Abstract. Let $L$ be the generator of an analytic semigroup whose kernels satisfy Gaussian upper bounds and Hölder’s continuity. Also assume that $L$ has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$. In this paper, we construct a frame decomposition for the functions belonging to the Hardy space $H^1_2(\mathbb{R}^n)$ associated to $L$, and for functions in the Lebesgue spaces $L^p$, $1 < p < \infty$. We then show that the corresponding $H^1_2(\mathbb{R}^n)$-norm (resp. $L^p(\mathbb{R}^n)$-norm) of a function $f$ in terms of the frame coefficients is equivalent to the $H^1_2(\mathbb{R}^n)$-norm (resp. $L^p(\mathbb{R}^n)$-norm) of $f$. As an application of the frame decomposition, we establish the radial maximal semigroup characterization of the Hardy space $H^1_2(\mathbb{R}^n)$ under the extra condition of Gaussian upper bounds on the gradient of the heat kernels of $L$.

1. Introduction and statement of results

Wavelet analysis has played an important role in many different branches of science and technology since it provides a simple and efficient way, in addition to Fourier series and integrals, to analyse functions and distributions. The wavelet series decompositions are effective expansion by unconditional bases in the standard Lebesgue spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$, as well as many other spaces such as Hardy spaces, BMO spaces, Besov spaces which arise in the theory of harmonic analysis. A function $f$ (may be tempered distributions in some cases) in these various spaces can thus be written in the form

$$f(x) = \sum_{\lambda \in \Lambda} \langle f, \psi_\lambda \rangle \psi_\lambda(x),$$

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and the series converges unconditionally to \( f(x) \) in the relevant norm. Here \( \Lambda = \{ \lambda = 2^{-j}k + \epsilon 2^{-j-1} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \epsilon \in E \} \), where \( E = \{ 0, 1 \}^n \) excluding \( (0, 0, \ldots, 0) \). The family \( \psi_{\lambda}, \lambda \in \Lambda \) is a wavelet basis arising from an \( r \)-regular multiresolution approximation of \( L^2(\mathbb{R}^n) \). Moreover, the norms of elements in these various spaces can be equivalently characterized by the corresponding norms via coefficients of the expansion in (1.1). To be more precise, taking \( L^p(\mathbb{R}^n), 1 < p < \infty \) and Hardy space \( H^1(\mathbb{R}^n) \) for example, we have

\[
\| f \|_{L^p(\mathbb{R}^n)} \approx \left\| \sum_{\lambda \in \Lambda} |\langle f, \psi_{\lambda} \rangle|^2 |Q_\lambda|^{-1} \chi_\lambda(\cdot) \right\|_p
\]

and

\[
\| f \|_{H^1(\mathbb{R}^n)} \approx \left\| \sum_{\lambda \in \Lambda} |\langle f, \psi_{\lambda} \rangle|^2 |Q_\lambda|^{-1} \chi_\lambda(\cdot) \right\|_1,
\]

where \( Q_\lambda \) is the dyadic cube defined by \( 2^j x - k \in [0, 1)^n \) and \( \chi_\lambda(x) \) is the characteristic function of \( Q_\lambda \). For more details about the wavelet theory, we refer to [9, 14, 25, 33, 34]. We note that wavelet theory has also been developed in many other settings including that wavelet bases being replaced by frames which offer the same service in many applications. The success of wavelet theory lies in the fact that it has had applications in widely differing areas of science, see for example [5, 6, 10, 11, 15, 16] and the references therein.

The classical theory of Hardy spaces on \( \mathbb{R}^n \) has been a great success and central to the estimates of singular integrals [37]. Since a number of characterizations of the classical Hardy space can be given via various estimates of the Laplace operator, one can say that the classical Hardy space is associated to the Laplace operator. We note that the Laplace operator has its heat kernel \( p_t(x, y) \) given explicitly by the Gaussian kernel, hence all the heat kernel regularity such as the time derivatives (to all order) and spacial derivatives can be computed explicitly. The Laplace operator also possesses the conservation property and is non-negative self-adjoint, therefore it has a bounded functional calculus on \( L^2(\mathbb{R}^n) \) for bounded measurable functions on \( [0, +\infty) \).

In the last decade, a theory of function spaces, and in particular Hardy spaces, associated to an operator \( L \) was developed and studied extensively. This theory has arisen from the need of studying singular integrals with non-smooth kernels which do not belong to the so-called class of Calderón-Zygmund operators. In this theory, the assumptions on the heat kernel of \( L \) and the functional calculus of \( L \) play a key role. The weaker these assumptions are, the more operators \( L \) are included in the theory but less features and characterizations of the spaces can be obtained. We now list a number of articles closely related to the development of this topic but our list is by no means exhaustive.

(i) In [2], P. Auscher, X.T. Duong and A. McIntosh introduced the Hardy space \( H^1_L(\mathbb{R}^n) \) associated to an operator \( L \), and obtained a molecular decomposition, assuming that \( L \) has a bounded holomorphic functional calculus on \( L^2(\mathbb{R}^n) \) and the kernel of the heat semigroup \( e^{-tL} \) has a pointwise Poisson upper bound. Under the same assumptions on \( L, X.T. \) Duong and L.X. Yan introduced the space \( BMO_L(\mathbb{R}^n) \) adapted to \( L \) and established the duality of \( H^1_L(\mathbb{R}^n) \) and \( BMO_L(\mathbb{R}^n) \) in [21, 22], where \( L^* \) denotes the adjoint operator of \( L \) in \( L^2(\mathbb{R}^n) \). Later, the Hardy spaces \( H^p_L(\mathbb{R}^n) \) for all \( 0 < p < 1 \) were established in [38].

(ii) P. Auscher, A. McIntosh and E. Russ [3] established the Hardy spaces \( H^p_L, p \geq 1 \), associated to the Hodge Laplacian on a Riemann manifold with doubling measure. S. Hofmann and S. Mayboroda [28] defined the Hardy spaces \( H^p_L, p \geq 1 \), associated to a second order divergence form elliptic operator on \( \mathbb{R}^n \) with complex coefficients. In these settings, pointwise heat kernel bounds may fail. By making use of the notion of “L-cancellation” of molecules, they studied the Hardy space \( H^1_L \) including a molecular decomposition, a square function characterization, its dual space and others properties.

(iii) Later, in [27], S. Hofmann et al developed the theory of \( H^1 \) and \( BMO \) spaces adapted to a non-negative, self-adjoint operator \( L \) whose heat kernel satisfies the weak Davies-Gaffney bounds, in
the setting of a space of homogeneous type $X$. For the Hardy space $H^1_f(X)$, they also obtained an atomic decomposition. X.T. Duong and J. Li [18] extended this line to develop Hardy spaces $H^p(X)$ for $0 < p \leq 1$, including a molecular decomposition, a square function characterization, duality of Hardy and Lipschitz spaces, and a Marcinkiewicz type interpolation theorem, where the operator $L$ needs not be a non-negative self-adjoint operator. R.J. Jiang and D.C. Yang [30] also extended this line to Orlicz–Hardy spaces.

(iv) X.T. Duong, J. Li and L.X. Yan [19] established a discrete characterization of weighted Hardy spaces $H^{p}_{L, S, w}(X)$ associated to $L$ in terms of the area function characterization, where $L$ is a second order non-negative self-adjoint operator on $L^2(X)$ satisfying the Moser-type condition, and the semigroup $e^{-tL}$ generated by $L$ satisfies Gaussian upper bounds.

(v) In [35], L. Song and L.X. Yan used a modification of technique due to A. Calderón [7] to give an atomic decomposition for the Hardy spaces $H^p_{L, \text{max}}(\mathbb{R}^n)$ in terms of the nontangential maximal functions associated with the heat semigroup of $L$, where $L$ is a second order non-negative self-adjoint operator on $L^2(X)$ and its heat semigroup satisfying Gaussian estimates on $L^2(\mathbb{R}^n)$. This leads eventually to characterizations of Hardy spaces associated to $L$, via atomic decomposition or the nontangential maximal functions. In term of the radial maximal function characterization of Hardy spaces, D.C. Yang and S.B. Yang [40] obtained it under the additional assumption that the kernel of the heat semigroup satisfies Hölder’s continuity. In [36], L. Song and L.X. Yan got rid of the additional assumption of [40] and proved the radial maximal function characterization of Hardy spaces associated to $L$.

(vi) Recently, G. Kerkyacharian and P. Petrushev [31] introduced a nice frame decomposition (associated to $L$) for the Schwarz functions and distributions and established the Besov and Triebel-Lizorkin spaces associated with $L$ in the framework of Dirichlet spaces with a doubling measure $\mu$ satisfying also the reverse doubling condition and the non-collapsing condition, where the operator $L$ is self-adjoint and satisfies the small time Gaussian upper bound, the Hölder continuity as well as the preservation property (Markov property), i.e., $e^{-tL}1 = 1$. Later, S. Dekel, G. Kerkyacharian, G. Kyriazis, P. Petrushev [17] obtained a compactly supported frames for spaces of distributions associated with non-negative self-adjoint operators satisfying the preservation property (Markov property) on a more general setting: spaces of homogeneous type. The development of such frames is important in a situation where no additional structures such as translation invariance or a dilation operator are present.

So far, the main characterisations of Hardy spaces associated to operators are obtained via area integral estimates, atomic or molecular decompositions, and maximal functions. We observe that the frame structure is absent so far for these Hardy spaces when $L$ is either non-selfadjoint or $e^{-tL}$ does not satisfy the preservation property (Markov property) or both.

The aim of this paper is to obtain a frame decomposition for the functions belonging to the Hardy space $H^1_f(\mathbb{R}^n)$ associated to $L$, as well as for the functions in the Lebesgue spaces $L^p$, $1 < p < \infty$, where the corresponding $L^1$ norm (resp. $L^p$-norm) of a function $f$ in terms of the frame coefficients is equivalent to the $H^1_f(\mathbb{R}^n)$-norm (resp. $L^p$-norm) of $f$ (see Theorems 1.1 and 1.3 below). As an application, we establish the radial maximal semigroup characterization of the Hardy space $H^1_f(\mathbb{R}^n)$ by using the frame decomposition (see Theorem 6.1 below).

We now state our assumptions and main results.

Let $L$ be a linear operator of type $\omega$ ($\omega < \pi/2$), which is one-one with dense range on $L^2(\mathbb{R}^n)$, hence $L$ generates a holomorphic semigroup $e^{-tL}$ : $0 \leq |\text{Arg}(z)| < \pi/2 - \omega$ (for more details we refer the readers to the beginning of Section 2). The following shall be assumed throughout the paper unless otherwise specified:
(H1) The operator $L$ has a bounded $H_{\infty}$-calculus on $L^2(\mathbb{R}^n)$. That is, there exists $c_{v,2} > 0$ such that $b(L) \in \mathcal{L}(L^2, L^2)$, and for $b \in H_{\infty}(S^0)$:

$$\|b(L)f\|_2 \leq c_{v,2}\|b\|_{\infty}\|g\|_2$$

for any $f \in L^2(\mathbb{R}^n)$. See Section 2 for the definition of $H_{\infty}(S^0)$ and for more details of this assumption.

(H2) The analytic semigroup $\{e^{-tL}\}$, $|\text{Arg}(z)| < \pi/2 - \omega$, is represented by the kernel $p_c(x,y)$ which satisfies the following Gaussian upper bound

$$(GE) \quad |p_c(x,y)| \leq C_{\theta} \frac{1}{|z|^{n/2}} \exp \left( -\frac{|x-y|^2}{c|z|} \right)$$

for all $x, y \in \mathbb{R}^n$, $|\text{Arg}(z)| < \pi/2 - \theta$ for $\theta > \omega$.

(H3) The analytic semigroup $\{e^{-tL}\}$, $|\text{Arg}(z)| < \pi/2 - \omega$, is represented by the kernel $p_c(x,y)$ which satisfies the following regularity

$$|p_c(x,y) - p_c(x',y)| + |p_c(y,x) - p_c(y,x')| \leq C_\delta \left( \frac{|x-x'|}{|z|^{1/2} + |x-y|} \right)^\gamma \frac{1}{|z|^{n/2}} \exp \left( -\frac{|x-y|^2}{c|z|} \right)$$

for some $\gamma \in (0, 1]$ and for all $x, x', y \in \mathbb{R}^n$ with $2|x-x'| \leq |z|^{1/2} + |x-y|$, $|\text{Arg}(z)| < \pi/2 - \theta$ for $\theta > \omega$.

Suppose $\zeta \in H(S^0)$ with $\zeta \not= 0$ and

$$|\zeta(z)| \leq C \frac{|z|^{\alpha}}{1 + |z|^\beta}$$

where $z \in S^0$, $\alpha > 0$, $\beta > \alpha + n + \gamma + 3$ in which $n$ is the dimension and $\gamma$ is the constant in the assumption (H3). Put

$$q(z) = z^2 \zeta^2(z), \quad q_t(z) = q(tz).$$

Let $\delta$ be a constant satisfying $1 < \delta < 2$. For each $j$, let $I_j$ denote the net of $\delta$-dyadic cubes with side-length $\delta^{-j-M}$ with a large fixed positive integer $M$, where one such cube in the net has the origin as the lower left vertex. And let $\tau$ be the index in $I_j$ and $Q_\tau^j$ denote the cube belong to $I_j$, and $y_{Q_\tau^j}$ denote the centre of the cube $Q_\tau^j$. Denote $q_j(x,y)$ the kernel of the operator $q_{\delta^{-j/2}}(L)$ with $q_j$ defined as in (1.2) above (where $\delta$ is to be determined later). Also, set

$$(1.3) \quad \psi_{j,t}(x) := \sqrt{\ln \delta}|Q_\tau^j|^{1/2}q_j(x,y_{Q_\tau^j}) \quad \text{and} \quad \psi_{j,t}^*(x) := \sqrt{\ln \delta}|Q_\tau^j|^{1/2}q_j(y_{Q_\tau^j}, x)$$

for any $y_{Q_\tau^j} \in Q_\tau^j$. We point out that $q_j(x,y)$ is continuous in both $x$ and $y$ (see Proposition 2.3 below), hence the functions $\psi_{j,t}(x)$ and $\psi_{j,t}^*(x)$ are well-defined for any $y_{Q_\tau^j} \in Q_\tau^j$.

Next for any $f \in L^2(\mathbb{R}^n)$, we define the auxiliary operator

$$(1.4) \quad T_\delta(f)(x) := \ln \delta \sum_{j} \sum_{\tau \in I_j} |Q_\tau^j| q_j(x,y_{Q_\tau^j}) q_{\delta^{-j/2}}(L)f(y_{Q_\tau^j}),$$

where $y_{Q_\tau^j}$ is any point in the cube $Q_\tau^j$ and

$$q_{\delta^{-j/2}}(L)(f)(y_{Q_\tau^j}) = \int_{\mathbb{R}^n} q_j(y_{Q_\tau^j},y)f(y)dy.$$

To see that $T_\delta$ is well-defined and bounded on $L^2(\mathbb{R}^n)$, we refer to Section 3 below.

The first main result in this paper is the following frame decomposition of the functions in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. 


Theorem 1.1. Assume that L satisfies (H1), (H2) and (H3). Let 1 < p < ∞ and f ∈ L^p(\mathbb{R}^n). Then we have the following frame decomposition of f
\begin{equation}
\langle T_\delta^{-1} f, \psi_{j_1, \ldots, j_n} \rangle \psi_{j_1, \ldots, j_n} \quad \text{in } L^p(\mathbb{R}^n).
\end{equation}
Furthermore, there exist positive constants C_1 and C_2, such that
\begin{equation}
C_1 \|f\|_p \leq \left\| \left( \sum_{j_1, \ldots, j_n} \langle T_\delta^{-1} f, \psi_{j_1, \ldots, j_n} \rangle^2 \right)^{1/2} \right\|_p \leq C_2 \|f\|_p.
\end{equation}

Next we recall the definition of Hardy space associated to L.

Definition 1.2 ([22, 27]). Assume that L satisfies (H1) and (H2). The Hardy space H^1_{L,S,L}(\mathbb{R}^n) is defined as the completion of \{f ∈ L^2(\mathbb{R}^n) : \|S_L f\|_{L^1(\mathbb{R}^n)} < \infty\}, with norm \|f\|_{H^1_{L,S,L}(\mathbb{R}^n)} := \|S_L f\|_{L^1(\mathbb{R}^n)}.

The second main result in this paper is the following frame decomposition of the functions in H^1_{L,\mathbb{R}^n}.

Theorem 1.3. Assume that L satisfies (H1), (H2) and (H3). For every f ∈ H^1_{L,S,L}(\mathbb{R}^n), we have the following frame decomposition
\begin{equation}
f = \sum_{j_1, \ldots, j_n} \langle T_\delta^{-1} f, \psi_{j_1, \ldots, j_n} \rangle \psi_{j_1, \ldots, j_n} \quad \text{in } H^1_{L,\mathbb{R}^n}.
\end{equation}
Furthermore, there exist positive constants C_1 and C_2, such that
\begin{equation}
C_1 \|f\|_{H^1_{L,\mathbb{R}^n}} \leq \left\| \left( \sum_{j_1, \ldots, j_n} \langle T_\delta^{-1} f, \psi_{j_1, \ldots, j_n} \rangle^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^n)} \leq C_2 \|f\|_{H^1_{L,\mathbb{R}^n}}.
\end{equation}

Remark 1.4. Operators which satisfy the assumptions (H1), (H2) and (H3) include the following:
(i) Laplace operator on the Euclidean spaces \mathbb{R}^n.
(ii) Second order elliptic divergence form operators with bounded, real coefficients on \mathbb{R}^n. See [4].
(iii) The Schrödinger operators −Δ + V on \mathbb{R}^n where the potentials V belong to a suitable Hölder class. See, for example, [23].

Remark 1.5. Note that in Theorems 1.1 and 1.3 above:
(i) The operators q_{δ^{-2}L}(L) can be replaced by \( (δ^{-2}L)^k e^{-δ^{-2}L} ) \) for any k ∈ \mathbb{N}, which can be observed from the strategy of our proofs. Hence the function \psi_{j_1, \ldots, j_n}(x) = \sqrt{\ln \delta} Q_j^0(x) (δ^{-2}L)^k e^{-δ^{-2}L}(x, y Q_j^0), where \( (δ^{-2}L)^k e^{-δ^{-2}L}(x, y Q_j^0) \) is the kernel of the operator \( (δ^{-2}L)^k e^{-δ^{-2}L} \).
(ii) If L is non-negative self-adjoint, then \psi_{j_1, \ldots, j_n} equals ψ_{j_1, \ldots, j_n}.

