Minimal Lipschitz and $\infty$-Harmonic Extensions of Vector-Valued Functions on Finite Graphs

Miroslav Bačák$^1$, Johannes Hertrich$^2$, Sebastian Neumayer$^2$, and Gabriele Steidl$^{2,3}$

$^1$Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig, Germany, bacak@mis.mpg.de
$^2$Department of Mathematics, Technische Universität Kaiserslautern, Paul-Ehrlich-Str. 31, 67663 Kaiserslautern, Germany, \{jhetric,neumayer,steidl\}@mathematik.uni-kl.de
$^3$Fraunhofer ITWM, Fraunhofer-Platz 1, 67663 Kaiserslautern, Germany

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Abstract

This paper deals with extensions of vector-valued functions on finite graphs fulfilling distinguished minimality properties. We show that so-called lex and $L$-lex minimal extensions are actually the same and call them minimal Lipschitz extensions. Then we prove that the solution of the graph $p$-Laplacians converge to these extensions as $p \to \infty$. Furthermore, we examine the relation between minimal Lipschitz extensions and iterated weighted midrange filters and address their connection to $\infty$-Laplacians for scalar-valued functions. A convergence proof for an iterative algorithm proposed by Elmoataz et al. (2014) for finding the zero of the $\infty$-Laplacian is given. Finally, we present applications in image inpainting.

Keywords: $p$-Laplacian, $\infty$-Laplacian, graph Laplacian, $\infty$-harmonic extension, absolutely minimal Lipschitz extension, midrange filter, image inpainting, nonlocal techniques

1 Introduction

In this paper, we study an $\infty$-harmonic variant of the Dirichlet problem for vector-valued functions defined on finite graphs. To be more specific, we assume that a vector-valued function is given on a subset of the vertex set (“boundary”) and our goal is to extend this function onto the whole vertex set so that the extension is $\infty$-harmonic. However, as observed by [38], $\infty$-harmonic extensions of vector-valued functions on finite graphs are not uniquely determined. To overcome this non-uniqueness issue they introduced the stronger notion of tight extensions and proved that such extensions exist and are unique. In the present paper, we approach these extensions from a different point of view and call them minimal Lipschitz extension, since its local Lipschitz constant (“oscillation”) is in some sense
optimal. We therefore aim at finding a solution to the above boundary problem which is minimal Lipschitz.

In [38, Section 2] an algorithm for computing these minimal Lipschitz extensions in the scalar-valued case was presented. To the best of our knowledge there exists no meaningful algorithm for computing minimal Lipschitz extensions of vector-valued functions and it is our goal to address their approximation in the present paper. Along the way we also clarify various aspects concerning minimal Lipschitz extensions and their relations to other concepts including iterated midrange filters, minimizers of $p$-energy functionals, $p$-Laplacians and $\infty$-Laplacians.

The topic of minimal Lipschitz extensions on graphs appears in various subfields of mathematics and computer science from different points of view and with different notations. These areas include approximation theory [12], discrete mathematics/graph theory [32], data and image processing [3, 14, 16, 39] including manifold-valued data [5], and mathematical morphology [37] to mention only a few. While many results on minimal Lipschitz extensions are available for scalar-valued functions, the vector-valued case has been less studied.

We work with weighted graphs which turns out to be crucial for the applications in image processing. Indeed, in Section 8 we present applications of minimal Lipschitz extensions for inpainting of vector-valued images which rely upon representing the image under consideration as a function on an appropriate weighted graph which is obtained by nonlocal patch-based techniques.

As a matter of fact, [38] define tight extensions also for functions defined on a bounded, open, connected subset $U \subset \mathbb{R}^n$. This is closely related to absolutely minimizing Lipschitz extensions (AMLE) and therefore also to $\infty$-harmonic functions. Indeed, [22], stimulated by the work [1], proved that for $g \in C(\partial U; \mathbb{R})$, the boundary-value problem

$$-\Delta_{\infty} f = 0 \quad \text{in } U, \quad f = g \quad \text{on } \partial U,$$

with the $\infty$-Laplacian defined by

$$\Delta_{\infty} f := |\nabla f|^{-2} \sum_{i,j=1}^{n} \partial_i f \partial_i f \partial_j f \partial_j f,$$

has a unique viscosity solution $f \in C(\bar{U}; \mathbb{R})$. [11] showed its equivalence to the AMLE of $g$. In connection with image interpolation algorithms and elliptic partial differential operators, AMLEs were studied by [10] inspired by the work of [9]. The operator was considered, e.g., for comparisons of image compression algorithms in [17, 36]. Even though we are aware of the existence of this deep theory of Lipschitz extensions in $\mathbb{R}^n$, we focus exclusively on functions defined on finite graphs in the present paper. For more information on the continuous case, the interested reader is also referred to the recent papers [23, 24, 28] and the references therein.

This paper is organized as follows. After fixing the notation in Section 2 we introduce lex and $L$-lex minimal extensions in Section 3 and prove that they actually coincide. In Section 4 we show that the minimizers of the grouped $p$-energy functionals $E_p$ converge to these extensions as $p \to \infty$. Section 5 deals with the relation between iterated midrange filters and minimal Lipschitz extensions. In Section 6 we consider $\infty$-Laplacians for scalar-valued functions and provide a convergence proof for an iterative algorithm of [15]. Section 7
The functionals

Lemma 2.1. Every cluster point of the sequence of minimizers \( \{f_p\}_p \) of \( E_p \) is a minimizer of \( E_\infty \). The same holds true for the minimizers \( \{f_\infty,p\}_p \) of \( E_\infty,p \).

Proof. Recall that the \( \ell_p \)-norms satisfy

\[ \|x\|_\infty \leq \|x\|_p \leq N^{1/p}\|x\|_\infty, \quad x \in \mathbb{R}^N. \]
In particular, \( \| \cdot \|_p \to \| \cdot \|_\infty \) as \( p \to \infty \) uniformly on bounded sets and the convergence is monotone. These properties are inherited by the convergence \( E^{1/p}_p \to E_\infty \) as \( p \to \infty \). This implies that \( E^{1/p}_p \) \( \Gamma \)-converges to \( E_\infty \) as \( p \to \infty \) and we obtain the desired results since the functionals \( E^{1/p}_p \) and \( E_p \) have the same minimizers. Similar arguments can be applied for the functional \( E_{\infty,p} \).

\[ \square \]

3 Minimal Lipschitz Extensions

Let again \( U \subset V \) be a nonempty subset of the vertices of the graph \( G := (V, E, w) \) and let \( g: U \to \mathbb{R}^m \). We start by recalling the definitions of two types of minimal extensions of \( g \) and we then show in Theorem 3.2 below that they actually coincide. For \( u \in V \setminus U \), let

\[
lf(u,v) := w(u,v)|f(u) - f(v)| = |\partial_v f(u)|
\]

and \( \text{lex}(f) \in \mathbb{R}^M \) be the vector with entries \( (lf(u,v))_{v \sim u} \) in nonincreasing order. Note that we count the entries \( lf(u,v) = l\text{f}(v,u) \) only once. For \( u \in V \setminus U \), define

\[
Lf(u) := \max_{v \sim u} lf(u,v) = \|\nabla_w f(u)\|_{2,\infty},
\]

and the vector \( L\text{lex}(f) \in \mathbb{R}^{|V \setminus U|} \) with entries \( (Lf(u))_{u \in V \setminus U} \) in nonincreasing order. Denoting by \( \leq \) the lexicographical ordering, a function \( f \in \mathcal{H}_g(V) \) is called

(i) \( \text{lex} \) minimal extension (of \( g \)) if \( \text{lex}(f) \leq \text{lex}(h) \) for every \( h \in \mathcal{H}_g(V) \), and

(ii) \( L\text{-lex} \) minimal extension (of \( g \)) if \( L\text{lex}(f) \leq L\text{lex}(h) \) for every \( h \in \mathcal{H}_g(V) \).

The first notation can be found for instance in [26]. The \( L\text{-lex} \) minimal extension was called tight extension in the paper [38].

The existence of \( L\text{-lex} \) minimal extensions in the non-weighted case was shown in [38, Theorem 1.2]. The existence in the weighted setting as well as for \( L\text{-lex} \) minimal extensions can be proved similarly; see [19]. By definition we clearly have \( \text{lex}_1(f) = L\text{lex}_1(f) \), where the subscript denotes the coordinate index. In this section, we want to show that \( \text{lex} \) and \( L\text{-lex} \) minimal extensions indeed coincide. To this end, we need the following lemma.

**Lemma 3.1.** Let \( f \in \mathcal{H}_g(V) \) be a \( \text{lex} \) minimal extension and \( \tilde{f} \in \mathcal{H}_g(V) \) with \( f \neq \tilde{f} \) and \( \text{lex}_1(f) = \text{lex}_1(\tilde{f}) \). Let \( k \in \{1, \ldots, M\} \) be the largest index such that \( (\text{lex}_i(f))_{i=1}^k = (\text{lex}_i(\tilde{f}))_{i=1}^k \) and \( K := \text{lex}_k(f) \). Then \( f \) and \( \tilde{f} \) coincide on the set

\[ W := \{ u \in V \setminus U : \exists v \sim u \text{ such that } lf(u,v) \geq K \} . \]

Note that for a \( \text{lex} \) minimal extension \( f \in \mathcal{H}_g(V) \) the case \( k = M \), i.e. \( \text{lex}(f) = \text{lex}(\tilde{f}) \) implies \( f = \tilde{f} \). This means that the \( \text{lex} \) minimal extension is unique.

