Research Article

An Integral Equation Formalism for Integrating a Nonlinear Initial-Boundary Value Problem for a Boussinesq Equation

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In this paper, a new nonlinear initial-boundary value problem for a Boussinesq equation is formulated. And a coupled system of nonlinear integral equations, equivalent to the new initial-boundary value problem, is constructed for integrating the initial-boundary value problem, but which is inherently different from other conventional formulations for integral equations. For the numerical solutions, successive approximations are applied, which leads to a functional iterative formula. A propagating solitary wave is simulated via iterating the formula, which is in good agreement with the known exact solution.

1. Introduction

We begin with formulating a nonlinear initial-boundary value problem for the nonlinear partial differential equation (PDE) [1]:

\[ u_{tt} - c_0^2 u_{xx} + \frac{1}{2} \left( u^2 \right)_{xx} - \frac{1}{3} h_0^2 u_{xxtt}, \quad x > 0, \quad (1) \]

in which constant \( h_0 \) denotes water depth and constant \( c_0 \) characteristic velocity, i.e.,

\[ c_0 = \sqrt{gh_0}, \quad (2) \]

with gravitation acceleration denoted by \( g \). The nonlinear PDE (1) describes evolution of nonlinear dispersive waves in shallow water propagating in a uniform channel of water depth \( h_0 \) under gravitation \( g \), where \( u \) in (1) represents depth-average velocity in horizontal direction \( x \). In fact, the model equation (1) is chosen for mathematical simplicity, as a starting point in this paper, which is the time evolution equation of the lowest order mathematical model for nonlinear and dispersive long wave; however, the physics represented by (1) is clearly included in higher order Boussinesq’s models [2]. Generally, the differential equations of the Boussinesq type such as (1) indicate essential model equations in physics and applied mathematics, the time evolution of which is an issue of scientific as well as engineering importance.

Next, we would like to solve the nonlinear initial-boundary value problem formulated above through the construction of an (equivalent) integral equation formalism, a primary purpose of this paper. To this end, we first build an auxiliary (initial-boundary value) problem, associated with the (original) initial-boundary value problem. Thereby, it is possible to build (regular) coupled nonlinear integral equations of second kind, being equivalent to the present initial-boundary value problem. Applying the method of successive approximation to the integral equations, we arrive at a functional iterative formula. A propagating solitary wave is simulated via iterating the formula, which is in good agreement with the known exact solution.

In referring to the Boussinesq type equations such as (1), a considerable number of studies has been conducted over the past few decades, but most of them are limited to just
their “initial” value problems (e.g., [3]), while the present study concerns the initial-“boundary” value problem as discussed at the beginning of this section. To name a few, El-Zoheiry [4] performed a numerical study of the classical Boussinesq equation using a three-level iterative scheme based on the compact implicit method, where a solitary wave solution of the equation was examined for the accuracy and efficiency. Wang et al. [5] analyzed the improved Boussinesq equation using an energy-preserving finite volume element method. Finally, in a recent year, Jang [6] investigated the classical Boussinesq equation by proposing a new dispersion-relation preserving (DRP) method and further improved the convergence characteristic of the DRP method for integrating the Boussinesq equation [7].

2. Initial-Boundary Value Problem and Auxiliary Problem

We consider the nonlinear PDE of fourth order (1), as discussed in the preceding section, but on the half interval \( R^+ = (0, \infty) \), with the dispersion relation [6, 7] between wave frequency \( \omega_B \) and wave number \( k \):

\[
\omega_B^2 = \frac{c_1^2 k^2}{1 + h_k k^2/3}.
\]

For PDE (1), we shall formulate a (initial-boundary value) problem subject to initial conditions:

\[
\begin{align*}
  u(x, 0) &= 0, \\
  u_t(x, 0) &= 0, \\
  0 < x < \infty,
\end{align*}
\]

(4)

together with boundary condition at \( x = 0^+ \):

\[
  u(0^+, t) = U(t), \quad t > 0,
\]

(5)

and at (positive) infinity,

\[
  u, u_x \rightarrow 0, \quad \text{as } x \rightarrow \infty.
\]

(6)

Associated with the initial-boundary value problem formulated above, we will next build an auxiliary problem.

Definition 1. As a counterpart of the original initial-boundary problem of (1) with (4)–(6), let an auxiliary initial-boundary value problem for \( u_{\text{Aux}} \) be defined as

\[
(u_{\text{Aux}})_{tt} - c_0^2 (u_{\text{Aux}})_{xx} + \frac{1}{2} (u_{\text{Aux}})_{xt} = \frac{1}{3} h_0^2 (u_{\text{Aux}})_{xxtt},
\]

\[-\infty < x < 0, \quad t > 0,
\]

subject to initial conditions at \( t = 0 \):

\[
\begin{align*}
  u_{\text{Aux}}(x, 0) &= 0, \\
  (u_{\text{Aux}})_t(x, 0) &= 0, \quad -\infty < x < 0,
\end{align*}
\]

(8)

as well as, boundary conditions at \( x = 0^- \):

\[
  u_{\text{Aux}}(0^-, t) = -U(t), \quad t > 0,
\]

(9)

and at negative infinity:

\[
u_{\text{Aux}}, (u_{\text{Aux}})_x \rightarrow 0, \quad \text{as } x \rightarrow -\infty.
\]

(10)

Remark 1. The auxiliary initial-boundary value problem in Definition 1, defined on the half interval \( R^+ = (-\infty, 0) \), is in fact constructed in such a way that

\[
-u_{\text{Aux}}(x, t) = u(-x, t), \quad -\infty < x < 0, t > 0.
\]

(11)

The original and auxiliary problems may be combined to form a new problem through the next definition.

Definition 2. \( u_{\text{Ext}} \) is a function from \( \mathbb{R} \times [0, T] \) to \( \mathbb{R} \) for \( T \in \mathbb{R}^+ \) satisfying a new (extended) initial-boundary value problem, being governed by the same Boussinesq PDE (1) but in a weak (derivative) sense:

\[
(u_{\text{Ext}})_{tt} - c_0^2 (u_{\text{Ext}})_{xx} + \frac{1}{2} (u_{\text{Ext}})_{xt} = \frac{1}{3} h_0^2 (u_{\text{Ext}})_{xxtt},
\]

\[-\infty < x < \infty, \quad t > 0,
\]

(12)

together with (null) initial conditions

\[
(u_{\text{Ext}})(x, 0) = 0,
\]

(13)

jump discontinuity at \( x = 0 \)

\[
  u_{\text{Ext}}(0^+, t) = U(t),
\]

\[
  u_{\text{Ext}}(0^-, t) = -U(t), \quad t > 0,
\]

(14)

and localized conditions

\[
u_{\text{Ext}}(x, t) \rightarrow 0, \quad \text{as } x \rightarrow \pm \infty.
\]

(15)

Remark 2. For \( x > 0 \), the new initial-boundary value problem for \( u_{\text{Ext}} \) in Definition 2 reduces to the original initial-boundary value problem for \( u \); while, for \( x < 0 \), the problem for \( u_{\text{Ext}} \) remains the same as the auxiliary problem for \( u_{\text{Aux}} \).

The above remarks immediately give the following lemma.

Lemma 1. \( u_{\text{Ext}} \) in Definition 2 is the odd extension of \( u \) with respect to \( x \) in the original problem of (1) with (4)–(6); thus, \( u_{\text{Ext}} \) is antisymmetric with respect to \( x = 0 \) (See Figure 1).

Proof. \( u_{\text{Ext}}(x, t) \) becomes \( u(x, t) \) when \( x > 0 \) from Remark 2, i.e., \( u_{\text{Ext}}(x, t) = u(x, t) \), for \( x > 0 \). However, for \( x < 0 \),

\[
  u_{\text{Ext}}(x, t) = u_{\text{Aux}}(x, t) = -u(-x, t),
\]

(16)
due to Remarks 1 and 2. This completes the proof.
It would be instructive to note that, as indicated in Figure 1, the original initial-boundary value problem for \( u(x > 0) \) corresponds to a physical problem, where a wavemaker, located at \( x = 0 \), generates a propagating (velocity) wave with specified wavemaker-velocity \( U(t) \).

3. Integral Formalism for \( u_{\text{Ext}} \)

This section is devoted to the construction of an integral formulation which is equivalent to the extended initial-boundary value problem (appearing in Definition 2) in the previous section.

3.1. Pseudodissipative Differential Formulation. We start by introducing two pseudoparameters of \( \alpha, \beta > 0 \) in (12), adding the sum \( \alpha \cdot (u_{\text{Ext}})_t + \beta \cdot u_{\text{Ext}} \) to both sides of (12) and arranging, which leaves us with

\[
(u_{\text{Ext}})_t - c_0^2 (u_{\text{Ext}})_{xx} - \frac{1}{3} h_0^2 (u_{\text{Ext}})_{xxt} + \alpha \cdot (u_{\text{Ext}})_t + \beta \cdot u_{\text{Ext}} = \varphi, \quad -\infty < x < \infty, \tag{17}
\]

and a forcing term \( \varphi \) with the functional form

\[
\varphi \equiv -\frac{1}{2} (u_{\text{Ext}}^2)_t + \alpha \cdot (u_{\text{Ext}})_t + \beta \cdot u_{\text{Ext}}. \tag{18}
\]

Notice that the pair of (17) and (18) is a pseudodissipative differential formulation, but which is mathematically still equivalent to the (extended) Boussinesq equation (12); both the parameters, \( \alpha, \beta > 0 \), are directly related to a pseudodissipation (which will be discussed later). Thereby, we would like to build an equivalent (nonlinear) pseudodissipation integral formalism in Section 3.5. We begin with integral transforms for (17), which are explained in the next section.

3.2. Image of \( u_{\text{Ext}} \) under Laplace–Fourier Transform. We first discover the image of Laplace transform \( \mathcal{L} \) of \( u_{\text{Ext}} \) in (17), with respect to time \( t \),

\[
\mathcal{L}[u_{\text{Ext}}] \equiv \int_0^\infty u_{\text{Ext}}(x, t) \cdot e^{-st} \, dt \equiv u_{\text{Ext}}^s(x, s), \tag{19}
\]

for a (Laplace transform) parameter \( s \). Here, we assume that \( u_{\text{Ext}} \) is Laplace transformable.

