Subgroup decomposition in $\text{Out}(F_n)$
Part IV: Relatively irreducible subgroups

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Abstract

This is the fourth and last in a series of four papers, announced in [HM13a], that
develop a decomposition theory for subgroups of $\text{Out}(F_n)$.

In this paper we develop general ping-pong techniques for the action of $\text{Out}(F_n)$ on
the space of lines of $F_n$. Using these techniques we prove the main results stated in
[HM13a], Theorem C and its special case Theorem I, the latter of which says that for
any finitely generated subgroup $H < \text{Out}(F_n)$ that acts trivially on homology with $\mathbb{Z}/3$
coefficients, and for any free factor system $\mathcal{F}$ that does not consist of (the conjugacy
classes of) a complementary pair of free factors of $F_n$ nor of a rank $n-1$ free factor,
if $H$ is fully irreducible relative to $\mathcal{F}$ then $H$ has an element that is fully irreducible
relative to $\mathcal{F}$. We also prove Theorem J which, under the additional hypothesis that
$H$ is geometric relative to $\mathcal{F}$, describes a strong relationship between $H$ and a mapping
class group of a surface.

Recall from the introduction [HM13a] the main theorem of this series, and recall also a
separately stated special case to which the general case will be reduced in Section 2.1:

**Theorem C.** For each finitely generated subgroup $H < \text{IA}_n(\mathbb{Z}/3)$ and each $H$-invariant
multi-edge extension of free factor systems $\mathcal{F} \sqsubset \mathcal{F}'$, if $H$ is irreducible relative to
this extension then then there exists $\phi \in H$ which is fully irreducible relative to the
extension $\mathcal{F} \sqsubset \mathcal{F}'$.

**Theorem I.** For each finitely generated subgroup $H < \text{IA}_n(\mathbb{Z}/3)$ and each $H$-invariant
free factor system $\mathcal{F}$, if $\mathcal{F} \sqsubset \{[F_n]\}$ is a multi-edge extension and if $H$ is irreducible
relative to $\mathcal{F}$ then then there exists $\phi \in H$ which is fully irreducible relative to $\mathcal{F}$.

In this paper we prove these theorems, and the closely related Theorem J which is a version
of Theorem I that applies under the additional hypothesis that $H$ is “geometric relative
to $\mathcal{F}$”, meaning that every non-geometric lamination pair of every element of $H$ is supported
by $\mathcal{F}$ (Definition 1.2).

Recall that for $H$ or $\phi$ to be irreducible relative to $\mathcal{F} \sqsubset \mathcal{F}'$ (when $\mathcal{F}' = \{[F_n]\}$ it is
dropped from the notation) means that there is no $H$-invariant free factor system strictly be-
tween $\mathcal{F}$ and $\mathcal{F}'$; and to be fully irreducible means irreducibility of all finite index subgroups
or finite powers, respectively, of $H$ or $\phi$. Throughout this paper we shall, in the context of
$\text{IA}_n(\mathbb{Z}/3)$, drop the adjective “fully” and simply write “irreducible” — this is justified by
applying Theorem B aka Theorem II.3.1, which says that for each \( \phi \in \text{IA}_n(\mathbb{Z}/3) \), a free factor system \( \mathcal{F} \) is \( \phi \)-periodic if and only if it is \( \phi \)-invariant, and so a subgroup or element of \( \text{IA}_n(\mathbb{Z}/3) \) is fully irreducible relative to an invariant extension of free factor systems \( \mathcal{F} \sqsubseteq \mathcal{F}' \) if and only if it irreducible relative to that extension.

We now give an overview of the paper, accompanied with a somewhat detailed sketch of the proof of Theorem I. The reader may find it convenient to skip this sketch and instead go right to main body of the paper beginning in Section 1.1, and to refer back to this introduction as needed for overview. Alternatively, the reader may wish to go back to [HM13a], review the brief introduction to Part IV found there, and then follow up with this more detailed overview here.

**Section 1.2. The ping-pong argument.** The proof of Theorem I depends on a ping-pong game described in Proposition 1.3, which in turn is ultimately based on the weak attraction theory developed in Part III [HM13d]. Given \( \mathcal{H} \) and \( \mathcal{F} \) as in Theorem I, consider \( \phi, \psi \in \mathcal{H} \) and laminations pairs \( \Lambda^\pm_\phi \in \mathcal{L}^\pm(\phi) \), \( \Lambda^\pm_\psi \in \mathcal{L}^\pm(\psi) \), such that their nonattracting subgroup systems \( A_{na}\Lambda^\pm_\phi \), \( A_{na}\Lambda^\pm_\psi \) each carry \( \mathcal{F} \). The conclusion of Proposition 1.3 is the existence of large values of the exponents \( l, m > 0 \) so that \( \zeta = \psi^l \phi^m \) has a lamination pair \( \Lambda^\pm_\xi \) whose nonattracting subgroup system \( A_{na}\Lambda^\pm_\xi \) also carries \( \mathcal{F} \), and such that the following additional properties hold: \( A_{na}\Lambda^\pm_\xi \) is carried by each of \( A_{na}\Lambda^\pm_\phi \) and \( A_{na}\Lambda^\pm_\psi \); the laminations \( \Lambda^\pm_\xi \) are close in the weak topology; and the laminations \( \Lambda^\pm_\xi \), \( \Lambda^\pm_\phi \), \( \Lambda^\pm_\psi \) are also close.

Proposition 1.3 achieves these conclusions under the hypothesis that the positive laminations are weakly attracted to each other under forward iteration and the negative laminations are weakly attracted to each other under negative iteration; for example, a generic leaf of \( \Lambda^+_\phi \) is weakly attracted to a generic leaf of \( \Lambda^+_\psi \) under positive iterates of \( \psi \). Proposition 1.3 has further hypotheses and conclusions designed to control geometric behavior, and to give quantitative control over the “closeness” of \( \Lambda^\pm_\xi \) to \( \Lambda^\pm_\phi \) and of \( \Lambda^\pm_\xi \) to \( \Lambda^\pm_\psi \).

**Section 2.2. Constructing a conjugator.** Ping-pong is a game with two players, and sometimes one at first has only a single player, consisting of \( \phi \in \mathcal{H} \) and \( \Lambda^\pm_\phi \in \mathcal{L}^\pm(\phi) \) such that \( A_{na}\Lambda^\pm_\phi \) carries \( \mathcal{F} \). The second player will be a conjugate \( \psi = \zeta \phi \zeta^{-1} \) with lamination pair \( \Lambda^\pm_\psi = \zeta(\Lambda^\pm_\phi) \). The conjugator \( \zeta \in \mathcal{H} \) is constructed by applying Proposition 2.1, the conclusion of which is the existence of a \( \zeta \) that scrambles up the given data regarding \( \phi \): \( \zeta \) does not preserve \( A_{na}\Lambda^\pm_\psi \), \( \zeta \) does not map generic leaves of \( \Lambda^\pm_\phi \) into the nonattracting subgroup system \( A_{na}\Lambda^\pm_\phi \), and a few other useful properties of \( \zeta \) are attained. One cannot expect such a \( \zeta \) to exist in general, certainly not if \( \mathcal{H} \) stabilizes \( A_{na}\Lambda^\pm_\phi \), and so Lemma 2.1 has a strong hypothesis, saying that the subgroup of \( \mathcal{H} \) that stabilizes \( A_{na}(\Lambda^\pm_\phi) \) has infinite index in \( \mathcal{H} \).

**Section 2.3. Driving down \( A_{na}\Lambda^\pm_\phi \).** Proving Theorem I requires two ping-pong tournaments. Each round of the first tournament applies Proposition 2.2 to inductively drive down the nonattracting subgroup system: the conclusion of that proposition is the existence of \( \phi \in \mathcal{H} \) and \( \Lambda^\pm_\phi \) such that \( A_{na}\Lambda^\pm_\phi \) takes on its minimal value subject to the requirement that it carries \( \mathcal{F} \). The meaning of “minimal value” depends on a dichotomy for the subgroup \( \mathcal{H} \) given in Definition 1.2: \( \mathcal{H} \) is geometric above \( \mathcal{F} \) if, for every element of \( \mathcal{H} \),

\[\text{Cross references such as “Theorem II.X.Y” refer to Theorem X.Y of Part II [HM13c]. Cross references to the Introduction [HM13a], to Part I [HM13b], and to Part II [HM13c] are to the June 2013 versions.}\]
every lamination pair not supported by \( \mathcal{F} \) is geometric (see Part I [HM13b] for material on geometric lamination pairs, particularly Definition I.2.19). If \( \mathcal{H} \) is not geometric above \( \mathcal{F} \) then the minimal value of \( \Lambda_{\mathsf{na}}^\pm \) is simply the free factor system \( \mathcal{F} \). If \( \mathcal{H} \) is geometric above \( \mathcal{F} \) then \( \Lambda_{\mathsf{na}}^\pm \) can never be a free factor system (Proposition III.1.4 (2)), and its minimal value is a vertex group system of the form \( \Lambda_{\mathsf{na}}^\pm = \mathcal{F} \cup \{C\} \) for some infinite cyclic subgroup \( C \) of \( \mathcal{F} \mathcal{F} \) (see Definition III.1.2 for this form for the nonattracting subgroup system of a "top stratum" geometric lamination pair).

The proof of Proposition 2.2 is in two cases, depending on whether the stabilizer of \( \Lambda_{\mathsf{na}}^\pm \) has finite or infinite index in \( \mathcal{H} \). When this index is finite we show directly that \( \Lambda_{\mathsf{na}}^\pm \) must have the minimal value described above. When the index is infinite we apply Lemma 2.1 to obtain a conjugator \( \xi \); and then we apply the ping-pong result, Proposition 1.3, to \( \phi \) and \( \psi = \zeta \phi \zeta^{-1} \), producing \( \xi \in \mathcal{H} \) such that \( \Lambda_{\mathsf{na}}^\pm \) is carried by \( \Lambda_{\mathsf{na}}^\pm \) and in \( \Lambda_{\mathsf{na}}^\pm \). The latter two subgroup systems being unequal by the conclusion of Lemma 2.1, but being of same "complexity" by construction, it follows that \( \Lambda_{\mathsf{na}}^\pm \) is properly carried by \( \Lambda_{\mathsf{na}}^\pm \). We then apply induction using the nested chain condition for vertex group systems, Proposition I.3.2, to show that after finitely many rounds of ping-pong we have reduced to the finite index case, which completes the first tournament.

Section 2.4. Proof of Theorem I: Driving up \( \mathcal{F}_{\mathsf{supp}}(\mathcal{F}, \Lambda^\pm_{\mathsf{na}}) \). Application of Proposition 2.2 produces \( \phi \in \mathcal{H} \) and \( \Lambda^\pm_{\mathsf{na}} \) so that the nonattracting subgroup system \( \Lambda_{\mathsf{na}}^\pm \) takes on its minimal value. Each round of the second ping-pong tournament will drive up the joint free factor support of \( \mathcal{F} \) and \( \Lambda^\pm_{\mathsf{na}} \), denoted \( \mathcal{F}_{\mathsf{supp}}(\mathcal{F}, \Lambda^\pm_{\mathsf{na}}) \), towards its maximal value \( \{\{F_n\}\} \).

In the case that \( \mathcal{H} \) is geometric above \( \mathcal{F} \), the second ping-pong tournament is not necessary: having minimized \( \Lambda_{\mathsf{na}}^\pm \), the free factor system \( \mathcal{F}_{\mathsf{supp}}(\mathcal{F}, \Lambda^\pm_{\mathsf{na}}) \) is automatically maximized, and \( \phi \) is automatically irreducible rel \( \mathcal{F} \). This case is very closely analogous to a step in the proof of the subgroup classification theorem for the mapping class group \( \mathcal{MCG}(S) \) of a finite type surface [Iva92]. Given a subsurface \( A \subset S \) with connected complement \( S - A \), and given a mapping class leaving \( A \) and \( S - A \) invariant up to isotopy and which is pseudo-Anosov on some subsurface of \( S - A \) with a certain lamination pair, if \( A \) is the "nonattracting subsurface" for that lamination pair (meaning that a simple closed curve is not attracted to the laminations if and only if that curve is isotopic into \( A \)) then the mapping class is automatically pseudo-Anosov on \( S - A \).

In the case that \( \mathcal{H} \) is not geometric above \( \mathcal{F} \), the second ping-pong tournament is necessary: even when \( \Lambda_{\mathsf{na}}^\pm \) takes on its minimal value \( \mathcal{F} \), it does not follow that \( \mathcal{H} \) is irreducible rel \( \mathcal{F} \); counterexamples where \( \mathcal{F} = \emptyset \) are easily constructed using relative train tracks. Each round of ping-pong again starts with a conjugator \( \xi \), chosen simply so that it does not preserve the free factor system \( \mathcal{F}_{\mathsf{supp}}(\mathcal{F}, \Lambda^\pm_{\mathsf{na}}) \). One again applies Proposition 1.3 to \( \phi \) and \( \psi = \zeta \phi \zeta^{-1} \) producing \( \xi \in \mathcal{H} \) such that \( \Lambda_{\mathsf{na}}^\pm \) still equals \( \mathcal{F} \). We then use the conclusion of Proposition 1.3, which says that \( \Lambda^+ = \Lambda^\pm_{\mathsf{na}} \) and \( \Lambda^- = \Lambda^-_{\mathsf{na}} \), to prove that \( \mathcal{F}_{\mathsf{supp}}(\mathcal{F}, \Lambda^\pm_{\mathsf{na}}) \) supports both of \( \mathcal{F}_{\mathsf{supp}}(\mathcal{F}, \Lambda^+_{\mathsf{na}}) \) and \( \mathcal{F}_{\mathsf{supp}}(\mathcal{F}, \Lambda^-_{\mathsf{na}}) \). The latter two free factor systems being unequal but of equal complexity by construction, it follows that \( \mathcal{F}_{\mathsf{supp}}(\mathcal{F}, \Lambda^\pm_{\mathsf{na}}) \) properly supports \( \mathcal{F}_{\mathsf{supp}}(\mathcal{F}, \Lambda^\pm_{\mathsf{na}}) \). We then apply induction, using the chain condition of free factor systems, to show that after finitely many rounds of ping-pong the
free factor system $\mathcal{F}_{\text{supp}}(\mathcal{F}, \Lambda^\pm)$ attains its maximal value $\{(F_n)\}$.

Section 2.5. Theorem J: Relatively geometric irreducible subgroups. We prove the general, relative version of Theorem J, the absolute version of which is stated in [HM13a].

To do this, we go one step further in the analysis of the case where $\mathcal{H}$ is geometric above $\mathcal{F}$, and in which we produced $\phi \in \mathcal{H}$ and a geometric lamination pair $\Lambda^\pm \in \mathcal{L}^\pm(\phi)$ for which $A_{\text{na}}\Lambda^\pm \phi$ takes on its minimal value, of the form $\mathcal{F} \cup \{[C]\}$. In this case, $[C]$ is represented by the top boundary curve $\partial_0 S$ of the surface $S$ associated to a geometric model for $\phi$ and $\Lambda^\pm \phi$. We use the logic of the proof of Theorem I to conclude that the stabilizer in $\mathcal{H}$ of $A_{\text{na}}\Lambda^\pm \phi = \mathcal{F} \cup \{[C]\}$ has finite index in $\mathcal{H}$, and therefore must equal $\mathcal{H}$. We then apply Proposition I.2.20 to conclude that the entire subgroup $\mathcal{H}$ preserves the surface $S$ and its boundary components, inducing a homomorphism $\mathcal{H} \to \text{MCG}(S)$ under which the image of $\phi$ is pseudo-Anosov.

