DO WE UNDERSTAND WHAT IS DECONFINEMENT?

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Abstract
An overview is given of different approaches to describing the process of deconfinement in quantum chromodynamics. The analysis of the known approaches demonstrates that the detailed picture of how deconfinement really occurs has not yet been understood. Therefore, one has to be rather cautious when interpreting experimental signals as attributed to deconfinement.

Key-words: deconfinement, phase transitions, crossovers, quark-hadron matter.

1 Introduction
When talking about deconfinement in quantum chromodynamics, one should distinguish two things: deconfinement as a general phenomenon and deconfinement as a concrete process. What is, in general, the phenomenon of deconfinement is well known - this is the transformation of hadron matter into quark-gluon plasma occurring with increasing temperature or density of matter (see reviews [1–5]). But how this process occurs in reality? The answer to the question is crucially important for the correct interpretation of experimental signals, that are attributed to deconfinement, such as the suppression of charmonium production ($\text{J}/\psi$ suppression [5,6]), the enhancement of strangeness production [7,8], the enhancement of dilepton production [7,8] or other signals.

The process of deconfinement can be considered from two points of view, stationary and nonstationary. In the stationary picture, one considers the transformation of equilibrium infinite hadron matter into quark-gluon plasma, when rising temperature or density. In the nonstationary picture, one takes into account the peculiarities related to heavy ion reactions, which involves the consideration of deconfinement in nonequilibrium finite objects [9]. As is evident, before going to the complications of the nonstationary picture, it is necessary to have a more or less complete understanding of what happens in the stationary picture, which the nonstationary one is based on. In what follows, we shall analyze only the stationary case and will show that even this has not yet been completely understood.

2 Numerical Lattice Simulations
Lattice simulations of high-temperature QCD provide nonperturbative theoretical insights into the phenomenology of the transition from hadronic matter to the quark-gluon plasma.
One of the basic goals of lattice QCD calculations at finite $T$ is to provide quantitative results for the deconfinement transition temperature. Up to now, this goal has been achieved only in the pure gauge sector [10]. The value of the critical temperature for the deconfining phase transition in a $SU(3)$ pure gauge theory (without quarks) is known with small errors of the order of 3%. This temperature $T_c \approx 270$ MeV, and deconfinement is a real phase transition, either of first order or a rather sharp phase transition of second order [10].

Unlike in the pure gluodynamics, the transition temperature for $SU(3)$ chromodynamics with finite quark masses is not well defined. Even at vanishing baryon number density, there is no yet a satisfactory understanding of the critical behaviour in QCD [10]. In zero-density chromodynamics at physical values of quarks, deconfinement is rather a rapid crossover than a pure phase transition [10,11]. The crossover temperature can conditionally be defined as that corresponding to a maximum of some thermodynamic characteristics [12], such as a susceptibility [10,11]. Different lattice calculations for the zero-density QCD give the crossover temperatures in the interval $T_c \approx (140 - 190)$ MeV [10,11].

Finite baryon density calculations in QCD are affected by the so-called sign problem, when the fermion determinant becomes complex for nonzero values of the chemical potential and the partition function fails to be positive [10,13-15], which prohibits the use of the conventional numerical algorithms. Because numerical simulations of QCD at finite baryon density are plagued by the principal technical difficulties, the present understanding of deconfinement is not satisfactory. However, the available data show no signals of phase transition [13–15]. The present situation is rather pessimistic - it seems that there is no reliable hope to get important improvements in the knowledge of QCD at finite density from lattice simulations [14].

3 Pure Phase Models

Because of the principal difficulties and uncertainty in the finite-density lattice simulations, several phenomenological models of deconfinement have been suggested. The most often employed are the pure phase models, when hadron matter and quark-gluon plasma are treated as different pure thermodynamic phases. Each phase is supposed to possess its own thermodynamic potential.

It is convenient to work with the grand potential $\Omega = -PV$, which is a function of temperature $T$, volume $V$, and chemical potentials $\mu_i$ of different particles, each particle sort being enumerated by the index $i = 1, 2, \ldots$. Each type of particles is characterized by a set of quantum numbers, such as the baryon number $B_i$, strangeness $S_i$, and others. The related baryon density $n_B$ and strangeness density $n_S$ are

$$n_B = \sum_i B_i \rho_i \quad n_S = \sum_i S_i \rho_i,$$

where $\rho_i$ is the density of particles of type $i$. Then the chemical potential $\mu_i$ can be expressed as

$$\mu_i = \mu_B B_i + \mu_S S_i,$$

with the baryon potential $\mu_B$ and strangeness potential $\mu_S$. Therefore, the grand potential can be considered as a function $\Omega = \Omega(T,V,\mu_B,\mu_S)$. Consequently, the pressure is a function $P = P(T,\mu_B,\mu_S)$. The baryon and strangeness densities (1) can be written as the derivatives

$$n_B = \frac{\partial P}{\partial \mu_B}, \quad n_S = \frac{\partial P}{\partial \mu_S}.$$
In what follows, we shall consider, for simplicity, the case with fixed $\mu_S$ and will analyze the dependence of pressure on temperature and $\mu_B$, writing $P = P(T, \mu_B)$. Respectively, the baryon density $n_B = n_B(T, \mu_B)$.