Lemma 2. Given two positive pseudoparameters, \( \alpha, \beta > 0 \), PDE (17) for \( u_{\text{Ext}} \) can be converted into the following ordinary differential equation (ODE) for \( u_{\text{Ext}}^s \):

\[
\left( s^2 \cdot u_{\text{Ext}}^s \right) - c_0^2 \frac{d^2 u_{\text{Ext}}^s}{dx^2} - \frac{h_0^2}{3} \frac{d^2}{dx^2} \left( s^2 \cdot u_{\text{Ext}}^s \right) + \alpha (s \cdot u_{\text{Ext}}^s) + \beta \cdot u_{\text{Ext}}^s = \varphi^s, \tag{20}
\]

for a parameter \( s \).

Proof. We remind the differentiation properties of Laplace transform

\[
\begin{aligned}
(u_{\text{Ext}})_t &= s \cdot u_{\text{Ext}} - u_{\text{Ext}}(x, 0) = s \cdot u_{\text{Ext}}^s, \\
(u_{\text{Ext}})_t &= s^2 \cdot u_{\text{Ext}}(x, 0) - (u_{\text{Ext}})_t(x, 0) = s^2 \cdot u_{\text{Ext}}^s,
\end{aligned}
\]

due to the null initial conditions (13). Applying Laplace transform to (17), combined with the properties of (21) and (22), leads to (20), which proves the lemma. \( \square \)

Next, we denote Fourier transform \( \mathcal{F} \) of \( u_{\text{Ext}}^s(x, s) \), with respect to space \( x \), by

\[
\mathcal{F}[u_{\text{Ext}}^s] = \int_{-\infty}^{\infty} u_{\text{Ext}}^s(x, s) \cdot e^{-ikx} \, dx \equiv \tilde{u}_{\text{Ext}}^s(k, s) \tag{23}
\]

for a parameter \( k \) and its inverse by \( \mathcal{F}^{-1} \), i.e.,

\[
u_{\text{Ext}}^s(x, s) = \mathcal{F}^{-1}[\tilde{u}_{\text{Ext}}^s] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}_{\text{Ext}}^s(k, s) \cdot e^{-ikx} \, dk, \tag{24}
\]

in which \( u_{\text{Ext}}^s(x, s) \) and \( \tilde{u}_{\text{Ext}}^s(k, s) \) are assumed to be Fourier transformable and inverse Fourier transformable, respectively. Then, applying Fourier transform \( \mathcal{F} \) to ODE (20), we find that

\[
\left( s^2 \cdot \tilde{u}_{\text{Ext}}^s \right) - c_0^2 \mathcal{F} \left( \frac{d^2 u_{\text{Ext}}^s}{dx^2} \right) - \frac{h_0^2}{3} \mathcal{F} \left( s^2 \cdot \frac{d^2 u_{\text{Ext}}^s}{dx^2} \right) + \alpha (s \cdot \tilde{u}_{\text{Ext}}^s) + \beta \cdot \tilde{u}_{\text{Ext}}^s = \tilde{\varphi}^s, \tag{25}
\]

which becomes an algebraic equation for \( \tilde{u}_{\text{Ext}}^s \) because of the following lemma.

Lemma 3. The boundary velocity \( U \) in (5) appears in Fourier transform of the second derivative, \( d^2 u_{\text{Ext}}^s/dx^2 \), as follows:

\[
\mathcal{F} \left( \frac{d^2 u_{\text{Ext}}^s}{dx^2} \right) = (-ik)^2 \tilde{u}_{\text{Ext}}^s + 2ikU^s, \tag{26}
\]

where \( U^s \) stands for Laplace transform of (5), i.e.,

\[
U^s(s) \equiv u_{\text{Ext}}^s(x = 0^+, s). \tag{27}
\]

Proof. At first, we consider,
\[
\mathcal{F}\left(\frac{d^2 u_{\text{Ext}}^*}{dx^2}\right) = \int_{-\infty}^{\infty} \frac{d^2}{dx^2} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \, dx = \int_{0}^{\infty} \frac{d}{dx} u_{\text{Ext}}(x, s) \cdot e^{ix} \, dx + \int_{-\infty}^{0} \frac{d}{dx} u_{\text{Ext}}(x, s) \cdot e^{ix} \, dx
\]

\[
= u_{\text{Ext}}^*(x, s) \cdot e^{ix} \bigg|_{0}^{\infty} - ik \int_{0}^{\infty} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \, dx + u_{\text{Ext}}^*(x, s) \cdot e^{ix} \bigg|_{-\infty}^{0} - ik \int_{-\infty}^{0} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \, dx
\]

\[
= -ik \int_{-\infty}^{\infty} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \, dx - 2U^*(s) = -ik \cdot \mathcal{F}(u_{\text{Ext}}^*) - 2U^*(s),
\]

where integration by parts is used with (14).

Similarly, we further have

\[
\mathcal{F}\left(\frac{d^2 u_{\text{Ext}}^*}{dx^2}\right) = \int_{-\infty}^{\infty} \frac{d^2}{dx^2} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \, dx = \int_{0}^{\infty} \frac{d}{dx} u_{\text{Ext}}(x, s) \cdot e^{ix} \, dx + \int_{-\infty}^{0} \frac{d}{dx} u_{\text{Ext}}(x, s) \cdot e^{ix} \, dx
\]

\[
= \frac{d}{dx} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \bigg|_{0}^{\infty} - ik \int_{0}^{\infty} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \, dx + \frac{d}{dx} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \bigg|_{-\infty}^{0} - ik \int_{-\infty}^{0} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \, dx
\]

\[
= -ik \int_{-\infty}^{\infty} \frac{d}{dx} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \, dx
\]

\[
= -ik \cdot \mathcal{F}\left(\frac{du_{\text{Ext}}^*}{dx}\right).
\]

Here, note that

\[
\frac{d}{dx} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \bigg|_{0}^{\infty} + \frac{d}{dx} u_{\text{Ext}}^*(x, s) \cdot e^{ix} \bigg|_{-\infty}^{0} = -\frac{d}{dx} u_{\text{Ext}}^*(0^+, s) + \frac{d}{dx} u_{\text{Ext}}^*(0^-, s) \quad (\because \text{localized conditions (15)}),
\]

\[
= \lim_{\epsilon_1, \epsilon_2 \to 0^+} \mathcal{L}\left[-u_{\text{Ext}}(\epsilon_1, t) + u_{\text{Ext}}(\epsilon_2, t)\right]
\]

\[
= \lim_{\epsilon_1, \epsilon_2 \to 0^+} \mathcal{L}\left[-u_{\chi}(\epsilon_1, t) + u_{\chi}(\epsilon_2, t)\right],
\]

\[
= \mathcal{L}\left[-u_{\chi}(0^+, t) + u_{\chi}(0^-, t)\right] = 0,
\]

because of the property of odd extension by Lemma 1, i.e.,

\[
(u_{\text{Ext}})_{x}(x, t) = u_{\chi}(x, t), \quad \text{for } x > 0,
\]

and the chain rule

\[
(u_{\text{Ext}})_{x}(x, t) = \frac{\partial}{\partial x} [-u(-x, t)] = \frac{\partial}{\partial (-x)} u(-x, t)
\]

\[
= \frac{\partial}{\partial X} u(X, t) = u_{x}(x, t) \bigg|_{x=X}
\]

\[
= u_{x}(-x, t), \quad \text{for } x < 0,
\]

from (16).

Plugging (28) into (29) gives

\[
\mathcal{F}\left(\frac{d^2 u_{\text{Ext}}^*}{dx^2}\right) = -ik\left[-ik \cdot \mathcal{F}(u_{\text{Ext}}^*) - 2U^*(s)\right]
\]

\[
= (-ik)^2 u_{\text{Ext}}^* + 2ikU^*,
\]

which completes the proof. □

With the use of Lemma 3, (25) is rewritten as

\[
\frac{\partial^2}{\partial X^2} \left[\frac{\mu}{\rho} \cdot \nabla_{\text{Ext}}^2 \mathbf{n}_{\text{Ext}}^* + 2ikU^*\right]
\]

\[
= \frac{\mu^2}{\rho^2} \left[\frac{\mu}{\rho} \cdot \nabla_{\text{Ext}}^2 \mathbf{n}_{\text{Ext}}^* + 2ikU^*\right] + \alpha(s \cdot \nabla_{\text{Ext}}^* + \beta \cdot \mathbf{n}_{\text{Ext}}^*) = \mathbf{F},
\]

which can be solved for $\mathbf{n}_{\text{Ext}}^*$, i.e.,
\[ \tilde{u}_{\text{Ext}} = \frac{1}{1 + (k^2 h_0^2/3)} \cdot \frac{\psi^* + U^* (s) \cdot (2ikc_0^2 + (h_0^2/3) 2ik^2)}{s^2 \left( (c_0^2 k^2 + \alpha + \beta)/(1 + (k^2 h_0^2/3)) \right)} \]

(35)

Even though the above expression presents the (desired) exact result for the image of \( u_{\text{Ext}} \) under the Laplace-Fourier transform, however, the content seems to be hard to see. To overcome the difficulty, the next section discusses an appropriate arrangement of (35) suitable for finding the image of \( u_{\text{Ext}} \) under Laplace-Fourier transform.

3.3. Pseudodissipative Parameter and Frequency. This section gives an appropriate decomposition of the algebraic expression for \( \tilde{u}_{\text{Ext}}^* \) in (35) derived in the previous section, being realized by introducing a new concept of pseudodissipation. The new concept will make it more efficient (i.e., concise and clear) to analyze the problem both physically as well as mathematically. We need some definitions.

