Output radiation from a degenerate parametric oscillator

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We study the squeezing as well as the statistical properties of the output radiation from a degenerate parametric oscillator coupled to a squeezed vacuum reservoir employing the stochastic differential equations associated with the normal ordering. It is found that the degree of squeezing of the output radiation is less than the corresponding cavity radiation. However, for output radiation the correlation of the quadrature operators evaluated at different times also exhibits squeezing, which is the reason for quenching of the overall noise in one of the quadrature components of the squeezing spectrum even when the oscillator is coupled to a vacuum reservoir. Moreover, coupling the oscillator to the squeezed vacuum reservoir enhances the squeezing exponentially and it also increases the mean photon number.

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I. INTRODUCTION

Optical degenerate parametric oscillator is one of the most interesting and well studied devices in the nonlinear quantum optics [1, 2, 3, 4, 5, 6, 7]. In a degenerate parametric oscillator, a pump photon of frequency 2ω is down converted by nonlinear crystal into a pair of signals photons each of frequency ω. Due to this inherent two-photon nature of the interaction, the parametric oscillator is found to be a good source of a squeezed light. The maximum degree of squeezing of the cavity radiation coupled to an ordinary vacuum is found to be 50% at the critical point by many authors using various approaches [1, 2, 3]. This limitation in squeezing is associated with the leakage through the mirror and the inevitable amplification of the quantum fluctuations in the cavity due to thermal heating. Since the vacuum field has no definite phase, squeezing of the cavity radiation when coupled to a vacuum reservoir would be degraded. However, if the ordinary vacuum is replaced by a squeezed vacuum reservoir, it is expected that the fluctuations entering the cavity is biased and hence the squeezing of the cavity radiation could be enhanced [2, 3, 4, 5] provided that the reservoir is squeezed in the right quadrature.

Theoretical analysis of the quantum fluctuations and photon statistics of the cavity radiation of a degenerate parametric oscillator coupled to a squeezed vacuum reservoir has been exhaustively made by many authors [1, 2, 3], although it is the output radiation which is practically accessible for an experiment. To our knowledge, there is no a thorough study of the squeezing and statistical properties of the output radiation except for some discussions related with the squeezing spectrum [1, 3, 4]. For instance, Collett and Gardiner [2] claimed, after comparing the variance of the quadrature operators for the cavity radiation with the squeezing spectrum for the output radiation, that the squeezing of the output radiation is much better than the cavity radiation. Since the output squeezing spectrum corresponds to the correlation of the output quadrature operators evaluated at different times, t and t + τ, in the frequency domain, we believe that it would not be right comparing the squeezing spectrum with the variance of the cavity radiation.

We hence devoted this work to the analysis of the output radiation of a degenerate parametric oscillator coupled to a squeezed vacuum reservoir using the input-output relation introduced by Gardiner and Collett [9]. We devise a means of evaluating a correlation between the noise forces arising from the reservoir and vacuum fluctuations in the cavity that required in studying the properties of the radiation outside the cavity. Moreover, we followed an adhoc approach in casting the characteristic function describing the cavity radiation onto the corresponding output variables. On the basis of these approaches we study the squeezing as well as the statistical properties of the output radiation employing the stochastic differential equations associated with the normal ordering. In particular, we calculate the quadrature variance, squeezing spectrum, photon number distribution, and power spectrum for the output radiation. We also compare the results we obtained with the corresponding values for the cavity radiation. We believe that the approach followed in this work could also be used to study the output radiation of other quantum optical systems.

