Quantum criticality of granular SYK matter

Alexander Altland,1 Dmitry Bagrets,1 and Alex Kamenev2

1Institut für Theoretische Physik, Universität zu Köln, Zülpicher Straße 77, 50937 Köln, Germany
2W. I. Fine Theoretical Physics Institute and School of Physics and Astronomy, University of Minnesota, Minneapolis, MN 55455, USA

(Dated: March 25, 2019)

We consider granular quantum matter defined by Sachdev-Ye-Kitaev (SYK) dots coupled via random one-body hopping. Within the framework of Schwarzian field theory, we identify a zero temperature quantum phase transition between an insulating phase at weak and a metallic phase at strong hopping. The critical hopping strength scales inversely with the number of degrees of freedom on the dots. The increase of temperature out of either phase induces a crossover into a regime of strange metallic behavior.

Introduction: Despite decades of research, our understanding of strongly correlated (‘non-Fermi liquid’) quantum matter with metallic parent states remains incomplete. A universal feature of these materials is that seemingly incongruent phases of matter — superconducting, insulating, poorly conducting, metallic, etc. — coexist in close parametric proximity to each other[1]. The understanding of this diversity of competing phases, which finds its most prominent manifestations in the physics of the cuprates[2] or heavy fermion materials[3], requires universal blueprints of correlated fermion matter transcending the Landau quasiparticle paradigm. Recently, systems of coupled Sachdev-Ye-Kitaev (SYK)[4–15] quantum dots have gained popularity in this regard. What makes these systems interesting is that a hallmark of many correlated fermion materials — crossover from a strange metal (SM) phase to a Fermi liquid (FL) upon lowering temperatures — is generated within a very simple mean field picture[6], which assumes the individual SYK cells to contain a thermodynamically large number \( N \to \infty \) of quantum particles. In this paper, we do not take this limit and explore what happens in ‘mesoscopic’ setups where \( N \) is large but finite. Our main finding is that the phase diagram becomes significantly more interesting and now features a zero temperature insulator–FL transition at a critical value of the inter-dot coupling inversely proportional to \( N \). Extending the analysis to finite temperatures, we find an insulator/SM/FL phase separation as shown in (see Fig.1). Competitions of this type are seen in many contexts, indicating that the mesoscopic SYK network may capture essential ingredients for the phenomenological description of the correlated fermion matter.

The SYK model[4–5] is a system of \( N \) Majorana fermions, \( \eta_i, i = 1 \ldots N \), subject to an all-to-all four fermion interaction

\[
H_{\text{SYK}} = (1/4!) \sum_{ijk} J_{ijk} \eta_i \eta_j \eta_k \eta_l
\]

with Gaussian distributed matrix elements \( J_{ijk} \) of variance \( 3 J^2 / N^3 \). The system can be seen as a spatially local, zero dimensional paradigm of strongly interacting quantum matter: In the limit \( N \to \infty \), the absence of a single particle term in the Hamiltonian implies that the fermion operators carry dimension \( \text{time}^{-1/4} \), in marked distinction to the FL dimension \(-1/2\). This motivates the extension to a \( d \)-dimensional array of nearest neighbor coupled non Fermi liquid cells. In view of the inherent randomness, it is natural to model the coupling by one-body operators

\[
H_T = (i/2) \sum_{\langle ab \rangle, ij} V_{ij}^{ab} \eta_i^a \eta_j^b,
\]

where \( a, b \) label the individual dots, and \( V_{ij}^{ab} \) are Gaussian distributed with variance \( V^2 / N \). Importantly, this coupling is a relevant perturbation of dimension \( \int d \tau \eta_i \eta_i \) = \( 1 - 2 \times 1/4 = +1/2 \). It implies a crossover from a non-FL ‘strange’ metal at high temperatures to a conventional, yet strongly renormalized, FL metal at low temperatures[6].

