Proton neutron Star Convection Simulated with a New General Relativistic Boltzmann Neutrino Radiation Hydrodynamics Code

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Abstract

We investigate proton neutron star (PNS) convection using our newly developed general relativistic Boltzmann neutrino radiation hydrodynamics code. This is a pilot study for more comprehensive investigations later. As such, we take a snapshot of a PNS at 2.3 s after bounce from a 1D PNS cooling calculation and run our simulation for ~160 ms in 2D under axisymmetry. The original PNS cooling calculation neglected convection entirely and the initial conditions were linearly unstable to convection. We find in our 2D simulation that convection is instigated there indeed and expands inward after being full-fledged. The convection then settled to a quasi-steady state after ~100 ms, being sustained by the negative $Y_e$ gradient, which is in turn maintained by neutrino emissions. It enhances the luminosities and mean energies of all species of neutrinos compared to 1D. Taking advantage of the Boltzmann solver, we analyse the possible occurrence of neutrino fast flavor conversion (FFC). We found that FFC is likely to occur in regions where $Y_e$ is lower, and that the growth rate can be as high as $\sim 10^{-1} \text{ cm}^{-1}$.

Unified Astronomy Thesaurus concepts: Neutron stars (1108); Core-collapse supernovae (304); Supernova neutrinos (1666); Radiative transfer simulations (1967); Neutrino oscillations (1104)

1. Introduction

Massive stars heavier than $\sim 8 M_\odot$ will undergo core-collapse supernovae (CCSNe) and eventually form neutron stars (NSs) or black holes (BHs). Sophisticated CCSN simulations have been performed in the last decades, and successful explosions have been indeed obtained (see Burrows & Vartanyan 2021 for a recent review). However, the explosion energies and the Ni yields suggested by observations remain to be reproduced systematically by 3D simulations (Bollig et al. 2021).

Recently, the subsequent phase, i.e., the proton neutron star (PNS) cooling, is also getting attention. For Galactic CCSNe, a large amount of neutrinos will be observed for $\sim 100$ s by current and future detectors (Li et al. 2021). This phase will be hence useful for tightly constraining the nuclear matter equation of state (EOS) (Pons et al. 2001a, 2001b; Roberts et al. 2012; Nakazato & Suzuki 2020; Nagakura & Vartanyan 2022; Nakazato et al. 2022). Putting aside the initial conditions, the evolution of a PNS in its cooling phase may be (and has been indeed) studied independently of the SN explosion preceding it. The theoretical study of the PNS cooling phase has its own difficulties, though. Its long duration of $\sim 10$ s to $\sim 100$ s makes it difficult to simulate numerically with detailed microphysics taken into account.

In fact, long-term ($\gtrsim 10$ s) PNS cooling calculations have been performed in 1D under the assumption of spherical symmetry (see Roberts & Reddy 2017 for a summary of recent studies). The largest drawback of 1D calculations, however, is that convection cannot be treated self-consistently. In previous 1D simulations, it was either simply ignored (Nakazato et al. 2013; Nakazato & Suzuki 2019), or mixing-length treatment was employed to model the matter mixing by convection (Roberts et al. 2012). Other multidimensional features such as rotation, if any, cannot be treated, either (but see also Margalit et al. 2022).

Multidimensional simulations of PNS cooling are very much scarce mainly because of the difficulties mentioned above. For example, Keil et al. (1996) studied PNS convection up to $\sim 1$ s after core bounce in 2D under axisymmetry with radial, gray neutrino transport taken into account. The initial conditions were constructed by extracting a central portion (1.1 $M_\odot$) of the SN core at 25 ms post bounce in their 2D simulation for a 15 $M_\odot$ progenitor. They showed that lepton-driven convection occurs inside the PNS, and the convective zone is extended inward with time as the positive entropy gradient gets weaker. They also found that the neutrino luminosity and the mean energy are both enhanced by convection. Mezzacappa et al. (1998), employing 1D multi-group flux-limited diffusion transport coupled with 2D hydrodynamics, reported that PNS convection is suppressed by neutrino transport in their simulations up to $\sim 100$ ms after bounce. This may be an artifact of the angle-averaged 1D neutrino transport, though, in which the lateral transfer will be overestimated. In fact, the simulations with the 2D neutrino transport performed by Buras et al. (2006) and Dessart et al. (2006) observed convection, which enhances the neutrino luminosities. Note that these studies are all limited to the very

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early phase up to a few 100 ms after bounce, since their main focus is the explosion mechanism. To our knowledge, the paper by Nagakura et al. (2020) is the only one that is devoted to the study of PNS convection in 3D. It is based on their 3D SN simulations, which obtained shock revival. They demonstrated that convection enhances the luminosity of \( \nu_e \), but not of \( \nu_x \) and \( \bar{\nu}_e \). There are even longer simulations in 2D (Burrows & Vartanyan 2021) and 3D (Bollig et al. 2021) up to several seconds post bounce with the two-moment approximation employed for neutrino transfer. Unfortunately, PNS convection was not discussed there. Hence there has been no detailed multidimensional numerical investigation of PNS convection after \( \sim 1 \) s. That is what we want to do in this and later papers.

In so doing, neutrino transport plays an important role, since the PNS is mostly optically thick but the notable exception is near the surface, where neutrinos decouple from matter and their luminosity and spectra are formed. The latter region is a part of the convectively unstable zone, as shown later, and multidimensional neutrino transfer coupled with hydrodynamics is required. Since neutrinos are neither in chemical nor in thermal equilibrium with matter there, the neutrino distribution in phase space should be calculated in general. There are broadly two categories of methods to solve the transfer equation without artificial approximations (see, e.g., Richers et al. 2017 for their comparison): (1) deterministic methods, which directly solve the Boltzmann equation using the \( S_n \) method (Mezzacappa & Bruenn 1993; Sumiyoshi et al. 2005; Sumiyoshi & Yamada 2012) or a spectral method (Peres et al. 2014) and (2) probabilistic methods, of which the Monte Carlo method is a representative (Abdikamalov et al. 2012; Kato et al. 2020).

Since these rigorous transport schemes are computationally demanding, various approximate transport schemes have been also employed in SN simulations: the two-moment transport method (Thorne 1981; Shibata et al. 2011; Cardall et al. 2013), the flux-limited diffusion (FLD) approximation (Arnett 1977), the isotropic diffusion source approximation (IDSA) method (Liebendörfer et al. 2009), and the ray-by-ray plus approximation (Buras et al. 2006). Comparisons of these methods were also conducted and some differences were observed (Skinner et al. 2016; Cabezón et al. 2018; Just et al. 2018).

In this paper we employ the \( S_n \) method, which we have developed over the years for SN simulations (Nagakura et al. 2018, 2019; Harada et al. 2019, 2020; Iwakami et al. 2020). Unlike the computations cited, we use a general relativistic hydrodynamics code coupled to a Boltzmann solver, which is already fully general relativistic (Akaho et al. 2021). It is needless to say that general relativity is indispensable in CCSNe (Bruenn et al. 2001; Lentz et al. 2012; O’Connor & Couch 2018), particularly at late times, when the PNS becomes more compact. Note that many of recent state-of-the-art SN simulations are still not fully general relativistic but use effective potentials (Gläs et al. 2019; Burrows et al. 2020; Mezzacappa et al. 2022; Vartanyan et al. 2022). In this paper, the time evolution of spacetime is still not considered and its metric is fixed. The use of a Boltzmann solver is also motivated by our interest in the collective neutrino oscillation via the fast flavor conversion (FFC). Since FFC is driven by the crossing of the neutrino flavor lepton numbers in the angular distributions in momentum space (Morinaga 2022), Boltzmann solvers, which keep track of the neutrino angular distributions in momentum space faithfully, are best suited for the study of FFC.

As we saw above, the long-term multidimensional study of PNS cooling at \( t \gtrsim 1 \) s is absent. We have started a project to tackle this issue. We have conceived a new Lagrangian formulation in full general relativity to construct numerically equilibrium configurations, possibly with rotation in 2D under axisymmetry (Okawa et al. 2022), to be used for their long-term evolutions. In this paper, we study convection in a PNS, particularly its influence on the neutrino emission, by rather short-term (\( \sim 100 \) ms) 2D simulations conducted for a snapshot at \( t = 2.3 \) s provided by a conventional 1D simulation of PNS cooling (Nakazato & Suzuki 2019). Note that this is a first ever attempt of its kind and meant to be a pilot study for more systematic explorations later. It is known that the convection and turbulent motions it produces in 2D are qualitatively different from those in 3D. Hence we need to be cautious when considering the implications of this study for realistic 3D PNS cooling. However, the difference between 1D and 2D is much larger than that between 2D and 3D (Vartanyan et al. 2018) and we think that the 2D study is still useful.