**Proof.** 1. First, we prove that \( f(u) - f(v) = \tilde{f}(u) - \tilde{f}(v) \) for all \( u \sim v \) with \( lf(u,v) \geq K \). Consider \( u \sim v \) with \( lf(u,v) \geq K \). We suppose that if \( lf(u,v) = l\text{f}(\tilde{u}, \tilde{v}) = l\text{f}(u,v) = l\text{f}(\tilde{u}, \tilde{v}) \) and those values appear at positions \( i \) and \( j \) in \( \text{lex}(f) \) and \( \text{lex}(\tilde{f}) \), then the corresponding values \( l\text{f}(u,v), l\text{f}(\tilde{u}, \tilde{v}) \) and \( l\text{f}(\tilde{u}, \tilde{v}), l\text{f}(\tilde{u}, \tilde{v}) \) have the same position.
For a contradiction, assume it is not the case and let thus $C$ be the largest value in $(\text{lex}_i(f))_{k=1}^5$ with $f(u) - f(v) \neq \tilde{f}(u) - \tilde{f}(v)$. For $u \sim v$ with $lf(u,v) > C$, we have $f(u) - f(v) = \tilde{f}(u) - \tilde{f}(v)$ and for $lf(u,v) = C$ the relation $|f(u) - f(v)| \geq |\tilde{f}(u) - \tilde{f}(v)|$, where at least one $u \sim v$ with $f(u) - f(v) \neq \tilde{f}(u) - \tilde{f}(v)$ exists. For $u \sim v$ with $lf(u,v) < C$ we have $lf(u,v) \leq C$. For $h := 1/2(f + \tilde{f}) \in \mathcal{H}_g(V)$ and $u \sim v$, we obtain

$$|h(u) - h(v)| = \frac{1}{2}|f(u) - f(v) + \tilde{f}(u) - \tilde{f}(v)| \leq \frac{1}{2} \left( |f(u) - f(v)| + |\tilde{f}(u) - \tilde{f}(v)| \right).$$

(2)

Consequently, by the previous considerations, $lh(u,v) \leq lf(u,v)$ whenever $lf(u,v) \geq C$.

Consider $u \sim v$ with $lf(u,v) = C$ with $f(u) - f(v) \neq \tilde{f}(u) - \tilde{f}(v)$. Then either $|f(u) - f(v)| > |\tilde{f}(u) - \tilde{f}(v)|$ which implies $lh(u,v) \leq lf(u,v)$ or $|f(u) - f(v)| = |\tilde{f}(u) - \tilde{f}(v)|$. For two vectors $a, b \in \mathbb{R}^n$ with $|a| = |b|$ we have $|a + b| = |a| + |b|$ if and only if $a = b$. Thus we have strict inequality in (2) which results again in the strict inequality $lh(u,v) < lf(u,v)$. Finally we conclude $\text{lex}(h) < \text{lex}(f)$ which contradicts the lex minimality of $f$. Hence $f(u) - f(v) = \tilde{f}(u) - \tilde{f}(v)$ for all $u \sim v$ with $lf(u,v) \geq \text{lex}_i(f)$.

2. Next, we show that there exists $u \sim v$ with $v \in U$ such that $lf(u,v) = \text{lex}_1(f)$. Assume in contrary that this is not the case, i.e., there exists $\delta > 0$ such that $lf(u,v) \leq \text{lex}_1(f) - \delta$ for all $u \sim v$ with $v \in U$. For $\varepsilon > 0$, consider the function $\hat{f} \in \mathcal{H}_g(V)$ with $\hat{f}(u) := (1 - \varepsilon)f(u)$, $u \in V \setminus U$. Then we have for all $u_1, u_2 \in V \setminus U$ with $u_1 \sim u_2$ that $l\hat{f}(u_1, u_2) = (1 - \varepsilon)lf(u_1, u_2) < lf(u_1, u_2)$ and for all $u \in V \setminus U$ and $v \in U$ that $l\hat{f}(u,v) = w(u,v)|(1 - \varepsilon)f(u) - f(v)| \leq lf(u,v) + \varepsilon w(u,v)|f(u)| < \text{lex}_1 f$

for $\varepsilon < \delta/\max_{w} w(u,v)|f(u)|$. Thus $\text{lex}\hat{f} < \text{lex} f$ which is a contradiction.

3. Let $\tilde{f} \in \mathcal{H}_g(V)$ with $\text{lex}_1(f) = \text{lex}_1(\tilde{f})$. For $u \sim v$ with $v \in U$ such that $lf(u,v) = \text{lex}_1(f)$, Part 1 of the proof implies that $f(u) = \tilde{f}(u)$. Choose one such $u$ and set $U_1 := U \setminus \{u\}$, extend $g$ to $g_1$ on $U_1$ by $g_1(u) := f(u)$. Cut all edges $u \sim v$ with $v \in U$ and remove the corresponding entries in lex($f$) and lex($\tilde{f}$). Note that only entries with the same value are removed, including the first one. Consider $f$ and $\tilde{f}$ as extensions of $g_1$, where $f$ is still the lex minimal extension of $g_1$.

The whole procedure is repeated with respect to the new first component $\text{lex}_1(f)$ and so on until all edges with $lf(u,v) \geq K$ are removed. This yields the assertion.

Now we can prove the equivalence between lex and $L$-lex minimal extensions.

**Theorem 3.2.** There exists a unique lex/$L$-lex minimal extension $f \in \mathcal{H}_g(V)$ and both extensions coincide.

**Proof.**

1. The uniqueness of the $L$-lex minimal extension in the unweighted case was shown in [35, Theorem 1.2] and it is not hard to extend the arguments to the weighted setting; see [19]. The uniqueness of the lex minimal extension follows directly from Lemma 3.1.

2. Let $\tilde{f} \in \mathcal{H}_g(V)$ be the lex minimal extension and $\tilde{f} \in \mathcal{H}_g(V)$ the $L$-lex minimal extension. Assume that $f \neq \tilde{f}$. Then $\text{lex}(f) < \text{lex}(\tilde{f})$ and $L$-lex$(f) > L$-lex$(\tilde{f})$. If $L$-lex$(f) = L$-lex$(\tilde{f}) = L$-lex$(\hat{f})$ we have a contradiction. Thus, lex$(f) = lex(\hat{f})$ and we can apply Lemma 3.1 which implies that $f$ and $\hat{f}$ coincide on the set $W$. Now we can consider $f$ and $\hat{f}$ as extensions of $g$ extended to $U \cup W$, where the edges between vertices in $U \cup W$ and the corresponding entries in lex($f$) and lex($\hat{f}$) are removed. But for this new constellation we have lex$(f) < lex(\hat{f})$ which again contradicts the $L$-lex minimality of $\hat{f}$. 

$\square$
We call the minimal lex/L-lex extension of $g$ on $G$ the *minimal Lipschitz extension* of $g$, or sometimes shortly the *minimal extension* of $g$.

## 4 Approximations of Minimal Lipschitz Extensions

In this section, we focus on algorithms for computing minimal Lipschitz extensions. In the scalar-valued case minimal Lipschitz extensions coincide with $\infty$-harmonic functions and one can therefore use algorithms for $\infty$-harmonic functions. Such an algorithm (for non-weighted graphs) can be found in [38, Section 2.1], where the authors reference earlier papers [27, 32]. Note that it is a polynomial-time algorithm. For weighted graphs, a polynomial-time algorithm for computing $\infty$-harmonic functions was given in [26] along with a faster variant for practical computations.

To the best of our knowledge there exist no methods for computing minimal Lipschitz extensions of general vector-valued functions; this problem is also implicitly formulated in [38, Section 2.1]. Our aim is to address this issue. To this end recall the following theorem, which is just a simple generalization of a result in [38, Theorem 1.3] for weighted graphs.

**Theorem 4.1.** For any $m \geq 1$, the sequence $\{f_{\infty,p}\}^\infty_{p=1}$ of minimizers $f_{\infty,p} \in H_g(V)$ of $E_{\infty,p}$ converges to the minimal Lipschitz extension of $g$, as $p \to \infty$.

**Proof.** See [19] for the weighted-graph case and [38, Theorem 1.3] for the non-weighted graph case.

**Remark 4.2.** Let us mention that instead of considering $E_{\infty,p}$ we can also prove convergence for the minimizers of other functionals $E_{\infty,\varphi} f := \varphi_p(\|\nabla w f\|_{\infty})$, where $\varphi_p$ has to fulfill certain properties. For example, the function

$$
\varphi_p(x) := \frac{1}{p} \log \exp(px) = \frac{1}{p} \log \left( \sum_{i=1}^N \exp(px_i) \right)
$$

is an appropriate choice since the asymptotic function of the function $\log \exp$ is the vecmax function, that is, $\lim_{p \to \infty} \frac{1}{p} \log \left( \sum_{i=1}^N \exp(px_i) \right) = \max_{i=1,\ldots,N} x_i$.

