SOME PROPERTIES OF ANTI-KÄHLER MANIFOLDS EQUIPPED WITH QUARTER-SYMMETRIC $F$-CONNECTIONS

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Abstract. We construct quarter-symmetric metric and nonmetric $F$-connections on anti-Kähler manifolds and analyze some properties of torsion and curvature tensors of these connections.

1. Introduction

The idea of metric connection with torsion tensor on a Riemannian manifold was introduced by Hayden [3]. Later, Yano [13] considered a semi-symmetric metric connection and studied some of its properties. Golab [2] defined and studied quarter-symmetric linear connections on differentiable manifolds, which generalizes the idea of semi-symmetric connection. After that, Rastogi [7, 8, 9] continued the systematic study of quarter-symmetric metric connection. In 1980, Mishra and Pandey [6] defined a quarter-symmetric $F$-connection and studied the conditions for Einstein manifolds, Sasakian manifolds and Kähler manifolds equipped with this connection to be flat, projectively flat or conharmonically flat. In 1982, Yano and Imai [12] obtained the most general expression for quarter-symmetric metric connections on Riemannian, Hermitian and Kählerian manifolds and gave some special examples of them. Recently, Chaubey and Ojha [1] proved that an Einstein manifold admitting a quarter-symmetric $F$-connection whose Ricci tensor vanishes is conformally flat.

Our aim is to systematically study quarter-symmetric metric and nonmetric $F$-connections on anti-Kähler manifolds by focusing on properties of their torsion and curvature tensors.

2. Preliminaries

Let $M_n$ be an $n = 2m$-dimensional differentiable manifold of class $C^\infty$ covered by any system of coordinate neighbourhoods $(x^h)$, where here and in the sequel the 2010 Mathematics Subject Classification: Primary 53B20, 53B15; Secondary 53C15.

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indices $h, i, j, k, \ldots$ run over the range $1, 2, \ldots, n$. Also note that summation over repeated indices is always implied.

An almost complex structure $F = (F_i^k)$ on $M_n$ is a $(1, 1)$-tensor field on $M_n$ such that $F_i^k F_k^l = -\delta_i^l$. The pair $(M_n, F)$ is called an almost complex manifold. When the almost complex structure $F$ is integrable, it is called a complex structure, and $(M_n, F)$ is a complex manifold. A pseudo-Riemannian manifold $(M_n, g)$, endowed with an almost complex structure $F$, satisfying the relations

$$F_{ij}^k g_{kj} = F_j^k g_{ki}$$

and

$$\nabla_k F_{ij} = 0$$

is called an anti-Kähler manifold (or a Kähler-Norden manifold), where $\nabla_k$ denotes the operator of covariant derivation w.r.t. the Levi-Civita connection of $g$. It is well known that the condition $\nabla_k F_{ij} = 0$ is equivalent to holomorphicity (analyticity) of the pseudo-Riemannian metric $g$ [4, 10], i.e., $\phi_F g = 0$, where $\phi_F$ is the Tachibana operator. Also note that the first condition in (2.1) is the purity condition of the pseudo-Riemannian metric $g$ w.r.t. the almost complex structure $F$.

In general, any $(p, q)$-tensor $K$ with components $K_{i_1 i_2 \ldots i_q}^{j_1 j_2 \ldots j_p}$ on an almost complex manifold $(M_n, F)$ satisfying the relations

$$K_{i_1 i_2 \ldots i_q}^{j_1 j_2 \ldots j_p} F_{i_1}^m F_{i_2}^m \ldots = K_{i_1 i_2 \ldots i_q}^{j_1 j_2 \ldots j_p} F_{i_1}^m F_{i_2}^m \ldots = K_{i_1 i_2 \ldots i_q}^{j_1 j_2 \ldots j_p} F_{i_1}^m F_{i_2}^m \ldots = K_{i_1 i_2 \ldots i_q}^{j_1 j_2 \ldots j_p} F_{i_1}^m F_{i_2}^m \ldots$$

(purity condition) and

$$(\phi_F K)_{i_1 \ldots i_q}^{j_1 \ldots j_p} = F_{i_1}^m \partial_m (K_{i_1 \ldots i_q}^{j_1 \ldots j_p})$$

$$(\phi_F K)_{i_1 \ldots i_q}^{j_1 \ldots j_p} = F_{i_1}^m \partial_m (K_{i_1 \ldots i_q}^{j_1 \ldots j_p})$$

$$(\phi_F K)_{i_1 \ldots i_q}^{j_1 \ldots j_p} = F_{i_1}^m \partial_m (K_{i_1 \ldots i_q}^{j_1 \ldots j_p})$$

(Tachibana operator) is called as a holomorphic (analytic) tensor w.r.t. the almost complex structure $F$ [11] (see also [10] and [14]). We recall that the Riemannian curvature tensor $R$ of an anti-Kähler manifold is a holomorphic tensor [4, 10].