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1 Ping-pong on geodesic lines

1.1 Finding attracting laminations

Given $\phi \in \text{Out}(F_n)$ there are several methods for finding an attracting lamination of $\phi$. The most concrete method is to take a relative train track representative $f: G \to G$ and check the existence of an EG stratum $H_r \subset G$ (Fact I.1.55). Another method—whose statement, at least, does not mention relative train track maps—is to check existence of a nontrivial conjugacy class $c$ such that for some (any) marked graph $G$, the length in $G$ of the conjugacy class represented by $\phi^i(c)$ is bounded below by an exponentially growing function of the exponent $i$; the proof does use a relative train track representative $f: G \to G$ of $\phi$, by noting that if $f$ has no EG strata then for any circuit $\gamma$ in $G$ the number of edges in $f^k(\gamma)$ has a polynomial upper bound in $k$.

In Lemma 1.1 we give a third method for finding attracting laminations of $\phi$ in terms of a topological representative: from any path which maps over itself three times (in the sense of the ## Lemma I.1.6) one can obtain an attracting lamination. Neither the statement
nor the proof invokes relative train track maps: the proof uses the definition of attracting laminations directly.

**Remarks.** The three methods just described give different amounts of information on the side regarding the attracting lamination that is produced. Relative train track maps give the most information: using filtration elements one obtains certain free factor systems which do and do not support the lamination; using edges of the EG stratum $H_r$ one produces attracting neighborhoods of the lamination; and one can construct the nonattracting subgroup system of the lamination (Definitions III.1.2 and Corollary III.1.9 (2)). The other two methods, including Lemma 1.1, are useful when no relative train track map is available and when less extra information on the side is needed, although Lemma 1.1 will produce a useful attracting neighborhood.

Recall from Section I.1.1.6 that any $\pi_1$-injective map $f : K \to G$ of marked graphs naturally induces two path maps $f_\#, f_\# : B(K) \to B(G)$ as follows: the path $f_\#(\gamma)$ is obtained by straightening the $f$ image of $\gamma$; and, roughly speaking, $f_\#(\gamma)$ is the largest common subpath of all $f_\#$-images of paths containing $\gamma$. Recall also from Section I.1.1.5 the notation $V(G, \gamma)$ for the basis element of the weak topology on $B(G)$ associated to a finite path $\gamma$ in a finite graph $G$.

**Lemma 1.1.** Given $\phi \in \text{Out}(F_n)$, a marked graph $G$, a topological representative $f : G \to G$ of $\phi \in \text{Out}(F_n)$, and a finite path $\beta \subset G$, if the path $f_\#(\beta)$ contains three disjoint copies of $\beta$ then there exists $\Lambda \in \mathcal{L}(\phi)$ and a generic leaf $\lambda$ of $\Lambda$ such that $\phi$ fixes $\Lambda$, $\phi$ fixes $\lambda$ preserving orientation, and $V(G, \beta)$ is an attracting neighborhood for $\Lambda$. Furthermore for any $i \geq 0$ each generic leaf of $\Lambda$ contains $f_\#^i(\beta)$ as a subpath.

**Proof.** For any lift $\tilde{\beta}$ of $\beta$ to the universal cover $\tilde{G}$ and for any lift $\tilde{f} : \tilde{G} \to \tilde{G}$ of $f$, the hypothesis can be restated to say that

$$\tilde{f}_\#(\tilde{\beta}) = \tilde{\alpha}_1 \tilde{\beta}_L \tilde{\alpha}_2 \tilde{\beta}_C \tilde{\alpha}_3 \tilde{\beta}_R \tilde{\alpha}_4$$

where $\tilde{\beta}_L, \tilde{\beta}_C, \tilde{\beta}_R$ are translates of $\tilde{\beta}$.

For inductive reasons we write $\beta_0 = \beta$. Choosing a lift $\tilde{\beta}_0$ of $\beta_0$ to the universal cover $\tilde{G}$, there exists a lift $\tilde{f} : \tilde{G} \to \tilde{G}$ of $f$ and lifts $\tilde{\beta}_L, \tilde{\beta}_R$ of $\tilde{\beta}_0$ such that

$$\tilde{f}_\#(\tilde{\beta}_0) = \tilde{\alpha}_{0,1} \tilde{\beta}_{0,L} \tilde{\alpha}_{0,2} \tilde{\beta}_0 \tilde{\alpha}_{0,3} \tilde{\beta}_{0,R} \tilde{\alpha}_{0,4}$$

Define

$$\tilde{\beta}_1 = \tilde{\beta}_{0,L} \tilde{\alpha}_{0,2} \tilde{\beta}_0 \tilde{\alpha}_{0,3} \tilde{\beta}_{0,R} \subset \tilde{f}_\#(\tilde{\beta}_0).$$

Combining the definition of $\tilde{\beta}_1$, the hypothesis, and Lemma I.1.6 (5), we may write $\tilde{f}_\#(\tilde{\beta}_1)$ as

$$\tilde{f}_\#(\tilde{\beta}_1) = \tilde{\alpha}_{1,1} \tilde{\beta}_{1,L} \tilde{\alpha}_{1,2} \tilde{\beta}_1 \tilde{\alpha}_{1,3} \tilde{\beta}_{1,R} \tilde{\alpha}_{1,4}$$

where $\tilde{\beta}_{1,L} \subset \tilde{f}_\#(\tilde{\beta}_{0,L})$ and $\tilde{\beta}_{1,R} \subset \tilde{f}_\#(\tilde{\beta}_{0,R})$ are translates of $\tilde{\beta}_1 \subset \tilde{f}_\#(\tilde{\beta}_0)$. Assuming by induction that

$$\tilde{f}_\#(\tilde{\beta}_i) = \tilde{\alpha}_{i,1} \tilde{\beta}_{i,L} \tilde{\alpha}_{i,2} \tilde{\beta}_i \tilde{\alpha}_{i,3} \tilde{\beta}_{i,R} \tilde{\alpha}_{i,4}$$

where $\tilde{\beta}_{i,L} \subset \tilde{f}_\#(\tilde{\beta}_{i-1,L})$ and $\tilde{\beta}_{i,R} \subset \tilde{f}_\#(\tilde{\beta}_{i-1,R})$ are translates of $\tilde{\beta}_i \subset \tilde{f}_\#(\tilde{\beta}_{i-1})$, define

$$\tilde{\beta}_{i+1} = \tilde{\beta}_{i,L} \tilde{\alpha}_{i,2} \tilde{\beta}_i \tilde{\alpha}_{i,3} \tilde{\beta}_{i,R} \subset \tilde{f}_\#(\tilde{\beta}_i)$$
and apply Lemma I.1.6 (5) to complete the induction step.

The union of the nested sequence $\tilde{\beta}_0 \subset \tilde{\beta}_1 \subset \tilde{\beta}_2 \subset \cdots$ is line $\tilde{\lambda} \in \tilde{B}(G)$ which $\tilde{f}_{##}$ fixes preserving orientation, and so determines a line $\lambda \in B$ which $\phi$ fixes preserving orientation. Each ray $\tilde{R}$ in $\tilde{\lambda}$ contains a translate of $\tilde{\beta}_i$ for all sufficiently large $i$ and so contains a translate of $\tilde{\beta}_i$ for all $i$. Thus $\lambda$ is birecurrent. If a line $\tilde{\gamma}$ contains $\tilde{\beta}_0$ as a subpath then $\tilde{f}_{##}(\tilde{\gamma})$ contains $\tilde{f}_{##}(\tilde{\beta}_0)$ by Lemma I.1.6 (3) and so contains $\tilde{\beta}_1$. The obvious induction argument shows that $\tilde{f}_{##}(\tilde{\gamma})$ contains $\tilde{\beta}_i$ for all $i$. This proves that $V(G, \beta) \subset B(G) \approx B$ is an attracting neighborhood for $\lambda$ in $B$ with respect to the action of $\phi$. Since the length of $\tilde{f}_{##}(\tilde{\beta}_i)$ is at least three times the length of $\tilde{\beta}_i$, the line $\lambda$ is not the axis of a covering translation. By the definition of attracting laminations (Definition 3.1.5 of [BFH00], or see Definition I.1.13) the weak closure $\Lambda \subset B$ of the line $\lambda$ is an attracting lamination for $\phi$ and $\lambda$ is a generic leaf of $\Lambda$. Since $V(G, \beta)$ is an attracting neighborhood for $\lambda$, it follows that $V(G, \beta)$ is an attracting neighborhood for $\Lambda$.

**1.2 The ping-pong argument**

In this section we state and prove Proposition 1.3, a technical statement in which our ping-pong arguments are packaged.

For stating the proposition and several later results including Theorem J, we need the following definition:

**Definition 1.2.** Given a free factor system $\mathcal{F}$ and a subgroup $\mathcal{H} < \text{Out}(F_n)$ that preserves $\mathcal{F}$, we say that $\mathcal{H}$ is geometric above $\mathcal{F}$, or that $\mathcal{H}$ is geometric relative to $\mathcal{F}$, if for each rotationless $\phi \in \mathcal{H}$, each nongeometric lamination pair in $\mathcal{L}^\pm(\phi)$ is supported by $\mathcal{F}$.

**Proposition 1.3.** Suppose that $\mathcal{F}$ is a (possibly empty) free factor system, that $\phi, \psi \in \text{Out}(F_n)$ are rotationless and preserve $\mathcal{F}$, that $\Lambda_\phi^\pm, \Lambda_\psi^\pm \in \mathcal{L}^\pm(\phi)$, $\Lambda_\psi^\pm \in \mathcal{L}^\pm(\psi)$ are lamination pairs. Suppose that the laminations $\Lambda_\phi^\pm, \Lambda_\psi^\pm$ each have a generic leaf $\lambda_\phi^\pm, \lambda_\psi^\pm$ that is fixed by $\phi^\pm, \psi^\pm$, respectively, with fixed orientation. Suppose also that the following hold:

- $\mathcal{F} \subset \mathcal{A}_{na}\Lambda_\phi^\pm$ and $\mathcal{F} \subset \mathcal{A}_{na}\Lambda_\psi^\pm$;

- Either both pairs $\Lambda_\phi^\pm$ and $\Lambda_\psi^\pm$ are non-geometric, or the subgroup $(\phi, \psi) < \text{Out}(F_n)$ is geometric above $\mathcal{F}$;

and suppose that the following hold:

(i) $\Lambda_\phi^+$ is weakly attracted to $\Lambda_\phi^+$ under iteration by $\phi$.
(ii) $\Lambda_\phi^-$ is weakly attracted to $\Lambda_\phi^-$ under iteration by $\phi^{-1}$.
(iii) $\Lambda_\psi^+$ is weakly attracted to $\Lambda_\psi^+$ under iteration by $\psi$.
(iv) $\Lambda_\psi^-$ is weakly attracted to $\Lambda_\psi^-$ under iteration by $\psi^{-1}$.

Under these suppositions, there exist attracting neighborhoods $V_\phi^\pm, V_\psi^\pm$ of $\Lambda_\phi^\pm, \Lambda_\psi^\pm$, respectively, and there exists an integer $M$, such that for any $m, n \geq M$ the outer automorphism $\xi = \psi^m \phi^n$ has a $\xi$-invariant lamination pair $\Lambda_\xi^\pm$ such that $\Lambda_\xi^\pm$ is non-geometric if $\Lambda_\phi^\pm$ and $\Lambda_\psi^\pm$ are non-geometric, and the following hold:

1. $\mathcal{F}$ is carried by $\mathcal{A}_{na}\Lambda_\xi^\pm$, and $\mathcal{A}_{na}\Lambda_\xi^\pm$ is carried by each of $\mathcal{A}_{na}\Lambda_\phi^\pm$ and $\mathcal{A}_{na}\Lambda_\psi^\pm$.  

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(2⁺) $\psi^m(V^+_{\phi}) \subset V^+_\psi$
(2⁻) $\phi^{-n}(V^-_{\psi}) \subset V^-_{\phi}$
(3⁺) $\phi^n(V^+_{\phi}) \subset V^+_\psi$
(3⁻) $\psi^{-m}(V^-_{\phi}) \subset V^-_{\psi}$
(4⁺) $V^+_{\xi} := V^+_\psi$ is an attracting neighborhood of $\Lambda^+_{\xi}$.
(4⁻) $V^-_{\xi} := V^-_{\phi}$ is an attracting neighborhood for $\Lambda^-_{\xi}$.
(5⁺) For any neighborhood $U^+_{\psi}$ of a generic leaf of $\Lambda^+_{\phi}$ there exists an integer $M(U^+_{\psi})$ such that if $m,n \geq M(U^+_{\psi})$ then a generic leaf of $\Lambda^+_{\phi}$ is in $U^+_{\psi}$.
(5⁻) For any neighborhood $U^-_{\phi}$ of a generic leaf of $\Lambda^-_{\phi}$ there exists an integer $M(U^-_{\phi})$ such that if $m,n \geq M(U^-_{\phi})$ then a generic leaf of $\Lambda^-_{\phi}$ is in $U^-_{\phi}$.

For the proof we shall need the following lemma which says, roughly speaking, that for any line which is weakly attracted to some $\Lambda^+ \in L(\phi)$, the realization of that line in any marked graph contains a finite segment which is uniformly attracted to $\Lambda^+$ in an appropriate sense. The proof uses the “buffered splitting argument”, [BFH00] Lemma 4.2.2, to obtain finite subpaths of generic leaves which survive under iteration in a very strong sense.

**Lemma 1.4.** Consider $\phi \in \text{Out}(F_n)$ and $\Lambda \in L(\phi)$ so that $\phi(\Lambda) = \Lambda$, a relative train track representative $f: G \to G$ with $E \subset H_r \subset G$ corresponding to $\Lambda$, and a generic leaf $\lambda \in \Lambda$ realized in $G$. Consider also a marked graph $K$, a homotopy equivalence $h: K \to G$ that preserves marking, and a line $\ell \in $ realized in $K$. If $\ell$ is weakly attracted to $\Lambda$ then for any finite subpath $\tau$ of $\lambda$ in $G$ that begins and ends with edges of $H_r$ there exists $k \geq 0$ and a finite subpath $\alpha$ of $\ell$ in $K$ such that:

1. For any $i \geq 0$ the path $(f^{k+i} \circ h)_{\#}(\alpha)$ contains $f^i_{\#}(\tau)$ as a subpath.
2. For any $i \geq 0$ and any path $\beta$ in $K$ containing $\alpha$ as a subpath, the path $(f^{k+i} \circ h)_{\#}(\beta)$ contains $f^i_{\#}(\tau)$ as a subpath.

**Proof.** Item (2) follows from item (1) by the $\#\#$ Lemma I.1.6 (3).

Consider a finite subpath of $\lambda^+ \subset G$ of the form $\tau^- \tau^+ \tau^+$ and consider another finite path $\gamma$ in $G$ that contains $\tau^- \tau^+$ as a subpath and so can be written in the form $\gamma = \gamma^- \tau \gamma^+$ where $\tau^-$ is a terminal segment of $\gamma^-$ and $\tau^+$ is an initial segment of $\gamma^+$. By Lemma 4.2.2 of [BFH00] there exists a constant $C_1$ (depending only on $f$, independent of $\tau^- \tau^+$ and of $\gamma$) such that $\tau^-$, $\tau^+$ each cross at least $C_1$ edges of $H_r$ then for each $i \geq 0$ the path $f^i_{\#}(\gamma)$ decomposes as $f^i_{\#}(\gamma) = f^i_{\#}(\gamma^-) f^i_{\#}(\tau) f^i_{\#}(\gamma^+)$. Since $\lambda^+$ is a generic leaf we may choose $\tau^-$, $\tau^+$ so that this is so.