Our strategy of proof is the following:
Step 1. We first develop the following two key technical results:
(i) Using the holomorphic functional calculi of operators, we obtain the almost orthogonality estimates for the operators \{q(tL)\}_{t > 0}, where the function q(λ) is defined as in (1.2). See more details in Proposition 2.4.
(ii) We introduce four different versions of the discrete Littlewood–Paley g-functions associated to the operator L, and by applying the almost orthogonality estimates for the operators \{q(tL)\}_{t > 0} above, we prove that these g-functions are bounded on L^p(\mathbb{R}^n), 1 < p < ∞, and on H^1_{L,\mathbb{R}^n}. See Lemmas 4.1 and 5.2.
Step 2. Let the operator $R_\delta = I - T_\delta$, where $I$ is the identity operator and $T_\delta$ is defined as in (1.4). Using the two key technical results above, we can obtain the operator norms of $R_\delta$ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and on $H^1_\delta(\mathbb{R}^n)$. By choosing $\delta > 1$ and close to 1, we get that the operator norms of $R_\delta$ are strictly less than 1. This shows that $T_\delta$ is invertible and the inverse operator $T_\delta^{-1}$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and on $H^1_\delta(\mathbb{R}^n)$. See more details in Theorems 4.5 and 5.4.

In Section 5, as an application of the frame decomposition, we apply our main result, Theorem 1.3 and to obtain the radial maximal function characterization of the Hardy space $H^1_\delta(\mathbb{R}^n)$ under the extra assumption of Gaussian upper bounds on the gradient of the heat kernels of $L$.

The paper is organised as follows. In Section 2 we prove the almost orthogonality estimates for the operators $\{q_i(L)\}_{i>0}$, where the function $q_i(z)$ is defined as in (1.2). In Section 3 we prove Theorem 1.1 by showing that $T_\delta^{-1}$ exists and is bounded on $L^p(\mathbb{R}^n)$. In Section 4 we prove Theorem 1.3 by showing that $T_\delta^{-1}$ exists and is bounded on $H^1_\delta(\mathbb{R}^n)$. The last section is devoted to the proof of the radial maximal function characterization of the Hardy space $H^1_\delta(\mathbb{R}^n)$.

2. Notation and preliminaries

We first recall some preliminaries on holomorphic functional calculi of operators. See [32]. Let $0 \leq \omega < \nu < \pi$. We define the closed sector in the complex plane $\mathbb{C}$

$$S_\omega = \{z \in \mathbb{C} : |\arg z| \leq \omega\} \cup \{0\}$$

and denote the interior of $S_\omega$ by $S^0_\omega$. We define the following subspaces of the space $H(S^0_\nu)$ of all holomorphic functions on $S^0_\nu$:

$$H_\omega(S^0_\nu) = \{b \in H(S^0_\nu) : \|b\|_\omega < \infty\},$$

where $\|b\|_\omega = \sup\{|b(z)| : z \in S^0_\nu\}$, and

$$\Psi(S^0_\nu) = \{\psi \in H(S^0_\nu) : \exists s > 0, \ |\psi(z)| \leq c\frac{|z|^\nu}{1 + |z|^2s}\}.$$

Let $0 \leq \omega < \pi$. A closed operator $L$ in $L^2(\mathbb{R}^n)$ is said to be of type $\omega$ if $\sigma(L) \subset S_\omega$, and for each $\nu > \omega$, there exists a constant $c_\nu$ such that $\|(L - \lambda I)^{-1}\| \leq c_\nu|\lambda|^{-1}$, $\lambda \not\in S_\nu$. If $L$ is of type $\omega$ and $\psi \in \Psi(S^0_\nu)$, we define $\psi(L) \in \mathcal{L}(L^2, L^2)$ by

$$\psi(L) = \frac{1}{2\pi i} \int_{\Gamma} (L - \lambda I)^{-1}\psi(\lambda)d\lambda,$$

where $\Gamma$ is the contour $\{\xi = re^{\pm i\theta} : r \geq 0\}$ parametrized clockwise around $S_\omega$, and $\omega < \theta < \nu$. Clearly, this integral is absolutely convergent in $\mathcal{L}(L^2, L^2)$, and it is straightforward to show that, using Cauchy’s theorem, the definition is independent of the choice of $\theta \in (\omega, \nu)$. If, in addition, $L$ is one-one and has dense range and if $b \in H_\omega(S^0_\nu)$, then $b(L)$ can be defined by

$$b(L) = (\psi(L))^{-1}(b\psi)(L),$$

where $\psi(z) = z(1 + z)^{-2}$. It can be shown that $b(L)$ is a well-defined linear operator in $L^2(\mathbb{R}^n)$.

We point out that if a closed operator $L$ in $L^2(\mathbb{R}^n)$ is of type $\omega$, then $L^*$ is also of type $\omega$, see [1, Page 20].

We say that $L$ has a bounded $H_\infty$ calculus in $L^2$ if there exists $c_{\nu, 2} > 0$ such that $b(L) \in \mathcal{L}(L^2, L^2)$, and for $b \in H_\infty(S^0_\nu)$,

$$\|b(L)\| \leq c_{\nu, 2}\|b\|_\omega.$$
In [32] it was proved that $L$ has a bounded $H_\infty$-calculus in $L^2(\mathbb{R}^n)$ if and only if for any non-zero function $\psi \in \Psi(S^0_\nu)$, $L$ satisfies the square function estimate and its reverse

$$c_1\|g\|_2 \leq \left( \int_0^\infty \|\psi_t(L)g\|_2^2 \frac{dt}{t} \right)^{1/2} \leq c_2\|g\|_2$$

for some $0 < c_1 \leq c_2 < \infty$, where $\psi_t(\xi) = \psi(t\xi)$. Note that different choices of $\nu > \omega$ and $\psi \in \Psi(S^0_\nu)$ lead to equivalent quadratic norms of $g$.

Note that by Corollary E in [1, Page 22], if $L$ satisfies (2.2), then $L^*$ also satisfies (2.2).

As noted in [32], non-negative self-adjoint operators satisfy the quadratic estimate (2.2). So do normal operators with spectra in a sector, and maximal accretive operators. For further study of holomorphic functional calculi on Banach spaces, see [32] and [13].

**Proposition 2.1.** Suppose $\psi \in H(S^0_\nu)$ with two parameters $\alpha > 0, \beta > \alpha$ such that

$$|\psi(z)| \leq C \frac{|z|^{\alpha}}{1 + |z|^\beta}.$$

Then for each fixed $k \in \mathbb{N}$, we have $\psi^{(k)}$ is holomorphic in $S^0_\omega$ for some $\omega$ with $\omega < \nu$, and

$$|\psi^{(k)}(z)| \leq C \frac{1}{|z|^k} \frac{|z|^{\alpha}}{1 + |z|^\beta}.$$

**Proof.** For each $\psi \in \Psi(S^0_\nu)$, we have $|\psi(z)| \leq C \frac{|z|^{\nu}}{1 + |z|^\nu}$. Now fix $\epsilon > 0$ such that $\sin(k\epsilon) < \frac{1}{100}$, we consider the sector $S^0_{\nu - \epsilon}$. For every $z \in S^0_{\nu - \epsilon}$ and $z \neq 0$, we consider the ball $B(z, r)$, centered at $z$, with radius $r = |z| \sin \epsilon$, such that $B(z, r)$ is contained in $S^0_\nu$. Then by Cauchy’s formula we obtain that

$$\psi^{(1)}(z) = \frac{1}{2\pi i} \oint_{\partial B} \frac{\psi(\lambda)}{(\lambda - z)^2} d\lambda,$$

where $\partial B$ is the boundary of the ball $B(z, r)$.

Then we have

$$|\psi^{(1)}(z)| \leq \frac{1}{2\pi} \oint_{\partial B} \frac{|\psi(\lambda)|}{|\lambda - z|^2} |d\lambda| \leq \frac{1}{2\pi r^2} \oint_{\partial B} |\psi(\lambda)||d\lambda|.$$

Next, note that $|\lambda| \leq |\lambda - z| + |z| \leq \frac{101}{100}|z|$, and $|\lambda| \geq |z| - |\lambda - z| \geq \frac{99}{100}|z|$

$$|\psi(\lambda)| \leq C \frac{|\lambda|^{\alpha}}{1 + |\lambda|^\beta} \leq C \left( \frac{101}{100} \right)^\alpha \left( \frac{100}{99} \right)^\beta \frac{|z|^{\alpha}}{1 + |z|^\beta}.$$

Thus,

$$|\psi^{(1)}(z)| \leq \frac{C'}{2\pi r} \frac{|z|^{\alpha}}{1 + |z|^\beta} \leq C \frac{1}{|z|} \frac{|z|^{\alpha}}{1 + |z|^\beta}.$$

As a consequence, we get that $\phi^{(1)}(z)$ is in $\Psi(S^0_{\nu - \epsilon})$. By induction, we can obtain that

$$|\psi^{(k)}(z)| \leq C \frac{1}{|z|^k} \frac{|z|^{\alpha}}{1 + |z|^\beta}.$$

The proof of Proposition 2.1 is complete. \qed
Proposition 2.2. Suppose that \( L \) satisfies (H1) and (H2). Suppose \( \psi \in H(S^0_\nu) \) with two parameters \( \alpha > 0, \beta > n/2 + \alpha \) such that

\[
|\psi(z)| \leq C \frac{|z|^{\alpha}}{1 + |z|^\beta}.
\]

Then there exists a positive constant \( C = C(n, \nu, \alpha, \beta) \) such that the kernel \( K_{\psi(tL)}(x,y) \) of \( \psi(tL) \) satisfies

\[
|K_{\psi(tL)}(x,y)| \leq C t^{-n/2} \left(1 + \frac{|x-y|^2}{t}\right)^{-(\beta+\alpha)}
\]

for all \( t > 0 \) and \( x, y \in \mathbb{R}^n \).

Proof. To prove (2.5), it suffices to show the following estimates:

\[
|K_{\psi(tL)}(x,y)| \leq C t^{-n/2}
\]

and

\[
|K_{\psi(tL)}(x,y)| \leq C t^{-n/2} \left(\frac{t}{|x-y|^2}\right)^{\frac{\beta+\alpha}{2}}.
\]

Let us verify (2.6). Note that for any \( m \in \mathbb{N} \) and \( t > 0 \), we have the relationship

\[
(I + tL)^{-m} = \frac{1}{(m - 1)!} \int_0^\infty e^{-tsL}e^{-s} s^{m-1} ds
\]

and so when \( m > n/4 \),

\[
\left\| (I + tL)^{-m} \right\|_{L^2(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} \leq \frac{1}{(m - 1)!} \int_0^\infty \left\| e^{-tsL} \right\|_{L^2(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} e^{-s} s^{m-1} ds \leq Ct^{-n/4}
\]

for all \( t > 0 \). Similarly, we have that \( \left\| (I + tL^*)^{-m} \right\|_{L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} = \left\| (I + tL)^{-m} \right\|_{L^2(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} \leq Ct^{-n/4} \) where \( L^* \) is the adjoint operator of \( L \). Next we choose a constant \( m \) in (2.8) such that \( n/4 < m < (\beta - \alpha)/2 \). One can write

\[
\psi(tL) = (I + tL)^{-2m}[(I + tL)^{2m} \psi(tL)]
= (I + tL)^{-m}[(I + tL)^{2m} \psi(tL)](I + tL)^{-m},
\]

and then

\[
\left\| \psi(tL) \right\|_{L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} \leq \left\| (I + tL)^{-m} \right\|_{L^2(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} \times \left\| (I + tL)^{2m} \psi(tL) \right\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \left\| (I + tL)^{-m} \right\|_{L^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}
\]

We see that \((1 + z)^{2m} \psi(z) \in H_{\infty}(S^0_\nu)\) with \((1 + z)^{2m} \psi(z) \leq C|z|^\alpha/(1 + |z|^{\beta-2m}) \leq C < +\infty\). From condition (H1), \( L \) has a bounded \( H_{\infty} \)-calculus on \( L^2(\mathbb{R}^n) \). This implies that the \( L^2 \) operator norm of the term \((I + tL)^{2m} \psi(tL)\) is uniformly bounded in \( t > 0 \). Hence, estimate (2.6) holds.

To prove (2.7), we first represent the operator \( \psi(tL) \) by using the semigroup \( e^{-zL} \). As in [20], \( \psi(tL) \) (acting on \( L^2(\mathbb{R}^n) \)) is given by

\[
\psi(tL) = \frac{1}{2\pi i} \int_{\Gamma} (L - \lambda I)^{-1} \psi(t\lambda) d\lambda,
\]

where the contour \( \Gamma = \Gamma_+ \cup \Gamma_- \) is given by \( \Gamma_+(t) = te^{i\mu} \) for \( t \geq 0 \) and \( \Gamma_-(t) = -te^{-i\nu} \) for \( t < 0 \), and \( \mu > \nu > \omega \). For \( \lambda \in \Gamma \), substitute

\[
(L - \lambda I)^{-1} = \int_{\gamma} e^{lz} e^{-zL} dz,
\]
where \( \gamma \) is the ray \( \{re^{i\theta} : 0 < r < \infty \} \) with \( \theta \) chosen to satisfy \( |\arg(\lambda z)| > \pi/2. \)

Changing the order of integration gives

\[
\psi(tL) = \int_{\gamma} e^{-zL} n(z) \, dz,
\]

where

\[
n(z) = \frac{1}{2\pi i} \int_{\Gamma} e^{zL} \psi(t\lambda) \, d\lambda.
\]

Consequently, the kernel \( K_{\psi(tL)}(x,y) \) of \( \psi(tL) \) is given by

\[
K_{\psi(tL)}(x,y) = \int_{\gamma} p_z(x,y)n(z) \, dz.
\]

It follows from (H2) that

\[
|K_{\psi(tL)}(x,y)| \leq C \int_{0}^{\infty} |z|^{-\frac{\alpha}{2}} e^{-\frac{|x-y|^2}{|z|}} \left( \int_{0}^{\infty} e^{-\eta \psi(w)} \frac{(tw)^\gamma}{1+(tw)^\beta} \, dw \right) \, dz
\]

with \( \eta > 0. \) Changing variables \( tw \to w \) and \( s/t \to s, \) we have

\[
|K_{\psi(tL)}(x,y)| \leq Ct^{-\frac{\alpha}{2}} \int_{0}^{\infty} s^{-\frac{\alpha}{2}} e^{-c \frac{|x-y|^2}{s}} \left( \int_{0}^{\infty} e^{-\eta \psi(w)} \frac{w^{\alpha}}{1+w^{\beta}} \, dw \right) \, ds
\]

\[
\leq Ct^{-\frac{\alpha}{2}} \left( \frac{t}{|x-y|^2} \right)^{\frac{\alpha}{2} + \gamma} \int_{0}^{\infty} s^{-\left(\frac{\alpha}{2} + \gamma + 1\right)} e^{-\frac{s}{2}} \, ds
\]

\[
\leq Ct^{-\frac{\alpha}{2}} \left( \frac{t}{|x-y|^2} \right)^{\frac{\alpha}{2} + \gamma}.
\]

Estimate (2.7) follows readily. The proof of Proposition 2.2 is complete.

**Proposition 2.3.** Suppose that \( L \) satisfies (H1), (H2) and (H3) with some \( \gamma > 0. \) Suppose \( \psi \in H(S^0) \) with two parameters \( \alpha > 0, \beta > n + \alpha + \frac{\gamma}{2} \) such that

\[
|\psi(z)| \leq C \frac{|z|^\alpha}{1+|z|^{\beta}},
\]

Then there exists a positive constant \( C = C(n, \gamma, \alpha, \beta) \) such that the kernel \( K_{\psi(tL)}(x,y) \) of \( \psi(tL) \) satisfies

\[
|K_{\psi(tL)}(x+h,y) - K_{\psi(tL)}(x,y)| + |K_{\psi(tL)}(x,y+h) - K_{\psi(tL)}(x,y)|
\]

\[
\leq C \left( \frac{|h|}{\sqrt{t} + |x-y|} \right)^{\gamma} \frac{r^\alpha}{(t + |x-y|^2)^{\frac{\alpha}{2} + \beta}}
\]

whenever \( 2|h| \leq t^{1/2} + |x-y|, \) and for all \( t > 0 \) and \( x, y \in \mathbb{R}^n. \)

**Proof.** To prove (2.10), it suffices to consider the part \( |K_{\psi(tL)}(x+h,y) - K_{\psi(tL)}(x,y)| \) since the proof of

\[
|K_{\psi(tL)}(x,y+h) - K_{\psi(tL)}(x,y)|
\]

is similar.

To prove this, it suffices to verify the following: there exists a positive constant \( C \) such that such that for all \( t > 0 \) and \( x, y, h \in \mathbb{R}^n, \)

\[
|h|^{-\gamma} |K_{\psi(tL)}(x+h,y) - K_{\psi(tL)}(x,y)| \leq Ct^{-\left(n+\gamma\right)/2},
\]

where \( \gamma \) is the ray \( \{re^{i\theta} : 0 < r < \infty \} \) with \( \theta \) chosen to satisfy \( |\arg(\lambda z)| > \pi/2. \)
and

\begin{equation}
|h|^{-\gamma} |K_{\psi(tL)}(x + h, y) - K_{\psi(tL)}(x, y)| \leq C \frac{t^\alpha}{|x - y|^{n+2\alpha + \gamma}}.
\end{equation}

Let us prove (2.11). It is well known that this inequality is equivalent to the boundedness of \(\psi(tL)\) from \(L^1\) to the homogeneous space \(C^\gamma\) with the right hand side of (2.11) being its operator norm. From (2.8), we see that when \(m > (n + \gamma)/2\),

\[
\left\|(I + tL)^{-m}\right\|_{L^1(\mathbb{R}^n) \rightarrow C^\gamma(\mathbb{R}^n)} \leq \frac{1}{(m - 1)!} \int_0^\infty \left\|e^{-tsL}\right\|_{L^1(\mathbb{R}^n) \rightarrow C^\gamma(\mathbb{R}^n)} e^{-s} s^{m-1} ds
\leq C \int_0^\infty (ts)^{-(n+\gamma)/2} e^{-s} s^{m-1} ds
\leq Ct^{-(n+\gamma)/2}
\]

for all \(t > 0\).

