Rational Points on the Intersection of Three Quadrics

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1 Introduction

Let $Q_1(x), Q_2(x), Q_3(x) \in K[x]$ be three quadratic forms in $n$ variables $x = (x_1, \ldots, x_n)$, defined over a number field $K$. This paper will be concerned with the Hasse principle and weak approximation for points over $K$ on the intersection

$$R : Q_1 = Q_2 = Q_3 = 0.$$ 

We begin by reviewing the situation for individual quadrics and pairs of quadrics. In the case of a single quadratic form the Hasse principle is always valid, while weak approximation holds for all nonsingular quadratic forms in $n \geq 3$ variables. When one has a pair of forms the Hasse principle may fail, even when the variety $Q_1 = Q_2 = 0$ is nonsingular, as is shown by the example

$$Q_1 = X_1X_2 - (X_2^3 - 5X_3^2), \quad Q_2 = (X_1 + X_2)(X_1 + 2X_2) - (X_3^2 - 5X_4^2)$$

(with $K = \mathbb{Q}$) due to Birch and Swinnerton-Dyer [3]. However it is known that for a nonsingular intersection defined by a pair of forms in 8 or more variables both the Hasse principle and weak approximation hold, see Heath-Brown [11, Theorem 1]. One cannot dispense with the smoothness condition here. The example

$$6X_1^2 - X_2^2 - X_3^2 = X_4^2 + \ldots + X_n^2 = 0$$

over the field $\mathbb{Q}$, has points in every completion, as soon as $n \geq 6$, but has no rational points.

The difficulty with the above example arises from the real completions, and it was shown by Colliot-Thélène, Sansuc and Swinnerton-Dyer [7] Theorem C] that any pair of quadratic forms defined over a totally complex number field $K$ has a common zero over $K$ (and hence satisfies the Hasse principle) as soon as the number of variables $n$ is at least 9. This enables us to handle an intersection $R$ of three quadrics over a totally complex number field $K$, using the method of Leep [13]. If $n \geq 21$, then the projective variety $Q_3 = 0$ automatically contains a linear space of dimension 8 defined over $K$, since it must split off 9 hyperbolic planes. However the pair of forms $Q_1, Q_2$ will have a common zero over $K$ in this linear space, by the above mentioned result of Colliot-Thélène, Sansuc and Swinnerton-Dyer, and we deduce that any triple of quadratic forms in at least 21 variables, defined over a totally complex number field $K$, will have a common zero in $K$. 
However we wish to handle general number fields, and so we will assume that the variety $\mathcal{R}$ is nonsingular. To be more precise, we shall require that the matrix

$$\begin{pmatrix}
\nabla Q_1(x) \\
\nabla Q_2(x) \\
\nabla Q_3(x)
\end{pmatrix}$$

has rank 3 for every point $[x]$ in $\mathcal{R}(\overline{K})$. When this condition holds we will say that $Q_1, Q_2, Q_3$ is a “nonsingular system” of quadratic forms. According to Lemma 3.2 of Browning and Heath-Brown [6], this will ensure that $\mathcal{R}$ is an absolutely irreducible variety of codimension 3 and degree 8. For nonsingular systems over $\mathbb{Q}$ one can apply the well-known result of Birch [3], which was proved using the Hardy–Littlewood circle method. Indeed Birch’s work was generalized to arbitrary number fields by Skinner [15], who explicitly considers the question of weak approximation. The outcome is that, for a nonsingular system, $\mathcal{R}$ will satisfy the Hasse principle and weak approximation provided that $n > d + 24$, where $d$ is the dimension of the “Birch singular locus”. In fact, for a nonsingular system of 3 quadratic forms the Birch singular locus will have dimension at most 2, so that it suffices to have $n \geq 27$.

Having described the relevant background we are ready to state our principal result.

**Theorem** Let $Q_1, Q_2, Q_3$ be a nonsingular system of quadratic forms in $n$ variables, defined over a number field $K$. Then the variety

$$\mathcal{R} : Q_1 \cap Q_2 \cap Q_3,$$

where $Q_i$ is the quadric $Q_i = 0$, satisfies both the Hasse principle and weak approximation, as soon as $n \geq 19$.

Thus we get both an improvement on the range $n \geq 21$ mentioned above for the case of totally imaginary fields, and on the range $n \geq 27$ coming from the methods of Birch and Skinner. We should also observe at this point that $\mathcal{R}$ will have points over any completion $K_v$ at a finite place $v$, as soon as $n \geq 17$. This is established by Heath-Brown [10] page 138] when $K = \mathbb{Q}$, and the proof for general number fields $K$ is completely analogous. Thus as far as the Hasse principle is concerned our theorem only requires solvability in the real completions of $K$. In the same connection we mention that $\mathcal{R}$ has local points over $K_v$ for finite places over primes $p \geq 37$ as soon as $n \geq 13$, see Heath-Brown [9] Corollary 1].

Our basic strategy for proving the theorem will be to try and find a linear space $L$ of dimension 7, defined over $K$ and lying in the quadric hypersurface $Q_3$. If we can ensure that the variety $Q_1 \cap Q_2 \cap L$ is nonsingular, we can then apply Theorem 1 of Heath-Brown [11], as mentioned above. This approach will require us firstly to establish the smoothness condition, and secondly to ensure that $Q_1 \cap Q_2 \cap L$ has points everywhere locally.

A comment is required about “smoothness” requirements, where we shall adopt a significant abuse of terminology. When we say that the variety $Q_1 \cap Q_2 \cap Q_3$ is nonsingular, for example, we mean that the corresponding system of 3 quadratic forms is a nonsingular system, as described above. It is of course quite possible for the variety $Q_1 \cap Q_2 \cap Q_3$ to be nonsingular in the usual sense, without the corresponding system of forms being nonsingular. For example, if all
the forms $Q_i$ vanish identically then $Q_1 \cap Q_2 \cap Q_3 = \mathbb{P}^{n-1}$, which is nonsingular. When we say that $Q_1 \cap Q_2 \cap L$ is nonsingular we will have in mind a system consisting of 2 quadratic forms and a (minimal) set of linear forms defining $L$. A variety can be defined in many ways via a system of forms, and the reader will have to decide from the context what constitutes an appropriate system to use. Since all the relevant varieties will be complete intersections there should be little difficulty with this.

2 Geometric Considerations

We begin by replacing the forms $Q_1, Q_2, Q_3$ by more convenient ones. We will write $\bar{Q} = (Q_1, Q_2, Q_3)$ for our triple of quadratic forms, and proceed to consider linear combinations $t \bar{Q}$.

$$t \bar{Q} = t_1 Q_1 + t_2 Q_2 + t_3 Q_3.$$ The determinant $d_1(t) := \det(t \bar{Q})$ is a form in $t_1, t_2, t_3$ of degree $n$. We also define the determinant $\delta(X, Y; t \bar{Q}(1), t \bar{Q}(2)) := \det(X t \bar{Q}(1) + Y t \bar{Q}(2))$ and the discriminant $d_2(t \bar{Q}(1), t \bar{Q}(2)) := \text{Disc}(\delta(X, Y; t \bar{Q}(1), t \bar{Q}(2)))$.

