Global and exponential attractors for a Ginzburg-Landau model of superfluidity

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Abstract

The long-time behavior of the solutions for a non-isothermal model in superfluidity is investigated. The model describes the transition between the normal and the superfluid phase in liquid ⁴He by means of a non-linear differential system, where the concentration of the superfluid phase satisfies a non-isothermal Ginzburg-Landau equation. This system, which turns out to be consistent with thermodynamical principles and whose well-posedness has been recently proved, has been shown to admit a Lyapunov functional. This allows to prove existence of the global attractor which consists of the unstable manifold of the stationary solutions. Finally, by exploiting recent techniques of semigroups theory, we prove the existence of an exponential attractor of finite fractal dimension which contains the global attractor.

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1 Introduction

In this paper we study the asymptotic behavior of the solutions of a Ginzburg-Landau model for superfluidity. This model describes the phase transition between the normal and the superfluid state occurring in liquid helium II when the temperature overcomes a critical value of about $2.2 \, K$. The phenomenon can be interpreted as a second-order phase transition and accordingly set into the framework of the Ginzburg-Landau theory (see *e.g.* [6, 11]). The derivation of this model, its consistency with thermodynamics and the interpretation of some physical aspects related to superfluidity can be found in [12]. In agreement with Landau’s viewpoint, the main matter is to consider each particle of the superfluid as a pair endowed with two different excitations, normal and superfluid, represented respectively by two components $v_n$ and $v_s$ of the velocity. The differential system describing the behavior of the superfluid involves three unknowns: the concentration $f$ of the superfluid phase, whose evolution is governed by the Ginzburg-Landau equation, the absolute temperature $u$ which induces the transition and the superfluid component $v_s$. The normal component $v_n$ is supposed to be expressed in terms of the superfluid velocity through the constitutive equation (see [12])

$$v_n = \nabla \times v_s.$$ 

By means of a suitable decomposition of the state variables the differential system ruling the evolution of the superfluid assumes the form

$$\gamma \psi_t = \frac{1}{\kappa^2} \Delta \psi - \frac{2i}{\kappa} A \cdot \nabla \psi - \psi |A|^2 + i\beta (\nabla \cdot A) \psi - \psi (|\psi|^2 - 1 + u)$$  

$$A_t = \nabla (\nabla \cdot A) - \mu \nabla \times \nabla \times A - |\psi|^2 A + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) - \nabla u - g$$  

$$c_0 u_t = \frac{1}{2} (\psi \bar{\psi} + \bar{\psi} \psi_t) + k_0 \Delta u$$  

$$+ \nabla \cdot \left[ -|\psi|^2 A + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \right] + r$$  

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where $\psi$ is a complex-valued function whose modulus coincides with the concentration of the superfluid phase, and $A$ is related to the component $v_s$ by

$$\nabla \times A = -\nabla \times v_s.$$

Equations (1.1)-(1.2) have the same structure of the Ginzburg-Landau equations of superconductivity ([24]). Indeed, as pointed out by several authors (see e.g. [18, 23]), there are evident analogies between the phenomena of superfluidity and superconductivity. In this framework, the choice of the decomposition for the unknown variables corresponds to a choice of the gauge for the Ginzburg-Landau equations [5, 14, 17].

Existence and uniqueness of the global solutions to problem (1.1)-(1.3) completed with initial and boundary conditions have been proved in [4]. In this paper we analyze the asymptotic behavior of the solution, by proving first existence of the global attractor and then of exponential attractors. In the context of superconductivity, the same problem has been treated in [21], where the authors prove existence of the global attractor. Later, Rodriguez-Bernal et al. [20] show that the semigroup generated by the system admits finite-dimensional exponential attractors. The main difference and difficulty in our problem is due to the presence of the absolute temperature which does not appear in the traditional Ginzburg-Landau equations of superconductivity, where an isothermal model is analyzed. In particular, even if from a physical point of view $u > 0$, such a bound cannot be proved a-priori from equations (1.1)-(1.3). The positivity of the temperature would guarantee the boundedness

$$|\psi| \leq 1,$$

which can be proved in the same way as in superconductivity ([8]), provided that this inequality holds at the initial instant. Relation (1.4) is widely exploited in [3] and [5] to prove that the Ginzburg-Landau system of superconductivity admits absorbing sets, global and exponential attractors. As a matter of facts the inequality (1.4) is not used neither in [20] nor in [21], where existence of the global attractor is proved by means of a Lyapunov functional and exponential attractors are obtained
as a consequence of the squeezing property of the solutions ([9]). Therefore, in this paper we construct a Lyapunov functional for system (1.1)-(1.3) which allows to show existence of the global attractor consisting of the unstable manifold of the stationary solutions. Furthermore, by means of more recent results devised in [15], we prove that the semigroup generated by (1.1)-(1.3) possesses an exponential attractor.

The plan of the paper is the following. The model describing the behavior of the superfluid is recalled in section 2. In section 3 we state the existence and uniqueness result obtained in [4] and prove a-priori estimates and continuous dependence of solutions on the initial data which ensures that the system generates a strongly continuous semigroup on the phase space. Section 4 is devoted to the construction of a Lyapunov functional and to the proof of existence of the global attractor. Finally in section 5, we show that the semigroup admits an exponential attractor.

2 Statement of the problem

In this section, we briefly recall the model proposed in [12] describing the behavior of a superfluid. Let \( \Omega \subset \mathbb{R}^3 \) be the domain occupied by the material. We suppose that \( \Omega \) is bounded with a smooth boundary \( \partial \Omega \), whose unit outward normal will be denoted by \( \mathbf{n} \). The state variables are identified with the triplet \((f, v_s, u)\) representing the concentration of the superfluid phase, the velocity of the superfluid component and the ratio between the absolute temperature and the transition temperature. The evolution of \( f \) is ruled by the Ginzburg-Landau equation typical of second order phase transitions ([11]), i.e.

\[
\gamma f_t = \frac{1}{\kappa^2} \Delta f - f(f^2 - 1 + u + v_s^2),
\]

where \( \gamma, \kappa \) are positive constants. The term \( v_s^2 \) allows to prove the existence of a critical velocity above which superfluid properties disappear. Indeed, if \( v_s \) overcomes a threshold value, the unique solution to (2.1) with boundary and initial conditions

\[
\nabla f \cdot \mathbf{n}|_{\partial \Omega} = 0 \quad f(x, 0) = f_0(x)
\]
is \( f = 0 \) that corresponds to the normal phase.

The superfluid component is assumed to solve the equation

\[
(v_s)_t = -\nabla \phi_s - \mu \nabla \times \nabla \times v_s - f^2 v_s + \nabla u + g, \tag{2.2}
\]

where \( \mu \) is a positive constant, \( g \) is a known function related to the body force and \( \phi_s \) is a suitable scalar function satisfying

\[
\nabla \cdot (f^2 v_s) = -\kappa^2 \gamma f^2 \phi_s. \tag{2.3}
\]

Equations (2.2) and (2.3) are completed by boundary and initial conditions

\[
v_s \cdot n|_{\partial \Omega} = 0, \quad (\nabla \times v_s) \times n|_{\partial \Omega} = \omega, \quad v_s(x, 0) = v_{s0}(x).
\]

We notice that (2.2) and (2.3) are similar to equations governing the motion of the superconducting electrons in the framework of superconductivity [24]. However, in order to account for the thermomechanical effect, the further term \( \nabla u \) enters equation (2.2). Indeed, since \( \nabla u \) has the same sign of the acceleration, an increase of the temperature yields a superfluid flow in the direction of the heat flux. In this model, we assume that the heat flux \( q \) satisfies the Fourier constitutive equation

\[
q = -k(u) \nabla u,
\]

where the thermal conductivity \( k \) depends linearly on the temperature, namely

\[
k(u) = k_0 u, \quad k_0 > 0.
\]

The thermal balance law and the first principle of Thermodynamics lead to the heat equation [12]

\[
c_0 u_t - f f_t = k_0 \Delta u + \nabla \cdot (f^2 v_s) + r, \tag{2.4}
\]

where \( c_0 > 0 \) is related to the specific heat and \( r \) is the heat supply. The temperature is required to verify the boundary and initial conditions

\[
u|_{\partial \Omega} = u_b, \quad u(x, 0) = u_0(x).
\]
The differential system introduced is proved to be compatible with second law of thermodynamics, since the Clausius-Duhem inequality is satisfied ([4]).

The functional setting of the differential problem is more convenient if we introduce a suitable decomposition of the variables \(v_s, \phi_s\), namely

\[
v_s = -A + \frac{1}{\kappa} \nabla \varphi, \quad \phi_s = \nabla \cdot A - \frac{1}{\kappa} \varphi_t,
\]  

(2.5)

where \(A\) and \(\varphi\) satisfy

\[
A \cdot n|_{\partial \Omega} = 0, \quad \nabla \varphi \cdot n|_{\partial \Omega} = 0.
\]  

(2.6)

In addition, by means of the complex valued function

\[
\psi = f e^{i\varphi},
\]

equations (2.1)-(2.4) can be written in the form

\[
\begin{align*}
\gamma \psi_t &= \frac{1}{\kappa^2} \Delta \psi - \frac{2i}{\kappa} A \cdot \nabla \psi - \psi |A|^2 + i \beta (\nabla \cdot A) \psi \\
&\quad - \psi (|\psi|^2 - 1 + u) \quad (2.7) \\
A_t &= \nabla (\nabla \cdot A) - \mu \nabla \times \nabla \times A - |\psi|^2 A + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \\
&\quad - \nabla u - g \\
c_0 u_t &= \frac{1}{2} (\psi_t \bar{\psi} + \psi \bar{\psi}_t) + k_0 \Delta u \\
&\quad + \nabla \cdot \left[ -|\psi|^2 A + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \right] + r \quad (2.8)
\end{align*}
\]

where \(\beta = \kappa \gamma - 1/\kappa\) and \(\bar{\psi}\) denotes the complex conjugate of \(\psi\).

We associate to (2.7)-(2.9) the boundary conditions

\[
\begin{align*}
A \cdot n|_{\partial \Omega} &= 0 \quad (\nabla \times A) \times n|_{\partial \Omega} = -\omega \\
\nabla \psi \cdot n|_{\partial \Omega} &= 0 \quad u|_{\partial \Omega} = u_b
\end{align*}
\]

and initial data

\[
\begin{align*}
\psi(x, 0) &= \psi_0(x) \quad A(x, 0) = A_0(x) \quad u(x, 0) = u_0(x).
\end{align*}
\]  

(2.10)

Furthermore, we assume that \(g, r, \omega, u_b\) are time independent.
2.1 Notation and functional spaces

In order to obtain a precise formulation of the problem, we introduce here some notation and recall the main inequalities used in the sequel.

For each $p \geq 1$ and $s \in \mathbb{R}$, we denote by $L^p(\Omega)$ and $H^s(\Omega)$ the Lebesgue and Sobolev spaces of real valued, complex valued or vector valued functions, according to the context. Let $\| \cdot \|_p$ and $\| \cdot \|_{H^s}$ be the standard norms of $L^p(\Omega)$ and $H^s(\Omega)$, respectively. In particular $\| \cdot \|$ stands for the $L^2(\Omega)$-norm. The space $H^1_0(\Omega)$ is the closure of $C^\infty$ functions with compact support with respect to the norm $\| \cdot \|_{H^1}$.

Finally, we denote by

$$H^1_{0m} = \left\{ w \in H^1(\Omega) : \int_{\Omega} w \, dv = 0 \right\}.$$

Here and henceforth we denote by $C$ any constant depending only on the domain $\Omega$ which is allowed to vary even in the same formula. Further dependencies will be specified.

