Topological Entropy of Quantum Hall States in Rotating Bose Gases

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Through exact numerical diagonalization, the von Neumann entropy is calculated for the Laughlin and Pfaffian quantum Hall states in rotating interacting Bose gases at zero temperature in the lowest Landau level limit. The particles comprising the states are indistinguishable, so the required spatial bipartitioning is effected by tracing over a subset of single-particle orbitals. The topological entropy is then extracted through a finite-size scaling analysis. The results for the Laughlin and the Pfaffian states agree with the expected values of \( \ln \sqrt{2} \) and \( \ln \sqrt{4} \), respectively.

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Quantum Hall states are characterized by topological order, in that they can be described by a set of quantum numbers that are robust against local perturbations. Examples of such topological invariants include the ground-state degeneracy [1], the Chern number [2], and the braiding statistics of the quasiparticle excitations [3]. Interest in the topological underpinnings of the quantum Hall effect has surged recently due to the possibility of using these states for performing intrinsically fault-tolerant quantum computation [3–5]. The potential usefulness of certain fractional quantum Hall (FQH) states for quantum computation stems from the expectation that their quasiparticle excitations possess non-Abelian fractional statistics, which can be braided in order to perform topologically-protected logical operations. Of the few FQH states that are thought to possess non-Abelian excitations (though these alone would be insufficient to perform universal quantum computation), the most experimentally accessible is the so-called Pfaffian state that occurs at a filling factor of \( \nu = 5/2 \) in electronic FQH systems. For small particle numbers, calculation of the overlap between the Pfaffian wavefunction and the exact ground state is very good [6–9], though some doubt has been recently cast over the validity of the Pfaffian wavefunction description [10, 11].

FQH states have also been predicted to occur in rotating neutral Bose gases confined in harmonic traps due to the formal equivalence between the Hamiltonians describing these systems and two-dimensional (2D) electron gases in transverse magnetic fields [12–15]. Notably, the bosonic Laughlin state occurs at a filling factor \( \nu_B = 1/2 \) while for the bosonic Pfaffian the filling factor is \( \nu_B = 1 \). Here the filling factor is defined as the ratio between number of particles and vortices. Although such states have not yet been experimentally observed due to the difficulty in achieving the required high rotation rates [16], bosonic FQH states have a distinct advantage for topological quantum computing in that quasiparticle excitations could potentially be simple to excite and control [17]. Previous exact diagonalization calculations yield a strong overlap between the bosonic Pfaffian wavefunction and the exact ground state, though this decreases with increasing particle number [18, 19]. It is therefore important to calculate global properties such as topological quantum numbers in order to provide further evidence for the Pfaffian description of the ground state.

In this work, we focus on one such quantity called the topological entanglement entropy \( \gamma \) [20–22]. Statistical mechanics defines the classical entropy as proportional to the logarithm of a state’s multiplicity. In a similar manner, \( \gamma \) is defined as \( \gamma = \ln D \), where \( D \geq 1 \) is the total quantum dimension of the topological phase [23]. A general quantum Hall state in the Laughlin sequence with \( \nu = 1/p \) has \( D = \sqrt{p} \), while the bosonic Pfaffian state has \( D = \sqrt{4} \). (Of course, the \( \nu = 1/4 \) Laughlin state is unambiguously distinguished from the Pfaffian state by the filling factor). The topological entropy thus provides a powerful tool for classifying different quantum Hall states, as long as it can readily be calculated.

Fortunately, a connection between the von Neumann entropy \( S \) and topological order has recently been demonstrated, from which \( \gamma \) can be extracted in principle. Suppose a topologically ordered state is separated into two partitions \( A \) and \( B \) by a circle of radius \( R \). The von Neumann entropy is defined as \( S = S_A = -\text{Tr} (\rho_A \ln \rho_A) \), where \( \rho_A \) is the reduced density matrix obtained after tracing over region \( B \). In large systems, under conditions in which \( B \gg A \) or vice versa, the von Neumann entropy will scale with the length \( 2\pi R \) of the boundary delimiting both sections as

\[
S = \alpha (2\pi R) - \gamma + O(1/R),
\]

where \( \alpha \) is a nonuniversal coefficient. Cutting the system into multiple subsections and judiciously combining the resulting von Neumann entropies, the terms proportional to the boundary length can be cancelled, leaving only the universal topological entropy \( \gamma \) [20, 21]. In the present work, we use exact calculations for small number of particles and a finite-size scaling analysis to obtain \( \gamma \) for both the bosonic Laughlin and Pfaffian states.