Thus $\delta(X, Y; t \bar{Q})$ is a form of degree $n$ in $X$ and $Y$, and $d_2(t \bar{Q}(1), t \bar{Q}(2))$ is bihomogeneous in the entries of $t \bar{Q}$.

It will be convenient to record some properties of nonsingular systems of two quadratic forms. The following lemma follows from Heath-Brown and Pierce [12, Proposition 2.1], for example.

**Lemma 2.1** Let $R_1(x_1, \ldots, x_m)$ and $R_2(x_1, \ldots, x_m)$ be quadratic forms over an algebraically closed field $k$ of characteristic zero, and suppose that they constitute a nonsingular system, so that $\nabla R_1(x)$ and $\nabla R_2(x)$ are linearly independent for any non-zero $x$ satisfying $R_1(x) = R_2(x) = 0$. Then every non-trivial linear combination $aR_1 + bR_2$ has rank at least $m - 1$. Moreover $\det(X R_1 + Y R_2)$ is not identically zero, and has distinct linear factors over $k$. Conversely, this last condition is equivalent to the statement that $R_1$ and $R_2$ form a nonsingular system.

We now use the following result of Aznar [1, §2].

**Lemma 2.2** Let $V \subset \mathbb{P}^{n-1}$ be a nonsingular complete intersection of codimension $r$, which is defined over a field $k$ of characteristic zero. Then there is a system of generators $F_1, \ldots, F_r \in k[\bar{x}]$ of the ideal of $V$, with

$$\deg F_1 \geq \cdots \geq \deg F_r,$$

such that the varieties

$$W_i : F_1 = \cdots = F_i = 0, \quad (i \leq r),$$

are all non-singular. Moreover if $r < n - 1$ then $V$ will be irreducible with degree equal to the product of the degrees of the $F_i$.  

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The last statement in the lemma follows from Heath-Brown and Browning [6, Lemma 3.2]. When $V$ is originally defined by $r$ forms $G_j$ of the same degree $d$, the proof of Aznar’s result shows that one can take the $F_i$ to be linear combinations of the $G_j$.

In our case Lemma 2.2 implies in particular that there are linear combinations $m^{(1)}_Q, m^{(2)}_Q$, with $m^{(1)}_Q, m^{(2)}_Q \in \overline{K}^3$, such that both the hypersurface $m^{(1)}_Q = 0$ and the intersection $m^{(1)}_Q = m^{(2)}_Q = 0$ are nonsingular. In particular we will have $d_1(m^{(1)}) \neq 0$, so that the form $d_1(t)$ does not vanish identically. Moreover it follows from Lemma 2.1 that the intersection $m^{(1)}_Q = m^{(2)}_Q = 0$ is nonsingular if and only if $d_2(m^{(1)}, m^{(2)})$ is non-zero, and so we deduce that the form $d_2(t^{(1)}, t^{(2)})$ does not vanish identically.

We therefore see that if we choose any three vectors $m^{(1)}_Q, m^{(2)}_Q, m^{(3)}_Q \in K^3$ such that none of

\[ d_1(m^{(3)}), \quad d_2(m^{(1)}, m^{(2)}), \quad d_2(m^{(1)}, m^{(3)}), \quad \text{or} \quad \det(m^{(1)}|m^{(2)}|m^{(3)}), \]

vanishes, then the three forms $Q_i'(x) = m^{(i)}_Q Q(x)$, for $i = 1, 2, 3$ are defined over $K$ and generate the same linear system as do $Q_1, Q_2, Q_3$. Moreover $Q'_1 = 0$ is nonsingular, as are the varieties $Q'_1 = Q'_2 = 0$ and $Q'_1 = Q'_3 = 0$. Thus without loss of generality we will assume that our original forms satisfy these conditions.

We will also require $Q_3$ to contain suitable linear spaces defined over the real completions of $K$. Suppose as above that $Q_1$ and $Q_3$ form a nonsingular system, and that $K_v$ is real. Then the argument of Heath-Brown [11, Lemma 12.1] shows that there is a real $\theta_v$ for which $(\cos \theta_v)Q_1 + (\sin \theta_v)Q_3 = 0$ is nonsingular and contains a linear space of dimension at least $(n-4)/2$ over $K_v$. (The argument of [11] does not explicitly produce a nonsingular quadratic form, but since the functions $n_+$ and $n_-$ are everywhere locally minimal one can change $\theta_v$ slightly, if necessary.) By weak approximation in $\mathbb{P}^1(K)$ we deduce that there exists $c \in K$ for which $cQ_1 + Q_3 = 0$ is also nonsingular and contains linear spaces of dimension at least $(n-4)/2$ over each real completion $K_v$. We now replace $Q_3$ by $cQ_1 + Q_3$ so that $Q_3$ is nonsingular and has linear spaces over each real completion, of dimension at least $(n-4)/2$. For finite places $v$ it is automatic that $Q_3$ contains linear spaces over $K_v$, of dimension at least $(n-5)/2$. We can therefore conclude that $Q_3$ has a linear space over $K$, with dimension at least $(n-5)/2$. The existence of a single such linear space is enough to ensure that there is one through every $K$-point of $Q_3$. An alternative way to express the above facts is to say that $Q_3$ splits off at least $(n-5)/2$ hyperbolic planes over $K$.

From now on we will fix the forms $Q_1, Q_2$ and $Q_3$. We remind the reader that the varieties $R, Q_1 \cap Q_2$ and $Q_3$ are all nonsingular, and that we have arranged that $Q_3$ contains a linear space, defined over $K$, of dimension at least $(n-5)/2$.

For the remainder of this section we work over an arbitrary algebraically closed field $k$ of characteristic zero. After the above preliminary manoeuvres we are ready to prove our first key result.

It will be convenient to define

\[ F_i = F_i(Q_3) := \{ L \in \mathbb{G}(t, n-1) : L \subset Q_3 \}. \]
Lemma 2.3 Suppose that the forms \( Q_1, Q_2, Q_3 \) are such that the varieties \( \mathcal{R} = Q_1 \cap Q_2 \cap Q_3 \), \( Q_1 \cap Q_2 \), and \( Q_3 \) are all nonsingular. Then for every integer \( t \) in the range \( 3 \leq t \leq (n-5)/2 \) there is an \( L \in F_t \), such that \( L \cap Q_1 \cap Q_2 \) is nonsingular.

We should perhaps be more specific as to this last condition. If \( L \) has codimension \( c \) say, and is given by linear equations \( \ell_1(x) = \ldots = \ell_c(x) = 0 \), then according to our convention, \( L \cap Q_1 \cap Q_2 \) is nonsingular if the partial derivatives of \( Q_1, Q_2 \) and \( \ell_1, \ldots, \ell_c \) are linearly independent at any non-zero vector \( x \in k^n \) such that

\[
Q_1(x) = Q_2(x) = \ell_1(x) = \ldots = \ell_c(x) = 0.
\]

Thus for example, if \( L \subset Q_3 \) is a line disjoint from \( Q_1 \cap Q_2 \), then \( L \) fulfils the condition vacuously.