Since $\ell$ is weakly attracted to $\lambda^+$, there exists $k \geq 0$ such that the line $(f^k \circ h)_{\#}(\ell)$, which is the realization of $\phi^k(\ell)$ in $G$, contains $\tau^- \tau^+$ as a subpath. Let $C_2$ be a bounded cancellation constant for the map $f^k \circ h: K \to G$. Choose $\alpha$ to be a subpath of $\ell$ such that $(f^k \circ h)_{\#}(\alpha)$ decomposes as an initial subpath of length at least $C_2$ followed by $\tau^- \tau^+$ followed by a terminal subpath of length at least $C_2$. For any subpath $\alpha'$ of $\ell$ that contains $\alpha$ as a subpath, it follows by the bounded cancellation lemma that $\gamma = (f^k \circ h)_{\#}(\alpha')$ contains $\tau^- \tau^+$ as a subpath, and so for any $i \geq 0$ the path $(f^{k+i} \circ h)_{\#}(\alpha') = f^i_{\#}(\gamma)$ contains $f^i_{\#}(\tau)$
as a subpath. Since this is true for any such \( \alpha' \), it follows by definition of the \( \#\# \) operator that \((f^{k+i} \circ h)\#\#(\alpha)\) contains \( f^i(\tau) \) as a subpath.

The proof of Proposition 1.3 takes up the rest of the section. As the proof proceeds, we will impose finitely many lower bounds constraining \( M \); in the end we take \( M \) to be the minimum integer satisfying these constraints.

Choose \( \text{CT} \) representatives
\[
g_\phi : G_\phi \to G_\phi \quad \text{and} \quad g_\psi : G_\psi \to G_\psi
\]
of \( \phi \) and \( \psi \), respectively, in which \( \mathcal{F} \) is realized by a filtration element. Let \( H_\phi \subset G_\phi \), \( H_\psi \subset G_\psi \) denote the \( \text{EG} \) strata corresponding to \( \Lambda^+_\phi, \Lambda^+_\psi \), respectively. Applying Fact I.1.62 pick an attracting neighborhood basis \( \mathcal{V}_{\psi}^+ = \{V(G_\psi, \beta_k)\} \) of \( \Lambda^+_\psi \) where \( \beta_k \) is a nested sequence of finite subpaths exhausting \( \lambda^+_\psi \), and each \( \beta_k \) begins and ends with edges of \( H_\psi \).

Similarly pick \( \mathcal{V}_{\phi}^+ = \{V(G_\phi, \gamma_k)\} \) with respect to \( \Lambda^+_\phi \) and \( \lambda^+_\phi \). Pick homotopy equivalences \( h_\psi : G_\phi \to G_\psi \) and \( h_\phi : G_\psi \to G_\phi \) that respect the markings.

In the arguments to follow, in phrases like “this path contains that path as a subpath” we often drop the words “as a subpath”.

Let \( \tau \) be an edge of \( H_\phi \). Applying Lemma 1.4 with \( f = g_\phi \) and \( h = \text{Id}_{G_\phi} \) there exists a finite subpath \( \alpha \) of \( \lambda^+_\phi \) and an \( m_0 \geq 0 \) such that for all \( i \geq 0 \) the path \((g_\phi^{m_0+i})\#\#(\alpha)\) contains \((g_\phi^i)\#(\tau)\) as a subpath—in this special case one could choose \( \alpha \) so that \( m_0 = 0 \) but this extra precision makes no difference to the argument. By another application of Lemma 1.4, there exists \( m_1 \geq 0 \) and a finite subpath \( \beta \) of \( \lambda^+_\psi \) such that for all \( i \geq 0 \) the path \((g_\phi^{m_1+i}h_\phi)\#\#(\beta)\) in \( G_\phi \) contains \((g_\phi^i)\#(\alpha)\) as a subpath. Setting \( i = m_0 + k \), for any \( k \geq 0 \) we have:

(*) the path \((g_\phi^{m_1+m_0+k}h_\phi)\#\#(\beta)\) in \( G_\phi \) contains \((g_\phi^{m_0+k})\#\#(\alpha)\) as a subpath, which contains \((g_\phi^{m_0+k})\#\#(\alpha)\) (by the \#\# Lemma I.1.6), which contains \((g_\phi^k)\#(\tau)\).

Furthermore, Lemma 1.4 allows us to lengthen \( \beta \) arbitrarily, and so by the description above of the attracting neighborhood basis \( V_\psi^+ \) we may choose \( \beta \) so that \( V_\psi^+ \equiv V(G_\psi, \beta) \in V_\psi^+ \) is an attracting neighborhood of \( \Lambda^+_\psi \). As a consequence the path \((g_\phi^i)\#(\beta)\) contains \( \beta \) as a subpath for each \( \ell \geq 0 \).

By a similar argument applying (iii) and Lemma 1.4 with the roles of \( \phi \) and \( \psi \) reversed, there exists \( m_2 \geq 0 \) and a subpath \( \gamma \) of \( \lambda^+_\psi \) such that \( V_\phi^+ \equiv V(G_\phi, \gamma) \) is an attracting neighborhood of \( \Lambda^+_\phi \), and such that for all \( l \geq 0 \) the path \((g_\psi^{m_2+l}h_\psi)\#\#(\gamma)\) in \( G_\phi \) contains \((g_\psi^l)\#(\beta)\) as a subpath. Note in particular that for all \( m \geq m_2 \) we have verified that the path \((g_\psi^m h_\psi)\#\#(\gamma)\) contains \( \beta \) as a subpath, and so \( \psi^m(V_\phi^+) \subset V_\psi^+ \) if \( m \geq m_2 \). We now constrain \( M \) so that \( M \geq m_2 \) (which verifies \( (2^+)) \).

Since \((g_\phi^k)\#(\tau)\) converges weakly to the birecurrent line \( \lambda^+_\phi \) we may choose \( m_3 \) so that for all \( j \geq 0 \) the path \((g_\phi^{m_3+j})\#(\tau)\) contains three disjoint copies of \( \gamma \). We further constrain \( M \) so that \( M \geq m_0 + m_1 + m_3 \). For any \( m, n \geq M \), we have

\[
m \geq m_2 \\
n = m_1 + m_0 + m_3 + j, \quad \text{with } j \geq 0
\]
It follows from (*) that the path \((g^n h_{\phi})_{##}(\beta)\) contains \((g^{m+j}_{\phi})_{##}(\tau)\), which in turn contains three disjoint copies of \(\gamma\) (which verifies \((3^+)\)). It follows furthermore, applying the \#\# Lemma I.1.6, that the path \((g^{m} h_{\psi} g^{n}_{\phi} h_{\phi})_{##}(\beta)\) contains three disjoint copies of \((g^{n} h_{\psi})_{##}(\gamma)\) which contains three disjoint copies of \(\beta\). The homotopy equivalence \(f_\xi = g^{m}_{\psi} h_{\psi} g^{n}_{\phi} h_{\phi} : G_{\psi} \to G_{\psi}\) represents \(\xi = \psi^{m} \phi^{n}\), and by applying Lemma 1.1 it follows that \(V^{+}_\xi := V^{+}_{\psi} = V(G_{\psi}, \beta)\) is an attracting neighborhood of an attracting lamination \(\Lambda^{+}_\xi\) of \(\xi\) (which verifies \((4^+)\)).

To verify \((5^+)\), in the previous paragraph we could have taken \(m = M\); it follows that \((g^{M} h_{\psi} g^{n}_{\phi} h_{\phi})_{##}(\beta)\) contains \(\beta\) as a subpath. For arbitrary \(m = M + l \geq M\) it follows that \((f_{\xi})_{##}(\beta)\) contains \((g^{M}_{\psi} h_{\psi} g^{n}_{\phi} h_{\phi})_{##}(\beta)\) (by the ##-Lemma I.1.6 (4)), which contains \((g^{l}_{\psi})_{##}(\beta)\). Lemma 1.1 has the additional conclusion that a generic leaf \(\lambda^{+}_\xi\) of \(\Lambda^{+}_\xi\) contains \((f_{\xi})_{##}(\beta)\), and \(\lambda^{+}_\xi\) also contains \((g^{l}_{\psi})_{##}(\beta)\). Since \(V(G_{\psi}, \beta)\) is an attracting neighborhood for \(\Lambda^{+}_\psi\), for any \(U^{+}_{\psi}\) as in \((5^+)\) it follows that if \(l\) is sufficiently large, say \(l \geq L(U^{+}_{\psi})\), then any line containing \((g^{l}_{\psi})_{##}(\beta)\) is in \(U^{+}_{\psi}\). In particular \(\lambda^{+}_\xi \in U^{+}_{\psi}\), and so \((5^+)\) follows with \(M(U^{+}_{\psi}) = M + L(U^{+}_{\psi})\).

By similar arguments, with the roles of \(\phi, \psi\) played by \(\psi^{-1}, \phi^{-1}\) respectively, applying \((ii), (iv)\) in place of \((i), (iii)\), and after constraining \(M\) with further lower bounds as necessary, we obtain attracting neighborhoods \(V^{+}_{\phi} \subset U^{+}_{\phi}\) of \(\Lambda^{-}_{\phi}\) and \(V^{-}_{\psi} \subset U^{-}_{\psi}\) of \(\Lambda^{-}_{\psi}\) so that if \(m, n \geq M\) then \(\xi^{-1} = \phi^{-n} \psi^{-m}\) has an attracting lamination \(\Lambda^{-}_{\xi}\) that satisfies \((2^{-}), (3^{-}), (4^{-})\) and \((5^{-})\).

By item (2) of Corollary III.2.17 (Theorem H) there exists \(m_4\) so that if \(\nu\) is a line that is neither an element of \(V^{-}_{\phi} = V^{-}_{\psi}\) nor carried by \(A_{na} \Lambda^{\pm}_{\phi}\) then \(\phi^{k}_{\phi}(\nu) \in V^{+}_{\phi}\) for all \(k \geq m_4\). Imposing the further constraint \(M \geq m_4\), it follows that \(\xi^{-1}(\nu) = \psi^{m}_{\psi} \phi^{n}_{\phi}(\nu) \in V^{+}_{\psi} \subset V^{+}_{\psi} = V^{+}_{\xi}\) which proves that \(\nu\) is weakly attracted to \(\Lambda^{+}_{\xi}\). We will use this in the following form: every line that is not contained in \(V^{-}_{\psi}\) and is not weakly attracted to \(\Lambda^{+}_{\xi}\) under iteration of \(\xi\) is carried by \(A_{na} \Lambda^{\pm}_{\psi}\).

By a completely symmetric argument we may assume, after constraining \(M\) with further lower bounds as necessary, that every line that is not contained in \(V^{+}_{\xi}\) and \(V^{-}_{\xi}\) is disjoint from both \(V^{+}_{\psi}\) and \(V^{-}_{\psi}\) and so is carried by both \(A_{na} \Lambda^{\pm}_{\phi}\) and \(A_{na} \Lambda^{\pm}_{\psi}\).

This completes the description of all lower bounds constraining \(M\). We note that these lower bounds are all determined by the choices of CTs representing \(\phi^{\pm 1}\) and \(\psi^{\pm 1}\) and by choices of homotopy equivalences preserving marking amongst the domains of those CTs.

Note that every line that is contained in \(V^{+}_{\psi}\) is weakly attracted to \(\Lambda^{+}_{\psi}\), and every line contained in \(V^{-}_{\psi}\) is weakly attracted to \(\Lambda^{-}_{\psi}\). As a consequence we have shown:

(A) Every line that is weakly attracted to neither \(\Lambda^{+}_{\xi}\) nor \(\Lambda^{-}_{\xi}\) is disjoint from both \(V^{+}_{\psi}\) and \(V^{-}_{\psi}\) and so is carried by both \(A_{na} \Lambda^{\pm}_{\phi}\) and \(A_{na} \Lambda^{\pm}_{\psi}\).

(B) Restricting to periodic lines, every conjugacy class carried by both \(A_{na} \Lambda^{+}_{\xi}\) and \(A_{na} \Lambda^{-}_{\xi}\) is carried by both \(A_{na} \Lambda^{\pm}_{\phi}\) and \(A_{na} \Lambda^{\pm}_{\psi}\).

Assuming for the moment that \(\Lambda^{+}_{\xi}\) and \(\Lambda^{-}_{\xi}\) are indeed a dual lamination pair for \(\xi\), what remains to complete the proof of Proposition 1.3 is to verify conclusion (1) and to verify
that if the dual laminations pairs $\Lambda_\phi^\pm$ and $\Lambda_\psi^\pm$ are nongeometric then the pair $\Lambda_\xi^\pm$ is also nongeometric.

By Corollary III.1.9 it follows that $A_{na}\Lambda_\xi^\pm = A_{na}\Lambda_\xi^\mp$ which is denoted $A_{na}\Lambda_\xi^{\mp}$. Since the set of lines carried by any subgroup system is the closure of the axes of the conjugacy classes that it carries, it follows from (B) that $A_{na}\Lambda_\xi^{\mp}$ is carried by both $A_{na}\Lambda_\phi^{\pm}$ and $A_{na}\Lambda_\psi^{\pm}$. Since a generic leaf of $\Lambda_\xi^\pm$ realized in $G_\psi$ contains the path $\beta$ which begins and ends with edges of $H_\psi$, and since the filtration element of $G_\psi$ corresponding to $F$ is below the stratum $H_\psi$, it follows that a generic leaf of $\Lambda_\xi^\pm$ is not carried by $F$. Since $F$ is fixed by each of $\phi$ and $\psi$, it is also fixed by $\xi = \psi^m \phi^n$, and so the closure of the $\xi$-orbit of any conjugacy class supported by $F$ is also supported by $F$ and therefore does not contain a generic leaf of $\Lambda_\xi^\pm$. It follows that no conjugacy class supported by $F$ is weakly attracted to $\Lambda_\xi^\pm$ under iteration by $\xi$, and so $F \subset A_{na}(\Lambda_\xi^\mp)$. The proves conclusion (1).

The only other thing to show is that if the dual laminations pairs $\Lambda_\phi^\pm$ and $\Lambda_\psi^\pm$ are nongeometric then the pair $\Lambda_\xi^\mp$ is also nongeometric. Otherwise $\Lambda_\xi^\pm$ is geometric and by Proposition I.2.18 there is a finite set of $\xi$-invariant conjugacy classes whose free factor support carries $\Lambda_\xi^\pm$. But $\xi$-invariant conjugacy classes are carried by both $A_{na}\Lambda_\xi^\pm$ and $A_{na}\Lambda_\xi^\pm$ and hence carried by $A_{na}\Lambda_\psi^{\mp}$. It follows that $\Lambda_\xi^\pm$ is carried by $A_{na}(\Lambda_\psi^{\mp})$. But the realization of a generic leaf of $\Lambda_\xi^\pm$ in $G_\psi$ contains $\beta$ as a subpath, and so it is not carried by $A_{na}\Lambda_\psi^{\mp}$, a contradiction.

All that remains is to show that $\Lambda_\xi^\pm$ and $\Lambda_\xi^\mp$ are a dual lamination pair for $\xi$. Consider the subset of $L^\pm(\xi)$ consisting of all lamination pairs that are not supported by $F$, and let this subset be indexed as $\{\Lambda_i^{\pm}\}_{i \in I}$. Assuming that $\Lambda_\xi^\pm$ and $\Lambda_\xi^\mp$ are not dual, we have

$$\Lambda_\xi^+ = \Lambda_i^+ \quad \text{and} \quad \Lambda_\xi^- = \Lambda_j^- \quad \text{for some } i \neq j \in I$$

From this we derive a contradiction. There are two cases, depending on whether $\Lambda_i^+ \subset \Lambda_j^+$.

**Case 1:** $\Lambda_i^+ \not\subset \Lambda_j^+$. It follows that a generic leaf $\sigma \in \Lambda_j^+$ is not weakly attracted to $\Lambda_i^+ = \Lambda_\xi^+$ by forward iteration of $\xi$. Lemma III.2.1 (3) implies that $\sigma$ is not weakly attracted to $\Lambda_j^- = \Lambda_\xi^-$ by backward iteration of $\xi$. It follows by (A) that $\sigma$ is carried by $A_{na}\Lambda_\phi^{\pm}$, and so the lamination $\Lambda_j^+$ is carried by $A_{na}\Lambda_\phi^{\pm}$ (by Fact I.1.8 (1)).