This paper is organized as follows. Section 2 describes the numerical method, and the simulation setups are summarized in Section 3. The simulation results are presented in Section 4. We summarize our findings and discuss future prospects in Section 5. Details of the newly developed hydrodynamics code are given in Appendix A, and code verification tests are performed in Appendix B. Throughout the paper, we use the metric signature \(-+++\) and the Greek \((\alpha, \beta, \mu, \phi)\) and Latin \((i, j, k)\) indices run over 0–3 and 1–3, respectively.

2. Numerical Method

We use a Boltzmann radiation hydrodynamics code. It simultaneously solves the Boltzmann neutrino transport and hydrodynamics equations in general relativity. The details are summarized in the following subsections.

2.1. Boltzmann Solver

For the neutrino transport, we solve the Boltzmann equation with respect to the distribution function \( f \) in phase space (Lindquist 1966; Ehlers 1971). The equation is written in conservative form (Shibata et al. 2014):

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left[ e^{\nu(0)} \sum_{i=1}^{3} l_i e_i^\mu \sqrt{-g} f \right] - \frac{1}{e^2} \frac{\partial}{\partial \epsilon} (\epsilon g f) + \frac{1}{\sin^2 \theta_e} \frac{\partial}{\partial \theta_e} (\sin \theta_e f \omega(i)) - \frac{1}{\sin^2 \theta_e} \frac{\partial}{\partial \phi_e} (f \omega(i)) = S_{rad},
\]

where \( \epsilon \equiv -\nu(0) \) is the neutrino energy and \( \theta_e \) and \( \phi_e \) are the zenith and azimuth angles in momentum space, respectively. The symbol \( q_i \) represents the momentum space coordinates as \( q_1 = \epsilon, q_2 = \theta_e \), and \( q_3 = \phi_e \). The symbol \( g \) denotes the determinant of the metric tensor. The directional cosines in
momentum space $l_{(i)}$ are given as:

$$l_{(1)} = \cos \theta, \quad l_{(2)} = \sin \theta \cos \phi, \quad l_{(3)} = \sin \theta \sin \phi.$$  

The factors $\omega_{(0)}$, $\omega_{(\theta)}$, and $\omega_{(\phi)}$ are defined as:

$$\omega_{(0)} = e^{-2\rho u^2} \nabla e_{(0)}, \\
\omega_{(\theta)} = \sum_{i=1}^{3} \frac{\omega_{(0)}}{\theta}, \\
\omega_{(\phi)} = \sum_{i=2}^{3} \frac{\omega_{(0)}}{\phi}, \\
\omega_l = e^{-2\rho u^2} \nabla e_{(0)}.$$

In configuration space, spherical polar coordinates are deployed, where $r$, $\theta$, and $\phi$ corresponds to the radius, the zenith, and azimuth angles, respectively. The tetrad bases are written in 3 decomposition:

$$e_{(0)}^{il} = (1/\alpha, \beta^l/\alpha), \\
e_{(1)}^{il} = \gamma_{rr}^{-1/2} \left( \frac{\partial}{\partial r} \right)^{il}, \\
e_{(2)}^{il} = -\frac{\gamma_{\theta \phi}}{\sqrt{\gamma_{\theta \theta} \gamma_{\phi \phi} - \gamma_{\theta \phi}^2}} \left( \frac{\partial}{\partial \theta} \right)^{il} + \frac{\gamma_{\theta \theta}}{\sqrt{\gamma_{\theta \theta}}} \left( \frac{\partial}{\partial \phi} \right)^{il}, \\
e_{(3)}^{il} = \frac{\gamma_{\theta \phi}}{\sqrt{\gamma_{\theta \theta}}} \left( \frac{\partial}{\partial \theta} \right)^{il} + \frac{\gamma_{\phi \phi}}{\sqrt{\gamma_{\phi \phi}}} \left( \frac{\partial}{\partial \phi} \right)^{il},$$

where the symbols $\alpha$, $\beta^l$, and $\gamma_{ij}$ stand for the lapse function, the shift vector, and the spatial metric on the time-constant hypersurface, respectively. Further details and the code tests can be found in Akaho et al. (2021).

### 2.2. Hydrodynamics Solver

In this paper, we newly developed a general relativistic hydrodynamics solver coupled with a Boltzmann neutrino transport solver. It is a general relativistic extension of our previous Newtonian hydrodynamics code. Code verification tests are performed in Appendix B.

We solve the general relativistic hydrodynamics equations written in $3+1$ decomposition (see Section 4 in Shibata 2016):

$$\partial_t \rho + \partial_j (\rho u^j) = 0,$$

$$\partial_t S_i + \partial_j (S_i u^j + \alpha \sqrt{\gamma} P c^2 b_i) = -S_0 c^2 \partial_i \alpha + S_i c \partial_j \beta^j - \frac{1}{2} \alpha c \sqrt{\gamma} S_0 \partial \gamma_{ij}^{hk} - \alpha \sqrt{\gamma} \gamma_{il} G_{il},$$

$$\partial_t (S_0 - \rho c^2) + \partial_j ((S_0 - \rho c^2) u^j + \sqrt{\gamma} P (u^j + c \beta^j)) = \alpha \sqrt{\gamma} S^j G_{ij} - S_i D^j\alpha + \alpha \sqrt{\gamma} u^i G_{ii},$$

where:

$$v^i \equiv \frac{u^i}{u^0},$$

$$\rho_\alpha = \alpha \sqrt{\gamma} \rho_0 u^i, \quad \rho_\phi = \rho_\mu u^\mu,$$

$$S_i = \rho_h u_i c, \quad S_0 = \sqrt{\gamma} (\rho hw^2 - P), \quad w = cu^\mu,$$

and $\rho$, $P$, $u^\mu$, and $h$ represent the density, the pressure, the four-velocity, and the specific enthalpy, respectively. The symbol $G^{\mu\nu}$ stands for feedback from neutrinos, which is calculated in the same way as in our previous Newtonian hydrodynamics code (Nagakura et al. 2017).

We need special care when we discretize the equations. In a near-equilibrium situation, there is severe cancellation between the source terms and the derivatives of the fluxes. However, unlike analytical formulae, it is not necessarily guaranteed in the finite difference. The mismatch between the terms to be canceled will cause unphysical matter motions. We indeed found in the stability test of the relativistic star in Appendix B.2 that ensuring this cancellation as much as possible is crucially important. We give in Appendix A the explicit formulation that satisfies this cancellation exactly in flat spacetime. Although the cancellation is not perfect in curved spacetimes in general, it turns out that the formulation still works well also in such cases as demonstrated in the tests presented in Appendices B.2 and B.3. Further details of the hydrodynamics code are described in Appendix A.

### 3. Setup

We use a snapshot taken from a 1D PNS model (Nakazato & Suzuki 2019) as our initial conditions. In their paper, the collapse of a 15 $M_\odot$ progenitor (Woosley & Weaver 1995) was followed with the Lagrangian radiation hydrodynamics code by Sumiyoshi et al. (2005) up to 300 ms. Although this simulation did not lead to shock revival, they extracted the central portion of the SN core that includes a PNS and its subsequent evolution was calculated for $\sim 80$ s by another code for quasi-static PNS cooling developed by the same authors. This cooling computation was done under the assumption of spherical symmetry, and the EOS of Togashi et al. (2017) was employed. We pick up the snapshot at 2.3 s after bounce and use it as the initial data for our own calculations. Note that the original PNS cooling by Nakazato & Suzuki (2019) employed the FLD approximation. The use of the Boltzmann transport in this paper is hence inconsistent, resulting in some differences in the luminosities and spectra of neutrinos obtained in the two calculations. Since the neutrino reactions incorporated and their treatments are also different between the two simulations, a detailed comparison is not very useful and indeed not needed for the current purpose of the paper, i.e., the study of the dynamics of PNS convection and the difference it could make in the neutrino luminosity and energy. We will hence not consider this issue further.

