Topological Roots of Black Hole Entropy*

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Abstract

We review the insights into black hole entropy that arise from the formulation of gravitation theory in terms of dimensional continuation. The role of the horizon area and the deficit angle of a conical singularity at the horizon as canonically conjugate dynamical variables is analyzed. The path integral and the extension of the Wheeler-De Witt equation for black holes are discussed.

1 Introduction

Boltzmann’s formula

\[ S = \log W \]  

(1)

is a cornerstone of statistical mechanics. It relates \( S \), the macroscopic entropy of a system, to \( W \), the number of microscopic states of the system which have the same given macroscopic properties.

An outstanding problem in gravitation theory is to express the black hole entropy of Bekenstein and Hawking

\[ S = \frac{1}{4G\hbar} \text{(horizon area)} \]  

(2)

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in terms of (1). This poses two questions, namely,

(i) What are the microscopic states?

(ii) How many are there?

It turns out that formulating gravitation theory in terms of dimensional continuation provides an answer to the first question and suggests an answer to the second.

The idea is the following. One analyzes the gravitational action keeping in mind that in two spacetime dimensions it reduces to a topological invariant, the Euler class. This has two rewards. First, the dimensional continuation of the Gauss–Bonnet theorem shows that the black hole entropy itself is the dimensional continuation of the Euler class of a small disk centered at the horizon. Second, since the Euler class of the small disk is still well defined when one allows for a cusp (conical singularity) within the disk, it is natural to allow for new degrees of freedom to be admitted in the path integral that correspond to the possibility of a conical singularity.

One then finds that the “deficit angle” of the cusp and the horizon area are canonically conjugate. Summing over all horizon areas yields the black hole entropy. This provides an answer to question (i) above. However, a “microscopic explanation” for the exponential weight in the integration measure for the surface degrees of freedom, or equivalently for the $\hbar^{-1}$ dependence in (2), is still lacking.

Thus the answer to the second question is not provided but only suggested by the present analysis: It would seem natural to attempt to obtain the black hole entropy as the “number of states within a very small two-dimensional disk”. This has not been done at the moment of this writing.

The plan of this report is the following. Sections 2, 3, and 4 review the treatment of the action and the entropy in terms of dimensional continuation. These sections are based on joint work with M. Bañados and J. Zanelli and follow closely Ref. [1]. Section 5 discusses the relationship of the surface degrees of freedom with the propagator whose trace is the partition function. This section is based on joint work with S. Carlip and follows Ref. [2].
2 The Action as the Dimensional Continuation of the Euler Class

If one considers a two dimensional manifold \( M \) with boundary \( \partial M \), the Gauss–Bonnet theorem reads

\[
\frac{1}{2} \int_M \sqrt{g} g^\mu\nu R^\alpha_\mu\alpha\nu d^2 x - \int_{\partial M} \sqrt{g} K d^1 x = 2\pi \chi(M) .
\]

(3)

The integer \( \chi(M) \) on the right hand side of (3) is the Euler number of \( M \) and depends solely on its topology. One has \( \chi = 1 \) for a disk and \( \chi = 0 \) for an annulus. We will refer to the sum of integrals appearing on the left side of (3) as the Euler class of \( M \). The Gauss–Bonnet theorem then says that the Euler class of \( M \) is equal to \( 2\pi \) times its Euler number.

If one varies the integral over \( M \) in (3) one finds, by virtue of the Bianchi identity, that the piece coming from the variation of the Riemann tensor yields a surface term. This surface term exactly cancels the variation of the surface integral appearing in the Euler class. On the other hand, because of the special algebraic properties of the Riemann tensor in two spacetime dimensions, the contribution of the variation of \( \sqrt{g} g^\mu\nu \) is identically zero. This is a poor man’s way to put into evidence that the Euler class is “a topological invariant”, the real work is to show that the actual value of the sum of integrals is \( 2\pi \chi \).