In the subcase that the pair $A_{na}\Lambda_\phi^{\pm}$ is nongeometric, $A_{na}\Lambda_\phi^{\pm}$ is a free factor system, and since $\mathcal{F}_{\supp}(\Lambda_j^+) = \mathcal{F}_{\supp}(\Lambda_j^-)$ it follows that $\Lambda_j^-$ is carried by $A_{na}\Lambda_\phi^{\pm}$.

In the subcase that the pair $A_{na}\Lambda_\phi^{\pm}$ is geometric, $A_{na}\Lambda_\phi^{\pm}$ is not a free factor system, but it is malnormal (Proposition III.1.4 (3)). By hypothesis the group $\langle \phi, \psi \rangle$ is geometric above $F$, and so $\Lambda_j^+$ is geometric. We have verified the hypotheses of Fact I.2.17, the conclusion of which says that $\Lambda_j^-$ is carried by $A_{na}\Lambda_\phi^{\mp}$.

In either case, it follows that any generic leaf $\tau$ of $\Lambda_j^- = \Lambda_\xi^-$ is carried by $A_{na}\Lambda_\phi^{\mp}$, contradicting that $\tau$ is contained in $V_\xi^- = V_\phi^-$ which is an attracting neighborhood of $\Lambda_\phi^-$.  

**Case 2:** $\Lambda_i^+ \subset \Lambda_j^+$. By Lemma I.2.16 we have $\Lambda_i^- \subset \Lambda_j^-$ which implies that a generic leaf $\sigma$ of $\Lambda_i^-$ is not weakly attracted to $\Lambda_j^- = \Lambda_\xi^-$ under iteration of $\xi^{-1}$. Lemma III.2.1 (3) implies that $\sigma$ is not weakly attracted to $\Lambda_i^+ = \Lambda_\xi^+$ under iteration of $\xi$. It follows by (A)
that $\sigma$ is carried by $A_{na}\Lambda_{\psi}^\pm$, and so $\Lambda^{-}_{\xi}$ is carried by $A_{na}\Lambda_{\psi}^\pm$. As in Case 1, arguing in two subcases depending on geometricity of $A_{na}\Lambda_{\psi}^\pm$, we conclude that $\Lambda^{+}_{\xi} = \Lambda^{-}_{\xi}$ is carried by $A_{na}\Lambda_{\psi}^\pm$. A generic leaf $\tau$ of $\Lambda^{+}_{\xi}$ is therefore carried by $A_{na}\Lambda_{\psi}^\pm$, contradicting that $\tau$ is contained in $V_{\psi}^+ = V_{\psi}^+$ which is an attracting neighborhood of $\Lambda_{\psi}^+$. 

This completes the proof of Proposition 1.3.

**Remark 1.5.** The proof that $\Lambda^{\pm}_{\xi} \in \mathcal{L}^{\pm}(\xi)$ is a dual lamination pair depends on applying Fact I.2.17 to a certain lamination pair of $\xi$—namely $\Lambda^{\pm}_{j}$ in Case 1 and $\Lambda^{\pm}_{i}$ in Case 2—and to the subgroup system $A_{na}(\Lambda^{\pm}_{\phi})$. This application fails apart when the $\xi$-lamination pair is nongeometric and $\Lambda^{\pm}_{\phi}$ is geometric, but that possibility has been ruled out by applying the second bulleted hypothesis of Proposition 1.3 to $\xi = \psi^{m}\phi^{n} \in \langle \phi, \psi \rangle$: either $\Lambda^{\pm}_{\phi}$ is not geometric, or every lamination pair of every element of $\langle \phi, \psi \rangle$ that is not supported by $\mathcal{F}$ is a geometric lamination pair. The proof of duality of $\Lambda^{+}_{\xi}, \Lambda^{-}_{\xi}$ in the case that all relevant lamination pairs are geometric, via the application of Fact I.2.17, ultimately depends on the span construction of [BFH00] Lemma 7.0.7. We do not know of any analogue of the span construction for nongeometric laminations, and we do not know if the duality of $\Lambda^{+}_{\xi}$ and $\Lambda^{-}_{\xi}$ would hold without the second bulleted hypothesis of Proposition 1.3.

## 2 Proof of the Main Theorem C

In Section 2.1 we shall reduce Theorem C to its special case Theorem I; for the statements of Theorems C and I we refer the reader to the Introduction of this series [HM13a], or to the early passages of this Part IV. Section 2.2 contains the construction of conjugators needed for application to ping-pong arguments. Section 2.3 contains the argument used to drive down the nonattracting subgroup system of an attracting lamination pair, by applying ping-pong. Section 2.4 contains the argument used to drive up the relative free factor support of an attracting lamination pair, again by applying ping-pong, and also puts the pieces together to prove Theorem I. Section 2.5 contains the general statement and the proof of Theorem J, the absolute case of which was stated in the Introduction of this series [HM13a].

### 2.1 Reduction to Theorem I.

For proving Theorem C, consider a finitely generated subgroup $\mathcal{H} < \text{IA}_{n}(\mathbb{Z}/3)$ and an $\mathcal{H}$-invariant multi-edge extension $\mathcal{F} \sqsubseteq \mathcal{F}'$ relative to which $\mathcal{H}$ is irreducible. It follows by Lemma II.4.2 that each component of $\mathcal{F}$ and of $\mathcal{F}'$ is $\mathcal{H}$-invariant.

We claim that there exists exactly one component $[F'] \in \mathcal{F}'$ that is not a component of $\mathcal{F}$. The existence of at least one such component follows because the extension $\mathcal{F} \sqsubseteq \mathcal{F}'$ is proper. If there are two such components $[F']_1 \neq [F']_2$ then, letting $\mathcal{F}_1 = \mathcal{F} \wedge [F']_1$, which is just the maximal subset of $\mathcal{F}$ such that $\mathcal{F}_1 \supseteq \{[F']_1\}$, it follows that the free factor system $(\mathcal{F} - \mathcal{F}_1) \cup \{[F']_1\}$ is $\mathcal{H}$-invariant, and it is nested strictly between $\mathcal{F}$ and $\mathcal{F}'$ because $[F']_1$ is a component but $[F']_2$ is not. This contradicts that $\mathcal{H}$ is irreducible relative to $\mathcal{F} \sqsubseteq \mathcal{F}'$.

Consider $\widehat{\mathcal{A}} = \mathcal{F} \wedge [F']$, which equals the maximal subset of $\mathcal{F}$ such that $\widehat{\mathcal{A}} \sqsubseteq [F']$. We may represent $\widehat{\mathcal{A}} = \{[A_1], \ldots, [A_K]\}$ where $A_k < F'$ for each $k$. Letting $[\cdot]'$ denote conjugacy classes in the group $F'$, it follows that $\mathcal{A} = \{[A_1]', \ldots, [A_K]'\}$ is a free factor.
system in $F'$ and that $A \subset \{[F']\}$ is a multi-edge extension. Let $\hat{H} < \text{Out}(F')$ be the image of $H$ under the restriction homomorphism $\text{Stab}_{\text{Out}(F')}([F']) \to \text{Out}(F')$ (see Fact I.1.4). By construction $\hat{H}$ is finitely generated, $A$ is $H$-invariant, and $H$ is irreducible relative to $A$. Also, since $H_1(F'; \mathbb{Z}/3)$ is an $H$-invariant free factor of $H_1(F_n; \mathbb{Z}/3)$ and since $H$ acts trivially on $H_1(F_n; \mathbb{Z}/3)$ it follows that $\hat{H}$ acts trivially on $H_1(F'; \mathbb{Z}/3)$. Theorem I produces some $\hat{\phi} \in \hat{H}$ that is fully irreducible relative to $A$, and any of its pre-images $\phi \in H$ is fully irreducible relative to $F \supset F'$, completing the reduction.

### 2.2 Constructing a conjugator

Ping pong arguments often use two group elements $\phi, \psi$ which are “independent” in some sense which guarantees that words in high powers of $\phi$ and $\psi$ produce other interesting group elements. “Independence” means different things in different contexts, depending on the application. When the ambient group is acting on $H^n$, independence might mean that $\phi, \psi$ are loxodromic and their axes have disjoint endpoints. When the ambient group is the mapping class group of a surface, independence might mean that the stable/unstable laminations of the pseudo-Anosov components of $\phi$ and $\psi$ are mutually transverse and fill the surface.

Often one is handed only $\phi$, and $\psi$ is then constructed as a conjugate $\psi = \zeta \phi \zeta^{-1}$. In order to guarantee that $\phi$ and $\psi$ are “independent”, the conjugating element $\zeta$ must somehow move $\phi$ “away from itself” or make $\phi$ “transverse to itself”. Examples of this train of thought can be seen in the proof of the Tits alternative in various settings [BFH00], [McC85], [Iva92] and in the proofs of subgroup decomposition theorems for mapping class groups [Iva92].

Here is our conjugator constructor lemma, which starts with an element $\phi \in \text{IA}_n(\mathbb{Z}/3)$ and a lamination pair. Under a certain group theoretic hypothesis, the conclusion states the existence of a conjugator $\zeta$ satisfying several properties (1)–(4) which in some sense describe how $\zeta$ “moves $\phi$ away from itself” or “makes $\phi$ transverse to itself”. The proof of this lemma borrows heavily from the proof of Lemma 7.0.3 of [BFH00], which plays a similar role in the ping-pong argument of [BFH00] Proposition 7.0.2.

**Lemma 2.1.** Consider $H < \text{IA}_n(\mathbb{Z}/3)$, an outer automorphism $\phi \in H$, and a lamination pair $\Lambda^\pm_\phi \in \mathcal{L}^\pm_{\text{per}}$. If $\text{Stab}_H(\mathcal{A}_{\text{na}} \Lambda^\pm_\phi)$ has infinite index in $H$ then, letting $\lambda^\pm_\phi$ be generic leaves of $\Lambda^\pm_\phi$, respectively, there exists $\zeta \in H$ such that the following hold:

1. Neither $\zeta(\lambda^+_\phi)$ nor $\zeta(\lambda^-_\phi)$ is carried by $\mathcal{A}_{\text{na}} \Lambda^\pm_\phi$,
2. $\zeta(\mathcal{A}_{\text{na}} \Lambda^\pm_\phi) \neq \mathcal{A}_{\text{na}} \Lambda^\pm_\phi$
3. $\zeta(\Lambda^+_\phi) \neq \Lambda^-_\phi$
4. $\zeta(\Lambda^-_\phi) \neq \Lambda^+_\phi$.

**Proof.** We quickly reduce the proof to two sublemmas which we afterwards prove. The first sublemma establishes (1). Its proof will depend upon the proof of Lemma 7.0.3 of [BFH00].

**First Sublemma:** There exists a finite index subgroup $H_0 < H$ such that for any $\zeta \in H_0$, neither $\zeta(\lambda^+_\phi)$ nor $\zeta(\lambda^-_\phi)$ is carried by $\mathcal{A}_{\text{na}} \Lambda^\pm_\phi$. 

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Applying the First Sublemma, and after passing to a further finite index subgroup of \( \mathcal{H} \) still called \( \mathcal{H}_0 \), we may assume that (1) holds for any \( \zeta \in \mathcal{H}_0 \), and that either that \( \mathcal{H}_0 < \text{Stab}(\Lambda^+_\phi) \) or that \( \text{Stab}_{\mathcal{H}_0}(\Lambda^+_\phi) = \mathcal{H}_0 \cap \text{Stab}(\Lambda^+_\phi) \) has infinite index in \( \mathcal{H}_0 \). Notice that in proving (3) we need only be concerned with the case that \( \text{Stab}_{\mathcal{H}_0}(\Lambda^+_\phi) \) has infinite index, because if \( \mathcal{H}_0 < \text{Stab}(\Lambda^+_\phi) \) then for any choice of \( \zeta \in \mathcal{H}_0 \) item (3) obviously holds, because \( \Lambda^+_\phi \neq \Lambda^-\phi \). Similarly, passing to still a further finite index subgroup, we may assume either that \( \mathcal{H}_0 < \text{Stab}(\Lambda^-\phi) \) or that \( \text{Stab}_{\mathcal{H}_0}(\Lambda^-\phi) = \mathcal{H}_0 \cap \text{Stab}(\Lambda^-\phi) \) has infinite index in \( \mathcal{H}_0 \), and in proving item (4) we need only be concerned about the infinite index case.

The second sublemma is a simple result about group actions on sets:

**Second Sublemma:** Suppose that a group \( H \) acts on sets \( X_1, \ldots, X_M \) and that \( x_m \in X_m \) are points whose stabilizers in \( H \) have infinite index. Then there is an infinite sequence \( g_1, g_2, \ldots \in H \) such that for all \( 1 \leq m \leq M \) and for all \( k \neq l \geq 1 \) we have \( g_k(x_m) \neq g_l(x_m) \).

We prove Lemma 2.1 as follows. The group \( \mathcal{H}_0 \) acts on the set \( X_1 = \{ \text{vertex groups systems of } F_n \} \), and by hypothesis the stabilizer subgroup in \( \mathcal{H}_0 \) of the vertex group system \( x_1 = A_{na}(\Lambda^+_\phi) \) is the infinite index subgroup \( \text{Stab}_{\mathcal{H}_0}(A_{na}(\Lambda^+_\phi)) \). Also, \( \mathcal{H}_0 \) acts on the set \( C(B) = \{ \text{closed subsets of } B \} \) and the two laminations \( \Lambda^+_\phi, \Lambda^-\phi \) are each elements of \( C(B) \); we focus solely on which of the two stabilizer subgroups \( \text{Stab}_{\mathcal{H}_0}(\Lambda^+_\phi), \text{Stab}_{\mathcal{H}_0}(\Lambda^-\phi) \) has infinite index in \( \mathcal{H}_0 \) — both of them, one of them, or none of them — we take \( M = 3, 2, \) or \( 1 \) respectively, and for \( 2 \leq i \leq M \) we take \( X_i = C(B) \) and we let \( x_i \) be the appropriate one of \( \Lambda^+_\phi, \Lambda^-\phi \). If for instance both of these subgroups have infinite index in \( \mathcal{H}_0 \) then \( X_2 = X_3 = C(B), x_2 = \Lambda^+_\phi, \) and \( x_3 = \Lambda^-\phi \). Applying the Second Sublemma, we may choose \( \zeta \in \mathcal{H}_0 \) so as to satisfy conclusions (2), (3) and (4).

**Proof of First Sublemma.** Passing to a positive power we assume \( \phi \) is rotationless. Let \( f : G \rightarrow G \) be a CT representing \( \phi \) with EG stratum \( H_r \) corresponding to \( \Lambda^+_\phi \) chosen so that \( [G_r] = \mathcal{F}_{\text{supp}}(\Lambda^+_\phi) \). Recall the following notions from Definition III.1.2: the subgraph \( Z \subset G \), the path \( \rho_r \), which is a trivial path if \( H_r \) is nongeometric and is the unique closed height \( r \) Nielsen path \( \rho_r \) if \( H_r \) is geometric, and the subset \( \langle Z, \rho_r \rangle \subset \mathcal{B}(G) \) consisting of all lines which are concatenations of edges of \( Z \) and copies of \( \rho_r \). Recall also from Lemma III.1.5 (3) and Corollary III.1.9 that a line is in the set \( \langle Z, \rho_r \rangle \) if and only if it is carried by \( A_{na}(\Lambda^+_\phi) \). It therefore suffices to prove the following statement:

- There exists a finite index subgroup \( \mathcal{H}_0 < \mathcal{H} \) such that for any \( \theta \in \mathcal{H}_0 \) and any generic lines \( \lambda^\pm \) for \( \Lambda^\pm \), the realizations of \( \theta(\lambda^+_\phi) \) and \( \theta(\lambda^-\phi) \) in \( G \) are not in the subset \( \langle Z, \rho_r \rangle \).

