FINITE TIME BLOW UP FOR A FLUID MECHANICS MODEL WITH NONLOCAL VELOCITY

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ABSTRACT. We study a 1D fluid mechanics model with nonlocal velocity. The equation can be viewed as a fractional porous medium flow, a 1D model of quasi-geostrophic equation, and also a special case of Euler-Alignment system. For strictly positive smooth initial data, global regularity has been proved in [9]. We construct a family of non-negative smooth initial data so that solution loses regularity in finite time. Our result indicates that strict positivity is a critical condition to ensure global regularity of the system. We also extend our construction to the corresponding models in multi-dimensions.

1. Introduction

We are interested in the following 1D continuity equation
\[ \partial_t \rho + \partial_x (\rho u) = 0, \]  \hspace{1cm} (1)
with a nonlocal velocity field
\[ u = H \Lambda^{\alpha-1} \rho, \quad 0 < \alpha < 2, \]  \hspace{1cm} (2)
where \( H \) is the Hilbert transform, and \( \Lambda^s = (-\Delta)^{s/2} \) denotes the nonlocal fractional Laplacian operator. The initial density is set to be non-negative
\[ \rho(x, t)|_{t=0} = \rho_0(x) \geq 0. \]  \hspace{1cm} (3)

The dynamics of \( \rho \) in the system (1)-(3) can be alternatively written as
\[ \partial_t \rho + u \partial_x \rho = -\rho \Lambda^\alpha \rho. \]  \hspace{1cm} (4)

It consists of a nonlocal transport term \( u \partial_x \rho \), and a dissipation term \( -\rho \Lambda^\alpha \rho \) which is nonlinear and nonlocal.

Without the dissipation term, the equation is an active scaler
\[ \partial_t \rho + u \partial_x \rho = 0, \]  \hspace{1cm} (5)
with the velocity \( u \) defined in (2). It arises as 1D simplified models for 2D surface quasi-geostrophic equations. For \( \alpha = 1 \), equation (5) was studied by Córdoba, Córdoba and Fontelos [7], where a finite time loss of \( C^1 \) regularity is shown for some initial data. Silvestre and Vicol [15] proved the similar behavior for \( \alpha \in (0, 2) \). Both results indicate that the transport term intends to drive the dynamics into singularity in finite time.

\begin{footnotesize}
\begin{itemize}
\item \textit{Date:} August 31, 2017.
\item 2010 \textit{Mathematics Subject Classification.} 35Q35, 35Q92.
\item \textit{Key words and phrases.} porous medium flow, quasi-geostrophic equations, Euler-Alignment system, finite time blow up.
\end{itemize}
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With the dissipation term, the equation (4) appears in many models in fluid mechanics. Since the dissipation term has a possible regularizing effect, the understanding of the competition between the transport term and the dissipation term attracts a lot of attentions in recent years.

**Fractional porous medium flow.** The main system (1)-(3) can be viewed as a porous medium equation with fractional potential pressure, where $\rho$ represents the density of the fluid. It was introduced by Caffarelli and Vázquez [2], where an existence theory for weak solutions was established, for $\rho_0 \in L^1$. The regularizing effect was discussed in a series of successive works: [1] for $\alpha \in (0,1) \cup (1,2)$, and [3] for $\alpha = 1$. Their result states that weak solutions of the system with any $L^1$ initial data instantly becomes Hölder continuous, and stays in $C^\gamma$ for all time, with some $\gamma \in (0,1)$. Such regularizing effect is proved in higher dimensions as well.

For $\alpha = 1$, Carrillo, Ferreira and Precioso [5] studied the system in the space of probability measures with bounded second moment. They established a global wellposedness theory by taking advantage of the gradient flow structure of the system in 1D.

**1D model of quasi-geostrophic equation.** Chae, Córdoba, Córdoba and Fontelos [6] considered (1)-(3) with $\alpha = 1$. They interpreted the system as a 1D simplified model of 2D quasi-geostrophic equation in atmospheric science, where $\rho$ represents the temperature of the air subject to a shift ($\rho = \theta + \kappa$ in their notations). They studied the system in the periodic domain $\mathbb{T} = [-1/2, 1/2]$, and focused on propagation of regularity with smooth initial data. The result consists two parts. First, they showed that if $\rho_0 > 0$, then all $H^3$ initial data stays in $H^3$ in all time. Second, they proved that the system loses $C^1$ regularity in finite time, with the initial data chosen as

$$\rho_0(x) = 1 - \cos(2\pi x), \quad x \in \mathbb{T}. \quad (6)$$

The main difference between the two types of initial data is that $\rho_0(x) = 0$ is attained in the latter case. It indicates that the preservation of $C^1$ regularity critically depends on the strict positivity of the initial data.

It is worth noting that $u = H\rho$ when $\alpha = 1$. Some properties and identities of Hilbert transform were crucially used in their proof. So, the extension of the result to general $\alpha \in (0,2)$ is far from trivial.

**Euler-Alignment system.** System (1)-(3) is also related to a biologically motivated complex interacting system modeling collective behaviors. The Cucker-Smale model [8] is an agent-based model governed by Newton’s second law

$$\dot{x}_i = v_i, \quad m\dot{v}_i = F_i := \frac{1}{N} \sum_{j=1}^{N} \psi(|x_i - x_j|)(v_j - v_i), \quad (7)$$

where $(x_i, v_i)_{i=1}^{N}$ represent the position and velocity of agent $i$. The force $F_i$ describes the alignment interaction on velocity, where the influence function $\psi$ characterizes the strength of the velocity alignment between two agents. Naturally, it is a decreasing function of the distance between the agents.

