DIMENSION DROP OF CONNECTED PART OF SLICING SELF-AFFINE SPONGES

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ABSTRACT. The connected part of a metric space $E$ is defined to be the union of non-trivial connected components of $E$. We proved that for a class of self-affine sets called slicing self-affine sponges, the connected part of $E$ either coincides with $E$, or is essentially contained in the attractor of a proper sub-IFS of an iteration of the original IFS. This generalizes an early result of Huang and Rao [L. Y. Huang, H. Rao. A dimension drop phenomenon of fractal cubes, J. Math. Anal. Appl. 497 (2021), no. 2] on a class of self-similar sets called fractal cubes. Moreover, we show that the result is no longer valid if the slicing property is removed. Consequently, for a Barański carpet $E$, the Hausdorff dimension and the box dimension of the connected part of $E$ are strictly less than the Hausdorff dimension and the box dimension of $E$, respectively. For slicing self-affine sponges in $\mathbb{R}^d$ with $d \geq 3$, whether the attractor of a sub-IFS has strictly smaller dimensions is an open problem.

1. Introduction

Usually a fractal set contains both trivial connected components and non-trivial connected components, where a connected component is called trivial if it is a singleton. Let $X$ be a metric space. Let $X_c$ be the union of non-trivial connected components and we call it the connected part of $X$. Huang and Rao [9] introduced a notion of connectedness index of $X$ as

\begin{equation}
\text{ind}_H X = \dim_H X_c,
\end{equation}

where $\dim_H$ denotes the Hausdorff dimension. In the following, we will call $\text{ind}_H X$ the Hausdorff connectedness index of $X$. It is shown that (9) if $X$ is a fractal cube, that is, $X \subset \mathbb{R}^d$ is the non-empty compact set satisfying the set equation $X = \bigcup_{d \in D} (X + d)/n$, where $n \geq 2$ is an integer and $D \subset \{0, 1, \ldots, n-1\}^d$, then either $X_c = X$ or $\text{ind}_H X = \dim_H X_c < \dim_H X$.

This result has several interesting consequences. Let $X$ be a fractal cube possessing trivial points. First, the topological Hausdorff dimension of $X$ is less than $\dim_H X$, since the former one is no larger than $\text{ind}_H$, see [9] and [25]. (The topological Hausdorff dimension is a new dimension introduced by [1].) Secondly, the
gap sequence of \( X \) is comparable to \((k^{-\beta})_{k \geq 1}\) where \( \beta = \dim_H X \), see [10]. Thirdly, \( \text{ind}_H X \) provides a new Lipschitz invariant.

The aim of the present paper is to generalize the results of [9] to a larger class of fractals called slicing self-affine sponges.

**Definition 1.1** (Diagonal IFS ([5])). Fix \( d \geq 1 \). For each \( i \in \{1, \ldots, d\} \), let \( A_i = \{0, 1, \ldots, n_i - 1\} \) with \( n_i \geq 2 \), and let \( \Phi_i = (\phi_{a,i})_{a \in A_i} \) be a collection of contracting similarities of \([0, 1]\), called the base IFS in coordinate \( i \). Let \( A = \prod_{i=1}^d A_i \), and for each \( a = (a_1, \ldots, a_d) \in A \), consider the contracting affine maps \( \phi_a : [0, 1]^d \to [0, 1]^d \) defined by the formula

\[
\phi_a(x_1, \ldots, x_d) = (\phi_{a,1}(x_1), \ldots, \phi_{a,d}(x_d)),
\]

where \( \phi_{a,i} \) is shorthand for \( \phi_{a,i} \) in the formula above. Then we can get

\[
\phi_a([0, 1]^d) = \prod_{i=1}^d [0, 1] \subset [0, 1]^d.
\]

Given \( D \subset A \), we call the collection \( \Phi = (\phi_a)_{a \in D} \) a diagonal IFS, and we call its invariant set \( \Lambda_\Phi \) a diagonal self-affine sponge.

A diagonal self-affine IFS \( \Phi \) is called a slicing self-affine IFS, if for each \( 1 \leq i \leq d \),

\[
[0, 1) = \phi_{0,i}[0, 1) \cup \cdots \cup \phi_{n_i-1,i}[0, 1)
\]

is a partition of \([0, 1)\) from left to right; in this case, we call \( \Lambda_\Phi \) a slicing self-affine sponge. In the two dimensional case, \( \Lambda_\Phi \) is called a Barański carpet ([5]).

Let \( \Lambda_\Phi \) be a slicing self-affine sponge. If for each \( i \), the maps in \( \Phi_i \) have equal contraction ratio \( 1/n_i \), then \( \Lambda_\Phi \) is called a Sierpiński self-affine sponge, see Kenyon and Peres [14] and Olsen [22]. There is a simple way to define a Sierpiński self-affine sponge. Let \( M = \text{diag}(n_1, \ldots, n_d) \) and let \( D \subset \prod_{i=1}^d \{0, 1, \ldots, n_i - 1\} \). Then \( M \) and \( D \) determine an IFS

\[
\lambda_d(z) = M^{-1}(z + d) \text{ for } d \in D.
\]

Then its invariant set \( K = K(M, D) \) is a Sierpiński self-affine sponge. If \( M \) and \( D \) are obtained from a slicing IFS \( \Phi \), then we call it the associated Sierpiński self-affine sponge of \( \Phi \).

**Remark 1.2.** There are a lot of works on dimensions, multifractal analysis and other topics of diagonal self-affine sponges, see for instance, McMullen [20], Bedford [4], Lalley and Gatzouras [17], King [16], Kenyon and Peres [14][15], Feng and Wang [7], Barański [2], Olsen [22], Barral and Mensi [3], Jordan and Rams [12], Mackay [18], Das and Simmons [5], Fraser [8], Miao, Xi and Xiong [21], Rao, Yang and Zhang [23].

