Quantum Space-Time as a Quantum Causal Set

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A recently proposed algebraic representation of the causal set model of the small-scale structure of space-time of Sorkin et al. is briefly reviewed and expanded. The algebraic model suggested, called quantum causal set, is physically interpreted as a locally finite, causal and quantal version of the kinematical structure of general relativity: the 4-dimensional Lorentzian space-time manifold and its continuous local orthochronous Lorentz symmetries. We discuss various possible dynamical scenarios for quantum causal sets mainly by using sheaf-theoretic ideas, and we entertain the possibility of constructing an inherently finite and genuinely $C^\infty$-smooth space-time background free quantum theory of gravity. At the end, based on the quantum causal set paradigm, we anticipate and roughly sketch out a potential future development of a noncommutative topology, sheaf and topos theory suitable for quantum space-time structure and its dynamics.

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The causal set approach to quantum gravity was initially proposed by Sorkin and coworkers almost a decade and a half ago [1]. At the heart of this theoretical scenario lies the proposal that the structure of space-time at quantum scales should be modelled after a locally finite partially ordered set (poset) of elements, the causal set, whose partial order is the small-scale correspondent of the relation that defines past and future distinctions between events in the space-time continuum of macroscopic physics. It is remarkable indeed that from so simple an assumption one can recover the basic kinematical features of the classical space-time manifold of general relativity, namely, its topological (i.e., $C^0$-continuous), its differential (i.e., $C^\infty$-smooth) and its conformal Lorentzian metric structures, as well as, in a statistical and scale dependent sense, its dimensionality. That alone should suffice for taking the causal set scenario seriously.

However, by abiding to a Wheeler-type of principle holding that no theory can qualify as a physical theory proper unless it is a dynamical theory, one could maintain that causal set theory would certainly be able to qualify as a physically sound theoretical scheme for quantum gravity if it somehow offered a plausible dynamics for causal sets. Thus, `how can one vary a locally finite poset?' has become the central question underlying the quest for a dynamical theory of causal sets [2]. To be sure, a classical stochastic sequential growth dynamics for causal sets, supported and guided by a discrete analogue of the principle of general covariance of general relativity, has been proposed recently [3] and it has been regarded as the stepping stone to a deeper quantum dynamics which, in turn, has already been anticipated to involve an as yet unknown `sum-over-causal set-histories' argument [4].

In this paper we briefly revisit a recently proposed algebraic picture of the causal set model of the kinematical structure of space-time in the quantum deep, which is coined quantum causal set and bears a sound quantum physical interpretation. Some possible dynamical scenarios for these finite dimensional noncommutative algebraic structures are sketched mainly along sheaf-theoretic lines. For expository fluency and continuity, we have decided to present the basic tenets of quantum causal set theory by means of a brief history of the central mathematical and conceptual developments that led to the quantum causal set idea. At the end we venture into some novel mathematical structures that are motivated by the quantum causal set paradigm, which are cumulatively referred to here as `noncommutative topologies', and we discuss how they may be applied to the problem of the quantum structure and dynamics of space-time.

The topological ancestors of causal sets are the so-called finitary substitutes of continuous (i.e., $C^0$) manifolds, which are $T_0$-topological spaces associated with locally finite open covers of a bounded region $X$ in a topological manifold space-time $M$, and which have the structure of posets [5]. To recall briefly how such posets arise, for $X$ a bounded region in the $C^0$-manifold $M$ and $\mathcal{U}$ a locally finite open cover of it, define the following preorder (i.e., reflexive and transitive) binary relation `$\rightarrow$' between its points

\[ x \rightarrow y \iff x \in \Lambda(y) \iff \Lambda(x) \subseteq \Lambda(y) \quad (1) \]

with $\Lambda(x)|_{\mathcal{U}} := \bigcap\{ U \in \mathcal{U} : x \in U \}$. The preorder `$\rightarrow$' becomes a partial order `$\leftrightarrow$' in the quotient space $P := X/\leftrightarrow$, where `$\leftrightarrow$' is the following equivalence relation between $X$'s points
\[ x \leftrightarrow y \Leftrightarrow (x \rightarrow y) \wedge (y \rightarrow x) \Leftrightarrow \Lambda(x)|_U = \Lambda(y)|_U \quad (2) \]

The relation \( x \rightarrow y \) in \( P \) (which \( P \) really consists of \( \leftrightarrow \)-equivalence classes \([x]\) of points in \( X \)) can be literally interpreted as the confluence of the constant sequence \((x)\) to \( y \) in the \( T_0 \)-topology of \( P \).

Such posets can be also viewed as simplicial complexes in the homological sense of Čech-Alexandroff as nerves of open coverings of manifolds, and the substitutions of \( X \) by them are regarded as cellular approximations of a \( C^0 \)-manifold—they are discretizations of the locally Euclidean topology of \( X \subset M \). Indeed, that they qualify as approximations proper of the topological space-time manifold rests on the fact that an inverse system of those posets ‘converges’, at the projective limit of infinite localization of \( X \)’s points by ‘infinitely small’ open sets about them, to a space that is homeomorphic to \( X \) itself.