The macroscopic representation of Cucker-Smale model (7), derived through a kinetic system (see [10]), is called Euler-Alignment system. In 1D, it reads

$$\partial_t \rho + \partial_x (\rho u) = 0, \quad (8)$$

$$\partial_t u + u \partial_x u = \int_{\mathbb{R}} \psi(|x - y|)(u(y,t) - u(x,t))\rho(y,t)dy. \quad (9)$$
For the case when $\psi$ is Lipschitz, the system was studied in [16, 4]. A critical threshold phenomenon was discovered: preservation of $C^1$ regularity depends on the choice of initial data. Subcritical initial data lead to global regularity, while supercritical initial data lead to finite time shock formation.

Another case is when $\psi$ is singular, taking the form

$$
\psi(|x|) = \frac{c_\alpha}{|x|^{1+\alpha}}, \quad 0 < \alpha < 2,
$$

with $c_\alpha$ be a positive constant such that

$$
\Lambda^\alpha f = c_\alpha \int_\mathbb{R} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} dy.
$$

One interesting feature of such choice of $\psi$ is that, equation (9) becomes closely related to the Burgers equation with fractional dissipation

$$
\partial_t u + u \partial_x u = -\Lambda^\alpha u,
$$

by enforcing $\rho \equiv 1$. Kiselev, Nazarov and Shterenberg [11] studied (11): when $0 < \alpha < 1$, there exists initial data leading to finite time blow up; when $\alpha \in [1, 2)$, all smooth initial data lead to global regularity.

The Euler-Alignment system (8)-(9) with singular influence function (10) was studied in [9] in the periodic domain. It was shown that for $\alpha \in (0, 2)$, all smooth initial data $\rho_0 > 0$ leads to global regularity. In particular, in the range of $\alpha \in (0, 1)$, the behavior of the solution is very different from the Burgers equation with fractional dissipation, despite their similarity. The global regularity result is extended to more general singular influence function in [12]. See also [13, 14] for a different approach.

As discussed in [9], a useful reformulation of the Euler-Alignment system for $\rho$ and $G = \partial_x u - \Lambda^\alpha \rho$ has the form

$$
\partial_t \rho + \partial_x (\rho u) = 0, \quad \partial_t G + \partial_x (G u) = 0, \quad \partial_x u = \Lambda^\alpha \rho + G.
$$

In particular, if we pick the initial data such that $G_0(x) = \partial_x u_0(x) - \Lambda^\alpha \rho_0(x) \equiv 0$, then $G \equiv 0$ for all $t > 0$, and the dynamics of $\rho$ becomes our main system (11)-(12).

Therefore, the result in [9] implies that for $\alpha \in (0, 2)$, system (11)-(12) with smooth initial data $\rho_0 > 0$ stays smooth in all time. It serves as an extension to the first part of the result in [6] with general $\alpha$.

The main result. In this paper, we focus on (11)-(12) with non-negative initial data $\rho_0$ which is not strictly positive. We construct initial data which lead to a finite time blow up.

**Theorem 1.1.** Consider the system (11)-(12) in the periodic domain $\mathbb{T}$. There exists a family of smooth initial data $\rho_0$ such that the solution loses $C^\alpha$ regularity in finite time.

We compare the theorem with the results discussed above. In [11, 13], it was shown that the dissipation regularize the solution to $C^7$. Theorem 1.1 says that the regularization can not be improved to $C^\alpha$ or smoother. Also, Theorem 1.1 extends the second part of the result in [6] to the general case $\alpha \in (0, 2)$. Finally, together with the result in [9], we know that the assumption $\rho_0 > 0$ is sharp in order to obtain global regularity for our system.

As a direct consequence, we have the following result for Euler-Alignment system.

**Corollary 1.2.** Consider the initial value problem of Euler-Alignment system (8)-(9) with singular influence function $\psi$ defined in (10). There exists smooth initial data $\rho_0 \geq 0$ and $u_0$ such that the solution lose smoothness in finite time.
The choice of initial data could be \( \rho_0 \) from Theorem 1.1 and \( u_0 = H\Lambda^{\alpha-1}\rho_0 \). In finite time, \( \rho \) loses \( C^\alpha \) regularity, and as a consequence \( u \) loses \( C^1 \) regularity.

The rest of the paper is organized as follows. In section 2, we show apriori bounds for the system with some proposed symmetry. In section 3, we construct initial data \( \rho_0 \) and prove finite time blow up. In section 4, we extend the result to systems in multi-dimensional spaces. Finally, in section 5, we make some remarks on related topics for further investigation.

2. Apriori estimates

In this section, we derive some useful estimates for our main system (1)-(3), which will help us to construct initial data and obtain finite time blow up.

We first propose the following even symmetry condition to \( \rho_0 \)

\[
\rho_0(x) = \rho_0(-x). \tag{H1}
\]

Since we consider periodic data, \( \rho_0 \) can be determined by its value in \( x \in [0, 1/2] \). We also note that periodicity and even symmetry preserves in time.