In order to mitigate the possible unphysical transient at the beginning of the 2D simulation, we first perform a 1D calculation with our code until the initial wobbling is settled. Then we start the 2D calculation, using the resultant...
configuration as the initial conditions. We impose the following perturbations to the four-velocity of matter by hand to instigate convection:

\[
\delta u_r = V \cos \left( m_\pi \frac{r - r_{\min}}{r_{\max} - r_{\min}} \right) \cos(n\theta),
\]

\[
\delta u_\theta = Vr \sin \left( m_\pi \frac{r - r_{\min}}{r_{\max} - r_{\min}} \right) \sin(n\theta),
\]

where \( V \equiv 1 \times 10^{-4} \times c \), \( r_{\min} \equiv 3 \, \text{km} \), \( r_{\max} \equiv 13 \, \text{km} \), \( m \equiv 10 \), and \( n \equiv 10 \).

The mesh setup is the same for the 1D and 2D calculations except that the \( \theta \) and \( \phi_\theta \) degrees of freedom are excluded in 1D. The spatial coordinates are divided into \( N_r = 384 \) and \( N_\theta = 128 \) cells in the \( r \) and \( \theta \) directions, respectively. The radial grid covers the range \( r \in [0:50] \, \text{km} \), where it has a finer resolution around the PNS surface. As for the momentum space, the energy, \( \theta_\nu \), and \( \phi_\nu \) grids have \( N_\nu = 20 \), \( N_\theta = 10 \), and \( N_\phi = 6 \), respectively. The energy grid covers the range \( \epsilon \in [0:300] \, \text{MeV} \), where it is logarithmically spaced. In order to see the resolution dependence of the numerical results, we also run a simulation with a higher spatial resolution: \( N_r = 512 \) and \( N_\theta = 192 \) (referred to as model \( \text{HR-S} \)), and another with a higher angular resolution in momentum space: \( N_\nu = 14 \) and \( N_\theta = 10 \) (model \( \text{HR-M} \)). The boundary conditions are imposed as follows: the reflective condition at the center \( r = 0 \) and on the poles \( \theta = 0, \pi \), the free-streaming condition at the outer boundary located at \( r = 50 \, \text{km} \), and the periodic condition for \( \phi_\nu = 0, 2\pi \).

In this study, we employ the Furusawa–Togashi EOS (Furusawa et al. 2017). Although it is another inconsistency with the initial data prepared with the Togashi EOS, this difference is minor. We employ the neutrino reactions based on Brueg (1985) with some extensions: energy-changing electron scattering and nucleon–nucleon bremsstrahlung is incorporated. The spacetime metric is determined in the following manner. We adopt the metric ansatz:

\[
ds^2 = -e^{2\nu(r)}c^2dt^2 + (X(r))^2dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2.
\]

Then the Einstein equation gives the function \( X(r) \) as (Oppenheimer & Volkoff 1939):

\[
X(r) = \left( 1 - \frac{2GM(r)}{r} \right)^{-\frac{3}{2}},
\]

in which \( M(r) \) is the enclosed mass and is given by the following equation with the density \( \rho \):

\[
\frac{dM(r)}{dr} = 4\pi r^2 \rho(r).
\]

The function \( \Phi(r) \), on the other hand, is derived as (Oppenheimer & Volkoff 1939):

\[
\frac{d\Phi(r)}{dr} = \frac{G}{r^2 c^2} \left( M(r) + \frac{4\pi r^2 \rho(r)}{c^2} \left( 1 - \frac{2GM(r)}{rc^2} \right)^{-1} \right),
\]

where \( P(r) \) denotes the pressure. The subsequent evolution of the metric is not considered and the spacetime geometry is fixed in this paper. Since the density distribution is not globally asymmetric and the fluid velocities are orders of magnitude smaller than the speed of light, this assumption is reasonable.

**4. Results**

**4.1. Basic Characteristics**

We first perform the 1D calculations with our own code until the initial transients subside as stated above. Figure 2 compares the radial profiles of some hydrodynamical variables at 50 ms with those at the initial time. Note that the computational range in this calculation is extended from the original one: \( r \lesssim 17 \, \text{km} \) to \( r < 50 \, \text{km} \) for numerical convenience by adding low-density fluid velocities are orders of magnitude smaller than the speed of light, this assumption is reasonable.

The nontrivial metric components, \( g_\nu \) and \( g_{rr} \), are shown in Figure 1.

![Figure 1. Radial profiles of metric components \( g_\nu \) (red) and \( g_{rr} \) (blue).](image-url)
In order to see the vigor of convection, we show the time evolution of the kinetic energy in Figure 4. The perturbation given in Equation (12) instigates violent convective motions in several milliseconds. The kinetic energy decreases until 30 ms and settles gradually down to a fluctuation around a constant value thereafter. The initial violent turbulence is an artifact produced by the transition from 1D to 2D. After $\sim 100$ ms, however, the transient is subsided and the convection has reached a (quasi) steady state, in which the angle-averaged matter profiles change much more slowly on the secular timescale. Regarding this state as representative of PNS convection around this time of the cooling phase, we analyse it further in the following.

We first look at where the PNS convection occurs. Figure 5 is a 2D color map of the kinetic energy in the meridian section at different times. The velocity field inside the PNS is superimposed. Note that the PNS is defined as the region where the density is higher than $10^{11}$ g cm$^{-3}$. The kinetic energy and velocity distributions are nonspherical from the center up to $r \sim 17$ km, with the lateral motions dominant over

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**Figure 2.** Radial profiles of the density (top), the temperature (middle), and the electron fraction (bottom) are shown with red lines. The original data (provided by another code) is shown with blue lines.

**Figure 3.** Radial profiles of temperature (top), the electron fraction (middle), and the entropy per baryon (bottom). The green lines denote 2Dinitial, and the rest denote the 2D data for time snapshots at 10, 30, 60, 90, 120, and 150 ms, respectively. Note that $\theta$-averaged quantities are shown for the 2D data.
the radial ones (see also Figure 4). This seems to be consistent with the convection that Nagakura et al. (2020) investigated. This is in contrast to other earlier works, though, in which either almost no convection was found (Lattimer & Mazurek 1981; Mezzacappa & Bruenn 1993) or it was observed in a rather limited region (Dessart et al. 2006; Nagakura & Johns 2021). This discrepancy may be attributed to the differences in the treatment of neutrino transfer as well as to the difference in the phase on which we focus. As we saw in Figure 3, the entropy gradient is positive in the entire region while the $Y_e$ gradient remains negative in the convective region at this rather late time of $t \sim 2$ s. Although our simulation is not fully self-consistent as the initial matter profile is taken from a 1D PNS cooling calculation (Nakazato & Suzuki 2019), in which convection was ignored, the entropy and $Y_e$ distributions are self-adjusted and sustained once the (quasi) steady state is established, and hence the results obtained in this paper will have some generality.

In order to see the convectively unstable region more quantitatively, we evaluate the relativistic Brunt–Väisälä frequency:

$$N^2 = \frac{\partial \alpha}{\partial r} \alpha \left( \frac{1}{c_s^2} \frac{\partial P}{\partial r} - \frac{\partial \rho (1 + \epsilon)}{\partial r} \right),$$

(17)

where $N^2 < 0$ indicates the linear convective instability. We adopted this formula from Müller et al. (2013), in which conformal flatness is assumed, by simply replacing the conformal factor with $\gamma_{rr}$. It is conventional (Gossan et al. 2020) to define the following quantity:

$$f_{BV} = \text{sign}(N^2) \sqrt{|N^2|}.$$  

(18)

Figure 6 shows the spatial distributions of $f_{BV}$ in the meridian section for some time snapshots. The leftmost panel presents the result for the initial time. We can see clear boundaries between the (linearly) unstable region with bluish colors and the stable regions with reddish colors. It is found that the inner region, $r < 9$ km, is linearly stable although it has negative $Y_e$ gradients (see Figure 3). Convection initially occurs in this linearly unstable region. As it grows with time and becomes nonlinear, the convective region is extended inward by overshooting and the inner stable region becomes extinct eventually. At the same time, the value of $f_{BV}$ gets smaller inside the convective region as can be observed in the other panels to the right. This happens because the convection mixes the matter up until its profiles become marginally stable, since the neutrino emission still works in the opposite way to produce unstable $Y_e$ profiles. As a result, the convection is sustained even if the Brunt–Väisälä frequency is marginally positive.

