On the validity of SLAC fermions for the 1+1D helical Luttinger liquid

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The Nielson-Ninomiya theorem states that a chirally invariant free fermion lattice action, which is local, translation invariant, and real necessarily has fermion doubling [1]. How should one then carry out simulations of a single Dirac cone? A possible route is to consider higher dimensions. A single Dirac cone in say 1+1-dimensions can be realized as a surface state of a 2+1-dimensional topological insulator. The other Dirac cone lies on the other surface and as the system size grows mixing between the cones will vanish such that the physics of a single cone can be studied. In the realm of high-energy physics, this construction is referred to as domain wall fermions [2]. In the domain of the solid state, this idea has been used to study correlation effects in helical Luttinger liquids [3, 4].

Alternatively one can violate one of the assumptions of the Nielson-Ninomiya theorem. SLAC fermions are subject to long range hopping and thereby violate the locality condition. They have been used in a number of solid state setups and high energy physics setups and seem to provide a simple route to simulate a single Dirac cone in a lattice model with finite lattice constant $a$. In particular, it avoids the potentially expensive step of dealing with higher dimensional systems. SLAC fermions come with a singularity at the Brillouin zone boundary at $k = \pm \pi/a$ in one dimension. The question we will ask in this article is how the non-locality and concomitant singularity at the zone edge affects the physical results, in comparison to a domain wall fermion approach.

To do so, we will consider the simplest possible model, the helical Luttinger liquid emerging at the boundary of a 2D quantum spin Hall insulator as realized by the Kane-Mele model [12]. In particular we will consider a setup with U(1) symmetry, corresponding to conservation of total spin. This choice is challenging for SLAC fermions. For short ranged interactions the Mermin-Wagner theorem states that continuous symmetries cannot be spontaneously broken in 1+1-dimension even in the zero temperature limit. In fact, in conjunction with the intrinsic nesting instabilities of 1+1-dimensional systems this impossibility of ordering leads to the fluctuation dominated physics of the Luttinger liquid [13, 14]. The non-locality of SLAC fermions violate the assumptions of the Mermin-Wagner theorem and can hence lead to artifacts especially in the strong coupling limit. We note that this has recently been pointed out in Ref. [8].

Another reason for the choice of this model is that it can be solved exactly since only forward scattering is allowed. The results of the bosonization approach have been favorably compared to calculations based on domain wall fermions [3, 4]. Hence the bosonization results will be our gold standard. In this article we formulate a SLAC lattice regularization of the helical Luttinger liquid that allows for negative sign free auxiliary field quantum Monte Carlo (QMC) simulations. Our aim is to assess if the SLAC fermion approach has the ability to reproduce the continuum bosonization results.

The article is organized as follows. In the next section, II, we will discuss the SLAC formulation of the helical Luttinger liquid. Before discussing our results for the non-interacting and interacting cases in Sec. V, we will summarize the bosonization results in Sec. III and the technicalities of the Monte Carlo simulations in Sec. IV. In Sec. VI we discuss a simple model to understand our strong coupling results. In Sec. VII we summarize the implications of our results.

Keywords: SLAC fermions, Quantum Monte Carlo, Helical Luttinger liquid
II. SLAC FORMULATION OF THE HELICAL LIQUID

We consider the following one-dimensional model of length $L$ and lattice constant $a$:

$$
\hat{H} = -v_F \sum_{i=1}^{L} \sum_{r=-L/2}^{L/2} t(r) \left( \hat{a}_{i+1}^{\dagger} \hat{b}_{i+r} + \hat{b}_{i+r}^{\dagger} \hat{a}_{i} \right) + U \sum_{i} \left( \hat{n}_{i}^{a} - \frac{1}{2} \right) \left( \hat{n}_{i}^{b} - \frac{1}{2} \right)
$$

with

$$
t(r) = (-1)^r \frac{\pi}{L \sin\left(\pi r/L\right)} \text{ for } r \neq 0 \text{ and } t(0) = 0. \quad (2)
$$

Each unit cell harbors two orbitals, and $\hat{a}_{i}^{\dagger}$, $\hat{b}_{i}^{\dagger}$, are spinless fermion creation operators.