II. STOCHASTIC DIFFERENTIAL EQUATIONS

The degenerate parametric oscillator can be described in the interaction picture and in the rotating-wave approximation, upon treating the pump radiation classically, by the Hamiltonian of the form

\[ \hat{H} = \frac{i\varepsilon}{2}(\hat{a}^{\dagger 2} - \hat{a}^2), \]  

where ε is proportional to the amplitude of the external coherent radiation which is taken to be a real-positive
constant and \( \hat{a} \) is the annihilation operator for the signal mode. We consider the case in which a continuum mode of squeezed vacuum centered at frequency \( \omega \) is allowed to enter the cavity through one of the coupler mirrors. In this case, the master equation describing the degenerate parametric oscillator coupled to a broadband squeezed vacuum reservoir in the interaction picture is found using the standard procedure \[10\] to be

\[
\frac{d\hat{\rho}}{dt} = \frac{\epsilon}{2} \left( \hat{a}^\dagger \hat{\rho} - \hat{\rho} \hat{a} + \hat{\rho} \hat{a} - \hat{a}^\dagger \hat{\rho} \right) + \frac{\kappa(N+1)}{2} \left( 2\hat{a}^\dagger \hat{\rho} \hat{a} - \hat{a}^\dagger \hat{\rho} + \hat{\rho} \hat{a} \right) + \frac{\kappa N}{2} \left( 2\hat{a}^\dagger \hat{\rho} - \hat{a}^\dagger \hat{\rho} + \hat{\rho} \hat{a} \right) + \frac{\kappa M}{2} \left( \hat{a}^2 \hat{\rho} - 2\hat{a} \hat{\rho} - \hat{\rho} \hat{a} + \hat{a}^\dagger \hat{\rho} + \hat{\rho} \hat{a}^\dagger \hat{\rho} \right),
\]

where \( N \) and \( M \) represent the squeezed vacuum reservoir, with \( N = \sinh^2(r), M = \sinh(r) \cosh(r) \), \( \kappa \) is the cavity damping constant, and \( r \) is the squeeze parameter.

We notice that the operators in Eqs. (4), (5), and (6) are in the normal order. Hence the corresponding expressions in the c-number variables associated with the normal ordering take the form

\[
\frac{d}{dt} \langle \hat{a}(t) \rangle = \frac{\epsilon}{2} \langle \hat{a}(t) \rangle + \langle \hat{a}^\dagger(t) \rangle, \quad \frac{d}{dt} \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle = -\kappa \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle + \epsilon \left[ \langle \hat{a}^2(t) \rangle + \langle \hat{a}^\dagger \hat{a}(t) \rangle \right] + \kappa N, \quad \frac{d}{dt} \langle \hat{a}^2(t) \rangle = -\kappa \langle \hat{a}^2(t) \rangle + 2\epsilon \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle + \epsilon + \kappa M.
\]

We next proceed to obtain the stochastic differential equations pertinent to the cavity mode variables using the master equation \[2\]. To this end, employing the fact that

\[
\frac{d}{dt} \langle \hat{a}(t) \rangle = Tr \left( \frac{d\hat{\rho}}{dt} \hat{a} \right),
\]

it is possible to see that

\[
\frac{d}{dt} \langle \hat{a}(t) \rangle = -\frac{\kappa}{2} \langle \hat{a}(t) \rangle + \epsilon \langle \hat{a}^\dagger(t) \rangle, \quad \frac{d}{dt} \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle = -\kappa \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle + \epsilon \left[ \langle \hat{a}^2(t) \rangle + \langle \hat{a}^\dagger \hat{a}(t) \rangle \right] + \kappa N, \quad \frac{d}{dt} \langle \hat{a}^2(t) \rangle = -\kappa \langle \hat{a}^2(t) \rangle + 2\epsilon \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle + \epsilon + \kappa M.
\]

On the basis of Eq. \(7\), one can write that

\[
\frac{d}{dt} \alpha(t) = -\frac{\kappa}{2} \alpha(t) + \epsilon \alpha^\ast(t) + F(t), \quad \frac{d}{dt} \alpha^\ast(t) = \frac{\kappa}{2} \alpha^\ast(t) + \epsilon \alpha(t) + F^\ast(t),
\]

where \( F(t) \) is a complex Gaussian white noise force the mean and correlation properties of which remain to be determined. It is not difficult to see that Eq. \(7\) will be equal to the expectation value of Eq. \(10\) provided that

\[
\langle F(t) \rangle = 0, \quad \langle F^\ast(t) \rangle = \kappa N \delta(t - t'), \quad \langle F(t) F^\ast(t') \rangle = \kappa N \delta(t - t').
\]

On the other hand, one can verify applying Eqs. \(7\), \(9\), and \(10\) along with the fact that the noise force at time \( t \) does not correlate with the cavity mode variables at earlier times that

\[
\langle F(t') F(t) \rangle = \kappa N \delta(t - t'), \quad \langle F^\ast(t') F^\ast(t) \rangle = \kappa N \delta(t - t').
\]

We observe that Eqs. \(11\), \(12\), and \(13\) represent the mean and correlation properties of the noise force.