Next, we discuss the properties of neutrino emission from the PNS. Figure 7 shows the time evolution of the luminosity and mean energy of the neutrinos. Similarly to the kinetic energy, we found a sudden increase of the $\nu_e$ luminosity right after the start of the simulation, which is followed by a gradual decline as the initial transient subsides, and is settled to a roughly constant value in $\sim 70$ ms. The rise of the $\bar{\nu}_e$ luminosity then starts to decline very quickly, followed by a gradual increase to a constant value over several tens of milliseconds. It is interesting that the time variations are anticorrelated with those of $\nu_e$. The $\nu_e$ luminosity, on the other hand, increases monotonically and approaches an asymptotic value a bit more quickly than other neutrinos without producing a pronounced peak. These different behaviors reflect the differences in their decoupling with matter. This early time evolution is an artifact, though, caused by the switch from 1D to 2D. We hence focus on the asymptotic phase after (quasi) steady convection is established. The luminosity and the mean energy in 2D are both higher than those in 1D (compare the solid lines with the dashed lines). Such an enhancement was also observed in the previous studies on the earlier phase (Keil et al. 1996; Buras et al. 2006; Dessart et al. 2006; Roberts et al. 2012; Pascal et al. 2022).

The enhancement of the mean energy in 2D can be understood from the angle-averaged radial profiles of the temperature near the neutrino sphere shown in Figure 8. The vertical lines indicate the positions of the neutrino spheres, which are defined to be the surfaces with an optical depth of 2/3, for individual neutrino species at an energy of 12.8 MeV. The dashed–dotted and dashed lines indicate the positions of the neutrino spheres for 1D and 2D, respectively. It is evident that the temperature is higher in 2D than in 1D. This is due to the dredge-up by convection of the hotter matter located originally deeper inside. If the location of a neutrino sphere were unchanged, the mean energy would be even higher. As should be also apparent in Figure 8, the neutrino spheres are all shifted outwards to lower temperatures. This is due to the density rise at $r \gtrsim 15$ km (see Figure 9) associated with the expansion of the PNS, which is in turn driven by convection. For all flavors, the temperatures at the neutrino spheres are higher, which naturally leads to their greater mean energies as well as luminosities as observed in Figure 7. In addition, the smaller neutrino spheres in 1D means that they are located deeper in the gravitational well, and the neutrinos emitted from it experience a greater gravitational redshift. This effect further lowers the luminosity and the mean energy for all flavors of neutrinos in 1D.

4.2. Asymmetric Properties

In this subsection, we discuss the asymmetry caused by convection and its effect on neutrino emission. Since we assume axisymmetry in this study, the results are inevitably affected by the artifacts unique to 2D. Nevertheless, the results will be useful as a reference for 3D studies in the future.
Figures 10 and 11 show the angle-dependence of the density, temperature, and electron fraction at different times. The density variation is rather small except for the outer low-density layer at \( r \gtrsim 20 \) km, which we do not consider. The temperature distribution shows remarkable variations at \( r \approx 13 \) km due to convection, whereas it is almost symmetric at larger radii. The temperature profiles are especially bumpy at the north (\( \theta = 0 \)) and the south (\( \theta = \pi \)) poles, suggesting that the convective motion is more violent there. The \( Ye \) profile is also highly asymmetric in a more extended region, in which the maximum deviation from the average, \( \Delta Ye \gtrsim 0.05 \), occurs around \( r = 10 \) km. In addition, \( Ye \) tends to be lower on the poles than on the equator. This again indicates that the convection is stronger near the poles than around the equator.

In order to see the convective pattern more clearly, we plot the time-averaged electron fraction and velocity in Figure 12. One can see three large vortices: one near the north pole, another centered at an intermediate latitude in the northern hemisphere, and the other covering most of the southern hemisphere. Note that we do not impose equatorial symmetry. They are extended radially from \( \sim 7 \) km to \( \sim 16 \) km. Near the both poles matter is moving downwards, causing lower values of \( Ye \) to prevail at \( r \approx 15 \) km. At the same time, the central region with higher values (\( \gtrsim 0.15 \)) of \( Ye \) becomes a bit oblate. This is why we observed that \( Ye \) tends to be lower near the poles in Figures 10 and 11. The converging flows observed at the poles are mostly due to an artifact that is well-known in axisymmetric simulations. However, a similar \( Ye \) anisotropy was observed by Keil et al. (1996) (Figure 3), in their 2D simulation, in which the poles were avoided by choosing a 45° wedge region centered at the equator as their computational zone. They found that the zone at \( 15 \lesssim r \lesssim 20 \) km is divided
into two large vortices with lepton-rich matter rising and deleptonized matter sinking. We hence think that such configurations are rather generic and expect that the downdraft of low-$Y_e$ matter and the updraft of high-$Y_e$ matter will occur also in 3D at several points. Moreover, if the PNS is rotating rapidly, there may occur converging flows at the poles indeed. We have to wait for 3D studies, however, to confirm this.

Bearing this possible caveat in mind, we discuss the directional dependence of the neutrino emission caused by the asymmetric matter distribution derived above. Figure 13 shows a comparison of the luminosities at several angles in space. Both $\nu_e$ and $\bar{\nu}_e$ show large angular variations, where the maximum value becomes twice as large as the minimum value. The temporal changes at different angles for $\bar{\nu}_e$ are inversely correlated with that of $\nu_e$. The $\nu_e$ luminosities near the south pole ($\theta = 5\pi/6$, $\pi$) tend to be lower than at the other angles. This is because the low-$Y_e$ environment there (Figures 10 and 11) is preferable for the absorption of $\nu_e$ and emission of $\bar{\nu}_e$. Although the $\nu_e$ luminosities near the north pole ($\theta = 0$, $\pi/6$) are higher than those near the south pole at early times, they become similar later as the transient subsides and the convection becomes quasi-steady. It is important that there is $\sim 30\%$ anisotropy existent even if these pole regions are excluded. The $\nu_x$ luminosity, on the other hand, shows much smaller angular variations ($\lesssim 10\%$) compared to those of $\nu_e$ and $\bar{\nu}_e$ in the late phase. This is because the $\nu_x$ emission is barely affected by the anisotropy of $Y_e$ and the temperature variations are smaller.

4.3. Resolution Dependence

Here we investigate the numerical resolution in our 2D simulation. As mentioned earlier in Section 3, we run additional simulations either with a higher spatial resolution (model HR-S) or with a higher angular resolution in momentum space (model HR-M). Since we are interested in the (quasi) steady convection, we start these runs from the normal-resolution result at 110 ms.

Figure 14 shows the time evolutions of the kinetic energy for the three runs. Since the convection is stochastic, the time variations are a bit different from model to model but they show the same trend. It is apparent that model HR-S has larger values of the kinetic energy than the other two by $\sim 20\%$ for most of the time. The enhancement of turbulence due to higher spatial resolution is already reported in the context of CCSN simulations (Nagakura et al. 2019a). Model HR-M also tends to have higher values than the normal-resolution model but the
difference is much smaller and it may be temporary. The period of the oscillation at $\sim 1$ ms is the same for the different resolution models.

Figure 15 shows the resolution dependence of the neutrino luminosity. The $\nu_e$ luminosity for model HR-S is slightly higher than that for the normal-resolution whereas the $\bar{\nu}_e$ luminosity shows opposite behavior. The $\nu_x$ luminosity gets greater for the higher spatial resolution model but the difference is even smaller. The luminosities in model HR-M are almost the same as those in the normal-resolution model. Figure 16 shows a comparison of the (angle-averaged) mean neutrino energy among the three runs. For all flavors, the resolution dependence is very minor. We can say that although the turbulence induced by the convection is under-resolved spatially by a few tens of percent in our 2D simulation, the enhancement in the neutrino luminosities and mean energies is much less affected by the resolution.

### 4.4. Occurrence of Neutrino FFC

Since we directly solve the Boltzmann equation, we have an advantage that full information of the neutrino distribution in phase space can be obtained. This allows us to analyse, albeit
linearly, the occurrence of neutrino FFC. In fact, our group has already reported the possibility of FFC (Nagakura et al. 2019b; Delfan Azari et al. 2020; Harada & Nagakura 2022) for the results of their CCSN simulations performed with our Boltzmann radiation hydrodynamics code. We extend them to a later time in the PNS cooling phase here.