Subsequently, the aforesaid finitary substitutes were represented by finite dimensional, complex, noncommutative, associative incidence Rota algebras and the resulting algebraic structures were interpreted quantum mechanically. The standard representation of a general poset \( P = (S, \rightarrow) \) by an incidence algebra \( \Omega \), where \( S \) is a set of elements and \( \rightarrow \) a reflexive, antisymmetric and transitive binary relation (ie, a partial order) between them, is given by

\[
\mathcal{P} = \{ p \rightarrow q : p, q \in S \} \longrightarrow \Omega(\mathcal{P}) := \text{span}_\mathbb{C}\{p \rightarrow q\},
(p \rightarrow q) \cdot (r \rightarrow s) = \begin{cases} p \rightarrow s & , \text{if } q = r \\ 0 & , \text{otherwise} \end{cases}
\quad (3)
\]

which depicts the defining \( \mathbb{C} \)-linear and (associative) multiplication structure of \( \Omega(\mathcal{P}) \), noting also that associativity is secured by the transitivity of \( \rightarrow \).

The interpretation in \( \mathcal{P} \) of the incidence algebras \( \Omega(P) \) associated with the finitary \( T_0 \)-posets \( P \) above as discrete quantum topological spaces rests essentially on the following four structural issues:

- In the Rota algebraic environment, the partial order arrow-connections \( \rightarrow \) between the elements of the posets, that actually define the aforementioned \( T_0 \)-topologies on them, can superpose coherently with each other. This possibility for coherent quantum interference of topological connections, which is encoded in the \( \mathbb{C} \)-linear structure of the incidence algebras, is manifestly absent from the corresponding posets which are merely associative multiplication structures (ie, arrow semigroups or monoids, or even small poset categories).

- The incidence algebras are noncommutative.

- The points extracted from these non-abelian algebras are, in a technical sense, quantum and so are the topological spaces that they constitute.

On the one hand, the qualification of points as being quantum comes from their being identified with the kernels of (equivalence classes of) irreducible \( \text{(} \end{align*} \) finite dimensional Hilbert space representations of the noncommutative incidence algebras, which kernels are in turn primitive ideals in those algebras. On the other hand, the (Rota) topology defined on these points can be thought of as being quantum, because in the very definition of its generating relation the noncommutativity of the algebras’ product structure is crucially involved. To recall briefly the relevant concepts and constructions which are quite standard in algebraic geometry, and which in the finitary context are cumulatively referred to as Gelfand spatialization, one first defines points in the incidence algebras, as follows

\[
\{ \text{points in } \Omega(\cdot) \} = \{ \text{kernels of irreps of } \Omega(\cdot) \} = \{ \text{primitive ideals in } \Omega(\cdot) \}
\]

Then one defines the generating relation ‘\( p' \)’ for the so-called Rota topology on the primitive spectrum \( \text{Spec}(\Omega) \) of \( \Omega \) in terms of \( \Omega \)’s noncommutative product, as follows

\[
I_x, I_y \in \text{Spec}(\Omega) : I_x \rho I_y \Leftrightarrow I_x \cdot I_y (\neq I_y \cdot I_x) \subseteq I_x \cap I_y
\quad (5)
\]

where \( I_x \) and \( I_y \) are \( (\text{kernels of}) \) irreducible \( \text{(finite dimensional Hilbert space matrix)} \) representations \( x \) and \( y \) of \( \Omega \). In \( \mathcal{P} \), \( I_x \cdot I_y \) is the product ideal, while \( I_x \cap I_y \) is the intersection ideal. For the incidence algebras \( \Omega(P) \) associated with Sorkin’s finitary \( T_0 \)-topological posets \( P \), one can identify the indices of the ideals \( I_x \) and \( I_y \) in \( \text{Spec}(\Omega(P)) \) with the vertices (points) \( x \) and \( y \) of \( P \) by defining

\[
I_x := \text{span}_\mathbb{C}\{x \rightarrow y : (x \rightarrow y) \neq (x \rightarrow x)\}
\quad (6)
\]

and verify that they are indeed ideals in \( \Omega(P) \). It can be shown that the Rota topology is the weakest one in which \( I_x \rho I_y \) implies the convergence \( x \rightarrow y \) of the point \( x \) to the point \( y \) in \( P \). This means that ‘\( p' \) is the transitive reduction of the partial order ‘\( \rightarrow \)’ defining the \( T_0 \)-topology on \( P \), ‘\( p' \) corresponds to the so-called covering relations in \( P \). For more about the Gelfand spatialization procedure and the non-standard Rota topology, the reader should refer to \( \mathcal{P} \).

Quantum points and topological spaces in the sense above have been studied from the more general and abstract perspective of mathematical structures.
representing ‘quantal topological spaces’ known as quantales. The latter were originally invented in order to formulate a noncommutative version of the usual Gelfand-Naimark representation theorem for abelian C*-algebras A involving their maximal spectra Max(A) (ie, the set of maximal ideals of A) which are general, pointless, classical (ie, commutative) topological spaces commonly known as locales.

- The incidence Rota algebras and their finite dimensional Hilbert space matrix representations have been seen to stand for sound finitary-algebraic models of space-time foam. The latter refers to the conception of the topology of space-time as a quantum observable: a quantally fluctuating, dynamically variable and in principle measurable structure.

At this point it must be stressed that the incidence algebras associated with Sorkin’s finitary topological posets were shown to encode combinatorial information not only about the topology of the space-time manifold, but also about its differential structure. In fact, they were seen to be Z_+ -graded discrete differential manifolds in the sense that every Ω(P) splits into the following direct sum of linear subspaces

\[ Ω(P) = Ω^0 ⊕ Ω^1 ⊕ Ω^2 ⊕ \cdots \]

\[ Ω^i := \text{span}_C \{ p \xrightarrow{\min} q : \deg(p \xrightarrow{\min} q) = i \in \mathbb{N} \} \]  

(7)

where \( \deg(p \xrightarrow{\min} q) \) is the length of the shortest path \( p \xrightarrow{\min} q \) connecting \( p \) with \( q \) (\( \forall p, q \in P \)) in the Hasse diagram of \( P \) and it corresponds to the homological degree when the \( Ps \) are viewed as simplicial complexes (nerves) as alluded to above.