Lemma 7.0.3 of [BFH00], proved on pages 615–620 of [BFH00], is the special case of this statement under the additional hypothesis that the lamination pair \( \Lambda^\pm \) is topmost in \( \mathcal{L}^\pm(\phi) \), that being a requirement for defining the subgraph \( Z \) in [BFH00]. But the proof given there works exactly as stated in our present setting, with the following minor changes. One uses our general definition of \( Z \) given in Definition III.1.2, rather than the special definition in the “topmost” case given in the proof of [BFH00] Proposition 6.0.4; the only property of \( Z \) needed to make the proof of Lemma 7.0.3 work is that \( Z \cap G_r = G_{r-1} \), which holds here.
as it does in [BFH00]. And in the geometric case: one uses our (strong) geometric model for \(f\) and \(H_r\) given in Definition I.2.4, rather than the weak geometric model which suffices for the topmost case; and one uses our generalized span argument contained in Fact I.2.17 (together with malnormality of \(A_{na}(\Lambda^\pm)\) proved in Proposition III.1.4 (3)), in place of the topmost case of the span argument contained in [BFH00] Corollary 7.0.8.

**Proof of Second Sublemma.** The proof is by induction on \(M\) with the \(M = 1\) case being an immediate consequence of the assumption that the stabilizer of \(x_1\) in \(H\) has infinite index.

For the inductive step, assume that there is an infinite sequence \(\hat{g}_1, \hat{g}_2, \ldots \in H\) such that \(\hat{g}_k(x_m) \neq \hat{g}_k(x_m)\) for \(1 \leq m \leq M - 1\) and for \(k \neq l\). If \(\{\hat{g}_l(x_M)\}\) is an infinite set then, after passing to a subsequence, we may assume that \(\hat{g}_k(x_M) \neq \hat{g}_l(x_M)\) for \(k \neq l\). In this case we define \(g_k = \hat{g}_k\) for all \(k\).

We may therefore assume, after passing to a subsequence, that \(\hat{g}_k(x_M) = \bar{x}_M\) is independent of \(k\). Since the \(H\)-orbit of \(\bar{x}_M\) equals the \(H\)-orbit of \(x_M\), there is an infinite sequence \(\{h_s\}\) in \(H\) such that \(h_s(\bar{x}_M) \neq h_t(\bar{x}_M)\) for \(s \neq t\).

We define by induction an increasing function \(\alpha : \mathbb{N} \to \mathbb{N}\) such that \(h_j\hat{g}_{\alpha(j)}(x_m) \neq h_j\hat{g}_{\alpha(j)}(x_m)\) for \(j < J\) and \(1 \leq m \leq M - 1\). Assume that \(\alpha(1), \ldots, \alpha(J - 1)\) are defined. For \(1 \leq m \leq M - 1\), the points \(\hat{g}_{\alpha(j - 1) + k}(x_m)\) take infinitely many values as \(k \geq 1\) varies, and so the points \(h_j\hat{g}_{\alpha(j - 1) + k}(x_m)\) take infinitely many values. We may therefore pick \(k \geq 1\) and set \(\alpha(J) = \alpha(J - 1) + k\) so that for \(1 \leq m \leq M - 1\), the point \(h_j\hat{g}_{\alpha(j)}(x_m)\) is different from each of \(h_1\hat{g}_{\alpha(1)}(x_m), \ldots, h_{J - 1}\hat{g}_{\alpha(J - 1)}(x_m)\). This completes the definition of \(\alpha\).

Setting \(g_j = h_j\hat{g}_{\alpha(j)}\) completes the proof. \(\square\)

### 2.3 Driving down \(A_{na}(\Lambda^\mp)\)

In the setting of Theorem I, where \(\mathcal{H} < \mathcal{I}(\mathbb{Z}/3)\) is finitely generated and irreducible with respect to an \(\mathcal{H}\)-invariant free factor system \(\mathcal{F}\) such that \(\mathcal{F} \sqsubseteq \{[F_n]\}\) is a multi-edge extension, the desired conclusion is the existence of \(\phi \in \mathcal{H}\) which is irreducible rel \(\mathcal{F}\). From the Weak Attraction Theory developed in Part III [HM13d], this conclusion follows if one can exhibit a dual lamination pair \(\Lambda^\pm_\phi \in \mathcal{L}^\pm_\phi\) such that that the joint free factor support of \(\mathcal{F}\) and \(\Lambda^\pm_\phi\) is “large enough”—namely is equal to \(\{[F_n]\}\)—and such that the nonattracting subgroup system \(A_{na}(\Lambda^\pm_\phi)\) is “small enough”—namely is equal to either \(\mathcal{F}\) (in the nongeometric case) or to the union of \(\mathcal{F}\) and a single rank 1 component that together with \(\mathcal{F}\) fills \(\{[F_n]\}\) (in the geometric case).

In this section we focus on the problem of making \(A_{na}(\Lambda^\pm_\phi)\) small enough, with particular attention on attaining the equation \(A_{na}(\Lambda^\mp_\phi) = \mathcal{F}\). Of course this equation implies that \(\Lambda^\pm_\phi\) is a nongeometric lamination pair and so it would be impossible to attain if it so happened that the subgroup \(\mathcal{H}\) is geometric above \(\mathcal{F}\) (Definition 1.2).

Fortunately, as the next proposition shows, in the nongeometric case one can attain the desired equation \(A_{na}(\Lambda^\mp_\phi) = \mathcal{F}\) for some \(\phi\), and in the geometric case one can just finish off the conclusion of Theorem I in its entirety, with even stronger conclusions that will be used in Theorem J (see Section 2.5). Recall that the vertex group system \(A_{na}(\Lambda^\mp_\phi)\) is a free factor system system if and only if \(\Lambda^\mp_\phi\) is nongeometric; in particular, if \(\Lambda^\pm_\phi\) is geometric then \(A_{na}(\Lambda^\pm_\phi)\) is properly carried by its free factor support \(\mathcal{F}_{supp}(A_{na}(\Lambda^\pm_\phi))\).
Proposition 2.2. Let $H < IA_n(\mathbb{Z}/3)$ be finitely generated, and let $F$ be a proper, $H$-invariant free factor system such that $H$ is irreducible rel $F$ and $F \subseteq \{[F_n]\}$ is a multi-edge extension.

1. If $H$ is not geometric above $F$ then there exists $\phi \in H$ and a nongeometric $\Lambda_{\phi}^\pm \in \mathcal{L}^\pm(\phi)$ such that $\mathcal{A}_{na}(\Lambda_{\phi}^\pm) = F$.

2. If $H$ is geometric above $F$ then there exists $\phi \in H$, $\Lambda_{\phi}^\pm \in \mathcal{L}^\pm(\phi)$, and a root free $\gamma \in F_n$ such that
   - (a) $F \cup \{[\gamma]\} = \mathcal{A}_{na}(\Lambda_{\phi}^\pm)$,
   - (b) $F_{supp}(F, \Lambda_{\phi}^\pm) = \{[F_n]\},$
   - (c) The subgroup system $\mathcal{A}_{na}(\Lambda_{\phi}^\pm)$ is invariant under the entire group $H$.

Remark. Only items (1), (2a) and (2b) will be used in the proof of Theorem I in Section 2.4. Item (2c) is used in the proof of Theorem J to follow in Section 2.5.

Proof. Using the hypotheses and applying the Relative Kolchin Theorem II.1.1, it follows that there exists $\phi \in H$ and a lamination pair $\Lambda_{\phi}^\pm \in \mathcal{L}^\pm(\phi)$ which is not carried by $F$. We may choose $\phi$ and $\Lambda_{\phi}^\pm$ so that the following also holds:

Geometricity Alternative: If $H$ is not geometric above $F$ then $\Lambda_{\phi}^\pm$ is not geometric.

Passing to a power if necessary we may assume that $\phi$ is rotationless. For the proof we choose:

- A CT $f: G \to G$ representing $\phi$, with EG stratum $H_s$ corresponding to $\Lambda_{\phi}^+$, and with core filtration element $G_t$ satisfying $[\pi_1 G_t] = F$ (Theorem I.1.30).

With this choice, we note that the following properties hold:
   - $s > t$ — because $\Lambda_{\phi}^+$ is not carried by $F$ (Fact I.1.55).
   - $F \subseteq \mathcal{A}_{na}(\Lambda_{\phi}^+)$ — because $F \subseteq [\pi_1 G_{s-1}] \subseteq \mathcal{A}_{na}(\Lambda_{\phi}^+)$ (Definition III.1.2).

We break into cases depending on the index of the subgroup $\text{Stab}_H(\mathcal{A}_{na}(\Lambda_{\phi}^+))$ in the group $H$.

Case 1: $\text{Stab}_H(\mathcal{A}_{na}(\Lambda_{\phi}^+))$ has finite index in $H$. We prove in two subcases that $\phi$ satisfies the two conclusions (1) and (2) of Proposition 2.2. We break into subcases depending on whether $H$ is geometric above $F$. In the geometric subcase, much of the hard work in verifying (2a) is subsumed in the result from Part II [HM13c], Theorem II.4.1, which says that for outer automorphisms contained in $IA_n(\mathbb{Z}/3)$, periodic conjugacy classes are fixed.

Subcase 1a: $H$ is not geometric above $F$. By the Geometricity Alternative the lamination pair $\Lambda_{\phi}^\pm$ is nongeometric and so $\mathcal{A}_{na}(\Lambda_{\phi}^\pm)$ is a free factor system (Proposition III.1.4). We obtain a chain of inclusions of free factor systems as follows:

$$F \subseteq F' = \mathcal{A}_{na}(\Lambda_{\phi}^\pm) \subseteq \{[F_n]\}$$

Note that the second inclusion of this chain is not proper, that is $F = F'$, and therefore $\phi$ satisfies conclusion (1).
Sublemma 2.3. If $\mathcal{H} < \mathcal{I}_n(\mathbb{Z}/3)$ is irreducible rel $\mathcal{F}$ and if $\mathcal{F} \subset \mathcal{F}' \subset \{[F_n]\}$ is a chain of proper inclusions of free factor systems then the subgroup $\text{Stab}_\mathcal{H}(\mathcal{F}')$ has infinite index in $\mathcal{H}$.

Proof. Since $\mathcal{H}$ is irreducible rel $\mathcal{F}$, there exists $\phi \in \mathcal{H}$ that does not fix $[\mathcal{F}']$. Since $\phi \in \mathcal{I}_n(\mathbb{Z}/3)$, it follows by Theorem II.3.1 that no nontrivial power $\phi^k$ fixes $[\mathcal{F}']$. This shows that $\langle \phi \rangle$ is an infinite cyclic subgroup of $\mathcal{H}$ having trivial intersection with $\text{Stab}_\mathcal{H}(\mathcal{F}')$, and so the latter has infinite index in $\mathcal{H}$.

Subcase 1b: $\mathcal{H}$ is geometric above $\mathcal{F}$. It follows that the lamination pair $\Lambda^\pm$ is geometric, so $\Lambda_{\text{na}} \Lambda^\pm$ is a vertex group system but not a free factor system (Proposition III.1.4), and $H_s \subset G$ is a geometric stratum. Choose a geometric model for the CT $f: G \to G$ and the stratum $H_s$ having associated surface subgroup system $[\pi_1 S]$ and rank 1 subgroup systems $[\partial_0 S], \ldots, [\partial_m S]$ associated to the components of $\partial S$, with the notation chosen so that $\partial_0 S$ represents the same conjugacy class in $F_n$ as the height $s$ closed indivisible Nielsen path $\rho_s$ (Definition I.2.4), and so that each of $[\partial_0 S], \ldots, [\partial_m S]$ is carried by $[\pi_1 G_{s-1}]$. By Proposition I.2.20, the action of $\text{Stab}_\mathcal{H}(\Lambda_{\text{na}} \Lambda^\pm)$ on conjugacy classes of elements and subgroups fixes $[\pi_1 S]$ and permutes $[\partial_0 S], [\partial_1 S], \ldots, [\partial_m S]$ amongst themselves. By Theorem II.4.1, periodic conjugacy classes are fixed for each element of $\mathcal{I}_n(\mathbb{Z}/3)$, and so each of $[\partial_0 S], [\partial_1 S], \ldots, [\partial_m S]$ is fixed by each element of $\text{Stab}_\mathcal{H}(\Lambda_{\text{na}} \Lambda^\pm)$. By Proposition I.2.18 (5) we have

$$\mathcal{F} \supset \{[\partial_0 S], [\partial_1 S], \ldots, [\partial_m S]\} = \mathcal{F} \supset \{[\pi_1 S]\} = \mathcal{F} \supset \Lambda^\pm,$$

Consider the following two free factor systems, both of which are stabilized by $\text{Stab}_\mathcal{H}(\Lambda_{\text{na}} \Lambda^\pm)$:

$$\mathcal{F}_1 = \mathcal{F} \supset \{[\partial_0 S], [\partial_1 S], \ldots, [\partial_m S]\}\}
\mathcal{F}_2 = \mathcal{F} \supset \{[\pi_1 S]\}
\mathcal{F}_2 = \mathcal{F} \supset \Lambda^\pm.$$

By construction we have three inclusions of free factor systems $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \{[F_n]\}$. The middle inclusion is proper because $\mathcal{F} \supset \{[\partial_0 S]\} \not\subset [\pi_1 G_{s-1}]$ (Lemma I.2.5 (2)) but $\mathcal{F}_1 \subset [\pi_1 G_{s-1}]$. If either the first or third inclusion is proper then, taking $\mathcal{F}' = \mathcal{F}_1$ or $\mathcal{F}' = \mathcal{F}_2$ respectively, and applying Sublemma 2.3, we conclude that $\text{Stab}_\mathcal{H}(\mathcal{F}')$ has infinite index in $\mathcal{H}$, and therefore the subgroup $\text{Stab}_\mathcal{H}(\Lambda^\pm) < \text{Stab}_\mathcal{H}(\mathcal{F}')$ also has infinite index, contradicting the hypothesis of Case 1. We have verified the two equations

$$\mathcal{F}_1 = \mathcal{F}_2 = \{[F_n]\}$$
from the second of which it follows that $\mathcal{F} \supset \{[\pi_1 G_1]\} = \mathcal{F} \supset \{[\pi_1 G_{s-1}]\}$ which is (2b). It also follows that $H_s$ is the top stratum because $\mathcal{F} \supset \{[\pi_1 G_1]\}$ and so $\mathcal{F} = [\pi_1 G_1] = [\pi_1 G_{s-1}]$, for if not then we may apply Lemma I.2.5 (5) with the conclusion that $[\pi_1 G_{s-1}] \not\subset \mathcal{F} \supset \{[\pi_1 G_1], [\pi_1 S]\} = \{[F_n]\}$, a contradiction. Combining this with Definition III.1.2 and Corollary III.1.9 we have

$$\Lambda_{\text{na}}(\Lambda^\pm) = [\pi_1 G_{s-1}] \cup \{[\partial_0 S]\} = \mathcal{F} \cup \{[\gamma]\}.$$
where \( \gamma \in F_n \) is represented by \( \rho_s \). This proves (2a).

To verify (2c), we have already shown that \( A_{na}(\Lambda^{\pm}_\phi) \) is invariant under the finite index subgroup \( Stab_H(A_{na}(\Lambda^{\pm}_\phi)) \). Consider any \( \psi \in H \). Since \( \psi \) has a positive power in the subgroup \( Stab_H(A_{na}(\Lambda^{\pm}_\phi)) \) it follows that \( A_{na}(\Lambda^{\pm}_\phi) \) is periodic under \( \psi \). Since \( A_{na}(\Lambda^{\pm}_\phi) = F \cup \{[\partial_0 S]\} \) it follows that each of \( F \) and \( [\partial_0 S] \) is \( \psi \)-periodic. Since \( \psi \in IA_\mu(\mathbb{Z}/3) \), we may apply Theorem II.4.1 to \( F \) and Theorem II.4.1 to \( [\partial_0 S] \) and so each of \( F \) and \( [\partial_0 S] \) is fixed by \( \psi \). Since \( \psi \in H \) is arbitrary, the vertex group system \( A = A_{na}(\Lambda^{\pm}_\phi) \) is invariant under the whole group \( H \).