2.1. Maximum principle. Let us assume the initial data is bounded, satisfying

\[
0 \leq \rho_0(x) \leq 1, \quad \forall \ x \in \mathbb{T}. \tag{H2}
\]

Then, \( \rho(\cdot, t) \) satisfies (H2) for all \( t \geq 0 \), due to maximum principle.

**Proposition 2.1** (Maximum principle). Let \( \rho \) be a smooth solution of (1) with initial data \( \rho_0 \) satisfying (H2). Then, \( \rho(\cdot, t) \) satisfies (H2) for all \( t \geq 0 \).

**Proof.** Suppose \( \rho(x, t) \leq 1 \) does not hold for all \( (x, t) \). Then, there exists \( x_0 \) and \( t_0 \) such that

\[
\rho(x_0, t_0) = 1, \quad \rho(x, t_0) \leq 1, \quad \forall \ x \in \mathbb{T}, \quad \text{and} \quad \partial_t \rho(x_0, t_0) > 0.
\]

So the violation first occurs at \( x_0 \) at time \( t_0^+ \).

Since \( \rho(\cdot, t_0) \) attains its maximum at \( x_0 \), we know

\[
\partial_x \rho(x_0, t_0) = 0, \quad \text{and} \quad \Lambda^\alpha \rho(x_0, t_0) \geq 0.
\]

Therefore, from (4) we obtain

\[
\partial_t \rho(x_0, t_0) = -u(x_0, t_0)\partial_x \rho(x_0, t_0) - \rho(x_0, t_0)\Lambda^\alpha \rho(x_0, t_0) \leq 0.
\]

This leads to a contradiction. Therefore, \( \rho(x, t) \leq 1 \) holds for all \( x \in \mathbb{T} \) and \( t \geq 0 \).

Positivity preserving property \( \rho(x, t) \geq 0 \) can be proved similarly. \( \square \)

2.2. Preservation of monotonicity. We make another assumption on \( \rho_0 \).

\[
\rho_0(0) = 0, \quad \partial_x \rho_0(x) \geq 0, \quad \forall \ x \in [0, 1/2], \tag{H3}
\]

namely \( \rho_0 \) is increasing in \( [0, 1/2] \).

The following proposition shows that such monotonicity is preserved in time.

**Proposition 2.2** (Monotonicity). Assume that \( \rho_0 \) is smooth and satisfies (H1)-(H3). Let \( \rho \) be a classical solution of (1)-(3). Then, \( \rho(\cdot, t) \) satisfies (H3) for any \( t \geq 0 \).
\textbf{Proof.} Let us denote $\zeta := \partial_x \rho$, and write down its dynamics by differentiating (13) in $x$

$$\partial_t \zeta = -u \partial_x \zeta - 2\zeta \partial_x u - \rho \partial_x^2 u = -u \partial_x \zeta - 2\zeta \Lambda^\alpha \rho - \rho \Lambda^\alpha \zeta.$$  \hspace{0.5cm} (13)

By periodicity and (11), we know $\zeta(\cdot, t)$ is odd, and so

$$\zeta(0, t) = \zeta(1/2, t) = 0.$$

Our goal is to prove $\zeta(x, t) \geq 0$, for all $x \in [0, 1/2]$ and $t \geq 0$. Assume the argument is false, then there exist at time $t_0$ and position $x_0 \in (0, 1/2)$ such that the solution satisfies

$$\zeta(x_0, t_0) = 0, \quad \zeta(x, t_0) \geq 0, \ \forall \ x \in [0, 1/2], \quad \text{and} \quad \partial_t \zeta(x_0, t_0) < 0,$$

so that the break down first happens at $(x_0, t_0^+)$.

Since $\zeta(\cdot, t_0)$ reaches a local minimum at $x_0$, clearly $\partial_x \zeta(x_0, t_0) = 0$. Therefore, the dynamics (13) at $(x_0, t_0)$ becomes

$$\partial_t \zeta(x_0, t) = -\rho(x_0, t_0) \Lambda^\alpha \zeta(x_0, t_0).$$

From Proposition 2.1 we know $\rho(x_0, t_0) \geq 0$. So, we are left to estimate $\Lambda^\alpha \zeta(x_0, t_0)$.

\begin{align*}
\Lambda^\alpha \zeta(x_0, t_0) &= c_\alpha \int_{R} \frac{\zeta(x_0, t_0) - \zeta(y, t_0)}{|x_0 - y|^{1+\alpha}} \, dy = -c_\alpha \sum_{m \in \mathbb{Z}} \int_{-1/2}^{1/2} \frac{\zeta(y, t_0)}{|x_0 - y - m|^{1+\alpha}} \, dy \\
&= -c_\alpha \left[ \sum_{m \in \mathbb{Z}} \int_{0}^{1/2} \frac{\zeta(-y, t_0)}{|x_0 + y - m|^{1+\alpha}} \, dy + \sum_{m \in \mathbb{Z}} \int_{0}^{1/2} \frac{\zeta(y, t_0)}{|x_0 - y - m|^{1+\alpha}} \, dy \right] \\
&= -c_\alpha \int_{0}^{1/2} \zeta(y, t_0) \sum_{m \in \mathbb{Z}} \left( \frac{1}{|x_0 - y - m|^{1+\alpha}} - \frac{1}{|x_0 + y - m|^{1+\alpha}} \right) \, dy.
\end{align*}

