SINGULARITIES OF MAXIMAL SURFACES

S. FUJIMORI, K. SAJI, M. UMEHARA, AND K. YAMADA

Abstract. We show that the singularities of spacelike maximal surfaces in
Lorentz-Minkowski 3-space generically consist of cuspidal edges, swallowtails
and cuspidal cross caps. The same result holds for spacelike mean curvature
one surfaces in de Sitter 3-space. To prove these, we shall give a simple criterion
for a given singular point on a surface to be a cuspidal cross cap.

Introduction

In [UY2], a notion of maxface was introduced as a class of spacelike maximal
surfaces in Lorentz-Minkowski 3-space with singularities (see Section 2). On
a neighborhood of a singular point, a maxface is represented by a holomorphic
function using a Weierstrass type representation given in [K]. In fact, for a holomorphic
function \( h \in \mathcal{O}(U) \) defined on a simply connected domain \( U \subset C \), there is a maxface
\( f_h \) with Weierstrass data \( (g = e^h, \omega = dz) \), where \( \mathcal{O}(U) \) is the set of holomorphic
functions on \( U \) and \( z \) is a complex coordinate of \( U \). Conversely, for a neighborhood
of a singular point of a maxface \( f \), there exists an \( h \in \mathcal{O}(U) \) such that \( f = f_h \). For
precise descriptions, see Section 2 and [UY2].

On the other hand, in [F], a notion of CMC-1 face was introduced as a class
of spacelike surfaces of constant mean curvature one in de Sitter 3-space (see Sec-
tion 3), and their global properties are investigated (see also [FRUY]). Like the
case of maxfaces, such surfaces near a singular point are represented by holomorphic
functions; that is, for \( h \in \mathcal{O}(U) \), there is a CMC-1 face \( f_h \).

In this paper, we endow the set \( \mathcal{O}(U) \) of holomorphic functions on \( U \) with the
compact open \( C^\infty \)-topology.

Then we shall show that cuspidal edges, swallowtails and cuspidal cross caps are
generic singularities of maxfaces in Lorentz-Minkowski 3-space or CMC-1 faces in
de Sitter 3-space; that is:

**Theorem A.** Let \( U \subset C \) be a simply connected domain and \( K \) an arbitrary com-
 pact set, and let \( S(K) \) be the subset of \( \mathcal{O}(U) \) consisting of \( h \in \mathcal{O}(U) \) such that the
singular points of the maxface (resp. CMC-1 face) \( f_h \) are cuspidal edges, swallow-
tails or cuspidal cross caps. Then \( S(K) \) is an open and dense subset of \( \mathcal{O}(U) \).

We should remark that conelike singularities of maximal surfaces, although not
generic, are still important singularities, which are investigated by O. Kobayashi
[K], Fernández-López-Souam [FLS] and others.
Figure 1. The cuspidal cross cap

To prove the theorem, we shall give a criterion for a given singular point to be a cuspidal cross cap (See Theorem 1.4). It should be remarked that in [KRSUY], simple criteria for cuspidal edges and swallowtails have been given under the same spirit as this paper. These criteria in this paper and in [KRSUY] are both really useful: In fact, as an application, we shall show that a duality between swallowtails and cuspidal cross caps, that is, swallowtails on maxfaces correspond to cuspidal cross caps on their conjugate maxfaces. Moreover, an application of the criteria of [KRSUY] the last three authors (SUY) studied the behavior of the Gaussian curvature near cuspidal edges and swallowtails. Relating this result, we shall remark in this paper on how the behavior of the Gaussian curvature near a cuspidal cross cap is almost the same as that of a cuspidal edge (see Proposition 1.12). It should be also remarked that, analyzing the jet spaces of constant Gaussian curvature surfaces and using the criteria of [KRSUY], Ishikawa-Machida [IM] showed generic singularities on surfaces of constant Gaussian curvature in $\mathbb{R}^3$ consist of cuspidal edges and swallowtails.

Recently, [ISTa] gave simple criterions of cuspidal lips and beaks. As a consequence, the recognition problem of five singularities on surfaces in $\mathbb{R}^3$, that is, cuspidal edges, swallowtails, cuspidal cross caps, cuspidal lips and beaks are now solved completely. (See [KRSUY], Theorem 1.4 and [ISTa].)

