The Cohen-Macaulay Property of $f$-ideals*

A-Ming Liu†, Jin Guo‡
School of Science, Hainan University

Tongsu Wu§
School of Mathematical Sciences, Shanghai Jiao Tong University

Abstract For positive integers $d < n$, let $[n]_d = \{ A \in 2^{[n]} \mid |A| = d \}$ where $[n] =: \{1, 2, \ldots, n\}$. For a pure $f$-simplicial complex $\Delta$ such that $\dim(\Delta) = \dim(\Delta^c)$ and $\mathcal{F}(\Delta) \cap \mathcal{F}(\Delta^c) = \emptyset$, we prove that the facet ideal $I(\Delta)$ is Cohen-Macaulay if and only if it has linear resolution. For a $d$-dimensional pure $f$-simplicial complex $\Delta$ such that $\Delta' =: \langle F \mid F \in [n]_d \setminus \mathcal{F}(\Delta) \rangle$ is an $f$-simplicial complex, we prove that $I(\Delta^c)$ is Cohen-Macaulay if and only if $I(\Delta')$ has linear resolution.

Key Words $f$-ideal; Cohen-Macaulay; Newton complement dual; linear resolution

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1. Preliminaries

Throughout, let $\mathbb{R}$ be a field and let $S = \mathbb{R}[x_1, \ldots, x_n]$ be the polynomial ring over $\mathbb{R}$. For any square-free monomial ideal $I$ of $S$, let $G(I)$ be the set of minimal generator set of monomials, and let $\text{sm}(I)$ be the set of square-free monomials. For the ideal $I$, there exist two related simplicial complexes, i.e., the nonface simplicial complex

$$
\delta_X(I) =: \{ F \in 2^{[n]} \mid X_F \in \text{sm}(S) \setminus \text{sm}(I) \}
$$

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†aming8809@163.com
‡guojinecho@163.com
§Corresponding author. tswu@sjtu.edu.cn
of $I$ and the facet simplicial complex

$$\delta_F(I) =: \langle F \in 2^{[n]} \mid X_F \in G(I) \rangle$$

of the clutter $G(I)$. If they possess a same $f$-vector, then the ideal $I$ is called an $f$-ideal. For a graph $G$, if its edge ideal $I(G) =: \langle \{X_F \mid F \in E(G)\} \rangle$ is an $f$-ideal, then $G$ is called an $f$-graph. Refer to [1, 2, 9, 6, 8] for further related studies.

For a simplicial complex $\Delta$ on the vertex set $[n]$, let $\mathcal{F}(\Delta)$ be the clutter of facets in $\Delta$, let $\mathcal{N}(\Delta)$ be the set of all minimal nonfaces of $\Delta$. Let

$$\Delta^c =: \langle \{F \mid F^c \in \mathcal{F}(\Delta)\} \rangle \overset{\text{i.e.}}{=} \langle \{[n] - G \mid G \in \mathcal{F}(\Delta)\} \rangle.$$

$\Delta$ is called an $f$-simplicial complex if the facet ideal $I(\Delta)$ is an $f$-simplicial complex. Note that in defining an $f$-graph $G$, $G$ is regarded as a simplicial complex of dimension no more than 1, although we do have $I(G) = I_{\text{Ind}(G)}$, where $\text{Ind}(G)$ is the independence simplicial complex of the graph $G$. The definition of an $f$-simplicial complex seems to be reasonable with hindsight, due to the following two theorems on $f$-ideals.

**Theorem 1.1.** ([5, Theorem 2.3]) Let $S = K[x_1, \ldots, x_n]$, and let $I$ be a square-free monomial ideal of $S$ with the minimal generating set $G(I)$, where all monomials of $G(I)$ have a same homogeneous degree $d$. Then $I$ is an $f$-ideal if and only if, the set $G(I)$ is an LU-set and, $|G(I)| = \frac{1}{2} \binom{n}{d}$ holds true.

Note that $G(I)$ is said to be an LU-set if the set of all degree $d-1$ factors of elements of $G(I)$ has exactly $\binom{n}{d-1}$ elements, and the set of degree $d+1$ square-free monomials extended from elements of $G(I)$ has cardinality $\binom{n}{d+1}$.