Definition 3 (see [6]). Given \( \beta > 0 \), we define a frequency function \( \omega_B : \mathbb{R} \longrightarrow \mathbb{R}^+ \) such that
\[
\omega_B^2 (k) = \frac{c_0^2 k^2 + \beta}{1 + (k^2 h_0^2/3)} > 0. \tag{36}
\]

If we use \( \omega_B \) in the above Definition, the denominator of the second fraction in the right hand of (35) arranges to

\[
\tilde{u}_{\text{Ext}}^* = \frac{1}{1 + (k^2 h_0^2/3)} \cdot \frac{\psi^* + U^* (s) \cdot 2ik(c_0^2 + (h_0^2/3)^2)}{s^2 + \frac{2\alpha}{2(c_0^2 k^2 + \beta)} + \alpha^2 \omega_B^2 / (4(c_0^2 k^2 + \beta))} \tag{38}
\]

by substituting (37) into (35).

Definition 4. \( \zeta \) is defined as a (dimensionless) pseudodissipative parameter (dependent of frequency) whose absolute value is less than unity, relying on \( \alpha, \beta > 0 \):
\[
-1 < \zeta = \frac{\alpha \omega_B}{2(c_0^2 k^2 + \beta)} < 1. \tag{39}
\]

Definition 5. Given \( |\zeta| < 1 \), \( \omega_B^{\text{pd}} : \mathbb{R} \longrightarrow \mathbb{R}^+ \), termed as a pseudodissipative frequency, such that
\[
\omega_B^{\text{pd}} = \omega_B \sqrt{1 - \zeta^2}, \tag{40}
\]
where \( \omega_B \) denotes the frequency function in Definition 3.

The next lemma is immediately followed by the above definitions.

Lemma 4. Let \( A \) and \( B \) be denoted by
\[
A = \psi^* + 2ikc_0^2 \cdot U^* (s)/(1 + k^2 h_0^2/3), \tag{41}
\]
respectively. Then, the expression of \( \tilde{u}_{\text{Ext}}^* \) in (38) may be shortened to a simpler form via \( \xi \) and \( \omega_B^{\text{pd}} \) of Definitions 4 and 5, respectively, i.e.,
\[
\tilde{u}_{\text{Ext}}^* = \frac{A + s^2 \cdot B}{(s + \zeta \omega_B)^2 + (\omega_B^{\text{pd}})^2} \tag{42}
\]
Proof. Writing (38) in terms of \( A \) and \( B \) in the above as follows:
\[ \mathcal{F}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2 + b^2}, \]
\[ \mathcal{F}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2 + b^2}, \quad \text{for } a, b \geq 0. \]

**Lemma 5.** The first fraction involving \( A \) in the right hand of (42) can be expressed as
\[ \frac{A}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} = \frac{A}{\omega_B^{pd}} \mathcal{F}\{\exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{pd} \cdot t)\}. \]

**Proof.** It is noted that
\[ \frac{A}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} = \frac{A}{\omega_B^{pd}} \frac{\omega_B^{pd}}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} = \frac{A}{\omega_B^{pd}} \mathcal{F}\{\exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{pd} \cdot t)\}, \]
where (44) is used. This immediately proves the lemma. \(\square\)

**Lemma 6.** The second fraction involving \( B \) in the right hand of (42) decomposes into
\[ \frac{s^2 \cdot B}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} = B - B\zeta \omega_b \cdot \mathcal{F}\{\exp(-\zeta \omega_B \cdot t) \cdot \cos(\omega_B^{pd} \cdot t)\} + \frac{B\omega_b (2\zeta^2 - 1)}{\sqrt{1 - \zeta^2}} \mathcal{F}\{\exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{pd} \cdot t)\}. \]

**Proof.** Letting \( F \) denote the second fraction involving \( B \), we have
\[ F = \frac{s^2 \cdot B}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} = B \frac{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2 - 2s \zeta \omega_b - \zeta^2 \omega_b^2 - (\omega_B^{pd})^2}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} = B \frac{1 - 2s \zeta \omega_b + \zeta^2 \omega_b^2 + (\omega_B^{pd})^2}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} \]
\[ = B - 2B\zeta \omega_b \cdot \frac{(s + \zeta \omega_b) - \zeta \omega_b}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} \]
\[ = B - 2B\zeta \omega_b \cdot \frac{\zeta^2 \omega_b^2}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} - B \cdot \frac{(\omega_B^{pd})^2}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} \]
\[ = B - 2B\zeta \omega_b \cdot \frac{s + \zeta \omega_b}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} - B \cdot \frac{(\omega_B^{pd})^2}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} \]
\[ \quad + B \cdot \frac{\zeta^2 \omega_b^2}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} - B \cdot \frac{(\omega_B^{pd})^2}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} \]
\[ = 2B\zeta \omega_b \cdot \frac{s + \zeta \omega_b}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} - B \cdot \frac{(\omega_B^{pd})^2}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2}. \]

Here, we note that
\[ -2B\zeta \omega_b \cdot \frac{s + \zeta \omega_b}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} = -2B\zeta \omega_b \cdot \mathcal{F}\{\exp(-\zeta \omega_B \cdot t) \cdot \cos(\omega_B^{pd} \cdot t)\}, \]
\[ \text{but} \quad \mathcal{F}\{\exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{pd} \cdot t)\} = \mathcal{F}\{\exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{pd} \cdot t)\}. \]

(49)

\[ B \cdot \frac{\zeta^2 \omega_b^2}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} = B \cdot \frac{\omega_B^{pd} \cdot \mathcal{F}\{\exp(-\zeta \omega_B \cdot t) \cdot \cos(\omega_B^{pd} \cdot t)\}}{\sqrt{1 - \zeta^2}} \]
\[ = B \omega_b \cdot \mathcal{F}\{\exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{pd} \cdot t)\}, \]
\[ B \cdot \frac{(\omega_B^{pd})^2}{(s + \zeta \omega_b)^2 + (\omega_B^{pd})^2} = B \omega_B^{pd} \cdot \mathcal{F}\{\exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{pd} \cdot t)\}. \]

(50)

(47)

From (40) and (44). And, (50) can be combined to (51), yielding
\[
\frac{B_k^2 \bar{\omega}_B}{\sqrt{1 - \zeta^2}} \mathcal{L} \left[ \exp(-\zeta \bar{\omega}_B \cdot t) \cdot \sin(\bar{\omega}_B^{PD} \cdot t) \right] - B \bar{\omega}_B^{PD} \cdot \mathcal{L} \left[ \exp(-\zeta \bar{\omega}_B \cdot t) \cdot \sin(\bar{\omega}_B^{PD} \cdot t) \right]
\]

\[
= B \left( \frac{\zeta^2 \bar{\omega}_B}{\sqrt{1 - \zeta^2}} - \bar{\omega}_B^{PD} \right) \cdot \mathcal{L} \left[ \exp(-\zeta \bar{\omega}_B \cdot t) \cdot \sin(\bar{\omega}_B^{PD} \cdot t) \right]
\]

\[
= B \left( \frac{\zeta^2 \bar{\omega}_B}{\sqrt{1 - \zeta^2}} - \bar{\omega}_B \frac{(1 - \zeta^2)}{\sqrt{1 - \zeta^2}} \right) \cdot \mathcal{L} \left[ \exp(-\zeta \bar{\omega}_B \cdot t) \cdot \sin(\bar{\omega}_B^{PD} \cdot t) \right] \quad \left( \therefore \bar{\omega}_B^{PD} = \bar{\omega}_B \sqrt{1 - \zeta^2} = \frac{\bar{\omega}_B (1 - \zeta^2)}{\sqrt{1 - \zeta^2}} \right)
\]

\[
= \frac{B \bar{\omega}_B (2\zeta^2 - 1)}{\sqrt{1 - \zeta^2}} \cdot \mathcal{L} \left[ \exp(-\zeta \bar{\omega}_B \cdot t) \cdot \sin(\bar{\omega}_B^{PD} \cdot t) \right].
\]

Therefore, substituting (49) and (52) into \( F \) in (48) leads to (47), completing the proof.

It will be convenient, if we denote by \( f_i, i = 1, 2, 3, 4, \)

\[
f_1(k, s) = \frac{A}{\bar{\omega}_B^{PD}} \cdot \mathcal{L} \left[ \exp(-\zeta \bar{\omega}_B \cdot t) \cdot \sin(\bar{\omega}_B^{PD} \cdot t) \right],
\]

\[
f_2(k, s) = B,
\]

\[
f_3(k, s) = -2B \zeta \bar{\omega}_B \cdot \mathcal{L} \left[ \exp(-\zeta \bar{\omega}_B \cdot t) \cdot \cos(\bar{\omega}_B^{PD} \cdot t) \right],
\]

\[
f_4(k, s) = \frac{B \bar{\omega}_B (2\zeta^2 - 1)}{\sqrt{1 - \zeta^2}} \cdot \mathcal{L} \left[ \exp(-\zeta \bar{\omega}_B \cdot t) \cdot \sin(\bar{\omega}_B^{PD} \cdot t) \right],
\]

so that \( \pi^*_E \) in (42) can be simply represented as the sum of \( f_i \)'s from Lemmas 5 and 6, i.e.,

\[
\pi^*_E = \sum_{i=1}^{4} f_i(k, s).
\]