3. Quarter-symmetric metric $F$-connection

Let $(M_n, g, F)$ be an anti-Kähler manifold. A linear connection $\tilde{\nabla}$ with components $\tilde{\Gamma}_{ij}^k$ in the anti-Kähler manifold $(M_n, g, F)$ satisfying the relations

$$\tilde{\nabla}_k g_{ij} = 0,$$

and

$$\tilde{\nabla}_k F_{ij} = 0$$

is called a metric $F$-connection. When we consider a metric $F$-connection whose torsion tensor is in the form

$$(3.1) \quad \tilde{S}_{ij}^k = p_j F_i^k - p_i F_j^k + p_i F_j^l \delta_i^k - p_i F_i^l \delta_j^k,$$

standard calculations give the components $\tilde{\Gamma}_{ij}^k$ of the metric $F$-connection as follows

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + p_j F_i^k - p_i F_j^k + p_i F_j^l \delta_i^k - p_i F_i^l \delta_j^k,$$

where $p_i$ and $p^k$ are covariant and contravariant components of any covector field $p$ and $F_{ij} = F_i^k g_{kj}$. This covector field $p$ is called the generator of the
metric $F$-connection. We shall call such a connection a quarter-symmetric metric $F$-connection.

Let $f$ be a function on $M_n$. It is locally holomorphic if \[ (\phi_F(df))_{k\ell} = F_k^m(\partial_m f) - \partial_k(F_m^p f) + (\partial_i F_k^i)\partial_m f \]

Substituting (3.1) into the above relation, it follows that

\[
F_k^m(\partial_m f) - \partial_k(F_m^p f) = F_k^m(\nabla_m f) - F_m^s(\nabla_k f) = F_k^m(\nabla_m f) - F_m^s(\nabla_k f) = 0.
\]

For the sake of simplicity, we specialize the quarter-symmetric metric $F$-connection in such a way that its generator $p_i$ is gradient of a locally holomorphic function $f$ on $M_n$. In what follows, we call such a quarter-symmetric metric $F$-connection simply a quarter-symmetric metric $F$-connection.

3.1. Properties of the torsion tensor. The section deals with some properties concerning the torsion tensor of the quarter-symmetric metric $F$-connection.

Theorem 3.1. The torsion tensor $\tilde{S}$ of the quarter-symmetric metric $F$-connection is a pure tensor w.r.t. the complex structure $F$.

Proof. The result follows by a straightforward calculation.

In [10], it was proved that an $F$-connection is pure if and only if its torsion tensor is pure. Thus we can say that the quarter-symmetric metric $F$-connection is pure w.r.t. the complex structure $F$.

Theorem 3.2. The torsion tensor $\tilde{S}$ of the quarter-symmetric metric $F$-connection is a holomorphic tensor.

Proof. In the anti-Kähler manifold $(M_n, g, F)$, denote by $\nabla$ the Levi-Civita connection of $g$. It is well-known that a torsion-free $F$-connection is always pure. Hence the Levi-Civita connection $\nabla$ on $M_n$ is pure w.r.t. the complex structure $F$.

The Tachibana operator $\phi_F$ applied to the torsion tensor $\tilde{S}$ can be expressed in the form

\[
(\phi_F \tilde{S})_{k\ell} = F_k^m(\partial_m \tilde{S}_{\ell ij}) - \partial_k(\tilde{S}_m^{\ell ji} F_m^l).
\]

Substituting (3.1) into the above relation, it follows that

\[
(\phi_F \tilde{S})_{k\ell} = F_k^m(\nabla_m p_j) - F_m^s(\nabla_k p_m) = F_k^m(\nabla_m p_j) - F_m^s(\nabla_k p_m) = 0.
\]

Using the fact that the generator $p_i$ is a gradient, we get $(\phi_F \tilde{S})_{k\ell} = 0$ i.e., the torsion tensor $\tilde{S}$ is holomorphic. □
Theorem 3.3. The torsion tensor \( \bar{S} \) w.r.t. the quarter-symmetric metric \( F \)-connection is recurrent, i.e., \( \bar{\nabla}_k S^i_{jkl} = \omega_k S^i_{jkl} \) if and only if the generator \( p \) is recurrent w.r.t. the quarter-symmetric metric \( F \)-connection, where \( \omega_k \) is the recurrence covector field.