The above scenario makes reference to the engineering dimensions of the fermion operators and becomes valid in the thermodynamic limit. We here consider what happens in ‘mesoscopic’ models with large but finite \( N \), where very different behavior at low temperatures is expected. The non-FL nature of an isolated SYK dot manifests itself in an infinite dimensional ‘conformal’ symmetry[16–20] under continuous reparameterizations of time. The above scaling dimension \(-1/4\) reflects the breaking of this symmetry at the large \( N \).
mean-field level. However, as temperature is lowered below the energy scale \(J/N\), strong Goldstone fluctuations associated to the conformal symmetry ensue, and effectively change the dimension of the fermion operator to \(-3/4\)\(^t\left[18\right]\left[19\right]\left[21\right]\left[22\right]\). In this low energy regime, a single particle perturbation has dimension \(1 - 2 \times 3/4 = -1/2\) and now is RG irrelevant\(^{23}\).

This dimensional crossover implies a competition between inter-dot couplings and intra-dot quantum fluctuations: depending on the bare strength of the coupling, Goldstone modes are either suppressed, or render the inter-dot coupling irrelevant. This implies the existence of a metal-insulator quantum phase transition (QPT) separating a phase of a strongly coupled FL from an insulating phase of essentially isolated dots. Below, we will explore this QPT within the framework of an effective low energy field theory characterizing granular SYK matter in terms of two coupling constants, representing intra-dot interaction and inter-dot coupling strength, respectively. We will demonstrate the renormalizability of the theory and from the flow of coupling constants (cf. Fig. 3 below) derive the manifestations of quantum criticality in two temperature scales marking an insulator/SM and FL/SM crossover at weak and strong coupling, respectively, cf. Fig. 1.

On general grounds we expect this scenario to extend to models of interacting complex fermions, the SY model\(^{11,16}\). However, the presence of U(1)-mode associated with particle number conservation in the SY system makes the theory more complicated. We here prefer to sidestep this complication and expose the relevant physics within the SYK framework, unmasked by the U(1) phase fluctuations\(^{24}\). In this system of electrically neutral Majorana fermions, thermal conductivity, \(\kappa(T)\), is the main signature of transport, and from the Wiedemann-Franz law we infer that the ratio \(T/\kappa\) plays a role analogous to the electrical resistivity of complex fermion matter. We find that in the insulating phase it exhibits a minimum before diverging at small \(T\) as \(T/\kappa(T) \propto 1/T\) (cf. bottom left inset in Fig. 1). In the SM (FL) phase \(T/\kappa\) ratio exhibits \(T\)-linear (approximately \(T\)-independent) behavior, respectively.

The model: we consider a system described by the Hamiltonian\(^{6}\)

\[
H = \frac{1}{4!} \sum_{ijkl} J^{a}_{ijkl} \eta_{ij}^{a} \eta_{kl}^{a} + \frac{i}{2} \sum_{ab} \sum_{ij} V_{ij}^{ab} \eta_{ij}^{a} \eta_{ij}^{b}
\]

where the mutually uncorrelated Gaussian distributed coefficients \(J^{a}_{ijkl}\) and \(V_{ij}^{ab}\) have been specified above. Following a standard procedure\(^{16,20}\), the theory averaged over the coupling constant distributions is described by an imaginary time functional \(Z = \int D(G, \Sigma) \exp(-S[G, \Sigma])\), where \(G = \{G_{\tau_1, \tau_2}\}\) and \(\Sigma = \{\Sigma_{\tau_1, \tau_2}\}\) are time bi-local integration fields playing the role of the on-site SYK Green function and self-energy, respectively. The action \(S[G, \Sigma] = \sum_{a} S_{0}[G^{a}, \Sigma^{a}] + \sum_{(ab)} S_{T}[G^{a}, G^{b}]\), contains the ‘\(\Sigma\)-action’, \(S_{0}\), of the individual dots, and a tunneling action \(S_{T}[G^{a}, G^{b}] = \frac{1}{4} N V^{2} \int d\tau_{1} d\tau_{2} G_{\tau_1, \tau_2} G_{\tau_2, \tau_1}^{b}\) describing the nearest neighbor hopping. Here, we omit a replica structure\(^{22}\) technically required to perform the averaging, but inessential in the present context.