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1. Cuspidal cross caps

Let $U$ be a domain in $\mathbb{R}^2$ and $f : U \to (N^3, g)$ a $C^\infty$-map from $U$ into a Riemannian 3-manifold $(N^3, g)$. The map $f$ is called a frontal if there exists a unit vector field $\nu$ on $N^3$ along $f$ such that $\nu$ is perpendicular to $f_*(TU)$. We call this $\nu$ the unit normal vector field of a frontal $f$. Identifying the unit tangent bundle $T_1N^3$ with the unit cotangent bundle $T^*_1N^3$, the map $\nu$ is identified with the map $L = g(\nu, \ast) : U \to T^*_1N^3$.

The unit cotangent bundle $T^*_1N^3$ has a canonical contact form $\mu$ and $L$ is an isotropic map, that is, the pull back of $\mu$ by $L$ vanishes. Namely, a frontal is the projection of an isotropic map. We call $L$ the Legendrian lift (or isotropic lift) of $f$. If $L$ is an immersion, the projection $f$ is called a front. Whitney [W] proved that the generic singularities of $C^\infty$-maps of 2-manifolds into 3-manifolds can only be cross caps. (For example, $f_{CR}(u, v) = (u^2, v, uv)$ gives a cross cap.) On the other hand, a cross cap is not a frontal, and it is also well-known that cuspidal edges and swallowtails are generic singularities of fronts (see, for example, [AGV], Section 21.6, page 336). The typical examples of a cuspidal edge $f_C$ and a swallowtail $f_S$ are given by

$$f_C(u, v) := (u^2, u^3, v), \quad f_S(u, v) := (3u^4 + u^2v, 4u^3 + 2uv, v).$$

A cuspidal cross cap is a singular point which is $A$-equivalent to the $C^\infty$-map (see
Figure 1

\[ f_{\text{CCR}}(u, v) := (u, v^2, uv^3), \]
which is not a front but a frontal with unit normal vector field
\[ \nu_{\text{CCR}} := \frac{1}{\sqrt{4 + 9u^2v^2 + 4v^6}}(-2v^3, -3uv, 2). \]

Here, two \( C^\infty \)-maps \( f: (U, p) \to N^3 \) and \( g: (V, q) \to N^3 \) are \( A \)-equivalent (or right-left equivalent) at the points \( p \in U \) and \( q \in V \) if there exists a local diffeomorphism \( \varphi \) of \( \mathbb{R}^2 \) with \( \varphi(p) = q \) and a local diffeomorphism \( \Phi \) of \( N^3 \) with \( \Phi(f(p)) = g(q) \) such that \( g = \Phi \circ f \circ \varphi^{-1} \).

In this section, we shall give a simple criterion for a given singular point on the surface to be a cuspidal cross cap. Let \( (N^3, g) \) be a Riemannian 3-manifold and \( \Omega \) the Riemannian volume element on \( N^3 \). Let \( f: U \to N^3 \) be a frontal defined on a domain \( U \) in \( \mathbb{R}^2 \). Then we can take the unit normal vector field \( \nu: U \to T_pN^3 \) of \( f \) as mentioned above. The smooth function \( \lambda: U \to \mathbb{R} \) defined by

\[ \lambda(u, v) := \Omega(f_u, f_v, \nu) \]

is called the signed area density function, where \( (u, v) \) is a local coordinate system of \( U \). The singular points of \( f \) are the zeros of \( \lambda \). A singular point \( p \in U \) is called non-degenerate if the exterior derivative \( d\lambda \) does not vanish at \( p \). (It can be easily checked that this non-degeneracy condition is independent of the choice of a local coordinate system.)

When \( p \) is a non-degenerate singular point, the singular set \( \{ \lambda = 0 \} \) consists of a regular curve near \( p \), called the singular curve, and we can express it as a parametrized curve \( \gamma(t) : (-\varepsilon, \varepsilon) \to U \) such that \( \gamma(0) = p \) and \( \lambda(\gamma(t)) = 0 \quad (t \in (-\varepsilon, \varepsilon)) \).

