COMMUTATORS CANNOT BE PROPER POWER IN METRIC SMALL-CANCELLATION TORSION-FREE GROUPS

Elizaveta V. Frenkel\textsuperscript{b} \hspace{1cm} Anton A. Klyachko\textsuperscript{f}

\textsuperscript{b} Faculty of Further Education, \hspace{1cm} \textsuperscript{f} Faculty of Mechanics and Mathematics
Moscow State University, Moscow 119991, Leninskie gory
lizzy.frenkel@gmail.com \hspace{1cm} klyachko@mech.math.msu.su

A nontrivial commutator cannot be a proper power in a torsion-free group satisfying \(C'(\lambda)\) small cancellation condition with sufficiently small \(\lambda\).

\section{Introduction}
It is well known that nontrivial commutators cannot be proper powers in free groups \cite{Sch59}. In free products, the situation is more complicated but also completely studied \cite{CER94}; some partial results are known about amalgamated products \cite{FRR11}. We prove the following fact.

\textbf{Theorem.} If a torsion-free group satisfies the \(C'(\lambda)\) small cancellation condition for sufficiently small \(\lambda\), then no nontrivial commutator can be a proper power in this group.

Recall that a presentation \(\langle X | R \rangle\) satisfies the \(C'(\lambda)\) small cancellation condition if it is symmetrised (i.e. \(R\) is closed under taking cyclic permutations and inverses) and, for any two different relators \(r_1, r_2 \in R\), the length of their common initial segment is less than \(\lambda |r_1|\) (see, e.g., \cite{LS80}).

To prove the theorem we use van Kampen diagrams and the car-crash lemma \cite{Kl93} (see also \cite{FeR96}); this lemma was applied earlier to study quite different problems such as equations over groups and relative presentations (see, e.g., \cite{CG95}, \cite{FeR96}, \cite{Kl97}, \cite{FeR98}, \cite{CG00}, \cite{CR01}, \cite{For05}, \cite{Kl05}, \cite{Kl06a}, \cite{Kl06b}, \cite{Kl07}, \cite{Kl09}, \cite{Le09} and \cite{Kil12}).

Let us explain our approach on the following toy example. Suppose we want to show that a nontrivial commutator cannot be a cube in the free group \(F(a, b)\). Suppose the contrary. By the Wicks theorem \cite{Wic62}, a cyclically reduced word is a commutator if and only if a cyclic permutation of this word is graphically equal to \(xyzx^{-1}y^{-1}z^{-1}\), where \(x\), \(y\), and \(z\) are some reduced words. Therefore, we have a graphical equality of the form \(xyzx^{-1}y^{-1}z^{-1} = www\). This can be described geometrically as follows. There is a graph on a torus, and all vertices have degree two, except two vertices of degree three, or except one vertex of degree four (the latter means that one of the words \(x\), \(y\), or \(z\) is empty). The edges of the graph are directed and labeled by letters \(a\) and \(b\). The complement to this graph is homeomorphic to a disk:

\begin{center}
\begin{tikzpicture}
\node (x) at (0,0) {}; \node (y) at (0.5,0.5) {}; \node (z) at (0.5,-0.5) {};
\draw (x) edge[->] (y) edge[->] (z) edge[->] cycle;
\end{tikzpicture}
\end{center}

(on this figure, the torus is presented as a rectangle with identified opposite sides). Three cars move counterclockwise along the boundary of this disk with a constant speed of one edge per minute. This motion is periodic with period \(|w|\), i.e. each \(|w|\) minutes the cars cyclically interchange (each car “reads” the word \(w\) in \(|w|\) minutes). According to the car-crash lemma (see Section 2), every such periodic motion on a torus leads to a collision. This collision can occur only at a vertex. Indeed, if a car is driving along an edge labeled, say \(a\) in the positive direction (with respect to the direction of this edge), then all remaining cars are also driving edges labeled \(a\) in the positive direction at this moment (so, they cannot collide). Also, it is easy to see that a collision at a vertex contradicts the irreducibility of the boundary label of the disk (i.e. of the word \(xyzx^{-1}y^{-1}z^{-1}\)).

The case of a non-free group is more complicated due to different structure of the picture: instead of the simple graph shown above, we have a van Kampen diagram. However, for small cancellation groups, this diagram turns out to be very thin and similar to the graph shown above (see Fig. 2). This allows us to apply “automobile technique”, although the arguments become more complicated.

We also utilise the lowest parameter principle (as in \cite{Ols89}), i.e. we assume that there are fixed small positive numbers \(\lambda < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < 1\) and each time an inequality of the form, e.g., \(2014\lambda_2 < \lambda_3\) arises, we conclude that it is automatically fulfilled by the choice of \(\lambda_4\). For reader’s convenience, we give here a brief

\begin{thebibliography}{99}

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where the first sum is over all corners at the vertex $v$. Let us continue the proof of Lemma 1. To each corner $c$ of a map on a closed surface $S$ is assigned a number $\nu(c)$ (called the weight or the value of the corner $c$), then
\[
\sum_v K(v) + \sum_D K(D) + \sum_e K(e) = 2\chi(S).
\]
Here, the summations are over all vertices $v$ and all cells $D$ of the map, and the values $K(v)$, $K(D)$, and $K(e)$, called the curvatures of the corresponding vertex, cell, and edge, are defined by the formulæ
\[
K(v) \overset{\text{def}}{=} 2 - \sum_c \nu(c), \quad K(D) \overset{\text{def}}{=} 2 - \sum_c (1 - \nu(c)), \quad K(e) \overset{\text{def}}{=} 0,
\]
where the first sum is over all corners at the vertex $v$, and the second sum is over all corners of the cell $D$.

