Markovian Statistics on Evolving Systems

Ulrich Faigle 1, Gerhard Gierz 2

1 Mathematisches Institut
Universität zu Köln
Weyertal 80, 50931 Köln, Germany
faigle@zpr.uni-koeln.de

2 Department of Mathematics
University of California at Riverside
Riverside, CA 92521, USA
gierz@math.ucr.edu

Abstract. A novel framework for the analysis of observation statistics on time
discrete linear evolutions in Banach space is presented. The model differs from
traditional models for stochastic processes and, in particular, clearly distinguishes
between the deterministic evolution of a system and the stochastic nature of ob-
servations on the evolving system. General Markov chains are defined in this
context and it is shown how typical traditional models of classical or quantum
random walks and Markov processes fit into the framework and how a theory
of quantum statistics (sensu Barndorff-Nielsen, Gill and Jupp) may be devel-
oped from it. The framework permits a general theory of joint observability of
two or more observation variables which may be viewed as an extension of the
Heisenberg uncertainty principle and, in particular, offers a novel mathematical
perspective on the violation of Bell’s inequalities in quantum models. Main re-
sults include a general sampling theorem relative to Riesz evolution operators in
the spirit of von Neumann’s mean ergodic theorem for normal operators in Hilbert
space.

Keywords: Banach space, Bell inequality, ergodic theorem, evolution, Heisenberg un-
certainty, Hilbert space, Markov chain, observable, quantum statistics, random walk,
sampling, Riesz operator, stochastic process

1 Introduction

Consider a system S that is observed at discrete times t = 1, 2, 3, . . . relative to a pre-
specified event S that may or may not occur at time t. A statistical analysis is interested
in the relative frequency of the event S. In particular, one would like to know whether
the sample frequency of its occurrence converges to a definite limiting value. For a
mathematical formulation of this problem, let us define the indicator functions I_S(t) as
(0, 1)-variables with the event notation {I_S(t) = 1} meaning that S is observed at time
t. The sampled frequency up to time t would then be

\[ I_S(t) = \frac{1}{t} \sum_{m=1}^{t} I_S^{(m)} \]
We call the study of the limiting behavior of the sample frequency of an event $S$ as the Markovian problem.

Mathematical approaches to this problem typically think of $S$ as a random source that emits symbols from some (finite or infinite) alphabet $A$ and thus gives rise to an $A$-valued stochastic process $(X_t)$. $S$ is assumed to represent a particular event relative to this process that may or may not materialize at time $t$. So the observation variables $I_S^{(t)}$ reflect an associated stochastic process in their own right and the expected average number of observations of $S$ at time $t$ is

$$E(I_S^{(t)}) = \frac{1}{t} \sum_{m=1}^{t} E(I_S^{(m)}) = \frac{1}{t} \sum_{m=1}^{t} I^{(m)} \Pr\{I_S^{(m)} = 1\}.$$ 

No matter what the nature of the underlying stochastic process $(X_t)$ is, the mathematical analysis of the statistical problem relative to the event $S$, i.e., the Markovian problem, will actually be on the associated binary process $(I_S^{(t)})$.

The present investigation is concerned with the Markovian problem of the relative occurrence of some event $S$ rather than with the mathematical analysis of general stochastic processes per se. Therefore, there is no loss in generality when we restrict ourselves to a model where $S$ is viewed as some source that emits symbols from some alphabet $A$ of finite cardinality $|A| < \infty$. In fact, the assumption of $A$ as a binary alphabet (i.e., $|A| = 2$) would theoretically suffice. However, it is convenient to also consider more general alphabets occasionally.

A stochastic process $(X_t)$ is usually understood to be a sequence of stochastic variables $X_t$ that are defined on some probability space and one is interested in the expected limiting behavior of $(X_t)$. However, if one takes statistics on $(X_t)$, i.e., averages the observation of special events over time, there is not always a clear asymptotic behavior. Indeed, there are examples of even completely deterministic processes $(X_t)$ where observation statistics do not converge. It is the limiting behavior of observation averages we are concerned with here.

Our approach is motivated by a Markovian interpretation of $(X_t)$: A random source $S$ produces symbols $a$ of an alphabet $A$ over time $t$ with $\{X_t = a\}$ denoting these events. $S$ is thought to change over discrete time $t$, with the probability $\Pr\{X_t = a\}$ depending on the current state of $S$. The classical example goes back to Markov’s [21] model of a system that admits a set $N$ of ground states and is subject to a random walk on $N$ with transition probabilities $p_{ij}$. The system states are then the probability distributions $p^{(t)}$ of the positions of the random walk at times $t$. The Markov model has been very successful in application modeling. Statistical mechanics in physics, for example, describes the behavior of ideal gases in this way. But also the behavior of economic and social systems is often viewed as following Markovian principles. Internet search engines successfully organize and rank their search according to Markovian statistics.

The situation seems to be more complicated with quantum systems that do not admit a classical analysis. For example, the result of a quantum measurement is not a deterministic function of the state of the system and the measuring instrument applied but
rather an expected value relative to some (state dependent) probability distribution on
the possible measurement outcomes. Moreover, Heisenberg’s uncertainty principle says
that observations may not be simultaneously feasible unless they conform to a special
condition. Experimental evidence with spin correlations (Aspect et al. [2]) furthermore
exhibits a definite violation of classical statistical principles as expressed in Bell’s [4,5]
inequalities. While the Schrödinger picture of quantum states being described by wave
functions yields a special theory of quantum probabilities with applications also to
quantum computing, active current research effort is devoted to the quantum analogs
of classical Markov random walks. In this spirit, Barndorff-Nielsen, Gill and Jupp [3]
have put forward a theory of quantum statistical inference.

The present investigation proposes a quite general model for Markovian statistical
analysis. Rather than following the standard approach to stochastic processes, our
model is linear and motivated by the linear algebraic analysis of classical Markov chains
of Gilbert [14], Dharmadhari [8] and Heller [16], which has led to the identification of
more general Markov type processes (e.g., Jaeger [19]). Addressing the issue of the
"dimension" of a stochastic evolution, the asymptotic behavior of even more general
stochastic processes could be clarified (Faigle and Schönhuth [11]). Generalizing these
previous models, our setting is in Banach space and focuses on the evolution of linear
operators, which allows us to deal also with the statistics of discrete quantum type
evolutions appropriately.

There are several advantages and novel aspects in our approach. Not only does our
model include typical Markovian models proposed so far (see the examples in Sec-
tion 4), but its generality allows us to develop a meaningful notion of jointly observable
statistical measuring instruments on an evolving system. Sets of classical stochastic
variables are always jointly observable (for the simple reason that they are mathemat-
ically based on the same underlying probability space). The Heisenberg uncertainty
principle, on the other hand, makes it clear that this property is no longer guaranteed
for statistical observations on quantum systems.

While the Heisenberg principle is formulated for pairs of self-adjoint operators, our
model allows us to deal with three (or more) operators as well. We show that the Heisen-
berg principle corresponds to a very special case in our setting (Section 5). In fact, a
careful mathematical analysis of the joint observability of 3 measurement operators may
offer a straightforward key to the understanding of Bell’s inequalities (Section 5.2).

These advantages are the result of a clear separation of the aspect of the (determin-
istic) evolution of a system from the aspect of statistical observations on the evolving
system in the mathematical model.

