Uncertainty principle and geometry of the infinite Grassmann manifold

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Abstract

We study the pairs of projections

\[ P_I f = \chi_I f, \quad Q_J f = \left( \chi_J \hat{f} \right) \hat{\circ}, \quad f \in L^2(\mathbb{R}^n), \]

where \( I, J \subset \mathbb{R}^n \) are sets of finite Lebesgue measure, \( \chi_I, \chi_J \) denote the corresponding characteristic functions and \( \hat{\cdot}, \hat{\circ} \) denote the Fourier-Plancherel transformation \( L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) and its inverse. These pairs of projections have been widely studied by several authors in connection with the mathematical formulation of Heisenberg’s uncertainty principle. Our study is done from a differential geometric point of view. We apply known results on the Finsler geometry of the Grassmann manifold \( \mathcal{P}(\mathcal{H}) \) of a Hilbert space \( \mathcal{H} \) to establish that there exists a unique minimal geodesic of \( \mathcal{P}(\mathcal{H}) \), which is a curve of the form

\[ \delta(t) = e^{itX_{I,J}} P_I e^{-itX_{I,J}}, \]

which joins \( P_I \) and \( Q_J \) and has length \( \pi/2 \). As a consequence we obtain that if \( H \) is the logarithm of the Fourier-Plancherel map, then

\[ \|[H, P_I]\| \geq \pi/2. \]

The spectrum of \( X_{I,J} \) is denumerable and symmetric with respect to the origin, it has a smallest positive eigenvalue \( \gamma(X_{I,J}) \) which satisfies

\[ \cos(\gamma(X_{I,J})) = \|P_I Q_J\|. \]

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1 Introduction

Consider the following example:

Example 1.1. Let \( I, J \subset \mathbb{R}^n \) be Lebesgue-measurable sets of finite measure. Let \( P_I, Q_J \) be the projections in \( L^2(\mathbb{R}^n, dx) \) given by

\[ P_I f = \chi_I f \quad \text{and} \quad Q_J f = \left( \chi_J \hat{f} \right) \hat{\circ}, \]
where $\chi_L$ denotes the characteristic function of the set $L$. Equivalently, denoting by $U_{\mathcal{F}}$ the Fourier transformation regarded as a unitary operator acting in $L^2(\mathbb{R}^n, dx)$ and by $M_\varphi$ the multiplication by $\varphi$, then

$$P_I = M_{\chi_I} \quad \text{and} \quad Q_J = U_{\mathcal{F}}^* P_J U_{\mathcal{F}}.$$

The operator $P_I Q_J$ is Hilbert-Schmidt (see for instance [11], Lemma 2).

An intuitive formulation of Heisenberg’s uncertainty principle says that a nonzero function and its Fourier transform cannot be (simultaneously) sharply localized (see [13], page 207). We give more precision to this statement below (see for instance [11], page 906).

According to Folland and Sitaram [13], the idea of using projections $P_I$ and $Q_J$ to obtain a form of the uncertainty principle is due to Fuchs [14], and it was developed later in a series of papers by Landau, Pollack and Slepian [20], [21], [25]. See the survey by Folland and Sitaram [13].

Donoho and Stark [11] proved that if $I, J \subset \mathbb{R}^n$ with finite Lebesgue measure and $f \in L^2(\mathbb{R}^n)$ with $\|f\|_2 = 1$ satisfy that

$$\int_{\mathbb{R}^n - I} |f(t)|^2 dt < \epsilon_I \quad \text{and} \quad \int_{\mathbb{R}^n - J} |\hat{f}(w)|^2 dw < \epsilon_J$$

then

$$|I||J| \geq (1 - (\epsilon_I + \epsilon_J))^2.$$

Donoho and Stark showed several applications of these ideas to signal processing (and the obstruction to the existence of an instantaneous frequency). Smith [26] generalized these results to a locally compact abelian group $G$ where $I \subset G$ and $J \subset \hat{G}$, the dual group of $G$. The books by Havin and Jöricke [17], Hogan and Lakey [18], and Gröchenig [15] among many others, contain further applications, generalizations and history of the different uncertainty principles.