Fix \(m\) in (2.13) such that \(m \in ((n + \gamma)/2, \beta - (\frac{n}{2} + \alpha))\). We see that \((1 + z)^m \psi(z) \in H(S_0^0)\) and \(|(1 + z)^m \psi(z)| \leq C|z|^{\alpha}(1 + |z|)^{m-\beta}\). By Proposition 2.2, we have

\begin{equation}
\left|K_{(1+tL)^m\psi(tL)}(x, y)\right| \leq Ct^{-n/2} \left(1 + \frac{|x - y|^2}{t}\right)^{-(\frac{n}{2} + \alpha)}
\end{equation}

for all \(t > 0\) and \(x, y \in \mathbb{R}^n\). From this, we have that \(\left\|(I + tL)^m \psi(tL)\right\|_{L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)} \leq C\). Therefore,

\[
\left\|\psi(tL)\right\|_{L^1(\mathbb{R}^n) \rightarrow C^\gamma(\mathbb{R}^n)} \leq \left\|(I + tL)^{-m}\right\|_{L^1(\mathbb{R}^n) \rightarrow C^\gamma(\mathbb{R}^n)} \left\|(I + tL)^m \psi(tL)\right\|_{L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)} \leq Ct^{-(n+\gamma)/2},
\]

which yields (2.11).

We now prove (2.12). From the proof of Proposition 2.2, we have

\[K_{\psi(tL)}(x, y) = \int_{\gamma} p_z(x, y)n(z) \, dz,\]

which implies that

\[
|K_{\psi(tL)}(x + h, y) - K_{\psi(tL)}(x, y)| = \left|\int_0^\infty [p_z(x + h, y) - p_z(x, y)]n(z) \, dz\right|
\leq C \int_0^\infty \left(\frac{|h|}{\sqrt{s}}\right)^\gamma s^{-\frac{n}{2}} e^{-\frac{(\frac{n}{2} + \gamma + 1)\nu}{s}} \left(\int_0^\infty e^{-\eta w} \frac{(tw)^\alpha}{1 + (tw)^\beta} \, dw\right) ds
\leq C \frac{|h|^\gamma}{t^{n/2}} \int_0^\infty s^{-\frac{n}{2} + \frac{n}{2} + \alpha + 1} e^{-\frac{(n + 1)\nu}{s}} \left(\int_0^\infty e^{-\eta w} w^\alpha \, dw\right) ds
\leq C \frac{|h|^\gamma}{t^{n/2}} \frac{t^\alpha}{(|x - y|^2)^{\frac{n}{2} + \alpha}} \int_0^\infty s^{-\frac{n}{2} + \frac{n}{2} + \alpha + 1} e^{-\frac{s}{|x - y|^2}} \, ds
\leq C\left(\frac{|h|}{|x - y|}\right)^\gamma \frac{t^\alpha}{(|x - y|^2)^{\frac{n}{2} + \alpha}}.
\]

The proof of Proposition 2.3 is end. \(\Box\)

Recall that \(\zeta \in H(S_\nu^0)\) with two parameters \(\alpha > 0, \beta > n + \alpha + 3 + \gamma\) such that \(\zeta\) satisfies (2.4), and \(q(\lambda) = \lambda^2 \zeta^2(\lambda)\). Set \(\varphi(\lambda) = \lambda^2 \zeta(\lambda)\). Then \(q(\lambda) = \zeta(\lambda)\varphi(\lambda)\). Denote \(q_t(x, y)\) the kernel of the operator \(q(tL)\) where \(t > 0\).
Proposition 2.4. Suppose that $L$ satisfies (H1) – (H3). Suppose that $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. For any $t > 0$, $s > 0$, and $x, y \in \mathbb{R}^n$, the following results hold.

\begin{equation}
|K_{q(t^2L)q(s^2L)}(x, y)| \leq C \left( \frac{t}{s} \wedge \frac{s}{t} \right) \frac{(t + s)^\alpha}{(t + s + |x - y|)^{n+\alpha}},
\end{equation}

\begin{equation}
|K_{(t^2Lq(t^2L))q(s^2L)}(x, y)| \leq C \left( \frac{t}{s} \wedge \frac{s}{t} \right) \frac{(t + s)^\alpha}{(t + s + |x - y|)^{n+\alpha}},
\end{equation}

where we use $a \wedge b$ to denote $\min\{a, b\}$ for every positive numbers $a$ and $b$.

\begin{equation}
|q(t^2L)f(x) - q(t^2L)f(y)| \leq C \left( \frac{|x - y|}{t} \right)^\gamma \inf_{|u| < t} M(\varphi(t^2L)f)(u),
\end{equation}

when $|x - y| < t/2$, where we use $M$ to denote the Hardy–Littlewood maximal operator.

\begin{equation}
|q(s^2L)f(x) - q(s^2L)f(x)| \leq 2 \left( \ln \frac{s_2}{s_1} \right)^{1/2} \left( \int_{s_1}^{s_2} |r^2Lq(t^2L)f(x)|^2 \frac{dr}{r} \right)^{1/2},
\end{equation}

when $0 < s_1 \leq s_2$.

\textbf{Proof.} To prove (2.15), it suffices to prove that if $s \leq t$, then

\begin{equation}
|K_{q(t^2L)q(s^2L)}(x, y)| \leq C \left( \frac{t}{s} \right) \frac{t^\alpha}{(t + |x - y|)^{n+\alpha}},
\end{equation}

In fact,

$q(s^2L)q(t^2L) = s^4L^2 \zeta^2(s^2L)t^4L^2 \zeta^2(t^2L) = \left( \frac{s^4}{t^4} \right) \zeta^2(s^2L)t^4L^4 \zeta^2(t^2L)$.

Note that $\zeta^2(x)$ and $x^4\zeta^2(x)$ satisfy the condition of Proposition 2.2. Then by Proposition 2.2, we obtain

\begin{align*}
|K_{q(t^2L)q(t^2L)}(x, y)| &\leq \left( \frac{s^4}{t^4} \right) \int_{\mathbb{R}^n} |K_{\zeta^2(s^2L)}(x, u)||K_{\zeta^2(t^2L)}(u, y)| du \\
&\leq C \left( \frac{s^4}{t^4} \right) \int_{\mathbb{R}^n} (s + |u|)^{n+\alpha} \frac{t^\alpha}{(t + |u - y|)^{n+\alpha}} du \\
&\leq C \left( \frac{s}{t} \right) \frac{t^\alpha}{(s + t + |x - y|)^{n+\alpha}},
\end{align*}

which have proved (2.19), and hence (2.15) follows.

Similarly we can obtain (2.16) by the argument above and Propositions 2.1 and 2.2.

Next, let us prove (2.17). One can write

\[ q(t^2L)f(x) = \int_{\mathbb{R}^n} K_{\zeta(t^2L)}(x, z) \varphi(t^2L)f(z) dz. \]

Hence, if $|x - y| \leq t/2$, we use Proposition 2.3 to obtain

\begin{align*}
|q(t^2L)f(x) - q(t^2L)f(y)| &\leq \int_{\mathbb{R}^n} |K_{\zeta(t^2L)}(x, z) - K_{\zeta(t^2L)}(y, z)||\zeta(t^2L)f(z)| dz \\
&\leq \int_{\mathbb{R}^n} \left( \frac{|x - y|}{t} \right)^\gamma \frac{t^\alpha}{(t + |x - z|)^{n+\alpha}} |\varphi(t^2L)f(z)| dz.
\end{align*}
\[
\left| q(t^2L)f(x) - q(s^2L)f(x) \right| \leq 2 \left( \ln \frac{t}{s} \right)^{1/2} \left( \int_s^t \left| r^2Lq^{(1)}(r^2L)f(x) \right|^2 \frac{dr}{r} \right)^{1/2}.
\]

Remark 2.5. For any complex function \( \eta(\lambda) \), we denote \( \overline{\eta}(\lambda) := \overline{\eta(\lambda)} \). Recall that \( \zeta \in H(S^0) \) with two parameters \( \alpha > 0, \beta > n + \alpha + 3 + \gamma \) such that \( \zeta \) satisfies (2.4), and \( q(\lambda) = \lambda^2 \zeta^2(\lambda) \). Then \( \overline{q}(\lambda) = \lambda^2 \overline{\zeta}^2(\lambda) \), and \( \overline{\zeta}(\lambda) \) satisfy the same condition as that of \( \zeta \). We remark that the above estimates (2.15)–(2.18) also hold when we replace \( L \) and \( q \) by \( L^* \) and \( \overline{q} \) respectively, since when \( L \) satisfies (H1)–(H3), \( L^* \) also satisfies (H1)–(H3) and \( \overline{q} \) satisfy the same conditions as that of \( q \).

Next we recall the Littlewood–Paley theory as follows. For the proof, we refer to Theorem 6 of [2], see also (3.8) of [22], as well as [39].

Lemma 2.6. Suppose that \( L \) satisfies (H1)-(H2). Let \( \psi \in H(S^0) \) and there exist \( \alpha > 0 \) and \( \beta > \alpha \), such that
\[
|\psi(z)| \leq C \frac{|z|^\alpha}{1 + |z|^\beta}, \quad \text{for any} \ z \in S^0.
\]
Then for any \( 1 < p < +\infty \), there exists constants \( C_p, \) such that
\[
\left\| \left( \int_0^\infty |\psi(t^2L)|^2 \frac{dt}{t} \right)^{1/2} \right\|_p \leq C_p \|f\|_p.
\]

3. Boundedness of the operator \( T_\delta \) on \( L^2(\mathbb{R}^n) \)

Let \( f(z) \) be the same as in (1.2). Recall that for any \( f \in L^2(\mathbb{R}^n) \), the operator \( T_\delta \) in (1.4) is defined as
\[
T_\delta(f)(x) := \ln \delta \sum_j \sum_{t \in I_j} |Q_t^j| q_j(x, y_{Q_t^j}) q_{\delta^{-2j}}(L)(f)(y_{Q_t^j}),
\]
where \( y_{Q_t^j} \) is any point in the cube \( Q_t^j \).

The main aim in this section is to show that \( T_\delta(f)(x) \) is well-defined and bounded on \( L^2(\mathbb{R}^n) \).

First, we point out that from Proposition 2.3, \( q_j(x, y) \) is continuous in both \( x \) and \( y \). Hence, we see that \( q_j(x, y_{Q_t^j}) \) is well-defined for any \( y_{Q_t^j} \) in the cube \( Q_t^j \).

Second, we consider the term \( q_{\delta^{-2j}}(L)(f)(y_{Q_t^j}) \), which is defined as
\[
q_{\delta^{-2j}}(L)(f)(y_{Q_t^j}) = \int_{\mathbb{R}^n} q_j(y_{Q_t^j}, y) f(y) dy, \quad f \in L^2(\mathbb{R}^n).
\]
We point out that for every $f \in L^2(\mathbb{R}^n)$, $q_{\delta^{-j}}(L)(f)(y_Q^j)$ is well-defined for every $y_Q^j \in Q^j_t$. In fact, since $q_{\delta^{-j}}(L)$ is a bounded operator on $L^2(\mathbb{R}^n)$, we get that for $f \in L^2(\mathbb{R}^n)$, $q_{\delta^{-j}}(L)(f)$ is also in $L^2(\mathbb{R}^n)$, and hence $q_{\delta^{-j}}(L)(f)(x)$ is defined for a.e. $x \in \mathbb{R}^n$. Moreover, since $q_1(x, y)$ is continuous in $x$ and satisfies (2.10), we see that $q_{\delta^{-j}}(L)(f)(x)$ is also continuous in $x$. Hence $q_{\delta^{-j}}(L)(f)(y_Q^j)$ is well-defined for every $y_Q^j \in Q^j_t$.

**Theorem 3.1.** Let all the notation be the same as above. We have that $T_\delta$ is well-defined and bounded on $L^2(\mathbb{R}^n)$.

Before proving this theorem, we first establish the following Littlewood–Paley estimate on $L^2(\mathbb{R}^n)$.

**Lemma 3.2.** There exists a positive constant $C$ such that for every $1 < \delta < 2$,

\[
(3.2) \quad \left\| \sqrt{\ln \delta} \sum_j \sum_{t \in I_j} |q_{\delta^{-j}}(L)f(y_Q^j)|^2 \chi_{Q^j_t}(x) \right\|_{L^2(\mathbb{R}^n)}^{1/2} \leq C \left\| f \right\|_{L^2(\mathbb{R}^n)}.
\]

**Proof.** First, we need a Calderón type reproducing formula, which is inspired from the $H_\infty$-calculus for $L$. We start from the following fact: for $q(z)$ defined as in (1.2),

\[
\frac{1}{2} \int_0^\infty q(t) \cdot q(t) \frac{dt}{t} =: c,
\]

it is direct to see that $c \neq 0$.

Then, by $H_\infty$-functional calculus ([32]), for every $f \in L^2(\mathbb{R}^n)$,

\[
(3.3) \quad f = c^{-1} \int_0^\infty q_\tau(L)q_\tau(L)f \frac{dt}{t}
\]

in the sense of $L^2(\mathbb{R}^n)$. To be more precise, we have

\[
(3.4) \quad f = \lim_{N \to \infty} F_N \quad \text{in the sense of } L^2(\mathbb{R}^n), \quad \text{where } F_N := c^{-1} \int_0^N q_\tau(L)q_\tau(L)f \frac{dt}{t}.
\]

Then for any fixed $1 < \delta < 2$, from the reproducing formula (3.3), (3.1) and the fact that $q_{\delta^{-j}}(L)f(x)$ is a continuous function (see the explanation below (3.1)), we have that

\[
(3.5) \quad q_{\delta^{-j}}(L)f(y_Q^j) = q_{\delta^{-j}}(L) \left( c^{-1} \int_0^\infty q_\tau(L)q_\tau(L)f \frac{dt}{t} \right) (y_Q^j),
\]

where $y_Q^j$ is any point in the cube $Q^j_t$.

Next, from (3.5) and (3.4), by noting that $q_{\delta^{-j}}(L)$ is a bounded, linear operator on $L^2(\mathbb{R}^n)$, we have that

\[
q_{\delta^{-j}}(L)f(y_Q^j) = q_{\delta^{-j}}(L) \left( \lim_{N \to \infty} F_N \right) (y_Q^j)
= \lim_{N \to \infty} q_{\delta^{-j}}(L)(F_N) (y_Q^j)
= \lim_{N \to \infty} c^{-1} \int_{N^{-1}}^N q_{\delta^{-j}}(L)q_\tau(L)f(y_Q^j) \frac{dt}{t}
= c^{-1} \int_0^\infty q_{\delta^{-j}}(L)q_\tau(L)f(y_Q^j) \frac{dt}{t},
\]

where the third equality follows from the size estimate of the kernels of $q_{\delta^{-j}}(L)$ and $q_\tau(L)q_\tau(L)$ (see Proposition 2.4) and Fubini’s theorem.

Note that from the almost orthogonality estimates in Section 2 (Proposition 2.4), we have

\[
|q_{\delta^{-j}}(L)q_\tau(L)(x, z)| \leq C \left( \frac{\delta^{-j}}{t} \right) \left( \frac{t}{\delta^{-j}} \right) \left( \frac{t + \delta^{-j}}{t + \delta^{-j} + |x - z|^{n+1}} \right),
\]
where \( q_{\delta^{-2}}(L)q_{z}(L)(x, z) \) is the kernel of \( q_{\delta^{-2}}(L)q_{z}(L) \).

Hence
\[
|q_{\delta^{-2}}(L)q_{z}(L)q_{z}(L)f(y_{Q'})| = \left| \int_{\mathbb{R}^n} q_{\delta^{-2}}(L)q_{z}(L)(x, z) q_{z}(L)f(z)dz \right|
\leq C \int_{\mathbb{R}^n} \left( \frac{\delta^{-j}}{t} \right) \wedge \left( \frac{t}{\delta^{-j}} \right) \frac{(t + \delta^{-j})(t + \delta^{-j})}{(t + \delta^{-j}) + |y_{Q'} - z|^{n+1}}|q_{z}(L)f(z)|dz
\]
(3.7)
\[
\leq C \left( \frac{\delta^{-j}}{t} \right) \wedge \left( \frac{t}{\delta^{-j}} \right) \inf_{y \in Q'} M(q_{z}(L)f)(y).
\]

By substituting (3.7) into (3.6), we have
\[
|q_{\delta^{-2}}(L)f(y_{Q'})| \leq C \int_{0}^{\infty} \left( \frac{\delta^{-j}}{t} \right) \wedge \left( \frac{t}{\delta^{-j}} \right) \inf_{y \in Q'} M(q_{z}(L)f)(y) \frac{dt}{t}. \tag{3.8}
\]

Observe that
\[
\int_{0}^{\infty} \left( \frac{\delta^{-j}}{t} \right) \wedge \left( \frac{t}{\delta^{-j}} \right) \frac{dt}{t} = 2, \quad \sum_{j} \left( \frac{\delta^{-j}}{t} \right) \wedge \left( \frac{t}{\delta^{-j}} \right) \leq \frac{2\delta}{\delta - 1}.
\]

We then apply Hölder’s inequality, Lebesgue’s theorem and (3.8) to obtain
\[
\ln \delta \sum_{j} \sum_{\tau \in I_{j}} |q_{\delta^{-2}}(L)f(y_{Q'})|^{2} \chi_{Q'}(x)
\leq C \ln \delta \sum_{j} \sum_{\tau \in I_{j}} \left( \int_{0}^{\infty} \left( \frac{\delta^{-j}}{t} \right) \wedge \left( \frac{t}{\delta^{-j}} \right) \inf_{y \in Q'} M(q_{z}(L)f)(y) \frac{dt}{t} \right)^{2} \chi_{Q'}(x)
\leq C \ln \delta \sum_{j} \sum_{\tau \in I_{j}} \int_{0}^{\infty} \left( \frac{\delta^{-j}}{t} \right) \wedge \left( \frac{t}{\delta^{-j}} \right) \left( M(q_{z}(L)f)(x) \right)^{2} \frac{dt}{t} \chi_{Q'}(x)
\leq C \ln \delta \int_{0}^{\infty} \left( \frac{\delta^{-j}}{t} \right) \wedge \left( \frac{t}{\delta^{-j}} \right) \left( M(q_{z}(L)f)(x) \right)^{2} \frac{dt}{t}
\leq C \int_{0}^{\infty} \left( M(q_{z}(L)f)(x) \right)^{2} \frac{dt}{t}.
\]

Therefore, it follows from the \( L^{2} \)-boundedness of the Hardy–Littlewood maximal operator and Lemma 2.6 that
\[
\left\| \sqrt{\ln \delta} \left( \sum_{j} \sum_{\tau \in I_{j}} |q_{\delta^{-2}}(L)f(y_{Q'})|^{2} \chi_{Q'}(x) \right) \right\|_{L^{2}(\mathbb{R}^{n})}^{1/2}
\leq C \left\| \left( \int_{0}^{\infty} \left( M(q_{z}(L)f) \right)^{2} \frac{dt}{t} \right)^{1/2} \right\|_{L^{2}(\mathbb{R}^{n})} \leq C \left\| \left( \int_{0}^{\infty} |q_{z}(L)f|^{2} \frac{dt}{t} \right)^{1/2} \right\|_{L^{2}(\mathbb{R}^{n})} \leq C \| f \|_{L^{2}(\mathbb{R}^{n})},
\]
which shows that (3.2) holds. \( \square \)

We now start to provide the proof for Theorem 3.1.

