Determinantal Point Processes, 
Stochastic Log-Gases, and Beyond *

Makoto Katori †

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Abstract

A determinantal point process (DPP) is an ensemble of random nonnegative-integer-valued Radon measures, whose correlation functions are all given by determinants specified by an integral kernel called the correlation kernel. First we show our new scheme of DPPs in which a notion of partial isometries between a pair of Hilbert spaces plays an important role. Many examples of DPPs in one-, two-, and higher-dimensional spaces are demonstrated, where several types of weak convergence from finite DPPs to infinite DPPs are given. Dynamical extensions of DPP are realized in one-dimensional systems of diffusive particles conditioned never to collide with each other. They are regarded as one-dimensional stochastic log-gases, or the two-dimensional Coulomb gases confined in one-dimensional spaces. In the second section, we consider such interacting particle systems in one dimension. We introduce a notion of determinantal martingale and prove that, if the system has determinantal martingale representation (DMR), then it is a determinantal stochastic process (DSP) in the sense that all spatio-temporal correlation function are expressed by a determinant. In the last section, we construct processes of Gaussian free fields (GFFs) on simply connected proper subdomains of $\mathbb{C}$ coupled with interacting particle systems defined on boundaries of the domains. There we use multiple Schramm–Loewner evolutions (SLEs) driven by the interacting particle systems. We prove that, if the driving processes are time-changes of the log-gases studied in the second section, then the obtained GFF with multiple SLEs are stationary. The stationarity defines an equivalence relation of GFFs, which will be regarded as a generalization of the imaginary surface studied by Miller and Sheffield.

Keywords  Determinantal point processes · Partial isometries and dualities · Stochastic log-gases · Determinantal stochastic processes · Determinantal martingale representations · Gaussian free fields · multiple Schramm–Loewner evolutions · Imaginary surfaces

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†Department of Physics, Faculty of Science and Engineering, Chuo University, Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan; e-mail: katori@phys.chuo-u.ac.jp
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1 Determinantal Point Processes (DPPs)

A determinantal point process (DPP) is an ensemble of random nonnegative-integer-valued Radon measures \( \Xi \) on a space \( S \) with measure \( \lambda \), whose correlation functions are all given by determinants specified by an integral kernel \( K \) called the correlation kernel. We consider a pair of Hilbert spaces, \( H_\ell, \ell = 1, 2 \), which are assumed to be realized as \( L^2 \)-spaces, \( L^2(S_\ell, \lambda_\ell) \), \( \ell = 1, 2 \), and introduce a bounded linear operator \( W : H_1 \to H_2 \) and its adjoint \( W^* : H_2 \to H_1 \). We show that if \( W \) is a partial isometry of locally Hilbert–Schmidt class, then we have a unique DPP on \( (\Xi_1, K_1, \lambda_1) \) associated with \( W^*W \). In addition, if \( W^* \) is also of locally Hilbert–Schmidt class, then we have a unique pair of DPPs, \( (\Xi_\ell, K_\ell, \lambda_\ell), \ell = 1, 2 \).

We also give a practical framework which makes \( W \) and \( W^* \) satisfy the above conditions. Our framework to construct pairs of DPPs implies useful duality relations between DPPs making pairs.
For a correlation kernel of a given DPP our formula can provide plural different expressions, which reveal different aspects of the DPP.

In order to demonstrate these advantages of our framework as well as to show that the class of DPPs obtained by this method is large enough to study universal structures in a variety of DPPs, we report plenty of examples of DPPs in one-, two-, and higher-dimensional spaces $S$, where several types of weak convergence from finite DPPs to infinite DPPs are given.

This section is based on the collaborations with Tomoyuki Shirai (Kyushu University) [53].

### 1.1 Definition and existence theorem of DPP

Let $S$ be a base space, which is locally compact Hausdorff space with countable base, and $\lambda$ be a Radon measure on $S$. The configuration space over $S$ is given by the set of nonnegative-integer-valued Radon measures:

$$\text{Conf}(S) = \left\{ \xi = \sum_{j} \delta_{x_j} : x_j \in S, \xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset S \right\}.$$  

Conf$(S)$ is equipped with the topological Borel $\sigma$-fields with respect to the vague topology; we say $\xi_n, n \in \mathbb{N} := \{1, 2, \ldots\}$ converges to $\xi$ in the vague topology, if $\int_S f(x)\xi_n(dx) \to \int_S f(x)\xi(dx)$, $\forall f \in C_c(S)$, where $C_c(S)$ is the set of all continuous real-valued functions with compact support. A point process on $S$ is a Conf$(S)$-valued random variable $\Xi = \Xi(\cdot, \omega)$ on a probability space $(\Omega, \mathcal{F}, P)$. If $\Xi(\{x\}) \in \{0, 1\}$ for any point $x \in S$, then the point process is said to be simple.

Assume that $\Lambda_j, j = 1, \ldots, m$, $m \in \mathbb{N}$ are disjoint bounded sets in $S$ and $k_j \in \mathbb{N}_0 := \{0, 1, \ldots\}$, $j = 1, \ldots, m$ satisfy $\sum_{j=1}^m k_j = n \in \mathbb{N}_0$. A symmetric measure $\lambda^n$ on $S^n$ is called the $n$-th correlation measure, if it satisfies

$$E \left[ \prod_{j=1}^m \frac{\Xi(\Lambda_j)!}{(\Xi(\Lambda_j) - k_j)!} \right] = \lambda^n(\Lambda_1^{k_1} \times \cdots \times \Lambda_m^{k_m}),$$

where if $\Xi(\Lambda_j) - k_j \leq 0$, we interpret $\Xi(\Lambda_j)!/(\Xi(\Lambda_j) - k_j)! = 0$. If $\lambda^n$ is absolutely continuous with respect to the $n$-product measure $\lambda^n$, the Radon–Nikodym derivative $\rho^n(x_1, \ldots, x_n)$ is called the $n$-point correlation function with respect to the background measure $\lambda$:

$$\lambda^n(dx_1 \cdots dx_n) = \rho^n(x_1, \ldots, x_n)\lambda^n(dx_1 \cdots dx_n).$$

Determinantal point process (DPP) is defined as follows [72, 97, 98, 94, 44].

**Definition 1.1** A simple point process $\Xi$ on $(S, \lambda)$ is said to be a determinantal point process (DPP) with correlation kernel $K : S \times S \to \mathbb{C}$ if it has correlation functions $\{\rho^n\}_{n \in \mathbb{N}}$, and they are given by

$$\rho^n(x_1, \ldots, x_n) = \det_{1 \leq j, k \leq n} [K(x_j, x_k)] \text{ for every } n \in \mathbb{N}, \text{ and } x_1, \ldots, x_n \in S. \quad (1.1)$$

The triplet $(\Xi, K, \lambda(dx))$ denotes the DPP, $\Xi \in \text{Conf}(S)$, specified by the correlation kernel $K$ with respect to the measure $\lambda(dx)$. 

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If the integral projection operator $K$ on $L^2(S, \lambda)$ with a kernel $K$ is of rank $N \in \mathbb{N}$, then the number of points is $N$ a.s. If $N < \infty$ (resp. $N = \infty$), we call the system a finite DPP (resp. an infinite DPP). The density of points with respect to the background measure $\lambda(dx)$ is given by

$$\rho(x) := \rho^1(x) = K(x, x).$$

The DPP is negatively correlated as shown by

$$\rho^2(x, x') = \det \begin{bmatrix} K(x, x) & K(x, x') \\ K(x', x) & K(x', x') \end{bmatrix} = K(x, x)K(x', x') - |K(x, x')|^2 \leq \rho(x)\rho(x'), \quad x, x' \in S, \quad (1.2)$$

provided that $K$ is Hermitian.

Let $H$ be a separable Hilbert space. For operators $A, B$ on $H$, we say that $A$ is positive definite and write $A \geq O$ if $\langle Af, f \rangle_H \geq 0$ for any $f \in H$, and write $A \geq B$ if $A - B \geq O$. For any bounded operator $A$, the operator $A^*A$ is positive definite. Then it admits a unique positive definite square root $\sqrt{A^*A}$ and is denoted by $|A|$. Let $(\phi_n)_{n \geq 1}$ be an orthonormal basis of $H$. For $A \geq O$, we define the trace of $A$ by

$$\text{Tr} A := \sum_{n=1}^{\infty} \langle A\phi_n, \phi_n \rangle_H,$$

which does not depend on the choice of an orthonormal basis. An operator $A$ is said to be of trace class or a trace class operator if the trace norm $||A||_1 := \text{Tr}|A|$ is finite. The trace $\text{Tr} A$ is defined whenever $||A||_1 < \infty$.

Now, we consider the case $H = L^2(S, \lambda)$. For a compact set $\Lambda \subset S$, the projection from $L^2(S, \lambda)$ to the space of all functions vanishing outside $\Lambda$ $\lambda$-a.e. is denoted by $P_\Lambda$. $P_\Lambda$ is the operation of multiplication of the indicator function $1_\Lambda$ of the set $\Lambda$: $1_\Lambda(x) = 1$ if $x \in \Lambda$, and $1_\Lambda(x) = 0$ otherwise. We say that the bounded self-adjoint operator $A$ on $L^2(S, \lambda)$ is of locally trace class or a locally trace class operator, if the restriction of $A$ to each compact subset $\Lambda$, is of trace class; that is,

$$\text{Tr} A_\Lambda < \infty \quad \text{with} \quad A_\Lambda := P_\Lambda AP_\Lambda \quad \text{for any compact set} \quad \Lambda \subset S. \quad (1.3)$$

The totality of locally trace class operators on $L^2(S, \lambda)$ is denoted by $\mathcal{I}_{1,\text{loc}}(S, \lambda)$.

Let $(S, \lambda)$ be a $\sigma$-finite measure space. We assume that $\mathcal{K} \in \mathcal{I}_{1,\text{loc}}(S, \lambda)$. If, in addition, $\mathcal{K} \geq O$, then it admits a Hermitian integral kernel $K(x, x')$ such that (cf. [34])

(i) $\det_{1 \leq j, k \leq n} [K(x_j, x_k)] \geq 0$ for $\lambda^\otimes n$-a.e. $(x_1, \ldots, x_n)$ for every $n \in \mathbb{N}$,

(ii) $K_{x'} := K(\cdot, x') \in L^2(S, \lambda)$ for $\lambda$-a.e. $x'$,

(iii) $\text{Tr} \mathcal{K}_\Lambda = \int_\Lambda K(x, x)\lambda(dx)$, $\Lambda \subset S$ and

$$\text{Tr}(P_\Lambda \mathcal{K}^n P_\Lambda) = \int_\Lambda \langle K_{x''}, \mathcal{K}^{n-2} K_{x'} \rangle_{L^2(S, \lambda)}\lambda(dx'), \quad \forall n \in \{2, 3, \ldots\}.$$

The following is the existence theorem of DPP.

**Theorem 1.2 ([97] [93] [94])** Assume that $\mathcal{K} \in \mathcal{I}_{1,\text{loc}}(S, \lambda)$ and $O \leq \mathcal{K} \leq I$. Then there exists a unique DPP $(\Xi, K, \lambda)$ on $S$. 

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If $K \in \mathcal{L}_{1,\text{loc}}(S,\lambda)$ is a projection onto a closed subspace $H \subset L^2(S,\lambda)$, one has the DPP associated with $K$ and $\lambda$, or one may say the DPP associated with the subspace $H$.

For $K$ its kernel space is denoted as $\ker K$ and the orthogonal complement of $\ker K$ is written as $(\ker K)^\perp$. In this section, we consider the case that $Kf = f$ for all $f \in (\ker K)^\perp \subset L^2(S,\lambda)$ $\iff$ $K$ is an orthogonal projection.

By definition, it is obvious that the condition $O \leq K \leq I$ is satisfied. The purpose of the present section is to introduce a useful method to provide orthogonal projections $K$ and DPPs whose correlation kernels are given by the Hermitian integral kernels of $K$, $K(x,x')$, $x,x' \in S$.

1.2 Partial isometries, locally Hilbert–Schmidt operators, and DPPs

Let $H_\ell$, $\ell = 1,2$ be separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{H_\ell}$. For a bounded linear operator $W : H_1 \to H_2$, the adjoint of $W$ is defined as the operator $W^* : H_2 \to H_1$, such that

$$\langle Wf, g \rangle_{H_2} = \langle f, W^*g \rangle_{H_1} \quad \text{for all } f \in H_1 \text{ and } g \in H_2. \quad (1.4)$$

A linear operator $W$ is called an isometry if

$$\|Wf\|_{H_2} = \|f\|_{H_1} \quad \text{for all } f \in H_1.$$ 

The kernel space of $W$ is denoted as $\ker W$ and its orthogonal complement is written as $(\ker W)^\perp$.

A linear operator $W$ is called a partial isometry, if

$$\|Wf\|_{H_2} = \|f\|_{H_1} \quad \text{for all } f \in (\ker W)^\perp.$$ 

For the partial isometry $W$, $(\ker W)^\perp$ is called the initial space and the range of $W$, $\text{ran} W$, is called the final space. By the definition (1.4), $\|Wf\|_{H_2}^2 = \langle Wf, Wf \rangle_{H_2} = \langle f, W^*Wf \rangle_{H_1}$. As is suggested from this equality, we have the following fact for partial isometries. Although this might be known, we give a proof below.

Lemma 1.3 Let $H_1$ and $H_2$ be separable Hilbert spaces and $W : H_1 \to H_2$ be a bounded operator. Then, the following are equivalent.

(i) $W$ is a partial isometry.

(ii) $W^*W$ is a projection on $H_1$, which acts as the identity on $(\ker W)^\perp$.

(iii) $W = WW^*W$.

Moreover, $W$ is a partial isometry if and only if so is $W^*$.

Assumption 1 $W$ is a partial isometry.

By Lemma 1.3 under Assumption 1, $W^*$ is also a partial isometry and hence the operator $W^*W$ (resp. $WW^*$) is the projection onto the initial space of $W$ (resp. the final space of $W$).

Now we assume that $H_1$ and $H_2$ are realized as $L^2$-spaces, $L^2(S_1,\lambda_1)$ and $L^2(S_2,\lambda_2)$, respectively.
A bounded linear operator $A : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ is a Hilbert–Schmidt operator if Hilbert–Schmidt norm is finite: $\|A\|_{HS} := \text{Tr}(A^*A) < \infty$. We say that $A$ is a locally Hilbert–Schmidt operator or of locally Hilbert–Schmidt class, if $AP_A$ is a Hilbert–Schmidt operator for any compact set $\Lambda \subset S$. It is known as the kernel theorem that every Hilbert–Schmidt operator $A : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ is defined as an integral operator with kernel $k \in L^2(S_1 \times S_2, \lambda_1 \otimes \lambda_2)$ (cf. Theorem 12.6.2 [8]).

We put the second assumption.

**Assumption 2**

(i) $\mathcal{W}$ is a locally Hilbert–Schmidt operator,

(ii) $\mathcal{W}^*$ is a locally Hilbert–Schmidt operator.

We note that for any compact set $\Lambda_1 \subset S_1$, the operator $\mathcal{W}_\Lambda$ is of Hilbert–Schmidt class if and only if the operator $\mathcal{P}_\Lambda \mathcal{W}^* \mathcal{W}_\Lambda$ is of trace class since

$$\|\mathcal{W}_\Lambda\|_{HS}^2 := \text{Tr} \left( (\mathcal{W}_\Lambda)^* \mathcal{W}_\Lambda \right) = \text{Tr} \left( \mathcal{P}_\Lambda \mathcal{W}^* \mathcal{W}_\Lambda \right) < \infty.$$  

Therefore, Assumption 2 (i) (resp. Assumption 2 (ii)) is equivalent to the following Assumption 2’ (i) (resp. Assumption 2’ (ii)), which guarantees the existence of DPP associated with $\mathcal{W}^* \mathcal{W}$ (resp. $\mathcal{W} \mathcal{W}^*$).

**Assumption 2’**

(i) $\mathcal{W}^* \mathcal{W} \in \mathcal{T}_{1,loc}(S_1, \lambda_1),$

(ii) $\mathcal{W} \mathcal{W}^* \in \mathcal{T}_{1,loc}(S_2, \lambda_2).$

Given a measure space $(S, \lambda)$, if $f \in L^2(\Lambda, \lambda)$ for all compact subsets $\Lambda$ of $S$, then $f$ is said to be locally $L^2$-integrable. The set of all such functions is denoted by $L^2_{loc}(S, \lambda)$. By this definition if $\mathcal{P}_\Lambda f \in L^2(S, \lambda)$ for any compact set $\Lambda \subset S$, then $f \in L^2_{loc}(S, \lambda)$. The following proposition is a local version of the kernel theorem for Hilbert–Schmidt operators.

**Proposition 1.4** Suppose Assumption 2 (i) holds. Then, $\mathcal{W}$ is regarded as an integral operator associated with a kernel $W : S_2 \times S_1 \to \mathbb{C}$;

$$Wf(y) = \int_{S_1} W(y,x)f(x)\lambda_1(dx), \quad f \in L^2(S_1, \lambda_1), \quad (1.5)$$

such that $\Psi_1 \in L^2_{loc}(S_1, \lambda_1)$, where $\Psi_1(x) := \|W(\cdot,x)\|_{L^2(S_2, \lambda_2)}, x \in S_1$.

From Proposition [1.4] under Assumption 2 (ii), the dual operator $\mathcal{W}^*$ also admits an integral kernel $\Lambda \times S_2 \rightarrow \mathbb{C}$ such that $\Psi_2 \in L^2_{loc}(S_2, \lambda_2)$, where $\Psi_2(y) := \|\mathcal{W}^*(\cdot,y)\|_{L^2(S_1, \lambda_1)}, y \in S_2$. It is easy to see that $\mathcal{W}^*(x,y) = \overline{W(y,x)}$ for $\lambda_1 \otimes \lambda_2$-a.e. $(x,y)$. Then

$$(\mathcal{W}^*g)(x) = \int_{S_2} \overline{W(y,x)}g(y)\lambda_2(dy), \quad g \in L^2(S_2, \lambda_2). \quad (1.6)$$
Following (1.5) and (1.6), we have
\[
(W^*Wf)(x) = \int_{S_1} K_{S_1}(x,x')f(x')\lambda_1(dx'), \quad f \in L^2(S_1, \lambda_1),
\]
\[
(WW^*g)(y) = \int_{S_2} K_{S_2}(y,y')g(y')\lambda_2(dy'), \quad g \in L^2(S_2, \lambda_2),
\]
with the integral kernels,
\[
K_{S_1}(x,x') = \int_{S_2} W(y,x)W(y,x')\lambda_2(dy) = \langle W(\cdot, x'), W(\cdot, x) \rangle_{L^2(S_2, \lambda_2)},
\]
\[
K_{S_2}(y,y') = \int_{S_1} W(y,x)W(y,x')\lambda_1(dx) = \langle W(y, \cdot), W(y', \cdot) \rangle_{L^2(S_1, \lambda_1)}. \tag{1.7}
\]

We see that \(K_{S_1}(x', x) = K_{S_1}(x, x')\) and \(K_{S_2}(y', y) = K_{S_2}(y, y')\).

Under Assumptions 1 and 2, we obtain the following theorem as an immediate consequence of the well-known existence theorem of DPP (Theorem 1.2). This is a starting-point for our discussion in the present section.

**Theorem 1.5** Under Assumptions 1 and 2, associated with \(W^*W\) and \(WW^*\), there exists a unique pair of DPPs; \((\Xi_1, K_{S_1}, \lambda_1(dx))\) on \(S_1\) and \((\Xi_2, K_{S_2}, \lambda_2(dy))\) on \(S_2\). The correlation kernels \(K_{S_1}, \ell = 1, 2\) are Hermitian and given by (1.7).

Note that the densities of the DPPs, \((\Xi_1, K_{S_1}, \lambda_1(dx))\) and \((\Xi_2, K_{S_2}, \lambda_2(dy))\), are given by
\[
\rho_1(x) = K_{S_1}(x, x) = \int_{S_2} |W(y, x)|^2\lambda_2(dy) = \|W(\cdot, x)\|_{L^2(S_2, \lambda_2)}^2, \quad x \in S_1,
\]
\[
\rho_2(y) = K_{S_2}(y, y) = \int_{S_1} |W(y, x)|^2\lambda_1(dx) = \|W(y, \cdot)\|_{L^2(S_1, \lambda_1)}^2, \quad y \in S_2,
\]
with respect to the background measures \(\lambda_1(dx)\) and \(\lambda_2(dy)\), respectively.

We say that a pair of DPPs \((\Xi_1, K_{S_1}, \lambda_1(dx))\) on \(S_1\) and \((\Xi_2, K_{S_2}, \lambda_2(dy))\) on \(S_2\) is associated with \(W\).

### 1.3 Basic properties of DPPs

For \(v = (v^{(1)}, \ldots, v^{(d)}) \in \mathbb{R}^d, y = (y^{(1)}, \ldots, y^{(d)}) \in \mathbb{R}^d, d \in \mathbb{N}\), the inner product of them is given by \(v \cdot y := \sum_{a=1}^{d} v^{(a)}y^{(a)}\), and \(|v|^2 := v \cdot v\). When \(S \subset \mathbb{C}^d, d \in \mathbb{N}\), \(x \in S\) has \(d\) complex components; \(x = (x^{(1)}, \ldots, x^{(d)})\) with \(x^{(a)} = \text{Re}x^{(a)} + \sqrt{-1}\text{Im}x^{(a)}, a = 1, \ldots, d\). In order to describe clearly such a complex structure, we set \(x_R = (\text{Re}x^{(1)}, \ldots, \text{Re}x^{(d)}) \in \mathbb{R}^d, x_I = (\text{Im}x^{(1)}, \ldots, \text{Im}x^{(d)}) \in \mathbb{R}^d\), and write \(x = x_R + \sqrt{-1}x_I\) in this manuscript. The Lebesgue measure is written as \(dx = dx_R dx_I := \prod_{a=1}^{d} d\text{Re}x^{(a)}d\text{Im}x^{(a)}\). The complex conjugate of \(x = x_R + \sqrt{-1}x_I\) is defined as \(\overline{x} = x_R - \sqrt{-1}x_I\). For \(x = x_R + \sqrt{-1}x_I, x' = x_R' + \sqrt{-1}x_I' \in \mathbb{C}^d\), we use the **Hermitian inner product**;
\[
x \cdot \overline{x}' := (x_R + \sqrt{-1}x_I) \cdot (x'_R - \sqrt{-1}x'_I) = (x_R \cdot x'_R + x_I \cdot x'_I) - \sqrt{-1}(x_R \cdot x'_I - x_I \cdot x'_R)
\]
and define
\[
|x|^2 := x \cdot \overline{x} = |x_R|^2 + |x_I|^2, \quad x \in \mathbb{C}^d.
\]

For \((\Xi, K, \lambda(dx))\) defined on \(S = \mathbb{R}^d, S = \mathbb{C}^d\), or on the space having appropriate periodicities, we introduce the following operations.
For a measurable function \( S_u \Xi := \sum_j \delta_{x_j - u} \),
\[
S_u K(x, x') = K(x + u, x' + u),
\]
and \( S_u \lambda(dx) = \lambda(u + dx) \). We write \((S_u \Xi, S_u K, S_u \lambda(dx))\) simply as \( S_u(\Xi, K, \lambda(dx)) \).

**Dilatation** For \( c > 0 \), we set \( c \circ \Xi := \sum_j \delta_{cx_j} \)
\[
c \circ K(x, x') := K \left( \frac{x}{c}, \frac{x'}{c} \right), \quad x, x' \in cS := \{cx : x \in S\},
\]
and \( c \circ \lambda(dx) := \lambda(dx/c) \). We define \( c \circ (\Xi, K, \lambda(dx)) := (c \circ \Xi, c \circ K, c \circ \lambda(dx)) \).

**Gauge transformation** For non-vanishing \( u : S \to \mathbb{C} \), a gauge transformation of \( K \) by \( u \) is defined as
\[
K(x, x') \mapsto \hat{K}_u(x, x') := u(x)K(x,x')u(x')^{-1}.
\]

In particular, when \( u : S \to U(1) \), the \( \text{U}(1) \)-gauge transformation of \( K \) is given by
\[
K(x, x') \mapsto \hat{K}_u(x, x') := u(x)K(x, x')u(x')\bigg|_{\lambda=1}.
\]

We will use the following basic properties of DPP.

**Gauge invariance** For any \( u : S \to \mathbb{C} \), a gauge transformation does not change the probability law of DPP;
\[
(\Xi, K, \lambda(dx)) \overset{\text{law}}{=} (\hat{\Xi}, \hat{K}_u, \lambda(dx)).
\]

**Measure change** For a measurable function \( g : S \to \mathbb{R}_{>0} \),
\[
(\Xi, K, \lambda(dx)) \overset{\text{law}}{=} (\Xi, \sqrt{g(x)} K(x, x') \sqrt{g(x')}, \lambda(dx)). \tag{1.8}
\]

**Mapping and scaling** For a one-to-one measurable mapping \( h : S \to \hat{S} \), if we set
\[
\hat{\Xi} = \sum_j \delta_{h(x_j)}, \quad \hat{K}(x, x') = K(h^{-1}(x), h^{-1}(y)), \quad \hat{\lambda}(dx) = (\lambda \circ h^{-1})(dx),
\]
then \((\hat{\Xi}, \hat{K}, \hat{\lambda}(dx))\) is a DPP on \( \hat{S} \). In particular, when \( h(x) = x - u, u \in S \), \((\hat{\Xi}, \hat{K}, \hat{\lambda}(dx)) = S_u(\Xi, K, \lambda(dx)) \), when \( h(x) = cx, c > 0 \), \((\hat{\Xi}, \hat{K}, \hat{\lambda}(dx)) = c \circ (\Xi, K, \lambda(dx)) \), and when \( h(x) = \sqrt{x} \) for \( S = [0, \infty) \), \((\hat{\Xi}, \hat{K}, \hat{\lambda}(dx)) = (\Xi, K, \lambda(dx)) \) \(^{1/2} \). If \( c \circ \lambda(dx) = c^{-d}\lambda(dx) \), then \((\hat{\Xi}, \hat{K}, \hat{\lambda}(dx)) = (\Xi, K, \lambda(dx)) \) \(^{1/2} \) with \( g(x) := c > 0 \) gives
\[
c \circ (\Xi, K, \lambda(dx)) \overset{\text{law}}{=} (c \circ \Xi, K_c, \lambda(dx)), \quad c > 0,
\]
with
\[
K_c(x, x') = \frac{1}{cd} K \left( \frac{x}{c}, \frac{x'}{c} \right),
\]
where the base space is given by \( cS \).
We will give some limit theorems for DPPs in this manuscript. Consider a DPP which depends on a continuous parameter, or a series of DPPs labeled by a discrete parameter (e.g., the number of points $N \in \mathbb{N}$), and describe the system by $(\Xi, K_p, \lambda_p(dx))$ with the continuous or discrete parameter $p$. If $(\Xi, K_p, \lambda_p(dx))$ converges to a DPP, $(\Xi, K, \lambda(dx))$, as $p \to \infty$, weakly in the vague topology, we write this limit theorem as $(\Xi, K_p, \lambda_p(dx)) \xrightarrow{p \to \infty} (\Xi, K, \lambda(dx))$. The weak convergence of DPPs is verified by the uniform convergence of the kernel $K_p \to K$ on each compact set $C \subset S \times S$.

1.4 Duality relations

For $f \in C_c(S)$, the Laplace transform of the probability measure $P$ for a point process $\Xi$ is defined as

$$\Psi[f] = \mathbb{E} \left[ \exp \left( \int_S f(x) \Xi(dx) \right) \right].$$

(1.9)

For the DPP, $(\Xi, K, \lambda(dx))$, this is given by the Fredholm determinant on $L^2(S, \lambda)$ [95],

$$\text{Det}_{L^2(S, \lambda)} \left[ I - (1 - e^{f})K \right] := 1 + \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n!} \int_{S^n} \det \left[ K(x_j, x_k) \right] \prod_{\ell=1}^{n} (1 - e^{f(x_\ell)}) \lambda^\otimes n(dx).$$

Lemma 1.6 Between two DPPs, $(\Xi_1, K_{S_1}, \lambda_{S_1}(dx))$ on $S_1$ and $(\Xi_2, K_{S_2}, \lambda_{S_2}(dy))$ on $S_2$, given by Theorem 1.5, the following equality holds with an arbitrary parameter $\alpha \in \mathbb{C}$,

$$\text{Det}_{L^2(S_1, \lambda_1)} \left[ I + \alpha K_{S_1} \right] = \text{Det}_{L^2(S_2, \lambda_2)} \left[ I + \alpha K_{S_2} \right].$$

(1.10)

Proof We recall that if $AB$ and $BA$ are trace class operators on a Hilbert space $H$ then [95]

$$\text{Det}_H[I + BA] = \text{Det}_H[I + AB].$$

(1.11)

Now we have $A : H_1 \to H_2$ and $B : H_2 \to H_1$ between two Hilbert spaces $H_1$ and $H_2$. Let $\tilde{A}$ and $\tilde{B}$ be two operators on $H_1 \oplus H_2$ defined by

$$\tilde{A} = \begin{pmatrix} O & 0 \\ A & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & B \\ O & O \end{pmatrix}.$$ 

Then, $\tilde{A}\tilde{B}$ and $\tilde{B}\tilde{A}$ are diagonal operators $O \oplus AB$ and $BA \oplus O$, respectively, and hence also they are trace class operators. By applying (1.11) to $\tilde{A}$ and $\tilde{B}$ with $H := H_1 \oplus H_2$, we obtain

$$\text{Det}_{H_1}[I + BA] = \text{Det}_{H_2}[I + AB].$$

Consequently, taking $A = \sqrt{\alpha}W$, $B = \sqrt{\alpha}W^*$, $H_1 = L^2(S_1, \lambda_1)$, and $H_2 = L^2(S_2, \lambda_2)$ yields (1.10).

For $A_\ell \subset S_\ell$, $\ell = 1, 2$, let

$$\tilde{\mathcal{W}} := \mathcal{P}_{A_2} \mathcal{W} \mathcal{P}_{A_1}, \quad K^{(A_2)}_{S_1} := \mathcal{W}^* \mathcal{P}_{A_2} \mathcal{W}, \quad K^{(A_1)}_{S_2} := \mathcal{W} \mathcal{P}_{A_1} \mathcal{W}^*.$$ 

(1.12)
They admit the following integral kernels,
\[
\tilde{W}(y, x) = 1_{A_2}(y)W(y, x)1_{A_1}(x),
\]
\[
K_{S_1}^{(A_2)}(x, x') = \int_{A_2} \tilde{W}(y, x)\tilde{W}(y, x')\lambda_2(dy),
\]
\[
K_{S_2}^{(A_1)}(y, y') = \int_{A_1} W(y, x)\tilde{W}(y', x)\lambda_1(dx).
\]
(1.13)

Using Lemma 1.6, the following theorem is proved.

**Theorem 1.7** Let \((\Xi_1^{(A_2)}, K_{S_1}^{(A_2)}(\lambda_1(dx))) \) and \((\Xi_2^{(A_1)}, K_{S_2}^{(A_1)}(\lambda_2(dy))) \) be DPPs associated with the kernels \(K_{S_1}^{(A_2)}\) and \(K_{S_2}^{(A_1)}\) given by (1.13), respectively. Then, \(\Xi_1^{(A_2)}(\lambda_1) \overset{\text{(law)}}{=} \Xi_2^{(A_1)}(\lambda_2)\), i.e.,
\[
P(\Xi_1^{(A_2)}(\lambda_1) = m) = P(\Xi_2^{(A_1)}(\lambda_2) = m), \quad \forall m \in \mathbb{N}_0.
\]

**Proof** As a special case of (1.9) with \(f(x) = 1_{A_1}(x)\log z\) for \(\Xi = \Xi_1^{(A_2)}\), \(z \in \mathbb{C}\), we have the equality,
\[
E \left[ z^{\Xi_1^{(A_2)}(A_1)} \right] = \frac{\text{Det}}{L^2(S_1, \lambda_1)} \left[ I - (1 - z)\mathcal{P}_{A_1} K_{S_1}^{(A_2)}/\mathcal{P}_{A_1} \right],
\]
(1.14)
where \(K_{S_2}^{(A_2)}\) is defined by (1.12). Here \(\text{LHS}\) is the moment generating function of \(\Xi_1^{(A_2)}(\lambda_1)\) and \(\text{RHS}\) gives its Fredholm determinantal expression. By replacing \(W\) by \(\tilde{W}\) and letting \(\alpha = -(1 - z)\) in the proof of Lemma 1.6, we obtain the equality,
\[
\frac{\text{Det}}{L^2(S_1, \lambda_1)} \left[ I - (1 - z)\mathcal{P}_{A_1} K_{S_1}^{(A_2)}/\mathcal{P}_{A_1} \right] = \frac{\text{Det}}{L^2(S_2, \lambda_2)} \left[ I - (1 - z)\mathcal{P}_{A_1} K_{S_2}^{(A_1)}/\mathcal{P}_{A_1} \right].
\]

Through (1.14) and the similar equality for \(E \left[ z^{\Xi_2^{(A_1)}(A_2)} \right] \), we obtain the corresponding equivalence between the moment generating functions of \(\Xi_1^{(A_2)}(\lambda_1)\) and \(\Xi_2^{(A_1)}(\lambda_2)\), and hence the statement of the proposition is proved. ■

### 1.5 Orthonormal functions and correlation kernels

In addition to \(L^2(S_\ell, \lambda_\ell)\), \(\ell = 1, 2\), we introduce \(L^2(\Gamma, \nu)\) as a parameter space for functions in \(L^2(S_\ell, \lambda_\ell)\), \(\ell = 1, 2\). Assume that there are two families of measurable functions \(\{\psi_1(x, \gamma) : x \in S_1, \gamma \in \Gamma\} \) and \(\{\psi_2(y, \gamma) : y \in S_2, \gamma \in \Gamma\} \) such that two bounded operators \(\mathcal{U}_\ell : L^2(S_\ell, \lambda_\ell) \rightarrow L^2(\Gamma, \nu)\) given by
\[
(\mathcal{U}_\ell f)(\gamma) := \int_{S_\ell} \psi_\ell(x, \gamma)f(x)\lambda_\ell(dx), \quad \ell = 1, 2,
\]
are well-defined. Then, their adjoints \(\mathcal{U}_\ell^* : L^2(\Gamma, \nu) \rightarrow L^2(S_\ell, \lambda_\ell)\), \(\ell = 1, 2\) are given by
\[
(\mathcal{U}_\ell^* F)(\cdot) = \int_{\Gamma} \psi_\ell(\cdot, \gamma)F(\gamma)\nu(d\gamma).
\]

A typical example of \(\mathcal{U}_1\) is the Fourier transform, i.e., \(\psi_1(x, \gamma) = e^{\sqrt{-1}x\gamma}\). In this case, for any \(\gamma\), the function \(\psi_1(\cdot, \gamma)\) is not in \(L^2(\mathbb{R}, dx)\). Now we define \(\mathcal{W} : L^2(S_1, \lambda_1) \rightarrow L^2(S_2, \lambda_2)\) by \(\mathcal{W} = \mathcal{U}_2^*\mathcal{U}_1\), i.e.,
\[
(\mathcal{W}f)(y) = \int_{\Gamma} \psi_2(y, \gamma)(\mathcal{U}_1 f)(\gamma)\nu(d\gamma).
\]
(1.15)
Let \(I_{\Gamma}\) be an identity in \(L^2(\Gamma, \nu)\). We can see the following.
Lemma 1.8 If $U_\ell U_\ell^* = I_\Gamma$ for $\ell = 1, 2$, then both $W$ and $W^*$ are partial isometries.

