Collisional matter-phase damping in Bose-condensed gas

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Collisional damping of the excitations in a Bose-condensed gas is investigated over the wide range of energies and temperatures. Numerical results for the damping rate are presented and a number of asymptotic and interpolating expressions for it are derived.

The experimental realisation of Bose-Einstein condensation in dilute atomic gases [1] has led to a new range of experiments investigating the properties of the condensate. Recently, low energy excitations of a Bose-gas in a magnetic trap have been studied by modulating the trap potential [2]. The frequencies of the lowest modes agreed well with theoretical predictions based on mean field theory [3]. Higher frequency excitations could be studied, for example, by light scattering [4]. Very recent experiments [5] have extended the study of low-lying collective excitations to include higher temperatures where a finite non-condensate component may interact with the condensate.

A full study of the excitation spectrum, relevant to the recent experimental advances, should include lifetimes of excitations at different temperatures and over the full energy range. In particular the behaviour of the damping rate as a function of temperature needs to be calculated.

There have been a number of calculations of the condensate excitations for homogeneous systems, dating back to the sixties and seventies [6-9]. These calculations were mainly concerned with understanding the physics of the phase transition. Explicit expressions where found for low-momentum asymptotics of the damping rate. Its high-momentum asymptotics at $T = 0$ were calculated by Beliaev [10].

In this paper, we present numerical results for the relaxation of the condensate excitations which encompasses the whole range of energies and temperatures, and derive a number of asymptotic and interpolating expressions for the damping rate. Our results are restricted to the homogeneous case.

We consider Bogoliubov’s quasiparticles, described by the field operator $\hat{b}_p$ in the momentum representation, $\hat{a}_p = (\hat{b}_p - \alpha_p \hat{b}^\dagger_p) / \sqrt{1 - \alpha^2_p}$. Here, $\alpha_p$ is a particle field operator, and $\alpha_p$ is the parameter of Bogoliubov’s transformation, $\alpha_p = 1 + p^2 / (2 \hbar \omega_0) - \Omega_p / (\sqrt{n_0} U_0)$, where $\Omega_p = \sqrt{p^2 / 2m (p^2 / 2m + 2 U_0 n_0)}$ is the quasiparticle energy. $U_0$ is the parameter of the collisional interaction of particles, written in the S-wave scattering approximation, $H_{\text{coll}} = (U_0 / 2) \sum_p \Delta_p \Delta_p^\dagger \hat{a}_p \hat{a}_p^\dagger$, where $U_0 = 4 \pi a / m$, $a$ is the scattering length and $m$ is the mass of the particle. We use units where $\hbar = k_B = 1$.

After the Bogoliubov transformation, and neglecting interactions of the quasiparticles not involving the condensate, we find the Hamiltonian in the form $H = H_0 + H_{\text{int}}$, where $H_0 = \sum_p \Omega_p \hat{b}_p \hat{b}_p^\dagger$ and

$$H_{\text{int}} = U_0 \sqrt{n_0} \sum_{p_1 + p_2 = p_3} \hat{a}_{p_1} \hat{a}_{p_2} \hat{a}_{p_3} + \text{H. c.}$$

$$= U_0 \sqrt{n_0} \sum_{p_1 + p_2 = p_3} \left[ \kappa_n \left( \omega_{p_1}, \omega_{p_2}, \omega_{p_3} \right) \hat{b}_{p_1} \hat{b}_{p_2} \hat{b}_{p_3}^\dagger + \kappa_0 \left( \omega_{p_1}, \omega_{p_2}, \omega_{p_3} \right) \hat{b}_{p_1} \hat{b}_{p_2} \hat{b}_{p_3} \right] + \text{H. c.}$$

(1)

(in all sums over quasiparticle states zero momenta are excluded). For brevity, we omit explicit formulae for the normal and anomalous interaction formfactors $\kappa_n$ and $\kappa_0$ [11]. They are expressed in terms of the three scaled quasiparticle energies, $\omega_{p_0} = \Omega_p / \hbar \omega_0$, $k = 1, 2, 3$, using $\alpha_p = \sqrt{1 + \omega^2_p - \omega^2_{p_0}}$, where $\Omega_0 = U_0 n_0 = 4 \pi a n_0 / m$ is the energy scale introduced by the Bogoliubov’s transformation.