From (14) and the following Lemma 2.3 we conclude that $\Lambda^\alpha \zeta(x_0, t_0) \leq 0$ and hence $\partial_t \zeta(x_0, t_0) \geq 0$. This contradicts with the last inequality in (14).  \hfill \Box

\textbf{Lemma 2.3.} Suppose $x, y \in [0, 1/2]$ and $\alpha > 0$. Then

$$\sum_{m \in \mathbb{Z}} \left( \frac{1}{|x - y - m|^{1+\alpha}} - \frac{1}{|x + y - m|^{1+\alpha}} \right) \geq 0.$$

\textbf{Proof.} We first consider the case when $y \leq x$. The sum can be rewritten as

$$\sum_{m \geq 1} \left[ \left( \frac{1}{(m-1+x-y)^{1+\alpha}} - \frac{1}{(m-x-y)^{1+\alpha}} \right) - \left( \frac{1}{(m-1+x+y)^{1+\alpha}} - \frac{1}{(m-x+y)^{1+\alpha}} \right) \right].$$

Define

$$H_m(z) = \frac{1}{(m-1+x-z)^{1+\alpha}} - \frac{1}{(m-x-z)^{1+\alpha}}.$$

Then, the sum can be represented as

$$\sum_{m \geq 1} (H_m(y) - H_m(-y)).$$

Since we have

$$H'_m(z) = (1 + \alpha) \left[ \frac{1}{(m-1+x-z)^{2+\alpha}} - \frac{1}{(m-x-z)^{2+\alpha}} \right] \geq 0, \ \forall \ z \in [-1/2, 1/2],$$

we get $H_m(y) - H_m(-y) \geq 0$ for any $y \in [0, x]$. It implies that the sum is non-negative.

The case when $y > x$ can be treated in the same way.  \hfill \Box
2.3. **An estimate on velocity.** The velocity $u$ defined in (2) can be expressed in the integral form as follows:

$$ u(x, t) = c_\alpha \int_\mathbb{R} \frac{\rho(y, t) - \rho(x, t)}{\text{sgn}(x-y)|x-y|^\alpha} \, dy. \quad (15) $$

Fix $x \in [0, 1/2]$ and $t \geq 0$. We decompose the integrand and use (H1) to get

$$ \frac{1}{c_\alpha} u(x, t) = \int_0^\infty \frac{\rho(y, t) - \rho(x, t)}{|y|+y|^\alpha} \, dy + \int_0^x \frac{\rho(y, t) - \rho(x, t)}{|x-y|^\alpha} \, dy - \int_x^\infty \frac{\rho(y, t) - \rho(x, t)}{|x-y|^\alpha} \, dy $$

$$ = \int_0^x (\rho(y, t) - \rho(x, t)) \left( \frac{1}{|x+y|^\alpha} + \frac{1}{|x-y|^\alpha} \right) \, dy $$

$$ + \int_x^\infty (\rho(y, t) - \rho(x, t)) \left( \frac{1}{|x+y|^\alpha} - \frac{1}{|y-x|^\alpha} \right) \, dy =: I + II. $$

Due to monotonicity condition of $\rho(\cdot, t)$ (H3), we know that the first term $I \leq 0$. For the second term $II$, observe that

$$ \frac{1}{|x+y|^\alpha} < 0, \quad \forall \ y > x > 0. $$

So, the integral in $II$ can be decompose into two parts:

$$ \int_x^\infty = \sum_{m=0}^\infty \int_{m+x}^{m+1-x} + \sum_{m=1}^\infty \int_{m-x}^{m+x}. $$

Again, condition (H3) implies that for the first part $\rho(y, t) - \rho(x, t) \geq 0$, and for the second part $\rho(y, t) - \rho(x, t) \leq 0$. Let us denote $II = II_1 + II_2$ where $II_1$ and $II_2$ represents the corresponding integrals. Then, $II_1 \leq 0$ and $II_2 \geq 0$.

The next lemma shows $I + II_2 \leq 0$, at least when $x$ is sufficiently small.

**Lemma 2.4.** There exists a $\delta = \delta(\alpha) > 0$, such that for all $x \in [0, \delta]$, $I + II_2 \leq 0$.

**Proof.** Let us first write

$$ II_2 = \int_{-x}^x (\rho(x, t) - \rho(y, t)) \sum_{m=1}^\infty \left( \frac{1}{(y+m-x)^\alpha} - \frac{1}{(y+m+x)^\alpha} \right) \, dy. $$

Using mean value theorem, we have for $y \in (-x, x)$,

$$ \frac{1}{(y+m-x)^\alpha} - \frac{1}{(y+m+x)^\alpha} \leq \alpha (m-2x)^{-1-\alpha} \cdot (2x). $$

Therefore,

$$ \sum_{m=1}^\infty \frac{1}{(y+m-x)^\alpha} - \frac{1}{(y+m+x)^\alpha} \leq 2\alpha x \left[ (1 - 2z)^{-1-\alpha} + \int_1^\infty (z - 2z)^{-1-\alpha} \, dz \right] \leq Cx. $$

For $x \leq 1/4$, the last inequality holds with the choice of $C = 2^\alpha + 1(1 + 2\alpha)$.