Using periodic boundary conditions and Fourier transformation,

$$
\left( \begin{array}{c} \hat{a}_{k} \\ \hat{b}_{k} \end{array} \right) = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{ikj} \left( \begin{array}{c} \hat{a}_{j} \\ \hat{b}_{j} \end{array} \right) \quad (3)
$$

gives, up to a constant,

$$
\hat{H} = -v_F \sum_{k} t(k) \left( \hat{a}_{k}^{\dagger} \hat{b}_{k}^{\dagger} \right) \sigma^{y} \left( \begin{array}{c} \hat{a}_{k} \\ \hat{b}_{k} \end{array} \right) - \frac{U}{2} \sum_{i} \left[ \left( \hat{a}_{i}^{\dagger} \hat{b}_{i}^{\dagger} \right) \sigma^{x} \left( \begin{array}{c} \hat{a}_{i} \\ \hat{b}_{i} \end{array} \right) \right]^{2} \quad (4)
$$

with

$$
t(k) = i \sum_{r=-L/2}^{L/2} e^{-ikr} t(r). \quad (5)
$$

For any lattice size, $t(k)$ is a real and odd function. It is plotted in Fig. 1 and as apparent scales to $t(k) = k$ in the thermodynamic limit. One will also notice the Gibbs phenomenon (on even lattices) at the zone boundary associated to the discontinuity of $t(k)$.

The rotation:

$$
\left( \begin{array}{c} \hat{a}_{i} \\ \hat{b}_{i} \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \hat{c}_{i}^{\uparrow} \\ \hat{c}_{i}^{\downarrow} \end{array} \right) \quad (6)
$$

gives

$$
\hat{H} = -v_F \sum_{k,\sigma} \sigma k_{F}^{\dagger} \hat{c}_{k,\sigma}^{\dagger} \hat{c}_{k,\sigma} + U \sum_{i} \left( \hat{n}_{i,\uparrow} - 1/2 \right) \left( \hat{n}_{i,\downarrow} - 1/2 \right) \quad (7)
$$

corresponding to a helical liquid with Hubbard interaction. Our goal is to investigate if the SLAC approach indeed reproduces the expected results obtained from bosonization of the helical liquid.

III. RESULTS FROM BOSONIZATION

Let us start by stating the bosonization [4, 14] results valid in the continuum limit, $a \to 0$. In this limit, charge fluctuations obey boson commutation rules. Owing to time reversal symmetry as well as $U(1)$ spin symmetry only forward scattering processes are allowed. As a consequence the model, within bosonization can be solved exactly and the correlation functions follow the universal form:

$$
C_{n}(r) \equiv \langle \hat{n}(r) \hat{n}(0) \rangle \propto \frac{1}{r^2}
$$

$$
C_{S^{r}}(r) \equiv \langle \hat{S}^{r}(r) \hat{S}^{r}(0) \rangle \propto \frac{1}{r^2}
$$

$$
C_{S^{z}}(r) \equiv \langle \hat{S}^{z}(r) \hat{S}^{z}(0) \rangle \propto \frac{\cos(2k_{F}r)}{r^2 K_{p}}
$$

$$
C_{\Delta}(r) \equiv \langle \text{Re} \hat{\Delta}_{r}^{\dagger} \text{Re} \hat{\Delta}_{r} \rangle \propto \frac{1}{r^2/K_{p}} \quad (8)
$$

In the above $K_{p}$ is a interaction strength dependent Luttinger liquid exponent, $\hat{n}(r) = \sum_{\sigma} \hat{c}_{r,\sigma}^{\dagger} \hat{c}_{r,\sigma}$, $S(r) = \frac{1}{2} c_{r}^{\dagger} \sigma c_{r}$ with $\sigma$ a vector of Pauli spin matrices and $\hat{\Delta}^{r} = \hat{c}_{r,\uparrow}^{\dagger} \hat{c}_{r,\downarrow}^{\dagger}$. Here, $2k_{F}$ denotes the momentum difference of the left spin up and right spin down movers. This wave vector is naturally picked up in $C_{S^{z}}(r)$ since it involves scattering between the two branches. In our construction, $k_{F} = 0$. Before proceeding and as mentioned earlier the bosonization results are consistent with the domain wall fermion approach of [3, 4] even in the rather strong coupling limit. The above will be our reference result and we will ask the question under which conditions we can reproduce it with the SLAC lattice regularization.
IV. QUANTUM MONTE CARLO SIMULATIONS

The Hamiltonian of Eq. 1 does not suffer from a negative sign problem in QMC simulations [15, 16]. To see this, one will rewrite the model as,

$$\hat{H} = -v_F \sum_{i=1}^{L} \sum_{r=-L/2}^{L/2} t(r) \left( \hat{a}_i^\dagger \hat{b}_{i+r} + \hat{b}_{i+r}^\dagger \hat{a}_i \right) - \frac{U}{2} \sum_i \left( \hat{a}_i^\dagger \hat{b}_i + \hat{b}_i^\dagger \hat{a}_i \right)^2 \quad (9)$$

where we have omitted a constant and adopt a Majorana representation,

$$\hat{a}_i = \frac{1}{2} (\hat{\gamma}_{i,1,1} + i \hat{\gamma}_{i,2,1}) , \quad \hat{b}_i = -i \left( \frac{1}{2} (\hat{\gamma}_{i,1,2} + i \hat{\gamma}_{i,2,2}) \right). \quad (10)$$