While the explicit form of the \(\Sigma\)-action\(^{20}\) will not be needed, the following points are essential: (i) the action \(S_{0}\) is approximately invariant under reparameterizations of time\(^{11,16,20}\), \(h : S^{1} \rightarrow S^{1}, \tau \mapsto h(\tau)\), where \(h\) is a diffeomorphism of the circle, \(S^{1}\), defined by imaginary time with periodic boundary conditions onto itself. The infinite dimensional symmetry group \(\text{diff}(S^{1})\) of these transformations is generated by a Virasoro algebra, hence the denotation ‘conformal’. (ii) The symmetry is subject to a weak explicit breaking by the time derivatives present in the action \(S_{0}\). For low energies, the corresponding action cost is given by\(^{11,16,20,27,29}\)

\[
S_{0}[h] = -m J^{\beta} d\tau \{h, \tau\}, \quad \{h, \tau\} = \frac{(h''(\tau))^{\beta} - \frac{1}{2} h''(\tau)^{2}}{4}\]

The Schwarzian derivative, and \(m \propto N/J\) the first of two coupling constants defining the model. For temperature scales \(T < m^{-1}\) even large deviations, \(h\), away from \(h(\tau) = \tau\) may have low action. This marks the entry into a low temperature regime dominated by strong reparameterization fluctuations. Finally, (iv) the mean-field Green function \(G_{\tau_1, \tau_2} = \{\tau_1 - \tau_2\}^{-1/2}\) (the square root dependence reflects the non-FL dimension of the fermions) transforms under reparameterizations as

\[
G_{\tau_1, \tau_2} \rightarrow G_{\tau_1, \tau_2}[h] = \left( \frac{h_{1}^{1/2} h_{2}^{1/2}}{(h_{1} - h_{2})^{2}} \right)^{1/4}, \quad (2)
\]

where e.g. \(h_{1} \equiv h(\tau_{1})\) and \(h_{1}' \equiv dh(\tau)/d\tau|_{\tau=\tau_{1}}\). For an isolated dot, integration over the \(h\)-fluctuations effectively changes the Green function to \(\langle G_{\tau_1, \tau_2}[h]\rangle_{h \frac{m T^{2}}{4} < 1 m|\tau_1 - \tau_2|^{-3/2}}\), corresponding to a change of the fermion operator dimension to \(3/4\)\(^{18,19,21,23}\). The effective low-energy lattice Schwarzian theory is formulated in terms of the reparameterizations \(h(\tau)\) effective on different dots. Its action \(S[h] = S_{0}[h] + S_{T}[h]\), is defined through

\[
S_{0}[h] = -m \sum_{a} \int d\tau \{h^{a}, \tau\}, \quad (3)
\]

\[
S_{T}[h] = -w \sum_{(ab)} \int d\tau_{1} d\tau_{2} \left\{ \left( \frac{h_{1}^{a} h_{2}^{a}}{h_{1}^{a} - h_{2}^{a}} \right)^{2} \times \left( \frac{h_{1}^{b} h_{2}^{b}}{h_{1}^{b} - h_{2}^{b}} \right)^{2} \right\}^{1/4},
\]

where \(m\) and \(w\) are parameters with dimensions of [time] and [energy], and bare values \(m \propto N/J\) and \(w \propto N V^{2}/J\).

A hallmark of the lattice Schwarzian action, \(S[h]\), is its invariance under actions of \(\text{SL}(2, R)\), where the group is represented via the Möbius transformations \(h(\tau) = \frac{\alpha \tau + \beta}{\gamma \tau + \delta}\) with \(\alpha \delta - \beta \gamma = 1\). This shows that the \(h\)-transformations to be integrated over the coset space \(\text{diff}(S^{1})/\text{SL}(2, R)\).
The action itself is built from the two simplest SL(2, R) invariant blocks: local \( \{h, \tau\} \) and bi-local \( h'_i h'_j / [h_1 - h_2]^2 \). Maintained SL(2, R) symmetry imposes a stringent condition on the behavior of the theory under renormalization. A successive integration over \( h \)-transformations must leave the local and bi-local terms form invariant (multi-point terms may be generated but are irrelevant).

The invariance condition thus implies that the renormalization results in a flow of the two couplings \( m \) and \( w \).