Recall also the following recently discovered result:

**Theorem 1.2.** ([4, Theorem 4.1]) $\Delta$ is an $f$-simplicial complex, if and only if $\Delta^c$ is an $f$-simplicial complex.

Equivalently, a square-free monomial ideal $I$ of $S$ is an $f$-ideal if and only if the following Newton complement dual ideal

$$\hat{I} = \langle x_1x_2 \cdots x_n / u \mid u \in G(I) \rangle$$

of $I$ is an $f$-ideal.

It is clear that Theorem 1.2 follows easily from Theorem 1.1 for a pure simplicial complex $\Delta$. 

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2. Well-distributed $f$-simplicial complexes

We begin with the following interesting example.

**Example 2.1.** Consider the simplicial complex $\Delta$ whose facet set is

$$\{123, 125, 136, 145, 146, 234, 246, 256, 345, 356\}.$$  

By [3, Example 7.7], it is not shellable. It is direct to check that $F(\Delta)$ is a LU-set and $|G(I)| = \frac{1}{2}(6)$. Thus the facet ideal $I := I(\Delta)$ is an $f$-ideal, hence $\hat{I} := I(\Delta^c)$ is also an $f$-ideal. Note that $F(\Delta) \cap F(\Delta^c) = \emptyset$.

By taking advantage of CoCoA, we get the primary decomposition of $I$ as follows:

$$I(\Delta) = \langle x_3, x_5, x_6 \rangle \cap \langle x_2, x_4, x_6 \rangle \cap \langle x_2, x_5, x_6 \rangle \cap \langle x_1, x_4, x_6 \rangle \cap \langle x_1, x_3, x_6 \rangle \cap \langle x_3, x_4, x_5 \rangle$$

$$\cap \langle x_1, x_4, x_5 \rangle \cap \langle x_1, x_2, x_5 \rangle \cap \langle x_2, x_3, x_4 \rangle \cap \langle x_1, x_2, x_3 \rangle,$$

hence $I$ is unmixed. Furthermore, we get the following same $3$-linear resolution for both $I$ and $\hat{I}$ by CoCoA:

$$0 \rightarrow S^6(-5) \rightarrow S^{15}(-4) \rightarrow S^{10}(-3) \rightarrow S.$$  

Hence by $I(\Delta) = I(\Delta^c)^\vee$, both ideals $I(\Delta)$ and $I(\Delta^c)$ are Cohen-Macaulay by Eagon-Reiner theorem.

In order to seek more information of the example, we introduce the following concept:

**Definition 2.2.** Let $n = 2d$, and let $\Delta$ be a pure $d - 1$-dimensional simplicial complex over vertex set $[n]$. If $F(\Delta) \cap F(\Delta^c) = \emptyset$ holds, then $\Delta$ is called a well-distributed simplicial complex.

Let $\pi$ be a permutation on $[n]$, and let $\Delta$ be a pure simplicial complex with vertex set $[n]$. By Theorem 1.1, $\pi(\Delta) := \{\pi(F) \mid F \in \Delta\}$ is an $f$-simplicial complex if $\Delta$ is. For a $(2d - 1)$-dimensional pure $f$-simplicial complex $\Delta$, note that $\Delta$ is well-distributed if and only if there exists no facet $F \in \Delta$ such that $[n] - F \in F(\Delta)$, thus $\pi(\Delta)$ is still well-distributed if $\Delta$ is.

We have the following observation.

**Theorem 2.3.** For a well-distributed $f$-simplicial complex $\Delta$, $N(\Delta^c) = F(\Delta)$ holds true.

**Proof.** Assume $\dim(\Delta) = d - 1$ and $V(\Delta) = [n]$. By Theorem 1.2 and 1.1, $F(\Delta^c)$ is a L-set, thus $\text{sm}(S)_{d-1} \subseteq \Delta^c$. Then the condition $F(\Delta) \cap F(\Delta^c) = \emptyset$ implies $F(\Delta) \subseteq N(\Delta^c)$. Conversely, for any $U \subseteq [n]$ with $|U| > d$, since $F(\Delta)$ is a U-set, we have $H \in F(\Delta)$ such that $H \subseteq U$, hence $U \notin N(\Delta^c)$. Finally, the condition $F(\Delta) \cap F(\Delta^c) = \emptyset$ implies $N(\Delta^c) = F(\Delta)$. \hfill $\blacksquare$
Corollary 2.4. For a well-distributed f-simplicial complex $\Delta$, $\Delta^\vee = \Delta$ holds true.