By an elementary computation using Fubini’s theorem, Donoho and Stark prove that

$$\|P_I Q_J\|_{HS} = \sqrt{|I||J|},$$

where $\|\|_{HS}$ is Hilbert-Schmidt norm. Next they prove that

$$\|P_I Q_J\| \geq 1 - \epsilon_I - \epsilon_J.$$

The fact that $\|P_I Q_J\| \leq \|P_I Q_J\|_{HS}$ is well known.

They argue that any bound $c$ such that

$$\|P_I Q_J\| \leq c < 1$$

is an expression of the uncertainty principle ([11], page 912).

Denote by $\mathcal{P}(\mathcal{H})$ the set of orthogonal projections of the Hilbert space $\mathcal{H}$, also called the Grassmann manifold of $\mathcal{H}$. It is indeed a differentiable manifold of $B(\mathcal{H})$ (also in the infinite dimensional setting), with rich geometric structure (see for instance [24] or [7]). The pairs $(P_I, Q_J)$ might be put in the broader context of the sets

$$C = \{(P, Q) : P, Q \text{ are orthogonal projections and } PQ \text{ is compact}\}.$$

This set is a $C^\infty$-submanifold of $\mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H})$. 


An application of these geometrical results facts is a form of the uncertainty principle (see Theorem 3.6 below).

Let us describe the content of the paper.

In Section 2 we recall the known facts on the geometry of \( \mathcal{P}(\mathcal{H}) \). In section 3 we apply known results [24], [7], [2] on the Finsler geometry of the Grassmann manifold of \( \mathcal{H} \) to the special case of pairs \( P_I, Q_J \). We prove that there exists a unique minimal geodesic of the Grassmann manifold of length \( \pi/2 \) which joins \( P_I \) and \( Q_J \). That is, there exists a unique selfadjoint operator \( X_{I,J} \) of norm \( \pi/2 \), which is co-diagonal with respect both to \( P_I \) and \( Q_J \), such that

\[ e^{iX_{I,J}} P_I e^{-iX_{I,J}} = Q_J. \]

The spectrum of the operator \( X_{I,J} \) is denumerable and symmetric with respect to the origin. The smallest positive eigenvalue \( \gamma(X_{I,J}) \) verifies

\[ \cos(\gamma(X_{I,J})) = \|P_I Q_J\|. \]

As a consequence from the fact that the minimal geodesic has length \( \pi/2 \), we prove that if \( H \) is the logarithm of the Fourier transform in \( L^2(\mathbb{R}^n) \), and \( I \subset \mathbb{R}^n \) is a set of finite Lebesgue measure, then

\[ \|[H, P_I]\| = \|[H, Q_J]\| \geq \pi/2. \]

In Section 4 we show that for any pair of sets \( I, J \subset \mathbb{R}^n \) of finite measure, one has

\[ N(P_I) + N(Q_J) = L^2(\mathbb{R}^n), \]

where the sum is non-direct (the subspaces have infinite dimensional intersection).

2 Basic properties

2.1 Halmos decomposition

Let \( \mathcal{H} \) be a Hilbert space, \( \mathcal{B}(\mathcal{H}) \) the algebra of bounded linear operators in \( \mathcal{H} \), \( \mathcal{K}(\mathcal{H}) \) the ideal of compact operators and \( \mathcal{P}(\mathcal{H}) \) the set of selfadjoint (orthogonal) projections, and \( \mathcal{P}_\infty(\mathcal{H}) \) the subset of projections whose nullspaces and ranges have infinite dimension.

A tool that will be useful in the study of the pairs \( P_I, Q_J \) is Halmos decomposition [16], which is the following orthogonal decomposition of \( \mathcal{H} \): given a pair of projections \( P \) and \( Q \), consider

\[ \mathcal{H}_{11} = R(P) \cap R(Q), \quad \mathcal{H}_{00} = N(P) \cap N(Q), \quad \mathcal{H}_{10} = R(P) \cap N(Q), \quad \mathcal{H}_{01} = N(P) \cap R(Q) \]

and \( \mathcal{H}_0 \) the orthogonal complement of the sum of the above. This last subspace is usually called the generic part of the pair \( P, Q \). Note also that

\[ N(P - Q) = \mathcal{H}_{11} \oplus \mathcal{H}_{00}, \quad N(P - Q - 1) = \mathcal{H}_{10} \quad \text{and} \quad N(P - Q + 1) = \mathcal{H}_{01}, \]

so that the generic part depends in fact of the difference \( P - Q \).

