Numerical Methods for the Fractional Laplacian
Part I: a Finite Difference-quadrature Approach

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Abstract

The fractional Laplacian is a non-local operator which depends on the parameter α and recovers the usual Laplacian as α → 2. A numerical method for the fractional Laplacian is proposed, based on the singular integral representation for the operator. The method combines finite difference with numerical quadrature, to obtain a discrete convolution operator with positive weights. The accuracy of the method is shown to be \(O(h^{3-\alpha})\). Convergence of the method is proven. The treatment of far field boundary conditions using an asymptotic approximation to the integral is used to obtain an accurate method. Numerical experiments on known exact solutions validate the predicted convergence rates. Computational examples include exponentially and algebraically decaying solution with varying regularity. The generalization to nonlinear equations involving the operator is discussed: the obstacle problem for the fractional Laplacian is computed.

1 Introduction

The fractional Laplacian is the prototypical operator to model non-local diffusions. These non-local (or anomalous) diffusions, which incorporate long range interactions, have been the subject of much interest in recent years [OS74, Hil00, Das11, Her11]. Despite the diversity of the numerical methods in use, the modern tools of numerical analysis have not been applied to the study of the operator. In particular, convergence proofs and even error estimates have not been firmly established for the majority of the related numerical methods.

In this paper, we derive a finite difference/quadrature method for the fractional Laplacian. We obtain the accuracy of the method (at least on smooth solutions) and the convergence proof for the extended Dirichlet problem, while in a companion paper [HO], we discuss in detail the different related finite difference methods, and perform a comparative numerical study.

On the entire space \(\mathbb{R}^n\), the fractional Laplacian \((-\Delta)^{\alpha/2}\) with \(0 < \alpha < 2\) can be defined in many equivalent ways. It is defined simply as a pseudo-differential operator with symbol \(|\xi|^\alpha\) [Ste70], that is via the Fourier transform \(\mathcal{F}\),

\[
\mathcal{F}((-\Delta)^{\alpha/2}u)(\xi) = |\xi|^\alpha \mathcal{F}[u](\xi).
\]

(The definition (1) provides a simple method for solving \((-\Delta)^{\alpha/2}u = f\) (assuming \(f\) decays quickly enough at infinity). An equivalent definition of the operator is given by a singular integral [Lan72],

\[
(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy.
\]

Here the constant \(C_{n,\alpha}\) is given by

\[
C_{n,\alpha} = \frac{\alpha 2^{\alpha-1} \Gamma\left(\frac{\alpha+n}{2}\right)}{\pi^{n/2} \Gamma\left(\frac{\alpha+n}{2}\right)},
\]
and $\Gamma(x)$ is the Gamma function. In probability theory, the fractional Laplacian is also the infinitesimal generator of symmetric $\alpha$-stable Lévy process [App09]. It appears in financial mathematics, as an alternative model to Brownian motion, incorporating the jumps in asset prices [CT04, Rai00]. These different definitions of the fractional Laplacian reflect its extensive usages in applications, and they are indeed equivalent as shown in [Lan72, Ste70, Val09].

On a bounded domain, the study of the operator becomes more complicated. For the (standard) Laplacian operator, Dirichlet and Neumann boundary conditions are well understood, and have simple interpretations at the particle or probabilistic level. However, for the fractional Laplacian, boundary conditions are still the subject of current study, see for example, [CMG+03, ZRK07]. As a result, not all existing definitions of the fractional Laplacian are equivalent, mainly due to their boundary conditions. For periodic boundary conditions, the operator can still be defined via (1). On an interval $[-L, L]$, the fractional Laplacian can be defined in terms of the left and right Riemann-Liouville fractional derivatives $-L D_x^\alpha u(x)$ and $x D_x^\alpha u(x)$ [SKM93, GM98], i.e.,

$$(-\Delta)^{\alpha/2} u(x) = \frac{-L D_x^\alpha u(x) + x D_x^\alpha u(x)}{2 \cos(\alpha \pi/2)}, \quad \alpha \neq 1. \quad (4)$$

The spectral decomposition gives another way to define the fractional Laplacian on a bounded domain $D \subset \mathbb{R}^n$. Let $(\lambda_k, \phi_k)_k$ be the eigenpairs of the (negative) Laplacian operator $-\Delta$,

$$-\Delta \phi_k = \lambda_k \phi_k,$$

subject to appropriate boundary conditions such that $\lambda_k$ is nonnegative and $\{\phi_k\}$ is a complete orthonormal basis. Then if $u$ has the expansion $u(x) = \sum_k c_k \phi_k(x)$, we can define

$$(-\Delta)^{\alpha/2} u(x) = \sum_k c_k \lambda_k^{\alpha/2} \phi_k(x).$$

However, it is not clear how to interpret the spectral definition at the particle level or, more rigorously, in terms of the underlying Lévy process, although most of these definitions formally converge to the fractional Laplacian operator in $\mathbb{R}^n$ as the domain is extended to the whole space.

While the appropriate treatment of boundary conditions in practical modeling is still an open problem, the most natural case is the extended Dirichlet boundary value problem given by

$$(-\Delta)^{\alpha/2} u = f, \quad \text{for } x \in D$$

$$u = g, \quad \text{for } x \in \mathbb{R}^n \setminus D. \quad \text{(FLD)}$$

Here the functions $f$ and $g$ are given, with appropriate smoothness and decaying conditions. Note that now (1) cannot be applied directly, since $f$ is defined only on the bounded domain $D$.

Using the singular integral definition (2), the operator splits into

$$(-\Delta)^{\alpha/2} u(x) = C_{n, \alpha} \left( \int_D \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy + \int_{\mathbb{R}^n \setminus D} \frac{u(x) - g(y)}{|x - y|^{n+\alpha}} dy \right), \quad x \in D$$

where the representation now makes it clear that the unknowns are only in $D$, despite the fact that we have an integral over $\mathbb{R}^n$. In contradistinction to the Dirichlet problem for the standard Laplacian, the values of $g$ are required on the entire complement of the domain $D$ instead of the boundary $\partial D$. The problem (FLD) is natural for us, and treating problems of this type is necessary for more general nonlinear problems, for example, the obstacle problem [Sil07].

In the special case $g = 0$, $f = 1$, (FLD) corresponds to the expected first passage time of the symmetric $\alpha$-stable Lévy process from a given domain $D$ [Get61]. In the case $D$ is a ball about the origin, there is an explicit expression (“balayage problem”) for the solution, which is an integral involving $f$ and $g$ ([Lan72, Chapter I] or [Sil07, Section 5.1]). The balayage integral generalizes the classical Poisson formula for the Laplace operator.

More general Parabolic Integro-Differential Equations (PIDE) driven by Lévy processes appear in mathematical finance [CV05]. The infinitesimal generators $L^X$ of Lévy processes are of the form

$$L^X u(x) = \frac{\sigma^2}{2} u_{xx}(x) + \gamma u_x(x) + I[u](x)$$
where the nonlocal operator reads
\[
I[u](x) = \int_{\mathbb{R}} \left( u(x + y) - u(x) - y\chi_{|y|\leq 1}(y)u_x(x) \right) \nu(y) \, dy
\]  
(5)

Here the so called Lévy measure \( \nu(x) \) satisfies \( \int_{\mathbb{R}} \min(1, x^2) \nu(x) \, dx < \infty \). In the special case \( \nu(x) = \nu^\alpha(x) = C_{1,\alpha} |x|^{-1-\alpha} \) we recover the fractional Laplacian operator (since by symmetry the third term appearing in the integral vanishes). The finite difference schemes presented in this article are derived for a general Lévy measure \( \nu(x) \), although our focus is on the special case \( \nu^\alpha \) of fractional Laplacian. We discuss this method more in section 6.1, where we present a comparison of the method with our method.

Once we are equipped with a suitable discretization of the fractional Laplacian, the extension to general linear or nonlinear partial differential equations (PDEs) involving the operator can be accomplished easily. For example, linear elliptic or parabolic PDEs involving the Laplacian, combined with convex combinations of \(( -\Delta )^{\alpha/2}\) can be treated. Below, we discuss the obstacle problem for the Fractional Laplacian, and in a companion paper [HO] we solve a number of PDEs (nonlinear as well) involving the operator.