The Sobolev embedding theorem implies ([1])

$$\begin{align*}
\| w \|_p &\leq C \| w \|_{H^1}, & 1 \leq p \leq 6, & w \in H^1(\Omega), \\
\| w \|_\infty &\leq C \| w \|_{H^2}, & w \in H^2(\Omega).
\end{align*}$$

and the following interpolation inequality holds

$$\| w \|_2^2 \leq C \| w \|_{H^1} \| w \|_{H^2}, \quad w \in H^1(\Omega).$$

(2.13)

If $w \in H^1_0(\Omega)$ or $w \in H^1_{0m}(\Omega)$, Poincaré inequality provides ([10])

$$\| w \| \leq C \| \nabla w \|.$$

(2.14)

Every $w \in H^2(\Omega)$ satisfies

$$\| w \|_{H^2} \leq C( \| w \| + \| \Delta w \|).$$

(2.15)

Furthermore, for every $v \in H^1(\Omega)$, $w \in H^2(\Omega)$ the following interpolation inequality holds ([6])

$$\| vw \|_{H^1} \leq C \| v \|_{H^1} \| w \|_{H^2}.$$  

(2.16)
For vector valued functions we introduce the Hilbert spaces
\[
    \mathcal{H}^1(\Omega) = \{ w \in H^1(\Omega) : w \cdot n|_{\partial \Omega} = 0 \}, \\
    \mathcal{H}^2(\Omega) = \{ w \in H^2(\Omega) : w \cdot n|_{\partial \Omega} = 0, \ (\nabla \times w) \times n|_{\partial \Omega} = 0 \}.
\]

**Lemma 2.1** The spaces \( \mathcal{H}^1(\Omega), \mathcal{H}^2(\Omega) \) are Hilbert spaces with respect to the norms
\[
    \| w \|_{\mathcal{H}^1}^2 = \| \nabla \cdot w \|^2 + \| \nabla \times w \|^2, \\
    \| w \|_{\mathcal{H}^2}^2 = \| \nabla(\nabla \cdot w) \|^2 + \| \nabla \times \nabla \times w \|^2.
\]

In particular, the following estimates hold
\[
    C_1 \| w \|_{\mathcal{H}^1}^2 \leq \| w \|_{\mathcal{H}^1}^2 \leq C_2 \| w \|_{\mathcal{H}^1}^2, \quad w \in \mathcal{H}^1(\Omega) \quad (2.19) \\
    C_3 \| w \|_{\mathcal{H}^2}^2 \leq \| w \|_{\mathcal{H}^2}^2 \leq C_4 \| w \|_{\mathcal{H}^2}^2, \quad w \in \mathcal{H}^2(\Omega) \quad (2.20)
\]

**Proof.** The inequalities (2.19) follow from [19, Prop. 3.2].

Let \( w \in \mathcal{H}^2(\Omega) \). The identity
\[
    \Delta w = \nabla(\nabla \cdot w) - \nabla \times \nabla \times w
\]
and (2.15) yield
\[
    \| w \|_{\mathcal{H}^2}^2 \leq C (\| w \|^2 + \| \nabla(\nabla \cdot w) \|^2 + \| \nabla \times \nabla \times w \|^2). \quad (2.21)
\]

Moreover, by means of (2.11), (2.17) and (2.19), we obtain
\[
    \| w \|^2 \leq \| w \|_{\mathcal{H}^1}^2 \leq C_2 \left( \| \nabla \cdot w \|^2 + \| \nabla \times w \|^2 \right).
\]

The boundary condition \( w \cdot n|_{\partial \Omega} = 0 \) ensures that \( \nabla \cdot w \in H^1_{0m}(\Omega) \). Hence, (2.14) yields
\[
    \| \nabla \cdot w \| \leq C \| \nabla(\nabla \cdot w) \|.
\]

Finally, by applying (2.17) and [19, Prop. 2.2], we prove
\[
    \| \nabla \times w \| \leq \| \nabla \times w \|_{\mathcal{H}^1} \leq C \| \nabla \times \nabla \times w \|.
\]

Substitution into (2.21) leads to (2.20).
3 Well-posedness of the problem

3.1 Existence and uniqueness

In order to deal with homogeneous boundary conditions, we consider the new variables

\[
\hat{A} = A - A_H, \quad \hat{u} = u - u_H,
\]

where \(A_H\) and \(u_H\) are solutions of the problems

\[
\begin{align*}
\nabla \times \nabla \times A_H &= 0 \\
\nabla \cdot A_H &= 0 \\
A_H \cdot n|_{\partial \Omega} &= 0 \\
(\nabla \times A_H) \times n|_{\partial \Omega} &= -\omega
\end{align*}
\]

From the standard theory of linear partial differential equations, it follows that if \(\omega \in H^{1/2}(\partial \Omega), u_b \in H^{1/2}(\partial \Omega)\), then

\[
\|A_H\|_{H^1} \leq C\|\omega\|_{H^{1/2}(\partial \Omega)}, \quad \|u_H\|_{H^1} \leq C\|u_b\|_{H^{1/2}(\partial \Omega)}.
\]

Accordingly, system (2.7)-(2.10) can be written as

\[
\begin{align*}
\gamma \psi_t &= \frac{1}{\kappa^2} \Delta \psi - \frac{2i}{\kappa} (\hat{A} + A_H) \cdot \nabla \psi - \psi |\hat{A} + A_H|^2 + i\beta (\nabla \cdot \hat{A}) \psi \\
&\quad - \psi (|\psi|^2 - 1 + \hat{u} + u_H) \\
\hat{A}_t &= \nabla (\nabla \cdot \hat{A}) - \mu \nabla \times \nabla \times \hat{A} - |\psi|^2 (\hat{A} + A_H) + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \\
&\quad - \nabla \hat{u} - \nabla u_H - g \\
c_0 \hat{u}_t &= \frac{1}{2} (\psi \bar{\psi} + \psi \bar{\psi}_t) + k_0 \Delta \hat{u} \\
&\quad + \nabla \left[ -|\psi|^2 (\hat{A} + A_H) + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \right] + r
\end{align*}
\]

with boundary conditions

\[
\begin{align*}
\hat{A} \cdot n|_{\partial \Omega} &= 0 \quad (\nabla \times \hat{A}) \times n|_{\partial \Omega} = 0 \\
\nabla \psi \cdot n|_{\partial \Omega} &= 0 \quad \hat{u}|_{\partial \Omega} = 0
\end{align*}
\]
and initial data

\[ \psi(x, 0) = \psi_0(x) \quad \hat{A}(x, 0) = \hat{A}_0(x) \quad \hat{u}(x, 0) = \hat{u}_0(x), \]  

(3.6)

where \( \hat{A}_0(x) = A_0(x) - A_H(x) \) and \( \hat{u}_0(x) = u_0(x) - u_H(x) \).

We denote by \( z = (\psi, \hat{A}, \hat{u}) \) and introduce the functional spaces

\[
\begin{align*}
Z^1(\Omega) & = H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega), \\
Z^2(\Omega) & = H^2(\Omega) \times H^2(\Omega) \times H^1_0(\Omega),
\end{align*}
\]

endowed respectively with the norms

\[
\|z(t)\|_{Z^1} = (\|\psi(t)\|_{H^1}^2 + \|\hat{A}(t)\|_{H^1}^2 + \|\hat{u}(t)\|_{H^1}^2)^{1/2},
\]

\[
\|z(t)\|_{Z^2} = (\|\psi(t)\|_{H^2}^2 + \|\hat{A}(t)\|_{H^2}^2 + \|\hat{u}(t)\|_{H^1_0}^2)^{1/2}.
\]

Existence and uniqueness of solutions to problem (3.1)-(3.6) have been shown in [4]. For convenience we recall this result.

**Theorem 3.1** Let \( z_0 = (\psi_0, \hat{A}_0, \hat{u}_0) \in Z^1(\Omega), \ A_H \in H^1(\Omega), \ u_H \in H^1(\Omega), \ g, r \in L^2(\Omega). \) Then, for every \( T > 0, \) there exists a unique solution \( z \) of the problem (3.8)-(3.13) such that

\[
\begin{align*}
\psi & \in L^2(0, T, H^1(\Omega)) \cap H^1(0, T, L^2(\Omega)), \\
\hat{A} & \in L^2(0, T, H^1(\Omega)) \cap H^1(0, T, L^2(\Omega)), \\
\hat{u} & \in L^2(0, T, H^1_0(\Omega)) \cap H^1(0, T, H^{-1}(\Omega)).
\end{align*}
\]

Moreover \( \psi \in L^2(0, T, H^2(\Omega)) \cap C(0, T, H^1(\Omega)), \ \hat{A} \in L^2(0, T, H^2(\Omega)) \cap C(0, T, H^1(\Omega)), \ \hat{u} \in C(0, T, L^2(\Omega)). \)

### 3.2 A-priori estimates

Henceforth we assume that \( g \in H^1(\Omega) \) and

\[
\nabla \cdot g = 0, \quad r = 0.
\]
In particular, since $\Delta u_H = 0$, there exists a vector-valued function $G$ such that

$$\nabla u_H + g = \nabla \times G.$$  

Moreover, $G$ is defined to within the gradient of an arbitrary scalar function. Therefore it is not restrictive to assume the boundary condition

$$G \times n|_{\partial \Omega} = 0. \tag{3.7}$$

With these assumptions, system (3.1)-(3.6) reduces to

\begin{align*}
\gamma \psi_t & = \frac{1}{\kappa^2} \Delta \psi - \frac{2i}{\kappa} (\hat{A} + A_H) \cdot \nabla \psi - \psi |\hat{A} + A_H|^2 + i\beta (\nabla \cdot \hat{A}) \psi \\
& \quad - \psi (|\psi|^2 - 1 + \hat{u} + u_H) \tag{3.8}
\end{align*}

\begin{align*}
\hat{A}_t & = \nabla (\nabla \cdot \hat{A}) - \mu \nabla \times \nabla \times \hat{A} - |\psi|^2 (\hat{A} + A_H) + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \\
& \quad - \nabla \hat{u} - \nabla \times G \tag{3.9}
\end{align*}

\begin{align*}
c_0 \hat{u}_t & = \frac{1}{2} (\psi \bar{\psi} + \psi \bar{\psi}_t) + k_0 \Delta \hat{u} \\
& \quad + \nabla \left[ -|\psi|^2 (\hat{A} + A_H) + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \right] \tag{3.10}
\end{align*}

with boundary conditions

\begin{align*}
\hat{A} \cdot n|_{\partial \Omega} & = 0 \quad (\nabla \times \hat{A}) \times n|_{\partial \Omega} = 0 \tag{3.11}
\end{align*}

\begin{align*}
\nabla \psi \cdot n|_{\partial \Omega} & = 0 \quad \hat{u}|_{\partial \Omega} = 0 \tag{3.12}
\end{align*}

and initial data

$$\psi(x, 0) = \psi_0(x) \quad \hat{A}(x, 0) = \hat{A}_0(x) \quad \hat{u}(x, 0) = \hat{u}_0(x). \tag{3.13}$$

**Proposition 3.1** The solution of (3.8)-(3.13) with initial datum $z_0 \in Z^1(\Omega)$ such that $\|z_0\|_{Z^1} \leq R$, satisfies the following a-priori estimates

\begin{align*}
\sup_{t \geq 0} (\|\psi(t)\|_{H^1} + \|\hat{A}(t)\|_{H^1} + \|\hat{u}(t)\|) & \leq C_R, \tag{3.14}
\end{align*}

\begin{align*}
\sup_{t \geq 0} \int_0^t \left( \|\psi\|^2 + \|\hat{A}\|^2 \right) ds & \leq C_R, \tag{3.15}
\end{align*}

\begin{align*}
\int_0^t \left( \|\psi\|^2_{H^2} + \|\hat{A}\|^2_{H^2} + \|\hat{u}\|^2_{H^2} \right) ds & \leq C_R (1 + t), \quad t > 0, \tag{3.16}
\end{align*}

\begin{align*}
\int_t^{t+1} \left( \|\psi\|^2_{H^2} + \|\hat{A}\|^2_{H^2} + \|\hat{u}\|^2_{H^2} \right) ds & \leq C_R, \quad t > 0, \tag{3.17}
\end{align*}

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where $C_R$ depends increasingly on $R$.

**Proof.** Let

\[
\mathcal{L}(\psi, \dot{A}, \ddot{u}) = \frac{1}{2} \int_\Omega \left\{ \left| \frac{i}{\kappa} \nabla \psi + \psi \dot{A} + \psi A_H \right|^2 + \frac{1}{2} (|\psi|^2 - 1)^2 + |\psi|^2 u_H \\
+ \mu |\nabla \times \dot{A}|^2 + \eta (\nabla \cdot \dot{A})^2 + 2 \nabla \times \mathbf{G} \cdot \dot{A} + c_0 \ddot{u}^2 \right\} dv, \quad (3.18)
\]

where $\eta = 2k_0/(k_0 + 1)$.