We call the tangential direction \( \gamma'(t) \) the singular direction. Since \( d\lambda \neq 0 \), \( f_u \) and \( f_v \) do not vanish simultaneously. So the kernel of \( df \) is 1-dimensional at each singular point \( p \). A nonzero tangential vector \( \eta \in T_pU \) belonging to the kernel is called the null direction. There exists a smooth vector field \( \eta(t) \) along the singular curve \( \gamma(t) \) such that \( \eta(t) \) is the null direction at \( \gamma(t) \) for each \( t \). We call it the vector field of the null direction. In \([\text{KRSUY}]\), the following criteria for cuspidal edges and swallowtails are given:

**Fact.** Let \( f: U \to N^3 \) be a front and \( p \in U \) a non-degenerate singular point. Take a singular curve \( \gamma(t) \) with \( \gamma(0) = 0 \) and a vector field of null directions \( \eta(t) \). Then

1. The germ of \( f \) at \( p = \gamma(0) \) is \( A \)-equivalent to a cuspidal edge if and only if the null direction \( \eta(0) \) is transversal to the singular direction \( \gamma'(0) \).
2. The germ of \( f \) at \( p = \gamma(0) \) is \( A \)-equivalent to a swallowtail if and only if the null direction \( \eta(0) \) is proportional to the singular direction \( \gamma'(0) \) and

\[ \frac{d}{dt} \Big|_{t=0} \det(\eta(t), \gamma'(t)) \neq 0, \]

where \( \eta(t) \) and \( \gamma'(t) \in T_{\gamma(t)}U \) are considered as column vectors, and \( \det \) denotes the determinant of a \( 2 \times 2 \)-matrix.

We denote by \((T^*N^3)^c\) the complement of the zero section in \( T^*N^3 \).

**Lemma 1.1.** Let \( f: U \to (N^3, g) \) be a frontal. Then there exists a \( C^\infty \)-section \( L: U \to (T^*N^3)^c \) along \( f \) such that \( (\pi \circ L)_*(T_pU) \subset \ker L_p \) for all \( p \in U \), where \( \pi: (T^*N^3)^c \to N^3 \) is the canonical projection, and \( \ker L_p \subset T_{f(p)}N^3 \) is the kernel of \( L_p : T_pN^3 \to \mathbb{R} \). We shall call such a map \( L \) the admissible lift of \( f \). Conversely, let \( L: U \to (T^*N^3)^c \) be a smooth section satisfying \( (\pi \circ L)_*(T_pU) \subset \ker L_p \). Then \( f := \pi \circ L \) is a frontal and \( L \) is a lift of \( f \).
By this lemma, we know that the concept of frontal does not depend on the Riemannian metric of $N^3$. Frontal can be interpreted as a projection of a mapping $L$ into $N^3$ satisfying $(\pi \circ L)_*(T_pU) \subset \text{Ker} \ L_p$ for all $p \in U$. (The projection of such an $L$ into the unit cotangent bundle $T^*_{\text{unit}}N^3$ gives the Legendrian lift of $f$. An admissible lift of $f$ is not uniquely determined, since multiplication of $L$ by non-constant functions also gives admissible lifts.)

Proof of Lemma 1.1. Let $\nu$ be the unit normal vector field of $f$. Then the map

$$L : U \ni p \mapsto g_p(\nu, \ast) \in T^*_{\text{unit}}N^3$$

gives an admissible lift of $f$. Conversely, let $L : U \to (T^*N^3)^*$ be a non-vanishing smooth section with $(\pi \circ L)_*(T_pU) \subset \text{Ker} \ L_p$. Then a non-vanishing section of the orthogonal complement $(\text{Ker} \ L)^\perp$ gives a normal vector field of $f$.

Let $TN^3|_{f(U)}$ be the restriction of the tangent bundle $TN^3$ to $f(U)$. The subbundle of $TN^3|_{f(U)}$ perpendicular to the unit normal vector $\nu$ is called the limiting tangent bundle.