Proof of Theorem 3.1. Let \( \Lambda_{\text{finite}} \) be an arbitrary finite subset of the integers \( \mathbb{Z} \). For every \( j \in \mathbb{Z} \), let \( I_{j, \text{finite}} \) be an arbitrary finite subset of the index \( I_{j} \).
For every $f \in L^2(\mathbb{R}^n)$, we consider the following auxiliary operator

\begin{equation}
T_{\delta,j}(f)(x) := \ln \delta \left| Q^j_\delta(x) \right| q_j(x, y_{Q_j}) q_{\delta^{-2j}}(L)(f)(y_{Q_j}).
\end{equation}

First, it is easy to see that for every $h \in L^2(\mathbb{R}^n)$,

\[
\left\langle \sum_{j \in \Lambda_{finite}} \sum_{\tau \in I_{j, finite}} T_{\delta,j}(f), h \right\rangle = \ln \delta \left| Q^j_\delta(L)(h)(y_{Q_j}) \right|^2 \left| q_{\delta^{-2j}}(L)(f)(y_{Q_j}) \right|^2, h(\cdot)
\]

where the last equality follows from the fact that the sums on $j$ and $\tau$ are finite. Then, by using Hölder’s inequality, we obtain that

\[
\left| \left\langle \sum_{j \in \Lambda_{finite}} \sum_{\tau \in I_{j, finite}} T_{\delta,j}(f), h \right\rangle \right| \leq \ln \delta \left( \sum_{j \in \Lambda_{finite}} \sum_{\tau \in I_{j, finite}} \left| q_{\delta^{-2j}}(L)(h)(y_{Q_j}) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{j \in \Lambda_{finite}} \sum_{\tau \in I_{j, finite}} \left| q_{\delta^{-2j}}(L)(f)(y_{Q_j}) \right|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \left\| \ln \delta \left( \sum_{j \in \Lambda_{finite}} \sum_{\tau \in I_{j, finite}} \left| q_{\delta^{-2j}}(L)(h)(y_{Q_j}) \right|^2 \right)^{\frac{1}{2}} \left\| L^2(\mathbb{R}^n) \right\| \sum_{j \in \Lambda_{finite}} \sum_{\tau \in I_{j, finite}} \left| q_{\delta^{-2j}}(L)(f)(y_{Q_j}) \right|^2 \right\| L^2(\mathbb{R}^n) \right\|
\]

\[
\leq C \| f \|_{L^2(\mathbb{R}^n)} \| h \|_{L^2(\mathbb{R}^n)},
\]

where the last inequality follows from Lemma 3.2, and hence it is clear that the constant $C$ is independent of $\delta, f, h, \Lambda_{finite}$ and $I_{j, finite}$.

This implies that

\begin{equation}
\left\| \sum_{j \in \Lambda_{finite}} \sum_{\tau \in I_{j, finite}} T_{\delta,j}(f) \right\|_{L^2(\mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)}.
\end{equation}

Next we use the Rademacher functions $r_j$ of Appendix C.1 in [26]. These functions are defined for nonnegative integers $j$, but we now reindex them so that the subscript are represented by $\{j, \tau\}$, where $j \in \mathbb{Z}$ and $\tau \in I_j$. The fundamental property of these functions is their orthogonality, that is

\[
\int_0^1 r_{j,\tau}(\omega) r_{j',\tau'}(\omega) d\omega = 0
\]

when $j \neq j'$ or $\tau \neq \tau'$.

Now for every $\Lambda_{finite}$ and $I_{j,finite}$ and for every $f \in L^2(\mathbb{R}^n)$, we obtain that

\begin{equation}
\int_0^1 \left\| \sum_{j \in \Lambda_{finite}} \sum_{\tau \in I_{j, finite}} r_{j,\tau}(\omega) T_{\delta,j,\tau}(f) \right\|^2_{L^2(\mathbb{R}^n)} d\omega
\end{equation}

\[
= \sum_{j \in \Lambda_{finite}, \tau \in I_{j, finite}} \left\| T_{\delta,j,\tau}(f) \right\|^2_{L^2(\mathbb{R}^n)}
\]

\[
+ \int_0^1 \sum_{j \in \Lambda_{finite}, \tau \in I_{j, finite}} \sum_{j' \in \Lambda_{finite}, \tau' \in I_{j', finite}} r_{j,\tau}(\omega) r_{j',\tau'}(\omega) \left\langle T_{\delta,j,\tau}(f), T_{\delta,j',\tau'}(f) \right\rangle d\omega
\]
\[ = \sum_{j \in \Lambda_{finite}, r \in I_{finite}} \left\| T_{\delta,j,r}(f) \right\|_{L^2(\mathbb{R}^n)}^2. \]

For any fixed \( \omega \in [0, 1] \) we now repeat the proof of (3.11) for the operators \( r_{j,r}(\omega) T_{\delta,j,r} \), and use the fact that \( r_{j,r}(\omega) = \pm 1 \) to obtain that
\[ \left\| \sum_{j \in \Lambda_{finite}, r \in I_{finite}} r_{j,r}(\omega) T_{\delta,j,r}(f), h \right\| \leq C \| f \|_{L^2(\mathbb{R}^n)} \| h \|_{L^2(\mathbb{R}^n)}, \quad \forall h \in L^2(\mathbb{R}^n), \]
which implies that
\[ (3.13) \quad \left\| \sum_{j \in \Lambda_{finite}, r \in I_{finite}} r_{j,r}(\omega) T_{\delta,j,r}(f) \right\|_{L^2(\mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)}. \]
Combining the estimates of (3.12) and (3.13), we get that
\[ \sum_{j \in \Lambda_{finite}, r \in I_{finite}} \left\| T_{\delta,j,r}(f) \right\|_{L^2(\mathbb{R}^n)}^2 = \int_0^1 \left\| \sum_{j \in \Lambda_{finite}, r \in I_{finite}} r_{j,r}(\omega) T_{\delta,j,r}(f) \right\|_{L^2(\mathbb{R}^n)}^2 d\omega \leq C \int_0^1 \| f \|_{L^2(\mathbb{R}^n)}^2 d\omega = C \| f \|_{L^2(\mathbb{R}^n)}^2. \]
By taking the limit of \( \tau \) and \( j \), we obtain that
\[ (3.14) \quad \sum_{j \in \mathbb{Z}, r \in I_j} \left\| T_{\delta,j,r}(f) \right\|_{L^2(\mathbb{R}^n)}^2 \leq C \| f \|_{L^2(\mathbb{R}^n)}^2. \]
Next, we show that for every \( f \in L^2(\mathbb{R}^n) \), the sequence
\[ \left\{ \sum_{j=-N}^{N} \sum_{r=0}^{N} T_{\delta,j,r}(f) \right\} \]
is a Cauchy sequence in \( L^2(\mathbb{R}^n) \). Suppose that this is not the case. This means that there is some \( \epsilon > 0 \) and a subsequence of integers \( 1 < N_1 < N_2 < N_3 < \cdots \) such that
\[ (3.15) \quad \left\| \tilde{T}_{\delta,k}(f) \right\|_{L^2(\mathbb{R}^n)} \geq \epsilon, \]
where
\[ \tilde{T}_{\delta,k}(f) := \sum_{j=-N_k+1}^{N_k+1} \sum_{r=0}^{N_k} T_{\delta,j,r}(f) - \sum_{j=-N_k}^{N_k} \sum_{r=0}^{N_k} T_{\delta,j,r}(f). \]
For any fixed \( \omega \in [0, 1] \), we repeat the proof of (3.11) to the operator \( r_k(\omega) \tilde{T}_{\delta,k} \) to obtain that
\[ \left\| \sum_{k=1}^{N} r_k(\omega) \tilde{T}_{\delta,k}(f) \right\|_{L^2(\mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)}. \]
Squaring and integrating this inequality with respect to \( \omega \in [0, 1] \), and using (3.12) with \( \tilde{T}_{\delta,k} \) in the place of \( T_{\delta,j,r} \) and \( k \in \{1, 2, \ldots, K\} \) in the place of \( j \in \Lambda_{finite}, \tau \in I_{finite} \), we get that
\[ \sum_{k=1}^{K} \left\| \tilde{T}_{\delta,k}(f) \right\|_{L^2(\mathbb{R}^n)}^2 \leq C \| f \|_{L^2(\mathbb{R}^n)}^2. \]
But this contradicts (3.15) as \( K \to \infty \).
So we conclude that every sequence
\[ \left\{ \sum_{j=-N}^{N} \sum_{\tau=0}^{N} T_{\delta,j,\tau}(f) \right\} \]
is a Cauchy sequence in \( L^2(\mathbb{R}^n) \), and thus it converges to \( T_\delta \).

This, together with (3.11), implies that \( T_\delta \) is a bounded operator on \( L^2(\mathbb{R}^n) \) with norm at most some constant \( C \). \( \Box \)

4. Frame decompositions on \( L^p(\mathbb{R}^n), 1 < p < \infty \)

Suppose \( \zeta \in H(S^0) \) with two parameters \( \alpha > 0, \beta > n + \alpha + 3 + \gamma \) such that \( \zeta \) satisfies (2.4). For the sake of simplicity, in the rest of the paper, we take \( \alpha = \gamma = 1 \). Recall that \( q(z) = z^2 \zeta^2(z) \). Similar to Section 3, we denote \( q_\delta(x,y) \) the kernel of the operator \( q(tL) \), where \( t > 0 \), and denote
\[ q_j(x,y) := q_{\delta^{-1}}(x,y), \]
where \( j \in \mathbb{Z} \).

4.1. Littlewood–Paley \( g \) functions on \( L^p(\mathbb{R}^n), 1 < p < \infty \). We introduce four discrete Littlewood–Paley \( g \)-functions. For any fixed \( \delta > 1 \), we define
\[ g_{1,\delta}(L)f(x) := \sqrt{\ln \delta} \left( \sum_j \sum_{\tau \in I_j} |q_{\delta^{-1}}(L)f(y_{Q_j})|^2 \chi_{Q_j}(x) \right)^{1/2}; \]
\[ g_{2,\delta}(L)f(x) := \left( \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} |q_{\delta^{-1}}(L)f(y_{Q_j})|^2 \frac{dt}{t} \chi_{Q_j}(x) \right)^{1/2}; \]
\[ g_{3,\delta}(L)f(x) := \left( \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \int_{Q_j} |q_{\delta^{-1}}(L)f(y)|^2 \frac{dy}{t} \frac{1}{|Q_j|} \chi_{Q_j}(x) \right)^{1/2}; \]
\[ g_{4,\delta}(L)f(x) := \left( \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} |s^2 L q_{\delta^{-1}}(L)f(y_{Q_j})|^2 \frac{ds}{s} \chi_{Q_j}(x) \right)^{1/2}. \]

**Lemma 4.1.** Suppose \( 1 < p < +\infty \). There exists a positive constant \( C_p \) such that for every \( 1 < \delta < 2 \),
\[ \|g_{1,\delta}(L)f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \]
where \( i = 1, 2, 3, 4 \).

**Proof.** We first estimate \( g_{1,\delta}(f) \). Following the same estimate as in the proof of Lemma 3.2 we can obtain that
\[ (g_{1,\delta}(L)f(x))^2 \leq C \int_0^\infty (\mathcal{M}(q_{\delta^{-1}}(L)f))(x)^2 \frac{dt}{t}. \]
Therefore, it follows from the vector-value maximal theorem (see Proposition 4.5.11, [26]) and Lemma 2.6 that
\[ \|g_{1,\delta}(L)f\|_p \leq C \left\| \int_0^\infty (\mathcal{M}(q_{\delta^{-1}}(L)f))^2 \frac{dt}{t} \right\|_p^{1/2} \leq C \left\| \int_0^\infty |q_{\delta^{-1}}(L)f|^2 \frac{dt}{t} \right\|_p^{1/2} \leq C \|f\|_p, \]
which shows that (4.1) holds for \( i = 1 \).

The proofs of (4.1) for \( i = 2, 3 \) are similar to that for \( i = 1 \). The proof of (4.1) for \( i = 4 \) is similar to that for \( i = 1 \), but via the almost orthogonality estimate (2.16) in Proposition 2.4. We omit the details here. \( \Box \)
Remark 4.2. For $i = 1, 2, 3, 4$, we define $g_{i, \delta}^{\ast}(L^\ast)$ following the same way as $g_{i, \delta}(L)$ with $L$ and $q$ replaced by $L^\ast$ and $\bar{q}$, respectively. Then, by Remark 2.5, the above estimates (4.1) also hold for $g_{i, \delta}^{\ast}(L^\ast)$ for $i = 1, 2, 3, 4$.

4.2. Proof of frame decomposition on Lebesgue spaces. In the next two results, we obtain estimates on the norms of the operators $T_\delta$ and $I - T_\delta$ on Lebesgue spaces.

Theorem 4.3. For every $1 < p < \infty$, there exists a constant $C_p > 0$ such that

$$\|T_\delta(f)\|_p \leq C_p \|f\|_p.$$  

Proof. To see this, we first recall from Theorem 3.1, $T_\delta$ is well-defined and bounded on $L^2(\mathbb{R}^n)$. Hence, for every $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $h \in L^{p'}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, from the definition of $T_\delta$ as in (1.4), we have

$$\langle T_\delta(f), h \rangle = \ln \delta \sum_{j} \sum_{r \in I_j} |Q^j_r| \langle q_j(\cdot, y^j_{Q^j_r}), h(\cdot) \rangle q_{\delta^{-2}/L}(L) f(y^j_{Q^j_r}).$$

We first claim that

$$\langle q_j(\cdot, y^j_{Q^j_r}), h(\cdot) \rangle = \overline{\bar{q}(\delta^{-2} L^\ast) h(y^j_{Q^j_r})}.$$  

In fact, for every $h \in L^{p'}$, one can write

$$\langle q_j(\cdot, y^j_{Q^j_r}), h(\cdot) \rangle = \int_{\mathbb{R}^n} K_{q(\delta^{-2}/L)}(x, y^j_{Q^j_r}) \overline{h(x)} \, dx = \int_{\mathbb{R}^n} K_{q(\delta^{-2}/L^\ast)}(y^j_{Q^j_r}, x) h(x) \, dx \quad (4.2)$$

where we use $K_{\psi(tL)}(x, y)$ to denote the kernel of the operator $\psi(tL)$.

Then we combine (4.2), Lemma 4.1, Remark 4.2 and the Hölder inequality to obtain

$$\left| \langle T_\delta(f), h \rangle \right| \leq \ln \delta \sum_{j} \sum_{r \in I_j} |Q^j_r| \|q_{\delta^{-2}/L}(L) f(y^j_{Q^j_r})\| \|\bar{q}(\delta^{-2} L^\ast) h(y^j_{Q^j_r})\|$$

$$\leq \ln \delta \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |q_{\delta^{-2}/L}(L) f(y^j_{Q^j_r})| \|\bar{q}(\delta^{-2} L^\ast) h(y^j_{Q^j_r})\| \chi_{Q^j_r}(x) \, dx \quad d\bar{r}$$

$$\leq \|g_{1, \delta}(L)f\|_{p'} \|g_{1, \delta}^{\ast}(L^\ast) h\|_{p'} \quad \leq \|f\|_p \|h\|_{p'},$$

which, together with the fact that $L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ and $L^{p'}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^{p'}(\mathbb{R}^n)$, implies that

$$\|T_\delta(f)\|_p = \sup_{\|h\|_{p'} \leq 1} \left| \langle T_\delta(f), h \rangle \right| \leq C \|f\|_p.$$  

This finishes the proof of Theorem 4.3. \hfill \□

We now introduce the remainder operator $R_\delta$.

Definition 4.4. Let $T_\delta$ be the same as in (1.4). We now set

$$R_\delta := I - T_\delta,$$

where $I$ is the identity operator on $L^2(\mathbb{R}^n)$.

Theorem 4.5. Then there exists a constant $1 < \delta < 2$ such that $\|R_\delta(f)\|_p \leq \frac{1}{2} \|f\|_p$, for all $1 < p < \infty$.  

\hfill \□
Proof. For any \( f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \), one can write (by using \( H_\infty \)-functional calculus \([32]\))
\[
f(x) = \int_0^\infty \int_{\mathbb{R}^n} q^2(x,y)q^2(L)f(y)\frac{dy\,dt}{t} = \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \int_{Q^c_\tau} q^2(x,y)q^2(L)f(y)\frac{dy\,dt}{t},
\]
where the last equality follows from the argument as in (3.6) in the sense of \( L^2(\mathbb{R}^n) \). We then decompose \( R_\delta(f) \) into four terms: \( R_\delta(f) = \sum_{i=1}^4 R_{\delta,i}(f) \), where for every \( f \in L^2(\mathbb{R}^n) \),
\[
R_{\delta,1}(f) := \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \int_{Q^c_\tau} [q^2(x,y) - q^2(x,y_{Q^c_1})]q^2(L)f(y)\frac{dy\,dt}{t};
\]
\[
R_{\delta,2}(f) := \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \int_{Q^c_\tau} [q^2(x,y_{Q^c_1}) - q^2(x,y_{Q^c_1})]q^2(L)f(y)\frac{dy\,dt}{t};
\]
\[
R_{\delta,3}(f) := \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \int_{Q^c_\tau} q^2(x,y_{Q^c_1})[q^2(L)f(y) - q^2(L)f(y_{Q^c_1})]dy\frac{dt}{t};
\]
\[
R_{\delta,4}(f) := \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \int_{Q^c_\tau} q^2(x,y_{Q^c_1})[q^2(L)f(y_{Q^c_1}) - q^2(L)f(y_{Q^c_1})]dy\frac{dt}{t}.
\]

We point out that, by repeating the argument as in the proof of Theorem 3.1, we obtain that all the above operators \( R_{\delta,i}, i = 1, 2, 3, 4 \), are well-defined and the series converges in the sense of \( L^2(\mathbb{R}^n) \).