This yields:

$$\hat{H} = v_F \sum_{i=1}^{L} \sum_{r=-L/2}^{L/2} t(r) \frac{i}{4} \hat{\gamma}_i^T \tau_x \hat{\gamma}_{i+r} - \frac{U}{2} \sum_i \left( \frac{1}{4} \hat{\gamma}_i^T \tau_y \hat{\gamma}_i \right)^2 \quad (11)$$

where $\hat{\gamma}_i^T = (\hat{\gamma}_{i,1,1}, \hat{\gamma}_{i,2,1}, \hat{\gamma}_{i,1,2}, \hat{\gamma}_{i,2,2})$. Here we have used the fact that $t(r)$ is an odd function of $r$ and adopted the notation $\hat{\gamma}_i$ where the Pauli $\tau$ ($\sigma$)-matrices act of the $\tau$ ($\sigma$) indices. A global $O(2)$ symmetry in the $\sigma$-indices now becomes apparent. After a real Hubbard-Stratonovich transformation of the perfect square, the Fermion determinant will be given by the square of a Pfaffian. Since one will show that the Pfaffian is real, we will conclude in the absence of the negative sign problem. Hence the absence of sign problem for this SLAC model of the helical liquid follows the same logic as for the so called spin-less t-V model [17, 18]. In the appendix we show the absence of sign problem for the generic model:

$$\hat{H} = -v_F \sum_{i=1}^{L} \sum_{r=-L/2}^{L/2} t(r) \left( \hat{a}_i^\dagger \hat{b}_{i+r} + \hat{b}_{i+r}^\dagger \hat{a}_i \right) - \frac{U}{2} \sum_i \left[ \sigma_\alpha \left( \hat{b}_i, \hat{a}_i \right)^T \right]^2 \quad (12)$$

where $\sigma_\alpha$ is a Pauli spin matrix. Note that after computing the square one will explicitly see that the Hamiltonian is $\alpha$-independent. For any value of $\alpha$, we can use the ALF [19] implementation of the finite temperature auxiliary field QMC algorithm [20–23]. In fact, Eq. 12 that formulates the interaction in terms of a perfect square has the required form for usage of the ALF-library, and concomitant Hubbard-Stratonovich transformation.

As mentioned above, the results are $\alpha$-independent. However the Monte Carlo Markov chain will have a strong $\alpha$-dependence. We have seen that we obtain the best results when considering the $\sigma_y$ formulation. This stems form the fact that after the rotation of Eq. 6 the $U(1)$ symmetry of the helical liquid is satisfied for each Hubbard-Stratonovich field configuration.

We also would like to stress that since we are working in the Hamiltonian formulation, the resulting Lagrangian has SLAC hoppings only in the spatial direction and is local along the Euclidean time direction.

We used the interaction strength (band width) as the energy unit for simulations at large (small) values of $U/v_F$. For $U/v_F \leq 4$ we choose $v_F \beta = L$ and $v_F \Delta \tau = 0.1$; whereas for $U/v_F > 4$ we considered $U\beta/4 = L$ and $U\Delta \tau/4 = 0.1$.

V. RESULTS

We will show that the SLAC approach suffers from two basic issues.

The first one can be seen already in the non interacting limit and originates from processes with large momentum transfer. This deficiency can be illustrated as the violation of the anomaly relation in lattice Schwinger model with SLAC fermions [24]. If we consider the continuum theory and turn on a constant electrical field pointing to the right, the right movers will acquire momentum and energy and will fill their branch of the dispersion relation up to some positive level. At the same time the left movers will lose energy and hence their branch of the cone will be filled only up to the same but negative level. Thus the axial charge will appear as an imbalance between the right and left movers. However, this is not true for the SLAC fermions due to the finite size of the Brillouin zone and finite depth of the Dirac sea. E.g. the right movers at the bottom of the Dirac sea will also acquire energy thus the very bottom of this branch of the dispersion relation will not be filled any more. These effects will compensate the difference between right- and left-movers in the low momentum modes leading to the axial charge being always zero. Though this is only qualitative illustration, it shows the presence of the non trivial dynamics near the edge of the Brillouin zone.

Another issue can be seen upon switching on intermediate to strong correlations as measured in unit of the band-width. In this case we observe a long ranged order again contradicting the results from continuum theory.

Both points will be carefully studied in the subsequent sub-sections.