Halmos proved that there is an isometric isomorphism between \( \mathcal{H}_0 \) and a product Hilbert space \( \mathcal{L} \times \mathcal{L} \) such that in the above decomposition (putting \( \mathcal{L} \times \mathcal{L} \) in place of \( \mathcal{H}_0 \)), the projections are

\[ P = 1 \oplus 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

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and
\[ Q = 1 \oplus 0 \oplus 0 \oplus 0 \oplus 1 \oplus \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}, \]

where \( C = \cos(X) \) and \( S = \sin(X) \) for some operator \( 0 < X \leq \pi/2 \) in \( \mathcal{L} \) with trivial nullspace.

Apparently, the pair \((P, Q)\) belongs to \( \mathcal{C} \) if and only if \( \mathcal{H}_{11} \) is finite dimensional and \( C = \cos(X) \) is compact.

**Remark 2.1.** If \((P, Q) \in \mathcal{C}\), then the spectral resolution of \( X \) can be easily described. Since \( 0 < \cos(X) \) is compact, it follows that
\[ X = \sum_n \gamma_n P_n + \frac{\pi}{2} E, \]

where \( 0 < \gamma_n < \pi/2 \) is an increasing (finite or infinite) sequence. For all \( n \), \( \dim R(P_n) < \infty \), and
\[ R(E) \oplus (\oplus_{n \geq 1} R(P_n)) = \mathcal{L}. \]

### 2.2 Finsler geometry of the Grassmann manifold of \( \mathcal{H} \)

Let us recall some basic facts on the differential geometry of the set \( \mathcal{P}(\mathcal{H}) \) (see for instance [7], [24], [2]).

1. The space \( \mathcal{P}(\mathcal{H}) \) is a homogeneous space under the action of the unitary group \( \mathcal{U}(\mathcal{H}) \) by inner conjugation: if \( U \in \mathcal{U}(\mathcal{H}) \) and \( P \in \mathcal{P}(\mathcal{H}) \), the action is given by
\[ U \cdot P = UPU^*. \]

This action is locally transitive: it is well known that two projections \( P_1, P_2 \) such that \( \|P_1 - P_2\| < 1 \), are conjugate. Therefore, since the unitary group \( \mathcal{U}(\mathcal{H}) \) is connected, the orbits of the action coincide with the connected components of \( \mathcal{P}(\mathcal{H}) \), which are: for \( n \in \mathbb{N} \), \( \mathcal{P}_{n,\infty}(\mathcal{H}) \) (projections of nullity \( n \)), \( \mathcal{P}_{\infty,n}(\mathcal{H}) \) (projections of rank \( n \)) and \( \mathcal{P}_{\infty}(\mathcal{H}) \) (projections of infinite rank and nullity). These components are \( C^\infty \)-submanifolds of \( \mathcal{B}(\mathcal{H}) \).

2. There is a natural linear connection in \( \mathcal{P}(\mathcal{H}) \). If \( \dim \mathcal{H} < \infty \), it is the Levi-Civita connection of the Riemannian metric which consists of considering the Frobenius inner product at every tangent space. It is based on the diagonal / co-diagonal decomposition of \( \mathcal{B}(\mathcal{H}) \).

To be more specific, given \( P_0 \in \mathcal{P}(\mathcal{H}) \), the tangent space of \( \mathcal{P}(\mathcal{H}) \) at \( P_0 \) consists of all self-adjoint co-diagonal matrices (in terms of \( P_0 \)). The linear connection in \( \mathcal{P}(\mathcal{H}) \) is induced by a reductive structure, where the horizontal elements at \( P_0 \) (in the Lie algebra of \( \mathcal{U}(\mathcal{H}) \): the space of antihermitian elements of \( \mathcal{B}(\mathcal{H}) \)) are the co-diagonal antihermitian operators. The geodesics of \( \mathcal{P} \) which start at \( P_0 \) are curves of the form
\[ \delta(t) = e^{itX} P_0 e^{-itX}, \]
with \( X^* = X \) co-diagonal with respect to \( P_0 \). Observe that \( X \) is co-diagonal with respect to any \( P \). It was proved in [24] that if \( P_0, P_1 \in \mathcal{P}(\mathcal{H}) \) satisfy \( \|P_0 - P_1\| < 1 \), then there exists a unique geodesic (up to reparametrization) joining \( P_0 \) and \( P_1 \). This condition is not necessary for the existence of a unique geodesic.
3. There exists a unique geodesic joining two projections $P$ and $Q$ if and only if
\[ R(P) \cap N(Q) = N(P) \cap R(Q) = \{0\}, \]
(see [2]).

