SIMULTANEOUS CONTROLLABILITY OF TWO VIBRATING STRINGS WITH VARIABLE COEFFICIENTS

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Abstract. We study the simultaneous exact controllability of two vibrating strings with variable physical coefficients and controlled from a common endpoint. We give sufficient conditions on the physical coefficients for which the eigenfrequencies of both systems do not coincide and the associated spectral gap is uniformly positive. Under these conditions, we show that these systems are simultaneously exactly controllable.

1. Introduction. Boundary controllability of elastic systems has been extensively investigated for the last several decades. In the present paper we focus on a particular case of the following general question: if we consider two exactly controllable systems, find the assumptions permitting simultaneous control of all systems using the same input function. This property is called \textit{simultaneous controllability}. More precisely, for $0 < \xi < 1$, we consider the following problems

$$\begin{cases}
\rho_1(x)u_{tt} = (\sigma_1(x)u_x)_x - q_1(x)u, \\
u(0,t) = 0, \quad u(\xi,t) = f(t), \quad t > 0, \\
u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), 
\end{cases} \quad x \in (0,\xi), \quad t > 0,$$

and

$$\begin{cases}
\rho_2(x)v_{tt} = (\sigma_2(x)v_x)_x - q_2(x)v, \\
v(\xi,t) = f(t), \quad v_x(1,t) = 0, \quad t > 0, \\
v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x), 
\end{cases} \quad x \in (\xi,1),$$

where the functions $\rho_i(x)$ and $\sigma_i(x)$ ($i = 1, 2$) represent respectively the density and the tension of each string. The elasticity of each string is denoted by the functions $q_1(x)$ and $q_2(x)$. Throughout the rest of the paper the coefficients $\rho_i$ and $\sigma_i$ are assumed to be positive, $q_i$ is nonnegative ($i=1,2$) and

$$\rho_1, \quad \rho_1 \in H^2(0,\xi), \quad q_1 \in L^1(0,\xi),$$

$$\rho_2, \quad \sigma_2 \in H^2(\xi,1), \quad q_2 \in L^1(\xi,1).$$

The systems above describe the vibrations of two strings joined at a common endpoint $x = \xi$.

This kind of problems was first considered by Russell in [13] and by Lions in [12, Chapter 5]. These problems were considered for two elastic strings in [2, 5, 14] and

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for two beams in [5]. For several elastic strings and beams, we can cite [1, 3, 4, 6]. Note that all the papers mentioned previously deal with the case of constant physical coefficients.

In [14] simultaneous exact controllability of two abstract systems, where one is infinite-dimensional and the other is finite-dimensional was studied. In that paper, it was shown that if both systems are exactly controllable in time $T_0$ and the associated generators have no common eigenvalues, then they are simultaneously exactly controllable in any time $T > T_0$. At the end of that paper, it was studied the simultaneous exact controllability of Systems (1) and (2) with Dirichlet boundary condition at $x = 1$ and the coefficients $\rho_i = \sigma_i \equiv 1$, $q_i \equiv 0$ ($i = 1, 2$). In particular, it was proved the lack of simultaneous exact controllability and under the condition that $\xi$ is an irrational quadratic number, it was given the space of simultaneously reachable states for a control time $T > \max\{4\xi, 4(1 - \xi)\}$. Similar results were announced in [5]. This result was generalized in [2, 6, 7] for $T > 2$. In these references, the main difficulty consists in the fact that between the elements of the sequences of the corresponding eigenvalues there is no uniform gap. This feature implies that the system is not exactly controllable in any time and so, it is necessary to restrict themselves to prove the controllability for small subspaces of initial states of the strings.

In the present paper we give sufficient conditions on the physical coefficients for which the eigenfrequencies associated with Systems (1) and (2), do not coincide and the associated spectral gap is uniformly positive (see Theorem 2.1). Under these conditions, we prove the simultaneous exact controllability of Systems (1) and (2) for a control time $T > \left(\int_{0}^{\xi} \sqrt{\frac{\rho_1(x)}{\sigma_1(x)}} dx + \int_{\xi}^{1} \sqrt{\frac{\rho_2(x)}{\sigma_2(x)}} dx\right)^2$.