V.1. The non-interacting case

SLAC fermions become very transparent when introducing a length scale $\xi$ in the hopping:

$$t_\xi(r) = t_0(\xi) \frac{(-1)^{r/\alpha} \pi}{\sin(r\pi/L)} e^{-\sin(r\pi/L)/\xi}. \quad (13)$$
In the above, we adjust $t_0(\xi)$ so as to fix the bandwidth to $2\pi$. Clearly, the Fourier transform of a short range function has to be smooth and one will see that for any finite value of $\xi$ we observe two crossings of the Fermi surface albeit with very different values of the velocity. In fact the velocity at the zone boundary diverges with growing values of $\xi$. In principle, for any finite value of $\xi$ we expect Umklapp processes to be relevant such that any finite value of $U$ should lead to an insulating state. However, since the velocity at the zone boundary diverges as $\xi$, the phase space available to these Umklapp process will vanish in the $\xi \to \infty$ limit. Another consequence of the singularity at the zone boundary is that large momentum transfer will always provide a discrepancy with the bosonization even in the non-interacting case. One can illustrate this by considering the charge-charge correlation functions for SLAC Hamiltonian for the half-filled case, $\mu = 0$, at zero temperature:

$$\langle \hat{n}(r) \hat{n}(0) \rangle = \frac{1}{2\pi} \frac{\cos(\pi r) - 1}{r^2}. \quad (14)$$

In the above $\hat{n}(r) = \sum_{s} \hat{c}^\dagger_{r,s} \hat{c}_{r,s}$. This result is independent on the value of $\xi$ and merely relies on the fact that the dispersion relation intersects the Fermi energy at wave vectors $k = 0$ and $k = \pi$. The above expression already deviates from the bosonization result 8 and shows that already at this level one will obtain the same result as for the continuum model, where the zone edge diverges, only if one blocks large momentum transfers. This can be done introducing point-splitting operators on the lattice, as was already suggested in [11, 25].

V.2. Monte Carlo results

We have computed structure factors:

$$S_\bullet(k) = \sum_r e^{ikr} C_\bullet(r) \quad (15)$$

where the bullet refers to charge, spin along the z- or x-spin quantization axis or paring correlations (see Eq. 8). To obtain an estimate of the power law decay at a given wave vector, one can consider:

$$b_\bullet(k,L/2) = 2C_\bullet(k) - C_\bullet(k + 2\pi/L) - C_\bullet(k - 2\pi/L). \quad (16)$$

$$\frac{L}{2\pi} b_\bullet(k,L/2) \quad (17)$$

such that the scaling of $b_\bullet(k,L/2)$ at wave vectors $k$ where one observes a cusp will reflect the decay of the correlation function [26] at this wave vector.

Fig. 3 plots the real and k-space correlation functions for the above mentioned quantities. To better understand the results, we consider the behavior of the cusps in the corresponding structure factors by plotting $b_\bullet(k = 0, L/2)$ as a function of system size and coupling strength.

Let us start with the charge. From Fig. 3(a) we see that irrespective of the coupling constant in the range $U \in [0, 10]$ the real space charge correlation decays as $1/r^2$. In the weak coupling limit we observe a $(-1)^r$ modulation alongside the uniform decay. This weak coupling behavior gives way to a uniform decay at strong coupling. In k-space, Fig. 3(f) we see that the cusp at $k = \pi$ rounds off as a function of growing interaction strength but that the cusp at $q = 0$ remains robust. We also notice that as a function of growing interaction strength the charge response is suppressed. We can pin down the charge exponent by analyzing $Lb_\bullet(k = 0, L/2)$ in Fig. 4(a). Irrespective of the interaction strength, it is to an accurate degree $L$-independent thus reflecting a $1/r^2$ decay of the charge correlations.

At weak coupling the z-component of spin is very similar to the charge (at $U = 0$ they are identical), see Figs. 3(c,h). In contrast however, the cusp at $k = 0$ becomes more pronounced at strong coupling. Fig. 4(c), shows that the z-spin correlations acquire a non-trivial exponent in the strong coupling limit. This stands at odds with the bosonization result of Eq. 8.

The spin-correlations in the x-spin quantization, Figs. 3(b,g), direction are most intriguing results. At weak coupling and on our system sizes, this correlation function follows roughly a $1/r^2$ form consistent with
FIG. 3. Real space correlation functions (a)-(e) and corresponding structure factors (f)-(j). Here we consider the charge, $C_n$, $x$ ($z$)-component of spin $C_{Sx}$ ($C_{Sz}$), pairing $C_\Delta$ and single particle $G_\uparrow$ correlation functions. All the subfigures share the same legend color as the one in (a). For $C_n$, $C_{Sx}$, $C_\Delta$ and $G_\uparrow$, we took $L = 243$; whereas for $C_{Sz}$, we took $L = 203$. The reason of this mismatch is the large QMC fluctuation of $C_{Sz}$ for large sizes.
Long ranged order along the x-spin quantization axis breaks time reversal symmetry and allows for elastic scattering between the right-moving spin down and left moving spin up electrons. At the single particle, mean-field level we expect:

