Modeling wave interaction in long wave theory during its spread over a stratified porous bottom

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Abstract. In this work, we are interested in the study of the interaction between a gravity wave and a porous media made up of multi layers of different geometrical characteristics. The dispersion relation and the reflection coefficient are calculated and analysed according to the porosity of the porous media. A generalization of these relations is proposed in the case of long waves. In terms of application, we analyse the most adequate configuration for better behaviour of the porous media under the gravity wave. The wave is calculated within the potential theoretical framework and the Forchheimer model is selected in order to take into account the inertial aspect of the flow in the porous media.

Keywords: Interaction; Wave; Porous media; Forchheimer model; Analytical Formulation; Long Wave Theory

1. Introduction
When the swell propagates on seabed’s, considered as porous media, which permeability induces internal flow and non-negligible flux through the porous medium interface - external medium. This type of problem has been the subject of several investigations in coastal engineering. The theory of SOLITT & CROSS (1972) [1] served as a theoretical basis for modeling diverse problems in this area. This approach has also been applied to long wave analysis by MADSEN (1983) [2]. LEE and LAN (1995) [3] have developed a generalized method for multi-layers and multi-region porous media. BROSSARD and al. (2004) [4] have proposed, by an analytical approach validated experimentally in a wave flume, the dispersion relation of the flow in a porous monodisperse and rigid media. In this work, the dispersion relation is generalized to a multi-layer laminated (stratified) porous media based on the hypothesis of the long wave propagation hypothesis.
Forchheimer model is used as a theoretical model. This model is adapted to unsteady flows and it allows to take into account the friction forces and the terms of inertia. We proceed by a first-order linear analytical approach by adopting the Lorentz principle (SOLITT & CROSS, 1972).

2. Theoretical formula
A porous medium consisting of several horizontal layers in the presence of the swell is shown in Figure 1. This porous medium consists of porous multilayers of different characteristics. The orthogonal axes Ox and Oz coincide with the free surface at rest. The depths of the fluid and the porous layers are respectively \( h_0, h_1, h_2 \ldots h_i \).

\[ \text{Figure 1. Schemes of study} \]

We consider a monochromatic swell propagating in the presence of a porous bottom of different characteristics. The field of study consists of an external flow:
\[ -\infty < x < +\infty ; -h_0 \leq z \leq 0 \]
The forced flow is defined in the domain defining the porous bottom: \( -\infty < x < +\infty ; -h_i \leq z \leq -h_{i-1} \); (\( i = 1,2,3,4 \)). The index used takes the values 1,2,3,4.

Note:
\[ Z(h_i) = s(h_i) - i f(h_i) \]  
\[ Z(h_{b_i}) = s(h_{b_i}) - i f(h_{b_i}) = 1 - i 0 \]

\( i \) is the complex number; \( \varepsilon \) the porosity of the medium; \( \phi \) the velocity potential; \( \eta \) the elevation of the free surface; \( \rho \) the density of the fluid; \( z \) the vertical rating; \( g \) gravity; \( \nabla \) designates the gradient operator; \( \vec{n} \) the normal vector at the interface; \( s \) is the coefficient of inertia; \( f \) can be assimilated to a coefficient of friction or damping; \( \omega \) represents the pulsation of the harmonic motion of period \( T \); \( Z \) is a coefficient characterizing the porous layer in terms of added mass and coefficient of inertia.

Equations of the movement
\[ \Delta \phi = 0 \quad -h_0 \leq z \leq 0 \quad (1) \]
\[ \nabla (\varepsilon \nabla \phi) = 0 \quad -h_i \leq z \leq -h_{i-1} \quad (2) \]
\[ \frac{\partial \phi}{\partial z} \frac{\partial^2 \phi}{\partial t^2} - g \phi = 0 \]

**Boundary condition**

\[ \frac{\partial \phi}{\partial z} \frac{\partial^2 \phi}{\partial t^2} - g \phi = 0 \]

\[ \eta(x,t) = -\frac{1}{g} \frac{\partial \phi}{\partial t} \]

\[ Z(-h_i^{-}) \phi(-h_i^{-}) = Z(-h_i^{+}) \phi(-h_i^{+}) \]

\[ \nabla \phi(-h_i^{-}) \cdot \vec{n}_i = \nabla \phi(-h_i^{+}) \cdot \vec{n}_i \]

\[ \nabla \phi \cdot \vec{n}_j = 0 \]