Now, let us put together $I$ and $II_2$.

$$ I + II_2 = \int_0^x (\rho(x, t) - \rho(y, t)) \left[ -\frac{1}{(x+y)^\alpha} - \frac{1}{(x-y)^\alpha} \right. $$

$$ + \sum_{m=1}^\infty \left( \frac{1}{(y+m-x)^\alpha} - \frac{1}{(y+m+x)^\alpha} + \frac{1}{(-y+m-x)^\alpha} - \frac{1}{(-y+m+x)^\alpha} \right) \, dy $$
\[
\int_0^x (\rho(x, t) - \rho(y, t)) \left[ -\frac{1}{(x-y)\alpha} + 0 + 2Cx \right] dy \\
\leq (-x^{-\alpha} + 2Cx) \int_0^x (\rho(x, t) - \rho(y, t)) dy.
\]

We pick a small enough \( \delta \) as follows
\[
\delta = \min \left\{ \frac{1}{4}, \left( \frac{1}{3C} \right)^{\frac{1}{1+\alpha}} \right\}, \tag{16}
\]
Then, for any \( x \in (0, \delta] \), we have \(-x^{-\alpha} + 2Cx \leq -Cx < 0\).
Also, the monotonicity condition \( (H3) \) implies that
\[
\int_0^x (\rho(x, t) - \rho(y, t)) dy \geq 0.
\]
Therefore, conclude that \( I + II_2 \leq 0 \) for all \( x \in [0, \delta] \). \( \square \)

Lemma 2.4 directly implies the following estimate on \( u \).

**Theorem 2.5.** Let \( \rho \) be a classical solution of \( (1) - (3) \), with periodic initial data \( \rho_0 \) satisfying \( (H1)-(H3) \). Let \( \delta \) be defined as \( (16) \). Then, the velocity
\[
u(x, t) \leq 0, \quad \forall \ x \in [0, \delta], \ t \geq 0.
\]

One may remove the smallness assumption on \( x \) in Theorem 2.5 by a more careful estimate on \( II_2 \). For our purpose, it is enough to consider small \( x \).

3. **Finite time blow up**

In this section, we construct a family of initial data \( \rho_0 \) and prove Theorem 1.1. The following figure illustrates the choice of \( \rho_0 \). To make use of the apriori estimates, \( \rho_0 \) satisfies \( (H1)-(H3) \).

![Figure 1. The choice of Initial data \( \rho_0 \)](image)

The main feature of the initial data is
\[
\rho_0(x) = 0, \quad \forall \ x \in [-\delta/2, \delta/2], \ t \geq 0. \tag{H4}
\]
It differs from \( (6) \) as \( \rho_0(x) = 0 \) is attained in an interval instead of a single point. Such property plays a crucial role in our proof of finite time blow up. It is worth noting that \( (H4) \) holds for \( \rho(\cdot, t) \) as long as the solution is smooth.
One final assumption on the initial data is that the total mass is big enough.

\[ \int_T \rho_0(x) dx \geq \frac{2}{3}. \]  \hspace{1cm} (H5)

Note that by integrating (1) in \( x \), we get

\[ \frac{d}{dt} \int_T \rho(x,t) dx = 0. \]

It implies conservation of mass,

\[ \int_T \rho(x,t) dx = \int_T \rho_0(x) dx. \]

Therefore, (H5) holds for \( \rho(\cdot, t) \) as well.

We will show that all smooth initial data which satisfies (H1)-(H5) lead to a finite time blow up. Clearly, such initial data exists if \( \delta \) is small enough.

### 3.1. Enhanced estimate on velocity

In this part, we improve the estimate on \( u \) in Theorem 2.5, taking advantage of the additional assumptions (H4)-(H5).

**Theorem 3.1.** Let \( \rho \) be a classical solution of (1)-(3), with periodic initial data \( \rho_0 \) satisfying (H1)-(H5). Then, there exists a positive constant \( A = A(\alpha) \), such that the velocity

\[ u(x,t) \leq -Ax, \quad \forall x \in [0, \delta], \quad t \geq 0. \]

Let us explain the main idea of the proof. Recall that for \( x \in [0, \delta] \) and \( t \geq 0 \),

\[ u(x,t) = (I + II_2) + II_1, \]

where both terms \( I + II_2 \) and \( II_1 \) are non-positive. To improve the estimate, we need to obtain better bounds on either term.

Consider the following two extreme scenarios, illustrated by the graph below.

**Case 1:** \( II_1 = 0 \)

\[ \rho(y,t) \quad \text{even} \]

**Case 2:** \( I + II_2 = 0 \)

\[ \rho(y,t) \quad \text{even} \]

The first scenario is when \( \rho(y,t) = \rho(x,t) \), for any \( y \in [x, 1/2] \). Clearly, \( II_1 = 0 \) by definition. The second scenario is when \( \rho(y,t) = \rho(x,t) \), for any \( y \in [0, x] \). In this case, \( I + II_2 = 0 \).

Therefore, we should improve the bound on \( I + II_2 \) if \( \rho(\cdot, t) \) is closer to the first scenario. If \( \rho(\cdot, t) \) is closer to the second scenario, then we improve the bound on \( II_1 \).