The occurrence of FFC can be judged by the existence of the electron-lepton number (ELN) crossing, as proved by Morinaga (2022). We estimate the growth rate of FFC using the following approximate formulae, motivated by the two-beam model (Morinaga et al. 2020):

\[
\sigma \sim \sqrt{\int_{\Delta G > 0} d\Omega G(\Delta G) \int_{\Delta G < 0} d\Omega G(\Delta G)},
\]

where:

\[
\Delta G = \frac{\sqrt{2} G_F}{2\pi^2} \int (f_\nu - f_{\bar{\nu}}) \mu^2 dv.
\]

We show the results in Figure 17 at four different times. In the green regions \(\sigma\) is real and FFC is expected to occur therein, whereas in the dark regions \(\sigma\) is actually imaginary and FFC is not expected. At early times (\(<\sim 100\) ms), the FFC regions are confined in the southern hemisphere. At later times, when the initial transient is subsided and the convection becomes quasi-stationary, FFC regions appear also in the northern hemisphere and extend horizontally. The growth rates are as high as \(\gtrsim 10^{-1}\) cm\(^{-1}\).

Figure 18 shows a comparison of the energy-integrated distribution functions for the inward- and outward-going \(\nu_e\) and \(\bar{\nu}_e\) between the equator and the intermediate latitude (\(\theta = \pi/4\)) for a snapshot at \(t = 160\) ms. At the equator, where no crossing is observed, the abundance of \(\bar{\nu}_e\) is clearly smaller than that of \(\nu_e\) both for the outgoing and incoming directions, which makes it impossible to realize a crossing. At \(\theta = \pi/4\), on the other hand, the abundance of \(\bar{\nu}_e\) is comparable or sometimes larger than that of \(\nu_e\). As discussed in Section 4.2, this is caused by the low-\(Y_e\) environment produced by the downdraft in the convective motion. At \(r \gtrsim 18.5\) km, \(\bar{\nu}_e\) dominates over \(\nu_e\) for the outward direction, whereas the opposite holds for the inward direction (the so-called type-II crossing in the language of Nagakura et al. 2021). This reflects the well-known difference in their interactions with matter between the two; \(\bar{\nu}_e\) is decoupled from matter deeper inside compared to \(\nu_e\), which produces a region where \(\bar{\nu}_e\) is more forward-peaked than \(\nu_e\) is (see the magnified figure).

The downdrafts observed near the poles and the resultant low-\(Y_e\) regions are likely to be exaggerated by 2D artifacts; such structures are probably generic as we argued earlier. We
hence think that FFC is likely to occur also in 3D PNS cooling via the mechanism mentioned above. It should be also added that ELN crossing was also observed in the post-shock region by Harada & Nagakura (2022) in their 2D simulations of a rapidly rotating CCSN, although the time and the location are different from ours and centrifugal forces play an important role in the prevalence of the low-$Y_e$ region.

5. Summary and Conclusions

We investigated PNS convection in 2D under axisymmetry, using our newly developed general relativistic Boltzmann neutrino radiation hydrodynamics code. This is meant to be a pilot study for more comprehensive explorations of PNS cooling in multidimensions. To our knowledge, such a study has not been conducted yet in the literature for a couple of reasons. In fact, most of the previous works on PNSs were done either in 1D (Roberts 2012; Roberts et al. 2012; Nakazato & Suzuki 2019, 2020; Li et al. 2021; Nakazato et al. 2022; Pascal et al. 2022) or in the early phase ($\lesssim 1$ s) of CCSN explosion (Mezzacappa et al. 1998; Buras et al. 2006; Dessart et al. 2006; Nagakura et al. 2020; but see Bollig et al. 2021). For the purpose of this paper, we extracted from a conventional 1D PNS cooling calculation in spherical symmetry by Nakazato & Suzuki (2019) a snapshot of the PNS at 2.3 s post bounce, mapped it onto a 2D grid and ran our code, adding some perturbations initially, to see the subsequent convective activity.

The Brunt–Väisälä frequency calculated at each grid point shows that this model has indeed a radially extended zone inside the PNS that is linearly unstable against convection. We observed that PNS convection is actually instigated in that region. The convective motion is particularly violent in the first $\sim 100$ ms, extending itself inward by overshooting and rendering the entropy gradient positive there. We also saw a rapid rise of the neutrino luminosities and mean energies. These are all transients, though, which are induced by the switch from 1D to 2D and the subsequent growth of convective motions. They subside gradually in $\sim 100$ ms. Then the convection enters a quasi-steady phase sustained up to the end of the simulation at $\sim 160$ ms by the negative radial gradient of $Y_e$, which remains thanks to the lasting neutrino emission from the PNS surface. The PNS, on the other hand, has settled to a new expanded configuration, emitting neutrinos at higher luminosities and mean energies. The density, temperature, and $Y_e$ as well as the neutrino luminosities and spectra change much more slowly on the secular timescale thereafter. The sustained convection with an extension of the convective zone inward is consistent with Keil et al. (1996) but is at odds with Mezzacappa et al. (1998). The latter is due probably to their approximation of neutrino transport as well as to the differences in the considered phases. The higher neutrino luminosities and mean energies in 2D than in 1D were also

Figure 14. Time evolution of the matter kinetic energy for the normal-resolution (purple), HR-S (blue), and HR-M (red) models.

Figure 15. Time evolution of the energy luminosity for three different flavors: $\nu_e$ (top), $\bar{\nu}_e$ (middle), and $\nu_x$ (bottom). The different colors denote the different resolutions in the models: normal-resolution (purple), HR-S (blue), and HR-M (red).
observed by other earlier works done for CCSN simulations (Dessart et al. 2006).

Supposing that the self-sustained state above is representative of the PNS around this time during its cooling, we investigated it further. Note that the difference between the original matter profile obtained in the 1D PNS cooling calculation and the self-sustained state in our 2D simulation indicates clearly that the multidimensional PNS calculations should be done from much earlier on, possibly from right after the successful launch of the shock wave from the SN core.

We analysed angular variations in the matter distributions that are produced by convective motions. We found that the temperature and \( Y_e \) showed larger deviations from spherical symmetry than the density. In particular, \( Y_e \) tends to be lower near the poles and higher around the equator. This asymmetry in the \( Y_e \) distribution in turn gives rise to anisotropic emissions of \( \nu_e \) and \( \bar{\nu}_e \). The time-averaged convective pattern revealed that this \( Y_e \) distribution is generated by the subduction of low-\( Y_e \) matter and the dredge-up of high-\( Y_e \) matter by convective motion. In our 2D simulation with axisymmetry imposed, the former occurs predominantly in the polar region and is most likely to be exaggerated. On the other hand, the downdraft of low-\( Y_e \) matter and the updraft of high-\( Y_e \) matter were also observed in other simulations (Keil et al. 1996) that avoided the polar artifact. Hence we think it is a rather generic feature and expect that a similar asymmetry will occur also in 3D.

This paper is also meant to check our newly developed code. In fact, before starting the 2D run, we ran our code in 1D mode to check for possible inconsistencies between the 1D PNS cooling code of Nakazato & Suzuki (2019) and our code. It turns out that the resultant matter profiles were not much different from the original ones, showing that our code is working as expected. We also investigated the resolution dependence of our findings of the convection and its consequences. We increased the spatial resolution and the angular resolution in momentum space. We found that the latter has little effect on the results. On the other hand, roughly doubling the number of grid points in space increased the turbulence kinetic energy by \( \sim 20\% \).

Finally, taking advantage of the Boltzmann solver, which can provide full information of the angular distribution of neutrinos in momentum space, we analysed the possible occurrence of FFC in our model by searching for ELN crossing. We found that FFC is unlikely to occur in the high-\( Y_e \) region, i.e., around the equator in our model, where \( \nu_e \) dominates \( \bar{\nu}_e \) in all directions in momentum space. In the low-\( Y_e \) region, on the other hand, we found type-II ELN crossing, since the \( \bar{\nu}_e \) distribution becomes more forward-peaked than the \( \nu_e \) distribution because \( \bar{\nu}_e \) decouples from matter deeper inside.