We proceed by studying the sequence $\{f_p\}^\infty_{p=1}$ of minimizers of $E_p$. For scalar-valued functions, that is, $m = 1$, its convergence to the minimal extension of $g$ can be shown by applying the following classical result from approximation theory concerning the convergence of Pólya's algorithm [34]: In [12] it was proved that, given an affine subspace $K \subset \mathbb{R}^M$ and $z \in \mathbb{R}^M$, the sequence of $L_p$ approximations $\{x_p\}^\infty_{p=1}$ defined by

$$
x_p := \arg \min_{x \in K} \|x - z\|_p, \quad p \in (1, \infty),
$$

converges (as $p \to \infty$) to the minimizer $x_\infty \in K$ of $\|x - z\|_\infty$ with the following property: For every minimizer $x_\infty \in K$ of $\|x - z\|_\infty$ consider the vector $\sigma(x_\infty)$ whose coordinates $|x_{\infty,i} - z_i|$ are arranged in nonincreasing order. Then $\hat{x}_\infty$ is the minimizer with smallest
lexicographical ordering with respect to $\sigma$. Indeed this vector is uniquely determined and called \textit{strict uniform approximation} of $z$. Concerning the convergence rate it was shown in \cite{35}, see also \cite{13}, that there exist constants $0 < C_1, C_2 < \infty$ and $a \in [0, 1]$ depending on $K$, such that

$$C_1 \frac{1}{p} a^p \leq \|x_p - \hat{x}_\infty\|_\infty \leq C_2 \frac{1}{p} a^p.$$  

To apply this result in to minimal Lipschitz extensions, we rewrite $u$ with $v = \delta$ Setting $f \in H$ be deduced in this way and we therefore prove the following theorem.

**Theorem 4.3.** For any $m \geq 1$, the sequence $\{f_p\}_p$ of minimizers $f_p \in H_g(V)$ of $E_p$ converges to the minimal Lipschitz extension of $g$, as $p \to \infty$.

**Proof.** The sequence $\{f_p\}_p$ is bounded and hence there exists some accumulation point. Let $f \in H_g(V)$ be the minimal extension and assume that there exists a convergent subsequence $\{f_{p_j}\}_j$ of $\{f_p\}_p$ with limit $f \neq f$. Choose $k \in \{1, \ldots, M\}$ to be the largest index such that $(\text{lex}_i(f))_{i=1}^k = (\text{lex}_i(\tilde{f}))_{i=1}^k$, which exists since $\tilde{f}$ is a minimizer of $E_\infty$. By Lemma 3.1 the functions $f$ and $\tilde{f}$ coincide on the set $W := \{u \in V \setminus U : \exists v \sim u \text{ such that } f(u, v) \geq \text{lex}_k(f)\}$. Define

$$h_p(u) := \begin{cases} f_p(u) & \text{if } u \in W \cup U, \\ f(u) & \text{otherwise.} \end{cases}$$

Since $f(u) = \tilde{f}(u)$ for $u \in W$, we have that $h_{p_j} \to f$ as $j \to \infty$. For $u \in W \cup U$ and $v \in W$ with $u \sim v$ it holds

$$w(u, v)|f_{p_j}(u) - f_{p_j}(v)| = w(u, v)|h_{p_j}(u) - h_{p_j}(v)|,$$

and for $u \in W$ and $v \in V \setminus (W \cup U)$ with $u \sim v$,

$$w(u, v)|h_{p_j}(u) - h_{p_j}(v)| \to w(u, v)|f(u) - f(v)| \leq \text{lex}_{k+1}(f).$$

Setting $\delta := \text{lex}_{k+1}(\tilde{f}) - \text{lex}_{k+1}(f) > 0$, we obtain, for sufficiently large $j$ and $u \in V$, $v \in V \setminus (W \cup U)$ with $u \sim v$, that

$$w(u, v)|h_{p_j}(u) - h_{p_j}(v)| \leq \text{lex}_{k+1}(f) + \frac{\delta}{4}.$$  

On the other hand, there exist $u \in V$ and $v \in V \setminus (W \cup U)$ with $u \sim v$ such that

$$w(u, v)|f_{p_j}(u) - f_{p_j}(v)| \to w(u, v)|\tilde{f}(u) - \tilde{f}(v)| = \text{lex}_{k+1}(\tilde{f})$$

so that, for sufficiently large $j$,

$$w(u, v)|f_{p_j}(u) - f_{p_j}(v)| \geq \text{lex}_{k+1}(\tilde{f}) - \frac{\delta}{4}.$$  

7
Then we conclude
\[
E_{p_j}(f_{p_j}) - E_{p_j}(h_{p_j}) = \sum_{u \in V} \sum_{v \sim u} w(u, v)^{p_j} (|f_{p_j}(v) - f_{p_j}(u)|^{p_j} - |h_{p_j}(v) - h_{p_j}(u)|^{p_j})
\]
\[
= \sum_{u \in W \cup U} \sum_{v \sim u} w(u, v)^{p_j} |f_{p_j}(v) - f_{p_j}(u)|^{p_j}
\]
\[
+ \sum_{u \in V \setminus (W \cup U)} \sum_{v \sim u} w(u, v)^{p_j} |f_{p_j}(v) - f_{p_j}(u)|^{p_j}
\]
\[
- \sum_{u \in W \cup U} \sum_{v \sim u} w(u, v)^{p_j} |h_{p_j}(v) - h_{p_j}(u)|^{p_j}
\]
\[
- \sum_{u \in V \setminus (W \cup U)} \sum_{v \sim u} w(u, v)^{p_j} |h_{p_j}(v) - h_{p_j}(u)|^{p_j},
\]
and further,
\[
E_{p_j}(f_{p_j}) - E_{p_j}(h_{p_j}) \geq \left( \text{lex}_{k+1}(\tilde{f}) - \frac{\delta}{4} \right)^{p_j} - C \left( \text{lex}_{k+1}(f) + \frac{\delta}{4} \right)^{p_j}
\]
\[
= \left( \text{lex}_{k+1}(\tilde{f}) - \frac{\delta}{4} \right)^{p_j} \left( 1 - C \left( \frac{\text{lex}_{k+1}(f) + \frac{\delta}{4}}{\text{lex}_{k+1}(f) - \frac{\delta}{4}} \right)^{p_j} \right).
\]

By the definition of \( \delta \), the quotient is smaller than 1 and we thus obtain, for sufficiently large \( j \), that \( E_{p_j}(f_{p_j}) - E_{p_j}(h_{p_j}) > 0 \). This contradicts that \( f_{p_j} \) is the minimizer of \( E_{p_j} \) and the proof is complete. \( \square \)

5 Midrange Filters and \( \infty \)-Harmonic Extensions

This section relates midrange filters, \( \infty \)-harmonic extensions and minimal Lipschitz extensions. We then arrive at the Krasnoselskii–Mann algorithm which provides us with an approximation algorithm for minimal Lipschitz extensions. Let us start by recalling the concept of midrange filters.

5.1 Midrange Filters

Consider the following minimization problem: For \( x = (x_1, \ldots, x_n)^T \), \( x_i \in \mathbb{R}^m \) and \( w = (w_1, \ldots, w_n)^T \in (0, 1]^n \) with \( w_1 \geq \ldots \geq w_n \), define the weighted midrange filter \( \text{midr}_w : \mathbb{R}^{mN} \to \mathbb{R}^m \) given by

\[
\text{midr}_w x = \arg \min_{a \in \mathbb{R}^m} \left\{ \max_i (w_i |x_i - a|) \right\}.
\]  \( \text{(3)} \)

If \( w_i |x_i - a| \) is replaced by \( w_i^2 |x_i - a|^2 \), \( i = 1, \ldots, n \), the minimizer remains the same. As the pointwise maximum of strongly convex functions with modulus \( 2w_i^2 \), the functional \( \max_i (w_i^2 |x_i - a|^2) \) is strongly convex with modulus \( 2w_i^2 \). Hence \( \text{midr}_w x \) is uniquely determined.
In the scalar-valued case, that is \( m = 1 \), in was shown in [31, Theorem 5] that the midrange filter can be expressed as
\[
\text{midr}_w x = \frac{w_i x_i + w_j x_j}{w_i + w_j}, \quad (i, j) \in \arg \max_{k, l} \frac{|x_k - x_l|}{1/w_k + 1/w_l}, \tag{4}
\]
which further simplifies to
\[
\text{midr}_w x = \frac{1}{2} \left( \min_i x_i + \max_i x_i \right),
\]
for the non-weighted filter.

Note that we call a midrange filter from the signal processing point of view, is quite classic and can be found in the literature under various names including the *smallest circle/bounding sphere center*, or *Chebyshev center*, or *circumcenter*. The problem is known to be solvable by an \( \mathcal{O}(n) \) linear programming algorithm, where the factor in the \( \mathcal{O}(n) \) term depends sub-exponentially on the dimension \( m \); see [29]. For a comparison of several algorithms, see [32].