Proof. Let the torsion tensor \( \bar{S} \) be recurrent w.r.t. the quarter-symmetric metric \( F \)-connection, i.e., \( \bar{\nabla}_k S^i_{jkl} = \omega_k S^i_{jkl} \). By contracting this w.r.t. \( i \) and \( l \), we obtain
\[
\bar{\nabla}_k \bar{S}^i_{jkl} = \omega_k \bar{S}^i_{jkl}, \quad \bar{\nabla}_k [(n-2)p_i F^i_j] = \omega_k [(n-2)p_i F^i_j],
\]
\[
(\bar{\nabla}_k p_i) F^i_j = \omega_k p_i F^i_j, \quad \bar{\nabla}_k p_j = \omega_k p_j
\]
which means that the generator \( p_i \) is recurrent.

Conversely, let the generator \( p_i \) be recurrent. On taking covariant derivative of (3.3) w.r.t. the quarter-symmetric metric \( F \)-connection, we get
\[
\bar{\nabla}_k \bar{S}^i_{jkl} = (\bar{\nabla}_k p_i) F^i_j - (\bar{\nabla}_k p_i) F^i_j \delta^i_j + (\bar{\nabla}_k p_i) F^i_j \delta^i_j
\]
\[
= \omega_k p_i F^i_j - \omega_k p_i F^i_j - \omega_k p_i F^i_j \delta^i_j + \omega_k p_i F^i_j \delta^i_j = \omega_k \bar{S}^i_{jkl}.
\]
\[
\square
\]

3.2. Properties of the curvature tensor. If \((M_n, g, F)\) is the anti-Kähler manifold, then denote by \( \bar{R}_{ijk}^{l} \) and \( R_{ijk}^{l} \) its curvature tensors of the quarter-symmetric metric \( F \)-connection and the Levi-Civita connection, respectively. The curvature \((0, 4)\)-tensor \( \bar{R}_{ijkl} = \bar{\nabla}_{ijkl} g^{kl} \) can be expressed in the form
\[
\bar{R}_{ijkl} = R_{ijkl} + F_j \sigma_{ik} - F_i \sigma_{jk} + F_k \sigma_{ij} - F_{jk} \sigma_{il} + F_j \sigma_{ik} - F_i \sigma_{jk} + F_k \sigma_{ij} - F_{jk} \sigma_{il},
\]
where we use the following abbreviation
\[
(\bar{\nabla}_p F^m) g_{jk} = \bar{\nabla}_p F^m g_{jk} - p_j p_m F^m_k - p_k p_m F^m_j + \frac{1}{2} p_m p_t F^m t g_{jk}.
\]
It is easy to see that \( \sigma_{jk} - \sigma_{kj} = \bar{\nabla}_f \bar{\nabla}_j f - \bar{\nabla}_k \bar{\nabla}_j f = 0 \). Also, it is not too hard to see that the curvature \((0, 4)\)-tensor \( \bar{R} \) satisfies \( \bar{R}_{ijkl} = -\bar{R}_{ijlk} \), \( \bar{R}_{ijkl} = -\bar{R}_{ikjl} \), \( \bar{R}_{ijk} k = 0 \), \( \bar{R}_{ijkl} = \bar{R}_{kijl} \), \( \bar{R}_{ijkl} + \bar{R}_{kijl} + \bar{R}_{ijkl} = 0 \).

Proposition 3.1. The \((0, 2)\)-tensor \( \sigma \) given by (3.3) is a holomorphic tensor.

Proof. Taking into account (3.3), we infer
\[
F_k \sigma_{il} - F_i \sigma_{lk} = (\bar{\nabla}_p F^m) F^m_{kj} - (\bar{\nabla}_i p_k) F^i_{j} = (\bar{\nabla}_i \bar{\nabla}_j f) F_k^i - (\bar{\nabla}_i \bar{\nabla}_k f) F_i^j = 0,
\]
i.e., the \((0, 2)\)-tensor \( \sigma \) is pure w.r.t. the complex structure \( F \).

The Tachibana operator applied to the \((0, 2)\)-tensor \( \sigma \) in the anti-Kähler manifold \((M_n, g, F)\) is given by
\[
(\phi_F \sigma)_{kij} = (\bar{\nabla}_m \sigma_{ij}) F^m_k - (\bar{\nabla}_k \sigma_{im}) F^m_j.
\]
Substituting (3.3) into the last relation, we find
\[
(\phi_F \sigma)_{kij} = (\bar{\nabla}_m \bar{\nabla}_p F^p_j) F^m_k - (\bar{\nabla}_k \bar{\nabla}_m p_j) F^m_p.
\]
On the other hand, we write down the Ricci identity for the generator $p_i$

$$(\nabla_m \nabla_i p_j) F^m_k = (\nabla_i \nabla_m p_j) F^m_k - \frac{1}{2} p_s R^s_{mij} F^m_k,$$

$$(\nabla_k \nabla_i p_m) F^m_j = (\nabla_i \nabla_k p_m) F^m_j - \frac{1}{2} p_s R^s_{kim} F^m_j.$$ 

Finally, in view of the above two relations, (3.4) becomes

$$(\phi_F \sigma)_{kij} = -\frac{1}{2} p_s (R^s_{mij} F^m_k - R^s_{kim} F^m_j) = 0. \qed$$

**Theorem 3.4.** The curvature $(0,4)$-tensor $\tilde{R}$ of the quarter-symmetric metric $F$-connection is a holomorphic tensor.