Our main results are as follows. First, we prove that a slicing self-affine sponge is always homeomorphic to its associated Sierpiński self-affine sponge. We use \( \partial E \) to denote the boundary of a set \( E \).

**Theorem 1.1.** Let \( \Lambda_\Phi \) be a \( d \)-dimensional slicing self-affine sponge generated by \( \Phi = (\phi_a)_{a \in D} \), and \( K(M, D) \) be the associated Sierpiński self-affine sponge of \( \Phi \).
Then there exists a map $F : \Lambda_{\Phi} \rightarrow K(M, D)$ which is bi-Hölder continuous, and $F(\Lambda_{\Phi} \cap \partial[0, 1]^d) \subset \partial[0, 1]^d$.

A slicing self-affine sponge is said to be degenerated, if it is contained in a $(d - 1)$-face of $[0, 1]^d$. (For definition of face of a convex body, see Section 3.) Next, we show that

**Theorem 1.2.** Let $\Lambda_{\Phi}$ be a slicing self-affine sponge. If $\Lambda_{\Phi}$ is non-degenerated and possesses trivial points, then $\Lambda_{\Phi}$ has a trivial point in $(0, 1)^d$.

**Remark 1.3.** Thanks to Theorem 1.1 we only need to proof Theorem 1.2 for Sierpiński self-affine sponge, which generalizes the main result of [9]. The key point in our proof is that we build up Lemma 3.3, which allows us to generalize and simplify the approach of [9]. Besides, our definition of degeneration of self-affine sponge is more suitable for the purpose than that in [9].

For $\omega = \omega_1 \ldots \omega_k \in D^k$, we denote $\phi_\omega = \phi_{\omega_1} \circ \ldots \circ \phi_{\omega_k}$. Thanks to Theorem 1.2 we can show that

**Theorem 1.3.** Let $\Lambda_{\Phi}$ be a slicing self-affine sponge with digit set $D$. If $\Lambda_{\Phi}$ possesses trivial points, then there exists a proper sub-IFS of an iteration of $\Phi$ with invariant set $\Lambda'$ such that the connected part of $\Lambda_{\Phi}$ is contained in $\bigcup_{\omega \in U_k \geq 0} D^k \phi_\omega(\Lambda')$

The example below shows that the ‘slicing property’ in Theorem 1.2 cannot be removed.

**Example 1.4.** In this example, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Let $E$ be the self-similar set generated by the IFS \{f_i\}_{i=1}^{7}, where

\[
\begin{align*}
f_1(z) &= \frac{2z}{3}, \quad f_2(z) = \frac{z}{3} + \frac{2i}{3}, \quad f_3(z) = \frac{z}{3} + \frac{1 + 2i}{3}, \quad f_4(z) = \frac{z}{6} + \frac{4 + 5i}{6}, \\
f_5(z) &= \frac{z}{6} + \frac{5 + 5i}{6}, \quad f_6(z) = \frac{z}{3} + \frac{2 + i}{3}, \quad f_7(z) = \frac{z}{12} + \frac{22 + 17i}{24}.
\end{align*}
\]

We will show that the trivial points of $E$ are all located on the line segment $\{1\} \times [1/2, 1]$ (Theorem 6.1).
Let $X$ be a metric space. Similar to the Hausdorff connectedness index, we define the box connectedness index of $X$ to be

\begin{equation}
\text{ind}_B X = \dim_B X_c,
\end{equation}

where $X_c$ is the connected part of $X$.

Let $\Phi_i, i = 1, \ldots, d$ be the base IFSes in definition of slicing IFS. We denote by $\Phi(\mathcal{D})$ the slicing IFS determined by $\mathcal{D}$. There is an important open question in dynamical system that if $\mathcal{D}'$ is a proper subset of $\mathcal{D}$, is it true that

\begin{equation}
\dim_H \Lambda_{\Phi(\mathcal{D}')} < \dim_H \Lambda_{\Phi(\mathcal{D})} \text{ and } \dim_B \Lambda_{\Phi(\mathcal{D}')} < \dim_B \Lambda_{\Phi(\mathcal{D})}?
\end{equation}

(See for example, Kanemaki [13].)

Till now we can only show that the above inequalities hold for Barański carpets. In the box dimension case, Barański [2] gave an implicit formula for box dimension, which confirms the second inequality of (1.4) easily. As for the Hausdorff dimension, Barański showed that a certain variation principle holds where only the Bernoulli measures are involved. Hence, it is not hard to show that the maximum cannot occur at a boundary point of the (finite dimensional) parameter space (It is proved in a manuscript of Dejun Feng [6]). Hence, we have

**Corollary 1.5.** If a Barański carpet $\Lambda_{\Phi}$ has a trivial point, then

$$\text{ind}_H(\Lambda_{\Phi}) < \dim_H(\Lambda_{\Phi}) \text{ and } \text{ind}_B(\Lambda_{\Phi}) < \dim_B(\Lambda_{\Phi}).$$

But when $d \geq 3$, Das and Simmons [5] proved that the Hausdorff dimension can be obtained by a variation principle which involves a larger class of measures called pseudo Bernoulli measures. So it is not clear whether the first inequality of (1.4) holds. Also, till now, we do not have a formula of box dimension of slicing self-affine sponge with $d \geq 3$.

**Example 1.6.** Let $E$ and $E'$ be two Bedford-McMullen carpets indicated by Figure 2. They have the same Hausdorff dimension and also have the same box dimension. We will calculate the connectedness indices of $E$ and $E'$ in Section 5, see Table 1. In fact, the connected part of $E$ is essentially a sofic self-affine carpet studied by Kenyon and Peres [15]. It is seen that $\text{ind}_H(E) \neq \text{ind}_H(E')$ and $\text{ind}_B(E) \neq \text{ind}_B(E')$, so $E$ and $E'$ are not Lipschitz equivalent.