Firstly, we show that $\mathcal{L}$ is non-increasing. By differentiating (3.18) with respect to $t$, we obtain

\[
\frac{d\mathcal{L}}{dt} = \int_\Omega \left\{ \frac{1}{2\kappa^2} (\nabla \psi_t \cdot \nabla \bar{\psi} + \nabla \bar{\psi}_t \cdot \nabla \psi) + \frac{i}{2\kappa} (\bar{\psi} \nabla \psi_t - \psi \nabla \bar{\psi}_t) \cdot (\dot{A} + A_H) \\
- \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \cdot (\dot{A} + A_H) - \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \cdot \dot{A}_t \\
+ \frac{1}{2} (\psi \bar{\psi} + \bar{\psi} \psi) |\dot{A} + A_H|^2 - |\psi|^2 |\dot{A}_t| \cdot (\dot{A} + A_H) \\
+ \frac{1}{2} (|\psi|^2 - 1 + u_H)(\psi \bar{\psi}_t + \bar{\psi} \psi_t) + \mu \nabla \times \dot{A} \cdot \nabla \times \dot{A}_t \\
+ \eta (\nabla \cdot \dot{A})(\nabla \cdot \dot{A}_t) + \nabla \times \mathbf{G} \cdot \dot{A}_t + c_0 \ddot{u}_t \right\} dv
\]

By integrating by parts and using (3.11)-(3.12), the terms in the previous expression can be written in the form

\[
\frac{d\mathcal{L}}{dt} = \int_\Omega \left\{ \frac{1}{2} \psi_t \left[ -\frac{1}{\kappa^2} \Delta \bar{\psi} - \frac{2i}{\kappa} \nabla \bar{\psi} \cdot (\dot{A} + A_H) + \bar{\psi} |\dot{A} + A_H|^2 \\
+ (|\psi|^2 - 1 + u_H) \psi \right] + \frac{1}{2} \bar{\psi}_t \left[ -\frac{1}{\kappa^2} \Delta \psi + \frac{2i}{\kappa} \nabla \psi \cdot (\dot{A} + A_H) \\
+ \psi |\dot{A} + A_H|^2 + (|\psi|^2 - 1 + u_H) \psi \right] + \dot{A}_t \left[ -\frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \\
+ |\psi|^2 (\dot{A} + A_H) + \mu \nabla \times \nabla \times \dot{A} + \nabla \times \mathbf{G} \right] \\
- \frac{i}{2\kappa} (\psi_t \bar{\psi} - \bar{\psi}_t \psi) \nabla \cdot \dot{A} + \eta (\nabla \cdot \dot{A})(\nabla \cdot \dot{A}_t) + c_0 \ddot{u}_t \right\} dv.
\]
By substituting (3.8)-(3.10), we obtain

\[
\frac{dL}{dt} = \int_{\Omega} \left\{ -\gamma|\psi_t|^2 - \frac{i}{2\kappa}(1 + \kappa\beta)(\psi_t \bar{\psi} - \bar{\psi}_t \psi)\nabla \cdot \hat{A} - |\hat{A}_t|^2 \\
+ \hat{A}_t \cdot \left[ \nabla(\nabla \cdot \hat{A}) - \nabla \bar{u} \right] - k_0|\nabla \bar{u}|^2 + \eta(\nabla \cdot \hat{A})(\nabla \cdot \hat{A}_t) \\
- \nabla \bar{u} \cdot \left[ -|\psi|^2(\hat{A} + A_H) + \frac{i}{2\kappa}(\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \right] \right\} dv \\
= \int_{\Omega} \left\{ -\gamma|\psi_t|^2 - \frac{i\kappa\gamma}{2}(\psi_t \bar{\psi} - \bar{\psi}_t \psi)\nabla \cdot \hat{A} - |\hat{A}_t|^2 - k_0|\nabla \bar{u}|^2 \\
- \nabla \bar{u} \cdot \left[ 2\hat{A}_t - \nabla(\nabla \cdot \hat{A}) + \nabla \bar{u} \right] - (\eta - 1)\hat{A}_t \cdot \nabla(\nabla \cdot \hat{A}) \right\} dv, \
\]

(3.19)

since \( \beta = \kappa\gamma - 1/\kappa \).

We let

\[
q(\hat{A}_t, \nabla(\nabla \cdot \hat{A}), \nabla \bar{u}) = |\hat{A}_t|^2 + |\nabla(\nabla \cdot \hat{A})|^2 + (k_0 + 1)|\nabla \bar{u}|^2 \\
+ (\eta - 2)\hat{A}_t \cdot \nabla(\nabla \cdot \hat{A}) + 2\hat{A}_t \cdot \nabla \bar{u} - 2\nabla(\nabla \cdot \hat{A}) \cdot \nabla \bar{u}.
\]

A direct check proves that \( q \) is a positive definite quadratic form, since \( \eta = 2k_0/(k_0 + 1) \).

Owing to the identity

\[
|\psi_t|^2 = |\psi_t - i\kappa \psi \nabla \cdot \hat{A}|^2 - \kappa^2|\psi|^2(\nabla \cdot \hat{A})^2 - i\kappa(\psi_t \bar{\psi} - \bar{\psi}_t \psi)\nabla \cdot \hat{A},
\]

equation (3.19) reads

\[
\frac{dL}{dt} = \int_{\Omega} \left\{ -\gamma|\psi_t|^2 - i\kappa\psi \nabla \cdot \hat{A} + \frac{i\kappa\gamma}{2}(\psi_t \bar{\psi} - \bar{\psi}_t \psi)\nabla \cdot \hat{A} + \kappa^2|\psi|^2(\nabla \cdot \hat{A})^2 \\
- q(\hat{A}_t, \nabla(\nabla \cdot \hat{A}), \nabla \bar{u}) - \nabla \bar{u} \cdot \nabla(\nabla \cdot \hat{A}) - \hat{A}_t \cdot \nabla(\nabla \cdot \hat{A}) + |\nabla(\nabla \cdot \hat{A})|^2 \right\} dv.
\]

Taking (3.8) into account, we obtain

\[
\frac{dL}{dt} = \int_{\Omega} \left\{ -\gamma|\psi_t|^2 - i\kappa\psi \nabla \cdot \hat{A} + \frac{i\kappa}{2\kappa}(\bar{\psi}\Delta \psi - \psi\Delta \bar{\psi}) \\
- \frac{2i}{\kappa}(\hat{A} + A_H) \cdot (\psi \nabla \psi + \bar{\psi} \nabla \bar{\psi}) + 2i\beta|\psi|^2\nabla \cdot \hat{A} \right\} dv \\
+ \kappa^2|\psi|^2(\nabla \cdot \hat{A})^2 - q(\hat{A}_t, \nabla(\nabla \cdot \hat{A}), \nabla \bar{u}) \\
- \left[ \nabla \bar{u} + \hat{A}_t - \nabla(\nabla \cdot \hat{A}) \right] \cdot \nabla(\nabla \cdot \hat{A}) \right\} dv
\]
Furthermore, in view of (3.9), the previous equation can be reduced to
\[
\frac{d\mathcal{L}}{dt} = \int_{\Omega} \left\{ -\gamma |\psi_t - i\kappa \psi \nabla \hat{A}|^2 + \left[ \frac{i}{2\kappa}(\hat{\psi}\Delta \psi - \psi \Delta \hat{\psi}) \\
+ (\hat{\mathbf{A}} + \mathbf{A}_H) \cdot (\hat{\psi} \nabla \psi + \psi \nabla \hat{\psi}) + |\psi|^2 \nabla \cdot \hat{\mathbf{A}} \right] \nabla \cdot \hat{\mathbf{A}} \\
- q(\hat{\mathbf{A}}_t, \nabla (\nabla \cdot \hat{\mathbf{A}}), \nabla \hat{u}) \\
+ \left[ |\psi|^2 (\hat{\mathbf{A}} + \mathbf{A}_H) - \frac{i}{2\kappa}(\psi \nabla \hat{\psi} - \hat{\psi} \nabla \psi) \right] \cdot \nabla (\nabla \cdot \hat{\mathbf{A}}) \right\} dv.
\]

The terms involving $\nabla \times \nabla \times \hat{\mathbf{A}}$ and $\nabla \times \mathbf{G}$ vanish by means of an integration by parts owing to (3.11) and (3.7).

Finally, a further integration by parts leads to
\[
\frac{d\mathcal{L}}{dt} = \int_{\Omega} \left[ -\gamma |\psi_t - i\kappa \psi \nabla \hat{\mathbf{A}}|^2 - q(\hat{\mathbf{A}}_t, \nabla (\nabla \cdot \hat{\mathbf{A}}), \nabla \hat{u}) \right] dv \leq 0. \tag{3.20}
\]

Accordingly, $\mathcal{L}$ is non-increasing.

We define
\[
\mathcal{F}_1(\psi, \hat{\mathbf{A}}, \hat{u}) = \left\| \frac{i}{\kappa} \nabla \psi + \psi \hat{\mathbf{A}} + \psi \mathbf{A}_H \right\|^2 + \frac{1}{2} \| |\psi|^2 - 1 \|^2 \\
+ \int_{\Omega} |\psi|^2 u_H dv + \| \hat{\mathbf{A}} \|_{\mathfrak{H}_1}^2 + \| \hat{u} \|^2.
\]

An application of Hölder's and Young's inequality leads to
\[
c_1 \mathcal{F}_1 - c_2 \leq \mathcal{L} \leq c_3 \mathcal{F}_1 + c_2, \tag{3.21}
\]
where $c_1, c_2, c_3$ are suitable positive constants.

Moreover,
\[
\left\| \frac{i}{\kappa} \nabla \psi \right\| \leq \left\| \frac{i}{\kappa} \nabla \psi + \psi \hat{\mathbf{A}} + \psi \mathbf{A}_H \right\| + \|(\hat{\mathbf{A}} + \mathbf{A}_H)\psi\| \\
\leq \left\| \frac{i}{\kappa} \nabla \psi + \psi \hat{\mathbf{A}} + \psi \mathbf{A}_H \right\| + C \left( \|\hat{\mathbf{A}}\|_6 + \|\mathbf{A}_H\|_6 \right) \|\psi\|_3,
\]
so that, by means of (2.11), (2.13) and Young’s inequality, we obtain
\[
\frac{1}{\kappa} \left\| \nabla \psi \right\| \leq \left\| \frac{i}{\kappa} \nabla \psi + \psi \hat{\mathbf{A}} + \psi \mathbf{A}_H \right\| + C \left( \|\hat{\mathbf{A}}\|_{\mathfrak{H}_1}^2 + \|\mathbf{A}_H\|_{\mathfrak{H}_1}^2 + 1 \right) \|\psi\| \\
+ \frac{1}{2\kappa} \|\nabla \psi\|,
\]

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which leads to the estimate
\[
\|\nabla \psi\|^2 \leq C \left[ \left\| \frac{\dot{A}}{\kappa} \nabla \psi + \psi \dot{A} + \psi A_{\mathcal{H}} \right\|^2 + (\| \dot{A} \|^4_{\mathcal{H}^1} + 1) \| \psi \|^2 \right].
\] (3.22)

In addition, Hölder’s inequality yields
\[
\|\psi\|^2 \leq C \|\psi^2\| = C (\|\psi\|^2 - 1 + 1).
\] (3.23)

From the definition of \( F_1 \) and relations (3.22)-(3.23) we deduce
\[
\|z(t)\|^2_{Z^1} \leq C \left[ 1 + F_1(z(t)) + F_2^2(z(t)) + F_3(z(t)) \right].
\] (3.24)

Since \( \mathcal{L}(z(t)) \leq \mathcal{L}(z(0)) \), (3.21) and (3.24) yield (3.14).

By integrating (3.20) with respect to \( t \) we obtain
\[
\int_0^t [\|\dot{A}_t\|^2 + \|\nabla(\nabla \cdot \dot{A})\|^2 + \|\nabla \dot{u}\|^2] ds \leq C_R,
\] (3.25)
\[
\int_0^t \|\psi_t - i\kappa \psi \nabla \cdot \dot{A}\|^2 ds \leq C_R.
\] (3.26)

In view of Hölder’s inequality and the Sobolev embedding theorem, we have
\[
\int_0^t \|\psi_t\|^2 ds \leq 2 \int_0^t \|\psi_t - i\kappa \psi \nabla \cdot \dot{A}\|^2 + \kappa^2 \|\psi \nabla \cdot \dot{A}\|^2 |ds
\leq C_R + C \int_0^t \|\psi\|^2_{\mathcal{H}^1} \|\nabla(\nabla \cdot \dot{A})\|^2 |ds \leq C_R
\]
where the last inequality follows from (3.14) and (3.25). Hence (3.15) holds.