Now, the Hilbert action for the gravitational field in \( d \) Euclidean spacetime dimensions may be written as

\[
I_H = \frac{1}{2} \int_M \sqrt{g} g^\mu\nu R^\alpha_\mu\alpha\nu d^d x - \int_{\partial M} \sqrt{g} K d^{d-1} x .
\]

(4)

[One integrates \( \exp(+I) \) in the Euclidean path integral. We have set \( 8\pi G = 1 \).] This action has the same form as the Euler class of two dimensions, with the change that now the integrals, and the geometric expressions appearing in them, refer to a spacetime of dimension \( d > 2 \). For this reason, one says that the Hilbert action is the dimensional continuation of the Euler class of two dimensions. After dimensional continuation, the Euler class ceases to be a topological invariant. While it is still true that the variation of the Riemann tensor in (3) yields a surface term, this surface term no longer cancels the variation of the integral of the extrinsic curvature. Rather, the sum of the two variations vanishes only when the intrinsic geometry of the boundary is held fixed. Moreover, the contribution to the variation coming from \( \sqrt{g} g^\mu\nu \) gives the Einstein tensor, which is no longer identically zero,
and hence the demand that it vanishes is not empty but gives the Einstein equations.

This reasoning applies also to the natural generalization of the Hilbert action to higher spacetime dimensions, the Lovelock action \[3\]. This action, which keeps the field equations for the metric of second order and hence does not change the degrees of freedom, can also be understood in terms of dimensional continuation \[4, 5\]. For a spacetime of dimension \(d\), the generalized action contains the dimensionally continued Euler classes of all even dimensions \(2p < d\). Thus, the Hilbert action with a cosmological constant may be thought of as coming from dimensions \(2p = 2\) and \(2p = 0\), respectively.

The analog of the Hilbert action given by (4) is

\[
I_L = \sum_{2p<d} \frac{\alpha_p}{2^{2p} p!} (I_L^p + B^p),
\]

with

\[
I_L^p = \int_M \sqrt{g} \delta^{[\beta_1\ldots\beta_{2p}]} R^{\alpha_1\alpha_2} \cdots R^{\alpha_{2p-1}\alpha_{2p}} \, d^d x.
\]

(Here the totally antisymmetrized Kronecker symbol is normalized so that it takes the values \(0, \pm 1\).)

The boundary term \(B^p\) is the generalization of the integrated trace of the extrinsic curvature in (4). It is given by

\[
B^p = -\frac{2}{d-2p} \int_{\partial M} d^{d-1} x g_{ij} \pi^{ij}_{(p)}.
\]

Here \(\pi^{ij}_{(p)}\) is the contribution of (4) to the momentum canonically conjugate to the metric \(g_{ij}\) of \(\partial M\). It may be expressed as a function of the intrinsic and extrinsic curvatures of the boundary \(\mathbb{R}\).

3 Covariant Action versus Canonical Action. Entropy as Dimensional Continuation

There is another action, which differs from the \(I_H\) by boundary terms. It is the canonical action

\[
I_C = \int (\pi^{ij} \dot{g}_{ij} - N \mathcal{H} - N^i \mathcal{H}_i).
\]
When one studies black holes $I_C$ has a significant advantage over the Hilbert action. It vanishes on the black hole due to the constraint equations $\mathcal{H} = 0 = \mathcal{H}_i$ and the time independence of the spatial metric. The black hole entropy and its relation with the Gauss-Bonnet theorem will arise through the difference between the Hilbert and the canonical actions.

In the Euclidean formalism for black holes, it is useful to introduce a polar system of coordinates in the $\mathbb{R}^2$ factor of $\mathbb{R}^2 \times S^{d-2}$. The reason is that the black hole will have a Killing vector field—the Killing time—whose orbits are circles centered at the horizon. But, it should be stressed that the discussion that follows is valid for a system of polar coordinates centered anywhere in $\mathbb{R}^2$. Indeed the Killing vector exists only on the extremum and not for a generic spacetime admitted in the action principle.

Take now a polar angle in $\mathbb{R}^2$ as the time variable in a Hamiltonian analysis. An initial surface of time $t_1$ and a final surface of time $t_2$ will meet at the origin, which is a fixed point of the time vector field. There is nothing wrong with the two surfaces intersecting. The Hamiltonian formalism can handle that. Next, divide $\mathbb{R}^2$ into a small disk $D_\epsilon$ of radius $\epsilon$ around the origin, and an annulus of inner radius $\epsilon$ and outer radius that will tend to infinity. Analysis of the boundary terms—which will not be given here—shows that, in the limit $\epsilon \to 0$, the Hilbert action for the annulus and the canonical action differ only by a local surface integral at $r = \infty$. Thus we have

$$I_H = \lim_{\epsilon \to 0} I_H[D_\epsilon \times S^{d-2}] + I_C + B_\infty. \tag{9}$$

Here $I_C$ is the canonical action (8) for the annulus in the limit $\epsilon \to 0$.