The minimization problem (3) can be generalized from \( \mathbb{R}^m \) into Hadamard spaces (see [2, Example 2.2.18]), which is of interest when one works, for instance, with the Hadamard manifold of symmetric positive definite matrices; see [41].

**Lemma 5.1.** For scalar-valued functions, that is, \( m = 1 \), the operator \( \text{midr}_w \) is Lipschitz continuous with constant \( L \leq 1 \) with respect to the \( \infty \)-norm.

**Proof.** Given \( x, y \in \mathbb{R}^n \), consider the line segment \( x(t) := (1 - t)x + ty \), where \( t \in [0, 1] \). Then the functions \( L_{i,j} : [0,1] \to \mathbb{R} \), where \( i, j = 1, \ldots, n \), defined by
\[
L_{i,j}(t) = \frac{|x_i(t) - x_j(t)|}{1/w_i + 1/w_j}
\]
are piecewise linear and hence \( L(t) := \sup_{i,j} L_{i,j}(t) \) is also piecewise linear. Therefore, \( [0,1] \) can be split into a finite number of intervals \( [t_{k-1}, t_k] \), \( k = 1, \ldots, K \) with \( t_0 := 0 \) and \( t_K := 1 \) with corresponding maximizing indices \((i_k, j_k)\) on \([t_{k-1}, t_k], k = 1, \ldots, K \). By the triangle inequality and (4), we get
\[
|\text{midr}_w x - \text{midr}_w y| \leq \sum_{k=1}^{K} |\text{midr}_w x(t_{k-1}) - \text{midr}_w x(t_k)| = \sum_{k=1}^{K} \frac{|w_i x_{i_k}(t_{k-1}) + w_j x_{j_k}(t_{k-1}) - w_i x_{i_k}(t_k) - w_j x_{j_k}(t_k)|}{w_i + w_j} = \sum_{k=1}^{K} \frac{|w_i (x_{i_k}(t_{k-1}) - x_{i_k}(t_k)) + w_j (x_{j_k}(t_{k-1}) - x_{j_k}(t_k))|}{w_i + w_j}.
\]

Finally, the definition of \( x(t) \) implies
\[
|\text{midr}_w x - \text{midr}_w y| \leq \sum_{k=1}^{n} \frac{w_i (t_k - t_{k-1})}{w_i + w_j} \|x - y\|_\infty + \frac{w_j (t_k - t_{k-1})}{w_i + w_j} \|x - y\|_\infty = \|x - y\|_\infty,
\]
which concludes the proof. \( \square \)
Unfortunately, the operator $\text{midr}_w$ is not Lipschitz in dimensions $m \geq 2$. As a matter of fact, one can find a counterexample already in the non-weighted case for $m = 2$ and $n = 3$. In general we can only show that $\text{midr}_w$ is locally $\frac{1}{2}$-Hölder continuous.

**Lemma 5.2.** The operator $\text{midr}_w$ is locally $\frac{1}{2}$-Hölder continuous.

**Proof.** Let $x, y \in \mathbb{R}^{nm}$ with distance $\|x - y\|_{2,\infty} \leq r$ be given. Then $\hat{x} := \text{midr}_w x$ and $\hat{y} := \text{midr}_w y$ fulfill $|\hat{x} - \hat{y}| \leq C_r$ with a constant $C_r$ depending only on $r$. Set $R(x, a) := \max_i (w_i|x_i - a|)$. Without loss of generality we can and do assume $R(x, a) = R(y, \hat{y})$. We then obtain

$$R(x, \hat{y}) \leq R(y, \hat{y}) + w_1\|x - y\|_{2,\infty} \leq R(x, \hat{x}) + w_1\|x - y\|_{2,\infty}.$$

Since the function $R(x, \cdot)^2$ is strongly convex with parameter $2w_n^2$ we have

$$R(x, \hat{x})^2 + w_n^2|\hat{x} - \hat{y}|^2 \leq R(x, \hat{y})^2,$$

which together with (5) this results in

$$\|x - y\|_{2,\infty} \geq \frac{\sqrt{R(x, \hat{x})^2 + w_n^2|\hat{x} - \hat{y}|^2} - R(x, \hat{x})}{w_1} = \frac{w_n^2|\hat{x} - \hat{y}|^2}{w_1(\sqrt{R(x, \hat{x})^2 + w_n^2|\hat{x} - \hat{y}|^2} + R(x, \hat{x}))},$$

and we arrive at

$$|\hat{x} - \hat{y}|^2 \leq \frac{w_1}{w_n^2} (\sqrt{R(x, \hat{x})^2 + w_n^2C_r^2} + R(x, \hat{x})) \|x - y\|_{2,\infty}.$$

Hence, $\text{midr}_w$ is locally $\frac{1}{2}$-Hölder continuous.

**Remark 5.3.** For the non-weighted case, a detailed discussion on continuity properties of midrange filters can be found in [21]. In particular, it was proved that for pairwise different $x_i \in \mathbb{R}^m$, there exists a ball around $x \in \mathbb{R}^{mn}$ such that the midrange operator is Lipschitz continuous on this ball.

### 5.2 $\infty$-Harmonic Extensions

Next, we are interested in applying the midrange filter to given functions $f \in \mathcal{H}_g(V)$. Define the operator $\text{Midr}_w : \mathcal{H}_g(V) \to \mathcal{H}_g(V)$ by

$$\text{Midr}_w f(u) := \text{midr}_w(u) x(u) = \arg \min_{a \in \mathbb{R}^m} \left\{ \max_{v \sim u} w(u, v) |f(v) - a| \right\}, \quad u \in V \setminus U,$$

where $x(u) := (f(v))_{v \sim u}$ and $w(u) := (w(u, v))_{v \sim u}$. A function $f \in \mathcal{H}_g(V)$ is called an $\infty$-harmonic extension of $g$ if it is a fixed point of $\text{Midr}_w$, i.e.,

$$f = \text{Midr}_w f.$$

The relation between $\infty$-harmonic extensions and minimal Lipschitz ones is given by the following lemma. The first part of the lemma also guarantees the existence of $\infty$-harmonic extensions.
Figure 1: Discrete ∞-harmonic extensions of the points (0, 0, 0) (white), (1, 0, 0) (red) and (0, 1, 0) (green) in the RGB color cube visualized in the triangle plane. Left: minimal Lipschitz extension, Middle: some ∞-harmonic extension, Right: set of all ∞-harmonic extensions.

Lemma 5.4. Let \( f \in \mathcal{H}_g(V) \).

(i) If \( f \) is a minimal Lipschitz extension, then it is ∞-harmonic.

(ii) For \( m = 1 \), if \( f \) is an ∞-harmonic extension, then it is uniquely determined and coincides therefore with the minimal Lipschitz extension.

Proof. Let \( f \in \mathcal{H}_g(V) \) be the minimal Lipschitz extension. Assume that \( f \) is not ∞-harmonic. Then there exist \( u_0 \in V \setminus U \) such that

\[
f(u_0) \neq a_0 := \arg \min_{a \in \mathbb{R}^m} \max_{v \sim u_0} w(u_0, v) |a - f(v)|.
\]

Define \( h \in \mathcal{H}_g(V) \) as

\[
h(u) := \begin{cases} f(u) & \text{if } u \neq u_0, \\ a_0 & \text{if } u = u_0. \end{cases}
\]

Then we have \( Lf(u_0) > Lh(u_0) \) and \( Lf(u) = Lh(u) \) for all \( u \in V \setminus U \) with \( u \sim u_0 \). Assume that \( Lf(u) < Lh(u) \) for some \( u \neq u_0, u \sim u_0 \). Since

\[
w(u, v)|f(u) - f(v)| = w(u, v)|h(u) - h(v)|,
\]

for every \( v \sim u, v \neq u_0 \), we have \( u_0 = \arg \max_{v \sim u} w(u, v)|h(u) - h(v)| \). Thus, \( Lh(u) \leq Lh(u_0) \).

Now we get

\[
\max \{Lf(u) : u \in V \setminus U, Lf(u) > Lh(u) \} \geq Lf(u_0) > Lh(u_0) \geq \max \{Lf(u) : u \in V \setminus U, Lh(u) > Lf(u) \}.
\]

This yields the contradiction \( L\text{-}\text{lex}(h) < L\text{-}\text{lex}(f) \).

(i) Follows from Theorem 6.1. For the non-weighted case, see also [32].

The second statement of the lemma is in general not true for \( m \geq 2 \). Moreover, in the vector-valued case, an ∞-harmonic extension \( f \in \mathcal{H}_g(V) \) is not necessarily a minimizer of \( E_\infty \). An example is given in Fig. 1 and a more sophisticated one in [35]. Also, if an ∞-harmonic extension is a minimizer of \( E_\infty \) it doesn’t have to be Lipschitz minimal. For the scalar-valued case, we can use the fact that \( \text{Mid}_w \) is nonexpansive to deduce
an iterative algorithm for computing the minimal Lipschitz extension of \( g \). To this end, we use Krasnoselskii–Mann iterations. This method is in general known to converge in Hilbert spaces, but cannot be generalized to Banach spaces without additional constraints. Fortunately, the Krasnoselskii–Mann method works for finite dimensional normed spaces [6, Corollaries 10,11].