\[ \hat{H}_{MF} = \sum_k \hat{c}_k^{\dagger} [-v_f k \sigma_z + m_x \sigma_x] \hat{c}_k \]  

where \( m_x \) denotes the ordered moment. This symmetry breaking generates a mass gap at the Fermi momentum \( k_f = 0: E_k = \pm \sqrt{(v_f k)^2 + m_0^2} \). Fig. 3(j) plots the single particle equal time Green function, \( G_\sigma(k) = \langle \hat{c}_{k,\sigma}^{\dagger} \hat{c}_{k,\sigma} \rangle \). As \( U \) grows, the singularity at \( k = 0 \) evolves to a smooth feature. Fig. 3(e) confirms this: at weak coupling, \( G_\sigma(r) \propto 1/r \) as expected for Dirac electrons in 1+1D, and in the strong coupling limit the \( L \)-independent form of \( LB(k = 0, L) \) is consistent with a mass gap. \( G_\sigma(k) \) has another singularity at \( k = \pi \) that dominates the long-ranged real space behavior. Putting all together, the data in the strong coupling limit is consistent with the form: \( G_\sigma(r) \propto (ae^{-r/\xi} + (-1)^r) /r \) where \( \xi \) is set by the inverse ordered moment \( m_x \). We also notice that the overall amplitude of \( G_\sigma(r) \) diminishes as a function of \( U \) in the strong coupling.

Finally, we consider the pairing correlations in Figs. 3(d,i) as well as in Fig. 4(d). We again observe non-analyticities at \( k = 0 \) and \( k = \pi \) in the structure factor. The non-analytical behavior at \( k = 0 \) survives the strong coupling limit, whereas \( C_\Delta(k) \) evolves towards a smooth function in the vicinity of \( k = \pi \). The singularity at \( k = 0 \) leads to a \( 1/r^2 \) decay of the pair correlation, and again the overall amplitude of the correlation function decreases as as function of increasing \( U \).

**VI. INTERPRETATION OF THE STRONG COUPLING LIMIT**

In this section, we provide consistent interpretation of the strong coupling limit. In this limit the QMC data shows long ranged magnetic ordering along the x-spin
quantum. The single particle Green function decays as \((-1)^r/r\), and the density as well as the pairing correlations follow a \(1/r^2\) law. On the other hand the \(\alpha\)-component of spin correlations have a power-law decay with exponent depending on the interaction strength. Clearly this behavior lies at odds with the bosonization results. A simple model can account for most of the above. Let us start with a mean-field representation of the strong coupling ground state:

\[
|\psi_0\rangle = \prod_r \frac{1}{\sqrt{2}} \left( \hat{c}^\dagger_{r,\uparrow} + \hat{c}^\dagger_{r,\downarrow} \right) |0\rangle \tag{19}
\]

Since the above wave function has no charge fluctuations and precisely one electron per site it is the ground state of the Hubbard interaction term, \(\hat{H}_U\) with energy \(E_0\). Consider the small hopping limit such that the ground state wave function can be estimated perturbatively in the hopping \(\hat{H}_t\) \cite{28}:

\[
|\psi\rangle = |\Psi_0\rangle + \hat{Q}_0 \frac{1}{\hat{H}_U - E_0} \hat{Q}_0 \hat{H}_t |\Psi_0\rangle. \tag{20}
\]

In the above \(\hat{Q}_0 = 1 - |\Psi_0\rangle \langle \Psi_0|\). Let us now compute the charge fluctuations, \(C_n(r) = \langle \Psi | (\hat{n}_r - 1) (\hat{n}_0 - 1) |\Psi\rangle\) for \(r \neq 0\). The sole contribution reads:

\[
C_n(r) = \langle \Psi | \hat{H}_t \hat{Q}_0 \frac{1}{\hat{H}_U - E_0} \hat{Q}_0 (\hat{n}_r - 1) (\hat{n}_0 - 1) \times \hat{Q}_0 \frac{1}{\hat{H}_U - E_0} \hat{Q}_0 \hat{H}_t |\Psi_0\rangle. \tag{21}
\]

Since \(\hat{H}_t\) has hopping processes on all scales it contains the operator \(t(r) \sum_\sigma \hat{c}^\dagger_{r,\sigma} \hat{c}_{0,\sigma}\). Applied on \(|\Psi_0\rangle\) it will generate a doublon and holon at distance \(r\) with an energy cost with respect to \(E_0\) set by \(U\). This charge fluctuation will be picked up by \((\hat{n}_r - 1) (\hat{n}_0 - 1)\). Finally the doublon holon pair will be destroyed by the operator \(t(r) \sum_\sigma \hat{c}^\dagger_{0,\sigma} \hat{c}_{r,\sigma}\) again contained in \(\hat{H}_t\). As a result, we estimate:

\[
C_n(r) \approx \frac{t^2(r)}{U^2} \propto \frac{1}{U^2 r^2}. \tag{22}
\]

The power-law is confirmed by the QMC data of Fig. 3(a). It is also interesting to note that the magnitude of the charge-charge correlations are predicted to scale as \(1/U^2\). Comparison between the \(U = 6\) and \(U = 10\) data in Fig. 3(f) supports this scaling.