2. The eigenvalue gap. In this section we prove that the eigenvalues associated with Systems (1) and (2) do not coincide and the associated spectral gap is uniformly positive. Firstly we give some preliminary results which will be used along this section.

It is well known that, if $f \in L^2(0, T)$, Systems (1) and (2) are well posed in $L^2(0, \xi) \times H^{-1}(0, \xi)$ and $L^2(\xi, 1) \times V'$, respectively, where $V'$ is the dual space of $V$ and

$$V = \{v \in H^1(\xi, 1) \mid v(\xi) = 0\}$$

endowed with the norm

$$\|u\|_V^2 = \int_{\xi}^{1} |v_x(x)|^2 dx,$$

(e.g., see [12]). Let us introduce the operators $A_1$ and $A_2$ defined in $L^2(0, \xi)$ and $L^2(\xi, 1)$, respectively by setting

$$A_1 u = \frac{1}{\rho_1} (-\sigma_1 u')' + q_1 u, \quad x \in (0, \xi),$$

$$A_2 v = \frac{1}{\rho_2} (-\sigma_2 v')' + q_2 v, \quad x \in (\xi, 1),$$

on the domains

$$D(A_1) = H^2(0, \xi) \cap H^1_0(0, \xi)$$

and

$$D(A_2) = \{v \in H^2(\xi, 1) \cap V \text{ such that } v_x(1) = 0\}.$$
basis in $L^2(0, \xi)$ (resp. in $L^2(\xi, 1)$). Consider now the following eigenvalue problems associated with System (1) with $u(\xi, t) = 0$ and System (2) with $v(\xi, t) = 0$, respectively

\[
\begin{cases}
- (\sigma_1(x) \ddot{u})' + q_1(x) \dot{u} = \lambda \rho_1(x) \dot{u}, \quad x \in (0, \xi), \\
\dot{u}(0) = \ddot{u}(\xi) = 0
\end{cases}
\] (4)

and

\[
\begin{cases}
- (\sigma_2(x) \ddot{v})' + q_2(x) \dot{v} = \lambda' \rho_2(x) \dot{v}, \quad x \in (\xi, 1), \\
\dot{v}(\xi) = \ddot{v}'(1) = 0.
\end{cases}
\] (5)

The spectrum of Problem (4) (resp. (5)) is discrete and real. It consists of an increasing sequence of positive and simple eigenvalues $(\lambda_n)_{n \in \mathbb{N}^*}$ (resp. $(\lambda'_n)_{n \in \mathbb{N}^*}$) tending to $+\infty$:

\[
0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \xrightarrow{n \to +\infty} +\infty,
\]

(resp. $0 < \lambda'_1 < \lambda'_2 < \ldots < \lambda'_n < \ldots \xrightarrow{n \to +\infty} +\infty$).

Using the following Liouville transformation $s = \frac{\xi}{\gamma_1} \int_0^x \frac{\rho_1(t)}{\sigma_1(t)} dt$ (e.g., see [11]), System (4) can be written in the following form

\[
\begin{cases}
- \ddot{y} + Q_1 y = \mu y, \quad s \in (0, \xi), \\
y(0) = y(\xi) = 0,
\end{cases}
\] (6)

where

\[
Q_1 = \left( \frac{\gamma_1}{\xi} \right)^2 \left( \frac{q_1}{\rho_1} - \frac{\sigma_1}{\rho_1^2} \right) \left( \sigma_1 \left( \sigma_1 \rho_1 \right)^{-\frac{3}{2}} \right) \left( \sigma_1 \left( \sigma_1 \rho_1 \right)^{-\frac{3}{2}} \right)
\] (7)

and

\[
\gamma_1 = \int_0^\xi \sqrt{\frac{\rho_1(x)}{\sigma_1(x)}} \, dx.
\] (8)