In pure-phase models of deconfinement, one divides all particles into two groups, one group consisting of hadrons and another group consisting of quarks and gluons. The first group is assumed to form the hadron phase and the second one, the quark-gluon phase. The corresponding pressures of pure phases, $P_h$ and $P_p$, are calculated in different approximations [1–5]. As a result, the baryon densities of pure hadron and pure plasma phases are different,

$$n_{ Bh} \equiv \sum_i B_{ih} \rho_{ih} = \frac{\partial P_h}{\partial \mu_B}, \quad n_{ Bp} \equiv \sum_i B_{ip} \rho_{ip} = \frac{\partial P_p}{\partial \mu_B},$$

(4)

where the summations include, respectively, either only hadrons (and antihadrons) or only quarks, antiquarks, and gluons. The deconfinement temperature is given by the equality

$$P_h(T_c, \mu_B) = P_p(T_c, \mu_B)$$

(5)

yielding a uniquely defined transition line $T_c = T_c(\mu_B)$. However, the transition temperature as a function of baryon density is not uniquely defined. This is because there are two different baryon densities (4), which also are different at the transition temperature,

$$n_h \equiv n_{ Bh}(T_c, \mu_M), \quad n_p \equiv n_{ Bp}(T_c, \mu_B).$$

(6)

These densities are different at $T_c$ since at this point the pressures of hadron phase and of plasma phase intersect. For the deconfinement transition, one has $n_h > n_p$. Treating the relations (6) as the equations for the baryon potential, one gets two different potentials $\mu_h \equiv \mu_B(T_c, n_h)$ and $\mu_p \equiv \mu_B(T_c, n_p)$. Substituting these into the dependence $T_c(\mu_B)$, one obtains two lines

$$T_h(n_h) \equiv T_c(\mu_h), \quad T_p(n_p) \equiv T_c(\mu_p).$$

(7)

These two lines environ the region on the plane $T_c - n_B$ where the hadron and plasma phases coexist. This situation is completely analogous to the standard case of first-order phase transition, when coexisting phases are treated in different approximations or when the pressure as a function of density contains an instability interval [16,17]. In the considered case, the baryon potential of the hadron-plasma mixture is

$$\mu_B(T_c, n_B) = x_h \mu_h + x_p \mu_p,$$

(8)

where the phase concentrations are defined by the equations

$$x_h n_h + x_p n_p = n_B, \quad x_h + x_p = 1,$$

(9)

which yield

$$x_h = \frac{n_B - n_p}{n_h - n_p}, \quad x_p = \frac{n_h - n_B}{n_h - n_p}.$$  

(10)

Let us stress that the coexisting phases, characterized by the linear combination (8), are both macroscopic. Such a mixture of macroscopic phases is called the Gibbs mixture, in order to distinguish it from a mixture of mesoscopic phases [18].
In the region of the existence of the Gibbs mixture, when $n_B$ changes from $n_p$ to $n_h$, the transition line is represented by a horizontal line

$$T_c(n_B) = \text{const} \quad (n_p \leq n_B \leq n_h)$$

(11)

connecting the points $T_h$ and $T_p$ defined in the relations (7). Along this line, the pressure

$$p(T,n_B) \equiv P(T,\mu_B(T,n_B))$$

(12)

is given by a horizontal line connecting the points $P_h$ and $P_p$, according to equality (5),

$$p(T_c,n_B) = \text{const} \quad (n_p \leq n_B \leq n_h).$$

(13)

Then the compression modulus

$$\kappa^{-1}_T(T,n_B) \equiv n_B \frac{\partial p}{\partial n_B}$$

(14)

remains zero on the transition line,

$$\kappa^{-1}_T(T_c,n_B) = 0 \quad (n_p \leq n_B \leq n_h).$$

(15)

Hence, the compressibility $\kappa_T = \infty$ diverges exhibiting instability. Thus, the Gibbs mixture is, actually, unstable.

Constructing the pressure of a pure phase, one invokes phenomenological arguments of the mean-field type. The most popular is a kind of a quasiparticle description, when each sort of particles is characterized by an effective spectrum $\omega_i(k)$, the interparticle interactions being included in the spectrum as mean-field parts. For example, one may employ a relativistic spectrum

$$\omega_i(k) = \sqrt{k^2 + m_i^2 + \Pi_i}$$

or a semi-relativistic one

$$\omega_i(k) = \sqrt{k^2 + m_i^2 + U_i},$$

where $m_i$ is a bare particle mass and $\Pi_i$ or $U_i$ are the real parts of self-energy playing the role of mean fields. The interparticle interactions are taken sometimes in the excluded volume approximation, as it is done in the statistical bootstrap models [19] of hadrons. The interactions between hadrons can be modelled by various linear [4,12] or nonlinear [20] functions of density.

The quasiparticle picture is also used for the quark-gluon plasma phase [4,12,21,22] giving the description of pure gluodynamics and of zero-density chromodynamics in good agreement with numerical lattice calculations for the region of pure deconfined phase. However, the pure-phase models cannot correctly describe the whole process of deconfinement, always predicting a first-order phase transition, contrary to lattice simulations (see discussion in [4]).

4 Extrapolating Equations of State

Instead of invoking phenomenological arguments for describing the whole process of deconfinement, one sometimes resorts to the following method. One starts with considering the properties of a pure phase far from deconfinement, where the equation of state for this pure
phase can be found with a reliable certainty, and then one extrapolates the found equation to the region close to deconfinement. As is evident, analyzing the properties of a sole pure phase, it is impossible to get a correct description of the whole phase transition, but only the region of stability can be determined in this way. Nevertheless, determining an instability point can give a reasonable estimate for that of deconfinement, and the extrapolated equation of state could provide an approximation for the region close to deconfinement. For example, examining the stability boundaries for nucleons inside nuclear matter yields [23] quite reasonable values for the instability temperature of 200 MeV and for the instability of 2 normal densities, which are close to the estimates for the deconfinement temperature and density [1–5].