We note that the very same argument can be carried out for the pairing correlations. Let us pick up the above argument at the point where a doublon is created on site \(r\) and a holon on site \(0\). Applying the pairing operator \(\Delta_r \Delta^*_0\) on this state, will yield a non-zero result and transfer the doublon (holon) to the origin (site \(r\)). The operator \(t(r) \sum_\sigma \sigma \hat{c}^\dagger_{r,\sigma} \hat{c}_{0,\sigma}\) will then annihilate the doublon-holon pair, and we will obtain a finite overlap with the mean-field ground state. Hence we also expect:

\[
C_\Delta(r) \approx \frac{t^2(r)}{U^2} \propto \frac{1}{U^2 r^2} \tag{23}
\]

in the strong coupling limit, which is consistent with our QMC data, but inconsistent with the bosonization results Eq. 8.

Doublons and holons are spin singlets. In the above approximation \(C_\sigma(r)\) will hence vanish identically. The data for the \(\alpha\)-component of spin have to be understood in terms of the transverse fluctuations of the ordered moment along the spin-x quantization axis.

We now consider the single particle Green function. Here, the relevant terms in \(\langle \Psi | \hat{c}^\dagger_{r,\sigma} \hat{c}_{0,\sigma} |\Psi\rangle\) are the mixed terms of the form:

\[
\langle \Psi_0 | \hat{c}^\dagger_{r,\sigma} \hat{c}_{0,\sigma} \hat{Q}_0 \frac{1}{\hat{H}_U - E_0} \hat{Q}_0 \hat{H}_t |\Psi_0\rangle. \tag{24}
\]

The doublon-holon pair created by \(\hat{H}_t\) will be annihilated by the single particle transfer \(\hat{c}^\dagger_{r,\sigma} \hat{c}_{0,\sigma}\). In accordance with the QMC results this approximation gives:

\[
G_\sigma(r) \propto \frac{t(r)}{U} \propto \frac{1}{U} \tag{25}
\]

Finally we comment on the nature, metallic or insulating, of the strong coupling wave function. The very fact that the charge correlations follow a power-law, suggest a metallic ground state. An accepted definition of an insulating or metallic state is the Drude weight \cite{29}, that probes the localization of the wave function. Here, one considers a ring geometry and threads a magnetic flux \(\Phi\) through the center of the ring. Such a flux will have an effect if the charge carriers are delocalized and can circle around it and, owing to the Aharonov-Bohm effect, acquire a phase factor \(e^{2\pi i\Phi/\Phi_0}\) where \(\Phi_0\) is the flux quanta. Here we assume that the charge carriers have the electron charge. The Drude weight in d-spatial dimensions is defined as:

\[
D(L) = \frac{1}{L^{d-2}} \left\langle \frac{\partial^2 E_0(\Phi)}{\partial \Phi^2} \right|_{\Phi=0}. \tag{26}
\]