Many numerical methods have been proposed to solve equations with fractional Laplacians, either on the whole space or bounded domains. A majority of them are related to fractional derivatives (either Riemann-Liouville or Caputo type) on bounded domains, termed fractional diffusion, as summarized in [DFFL05]. However, the theoretical aspects of most of these equations with fractional derivatives, especially nonlinear ones, are still poorly understood. In contrast, during the last decade, equations involving the fractional Laplacian on the whole space have been studied intensively, for instance the fractional Burgers equation [BFW98, TW06] and fractional porous medium equations [dPQRV11, BIK11, CV11a].

Compared with the advances in theoretical analysis, there are not many state-of-the-art numerical methods designed for equations with fractional Laplacian. Somewhat surprisingly, the nonlinear theory is ahead of the linear theory, in the sense that once a suitable numerical method is obtained for the fractional Laplacian, it can be extended to nonlinear problems. For example, in the case of nonlinear elliptic operators involving the fractional Laplacian, the requirement of a monotone scheme is that the (consistent) difference scheme has positive weights [BJK10]. In this case, the scheme of nonlinear equation converges to the generalized viscosity solution [AT96].

Similarly, for the fractal conservation laws considered by Droniou [Dro10], a numerical method with several desired properties are obtained, again using the building block of a suitable scheme for the fractional Laplacian, in which the nonlinear scheme converges to the corresponding entropy solution. This work is followed by Cifani, Jakobsen and their colleagues [CJ11, CJK11a, CJK11b] for other equations with convection or degenerate diffusion. However, the focus of these schemes are more on the right limiting entropy solutions, while the accuracy and the fast implementation of the schemes may not be essential. In financial mathematics, a discussion of the related numerical methods can be found in [CT04, Chapter 12]. Numerical methods for those parabolic obstacle Integro-Differential Equations have been studied in [CV05], including a proof of convergence to the viscosity solution. Our methods are derived in a similar spirit as in [CV05], but with several key improvements in the order of accuracy and the treatment of boundary conditions.

In theory, the fractional Laplacian operator can be approximated numerically by any of the different yet equivalent definitions above, (1), (2) or (4), though with their own difficulties. The spectral methods based on (1), usually implemented with FFT, are effective for periodic domains. However in the whole space, the relatively slow decay of the functions either requires a large number of modes or introduces significant aliasing errors. Schemes based on the integral representation must be designed carefully to avoid large errors near the singularity and to take into account the proper contribution on the unbounded space. The popular Grünwald-Letnikov type difference methods based on (4) become singular when \( \alpha \approx 1 \) [TMS06], already observed from the denominator of the definition. In addition, there is no isotropic extension of the fractional derivatives to higher dimensions. The fractional Laplacian defined by an extension problem [CS07] also motivate a finite difference method [dTV13]. This approach can be generalized to higher dimensions easily, but the zero Dirichlet boundary condition on a truncated domain may be improved with a more effective numerical boundary condition to reduce the overall error.

In this paper, we derive a finite difference/quadrature discretization of the fractional Laplacian based on the singular integral definition (2) in one dimension, which is represented as a discrete convolution of the
function value $u_i$ with positive weights $w_j$. We write the discretization as

$$(−Δ_h)^{α/2}u_𝑖 = \sum_{j=−∞}^{\infty} (2u_𝑖 − u_{𝑖+j} − u_{𝑖−j})w_j = \sum_{j=−∞}^{\infty} (u_𝑖 − u_{𝑖−j})w_j.$$ (FLh)

This discrete fractional Laplacian inherits many desired properties from its continuous counterpart such as positivity and decaying rate. Despite the non-locality, the discrete operator can be computed efficiently using a fast convolution algorithm. The discretization is derived in Section 2, with special attention to the singularity in the integral, and the explicit weights $w_j$ are given in Section 3. The convergence proof of the extended Dirichlet problem is given in Section 4. The far field boundary condition are treated in Section 5, with some numerical convergence tests and numerical experiments in Section 6.

2 Quadrature and finite difference discretization of the fractional Laplacian

In this section we derive the combined finite difference/exact quadrature discretization of the fractional Laplacian operator in one dimension. The weights $w_j$ in the general scheme (FLh) are collected from approximations of the integral (2), by splitting it into two parts. In the singular part of the integral, we obtain a (rescaled) second derivative which is then discretized using a standard centered finite difference. In the tail of the integral, because a direct quadrature method would lead to large errors, we perform a non-standard quadrature. This semi-exact quadrature uses exact integration of the weight function $ν(y)$, multiplied by an interpolation of the unknown function $u$, giving explicit weights $w_j$ in (FLh).

2.1 The singular integral operator

We consider the slightly more general singular integral operator

$$I^ν[u](x) = \text{P.V.} \int_\mathbb{R} (u(x) − u(x − y))ν(y)dy.$$ (SI)

Here the nonnegative measure $ν$ satisfies the conditions:

$$\int_{−1}^{1} y^2ν(y)dy < \infty, \quad \int_{|y|>1} ν(y)dy < \infty.$$ (6)

In the special case $ν(y) = ν^α(y) := C_{1,α}|y|^{-1−α}$, the definition (SI) recovers the singular integral representation (2) of the fractional Laplacian.

We immediately divide the integral (SI) into two parts, the singular part, and the tail.

Definition 2.1. Take $h < 1$ and define the singular part, and the tail, of the integral to be, respectively

$$I^ν_S[u](x) = \text{P.V.} \int_{|y|≤h} (u(x) − u(x − y))ν(y)dy,$$ (7)

$$I^ν_T[u](x) = \int_{|y|>h} (u(x) − u(x − y))ν(y)dy.$$ (8)

Later we will specialize to functions on the grid $\mathbb{Z}_h = \{x_i = ih \mid i = 0, \pm 1, \pm 2, \ldots\}$ with uniform spacing $h$, and derive an expression for the discrete fractional Laplacian at $x = x_i$.

2.2 Approximation of the singular part of the integral

In this section, we deal with the singular integral (7), whose approximation is shown to a rescaled second derivative at $x$. 
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First, we can symmetrize (7) to obtain,

\[ I_μ^α [u](x) = \int_0^h (2u(x) - u(x + y) - u(x - y)) \nu(y) dy, \]

(9)

which is no longer a principal value integral for smooth function \( u \).

Assuming that \( u \in C^4 \), substitute the Taylor expansion with exact remainder

\[ u(x \pm y) = u(x) \pm u'(x)y + \frac{y^2}{2}u''(x) \pm \frac{y^3}{6}u'''(x) + \frac{y^4}{24}u''''(\xi(y)) \]

into (9). The odd terms cancel, giving

\[ I_μ^α [u](x) = -u''(x) \int_0^h y^2 \nu(y) dy - \frac{1}{12} \int_0^h u^{(4)}(\xi(y)) y^4 \nu(y) dy \]

(10)

for some \( \xi \) in the interval \((x - h, x + h)\). When \( \nu(y) = \nu^α(y) \) we obtain

\[ I_μ^α [u](x) = -C_{1,α} \left( \frac{h^2 - α}{2 - α} u''(x) + \frac{u'''(\xi)}{12} h^4 - α \right). \]

(11)

Therefore, the approximation above replaces the singular integral (7) by a second derivative of \( u(x) \). For a fully discrete approximation, we next replace \( u''(x) \) by central differences,

\[ u''(x) = \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} + \frac{u''''(\xi)}{12} h^2 \]

to obtain

\[ I_μ^α [u](x) = -C_{1,α} \frac{u(x + h) - 2u(x) + u(x - h)}{(2 - α)h^α} - M_4 \frac{h^{4 - α}}{12 (2 - α)}, \]

(12)

where \( M_4 \) is a bound on the fourth derivative of \( u \). This last equation leads to the fully discrete approximation of (7) at \( x_i \),

\[ I_μ^α [u](x_i) \approx -C_{1,α} \frac{u_{i+1} - 2u_i + u_{i-1}}{(2 - α)h^α} = C_{1,α} h^{-α} \frac{u_{i+1} - u_{i-1}}{2 - α} + \frac{C_{1,α} h^{-α}}{2 - α} (u_i - u_{i-1}). \]

(13)

2.3 The tail of the integral

Next we focus on the tail of the integral (8). A natural idea is to use a simple quadrature rule to approximate it, for example the trapezoidal rule for the integrand \((u(x) - u(x - y))\nu(y)\). However, the errors from the trapezoidal rule, which depend on a derivative of the integrand, blow up as \( h \to 0 \). Even for finite \( h \) these errors are too large to be used in practice. Instead, we take some extra effort to integrate exactly the weight function \( \nu(y) \) multiplied by a polynomial interpolant of \( \nu(y) := u(x_i) - u(x_i - y) \).