Finally, (3.14), (3.15) and a comparison with (3.8)-(3.9) lead to (3.16). By repeating the same arguments, one can easily prove (3.17).

3.3 Continuous dependence

The following theorem proves the continuous dependence of the solutions to (3.8)-(3.13) on the initial data.
Theorem 3.2 Let $z_i = (\psi_i, \hat{A}_i, \hat{u}_i), i = 1, 2$ be two solutions of (3.8)-(3.13) with data $(A_H, u_H, G) \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ and $z_{0i} = (\psi_{0i}, A_{0i}, u_{0i}) \in Z^1(\Omega), i = 1, 2.$ Then, there exists a constant $C_R$ such that

$$\|z_1(t) - z_2(t)\|_{Z^1}^2 \leq C_R e^{C_R t} \|z_{01} - z_{02}\|_{Z^1}^2.$$  

Moreover, inequality

$$\int_0^t \|z_1(t) - z_2(t)\|_{Z^1}^2 \, ds \leq C(t) \|z_{01} - z_{02}\|_{Z^1}^2$$  

holds, where $C(t)$ is a suitable function depending on $t.$

Proof. We denote by $\psi = \psi_1 - \psi_2, \hat{A} = \hat{A}_1 - \hat{A}_2, \hat{u} = \hat{u}_1 - \hat{u}_2.$ Equations (3.8)-(3.10) lead to

$$\gamma \psi_t - \frac{1}{\kappa^2} \Delta \psi + \frac{2i}{\kappa} [\hat{A} \cdot \nabla \psi + (\hat{A}_2 + A_H) \cdot \nabla \psi] + |\hat{A}_1 + A_H|^2 \psi +$$

$$+ \psi_2 (\hat{A}_1 + A_H) \cdot \hat{A} - i \beta (\psi \nabla \cdot \hat{A}_1 + \psi_2 \nabla \cdot \hat{A}) - \psi$$

$$+ \psi_1 |2 + \psi_2 (\hat{1}_1 \psi + \psi_2 \hat{\psi}) + \psi (\hat{u}_1 + u_H) + \psi_2 \hat{u} = 0$$  

(3.28)

$$\hat{A}_t - \nabla (\nabla \cdot \hat{A}) + \mu \nabla \times \nabla \times \hat{A} + |\psi_1|^2 \hat{A} + (\hat{\psi}_1 \psi + \psi_2 \hat{\psi})(\hat{A}_2 + A_H)$$

$$- \frac{i}{2 \kappa} (\psi \nabla \hat{\psi}_1 - \hat{\psi} \nabla \psi_1 + \psi_2 \nabla \hat{\psi} - \hat{\psi}_2 \nabla \psi) + \nabla \hat{u} = 0$$  

(3.29)

$$c_0 \hat{u}_t - k_0 \hat{D}_u - \frac{1}{2} (\hat{\psi} \nabla \hat{\psi}_1 + \hat{\psi} \nabla \psi_1 + \psi_2 \nabla \hat{\psi} + \hat{\psi}_2 \nabla \psi) - \nabla \cdot \left[ - |\psi_1|^2 \hat{A} \right]$$

$$- (\hat{\psi}_1 \psi + \psi_2 \hat{\psi})(\hat{A}_2 + A_H) + \frac{i}{2 \kappa} (\psi \nabla \hat{\psi}_1 - \hat{\psi} \nabla \psi_1 + \psi_2 \nabla \hat{\psi} - \hat{\psi}_2 \nabla \psi) \right] = 0$$  

(3.30)

Let us multiply (3.28) by $1/2 (\hat{\psi} + \hat{\psi}_t),$ its conjugate by $1/2 (\psi + \psi_t),$ (3.29) by $\hat{A}_t,$ (3.30) by $\hat{u}$ and add the resulting equations. An integration over $\Omega$ yields the equality

$$\frac{1}{2} \frac{d}{dt} \left[ \gamma \|\psi\|^2 + \frac{1}{\kappa^2} \|\nabla \psi\|^2 + \|\nabla \cdot \hat{A}\|^2 + \mu \|\nabla \times \hat{A}\|^2 + c_0 \|\hat{u}\|^2 \right] + \frac{1}{\kappa^2} \|\nabla \psi\|^2$$

$$+ \gamma \|\psi_t\|^2 + \|\hat{A}_t\|^2 + k_0 \|\nabla \hat{u}\|^2 + \int_{\Omega} \left[ |\hat{A}_1 + A_H|^2 \psi|^2 + |\psi_1|^2 |\psi|^2 \right] dv$$

$$= \|\psi\|^2 - \sum_{h=1}^8 I_h,$$  

(3.31)
where

\[
I_1 = i \int_{\Omega} \left[ \hat{\mathbf{A}} \cdot (\bar{\psi} \nabla \psi_1 - \psi \nabla \bar{\psi}_1) + (\hat{\mathbf{A}}_2 + \mathbf{A}_h) \cdot (\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi}) \right] \, dv
\]

\[
I_2 = \frac{1}{2} \int_{\Omega} (\psi_2 \bar{\psi} + \bar{\psi}_2 \psi) \left[ (\hat{\mathbf{A}}_1 + \hat{\mathbf{A}}_2 + 2\mathbf{A}_h) \cdot \hat{\mathbf{A}} + \bar{\psi}_1 \psi + \psi_2 \bar{\psi} + \bar{\psi}_2 \psi \right] \, dv
\]

\[
I_3 = \frac{-i\beta}{2} \int_{\Omega} \left[ (\psi \bar{\psi}_t - \bar{\psi} \psi_t) \nabla \cdot \hat{\mathbf{A}}_1 + (\psi_2 \bar{\psi} - \bar{\psi}_2 \psi + \psi_2 \bar{\psi}_t - \bar{\psi}_2 \psi_t) \nabla \cdot \hat{\mathbf{A}} \right] \, dv
\]

\[
I_4 = \int_{\Omega} \left\{ (\bar{u}_1 + u_h) |\psi|^2 + \frac{1}{2} (\bar{\psi}_t + \psi \bar{\psi}_t) \left[ |\hat{\mathbf{A}}_1 + \mathbf{A}_h|^2 - 1 + |\psi_1|^2 + \bar{u}_1 + u_h \right] \right\} \, dv
\]

\[
I_5 = \frac{i}{\kappa} \int_{\Omega} \left[ \hat{\mathbf{A}} \cdot (\bar{\psi} \nabla \psi_1 - \psi \nabla \bar{\psi}_1) + (\hat{\mathbf{A}}_2 + \mathbf{A}_h) \cdot (\bar{\psi}_t \nabla \psi - \psi \nabla \bar{\psi}) \right] \, dv
\]

\[
I_6 = \frac{1}{2} \int_{\Omega} \left\{ (\psi_2 \bar{\psi}_t + \bar{\psi}_2 \psi_t) (\hat{\mathbf{A}}_1 + \hat{\mathbf{A}}_2 + 2\mathbf{A}_h) \cdot \hat{\mathbf{A}} \right. \\
\left. + (\psi_2 \bar{\psi}_t + \bar{\psi}_2 \psi_t) (\bar{\psi}_1 \psi + \psi_2 \bar{\psi}) - (\bar{\psi}_2 \psi + \bar{\psi}_1 \psi) \bar{u} \right\} \, dv
\]

\[
I_7 = \int_{\Omega} \left[ |\psi_1|^2 \hat{\mathbf{A}} + (\psi_1 \bar{\psi} + \psi_2 \bar{\psi}) (\hat{\mathbf{A}}_2 + \mathbf{A}_h) + (\bar{\psi}_t \nabla \psi - \psi \nabla \bar{\psi}) \right] \cdot (\nabla \bar{u} - \hat{\mathbf{A}}_t) \, dv
\]

\[
I_8 = \int_{\Omega} \left[ \frac{i}{2\kappa} (\psi \bar{\nabla} \psi_1 - \bar{\psi} \nabla \psi_1 + \psi_2 \nabla \bar{\psi} - \bar{\psi}_2 \nabla \psi) \cdot (\nabla \bar{u} - \hat{\mathbf{A}}_t) + \nabla \bar{u} \cdot \hat{\mathbf{A}}_t \right] \, dv.
\]

By recalling that the solution of (3.8)-(3.10) satisfies the a-priori estimate (3.14), the previous integrals can be estimated by means of the Hölder’s and Young’s inequalities the Sobolev embedding theorem as

\[
\sum_{k=1}^{8} I_k \leq \varphi_1 \|\psi\|_{H^1}^2 + \varphi_2 \|\hat{\mathbf{A}}\|_{H^1}^2 + C\|\bar{u}\|^2 + \frac{1}{2} (\gamma \|\psi_t\|^2 + k_0 \|\nabla \bar{u}\|^2 + \|\hat{\mathbf{A}}_t\|^2),
\]

(3.32)

where

\[
\varphi_1 = C_R (1 + \|\psi_1\|_{H^1}^2 + \|\psi_2\|_{H^1}^2 + \|\hat{\mathbf{A}}_1\|_{H^2}^2 + \|\hat{\mathbf{A}}_2\|_{H^2}^2 + \|\hat{\mathbf{A}}_{1t}\|_{H^1}^2 + \|\hat{\mathbf{A}}_{2t}\|_{H^1}^2 + \|\psi_1\|_{H^2}^2)
\]

\[
\varphi_2 = C_R (1 + \|\psi_1\|_{H^2}^2 + \|\psi_2\|_{H^2}^2).
\]

Substitution into (3.31) yields the inequality

\[
\frac{1}{2} \frac{d}{dt} \left[ \gamma \|\psi\|^2 + \frac{1}{\kappa} \|\nabla \psi\|^2 + \|\nabla \cdot \hat{\mathbf{A}}\|^2 + \mu \|\nabla \times \hat{\mathbf{A}}\|^2 + c_0 \|\bar{u}\|^2 \right] \leq \varphi_1 \|\psi\|_{H^1}^2 + \varphi_2 \|\hat{\mathbf{A}}\|_{H^1}^2 + C\|\bar{u}\|^2.
\]

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In view of (3.15), (3.16), Gronwall’s inequality leads to
\[
\| z_1(t) - z_2(t) \|^2_{Z_1} \leq C R e^{C R t} \| z_{01} - z_{02} \|^2_{Z_1}.
\] (3.33)

Now we prove inequality (3.27). We substitute (3.32) into (3.31) and integrate over \( t \), thus obtaining
\[
\int_0^t \left( \gamma \| \psi_t \|^2 + \| \hat{A}_t \|^2 + k_0 \| \nabla \hat{u} \|^2 \right) ds \\
\leq \int_0^t \left( \varphi_1 \| \psi \|^2_{H^1} + \varphi_2 \| \hat{A} \|^2_{H^1} + C \| \hat{u} \|^2 \right) ds + C \| z_{01} - z_{02} \|^2_{Z_1}.
\]

A-priori estimates (3.14), (3.15) and inequality (3.33) provide
\[
\int_0^t \left( \| \Delta \psi \|^2 ds \leq C(t) \| z_{01} - z_{02} \|^2_{Z_1}.
\]

From (3.28) we obtain the estimate of \( \| \Delta \psi \| \) by means of Hölder’s inequality, (3.14)-(3.16) and (3.33), namely
\[
\int_0^t \| \Delta \psi \|^2 ds \leq C(t) \| z_{01} - z_{02} \|^2_{Z_1}.
\]

Likewise, multiplying (3.29) by \( \nabla \times \nabla \times \hat{A} \) and integrating over \( \Omega \), we deduce
\[
\| \nabla \times \nabla \times \hat{A} \|^2 \leq C \| \nabla \times \nabla \times \hat{A} \| \| \hat{A}_t \| + C(t) \| \nabla \times \nabla \times \hat{A} \| \| z_{01} - z_{02} \|^2_{Z_1},
\]
which implies
\[
\int_0^t \| \nabla \times \nabla \times \hat{A} \|^2 ds \leq C(t) \| z_{01} - z_{02} \|^2_{Z_1}.
\]

Finally, by comparison with (3.29) we reach the conclusion. \( \square \)

Theorems 3.1 and 3.2 ensure that there exists a unique solution of problem (3.8)-(3.13) depending continuously on the initial data. In other words, (3.8)-(3.13) generate a strongly continuous semigroup \( S(t) \) on the phase space \( Z^1(\Omega) \) (see e.g. [22]).