4. If $\mathcal{H}$ is infinite dimensional, the Frobenius metric is not available. However, if one endows each tangent space of $\mathcal{P}(\mathcal{H})$ with the usual norm of $\mathcal{B}(\mathcal{H})$, one obtains a continuous (non-regular) Finsler metric,
\[ d(P_0, P_1) = \inf \{ \ell(\gamma) : \gamma \text{ a continuous piecewise smooth curve in } \mathcal{P}(\mathcal{H}) \text{ joining } P_0 \text{ and } P_1 \} \]
where $\ell(\gamma)$ denotes the length of $\gamma$ (parametrized in the interval $I$):
\[ \ell(\gamma) = \int_I \| \dot{\gamma}(t) \| dt. \]
In [24] it was shown that the geodesics (1) remain minimal among their endpoints for all $t$ such that
\[ |t| \leq \frac{\pi}{2\|X\|}. \]
It can be shown that $d(P_0, P_1) < \pi/2$ if and only if $\|P_0 - P_1\| < 1$. In other words, $\|P_0 - P_1\| = 1$ if and only if $d(P_0, P_1) = \pi/2$.

3 Geometry of the pairs $P_I$, $Q_J$

Lenard proved in [22] that the projections $P_I, Q_J \in \mathcal{P}(L^2(\mathbb{R}^n, dx))$ defined in Example (1.1), satisfy
\[ R(P_I) \cap N(Q_J) = N(P_I) \cap R(Q_J) = \{0\}. \]
(2)
Moreover, $\|P_I - Q_J\| = 1$.
Therefore one obtains the following:

**Theorem 3.1.** Let $I, J$ be measurable subsets of $\mathbb{R}^n$ of finite measure, and $P_I, Q_J$ the above projections. Then there exists a unique selfadjoint operator $X_{I,J}$ satisfying:

1. $\|X_{I,J}\| = \pi/2$.
2. $X_{I,J}$ is $P_I$ and $Q_J$ co-diagonal. In other words, $X_{I,J}$ maps functions in $L^2(\mathbb{R}^n, dx)$ with support in $I$ to functions with support in $\mathbb{R}^n - I$, and functions such that $\hat{f}$ has support in $J$ to functions such that the Fourier transform has support in $\mathbb{R}^n - J$.
3. $e^{iX_{I,J}}P_I e^{-iX_{I,J}} = Q_J$.
4. If $P(t)$, $t \in [0,1]$ is a smooth curve in $\mathcal{P}(\mathcal{H})$ with $P(0) = P_I$ and $P(1) = Q_J$, then
\[ \ell(P) = \int_0^1 \| \dot{P}(t) \| dt \geq \pi/2. \]
Proof. By the condition (2) above ([22]), it follows from [2] that there exists a unique minimal geodesic of $\mathcal{P}(\mathcal{H})$, of the form

$$\delta_{I,J}(t) = e^{itX_{I,J}} P_I e^{itX_{I,J}}$$

with $X^*_{I,J} = X_{I,J}$ co-diagonal with respect to $P_I$ (and $Q_J$) such that

$$\delta_{I,J}(1) = Q_J.$$

Condition 4. above is the minimality property of $\delta_{I,J}$. Finally, the fact that $\|P_I - Q_J\| = 1$ means that $\|X_{I,J}\| = \pi/2$.

Remark 3.2. It is known [13] that $\lambda_1 = \|P_I Q_J P_I\| = \|P_I Q_J\|^2 < 1$, and moreover $\sqrt{\lambda_1}$ equals the cosine of the angle between the subspaces $R(P_I)$ and $R(Q_J)$.