For the proof we will need some information about \( t \)-planes lying in quadric hypersurfaces. We begin by introducing some notation. Let \( Q(x_1, \ldots, x_n) \) be a quadratic form of rank \( r \), over \( k \), and let \( Q \) be the quadric \( Q = 0 \), with dimension \( n - 2 \). Let \( F(n, r, t) \) be the Fano variety of \( t \)-planes in \( Q \), and let \( F(n, r; t; P) \) be the subvariety of such planes passing through a given point \( P \in \mathcal{Q} \). The variety \( F(n, r, t) \) will be non-empty when \( t \leq n - r/2 - 1 \). Write \( d_0(n, t, r) = \dim F(n, t, r) \), which will be independent of the particular quadratic form \( Q \). Similarly let \( \dim F(n, t, r; P) = d_1(n, t, r) \) for \( P \) a smooth point of \( \mathcal{Q} \). These dimensions are given by the following lemma.

Lemma 2.4 We have the following statements.

(i) \( d_1(n, t, r) = d_0(n - 2, t - 1, r - 2) \) if \( t \geq 1 \) and \( r \geq 2 \).

(ii) \( d_0(n, t, r) = n - 2 - t + d_0(n - 2, t - 1, r - 2) \) if \( t \geq 1 \) and \( 2 \leq r \leq 2n - 2t - 2 \).

(iii) \( d_0(n, t, r) = (t + 1)(n - 2 - 3t/2) \) if \( 2t + 2 \leq r \leq 2n - 2t - 2 \).

We take \( P = [e_1] \) and extend to a basis \( e_1, \ldots, e_n \) of \( k^n \). Then \( Q \) takes the shape \( x_1 L(x_2, \ldots, x_n) + Q'(x_2, \ldots, x_n) \) with respect to this basis, with \( L \neq 0 \). A further change of basis simplifies this to \( x_1 x_2 + Q''(x_3, \ldots, x_n) \). Here \( Q'' \) will have rank \( r - 2 \). One then sees that \( t \)-planes in \( Q \) containing \( e_1 \) correspond to \( (t - 1) \)-planes in \( Q'' \), and the result (i) follows.

For part (ii) we consider the incidence correspondence

\[
I = \{(P, L) : P \in L \in F(n, t, r)\}.
\]

The projection \( \pi_2 \) onto the second factor takes \( I \) onto \( F(n, t, r) \), and each fibre has dimension \( t \), so that \( d_0(n, t, r) = \dim(I) - t \). The condition \( r \leq 2n - 2t - 2 \) ensures that \( d_0(n-2, t-1, r-2) \geq 0 \), whence \( d_1(n, t, r) \geq 0 \) by part (i). Thus, for the projection \( \pi_1 \) onto the first factor, we see that \( \pi_1(I) \) contains every smooth point of \( \mathcal{Q} \). Hence \( \pi_1(I) = \mathcal{Q} \), and \( \pi_1^{-1}(P) \) will have dimension \( d_1(n, t, r) \) for smooth points \( P \). Thus \( \dim(I) = \dim(Q) + d_1(n, t, r) \), and the result follows from part (i).

Finally, part (iii) follows from part (ii) by induction on \( t \), the result being clearly true for \( t = 0 \).

We can now move to the proof of Lemma 2.3 By Lemma 2.4 we have

\[
\dim(F_t) = d_0(n, t, n) = (t + 1)(n - 2 - 3t/2)
\]
Lemma 2.3
Suppose that the forms $F$ so that $t \leq 2$ if $2t + 2 \leq n$. We proceed to consider the variety $F^\dagger$ defined to be the set of $t$-planes $L \in F$ for which $L \cap Q_1 \cap Q_2$ is singular. Let

\[ I := \{(L, [x], [t]) \in F_1 \times \mathcal{R} \times \mathbb{P}^1 : [x] \in L, \ x^T(t_1Q_1 + t_2Q_2)x = 0 \forall [y] \in L \}. \]

If $L \cap Q_1 \cap Q_2$ has a singular singular point $x$, then $x \in L \cap \mathcal{R}$ and there is some $[t] \in \mathbb{P}^1$ such that $x^T(t_1Q_1 + t_2Q_2)x = 0$ for every $[y] \in L$. Thus if $\pi_1$ is the projection from $I$ onto its first factor, one has $\pi_1(I) = F^\dagger$. It follows that $\dim(F^\dagger) \leq \dim(I)$.

Now consider the projection $\pi_{2,3}$ onto the second and third factors. The fibre above the pair $([x], [t])$ will be

\[ \{L \in F_1 : [x] \in L, \ x^T(t_1Q_1 + t_2Q_2)x = 0 \forall [y] \in L \}. \quad (2.1) \]

We write $Q = t_1Q_1 + t_2Q_2$ and $H(x,t) = \{ [y] : y^TQy = 0 \}$ for convenience. Thus the fibre (2.1) may be written as

\[ \{L \in F_1 : L \subseteq H(x,t), [x] \in L \}. \]

Since $[x] \in \mathcal{R}$, and $\mathcal{R}$ is nonsingular, we must have $Qx \neq 0$, so that $H(x,t)$ is a hyperplane. Hence $H(x,t)$ intersects $Q_1$ to produce a quadric hypersurface $Q'$ in $\mathbb{P}^{n-2}$, with rank, $r$ say, at least $n - 2$. The point $[x]$ must be a smooth point of $Q'$, since otherwise $Qx$ and $Q_1x$ would be proportional, contradicting the fact that $[x]$ is a smooth point of $\mathcal{R}$.

It follows that the dimension of the fibre (2.1) will be $d_1(n-1,t,r)$. According to Lemma [2.3] we have

\[ d_1(n-1,t,r) = d_0(n-3,t-1,r-2) = t(n-5-3(t-1)/2) \]

if $2(t-1) + 2 \leq r - 2 \leq 2(n-3) - 2(t-1) - 2$. Since we are assuming that $t \leq (n-5)/2$ the required condition on $r$ certainly holds. We therefore deduce that

\[ \dim(F^\dagger) \leq \dim(I) = \dim(\mathcal{R} \times \mathbb{P}^1) + \dim \pi_{2,3}^{-1}([x], [t]) \]

\[ = (n-4) + 1) + t(n-5-3(t-1)/2) \]

\[ = (t+1)(n-2-3t/2) - 1 \]

\[ = \dim(F_1) - 1 \]

so that $F^\dagger$ must be a proper subvariety of $F_1$. This completes the proof of Lemma [2.3].

We turn now to our second key result.

**Lemma 2.5** Suppose that the forms $Q_1, Q_2, Q_3$ are such that the varieties $\mathcal{R} = Q_3 \cap Q_2 \cap Q_3, Q_1 \cap Q_2, \text{and } Q_3$ are all nonsingular. Then for every non-negative integer $t \leq (n-5)/2$ there is an $L \in F_1$ such that $(Q_3L)^{\dagger} \cap \mathcal{R}$ is nonsingular.
Here we define
\[
(Q_3L)^\perp := \{[y]: y^T Q_3 x = 0, \forall [x] \in L\}.
\]

We prove this by induction on \(t\). For the base case \(t = 0\) the space \(L\) is a single point \(P\), say. It then suffices that \(P \in Q_3\) and \(P \not\in Q_3^{-1}(R^*)\), where \(R^*\) is the dual variety to \(R\), and \(Q_3^{-1}(R^*)\) is its image under the linear map \(Q_3^{-1}\). There will always be a suitable point \(P\) if \(Q_3 \not\subseteq Q_3^{-1}(R^*)\). However \(R^*\) is a proper subvariety of \(\mathbb{P}^{n-1}\). We claim that it cannot be a nonsingular quadric, whence we cannot have \(Q_3 = Q_3^{-1}(R^*)\). To prove the claim we merely observe that if \(R^* = Q\), say, then \(R = R^{**} = Q^*\). However \(Q^*\) is itself a nonsingular quadric, whereas \(R\) has degree 8, by Lemma 2.2. The lemma then follows in the case of dimension zero.