\( Ω^0 \) is an abelian subalgebra of \( Ω \) called the commutative algebra \( A \) of coordinates in \( Ω \), while \( R := \bigoplus_{i \geq 1} Ω^i \) is called the A-module of discrete differential forms in \( Ω \). There is also a nilpotent Kähler-Cartan differential operator \( d \), which is defined in terms of the homological boundary \( δ \) and coboundary \( δ^* \) operators when the \( Ps \) are viewed as simplicial complexes, satisfying a graded Leibniz rule and effecting linear maps of the following sort

\[ d : Ω^i \longrightarrow Ω^{i+1} \]  

(8)

But for more details the reader may again have to refer to [12] and references therein.

In totto, the \( Ω(P) \)s have been physically interpreted as the reticular substitutes of bounded regions \( X \) of the \( C^\infty \) smooth space-time manifold \( M \) of macroscopic physics. Moreover, it has been argued that, similarly to the case of the finitary posets, an inverse system of incidence algebras yields at the projective limit of infinite localization of \( X \) about its points the commutative algebra \( A = C^\infty(X) \) of smooth complex-valued coordinates labelling \( X \)’s point events (as it were, \( A \) ‘converges’ to \( A \)), as well as the \( Z_+ \)-graded \( A \)-bimodule \( ^A Ω \) of smooth complex differential forms cotangent to every point event in the differential manifold space-time (as it were, \( R \) ‘converges’ to the module of complex smooth exterior forms over \( A \)) [11,12,13].

At the same time, the aforementioned recovery of the classical smooth space-time manifold, in the ideal limit of infinite localization, from the quantal incidence algebraic substrata was interpreted physically as Bohr’s correspondence principle, or equivalently, as a classical limit, and it represented the emergence of the classical local differential space-time macrostructure from the ‘decoherence’ of (ie, the breaking of coherent quantum superpositions in) an ensemble of these characteristically alocal quantum space-time structures [13]. Hence, the finite dimensional incidence algebras associated with Sorkin's finitary poset substitutes of the space-time continuum encode information about both the topological (\( C^0 \)) and the differential (\( C^\infty \)) structure of the manifold. Arguably, this is an algebraic representation of Sorkin et al.’s thesis mentioned earlier in connection with causal sets that a partial order effectively determines both the topological and the differential structure of the space-time continuum.

On the other hand, in quite a dramatic change of physical interpretation and philosophy nicely recollected in [12], Sorkin stopped thinking of the locally finite posets above as representing finitary discretizations or simplicial decompositions (with a strong operational flavor) of the space-time continuum and, as noted in the opening paragraph, he regarded them as causal sets. Then, he and coworkers posited that the deep structure of space-time is, in reality, a causal set [1,4]. Thus, partial orders stand now for causal relations between events in the quantum deep and not for topological relations proper, as it were, the original ‘spatial’ conception of a poset gave way to a more ‘temporal’ one. Of course, the ideas to model causality after a partial order and that at both the classical and the quantum level of description of relativistic space-time structure, partial order as causality is a more physical notion than partial order as topology have a long and noble ancestry [11]. So, as a consequence of Sorkin’s ‘semantic switch’, the macroscopic space-time manifold should be thought of as a coarse approximation of the fundamental causal set substrata in contrast to the finitary topological posets which, as it was noted earlier, were regarded as being coarse approximations of the space-time continuum. Also, the local finiteness of the open covers of the bounded region \( X \) in the continuous space-time manifold \( \bar{M} \) relative to which the topological posets were defined in [1] was replaced by the slightly different local finiteness of the causal sets, although the underlying mathematical structure, the poset, remained the same [12].

In the sequel, after having replaced the finitary topological posets \( P \) by causal sets \( \bar{P} \) as the truly fundamental structures underlying the classical space-time mani-
fold of macroscopic experience, it was a rather natural step to associate with the latter incidence Rota algebras \(\Omega\) and, like in the case of the incidence algebras \(\Omega(P)\) representing discrete quantum topological spaces \([7,14]\), to interpret the resulting structures as *quantum causal sets* \([14]\). Hence, in the new non-abelian algebraic realm of quantum causal sets, the causal arrows \(\rightarrow\) defining their causal set correspondents can superpose or interfere quantum mechanically with each other. This lies at the heart of the conception of *quantum causality* and qualifies the epithet ‘quantum’ in front of ‘causal sets’ used here \([14]\). Here too, quantum causal sets, like their causal set counterparts, stand for locally finite, causal and *quantal* models of the kinematics of the classical space-time of general relativity.

Having in hand sound reticular and quantal models of the kinematics of gravitational space-time, and by abiding to the aforesaid Wheeler-type of criterion for a physical theory, the next stage in the development of quantum causal sets was the search for a plausible dynamics for them. The basic idea in \([15]\) was that in order to construct a dynamical theory of quantum causality, and since Zeeman had shown that causality as a partial order determines the conformal metric structure of the flat (ie, non-dynamical) Minkowski space of special relativity and its global orthochronous Lorentz symmetries up to conformal transformations \([13]\), one should somehow look for a scenario to *curve quantum causality*. Then, the main intuition, principally motivated by the fact that the curved space-time of general relativity is Minkowski space localized or gauged, was that we could possibly arrive at dynamical variations of quantum causal sets by localizing (or gauging) them and, concomitantly, by mathematically implementing this localization process sheaf-theoretically. In other words, a possible reply to Sorkin’s question ‘how can one vary a poset?’ mentioned earlier is ‘by sheaf-theoretic means’ \([14]\). To this end, we first needed to adapt basic sheaf-theoretic notions and constructions to a locally finite setting. With that need in mind, *finitary space-time sheaves* of (algebras of) continuous observables on Sorkin’s finitary substitutes were first defined in \([17]\) as *étalés* spaces that are *locally homeomorphic* to those \(T_0\)-posets \(P\)—the latter serving as the topological base spaces for the sheaves.