One can also relate this number $\lambda_1$ with the operator $X_{I,J}$. Using Halmos decomposition (recall that it consists only of $\mathcal{H}_{00}$ and the generic part $\mathcal{H}_0$ in this case),

$$P_I Q_J P_I = 0 \oplus \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and thus $\lambda_1 = \|\cos(X)\|^2$. We shall see below that the spectrum of $X$ is a strictly increasing sequence of positive eigenvalues $\gamma_n \to \pi/2$, with finite multiplicity. Moreover, since $P_I Q_J P_I$ belongs to $\mathcal{B}_1(\mathcal{H})$, it follows that $C \in \mathcal{B}_2(\mathcal{L})$. Thus

$$\{\cos(\gamma_n)\} \in \ell^2.$$

For a given $P \in \mathcal{P}(\mathcal{H})$, let $A_P$ be

$$A_P = \{X \in \mathcal{B}(\mathcal{H}) : [X, P] \text{ is compact}\}.$$

Apparently $A_P$ is a $C^*$-algebra.

Theorem 3.3. Let $I, J$ be measurable subsets of $\mathbb{R}^n$ of finite Lebesgue measure.

1. The selfadjoint operator $X_{I,J}$ has closed infinite dimensional range, in particular it is not compact.

2. Let $I_0$ be another measurable set with finite measure such that $|I \cap I_0| = 0$, and let $P_0 = P_{I_0}$. Then, the commutant $[X_{I,J}, P_0]$ is compact.

Proof. Easy matrix computations ([2]) show that, in the decomposition $\mathcal{H} = \mathcal{H}_{00} \oplus (\mathcal{L} \times \mathcal{L})$, $X_{I,J}$ is of the form

$$X_{I,J} = 0 \oplus \begin{pmatrix} 0 & -iX \\ iX & 0 \end{pmatrix}.$$

Note that the spectrum of this operator is symmetric with respect to the origin. Indeed, if $V$ equals the symmetry

$$V = 1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then apparently $V X_{I,J} V = -X_{I,J}$. Also note that

$$X^2_{I,J} = 0 \oplus \begin{pmatrix} X^2 & 0 \\ 0 & X^2 \end{pmatrix}.$$
Therefore the spectrum of $X_{I,J}$ is

$$\sigma(X_{I,J}) = \{0\} \cup \{\gamma_n : n \geq 1\} \cup \{-\gamma_n : n \geq 1\},$$

with 0 of infinite multiplicity, and the multiplicity of $\gamma_n$ equal to the multiplicity of $-\gamma_n$, and finite. What matters here, is that the set $\{\gamma_n : n \geq 1\}$ is infinite, and is therefore an increasing sequence converging to $\pi/2$. This holds because otherwise, the operator $C$ would have finite rank, and therefore $P_I Q_J P_I$ would be of finite rank, which is not the case (see [22]). Thus $X_{I,J}$ has closed range. of infinite dimension.

Note that $P_I$ and $Q_J$ satisfy that $P_I P_0 = 0$ and $Q_J P_0 = Q_J P_I$ is compact, and therefore $P_I, Q_J \in \mathcal{A}_{P_0}$. Thus the symmetries $S_{P_I}, S_{Q_J}$ belong to $a_{P_0}$. Since $S_{Q_J} = e^{i2X_{I,J}} S_{P_I}$, this implies that

$$e^{i2X_{I,J}} \in \mathcal{A}_{P_0}.$$ 

By the spectral picture of $X_{I,J}$ it is clear that $X_{I,J}$ can be obtained as an holomorphic function of $e^{i2X_{I,J}}$. Since $\mathcal{A}_{P_0}$ is a C*-algebra, this implies that $X_{I,J} \in \mathcal{A}_{P_0}$. \hfill \Box

Let us relate the operator $X_{I,J}$ with the mathematical version of the uncertainty principle, according to [11] and [13].