The boundary term $B_\infty$, which need not be explicitly written, appears because of the different boundary conditions at infinity for $I_H$ and $I_C$. Indeed, as stated above, the Hilbert action (4) needs the intrinsic geometry of the boundary at $r = \infty$ to be fixed. On the other hand, for the Hamiltonian action (8) one must fix at infinity the mass $M$ and angular momentum $J$—with a precise rate of fall off for the fields (see, for example (4)). If instead of $M$ one fixes its conjugate, the asymptotic Killing time difference $\beta$, while still keeping $J$ fixed, one must subtract $\beta M$ from (8).

The contribution at the origin in (9) appears precisely because there is no boundary there in the topological sense. Indeed, the canonical action introduces an additional structure, the time vector field which has a fixed point at the origin. This makes it not covariant. The boundary term is brought in in order to restore covariance.
Thus, if we drop $B_\infty$, we obtain the improved covariant action,

$$I = \lim_{\epsilon \to 0} I_H[D_\epsilon \times S^{d-2}] + I_C,$$

which is suited for fixing $M$ and $J$ at infinity. The action (10) differs from expression (9) only by a local surface term at infinity due to the different boundary condition there, and it is therefore as covariant as (4). Furthermore, (10) is finite on the black hole and thus it is “already regularized”. [The Hilbert action (4) is infinite on the black hole because $B_\infty$ diverges.]

A short analysis reveals that the first term in (10) factorizes into the product of the Euler class (3) for $D_\epsilon$ and the area of the $S^{d-2}$ at the origin. Thus one finds

$$\lim_{\epsilon \to 0} I_H[D_\epsilon \times S^{d-2}] = 2\pi \times (\text{area of } S^{d-2})_{\text{origin}}. \quad (11)$$

Consider now the value of the action on the extremum. Then it is convenient to take the polar angle to be the Killing time, for—in that case—the spatial geometry $g_{ij}$ is time independent. Furthermore, since the Hamiltonian contraints $\mathcal{H} = \mathcal{H}_i = 0$ hold on the extremum, the value of the improved action (10) for the black hole is just the contribution of the disk at the horizon,

$$S = 2\pi \times (\text{area of } S^{d-2})_{\text{horizon}}. \quad (12)$$

This is the standard expression for the black hole entropy in Einstein’s theory, in the semiclassical approximation (“tree level”). This should be the case since in (10) $M$ and $J$ are fixed, which corresponds to the microcanonical ensemble.

Note that the overall factor in front of the area, usually quoted as one fourth in units where Newton’s constant is unity, is really the Euler class of the two-dimensional disk.

4 Deficit Angle as Off–Shell Degree of Freedom. Partition Function

On account of the Gauss–Bonnet theorem the value of the Euler class for a disk is equal to $2\pi$ even if there is a conical singularity (curvature localized at a point). It is therefore natural to allow for that possibility. If there is a “cusp of deficit angle $\alpha$” at the origin of $\mathbb{R}^2$, the value of the two–dimensional integral in the Euler class (3) is equal to $\alpha$, whereas the line integral over
the boundary has the value $2\pi - \alpha$. The full action (10) depends on $\alpha$. This is most directly seen by recalling that—as stated in (9)—the action (10) differs from the Hilbert action (4) by a local boundary term at infinity. As a consequence, if the geometry of the $S^{d-2}$ at the cusp is varied, one finds that the action changes by

$$\delta I = \alpha \delta (\text{area of } S^{d-2} \text{ at cusp}).$$

(13)

Equation (13) shows that the deficit angle, which is a property of the intrinsic Riemannian geometry of $\mathbb{R}^2$, is canonically conjugate to the area of the $S^{d-2}$ attached to that point—an extrinsic property.

Observe that one could incorrectly believe, due to (11), that the action (10) (and hence its variation) is independent of the deficit angle $\alpha$. What happens is that there is a boundary term in the variation of the canonical action, coming from space derivatives in $\mathcal{H}$, which cancels the variation of the surface term in the Euler class leaving (13) as the net change [7].