To justify the S-wave scattering approximation, one needs $a^3 n_0 \ll 1$. Since $U_0 n_0^{1/2} = (4 \pi / m)^{3/4} (\Omega_0 n_0)^{1/4}$, this also justifies the limit of non-interacting quasiparticles, when $H_{\text{int}} \to 0$ while $\Omega_0$ is fixed, and allows one to consider $H_{\text{int}}$ as a small perturbation. Note that, in practice, the density of the condensed phase and the temperature of the sample are functions of experimental conditions [5], rather than the former being a function of the latter. We therefore regard $n_0$ as an independent parameter.

The system can be characterised by the normal average and the linear response function [11],

$$N_p(t-t') = \left\langle \hat{b}_p(t) \hat{b}_p(t') \right\rangle, \quad (2a)$$

$$K_p(t-t') = -i \theta(t) \left\langle \left[ \hat{b}_p(t), \hat{b}_p(t')^\dagger \right] \right\rangle. \quad (2b)$$

We neglect anomalous averages (due to the anomalous $b b b$ interaction) because they can be important only at extremely low energies.

For non-interacting quasiparticles,

$$K_p^{(0)}(t) = -i \theta(t) e^{-i \Omega_p t}, \quad N_p^{(0)}(t) = n_p e^{-i \Omega_p t},$$

(3)
where \( n_p = 1/(e^{\beta p/T} - 1) \). The chemical potential of the quasiparticles is zero since their number is not an integral of the motion. \( K_p^{(0)}(t) \) is the retarded Green’s function of the ‘free’ Schrödinger equation,

\[
(i\partial/\partial t - \Omega_p)K_p^{(0)}(t) = \delta(t).
\]

In order to derive equations for \( K_p \) and \( N_p \), consider the Dyson equation for the two-point quantum averages in Perel-Keldish techniques:

\[
\hat{G}_{p}^{\alpha\alpha'} = \hat{G}_{p}^{(0)\alpha\alpha'} + \sum_{\alpha''\alpha'''} \varepsilon_{\alpha''\alpha'} \hat{G}_{p}^{(0)\alpha''\alpha'''} \Sigma_{p}^{\alpha''\alpha'''} \hat{G}_{p}^{\alpha''\alpha'}.
\]

Here, \( \alpha, \alpha', \alpha'', \alpha''' = +, - \) are the C-continuity indices, \( \varepsilon_{\pm} = \mp i \),

\[
G_{p}^{\alpha\alpha'}(t - t') = \left\langle T_{c} \hat{h}_{p}(t_{\alpha}) \hat{h}_{p}^{\dagger}(t_{\alpha'}) \right\rangle,
\]

where \( T_{c} \) is the C-contour ordering of the field operators, and \( G_{p}^{(0)} = G_{p}|_{H_{\text{int}} = 0} \). \( \Sigma_{p}^{\alpha\alpha'}(t) \) is the exact self-energy. For brevity, we omit time integrations, regarding the Green’s functions and self-energies as kernels of integral operators in respect of their time arguments. Since \( \left\langle \left\{ \hat{h}_{p}(t_{\alpha}), \hat{h}_{p}^{\dagger}(t'_{\alpha'}) \right\} \right\rangle = i \left[ K_{p}(t - t') - K_{p}^{\dagger}(t' - t) \right] \), the Green’s functions \( G_{p}^{\alpha\alpha'} \) can be expressed as linear combinations of \( K_{p} \) and \( N_{p} \). The Dyson equations then become,

\[
\dot{K}_{p} = \dot{K}_{p}^{(0)} + K_{p}^{(0)} \kappa_{p} K_{p},
\]

\[
\dot{N}_{p} = \dot{N}_{p}^{(0)} + K_{p}^{(0)} \kappa_{p} N_{p} + K_{p}^{(0)} \sigma_{p} K_{p}^{\dagger} + \dot{N}_{p}^{(0)} \hat{\kappa}_{p}^{\dagger} \dot{K}_{p}^{\dagger},
\]

where

\[
\kappa_{p}(t) = -i \left[ \Sigma_{p}^{++}(t) - \Sigma_{p}^{+-}(t) \right] \propto \theta(t),
\]

and \( \sigma_{p}(t) \) is also a certain linear combination of the self-energy components.