We observe that \( \rho(x,t) \) is large in the first case, while \( \rho(x,t) \) is small in the second case. Therefore, the value of \( \rho(x,t) \) could be used to determine which scenario is closer to the configuration.
Case 1: \(\rho(x,t) > 1/2\). We shall obtain a better bound on \(I + II_2\). Recall the estimate in Lemma 2.4
\[
I + II_2 \leq -Cx \int_0^x (\rho(x,t) - \rho(y,t))dy.
\]
Since \(\rho(x,t) > 1/2\), we have
\[
\int_0^x (\rho(x,t) - \rho(y,t))dy \geq \int_{\delta/2}^x (\rho(x,t) - \rho(y,t))dy \geq \frac{\delta}{2}\rho(x,t) > \frac{\delta}{4}.
\]
Therefore, we obtain
\[
I + II_2 \leq -\frac{\delta C}{4}x.
\] (17)

Case 2: \(\rho(x,t) \leq 1/2\). We improve the bound on \(II_1\). Recall
\[
II_1 = -\sum_{m=0}^{\infty} \int_{m+1}^{m+1-x} (\rho(y,t) - \rho(x,t)) \left(\frac{1}{(y-x)^\alpha} - \frac{1}{(y+x)^\alpha}\right) dy
\]
\[
\leq -\int_x^{1/2} (\rho(y,t) - \rho(x,t)) \left(\frac{1}{(y-x)^\alpha} - \frac{1}{(y+x)^\alpha}\right) dy.
\]
We are going to get a better bound on \([x, 1/2]\), and use the rough bound by zero for the rest of the integrand. Denote the term that we concern by \(III\).
\[
III = \int_x^{1/2} (\rho(y,t) - \rho(x,t)) h(x,y) dy, \quad h(x,y) = \frac{1}{(y-x)^\alpha} - \frac{1}{(y+x)^\alpha}.
\]
To obtain a lower bound on \(III\), we need several observations. First, for a fixed \(x \in [0, \delta]\), \(h(x,y) \geq 0\) for any \(y \in (x, 1/2]\). Moreover,
\[
\partial_y h(x,y) = -\alpha \left[\frac{1}{(y-x)^{\alpha+1}} - \frac{1}{(y+x)^{\alpha+1}}\right] \leq 0.
\] (18)
Next, we apply (H2) (H3), and get
\[
0 \leq \rho(y,t) - \rho(x,t) \leq 1 - \rho(x,t), \quad \forall y \in (x, 1/2].
\] (19)
Moreover, condition (H5) implies
\[
\int_x^{1/2} (\rho(y,t) - \rho(x,t)) dy \geq \int_0^{1/2} (\rho(y,t) - \rho(x,t)) dy \geq \frac{1}{3} - \frac{1}{2}\rho(x,t) > \frac{1}{12}.
\] (20)
The following lemma is helpful to get a lower bound of \(III\).

Lemma 3.2. Let \(f\) be a positive decreasing function on \([a,b]\). \(m\) and \(M\) are positive constant such that \(m < M(b - a)\). Then,
\[
\min_\omega \left\{ \int_a^b \omega(x)f(x)dx \mid 0 \leq \omega(x) \leq M, \int_a^b \omega(x)dx \geq m \right\} = M \int_{b - \frac{m}{M}}^b f(x)dx.
\]
The minimum is attained at
\[
\omega_m(x) = \begin{cases} 0 & a \leq x < b - \frac{m}{M} \\ M & b - \frac{m}{M} \leq x \leq b \end{cases}.
\]
Proof. First, it is easy to check $\omega$ satisfies
\[ 0 \leq \omega(x) \leq M, \quad \int_a^b \omega(x) \, dx \geq m. \] (21)
We will prove that for any $\omega$ which satisfies (21), \[ \int_a^b (\omega(x) - \omega_m(x)) f(x) \, dx \geq 0. \] Compute
\[ \int_a^b (\omega(x) - \omega_m(x)) f(x) \, dx = \int_a^{b - \frac{M}{M}} \omega(x) \, dx + \int_{b - \frac{M}{M}}^b (\omega(x) - M) f(x) \, dx. \]
From the first condition in (21), we know $\omega(x) \geq 0$ and $\omega(x) - M \leq 0$. Together with the assumption that $f$ is positive and decreasing, we obtain
\[ \int_a^b (\omega(x) - \omega_m(x)) f(x) \, dx \geq f(b - \frac{M}{M}) \int_{b - \frac{M}{M}}^b \omega(x) - M \cdot \frac{m}{M} \, dx \geq f(b - \frac{M}{M})(m - m) = 0. \]
Hence, we conclude
\[ \min_{\omega \text{ satisfies } (21)} \int_a^b \omega(x) f(x) \, dx = \int_a^b \omega_m(x) f(x) \, dx = M \int_{b - \frac{M}{M}}^b f(x) \, dx. \]
\[ \square \]
Putting together (18), (19) and (20), we can apply Lemma 3.2 with $f(y) = h(x, y), \omega(y) = \rho(y, t) - \rho(x, t), m = \frac{1}{12}, M = 1 - \rho(x, t), a = x, b = \frac{1}{2}$.
Then,
\[ III \geq (1 - \rho(x, t)) \int_{\frac{1}{2} - \frac{1}{2}t(1 - \rho(x, t))}^{\frac{1}{2}} h(x, y) \, dy \geq \frac{1}{2} \int_{\frac{1}{2}}^{1/2} h(x, y) \, dy. \]
Using mean value theorem, we have
\[ h(x, y) \geq \frac{\alpha}{(y + x)^{1 + \alpha}} \cdot (2x) \geq 2\alpha x. \]
Finally, we obtain
\[ III \geq \frac{1}{2} \cdot \frac{1}{6} \cdot (2\alpha x) = \frac{\alpha}{6} x, \]
and therefore
\[ II_1 \leq -\frac{\alpha}{6} x. \] (22)
Combining the two cases (17) and (22) together, we arrive at an enhanced estimate on $u$ as follows:
\[ u(x, t) \leq -c_\alpha \min \left\{ \frac{\delta C}{4}, \frac{\alpha}{6} \right\} x. \]
This concludes the proof of Theorem 3.1 with $A = c_\alpha \min \left\{ \frac{\delta C}{4}, \frac{\alpha}{6} \right\} > 0$, which only depends on $\alpha$. 