**Proof.** With the help of the purity of the $(0,2)$-tensor $\sigma$, it follows immediately that

$$\tilde{R}_{mijl} F^m_{ij} = \tilde{R}_{smkl} F^m_{jl} = \tilde{R}_{ijkl} F^m_{lm} = \tilde{R}_{ijkl} F^m_{lm}.$$ 

In the anti-Kähler manifold $(M_\alpha, g, F)$, the Tachibana operator $\phi_F$ applied to the curvature $(0,4)$-tensor $\tilde{R}$ is

$$(\phi_F \tilde{R})_{kijl} = F^m_i (\partial_m \tilde{R}_{ijkl}) - \partial_k (\tilde{R}_{ijlm} F^m_l)$$

$$= F^m_i (\nabla_m \tilde{R}_{ijkl} + \Gamma^s_{ml} \tilde{R}_{kjsl} + \Gamma^s_{mj} \tilde{R}_{klsi} + \Gamma^s_{mi} \tilde{R}_{kjsl})$$

$$- F^m_i (\nabla_k \tilde{R}_{ijlm} + \Gamma^s_{kl} \tilde{R}_{ijms} + \Gamma^s_{js} \tilde{R}_{iklm} + \Gamma^s_{iks} \tilde{R}_{ijlm})$$

$$= (\nabla_m \tilde{R}_{ijkl}) F^m_i - (\nabla_k \tilde{R}_{ijlm}) F^m_l.$$ 

By Proposition 3.2, we find

$$(\phi_F \tilde{R})_{kijl} = (\phi \tilde{R})_{kijl}$$

$$+ [(\nabla_m \sigma_{ij}) F^m_k - (\nabla_k \sigma_{ij}) F^m_i] F^m_j - [(\nabla_m \sigma_{jl}) F^m_k - (\nabla_k \sigma_{jm}) F^m_i] F^m_l$$

$$+ [(\nabla_m \sigma_{jl}) F^m_k - (\nabla_k \sigma_{jm}) F^m_i] F^m_i - [(\nabla_m \sigma_{ij}) F^m_k - (\nabla_k \sigma_{jm}) F^m_i] F^m_j$$

$$+ [(\nabla_m \sigma_{ij}) F^m_k - (\nabla_k \sigma_{jm}) F^m_i] F^m_i + (\nabla_k \sigma_{ij}) F^m_i - (\nabla_m \sigma_{ij}) F^m_i$$

Since the $(0,2)$-tensor $\sigma$ is holomorphic, it satisfies

$$(\nabla_m \sigma_{ij}) F^m_k = (\nabla_k \sigma_{ij}) F^m_i = (\nabla_k \sigma_{jm}) F^m_i$$

(see the proof of Proposition 3.1). Hence, the last relation becomes $(\phi_F \tilde{R})_{kijl} = 0$, i.e., the curvature $(0,4)$-tensor $\tilde{R}$ is a holomorphic tensor. \qed

**Example 3.1.** The pseudo-Euclidean space $\mathbb{R}^{2n}$ is given by pseudo-Euclidean metric $(g_{\alpha \beta}) = \left( \begin{array}{cc} I_n & 0 \\ 0 & -I_n \end{array} \right)$ Let $\mathbb{C}^n$ be the complex space. The usual identification $r$ of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ is given by

$$r : z = (z^1, z^2, \ldots, z^n) \in \mathbb{C}^n \rightarrow r(z) = (x^1, x^2, \ldots, x^n; y^1, y^2, \ldots, y^n) \in \mathbb{R}^{2n}$$

where $z^k = x^k + iy^k$, $k = 1, \ldots, n$. The canonical complex structure $F$ on $\mathbb{R}^{2n}$ is determined by the matrix

$$(F^\alpha_{\beta}) = \left( \begin{array}{cc} 0 & \delta^i_j \\ -\delta^i_j & 0 \end{array} \right) \quad \text{or} \quad (F_{\alpha \beta}) = \left( \begin{array}{cc} 0 & \delta_{ij} \\ \delta_{ij} & 0 \end{array} \right)$$
are curvature tensor \(\tilde{R}\) w.r.t. the natural basis of \(\mathbb{R}^{2n}\). In the example, Greek indices take on values 1 to 2\(n\). For all \(Z, W\) on \(\mathbb{R}^{2n}\) the metric \(g\) and the complex structure \(F\) on \(\mathbb{R}^{2n}\) are related by the equality \(g(FZ, FW) = -g(Z, W)\), that is, \(g\) is pure w.r.t. \(F\). Hence \((\mathbb{R}^{2n}, g, F)\) is an anti-Kähler Euclidean space.