We now first estimate the norm of \( R_{\delta,1}(f) \). Applying (4.2), we have that for every \( f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \),
\[
\|R_{\delta,1}(f)\|_p = \sup_{g \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)} |\langle R_{\delta,1}(f), g \rangle| = \sup_{\|g\|_{L^p(\mathbb{R}^n)} \leq 1} \left| \sum_j \sum_{\tau \in I_j} \left( \int_{\delta^{-j}}^{\delta^{-j+1}} \int_{Q^c_\tau} [q^2(x,y) - q^2(x,y_{Q^c_1})]q^2(L)f(y)\frac{dy\,dt}{t}, g \right) \right| \leq \sup_{\|g\|_{L^p(\mathbb{R}^n)} \leq 1} \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \int_{Q^c_\tau} \left| \tilde{q}(t^2L^*)g(y) - \tilde{q}(t^2L^*)g(y_{Q^c_1}) \right| q^2(L)f(y)\frac{dy\,dt}{t},
\]
where the second equality follows from the fact that \( R_{\delta,1} \) is well-defined, and the series converges in the sense of \( L^2(\mathbb{R}^n) \), and the inequality follows from Fubini’s theorem.

Note that \( y \in Q^c_\tau, \delta^{-j} \leq t \leq \delta^{-j+1} \), we use (2.17) to get
\[
|\tilde{q}(t^2L^*)g(y) - \tilde{q}(t^2L^*)g(y_{Q^c_1})| \leq C \left( \frac{|y - y_{Q^c_1}|}{t} \right) \inf_{u \in Q^c_\tau} M(\tilde{q}(t^2L^*)g)(u) \leq C \sigma^{(-M+1)} \inf_{u \in Q^c_\tau} M(\tilde{q}(t^2L^*)g)(u).
\]

By Hölder’s inequality, vector-value maximal theorem and Lemma 2.6, we obtain
\[
\|R_{\delta,1}(f)\|_p \leq C \sigma^{(-M+1)} \sup_{\|g\|_{L^p(\mathbb{R}^n)} \leq 1} \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \int_{Q^c_\tau} \inf_{u \in Q^c_\tau} M(\tilde{q}(t^2L^*)g)(u) \left| q^2(L)f(y) \right| \frac{dy\,dt}{t} \leq C \sigma^{(-M+1)} \sup_{\|g\|_{L^p(\mathbb{R}^n)} \leq 1} \int_0^\infty M(\tilde{q}(t^2L^*)g)(y) \left| q^2(L)f(y) \right| \frac{dy\,dt}{t}.
\]
\[
\begin{align*}
(4.3) \quad &\leq C\delta^{(-M+1)} \sup_{\|g\|_{p'} \leq 1} \left\| \left( \int_0^\infty |q_{\delta}(L)f(y)|^2 \frac{dt}{t} \right)^{1/2} \right\|_p \left\| \left( \int_0^\infty M(\bar{\varphi}(\tau^2 L^*)g)(y)^2 \frac{dt}{t} \right)^{1/2} \right\|_{p'} \\
&\leq C\delta^{(-M+1)} \|f\|_p.
\end{align*}
\]

By similar argument, we have
\[
\begin{align*}
\|R_{\delta,3}(f)\|_p &\leq C\delta^{(-M+1)} \sup_{\|g\|_{p'} \leq 1} \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j+1}}^{\delta^{-j}} \int_{Q_t'} \inf_{u \in Q_t'} M(\varphi_{\delta}(L)(f))(u)|q(\delta^{-2j}L^*)g(y_{Q_t'})|dy \frac{dt}{t} \\
&= C\delta^{(-M+1)} \sup_{\|g\|_{p'} \leq 1} \sum_j \sum_{\tau \in I_j} |Q_t'| \int_{\delta^{-j+1}}^{\delta^{-j}} \inf_{u \in Q_t'} M(\varphi_{\delta}(L)(f))(u)|q(\delta^{-2j}L^*)g(y_{Q_t'})| \frac{dt}{t}
\end{align*}
\]

To continue, one can write
\[
\begin{align*}
\sum_j \sum_{\tau \in I_j} |Q_t'| \int_{\delta^{-j+1}}^{\delta^{-j}} \inf_{y \in Q_t'} M(\varphi_{\delta}(L)(f))(y)|q(\delta^{-2j}L^*)g(y_{Q_t'})| \frac{dt}{t} \\
&\leq C \int_{\mathbb{R}^n} \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j+1}}^{\delta^{-j}} \inf_{y \in Q_t'} M(\varphi_{\delta}(L)(f))(y)|q(\delta^{-2j}L^*)g(y_{Q_t'})| \chi_{Q_t'}(x) \frac{dt}{t} dx \\
&\leq C \left\| \left( \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j+1}}^{\delta^{-j}} |M(\varphi_{\delta}(L)(f))(x)|^2 \frac{dt}{t} \chi_{Q_t'}(x) \right) \right\|_{1/2}^{1/2} \\
&\quad \times \left\| \left( \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j+1}}^{\delta^{-j}} |q(\delta^{-2j}L^*)g(y_{Q_t'})|^2 \frac{dt}{t} \chi_{Q_t'}(x) \right) \right\|_{p'}^{1/2} \\
&\leq C \|f\|_p \|g\|_{p'},
\end{align*}
\]

where in the last inequality above we have used Remark 4.2. Therefore, we show that
\[
(4.4) \quad \|R_{\delta,3}(f)\|_p \leq C\delta^{(-M+1)} \|f\|_p.
\]

As for \(R_{\delta,4}(f)\), we apply dual argument and (4.2) to write
\[
\begin{align*}
\|R_{\delta,4}(f)\|_p &\leq \sup_{\|g\|_{p'} \leq 1} \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j+1}}^{\delta^{-j}} \int_{Q_t'} |q(\delta^{-2j}L^*)g(y_{Q_t'})| |q_{\delta}(L)f(y_{Q_t'}) - q_{\delta^{-2j}}(L)f(y_{Q_t'})| dy \frac{dt}{t} \\
&= \sup_{\|g\|_{p'} \leq 1} \int_{\mathbb{R}^n} \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j+1}}^{\delta^{-j}} |q(\delta^{-2j}L^*)g(y_{Q_t'})| |q_{\delta}(L)f(y_{Q_t'}) - q_{\delta^{-2j}}(L)f(y_{Q_t'})| \chi_{Q_t'}(x) \frac{dt}{t} dx \\
&\leq \sup_{\|g\|_{p'} \leq 1} \sqrt{\ln \delta} \left\| \left( \sum_j \sum_{\tau \in I_j} |q(\delta^{-2j}L^*)g(y_{Q_t'})|^2 \chi_{Q_t'} \right) \right\|_{p'}^{1/2} \\
&\quad \times \left\| \left( \sum_j \sum_{\tau \in I_j} |q_{\delta}(L)f(y_{Q_t'}) - q_{\delta^{-2j}}(L)f(y_{Q_t'})|^2 \frac{dt}{t} \chi_{Q_t'} \right) \right\|_p^{1/2} 
\end{align*}
\]
where in the last inequality we have used Remark 4.2.

For \( \delta^{-j} \leq t < \delta^{-j+1} \), we use (2.18) to get

\[
(4.6) \quad |q_{\delta}(L)f(y_{Q_j^t}) - q_{\delta^{-2j}}(L)f(y_{Q_j^t})| \leq C \sqrt{\ln \delta} \left( \int_{\delta^{-j}}^{\delta^{-j+1}} |s^2 Lq^{(1)}(s^2 L)f(y_{Q_j^t})|^2 \frac{ds}{s} \right)^{1/2}.
\]

Substituting (4.6) into (4.5) and applying Lemma 4.1, we have

\[
\left\| \left( \sum_j \left\| \left( \sum_{r \in J_j} \int_{\delta^{-j}}^{\delta^{-j+1}} |q_{\delta}(L)f(y_{Q_j^t}) - q_{\delta^{-2j}}(L)f(y_{Q_j^t})|^2 \frac{dt}{t} \chi_{Q_j^t} \right) \right\|_p \right\|_p^2
\leq C \sqrt{\ln \delta} \left( \sum_j \left\| \left( \sum_{r \in J_j} \int_{\delta^{-j}}^{\delta^{-j+1}} |s^2 Lq^{(1)}(s^2 L)f(y_{Q_j^t})|^2 \frac{ds}{s} \chi_{Q_j^t} \right) \right\|_p \right)^{1/2}.
\]

Observe that if \( 1 < \delta < 2 \), then \( \ln \delta \leq (\delta - 1) \). Thus, we have

\[
(4.8) \quad \left\| R_{\delta;4}(f) \right\|_p \leq C \sqrt{(\delta - 1)}\|f\|_p.
\]

For the term \( R_{\delta;2}(f) \). We note that for every \( y \in Q_j^t \) and \( \delta^{-j} \leq t < \delta^{-j+1} \),

\[
|q_{\delta}(L)f(y)| = 2 \left| \int_{\mathbb{R}^n} K_{(t^2; L_j)}(y, z) \varphi(t^2) f(z) dz \right| \leq C \inf_{u \in Q_j^t} M(\varphi_{\delta} f)(u).
\]

By dual argument and (4.2), we have

\[
\left\| R_{\delta;2}(f) \right\|_p \leq \sup_{\|\varphi\|_p \leq 1} \left\| \left( \sum_j \left\| \left( \sum_{r \in J_j} \int_{\delta^{-j}}^{\delta^{-j+1}} |q_{\delta}(L)f(y_{Q_j^t})|^2 \frac{dt}{t} \chi_{Q_j^t} \right) \right\|_p \right\|_p
\leq C \sup_{\|\varphi\|_p \leq 1} \left\| \left( \sum_j \left\| \left( \sum_{r \in J_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \inf_{u \in Q_j^t} M(\varphi_{\delta} f)(u)\bar{q}(t^2) f(y_{Q_j^t}) - \bar{q}(\delta^{-2j} L^*) f(y_{Q_j^t}) d\frac{dt}{t} \chi_{Q_j^t}(x) dx \right) \right\|_p \right\|_p
\leq C \sup_{\|\varphi\|_p \leq 1} \left\| \left( \sum_j \left\| \left( \sum_{r \in J_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \inf_{u \in Q_j^t} M(\varphi_{\delta} f)(u)\bar{q}(t^2) f(y_{Q_j^t}) - \bar{q}(\delta^{-2j} L^*) f(y_{Q_j^t}) d\frac{dt}{t} \chi_{Q_j^t}(x) dx \right) \right\|_p \right\|_p
\times \left\| \left( \sum_j \left\| \left( \sum_{r \in J_j} \int_{\delta^{-j}}^{\delta^{-j+1}} |q_{\delta}(L)f(y_{Q_j^t})|^2 \frac{dt}{t} \chi_{Q_j^t} \right) \right\|_p \right\|_p
\leq C \sup_{\|\varphi\|_p \leq 1} \left\| \left( \sum_j \left\| \left( \sum_{r \in J_j} \int_{\delta^{-j}}^{\delta^{-j+1}} |q_{\delta}(L)f(y_{Q_j^t})|^2 \frac{dt}{t} \chi_{Q_j^t} \right) \right\|_p \right\|_p^2
\leq C \sqrt{(\delta - 1)}\|f\|_p.
\]
Combining (4.3), (4.4), (4.8) and (4.9), we have, there exists a constant $C_1 > 0$ such that
\[
\|R_\delta(f)\|_p \leq C_1(\delta^{-M} + \sqrt{(\delta - 1)})\|f\|_p.
\]
We can choose $\delta$ close to 1 and $M$ large enough, such that $C_1(\delta^{-M} + \sqrt{(\delta - 1)}) < 1/2$, which completes the proof.

We now apply the estimates of Theorems 4.3 and 4.5 to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Theorem 4.5, we have that for each $k = 1, 2, \cdots$, $\|R_\delta^k(f)\|_p \leq 2^{-k}\|f\|_p$. Therefore, the operator $T_\delta(f)$ is invertible, and
\[
\|T_\delta^{-1}f\|_p = \|(I - R_\delta)^{-1}f\|_p \leq \sum_{k=0}^{\infty}\|R_\delta^k(f)\|_p \leq 2\|f\|_p.
\]

For every $f \in L^p(\mathbb{R}^n)$, one can write
\[
f = T_\delta \circ T_\delta^{-1}f = \sum_j \sum_{\tau \in I_j} \langle T_\delta^{-1}f, \psi_{j,\tau}^* \rangle \psi_{j,\tau} \quad \text{in } L^p(\mathbb{R}^n).
\]

Applying Lemma 4.1, we have
\[
\left\| \left( \sum_j \sum_{\tau \in I_j} \langle T_\delta^{-1}f, \psi_{j,\tau}^* \rangle \frac{1}{|Q_l^j|} \chi_{Q_l^j} \right)^{\frac{1}{2}} \right\|_p = \left\| R_{1,\delta}(L)(T_\delta^{-1}f) \right\|_p \leq C\|T_\delta^{-1}f\|_p \leq C\|f\|_p.
\]

For the left inequality, we use the dual argument to write
\[
\|f\|_p = \sup_{\|g\|_{p'} \leq 1} |\langle f, g \rangle|,
\]
where $p'$ is the adjoint number of $p$.

For each $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$, using the equality (4.11) we have
\[
|\langle f, g \rangle| = \ln \delta \left| \sum_j \sum_{\tau \in I_j} \langle T_\delta^{-1}f, \psi_{j,\tau}^* \rangle \langle \psi_{j,\tau}, g \rangle \right|
\]
\[
= \ln \delta \left| \int_{\mathbb{R}^n} \sum_j \sum_{\tau \in I_j} \langle T_\delta^{-1}f, \psi_{j,\tau}^* \rangle \langle \psi_{j,\tau}, g \rangle \frac{1}{|Q_l^j|} \chi_{Q_l^j}(x) dx \right|.
\]

Applying Hölder inequality and Lemma 4.1, one writes
\[
|\langle f, g \rangle| \leq \left\| \left( \sum_j \sum_{\tau \in I_j} \langle T_\delta^{-1}f, \psi_{j,\tau}^* \rangle \frac{1}{|Q_l^j|} \chi_{Q_l^j} \right)^{\frac{1}{2}} \right\|_p \cdot \left\| \left( \sum_j \sum_{\tau \in I_j} \langle g, \psi_{j,\tau} \rangle \frac{1}{|Q_l^j|} \chi_{Q_l^j} \right)^{\frac{1}{2}} \right\|_{p'}
\]
\[
\leq C \left\| \left( \sum_j \sum_{\tau \in I_j} \langle T_\delta^{-1}f, \psi_{j,\tau}^* \rangle \frac{1}{|Q_l^j|} \chi_{Q_l^j} \right)^{\frac{1}{2}} \right\|_p \|g\|_{p'}.
\]

Therefore, we obtain that
\[
\|f\|_p \leq C \left\| \left( \sum_j \sum_{\tau \in I_j} \langle T_\delta^{-1}f, \psi_{j,\tau}^* \rangle \frac{1}{|Q_l^j|} \chi_{Q_l^j} \right)^{\frac{1}{2}} \right\|_p,
\]
which completes the proof. \qed
5. Frame decompositions on $H^1_L(\mathbb{R}^n)$

We first recall the tent space $T^1_2(\mathbb{R}^{n+1})$ and the molecules for the Hardy space $H^1_L(\mathbb{R}^n)$.

In [12], Coifman, Meyer and Stein introduced and studied a new family of function spaces, the so-called “tent spaces”. For any function $f(y, t)$ defined on $\mathbb{R}^{n+1}$ we will denote

$$\mathcal{A}(f)(x) = \left( \int_0^\infty \int_{|x-y|<t} |f(y, t)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.$$ 

As in [12], the tent space $T^1_2$ is defined as the space of functions $f$ such that $\mathcal{A}(f) \in L^1(\mathbb{R}^n)$. The resulting equivalence classes are then equipped with the norm $\|f\|_{T^1_2} = \|\mathcal{A}(f)\|_1$.

Next, a function $a(x, t)$ is called a $T^1_2$-atom if

(i) the function $a(x, t)$ is supported in $\hat{B}$ (for some ball $B \subset \mathbb{R}^n$);

(ii) $\int_B |a(x, t)|^2 \frac{dxdy}{t} \leq |B|^{-1}$,

where $\hat{B}$ is the tent of the ball $B$, defined as $\hat{B} = \{(y, t) \in \mathbb{R}^n \times \mathbb{R}_+ : B(y, t) \subset B\}$, and $B(y, t)$ is the ball in $\mathbb{R}^n$ centered at $y$ with radius $t$.

Recall that in [2], a function $m(x)$ is called an $L$-molecule if

$$m(x) = \int_0^\infty t^2 L e^{-t^2} (a(\cdot, t))(x) \frac{dt}{t},$$

where $a(t, x)$ is a $T^1_2$-atom as defined above. An $L$-molecule decomposition of $f$ in the space $H^1_L$ is first obtained in Theorem 7 of [2]. Here we refer to the following statement as in [22].

**Proposition 5.1 ([22]).** Let $f \in H^1_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. There exist $L$-molecules $m_k$ and numbers $\lambda_k$ for $k = 0, 1, 2, \ldots$ such that

$$f(x) = \sum_k \lambda_k m_k(x).$$

The sequence $\{\lambda_k\}$ satisfies $\sum_k |\lambda_k| \leq C \|f\|_{H^1_L(\mathbb{R}^n)}$. Conversely, the decomposition (5.2) satisfies

$$\|f\|_{H^1_L(\mathbb{R}^n)} \leq C \sum_k |\lambda_k|.$$

We point out that the equality (5.2) holds in the sense of $L^2(\mathbb{R}^n)$, for more detail of explanation, we refer to Proposition 3.23 in [18].

5.1. Littlewood–Paley $g$ functions on $H^1_L(\mathbb{R}^n)$. Next we prove that the four auxiliary Littlewood–Paley $g$ functions as defined in Section 3.1 are bounded from $H^1_L(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

**Lemma 5.2.** Assume that $L$ satisfies (H1), (H2) and (H3). There exists a positive constant $C$, such that for any fixed $1 < \delta \leq 2$ and every $f \in H^1_L(\mathbb{R}^n)$,

$$\|g_{i, \delta}(L)(f)\|_{L^1(\mathbb{R}^n)} \leq C \|f\|_{H^1_L(\mathbb{R}^n)},$$

where $i = 1, 2, 3, 4$.

**Proof.** We now verify (5.3) for $i = 1$.

Note that $g_{1, \delta}(L)$ is non-negative, sublinear, and bounded on $L^2(\mathbb{R}^n)$ and note also that for every $f \in H^1_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $f$ has the following molecular decomposition

$$f = \sum_{k=1}^\infty \lambda_k m_k$$

in the sense of $L^2(\mathbb{R}^n)$ with $\sum_k |\lambda_k| \approx \|f\|_{H^1_L(\mathbb{R}^n)}$. 