As shown by (11), the actions (8) and (10) differ by a contact transformation which depends only on the intrinsic geometry of the $S^{d-2}$ at the origin. Therefore, if that geometry were held fixed, both actions correctly yield Einstein’s equations and, on this basis, they would be equally good. However, in the calculation of the partition function (see below) one must integrate over all “closed Euclidean histories” keeping fixed only the data at infinity. This means that in the semiclassical approximation one must extremize with respect to the geometry of the $S^{d-2}$ at the origin, instead of keeping it fixed.

For that problem, the improved action (10) and the canonical action (8) are not equivalent. The black hole will be an extremum for the covariant action (10), because the demand that the variation (13) vanishes yields $\alpha = 0$ at all points, which is the condition for the manifold to be metrically smooth. This is a property that the Euclidean black hole indeed possesses, since the empty space Einstein equations are obeyed everywhere. On the other hand, the demand that the canonical action should have an extremum with respect to variations of the area of the $S^{d-2}$, would yield $\alpha = 2\pi$ at the origin, which would introduce a sort of source at the origin.

Thus, adding the Hilbert action for a small disk around the origin to the canonical action restores covariance without introducing sources. This addition ensures that the fixed point can be located anywhere. This must be so since the manifold has only one boundary, that at infinity. In this sense the presence of a non–vanishing black hole entropy given by (12) is a consequence of general covariance.
The preceding analysis goes through step by step for the Lovelock theory [3]. For Euclidean black holes in $d$-spacetime dimensions, again with topology $\mathbb{R}^2 \times S^{d-2}$ [8], the action (10) now reads

$$I = \lim_{\epsilon \to 0} I_L[D_\epsilon \times S^{d-2}] + I_C,$$

and the entropy becomes

$$S = \lim_{\epsilon \to 0} I_L[D_\epsilon \times S^{d-2}].$$

The limit (15) factorizes into the Euler class of the disk, equal to $2\pi$, and a sum of dimensional continuations to $S^{d-2}$ of the Euler classes of all even dimensions below $d - 2$,

$$S = 2\pi \times \sum_{2p < d} \frac{\alpha_p}{2^{2(p-1)}[2(p-1)]!} S_p^{p-1}$$

with

$$S_p = \int \sqrt{g} \delta^{[\beta_1 \ldots \beta_{2p}]} R^\alpha_{\beta_1 \alpha_2} \cdots R^a_{\beta_2 \cdots \beta_{2p}} d^{d-2}x,$$

where the integral is taken over the $(d - 2)$-sphere at the horizon.

The Hilbert action corresponds to $2p = 2$ and the corresponding entropy is $2\pi$ times the area. The cosmological constant term corresponds to $2p = 0$ and gives no contribution to the entropy. Expression (14) was first given in [3].

## 5 Partition Function as Trace of Propagator over Horizon Degrees of Freedom

In the previous sections the attention was focused on the complete black hole spacetime. To identify more precisely the horizon degrees of freedom it is necessary to analyze the dynamics for a wedge between $t_1$ and $t_2$.

The first observation is that the action for the wedge will again be given by Eq. (10). This is just because we want to obtain the partition function as a trace of the propagation amplitude. Equation (11) then shows that the integration measure over the horizon geometries has a contribution of classical order that appears as the difference between the Hilbert and the canonical action for a disk of vanishing radius.

It should be noted that the Hilbert action for the wedge between $t_1$ and $t_2$ is given by $I_H(\text{wedge}) = I_C + \pi(\text{area of } S^{d-2} \text{ at horizon}) + B_\infty +$
\( \pi (\text{area of } S^{d-2} \text{ at infinity}) \). It differs from (10) and is not the correct action for the wedge. This means that, after dimensional continuation, the Hilbert action for a “full turn wedge” is not the same as that for a disk. Before dimensional continuation the \( S^{d-2} \) factors are absent and the action is the same for both configurations.