The components of the quarter semisymmetric metric \(F\)-connection in \((\mathbb{R}^{2n}, g, F)\) are
\[
\tilde{g}^{ij} = \tilde{g}^{ik}_{
 \frac{\partial}{\partial x^i}} = \tilde{g}^i_{
 \frac{\partial}{\partial x^i}} = -(\partial_f \delta^k_i - (\partial_h \delta^k_i) \delta^m_i \delta_{ij},
\]
\[
\tilde{g}_{ij} = \tilde{g}_{ij} = \tilde{g}_{ij} = -(\partial_f \delta^k_i - (\partial_h \delta^k_i) \delta^m_i \delta_{ij}.
\]

The torsion tensor of the complex symmetric metric \(F\)-connection has the components
\[
\tilde{S}^{k}_{ij} = \tilde{S}^{k}_{ij} = \tilde{S}^{k}_{ij} = -(\partial_f \delta^k_i - (\partial_h \delta^k_i) \delta^m_i \delta_{ij},
\]
\[
\tilde{S}^{k}_{ij} = \tilde{S}^{k}_{ij} = \tilde{S}^{k}_{ij} = -(\partial_f \delta^k_i - (\partial_h \delta^k_i) \delta^m_i \delta_{ij}.
\]
One verifies that the torsion tensor \(\tilde{S}\) is pure w.r.t. \(F\) and moreover \((\phi F \tilde{S})^\gamma_{\sigma \alpha \beta} = 0\), i.e., \(\tilde{S}\) is holomorphic.

The components of the curvature \((0, 4)\)-tensor \(\tilde{R}\) of the quarter semi-symmetric metric \(F\)-connection are
\[
\tilde{R}^{ijkl} = \tilde{R}^{ijkl} = \tilde{R}^{ijkl} = \tilde{R}^{ijkl} = \tilde{R}^{ijkl} = \tilde{R}^{ijkl} = \delta_{ij} \sigma_{kl} - \delta_{kl} \sigma_{ij} + \delta_{ki} \sigma_{lj} - \delta_{kij} \sigma_{l},
\]
\[
\tilde{R}^{ijkl} = \tilde{R}^{ijkl} = \tilde{R}^{ijkl} = \tilde{R}^{ijkl} = \tilde{R}^{ijkl} = \tilde{R}^{ijkl} = \delta_{ij} \sigma_{kl} - \delta_{kl} \sigma_{ij} + \delta_{ki} \sigma_{lj} - \delta_{kij} \sigma_{l},
\]
where
\[
\sigma_{ij} = \sigma_{ij} = \delta_{ij} \partial_f + (\partial_f \partial_f) - (\partial_k \partial_f) \delta_{ij} + \frac{1}{2} \delta^h \phi \delta^m \delta_{ij} \frac{1}{2} \delta^h \phi \delta^m \delta_{ij},
\]
\[
\sigma_{kj} = -\sigma_{ij} = \delta_{kj} \partial_f + (\partial_f \partial_f) - (\partial_k \partial_f) \delta_{ij} + \frac{1}{2} \delta^h \phi \delta^m \delta_{ij} \frac{1}{2} \delta^h \phi \delta^m \delta_{ij}.
\]
Simple calculations show that \((\phi F \sigma)_{\sigma \alpha \beta} = 0\). Using this, one checks that the curvature tensor \(\tilde{R}\) is pure w.r.t. \(F\) and furthermore \((\phi F \tilde{R})_{\sigma \alpha \beta} = 0\), i.e., \(\tilde{R}\) is holomorphic.

Denote by \(\tilde{R}_{jk} = \tilde{R}_{ijkl} g^{il}\) the Ricci tensor of the quarter-symmetric metric \(F\)-connection and \(\tilde{R}_{jk}\) the Ricci tensor of the Levi-Civita connection. Then the Ricci tensor has the components
\[
(3.5) \quad \tilde{R}_{jk} = R_{jk} + (4 - n) F_k^l \sigma_{lj} - g_{jk} F_i^m \sigma^m_l - F_{jk} (\text{trace } \sigma).
\]
It is easy to see that \(\tilde{R}_{jk} = \tilde{R}_{kj}\). In fact,
\[
\tilde{R}_{jk} = \tilde{R}_{kj} = (4 - n) F_k^l (\sigma_{lj} - \sigma_{lj}).
\]
The scalar curvature $\tilde{\tau} = \tilde{R}_{ik}\delta^{ik}$ of the quarter-symmetric metric $F$-connection is given by $\tilde{\tau} = \tau + 2(2 - n)F_{l}^{m}\sigma_{m}^{l}$, where $\tau$ is the Riemann scalar curvature of $g$.