Note that $\sum_{k=1}^{\infty} \lambda_k m_k = f$ in the sense of $L^2(\mathbb{R}^n)$. So we have that
\[
\lim_{N \to \infty} F_N = 0 \quad \text{in} \quad L^2(\mathbb{R}^n),
\]
where $F_N := \sum_{k=N+1}^{\infty} \lambda_k m_k$. So we have
\[
\lim_{N \to \infty} g_{1,\delta}(L)(F_N) = 0 \quad \text{in} \quad L^2(\mathbb{R}^n).
\]

So there exists a subsequence $\{g_{1,\delta}(L)(F_{N_j})\}$ such that
\[
\lim_{j \to \infty} g_{1,\delta}(L)(F_{N_j}) = 0 \quad \text{a.e. in} \quad \mathbb{R}^n.
\]
Then for almost every $x \in \mathbb{R}^n$, for any $\varepsilon > 0$, there exists $J > 0$ sufficiently large such that for every integer $j > J$, we have
\[
|g_{1,\delta}(L)f(x)| = |g_{1,\delta}(L)\left(\sum_{k=1}^{\infty} \lambda_k m_k\right)(x)| = |g_{1,\delta}(L)\left(\sum_{k=1}^{N_j} \lambda_k m_k + \sum_{k=N_j+1}^{\infty} \lambda_k m_k\right)(x)|
\leq |g_{1,\delta}(L)\left(\sum_{k=1}^{N_j} \lambda_k m_k\right)(x)| + |g_{1,\delta}(L)\left(\sum_{k=N_j+1}^{\infty} \lambda_k m_k\right)(x)|
\leq \sum_{k=1}^{N_j} |\lambda_k| |g_{1,\delta}(L)(m_k)(x)| + \varepsilon,
\]
which gives
\[
|g_{1,\delta}(L)f(x)| \leq \sum_{k=1}^{\infty} |\lambda_k| |g_{1,\delta}(L)(m_k)(x)|, \quad \text{a.e. } x \in \mathbb{R}^n.
\]
Hence we obtain that
\[
\|g_{1,\delta}(L)f\|_{L^1(\mathbb{R}^n)} \leq \sum_{k=1}^{\infty} |\lambda_k| \|g_{1,\delta}(L)(m_k)\|_{L^1(\mathbb{R}^n)}.
\]

As a consequence, to prove (5.3), it suffices to prove that there exists a positive constant $C$ independent of $\delta$ such that for every molecule $m$ as defined in (5.1), the following estimate
\[
\|g_{1,\delta}(L)(m)\|_{L^1(\mathbb{R}^n)} \leq C.
\]
holds. To verify this, we first see that
\[
\|g_{1,\delta}(L)(m)\|_{L^1(\mathbb{R}^n)} = \int_{4B} |g_{1,\delta}(L)(m)(x)|dx + \int_{(4B)^c} |g_{1,\delta}(L)(m)(x)|dx =: I + II,
\]
where $B$ is the ball associated to $m$.

As for $I$, we have
\[
I = \int_{4B} |g_{1,\delta}(L)(m)(x)|dx \leq |4B|^{1/2} \|g_{1,\delta}(L)(m)\|_{L^2(\mathbb{R}^n)}
\]
\[
\leq C|4B|^{1/2} \|m\|_{L^2(\mathbb{R}^n)} \leq C|4B|^{1/2} \left(\int_0^{|B|} \int_B \alpha(x,t)^2 dt\right)^{1/2} \leq C,
\]
where in the second inequality we have used Lemma 4.1 and in the third inequality we have used Lemma 4.3 in [22] and the definition of the molecule $m$.

Now we turn to $II$.
\[
II = \sqrt{\ln \delta} \int_{(4B)^c} \left(\sum_{j \in I_j} \sum_{\tau \in I_j} |q_{ij}(m)(y_{ij})|^2 \chi_{Q_i}(x)\right)^{1/2} dx
\]
\[ \delta^{-j} + t + |y_{Q^c_i} - y| \geq C(\delta^{-j} + t + |y_B - x|). \]

In fact, if \( \delta^{-j} > r_B \), then
\[ |y_B - x| \leq |x - y_{Q^c_i}| + |y_{Q^c_i} - y| + |y - y_B| \leq 2\delta^{-j} + |y_{Q^c_i} - y|; \]
if \( \delta^{-j} \leq r_B \), then \( |y - y_B| \leq \frac{1}{2}|y_B - x| \), which implies that
\[ |y_B - x| \leq |x - y_{Q^c_i}| + |y_{Q^c_i} - y| + \frac{1}{2}|y_B - x|, \]
and thus
\[ |y_B - x| \leq 2(|x - y_{Q^c_i}| + |y_{Q^c_i} - y|) \leq 2(\delta^{-j} + |y_{Q^c_i} - y|). \]

We insert the inequality (5.7) into (5.6), and get
\[
II \leq C \sqrt{\ln \delta} \int_{(4B)^c} \left( \sum_{j} \int_{0}^{\infty} \left( \frac{\delta^{-j}}{t} \right)^{t} |a(y, t)| \left( \frac{\delta^{-j} \vee t}{|x - y_B|^{2n+1}} \right) \right)^{1/2} dx
\]
\[
\leq C \sqrt{\ln \delta} \int_{(4B)^c} \left( \sum_{j} \int_{0}^{\infty} \left( \frac{\delta^{-j}}{t} \right)^{t} |a(y, t)| \left( \frac{\delta^{-j} \vee t}{|x - y_B|^{2n+1}} \right) \right)^{1/2} dx
\]
\[
\leq C \sqrt{\ln \delta} \int_{(4B)^c} \left( \sum_{j} \int_{0}^{\infty} \left( \frac{\delta^{-j}}{t} \right)^{t} |a(y, t)| \left( \frac{\delta^{-j} \vee t}{|x - y_B|^{2n+1}} \right) \right)^{1/2} dx
\]
\[
\leq C \sqrt{\ln \delta} \int_{(4B)^c} \left( \sum_{j} \int_{0}^{\infty} \left( \frac{\delta^{-j}}{t} \right)^{t} |a(y, t)| \left( \frac{\delta^{-j} \vee t}{|x - y_B|^{2n+1}} \right) \right)^{1/2} dx
\]

where the fourth inequality follows from property (ii) of the \( T^1_2 \) atom \( a \), and we use \( s \vee t \) to denote \( \max\{s, t\} \) for every positive numbers \( s \) and \( t \).

One can compute
\[
\sum_{j} \int_{0}^{\infty} \left( \frac{\delta^{-j}}{t} \right)^{t} |a(y, t)| \left( \frac{\delta^{-j} \vee t}{|x - y_B|^{2n+1}} \right) \frac{dt}{t} = (\sum_{j: \delta^{-j} \leq r_B} + \sum_{j: \delta^{-j} > r_B}) \int_{0}^{\infty} \left( \frac{\delta^{-j}}{t} \right)^{t} |a(y, t)| \left( \frac{\delta^{-j} \vee t}{|x - y_B|^{2n+1}} \right) \frac{dt}{t}
\]
\[= II_1 + II_2. \]
For the term $I_1$, we have
\[
I_1 = \sum_{j: \delta^{-j} \leq r_B} \left( \int_0^{\delta^{-j}} + \int_{\delta^{-j}}^{r_B} \right) \left( \frac{\delta^{-j}}{t} \wedge \frac{t}{\delta^{-j}} \right)^2 (\delta^{-j} \vee t) \frac{dt}{t}
\]
\[
\leq \sum_{j: \delta^{-j} \leq r_B} \int_0^{\delta^{-j}} \frac{t}{\delta^{-j}} \delta^{-j} \frac{dt}{t} + \sum_{j: \delta^{-j} \leq r_B} \int_{\delta^{-j}}^{r_B} \frac{t}{\delta^{-j}} \delta^{-j} \frac{dt}{t}
\]
\[
\leq C \sum_{j: \delta^{-j} \leq r_B} \delta^{-j} \leq C \frac{r_B}{\delta - 1}.
\]

For the term $I_2$, we have
\[
I_2 \leq \sum_{j: \delta^{-j} > r_B} \int_0^{r_B} \left( \frac{t}{\delta^{-j}} \right)^2 \frac{dt}{t} \leq C(r_B)^2 \sum_{j: \delta^{-j} > r_B} \delta^{-j} \leq C \frac{r_B}{\delta - 1}.
\]

Thus, we obtain that for any $1 < \delta \leq 2$,
\[
I \leq C \sqrt{\ln \delta} \frac{1}{\sqrt{\delta - 1}} \int \frac{r_B}{|x - y|^{n+\frac{1}{2}}} \, dx \leq \tilde{C},
\]
where $\tilde{C}$ is independent of $\delta$. Combining the estimate of $I$ and $II$, we can obtain that (5.4) holds.

The proofs of (5.3) for $i = 2, 3$ are similar to that for $i = 1$. The proof of (5.3) for $i = 4$ is similar to that for $i = 1$, but via the almost orthogonality estimate (2.16) in Proposition 2.4. We omit the details here.

5.2. **Proof of frame decomposition on Hardy spaces.** The next two results give the estimates on the norms of the operators $T_\delta$ and $I - T_\delta$ on the Hardy spaces associated with the operator $L$.

**Theorem 5.3.** For every $\frac{n}{n+1} < r < 1$, there exists a positive constant $C(n, r)$, such that for every $1 < \delta < 2$ and $f \in H^1_L(\mathbb{R}^n)$,
\[
||T_\delta(f)||_{H^1_L(\mathbb{R}^n)} \leq C(n, r)\delta^{M_n(\frac{1}{r})-1}||f||_{H^1_L(\mathbb{R}^n)}.
\]

**Proof.** For any $f \in H^1_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),
\[
||T_\delta(f)||_{H^1_L(\mathbb{R}^n)}
\]
\[
= \left\{ \int_{\Gamma(x)} \left| q_s(L)(T_\delta(f))(y) \right|^2 \frac{dy \, ds}{s^{n+1}} \right\}^{\frac{1}{2}}_{L^1(\mathbb{R}^n)}
\]
\[
= \ln \delta \left\{ \int_0^\infty \left( \int_{|y-z|<s} \left| q_s(L) \left( \sum_j \sum_{t \in I_j} \left| Q^s \right| q_s(L)(y, y_{Q_j}) q_{\frac{1}{2}}(L)(f)(y_{Q_j}) \right) \right|^2 \frac{dy \, ds}{s^{n+1}} \right)^{\frac{1}{2}} \right\}_{L^1(\mathbb{R}^n)}
\]
\[
= \ln \delta \left\{ \int_0^\infty \left( \int_{|y-z|<s} \left| q_s(L) \left( \sum_j \sum_{t \in I_j} \left| Q^s \right| q_s(L) q_{\frac{1}{2}}(L)(y, y_{Q_j}) q_{\frac{1}{2}}(L)(f)(y_{Q_j}) \right) \right|^2 \frac{dy \, ds}{s^{n+1}} \right)^{\frac{1}{2}} \right\}_{L^1(\mathbb{R}^n)}
\]
\[
\leq \ln \delta \left\{ \int_0^\infty \left( \int_{|y-z|<s} \left| \sum_j \sum_{t \in I_j} \left| Q^s \right| \delta^{-\frac{j}{2}} \frac{dy \, ds}{s^{n+1}} \right|^2 \right) \right\}_{L^1(\mathbb{R}^n)}
\]
\[
\leq (\ln \delta)^2 \left\{ \int_0^\infty \left( \int_{|y-z|<s} \left| \sum_j \sum_{t \in I_j} \left| Q^s \right| \delta^{-\frac{j}{2}} \frac{dy \, ds}{s^{n+1}} \right|^2 \right) \right\}_{L^1(\mathbb{R}^n)}.
\]
Next we claim that

\[ \sum_{\tau \in I_j} |Q_\tau^l| \frac{\delta^{- (j/k)r}}{(\delta^{- (j/k)r} + |x - y_{Q_\tau^l}|^{(n+1)r})} \left| q_{\delta^{- 2j}}(L)(f)(y_{Q_\tau^l}) \right|^r \]

\[ \leq C \delta^{M_{n+1}} \delta^{j \cdot (1/k)} \sum_{\tau \in I_j} \left| q_{\delta^{- 2j}}(L)(f)(y_{Q_\tau^l}) \right|^r \chi_{Q_\tau^l}(x), \]

where \( \mathcal{M} \) is the Hardy–Littlewood maximal function and \( \frac{n}{n+1} < r < 1 \).

We now prove this claim (5.9). We point out that this type of inequality is first proved by Frazier and Jawerth in the Euclidean setting (See \cite{24}, pp.147–148).

To prove (5.9), we first point out that for all \( 0 < r < 1, \) \( \sum_j |a_j|^r \leq \left( \sum_j |a_j|^p \right)^{1/r} \). As a consequence, the left-hand side of the inequality (5.9) is controlled by

\[ \left( \sum_{\tau \in I_j} \frac{|Q_\tau^l|^r}{(\delta^{- (j/k)r} + |x - y_{Q_\tau^l}|^{(n+1)r})} \left| q_{\delta^{- 2j}}(L)(f)(y_{Q_\tau^l}) \right|^r \right)^{1/r} \]

\[ \leq \left( \sum_{\tau \in A_0} \frac{|Q_\tau^l|^r}{(\delta^{- (j/k)r} + |x - y_{Q_\tau^l}|^{(n+1)r})} \left| q_{\delta^{- 2j}}(L)(f)(y_{Q_\tau^l}) \right|^r \right)^{1/r} \]

\[ \leq \left( \sum_{\tau \in A_0} \frac{|Q_\tau^l|^r}{(\delta^{- (j/k)r} + |x - y_{Q_\tau^l}|^{(n+1)r})} \left| q_{\delta^{- 2j}}(L)(f)(y_{Q_\tau^l}) \right|^r \right)^{1/r} \]

\[ \leq C \delta^{j \cdot (1/k)} \delta^{M_{n+1}} \int_{B(x, \delta^{- 1} \delta^{j \cdot (1/k)})} \left( \sum_{\tau \in A_0} \left| q_{\delta^{- 2j}}(L)(f)(y_{Q_\tau^l}) \right|^r \chi_{Q_\tau^l}(y) \right) dy \]

\[ \leq C \delta^{j \cdot (1/k)} \delta^{M_{n+1}} \int_{B(x, 2^j \delta^{- 1} \delta^{j \cdot (1/k)})} \left( \sum_{\tau \in A_0} \left| q_{\delta^{- 2j}}(L)(f)(y_{Q_\tau^l}) \right|^r \chi_{Q_\tau^l}(y) \right) dy \]

where in the last inequality we use the fact that \( \frac{n}{n+1} < r \). This implies that the claim (5.9) holds.
As a consequence, we can obtain that
\[
\|T_\delta(f)\|_{H^1_1(\mathbb{R}^n)} \leq C_\delta^{M(n-\frac{1}{2})}(\ln \delta)^\frac{3}{2}\left(\sum_k \left(\sum_j \delta^{-k-j}\delta^{j-(k\wedge j)}|n|^{\frac{1}{r}-1}\right)\right)
\times \left\| \mathcal{M}\left(\sum_{\tau \in I_j} |q_{\delta^{-j}}(L)(f)(y_\tau)| \chi_{Q_\tau}(\cdot)\right)^{\frac{1}{r}}(x) \right\|_{L^1(\mathbb{R}^n)}
\leq C_\delta^{M(n-\frac{1}{2})}(\ln \delta)^\frac{3}{2}\left(\sum_k \left(\sum_j \delta^{-k-j}\delta^{j-(k\wedge j)}|n|^{\frac{1}{r}-1}\right)\right)
\times \left(\sum_j \delta^{-k-j}\delta^{j-(k\wedge j)}|n|^{\frac{1}{r}-1}\right)\left\| \mathcal{M}\left(\sum_{\tau \in I_j} |q_{\delta^{-j}}(L)(f)(y_\tau)| \chi_{Q_\tau}(\cdot)\right)^{2/r}(x) \right\|_{L^1(\mathbb{R}^n)}
\]

Observe that
\[
\sum_j \delta^{-k-j}\delta^{j-(k\wedge j)}|n|^{\frac{1}{r}-1} + \sum_k \delta^{-k-j}\delta^{j-(k\wedge j)}|n|^{\frac{1}{r}-1} \leq \frac{C(n, r)}{\delta - 1}.
\]

It follows that
\[
\|T_\delta(f)\|_{H^1_1(\mathbb{R}^n)} \leq C_\delta^{M(n-\frac{1}{2})}(\ln \delta)^\frac{3}{2}(\delta - 1)^{-\frac{1}{2}}\left(\sum_k \left(\sum_j \delta^{-k-j}\delta^{j-(k\wedge j)}|n|^{\frac{1}{r}-1}\right)\right)
\times \left\| \mathcal{M}\left(\sum_{\tau \in I_j} |q_{\delta^{-j}}(L)(f)(y_\tau)| \chi_{Q_\tau}(\cdot)\right)^{\frac{1}{r}}(x) \right\|_{L^1(\mathbb{R}^n)}
\leq C_\delta^{M(n-\frac{1}{2})}(\ln \delta)^\frac{3}{2}\left(\sum_j \left(\sum_{\tau \in I_j} |q_{\delta^{-j}}(L)(f)(y_\tau)| \chi_{Q_\tau}(\cdot)\right)^{2/r}(x) \right)\left\| \mathcal{M}\left(\sum_{\tau \in I_j} |q_{\delta^{-j}}(L)(f)(y_\tau)| \chi_{Q_\tau}(\cdot)\right)^{\frac{1}{r}}(x) \right\|_{L^1(\mathbb{R}^n)}
\leq C_\delta^{M(n-\frac{1}{2})}(\ln \delta)^\frac{3}{2}\left\| \mathcal{M}\left(\sum_{\tau \in I_j} |q_{\delta^{-j}}(L)(f)(y_\tau)| \chi_{Q_\tau}(\cdot)\right)^{\frac{1}{r}}(x) \right\|_{L^1(\mathbb{R}^n)}
\leq C_\delta^{M(n-\frac{1}{2})}(\ln \delta)^\frac{3}{2}\|1_{\mathbb{R}^n}\|_{L^1(\mathbb{R}^n)}\|g_1_{\delta}(f)\|_{L^1(\mathbb{R}^n)},
\]

where the last inequality follows from boundedness of the vector-valued maximal function.

Now applying (5.3) in Lemma 5.2, we obtain that
\[
\|T_\delta(f)\|_{H^1_1(\mathbb{R}^n)} \leq C_\delta^{M(n-\frac{1}{2})}\|f\|_{H^1_1(\mathbb{R}^n)}.
\]

The proof of this theorem is complete.

\[\Box\]

**Theorem 5.4.** Define \(R_\delta(f)(x) = f(x) - T_\delta(f)(x)\). Then there exists positive constants 1 < \(\delta < 2\) and \(M \geq 1\) such that \(\|R_\delta(f)\|_{H^1_1(\mathbb{R}^n)} \leq \frac{1}{2}\|f\|_{H^1_1(\mathbb{R}^n)}\).