As pointed out in [SUY], the non-degeneracy of the singular points is also independent of the Riemannian metric $g$ of $N^3$. In fact, Proposition 1.3 in [SUY] can be proved under the weaker assumption that $f$ is only a frontal. In particular, we can show the following:

**Proposition 1.2.** Let $f : U \to N^3$ be a frontal and $p \in U$ a singular point of $f$. Let $\Omega$ be a nowhere vanishing 3-form on $N^3$ and $E$ a vector field on $N^3$ along $f$ which is transversal to the limiting tangent bundle. Then $p$ is a non-degenerate singular point of $f$ if and only if $\lambda = \Omega(f_u, f_v, E)$ satisfies $d\lambda \neq 0$.

Before describing the criterion for cuspidal cross caps, we shall recall the covariant derivative along a map. Let $D$ be an arbitrarily fixed linear connection on $N^3$ and $f : U \to N^3$ a $C^\infty$-map. We take a local coordinate system $(V; x^1, x^2, x^3)$ on $N^3$ and write the connection as

$$D_{\frac{\partial}{\partial x^l}} = \sum_{k=1}^3 \Gamma_{kl} \frac{\partial}{\partial x^k}.$$  

We assume that $f(U) \subset V$. Let $X : U \to TN^3$ be an arbitrary vector field of $N^3$ along $f$ given by

$$X = \xi^1(u, v) \left( \frac{\partial}{\partial x^1} \right)_{f(u,v)} + \xi^2(u, v) \left( \frac{\partial}{\partial x^2} \right)_{f(u,v)} + \xi^3(u, v) \left( \frac{\partial}{\partial x^3} \right)_{f(u,v)}.$$  

Then its covariant derivative along $f$ is defined by

$$D^l_{\partial/\partial x^l} X := \sum_{k=1}^3 \left( \frac{\partial \xi^k}{\partial u^l} + \sum_{i,j=1}^3 (\Gamma^k_{ij} \circ f) \xi^j \frac{\partial f^i}{\partial u^l} \right) \frac{\partial}{\partial x^k}, \quad (l = 1, 2),$$

where $(u^1, u^2) = (u, v)$ is the coordinate system of $U$ and $f = (f^1, f^2, f^3)$. Let

$$\eta = \eta^1 \frac{\partial}{\partial u^1} + \eta^2 \frac{\partial}{\partial u^2} \in TU$$

be a null vector of $f$, that is, $f_* \eta = 0$. In this case, we have $\eta^1 f^k_u + \eta^2 f^k_v = 0$ for $k = 1, 2, 3$, and thus

$$D^l_{\partial/\partial x^l} X = \sum_{k=1}^3 \left( \sum_{l=1}^2 \eta^l \frac{\partial \xi^k}{\partial u^l} \right) \frac{\partial}{\partial x^k}$$

holds, which implies the following:
Lemma 1.3. The derivative $D^f\gamma$ does not depend on the choice of the linear connection $D$ if $\eta$ is a null vector of $f$.

The purpose of this section is to prove the following assertion:

**Theorem 1.4.** Let $f: U \to N^3$ be a frontal and $L: U \to (T^*N^3)\circ$ an admissible lift of $f$. Let $D$ be an arbitrary linear connection on $N^3$. Suppose that $\gamma(t)$ ($|t| < \epsilon$) is a singular curve on $U$ passing through a non-degenerate singular point $p = \gamma(0)$, and that $X: (-\epsilon, \epsilon) \to TN^3$ is an arbitrarily fixed vector field along $\gamma$ such that

1. $L(X)$ vanishes on $U$, and
2. $X$ is transversal to the subspace $f_*(T_pU)$ at $p$.

We set

$$\tilde{\psi}(t) := L(D^f_{\gamma(t)} X_{\gamma(t)}),$$

where $\gamma(t)$ is the singular curve at $p$, $\eta(t)$ is a null vector field along $\gamma$. Then the germ of $f$ at $p = \gamma(0)$ is $A$-equivalent to a cuspidal cross cap if and only if

(i) $\eta(0)$ is transversal to $\gamma'(0)$, and
(ii) $\psi(0) = 0$ and $\psi'(0) \neq 0$

hold, where $\cdot' = d/dt$.