In general, it is possible to express Eq. \(10\) in a more convenient from, upon introducing two variables defined by

\[
\alpha_\pm(t) = \alpha^\ast(t) \pm \alpha(t),
\]

as

\[
\frac{d}{dt} \alpha_\pm(t) = \frac{\lambda_\pm}{2} \lambda_\pm \alpha_\pm(t) + F_\pm(t),
\]

with

\[
\lambda_\pm = \kappa \mp 2\epsilon,
\]

\[
F_\pm(t) = F^\ast(t) \pm F(t),
\]

where the formal solution of Eq. \(15\) can be written as

\[
\alpha_\pm(t + \tau) = \alpha_\pm(t) e^{\frac{\lambda_\pm}{2} \tau} + \int_0^\tau e^{-\frac{\lambda_\pm}{2} \tau'} F_\pm(t + \tau') d\tau',
\]

Now in view of Eqs. \(14\) and \(18\) into account, we get

\[
\alpha(t + \tau) = A_\pm(\tau) \alpha(t) + A_\pm(\tau) \alpha^\ast(t) + B_\pm(t + \tau) - B_\pm(t + \tau),
\]

in which

\[
A_\pm(\tau) = \frac{1}{2} \left[ e^{-\frac{\lambda_\pm}{2} \tau} \pm e^{\frac{\lambda_\pm}{2} \tau} \right],
\]

\[
B_\pm(t + \tau) = \frac{1}{2} \int_0^\tau e^{-\frac{\lambda_\pm}{2} \tau'} F_\pm(t + \tau') d\tau'.
\]

Next upon setting \( t = 0 \) in Eq. \(19\) and then replacing \( \tau \) by \( t \), we have

\[
\alpha(t) = A_\pm(t) \alpha(0) + A_\pm(t) \alpha^\ast(0) + B_\pm(t) - B_\pm(t). \quad (22)
\]

It perhaps worth mentioning that Eqs. \(14\) and \(22\) is applied in calculating various quantities of interest. We also realize that these solutions would be well-behaved functions at steady state, if \( \kappa > 2\epsilon \). Hence we designate \( \kappa = 2\epsilon \) as a critical point.
III. QUADRATURE FLUCTUATIONS

The squeezing properties of a single-mode output radiation can be described with the aid of the quadrature operators defined by

$$\hat{a}_+^{\text{out}} = \hat{a}_+^{\dagger} + \hat{a}_{\text{out}} \quad (23)$$

and

$$\hat{a}_-^{\text{out}} = i(\hat{a}_-^{\dagger} - \hat{a}_{\text{out}}). \quad (24)$$

We next seek to determine the correlations involved in Eq. (24) can be put, using the boson commutation relations for the output radiation, in the form

$$\Delta a^2_{\pm(\text{out})} = 1 \pm [\langle \hat{a}_+^{\dagger}(t) \rangle + \langle \hat{a}_+^{\dagger}(t) \rangle + 2\langle \hat{a}_+^{\dagger}(t) \hat{a}_{\text{out}}(t) \rangle + \langle \hat{a}_+^{\dagger}(t) \rangle^2 + \langle \hat{a}_+^{\dagger}(t) \rangle^2 + 2\langle \hat{a}_+^{\dagger}(t) \hat{a}_{\text{out}}(t) \rangle]. \quad (25)$$