The QCD pressure is known to be presentable as an asymptotic expansion in powers of the coupling parameter $g$ at high temperature and zero chemical potential, when the coupling parameter is small. This expansion is known [24,25] to the order of $O(g^6 \ln g)$,

$$P(g) \simeq \frac{8\pi^2}{45} T^4 \left( a_0 + a_2 g^2 + a_3 g^3 + a_4 g^4 + a_4' g^4 \ln g + a_5 g^5 \right),$$  \hspace{1cm} (16)$$

with the coefficients

$$a_0 = 1 + \frac{21}{32} N_f, \quad a_2 = -0.09499 \left(1 + \frac{5}{12} N_f\right), \quad a_3 = 0.12094 \left(1 + \frac{1}{6} N_f\right)^{3/2},$$

$$a_4 = 0.04331 \left(1 + \frac{1}{6} N_f\right) \ln \left(1 + \frac{1}{6} N_f\right) + 0.01733 - 0.00763 N_f - 0.00088 N_f^2 - 0.01323 \left(1 + \frac{5}{12} N_f\right) \left(1 - \frac{2}{33} N_f\right) \ln \frac{\mu}{T}, \quad a_4' = 0.08662 \left(1 + \frac{1}{6} N_f\right),$$

$$a_5 = - \left(1 + \frac{1}{6} N_f\right)^{1/2} \left(0.12806 + 0.00717 N_f - 0.00027 N_f^2 + 0.02527 \left(1 + \frac{1}{6} N_f\right)^{3/2} \left(1 - \frac{2}{33} N_f\right) \ln \frac{\mu}{T}, \right.$$ 

where $N_f$ is the number of flavours. The dimensional regularization is used here, and the renormalization scale $\mu$ corresponds to the modified minimal subtraction scheme $\overline{MS}$.

This expansion (16) is not convergent, but it is merely asymptotic, being valid only for $g \to 0$. Accepting the high-temperature dependence

$$g^2(T) \simeq \frac{24\pi^2}{(11N_c - 2N_f) \ln(T/\Lambda)} \quad (T \to \infty),$$

in which $N_c = 3$ is the number of colours and $\Lambda \approx 200$ MeV is the QCD scale parameter, one sees that the condition $g \ll 1$ corresponds to the very high temperatures $T > 10^3$ MeV. This is why the form (16) does not agree with the lattice simulations for lower temperatures [26].

To extrapolate expression (16) to the region of finite $g$, Padé approximants have been used [27,28]. The latter are often employed in the attempt of improving perturbative results of field theory [29]. However, the constructed Padé approximants exhibit unnatural features, containing terms proportional to $g$ both in the numerator and denominator [28]. The most important is that the Padé approximants do not converge, but some turn out to develop unphysical poles [27,28]. At large $g$, Padé approximants exhibit chaotic behaviour,
since $P_{MN} \sim g^{M-N}$ as $g \to \infty$, so that $P_{MN} \to \pm \infty$ if $M > N$, $P_{MN} \to \text{const}$ for $M = N$, and $P_{MN} \to 0$ when $M < N$.

Another method of deriving expressions, valid at finite values of the coupling parameter, from asymptotic expansions having sense only in the vicinity of zero coupling parameter, is based on the **Self-Similar Approximation Theory** [30–39]. Below, we give a survey of the method allowing us to obtain the equation of state in QCD for finite values of the coupling parameter and for temperatures in a wide diapason [40]. The approach employs the **self-similar exponential approximants** [38], which, contrary to Padé approximants, contain no poles and possess good convergence.

It is convenient to introduce the dimensionless function

$$P(g) \equiv \frac{P(g)}{P(0)}, \quad P(0) = \frac{8\pi^2}{45} \left(1 + \frac{21}{32} N_f\right) T^4,$$

(normalizing pressure $P(g)$ by the Stefan-Boltzmann limit $P(0)$). Then, expansion (16) reduces to the set of the approximants

$$\mathcal{P}_k(g) = \sum_{n=0}^{k} \bar{a}_n g^n,$$

where $k = 0, 1, 2, 3, 4, 5$ and the reduced coefficients are

$$\bar{a}_0 = 1, \quad \bar{a}_1 = 0, \quad \bar{a}_4 = \frac{a_4 + a'_4 \ln g}{a_0}, \quad \bar{a}_n = \frac{a_n}{a_0} \quad (n = 2, 3, 5).$$

According to the idea of the *optimized perturbation theory* [30], the renormalization scale can be treated as a control function defined by the minimal difference condition. For the present case, we require that the approximation (18), where $\mu$ appears first, be equal to the precedent approximation. This leads to the equations

$$\mathcal{P}_4(g) = \mathcal{P}_3(g), \quad \bar{a}_4 = 0.$$

Such minimal difference conditions are often employed in theoretical calculations [30,41–47]. The meaning of this condition has been explained in the frame of the self-similar approximation theory [30–39] as a kind of a fixed-point condition for an approximation cascade. Another type of fixed-point conditions is the minimal sensitivity condition [48–55] that is also often used in calculations. But the latter condition cannot be directly applied to the expansion (18). It is worth noting that the optimized perturbation theory [30] should not be confused with the variational minimization of free energy, common in statistical mechanics [56,57]. The optimized perturbation theory is a *systematic procedure* yielding a convergent sequence of approximants, while the variational minimization of free energy is a one-step procedure giving just a single estimate. In addition, the latter is valid solely for free energy or some other thermodynamic potential, since this variation is based on minimizing the right-hand side of the Gibbs-Bogolubov inequality, while the optimized perturbation theory can be developed for any quantity of interest [30].