Let $A \in \mathcal{B}(\mathcal{H})$ be an operator with closed range, the reduced minimum modulus $\gamma_A$ of $A$ is the positive number

$$\gamma_A = \min\{\|A\xi\| : \xi \in N(A)^\perp, \|\xi\| = 1\} = \min\{|\lambda| : \lambda \in \sigma(A), \lambda \neq 0\}.$$ 

Donoho and Stark [11] underline the role of the number $\|Q_J P_I\|$ and consider any constant $c$ such that $\|Q_J P_I\| \leq c$ a manifestation of the (mathematical) uncertainty principle. By the above Remark, we have:

**Corollary 3.4.** With the current notations,

$$\|Q_J P_I\| = \cos(\gamma_{X_{I,J}}).$$

*Proof.* Indeed, in the above description of the spectrum of $X_{I,J}$, the reduced minimum modulus $\gamma_{X_{I,J}}$ of $X_{I,J}$ coincides with $\gamma_I$. \hfill \Box

Let $X^0_{I,J}$ be the restriction of $X_{I,J}$ to the generic part of $P_I$ and $Q_J$, i.e., its restriction to $N(X_{I,J})^\perp$. In Halmos decomposition

$$X^0_{I,J} = \begin{pmatrix} 0 & -iX \\ iX & 0 \end{pmatrix}.$$ 

Recall the formula by Donoho and Stark [11]

$$\|P_I Q_J\|_{HS} = |I|^{1/2} |J|^{1/2}.$$ 

From the preceeding facts, it also follows:

**Corollary 3.5.** With the current notations

$$|I|^{1/2} |J|^{1/2} = \|\cos(X)\|_{HS} = \frac{1}{\sqrt{2}} \|\cos(X^0_{I,J})\|_{HS} = \left\{\sum_{n=1}^{\infty} \frac{1}{2} \cos(\gamma_n^2)\right\}^{1/2}. $$
Remark 3.7.

1. We may write $H$ in terms of $U_F$ using the well known formulas

$$E_{-1} = \frac{1}{4}(1 - U_F + \bar{U}_F^2 - \bar{U}_F^3), \quad E_i = \frac{1}{4}(1 - iU_F - \bar{U}_F^2 + i\bar{U}_F^3), \quad E_{-i} = \frac{1}{4}(1 + iU_F - \bar{U}_F^2 - i\bar{U}_F^3),$$

and thus

$$H = \frac{\pi}{4}\{-1 + (1 + i)U_F - \bar{U}_F^2 + (1 + i)\bar{U}_F^3\}.$$

Then

$$[H,P_I] = \frac{\pi}{4}\{(1 + i)[U_F,P_I] - [U_F^2,P_I] + (1 + i)[U_F^3,P_I]\}.$$

The inequality in Corollary 3.6 can be written

$$\|(1 + i)[U_F,P_I] - [U_F^2,P_I] + (1 + i)[U_F^3,P_I]\| \geq 2.$$
2. In the special case when the set $I$ is (essentially) symmetric with respect to the origin, $P_I$ commutes with $U_F^2$, so that

$$[U_F^2, P_I] = 0 \quad \text{and} \quad [U_F^2, P_I] = [U_F, P_I]U_F^2 = U_F^2[U_F, P_I]$$

one has

$$[H, P_I] = \frac{(1 + i)\pi}{4}[U_F, P_I](1 + U_F^2).$$

The operator $U_F^2 f(x) = f(-x)$ is a symmetry, then $\frac{1}{2}(1 + U_F^2)$ is the orthogonal projection $E_e$ onto the subspace of essentially even functions ($f(x) = f(-x)$ a.e.). Then one can write

$$[H, P_I] = \frac{(1 + i)\pi}{2}[U_F, P_I]E_e = \frac{(1 + i)\pi}{2}E_e[U_F, P_I].$$

**Corollary 3.8.** Suppose that $I$ is essentially symmetric, with finite measure.

1. \[\|E_e[U_F, P_I]\| = \|E_e[U_F, P_I]E_e\| \geq \frac{1}{\sqrt{2}}.\]

2. \[\|E_eP_I - E_eQ_I\| \geq \frac{1}{\sqrt{2}}.\]

where $E_eP_I = P_IE_e$ and $E_eQ_I = Q_IE_e$ are orthogonal projections.