Let us continue the proof of Lemma 1. To each corner $c$, we assign the value $\nu(c)$ by the following rule:
- all corners of the hole have value 1;
- each corner of an interior cell adjacent to a vertex of degree $k + l$ with $k$ corners of the hole and $l > 0$ corners of interior cells has value $\nu(c) = \frac{2 - k}{2}$.

Thus, the curvature of the hole is 2, and each vertex has nonpositive curvature. This curvature is zero if the vertex is adjacent to at least one corner of an interior cell. Since the Euler characteristic of the torus is 0, the weight test implies that

the total curvature of all interior cells is at most minus two.

**Step 1.** The curvature of an interior cell cannot be positive.

Let $\Gamma$ be a hypothetical interior cell of positive curvature. If the boundary of this cell consists of $d$ pieces, then its curvature is at most $2 - \frac{1}{d}$ (see Fig. 0). Hence, the curvature of $\Gamma$ can be positive only if $d \leq 5$. Such a cell must be adjacent to the hole, because otherwise we obtain a contradiction with the small cancellation condition. If only one piece of $\Gamma$ lies on the hole boundary, then
\[
0 < K(\Gamma) \leq 2 - \frac{1}{2} - \frac{1}{2} - \frac{1}{3}(d - 2) = 1 - \frac{1}{3}(d - 2), \quad \text{i.e.} \quad d \leq 4.
\]

By definition of a $C'(\lambda)$-map, the length of a common piece of the boundaries of $\Gamma$ and the hole does not exceed approximately a half of the perimeter of this cell, and the length of each other three or less pieces of the boundary is less than $\lambda|\partial\Gamma|$, which is impossible since $\lambda$ is small enough. If the boundary of $\Gamma$ has at least two common pieces with the hole boundary, then the endpoints of these pieces of $\Gamma$ have either four corners of weights $\leq \frac{1}{2}$, two corners of weights $\leq \frac{1}{2}$ and one corner of weight $\leq 0$, or two corners of weight $\leq 0$ (Fig. 0). Thus, $K(\Gamma) \leq 0$. The authors thank G. O. Astafurov for useful remarks.

### Parameter Notion

| Parameter | Notion | § |
|-----------|--------|---|
| $\lambda$ | $C'(\lambda)$ | 0, 1, 3 |
| $\lambda_1$ | Model | 1 |
| $\lambda_2$ | Distance along streets | 1 |
| $\lambda_3$ | Very near | 4 |
| $\lambda_4$ | Near | 4 |
| $\lambda_5$ | Substantially special town3 | |
| $\lambda_6$ | Globally very near | 4 |
| $\lambda_7$ | Globally near | 4 |

1. Small-cancellation maps on a torus with one hole

A map on a closed surface is a finite graph on this surface that divides the surface into simply connected domains, called cells or faces. Some cells (possibly none) are distinguished and called exterior or holes; the remaining cells are called interior. The boundary $\partial\Gamma$ and the perimeter $|\partial\Gamma|$ of a cell $\Gamma$ are defined naturally.

A piece is a simple path of positive length in the map (i.e. in the graph) that joins two vertices (possibly coinciding) of degree other than two, and does not pass through other vertices of degree other than two.

We say that a map without vertices of degree one satisfies $C'(\lambda)$ condition or is a $C'(\lambda)$-map, where $\lambda$ is a nonnegative real number if the length of any common piece of the boundaries of two interior cells $\Gamma_1$ and $\Gamma_2$ is less than $\lambda|\partial\Gamma_1|$, and the length of any piece that separates an interior cell $\Gamma$ and a hole is less than $(\frac{1}{2} + \lambda)|\partial\Gamma|$.

In what follows, we consider only maps on a torus with one hole.

**Lemma 1.** If $\lambda \ll 1$, then the boundary of each interior cell of a $C'(\lambda)$-map on the torus with a hole contains at least two pieces lying on the hole boundary.

**Proof.** We need also the following simple but useful fact, sometimes called the combinatorial Gauss–Bonnet formula.

**Weight test** [Ger87], [Pri88], see also [MCW02]. If each corner $c$ of a map on a closed surface $S$ is assigned a number $\nu(c)$ (called the weight or the value of the corner $c$), then
\[
\sum K(v) + \sum K(D) + \sum K(e) = 2\chi(S).
\]

Let us continue the proof of Lemma 1. To each corner $c$, we assign the value $\nu(c)$ by the following rule:
- all corners of the hole have value 1;
- each corner of an interior cell adjacent to a vertex of degree $k + l$ with $k$ corners of the hole and $l > 0$ corners of interior cells has value $\nu(c) = \frac{2 - k}{2}$.