**Case 2:** \( Stab_H(A_{na}(\Lambda^{\pm}_\phi)) \) has infinite index in \( H \).

In this case we carry out an induction argument. We will construct \( \xi \in H \) and a lamination pair \( \Lambda^{\pm}_\xi \in L^{\pm}(\xi) \) such that the following hold:

- \( F \subseteq A_{na}(\Lambda^{\pm}_\xi) \subseteq A_{na}(\Lambda^{\pm}_\phi) \), and the second of these inclusions is proper;
- \( \Lambda^{\pm}_\xi \) is non-geometric if \( \Lambda^{\pm}_\phi \) is non-geometric, and therefore \( \xi \) and \( \Lambda^{\pm}_\xi \) continue to obey the Geometricity Alternative.

Once \( \xi \) and \( \Lambda^{\pm}_\xi \) have been constructed, if \( Stab_H(A_{na}(\Lambda^{\pm}_\xi)) \) has finite index in \( H \) then we are reduced to Case 1 with \( \xi \) in place of \( \phi \). If \( Stab_H(A_{na}(\Lambda^{\pm}_\xi)) \) has infinite index in \( H \) then we may iterate this process, returning to Case 2 with \( \xi \) in place of \( \phi \). Since nonattracting subgroup systems are vertex group systems (Proposition III.1.4 (1)), this iteration must eventually stop because of the chain condition for vertex group systems (Proposition I.3.2) which bounds the length of any properly nested chain of vertex group systems. By induction we are eventually reduced to Case 1, completing the proof of Proposition 2.2.

We turn to the construction of \( \xi \) and \( \Lambda^{\pm}_\xi \), for which we will apply our main ping-pong argument, Proposition 1.3. Verifying the hypotheses of that proposition requires some further work.

After passing to a power of \( \phi \), the laminations \( \Lambda^{\pm}_\phi \) have generic leaves \( \gamma^{\pm}_\phi \) that are fixed by \( \phi, \phi^{-1} \), respectively, with fixed orientations. The hypothesis of Case 2 matches the hypothesis of Lemma 2.1, and so by applying that lemma we may choose \( \zeta \in H \) so that neither of \( \zeta(\gamma^+_\phi) \), \( \zeta(\gamma^-_\phi) \) is carried by \( A_{na}(\Lambda^{\pm}_\phi) \) and so that \( \zeta(A_{na}(\Lambda^{\pm}_\phi)) \neq A_{na}(\Lambda^{\pm}_\phi) \), \( \zeta(\Lambda^{\pm}_\phi) \neq \Lambda^{\pm}_\phi \), and \( \zeta(\Lambda^{\pm}_\phi) \neq \Lambda^{\pm}_\phi \). Define \( \psi = \zeta \circ \phi \circ \zeta^{-1} \), and so we have a dual lamination pair \( \Lambda^{\pm}_\psi = \zeta(\Lambda^{\pm}_\phi) \in L^{\pm}(\psi) \) with generic leaves \( \gamma^{\pm}_\psi = \zeta(\gamma^{\pm}_\phi) \) and \( \gamma^{\pm}_\psi = \zeta(\gamma^{\pm}_\phi) \) that are fixed by \( \psi \), \( \psi^{-1} \) respectively with fixed orientations, and the nonattracting subgroup system of this lamination pair satisfies

\[
F = \psi(F) \subseteq A_{na}(\Lambda^{\pm}_\psi) = \zeta(A_{na}(\Lambda^{\pm}_\phi)) \neq A_{na}(\Lambda^{\pm}_\phi)
\]

Note that \( F_{\supp}(\Lambda^{\pm}_\psi) = \zeta(F_{\supp}(\Lambda^{\pm}_\phi)) \) and so the connected free factor systems \( F_{\supp}(\Lambda^{\pm}_\psi) \) and \( F_{\supp}(\Lambda^{\pm}_\phi) \) have the same rank. Noting also that \( \Lambda^{\pm}_\psi \) is non-geometric if \( \Lambda^{\pm}_\psi \) is non-geometric, and that \( (\phi, \psi) < H \), the two bulleted hypotheses of Proposition 1.3 are therefore satisfied, the second one following from the Geometricity Alternative.

We next verify the hypotheses (i)–(iv) of Proposition 1.3. Hypothesis (i) says that \( \Lambda^{\pm}_\psi \) is weakly attracted to \( \Lambda^{\pm}_\phi \) under iteration by \( \phi \). Note that \( \Lambda^{\pm}_\phi \not\subset \Lambda^{\pm}_\psi \), for otherwise, since \( F_{\supp}(\Lambda^{\pm}_\phi) \) and \( F_{\supp}(\Lambda^{\pm}_\psi) \) have the same rank, Lemma 3.1.15 of [BFH00] would imply that
\[ \Lambda^-_\phi = \Lambda^+_\psi, \text{ a contradiction.} \]

We conclude that \( \Lambda^+_\psi \) does not contain a generic leaf of \( \Lambda^-_\phi \) and hence the generic leaf \( \gamma^+_\psi \) is not contained in every neighborhood of a generic leaf of \( \Lambda^-_\phi \). By the choice of \( \zeta \in \mathcal{H} \), the line \( \gamma^+_\psi = \zeta(\gamma^+_\phi) \) is not carried by \( A_{na}\Lambda^+_\phi \). By our weak attraction result, Corollary III.2.17 (Theorem H), it follows that \( \gamma^+_\psi \) is weakly attracted to \( \Lambda^+_\phi \) under iteration by \( \phi \). This verifies hypothesis (i). The symmetric arguments show that hypotheses (ii), (iii) and (iv) are also satisfied.

We have now verified all of the hypotheses of Proposition 1.3. From the conclusion of that proposition, there exists \( \xi \in \mathcal{H} \) and a lamination pair \( \Lambda^+_\xi \) that is non-geometric if \( \Lambda^+_\phi \) is non-geometric such that \( \mathcal{F} \sqsubset A_{na} \Lambda^+_\xi \), and such that \( A_{na} \Lambda^+_\xi \sqsubset A_{na} \Lambda^+_\phi \) and \( A_{na} \Lambda^+_\xi \sqsubset A_{na} \Lambda^+_\phi \).

Since \( \zeta(A_{na} \Lambda^+_\phi) = A_{na} \Lambda^+_\psi \) it follows that the maximum length of a strictly decreasing sequence of vertex group systems beginning with \( A_{na} \Lambda^+_\phi \) is the same as the maximum length of a strictly decreasing sequence of vertex group systems beginning with \( A_{na} \Lambda^+_\phi \), from which it follows that \( A_{na} \Lambda^+_\phi \) is not strictly contained in \( A_{na} \Lambda^+_\psi \), and so it is not contained at all. The containment \( A_{na} \Lambda^+_\phi \sqsubset A_{na} \Lambda^+_\psi \) is therefore proper. \( \square \)

### 2.4 Proof of Theorem I. Driving up \( \mathcal{F}_{\text{supp}}(\mathcal{F}, \Lambda^+_\phi) \)

In this section we prove Theorem I, so let \( \mathcal{H} < IA_n(Z/3) \) be finitely generated, and let \( \mathcal{F} \) be a proper, \( \mathcal{H} \)-invariant free factor system so that \( \mathcal{F} \sqsubset \{[F_n]\} \) is a multi-edge extension and \( \mathcal{H} \) is irreducible rel \( \mathcal{F} \).

By Proposition 2.2, there exists a rotationless \( \phi \in \mathcal{H} \) and a lamination pair \( \Lambda^+_\phi \in \mathcal{L}^+(\phi) \) such that one of two conclusions holds:

\( \Lambda^+_\phi \) is **nongeometric**, in which case \( A_{na}(\Lambda^+_\phi) = \mathcal{F} \).

\( \Lambda^+_\phi \) is **geometric**, in which case there exists a root free \( \gamma \in F_n \) such that

\[ A_{na}(\Lambda^+_\phi) = \mathcal{F} \cup \{[\langle \gamma \rangle]\}, \text{ and } \mathcal{F}_{\text{supp}}(\mathcal{F}, \Lambda^+_\phi) = \{[F_n]\}. \]

We now prove Theorem I in two separate cases. As we shall see, what is left in the geometric case will be very quick, because the conditions above are sufficient to prove that \( \phi \) is irreducible rel \( \mathcal{F} \). But in the nongeometric case notice that the conditions above make no mention of the \( \phi \)-invariant free factor system \( \mathcal{F}_{\text{supp}}(\mathcal{F}, \Lambda^+_\phi) \), which properly contains \( \mathcal{F} \), and which might be proper in \( F_n \), and so \( \phi \) might be reducible rel \( \mathcal{F} \). The problem in the nongeometric case then becomes to “drive up” the free factor system \( \mathcal{F}_{\text{supp}}(\mathcal{F}, \Lambda^+_\phi) \).

**Proof of Theorem I in the Geometric Case.** Let \( \phi, \Lambda^+_\phi \in \mathcal{L}^+(\phi) \) satisfy the above conclusions in the geometric case. We prove that \( \phi \) is irreducible rel \( \mathcal{F} \) by a CT argument.

Choose a CT \( f : G \to G \) with a core filtration element \( G_t \) such that \( [\pi_1 G_t] = \mathcal{F} \) and with a stratum \( H_s \) corresponding to \( \Lambda^+_\phi \). Since \( \mathcal{F}_{\text{supp}}(\mathcal{F}, \Lambda^+_\phi) = \{[F_n]\} \) it follows that \( H_s \) is the top stratum, that is, \( G = G_s \). By Definition III.1.2, which defines \( A_{na}(\Lambda^+_\phi) \) in terms of \( f \), there exists \( \rho \in F_n \) such that \( A_{na}(\Lambda^+_\phi) = [\pi_1 G_{s-1}] \cup \{[\rho]\} \) and so

\[ \mathcal{F} \cup \{[\langle \gamma \rangle]\} = [\pi_1 G_{s-1}] \cup \{[\rho]\} \supseteq [\pi_1 G_{s-1}] \supset [\pi_1 G_t] = \mathcal{F} \]

The final \( \supseteq \) relation holds because \( G_{s-1} \supset G_t \). It follows that \( \mathcal{F} \) consists of all but one component of the subgroup system \( [\pi_1 G_{s-1}] \cup \{[\rho]\}] = A_{na}(\Lambda^+_\phi) \) which, being a vertex
group system, is malnormal (Lemma I.3.1). Since \( F \subset [\pi_1 G_{s-1}] \) it follows by malnormality that the excluded component is \([\langle \rho \rangle ] = [\langle \gamma \rangle ]\), and that \( F = [\pi_1 G_{s-1}] \). Applying (Filtration) in the definition of a CT (Definition I.1.29), it follows that \( \phi \) is irreducible relative to \( [\pi_1 G_{s-1}] = F \).

For the rest of this section we turn to:

**Proof of Theorem I in the Nongeometric Case.** We shall prove:

(*) If \( \Lambda_\phi^\pm \) is nongeometric, and if \( \mathcal{F}_{\text{supp}}(F, \Lambda_\phi^\pm) \) is a proper free factor system, then there exists \( \xi \in \mathcal{H} \) and a non-geometric lamination pair \( \Lambda_\phi^\pm \in \mathcal{L}^\pm(\xi) \) such that \( \mathcal{A}_{\text{na}}(\Lambda_\phi^\pm) = \mathcal{F} \) and such that \( \mathcal{F}_{\text{supp}}(F, \Lambda_\phi^\pm) \) is properly contained in \( \mathcal{F}_{\text{supp}}(F, \Lambda_\phi^\pm) \).

Assuming (*) for the moment, it follows by an obvious induction argument that:

(**) If \( \Lambda_\phi^\pm \) is nongeometric then there exists \( \zeta \in \mathcal{H} \) and a non-geometric lamination pair \( \Lambda_\zeta^\pm \in \mathcal{L}^\pm(\zeta) \) such that \( \mathcal{A}_{\text{na}}(\Lambda_\zeta^\pm) = \mathcal{F} \) and \( \mathcal{F}_{\text{supp}}(F, \Lambda_\zeta^\pm) = \{[F_n]\}, \)

and using (** we prove that \( \zeta \) is irreducible rel \( F \) by a CT argument similar to but simpler than in the Geometric Case. Choose a CT \( f: G \to G \) representing \( \zeta \) with core filtration element \( G_t \) such that \([\pi_1 G_t] = F \) and with stratum \( H_s \) corresponding to \( \Lambda_\zeta^\pm \).

Since \( \mathcal{F}_{\text{supp}}(F, \Lambda_\zeta^\pm) = \{[F_n]\} \) it follows that \( H_s \) is the top stratum, that is, \( G_s = G \). By Definition III.1.2 it follows that \([\pi_1 G_{s-1}] = \mathcal{A}_{\text{na}}(\Lambda_\phi^\pm) = \mathcal{F} = [\pi_1 G_t] \). By (Filtration) in the definition of a CT it follows that \( \zeta \) is irreducible rel \([\pi_1 G_{s-1}] = F \).

It remains to prove (*). For the rest of the proof we denote \( \mathcal{F}_\phi = \mathcal{F}_{\text{supp}}(F, \Lambda_\phi^\pm) \), and the method of proof of (*) is to drive up \( \mathcal{F}_\phi \), using our ping-pong argument, Proposition 1.3, to produce \( \xi \). After replacing \( \phi \) by some positive power we may assume that \( \phi \) and \( \phi^{-1} \) are rotationless and that there are generic leaves \( \lambda^+ \in \Lambda_\phi^+ \) and \( \lambda^- \in \Lambda_\phi^- \), fixed by \( \phi \), \( \phi^{-1} \), respectively, preserving orientation.

Since the free factor support of any attracting lamination has a single component (Fact I.1.14 (2)), it follows that the free factor system \( \mathcal{F}_\phi \) has a single component denoted \([F_\phi]\) that supports \( \Lambda_\phi^\pm \). Since \([F_\phi]\) is the unique component of \( \mathcal{F}_\phi \) that is not a component of \( F \), components of \( \mathcal{F}_\phi \) other than \([F_\phi]\) are components of \( F \). By applying Theorem II.3.1 it therefore follows for any \( \eta \in \mathcal{H} \) that \( \mathcal{F}_\phi \) is \( \eta \)-invariant if and only if \([F_\phi]\) is \( \eta \)-invariant. Since \( \mathcal{H} \) is irreducible rel \( F \) we may choose \( \eta \in \mathcal{H} \) so that \( \eta([F_\phi]) \neq [F_\phi] \), and so \( \eta([F_\phi]) \neq [F_\phi] \). Let \( \psi = \eta \phi \eta^{-1} \) and \( \Lambda_\psi^\pm = \eta(\Lambda_\phi^\pm) \). Then \( \mathcal{A}_{\text{na}}(\Lambda_\psi^\pm) = \eta(\mathcal{A}_{\text{na}}(\Lambda_\phi^\pm)) = \eta(\mathcal{F}) = \mathcal{F} \) and \([F_\psi]\) is contained in \([F_\phi]\). In particular, there are no inclusions among the four non-geometric laminations \( \{\Lambda_\phi^+, \Lambda_\phi^- \} \).

The two bulleted items in the hypotheses of Proposition 1.3 are satisfied. Since \( \Lambda_\psi^+ \) does not contain \( \Lambda_\phi^- \), there is a neighborhood \( V^-_\phi \) of a generic leaf of \( \Lambda_\phi^- \) that does not contain any generic leaf of \( \Lambda_\psi^+ \). Corollary III.2.17 (Theorem H) therefore implies that \( \Lambda_\psi^+ \) is weakly attracted to \( \Lambda_\phi^+ \) under iteration by \( \phi \). This is item (i) in the hypotheses of Proposition 1.3; items (ii), (iii) and (iv) are proved similarly.