The next step is to give the boundary conditions which characterize a wedge of an “off–shell” black hole. At infinity they will be the usual conditions expressing a localized distribution of matter (see, for example, [6]). At the origin, although it is unnecessarily complicated, we will conform to standard practice and use Schwarzschild coordinates near \( r_+ \). That is, we write the generic Euclidean metric as

\[
d s^2 = N^2(r) d t^2 + N^{-2}(r) d r^2 + \gamma_{mn}(r, x^p) d x^m d x^n
\]

(18)

up to terms of order \( O(r - r_+) \), with

\[
(t_2 - t_1) N^2 = 2 \Theta(r - r_+) + O(r - r_+)^2 .
\]

(19)

Here the \( x^m \) are coordinates on the two–sphere \( S^2 \). The parameter \( \Theta \) is the total proper angle (proper length divided by proper radius) of an arc of very small radius and coordinate angular opening \( t_2 - t_1 \). For this reason it will be called the “opening angle.” If one identifies the surfaces \( t = t_1 \) and \( t = t_2 \), thus considering a disk in \( \mathbb{R}^2 \), then the deficit angle \( 2\pi - \Theta \) is the strength of a conical singularity in \( \mathbb{R}^2 \) at \( r_+ \). For the moment, we assume for simplicity that \( \Theta \) is independent of \( x^m \); we shall see below that this restriction may be lifted without changing the conclusions. It is important to emphasize that no a priori relation between \( \Theta \) and the asymptotic geometry is assumed.

Besides \( \Theta \) and \( N(\infty) \) we fix, as usual, the spatial geometries \( G_1 \) and \( G_2 \) at \( t_1 \) and \( t_2 \). The transition amplitude depends on what is fixed in the action principle, that is, it takes the form

\[
K[G_2, G_1; \Theta; \beta]
\]

(20)

with

\[
\beta = N(\infty)(t_2 - t_1) .
\]

(21)

The asymptotic Killing time separation \( \beta \) is conjugate to the total mass whereas the opening angle is conjugate to the horizon area \( A \). (Equation (13) remains valid for the wedge with \( \alpha = 2\pi - \Theta \).)
The propagator (20), regarded as a functional of \( \mathcal{G}_2 \), obeys the differential equations

\[
\hbar \frac{\partial K}{\partial T} + MK = 0 , \tag{22}
\]

\[
\hbar \frac{\partial K}{\partial \Theta} - AK = 0 , \tag{23}
\]

in addition to the Hamiltonian constraints

\[
\mathcal{H} K = \mathcal{H}_i K = 0 . \tag{24}
\]

The amplitude

\[
K[\mathcal{G}_2, M_2, A_2; \mathcal{G}_1, M_1, A_1] \tag{25}
\]

to propagate from \((\mathcal{G}_1, M_1, A_1)\) to \((\mathcal{G}_2, M_2, A_2)\) is related to the Laplace transform of (20) in \( \Theta \) and \( \beta \),

\[
K[\mathcal{G}_2, \mathcal{G}_1; M, A] , \tag{26}
\]

by

\[
K[\mathcal{G}_2, M_2, A_2; \mathcal{G}_1, M_1, A_1] = \delta(M_2 - M_1)\delta(A_2 - A_1)K[\mathcal{G}_2, \mathcal{G}_1; M_2, A_2] . \tag{27}
\]

The (microcanonical) partition function is obtained by integrating (26) over \( \mathcal{G} = \mathcal{G}_1 = \mathcal{G}_2 \) and \( A \) for fixed \( M \). In the semiclassical approximation the integral over \( \mathcal{G} \) gives unity because the canonical action is zero on–shell, whereas the integral over \( A \) yields

\[
Z = e^{S} \tag{28}
\]

with \( S \) given by (2). If one allows for a dependence of \( \Theta \) on the coordinates \( x^m \) of the two–sphere at \( r_+ \) then \( \Theta(x) \) becomes canonically conjugate to the local area element \( \gamma^{1/2}(x) \) on the two–sphere. Summing over all \( \gamma^{1/2}(x) \) gives back (28). Thus the entropy associated with a small disk in \( \mathbb{R}^2 \) at a given horizon location \( x^m \) coincides with the entropy per unit of area obtained from (2).

The above analysis shows that one may regard the black hole entropy as arising from summing over all horizon geometries. We still lack a “microscopic” explanation for the exponential weight in the integration measure for the surface degrees of freedom, or equivalently for the \( \hbar^{-1} \) dependence in (2).
Note, however, that the factor multiplying the area in (12) comes from the action of a small disk in $\mathbb{R}^2$. This would suggest that the entropy per unit area may arise from counting the two-dimensional geometries within the small disk. This would be satisfactory from the point of view of dimensional continuation: the theory would be sending us back to its two-dimensional roots.

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