Shortly after, and in a way analogous to the aforesaid transition from finitary substitutes to causal sets and then to quantum causal sets, *finitary space-time sheaves of the incidence algebras modelling quantum causal sets* were defined in \([17]\) along the lines of Mallios’ Abstract Differential Geometry (ADG). The latter pertains to the geometry of vector, (differential) module and algebra sheaves which generalizes the usual differential calculus on \(C^\infty\)-manifolds to an abstract differential geometry on, in principle, *any* (topological) base space \([18]\). *Principal finitary space-time sheaves of quantum causal sets* over the causal sets \(\mathcal{F}\) of Sorkin *et al.*, having for structure group reticular versions of the continuous local orthochronous Lorentz group manifold \(L^+ = SO(1,3)\) of general relativity, together with non-flat finitary \(\ell^+ = so(1,3)^\ast \simeq sl(2,\mathbb{C})\)-valued spin-Lorentzian connection 1-forms \(A\), were seen to be sound models of a locally finite, causal and quantal version of the kinematics of classical Lorentzian gravity in its gauge-theoretic formulation in terms of the spin-Lorentzian connection \(A\) on a principal fiber bundle over the \(C^\infty\)-smooth space-time manifold \(M\) having \(L^+\) as its structure group. Furthermore, finitary-algebraic versions, of strong categorical (ie, functorial) character, of the principles of locality, equivalence and general covariance were formulated entirely in sheaf-theoretic terms mainly due to the fact that the \(\mathcal{A}\)s were represented by suitable *finitary space-time sheaf morphisms*.

The next step in the development of quantum causal set theory was to carry the entire de Rham theory of differential forms on the classical \(C^\infty\)-smooth space-time manifold, virtually unaltered, to the reticular realm of the curved finitary space-time sheaves of quantum causal sets with the help of some rather ‘universal’ sheaf-theoretic techniques, mainly based on sheaf cohomology, borrowed from ADG \([10]\). One of the main achievements of that transcription was the *sheaf-cohomological classification of the non-flat reticular spin-Lorentzian connections* \(A\) the quanta of which, originally coined *causons* in \([14]\), were taken to represent *dynamical quanta of causality*—elementary particle-like entities, anticipated to be closely related to gravitons, that dynamically propagate in a discrete manner in the ‘*curved quantum space-time vacuum*’ represented by the curved finitary space-time sheaves of quantum causal sets.

We also showed how basic differential geometric ideas and results usually thought of as being vitally dependent on \(C^\infty\)-smooth manifolds for their realization, as for example the standard de Rham cohomology, carry through unchanged to the finitary regime of the curved finitary space-time sheaves of quantum causal sets. For instance, we gave reticular versions of central \(C^\infty\)-theorems such as de Rham’s, Weil’s integrality, as well as the Chern-Weil theorem. By this essentially complete transcription of the basic \(C^\infty\)-constructions, concepts and results to the locally finite and quantal regime of the curved finitary sheaves quantum causal sets, we highlighted that for the formulation of fundamental differential geometric notions the classical smooth background space-time continuum is of no significantly contributing value. Quite on the contrary, we argued that since the \(C^\infty\)-smooth space-time manifold can be regarded as the main culprit for the singularities that plague general relativity and the weaker but still troublesome infinities that assail the flat quantum field theories of matter, its evasion—especially by the finitistic-algebraic means that we employed—should be most welcome for the formulation of a ‘calculationally’ and, in a sense to be explained below, ‘heterenly’ finite and \(C^\infty\)-smooth space-time background independent quantum theory of gravity.

However, it is fair to say that, the aforementioned
kinematical developments aside, an explicit dynamics for quantum causal sets has not been proposed yet. The formulation of a finitary, causal and quantal analogue of the classical vacuum Einstein equations for gravity, which in turn can be interpreted as the dynamical equations for the reticular sort of Ashtekar’s (self-dual) spin-Lorentzian connection variable $\mathcal{A}$ inhabiting the relevant curved principal finitary space-time (line) sheaves of quantum causal sets, is currently under way again along the lines of ADG [20]. A ‘covariant’ path integral over the space $\mathcal{A}$ of connections $\mathcal{A}$, or an even more finitistic, as befits our discrete models, ‘sum-over-quantum causal set-histories’ scenario appears to be prima facie more well suited for such a quantum dynamics than a canonical (ie, Hamiltonian) approach mainly due to the characteristic absence in the innately granular realm of the causal or the quantum causal sets of (a generator of) infinitesimal time increments $\text{d}t$. This also seems to accord with Sorkin et al.’s anticipation noted earlier of a ‘sum-over-causal set-histories’ dynamical scheme for causal sets. Moreover, we anticipate that precisely because of the discrete character of the spin-Lorentzian connections dwelling on the principal finitary space-time sheaves of the finite dimensional quantum causal sets, the problem of finding a well-defined measure for the aforesaid path integral over the infinite dimensional coset space (orbifold or ‘moduli space’) $[\mathcal{A}]/{\text{Diff}(M)}$ of diffeomorphism-equivalent connections defined on the $C^\infty$-smooth space-time continuum, may simply disappear in our finitary model.