As repeatedly mentioned, convection in 2D under axisymmetry is different from that in 3D qualitatively (Lentz et al. 2015) and the converging flows in the polar regions are most likely due to 2D artifacts. The results of FFC are hence also affected by them. As argued above, however, the convective pattern with low-\( Y_e \) matter sinking and high-\( Y_e \) matter rising will be rather generic and likely to occur also in 3D. Then the possibility of FFC in the low-\( Y_e \) region may also survive in 3D. It is encouraging that the FFC region is much wider than the converging flows. In a rapidly rotating PNS, converging flows may be realized indeed. FFC is important as it will change the neutrino signals observed by terrestrial detectors and may also affect PNS cooling. With the Boltzmann solver, we will be able to study these issues in detail.

As we mentioned at the beginning, we are planning to conduct a comprehensive study of long-term PNS cooling in multidimensions. We are particularly interested in rapidly rotating PNSs, which may be also produced by a NS merger. For that purpose we have developed a code to build a general relativistic rotational equilibrium configuration in Lagrangian coordinates (Okawa et al. 2022). We will run the Boltzmann–\( \nu_e \)-radiation–hydrodynamics code on top of those configurations and, based on these models, convection, neutrino emission and oscillation, as well as gravitational waves from neutrinos (Fu & Yamada 2022) will be investigated quantitatively. The results will be reported elsewhere in the near future.

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Appendix A
Formulation of the Hydrodynamics Solver

In this appendix, we describe the formulation of the hydrodynamics equations solver. It is a general relativistic extension of the Newtonian counterpart employed in our previous papers (Nagakura et al. 2014, 2017, 2019). By following Kawaguchi et al. (2021), we first decompose the variables as follows:

\[
\Lambda^i_{(r)} \equiv 1, \quad \Lambda^j_{(\theta)} \equiv \frac{1}{r}, \quad \Lambda^l_{(\phi)} \equiv \frac{1}{r \sin \theta},
\]

\[
\Lambda^i_{(j)} \equiv 0(i \neq j), \quad \Lambda^j_{(i)} \equiv 0(i = j),
\]

\[
\gamma^i_{(j)} \equiv \Lambda^i_{(j)} \Lambda^j_{(i)} \gamma_{ij},
\]

\[
\tilde{K}^i_{(j)} \equiv \Lambda^i_{(j)} \Lambda^j_{(i)} K_{ij},
\]

\[
\gamma_{ij} \equiv \frac{1}{r^2 \sin \theta \sqrt{\gamma}},
\]

\[
\tilde{\rho}_s \equiv \frac{1}{r^2 \sin \theta \rho_s},
\]

\[
\gamma^i \equiv \Lambda^i_{(i)} \gamma_{ii},
\]

\[
\beta^i \equiv \Lambda^i_{(i)} \beta_i,
\]

\[
\tilde{S}^i \equiv \frac{1}{r^2 \sin \theta \Lambda^i_{(i)} S_i},
\]

Figure 17. Growth rate of FFC for some time snapshots. From left to right: 60 ms, 100 ms, 140 ms, and 160 ms. The white lines denote the PNS surface at each time.

Figure 18. Radial profile of the energy-integrated distribution function at the equator (top) and the angular slice $\theta = \pi/4$ (bottom) for the time $t = 160$ ms. The red and blue lines denote the $\nu_e$ and $\bar{\nu}_e$ distributions, respectively. The solid and the dashed lines denote incoming and outgoing ones, respectively. The gray-shaded region denotes where the crossing occurs.
\[
S_{(i)(j)} = \Lambda^{(i)}_{(i)} \Lambda^{(j)}_{(j)} S_{ij}. \tag{A10}
\]

Note that if spacetime is flat, the parenthesis indices \((r), (\theta), \text{ and } (\phi)\) means the orthonormal components of flat spacetime. Therefore, similar to the philosophy of Baumgarte et al. (2013), we factor out the trivial coordinate dependence from the coordinate components of the tensors. This decomposition gives better accuracy of the interface value reconstruction.

With the variables defined above, we discretized Equations (3)–(5) into the following form:

\[
\partial_t(\rho_{st}) = -\frac{r_i^2}{r_i^2/3 - r_i^{-1}/3} \Delta_r \left( r_i^2 \sin \theta_j \rho_{st} Y_{ij} \right) \tag{A11}
\]

\[
+ \frac{\sin \theta_j}{\mu_j - \mu_{j-1}} \Delta_\rho(\rho_{st} \sin \theta_j \rho_{st} Y_{ij}) \tag{A12}
\]

\[
- \frac{1}{\phi_K - \phi_{K-1}} \Delta_\phi(\rho_{st} \phi_{K-1} \phi_{K} Y_{ij}) = 0, \tag{A13}
\]

\[
\partial_t(S_{ij}) = -\frac{r_i^2}{r_i^2/3 - r_i^{-1}/3} \Delta_r \left( r_i^2 \sin \theta_j (\tilde{S}_{ij} + P_{ij} \sqrt{\gamma_{ij}}) \right) \tag{A14}
\]

\[
+ \frac{r_i^2 - r_i^{-1}}{r_i^2/3 - r_i^{-1}/3} \left( \alpha_i \sqrt{\gamma_{ij}} P_{ij} + \frac{1}{2} (S_{ij} \rho_{st} \rho_{st} + S_{ij} Y_{ij}) \right) \tag{A15}
\]

\[
- \alpha_i \sqrt{\gamma_{ij}} G_{ij}, \tag{A16}
\]

\[
\partial_t(S_{0i}) = -\frac{r_i^2}{r_i^2/3 - r_i^{-1}/3} \Delta_r (r_i^2 \sin \theta_j \tilde{S}_{0i} Y_{ij}) \tag{A17}
\]

\[
+ \frac{\sin \theta_j}{\mu_j - \mu_{j-1}} \Delta_\rho(\rho_{st} \sin \theta_j \tilde{S}_{0i} Y_{ij}) + P_{ij} \alpha_j \sqrt{\gamma_{ij}} \tag{A18}
\]

\[
- \frac{1}{\phi_K - \phi_{K-1}} \Delta_\phi(\rho_{st} \phi_{K-1} \phi_{K} Y_{ij}) = 0, \tag{A19}
\]

\[
\partial_t(S_{0i}) = -\frac{r_i^2}{r_i^2/3 - r_i^{-1}/3} \Delta_r (r_i^3 \sin^2 \theta_j \tilde{S}_{0i} Y_{ij}) \tag{A20}
\]

\[
+ \frac{\sin \theta_j}{\mu_j - \mu_{j-1}} \Delta_\rho(\rho_{st} \sin^2 \theta_j \tilde{S}_{0i} Y_{ij}) \tag{A21}
\]

\[
- \alpha_j \sqrt{\gamma_{ij}} G_{ij}. \tag{A22}
\]

\[
\partial_t(S_{0i}) = -\frac{r_i^2}{r_i^2/3 - r_i^{-1}/3} \Delta_r (r_i^2 \sin \theta_j \tilde{S}_{0i} Y_{ij}) \tag{A23}
\]

\[
+ \frac{\sin \theta_j}{\mu_j - \mu_{j-1}} \Delta_\rho(\rho_{st} \sin \theta_j \tilde{S}_{0i} Y_{ij} + P_{ij} \alpha_j \sqrt{\gamma_{ij}}) \tag{A24}
\]

\[
- \frac{1}{\phi_K - \phi_{K-1}} \Delta_\phi(\rho_{st} \phi_{K-1} \phi_{K} Y_{ij}) = 0, \tag{A25}
\]

\[
\partial_t(S_{0i}) = -\frac{r_i^2}{r_i^2/3 - r_i^{-1}/3} \Delta_r (r_i^3 \sin^2 \theta_j \tilde{S}_{0i} Y_{ij} + P_{ij} \alpha_j \sqrt{\gamma_{ij}}) \tag{A26}
\]

\[
+ \frac{\sin \theta_j}{\mu_j - \mu_{j-1}} \Delta_\rho(\rho_{st} \sin^2 \theta_j \tilde{S}_{0i} Y_{ij}) \tag{A27}
\]

\[
- \alpha_j \sqrt{\gamma_{ij}} G_{ij}. \tag{A28}
\]