**Lemma 7.** Given \( f_i, i = 1, 2, 3, 4, \) in (53)--(56), their inverse Laplace transforms are calculated as

\[
\mathcal{L}^{-1} (f_1) = \frac{1}{\bar{\omega}_B^{PD} \left( 1 + k^2 h^2_{\phi}/3 \right)} \int_0^t \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \cdot \sin \left[ \bar{\omega}_B^{PD} \cdot (t - \tau) \right] \cdot \varphi(k, \tau) d\tau
\]

\[
+ \frac{2ik\zeta^2}{\bar{\omega}_B^{PD} \left( 1 + k^2 h^2_{\phi}/3 \right)} \int_0^t \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \cdot \sin \left[ \bar{\omega}_B^{PD} \cdot (t - \tau) \right] \cdot U(\tau) d\tau,
\]

\[
\mathcal{L}^{-1} (f_2) = \frac{2ikh^2_{\phi}}{3 \left( 1 + k^2 h^2_{\phi}/3 \right)} \cdot U(t),
\]

\[
\mathcal{L}^{-1} (f_3) = -\frac{4ikh^2_{\phi} \zeta \bar{\omega}_B}{3 \left( 1 + k^2 h^2_{\phi}/3 \right)} \int_0^t \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \cdot \cos \left[ \bar{\omega}_B^{PD} \cdot (t - \tau) \right] \cdot U(\tau) d\tau,
\]

\[
\mathcal{L}^{-1} (f_4) = \frac{2ikh^2_{\phi} \bar{\omega}_B (2\zeta^2 - 1)}{3 \left( 1 + k^2 h^2_{\phi}/3 \right) \sqrt{1 - \zeta^2}} \int_0^t \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \cdot \sin \left[ \bar{\omega}_B^{PD} \cdot (t - \tau) \right] \cdot U(\tau) d\tau.
\]
Proof. To prove (58), note that \( \mathcal{L}^{-1}(f_1) \) has the explicit expression:

\[
\mathcal{L}^{-1}(f_1) = \mathcal{L}^{-1} \left[ \frac{A}{\omega_B^2} \cdot \mathcal{L} \left[ \exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{PD} \cdot t) \right] \right] \quad (\cdot (53))
\]

\[
= \mathcal{L}^{-1} \left[ \psi' + 2ikC_0^2 \cdot U^* (s) \cdot \mathcal{L} \left[ \exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{PD} \cdot t) \right] \right] \quad (\cdot (41))
\]

\[
= \frac{1}{\omega_B^{PD} (1 + k^2 \omega_B^2/3)} \cdot \psi (k, t) \circ \left[ \exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{PD} \cdot t) \right] + \frac{2ikC_0^2}{\omega_B^{PD} (1 + k^2 \omega_B^2/3)} \cdot U (t) \circ \left[ \exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{PD} \cdot t) \right] 
\]

\[
= \frac{1}{\omega_B^{PD} (1 + k^2 \omega_B^2/3)} \int_0^t \exp[-\zeta \omega_B \cdot (t - r)] \cdot \sin[\omega_B^{PD} \cdot (t - r)] \cdot \psi (k, r) \, dr 
\]

\[
+ \frac{2ikC_0^2}{\omega_B^{PD} (1 + k^2 \omega_B^2/3)} \int_0^t \exp[-\zeta \omega_B \cdot (t - r)] \cdot \sin[\omega_B^{PD} \cdot (t - r)] \cdot U (r) \, dr,
\]

where the notation \( \circ \) denotes Laplace convolution and \( \mathcal{L}^{-1} \mathcal{L} \) is the identity. This completes the proof of (58). \( \Box \)

The inverse Laplace transform of \( f_2 \) is written as (simply through \( B \) in (41))

\[
\mathcal{L}^{-1}(f_2) = \mathcal{L}^{-1}[B] \quad (\cdot (54))
\]

\[
= \mathcal{L}^{-1} \left[ \frac{2ikh_0^2 \cdot U^* (s)}{3(1 + k^2 \omega_B^2/3)} \right] \quad (\cdot (41))
\]

\[
= \frac{2ikh_0^2 \cdot U (t)}{3(1 + k^2 \omega_B^2/3)},
\]

by linearity of \( \mathcal{L}^{-1} \), which proves (59). \( \Box \)

Similarly as above,

\[
\mathcal{L}^{-1}(f_3) = \mathcal{L}^{-1} \left[ -2B \zeta \omega_B \cdot \mathcal{L} \left[ \exp(-\zeta \omega_B \cdot t) \cdot \cos(\omega_B^{PD} \cdot t) \right] \right] \quad (\cdot (54))
\]

\[
= \frac{-4ikh_0^2 \zeta \omega_B}{3(1 + k^2 \omega_B^2/3)} \cdot \mathcal{L}^{-1}[U^* (s)] \circ \mathcal{L}^{-1}[\mathcal{L} \left[ \exp(-\zeta \omega_B \cdot t) \cdot \cos(\omega_B^{PD} \cdot t) \right]] \quad (\cdot (41))
\]

\[
= \frac{-4ikh_0^2 \zeta \omega_B}{3(1 + k^2 \omega_B^2/3)} \cdot U (t) \circ \left[ \exp(-\zeta \omega_B \cdot t) \cdot \cos(\omega_B^{PD} \cdot t) \right] \quad (\cdot (41))
\]

\[
= \frac{-4ikh_0^2 \zeta \omega_B}{3(1 + k^2 \omega_B^2/3)} \int_0^t \exp[-\zeta \omega_B \cdot (t - r)] \cdot \cos[\omega_B^{PD} \cdot (t - r)] \cdot U (r) \, dr,
\]

(64)

which completes the proof of (60). \( \Box \)

Finally, \( \mathcal{L}^{-1}(f_4) \) has the form

\[
\mathcal{L}^{-1}(f_4) = \mathcal{L}^{-1} \left[ \frac{B \omega_B (2 \zeta^2 - 1)}{\sqrt{1 - \zeta^2}} \cdot \mathcal{L} \left[ \exp(-\zeta \omega_B \cdot t) \right] \cdot \sin(\omega_B^{PD} \cdot t) \right] \quad (\cdot (56))
\]

\[
= \frac{2ikh_0^2 \omega_B (2 \zeta^2 - 1)}{3(1 + k^2 \omega_B^2/3) \sqrt{1 - \zeta^2}} \cdot \mathcal{L}^{-1}[U^* (s)] \circ \mathcal{L}^{-1}[\mathcal{L} \left[ \exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{PD} \cdot t) \right] \quad (\cdot (41))
\]

\[
= \frac{2ikh_0^2 \omega_B (2 \zeta^2 - 1)}{3(1 + k^2 \omega_B^2/3) \sqrt{1 - \zeta^2}} \cdot U (t) \circ \left[ \exp(-\zeta \omega_B \cdot t) \cdot \sin(\omega_B^{PD} \cdot t) \right] \quad (\cdot (41))
\]

\[
= \frac{2ikh_0^2 \omega_B (2 \zeta^2 - 1)}{3(1 + k^2 \omega_B^2/3) \sqrt{1 - \zeta^2}} \int_0^t \exp[-\zeta \omega_B \cdot (t - r)] \cdot \sin[\omega_B^{PD} \cdot (t - r)] \cdot U (r) \, dr.
\]

(65)

This finishes proving (61). \( \Box \)

We are now in a position to take the inverse Laplace–Fourier transform of \( \overline{\tilde{p}}_{\text{ext}} \) in (42).

**Lemma 8.** The inverse Laplace–Fourier transforms of \( f_i \), \( i = 1, 2, 3, 4 \), in (53)–(56), are evaluated in the forms
\( \mathcal{F}^{-1} \mathcal{L}^{-1} (f_1) = \frac{1}{2 \pi} \int_{-\infty}^{\infty} \left\{ \frac{2ikc_0^2}{\bar{\omega}_B^{pD} (1 + k^2 h_0^2/3)} \cdot \exp\left[ -\bar{\omega}_B \cdot (t - \tau) \right] \cdot \sin\left[ \bar{\omega}_B^{pD} \cdot (t - \tau) \right] \cdot \frac{1}{\bar{\omega}_B^{pD} (1 + k^2 h_0^2/3)} \cdot \cos[k(\xi - x)] \cdot \varphi(\xi, \tau) d\xi d\tau \right\} e^{-ikx} dk \) \quad (\because (24))

\[
\mathcal{F}^{-1} \mathcal{L}^{-1} (f_2) = \frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{2kh_0^2}{3 (1 + k^2 h_0^2/3)} \cdot U(t) \cdot e^{-ikx} dk \quad (\because (59) and (24))
\]

Here, we note that the \( \mathcal{F}^{-1} \mathcal{L}^{-1} (f_1) \) remains real because the real parts of the integrands in the last two integrals in the above are even in \( k \), whereas the imaginary parts are odd. We are thus led to (66), completing the proof. \( \square \)

\( \mathcal{F}^{-1} \mathcal{L}^{-1} (f_3) = \frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{2kh_0^2}{3 (1 + k^2 h_0^2/3)} \cdot \sin kx \cdot U(t) dk \)
where we have used the fact that the imaginary part of the integrand in the above is odd in $k$ while the real part is even. An analytic expression is obtained for the above integral, that is,

$$F^{-1}F^{-1}(f_3) = U(t) \cdot \frac{2}{\pi} \int_0^\infty \left( \frac{h_0^2/3}{1 + k^2 h_0^2/3} \right) \cdot k \sin kx \, dk$$

$$= U(t) \cdot \frac{2}{\pi} \left( \frac{\pi x}{2 |x|} \cdot \exp \left[ \frac{-|x|\sqrt{3}}{h_0} \right] \right)$$

$$= \frac{x}{|x|} \cdot U(t) \cdot \exp \left[ \frac{|x|\sqrt{3}}{h_0} \right].$$

3.5. Integral Formulation Involving $u_{\text{Ext}}$. Recalling that the $u_{\text{Ext}}$ in Definition 2 is recovered formally through the identities by noting from (mathematical integral) table that

$$\int_0^\infty \frac{b k \sin kx}{1 + b k^2} \, dk = \frac{\pi x}{2 |x|} \exp \left[ \frac{-|x|}{\sqrt{b}} \right], \quad \text{for } b \geq 0. \quad (73)$$

This proves (67). $\square$

For the proof of (68), we calculate $F^{-1}F^{-1}(f_3)$, being written as

$$_{F^{-1}F^{-1}(f_3)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ -\frac{4ik h_0^2 \omega_B}{3(1 + k^2 h_0^2/3)} \right\} \cdot \int_0^t \exp \left[ -\zeta \omega_B \cdot (t - \tau) \right] \cdot \cos \left[ \omega_B^D \cdot (t - \tau) \right] \cdot U(\tau) \, d\tau \cdot e^{-ikx} \, dk$$

$$= \frac{1}{\pi} \int_0^t \int_{-\infty}^{\infty} \frac{2ik h_0^2 \omega_B}{3(1 + k^2 h_0^2/3)} \left( 2\zeta^2 - 1 \right) \cdot \exp \left[ -\zeta \omega_B \cdot (t - \tau) \right] \cdot \sin \left[ \omega_B^D \cdot (t - \tau) \right] \cdot U(\tau) \cdot e^{-ikx} \, d\tau \, dk$$

$$= \frac{1}{\pi} \int_0^t \int_{-\infty}^{\infty} \frac{2k h_0^2 \omega_B}{3(1 + k^2 h_0^2/3)} \left( 2\zeta^2 - 1 \right) \cdot \exp \left[ -\zeta \omega_B \cdot (t - \tau) \right] \cdot \sin \left[ \omega_B^D \cdot (t - \tau) \right] \cdot \sin kx \cdot U(\tau) \, d\tau \, dk,$$

where the imaginary part in the above vanishes because the imaginary part of the integrand (in the second line in the above) is odd in $k$, but the real part of the integrand is even. $\square$

To show that (69) is true, recall the Fourier inverse (24) together with (61), i.e.,

$$u_{\text{Ext}} = F^{-1}F^{-1}(\Pi_{\text{Ext}}) = \sum_{i=1}^{4} F^{-1}F^{-1}(f_i),$$

because of (57), we readily arrive at the following theorem from the concerning lemmas without proof, but with a short note that the sum of the last line of (66) and (69) becomes
where (40) is utilized.