Our presentation is organized as follows. Section 2 introduces evolution operators
on Banach spaces and discusses their ergodicity. Then sampling functions are studied
and their convergence behavior is characterized in the Sampling Theorem (Theorem 3)
relative to finitary evolutions, which include all evolutions based on Riesz operators, for
example. Observables and generalized Markov chains are defined in Section 3. These
notions are illustrated by the examples in Section 4 with particular emphasis on ran-
dom walks and quantum statistics. The proofs of the main results are deferred into the
Appendix.
2 Evolutions of systems

Let $S$ be some system that is in a certain state $S_t$ at any time $t$. Observing $S$ at discrete times $t = 0, 1, 2, \ldots$, we refer to the sequence $\epsilon = (S_t)_{t \geq 0}$ as an evolution of $S$. For a mathematical analysis, the evolution needs to be represented in some (mathematical) universe $U$. In the present investigation, we will always assume $U$ to be a vector space over the complex field $\mathbb{C}$. A representation of the evolution $\epsilon$ in $U$ is then a map $t \mapsto s^{(t)} \in U$ such that there is a linear operator $\psi$ on $U$ with the property

$$s^{(t+1)} = \psi s^{(t)} \quad (t = 0, 1, 2, \ldots).$$

We think of the vector $s^{(t)} \in U$ as the representation of the state $S_t$ of $S$ at time $t$ and call $\psi$ an evolution operator. Clearly, any evolution $\epsilon$ of $S$ admits such a representation.

In a practical system analysis, it is the first task of the modeler consists in the determination of an appropriate representation of the evolution of the system $S$ under consideration. Here, however, we will assume that the evolution is already represented in some universe $U$ so that the evolutions are vector sequences $\Psi$ of the form

$$\Psi = (\psi, s) = (\psi^t s \mid t = 0, 1, 2, \ldots)$$

where $\psi$ is an operator on $U$. We furthermore assume that $U$ is endowed with some norm $\| \cdot \|$ and is complete with respect to this norm (otherwise we replace $U$ by its completion $\overline{U}$).

**Remark 1** By standard complexification arguments in functional analysis (e.g., [7]), the results we obtain in this section apply to universes over the real field $\mathbb{R}$ as well. We choose $\mathbb{C}$ for mathematical convenience, without loss of generality.

The evolution space of the evolution $\Psi = (\psi, s)$ in $U$ is the linear subspace $U_\Psi$ generated by $\Psi$, i.e.,

$$U_\Psi = \text{lin}\{\psi^t s \mid t = 0, 1, \ldots\}.$$  

The parameter $\dim \Psi = \dim U_\Psi$ is the dimension of the evolution $\Psi$. We will refer to the vectors $s^{(t)} = \psi^t s$ as the states of $\Psi$.

Notice that $U_\Psi$ is $\psi$-invariant (i.e., $\psi(U_\Psi) \subseteq U_\Psi$). So the restriction of $\psi$ to $U_\Psi$ is an operator on the normed space $U_\Psi$. Let $\overline{U_\Psi}$ be the closure of $U_\Psi$ in $U$ and recall from general operator theory\footnote{e.g., [7,9]} that $\psi$ extends to a unique norm bounded (and hence continuous) operator $\overline{\psi} : \overline{U_\Psi} \to \overline{U_\Psi}$ with the same (finite) norm, provided $\psi$ is norm bounded on $U_\Psi$. The norm of $\psi$ on $U_\Psi$ is

$$\|\psi\|_s = \inf\{c \in \mathbb{R} \mid \|\psi u\| \leq c \|u\| \text{ for all } u \in U_\Psi\}.$$
In the case of a finite-dimensional evolution (i.e., $\dim \Psi < \infty$), for example, $\|\psi\|_s$ is necessarily finite. The norm of the evolution $\Psi = (\psi, s)$ is defined as

$$\|\Psi\| = \inf \{ c \in \mathbb{R} \mid \|\psi^t s\| \leq c \|s\|, \forall t \geq 0 \}$$

and $\Psi$ said to be stable if $\|\Psi\| < \infty$. Thus $\Psi$ is stable if $\|\psi\|_s \leq 1$, for example.

**Lemma 1.** If $\Psi = (\psi, s)$ is stable, then $\Psi' = (\psi, s')$ is stable for every $s' \in \mathcal{U}_\Psi$. Hence the restriction of $\psi$ to $\mathcal{U}_\Psi$ does not admit any eigenvalue $\lambda$ with $|\lambda| > 1$.

**Proof.** Consider any $s' = \sum_{j=1}^k a_j \psi^j s \in \mathcal{U}_\Psi$. Then the triangle inequality yields

$$\|\psi^t s'\| \leq (\|\Psi\| \sum_{j=1}^k |a_j|) \|s\| \text{ for all } t \geq 0.$$ 

The evolution $\Psi = (\psi, s)$ is ergodic if its states $s^{(t)} = \psi^t s$ converge in the norm. $\Psi$ is mean ergodic if the state averages

$$\bar{s}^{(t)} = \frac{1}{t} \sum_{m=1}^t s^{(m)}$$

converge to some limit state $\bar{s}^{(\infty)} \in \overline{\mathcal{U}_\Psi}$. Clearly, if $\bar{s}^{(\infty)}$ exists (if and $\|\psi\|_s < \infty$ holds), $\Psi$ is stationary in the sense

$$\frac{\psi}{\bar{s}}^{(\infty)} = \bar{s}^{(\infty)}.$$  

Moreover, an ergodic evolution is also mean ergodic, while the converse conclusion is generally false.

### 2.1 Equivalent evolutions

Let us call two evolutions $\Phi = (\varphi, v)$ and $\Psi = (\psi, w)$ in $\mathcal{U}$ equivalent if

$$\lim_{t \to \infty} \|\psi^t w - \varphi^t v\| = 0.$$  

By Cauchy’s Theorem, equivalence implies

$$\lim_{t \to \infty} \frac{1}{t} \sum_{m=1}^t \|\psi^m w - \varphi^m v\| = 0.$$  

So, assuming equivalence, $\Phi$ is mean ergodic exactly when $\Psi$ is mean ergodic and in either case, one has

$$\lim_{t \to \infty} \frac{1}{t} \sum_{m=1}^t \psi^m w = \lim_{t \to \infty} \frac{1}{t} \sum_{m=1}^t \varphi^m v.$$  

(1)
Finitary evolutions  We say that the evolution $\Psi = (\psi, s)$ is finitary if $\Psi$ is equivalent to a finite-dimensional evolution $\Phi = (\varphi, v)$. A characterization of the mean ergodicity of a finite-dimensional evolution $\Phi$ follows from the analysis of Faigle and Schönhuth [11] and says in essence: $\Phi$ is mean ergodic precisely when $\Phi$ is equivalent to an evolution $\Pi = (\pi, v)$, where $\pi$ is a projection operator on $\overline{\mathcal{U}}_{\Phi} = \mathcal{U}_{\Phi}$.

**Proposition 1 ([11]).** Let $\Phi = (\varphi, v)$ be a finite-dimensional evolution. Then $\Phi$ is mean ergodic if and only if $\Phi$ is stable. Moreover, if $\Phi$ is stable, one has

$$
\lim_{t \to \infty} \frac{1}{t} \sum_{m=1}^{t} \varphi^m v = \begin{cases} 
0 & \text{if } \lambda = 1 \text{ is not an eigenvalue of } \varphi \\
\Pi v & \text{if } \lambda = 1 \text{ is an eigenvalue of } \varphi,
\end{cases}
$$

where $\Pi$ is a projection operator onto the eigenspace $E_1 = \{ x \in \mathcal{U}_{\Phi} \mid x = \varphi x \}$.

**Proof.** By Lemma [11] $\Phi$ is stable if and only if $(\phi, v')$ is stable for all $v' \in \mathcal{U}_{\Phi}$. So Proposition [11] is a direct consequence of Theorem 2 and its proof in [11].

Riesz evolutions  Recall that the spectrum $\sigma(T)$ of a (linear) operator $T : \mathcal{V} \to \mathcal{V}$ on a complex normed vector space $\mathcal{V}$ consists of those $\lambda \in \mathbb{C}$ such that the operator $L_\lambda = T - \lambda$ (with values $L_\lambda v = Tv - \lambda v$) is not invertible. $T$ is called a Riesz operator if $T$ is bounded and

(a) each $\lambda \in \sigma(T) \setminus \{0\}$ is an eigenvalue of $T$ with finite algebraic multiplicity;

(b) 0 is the only possible accumulation point of $\sigma(T)$.

