup of $G_0$ where $K_1 = K_0 \cap G_1$, resp. $K_2 = K_0 \cap G_2$ is a maximal compact subgroup of $G_1$, resp. $G_2$.

Proof: Clearly $K_i \subset G_i$, $i = 1, 2$ is a maximal compact subgroup and thus also their semi-direct product is compact. Suppose that $K_3$ is a compact subgroup of $G_0$ such that $K_1 \rtimes K_2 \subset K_3$. Then $K_3/(K_3 \cap G_1) \subset K_2$ and also $K_3 \cap G_1 \subset K_1$ but this implies that $K_3 \subset K_1 \rtimes K_2$. $\square$

0.4.5 More on $G_2$-structure.

A $G_2$-structure on a closed connected 7-dimensional Riemannian manifold $M$ is a reduction of structure group of the tangent bundle $TM$ of $M$ to $G_2$, i.e. it is given by a following commutative diagram of principle bundles

$$
\begin{array}{cc}
G_2 & \longrightarrow \mathcal{O} & \longrightarrow & M \\
\downarrow & & \downarrow l_d \\
\text{SO}(7) & \longrightarrow \mathcal{P} & \longrightarrow & M
\end{array}
$$

where $\mathcal{P}$ is the principle bundle of orthonormal frames with respect to some Riemannian metric on $M$. It is a well known fact that existence of a reduction to $G_2$ is equivalent to vanishing of the second Stiefel-Whitney class $w_2$ of $TM$, see [Gr].

**Octonionic structure.** An octonionic structure on a 7-dimensional Riemannian manifold $(M, g)$ is a smooth bundle map

$$
\mu : (TM \oplus R) \otimes (TM \oplus R) \rightarrow (TM \oplus R), \quad (48)
$$

where $R$ is a trivial real line bundle over $M$ such that $\forall x \in M$ there exists an algebra isomorphism between $(\mathcal{O}, \cdot)$ and $(\mathbb{R}_x \oplus T_x M, \mu_x)$ compatible with the metric structures.

**Lemma 10.** Let $M$ be a 7-dimensional closed connected manifold. Then the following are equivalent:

1. there exists a $G_2$-structure on $M$.
2. there exists an octonionic structure in the fibers of $TM$.
Proof: See [D].
In the paper [FKMS] can be found the following theorem.

**Theorem 12.** Let \( M \) be a 7-dimensional closed connected manifold. Then following are equivalent:
1. there exists a \( G_2 \)-structure on \( M \).
2. there exists a \( SU(2) \)-structure on \( M \), i.e. \( TM \cong \xi^3 \oplus \eta^2 \) where \( \xi^3 \) is a trivial 3-dimensional bundle and \( \eta^2 \) is a complex 2-dimensional bundle with \( SU(2) \)-structure.

Proof: See [FKMS]. The assumption on compactness of \( M \) is not necessary. □

**Remark 1.** A \( SU(2) \)-structure coming from a \( G_2 \)-structure can be viewed in the following way. By a result from [T], any orientable 7-dimensional manifold admits two everywhere linearly independent vector fields. Let us denote them \( \zeta_1, \zeta_2 \). We may assume that \( \zeta_1, \zeta_2 \) are orthonormal. Since \( G_2 \)-structure is equivalent to an octonionic structure, the vector field \( \zeta_1, \zeta_2 = \zeta_3 \) is also of unit length and orthogonal to \( \zeta_1, \zeta_2 \). The vector fields \( \{ \zeta_1, \zeta_2, \zeta_3 \} \) span a 3-dimensional vector subbundle \( \xi^3 \) of the tangent bundle. The orthogonal complement to \( \xi^3 \) is a four-dimensional real vector bundle. The multiplication by \( \zeta_1, \zeta_2, \zeta_3 \) gives quaternionic structure on this bundle and thus also reduction to \( SU(2) \).