For the insulating state \(D(L)\) vanishes exponentially with \(L\) reflecting the localization length of the wave function. For a metallic state the Drude weight is finite. Let us now use this accepted criterion to the SLAC fermions, in the strong coupling limit. A glimpse at the wave function in second order perturbation theory (see Eq. 20) shows that it contains holon-doublon excitations, at all length scales. The fact that they are costly in energy, means that they are short lived, but during this short time, they can propagate over large distances due to the non-locality of the hopping. Hence we expect the Drude weight to be finite. To substantiate this statement we carry out the
following estimations. The flux leads to a twist in the boundary condition:
\[ \hat{c}_{r+L,\sigma} = e^{2\pi i \frac{\Phi}{\Phi_0}} \hat{c}_{r,\sigma} \] (27)
that we can rid of with the canonical transformation:
\[ \hat{d}_{r,\sigma} = e^{-2\pi i \frac{\Phi}{\Phi_0} \frac{x}{L}} \hat{c}_{r,\sigma}. \] (28)
Under this canonical transformation, the Hubbard term remains invariant, the hopping reads,
\[ \hat{H}_i(\Phi) = v_F i \sum_{r,\sigma} \tau(r) \hat{d}_{i,\sigma}^\dagger \hat{d}_{i+r,\sigma} e^{2\pi i \frac{\Phi}{\Phi_0} \frac{x}{L}}, \] (29)
and \( \hat{d}_{r,\sigma} \) satisfies periodic boundary conditions: \( \hat{d}_{r,\sigma} = \hat{d}_{r+L,\sigma} \). Let us now compute the second order contribution to the energy that will pick up the dependence on the flux:
\[ E_2(\Phi) = \langle \Psi_0 | \hat{H}_i(\Phi) \hat{Q}_0 \frac{1}{\hat{H}_U - \hat{E}_0} \hat{Q}_0 \hat{H}_i(\Phi) | \Psi_0 \rangle \] (30)
Starting from \( |\Psi_0 \rangle \) one will create for example a holon at position \( i \) and a doublon at position \( i + r \) by applying the hopping. This process has matrix element \( iv_F \tau(r) e^{-2\pi i \frac{\Phi}{\Phi_0} \frac{x}{L}} \) and energy cost set by \( U \). The only way to perceive the flux is for the charge excitation to encircle it. Hence the second hopping process should destroy the doublon in favor of single occupancy at site \( i + r \) and create an electron at site \( i + L \equiv i \) thereby restoring single occupancy on this site such that the overlap with \( |\Psi_0 \rangle \) does not vanish. This second process comes with matrix element: \( iv_F \tau(L - r) e^{-2\pi i \frac{\Phi}{\Phi_0} \frac{x}{L}} \). Putting everything together one obtains:
\[ E_2(\Phi) \propto \frac{v^2}{U} \sum_i \sum_r \tau(r) \tau(L - r) \cos \left( 2\pi \frac{\Phi}{\Phi_0} \right). \] (31)
We hence see that in this approximation, the Drude weight reads:
\[ D \propto L^2 \left( \frac{2\pi}{\Phi_0} \right)^2 \frac{v^2}{U} \sum_r \tau(r) \tau(L - r). \] (32)
One will check that \( \sum_r \tau(r) \tau(L - r) \) takes a finite value. Hence we obtain the result that the Drude weight actually diverges as \( L^2 \), and only at \( U = \infty \) will we have an insulating state on any finite lattice.

VII. SUMMARY AND CONCLUSIONS

One dimensional systems are generically nested. For the helical Luttinger liquid this leads to \( \chi(k = 0, \omega = 0) \propto \log \frac{v_F}{\omega} \). As a consequence, a mean-field approach to correlation effects will generate long-ranged magnetic order along the spin-x quantization axis and a charge gap. Both the charge gap and the ordered moment will follow an essentially singularity in the weak coupling limit.

For generic local one-dimensional models we know that the above Stoner arguments cannot be made due to the Mermin-Wagner theorem that tells us that fluctuations will destroy the ordering even in the ground state. For our specific case, continuous U(1) spin-symmetry breaking is not allowed. This competition between the Stoner instability and the Mermin-Wagner theorem is at the very origin of Luttinger liquid behavior generic to 1+1D interacting systems. This is exemplified by the helical Luttinger liquid: a metallic state with no single particle gap and an interaction strength dependent power-law decay of the spin-spin correlations in the transverse direction. We note that due to U(1) spin-symmetry Umklapp processes are symmetry forbidden such that the system will remain metallic for arbitrary large interactions. This understanding of the helical Luttinger liquid, has been confirmed numerically within a domain-wall fermion approach [3, 4] in which interaction effects are included only on one set of domain-wall fermions.

The non-locality of the SLAC fermion approach brings major differences to the above picture. The key-point is that it violates the assumptions of the Mermin-Wagner theorem and allows for Stoner type physics. This aspect of SLAC fermions was recently pointed out in Ref. [8]. Our numerical results explicitly confirm this in the strong coupling limit where long ranged magnetic order along the x-spin quantization axis and global U(1) spin symmetry breaking is observed. This allows for a mass term and in fact we observe a single particle gap opening at the Fermi wave vector. Generically, spin-orbit coupling will reduce the U(1) continuous symmetry to a \( Z_2 \) discrete one. In this case, the aforementioned issues of SLAC action with the Mermin-Wagner theorem will be waived. However, some artefacts will likely survive even for the discrete \( Z_2 \) symmetry. In particularly, the deviation of the behaviour of the correlation functions in Eq. 23 from the strong coupling limit does not involve a continuous symmetry for its derivation. Hence these discrepancies will remain even for models with discrete symmetries. Another important point is the nature of the ordered state observed in our QMC simulations. In contrast to Dirac systems where magnetic mass terms are generated spontaneously [30], this ordered state remains metallic. This is again a consequence of non-locality inherent to SLAC approach that produces doublon-holon pairs at any length scale. Equivalently, the current operator becomes long-ranged. Strictly speaking Gross-Neveu transitions that have been studied in the realm of SLAC fermions [5, 7, 8] are not metal to insulator transitions but metal-to-metal ones.