Proof By the assumption, we see that
\[ W W^* W = (U_1^* U_1)(U_2^* U_2)(U_2^* U_1) = U_2^* U_1 = W. \]
From Lemma [1.3], $W$ is a partial isometry. By symmetry, the assertion for $W^*$ also follows. \(\blacksquare\)

We often use these relations below.

Lemma 1.9 Let $\Psi_1(x) := \|\psi_1(x, \cdot)\|_{L^2(\Gamma, \nu)}$, $x \in S_1$ and assume that $\Psi_1 \in L^2_{\text{loc}}(S_1, \lambda_1)$. Then, the operator $U_1$ is of locally Hilbert–Schmidt class.

Proof For a compact set $\Lambda \subset S_1$, we see that
\[
|\mathcal{P}_\Lambda U_1^* U_1 \mathcal{P}_\Lambda f(x)| = \left| 1_\Lambda(x) \int_{\Gamma} \nu(d\gamma) \psi_1(x, \gamma) \int_{S_1} \overline{\psi_1(x', \gamma)} 1_\Lambda(x') f(x') \lambda_1(dx') \right|
\leq 1_\Lambda(x) \Psi_1(x) \int_{S_1} 1_\Lambda(x') \Psi_1(x') |f(x')| \lambda_1(dx')
\leq \mathcal{P}_\Lambda \Psi_1(x) \mathcal{P}_\Lambda \Psi_1 \mathcal{P}_\Lambda f \mathcal{P}_\Lambda f \mathcal{P}_\Lambda f
\|_{L^2(S_1, \lambda_1)}.
\]

By Fubini’s theorem, we have
\[
\mathcal{P}_\Lambda U_1^* U_1 \mathcal{P}_\Lambda f(x) = \int_{S_1} \lambda_1(dx') f(x') \left( \int_{\Gamma} 1_\Lambda(x) \psi_1(x, \gamma) \overline{1_\Lambda(x')} \overline{\psi_1(x', \gamma)} \nu(d\gamma) \right)
\]
and hence
\[
\|U_1 \mathcal{P}_\Lambda\|^2_{HS} = \text{Tr}(\mathcal{P}_\Lambda U_1^* U_1 \mathcal{P}_\Lambda) = \int_{S_1} \lambda_1(dx) 1_\Lambda(x) \left( \int_{\Gamma} |\psi_1(x, \gamma)|^2 \nu(d\gamma) \right) = \|\mathcal{P}_\Lambda \Psi_1\|^2_{L^2(S_1, \lambda_1)} < \infty.
\]
This completes the proof. \(\blacksquare\)

Now we put the following.

Assumption 3 For $\ell = 1, 2$,

(i) $U_\ell U_\ell^* = I_\Gamma$,

(ii) $\Psi_\ell \in L^2_{\text{loc}}(S_\ell, \lambda_\ell)$, where $\Psi_\ell(x) := \|\psi_\ell(x, \cdot)\|_{L^2(\Gamma, \nu)}$, $x \in S_\ell$.

Assumption 3(i) can be rephrased as the following orthonormality relations:
\[
\langle \psi_\ell(\cdot, \gamma), \psi_\ell(\cdot, \gamma') \rangle_{L^2(S_\ell, \lambda_\ell)} \nu(d\gamma) = \delta(\gamma - \gamma') \nu(\gamma), \quad \gamma, \gamma' \in \Gamma, \quad \ell = 1, 2.
\]
We often use these relations below.

The following is immediately obtained as a corollary of Theorem [1.5].
Corollary 1.10 Let $\mathcal{W} = \mathcal{U}_2^* \mathcal{U}_1$ as in the above. We assume Assumption 3. Then, there exist unique pair of DPPs; $(\Xi_1, K_{S_1}, \lambda_1(dx))$ on $S_1$ and $(\Xi_2, K_{S_2}, \lambda_2(dy))$ on $S_2$. Here the correlation kernels $K_{S_\ell}, \ell = 1, 2$ are given by

$$
K_{S_1}(x, x') = \int_{\Gamma} \psi_1(x, \gamma) \overline{\psi_1(x', \gamma)} \nu(d\gamma) = \langle \psi_1(x, \cdot), \psi_1(x', \cdot) \rangle_{L^2(\Gamma, \nu)},
$$

$$
K_{S_2}(y, y') = \int_{\Gamma} \psi_2(y, \gamma) \overline{\psi_2(y', \gamma)} \nu(d\gamma) = \langle \psi_2(y, \cdot), \psi_2(y', \cdot) \rangle_{L^2(\Gamma, \nu)}.
$$

(1.16)

In particular, the densities of the DPPs are given by $\rho_1(x) = K_{S_1}(x, x) = \Psi_1(x)^2, x \in S_1$ and $\rho_2(y) = K_{S_2}(y, y) = \Psi_2(y)^2, y \in S_2$ with respect to the background measures $\lambda_1(dx)$ and $\lambda_2(dy)$, respectively.

Remark 1.11 Consider the symmetric case such that $L^2(S_1, \lambda_1) = L^2(S_2, \lambda_2) =: L^2(S, \lambda)$, $\psi_1 = \psi_2 =: \psi, \nu = \lambda|\Gamma, \Gamma \subseteq S$. In this case, $\mathcal{W} = \mathcal{U}^* \mathcal{U}$ with

$$(\mathcal{U}f)(\gamma) = \int_S \overline{\psi(x, \gamma)} f(x) \lambda(dx).$$

Then $K_{S_1} = K_{S_2} = \mathcal{W} =: K$ is given by

$$
K(x, x') = \int_{\Gamma} \psi(x, \gamma) \overline{\psi(x', \gamma)} \lambda(d\gamma).
$$

(1.17)

This is Hermitian; $\overline{K(x', x)} = K(x, x')$, and satisfies the reproducing property

$$
K(x, x') = \int_S K(x, \zeta) K(\zeta, x') \lambda(d\zeta).
$$

Now we consider a simplified version of the preceding setting. Let $\Gamma \subseteq S_2$ and $\nu = \lambda_2|\Gamma$. We define $\mathcal{U}_2 : L^2(S_2, \lambda_2) \to L^2(\Gamma, \nu)$ as the restriction onto $\Gamma$, and then its adjoint $\mathcal{U}_2^*$ is given by $(\mathcal{U}_2^* F)(y) = F(y)$ for $y \in \Gamma$, and by $0$ for $y \in S_2 \setminus \Gamma$. We write the extension $F = \mathcal{U}_2^* F$ for $F \in L^2(\Gamma, \nu)$. It is obvious that $\mathcal{U}_2 \mathcal{U}_2^* = I_{\Gamma}$ and hence $\mathcal{U}_2$ is a partial isometry.

For $\Gamma \subseteq S_2$, we assume that there is a family of measurable functions $\{\psi_1(x, y) : x \in S_1, y \in \Gamma\}$ such that a bounded operator $\mathcal{U}_1 : L^2(S_1, \lambda_1) \to L^2(\Gamma, \nu)$ given by

$$(\mathcal{U}_1 f)(\gamma) := \int_{S_1} \overline{\psi_1(x, \gamma)} f(x) \lambda_1(dx) \quad (\gamma \in \Gamma)$$

is well-defined.

Assumption 3’

(i) $\mathcal{U}_1 \mathcal{U}_1^* = I_{\Gamma}$,

(ii) $\Psi_1 \in L^2_{loc}(S_1, \lambda_1)$, where $\Psi_1(x) := \|\psi_1(x, \cdot)\|_{L^2(\Gamma, \nu)}, x \in S_1$. 

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Assumption 3’(i) can be rephrased as the following orthonormality relation:

\( \langle \psi_1(\cdot, y), \psi_1(\cdot, y') \rangle_{L^2(S_1, \lambda_1)} \lambda_2(dy) = \delta(y - y')dy, \quad y, y' \in \Gamma. \)

Now we define \( W : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2) \) by \( W = U_2^* U_1 \) as before. In this case, we have

\[
(Wf)(y) = 1_\Gamma(y) \int_{S_1} \psi_1(x, y) f(x) \lambda_1(dx),
\]

and hence

\[
W(y, x) = \overline{\psi_1(x, y)} 1_\Gamma(y). \tag{1.18}
\]

It follows from Assumption 3’ that \( W \) is a partial isometry. Corollary 1.10 is reduced to the following.

**Corollary 1.12** Let \( W = U_2^* U_1 \) as in the above. We assume Assumption 3’. Then there exists a unique DPP, \( (\Xi, K, \lambda_1) \) on \( S_1 \) with the correlation kernel

\[
K_{S_1}(x, x') = \int_\Gamma \psi_1(x, y) \overline{\psi_1(x', y)} \lambda_2(dy) = \langle \tilde{\psi}_1(x, \cdot), \tilde{\psi}_1(x', \cdot) \rangle_{L^2(\Gamma, \lambda_2)}. \tag{1.19}
\]

In particular, the density of the DPP is given by \( \rho_1(x) = K_{S_1}(x, x) = \Psi_1(x)^2, x \in S_1 \) with respect to the background measures \( \lambda_1(dx) \).

### 1.6 Examples in one-dimensional spaces

#### 1.6.1 Finite DPPs in \( \mathbb{R} \) associated with classical orthonormal polynomials

Let \( S_1 = S_2 = \mathbb{R} \). Assume that we have two sets of orthonormal functions \( \{ \varphi_n \}_{n \in \mathbb{N}_0} \) and \( \{ \phi_n \}_{n \in \mathbb{N}_0} \) with respect to the measures \( \lambda_1 \) and \( \lambda_2 \), respectively,

\[
\langle \varphi_n, \varphi_m \rangle_{L^2(\mathbb{R}, \lambda_1)} = \int_\mathbb{R} \varphi_n(x) \overline{\varphi_m(x)} \lambda_1(dx) = \delta_{nm},
\]

\[
\langle \phi_n, \phi_m \rangle_{L^2(\mathbb{R}, \lambda_2)} = \int_\mathbb{R} \phi_n(y) \overline{\phi_m(y)} \lambda_2(dy) = \delta_{nm}, \quad n, m \in \mathbb{N}_0. \tag{1.20}
\]

Then for an arbitrary but fixed \( N \in \mathbb{N} \), we set \( \Gamma = \{ 0, 1, \ldots, N - 1 \} \subseteq \mathbb{N}_0 \), \( \psi_1(\cdot, \gamma) = \varphi_\gamma(\cdot), \psi_2(\cdot, \gamma) = \phi_\gamma(\cdot), \gamma \in \Gamma \), and consider \( l^2(\Gamma) \) as \( L^2(\Gamma, \nu) \) in the setting of Section 1.5.

**Remark 1.13** If \( \Gamma \) is a finite set, \( |\Gamma| = N \in \mathbb{N} \), and the parameter space is given by \( l^2(\Gamma) \), Assumption 3(ii) (resp. Assumption 3'(ii)) is concluded from 3(i) (resp. 3'(i)) as shown below. Since

\[
\Psi(x)^2 := \| \varphi(x) \|^2_{l^2(\Gamma)} = \sum_{n \in \Gamma} |\varphi_n(x)|^2, \quad x \in S,
\]

we have

\[
\int_S \Psi(x)^2 \lambda(dx) = \sum_{n \in \Gamma} \| \varphi_n \|^2_{L^2(S, \lambda)}.
\]

Then, if \( \{ \varphi_n \}_{n \in \Lambda} \) are normalized, the above integral is equal to \( |\Gamma| = N < \infty \). This implies \( \Psi \in L^2(S, \lambda) \subset L^2_{\text{loc}}(S, \lambda) \).

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Hence Assumption 3 is satisfied for any $N \in \mathbb{N}$. Then the integral kernel for $\mathcal{W}$ defined by (1.15) is given by

$$W(y, x) = \sum_{n=0}^{N-1} \varphi_n(x)\phi_n(y).$$

By Corollary 1.10 we have a pair of DPPs on $\mathbb{R}$, $(\Xi_1, K^{(N)}_\varphi, \lambda_1(dx))$ and $(\Xi_2, K^{(N)}_\phi, \lambda_2(dy))$, where the correlation kernels are given by

$$K^{(N)}_\varphi(x, x') = \sum_{n=0}^{N-1} \varphi_n(x)\varphi_n(x'), \quad K^{(N)}_\phi(y, y') = \sum_{n=0}^{N-1} \phi_n(y)\phi_n(y'),$$

(1.21)

respectively. Here $N$ gives the number of points for each DPPs. If we can use the three-term relations in $\{\varphi_n\}_{n \in \mathbb{N}_0}$ or $\{\phi_n\}_{n \in \mathbb{N}_0}$, (1.21) can be written in the Christoffel–Darboux form (see, for instance, Proposition 5.1.3 in [32]). As a matter of course, if we have three or more than three, say $M$ distinct sets of orthonormal functions satisfying Assumption 3 with a common $\Gamma$, then by applying Corollary 1.10 to every pair of them, we will obtain $M$ distinct finite DPPs.

Even if we have only one set of orthonormal functions, for example, only the first one $\{\varphi_n\}_{n \in \mathbb{N}_0}$ in (1.20), we can obtain a DPP (labeled by the number of particles $N \in \mathbb{N}$) following Corollary 1.12. In such a case, we set

$$W(n, x) = \varphi_n(x)\mathbb{1}_{\Gamma(n)}$$

(1.22)

with $\Gamma = \{0, 1, \ldots, N - 1\}$ for (1.18). Then we have a DPP, $(\Xi, K^{(N)}_\varphi, \lambda_1(dx))$.

Now we give classical examples of DPPs associated with real orthonormal polynomials. Let $\lambda_{N(m, \sigma^2)}(dx)$ denote the normal distribution,

$$\lambda_{N(m, \sigma^2)}(dx) = \frac{1}{\sqrt{2\pi\sigma}}e^{-(x-m)^2/(2\sigma^2)}dx, \quad m \in \mathbb{R}, \quad \sigma > 0,$$

and $\lambda_{\Gamma(a, b)}(dy)$ do the Gamma distribution,

$$\lambda_{\Gamma(a, b)}(dy) = \frac{b^a}{\Gamma(a)}y^{a-1}e^{-by}\mathbb{1}_{\mathbb{R}_{\geq 0}}(y)dy, \quad a > 0, \quad b > 0,$$

with the Gamma function $\Gamma(z) := \int_0^\infty u^{z-1}e^{-u}du, \text{Re}z > 0$. We set

$$\lambda_1(dx) = \lambda_{N(0, 1/2)}(dx) = \frac{1}{\sqrt{\pi}}e^{-x^2}dx,$$

$$\varphi_n(x) = \frac{1}{\sqrt{2^nn!}}H_n(x), \quad n \in \mathbb{N}_0,$$

(1.23)

and

$$\lambda_2(dy) = \lambda_{\Gamma(\nu+1,1)}(dy) = \frac{1}{\Gamma(\nu+1)}y^{\nu}e^{-y}\mathbb{1}_{\mathbb{R}_{\geq 0}}(y)dy,$$

$$\phi_n(y) = \phi_n^{(\nu)}(y) = \sqrt{\frac{n\Gamma(\nu+1)}{\Gamma(n+\nu+1)}}L_n^{(\nu)}(y), \quad n \in \mathbb{N}_0,$$

(1.24)
with parameter \( \nu \in (-1, \infty) \). Here \( \{H_n(x)\}_{n \in \mathbb{N}_0} \) are the Hermite polynomials,

\[
H_n(x) := (-1)^n e^{2x} \frac{d^n}{dx^n} e^{-x^2} = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^n - 2k}{k! (n-2k)!}, \quad n \in \mathbb{N}_0,
\]

where \([a]\) denotes the largest integer not greater than \( a \in \mathbb{R} \), and \( \{L_n^{(\nu)}(x)\}_{n \in \mathbb{N}_0} \) are the Laguerre polynomials,

\[
L_n^{(\nu)}(x) := \frac{1}{n!} x^{-\nu} e^{x} \frac{d^n}{dx^n} (x^{\nu+n} e^{-x}) = \sum_{k=0}^{n} \frac{(-\nu+k+1)_{n-k}}{(n-k)! k!} (-x)^k, \quad n \in \mathbb{N}_0, \quad \nu \in (-1, \infty),
\]

where \((\alpha)_n := \alpha(\alpha+1) \cdots (\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha), n \in \mathbb{N}, (\alpha)_0 := 1\).

The correlation kernels \((1.21)\) are written in the Christoffel–Darboux form as,

\[
K_\varphi^{(N)}(x, x') = K_\text{Hermite}(x, x') = \sum_{n=0}^{N-1} \varphi_n(x) \varphi_n(x') = \sqrt{\frac{N}{2}} \varphi_N(x) \varphi_{N-1}(x') - \varphi_N(x') \varphi_{N-1}(x), \quad x, x' \in \mathbb{R},
\]

and

\[
K_\phi^{(N)}(y, y') = K_\phi^{(\nu, N)}(y, y') = \frac{1}{N} \sum_{n=0}^{N-1} \phi_n^{(\nu)}(y) \phi_n^{(\nu)}(y') = -\sqrt{\frac{N}{N+N+\nu}} \frac{\phi_N^{(\nu)}(y) \phi_N^{(\nu)}(y') - \phi_N^{(\nu)}(y') \phi_N^{(\nu)}(y)}{y-y'}, \quad y, y' \geq 0.
\]

When \( x = x' \) or \( y = y' \), we make sense of the above formulas by using L’Hôpital’s rule. The former is called the Hermite kernel and the latter is the Laguerre kernel.

By definition, for a finite DPP \( (\Xi, K^{(N)}_\text{Hermite}, \lambda(dx)) \) with \( N \) points in \( S \), the probability density with respect to \( \lambda^{N} (dx_1 \cdots dx_N) \) is given by \( \rho^N(x_1, \ldots, x_N) = \det_{1 \leq j, k \leq N} [K(x_j, x_k)], x = (x_1, \ldots, x_N) \in S^N \). Using the Vandermonde determinantal formula, \( \det_{1 \leq j, k \leq N} (z_k - z_j) \), which will be given also as the type \( A_{N-1} \) of Weyl denominator formula \((1.31)\) below, we can verify that the probability densities of the DPPs \( (\Xi, K^{(N)}_\text{Hermite}, \lambda^{N(0, 1/2)}(dx)) \) and \( (\Xi, K^{(N)}_\text{Laguerre}, \lambda^{(\nu+1/2)}_\nu(dy)) \) with respect to the Lebesgue measures \( dx = \prod_{j=1}^{N} dx_j \) and \( dy = \prod_{j=1}^{N} dy_j \) are given as

\[
P^{(N)}_\text{Hermite}(x) = \frac{1}{Z^{(N)}_\text{Hermite}} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2 \prod_{\ell=1}^{N} e^{-x_{\ell}^2}, \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N,
\]

\[
P^{(\nu, N)}_\text{Laguerre}(y) = \frac{1}{Z^{(\nu, N)}_\text{Laguerre}} \prod_{1 \leq j < k \leq N} (y_k - y_j)^2 \prod_{\ell=1}^{N} y_{\ell}^\nu e^{-y_\ell}, \quad \nu \in (-1, \infty), \quad y \in \mathbb{R}^N_\geq.
\]
with the normalization constants $Z_{\text{Hermite}}^{(N)}$ and $Z_{\text{Laguerre}}^{(\nu,N)}$.

The DPP $(\Xi, K_{\text{Hermite}}^{(N)}, \lambda_{N(0,1/2)}(dy))$ describes the eigenvalue distribution of $N \times N$ Hermitian random matrices in the Gaussian unitary ensemble (GUE). When $\nu \in \mathbb{N}_0$, the DPP $(\Xi, K_{\text{Laguerre}}^{(N)}, \lambda_{(\nu+1,1)}(dx))$ describes the distribution of the nonnegative square roots of eigenvalues of $M^\dagger M$, where $M$ is $(N + \nu) \times N$ complex random matrix in the chiral Gaussian ensemble (chGUE) and $M^\dagger$ is its Hermitian conjugate. The probability density (1.29) can be extended to any $\nu \in (-1, \infty)$ and it is called the complex Laguerre ensemble or the complex Wishart ensemble. Many other examples of one-dimensional DPPs are given as eigenvalue ensembles of Hermitian random matrices in the literatures of random matrix theory (see, for instance, [73, 32, 54]).

**Remark 1.14** It should be noted that we can regard (1.29) as the Gibbs measures

$$P_{\text{Hermite}}^{(N)}(x) = \frac{1}{Z_{\text{Hermite}}^{(N)}} e^{-\beta V_{\text{Hermite}}^{(N)}(x)}, \quad x \in \mathbb{R}^N,$$

$$P_{\text{Laguerre}}^{(\nu,N)}(y) = \frac{1}{Z_{\text{Laguerre}}^{(\nu,N)}} e^{-\beta V_{\text{Laguerre}}^{(\nu,N)}(y)}, \quad \nu \in (-1, \infty), \quad y \in \mathbb{R}_{\geq 0}^N,$$

with the inverse temperature $\beta = 2$ associated with the potentials,

$$V_{\text{Hermite}}^{(N)}(x) = - \sum_{1 \leq j < k \leq N} \log |x_k - x_j| + \frac{1}{2} \sum_{j=1}^N x_j^2, \quad x \in \mathbb{R}^N,$$

$$V_{\text{Laguerre}}^{(\nu,N)}(y) = - \sum_{1 \leq j < k \leq N} \log |y_k - y_j| + \frac{1}{2} \sum_{j=1}^N (-\nu \log y_j + y_j), \quad \nu \in (-1, \infty), \quad y \in \mathbb{R}_{\geq 0}^N.$$

*Here the interactions are given by logarithmic two-body potentials.*

**1.6.2 Duality relations between DPPs in continuous and discrete spaces**

We consider the simplified setting (1.22) of $W$ with $\Gamma = \mathbb{N}_0$. If we set $\Lambda_1 = [r, \infty) \subset S_1 = \mathbb{R}$, $r \in \mathbb{R}$ and $\Lambda_2 = \{0, 1, \ldots, N - 1\} \subset S_2 = \Gamma = \mathbb{N}_0$, $N \in \mathbb{N}$ in (1.13), we obtain

$$K_{\mathbb{R}}^{\{0,1,\ldots,N-1\}}(x, x') = \sum_{n=0}^{N-1} \varphi_n(x) \overline{\varphi_n(x')}, \quad x, x' \in \mathbb{R},$$

$$K_{\mathbb{N}_0}^{[r,\infty)}(n, n') = \int_r^{\infty} \varphi_n(x) \varphi_n'(x) \lambda_1(dx), \quad n, n' \in \mathbb{N}_0. \quad (1.30)$$

When $\lambda_1(dx)$ and $\{\varphi_n\}_{n \in \mathbb{N}_0}$ are given by (1.23) or by (1.24), the kernels (1.30) are given by

$$K_{\text{DHermite}}^{+(r)}(n, n') = (\pi 2^{n+n'+1} n! n')^{-1/2} \int_r^{\infty} H_n(x) H_n'(x) e^{-x^2} dx$$

$$= - (\pi n! n' 2^{n+n'+2})^{-1/2} e^{-r^2} H_{n+1}(r) H_{n'+1}(r) - H_n(r) H_{n'+1}(r),$$
and, provided $r > 0$,

$$
K_{DLaguerre}^{+(r, ν+1)}(n, n') = \left( \frac{n!n'}{\Gamma(n + ν + 1)\Gamma(n' + ν + 1)} \right)^{1/2} \int_{r}^{\infty} L_n^{(ν)}(x)L_{n'}^{(ν)}(x)e^{-x}dx
$$

with the convention that $L_0^{(ν)}(r) = 0$, respectively (see Propositions 3.3 and 3.4 in [15]). Borodin and Olshanski called the correlation kernels $K_{DHermite}^{+(r)}$ and $K_{DLaguerre}^{+(r, ν+1)}$ the discrete Hermite kernel and the discrete Laguerre kernel, respectively. Theorem 1.7 gives

$$
P(Ξ_1^{(0, 1, ..., N-1)}([r, \infty)) = m) = P(Ξ_2^{([r, \infty)}}([0, 1, ..., N-1]) = m), \quad \forall m \in \mathbb{N}_0,
$$

where LHS denotes the probability that the number of points in the interval $[r, \infty)$ is $m$ for the $N$-point continuous DPP on $\mathbb{R}$ such as $(Ξ_1, K_{Hermite}^{(N)}, \lambda_{N(0, 1/2)}(dx))$ or $(Ξ_1, K_{Laguerre}^{(N)}, \lambda_{(ν, 1, 1)}(dx))$, $ν ∈ (-1, \infty)$, while RHS does the probability that the number of points in $\{0, 1, ..., N-1\}$ is $m$ for the discrete DPP on $\mathbb{N}_0$ such as $(Ξ_2, K_{DHermite}^{+(r)})$ or $(Ξ_2, K_{DLaguerre}^{+(r, ν+1)})$, $ν ∈ (-1, \infty)$. The duality between continuous and discrete ensembles of Borodin and Olshanski (Theorem 3.7 in [15]) is a special case with $m = 0$ of the equality (1.31).

### 1.6.3 Finite DPPs in intervals related with classical root systems

Let $N ∈ \mathbb{N}$ and consider the four types of classical root systems denoted by $A_{N-1}, B_N, C_N,$ and $D_N$. We set $S^{A_{N-1}} = S^1 = [0, 2\pi)$, the unit circle, with a uniform measure $\lambda^{A_{N-1}}(dx) = \lambda_{[0, 2\pi]}(dx) := dx/(2\pi)$, and $S^{R_N} = [0, π]$, the upper half-circle, with $\lambda^{R_N}(dx) = \lambda_{[0, \pi]}(dx) := dx/π$ for $R_N = B_N, C_N, D_N$.

For a fixed $N ∈ \mathbb{N}$, we introduce the four sets of functions $\{ϕ_n^{R_N}\}_{n=1}^{N}$ on $S^{R_N}$ defined as

$$
ϕ_n^{R_N}(x) = \begin{cases} 
e^{-i(\lambda^{A_{N-1}} - 2J^{R_N}(n))x/2}, & R_N = A_{N-1}, \\ \sin((N^{R_N} - 2J^{R_N}(n))x/2), & R_N = B_N, C_N, \\ \cos((N^{R_N} - 2J^{D_N}(n))x/2), & R_N = D_N, \end{cases}
$$

where

$$
N^{R_N} = \begin{cases} N, & R_N = A_{N-1}, \\ 2N - 1, & R_N = B_N, \\ 2(N + 1), & R_N = C_N, \\ 2(N - 1), & R_N = D_N. \end{cases}
$$

and

$$
J^{R_N}(n) = \begin{cases} n - 1/2, & R_N = A_{N-1}, \\ n - 1, & R_N = B_N, D_N, \\ n, & R_N = C_N. \end{cases}
$$

It is easy to verify that they satisfy the following orthonormality relations,

$$
\langle ϕ_n^{A_{N-1}}, ϕ_m^{A_{N-1}} \rangle_{L^2(S^1, \lambda_{[0, 2\pi]})} = δ_{nm},
$$

$$
\langle ϕ_n^{R_N}, ϕ_m^{R_N} \rangle_{L^2(\{0, \pi\}, \lambda_{[0, \pi]})} = δ_{nm}, \quad R_N = B_N, C_N, D_N, \quad \text{if } n, m ∈ \{1, ..., N\}.
$$

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We put $\Gamma = \{1, \ldots, N\}, N \in \mathbb{N}$ and $L^2(\Gamma, \nu) = \ell^2(\Gamma)$. By the argument given in Remark 1.13 Assumption 3 is verified, and hence Corollary 1.10 gives the four types of DPPs; $(\Xi, K^{A_{N-1}}, \lambda_{0,2\pi}(dx))$ on $S^1$, and $(\Xi, K^{R_N}, \lambda_{[0,\pi]}(dx))$ on $[0, \pi]$, $R_N = B_N, C_N, D_N$, with the correlation kernels,

$$K^{R_N}(x, x') = \sum_{n=1}^{N} \varphi_n^{R_N}(x) \varphi_n^{R_N}(x')$$

$$= \begin{cases} 
\sin\{N(x-x')/2\}, & R_N = A_{N-1}, \\
\frac{1}{2} \left[ \sin\{N(x-x')\} - \sin\{N(x+x')\} \right], & R_N = B_N, \\
\frac{1}{2} \left[ \sin\{(2N+1)(x-x'/2)\} - \sin\{(2N+1)(x+x'/2)\} \right], & R_N = C_N, \\
\frac{1}{2} \left[ \sin\{(2N-1)(x-x'/2)\} + \sin\{(2N-1)(x+x'/2)\} \right], & R_N = D_N. 
\end{cases}$$

The Weyl denominator formulas for classical root systems play a fundamental role in Lie theory and related area. For a reduced root systems they are given in the form,

$$\sum_{w \in W} \det(w)e^{w(\rho) - \rho} = \prod_{\alpha \in R_+} (1 - e^{-\alpha}),$$

where $W$ is the Weyl group, $R_+$ the set of positive roots and $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$. For classical root systems $A_{N-1}, B_N, C_N$ and $D_N, N \in \mathbb{N}$, the explicit forms are given as follows,

(type $A_{N-1}$) $\det_{1 \leq j, k \leq N} (z_k^{-1}) = \prod_{1 \leq j < k \leq N} (z_k - z_j),$

(type $B_N$) $\det_{1 \leq j, k \leq N} (z_k^{j-N} - z_k^{N+1-j}) = \prod_{\ell=1}^{N} z_{\ell}^{1-N}(1-z_{\ell}) \prod_{1 \leq j < k \leq N} (z_k - z_j)(1-z_jz_k),$

(type $C_N$) $\det_{1 \leq j, k \leq N} (z_k^{j-N-1} - z_k^{N+1-j}) = \prod_{\ell=1}^{N} z_{\ell}^{-N}(1-z_{\ell}^2) \prod_{1 \leq j < k \leq N} (z_k - z_j)(1-z_jz_k),$

(type $D_N$) $\det_{1 \leq j, k \leq N} (z_k^{j-N} + z_k^{N-j}) = 2 \prod_{\ell=1}^{N} z_{\ell}^{1-N} \prod_{1 \leq j < k \leq N} (z_k - z_j)(1-z_jz_k)$ (1.34)

respectively. See, for instance, [84]. If we change the variables as

$$z_k = e^{-2\sqrt{-1} \zeta_k}, \quad \zeta_k \in \mathbb{C}, \quad k = 1, \ldots, N,$$

then, the following equalities are derived from the above.

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Lemma 1.15 For $\zeta_k \in \mathbb{C}, k = 1, \ldots, N$, the following equalities are established.

\[
\begin{align*}
\text{(type A}_{N-1}&\text{)} \quad \det_{1 \leq j, k \leq N} \left[ e^{-\sqrt{-1} \zeta_j A_{N-1}^T - 2 \zeta_j A_{N-1}} \right] = (2i)^{N(N-1)/2} \prod_{1 \leq j < k \leq N} \sin(\zeta_k - \zeta_j). \\
\text{(type B}_{N}&\text{)} \quad \det_{1 \leq j, k \leq N} \left[ \sin\left( (\zeta_j B_N - 2 \zeta_j B_N) \zeta_j \right) \right] \\
&= 2^{N(N-1)} \prod_{\ell=1}^{N} \sin(2\zeta_{\ell}) \prod_{1 \leq j < k \leq N} \sin(\zeta_k - \zeta_j) \sin(\zeta_k + \zeta_j), \\
\text{(type C}_{N}&\text{)} \quad \det_{1 \leq j, k \leq N} \left[ \sin\left( (\zeta_j C_N - 2 \zeta_j C_N) \zeta_j \right) \right] \\
&= 2^{N(N-1)} \prod_{\ell=1}^{N} \sin(2\zeta_{\ell}) \prod_{1 \leq j < k \leq N} \sin(\zeta_k - \zeta_j) \sin(\zeta_k + \zeta_j), \\
\text{(type D}_{N}&\text{)} \quad \det_{1 \leq j, k \leq N} \left[ \cos\left( (\zeta_j D_N - 2 \zeta_j D_N) \zeta_j \right) \right] \\
&= 2^{(N-1)^2} \prod_{1 \leq j < k \leq N} \sin(\zeta_k - \zeta_j) \sin(\zeta_k + \zeta_j),
\end{align*}
\]

where $N^{R_N}$ and $J^{R_N}(j)$, $R_N = A_{N-1}, B_N, C_N, D_N$, are given by (1.32) and (1.33).

By Lemma 1.15, the probability densities for these finite DPPs with respect to the Lebesgue measures, $dx = \prod_{j=1}^{N} dx_j$, are given as

\[
\begin{align*}
p^{A_{N-1}}(x) &= \frac{1}{Z^{A_{N-1}}} \prod_{1 \leq j < k \leq N} \sin^2 \frac{x_k - x_j}{2}, \quad x \in [0, 2\pi]^N, \\
p^{B_N}(x) &= \frac{1}{Z^{B_N}} \prod_{1 \leq j < k \leq N} \left( \sin^2 \frac{x_k - x_j}{2} \sin^2 \frac{x_k + x_j}{2} \right) \prod_{\ell=1}^{N} \sin^2 \frac{x_{\ell}}{2}, \quad x \in [0, \pi]^N, \\
p^{C_N}(x) &= \frac{1}{Z^{C_N}} \prod_{1 \leq j < k \leq N} \left( \sin^2 \frac{x_k - x_j}{2} \sin^2 \frac{x_k + x_j}{2} \right) \prod_{\ell=1}^{N} \sin^2 x_{\ell}, \quad x \in [0, \pi]^N, \\
p^{D_N}(x) &= \frac{1}{Z^{D_N}} \prod_{1 \leq j < k \leq N} \left( \sin^2 \frac{x_k - x_j}{2} \sin^2 \frac{x_k + x_j}{2} \right), \quad x \in [0, \pi]^N,
\end{align*}
\]

with the normalization constants $Z^{R_N}$.

The DPP, $(\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi]}(dx))$ is known as the circular unitary ensemble (CUE) in random matrix theory (see Section 11.8 in [73]). These four types of DPPs, $(\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi]}(dx))$, $(\Xi, K^{R_N}, \lambda_{[0,\pi]}(dx))$, $R_N = B_N, C_N, D_N$ are realized as the eigenvalue distributions of random matrices in the classical groups, $U(N)$, $SO(2N + 1)$, $Sp(N)$, and $SO(2N)$, respectively. (See Section 2.3 c) in [97] and Section 5.5 in [32].)

As mentioned in Remark 1.11, $p^{R_N}(x)$ can be regarded as Gibbs measures $e^{-\beta V^{R_N}(x)}/Z^{R_N}$ with $\beta = 2$, $R_N = A_{N-1}, B_N, C_N, D_N$. For example, for type $A_{N-1}$ the potential is given as

\[
\begin{align*}
V^{A_{N-1}}(x) &= -\sum_{1 \leq j < k \leq N} \log \left| \sin \frac{x_k - x_j}{2} \right|, \quad x \in (0, 2\pi]^N.
\end{align*}
\]
1.6.4 Infinite DPPs in $\mathbb{R}$ associated with classical orthonormal functions

Here we give examples of infinite DPPs obtained by Corollary [1.12]

(i) DPP with the sinc kernel: We set $S_1 = \mathbb{R}$, $\lambda_1(dx) = dx$, $\Gamma = (-1, 1)$, $\nu(dy) = \lambda_2(dy) = dy$, and put

$$\psi_1(x, y) = \frac{1}{\sqrt{2\pi}} e^{-xy}. $$

We see that

$$\Psi_1(x)^2 := ||\psi_1(x, \cdot)||^2 = \frac{1}{\pi}, \quad x \in \mathbb{R},$$

and thus Assumption 3'(ii) is satisfied. The correlation kernel $K_{S_1}$ is given by

$$K_{\text{sinc}}(x, x') = \frac{1}{2\pi} \int_{-1}^{1} e^{-y(x-x')} dy = \frac{\sin(x-x')}{\pi(x-x')}, \quad x, x' \in \mathbb{R}. $$

(ii) DPP with the Airy kernel: We set $S_1 = \mathbb{R}$, $\lambda_1(dx) = dx$, $\Gamma = \mathbb{R}_{\geq 0}$, $\nu(dy) = \lambda_2(dy) = dy$, and put

$$\psi_1(x, y) = \text{Ai}(x+y),$$

where $\text{Ai}(x)$ denotes the Airy function [80].

$$\text{Ai}(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos \left( \frac{k^3}{3} + kx \right) dk. $$

We see that

$$\Psi_1(x)^2 := ||\psi_1(x, \cdot)||^2 = -x\text{Ai}(x)^2 + \text{Ai}'(x)^2, \quad x \in \mathbb{R},$$

and thus Assumption 3(ii) is satisfied. The correlation kernel $K_{S_1}$ is given by

$$K_{\text{Airy}}(x, x') = \int_{0}^{\infty} \text{Ai}(x+y)\text{Ai}(x'+y) dy = \frac{\text{Ai}(x)\text{Ai}'(x') - \text{Ai}(x')\text{Ai}(x)}{x-x'}, \quad x, x' \in \mathbb{R},$$

where $\text{Ai}'(x) = d\text{Ai}(x)/dx$.