Another possible quantum dynamics for quantum causal sets, now perhaps one with a more canonical flavor, arises when, instead of working with the individual incidence algebras representing quantum causal sets per se, one works with the ‘universal’ incidence algebra of all finitary incidence algebras that are partially ordered by inclusion $\subseteq$—again itself a poset representing ‘quantum causal refinement’ [15]. This poset may be thought of as the incidence algebraic analogue of the lattice of all (finite) topologies studied in [21] in the context of quantum topology, and it may be interpreted as some kind of kinematical state space for quantum causal sets. One could then try to define suitable generalized position and momentum-like observables of quantum causal topology à la Isham, impose some non-trivial canonical commutation relations on them, and look for a Hamiltonian-like operator in terms of them that effects contiguous dynamical transitions between quantum causal sets, as it were, dynamical quantum (causal) topology changes in the universal incidence algebra. This could also be done ‘by hand’, that is, one could first define creators and annihilators of vertices in causal sets, look for their incidence algebraic correspondents in the associated quantum causal sets and their finite dimensional Hilbert space representations as in [6], then consider their ‘canonical’ commutation relations, and finally proceed in a Hamiltonian fashion to implement dynamical quantum causal topology changes as briefly alluded to above [22].

In parallel with these canonical scenarios, one could also try to define up-front the notion of quantum causal histories [23] and the finitary sheaves thereof in the universal incidence algebra [24], and proceed in a consistent-histories-like fashion to look for a quantum measure for quantum causal histories, relative to which the ‘sum-over-quantum causal set-histories’ is going to be weighed, in the form of some kind of decoherence functional on them [4]. Alternatively, one could seek directly for a discrete quantum causal dynamics and for functorial expressions of discrete causal covariance and quantum entanglement by applying ideas from linear logic and the theory of polycategories to quantum causal histories [25].

Certainly though, and in spite of the prominent absence of an explicit quantum dynamics, whether covariant, canonical or other, for either the causal or the quantum causal sets, the finitary-algebraic and sheaf-theoretic approach to quantum space-time structure supporting quantum causal set theory has provided us with strong clues on how to construct an intrinsically finite quantum gravity that is genuinely $C^\infty$-smooth space-time background independent. For one thing, it has shown us that one can define the usual differential geometric objects and carry out the standard smooth constructions that are of vital importance for the formulation of general relativity, such as connection, curvature and de Rham cohomology to name a few, literally without the use of any Calculus and over reticular base spaces that may appear to be unmanageably singular and incurably pathological from the perspective of the classical $C^\infty$-smooth space-time manifold. This is one of the benefits we get from applying ADG to the finitary-algebraic setting, for ADG has time and again proven itself when it comes to evading $C^\infty$-smoothness and the singularities that go with it [19,20,26]. If anything, such an independence of quantum causal set theory (and of a future dynamics for it) from the smooth space-time continuum is more than welcome from the point of view of modern research in (canonical) quantum gravity, since the theory appears to avoid ab initio the infinite dimensional diffeomorphism group $\text{Diff}(M)$ of general relativity, which comes hand in hand with the $C^\infty$-smoothness of the space-time manifold $M$, and the two notorious problems associated with it, namely, the ‘inner product problem’ and the ‘problem of time’ [27], as well as the path integral measure problem in the $\text{Diff}(M)$-covariant approach mentioned earlier.

We conclude the present review by elaborating on how the quantum causal set model and its sheaf-theoretic representations may substantially motivate the development of a noncommutative topology and an associated sheaf as well as topos theory for quantum space-time structure and its dynamics [28]. We first note that it seems theoretically short-sighted and lame to think that only a high-level structure such as the geometry of space-time should be subject to dynamics and quantization, while that a deeper structure like its topology should be fixed once and forever by the theoretician to the immutable classical continuous manifold. As noted earlier, the incidence Rota algebras which have been employed to model
quantum causal sets have also been used to represent space-time foam—Wheeler’s original insight that not only the geometry but also the topology of space-time should partake into dynamical changes and coherent quantum superpositions [3].

It must be also emphasized that it appears unreasonably limited to possess a well developed noncommutative differential geometry [29], enjoying numerous applications to quantum space-time and gravity, while the topology, which again is a structure deeper than the differential, to be treated essentially commutatively (classically) [30]. Another possible way to arrive at a suitable noncommutative topology for quantum space-time and its dynamics, now by using the sheaf-theoretic localizations of quantum causal sets as in [13], could be the following: Finkelstein and coworkers, as part of an ongoing effort to find a quantum replacement for the space-time manifold of macroscopic physics, have developed a theory of quantum sets, which in a sense represents a quantization of ordinary ‘classical’ set theory [31]. The basic idea is that space-time at small scales should really be viewed as a ‘quantum’ set, not a classical one. This is supposed to be a step on the path to a ‘correct’ version of quantum space-time topology and quantum gravity. Now, a question which may occur to a modern logician (or topos theorist) is the following:

what is so special about the category Set of classical sets, since there are other logical universes just as good, and possibly better, namely ‘topoi’ [32].

Perhaps it would be a better idea to try and quantize these more general categories, since the use of Set may be prey to classical chauvinism. That is, the underlying logical model may turn out to be some kind of ‘quantum topos’ [33] different from the category of sets whose existence is after all predicated upon the preconceptions of macroscopic thinkers.