**Proof.** We decompose \(R_\delta(f)\) into four terms: \(R(f) = R_{\delta, 1}(f) + R_{\delta, 2}(f) + R_{\delta, 3}(f) + R_{\delta, 4}(f)\), where
\[
R_{\delta, 1}(f) := \sum_j \sum_{\tau \in I_j} \left| \int_{Q_{\tau}} q_{\tau}(x, y) - q_{\tau}(x, y_\tau) |q_{\tau}(L)f(y)dy \right| dt;
\]
\[
R_{\delta, 2}(f) := \sum_j \sum_{\tau \in I_j} \left| \int_{Q_{\tau}} q_{\tau}(x, y_\tau) - q_{\tau}(x, y_{\tau'}) |q_{\tau}(L)f(y)dy \right| dt;
\]
\[
R_{\delta, 3}(f) := \sum_j \sum_{\tau \in I_j} \left| \int_{Q_{\tau}} q_{\tau}(x, y_\tau) |q_{\tau}(L)f(y) - q_{\tau}(L)f(y_\tau)dy \right| dt;
\]
\[
R_{\delta, 4}(f) := \sum_j \sum_{\tau \in I_j} \left| \int_{Q_{\tau}} q_{\tau}(x, y_\tau) (|q_{\tau}(L)f(y) - q_{\tau}(L)f(y_\tau)| dt;
\]
$$R_{δ,4}(f) := \sum_{j} \sum_{τ \in I_j} \int_{δ^{-j}}^{δ^{-j+1}} \int_{Q_τ} q_ζ(x, y; Q_τ) |q_τ(L)f(y; Q_τ)| - q_δ^{-2j}(L)f(y; Q_τ)|dy\frac{dt}{t}.$$ 

We first estimate $R_{δ,1}(f)$. For every $f \in H^1_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, from the definition of $R_{δ,1}(f)$, we have

(5.11) $$\|R_{δ,1}(f)\|_{H^1_L(\mathbb{R}^n)} = \|\{ \int_{\Gamma(x)} |q_ζ(L)(R_{δ,1}(f))(y)|^2 dyds \}^{\frac{1}{2}}\|_{L^1(\mathbb{R}^n)}.$$ 

$$= \|\{ \sum_k \int_{δ^{-k}}^{δ^{-k+1}} \int_{|x-y|<s} \sum_j \sum_{τ \in I_j} \int_{δ^{-j}}^{δ^{-j+1}} \int_{Q_τ} |q_ζ(L)q_τ(L)(y, z) - q_ζ(L)q_τ(L)(y, y; Q_τ)|q_τ(L)f(z)dz \frac{dt}{t} \frac{dy}{s} \frac{ds}{s} \}^{\frac{1}{2}}\|_{L^1(\mathbb{R}^n)}.$$ 

Note that for $s \in (δ^{-k}, δ^{-k+1})$ and $t \in (δ^{-j}, δ^{-j+1})$, we have

$$|q_ζ(L)q_τ(L)(y, z) - q_ζ(L)q_τ(L)(y, y; Q_τ)|$$

$$= \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} q_ζ(y, w)φ_ζ(w, v)[ζ_τ(v, z) - ζ_τ(v, y; Q_τ)]dwdv \right|$$

$$\leq \int_{\mathbb{R}^n} |q_ζ(L)φ_ζ(L)(y, v)||ζ_τ(v, z) - ζ_τ(v, y; Q_τ)|dv$$

$$\leq Cδ^{-j-k} \frac{|z - y; Q_τ|}{δ^{-j}} \int_{\mathbb{R}^n} \frac{δ^{-j-k}}{(δ^{-j-k} + |y - v|)^{1+1}} \frac{δ^{-j}}{δ^{-j-k} + |y - y; Q_τ|^1} dv$$

where in the second inequality above we have used the similar argument of (2.15) and (2.17). Thus, by substituting the above estimate into the right-hand side of (5.11), we get

$$\|R_{δ,1}(f)\|_{H^1_L(\mathbb{R}^n)} \leq C \left\{ \sum_k \int_{δ^{-k}}^{δ^{-k+1}} \left| \sum_j \sum_{τ \in I_j} \int_{δ^{-j}}^{δ^{-j+1}} \int_{Q_τ} \frac{|z - y; Q_τ|}{δ^{-j}} \right|^2 \frac{dt}{t} \frac{dy}{s} \frac{ds}{s} \right\}^{\frac{1}{2}} \|_{L^1(\mathbb{R}^n)}.$$ 

$$\leq Cδ^{-M} \sqrt{\ln δ} \left\{ \sum_k \int_{δ^{-k}}^{δ^{-k+1}} \left| \sum_j \sum_{τ \in I_j} \frac{|Q_τ|}{(δ^{-j-k} + |x - y; Q_τ|)^{1+1}} \right|^2 \frac{dt}{t} \frac{dy}{s} \frac{ds}{s} \right\}^{\frac{1}{2}} \|_{L^1(\mathbb{R}^n)}.$$ 

$$\leq Cδ^{-M} \sqrt{\ln δ} \left\{ \sum_k \int_{δ^{-j}}^{δ^{-j+1}} \left| \sum_j \sum_{τ \in I_j} \frac{|Q_τ|}{(δ^{-j-k} + |x - y; Q_τ|)^{1+1}} \right|^2 \frac{dt}{t} \frac{dy}{s} \frac{ds}{s} \right\}^{\frac{1}{2}} \|_{L^1(\mathbb{R}^n)}.$$ 

$$\leq Cδ^{-M} \sqrt{\ln δ} \left\{ \sum_k \int_{δ^{-j}}^{δ^{-j+1}} \left( \sum_j \sum_{τ \in I_j} \frac{|Q_τ|}{(δ^{-j-k} + |x - y; Q_τ|)^{1+1}} \right)^2 \frac{dt}{t} \frac{dy}{s} \frac{ds}{s} \right\}^{\frac{1}{2}} \|_{L^1(\mathbb{R}^n)}.$$ 

$$\leq Cδ^{-M} \sqrt{\ln δ} \left\{ \sum_k \int_{δ^{-j}}^{δ^{-j+1}} \left( \sum_j \sum_{τ \in I_j} |Q_τ| \frac{1}{(δ^{-j-k} + |x - y; Q_τ|)^{1+1}} \right)^2 \frac{dt}{t} \frac{dy}{s} \frac{ds}{s} \right\}^{\frac{1}{2}} \|_{L^1(\mathbb{R}^n)}.$$ 

$$\leq Cδ^{-M} \sqrt{\ln δ} \left\{ \sum_k \int_{δ^{-j}}^{δ^{-j+1}} \left( \sum_j \sum_{τ \in I_j} |Q_τ| \frac{1}{(δ^{-j-k} + |x - y; Q_τ|)^{1+1}} \right)^2 \frac{dt}{t} \frac{dy}{s} \frac{ds}{s} \right\}^{\frac{1}{2}} \|_{L^1(\mathbb{R}^n)}.$$
where the last inequality follows from the claim (5.9) and \( r \) can be any number in \( (\frac{m}{n+1}, 1) \). Then using Hölder’s inequality, we have

\[
\left| \int_{\delta^{-j+1}}^{\delta^{-j}} \frac{1}{|Q_t^r|} \int_{Q_t^r} |q_r(z)(L)f(z)|dz dt \right|^\frac{1}{r} \leq (\ln \delta)^\frac{1}{r} \left( \int_{\delta^{-j}}^{\delta^{-j+1}} \frac{1}{|Q_t^r|} \int_{Q_t^r} |q_r(z)(L)f(z)|^2 dz dt \right)^\frac{1}{2}.
\]

Thus,

\[
\|R_{\delta \lambda}(f)\|_{H^1_1(\mathbb{R}^n)} \leq C \delta^{-M} \delta^{M_n(\frac{1}{r}-1)} (\ln \delta) \left\| \sum_j \sum_k \delta^{-|j-k|} \delta^{l(j,k)} \right\|_{L^1(\mathbb{R}^n)}
\]

\[
\times \left[ \mathcal{M} \left( \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \frac{1}{|Q_t^r|} \int_{Q_t^r} |q_r(z)(L)f(z)|^2 dz dt \right)^\frac{1}{2} \right]^\frac{1}{r} \left( \int_{\delta^{-j}}^{\delta^{-j+1}} \frac{1}{|Q_t^r|} \int_{Q_t^r} |q_r(z)(L)f(z)|^2 dz dt \right)^\frac{1}{2}.
\]

\[
\leq C \delta^{-M} \delta^{M_n(\frac{1}{r}-1)} (\ln \delta) \left\| \sum_j \sum_k \delta^{-|j-k|} \delta^{l(j,k)} \right\|_{L^1(\mathbb{R}^n)}
\]

\[
\times \left[ \left( \int_{\delta^{-j}}^{\delta^{-j+1}} \frac{1}{|Q_t^r|} \int_{Q_t^r} |q_r(z)(L)f(z)|^2 dz dt \right)^\frac{1}{2} \right]^\frac{1}{r} \left( \int_{\delta^{-j}}^{\delta^{-j+1}} \frac{1}{|Q_t^r|} \int_{Q_t^r} |q_r(z)(L)f(z)|^2 dz dt \right)^\frac{1}{2}.
\]

where in the second inequality above we have used (5.10) and in the last inequality we have used the boundedness of the vector-valued maximal function.

Now applying Lemma 5.2, we can obtain that

\[
\|R_{\delta \lambda}(f)\|_{H^1_1(\mathbb{R}^n)} \leq C \delta^{-M} \delta^{M_n(\frac{1}{r}-1)} \|f\|_{H^1_1(\mathbb{R}^n)}.
\]

Next we only need to estimate \( R_{\delta \lambda}(f) \) since the terms \( R_{\delta \lambda}(f) \) and \( R_{\delta \lambda}(f) \) can be obtained by following similar steps as in \( R_{\delta \lambda}(f) \) and \( R_{\delta \lambda}(f) \), respectively.

For any \( f \in H^1_1(\mathbb{R}^n) \), we have

\[
\|R_{\delta \lambda}(f)\|_{H^1_1(\mathbb{R}^n)} = \left\| \left( \int_{\delta^{-j+1}}^{\delta^{-j}} \frac{1}{|Q_t^r|} \int_{Q_t^r} |q_r(z)(R_{\delta \lambda}(f))(y)|^2 dz dt \right)^\frac{1}{2} \right\|_{L^1(\mathbb{R}^n)}
\]

\[
= \left\| \left( \sum_k \int_{\delta^{-j}}^{\delta^{-j+1}} \int_{|x-y|<s} |q_r(z)(R_{\delta \lambda}(f))(y)|^2 dz dt \right)^\frac{1}{2} \right\|_{L^1(\mathbb{R}^n)}
\]

\[
= \left\| \left( \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \int_{Q_t^r} |q_r(z)(L)(y, y_{Q_t^r})(y)|^2 dz dt \right)^\frac{1}{2} \right\|_{L^1(\mathbb{R}^n)}.
\]

Applying (4.6) we obtain that

\[
\|R_{\delta \lambda}(f)\|_{H^1_1(\mathbb{R}^n)} \leq C \ln \delta \left\| \sum_k \sum_j \sum_{\tau \in I_j} \int_{\delta^{-j}}^{\delta^{-j+1}} \delta^{-|j-k|} \right\|_{L^1(\mathbb{R}^n)}
\]

\[
\times \left[ \frac{\delta^{-l(j,k)}}{(\delta^{-l(j,k)} + |x-y_{Q_t^r}|)^n + 1} \left( \int_{\delta^{-j}}^{\delta^{-j+1}} \|r^2Lq^{(1)}(r^2Lf(y_{Q_t^r}))^2 dz dt \right)^\frac{1}{2} \right]^\frac{1}{r} \left( \int_{\delta^{-j}}^{\delta^{-j+1}} \|r^2Lq^{(1)}(r^2Lf(y_{Q_t^r}))^2 dz dt \right)^\frac{1}{2} \right]^{\frac{1}{r}} \left\| \sum_j \delta^{-l(j,k)} \sum_{\tau \in I_j} |Q_t^r| \right\|_{L^1(\mathbb{R}^n)}
\]

\[
\times \left[ \frac{\delta^{-l(j,k)}}{(\delta^{-l(j,k)} + |x-y_{Q_t^r}|)^n + 1} \left( \int_{\delta^{-j}}^{\delta^{-j+1}} \|r^2Lq^{(1)}(r^2Lf(y_{Q_t^r}))^2 dz dt \right)^\frac{1}{2} \right]^\frac{1}{r} \left( \int_{\delta^{-j}}^{\delta^{-j+1}} \|r^2Lq^{(1)}(r^2Lf(y_{Q_t^r}))^2 dz dt \right)^\frac{1}{2} \right]^{\frac{1}{r}} \right\|_{L^1(\mathbb{R}^n)}
\]

\[
\leq C(\ln \delta)^2 \left\| \sum_k \sum_j \sum_{\tau \in I_j} \delta^{-|j-k|} \right\|_{L^1(\mathbb{R}^n)}
\]

\[
\times \left[ \frac{\delta^{-l(j,k)}}{(\delta^{-l(j,k)} + |x-y_{Q_t^r}|)^n + 1} \left( \int_{\delta^{-j}}^{\delta^{-j+1}} \|r^2Lq^{(1)}(r^2Lf(y_{Q_t^r}))^2 dz dt \right)^\frac{1}{2} \right]^\frac{1}{r} \left( \int_{\delta^{-j}}^{\delta^{-j+1}} \|r^2Lq^{(1)}(r^2Lf(y_{Q_t^r}))^2 dz dt \right)^\frac{1}{2} \right]^{\frac{1}{r}} \right\|_{L^1(\mathbb{R}^n)}
\]
\[
\leq C(\ln \delta)^2 \left\| \sum_k \left( \sum_j \delta^{-[j-k]} \delta^{Mn(j-(k\wedge j)n)} \right) \right\|^{1/2}_{L^1(\mathbb{R}^n)} \\
\times \left\{ \mathcal{M} \left( \sum_{\tau \in I_j} \left( \int_{\delta^{-j}} \left| r^2 L_q^{(1)}(r^2 L f(y_{\ell \delta})^2 \frac{dr}{r} \right)^{1/2} \mathcal{X}_{\ell \delta}(\cdot) \right) \right)^{1/2} \right\}^{1/2}_{L^1(\mathbb{R}^n)} \\
\leq C(\ln \delta)^2 \delta^{Mn(j-(k\wedge j)n)} \left\| \sum_k \left( \sum_j \delta^{-[j-k]} \delta^{Mn(j-(k\wedge j)n)} \right) \right\|^{1/2}_{L^1(\mathbb{R}^n)} \\
\times \left\{ \mathcal{M} \left( \sum_{\tau \in I_j} \left( \int_{\delta^{-j}} \left| r^2 L_q^{(1)}(r^2 L f(y_{\ell \delta})^2 \frac{dr}{r} \right)^{1/2} \mathcal{X}_{\ell \delta}(\cdot) \right) \right)^{1/2} \right\}^{1/2}_{L^1(\mathbb{R}^n)} \\
\leq C(\ln \delta)\delta^{Mn(j-(k\wedge j)n)} \left\| \sum_j \left( \mathcal{M} \left( \int_{\delta^{-j}} \left| r^2 L_q^{(1)}(r^2 L f(y_{\ell \delta})^2 \frac{dr}{r} \right)^{1/2} \mathcal{X}_{\ell \delta}(\cdot) \right) \right)^{1/2} \right\|_{L^1(\mathbb{R}^n)} \\
\leq C(\ln \delta)\delta^{Mn(j-(k\wedge j)n)} \left\| \sum_j \left( \mathcal{M} \left( \sum_{\tau \in I_j} \int_{\delta^{-j}} \left| r^2 L_q^{(1)}(r^2 L f(y_{\ell \delta})^2 \frac{dr}{r} \right)^{1/2} \mathcal{X}_{\ell \delta}(\cdot) \right) \right)^{1/2} \right\|_{L^1(\mathbb{R}^n)} \\
= C(\ln \delta)\delta^{Mn(j-(k\wedge j)n)} \left\| g_{4,\delta}(f) \right\|_{L^1(\mathbb{R}^n)}.
\]

By applying Lemma 5.2, we can obtain that
\[
\|R_{\delta,4}(f)\|_{H^1_{L_q}(\mathbb{R}^n)} \leq C(\delta - 1)\delta^{Mn(j-(k\wedge j)n)}\|f\|_{H^1_{L_q}(\mathbb{R}^n)}. \]

Similarly we can verify the $H^1_{L_q}(\mathbb{R}^n)$-norm of the terms $R_{\delta,2}(f)$ and $R_{\delta,3}(f)$ with constants $C\delta^{Mn(j-(k\wedge j)n)}(\delta - 1)$ and $C\delta^{-M}\delta^{Mn(j-(k\wedge j)n)}$, respectively.

Thus, combining the estimates of $R_{\delta,1}(f)$, $R_{\delta,2}(f)$, $R_{\delta,3}(f)$ and $R_{\delta,4}(f)$, we have, there exists a constant $C_2 > 1$ such that
\[
\|R_{\delta}(f)\|_{H^1_{L_q}(\mathbb{R}^n)} \leq C_2\left( \delta^{Mn(j-(k\wedge j)n)}(\delta - 1) + \delta^{-M(1-n(\frac{1}{2}-1))} \right)\|f\|_{H^1_{L_q}(\mathbb{R}^n)} \\
\leq C_2\left( \delta^{M(\delta - 1)} + \delta^{-M(1-n(\frac{1}{2}-1))} \right)\|f\|_{H^1_{L_q}(\mathbb{R}^n)},
\]

since $n/(n + 1) < r < 1$. Let us choose $\delta$ close to 1, such that
\[
2C_2(4C_2)^{(1/(2-n))}(\delta - 1) < \frac{1}{4}
\]
and choose $\hat{M} \in \mathbb{R}^+$ such that $C_2\delta^{-\hat{M}(1-n(\frac{1}{2}-1))} = \frac{1}{4}$. Let $M = [\hat{M}] + 1$, where we use $[x]$ to denote the maximal integer which is not greater than $x$. It follows that $2^{-3-n(\frac{1}{2}-1))} \leq C_2\delta^{-M(1-n(\frac{1}{2}-1))} \leq \frac{1}{4}$. It implies that $\delta^{M} \leq 2(4C_2)^{(1/(2-n))}$, thus $C_2\delta^{M(\delta - 1)} \leq \frac{1}{4}$. It follows that $\|R_{\delta}(f)\|_{H^1_{L_q}(\mathbb{R}^n)} \leq \frac{1}{2}\|f\|_{H^1_{L_q}(\mathbb{R}^n)}$. \hfill \square

We now apply the results in Theorems 5.3 and 5.4 to prove our main result, Theorem 1.3.

