Schmidt decomposition for non-collinear biphoton angular wave functions*

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Abstract

Schmidt modes of non-collinear biphoton angular wave functions are found analytically. The experimentally realizable procedure for their separation is described. Parameters of the Schmidt decomposition are used to evaluate the degree of the biphoton’s angular entanglement.

Keywords: Schmidt modes, Schmidt decomposition, entanglement, non-collinear biphotons

(Some figures may appear in colour only in the online journal)

1. Introduction

It is well known that the Schmidt decomposition is a powerful instrument for the analysis of the correlations (entanglement) of pure bipartite states [1]. The most common systems of this kind are states of two photons produced in the spontaneous parametric down-conversion (SPDC) process, in which some photons of the pump decay in a nonlinear crystal for pairs of photons of smaller frequencies. Regimes of the SPDC depend on features of the pump and nonlinear crystals. The simplest collinear degenerate regime is that of a plane monochromatic wave giving rise to collinearly propagating SPDC photons with coinciding frequencies equal to half of the pump frequency. In this case, the only degree of freedom that the SPDC photons can be entangled in is their polarization. Such states are known as polarization biphoton qutrits. Their features have been investigated by many authors in a number of papers, including [2–8]. In particular, the Schmidt decomposition of biphoton polarization qutrits was discussed [1, 8, 9], and some of these results from these papers are reproduced briefly in the following section. In the same collinear and degenerate regime, but with a transversely spreading pump and SPDC photon beams, the angular Schmidt modes and decomposition were found experimentally in [10, 11]. In section 3 we will describe the Schmidt decomposition and the separation method of the angular Schmidt modes for the noncollinear degenerate regime of the SPDC, while ignoring the spreading of the pump and the SPDC photons. Section 4 is devoted to the derivation of the angular Schmidt modes in the same noncollinear degenerate regime, with spearing of the photon beams completely taken into account. The derivation will be strongly related to and based on the proposal of a scheme in which the Schmidt modes can be found and separated experimentally.

2. Polarization biphoton qutrits

The most general form of the state vector of biphoton polarization qutrits is given by

$$|\Psi\rangle = C_1 |2_H\rangle + C_2 |1_H, 1_V\rangle + C_3 |2_V\rangle,$$

(1)

where $H$ and $V$ refer to the horizontal and vertical polarizations of photons and $C_{1,2,3}$ are arbitrary complex constants obeying the normalization condition, $|C_1|^2 + |C_2|^2 + |C_3|^2 = 1$. As shown in [1, 8], at any values of the constants $C_i$, state vectors of biphoton polarization qutrits can be presented in the Schmidt-decomposition form

$$|\Psi\rangle = \sqrt{\lambda_+} |2_+\rangle + \sqrt{\lambda_-} |2_-\rangle,$$

(2)

where $|2_+\rangle$ and $|2_-\rangle$ are state vectors of two photons in two orthogonal Schmidt modes, $|1_+\rangle$ and $|1_-\rangle$. There are several ways to find explicitly Schmidt modes in terms of the constants, $C_i$ [1, 8, 9]. The conventional method [7] is related to the use of polarization wave functions,
\[ \Psi'(\sigma_1, \sigma_2) = \langle \sigma_1, \sigma_2 | \Psi' \rangle, \] where \( \sigma_1 \) and \( \sigma_2 \) are the polarization variables of two photons, such that \( (\sigma_1, 2 | 1_H) = \delta_{\sigma_2, H} \) and \( (\sigma_1, 2 | 1_V) = \delta_{\sigma_2, V} \), construction of the total density matrix \( \Psi'(\sigma_1, \sigma_2) \Psi'^*(\sigma_1', \sigma_2') \), and its reduction over one of the variables, \( \sigma_2 = \sigma_2' \) or \( \sigma_1 = \sigma_1' \). Eigvenfunctions and eigenvalues of the reduced density matrices are just the Schmidt modes and parameters \( \lambda_+ \) and \( \lambda_- \) determining the Schmidt decomposition (2). These parameters obey the normalization condition \( \lambda_+ + \lambda_- = 1 \), and their variation intervals are \( 1 \geq \lambda_+ \geq 0.5 \) and \( 0.5 \geq \lambda_- \geq 0 \). As shown in [7], these parameters determine completely the qutrit’s degree of polarization, \( P \), and such entanglement quantifiers as the concurrence, \( C \), Schmidt number \( K \), and the entropy of the reduced states
\[
P = \lambda_+ - \lambda_-, \quad K = \frac{1}{\lambda_+^2 + \lambda_-^2},
\]
\[
C = 2 \sqrt{\lambda_+ \lambda_-},
\]
\[
S_r = -\log_2 \lambda_+ - \log_2 \lambda_-.
\]
Qutrits are maximally entangled in the case \( \lambda_+ = \lambda_- = 0.5 \) and disentangled when \( \lambda_+ = 1, \lambda_- = 0 \).