For, arguably, all the usual flat classical and quantum field theories of matter are conveniently formulated in Set, or more precisely, in Shv(M)—the ‘classical’ topos of sheaves of sets over the classical space-time manifold M [22,33]. However, as noted earlier, these continuum theories suffer from non-renormalizable infinities coming from singularities that plague the classical smooth manifold M. These unphysical infinities are on the one hand due to the fact that one can pack an uncountable infinity of events into a finite space-time volume, and on the other, due to the geometric point-like nature of the latter. The manifold model, as an inert classical geometric background stage on which fields propagate and interact, must at least be revised in view of the pathological nature of quantum gravity when treated as another quantum field theory. Topoi and their relatives, locales, which are pointless topological spaces modelling topological theories about regions rather than points [34], are structures well-suited not to significantly commit themselves to the pathological geometric point-like character of a base space-time manifold. Perhaps one could arrive at the ‘true’ quantum topos of Nature, on which a finite quantum theory of gravity can be founded, by considering the pointless topos of the curved finitary space-time sheaves of quantum causal sets instead of its classical continuum relative Shv(M). In the same way that Shv(M) may be thought of as a universe of continuously variable sets, so the topos of sheaves of quantum causal sets may be thought of as a universe of dynamically variable quantum causal sets—ones that vary due to a locally finite, causal and quantal version of Lorentzian gravity.

Moreover, and in connection with our anticipation of the development of a noncommutative topology and, as a result, of a noncommutative sheaf theory adapted to it, the aforesaid quantum topos of finitary sheaves of quantum causal sets is also expected to provide a model of the missing structure in the following analogy that has puzzled mathematicians for quite some time now [34]:

\[
\text{locales } \quad \text{quantales } \quad \text{topoi } \quad ?
\]

For we have seen how the building of the Rota topology on the quantum causal sets is akin to the definition of noncommutative topological spaces par excellence, namely, quantales [1], and there is a strong feeling among mathematicians that the category of sheaves over a quantale is a quantum version of the archetypal topos—the category of sheaves over a locale [22,33].

Further to our noncommutative topological and sheaf-theoretic epilogue, we mention that, since the incidence algebras modelling quantum causal sets are graded non-abelian Polynomial Identity (PI) rings, it would in principle be possible to develop a noncommutative sheaf or scheme type of theory for the non-abelian PI-ring (and differential module) localizations in [22,32,33,28]. Rigorous mathematical results, cast in a general categorical setting, from the non-commutative algebraic geometry of similar non-abelian schematic algebras and their localizations [36] are expected to deepen our physical understanding of the dynamically variable non-commutative quantum causal Rota topologies defined on the primitive spectra of quantum causal sets [22].

However, such a possible noncommutative topological, scheme and topos-theoretic application of quantum causal sets and their sheaves to the problem of quantum gravity is still at its birth [15].

We close the present paper with some remarks of Steve Selesnick [28] about the importance of developing a non-commutative topological and sheaf-theoretic perspective on quantum space-time structure and gravity:

...One of the primary technical hurdles which must be overcome by any theory that purports to account, on the basis of microscopic quantum principles, for macroscopic effects (such as the large-scale structure of what appears to us as space-time, ie, gravity) is the handling of
the transition from ‘localness’ to ‘globalness’. In the ‘classical’ world this kind of maneuver has been traditionally effected either measure-theoretically—by evaluating largely mythical integrals, for instance—or geometrically, through the use of sheaf theory, which, surprisingly, has a close relation to topos theory. The failure of integration methods in traditional approaches to quantum gravity may be ascribed in large measure to the inappropriateness of maintaining a manifold—a ‘classical’ object—as a model for space-time, while performing quantum operations everywhere else. If we give up this classical manifold and replace it by a quantal structure, then the already considerable problem of mediating between local and global (or micro and macro) is compounded with problems arising from the appearance of subtle effects like quantum entanglement, and more generally by the problems arising from the non-objective nature of quantum ‘reality’. Although there is a rich and now highly developed mathematical theory of ‘noncommutative geometry’ (which has had considerable success in application to traditional quantum field theories), a concomitant noncommutative sheaf theory seems to have been slow in coming...