4 The global attractor

This section is devoted to prove existence of the global attractor for the semigroup \( S(t) \). For reader’s convenience, we recall its definition.
Definition 4.1 The global attractor $\mathcal{A} \subset Z^1(\Omega)$ is the unique compact set enjoying the following properties:

(i) $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$;

(ii) $\lim_{t \to \infty} \text{dist}_{Z^1}(S(t)\mathcal{B}, \mathcal{A}) = 0$ for every bounded set $\mathcal{B} \subset Z^1(\Omega)$, where $\text{dist}_{Z^1}$ denotes the usual Hausdorff semidistance in $Z^1(\Omega)$.

Usually, existence of the global attractor is established by showing that the semigroup admits a bounded absorbing set and that the operators $S(t)$ are uniformly compact for large values of $t$ ([22, Theor. 1.1]). However, we are unable to obtain directly the estimates that guarantee the dissipativity of the semigroup. This prevent us from proving existence of an absorbing set. Thus we deduce that $S(t)$ possesses a global attractor by means of a Lyapunov functional which leads to existence of a bounded absorbing set as a consequence.

We denote by $\mathcal{S}$ the set of stationary solutions of problem (3.8)-(3.12). In other words, every steady solution satisfies the equations

$$
0 = \frac{1}{\kappa^2} \Delta \psi - \frac{2i}{\kappa} (\hat{A} + A_H) \cdot \nabla \psi - \psi |\hat{A} + A_H|^2 + i \beta (\nabla \cdot \hat{A}) \psi - \psi (|\psi|^2 - 1 + \hat{u} + u_H) \quad (4.1)
$$

$$
0 = \nabla (\nabla \cdot \hat{A}) - \mu \nabla \times \nabla \times \hat{A} - |\psi|^2 (\hat{A} + A_H) + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) - \nabla \hat{u} - \nabla \times G \quad (4.2)
$$

$$
0 = k_0 \Delta \hat{u} + \nabla \cdot \left[ -|\psi|^2 (\hat{A} + A_H) + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \right] \quad (4.3)
$$

Definition 4.2 A continuous function $\mathcal{L} : Z^1(\Omega) \to \mathbb{R}$ is said a Lyapunov functional if

(i) $t \to \mathcal{L}(S(t)z)$ is non-increasing for any $z \in Z^1(\Omega)$;

(ii) $\mathcal{L}(z) \to \infty \Leftrightarrow \|z\|_{Z^1} \to \infty$;

(iii) $\mathcal{L}(S(t)z) = \mathcal{L}(z)$, $\forall t > 0 \Rightarrow z \in \mathcal{S}$.
In order to prove the existence of the global attractor, we will exploit the following result (see e.g. [2, 16]).

**Theorem 4.1** Let the semigroup \( S(t), t > 0 \) satisfy the following conditions:

(a) \( S(t) \) admits a continuous Lyapunov functional \( \mathcal{L} \);

(b) the set \( \mathcal{S} \) of the stationary solutions is bounded in \( Z^1(\Omega) \);

(c) for any bounded set \( B \subset Z^1(\Omega) \), there exists a compact set \( K_B \subset Z^1(\Omega) \) such that \( S(t)B \subset K_B, t > 0 \).

Then, \( S(t) \) possesses a connected global attractor \( \mathcal{A} \) which coincides with the unstable manifold of \( S \), namely

\[
\mathcal{A} = \{ z \in Z^1(\Omega) : z \text{ belongs to a complete trajectory } S(t)z, t \in \mathbb{R}, \lim_{t \to -\infty} \text{dist}_{Z^1}(S(t)z, \mathcal{S}) = 0 \}.
\]

The next subsections will be devoted to the proof of conditions (a), (b), (c).

### 4.1 Lyapunov functional

**Proposition 4.1** The function

\[
\mathcal{L}(\psi, \hat{A}, \hat{u}) = \frac{1}{2} \int_{\Omega} \left\{ \left| \frac{i}{k} \nabla \psi + \psi \hat{A} + \psi A_H \right|^2 + \frac{1}{2}(|\psi|^2 - 1)^2 + |\psi|^2 u_H \right. \\
+ \mu |\nabla \times \hat{A}|^2 + \eta (\nabla \cdot \hat{A})^2 + 2 \nabla \cdot G \cdot \hat{A} + c_0 \hat{u}^2 \left. \right\} dv,
\]

where \( \eta = 2k_0/(k_0 + 1) \), is a Lyapunov functional.

**Proof.** The non-increasing character of \( \mathcal{L} \) has been proved in proposition 3.1. Moreover, the inequalities

\[
c_1 \mathcal{F}_1 - c_2 \leq \mathcal{L} \leq c_3 \mathcal{F}_1 + c_2
\]

\[
\|z\|_{Z^1}^2 \leq C \left[ 1 + \mathcal{F}_1(z) + \mathcal{F}_1^2(z) + \mathcal{F}_1^3(z) \right]
\]
hold. With similar arguments one can show that
\[ \mathcal{F}_1(z(t)) \leq C(1 + \|z(t)\|_{Z^1}^2 + \|z(t)\|_{Z^1}^4). \]
Hence, we deduce that
\[ \mathcal{F}_1(z) \to \infty \iff \mathcal{L}(z) \to \infty \iff \|z\|_{Z^1} \to \infty. \]

Finally, we show (iii). We suppose that \( \mathcal{L}(S(t)z) = \mathcal{L}(z) \) for every \( t > 0 \). Then, from (3.20) and the positive definiteness of \( q \) we deduce that
\[ \psi_t - i\kappa \psi \nabla \cdot \hat{A} = 0 \quad (4.4) \]
\[ \nabla (\nabla \cdot \hat{A}) = 0 \quad (4.5) \]
\[ \nabla \hat{u} = 0 \quad (4.6) \]
\[ \hat{A}_t = 0 \quad (4.7) \]
In particular, (4.5) guarantees that there exists a constant \( c \) such that \( \nabla \cdot \hat{A} = c \). Since \( \nabla \cdot \hat{A} \in H^1_{\text{loc}}(\Omega) \), we have
\[ \nabla \cdot \hat{A} = 0, \]
which, in view of (4.4) implies
\[ \psi_t = 0. \quad (4.8) \]
Finally, by substituting the previous relations into (3.9), (3.10), we obtain
\[ \hat{u}_t = 0. \]
Thus, \( z_t = 0 \), namely \( z \in S \).

### 4.2 Stationary solutions

**Proposition 4.2** The set of stationary solutions is bounded in \( Z^1(\Omega) \), namely there exists \( R > 0 \) such that
\[ \|z\|_{Z^1} \leq R, \]
for every \( z \in S \).
Proof. Let $z \in S$. Then, $\frac{dC}{dt} = 0$ and hence

$$\nabla \hat{u} = 0, \quad \nabla (\nabla \cdot \hat{A}) = 0.$$ 

In particular, the boundary conditions (3.11) and (3.12) lead to $\hat{u} = 0$ and $\nabla \cdot \hat{A} = 0$. By substituting into (4.1)-(4.2) we obtain

$$\frac{1}{\kappa^2} \Delta \psi - \frac{2i}{\kappa}(\hat{A} + A_H) \cdot \nabla \psi - \psi|\hat{A} + A_H|^2 - \psi(|\psi|^2 - 1 + u_H) = 0 \quad (4.9)$$

$$\mu \nabla \times \nabla \times \hat{A} + |\psi|^2(\hat{A} + A_H) - \frac{i}{2\kappa}(\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) + \nabla \times G = 0 \quad (4.10)$$

By multiplying (4.9) by $1/2\bar{\psi}$, its conjugate by $1/2\psi$ and integrating over $\Omega$ we obtain

$$\left\| \frac{i}{\kappa} \nabla \psi + (\hat{A} + A_H)\bar{\psi} \right\|^2 + \|\psi\|^4 = \|\psi\|^2 - \int_\Omega |\psi|^2 u_H dv;$$

Hölder’s inequality yields

$$\|\psi\|^2 \leq \varepsilon \|\psi\|^4 + C, \quad (4.11)$$

where $\varepsilon > 0$ is a suitable (small) constant. Therefore, we have

$$\|\psi\|^4 \leq C, \quad (4.12)$$

$$\left\| \frac{i}{\kappa} \nabla \psi + (\hat{A} + A_H)\bar{\psi} \right\| \leq C. \quad (4.13)$$

We multiply (4.10) by $\hat{A}$ and we integrate over $\Omega$, thus obtaining

$$\mu \|\nabla \times \hat{A}\|^2 = -\int_\Omega \left[ |\psi|^2(\hat{A} + A_H) - \frac{i}{2\kappa}(\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) + \nabla \times G \right] \cdot \hat{A} dv$$

$$= -\int_\Omega \left[ \frac{1}{2} \left[ \frac{i}{\kappa} \nabla \psi + (\hat{A} + A_H)\bar{\psi} \right] \bar{\psi} \right.$$

$$\left. + \frac{1}{2} \left[ -\frac{i}{\kappa} \nabla \bar{\psi} + (\hat{A} + A_H)\bar{\psi} \right] \psi + \nabla \times G \right] \cdot \hat{A} dv$$

$$\leq \left\| \frac{i}{\kappa} \nabla \psi + (\hat{A} + A_H)\bar{\psi} \right\| \|\psi\| + \|\nabla \times G\| \|\hat{A}\|$$

Hence thanks to (4.12) and (4.13) we deduce

$$\mu \|\nabla \times \hat{A}\|^2 \leq C\|\hat{A}\|_4 + \|\nabla \times G\| \|\hat{A}\| \leq C\|\hat{A}\|_{H^1} = C\|\nabla \times \hat{A}\|,$$

where last identity holds since $\nabla \cdot \hat{A} = 0$. Thus,

$$\|\hat{A}\|_{H^1} \leq C. \quad (4.14)$$
Finally, in view of the inequalities (4.11)-(4.14) we obtain

\[ \| \psi \| \leq C \]
\[ \frac{1}{\kappa} \| \nabla \psi \| \leq \left| \frac{\dot{\psi}}{\kappa} \nabla \psi + (\dot{\psi} + \mathbf{A}_\mathbf{H}) \psi \right| + \left| \mathbf{A} + \mathbf{A}_\mathbf{H} \right| \| \psi \|_4 \leq C, \]

namely

\[ \| \psi \|_{H^1} \leq C. \]

This concludes the proof.

4.3 Existence of the global attractor

Existence of the global attractor for the semigroup \( S(t) \) is established once we prove condition (c) of Theorem 4.1.

Proposition 4.3 Let \( S(t)z, \; t > 0, \) be a solution to problem (3.8)-(3.13) with initial datum \( z \in Z^1(\Omega) \) such that \( \|z\|_{Z^1} \leq R \). Then, \( S(t)z, \; t > 0, \) belong to a compact set \( \mathcal{K} \subset Z^1(\Omega) \).

Proof. In view of the compact embedding \( Z^2(\Omega) \hookrightarrow Z^1(\Omega) \), our goal consists in proving the existence of a positive constant \( C_R \) depending on \( R \) and \( \mathbf{A}_\mathbf{H}, u_\mathbf{H}, \mathbf{G} \) such that

\[ \|z(t)\|_{Z^2} \leq C_R. \] \hspace{1cm} (4.15)

Let us multiply (3.8) by \( 1/2 \Delta \bar{\psi}_t \) and its conjugate by \( 1/2 \Delta \psi_t \). Adding the resulting equations and integrating over \( \Omega \), thanks to the boundary condition (3.12)\(_1\), we obtain

\[ \frac{1}{2\kappa^2} \frac{d}{dt} \| \Delta \psi \|^2 + \gamma \| \nabla \psi_t \|^2 = \sum_{h=1}^4 J_h, \]
where

\begin{align*}
J_1 &= \frac{i}{\kappa} \int_\Omega (\hat{A} + A_H) \cdot (\nabla \psi \Delta \bar{\psi}_t - \nabla \bar{\psi} \Delta \psi_t) \, dv \\
J_2 &= \frac{1}{2} \int_\Omega |\hat{A} + A_H|^2 (\bar{\psi} \Delta \psi_t + \psi \Delta \bar{\psi}_t) \, dv \\
J_3 &= \frac{i\beta}{2} \int_\Omega \nabla \cdot (\bar{\psi} \Delta \psi_t - \psi \Delta \bar{\psi}_t) \, dv \\
J_4 &= \frac{1}{2} \int_\Omega (|\psi|^2 - 1 + \bar{u} + u_H)(\bar{\psi} \Delta \psi_t + \psi \Delta \bar{\psi}_t) \, dv.
\end{align*}

An integration by parts leads to

\[ |J_1| \leq C \int_\Omega \left[ |\nabla (\hat{A} + A_H)| |\nabla \psi| + |\nabla \psi| |\hat{A} + A_H| \right] |\nabla \psi_t| \, dv. \]

Hölder’s and Young’s inequalities and (2.12) yield

\begin{align*}
|J_1| &\leq C \|\nabla \psi_t\| \left[ \|\hat{A} + A_H\|_{\mathcal{H}^2} \|\psi\|_{\mathcal{H}^2} + \|\hat{A} + A_H\|_{\mathcal{H}^2} \|\psi\|_{\mathcal{H}^2} \right] \\
&\leq \varepsilon \|\nabla \psi_t\|^2 + \frac{C}{4\varepsilon} \|\hat{A} + A_H\|_{\mathcal{H}^2}^2 \|\psi\|_{\mathcal{H}^2}^2.
\end{align*}

for any \( \varepsilon > 0 \).