**Remark.** This criterion for cuspidal cross caps is independent of the metric and the linear connection of the ambient space. This property will play a crucial role in Section 3 where we investigate singular points on constant mean curvature one surfaces in de Sitter 3-space.

As a corollary, we shall prove the following:

**Corollary 1.5.** Let $f: U \to (N^3, g)$ be a frontal with unit normal vector field $\nu$ with the Riemannian volume element $\Omega$ on $N^3$, and $\gamma(t)$ a singular curve on $U$ passing through a non-degenerate singular point $p = \gamma(0)$. We set

$$\psi(t) := \Omega(\tilde{\gamma}', D^f_0 \nu, \nu),$$

where $\tilde{\gamma} = f \circ \gamma$, $D^f_0 \nu$ is the canonical covariant derivative along a map $f$ induced from the Levi-Civita connection on $(N^3, g)$, and $'= d/dt$. Then the germ of $f$ at $p = \gamma(0)$ is $A$-equivalent to a cuspidal cross cap if and only if

(i) $\eta(0)$ is transversal to $\gamma'(0)$,
(ii) $\psi(0) = 0$ and $\psi'(0) \neq 0$.

We remark that an application of this criterion is given in Izumiya-Saji-Takeuchi [IST0].

**Proof of Corollary 1.5.** We set $X_0 := \tilde{\gamma}' \times_g \nu$, where $\tilde{\gamma} = f \circ \gamma$, $'= d/dt$ and $\times_g$ is the vector product of $TN^3$ with respect to the Riemannian metric $g$. Since $X_0$ is perpendicular to $\nu$, we have $L(X_0) = 0$. Moreover, $X_0$ is obviously transversal to $\tilde{\gamma}'$, and then it satisfies the conditions (1) and (2) in Theorem 1.4. On the other hand, $L := g(\nu, \bullet)$ gives an admissible lift of $f$ and we have

$$\psi(t) = \Omega(\tilde{\gamma}', D^f_0 \nu, \nu) = g(\nu \times_g \tilde{\gamma}', D^f_0 \nu)$$
$$= -g(X_0, D^f_0 \nu) = g(D^f_0 X_0, \nu) = L(D^f_0 X_0) = \tilde{\psi}(t).$$

This proves the assertion. \qed

To prove Theorem 1.4 we prepare two lemmas. The first one is obvious because the limiting tangent bundle is defined as the orthogonal complement of $\tilde{v}$, where $\tilde{v}$ is a (not necessarily unit) normal vector field.
Lemma 1.6. Let $f: U \to (N^3, g)$ be a frontal and $p \in U$ a singular point such that $\text{rank}(df)_p = 1$ (that is, $p$ is a corank-one singular point). Take a (not necessarily unit) normal vector field $\nu$ of $f$, that is, the limiting tangent bundle is defined as $\{X; \, g(X, \nu) = 0\}$. Then $f$ is a front on a neighborhood of corank-one singular point $p$ if and only if $D^f_\nu \nu$ is not proportional to $\nu$ at $p$.

Remark. Consider a frontal $f: U \to (N^3, g)$ in a pseudo-Riemannian manifold with an indefinite metric $g$. In this case, the unit normal vector may not be defined, but there exists a vector field $\nu$ along $f$ such that the limiting tangent bundle is obtained as $\{X; \, g(\nu, X) = 0\}$. Then Lemma 1.6 holds for $\nu$ in this case.

Corollary 1.7. Under the same assumptions as in Theorem 1.4 and the null direction is not proportional to the singular direction. Then $\psi(0) \neq 0$ holds if and only if $f$ is a front on a sufficiently small neighborhood of $p$.

Proof. Take the unit normal vector field $\nu$ and the Levi-Civita connection $D$. Then $L(D^f_\nu X) = g(D^f X, \nu) = -g(X, D^f_\nu \nu)$. Here, $D_\nu \nu$ is perpendicular to $\nu$ and $df(\gamma)$ under the assumptions, we have the conclusion. Since $D^f_\nu \nu$ is perpendicular to both $\nu$ and $df(\gamma)$, Lemma 1.6 implies the conclusion. \hfill $\Box$

Lemma 1.8. Let $f: U \to N^3$ be a frontal and a non-degenerate singular point of $f$ satisfying \[(1)\] of Theorem 1.4. Then the condition $\psi(0) = \bar{\psi}(0) = 0$ is independent of the choice of vector field $X$ along $f$ satisfying \[(1)\] and \[(2)\].