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[1] L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, Phys. Rev. Lett. 59, 521 (1987); R. D. Sorkin, in Proceedings of the Third Canadian Conference on General Relativity and Relativistic Astrophysics, edited by F. Cooperstock and B. Tupper (World Scientific, Singapore 1990); in Relativity and Gravitation: Classical and Quantum, edited by J. D’Olivo et al. (World Scientific, Singapore 1991), and Int. J. Theor. Phys. 36, 2759 (1997).
[2] Rafael Sorkin in private communication.
[3] D. P. Rideout and R. D. Sorkin, Phys. Rev. D 61, 024002 (2000).
[4] Two results anticipated to come from the formulation of a quantum dynamics for causal sets would be the derivation of the law for the entropy of the horizon of a black hole, as well as an estimate and an explanation of the (small) value of the cosmological constant (R. D. Sorkin, research seminar, Imperial College 4/12/2001).
[5] R. D. Sorkin, in General Relativity and Gravitation, Eds. B. Bertotti et al. (CNR, Roma 1983), and Int. J. Theor. Phys. 30, 923 (1991). Of these two references, the second is the definitive and complete treatment of finitary substitutes of continuous topological spaces.
[6] I. Raptis and R. R. Zapatrin, Int. J. Theor. Phys. 39, 1 (2000); R. R. Zapatrin, Pure Math. Appl. (2002) (to appear). [math.CO/0001063]
[7] I. Raptis and R. R. Zapatrin, Class. Quant. Grav. 20, 4187 (2001).
[8] Gelfand spatialization is a particular instance of Gelfand duality—the general ‘functional philosophy’ whereby, loosely speaking, ‘the variable becomes function and the function variable’; one writes: \( f(x) \rightarrow \hat{f} \) (Tasos Mallios in private communication). In the context of incidence Rota algebras (modelling finitary topological spaces, not quantum causal sets), Gelfand duality and, in particular, spatialization, was first applied in R. R. Zapatrin, Int. J. Theor. Phys. 37, 799 (1998). In the latter paper one also encounters for the first time finite dimensional Hilbert space matrix representations of incidence algebras \( \Omega \) associated with Sorkin’s finitary topological posets \( P \).
[9] C. J. Mulvey, Suppl. Rend. Circ. Mat. Palermo II (12), 99 (1986); C. J. Mulvey and J. W. Pelletier, J. Pure Appl. Algebra 159, 231 (2001), and On the Quantisation of Spaces, J. Pure Appl. Algebra (2002) (to appear). Quantales and their classical (commutative) analogues, locales, will be mentioned again towards the end when we discuss the possibility of developing a noncommutative topology and its corresponding sheaf (and topos) theory suitable for quantum space-time structure and dynamics.
[10] Starting from incidence algebras over \( \mathbb{C} \), complex smooth coordinate algebras and modules of differential forms over them are expected to emerge at the projective limit of infinite localization. To recover the real space-time continuum of general relativity, some sort of reality conditions must be imposed after the limit, the details of which have not been explored yet. However, perhaps starting from incidence algebras over \( \mathbb{R} \) (and their real Hilbert space representations) would immediately remedy this, but then we would not be faithful to the conventional quantum theory with its continuous superpositions over \( \mathbb{C} \) (i.e., in a complex Hilbert space). On the other hand, altogether it seems to be begging the question to claim that we have an inherently discrete model for quantum space-time structure when its algebraic representation employs \( ab \text{ initio} \) the continuum of complex numbers (or the reals \( \mathbb{R} \) to the same effect) as the field of amplitudes (\( \mathbb{c} \)-numbers). Perhaps the use of one of the discrete prime number fields \( \mathbb{Z}_p \) \( (p \text{ a prime integer}) \) would be more suitable to our reticular models, but then again, what kind of quantum theory can one make out of them?
[11] In connection with the Gelfand duality mentioned in [3], we note that the \( \Omega \)s are objects dual to Sorkin’s topological posets in the sense that there is a contravariant functor from the category \( \mathcal{P} \) of finitary posets and poset morphisms (monotone maps which are continuous as they
respect $P$s' $T_0$-topology), and the category $\mathfrak{R}$ of Rota algebras and incidence algebra homomorphisms $\mathfrak{R}$.

This duality is reflected in the localization processes of the $P$s and the $\mathfrak{R}s$ in that with the former a projective limit (categorical limit) is involved (on an inverse system of posets), while with the latter an inductive limit (categorical colimit) is employed (on a direct system of incidence algebras); see [13] below.

[12] R. D. Sorkin, in *The Creation of Ideas in Physics*, Ed. J. Leplin (Kluwer, Dordrecht 1995).

[13] To recall a few references: A. A. Robb, “Geometry of Space and Time”, Cambridge University Press, Cambridge (1914); E. C. Zeeman, J. Math. Phys. 5, 490 (1964); A. D. Alexandrov, Can. J. Math. 19, 1119 (1967); D. Finkelstein, Phys. Rev. 184, 1261 (1969).

[14] I. Raptis, Int. J. Theor. Phys. 39, 1233 (2000).

[15] A. Mallios and I. Raptis, Int. J. Theor. Phys. 40, 1885 (2001).

[16] Tasos Mallios in private communication.

[17] I. Raptis, Int. J. Theor. Phys. 39, 1703 (2000).

[18] A. Mallios, “Geometry of Vector Sheaves: An Axiomatic Approach to Differential Geometry”, vols. 1-2, Kluwer, Dordrecht (1998), Math. Japonica (International Plaza), 48, 93 (1998), and “Gauge Theories from the Point of View of Abstract Differential Geometry”, 2-volume continuation of the 2-volumes book above (2002) (in press).

[19] A. Mallios and I. Raptis, Int. J. Theor. Phys. (2002) (to appear), gr-qc/0110033.

[20] A. Mallios and I. Raptis, *Finitary, Causal and Quantal Vacuum Einstein Gravity* (2002) (in preparation).

[21] C. J. Isham, Class. Quant. Grav. 6, 1509 (1989).

[22] Chris Isham in private communication.

[23] F. Markopoulou, Class. Quant. Grav. 17, 2059 (2000).

[24] I. Raptis (2001), quant-ph/0107037.

[25] R. Blute, E. T. Ivanov, and P. Panangaden (2001), gr-qc/0109053, and (2001), gr-qc/0111020.

[26] A. Mallios and E. E. Rosinger, Acta Appl. Math. 55, 231 (1999), and Acta Appl. Math. 67, 59 (2001); A. Mallios, in *Unresolved Problems in Mathematics for the 21st Century: a tribute to Kiyoshi Iseki’s 80th birthday*, IOS Press, Amsterdam (2001), and gr-qc/0202028.

[27] The ‘inner product problem’ is the problem of fixing the inner product in the Hilbert space of states by requiring that it is invariant under $\text{Diff}(M)$, while the ‘problem of time’ is the problem of requiring that the dynamics is encoded in the action of $\text{Diff}(M)$ on the space of states.