**Theorem 5.5 (Krasnoselskii–Mann Iteration).** Let \( T: \mathbb{R}^N \to \mathbb{R}^N \) be a nonexpansive mapping with nonempty fixed point set, where \( \mathbb{R}^N \) is equipped with an arbitrary norm. Then, for every starting point \( f^{(0)} \in \mathbb{R}^N \), the sequence of iterates \( \{f^{(r)}\}_{r \in \mathbb{N}} \) generated by

\[
 f^{(r+1)} := ((1 - \tau_r)I + \tau_rT) f^{(r)}, \quad \tau_r \in (0, 1)
\]

converges to a fixed point of \( T \) provided that \( \sum_{r=1}^{\infty} \tau_r = \infty \) and \( \limsup_{r \to \infty} \tau_r < 1 \).

We apply Theorem 5.5 with \( T := \text{Midr}_w \). For given \( g: U \to \mathbb{R}^m \), \( f^{(0)} \in \mathcal{H}_g(V) \) and \( \tau_r = \tau \in (0, 1) \), we obtain the following iteration scheme:

\[
 f^{(r+1)} := f^{(r)} + \tau \left( \text{Midr}_w f^{(r)} - f^{(r)} \right).
\] (6)

In the scalar-valued case, we have an immediate convergence result.

**Corollary 5.6.** Let \( g: U \to \mathbb{R} \) be given. Then, for every \( f^{(0)} \in \mathcal{H}_g(V) \) and \( \tau \in (0, 1) \), the sequence of iterates \( \{f^{(r)}\}_{r \in \mathbb{N}} \) generated by (6) converges to the minimal Lipschitz extension of \( g \).

**Proof.** By Lemma 5.1, we have for \( f_1, f_2 \in \mathcal{H}_g(V) \) that

\[
 \| \text{Midr}_w f_1 - \text{Midr}_w f_2 \|_{\infty} = \max_{u \in V \setminus U} | \text{Midr}_w f_1(u) - \text{Midr}_w f_2(u) |
\]

\[
 = \max_{u \in V \setminus U} | \text{midr}_w(u) x_1(u) - \text{midr}_w(u) x_2(u) |
\]

\[
 \leq \max_{u \in V \setminus U} \| x_1(u) - x_2(u) \|_{\infty} = \| f_1 - f_2 \|_{\infty}.
\]

Hence \( \text{Midr}_w: \mathbb{R}^N \to \mathbb{R}^N \) is nonexpansive with respect to the \( \infty \)-norm and by Theorem 5.5 the sequence converges to a fixed point of \( \text{Midr}_w \). By Lemma 5.4 there is exactly one fixed point which is the minimal Lipschitz extension of \( g \). \( \square \)

For the vector-valued case, we apply the iteration scheme (6) as a heuristic method and obtain promising results in Section 8. However, we did not succeed in proving a convergence theorem. On the positive side, we have the following result.

**Corollary 5.7.** Let \( g: U \to \mathbb{R}^m \), \( m \geq 1 \) be given. If for \( f^{(0)} \in \mathcal{H}_g(V) \) the sequence of iterates \( \{f^{(r)}\}_{r \in \mathbb{N}} \) generated by (6) is asymptotically regular, that is, it satisfies \( \lim_{r \to \infty} \| f^{(r+1)} - f^{(r)} \|_{2, \infty} = 0 \), then every cluster point is an \( \infty \)-harmonic extension of \( g \).

**Proof.** Assume that we have a convergent subsequence \( \{f^{(r_j)}\} \) with limit \( \tilde{f} \). Since the iterates \( \{f^{(r_j+1)}\} \) are bounded, we can choose a subsequence of \( \{f^{(r_j+1)}\} \) (again denoted with the same indices) such that also \( \{f^{(r_j+1)}\} \) converges with limit \( \tilde{f} \). Since \( \lim_{r \to \infty} \| f^{(r+1)} - f^{(r)} \|_{2, \infty} = 0 \), it follows \( \tilde{f} = \tilde{f} \). Let \( u \in V \setminus U \). Using the continuity of \( \text{midr}_w(u) \), we obtain

\[
 \tilde{f}(u) = \lim_{j \to \infty} f^{(r_j+1)}(u) = \lim_{j \to \infty} (1 - \tau) f^{(r_j)}(u) + \tau \text{midr}_w(u) f^{(r_j)}(u)
\]

\[
 = (1 - \tau) \tilde{f}(u) + \tau \text{midr}_w(u) \tilde{f}(u).
\]

Rearranging the terms yields the assertion. \( \square \)
6 $\infty$-Laplacians on Scalar-Valued Functions

The minimizers of $E_p$ in $[1]$ are determined by the zero of the anisotropic $p$-Laplacian operator
\[ \Delta_{w,p}f(u) := \sum_{v \in V} w(u,v) |f(u) - f(v)|^{p-2} (f(u) - f(v)) = 0, \]  
for $u \in V \setminus U$. To the best of our knowledge there exists no satisfactory definition of an $\infty$-Laplacian for vector-valued functions. There was an attempt in this direction in $[4]$ which is unfortunately not well-defined.

For scalar valued-functions, the $\infty$-Laplacian $\Delta_{w,\infty} := \mathcal{H}_g(V) \rightarrow \mathcal{H}_g(V)$ is given by
\[ \Delta_{w,\infty}f(u) := \frac{1}{2} \max_{v \sim u} w(u,v) (f(v) - f(u)) + \min_{v \sim u} w(u,v) (f(v) - f(u)) \]
for $u \in V \setminus U$. This definition is due to $[32]$ and for weighted graphs due to $[15, 16]$. Note that $w(u,u) > 0$ implies $\max_{v \sim u} w(u,v) (f(v) - f(u)) \geq 0$. For non-weighted graphs, we have
\[ \Delta_{\infty}f(u) = \frac{1}{2} \max_{v \sim u} (f(v) - f(u)) + \frac{1}{2} \min_{v \sim u} (f(v) - f(u)) = \text{midr} x(u) - f(u). \]  
Hence, on $\mathcal{H}_g(V)$, the zero of $\Delta_{\infty}$ coincide with the fixed point of Midr, that is, with the unique $\infty$-harmonic extension of $g$. This observation extends into the weighted case as well.

Indeed, it was showed already in $[15]$ that, given $g: U \rightarrow \mathbb{R}$, there exists a unique extension $f \in \mathcal{H}_g(V)$ with $\Delta_{w,\infty}f(u) = 0$ for every $u \in V \setminus U$. We now obtain the desired statement.

**Theorem 6.1.** Let $m = 1$ and $g: U \rightarrow \mathbb{R}$. Then $f \in \mathcal{H}_g(V)$ satisfies $\Delta_{w,\infty}f(u) = 0$ for every $u \in V \setminus U$ if and only if it is an $\infty$-harmonic extension of $g$ and it is the case if and only if it is the minimal Lipschitz extension of $g$.

**Proof.** On account of the above discussion it remains to prove the following. If $f \in \mathcal{H}_g(V)$ fulfills $\Delta_{w,\infty}f(u) = 0$ for all $u \in V \setminus U$, then $f$ is $\infty$-harmonic. Since
\[ \max_{v \sim u} w(u,v) (f(v) - f(u)) = \max_{v \sim u} w(u,v) (f(u) - f(v)) = \max_{v \sim u} w(u,v) |f(v) - f(u)|, \]
we obtain for $a \geq 0$ that
\[ \max_{v \sim u} w(u,v) |f(v) - (f(u) + a)| \geq \max_{v \sim u} w(u,v) \max \{0, f(u) - f(v) + a\} \]
\[ \geq \max_{v \sim u} w(u,v) (f(u) - f(v)) \]
\[ = \max_{v \sim u} w(u,v) |f(v) - f(u)|. \]
Similarly, for $a \leq 0$, we obtain the inequality
\[ \max_{v \sim u} w(u,v) |f(v) - (f(u) + a)| \geq \max_{v \sim u} w(u,v) |f(v) - f(u)|. \]
This finishes the proof. \qed
For given $g: U \to \mathbb{R}$ and $f^{(0)} \in \mathcal{H}_g(V)$ we consider the iteration scheme

$$f^{(r+1)} := f^{(r)} + \tau \Delta_{w,\infty} f^{(r)}. \tag{9}$$

This scheme was proposed in \cite{15,16} without a convergence proof. The authors only proved Corollary 6.2.

In the non-weighted case, this can be rewritten by \cite{5} as

$$f^{(r+1)}(u) = (1 - \tau)f^{(r)}(u) + \tau \mid w, x^{(r)}(u), \quad u \in V \setminus U.$$ 

By \cite{6} and Corollary 5.5 we see that the sequence of iterates (9) converges to the minimal Lipschitz extension of $g$ for $\tau \in (0,1)$.

For the weighted setting, this is also true by the following corollary.

**Corollary 6.2.** Let $g: U \to \mathbb{R}$ be given. Then, for every $f^{(0)} \in \mathcal{H}_g(V)$ and $\tau \in (0,1)$, the sequence of iterates $\{f^{(r)}\}_{r \in \mathbb{N}}$ generated by (9) converges to the minimal Lipschitz extension of $g$.