**Table 1.** Dimensions and indices of $E$ and $E'$

|       | $\dim_H$ | $\dim_B$ | $\text{ind}_H$ | $\text{ind}_B$ |
|-------|----------|----------|----------------|----------------|
| $E$   | $\log_5 (12 + \sqrt{5}) \approx 1.627$ | $1 + \log_8 4 \approx 1.667$ | $\approx 1.61$ | $\frac{\log 19 + (\sigma - 1) \log 5}{\log 8} \approx 1.662$ |
| $E'$  | $\log_5 (12 + \sqrt{5})$ | $1 + \log_8 4$ | $\approx 1.54$ | $\frac{\log 19 + (\sigma - 1) \log 5}{\log 8} \approx 1.640$ |

*In the above table, $\lambda = \frac{22 + \sqrt{312}}{2}$ and $\sigma = \log 8 / \log 5$.*
This article is organized as follows. Theorem 1.1 is proved in section 2. In section 3, we recall some basic facts about faces of the polytope \([0, 1]^d\). Theorem 1.2 and Theorem 1.3 are proved in section 4. Example 1.6 and Example 1.4 are discussed in Section 5 and Section 6, respectively.

2. Proof of Theorem 1.1

First, let us give some notations about IFS. Let \(A\) be a finite set. An iterated function system (IFS) on \(\mathbb{R}^d\) is a family of contractions \(\Phi = \{\phi_a\}_{a \in A}\) on \(\mathbb{R}^d\), and the attractor of the IFS is the unique nonempty compact set \(K = \bigcup_{a \in A} \phi_a(K)\), and it is called a self-similar set if all the contractions are similitudes. (See [11].) Let \(\pi : A^\infty \to K\) be the coding map defined by

\[
\{\pi \Phi((a_k)_{k \geq 1})\} = \bigcap_{k \geq 1} \phi_{a_1...a_k}(K).
\]

We say \((a_k)_{k \geq 1}\) is a coding of \(x \in K\) if \(x = \pi \Phi((a_k)_{k \geq 1})\).

Let \(\Phi\) be a slicing IFS with digit set \(D\). Let \(\Lambda := \Lambda_{\Phi}\) be a slicing self-affine sponge. Let \(\pi : \mathcal{D}^\infty \to \Lambda_{\Phi}\) be the corresponding coding map. Let \(K := K(M, \mathcal{D})\) be the associated Sierpiński sponge of \(\Phi\), let \((\lambda_d)_{d \in \mathcal{D}}\) be an generating IFS of \(K\) defined by (1.2), and \(\pi' : \mathcal{D}^\infty \to K(M, \mathcal{D})\) be the corresponding coding map.

First, we build a simple lemma. Let \(n \geq 2\) and let

\[
(2.1) \quad \Psi := (\psi_j = r_j x + b_j)^{n-1}_{j=0}
\]

be a family of contractions on \(\mathbb{R}\) with \(0 < r_j < 1\), such that \(\psi_0([0, 1)) \cup \cdots \cup \psi_{n-1}([0, 1))\) is a partition of \([0, 1)\) from left to right.

For \(h = (h_k)_{k \geq 1} \in \{0, 1, \ldots, n-1\}^\infty\), we define two functions

\[
(2.2) \quad f(h) = \sum_{k \geq 1} \frac{h_k}{n^k},
\]

and

\[
(2.3) \quad g(h) = \sum_{k \geq 1} r_{h_1} \cdots r_{h_{k-1}} b_{h_k}.
\]

Note that the first term of the right hand side of (2.3) is \(b_{h_1}\).
Lemma 2.1. There exists $\alpha > 0$ such that for any $h, h' \in \{0, 1, \ldots, n - 1\}^\infty$,
\[
1/2 \cdot |g(h) - g(h')|^{1/\alpha} \leq |f(h) - f(h')| \leq 2|g(h) - g(h')|^{\alpha}.
\]

Proof. Denote $r^* = \max\{r_i; 0 \leq i \leq n - 1\}$ and $r_* = \min\{r_i; 0 \leq i \leq n - 1\}$.

Write $h = (h_k)_{k \geq 1}$ and $h' = (h'_k)_{k \geq 1}$. Let $k$ be the smallest integer such that $h_k \neq h'_k$. Without loss of generality, we assume that $h_k > h'_k$.

If $h_k - h'_k \geq 2$, we set $k^* = k$, otherwise, that is, if $h_k - h'_k = 1$, we set
\[
k^* = \max \left\{ \ell \geq k + 1; \left( \begin{array}{c} h_{k+1} \ldots h_{\ell-1} \\ h'_{k+1} \ldots h'_{\ell-1} \end{array} \right) = \left( \begin{array}{c} 0 \\ n-1 \end{array} \right) \right\}.
\]

Then
\[
n^{-k^*} \leq |g(h) - g(h')| \leq 2n^{-k^*+1}.
\]

Similarly, we have
\[
(r_*)^{-k^*} \leq |f(h) - f(h')| \leq 2(r^*)^{-k^*+1}.
\]

Setting $\alpha = \min \{-\log r^*/\log n, -\log n/\log r_*\}$, we obtain the lemma. □

Write $\pi = (\pi_1, \ldots, \pi_d)$ and $\pi' = (\pi'_1, \ldots, \pi'_d)$.

Lemma 2.2. Let $\overrightarrow{a} = (a_k)_{k \geq 1}$, $\overrightarrow{a'} = (a'_k)_{k \geq 1} \in D^\infty$. Then $\pi(\overrightarrow{a}) = \pi(\overrightarrow{a'})$ if and only if $\pi'(\overrightarrow{a}) = \pi'(\overrightarrow{a'})$.