In the tail region \(|y| \geq h\), we approximate the regular part of the integrand \( v(y) = u(x_i) - u(x_i - y) \) by the interpolant,

\[ \mathcal{P} v(y) = \sum_{j \in \mathbb{Z}} v(x_j) P_j(y - x_j) = \sum_{j \in \mathbb{Z}} (u_i - u_{i-1}) P_j(y - x_j), \]

(14)

for some functions \( P_j \) such that \( P_j(0) = 1 \) and \( P_j(x_k) = 0 \) for \( k \neq 0 \). Here these basis functions \( P_j \) can be piece-wise polynomials (like the tent function defined in Section 2.3). In fact, they are Lagrange basis polynomials extended to finite overlapping domains.

Substituting the interpolant (14) into (8), we obtain

\[ I_μ^α [u](x_i) \approx \int_{|y| \geq h} \mathcal{P} v(y) \nu(y) dy = \sum_{j \neq 0} (u_i - u_{i-1}) \int_{|y| \geq h} P_j(y - x_j) \nu(y) dy \]

(15)
Now we can evaluate the last integral above exactly, since both the weight function \( \nu(y) \) and the basis polynomials \( P_j \) are known, avoiding the additional error from a direct numerical quadrature involving \( \nu(y) \).

The approximation (15) defines, for fixed \( x_i \), the weight \( \tilde{w}_{i,j} = \int_{|y| \geq h} P_j(y - x_j) \nu(y) dy. \) Since the basis function \( P_j(y) \) is chosen to be symmetric and independent of \( i \) below in this paper, so is the weight \( \tilde{w}_{i,j} \). Therefore the weight \( w_j \) in (FLh) is defined by

\[
 w_j = w_{-j} = \int_{|y| \geq h} P_j(y - x_j) \nu(y) dy. \tag{16}
\]

**Remark 1.** We must be careful evaluating (16) on the boundary of intervals, both near the singular interval \( (|y| \approx h \text{ or } |y| = 1) \), and near \( |y| \approx L \) when the computational domain is truncated to \([-L, L]\), in order to avoid double counting of the weights.

**Remark 2.** We have to isolate \( I_{\nu}^w[u] \) from (SI), because of the singularity of \( \nu(y) \) at the origin. If (16) is integrated on \( \mathbb{R} \) instead of the interval \((-\infty, -h) \cup (h, \infty)\), then \( w_{\pm 1} \) may diverge for most functions \( P_{\pm 1} \) with \( \alpha \in (1, 2) \). The splitting of \( I_{\nu}^w[u] \) and \( I_{\nu}^p[u] \) avoids this problem at the singularity.

Using the formula (16), the weights are obtained when \( P_j(y) \) is replaced by piecewise polynomials. These functions are local Lagrange basis polynomials on each interval. The first order basis function are piecewise linear “tent” functions; the second order polynomials comprise two families of piecewise quadratic functions (see Figure 1).

**Remark 3.** The interpolating functions can also be global functions like \( \sin(x/h) = \frac{\sin \pi x / h}{\pi x / h} \), which is the limit of the Lagrange basis polynomials as the number of interpolation points tends to infinity. This leads to a spectral-like finite difference discretization discussed in [HO].

Before giving the explicit weights for specific interpolations in next section, we discuss the errors in the approximations first.

### 2.4 Error from the quadrature in the tail and the overall error

We give the error arising from approximating \( I_{\nu}^w[u] \) using (13), which is shown to be \( O(h^{4-\alpha}) \). In this section we estimate the errors from the approximation of \( I_{\nu}^p[u] \) using (15), with the weights defined by (16). First we have the following lemma related to the error in quadrature using Lagrange interpolation.

**Lemma 1.** Let \( \mathcal{P}^k f \) be the Lagrange polynomial interpolant with order \( k \) of the function \( f \) on the interval \([a, b]\), using equally spaced nodes which include the endpoints. Then for any nonnegative measure \( \nu(t) \),

\[
 \left| \int_a^b (\mathcal{P}^k f(t) - f(t)) \nu(t) dt \right| \leq (b - a)^{k+1} \frac{M_{k+1}}{2k+1(k+1)!} \int_a^b \nu(t) dt, \tag{17}
\]

where \( M_{k+1} \) is a bound on the \((k+1)th\) derivative of \( f \) on \([a, b]\).

**Proof.** The Lagrange interpolant on \([a, b]\) satisfies

\[
 \mathcal{P}^k f(t) - f(t) = \frac{f^{(k+1)}(\xi(t))}{(k+1)!} (t - t_0) \cdots (t - t_k),
\]

for some \( \xi(t) \) in the interval \((a, b)\), where \( a = t_0 < t_1 < \cdots < t_k = b \) are the interpolation nodes. So

\[
 \left| \int_a^b (\mathcal{P}^k f(t) - f(t)) \nu(t) dt \right| = \left| \int_a^b \frac{f^{(k+1)}(\xi(t))}{(k+1)!} (t - t_0) \cdots (t - t_k) \nu(t) dt \right| \leq \frac{M_{k+1}}{(k+1)!} \int_a^b |(t - t_0) \cdots (t - t_k)| \nu(t) dt.
\]

For any \( t \in (a, b) \), we have the bound \( |(t - t_0) \cdots (t - t_k)| \leq (b - a)^{k+1}/2^{k+1} \). Combine these estimates to obtain (17). \( \square \)
The Composite Exact Quadrature Rule is given by piecewise polynomial interpolation of degree $k$ on the each sub-interval of size $kh$ in the interval $(h, +\infty)$ or $(-\infty, -h)$. In other words, the piecewise interpolations $P^k \nu$ in (15) are polynomials of degree $k$ on the sub-intervals $[h, (k+1)h], [(k+1)h, (2k+1)h], \cdots$. The error in this approximation is summarized in the following lemma.

**Lemma 2.** The error for the Composite Exact Quadrature Rule is given by

\[
\left| \int_{|t| \geq h} (P^k f(t) - f(t)) \nu(t) dt \right| \leq \frac{k^{k+1} M_{k+1}}{2^{k+1}(k+1)!} h^{k+1} \int_h^\infty [\nu(t) + \nu(-t)] dt
\]

where $M_{k+1}$ is a bound on the $(k+1)$-th order derivative of $f$.

**Proof.** Write the integral as a sum over sub-intervals of length $kh$ and apply Lemma 1 on each sub-interval. Then the combined error is bounded by

\[
(kh)^{k+1} \frac{M_{k+1}}{2^{k+1}(k+1)!} \left( \int_h^\infty \nu(t) dt + \int_{-\infty}^{-h} \nu(t) dt \right)
\]

which gives the result (18).

In particular, from the conditions on the measure (6) and the additional assumption $\nu(y) \sim |y|^{-1-\alpha}$ near the origin,

\[
\int_h^\infty [\nu(y) + \nu(-y)] dy = \int_{|y| \geq 1} \nu(y) dy + \int_{h \leq |y| \leq 1} \nu(y) dy \sim h^{-\alpha}.
\]

So the error in approximating the tail of the integral (8) is bounded by

\[
M_{k+1} h^{k+1-\alpha}.
\]

This, together with the error $O(h^{4-\alpha})$ from (12), implies the following overall error of the method.

**Lemma 3.** Suppose $u \in C^4$. Then the combined error for the finite difference scheme (FLh) is $O(h^{2-\alpha})$ for weights using linear interpolation and $O(h^{3-\alpha})$ for weights using quadratic interpolation.

**Remark 4.** In fact, a careful examination of the errors from the singular part and the tail above suggests at least two improvements. Higher order interpolation polynomials can be used, but the corresponding weights computed from (16) can be negative and many desired properties like maximum principle are lost. Another possible improvement is to choose a wider interval for the singular part of the integral $I^k_S[u]$, say from $h$ to $h_0$. The leading order errors now become $h_0^{4-\alpha}$ (from $I^k_S[u]$) and $h^{k+1} h_0^{-\alpha}$ (from $I^k_T[u]$). The balance of these two errors leads to the optimal choice $h_0 \sim h^{(k+1)/4}$ and consequently the overall error $O(h^{(k+1)(4-\alpha)/4})$.

Since the improvement of the order for linear or quadratic interpolation is just a fractional of $\alpha$, we don’t pursue this direction here for simplicity.