Thus, the curvature of the hole is 2, and each vertex has nonpositive curvature. This curvature is zero if the vertex is adjacent to at least one corner of an interior cell. Since the Euler characteristic of the torus is 0, the weight test implies that

the total curvature of all interior cells is at most minus two.

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\[
0 < K(\Gamma) \leq 2 - \frac{1}{2} - \frac{1}{2} - \frac{1}{3}(d - 2) = 1 - \frac{1}{3}(d - 2), \quad \text{i.e.} \quad d \leq 4.
\]

By definition of a $C'(\lambda)$-map, the length of a common piece of the boundaries of $\Gamma$ and the hole does not exceed approximately a half of the perimeter of this cell, and the length of each other three or less pieces of the boundary is less than $\lambda|\partial\Gamma|$, which is impossible since $\lambda$ is small enough. If the boundary of $\Gamma$ has at least two common pieces with the hole boundary, then the endpoints of these pieces of $\Gamma$ have either four corners of weights $\leq \frac{1}{2}$, two corners of weights $\leq \frac{1}{2}$ and one corner of weight $\leq 0$, or two corners of weight $\leq 0$ (Fig. 0). Thus, $K(\Gamma) \leq 0$. The authors thank G. O. Astafurov for useful remarks.
Step 2. Completion of the proof. At least two pieces of each interior cell are adjacent to the hole.

Indeed, otherwise the cell would have a very big negative curvature (as the second inequality of \((*)\) shows, because \(d \gg 1\) by the small cancellation condition), which is a contradiction, because the interior cells have nonpositive curvature and the sum of their curvatures is at least minus two. This completes the proof of Lemma 1.

We call an interior cell \textit{ordinary} if its boundary has two common pieces with the hole boundary. An interior cell is called \textit{1-special} if its boundary has three common pieces with the hole. An interior cell is \textit{2-special} if its boundary has four common pieces with the boundary of the hole.

\textbf{Lemma 2.} If \(\lambda \ll 1\), then each interior cell of a \(C'(\lambda)\)-map on a torus with one hole is either ordinary, 1-special, or 2-special. The number of all special cells is at most two; if the map has a 2-special cell, then there are no other special cells. All vertices, except may be two, belong to one of the following classes: vertices of degree two; vertices of degree three lying on the hole boundary; vertices of degree four with exactly two corners of the hole, and these corners are not adjacent.

\textbf{Proof.} Let us use the weight test again, but now we assign the weights to the corners in a slightly different manner:

- a corner of an interior cell at a vertex of degree \(> 2\) adjacent to one corner of the hole has value \(\frac{1}{3}\);
- a corner of an interior cell at a vertex of degree \(> 2\) adjacent to two corners of the hole has value 0;
- all remaining corners have value 1.

The curvature of each vertex is nonpositive (if this vertex lies on the hole boundary, then, at this vertex, there are one corner of value 1 and either two corners of value \(\frac{1}{3}\), or another corner of value 1; if this vertex does not belong to the hole boundary, then all corners at this vertex have value one). Since all corners of the hole have value one, the curvature of the hole is two. The curvature of an interior face is nonpositive by Lemma 1. The curvatures of ordinary, 1-special, and 2-special cell are 0, \(-1\), and \(-2\), respectively. The curvatures of other hypothetical cells (“more special”) do not exceed \(-3\).

The curvature of each vertex that belongs to none of the classes listed is at most \(-1\). But according to the weight test, the total curvature must be zero. This contradiction completes the proof.

\textbf{Lemma 3.} The edges not lying on the hole boundary form a forest. This forest has at most two vertices of degree higher than two.

\textbf{Proof.} If some edges lying outside of the hole boundary form a cycle, then, cutting the torus along this cycle, one can obtain either a sphere with three holes, a torus with a hole and a sphere with two holes, or a torus with two holes and a sphere with a hole. Each cell of the spherical part shares only a small segment of its boundary with each hole, except one, by the small cancellation condition (because the cut was done along a cycle consisting of at most four pieces by Lemma 2).

Contracting the new holes, we obtain a \(C'(\lambda_1)\)-map on a sphere with at most one hole, which is impossible for small \(\lambda_1\). Thus, there is no cycles, and we have a forest. The number of vertices of degree higher than two in this forest is at most two by Lemma 2, which completes the proof.

If we contract each edge lying outside of the hole boundary, the surface remains a torus by Lemma 3. The obtained map is called the \textit{model} of the initial map. The interior cells of the model are called \textit{towns}, the vertices having at least three corners of the hole are called \textit{junctions}. Pieces of the hole boundary not lying on towns boundaries are called \textit{highways}. Each highway connects either two towns, two junctions, or a town and a junction. Adding to this model “zero length highways”, we obtain a map all whose vertices have degree two or three; e.g., if, at some vertex of degree four, there are two corners of towns, then we assume that these two towns are connected by a highway of zero length. Similarly, we assume that a town and a junction are connected by a highway of zero length if this junction lies on the boundary of the town; a junction of degree four is a pair of triple junctions connected by a highway of zero length.
Thus, a model is a map on the torus, and the cells of this map are the hole and towns. By Lemma 2, from each town, two (ordinary town), three (1-special town), or four (2-special town) highways go; each of these highways leads to another (or the same) town or to a junction (of degree three).