Having verified the hypotheses of Proposition 1.3, we may adopt notation from its conclusions and for all sufficiently large \( M \) we obtain the following objects:
(1) an outer automorphism of the form $\xi_M = \phi^m(M)\psi^n(M)$ where $m(M), n(M) \geq M$;
(2) lamination pairs $\Lambda^\pm_M \in \mathcal{L}^\pm(\xi_M)$;
(3) an attracting neighborhood $V^+_M := V^+_\phi$ of a generic leaf of $\Lambda^+_M$ for the action of $\xi_M$;
(4) an attracting neighborhood $V^-_M := V^-_\phi$ of a generic leaf of $\Lambda^-_M$ for the action of $\xi^-_M$;

such that the following hold:

(5) $\mathcal{A}_{\text{na}}(\Lambda^+_M) = \mathcal{F}$;
(6) $\psi^m(M)(V^+_\phi) \subset V^+_\phi = V^+_M$;
(7) $\phi^{-n}(M)(V^-_\phi) \subset V^-_\phi = V^-_M$;
(8) For any neighborhoods $U^+_\psi, U^-_\psi \subset \mathcal{B}$ of generic leaves of $\Lambda^+_\psi, \Lambda^-_\psi$ respectively, if $M$ is sufficiently large then generic leaves of $\Lambda^+_M$ are contained in $U^+_\psi$ and generic leaves of $\Lambda^-_M$ are contained in $U^-_\psi$.

Consider a sequence $(\gamma^-_M)$ of generic leaves of the laminations $\Lambda^-_M$, once such leaf for each $M$, and let $\text{Acc}(\gamma^-_M)$ denote its weak accumulation set, meaning the set of weak limits of subsequences; this is a closed subset of $\mathcal{B}$, since $\mathcal{B}$ has a countable basis. Consider similarly a sequence $(\gamma^+_M)$ of generic leaves of the laminations $\Lambda^+_M$, and its weak accumulation set $\text{Acc}(\gamma^+_M)$. We claim that:

(9) The closed set $\text{Acc}(\gamma^-_M) \cap \text{Acc}(\gamma^+_M)$ is carried by $\mathcal{F}$.

Arguing by contradiction, suppose that there exists $\ell \in \text{Acc}(\gamma^-_M) \cap \text{Acc}(\gamma^+_M)$ which is not carried by $\mathcal{F}$. First we show that the weak closure of $\ell$ contains a generic leaf $\sigma^-_\phi$ of $\Lambda^-_\phi$. If not then $\ell$ is disjoint from some weak neighborhood of $\sigma^-_\phi$. Applying Corollary III.2.17 (2) (Theorem H), and using that $\ell$ is not carried by $\mathcal{F} = \mathcal{A}_{\text{na}}(\Lambda^\pm_\phi)$, it follows that $\phi^N(\ell) \in V^+_\phi$ for some $N > 0$. Since $\ell \in \text{Acc}(\gamma^-_M)$ it follows that there exists $M$ such that $n(M) \geq M \geq N$ and such that $\phi^N(\gamma^-_M) \subset V^+_\phi$, and so $\phi^n(M)(\gamma^-_M) \subset V^+_\phi$. By item (6) we have $\xi_M(\gamma^-_M) = \psi^m(M)\phi^n(M)(\gamma^-_M) \subset V^+_\phi$. This contradicts the fact that $\xi_M(\gamma^-_M)$, which is a generic leaf of $\Lambda^-_M$, is not contained in any attracting neighborhood for $\Lambda^-_M$. Having shown that the weak closure of $\ell$ contains $\sigma^-_\phi$, it follows that $\sigma^-_\phi \in \text{Acc}(\gamma^-_M) \cap \text{Acc}(\gamma^+_M)$, and we also know that $\sigma^-_\phi$ is not carried by $\mathcal{F} = \mathcal{A}_{\text{na}}(\Lambda^\pm_\phi)$. By a completely symmetric argument, using $\sigma^+_\phi$ instead of $\ell$, using $\xi^+_M = \phi^-\psi^-n$ instead of $\xi_M = \psi^m\phi^n$, and using (7) instead of (6), it follows that the weak closure of $\sigma^+_\phi$ contains $\sigma^+_\psi$, and so $\Lambda^+_\psi \subset \Lambda^-_\phi$, a contradiction that proves Claim (9).

To complete the proof of Theorem 2.2, we shall show that if $M$ is sufficiently large then, letting $\mathcal{F}_M = \mathcal{F}_{\xi_M} = \mathcal{F}_{\text{supp}(\mathcal{F}, \Lambda^\pm_M)}$, we have $\mathcal{F}_\phi, \mathcal{F}_\psi \subset \mathcal{F}_M$ and so $\mathcal{F}_{\text{supp}(\mathcal{F}_\phi, \mathcal{F}_\psi)} \subset \mathcal{F}_M$. Once we have shown this, since $\mathcal{F}_\phi, \mathcal{F}_\psi$ are unequal but have components in bijective correspondence with equal ranks, it follows that the containment $\mathcal{F}_\phi \subset \mathcal{F}_{\text{supp}(\mathcal{F}_\phi, \mathcal{F}_\psi)}$ is proper, and so the containment $\mathcal{F}_\phi \subset \mathcal{F}_M$ is proper, and the proof is completed by taking $\xi = \xi_M$ and $\Lambda^\pm_\xi = \Lambda^\pm_M$. 

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Choose a marked graph $H$ with a core subgraph $H_0$ realizing $\mathcal{F}$. For each $M \geq M_0$ let $[F_M] \in \mathcal{F}_M$ denote the component of $\mathcal{F}_M$ that supports $\Lambda_M^\pm$, and note that the free factor system $\mathcal{F} \setminus [F_M]$ is precisely the set of components of $\mathcal{F}$ that are contained in $[F_M]$. Let $K_M$ be the Stallings graph determined by $H$ and $F_M$, i.e. the core of the covering space of $H$ corresponding to the subgroup $F_M$. The restriction $p_M: K_M \rightarrow H$ of the covering map is an immersion such that $[p_*(\pi_1(K_M))]= [F_M]$.

Since $[F_M]$ is a free factor system of $F_n$, there exists an embedding of $K_M$ into a marked graph $G_M$ and an extension of $p_M$ to a homotopy equivalence $q_M: G_M \rightarrow H$ that preserves marking. For every line $\ell \in \mathcal{B}$, letting $\ell_M$, $\ell_H$ be its realizations in $G_M$, $H$ respectively, note that $\ell$ is supported by $[F_M]$ if and only if $\ell_M$ is contained in $K_M$; furthermore, if these equivalent statements hold then the restriction of $p_M$ to $\ell_M$ is an immersion whose image is $\ell_H$, and conversely $\ell_H$ lifts uniquely via $p_M$ to a line in $K_M$, that line being $\ell_M$. By construction these statements hold whenever $\ell$ is a leaf of $\Lambda_M^+$ or $\Lambda_M^-$. The graph $K_M$ has a natural cell structure in which each vertex has valence $\geq 3$. The natural cell structure on $K_M$ also has a subdivided cell structure with respect to which $p_M: K_M \rightarrow H$ is a cellular map, taking each vertex to a vertex and each edge to an edge; the edges with respect to this subdivision are called edgelets of $K_M$, and we label each edgelet by its image in $H$. There is a unique subgraph $\tilde{K}_M$ of $K_M$ such that $p \mid \tilde{K}_M$ is a homeomorphism onto the union of the components of $H_0$ corresponding to $\mathcal{F} \setminus [F_M]$, and in fact this homeomorphism is an edgelet cellular isomorphism. Note also that

\begin{equation}
[K_M]=[F_M]=\mathcal{F}_{\text{supp}}(\mathcal{F} \setminus [F_M], \Lambda_M^+) = \mathcal{F}_{\text{supp}}(\tilde{K}_M, \Lambda_M^+) \tag{10}
\end{equation}

For each $M$ we shall identify the leaves $\gamma^+_M$ with their realizations in $G_M$, each of which is contained in the subgraph $K_M$ and so is immersed by $p_M$ with image being their realizations in $H$. By (10) it follows that

\begin{equation}
\text{(11) Each natural edge of } K_M \text{ that is not contained in } \tilde{K}_M \text{ is crossed by both } \gamma^+_M \text{ and } \gamma^-_M.
\end{equation}

For each integer $C \geq 0$ let $Y_{M,C} \subset K_M$ be the $C$-neighborhood of the set of natural vertices of $K_M$, with respect to a path metric in which every edgelet of $K_M$ has length 1. We claim next that:

\begin{equation}
\text{(12) There exists a constant } C \text{ independent of } M \text{ such that each edgelet of } K_M \text{ that is labelled by an edge in } H \setminus H_0 \text{ is contained in } Y_{M,C}, \text{ equivalently each edgelet of } K_M \setminus Y_{M,C} \text{ is labelled by an edge in } H_0.
\end{equation}

To prove this, if no such $C$ exists then there is a subsequence $(M_i)_{i \geq 1}$ diverging to $+\infty$, and for each $i \geq 1$ there is an edgelet $e_i \subset K_{M_i}$ projecting to an edge of $H \setminus H_0$, such that $e_i$ is the central edgelet of an embedded edgelet path $\eta_i \subset K_{M_i}$ of length $2i+1$ and $\eta_i$ contains no natural vertex of $K_{M_i}$. For some subsequence of $(M_i)$, the projection to $H$ of the edgelet $e_i$ is constant independent of $i$; for some further subsequence of $(M_i)$, the projection to $H$ of the central length 3 subpath of $\eta_i$ is constant, independent of $i$; for some further subsequence the projection of the central length 5 subpath is constant; ... . By continuing inductively and then diagonalizing, we obtain a subsequence of $(M_i)$ such that for each $i$ the projection to $H$ of the central $2i+1$ subsequence of $\eta_j$ is constant independent of $j \geq i$. Noting that $e_i \not\subset \tilde{K}_{M_i}$, it follows by (11) that $\gamma^+_M$ and $\gamma^-_M$ each cross $\eta_i$. The nested union
of the projections to $H$ of the paths $\eta_i$ is therefore a line $\ell \in \text{Acc}(\gamma_M^+) \cap \text{Acc}(\gamma_M^-)$ realized in $H$. But $\ell$ crosses an edge of $H \setminus H_0$, namely the projection of $e_1$, and so $\ell$ is not carried by $[H_0] = F$, contradicting (9) and therefore proving (12).

As a consequence of the fact that $K_M$ has a uniformly bounded number of natural vertices, it follows that the graph $Y_M = Y_{M,C}$ has a uniformly bounded number of edgelets. Note also that the set of edgelet labels—namely, the edges of $H$—is finite, and that $K_M$ has uniformly bounded rank. We may therefore assume, after passing to a further subsequence, that for all $M, M'$ there is a homeomorphism $h_{M,M'} : (K_M, Y_M) \to (K_{M'}, Y_{M'})$ whose restriction to $Y_M$ maps edgelet to edgelet and preserves labels. In other words, as an unlabelled natural graph $K_M$ is independent of $M$, and furthermore its labelled edgelet subgraph $Y_M$ is independent of $M$. The components of $K_M \setminus Y_M$ are central subpaths of natural edges, and all the edgelet labels along these subpaths are in $H_0$. After passing to another subsequence and perhaps enlarging $Y_M$ we may assume that the edgelet length of each component of $K_M \setminus Y_M$ goes to infinity with $M$.

To complete the proof, letting $\sigma_\psi^+ \in \Lambda_\psi^+$, $\sigma_\phi^- \in \Lambda_\phi^-$ be generic leaves, it suffices to show that if $M$ is sufficiently large then the realizations in $H$ of both $\sigma_\psi^+$ and $\sigma_\phi^-$ lift into $Y_M$, for that implies that both $\Lambda_\psi^+$ and $\Lambda_\phi^-$ are carried by $[F_M]$, and so $F_\phi, F_\psi \subset F_M$ as desired. Suppose that $\rho$ is a finite subpath of the realization of $\sigma_\psi^+$ in $H$ such that $\rho$ begins and ends with edges in $H \setminus H_0$. Letting $U_\psi^+ = B(H, \rho)$, it follows by (8) that there exists $M_\rho$ such that $\rho$ is a subpath of the realization of $\gamma_M^+$ in $H$, for all $M \geq M_\rho$. Lifting $\rho$ along with $\gamma_M^+$ into $K_M$ we obtain a lift $\rho' \subset K_M$ with first and last edgelets in $Y_M$ and edgelet length $L$ that is independent of $M$. For sufficiently large $M$, the edgelet length of each component of $K_M \setminus Y_M$ is greater than $L$ and so $\rho' \subset Y_M$. Since $Y_M$ is independent of $M$, each such path $\rho$, and hence the whole line $\sigma_\psi^+$, lifts into $Y_M$ as desired. The symmetric argument for $\sigma_\phi^-$ completes the proof of Theorem I in the Nongeometric Case.

\[ \square \]

2.5 Theorem J: Relatively geometric irreducible subgroups

In the case of Proposition 2.2 where $\mathcal{H}$ is geometric above $F$, stronger conclusions follow as explained in Theorem J, the absolute case of which was stated in the introduction. Here we state and prove Theorem J in its full generality, which will be very quick after we review from Part I [HM13b] concepts of geometric models needed for the general statement of the theorem.

First we review the definition of a weak geometric model (Definition I.2.1), which applies to any top EG stratum of any CT. Consider $\phi \in \text{Out}(F_n)$ represented by a CT $f : G \to G$ with top EG stratum $H_r$, and let $\Lambda \in L(\phi)$ be the attracting lamination that corresponds to $H_r$. Recall from Definition I.2.19, Proposition I.2.18, and Definition I.2.2 that $\Lambda$ is geometric if and only if $H_r$ is geometric if and only if there exists a weak geometric model of the CT $f$ for the stratum $H_r$, the definition of which is as follows. The static data of a weak geometric model consists of a 2-complex $Y$ formed as the quotient of a compact surface $S$ and the graph $G_{r-1}$, where $S$ has one “upper” boundary component $\partial_0 S$ and remaining “lower” boundary components $\partial_i S$, $i = 1, \ldots, m$ ($m \geq 0$), and where the quotient is formed by gluing each lower boundary circle $\partial_i S$ to $G_{r-1}$ using a homotopically nontrivial closed edge path $\alpha_i : \partial_i S \to G_{r-1}$. The static data also includes an embedding $G \hookrightarrow Y$ extending
the embedding of $G_{r-1}$, and a deformation retraction $d: Y \rightarrow G$, such that $G \cap \partial_0 S = \{p_r\}$ is a single point, and such that $d \mid \partial_0 S$ is an immersion. The dynamic data of a weak geometric model consists of a homotopy equivalence $h: Y \rightarrow Y$ and a homeomorphism $\Psi: (S, \partial_0 S) \rightarrow (S, \partial_0 S)$ with pseudo-Anosov mapping class $\psi \in \mathcal{MCG}(S)$ such that the maps $(f \mid G_r) \circ d, d \circ h: Y \rightarrow G_r$ are homotopic, and the maps $j \circ \Psi, h \circ j: S \rightarrow Y$ are homotopic.