Proof. Notice that both $\phi_d(z)$ and $\lambda_d(z)$ are variable separation functions. Clearly $\pi(\overrightarrow{a}) = \pi(\overrightarrow{a'})$ if and only if
\[
\pi_j(\overrightarrow{a}) = \pi_j(\overrightarrow{a'}), \quad \text{for } 1 \leq j \leq d.
\]

By Lemma 2.1, (2.6) holds if and only if
\[
\pi'_j(\overrightarrow{a}) = \pi'_j(\overrightarrow{a'}), \quad \text{for } 1 \leq j \leq d,
\]

which is equivalent to $\pi'(\overrightarrow{a}) = \pi'(\overrightarrow{a'})$. The lemma is proved. □

Proof of Theorem 1.1. According to Lemma 2.2 we define a coding preserving map $F : \Lambda_\Phi \to K(M, D)$ by
\[
F(z) = \pi' \circ \pi^{-1}(z).
\]

Write $F(z) = (F_1(z), \ldots, F_d(z))$. Then $F_j(z) = \pi'_j \circ \pi^{-1}_j(z)$ for $1 \leq j \leq d$. By Lemma 2.1 all $F_j$ are bi-Hölder continuous, so $F$ is also bi-Hölder continuous.

Suppose $z \in \Lambda_\Phi \cap \partial[0, 1]^d$. Denote $(a_k)_{k \geq 1} = \pi^{-1}(z)$. Then there exists $j$ such that $\phi_{a_{k,j}} = \phi_{0,j}$ for all $k \geq 1$, or $\phi_{a_{k,j}} = \phi_{n_k-1,j}$ for all $k \geq 1$. It follows that $a_{k,j} = 0$ for all $k \geq 1$, or $a_{k,j} = n_j - 1$ for all $k \geq 1$. Therefore, $\pi'_j((a_k)_{k \geq 1}) = 0$ or $1$, which implies that $F(z) \in \partial[0, 1]^d$. The theorem is proved. □
3. Preliminaries on faces of $[0,1]^d$

We recall some notions about convex polytopes, see [26] or [24]. Let $C \subset \mathbb{R}^d$ be a convex polytope, let $F$ be a convex subset of $C$. We say $F$ is a face of $C$, if any closed line segment $I \subset C$ with a relative interior point in $F$ has both endpoints in $F$ (see [24]). The dimension of a face $F$, denoted by $\text{dim} F$, is the dimension of the smallest affine subspace containing $F$; moreover, $F$ is called an $r$-face of $C$ if $\text{dim} F = r$. We note that $C$ is a $d$-face of itself if $\text{dim} C = d$. For $z \in C$, a face $F$ of $C$ is called the containing face of $z$ if $z$ is a relative interior point of $F$.

In this section, we list some simple facts about faces of $[0,1]^d$ we need later. We call a pair $(A,B)$ an ordered partition of $\{1, \ldots, d\}$ if $A \cap B = \emptyset$ and $A \cup B = \{1, \ldots, d\}$. Let $e_1, \ldots, e_d$ be the canonical basis of $\mathbb{R}^d$. The following lemma is obvious, see Chapter 2 of [26].

**Lemma 3.1.** (i) Let $(A,B)$ be an ordered partition of $\{1, \ldots, d\}$ with $\#A = r$. Then the set

$$F = \left\{ \sum_{j \in A} c_j e_j; \ c_j \in [0,1] \right\} + b$$

is an $r$-face of $[0,1]^d$ if and only if $b \in T$, where

$$T := \left\{ \sum_{j \in B} \varepsilon_j e_j; \ \varepsilon_j \in \{0,1\} \right\}.$$

(ii) For any $r$-face $F$ of $[0,1]^d$, there exists an ordered partition $(A,B)$ of $\{1, \ldots, d\}$ with $\#A = r$ such that $F$ can be written as (3.1).

We will call $F_0 = \{ \sum_{j \in A} c_j e_j; \ c_j \in [0,1] \}$ a basic face related to the ordered partition $(A,B)$. If $A = \emptyset$, we set $F_0 = \{0\}$ by convention. Let $x = \sum_{j \in A} \alpha_j e_j + \sum_{i \in B} \beta_i e_i \in [0,1]^d$, we define two projection maps as follows:

$$\pi_A(x) = \sum_{j \in A} \alpha_j e_j, \quad \pi_B(x) = \sum_{i \in B} \beta_i e_i.$$

If $F$ is an $r$-face of $[0,1]^d$, we denote by $\hat{F}$ the relative interior of $F$.

**Lemma 3.2** (Huang and Rao [9]). Let $F = F_0 + b$ be an $r$-face of $[0,1]^d$ given by (3.1). Let $u \in \mathbb{Z}^d$. Then $\hat{F} \cap (u + [0,1]^d) \neq \emptyset$ if and only if $u \in b - T$ where $T$ is defined in (3.2).

The following lemma strengthens Lemma 3.1 in [9], and allows us to give simpler arguments in Section 4 comparing to [9].

**Lemma 3.3.** Let $z_0 \in \partial [0,1]^d$ and let $F$ be the containing face of $z_0$. Let

$$g(x_1, \ldots, x_d) = (a_1 x_1 + t_1, \ldots, a_d x_d + t_d)$$

be a map such that $a_j \in (0,1)$ and $g([0,1]^d) \subset [0,1]^d$. Let $F'$ be the containing face of $g(z_0)$. Then
(i) \(g(F) \subset F'\);
(ii) either \(F' = F\) or \(\dim F' \geq \dim F + 1\).

**Proof.** Notice that the assumptions imply that \(t_j \in [0, 1)\) for all \(1 \leq j \leq d\).

Let \((A, B)\) be an ordered partition in Lemma 3.1 which defines \(F\). By the definition of containing face, we have \(z_0 \in F\). Suppose that \(g(z_0) \notin F\).

First, we prove (i). If \(F = \{z_0\}\), the assertion holds trivially. Now we suppose that \(\dim F \geq 1\). Take any point \(x \in F \setminus \{z_0\}\) and let \(I\) be a closed line segment in \(F\) such that \(x\) is an endpoint of \(I\) and \(z_0\) is a relative interior point of \(I\). It is clear that \(g(I) \subset g([0, 1]^d) \subset [0, 1]^d\). Since \(g(z_0) \notin F\), we have \(g(I) \subset F'\). By the arbitrariness of \(x\) we obtain that \(g(F) \subset F'\). Especially, we have \(\dim F' \geq \dim F\).

Next, we prove (ii). Denote \(r = \dim F\) and write \(F \) as \(F = F_0 + b\), where \(F_0\) is a basic \(r\)-face, and \(b \in T := \{\sum_{j \in B} \varepsilon_j e_j; \varepsilon_j \in \{0, 1\}\}\).

Suppose that \(F'\) is an \(r\)-face of \([0, 1]^d\). Then there exists an ordered partition \((A', B')\) of \([1, \ldots, d]\) such that \(F' = F_0' + b'\), where \(F_0' = \{\sum_{j \in A'} c_j e_j; c_j \in [0, 1]\}\) and \(b' \in T' = \{\sum_{j \in B'} \varepsilon_j e_j; \varepsilon_j \in \{0, 1\}\}\).

\[\text{(3.4)} \quad \text{diag}(a_1, \ldots, a_d)(F_0 + b) + g(0) = g(F) \subset F' = F_0' + b', \]

we have \(F_0' = F_0\). Hence \(A' = A\) and \(B' = B\), and it follows that \(T' = T\). Applying \(\pi_B\) to both sides of (3.4), we have

\[\text{(3.5)} \quad b' = \pi_B(F') = \pi_B(g(F)) = \pi_B(\text{diag}(a_1, \ldots, a_d)F_0) + \pi_B(g(b)) = \pi_B(g(b)).\]

Denote \(b = \sum_{j \in B} b_j e_j\), where all \(b_j \in \{0, 1\}\), then

\[\text{(3.6)} \quad b' = \sum_{j \in B} (a_j b_j + t_j) e_j \in T.\]

If \(b_j = 1\), then \(a_j b_j + t_j > 0\) and it forces \(a_j b_j + t_j = 1\); if \(b_j = 0\), then \(a_j b_j + t_j = t_j < 1\) and it forces that \(a_j b_j + t_j = 0\). Hence \(b' = b\) and it follows that \(F' = F\). This confirms (ii) and the lemma is proved. \(\square\)

4. **Proof of Theorem 1.2 and Theorem 1.3**

Let \(K = K(M, D)\) be a \(d\)-dimensional Sierpiński sponge defined in Section 1. We denote the IFS generating \(K\) by \(\{\Phi_i\}_{i=1}^N\). Let \(\Sigma = \{1, 2, \ldots, N\}\). Denote by \(\Sigma^\infty\) and \(\Sigma^k\) the sets of infinite words and words of length \(k\) over \(\Sigma\) respectively. Let \(\Sigma^* = \bigcup_{k \geq 0} \Sigma^k\) be the set of all finite words. For \(k \geq 1\), denote \(D_k = D + M D + \cdots + M^{k-1} D\).

We call

\[K_k = M^{-k}([0, 1]^d + D_k)\]

the \(k\)-th approximation of \(K\). Clearly, \(K_k \subset K_{k-1}\) and \(K = \bigcap_{k=0}^\infty K_k\). For each \(\sigma = \sigma_1 \ldots \sigma_k \in \Sigma^k\), we call \(\Phi_\sigma([0, 1]^d)\) a \(k\)-th cell.

For a point \(z \in K\), we say \(F\) is the containing face of \(z\) if \(F\) is a face of \([0, 1]^d\) and it is the containing face of \(z\). From now on, we always assume that

\[\text{(4.1)} \quad z_0 \text{ is a trivial point of } K \text{ and } F \text{ is the containing face of } z_0.\]
Let \((A, B)\) be an ordered partition in Lemma 3.1 which defines \(F\). Recall \(\pi_A\) and \(\pi_B\) are defined in (3.3). Denote
\[
\Sigma_\sigma = \{\omega \in \Sigma^k; \pi_A(\Phi_\omega(0)) = \pi_A(\Phi_\sigma(0))\},
\]
(4.2)
\[
H_\sigma = \bigcup_{\omega \in \Sigma_\sigma} \Phi_\omega([0, 1]^d).
\]
(4.3)
Indeed, \(H_\sigma\) is the union of all \(k\)-th cells having the same projection with \(\Phi_\sigma([0, 1]^d)\) under \(\pi_A\). The following lemma is a generalization of Lemma 3.2 in [9].

**Lemma 4.1.** Let \(k > 0\) and let \(\sigma \in \Sigma^k\). If \(H_\sigma\) is not connected or \(H_\sigma \cap F = \emptyset\), then there exists \(\omega^* \in \Sigma_\sigma\) such that \(\Phi_{\omega^*}(z_0) \notin F\) and it is a trivial point of \(K\).

**Proof.** Let \(\dim F = r\) and let \((A, B)\) be the ordered partition which defines \(F\). Then \(F = F_0 + b\), where \(F_0\) is a basic \(r\)-face and \(b = \sum_{j \in B} b_j e_j\).

We claim that if \(H_\sigma \cap F \neq \emptyset\), then there is only one \(k\)-th cell \(\Phi_\omega([0, 1]^d)\) in \(H_\sigma\) which intersects \(F\). Let \(\Phi_\omega([0, 1]^d)\) be such a cell. Write
\[
\Phi_\omega(z) = \text{diag}(1/n^1_1, \ldots, 1/n^k_d)z + (t_1, \ldots, t_d).
\]
By Lemma 3.3, we have \(\Phi_\omega(F) \subset F\); especially, we have \(\Phi_\omega(b) = b\), which implies
\[
\frac{b_j}{n_j^k} + t_j = b_j \quad \text{for } j \in B.
\]
It follows that \(t_j = 1 - 1/n^k_j\) if \(b_j = 1\) and \(t_j = 0\) if \(b_j = 0\). As for \(j \in A\), the maps \(\Phi_\omega, \tilde{\omega} \in H_\sigma\), share the same \(j\)-th component with \(\Phi_\sigma\), so \(t_j\) are determined. Hence \(\omega\) is unique in \(\Sigma_\sigma\), and the claim is proved.

The assumptions of the lemma and the above claim imply that there is a connected component \(U\) of \(H_\sigma\) such that \(U \cap F = \emptyset\). Let \(W = \{\omega \in \Sigma_\sigma; \Phi_\omega([0, 1]^d) \subset U\}\).

First, set \(B_0 = \{j \in B; \text{the } j\text{-th coordinate of } b \text{ is } 0\}\), \(B_1 = B \setminus B_0\), and for each \(\omega \in W\), write
\[
\pi_B(\Phi_\omega(0)) = \sum_{j \in B_0} \alpha_j(\omega)e_j + \sum_{j \in B_1} \beta_j(\omega)e_j.
\]
Secondly, set \(W' = \{\omega \in W; \sum_{j \in B_0} \alpha_j(\omega) \text{ attains the minimum}\}\), and take \(\omega^* \in W'\) such that \(\sum_{j \in B_1} \beta_j(\omega^*) \text{ attains the maximum in } W'\). Then by exactly the same argument as Lemma 3.2 of [9], one can show that \(\Phi_{\omega^*}(z_0)\) is a trivial point of \(K\). The lemma is proved.

Recall that a \(d\)-dimensional Sierpiński sponge is degenerated if it is contained in a \((d - 1)\)-face of \([0, 1]^d\).

**Theorem 4.1.** Let \(K\) be a non-degenerated Sierpiński self-affine sponge in \(\mathbb{R}^d\). Let \(z_0\) be a trivial point of \(K\). Then there exists another trivial point \(z^* \in (0, 1)^d\).
Proof. Let \( F \) be the containing face of \( z_0 \) and suppose that \( \dim F = r < d \). Let \((A, B)\) be the ordered partition which defines \( F \). Let \( \sigma = (\sigma_i)_{i \geq 1} \in \Sigma^\infty \) be a coding of \( z_0 \), and let \( \sigma|_k \) denotes the prefix of \( \sigma \) with length \( k \). Then for each \( k > 0 \), \( z_0 \in H_{\sigma|_k} \cap F \), where \( H_{\sigma|_k} \) is defined in \((4.3)\). We claim that \( K \) contains a trivial point of the form \( \Phi_\omega(z_0) \), \( \omega \in \Sigma^* \), and it is not in \( F \).

Now we assume that \( H_{\sigma|_k} \) is connected for all \( k \geq 1 \), for otherwise, the claim holds by Lemma \([4.1]\). Let \( p \) be an integer such that \( C_p \), the connected component of \( K \) containing \( z_0 \), satisfies \( \text{diam}(C_p) < \frac{1}{3} \). Then

\[
C_p \cap (C_p + a) = \emptyset \quad \text{for all } a \in \mathbb{Z}^d \setminus \{0\}.
\]

That \( H_{\sigma|_p} \) is connected implies that \( H_{\sigma|_p} \subset C_p \). Since \( K \) is not degenerated, there exist \( j \in \Sigma \) such that

\[
(4.4) \quad \Phi_j([0, 1]^d) \cap F = \emptyset.
\]

We consider the set \( H_{\sigma_1 \ldots \sigma_p} \). Let \( \Sigma_j = \{ i \in \Sigma; \pi_A(\Phi_i(0)) = \pi_A(\Phi_j(0)) \} \). It is easy to see that \( H_{\sigma_1 \ldots \sigma_p} = \bigcup_{i \in \Sigma_j} \Phi_i(H_{\sigma|_p}) \). By \((4.4)\) and \((4.5)\), we infer that \( U := \Phi_j(H_{\sigma|_p}) \) is a connected component of \( H_{\sigma_1 \ldots \sigma_p} \) and \( U \cap F = \emptyset \). Therefore, by Lemma \([3.3]\) there exists \( \omega^* \in \Sigma_{\sigma_1 \ldots \sigma_p} \) such that \( \Phi_{\omega^*}(z_0) \notin F \) and it is a trivial point of \( E \). The claim is proved.

Now by Lemma \([3.3]\) the containing face of \( \Phi_{\omega^*}(z_0) \) has dimension no less than \( r + 1 \). Inductively, we can find a trivial point \( z^* \) whose containing face is \([0, 1]^d \). \( \square \)

**Proof of Theorem 1.2.** It follows directly from Theorem \([1.1]\) and Theorem \([4.1]\) \( \square \)

Let \( \Lambda = \Lambda_{\Phi(D)} \) be a slicing self-affine sponge in \( \mathbb{R}^d \). Let

\[
A_k = \bigcup_{\omega \in D^k} \phi_\omega([0, 1]^d)
\]

be the \( k \)-th approximation of \( \Lambda \). Let \( U \) be a connected component of \( A_k \), following \([9]\), we call \( U \) a \( k \)-th island if \( U \cap \partial [0, 1]^d = \emptyset \).

**Proof of Theorem 1.3.** First, let us assume \( \Lambda_{\Phi(D)} \) is not degenerated.

Since \( \Lambda_{\Phi(D)} \) contains a trivial point, by Theorem \([1.2]\) \( \Lambda_{\Phi(D)} \cap (0, 1)^d \) also possesses trivial points. Consequently, \( \Lambda_k \) has a \( k \)-th island for \( k \) large; for a careful proof of this fact, we refer to \([9]\). Without loss of generality, we assume that \( \Lambda_{\Phi(D)} \) has a 1-island and we denote it by \( U \). Write \( U = \bigcup_{d \in J} \phi_d([0, 1]^d) \), where \( J \subset D \).

Let \( D' = D \setminus J \) and let \( \Lambda_{\Phi(D')} \) be the slicing self-affine sponge determined by \( D' \). Clearly, if a coding of a point \( x \) having infinitely many entries in \( J \), then \( x \) is a trivial point. Let \( P \) be the connected part of \( \Lambda_{\Phi(D')} \), then \( P \subset \bigcup_{\omega \in \bigcup_{\omega \in D^k} \phi_\omega(\Lambda_{\Phi(D')} \big) \).

Next, if \( \Lambda_{\Phi(D)} \) is degenerated, then it can be identified with a lower dimensional slicing self-affine sponge which is non-degenerated, so \((1.4)\) still holds. \( \square \)

5. **Connected indices of fractals in Example 1.6**

Let \( E \) and \( E' \) be the Bedford-McMullen carpets in Example \([1.6]\). In this section we calculate the connected indices of \( E \) and \( E' \).
5.1. **Graph-directed IFS.** Let \((V, \Gamma)\) be a directed graph with a vertex set \(V\) and directed-edge set \(\Gamma\) where both \(V\) and \(\Gamma\) are finite. The set of edges from \(i\) to \(j\) is denoted by \(\Gamma_{i,j}\), and we assume that for any \(i \in V\), there is at least one edge starting from vertex \(i\). For each edge \(e \in \Gamma\), there is a corresponding contraction \(T_e : \mathbb{R}^n \to \mathbb{R}^n\). We call \(\{T_e\}_{e \in \Gamma}\) a **graph-directed IFS** (see [19]). Its invariant sets, called **graph-direct sets**, are the unique non-empty compact sets \((K_i)_{i \in V}\) satisfying

\[
K_i = \bigcup_{j \in V} \bigcup_{e \in \Gamma_{i,j}} T_e(K_j), \quad i \in V.
\]

Take any order of the vertex set \(V\) and fix it. The the **adjacency matrix** \(A\) of a graph \((V, \Gamma)\) is defined as for any two vertices \(v, w\) in \(V\), \(A_{v,w}\) is the number of edges in \(G\) from \(v\) to \(w\).

5.2. **Connected part of \(E\).** Let \(M = \begin{pmatrix} 8 & 0 \\ 0 & 5 \end{pmatrix}\). Let \(D \subset \{0, 1, \ldots, 7\} \times \{0, 1, \ldots, 4\}\) be the digit set illustrated in Figure 3 (a). Then \(E = K(M, D)\) is the Bedford-McMullen carpet generated by the IFS \(\{\varphi_d\}_{d \in D}\) where \(\varphi_d(z) = M^{-1}(z + d)\).

First, we determine the connected component of \(K\) containing \(0\). Denote

\[
\begin{align*}
J_{XY} &= \{(0, 1), (0, 2), (0, 3)\} , \quad J_{XX} = D \setminus J_{XY}; \\
J_{YX} &= \{(7, 0), (0, 1), (0, 2), (0, 3), (7, 4)\}, \quad J_{YY} = D \setminus J_{YX}.
\end{align*}
\]

(See Figure 3) Let \((X, Y)\) be the invariant sets of the graph-directed IFS given by Figure 4. Then \(X\) and \(Y\) satisfy the set equations

\[
X = \left( \bigcup_{d \in J_{XX}} \varphi_d(X) \right) \cup \left( \bigcup_{d \in J_{XY}} \varphi_d(Y) \right), \quad Y = \left( \bigcup_{d \in J_{YX}} \varphi_d(X) \right) \cup \left( \bigcup_{d \in J_{YY}} \varphi_d(Y) \right).
\]

(a) The first iteration of \(X\).  
(b) The first iteration of \(Y\).  

**Figure 3.**
Figure 4. The directed graph \((V, \Gamma)\), where \(V = \{S_X, S_Y\}\). Each \(d \in J_{XY}\) corresponds to an edge from \(S_X\) to \(S_Y\), and the corresponding is \(M^{-1}(z + d)\). The same holds for \(J_{XX}, J_{YX}\) and \(J_{YY}\).

**Lemma 5.1.** (i) \(Y\) is the connected component of \(K\) containing 0;
(ii) for any non-trivial connected component \(C \neq Y\) of \(K\), there exists \(\omega \in \Sigma^*\) such that \(C = \Phi_\omega(Y)\).

The prove of the above lemma is exactly the same as the proof of Lemma 5.1 in [9], so we omit it.

5.3. **Connectedness indices of** \(E\). Kenyon and Peres [15] studied the Hausdorff dimension and box dimension of such graph-directed sets related to Bedford-McMullen carpets. The adjacency matrix of the graph IFS \((5.1)\) is

\[
A = \begin{bmatrix}
#J_{XX} & #J_{YX} \\
#J_{XY} & #J_{YY}
\end{bmatrix} = \begin{bmatrix}
17 & 14 \\
3 & 5
\end{bmatrix},
\]

where \(#E\) denotes the cardinality of a set \(E\). The box dimension of \(X\) and \(Y\) are given by

\[
\dim_B X = \dim_B Y = \frac{\log \lambda}{\log n} + \left(\frac{1}{\log m} - \frac{1}{\log n}\right) \log s
\]

where \(n, m\) is the expanding factors, \(\lambda\) is the spectral radius of \(A\), and \(s\) is the number of non-vacate rows of the digit set \(D\). In our example,

\[
\text{ind}_B(E) = \dim_B Y = \frac{\log \lambda}{\log 8} + \left(\frac{1}{\log 5} - \frac{1}{\log 8}\right) \times \log 5 \approx 1.662
\]

where \(\lambda = \frac{22 + \sqrt{312}}{2} \approx 19.83\).

As for the Hausdorff dimension of \(X\) and \(Y\), [15] proves that

\[
\dim_H X = \dim_H Y = \lim_{k \to \infty} \frac{1}{k} \log m \sum_{0 \leq i_1, \ldots, i_k \leq m-1} \left\| A_{i_k} \cdot A_{i_{k-1}} \cdots A_{i_1} \right\|^{1/\sigma}
\]

where \(A_j\) is the adjacent matrix with respect to the \(j\)-th row and \(\sigma = \log n / \log m\). In our example,

\[
A_0 = A_4 = \begin{bmatrix} 8 & 7 \\ 0 & 1 \end{bmatrix}, \quad A_1 = A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.
\]

By numerical calculation, we have that \(\text{ind}_HE = \dim_H Y \approx 1.61\).
5.4. **Connected indices of** \( E' \). It is easy to show that the connected part of \( E' \) is a Bedford-McMullen carpet indicated in Figure 5. Hence

\[
\text{ind}_H(E') = \log_5 13 \approx 1.54, \quad \text{ind}_B(E') = \frac{\log (19 + (1/\sigma - 1) \log 5)}{\log 8} \approx 1.640.
\]

**Figure 5.** The connected part of \( E' \).

6. **Trivial points of the fractal in Example 1.4**

Let \( E \) be the self-similar set in Example 1.4. Denote \( E^* = E \cup L \) where \( L = \{1\} \times [1/2, 1] \). Let \( H \) be the boundary of the convex hull of \( E \), that is, \( H \) is the trapezoid with vertices \( 0, i, 1 + i \) and \( 1 + i/2 \). Denote \( \Sigma = \{1, \ldots, 7\} \), see Figure 6.

**Figure 6.** The first iteration of \( E \).

**Lemma 6.1.** \( E^* \) is connected.

**Proof.** It is seen that \( H_1 = \bigcup_{j=1}^7 f_j(H) \) is connected and it is a subset of \( E^* \). Inductively, we see that for every \( n \geq 1 \), \( H_n = \bigcup_{\omega \in \Sigma^n} f_\omega(H) \) is connected and it is a subset of \( E^* \). Therefore, \( E^* \), as the limit of \( H_n \) in Hausdorff metric, is connected. \( \square \)

Let \( \Sigma = A \cup B \cup C \) be a partition where \( A = \{1, 2, 3, 4\} \), \( B = \{5, 6\} \) and \( C = \{7\} \). Let \( X \) be the connected component of \( E \) containing 0.

**Lemma 6.2.** Let \( \omega = \omega_1 \ldots \omega_k \in \Sigma^k \) such that \( \omega_j \in B \) for \( j < k \) and \( \omega_k \in A \). Then

\[
f_\omega(E^*) \subset X.
\]
Proof. We prove the lemma by induction on the length of \( \omega \). By Lemma 6.1 \( \bigcup_{j \in A} f_j(E^*) \) is connected, so \( \bigcup_{j \in A} f_j(E^*) \subset X \). This proves that (6.1) holds for \( k = 1 \).

Let \( \omega_{k-1} = 4 \) if \( \omega_{k-1} = 5 \) and \( \omega_{k-1} = 1 \) if \( \omega_{k-1} = 6 \). By induction hypothesis,

\[
f_{\omega_1...\omega_{k-2}\omega_{k-1}}(E^*) \subset X.
\]

On one hand, \( f_\omega(E^*) \subset f_{\omega_1...\omega_{k-1}}(X) \), on the other hand, the intersection of \( f_{\omega_1...\omega_{k-1}}(X) \) and \( f_{\omega_1...\omega_{k-2}\omega_{k-1}}(E^*) \) is not empty. Therefore, \( f_{\omega_1...\omega_{k-1}}(X) \), and also \( f_\omega(E^*) \), are subsets of \( X \). The lemma is proved.

Theorem 6.1. If \( x \) is a trivial point of \( E \), then \( x \in L \).

Proof. Let \( (\omega_k)_{k \geq 1} \) be a coding of \( x \). We are going to show that \( \omega_k \not\in A \) for all \( k \geq 1 \), which implies that \( x \in L \).

Suppose on the contrary that \( \omega_k \in A \) and \( k \) is the least integer with this property. If \( \omega_j \not\in C \) for all \( 1 \leq j < k \), then \( \omega_j \in B \) for \( 1 \leq j < k \). By Lemma 6.2 \( x \in f_\omega(E^*) \subset X \), which contradicts the fact that \( x \) is a trivial point.

If \( \omega_j \in C \) for some \( 1 \leq j < k \), we set \( i \) to be the greatest integer such that \( \omega_i \in C \). Notice that \( x \) is a trivial point of \( f_{\omega_1...\omega_i}(E) \) since the above cylinder is disjoint from other cylinders of rank \( i \). Moreover, \( f_{\omega_1...\omega_i} : E \rightarrow f_{\omega_1...\omega_i}(E) \) is a bijection, so \( y = f_{\omega_1...\omega_i}^{-1}(x) \) is a trivial point of \( E \). A coding of \( y \) is \( (\omega_j)_{j \geq i+1} \). By Lemma 6.2 \( y \in f_{\omega_{i+1}...\omega_k}(E^*) \subset X \), which contradicts the fact that \( y \) is a trivial point of \( E \). The theorem is proved.

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