Note that, by Lemma 3, the perimeter of each town approximately equals (up to $\lambda_1$ multiplied by the perimeter) the perimeter of the initial cell, from which this town was obtained by contraction of edges.

**Lemma 4.** The model has either
- two 1-special towns and no junctions,
- one 2-special town and no junctions,
- one 1-special town and one triple (i.e. of degree three) junction,
- or no special towns and two triple junctions (see Fig. 1).

On Figure 1, the torus is represented as a square with identified opposite sides. Special towns are labeled by letter “o”. The dashed lines will be explain later.

**Proof.** Let us assign the weights as in the proof of Lemma 2. Then, junctions and 1-special towns have curvature $-1$; 2-special towns have curvature $-2$; the hole has curvature $2$; and all remaining towns and vertices have zero curvature. So, the assertion of the lemma follows immediately from the weight test.

The structure of an initial $C'(\lambda)$-map is shown on Figure 2, where the joints of “chains” are covered with black circles; there are some special cells and junctions behind these circles.
2. Motions

All definitions and facts mentioned in this section are taken from [Kl05].

Consider a map $M$ on a closed oriented surface $S$. A car moving around a face $D$ of this map is an orientation preserving covering of the boundary $\partial D$ of the face $D$ by an oriented circle $R$ (the circle of time).

Roughly speaking, a car moves along the boundary of its face counterclockwise (the interior of the face remains on the left from the car), without U-turns and stops. This motion is periodic.

If the number of cars being at a moment of time $t$ at a point $p$ of the 1-skeleton of $M$ equals the degree of this point, then we say that at the point $p$ at the moment $t$ a complete collision occurs; the point $p$ is called a point of complete collision. Points of complete collision lying on edges are called simply points of collision.

A multiple motion of period $T$ on a map $M$ is a set of cars $\alpha_{D,j}: R \to \partial D$, where $j = 1, \ldots, d_D$, such that
1) $d_D \geq 1$ (i.e. each face is moved around by at least one car);
2) $\alpha_{D,j}(t + T) = \alpha_{D,j+1}(t)$ for any $t \in R$ and $j = \{1, \ldots, d_D\}$ (subscripts modulo $d_D$, and the addition of points of the circle $R$ is defined naturally: $R = \mathbb{R}/\mathbb{Z}$);
3) there exists a partition of each circle $\partial D$ into $d_D$ arcs (with disjoint interiors) such that during the time interval $[0, T)$ each car $\alpha_{D,j}$ moves along the $j$-th arc.

**Car-crash lemma** [Kl05], [Kl97]. For any multiple motion on a map $M$ on a closed oriented surface $S$, the number of points of complete collision is at least

$$\chi(S) + \sum_D (d_D - 1),$$

where the sum is over all faces $D$ of $M$.

3. Van Kampen diagrams, buses, and cabs

Let us continue the proof of the main theorem. Suppose that there exists a proper power $w^n \neq 1$ which is a commutator in a torsion-free $C'(\lambda)$-group $G = \langle X \mid R \rangle$. The torsion-freeness of a small cancellation groups means that all relators in $R$ are not proper powers (see, e.g., [LS80]). It is well known (and can be easily proved) that a word is a commutator in $G = \langle X \mid R \rangle$ if and only if it can be read on the hole boundary of some van Kampen diagram on a torus with one hole. If the group satisfies the $C'(\lambda)$ condition, then this van Kampen diagram is a $C'(\lambda)$-map as the following (probably) well-known lemma shows. We provide the proof of this lemma for the sake of completeness.

**Lemma about powers.** Suppose that $G = \langle X \mid R \rangle$ is a presentation satisfying the $C'(\lambda)$ condition, where $\lambda \ll 1$, $G$ is a torsion-free group, $w$ is a word not conjugate in $G$ to a shorter word, and $n \in \mathbb{N}$. If a word $v$ is a common initial subword of a relator $r \in R$ and $w^n$, then $|v| < \frac{1}{2\lambda} |r|$.

**Proof.** Since $v$ is a subword of $w^n$, it has a form $v = w^kt$, where $t$ is an initial segment of $w$. If $k = 0$, then $|v| \leq \frac{1}{2} |r|$, since $w$ and its initial subword $t$ are irreducible. Suppose that $k \geq 1$. Then $w^{k-1}$ is a piece (it is contained twice in $r$). Therefore, the small cancellation condition implies that $|w^{k-1}| = (k - 1)|w| < \lambda |r|$. If $k > 1$, we obtain $|w| < \lambda |r|$ and $|v| = |w^kt| = |w^{k-1}| + |w| + |t| < 3\lambda |r|$. Hence, the inequality holds for all sufficiently small $\lambda$. If $k = 1$, i.e. $v = wt$, then $t$ is a piece, and, therefore, $|t| < \lambda |r|$. Now the required inequality is fulfilled, because $|w| \leq \frac{1}{2}\lambda |r|$ by the irreducibility of $w$.