We consider a zero-temperature gas of bosons confined in a cylindrically symmetric harmonic trap that is rapidly rotated around the \( z \)-axis with a frequency \( \Omega \). The axial trapping frequency \( \omega_z \) is assumed to be much larger than that along the radial direction \( \omega \), so that the gas can by considered quasi-2D. This corresponds to the disk
geometry that has been used in previous studies of electronic FQH systems [24]. Particles interact via a standard delta-function pseudopotential whose strength is controlled by a 2D coupling constant \( \tilde{g} = \sqrt{8\pi \hbar \omega \ell^2 a / \ell_z} \). Variables \( \ell = \sqrt{\hbar / M \omega} \) and \( \ell_z = \sqrt{\hbar / M \omega_z} \) are the characteristic oscillator lengths along the radial and axial directions, respectively, and \( a \) is the three-dimensional scattering length. Expressing all lengths in units of \( \ell \), energies in terms of \( \hbar \omega \), and frequencies in terms of \( \omega \), the effective 2D Hamiltonian in the frame co-rotating with the atoms at frequency \( \Omega \) is

\[
\hat{H} = \sum_{i} \left( -\frac{1}{2} \nabla_i^2 + \frac{1}{2} \rho_i^2 \right) - \Omega L + g \sum_{i,j} \delta(\mathbf{r}_i - \mathbf{r}_j),
\]

where \( g = \tilde{g} / \ell^2 \) is the dimensionless coupling constant, \( L \) is the number of bosons of mass \( M \), \( \hbar \omega L = \sum_i^N m_i \hbar \Omega \) is the \( z \)-projection of the total angular momentum (which is a conserved quantity in this axisymmetric potential), and \( \mathbf{r} = (\rho, \phi) \) is a particle’s position in polar coordinates. For the remainder of this work, we set \( g = 1 \).

Obtaining the topological entanglement entropy consists of three main steps: evaluation of the Hamiltonian, its diagonalization, and the calculation of the ground state’s von Neumann entropy. The Hamiltonian is expressed in a lowest Landau level (LLL) approximation [25] Fock basis of the form

\[
|N_0, N_1, \ldots, N_L\rangle = \prod_{m=0}^{L} \frac{(\hat{b}^+_m)^{N_m}}{\sqrt{N_m!}} |0\rangle,
\]

where \( \hat{b}^+_m \) creates a boson with \( m \) units of angular momentum, and the \( N \)'s are the occupation numbers. The Hilbert space size is determined by the number of unique ways to distribute \( L \) units of angular momentum among \( N \) particles. This number grows very rapidly which limits our study to \( N \leq 10 \) for the Laughlin state and \( N \leq 13 \) for the Pfaffian state. The Hamiltonian is evaluated in this basis by writing it in a second quantized form using the bosonic field operators \( \hat{\psi}(\mathbf{r}) = \sum_m \hat{b}_m \Phi_m(\mathbf{r}) \) expanded in terms of 2D harmonic oscillator orbitals in the lowest Landau level approximation (i.e. zero principal quantum number), \( \Phi_m(\mathbf{r}) = \sqrt{\frac{1}{m!}} \rho^m e^{-\rho^2 / 2} e^{im\phi} \).

Using these field operators, the Hamiltonian reduces to

\[
\hat{H} = \sum_i \hat{b}^+_i \hat{b}_i \epsilon_i + \frac{g}{2} \sum_{ijkl} \hat{b}^+_i \hat{b}^+_j \hat{b}_k \hat{b}_l D_{ijkl}
\]

where \( \epsilon_i = \hbar \omega [N + L(1 - \Omega)] \) and

\[
D_{ijkl} = \frac{\pi}{2L+1} \frac{(i+j)!}{\sqrt{i!j!k!l!}} \delta_{i+j,k+l}.
\]

In this Fock basis, the first term in \( \hat{H} \) is equivalent to the identity matrix and simply represents an energy offset. The problem is then reduced to finding the eigenstates of the interaction matrix \( \hat{H}_{int} \). These are obtained by exact diagonalization using a Lanczos algorithm.

A particular quantum Hall state is selected by specifying the total angular momentum according to the relationships between \( L \) and particle number \( N \) established in Ref. 14. The Laughlin state is selected by requiring that \( L = N(N-1) \) while for the Pfaffian state the relation is \( L = (N-1)^2 / 2 \) and \( L = (N-1)^2 / 2 \), for even and odd \( N \) respectively. As an example, consider the 2-particle Laughlin state. It occurs when \( L = 2(2-1) = 2 \) which corresponds to a Hilbert space spanned by the following two states: \( |101\rangle = b^+_1 b^+_0 |0\rangle \) and \( |020\rangle = \sqrt{2} b^+_1 b^+_0 |0\rangle \).