(iii) DPP with the Bessel kernel: We set $S_1 = [0, \infty)$, $\lambda_1(dx) = dx$, $\Gamma = [0, 1]$, $\nu(dy) = \lambda_2(dy) = dy$. With parameter $\nu \in (-1, \infty)$ we put

$$\psi_1(x, y) = \sqrt{xy} J_{\nu}(xy),$$

where $J_{\nu}$ is the Bessel function of the first kind defined by

$$J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left( \frac{x}{2} \right)^{2n+\nu}, \quad x \in \mathbb{C} \setminus (-\infty, 0).$$

We see that

$$\Psi_1(x)^2 := ||\psi_1(x, \cdot)||^2 = x\{J_{\nu}(x)^2 - J_{\nu-1}(x)J_{\nu+1}(x)\}/2, \quad x \in \mathbb{R}_{\geq 0}. $$

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and thus Assumption 3'(ii) is satisfied. The correlation kernel $K_{S_1}$ is given by
\[
K_{\text{Bessel}}^{(\nu)}(x, x') = \int_0^1 \frac{\sqrt{x}y J_\nu(xy) \sqrt{x'}y J_\nu(x'y)dy}{x^2 - (x')^2} \{ J_\nu(x)x'J'_\nu(x') - xJ'_\nu(x)J_\nu(x') \}, \quad x, x' \in \mathbb{R}_0,
\] (1.37)
where $J'_\nu(x) = dJ_\nu(x)/dx$.

These three kinds of infinite DPPs, $(\Xi, K_{\text{sinc}}(dx))$, $(\Xi, K_{\text{Airy}}(dx))$, and $(\Xi, K_{\text{Bessel}}^{(\nu)}, 1_{\mathbb{R}_0}(x)dx)$, are obtained as the scaling limits of the finite DPPs, $(\Xi, K^{(N)}_{\text{Hermite}, \lambda N(0,1/2)}(dx))$ and $(\Xi, K^{(N)}_{\text{Laguerre}, \lambda \Gamma(\nu+1,1)}(dx))$, given in Section 1.6.1 as follows.

(i) **Bulk scaling limit,**
\[
\sqrt{2N} \circ (\Xi, K^{(N)}_{\text{Hermite}, \lambda N(0,1/2)}(dx)) \xrightarrow{N \to \infty} (\Xi, K_{\text{sinc}, dx}).
\]

(ii) **Soft-edge scaling limit,**
\[
\sqrt{2}N^{1/6} \circ S_{\sqrt{2}N} (\Xi, K^{(N)}_{\text{Hermite}, \lambda N(0,1/2)}(dx)) \xrightarrow{N \to \infty} (\Xi, K_{\text{Airy}, dx}).
\]

(iii) **Hard-edge scaling limit,** for $\nu \in (-1, \infty)$,
\[
4N \circ \left( (\Xi, K^{(\nu,N)}_{\text{Laguerre}, \lambda \Gamma(\nu+1,1)}(dx))^{(1/2)} \right) \xrightarrow{N \to \infty} (\Xi, K_{\text{Bessel}}^{(\nu)}, 1_{\mathbb{R}_0}(x)dx).
\]

See, for instance, [73, 52, 6, 48], for more details.

The DPPs with the sinc kernel and the Bessel kernel with the special values of parameter $\nu$ can be obtained as the bulk scaling limits of the DPPs, $(\Xi, K^{R_N, \lambda N}(dx))$, $R_N = A_{N-1}, B_N, C_N, D_N$ given in Section 1.6.3 as
\[
\frac{N}{2} \circ (\Xi, K^{A_{N-1}, \lambda_{[0,2\pi]}(dx))) \xrightarrow{N \to \infty} (\Xi, K_{\text{sinc}, dx}),
\]
\[
N \circ (\Xi, K^{B_N, \lambda_{[0,\pi]}(dx)}) \xrightarrow{N \to \infty} (\Xi, K^{(1/2)}_{\text{Bessel}}, 1_{\mathbb{R}_0}(x)dx),
\]
\[
N \circ (\Xi, K^{C_N, \lambda_{[0,\pi]}(dx)}) \xrightarrow{N \to \infty} (\Xi, K^{-1/2}_{\text{Bessel}}, 1_{\mathbb{R}_0}(x)dx),
\]
\[
N \circ (\Xi, K^{D_N, \lambda_{[0,\pi]}(dx)}) \xrightarrow{N \to \infty} (\Xi, K^{(1/2)}_{\text{Bessel}}, 1_{\mathbb{R}_0}(x)dx),
\] (1.38)
where
\[
K^{(1/2)}_{\text{Bessel}}(x, x') = \frac{\sin(x - x')}{\pi(x - x')}, \quad x, x' \in \mathbb{R}_0,
\]
\[
K^{(-1/2)}_{\text{Bessel}}(x, x') = \frac{\sin(x - x')}{\pi(x - x')} + \frac{\sin(x + x')}{\pi(x + x')}, \quad x, x' \in \mathbb{R}_0.
\]
Since $J_{1/2}(x) = \sqrt{2/(\pi x)} \sin x$ and $J_{-1/2}(x) = \sqrt{2/(\pi x)} \cos x$, the above correlation kernels are readily obtained from (1.37) by setting $\nu = 1/2$ and $-1/2$, respectively.
1.7 Examples in two-dimensional spaces

1.7.1 Infinite DPPs on $\mathbb{C}$: Ginibre and Ginibre-type DPPs

Let $\lambda_{N(m,\sigma^2;\mathbb{C})}(dx)$ denote the complex normal distribution,

$$
\lambda_{N(m,\sigma^2;\mathbb{C})}(dx) := \frac{1}{\pi \sigma^2} e^{-|x-m|^2/\sigma^2} dx = \frac{1}{\pi \sigma^2} e^{-(x_\Re-m_\Re)^2/\sigma^2-(x_\Im-m_\Im)^2/\sigma^2} dx d\Im,
$$

$m \in \mathbb{C}, m_\Re := \Re m, m_\Im := \Im m, \sigma > 0$. We set $S = \mathbb{C}$,

$$
\lambda(dx) = \lambda_{N(0,1;\mathbb{C})}(dx) = \frac{1}{\pi} e^{-|x|^2} dx = \lambda_{N(0,1/2)}(dx)\lambda_{N(0,1/2)}(dx_1),
$$

and

$$
\psi^A(x, \gamma) = e^{-(x_\Re^2-x_\Im^2)/2+2x\gamma},
\psi^C(x, \gamma) = \sqrt{2}\sinh(2x\gamma)e^{-(x_\Re^2-x_\Im^2)/2},
\psi^D(x, \gamma) = \sqrt{2}\cosh(2x\gamma)e^{-(x_\Re^2-x_\Im^2)/2}.
$$

It is easy to confirm that

$$
\frac{1}{\pi} \int_{\mathbb{R}} \psi^A(x, \gamma)\overline{\psi^A(x, \gamma')} e^{-x_\Re^2} dx_1 = e^{-\frac{1}{\pi}(4x\gamma)}\delta(\gamma - \gamma'), \ R = C,
$$

$$
\frac{1}{\pi} \int_{\mathbb{R}} \psi^R(x, \gamma)\overline{\psi^R(x, \gamma')} e^{-x_\Re^2} dx_1 = e^{-\frac{1}{\pi}(4x_\Re^2)} \cosh(4x_\Re\gamma) \times \left\{ \delta(\gamma - \gamma') - \delta(\gamma + \gamma'), \ R = C, \right.
$$

$$
\left. \delta(\gamma - \gamma') + \delta(\gamma + \gamma'), \ R = D. \right\}
$$

Therefore, we have

$$
\langle \psi^A(\cdot, \gamma), \psi^A(\cdot, \gamma') \rangle_{L^2(\mathbb{C},\lambda_{N(0,1;\mathbb{C})})} \nu(d\gamma) = \delta(\gamma - \gamma') d\gamma, \ \gamma, \gamma' \in \Gamma^A := \mathbb{R},
$$

$$
\langle \psi^R(\cdot, \gamma), \psi^R(\cdot, \gamma') \rangle_{L^2(\mathbb{C},\lambda_{N(0,1;\mathbb{C})})} \nu(d\gamma) = \delta(\gamma - \gamma') d\gamma, \ \gamma, \gamma' \in \Gamma^R := (0, \infty), \ R = C, D,
$$

with $\nu(d\gamma) = \lambda_{N(0,1/4)}(d\gamma)$. We also see that

$$
\Psi^A(x) := \|\psi^A(x, \cdot)\|_{L^2(\Gamma,\nu)}^2 = e^{||x||^2},
$$

$$
\Psi^C(x) := \|\psi^C(x, \cdot)\|_{L^2(\Gamma,\nu)}^2 = \sinh ||x||^2,
$$

$$
\Psi^D(x) := \|\psi^D(x, \cdot)\|_{L^2(\Gamma,\nu)}^2 = \cosh ||x||^2, \quad x \in \mathbb{C}.
$$

Thus Assumption 3 is satisfied and we can apply Corollary 1.10 The kernels (1.16) of obtained DPPs are given as

$$
K^A(x, x') = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}(x_\Re^2-x_\Im^2)+(x_\Re^2-x_\Im^2)/2} \int_{-\infty}^{\infty} e^{-2(\gamma^2-(x+x'\gamma))} d\gamma,
$$

$$
K^C(x, x') = \frac{2}{\pi} e^{-\frac{1}{2}(x_\Re^2-x_\Im^2)+(x_\Re^2-x_\Im^2)/2} \int_{0}^{\infty} e^{-2\gamma^2} \sinh(2x\gamma) \sinh(2x'\gamma) d\gamma,
$$

$$
K^D(x, x') = \frac{2}{\pi} e^{-\frac{1}{2}(x_\Re^2-x_\Im^2)+(x_\Re^2-x_\Im^2)/2} \int_{0}^{\infty} e^{-2\gamma^2} \cosh(2x\gamma) \cosh(2x'\gamma) d\gamma.
$$
The integrals are performed and we obtain \( K^R(x, x') = e^{\sqrt{\Gamma} x x'} K^R_{\text{Ginibre}}(x, x') e^{-\sqrt{\Gamma} x x'}, R = A, C, D, \) with
\[
K^A_{\text{Ginibre}}(x, x') = e^{\pi x}, \\
K^C_{\text{Ginibre}}(x, x') = \sinh(x x'), \\
K^D_{\text{Ginibre}}(x, x') = \cosh(x x'), \quad x, x' \in \mathbb{C}.
\]

Due to the gauge invariance of DPP mentioned in Section 1.3, the obtained three types of infinite DPPs on \( \mathbb{C} \) are written as \( (\Xi, K^R_{\text{Ginibre}}, \lambda_{N(0;1;\mathbb{C})}(dx)), R = A, C, D. \) The DPP, \( (\Xi, K^A_{\text{Ginibre}}, \lambda_{N(0;1;\mathbb{C})}(dx)) \) with (1.39) describes the eigenvalue distribution of the Gaussian random complex matrix in the bulk scaling limit, which is called the complex Ginibre ensemble \([35, 73, 44, 32, 92]\). This density profile is uniform with the Lebesgue measure \( dx \) on \( \mathbb{C} \) as
\[
\rho_{\text{Ginibre}}(x) dx = K^A_{\text{Ginibre}}(x, x) \lambda_{N(0;1;\mathbb{C})}(dx) = \frac{1}{\pi} dx dx_1, \quad x \in \mathbb{C}.
\]

On the other hands, the Ginibre DPPs of types \( C \) and \( D \) with the correlation kernels (1.40) and (1.41) are rotationally symmetric around the origin, but non-uniform on \( \mathbb{C} \). The density profiles with the Lebesgue measure \( dx \) on \( \mathbb{C} \) are given by
\[
\rho^C_{\text{Ginibre}}(x) dx = K^C_{\text{Ginibre}}(x, x) \lambda_{N(0;1;\mathbb{C})}(dx) = \frac{1}{2\pi} (1 - e^{-2|x|^2}) dx dx_1, \quad x \in \mathbb{C}, \\
\rho^D_{\text{Ginibre}}(x) dx = K^D_{\text{Ginibre}}(x, x) \lambda_{N(0;1;\mathbb{C})}(dx) = \frac{1}{2\pi} (1 + e^{-2|x|^2}) dx dx_1, \quad x \in \mathbb{C}.
\]

They were first obtained in [50] by taking the limit \( W \to \infty \) keeping the density of points of the infinite DPPs in the strip on \( \mathbb{C} \), \( \{z \in \mathbb{C} : 0 \leq \text{Im} \ z \leq W\} \).

1.7.2 Representations of Ginibre and Ginibre-type kernels in the Bargmann–Fock space and the eigenspaces of Landau levels

One of the advantages of our framework is that the we can obtain pairs of DPPs satisfy useful duality relations. Now we concentrate on one of a pair of DPPs constructed in our framework, \( (\Xi_1, K_{S_1}, \lambda_1) \). The correlation kernel \( K_{S_1} \) is given by the first equation of (1.7), that is,
\[
K_{S_1}(x, x') = \int_{S_2} \overline{W}(y, x) W(y, x') \lambda_2(dy) = (W(\cdot, x'), W(\cdot, x))_{L^2(S_2, \lambda_2)}, \quad x, x' \in S_1,
\]
which is an integral kernel for \( f \in L^2(S_1, \lambda_1) \). We can regard this equation as a decomposition formula of \( K_{S_1} \) by a product of \( W \) and \( \overline{W} \). Since \( W \) is an integral kernel for an isometry \( L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2) \), as a matter of course, it depends on a choice of another Hilbert space \( L^2(S_2, \lambda_2) \). We note that a given DPP, \( (\Xi_1, K_{S_1}, \lambda_1) \), choice of \( L^2(S_2, \lambda_2) \) is not unique. Such multivalency gives plural different expressions for one correlation kernel \( K_{S_1} \) and they reveal different aspects of the DPP.

Here we demonstrate this fact using the three kinds of Ginibre DPPs associated with \( L^2(\mathbb{C}, \lambda_{N(0;1;\mathbb{C})}) \). In the previous section we have chosen the parameter spaces as \( L^2(\Gamma^R, \lambda_{N(0;1/4)}) \) with \( \Gamma^A = \mathbb{R} \) and \( \Gamma^C = \Gamma^D = (0, \infty) \). We will choose another parameter spaces below.
Let $S_1 = \mathbb{C}$ and $S_2 = \mathbb{N}_0$ with $\lambda_1(dx) = \lambda_{N(0,1;\mathbb{C})}(dx)$. We put
\[
\varphi_n(x) = \frac{x^n}{\sqrt{n!}}, \quad n \in \mathbb{N}_0.
\] (1.42)

Note that $\{\varphi_n(x)\}_{n \in \mathbb{N}_0}$ forms a complete orthonormal system of the Bargmann–Fock space, which is the space of square-integrable analytic functions on $\mathbb{C}$ with respect to the complex Gaussian measure;
\[
\langle \varphi_n, \varphi_m \rangle_{L^2(\mathbb{C}, \lambda_{N(0,1;\mathbb{C})})} = \delta_{nm}, \quad n, m \in \mathbb{N}_0.
\]
We assume that $\Gamma = S_2 = \mathbb{N}_0$. We put
\[
\|\varphi_n\|_{\ell^2(\Gamma)} = \sum_{n \in \mathbb{N}_0} \frac{|x|^2 n^n}{n!} = e^{|x|^2}, \quad x \in \mathbb{C}.
\]

Hence Assumption 3' is satisfied. By Corollary 1.12, we obtain the DPP on $\mathbb{C}$ in which the correlation kernel with respect to $\lambda_{N(0,1;\mathbb{C})}$ is given by
\[
K_{\text{BF}}(x,x') = \sum_{n \in \mathbb{N}_0} \varphi_n(x) \overline{\varphi_n(x')} = e^{xx'}, \quad x, x' \in \mathbb{C}.
\]

This is the reproducing kernel in the Bargmann–Fock space and obtained DPP is identified with $(\Xi, K^{A}_{\text{Ginibre}}, \lambda_{N(0,1;\mathbb{C})}(dx))$. See [92, 19, 2].

If we set $\Gamma = 2\mathbb{N}_0 + 1 = \{1, 3, 5, \ldots\}$ or $\Gamma = 2\mathbb{N}_0 = \{0, 2, 4, \ldots\}$, we will obtain the DPPs with the following kernels
\[
K^{\text{odd}}_{\text{BF}}(x,x') = \sum_{k=0}^{\infty} \frac{(xx')^{2k+1}}{(2k+1)!} = \sinh(xx'), \quad x, x' \in \mathbb{C}.
\]
\[
K^{\text{even}}_{\text{BF}}(x,x') = \sum_{k=0}^{\infty} \frac{(xx')^{2k}}{(2k)!} = \cosh(xx'), \quad x, x' \in \mathbb{C}.
\]

The obtained DPPs are identified with $(\Xi, K^{A}_{\text{Ginibre-type}}, \lambda_{N(0,1;\mathbb{C})}(dx))$ and $(\Xi, K^{D}_{\text{Ginibre}}, \lambda_{N(0,1;\mathbb{C})}(dx))$, respectively.

The Ginibre DPP of type A is extended to Ginibre-type DPPs indexed by $q \in \mathbb{N}_0, (\Xi, K^{(q)}_{\text{Ginibre-type}}, \lambda_{N(0,1;\mathbb{C})}(dx)), q \in \mathbb{N}_0$, which are introduced in [92] and also known as the infinite pure polyanalytic ensembles (cf. [2]). Each Ginibre-type DPP with index $q \in \mathbb{N}_0$ is associated with the correlation kernel
\[
K^{(q)}_{\text{Ginibre-type}}(x,x') := L_q^{(0)}(|x-x'|^2)K^{A}_{\text{Ginibre}}(x,x'), \quad x, x' \in \mathbb{C},
\] (1.43)
where $L_q^{(0)}$ is the $q$-th Laguerre polynomial (1.26) with parameter $\nu = 0$ and $K^{A}_{\text{Ginibre}}$ is defined by (1.39). The correlation kernel (1.43) admits the similar representation in terms of the complex Hermite polynomials defined by
\[
H_{p,q}(\zeta, \overline{\zeta}) = (-1)^{p+q} e^{\zeta \overline{\zeta}} \frac{\partial^p}{\partial \overline{\zeta}^p} \frac{\partial^q}{\partial \zeta^q} e^{-\overline{\zeta} \zeta}, \quad \zeta \in \mathbb{C}, \quad p, q \in \mathbb{N}_0,
which were introduced by Itô [15]. We note that their generating function is given by

\[
\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} H_{p,q}(\zeta, \bar{\zeta}) \frac{s^p q^q}{p! q!} = \exp(\zeta s + \bar{\zeta} t - st)
\]

and the set \( \{ H_{p,q}(\zeta, \bar{\zeta})/\sqrt{p! q!} : p, q \in \mathbb{N}_0 \} \) forms a complete orthonormal system of \( L^2(\mathbb{C}, \lambda_N(0,1;\mathbb{C})(d\zeta)) \).

Let \( S_1 = \mathbb{C} \) and \( S_2 = \mathbb{N}_0 \) with \( \lambda_1(dx) = \lambda_N(0,1;\mathbb{C})(dx) \), and for fixed \( q \in \mathbb{N}_0 \), define

\[
\varphi_n^{(q)}(x) := \frac{1}{\sqrt{n! q!}} H_{n,q}(x, \bar{x}), \quad x \in \mathbb{C}, \quad n \in \mathbb{N}_0.
\]

Then \( \{ \varphi_n^{(q)}(x) \}_{n \in \mathbb{N}_0} \) forms a complete orthonormal system of the eigenspace corresponding to the \( q \)-th Landau level, which coincides with the Bargmann–Fock space when \( q = 0 \). Since the following formula is known

\[
L_q^{(0)}(|\zeta - \eta|^2)e^{\bar{\zeta} \eta} = \sum_{p=0}^{\infty} \frac{1}{p! q!} H_{p,q}(\zeta, \bar{\zeta}) H_{p,q}(\eta, \bar{\eta}), \quad \zeta, \eta \in \mathbb{C}, \quad q \in \mathbb{N}_0,
\]

we obtain the following expansion formula for (1.43),

\[
K_{\text{Ginibre-type}}^{(q)}(x, x') = \sum_{n=0}^{\infty} \varphi_n^{(q)}(x) \overline{\varphi_n^{(q)}(x')}, \quad x, x' \in \mathbb{C}, \quad q \in \mathbb{N}_0.
\]

### 1.7.3 Application of duality relations

We consider the simplified setting (1.22) of \( W \) with (1.32) and \( \Gamma = \mathbb{N}_0 \). If we set \( \lambda_1(dx) = \lambda_N(0,1;\mathbb{C})(dx) \), \( \Lambda_1 \) be a disk (i.e., two-dimensional ball) \( \mathbb{B}_r^2 \) with radius \( r \in (0, \infty) \) centered at the origin in \( S_1 = \mathbb{C} \cong \mathbb{R}^2 \) and \( \Lambda_2 = S_2 = \mathbb{N}_0 \) in (1.43), we obtain

\[
K_{\mathbb{C}}^{(N_0)}(x, x') = \sum_{n=0}^{\infty} \varphi_n(x) \overline{\varphi_n(x')} = e^{\bar{\zeta} x'}
\]

where \( K_{\text{Ginibre}}^A(x, x') \) denotes the correlation kernel of the Ginibre DPP of type \( A \), and

\[
K_{\mathbb{B}_r^2}^{(N_0)}(n, n') = \int_{\mathbb{B}_r^2} \varphi_n(x) \varphi_{n'}(x) \lambda_N(0,1;\mathbb{C})(dx) = \frac{1}{\pi \sqrt{n n'}} \int_0^r ds e^{-s^2} s^{n+n'+1} \int_0^{2\pi} d\theta e^{i\theta (n'-n)}
\]

\[
= 2\delta_{n n'} \int_0^r s^{2n+1} e^{-s^2} n! ds = \delta_{n n'} \int_0^r \lambda_{\Gamma(n+1,1)}(du), \quad n, n' \in \mathbb{N}_0.
\]

Define

\[
\lambda_n(r) := \int_0^r \frac{u^n e^{-u}}{n!} du = \sum_{k=n+1}^{\infty} \frac{r^k e^{-r^2}}{k!}, \quad n \in \mathbb{N}_0, \quad r \in (0, \infty),
\]

where the second equality is due to Eq.(4.1) in [92]. That is, if we write the Gamma distribution with parameters \((a, b)\) as \( \Gamma(a, b) \) (see Section 1.6.1) and the Poisson distribution with parameter \( c \) as \( \text{Po}(c) \),

\[
\lambda_n(r) := P(R_n \leq r^2) = P(Y_{r^2} \geq n + 1),
\]

25
provided $R_n \sim \Gamma(n+1,1)$ and $Y_{r^2} \sim \text{Po}(r^2)$. Then DPP $(\Xi_2^{(B^2)}, K^{(B^2)}_{n_0})$ on $\mathbb{N}_0$ is the product measure $\bigotimes_{n \in \mathbb{N}_0} \mu^{\text{Bernoulli}}_{n(r)}$ under the natural identification between $\{0,1\}^{\mathbb{N}_0}$ and the power set of $\mathbb{N}_0$, where $\mu^{\text{Bernoulli}}_p$ denotes the Bernoulli measure of probability $p \in [0,1]$. Theorem 1.7 gives the duality relation
\[ P(\Xi_1^{(B_r^2)} = m) = P(\Xi_2^{(B_2)}(\mathbb{N}_0) = m), \quad \forall m \in \mathbb{N}_0, \]
where we have identified the DPP, $(\Xi_1^{(N_0)}, K^{(N_0)}_{\mathbb{C}_1}, \lambda_1(dx))$ with the Ginibre DPP of type A, $(\Xi_1^{(G_{\mathbb{C}})}, K^{(G_{\mathbb{C}})}_{\mathbb{C}}, \lambda_{N(0,1,\mathbb{C})})$. If we introduce a series of random variables $X_n^{(r)} \in \{0,1\}, n \in \mathbb{N}_0$, which are mutually independent and $X_n^{(r)} \sim \mu^{\text{Bernoulli}}_{n(r)}$, then the above implies the equivalence in probability law
\[ \Xi_1^{(G_{\mathbb{C}})}(law) \Xi_2^{(B_2)}(law) = \sum_{n \in \mathbb{N}_0} X_n^{(r)}, \quad r \in (0, \infty). \]

Similarly, we have the following equalities by the results Section 1.7.2 and Theorem 1.7,
\[ \Xi_1^{(G_{\mathbb{C}})}(law) = \sum_{n \in 2\mathbb{N}_0+1} X_n^{(r)}, \quad \Xi_2^{(B_2)}(law) = \sum_{n \in 2\mathbb{N}_0} X_n^{(r)}, \quad r \in (0, \infty). \]

The argument above is valid for general radially symmetric DPPs associated with radially symmetric finite measure $\lambda_1(dx) = p(|x|)dx$ on $\mathbb{C}$. Let $\varphi_n(x) = a_nx^n, n \in \mathbb{N}_0$ be an orthonormal system in $L^2(\mathbb{C}, \lambda_1)$ where $a_n > 0, n \in \mathbb{N}_0$ are the normalization constants, and we set
\[ K^{(N_0)}_{\mathbb{C}}(x,x') = \sum_{n=0}^{\infty} \varphi_n(x)\overline{\varphi_n(x')} = \sum_{n=0}^{\infty} a_n(x\overline{x'})^n, \quad x, x' \in \mathbb{C}, \]
\[ K^{(B_2)}_{N_0}(n,n') = \int_{B^2_r} \varphi_n(x)\overline{\varphi_n'(x)} \lambda_1(dx) = \delta_{nn'}\lambda_n(r), \quad n, n' \in \mathbb{N}_0, \]
where
\[ \lambda_n(r) := \frac{1}{Z_n} \int_0^{2} u^n p(\sqrt{u}) du \]
with $Z_n = \int_0^\infty u^n p(\sqrt{u}) du$. Then DPP $(\Xi_1^{(N_0)}, K^{(N_0)}_{\mathbb{C}}, p(|x|)dx)$ on $\mathbb{C}$ is radially symmetric and DPP $(\Xi_2^{(B_2)}, K^{(B_2)}_{N_0})$ on $\mathbb{N}_0$ is again identified with the product measure $\bigotimes_{n \in \mathbb{N}_0} \mu^{\text{Bernoulli}}_{n(r)}$. For example, if $p(s) = \pi^{-1}e^{-s^2}$ and $a_n = 1/\sqrt{n!}$, then $(\Xi_1^{(N_0)}, K^{(N_0)}_{\mathbb{C}}, p(|x|)dx)$ is the Ginibre DPP of type A. The function $\lambda_n(r)$ is considered as a probability distribution function on $[0, \infty)$ and hence there exist independent random variables $R_n, n \in \mathbb{N}_0$ such that
\[ \lambda_n(r) = P(R_n \leq r^2). \]

If we define $X_n^{(r)} = 1_{\{R_n \leq r^2\}}$ for each $n \in \mathbb{N}_0$, then Theorem 1.7 gives the duality relation
\[ \Xi_1^{(N_0)}(law) = \Xi_2^{(B_2)}(law) = \sum_{n \in \mathbb{N}_0} X_n^{(r)}, \quad r \in (0, \infty). \]

Indeed, $\{X_n^{(r)}, n \in \mathbb{N}_0\}$ are mutually independent $\{0,1\}$-valued random variables whose laws are given by $\{\mu^{\text{Bernoulli}}_{n(r)} \in \mathbb{N}_0\}$. If we take a set $\Lambda_2 \subset \mathbb{N}_0$, then DPP $(\Xi_1^{(A_2)}, K^{(A_2)}_{\mathbb{C}}, p(|x|)dx)$ satisfies
\[ \Xi_1^{(A_2)}(law) = \Xi_2^{(B_2)}(law) = \sum_{n \in \Lambda_2} X_n^{(r)}, \quad r \in (0, \infty). \]
We note that if we write $\Xi_{1}^{(N_{0})} = \sum_{i} \delta_{x_{i}}$, then $\sum_{i} \delta_{|x_{i}|^{2}}$ is equal to $\sum_{n \in N_{0}} \delta_{R_{n}}$ in law, which was discussed in Theorem 4.7.1 in [43] by constructing $\{R_{n}\}_{n \in N_{0}}$ in terms of size-biased sampling.

1.8 Examples in spaces with arbitrary dimensions

1.8.1 Euclidean family of infinite DPPs on $\mathbb{R}^{d}$

For $d \in \mathbb{N}$, let $S_{1} = S_{2} = \mathbb{R}^{d}$, $\lambda_{1}(dx) = dx$, $\lambda_{2}(dy) = \nu(dy) = e^{\sqrt{2}r}y / (2\pi)^{d/2}$, and $\Gamma = B \subseteq \mathbb{R}^{d}$, where $B$ denotes the unit ball centered at the origin; $B := \{ y \in \mathbb{R}^{d} : |y| \leq 1 \}$. We see

$$\Psi_{1}(x)^{2} := \|\psi_{1}(x)\|_{L_{2}(\Gamma, d\nu)}^{2} = |B|/(2\pi)^{d}, \quad x \in \mathbb{R}^{d},$$

where the volume of $B$ is denoted by $|B| = \pi^{d/2}/\Gamma((d + 2)/2)$. Then Assumption 3’ is satisfied and Corollary [1,12] gives the DPP in $S_{1} = \mathbb{R}^{d}$ whose correlation kernel with respect to $\lambda_{1}(dx) = dx$ is given by

$$K^{(d)}(x, x') = \frac{1}{(2\pi)^{d}} \int_{\mathbb{R}^{d}} e^{\sqrt{2}r(x-x')}\cdot y\, dy. \quad (1.44)$$

The kernels $K^{(d)}$ on $\mathbb{R}^{d}, d \in \mathbb{N}$ have been studied by Zelditch and others (see 103, 96, 104, 20 and references therein), who regarded them as the Szegő kernels for the reduced Euclidean motion group. Here we call the DPPs associated with the correlation kernels in this form the Euclidean family of DPPs on $\mathbb{R}^{d}, d \in \mathbb{N}$. We can verify other expressions of $K^{(d)}, d \in \mathbb{N}$ using the Bessel function of the first kind [130] as follows [53].

**Definition 1.16** The Euclidean family of DPP on $\mathbb{R}^{d}, d \in \mathbb{N}$ is defined by $\left( \Xi, K^{(d)}_{\text{Euclidean}}, dx \right)$ with the correlation kernel

$$K^{(d)}_{\text{Euclid}}(x, x') = \frac{1}{(2\pi)^{d}} \int_{\mathbb{R}^{d}} 1_{B}(y) e^{\sqrt{2}r(x-x')}\cdot y\, dy = \frac{1}{(2\pi)^{d}} \int_{B} e^{\sqrt{2}r(x-x')}\cdot y\, dy,

= \frac{1}{(2\pi)^{d/2} |x-x'|^{(d-2)/2}} \int_{0}^{1} s^{d/2} J_{d-2}(s)\cdot ds,

= \frac{1}{(2\pi)^{d/2} |x-x'|^{d/2}}, \quad x, x' \in \mathbb{R}^{d}.$$

We see that

$$K^{(d)}_{\text{Euclid}}(x, x) = \lim_{r \to 0} \frac{1}{(2\pi)^{d/2} r^{d/2}} \frac{J_{d/2}(r)}{r^{d/2}} = \frac{1}{2^{d} \pi d/2 \Gamma((d + 2)/2)}.$$

Then the Euclidean family of DPP is uniform on $\mathbb{R}^{d}$ with the density

$$\rho^{(d)}_{\text{Euclid}} = \frac{1}{2^{d} \pi^{d/2} \Gamma((d + 2)/2)}$$

with respect to the Lebesgue measure $dx$ of $\mathbb{R}^{d}$.
For lower dimensions, the correlation kernels and the densities are given as follows,

\[ K^{(1)}_{\text{Euclid}}(x, x') = \frac{\sin(x-x')}{\pi (x-x')} = K_{\text{sinc}}(x, x') \quad \text{with} \quad \rho^{(1)}_{\text{Euclid}} = \frac{1}{\pi}, \]

\[ K^{(2)}_{\text{Euclid}}(x, x') = \frac{J_1(|x-x'|)}{2\pi|x-x'|} \quad \text{with} \quad \rho^{(2)}_{\text{Euclid}} = \frac{1}{4\pi}, \]

\[ K^{(3)}_{\text{Euclid}}(x, x') = \frac{1}{2\pi^2 |x-x'|^2} \left( \frac{\sin|x-x'|}{|x-x'|} - \cos|x-x'| \right) \quad \text{with} \quad \rho^{(3)}_{\text{Euclid}} = \frac{1}{6\pi^2}. \]

This family of DPPs includes the DPP with the sinc kernel \( K_{\text{sinc}} \) as the lowest dimensional case with \( d = 1 \). Note that, if \( d \) is odd,

\[ K^{(d)}_{\text{Euclid}}(x, x') = \left( -\frac{1}{2\pi} \frac{d}{dr} \right)^{(d-1)/2} \frac{\sin r}{\pi r} \quad \text{with} \quad r = |x-x'|. \]

This is proved by Rayleigh’s formula for the spherical Bessel function of the first kind (Eq. (10.49.14) in [80]):

\[ j_m(x) := \sqrt{\frac{\pi}{2x}} J_{m+1/2}(x) = x^m \left( -\frac{1}{x} \frac{d}{dx} \right) \frac{\sin x}{x}, \quad m \in \mathbb{N}. \]

### 1.8.2 Heisenberg family of infinite DPPs on \( \mathbb{C}^d \)

The Ginibre DPP of type A on \( \mathbb{C} \) given in Section 1.7.1 can be generalized to the DPPs on \( \mathbb{C}^d \) for \( d \geq 2 \). This generalization was done by [11, 3, 2] as the family of DPP called the Weyl–Heisenberg ensembles, but here we derive the DPPs on \( \mathbb{C}^d, \quad d \in \mathbb{N}, \) following Corollary 1.12 given in Section 1.5.