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3.2. **Finite time blowup.** We assume $\rho$ is a smooth solution and argue by a contradiction. Consider the characteristic path originated from $x = \delta$.

$$\frac{d}{dt} X(t) = u(X(t), t), \quad X(0) = \delta,$$

From Theorem 2.5, we know $X(t) \leq X(0) = \delta$. Hence, we are able to apply Theorem 3.1 and get an improved estimate

$$X(t) \leq \delta e^{-At}.$$ 

Clearly, $X(t)$ will reach $\delta/2$ in a finite time $t_0 \leq \ln 2/A$. From (H4), $\rho(\delta/2, t_0) = 0$. Therefore, we must have

$$\rho(X(t_0), t_0) = \rho(\delta/2, t_0) = 0.$$  

(23)

Otherwise, the solution will form a discontinuity.

On the other hand, from (4), we know

$$\frac{d}{dt} \rho(X(t), t) = -\rho(X(t), t) \Lambda^\alpha \rho(X(t), t).$$

It implies,

$$\rho(X(t_0), t_0) = \rho_0(\delta) \exp \left[ - \int_{t_0}^{t} \Lambda^\alpha \rho(X(t), t) dt \right].$$

We assume that $\rho_0(\delta) > 0$. To ensure (23), we need

$$\int_{t_0}^{t} \Lambda^\alpha \rho(X(t), t) dt = \infty.$$ 

Therefore, the solution $\rho$ will lose $C^\alpha$ regularity at $t_0$, which contradicts with the assumption that $\rho$ is smooth.

Let us summarize our construction of $\rho_0$. First, pick a small $\delta > 0$ satisfying (16). Then, pick $\rho_0$ to be smooth, periodic, such that $\rho_0(\delta) > 0$, and $\rho_0$ satisfies (H1)-(H5). Such $\rho_0$ exists and is illustrated in Figure 1. We have proved Theorem 1.1 with this choice of $\rho_0$.

4. Extension to systems in multi-dimensions

In this section, we extend our main result to systems in higher dimensions. The main idea is to consider $\rho_0(x) = \rho_0(x_1)$ and reduce the system to 1D so that our construction can be used.

4.1. **Fractional porous medium flow.** Let us recall the fractional porous medium flow in multi-dimension

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad u = \nabla \Lambda^{\alpha-2} \rho,$$

with $x = (x_1, \cdots, x_n) \in \mathbb{T}^n$ and $0 < \alpha < 2$.

Fix any time $t$ and drop the time dependence for simplicity. Assume $\rho(x) = \rho(x_1)$, namely $\rho$ is a constant in $(x_2, \cdots, x_n)$ variables. We calculate the velocity field $u$, starting with

$$\Lambda^{\alpha-2} \rho = c_{n,\alpha} \int_{\mathbb{R}^n} \rho(x - y) \frac{1}{|y|^{n+\alpha-2}} dy.$$ 

Then, we obtain $u$ by taking the gradient of the potential

$$u_i(x) = \partial_{x_i} \Lambda^{\alpha-2} \rho = c_{n,\alpha} \int_{\mathbb{R}^n} \left( \rho(x_1 - y_1) - \rho(x_1) \right) \frac{y_i}{|y|^{n+\alpha}} dy.$$
For \( i = 2, \cdots, n \), we have
\[
  u_i(x) = c_{n,\alpha} \int_{\mathbb{R}} \left( \rho(x_1 - y_1) - \rho(x_1) \right) \left[ \int_{\mathbb{R}^{n-1}} \frac{y_i}{|y|^{n+\alpha}} dy_2 \cdots dy_n \right] dy_1 = 0. \tag{25}
\]

The last equality is due to oddness of the inside integral with respect to \( y_i \).

For \( i = 1 \),
\[
  u_1(x) = c_{n,\alpha} \int_{\mathbb{R}} \left( \rho(x_1 - y_1) - \rho(x_1) \right) y_1 \left[ \int_{\mathbb{R}^{n-1}} \frac{1}{|y|^{n+\alpha}} dy_2 \cdots dy_n \right] dy_1.
\]

Compute the integral inside,
\[
  \int_{\mathbb{R}^{n-1}} \frac{1}{|y|^{n+\alpha}} dy_2 \cdots dy_n = \int_{\mathbb{R}^{n-1}} \left( y_1^2 + y_2^2 + \cdots + y_n^2 \right)^{-\frac{n+\alpha}{2}} dy_2 \cdots dy_n
  = |y_1|^{-\frac{(n+\alpha)}{2}} \omega_{n-1} \int_0^\infty (1 + r^2)^{-\frac{n+\alpha}{2}} r^{n-2} dr = c'_{n,\alpha} |y_1|^{-1-\alpha}.
\]

Here, \( \omega_n \) denotes the area of the unit sphere in \( n \) dimension. The constant \( c'_{n,\alpha} \) is clearly positive, finite, and only depend on \( n \) and \( \alpha \).