Condition (19) results in the renormalization scale

$$\mu = \gamma T g^\nu = \mu(T, g)$$

as a function of temperature and the coupling parameter, where

$$0.01323 \left(1 + \frac{5}{12} N_f\right) \left(1 - \frac{2}{33} N_f\right) \ln \gamma = \ldots$$
\[ = 0.04331 \left(1 + \frac{1}{6} N_f\right) \ln \left(1 + \frac{1}{6} N_f\right) + 0.01733 - 0.00763 N_f - 0.00088 N_f^2 \]

and

\[ \nu \equiv \frac{0.08662 \left(1 + \frac{1}{6} N_f\right)}{0.01323 \left(1 + \frac{5}{12} N_f\right) \left(1 - \frac{1}{32} N_f\right)} . \]

In particular, for \( N_f = 6 \), one has \( \gamma = 0.996964 \) and \( \nu = 5.879155 \). With the scale (20), the form (18) reduces to the expansion

\[ \overline{P}(g) \simeq 1 + \overline{a}_2 g^2 + \overline{a}_3 g^3 + \overline{a}_5 g^5 , \]

valid for \( g \to 0 \).

The extrapolation of the asymptotic expansion (21) to the region of finite \( g \), by means of the self-similar exponential approximants [38,40], leads to the sequence

\[ \overline{P}_2(g) = \exp \left(c_2 g^2\right) , \quad \overline{P}_3(g) = \exp \left(c_2 g^2 \exp(c_3 g)\right) , \]
\[ \overline{P}_5(g) = \exp \left(c_2 g^2 \exp(c_3 g \exp(c_5 g^2))\right) , \]

in which the coefficients

\[ c_2 = \frac{a_2}{a_0} \tau_2 , \quad c_3 = \frac{a_3}{a_2} \tau_3 , \quad c_5 = \frac{a_5}{a_3} \tau_5 \]

are connected with the control functions \( \tau_i \). The latter are to be defined from fixed-point conditions or from the minimization of a cost functional.

The running QCD coupling \( \alpha_s = \alpha_s(\mu) \) satisfies the renormalization-group equation

\[ \mu \frac{\partial \alpha_s}{\partial \mu} = \beta(\alpha_s) , \]

which, because of the relation \( g^2 = 4\pi \alpha_s \), defines the dependence of \( g \) on \( \mu \). The renormalization function \( \beta(\alpha) \) is known for \( \alpha \to 0 \) in the four-loop order [58] as the asymptotic expansion

\[ \beta(\alpha) \simeq b_2 \alpha^2 + b_3 \alpha^3 + b_4 \alpha^4 + b_5 \alpha^5 , \]

with the coefficients

\[ b_2 = -\frac{1}{2\pi} \left(11 - \frac{2}{3} N_f\right) , \quad b_3 = -\frac{4}{(4\pi)^2} \left(51 - \frac{19}{3} N_f\right) , \]
\[ b_4 = -\frac{1}{(4\pi)^3} \left(2857 - \frac{5033}{9} N_f + \frac{325}{27} N_f^2\right) , \]
\[ b_5 = -\frac{2}{(4\pi)^4} \left(29243 - 6946.3 N_f + 405.089 N_f^2 + 1.49931 N_f^3\right) . \]

In particular, for \( N_f = 6 \), we have

\[ b_2 = -\frac{7}{2\pi} , \quad b_3 = -\frac{52}{(4\pi)^2} , \quad b_4 = \frac{65}{(4\pi)^3} , \quad b_5 = -\frac{4944.50992}{(4\pi)^4} = -0.198282 . \]

Generally, the signs of the coefficients \( b_i \) depend on the number of flavours \( N_f \) in the following way:

\[ b_2 < 0 , \quad b_3 < 0 , \quad b_4 < 0 , \quad b_5 < 0 \quad (0 \leq N_f \leq 5) , \]
The qualitative change in the behaviour of $\beta(\alpha)$ happens at $N_f$ where the coefficient $b_2$ from negative becomes positive [59]. The coefficients $b_2$ and $b_3$ are renorm-scheme independent, but the higher coefficients $b_4$ and $b_5$ depend on the renorm-scheme employed in their calculation. The expansion (24) is obtained [58] within the minimal subtraction scheme. But since the $\beta$- function does not depend explicitly on $\mu$, this function is the same in $\overline{MS}$ scheme.