Let \( S_1 = \mathbb{C}^d, S_2 = \Gamma = \mathbb{R}^d, \)

\[ \lambda_1(dx) = \prod_{a=1}^{d} \lambda_{N(0,1;\mathbb{C})}(dx^{(a)}) = \frac{1}{\pi^d} e^{-|x|^2} = \frac{1}{\pi^d} e^{-(|x_R|^2 + |x_I|^2)} =: \lambda_{N(0,1;\mathbb{C}^d)}(dx), \]

\[ \lambda_2(dy) = \nu(dy) = \prod_{a=1}^{d} \lambda_{N(0,1/4)}(dy^{(a)}) = \left( \frac{2}{\pi} \right)^{d/2} e^{-2|y|^2}, \]

and

\[ \psi_1(x,y) = e^{-|x_R|^2 - |x_I|^2}/2 + 2(x_R y + \sqrt{-1} x_I y), \quad x = x_R + \sqrt{-1} x_I \in \mathbb{C}^d, \quad y \in \mathbb{R}^d. \]

We see that

\[ \Psi_1(x)^2 := \| \psi_1(x, \cdot) \|^2_{L^2(\mathbb{R}^d, \nu)} = e^{2|x|^2}, \quad x \in \mathbb{C}^d. \]

Hence Assumptions 3’ is satisfied, and then, by Corollary 1.12, we obtain the DPP on \( \mathbb{C}^d \) with the correlation kernel,

\[ K^{(d)}(x, x') = \left( \frac{2}{\pi} \right)^{d/2} e^{-((|x_R|^2 - |x_I|^2)+(|x'_R|^2 - |x'_I|^2))/2} \int_{\mathbb{R}^d} e^{-2|y|^2 - ((x_R + \sqrt{-1} x_I) + (x'_R - \sqrt{-1} x'_I)) y} dy \]

\[ = \frac{e^{\sqrt{-1} x_R x_I} e^{\sqrt{-1} x'_R x'_I}}{e^{\sqrt{-1} x'_R x'_I}} K^{(d)}_{\text{Heisenberg}}(x, x'). \]
with

\[ K^{(d)}_{\text{Heisenberg}}(x, x') = e^{x \cdot x'}, \quad x, x' \in \mathbb{C}^d. \]

The kernels in this form on \( \mathbb{C}^d, d \in \mathbb{N} \) have been studied by Zelditch and his coworkers (see [103], [14] and references therein), who identified them with the Szeg"o kernels for the reduced Heisenberg group. Here we call the DPPs associated with the correlation kernels in this form the \textit{Heisenberg family of DPPs} on \( \mathbb{C}^d, d \in \mathbb{N} \). This family includes the Ginibre DPP of type A as the lowest dimensional case with \( d = 1 \).

**Definition 1.17** The Heisenberg family of DPP on \( \mathbb{C}^d, d \in \mathbb{N} \) is defined by \((\Xi, K^{(d)}_{\text{Heisenberg}}, \lambda_{N(0,1;\mathbb{C}^d)}(dx))\) with

\[ K^{(d)}_{\text{Heisenberg}}(x, x') = e^{x \cdot x'}, \quad x, x' \in \mathbb{C}^d. \]

Since

\[ K^{(d)}_{\text{Heisenberg}}(x, x) \lambda_{N(0,1;\mathbb{C}^d)}(dx) = \frac{1}{\pi^d} dx, \quad x \in \mathbb{C}^d, \]

every DPP in the Heisenberg family is uniform on \( \mathbb{C}^d \) and the density with respect to the Lebesgue measure \( dx \) is given by \( 1/\pi^d \).

1.9 Open problems

With \( L^2(S, \lambda) \) and \( L^2(\Gamma, \nu) \), we can consider the system of \textit{biorthonormal functions}, which consists of a pair of distinct families of measurable functions \( \{\psi(x, \gamma) : x \in S, \gamma \in \Gamma\} \) and \( \{\varphi(x, \gamma) : x \in S, \gamma \in \Gamma\} \) satisfying the biorthonormality relations

\[ \langle \psi(\cdot, \gamma), \varphi(\cdot, \gamma') \rangle_{L^2(S, \lambda)} \nu(d\gamma) = \delta(\gamma - \gamma')d\gamma, \quad \gamma, \gamma' \in \Gamma. \]  \hspace{1cm} (1.45)

If the integral kernel defined by

\[ K^{\text{bi}}(x, x') = \int_{\Gamma} \psi(x, \gamma) \overline{\varphi(x', \gamma)} \nu(d\gamma), \quad x, x' \in S, \]  \hspace{1cm} (1.46)

is of finite rank, we can construct a finite DPP on \( S \) whose correlation kernel is given by (1.46) following a standard method of random matrix theory (see, for instance, Appendix C in [49]). By the biorthonormality (1.45), it is easy to verify that \( K^{\text{bi}} \) is a projection kernel, but it is not necessarily an orthogonal projection. This observation means that such a DPP is not constructed by the method reported in this manuscript. Generalization of the present framework in order to cover such DPPs associated with biorthonormal systems is required.

Moreover, the dynamical extensions of DPPs called \textit{determinantal processes} (see, for instance, [48]) shall be studied in the context of the present section.
2 One-Dimensional Stochastic Log-Gases

2.1 Eigenvalue and singular-value processes

For \( N \in \mathbb{N} := \{1, 2, \ldots \} \), let \( H(N) \) and \( U(N) \) be the space of \( N \times N \) Hermitian matrices and the group of \( N \times N \) unitary matrices, respectively. The space of \( N \times N \) real symmetric matrices and the group of \( N \times N \) orthogonal matrices are denoted by \( S(N) \) and \( O(N) \), respectively. As a matter of course, \( S(N) \subset H(N) \) and \( O(N) \subset U(N) \). In the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), we consider complex-valued processes \( M_{ij}(t) \in \mathbb{C}, t \geq 0, 1 \leq i, j \leq N \) with the condition \( M_{ij}(t) = M_{ji}(t) \), where \( \overline{z} \) denotes the complex conjugate of \( z \in \mathbb{C} \). We consider an \( H(N) \)-valued process \( M(t) = (M_{ij}(t))_{1 \leq i,j \leq N}, t \geq 0 \). Let \( \mathbb{R}_{\geq 0} := \{ x \in \mathbb{R} : x \geq 0 \} \). For \( S = \mathbb{R} \) and \( \mathbb{R}_{\geq 0} \), we define the Weyl chambers in \( S^N \) as

\[
W_N(S) := \left\{ x = (x_1, \ldots, x_N) \in S^N : x_1 < \cdots < x_N \right\}, \tag{2.1}
\]

and their closures as \( \overline{W}_N(S) := \{ x \in S^N : x_1 \leq \cdots \leq x_N \} \). For each \( t \geq 0 \), there exists \( U(t) = (U_{ij}(t))_{1 \leq i,j \leq N} \in U(N) \) such that it diagonalizes \( M(t) \) as

\[
U(t)^* M(t) U(t) = \Lambda(t) := \text{diag}(\Lambda_1(t), \ldots, \Lambda_N(t)),
\]

with the eigenvalues \( \{ \Lambda_i(t) \}_{i=1}^{N} \) of \( M(t) \) which are assumed to be in the non-decreasing order,

\[
\Lambda_1(t) \leq \cdots \leq \Lambda_N(t) \iff \Lambda := (\Lambda_1(t), \ldots, \Lambda_N(t)) \in \overline{W}_N(\mathbb{R}).
\]

For \( dM(t) := (dM_{ij}(t))_{1 \leq i,j \leq N}, t \geq 0 \), define a set of quadratic variations,

\[
\Gamma_{ij,kl}(t) := \langle (U^* dMU)_{ij}, (U^* dMU)_{kl} \rangle_t, \quad 1 \leq i,j,k,\ell \leq N, \quad t \geq 0.
\]

We denote by \( 1(\omega) \) the indicator function of a condition \( \omega \); \( 1(\omega) = 1 \) if \( \omega \) is satisfied, and \( 1(\omega) = 0 \) otherwise. In particular, given a subspace \( \Gamma \subset \mathbb{R}^N \) we define \( 1_{\Gamma}(x) := 1(x \in \Gamma) \) for \( x \in \mathbb{R}^N \), and \( \delta_{ij} := 1(i = j) \). The following is proved \([17, 54, 48]\). See Section 4.3 of \([6]\) for details of proof.

**Proposition 2.1** Assume that \( (M_{ij}(t))_{t \geq 0}, 1 \leq i, j \leq N \) are continuous semi-martingales. The eigenvalue process \( (\Lambda(t))_{t \geq 0} \) satisfies the following system of stochastic differential equations (SDEs),

\[
d\Lambda_i(t) = dM_i(t) + dJ_i(t), \quad t \geq 0, \quad 1 \leq i \leq N,
\]

where \( (M_i(t))_{t \geq 0}, 1 \leq i \leq N \) are martingales with quadratic variations

\[
\langle M_i, M_j \rangle_t = \int_0^t \Gamma_{ii,jj}(s) ds, \quad t \geq 0, \quad 1 \leq i,j \leq N,
\]

and \( (J_i(t))_{t \geq 0}, 1 \leq i \leq N \) are the processes with finite variations given by

\[
dJ_i(t) = \sum_{j=1}^{N} \frac{1}{\Lambda_i(t) - \Lambda_j(t)} 1(\Lambda_i(t) \neq \Lambda_j(t)) \Gamma_{ij,ji}(t) dt + d\Upsilon_i(t), \quad t \geq 0, \quad 1 \leq i \leq N.
\]

Here \( (d\Upsilon_i(t))_{t \geq 0}, 1 \leq i \leq N \) are the finite-variation parts of \( (U(t)^* dM(t) U(t))_{ii} \).
This proposition is given in [54] as a generalization of Theorem 1 in Bru [17]. A proof is given at Section 3.2 in [48].

We will show four basic examples of $M(t) \in \mathbb{H}(N), t \geq 0$ and applications of Proposition 2.1 in [54]. Let $\nu \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $(B_{ij}(t))_{t \geq 0}, (\bar{B}_{ij}(t))_{t \geq 0}, 1 \leq i \leq N + \nu, 1 \leq j \leq N$ be independent one-dimensional standard Brownian motions. For $1 \leq i \leq j \leq N$, put

$$S_{ij}(t) = \left\{ \begin{array}{ll} B_{ij}(t)/\sqrt{2}, & (i < j), \\ B_{ii}(t), & (i = j), \\ 0, & (i > j), \end{array} \right.$$ 

and put $S_{ij}(t) = S_{ji}(t)$ and $A_{ij}(t) = -A_{ij}(t), t \geq 0$ for $1 \leq j < i \leq N$.

**Example 2.2**

(i) Put $M_{ij}(t) = S_{ij}(t) + \sqrt{-1}A_{ij}(t), t \geq 0, 1 \leq i, j \leq N$. By definition $(dM_{ij}, dM_{kl})_t = \delta_{ij}\delta_{jk}dt, t \geq 0, 1 \leq i, j, k, \ell \leq N$. Hence, by unitarity of $U(t), t \geq 0$, we see that $\Gamma_{ij,kl}(t) = \delta_{il}\delta_{jk}$, which gives $(dM_{ij}, dM_{ij})_t = \Gamma_{ii,ij}(t)dt = \delta_{ij}dt$ and $\Gamma_{ij,ji}(t) = 1$, $t \geq 0, 1 \leq i, j \leq N$. Then Proposition 2.1 proves that the eigenvalue process $\Lambda(t) = (\Lambda_1(t), \ldots, \Lambda(t)), t \geq 0$, satisfies the following system of SDEs with $\beta = 2$,

$$d\Lambda_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{1 \leq j \leq N, j \neq i} \frac{dt}{\Lambda_i(t) - \Lambda_j(t)}, \quad t \geq 0, \quad 1 \leq i \leq N. \quad (2.2)$$

Here $(B_{ii}(t))_{t \geq 0}, 1 \leq i \leq N$ are independent one-dimensional standard Brownian motions, which are different from $(B_{ij}(t))_{t \geq 0}$ and $(\bar{B}_{ij}(t))_{t \geq 0}$ used above to define $(S_{ij}(t))_{t \geq 0}$ and $(A_{ij}(t))_{t \geq 0}, 1 \leq i, j \leq N$.

(ii) Put $M(t) = (S_{ij}(t))_{1 \leq i, j \leq N} \in \mathbb{S}(N), t \geq 0$. In this case $(dM_{ij}, dM_{kl})_t = (\delta_{ij}\delta_{jk} + \delta_{ik}\delta_{jl})dt/2, t \geq 0, 1 \leq i, j, k, \ell \leq N$ and $(U(t))_{t \geq 0}$ is $O(N)$-valued. Then we see that $\Gamma_{ij,kl}(t) = (\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})/2, t \geq 0, 1 \leq i, j \leq N$, and hence Proposition 2.1 proves that the eigenvalue process satisfies (2.2) with $\beta = 1$.

(iii) Consider an $(N + \nu) \times N$ rectangular-matrix-valued process given by $K(t) = (B_{ij}(t) + \sqrt{-1}\bar{B}_{ij}(t))_{1 \leq i \leq N + \nu, 1 \leq j \leq N}, t \geq 0$, and define an $\mathbb{H}(N)$-valued process by

$$M(t) = K^*(t)K(t), \quad t \geq 0,$$

where $K^*(t)$ is the Hermitian conjugate of $K(t)$, that is, $K^*_{ij}(t) = \overline{K_{ij}(t)}, 1 \leq i \leq N + \nu, 1 \leq j \leq N$. The matrix $M$ is positive definite and hence the eigenvalues are non-negative; $\Lambda_i(t) \in \mathbb{R}_{\geq 0}, t \geq 0, 1 \leq i \leq N$. We see that the finite-variation part of $dM_{ij}(t)$ is equal to $2(N + \nu)\delta_{ij}dt, t \geq 0$, and $(dM_{ij}, dM_{kl})_t = 2(M_{ij}(t)\delta_{jk} + M_{ik}(t)\delta_{jl})dt, t \geq 0, 1 \leq i, j, k, \ell \leq N$, which implies that $d\Gamma(t) = (2(N + \nu)dt, \Gamma_{ij,ji} = 2(\Lambda_i(t) + \Lambda_j(t))$, and $(dM_{ij}, dM_{ij})_t = \Gamma_{ii,ij}(t)dt = 4\Lambda_i(t)\delta_{ij}dt, t \geq 0, 1 \leq i, j \leq N$. Then we have the SDEs for eigenvalue

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It was proved that when $\beta \lambda$ for $(GUE)$, and the ensemble starting from the configuration following the SDEs (2.2) with by the eigenvalue distributions of the Gaussian orthogonal ensemble terms of the Ornstein–Uhlenbeck type so that the stationary measures of the processes are given (iv) Put $K(t) = (B_{ij}(t))_{1 \leq i \leq N+\nu, 1 \leq j \leq N}, t \geq 0$ and consider the process in $S(N), M(t) = K^t(t), t \geq 0$. In this case the finite-variation part of $dM_{ij}(t)$ is $(N+\nu)\delta_{ij} dt$, $t \geq 0$, and \( (dM_{ij}, dM_{k\ell})_t = (M_{ik}(t)\delta_{k\ell} + M_{i\ell}(t)\delta_{ik} + M_{jk}(t)\delta_{\ell i} + M_{j\ell}(t)\delta_{ki}) dt, t \geq 0, 1 \leq i, j, k, \ell \leq N, \) which imply that \( d\Gamma_{ij}(t) = (N+\nu) dt, \Gamma_{ij,jj} = (\Lambda_i(t) + \Lambda_j(t))(1 + \delta_{ij}), \) and \( (dM_i, dM_j)_t = \Gamma_{ii,jj}(t) = 4\Lambda_i(t)\delta_{ij} dt, t \geq 0, 1 \leq i, j \leq N. \) Then we have (2.3) with $\beta = 1, \nu \in \mathbb{N}_0$ as the SDEs for the eigenvalue process of $(M(t))_{t \geq 0}$, and (2.4) with $\beta = 1, \nu \in \mathbb{N}_0$ as the SDEs for the singular-value process of $(K(t))_{t \geq 0}$.

Other examples of $M(t) \in H(N), t \geq 0$ are shown in [54], in which the eigenvalue processes following the SDEs (2.2) with $\beta = 4$ are also shown.

Dyson [30] introduced the system of SDEs similar to (2.2) with $\beta = 1, 2, 4$, and with drift terms of the Ornstein–Uhlenbeck type so that the stationary measures of the processes are given by the eigenvalue distributions of the Gaussian orthogonal ensemble (GOE), the Gaussian unitary ensemble (GUE), and the Gaussian symplectic ensemble (GSE) studied in random matrix theory [73] for $\beta = 1, 2$, and $\beta = 4$, respectively. For general $\beta \in \mathbb{R}$, the system of SDEs (2.2) starting from the configuration $\lambda := \Lambda(0) \in \mathcal{W}_N(\mathbb{R})$ can be defined up to the first collision time,

$$T^\lambda := \inf\{t > 0 : \Lambda_i(t) = \Lambda_j(t) \text{ for some } 1 \leq i \neq j \leq N\}.$$  

It was proved that when $\beta \geq 1$, $T^\lambda = \infty$ a.s. and (2.2) has a strong and pathwise unique non-colliding solution for $\lambda \in \mathcal{W}_N(\mathbb{R})$ [83]. The statement was extended to general initial configuration
The determinantal stochastic processes are proved to be boundary condition will be assumed at the origin for (2.4). These noncolliding diffusion processes never to collide with each other and Laguerre polynomials if the initial configurations called the spatio-temporal correlation kernels, which can be simply expressed using the Hermite functions are explicitly expressed by determinants. The determinants are governed by the functions $\nu$-dimensional squared Bessel processes, and the $2(\nu + 1)$-dimensional Bessel processes conditioned never to collide with each other. Here $\nu \in (-1, \infty)$, and if $\nu \in (-1, 0]$ the reflection boundary condition will be assumed at the origin for (2.4). These noncolliding diffusion processes are proved to be determinantal stochastic processes and all spatio-temporal correlation functions are explicitly expressed by determinants. The determinants are governed by the functions called the spatio-temporal correlation kernels, which can be simply expressed using the Hermite and Laguerre polynomials if the initial configurations $\lambda$ are given by $0$. We will show these facts in Sections 2.2–2.5. For more detail, see [58, 59, 47, 48].

In the last section, we will consider the Schramm–Loewner evolution (SLE). Schramm used a strong and pathwise unique non-colliding solution for general initial configurations $\lambda$ in order to parametrize time change of the Brownian motion. Accordingly, we change the parameter $\beta$ by setting

$$\beta = \frac{8}{\kappa},$$  \hspace{1cm} (2.5)

and perform the time change $t \to \kappa t$. Since $(B(\kappa t))_{t \geq 0} \overset{\text{(law)}}{=} \sqrt{\kappa} B(t)_{t \geq 0}$, if we put $Y_i^R(t) := \Lambda_i(\kappa t)$, $Y_i^{R \geq 0}(t) := S_i(\kappa t)$, $t \geq 0$, $1 \leq i \leq N$, the systems of SDEs (2.2) and (2.3) are written as

$$dY_i^R(t) = \sqrt{\kappa dB_i(t)} + 4 \sum_{1 \leq j \leq N, j \neq i} \frac{dt}{Y_i^R(t) - Y_j^R(t)}, \hspace{1cm} t \geq 0, \hspace{0.5cm} 1 \leq i \leq N,$$ \hspace{1cm} (2.6)

$$dY_i^{R \geq 0}(t) = \sqrt{\kappa dB_i(t)} + 4 \sum_{1 \leq j \leq N, j \neq i} \left( \frac{1}{Y_i^{R \geq 0}(t) - Y_j^{R \geq 0}(t)} + \frac{1}{Y_i^{R \geq 0}(t) + Y_j^{R \geq 0}(t)} \right) dt \hspace{0.5cm} + \left\{ 4(1 + \nu) - \frac{\kappa}{2} \right\} \frac{dt}{Y_i^{R \geq 0}(t)}, \hspace{1cm} t \geq 0, \hspace{0.5cm} 1 \leq i \leq N.$$ \hspace{1cm} (2.7)

In the last section, we will call $Y^R(t) = (Y_1^R(t), \ldots, Y_N^R(t)) \in \mathbb{R}^N, t \geq 0$, the $(8/\kappa)$-Dyson model and $Y^{R \geq 0}(t) = (Y_1^{R \geq 0}(t), \ldots, Y_N^{R \geq 0}(t)) \in (\mathbb{R} \geq 0)^N, t \geq 0$, the $(8/\kappa, \nu)$-Bru–Wishart process, respectively.
The above stochastic processes $Y^S(t) = (Y^S_1(t), \ldots, Y^S_N(t))$, $t \geq 0$, $S = \mathbb{R}$ or $\mathbb{R}_{\geq 0}$ can be written in the form,

$$dY^S_i(t) = \sqrt{\kappa} dB_i(t) + \frac{\partial \phi^S(x)}{\partial x_i} \bigg|_{x = Y^S(t)} dt, \quad t \geq 0, \quad 1 \leq i \leq N, \quad (2.8)$$

if we introduce the following potential energies,

$$\phi^S(x) := \begin{cases} 
4 \sum_{1 \leq i < j \leq N} \log(x_j - x_i), & \text{for } S = \mathbb{R}, \\
4 \sum_{1 \leq i < j \leq N} \left[ \log(x_j - x_i) + \log(x_j + x_i) \right] + \left\{ 4(\nu + 1) - \frac{\kappa}{2} \right\} \sum_{i=1}^N \log x_i, & \text{for } S = \mathbb{R}_{\geq 0}.
\end{cases} \quad (2.9)$$

In both cases of $(2.9)$, the potentials are given by logarithmic functions, and the drift terms are gradient forces of these potentials. (See Remark 1.14 in Section 1.6.1.) In this sense the $(8/\kappa)$-Dyson model and the $(8/\kappa, \nu)$-Bru–Wishart process are regarded as stochastic log-gases in $\mathbb{R}$ [32]. Since the logarithmic potential describes the two-dimensional Coulomb law in electrostatics, the present processes are also considered as stochastic models of two-dimensional charged-particles ($2D$-Coulomb gas) confined on a line $\mathbb{R}$ or to a half-line $\mathbb{R}_{\geq 0}$.

### 2.3 Determinantal martingales and determinantal stochastic processes (DSPs)

We consider the same setting as in Section 1. Let $S$ be a base space, which is locally compact Hausdorff space with countable base, and $\lambda$ be a Radon measure on $S$. The configuration space over $S$ is given by the set of nonnegative-integer-valued Radon measures;

$$\text{Conf}(S) = \left\{ \xi = \sum_j \delta_{x_j} : x_j \in S, \xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset S \right\}.$$ 

Conf$(S)$ is equipped with the topological Borel $\sigma$-fields with respect to the vague topology.

We consider interacting particle systems as $\text{Conf}(S)$-valued continuous-time processes and write them as

$$\Xi(t) = \sum_{j=1}^N \delta_{X_j(t)}, \quad t \geq 0, \quad (2.10)$$

where $X(t) = (X_1(t), \ldots, X_N(t)), t \geq 0$ are defined by a solution of a given SDEs. We call $x \in S^N$ a labeled configuration and $\xi \in \text{Conf}(S)$ an unlabeled configuration. The probability law of $(\Xi(t))_{t \geq 0}$ starting from a fixed configuration $\xi \in \text{Conf}(S)$ is denoted by $\mathbf{P}_\xi$ and the process specified by the initial configuration is expressed by $((\Xi(t))_{t \geq 0}, \mathbf{P}_\xi)$. The expectations with respect to $\mathbf{P}_\xi$ is denoted by $\mathbf{E}_\xi$. We introduce a filtration $(\mathcal{F}_\Xi(t))_{t \geq 0}$ generated by $(\Xi(t))_{t \geq 0}$ satisfying the usual conditions (see, for instance, p.45 in [31]). We set

$$\text{Conf}_0(S) = \{ \xi \in \text{Conf}(S) : \xi(\{x\}) \leq 1 \text{ for any } x \in S \},$$

which gives a collection of configurations of simple point processes (i.e., without multiple points).
Let $0 \leq T < \infty$. We consider the expectation of an $\mathcal{F}_t(T)$-measurable bounded function $F$, $E_\xi[F(\Xi(t))]$. It is sufficient to consider the case that $F$ is given as

$$F(\Xi(t)) = \prod_{m=1}^M g_m(X(t_m))$$

for an arbitrary $M \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_M \leq T < \infty$ with bounded measurable functions $g_m$ on $S^N$, $1 \leq m \leq M$. Since the particles are unlabeled in the process $((\Xi(t))_{t \geq 0}, \mathbf{P}_\xi)$, $g_m$‘s are symmetric functions.

We consider a continuous-time Markov process $(Y(t))_{t \geq 0}$ on $S$, which is a connected open set in $\mathbb{R}$. It is a diffusion process in $S$ or a process showing a position on the circumference $S = [0, 2\pi r)$ of a diffusion process moving around on the circle with a radius $r > 0$; $S^1(r) := \{x \in [0, 2\pi r) : x + 2\pi r = x\}$. The probability space is denoted by $(\Omega, \mathcal{F}, P^\nu)$ with expectation $E^\nu$, when the initial state is fixed to be $v \in S$. When $v$ is the origin, the subscript is omitted. We introduce a filtration $\{\mathcal{F}(t) : t \geq 0\}$ generated by $Y$ so that it satisfies the usual conditions (see, for instance, p.45 in [1]). We assume that the process has a transition density, $p(t, y|x)$, $t \in [0, \infty), x, y \in S$ such that for any measurable bounded function $f(t, x), (t, x) \in [0, \infty) \times S$,

$$E[f(t, Y(t))|\mathcal{F}(s)] = \int_S dy f(t, y)p(t - s, y|Y(s)) \text{ a.s., } 0 \leq s \leq t < \infty. \quad (2.12)$$

Recall that $\mathcal{W}_N(S), N \in \mathbb{N}$ denotes the Weyl chamber in $S$;

$$\mathcal{W}_N(S) = \{x = (x_1, \ldots, x_N) \in S^N : x_1 < x_2 < \cdots < x_N\}.$$ Given $u = (u_1, \ldots, u_N) \in \mathcal{W}_N(S)$, we have a measure $\xi(\cdot) = \sum_{j=1}^N \delta_{u_j}(\cdot) \in \text{Conf}_0(S)$. Depending on $\xi$, we assume that there is a one-parameter family of continuous functions

$$\mathcal{M}_\xi^v(\cdot, \cdot) : [0, \infty) \times S \to \mathbb{R}$$

with parameter $v \in S$, such that the processes $\mathcal{M}_\xi(t, Y(t)) = \{\mathcal{M}_\xi^v(t, Y(t)) : v \in \{u_1, \ldots, u_N\}\}$, $t \geq 0$ satisfy the following conditions.

**M1**  $(\mathcal{M}_\xi^u(t, Y(t)))_{t \geq 0}, 1 \leq k \leq N$ are continuous martingales;

$$E[\mathcal{M}_\xi^u(t, Y(t))|\mathcal{F}(s)] = \mathcal{M}_\xi^u(s, Y(s)) \text{ a.s. for all } 0 \leq s \leq t < \infty.$$

**M2**  For any time $t \geq 0$, $\mathcal{M}_\xi^u(t, x), 1 \leq k \leq N$ are linearly independent functions of $x$.

**M3**  For $1 \leq j, k \leq N$,

$$\lim_{t \to 0} E^\nu[\mathcal{M}_\xi^u(t, Y(t))] = \delta_{jk}.$$ We call $\{\mathcal{M}_\xi^u(t, x) : v \in \{u_1, \ldots, u_N\}\}$ martingale functions.

Let $(Y_j(t))_{t \geq 0}, 1 \leq j \leq N$ be a collection of $N$ independent copies of $(Y(t))_{t \geq 0}$. We consider the $N$-component vector-valued Markov process $Y(t) = (Y_1(t), \ldots, Y_N(t))$, $t \geq 0$, for which the initial values are fixed to be $Y_j(0) = u_j \in S, 1 \leq j \leq N$, provided $u = (u_1, \ldots, u_N) \in \mathcal{W}_N(S)$. We consider a determinant of the martingales

$$D_\xi(t, Y(t)) = \det_{1 \leq j, k \leq N} \mathcal{M}_\xi^{u_j}(t, Y_j(t)), \ t \geq 0. \quad (2.13)$$

The condition (M2) is necessary so that it is not zero constantly. This determinant is a continuous martingale and we call it a determinantal martingale.
Definition 2.3 Given $\xi \in \text{Conf}_0(S)$, consider a process $((\Xi(t))_{t \geq 0}, P_\xi)$. If there exists a pair $(Y, \mathcal{M}_\xi)$ defining $D_\xi$ by (2.13) such that for any $F_{\Xi}(t)$-measurable bounded function $F$, $0 \leq t \leq T < \infty$, the equality
\[
E_\xi[F(\Xi(\cdot))] = E^{\mu} \left[ F \left( \sum_{j=1}^{N} \delta_{Y_j(\cdot)} \right) D_\xi(T, Y(T)) \right]
\]
holds, then we say $((\Xi(t))_{t \geq 0}, P_\xi)$ has a determinantal-martingale representation (DMR) associated with $(Y, \mathcal{M}_\xi)$.

Consider an arbitrary but fixed $M \in \mathbb{N}$. Assume that we have a sequence of times $t = (t_1, \ldots, t_M)$ with $0 \leq t_1 < \cdots < t_M < \infty$, and a sequence of functions $\mathbf{f} = (f_{t_1}, \ldots, f_{t_M}) \in \mathcal{C}_c(S)^M$. Then the multitime Laplace transform of probability measure $P_\xi$ is defined by
\[
\Psi^t_\xi[\mathbf{f}] := E_\xi \left[ \exp \left( \sum_{m=1}^{M} \int_{S} f_{t_m}(x) \Xi(t_m, dx) \right) \right]. \tag{2.14}
\]
We assume that it is expanded with respect to $\{1 - e^{f_{t_m}} : 1 \leq m \leq M\}$ as
\[
\Psi^t_\xi[\mathbf{f}] = 1 + \prod_{m=1}^{M} \sum_{1 \leq n_m \leq N} \frac{(-1)^{n_m}}{n_m!} \int_{S^{n_m}} \lambda^{\otimes n_m}(dx_{n_m}^{(m)}) \prod_{\ell=1}^{n_m} (1 - e^{f_{t_m}(x_{\ell}^{(m)}))} \times \rho_\xi(t_1, x_{n_1}^{(1)}; \ldots; t_M, x_{n_M}^{(M)}), \tag{2.15}
\]
where $(\lambda_\ell)_{\ell \geq 0}$ is a time-dependent measure on $S$, $x_{n_m}^{(m)}$ denotes $(x_{1,m}^{(m)}, \ldots, x_{n_m}^{(m)})$, $1 \leq m \leq M$, $1 \leq n_m \leq N$, and $dx_{n_m}^{(m)} = \prod_{j=1}^{n_m} dx_{j}^{(m)}$, $1 \leq m \leq M$. This expansion formula of $\Psi^t_\xi[\mathbf{f}]$ defines the spatio-temporal correlation functions by determinants as
\[
\rho_\xi(t_1, x_{n_1}^{(1)}; \ldots; t_M, x_{n_M}^{(M)}) = \det_{1 \leq j \leq n_m, 1 \leq k \leq n_n} K_\xi(t_m, x_j^{(m)}; t_n, x_k^{(n)}), \tag{2.16}
\]
$0 \leq t_1 < \cdots < t_M < \infty$, $1 \leq n_m \leq N$, $x_{n_m}^{(m)} \in S^{n_m}$, $1 \leq m \leq M \in \mathbb{N}$. The DSP is denoted by $((\Xi(t))_{t \geq 0}, K_\xi, (\lambda_\ell)_{\ell \geq 0})$.

By Definition 2.4, the multitime Laplace transform of $P_\xi$ (2.14) is written as follows. We regard this as a definition of the multitime Fredholm determinant of an integral kernel $K_\xi$ specified by $K_\xi$, and
\[
1 + \prod_{m=1}^{M} \sum_{1 \leq n_m \leq N} \frac{(-1)^{n_m}}{n_m!} \int_{S^{n_m}} \lambda^{\otimes n_m}(dx_{n_m}^{(m)}) \prod_{\ell=1}^{n_m} (1 - e^{f_{t_m}(x_{\ell}^{(m)}))} \times \det_{1 \leq j \leq n_m, 1 \leq k \leq n_n} K_\xi(t_m, x_j^{(m)}; t_n, x_k^{(n)})) =: \det_{\mathcal{S}_m}^{L^2(S, \mathcal{M}_\xi)} \left[ I - (1 - e^{f}) K_\xi \right]. \tag{2.17}
\]
The main theorem in this section is the following.

**Theorem 2.5** If \(((\Xi(t))_{t \geq 0}, P_\xi)\) has DMR associated with \((Y, M_\xi)\), then it is a DSP \(((\Xi(t))_{t \geq 0}, K_\xi, dx)\) with the spatio-temporal correlation kernel

\[
K_\xi(s, x; t, y) = \int_S \xi(dv)p(s, x|v)M_\xi^t(t, y) - 1(s > t)p(s - t, x|y),
\]

(2.18)

\((s, x), (t, y) \in [0, \infty) \times S\), where \(p\) is the transition density of the process \(Y\).

This type of correlation kernel (2.18) was first obtained by Eynard and Mehta for a multi-matrix model [31] and by Nagao and Forrester [78] for the noncolliding Brownian motion started at a special initial distribution \(p_{\text{Hermite}}\) given by (1.29) (the GUE eigenvalue distribution), and has been extensively studied [33, 101, 73, 16, 32, 58, 61]. Note that, while all correlation kernels obtained discussed in Section 1 are symmetric, the present spatio-temporal correlation kernel is asymmetric with respect to the exchange of two points \((s, x)\) and \((t, y)\) on the spatio-temporal plane \([0, \infty) \times S\) and shows causality in the system.

In the following, first we states a lemma and a proposition, and then prove Theorem 2.5 using them. For \(n \in \mathbb{N}\), an index set \(\{1, 2, \ldots, n\}\) is denoted by \(\mathbb{I}_n\). Fixing \(N \in \mathbb{N}\) with \(N' \subseteq \mathbb{I}_N\), we write

\[
\mathbb{J} \subseteq \mathbb{I}_N, |\mathbb{J}| = N' \iff \mathbb{J} = \{j_1, \ldots, j_{N'}\}, \quad 1 \leq j_1 < \cdots < j_{N'} \leq N.
\]

For \(x = (x_1, \ldots, x_N) \in \mathbb{R}^N\), put \(x_J := (x_{j_1}, \ldots, x_{j_{N'}})\). In particular, we write \(x_{N'} := x_{J_{N'}}\), \(1 \leq N' \leq N\). (By definition \(x_N = x\).) A collection of all permutations of elements in \(\mathbb{J}\) is denoted by \(\Sigma(\mathbb{J})\). In particular, we write \(\Sigma_{N'} := \Sigma(\mathbb{I}_{N'}), 1 \leq N' \leq N\).

The following lemma shows the reducibility of the determinantal martingale in the sense that, if we observe a symmetric function depending on \(N'\) variables, \(N' \leq N\), then the size of determinantal martingale can be reduced from \(N\) to \(N'\).