The conharmonic curvature tensor w.r.t. the quarter-symmetric metric $F$-connection is

$$
\tilde{V}_{ijkl} = \tilde{R}_{ijkl} - \frac{1}{n - 2}[\tilde{R}_{ik}g_{jl} - \tilde{R}_{ik}g_{jl} - \tilde{R}_{jl}g_{ik} + \tilde{R}_{id}g_{jk}].
$$

It follows from (3.5) that

$$
(3.6) \quad \tilde{V}_{ijkl} = V_{ijkl} + F_{jlt}\sigma_{ik} - F_{it}\sigma_{jk} + F_{ik}\sigma_{jl} - F_{jl}\sigma_{ik}
$$

where $\tilde{V}_{ijkl}$ is the conharmonic curvature tensor w.r.t. the Levi-Civita connection. It is easy to see that it satisfies the relations

$$
\tilde{V}_{ijkl} = -\tilde{V}_{jikl}, \quad \tilde{V}_{ijkl} = -\tilde{V}_{klij}, \quad \tilde{V}_{ijkl} + \tilde{V}_{klij} + \tilde{V}_{ijkl} = 0.
$$

**Theorem 3.5.** If the conharmonic curvature tensor w.r.t. the quarter-symmetric metric $F$-connection vanishes, then

$$
(\nabla_{i}\nabla_{l}f)F_{l}^{i} + \frac{\tau}{2(2 - n)} + \frac{4 - n}{2}(\nabla_{i}f)(\nabla^{i}f) = 0
$$

where $\tau$ is the Riemann scalar curvature of $g$.

**Proof.** Suppose that $\tilde{V}_{ijkl} = 0$; from (3.6) we get

$$
0 = V_{ijkl} + F_{jlt}\sigma_{ik} - F_{it}\sigma_{jk} + F_{ik}\sigma_{jl} - F_{jl}\sigma_{ik}
$$

where $\tilde{V}_{ijkl}$ is the conharmonic curvature tensor w.r.t. the Levi-Civita connection. It is easy to see that it satisfies the relations

$$
\tilde{V}_{ijkl} = -\tilde{V}_{jikl}, \quad \tilde{V}_{ijkl} = -\tilde{V}_{klij}, \quad \tilde{V}_{ijkl} + \tilde{V}_{klij} + \tilde{V}_{ijkl} = 0.
$$

Theorem 3.5. If the conharmonic curvature tensor w.r.t. the quarter-symmetric metric $F$-connection vanishes, then

$$
(\nabla_{i}\nabla_{l}f)F_{l}^{i} + \frac{\tau}{2(2 - n)} + \frac{4 - n}{2}(\nabla_{i}f)(\nabla^{i}f) = 0
$$

where $\tau$ is the Riemann scalar curvature of $g$.

**Proof.** Suppose that $\tilde{V}_{ijkl} = 0$; from (3.6) we get

$$
0 = V_{ijkl} + F_{jlt}\sigma_{ik} - F_{it}\sigma_{jk} + F_{ik}\sigma_{jl} - F_{jl}\sigma_{ik}
$$

where $\tilde{V}_{ijkl}$ is the conharmonic curvature tensor w.r.t. the Levi-Civita connection. It is easy to see that it satisfies the relations

$$
\tilde{V}_{ijkl} = -\tilde{V}_{jikl}, \quad \tilde{V}_{ijkl} = -\tilde{V}_{klij}, \quad \tilde{V}_{ijkl} + \tilde{V}_{klij} + \tilde{V}_{ijkl} = 0.
$$

4. Quarter-symmetric nonmetric $F$-connections

We define a linear connection $\nabla$ in the anti-Kähler manifold $(M_{\ast}, g, F)$ whose torsion is in the form (3.1), as follows:

$$
T_{ij}^{k} = \Gamma_{ij}^{k} + (1 - \lambda)(p_{j}F_{i}^{k} + p_{i}F_{j}^{k}) - \lambda(p_{j}F_{j}^{k} + p_{i}F_{j}^{k}).
$$
where the generator $p_i$ is gradient of a locally holomorphic function $f$ on $M_n$ and $\lambda \neq 1 \in \mathbb{R}$. Calculating the covariant derivative of $g$ and $F$ w.r.t. the connection, we obtain

$$\nabla_k g_{ij} = (\lambda - 1)(p_i F_{jk} + p_j F_{ik} + p_t F_{i}^t g_{kj} + p_i F_{j}^t g_{ki}) - 2\lambda(p_k F_{ij} + p_j F_{ik}^t g_{ij}) \neq 0$$

and $\nabla_k F_{ij} = 0$. We shall call the connection a quarter-symmetric nonmetric $F$-connection.