At vanishing coupling strength, the results of the
Quantum physics in one dimension

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Appendix A: Absence of negative sign problem

There are various ways of formulating the QMC. Although in principle equivalent, the various formulations will lead to different Markov time series. Thereby the autocorrelation time for a given observable may be formulation dependent. The various formulations stem from different ways writing the interaction term:

$$\hat{H}_U = -\frac{U}{2} \sum_i \left[ \left( a_i^\dagger, b_i^\dagger \right) \sigma_\alpha \left( b_i, a_i \right)^T \right]^2. \quad (A1)$$

In the above, $\hat{H}_U$ is independent on the choice of the Pauli spin matrix $\sigma_\alpha$. For the repulsive values of $U > 0$, we can carry out an HS decomposition of the perfect square term to obtain:

$$Z \propto \int D \{ \phi(i, \tau) \} e^{-S_\alpha(\phi(i, \tau))} \quad (A2)$$

with

$$S_\alpha(\phi(i, \tau)) = \int_0^\beta d\tau \sum_i \frac{\phi^2(i, \tau)}{2U} - \log \text{Tr}T e^{-\int_0^\beta d\tau \sigma_\alpha(\tau)} \quad (A3)$$
\[ \hat{h}_\alpha(\tau) = \hat{H}_0 + \sum_i \phi(i, \tau) \left( \hat{a}_i^\dagger \hat{a}_i \right) \sigma_\alpha \left( \hat{b}_i, \hat{\bar{b}}_i \right)^T, \]  
(A4)

and

\[ \hat{H}_0 = -v_F \sum_{i=1}^{L} \sum_{r=-L/2}^{L/2} t(r) \left( \hat{a}_i^\dagger \hat{b}_{i+r} + \hat{\bar{b}}_{i+r}^\dagger \hat{a}_i \right). \]  
(A5)

We again stress that the partition function is \( \alpha \)-independent but that \( \hat{h}_\alpha(\tau) \) has an explicit \( \alpha \)-dependency.

Using the Majorana representation of Eq. 10 we obtain different expressions for various choices of \( \alpha \). Below we go through them one by one.

**• \( \sigma_x \)**

In this case,

\[ \hat{h}_x(\tau) = v_F \sum_{i=1}^{L} \sum_{r=-L/2}^{L/2} \frac{it(r)}{4} \hat{a}_i^\dagger \tau_x \hat{a}_{i+r}^\dagger \phi(i, \tau) \frac{1}{4} \hat{\gamma}_i^T \tau_y \hat{\gamma}_i. \]  
(A6)

We see that the operators \( \hat{T}^+ \) and \( \hat{T}^- \) with

\[ \hat{T}^+ \gamma_i \hat{T}^+ \gamma_i^{-1} = T^+ \gamma_i \quad \text{with} \quad T^+ = i \tau_y \tau_x \sigma_x K \]  
(A7)

and

\[ \hat{T}^- \gamma_i \hat{T}^- \gamma_i^{-1} = T^- \gamma_i \quad \text{with} \quad T^- = \tau_z i \sigma_y K \]  
(A8)

where \( K \) denotes complex conjugation leaves \( \hat{h}(\tau) \) invariant. Furthermore \( (T^\pm)^2 = \pm 1 \) and \( \{T^+, T^-\} = 0 \). Hence according to Ref. [15] the Hamiltonian falls into the so-called Majorana class and does not suffer for the negative sign problem.

**• \( \sigma_y \)**

For this choice,

\[ \hat{h}_y(\tau) = v_F \sum_{i=1}^{L} \sum_{r=-L/2}^{L/2} \frac{it(r)}{4} \hat{a}_i^\dagger \tau_y \hat{a}_{i+r}^\dagger \phi(i, \tau) \frac{1}{4} \hat{\gamma}_i^T \sigma_y \tau_z \hat{\gamma}_i. \]  
(A9)

such that we have to choose:

\[ T^- = i \tau_y K \quad \text{and} \quad T^+ = \tau_z K \]  
(A10)

to show that the model is in the Majorana class.

**• \( \sigma_z \)**

For this choice,

\[ \hat{h}_z(\tau) = v_F \sum_{i=1}^{L} \sum_{r=-L/2}^{L/2} \frac{it(r)}{4} \hat{a}_i^\dagger \tau_z \hat{a}_{i+r}^\dagger \phi(i, \tau) \frac{1}{4} \hat{\gamma}_i^T \sigma_y \tau_z \hat{\gamma}_i. \]  
(A11)

such that we have to choose:

\[ T^- = i \tau_y K \quad \text{and} \quad T^+ = \tau_z \sigma_x K \]  
(A12)

to show that the model is in the Majorana class.