### 0.4.6 Extensions.

**Theorem 13.** Let \( G \) be a topological group and let \( H \leq G \) be a closed subgroup. Let \( P \) be a principal \( H \)-bundle over a topological space \( X \). Then there exists a principal \( G \)-bundle \( P' \) over \( X \) such that the diagram

\[
\begin{array}{ccc}
H & \rightarrow & P \\
\downarrow & & \downarrow \text{Id} \\
G & \rightarrow & P'
\end{array}
\]

commutes.

Proof: See [Hu].

**Extension of a lift from subcomplex.** Let \( p : X \rightarrow Y \) be a fibration with fiber \( F \) with a Postnikov tower \( \{X_n, q_n\} \) and let \((W,A)\) be a CW-complex and let \( i : A \rightarrow W \) be the canonical inclusion. Suppose that we have a map \( f : W \rightarrow Y \) and a lift \( F' : A \rightarrow Y \) such that \( p \circ F' = f \circ i \). The picture is given in the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{F'} & X \\
\downarrow \text{p} & & \downarrow \text{q}_1 \\
W & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \text{q}_2 \\
X_1 & \xrightarrow{p_1} & K(\pi_1(F), 2) \\
\downarrow \text{q}_3 \\
X_2 & \xrightarrow{p_2} & K(\pi_2(F), 3) \\
\end{array}
\]

**Lemma 11.** Let us keep the notation as above. Suppose that \( H^{i+1}(W, A, \pi_i(F)) = 0 \) for all \( i \geq 0 \). Then there exist a lift \( F : W \rightarrow X \) such that \( F \circ i = F' \) and \( p \circ F = f \).

Proof: Suppose that we have a lift \( F_i : W \rightarrow X_i \) such that \( q_1 \circ \ldots \circ q_i \circ F_i = f \) and \( p_i \circ F' = F_i \circ i \). Then the obstruction of lifting \( F_i \) to \( F_{i+1} : W \rightarrow X_{i+1} \) such that \( q_{i+1} \circ F_{i+1} = F_i \) and \( p_{i+1} \circ F' = F_{i+1} \circ i \) is a class \( \omega_i \in H^{i+1}(W, A, \pi_i(F)) \). Thus if \( \omega_i = 0 \) for all \( i \), we can find an extension \( W \rightarrow \lim X_i \) and further to \( W \rightarrow X \). Full discussion is in [H]. □
Bibliography

[BV] Jarolím Bureš; Jiří Vanžura: Multisymplectic forms of degree three in dimension seven, Proceedings of the 22nd Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2003. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 71. pp. [73]–91

[CV] Martin Čadek; Jiří Vanžura: On the classification of oriented vector bundles over 9-complexes, Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1994. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 37. pp. [33]–40.

[D] Martin Doubek: Structures defined by 3-forms on 7-dimensional manifolds, Diploma Thesis, Charles University 2008

[FKMS] T. Friedrich, I. Kath, A. Moroianu, U. Semmelmann: On nearly parallel $G_2$-structures, Journal of Geometry and Physics 23 (1997) 259-286 DOI: 10.1016/S0393-0440(97)80004-6

[Gr] A. Gray: Vector cross product on manifolds, Trans. Amer. Math. Soc. 141 (1969), pp. 465-504

[G] Robert Greenblatt: Homology with local coefficients and characteristic classes, Homology, Homotopy and Applications, vol. 8(2), 2006, pp.91-103

[H] Allan Hatcher: Algebraic topology, Cambridge University Press, 2002

[Hu] Husseinzadeh: Fibre Bundles, Graduate Texts in Mathematics, Springer-Verlag

[Le] Hông-Vân Lê: Manifolds admitting a $G_2$-structure, arXiv:0704.0503v1

[MS] Milnor, Stasheff: Characteristic classes, Princeton University Press and University of Tokyo Press

[T] E. Thomas: Vector Fields on Low Dimensional Manifolds, Math. Zeitschr. 103, 85–93 (1968)

[Th] E. Thomas: Seminar on Fibre Bundles, Lecture Notes in Mathematics, Springer-Verlag

[Y] Ichiro Yokota: Exceptional Lie groups, arXiv:0902.0431v1 [math.DG] 3 Feb 2009.