Similarly, using the modified Liouville transformation $s = \frac{1-\xi}{\gamma_2} \int_\xi^x \frac{\rho_2(t)}{\sigma_2(t)} dt + \xi$, System (5) is transformed into the following problem

\[
\begin{cases}
- \ddot{z} + Q_2 z = \mu' z, \quad s \in (\xi, 1), \\
z(\xi) = 0, \quad \dot{z}(1) = az(1),
\end{cases}
\] (9)

where

\[
Q_2 = \left( \frac{\gamma_2}{1-\xi} \right)^2 \left( \frac{q_2}{\rho_2} - \frac{\sigma_2}{\rho_2^2} \right) \left( \sigma_2 \left( \sigma_2 \rho_2 \right)^{-\frac{3}{2}} \right) \left( \sigma_2 \left( \sigma_2 \rho_2 \right)^{-\frac{3}{2}} \right)
\] (10)

and

\[
\gamma_2 = \int_\xi^1 \sqrt{\frac{\rho_2(x)}{\sigma_2(x)}} \, dx.
\] (11)
For a function \( g \) defined on a bounded interval \( \Omega \subset \mathbb{R} \), we denote \( g^- = \inf_{x \in \Omega} g(x) \) and \( g^+ = \sup_{x \in \Omega} g(x) \). Using the variational principle, yields

\[
\left(\frac{n\pi}{\xi}\right)^2 + Q^- \leq \mu_n \leq \left(\frac{n\pi}{\xi}\right)^2 + Q^+, \quad \text{for all } n \geq 1
\]

and

\[
\omega_n + Q^-_2 \leq \mu'_n \leq \omega_n + Q^+_2, \quad \text{for all } n \geq 1,
\]

where \( \left(\frac{n\pi}{\xi}\right)^2 \) and \( \omega_n \) are the eigenvalues of Problems (6) and (10), respectively with \( Q_i \equiv 0, \ (i = 1, 2) \). It is clear that, if \( a = 0 \), then the eigenvalues \( \omega_n \) coincide with \( \left(\frac{(n-\frac{1}{2})\pi}{1-\xi}\right)^2 \). If not, then the eigenvalues \( \omega_n \) satisfy the following equation

\[
\sqrt{\omega} = \tan(\sqrt{\omega}(1 - \xi)).
\]

For \( a < 0 \), we have for all \( n \geq 1 \),

\[
\frac{(n - \frac{1}{2})\pi}{1 - \xi} < \sqrt{\omega_n} < \frac{(n + \frac{1}{2})\pi}{1 - \xi}.
\]

Let us define the following function

\[
Q(x) = \begin{cases} 
\left(\frac{x}{\gamma_1}\right)^2 Q_1(x), & x \in (0, \xi), \\
\left(\frac{1-x}{\gamma_2}\right)^2 Q_2(x), & x \in (\xi, 1),
\end{cases}
\]

where \( Q_i \) and \( \gamma_i \ (i = 1, 2) \) are defined by (7), (11), (9) and (14), respectively. Now, we can state our main result in this section:

**Theorem 2.1.** Let \( \gamma_i \ (i = 1, 2) \), \( a \leq 0 \) and \( Q \) be defined by (9), (14), (13) and (19), respectively. If the following conditions

- \( \frac{\gamma_1}{\gamma_2} \) is a rational number such that
  \[
  \frac{\gamma_1}{\gamma_2} = \frac{p}{q}, \ \text{with } p \text{ is odd},
  \]

- \[
  -a < \frac{\pi^2}{4p(1 - \xi)}
  \]

and

- \[
  |Q^+ - Q^-| < \frac{(\pi^2 + 4ap(1 - \xi))(2q + p)}{(2q\gamma_1)^2}
  \]

are satisfied, then there exists a constant \( \delta > 0 \) such that

\[
\left| \sqrt{\lambda_n} - \sqrt{\lambda_m^+} \right| \geq \delta, \quad \text{for all } n, m \in \mathbb{N}^*.
\]
Proof. Let us prove the case when \( a < 0 \). Using (8) and (12) together with (15) and (16),

\[
\left( \frac{n\pi}{\gamma_1} \right)^2 - \omega_m \left( \frac{1 - \xi}{\gamma_2} \right)^2 - \left( \frac{1 - \xi}{\gamma_2} \right)^2 Q_2^+ - \left( \frac{\xi}{\gamma_1} \right)^2 Q_1^- \leq \lambda_m - \lambda_m'
\]

(24)