Defining the reduced function

$$\overline{\beta}(\alpha) \equiv \frac{\beta(\alpha)}{b_2 \alpha^2},$$

we find from expansion (24)

$$\overline{\beta}(\alpha) \simeq 1 + \overline{b}_3 \alpha + \overline{b}_4 \alpha^2 + \overline{b}_5 \alpha^3,$$

with the reduced coefficients

$$\overline{b}_n \equiv \frac{b_n}{b_2}, \quad (n = 3, 4, 5).$$

The self-similar exponential approximants extrapolating Eq. (24) to finite $\alpha$ are

$$\beta_3^\ast(\alpha) = b_2 \alpha^2 \exp(d_3 \alpha), \quad \beta_4^\ast(\alpha) = b_2 \alpha^2 \exp(d_3 \alpha \exp(d_4 \alpha)),$$

$$\beta_5^\ast(\alpha) = b_2 \alpha^2 \exp(d_3 \alpha \exp(d_4 \alpha \exp(d_5 \alpha))),$$

where the coefficients

$$d_n \equiv \frac{b_n}{b_{n-1}} t_n, \quad (n = 3, 4, 5)$$

are expressed through the control functions $t_n$, which again have to be defined either from fixed-point conditions or from the minimization of a cost functional.

Substituting the approximants (27) into the renorm-group equation (23), we solve the latter obtaining the corresponding approximations for $\alpha_s(\mu)$. As an initial condition, we may take the value

$$\alpha_s(m_Z) = 0.1185, \quad m_Z = 91.1882 \text{ GeV}$$

at the $Z^0$ boson mass [60].

The renorm-group equation (23), with the right-hand side defined by one of the forms (27), gives $\alpha_s = \alpha_s(\mu)$. The relation $g^2 = 4\pi\alpha_s$, together with the scale (20), leads to the equation

$$g^2 = 4\pi\alpha_s(\mu(T, g))$$

determining the function $g = g(T)$. Substituting the latter in the approximants (22) results in the reduced pressure

$$\overline{p}_k(T) \equiv \overline{p}_k^\ast(g, T)$$

as a function of temperature.
Calculations show [40] that the behaviour of the reduced pressure \( p(T) \) is in reasonable agreement with lattice simulations [4]. At the temperature \( T_c \approx 200 \text{ MeV} \), the pressure sharply drops down, when decreasing \( T \), which can be interpreted as confinement. However, the details of the confinement-deconfinement process cannot be accurately described by such an extrapolation approach based on the consideration of the quark-gluon plasma only. In addition, the self-similar exponential approximants (22) or (27) provide an accurate extrapolation to the region of finite \( g \) or \( \alpha_s \), where these parameters are of order one. In the vicinity of confinement, when \( \alpha_s \) fastly grows becoming much more than one, any extrapolation procedure would be quantitatively unreliable.

5 Effective Coupling under Confinement

The running coupling \( \alpha_s(\mu) \) as a function of the scale \( \mu \) is experimentally studied only for \( \mu \geq 2 \text{ GeV} \), where \( \alpha_s < 0.4 \) [60,61]. In the low-momentum region, the behaviour of the effective coupling is poorly known not because of the limited knowledge of higher order effects, but because of an essentially different physical phenomenon that enters the game, the one that is referred to as confinement.

For large \( \mu \), perturbation theory gives

\[
\alpha_s(\mu) \simeq \frac{2\pi}{\beta_0 \ln(\mu/\Lambda)} \quad (\mu \to \infty) ,
\]

where \( \Lambda \approx 200 \text{ MeV} \) is the QCD scale parameter and \( \beta_0 \equiv -2\pi b_2 = 11 - \frac{2}{3}N_f \). The form (31) is valid if \( \alpha_s \ll 1 \), that is when

\[
\mu \gg \Lambda \exp \left( \frac{2\pi}{\beta_0} \right) > \Lambda .
\]

This implies, as far as \( \Lambda \approx T_c \approx 200 \text{ MeV} \), that expression (31) is applicable only for \( \mu \gg T_c \). If, nevertheless, one formally considers (31) at lower \( \mu \), then the coupling (31) diverges at \( \mu = \Lambda \). There exist arguments [62] that \( \alpha_s(\mu) \) is finite at all \( \mu \), and satisfies the sum rule

\[
\frac{1}{\pi} \int_0^{2\text{GeV}} \alpha_s(k) \, dk \approx 0.38 \text{ GeV} .
\]

There are several models [62,63] constructing \( \alpha_s(\mu) \) for arbitrary \( \mu \).

The simplest way of a phenomenological construction of finite coupling could be by means of the pole-removal trick. The idea of the latter is as follows. Suppose that a function \( f(x) \) has a pole at \( x_0 \), which implies that in the vicinity of the pole the function can be presented as the sum

\[
f(x) \simeq f_{\text{reg}}(x) + f_{\text{sin}}(x) \quad (x \to x_0)
\]

of a regular and singular parts. Let us define a regularized function

\[
\tilde{f}(x) \equiv f(x) - f_{\text{sin}}(x)
\]

as the function \( f(x) \), with the removed singular part.

In applying this trivial trick to the running coupling, we may postulate that the latter is defined by extrapolating the perturbative approximation (31) to all \( \mu \) by means of the
The perturbative expression (31) can be identically rewritten as
\[ \alpha_s(\mu) \simeq \frac{2\pi n}{\beta_0 \ln(\mu/\Lambda)^n}, \]
with any positive \( n > 0 \). Taking into account the asymptotic equality
\[ \frac{1}{\ln x} \simeq \frac{1}{2} - \frac{1}{1-x} \quad (x \to 1), \]
we immediately obtain the regularized coupling
\[ \tilde{\alpha}_s(\mu) = \frac{2\pi n}{\beta_0} \left[ \frac{1}{\ln(\mu/\Lambda)^n} + \frac{1}{1 - (\mu/\Lambda)^n} \right], \] \( \text{(35)} \)
which is finite for any \( \mu \), including \( \mu = \Lambda \) and \( \mu = 0 \), where
\[ \tilde{\alpha}_s(0) = \frac{2\pi n}{\beta_0}, \quad \tilde{\alpha}_s(\Lambda) = \frac{\pi n}{\beta_0}. \] \( \text{(36)} \)

Similarly, one can construct regularized couplings from perturbative expressions of higher orders. But let us note that, as follows from Eq. (35), the pole-removal trick does not define the regularized functions in a unique way.