Proof. By Lemma 2.3, we have $\mathcal{N}(\Delta) = \mathcal{F}(\Delta^c)$. By definition of $\Delta^\vee$, we have

$$\mathcal{F}(\Delta^\vee) = \{[n] - F \mid F \in \mathcal{N}(\Delta)\} = \{[n] - F \mid F \in \mathcal{F}(\Delta^c)\} = \mathcal{F}(\Delta).$$

Thus $\Delta^\vee = \Delta$. ■

With the observations, Eagon-Reiner theorem has a stronger form for the small class of well-distributed f-simplicial complexes:

Theorem 2.5. For a well-distributed f-simplicial complex $\Delta$, let $J = I(\Delta)$. Then the following statements are equivalent:

(1) The ideal $J$ is Cohen-Macaulay.
(2) $J$ has linear resolution.

Proof. By Lemmas 2.3, we have $I(\Delta) = I_{\Delta^c}$. Then apply Eagon-Reiner theorem to the equality $I(\Delta) = I_{(\Delta^c)^\vee}$, $\Delta^c$ is Cohen-Macaulay if and only if $I(\Delta)$ has linear resolution. On the other hand, $\Delta^c$ is Cohen-Macaulay if and only if $I_{(\Delta^c)^\vee}$ is a Cohen-Macaulay ideal. Since $\Delta^c$ is also a well-distributed f-simplicial complex, we get $I(\Delta) = \Delta^c = (\Delta^c)^\vee$ by Lemma 2.4. This completes the proof. ■

Now we can go back to answer the question posed after Example 2.1. Note that both $\Delta$ and $\Delta^c$ are well-distributed f-simplicial complexes, thus Theorem 2.5 applies. Hence both $I(\Delta)$ and $I(\Delta^c)$ are Cohen-Macaulay ideals. Note that though both $\Delta$ and $\Delta^c$ are Cohen-Macaulay, neither is shellable.

Recall that in [5], a characterization (in fact, a complete classification) of f-graphs was presented. Based on the classification, it was proved in [5] that all f-graphs are connected, thus well-covered and vertex-decomposable and hence, pure shellable. In particular, all f-graphs are Cohen-Macaulay. In contrast, there exist a lot of nonpure f-simplicial complexes of dimension greater than 1 ([5, 6]). Example 2.1 gives more evidence showing that the higher dimensional f-simplicial complexes are a little bit complicated.

3. Strong f-simplicial complexes

Throughout this section, let $\Delta$ be a $(d - 1)$-dimensional pure simplicial complex over vertex set $[n]$. Let $[n]_d = \{A \subseteq 2^n \mid |A| = d\}$. Let $\Delta' = \langle F \mid F \in [n]_d \setminus \mathcal{F}(\Delta) \rangle$ be the homogeneous complement of $\Delta$. 

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Inspired by the concept of well-distributed $f$-simplicial complex, we now introduce the following definitions:

**Definition 3.1.** For a pure simplicial complex $\Delta$, if both $\Delta$ and its homogeneous complement complex $\Delta'$ are $f$-simplicial complexes, then $\Delta$ is said to be **strong**. A square-free monomial ideal $I$ is called a strong $f$-ideal, if its facet simplicial complex $I(\Delta)$ is strong.

Consider all $(2d-1)$-dimensional pure simplicial complexes with $|F(\Delta)| = d$. $f$-ideals are abundant among the kind of complexes, but only a few are well-distributed. For example, when $d = 2$, there are totally 20 such complexes, among them 14 are $f$-simplicial complexes and 2 are well-distributed.