Now we consider \( J_2 \). We obtain

\begin{align*}
|J_2| &\leq \int_\Omega |\nabla \psi_t| \left[ 2|\nabla (\hat{A} + A_H)| |\hat{A} + A_H| |\psi| + |\hat{A} + A_H|^2 |\nabla \psi| \right] \, dv \\
&\leq C \|\nabla \psi_t\| \left[ \|\hat{A} + A_H\|_{\mathcal{H}^2} \|\psi\|_{\mathcal{H}^2} + \|\hat{A} + A_H\|_{\mathcal{H}^2} \|\psi\|_{\mathcal{H}^2} \right].
\end{align*}

The assumption \( \|z\|_{z^1} \leq R \) together with (3.14) give

\begin{align*}
|J_2| &\leq C_R \|\nabla \psi_t\| \left[ \|\hat{A} + A_H\|_{\mathcal{H}^2} \|\psi\|_{\mathcal{H}^2} + \|\hat{A} + A_H\|_{\mathcal{H}^2}^2 \right] \\
&\leq \varepsilon \|\nabla \psi_t\|^2 + \frac{C_R}{4\varepsilon} \|\hat{A} + A_H\|_{\mathcal{H}^2}^2 (\|\psi\|_{\mathcal{H}^2}^2 + \|\hat{A} + A_H\|_{\mathcal{H}^2}^2).
\end{align*}

Similarly, we have

\begin{align*}
|J_3| &\leq \varepsilon \|\nabla \psi_t\|^2 + \frac{C}{4\varepsilon} \|\hat{A} + A_H\|_{\mathcal{H}^2}^2 \|\psi\|_{\mathcal{H}^2}^2 \\
|J_4| &\leq \varepsilon \|\nabla \psi_t\|^2 + \frac{C_R}{4\varepsilon} (\|\psi\|_{\mathcal{H}^2}^4 + \|\psi\|_{\mathcal{H}^2}^2 \|\bar{u}\|_{\mathcal{H}^2}^2 + 1).
\end{align*}
Therefore,

\[
\frac{1}{2\kappa^2} \frac{d}{dt} \|\Delta \psi\| + \gamma \|\nabla \psi_t\|^2 \leq 4\varepsilon \|\nabla \psi_t\|^2 \\
+ C_R \left[ \|\hat{A} + A_H\|^2_{L^2} \|\psi\|^2_{H^2} + \|\hat{A} + A_H\|^4_{H^2} + \|\psi\|^4_{H^2} + \|\hat{A}\|^2_{H^2} + 1 \right].
\]

(4.16)

Let us multiply (3.9) by \(\nabla \times \nabla \times \hat{A}_t\). Keeping (3.11) into account, an integration over \(\Omega\) provides

\[
\frac{\mu}{2} \frac{d}{dt} \|\nabla \times \nabla \times \hat{A}\|^2 + \|\nabla \times \hat{A}_t\|^2 = \sum_{h=1}^3 L_h,
\]

where

\[
L_1 = -\int_\Omega \nabla \times [\|\psi\|^2(\hat{A} + A_H)] \cdot \nabla \times \hat{A}_t \, dv \\
L_2 = \frac{i}{2\kappa} \int_\Omega \nabla \times (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \cdot \nabla \times \hat{A}_t \, dv \\
L_3 = -\int_\Omega \nabla \times \nabla \times G \cdot \nabla \times \hat{A}_t \, dv.
\]

Thus, we obtain

\[
|L_1| \leq \varepsilon \|\nabla \times \hat{A}_t\|^2 + \frac{C_R}{4\varepsilon} \left[ \|\psi\|^2_{H^2} + \|\hat{A} + A_H\|^2_{L^2} \right] \\
|L_2| \leq \varepsilon \|\nabla \times \hat{A}_t\|^2 + \frac{C}{4\varepsilon} \|\psi\|^2_{H^2} \\
|L_3| \leq \varepsilon \|\nabla \times \hat{A}_t\|^2 + \frac{1}{4\varepsilon} \|\nabla \times \nabla \times G\|^2.
\]

Therefore,

\[
\frac{\mu}{2} \frac{d}{dt} \|\nabla \times \nabla \times \hat{A}\|^2 + \|\nabla \times \hat{A}_t\|^2 \leq 3\varepsilon \|\nabla \times \hat{A}_t\|^2 + \frac{C_R}{4\varepsilon} \left( \|\psi\|^2_{H^2} + \|\hat{A} + A_H\|^2_{H^2} + 1 \right).
\]

(4.17)

Let us multiply (3.9) by \(\nabla (\nabla \cdot \hat{A}_t) - \nabla \hat{u}_t\). An integration by parts and boundary conditions (3.11)-(3.12) lead to

\[
\frac{1}{2} \frac{d}{dt} \|\nabla (\nabla \cdot \hat{A}) - \nabla \hat{u} - \nabla \times G\|^2 + \|\nabla \cdot \hat{A}_t\|^2 = \sum_{h=1}^4 M_h
\]

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with

\[ M_1 = \int \hat{u}_t \nabla \cdot \hat{A}_t \, dv \]
\[ M_2 = -\int \left[ \nabla (|\psi|^2) \cdot (\hat{A} + A_H) + |\psi|^2 \nabla \cdot \hat{A} \right] \nabla \cdot \hat{A}_t \, dv \]
\[ M_3 = \frac{i}{2\kappa} \int (\psi \Delta \bar{\psi} - \bar{\psi} \Delta \psi) \nabla \cdot \hat{A}_t \, dv \]
\[ M_4 = \int \left[ \nabla (|\psi|^2) \cdot (\hat{A} + A_H) + |\psi|^2 \nabla \cdot \hat{A} - \frac{i}{2\kappa} (\psi \Delta \bar{\psi} - \bar{\psi} \Delta \psi) \right] \hat{u}_t \, dv. \]

We obtain

\[ |M_1| \leq \varepsilon \|\nabla \cdot \hat{A}_t\|^2 + \frac{1}{4\varepsilon} \|\hat{u}_t\|^2 \]
\[ |M_2| \leq \varepsilon \|\nabla \cdot \hat{A}_t\|^2 + \frac{C_R}{4\varepsilon} \left[ \|\hat{A} + A_H\|^2_{H^2} + \|\psi\|^2_{H^2} \right] \]
\[ |M_3| \leq \varepsilon \|\nabla \cdot \hat{A}_t\|^2 + \frac{C}{4\varepsilon} \|\psi\|^4_{H^2} \]
\[ |M_4| \leq \varepsilon \|\hat{u}_t\|^2 + \frac{C_R}{4\varepsilon} \left[ \|\hat{A} + A_H\|^2_{H^2} + \|\psi\|^4_{H^2} \right]. \]

Therefore, we have

\[ \frac{1}{2} \frac{d}{dt} \| \nabla (\nabla \cdot \hat{A}) - \nabla \hat{u} - \nabla \times G \|^2 + \| \nabla \cdot \hat{A}_t \|^2 \leq \varepsilon \|\nabla \cdot \hat{A}_t\|^2 \]
\[ + \left( \varepsilon + \frac{1}{4\varepsilon} \right) \|\hat{u}_t\|^2 + C_R \left[ 1 + \|\hat{A} + A_H\|^2_{H^2} + \|\psi\|^4_{H^2} \right]. \] (4.18)

Let us multiply (3.10) by \( \hat{u}_t \) and integrate over \( \hat{A}_t \) and \( \hat{u}_t \).
Since $N_2 = -M_4$, we deduce that

$$
\frac{k_0}{2} \frac{d}{dt} \| \nabla \hat{u} \|^2 + c_0 \| \hat{u}_t \|^2 \leq 2\varepsilon \| \hat{u}_t \|^2 + C_R \left[ \| \psi \|_{H^2}^2 \| \psi_t \|^2 + \| \psi \|_{H^2}^4 \| \hat{A} + \mathbf{A}_H \|^2 \right].
$$

(4.19)

We multiply (4.19) by $1/(2\varepsilon c_0)$. By adding the resulting inequality with (4.16)-(4.18), we obtain

$$
\frac{d}{dt} \left[ \frac{1}{\kappa^2} \| \nabla \psi \|^2 + \frac{k_0}{2\varepsilon c_0} \| \nabla \hat{u} \|^2 \right] + (\gamma - 4\varepsilon) \| \nabla \psi_t \|^2 + (1 - 3\varepsilon) \| \nabla \hat{A}_t \|^2

\leq C_R \left[ 1 + \| \hat{A} + \mathbf{A}_H \|_{H^2}^2 \| \psi \|_{H^2}^2 + \| \hat{A} + \mathbf{A}_H \|_{H^2}^4 \| \psi \|_{H^2}^4 + \| \psi \|_{H^2}^4 \| \hat{u} \|_{H^2}^2 \right.

\left. + \| \psi \|_{H^2}^4 \| \psi_t \|^2 \right].
$$

(4.20)

We choose

$$
\varepsilon = \frac{1}{2} \min \left( \frac{\gamma}{4}, \frac{1}{3}, \frac{\sqrt{1 + c_0^2} - 1}{2c_0} \right)
$$

and we let

$$
\mathcal{F}_2 = \frac{1}{\kappa^2} \| \nabla \psi \|^2 + \frac{k_0}{2\varepsilon c_0} \| \nabla \hat{u} \|^2

\xi(t) = C_R \left[ 1 + \| \hat{A} + \mathbf{A}_H \|_{H^2}^2 \| \psi \|_{H^2}^2 + \| \hat{A} + \mathbf{A}_H \|_{H^2}^4 \| \psi \|_{H^2}^4 \right].
$$

Therefore, we have

$$
\frac{d}{dt} \mathcal{F}_2 \leq \xi(t) \mathcal{F}_2 + \xi(t).
$$

(4.21)

A-priori estimates (3.14)-(3.16) and Gronwall’s uniform lemma ([22]) guarantee that $\mathcal{F}_2$ is bounded.

Thus, $\| z(t) \|_{Z^2} < C_R$. □

**Remark 4.1** By comparison with (3.8), on account of the Hölder’s inequality and
the Sobolev embedding theorem, from (4.15) we prove

\[
\|\psi_t\| \leq C(\|\Delta \psi\| + \|\hat{\mathbf{A}} + \mathbf{A}_H\|_{\mathcal{H}^1} \|\psi\|_{H^2} + \|\hat{\mathbf{A}} + \mathbf{A}_H\|_{\mathcal{H}^1}^2 \|\psi\|_{H^1} + \|\psi\|_{H^1}^3 + \|\hat{\psi} + u_H\|_{H^2}) \leq C_R.
\]

Propositions 4.1-4.3 allow to apply Theorem (4.1) and to prove existence of the global attractor. As a consequence ([7]), \(S(t)\) possesses a bounded absorbing set \(B_1 \subset \mathcal{Z}^1(\Omega)\) of radius

\[ R_1 = 1 + \sup\{\|z\|_{\mathcal{Z}^1}, \mathcal{L}(z) \leq K\}, \]

where \(K = 1 + \sup_{z \in S} \mathcal{L}(z)\).