where \(\mu \equiv \cos \theta\), and lower (e.g., \(i\)) and upper (e.g., \(I\)) case subscripts denote the cell center and interface values, respectively; \(i, j, k\) indicate the grid ID numbers of the radial \((r)\), zenith \((\theta)\), and azimuthal \((\phi)\) coordinates, respectively. The symbols \(\Delta_r, \Delta_\rho, \text{ and } \Delta_\phi\) mean the difference between the interface values along the indicated coordinates, e.g., \(\Delta_r(\rho_{st}) = \rho_{st,i} - \rho_{st,i-1}\) for any functions \(f, g\). The interface values of the decomposed hydrodynamic variables (rest mass density \(\rho_{st}\), temperature \(T\), pressure \(P\), electron fraction \(Y_e\), and the spatial components of four-velocity \(u^{(i)} \equiv u^{(i)}(t)\)) are evaluated by the piecewise parabolic method (PPM; Colella & Woodward 1984) with a minmod flux limiter. The interface values of the decomposed metric variables (lapse \(\alpha\), shift \(\beta^{(i)}\), and spatial metric \(\tilde{g}_{(i)(j)}\)) are evaluated by third-order Lagrange interpolation. The coordinate values \(r_i, \theta_j, \text{ and } \phi_K\) is exactly evaluated. The numerical flux is evaluated by the Harten–Lax–van Leer (HLL) method (Harten et al. 1983) and the time evolution is solved by the fourth-order Runge–Kutta scheme. By evaluating the curvature terms such as \(\alpha \sqrt{\gamma} P/r\) and so on in these ways, the steady state of uniform matter in flat spacetime is guaranteed.

\section*{Appendix B}

\textbf{Code Verification Tests of the Hydrodynamics Solver}

In this section, we performed three kinds of code tests to verify the ability of our newly developed general relativistic hydrodynamics code. Special relativistic shock tube tests are performed in Appendix B.1. In Appendices B.2 and B.3, we perform stability tests of the relativistic star, and accreting matter onto BH, respectively.

\subsection*{B.1. Special Relativistic Shock Tube Test}

Since our previous code was for Newtonian hydrodynamics, matter with relativistic motion could not be treated properly. Here, we perform special relativistic shock tube tests. As the initial state, we employ the first case of Nagakura et al. (2011).
The initial left state is \((\rho, v, p) = (10, 0, 13.3)\) and the right state is \((\rho, v, p) = (1, 0, 10^{-6})\). Since those values are for geometrical units, the input hydrodynamics quantities should be rescaled for our code written in cgs units. If the length conversion factor is defined as \(L\), the conversions are \(\rho_{\text{cgs}} = \rho \times c^2 / G \times L^2\), \(p_{\text{cgs}} = p \times c^4 / G \times L^2\), \(v_{\text{cgs}} = v \times c\), and \(t = L / c\). We choose the value \(L = 1 \times 10^6\) cm. We employ the gamma-law EOS with an adiabatic index of \(\Gamma = 5/3\). Analytical solutions can be exactly calculated following Pons et al. (2000).

Since our code employs polar coordinates, in principle, we cannot perform this test meant for Cartesian coordinates. Similarly to Yamada et al. (1999), we simulate in a very thin shell where curvature can be ignored. An initial discontinuity is placed at a radius of \(r = 10^3\) km, and the width of the computational domain is 10 km, which is four orders of magnitude smaller than the distance from the origin. In order to check the resolution dependence, we test three mesh cases: 100, 200, and 400 grid points. The computational region is equally divided.

Figure B1 shows the hydrodynamic quantities at the time snapshot \(t = 1.5 \times 10^{-5}\) s. The analytical solution is well reproduced for all resolutions. In addition, raising the resolution improves the result, which indicates the resolution convergence of our code.

**B.2. Stability Test of a Relativistic Star**

We check the code’s ability to test matter under a strong gravitational field. By starting from the stable NS model as the initial conditions, we check that the initial state is maintained by time evolution.

We constructed a NS model with a central density of \(1 \times 10^{15} \text{ g cm}^{-3}\) and a central pressure of \(p = 1.35 \times 10^{35} \text{ g cm}^{-1} \text{ s}^{-2}\). We employ the gamma-law EOS with an adiabatic index of \(\Gamma = 2\). We solve the Tolman–Oppenheimer–Volkoff equations by using the fourth-order explicit Runge–Kutta method. This results in a NS with a mass of \(1.42 M_\odot\) and a radius of 13.2 km. The density and the pressure distributions are shown in Figure B2. In order to check the resolution dependence, we test three mesh cases: \(N_r = 128, 256, \text{ and } 384\) grid points. The computational region is \(r \in [0: 15]\) km, and it is equally divided.

Figure B3 shows the time evolution of the relative error of the density for the different radii. The error for \(N_r = 128\) increases with time. On the other hand, the errors for \(N_r = 256\) and 384 show oscillations around the initial data. The amplitude of the oscillation is smaller for the highest-resolution case. Hence we can conclude that resolution convergence is obtained for this test.
\[ \dot{\rho} + \rho u^r \dot{u}^r = 0, \]  
\[ \dot{p} + \rho u^r \dot{u}^r \dot{h}_u = \frac{1}{2} \dot{h}_0 \dot{h}, \]  
\[ \times \{ \partial_r g_{rr}(u^r)^2 + \partial_r g_{\phi\phi}(u^\phi)^2 + \partial_r g_{tt}(u^t)^2 + 2 \partial_r g_{t\phi} u^\phi \}, \]  
\[ \dot{h}_u = 0, \]  
\[ \dot{h}_t = 0. \]  

Note that Boyer–Lindquist coordinates are used for the Kerr metric.

By explicitly expanding the specific enthalpy by the pressure, Equation (B2) can be rewritten as an equation with a single pressure derivative term as follows:

\[ \frac{dp}{dr} = \frac{\rho u^r u_r - \rho \dot{h}_u u^r S_t/r^3}{1 + u^r u_r - u^\phi h_\phi S_t/(r^2 \Gamma p)}, \]  

where \( S_t \equiv r^2 \rho u^r \). We solve Equation (B5) using the fourth-order explicit Runge–Kutta method. The other equations, Equations B1, B3, and B4 are used to calculate the four-velocity from the pressure.

In this test, we employ a BH with a mass of \( M_{BH} = 5 M_\odot \) with a dimensionless spin parameter of 0.5. With these parameters, the horizon is located at 13.8 km. We do not solve hydrodynamic equations inside the horizon, where the metric is singular. We employ the gamma-law EOS with \( \Gamma = 4/3 \) in this test.

The parameters for the hydrodynamics variables are as follows. We assume that matter with a density of \( \rho = 1 \times 10^7 \, \text{g cm}^{-3} \) is constantly injected from a radius of \( r = 300 \, \text{km} \) and with a supersonic velocity of \( u' = -1.2 c_s \), where \( c_s \) is the speed of sound \( c_s \equiv \sqrt{\Gamma p/\rho} \). As for the specific angular momentum \( \lambda = u_\phi / u_r \), we test three cases: \( \lambda = 0 \) (no rotation), \( GM_{BH}/c^2 \) (prograde), and \( GM_{BH}/c^2 \) (retrograde). Figure B4 shows the initial hydrodynamics profiles for the three cases.

In the hydrodynamics calculations, the outermost meshes are fixed to inject the matter constantly. Same as the previous tests, we test three resolutions: \( N_r = 128, 256, \) and 384 grid points. The computational region is \( r \in [0:300] \, \text{km} \), and the mesh width is varied exponentially so that the resolution gets finer for smaller radii.

Figure B5 shows the time evolution of the relative error of the density. The error at 20 km is of the order \( o(10^{-5}) \), and the error at 50 km is of the order \( o(10^{-4}) \). All calculations show reasonable agreement. In addition, the highest-resolution case \( N_r = 384 \) gives the most accurate result, which indicates resolution convergence.
Figure B5. Relative error of the density for different specific angular momenta: non-rotating (left), prograde (middle), and retrograde (right). The spatial positions are 20 km (top) and 50 km (bottom). The red, purple, and blue lines denote the results for 384 mesh, 256 mesh, and 128 mesh resolutions, respectively.