The evaluation of the coefficient \( f \) is based on the Lorentz hypothesis of equivalent work which consists in equalizing over a period, the work of the non-linear term with that of a linear term. The term \( f \) is used (Dingemans, 1996) [5]:

\[ f \omega = \frac{\nu}{K} (1 + \frac{C_i \sqrt{K}}{\nu} \tilde{f}) \]

\[ \tilde{f} = \frac{\int_{t_0}^{t} \int_{t_0}^{\tau} \tilde{q}^2 \, dt \, d\tau}{\int_{t_0}^{T} \int_{t_0}^{\tau} \tilde{q}^2 \, dt \, d\tau} = \frac{8\sqrt{a+b}}{5\pi} \left\{ E(m) - \frac{1}{4} \left(1 - \frac{b}{a}\right) K(m) \right\} \]

Or:

\[ a = \frac{1}{T} \int_{t_0}^{T} \tilde{q}^2 \, dt \quad b = \text{Max}[\tilde{q}^2 - a] \quad m = \frac{2b}{a+b} \]

\( \tilde{q} \) is the actual interstitial velocity or mean velocity of the fluid, \( K(m) \) and \( E(m) \) represent elliptic integrals of first and second species respectively. It is therefore sufficient to use an iterative method combining the two expressions of \( f \) and the Forchheimer equation to obtain the velocity field in the porous medium.

In the following, we note \( j \) the variable relative to each layer \( j \), where the index \( j \) denotes the \( j \) multi porous layer \( (j=1,2) \).

### 3. Analytical resolution

The procedure of the analytical calculation (The analytical calculation process) developed in this work, consists of three steps:

- **Subdivision of the field of study into several sub-domains.** The swell propagates in the external domain. The other sub-domains represent the three layers of the porous media (fig.1).
- **Analytical resolution in each subdomain.** In the external domain, the form of the general solution is obtained by solving the problem in the first order with long wave approximation. In the porous media, the theory of SOLLITT & CROSS (1972) will be considered as a basis to model the form of the general solution.
- **In the third step, the solutions are connected between all the subdomains using the continuity of flows and pressures.**

By this procedure, we calculate the dispersion relation of the wave parameters considering a porous multilayer bottom in long wave theory as well as the kinematics of the flow generated by the swell in the porous media.

In particular, we are interested in the depth of closure of the swell’s impact in the porous media. The analysis will be done according to two parameters: The porosity and the thickness of the laminated layers.
3.1. Dispersion relation and velocity potential

**External media**
- Dispersion relation
  \[
  \frac{\omega^2}{K_0 g} = t_0 = \frac{t_0}{Z_1}
  \]
  (11)
- Velocity potential
  \[
  \phi_0(x, z, t) = \text{Re} \left\{ \frac{jag}{gK_0} \left[ \frac{\omega^2}{gK_0} sh(K_0 z) + ch(K_0 z) \right] e^{i(\omega - K_0)} \right\}
  \]
  (12)

**Single layer**
- Dispersion relation
  \[
  \frac{\omega^2}{K_1 g} = \frac{t_0 + t_{10}}{Z_1} + \frac{t_21}{Z_2} + \frac{Z_1}{Z_2} t_0 t_{10} t_{21}
  \]
  (13)
- Velocity potential (external media)
  \[
  \phi_{10}(x, z, t) = \left[ \frac{jag}{gK_1} \left[ \frac{\omega^2}{gK_1} sh(K_1 z) + ch(K_1 z) \right] e^{i(\omega - K_1)} \right] = \psi_0(K_1)
  \]
  (14)
- Velocity potential (Porous media)
  \[
  \phi_1(x, z, t) = \text{Re} \left\{ \frac{jagc}{gK_1} \left[ Z_1 \left( \frac{\omega^2}{gK_1} - t_0 \right) sh(K_1 z + h_0) - \left( \frac{\omega^2}{gK_1} t_0 - 1 \right) ch(K_1 z + h_0) \right] e^{i(\omega - K_1)} \right\}
  \]
  (15)