A deeper insight shows that, ultimately, separation of the equation for \( K_{p} \) is due to microscopic causality. It is also very important from a more practical viewpoint. The initial condition \( K_{p}^{(0)}(0) = -i \) is independent of the interaction. Solving \( \dot{K}_{p} \) hence implies evolving the system during a finite time, and a certain simple approximation (e.g., one-loop) to the susceptibility \( \kappa_{p} \) may well suffice. Conversely, the knowledge of \( N_{p} \) implies that of the steady-state solution, so that simple approximations to \( \sigma_{p} \) are hopeless (cf the fact that noise sources in kinetic equations cannot be found perturbatively). The way around this problem is to find \( K_{p} \) and then recover \( N_{p} \) using Kubo’s fluctuation-dissipation theorem, thus making \( \dot{K}_{p} \) redundant. In this paper, we confine our attention to \( K_{p} \).

By making use of (8), Eq. (8) may be re-written in the Markov approximation (discussed below) as,

\[
\left(i\partial/\partial t - \Omega_p - i\frac{\gamma_p}{2} - \Delta_p\right)K_p = \delta(t).
\]

where

\[
\Delta_p - i\frac{\gamma_p}{2} = \lim_{\varepsilon\to0} \int_{0}^{\infty} dt e^{i(\Omega_{p} - \varepsilon)t} K_p(t).
\]

Taking the self-energy in the one-loop approximation and expressing \( G_{p}^{(0)\alpha\alpha'} \) by \( K_{p}^{(0)} \) and \( N_{p}^{(0)} \), after some algebra we find

\[
\gamma_p = \gamma_p^{0} + \gamma_p^{T} = \gamma_p^{0} + \gamma_p^{T\dagger} + \gamma_p^{T\prime},
\]

\[
\gamma_p^{0} = \frac{\gamma_0 p_0}{2} \int_{0}^{\infty} d\omega g(\omega_p - \omega, \omega_p),
\]

\[
\gamma_p^{T\dagger} = \frac{\gamma_0 p_0}{p} \int_{0}^{\infty} d\omega g(\omega_p - \omega, \omega_p),
\]

\[
\gamma_p^{T\prime} = \frac{\gamma_0 p_0}{p} \int_{0}^{\infty} d\omega \left[ g(\omega, \omega_p) - \frac{g(\omega_p, \omega_p)}{e^{\omega/\theta} - 1} \right],
\]

where

\[
g(\omega, \omega_p) = \frac{\omega(\omega_p + \omega) e^{\omega_p + \omega}}{\sqrt{(\omega^2 + 1)(\omega_p^2 + 1)}},
\]

\[
p = |p|, p_0 = \sqrt{2m T_{\text{det}}} = \sqrt{8\pi a m_0}, \quad \gamma_0 = U_{0} p_{0}^{3}/2\pi = \Omega_{0} \sqrt{128\pi a m_0}, \quad T = T_{\text{det}} = 2m/p_{0}^{2}.
\]

Two types of collision processes contribute to \( \gamma_p \). The ‘probe’ quasiparticle can collide with a condensate particle, producing two quasiparticles. If the thermal population of the final states is neglected, this is in essence a classical collision, responsible for \( \gamma_p^{0} \). Bosonic stimulation of this process by the thermal population of the final states results in \( \gamma_p^{T\prime} \). The ‘probe’ quasiparticle can also collide with another quasiparticle, the final state being a condensate particle and a quasi-particle. This process is due to the bosonic attraction of the condensate, and results in \( \gamma_p^{T\dagger} \). Note that both processes contributing to \( \gamma_p^{T} \) are purely quantum.