**RG analysis:** we decompose fluctuations into ‘fast’ and ‘slow’ as \( h(\tau) = f(s(\tau)) \equiv (f \circ s)(\tau), \) where \( f \) and \( s \) are fluctuations in the frequency range \( \lfloor \Lambda, J \rfloor \) and \( 0, \Lambda \), and \( \Lambda \) is a running cutoff energy [30]. We then integrate out the fast modes \( f(s) \), and rescale time \( \tau \rightarrow \tau J/\Lambda \) to restore the UV cutoff \( \Lambda \rightarrow J \). Consider first the case \( m^{-1} < \Lambda < J \), where the reparameterization fluctuations are suppressed. The RG flow is then governed by the ‘engineering’ dimensions, resulting in:

\[
\frac{d \ln m}{dl} = -1; \quad \frac{d \ln w}{dl} = +1, \tag{4}
\]

where \( l = \ln(J/\Lambda) \). For \( T > J/N \) this flow should be terminated when either \( \Lambda \) reaches \( T \), or \( V(l) \sim \sqrt{w(l)} \) reaches UV cutoff \( J \). This defines the temperature scale \( T_F = V^2/J \), which separates the high temperature SM and low temperature FL. In SM phase \( w(T) = N V^2/T \) and \( T/\kappa(T) \propto J/w(T) \propto J/(NT_F) \) [3], while in FL the thermal resistivity saturates at \( T/\kappa(T) \propto 1/N \).

We turn now to the regime of strong reparameterization fluctuations. By the time \( \Lambda \) reaches \( J/N \), \( m(l) = m(0)e^{-l} \) reaches the inverse UV cutoff \( m(l) \approx 1/J \). To proceed with the further renormalization, we employ the Schwarzian chain rule

\[
\{f \circ s, \tau\} = (s')^2 \{f, s\} + \{s, \tau\}, \tag{5}
\]

to obtain the action: \( S_0[f \circ s] = S_0^{\text{fast}}[f, s] + S_0[s] \), where \( S_0^{\text{fast}}[f, s] = -\sum_n f(m^n(s) \{f^n, s\}) ds \), and \( m^n(s) \equiv m^{-n} \).

At lowest order in \( w \) one needs to average the coupling action \( S_T[f \circ s] \) over the fast fluctuations with the weight \( e^{-s^{\text{fast}}[f, s]} \). A straightforward application of the chain rule to the Green functions, Eq. (3), shows that

\[
G_{\tau_1, \tau_2}[f \circ s] = G_{s_1, s_2}[f](s_1' s_2')^{1/4}, \tag{6}
\]

so that \( \langle S_T[f \circ s]\rangle_f \propto \langle G_{s_1, s_2}[f^2]\rangle_f \propto \langle G_{s_1, s_2}[f^3]\rangle_f \) splits into two fast averages. These expressions can be evaluated with the help of exact results [18] [21] for \( \langle G_{s_1, s_2}[f]\rangle_f \). Referring to the supplementary material for details, we note the asymptotic expressions \( s_{12} \equiv s_1 - s_2 \):

\[
\langle G_{s_1, s_2}[f]\rangle_f \approx \begin{cases} 
|s_{12}|^{-1/2}, & s_{12} < m; \\
\sqrt{m(s_1) m(s_2)} |s_{12}|^{-3/2}, & m < s_{12} < \Lambda^{-1}; \\
m\Lambda |s_{12}|^{-1/2}, & \Lambda^{-1} < s_{12}.
\end{cases} \tag{7}
\]

The first line states that at time scales shorter than \( m \) reparameterization fluctuations are suppressed and \( G \) retains its mean-field form. For intermediate time scales, they change the time dependence of \( G \) to a \( -3/2 \) power law. At yet longer times, fast reparameterization fluctuations cease to be effective and \( G \) is back to \(-1/2\) decay.

Equation (7) implies that the double time integral in the averaged tunneling action \( \langle S_T[f \circ s]\rangle_f = S_{\text{int}} + S_{\text{long}} \) gets different contributions from intermediate \( m < \tau_{12} < \Lambda^{-1} \) and long time differences \( \tau_{12} > \Lambda^{-1} \). The former is handled with the help of the expansion

\[
\left( \frac{s_{12}'}{s_{12}} \right)^{\Delta} \approx \frac{1}{|\tau_1 - \tau_2|^2} + \frac{\Delta}{6} \left[ \frac{s(\tau)}{\tau - \tau_{12}} \right]^{2\Delta - 2} + \cdots, \tag{8}
\]