Proof. By \[(1)\], we may assume that the null vector field $\eta(t)$ ($|t| < \varepsilon$) is transversal to $\gamma(t)$. Then we may take a coordinate system $(u, v)$ with the origin at $p$ such that the $u$-axis corresponds to the singular curve and $\eta(u) = (\partial/\partial v)(\gamma(u, 0))$. We fix an arbitrary vector field $X_0$ satisfying \[(1)\] and \[(2)\]. By \[(2)\], $X_0$ is transversal to the vector field $V := f_*(\partial/\partial u)(\neq 0)$ along $f$. Take an arbitrary vector field $X$ along $f$ satisfying \[(1)\] and \[(2)\] Then it can be expressed as a linear combination

$$X = a(u, v)X_0 + b(u, v)V \quad (a(0, 0) \neq 0).$$

Then we have

$$D^f_\eta X = da(\eta)X_0 + db(\eta)V + aD^f_\eta X_0 + bD^f_\eta V.$$

Now $L(V) = 0$ holds, since $L$ is an admissible lift of $f$. Moreover, \[(1)\] implies that $L(X) = 0$, and we have

$$L(D^f_\eta X) = aL(D^f_\eta X_0) + bL(D^f_\eta V).$$

Since $D^f_\eta$ does not depend on the choice of a connection $D$, we may assume that $D$ is a torsion-free connection. Then we have

$$D^f_\eta V = D^f_\eta f_* \left( \frac{\partial}{\partial u} \right) = D^f_\eta f_* \left( \frac{\partial}{\partial v} \right) = 0,$$

since $f_*(\partial/\partial v) = f_* \eta = 0$. Thus we have

$$L(D^f_\eta X) = aL(D^f_\eta X_0) = a\bar{\psi}(u).$$

Since $a(0, 0) \neq 0$, the conditions \[(1)\] and \[(2)\] for $X$ are the same as those of $X_0$. \hfill $\Box$

The following two lemmas are well-known (see [GG]). They play a crucial role in Whitney [W] to give a criterion for a given $C^\infty$-map to be a cross cap. Let $h(u, v)$ be a $C^\infty$-function defined around the origin.

Fact 1.9 (Division Lemma). If $h(u, 0)$ vanishes for sufficiently small $u$, then there exists a $C^\infty$-function $h(u, v)$ defined around the origin such that $h(u, v) = v\tilde{h}(u, v)$ holds.
Fact 1.10 (Whitney Lemma). If \( h(u,v) = h(-u,v) \) holds for sufficiently small \((u,v)\), then there exists a \( C^\infty \)-function \( \tilde{h}(u,v) \) defined around the origin such that \( h(u,v) = \tilde{h}(u^2,v) \) holds.

Proof of Theorem 1.4. As the assertion is local in nature, we may assume that \( N^3 = \mathbb{R}^3 \) and let \( g_0 \) be the canonical metric. We denote the inner product associated with \( g_0 \) by \( \langle \cdot, \cdot \rangle \). The canonical volume form \( \Omega \) is nothing but the determinant: \( \Omega(X,Y,Z) = \det(X,Y,Z) \). Then the signed area density function \( \lambda \) defined in (1.2) in the introduction is written as

\[
\lambda(u,v) = \det(f_u,f_v,\nu).
\]

Let \( f : U \to \mathbb{R}^3 \) be a frontal and \( \nu \) the unit normal vector field of \( f \). Take a coordinate system \((u,v)\) centered at the singular point \( p \) such that the \( u \)-axis is a singular curve and the vector field \( \partial \)/\( \partial v \) gives the null direction along the \( v \)-axis.