Then, we obtain
\[
  u_1(x) = c_{n,\alpha} c'_{n,\alpha} \int_{\mathbb{R}} \frac{\rho(x_1 - y_1) - \rho(x_1)}{|y_1|^{\alpha}} dy_1. \tag{26}
\]

So, \( u_1(x) = u_1(x_1) \) is also a constant in \( (x_2, \cdots, x_n) \). Moreover, as a function of \( x_1 \), the expression of \( u_1 \) is the same as (15), except the constant \( c_\alpha \) might be different.

From (25) and (26), we have
\[
  \nabla \cdot (\rho(x)u(x)) = \partial_{x_1}(\rho(x_1)u_1(x_1)).
\]

This implies if \( \rho_0(x) = \rho_0(x_1) \), then \( \rho(x, t) = \rho(x_1, t) \). Moreover, \((\rho, u_1)\) as functions of \( x_1 \), will be the solution of the 1D system (1)-(3). Hence, Theorem 1.1 can be extended to multi-dimension, with the choice of initial data \( \rho_0(x) = \rho_0(x_1) \), where \( \rho_0 \) as a function of \( x_1 \) is chosen the same way as in the 1D case. The different constant in (26) mentioned above will only affect the choice of \( \delta \) throughout the proof.

We summarize the discussion to the following theorem.

**Theorem 4.1.** Consider the initial value problem of system (24) in the periodic domain \( \mathbb{T}^n \). There exists a family of smooth initial data \( \rho_0 \) such that the solution loses \( C^\alpha \) regularity in finite time.

### 4.2. Fractional Euler-Alignment system

The multi-dimensional Euler-Alignment system with singular influence function takes the form
\[
  \partial_t \rho + \nabla \cdot (\rho u) = 0, \quad \partial_t u + u \cdot \nabla u = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{u(y, t) - u(x, t)}{|y - x|^{n+\alpha}} \rho(y, t) dy. \tag{27}
\]

Let \( G = \nabla \cdot u - \Lambda^\alpha \rho \). Then, the dynamics of \( G \) reads
\[
  \partial_t G + \nabla \cdot (Gu) = \text{tr}(\nabla u \otimes u) - (\nabla \cdot u)^2.
\]

Note that In the 1D case, the right hand side becomes \( (\partial_x u)^2 - (\partial_x u)^2 = 0 \). Then, the dynamics becomes (12), and as a special case of \( G \equiv 0 \), we reach our system (1)-(2).

However, the right hand side is not necessarily zero in higher dimensions. This quantity is known as spectral gap. In particular, it destroys the maximum principle on \( G \), and hence \( G_0 \equiv 0 \) does not imply \( G(\cdot, t) \equiv 0 \).
Therefore, fractional porous median flow (24) is not a special case of the Euler-Alignment system, except in 1D. The global regularity on (27) for $\rho_0 > 0$ is an open problem. The main difficulty is the lack of apriori control of the spectral gap.

To construct $\rho_0 \geq 0$ which leads to finite time blowup, we can avoid the difficulty by select a special family of initial data such that the spectral gap is zero in all time.

The choices of $(\rho_0, u_0)$ is the same as Section 4.1

$$
\rho_0(x) = \rho_0(x_1), \quad (u_0)_1(x) = (u_0)_1(x_1), \quad (u_0)_i(x) = 0, \ \forall \ i = 2, \ldots, n.
$$

By the same argument, we know such structure preserves in time. So,

$$
\text{tr}(\nabla u^{\otimes 2}) - (\nabla \cdot u)^2 = (\partial_{x_1} u_1)^2 - (\partial_{x_1} u_1)^2 = 0.
$$

Therefore, we pick $\rho_0$ the same as in Theorem 4.1 and $u_0 = \nabla \Lambda^{\alpha-2} \rho_0$. The solution will blow up the same way as (24).

**Corollary 4.2.** Consider the initial value problem of system (27) in the periodic domain $\mathbb{T}^n$. There exists a family of smooth initial data $(\rho_0, u_0)$ such that the solution loses smoothness in finite time.

5. Further discussions

In this section, we discuss another possible type of initial data which leads to finite time blowup.

![Initial data with a point vacuum](image)

**Figure 2.** Initial data with a point vacuum

As illustrated in Figure 2, the main difference compared with the setup in Section 3 is that $\rho_0(x) = 0$ is attained only at one point $x = 0$. So, (H4) is violated. Note that the initial data (6) belongs to this type. Therefore, the result in [6] would support the conjecture that there exists initial data of such type which leads to a finite time blowup.

Without condition (H4), one can only argue by contradiction when the characteristic path $X(t)$ reaches zero, instead of a positive number $\delta/2$. It can not be guaranteed by the estimate in Theorem 3.1 that $X(t)$ reaches zero in finite time. Therefore, a stronger estimate on $u$ is required to prove the conjecture. We will leave it for future investigations.

**Acknowledgments.** The author would like to thank Tam Do, Alexander Kiselev and Xiaoqian Xu for valuable discussions.
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