**Proof.** Recall that $E_e$ and $U_F$ commute. Then

$$E_e[U_F, P_I]E_e = E_e(UP_I - P_IU_F)E_e = U_FE_e(P_I - U_F^2P_IU_F)E_e$$

$$= U_FE_e(P_I - Q_I)E_e.$$

where $E_e$, as well as $U_F$, and thus also $Q_I = U_F^2P_IU_F$ commute with $E_e$. \hfill \Box

The ranges of these two orthogonal projections $E_eP_I$ and $E_eQ_I$ consist of the elements of $L^2$ which are essentially even and vanish (essentially) outside $I$, and the analogous subspace for the Fourier transform.

### 4 Spatial properties of $P_I$ and $Q_J$

Let us return to the general setting ($I$ not necessarily equal to $J$). The ranges and nullspaces of $P_I$ and $Q_J$ have several interesting properties. First we need the following lemma:

**Lemma 4.1.** Let $P, Q$ be orthogonal projections such that $\|P - Q\| = 1$. Then one and only one of the following conditions hold:

1. $N(P) + R(Q) = \mathcal{H}$, with non direct sum (and this is equivalent to $R(P) + N(Q)$ being a direct sum and a closed proper subspace of $\mathcal{H}$).

2. $R(P) + N(Q) = \mathcal{H}$, with non direct sum (and this is equivalent to $N(P) + R(Q)$ being a direct sum and a closed proper subspace of $\mathcal{H}$).
3. \( R(P) + N(Q) \) is non closed (and this is equivalent to \( N(P) + R(Q) \) being non closed).

Proof. By the Krein-Krasnoselskii-Milman formula (see for instance [19])
\[
\|P - Q\| = \max\{\|P(1 - Q)\|, \|Q(1 - P)\|\},
\]
we have that one and only one of the following hold:

1. \( \|P(1 - Q)\| < 1 \) and \( \|Q(1 - P)\| = 1 \),
2. \( \|P(1 - Q)\| = 1 \) and \( \|Q(1 - P)\| < 1 \), or
3. \( \|P(1 - Q)\| = 1 \) and \( \|Q(1 - P)\| = 1 \).

This alternative corresponds precisely with the three conditions in the Lemma. It is known [9] that for two orthogonal projections \( E \) and \( F \), \( \|EF\| < 1 \) holds if and only if \( R(E) \cap R(F) = \{0\} \) and \( R(E) + R(F) \) closed. The sum \( M + N \) of two subspaces is closed if and only if the sum \( M^\perp + N^\perp \) is closed (see [9]). Therefore, \( \|EF\| < 1 \) is also equivalent to \( N(E) + N(F) = \mathcal{H} \).

If we apply these facts to \( E = P \) and \( F = 1 - Q \), we obtain that the first alternative is equivalent to \( R(P) \cap N(Q) = \{0\} \) and \( R(P) + N(Q) \) closed, or to \( N(P) + R(Q) = \mathcal{H} \).

Analogously, the second alternative is equivalent to \( R(Q) \cap N(P) = \{0\} \) and \( R(Q) + N(P) \) closed, or to \( N(P) + R(Q) = \mathcal{H} \).

Note that in the first case, \( R(P) + N(Q) \) is proper, otherwise its orthogonal complement would be \( N(P) \cap R(Q) = \{0\} \), which together with the fact that \( N(P) + R(Q) = \mathcal{H} \) (closed!), would lead us to the second alternative.

Analogously in the second alternative, \( N(P) + R(Q) \) is proper.

If neither of these two happen, it is clear that neither \( R(P) + N(Q) \) nor (equivalently) the sum of the orthogonals \( N(P) + R(Q) \) is closed. □

We have the following:

**Theorem 4.2.** Let \( I, J \subset \mathbb{R}^n \) with finite Lebesgue measure. Then

1. \( R(P_I) + R(Q_J) \) is a closed proper subset of \( L^2(\mathbb{R}^n) \), with infinite codimension. The sum is direct \( (R(P_I) \cap R(Q_J) = \{0\}) \).
2. \( N(P_I) + N(Q_J) = L^2(\mathbb{R}^n) \), and the sum is not direct \( (N(P_I) \cap N(Q_J) \) is infinite dimensional).
3. \( R(P_I) + N(Q_J) \) and \( N(P_I) + R(Q_J) \) are proper dense subspaces of \( L^2(\mathbb{R}^n) \), and \( R(P_I) \cap N(Q_J) = N(P_I) \cap R(Q_J) = \{0\} \).