Another way of regularization, leading to the same result, would be by means of analytical continuation. Consider again a function \( f(x) \), with \( x \geq 0 \), having one or several poles on the positive semiaxis. Assume that \( f(-x) \) has no poles. Define the analytic continuation \( \tilde{f}(z) \) to the complex plane, except the cut along the negative semiaxis, so that
\[ \tilde{f}(-x \pm i0) = f(-x \pm i0) \quad (x \geq 0). \] \( \text{(37)} \)

In the region of analyticity of \( \tilde{f}(z) \), the spectral representation
\[ \tilde{f}(z) = \frac{1}{\pi} \int_0^\infty \frac{J(x)}{x + z} \, dx \] \( \text{(38)} \)
is valid, from where the spectral function is
\[ J(x) = \frac{i}{2} \left[ \tilde{f}(-x + i0) - \tilde{f}(-x - i0) \right]. \] \( \text{(39)} \)
In the latter, condition (37) is to be used.

Applying the analytic continuation method to the perturbative coupling (31), we take into account that \( \ln(-x \pm i0) = \ln |x| \pm i\pi \) and
\[ \frac{1}{\ln(-x \pm i0)} = \frac{\ln |x| + i\pi}{\ln^2 |x| + \pi^2}. \]
Then the corresponding spectral function is
\[ J(x) = \frac{2\pi^2 n}{\beta_0 (\ln^2 |x| + \pi^2)} \quad x \equiv \left( \frac{\mu}{\Lambda} \right)^n. \] \( \text{(40)} \)

The spectral representation (38) gives the same regularized coupling (35). This way of regularization was employed in Refs. [64,65] for the case \( n = 2 \).

The weakest point in any regularization procedure, based on the perturbative expression (31), is that the latter is valid only under condition (32), hence a regularized function has sense solely for \( |\mu/\Lambda| \gg 1 \). And there is no any reason of extrapolating \( \alpha_s(\mu) \) to the region \( \mu/\Lambda \leq 1 \), where confinement is expected.
6 Clustering Quark-Hadron Matter

Any attempt of treating deconfinement from the point of view of pure thermodynamic phases contains the following principal contradiction. Hadrons are believed to present bound states of quarks and gluons, while quark-gluon plasma represents their unbound states. This implies that the system of quarks and gluons possesses, in general, both a discrete spectrum corresponding to bound states and a continuous spectrum associated with unbound states. When a many-body system possesses an energy spectrum $E_n$, then the distribution of particles over the energy levels is described by the Gibbs probability $p_n \sim e^{-\beta E_n}$, where $\beta T \equiv 1$. If the energy spectrum contains both discrete as well as continuous parts, at each moment of time there exists a probability for particles to form bound states or to pertain to unbound states. In the standard quantum picture, bound and unbound states do coexist, with the related probability weights. Hence, hadrons must coexist with quark-gluon plasma. Let us stress that this is a direct logical conclusion immediately resulting from the treatment of hadrons as describing bound quark-gluon states. If a quantum system possesses different parts of spectra, all of them are to be taken into account by calculating the corresponding probability weights. It is not correct to separate the whole spectrum onto particular sections, prohibiting the existence of some of its parts. It is also incorrect to identify separate sections of the quantum spectrum with different thermodynamic phases. Hadron states and plasma states are quantum states but not thermodynamic phases.

This situation is similar to that of electron-ion plasma. Electrons and ions can form bound states, i.e. neutral atoms, or unbound states of electrons and charged ions. In the system of electrons and ions, under given conditions, there is a fraction of neutral atoms and a portion of separated ions and electrons. Changing conditions varies the fractional concentrations of neutral and ionized atoms. Ionization in the system of predominantly neutral atoms is the direct analog of deconfinement in predominantly hadronic matter. Ionization as well as deconfinement can occur, depending on circumstances, either as a sharp transition or as a gradual crossover.

The description of statistical properties of a quantum many-body system possessing several qualitatively different quantum states, such as bound and unbound, is not a trivial task. For describing such systems, Theory of Clustering Matter has been elaborated [4,12]. The approach is based on three pivotal concepts: Cluster Representation, Statistical Correctness, and Potential Scaling.

The idea of the quasiparticle cluster representation goes back to the authors who analyzed the abundances of chemical elements on Earth by treating each element as a quasiparticle characterized by the corresponding atomic weight and the binding energy, with the related chemical potentials taking into account the allowed interparticle reactions. Such approaches are reviewed in Refs. [66–68]. The same idea was applied to considering nuclear multifragmentation [69]. A more accurate mathematical formulation for the problem of constructing the quasiparticle representation for composite particles was initiated by Weinberg [70–72]. Such a representation could be unambiguously defined provided that a transformation from the state space of elementary particles to that of the system containing composite particles, together with unbound elementary particles, would be given [73–75]. For this purpose, different Boson realizations of Lie algebras [76] were employed [77–81]. The most general approach, based on the Tani transformation [82], has been developed by Girardeau [83–86] who coined the term Fock-Tani representation. This was applied to various systems containing bound clusters, including the quark-hadron matter [87].