Consider the model (Fig. 1) of a map. On the hole boundary of this model, the motion of $n$ cars can be specified in a natural way; these cars are called **buses**. Each bus moves with a speed of one edge per minute, and, during an $i$-th
minute, it drives along an edge labeled by the $i$-th letter of $w$ (where $i$ is modulo $|w|$). This motion is periodic with period $|w|$.

Let us draw additional edges called streets inside each town in such a way that the streets of each town form a tree, connecting the exits from the town (i.e. the set of vertices of degree one of this tree coincides with the set of endpoints of highways lying on the boundary of the town). Moreover, we draw the streets in such a way that the distance along streets between any two neighbouring exits from a town is approximately equal (up to $\lambda_2$ multiplied by the perimeter of the town) to the distance between these exits along the boundary of the town. Clearly, this can be done, because each part of the boundary of a town is not significantly greater than a half of its perimeter (i.e. $\leq (\frac{1}{2} + \lambda_2)|\partial \Gamma|$). The streets and the highways form a map (with one cell) on the torus (see Fig. 1 (lower left), where streets are drawn as dashed lines). The vertices of degree higher than two in special towns are also called junctions henceforth. A junction (of streets) of degree four is considered as a pair of junctions of degree three joined by a highway of zero length. A special town $\Gamma$ is called substantially special if all street junctions in this town are further than $\lambda_5|\partial \Gamma|$ from the boundary of the town.

Now, we define a motion of $n$ cars called cabs on this map. A cab drives along a highway together with the corresponding bus. When a cab enters a town, it moves along the streets with a constant (approximately unit) speed in such a way that it leaves the town simultaneously with the corresponding bus (which drives along the boundary of the town). On Figure 3, we draw a special town and positions of a bus (black circle) and a cab (white circle) at four moments.
4. Nearness

Let $x$ and $y$ be points of the graph formed by the highways, streets, and boundaries of towns. We say that $x$ is (locally) near $y$ if either $x = y$, or $y$ is in a town $\Gamma$ and the distance between $x$ and $y$ is at most $\lambda_1|\partial\Gamma|$. If this distance is at most $\lambda_3|\partial\Gamma|$, then we say that $x$ is (locally) very near $y$.

We say that $x$ and $y$ are globally near each other if the distance between them is at most $\lambda_7|w|$, where $|w|$ is the period of the motion. If this distance is at most $\lambda_6|w|$, then these points are said to be globally very near each other. Note that, if $x$ is near $y$, then these points are globally very near each other.

5. Where do collisions happen?

The car-crash lemma implies that a complete collision of cabs occurs at least at $n - 1$ points. Indeed, the map formed by all streets and highways has one cell; along the boundary of this cell $n$ cabs move regularly and the Euler characteristic of the torus is zero. It remains to understand where can collisions occur.

**Lemma about far-from-junction collisions.** Suppose that some cabs collide at a point $p$ which is not a junction. Then $p$ is in a town and either there is a junction very near to $p$ or $p$ is on a highway of zero length connecting two different special towns with equal labels (i.e. the labels of the corresponding cells of the initial diagram are equal if we read them starting from the point $p$).

**Proof.** Consider several cases.

I. **Collisions cannot occur on a highway of nonzero length outside junctions.**

Indeed, suppose a collision of cabs occurred on an edge labeled by a letter $x$. Since cabs move along highways in the same way as buses, this means that a bus moves along an edge labeled by $x$ at the moment of collision and another bus moves along this edge in the opposite direction, i.e. moves along an edge labeled by $x^{-1}$. This contradicts the definition of the motion, because, at each moment of time, all buses drive along edges with the same labels. A collision at some vertex on a highway is also impossible, because, by definition of the motion of buses, this would mean that the word $w$ has a subword $xx^{-1}$ that contradicts the irreducibility of $w$.

II. If two cabs collide in a town $\Gamma$, then at least one of them leaves the town in at most $\lambda_1|\partial\Gamma|$ minutes after the collision.

The $C'(\lambda_1)$ condition implies that the boundary of $\Gamma$ does not have two segments with equal labels of length $\lambda_1|\partial\Gamma|$. Therefore, two cabs cannot stay in one town longer than $\lambda_1|\partial\Gamma|$ minutes simultaneously.

III. If two cabs collide in a town $\Gamma$ and this collision occurred further than $\lambda_2|\partial\Gamma|$ from the junctions, then, during $\lambda_1|\partial\Gamma|$ minutes after the collision, precisely one of the cabs leaves $\Gamma$.

If both cabs leave the town during $\lambda_1|\partial\Gamma|$ minutes, then the distance between two entrances to this town is at most $\lambda_2|\partial\Gamma|$. This is possible only in a special town near a junction (because the distance between two entrances in an ordinary town is approximately a half of the perimeter, and there is always a street junction between two entrances in a special town).

IV. The length of a highway $r$ along which one of the colliding cabs leaves the town $\Gamma$ during $\lambda_1|\partial\Gamma|$ minutes is at most $\lambda_2|\partial\Gamma|$.