To establish the induction step we suppose we have a suitable space \(L\) of dimension \(t\), and look for an appropriate \(L'\) of dimension \(t + 1\). Indeed we shall restrict our attention to linear spaces satisfying
\[
L \subset L' \subseteq Q_3 \cap (Q_3L)^\perp.
\]
This requirement on \(L'\) allows us to restrict all our varieties to the subspace \((Q_3L)^\perp\). We write \(Q_i' = Q_i \cap (Q_3L)^\perp\) for \(i = 1, 2, 3\), and \(R' = R \cap (Q_3L)^\perp\). Thus \(R'\) is nonsingular. More concretely, we choose \([e_0], \ldots, [e_1]\) spanning \(L\), and extend these to a set of points \([e_0], \ldots, [e_{n-t-2}]\) spanning \((Q_3L)^\perp\), and then to a set \([e_0], \ldots, [e_{n-1}]\) spanning \(\mathbb{P}^{n-1}\). We proceed to write
\[
Q_i'(x_1, \ldots, x_{n-t-1}) = Q_i(\sum_{j=1}^{n-t-1} x_j e_{j-1}) \quad (i = 1, 2, 3)
\]
so that these quadratic forms correspond to the quadrics \(Q_i'\), seen as varieties in \(\mathbb{P}^{n-t-2}\). We note in particular that \(Q_3'\) is singular, with rank \(n - 2t - 2\). Indeed, as a subvariety of \(\mathbb{P}^{n-t-2}\) the quadric \(Q_3'\) hypersurface will be a cone with vertex set \(L \subset \mathbb{P}^{n-t-2}\).

We construct \(L'\) as \(< L, [x] > \) with \([x] \in Q_3'\) but \([x] \not\in L\). This will ensure that \(L' \subseteq Q_3'\) and that \(\text{dim}(L') = t + 1\). Moreover we will have \((Q_3L')^\perp = (Q_3L)^\perp \cap (Q_3x)^\perp\), so that \((Q_3L)^\perp \cap R'\) will be nonsingular provided that \([Q_3x]\) is not in \((R')^*\). Since \(\text{dim}(L) = t < \text{dim}(Q_3') = n - t - 3\), the generic \([x] \in Q_3'\) will satisfy \([x] \not\in L\). Moreover it will also satisfy \([Q_3x] \not\in (R')^*\) by a similar reasoning to that given in the dimension zero case above. Specifically, \((R')^*\) is a proper subvariety of \(\mathbb{P}^{n-t-2}\), so the only situation to rule out is that in which it is equal to \(Q_3'\). However if \(Q_3'\) has rank \(n - 2t - 2\) then the dual \((Q_3')^*\) will be a quadric of dimension \(n - 2t - 4\), which cannot possibly be the variety \(R'\), since the latter will have degree 8.

It follows that we can take \(L' = < L, [x] >\) for a generic \([x] \in Q_3'\). This completes the induction step and so establishes the lemma.

Following on from Lemmas 2.3 and 2.5 we make the following definitions.

**Definition.** Let \(Q_1, Q_2, Q_3\) be a nonsingular system of quadratic forms in \(n\) variables as above, and let \(L \subseteq \mathbb{P}^{n-1}\) be a linear space contained in the quadric \(Q_3\). Then we say that \(L\) is "admissible" if and only if both \(L \cap Q_1 \cap Q_2\) and \((Q_3L)^\perp \cap R\) are nonsingular. If \(L\) has dimension \(t\) in the range \(3 \leq t \leq 7\) we say that \(L\) is "chain-admissible" if there exist admissible linear spaces \(L = L_1 \subset L_{t+1} \subset \cdots \subset L_7\) with \(\text{dim}(L_i) = i\) for \(t \leq i \leq 7\).
We then have the following result.

**Lemma 2.6** For each \( t \in [3, 7] \) there is a Zariski-closed proper subset \( Z_t \subset F_t \) such that \( L \in F_t \) is chain-admissible if and only if \( L \notin Z_t \).

It is clear that the \( L \in F_t \) for which \( L \cap Q_1 \cap Q_2 \) is singular form a closed subset \( A_t \) say, of \( F_t \). Similarly those for which \( (Q_3 L)^\perp \cap R \) is singular form a closed subset \( B_t \) say. By Lemmas 2.3 and 2.5 these are proper closed subsets of \( F_t \). Moreover, since \( F_t \) is absolutely irreducible, the union \( A_t \cup B_t = C_t \), say, is also a closed proper subset of \( F_t \). A linear subspace \( L \in F_t \) is admissible if and only if \( L \notin C_t \).

We proceed to prove Lemma 2.6 by downwards induction, and it is clear that we can take \( Z_7 = C_7 \).

For \( t < 7 \) if \( L \in F_t \) fails to be chain-admissible then either \( L \in C_t \) (because \( L \) is not itself admissible) or \( L' \) fails to be chain-admissible for every \( L' \in F_{t+1} \) containing \( L \). This latter case holds precisely when \( L' \in Z_{t+1} \) for every such \( L' \). Write \( D_t \) for the set of \( L \in F_t \) with the property that \( L' \in Z_{t+1} \) for every \( L' \in F_{t+1} \) containing \( L \). We claim that \( D_t \) is a proper closed subset of \( F_t \).

To handle \( D_t \) we consider

\[
I := \{(L, L') \in F_t \times Z_{t+1} : L \subset L'\}.
\]

For the projection onto the first factor we have

\[
\dim(\pi_1^{-1}(L)) = \dim\{L' \in Z_{t+1} : L \subset L'\} \leq n - 2t - 4,
\]

with equality exactly when \( L' \in Z_{t+1} \) for every linear space \( L' \in F_{t+1} \) containing \( L \). Thus \( D_t \) is the set of \( L \) for which \( \dim(\pi_1^{-1}(L)) \) is maximal, whence \( D_t \) is Zariski-closed. Moreover we have

\[
\dim(D_t) \leq \dim(I) - (n - 2t - 4).
\]

On the other hand, for the projection onto the second factor we have

\[
\dim(\pi_2^{-1}(L')) = t + 1,
\]

whence

\[
\dim(I) = \dim(Z_{t+1}) + t + 1.
\]

Since \( Z_{t+1} \) is a proper subset of \( F_{t+1} \) by the downward induction hypothesis we deduce that

\[
\dim(D_t) \leq \dim(I) - (n - 2t - 4) = \dim(Z_{t+1}) + t + 1 - (n - 2t - 4) = \dim(F_{t+1}) - n + 3t + 5 = (t + 2)(n - 2 - 3(t + 1)/2) - n + 3t + 5 = (t + 1)(n - 2 - 3t/2) = \dim(F_t).
\]

Here we have used Lemma 2.4 to compute

\[
\dim(F_{t+1}) = d_0(n, t + 1, n) = (t + 2)(n - 2 - 3(t + 1)/2).
\]
and
\[ \dim(F_t) = d_0(n, t, n) = (t + 1)(n - 2 - 3t/2). \]
The above calculation shows that \( \dim(D_t) < \dim(F_t) \), so that \( D_t \) is a proper subset of \( F_t \), as required.