**Example 3.2.** We list all the 3-dimensional pure $f$-simplicial complexes with $|F(\Delta)| = 3$:

1. Balanced $f$-simplicial complexes:
   \[
   \Delta_1 = \langle 12, 13, 14 \rangle \text{ and its homogeneous complement } \Delta'_1 =: \langle 34, 24, 23 \rangle = \Delta^c_1.
   \]

2. Non-well-distributed $f$-simplicial complexes:
   \[
   \langle 12, 13, 14 \rangle, \langle 12, 13, 24 \rangle, \langle 12, 14, 23 \rangle, \langle 12, 14, 34 \rangle, \langle 12, 23, 34 \rangle, \langle 12, 24, 34 \rangle,
   \]
   \[
   \langle 14, 23, 34 \rangle, \langle 14, 23, 24 \rangle, \langle 13, 24, 34 \rangle, \langle 13, 23, 24 \rangle, \langle 13, 14, 24 \rangle, \langle 13, 14, 23 \rangle.
   \]

Note that homogeneous complements of the $f$-simplicial complexes are all $f$-simplicial complexes, thus all are strong $f$-simplicial complexes.

Examples of strong $f$-simplicial complexes are abundant, see [6] for all the constructions of several classes of pure $f$-simplicial complexes. Actually, it is still difficult to construct an unmixed $f$-ideal that is not strong.

First, we show that $'$ and $^c$ commutes:

**Proposition 3.3.** For any pure simplicial complex $\Delta$, $(\Delta'^c) = (\Delta^c)'$ holds true.

**Proof.** Let $[n]$ be the vertex set of $\Delta$ and assume $\dim(\Delta) = d$. Then for any $F \in 2^{[n]}$ with $|F| = n - d$, we have

\[
F \in \mathcal{F}((\Delta'^c) \iff [n] - F \in \Delta' \iff [n] - F \not\in \mathcal{F}(\Delta) \iff F \not\in \mathcal{F}(\Delta^c) \iff F \in \mathcal{F}((\Delta^c)').
\]

Thus $(\Delta'^c) = (\Delta^c)'$ holds true. ■

Clearly, the map $\varphi: [n]_d \to [n]_{n-d}$, $F \mapsto [n] - F$ is bijective, thus the decomposition $[n]_d = \mathcal{F}(\Delta) \cup \mathcal{F}(\Delta')$ implies $\varphi(\mathcal{F}(\Delta')) = \mathcal{F}((\Delta^c)')$. Thus by Theorem 1.2, we have

**Proposition 3.4.** A simplicial complex $\Delta$ is strong if and only if $\Delta^c$ is strong.
We have the following observation:

**Proposition 3.5.** Let $\Delta$ be a strong $f$-simplicial complex. Then

1. $N(\Delta) = F(\Delta')$, hence we have $I_\Delta = I(\Delta')$.
2. $\Delta^\vee = (\Delta^c)'$.

**Proof.** (1) The equality $I_\Delta = I(\Delta')$ follows by definition of $\Delta'$ and the proof of Theorem 2.3.

(2) Since $N(\Delta) = F(\Delta')$, by definition of $\Delta^\vee$, we have

$$F(\Delta^\vee) = \{[n] - F | F \in N(\Delta)\} = \{[n] - F | F \in F(\Delta')\} = F((\Delta^c)^c).$$

Thus by Proposition 3.3, $\Delta^\vee = (\Delta^c)'$ holds true.

We have the following main result of this section:

**Theorem 3.6.** For a strong $f$-simplicial complex $\Delta$, the following statements are equivalent:

1. The ideal $I(\Delta^c)$ is Cohen-Macaulay.
2. $(\Delta^c)'$ is a Cohen-Macaulay simplicial complex.
3. The ideal $I(\Delta')$ has linear resolution.

**Proof.** By Eagon-Reiner theorem, $I_\Delta$ is Cohen-Macaulay if and only if $\Delta$ is Cohen-Macaulay, the latter holds true if and only if $I_{\Delta^\vee}$ has linear resolution. Then the result follows by applying Proposition 3.5 and 3.3.

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