**Two layers**
- Dispersion relation
  \[
  \frac{\omega^2}{K_2 g} = \frac{t_0 + t_{10}}{Z_1} + \frac{t_{21}}{Z_2} + \frac{Z_1}{Z_2} t_0 t_{10} t_{21}
  \]
  (16)
- Velocity potential (external media)
  \[
  \phi_{20}(x, z, t) = \left[ \frac{jag}{gK_2} \left[ \left( \frac{\omega^2}{gK_2} - t_0 \right) sh(K_2 z) + ch(K_2 z) \right] e^{i(\omega - K_2)} \right]
  \]
  (17)
- Velocity potential (Mono-layer)
  \[
  \phi_{21}(x, z, t) = \text{Re} \left\{ \frac{jagc}{gK_2} \left[ \left( \frac{\omega^2}{gK_2} - t_0 \right) sh(K_1 z + h_0) - \left( \frac{\omega^2}{gK_2} t_0 - 1 \right) ch(K_1 z + h_0) \right] e^{i(\omega - K_2)} \right\}
  \]
  (18)
- Velocity potential (two-layers)
  \[
  \phi_2(x, z, t) = \text{Re} \left\{ \frac{jagc}{gK_2} \left[ \left( \frac{\omega^2}{gK_2} - t_0 \right) sh(K_1 z + h_0) - \left( \frac{\omega^2}{gK_2} t_0 - 1 \right) ch(K_1 z + h_0) \right] e^{i(\omega - K_2)} \right\}
  \]
  (19)

**Three layers**
- Dispersion relation
\[ K_g \omega^2 = \frac{t_0 + t_1 Z_1^T + t_2 Z_1 Z_2^T + \frac{Z_1 Z_2}{Z_1} t_3 Z_1 Z_3^T + \frac{Z_1 Z_2}{Z_1} Z_4 Z_3^T + \frac{Z_1 Z_2}{Z_1} Z_4}{1 + t_0 + t_1 Z_1^T + t_2 Z_1 Z_2^T + \frac{Z_1 Z_2}{Z_1} t_3 Z_1 Z_3^T + \frac{Z_1 Z_2}{Z_1} Z_4 Z_3^T + \frac{Z_1 Z_2}{Z_1} Z_4} \]  

(20)

- **Velocity potential (external media)**

\[ \phi_{\text{ex}}(x, z, t) = \text{Re} \left\{ \frac{j \omega^2}{\omega Z_0} \text{sh}(K_g z) + ch(K_g z) e^{i(\omega - \kappa z)} \right\} \]

(21)

- **Velocity potential (Mono-layer)**

\[ \phi_1(x, z, t) = \text{Re} \left\{ \frac{j \omega^2}{g K_1} \left( t_0 \left( \frac{\omega^2}{g K_1} - t_0 \right) \right) \right\}

(22)

- **Velocity potential (Two-layers)**

\[ \phi_2(x, z, t) = \text{Re} \left\{ \frac{j \omega^2 e^{-t_0}}{g K_2} \left( t_0 \left( \frac{\omega^2}{g K_2} - t_0 \right) \right) \right\}

(23)

- **Velocity potential (Three-layers)**

\[ \phi_3(x, z, t) = \text{Re} \left\{ \frac{j \omega^2 e^{-t_0}}{g K_3} \left( t_0 \left( \frac{\omega^2}{g K_3} - t_0 \right) \right) \right\}

(24)

**Four layers**

- **Dispersion relation**

\[ \frac{\omega^2}{K_g} = \frac{t_0 + t_1 Z_1^T + t_2 Z_1 Z_2^T + \frac{Z_1 Z_2}{Z_1} t_3 Z_1 Z_3^T + \frac{Z_1 Z_2}{Z_1} Z_4 Z_3^T + \frac{Z_1 Z_2}{Z_1} Z_4}{1 + t_0 + t_1 Z_1^T + t_2 Z_1 Z_2^T + \frac{Z_1 Z_2}{Z_1} t_3 Z_1 Z_3^T + \frac{Z_1 Z_2}{Z_1} Z_4 Z_3^T + \frac{Z_1 Z_2}{Z_1} Z_4} \]

(25)

- **Velocity potential (external media)**

\[ \phi_{\text{ex}}(x, z, t) = \text{Re} \left\{ \frac{j \omega^2}{\omega Z_0} \text{sh}(K_g z) + ch(K_g z) e^{i(\omega - \kappa z)} \right\} \]

(26)

- **Velocity potential (Mono-layer)**

\[ \phi_1(x, z, t) = \text{Re} \left\{ \frac{j \omega^2}{g K_1} \left( Z_1 \left( \frac{\omega^2}{g K_1} - t_1 \right) \right) \right\}

(27)

- **Velocity potential (Two-layers)**
\[ \phi_{g}(x, z, t) = \text{Re} \left\{ \frac{jagc_{g}c_{0}}{\omega Z} \left[ (t_{0} + t_{h}) \phi_{g}(x, z, t) \right] + \frac{1}{Z_{2}} \left[ (t_{0} + t_{h}) \phi_{g}(x, z, t) \right] \right\} \]