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References

Abdikamalov, E., Burrows, A., Ott, C. D., et al. 2012, ApJ, 755, 111
Akaho, R., Harada, A., Nagakura, H., et al. 2021, ApJ, 909, 210
Arnett, W. D. 1977, ApJ, 218, 815
Baumgarte, T. W., Montero, P. J., Cordero-Carrion, I., & Muller, E. 2013, PhRvD, 87, 044026
Bollig, R., Yadav, N., Kresse, D., et al. 2021, ApJL, 915, 28
Bruenn, S. W. 1985, ApJS, 58, 771
Bruenn, S. W., De Nico, K. R., & Mezzacappa, A. 2001, ApJ, 560, 326
Burrows, A., Janka, H. T., Rampp, M., & Kifonidis, K. 2006, A&A, 457, 281
Burrows, A., Radice, D., Vartanyan, D., et al. 2020, MNRAS, 491, 2715
Burrows, A., & Vartanyan, D. 2021, Nature, 589, 29
Cabezón, R. M., Pan, K. C., Liebendörfer, M., et al. 2018, A&A, 619, A118
Cardall, C. Y., Endeve, E., & Mezzacappa, A. 2013, PhRvD, 87, 103004
Colella, P., & Woodward, P. R. 1984, JCoPh, 54, 174
Delfan Azari, M., Yamada, S., Morinaga, T., et al. 2020, PhRvD, 101, 023018
Dessart, L., Burrows, A., Livne, E., & Ott, C. D. 2006, ApJ, 645, 534
Ehlers, J. 1971, In Proc. of the Int. School of Physics “Enrico Fermi,” Course 47, General Relativity and Cosmology, ed. R. K. Sachs (New York: Academic Press)
Fu, L., & Yamada, S. 2022, PhRvD, 105, 123028
Furusawa, S., Togashi, H., Nagakura, H., et al. 2017, JPhG, 44, 094001
Glas, R., Just, O., Janka, H.-T., & Obergaulinger, M. 2019, ApJ, 873, 45
Gossan, S. E., Fuller, J., & Roberts, L. F. 2020, MNRAS, 491, 5376
Harada, A., & Nagakura, H. 2022, ApJ, 924, 109
Harada, A., Nagakura, H., Iwakami, W., et al. 2019, ApJ, 872, 181
Harada, A., Nagakura, H., Iwakami, W., et al. 2020, ApJ, 902, 150
Harten, A., Lax, P. D., & van Leer, B. 1983, SIAMR, 25, 35
Iwakami, W., Okawa, H., Nagakura, H., et al. 2020, ApJ, 903, 82
Just, O., Bollig, R., Janka, H.-T., et al. 2018, MNRAS, 481, 4786
Kato, C., Nagakura, H., Hori, Y., & Yamada, S. 2020, ApJ, 897, 43
Kawaguchi, K., Fujibayashi, S., Shibata, M., Tanaka, M., & Watanabe, S. 2021, ApJ, 913, 100
Keil, W., Janka, H. T., & Muller, E. 1996, ApJL, 473, L111
Lattimer, J. M., & Mazurek, T. J. 1981, ApJ, 246, 955
Lentz, E. J., Bruenn, S. W., Hix, W. R., et al. 2015, ApJL, 807, L31
Lentz, E. J., Mezzacappa, A., Messer, O. B. E., et al. 2012, ApJ, 747, 73
Li, S. W., Roberts, L. F., & Beacom, J. F. 2021, PhRvD, 103, 023016
Liebendörfer, M., Whitehouse, S. C., & Fischer, T. 2009, ApJ, 698, 1174
Lindquist, R. W. 1966, ApPhy, 37, 487
Margalit, B., Jermy, A. S., Metzger, B. D., Roberts, L. F., & Quataert, E. 2022, ApJ, 939, 51
Mezzacappa, A., & Bruenn, S. W. 1993, ApJ, 405, 669
Mezzacappa, A., Calder, A. C., Bruenn, S. W., et al. 1998, ApJ, 493, 848
Mezzacappa, A., Marronetti, P., Landfield, R. E., et al. 2022, arXiv:2208.10643
Morinaga, T. 2022, PhRvD, 105, L101301
Morinaga, T., Nagakura, H., Kato, C., & Yamada, S. 2020, PhRvR, 2, 012046
Müller, B., Janka, H. T., & Marek, A. 2013, ApJ, 766, 43
Nagakura, H., Burrows, A., Johns, L., & Fuller, G. M. 2021, PhRvD, 104, 083025
Nagakura, H., Burrows, A., Radice, D., & Vartanyan, D. 2019a, MNRAS, 490, 4622
Nagakura, H., Burrows, A., Radice, D., & Vartanyan, D. 2020, MNRAS, 492, 5764
Nagakura, H., Ito, H., Kiuchi, K., & Yamada, S. 2011, ApJ, 731, 80
Nagakura, H., Iwakami, W., Furusawa, S., et al. 2017, ApJS, 229, 42
Nagakura, H., Iwakami, W., Furusawa, S., et al. 2018, ApJ, 854, 136
Nagakura, H., & Johns, L. 2021, PhRvD, 103, 123025
Nagakura, H., & Morinaga, T. 2022, MNRAS, 511, 356
Nagakura, H., Sumiyoshi, K., & Yamada, S. 2014, ApJS, 214, 16
Nagakura, H., Sumiyoshi, K., & Yamada, S. 2019, ApJ, 878, 160
Nagakura, H., & Vartanyan, D. 2022, MNRAS, 512, 2806
Nagakura, H., & Yamada, S. 2009, ApJ, 696, 2026
Nakazato, K., Nakashima, K., Harada, M., et al. 2022, ApJ, 925, 98
Nakazato, K., Sumiyoshi, K., Suzuki, H., et al. 2013, ApJS, 205, 2
Nakazato, K., & Suzuki, H. 2019, ApJ, 878, 25
Nakazato, K., & Suzuki, H. 2020, ApJ, 891, 156
O’Connor, E. P., & Couch, S. M. 2018, ApJ, 854, 63
Okawa, H., Fujisawa, K., Yasutake, N., et al. 2022, arXiv:2204.09943
Oppenheimer, J. R., & Volkoff, G. M. 1939, PhRv, 55, 574
Pascal, A., Novak, J., & Oertel, M. 2022, MNRAS, 511, 356
Peres, B., Penner, A. J., Novak, J., & Bonazzola, S. 2014, CQGra, 31, 045012
Pons, J. A., Ma Martí, J., & Müller, E. 2000, JFM, 422, 125
Pons, J. A., Miralles, J. A., Prakash, M., & Lattimer, J. M. 2001a, ApJ, 553, 382
Pons, J. A., Steiner, A. W., Prakash, M., & Lattimer, J. M. 2001b, PhRvL, 86, 5223
Richers, S., Nagakura, H., Ott, C. D., et al. 2017, ApJ, 847, 133
Roberts, L. F. 2012, ApJ, 755, 126
Roberts, L. F., & Reddy, S. 2017, in Handbook of Supernovae, ed. A. W. Alsabti & P. Murdin (Cham: Springer), 1605
Roberts, L. F., Shen, G., Cirigliano, V., et al. 2012, PhRvL, 108, 061103
Shibata, M. 2016, Numerical Relativity (Singapore: World Scientific)
Shibata, M., Kiuchi, K., Sekiguchi, Y.-i., & Suwa, Y. 2011, PThPh, 125, 1255
Shibata, M., Nagakura, H., Sekiguchi, Y., & Yamada, S. 2014, PhRvD, 89, 084073
Skinner, M. A., Burrows, A., & Dolence, J. C. 2016, ApJ, 831, 81
Sumiyoshi, K., & Yamada, S. 2012, ApJS, 199, 17
Sumiyoshi, K., Yamada, S., Suzuki, H., et al. 2005, ApJ, 629, 922
Thorne, K. S. 1981, MNRAS, 194, 439
Togashi, H., Nakazato, K., Takehara, Y., et al. 2017, NuPhA, 961, 78
Vartanyan, D., Burrows, A., Radice, D., Skinner, M. A., & Dolence, J. 2018, MNRAS, 482, 351
Vartanyan, D., Coleman, M. S. B., & Burrows, A. 2022, MNRAS, 510, 4689
Woosley, S. E., & Weaver, T. A. 1995, ApJS, 101, 181
Yamada, S., Janka, H.-T., & Suzuki, H. 1999, A&A, 344, 533