**Lemma 2.6** Let \(u = (u_1, \ldots, u_N) \in \mathbb{W}_N(S)\) and \(\xi = \sum_{j=1}^N \delta_{u_j} \in \text{Conf}_0(S)\). Assume that there exists a pair \((Y, M_\xi)\) satisfying conditions (M1)–(M3) and \(\mathcal{D}_\xi\) is defined by (2.13). Let \(1 \leq N' \leq N\). For \(0 < t \leq T < \infty\) and an \(\mathcal{F}_t\)-measurable symmetric function \(F_{N'}\) on \(\mathbb{R}^{N'}\),

\[
\sum_{J \subseteq \mathbb{I}_N, |J| = N'} \mathbb{E}^x[F_{N'}(Y_J(t))\mathcal{D}_\xi(T, Y(T))] = \int_{\mathbb{W}_N^{N'}} \xi^{(N')} (dv)\mathbb{E}^v[F_{N'}(Y_{N'}(t))\mathcal{D}_\xi(T, Y_{N'}(T))].
\]

(2.19)

**Proof** By the definition (2.13), LHS of (2.19) is equal to

\[
\sum_{J \subseteq \mathbb{I}_N, |J| = N'} \mathbb{E}^u[F_{N'}(Y_J(t)) \det[M_{\xi}(T, Z_i(T))] \]

\[
= \sum_{J \subseteq \mathbb{I}_N, |J| = N'} \mathbb{E}^u \left[ F_{N'}(Y_J(t)) \sum_{\sigma \in \Sigma_N} \text{sgn}(\sigma) \prod_{i=1}^N M_{\xi}^{\sigma(i)}(T, Y_i(T)) \right]
\]

\[
= \sum_{J \subseteq \mathbb{I}_N, |J| = N'} \sum_{\sigma \in \Sigma_N} \text{sgn}(\sigma)
\times \mathbb{E}^u \left[ F_{N'}(Y_J(t)) \prod_{i \in J} M_{\xi}^{\sigma(i)}(T, Y_i(T)) \prod_{j \in \mathbb{I}_N \setminus J} M_{\xi}^{\sigma(i)}(T, Y_j(T)) \right]
\]

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\[ \sum_{J \in \mathcal{J}_N, \delta = N'} \sum_{\sigma \in \mathcal{S}_N} \text{sgn}(\sigma) E^u \left[ F_{N'}(Y_j(t)) \prod_{i \in J} \mathcal{M}_{\xi}^{u_{\sigma(i)}}(T, Y_i(T)) \right] \times \prod_{j \in \mathcal{I}_N \setminus J} E^u \left[ \mathcal{M}_{\xi}^{u_{\sigma(j)}}(T, Y_j(T)) \right]. \] (2.20)

By the condition \((M1)\) of \(\mathcal{M}_\xi\),
\[ \prod_{j \in \mathcal{I}_N \setminus J} E^u \left[ \mathcal{M}_{\xi}^{u_{\sigma(j)}}(T, Y_j(T)) \right] = \prod_{j \in \mathcal{I}_N \setminus J} E^u \left[ \mathcal{M}_{\xi}^{u_{\sigma(j)}}(t, Y_j(t)) \right], \quad \forall t \in [0, T], \]
and by the condition \((M3)\) of \(\mathcal{M}_\xi\), this is equal to \(\prod_{j \in \mathcal{I}_N \setminus J} \delta_{\sigma(j)}\). Then (2.20) becomes
\[ \sum_{J \in \mathcal{J}_N, \delta = N'} \sum_{\sigma \in \mathcal{S}_N} \text{sgn}(\sigma) E^u \left[ F_{N'}(Y_j(t)) \prod_{i \in J} \mathcal{M}_{\xi}^{u_{\sigma(i)}}(T, Y_i(T)) \right] = \sum_{J \in \mathcal{J}_N, \delta = N'} E^u \left[ F_{N'}(Y_j(t)) \det \left[ \mathcal{M}_{\xi}^{u_{\sigma(j)}}(T, Y_i(T)) \right] \right] \]
\[ = \int_{\mathcal{W}_N^{\sigma N} (d\nu)} E^u \left[ F_{N'}(Y_j(t)) \det \left[ \mathcal{M}_{\xi}^{u_{\sigma(j)}}(T, Y_i(T)) \right] \right], \]
where equivalence in probability law of \((Y_i(t))_{t \geq 0}, 1 \leq i \leq N\) is used. This is RHS of (2.19) and the proof is completed. \(\blacksquare\)

**Proposition 2.7** Let \(u = (u_1, \ldots, u_N) \in \mathcal{W}_N(S)\) and \(\xi = \sum_{j=1}^N \delta_{u_j} \in \text{Conf}_0(S)\). Assume that there exists a pair \((Y, \mathcal{M}_\xi)\) satisfying conditions \((M1)\)--\((M3)\) and \(\mathcal{D}_\xi\) is defined by (2.13). Then for any \(M \in \mathbb{N}, 0 \leq t_1 < \cdots < t_M \leq T < \infty, f_{t_m} \in C_c(S), 1 \leq m \leq M, \) the equality
\[ E^u \left[ \prod_{m=1}^M \prod_{j=1}^N \left( 1 - (1 - e^{f_{t_m}(Y_j(t_m))}) \right) \mathcal{D}_\xi(T, Y(T)) \right] = \text{Det} \left[ \bigotimes_{m=1}^M L^{2(S, \lambda_{t_m})} \left( I - (1 - e^f)K_\xi \right) \right] \] (2.21)
holds, where RHS is the multitime Fredholm determinant of \(K_\xi\) specified by the spatio-temporal correlation kernel (2.13).

Let \(\chi_{\xi}(\cdot) := 1 - e^{f_{t_m}(\cdot)}, t \geq 0\). LHS of (2.21) is an expectation of a usual determinant multiplied by \(\prod_{m=1}^M \prod_{j=1}^N (1 - \chi_{t_m}(Y_j(t_m)))\), while RHS is a multitime Fredholm determinant. First we expand LHS with respect to \(\{\chi_{t_m}(Y_j(t_m)) : 1 \leq m \leq M, 1 \leq j \leq N\}\) and apply Lemma 2.6. The expectation of each term in LHS will be calculated by performing integrals using the transition density \(p\) of the process \(Y\) as an integral kernel, while \(p\) is involved in the integral representation (2.13) of the spatio-temporal correlation kernel \(K_\xi\) for the multitime Fredholm determinant in RHS. Therefore, simply to say, this equality is just obtained by applying Fubini’s theorem. Since the quantities in (2.21) are multivariate and multitime joint distribution is considered, however, we also need combinatorics arguments to prove Proposition 2.7. The proof was given in [61] [47]. Here we omit the proof of this lemma.
Proof of Theorem 2.5

By (2.10), the multitime Laplace transform of \( P_\xi \) \((2.14)\) is written in the form
\[
\Psi_t[\xi] = E_\xi \left[ \prod_{m=1}^N \prod_{j=1}^N \left( 1 - (1 - e^{f(t_m)(X_j(t_m))}) \right) \right].
\]

By assumption of the theorem, it has DMR associated with \((Y, M_\xi)\),
\[
\Psi_t[\xi] = E_\xi \left[ \prod_{m=1}^N \prod_{j=1}^N \left( 1 - (1 - e^{f(t_m)(Y_j(t_m))}) \right) \right] D_\xi(T, Y(T)).
\]

Then Proposition 2.7 gives a multitime Fredholm determinant expression to this as
\[
\Psi_t[\xi] = \text{Det} \left( \bigotimes_{m=1}^M L^2(S, \lambda_{tm}) \right) \left[ I - (1 - e^{f}) K_\xi \right]
\]
with (2.18). By Definition 2.4 the proof is completed. 

2.4 Three applications

In order to show applications of Theorem 2.5, we consider the following three kinds of interacting particle systems, \((\Xi(t))_{t \geq 0}, P_\xi, \xi = \sum_{j=1}^N \delta_{u_j} \in \text{Conf}_0(S)\) with \(\Xi(t, \cdot) = \sum_{j=1}^N \delta_{X_j(t)}(\cdot), t \geq 0\). For each system of SDEs, \((B_j(t))_{t \geq 0}, 1 \leq j \leq N\) denote a set of independent one-dimensional standard Brownian motions (BMs) started at 0. (From now on, BM means a one-dimensional standard Brownian motion unless specially mentioned.)

Process 1: Noncolliding Brownian motions (the Dyson model with \(\beta = 2\));
\[
S = \mathbb{R},
X_j(t) = u_j + B_j(t) + \sum_{1 \leq k \leq N, k \neq j} \int_0^t \frac{ds}{X_j(s) - X_k(s)}, \quad 1 \leq j \leq N, \quad t \geq 0.
\]
(2.22)

Process 2: Noncolliding squared Bessel processes (BESQ\((\nu)\)) with \(\nu \in (-1, \infty)\) (the Bru–Wishart process with \(\beta = 2\));
\[
S = \mathbb{R}_{\geq 0} := \{ x \in \mathbb{R} : x \geq 0 \},
X_j(t) = u_j + \int_0^t 2 \sqrt{X_j(s)} dB_j(s) + 2(\nu + 1)t
\]
\[
+ \sum_{1 \leq k \leq N, k \neq j} \int_0^t \frac{4X_j(s)ds}{X_j(s) - X_k(s)}, \quad 1 \leq j \leq N, \quad t \geq 0.
\]
(2.23)

where if \(-1 < \nu < 0\) the reflection boundary condition is assumed at the origin.
**Process 3**: Noncolliding BM on a circle with a radius $r > 0$; (a trigonometric extension of the Dyson motion model with $\beta = 2$);

We solve the SDEs

$$
\dot{X}_j(t) = u_j + B_j(t) + \frac{1}{2r} \sum_{1 \leq k \leq N, k \neq j} \int_0^t \cot \left( \frac{\dot{X}_j(s) - \dot{X}_k(s)}{2r} \right) ds, \quad 1 \leq j \leq N, \quad t \geq 0,
$$

(2.24)

on $\mathbb{R}$ and then define the process on $S = [0, 2\pi r)$ by

$$
X_j(t) = \dot{X}_j(t) \mod 2\pi r, \quad 1 \leq j \leq N, \quad t \geq 0.
$$

(2.25)

Note that $\cot(x/2r)$ is a periodic function of $x$ with period $2\pi r$. By the definition (2.25), measurable functions for Process 3 should be periodic with period $2\pi r$ in the following sense.

For $0 \leq t < \infty$, if an $\mathcal{F}_t$-measurable function $F$ is given in the form (2.11), then for any $n \in \mathbb{Z},$

$$
g_m((x_j + 2\pi r n)_{j=1}^N) = g_m(x), \quad 1 \leq m \leq M.
$$

The system (2.24) with the identification (2.25) can be regarded as a dynamical extension of the circular unitary ensemble (CUE) of random matrix theory (see Section 11.8 in [73] and Chapter 11 in [32]). The dynamics was studied in [79] and papers cited therein. Process 3 is a trigonometric extension of Process 1 and in the limit $r \to \infty$ Process 3 should be reduced to Process 1. Since functions used to represent Process 3 in the present manuscript are all analytic with respect to $r$, the hyperbolic extension will be similarly discussed [24].

Corresponding to the three interacting $N$-particle systems, we consider the following three kinds of one-dimensional processes. The first one is BM on $S = \mathbb{R}$, whose transition density started at $x \in \mathbb{R}$ is given by

$$
p(t, y|x) = p_{BM}(t, y|x) = \begin{cases} 
\frac{e^{-(y-x)^2/(2t)}}{\sqrt{2\pi t}}, & t > 0, y \in \mathbb{R} \\
\delta_x(\{y\}), & t = 0, y \in \mathbb{R}.
\end{cases}
$$

(2.26)

The second one is BESQ$(\nu)$ with $\nu > \in (-1, \infty)$ on $S = \mathbb{R}_{\geq 0}$, which is given by the solution of the SDE,

$$
Y(t) = u + \int_0^t 2\sqrt{Y(s)}dB(s) + 2(\nu + 1)t, \quad t \geq 0, \quad u > 0,
$$

where $B$ is BM, and if $-1 < \nu < 0$ a reflecting wall is put at the origin. The transition density is given by

$$
p(t, y|x) = p^{(\nu)}(t, y|x)
= \begin{cases} 
\frac{\Gamma(\nu+\nu)}{2\Gamma(\nu+1)} I_{\nu} \left( \frac{\sqrt{xy}}{t} \right), & t > 0, x > 0, y \in \mathbb{R}_{\geq 0}, \\
y^\nu e^{-y/(2t)} I_{\nu} \left( \frac{\sqrt{xy}}{t} \right), & t > 0, x = 0, y \in \mathbb{R}_{\geq 0}, \\
\delta_x(\{y\}), & t = 0, x, y \in \mathbb{R}_{\geq 0},
\end{cases}
$$

(2.27)
where $I_{\nu}(x)$ is the modified Bessel function of the first kind \[80\]
\[
I_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+\nu}}{n!\Gamma(n+1+\nu)}, \quad x \in \mathbb{R}_{\geq 0}.
\]

The third one is a Markov process on $S = [0, 2\pi r)$, $r > 0$, whose the transition density is given by

\[
p(t, y|x) = p^r(t, y|x; N) = \begin{cases} 
\sum_{\ell \in \mathbb{Z}} p_{BM}(t, y + 2\pi r \ell|x), & \text{if } N \text{ is odd}, \\
\sum_{\ell \in \mathbb{Z}} (-1)^{\ell} p_{BM}(t, y + 2\pi r \ell|x), & \text{if } N \text{ is even},
\end{cases} \tag{2.28}
\]

$x, y \in [0, 2\pi r), t \geq 0$, where $N$ is the number of particles of Process 3. If we introduce the notation

\[
\sigma_N(m) = \begin{cases} 
m, & \text{when } N \text{ is odd}, \\
m - 1/2, & \text{when } N \text{ is even},
\end{cases} \tag{2.29}
\]

for $m \in \mathbb{Z}$, then (2.28) is written as

\[
p^r(t, y|x; N) = \frac{1}{2\pi r} \sum_{\ell \in \mathbb{Z}} e^{-\sigma_N(\ell)^2 t/2r^2 + \sqrt{-1}\sigma_N(\ell)(y-x)/r}. \tag{2.30}
\]

It should be noted that this expression is found in Nagao and Forrester [79]. This process shows a position on the circumference $S = [0, 2\pi r)$ of a Brownian motion moving around on the circle $S^1(r)$ (with alternating signed densities if $N$ is even). In the following, we call this one-dimensional Markov process $Y(t) \in [0, 2\pi r), t \geq 0$ simply ‘BM on $[0, 2\pi r)$’.

We introduce integral transformations of function $f = f(W)$,

\[
\mathcal{I}[f(W)|(t, x)] = \begin{cases} 
\int_{\mathbb{R}} dw f(\sqrt{w}) q(t, w|x), & \text{for Processes 1 and 3}, \\
\int_{\mathbb{R}_{\geq 0}} dw f(-w) q^{(\nu)}(t, w|x), & \text{for Process 2},
\end{cases} \tag{2.31}
\]

with

\[
q(t, w|x) = \frac{e^{-(ix+w)^2/2t}}{\sqrt{2\pi t}}, \tag{2.32}
\]

\[
q^{(\nu)}(t, w|x) = \left(\frac{w}{x}\right)^{\nu/2} \frac{e^{(x-w)^2/2t}}{2t} J_{\nu}\left(\frac{\sqrt{w}x}{t}\right), \quad \nu > \in (-1, \infty), \tag{2.33}
\]

where $J_{\nu}$ is the Bessel function defined by (1.36).

Note that $W$ in LHS of (2.31) is a dummy variable, but it will be useful in order to specify a function $f$. For example, we can verify the following [44]:

\[
m_n(t, x) := \mathcal{I}[W^n|(t, x)] = \left(\frac{t}{2}\right)^{n/2} H_n\left(\frac{x}{\sqrt{2t}}\right), \quad t \geq 0, \quad n \in \mathbb{N}_0, \quad \text{for Process 1}, \tag{2.34}
\]

where $\{H_n\}_{n \in \mathbb{N}_0}$ are the Hermite polynomials [125], and for $\nu \in (-1, \infty),$

\[
m_n^{(\nu)}(t, x) := \mathcal{I}[W^n|(t, x)] = (-1)^n n!(2t)^n I^{(\nu)}_n\left(\frac{x}{2t}\right), \quad t \geq 0, \quad n \in \mathbb{N}_0, \quad \text{for Process 2}, \tag{2.35}
\]

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where \( \{ L_n^{(v)} \}_{n \in \mathbb{N}_0} \) are the Laguerre polynomials \( (1.20) \).

We introduce sets of entire functions of \( z \in \mathbb{C} \) \( [70] \), 1 \( \leq k \leq N \), for \( u \in \mathbb{W}_N(S) \), \( \xi = \sum_{j=1}^N \delta_{u_j} \), \( v \in S \), \( r > 0 \),

\[
\Phi_{\xi}^v(z) = \begin{cases} 
\prod_{1 \leq \ell \leq N, \ u_\ell \neq v} \frac{z - u_\ell}{v - u_\ell} & \text{for Processes 1 and 2,} \\
\prod_{1 \leq \ell \leq N, \ u_\ell \neq v} \frac{\sin((z - u_\ell)/2r)}{\sin((v - u_\ell)/2r)} & \text{for Process 3.} 
\end{cases} \tag{2.36}
\]

For \( v \in S \), set

\[
\mathcal{M}_\xi^v(t,x) = \mathcal{I}[\Phi_{\xi}^v(W)|(t,x)], \quad (t,x) \in [0, \infty) \times S. \tag{2.37}
\]

Then the following is proved \( [47] \).

**Proposition 2.8** For Processes 1, 2, and 3, if \( \xi = \sum_{j=1}^N \delta_{u_j} \in \text{Conf}_0(S) \), then \( \mathcal{M}_\xi = \{ \mathcal{M}_\xi^v(\cdot, \cdot) : v \in \{ u_1, \ldots, u_N \} \} \) satisfies the conditions (M1)–(M3).

We can prove the following \( [47] \).

**Theorem 2.9** For any \( \xi \in \text{Conf}_0(S) \), the three processes have DMRs associated with \((Y, \mathcal{M}_\xi)\) such that

\[
Y \text{ is given by } \begin{cases} 
\text{BM on } \mathbb{R}, & \text{for Process 1,} \\
\text{BESQ}^{(v)} \text{ on } \mathbb{R}_{\geq 0}, & \text{for Process 2,} \\
\text{BM on } [0, 2\pi r), & \text{for Process 3,}
\end{cases}
\]

and \( \mathcal{M}_\xi = \{ \mathcal{M}_\xi^v(\cdot, \cdot) \}, v \in S \) is given by \( (2.37) \). Then they are all DSPs with the spatio-temporal correlation kernels

\[
K_{\xi}(s,x; t,y) = \int_S \xi(dv)p(s,x|v)\mathcal{M}_\xi^v(t,y) - 1(s > t)p(s-t,x|y), \quad (s,x),(t,y) \in [0, \infty) \times S,
\]

with \( \mathcal{M}_\xi^v(t,x) = \mathcal{I}[\Phi_{\xi}^v(W)|(t,x)], \quad (t,x) \in [0, \infty) \times S \),

where \( p \) is the transition density of the process \( Y \).

### 2.5 Martingales for configurations with multiple points

For general \( \xi \in \text{Conf}(S) \) with \( \xi(S) = N < \infty \), define \( \text{supp} \xi = \{ x \in S : \xi(x) > 0 \} \) and let \( \xi_s(\cdot) = \sum_{v \in \text{supp} \xi} \delta_{v}(\cdot) \). For \( s \in [0, \infty) \), \( v, x \in S \), \( z, \zeta \in \mathbb{C} \), let

\[
\phi_{\xi}^v((s,x);z,\zeta) = \begin{cases} 
\frac{p(s,x|v)}{p(s,x|v)} \frac{1}{z - \zeta} \prod_{\ell=1}^N \frac{z - u_\ell}{\zeta - u_\ell} & \text{for Processes 1 and 2,} \\
\frac{p(s,x|v)}{p(s,x|v)} \frac{1}{z - \zeta} \prod_{\ell=1}^N \frac{\sin((z - u_\ell)/2r)}{\sin((\zeta - u_\ell)/2r)} & \text{for Process 3,} \tag{2.38}
\end{cases}
\]

and

\[
\Phi_{\xi}^v((s,x);z) = \frac{1}{2\sqrt{\pi} - 1} \oint_{C(\delta_v)} d\zeta \phi_{\xi}^v((s,x);z,\zeta) = \text{Res} \left[ \phi_{\xi}^v((s,x);z,\zeta) ; \zeta = v \right], \tag{2.39}
\]

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where $C(\delta_v)$ is a closed contour on the complex plane $\mathbb{C}$ encircling a point $v$ on $S$ once in the positive direction. Define

$$
\mathcal{M}_\xi^\nu((s, x)|(t, y)) = \mathcal{I} \left[ \Phi_\xi^\nu((s, x); W)|_{(t, y)} \right], \quad (s, x), (t, y) \in [0, \infty) \times S. \quad (2.40)
$$

Then it is easy to see that (2.48) is rewritten as

$$
K_\xi(s, x; t, y) = \int_S \xi_* (dv)p(s, x|v)\mathcal{M}_\xi^\nu((s, x)|(t, y)) - 1(s > t)p(s - t, x|y), \quad (2.41)
$$

$(s, x), (t, y) \in [0, \infty) \times S$.

We note that, even though the systems of SDEs (2.22), (2.23), cannot be solved for any initial configuration with multiple points, $\xi \in \text{Conf}(S) \setminus \text{Conf}_0(S)$, the kernel (2.41) with (2.40) is bounded and integrable also for $\xi \in \text{Conf}(S) \setminus \text{Conf}_0(S)$. Therefore, spatio-temporal correlations are given by (2.16) for any $0 \leq t_1 < \cdots < t_M < \infty, M \in \mathbb{N}$ and finite-dimensional distributions are determined.

**Proposition 2.10** Also for $\xi \in \text{Conf} \setminus \text{Conf}_0$, the DSPs with the spatio-temporal correlation kernels (2.41) are well-defined. They provide the entrance laws for the processes $(\Xi(t))_{t \geq 0, P_\xi}$.

In order to give examples of Proposition 2.10, here we study the extreme case such that all $N$ points are concentrated on an origin,

$$
\xi = N\delta_0 \iff \xi_* = \delta_0 \quad \text{with} \quad \xi(\{0\}) = N. \quad (2.42)
$$

We consider Processes 1 and 2. For (2.22), (2.23) and (2.24) become

$$
\phi^0_{N\delta_0}((s, x); z, \zeta) = \frac{p(s, x|\zeta)}{p(s, x|0)} \frac{1}{z - \zeta} \left( \frac{z}{\zeta} \right)^N
$$

and

$$
\Phi^0_{N\delta_0}((s, x); z) = \frac{1}{p(s, x|0)} \sum_{\ell=0}^{N-1} z^{N-\ell-1} \frac{1}{2\pi \sqrt{-1}} \oint_{C(\delta_0)} d\zeta \frac{p(s, x|\zeta)}{\zeta^{N-\ell}}, \quad (2.43)
$$

since the integrands are holomorphic when $\ell \geq N$, where we assume $\nu \in (-1, \infty)$ and $C(\delta_0)$ is interpreted as $\lim_{\epsilon \downarrow 0} C(\delta_v)$ for BESQ($\nu$).

For BM with the transition density (2.20), (2.21) gives

$$
\Phi^0_{N\delta_0}((s, x); z) = \sum_{\ell=0}^{N-1} \left( \frac{z}{\sqrt{2s}} \right)^{N-\ell-1} \frac{1}{2\pi \sqrt{-1}} \oint_{C(\delta_0)} d\zeta \frac{e^{\zeta z - \zeta^2/2s}}{\zeta^{N-\ell}}
$$

$$
= \sum_{\ell=0}^{N-1} \left( \frac{z}{\sqrt{2s}} \right)^{N-\ell-1} \frac{1}{2\pi \sqrt{-1}} \oint_{C(\delta_0)} d\eta \frac{e^{2(z/x\sqrt{2s})\eta - \eta^2}}{\eta^{N-\ell}}
$$

$$
= \sum_{\ell=0}^{N-1} \left( \frac{z}{\sqrt{2s}} \right)^{N-\ell-1} \frac{1}{(N - \ell - 1)!} H_{N-\ell-1} \left( \frac{x}{\sqrt{2s}} \right),
$$

where $H_k(x)$ is the $k$th Hermite polynomial.
Thus by (2.34) its integral transformation is calculated as

\[ I[\Phi^{(0)}_{N\delta_0}((s,x);W)|(t,y)] \]

\[ = \sum_{\ell=0}^{N-1} \frac{1}{(N-\ell-1)!} H_{N-\ell-1} \left( \frac{x}{\sqrt{2s}} \right) \frac{1}{(2s)^{(N-\ell-1)/2}} I[W^{N-\ell-1}|(t,y)] \]

\[ = \sum_{\ell=0}^{N-1} \frac{1}{(N-\ell-1)!} H_{N-\ell-1} \left( \frac{x}{\sqrt{2s}} \right) \frac{1}{(2s)^{(N-\ell-1)/2}} m_{N-\ell-1}(t,y) \]

\[ = \sum_{\ell=0}^{N-1} \frac{1}{(N-\ell-1)!2^{N-\ell-1}} \left( \frac{t}{s} \right)^{(N-\ell-1)/2} H_{N-\ell-1} \left( \frac{x}{\sqrt{2s}} \right) H_{N-\ell-1} \left( \frac{y}{\sqrt{2t}} \right). \]

Then we obtain the following,

\[ \mathcal{M}^0_{N\delta_0}((s,x)|(t,Y(t))) = \sum_{n=0}^{N-1} \frac{1}{n!} m_n(s,x) m_n(t,Y(t)) \]

\[ = \sum_{n=0}^{N-1} \left( \frac{t}{s} \right)^{n/2} \varphi_n \left( \frac{x}{\sqrt{2s}} \right) \varphi_n \left( \frac{y}{\sqrt{2t}} \right), \tag{2.44} \]

where

\[ \varphi_n(x) = \frac{1}{\sqrt{2^n n!}} H_n(x), \quad n \in \mathbb{N}, \quad x \in \mathbb{R}. \]

Similarly, for BESQ\((\nu), \nu \in (-1, \infty)\) with the transition density (2.27), we obtain

\[ \Phi^{(0)}_{N\delta_0}((s,x);z) = \frac{(2s)^\nu \Gamma(\nu + 1)}{x^{\nu/2}} \sum_{\ell=0}^{N-1} z^{-\ell-N-1} \frac{1}{2\pi \sqrt{-1}} \oint_{C(\delta_0)} d\zeta \frac{e^{-\zeta/2s}}{\zeta^{N-\ell+\nu/2}} I_\nu \left( \frac{\sqrt{x \zeta}}{s} \right) \]

\[ = \Gamma(\nu + 1) \sum_{\ell=0}^{N-1} \left( -\frac{z}{2s} \right)^{-\ell-N-1} \frac{1}{\Gamma(N-\ell+\nu)} L^\nu_{N-\ell-1} \left( \frac{x}{2s} \right), \]

where we have used the contour integral representation of the Laguerre polynomials

\[ L^\nu_n(x) = \frac{\Gamma(n+\nu+1)}{x^{\nu/2}} \frac{1}{2\pi \sqrt{-1}} \oint_{C(\delta_0)} d\eta \frac{e^{\eta}}{\eta^{n+\nu+1/2}} J_\nu \left( 2\sqrt{\eta x} \right). \]

By (2.33), we have

\[ \mathcal{M}^0_{N\delta_0}((s,x)|(t,Y(t))) = I^{(\nu)} \left[ \Phi^{(0)}_{N\delta_0}((s,x);W)|(t,Y(t)) \right] \]

\[ = \Gamma(\nu + 1) \sum_{n=0}^{N-1} \frac{1}{n! \Gamma(n+\nu+1)(2s)^{2n}} m^{(\nu)}_n(s,x) m^{(\nu)}_n(t,Y(t)) \]

\[ = \sum_{n=0}^{N-1} \left( \frac{t}{s} \right)^{n} \phi^{(\nu)}_n \left( \frac{x}{2s} \right) \phi^{(\nu)}_n \left( \frac{Y(t)}{2t} \right), \tag{2.45} \]
where
\[ \phi_n^{(\nu)}(x) = \sqrt{n!\Gamma(\nu + 1)} \frac{1}{\Gamma(n + \nu + 1)} L_n^{(\nu)}(x), \quad n \in \mathbb{N}_0, \quad x \in \mathbb{R}_0. \]

The processes (2.44) and (2.45) are continuous martingales. Then we see
\[ E \left[ \mathcal{M}_{\mathbb{N}_0}^0((s, x)|(t, Y(t))) \right] = E \left[ \mathcal{M}_{\mathbb{N}_0}^0((s, x)|(0, Y(0))) \right] = 1 \]
for \((s, x) \in [0, T] \times S, 0 \leq t < \infty.

By the formula (2.11), we obtain the spatio-temporal correlation kernels for Process 1 as
\[ K_{\mathbb{N}_0}^1(s, x; t, y) = p_{BM}(s, x|0)\mathcal{M}_{\mathbb{N}_0}^0((s, x)|(t, y)) - 1(s > t)p_{BM}(s - t, x|y) \]
\[ = \frac{e^{-x^2/4s/s^{1/4}}}{e^{-y^2/4t/t^{1/4}}} K^{(N)}_{\text{Hermite}}(s, x; t, y) \left( \frac{1}{\sqrt{2s}} e^{-x/\sqrt{2s}} \right) \left( \frac{1}{\sqrt{2t}} e^{-y/\sqrt{2t}} \right)^{1/2} \]
with
\[ K^{(N)}_{\text{Hermite}}(s, x; t, y) = \sum_{n=0}^{N-1} \left( \frac{t}{s} \right)^{n/2} \varphi_n \left( \frac{x}{\sqrt{2s}} \right) \varphi_n \left( \frac{y}{\sqrt{2t}} \right) - 1(s > t) \sum_{n=0}^{N-1} \left( \frac{t}{s} \right)^{n/2} \varphi_n \left( \frac{x}{\sqrt{2s}} \right) \varphi_n \left( \frac{y}{\sqrt{2t}} \right), \]
and for Process 2
\[ K_{\mathbb{N}_0}^2(s, x; t, y) = p^{(\nu)}(s, x|0)\mathcal{M}_{\mathbb{N}_0}^0((s, x)|(t, y)) - 1(s > t)p^{(\nu)}(s - t, x|y) \]
\[ = \frac{(x/2s)\nu e^{-x^2/4s/s^{1/2}}}{(y/2t)\nu e^{-y^2/4t/t^{1/2}}} K^{(\nu, N)}_{\text{Laguerre}}(s, x; t, y) \left( \frac{(x/2s)\nu e^{-x/2s}}{\Gamma(\nu + 1)} 1^{1/2} \left( \frac{(y/2t)\nu e^{-y/2t}}{\Gamma(\nu + 1)} 1^{1/2} \right) \right), \]
with
\[ K^{(\nu, N)}_{\text{Laguerre}}(s, x; t, y) = \sum_{n=0}^{N-1} \left( \frac{t}{s} \right)^{n/2} \phi_n^{(\nu)} \left( \frac{x}{2s} \right) \phi_n^{(\nu)} \left( \frac{y}{2t} \right) - 1(s > t) \sum_{n=0}^{N-1} \left( \frac{t}{s} \right)^{n/2} \phi_n^{(\nu)} \left( \frac{x}{2s} \right) \phi_n^{(\nu)} \left( \frac{y}{2t} \right). \]

The above results are summarized as follows.

**Proposition 2.11**  \((i)\) The process 1 (the noncolliding Brownian motions = Dyson model with \(\beta = 2\)) on \(\mathbb{R}\) starting from \(\mathbb{N}_0\) is the DSP, \(((\Xi(t))_{t \geq 0}, K^{(N)}_{\text{Hermite}}, (d\lambda_t(dx))_{t \geq 0})\), with the spatio-temporal correlation kernel
\[ K^{(N)}_{\text{Hermite}}(s, x; t, y) = \begin{cases} \sum_{n=0}^{N-1} \left( \frac{t}{s} \right)^{n/2} \varphi_n \left( \frac{x}{\sqrt{2s}} \right) \varphi_n \left( \frac{y}{\sqrt{2t}} \right), & \text{if } s < t, \\ \sum_{n=0}^{N-1} \varphi_n \left( \frac{x}{\sqrt{2s}} \right) \varphi_n \left( \frac{y}{\sqrt{2t}} \right), & \text{if } s = t, \\ -\sum_{n=N}^{\infty} \left( \frac{t}{s} \right)^{n/2} \varphi_n \left( \frac{x}{\sqrt{2s}} \right) \varphi_n \left( \frac{y}{\sqrt{2t}} \right), & \text{if } s > t, \end{cases} \]

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(s, x), (t, y) ∈ [0, ∞) \times \mathbb{R}, where \( \varphi(x) = H_n(x)/\sqrt{2^nn!} \), \( n \in \mathbb{N}_0 \), and with the time-dependent background measure

\[
d\lambda_t(dx) = \sqrt{2t} \circ \lambda_{N(0,1/2)}(dx) = \frac{e^{-x^2/2t}}{\sqrt{\pi}} \frac{dx}{\sqrt{2t}}
\]

\[
= \frac{e^{-x^2/(2t)}}{\sqrt{2\pi t}} dx = p_{BM}(t, x|0)dx, \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]

(ii) The process 2 (the noncolliding BESQ\(^{(\nu)}\)) = Bru–Wishart process with \( \beta = 2 \), \( \nu \in (-1, \infty) \), on \( \mathbb{R}_{\geq 0} \) starting from \( N \delta_0 \) is the DSP, \((\Xi(t))_{t \geq 0}, K^{(N)}_{\text{Laguerre}}, (d\lambda_t(dx))_{t \geq 0}\), with the spatio-temporal correlation kernel

\[
K^{(N)}_{\text{Laguerre}}(s, x; t, y) = \begin{cases} \sum_{n=0}^{N-1} \frac{n}{s} \phi_n \left( \frac{x}{2s} \right) \phi_n \left( \frac{y}{2t} \right), & \text{if } s < t, \\ \sum_{n=0}^{N-1} \phi_n \left( \frac{x}{2t} \right) \phi_n \left( \frac{y}{2t} \right), & \text{if } s = t, \\ -\sum_{n=N}^{\infty} \frac{n}{s} \phi_n \left( \frac{x}{2s} \right) \phi_n \left( \frac{y}{2t} \right), & \text{if } s > t, \end{cases}
\]

\((s, x), (t, y) \in [0, \infty) \times \mathbb{R}_{\geq 0}, where \( \phi(x) = \sqrt{n!/\Gamma(\nu+1)/\Gamma(n+\nu+1)}L^{(\nu)}_n, n \in \mathbb{N}_0 \), and with the time-dependent background measure

\[
d\hat{\lambda}_t(dx) = (2t) \circ \lambda_{\Gamma(\nu+1)}(dx) = \frac{1}{\Gamma(\nu+1)} \left( \frac{x}{2t} \right)^\nu e^{-x^2/(2t)} \frac{dx}{2t}
\]

\[
= \frac{x^\nu e^{-x^2/(2t)}}{(2t)^{\nu+1} \Gamma(\nu+1)} dx = p^{(\nu)}(t, x|0)dx, \quad (t, x) \in [0, \infty) \times \mathbb{R}_{\geq 0}.
\]

Note that, for each time \( t \in (0, \infty) \),

\[
K^{(N)}_{\text{Hermite}}(t, x; t, y) = K^{(N)}_{\text{Hermite}} \left( \frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}} \right) = \sqrt{2t} \circ K^{(N)}_{\text{Hermite}}, \quad x, y \in \mathbb{R},
\]

\[
K^{(N)}_{\text{Laguerre}}(t, x; t, y) = K^{(N)}_{\text{Laguerre}} \left( \frac{x}{2t}, \frac{y}{2t} \right) = (2t) \circ K^{(N)}_{\text{Laguerre}}, \quad x, y \in \mathbb{R}_{\geq 0}.
\]

The spatio-temporal correlation kernels \( K^{(N)}_{\text{Hermite}} \) and \( K^{(N)}_{\text{Laguerre}} \) are called the extended Hermite and Laguerre kernels, respectively, in random matrix theory (see, for instance, [32]). Hence at each time \( t \in (0, \infty) \),

\[
(\Xi(t), K^{(N)}_{\text{Hermite}}, \lambda_t(dx)) = (\Xi, \sqrt{2t} \circ K^{(N)}_{\text{Hermite}}, \sqrt{2t} \circ \lambda_{N(0,1/2)}(dx)),
\]

\[
(\widehat{\Xi}(t), K^{(N)}_{\text{Laguerre}}, \hat{\lambda}_t(dx)) = (\Xi, (2t) \circ K^{(N)}_{\text{Laguerre}}, (2t) \circ \lambda_{\Gamma(\nu+1,\nu)}(dx)),
\]

where \( (\Xi, K^{(N)}_{\text{Hermite}}, \lambda_{N(0,1/2)}(dx)) \) and \( (\Xi, K^{(N)}_{\text{Laguerre}}, \lambda_{\Gamma(\nu+1,\nu)}(dx)) \) and the DPPs studied in Section 1.6.4.