**Proof.** Consider the operator $\Phi: \mathcal{H}_g(V) \to \mathcal{H}_g(V)$ defined by

$$\Phi f := f + \Delta_{w,\infty} f.$$ 

By virtue of Theorem 6.1, the mapping $\Phi$ has a unique fixed point determined by the zero of the $\infty$-Laplacian. We show that $\Phi$ is nonexpansive. Then it follows immediately from Theorem 5.5 that the sequence $\{f^{(r)}\}_{r \in \mathbb{N}}$ converges to this fixed point. For $u \in V \setminus U$ we rewrite

$$\Delta_{w,\infty} f(u) = \frac{1}{2} \max_{v \sim u} w(u,v) (f(v) - f(u)) - \frac{1}{2} \max_{v \sim u} w(u,v) (f(u) - f(v)).$$ 

For $i = 1, 2$, let $f_i \in \mathcal{H}_g(V)$. Now, define $y_i := \arg \max_{v \sim u} w(u,v) (f_i(v) - f_i(u))$ together with $z_i := \arg \max_{v \sim u} w(u,v) (f_i(u) - f_i(v))$. Then, for $u \in V \setminus U$, we obtain

$$\begin{align*}
\Phi f_1(u) - \Phi f_2(u) &= f_1(u) - f_2(u) + \Delta_{w,\infty} f_1(u) - \Delta_{w,\infty} f_2(u) \\
&= f_1(u) - f_2(u) + \frac{w(u,y_1)}{2} (f_1(y_1) - f_1(u)) - \frac{w(u,z_1)}{2} (f_1(u) - f_1(z_1)) \\
&\quad - \frac{w(u,y_2)}{2} (f_2(y_2) - f_2(u)) + \frac{w(u,z_2)}{2} (f_2(u) - f_2(z_2)) \\
&\leq f_1(u) - f_2(u) + \frac{w(u,y_1)}{2} (f_1(y_1) - f_1(u)) - f_2(y_1) + f_2(u) \\
&\quad - \frac{w(u,z_2)}{2} (f_1(u) - f_1(z_2)) - f_2(u) + f_2(z_2) \\
&= \left(1 - \frac{w(u,y_1)}{2} + \frac{w(u,z_2)}{2}\right) (f_1(u) - f_2(u)) \\
&\quad + \frac{w(u,y_1)}{2} (f_1(y_1) - f_2(y_1)) + \frac{w(u,z_2)}{2} (f_1(z_2) - f_2(z_2))
\end{align*}$$

and with $\|f_1 - f_2\|_\infty = \max_{u \in V} |f_1(u) - f_2(u)|$ further $\Phi f_1(u) - \Phi f_2(u) \leq \|f_1 - f_2\|_\infty$. Analogously, we get $\Phi f_1(u) - \Phi f_2(u) \geq -\|f_1 - f_2\|_\infty$ and thus $\|\Phi f_1 - \Phi f_2\|_\infty \leq \|f_1 - f_2\|_\infty$. \hfill \Box
7 Numerical Aspects of Approximating Algorithms

In this section, we comment on numerical computations of the minimizer of $E_p$ for large $p \in \mathbb{N}$. By Theorem 4.3 such a minimizer can be considered as an approximation of the minimal Lipschitz extension of $g$. As in the scalar-valued case, we refer to it as Polya’s method. Using a preconditioned Newton method, computations are performed up to $p = 2400$ in Section 8.

Let us mention that we could also compute the minimizer of $E_{\infty,p}$, e.g., by the ADMM method described in [19]. However, this method is more time consuming and works so far only for moderate sizes of $p \leq 25$. We also apply the iterated midrange filter from Section 5 but in a Gauss-Seidel like fashion.

7.1 Preconditioned Newton method

Since $E_p$ is a smooth functional, the minimizers can be computed using Newton’s method. This concept was also pursued for the continuous setting in [25].

**Theorem 7.1.** [30, Theorem 3.7]

Let $F \in C^2(\mathbb{R}^N, \mathbb{R})$ have a unique minimizer $x^*$ with Lipschitz continuous Hessian $\nabla^2 F$ in a neighborhood of $x^*$. Further, assume that $\nabla^2 F$ is positive definite at $x^*$. If the initial guess $x^{(0)}$ is sufficiently close to $x^*$, the iteration

$$\nabla^2 F(x^{(r)}) h^{(r)} = \nabla F(x^{(r)}), \quad x^{(r+1)} = x^{(r)} - h^{(r)}$$

(10)

converges quadratically to $x^*$.

A global convergence result under some stronger assumptions can be found in [30, Theorem 6.3] or [7, Section 9.5.3]. The idea is to change the Newton scheme to

$$x^{(r+1)} = x^{(r)} - \alpha_r \nabla^2 F(x^{(r)})^{-1} \nabla f(x^{(r)}),$$

where $\alpha_r$ is computed with an Armijo backtracking line search which always tries the step size $\alpha_r = 1$ first. If $K = \{ x \in \mathbb{R}^N : F(x) \leq F(x_0) \}$ is compact and there exists a constant $C$ such that the condition number

$$\text{cond}(\nabla^2 f(x^{(r)})) \leq C$$

(11)

for all $r$, then the scheme is globally convergent to $x^*$. For $r$ large enough, the step size is always chosen as $\alpha_r = 1$ so that the convergence becomes quadratic. In our numerical examples, we have observed (11).

In order to apply Newton’s method to $E_p$, we need to compute its Hessian. Differentiating the gradient of $E_p$ in [7], we observe that the Hessian is block structured with diagonal...
blocks \( \left( \frac{\partial^2 E_p}{\partial f_i(u) \partial f_j(u)} \right)_{i,j=1}^m \) and non-diagonal blocks \( \left( \frac{\partial^2 E_p}{\partial f_i(u) \partial f_j(u)} \right)_{i,j=1}^m \) for \( u \sim v, u \neq v \), where

\[
\frac{\partial^2 E_p}{\partial f_i(u)^2} = \sum_{v \in V} w(u,v) \left( |f(u) - f(v)|^{p-2} + (p - 2)|f(u) - f(v)|^{p-4}(f_i(u) - f_i(v))^2 \right),
\]

\[
\frac{\partial^2 E_p}{\partial f_i(u) \partial f_j(u)} = \sum_{v \in V} w(u,v) (p-2)|f(u) - f(v)|^{p-4}(f_i(u) - f_i(v))(f_j(u) - f_j(v)),
\]

\[
\frac{\partial^2 E_p}{\partial f_i(u) \partial f_j(v)} = w(u,v)(p-2)(|f(u) - f(v)|^{p-4}(f_i(u) - f_i(v))^2 + |f(u) - f(v)|^{p-2}),
\]

\[
\frac{\partial^2 E_p}{\partial f_i(v)^2} = -w(u,v)(p-2)(|f(u) - f(v)|^{p-4}(f_i(u) - f_i(v))f_j(u) - f_j(v)).
\]

For large \( p \) the factor \( w(u,v)|f(u) - f(v)|^{p-4} \) can get very large, resp., very small, possibly causing a bad condition number of the Hessian. Therefore, the choice of a suitable preconditioner is crucial for solving the linear system of equations in \( 10 \). One possible preconditioner choice is the diagonal matrix \( D \) with entries

\[
D_{f_i(u),f_i(u)} = \frac{c}{\max_{v \in V} (w(u,v)|f(u) - f(v)|^{p-2})}, \quad c \in (0, 1]
\]

in the diagonal block related to \( u \in V \setminus U \). This matrix ensures that for every \( f(u) \) at least one edge has numerically reasonable values. Then, we solve

\[
D \nabla^2 E_p h = -D \nabla E_p
\]

instead of \( 10 \). In our implementation, the minimizers of \( E_p \) are computed for increasing \( p \) with the previous minimizer as an initialization. We always choose \( p = 2 \) as a starting point, since this problem reduces to solving a linear system. It was not necessary to update the preconditioner after every Newton step and the one from the first step is used for all iterations. Since \( 12 \) is non symmetric, we choose \textit{bicgstab} with Gauss-Seidel or Jacobi preconditioning as a linear solver. For better performance the problem can be solved on a GPU. This naturally rises the question if a symmetric preconditioning of \( 10 \) exists which circumvents the numerical instabilities.

### 7.2 Iterated Midrange Filter

Based on the ideas in Section 5, a second approach to ,,compute” the minimal Lipschitz extension of \( g \) is to apply the iterated midrange filter with an appropriate starting point. As starting point \( f^{(0)} \) we use again the minimizer of \( E_2 \), i.e. the zero of the 2-Laplacian. In contrast to Section 5, we apply the midrange filter in a cyclic or Gauss-Seidel like fashion. If the vertices in \( V \setminus U \) are numbered from 0 to \( N \) with \( c(k) := \text{mod}(r,N) \), one cycle of the algorithm reads for \( \tau \in (0, 1] \) as follows

\[
f^{(k+1)}(v) := \begin{cases} f^{(k)}(v) + \tau (\text{midr}_{w(v)} x^{(k)}(v) - f^{(k)}(v)) & \text{if } v = u_{c(k)}, \\ f^{(k)}(v) & \text{otherwise}. \end{cases}
\]

Clearly, the vertices can be also visited in a random cyclic order. For the case \( m = 1 \), a convergence result follows similar as in Corollary 5.6 with minor modifications due to the
cyclic update. Unfortunately, the proof does not generalize to the vector-valued case, because the midrange filter is not non-expansive for \( m \geq 2 \). However, if the sequence converges, we can adapt Corollary 7.1 to the cyclic setting and observe that the scheme provides an \( \infty \)-harmonic extension of \( g \). Additionally, we have the following „descent” property.