We conclude this section with two easy results in a similar vein.

**Lemma 2.7** There is a Zariski-closed proper subset \( \mathcal{R}_0 \subset \mathcal{R} \) such that every \( P \in \mathcal{R} - \mathcal{R}_0 \) is contained in a chain-admissible linear space \( L \in F_3 - Z_3 \).

Let \( \mathcal{R}_0 \) be the set of points \( P \in \mathcal{R} \) such that \( L \in Z_3 \) for every \( L \in F_3 \) which contains \( P \). We need to show that \( \mathcal{R}_0 \) is a proper closed subset of \( \mathcal{R} \).

It follows from Lemma 2.6 that there is at least one \( L \in F_3 - Z_3 \). Since \( \dim(L) + \dim(\mathcal{R}) = 3 + (n - 4) = n - 1 \) it follows that \( L \cap \mathcal{R} \) is non-empty, containing \( P \) say. Then \( P \in \mathcal{R} - \mathcal{R}_0 \), so that \( \mathcal{R}_0 \) is a proper subset of \( \mathcal{R} \).

To show that \( \mathcal{R}_0 \) is Zariski-closed we consider
\[ I = \{(P, L) \in \mathcal{R} \times Z_3 : P \in L \}. \]
For the projection onto the first factor we have
\[ \dim(\pi_1^{-1}(P)) \leq d_4(n, 3, n), \]
in the notation of Lemma 2.3 with equality exactly when \( L \in Z_3 \) for every \( L \in F_3 \) which contains \( P \). Thus \( \mathcal{R}_0 \) is the set of \( P \) for which \( \dim(\pi_1^{-1}(P)) \) is maximal, and it follows that \( \mathcal{R}_0 \) is Zariski-closed, as claimed.

**Lemma 2.8** Given \( L \in F_3 - Z_3 \), there is a Zariski-closed proper subset \( \mathcal{R}_1 \) of \( \mathcal{R} \cap (Q_3 L) \perp \), such that \( < L, P > \in F_4 - Z_4 \) for every \( P \in \mathcal{R} \cap (Q_3 L) \perp - \mathcal{R}_1 \). Similarly, given \( L \in F_t - Z_t \), for some \( t \in [3, 6] \), there is a Zariski-closed proper subset \( \mathcal{E}_t \) of \( Q_3 \cap (Q_3 L) \perp \), such that \( < L, P > \in F_{t+1} - Z_{t+1} \) for every \( P \in Q_3 \cap (Q_3 L) \perp - \mathcal{E}_t \).

To prove the first part of the lemma we let
\[ I := \{(P, L') \in (\mathcal{R} \cap (Q_3 L) \perp) \times Z_4 : P \in L', L \subset L' \}, \]
and take \( \mathcal{R}_1 = \pi_1(I) \), so that \( \mathcal{R}_1 \subset \mathcal{R} \cap (Q_3 L) \perp \) is clearly Zariski-closed. Since \( L \in F_3 - Z_3 \) is chain-admissible there is at least one linear space \( L_0 \in F_4 - Z_4 \) containing \( L \). Then \( L_0 \cap \mathcal{R} = L_0 \cap Q_1 \cap Q_2 \) has dimension 2, since \( L_0 \) is admissible, and similarly \( L \cap \mathcal{R} = L \cap Q_1 \cap Q_2 \) has dimension 1. We may therefore find a point \( P \in L_0 \cap \mathcal{R} - L \cap \mathcal{R} \). We claim that \( P \in \mathcal{R} \cap (Q_3 L) \perp - \mathcal{R}_1 \), which shows that \( \mathcal{R}_1 \) is a proper subset of \( \mathcal{R} \cap (Q_3 L) \perp \), and so proves the first part of the lemma. Since \( L \subset L_0 \subset Q_3 \) it follows that \( L_0 \subset (Q_3 L) \perp \), whence \( P \in L_0 \cap \mathcal{R} \subset (Q_3 L) \perp \cap \mathcal{R} \). On the other hand if we had \( P \in \mathcal{R}_1 \) there would be a linear space \( L' \) such that \( (P, L') \in I \). Then we would have \( P \in L' \) and \( L \subset L' \), and therefore \( L' = < L, P > \), since \( P \notin L \) by our choice of \( P \). However the same reasoning shows that \( L_0 \) is also equal to \( < L, P > \), so that \( L' = L_0 \). This gives us a contradiction, since \( L' \in Z_4 \) while \( L_0 \in F_4 - Z_4 \).

Turning to the second part of the lemma, we consider
\[ I := \{(P, L') \in (Q_3 \cap (Q_3 L) \perp) \times Z_{t+1} : P \in L', L \subset L' \}, \]
and take $E_t = \pi_1(I)$, so that $E_t \subseteq Q_3 \cap (Q_3L)^{\perp}$ is clearly Zariski-closed. Since $L \in F_t - Z_t$ is chain-admissible there is at least one linear space $L_0 \subseteq F_{t+1} - Z_{t+1}$ containing $L$. We claim that $P \in Q_3 \cap (Q_3L)^{\perp} - E_t$ for any point $P \in L_0 - L$, which will show that $E_t$ is a proper subset of $Q_3 \cap (Q_3L)^{\perp}$. Since $L \subseteq L_0 \subset Q_3$ it follows that $L_0 \subseteq (Q_3L)^{\perp}$, whence $P \in Q_3 \cap (Q_3L)^{\perp}$. On the other hand if we had $P \in E_t$ there would be a linear space $L'$ such that $(P, L') \in I$. Then we would have $P \in L'$ and $L \subseteq L'$, and therefore $L' = < L, P >$, since $P \notin L$ by our choice of $P$. However the same reasoning shows that $L_0$ is also equal to $< L, P >$, so that $L' = L_0$. This gives us a contradiction, since $L' \subseteq Z_{t+1}$ while $L_0 \subseteq F_{t+1} - Z_{t+1}$.

### 3 Global 3-planes in $Q_3$

In this section we shall make repeated use of three key principles. The first of these is the fact that we have weak approximation on quadrics. The second is that if $V$ is an absolutely irreducible projective variety defined over the number field $K$, with a smooth point $P_v$ in some completion $K_v$, then the $K_v$-points of $V$ are Zariski-dense in any given neighbourhood of $P_v$. (See Browning, Dietmann and Heath-Brown [5, Lemma 3.4], for example. The proof is an application of the implicit function theorem.)

The third general principle is embodied in the following lemma.

**Lemma 3.1** Let $V$ be a projective algebraic variety defined over a completion $K_v$ of $K$ by equations

$$f_1(x_0, \ldots, x_m) = \ldots = f_r(x_0, \ldots, x_m) = 0,$$

and suppose that $P$ is a $K_v$-point on $V$ at which the vectors $\nabla f_1, \ldots, \nabla f_r$ are linearly independent. Suppose we are given varieties $V^{(j)}$ defined over $K_v$ by equations

$$f_{1}^{(j)}(x_0, \ldots, x_m) = \ldots = f_{r}^{(j)}(x_0, \ldots, x_m) = 0,$$

of bounded degree, in which $f_{1}^{(j)} \to f_1$ (under the metric induced from $K_v$) as $j \to \infty$. Then for sufficiently large $j$ there are $K_v$-points $P^{(j)}$ on $V^{(j)}$ such that $P^{(j)} \to P$ as $j \to \infty$. Moreover $P^{(j)}$ will be a nonsingular point of $V^{(j)}$, in the sense above.