Suppose that this highway is longer. Then, one can choose a segment $s$ of length at least $\lambda_1|\partial\Gamma|$ on this highway such that, before the collision, one of the buses is driving along $s$ while the other one is driving along $\partial\Gamma$; and, shortly after the collision, one of the buses is driving along $s$ (in the opposite direction), while the other bus is driving along $\partial\Gamma$. This means that the label of the boundary of the cell $\Gamma$ contains both $f$ and $f^{-1}$ as subwords, where $f$ is the label of $s$. This is impossible by the small cancellation condition.

V. The highway $r$ leads to a town $\Delta$ with a label equals the label of the town $\Gamma$.

Indeed, shortly after the collision one of the buses passes through the highway $r$ and then is driving along the boundary of a town $\Delta$ during at least $\lambda_1|\partial\Gamma|$ minutes (because we suppose that there are no junctions near $p$), while the other bus is driving along the boundary of $\Gamma$. The small cancellation condition implies that the labels of $\Gamma$ and $\Delta$ are equal.

VI. The highway $r$ has zero length.

As it was shown above, one of the buses goes from a point $A \in \partial\Gamma$ to a point $B \in \partial\Delta$, while the other bus goes from a point $A' \in \partial\Delta$ to a point $B' \in \partial\Gamma$ (Fig. 4). Each of them is driving along a path with the same label $u$ during this time. Thus, the labels of $\Gamma$ and $\Delta$ are equal if we read them starting from the points $A$ and $A'$; and the same is true, if we read these labels starting from points $B'$ and $B$. Therefore, the label $uv$ of the part of the boundary of $\Gamma$ from $B'$ to $A$ (counterclockwise) equals the label of the part of the boundary of $\Delta$ from $B$ to $A'$. Hence, $(uv)^2 = 1$ in the free group. This means that $uv = 1$. Therefore, the highway $r$ has zero length by the irreducibility.

*) Points in zero distance from a town are also considered as being in the town.

**) The label of a town is the label of the corresponding cell of the initial diagram, i.e. one of the defining relators of $G$. 
VII. The towns $\Delta$ and $\Gamma$ are special.

Suppose the contrary. The length of each piece of the boundary of an ordinary town is approximately the half of the perimeter; the pieces of the boundary of a special town also have a big length ($\geq \lambda_4 |\partial \Gamma|$), because we assume that there are no junctions near the point $p$ (otherwise, there is nothing to prove). The labels of towns $\Gamma$ and $\Delta$ are equal, if we read them from the point $p$. Therefore, we have a segment of the hole boundary with a label of length much more than the length of a defining relator (the part $uv$ on Fig. 5), which is impossible by the lemma about powers. This completes the proof.

Lemma about uniqueness. At most one collision can happen globally far from junctions.

Proof. By the lemma about far-from-junction collisions, it suffices to prove that there cannot be two towns $\Gamma$ and $\Delta$ connected by two zero-length highways $p$ and $q$ with collisions happening on each of these highways (Fig. 6).

Three exits from the town $\Gamma$ divides its boundary into three parts $a$, $b$, and $c$ (listed counterclockwise starting with the piece between $p$ and $q$). Similar fragments of the boundary of $\Delta$ are denoted $a'$, $b'$, and $c'$. The labels of $\Gamma$ and $\Delta$ are equal if are read starting from $p$ (or $q$) by lemma about far-from-junction collisions. Therefore, the labels of segments $a \cup b'$, $a' \cup b$, $c \cup a'$ and $c' \cup a$ of the hole boundary are equal to subwords of the label of $\Gamma$. Hence, by lemma about powers, these fragments cannot be very long:

$$|a| + |b'| \leq \frac{1}{2} |\partial \Gamma|, \quad |a'| + |b| \leq \frac{1}{2} |\partial \Gamma|, \quad |c| + |a'| \leq \frac{1}{2} |\partial \Gamma|, \quad |c'| + |a| \leq \frac{1}{2} |\partial \Gamma|.$$
where $x \preceq y$ means $x < y + k\lambda_1|\partial\Gamma|$ for an absolute constant $k$. Summing up these four inequalities we obtain $2|a| + 2|a'| + |b| + |b'| + |c| + |c'| \leq 2|\partial\Gamma|$. Taking into account the equality $|a| + |b| + |c| = |a'| + |b'| + |c'| = |\partial\Gamma|$, we get $|a| + |a'| \preceq 0$ that contradicts the fact that collision points $p$ and $q$ are far from a junction (lying in $\Gamma$).

6. Neighborhoods of junctions and the proof of the theorem for $n \neq 3$

As we already know, there is at most one point of collision lying (globally) far from junctions, and there are at most two junctions. Therefore, if (globally) near each junction, there was at most one collision point, then the total number of collision points would not exceed three. By the car-crash lemma, this means that the multiplicity of the motion (i.e. $n$) is at most four which proves the theorem for all $n \geq 5$. However, a nontrivial commutator in a torsion-free small cancellation group cannot be a square (see [Sch80], [Gu89]), and, therefore, cannot be a fourth power. Thus, the theorem would be proven for all $n$, except 3.

The problem is that we do not know how many collision-points are near junctions. The following lemma allows us to overcome this difficulty.