Here we would like to emphasize the fact that these spatio-temporal kernels have been derived here by not following the ‘bi(multiple)-orthogonal-function method’ [58], but by only using proper martingales determined by the chosen initial configuration [224]. The above kernels determine the finite-dimensional distributions and specify the entrance laws for the systems of SDEs (2.22) and (2.23) from the state \( N \delta_0 \) [54].
2.6 Relaxation phenomenon in Process 3

As an application of Theorem 2.9, here we study a typical non-equilibrium dynamics of Process 3, that is, a relaxation phenomenon to the equilibrium. For Processes 1 and 2, see [57, 58, 59].

For Process 3 we consider the following special initial configuration

$$\eta(\cdot) = \sum_{j=1}^{N} \delta_{w_j}(\cdot) \quad \text{with} \quad w_j = \frac{2\pi r}{N}(j-1), \quad 1 \leq j \leq N.$$  \hspace{1cm} (2.46)

It is an unlabeled configuration with equidistant spacing on $[0, 2\pi r)$. In this case the entire function for Process 3 given by (2.36) becomes

$$\Phi_{\eta}^{w_k}(z) = \prod_{1 \leq \ell \leq N, \ell \neq k} \frac{\sin(z/2r - (\ell - 1)\pi/N)}{\sin((k - \ell)\pi/N)} = \prod_{n=1}^{N-1} \frac{\sin\{z/2r - (k - 2)\pi/N\} + (n-1)\pi/N]}{\sin(n\pi/N)}.$$

We use the product formulas

$$\prod_{n=1}^{N-1} \sin\left(\frac{n\pi}{N}\right) = \frac{N}{2^{N-1}}, \quad \prod_{n=1}^{N} \sin\left[x + \frac{(n-1)\pi}{N}\right] = \frac{\sin(Nx)}{2^{N-1}},$$

and obtain

$$\Phi_{\eta}^{w_k}(z) = \frac{1}{N} \sum_{|\sigma_N(m)| \leq (N-1)/2} e^{2\sqrt{-1}\sigma_N(m)x}, \quad N \in \mathbb{N},$$

where $\sigma_N$ is defined by (2.29). Then, the martingale function is given by

$$\mathcal{M}_{\eta}^{w_k}(t, y) = \int_{\mathbb{R}} d\tilde{y} \frac{1}{N} \sum_{m \in \mathbb{Z}, |\sigma_N(m)| \leq (N-1)/2} e^{2\sqrt{-1}\sigma_N(m)(y+\sqrt{-1}\tilde{y}-2\pi r(k-1)/N)/2r} p_{BM}(t, \tilde{y}|0)$$

$$= \frac{1}{N} \sum_{m \in \mathbb{Z}, |\sigma_N(m)| \leq (N-1)/2} e^{\sigma_N(m)^2t/2r^2 - \sqrt{-1}\sigma_N(m)y/r + 2\sqrt{-1}(k-1)\sigma_N(m)\pi/N},$$

where $p_{BM}$ is given by (2.26).

Now the spatio-temporal correlation kernel with respect to the Lebesgue measure $dx$ on $[0, 2\pi r)$ is obtained by the formula (2.18),

$$\tilde{K}_{\eta}(s, x; t, y) = G_{\eta}(s, x; t, y) - 1(s > t) p^T(s - t, x|y; N).$$
with
\[
G_\eta(s, x; t, y) = \sum_{k=1}^{N} p_\eta^k(s, x|w_k; N) \mathcal{M}_\eta^u(t, y)
\]
\[
= \frac{1}{N} \frac{1}{2\pi r} \sum_{k=1}^{N} \sum_{\ell \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, |\sigma_N(m)| \leq (N-1)/2} h_{k, \ell, m}(s, x; t, y),
\]
where we use the expression (2.30) for \( p_\eta \) and obtain
\[
h_{k, \ell, m}(s, x; t, y) = e^{\sigma_N(m)^2 t - \sigma_N(\ell)^2 s} / 2r^2 \sqrt{1 - (\sigma_N(m) - \sigma_N(\ell)) x / r}.
\]
By the equality
\[
\sum_{k=1}^{N} e^{2\sqrt{1-(k-1)\sigma_N(m)-\sigma_N(\ell)} x / N} = N \sum_{k \in \mathbb{Z}} 1, \quad \text{if} \quad k \neq 0,
\]
we obtain the following decomposition,
\[
G_\eta(s, x; t, y) = \sum_{k \in \mathbb{Z}} G_\eta^{(k)}(s, x; t, y)
\]
with
\[
G_\eta^{(k)}(s, x; t, y) = \frac{1}{2\pi r} \sum_{m \in \mathbb{Z}, \sigma_N(m) \leq (N-1)/2} e^{\sigma_N(m)^2 t - \sigma_N(m)^2 s / 2r^2}
\]
\[
\times e^{\sigma_N(m)^2 (t-s) / 2r^2 - \sqrt{1-1} \sigma_N(m) y - \sigma_N(m+kN)x / r}.
\]
Since \( \sigma_N(m+kN)^2 > \sigma_N(m)^2 \) if \( m \in \mathbb{Z}, \sigma_N(m) \leq (N-1)/2 \) and \( k \neq 0 \), we see that for \((s, x), (t, y) \in [0, \infty) \times [0, 2\pi r)\)
\[
\lim_{T \to \infty} G_\eta^{(k)}(s+T, x; t+T, y) = \begin{cases} G_{eq}(t-s, y-x), & \text{if } k = 0, \\ 0, & \text{otherwise}, \end{cases}
\]
where
\[
G_{eq}(t, x) = \frac{1}{2\pi r} \sum_{m \in \mathbb{Z}, \sigma_N(m) \leq (N-1)/2} e^{\sigma_N(m)^2 t / 2r^2 - \sqrt{1} \sigma_N(m) x / r}.
\]
In particular, when \( s = t \) we have
\[
G_{eq}(0, x) = \frac{1}{2\pi r} \sum_{m \in \mathbb{Z}, \sigma_N(m) \leq (N-1)/2} e^{-\sqrt{1} \sigma_N(m) x / r} = \frac{1}{2\pi r} \sin(Nx/2r) / \sin(x/2r).
\]
The results are summarized as follows. Define
\[
e_n(t, x) := e^{n^2 t - \sqrt{1} nx}.
\]
Proposition 2.12 Let \( ((\Xi(t))_{t \geq 0}, P_\eta) \) be the Process 3 started at the configuration \([2.46]\). It is a DSP, \( ((\Xi(t))_{t \geq 0}, K_\eta, \lambda_{[0,2\pi r]}(dx)) \), with the spatio-temporal kernel

\[
K_\eta(s, x; t, y) = \frac{1}{N} \sum_{k=1}^{N} \sum_{\ell \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \frac{e_{\sigma_N(m)}(t/2r^2, y/r - 2\pi( k-1)/N)}{e_{\sigma_N(i)}(s/2r^2, x/r - 2\pi( k-1)/N)}
\]

\[
- \mathbf{1}(s > t) \sum_{\ell \in \mathbb{Z}} \frac{e_{\sigma_N(i)}(t/2r^2, y/r)}{e_{\sigma_N(i)}(s/2r^2, x/r)}
\]

and with the uniform measure on the circle \([0, 2\pi r], \lambda_{[0,2\pi r]}(dx) = dx/(2\pi r) = r \circ \lambda^{A_{N-1}}(dx)\). This DSP shows a relaxation phenomenon to an equilibrium DSP, \( ((\Xi(t))_{t \geq 0}, K^{eq}_{\eta}, \lambda_{[0,2\pi r]}(dx)) \), where

\[
K^{eq}_{\eta}(s, x; t, y) = \begin{cases} 
\sum_{m \in \mathbb{Z}, \ |\sigma_N(m)| \leq (N-1)/2} \frac{e_{\sigma_N(m)}(t/2r^2, y/r)}{e_{\sigma_N(m)}(s/2r^2, x/r)}, & \text{if } s < t, \\
\frac{\sin[N(y-x)/2r]}{\sin[(y-x)/2r]}, & \text{if } s = t, \\
- \sum_{m \in \mathbb{Z}, \ |\sigma_N(m)| > (N-1)/2} \frac{e_{\sigma_N(m)}(t/2r^2, y/r)}{e_{\sigma_N(m)}(s/2r^2, x/r)}, & \text{if } s > t
\end{cases}
\]

for \((s, x), (t, y) \in [0, \infty) \times [0, 2\pi r)\).

Note that

\[
K^{eq}_{\eta}(t, x; t, y) = r \circ K^{A_{N-1}}(x, y), \quad t \geq 0, \quad x, y \in [0, 2\pi r),
\]

where \((\Xi, K^{A_{N-1}}, \lambda_{[0,2\pi r]}(dx))\) was studied in Section \([1.6.3]\). The above equilibrium DSP, \( ((\Xi(t))_{t \geq 0}, K^{eq}_{\eta}, \lambda_{[0,2\pi r]}(dx)) \), is reversible with respect to the DPP, \((\Xi, r \circ K^{A_{N-1}}, r \circ \lambda^{A_{N-1}}(dx))\).
3 Multiple Schramm–Loewner Evolutions (SLEs) and Gaussian Free Fields (GFFs)

3.1 Imaginary surface and SLE

The present study has been motivated by the recent work by Sheffield on the quantum gravity zipper and the AC geometry \[ 91 \] and a series of papers by Miller and Sheffield on the imaginary geometry \[ 74, 75, 76, 77 \]. In both of them, a Gaussian free field (GFF) on a simply connected proper subdomain \( D \) of the complex plane \( \mathbb{C} \) (see, for instance, \[ 90 \]) is coupled with a Schramm–Loewner evolution (SLE) driven by a Brownian motion moving on the boundary \( \partial D \) \[ 87, 69, 68 \].

Consider a simply connected domain \( D \subseteq \mathbb{C} \) and write \( \mathcal{C}_c^\infty(D) \) for the space of real smooth functions on \( D \) with compact support. Assume \( h \in \mathcal{C}_c^\infty(D) \) and consider a smooth vector field \( e^{\sqrt{-1}(h/\chi+\theta)} \) with parameters \( \chi, \theta \in \mathbb{R} \). Then a flow line along this vector field, \( \eta : (0, \infty) \ni t \mapsto \eta(t) \in D \) starting from \( \lim_{t \to 0} \eta(t) =: \eta(0) = x \in \partial D \) is defined (if exists) as the solution of the ordinary differential equation (ODE) \[ 91, 74 \]

\[
\frac{d\eta(t)}{dt} = e^{\sqrt{-1}(h/\chi+\theta)}, \quad t \geq 0, \quad \eta(0) = x.
\] (3.1)

Let \( \tilde{D} \subseteq \mathbb{C} \) be another simply connected domain and consider a conformal map \( \varphi : \tilde{D} \to D \). Then we define the pull-back of the flow line \( \eta \) by \( \varphi \) as \( \tilde{\eta}(t) = (\varphi^{-1} \circ \eta)(t) \). That is, \( \varphi(\tilde{\eta}(t)) = \eta(t) \), and the derivatives with respect to \( t \) of the both sides of this equation gives \( \varphi'(\tilde{\eta}(t))d\tilde{\eta}(t)/dt = d\eta(t)/dt \) with \( \varphi'(z) := d\varphi(z)/dz \). We use the polar coordinate \( \varphi'(\cdot) = |\varphi'(\cdot)|e^{\sqrt{-1}\arg \varphi'(\cdot)} \), where \( \arg \zeta \) of \( \zeta \in \mathbb{C} \) is a priori defined up to additive multiples of \( 2\pi \), and hence we have \( d\tilde{\eta}(t)/dt = e^{\sqrt{-1}(\varphi' - \chi \arg \varphi')(\tilde{\eta}(t))/\chi+\theta)}/|\varphi'(\tilde{\eta}(t))|, t \geq 0 \). If we perform a time change \( t \to t/\tau = \tau(t) \) by putting \( t = \int_0^\tau ds/|\varphi'(\tilde{\eta}(s))| \) and \( \tilde{\eta}(t) := \tilde{\eta}(\tau(t)) \), then the above equation becomes

\[
\frac{d\tilde{\eta}(t)}{dt} = e^{\sqrt{-1}(\varphi' - \chi \arg \varphi')(\tilde{\eta}(t))/\chi+\theta)}, \quad t \geq 0.
\]

Since a time change does not affect the geometry of a flow line, we can identify \( h \) on \( D \) and \( h \circ \varphi - \chi \arg \varphi' \) on \( D = \varphi^{-1}(D) \). In \[ 91, 74, 75, 76, 77 \], such a flow line is considered also in the case that \( h \) is given by an instance of a GFF defined as follows.

**Definition 3.1** Let \( D \subseteq \mathbb{C} \) be a simply connected domain and \( H \) be the Dirichlet boundary GFF following the probability law \( \mathbb{P} \) (constructed in Section 3.3). A GFF on \( D \) is a random distribution \( h \) of the form \( h = H + u \), where \( u \) is a deterministic harmonic function on \( D \).

Since a GFF is not function-valued, but it is a distribution-valued random field (see Remark 3.10 in Section 3.3), the ODE in the form (3.1) no longer makes sense mathematically in general. Using the theory of SLE, however, the notion of flow lines has been generalized as follows.

Consider the collection

\[
S := \left\{ (D, h) \middle| D \subseteq \mathbb{C}; \text{ simply connected } \begin{array}{c} h: \text{GFF on } D \end{array} \right\}.
\]

Fixing a parameter \( \chi \in \mathbb{R} \), we define the following equivalence relation in \( S \).
Definition 3.2 Two pairs \((D, h)\) and \((\tilde{D}, \tilde{h})\) in \(S\) are equivalent if there exists a conformal map \(\varphi : \tilde{D} \to D\) and \(\tilde{h} \overset{\text{law}}{=} h \circ \varphi - \chi \text{arg } \varphi\) in \(\mathbb{P}\). In this case, we write \((D, h) \sim (\tilde{D}, \tilde{h})\).

We call each orbit belonging to \(S/ \sim\) an imaginary surface \([74]\) (or an AC surface \([91]\)). That is, in this equivalence class, a conformal map \(\varphi\) causes not only a coordinate change of a GFF as \(h \mapsto h \circ \varphi\) associated with changing the domain of definition of the field as \(D \mapsto \varphi^{-1}(D)\), but also an addition of a deterministic harmonic function \(-\chi \text{arg } \varphi\) to the field. Notice that this definition includes one parameter \(\chi \in \mathbb{R}\). Then the collection of its flow lines is named as the imaginary geometry \([74]\) (or the AC geometry \([91]\)).

Consider the case in which \(D\) is given by the upper half-plane \(\mathbb{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}\) with \(\partial \mathbb{H} = \mathbb{R} \cup \{\infty\}\). Let \((B(t))_{t \geq 0}\) be a one-dimensional standard Brownian motion starting from the origin following the probability law \(\mathbb{P}\). We consider a chordal SLE\(\kappa\) (or the AC geometry \([74]\)).

Consider the case in which \(D\) is given by the upper half-plane \(\mathbb{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}\) with \(\partial \mathbb{H} = \mathbb{R} \cup \{\infty\}\). Let \((B(t))_{t \geq 0}\) be a one-dimensional standard Brownian motion starting from the origin following the probability law \(\mathbb{P}\). We consider a chordal SLE\(\kappa\) (or the AC geometry \([74]\)).

We think that this equivalence in probability law \((3.3)\) realizes the equivalence relation defined by \((3.2)\). Let \(\eta_B(t) := \chi \text{arg } \eta\) be an instance of the free boundary GFF on \(\mathbb{H}\) with the Dirichlet boundary condition on \(\mathbb{R}\) following the probability law \(\mathbb{P}\), which is independent of \((\chi - \sqrt{k}B(t))_{t \geq 0}\) and hence of \((g_{\ii x}^\kappa)_{t \geq 0}\).

Instead of \(H(\cdot)\) itself, we consider the following GFF on \(\mathbb{H}\) by adding a deterministic harmonic function,

\[
h(\cdot) := H(\cdot) - \frac{2}{\sqrt{k}} \text{arg } (\cdot).
\]

\(\forall f \in C^\infty_c(\mathbb{H})\), at each \(t \geq 0\), where the pairing \(\langle \cdot, \cdot \rangle\) is defined by \(3.22\) below. See also \([69]\).

We think that this equivalence in probability law \((3.3)\) realizes the equivalence relation defined by \(3.2\) where conformal maps \(\varphi\) are chosen from shifts of the chordal SLE\(\kappa\) \((\mathbb{H}^\kappa : t \geq 0)\). In other words, an imaginary surface whose representative is given by \((\mathbb{H}, h)\) with \(3.2\) is constructed as a pair of time-evolutionary domains, \(f_{\mathbb{H}^\kappa}^{-1}(\mathbb{H}) = \mathbb{H}^\kappa_0 - \sqrt{k}B(t), t \geq 0\), and a stationary process of GFF, \(h \circ f_{\mathbb{H}^\kappa}^{-1} - \chi \text{arg } f_{\mathbb{H}^\kappa}^{-1}, t \geq 0\), defined on it. It was proved \([91, 74, 75, 76, 77]\) that the ray of this imaginary geometry starting from the origin is realized as the chordal SLE\(\kappa\) curve \(\eta\) when \(\kappa \in (0, 4]\). Moreover, it was argued that, if \(\chi = 0\) (i.e., \(\kappa = 4\)), the flow lines are identified with the zero contour lines of the GFF \(h\) \([89]\).

Notice that \(\text{arg } z\) in \(3.2\) is the imaginary part of the complex analytic function \(\log z\). Sheffield \([91]\) studied another type of distribution-valued random field on \(\mathbb{H}\) given by \(\tilde{h}(\cdot) := H(\cdot) + (2/\sqrt{k}) \text{Re } \log (\cdot) = \tilde{H}(\cdot) + (2/\sqrt{k}) \text{log } | \cdot |\), where \(\tilde{H}(\cdot)\) is an instance of the free boundary GFF on \(\mathbb{H}\). An equivalence class of pairs represented by \((D, \tilde{h})\) is called a quantum surface, which gives a mathematical realization of the quantum gravity \([29]\). In \([91]\), this quantum surface was shown to be stationary under a backward SLE.

In the last section, we generalize some of the above results to the case in which the conformal
maps are generated by a multiple Loewner equation associated with a multi-slit. This section is based on the collaborations with Shinji Koshida (Chuo University) [51, 52]. See also [65].

3.2 Multiple SLEs

3.2.1 Loewner equation for single-slit and multi-slit

Let \( D \) be a simply connected domain in \( \mathbb{C} \) which does not complete the plane; \( D \subseteq \mathbb{C} \). Its boundary is denoted by \( \partial D \). We consider a slit in \( D \), which is defined as a trace \( \eta = \{ \eta(t) : t \in (0, \infty) \} \) of a simple curve \( \eta(t) \in D, 0 < t < \infty; \eta(s) \neq \eta(t) \) for \( s \neq t \). We assume that the initial point of the slit is located in \( \partial D \), \( D' := \lim_{t \to 0} \eta(t) \in \partial D \). Let \( \eta(0,t] := \{ \eta(s) : s \in [0,t] \} \) and \( D^\eta := D \setminus \eta(0,t], t \in (0,\infty) \). The Loewner theory [71] describes a slit \( \eta \) by encoding the curve into a time-dependent analytic function \( g_D^\eta : t \in (0,\infty) \) such that

\[
g_D^\eta : \text{conformal map } D^\eta \to D, \quad t \in (0,\infty).
\]

By the Riemann mapping theorem (see, for instance, Section 6 in [41]), for \( D \subseteq \mathbb{C} \) and a point \( z_0 \in D \), there exists a unique analytic function \( \varphi(z) \) in \( D \), normalized by \( \varphi(z_0) = 0, \varphi'(z_0) > 0 \), such that

\[
\varphi : \text{conformal map } D \to D,
\]

where \( \mathbb{D} \) denotes a unit disk; \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \). Loewner gave differential equation for \( g_D^\eta \) in the case \( D = \mathbb{D} \), which is called the Loewner equation [71]. Since a special case of the Möbius transformation

\[
m(z) := \sqrt{-1} \frac{\alpha - z}{\alpha + z}, \quad |\alpha| = 1,
\]

maps \( \mathbb{D} \) to the upper half plane \( \mathbb{H} := \{ z \in \mathbb{C} : \text{Im } z > 0 \} \) with \( m(0) = \sqrt{-1}, m(\infty) = -\sqrt{-1} \), we can apply the Loewner theory to the case with \( D = \mathbb{H} \), in which \( \mathbb{H} \) is a simply connected domain in \( \mathbb{C} \) and there exists a unique analytic function \( g_H^\eta \) such that

\[
g_H^\eta : \text{conformal map } H^\eta \to \mathbb{H},
\]

which satisfies the condition

\[
g_H^\eta(z) = z + \frac{c_t}{z} + O(|z|^{-2}) \quad \text{as } z \to \infty
\]

for some \( c_t > 0 \), in which the coefficient of \( z \) is unity and no constant term appears. This is called the hydrodynamic normalization. The coefficient \( c_t \) gives the half-plane capacity of \( \eta(0,t] \) and denoted by \( \text{hcap}(\eta(0,t]) \). The following has been shown (see [67, 68, 25]).

**Theorem 3.3** Let \( \eta \) be a slit in \( \mathbb{H} \) such that \( \text{hcap}(\eta(0,t]) = 2t, t \in (0,\infty) \). Then there exists a unique continuous driving function \( V(t) \in \mathbb{R}, t \in (0,\infty) \) such that the solution \( g_t \) of the differential equation

\[
\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - V(t)}, \quad t \geq 0, \quad g_0(z) = z, \quad (3.4)
\]

gives \( g_t = g_H^\eta, t \in (0,\infty) \).
The equation (3.4) is called the chordal Loewner equation. Note that at each time \( t \in (0, \infty) \), the tip of slit \( \eta(t) \) and the value of \( V(t) \) satisfy the following relations,

\[
V(t) = \lim_{z \to 0} g_{\mathbb{H}_t^\eta}(\eta(t) + z) \iff \eta(t) = \lim_{z \to 0} g_{\mathbb{H}_t^\eta}^{-1}(V(t) + z).
\] (3.5)

Moreover, \( V(t) = \lim_{s<t, t \to \infty} g_{\mathbb{H}_s^\eta}(\eta(t)) \) and \( t \to V(t) \) is continuous (see, for instance, Lemma 4.2 in [68]). We write

\[
g_{\mathbb{H}_t^\eta}(\eta(t)) = V(t) \in \mathbb{R}, \quad t \geq 0
\]

in the sense of (3.5).

**Example 3.4** When the driving function is identically zero; \( V(t) \equiv 0, t \in (0, \infty) \), the chordal Loewner equation \( \frac{dg_{\mathbb{H}_t^\eta}(z)}{dt} = 2/g_{\mathbb{H}_t^\eta}(z), t \geq 0 \) is solved under the initial condition \( g_{\mathbb{H}_0^\eta}(z) = z \in \mathbb{H} \) as \( g_{\mathbb{H}_t^\eta}(z)^2 = 4t + z^2, t \geq 0 \). In this simple case, (3.4) gives \( \eta(t) = 2\sqrt{-t}t^{1/2}, t \geq 0 \). That is, the slit \( \eta(0, t], t > 0 \) is a straight line along the imaginary axis starting from the origin, \( \eta(0) = \lim_{t \to 0} \eta(t) = 0 \), and growing upward as time \( t \) is passing.

**Example 3.5** The above example can be extended by introducing one parameter \( \alpha \in (0, 1) \) as follows. Let \( \kappa = \kappa(\alpha) = 4(1 - 2\alpha)/\{\alpha(1 - \alpha)\} \), and consider the case such that

\[
V(t) = \begin{cases} \sqrt{\kappa t}, & \text{if } \alpha \leq 1/2, \\ -\sqrt{\kappa t}, & \text{if } \alpha > 1/2. \end{cases}
\]

In this case, the inverse of \( g_t \) is solved as \( g_{\mathbb{H}_t^\eta}^{-1}(z) = \left( z + 2\sqrt{\frac{\alpha}{1-\alpha}}\sqrt{t} \right)^{1-\alpha} \left( z - 2\sqrt{\frac{1-\alpha}{\alpha}}\sqrt{t} \right)^{\alpha} \), and the slit is obtained as

\[
\eta(t) = g_{\mathbb{H}_t^\eta}^{-1}(V(t)) = 2 \left( \frac{1-\alpha}{\alpha} \right)^{1/2-\alpha} e^{\sqrt{-t\alpha}t^{1/2}}, \quad t \geq 0.
\]

The slit grows from the origin along a straight line in \( \mathbb{H} \) which makes an angle \( \alpha \pi \) with respect to the positive direction of the real axis. When \( \alpha = 1/2 \), this is reduced to the result mentioned in Example 3.4. More detail for this example, see Example 4.12 in [68] and Section 2.2 in [48].

Theorem 3.3 can be extended to the situation such that \( \eta \) in \( \mathbb{H} \) is given by a multi-slit [86]. Let \( N \in \mathbb{N} := \{1, 2, \ldots \} \) and assume that we have \( N \) slits \( \eta_i = \{\eta_i(t) : t \in (0, \infty)\} \subset \mathbb{H}, 1 \leq i \leq N, \) which are simple curves, disjoint with each other, \( \eta_i \cap \eta_j = \emptyset, i \neq j, \) starting from \( N \) distinct points \( \lim_{t \to 0} \eta_i(t) =: \eta_i(0) \) on \( \mathbb{R}; \eta_1(0) < \cdots < \eta_N(0) \), and all going to infinity; \( \lim_{t \to \infty} \eta_i(t) = \infty, 1 \leq i \leq N. \) A multi-slit is defined as a union of them, \( \bigcup_{i=1}^N \eta_i \), and \( \mathbb{H}_t^{\eta} := \mathbb{H} \setminus \bigcup_{i=1}^N \eta_i(0, t) \) for each \( t > 0 \) with \( \mathbb{H}_0^\eta := \mathbb{H}. \) For each time \( t \in (0, \infty) \), \( \mathbb{H}_t^{\eta} \) is a simply connected domain in \( \mathbb{C} \) and then there exists a unique analytic function \( g_{\mathbb{H}_t^{\eta}} \) such that

\[
g_{\mathbb{H}_t^{\eta}} : \text{conformal map } \mathbb{H}_t^{\eta} \to \mathbb{H},
\]

satisfying the hydrodynamic normalization condition

\[
g_{\mathbb{H}_t^{\eta}}(z) = z + \frac{\text{hcap} \left( \bigcup_{i=1}^N \eta_i(0, t) \right)}{z} + O(|z|^{-2}) \quad \text{as } z \to \infty.
\]

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Theorem 3.6 ([86]) For $N \in \mathbb{N}$, let $\bigcup_{i=1}^{N} \eta_i$ be a multi-slit in $\mathbb{H}$ such that $\text{hcap}(\bigcup_{i=1}^{N} \eta(0, t)) = 2t, t \in (0, \infty)$. Then there exists a set of weight functions $\lambda_i(t) \geq 0, t \geq 0, 1 \leq i \leq N$ satisfying $\sum_{i=1}^{N} \lambda_i(t) = 1, t \geq 0$ and an $N$-variate continuous driving function $V(t) = (V_1(t), \ldots, V_N(t)) \in \mathbb{R}^N, t \in (0, \infty)$ such that the solution $g_t$ of the differential equation

$$\frac{dg_t(z)}{dt} = \sum_{i=1}^{N} \frac{2\lambda_i(t)}{g_t(z) - V_i(t)}, \quad t \geq 0, \quad g_0(z) = z, \quad (3.6)$$

gives $g_t = g_{\mathbb{H}^D}, t \in (0, \infty)$.

Roth and Schleissinger [86] called (3.6) the Loewner equation for the multi-slit $\bigcup_{i=1}^{N} \eta_i$. Similar to (3.5), the following relations hold,

$$V_i(t) = \lim_{\eta \to 0} g_{\mathbb{E}^D}(\eta(t) + z) \iff \eta_i(t) = \lim_{\mathbb{H} \to 0} g_{\mathbb{H}^D}^{-1}(V_i(t) + z), \quad 1 \leq i \leq N, \quad t \geq 0,$$

and we write $g_{\mathbb{E}^D}(\eta(t)) = V_i(t) \in \mathbb{R}, 1 \leq i \leq N, t \geq 0$ in this sense.

The Loewner equation for the multi-slit (3.6) given for $D = \mathbb{H}$ can be mapped to other simply connected domains by conformal maps. Here we consider a conformal transformation

$$\varphi(z) = \sqrt{z} : \mathbb{H} \to \mathbb{O},$$

where $\mathbb{O}$ denotes the first orthant in $\mathbb{C}$; $\mathbb{O} := \{z \in \mathbb{C} : \text{Re} z > 0, \text{Im} z > 0\}$. We set

$$\hat{g}_t(z) = \sqrt{g_t(z^2) + c(t)}, \quad t \geq 0, \quad z \in \mathbb{O}, \quad (3.9)$$

with a function of time $c(t), t \geq 0$. Then we can see that (3.6) is transformed to the following,

$$\frac{d\hat{g}_t(z)}{dt} = \sum_{i=1}^{N} \left( \frac{2\hat{\lambda}_i(t)}{\hat{g}_t(z) - \hat{V}_i(t)} + \frac{2\hat{\lambda}_i(t)}{\hat{g}_t(z) + \hat{V}_i(t)} \right) + \frac{2\hat{\lambda}_0(t)}{\hat{g}_t(z)}, \quad t \geq 0,$$

where $\hat{V}_i(t) = \sqrt{V_i(t) + c(t)}, t \geq 0, 1 \leq i \leq N$ and $2\sum_{i=1}^{N} \hat{\lambda}_i(t) + \hat{\lambda}_0(t) = (1/4)dc(t)/dt, t \geq 0$. Here we can assume that $\hat{V}_i(t) \in \mathbb{R}_+$ without loss of generality, since, even if we allow $\hat{V}_i(t) \in \mathbb{R}_+ \cup \sqrt{-1}\mathbb{R}_+ \cup \{0\}$, we can transform the whole system by a (possibly random) automorphism of $\mathbb{O}$ to the case that $\hat{V}_i(t) \in \mathbb{R}_+$.

The equation (3.10) can be regarded as the multi-slit version of the \textit{quadrant Loewner equation} considered in [99]. The solution of (3.10) gives the \textit{uniformization map} to $\mathbb{O}$:

$$\hat{g}_t = g_{\mathbb{O}^D} : \text{conformal map } \mathbb{O}^D \to \mathbb{O},$$

where $\mathbb{O}^D := \mathbb{O} \setminus \bigcup_{i=1}^{N} \eta_i(0, t], \quad g_{\mathbb{O}^D}(\eta(t)) = \hat{V}_i(t) \in \mathbb{R}_{\geq 0}, 1 \leq i \leq N, t \geq 0$.

3.2.2 SLE

So far we have considered the problem where, given time-evolution of a single slit $\eta(0, t], t \geq 0$ or a multi-slit $\sum_{i=1}^{N} \eta(0, t], t \geq 0$ in $\mathbb{H}$, time-evolution of the conformal map from $\mathbb{H}^D$ to $\mathbb{H}$, $t \geq 0$
For each case, the following statements hold with probability one.

**Proposition 3.8**

A continuous simple curve. 

He first asked a suitable family of driving stochastic processes \( (X(t))_{t \geq 0} \) on \( \mathbb{R} \). Then he asked the probability law of a random slit in \( \mathbb{H} \), which will be determined by the relations (3.5) from \( (X(t))_{t \geq 0} \) and the solution \( g_t = g_{\mathbb{H}^2}^t, t \geq 0 \) of the Loewner equation (3.4). Schramm argued that conformal invariance implies that the driving process \( (X(t))_{t \geq 0} \) should be a continuous Markov process which has in a particular parameterization independent increments. Hence \( X(t) \) can be a constant time change of a one-dimensional standard Brownian motion \( (B(t))_{t \geq 0} \), and it is expressed as \( (\sqrt{\kappa}B(t))_{t \geq 0} \) (law) \( (B(\kappa t))_{t \geq 0} \) with a parameter \( \kappa > 0 \). The solution of the Loewner equation driven by \( X(t) = \sqrt{\kappa}B(t), t \geq 0 \),

\[
\frac{dg_t^\kappa(z)}{dt} = \frac{2}{g_t^\kappa(z) - \sqrt{\kappa}B(t)}, \quad t \geq 0, \quad g_{0\mathbb{H}}(z) = z \in \mathbb{H}, \tag{3.11}
\]

is called the **chordal Schramm–Loewner evolution** (chordal SLE) with parameter \( \kappa > 0 \) and is written as \( \text{SLE}_\kappa \) for short.

The following was proved by Lawler, Schramm, and Werner [59] for \( \kappa = 8 \) and by Rohde and Schramm [82] for \( \kappa \neq 8 \).

**Proposition 3.7** By [37], a chordal \( \text{SLE}_\kappa \) \( g_{\mathbb{H}^2}^t, t \in (0, \infty) \) determines a continuous curve \( \eta = \{ \eta(t) : t \in (0, \infty) \} \subset \mathbb{H} \) with probability one.

The continuous curve \( \eta \) determined by an \( \text{SLE}_\kappa \) is called an **SLE\(_\kappa\) curve**. The probability law of an \( \text{SLE}_\kappa \) curve depends on \( \kappa \). As a matter of fact, \( \text{SLE}_\kappa \) curve becomes self-intersecting and can touch the real axis \( \mathbb{R} \) when \( \kappa > 4 \), so it is no more a slit, since a slit has been defined as a trace of a continuous simple curve.

There are three phases of an \( \text{SLE}_\kappa \) curve as shown by the follows.

**Proposition 3.8** For each case, the following statements hold with probability one.

(i) If \( 0 < \kappa \leq 4 \), then the \( \text{SLE}_\kappa \) curve is simple, \( \eta = \eta(0, \infty) \subset \mathbb{H} \), and \( \lim_{t \to \infty} |\eta(t)| = \infty \).

(ii) If \( 4 < \kappa < 8 \), the \( \text{SLE}_\kappa \) curve is self-intersecting, \( \eta \cap \mathbb{R} \neq \emptyset \), and hence at each time \( t \in (0, \infty) \) the hull of the \( \text{SLE}_\kappa \) curve can be defined as the union of \( \eta(0, t] \) and the finite domain in \( \mathbb{H} \) enclosed by any segment of \( \eta(0, t] \) and the real axis \( \mathbb{R} \), which is denoted by \( K_t \). Then \( \bigcup_{t > 0} K_t = \mathbb{H} := \mathbb{H} \cup \mathbb{R} \) and hence \( |\eta(t)| \to \infty \) as \( t \to \infty \), but \( \eta(0, \infty) \cap \mathbb{H} \neq \mathbb{H} \).

(iii) If \( \kappa \geq 8 \), then \( \eta \) is a space-filling curve. That is, if we put \( \eta[0, \infty) := \{0\} \cup \eta(0, \infty) \), then \( \eta[0, \infty) = \mathbb{H} \).