In [8], we outlined the method of direct experimental measurement of parameters \( \lambda_+ \) and \( \lambda_- \). The scheme of this experiment is shown in figure 1. The first step of this experiment should consist of transforming the polarizations of the orthogonal Schmidt modes \( |1_H \rangle \) and \( |1_L \rangle \), correspondingly, to the horizontal and vertical Schmidt modes to reduce the Schmidt decomposition (2) to the simplest form
\[
|\Psi'\rangle = \sqrt{\lambda_+} |2_H\rangle + e^{2i\phi} \sqrt{\lambda_-} |2_V\rangle,
\]
where \( \phi \) is some phase, which can be easily changed so it can be used to encode information [8], but which does not affect the degree of entanglement of the state (2), (4). In the experiment, the transformation (2) \( \rightarrow \) (4) can be achieved by appropriately installed half- and quarter-wavelength plates. Correct orientation of these plates can be found experimentally from the condition of the zero coincidence signal between two channels immediately after the polarization beam splitter (PBS), as shown in figure 1. Under this condition, after the PBS one gets two beams containing pairs of separated Schmidt modes: \( |2_H\rangle \) transmitted and \( |2_V\rangle \) reflected, or vice versa if the PBS is turned for 90° around the original propagation direction. By measuring the relative amounts of such pairs, one finds that the probabilities of their appearance are equal to \( \lambda_+ \) and \( \lambda_- \). Thus, if \( N_{HH} \) and \( N_{VV} \) are the amounts of clicks of the detectors registering horizontally and vertically polarized photons per some given time, the Schmidt-decomposition parameters are given by
\[
\lambda_+ = \frac{N_{HH}}{N_{HH} + N_{VV}}, \quad \lambda_- = \frac{N_{VV}}{N_{HH} + N_{VV}}.
\]

The described procedure is general and valid for any qutrits (1) with arbitrary unknown parameters, \( C_{1,2,3} \). But in many special cases this is not needed at all or can be significantly simplified. For example, in the case of the state with two photons of different polarizations, its Schmidt decomposition can be found by a simple transformation to the basis turned for 45°:
\[
|1_H, 1_V\rangle = \frac{1}{\sqrt{2}} (|2_{15}\rangle - |2_{135}\rangle).
\]

The right-hand side of this equation is just the Schmidt decomposition with the Schmidt modes \( |1_{25}\rangle \) and \( |1_{135}\rangle \). For experimental separation of the pairs of these Schmidt modes, one needs only the PBS turned for 45° around the propagation axis. Then the transmitted and reflected beams will contain pairs of photons in one or another Schmidt mode (i.e., both polarized either in the direction at 45° or 135° with respect to the horizontal axis). The amount of pairs to be measured in each channel will be equal to each other, which indicates that \( \lambda_+ = \lambda_- = 0.5 \), and the state \( |1_H, 1_V\rangle \) is maximally entangled [7].

Note that the polarization degree of freedom is very special. For example, a superposition of single-photon states with the horizontal and vertical polarizations also characterizes an experimentally detectable photon with some intermediate polarization. This is the reason for getting the Schmidt decomposition of the state \( |1_H, 1_V\rangle \) (6) to be easily realizalble experimentally. In contrast, in the case of other variables (angular or frequency) superpositions of states do not correspond to any experimentally detectable photons. For example, a superposition of states of spatially separated photons is not an experimentally observable photon propagating in any intermediate direction. This feature can cause problems with the experimental separation of Schmidt modes that depend on such variables. As shown below, for angular states of noncollinear biphotons, the problem can be solved by means of regrouping the photons. The main idea of this method lies in artificially created duplication of the characterization of angular modes with characterization by different polarizations. After this, by using manipulations with polarization variables, one can regroup photons in a way that ultimately provides separation of the angular Schmidt modes. This idea is formulated explicitly and described in detail in the following section on the example of a rather simple case of nonspreading, noncollinear biphoton states. Then, in section 4 the results are generalized for a more realistic case.
of noncollinear biphotons with finite angular widths of beams propagating in two different directions.