Riesz operators form a quite wide class of operators that includes the so-called compact operators. In particular, every operator $T$ with finite-dimensional range is Riesz. Further examples are the Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$, namely the bounded operators $T : \mathcal{H} \to \mathcal{H}$ such that

$$
\sum_{i \in I} \| Te_i \|^2 < \infty
$$

holds for some orthonormal basis $\{ e_i \mid i \in I \}$ of $\mathcal{H}$.

A Riesz evolution in our universe $\mathcal{U}$ is now an evolution $\Psi = (\psi, s)$ such that $\psi$ extends to a Riesz operator $\overline{\psi} : \overline{\mathcal{U}}_{\Phi} \to \overline{\mathcal{U}}_{\Phi}$. In particular, every evolution under a Riesz evolution operator on $\mathcal{U}$ is Riesz.

The characterization of mean ergodic finite-dimensional evolutions (Proposition [11]) extends to general Riesz evolutions.

**Theorem 1.** Let $\Psi = (\psi, s)$ be any Riesz evolution in $\mathcal{U}$. Then $\Psi$ is finitary. Moreover, $\Psi$ is mean ergodic if and only if $\Psi$ is stable. In particular, if $\Psi$ is stable, one has

$$
\lim_{t \to \infty} \frac{1}{t} \sum_{m=1}^{t} \psi^m v = \begin{cases} 
0 & \text{if } \lambda = 1 \text{ is not an eigenvalue of } \overline{\psi} \\
\Pi v & \text{if } \lambda = 1 \text{ is an eigenvalue of } \overline{\psi},
\end{cases}
$$

where $\Pi$ is a projection operator onto the eigenspace $E_1 = \{ x \in \mathcal{U}_{\Phi} \mid x = \overline{\psi} x \}$.
The essential part of the proof of Theorem 1 consists in showing that Riesz evolutions are finitary. We discuss the details in the Appendix (cf. Proposition 2 there).

**Normal evolutions in Hilbert space** Let \( H \) be a Hilbert space and recall that an operator \( T \) on \( H \) is normal if \( T \) commutes with its adjoint \( T^* \) (i.e., \( TT^* = T^*T \)).

**Theorem 2.** Let \( \psi \) be a bounded normal operator on \( H \) and \( \Psi = (\psi, s) \) an evolution. Then the following statements are equivalent:

(i) \( \Psi \) is stable.

(ii) \( \Psi \) is mean ergodic.

In this case, the averages \( \frac{1}{t} \sum_{m=1}^{t} f(\psi^m s) \) converge to the orthogonal projection of \( s \) onto the eigenspace \( E_1 = \{ x \in H \mid \psi x = x \} \).

We prove Theorem 2 in the Appendix. The implication "(i) \( \Rightarrow \) (ii)" is well-known and usually stated as von Neumann’s mean ergodic theorem:

**Corollary 1 (von Neumann).** If \( \psi \) is a normal operator on \( H \) of norm \( \| \psi \| \leq 1 \), then every evolution \( \Psi = (\psi, s) \) is mean ergodic.

An important special case is the evolution of a wave function \( v \in H \) of a quantum system in discrete time. According to Schrödinger’s differential equation, there is a unitary operator \( U \) (i.e., \( UU^* = I = U^*U \)) so that the discrete evolution of \( v \) is given as

\[
\psi(t) = U^t v \quad (t = 0, 1, \ldots).
\]

Clearly, the operator \( v \mapsto Uv \) is normal and bounded. Moreover, \((U, v)\) is stable for any \( v \in H \). So Schrödinger evolutions are mean ergodic.

### 2.2 Sampling

By a sampling function relative to the universe \( \mathcal{U} \) we understand a continuous linear map \( f : \mathcal{U} \to \mathcal{F} \), where \( \mathcal{F} \) is a normed vector space of samples. With respect to an evolution \( \psi = (\psi, s) \), the \( f_t = f(\psi^t s) \) are the sampling values with the corresponding sampling averages

\[
\mathcal{F}_t = \frac{1}{t} \sum_{m=1}^{t} f_m = \frac{1}{t} \sum_{m=1}^{t} f(\psi^m s) \quad (t = 1, 2, \ldots).
\]

In applications, a sampling function \( f \) will typically be a functional into the scalar field of \( \mathcal{U} \). But more general sample spaces \( \mathcal{F} \) may also be of interest.

The sampling averages will, of course, converge when \( \Psi \) is mean ergodic. Unitary evolutions in Hilbert space, for example, will guarantee converging sampling averages. But sampling averages may possibly also converge on evolutions that are not mean ergodic. The sampling convergence on finitary evolutions is characterized as follows.
Theorem 3 (Sampling Theorem). Let \( \Psi = (\psi, s) \) be an arbitrary finitary evolution and \( f : \mathcal{U} \rightarrow \mathcal{F} \) a sampling function. Then the following statements are equivalent:

(i) The sampling averages \( \overline{f_t} \) converge.

(ii) The sampling values \( f_t \) are bounded in norm.

Again, we defer the proof of Theorem 3 to the Appendix. Choosing \( \mathcal{F} = \mathcal{U} \) and \( f = I \) as the identity operator, we immediately note:

Corollary 2. Let \( \Psi = (\psi, s) \) be an arbitrary finitary evolution in \( \mathcal{U} \). Then

\( \Psi \) is mean ergodic \( \iff \Psi \) is stable.

\[ \square \]

3 Observables and Markov chains

Let \( A \) be a finite or countable set. An observable with range \( A \) on the evolution \( \Psi = (\psi, s) \) in \( \mathcal{U} \) is a collection \( X = \{ \chi_a \mid a \in A \} \) of continuous linear functionals \( \chi_a \) such that the \( \chi_a(s(t)) \) are real numbers with the property

\[ p_a(t) = \chi_a(s(t)) \geq 0 \quad \forall a \in A \quad \text{and} \quad \sum_{a \in A} p_a(t) = 1. \]

We think of \( X \) as producing the event \( \{ X_t = a \} \) at time \( t \) with probability

\[ \Pr\{ X_t = a \} = p_a(t). \]

So the observable \( X \) yields a sequence \( (X_t) \) of stochastic variables \( X_t \) with probability distributions \( p(t) \). We call \( (X_t) \) a (generalized) Markov chain on \( A \).

Remark 2 The probability distributions \( p_a(t) \) of \( X \) may be viewed as "stochastic kernel" of \( X \) and thus generalize the idea of a kernel of classical Markov chain theory (see, e.g. [13, 17]). A Markov chain in our sense, however, does not need to be a stochastic process nor does its stochastic kernel need to reflect any conditional probabilities with respect to state transitions.

Related to the (generalized) Markov chain \( (X_t) \) is are (statistical) sampling processes \( (Y_t^a) \) with respect to any \( a \in A \), where

\[ Y_t^a = \begin{cases} 1 & \text{if} \quad X_t = a \\ 0 & \text{otherwise.} \end{cases} \]

\( (Y_t^a) \) is a Markov chain on \( \Psi \) in its own right with binary alphabet \( \{0, 1\} \) and probability distributions

\[ \Pr\{ Y_t^a = 1 \} = p_a(t) \quad \text{and} \quad \Pr\{ Y_t^a = 0 \} = 1 - p_a(t). \]
Assuming that the evolution $Ψ$ is finitary, for example, Theorem 3 implies that the expectation of the observation averages

$$E\left(\frac{1}{t} \sum_{m=1}^{t} Y_m^a\right) = \frac{1}{t} \sum_{m=1}^{t} E(Y_m^a) = \frac{1}{t} \sum_{m=1}^{t} \Pr\{Y_m^a = 1\}$$

converges to a definite limit $\bar{p}_a^{(∞)} \geq 0$ since the numbers $\Pr\{Y_m^a = 1\}$ are bounded. In view of

$$\sum_{a \in A} p_a^{(m)} = 1 \quad \text{for all } m \geq 1,$$

we conclude that $\bar{p}_a^{(∞)} = \{\bar{p}_a^{(∞)} | a \in A\}$ is a probability distribution on $A$. It is in that sense that we refer to $\bar{p}_a^{(∞)}$ as the limit distribution of the Markov chain $(X_t)$.

4 Examples

4.1 Evolutions of stochastic processes

Let $(X_t)$ be a discrete stochastic process that takes values in the alphabet $A$. Without loss of generality, we assume that $A$ is binary, say $A = \{0, 1\}$. As usual, we denote the set of all finite length words over $A$ as

$$A^* = \bigcup_{n=0}^{∞} A^n,$$

where $A^0 = \{\Box\}$ and $\Box$ is the empty word. For any $v = v_1 v_2 \ldots v_n \in A^n$, $|v| = n$ is the length of $v$. Recall that $A^*$ is a semigroup with neutral element $\Box$ under the concatenation operation

$$(v_1 \ldots v_n)(w_1 \ldots w_k) = v_1 \ldots v_n w_1 \ldots w_k.$$

Moreover, we set

$$p(w_1 \ldots w_k | v_1 \ldots v_n) = \Pr\{X_{n+1} = w_1, \ldots, X_{n+k} = w_k | X_1 = v_1, \ldots, X_k = v_n\}$$

and $p(v) = p(v | \Box)$. If $v \in A^t$ has been produced, the process is in a state that is described by the prediction vector $P^v$ with the components

$$P^v_w = p(w | v) \quad (w \in A^*).$$

The expected coordinate values of the next prediction vector are then given by the components of the vector

$$\psi P^v = p(0 | v) P^v_0 + p(1 | v) P^v_1.$$

Binomial expansion, therefore, immediately shows that the expected prediction vector at time $t$ is given by

$$\sum_{v \in A^t} p(v) P^v = \psi^t \bar{P}.$$
Since the components of the prediction vectors $P^v$ are bounded, they generate a normed vector space $\mathcal{P}$ with respect to the supremum norm

$$\|g\|_\infty = \sup_{w \in A^*} |g_w|.$$  

Moreover, it is not difficult to see that $\psi$ extends to a unique linear operator on $\mathcal{P}$. The evolution $\Psi = (\psi, P^\square)$ is said to be the evolution of the stochastic process $(X_t)$.

For any $a \in A$, one has

$$(\psi_s P^\square)_a = \sum_{v \in A^s} p(v) P^v_a = \Pr\{X_{t+1} = a\}.$$  

Since coordinate projections are continuous linear functionals, we find that observations on $(X_t)$ yield observables in the sense of Section 3, which allows us to view the stochastic process $(X_t)$ as a (generalized) Markov chain on $A$.

Remark 3 The evolution of stochastic processes was first studied by Faigle and Schönhuth [11], to which be refer for further details. The stochastic evolution model generalizes earlier linear models for the analysis of Markov type stochastic processes (e.g., Gilbert [14], Dharmadhikari [8], Heller [16] and Jaeger [19]).

4.2 Finite-dimensional evolutions

Finite-dimensional evolution models are of particular interest in applications. An evolution $\Psi = (\psi, s)$ in $\mathbb{R}^n$ admits an $(n \times n)$-matrix $M$ such that $\psi_x = Mx$ holds for all $x \in \mathbb{R}^n$. As observed in [11], $\Psi$ is mean ergodic exactly when $\Psi$ is stable (cf. Corollary 2).

Letting $N = \{1, \ldots, n\}$ and assuming that $s$ and all columns of $M$ are probability distributions on $N$, $\Psi$ is clearly stable and hence mean ergodic. The Markov chain relative to $\{X_i | i \in N\}$, where the $X_i$ are the projections onto the $n$ components of $x \in \mathbb{R}^n$, yields the well-known model of a random walk on $N$ with the transition matrix $M$. Considering any map $X : N \rightarrow A$ into some alphabet $A$, induced Markov chain with the "kernel" functionals

$$\chi_a(x) = \sum_{X(i) = a} x_i \quad (x = (x_1, \ldots, x_n) \in \mathbb{R}^n)$$

is classically known as a hidden Markov chain on $A$. Hidden Markov models have proved very useful in practical applications.

It is important to note, however, that even finite-dimensional Markov chains in the general sense of Section 3 are not necessarily stochastic processes (see Example 1 in Section 5.2 below). Also quantum random walks (Section 4.4) are not necessarily stochastic processes.

\[\text{4 see, e.g., [610126]}\]
4.3 Quantum statistics

Let $\mathcal{H}$ be a complex Hilbert space of dimension $|N|$, where $N = \{1, \ldots, n\}$ is finite or $N = \mathbb{N}$, with inner product $\langle x | y \rangle$. Let $\mathcal{B} = \mathcal{B}(\mathcal{H})$ be the normed complex vector space of all continuous (linear) operators (i.e., operators $T$ with norm $||T|| < \infty$). We single out the set of normalized wave functions $\mathcal{W} = \{s \in \mathcal{H} | \|s\|^2 = \langle s | s \rangle = 1\}$.

Any $s \in \mathcal{W}$ gives rise to a (projection) operator $P_s \in \mathcal{B}$, where $P_su = \langle u | s \rangle s$ and hence $P_s^2 = P_s$.

The element $s \in \mathcal{W}$ furthermore defines a (linear) trace functional $\tau_s : \mathcal{B} \to \mathbb{C}$ via

$$\tau_s T = \langle Ts | s \rangle$$

for all $T \in \mathcal{B}$.

The trace functional is nonnegative real-valued on every projection operator $P_e$:

$$\tau_s P_e = \langle P_e s | s \rangle = \langle (s|e) | e|s \rangle = \langle s|e \rangle \langle e | s \rangle = ||s|e\rangle^2 \in \mathbb{R}_+. \quad (2)$$

We therefore obtain from any orthonormal basis $\{e_i | i \in N\}$ a probability distribution on $N$ with coefficients $p_i = \tau_s P_{e_i}$:

$$p_i = ||s|e_i\rangle|^2 \geq 0 \quad \text{and} \quad \sum_{i \in N} p_i = \sum_{i \in N} ||s|e_i\rangle|^2 = ||s||^2 = 1. \quad (3)$$

Switching viewpoints, one finds that the collection $X = \{\tau_{e_i} | i \in N\}$ of trace functionals $\tau_{e_i}$ yields an observable (in the sense of Section 3) for every evolution $\Psi$ of projection operators $P_s$ in the operator space $\mathcal{B}$.

Consider, for example, the Schrödinger evolution $\Phi_U = \{s(t) = U^t s | t \geq 0\}$ of the wave function $s \in \mathcal{H}$ with $||s|| = 1$ relative to the unitary operator $U$. $U$ induces the linear transformation

$$T \mapsto UTU^*$$

on $\mathcal{B}$. Notice that

$$P_s U^*(u) = \langle U^*(u) | s \rangle s = \langle u | Us \rangle U^*Us = U^*P_{Us}u \quad (4)$$

i.e. $P_s U^* = U^*P_{Us}$ and hence $UP_sU^* = P_{Us}$ holds. So $\Phi_U$ has the companion evolution

$$\forall U = \{P_{Us(t)} = (UP_{Us})^t | t \geq 0\}$$

of associated projection operators in $\mathcal{B}$ that can be observed under $X$. Slightly more generally, the states of a quantum system are thought to be described by densities, i.e., operators $D$ of the form

$$D = \sum_{i \in N} \lambda_i P_{e_i}, \quad (5)$$

where $\{e_i | i \in N\}$ is an orthonormal basis of $\mathcal{H}$ and $\{\lambda_i | i \in N\}$ a (real) probability distribution on $N$.

\[5\] or qbits in the terminology of quantum computing [22] if dim $\mathcal{H} < \infty$
Quantum measurements In the standard interpretation of quantum mechanics, a measurement is represented by an operator $M \in \mathcal{B}$ of the form

$$M = \sum_{i \in \Lambda} \lambda_i P_{e_i}$$

where $\{e_i \mid i \in N\}$ is an orthonormal basis, and the $\lambda_i$ are real (but not necessarily nonnegative) numbers and are the eigenvalues of $M$. Let $\Lambda$ be the set of different eigenvalues.

When a quantum system is in the state $P_s$ which is implied by the wave function $s \in \mathcal{W}$, the measurement is expected to produce the numerical value

$$E_M(s) = \sum_{i \in \Lambda} \lambda_i \tau_s P_{e_i} = \sum_{i \in \Lambda} \lambda_i p_i$$

(7)

where the $p_i$ are the probabilities as in (3). So the measurement comes down to the application of the $\Lambda$-valued observation variable $X$ that takes on a particular value $\lambda$ with probability

$$\Pr\{X = \lambda\} = \sum_{i : \lambda_i = \lambda} \tau_s P_{e_i}$$

and has the expectation

$$E_X(s) = \int_{\mathbb{R}} x dp = \sum_{\lambda \in \Lambda} \lambda \Pr\{X = \lambda\} = E_M(s).$$

In the finite-dimensional case $\dim \mathcal{H} = n < \infty$, the operators $M$ of the form (6) are precisely the self-adjoint operators and

$$E_M(s) = \text{tr}(MP_s)$$

is the usual trace of the product operator $MP_s$. If one restricts attention to Schrödinger evolutions, our quantum statistical model above becomes the quantum statistical inference model proposed by Barndorff-Nielsen, Gill and Jupp [3].

From a mathematical point of view, of course, there is no reason to restrict statistical inference theory to the analysis of Schrödinger evolutions. In the same way classical Markov chains generalize to hidden Markov chains, observable operator models etc., or more general evolutions in $\mathcal{H}$ or $\mathcal{B}(\mathcal{H})$ may be of interest as well. The statistics of such evolutions can be analyzed in the same way.

4.4 Quantum random walks

"Quantum random walks" and "quantum Markov chains" as generalizations of the classical models to the quantum model have received considerable recent interest. The models proposed in the literature are typically derived from Schrödinger type evolutions relative to a set $N$. The resulting random walk is then a particular $N$-valued Markov process in the sense of Section 4.
As an illustration, we outline a generalization of Gudder’s [15] model relative to the set \( N = \{1, \ldots, n\} \). Let \( d \geq 1 \) be some integer parameter and \( \mathbb{H}_d \) the real vector space of all self-adjoint \( d \times d \) matrices \( C \) with coefficients \( C_{ij} \in \mathbb{C} \). Define a state to be a collection \( S = \{ S_i \in \mathbb{H}_d \mid i \in N \} \) of self-adjoint matrices \( S_i \) with nonnegative eigenvalues such that \( \text{tr}(S) = \sum_{i \in N} \text{tr}(S_i) = 1 \). Assume to be further given a set \( E = \{ \epsilon_{ij} \mid i, j \in N \} \) of operators on \( \mathbb{H}_d \) that map densities onto densities. Consider a process that starts from a state \( S \) and iteratively effects state transitions as follows:

\[
S^{(t)} \rightarrow S^{(t+1)} \quad \text{with} \quad S_i^{(t+1)} = \sum_{j \in N} \epsilon_{ij} S_j^{(t)} \quad (i \in N).
\]

This process induces an evolution \( (S^{(t)}) \) in the universe \( U = \mathbb{H}_n^d \) with evolution matrix \( M \), say. Let \( \Pi_i \) be the projector that maps \( T \in \mathbb{H}_n^d \) onto its \( i \)th coordinate \( T_i \in \mathbb{H}_d \) and consider the set of linear operators

\[
\mathcal{M} = \{ M^{(i)} = \Pi_i M \mid i \in N \}.
\]

\( \mathcal{M} \) induces an \( N \)-valued stochastic process \( (X_t) \) with distribution

\[
\text{Pr}\{X_1 = i_1, \ldots, X_t = i_t\} = \text{tr}(M^{i_t} (M^{i_{t-1}} (\ldots (M^{i_1} S) \ldots))),
\]

which constitutes a quantum analog of a classical random walk on \( N \).

**Remark 4** The quantum random walk model proposed by Aharonov et al. [1] (see also [12, 20, 23, 24]) follows from the present approach by specializing the quantum evolution and observation further. For a generalization of the classical Metropolis random walk into a quantum context, see, e.g., Temme et al. [25].

### 5 Joint observations

We say that the \( k \) observables \( X^{(1)}, \ldots, X^{(k)} \) with alphabets \( A_1, \ldots, A_k \) are jointly observable on the evolution \( \Psi = (\psi, \psi) \) if there exists an observable \( X \) for \( \Psi \) with alphabet \( A = A_1 \times \ldots \times A_k \) such for all \( j = 1, \ldots, k \), \( a_j \in A_j \) and \( t \geq 0 \),

\[
\text{Pr}\{X_t^{(j)} = a_j\} = \sum_{(a_1, \ldots, a_j, \ldots, a_k) \in A} \text{Pr}\{X_t = (a_1, \ldots, a_j, \ldots, a_k)\},
\]

which means that \( X^{(j)} \) is the \( j \)th marginal of \( X \).

It is clear that joint observability of \( \{X^{(1)}, \ldots, X^{(k)}\} \) implies joint observability for any subset \( \{X^{i_1}, \ldots, X^{i_r}\} \). In particular, sums and products of jointly observable variables are observable and expected values, covariances etc. are well-defined.
5.1 Heisenberg uncertainty

We illustrate the concept of jointly observables with the example of an important measurement issue in the standard model of quantum theory. Let $X$ and $Y$ be two observation variables associated with two quantum measurements as in Section 4.3. So there are representative self-adjoint operators

$$A = \sum_{i \in N} \lambda_i P_{e_i} \quad \text{and} \quad B = \sum_{j \in N} \lambda_j' P_{f_j}$$

relative to orthonormal bases $\{e_i \mid i \in N\}$ and $\{f_j \mid j \in N\}$ of a complex Hilbert space $\mathcal{H}$. Let $A$ and $A'$ be the ranges of $X$ and $Y$.

If $X$ and $Y$ are jointly observable in this measurement model relative to the wave function $s$, there is an observable $Z$ with marginals $X$ and $Y$ and probability distribution

$$\Pr\{Z = (\lambda, \lambda')\} = \sum_{k \in \zeta^{-1}(\lambda, \lambda')} \tau_{s} P_{g_k}$$

for a suitable map $\zeta : N \to A \times A'$ and orthonormal basis $\{g_k \mid k \in N\}$. $Z$ admits the operator representation

$$C = \sum_{k \in N} \mu_k P_{g_k} \quad \text{with} \quad \mu_k = \lambda \cdot \lambda' \text{ if } \zeta(k) = (\lambda, \lambda').$$

Moreover, we have operator representations for $X$ and $Y$ with respect to the common basis $\{g_k\}$:

$$\tilde{A} = \sum_{k \in N} \lambda_k P_{g_k} \quad \text{and} \quad \tilde{B} = \sum_{k \in N} \lambda'_k P_{g_k} \quad \text{where} \quad (\lambda_k, \lambda'_k) = \zeta_k.$$

Hence $X$ and $Y$ are seen to satisfy the Heisenberg commutativity condition for observational compatibility:

$$\tilde{A}\tilde{B} = C = \tilde{B}\tilde{A}. \quad (10)$$

Conversely, if $X$ and $Y$ admit operator representations with respect to a common basis, it is clear that $X$ and $Y$ are jointly observable relative to every Schrödinger evolution.

**Theorem 4.** Quantum measurements $X$ and $Y$ are jointly observable on Schrödinger evolutions if and only if $X$ and $Y$ admit operator representations with respect to a common orthonormal basis. ■

Say that a quantum measurement $X$ has the Riesz property if it admits a finite or countable orthonormal basis of eigenvectors $e_i$ with eigenvalues $\lambda_i$ such that each eigenspace $E_\lambda$ is finite-dimensional and

$$X = \sum_i \lambda_i P_{e_i}. \quad (11)$$
Corollary 3. Riesz quantum measurements $X$ and $Y$ are jointly observable on Schrödinger evolutions if and only if they admit operator representations by commuting operators.

Proof. Let $A$ and $B$ be representations of $X$ and $Y$ as in (11). It remains to show that $AB = BA$ implies the existence of a common representative orthonormal basis for $X$ and $Y$.

Consider an arbitrary eigenvalue $\lambda \in A$ of $A$ with eigenspace $E_\lambda$. $A$ is $E_\lambda$-invariant. Moreover, for any $v \in E_\lambda$ one has

$$BAv = \lambda Bu = ABv$$

i.e., $E_\lambda$ is also $B$-invariant. $E_\lambda$ is finite-dimensional and therefore admits an orthonormal basis $G_\lambda$ of eigenelements of $B$ that are also eigenelements of $A$. So

$$G = \bigcup_{\lambda \in A} G_\lambda$$

is an orthonormal basis with the desired property. ■

In the same way, one finds:

Corollary 4. The Riesz quantum measurements $X^1, \ldots, X^k$ on Schrödinger evolutions in a Hilbert space are jointly observable if and only if they are pairwise observable. ■

5.2 A Bell-type inequality

We have seen that Riesz quantum measurements on Schrödinger evolutions are jointly observable if and only if they are pairwise observable (Corollary 3). This convenient criterion no longer applies to observables on arbitrary Markov chains (see Example 1 below). We now establish a necessary condition for joint observability of three observables in the spirit of Bell’s [5] inequalities in the standard quantum model.

Lemma 2 (Bell inequality). Let $X, Y, Z$ be pairwise observable on the (arbitrary) evolution $\Psi = (\psi, s)$, each taking values in $\{-1, +1\}$. Then the inequality

$$|E_t(XY) - E_t(YZ)| \leq 1 - E_t(XZ) \quad \text{holds for all } t \geq 0,$$  \hspace{1cm} (12)

where $E_t(XY)$ is the expected value of the product variable $X_i Y_t$ at time $t$.

Proof. Any choice of $x, y, z \in \{-1, +1\}$ satisfies the inequality

$$|xy - xz| \leq 1 - xz.$$
The probabilities \( p_t(x, y, z) = \Pr\{X_t = x, Y_t = y, Z_t = z\} \) are nonnegative real numbers that sum up to 1. So we conclude

\[
|E_t(XY) - E_t(YZ)| = \bigg| \sum_{x, y, z} (xy - yz)p_t(x, y, z) \bigg| \leq \sum_{x, y, z} |xy - yz|p_t(x, y, z) \\
\leq \sum_{x, y, z} (1 - xz)p_t(x, y, z) = 1 - E_t(XZ).
\]

\[
\blacksquare
\]

The inequality (12) may be violated by observables \( X, Y, Z \) that are pairwise but not jointly observable.

**Example 1** Consider the (stationary) evolution \( \Psi = (\psi, D) \) with \( D^{(t)} = D \) in the space \( \mathbb{H}_5 \) of all \( 5 \times 5 \) self-adjoint matrices, where

\[
D = \text{diag}(-1/3, 1/3, 1/3, 1/3, 1/3) \in \mathbb{H}_5
\]

and \( \text{diag}(v) \) denotes the diagonal matrix with diagonal vector \( v \). Let \( X, Y, Z \) be the measurements that are induced be the self-adjoint matrices

\[
A_X = \text{diag}(-1, +1, -1, -1, -1) \\
A_Y = \text{diag}(+1, +1, -1, +1, -1) \\
A_Z = \text{diag}(+1, +1, +1, -1, -1).
\]

Notice that \( A_X, A_Y, A_Z \) commute pairwise and thus satisfy the Heisenberg condition (10). Moreover, \( X, Y, Z \) are pairwise observable on \( \Psi \). The pairs of products have the expectations

\[
E(XY) = +1, \ E(YZ) = -1/3, \ E(XZ) = +1
\]

and violate the Bell inequality (12), which shows that \( X, Y, Z \) are not jointly observable on \( \Psi \).

**Remark 5** The experimental results of Aspect et al. suggest that measurements on real world quantum systems may violate Bell’s inequalities. In our Markov setting, these results can be explained as follows: the experiments were either not carried out with pairwise commuting observables (and thus subject to Heisenberg uncertainty) and/or the description of quantum states by ”densities” with only nonnegative eigenvalues is too restrictive for real world models.

6 Conclusion

A model for the Markovian statistical analysis of observations on evolving systems has been proposed that separates the evolution of the system states and the observation of system events clearly. This model not only generalizes classical views on homogeneous
Markov chains as random sources properly but allows a Markov type analysis of more
general observation processes which, in particular, include observations arising from
general underlying stochastic processes. This separation of the notions of system evo-
lution and system observation allows us to develop a general theory of joint observabil-
ity, which has no classical counterpart. It is compatible with the notion of Heisenberg
uncertainty relative to Schrödinger evolutions but it is not implied by it.

Several intriguing questions immediately raise themselves. For example, we do not
think that the model of Riesz evolutions is the most general in which Markovian con-
vergence can be proved. Do our convergence results hold relative to evolution operators
$T$ with $\lambda = 1$ being just an isolated eigenvalue of $T$? What is the general convergence
behavior of quantum densities? Is it true, for instance, that a normal operator $\psi$ of norm
$\|\psi\| \leq 1$ on a Hilbert space $\mathcal{H}$ not only yields a mean ergodic evolution in $\mathcal{H}$ itself
(Theorem 2) but also a mean-ergodic evolution of the associated densities?

7 Appendix: Proofs

For fundamental notions and facts on linear operators we refer to standard texts
for fundamentals on linear operators.

7.1 Proof of Theorem 2

Recall the spectral representation for a continuous normal operator in its multiplication
form:

**Theorem 5.** Let $\psi$ be a continuous operator on a complex Hilbert space $\mathcal{H}$ such that
$\psi^* \psi = \psi \psi^*$. Then there exists a measure space $(\Omega, \Sigma, \mu)$, an essentially bounded
measurable function $g : \Omega \to \mathbb{C}$ and a unitary operator $U : \mathcal{H} \to L^2(\mu)$ such that

$$\psi = U^* M_g U,$$

where $M$ is the multiplication operator $M_g f = f \cdot g$. Moreover,

$$\|\psi\| = \|M_g\| = \|g\|_\infty.$$  

By Theorem 5 we can assume w.l.o.g.:

- $\mathcal{H} = L^2(\mu)$ and $\psi$ is given as multiplication by a bounded measurable function $g$.

The stability of the evolution $\Psi = (\psi, s)$ implies that there is a constant $M$ so that
$|g^n(\omega)f(\omega)| \leq M$ holds almost everywhere for all positive integers $n$. It follows that
a.e., $|g(\omega)| \leq 1$ or $f(\omega) = 0$. Hence there is a measurable function $g_1$ so that a.e.,
$|g_1(\omega)| \leq 1$ and $g^n(\omega)f(\omega) = g_1^n(\omega)f(\omega)$. Setting

$$\bar{\psi}_t(f) = \left(\frac{1}{t} \sum_{m=1}^{t} g_1^m\right) f,$$

$6$ e.g., $[7,9]$
we therefore conclude
\[ |\psi_t(f)(\omega)|^2 \leq \left( \frac{1}{t} \sum_{m=1}^{t} |g_1(\omega)|^m \right)^2 |f(\omega)|^2 \leq |f(\omega)|^2. \]

The sequence \((\psi_t(f)(\omega))_{t \geq 0}\) converges to
\[ \pi(f)(\omega) = \begin{cases} f(\omega) & \text{if } g_1(\omega) = 1 \\ 0 & \text{otherwise.} \end{cases} \]

If \(g_1(\omega) = 1\), then \(\psi_t(f)(\omega) = f(\omega) = \pi(f)(\omega)\), and so
\[ \lim_{t \to \infty} ||\psi_t(f)(\omega)||_2 = \lim_{t \to \infty} \left( \int_{\Omega} |\psi_t(f) - \pi(f)(\omega)|^2 d\omega \right)^{1/2} = \lim_{t \to \infty} \left( \int_{g(\omega) \neq 1} |\psi_t(f)(\omega)|^2 d\omega \right)^{1/2}. \]

On the set \(\{\omega \in \Omega \mid g(\omega) \neq 1\}\), the functions \(|\psi_t(f)(\omega)|^2\) converge pointwise to 0 and are bounded by the integrable function \(|f(\omega)|^2\). The theorem of dominated convergence thus yields
\[ \lim_{t \to \infty} \left( \int_{g(\omega) \neq 1} |\psi_t(f)(\omega)|^2 d\omega \right)^{1/2} = \left( \int_{g(\omega) \neq 1} \lim_{t \to \infty} |\psi_t(f)(\omega)|^2 d\omega \right)^{1/2} = 0. \]

Clearly, \(\pi(f)\) is the orthogonal projection of \(f\) onto the eigenspace of \(\lambda = 1\). So stability is sufficient for mean ergodicity.

To see that stability is necessary for mean ergodicity, assume that
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k(f) \text{ exists for the operator } T(f) = \int g f d\mu. \]

We would like to show that \(|g(x)| \leq 1\) holds a.e. on the set \(\{x : f(x) \neq 0\}\).

Assume that the set
\[ M = \{x : f(x) \neq 0 \text{ and } |g(x)| > 1\} \quad (14) \]
has positive measure. Then for some integer \(r > 1\) the set
\[ M_r = \left\{ x : f(x) \neq 0 \text{ and } r > |g(x)| > 1 + \frac{1}{r} \right\} \quad (15) \]
has positive measure and for any \(x \in M_r\), we have
\[ \frac{1}{n} \sum_{k=0}^{n-1} T^k(f) = \frac{f(x)}{n} \sum_{k=0}^{n-1} g(x)^k = \frac{f(x)}{n} \frac{1-g(x)^r}{1-g(x)}. \quad (16) \]
Observing

\[ |1 - g(x)^n| \geq |g(x)|^n - 1 > \left(1 + \frac{1}{r}\right)^n - 1 \]

and \( |1 - g(x)| \leq 1 + |g(x)| \leq 1 + r \), we thus conclude

\[ \left| \frac{1}{n} \sum_{k=0}^{n-1} T^k(f)(x) \right| \geq \frac{|f(x)|}{n} \left(1 + \frac{1}{r}\right)^n - 1 \]

and hence

\[ \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k(f) \right\|_2 \geq \frac{1}{n} \left( \int_X \left| \sum_{k=0}^{n-1} T^k(f)(x) \right|^2 \, d\mu \right)^{1/2} \]

\[ \geq \frac{1}{n} \left( \int_{M_r} \sum_{k=0}^{n-1} T^k(f)(x) \right)^2 \frac{1}{n} \left( \int_{M_r} |f(x)|^2 \, d\mu \right)^{1/2} \]

\[ \geq \left(1 + \frac{1}{r}\right)^n - 1 \frac{1}{n} \left( \int_{M_r} |f(x)|^2 \, d\mu \right)^{1/2} \].

Since \( \int_{M_r} |f(x)| \, d\mu > 0 \) and \( \lim_{n \to \infty} \frac{(1 + \frac{1}{r})^n - 1}{n(1 + r)} = \infty \), it follows that \( \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k(f) \right\|_2 \) is unbounded. So the series \( \left\{ \frac{1}{n} \sum_{k=0}^{n-1} T^k(f) \right\}_{n>0} \) cannot converge. Consequently,

\[ |g(x)^n f(x)| \leq |f(x)| \] and hence \( \|T^n f\|_2 \leq \|f\|_2 \) holds for all \( n \).

\[ \square \]

### 7.2 Proof of Theorems 1 and 3

Throughout this section, let \( U \) be a fixed Banach space with a fixed element \( s \in U \). We further fix an operator \( T : U \to U \) that is bounded on \( U_s = \text{lin}\{T^n s : n \geq 0\} \) and denote by \( \hat{T} \) its (bounded) extension to \( U_s \). Without loss of generality, we can therefore assume \( U_s = U \) and \( \hat{T} = T \). \( \sigma(T) \) denotes the spectrum of \( T \). The spectral radius of \( T \) is

\[ r(T) = \max\{|\lambda| \mid \lambda \in \sigma(T)\} = \lim_{t \to \infty} \|T^t\|^{1/t} \leq \|T\|. \]

**Lemma 3.** If \( T \) is Riesz, then \( \sigma_1(T) = \{\lambda \in \sigma(T) \mid |\lambda| \geq 1\} \) is a finite set.

**Proof.** \( \sigma_{\varepsilon, \delta}(T) = \{\lambda \in \sigma(T) \mid \varepsilon \leq |\lambda| \leq \delta\} \) is a bounded subset of \( \mathbb{C} \) for any \( \varepsilon, \delta \geq 0 \). If \( T \) is Riesz, \( 0 \) is the only possible accumulation point of \( \sigma(T) \). So \( \sigma_{\varepsilon, \delta}(T) \) must be finite for every \( \varepsilon > 0 \). If \( T \) is bounded, \( \sigma_1(T) = \sigma_{\varepsilon, \delta}(T) \) holds for \( \varepsilon = 1 \) and \( \delta = \|T\| \) and the claim of the Lemma follows.

\[ \square \]
For any eigenvalue \( \lambda \) of \( T \) with finite algebraic multiplicity \( n_\lambda \), the Riesz decomposition of \( T \) with respect to \( \lambda \) guarantees:

**(R)** \( U \) admits the direct sum decomposition \( U = N_\lambda \oplus R_\lambda \), where

1. \( N_\lambda = \{ x \in U \mid (T - \lambda)^{n_\lambda} x = 0 \} \) is \( T \)-invariant with \( \dim N_\lambda < \infty \);
2. \( R_\lambda = (T - \lambda)^{n_\lambda} U \) is \( T \)-invariant.

If \( \sigma_1(T) \) is finite set of eigenvalues, repeated application of the Riesz decomposition (R) to some \( \lambda \in \sigma_1(T) \) and then to \( T : R_\lambda \to R_\lambda \) etc. and the other eigenvalues in \( \sigma_1(T) \) yields

**Lemma 4 (Riesz decomposition).** If \( \sigma_1(T) \) is a finite set of eigenvalues of \( T \) with finite algebraic multiplicities, then \( U \) admits a direct sum decomposition

\[
U = N \oplus W \ni \lambda \in \sigma_1(T)
\]

into \( T \)-invariant subspaces \( N \) and \( W \), where

\[
N = \bigoplus_{\lambda \in \sigma_1(T)} N_\lambda \quad \text{and} \quad \dim N < \infty.
\]

Moreover, \( |\lambda| < 1 \) holds for all eigenvalues \( \lambda \) of the restriction of \( T \) to \( W \).

The decomposition (R) implies that Riesz evolutions are finitary.

**Proposition 2.** Assume that \( T \) is a Riesz operator with decomposition \( U = N \oplus W \) into \( T \)-invariant subspaces \( N \) and \( W \) such that \( \dim N < \infty \) and the restriction of \( T \) to \( W \) has no eigenvalue in \( \sigma_1(T) \). Then the Riesz evolution \( (T, s) \) is equivalent to the finite-dimensional evolution \( (T, s_N) \), where \( s_N \in N \) is such that \( s = s_N + s_W \) holds for some \( s_W \in W \).