The basic point in the quasiparticle cluster picture is as follows. Consider a many-
body system, with the total space of quantum states being a Fock space $\mathcal{F}$. Let the algebra of observables, $\mathcal{A}$, be defined on $\mathcal{F}$. Assume that the particles of the system can form several types of bound states, e.g. corresponding to different hadron clusters. Enumerate all admissible types of bound states by the index $i = 2, 3, \ldots$, reserving the index $i = 1$ to unbound states. Each kind of bound clusters can be individualized by a set of characteristic parameters, such as the compositeness number $z_i$ showing the number of elementary particles bound into a cluster, effective mass of the cluster $m_i$, and a set of quantum numbers like spin, isospin, colour, baryon number, strangeness, and so on. And let us treat each type of bound clusters as a separate sort of particles, with the associated Fock space $\mathcal{F}_i$, called the ideal cluster space. The direct product

$$\tilde{\mathcal{F}} \equiv \otimes_i \mathcal{F}_i \quad (\mathcal{F}_1 = \mathcal{F})$$

(41)

composes the total cluster space. The formal relation between the Fock space of elementary-particle states and the cluster space (41) can be presented by means of a unitary transformation $\hat{U}$, such that

$$\mathcal{F} = \hat{U} \tilde{\mathcal{F}}, \quad \tilde{\mathcal{F}} = \hat{U}^+ \mathcal{F}. \quad (42)$$

Then the cluster algebra of observables is defined as

$$\tilde{\mathcal{A}} \equiv \hat{U}^+ \mathcal{A} \hat{U}.$$ 

(43)

With these definitions, all matrix elements of the algebra $\mathcal{A}$ in $\mathcal{F}$ are the same as those of $\tilde{\mathcal{A}}$ in $\tilde{\mathcal{F}}$, since $\tilde{\mathcal{F}} \tilde{\mathcal{A}} \tilde{\mathcal{F}} = \mathcal{F} \mathcal{A} \mathcal{F}$. Since the representations of $\tilde{\mathcal{A}}$ in $\tilde{\mathcal{F}}$ and $\mathcal{A}$ in $\mathcal{F}$ are isomorphic, all observables quantities are the same in the standard picture of elementary particles and in the quasiparticle picture of a clustering system.

Let us now delineate the mathematical structure of the Tani transformation. Let the field operators of elementary particles, say of quarks, be $q(x)$ defined on the Fock space $\mathcal{F}$, with $x$ being a set of spatial variables. Suppose $\varphi_i(x_1, x_2, \ldots, x_i)$ is a Schrödinger wave function describing a bound state of $i$ elementary particles. The field operator of this bound state can be presented as

$$\Psi_i(x) \equiv \int \varphi_i(x_1 - x, x_2 - x, \ldots, x_i - x) q(x_1) q(x_2) \ldots q(x_i) \, dx_1 dx_2 \ldots dx_i.$$ 

The image of the bound state in the ideal cluster space $\mathcal{F}_i$ is given by a cluster with the field operator $\psi_i(x)$. By definition, $\psi_i(x) \equiv q(x)$. The Tani transformation is described by the unitary operator

$$\hat{U} = \exp \left( \frac{\pi}{2} \hat{F} \right), \quad \hat{F} = \sum_i \int \left[ \psi_i^\dagger(x) \Psi_i(x) - \Psi_i^\dagger(x) \psi_i(x) \right] \, dx . \quad (44)$$

In the cluster representation, constructed on the cluster space (41), one defines the statistical state $< \mathcal{A} >$ for the algebra of observables (43). The density of the $i$-type clusters is

$$\rho_i = \frac{1}{V} < \hat{N}_i >, \quad \hat{N}_i = \int \psi_i^\dagger(x) \psi_i(x) \, dx .$$

The probability for this type of clusters to be formed can be characterized by the weight

$$w_i = z_i \frac{\rho_i}{\rho} \left( \rho \equiv \sum_i z_i \rho_i \right), \quad (45)$$
which may be called the *cluster probability*. The weight (45) satisfies the standard properties of probability, being nonnegative, $0 \leq w_i \leq 1$, and normalized, $\sum_i w_i = 1$.

The direct calculation of the *cluster Hamiltonian*

$$\tilde{H} = \hat{U}^+ H \hat{U}$$

(46)

is a rather complicated problem. Moreover, the actual form of the Hamiltonian (46) is written as an infinite series. Because of this, one usually simplifies the procedure by assuming an effective Hamiltonian $H_{\text{eff}}$, whose construction involves physical reasoning. The latter Hamiltonian is often written with an explicit dependence on thermodynamic parameters, such as the cluster densities $\rho_i$ and temperature $T$, so that $H_{\text{eff}} = H_{\text{eff}}(\{\rho_i\}, T)$. At this point the principle of *statistical correctness* [4,12] comes into play saying that the general form of the cluster Hamiltonian (46) has to be as

$$\tilde{H} = H_{\text{eff}} + CV,$$

(47)

where $C$ is a nonoperator term such that makes the Hamiltonian (47) statistically correct, which implies the validity of the equations

$$< \frac{\partial \tilde{H}}{\partial \rho_i} > = 0, \quad < \frac{\partial \tilde{H}}{\partial T} > = 0.$$  