**Corollary 4.1** The semigroup \(S(t)\) possesses a bounded absorbing set \(B_2 \in \mathcal{Z}^2(\Omega)\) of radius \(R_2\).

**Proof.** Let \(z \in \mathcal{Z}^2(\Omega)\) with \(\|z\|_{\mathcal{Z}^2} \leq R\). Then there exists \(t_1 = t_1(R) > 0\) such that

\[ S(t)z \in B_1, \quad t \geq t_1, \]

so that

\[ \|S(t)z\|_{\mathcal{Z}^1} \leq R_1, \quad t \geq t_1. \]

Inequality (4.15) implies

\[ \|S(t)z\|_{\mathcal{Z}^2} = \|S(t-t_1)S(t_1)z\|_{\mathcal{Z}^2} \leq C_{R_1}, \quad t \geq t_1. \]

If \(t < t_1\), the same inequality (4.15) leads to

\[ \|S(t)z\|_{\mathcal{Z}^2} \leq C_{Re^{t_1-t}}. \]

Therefore, we obtain

\[ \|S(t)z\|_{\mathcal{Z}^2} \leq C_{R_1} + C_{Re^{t_1-t}}, \quad t > 0. \]
By choosing $R_2 = 2C_{R_1}$ and $t_2 = \max \{ t_1 - \ln(C_{R_1}/C_R), 0 \}$, we prove

$$\|S(t)z\|_{Z^2} \leq R_2, \quad t > t_2.$$ 

\[\Box\]

5 Exponential attractors

In this section, we prove the existence of a regular exponential attractor $E$ for the semigroup $S(t)$, namely, a compact set of finite fractal dimension that exponentially attracts every bounded set in $Z^2(\Omega)$. Since the global attractor $A$ is the minimal compact attracting set, we have $A \subset E$. Accordingly, $A$ has finite fractal dimension.

We first recall the definition of the exponential attractor

**Definition 5.1** A compact subset $E \subset Z^2(\Omega)$ of finite fractal dimension is an exponential attractor for the semigroup $S(t)$ if

(i) $E$ is positively invariant, i.e. $S(t)E \subset E$ for all $t \geq 0$;

(ii) there exist $\omega > 0$ and a positive increasing function $J$ such that

$$\text{dist}_{Z^1}(S(t)B,E) \leq J(R)e^{-\omega t} \quad (5.1)$$

for any bounded $B \subset Z^1(\Omega)$ with $R = \sup\{\|z\|_{Z^1(\Omega)}, z \in B\}$.

The existence of an exponential attractor for the semigroup $S(t)$ is based on the following abstract result proved in [15].

**Lemma 5.1** Let $\mathcal{K}$ a bounded subset of $Z^2(\Omega)$, such that $S(t)\mathcal{K} \subset \mathcal{K}$ for each $t > t^*$. Suppose that

(i) the map

$$\Phi : [t^*, 2t^*] \times \mathcal{K} \rightarrow \mathcal{K}$$

$$(t, z) \mapsto S(t)z$$
is $1/2$-H"older continuous in time and Lipschitz continuous in the initial data, when $\mathcal{K}$ is endowed with the $Z^1(\Omega)$-topology;

(ii) there exist $\lambda \in (0, 1/2)$ and $\Lambda > 0$ such that

$$S(t^*) = L + K,$$

where

$$\|L(z_1) - L(z_2)\|_{Z^1} \leq \lambda\|z_1 - z_2\|_{Z^1},$$

$$\|K(z_1) - K(z_2)\|_{Z^2} \leq \Lambda\|z_1 - z_2\|_{Z^1}, \quad z_1, z_2 \in \mathcal{K}.$$

Then, there exists a set $\mathcal{E} \subset \mathcal{K}$, closed and of finite fractal dimension in $Z^1(\Omega)$, positively invariant for $S(t)$, such that

$$\text{dist}_{Z^1}(S(t)\mathcal{K}, \mathcal{E}) \leq J_0 e^{-\omega t}, \quad (5.2)$$

for some $\omega > 0$, $J_0 \geq 0$.

In order to prove that $\mathcal{E}$ is an exponential attractor for the semigroup $S(t)$, we have to show that the condition (5.2) holds replacing $\mathcal{K}$ with an arbitrary bounded set $\mathcal{B} \subset Z^1(\Omega)$. To this aim, we prove in the following lemma 5.2 that the absorbing set $\mathcal{B}_2$ exponentially attracts every bounded set $\mathcal{B} \subset Z^1(\Omega)$. Accordingly, owing to the transitivity property of exponential attraction, we prove the existence of an exponential attractor for $S(t)$.

**Lemma 5.2** The absorbing set $\mathcal{B}_2 \subset Z^2(\Omega)$ satisfies the following conditions

(i) there exists an increasing function $M$ such that for every bounded set $\mathcal{B} \subset Z^1(\Omega)$ we have

$$\text{dist}_{Z^1}(S(t)\mathcal{B}, \mathcal{B}_2) \leq M(R)e^{-\nu t}, \quad (5.3)$$

where $R = \sup_{z \in \mathcal{B}} \|z\|_{Z^1}$ and $\nu$ is a positive constant independent of $R$;
(ii) there exists \( t_2 > 0 \) such that

\[
S(t)B_2 \subset B_2, \quad \forall t \geq t_2.
\]

**Proof.** Let us split the solution \( S(t)z = z(t) \) as the sum

\[ z(t) = z^l(t) + z^k(t), \]

where \( z^l(t) = (\psi^l(t), \hat{A}^l(t), \hat{u}^l(t)) \) solves the differential problem

\[
\begin{align*}
\gamma \psi^l_t - \frac{1}{\kappa^2} \Delta \psi^l + \psi^l &= 0 \quad (5.4) \\
\hat{A}^l_t - \nabla(\nabla \cdot \hat{A}^l) + \mu \nabla \times \nabla \times \hat{A}^l + \nabla \hat{u}^l &= 0 \quad (5.5) \\
c_0 \hat{u}^l_t - k_0 \Delta \hat{u}^l &= 0 \quad (5.6) \\
\nabla \psi^l \cdot \mathbf{n} = 0, \quad \hat{A}^l \cdot \mathbf{n} = 0, \quad (\nabla \times \hat{A}^l) \times \mathbf{n} = 0, \quad \hat{u}^l = 0, \quad \text{on } \partial \Omega \\
\psi^l(0) = \psi_0, \quad \hat{A}^l(0) = \hat{A}_0, \quad \hat{u}^l(0) = \hat{u}_0 
\end{align*}
\]

with \( z^l(0) = z(0) = (\psi_0, \hat{A}_0, \hat{u}_0) \)

\[
\|z^l(0)\|_{L^1} \leq R.
\]

Moreover, \( z^k(t) = (\psi^k(t), \hat{A}^k(t), \hat{u}^k(t)) \) is a solution to

\[
\begin{align*}
\gamma \psi^k_t - \frac{1}{\kappa^2} \Delta \psi^k + \psi^k &= \Upsilon(\psi, \hat{A}, \hat{u}) \quad (5.9) \\
\hat{A}^k_t - \nabla(\nabla \cdot \hat{A}^k) + \mu \nabla \times \nabla \times \hat{A}^k + \nabla \hat{u}^k &= \Theta(\psi, \hat{A}, \hat{u}) \quad (5.10) \\
c_0 \hat{u}^k_t - k_0 \Delta \hat{u}^k &= \Gamma(\psi, \hat{A}, \hat{u}) \quad (5.11) \\
\nabla \psi^k \cdot \mathbf{n} = 0, \quad \hat{A}^k \cdot \mathbf{n} = 0, \quad (\nabla \times \hat{A}^k) \times \mathbf{n} = 0, \quad \hat{u}^k = 0, \quad \text{on } \partial \Omega \\
\psi^k(0) = 0, \quad \hat{A}^k(0) = 0, \quad \hat{u}^k(0) = 0 
\end{align*}
\]

where \( \Upsilon, \Theta, \Gamma \) are defined as

\[
\begin{align*}
\Upsilon &= -\frac{2i}{\kappa} (\hat{A} + A_H) \cdot \nabla \psi - \psi |\hat{A} + A_H|^2 + i \beta \psi \nabla \cdot \hat{A} \\
&\quad - \psi(|\psi|^2 - 2 + \hat{u} + u_H) \\
\Theta &= -|\psi|^2 (\hat{A} + A_H) + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) - \nabla \times \mathbf{G} \\
\Gamma &= \frac{1}{2} (\psi \bar{\psi} + \bar{\psi} \psi) + \nabla \cdot \left[ -|\psi|^2 (\hat{A} + A_H) + \frac{i}{2\kappa} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \right]
\end{align*}
\]
We prove that 
\[ \|z^f(t)\|_{Z^1} \leq m_1(R)e^{-\nu t}. \] (5.14)

To this aim, let us multiply (5.4) by \( \frac{1}{2}(\bar{\psi}^l + \psi^l) \), its conjugate by \( \frac{1}{2}(\bar{\psi}^l + \psi^l) \), (5.5) by \( \hat{A}^l + \hat{A}^l \) and (5.6) by \( \sigma \hat{u}^l \), where \( \sigma \) is a positive constant large enough. Adding the resulting equations and integrating over \( \Omega \) we obtain

\[ \frac{1}{2} \frac{d}{dt} \left[ \frac{1}{\kappa^2} \|\nabla \psi^l\|^2 + (\gamma + 1)\|\psi^l\|^2 + \|\nabla \cdot \hat{A}^l\|^2 + \|\hat{A}^l\|^2 \right] + \frac{1}{\kappa^2} \gamma \|\psi^l_t\|^2 + \frac{1}{\kappa^2} \|\nabla \psi^l_t\|^2 + \|\hat{A}^l_t\|^2 + \|\nabla \cdot \hat{A}^l_t\|^2 \\
+ \mu \|\nabla \cdot \hat{A}^l\|^2 + k_0 \sigma \|\nabla \hat{u}^l\|^2 \leq \frac{1}{2}(\|\hat{A}^l\|^2 + \|\hat{A}^l_t\|^2) + C\|\nabla \hat{u}^l\|^2. \]

Gronwall’s inequality implies the existence of a suitable constant \( \nu > 0 \) independent of \( R \) and of an increasing function \( m_1(R) \) such that (5.14) holds.

Now let us prove that \( z^k(t) \) belongs to a bounded set \( \tilde{B}_2 \subset Z^2(\Omega) \), namely

\[ \|z^k(t)\|_{Z^2} \leq m_2(R). \] (5.15)

Let us multiply (5.9) by \(-1/2(\Delta \bar{\psi}^k + \Delta \psi^k)\) and its conjugate by \(-1/2(\Delta \psi^k + \Delta \bar{\psi}^k)\).