**Proposition 7.2.** The update \( f^{(k+1)} \) in (13) is \( L \)-lex smaller than \( f^{(k)} \) if \( f^{(k+1)} \neq f^{(k)} \).

**Proof.** Assume \( f^{(k+1)} \neq f^{(k)} \) and let \( u \) denote the updated vertex in step \( k \). Then, it holds

\[
Lf^{(k+1)}(u) = \max_{v \sim u, v \neq u} w(u, v)|f^{(k+1)}(u) - f^{(k)}(v)| \leq \max_{v \sim u} w(u, v)|f^{(k+1)}(u) - f^{(k)}(v)|
\]

\[
\leq (1 - \tau)Lf^{(k)}(u) + \tau \max_{v \sim u} w(u, v) |\text{midr}_{w(u)} x^{(k)}(u) - f^{(k)}(v)|
\]

\[
= (1 - \tau)Lf^{(k)}(u) + \tau \max_{v \sim u} w(u, v) \arg \min_{a} (\max_{\tilde{v} \sim u} w(u, \tilde{v})|a - f^{(k)}(\tilde{v})|) - f^{(k)}(v)|
\]

\[
< Lf^{(k)}(u),
\]

where the last inequality follows by \( f^{(k+1)} \neq f^{(k)} \). For \( v \sim u, v \neq u \), we obtain by

\[
Lf^{(k+1)}(v) = \max \left\{ \max_{\tilde{v} \neq u, \tilde{v} \sim v} |f^{(k)}(v) - f^{(k)}(\tilde{v})|, w(v, u)|f^{(k)}(v) - f^{(k+1)}(u)| \right\}
\]

that \( Lf^{(k+1)}(v) \leq \max \{Lf^{(k)}(v), Lf^{(k+1)}(u)\} \). Hence, for all \( v \in V \setminus U \) with \( Lf^{(k)}(v) \geq Lf^{(k)}(u) \) it holds \( Lf^{(k+1)}(v) \leq Lf^{(k)}(v) \) and if \( Lf^{(k)}(v) < Lf^{(k)}(u) \) it holds \( Lf^{(k+1)}(v) < Lf^{(k)}(u) \). Since \( Lf^{(k+1)}(u) < Lf^{(k)}(u) \), this implies that \( f^{(k+1)} \) is \( L \)-lex smaller than \( f^{(k)} \). \( \square \)

8 Numerical Examples

In this section, proof-of-the-concept examples for the performance of the proposed approaches are provided. To outline the differences between the approximation of the minimal Lipschitz extension by Polya’s method, iterated midrange filters and componentwise minimal Lipschitz extensions, we start with an intuitive example, where the function is defined on a simple graph and maps into \( \mathbb{R}^2 \), i.e., \( m = 2 \). Then, we consider the inpainting of RGB images, i.e. of vector-valued functions with \( m = 3 \). An original image \( f \in [0, 1]^{M,N} \) has missing pixels, so that the function/image values are only known on a subset \( U \subset \{1, \ldots, M\} \times \{1, \ldots, N\} \). The aim is to reconstruct the complete image. In the first few examples, we assume that the function lives on a 4-neighborhood graph of the image grid. Of course, minimal Lipschitz extensions for inpainting tasks make more sense if the function values on neighboring vertices of the graph are similar. This can be achieved by applying nonlocal patch-based techniques. Starting with the pioneering work of [3] such methods were successfully refined and applied for different tasks in image processing. Concerning Polya’s method, we apply the method described in Section 7.1 with \( p = 2, 5, 10, 15, 20 \) and then further increase \( p \) in steps of 10 until the desired value of \( p \) is reached. In particular, we initialize with the with the result obtained from the previous \( p \) value. As initialization for computing the iterated midrange filter the 2-Laplacian is used. All algorithms are implemented in Matlab.
The solution of $p$-Laplacians for large $p$ as well as the iterated midrange filter approximate the minimal Lipschitz extension well. The componentwise, single-valued minimal Lipschitz extensions differ completely from those extensions.

**2D function.** The first example in Fig. 2 shows a simple graph $G = (V, E)$ with equal weights, where the blue lines indicate the edges $E$ between vertices $V$. The blue circles illustrate the fixed values of the function $g: U \rightarrow \mathbb{R}^2$ as spatial positions in $\mathbb{R}^2$. The red crosses are the inpainted values of $f \in \mathcal{H}_g(V)$. For this simple example, the minimal Lipschitz extension can be computed analytically and coincides with the result of the iterated midrange filter $\tilde{f}$ with $\tau = 0.95$. However, no convergence result is available in the vector-valued case so far and the result is only experimental. Note that for computing the individual midrange filters we applied the simple procedure described in [33, Section 4]. For larger size problems more sophisticated methods as described in Subsection 5.1 should be used. For Polya’s algorithm we observe that if $p$ increases, the solution of the $p$-Laplace indeed converges to the minimal Lipschitz extension. Finally, we computed the result of the componentwise minimal Lipschitz extension by $\tilde{f}$, see Corollary 6.2. The function differs completely from the minimal Lipschitz extension of the vector-valued function. This is not really surprising: Consider for example a graph consisting of four vertices, where one vertex is connected to the remaining three vertices by edges with equal weights. Let the function values on those three vertices be $(0,0), (1,0), (1/2, \sqrt{3}/2)$ (equilateral triangle), the minimal Lipschitz extension is $(1/2, \sqrt{3}/6)$ (the circumcenter of the triangle), while the componentwise minimal Lipschitz extensions yields the point $(1/2, \sqrt{3}/4)$. 

Figure 2: Comparison of different extensions of a 2D function with given blue circle values.
RGB inpainting on a local neighborhood graph. Next, we consider a simple RGB image with a missing square in the center, where the four color values form a square in the RGB cube. We use an equally weighted 4-neighborhood graph on the image grid. Fig. 3 shows the inpainting results with the same methods as in the previous example. Again the solution of the $p$-Laplacian for large $p$ and the iterated midrange filter lead to nearly the same images which differ from the comp. minimal Lipschitz solution. At first glance the results might look a bit unexpected since the 2-Laplacian appears to be smoother than the others. In order to check that the values of the Lipschitz constants in $L$-lex get indeed lexicographically smaller, we added a table with the 10 largest Lipschitz constants for increasing $p$ in Fig. 4. This table also contains the values for the comp. minimal Lipschitz extension in the last column, which are already worse than the values of the 40-Laplacian.

Nonlocal image inpainting (random mask). In our next two examples, 90% of the image pixels are missing, where the pixels are chosen randomly. We present inpainting results for a more sophisticated graph choice based on nonlocal patch similarities. To this end, we built a graph $G$ connecting the image grid points in a semi nonlocal way. Given some patch radius $r$, the local patch $p_{ij}$ around some pixel $(i, j)$ is defined as the quadratic part of $I$ with size $2r + 1 \times 2r + 1$ which is centered at the pixel $(i, j)$. Then, the distance
between two grid points \((i, j)\) and \((j, k)\) is defined as

\[
d((i, j), (k, l))^2 := \| p_{ij} - p_{kl} \|_F^2 + c \cdot \left( \frac{i - k}{m} \right)^2 + c \cdot \left( \frac{j - l}{n} \right)^2,
\]

where \(\| \cdot \|_F\) denotes the Frobenius norm. In order to reduce the computational effort, the distances are only computed in a neighborhood of radius \(R\) around \((i, j)\). The edge weights are defined as

\[
w((i, j), (k, l)) := \exp \left( -d((i, j), (k, l))^2 \left( \frac{1}{\sigma(i, j)^2} + \frac{1}{\sigma(k, l)^2} \right) \right),
\]

where \(\sigma(i, j)\) is the distance of \((i, j)\) to its 20th nearest neighbor. Note that the sum of \(\sigma(i, j)\) and \(\sigma(k, l)\) ensures symmetry of the adjacency matrix of \(G\). The weights are truncated to the \(K\) largest ones in order to make the adjacency matrix sparse. In our numerical experiments it turned out to be beneficial to add a small local 4-neighbor grid graph to \(G\) in the first few iterations.

At the beginning, the missing pixels are assigned random Gaussian numbers with mean and covariance of the known part of the image as proposed, e.g., in [40]. Then, the nonlocal graph is generated and the 200-Laplacian extension is computed as an approximation of the minimal Lipschitz extension. This step, including the grid generation, is repeated 15 times and the corresponding results are shown for two different images in Figs. 5 and 6, where we used \(c = 9, r = 5, R = 30\) and \(K = 40\). We observe that higher order Laplacians perform much better than the 2-Laplacian.