The result of direct numerical evaluation of expressions is shown in Fig. 1. We see that both \( \gamma_p^{0} \) and \( \gamma_p^{T} \) vanish if \( p \to 0 \), hence so does \( \gamma_p \). This is important for consistency, because (i) the condensate should not be damped and (ii) the low-energy excitations are physically indistinguishable from it. Note that for low energies the thermal contribution always prevails. For higher energies, \( \gamma_p^{0} \) grows monotonically while \( \gamma_p^{T\dagger} \) has a maximum at a certain momentum. It is interesting that, as the temperature grows, this maximum stabilises at \( \Omega_{p} \sim \Omega_{0} \), not at \( \Omega_{p} \sim T \), as might be expected. The maximal value of \( \gamma_p^{T} \) is close to \( \gamma_0 T = T \sqrt{128\pi a m_0} \). For energies high enough, the thermal contribution becomes negligible.

It is easy to check that \( \kappa_{p}^{(0)}(\omega_1, \omega_2, \omega_1 + \omega_2) \approx 9\omega_1 \omega_2 (\omega_1 + \omega_2)/32 \) if \( \omega_1 + \omega_2 \ll 1 \); \( \approx y_2 \omega_1 \) if \( \omega_1 \gg 1 \); and \( \approx \omega_1 y_2^2 (3 + y_2^2)^2 /8(1 + y_2^2)^2 \) if \( \omega_1 \ll \omega_2 \), where \( y_2 = (\sqrt{\omega_2^2 + 1} - 1)/\omega_2 \). We then have,
\[
\gamma^0 = \frac{3\gamma_0}{80} \left( \frac{p}{p_0} \right)^5 = \frac{3p^5}{320\pi m n_0}, \quad p \ll p_0, \quad (14)
\]
\[
\gamma^0_p = \frac{\gamma_0}{2} \frac{p}{p_0} (1 - \delta_p) = 8\pi a^2 n_0 \frac{p}{m} (1 - \delta_p), \quad p \gg p_0. \quad (15)
\]
where \(\delta_p = \frac{2\ln(\omega_p^2/\theta^2)}{p^2/\theta^2}\). Except \(\delta_p\), these are the well known results of Beliaev [4]. Since the expression for \(\delta_p\) is valid only for large momenta, we are free to ‘tune’ it at low momenta so as to improve its agreement with the exact numerical result. On ‘tuning’, \(\delta_p = \frac{2\ln(\omega_p^2/\theta^2) + 2\ln 2}{p^2/\theta^2 + 2\ln 2}\). In Fig. 2, we compare the results of the direct numerical calculation with the approximate expressions. We see that relation (13) gives a good approximation to the numerical result for \(p \gtrsim p_0\). (It even correctly reproduces the \(p^5\) law for low momenta.)

Consider now the thermal contribution to the width. For \(\omega_p \ll 1, \theta\), we find \(\gamma^T_p \approx \gamma_0 f(\theta)p/p_0\), where
\[
f(\theta) = \frac{2}{\theta} \int_0^\infty \frac{\omega^2 y^2 (3 + y^2)^2 e^{-\omega}}{8(\omega^2 + 1)(1 + y^2)^2 (e^{\omega/\theta} - 1)^2}, \quad (16)
\]
and \(y = (\sqrt{\omega^2 + 1} - 1)/\omega\). For low temperatures \(\theta \ll 1\) \((T \ll 8\pi a n_0)\), \(f(\theta) = (3\pi^4/20)^{1/2}\), and
\[
\gamma^T_p = \frac{3\pi^4 \theta^4 \gamma_0}{20} \frac{p}{p_0} = \frac{3\pi pT^4 m^3}{320a^2 n_0^3}. \quad (17)
\]
\(\gamma^T_p\) dominates if \(\Omega_p \lesssim T\). With the sound velocity \(u = 2\sqrt{\pi a n_0}/m\), Fig. 3 coincides with \(\gamma^T_p/2 = 3\pi^3 pT^4/40m n_0 u m^4\), found by Popov [4]. For high temperatures \(\theta \gg 1\), \(f(\theta) \approx 0.60\theta\) (cf (13)) and \(\gamma^T_p \sim 0.60\gamma_0 \theta p/p_0\). Note that if \(\theta \gg 1\), the thermal contribution prevails for low momenta \(p \ll p_0\) (cf Fig. 4).