We notice that the operators in Eq. (25) are in the normal order. Hence the corresponding expression in terms of the c-number variables associated with the normal ordering would be

$$\Delta a^2_{\pm(\text{out})} = 1 \pm [\langle \alpha_{\text{out}}^*(t) \rangle + \langle \alpha_{\text{out}}^*(t) \rangle + 2\langle \alpha_{\text{out}}^*(t)\alpha_{\text{out}}(t) \rangle + \langle \alpha_{\text{out}}^*(t) \rangle^2 + \langle \alpha_{\text{out}}^*(t) \rangle^2 + 2\langle \alpha_{\text{out}}^*(t) \alpha_{\text{out}}(t) \rangle]. \quad (26)$$

It is a well established fact that the output radiation can be defined in terms of the cavity mode variables \([9]\) as

$$\alpha_{\text{out}}(t) = \sqrt{\kappa}(t) - \frac{1}{\sqrt{\kappa}} F_{\text{R}}(t), \quad (27)$$

where \(F_{\text{R}}(t)\) is the noise force associated with the reservoir modes and satisfies the correlations:

$$\langle F_{\text{R}}(t) \rangle = 0, \quad (28)$$

$$\langle F_{\text{R}}^*(t) F_{\text{R}}(t') \rangle = \kappa N \delta(t - t'), \quad (29)$$

$$\langle F_{\text{R}}(t) F_{\text{R}}(t') \rangle = \kappa M \delta(t - t'). \quad (30)$$

If the cavity mode is taken to be initially in a vacuum state, Eq. (26) reduces to

$$\Delta a^2_{\pm(\text{out})} = 1 + 2\langle \alpha_{\text{out}}^*(t)\alpha_{\text{out}}(t) \rangle \pm [\langle \alpha_{\text{out}}^2(t) \rangle + \langle \alpha_{\text{out}}^2(t) \rangle]. \quad (31)$$

We observe that the noise force \(F(t)\) represents the contribution from the cavity vacuum fluctuations and reservoir, and hence can be expressed

$$F(t) = F_{\text{C}}(t) + F_{\text{R}}(t), \quad (33)$$

where \(F_{\text{C}}(t)\) is the noise force corresponding to the system in the cavity in the absence of damping. Since the noise force for the reservoir \(F_{\text{R}}(t)\) does not correlate with \(F_{\text{C}}(t)\) and the system variables at the earlier times, it is not difficult to verify that

$$\langle F_{\text{R}}(t)\alpha^*(t) \rangle + \langle F_{\text{R}}^*(t)\alpha(t) \rangle = \kappa N. \quad (34)$$

Thus, on account of Eqs. (29), (32), and (34), we get

$$\langle \alpha_{\text{out}}^*(t)\alpha_{\text{out}}(t) \rangle = \kappa \langle \alpha^*(t)\alpha(t) \rangle + N(1 - \kappa). \quad (35)$$

We are usually interested in steady state values in quantum optics. In this regard, it is easy to check applying Eqs. (5) and (9) that

$$\langle \alpha^*(t)\alpha(t) \rangle_{\text{ss}} = \frac{2\varepsilon^2}{\kappa^2 - 4\varepsilon^2} + \frac{\kappa(N\varepsilon + 2\varepsilon M)}{\kappa^2 - 4\varepsilon^2} \quad (36)$$

as a result the mean photon number for the output radiation at steady state takes the form

$$\langle \alpha_{\text{out}}^*(t)\alpha_{\text{out}}(t) \rangle_{\text{ss}} = N + \frac{2\kappa \varepsilon^2}{\kappa^2 - 4\varepsilon^2} + \frac{2\varepsilon \kappa (2N \varepsilon + \kappa M)}{\kappa^2 - 4\varepsilon^2} \quad (37)$$

We notice that the first term in Eq. (37) is the contribution from the squeezed vacuum radiation that exists outside the cavity, the second term corresponds to the light produced by the parametric oscillator and then escapes through the mirror, whereas the last term represents the correlation between the oscillator and reservoir due to the associated noises. We realize that the larger
the intensity of the external coherent radiation, the more
probable it is converted into the cavity radiation by the
crystal. It is a known fact that the squeezed parameter
is directly related to the mean photon number of the
squeezed vacuum modes. Thus the more the reservoir
modes are reflected from the mirror, the more the mean
photon number outside the cavity would be. We hence
clearly see from Fig. 2 that the mean photon number for
the output radiation increases with the amplitude of the
coherent radiation and squeeze parameter of the reser-
vior. However, in spite of the presence of the squeezed
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the output radiation increases with the amplitude of the
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in which
\[ F_{R \pm}(t) = F_{R \pm}(t) \pm F_{R}(t). \]  
(47)

On the basis of the fact that the noise force does not correlate with system variables at earlier times, it is possible to verify that
\[ \langle \alpha_{\pm}(t + \tau)\alpha_{\pm}(t) \rangle_{ss} = \frac{2}{\kappa + 2\varepsilon} [\kappa(M \pm N) + \varepsilon] e^{-\frac{\Delta}{2} \tau}, \]  
(48)
\[ \langle F_{R \pm}(t)F_{R \pm}(t + \tau) \rangle = 2\kappa(M \pm N)\delta(\tau), \]  
(49)
\[ \langle \alpha_{\pm}(t + \tau)\alpha_{\pm}(t) \rangle = 2\kappa(M \pm N)e^{-\frac{\Delta}{2} \tau}, \]  
(50)
as a result employing Eqs. (45), (46), (48), (49), and (50), we find
\[ S_{out}^{\pm}(\omega) = e^{\pm 2\tau} \left[ 1 \pm \frac{8\varepsilon \kappa}{(\kappa + 2\varepsilon)^2 + 4\omega^2} \right]. \]  
(51)

At critical point and \( \omega = 0 \), we see that \( S_{out}^{\pm}(0) = 0 \).

![Plot of the squeezing spectrum](image)

**FIG. 4:** Plots of the squeezing spectrum of the output radiation at steady state for \( \kappa = 0.8, \tau = 0.75 \), and different values of \( \varepsilon \).

We notice from Eq. [51] that for \( \varepsilon = 0 \), the squeezing spectrum is completely flat with its value depending on the squeeze parameter. However, for \( \varepsilon \neq 0 \) the depth of the squeezing spectrum at \( \omega = 0 \) increases with the amplitude of the coherent radiation and it would be zero at critical point irrespective of the values of the squeeze parameter. It is not difficult to see that the width of the squeezing spectrum decreases with the amplitude of the coherent radiation. We realize upon comparing the results shown in Figs. 3 and 4 that the squeezing spectrum can take smaller values than the corresponding quadrature variance of the output radiation for the same \( r \) and \( \varepsilon \). But this does not mean that the squeezing of the output radiation is greater than the cavity radiation, since the squeezing spectrum corresponds to the correlation of the quadrature operators evaluated at different times unlike the quadrature variance of the cavity radiation which is associated with the correlation of the quadrature variance at equal time.

**IV. PHOTON STATISTICS**

**A. Photon number distribution**

The probability for finding \( n \) photons in a single-mode radiation can be expressed in terms of the pertinent Q-function as
\[ P(n) = \frac{\pi}{n!} \frac{\partial^{2n}}{\partial \alpha^* \partial \alpha^n} [Q(\alpha) \exp(\alpha^* \alpha)]_{\alpha^* = \alpha = 0}, \]  
(52)
where the Q-function for the single-mode radiation can be defined as
\[ Q(\alpha, t) = \frac{1}{\pi} \int d^2 \phi(z, z^*, t) \exp[z^* \alpha - z \alpha^*], \]  
(53)
in which the antinormally ordered characteristic function \( \phi(z, z^*, t) \) is given by
\[ \phi(z, z^*, t) = Tr\{\hat{\rho}(0)e^{-z^* \hat{a}(t)}e^{z \hat{a}^*(t)}\}. \]  
(54)

Using the identity
\[ e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}[\hat{A}, \hat{B}], \]  
(55)
Eq. [54] can be written in terms of the c-number variables associated with the normal ordering as
\[ \phi(z, z^*, t) = e^{-z^*z}(\exp(z\alpha^*(t) - z^*\alpha(t))). \]  
(56)

We note that \( \alpha \) is a Gaussian variable with zero mean. Hence one can readily verify that [11]
\[ \langle \exp(z\alpha^*(t) - z^*\alpha(t)) \rangle = \exp \left[ \frac{1}{2} ((z\alpha^*(t) - z^*\alpha(t))^2 \right]. \]  
(57)

Therefore, it is possible to express the antinormally ordered characteristic function for the output radiation as
\[ \phi_{out}^{*}(z, z^*, t) = e^{-z^*z} \exp \left[ \frac{1}{2} (z^2 \langle \alpha_{out}^2(t) \rangle + z^* \langle \alpha_{out}^* \rangle) \right] \] 
\[ -2z^*z \langle \alpha_{out}^* \rangle \langle \alpha_{out}(t) \rangle], \]  
(58)
which can also be put at steady state in the form
\[ \phi_{out}(z, z^*) = \exp[-z^*za + \frac{b}{2}(z^2 + z^*z^2)], \]  
(59)
where \( a = 1 + \bar{n} \) and \( b = \langle \alpha_{out}^2(t) \rangle_{ss} \), in which \( \bar{n} \) represents the mean photon number for the output radiation at steady state [47]. Hence substituting Eq. [58] into [59] and then performing the integration, the Q-function describing the output radiation is found to have the form
\[ Q_{out}(\alpha) = \frac{\sqrt{u^2 - v^2}}{\pi} \exp \left[ -ua^* \alpha + \frac{v}{2}(\alpha^2 + \alpha^*2) \right], \]  
(60)
with \( u = \frac{a}{\sqrt{u^2 - v^2}} \) and \( v = \frac{b}{\sqrt{u^2 - v^2}} \).
On the other hand, insertion of Eqs. (60) into (52) results in

\[ P_{\text{out}}(n) = \frac{\sqrt{u^2 - v^2}}{n!} \frac{\partial^2 \rho}{\partial \alpha^* \partial \alpha} \left[ \exp \left( (1 - u) \alpha^* \alpha + \frac{v}{2} (\alpha^2 + \alpha^2) \right) \right]_{\alpha = \alpha^* = 0}, \]

from which follows

\[ P_{\text{out}}(n) = \frac{\sqrt{u^2 - v^2}}{n!} \sum_{i,j,k=0}^{\infty} \frac{(1 - u)^i v^{k+j}}{(2k+j)! j! k!} \times \frac{(2k+i)!}{(2j+i)! (2k+i-n)!} \times \alpha^{2k+i-n} \alpha^{2j+i-n} \left[ \alpha^{2k+i-n} \alpha^{2j+i-n} \right]_{\alpha = \alpha^* = 0}. \]

Now applying the condition, \( \alpha^* = \alpha = 0 \), we note that Eq. (62) would be different from zero provided that \( 2k + i - n = 0 \) and \( 2j + i - n = 0 \), as a result \( k = j \) and \( j = \frac{1}{2} \). Consequently, the photon number distribution for the output radiation finally takes the form

\[ P_{\text{out}}(n) = \frac{n!}{[1 + 2n + n^2 - (\alpha_{\text{out}}^2(t))^2]_{ss}^{2n+i}} \times \sum_{i=0}^{n} \frac{[n^2 + n - (\alpha_{\text{out}}^2(t))^2]_{ss}^{i}}{2n-i!(\alpha_{\text{out}}^2(t))^2}_{ss}. \]

We notice that there is a finite probability for counting odd number of photons outside the cavity, even though degenerate parametric oscillator generates pairs of photons. This could be related to the fact that odd number of photons can escape through the coupler mirror.

**B. Power spectrum**

The power spectrum for the output radiation can be expressed in terms of the c-number variables associated with the normal ordering as

\[ S_{\text{out}}(\omega) = 2Re \int_0^\infty \langle \alpha_{\text{out}}^*(t) \alpha_{\text{out}}(t + \tau) \rangle_{ss} e^{i\omega \tau} d\tau. \]

Here with the aid of Eqs. (12), (13), (19), (20), (21), (27), (28), and (30), we see that

\[ \langle \alpha_{\text{out}}^*(t) \alpha_{\text{out}}(t + \tau) \rangle_{ss} = \frac{\kappa}{2} \left( \langle \alpha^*(t) \alpha(t) \rangle_{ss} + \langle \alpha^2(t) \rangle_{ss} \right) - (M + N) e^{-\frac{\lambda}{2} \tau} + \frac{\kappa}{2} \left( \langle \alpha^*(t) \alpha(t) \rangle_{ss} - \langle \alpha^2(t) \rangle_{ss} \right) - (N - M) e^{-\frac{\lambda}{2} \tau} + N \delta(\tau), \]

as a result the power spectrum (63) for the output radiation turns out to be

\[ S_{\text{out}}(\omega) = N + 2\kappa \varepsilon \times \left[ \frac{1 + 2N + 2M}{(\kappa - 2\varepsilon)^2 + 4\omega^2} + 1 - 2N + 2M \right]. \]

In a similar manner the power spectrum for the cavity radiation is found to be

\[ S(\omega) = 2 \left[ \frac{\kappa(N + M) + \varepsilon}{(\kappa - 2\varepsilon)^2 + 4\omega^2} + \frac{\kappa(N - M) - \varepsilon}{(\kappa + 2\varepsilon)^2 + 4\omega^2} \right]. \]

![FIG. 5: Plots of the power spectrum of the output and cavity radiation at steady state for \( \kappa = 0.8, \tau = 0.5, \) and \( \varepsilon = 0.2. \)](image)

We realize from Eqs. (66) and (67) that the width of the cavity and output power spectra is independent of the squeeze parameter, but it decreases with the amplitude of the driving radiation. As clearly shown in Fig. 5 the power spectrum for cavity radiation is more picked and narrowed. However, the width for the cavity and output radiation is the same for the same values of \( \varepsilon. \)

**V. CONCLUSION**

In this paper, we present a thorough study of the squeezing and statistical properties of the output radiation generated by a degenerate parametric oscillator coupled to a squeezed vacuum reservoir using the input-output relation introduced by Gardiner and Collett [9]. We also make a comparison with the results of the cavity radiation. For both the cavity and output radiations a maximum squeezing is occured at a critical point, \( \kappa = 2\varepsilon. \) The maximum squeezing for a cavity radiation when the degenerate parametric oscillator is coupled to a vacuum reservoir is found to be 50%, irrespective of the cavity damping constant. The same result has been obtained by many authors [1-4] using various approaches. However, we see from Eq. (41) that the squeezing for the output radiation increases with the damping constant at
critical point. One can also clearly see from Fig. 3 that
the squeezing of the cavity radiation is greater than that
of the output radiation.

Coupling the degenerate parametric oscillator with the
squeezed vacuum reservoir is found to exponentially en-
hance the squeezing of the output as well as the cavity
radiation. Our calculation of the squeezing spectrum for
the output radiation indicates that a complete quench-
ing of the noise in one of the quadrature components is
possible at the critical point, even when the degenerate
parametric oscillator is coupled to the vacuum reservoir.
Since the squeezing spectrum corresponds to the corre-
lation of the quadrature operators at different times, we
note that the complete suppression of the noise could be
achieved if correlation at different times results squeezing
and there is a way of measuring the overall squeezing over
sufficiently long period of time. We also see from Fig. 4
that the width of the squeezing spectrum decreases with
the amplitude of the driving radiation.

As the degree of squeezing of the output and cavity
radiations increases with the amplitude of the coherent
radiation and squeeze parameter so does the mean pho-
ton number. This indicates that the degenerate para-
metric oscillator produces not only a light with high de-
gree of squeezing but also a quite intense light at the
critical point. Moreover, even though only even number
of photons are produced by the degenerate parametric
oscillator, there is a finite probability for counting odd
number of photons outside the cavity. Furthermore,
the power spectrum for the cavity and output radiations is
found to be picked at $\omega = 0$. As clearly shown in Fig.
5 the height of the power spectrum for the cavity radia-
tion is greater than that of the output radiation, and yet
have the same width. It is not difficult to see from Eqs.
(66) and (67) that the width of the power spectrum is
independent of $r$.

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