**Theorem 1.** Let \( u_{\text{Ext}} \) be satisfied by the Boussinesq equation (12) subject to (13)–(15); i.e., \( u_{\text{Ext}} \) is a solution for the (extended) initial-boundary value problem in Definition 2. Then, the following integral relation holds

\[
u_{\text{Ext}} = \frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{2K^2}{\omega_B^2 (1 + K^2h_B^2/3)} \cdot \exp\left[-2i\omega_B \cdot (t - \tau)\right] \cdot \sin\left[\frac{\omega_B^D \cdot (t - \tau)}{\omega_B}\right] \cdot \sin kt \cdot U(\tau) \, dk \, d\tau
\]

\[
+ \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{2k\omega_B^2 (2^2 - 1)}{\omega_B^2 (1 + K^2h_B^2/3)} \cdot \exp\left[-i\omega_B \cdot (t - \tau)\right] \cdot \sin\left[\frac{\omega_B^D \cdot (t - \tau)}{\omega_B}\right] \cdot \sin kt \cdot U(\tau) \, dk \, d\tau
\]

\[
= \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{2K^2}{\omega_B^2 (1 + K^2h_B^2/3)} \cdot \exp\left[-2i\omega_B \cdot (t - \tau)\right] \cdot \sin\left[\frac{\omega_B^D \cdot (t - \tau)}{\omega_B}\right] \cdot \sin kt \cdot U(\tau) \, dk \, d\tau
\]

\[
+ \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{2k\omega_B^2 (2^2 - 1)}{\omega_B^2 (1 + K^2h_B^2/3)} \cdot \exp\left[-i\omega_B \cdot (t - \tau)\right] \cdot \sin\left[\frac{\omega_B^D \cdot (t - \tau)}{\omega_B}\right] \cdot \sin kt \cdot U(\tau) \, dk \, d\tau
\]

\[
= \frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{3K^2}{\omega_B^2 (1 + K^2h_B^2/3)} \cdot \exp\left[-2i\omega_B \cdot (t - \tau)\right] \cdot \sin\left[\frac{\omega_B^D \cdot (t - \tau)}{\omega_B}\right] \cdot \sin kt \cdot U(\tau) \, dk \, d\tau
\]

where (40) is utilized.

4.1. Preliminaries. Here, we need to focus on examining the triple integral, denoted by \( I \), in (78) after substituting (18), i.e.,

\[
I = \frac{1}{\pi} \int_0^\infty \int_0^\infty \int_0^\infty K(x - \xi, t - \tau, k) \cdot \left[-\frac{1}{2} \left(u_{\text{Ext}}\right)_\tau \right] + \alpha \cdot (u_{\text{Ext}}) \cdot \beta \cdot u_{\text{Ext}} \, d\xi \, dk \, dr,
\]

with \( K \) denoted by

\[
K(x - \xi, t - \tau, k) = \frac{\exp\left[-2i\omega_B \cdot (t - \tau)\right] \cdot \sin\left[\frac{\omega_B^D \cdot (t - \tau)}{\omega_B}\right]}{\omega_B^D (1 + K^2h_B^2/3)} \cdot \cos[k(\xi - x)].
\]

The integration domain of the above integral \( I \) can be reduced by recalling that \( u_{\text{Ext}} \) is the odd extension of \( u \) with respect to \( x \) as was pointed out in Lemma 1, making the \( I \) depend upon \( u \)(instead of \( u_{\text{Ext}} \)).

**Lemma 9.** The integral \( I \) in (79) can be regarded as a functional of \( u \)(instead of \( u_{\text{Ext}} \)) through the identity

\[
I = \frac{1}{\pi} \int_0^\infty \int_0^\infty \int_0^\infty \tilde{K}(x, t - \tau, k) \cdot \left[-\frac{1}{2} \left(u^2\right)_\tau - \alpha \cdot u_{\tau} - \beta \cdot u\right] \, d\xi \, dk \, dr,
\]

where

\[
\tilde{K}(x, t - \tau, k) = \frac{-2 \sin \frac{k_x}{k} \cdot \sin kt}{\omega_B^D (1 + K^2h_B^2/3)} \cdot \exp\left[-2i\omega_B \cdot (t - \tau)\right] \cdot \sin\left[\frac{\omega_B^D \cdot (t - \tau)}{\omega_B}\right].
\]

**Proof.** We write \( I \) in (79) as the sum of two integrals:
\[ I = \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty K(x - \xi, t - \tau, k) \cdot \left\{ \frac{1}{2} [-u(-\xi, \tau)]^2_{\xi \tau} + \alpha \cdot [-u(-\xi, \tau)]_\tau + \beta \cdot [-u(-\xi, \tau)]_\tau \right\} d\xi \, dk \, dr \\
+ \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty K(x - \xi, t - \tau, k) \cdot \left\{ -\frac{1}{2} (u^2)_{\xi \tau} + \alpha \cdot u_{\tau} + \beta \cdot u \right\} d\xi \, dk \, dr, \tag{83} \]

based on the fact that \( u_{\text{ext}} \) is the odd extension of \( u \) with respect to \( x \) by Lemma 1.

Next, let us take a look at the first integral, denoted by \( J \), in the above and use the change of variables \( \sigma = -\xi \). We then have

\[ J = \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty K(x - \xi, t - \tau, k) \cdot \left\{ \frac{1}{2} [-u^2(-\xi, \tau)]_{\xi \tau} + \alpha \cdot [-u(-\xi, \tau)]_\tau + \beta \cdot [-u(-\xi, \tau)]_\tau \right\} d\xi \, dk \, dr \\
= \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty K(x + \sigma, t - \tau, \sigma) \cdot \left\{ -\frac{1}{2} [u^2(\sigma, \tau)]_{\sigma \tau} - \alpha \cdot u_{\tau}(\sigma, \tau) - \beta \cdot u(\sigma, \tau) \right\} (-d\sigma) \, dk \, dr \\
= \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty K(x + \sigma, t - \tau, \sigma) \cdot \left\{ -\frac{1}{2} [u^2(\sigma, \tau)]_{\sigma \tau} - \alpha \cdot u_{\tau}(\sigma, \tau) - \beta \cdot u(\sigma, \tau) \right\} d\sigma \, dk \, dr \\
= \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty K(x + \xi, t - \tau, k) \cdot \left\{ \frac{1}{2} [u^2(\xi, \tau)]_{\xi \tau} - \alpha \cdot u_{\tau}(\xi, \tau) - \beta \cdot u(\xi, \tau) \right\} d\xi \, dk \, dr, \tag{84} \]

where we utilize the chain rule

\[ \frac{\partial}{\partial \xi^i} [u^2(-\xi, \tau)]_\tau = \frac{\partial}{\partial \sigma} \left\{ \frac{\partial}{\partial \xi} [u^2(\sigma, \tau)]_\tau \right\} \]

\[ = \frac{\partial}{\partial \sigma} \left\{ \frac{\partial}{\partial \sigma} [u^2(\sigma, \tau)]_{\sigma \tau} \right\} = -[u^2(\sigma, \tau)]_{\sigma \tau}, \tag{85} \]

\[ I = \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty K(x + \xi, t - \tau, k) \cdot \left\{ \frac{1}{2} [u^2(\xi, \tau)]_{\xi \tau} - \alpha \cdot u_{\tau}(\xi, \tau) - \beta \cdot u(\xi, \tau) \right\} d\xi \, dk \, dr \\
+ \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty K(x - \xi, t - \tau, k) \cdot \left\{ -\frac{1}{2} (u^2)_{\xi \tau} + \alpha \cdot u_{\tau} + \beta \cdot u \right\} d\xi \, dk \, dr \\
= \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty [K(x + \xi, t - \tau, \xi) - K(x - \xi, t - \tau, \xi)] \cdot \left\{ \frac{1}{2} [u^2(\xi, \tau)]_{\xi \tau} - \alpha \cdot u_{\tau} - \beta \cdot u \right\} d\xi \, dk \, dr, \tag{86} \]

with a note on

\[ \bar{K}(x, t - \tau; k; \xi) = K(x + \xi, t - \tau, k) - K(x - \xi, t - \tau, k), \]

\[ = \frac{\exp[-\xi \bar{\omega}_B \cdot (t - \tau)] \cdot \sin[\bar{\omega}_B \cdot (t - \tau)]}{\bar{\omega}_B^{PD} (1 + k^2 h_0^2/3)} \cdot \cos[k(\xi - x)] - \cos[(\xi - x)], \tag{87} \]
because \( \cos(a + b) - \cos(a - b) = -2\sin a \sin b \) for \( a, b \in \mathbb{R} \).
This completes the proof.

4.2. Constructing Integral Equation. Using the result presented in the preceding section, we finally arrive at a nonlinear integral equation identical with the original initial-boundary value problem, i.e., the Boussinesq equation (1) subject to (4)–(6). We start with integration by parts.