A simple semi-quantitative expression for \(\gamma^T_p\), valid if \(\theta, \omega_p \gg 1\), can be obtained by dropping the factors \(g(\omega, \omega_p)\) and \(g(\omega_p - \omega, \omega_p)\) in the integrands (23), (24), which truncate the integrals in the low-energy region \(\omega \lesssim 1\), setting instead the lower limit to some \(\varepsilon \ll 0\). The integration is easily performed; we then find that we must choose \(\varepsilon = 2\) to match the asymptotical behaviour of \(\gamma^T_p\) at \(p \rightarrow \infty\). Then,
\[
\gamma^T_p \approx \frac{8mT}{p^2} \ln \left( \frac{Tm}{8\pi a n_0} \right) \frac{1 - e^{-p^2/2mT}}{8\pi a n_0}. \quad (18)
\]
Thus the thermal contribution dominates if \(\Omega_p \lesssim T \ln(Tm/8\pi a n_0) \lesssim T \ln(na^3/m)\), where \(n\) is the total density of particles. The last inequality is due to the fact that the temperature should be below the condensation point, \(T < T_0 \sim n^{2/3}/m\).

In Fig. 3 we compare the exact numerical results for the thermal component of the width with the approximate expression for low and high momenta. We have used a ‘tuned’ expression to give better accuracy at average momenta, \(p \gtrsim p_0\), namely,
\[
\gamma^T_p \approx 2\gamma_0 \theta \frac{p}{p_0} \omega_p \ln \frac{1 - e^{-2+\omega^2/p_0^2})/\theta}}{1 - e^{-2/\theta}}. \quad (19)
\]
This expression coincides, to a very good accuracy, with the numerics for \(\omega_p \geq 1\) and \(\theta \geq 10\). Together, expressions (14) and (19) provide a good approximation to \(\gamma^T_p\) if the temperature is not too low, \(\theta \geq 10\).

Consider now the validity of the Markov approximation. Assume that the temperature is not too low, \(T > \Omega_0\). The bosonic distribution \(\gamma_p\) is divergent at low energies. This is overcome by the ‘truncating’ factors \(g(\omega, \omega_p)\) and \(g(\omega_p - \omega, \omega_p)\) in the integrals, resulting in the major contribution to \(\gamma^T_p\) coming from the energies \(\sim \Omega_0\). Ipso facto, it can only be applicable to time scales longer than \(1/\Omega_0\). With the maximal value of \(\gamma^T_p \sim \gamma_0\), our results apply if \(\theta \ll \Omega_0/\gamma_0\). This can be written as
\[
mT \ll \left( \frac{n_0}{a} \right)^{1/2}. \quad (20)
\]
To understand what this restriction means in terms of the critical temperature \(T_c\), note that \(mT \sim n^{2/3}\), where \(n^\prime\) is the density of the non-condensate phase. Estimate (20) then is equivalent to \(n^\prime \ll n_0/(na^3)^{1/4}\). This does not contradict \(n^\prime \sim n_0\). Thus, although certainly \(T < T_c\), a stronger condition \(T \ll T_c\) does not seem necessary.

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FIG. 1. Results of the direct numerical calculation of the width components $\gamma_0$ (dashed line) and $\gamma_T$ (solid lines).

FIG. 2. Comparison of the results of the direct numerical calculation of $\gamma_0$ (solid line) with the approximate expressions: dashed lines – Beliaev’s asymptotics for small and large momenta, dash-dotted line – ‘tuned’ expression (15).

FIG. 3. Comparison of the results of the direct numerical calculation of $\gamma_T$ (solid lines) with approximate expressions (16) (dashed lines) and (19) (dash-dotted lines).
Fig 1:
Fig 2:

(a)

(b)
Fig 3: