Abstract. We provide a natural smooth projective compactification of the space of algebraic maps from \( \mathbb{P}^1 \) to \( \mathbb{P}^n \) by adding a divisor with simple normal crossings.

1. Introduction

Fix a vector space \( V \) of dimension \( n + 1 \). Let \( N_d \) be the Quot scheme parameterizing the exact sequences
\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \xrightarrow{f} V \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow Q \rightarrow 0,
\]
where \( Q \) is a coherent sheaf over \( \mathbb{P}^1 \) of degree \( d \) and rank \( n \). The locus of points of \( N_d \) where \( Q \) is locally free can be identified with the space \( \bar{N}_d \) of algebraic maps of degree \( d \) from \( \mathbb{P}^1 \) to \( \mathbb{P}(V) \). The boundary \( N_d \setminus \bar{N}_d \), consisting of points parameterizing the exact sequences above where \( Q \) is not locally free, has rather complicated singularities. One of the main themes of the current paper is to resolve the singularities of \( N_d \setminus \bar{N}_d \). For this, we find that \( N_d \setminus \bar{N}_d \) comes equipped with a natural filtration by subschemes
\[
Z_{d,0} \subset Z_{d,1} \subset \cdots \subset Z_{d,d-1} = N_d \setminus \bar{N}_d
\]
where \( Z_{d,k} \) is supported on the subset
\[
\{ [f] \in N_d \mid \text{the torsion part of } Q \text{ has degree } \geq d - k \}
\]
for all \( 0 \leq k \leq d - 1 \). As noted in pages 4738-39 of [8], it is expected that one can successively blow up \( N_d \) along the subschemes \( Z_{d,0}, Z_{d,1}, \ldots, Z_{d,d-1} \) such that the resulting final scheme \( M_d \) is smooth and the boundary \( M_d \setminus \bar{N}_d \) is a divisor with normal crossings. The main purpose of the current paper is to execute this conjectural construction in details.

We now briefly outline the main constructions. For each integer \( m \geq 0 \), we let
\[
\rho_{f,m} : \text{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(m)) \rightarrow \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}(m))
\]
be the homomorphism obtained by applying \( \text{Hom}(\cdot, \mathcal{O}_{\mathbb{P}^1}(m)) \) to \( f \). Fix any \( m \geq d - 1 \). We set
\[
W_m := \text{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(m)) \quad \text{and} \quad V_{d+m} := H^0(\mathcal{O}_{\mathbb{P}^1}(d + m)).
\]
Let \(0 \leq k \leq d - 1\). Then the exterior power \(\bigwedge^{k+2+m} \rho_{d,m}\) is a section of \(\text{Hom}(\bigwedge^{k+2+m} W_m, \bigwedge^{k+2+m} V_{d+m}) \otimes \mathcal{O}_{\tilde{N}_d}(k + 2 + m)\).

We proved that the scheme of zeros of \(\bigwedge^{k+2+m} \rho_{d,m}\) is independent of \(m \geq d - 1\) and it is by definition the subscheme \(Z_{d,k}\).

For any two vector spaces \(E\) and \(F\), we let \(S(E, F)\) denote \(P(\text{Hom}(E, F))\). Then our main theorem reads

**Theorem 1.1.** The variety \(M_d\) is a compactification of \(\tilde{N}_d\) such that the following hold.

1. \(M_d\) is isomorphic to the closure of the graph of the rational map

\[
N_d \to \prod_{l=0}^{d-1} S(\bigwedge^{m+2+l} W_m, \bigwedge^{m+2+l} V_{d+m})
\]

\[
[f] \mapsto ([\bigwedge^{m+2} \rho_{f,m}], [\bigwedge^{m+3} \rho_{f,m}], \ldots, [\bigwedge^{m+d+1} \rho_{f,m}])
\]

for all \(m \geq d - 1\);

2. \(M_d\) is a nonsingular projective variety;

3. The complement \(M_d \setminus \tilde{N}_d\) is a divisor with simple normal crossings.

In the course of proofs, we discover that our space \(M_d\) possesses structures strikingly similar to the classic and modern theories on complete collineations, complete correlations, and complete quadrics. Indeed, our proofs rely on Vainsencher’s construction of the spaces of the complete collineations \([21, 20]\). The beautiful stories on these complete objects went all the way back to the works of Schubert in 19th century, to the works of Severi, Van de Waerden, Semple, Tyrrell in the early and middle of the last century, and to the modern treatments, refinements and advances of Laskov \([12, 13]\), Vainsencher \([20, 21]\), Thorup-Kleiman \([19]\), De Concini-Procesi \([3]\), Demazure, and De Concini-Procesi-Goresky-MacPherson \([4]\) in the 1980’s. Needless to say, this way of producing good compactifications is nowadays very standard with the Fulton-MacPherson compactification \([5]\) and Procesi-MacPherson compactification \([14]\) being the prime examples. Related works in 1990’s include the works of De Concini-Procesi schools on hyperplane arrangements and the works of Bifet-De Concini-Procesi on regular embeddings. Lately there are further works in this direction by Wenchuan Hu and Li Li.

We believe that there are some lurking geometric objects, analogous to the above classic complete objects, for which our space \(M_d\) is a parameter space. This is being pursued in a forthcoming publication \([9]\).

Using Quot schemes of coherent sheaves of higher co-ranks, similar spaces can be constructed to provide good compactifications of the spaces of maps\footnote{At the risk of inadvertently omitting many authors who made important contributions to these areas, we mention only a few.}. 
from the smooth rational curve to Grassmannians. This has been carried out in the third author’s Ph.D dissertation [17].

Two further problems to consider are to generalize to higher genus curves and to compare our compactification with the Kontsevich moduli space of stable maps (see, for example, [1] [2] [6] [11]).

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Throughout the paper, we will work with a fixed algebraically closed base field of characteristic zero, unless otherwise stated.

2. Conventions and Terminology

2.1. From the introduction, \( N_d \) is the Quote scheme parameterizing the exact sequences

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \xrightarrow{f} V \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{Q} \rightarrow 0
\]

where \( \mathcal{Q} \) is a coherent sheaf over \( \mathbb{P}^1 \) of degree \( d \) and rank \( n \). We will denote the corresponding point of (2.1) by

\[[f] \in N_d.\]

Note that we have the following identification

\[N_d = \mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), V \otimes \mathcal{O}_{\mathbb{P}^1})).\]

The Quote scheme \( N_d \) comes equipped with the universal family which is an exact sequence of coherent sheaves on \( \mathbb{P}^1 \times N_d \):

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \otimes \mathcal{O}_{N_d}(-1) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^1 \times N_d} \rightarrow \mathcal{Q} \rightarrow 0.
\]

where \( \mathcal{Q} \) is a coherent sheaf of rank \( n \), of relative degree \( d \) and flat over \( N_d \). The restriction of (2.2) to the fiber of the projection \( \mathbb{P}^1 \times N_d \rightarrow N_d \) at the point \([f] \) is exactly the exact sequence (2.1).

2.2. Alternatively, we may realize \( N_d \) as

\[N_d = \{[f_0, \cdots, f_n] \mid f_0, \cdots, f_n \in H^0(\mathcal{O}_{\mathbb{P}^1}(d)), \text{ not all are zero } \}.\]

From this perspective, the boundary strata of \( N_d \setminus \hat{N}_d \) are

\[C_{d,k} = \{[f_0, \cdots, f_n] \mid f_0, \cdots, f_n \text{ have a common factor of degree } \geq d - k \}\]

for all \( 0 \leq k \leq d - 1 \). More details along this line is to be given in 3.2.

2.3. For any non-negative integer \( m \), we set

\[V_m := H^0(\mathcal{O}_{\mathbb{P}^1}(m)),\]

\[W_m := \text{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(m)).\]
In addition, we have the identifications
\[ V_{d+k} = \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}(k)), \]
\[ N_d = \mathbb{P}(V_d \otimes V). \]

3. Resultant Homomorphisms

3.1. Resultant homomorphism and degree of torsion.

Consider the exact sequence (2.1)
\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \xrightarrow{f} V \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow Q \rightarrow 0. \]
For each integer \( k \geq 0 \), recall from the introduction that
\[ (3.1) \quad \rho_{f,k} : \text{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(k)) \rightarrow \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}(k)) \]
is the map obtained by applying \( \text{Hom}(\cdot, \mathcal{O}_{\mathbb{P}^1}(k)) \) to \( f \). We call \( \rho_{f,k} \) the \( k \)th resultant homomorphism of \( f \).

**Proposition 3.2.** Let \( T \) be the torsion submodule of \( Q \). Then we have
\[ \text{rank} \ \rho_{f,k} \left\{ \begin{array}{ll}
= k + 1 + d - \deg T, & \text{if } k \geq d - \deg T - 1, \\
\geq 2(k + 1), & \text{if } 0 \leq k \leq d - \deg T - 1.
\end{array} \right. \]
In particular, \( \text{rank} \ \rho_{f,k} = 2(k + 1) \) when \( k = d - \deg T - 1 \).

**Proof.** First, we write \( Q = F \oplus T \) such that \( F \) is a locally free sheaf of rank \( n \) and degree \( (d - \deg T) \). By applying the functor \( \text{Hom}(\cdot, \mathcal{O}_{\mathbb{P}^1}(k)) \) to the exact sequence
\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow Q \rightarrow 0, \]
we get an exact sequence
\[ 0 \rightarrow \text{Hom}(Q, \mathcal{O}_{\mathbb{P}^1}(k)) \rightarrow \text{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(k)) \xrightarrow{\rho_{f,k}} \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(-d), \mathcal{O}_{\mathbb{P}^1}(k)). \]
Thus we have
\[ \text{rank} \ \rho_{f,k} = \dim \text{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(k)) - \dim \text{Hom}(Q, \mathcal{O}_{\mathbb{P}^1}(k)) \]
\[ = \dim H^0(V^\vee(k)) - \dim \text{Hom}(F, \mathcal{O}_{\mathbb{P}^1}(k)) \]
\[ = (n + 1)(k + 1) - \dim H^0(F^\vee(k)). \]

We write \( F \) as \( \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(d_i) \) with \( d_i \geq 0 \) and \( \sum_{i=1}^n d_i = d - \deg T \). Then
\[ H^0(F^\vee(k)) = H^0\left( \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(k - d_i) \right) = \bigoplus_{i=1}^n H^0(\mathcal{O}_{\mathbb{P}^1}(k - d_i)). \]
Consequently,
\[ h^0(F^\vee(k)) = \sum_{i=1}^n \max\{k - d_i + 1, 0\}. \]
If \( k \geq d - \deg T - 1 \), then \( k - d_i + 1 \geq 0 \) for all \( i \). In this case
\[ h^0(F^\vee(k)) = \sum_{i=1}^n (k - d_i + 1) = n(k + 1) - (d - \deg T) \]
and it implies
\[ \text{rank } \rho_{f,k} = k + 1 + d - \deg T. \]
This proves the first case of the proposition.
On the other hand, if \(0 \leq k \leq d - \deg T - 1\), then we claim that
\[ h^0(\mathcal{F}^\vee(k)) \leq (n - 1)(k + 1). \]
There are two cases to consider. First, \(k - d + 1 \geq 0\) for all \(i\). In this case,
\[ h^0(\mathcal{F}^\vee(k)) = \sum_{i=1}^{n} (k - d_i + 1) = n(k + 1) - (d - \deg T) \leq (n - 1)(k + 1). \]
Secondly, \(k - d_i + 1 < 0\) for some \(i\), say for \(i = 1\). In this case,
\[ h^0(\mathcal{F}^\vee(k)) = \sum_{i=2}^{n} \max\{k - d_i + 1, 0\} \leq \sum_{i=2}^{n} (k + 1) = (n - 1)(k + 1). \]
Thus, in either case, \(\text{rank } \rho_{f,k} \geq (n + 1)(k + 1) - (n - 1)(k + 1) = 2(k + 1). \)
This completes the proof. \(\square\)

**Corollary 3.3.** Let the notations be as in above.

1. Assume that \(\text{rank } \rho_{f,k} \leq 2k + 1\). Then we have
   \[ \deg T \geq d - k \quad \text{and} \quad \text{rank } \rho_{f,k} = k + 1 + d - \deg T. \]
   Further, for all \(m \geq k - 1\),
   \[ \text{rank } \rho_{f,m} - \text{rank } \rho_{f,k} = m - k; \]
2. For any fixed \(l \geq 0\),
   \[ \deg T = d - k \text{ if and only if } \text{rank } \rho_{f,k+l} = 2k + 1 + l; \]
3. For any fixed \(l \geq 0\),
   \[ \deg T \geq d - k \text{ if and only if } \text{rank } \rho_{f,k+l} \leq 2k + 1 + l. \]

**Proof.** (1). By Proposition 3.2, we must have \(k > d - \deg T - 1\), that is \(\deg T \geq d - k\). In this case, \(\text{rank } \rho_{f,k} = k + 1 + d - \deg T\). Further, when \(m \geq k - 1\), because \(k + 1 + d - \deg T = \text{rank } \rho_{f,k} \leq 2k + 1\), we must also have \(m \geq d - \deg T - 1\). Hence by Proposition 3.2 again,
\[ \rho_{f,m} - \rho_{f,k} = (m + 1 + d - \deg T) - (k + 1 + d - \deg T) = m - k. \]
(2). \(\deg T = d - k\) if and only if (by Proposition 3.2) \(\text{rank } \rho_{f,k} = 2k + 1\) if and only if (by (1)) \(\text{rank } \rho_{f,k+l} = 2k + 1 + l\).
(3). \(\deg T \geq d - k\) if and only if (by Proposition 3.2) \(\text{rank } \rho_{f,k} \leq 2k + 1\) if and only if (by (1)) \(\text{rank } \rho_{f,k+l} \leq 2k + 1 + l. \) \(\square\)
3.2. Resultant homomorphism and degree of common factor.

3.4. In this subsection, we reinterpret the results of §3.1 in terms of “common factors of polynomials”. For this, we use the following identification

$$\text{Hom}(\mathcal{O}_{p_1}(-d), V) = V \otimes \text{Hom}(\mathcal{O}_{p_1}(-d), \mathcal{O}_{p_1}) = V \otimes H^0(\mathcal{O}_{p_1}(d)).$$

Then we let \{x, y\} be a basis for $H^0(\mathcal{O}_{p_1}(1))$ and let \{e_0, \ldots, e_n\} be a basis for $V$. This way, any $f \in \text{Hom}(\mathcal{O}_{p_1}(-d), V)$ can be written as

$$f = e_0 \otimes f_0 + \cdots + e_n \otimes f_n$$

where $f_i = \sum_{j=0}^{d} a_{ij} x^{d-j} y^j$ are homogeneous polynomials in $x, y$ of degree $d$. Recall that under this view of points of the space $\mathcal{N}_d$, $\deg T$ is simply the degree of the greatest common factors of $f_0, \ldots, f_n$.

3.5. Consider the $k$th resultant homomorphism (3.1)

$$\rho_{f,k} : \text{Hom}(V, \mathcal{O}_{p_1}(k)) \rightarrow \text{Hom}(\mathcal{O}_{p_1}(-d), \mathcal{O}_{p_1}(k)).$$

We abbreviate $\rho_{f,k}$ as $\rho_{f,k} : W_k \rightarrow V_{d+k}$ where

$$W_k := \text{Hom}(V, \mathcal{O}_{p_1}(k)) = V^\vee \otimes H^0(\mathcal{O}_{p_1}(k))$$

$$V_{d+k} := \text{Hom}(\mathcal{O}_{p_1}(-d), \mathcal{O}_{p_1}(k)) = H^0(\mathcal{O}_{p_1}(d+k)).$$

Using the basis \{ $e_i^\vee \otimes x^k y^{k-j}$ : $i = 0, \ldots, n$; $j = 0, \ldots, k$ \} for $W_k$ and the basis \{ $x^{d+k-j} y^j$ : $j = 0, \ldots, d+k$ \} for $V_{d+k}$, the linear map $\rho_{f,k}$ is represented by the following $(k+1)(n+1) \times (d+k+1)$ matrix

$$A_{f,k} = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0d} & \vdots & \vdots & \cdots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nd} \\ a_{00} & a_{01} & \cdots & a_{0d} & \vdots & \vdots & \cdots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nd} \\ \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{00} & a_{01} & \cdots & a_{0d} \\ a_{n0} & a_{n1} & \cdots & a_{nd} \\ \vdots & \vdots & \cdots & \vdots \\ a_{00} & a_{01} & \cdots & a_{0d} & \vdots & \vdots & \cdots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nd} \end{pmatrix}$$

which acts on elements of $W_k$ by multiplication from the right.

Corollary 3.3 takes the following form in this setting.

**Corollary 3.6.** Let $g$ be a greatest common factor of $f_0, \cdots, f_n$.

1. Assume that $\text{rank} A_{f,k} \leq 2k + 1$. Then we have

$$\deg g \geq d - k \quad \text{and} \quad \text{rank} A_{f,k} = k + 1 + d - \deg g.$$  

Further, for all $m \geq k - 1$,

$$\text{rank} A_{f,m} - \text{rank} A_{f,k} = m - k;$$
(2) For any fixed \( l \geq 0 \),
\[
\text{deg } g = d - k \text{ if and only if } \text{rank } A_{f,k+l} = 2k + 1 + l;
\]
(3) For any fixed \( l \geq 0 \),
\[
\text{deg } g \geq d - k \text{ if and only if } \text{rank } A_{f,k+l} \leq 2k + 1 + l.
\]
We remark here that Proposition 3.2 implies Kakie’s Proposition 3, [10].

4. Determinantal Subschemes

4.1. The universal resultant homomorphisms.

4.1. Let \( \pi : \mathbb{P}^1 \times N_d \to N_d \) denote the second projection. For each integer \( m \geq 0 \), by applying the functor \( \pi_* \text{Hom}(\cdot, \mathcal{O}_{\mathbb{P}^1}(m)) \) to the homomorphism \( \mathcal{O}_{\mathbb{P}^1}(-d) \otimes \mathcal{O}_{N_d}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^1 \times N_d} \) (which comes from the universal family (2.2)), we obtain a nowhere zero \( \mathcal{O}_{N_d} \)-homomorphism
\[
\rho_{d,m} : W_m \to V_{d+m} \otimes \mathcal{O}_{N_d}(1).
\]
It is a nowhere zero section of \( \text{Hom}(W_m, V_{d+m}) \otimes \mathcal{O}_{N_d}(1) \) whose restriction to every point \([f] \in N_d\) is \( \rho_{f,m} \). We call \( \rho_{d,m} \) the \( m \)-th universal resultant homomorphism.

4.2. For \( m, k \geq 0 \), the exterior power \( \bigwedge^{k+2+m} \rho_{d,m} \) is a section of
\[
\text{Hom}
\left(
\bigwedge^{k+2+m} W_m, \bigwedge^{k+2+m} V_{d+m} \otimes \mathcal{O}_{N_d}(k + 2 + m)
\right).
\]
Fix any \( 1 \leq k \leq d-1 \). Then using Corollary 3.3 one checks that the scheme \( Z_{d,k;m} \) of zeros of \( \bigwedge^{k+2+m} \rho_{d,m} \) is supported on \( C_{d,k} \) whenever \( m \geq k \). The ideal sheaf \( I_{d,k;m} \) of \( Z_{d,k;m} \) is the image of the induced homomorphism
\[
\text{Hom}(\bigwedge^{k+2+m} W_m, \bigwedge^{k+2+m} V_{d+m})^\vee \otimes \mathcal{O}_{N_d}(-k - 2 - m) \to I_{d,k;m} \subset \mathcal{O}_{N_d}.
\]

4.3. We suspect that for any fixed \( 1 \leq k \leq d-1 \),
\[
I_{d,k;m} = I_{d,k;k}
\]
holds whenever \( m \geq k \). This would imply that \( I_{d,k;m} \) with \( m \geq k \) all endow the same scheme structure on \( C_{d,k} \). Rather than proving this, we will show the weaker Proposition 4.5 below, which already suffice for our purpose. To pave the way for its proof, we need some preparation.

4.4. Let \( R \) be a ring and \( A \) a \( p \times q \) matrix over \( R \). We let \( I_l(A) \) be the ideal generated by all \( l \times l \) minors of \( A \) with \( 1 \leq l \leq p, q \). Suppose \( B \) is an invertible \( p \times p \) matrix and \( C \) is an invertible \( q \times q \) matrix. Then one checks directly that
\[
I_l(A) = I_l(BA) = I_l(AC)
\]
for all \( 1 \leq l \leq p, q \). In more concrete terms, (4.2) means that the following three operations on the matrix \( A \) preserve the ideal \( I_l(A) \):

(1) multiply a row or a column by units;
(2) interchanging two rows or two columns;
(3) multiply one row (column) by an element of $R$ and add the result to another row (column).

**Proposition 4.5.** Fix $1 \leq k \leq d - 1$. Then

1. $I_{d,0;m} = I_{d,0;0}$ for all $m \geq 0$;
2. $I_{d,k;m} = I_{d,k;d-1}$ for all $m \geq d - 1$.

**Proof.** Consider any nonzero $f \in \text{Hom}(\mathcal{O}_{\mathbb{P}^d}(-d), V)$. Recall that using the bases as chosen in [3.5], we express it as $f = e_0 \otimes f_0 + \cdots + e_n \otimes f_n$ where $f_i = \sum_{j=0}^d a_{ij} x^{d-j} y^j$. This way, the coefficients $(a_{ij})$ become the homogeneous coordinates of $N_d$. Observe in addition that for any fixed $m \geq 0$, $(a_{ij})$ are also the entries in the first block of the matrix $A_{f,m}$ of the $m$-version of [3.2].

We regard $A_{f,m}$ as a matrix over the polynomial ring $k[a_{ij}]$. Observe that the ideal sheaf $I_{d,k;m}$ coincides with the sheaf $(I_{k+2+m}(A_{f,m}))^\sim$ associated to the module $I_{k+2+m}(A_{f,m})$. Our strategy of the proof is to cover $N_d$ by the standard affine open subsets and prove the statements over the open subsets.

We first localize to the affine open set $U_0 = (a_{00} \neq 0)$. By using the affine coordinates $b_{ij} = a_{ij}/a_{00}$, the matrix $A_{f,m}$ is reduced to $B_{f,m}$ such that all the entries $a_{ij}$ are replaced by $b_{ij}$ except that $a_{00}$ is replaced by 1. Then the localization of each ideal $I_{k+2+m}(A_{f,m})$ to $U_0$ is $I_{k+2+m}(B_{f,m})$. Now using $b_{00} = 1$, we can eliminate the entries $b_{i0}$ by some appropriate row operations and reduce the matrix $B_{f,m}$ to

$$C_{f,m} = \begin{pmatrix}
1 & c_{01} & \cdots & c_{0d} \\
0 & c_{11} & \cdots & c_{1d} \\
\vdots & \vdots & \ddots & \vdots \\
0 & c_{n1} & \cdots & c_{nd} \\
1 & c_{01} & \cdots & c_{0d} \\
0 & c_{11} & \cdots & c_{1d} \\
\vdots & \vdots & \ddots & \vdots \\
0 & c_{n1} & \cdots & c_{nd} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
1 & c_{01} & \cdots & c_{0d} \\
0 & c_{11} & \cdots & c_{1d} \\
\vdots & \vdots & \ddots & \vdots \\
0 & c_{n1} & \cdots & c_{nd}
\end{pmatrix}$$

where $c_{0i} = b_{0i}, 1 \leq i \leq d$ and $c_{ij} = b_{ij} - b_{i0} b_{0j}, 1 \leq i \leq n, 1 \leq j \leq d$.

We are now ready to prove (1) over the open subset $U_0$. One sees by direct calculations that

$$I_2(C_{f,0}) = \langle c_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq d \rangle.$$
For the ideal $I_{2+m}(C_{f,m})$ with $m \geq 1$, it is trivial that
$$I_{2+m}(C_{f,m}) \subset I_2(C_{f,0}).$$

On the other hand, observe that $C_{f,m}$ has a $(n + 1 + m) \times (d + m + 1)$ submatrix of the form
$$\begin{pmatrix}
1 & c_{01} & \cdots & c_{0d} \\
1 & c_{11} & \cdots & c_{1d} \\
\vdots & \vdots & \ddots & \vdots \\
1 & c_{n1} & \cdots & c_{nd}
\end{pmatrix}.$$ 

Using the $(m + 1)$ 1’s on the diagonal, one easily finds $(m + 2) \times (m + 2)$ minors such that their determinants are $c_{ij}$ for all $1 \leq i \leq n$ and $1 \leq j \leq d$. This implies that $I_2(C_{f,0}) \subset I_{2+m}(C_{f,m})$. Thus,
$$I_{2+m}(C_m) = I_2(C_0)$$
for all $m \geq 1$. This proves (1) over the open subset $U_0$.

We now turn to the statement (2) over $U_0$. Consider the matrix $C_{f,m}$ with $m \geq d$. It is routine to check that the following holds:

$$(4.3) \quad \text{row}_i + \sum_{j=1}^{d} (c_{ij}\text{row}_{j(n+1)+i-1} + c_{0j}\text{row}_{j(n+1)+i}) = 0$$

for all $2 \leq i \leq n + 1$. This means that we can eliminate row, for all $2 \leq i \leq n + 1$. We can also easily eliminate the entries $c_{0i}$ of the first row by using the first column. This implies that $I_{k+2+m}(C_{f,m}) = I_{k+1+m}(C_{f,m-1})$. This process can be repeated until we reach $I_{k+1+d}(C_{f,d-1})$. Thus we obtain
$$I_{k+2+m}(C_{f,m}) = I_{k+1+d}(C_{f,d-1})$$
for all $m \geq d - 1$. This completes the proof of (1) and (2) over the open subset $U_0$.

To investigate (1) and (2) over the rest of open charts of $N_d$, we use the symmetry of $N_d$. The group
$$\text{GL}(H^0(\mathcal{O}_{\mathbb{P}^1}(1))) \times \text{GL}(V) \cong \text{GL}_2 \times \text{GL}_{n+1}$$
acts on $N_d$. If $g \in \text{GL}(H^0(\mathcal{O}_{\mathbb{P}^1}(1)))$, it corresponds to a change of basis of $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$; since it induces bases changes in both $W_m$ and $V_{d+m}$, we see that $g$ acts on the matrix $A_{f,m}$ by multiplying invertible matrices from both the left and the right. Likewise, an element $g \in \text{GL}(V)$ acts on $A_{f,m}$ by multiplying an invertible matrix from the left. By $4.3$ $g^*(I_i(A_{f,m})) = I_i(A_{f,m})$ for any $g \in \text{GL}(H^0(\mathcal{O}_{\mathbb{P}^1}(1))) \times \text{GL}(V)$.

Now, for $[f] \in N_d \setminus U_0$, we have $a_{00} = 0$. If one of $a_{0i}$ is not zero, say $a_{0j} \neq 0$. It is routine to find a basis change of $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ such that under the new basis $a'_{00} \neq 0$. This means that there is $g \in \text{GL}(H^0(\mathcal{O}_{\mathbb{P}^1}(1)))$ such
that $g \cdot [f] \in U_0$. If all of $a_{qi}$ are zero, then there is $i \geq 1$ and $j$ such that $a_{ij} \neq 0$. Then let $g \in \text{GL}(V)$ correspond to interchanging $e_0$ and $e_j$, we see that $g \cdot [f]$ places us in the previous situation. In either case, by the invariance of the ideals $g^* (I_1(A_{f,m})) = I_1(A_{f,m})$, we conclude that the statements (1) and (2) hold everywhere in $N_d$.

This completes the proof. \hfill $\square$

4.2. Determinantal subschemes and their basic properties.

4.6. Let $1 \leq k \leq d - 1$. We set

$$I_{d,0} = I_{d,0;m}, \quad m \geq 0 \quad \text{and} \quad I_{d,k} = I_{d,k;m}, \quad m \geq d - 1.$$ 

By Proposition 4.5, these are well-defined.

**Definition 4.7.** For any $0 \leq k \leq d - 1$, we let $Z_{d,k}$ be the subscheme of $N_d$ defined by the ideal $I_{d,k}$.

By $4.2$, $Z_{d,k}$ is supported on $C_{d,k}$.

4.8. Since

$$N_d = \mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}_1}(-d), V \otimes \mathcal{O}_{\mathbb{P}_1})) \quad \text{and} \quad N_k = \mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}_1}(-k), V \otimes \mathcal{O}_{\mathbb{P}_1})),$$

using the identification $V_{d-k} = \text{Hom}(\mathcal{O}_{\mathbb{P}_1}(-d), \mathcal{O}_{\mathbb{P}_1}(-k))$, we obtain a natural morphism

$$\varphi_{d,k} : \mathbb{P}(V_{d-k}) \times N_k \longrightarrow N_d$$

$$( [h], [g]) \mapsto [g \circ h].$$

When $d = 0$, $N_0 = \mathbb{P}(V)$. Also we have the identification $N_d = \mathbb{P}(V_d \otimes V)$.

A direct computation shows that

**Proposition 4.9.** The morphism $\varphi_{d,0} : \mathbb{P}(V_d) \times \mathbb{P}(V) \longrightarrow N_d = \mathbb{P}(V_d \otimes V)$ is the Segre embedding and its image scheme is exactly $Z_{d,0}$.

In particular, this implies that $Z_{d,0}$ is smooth. For general $\varphi_{d,k}$, we have

**Proposition 4.10.** The restriction of $\varphi_{d,k}$ to $\mathbb{P}(V_{d-k}) \times (N_k \setminus Z_{k,k-1})$ gives rise to an isomorphism

$$\varphi'_{d,k} : \mathbb{P}(V_{d-k}) \times (N_k \setminus Z_{k,k-1}) \overset{\cong}{\longrightarrow} Z_{d,k} \setminus Z_{d,k-1}.$$ 

**Proof.** The idea of the proof is taken from the third author’s thesis [17]. We give sufficient sketch here.

We just need to produce the inverse to $\varphi'_{d,k}$.

First, recall that the Quot scheme $N_d$ comes equipped with a universal exact sequence of sheaves over $\mathbb{P}_1 \times N_d$

$$0 \rightarrow \mathcal{E} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}_1 \times N_d} \rightarrow \mathcal{D} \rightarrow 0$$

where $\mathcal{E} = \mathcal{O}_{\mathbb{P}_1}(-d) \otimes \mathcal{O}_{N_d}(-1)$ and $\mathcal{D}$ is a coherent sheaf of rank $n$, relative degree $d$ and flat over $N_d$. Similarly, $\mathbb{P}(V_{d-k})$ comes equipped with a universal exact sequence of sheaves over $\mathbb{P}_1 \times \mathbb{P}(V_{d-k})$

$$0 \rightarrow \mathcal{E}_{\mathbb{P}_1}(-d) \otimes \mathcal{O}_{\mathbb{P}(V_{d-k})}(-1) \rightarrow \mathcal{E}_{\mathbb{P}_1}(-k) \otimes \mathcal{O}_{\mathbb{P}(V_{d-k})} \rightarrow \mathcal{D} \rightarrow 0$$
where $\mathcal{F}$ is a torsion sheaf of relative degree $d - k$ and is flat over $\mathbb{P}(V_{d-k})$.

Next, taking dual of the exact sequence (4.6), we obtain

$$V_{d-k}^\vee \rightarrow \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathcal{O}_{N_d}(1) \rightarrow \mathcal{F} \rightarrow 0$$

where $\mathcal{G} = \mathcal{E}xt^1(\mathcal{Q}, \mathcal{O}_{\mathbb{P}^1 \times N_d})$. Here and below, we use $\mathcal{E}xt^1$ for $\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^1 \times N_d}}$.

Tensoring (4.8) by $\mathcal{O}_{\mathbb{P}^1}(m)$ for $m \gg 0$ and applying $\pi_*$ where $\pi : \mathbb{P}^1 \times N_d \rightarrow N_d$ is the projection map, we then obtain

$$\pi_* (V_{d-k}^\vee \otimes \mathcal{O}(d)) \rightarrow \pi_* (\mathcal{O}_{\mathbb{P}^1}(d + m) \otimes \mathcal{O}_{N_d}(1)) \rightarrow \pi_* (\mathcal{G}(m)) \rightarrow 0$$

where the first map is simply

$$\rho_{d,m} : W_m = \pi_* (V_{d-k}^\vee \otimes \mathcal{O}(d)) \rightarrow \pi_* (\mathcal{O}_{\mathbb{P}^1}(d + m) \otimes \mathcal{O}_{N_d}(1)) = V_{d+m} \otimes \mathcal{O}_{N_d}(1).$$

Since $Z_{d,k}$ is the scheme of zeros of $\bigwedge^{k+2+m} P_1$, we see that $\pi_* \mathcal{G}(m)$ pulls back to a locally free sheaf of rank $d - k$ over $Z_{d,k} \setminus Z_{d,k-1}$ for all $0 \leq k \leq d-1$. Set $Z_{d,-1} := \emptyset$. Then one checks directly that the disjoint union

$$\bigcup_{k=0}^{d-1} (Z_{d,k} \setminus Z_{d,k-1}) \bigcup (N_d \setminus Z_{d,d-1})$$

is exactly the flattening stratification of $\mathcal{G}$ (cf. Lecture 8, [15]).

Now, we let

$$\iota : \mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1}) \rightarrow \mathbb{P}^1 \times N_d$$

be the inclusion. Then, the torsion sheaf $\iota^* \mathcal{G}$ has relative degree $d - k$ and is flat over $Z_{d,k} \setminus Z_{d,k-1}$. Thus, by the universality of $\mathbb{P}(V_{d-k})$, we obtain a morphism

$$Z_{d,k} \setminus Z_{d,k-1} \rightarrow \mathbb{P}(V_{d-k}).$$

What remains is to get a morphism from $Z_{d,k} \setminus Z_{d,k-1}$ to $N_k \setminus Z_{k,k-1}$. For this, we pull back (4.6) to $\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})$. Since $\mathcal{Q}$ is flat over $N_d$, we get an exact sequence

$$0 \rightarrow \iota^* \mathcal{E} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})} \rightarrow \iota^* \mathcal{Q} \rightarrow 0.$$

Taking the dual to the above sequence, we obtain

$$0 \rightarrow (\iota^* \mathcal{Q})^\vee \rightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})} \rightarrow (\iota^* \mathcal{E})^\vee \rightarrow \mathcal{E}xt^1_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})}(\iota^* \mathcal{Q}, \mathcal{E}) \rightarrow 0,$$

where $\mathcal{E}xt^1_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})}(\iota^* \mathcal{Q}, \mathcal{E})$ is a torsion sheaf of relative degree $d - k$ and is flat over $\mathbb{P}(V_{d-k})$.

To see this, we first dualize the universal exact sequence (4.6) to obtain

$$0 \rightarrow \mathcal{Q}^\vee \rightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}^1 \times N_d} \rightarrow \mathcal{E}xt^1_{\mathbb{P}^1 \times N_d}(\mathcal{Q}, \mathcal{E}) \rightarrow 0.$$

Then, we pull it back to $\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})$ to obtain

$$V^\vee \otimes \mathcal{O}_{\mathbb{P}^1 \times (Z_{d,k} \setminus Z_{d,k-1})} \rightarrow \iota^* (\mathcal{E}^\vee) \rightarrow \iota^* \mathcal{E}xt^1_{\mathbb{P}^1 \times N_d}(\mathcal{Q}, \mathcal{E}) \rightarrow 0.$$
Since pulling-back and dualizing operations commute on locally free sheaves, we have a canonical identification \((ι^*E)∨ = ι^*(E∨)\) and hence a commutative diagram
\[
\begin{array}{c}
V^∨ \otimes O_{P^1 \times (Z_{d,k} \setminus Z_{d,k-1})} \\
\downarrow \\
V^∨ \otimes O_{P^1 \times (Z_{d,k} \setminus Z_{d,k-1})}
\end{array} \xrightarrow{\cdot} \begin{array}{c}
ι^*(E^∨) \\
\downarrow \\
ι^*(E^∨)
\end{array} \xrightarrow{\cdot} \begin{array}{c}
E^1(ι^*Q, O) \\
\downarrow \\
E^1(ι^*Q, O)
\end{array} \rightarrow 0.
\]
This gives the canonical identification in the third column, as desired.

We now break up the sequence (4.11) into two:
\[
0 \rightarrow (ι^*Q)^∨ \rightarrow V^∨ \otimes O_{P^1 \times (Z_{d,k} \setminus Z_{d,k-1})} \rightarrow \mathcal{K} \rightarrow 0
\]
and
\[
0 \rightarrow \mathcal{K} \rightarrow ι^*E^∨ \rightarrow ι^*G \rightarrow 0.
\]
Since \(ι^*G\) is flat over \(Z_{d,k} \setminus Z_{d,k-1}\), one checks that \(\mathcal{K}\) is locally free, hence \((ι^*Q)^∨\) is locally free. Now taking the dual of (4.12), we get an exact sequence of locally free sheaves
\[
0 \rightarrow \mathcal{K}^∨ \rightarrow V^\otimes O_{P^1 \times (Z_{d,k} \setminus Z_{d,k-1})} \rightarrow (ι^*Q)^∨ \rightarrow 0.
\]
One easily calculates that
\[
\text{rank}(ι^*Q)^∨ = n \quad \text{and} \quad \text{deg}(ι^*Q)^∨ = k.
\]
Thus, by the universality of the Quot scheme \(N_k\), we obtain a morphism
\[
Z_{d,k} \setminus Z_{d,k-1} \rightarrow N_k \setminus Z_{k,k-1}.
\]
Together with (4.10), this gives rise to a morphism
\[
ψ_{d,k} : Z_{d,k} \setminus Z_{d,k-1} \rightarrow \mathbb{P}(V_{d-k}) \times (N_k \setminus Z_{k,k-1})
\]
which is the inverse to \(ϕ'_{d,k}\). This completes the proof. \(\square\)

As a consequence of this proposition, we see that \(Z_{d,k} \setminus Z_{d,k-1}\) is a locally closed smooth subvariety of \(N_d\) for all \(k \geq 1\).

5. Statements of Main Results and Their Proofs

5.1. Describing the successive blowups.

We use induction to describe the iterated blowups of \(N_d\) along the \(Z_{d,0}, \ldots, Z_{d,d-1}\). We set \(N_d^{-1} := N_d\) and \(Z_d^{-1} := Z_{d,k}\). For any \(0 \leq l \leq d-1\), we let \(N_d^l\) be the blowup of \(N_d^{l-1}\) along \(Z_d^{l-1}\), \(Z_d^k\) the proper transform of \(Z_d^{l-1}\) when \(k ≠ l\), and \(Z_d^l\) the exceptional divisor of \(N_d^l \rightarrow N_d^{l-1}\). Observe that \(Z_d^l\) is a divisor when \(k ≤ l\) and \(Z_d^k\) is the blowup of \(Z_d^{l-1}\) along \(Z_d^l\) when \(k > l\).
5.2. We denote the final blowup $N_d^{l+1}$ by $M_d$. We aim to show that $M_d$ is smooth and the boundary $M_d \setminus N_d$ is a divisor with normal crossings. The smoothness of $M_d$ will follow by induction if the blowup center $Z^{l+1}_{d,l+1} \subset N^{l}_{d}$ is smooth. A technical key to prove this is to show that the total transform in $N^{l}_{d}$ of $Z^{l-1}_{d,l+1} \subset N^{l-1}_{d}$ is the scheme-theoretical union of a Cartier divisor with the proper transform $Z^{l}_{d,l+1} \subset N^{l}_{d}$, that is, locally we have

$$I_{Z^{l}_{d,l+1}} \cdot \mathcal{O}_{N^{l}_{d}} = P \cdot I_{Z^{l-1}_{d,l+1}}$$

where $P$ is a principal ideal. Here by $I_{Z}$, we mean the ideal sheaf of a subscheme $Z$. To this end, we will relate our blowups to spaces of complete collineations and apply the related results of Vainsencher [21] and [20].

5.2. Using the space of complete collineations.

5.3. Let $E$ and $F$ be vector spaces. Let $S(E, F) = \mathbb{P}(\text{Hom}(E, F))$ be the space of collineations from $E$ to $F$. It comes equipped with a universal homomorphism

$$u_{EF} : E \to F \otimes \mathcal{O}_{S(E, F)}(1)$$

For $k \geq 1$, set $D_{k}(E, F)$ to be the scheme of zeros of the section

$$\bigwedge^{k+1} u_{EF} : \text{Hom}(\bigwedge^{k+1} E, \bigwedge^{k+1} F) \otimes \mathcal{O}_{S(E, F)}(k+1).$$

5.4. Let $r + 1 = \min\{\dim E, \dim F\}$. Set $S^{0} := S(E, F)$, $D_{k}^{0} := D_{k}(E, F)$.

We define the following inductively. For $1 \leq l \leq r$, let $S^{l}$ be the blow up of $S^{l-1}$ along $D^{l-1}_{l}$, $D^{l}_{k}$ the proper transform of $D^{l}_{k}$ for $k \neq l$, and $D^{l}_{l}$ the exceptional divisor of $S^{l} \to S^{l-1}$.

5.5. Although $D_{k}(E, F)$ is singular for $k \geq 2$, Vainsencher [21] shows that $D^{l-1}_{l}$ is smooth, thus $S^{l}$ is smooth. In particular, the final blowup space $S^{r}$ is smooth. Further, he shows that $S^{r}$ parameterizes “complete collineations” from $E$ to $F$ (see [21] for more details). The property that we need from Vainsencher’s construction is the following

**Proposition 5.6.** (Theorem 2.4 (8), [21]) Assume $1 \leq l < k$.

$$I_{D^{l-1}_{l}} \cdot \mathcal{O}_{S^{l}} = I_{D^{l}_{k}} \cdot (I_{D^{l}_{l}})^{k-l+1}$$

5.7. The relations between our blowups as described in §5.1 and the spaces of “complete collineations” are as follows. Note that for any $m \geq 0$, the resultant homomorphism of [5.1] gives rise to an embedding

$$\rho_{f,m} : W_{m} \to V_{d+m}$$

Indeed, $N_{d} = S(W_{0}, V_{d})$. Further, $\rho_{d,m}$ is exactly the pullback to $N_{d}$ of the universal map $u = u_{W_{m}, V_{d+m}}$ on $S(W_{m}, V_{d+m})$. Consequently, $\bigwedge^{l} u$ pulls back
to $\bigwedge^l \rho_{d,m}$ for all $l$. Then it follows by definition that for all $0 \leq k \leq d - 1$ and $m \geq d - 1$

$$N_d \cap D_{k+1+m}(W_m, V_{d+m}) = Z_{d,k}$$

scheme-theoretically. Also it is easy to check that

$$\operatorname{rank} \rho_{f,m} \geq m + 1$$

for all $m \geq 0$ and $[f] \in N_d$. Therefore,

\begin{equation}
N_d \cap D_1(W_m, V_{d+m}) = \cdots = N_d \cap D_m(W_m, V_{d+m}) = \emptyset.
\end{equation}

Consequently, we have

**Proposition 5.8.** Fix any $m \geq d - 1$. $N^l_d$ is the proper transform of $N_d$ in $S^{m+1+l}(W_m, V_{d+m})$. In particular, $M_d$ is the proper transform of $N_d$ in $S^{d+m}(W_m, V_{d+m})$.

Further, we have

**Lemma 5.9.** Assume $1 \leq l < k$. Then

$$I_{Z_{d,k}} \cdot \mathcal{O}_{N^l_d} = I_{Z_{d,k}} \cdot (I_{Z^l_{d,k}})^{k-l+1}.$$ 

**Proof.** In the proof, we fix $m = d - 1$ and use the embedding

$$N_d \hookrightarrow S(W_{d-1}, V_{2d-1}).$$

We will use the notations introduced in 5.4 with $E = W_{d-1}$ and $F = V_{2d-1}$.

By (5.1), we have

$$N_d \cap D_1 = \cdots = N_d \cap D_{d-1} = \emptyset.$$

Thus, we have the induced embedding

$$N_d \longrightarrow S^{d-1},$$

and moreover, for $0 \leq k \leq d - 1$, we have

$$N_d \cap D^{d-1}_{d+k} = Z_{d,k}.$$ 

In other words,

$$I_{D^{d-1}_{d+k}} \cdot \mathcal{O}_{N_d} = I_{Z_{d,k}}.$$ 

Thus, we have the following blowing-up diagram

$$
\begin{array}{ccc}
N^l_d & \longrightarrow & S^{d+l} \\
\downarrow & & \downarrow \\
N^{l-1}_d & \longrightarrow & S^{d+l-1}
\end{array}
$$

for all $0 \leq l \leq d - 1$. Further, we have

$$N^l_d \cap D^{d+l}_{d+k} = Z^l_{d,k}, \quad \text{for } l \leq k.$$ 

Thus for $l < k$,

$$I_{Z^{l-1}_{d,k}} \cdot \mathcal{O}_{N^l_d} = (I_{D^{d+l}_{d+k}} \cdot \mathcal{O}_{N^{l-1}_d}) \cdot \mathcal{O}_{N^l_d} = (I_{D^{d+l}_{d+k}} \cdot \mathcal{O}_{S^{d+l}}) \cdot \mathcal{O}_{N^l_d}.$$
By Proposition 5.6, we have \( I_{D_{d+k}^{d+t-1}} \cdot \mathcal{O}_{S^{d+t}} = I_{D_{d+k}^{d+t}} \cdot (I_{D_{d+k}^{d+t}})^{k-1+t} \). It then follows that
\[
I_{Z_{d,k}^{d-1}} \cdot \mathcal{O}_{N_{d}^{d}} = I_{D_{d+k}^{d+t}} \cdot (I_{D_{d+k}^{d+t}})^{k-1+t} \cdot \mathcal{O}_{N_{d}^{d}} = I_{Z_{d,k}^{d+1}} \cdot (I_{Z_{d,k}^{d+1}})^{k-1+t}.
\]

Applying the above lemma repeatedly, we obtain

**Corollary 5.10.** For all \( 0 \leq l \leq d - 1 \),
\[
I_{Z_{d,l+1}} \cdot \mathcal{O}_{N_{d}^{d+1}} = (I_{Z_{d,l+1}})^{l+1} \cdot (I_{Z_{d,l+1}})^{l+1} \cdot (I_{Z_{d,l+1}})^{l+2}.
\]

We remark here that \( Z_{d,t}^{l} \) are Cartier divisors when \( t \leq l \). This implies that the blowup of \( N_{d}^{l} \) along the proper transform of \( Z_{d,l+1}^{l} \) is the same as the blowup of \( N_{d}^{l} \) along the total transform of \( Z_{d,l+1} \subset N_{d} \).

5.3. **Main theorems and proofs.**

**Theorem 5.11.** Let \( -1 \leq k \leq d - 1 \). Then

1. \( N_{d}^{k} \) is nonsingular;
2. \( N_{d}^{k} \) is isomorphic to the closure of the graph of the rational map
\[
N_{d} \to \prod_{l=0}^{k} S( \bigwedge_{m=0}^{m+2+l} W_{m}, \bigwedge_{m=0}^{m+2+l} V_{d+m}) \quad [f] \mapsto ([\bigwedge_{m=0}^{m+2+k} \rho_{f,m}], \bigwedge_{m=0}^{m+3} \rho_{f,m}, \ldots, [\bigwedge_{m=0}^{m+2+k} \rho_{f,m}])
\]

for all \( m \geq d - 1 \);
3. \( Z_{d,k+1}^{k} \) is isomorphic to \( \mathbb{P}(V_{d-k-1}) \times N_{k+1}^{k} \).