For the proof we suppose firstly that $v$ is a finite place. We let $P = [t]$ say, and we rescale $t$ and the polynomials $f_j$ so as to have $v$-adic integer coefficients. Since $f_{1}^{(j)} \to f_1$ it follows that $f_{1}^{(j)}$ also has integral coefficients if $j$ is large enough. By hypothesis, the matrix formed from the rows $\nabla f_{1}(t), \ldots, \nabla f_{r}(t)$ has rank $r$. We may therefore suppose without loss of generality that the determinant, $\Delta$, say, of the first $r$ columns is non-zero. Let $\Delta_j$ be the corresponding determinant formed from $\nabla f_{1}^{(j)}(t), \ldots, \nabla f_{r}^{(j)}(t)$. Thus $\Delta_j \to \Delta$, so that $|\Delta_j|_v = |\Delta|_v \neq 0$ if $j$ is large enough.

We also set

$$\delta_j = \max\{|f_{1}^{(j)}(t)|_v, \ldots, |f_{r}^{(j)}(t)|_v\},$$

and note that $\delta_j$ tends to

$$\max\{|f_{1}(t)|_v, \ldots, |f_{r}(t)|_v\} = 0,$$
since \( P \in V \). Thus if \( j \) is large enough we will have \( \delta_j < |\Delta_j|^2 \). This condition allows us to use Hensel’s Lemma, which provides a point \( P^{(j)} = [t^{(j)}] \) on \( V^{(j)} \), with
\[
\max_{i} |t^{(j)}_i - t_i|_v \leq \delta_j / |\Delta|_v. \quad (3.1)
\]
It follows that \( P^{(j)} \) tends to \( P \) as required. Since \( \Delta_j \neq 0 \) for large \( j \) the nonsingularity condition also holds.

When \( v \) is an infinite place we use a completely analogous argument, replacing Hensel’s Lemma by Newton Approximation. We begin by normalizing so that the entries of \( t\), and the coefficients of the \( f_j \), all have modulus at most 1. The condition \( \delta_j < |\Delta_j|^2 \) has to be replaced by \( \delta_j < C|\Delta_j|^2 \) with a constant \( C \) depending on \( m, r \) and the degrees of the polynomials involved. Similarly, the bound in (3.1) becomes \( C\delta_j / |\Delta| \) with a corresponding constant \( C' \). With these changes the proof goes through as before.

For our theorem we will assume that we are given local points \( [x_v] \in \mathcal{R}(K_v) \) for every place \( v \) of \( K \). We will also be given a finite set of places \( S \) and a (small) positive \( \varepsilon \), and our challenge will be to find a point \( [x] \in \mathcal{R}(K) \) such that
\[
|x - x_v|_v < \varepsilon \quad \text{for all } v \in S. \quad (3.2)
\]
Without loss of generality we will include all infinite places in \( S \), as well as all finite places above rational primes up to 37. In particular, \( S \) will be non-empty. From now on we will assume that the number \( n \) of variables in our quadratic forms satisfies \( n \geq 19 \).

The variety \( \mathcal{R} \) is nonsingular, and for each \( v \in S \) the \( K_v \)-points of \( \mathcal{R} \) are therefore Zariski-dense in every neighbourhood of \( [x_v] \), by the second principle above. It follows that if \( \mathcal{R}_0 \) is as in Lemma 2.7 then there is a point \( [x'_v] \in \mathcal{R} - \mathcal{R}_0 \), defined over \( K_v \), in the neighbourhood \( |x_v - x'_v|_v < \varepsilon / 2 \). Thus it suffices to find a \( K \)-point of \( \mathcal{R} \) with
\[
|x - x'_v|_v < \varepsilon / 2 \quad \text{for all } v \in S,
\]
where now there is a chain-admissible 3-plane \( L_v \in F_3 \) through \( [x_v] \). We therefore change our notation, replacing \( x'_v \) by \( x_v \) and \( \varepsilon \) by \( \varepsilon / 2 \) so that is still suffices to work with the condition (3.2). Note that \( L_v \) may be defined over \( K_v \), rather than \( K_v \). None the less the existence of a single chain-admissible \( L_v \) shows that the generic 3-plane \( L \subset Q_3 \) through \( [x_v] \) is also chain-admissible. Thus we can in fact assume that \( L_v \) is defined over \( K_v \).

Our plan now is to produce a sequence of 3-planes \( L^{(m)} \) defined over \( K \), which approximate \( L_v \) for each \( v \in S \), in the following sense.

**Lemma 3.2** For each \( v \in S \) let \( L_v \in F_3 - Z_3 \) be a chain-admissible 3-plane defined over \( K_v \), and suppose that \( [e_0,v], \ldots, [e_3,v] \) is a basis of \( L_v \). Then there are sequences of \( K \)-points \( \{e^{(m)}_0,v\}, \ldots, \{e^{(m)}_3,v\} \) spanning chain-admissible 3-planes \( L^{(m)} \subset Q_3 \), such that
\[
\lim_{m \to \infty} e^{(m)}_i = e_i,v \quad \text{for } 0 \leq i \leq 3, \quad (3.3)
\]
for every \( v \in S \).
We teach our students that a sequence can have at most one limit! There is of course some abuse of notation above, and strictly speaking we should have said that \( \iota_v(e^{(m)}_i) \) tends to \( e_{i, v} \), where \( \iota_v \) is the embedding of \( K^n \) into \( K^n_v \). In effect the result is saying that the Fano variety of 3-planes in \( Q_3 \) satisfies weak approximation. We will produce the sequences \( [e (m)] \) by induction on \( t \), the case \( t = 0 \) merely being an instance of weak approximation on \( Q_3 \).

We therefore consider the induction step, and suppose we already have suitable sequences of vectors \( e(0), \ldots, e^{(m)}_{t-1} \). Thus the conditions required for \( e^{(m)}_t \) are that

\[
[e^{(m)}_t] \in V^{(m)} := \{ [x] \in Q_3 : x^T Q_3 e^{(m)}_i = 0 \text{ for } 0 \leq i \leq t-1 \},
\]

and that

\[
\lim_{m \to \infty} e^{(m)}_t = e_{t, v} \tag{3.4}
\]

for every \( v \in S \). Notice that (3.3) (for \( i \leq t-1 \)) and (3.4) automatically ensure that the vectors \( e^{(m)}_0, \ldots, e^{(m)}_{t-1} \) are linearly independent, if \( m \) is large enough.