Lemma about junction neighborhoods. In $\lambda_7|w|$-neighborhoods of junctions (i.e. globally near them), the schedule of the motion can be modified in such a way that, in each connected component of the union of these neighborhoods, at most one collision occurs.

Proof. Consider the graph formed by the streets and highways; we do not need towns anymore. Let us consider neighborhoods of junctions of radius $\lambda_7|w|$ in this graph and then thicken them slightly to obtain open neighborhoods of junctions on the torus. The union $U$ of these neighborhoods is either two disks, one disk, or one annulus (see the upper part of Fig. 7). Now, we remove edges of the graph lying inside $U$ and add the boundary of $U$ to this graph; in the annulus case, we add also an edge cutting the annulus to make it simply connected (see the lower part of Fig. 7).

We obtain a map on the torus with three (in the first case) or two (in the second and third cases) faces. Let us specify a motion on this map. The cabs moving around the old face (the hole) drive almost as was defined before, but the time they spent inside $U$ they now spend moving along the boundary of $U$. Around the new faces, new cabs moves as follows. When there are no other cabs on the boundary of the corresponding face (such a moment exists, because old cabs spend little time on the boundary of the domain $U$, less than $\lambda_7$ multiplied by the period), a new cab drives quickly around almost the entire face, and then moves slowly along the remaining part of the boundary of the face until the end of period. This motion is periodic with the same period, the multiplicity of the motion on the new faces is 1 and each new cab collides at most once (during the slow motion). This completes the proof.

To complete the proof of the theorem (for $n \neq 3$), it remains to apply the previous lemma. As it was shown above, there is at most one point of collision globally far from junctions. Hence, the argument from the beginning of the section completes the proof of the theorem for all $n \neq 3$. 

Fig. 7
7. The proof of the theorem for \( n = 3 \)

**Lemma about close junctions.** If the junctions are globally very near each other, then all collisions occur globally near the junctions.

**Proof.** Suppose the contrary. By the lemma about far-from-junction collisions, these junctions are in different special towns \( \Gamma \) and \( \Delta \) with equal labels, and a collision occurs far from junctions at a point \( p \) lying on the intersection of the boundaries of these towns (Fig. 8). Moreover, the junctions are globally very near each other, i.e. they are globally very near exits from the towns to a (globally very short) highway \( r \). In particular, this means that the segments \( c \) and \( c' \) of town boundaries lying in opposite sides to the exits to \( r \) are equal to about a half of the perimeter:

\[
|c| \geq \frac{1}{2} |\partial \Gamma| - 10 \lambda_6 |w| \leq |c'|.
\]

![Fig. 8](image)

If the collision point \( p \) is globally far from junctions, then the segments \( a \) and \( a' \) of the boundaries of the towns connecting \( p \) with the endpoints of \( r \) are long:

\[
|a'| \geq \lambda_7 |w| \leq |a|.
\]

Summing up these inequalities we obtain

\[
|c| + |a'| \geq \frac{1}{2} |\partial \Gamma| + (\lambda_7 - 10 \lambda_6) |w| \leq |c'| + |a|.
\]

However, \( c \cup a' \) is a part of the hole boundary and its length cannot exceed \( \frac{1}{2} |\partial \Gamma| + \lambda_1 |\partial \Gamma| \). This contradiction completes the proof.

**Lemma about safe junctions.** Suppose that a collision occurs at a point \( p \); very near \( p \), there is exactly one junction, and at the moment of the collision, there are less than three cabs very near the point \( p \). Then the motion of cabs very near the point \( p \) can be modified in such a way that no collisions in this neighborhood of \( p \) will happen.

---

\( ^\ast \) The distance along streets between two exits is approximately equal to the distance between these exits along the boundary of the town. Therefore, if one of these streets is short, then the sum of lengths of the other two streets is approximately a half of the perimeter of the town.
Proof. Two cabs collide near the junction, and the remaining cabs are far from the junction at this moment (Fig. 9, the upper row). Let us slightly change the schedule of the motion. Namely, the first cab approaches the junction and decelerates a little bit to give to the second cab the possibility to pass this junction, then accelerates again and leaves the neighborhood of the junction according to its initial schedule (Fig. 9, the lower row).

Lemma about absence of dangerous junctions. Suppose that the junctions are not globally very near each other, a collision occurs at a point $p$, and there is a junction $j$ very near $p$. Then there cannot be three cabs at the moment of the collision very near $p$.

Proof. Suppose the contrary. There are five possible cases (Fig. 10).
I. The point $p$ is outside the towns.

By definition of the nearness, this means that $p = j$, and three buses collide at this point simultaneously. Therefore, all three edges leading to this junction have the same labels, and the word written on the hole boundary is reducible, which is impossible. In what follows, we assume that the point $p$ is in a town $\Gamma$. Note that the distance from $p$ to the second junction (different from $j$), even if it lies in $\Gamma$, is large (at least $\lambda|\partial|w|$), because these junctions are not globally very near each other.