The curvature $(0,4)$-tensor of the quarter-symmetric nonmetric $F$-connection can be written

$$\mathbf{R}_{ijkl} = R_{ijkl} + F_{ji} \theta_{ik} - F_{il} \theta_{jk} + g_{jl} F^t_k \theta_{it} - g_{il} F^t_k \theta_{jt},$$

where

$$\theta_{jk} = (1 - \lambda)\nabla_j p_k - (1 - \lambda)^2 p_k p_t F^t_j - (1 - \lambda)^2 p_j p_t F^t_k.$$

The $(0,2)$-tensor $\theta$ is a symmetric tensor and also pure w.r.t. the complex structure $F$.

**Proposition 4.1.** The $(0,2)$-tensor $\theta$ given by $(4.3)$ is a holomorphic tensor.

**Proof.** In the anti-Kähler manifold $(M_n, g, F)$, in view of $(4.3)$ we calculate

$$(\phi_F \theta)_{kij} = (\nabla_m \theta_{ij}) F^m_k - (\nabla_k \theta_{im}) F^m_j = (\lambda - 1) \left[ (\nabla_m \nabla_i f) F^m_k - (\nabla_k \nabla_m f) F^m_j \right].$$

Using the Ricci identity for the generator $p_i$, the above relation reduces to

$$(\phi_F \theta)_{kij} = \frac{1}{2} \lambda p_s (R_{ms} F^s_k F^m_j - R_{ks} F^s_m F^m_j)$$

from which we get $(\phi_F \theta)_{kij} = 0$. \qed

It is clear that the curvature $(0,4)$-tensor has the properties $\mathbf{R}_{ijkl} = -\mathbf{R}_{jikl}$ and $\mathbf{R}_{ijkl} + \mathbf{R}_{kijl} + \mathbf{R}_{jkl} = 0$.

**Theorem 4.1.** The curvature $(0,4)$-tensor $\mathbf{R}$ of the quarter-symmetric nonmetric $F$-connection is a holomorphic tensor.

**Proof.** By the purity of the $(0,2)$-tensor $\theta$, it follows from $(4.2)$ that

$$\mathbf{R}_{mjklt} F^m_i = \mathbf{R}_{imklt} F^m_j = \mathbf{R}_{ijml} F^m_k = \mathbf{R}_{ijkm} F^m_l.$$

The Tachibana operator $\phi_F$ applied to the curvature tensor $\mathbf{R}$ is in the form

$$(\phi_F \mathbf{R})_{kijlt} = (\nabla_m \mathbf{R}_{ijl}) F^m_k - (\nabla_k \mathbf{R}_{ijm}) F^m_l.$$

Substitution $(4.2)$ into the above relation gives

$$(\phi_F \mathbf{R})_{kijlt} = (\phi_F \mathbf{R})_{kijlt} + \left[ (\nabla_m \theta_{lt}) F^m_k - (\nabla_k \theta_{im}) F^m_l \right] F_{ij} - \left[ (\nabla_m \theta_{jt}) F^m_k - (\nabla_k \theta_{jm}) F^m_l \right] F_{it} + \left[ (\nabla_m \theta_{ls}) F^m_k F^l_s + \nabla_k \theta_{jl} \right] g_{it} - \left[ (\nabla_m \theta_{ls}) F^m_k F^l_s + \nabla_k \theta_{jl} \right] g_{it}.$$

Thus, by Proposition 4.1 the last relation reduces to $(\phi_F \mathbf{R})_{kijlt} = 0$. \qed
The Ricci tensor of the quarter-symmetric nonmetric $F$-connection has the form $\mathbf{R}_{jk} = R_{jk} + (2 - n) F_k^l \theta_{jl}$, wherefrom $\mathbf{R}_{jk} - \mathbf{R}_{kj} = (2 - n) F_k^l (\theta_{jl} - \theta_{lj}) = 0$ i.e., the Ricci tensor is symmetric. The scalar curvature is $\tau = \tau + (2 - n) F_l^t (\theta_{jt}^t - \theta_{tj}^t)$.