**Nonlocal image inpainting (hole mask).** In our final example, we apply an inpainting mask with larger holes. Here, the above approach is modified such that only pixels are compared which are known in both patches. If less than 10 percent of the pixels in the patches are known, the distance is defined as infinity. Further, the local part of the distance is omitted and \(\sigma\) is chosen as a fixed constant independent of \(i\) and \(j\). The algorithm uses the following initialization step: Iteratively, only pixels with at least one known neighbor in the grid graph are chosen for the inpainting and marked as known pixels afterwards. This
step is repeated until every pixel of the mask is filled, i.e. every pixel is treated exactly once. After this initialization, the same iterative procedure as for the random masks is applied and the result for $p = 200$ is shown in Fig. 5. The other parameters are $\sigma = 0.045$, $r = 7$, $R = 45$, and $K = 45$. Again, we observe quality differences between the 2-Laplacian and 200-Laplacian, especially in the zoomed parts of the image.

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**References**

[1] G. Aronsson. Extension of functions satisfying Lipschitz conditions. *Ark. Mat.*, 6:551–561, 1967.

[2] M. Bačák. *Convex Analysis and Optimization in Hadamard Spaces*, volume 22 of *De Gruyter Series in Nonlinear Analysis and Applications*. De Gruyter, Berlin, 2014.
Figure 6: Nonlocal inpainting using different extensions. (Simpsons image: https://frinkiac.com/img/S08E02/261944.jpg).

[3] M. Belkin and P. Niyogi. Laplacian eigenmaps for dimensionality reduction and data representation. *Neural Comput.*, 15:1373–1396, 2003.

[4] R. Bergmann and D. Tenbrinck. Nonlocal inpainting of manifold-valued data on finite weighted graphs. In *3rd Conference on Geometric Science of Information*, page 604–612. Springer International Publishing, 2017.

[5] R. Bergmann and D. Tenbrinck. A graph framework for manifold-valued data. *SIAM J. Imag. Sci.*, 11(1):325–360, 2018.

[6] J. Borwein, S. Reich, and I. Shafrir. Krasnoselski-Mann iterations in normed spaces. *Can. Math. Bull.*, 35(1):21–28, 1992.

[7] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge, 2004.

[8] A. Buades, B. Coll, and J.-M. Morel. A non-local algorithm for image denoising. In *IEEE Int. Conf. on Computer Vision and Pattern Recognition*, volume 2, pages 60–65, 2005.

[9] J. R. Casas and L. Torres. Strong edge features for image coding. In R. W. Schafer,
P. Maragos, and M. A. Butt, editors, *Mathematical Morphology and Its Applications to Image and Signal Processing*, pages 443–450. Kluwer, Boston, MA, 1996.

[10] V. Caselles, J.-M. Morel, and C. Sbert. An axiomatic approach to image interpolation. *IEEE Trans. Image Process.*, 7(3):376–386, 1998.

[11] M. G. Crandall, L. C. Evans, and R. F. Gariepy. Optimal Lipschitz extensions and the infinity Laplacian. *Calc. Var. Partial Differential Equations*, 13(2):123–139, 2001.

[12] J. Descloux. Approximations in $L^p$ and Chebychev approximations. *J. Soc. Ind. Appl. Math.*, 11:1017–1026, 1963.

[13] A. Egger and R. Huotari. Rate of convergence of the discrete Pólya algorithm. *J. Approx. Theory*, 60:24–30, 1990.

[14] C. Elion and L. Vese. An image decomposition model using the total variation and the infinity Laplacian. In *Proc. SPIE*, volume 6498, 02 2007.

[15] A. Elmoataz, X. Desquesnes, Z. Lakhdari, and O. Lézoray. Nonlocal infinity Laplacian equation on graphs with applications in image processing and machine learning. *Math. Comput. Simul.*, 102:153–163, 2014.

[16] A. Elmoataz, M. Toutain, and D. Tenbrinck. On the $p$-Laplacian and $\infty$-Laplacian on graphs with applications in image and data processing. *SIAM J. Imag. Sci.*, 8(4):2412–2451, 2015.

[17] I. Galić, J. Weickert, M. Welk, A. Bruhn, A. Belyaev, and H.-P. Seidel. Image compression with anisotropic diffusion. *J. Math. Imaging Vision*, 31:255–269, 2008.

[18] G. Gilboa and S. Osher. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.*, 7(3):1005–1028, 2008.

[19] J. Hertrich. *Infinity-Laplacians on scalar- and vector-valued functions and optimal Lipschitz extensions on graphs*. Bachelor Thesis, TU Kaiserslautern, 2018.

[20] J. Hertrich, S. Neumayer, G. Steidl, and M. Bačák. Minimal Lipschitz extensions for vector-valued functions on finite graphs. In M. Burger, J. Lellmann, and J. Modersitzki, editors, *Scale Space and Variational Methods in Computer Vision*, volume XX of *Lecture Notes in Computer Science*, page XX. Springer, 2019.

[21] P. N. Ivanshin and E. N. Sosov. Local Lipschitz property for the Chebyshev center mapping over $N$-nets. *Mat. Vesnik*, 60(1):9–22, 2008.

[22] R. Jensen. Uniqueness of Lipschitz extensions minimizing the sup-norm of the gradient. *Arch. Ration. Mech. Anal.*, 123:51–74, 1993.

[23] N. Katzourakis. $L^\infty$ variational problems for maps and the Aronsson PDE system. *J. Differential Equations*, 253(7):2123–2139, 2012.

[24] N. Katzourakis. On the structure of $\infty$-harmonic maps. *Comm. Partial Differential Equations*, 39(11):2091–2124, 2014.
[25] N. Katzourakis and T. Pryer. On the numerical approximation of ∞-harmonic mappings. *NoDEA. Nonlinear Differential Equations and Applications*, 23(6):Art. 51, 23, 2016.

[26] R. Kyng, A. Rao, S. Sachdeva, and D. A. Spielman. Algorithms for Lipschitz learning on graphs. In *Conference on Learning Theory*, pages 1190–1223, 2015.

[27] A. J. Lazarus, J. Loeb, D. E. Propp, W. R. Stromquist, and H. D. Ullman. Combinatorial games under auction play. *Games and Economical Behavior*, 27(2):229–264, 1999.

[28] P. Lindqvist. *Notes on the infinity Laplace equation*. Springer Briefs in Mathematics. BCAM Basque Center for Applied Mathematics, Bilbao; Springer, 2016.

[29] J. Matoušek, J. Sharir, and E. Welzl. A subexponential bound for linear programming. *Algorithmica*, 16:498–516, 1996.

[30] J. Nocedal and S. J. Wright. *Numerical Optimization*. Springer Series in Operations Research and Financial Engineering. Springer, New York, second edition, 2006.

[31] A. M. Oberman. A convergent difference scheme for the infinity Laplacian: construction of absolutely minimizing Lipschitz extensions. *Math. Comput.*, 74(251):1217–1230, 2005.

[32] Y. Peres, O. Schramm, S. Sheffield, and D. B. Wilson. Tug-of-war and infinity Laplacian. *J. Amer. Math. Soc.*, 22(1):167–210, 2005.

[33] T. V. Phan. Minimal Lipschitz extension. HAL Preprint 01303765, 2016.

[34] G. Pólya. Sur un algorithm touours convergent pour obtenir les polynomes de meilleur approximation de Tchebycheff pour une function continue quelconque. *Comptes Rendus de l’Académie des Sciences Paris*, 157:840–843, 1913.

[35] J. M. Quesada, J. Fernández-Ochoa, J. Martinez-Moreno, and J. Bustamante. The Pólya algorithm in sequence spaces. *J. Approx. Theory*, 135:245–257, 2005.

[36] C. Schmaltz, P. Peter, M. Mainberger, F. Ebel, J. Weickert, and A. Bruhn. Understanding, optimising, and extending data compression with anisotropic diffusion. *Int. J. Comput. Vision*, 108(3):222–240, 2014.

[37] J. Serra. *Image Analysis and Mathematical Morphology*. Academic Press, London, 1982.

[38] S. Sheffield and C. K. Smart. Vector-valued optimal Lipschitz extensions. *Commun. Pure Appl. Math.*, 65(1):128–154, 2012.

[39] J. Shi and J. Malik. Normalized cuts and image segmentation. *IEEE Trans. Pattern Anal. Mach. Intell.*, 22(8):888–905, 2000.

[40] Z. Shi, S. Osher, and W. Zhu. Weighted graph Laplacian and image inpainting. *J. Sci. Comput.*, 577, 2016.

[41] M. Welk, J. Weickert, F. Becker, C. Schnörr, C. Feddern, and B. Burgeth. Median and related local filters for tensor-valued images. *Signal Process.*, 87(2):291–308, 2007.
[42] S. Xu, R. M. Freund, and J. Sun. Solution methodologies for the smallest enclosing circle problem. *Comput. Optim. Appl.*, 25(1-3):283–292, 2003. A tribute to Elijah (Lucien) Polak.
Figure 7: Non-local inpainting using different extensions.