where \( \tau = (\tau_1 + \tau_2)/2 \). Collecting all factors \( s' \) from Eqs. (6) and (7), we encounter the l.h.s. of this equation with \( \Delta = 3/4 \), both for \( s = s_{\alpha \beta} \). Substituting the r.h.s. and noting that the contribution of lowest order in derivatives comes from the cross contribution of the first and the second term in the \( (a) \times (b) \)-product, we obtain a contribution of local Schwarzian form \( S_{\text{int}} \rightarrow \frac{2}{\pi} w m^2 l \sum_a \int_{-\Lambda}^\Lambda d\tau \{s^a, \tau\} \), where \( Z \) is the coordination number of the array and \( l \equiv \ln(1/m) = \int_m^{\Lambda^{-1}} d\tau \tau^{-2 - 4\Delta} \) from the time integration. Finally, rescaling the time variable \( \tau \rightarrow e^{\tau} \), to reset the cutoff \( \Lambda^{-1} \rightarrow m \), we observe that the action contains a term \( S_0[s] \) with the renormalized coefficient \( m(l) = -e^{-l}(m + \frac{2}{\pi} w m^2 l) \). We finally turn to the contribution, \( S_{\text{long}} \), from large time differences, where now the third line in Eq. (7) for the Green functions is to be used. After the rescaling of time, this generates an expression identical to the original \( S_T[s] \), but with a new constant \( w(l) = e^{l}(m \Lambda)^2 = e^l w e^{2 l} \).

From these results, RG equations are obtained by differentiation over \( l \) and putting \( l = 0 \). This leads to

\[
\frac{d \ln m}{dl} = -1 + \frac{Z}{4} w m; \quad \frac{d \ln w}{dl} = +1 - 2. \tag{9}
\]

The second equation reflects the aforementioned change of the dimension of \( w \) from \(+1\) to \(-1\). While Eqs. (1) are applicable for \( m J \gg 1 \), the new set of the RG equations (9) is derived in the opposite limit \( m J \ll 1 \). Indeed, this is the condition to employ the asymptotic expressions (7), as opposed to the exact expressions [18] [21], see the supplementary material for more information.

**Analysis of the RG:** we first note that the limiting forms of the scaling equations, Eqs. (1) and (9), admit a closed representation in the dimensionless variable \( \lambda \equiv w m \). In the regime \( m J \gg 1 \) one has \( d \ln \lambda / dl = 0 \), while for \( m J \ll 1 \):

\[
\frac{d \ln \lambda}{dl} = \frac{Z}{4} \left[ \lambda - 2 \right]. \tag{10}
\]

This equation exhibits an unstable fixed point \( \lambda_c = 8 Z \), marking a transition between a FL phase at \( \lambda > \lambda_c \) and
an insulating one at $\lambda < \lambda_c$. Since $\lambda(0) \sim (NV/J)^2$, one finds $V_c \sim J/\sqrt{ZN}$, inversely proportional to $N$, as stated in the introduction. Notice that according to Eq. (9), $d\ln m/dl|_{\lambda=\lambda_c} = +1$, opposite to Eq. (4). The only way to reconcile the two limits is to have another fixed point at $m_c \sim 1/J$. The resulting two parameter RG diagram in the plane $(\lambda, m)$ is shown in Fig. 2.

To first order in an expansion in $w$, but arbitrary $m$, this diagram may be derived from exact expressions for $\langle G_{s_1,s_2}[f] \rangle_f$, see supplementary material for details. The analysis of higher orders in $S_T$ shows that the actual small parameter of the perturbative expansion is $Z\lambda$. Therefore the fixed point is actually out of the perturbatively contolred regime and may not be used for quantitative evaluation of critical indices. However, second order calculations show that RG flow keeps its qualitative form, Fig. 2.

The FL part of the RG diagram, Fig. 2, is well described by Eqs. (4) and the physics of the array is the one discussed in Ref. [6]. The only addition is that the crossover temperature $T_{c,FL}(\lambda) \to 0$, when $\lambda \to \lambda_c$, Fig. 2. This is due to the fact that for $\lambda \approx \lambda_c$ the flow spends a long “time” in the vicinity of the $(\lambda_c, m_c)$ fixed point, thus reaching progressively lower $T$. In the insulating phase, $\lambda \to 0$ and thus according to $dw/dl = -w$ (Eq. [9]) and $w \sim V^2$, $V(T) \propto T^{1/2}$. The diminishing of the inter-dot coupling at low temperatures implies that second order perturbation theory in $V(T)$ may be applied to evaluate the thermal conductivity $\kappa(T)$. Therefore one finds $\kappa(T)/T \propto |V(T)|^2 \propto T$ in the insulating phase.