(28)

**Velocity potential (Three-layers)**

\[ \phi_{g}(x, z, t) = \text{Re} \left\{ \frac{jagc_{g}c_{0}}{\omega Z_{0}} \left[ (Z_{0} + Z_{1}) \phi_{g}(x, z, t) \right] + \frac{1}{Z_{2}} \left[ (Z_{0} + Z_{1}) \phi_{g}(x, z, t) \right] \right\} \]

(29)

**Velocity potential (Four-layers)**

\[ \phi_{g}(x, z, t) = \text{Re} \left\{ \frac{jagc_{g}c_{0}}{\omega Z_{0}} \left[ (Z_{0} + Z_{1} + Z_{2}) \phi_{g}(x, z, t) \right] + \frac{1}{Z_{2}} \left[ (Z_{0} + Z_{1} + Z_{2}) \phi_{g}(x, z, t) \right] \right\} = \psi_{g}(x, t) \]

(30)

3.2. Generalization of the dispersion relation and the velocity potential of a multi-layer medium

A generalized dispersion relationship of a multi-layer can be obtained in the following form:

\[ \frac{\omega^{2}}{K_{g}} = (\gamma_{0} \Gamma_{0} + \gamma_{l}(1 + \Gamma_{l})) \]

\[ \gamma_{0} = Z_{0}(1 + Z_{1}t_{f0}) ; \quad \gamma_{1} = Z_{1}(Z_{2}t_{f0} + t_{0}) ; \quad \gamma_{2} = Z_{2}(Z_{3}t_{f0} + t_{0}) ; \quad \gamma_{3} = Z_{3} + t_{f0} \]

(31)

\[ \Gamma_{0} = \sum_{n=0}^{N-1} \sum_{k=1}^{K-1} \sum_{l=1}^{L-1} \sum_{m=1}^{M-1} \sum_{p=1}^{P-1} \sum_{q=1}^{Q-1} \sum_{r=1}^{R-1} \left[ \prod_{j=1}^{J} \frac{Z_{j}}{Y_{j}} \right]^{2} \]

(32)

**Velocity potential**

\[ \phi_{g}(x, z, t) = \text{Re} \left\{ \frac{jagc_{g}}{\omega} \left[ \prod_{i=1}^{n} c_{i, j, l}(F_{i, j, l} \phi_{g}(z, t) + \frac{1}{Z_{i}} G_{i, j, l} \phi_{g}(z, t)) \right] \right\} \]

(33)

\[ \psi_{g}(x, t) = \psi_{g}(x, t) \]

Or

\[ \psi_{g}(x, t) = \psi_{g}(x, t) \]

\[ \Gamma_{1} = \sum_{n=0}^{N-1} \sum_{k=1}^{K-1} \sum_{l=1}^{L-1} \sum_{m=1}^{M-1} \sum_{p=1}^{P-1} \sum_{q=1}^{Q-1} \sum_{r=1}^{R-1} \left[ \prod_{j=1}^{J} \frac{Z_{j}}{Y_{j}} \right]^{2} \]

\[ Y_{i} = \frac{t_{i+1}}{Z_{i}} \]

\[ \phi_{g}(x, z, t) = \text{Re} \left\{ \frac{jagc_{g}}{\omega} \left[ \prod_{i=1}^{n} c_{i, j, l}(F_{i, j, l} \phi_{g}(z, t) + \frac{1}{Z_{i}} G_{i, j, l} \phi_{g}(z, t)) \right] \right\} \]

(34)

\[ P_{i, j} = 1 - t_{i}^{2} - (t_{i} + \frac{1}{Z_{i}} \sum_{j=1}^{J} t_{i, j}) \int_{j=1}^{J} f_{i, j} \frac{t_{i, j}}{Z_{i}} \]

(35)
3.3. Long wave theory approximation for a multilayer media

In the case of long wavelength waves \((kh<<1)\), (TEMPERVILLE, 1985) [6], using a limited development in the vicinity of the term \(K_0\) a neglecting order greater than one, the dispersion relation of a multilayer can be written in a homogeneous form:

\[
\frac{\omega^2}{K_0} \approx KH \quad \text{or} \quad H = h_0 + \sum_{i=1}^{n} \frac{h_{i-1}}{Z_i}
\]  

(33)

A dispersion relationship is then obtained with an equivalent medium (external medium + porous) of equivalent height \(H\) in order to homogenize the multilayer into a single layer.