Under the quotient map $j: S \coprod G_{r-1} \rightarrow Y$, the subset $j(G_{r-1} \coprod \partial_0 S) = Y - j(\text{int} S)$ is called the complementary subgraph $K \subset Y$ (Definition I.2.6). Since $G_{r-1}$ has no contractible components (by Fact I.2.3) it follows that $K$ has no contractible components; combining with Lemma I.2.7, the restricted map $d \mid K: K \rightarrow G$ is $\pi_1$-injective on each component of $K$ and there is a subgroup system $\{\pi_1 K\}$ having one component of the form $[d, \pi_1(K_i)]$ for each component $K_i$ of $K$. Letting $[\partial_0 S]$ denote the component of $[\pi_1 K]$ corresponding to the component $\partial_0 S$ of $K$, and since $G_{r-1}$—the union of the remaining components—represents $F$, we have the first equation in the following:

\[ [\pi_1 K] = \mathcal{F} \cup \{[\partial_0 S]\} = A_{\mathcal{F}}(\Lambda_{\phi}^+) \]  

(*)

For the second equation see Definition III.1.2; also, this subgroup system is a vertex group system but not a free factor system (Proposition III.1.4).

The restricted map $j \mid S: S \rightarrow Y$ and its composition with $d: Y \rightarrow G$ are $\pi_1$-injective, and the image subgroup $(d \circ j)_*(\pi_1 S)$ in $\pi_1 G \approx F_n$ is it own normalizer (Lemma I.2.7), so there is a well-defined homomorphism from the subgroup $\text{Stab}[\pi_1 S] < \text{Out}(F_n)$ to $\text{Out}(\pi_1 S)$ (Fact I.1.4). The Dehn-Nielsen-Baer Theorem [FM12] identifies $\mathcal{MCG}(S)$ with the subgroup of $\text{Out}(\pi_1 S)$ preserving the set of conjugacy classes in $\pi_1 S$ associated to oriented components of $\pi_1 S$ (see Proposition I.2.20 and the preceding paragraph).

**Theorem J (Relative, general version).** Given a finitely generated subgroup $\mathcal{H} < \text{IA}_n(\mathbb{Z}/3)$ and an $\mathcal{H}$-invariant free factor system $\mathcal{F}$, if $\mathcal{F} \cap \{[F_n]\}$ is a multi-edge extension, and if $\mathcal{H}$ is geometric irreducible rel $\mathcal{F}$, then there exists $\phi \in \mathcal{H}$ and $\Lambda \in \mathcal{L}(\phi)$ such that $\phi$ is irreducible rel $\mathcal{F}$ and $\mathcal{F}_{\text{supp}}(\mathcal{F}, \Lambda) = \{[F_n]\}$. Furthermore, for any such $\phi$ and $\Lambda$, for any CT $f: G \rightarrow G$ with top stratum $H_r$ corresponding to $\Lambda$, and for any geometric model of $f$ and $H_r$ as notated above, the following hold:

(1) $\mathcal{H}$ stabilizes the subgroup system $[\pi_1 S]$,

(2) the image of the induced homomorphism

\[ \mathcal{H} \hookrightarrow \text{Stab}[\pi_1 S] \twoheadrightarrow \text{Out}(\pi_1 S) \]

is contained in $\mathcal{MCG}(S)$, inducing a homomorphism $\xi: \mathcal{H} \rightarrow \mathcal{MCG}(S)$,

(3) $\xi(\phi)$ is a pseudo-Anosov mapping class on $S$.

**Remark.** In order to match the conclusions of the the general, relative case of Theorem I with the conclusion of the absolute case that was stated in the Introduction [HM13a], a few words are needed. In the absolute case the free factor system $\mathcal{F}$ is trivial, $G_{r-1} = 0$, $H_r = G$, and $Y = S$ has one boundary component $\partial_0 S$. In this case we have isomorphisms $\pi_1(S) = \pi_1(Y) \overset{d_r}{\rightarrow} \pi_1(G) \approx F_n$ well-defined up to inner automorphism; we have
a well-defined induced isomorphism $\text{Out}(\pi_1 S) \approx \text{Out}(F_n)$; the induced homomorphism $\text{Stab}[\pi_1 S] \to \text{Out}(F_n)$ is just the identity map; and the group $MCG(S)$ may be regarded as a subgroup of $\text{Out}(F_n)$. From the conclusions of general, relative version of Theorem J it follows that $\mathcal{H}$ is contained in the $MCG(S)$ subgroup of $\text{Out}(\pi_1 S)$, which is exactly the conclusion of the absolute case.

Proof. Proposition 2.2 (2) proves the existence of $\phi \in \mathcal{H}$ and $\Lambda^+_\phi \in \mathcal{L}^+(\phi)$ such that $\phi$ is irreducible rel $F$, $f_{\text{supp}}(\mathcal{F}, \Lambda^+_\phi) = \{[F_n]\}$, and the subgroup system $\mathcal{A}_{na}(\Lambda^+_\phi)$ is stabilized by $\mathcal{H}$. Fix such a $\phi$ and $\Lambda^+_\phi$. Choose a CT $f : G \to G$ representing $\phi$ with core filtration element $G_s$ such that $\mathcal{F} = [G_s]$ and with EG-stratum $H_r$ corresponding to $\Lambda^+_\phi$. It follows that $H_r$ is the top stratum, that $H_r$ is a geometric stratum, and from (Filtration) in the definition of a CT that $\mathcal{F} = [G_{r-1}]$, and so $G_s$ is the core of $G_{r-1}$. Choose a geometric model for $f$ and $H_r$ denoted as above, and so we have an equation of subgroup systems

$$\mathcal{A}_{na}(\Lambda^+_\phi) = [\pi_1 K] = [G_{r-1}] \cup [\partial_0 S]$$

It follows that $\mathcal{H} < \text{Stab}[\pi_1 K]$. Applying Proposition I.2.20 it follows further that $\text{Stab}[\pi_1 K] < \text{Stab}[\pi_1 S]$ and that the image of the induced homomorphism $\text{Stab}[\pi_1 K] \to \text{Out}(\pi_1 S)$ is contained in the $MCG(S)$ subgroup of $\text{Out}(\pi_1 S)$; and so the image of the composed homomorphism $\mathcal{H} \to \text{Out}(\pi_1 S)$ is contained in $MCG(S)$. It follows from the definition of a weak geometric model that the image of $\phi$ under the composed homomorphism $\xi : \mathcal{H} \to MCG(S)$ is pseudo-Anosov.

Remarks. We believe that further conclusions can be drawn in the context of Theorem J, which for each $\psi \in \mathcal{H}$ relate properties of the Thurston decomposition of $\xi(\psi)$ with properties of $\psi$. As an easy example, by applying Lemma I.1.64 it follows that $\psi$ is of polynomial growth relative to $\mathcal{F}$ if and only if the Thurston decomposition of $\xi(\psi)$ has no pseudo-Anosov components. We believe that the following also hold: $\psi$ is irreducible rel $\mathcal{F}$ if and only if $\xi(\psi)$ is pseudo-Anosov; more generally the subset of lamination pairs in $\mathcal{L}^+(\psi)$ that are not supported by $\mathcal{F}$ is in natural bijective correspondence with the unstable-stable lamination pairs of the Thurston decomposition of $\xi(\psi)$. We do not pursue this issue here because the proofs would require some strengthening of the already rather intricate technical results about geometric models developed in Part I [HM13b].

3 A filling lemma

In [HM09], the predecessor of this series of papers, we proved Theorem A which is the special case of Theorem I under the additional hypothesis that $\mathcal{F} = \emptyset$ (see [HM13a]). That proof follows the same structure of two ping-pong tournaments followed above in the proof of Theorem I. In the absolute case, the second tournament has the goal of driving up the “absolute” free factor support $f_{\text{supp}}(\Lambda^+_\phi)$ to its maximal value of $\{[F_n]\}$. The logic of that proof used a more complicated argument for driving up free factor supports, which is encapsulated in Proposition 8.1 of [HM09]. Although in proving Theorem I we have avoided these complications and produced an argument simpler than that in Theorem A, we nonetheless find that a relativization of [HM09] Proposition 8.1 is useful in other contexts [HM13e], and so we develop that relativization here.
**Proposition 3.1.** Suppose that $B_1, B_2 \subseteq \mathcal{B}$ are sets of lines, that $\mathcal{F}_0$ is a free factor system that carries $\text{cl}(B_1) \cap \text{cl}(B_2)$ and that for $l = 1, 2$:

1. neither end of any element of $B_l$ is carried by $\mathcal{F}_0$.
2. there does not exist a proper free factor system $\mathcal{F}$ that carries $B_l$.

Then there exist weak neighborhoods $U(b) \subseteq \mathcal{B}$, one for each $b \in B_1 \cup B_2$, such that for each proper free factor system $\mathcal{F}$ there exists $b_F \in B_1 \cup B_2$ such that $\mathcal{F}$ does not carry a line in $U(b_F)$.

**Remark.** In the absolute case where $\mathcal{F}_0 = \emptyset$ this statement is equivalent to Proposition 8.1 of [HM09].

**Proof.** Choose a marked graph $H$, all of whose vertices have valence at least three, with a subgraph $H_0$ realizing $\mathcal{F}_0$. Let $L(\cdot)$ denote the edge length of paths in $H$. Let each $b \in B_1 \cup B_2$ be realized by a line in $H$ with a chosen base point. For each integer $C$, let $b_C$ be the subpath of $b$ that contains the base point; starts and ends with edges not in $H_0$ and has exactly $2C$ edges not in $H_0, C$ before the base point and $C$ after it. The existence of $b_C$ follows from (1). Let $\hat{U}(b, C) \subseteq \hat{B}(H)$ denote the weak neighborhood of $b$ consisting of paths in $H$ that contain $b_C$ as a subpath. Let $U(b, C) = \hat{U}(b, C) \cap \mathcal{B}(H) \subset \mathcal{B}(H) \approx \mathcal{B}$.

Consider triples $(\mathcal{G}, S, \rho)$ consisting of a marked graph $\mathcal{G}$, a proper connected subgraph $S \subseteq \mathcal{G}$ with no valence 1 vertices, and a map $\rho: \mathcal{G} \rightarrow R_n$ which is a homotopy inverse of the marking of $\mathcal{G}$, such that $\rho$ takes vertices to vertices, $\rho$ is an immersion on each edge of $\mathcal{G}$, and $\rho$ is an immersion on the subgraph $S$; it follows that the restriction of $\rho$ to any path in $S$ is a path in $R_n$. Such a triple is called a representative of a proper free factor $F$ if $[F] = [S]$. We put a metric on each edge of $\mathcal{G}$ by pulling back the metric on $R_n$ under the map $\rho$, and we extend the length notation $L(\cdot)$ to this setting, and so for each edge $E \subseteq \mathcal{G}$ we have $L(E) = L(\rho(E))$. Note that a line $\ell \in \mathcal{B}$ is carried by $[F]$ if and only if the realization $\ell_{\mathcal{G}}$ in $\mathcal{G}$ is contained in $S$, in which case the restriction of $\rho$ to $\ell_{\mathcal{G}}$ is an immersion whose image is the realization of $\ell$ in $R_n$.

We claim that every proper free factor $F$ has a representative $(\mathcal{G}, S, \rho)$. To see why, it is evident that there exists a triple $(\mathcal{G}, S, \rho)$ that satisfies all the required properties except that $\rho$ need not be an immersion on $S$. Factor $\rho: \mathcal{G} \rightarrow R_n$ as a composition of folds $G = G_0 \xrightarrow{p_1} G_1 \xrightarrow{p_2} \ldots \xrightarrow{p_N} G_N = R_n$. Each intermediate graph $G_j$ is marked by a homotopy inverse of $\rho_j = p_{N\cdots j+1} : G_j \rightarrow R_n$. Let $P_j = p_jp_{j-1}\cdots p_1 : G \rightarrow G_j$. By giving precedence to folds involving two edges of $S$, we may assume that there exists $J \geq 1$ such that $S_J := P_J(S)$ is a proper subgraph of $G_J$, the map $P_J$ restricts to a homotopy equivalence from $S$ to $P_J(S)$, and the map $\rho_J | S_J$ is an immersion. After replacing $(\mathcal{G}, S, \rho)$ by $(G_j, S_J, \rho_J)$ we obtain a representative of $F$.

If the proposition fails then there exist $C_i \rightarrow \infty$ and proper free factor systems $\mathcal{F}^i$ represented by $(\mathcal{G}^i, S^i, \rho^i)$ such that for all $b \in B_1 \cup B_2$ there is a subpath $\beta^i \subseteq S^i$ such that $\rho^i(\beta^i) \subseteq H$ contains $b_{C_i}$ as a subpath. Assuming this to be the case, we argue to a contradiction.

By the natural simplicial structure on a marked graph, we mean the one in which each vertex has valence at least three.
If \( L(G^i) \not\to \infty \), then, after passing to a subsequence, \((G^i, S^i, \rho^i)\) is independent of \(i\). Since any path in \(S^i\) can be extended to a line in \(S^i\), this contradicts (2) applied to the free factor system determined by \(S^i\). We may therefore assume that \(L(G^i) \to \infty\).

Assume folding notation \(P^i_j : G^i \to G^i_j\) and \(\rho^i_j : G^i_j \to H\) as above with \(\rho^i = \rho^i_j P^i_j\). If \(i\) is sufficiently large, then (2) implies that \(\cup_{b \in B_l} \rho^i \circ (\beta^i) \cup \cup_{b \in B_l} \beta^i c_n\) covers every edge in \(H\) for both \(l = 1\) and \(l = 2\). We may therefore choose a minimal \(M_i\) such that \(\cup_{b \in B_l} P^i_{M_i} (\beta^i)\) covers every edge in \(G^i_{M_i}\) for both \(l = 1\) and \(l = 2\).

If \(L(G^i_{M_i}) \not\to \infty\) then, after passing to a further subsequence, we may assume that \(G^i_{M_i-1}\) is independent of \(i\) and that there is a natural edge \(E^i \subset G^i_{M_i-1}\), independent of \(i\), such that \(E^i\) is not covered by \(\cup_{b \in B_l} P^i_{M_i-1} (\beta^i)\) for at least one of \(l = 1, 2\). The path obtained from \(P^i_{M_i-1} (\beta^i)\) by removing any initial or terminal segments that are disjoint from the core of \(G^i_{M_i-1} \setminus E^i\) extends to a line in \(G^i_{M_i-1} \setminus E^i\). But then the free factor system determined by the core of \(G^i_{M_i-1} \setminus E^i\) carries \(B_l\) for at least one of \(l = 1, 2\) in contradiction to (2). We may therefore assume that \(L(G^i_{M_i}) \to \infty\).

For each natural edge \(E^i\) of \(G^i_{M_i}\), the path \(\rho^i_{M_i} (E^i) \subset H\) is a subpath of both an element of \(B_1\) and an element of \(B_2\). Since \(F_0\) carries \(\text{cl}(B_1) \cap \text{cl}(B_2)\), there is a constant \(K\) so that either \(E^i\) has length at most \(2K\) or \(\rho^i_{M_i} (E^i)\) decomposes as a concatenation of initial and terminal subpaths of length \(K\) and a central subpath that is contained in \(H_0\).

After passing to another subsequence, we may assume that for all \(i\) there is a subgraph \(Y^i \subset G^i_{M_i}\) such that each component of \(G^i_{M_i} \setminus Y^i\) is an interval (the central subpath of some natural edge) whose length tends to \(\infty\) with \(i\) and whose image under \(\rho^i_{M_i}\) is contained in \(H_0\). Moreover, for all \(i, i'\) there is a homeomorphism \(h_{i, i'} : (G^i_{M_i}, Y^i) \to (G^{i'}_{M_i}, Y^{i'})\) such that \(\rho^i_{M_i} | Y_i = (\rho^i_{M_i} | Y_i) \circ h_{i, i'} | Y_i\). Fix \(K\) and restrict to \(i > K\). Since \(b_{C_k}\) begins and ends with an edge not in \(H_0\), the subpath of \(P^i_{M_i} (\beta^i)\) whose \(\rho^i_{M_i}\)-image is \(b_{C_k}\) must be contained in \(Y^i\) for all sufficiently large \(i\). Since \(K\) is arbitrary, this contradicts (2) and the fact that the free factor system determined by \(Y^i\) is independent of \(i\).

\[\square\]

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