The variety \( V^{(m)} \) is a quadric of rank at least \( n - 2t \geq 5 \), so that it must have smooth points over \( K_v \) for every finite place \( v \). For every \( v \in S \), and in particular for every infinite place, the varieties \( V^{(m)} \) are approximations to \( V_v := \{ [x] \in Q_3 : x^T Q_3 e_{i, v} = 0 \text{ for } 0 \leq i \leq t-1 \} \), in the sense given by Lemma 3.1. Moreover \( [e^{(m)}_t] \) is a nonsingular point on \( V_v \), in the sense of the lemma. It follows that for large enough \( m \) the variety \( V^{(m)} \) has a smooth point \( [e^{(m)}_t] \) for every \( v \in S \), such that

\[
\lim_{m \to \infty} e^{(m)}_t = e_{t, v}.
\]

In particular \( V^{(m)} \) has points everywhere locally, and so has a \( K \)-point, by the Hasse principle.

We can now complete the induction step. Given \( \eta > 0 \) we choose \( m_0(\eta) \) so that

\[
|f^{(m)}_{i, v} - e_{i, v}|_v < \eta/2
\]

for all \( v \in S \), and all \( m \geq m_0(\eta) \). Moreover we can use weak approximation on \( V^{(m)} \) to find points \( e^{(m)}_t \) on \( V^{(m)}(K) \) such that

\[
|e^{(m)}_t - f^{(m)}_v|_v < \eta/2
\]

for all \( v \in S \), and all \( m \geq m_0(\eta) \). Then for all \( v \in S \) and all \( m \geq m_0(\eta) \) we will have

\[
|e^{(m)}_t - e_{t, v}|_v < \eta,
\]

whence (3.3) holds, as required. Finally, since \( L_v \notin Z_3 \) we will have \( L^{(m)} \notin Z_3 \) for large enough \( m \), by continuity, so that \( L^{(m)} \) is also chain-admissible.

## 4 Completion of the Argument

Our strategy now is to consider the intersection of \( Q_1 \cap Q_2 \) with \( L^{(m)} \). We begin with the following result.
Lemma 4.1 Let $\varepsilon > 0$ be given. If $m$ is large enough, for every $v \in S$ there will be a $K_v$-point $[y_v^{(m)}] \in Q_1 \cap Q_2 \cap L^{(m)}$ such that

$$|y_v^{(m)} - x_v|_v < \varepsilon/2.$$  

This is a further application of Lemma 3.1. The varieties

$$V^{(m)} := Q_1 \cap Q_2 \cap L^{(m)}$$

are approximations to $V := Q_1 \cap Q_2 \cap L_v$, in the sense of the lemma, and $P = [x_v]$ lies on $V$, since we have both $P \in R$ and $P \in L_v$. Moreover $L_v$ is chain-admissible, and hence in particular is admissible, whence $V$ is nonsingular. The lemma therefore produces appropriate points $[y_v^{(m)}]$. 

We now fix a suitable $m$ in Lemma 4.1 and write $L = L^{(m)}$ and $y_v = y_v^{(m)}$ accordingly. Thus $L$ is a chain-admissible 3-plane, defined over $K$. Moreover for each $v \in S$ we have $[y_v] \in Q_1 \cap Q_2 \cap L$. Finally, to prove our theorem it will suffice to find a point $[x] \in R(K)$ with

$$|y_v - x|_v < \varepsilon/2$$

for each $v \in S$.

For each $v \in S$ the variety $Q_1 \cap Q_2 \cap L$ has a local point, namely $[y_v]$. Moreover it is nonsingular, since $L$ is chain-admissible. Thus there are local points at all but finitely many places. Let $T$ be the set of places for which there are no local points. Thus $S$ and $T$ are disjoint, so that $T$ is a finite set of finite places $v$ each of which lies over a prime $p \geq 37$. To handle this remaining set of places it suffices to intersect $Q_1 \cap Q_2$ with a suitable 4-plane, as our next result shows.

Lemma 4.2 If $T$ is nonempty there is a chain-admissible 4-plane $L'$, defined over $K$, such that $L \subseteq L'$, for which $Q_1 \cap Q_2 \cap L'$ has $K_v$-points for every $v \in T$.

It follows of course that $Q_1 \cap Q_2 \cap L'$ has points over every completion of $K$. Naturally, if $T$ were empty the same would already be true for $Q_1 \cap Q_2 \cap L$.

To prove Lemma 4.2 we choose $K$-points $[e_0], \ldots, [e_3]$ spanning $L$, and look for an additional $K$-point $[x] = [e_4]$ such that

$$L' := \langle [e_0], \ldots, [e_4] \rangle$$

fulfils the necessary conditions. In order to have $L' \subseteq Q_3$ we will require $[x] \in Q_3 \cap (Q_3L)^+$. We would like the variety $L'$ to be chain-admissible, and so we will require that $L' \not\subseteq Z_4$. However $L$ itself is chain-admissible, so that there is at least one point $P_0$, (which might be defined over $K$) for which

$$< L, P_0 > \in F_4 - Z_4.$$  

Such a point $P_0$ will be smooth point of $Q_3 \cap (Q_3L)^+$. It follows that the set of points $P$ for which $< L, P >$ is a chain-admissible 4-plane, is a nonempty Zariski-open subset $(U, \text{say})$ of $Q_3 \cap (Q_3L)^+$.

To arrange that $Q_1 \cap Q_2 \cap L'$ has a $K_v$-point it will be helpful if $[x]$ is “near” to a $K_v$-point of $Q_1 \cap Q_2$. We would therefore like the variety

$$(Q_3 \cap (Q_3L)^+) \cap (Q_1 \cap Q_2) = (Q_3L)^+ \cap R$$

to be a $K_v$-point of $Q_1 \cap Q_2$. We would therefore like the variety

$$(Q_3 \cap (Q_3L)^+) \cap (Q_1 \cap Q_2) = (Q_3L)^+ \cap R$$


to contain a $K_v$-point $[x_v]$ for every $v \in T$. However $(Q_3L)^\perp$ has dimension $n - 5 \geq 14$, so that $(Q_3L)^\perp \cap R$ is the zero locus of a system of three quadratic forms in at least 15 variables. There is therefore a $K_v$-point whenever $v$ is a finite place above a prime $p \geq 37$, by Heath-Brown [3 Corollary 1]. (The reader should note that one only needs to know that some bound of the form $p \geq p_0$ suffices. This is a corollary of the famous Ax–Kochen Theorem [2]. However in the present situation one can now provide an explicit value of $p_0$. Indeed the work of Schuur [14] could also be used here.) Thus there are points $[x_v] \in R \cap (Q_3L)^\perp$ for every $v \in T$. Indeed, since $L$ is chain-admissible it is certainly admissible, so that $R \cap (Q_3L)^\perp$ must be nonsingular. If then follows that the $K_v$-points on $R \cap (Q_3L)^\perp$ will be Zariski-dense. In particular we can choose a $K_v$-point $[x_v]$ in $R \cap (Q_3L)^\perp - R_1$, with $R_1$ as in Lemma 2.8. Thus, if we set $L_v = L_v[x_v] > 0$, then $L_v$ is admissible so that $Q_1 \cap Q_2 \cap L_v$ is nonsingular. In particular $[x_v]$ will be a smooth point of $Q_1 \cap Q_2 \cap L_v$.