For proof of Proposition 3.8 and more detailed description of the probability laws of an \( \text{SLE}_\kappa \) curves at special values of \( \kappa \), see, for instance [68, 48, 62].
3.2.3 Multiple SLE

For simplicity, we assume that \( \lambda_i(t) \equiv 1/N, t \geq 0, 1 \leq i \leq N \) in (3.6) in Theorem 3.6. Then by a simple time change \( t/N \to t \) associated with a change of notation, \( g_{Nt} \to \hat{g}_t =: g_{\mathbb{H}^0} \), the Loewner equation for the multi-slit in \( \mathbb{H} \) is written as

\[
\frac{dg_{\mathbb{H}^0}(z)}{dt} = \sum_{i=1}^{N} \frac{2}{g_{\mathbb{H}^0}(z) - X^R_i(t)}, \quad t \geq 0, \quad g_{\mathbb{H}^0}(z) = z \in \mathbb{H}.
\]

(3.12)

Then we ask what is the suitable family of driving stochastic processes of \( N \) particles on \( \mathbb{R} \), \( X^R(t) = (X^R_1(t), \ldots, X^R_N(t)), t \geq 0 \).

The same argument with Schramm [87] will give that \( X^R(t) \) should be a continuous Markov process. Moreover, Bauer, Bernard, and Kytölä [11], Graham [39], and Dubédat [28] argued that \( (X^R_i(t))_{t \geq 0}, 1 \leq i \leq N \) are semi-martingales and the quadratic variations should be given by \( \langle dX^R_i, dX^R_j \rangle_t = \kappa \delta_{ij} \, dt \), \( t \geq 0 \), \( 1 \leq i, j \leq N \) with \( \kappa > 0 \). Then we can assume that the system of SDEs for \( (X^R(t))_{t \geq 0} \) is in the form,

\[
dX^R_i(t) = \sqrt{\kappa} dB_i(t) + F^R_i(X^R(t)) \, dt, \quad t \geq 0, \quad 1 \leq i \leq N,
\]

(3.13)

where \( (B_i(t))_{t \geq 0}, 1 \leq i \leq N \) are independent one-dimensional standard Brownian motions, \( \kappa > 0 \), and \( \{F^R_i(x)\}_{i=1}^{N} \) are suitable functions of \( x = (x_1, \ldots, x_N) \) which do not explicitly depend on \( t \).

In the orthant system (3.10), we put \( \hat{\lambda}_i(t) \equiv r/(2N), t \geq 0, r > 0, 1 \leq i \leq N \) and \( dc(t)/dt = 4, t \geq 0 \), and perform a time change \( rt/(2N) \to t \) associated with a change of notation \( \hat{g}_{2Nt/r} \to \hat{g}_t =: g_{Q_0^\infty} \). Then the Loewner equation in \( \mathbb{O} \) is written as

\[
\frac{dg_{Q_0^\infty}(z)}{dt} = \sum_{i=1}^{N} \left( \frac{2}{g_{Q_0^\infty}(z) - X^{R_{\geq 0}}_i(t)} + \frac{2}{g_{Q_0^\infty}(z) + X^{R_{\geq 0}}_i(t)} \right) + \frac{4\delta}{g_{Q_0^\infty}(z)}, \quad t \geq 0,
\]

(3.14)

where \( \delta := N(1-r)/r \in \mathbb{R} \). We assume that the system of SDEs for \( X^{R_{\geq 0}}(t) \in (\mathbb{R}_{\geq 0})^N, t \geq 0 \) is in the same form as (3.13),

\[
dX^{R_{\geq 0}}_i(t) = \sqrt{\kappa} dB_i(t) + X^{R_{\geq 0}}_i(X^{R}_{\geq 0}(t)) \, dt, \quad t \geq 0, \quad 1 \leq i \leq N.
\]

(3.15)

with a set of independent one-dimensional standard Brownian motions \( (B_i(t))_{t \geq 0}, 1 \leq i \leq N \).

3.3 GFF with Dirichlet boundary condition

3.3.1 Bochner–Minlos Theorem

Here we start with the classical Bochner theorem, which states that a probability measure on a finite dimensional Euclidean space is determined by a characteristic function which is a Fourier transform of the probability measure. Note that we have considered Laplace transforms of probability measures in Section 1 and multitime Laplace transforms of probability measures in Section 2. First we define a functional of positive type.
Definition 3.9 Let $\mathcal{V}$ be a finite or infinite dimensional vector space. A function $\psi : \mathcal{V} \to \mathbb{C}$ is said to be a functional of positive type if for arbitrary $N \in \mathbb{N}$, $\xi_1, \ldots, \xi_N \in \mathcal{V}$, and $z_1, \ldots, z_N \in \mathbb{C}$, we have

$$\sum_{n=1}^{N} \sum_{m=1}^{N} \psi(\xi_n - \xi_m)z_nz_m \geq 0.$$ 

Then the following is proved.

Lemma 3.10 Let $\psi : \mathcal{V} \to \mathbb{C}$ be a functional of positive type on a vector space $\mathcal{V}$. Then it follows that (i) $\psi(0) \geq 0$, (ii) $\psi(\xi) = \psi(-\xi)$ for all $\xi \in \mathcal{V}$, and (iii) $|\psi(\xi)| \leq \psi(0)$ for all $\xi \in \mathcal{V}$.

For $x, y \in \mathbb{R}^N$, the standard inner product is denoted by $x \cdot y$ and we write $|x| := \sqrt{x \cdot x}$. Let $\mathcal{B}^N$ be the family of Borel sets in $\mathbb{R}^N$. Then the following is known as the Bochner theorem.

Theorem 3.11 (Bochner theorem) Let $\psi : \mathbb{R}^N \to \mathbb{C}$ be a continuous functional of positive type such that $\psi(0) = 1$. Then there exists a unique probability measure $P$ on $(\mathbb{R}^N, \mathcal{B}^N)$ such that $\psi(\xi) = \int_{\mathbb{R}^N} e^{\sqrt{-1} x \cdot \xi} P(dx)$ for $\xi \in \mathbb{R}^N$.

If we consider the case that $\psi(\xi)$ is given by $\Psi(\xi) := e^{-||\xi||^2/2}, \xi \in \mathbb{R}^N$, then the probability measure $P$ given by the Bochner theorem is the finite-dimensional standard Gaussian measure,

$$P(dx) = \frac{1}{(2\pi)^{N/2}} e^{-|x|^2/2} dx = \prod_{i=1}^{N} \lambda_{N(0,1)}(dx_i), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N.$$ 

Hence we can say that the finite-dimensional standard Gaussian measure $P$ is determined by the characteristic function $\Psi(\xi)$ as

$$\Psi(\xi) = \int_{\mathbb{R}^N} e^{\sqrt{-1} x \cdot \xi} P(dx) = e^{-||\xi||^2/2} \quad \text{for} \quad \xi \in \mathbb{R}^N.$$ 

Now consider the case that $\mathcal{H}$ is an infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{H}}$ with $||x|| = ||x||_{\mathcal{H}} = \sqrt{\langle x, x \rangle_{\mathcal{H}}}$, $x \in \mathcal{H}$. The dual space of $\mathcal{H}$ will be denoted by $\mathcal{H}^*$. Suppose that there were a probability measure $P$ on $\mathcal{H}$ with a suitable $\sigma$-algebra such that

$$\psi(\xi) = \int_{\mathcal{H}} e^{\sqrt{-1} \langle x, \xi \rangle} P(dx) = e^{-||\xi||^2/2} \quad \text{for} \quad \xi \in \mathcal{H}.$$ 

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal system (CONS) of $\mathcal{H}$. If we set $\xi = te_n, t \in \mathbb{R}$ for an arbitrary $n \in \mathbb{N}$, then

$$\int_{\mathcal{H}} e^{\sqrt{-1} \langle x, e_n \rangle} P(dx) = e^{-t^2/2}, \quad t \in \mathbb{R}.$$ 

Since $x \in \mathcal{H}$, we have $\langle x, e_n \rangle \to 0$ as $n \to \infty$. Therefore in the limit $n \to \infty$, the above equality gives $e^{-t^2/2} = 1$, which is a contradiction. This observation suggests that the application of the Bochner theorem...
We write \( \| \cdot \| \). Using this fact, we define Example 3.13. When \( -\Delta \) is to an infinite dimensional space requires more consideration. The following arguments are base on [7] and a note given by Koshida [64]. Let \( D \subset \mathbb{C} \) be a simply connected domain that is bounded. We consider the case \( \mathcal{H} = L^2(D, \mu (dz)) \) with \( \langle f, g \rangle := \int_D f(z)g(z)d\mu (z), \ f, g \in L^2(D, \mu (dz)), \) where \( \mu (dz) \) is the Lebesgue measure on \( D \subset \mathbb{C}; \) \( \mu (dz) = dz \). Let \( \Delta \) be the Dirichlet Laplacian acting on \( L^2(D, \mu (dz)) \). Then \( -\Delta \) has positive discrete eigenvalues so that

\[
- \Delta e_n = \lambda_n e_n, \quad e_n \in L^2(D, \mu (dz)), \quad n \in \mathbb{N}.
\]

(3.16)

We assume that the eigenvalues are labeled in a non-decreasing order: \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \). The system of eigenvalue functions \( \{ e_n \}_{n \in \mathbb{N}} \) forms a CONS of \( L^2(D) \). The following is known as the Weyl formula

**Lemma 3.12** Let \( D \subset \mathbb{C} \) be a simply connected finite domain. The eigenvalues \( \{ \lambda_n \}_{n \in \mathbb{N}} \) of the operator \( -\Delta \) on \( D \) exhibit the following asymptotic behavior,

\[
\lim_{n \to \infty} \frac{\lambda_n}{n} = O(1).
\]

For two functions \( f, g \in C_c^\infty (D) \), their Dirichlet inner product is defined as

\[
\langle f, g \rangle_\nabla := \frac{1}{2\pi} \int_D (\nabla f)(z) \cdot (\nabla g)(z) \mu (dz).
\]

(3.17)

The Hilbert space completion of \( C_c^\infty (D) \) with respect to this Dirichlet inner product will be denoted by \( W(D) \). We write \( \| f \|_\nabla = \sqrt{\langle f, f \rangle_\nabla}, \ f \in W(D) \). If we set \( u_n = \sqrt{2\pi/\lambda_n} e_n, \ n \in \mathbb{N} \), then by integration by parts, we have

\[
\langle u_n, u_m \rangle_\nabla = \frac{1}{2\pi} \langle u_n, (-\Delta) u_m \rangle = \delta_{nm}, \quad n, m \in \mathbb{N}.
\]

Therefore \( \{ u_n \}_{n \in \mathbb{N}} \) forms a CONS of \( W(D) \).

Let \( \hat{\mathcal{H}}(D) \) be the space of formal real infinite series in \( \{ u_n \}_{n \in \mathbb{N}} \). This is obviously isomorphic to \( \mathbb{R}^\mathbb{N} \) by setting \( \hat{\mathcal{H}}(D) \ni \sum_{n \in \mathbb{N}} f_n u_n \mapsto (f_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} \). As a subspace of \( \hat{\mathcal{H}}(D) \), \( W(D) \) is isomorphic to \( \ell^2(\mathbb{N}) \subset \mathbb{R}^\mathbb{N} \). For two formal series \( f = \sum_{n \in \mathbb{N}} f_n u_n, \ g = \sum_{n \in \mathbb{N}} g_n u_n \in \hat{\mathcal{H}}(D) \) such that \( \sum_{n \in \mathbb{N}} |f_n g_n| < \infty \), we define their pairing as \( \langle f, g \rangle_\nabla := \sum_{n \in \mathbb{N}} f_n g_n \). In the case when \( f, g \in W(D) \), their pairing of course coincides with the Dirichlet inner product (3.17).

Notice that, for any \( a \in \mathbb{R} \), the operator \( (-\Delta)^a \) acts on \( \hat{\mathcal{H}}(D) \) as

\[
(-\Delta)^a \sum_{n \in \mathbb{N}} f_n u_n := \sum_{n \in \mathbb{N}} \lambda_n^a f_n u_n, \quad (f_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}.
\]

Using this fact, we define \( \mathcal{H}_a(D) := (-\Delta)^a W(D), \ a \in \mathbb{R}, \) each of which is a Hilbert space with inner product

\[
\langle f, g \rangle_a := \langle (-\Delta)^{-a} f, (-\Delta)^{-a} g \rangle_\nabla, \quad f, g \in \mathcal{H}_a(D).
\]

We write \( \| \cdot \|_a := \sqrt{\langle \cdot, \cdot \rangle_a}, \ a \in \mathbb{R} \).

**Example 3.13** When \( a = 1/2 \), we have

\[
\langle f, g \rangle_{1/2} = \langle (-\Delta)^{-1/2} f, (-\Delta)^{-1/2} g \rangle_\nabla = \frac{1}{2\pi} \langle f, g \rangle, \quad f, g \in \mathcal{H}_{1/2}(D).
\]

Therefore \( \mathcal{H}_{1/2}(D) = L^2(D, \mu (dz)) \).
We can prove the following two lemmas.

**Lemma 3.14** Assume $a < b$. Then $\mathcal{H}_a(D) \subset \mathcal{H}_b(D)$.

*Proof* Let $f = \sum_{n \in \mathbb{N}} f_n u_n \in \tilde{\mathcal{H}}(D)$ be a formal series. Then we have

$$\|f\|_b^2 = \sum_{n \in \mathbb{N}} \lambda_n^{-2b} f_n^2 \leq \sum_{n=1}^{N-1} (\lambda_n^{-2b} - \lambda_n^{-2a}) f_n^2 + \|f\|_a^2.$$  

Since the Weyl formula (Lemma 3.12) holds, we can take $N \in \mathbb{N}$ such that $\lambda_N > 1$. Then the desired inclusion follows.

**Lemma 3.15** Let $a \in \mathbb{R}$ and fix $h \in \mathcal{H}_a(D)$. Then the assignment

$$\langle h, \cdot \rangle_\nabla : \mathcal{H}_{-a}(D) \to \mathbb{R} \quad \text{such that} \quad \mathcal{H}_{-a}(D) \ni f \mapsto \langle h, f \rangle_\nabla \in \mathbb{R}$$

is well-defined and continuous. In particular, $\mathcal{H}_a(D)$ and $\mathcal{H}_{-a}(D)$ makes a dual pair of Hilbert spaces with respect to the Dirichlet inner product $\langle \cdot, \cdot \rangle_\nabla$.

*Proof* For $h = \sum_{n \in \mathbb{N}} h_n u_n$ and $f = \sum_{n \in \mathbb{N}} f_n u_n$, Cauchy’s inequality

$$\sum_{n \in \mathbb{N}} |h_n f_n| = \sum_{n \in \mathbb{N}} |(\lambda_n^{-a} h_n)(\lambda_n a f_n)| \leq \left( \sum_{n \in \mathbb{N}} |\lambda_n^{-a} h_n|^2 \right)^{1/2} \left( \sum_{n \in \mathbb{N}} |\lambda_n a f_n|^2 \right)^{1/2}$$

ensures that the pairing $\langle h, f \rangle$ is well-defined. Notice that

$$\langle h, f \rangle_\nabla = \langle (-\Delta)^{-a} h, (-\Delta)^a f \rangle_\nabla = \langle (-\Delta)^{-2a} h, f \rangle_{-a}.$$  

Since $(-\Delta)^{-2a} h \in \mathcal{H}_{-a}(D)$ by the assumption $h \in \mathcal{H}_a(D)$ and Lemma 3.14, then $\langle h, \cdot \rangle_\nabla$ is continuous on $\mathcal{H}_{-a}(D)$. Therefore $\mathcal{H}_a(D) \simeq \mathcal{H}_{-a}(D)^*$.

**Remark 3.16** Since $\mathcal{H}_{1/2}(D) = L^2(D, \nu(dz))$ as mentioned in Example 3.13, the members of $\mathcal{H}_a(D)$ with $a > 1/2$ cannot be functions, but are distributions.

Define

$$\mathcal{E}(D) := \bigcup_{a > 1/2} \mathcal{H}_a(D).$$  

Then its dual Hilbert space is identified with $\mathcal{E}(D)^* := \bigcap_{a < -1/2} \mathcal{H}_a(D)$ by Lemma 3.15 and

$$\mathcal{E}(D)^* \subset W(D) \subset \mathcal{E}(D)$$

is established (by definition and Lemma 3.14). Here $(\mathcal{E}(D)^*, W(D), \mathcal{E}(D))$ is called a *Gel’fand triple*. We set $\Sigma_{\mathcal{E}(D)} = \sigma(\{\langle \cdot, f \rangle_\nabla : f \in \mathcal{E}(D)^*\})$. On such a setting, the following is obtained. This theorem is the extension of the Bochner theorem (Theorem 3.11) and is called the *Bochner–Minlos theorem* (see, for instance, [42, 90, 7]).
We will give the proof in Section 3.5 below.

Assume that \( \psi \) is a continuous function of positive type on \( W(D) \) such that \( \psi(0) = 1 \). Then there exists a unique probability measure \( P \) on \( (\mathcal{E}(D), \Sigma_{\mathcal{E}(D)}) \) such that

\[
\psi(f) = \int_{\mathcal{E}(D)} e^{\sqrt{-1} \langle h, f \rangle_{\mathcal{E}}} P(dh) \quad \text{for } f \in \mathcal{E}(D)^*.
\]  

(3.19)

We will give the proof in Section 3.5 below.

Under certain conditions on \( \psi \), the domain of function \( f \) for (3.19) can be extended from \( \mathcal{E}(D)^* \) to \( W(D) \) (see Proposition 3.29 in Section 3.4 below). It is easy to verify that the functional \( \Psi(f) := e^{-\|f\|_{L^2}^2/2} \) satisfies the conditions. Then the following is established with a probability measure \( P \) on \( (\mathcal{E}(D), \Sigma_{\mathcal{E}(D)}) \),

\[
\Psi(f) = \int_{\mathcal{E}(D)} e^{\sqrt{-1} \langle h, f \rangle_{\mathcal{E}}} P(dh)
= e^{-\|f\|_{L^2}^2/2} \quad \text{for } f \in W(D).
\]  

(3.20)

**Definition 3.18 (Dirichlet boundary GFF)** A Gaussian free field (GFF) with Dirichlet boundary condition is defined as a pair \( ((\Omega, P), H) \) of a probability space \( (\Omega, P) \) and an isotopy \( H : W(D) \to L^2(\Omega, P) \) such that each \( H(f) \), \( f \in W(D) \) is a Gaussian random variable.

For each \( f \in W(D) \), we write \( (H, f)_{\mathcal{E}} \in L^2(\mathcal{E}(D), P) \) for the random variable defined by \( h \mapsto \langle h, f \rangle_{\mathcal{E}}, h \in \mathcal{E}(D) \). Then (3.20) ensures that the pair of \( ((\mathcal{E}(D), P), H) \) gives a GFF with Dirichlet boundary condition. We often just call \( H \) a Dirichlet boundary GFF without referring to the probability space \( (\mathcal{E}(D), P) \).

### 3.3.2 Conformal invariance of GFF

Assume that \( D, D' \subseteq \mathbb{C} \) are simply connected domains and let \( \varphi : D' \to D \) be a conformal map.

**Lemma 3.19** The Dirichlet inner product (3.17) is conformally invariant, that is,

\[
\int_D (\nabla f)(z) \cdot (\nabla g)(z) \mu(dz) = \int_{D'} (\nabla (f \circ \varphi))(z) \cdot (\nabla (g \circ \varphi))(z) \mu(dz) \quad \text{for } f, g \in \mathcal{C}_c^\infty(D).
\]

**Proof** 

For \( z \in D \) we write \( z = x + \sqrt{-1}y, x, y \in \mathbb{R} \) and put \( \varphi(z) = u(x, y) + \sqrt{-1}v(x, y) \) with real-valued functions \( u \) and \( v \). Owing to the Cauchy-Riemann identities, \( \partial u / \partial x = \partial v / \partial y, \partial u / \partial y = -\partial v / \partial x \), the Jacobian for the transformation \( \varphi \) is written as

\[
\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2.
\]

From the chain-rule and the Cauchy-Riemann identities again, we have the equality

\[
(\nabla f \circ \varphi)(z) \cdot (\nabla g \circ \varphi)(z) = \left( \frac{\partial f}{\partial u} \frac{\partial g}{\partial u} + \frac{\partial f}{\partial v} \frac{\partial g}{\partial v} \right) \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right).
\]

Therefore, the statement is proved. \( \blacksquare \)

From the above lemma, we see that \( \varphi^* : W(D) \ni f \mapsto f \circ \varphi \in W(D') \) is an isomorphism. This allows one to consider a GFF on an unbounded domain. Namely, if \( D' \) is bounded on which a
When we symbolically write Green's function $\langle \phi, H, f \rangle = \langle H, \phi^* f \rangle$, $f \in W(D)$ so as to have the following covariance structure,

$$\mathbb{E} \left[ \langle \phi, H, f \rangle \langle \phi, H, g \rangle \right] = \langle \phi^* f, \phi^* g \rangle = \langle f, g \rangle$$

for $f, g \in W(D)$. \hspace{1cm} (3.21)

Relying on the following formal computation

$$\langle \phi, H, f \rangle = \langle H, \phi^* f \rangle = \frac{1}{2\pi} \int_D (\nabla H)(z) \cdot (\nabla f \circ \varphi)(z) \mu(dz)$$

$$= \frac{1}{2\pi} \int_D (\nabla H \circ \varphi^{-1})(z) \cdot (\nabla f)(z) \mu(dz)$$

we understand the equality $\phi, H = H \circ \varphi^{-1}$. By the fact (3.21) such that the covariance structure does not change under a conformal map $\varphi$, we say that the GFF is conformally invariant.

### 3.3.3 The Green’s function of GFF

Assume that $D \subseteq \mathbb{C}$ is a simply connected domain. In the previous subsections, we have constructed a family $\{ \langle H, f \rangle : f \in W(D) \}$ of random variables whose covariance structure is given by

$$\mathbb{E} \left[ \langle H, f \rangle \langle H, g \rangle \right] = \langle f, g \rangle$$

for $f, g \in W(D)$.

By a formal integration by parts, we see that

$$\langle H, f \rangle = \frac{1}{2\pi} \int_D (\nabla H)(z) \cdot (\nabla f)(z) \mu(dz) = \frac{1}{2\pi} \int_D H(z)(-\Delta f)(z) \mu(dz)$$

$$= \frac{1}{2\pi} \langle H, (-\Delta) f \rangle.$$

Motivated by this observation, we define

$$\langle H, f \rangle := 2\pi \langle H, (-\Delta)^{-1} f \rangle \quad \text{for } f \in \mathcal{D}((-\Delta)^{-1})$$

(3.22)

where $\mathcal{D}((-\Delta)^{-1})$ denotes the domain of $(-\Delta)^{-1}$ in $W(D)$. Note that if $D$ is bounded, then $(-\Delta)^{-1}$ is a bounded operator, but if $D$ is unbounded, then $(-\Delta)^{-1}$ is not defined on $W(D)$. The action of $(-\Delta)^{-1}$ is expressed as an integral operator and the integral kernel is known as the Green’s function. Namely,

$$((-\Delta)^{-1} f)(z) = \frac{1}{2\pi} \int_D G_D(z, w) f(w) \mu(dw), \quad \text{a.e. } z \in D, \quad f \in \mathcal{D}((-\Delta)^{-1}),$$

where $G_D(z, w)$ denotes the Green’s function of $D$ under the Dirichlet boundary condition. Hence the covariance of $\langle H, f \rangle$ and $\langle H, g \rangle$ with $f, g \in \mathcal{D}((-\Delta)^{-1})$ is written as

$$\mathbb{E}[\langle H, f \rangle \langle H, g \rangle] = \int_{D \times D} f(z) G_D(z, w) g(w) \mu(dz) \mu(dw). \hspace{1cm} (3.23)$$

When we symbolically write

$$\langle H, f \rangle = \int_D H(z) f(z) \mu(dz), \quad f \in \mathcal{D}((-\Delta)^{-1}),$$

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the covariance structure can be expressed as
\[ \mathbb{E}[H(z)H(w)] = G_D(z,w), \quad z, w \in D, \quad n \neq w. \]

The conformal invariance of GFF implies that for a conformal map \( \varphi : D' \to D \), we have the equality,
\[ G_{D'}(z,w) = G_D(\varphi(z),\varphi(w)), \quad z, w \in D'. \tag{3.24} \]

Example 3.20 When \( D \) is the upper half plane \( \mathbb{H} \),
\[ G_{\mathbb{H}}(z,w) = \log \frac{|z-w|}{|\bar{z}-\bar{w}|} = \log |z-w| - \log |z-w| \]
\[ = \text{Re} \log(z-w) - \text{Re} \log(z-w), \]
\[ z, w \in \mathbb{H}, z \neq w. \]

Example 3.21 When \( D \) is the first orthant \( \mathbb{O} \),
\[ G_{\mathbb{O}}(z,w) = \log \frac{(z-w)(z+w)}{(z-w)(z+w)} \]
\[ = \log |z-w| + \log |z+w| - \log |z-w| - \log |z+w|, \]
\[ = \text{Re} \log(z-w) + \text{Re} \log(z+w) - \text{Re} \log(z-w) - \text{Re} \log(z+w), \]
\[ z, w \in \mathbb{O}, z \neq w. \]

From the formula (3.24), we see that \( C^\infty_c(D) \subset D((-\Delta)^{-1}) \). In the following, we will consider the family of random variables \( \{\langle H, f \rangle : f \in C^\infty_c(D) \} \) to characterize the GFF, \( H \).

3.4 GFFs coupled with stochastic log-gases

3.4.1 Dirichlet boundary GFF transformed by multiple SLE

Here we write the GFF with free boundary condition defined on a simply connected domain \( D \subseteq \mathbb{C} \) as \( H_D \). Consider the transformation of \( H_D \) by the multiple SLE,
\[ H_{D_t} := H_D \circ g_{D_t}, \quad t \geq 0. \]

By (3.24), the Green’s function of \( H_{D_t}, t \geq 0 \) is given by
\[ G_{D_t}(z,w) = G_D(g_{D_t}(z),g_{D_t}(w)), \quad z, w \in D_t := D \setminus \bigcup_{i=1}^{N} \eta_i(0,t], \quad t \geq 0. \]

Using the explicit expressions of the Greens’ functions for \( D = \mathbb{H} \) and \( \mathbb{O} \), the following is obtained.
**Lemma 3.22** For $D = \mathbb{H}$ and $\mathbb{Q}$, the increments of $G_{D^t}$ in time $t \geq 0$ are given as

\[
\begin{align*}
    dG_{\mathbb{H}^t}(z, w) &= -\sum_{i=1}^{N} \text{Im} \frac{2}{g_{\mathbb{H}^t}(z) - X^R_i(t)} - \text{Im} \frac{2}{g_{\mathbb{H}^t}(w) - X^R_i(t)} dt, \quad z, w \in \mathbb{H}^t, \quad t \geq 0, \\
    dG_{\mathbb{Q}^t}(z, w) &= -\sum_{i=1}^{N} \text{Im} \left( \frac{2}{g_{\mathbb{Q}^t}(z) - X^R_{i \geq 0}(t)} - \frac{2}{g_{\mathbb{Q}^t}(z) + X^R_{i \geq 0}(t)} \right) \\
    &\quad \times \text{Im} \left( \frac{2}{g_{\mathbb{Q}^t}(w) - X^R_{i \geq 0}(t)} - \frac{2}{g_{\mathbb{Q}^t}(w) + X^R_{i \geq 0}(t)} \right) dt, \quad z, w \in \mathbb{Q}^t, \quad t \geq 0.
\end{align*}
\]

**Proof** From the explicit expressions of $G_{\mathbb{H}}$ and $G_{\mathbb{Q}}$ given in Example 3.20 and 3.21, we have

\[
\begin{align*}
    dG_{\mathbb{H}^t}(z, w) &= \text{Re} \frac{dg_{\mathbb{H}^t}(z) - dg_{\mathbb{H}^t}(w)}{g_{\mathbb{H}^t}(z) - g_{\mathbb{H}^t}(w)} - \text{Re} \frac{dg_{\mathbb{H}^t}(z) - dg_{\mathbb{H}^t}(w)}{g_{\mathbb{H}^t}(z) - g_{\mathbb{H}^t}(w)}, \\
    dG_{\mathbb{Q}^t}(z, w) &= \text{Re} \frac{dg_{\mathbb{Q}^t}(z) - dg_{\mathbb{Q}^t}(w)}{g_{\mathbb{Q}^t}(z) - g_{\mathbb{Q}^t}(w)} - \text{Re} \frac{dg_{\mathbb{Q}^t}(z) - dg_{\mathbb{Q}^t}(w)}{g_{\mathbb{Q}^t}(z) - g_{\mathbb{Q}^t}(w)} \\
    &\quad + \text{Re} \frac{dg_{\mathbb{Q}^t}(z) + dg_{\mathbb{Q}^t}(w)}{g_{\mathbb{Q}^t}(z) + g_{\mathbb{Q}^t}(w)} - \text{Re} \frac{dg_{\mathbb{Q}^t}(z) + dg_{\mathbb{Q}^t}(w)}{g_{\mathbb{Q}^t}(z) + g_{\mathbb{Q}^t}(w)}.
\end{align*}
\]

By the multiple Loewner equation (3.12) in $\mathbb{H}$, we see that

\[
\begin{align*}
    dg_{\mathbb{H}^t}(z) - dg_{\mathbb{H}^t}(w) &= \sum_{i=1}^{N} \frac{2dt}{g_{\mathbb{H}^t}(z) - X^R_i(t)} - \sum_{i=1}^{N} \frac{2dt}{g_{\mathbb{H}^t}(w) - X^R_i(t)} \\
    &= -(g_{\mathbb{H}^t}(z) - g_{\mathbb{H}^t}(w)) \sum_{i=1}^{N} \frac{2dt}{g_{\mathbb{H}^t}(z) - X^R_i(t)}(g_{\mathbb{H}^t}(w) - X^R_i(t)).
\end{align*}
\]

Hence we have

\[
\begin{align*}
    dG_{\mathbb{H}^t}(z, w) &= \text{Re} \sum_{i=1}^{N} \frac{2dt}{g_{\mathbb{H}^t}(z) - X^R_i(t)}(g_{\mathbb{H}^t}(w) - X^R_i(t)) \\
    &\quad - \text{Re} \sum_{i=1}^{N} \frac{2dt}{g_{\mathbb{H}^t}(z) - X^R_i(t)}(g_{\mathbb{H}^t}(w) - X^R_i(t)).
\end{align*}
\]
For any two complex variables $\zeta$ and $\omega$, it is easy to verify the equality $\text{Re} \zeta \overline{\omega} - \text{Re} \zeta \omega = 2\text{Im} \zeta \text{Im} \omega$, and then the result is obtained. Similarly by the multiple Loewner equation (3.12) in $\mathcal{O}$, we have

$$
\text{Re} \frac{dg_{\gamma}^n(z) - dg_{\gamma}^n(w)}{g_{\gamma}^n(z) - g_{\gamma}^n(w)} = - \sum_{i=1}^{N} \text{Re} \frac{2dt}{(g_{\gamma}^n(z) - X_i^{R>0}(t)) (g_{\gamma}^n(w) - X_i^{R>0}(t))}
$$

$$
= - \sum_{i=1}^{N} \text{Re} \frac{2dt}{(g_{\gamma}^n(z) + X_i^{R>0}(t)) (g_{\gamma}^n(w) + X_i^{R>0}(t))} - \text{Re} \frac{4\delta dt}{g_{\gamma}^n(z) g_{\gamma}^n(w)},
$$

$$
\text{Re} \frac{dg_{\gamma}^n(z) + dg_{\gamma}^n(w)}{g_{\gamma}^n(z) + g_{\gamma}^n(w)} = \sum_{i=1}^{N} \text{Re} \frac{2dt}{(g_{\gamma}^n(z) - X_i^{R>0}(t)) (g_{\gamma}^n(w) - X_i^{R>0}(t))}
$$

$$
+ \sum_{i=1}^{N} \text{Re} \frac{2dt}{(g_{\gamma}^n(z) + X_i^{R>0}(t)) (g_{\gamma}^n(w) + X_i^{R>0}(t))} + \text{Re} \frac{4\delta dt}{g_{\gamma}^n(z) g_{\gamma}^n(w)}.
$$

Again we use the equality $\text{Re} \zeta \overline{\omega} - \text{Re} \zeta \omega = 2\text{Im} \zeta \text{Im} \omega$ for $\zeta, \omega \in \mathbb{C}$, and then we prove the lemma. \]

### 3.4.2 Complex-valued logarithmic potentials and martingales

We have remarked in Section 2.2 that the Dyson model and the Bru–Wishart processes studied in random matrix theory can be regarded as stochastic log-gasses defined on a line $S = \mathbb{R}$ and a half-line $S = \mathbb{R}_{\geq 0}$, respectively. There the logarithmic potential are given by (2.9). Here we consider a complex-valued logarithmic potential acting between a point $z$ in a two-dimensional domain $D \subseteq \mathbb{C}$ and $N$ points $x = (x_1, \ldots, x_N)$ on a part of the boundary $S \subset \partial D$. For $D = \mathbb{H}$ and $\mathcal{O}$, we put

$$
\Phi_{\mathbb{H}}(z, x) = \sum_{i=1}^{N} \log(z - x_i), \quad z \in \mathbb{H}, \quad x \in \mathbb{R}^N,
$$

$$
\Phi_{\mathcal{O}}(z, x) = \Phi_{\mathbb{O}}(z; x; q)
$$

$$
= \sum_{i=1}^{N} \{\log(z - x_i) + \log(z + x_i)\} + q \log z, \quad z \in \mathcal{O}, \quad x \in (\mathbb{R}_{\geq 0})^N,
$$

(3.25)

where the latter contains a real parameter $q \in \mathbb{R}$.

Now we consider time evolution of the complex-valued potential $\Phi_D$ by putting $x$ be the driving process $(X^S(t))_{t \geq 0}$ of the multiple SLE $(g_{D_i}^\gamma)_{t \geq 0}$ and map the function $\Phi_D(\cdot, X^S(t))$ by $(g_{D_i}^\gamma)_{t \geq 0}$. Note that by (3.7),

$$
X_i^S(t) = \lim_{\eta_i(t) \to 0} g_{D_i}^\gamma(\eta_i(t) + z) =: g_{D_i}^\gamma(\eta_i(t)), \quad 1 \leq i \leq N, \quad t \geq 0,
$$

and we will write

$$
\Phi_D(g_{D_i}^\gamma(z), X^S(t)) = \Phi_D\left(g_{D_i}^\gamma(z), g_{D_i}^\gamma(\eta_i(t))\right), \quad t \geq 0,
$$

where $\eta_i(t) := (\eta_1(t), \ldots, \eta_N(t))$ and $g_{D_i}^\gamma(\eta(t)) := (g_{D_i}^\gamma(\eta_1(t)), \ldots, g_{D_i}^\gamma(\eta_N(t)))$, $t \geq 0$. That is, we consider the complex-valued potentials representing interactions between images by the multiple SLE of a point $z$ in the domain $D$ and $N$ tips of a multi-slit, $\eta_i(t), 1 \leq i \leq N$ at each time $t \geq 0$. We obtain the following.