(48)

From Eqs. (47) and (48) one gets the equations

$$\frac{\partial C}{\partial \rho_i} = - \frac{1}{V} < \frac{\partial H_{\text{eff}}}{\partial \rho_i} >, \quad \frac{\partial C}{\partial T} = - \frac{1}{V} < \frac{\partial H_{\text{eff}}}{\partial T} >,$$

(49)

defining $C = C(\{\rho_i\}, T)$. These conditions guarantee the validity of the thermodynamic relations

$$P = - \frac{\partial \Omega}{\partial V} = - \frac{\Omega}{V}, \quad \varepsilon = T \frac{\partial P}{\partial T} - P + \mu_B n_B + \mu_S n_S = \frac{1}{V} < \hat{E} >,$$

$$s = \frac{\partial P}{\partial T} = \frac{1}{T} (\varepsilon + P - \mu_B n_B - n_S n_S), \quad n_B = \frac{\partial P}{\partial \mu_B} = \sum_i B_i \rho_i, \quad n_S = \frac{\partial P}{\partial \mu_S} = \sum_i S_i \rho_i,$$

in which $\varepsilon$ and $s$ are the energy and entropy densities, and

$$\hat{E} = \tilde{H} + \sum_i \mu_i \hat{N}_i, \quad \rho_i = \frac{\partial P}{\partial \mu_i} = \frac{1}{V} < \hat{N}_i >.$$

The cluster Hamiltonian (47) contains the terms describing effective interactions between different clusters. For defining the corresponding interaction potentials, the principle of *potential scaling* has been formulated [4,12]. According to the latter, the interaction potentials from the same class of universality are connected by the scaling relation

$$\frac{\Phi_{ij}(r)}{z_i z_j} = \frac{\Phi_{ab}(r)}{z_a z_b}.$$

(50)

This allows the definition of all qualitatively similar interaction potentials through one known potential. Another form of scaling (50) could be

$$\frac{\Phi_{ij}(r)}{m_i m_j} = \frac{\Phi_{ab}(r)}{m_a m_b}.$$
provided that \( m_i \sim z_i \).

The theory of clustering matter has been applied to clustering quark-hadron matter [4,12]. The appearance of multiquark clusters in nuclear matter is explained. The possibility of the dibaryon Bose condensation is advanced. Provisions for nuclear-matter lasers are estimated [88]. Thermodynamic characteristics for the SU(3) gluodynamics and zero-baryon-density chromodynamics are in good quantitative agreement with lattice simulations, displaying a first-order transition for pure gluodynamics and a crossover for chromodynamics. Deconfinement at finite baryon density, at conditions typical of heavy-ion collisions, is predicted to be a gradual crossover.

7 Discussion

The model of clustering quark-hadron matter [4,12] provides, to our mind, the most realistic approach to describing deconfinement at finite baryon density. Deconfinement is found to be a gradual crossover, but not a sharp transition. However, it would be yet too premature to state that all details of the deconfinement process are well understood. We do not imply here some technicalities that could always be varied in the frame of the same general approach to describing the clustering quark-hadron matter. For instance, one can take different interaction potentials or accept different cluster spectra \( \omega_i(k) \). Such technical variations do not change the general qualitative picture. But there are more principal questions that have not yet been properly addressed:

(i) The clustering quark-hadron matter has been treated in the mean-field approximation [4,12]. This seems to be reasonable especially because deconfinement is not a second-order phase transition but rather a crossover. The state where unbound quarks and gluons coexist with hadron clusters is shown to be thermodynamically stable with respect to the thermal and mechanical stability and with respect to the minimality of the thermodynamic potential as compared to those of pure phases corresponding either to quark-gluon plasma or to pure hadron states. However, the dynamic stability of the clustering system, which requires the positiveness of the collective-excitation spectrum, has not been checked. The latter would be suitable to consider as far as in the mean-field approximation the conditions of thermodynamic and dynamic stability do not coincide.

(ii) The finite-size effects in a clustering system have not been analyzed, while this would be useful keeping in mind that heavy ions colliding in realistic experiments are always finite. The finiteness of a system not only leads to quantitative corrections, as compared to an infinite matter, but may sometimes cause the existence of thermodynamic quasi-phases [89], not existing in thermodynamic limit.

(iii) It is possible that small static bubbles of quark-gluon plasma inside the predominantly hadron matter could arise, and vice versa, static hadron bubbles inside quark-gluon plasma could exist [90]. Also, static droplets of strange matter, the so-called strangelets could be formed. All such possibilities should be considered in the framework of the clustering matter.

(iv) Mesoscopic heterophase fluctuations may emerge even in a globally equilibrium system [18]. Such dynamically fluctuating germs of quark-gluon plasma or hadron droplets are principally different from static bubbles [90] and require a different theoretical approach [18].

(v) Finally, to be closer to collision experiments, one should consider a nonstationary
picture, analyzing all different possibilities mentioned above. In the nonequilibrium case, several scenarios of deconfinement could be feasible. Then the problem of pattern selection would arise, requiring the necessity of defining the probabilistic weights for the admissible deconfinement scenarios.

Summarizing the main material of this review, we come to the following conclusions:

1. For the correct description of deconfinement, it is necessary to employ the approach based on the clustering quark-hadron matter.

2. The process of deconfinement has not yet been completely understood.

3. One must be very cautious in trying to interpret observed experimental data as attributed to deconfinement.

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