3. Angular entanglement

Let us consider the state of two photons produced in the noncollinear degenerate regime of the SPDC with type-I phase matching. The latter means that polarizations of both photons are horizontal, and there is no polarization entanglement. Let the angles of propagation of these photons be $\theta_0$ and $-\theta_0$. These angles determine the two modes in which each of the two photons can be found, but the photons can never appear in one of these modes together. The state vector of such a state is $|1_{\theta_0}, 1_{-\theta_0}\rangle$. As the photons are indistinguishable, the localization of each photon in the modes $\theta_0$ and $-\theta_0$ is uncertain, and at the same time there is a specific correlation of localization: If one of the photons belongs to one of the two modes, there is 100% probability that the second photon belongs to the other mode. This uncertainty and correlation of localization indicate that the state $|1_{\theta_0}, 1_{-\theta_0}\rangle$ is entangled. The problem lies in finding the scheme of an experiment in which one could separate pairs of Schmidt modes of this state in a manner similar to that described above for the polarization-entangled state $|1_H, 1_V\rangle$ (equation (6)). The scheme of such an experiment is shown in figure 2. The first step of this scheme consists of changing the photon polarization in one of two channels, $H \rightarrow V$, with the help of $\lambda/4$ and $\lambda/2$ plates (green in figure 2). This change does not affect the degree of entanglement because it does not change the amount of modes accessible for the two photons. We still have two modes, but now they have a double labeling: $\theta_0$, $H$ and $-\theta_0$, $V$. The second step is merging two beams into a single beam with the help of a PBS. After the PBS, both photons propagate together in the same direction. However, there are still two modes corresponding to different polarizations, $H$ and $V$. As previously, because of the indistinguishability of the photons, belonging to any one of them any given mode remains uncertain, and hence, the system keeps the same degree of entanglement as it had before the performed manipulations. In fact, what is done now is the substitution of the angular entanglement by the equivalent polarization entanglement of the two photons, for which separation of the Schmidt modes is much easier and can be done as described above for the state $|1_H, 1_V\rangle$. Specifically, the merged state arising after the PBS in figure 2 has to be sent to the second PBS and turned for 45° around the propagation axis; the separating photons have polarizations along the directions 45° and 135° with respect to the horizontal axis. As follows from equation (6), after the second PBS in figure 2, pairs of photons will be either transmitted or reflected depending on their polarization (45° or 135°), but none of them will be split between two channels. The picture in figure 2(b) shows schematically that, if needed, after separating the pairs of photons, one can change both directions of their propagation and polarizations in each channel separately to return to the original geometry of two beams propagating in directions $\theta_0$ and $-\theta_0$ with the same horizontal polarization of all photons. The difference from the original state, $|1_{\theta_0}, 1_{-\theta_0}\rangle$, lies in regrouping the photons in such a way that both photons in each SPDC pair propagate together in the same direction. Mathematically, the arising state is characterized by the state vector in the Schmidt-decomposition form

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |2_{\theta_0}, U\rangle - |2_{-\theta_0}, U\rangle \right).$$

By counting the amount of photons in each channel after all manipulations, one can find the parameters of the Schmidt decomposition, $\lambda_+$ and $\lambda_-$ (5), which both must be close to 0.5 to indicate that the state of the two noncollinear SPDC photons with coinciding polarizations is maximally entangled.

4. Angular Schmidt modes of noncollinear spreading biphoton beams

Let us consider a more detailed form of the structure of photon angular distributions in the same noncollinear frequency-degenerate SPDC regime as in the previous section, but with finite transverse widths of both the pump and the SPDC photon beams. In the case of type-I phase matching, under some assumptions, the biphoton angular wave function can be taken in the form

$$\Psi(\theta_1, \theta_2) = N \exp\left(\frac{(\theta_1 + \theta_2)^2}{2\Delta^2_p}\right) \times \sin\left(\frac{(\theta_1 - \theta_2)^2 - 4\theta_0^2}{2\Delta^2_p}\right),$$

where $N$ is the normalization factor, $\sin\frac{\pi}{x/\alpha}; \theta_1$ and $\theta_2$ are angles between the wave vectors of two emitted photons and the central direction of propagation of the pump ($0z$-axis) in a free space after the crystal; $\theta_0$ and $-\theta_0$ are the angles between the central propagation directions of the
beams of emitted photons and the \( z \)-axis; \( \Delta \theta_p \) and \( \Delta \theta_L \) are, respectively, the angular width of the pump wave and the width of the angular distributions in the beams of emitted photons related to the finite length of the nonlinear crystal, \( L \)