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Lemma 3.23 For $D = \mathbb{H}$ and $\mathcal{O}$, the increments of the complex-valued potentials are given as follows,

$$
\begin{align*}
&d\Phi_{\mathbb{H}}(g_{\mathbb{H}}^\gamma(z), X^R(t)) = -\sum_{i=1}^{N} \frac{\sqrt{\kappa}dB_i(t)}{g_{\mathbb{H}}^\gamma(z) - X_i^R(t)} \\
&\quad - \sum_{i=1}^{N} \left( F_i^R(X^R(t)) - 4 \sum_{1 \leq j \leq N, j \neq i} \frac{1}{X_i^R(t) - X_j^R(t)} \right) dt \\
&\quad - \left( 1 - \frac{\kappa}{4} \right) d\log g'_{\mathbb{H}}(z), \\
&z \in \mathbb{H}_t^\gamma, \quad t \geq 0, \\
&d\Phi_{\mathcal{O}}(g_{\mathcal{O}}^\gamma(z), X^{R \geq 0}(t), q) = -\sum_{i=1}^{N} \left( \frac{1}{g_{\mathcal{O}}^\gamma(z) - X_i^{R \geq 0}(t)} - \frac{1}{g_{\mathcal{O}}^\gamma(z) + X_i^{R \geq 0}(t)} \right) \sqrt{\kappa}d\tilde{B}_i(t) \\
&\quad - \sum_{i=1}^{N} \left[ F_i^{R \geq 0}(X^{R \geq 0}(t)) - \left\{ 4 \sum_{1 \leq j \leq N, j \neq i} \left( \frac{1}{X_i^{R \geq 0}(t) - X_j^{R \geq 0}(t)} + \frac{1}{X_i^{R \geq 0}(t) + X_j^{R \geq 0}(t)} \right) \right\} \right] \left( \frac{1}{g_{\mathcal{O}}^\gamma(z) - X_i^{R \geq 0}(t)} - \frac{1}{g_{\mathcal{O}}^\gamma(z) + X_i^{R \geq 0}(t)} \right) dt \\
&\quad - 4\delta \left( 1 - \frac{\kappa}{4} - q \right) \frac{dt}{(g_{\mathcal{O}}^\gamma(z))^2} - \left( 1 - \frac{\kappa}{4} \right) d\log g'_{\mathcal{O}}(z), \\
&z \in \mathcal{O}_t^\gamma, \quad t \geq 0,
\end{align*}
$$

where $g'_{\mathcal{O}}(z) := d\Phi_{\mathcal{O}}^\gamma(z)/dz$.

Proof By Itô’s formula,

$$
\begin{align*}
&d\log(g_{D}^\gamma(z) \pm X_i^S(t)) = \frac{dg_{D}^\gamma(z) \pm dX_i^S(t)}{g_{D}^\gamma(z) \pm X_i^S(t)} - \frac{\kappa dt}{(2g_{D}^\gamma(z) \pm X_i^S(t))^2}, \\
&z \in \mathbb{H}_t^\gamma, \quad t \geq 0, \quad 1 \leq i \leq N,
\end{align*}
$$

for $(D, S) = (\mathbb{H}, \mathbb{R})$ and $(\mathcal{O}, \mathbb{R}_{\geq 0})$, and $d\log g_{\mathcal{O}}^\gamma(z) = d\Phi_{\mathcal{O}}^\gamma(z)/g_{\mathcal{O}}^\gamma(z)$. Put the multiple Loewner equation (3.12), (3.14) and the SDEs of their driving processes (3.13), (3.15). For $(D, S) = (\mathbb{H}, \mathbb{R})$, we obtain the equation,

$$
\begin{align*}
&d\Phi_{\mathbb{H}}(g_{\mathbb{H}}^\gamma(z), X^R(t)) = 2 \sum_{1 \leq i, j \leq N} \frac{(g_{\mathbb{H}}^\gamma(z) - X_i^R(t))(g_{\mathbb{H}}^\gamma(z) - X_j^R(t))}{(g_{\mathbb{H}}^\gamma(z) + X_i^R(t))^2} \\
&\quad - \sum_{i=1}^{N} \frac{\sqrt{\kappa}dB_i(t)}{g_{\mathbb{H}}^\gamma(z) - X_i^R(t)} - \sum_{i=1}^{N} F_i^R(X^R(t)) - \frac{1}{2} \sum_{i=1}^{N} \frac{\kappa dt}{(g_{\mathbb{H}}^\gamma(z) - X_i^R(t))^2}.
\end{align*}
$$

Here we use the equalities,

$$
\begin{align*}
&\sum_{1 \leq i, j \leq N} \frac{1}{(g - x_i)(g - x_j)} = \sum_{i=1}^{N} \frac{1}{(g - x_i)^2} + \sum_{1 \leq i \neq j \leq N} \frac{1}{(g - x_i)(g - x_j)} \\
&= \sum_{i=1}^{N} \frac{1}{(g - x_i)^2} + 2 \sum_{1 \leq i \neq j \leq N} \frac{1}{(g - x_i)(x_i - x_j)}, \quad 1 \leq i \leq N.
\end{align*}
$$
Since we obtain from (3.12) the equality,
\[ d \log g'_{H_1}(z) = -2 \sum_{i=1}^{N} \frac{dt}{(g_{H_1}(z) - X_i(t))^2}, \]
the equality (3.20) is verified. For \((D, S) = (\mathbb{O}, \mathbb{R}_{\geq 0})\), use the equalities
\[
\sum_{1 \leq i \neq j \leq N} \left( \frac{1}{g - x_i} + \frac{1}{g + x_i} \right) \left( \frac{1}{g - x_j} + \frac{1}{g + x_j} \right) = 2 \sum_{1 \leq i \neq j \leq N} \left( \frac{1}{g - x_i} + \frac{1}{g + x_i} \right) \left( \frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \right),
\]
\[
\left( \frac{1}{g - x_i} + \frac{1}{g + x_i} \right) \frac{1}{g} = \left( \frac{1}{g - x_i} - \frac{1}{g + x_i} \right) \frac{1}{x_i}, \quad 1 \leq i \leq N.
\]
and
\[
d \log g'_{\mathcal{O}}(z) = -2 \sum_{i=1}^{N} \left\{ \frac{1}{(g_{\mathcal{O}}(z) - X_i^{R_{>0}}(t))^2} + \frac{1}{(g_{\mathcal{O}}(z) + X_i^{R_{>0}}(t))^2} \right\} + \frac{2\delta}{(g_{\mathcal{O}}(z))^2} dt.
\]
Then the equality (3.27) is proved. \(\blacksquare\)

If we assume that \(X^{R}(t)\) is given by the \((8/\kappa)\)-Dyson model \((Y^{R}(t))_{t \geq 0}\) satisfying the SDEs (2.6), the second term in RHS of (3.26) vanishes. For (3.27), first we put \(a = 1 - \kappa/4\) to make the third term in RHS become zero. Then if we assume that \(\delta = \nu\) and \(X^{R_{>0}}(t)\) is given by the \((8/\kappa, \nu)\)-Bru–Wishart process \((Y^{R_{>0}}(t))_{t \geq 0}\) satisfying the SDEs (2.7), the second term in RHS of (3.27) vanishes.

We note that a multiple SLE driven by the Dyson model (or the Bru–Wishart process) is absolutely continuous with respect to multiple of independent SLEs (see Section 3 in [39]). Then the original SLE and multiple SLEs share many common properties. For example, if we define \(\tau_{D}^z := \sup\{t > 0 : z \in D_{\tau}^t\}\), then \(\tau_{D}^z < \infty\) a.s. for any \(z \in D\) [32]. Hence we obtain the following statements.

**Proposition 3.24** Assume that
\[ q = 1 - \frac{\kappa}{4}, \quad \delta = \nu, \quad (3.28) \]
and define
\[
\mathcal{M}_H(z, t) = -\Phi_H(g_{H_1}(z), Y^{R}(t)) - \left( 1 - \frac{\kappa}{4} \right) \log g'_{H_1}(z), \quad z \in \mathbb{H}_{\theta}^0, \quad t \geq 0,
\]
\[
\mathcal{M}_O(z, t) = -\Phi_O(g_{\mathcal{O}}(z), Y^{R_{>0}}(t); 1-\kappa/4) - \left( 1 - \frac{\kappa}{4} \right) \log g'_{\mathcal{O}}(z), \quad z \in \mathcal{O}_{\theta}^0, \quad t \geq 0. \quad (3.29)
\]
Then for each point \(z \in \mathbb{D}, \mathcal{M}_D(z, t \land \tau^z_{D}), D = \mathbb{H} and \mathcal{O}, provide local martingales with increments,
\[
d\mathcal{M}_H(z, t) = \sum_{i=1}^{N} \frac{\sqrt{k}dB_i(t)}{g_{H_1}(z) - Y_i^{R}(t)}, \quad z \in \mathbb{H}_{\theta}^0, \quad t \geq 0,
\]
\[
d\mathcal{M}_O(z, t) = \sum_{i=1}^{N} \left( \frac{1}{g_{\mathcal{O}}(z) - Y_i^{R_{>0}}(t)} - \frac{1}{g_{\mathcal{O}}(z) + Y_i^{R_{>0}}(t)} \right) \sqrt{k}dB_i(t), \quad z \in \mathcal{O}_{\theta}^0, \quad t \geq 0.
\]
3.4.3 Stationary evolution of GFFs coupled with stochastic log-gases

Now we consider a coupling of \((H_{D_t^n}(z))_{t\geq 0}\) and some functional of \((M_D(z, Y^S(t)))_{t\geq 0}\):

\[
H_{D_t^n}(z) + \alpha F[M_D(z, t)], \quad z \in D_t^n, \quad t \geq 0,
\]

where \(F[\cdot]\) denotes a functional and \(\alpha\) is a coupling constant.

Comparing Lemma 3.22 and Proposition 3.24 we observe the fact that

\[
d\left\{ \text{Im} M_D(z, \cdot), \text{Im} M_D(w, \cdot) \right\}_t = -\frac{\kappa}{4}dG_{D_t^n}(z, w), \quad z, w \in D_t^n, \quad t \geq 0,
\]

for \((D, S) = (\mathbb{H}, \mathbb{R})\) and \((\mathcal{O}, \mathbb{R}_{\geq 0})\). Hence we put

\[
F[\cdot] = \text{Im} [\cdot] \quad \text{and} \quad \alpha = \frac{2}{\sqrt{\kappa}},
\]

and define the time-dependent system of Gaussian field,

\[
H_D(z, t) := H_{D_t^n}(z) + \frac{2}{\sqrt{\kappa}}\text{Im} M_D(z, t)
= H_{D_t^n}(z) - \frac{2}{\sqrt{\kappa}}\text{Im} \Phi_D(g_{D_t^n}(z), Y^S(t)) - \chi \log g'_{D_t^n}(z), \quad z \in D_t^n, \quad Y^S(t) \in S^N, t \geq 0,
\]

with

\[
\chi = \alpha \left(1 - \frac{\kappa}{4}\right) = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}.
\]

Since \(\log \zeta = \log |\zeta| + \sqrt{-1}\text{Arg} \zeta\) for \(\zeta \in \mathbb{C}\), where \(\text{Arg} \zeta\) is a priori defined up to additive multiple of \(2\pi\), \(H_D(z, t)\) defined by (3.30) with (3.25) and (3.28) is written as follows,

\[
H_{\mathbb{H}}(z, t) = H_{\mathbb{H}_t^n}(z) - \frac{2}{\sqrt{\kappa}} \sum_{i=1}^N \arg (g_{\mathbb{H}_t^n}(z) - Y_i^R(t)) - \chi \arg g'_{\mathbb{H}_t^n}(z), \quad z \in \mathbb{H}_t^n, \quad t \geq 0,
\]

\[
H_{\mathcal{O}}(z, t) = H_{\mathcal{O}_t^n}(z) - \frac{2}{\sqrt{\kappa}} \sum_{i=1}^N \left\{ \arg (g_{\mathcal{O}_t^n}(z) - Y_i^R(t)) + \arg (g_{\mathcal{O}_t^n}(z) + Y_i^R(t)) \right\}
- \chi \arg g'_{\mathcal{O}_t^n}(z) - \chi \arg g'_{\mathcal{O}_t^n}(z), \quad z \in \mathcal{O}_t^n, \quad t \geq 0.
\]

Note that if we put \(t = 0\) in (3.30), we have

\[
H_D(z, 0) = H_D(z) + \frac{2}{\sqrt{\kappa}} \text{Im} M_D(z, 0)
= H_D(z) - \frac{2}{\sqrt{\kappa}} \text{Im} \Phi_D(z, Y^S)
= \begin{cases} 
H_{\mathbb{H}}(z) - \frac{2}{\sqrt{\kappa}} \sum_{i=1}^N \arg (z - y_i^R), & \text{for } H = \mathbb{H}, \\
H_{\mathcal{O}}(z) - \frac{2}{\sqrt{\kappa}} \left\{ \sum_{i=1}^N \arg (z - y_i^R + y_i^R) + \sum_{i=1}^N \arg (z + y_i^R) \right\} - \chi \arg z, & \text{for } H = \mathcal{O},
\end{cases}
\]

(3.33)
where \( y^S = Y^S(0) \in \mathcal{W}_N(S) \), since \( D_0^\eta = D \), \( g_{D_0^\eta}(z) = g_0(z) = z \) and \( g'_{D_0^\eta}(z) = g'_0(z) = 1, z \in D \).

It was argued in [9] that a GFF on a subdomain of \( D \) can be regarded as a GFF on \( D \). Following it, we regard \( H_D(\cdot, t), t > 0 \) as a GFF on \( \mathbb{H} \) so that the pairing \( \langle H_D(\cdot, t), f \rangle, t > 0 \) with \( f \in \mathcal{C}_c^\infty(D) \) makes sense.

**Theorem 3.25** Let \( \kappa \in (0, 4] \). Assume that \((D, S) = (\mathbb{H}, \mathbb{R}) \) or \((\mathbb{O}, \mathbb{R}_\geq 0) \) and \((Y^S(t))_{t \geq 0} \) is the \((8/\kappa)\)-Dyson model if \( S = \mathbb{R} \) and the \((8/\kappa, \nu)\)-Bru–Wishart process if \( S = \mathbb{R}_\geq 0 \). Then \( \{H_D(z, t)\}_{z \in D}, t \geq 0 \) is stationary in the sense that
\[
\langle H_D(\cdot, t), f \rangle \overset{\text{law}}{=} \langle H_D(\cdot, 0), f \rangle \quad \text{in} \quad \mathbb{P} \otimes \mathbb{P}, \quad \forall f \in \mathcal{C}_c^\infty(D) \quad \text{at each time} \ t \geq 0. \tag{3.34}
\]

**Proof** For any test function \( f \in \mathcal{C}_c^\infty(D) \subset D((-\Delta^{-1})) \), we have
\[
d\left\langle \left\langle \frac{2}{\sqrt{\kappa}} \text{Im} \mathcal{M}_D(\cdot, \cdot), f \right\rangle \right\rangle_t = -dE_t(f),
\]
where
\[
E_t(f) := \int_{D^\eta_t \times D^\eta_t} f(z)G_{D^\eta_t}(z, w)f(w)d\mu(z)d\mu(w),
\]
which is called the *Dirichlet energy*. Since \( D^\eta_t := D \setminus \bigcup_{i=1}^N \eta_i(0, t] \) is decreasing, \( E_t(f) \) is non-increasing in time \( t \geq 0 \). This implies that \( \langle(2/\sqrt{\kappa})\text{Im} \mathcal{M}_D(\cdot, t), f \rangle, t \geq 0 \) is a Brownian motion such that we can regard \( -E_t(f) \) as time. Let \( T \in (0, \infty) \). Then \( \langle(2/\sqrt{\kappa})\text{Im} \mathcal{M}_D(\cdot, T), f \rangle \) is normally distributed with mean \( \langle(2/\sqrt{\kappa})\text{Im} \mathcal{M}_D(\cdot, 0), f \rangle \) and variance \( -E_T(f) - (-E_0(f)) = -E_T(f) + E_0(f) \). On the other hand, the random variable \( \langle H_{D^\eta_t}, f \rangle := \langle H_D \circ g_{D^\eta_t}, f \rangle \) is also normally distributed with mean zero and variance \( E_t(f) \) by the conformal invariance of GFF. Since the random variable \( \langle H_{D^\eta_t}, f \rangle \) is conditionally independent of \( \langle(2/\sqrt{\kappa})\text{Im} \mathcal{M}_D(\cdot, T), f \rangle \), the sum
\[
\langle H_D(\cdot, T), f \rangle = \langle H_{D^\eta_t}, f \rangle + \left\langle \frac{2}{\sqrt{\kappa}} \text{Im} \mathcal{M}_D(\cdot, T), f \right\rangle
\]
is a normal random variable with mean \( \langle(2/\sqrt{\kappa})\text{Im} \mathcal{M}_D(\cdot, 0), f \rangle \) and variance \( (-E_T(f) + E_0(f)) + E_T(f) = E_0(f) \). These values of mean and variance coincide with those of \( \langle H_D(\cdot, 0), f \rangle = \langle H_D(\cdot) + (2/\sqrt{\kappa})\text{Im} \mathcal{M}_D(\cdot, 0), f \rangle \). Since \( T \in (0, \infty) \) is arbitrary, the statement is proved. \[ \square \]

Theorem 3.25 implies that for \((D, S) = (\mathbb{H}, \mathbb{R}) \) and \((\mathbb{O}, \mathbb{R}_\geq 0) \), by coupling the Dirichlet boundary GFF \( H \) on \( D \) with the stochastic log-gas \((Y^S(t))_{t \geq 0} \) on \( S \) via the multiple SLE driven by \((Y^S(t))_{t \geq 0} \), we have a new kind of family of stationary processes of GFF following the probability law \( \mathbb{P} \otimes \mathbb{P} \) on \( D \times S \). At the initial time, the process starts from
\[
H_D(\cdot, 0) = \begin{cases} H_\mathbb{H}(\cdot) - \frac{2}{\sqrt{\kappa}} \sum_{i=1}^N \arg(\cdot - y_i^\mathbb{R}), & \text{for} \ H = \mathbb{H}, \\ H_\mathbb{O}(\cdot) - \frac{2}{\sqrt{\kappa}} \left\{ \sum_{i=1}^N \arg(\cdot - y_i^{\mathbb{R}_\geq 0}) + \sum_{i=1}^N \arg(\cdot + y_i^{\mathbb{R}_\geq 0}) \right\} - \chi \arg \cdot, & \text{for} \ H = \mathbb{O}. \end{cases}
\]
Then we let the boundary points evolve according to the stochastic log-gas and, at each time $t > 0$, we consider the GFF $H_D(\cdot) + u_D(\cdot, t)$ on $D$, where

$$u_D(\cdot, t) = \begin{cases} \frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \text{arg} (\cdot - Y_i^R(t)), & \text{for } H = \mathbb{H}, \\ \frac{2}{\sqrt{\kappa}} \left\{ \sum_{i=1}^{N} \text{arg} (\cdot - Y_i^\mathcal{R}_\geq(0)) + \sum_{i=1}^{N} \text{arg} (\cdot + Y_i^\mathcal{R}_\leq(0)) \right\} - \chi \text{arg} , & \text{for } H = \mathcal{O}. \end{cases}$$

By definition, the pair $(D, H_D(\cdot) + u(\cdot, t))$ is equivalent to $(D^r, H_D(\cdot, t))$. Here the GFF $H_D(\cdot, t)$ can be extended to a GFF on $D$ \cite{89}. The stationary process $(H_D(\cdot, t))_{t \geq 0}$ can be regarded as a generalization of the one considered by Miller and Sheffield \cite{91, 74}, and, in particular, the equivalence class whose representative is given by $(D, H_D(\cdot, 0))$ is a generalization of the imaginary surfaces (the AC surfaces) studied by them. We note that $(\mathbb{H}, H_\mathbb{H}(\cdot, 0)) \sim (\mathcal{O}, H_\mathcal{O}(\cdot, 0))$ in the sense of Definition \ref{322}.

### 3.5 Proof of Theorem \ref{317}

We recall the Riesz–Markov–Kakutani theorem \cite{7}. Let $\mathcal{H}$ be a compact Hilbert space and write $\mathcal{B}_\mathcal{H}$ for the family of Borel sets in $\mathcal{H}$. Then the space of real-valued continuous functions denoted by $\mathcal{C}(\mathcal{H})$ is a real Banach space with respect to the supremum norm $\| \cdot \|_\infty$.

**Definition 3.26** A linear functional $\ell : \mathcal{C}(\mathcal{H}) \to \mathbb{C}$ is positive, if for an arbitrary non-negative function $f \in \mathcal{C}(\mathcal{H})$, we have $\ell(f) \geq 0$.

**Proposition 3.27** (Riesz–Markov–Kakutani theorem) Let $\ell : \mathcal{C}(\mathcal{H}) \to \mathbb{C}$ be a positive linear functional. Then there exists a unique finite measure $\mathbf{P}$ on $(\mathcal{H}, \mathcal{B}_\mathcal{H})$ such that

$$\ell(f) = \int_X f(x) \mathbf{P}(dx), \quad f \in \mathcal{C}(\mathcal{H}).$$

Moreover, $\mathbf{P}(\mathcal{H}) = \| \ell \|$ holds.

Let $\Sigma_a := \sigma(\{\cdot, g) : g \in \mathcal{H}_{-a}(D)\}$ be a $\sigma$-algebra of $\mathcal{H}_a(D)$. Then the following proposition is proved.

**Proposition 3.28** Let $\psi : W(D) \to \mathbb{C}$ be a continuous functional of positive type such that $\psi(0) = 1$. Then for each $a > 1/2$, there exists a probability measure $\mathbf{P}$ on $(\mathcal{H}_a(D), \Sigma_a)$ such that

$$\psi(f) = \int_{\mathcal{H}_a(D)} e^{\sqrt{\kappa} \langle h, f \rangle} \mathbf{P}(dh), \quad f \in \mathcal{H}_{-a}(D). \quad (3.35)$$

**Proof** The proof consists of two steps.

**Step 1.** Let $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ be a one-point compactification of $\mathbb{R}$. Then

$$Q := \hat{\mathbb{R}}^\mathbb{N} = \{h = (h_n)_{n \in \mathbb{N}} : h_n \in \hat{\mathbb{R}}, n \in \mathbb{N}\}$$

is a compact Hausdorff space. The family of Borel sets in $Q$ is denoted by $\mathcal{B}_Q$. Given $h = (h_n)_{n \in \mathbb{N}} \in Q$, we assign a real-valued function $q_n$ for each $n \in \mathbb{N}$ by

$$q_n(h) = \begin{cases} h_n, & h_n \neq \infty, \\ 0, & h_n = \infty. \end{cases}$$

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Then, it can be verified that \( q_n, n \in \mathbb{N} \) are Borel measurable. We write the space of real-valued continuous functions on \( Q \) as \( \mathcal{C}(Q) \), which is a real Banach space with respect to the supremum norm. Let \( \mathcal{C}_{\text{fin}}(Q) \) be the collection of continuous functions on \( Q \) that depend on finitely many \( f_n \)'s, that is,

\[
\mathcal{C}_{\text{fin}}(Q) := \{ f \in \mathcal{C}(Q) : \exists N \in \mathbb{N}, \exists \{ i_1, \ldots, i_N \} \subset \mathbb{N}, f = f(f_{i_1}, \ldots, f_{i_N}) \}.
\]

By a simple argument, it can be verified that \( \mathcal{C}_{\text{fin}}(Q) \) is dense in \( \mathcal{C}(Q) \).

Note that the space \( \hat{H}(D) \) of formal series is isomorphic to \( \mathbb{R}^N \), it can be identified with an open set in \( Q \). Let \( \mathcal{D} \) be a subspace in \( \hat{H}(D) \) defined by \( \mathcal{D} := \bigoplus_{n \in \mathbb{N}} \mathbb{R}u_n \). With \( h \in \mathcal{C}_{\text{fin}}(Q), f \in \mathcal{D} \), we associate a Borel measurable function on \( Q \),

\[
F_h(f) := \langle h, f \rangle_{\mathcal{V}} = \sum_{n \in \mathbb{N}} q_n(h)\langle u_n, f \rangle_{\mathcal{V}},
\]

which is a finite sum.

For \( N \in \mathbb{N} \) and \( \{ i_1, \ldots, i_N \} \subset \mathbb{N} \), we set \( \mathcal{D}_{\{ i_1, \ldots, i_N \}} := \bigoplus_{n=1}^N \mathbb{R}u_n \simeq \mathbb{R}^N \). Then we apply the Bochner theorem (Theorem 3.11) to \( \psi_{\{ i_1, \ldots, i_N \}} := \psi|_{\mathcal{D}_{\{ i_1, \ldots, i_N \}}} \) and obtain a probability measure \( \mathbf{P}_{\{ i_1, \ldots, i_N \}} \) on \( (\mathcal{D}_{\{ i_1, \ldots, i_N \}}, \mathcal{B}^N) \) such that

\[
\psi_{\{ i_1, \ldots, i_N \}}(f) = \int_{\mathcal{D}_{\{ i_1, \ldots, i_N \}}} e^{\sqrt{-1}F_h(f)} \mathbf{P}_{\{ i_1, \ldots, i_N \}}(dh), \quad f \in \mathcal{D}.
\]

Using this family \( \{ \mathbf{P}_{\{ i_1, \ldots, i_N \}} : N \in \mathbb{N}, \{ i_1, \ldots, i_N \} \subset \mathbb{N} \} \) of probability measures, we define a linear functional \( \ell : \mathcal{C}_{\text{fin}}(Q) \to \mathbb{C} \) by

\[
\ell(\varphi) := \int_{\mathcal{D}_{\{ i_1, \ldots, i_N \}}} \varphi(h_{i_1}, \ldots, h_{i_N}) \mathbf{P}_{\{ i_1, \ldots, i_N \}}(dh), \quad \varphi \in \mathcal{C}_{\text{fin}}(Q).
\]

Here we have chosen, for each \( \varphi \in \mathcal{C}_{\text{fin}}(Q) \), a finite set \( \{ i_1, \ldots, i_N \} \in \mathbb{N} \) such that \( \varphi \) depends on \( h_{i_1}, \ldots, h_{i_N} \). Then it can be verified that the functional \( \ell \) is well-defined independent of the choice of such finite sets. Moreover, it is extended to a positive functionals on \( \mathcal{C}(Q) \). Therefore, the Riesz–Markov–Kakutani theorem ensures that there exists a unique probability measure \( \mathbf{P} \) on \( (Q, \mathcal{B}Q) \) such that

\[
\ell(\varphi) = \int_Q \varphi(h) \mathbf{P}(dh), \quad \varphi \in \mathcal{C}(Q).
\]

In particular, if we take \( \varphi(h) = e^{\sqrt{-1}F_h(f)} \in \mathcal{C}_{\text{fin}}(Q) \) for \( f \in \mathcal{D} \), where \( F_h \) is defined by (3.36), we have

\[
\psi(f) = \int_Q e^{\sqrt{-1}F_h(f)} \mathbf{P}(dh), \quad f \in \mathcal{D}.
\]  

**Step 2.** By assumption, \( \psi \) is continuous. Therefore, for an arbitrary \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, if \( \| f \|_{\mathcal{V}} < \delta \), then \( |1 - \psi(f)| < \varepsilon \). Let us fix such \( \varepsilon \) and \( \delta \). Then, in particular, we have

\[
\text{Re}(\psi(f)) > 1 - \varepsilon, \quad \| f \|_{\mathcal{V}} < \delta.
\]

Since \( \psi \) is of positive type, we have \( |\psi(f)| \leq \psi(0) = 1 \), and in particular \( \text{Re}(\psi(f)) \geq -1, f \in W(D) \).

If \( \| f \|_{\mathcal{V}} \geq \delta \), then we have \(-1 > 1 - \varepsilon - 2\delta^{-2}\| f \|^2_{\mathcal{V}}\). Thus

\[
\text{Re}(\psi(f)) > 1 - \varepsilon - 2\delta^{-2}\| f \|^2_{\mathcal{V}}, \quad \| f \|_{\mathcal{V}} \geq \delta.
\]
The same inequality also hold when \( \|f\|_\nabla < \delta \). Therefore
\[
\Re(\psi(f)) > 1 - \varepsilon - 2\delta^{-2}\|f\|_\nabla^2, \quad f \in W(D).
\] (3.38)

In particular, if we set \( f = \sum_{n=1}^{N} f_n u_n \in D_{\{1,\ldots,N\}} \), we have
\[
\Re(\psi(f)) > 1 - \varepsilon - 2\delta^{-2} \sum_{n=1}^{N} f_n^2.
\]

For \( \alpha > 0 \), we introduce a probability measure \( P_{\alpha,N} \) on \( (D_{\{1,\ldots,N\}}, \mathcal{B}^N) \) as
\[
P_{\alpha,N}(df) = \prod_{n=1}^{N} \sqrt{\frac{\lambda_n^{2a}}{2\pi \alpha}} e^{-\lambda_n^{2a} f_n^2/2\alpha} df_n, \quad f = (f_1, \ldots, f_N) \in \mathbb{R}^N,
\]
where \( \{\lambda_n\}_{n \in \mathbb{N}} \) are eigenvalues of \( -\Delta \) as given by (3.16). When we put the integral expression of \( \psi(f) \) (3.37) with (3.36) for \( f \in D_{\{1,\ldots,N\}} \) into LHS of the inequality (3.38) and then integrate the both sides of it with respect to \( P_{\alpha,N}(df) \), we obtain
\[
\int_{Q} e^{-(\alpha/2) \sum_{n=1}^{N} \lambda_n^{-2a} q_n(h)^2} P(dh) > 1 - \varepsilon - 2\alpha\delta^{-2} \sum_{n=1}^{N} \lambda_n^{-2a}.
\] (3.39)

Now we take the limit \( N \to \infty \). Note that the integrand of LHS of (3.35) is supported on \( H_a(D) = \{h = \sum_{n \in \mathbb{N}} h_n u_n : \sum_{n \in \mathbb{N}} (\lambda_n^{-a} h_n)^2 < \infty\} \). The sum in RHS of (3.39) is shown to converge
\[
C := \lim_{N \to \infty} \sum_{n=1}^{N} \lambda_n^{-2a} \sim \sum_{n=1}^{\infty} n^{-2a} < \infty
\]
relying on the Weyl formula (Lemma 3.12) and the assumption \( a > 1/2 \). Therefore, we see that
\[
\int_{H_a(D)} e^{-(\alpha/2)\|h\|^2} P(dh) > 1 - \varepsilon - 2\alpha\delta^{-2} C.
\]

At the limit \( \alpha \to 0 \), this gives \( P(H_a(D)) > 1 - \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, we have \( P(H_a(D)) = 1 \), which allows us to restrict the measure \( P \) onto \( (H_a(D), \Sigma_a) \) and have
\[
\psi(f) = \int_{H_a(D)} e^{\sqrt{-1}(h,f)\nabla} P(dh), \quad f \in D.
\]

In this expression, it is obvious that the domain for \( f \) can be extended to \( H_{-a}(D) \), Therefore, the proof is complete.

By the definition (3.18), Theorem 3.17 is concluded from Proposition 3.28 proved above.

### 3.6 On the domain of functions for Theorem 3.17

We have constructed a family of random variables \( \{\langle H, f \rangle_{\nabla} : f \in \mathcal{E}(D)^*\} \) on \( D \subseteq \mathbb{C} \) so that the assignment \( f \mapsto \langle H, f \rangle_{\nabla} \) is almost surely continuous. We show here that, under certain conditions, the domain of test functions for the random field \( H \) can be extended from \( \mathcal{E}(D)^* \) to \( W(D) \) if we give up its continuity.
Proposition 3.29 Let $\psi : W(D) \to \mathbb{C}$ be a continuous functional of positive type such that $\psi(0) = 1$. Suppose that $\psi$ further satisfies the following assumptions.

(A.1) For an arbitrary $N \in \mathbb{N}$, the function $\psi(\sum_{n=1}^{N} t_n u_n), t_n \in \mathbb{R}, n = 1, \ldots, N$ is of $C^{2}$-class.

(A.2) For an arbitrary $f \in W(D)$, the infinite series

$$\sum_{n=1}^{N} \sum_{m=1}^{N} \frac{\partial^2 \psi(t_n u_n + t_m (1 - \delta_{nm}) u_m)}{\partial t_n \partial t_m} \bigg|_{t_n = t_m = 0} = \langle u_n, f \rangle \nabla \langle u_m, f \rangle \nabla : N \in \mathbb{N}$$

converges.

Then there exists a family of random variables $\{\langle H, f \rangle \nabla : f \in W(D)\}$ such that

(i) $\langle H, f \rangle \nabla \in L^2(\mathcal{E}(D), \mathbf{P})$ for $f \in W(D)$.

(ii) $\langle H, af + bg \rangle \nabla = a \langle H, f \rangle \nabla + b \langle H, g \rangle \nabla$ for $a, b \in \mathbb{R}, f, g \in W(D)$.

(iii) If $f \in \mathcal{E}(D)^{*}$, then $\langle H, f \rangle \nabla$ coincides with that given by Theorem 3.17.

(iv) The following is established

$$\psi(f) = \int_{\mathcal{E}(D)} e^{\sqrt{-1}(h,f)\nabla} \mathbf{P}(dh) \text{ for } f \in W(D). \quad (3.40)$$

Proof From the assumption (A.1), (3.19) gives

$$\frac{\partial^2 \psi(t_n u_n + t_m (1 - \delta_{nm}) u_m)}{\partial t_n \partial t_m} \bigg|_{t_n = t_m = 0} = - \int_{\mathcal{E}(D)} q_n(h)q_m(h) \mathbf{P}(dh), \; n, m \in \mathbb{N}.$$

Set

$$\langle h^{(N)}, f \rangle \nabla := \sum_{n=1}^{N} q_n(h)\langle u_n, f \rangle \nabla, \; f \in W(D), \; N \in \mathbb{N}.$$

Then for $N > M$,

$$\int_{\mathcal{E}(D)} \left| \langle h^{(N)}, f \rangle \nabla - \langle h^{(M)}, f \rangle \nabla \right|^2 \mathbf{P}(dh)$$

$$= \sum_{n=M+1}^{N} \sum_{m=M+1}^{N} \langle u_n, f \rangle \nabla \langle u_m, f \rangle \nabla \int_{\mathcal{E}(D)} q_n(h)q_m(h) \mathbf{P}(dh)$$

$$= - \sum_{n=M+1}^{N} \sum_{m=M+1}^{N} \frac{\partial \psi(t_n u_n + (1 - \delta_{nm}) t_m u_m)}{\partial t_n \partial t_m} \bigg|_{t_n = t_m = 0} \langle u_n, f \rangle \nabla \langle u_m, f \rangle \nabla.$$

By the assumption (A.2), this converges to 0 as $N, M \to \infty$. Therefore the sequence $\{\langle H^{(N)}, f \rangle \nabla : N \in \mathbb{N}\}$ is a Cauchy sequence in $L^2(\mathcal{E}(D), \mathbf{P})$ and the limit

$$\langle H, f \rangle \nabla := \lim_{N \to \infty} \langle H^{(N)}, f \rangle \nabla \in L^2(\mathcal{E}(D), \mathbf{P})$$

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exists and (i) is proved. The linearity (ii) is obvious. By construction (iii) is concluded. Since \( \psi \) is continuous, we have

\[
\psi(f) = \lim_{N \to \infty} \psi \left( \sum_{n=1}^{N} u_n \langle u_n, f \rangle \nabla \right) = \lim_{N \to \infty} \int_{E(D)} e^{\sqrt{-1} \langle h^{(N)}, f \rangle \nabla} P(dh), \quad f \in W(D).
\]

We see that

\[
\left| \int_{E(D)} \left( e^{\sqrt{-1} \langle h, f \rangle \nabla} - e^{\sqrt{-1} \langle h^{(N)}, f \rangle \nabla} \right) P(dh) \right| \leq \int_{E(D)} \left| e^{\sqrt{-1} \langle h, f \rangle \nabla} - e^{\sqrt{-1} \langle h^{(N)}, f \rangle \nabla} \right| P(dh)
\]

\[
\leq \int_{E(D)} \left| \langle h, f \rangle \nabla - \langle h^{(N)}, f \rangle \nabla \right| P(dh) \leq \left( \int_{E(D)} \left| \langle h, f \rangle \nabla - \langle h^{(N)}, f \rangle \nabla \right|^2 P(dh) \right)^{1/2} \to 0 \quad \text{as} \quad N \to \infty.
\]

This implies (iv). Then the proof is complete.

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