Proof. By the cited result [9], two projections \( P, Q \), satisfy that \( R(P) + R(Q) \) is closed and \( R(P) \cap R(Q) = \{0\} \) if and only if \( \|PQ\| < 1 \). It is also known (see above, [13]) that \( \|P_I Q_J\| < 1 \).

The intersection of these spaces is, in our case (using the notation of the Halmos decomposition)
\[
R(P_I) \cap R(Q_J) = \mathcal{H}_{11} = \{0\}.
\]

As remarked above, Lenard proved that \( \mathcal{H}_{11} = \mathcal{H}_{10} = \mathcal{H}_{01} = \{0\} \), and \( \mathcal{H}_{00} \) is infinite dimensional. The orthogonal complement of this sum is
\[
(R(P_I) + R(Q_J))^\perp = N(P_I) \cap N(Q_J) = \mathcal{H}_{00}.
\]
Thus the first assertion follows.
In our case $\|P_I - Q_J\| = 1$ ([13], [22]) thus we may apply the above Lemma.

The first condition cannot happen:

$$(N(P_I) + R(Q_J))^\perp = R(P_I) \cap N(Q_J) = \mathcal{H}_{10} = \{0\}.$$  

By a similar argument, neither the second condition can happen. Thus $R(P_I) + R(Q_J)$ is non closed, and its orthogonal complement is trivial. Thus the second and third assertions follow. □

**Remark 4.3.** It is known (see for instance [12]), that if $P, Q$ are projections with $P Q$ compact and $R(P) \cap R(Q) = \{0\}$, then

$$\|P Q\| < 1.$$  

In [6], the second named author and A. Maestripieri studied the set of operators $T \in \mathcal{B}(\mathcal{H})$ which are of the form $T = P Q$. Among other properties, they proved that $T$ may have many factorizations, but there is a minimal factorization (called canonical factorization of $T$), namely

$$T = P_{R(T)} R(T)^\perp,$$

which satisfies that if $T = P Q$, then $R(T) \subset R(P)$ and $N(T)^\perp \subset R(Q)$ (or equivalently $N(Q) \subset N(T)$). Following this notation,

**Proposition 4.4.** The factorization $P_I Q_J$ is canonical.

**Proof.** Put $T = P_I Q_J$. Using Halmos decomposition in this particular case ($\mathcal{H} = \mathcal{H}_{00} \oplus (\mathcal{L} \times \mathcal{L})$), apparently

$$P_I Q_J P_I = 0 \oplus \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix},$$

and thus $R(P_I Q_J P_I) = 0 \oplus (R(C) \times 0)$. Recall that $C^2 > 0$, and thus $C^2$ has dense range. It follows that

$$\overline{R(T)} = \overline{R(P_I Q_J)} = \overline{R(P_I Q_J P_I)} = 0 \oplus (\mathcal{L} \times 0),$$

which is precisely the range of $P_I$: $\overline{R(T)} = R(P_I)$. Note the following elementary fact:

$$N(PQ) = N(Q) \oplus (R(Q) \cap N(P)).$$

For the factorization $T = P_I Q_J$ it is known ([22]) that $R(Q_J) \cap N(P_I) = 0$. Thus

$$N(T) = N(P_I Q_J) = N(Q_J)$$

and the proof follows. □

In [6] it is proven that if $T = PQ = P_0 Q_0$, and the latter is the canonical factorization, then

$$\|P_0 f - Q_0 f\| \leq \|P f - Q f\|$$

for any $f \in L^2(\mathbb{R}^n)$. In particular $\|P_0 - Q_0\| \leq \|P - Q\|$. In our case we get the following result

**Corollary 4.5.** Let $P, Q$ projections in $L^2(\mathbb{R}^n)$ such that $PQ = P_I Q_J$. Then for any $f \in L^2(\mathbb{R}^n)$ one has

$$\|P_I f - Q_J f\|_2 \leq \|P f - Q f\|_2.$$  

In particular, $\|P_I - Q_J\| \leq \|P - Q\|$.  

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