**Lemma 10.** Given a kernel \( \bar{K} \) in (82), the following integral, denoted by \( I_1 \), is calculated as

\[
I_1 = \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty \bar{K}(x, t - \tau, k; \xi) \cdot \left[ \frac{1}{2} (u^2)_{\xi} \right] d\xi dk d\tau
\]

\[
= \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty k \cos k \xi \cdot \sin kx \cdot \frac{\alpha p D}{\tilde{\omega}_B (1 + k^2 h_0^2/3)} \exp[-\zeta \tilde{\omega}_B \cdot (t - \tau)]
\cdot \left[ \tilde{\omega}_B \cdot \sin[\tilde{\omega}_B^p \cdot (t - \tau)] - \tilde{\omega}_B^p \cdot \cos[\tilde{\omega}_B^p \cdot (t - \tau)] \right]
\cdot u^2 d\xi dk d\tau.
\]

**Proof.** By integration by parts with respect to spatial variable \( x \),

\[
I_1 = \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty \bar{K}(x, t - \tau, k; \xi) \cdot \left[ \frac{1}{2} (u^2) \right]_{\xi} d\xi dk d\tau
\]

\[
= \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty \bar{K}(x, t - \tau, k; \xi) \cdot \left[ \frac{1}{2} (u^2) \right]_{\xi}^{\xi = \infty} d\xi dk d\tau
\]

\[
- \int_0^\infty \bar{K}_t(x, t - \tau, k; \xi) \cdot \left[ \frac{1}{2} (u^2) \right]_{\tau} d\xi dk d\tau.
\]

The above is shortened to

\[
I_1 = \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty \bar{K}(x, t - \tau, k; \xi) \cdot \left[ \frac{1}{2} (u^2) \right]_{\xi} d\xi dk d\tau,
\]

because the far field condition (6) implies that \( u \) vanishes as \( \xi \rightarrow \infty \) and \( \bar{K}(x, t - \tau, k; \xi = 0) \) = 0 by (82).

Next, we interchange the order of integration in (90) due to Fubini’s theorem as

\[
I_1 = \frac{1}{\pi} \int_0^\infty \int_0^\infty \int_0^t \bar{K}_t(x, t - \tau, k; \xi) \cdot \left[ \frac{1}{2} (u^2) \right]_{\tau} d\xi dk d\tau.
\]

(91)

Then, we apply integration by parts with respect to time, leading to

\[
I_1 = \frac{1}{\pi} \int_0^\infty \int_0^\infty \int_0^t \bar{K}_t(x, t - \tau, k; \xi) \cdot \left[ \frac{1}{2} (u^2) \right]_{\tau} d\xi dk d\tau
\]

\[
- \int_0^t \bar{K}_t(x, t - \tau, k; \xi) \cdot \left[ \frac{1}{2} (u^2) \right]_{\tau} d\xi dk d\tau
\]

\[
= \frac{1}{\pi} \int_0^\infty \int_0^\infty \int_0^t \bar{K}_t(x, t - \tau, k; \xi) \cdot \left[ \frac{1}{2} (u^2) \right]_{\tau} d\xi dk d\tau
\]

\[
= \frac{1}{\pi} \int_0^\infty \int_0^\infty \int_0^t \bar{K}_t(x, t - \tau, k; \xi) \cdot \left[ \frac{1}{2} (u^2) \right]_{\tau} d\xi dk d\tau
\]

\[
\cdot \tilde{\omega}_B \sin[\tilde{\omega}_B^p \cdot (t - \tau)] - \tilde{\omega}_B^p \cdot \cos[\tilde{\omega}_B^p \cdot (t - \tau)]
\]

\[
\cdot u^2 d\xi dk d\tau,
\]

(92)

because \( u \) vanishes at \( t = 0 \) from initial condition (4), \( \tilde{K}(x, t - \tau, k; \xi) = 0 \) from the derivative

\[
\tilde{K}_t(x, t - \tau, k; \xi) = \frac{-2k \cos k \xi \cdot \sin kx}{\tilde{\omega}_B^p (1 + k^2 h_0^2/3)} \exp[-\zeta \tilde{\omega}_B \cdot (t - \tau)]
\cdot \tilde{\omega}_B^p \cdot (t - \tau),
\]

\[
\tilde{K}_t(x, t - \tau, k; \xi) = \frac{-2k \cos k \xi \cdot \sin kx}{\tilde{\omega}_B^p (1 + k^2 h_0^2/3)} \exp[-\zeta \tilde{\omega}_B \cdot (t - \tau)]
\cdot \tilde{\omega}_B^p \cdot (t - \tau),
\]

(93)

This completes the proof.

**Lemma 11.** The following triple integral, termed \( I_2 \), involving \( u_t \) is evaluated as

\[
I_2 = \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty \tilde{K}_t(x, t - \tau, k; \xi) \cdot (-\alpha \cdot u_t) d\xi dk d\tau
\]

\[
= \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty \tilde{K}_t(x, t - \tau, k; \xi) \cdot (-\alpha \cdot u_t) d\xi dk d\tau
\]

\[
\cdot \tilde{\omega}_B \sin[\tilde{\omega}_B^p \cdot (t - \tau)] - \tilde{\omega}_B^p \cdot \cos[\tilde{\omega}_B^p \cdot (t - \tau)]
\]

\[
\cdot (\alpha \cdot u) d\xi dk d\tau,
\]

(94)

in which the kernel \( \bar{K} \) is given in (82).

**Proof.** We begin by rewriting the triple integral \( I_2 \) as

\[
I_2 = \frac{1}{\pi} \int_0^\infty \int_0^\infty \int_0^t \tilde{K}_t(x, t - \tau, k; \xi) \cdot (-\alpha \cdot u_t) d\xi dk d\tau
\]

\[
- \int_0^t \tilde{K}_t(x, t - \tau, k; \xi) \cdot (-\alpha \cdot u_t) d\tau d\xi dk,
\]

(95)

due to Fubini’s theorem, where integration by parts with respect to time is used with interchange of the order of
integration. Here, notice that \( u \) equals zero when \( t = 0 \) from initial condition (4) and \( \bar{K}(x, t - \tau, k; \xi) = 0 \) from (82).

On the other hand, the differentiation of (82) with respect to \( \tau \) gives

\[
\bar{K}_\tau(x, t - \tau, k; \xi) = \frac{-2 \sin k \xi \cdot \sin k x}{\omega_B^{pD}(1 + k^2 \omega_B^{pD})} \cdot \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \\
\cdot \{ \zeta \bar{\omega}_B \cdot \sin[\omega_B^{pD}(t - \tau)] - \omega_B^{pD} \cdot \cos[\omega_B^{pD}(t - \tau)] \}.
\]

(96)

This results in

\[
I_2 = \frac{1}{\pi} \int_0^\infty \int_0^\infty \int_0^\infty \bar{K}_\tau(x, t - \tau, k; \xi) \cdot (\alpha \cdot u) \cdot d\xi \cdot dk \cdot dr
\]

\[
= \frac{1}{\pi} \int_0^\infty \int_0^\infty \int_0^\infty \frac{-2 \sin k \xi \cdot \sin k x}{\omega_B^{pD}(1 + k^2 \omega_B^{pD})} \cdot \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \\
\cdot \{ \zeta \bar{\omega}_B \cdot \sin[\omega_B^{pD}(t - \tau)] - \omega_B^{pD} \cdot \cos[\omega_B^{pD}(t - \tau)] \} \\
\cdot (\alpha \cdot u) \cdot d\xi \cdot dk \cdot dr.
\]

(97)

as required, which completes the proof.

\[\square\]

Remark 3. Noting that the integral \( I \) in (81) decomposes into the sum

\[
I = I_1 + I_2 + \frac{1}{\pi} \int_0^\infty \int_0^\infty \int_0^\infty \bar{K}(x, t - \tau, k; \xi) \cdot (-\beta \cdot u) \cdot d\xi \cdot dk \cdot dr,
\]

(98)

from (88) and (94), (81) may arrange to

\[
I = -\frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty k \cdot \cos k \xi \cdot \sin k x \\
\cdot \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \\
\cdot \{ \zeta \bar{\omega}_B \cdot \sin[\omega_B^{pD}(t - \tau)] - \omega_B^{pD} \cdot \cos[\omega_B^{pD}(t - \tau)] \} \\
\cdot (\alpha \cdot u) \cdot d\xi \cdot dk \cdot dr
\]

\[
- \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty 2 \sin k \xi \cdot \sin k x \\
\cdot \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \\
\cdot \{ \zeta \bar{\omega}_B \cdot \sin[\omega_B^{pD}(t - \tau)] - \omega_B^{pD} \cdot \cos[\omega_B^{pD}(t - \tau)] \} \\
\cdot (\beta \cdot u) \cdot d\xi \cdot dk \cdot dr,
\]

(99)

by Lemmas 10 and 11.

The following theorem is immediately directed from Theorem 1 plus Remarks 2 and 3.

**Theorem 2.** Let \( \xi \) be a dissipative parameter in (39) depending on two pseudoparameters \( \alpha, \beta > 0 \), and let \( \bar{\omega}_B \) and \( \omega_B^{pD} \) be the two pseudofrequencies in (36) and (40), respectively. Further, let \( u \) be a solution for the original Boussinesq equation (1) subject to (4)–(6). Then, \( u \) satisfies the following nonlinear integral equation:

\[
u(x, t) = \frac{2}{\pi} \int_0^t \int_0^\infty \frac{3 \xi^2 + 4 \xi^2 \omega_B^2}{3 \omega_B^{pD}(1 + k^2 \omega_B^{pD})} \cdot k \cdot \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \\
\cdot \sin[\omega_B^{pD}(t - \tau)] \cdot \sin k x \cdot U(\tau) \cdot d\xi \cdot dk \cdot dr
\]

\[
+ U(t) \cdot \exp \left[ \frac{-x \sqrt{3}}{h_0} \right]
\]

\[
- \frac{1}{\pi} \int_0^t \int_0^\infty \frac{4 k h_0^2 \zeta \bar{\omega}_B \cdot \sin k x}{3 (1 + k^2 \omega_B^{pD})} \\
\cdot \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \cdot \cos[\omega_B^{pD}(t - \tau)] \cdot U(\tau) \cdot d\xi \cdot dk \cdot dr
\]

\[
- \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty \frac{k \cdot \cos k \xi \cdot \sin k x}{\omega_B^{pD}(1 + k^2 \omega_B^{pD})} \cdot \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \\
\cdot \{ \zeta \bar{\omega}_B \cdot \sin[\omega_B^{pD}(t - \tau)] - \omega_B^{pD} \cdot \cos[\omega_B^{pD}(t - \tau)] \}
\]

\[
\cdot \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty 2 \sin k \xi \cdot \sin k x \\
\cdot \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \\
\cdot \{ \zeta \bar{\omega}_B \cdot \sin[\omega_B^{pD}(t - \tau)] - \omega_B^{pD} \cdot \cos[\omega_B^{pD}(t - \tau)] \}
\]

\[
\cdot (\alpha \cdot u) \cdot d\xi \cdot dk \cdot dr
\]

\[
+ \frac{1}{\pi} \int_0^t \int_0^\infty \int_0^\infty 2 \sin k \xi \cdot \sin k x \\
\cdot \exp[-\zeta \bar{\omega}_B \cdot (t - \tau)] \\
\cdot \{ \zeta \bar{\omega}_B \cdot \sin[\omega_B^{pD}(t - \tau)] - \omega_B^{pD} \cdot \cos[\omega_B^{pD}(t - \tau)] \}
\]

\[
\cdot (\beta \cdot u) \cdot d\xi \cdot dk \cdot dr
\]