An integration over \( \Omega \) leads to

\[ \frac{1}{2} \frac{d}{dt} \left[ \frac{1}{\kappa^2} \|\Delta \psi^k\|^2 + (\gamma + 1)\|\psi^k\|^2 \right] + \frac{1}{\kappa^2} \|\nabla \psi^k_t\|^2 + \gamma \|\psi^k_t\|^2 + \|\nabla \psi^k_t\|^2 \\
+ \mu \|\nabla \times \hat{A}^k\|^2 + \mu \|\nabla \times \hat{A}^k_t\|^2 \leq \frac{1}{2}(\|\hat{A}^k\|^2 + \|\hat{A}^k_t\|^2) + C\|\nabla \hat{u}^k\|^2. \]

Hölder’s and Young’s inequalities assure that

\[ \frac{d}{dt} \left[ \frac{1}{\kappa^2} \|\Delta \psi^k\|^2 + (\gamma + 1)\|\psi^k\|^2 \right] + \frac{1}{\kappa^2} \|\Delta \psi^k\|^2 + \gamma \|\psi^k_t\|^2 + 2\|\nabla \psi^k\|^2 \\
\leq C\|\psi^k\|^2_H^1. \] (5.16)

Next, the product in \( L^2(\Omega) \) of (5.10) by \( \nabla \times \nabla \times \hat{A}^k + \nabla \times \nabla \times \hat{A}^k \) yields the inequality

\[ \frac{d}{dt} \left[ \mu \|\nabla \times \nabla \times \hat{A}^k\|^2 + \|\nabla \times \hat{A}^k\|^2 \right] + \mu \|\nabla \times \nabla \times \hat{A}^k\|^2 + \|\nabla \times \hat{A}^k\|^2 \leq C\|\Theta\|^2_H^1. \] (5.17)
Similarly, by multiplying in $L^2(\Omega)$ the same equation (5.10) by $-\nabla(\nabla \cdot \hat{A}^k) + \nabla \hat{u}^k$, we infer
\begin{align*}
\frac{d}{dt} \left[ \| -\nabla(\nabla \cdot \hat{A}^k) + \nabla \hat{u}^k \|^2 + \|
abla \cdot \hat{A}^k \|^2 \right] \\
+ \| -\nabla(\nabla \cdot \hat{A}^k) + \nabla \hat{u}^k \|^2 + \|
abla \cdot \hat{A}^k \|^2 \leq C (\|\Theta\|^2_{H^1} + \|\hat{u}^k\|^2 + \|\nabla \hat{u}^k\|^2).
\end{align*}
(5.18)

Finally, multiplication in $L^2(\Omega)$ of (5.11) by $\sigma(\hat{u}^k + \hat{u}^k)$, with a (large enough) positive constant $\sigma$ leads to
\begin{align*}
\frac{d}{dt} \left[ c_0 \sigma \|\hat{u}^k\|^2 + k_0 \sigma \|
abla \hat{u}^k\|^2 \right] + c_0 \sigma \|\hat{u}^k\|^2 + k_0 \sigma \|
abla \hat{u}^k\|^2 \leq C \|\Gamma\|^2.
\end{align*}
(5.19)

From the definition of $\Upsilon, \Theta, \Gamma$, interpolation inequality (2.16) and (4.15) it follows
\begin{align*}
\|\Upsilon\|^2_{H^1} + \|\Theta\|^2_{H^1} + \|\Gamma\|^2 \leq C_R(1 + \|\psi_t\|^2).
\end{align*}

Therefore, in view of Remark 4.1, we obtain
\begin{align*}
\|\Upsilon\|^2_{H^1} + \|\Theta\|^2_{H^1} + \|\Gamma\|^2 \leq C_R.
\end{align*}
(5.20)

Summing up (5.16)-(5.19), with a properly choice of $\sigma$, on account of (5.20), we obtain
\begin{align*}
\frac{d}{dt} \mathcal{F}^k + \frac{1}{\kappa^2} \|\Delta \psi^k\|^2 + 2\|\nabla \psi^k\|^2 + \mu \|\nabla \times \nabla \times \hat{A}^k\|^2 \\
+ \| -\nabla(\nabla \cdot \hat{A}^k) + \nabla \hat{u}^k\|^2 + C \|\nabla \hat{u}^k\|^2 \leq C_R,
\end{align*}
(5.21)

where
\begin{align*}
\mathcal{F}^k &= \frac{1}{\kappa^2} \|\Delta \psi^k\|^2 + (\gamma + 1)\|\nabla \psi^k\|^2 + \mu \|\nabla \times \nabla \times \hat{A}^k\|^2 + \|\nabla \times \hat{A}^k\|^2 \\
&\quad + \|\nabla \cdot \hat{A}^k\|^2 + \| -\nabla(\nabla \cdot \hat{A}^k) + \nabla \hat{u}^k\|^2 + c_0 \sigma \|\hat{u}^k\|^2 + k_0 \sigma \|\nabla \hat{u}^k\|^2.
\end{align*}

By adding to both sides of (5.21) the terms $\varepsilon (\|\nabla \times \hat{A}^k\|^2 + \|\nabla \cdot \hat{A}^k\|^2 + \|\hat{u}^k\|^2)$, with a small positive constant $\varepsilon$, from the Poincaré inequality we prove
\begin{align*}
\frac{d}{dt} \mathcal{F}^k + \lambda \mathcal{F}^k \leq C_R,
\end{align*}

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where $\lambda > 0$. Owing to (5.13), an application of Gronwall’s lemma yields

$$\mathcal{F}^k \leq C_R.$$  

Moreover, from (4.15) and (5.14) it follows that

$$\|z^k(t)\|_{Z^1} \leq \|z(t)\|_{Z^1} + \|z'(t)\|_{Z^1} \leq C_R,$$

so that (5.15) is satisfied.

Relations (5.14) and (5.15) lead to

$$\text{dist}_{Z^1}(S(t)\tilde{B}_2, B_2) \leq m_1(R)e^{-\nu t}.$$  

(5.22)

Since $\tilde{B}_2$ is bounded in $Z^2(\Omega)$, we deduce that

$$S(t)\tilde{B}_2 \subset B_2, \quad \forall t > \tilde{t}_2 = \tilde{t}_2(R).$$

Accordingly, we obtain

$$\text{dist}_{Z^1}(S(t)\tilde{B}_2, B_2) = \begin{cases} 
\alpha(R) & t \leq \tilde{t}_2 \\
0 & t > \tilde{t}_2 
\end{cases}$$

where $\alpha(R)$ is a bounded function. Hence, there exists an increasing function $m_3 = m_3(R)$ such that

$$\text{dist}_{Z^1}(S(t)\tilde{B}_2, B_2) \leq \alpha(R)e^{\tilde{t}_2 - t} = m_3(R)e^{-t}.$$  

(5.23)

Inequalities (5.22), (5.23) and the transitivity property of exponential attraction (see [13, Theor 5.1]) prove (5.3).

Condition (ii) follows directly by the definition of absorbing set.

Now we are in a position to prove the main result of this section.

**Theorem 5.1** The semigroup $S(t)$ possesses an exponential attractor $\mathcal{E} \subset Z^2(\Omega)$.

**Proof.** We apply lemma 5.1 with $K = B_2$ and $t^* > t_2$. Firstly we prove condition (i). Let $t^* \leq t \leq \tau \leq 2t^*$ and $z_1, z_2 \in B_2$. Then we have

$$\|S(\tau)z_1 - S(t)z_2\|_{Z^1} \leq \|S(\tau)z_1 - S(\tau)z_2\|_{Z^1} + \|S(\tau)z_2 - S(t)z_2\|_{Z^1}.$$  

(5.24)
Theorem 3.2 implies
\[ \| S(\tau)z_1 - S(\tau)z_2 \|_{Z^1} \leq C(t^*) \| z_1 - z_2 \|_{Z^1}. \]

In order to estimate last term in (5.24), we integrate (4.20) with respect to \( t \). By taking (3.16) and (4.15) into account, we deduce
\[ \int_0^t \left[ \| \nabla \psi_t \|^2 + \| \nabla \times \hat{A}_t \|^2 + \| \nabla \cdot \hat{A}_t \|^2 + \| \hat{u}_t \|^2 \right] ds \leq C(t). \]

The latter, together with a-priori estimate (3.15), assures that
\[ \int_0^t \| z_t \|^2_{Z^1} ds \leq C(t). \]

An application of Hölder’s inequality leads to
\[ \| S(\tau)z_2 - S(t)z_2 \|_{Z^1} \leq \int_t^\tau \| z_{2t}(s) \|_{Z^1} ds \leq C(t^*) \sqrt{\tau - t}. \]

Hence, (5.24) reads
\[ \| S(\tau)z_1 - S(t)z_2 \|_{Z^1} \leq C(t^*) \left[ \| z_1 - z_2 \|_{Z^1} + \sqrt{\tau - t} \right]. \]

Now we show that condition (ii) of lemma 5.1 holds. We define
\[ L(z) = z^l(t^*), \]
with \( z^l(t) \) solution to (5.4)-(5.8) with initial datum \( z \). If \( z_1^l(t), z_2^l(t) \) are solutions of (5.4)-(5.8) with initial data \( z_1, z_2 \), their difference satisfies the same inequality (5.14), namely
\[ \| z_1^l(t) - z_2^l(t) \|_{Z^1} \leq C e^{-\nu t^*} \| z_1 - z_2 \|_{Z^1}. \]

By choosing a sufficiently large \( t^* > t_2 \), we prove
\[ \| L(z_1) - L(z_2) \|_{Z^1} \leq \lambda \| z_1 - z_2 \|_{Z^1}, \]
with \( \lambda \in (0, 1/2) \).

Let
\[ K(z) = z^k(t^*), \]
with $z^k(t^*)$ solution to (5.9)-(5.13). We denote by $\psi^k = \psi_1^k - \psi_2^k$, $\psi = \psi_1 - \psi_2$, $\hat{A}^k = \hat{A}_1^k - \hat{A}_2^k$, $\hat{A} = \hat{A}_1 - \hat{A}_2$, $\hat{u}^k = \hat{u}_1^k - \hat{u}_2^k$, $\hat{u} = \hat{u}_1 - \hat{u}_2$. Let us multiply (5.9) by $1/2 (\tilde{\psi}_1 + \Delta \tilde{\psi}_t)$, its conjugate by $1/2 (\psi_1 + \Delta \psi_t)$ and integrate over $\Omega$. Moreover, by proceeding like in the proof of lemma 5.2 for the remaining equations (5.10) and (5.11), from (5.9)--(5.13) we deduce

$$
\frac{1}{2} \frac{d}{dt} \left[ \frac{1}{\kappa^2} \| \Delta \psi^k \|^2 + \left( \frac{1}{\kappa^2} + 1 \right) \| \nabla \psi^k \|^2 + \| \nabla \times (\nabla \cdot \hat{A}^k) \|^2 + \mu \| \nabla \times \nabla \times \hat{A}^k \|^2 + k_0 \sigma \| \nabla \hat{u}^k \|^2 \right]
\leq C \left[ \| \Theta_1 - \Theta_2 \|^2_{H^1} + \| \Gamma_1 - \Gamma_2 \|^2 \right].
$$

where $\sigma > 0$ is a suitable constant and

$$
\begin{align*}
\Upsilon_1 - \Upsilon_2 & = -\frac{2i}{\kappa} (\hat{A} \cdot \nabla \psi_1 + (\hat{A}_2 + A_H) \cdot \nabla \psi) - \hat{A}_1 + A_H \big| \hat{\psi} \\
\Theta_1 - \Theta_2 & = -|\psi_1|^2 \hat{A} - (\tilde{\psi}_1 \psi + \psi_2 \tilde{\psi})(\hat{A}_2 + A_H) \\
\Gamma_1 - \Gamma_2 & = \frac{1}{2} (\tilde{\psi}_1 \psi + \psi_2 \tilde{\psi} + \tilde{\psi}_1 \psi + \psi_2 \tilde{\psi} + \psi_2 \tilde{\psi} + \psi_1 \psi) + \nabla \cdot \left[ -|\psi_1|^2 \hat{A} \\
& - (\tilde{\psi}_1 \psi + \psi_2 \tilde{\psi})(\hat{A}_2 + A_H) + \frac{i}{2\kappa} (\tilde{\psi}_1 \psi + \psi_2 \tilde{\psi} + \tilde{\psi}_1 \psi + \psi_2 \tilde{\psi} + \tilde{\psi}_1 \psi + \psi_2 \tilde{\psi} + \psi_2 \tilde{\psi} + \tilde{\psi}_1 \psi) \right]
\end{align*}
$$

Therefore, since $z_1, z_2 \in B_2$ in view of (2.16), we easily deduce

$$
\| \Upsilon_1 - \Upsilon_2 \|_{H^1} + \| \Theta_1 - \Theta_2 \|_{H^1} + \| \Gamma_1 - \Gamma_2 \| \leq C \left( \| \psi \|_{H^2} + \| \hat{A} \|_{H^2} + \| \hat{u} \|_{H^2} \right).
$$

Hence

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left[ \frac{1}{\kappa^2} \| \Delta \psi^k \|^2 + \left( \frac{1}{\kappa^2} + 1 \right) \| \nabla \psi^k \|^2 + \| \nabla \times (\nabla \cdot \hat{A}^k) \|^2 + \mu \| \nabla \times \nabla \times \hat{A}^k \|^2 + k_0 \sigma \| \nabla \hat{u}^k \|^2 \right]
\leq C \left( \| \psi \|^2_{H^2} + \| \hat{A} \|^2_{H^2} + \| \hat{u} \|^2_{H^2} \right),
\end{align*}
$$

which in view of (3.27) implies

$$
\| K(z_1, z_2) \|_{Z^2} \leq \Lambda \| z_1 - z_2 \|_{Z^1},
$$

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with a suitable constant $\Lambda > 0$.

Assumptions (i) and (ii) of lemma 5.1 hold, so that inequality (5.2) is satisfied. The inequality (5.1) follows by applying lemma 5.2 and the transitivity property of exponential attraction. \[\square\]

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