**Proof.** We prove it by induction on \( k \). When \( k = -1 \), the statements of the theorem are trivial (for all \( d > 0 \) and \( m \geq d - 1 \)). Assume the statements are true for all \( \leq k - 1 \) (for all \( d > 0 \) and \( m \geq d - 1 \)). We now prove the \( k \)-version of the theorem.

First, by the inductive assumption, \( N_{d}^{k-1} \) is nonsingular; also, \( Z_{d,k}^{k-1} \) is smooth because it is isomorphic to \( \mathbb{P}(V_{d-k-1}) \times N_{k-1}^{k-1} \). Hence, \( N_{d}^{k} \), as the blowup of \( N_{d}^{k-1} \) along \( Z_{d,k}^{k-1} \), is nonsingular. So (1) holds true for \( k \).

Next, we let
\[
\pi_{[k-1]} : N_{d}^{k-1} \to N_{d}
\]
be the iterated blowing-up morphism. By Corollary 5.10, the blowup of \( N_{d}^{k-1} \) along the proper transform \( Z_{d,k}^{k-1} \) is isomorphic to the blowup of \( N_{d}^{k-1} \) along the total transform \( \pi_{[k-1]}^{-1}(Z_{d,k}) \). Hence, by the definition of \( Z_{d,k}^{k} \) (see Definition 4.7 and Proposition 4.3), \( N_{d}^{k} \) is isomorphic to the closure of the graph of the rational map
\[
N_{d}^{k-1} \to S( \bigwedge_{m=0}^{m+2+k} W_{m}, \bigwedge_{m=0}^{m+2+k} V_{d+m}).
\]
By the induction hypothesis, $N_{d}^{k-1}$ is isomorphic to the closure of the graph of the rational map

$$N_{d} \to \prod_{l=0}^{k-1} S(\bigwedge W_{m}, \bigwedge V_{d+m}).$$

It follows that $N_{d}^{k}$ is isomorphic to the closure of the graph of the rational map

$$N_{d} \to \prod_{l=0}^{k} S(\bigwedge W_{m}, \bigwedge V_{d+m}).$$

Thus (2) also holds true for $k$.

Finally, to prove the $k$-version of (3), we introduce and establish the following commutative diagram

\[
\begin{array}{c}
Z_{d,k+1} \setminus Z_{d,k} \xrightarrow{\alpha} \mathbb{P}(V_{d-k-1}) \times N_{k+1}^{k-1} \times S(\bigwedge W_{m}, \bigwedge V_{k+m+1}) \\
\bigg| \\
Z_{d,k+1} \setminus Z_{d,k} \xrightarrow{\gamma} N_{d}^{k-1} \times S(\bigwedge W_{m}, \bigwedge V_{d+m}).
\end{array}
\]

Here $\gamma$ is the obvious embedding. The morphism $\alpha$ is the obvious embedding induced by the inverse of the morphism $\varphi'_{d,k+1}$ of Proposition 4.10. The morphism $\beta$ is defined as follows.

For any integers $r, s \geq 1$, using the following identifications

$$V_{r} = \text{Hom}(O_{P}^{1}(-r), O_{P}^{1}),$$

$$V_{s} = \text{Hom}(O_{P}^{1}, O_{P}^{1}(s)),$$

$$V_{r+s} = \text{Hom}(O_{P}^{1}(-r), O_{P}^{1}(s)),$$

we see that each nonzero $h \in V_{r}$ gives rise to an injective linear map

$$L_{h} : V_{s} \to V_{r+s},$$

$$g \mapsto g \circ h.$$

For any $1 \leq l \leq s + 1$, it induces an injective linear map

$$\bigwedge L_{h} : \bigwedge V_{s} \to \bigwedge V_{r+s},$$

which in turn induces a morphism

$$\mathbb{P}(V_{r}) \to S(\bigwedge V_{s}, \bigwedge V_{r+s}).$$

Now fix any $t \geq 1$. Then, by the means of composing with $\bigwedge L_{h}$, we obtain a morphism

$$S(\bigwedge W_{t}, \bigwedge V_{s}) \to S(\bigwedge W_{t}, \bigwedge V_{r+s}).$$
Since $\bigwedge^l L_h$ is injective, the above morphism is an embedding (in fact, a linear embedding).

Observe now that when $l = s + 1$, $\dim \bigwedge^l V_s = 1$. Hence

$$S(\bigwedge V_s, \bigwedge V_{r+s}) \cong \mathbb{P}(\bigwedge V_{r+s}).$$

Then one checks directly that the morphism

$$\mathbb{P}(V_r) \to S(\bigwedge V_s, \bigwedge V_{r+s})$$

is the Veronese embedding; further, the morphism

$$S(\bigwedge W_t, \bigwedge V_s) \times S(\bigwedge V_s, \bigwedge V_{r+s}) \to S(\bigwedge W_t, \bigwedge V_{r+s})$$

is the Segre embedding.

We are now ready to define $\beta$. It consists of two components ($\beta_1, \beta_2$). They are

$$\beta_1 : \mathbb{P}(V_{d-k-1}) \times N_{k+1}^{k-1} \to N_d^{k-1}
\beta_1([h], [g], \prod_{l=0}^{k-1} \bigwedge \rho_{g,m}) = ([g \circ h], \prod_{l=0}^{k-1} L_h \circ \bigwedge \rho_{g,m})$$

and

$$\beta_2 : \mathbb{P}(V_{d-k-1}) \times S(\bigwedge W_m, \bigwedge V_{k+m+1}) \to S(\bigwedge W_m, \bigwedge V_{d+m})$$

$$\beta_2([h], \bigwedge \rho_{g,m}) = [\bigwedge L_h \circ \bigwedge \rho_{g,m}].$$

Then one checks routinely that $\beta_2$ is the composition of the Veronese embedding

$$\mathbb{P}(V_{d-k+1}) \to S(\bigwedge V_{k+m+1}, \bigwedge V_{d+m})$$

followed by the Segre embedding

$$S(\bigwedge W_m, \bigwedge V_{k+m+1}) \times S(\bigwedge V_{k+m+1}, \bigwedge V_{d+m})$$

$$\to S(\bigwedge W_m, \bigwedge V_{d+m}),$$

hence an embedding itself. Since it is routine, we omit the details. For $\beta_1$, observe that when $[h]$ is fixed, it is injective on other factors. Together, this implies that $\beta$ is an embedding. Thus, we finally established the embedding diagram (5.2).

From (5.2), we see that the closure of $Z_{d,k+1} \setminus Z_{d,k}$ in

$$\mathbb{P}(V_{d-k-1}) \times N_{k+1}^{k-1} \times S(\bigwedge W_m, \bigwedge V_{k+m+1})$$


is contained in $\mathbb{P}(V_{d-k-1}) \times N_{k+1}^k$, and hence equals to $\mathbb{P}(V_{d-k-1}) \times N_{k+1}^k$ because $Z_{d+1} \setminus Z_{d,k}$ is an open subvariety in it. Because the embedding diagram (5.2) commutes, the above-mentioned closure is obviously isomorphic to the closure of $Z_{d,k+1}$ in

$$N_d^{k-1} \times S(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{d+m}),$$

which is by definition $Z_{d,k+1}^k$. This proves (3) for $k$.

By induction, the theorem is proved. $\square$

**Theorem 5.12.** Let $-1 \leq k \leq d - 1$. Then

1. $Z_{d,0}^k \cup \cdots \cup Z_{d,k}^k$ is a divisor of $N_d^k$ with simple normal crossings;
2. The scheme-theoretic intersection $Z_{d,k+1}^{k} \cap \bigcap_{j=1}^r Z_{d,j}^k$ is isomorphic to

$$\mathbb{P}(V_{d-k-1}) \times \bigcap_{j=1}^r Z_{k+1,i_j}^k$$

for any distinct integers $i_1, \ldots, i_r$ between 0 and $k$.

**Proof.** Again, we prove it by induction on $k$. When $k = -1$, both statements are trivial.

Assume that both statements are true for $k - 1$. Then, it implies that $Z_{d,0}^{k-1} \cup \cdots \cup Z_{d,k-1}^{k-1}$ is a divisor of $N_d^{k-1}$ with simple normal crossings. In addition, by statement (2) from the induction hypothesis, the scheme-theoretic intersection of $Z_{d,k}^{k-1}$ with an arbitrary intersection of $Z_{d,i_j}^{k-1}$'s is smooth.