Since $[x_v] \in R \cap (Q_3L)^\perp$ it follows in particular that $[x_v] \in Q_3 \cap (Q_3L)^\perp$. We now claim that $[x_v]$ cannot be a singular point of $Q_3 \cap (Q_3L)^\perp$. For otherwise the vectors $Q_3e_v, Q_3e_0, \ldots, Q_3e_3$

would be linearly dependent. Since $Q_3$ is nonsingular it would follow that $x_v \in <e_0, \ldots, e_3>$,

so that $[x_v] \in L$. However $[x_v]$ was chosen to lie in $R$, so that it would in particular be a $K_v$-point of $Q_1 \cap Q_2 \cap L$. We would therefore have a contradiction, since $T$ was defined to be the set of places where $Q_1 \cap Q_2 \cap L$ had no local points.

Thus $[x_v]$ is a smooth point of $Q_3 \cap (Q_3L)^\perp$. This variety certainly has at least one smooth $K_v$-point, since the quadratic form $Q_3$ was constructed to split off at least $(n - 5)/2 > 4$ hyperbolic planes over $K$. Thus we can use weak approximation on $Q_3 \cap (Q_3L)^\perp$ to produce a sequence of $K$-points $[x_v^{(m)}]$, such that $\lim_{m \to \infty} x_v^{(m)} = x_v$

for every $v \in T$. For large enough $m$ a continuity argument shows that if $L_v = L_v[x_v^{(m)}]$ then $L_v^{(m)} \in F_3 - Z_3$. We also see that $Q_1 \cap Q_2 \cap L_v^{(m)}$ approximates $Q_1 \cap Q_2 \cap L_v$ in the sense of Lemma 3.1 Moreover $[x_v]$ is a smooth point of $Q_1 \cap Q_2 \cap L_v$, whence Lemma 3.1 provides $K_v$-points $P^{(m)}$ on $Q_1 \cap Q_2 \cap L_v^{(m)}$ as soon as $m$ is large enough, for all $v \in T$. The lemma then follows on choosing $L' = L_v^{(m)}$ with a suitably large $m$.

As the final step in our argument we state the following result.

**Lemma 4.3** There is an admissible 7-plane $L''$ containing $L'$, defined over $K$, and such that $Q_1 \cap Q_2 \cap L''$ has points over every completion of $K$.

Before presenting the proof of the lemma, we show how it suffices for our theorem. Since $L'' \in F_7$ is admissible, the variety $Q_1 \cap Q_2 \cap L''$ is contained in $R$, and is a nonsingular intersection of two quadrics in $\mathbb{P}^7$. By construction it has points over $K_v$ for every place $v$. Indeed the subvariety $Q_1 \cap Q_2 \cap L$ has a $K_v$-point $[y_v]$ for every $v \in S$, and has $K_v$-points for all $v \notin T$, while $Q_1 \cap Q_2 \cap L'$ has $K_v$-points for any remaining places $v \in T$. We may therefore apply the author’s result [11 Theorem 1] described in the introduction, which
shows that $Q_1 \cap Q_2 \cap L''$ satisfies the Hasse principle and weak approximation. This allows us to conclude that $Q_1 \cap Q_2 \cap L'$ has $K$-points arbitrarily close to $[y_v]$ for each $v \in S$. Our theorem therefore follows.

It remains to establish Lemma 4.3. We have already observed that either $Q_3 \cap Q_2 \cap L$ has points everywhere locally, if $T$ is empty, or $Q_1 \cap Q_2 \cap L'$ does. The linear spaces $L$ and $L'$ are chain-admissible, and are defined over $K$. It is therefore enough to show that if $M$ is any chain-admissible linear space of dimension $t \in [3,6]$, defined over $K$, then there is a chain admissible space $M' \supset M$ of dimension $t+1$, also defined over $K$. Once this is proved we can use this repeatedly to go from $L$ or $L'$ to $L''$.

We recall that $Q_3$ contains at least one 7-plane defined over $K$, whence $M$ will be contained in such a 7-plane, $M^*$, say. We can choose a basis $e_1, \ldots, e_n$ for $K^n$ so that the 7-plane $Q_3$ is spanned by $[e_1], \ldots, [e_{t+1}]$ and $M^*$ by $[e_1], \ldots, [e_8]$, and such that

$$Q_3 \left(\sum_1^n X_i e_i\right) = X_1 X_9 + \ldots + X_8 X_{16} + Q(X_{17}, \ldots, X_n)$$

for a suitable nonsingular form $Q$. Then $(Q_3 M)^\perp$ is spanned by $[e_1], \ldots, [e_8]$ and $[e_{t+10}], \ldots, [e_n]$, and one therefore sees that $[e_8]$ will be a smooth $K$-point of $Q_3 \cap (Q_3 M)^\perp$. Having shown that there is at least one smooth $K$-point on $Q_3 \cap (Q_3 M)^\perp$ we deduce that the $K$-points are Zariski-dense, so that there is a $K$-point $P \in Q_3 \cap (Q_3 M)^\perp - E_t$, in the notation of Lemma 2.8. We can then complete the proof of Lemma 4.3 by taking $M' = < M, P >$.

References

[1] V.N. Aznar, On the Chern classes and the Euler characteristic for nonsingular complete intersections, *Proc. Amer. Math. Soc.*, 78 (1980), 143–148.

[2] J. Ax and S. Kochen, Diophantine problems over local fields. I, *Amer. J. Math.*, 87 (1965), 605–630.

[3] B.J. Birch, Forms in many variables, *Proc. Roy. Soc. Ser. A*, 265 (1961/1962), 245–263.

[4] B.J. Birch and H.P.F. Swinnerton-Dyer, The Hasse problem for rational surfaces, *J. Reine Angew. Math.*, 274/275 (1975), 164–174.

[5] T.D. Browning, R. Dietmann, and D.R. Heath-Brown, Rational points on intersections of cubic and quadric hypersurfaces, *J Inst. Math. Jussieu*, to appear. arXiv:1309.0147

[6] T.D. Browning, and D.R. Heath-Brown, Forms in many variables and differing degrees, *J. Eur. Math. Soc.*, to appear. arXiv:1403.5937.

[7] J.-L. Colliot-Thélène, J.-J. Sansuc, and H.P.F. Swinnerton-Dyer, Intersections of two quadrics and Châtelet surfaces. I, *J. Reine Angew. Math.*, 373 (1987), 37–107.

[8] J. Harris, *Algebraic geometry. A first course*, Graduate Texts in Mathematics, 133. (Springer-Verlag, New York, 1992).
[9] D.R. Heath-Brown, Zeros of systems of $p$-adic quadratic forms, *Compos. Math.*, 146 (2010), 271–287.

[10] D.R. Heath-Brown, $p$-adic zeros of systems of quadratic forms, *Diophantine methods, lattices, and arithmetic theory of quadratic forms*, 131–139, Contemp. Math., 587, (Amer. Math. Soc., Providence, RI, 2013).

[11] D.R. Heath-Brown, Zeros of pairs of quadratic forms, *J. reine Angew. Math.*, to appear. [arXiv:1304.3894]

[12] D.R. Heath-Brown and L.B. Pierce, Simultaneous integer values of pairs of quadratic forms, *J. reine Angew. Math.*, to appear. [arXiv:1309.6767]

[13] D.B. Leep, Systems of quadratic forms, *J. Reine Angew. Math.*, 350 (1984), 109–116.

[14] S. E. Schuur, On systems of three quadratic forms, *Acta Arith.* 36 (1980), 315–322.

[15] C.M. Skinner, Forms over number fields and weak approximation, *Compositio Math.*, 106 (1997), 11–29.

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