The dual connection $\nabla^*$ of any linear connection $\nabla$ on a differentiable manifold $M_n$ is given in [5] by

$$X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$$

for all vector fields $X, Y$ and $Z$ on $M_n$. In local coordinates, this equation can be written $\partial_k g_{ij} = \Gamma^m_{ki} g_{mj} + \Gamma^m_{kj} g_{im}$, where $\Gamma^m_{ki}$ and $\Gamma^m_{kj}$ are respectively the components of $\nabla$ and $\nabla^*$. The dual connection $\nabla^*$ of the quarter-symmetric nonmetric $F$-connection has the components (4.4)

$$*\Gamma^k_{ij} = \Gamma^k_{ij} + (\lambda - 1)(p^k F_{ij} + p^l F_{ik}^l g_{lj}) + \lambda(p_1 F_{ij}^k + p_1 F_{ij}^l \delta^k_l),$$

where $p^k = p_l g^t_l$. The torsion tensor $*\tilde{S}^k_{ij}$ of the dual connection (4.4) is $*\tilde{S}^k_{ij} = -\lambda \tilde{S}^k_{ij}$, where $\tilde{S}^k_{ij}$ is the torsion tensor of the quarter-symmetric metric $F$-connection. Taking the covariant derivatives of the pseudo-Riemannian metric $g$ and the complex structure $F$ w.r.t. the dual connection (4.4), we find $*\nabla_k g_{ij} \neq 0$ and $*\nabla_k F_{ij} = 0$, i.e., the dual connection (4.4) is another quarter-symmetric nonmetric $F$-connection.

Taking account of the relation between the curvature tensors of any linear connection and its dual connection (see, [5]), the curvature $(0, 4)$-tensor of the dual connection (4.4) has the components

$$*\mathbf{R}_{ijkl} = -\mathbf{R}_{ijlk}, \quad *\mathbf{R}_{ijkl} = R_{ijkl} + F_{ik} \theta_{jl} - F_{jk} \theta_{il} + g_{ik} F_{l}^t \theta_{jt} - g_{jk} F_{l}^t \theta_{it},$$

where $\mathbf{R}_{ijkl}$ is the curvature $(0, 4)$-tensor of the quarter-symmetric nonmetric $F$-connection.

In view of the relation between $*\mathbf{R}$ and $\mathbf{R}$, we directly state the following theorem.

**Theorem 4.2.** Let $(M_n, g, F)$ be an anti-Kähler manifold endowed with the dual connection (4.4). The curvature $(0, 4)$-tensor $*\mathbf{R}$ of the dual connection (4.4) satisfies:

$$*\mathbf{R}_{ijkl} = -*\mathbf{R}_{ijlk}, \quad *\mathbf{R}_{ijkl} + *\mathbf{R}_{kijl} + *\mathbf{R}_{jkil} = 0, \quad (\phi^*_F \mathbf{R})_{kijt} = 0,$$

i.e., the curvature $(0, 4)$-tensor $*\mathbf{R}$ is a holomorphic tensor.

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**References**

1. S. K. Chaubey, R. H. Ojha, *On a semi-symmetric non-metric and quarter symmetric metric connections*, Tensor (N.S.) 70(2) (2008), 202–213.
2. S. Golab, *On semi-symmetric and quarter-symmetric linear connections*, Tensor (N.S.) 29(3) (1975), 249–254.
3. H. A. Hayden, *Sub-spaces of a space with torsion*, Proc. London Math. Soc. S2-34 (1932), 27–50.
4. M. Iscan, A.A. Salimov, On Kahler-Norden manifolds, Proc. Indian Acad. Sci.(Math. Sci.) 119(1) (2009), 71–80.
5. S. Lauritzen, Statistical manifolds, in: Differential Geometry in Statistical Inference, Institute of Mathematical Statistics Lecture Notes—Monograph Series, 10 (Inst. Math. Statist., Hayward, CA, 1987), 163–216.
6. R. S. Mishra, S. N. Pandey, On quarter symmetric metric $F$-connections, Tensor (N.S.) 34(1) (1980), 1–7.
7. S. C. Rastogi, On quarter-symmetric metric connection, C. R. Acad. Bulgare Sci. 31(7) (1978), 811–814.
8. S. C. Rastogi, On quarter-symmetric metric connections, Tensor (N.S.) 44(2) (1987), 133–141.
9. S. C. Rastogi, A note on quarter-symmetric metric connections, Indian J. Pure Appl. Math. 18(12) (1987), 1107–1112.
10. A. Salimov, Tensor Operators and Their Applications, Mathematics Research Developments Series, Nova Science Publishers, New York, 2013, xii+186 pp.
11. S. Tachibana, Analytic tensor and its generalization, Tohoku Math. J. 12 (1960), 208–221.
12. K. Yano, T. Inai, Quarter-symmetric metric connections and their curvature tensors, Tensor (N.S.) 38 (1982), 13–18.
13. K. Yano, On semi-symmetric metric connection, Rev. Roumaine Math. Pures Appl. 15 (1970), 1579–1586.
14. K. Yano, M. Ako, On certain operators associated with tensor fields, Kodai Math. Sem. Rep. 20 (1968), 414–436.