Next, it is routine to check that

$$\text{codim} \left( Z_{d,k}^{k-1} \cap \bigcap_{j=1}^r Z_{d,i_j}^{k-1} \right) = \dim N_d^{k-1} - \dim \left( \mathbb{P}(V_{d-k}) \times \bigcap_{j=1}^r Z_{k,i_j}^{k-1} \right)$$

$$= \text{codim} Z_{d,k}^{k-1} + \sum_{j=1}^r \text{codim} Z_{d,i_j}^{k-1}.$$ 

Since it is routine, we omit further details. Thus, $Z_{d,k}^{k-1}$ meets the divisors $Z_{d,0}^{k-1}, \ldots, Z_{d,k-1}^{k-1}$ transversally. Since transversality is preserved under blowup along nonsingular center, we obtain the $k$-version of (1).

To prove (2), we consider again the commutative diagram in (5.2), which by now can also be expressed as

$$Z_{d,k+1}^k \xrightarrow{\alpha} \mathbb{P}(V_{d-k-1}) \times N_{k+1}^k \times S(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{k+m+1})$$

$$\bigg\|$$

$$Z_{d,k+1}^k \xrightarrow{\gamma} N_d^{k-1} \times S(\bigwedge^{k+m+2} W_m, \bigwedge^{k+m+2} V_{d+m}).$$
Then from the diagram we have that for any $0 \leq i \leq k$,

$$Z_{d,k+1}^k \cap Z_{d,i}^k = \mathbb{P}(V_{d-k-1}) \times Z_{k+1,i}^k$$

scheme-theoretically and further the scheme theoretic intersection

$$Z_{d,k+1}^k \cap \bigcap_{j=1}^r Z_{d,i_j}^k$$

is isomorphic to

$$\mathbb{P}(V_{d-k-1}) \times \bigcap_{j=1}^r Z_{k+1,i_j}^k.$$

Theorems 5.11 and 5.12 imply

Theorem 5.13. Fix any $m \geq d-1$. Then $M_d = N_d^{d-1}$ is a compactification of $\tilde{N}_d$ such that the following hold.

1. $M_d$ is isomorphic to the closure of the graph of the rational map

$$N_d \rightarrow \prod_{l=0}^{d-1} S(\bigwedge W_m, \bigwedge V_{d+m});$$

2. $M_d$ is a nonsingular projective variety;

3. The complement $M_d \setminus \tilde{N}_d = \bigcup_{k=0}^{d-1} Z_{d,k}^{d-1}$ is a divisor with simple normal crossings.

6. The Topology of the Compactification

6.1. In this section, we will work over the field of complex numbers and draw a few consequences on the topology of $M_d$. Recall that for any complex quasi-projective variety $V$ there is a (virtual Hodge) polynomial $e(V)$ in two variables $u$ and $v$ which is uniquely determined by the following properties

1. If $V$ is smooth and projective, then $e(V) = \sum h^{p,q}(-u)^p(-v)^q$;
2. If $U$ is a closed subvariety of $V$, then $e(V) = e(V \setminus U) + e(U)$;
3. If $V \rightarrow B$ is a Zariski locally trivial bundle with fiber $F$, then $e(V) = e(B)e(F)$.

6.2. Fix $n > 0$. For any $i > 0$, set

$$R_i(\lambda) = \frac{\lambda^{i+1} - 1}{\lambda - 1} \cdot \frac{\lambda^{ni} - \lambda}{\lambda - 1}.$$

First, we have the following recursive formula for $e_{M_d}$.

Proposition 6.3. Set $\lambda = uv$. Then

$$e_{M_d} = e_{N_d} + \sum_{k=0}^{d-1} e_{M_k} R_{d-k}$$

where $e_{N_d} = \frac{\lambda^{(d+1)(n+1)} - 1}{\lambda - 1}$. 
Proof. Since $M_d = N_d^{d-1}$ is the blowup of $N_d^{d-2}$ along $Z_{d,d-1}^{d-2}$, we have
\[ e_{M_d} = e_{N_d^{d-2}} + e_{Z_{d,d-1}^{d-2}} (e_{p}^{\operatorname{codim} Z_{d,d-1}} - 1), \]
that is
\[ e_{M_d} = e_{N_d^{d-2}} + e_{Z_{d,d-1}^{d-2}} \frac{\lambda \operatorname{codim} Z_{d,d-1} - \lambda}{\lambda - 1}. \]
Repeat the same arguments for $e_{N_d^{d-2}}$ and so on, we will eventually obtain
\[ e_{M_d} = e_{N_d} + d - 1 \sum_{k=0}^{d-1} e_{Z_{d,k}^{d-1}} \frac{\lambda \operatorname{codim} Z_{d,k} - \lambda}{\lambda - 1}. \]
Because $Z_{d,k}^{d-1} = \mathbb{P}(V_{d-k}) \times M_k$, $\mathbb{P}(V_{d-k}) \cong \mathbb{P}^{d-k}$ and $\operatorname{codim} Z_{d,k} = n(d-k)$, from here it is routine to obtain the formula as stated in the theorem. \qed

6.4. We can also derive a closed formula for $e_{M_d}$. For this consider $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r_{>0}$. Let $|\alpha|$ denote the sum $\sum_{j=1}^r \alpha_j$. Set
\[ R_\alpha = \prod_{j=1}^r R_{\alpha_j} \quad \text{and} \quad R_0 = R_{0,n} = 1. \]

Proposition 6.5. The Hodge polynomial $e_{M_d}(u, v)$ is given by
\[ e_{M_d} = \sum_{0 \leq |\alpha| \leq d} R_\alpha e_{N_d^{d-|\alpha|}} \]
where $e_{N_k} = e_{N_k^{k+1}}^{(\alpha+1) - 1}$ for all $k \geq 0$.

Proof. When $d = 0$, it is trivial. Assume that the formula holds for all $k < d$. By Theorem 6.3
\[ e_{M_d} = e_{N_d} + \sum_{k=0}^{d-1} R_k e_{M_k^{d-k}}. \]
By inductive assumption, for all $k < d$
\[ e_{M_k} = \sum_{0 \leq |\alpha| \leq k} R_\alpha e_{N_k^{k-|\alpha|}} = \sum_{0 \leq |\alpha| \leq k} R_\alpha e_{N_j}. \]
Substitute $e_{M_k}$ into the first formula, we have
\[ e_{M_d} = e_{N_d} + \sum_{k=0}^{d-1} \sum_{0 \leq |\alpha| \leq k} R_\alpha R_{d-k} e_{N_j}. \]
Let $\beta = (\alpha, d - k)$. Then $R_\beta = R_\alpha R_{d-k}$ and $\beta = |\alpha| + d - k = d - j$ if $|\alpha| = k - j$. Then a simple counting argument shows
\[ e_{M_d} = e_{N_d} + \sum_{|\beta| + j = d} R_\beta e_{N_j} = \sum_{0 \leq |\beta| \leq d} R_\beta e_{N_j} = \sum_{0 \leq |\beta| \leq d} R_\beta e_{N_d^{d-|\beta|}}, \]
as desired. □

In addition, we have

**Proposition 6.6.** Let $H$ be the pullback (to $M_d$) of the hyperplane class of $N_d$ and $T_k$ corresponds to the exceptional divisor $Z_{d,k}^{d-1}$ for all $0 \leq k \leq d - 1$. Then

$$A^1(M_d) = \mathbb{Z} \cdot H \oplus \bigoplus_{k=0}^{d-1} \mathbb{Z} \cdot T_k.$$ 

**Proof.** This follows from that fact that $M_d$ is a successive blowup of $N_d$ along nonsingular centers. □

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Department of Mathematics, University of Arizona, Tucson, AZ 85721 USA. E-mail address: yhu@math.arizona.edu

Department of Mathematics, SUNY at Canton, NY 13617, USA. E-mail address: linj@canton.edu

Department of Mathematics, University of Arizona, Tucson, AZ 85721 USA. E-mail address: yshao@math.arizona.edu