[28] Always keeping in mind however our following Sorkin’s semantic switch mentioned before, namely, that the noncommutative incidence algebras that we use have a direct (quantum) causal (ie, ‘temporal’ or time-like) rather than merely a topological (ie, ‘spatial’ or space-like) physical interpretation, although originally the latter was our aim in applying Rota algebras $\mathfrak{R}$. For instance, and this distinction cannot be overemphasized, while Sorkin [13] (Int. J. Theor. Phys. 30, 923 (1991)) notes that the finitary $T_0$-topological posets can be used to model some sort of ‘thickened space-like hypersurfaces in the continuous space-time manifold’, we have explicitly held in gr-qc/0110033 that our incidence Rota algebras are (quantum) discretizations of the causal topology of continuous space-time.

See I. Raptis (2001), gr-qc/0110082, for some preliminary remarks on noncommutative topology vis-à-vis our non-abelian incidence algebra sheaf-theoretic localizations modelling ‘curved quantum causality’ [13] (see also the concluding paragraphs about topos, quantales and noncommutative Pl-schemes).

[29] A. Connes, “Noncommutative Geometry”, Academic Press, New York (1994).

[30] Here, the epithet ‘commutative’ (hence, also ‘classical’) before the noun ‘topology’ has the following technical meaning: the usual topological spaces, abstractly referred to as locales (ie, complete distributive lattices), have the standard intersection of open sets ‘∩’ as their meet operation ‘∧’. On the other hand, noncommutative topologies employ a noncommutative intersection for their lattice meet operation. An example of a noncommutative topological space is the non-abelian $C^*$-quantale mentioned before [13]. There, ‘∧’ is noncommutative, because it derives from the noncommutative product in the underlying non-abelian $C^*$-algebra. In [13] we similarly saw how the noncommutative product enters in the very definition of the generating relation ‘ρ’ of the Rota topology. In this (weaker) sense, our non-abelian incidence Rota algebras may also be regarded as noncommutative (causal) topologies (see reference to I. Raptis in [29]).

[31] D. R. Finkelstein, “Quantum Relativity: A Synthesis of the Ideas of Einstein and Heisenberg”, Springer-Verlag, New York (1996): for further elaborations on Finkelstein’s quantum set theory and relativity along mathematical lines more similar to the ones presented here, see S. A. Selesnick, “Quanta, Logic and Spacetime: Variations on Finkelstein’s Quantum Relativity”, World Scientific, Singapore (1998).

[32] S. Mac Lane and I. Moerdijk, “Sheaves in Geometry and Logic: A First Introduction to Topos Theory”, Springer-Verlag, New York (1992).

[33] I. Raptis, “Axiomatic Quantum Timespace Structure: A Preamble to the Quantum Topos Conception of the (Minkowski) Vacuum”, Ph.D. Thesis, University of Newcastle upon Tyne, UK (1998).

[34] Jim Lambek in private communication. In view of the analogy [13], the caustic issue that must be resolved in order to arrive at the ‘right’ quantum topos structure for quantum space-time and gravity, is what is the ‘appropriate’ quantum version of the so-called subobject classifier object $\Omega$ in a ‘classical’ topos like $\text{Shv}(M)$. In the latter category, $\Omega$ is a Heyting algebra—a complete distributive lattice which, topologically speaking, is equivalent to a locale. This reflects the well known motto in classical topos theory that a (localic) topos, regarded as a generalized (pointless) topological space, is ‘locally’ a locale (equivalently, that its so-called internal logic is intuitionistic) [13]. This is one way in which $\text{Shv}(M)$, in contradistinction to $\text{Set}$, may be viewed as a universe of variable sets. Now, with [13] in mind, it is reasonable to expect that the envisioned quantum topos will turn
out to be, topologically speaking, ‘locally quantalic’ as it will consist object-wise of a sheaf over a quantale—a noncommutative topological space (see next). The hunting for the Ω in the elusive quantum topos for quantum space-time and gravity is still on; however, some anticipating remarks about the quantum Ω can already be found in.

[35] F. Borceaux and G. Van den Bossche, Order, 3, 61 (1986), and Topology and its Applications, 31, 203 (1989). Here, a ‘quantum topos’ is anticipated to be modelled after the category of sheaves over a quantale—the latter lattice being essentially endowed with a Grothendieck-type of topology, so that sheaves over the resulting site constitute in effect a Grothendieck topos. In this sense, the quantum topos may be viewed as a Grothendieck-type of topos.

[36] Freddy Van Oystaeyen in private communication, and in “Is the Topology of Nature Noncommutative?” and “A Grothendieck-type of Scheme Theory for Schematic Algebras”, research seminars given at the Mathematics Department of the University of Pretoria, Republic of South Africa on 2-3/2/2000 and at the Blackett Laboratory, Imperial College, London, United Kingdom on 2-3/12/2001 (notes available). Some pertinent references on noncommutative algebraic geometry are: F. Van Oystaeyen and A. Verschoren, “Non-Commutative Algebraic Geometry”, LNM, 887, Springer-Verlag, Berlin-Heidelberg (1981), and F. Van Oystaeyen, “Algebraic Geometry for Associative Algebras”, Marcel Dekker, New York (2000). In the second reference, a full-fledged noncommutative topology, different from the ‘C∗-quantalic’ ones briefly mentioned above, is developed and it may be viewed as the topological precursor to Connes’ noncommutative (differential) geometry.

[37] I. Raptis, Int. J. Theor. Phys. (2002) (to appear), gr-qc/0110064. See also J. Butterfield and C. J. Isham, Found. Phys., 30, 1707 (2000), gr-qc/9910003 for some alternative possible applications of topos theory to quantum gravity.

[38] Steve Selesnick in private communication.