(100)
Corollary 1. The nonlinear integral equation (100) in Theorem 2 reduces to
\[
u(x, t) = \frac{2}{\pi} \int_0^t \int_0^\infty \frac{3c_0^2 - h_0^2 \omega_B^2}{3\omega_B(1 + k^2h_0^2/3)} \cdot k \cdot \sin[k \cdot \omega_B \cdot (t - \tau)] \cdot \sin kx \cdot U(\tau) \, dk \, dr + U(t) \cdot \exp\left[-\frac{x\sqrt{3}}{h_0}\right] + 1 \cdot \frac{1}{\pi} \int_0^t \int_0^\infty \frac{k \cos k\xi \cdot \sin kx}{1 + k^2h_0^2/3} \cdot \cos[k \cdot \omega_B \cdot (t - \tau)] \cdot \cdot n^2 d\xi \,dk \,dr,
\]
(101)
as \alpha, \beta \to 0.

Proof. We first note that the last two integral terms in (100) vanish, as \( \alpha, \beta \to 0 \). It thus suffices to show that
\[
\frac{3c_0^2 - h_0^2 \omega_B^2}{3\omega_B(1 + k^2h_0^2/3)} \cdot \frac{k \cos k\xi \cdot \sin kx}{1 + k^2h_0^2/3} \to \frac{3c_0^2 - h_0^2 \omega_B^2}{3\omega_B(1 + k^2h_0^2/3)}, \quad \text{as } \alpha, \beta \to 0,
\]
in the first integral in (100), which is immediate because
\[
\zeta = \frac{\alpha}{2(c_0^2k^2 + \beta)} \cdot \sqrt{\frac{c_0^2k^2 + \beta}{1 + (k^2h_0^2/3)}} \to 0, \quad \text{for } k > 0, \text{as } \alpha \to 0,
\]
(103)

by (36) and (39). And then, the integrand in the fourth term in (100) has an asymptotic form:
\[
k \cos k\xi \cdot \sin kx \cdot \frac{k \cos k\xi \cdot \sin kx}{\omega_B^2(1 + k^2h_0^2/3)} \cdot \exp\left[-\zeta \omega_B \cdot (t - \tau)\right] \cdot \left\{\zeta \omega_B \cdot \sin kx \cdot \omega_B^P \cdot (t - \tau)\right\} \cdot \left\{-\omega_B \cdot \cos[k \cdot \omega_B \cdot (t - \tau)]\right\}\]
(104)
(105)
and
\[
= \frac{k \cos k\xi \cdot \sin kx}{1 + k^2h_0^2/3} \cdot \cos[k \cdot \omega_B \cdot (t - \tau)],
\]
as \( \alpha, \beta \to 0 \). This completes the proof. 

Remark 4. The kernels in integral equation (100) have singularities of (simple) poles at
\[
k = \pm \frac{\sqrt{3}}{c_0} i, \pm \frac{3}{h_0} \left(-1 \pm \frac{\alpha}{2}\right), \pm \frac{\sqrt{3}}{h_0} i,
\]
(106)
All the singularities become off the real line \( \mathbb{R} \) with a proper choice of \( \alpha, \beta > 0 \), which tend further to
\[
k = 0, \pm \frac{\sqrt{3}}{h_0} i
\]
as \( \alpha, \beta \to 0 \).

5. Application of Integral Formalism

In this section, we shall apply successive approximations to integral equation (100) constructed in the previous section. This produces an iterative strategy, whereby we investigate iterative numerical solutions of a solitary wave. Furthermore, we will examine the two pseudoparameters \( \alpha, \beta \) introduced artificially in this paper effect the convergence of the iterative solutions. For this reason, the two introduced parameters can be viewed as control ones for the iterative strategy.

5.1. Successive Approximations. Numerical solutions for the original initial-boundary value problem of the Boussinesq equation (1):
\[
u_{xt} - c_0^4u_{xx} + \frac{1}{2}(u^2)_{xt} = \frac{1}{2}h_0^2u_{xxtt}, \quad x > 0,
\]
(107)
subject to (4)–(6) may be found by applying successive approximations of the Banach fixed point theorem to (100). The theorem guarantees the existence and uniqueness of fixed points of certain self-operators of metric spaces, providing a constructive method, namely, the successive approximations, to find those fixed points [6–11]; in the present study, we choose the same space as the usual metric space [6–11] for the use of the theorem.

We bear in mind that (100) can be thought of a fixed form and we will seek its fixed point (solution) \( u \) through a recurrence relation with the pseudoparameters \( \alpha, \beta \) for \( n = 0, 1, 2, \ldots \), as expressed as
\[ u_{n+1}(x,t) = \frac{2}{\pi} \int_0^\infty \frac{3c_0^2 + h_0^2 \omega_b^2 (2c_0^2 - 1)}{3\omega_b^p (1 + k^2 h_0^2/3)} \cdot k \cdot \exp[-\xi \omega_b \cdot (t - \tau)] \cdot \sin[\omega_b^p \cdot (t - \tau)] \cdot \sin kx \cdot U(\tau) \, dk \, d\tau + U(t) \cdot \exp \left[ \frac{x \sqrt{3}}{h_0} \right] \]

\[ = \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{4kh_0^2 \xi \omega_b \cdot \sin kx}{3(1 + k^2 h_0^2/3)} \cdot \exp[-\xi \omega_b \cdot (t - \tau)] \cdot \cos[\omega_b^p \cdot (t - \tau)] \cdot U(\tau) \, dk \, d\tau \]

\[ = \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{2k \cos k \xi \cdot \sin kx}{\omega_b^p (1 + k^2 h_0^2/3)} \cdot \exp[-\xi \omega_b \cdot (t - \tau)] \cdot \left[ \xi \omega_b \cdot \sin[\omega_b^p \cdot (t - \tau)] - \omega_b^p \cdot \cos[\omega_b^p \cdot (t - \tau)] \right] \cdot \omega_b^q \, dk \, d\xi \, d\tau \]

\[ + \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{2k \cos k \xi \cdot \sin kx}{\omega_b^p (1 + k^2 h_0^2/3)} \cdot \exp[-\xi \omega_b \cdot (t - \tau)] \cdot \left[ \xi \omega_b \cdot \sin[\omega_b^p \cdot (t - \tau)] - \omega_b^p \cdot \cos[\omega_b^p \cdot (t - \tau)] \right] \cdot (\alpha \cdot u_n) \, dk \, d\xi \, d\tau, \]

with (zero) initial guess

\[ u_0 = 0. \tag{109} \]

It is noted that (108) is the recurrence directly obtained from (100).

The recurrence relation (108) and (109) is a new iterative formula for solving the original initial-boundary value problem, which defines recursively a functional sequence \{u_n\} of (108) for \( n = 0, 1, 2, \ldots \).

5.2. Numerical Solutions of a Moving Solitary Wave. It is worth discussing that the original governing equation (1) may be nondimensionalized as (see Appendix)

\[ u'_t' = u'_{xx'} - \left( u'^2 \right)_{x'} + u'_{x'\tau'}, \tag{110} \]

via appropriate dimensionless variables

\[ u' = \frac{u}{2 \sqrt{gh_0}} \]

\[ t' = t \sqrt{\frac{\beta h_0}{h_0}} \]

\[ x' = \frac{x \sqrt{3}}{h_0}. \tag{111} \]

The dimensionless PDE (110) is known to allow a moving solitary wave solution in the form [12, 13]

\[ u'_{exact}(x', t') = a' \cdot \text{sech}^2 \left[ \frac{a'}{6c} \left( x' - c' t' - x'_0 \right) \right], \tag{112} \]

constrained by amplitude dispersion

\[ c' = \frac{a'}{3} + \sqrt{1 + \frac{a'^2}{9}}, \tag{113} \]

in which the constants \( a' \), \( c' \), and \( x'_0 \) denote dimensionless amplitude, dimensionless wave speed, and initial dimensionless position of the solitary wave, respectively. Here, (112) and (113) can be written in original physical variables

\[ u_{exact}(x, t) = \frac{2ac_0 \sqrt{3}}{h_0} \cdot \text{sech}^2 \left[ \frac{ac_0 \sqrt{3}}{ch_0} \left( x - \frac{c}{2} t - x_0 \right) \right], \]

where \( a \) and \( c \) are such that

\[ a' = \frac{a}{h_0}, \tag{115} \]

\[ c' = \frac{c}{2 \sqrt{gh_0}} \]

consistent with (110); in fact, it is straightforward to check that (114) satisfies (1) by direct substitution of (114) into (1).