**Proof of Theorem 1.3.** By Theorem 5.4, we have that for every $k = 0, 1, \cdots$, $\|R_{\delta,k}^k(f)\|_{H^1_{L_q}(\mathbb{R}^n)} \leq \frac{1}{2^k}\|f\|_{H^1_{L_q}(\mathbb{R}^n)}$. Therefore, the operator $T_{\delta}$ as defined in (1.4) is invertible, and
\[
\|T_{\delta}^{-1}f\|_{H^1_{L_q}(\mathbb{R}^n)} \leq \sum_{k=0}^{\infty}\|R_{\delta,k}^k(f)\|_{H^1_{L_q}(\mathbb{R}^n)} \leq 2\|f\|_{H^1_{L_q}(\mathbb{R}^n)}.
\]

Thus, for every $f \in H^1_{L_q}(\mathbb{R}^n)$, by the definition of operator $T_{\delta}$, one can write
\[
f = T_{\delta}^{-1}f = \sum_{j \in I_j} \int_{\delta^{-j}} \left( T_{\delta}^{-1}f, \psi_{j,r}^* \right) \psi_{j,r} \quad \text{in} \quad H^1_{L_q}(\mathbb{R}^n).
\]
Applying Lemma 5.2, we have
\[
\left\| \left( \sum_j \sum_{\tau \in I_j} \langle T_{\delta}^{-1}f, \psi_{j,\tau}^* \rangle \right)^2 \frac{1}{|Q_{\tau}|} \right\|_{L^1(\mathbb{R}^n)} \leq \sqrt{\ln \delta} \left\| \left( \sum_j \sum_{\tau \in I_j} |q_{\delta^{-2}}(L)(T_{\delta}^{-1}f)(y_Q)|^2 \right)^{\frac{1}{2}} \right\|_{L^1(\mathbb{R}^n)} \\
\leq C \|T_{\delta}^{-1}f\|_{H^1_L(\mathbb{R}^n)} \leq C \|f\|_{H^1_L(\mathbb{R}^n)},
\]
For the left inequality in (1.8), using (5.13) we have
\[
\|f\|_{H^1_L(\mathbb{R}^n)} = \left\| \left\{ \int_0^\infty \int_{|x-y|<s} |q_{s}(\mathcal{L}(f)(y)) dy \d s \right\}^{1/2} \right\|_{L^1(\mathbb{R}^n)} \\
= \sqrt{\ln \delta} \left\| \left\{ \sum_k \int_0^{\delta^{-k}} \int_{|x-y|<s} \left| \sum_j \sum_{\tau \in I_j} \langle T_{\delta}^{-1}f, \psi_{j,\tau}^* \rangle \right| |Q_{\tau}|^{1/2} |q_{s}(y, y_Q)|^2 dy \d s \right\}^{1/2} \right\|_{L^1(\mathbb{R}^n)} \\
\leq C \ln \delta \left\| \left\{ \sum_k \left| \sum_j \sum_{\tau \in I_j} \delta^{-|j-k|} \delta^{-|j-k|/s^{n+1}} \langle T_{\delta}^{-1}f, |Q_{\tau}|^{-1/2} \psi_{j,\tau}^* \rangle \right| \right\}^{1/2} \right\|_{L^1(\mathbb{R}^n)} \\
\leq C \ln \delta \left\| \left\{ \sum_k \left| \sum_j \sum_{\tau \in I_j} \delta^{-|j-k|/s^{n+1}} \delta^{M(n-1)} \delta^{-|j-k|/\delta^{n+1}} \langle T_{\delta}^{-1}f, |Q_{\tau}|^{-1/2} \psi_{j,\tau}^* \rangle \right| \right\}^{1/2} \right\|_{L^1(\mathbb{R}^n)} \\
\leq C \ln \delta \left\| \left\{ \sum_j \left| \sum_{\tau \in I_j} \left\{ \left| \mathcal{M} \left( \sum_{\tau \in I_j} \langle T_{\delta}^{-1}f, |Q_{\tau}|^{-1/2} \psi_{j,\tau}^* \rangle \right) \right| \right\}^{1/2} \right\|_{L^1(\mathbb{R}^n)} \\
\leq C \ln \delta \left\| \left\{ \sum_j \left| \sum_{\tau \in I_j} \left| \mathcal{M} \left( \sum_{\tau \in I_j} \langle T_{\delta}^{-1}f, |Q_{\tau}|^{-1/2} \psi_{j,\tau}^* \rangle \right) \right| \right\}^{1/2} \right\|_{L^1(\mathbb{R}^n)} \\
\leq C \delta^{M(n-1)} \left\| \left\{ \sum_j \sum_{\tau \in I_j} \langle T_{\delta}^{-1}f, |Q_{\tau}|^{-1/2} \psi_{j,\tau}^* \rangle \right| \right\|_{L^1(\mathbb{R}^n)}^{1/2}.
\]
This completes the proof of Theorem 1.3.

6. Application: A maximal function characterization of $H^1_L(\mathbb{R}^n)$

In this section, we continue the discussion from Section 4 regarding a characterization of the Hardy space $H^1_L(\mathbb{R}^n)$ in terms of radial maximal function under the following conditions:

(H1)' $L$ is a second order non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$;

(H2)' The kernel of $e^{-tL}$, denoted by $p_t(x, y)$, is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies a Gaussian upper bound, that is
\[
|p_t(x, y)| \leq Ct^{-n/2} \exp \left( \frac{|x-y|^2}{ct} \right)
\]
for all $t > 0$, and $x, y \in \mathbb{R}^n$, where $C$ and $c$ are positive constants.

The space $H^1_L(\mathbb{R}^n)$ involves some different characterizations, see for examples, [2, 18, 19, 22, 27, 28, 29, 35, 36, 40]. If an operator $L$ satisfies conditions (H1)' and (H2)', then for any $M \geq 1$, $1 < q < \infty$, $f \in H^1_L(\mathbb{R}^n)$ implies $N_h f(x) := \sup_{|v| < ct} \left| e^{-tL}(y) \right| \in L^1(\mathbb{R}^n)$.
(6.1) \( f \) has an \((1, q, M)\) atomic decomposition \( f = \sum_{j=0}^{\infty} A_j a_j \) with \( \sum_{j=0}^{\infty} |A_j| < \infty. \)

Recall that a function \( a \in L^2(\mathbb{R}^n) \) is called an \((1, q, M)\)-atom associated to the operator \( L \) if there exist a function \( b \in D(L^M) \) and a ball \( B \subset \mathbb{R}^n \) such that (i) \( a = L^M b; \) (ii) \( \operatorname{supp} L^k b \subset B, k = 0, 1, \ldots, M; \) (iii) \( \| (r^2 L)^k b \|_{L^q(\mathbb{R}^n)} \leq r^2 B |B|^{1/q - 1/p}, k = 0, 1, \ldots, M. \)

Next, consider the following radial maximal operator associated to the heat semigroup generated by the operator \( L, \)

\[
(6.2) \quad f^+(x) = \sup_{r > 0} |e^{-r^2 L} f(x)|.
\]

Define the spaces \( H^1_{L,\max}(\mathbb{R}^n) \) as the completion of \( L^2(\mathbb{R}^n) \) in the norms given by the \( L^1(\mathbb{R}^n) \) norm of the maximal function, i.e., \( \| f \|_{H^1_{L,\max}(\mathbb{R}^n)} = \| f^+ \|_{L^1(\mathbb{R}^n)}. \) From (i) of (6.1), the following continuous inclusion holds:

\[
(6.3) \quad H^1_{L,\max}(\mathbb{R}^n) \subseteq H^1_1(\mathbb{R}^n).
\]

The aim of this section is to prove the following result.

**Theorem 6.1.** Suppose that an operator \( L \) satisfies \((H1)'\) and \((H2)'.\) In addition, we assume that the gradient estimate of the heat kernel of the semigroup \( \{e^{-tL}\}_{t>0} \) satisfies the pointwise bound

\[
(H3)' \quad |\nabla p_t(x, y)| \leq C t^{-(n+1)/2} \exp \left( -\frac{|x-y|^2}{ct} \right).
\]

Then we have that \( H^1_{L,\max}(\mathbb{R}^n) \subseteq H^1_1(\mathbb{R}^n) \) and hence by (6.3),

\[
H^1_{L,\max}(\mathbb{R}^n) \simeq H^1_1(\mathbb{R}^n).
\]

**Remark 6.2.** We should note that the equivalency of the radial maximal function characterization and the nontangential maximal function characterizations of \( H^1_1(\mathbb{R}^n) \) have been obtained in [40, 36]. Our Theorem 6.1 provides a different proof by using the frame decomposition.

The proof of Theorem 6.1 is based on the following lemma.

**Lemma 6.3.** Suppose that an operator \( L \) satisfies \((H1)', (H2)', and (H3)'.\) For any \( f \in H^1_{L,\max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \) there exists a constant \( C > 0 \) independent of \( f \) such that

\[
(6.4) \quad \left\| \sup_{|x-y| < 2t} |t \nabla e^{-r^2 L} f(y)| \right\|_{L^1(\mathbb{R}^n)} \leq C \| f \|_{H^1_{L,\max}(\mathbb{R}^n)}.
\]

**Proof.** First, we assume that \( f \in H^1_{L,\max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \) For any \( x, y \in \mathbb{R}^n \) and \( t > 0 \) with \( |x - y| < t, \) we apply Theorem 1.1 and Remark 1.5 with \( k > n \) to obtain

\[
|t \nabla e^{-r^2 L} f(y)| = \left| t \nabla e^{-r^2 L} \left( \sum_j \sum_{l \in I_j} (T_{\delta}^{-1} f, \psi_{j,l}) \psi_{j,l} \right)(y) \right|
\]

\[
= \ln \delta \left| \sum_j \sum_{l \in I_j} |Q_j| (T_{\delta}^{-1} f)(\cdot, (\delta^{-2} L)^k e^{-\delta^{-2} L} y_Q) t \nabla e^{-r^2 L} (\delta^{-2} L)^k e^{-\delta^{-2} L} (y, y_Q) \right|.
\]

From the property of the semigroup \( \{e^{-tL}\}_{t>0}, \) we can show that for every \( \eta > n, \)

\[
|t \nabla e^{-r^2 L} (\delta^{-2} L)^k e^{-\delta^{-2} L} (y, y_Q)| \leq C \left( \frac{\delta^{-2} L}{t^2} \right)^k \left( \frac{t}{\delta^{-2}} \right)^\eta \left( \frac{t}{\delta^{-2}} + |y - y_Q| \right)^{n+\eta}.
\]
Note also for some $\lambda < \min\{\eta, k\},$
\[
\sum_j \sum_{t \in I_j} \left(\frac{\delta^{-2j}}{t^2}\right)^k \wedge \left(\frac{t}{\delta^{-j}}\right) \frac{(t \vee \delta^{-j})^\eta}{(t \vee \delta^{-j} + |y - y_{Q_t^j}|)^{\eta+\eta}} \left(1 + \frac{|x - y_{Q_t^j}|}{\delta^{-j}}\right)^d |Q_t^j| \leq C.
\]
This yields
\[
\left| t \nabla e^{-r^2 L} f(y) \right| \\
\leq C \sum_j \sum_{t \in I_j} \left| \langle T_{\delta}^{-1} f(\cdot), (\delta^{-2j} L)^k e^{-\delta^{-2j} L} (\cdot, y_{Q_t^j}) \rangle \right| \left(\frac{\delta^{-2j}}{t^2}\right)^k \wedge \left(\frac{t}{\delta^{-j}}\right) \frac{(t \vee \delta^{-j})^\eta}{(t \vee \delta^{-j} + |y - y_{Q_t^j}|)^{\eta+\eta}} |Q_t^j| \\
\leq C \sup_{j \in I} \left| \langle T_{\delta}^{-1} f(\cdot), (\delta^{-2j} L)^k e^{-\delta^{-2j} L} (\cdot, y_{Q_t^j}) \rangle \right| \left(1 + \frac{|x - y_{Q_t^j}|}{\delta^{-j}}\right)^{-\lambda} \\
\times \sum_j \sum_{t \in I_j} \left(\frac{\delta^{-2j}}{t^2}\right)^k \wedge \left(\frac{t}{\delta^{-j}}\right) \frac{(t \vee \delta^{-j})^\eta}{(t \vee \delta^{-j} + |y - y_{Q_t^j}|)^{\eta+\eta}} \left(1 + \frac{|x - y_{Q_t^j}|}{\delta^{-j}}\right)^d |Q_t^j| \\
\leq C \sup_{j \in I} \left| \langle T_{\delta}^{-1} f(\cdot), (\delta^{-2j} L)^k e^{-\delta^{-2j} L} (\cdot, y_{Q_t^j}) \rangle \right| \left(1 + \frac{|x - y_{Q_t^j}|}{\delta^{-j}}\right)^{-\lambda}.
\]
Using Theorem 2.3 in [8], we decompose the cubes $\{Q_t^j\}$ into annuli according to the distance $|x - y_{Q_t^j}|$ with respect to $\delta^{-2j}$ to obtain
\[
\left\| \sup_{|x-y|<2r} |t \nabla e^{-r^2 L} f(y)| \right\|_{L^1(\mathbb{R}^n)} \leq C \left\| \sup_{j \in I} \left| \langle T_{\delta}^{-1} f(\cdot), (\delta^{-2j} L)^k e^{-\delta^{-2j} L} (\cdot, y_{Q_t^j}) \rangle \right| \left(1 + \frac{|x - y_{Q_t^j}|}{\delta^{-j}}\right)^{-\lambda} \right\|_{L^1(\mathbb{R}^n)} \\
\leq C \|N_h \left(T_{\delta}^{-1} f\right)\|_{L^1(\mathbb{R}^n)} \\
= C \|T_{\delta}^{-1} f\|_{H^k(\mathbb{R}^n)} \\
\leq C \|f\|_{H^k(\mathbb{R}^n)},
\]
since $\lambda > n.$ By a density argument, we obtain (6.4). Hence, the proof of Lemma 6.3 is complete. □

**Proof of Theorem 6.1.** By (i) of (6.1), it suffices to show
\[
\|N_h f\|_1 \leq C \|f^*_h\|_1.
\]
To prove this, we use the ideas in [8]. We claim that, there exists a constant $A$, independent of $f$ and $r$, such that the following inequality holds:
\[
\|N_h f\|_1 \leq 2Ar^{-n}\|f^*_h\|_1 + r \left\| \sup_{|x-y|<2r} |t \nabla e^{-r^2 L} f(y)| \right\|_1
\]
for any $r \in (0, 1].$ If the claim (6.6) is proven, we then apply Lemma 6.3 to get $\|N_h f\|_1 \leq 2Ar^{-n}\|f^*_h\|_1 + rC_2\|N_h f\|_1.$ Finally, we can choose $r$ small enough so that $C_2 r < 1$ to obtain
\[
\|N_h f\|_1 \leq \frac{2A}{1 - rC_2} r^{-n}\|f^*_h\|_1
\]
as desired.

To prove the claim (6.6), it suffices to show the following inequality:
\[
\left| \{N_h f > s\} \cap \{ \sup_{|x-y|<2r} |t \nabla e^{-r^2 L} f(y)| \leq sr^{-1} \} \right| \leq Ar^{-n}\|f^*_h\|_1.
\]
for any $0 < r \leq 1$ and $s > 0$. Indeed, if (6.7) is proven, one writes
\[
\|N_h f\|_1 = \int_0^\infty \|\{N_h f > s\}\| ds
\]
\[
\leq Ar^{-n} \int_0^\infty \|f^+ > s/2\| ds + \int_0^\infty \|\sup_{|x-y|<2r} |r\nabla e^{-r^2L} f(y)| > sr^{-1}\| ds
\]
\[
= 2Ar^{-n} \|f^+\|_1 + r \|\sup_{|x-y|<2r} |r\nabla e^{-r^2L} f(y)|\|_1.
\]

Now, let us verify (6.7). Let $x_0 \in \{N_h f > s\} \cap \{\sup_{|x-y|<2r} |r\nabla e^{-r^2L} f(y)| \leq s/r\}$. Then there exist $y_0 \in \mathbb{R}^n$ and $t_0 > 0$ such that $|x_0 - y_0| < t_0$, $|e^{-t_0^2L} f(y_0)| > s$ and $|t_0 \nabla e^{-t_0^2L} f(z)| \leq s/r$ whenever $|x_0 - z| < 2t_0$. Note that
\[
|e^{-t_0^2L} f(z)| \geq |e^{-t_0^2L} f(y_0)| - |e^{-t_0^2L} f(y_0) - e^{-t_0^2L} f(z)|
\]
\[
\geq |e^{-t_0^2L} f(y_0)| - |\nabla e^{-t_0^2L} f(\xi)||y_0 - z|
\]
for some $\xi$ which lies in between $y_0$ and $z$. If $|z - y_0| < \frac{s}{2}t_0$ and $r \in (0, 1]$, then $|z - x_0| < 2t_0$. It tells us that
\[
|e^{-t_0^2L} f(z)| \geq |e^{-t_0^2L} f(y_0)| - \frac{s}{t_0r}|z - y_0| \geq s - \frac{s}{2} = \frac{s}{2},
\]
which implies that $B(y_0, \frac{t_0r}{2}) \subset \{f^+ > s/2\}$. This, in combination with the fact that
\[
\frac{|B(x_0, 2t_0) \cap \{f^+ > s/2\}|}{|B(x_0, 2t_0)|} \geq \frac{|B(y_0, \frac{t_0r}{2})|}{|B(x_0, 2t_0)|} = \frac{r^n}{4^n},
\]
yields
\[
(6.8) \quad \{N_h f > s\} \cap \left\{\sup_{|x-y|<2r} |r\nabla e^{-r^2L} f(y)| \leq s/r\right\} \subset \{M_N f^+ > s/2\} > r^n/4^n.
\]
By the weak $(1,1)$ boundedness of $\mathcal{M}$,
\[
\left|\{N_h f > s\} \cap \left\{\sup_{|x-y|<2r} |r\nabla e^{-r^2L} f(y)| \leq s/r\right\}\right| \leq Ar^{-n} \|f^+ > s/2\|,
\]
which proves (6.7). The proof of Theorem 6.1 is complete. 

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Department of Mathematics, Macquarie University, NSW, 2109, Australia
E-mail address: xuan.duong@mq.edu.au

Department of Mathematics, Macquarie University, NSW, 2109, Australia
E-mail address: ji.li@mq.edu.au

Department of Mathematics, Sun Yat-sen University, China
E-mail address: songl@mail.sysu.edu.cn

Department of Mathematics, Sun Yat-sen University, China
E-mail address: mcsylx@mail.sysu.edu.cn