If the thicknesses of the porous layers are identical to a thickness \(e\), the multilayer is equivalent to a monolayer whose dispersion relation is:

\[
\frac{\omega^2}{K_0} Z_e \approx K(h_0 + ne) \approx KH_e
\]

(34)

An equivalent thickness \(h_e\) and an equivalent impedance \(Z_e\) (equivalent permeability) are defined as:

\[
h_e = \frac{ne}{Z_e} \quad \text{avec} \quad H = h_0 + h_e \quad \text{et} \quad \frac{1}{Z_e} = \frac{1}{n \sum_{i=1}^{n} Z_i}
\]

\[
Z_e = s - if \quad \text{or} \quad s = \frac{1}{\varepsilon} + \frac{1 - \varepsilon}{\varepsilon} C_m ; \quad f = \frac{1 - \varepsilon}{\omega \varepsilon}
\]

\(n\) is the number of layers and \(e\) is the index for “equivalent”.

3.4. Graphical results

The profile of the free surface in the case of a single-layer bottom and a bi-layer and tri-layer bottom are presented respectively for the following data:

\(T = 0.5\); \(a = 0.012\); \(h_0 = 0.15\); \(K = (16.287-6.95 \times 10^{-3})\); \(f_1 = 1\); \(\varepsilon_1 = 0.30\); \(C_{m1} = 1\); \(h_1 = 0.17\); \(f_2 = 0.8\); \(\varepsilon_2 = 0.40\); \(C_{m2} = 1\); \(h_2 = 0.185\); \(f_3 = 1\); \(\varepsilon_3 = 0.50\); \(C_{m3} = 1\); \(h_3 = 0.20\)

Figure 2. Free surface profile as a function of length (m)

The damping coefficient and the horizontal velocity profile are presented in different cases for the following data:

\(T = 1\); \(a = 0.012\); \(h_0 = 0.15\); \(K = 5.691-6.7856 \times 10^{-3}\); \(f_1 = 1\); \(\varepsilon_1 = 0.30\); \(C_{m1} = 0.86\); \(h_1 = 0.20\);
the horizontal velocity profile is presented in different cases for the following data:

\[ T = 1; \ a = 0.012; \ h_0 = 0.15; \ K = (5.64 - 1.56 \times 10^{-3}); \ f_1 = 1; \ \varepsilon_1 = 0.30; \ C_{m1} = 0.86; \ h_1 = 0.166; \]
\[ f_2 = 0.8; \ \varepsilon_2 = 0.40; \ C_{m2} = 1; \ h_2 = 1.183; \ f_3 = 0.8; \ \varepsilon_3 = 0.50; \ C_{m3} = 1; \ h_3 = 0.20; \ x = 0. \]

**Figure 3.** Damping Coefficient as a Function of Frequency (Hz)

**Figure 4.** Horizontal velocity profile in a different medium and its equivalent medium

**Concept of the depth of closure**

The analytical calculation of the velocity potential makes it possible to define the closing depth, that is to say the depth limit of remobilization of sediments in the porous medium.

We present the evolution of the damping coefficient as a function of the frequency in the presence of a porous medium composed of monolayers, two layers, three layers and four layers of different characteristics.
4. Conclusion

We modelled the interaction of the linear wave with a multi-layer porous medium by adapting the Forchheimer model in the instantaneous mode based on the flat wave hypotheses and the long wave approximation, perfect flow, incompressible and isothermal, the porous medium is rigid and undeformable. We have calculated analytically by the potential theory the dispersion relation and the velocity potentials. We note that:

- The wavelength decreases exponentially as the frequency increases, increases with the height of the outer medium and little influenced by the thickness of the porous layer;
- The variation of the coefficient of depreciation as a function of the frequency is not monotonous;
- The evolution of the free surface shows that a multilayer porous structure attenuates and shifts the free surface more than a single layer;
- The definition of an equivalent medium and an equivalent impedance (equivalent permeability) can influence the flow behavior in the two layers and three layers, replaced by this equivalent medium in order to homogenize the multi to a mono;
- The horizontal velocity profile presents a discontinuity at the interfaces, we will question the model of a perfect fluid, we had to take into account tangential constraints, a viscous fluid and the Forchheimer model is a more realistic approach;
- From the fourth porous layer, there is no longer any effect on the damping coefficient and therefore on the attenuation of the swell

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