Once that \( L \) is fixed, the Hamiltonian can be constructed and diagonalized. This produces a ground state wavefunction described in terms of the Fock basis states. In our example, the (Laughlin) state vector obtained is

\[
|\Psi_L\rangle = \frac{1}{\sqrt{2}} (|101\rangle - |020\rangle).
\]

It is straightforward to verify that this corresponds exactly to the theoretical Laughlin wavefunction

\[
\Psi_L = \prod_{i<j} (z_i - z_j)^2 e^{-\sum_i |z_i|^2 / 2},
\]

after expanding in terms of the bosonic field operators

\[
\Psi_L(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \frac{1}{\sqrt{N!}} |\psi(\mathbf{r}_1)\psi(\mathbf{r}_2) \ldots \psi(\mathbf{r}_N)| |\Psi_L\rangle
\]

and using \( z = \rho e^{i\phi} \).

Once the ground state wavefunction has been obtained, the von Neumann entropy must be calculated. We adopt the method of Refs. [26–28] that uses orbital partitioning (see Ref. [29] for particle partitioning). This consists of separating the state into regions delineated by a specific number \( n = m_{\text{max}} + 1 \) of single particle orbitals, so that region \( A \) would correspond to \( m \leq m_{\text{max}} \) and region \( B \) to \( m > m_{\text{max}} \). For the two-particle Laughlin state example considered above, the reduced density matrix \( \rho_A \) for \( n = 1 \) (meaning only the \( m = 0 \) orbital is in partition \( A \)) is

\[
\rho_A = Tr_B (|\Psi_L\rangle \langle \Psi_L|) = \frac{1}{2} (|1\rangle \langle 1| + |0\rangle \langle 0|) = \frac{1}{2} (|1\rangle \langle 1| + |0\rangle \langle 0|)
\]

which has a two-fold degenerate eigenvalue of \( 1/2 \). The von Neumann entropy is thus \( S_A = -2 \times \frac{1}{2} \ln \left( \frac{1}{2} \right) \approx 0.69 \).

Once \( S \) is obtained for \( n = 1 \), the process is repeated numerous times for increasing \( n \), and in principle Eq. (1) can be used to obtain the topological entropy \( \gamma \) from a plot of \( S \) versus \( R \). What remains to be specified is the partition boundary size. The single-particle orbital density is ring-shaped and centered at the origin of the trap, with radius \( \langle \rho \rangle_m = \int d\mathbf{r} |\Phi_m(\rho, \phi)|^2 \sim \sqrt{m+1} \) for
large $m$. We therefore consider the boundary between the two regions $A$ and $B$ for a particular choice of $n$ to correspond to a circle of size $2\pi \langle \rho \rangle_{n-1}$. Since the factor of $2\pi$ can simply be combined with $a$ in Eq. (1), we define the boundary size when the first $n$ orbitals are kept in partition $A$ as

$$R = \langle \rho \rangle_{n-1}.$$

A plot of $S(R)$ for $N = 5$ to 10 is shown in Fig. 1 for the bosonic Laughlin state. An initial linear increase of $S(R)$ is observed, as expected from Eq. (1); however, finite-size effects bring $S$ to zero for larger values of $R$. This is because in small-$N$ systems most of the particles occupy low-$m$ orbitals whose amplitudes are largest in the vicinity of the trap center; the ground-state wavefunction has a negligible overlap with higher angular momentum orbitals.

To perform a proper finite-size scaling analysis of the small-$N$ data, we follow the procedure introduced in Ref. 26. The value of $S(N \to \infty)$ is estimated by plotting $S$ as a function of $1/N$ for region $A$ containing different numbers of orbitals $n$, as shown in Fig. 2. Our calculations were restricted to the range $1 \leq n \leq 5$ because we have too little data for $n \geq 6$. To obtain the $N \to \infty$ values a linear regression was applied. A slight positive curvature in the data for the largest values of $N$ might indicate the emergence of asymptotic values, but the trend was not clear enough to enable a more sophisticated analysis. The results are plotted in the inset of Fig. 3 and another linear regression is made to provide the $y$-intercept. We find $\gamma_L = 0.30 \pm 0.02$, which is slightly lower than the expected result of 0.35 for the bosonic Laughlin state at filling $\nu = 1/2$; this might reflect the rather naïve linear analysis of the finite-size scaling.