\[
\Delta \theta_p = \frac{\lambda_p}{\pi d}, \quad \Delta \theta_L = 2 \sqrt{\frac{n_o \lambda_p}{\pi L}}, \quad \theta_0 = \sqrt{2n_o(n_o - n_e)}
\]

with \( n_o \) and \( n_e \) being the refracting indices of the ordinary and extraordinary waves in a crystal in the direction of the \( z \)-axis, and \( d \) being the waist of the pump wave. The widths and all characteristic angles are assumed to be small:

\[
\Delta \theta_p, \Delta \theta_L \ll \theta_0 \ll 1. \tag{10}
\]

Note also that the argument of the sinc-function in equation (8) does not contain linear terms in \( \theta_1 \) and \( \theta_2 \). This is correct in a general case for measurements in the plane perpendicular to that containing the optical axis of the crystal. For any other planes of measurements, the linear terms can be dropped only if the pump width, \( \Delta \theta_p \), is sufficiently small. Otherwise the linear terms can significantly affect the photon distributions, which has been demonstrated and widely discussed for the case of the collinear regime in [12, 13].

In a general case, the sinc-function in equation (8) can be approximated in its main part by the Gaussian function, sinc \( \approx \exp(-0.195 x^2) \), as shown in figure 3.

By applying this substitution to the wave function of equation (8), we get

\[
\Psi(\theta_1, \theta_2) = N \exp\left(-\frac{(\theta_1 + \theta_2)^2}{2\Delta \theta_p^2}\right) \exp\left(-\frac{0.195}{4\Delta \theta_L^2} \left( (\theta_1 - \theta_2)^2 - 4\theta_0^2 \right)^2 \right). \tag{11}
\]

In dependence on \( \theta_1 - \theta_2 \), the second exponent in this equation is a super-Gaussian function. As the width, \( \Delta \theta_L \), is assumed to be small compared to \( \theta_0 \) (10), in dependence on \( \theta_1 - \theta_2 \), the super-Gaussian function in equation (8) has two well-separated peaks at \( \theta_1 - \theta_2 = 2\theta_0 \) and \( \theta_1 - \theta_2 = -2\theta_0 \). For this reason, the super-Gaussian function in equation (11) can be approximated very well by the sum of two true Gaussian functions

\[
\Psi(\theta_1, \theta_2) \approx N \exp\left(\frac{0.195}{4\Delta \theta_L^2} \left(4\theta_0^2 - (\theta_1 - \theta_2)^2\right)^2\right) \exp\left(-\frac{0.195}{4\Delta \theta_L^2} \left( (\theta_1 - \theta_2)^2 - 4\theta_0^2 \right)^2 \right). \tag{12}
\]

A good quality of this approximation is illustrated by figure 4. With the substitution (12), the biphoton wave function of equations (8) and (11) takes the form of the sum of two products of Gaussian functions

\[
\Psi(\theta_1, \theta_2) = N \exp\left(-\frac{(\theta_1 + \theta_2)^2}{2\Delta \theta_p^2}\right) \left\{ \exp\left(-\frac{0.195}{4\Delta \theta_L^2} \left(4\theta_0^2 - (\theta_1 - \theta_2)^2\right)^2\right) \right\} \left\{ \exp\left(-\frac{0.195}{4\Delta \theta_L^2} \left( (\theta_1 - \theta_2)^2 - 4\theta_0^2 \right)^2 \right) \right\}. \tag{13}
\]

Three-dimensional plots of this function are shown in figure 5. The first of these pictures, 5(a), shows that the nonzero parts of the total biphoton wave function are localized only in two quadrants in the plane of the total biphoton wave function are localized only in two quadrants in the plane \( \theta_1, \theta_2 \), \( \theta_1 < 0, \theta_2 > 0 \) and \( \theta_1 > 0, \theta_2 < 0 \), and the photon distributions in these quadrants are perfectly symmetric. But individual distribution of each photon between the two quadrants is uncertain. The first term on the right-hand side of equation (13) describes photon localization in the region \( \theta_1 \approx \theta_0 \) and \( \theta_2 \approx -\theta_0 \), whereas the second term describes photon localization in the region \( \theta_1 \approx -\theta_0 \) and \( \theta_2 \approx \theta_0 \).