**Proof.** In view of

\[
T^m s = T^m s_N + T^m s_W ~ \text{for all} ~ m \geq 0,
\]

it suffices to establish the claim

\[
\lim_{n \to \infty} T^n s_W = 0.
\]

Since \( \sigma_\varepsilon(T) \) is a finite set for any \( \varepsilon > 0 \), the spectral radius \( r_W(T) \) of \( T \) on \( W \) must satisfy \( r_W(T) < 1 \). For clarity of notation, let \( T_W \) be the restriction of \( T \) to \( W \) and choose \( n_0 \) so large that \( \|T^n_W\|^{1/n} \leq r < 1 \) holds for all \( n \geq n_0 \). Then one has \( \|T^n_W\| \leq r^n \) and thus concludes

\[
\lim_{n \to \infty} \|T^n_W s_W\| = 0.
\]

The proof of Theorem is now immediate: The Riesz evolution \((\psi, s)\) is equivalent to the finite-dimensional evolution \((\psi, s_N)\). The ergodic properties stated in Theorem are directly obtained by applying Proposition to \((\psi, s_N)\).

\[\text{see, e.g., [9]}\]
For the proof of the sampling theorem (Theorem 3), let \((Q, x)\) be a finite-dimensional evolution that is equivalent to \((T, s)\). So we have \(\|T^n s - Q^n x\| \to 0\) and hence
\[
\lim_{n \to \infty} \|f(T^n s) - f(Q^n x)\| = \lim_{n \to \infty} \|f(T^n s - Q^n x)\| = 0
\]
since the sampling function \(f\) is continuous. This implies
\[
\lim_{t \to \infty} \frac{1}{t} \sum_{m=1}^{t} \|f(T^m s) - f(Q^m x)\| = 0,
\]
\(i.e.,\) the \(f\)-sample averages converge on \((T, s)\) exactly when they converge on \((Q, x)\). It is furthermore clear that \((T, s)\) is stable exactly when \((Q, x)\) is stable. For the proof, we can therefore assume without loss of generality that already \((T, s)\) is finite-dimensional and hence the sample space \(F = f(U_s)\) is finite-dimensional.

Passing to coordinates, we may thus assume: \(U_s = \mathbb{C}^n\) and \(F = \mathbb{C}^k\). Since \(f : \mathbb{C}^n \to \mathbb{C}^k\) is bounded on \((T, s)\) if and only if each component functional \(f_j\) of \(f\) is bounded on \((T, s)\), it suffices to consider the 1-dimensional case \(k = 1\).

With respect to the chosen coordinatization, \(T\) is an \(n \times n\) matrix, \(s\) a column vector and \(f\) a row vector of dimension \(n\). Assume first that the sequence \((f T^t s)\) is bounded. Then
\[
(f T^t u) \text{ is bounded for every } u \in U_s = \text{lin}\{T^t s \mid t = 0, 1, \ldots\} = \mathbb{C}^n.
\]
It follows that the sequence \((f T^t)\) of \(n\)-dimensional row vectors constitutes a bounded evolution. (The choice of \(u\) as the unit vector \(e_i\) in \(\mathbb{C}\) shows that the \(i\)th coordinate of the evolution is bounded.) In view of Proposition 1 (Section 2.1), this evolution is mean-ergodic. Consequently, the boundedness of \((f T^t s)\) implies the existence of
\[
\lim_{t \to \infty} \frac{1}{t} \sum_{m=1}^{t} f T^m = \mathbb{E} = \lim_{t \to \infty} \frac{1}{t} \sum_{m=1}^{t} f T^m s.
\]
To prove the converse implication, assume that \(\mathbb{E}\) exists. Then
\[
\frac{1}{t} \sum_{m=1}^{t} \lim_{l \to \infty} f T^m u \text{ exists for every } u \in U_s = \text{lin}\{T^t s \mid t = 0, 1, \ldots\} = \mathbb{C}^n.
\]
It follows that the evolution \((f T^t)\) of row vectors is mean ergodic and hence, again by Proposition 1 stable, \(i.e.,\) there is some constant \(c \in \mathbb{R}\) such that
\[
|f T^t s| \leq \|f T^t\| \cdot \|s\| \leq c \|s\| < \infty \text{ for all } t \geq 0,
\]
which establishes the claim of the sampling theorem.

■
References

1. D. Aharonov, A. Ambainis, J. Kempe, U. Vazirani: Quantum walks on graphs, Proc. 33th STOC, 60-69, ACM, New York, NY, 2001.
2. A. Aspect, J. Dalibard, G. Roger: Experimental tests of Bell’s inequalities using time-varying analyzers, Phys. Rev. Lett. 49 (1982), 1804.
3. O.E. Barndorff-Nielsen, R.D. Gill, P.E. Jupp: On quantum statistical inference, J. Roy. Statist. Soc. B (2003), 775-816.
4. J.S. Bell: On the Einstein Podolsky Rosen paradox, Physics 1 (1964), 195-200.
5. J.S. Bell: On the problem of hidden variables in quantum mechanics, Rev. Mod. Phys. 38 (1966), 447-452.
6. S.P.M. Choi, D.-Y. Yeung and N.L. Zhang: Hidden-Markov decision processes for nonstationary sequential decision making, in: Sequence Learning, R. Sun and C.L. Giles (eds.), Lecture Notes in Artificial Intelligence 1828, 264-287, Springer-Verlag, 2000.
7. J.B. Conway, A Course in Functional Analysis. Graduate Texts in Mathematics 96, Springer-Verlag, New York, 2nd ed., 1990.
8. S.W. Dharmadhikari: A characterization of a class of functions of finite markov chains. Annals of Mathematical Statistics 36 (1965), 524–528.
9. H.R. Dowson, Spectral Theory of Linear Operators. Academic Press, London, New York, San Francisco, 1978.
10. R.J. Elliot, L. Aggoun, J.B. Moore, Hidden Markov Models, Springer-Verlag, Heidelberg, 1995.
11. U. Faigle and A. Schönhuth: Asymptotic mean stationarity of sources with finite evolution dimension, IEEE Trans. Information Theory 53 (2007), 2342-2348.
12. U. Faigle and A. Schönhuth: Efficient tests for equivalence of hidden Markov processes and quantum random walks. IEEE Transactions on Information Theory, 57 (2011), 1746–1753.
13. W. Feller, An Introduction to Probability Theory and Its Applications II, Wiley, New York, 1971.
14. E.J. Gilbert: On the identifiability problem for functions of finite Markov chains, Ann. Math. Stat. 30 (1959), 688-697.
15. S. Gudder: Quantum Markov chains, J. Math. Physics 49 (2008) id.072105
16. A. Heller: On stochastic processes derived from Markov chains. Ann. Math. Satist. 36 (1965), 1286-1291.
17. O. Hernandez-Lerma and J.B. Lasserre, Markov Chains and Invariant Probabilities Theory, Birkaeuser, Basel, 2003.
18. H. Ito, S.-I. Amari, K. Kobayashi: Identifiability of hidden Markov information sources and their minimum degrees of freedom, IEEE Transactions on Information Theory 38, 324-333 (1992).
19. H. Jaeger: Observable operator models for discrete stochastic time series, Neural Computing 12, 1371-1398 (2000).
20. J. Kempe: Quantum random walks: an introductory overview, Contemorary Physics 44, 307-327 (2003).
21. A. A. Markoff, Wahrscheinlichkeitsrechnung. B. G. Teubner, Leipzig (Übersetzung der 2. russischen Auflage) 1912.
22. M. Nielsen and I. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, 2000.
23. R. Portugal, R.A.M. Santos, T.D. Fernandes, D.N. Goncalves: The staggered quantum walk model. To appear: Quantum Information Processing, arXiv:1505.04761 (2015).
24. M. Szegedy: Quantum speed-up of Markov chain based algorithms. In Proceedings 45th Symposium on Foundations of Computer Science (2004), 32-41.
25. K. Temme, T.J. Osborne, K.G. Vollbrecht, F. Verstraete: Quantum Metropolis sampling. *Nature* 471 (2011), 87-90.
26. M. Vidyasagar: (2011). *The complete realization problem for hidden Markov models: A survey and some new results*, Mathematics of Control, Signals and Systems 23 (2011), 1-65.