To conclude, we have seen that the renormalization procedure indeed preserves the form of the lattice Schwarzian field theory. This stability follows from the conformal relations and, but ultimately is required by the condition of maintained SL(2, R) symmetry. Our ability to deduce the entire RG flow (for $Z\lambda \lesssim 1$) is owed to the knowledge of the reparameterization averaged Green function $\langle G[f] \rangle_f$ for any $m$, which in turn follows from mapping of the local Schwarzian to Liouville quantum mechanics [18].

**Summary** — In this work we have shown that, regardless of dimensionality and geometric structure, an array of SYK dots coupled by one-body hopping exhibits a zero temperature metal-insulator transition. This phenomenon is rooted in the conformal invariance of the non-FL states supported by the individual SYK dots. The presence of this symmetry in turn is a direct consequence of an asymptotically strong dot-local interaction and may transcend the specific model employed here. A mutually suppressive competition between conformal fluctuations on the dots and the conformal symmetry breaking tunneling operators then enforces the presence of a transition between an insulating and a metallic phase, and crossover into a strange metal regime at finite temperatures. Read in this way, the main message of our study is that phenomenology present in many strongly correlated materials, may follow from a rather basic principle. This makes one optimistic that further features of quantum matter can be modeled via the SYK paradigm.

**Acknowledgements** — We are grateful to K. Tikhonov for useful discussions. Work of AA and DB was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Projektnummer 277101999 TRR 183 (project A03). AK was supported by the DOE contract DEFG02-08ER46482.

---

[1] E. Fradkin, S. A. Kivelson, and J. M. Tranquada, Rev. Mod. Phys. 87, 457 (2015)
[2] B. Keimer, S. A. Kivelson, M. R. Norman, S. Uchida, and J. Zaanen, Nature 518, 179 (2015)
[3] Q. Si and F. Steglich, Science 329, 1161 (2010). http://science.sciencemag.org/content/329/5996/1161.full.pdf
[4] A. Kitaev, “A simple model of quantum holography,” http://www.arXiv.org/cond-mat/9212030
[5] S. Sachdev and J. Ye, Phys. Rev. Lett. 70, 3339 (1993). arXiv:cond-mat/9210203
[6] X.-Y. Song, C.-M. Jian, and L. Balents, Phys. Rev. Lett. 119, 216601 (2017) arXiv:1705.00117
[7] R. A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen, and S. Sachdev, Phys. Rev. B 95, 155131 (2017) arXiv:1612.00849
[8] X. Chen, R. Fan, Y. Chen, H. Zhai, and P. Zhang, Phys. Rev. Lett. 119, 207603 (2017) arXiv:1705.03406
[9] M. Berkoos, P. Narayanan, M. Rozali, and J. Simon, JHEP 01, 138 (2017) arXiv:1610.02422
[10] A. Haldar, S. Banerjee, and V. B. Shenoy, Phys. Rev. B 97, 241106 (2018)
[11] X.-Y. G.-H. Cai, Wenheud Ge. Journal of High Energy Physics 2018, 76 (2018) arXiv:1711.07903
[12] D. V. Khveshchenko, Condensed Matter 3 (2018), 10.3390/condmat3000400
[13] R. A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen, and S. Sachdev, Phys. Rev. B 95, 155131 (2017)
Quantum criticality of granular SYK matter: supplementary material

In this supplementary material we provide some technical background on the lattice Schwarizian field theory discussed in the main text.

Effective action and stationary phase solutions — Averaging the Grassmann coherent state path integral representation of the SYK Hamiltonian over the Gaussian distributions of matrix elements $J_{ijkl}$ and $V_{ab}$ and subsequently integrating out the Grassmann variables, one obtains two contributions to the action:

$$ S_0[G^a, \Sigma^a] = -\frac{N}{2} \sum_a [\text{tr} \ln (\partial_\tau + \Sigma^a) - \frac{2}{\sqrt{3}} \int d\tau_1 d\tau_2 (G^a_{\tau_1, \tau_2} \Sigma^a_{\tau_2 - \tau_1} + \frac{J^2}{4} (G^a_{\tau_1, \tau_2})^4)] ;$$

and

$$ S_T[G^a, \Sigma^a] = \frac{1}{2} N \sum_{(ab)} V^2 \int d\tau_1 d\tau_2 G^a_{\tau_1, \tau_2} G^{b}_{\tau_2, \tau_1} ,$$

where we have suppressed the replica structure of the fields. The global factor $N$ upfront justifies a saddle point approach based on variational solutions for $G$ and $\Sigma$. To zeroth order in $\partial_\tau$ and $V^2$, one finds a family of conformally invariant solutions, parameterized by diffeomorphisms $h(\tau)$:

$$ G_{\tau_1, \tau_2} \propto \frac{(h_1' h_2')^{1/4}}{[h_1 - h_2]^{1/2}} ; \quad \Sigma_{\tau_1, \tau_2} \propto \frac{(h_1' h_2')^{3/4}}{[h_1 - h_2]^{3/2}} .$$

The terms coupled to $\partial_\tau$ and $V^2$ break the Diff($S^1$) symmetry down to SL(2, $R$), and the corresponding action cost is given by Eq. (3) of the main text. Eq. (3) thus defines the low-energy effective action of the SYK array.

RG analysis — We now decompose the $h(\tau)$ fluctuations into ‘fast’ and ‘slow’ as $h(\tau) = f(s(\tau)) \equiv (f \circ s)(\tau)$. The fast part, $f(s)$, includes fluctuations in the frequency range $[\Lambda, J]$, and the slow one, $s(\tau)$, the remaining modes with frequencies less than $\Lambda$. As long as $m^{-1} < \Lambda$, the action cost of fast modes is high and their integration has no bearing on the slow action. Effective renormalization sets as $\Lambda < m^{-1}$. To first order in $w$ one needs to consider $\langle S_T[(f \circ s)] \rangle_f$, which takes the following form (cf. Eq. (5) of the main text):

$$ \langle S_T \rangle_f = -w \sum_{(ab)} \int d\tau_1 d\tau_2 \langle G_{s_1^a, s_2^a}^a \rangle_f \times (a \rightarrow b) ,$$

where the Green function averaged over the fast degrees of freedom is

$$ \langle G_{s_1, s_2} \rangle_f \equiv \left\langle \frac{(f(s_1))^{1/4} - (f(s_2))^{1/2}}{[f(s_1) - f(s_2)]^{1/2}} \right\rangle_f ,$$

and $\langle \ldots \rangle_f$ stands for the integration over the functions $f(s)$ with the weight $S_0^{\text{fast}}[f, s]$. As discussed below, the

![FIG. 3. The log-log plot of the fast Green’s functions $G_f(s)$ versus time $s$ used in the RG analysis.](image-url)
resulting reparameterization-averaged Green function acquires the asymptotic form (see Fig. 3),
\[ G_f(s_1^a, s_2^a) \simeq \frac{m_\lambda^{1/2}(s_1^a)m_\lambda^{1/2}(s_2^a)}{|s_1^a - s_2^a|^{1/2}}, \quad m < |s_1^a - s_2^a| < \Lambda^{-1} \]
for long times. Indeed, the low-frequency spectrum of \( f(s) \) is cut off by \( \Lambda \), and one expects the Green function at longer times (\( > \Lambda^{-1} \)) to turn back to its mean-field form with the original fermion dimension 1/4. The suppression factor \( m_\lambda < 1 \) accounts for the drop of the Greens function in the intermediate time range. In Eq. (16) we have evenly split the mass \( m_a(s^a) = m a^a \) between the two times, \( s_1^a \) and \( s_2^a \), which is permissible on account of the assumed slowness of \( s^a \) and leads to manifest SL(2, R) invariance of the slow modes action.

The average action (14) acquires contributions from intermediate and long times ranges, \( \tau_1 - \tau_2 \), respectively,
\[ S_{\text{int}} = -w m^2 \int_{m < |\tau_1| < \Lambda^{-1}} d\tau_1 d\tau_2 \left( \frac{s^{a}(\tau_1) s^{a}(\tau_2)}{|s^a(\tau_1) - s^a(\tau_2)|^2} \right)^{3/4} \times (a \to b). \] (18)
and
\[ S_{\text{long}} = -w(m_\lambda)^2 \int_{|\tau_1| > \Lambda^{-1}} d\tau_1 d\tau_2 \left( \frac{s^{a}(\tau_1) s^{a}(\tau_2)}{|s^a(\tau_1) - s^a(\tau_2)|^2} \right)^{1/4} \times (a \to b). \] (19)
Rescaling time, \( \tau'_{1,2} = \tau_{1,2} m_\lambda = \tau e^{-l} \), and applying the procedure described in the main text, one obtains renormalized actions \( S_0[s] \) and \( S_T[s] \), where the coupling constants obey the RG equations
\[ \frac{d \ln m}{dl} = -1 + \frac{Z}{4} mw, \quad \frac{d \ln w}{dl} = -1, \quad m J \ll 1. \] (20)

On the other hand, when \( m J \gg 1 \) the renormalization is is only due to the engineering dimensions,
\[ \frac{d \ln m}{dl} = -1, \quad \frac{d \ln w}{dl} = +1, \quad m J \gg 1. \] (21)
A way to interpolate between the two limits (20) and (21) is to define an effective \( m \)-dependent scaling dimension of the fermion operators as, Fig. 4
\[ \Delta_\psi(m) = -\frac{1}{2} \left. \frac{d \ln G(s)}{d \ln s} \right|_{s=1/m}, \] (22)
where the exact two-point Green’s function is (18)
\[ G(s) \propto \frac{1}{\sqrt{m}} \int_0^{+\infty} dk \mathcal{M}_2(k) e^{-k^2 s/2 m}, \] (23)
\[ \mathcal{M}_2(k) = k \sinh(2\pi k)\Gamma^2(\frac{1}{4} + ik)\Gamma^2(\frac{1}{4} - ik). \]
The function \( \Delta_\psi(m) \) smoothly interpolates between \( \Delta_\psi = 3/4 \) at \( m J \ll 1 \) and \( \Delta_\psi = 1/4 \) at \( m J \gg 1 \), cf. Fig. 4. Using this representation of the two-point function, the RG equations may be derived along the same lines as above, leading to
\[ \frac{d \ln m}{dl} = -1 + \frac{Z}{4} mw(2\Delta_\psi(m) - \frac{1}{2}), \] (24)
\[ \frac{d \ln w}{dl} = 1 - 2(2\Delta_\psi(m) - \frac{1}{2}). \]
In terms of \( m \) and \( \lambda = mw \) they read
\[ \frac{d \ln m}{dl} = -1 + \frac{Z}{4} \lambda(2\Delta_\psi(m) - \frac{1}{2}), \] (25)
\[ \frac{d \ln \lambda}{dl} = \frac{Z}{4} \lambda - 2(2\Delta_\psi(m) - \frac{1}{2}). \]
These equations are valid to the first order in \( Z\lambda \), but for arbitrary \( m J \). They interpolate between the two limits elaborated in the main text. The corresponding RG flow diagram is presented in the main text as Fig. 2. It contains the non-trivial hyperbolic fixed point \( (\lambda_c, m_c) \), with \( \lambda_c = \frac{w}{2} \) and \( \Delta_\psi(m_c) = \frac{1}{2} \). Notice the FL fermion dimension at the critical point.
Linearizing Eqs. (25) around this fixed point one finds
\[ \frac{d}{dl} \left( \frac{\delta \lambda}{\delta m} \right) = \begin{pmatrix} 1 & 0 \\ m_c / \lambda_c & \kappa \end{pmatrix} \frac{d}{dl} \left( \frac{\delta \lambda}{\delta m} \right), \]
where
\[ \kappa \equiv \frac{4}{Z} \left. \frac{d \Delta_\psi(m)}{d \ln m} \right|_{m=m_c} = -0.41. \]
The two Lyapunov exponents corresponding to the relevant and irrelevant directions are thus found as \( \kappa_r = 1 \) and \( \kappa_i = \kappa \). The former specifies that the crossover scales \( T_1(\lambda) \propto (\lambda_c - \lambda) \) and \( T_\text{FL}(\lambda) \propto (\lambda - \lambda_c) \) behave linearly near the critical point, see Fig. 1 of the main text.