II. The junction $j$ is inside a substantially special town $\Delta$.

The point $p$ cannot lie in the same town, because this would mean that two different long (of length longer than $\lambda_1|\partial|\Gamma|$) parts of the boundary of the town $\Gamma = \Delta$ have equal labels, which is impossible by the assumption (similarly to the argument in II in the proof of the lemma about far-from-junction collisions). Therefore, $p \in \Gamma \neq \Delta \ni j$. In this case, repeating word-by-word the argument from V of the proof of the lemma about far-from-junction collisions, one can verify easily that the labels of $\Gamma$ and $\Delta$ are equal. This implies that

$$\lambda_5|\partial|\Delta| \leq \rho(p,j) \leq \lambda_3|\partial|\Gamma| = \lambda_3|\partial|\Delta|$$ (from now on $\rho$ means the distance).

(The first inequality holds, because the town $\Delta$ is substantially special; the second inequality holds because $j$ is very near $p$.) This is impossible, since $\lambda_3 \ll \lambda_5$.

III. There is only one town (i.e. $\Gamma$) near the point $p$.

Consider the $\lambda_4|\partial|\Gamma|$-neighborhood of $p$; remove the $\lambda_3|\partial|\Gamma$-neighborhood of this point, and intersect this difference with the 1-skeleton of the model. We obtain a disjoint union of four paths: $u_1, u_2, v,$ and $w$, where $u_1$ and $u_2$ lie on the boundary of $\Gamma$, and $v$ and $w$ lie on highways leading to the junction $j$. The label of paths $u_1, v,$ and $w$ are approximately equal (i.e. these three words contain a common subword of the length at least $(\lambda_4 - 10\lambda_3)|\partial|\Gamma|$), because, along these paths, buses are driving simultaneously while approaching the junction. On the other hand, the labels of paths $u_2^{-1}, v^{-1},$ and $w^{-1}$ are also approximately equal in the same sense, because, along these paths, buses are moving simultaneously while driving away from the junction. Therefore, the labels of paths $u_1$ and $u_2$ are approximately equal (contain a common subword of the length at least $(\lambda_4 - 20\lambda_3)|\partial|\Gamma|$), which contradicts the small cancellation condition.
IV. There are precisely two towns near the point $p$.

In this case, we obtain similarly two long segments with equal labels on the boundary of the town $\Gamma$, which is a contradiction.

V. There are precisely three towns near the point $p$.

Let us denote these towns $\Gamma$, $\Delta$, and $E$. Arguing as in the two previous cases, we obtain six long (of the length at least $(\lambda - 20\lambda_3)|\partial \Gamma|$) paths: $u_1$, $u_2$, $v_1$, $v_2$, $w_1$, and $w_2$. The first two lie on the boundary of $\Gamma$, the second two lie on the boundary of $\Delta$, and third two lie on the boundary of $E$; the labels of $u_1$, $v_1$, and $w_1$ coincide (along these paths buses approach the junction simultaneously) and labels of $u_2$, $v_2$, and $w_2$ coincide (along this paths buses drive away from the junctions simultaneously). By the small cancellation condition, this means that the cells of the initial $C'(\lambda)$-map on the torus corresponding to the three towns have equal labels, and these labels are equal, if one read them starting from paths $u_1$, $v_1$, and $w_1$ or starting from paths $u_2$, $v_2$, and $w_2$.

Thus, the paths of boundaries of the three cells between $[u_2, u_1^+]$, $[v_2, v_1^+]$, and $[w_2, w_1^+]$ also have a common label $g$, where $\pi^-$ and $\pi^+$ denote the origin and the endpoint of a path $\pi$ (we assume, that all paths are oriented counterclockwise with respect to the hole). Now, consider the following three paths in the model: from $u_1^+$ to $v_2^-$, from $v_1^+$ to $w_2^-$, and from $w_1^+$ to $u_2^-$. They also have a common label $h$, because the buses drive along these paths simultaneously. Moreover, these six paths form a closed contour in the initial diagram, and this contour does not have any cells inside. Therefore, $(gh)^3 = 1$ in the free group, i.e. $h = g^{-1}$.

However, the label of the hole in the initial diagram is a reduced word, and, therefore, the boundaries of the three cells have a common point and the labels of these three cells are equal if we read them starting from this point (Fig 11). This means that the junction $j$ lies outside of towns, i.e. only one of these three cells can be special (i.e. contains another junction). Arguing as in VII in the proof of lemma about far-from-junction collisions, we conclude that the label of the hole boundary contains a common subword with one of defining relators, and the length of this subword exceeds significantly a half of the length of the defining relator. If, for example, the towns $\Gamma$ and $\Delta$ on Figure 11 are ordinary, then the label of the part of the boundary of the hole from $x$ to $y$ contains almost a whole relator as a subword. This contradicts the lemma about powers and completes the proof.

Let us continue the proof of the theorem. By the car-crash lemma, it suffices to show that the number of collision points cannot exceed one.

If the junctions are globally very near each other, then, by the lemma about close junctions, all points of collision are globally near these junctions and, by the lemma about neighborhoods of junctions, we can assume that globally near junctions there is at most one point of collision. This means that there is at most one collision point, as required.

If the junctions are not globally very near each other, then, applying the lemma about absence of dangerous junctions and the lemma about safe junctions, we again obtain a motion with at most one collision point. This completes the proof of the theorem.

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