Figure 5(b) shows a more detailed scale of the photon distribution inside of one of these quadrants, \( \theta_1 > 0, \theta_2 < 0 \). As one can clearly see, this structure can be strongly asymmetric with respect to variables \( \theta_1 + \theta_2 \) and \( \theta_1 - \theta_2 \) in the case of the significantly differing widths of biphoton distributions in the directions along 45° and -45° in the \( (\theta_1, \theta_2) \)-plane.

Figure 3. Sinc-Gaussian approximation.

Figure 4. Super-Gaussian function (dashed line) and the sum of two Gaussian functions (solid line) on the left- and right-hand sides of equation (12); \( \Delta \theta_L/\theta_0 \approx 0.53 \).
As shown below, entanglement of the state (8), (11) and (13), can be determined by two factors: the uncertainty of photon localization in the two quadrants of (13), can be determined by two factors: the uncertainty of photon localization. For finding the degree of entanglement of the state (8), (11), and (13) as a whole and for finding its Schmidt modes, we can use the same procedure as in section 3 and the same manipulations as discussed above and shown in figure 2. As previously, the first step is changing the polarizations of the photons propagating in the region around $-\theta_0$. In equations (8), (11), and (13), the polarization parts of the wave function are not shown. As we consider the case of type-I phase matching, the polarizations of both photons are horizontal, and in terms of polarization variables $\sigma_1$ and $\sigma_2$, the polarization wave function of the two photons can be written as $\delta_{\sigma_1,H} \delta_{\sigma_2,H}$, with the numbers of polarization variables 1 and 2 associated with numbers of angular variables $\theta_1$ and $\theta_2$. This polarization wave function could be added as a factor to the angular wave function of equations (8), (11), and (13). With the horizontal polarization changed for the vertical polarization for parts of the photons moving in directions around $-\theta_0$, the total angular-polarization wave function of equation (13) takes the form

$$\Psi(\theta_1, \theta_2; \sigma_1, \sigma_2) = \sum_{n=0}^{\infty} \sqrt{\lambda_n} \psi_n(\theta_1)\psi_n(\theta_2),$$

where

$$\lambda_n = \frac{4ab - (a - b)^{2n}}{(a + b)^n(a + b)^{2n+1}}.$$
the same can be done for noncollinear spreading beams after they merge into a single beam. But this is not the end of the story. The next step is the same as that prescribed in figure 2: the merged beam must be sent to the PBS turned for 45° around the propagation axis to transform the wave function (15) to the form

$$\Psi = \sum_{n=0}^{\infty} \sqrt{\frac{\lambda_n}{2}} \left( y_n(\theta_1) y_n(\theta_2) \delta_{n,45} \delta_{m,45} \right. \left. + y_n(\theta_1 + 90^\circ) y_n(\theta_2 + 90^\circ) \delta_{n,135} \delta_{m,135} \right). \quad (20)$$

After transformation we get two beams consisting of many modes each, but with pairs of photons with coinciding polarizations (45° or 135°) propagating unsuppressed and in different directions. Directions of propagation can be changed by mirrors to the original directions (around θ₀ and −θ₀), and polarizations of photons in each beam can be changed to the horizontal polarizations, after which the polarization part of the wave function (2λ₁) for both beams) can be dropped. As a result, the Schmidt decomposition of the noncollinear angular wave function takes its final form

$$\Psi(\theta_1, \theta_2) = \sum_{n=0}^{\infty} \sqrt{\frac{\lambda_n}{2}} \left( y_n(\theta_1 - \theta_0) y_n(\theta_2 - \theta_0) \right. \left. + y_n(\theta_1 + \theta_0) y_n(\theta_2 + \theta_0) \right). \quad (21)$$

This result shows that all terms of the Schmidt decompositions are twice degenerate (i.e., there are two pairs of Schmidt modes corresponding to each λₙ). Eigenvalues of the reduced density matrices are equal to λₙ/2, and the normalization condition has the form 2 × ∑ₙ λₙ/2 = 1.