Taking into account that \( p \leq \alpha \), using (18), (20) and (28), one gets for all \( \alpha \in \left[ \frac{(m - \frac{1}{2})\pi}{1 - \xi}, \sqrt{\omega_m} \right] \) such that

\[
\frac{|a|}{\sqrt{\omega_m}} = \frac{(1 - \xi)}{\sin^{2}(\alpha(1 - \xi))} \left( \sqrt{\omega_m} - \frac{(m - \frac{1}{2})\pi}{1 - \xi} \right).
\]

(26)

Hence, by setting \( d = \sqrt{\omega_m} - \frac{(m - \frac{1}{2})\pi}{1 - \xi} \) for \( m \geq 1 \), we obtain

\[
d \leq \frac{|a|}{(1 - \xi)\sqrt{\omega_m}} \leq \frac{|a|}{\sqrt{\omega_m}}. \]

By (18),

\[
\sqrt{\omega_1} > \frac{\pi}{2(1 - \xi)}, \quad (27)
\]

whence

\[
d < \frac{2|a|}{\pi}. \quad (28)
\]

In view of (18), (20) and (28), one gets for all \( n, m \geq 1 \),

\[
\left| \frac{n\pi}{\gamma_1} - \frac{1 - \xi}{\gamma_2} \sqrt{\omega_m} \right| \geq \left| \frac{n\pi}{\gamma_1} - \frac{(m - \frac{1}{2})\pi}{\gamma_2} \right| - \left| \frac{1 - \xi}{\gamma_2} d \right|
\]

\[
> \frac{\pi}{2q\gamma_1} - 2|q|p + |q| > \frac{2|a|}{\sqrt{\omega_m}}. \quad (29)
\]

Taking into account that \( p \) is odd, then \( 2(nq - mp) + p \geq 1 \) for all \( n, m \geq 1 \). From this and (21), one obtains

\[
\frac{|n\pi}{\gamma_1} - \frac{1 - \xi}{\gamma_2} \sqrt{\omega_m} > \frac{\pi^2 - 4|a|p(1 - \xi)}{2q\pi\gamma_1} = \frac{\pi^2 + 4ap(1 - \xi)}{2q\gamma_1} > 0.
\]

Similarly, using (27), we have for all \( n, m \geq 1 \),

\[
\left| \frac{n\pi}{\gamma_1} + \frac{1 - \xi}{\gamma_2} \sqrt{\omega_m} \right| \geq \left| \frac{n\pi}{\gamma_1} + \frac{1 - \xi}{\gamma_2} \sqrt{\omega_1} \right| > \frac{\pi}{\gamma_1} + \frac{\pi}{\gamma_2} > \frac{\pi(p + 2q)}{2q\gamma_1}.
\]

Then, for all \( n, m \geq 1 \),

\[
\left| \left( \frac{n\pi}{\gamma_1} \right)^2 - \omega_m \left( \frac{1 - \xi}{\gamma_2} \right)^2 \right| > \frac{(\pi^2 + 4ap(1 - \xi))(2q + p)}{(2q\gamma_1)^2} > 0. \quad (30)
\]

In light of (30), suppose that

\[
\left( \frac{n\pi}{\gamma_1} \right)^2 - \omega_m \left( \frac{1 - \xi}{\gamma_2} \right)^2 < -\frac{(\pi^2 + 4ap(1 - \xi))(2q + p)}{(2q\gamma_1)^2} < 0. \quad (31)
\]

From (22), (25) and (31), one gets

\[
\lambda_n - \lambda_m' < -\left( \frac{(\pi^2 + 4ap(1 - \xi))(2q + p)}{(2q\gamma_1)^2} - |Q^+ - Q^-| \right) < 0. \quad (32)
\]
Now, suppose that
\[ \left( \frac{n\pi}{\gamma_1} \right)^2 - \omega_m \left( \frac{1}{\gamma_2} - \frac{1}{\gamma_1} \right)^2 > \frac{(\pi^2 + 4ap(1 - \xi))(2q + p)}{(2q\gamma_1)^2} > 0. \] (33)
Similarly, using (22), (24) and (33), one has
\[ \lambda_n - \lambda_m' > \frac{(\pi^2 + 4ap(1 - \xi))(2q + p)}{(2q\gamma_1)^2} - |Q^+ - Q^-| > 0. \] (34)
Therefore, from (32) and (34), there exists a constant \( \delta' > 0 \) such that we have for all \( n, m \in \mathbb{Z}^* \),
\[ |\lambda_n - \lambda_m'| > \delta' > 0. \]
Then \( \lambda_n \neq \lambda_m' \), for all \( n, m \in \mathbb{Z}^* \). On the other hand, it is known [10], [11, Chapter 1], that the eigenvalues \( \lambda_n \) and \( \lambda_m' \) satisfy the asymptotic estimate
\[
\begin{aligned}
\sqrt{\lambda_n} &= \frac{n\pi}{\gamma_1} + O \left( \frac{1}{n} \right), \\
\sqrt{\lambda_m'} &= \frac{(n-\frac{1}{2})\pi}{\gamma_2} + O \left( \frac{1}{n} \right).
\end{aligned}
\] (35)
Similarly to (29), one obtains for large enough \( n, m \)
\[ |\sqrt{\lambda_n} - \sqrt{\lambda_m'}| > \frac{\pi}{2q\gamma_1}. \]
Hence, (23) holds. The other case \( a = 0 \) can be proved in the same way. The theorem is proved.

We establish now the asymptotic behavior of the eigenfunctions of the eigenvalue problems (4) and (5), (for more details see [10]).

**Proposition 2.2.** The associated normalized eigenfunctions \( \phi_n(x) \) and \( \psi_n(x) \) of the eigenvalue problems (4) and (5), respectively, satisfy the following asymptotic estimates:
\[
\begin{aligned}
\phi_n(x) &= (\rho_1(x)\sigma_1(x))^{-\frac{1}{4}} \sin \left( \sqrt{\lambda_n} \int_0^x \sqrt{\frac{\rho_1(t)}{\sigma_1(t)}} dt \right) + O \left( \frac{1}{n} \right), \\
\phi'_n(x) &= \sqrt{\lambda_n}(\rho_1(x))^{-\frac{1}{4}} (\sigma_1(x))^{-\frac{1}{4}} \cos \left( \sqrt{\lambda_n} \int_0^x \sqrt{\frac{\rho_1(t)}{\sigma_1(t)}} dt \right) + O(1)
\end{aligned}
\] (36)
and
\[
\begin{aligned}
\psi_n(x) &= (\rho_2(x)\sigma_2(x))^{-\frac{1}{4}} \cos \left( \sqrt{\lambda_n} \int_0^x \sqrt{\frac{\rho_2(t)}{\sigma_2(t)}} dt \right) + O \left( \frac{1}{n} \right), \\
\psi'_n(x) &= \sqrt{\lambda_n}(\rho_2(x))^{-\frac{1}{4}} (\sigma_2(x))^{-\frac{1}{4}} \sin \left( \sqrt{\lambda_n} \int_0^x \sqrt{\frac{\rho_2(t)}{\sigma_2(t)}} dt \right) + O(1).
\end{aligned}
\] (37)

3. **Simultaneous exact controllability.** In this section we state our main result concerning the simultaneous exact controllability.

**Theorem 3.1.** Let \( \gamma_1 \) and \( \gamma_2 \) be defined by (9) and (14), respectively. Let \( \rho_1, \sigma_1 \in H^2[0, \xi], \rho_2, \sigma_2 \in H^2[\xi, 1], q_1 \in L^1[0, \xi], q_2 \in L^1[\xi, 1], \rho_i > 0, \sigma_i > 0 \) and \( q_i \geq 0, (i = 1, 2) \). Suppose that \( T > 2(\gamma_1 + \gamma_2) \), then the following holds:
If the conditions (20), (21) and (22) are satisfied, then Systems (1) and (2) are simultaneously controllable in \( L^2(0, \xi) \times H^{-1}(0, \xi) \times L^2(\xi, 1) \times V' \), where \( V' \) denotes the dual space of \( V \) which is defined by (3).

Applying HUM [12], the proof of this theorem is partially based on an observability inequality.
Lemma 3.2. For $T > 2(\gamma_1 + \gamma_2)$, we have

$$C_1 \left( \|(u^0, u^1)\|^2_{H^1_0(0, \xi) \times L^2(0, \xi)} + \|(v^0, v^1)\|^2_{V \times L^2(\xi, 1)} \right) \leq \int_0^T |v_x(\xi, t) - u_x(\xi, t)|^2 dt \leq C_2 \left( \|(u^0, u^1)\|^2_{H^1_0(0, \xi) \times L^2(0, \xi)} + \|(v^0, v^1)\|^2_{V \times L^2(\xi, 1)} \right).$$

Proof. Set for all $n \geq 1$,

$$\sqrt{\lambda_n} = -\sqrt{\lambda_n} \text{ and } \phi_n = \phi_n.$$

It is known that the solutions of Problems (1) and (2) with $u(\xi, t) = v(\xi, t) = 0$ are given by

$$u(x, t) = \sum_{n\in\mathbb{Z}^*} a_n \phi_n(x) e^{i \sqrt{\lambda_n} t}, \quad x \in (0, \xi), \quad (38)$$

$$v(x, t) = \sum_{n\in\mathbb{Z}^*} b_n \psi_n(x) e^{i \sqrt{\lambda_n} t}, \quad x \in (\xi, 1). \quad (39)$$

Let $(\sqrt{\lambda_n})$ denotes the increasing sequence formed by the elements of the set $\Gamma = \{\sqrt{\lambda_n} \}_{n \in \mathbb{Z}^*} \cup \{\sqrt{\lambda_n} \}_{n \in \mathbb{Z}^*}$, where $\lambda_n$ and $\lambda'_n$ are the eigenvalues of Problems (4) and (5), respectively. Let $n^+_r(n)$ the counting function of the sequence $(\sqrt{\lambda_n})$. It is the number of elements of the sequence contained on the interval $(0, r]$. Using (35), it is clear that $D^+ = \lim_{r \to +\infty} \frac{n^+_r(\xi)}{r} = \frac{\pi + \gamma_2}{\pi}$. In view of (23), (38) and (39), as an immediate consequence of Beurling’s theorem [8], the following holds: for $T > 2\pi D^+ = 2(\gamma_1 + \gamma_2)$, there exist two positive constants $C'$, $C'' > 0$, such that

$$C' \sum_{n\in\mathbb{Z}^*} (|a_n \phi_n'(\xi)|^2 + |b_n \psi_n'(\xi)|^2) \leq \int_0^T |v_x(\xi, t) - u_x(\xi, t)|^2 dt \leq \int_0^T \left( \sum_{n\in\mathbb{Z}^*} (a_n \phi_n'(\xi) e^{i \sqrt{\lambda_n} t} - b_n \psi_n'(\xi) e^{i \sqrt{\lambda_n} t}) \right)^2 dt \leq C'' \sum_{n\in\mathbb{Z}^*} (|a_n \phi_n(\xi)|^2 + |b_n \psi_n(\xi)|^2). \quad (40)$$

Since for all $n \in \mathbb{Z}^*$, $\phi'_n(\xi) \neq 0$ and $\psi'_n(\xi) \neq 0$, then by (36) and (37), we obtain the following estimates

$$c_1 \sqrt{\lambda_n} \leq |\phi_n(\xi)| \leq c_2 \sqrt{\lambda_n}$$

and

$$c_3 \sqrt{\lambda'_n} \leq |\psi_n(\xi)| \leq c_4 \sqrt{\lambda'_n},$$

for all $n \in \mathbb{Z}^*$ and $c_i > 0$, $i = 1, 2, 3, 4$. Substituting these estimates into (40), one gets

$$C_1 \left( \|(u^0, u^1)\|^2_{H^1_0(0, \xi) \times L^2(0, \xi)} + \|(v^0, v^1)\|^2_{V \times L^2(\xi, 1)} \right) \leq C' \sum_{n\in\mathbb{Z}^*} (\lambda_n |a_n|^2 + \lambda'_n |b_n|^2)$$

$$\leq \int_0^T |v_x(\xi, t) - u_x(\xi, t)|^2 dt \leq C'' \sum_{n\in\mathbb{Z}^*} (\lambda_n |a_n|^2 + \lambda'_n |b_n|^2)$$

$$\leq C_2 \left( \|(u^0, u^1)\|^2_{H^1_0(0, \xi) \times L^2(0, \xi)} + \|(v^0, v^1)\|^2_{V \times L^2(\xi, 1)} \right).$$

Hence the lemma is proved. \qed
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