Our aim here is to simulate the moving solitary wave (114) numerically, which enables us to demonstrate that the iterative strategy (108) works well for (114). For the simulation, we attempt to iterate (108) with initial guess (109) using the null initial conditions (4) as well as the boundary condition at \( x = 0 \) specified by

\[ U(t) = \frac{2ac_0 \sqrt{3}}{h_0} \cdot \text{sech}^2 \left[ \frac{ac_0 \sqrt{3}}{ch_0} \left( -\frac{c}{2} t - x_0 \right) \right], \tag{116} \]

from (114). Here, the Simpson and trapezoidal rules are employed (in space and time, respectively), by setting \( h_0 = \)
0.2 m and \( g = 9.80665 \) m/s\(^2\). The space and time domains are selected as \( 0 < x < 25 \) m (or \( 0 < x' < 216.51 \)) and \( 0 < t < 18 \) sec (or \( 0 < t' < 218.31 \)), respectively, with the increments, \( \Delta x = 0.10 \) m and \( \Delta t = 0.05 \) sec; we choose \( x_0 = -12 \) m (or \( x_0' = -103.92 \)).

Figure 2 shows typical convergence behaviors for the iterative wave solutions with two amplitudes, i.e., \( a' = 0.01 \) and \( a' = 0.02 \), where two pseudoparameters are taken as \( \alpha = 0.0700 \) rad/sec and \( \beta = 0.0490 \) rad/sec\(^2\), which are (alternatively) non-dimensionalized as \( \alpha' = 10^{-2} \) and \( \beta' = 10^{-3} \), respectively, according to the (dimensionless) scaling

\[
\alpha' = \alpha \frac{h_0}{g} \quad \beta' = \beta \frac{h_0}{g} \quad (117)
\]

The first iteration gives a numerical wave solution valid for the initial stage of time evolution and an almost converged solution seems to be reached at \( n = 10 \), which clearly shows a translating (moving) solitary wave.

We next inspect quantitative errors for the iterative wave solutions. Figure 3 plots the nature of convergence of (108).
for various $\beta'$’s, when $\alpha'$ is specified as $\alpha' = 10^{-2}, 10^{-1}$ (which corresponds to $\alpha = 0.0700, 0.7002$ [rad/sec], respectively). Here, errors are estimated by $L_{\infty}$:

$$L_{\infty} = \max_x |u_{\text{exact}} - u_n|,$$  \hspace{1cm} (118)  

where $L_{\infty}$ represents $L$-infinity norm taken over by the space variable $x$ only with reference time $t_{\text{ref}}$ being kept fixed; $t_{\text{ref}} = 18$ sec (i.e., end time). We see that the $L_{\infty}$ decreases as the number of iterations increases. For $\alpha' = 10^{-2}$ and $10^{-1}$, convergence characteristics depend on the pseudoparameter $\beta'$; to be specific, the smaller $\beta'$ would contribute to a higher convergence rate, with a specified $\alpha'$. This tendency is similar to that of Jang [6]. Finally, Table 1 tabulates the numerical values of errors plotted in Figure 3.

### 6. Discussions and Concluding Remarks

This work involves an initial value problem for the Boussinesq equation (1) for depth-average velocity together with an additional constraint of boundary condition, namely, an initial-boundary value problem for (1). In this paper, we have considered a nontrivial boundary condition, while being in contrast with other studies, where the null (or trivial) boundary condition has been usually imposed [6, 14, 15]. Physically, the nontrivial boundary condition considered here may be related with the wavemaker problem in fluid mechanics [16].

One of the main purposes of this paper is to construct an integral equation formalism, equivalent to the initial value problem for (1). Here, the derived formalism is required to be regular and different from the usual existing integral equation ones (e.g., see [17]). For the construction, as a counterpart of the original initial-boundary value problem (defined on the positive real axis), we have first built an auxiliary initial-boundary value problem (defined on the negative real axis). We also have introduced two pseudo-parameters on artificial dissipation to formulate a pseudo-dissipative formulation for the initial-boundary value problem for (1) [6]. This results in formulating a two-parameter family of the regular nonlinear integral equations, which is different from conventional integral-equation formalisms.
By the method of successive approximations, the regular integral equations derived above have immediately led to an iterative formula for solving the initial-boundary value problem. Further, the formula has enabled us to simulate the propagating solitary wave as illustrated in the numerical experiment. The (numerically) simulated results are compared to the exact solution and the agreement is shown to be excellent.

Appendix

Derivation of (110)

We use the chain rule

\[
\begin{align*}
\frac{\partial}{\partial t} &= \frac{\partial}{\partial t'} \frac{dt'}{dt} = \frac{\partial}{\partial t'} \sqrt{\frac{3g}{h_0}}, \\
\frac{\partial}{\partial x} &= \frac{\partial}{\partial x'} \frac{dx'}{dx} = \frac{\partial}{\partial x'} \sqrt{\frac{3}{h_0}}.
\end{align*}
\]  
(A.1)

giving

\[
\begin{align*}
\frac{\partial}{\partial t'} \left( \frac{\partial}{\partial t'} \right) &= \frac{\partial}{\partial t'} \left( \frac{\partial}{\partial t'} \frac{dt'}{dt} \right) = \frac{\partial^2}{\partial t' \partial t} = \frac{3g}{h_0} \frac{\partial^2}{\partial t'^2}, \\
\frac{\partial}{\partial x'} \left( \frac{\partial}{\partial x'} \right) &= \frac{\partial}{\partial x'} \left( \frac{\partial}{\partial x'} \frac{dx'}{dx} \right) = \frac{\partial^2}{\partial x' \partial x} = \frac{3}{h_0} \frac{\partial^2}{\partial x'^2}.
\end{align*}
\]  
(A.2)

because \( dt' = \sqrt{3g/h_0} dx' \) and \( dx' = \sqrt{3/h_0} dt' \) from (111). We are thus led to

\[
\frac{\partial^2 u}{\partial t'^2} = \left( \frac{3g}{h_0} \frac{\partial^2}{\partial t'^2} \right) \left( 2 \sqrt{gh_0} \cdot u' \right) = \frac{6g \sqrt{gh_0}}{h_0} u''_{xx},
\]  
(A.4)

\[
\frac{\partial^2}{\partial x'^2} u = \left( \frac{3}{h_0^2} \frac{\partial^2}{\partial x'^2} \right) \left( 2 \sqrt{gh_0} \cdot u' \right) = \frac{6g \sqrt{gh_0}}{h_0^2} u'_{xx}.
\]  
(A.5)

\[
\frac{1}{2} \left( u'' \right)_{xt} = \frac{1}{2} \left( \frac{6g \sqrt{gh_0}}{h_0} \right) \left( \frac{\partial}{\partial t'} \sqrt{\frac{3g}{h_0}} \right) = \frac{6g}{h_0} \left( \sqrt{\frac{3g}{h_0}} \right) u',
\]

\[
\text{from (A.1)--(A.3). Finally, we substitute (A.4)--(A.3) to (1) and arrive at the result}
\]

\[
\frac{6g \sqrt{gh_0}}{h_0} u''_{xx} - \left( \frac{6g \sqrt{gh_0}}{h_0} \right) u'_{xx} + \frac{6g}{h_0} \left( \frac{\partial}{\partial t'} \right)^2 u^2 = \frac{1}{3} \frac{h_0^2}{h_0} \left( 18g \frac{\sqrt{gh_0}}{h_0} u'_{xx} \right),
\]

(A.8)

where \( c_0 = \sqrt{gh_0} \) from (2), which proves (110).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] L. Debnath, Nonlinear Water Waves, Academic Press, Boston, MA, USA, 1994.
[2] M. Onorato, A. R. Osborne, P. A. E. M. Janssen, and D. Resio, “Four-wave resonant interactions in the classical quadratic Boussinesq equations,” Journal of Fluid Mechanics, vol. 618, pp. 263–277, 2009.
[3] T. S. Jang, “A regular integral equation formalism for solving the standard Boussinesq’s equations for variable water depth,” Journal of Scientific Computing, vol. 75, no. 3, pp. 1721–1756, 2018.
[4] H. El-Zoheiry, “Numerical study of the improved Boussinesq equation,” Chaos, Solitons & Fractals, vol. 14, no. 3, pp. 377–384, 2002.
[5] Q. Wang, Z. Zhang, X. Zhang, and Q. Zhu, “Energy-preserving finite volume element method for the improved Boussinesq equation,” Journal of Computational Physics, vol. 270, pp. 58–69, 2014.
[6] T. S. Jang, “A new dispersion-relation preserving method for integrating the classical Boussinesq equation,” Communications in Nonlinear Science and Numerical Simulation, vol. 43, pp. 118–138, 2017.
[7] T. S. Jang, “An improvement of convergence of a dispersion-relation preserving method for the classical Boussinesq equation,” Communications in Nonlinear Science and Numerical Simulation, vol. 56, pp. 144–160, 2018.
[8] I. Stakgold, Boundary value problems of mathematical physics: volume I,” Macmillien Series in Advance Mathematics and Theoretical Physics, SIAM, New York, NY, USA, 1967.
[9] P. Roman, Some Modern Mathematics for Physicists and Other Outsiders, Vol. 1, Pergamon Press, Inc., Oxford, UK, 1975.
[10] A. N. Kolmogorov, S. V. Fomin, and R. A. Silverman, Introductory Real Analysis, Dover Publications, Mineola, NY, USA, 1975.
[11] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons, Inc., Hoboken, NJ, USA, 1989.
[12] Y. Ucar, B. Karaagac, and A. Esen, “A new approach on numerical solutions of the Improved Boussinesq type equation using quadratic B-spline Galerkin finite element method,” Applied Mathematics and Computation, vol. 270, pp. 148–155, 2015.
[13] L. Iskandar and P. C. Jain, “Numerical solution of equations having inelastic solitary wave interaction,” Computers & Mathematics with Applications, vol. 6, no. 4, pp. 373–383, 1980.
[14] Q. Lin, Y. H. Wu, R. Loxton, and S. Lai, “Linear B-spline finite element method for the improved Boussinesq equation,” Journal of Computational and Applied Mathematics, vol. 224, no. 2, pp. 658–667, 2009.
[15] Z. Zhang and F. Lu, “Quadratic finite volume element method for the improved Boussinesq equation,” Journal of Mathematical Physics, vol. 53, no. 1, p. 013505, 2012.
[16] T. H. Havelock, “LIX Forced surface-waves on water,” The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, vol. 8, no. 51, pp. 569–576, 1929.
[17] T. S. Jang, “A new functional iterative algorithm for the regularized long-wave equation using an integral equation formalism,” Journal of Scientific Computing, vol. 74, no. 3, pp. 1504–1532, 2018.