The same procedure was then repeated for the Pfaffian state with $N = 5$ to 13. Unfortunately, a larger amount of scatter was present in the Pfaffian version of Fig. 2, preventing a clean extrapolation of the $S(N)$ data to the large-$N$ limit. Thus, instead of plotting $S(N \to \infty)$ as a function of $R$ to obtain $\gamma_p$, a linear regression is performed directly on the collection of data points shown in
with the predicted value of \( \ln(\gamma) \). However, corrections for finite-size effects have not been made. Fitting the aggregate Laughlin results in the same manner gives \( \gamma_L = 0.57 \pm 0.02 \) (see Fig. 3) instead of the previously obtained \( \gamma_L = 0.30 \pm 0.02 \), suggesting that the obtained \( \gamma_p \)'s accuracy is coincidental. The error ranges quoted above strictly reflects the scatter in the numerical data, and underestimates the actual uncertainty by neglecting systematic errors. In particular, each data point is treated as equally valid, whereas the \( \mathcal{O}(1/R) \) term in Eq. (1) clearly favors large-\( R \) results. Ideally, only the largest-\( n \) data would have been kept; however, this would not have provided enough data with which to extract values of \( \gamma \). In light of the discrepancy between both Laughlin results, a more reasonable estimate of the Pfaffian topological entropy is \( \gamma_p = 0.7 \pm 0.3 \).

In conclusion, we have calculated the topological entanglement entropy for both the bosonic Laughlin and Pfaffian states for a rotating Bose gas. For the Laughlin state, we obtain a result of \( 0.30 \pm 0.02 \) which is almost consistent with the expected value of \( \ln(\sqrt{2}) = 0.35 \). For the Pfaffian state we obtain \( 0.7 \pm 0.3 \). This value of the topological entropy is consistent with the expected value of \( \ln(\sqrt{4}) = 0.69 \), though the large amount of scatter present in Fig. 4 prevents a completely unambiguous identification of the state by the Pfaffian wavefunction.

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[1] X. G. Wen and Q. Niu, Phys. Rev. B 41, 9377 (1990).
[2] Y. Hatsugai, J. Phys.: Condens. Matter 9, 2507 (1997).
[3] L. Hormozi, G. Zikos, N. E. Bonesteel, and S. H. Simon, Phys. Rev. B 75, 165310 (2007).
[4] S. Das Sarma, M. Freedman, C. Nayak, S. H. Simon, and A. Stern, arXiv:0707.1889v1 (2007).
[5] A. Y. Kitaev, Ann. Phys. 303, 2 (2003).
[6] H. Morf, Phys. Rev. Lett. 80, 1505 (1998).
[7] E. H. Rezayi and F. D. M. Haldane, Phys. Rev. Lett. 84, 4685 (2000).
[8] V. W. Scarola, J. K. Jain, and E. H. Rezayi, Phys. Rev. Lett. 88, 216804 (2002).
[9] A. E. Feiguin, E. Rezayi, C. Nayak, and S. Das Sarma, Phys. Rev. Lett. 100, 166803 (2008).
[10] C. Töke and J. K. Jain, Phys. Rev. Lett. 96, 246805 (2006).
[11] C. Töke, N. Regnault, and J. K. Jain, Phys. Rev. Lett. 98, 036806 (2007).
[12] N. R. Cooper and N. K. Wilkin, Phys. Rev. B 60, R16279 (1999).
[13] S. Viefers, T. H. Hansson, and S. M. Reimann, Phys. Rev. A 62, 053604 (2000).
[14] N. K. Wilkin and J. M. F. Gunn, Phys. Rev. Lett. 84, 6 (2000).
[15] N. Regnault and T. Jolicoeur, Phys. Rev. Lett. 91, 030402 (2003).
[16] V. Schweikhard, I. Coddington, P. Engels, V. P. Morgenstern, and E. A. Cornell, Phys. Rev. Lett. 92, 040404 (2004).
[17] B. Paredes, P. Fedichev, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 87, 010402 (2001).
[18] C.-C. Chang, N. Regnault, T. Jolicoeur, and J. K. Jain, Phys. Rev. A 72, 013611 (2005).
[19] N. Regnault, C. C. Chang, T. Jolicoeur, and J. K. Jain, J. Phys. B 39, S89 (2006).
[20] M. Levin and X.-G. Wen, Phys. Rev. Lett. 96, 110405 (2006).
[21] A. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006).
[22] H. Li and F. D. M. Haldane, Physical Review Letters 101, 010504 (2008).
[23] P. Fendley, M. P. A. Fisher, and C. Nayak, J. Stat. Phys. 126, 1111 (2007).
[24] X. C. Xie, S. Das Sarma, and S. He, Phys. Rev. B 47, 15942 (1993).
[25] A. G. Morris and D. L. Feder, Phys. Rev. A 74, 033605 (2006).
[26] M. Haque, O. Zozulya, and K. Schoutens, Phys. Rev. Lett. 98, 060401 (2007).
[27] O. S. Zozulya, M. Haque, K. Schoutens, and E. H. Rezayi, Phys. Rev. B 76, 125310 (2007).
[28] B. A. Friedman and L. G. C., Physical Review B (2008).
[29] S. Blüth, R. J. Latorre, and R. Orus, Phys. Rev. Lett. 98, 060402 (2007).