Parameters characterizing the degree of entanglement of the states under consideration are the Schmidt number K and the entropy of the reduced density matrix Sᵣ, for which we get

$$K = \frac{1}{2} \times \sum_n (\lambda_n/2)^2 = \frac{\sum_n \lambda_n^2}{\sum_n \lambda_n^2} = \frac{a^2 + b^2}{ab} \geq 2 \quad (22)$$

and

$$S_r = -2 \times \sum_n (\lambda_n/2) \log_2(\lambda_n/2) = 1 - \sum_n \lambda_n \log_2 \lambda_n \geq 1. \quad (23)$$

As mentioned above, angular entanglement of noncollinear angular states of biphotons arises for two reasons: because of asymmetry of formations in each of the two occupied quadrants in the plane (θ₁, θ₂), and because of the quadrant-quadrant symmetry of the photon distributions. The first of these two reasons occurs if a ≠ b, and it disappears in the case where a = b (photon distributions in each occupied quadrant become symmetric). In this last case, λₙ = δₙ,0, but because of degeneracy, the state (21) remains entangled with the entanglement quantifiers (22), (23) equal to K = 2 and Sᵣ = 1. This remaining entanglement is related to the symmetry of two-boson wave functions, and the case λₙ = δₙ,0 corresponds exactly to that of section 3.

5. Conclusions

Thus, the main results obtained concern the derivation of the Schmidt decomposition (21) for the angular wave function of the noncollinear frequency-degenerate biphoton states (8) arising in the SPDC process, with type-I phase matching and angular widths of photon distribution completely taken into account. The derivation is based on the proposal of an experiment that can provide separation of Schmidt modes and measurement of parameters of the Schmidt decomposition. The derivation and the proposed experiment consist of three steps: (1) manipulation with photon polarization, which provides duplication of the symmetry of the angular wave functions by the symmetry of polarization states; (2) transformation/merging of the pair of noncollinear photon beams to a single collinear beam with the same degree of entanglement and the same amount of modes; and (3) the polarization-sensitive splitting/unmerging of the collinear beam into a pair of beams with photons regrouped in such a way that each of the two unmerged beams contains only unpaired pairs of photons. The uncertainty of the localization of photon pairs in these new beams is responsible for the entanglement related to the symmetry of biphoton states, and a true angular entanglement is determined by the amount of nonzero terms in the decomposition of the angular wave functions in each of the two channels into sums of products of Hermite–Gaussian one-photon angular functions (Schmidt modes). In the experiment, separation of terms corresponding to different products of Hermite-Gaussian angular functions can be performed at the stage of a merged beam between two PBSs in the scheme of figure 2.

Note that another type of experiment can be related to measuring the coincidence and single-particle angular distributions of photons. In a scheme with two detectors, one can install one detector to count photons moving strictly in the direction −θ₀, with the second detector scanning around the direction θ₀. The coincidence distribution found in this way is characterized by its width (e.g., the FWHM width), Δθᵣ(c). Measurements with the turned-off detector at −θ₀ will give a single-particle distribution, Δθᵣ(s). In accordance with the idea of [16], the ratio of these widths, Rᵣ(part1) = Δθᵣ(c)/Δθᵣ(s), can be considered as a measure of the degree of entanglement. For noncollinear beams this will be a partial entanglement (e.g., for the part of the biphoton angular wave function located in the quadrant (θ₁ > 0, θ₂ < 0) in figure 5(a)). As is known [17], for double-Gaussian bipartite wave functions, the parameter Rᵣ(part1) coincides exactly with the corresponding Schmidt number, Kᵣ(part1). If one changes the roles of the detectors by keeping the position of the detector constant at θ₀ and scanning the second detector around the direction −θ₀, one can measure the widths Δθᵣ(c) and Δθᵣ(s), as well as the parameters Rᵣ(part2) = Kᵣ(part2), for the second occupied quadrant of the angular wave function, (θ₁ > 0, θ₂ < 0), in figure 5(a). Because of the symmetry of the biphoton wave functions, the partial Schmidt numbers Kᵣ(part1) and Kᵣ(part2) must be equal to each other. Thus, in terms of the Schmidt number K, the total degree of entanglement of the angular
wave function as a whole is determined by the sum of partial contributions, \( K = K^{(\text{part1})} + K^{(\text{part2})} = 2K^{(\text{part1})} \geq 2 \).

It would be very interesting to perform two types of experiments together: (a) finding parameters, \( \lambda_n \), of the Schmidt decomposition (21) as described above and determining the Schmidt number \( K \) via equation (22); and (b) finding the same entanglement quanifier \( K \) by means of measuring the widths of the coincidence and single-particle distributions to find the width-ratio parameters, \( R^{(\text{part1,2})} \), and identifying their sum with the Schmidt number \( K \). Comparison of the results of these two experiments could be very interesting.

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