## 3 Explicit calculation of the weights

### 3.1 Piecewise Linear and quadratic interpolants

Piecewise polynomial interpolation is a standard topic in elementary numerical analysis. For our purpose, we recall the formulas for the first and second order interpolants. They will be used to derive explicit weights $w_j$ from the semi-exact quadrature rules against the weight function $\nu(y)$.

Given the values $\{\nu(x_j)\}$ on the grid $x_j = jh$ with spacing $h$, we defined the following piecewise linear and quadratic interpolants (See Figure 1).

The piecewise linear interpolant is given by

\[
P^1_h \nu(x) = \sum_{j \in \mathbb{Z}} \nu(x_j) H_h(x - x_j),
\]
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where \( T_h(t) \) is the “tent” function

\[
T_h(t) = \begin{cases} 
1 - |t|/h & |t| \leq h, \\
0 & \text{otherwise.}
\end{cases}
\]

The corresponding piecewise quadratic interpolant is given by

\[
\mathcal{P}_h^2 v(x) = \sum_{j-i \text{ even}} v(x_j) Q_h(x - x_j) + \sum_{j-i \text{ odd}} v(x_j) R_h(x - x_j),
\]

where \( Q_h(x) \) is the quadratic Lagrange polynomial which interpolates \((0, 1, 0)\) at \((-h, 0, h)\):

\[
Q_h(t) = \begin{cases} 
1 - t^2/h^2 & |t| \leq h, \\
0 & \text{otherwise,}
\end{cases}
\]

and \( R_h(x) \) is a piecewise quadratic Lagrange polynomial which interpolates \((0, 0, 1)\) on the left and \((1, 0, 0)\) on the right:

\[
R_h(t) = \begin{cases} 
1 - 3|t|/2h + t^2/2h^2 & |x| \leq 2h, \\
0 & \text{otherwise.}
\end{cases}
\]

### 3.2 Formulas for weights in semi-exact quadrature

To derive the explicit expressions for the weights, we need the following lemma, which is proved with elementary integration by parts.

**Lemma 4.** Let \( F(t) \) be a \( C^2 \) function and \( G(t) \) be a \( C^3 \) function on \([-2h, 2h]\), \( T_h(t), Q_h(t), R_h(t) \) be defined above. Then

\[
\int_{-h}^{h} T_h(t) F''(t) \, dt = \frac{1}{h} \left( F(h) - 2F(0) + F(-h) \right),
\]

\[
\int_{-h}^{h} Q_h(t) G'''(t) \, dt = 2 \frac{G'(h) + G'(-h)}{h^2} + \frac{G(h) - G(-h)}{h^2},
\]

\[
\int_{-2h}^{2h} R_h(t) G'''(t) \, dt = -\frac{G'(2h) + 6G'(0) + G'(-2h)}{2h} + \frac{G(2h) - G(-2h)}{h^2}.
\]
Also, we will need the following one-sided integrals at the boundaries
\begin{align}
\int_0^h T_h(t)F''(t)\,dt &= -F''(0) + \frac{F(h) - F(0)}{h}, \\
\int_0^{2h} R_h(t)G''(t)\,dt &= -G''(0) - \frac{G'(2h) + 3G'(0)}{2h} + \frac{G(2h) - G(0)}{h^2}.
\end{align}

We now combine the linear or quadratic interpolation and Lemma 4 to obtain the weights. First we rewrite (16) in a form consistent with the above lemma,
\[ w_j = \int_{|y| \geq h} P_j(y - x_j)\nu(y)\,dy = \int_{|x_j + t| \geq h} P_j(t)\nu(t + x_j)\,dt, \]
where \( P_j(t) \) is \( T_h(t), Q_h(t) \) or \( R_h(t) \) defined above.

**Definition 3.1.** Define the functions \( F(t) \) and \( G(t) \) to be primitives of the weight function: \( F''(t) = G''(t) = \nu(t) \). In particular in the case of fractional Laplacian \( \nu^\alpha(t) = C_1 t^{-\alpha-1}, \)
\[
F(t) = \begin{cases}
C_1 t^{-\alpha} |t|^{1-\alpha}, & \alpha \neq 1 \\
-C_1 t \log |t|, & \alpha = 1
\end{cases}, \quad G(t) = \begin{cases}
C_1 t^{\alpha-1} |t|^{\alpha-2}, & \alpha \neq 1 \\
C_1 t - t \log |t|, & \alpha = 1
\end{cases}.
\]

Therefore, from (16), for the piecewise "tent" functions, the weights are
\[
w_j^T = \int_{|x_j + t| \geq h} T_h(t)F''(x_j + t)\,dt = \frac{1}{h} (F(x_j + h) - 2F(x_j) + F(x_j - h)) = h^{-\alpha} (F(j + 1) - 2F(j) + F(j - 1)).
\]

For the quadratic functions, if \( j \) is even,
\[
w_j^Q = \int_{|x_j + t| \geq h} Q_h(t)G''(x_j + t)\,dt
= 2h^{-\alpha} \left[ G'(j + 1) + G'(j - 1) - G(j + 1) + G(j - 1) \right],
\]
and if \( j \) is odd,
\[
w_j^Q = \int_{|x_j + t| \geq h} R_h(t)G''(x_j + t)\,dt
= h^{-\alpha} \left[ -\frac{G'(j + 2) + 6G'(j) + G'(j - 2)}{2} + G(j + 2) - G(j - 2) \right].
\]

The above formulas are valid only for \(|j| > 1\). When \( j = 1 \), we need the one-sided integrals (23) or (24), as the other side falls into the singular interval \([-h, h]\) that is isolated in \( I_S^\nu[u]\). Combined with the contribution on the singular interval in (13), the weights for \( j = 1 \) become
\[
w_1^T = \frac{C_1}{2 - \alpha} h^{-\alpha} + \int_0^h T_h(t)F''(x_1 + t)\,dt = h^{-\alpha} \left[ \frac{C_1}{2 - \alpha} - F'(1) + F(2) - F(1) \right]
\]
and
\[
w_1^Q = h^{-\alpha} \left[ \frac{C_1}{2 - \alpha} - G''(1) - \frac{G'(3) + 3G'(1)}{2} + G(3) - G(1) \right].
\]

By the symmetry of the interpolation functions \( P_j \) we choose, \( w_{-j}^T = w_j^T \) and \( w_{-j}^Q = w_j^Q \).

We summarize the expression of the weights as follows.
defining properties of the weights

We collect some important properties of the weights here, focusing on the case ν = ν₀ of fractional Laplacian. The more general cases with positive Lévy measures ν can be studied in a similar way.

**Positivity of the weights.** The weights obtained from linear and quadratic interpolation, summarized in Definition 3.2, are positive. The positivity is easy to see for weights derived from linear interpolation, and for the even weights derived from quadratic interpolation, since their positivity has been verified numerically.

The positivity of the weights wᵢ for j ≠ 0 is important for the maximum principle, stability (or monotonicity) and convergence of the discrete solutions for the extended Dirichlet problem. Notice that the appearance of negative weights is exactly the reason we don’t use higher order polynomial interpolation, for instance w₃ < 0 with cubic interpolation polynomials and similarly for other higher order polynomials. In the extreme case of infinitely many interpolation nodes, the global basis function Pᵢ(x) = sincᵢ = sin(πx/h) / (πx/h) is used and the corresponding weights wᵢ with even indices j are in fact all negative.

**Scaling of the weights** The weights from both linear and quadratic interpolation have certain scaling properties inherited from the measure ν₀. The dependence on h is always h⁻α, reflecting the fact that the fractional Laplacian as a fractional derivative of order α. Furthermore, the weights wᵢ also decay at a rate j⁻¹⁻α, once again because of the fact ν₀(y) ∼ |y|⁻¹⁻α as |y| → ∞. The scaling rate can also be checked for both weights using the asymptotic expansions of the auxiliary functions F and G (and their derivatives) in j when j is large.

**Consistency when α → 2⁻**. In the limit α → 2⁻, we recover the standard three point central difference scheme for −∂ₓₓ. In fact, using the explicit expression C₁,α from (3) and the weights wᵢ either from linear or quadratic interpolation, we can get

\[ \lim_{α → 2⁻} wᵢ = \begin{cases} h^{-2}, & j = ±1, \\ 0, & \text{otherwise}, \end{cases} \]

and hence for both \( wᵀ \) and \( w^Q \),

\[ \lim_{α → 2⁻} \left( (−Δh)^{α/2} u \right)ᵢ = \frac{uᵢ₊₁ - uᵢ₋₁}{h^2} \frac{uᵢ₊₁ - uᵢ₋₁}{h^2} = - \frac{uᵢ₊₁ - 2uᵢ + uᵢ₋₁}{h^2}. \]

**CFL condition.** From the asymptotic decaying rate wᵢ = O(|j|⁻¹⁻αh⁻α) and the expression C₁,α given by (3), we expect an upper bound of the form Ch⁻α for the sum of the total weights (except the irrelevant \( w₀ \)). In fact, this sum can be worked out explicitly,

\[ \sum_{j ≠ 0} wᵀᵢ = 2h⁻α \frac{C₁,α}{2 - α} - F'(1) = \frac{2^α Γ((α + 1)/2)}{π^{1/2} Γ(2 - α/2)} h⁻α, \]
Lemma 5. Define
\[
v(x) = \begin{cases} 
4 - x^2, & |x| < 1 \\
0, & \text{otherwise.}
\end{cases}
\]
Then the grid function $v_i = v(ih)$ satisfies

$$(\Delta_h)^{\alpha/2} v_i \geq 1, \quad \text{for } i \in D_h,$$

$v_i = 0, \quad \text{for } i \in D_h^C,$

for $h$ small enough.

**Proof.** We prove the lemma in two steps.

Step 1. We will establish that

$$-\delta_j v_i \geq \min(2, 2(jh)^2), \quad \text{for } i \in D_h, \ j > 0. \quad (32)$$

Case (i). Suppose $i + j$ and $i - j$ are both in $D_h$. Then

$$-\delta_j v_i = h^2 ((i + j)^2 + (i - j)^2 - 2i^2) = 2(jh)^2.$$

Case (ii). Suppose $i + j$ and $i - j$ are both outside $D_h$. Then

$$-\delta_j v_i = 2v_i = 8 - 2(ih)^2 \geq 6.$$

Case (iii). Suppose exactly one of $i \pm j$ is in $D_h$. Set $y = (i \pm j)h$, choosing the sign so that $|y| < 1$. Then

$$-\delta_j v_i = -v(y) + 2u(x) = 4 - 2x^2 + y^2 \geq 4 - 2x^2 \geq 2.$$

So (32) holds in each case.

Step 2. Using (32), we have for $i \in D_h$,

$$(-\Delta_h)^{\alpha/2} v_i \geq \sum_{|jh| \geq 1} w_j \min(2, 2(jh)^2) = 2 \left( \sum_{|jh| \geq 1} w_j + \sum_{|jh| < 1} w_j (jh)^2 \right).$$

The two sums above can be estimated separately by integrals. For the first sum,

$$\sum_{|jh| \geq 1} w_j \approx 2C_{1,\alpha} \int_1^\infty y^{1-\alpha} dy = \frac{2C_{1,\alpha}}{\alpha},$$

with an error $O(h^{k+1})$ depending on the degree $k$ of the polynomial used to derive $w_j$. For the second sum,

$$\sum_{|jh| < 1} w_j \approx 2C_{1,\alpha} \int_0^1 y^{1-\alpha} dy = \frac{2C_{1,\alpha}}{2-\alpha},$$

with an error $O(h^{k+1-\alpha} + h^{4-\alpha})$. When $h$ is smaller enough,

$$(-\Delta_h)^{\alpha/2} v_i \geq C_{1,\alpha} \left( \frac{4}{2-\alpha} + \frac{4}{\alpha} \right) = \frac{2^{\alpha} \Gamma((\alpha + 1)/2)}{\Gamma(1/2)\Gamma(2 - \alpha/2)} > 1,$$

since $2^\alpha > 1$, $\Gamma((\alpha + 1)/2) > \pi^{1/2} = \Gamma(1/2)$ and $\Gamma(2 - \alpha/2) < \Gamma(2) = 1$. 

**Remark 5.** Continuous super-solutions as the one provided by Getoor, sometimes also called barrier functions, are needed in order to prove existence of solutions using Perron’s method, as this ensures that the boundary values are achieved. However, for the existence of the super-solutions alone, the continuity is not needed, and it is more convenient to use a simple discontinuous function. Of course, this simplicity is compensated by a larger (and thus cruder) constant in the estimates below.
4.2 Maximum principle and convergence proof

To prove the convergence of the discrete problem \((\text{FLD}^h)\), we need the following lemma.

**Lemma 6** (Maximum principle for \((\text{FLD}^h)\)). Let \( u \) satisfy \((-\Delta_h)^{\alpha/2} u_i \leq 0 \) for \( i \in D_h \). Then

\[
\max_{i \in D_h} u_i \leq \max_{i \in D_h^C} u_i.
\]

Similarly, if \((-\Delta_h)^{\alpha/2} u \geq 0 \) for \( i \in D_h \), then

\[
\min_{i \in D_h} u_i \geq \min_{i \in D_h^C} u_i.
\]

**Proof.** Suppose \((-\Delta_h)^{\alpha/2} u_i \leq 0 \) for \( i \in D_h \). If

\[
u_{i_0} = \max_{i \in D_h} u_i > \max_{i \in D_h^C} u_i,
\]

\( u_{i_0} \) is the global maximum on \( Z_h \). As a result,

\[
0 \geq (-\Delta_h)^{\alpha/2} u_{i_0} = \sum (u_{i_0} - u_{i_0-j}) u_j > 0,
\]

which is a contradiction. So \( u \) cannot have a global maximum on \( D_h \), and \( \max_{i \in D_h} u_i \leq \max_{i \in D_h^C} u_i \), as desired. The second result follows by a similar argument. \(\square\)

Define the notation for the maximum norm of a grid function on the set of indices \( I \):

\[
\|u\|_I = \max_{i \in I} |u_i|.
\]

**Lemma 7.** For any grid function \( u : Z_h \to \mathbb{R} \) and \( D_h = \{i : |ih| < 1\} \), we have

\[
\|u\|_{D_h} \leq \|u\|_{D_h^C} + 4\|(-\Delta_h)^{\alpha/2} u\|_{D_h}.
\]  \((35)\)

**Proof.** Use the function \( v \) from Lemma 5 and set

\[
z = u - \|(-\Delta_h)^{\alpha/2} u\|_{D_h} v.
\]

Then for any \( i \in D_h \),

\[
(-\Delta_h)^{\alpha/2} z_i = (-\Delta_h)^{\alpha/2} u_i - \|(-\Delta_h)^{\alpha/2} u\|_{D_h} (-\Delta_h)^{\alpha/2} v_i \leq 0.
\]

By the maximum principle in Lemma 6, \( \max_{D_h} z \leq \max_{D_h^C} z \). Since \( v = 0 \) on \( D_h^C \) and \( |v| \leq 4 \) on \( D_h \), we have

\[
\max_{D_h} z = \max_{D_h^C} u.
\]

and

\[
\max_{D_h} u \leq \max_{D_h^C} u + \|(-\Delta_h)^{\alpha/2} u\|_{D_h} \max_{D_h} v \leq \|u\|_{D_h^C} + 4\|(-\Delta_h)^{\alpha/2} u\|_{D_h}.
\]

Similarly, we can show that \( \min_{D_h} u \geq -\|u\|_{D_h} - 4\|(-\Delta_h)^{\alpha/2} u\|_{D_h} \) and this completes the proof of \((35)\). \(\square\)

**Definition 4.1** (Solution Error and Local Truncation Error). Let \( u \) be the solution of \((\text{FLD})\) and let \( u^h \) be the solution of \((\text{FLD}^h)\). Define the approximate solution error as

\[
e_j^h = u(x_j) - u_j^h,
\]

and the truncation error as

\[
r_j^h = (-\Delta)^{\alpha/2} u(x_j) - (-\Delta_h)^{\alpha/2} u_j.
\]

Next, we combine the previous results to prove the convergence of the Dirichlet problem \((\text{FLD}^h)\).
Theorem 4.1. Let \( u \) be the solution of (FLD) and let \( u^h \) be the solution of (FLD\(^h\)). Assume that \( u \) is smooth enough for the truncation error estimate to be valid. The maximum of the approximate solution error is bounded by (a constant times) the maximum of the truncation error:

\[
\|e^h\|_{D_h} \leq 4\|r^h\|_{D_h}.
\] (36)

Proof. Using the definition of the error \( e^h = u(x_j) - u_j^h \) and the residual we have

\[
(-\Delta_h)^{\alpha/2}e_j^h = (-\Delta_h)^{\alpha/2}u_j - (-\Delta_h)^{\alpha/2}u_j^h = r_j^h.
\]

Then, using (35), along with the fact that \( e^h = 0 \) on \( D_h^C \), we obtain (36).

In particular, the convergence rate is given by the global accuracy from Lemma 3, which is \( O(h^{2-\alpha}) \) for \( w^T \) and \( O(h^{3-\alpha}) \) for \( w^Q \).

5 Far field boundary conditions

In this section, we discuss some practical issues related to the truncation of the computational domain and the implementation of the boundary conditions.

5.1 Truncation of the domain

Even for the extended Dirichlet problem (FLD\(^h\)) with a finite number of unknowns, the discrete operator (FLH) involves an infinite sum. Therefore, if the function does not decay fast enough, the computational domain has to be truncated and the contribution of the integral (SI) outside this domain should be approximated. As we see from the numerical example below, the inclusion of the truncated boundary terms is essential for an accurate evaluation of the fractional Laplacian.

We can simply truncate (FLH) at a finite value \( M \) of the index \( j \), resulting in the operator

\[
(-\Delta_h)^{\alpha/2}_M u_i = \sum_{j=-M}^{M} (u_i - u_{i-j})w_j.
\] (37)

However, large errors can accumulate because of the slow decay of function \( u_i \) and the weights \( w_j \approx j^{-1-\alpha} \). Once again the singular integral (SI) allows us to take advantage of analytical formulas for the truncated terms, reducing the overall error significantly. More precisely, when \( (-\Delta)^{\alpha/2}u(x_i) \) is written as

\[
\int_{-L_W}^{L_W} (u(x_i) - u(x_i - y))\nu(y)dy + \int_{|y| > L_W} u(x_i)\nu(y)dy - \int_{|y| > L_W} u(x_i - y)\nu(y)dy,
\]

(I) (II) (III)

with \( L_W = Mh \), the truncated sum (37) approximates the first term (I), with an error \( O(h^{3-\alpha}) \) for \( w^Q \) and \( O(h^{2-\alpha}) \) for \( w^T \) that is independent of \( M \).

The second term \( \int_{|y| > L_W} u(x_i)\nu(y)dy = u(x_i)\int_{|y| > L_W} \nu(y)dy \) can often be evaluated explicitly, for instance when \( \nu = \nu^\alpha \),

\[
\text{(II)} = 2u(x_i)C_{1,\alpha}\int_{L_W}^{\infty} y^{-1-\alpha}dy = \frac{2C_{1,\alpha}}{\alpha(L_W)^\alpha}u(x_i).
\] (39)

This prefactor \( 2C_{1,\alpha}(L_W)^{-\alpha}/\alpha \) can also be approximated alternatively by the sum\(^1\) \( \sum_{|y| > L_W} w_j = \sum_{j \neq 0} w_j - \sum_{|y| > M} w_j \), with the total sum \( \sum_{j \neq 0} w_j \) given in (27) or (28).

The approximation of the last term (III) depends on the situation at hand. For the extended Dirichlet problem (FLD), assuming the domain \( D \) is the interval \([-L, L]\), then \( u(x_i + y) = g(x_i + y) \) is given on \( |y| > L_W \), if \( L_W \geq 2L \) (notice that we only need \( |x_i| \leq L \)). Therefore, this term is known and its evaluation

\(^1\)When the domain is truncated at \( L_W = Mh \), \( w \) are evaluated using the one-sided integrals (23) or (24), and they are different in the two sums \( \sum_{|j| \leq M} w_j \) and \( \sum_{|j| \geq M} w_j \).
Numerical methods for the fractional Laplacian

is independent of our scheme; in the special case $q \equiv 0$, this term is identically zero. As a result, below we focus on the important case of functions on the whole space $\mathbb{R}$, because of its prevalence in the recent theoretical advances of many evolution equations like the fractional Burgers equation \cite{BFW98, TW06} or the fractional porous medium equations \cite{dPQRV11, BIK11, CV11a}.

### 5.2 Approximations of the boundary term (III)

The key observation, for evolution equations posed on the whole space, is that the corresponding solutions usually develop an algebraic tail, with or without explicit exponents. This slowly decaying tail in the function $u_i$, together with that in the weight $w_j$, makes it impractical to choose a large $L_W$ so that the contribution of (III) in \eqref{eq:2.1} can be ignored. On the other hand, this algebraic tail can help us extract the leading order contribution of (III).

If $u(x)$ decays to zero algebraically, i.e., $u(y) \sim |y|^{-\beta}$, then we have $u(y) \approx u(L)L^\beta|y|^{-\beta}$ as $y \to \infty$ and $u(y) \approx u(-L)L^\beta|y|^{-\beta}$ as $y \to -\infty$. Substituting the asymptotic expansions into (III), in the case $\nu = \nu^\alpha$ and $L_W \geq 2L$,

$$
\int_{-L}^{L} u(x_i - y)u^\alpha(y)dy \approx C_{1,\alpha}u(-L)L^\beta \int_{-L}^{L} (y-x_i)^{-\beta} y^{-1-\alpha} dy \approx \frac{C_{1,\alpha}u(-L)L^\beta}{(\alpha + \beta)(L_W)^{\alpha + \beta}} 2F_1 \left(\beta, \alpha + \beta; \alpha + \beta + 1; \frac{x}{L_W} \right)
$$

(40a) and

$$
\int_{-L}^{L} u(x_i - y)u^\alpha(y)dy \approx \frac{C_{1,\alpha}u(L)L^\beta}{(\alpha + \beta)(L_W)^{\alpha + \beta}} 2F_1 \left(\beta, \alpha + \beta; \alpha + \beta + 1; -\frac{x}{L_W} \right),
$$

(40b)

where $2F_1$ is the (Gauss) hypergeometric function. These two terms (40a) and (40b) take into account the major contribution from (III).

One should notice that by choosing the limit of the integration in (I) from $-L_W$ to $L_W$, we need the values of $u$ on the interval $[-(L + L_W), L + L_W]$ instead of $[-L, L]$. The algebraic extension $u(y) \approx u(L)L^\beta|y|^{-\beta}$ as $y \to \infty$ and $u(y) \approx u(-L)L^\beta|y|^{-\beta}$ as $y \to -\infty$ can be used again to provide the information outside the interval $[-L, L]$. In the numerical experiments below, $L_W$ is chosen to be $2L$. When (I) is evaluated by fast convolution algorithm, the total computational cost is $O(N \log N)$, where $N$ is the number of grid points on the truncated computational domain $[-L, L]$.

![Figure 2](image)

**Figure 2**: (a): The comparison of different contributions (I), (II) and (III) in the finite difference method (using $w^Q$) for fractional Laplacian \cite{41}, with $\alpha = 0.4$, $L = 2.0$ and $h = 0.1$. (b): The $L^\infty$ error of the scheme for different domain sizes and grid points, where the error before the saturation is close to $O(h^{3-\alpha})$. 
The contributions of the three terms (I), (II) and (III) are illustrated in Figure 2(a) using the weights \( w^Q \), for
\[
u(x) = (1 + x^2)^{(1-\alpha)/2}\]
with the exact fractional Laplacian
\[
(-\Delta)^{\alpha/2} u(x) = 2^\alpha \Gamma \left( \frac{1+\alpha}{2} \right) \Gamma \left( \frac{1-\alpha}{2} \right) \left(1 + x^2\right)^{-1-(1+\alpha)/2}.
\]
When the computational domain is taken to be as small as \([-2, 2]\), neither (I) nor (I)+(II) gives an accurate result. However, the inclusion of the contribution \((40a)\) and \((40b)\) from (III), despite the crude approximation of an algebraic tail with the exact exponent \(\beta = 1 - \alpha\), reduces the overall error significantly.

The convergence (in \( L^\infty \) norm) of the fractional Laplacian for different size of the domain with different grid size \( h \) is shown in Figure 2(b). One salient feature is the saturation of the error: for a fixed size \( L \) of the domain, the error does not decrease any more when \( h \) is smaller than a certain critical value. This gives another indication on the importance of the far field boundary conditions. The convergence rate before the saturation seems to be \( 3 - \alpha \), consistent with the error analysis summarized in Lemma 3.

**Remark 6.** The method extends to other far field boundary conditions, such as \( u(x) \to c_\pm \) as \( x \to \pm \infty \) with an algebraic rate. (This is the case for certain traveling waves in fractional conservation laws).

**Remark 7.** When the exponent \( \beta \) is not available, it can still be estimated from the solution itself by data fitting, assuming that \( L \) is large enough to be in the algebraic decaying region.

### 6 Numerical experiments

In this section we perform numerical experiments, including convergence tests using exact solutions. The results validate the theoretically predicted convergence rates for smooth functions. These solutions, which have slow decay in the far field, reveal the need for controlling errors from far field boundary conditions.

#### 6.1 Convergence for smooth functions

The first computational example involves the solution \( u(x) = e^{-x^2} \), which covers the case of a smooth function with exponential decay. The fractional Laplacian of \( u \) at \( x = 0 \) can be obtained exactly by Fourier Transform,
\[
(-\Delta)^{\alpha/2} u(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty k^\alpha e^{-k^2/4} dk = 2^\alpha \Gamma \left( \frac{1+\alpha}{2} \right)/\sqrt{\pi}.
\]
Because \( u \) decays exponentially, there is no need for the boundary contribution from (III). The convergence rate is shown in Figure 3. In addition, in this figure, we compared our method with the method from [CV05] (for different values of \( \epsilon \)). In this context, the scheme from [CV05] reads
\[
(-\Delta)^{\alpha/2} u(0) = C_{1,\alpha} \int_{-\epsilon}^\epsilon (u(0) - u(y)) |y|^{1-\alpha} dy + C_{1,\alpha} \int_{|y|>\epsilon} (u(0) - u(y)) |y|^{-1-\alpha} dy 
\]
\[
\approx -\frac{C_{1,\alpha} \epsilon^{2-\alpha}}{2 - \alpha} u''(0) + C_{1,\alpha} \sum_{j=0}^\infty (u(0) - u(\epsilon + (j+1/2)h)) \int_{\epsilon+jh}^{\epsilon+(j+1)h} |y|^{1-\alpha} dy 
\]
\[
+ C_{1,\alpha} \sum_{j=0}^\infty (u(0) - u(-\epsilon - (j+1/2)h)) \int_{-\epsilon-(j+1)h}^{-\epsilon-jh} |y|^{-1-\alpha} dy,
\]
where \( u''(0) \) is also approximated by central difference.

Compared to our method using the weights \( w^Q \), the method from [CV05] is a slightly different, but the accuracy is the same, \( O(h^{2-\alpha}) \). For our method using the weights \( w^Q \), the accuracy is \( O(h^{3-\alpha}) \). For our case of the fractional Laplacian, the measure decays more slowly than the exponentially decaying measures in [CV05], so here the truncation of the integral in the far field is more challenging. We observed that the
accuracy of these methods is $O(h^{2-\alpha})$ which degenerates as $\alpha \to 2$. For this reason, the quadratic weights $w^Q$ are needed to obtain reasonable accuracy.

The second example is a smooth function which decays algebraically:

$$u(x) = (1 + x^2)^{-(1-\alpha)/2}.$$ 

This solution was discussed in Section 5.2, and the fractional Laplacian of $u$ is given by (41). In this case, the scheme recovers the asymptotic error $O(h^{2-\alpha})$ for $w^T$ and $O(h^{3-\alpha})$ for $w^Q$, provided the computational domain is large enough, so that the truncation error is controlled. We performed computations with computational domain size $L = 8$ and $L = 64$ to demonstrate the interaction of the domain size $L$ with the error. See Figure 4.

![Figure 3](image1.png)

Figure 3: The convergence of the fractional Laplacian of $u(x) = e^{-x^2}$, compared with the method in [CV05]. The second two lines are the [CV05] method with different values of $\epsilon$. The parameters are $L = 10$ and $\alpha = 0.8$.

![Figure 4](image2.png)

Figure 4: The convergence of the fractional Laplacian of $u(x) = (1 + x^2)^{-(1-\alpha)/2}$, with $\alpha = 0.8$.

### 6.2 Convergence for non-smooth solutions

When the solution is non-smooth the method still converges, but with a slower rate. The next example is for a solution $u$ is which is continuous, but the derivatives at $x = \pm 1$ are discontinuous, so $u \in C^{0}(\mathbb{R})$. It
comes from taking

\[ f(x) = (1 - x^2)^{1-\alpha/2}, \]

then the solution of \((-\Delta)^{\alpha/2} u = f\) is given by \(^2[BIK11]\),

\[
u(x) = \begin{cases} 
2^{-\alpha} \pi^{-1/2} \Gamma \left( \frac{1-\alpha}{2} \right) \Gamma \left( \frac{3-\alpha}{2} \right) \left( 1 - (1 - \alpha)x^2 \right) & \text{if } |x| \leq 1, \\
2^{-\alpha} \frac{\Gamma \left( \frac{1-\alpha}{2} \right) \Gamma \left( \frac{3-\alpha}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1-\alpha}{2} \right)} |x|^{-1 - \alpha} F_1 \left( \frac{1-\alpha}{2}, \frac{2-\alpha}{2}, \frac{5-\alpha}{2}; \frac{1}{|x|^2} \right) & \text{if } |x| \geq 1. 
\end{cases}
\]

(42)

As shown in Figure 5(a), the error is largest near \(x = \pm 1\), the points with discontinuous second order derivatives. Numerical experiments give an order of convergence \(O(h^{1-\alpha/2})\), see Figure 5(b). While consistent with the decreased regularity of the solution, this is much slower than the previous (more regular) example.

However, as the solution becomes smoother, the convergence rate improves. The next example is for a solution \(u\) which is \(C^1(\mathbb{R})\) function, but with \(u''\) is discontinuous at \(x = \pm 1\). This solution arise when we take (note the exponent is larger)

\[ f(x) = (1 - x^2)^{2-\alpha/2}. \]

Then the solution of \((-\Delta)^{\alpha/2} u = f\) is given by \([BIK11]\)

\[
u(x) = \begin{cases} 
2^{-\alpha-1} \pi^{-1/2} \Gamma \left( \frac{1-\alpha}{2} \right) \Gamma \left( \frac{3-\alpha}{2} \right) \left( 1 - (2 - 2\alpha)x^2 + (1 - \frac{4}{3}\alpha + \frac{1}{3}\alpha^2)x^4 \right) & \text{if } |x| \leq 1, \\
2^{-\alpha} \frac{\Gamma \left( \frac{1-\alpha}{2} \right) \Gamma \left( \frac{3-\alpha}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1-\alpha}{2} \right)} |x|^{-2 - \alpha} F_1 \left( \frac{1-\alpha}{2}, \frac{2-\alpha}{2}, \frac{7-\alpha}{2}; \frac{1}{|x|^2} \right) & \text{if } |x| \geq 1. 
\end{cases}
\]

(43)

The fractional Laplacian of \(u\) with \(\alpha = 0.8\) computed on \([-2, 2]\) is shown in Figure 6(a). For \(u^T\), the convergence rate is \(O(h^{2-\alpha})\), as expected. But for \(u^Q\), it is \(O(h^{2-\alpha/2})\), better than for \(u^T\), but not as good as the rate \(O(h^{3-\alpha})\) which requires more regularity of the solution. Refer to Figure 6(b). Again, the convergence rate is fast enough that the local truncation error is dominated by the error from the truncation of the far field, depending on the relative size of \(h\) and \(L\), the computation domain size.

![Figure 5: (a) The fractional Laplacian of \(u\) (dashed line) defined in (42) with \(L = 2\), \(\alpha = 0.5\) and \(h = 0.2\). (b) The \(L^\infty\) error of \((-\Delta_h)^{\alpha/2} u\) with different grid sizes \(h\) on computed on \([-4, 4]\) and \([-8, 8]\).](image)

![Figure 6: (a) The fractional Laplacian of \(u\) (dashed line) defined in (42) with \(L = 2\), \(\alpha = 0.5\) and \(h = 0.2\). (b) The \(L^\infty\) error of \((-\Delta_h)^{\alpha/2} u\) with different grid sizes \(h\) on computed on \([-4, 4]\) and \([-8, 8]\).](image)

### 6.3 Extended Dirichlet problem

Next we solve the extended Dirichlet problem

\[ (-\Delta)^{\alpha/2} u = 1 \text{ on } (-1, 1), \quad u = 0 \text{ on } (-\infty, -1] \cup [1, \infty). \]

(44)

\(^2\)The formula here is slightly different from the one in the reference, we have corrected what appeared to be two typos
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Figure 6: (a) The fractional Laplacian of $u$ defined in (43) with $L = 2$, $\alpha = 0.5$ and $h = 0.2$. (b) The $L^\infty$ error of $(-\Delta_h)^{\alpha/2} u$ with different grid sizes $h$ on computed on $[-4, 4]$ and $[-8, 8]$.

Figure 7: (a) The numerical solution for the extended Dirichlet problem (44) with parameters: $\alpha = 0.8, L = 1$, $h = 0.20$ or $0.05$; (b) The convergence (in $h$) of the solution to the exact one (45) for the weights $w^T$ and $w^Q$. 
The exact solution is the expected first exit time of the symmetric \( \alpha \)-stable Lévy process from the interval \([-1,1]\) and is given by [Get61]

\[
    u(x) = \frac{2^{-\alpha} \Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(1 + \frac{\alpha}{2}\right) \Gamma\left(\frac{1+\alpha}{2}\right)} (1 - x^2)^{\frac{\alpha}{2}}.
\] (45)

So in this case, the solution is \( C^{0,\alpha/2}([-1,1]) \). Despite the singularity, convergence to the exact solution is observed. Because of the singularity of the solution at the boundaries \( x = \pm 1 \), both weights \( u^Q \) and \( w^F \) have the same rate \( O(h^{\alpha/2}) \) of convergence, shown in Figure 7(b). The numerical solutions (we can always choose \( L = 1 \) for two grid sizes, \( h = 0.20 \), and \( h = 0.05 \), are shown in Figure 7(a). (In this case, the solution was obtained by directly solving a (non-sparse) linear equation, rather than by iteration). The resulting linear system is strictly diagonally dominant and is therefore well-conditioned.

### 6.4 The fractional obstacle problem

The fractional obstacle problem [CSS08, Sil06, Sil07] is a direct generalization of its elliptic counterpart: given a continuous function \( \varphi \) (the obstacle), consider the problem of determining a continuous function \( u \) satisfying

\[
\begin{align*}
    u & \geq \varphi, & \text{in } \mathbb{R}^n, \\
    (-\Delta)^{\alpha/2} u & \geq 0, & \text{in } \mathbb{R}^n, \\
    (-\Delta)^{\alpha/2} u(x) & = 0, & \text{on } \{ x \in \mathbb{R}^n \mid u(x) > \varphi(x) \}.
\end{align*}
\] (46)

The solution \( u(x) \) approaches zero when \( |x| \) goes to infinity, which requires the obstacle function \( \varphi \) to be compactly supported or rapidly decaying. This problem arises in financial mathematics as a pricing model [CT04] and also in the study of the long-time asymptotic behavior of a fractional porous medium equation [CV11a, CV11b].

The equations (46) can be written in the equivalent form

\[
    \min(u - \varphi, (-\Delta)^{\alpha/2} u) = 0,
\]

or as a steady state of the evolution equation

\[
    u_t + \min(u - \varphi, (-\Delta)^{\alpha/2} u) = 0.
\]

The latter suggests the following iterative scheme

\[
    u^{k+1}_j = \mathcal{L}[u^k]_j := u^k_j - \Delta t \min\left(u^k_j - \varphi_j, (-\Delta_h)^{\alpha/2} u^k_j\right).
\] (47)

When \( \Delta t \leq \min\left(1, \sum_{j \neq 0} w_j\right) \), the discrete evolution operator \( \mathcal{L}[u] \) is monotone in \( u \), and the iterative method is a contraction in the maximum norm [Obe06]. As a result, the extension to nonlinear elliptic operators involving the fractional Laplacian can be performed [BJK10], using the theory of viscosity solutions for these operators [AT96]. This allows us to conclude that the scheme converges to the unique viscosity solution of the equation. Moreover, if we start with the initial condition \( u_0 = \varphi \), then \( u^k_j \) is increasing in \( k \) for each \( j \) and converges to \( u^\infty_j \) from below.

In one dimension, when the obstacle is \( \varphi(x) = 2^{-\alpha} x^{-1/2} \Gamma\left(\frac{1+\alpha}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \left(1 - (1 - \alpha)x^2\right)^{\frac{\alpha}{2}} \), \( \alpha \in (0,1) \), one can check that the exact solution \( u \) is given by (42), with the coincident set \([-1,1]\) on which \( u = \phi \).

The convergence of the scheme (47) with \( \alpha = 0.5 \) and \( L = 4 \) is shown in Figure 8, for both the solution \( u \) and the fractional Laplacian \((-\Delta)^{\alpha/2} u\). The \( L^\infty \) errors on different domain sizes \( L \) and grid sizes are shown in Figure 9, which is shown to be \( O(h^{1+\frac{\alpha}{2}}) \) for the solution \( u \) and \( O(h^{1-\frac{\alpha}{2}}) \) for the fractional Laplacian \((-\Delta)^{\alpha/2} u\). Similar to the example shown in Figure 2, for fixed domain size \( L \), the error becomes saturated when the total number of grid points \( N \) is above a threshold.
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Figure 8: The convergence of the iterative scheme (47) for the obstacle problem: (a) comparing the numerical solution to the exact solution (given in (42)) (b) the discrete fractional Laplacian \((-\Delta_h)^{\alpha/2}u_i\) to \((1-x^2)^{1-\alpha/2}\). The parameters are \(\alpha = 0.5\), \(L = 4\), \(h = 0.1\) and \(\Delta t = 0.158 = h^{\alpha/2}\).

Figure 9: The convergence of the obstacle problem in \(L^\infty\) norm with different domain sizes \(L\) and grid sizes \(h\). (a) the error in the solution \(u\), (b) the error in the fractional Laplacian \((-\Delta_h)^{\alpha/2}u\).
In this paper, we used the singular integral representation of the operator \((\frac{2}{\alpha})\) to derive a finite difference/quadrature discretization of the fractional Laplacian in one dimension. The weights \(w_j\) in the discrete scheme \((\text{FLh})\) are obtained from approximations of \((\frac{2}{\alpha})\) by splitting it into two parts: one coming from the singular part of the integral, the other from the tail of the integral.

Two different sets of weights were obtained from using semi-exact quadrature in the tail of the integral. The weights \(w^T\) come from using linear interpolation functions, and the weights \(w^Q\) come from quadratic interpolation functions. The accuracy of the two schemes was obtained: \(O(h^{2-\alpha})\) using the weights \(w^T\) and \(O(h^{3-\alpha})\) using the weights \(w^Q\). (The weights \(w^T\) were included mainly because they are simpler to derive).

The discrete scheme obtained from the weights shares important properties with the continuous singular integral operator. The weights are positive, and they scale as expected in the parameter \(h\) and in the distance \(x_i = h_i\). The scheme is consistent in the limit \(\alpha \to 2^-\), in the sense that it recovers the standard centered finite difference approximation for the Laplacian in one dimension. A formula for the sum of the weights was given, which provides a CFL condition for stability of the explicit (e.g. forward Euler) discretizations of time dependent problem involving the operator.

In addition to the order of accuracy for the scheme, convergence of solutions to the extended Dirichlet problem was proved: for smooth solutions, the error (in the maximum norm) of the solution is given by the truncation error.

A practical issue is that the operator is posed on the entire line, and truncation of the computational domain can dominate the error. For solutions which decay algebraically, this require large domains. We developed an approximation for the truncated operator on finite domain, using asymptotic values of the boundary data. This led to an improvement of the accuracy by an order of magnitude.

Computational results of solutions of the fractional Laplacian were provided. We used known exact solutions to validate the predicted convergence rates. Several exact solutions were computed: exponentially decaying solutions could be computed on small domains. Algebraically decaying solution required larger domains, even with the approximation of the far field boundary condition. We also computed the singular Getoor solution, both on the line and for the extended Dirichlet problem on the interval. Despite the singularity, the numerical solutions converge, but with a slower rate, \(O(h^{\alpha/2})\).

Once equipped with a consistent, stable scheme with positive weights for the Fractional Laplacian, we can compute viscosity solutions of nonlinear elliptic and parabolic PDEs involving the operator. We computed solutions of the obstacle problem using a simple iterative method. Solutions were obtained, and the computed accuracy was \(O(h^{1+\alpha/2})\).

In the companion paper [HO], an extensive comparison is performed with other numerical methods, which can also be written in the form \((\text{FLh})\). In particular, we make comparisons with schemes using fractional derivatives and Fourier based methods. The scheme with quadratic weights is more accurate than that based on fractional derivatives, and the scheme based on Fourier method, while more accurate for small number of grid points, may have negative weights. So the scheme introduced here compares favorably with these other methods.

In future work we hope to explore numerical approximations of the fractional Laplacian in higher dimensions.

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