If we set \( X = V \times_{\nu} \nu \), then it satisfies (1) and (2) of Theorem 1.3 and by (1.3) we have

\[
\tilde{\psi}(u) = \det(g',D_u\nu,\nu) = \det(f_u,\nu_v,\nu).
\]

Thus we now suppose that \( \tilde{\psi}(0) = 0 \) and \( \tilde{\psi}'(0) \neq 0 \). It is sufficient to show that \( f \) is \( \mathcal{A} \)-equivalent to the standard cuspidal cross cap as in (1.1) in the introduction.

Without loss of generality, we may set \( f(0,0) = (0,0,0) \). Since \( f \) satisfies (1) \( f(u,0) \) is a regular space curve. Since \( f_u(u,0) \) \( \neq 0 \), we may assume \( f_u^3(u,0) \neq 0 \) for sufficiently small \( u \), where we set \( f = (f^1,f^2,f^3) \). Then the map

\[
\Phi : (y^1,y^2,y^3) \mapsto (f^1(y^1,0),f^2(y^1,0)+y^2,f^3(y^1,0)+y^3)
\]

is a local diffeomorphism of \( \mathbb{R}^3 \) at the origin. Replacing \( f \) by \( \Phi^{-1} \circ f(u,v) \), we may assume \( f(u,v) = (u,f^2(u,v),f^3(u,v)) \), where \( f^2 \) and \( f^3 \) are smooth functions around the origin such that \( f^2(u,0) = f^3(u,0) = 0 \) for sufficiently small \( u \).

Then by the division lemma (Fact 1.9), there exist \( C^\infty \)-functions \( \tilde{f}^2(u,v), \tilde{f}^3(u,v) \) such that \( \tilde{f}^j(u,v) = v\tilde{f}^j(u,v) \) (\( j = 2,3 \)). Moreover, since \( f_u = 0 \) along the \( u \)-axis, we have \( \tilde{f}^2(u,0) = \tilde{f}^3(u,0) = 0 \). Applying the division lemma again, there exist \( C^\infty \)-functions \( a(u,v) \), and \( b(u,v) \) such that

\[
f(u,v) = (u,v^2a(u,v),v^2b(u,v)).
\]

Since \( f_u(u,0) = 0, \lambda_u(u,0) = 0 \) and \( d\lambda \neq 0 \), we have

\[
0 \neq \lambda_u(u,0) = \det(f_{uu},f_{uv},\nu) + \det(f_u,f_{vu},\nu) + \det(f_u,f_{vv},\nu) = \det(f_u,f_{uv},\nu).
\]

In particular, we have

\[
0 \neq f_{uv}(0,0) = 2(0,a(0,0),b(0,0)).
\]

Hence, changing the \( y \)-coordinate to the \( z \)-coordinate if necessary, we may assume that \( a(0,0) \neq 0 \). Then the map \( (u,v) \mapsto (\tilde{u},\tilde{v}) = (u,v\sqrt{a(u,v)}) \) defined near the origin gives a new local coordinate around \( (0,0) \) by the inverse function theorem. Thus we may assume that \( a(u,v) = 1 \), namely

\[
f(u,v) = (u,v^2,v^2b(u,v)).
\]

Now we set

\[
\alpha(u,v) := \frac{b(u,v) + b(u,-v)}{2}, \quad \beta(u,v) := \frac{b(u,v) - b(u,-v)}{2}.
\]

Then \( b = \alpha + \beta \) holds, and \( \alpha \) (resp. \( \beta \)) is an even (resp. odd) function. By applying the Whitney lemma, there exist smooth functions \( \tilde{\alpha}(u,v) \) and \( \tilde{\beta}(u,v) \) such that

\[
\alpha(u,v) = \tilde{\alpha}(u,v^2), \quad \beta(u,v) = v\tilde{\beta}(u,v^2).
\]

Then we have

\[
f(u,v) = (u,v^2,v^2\tilde{\alpha}(u,v^2) + v^2\tilde{\beta}(u,v^2)).
\]
Thus (ii) of Theorem 1.4 holds if and only if

\[
\nu \quad \text{as} \quad f \quad \text{gives a local diffeomorphism at the origin. Replacing } f \quad \text{by } \Phi_1 \circ f, \quad \text{we may set}
\]

\[
f(u, v) = (u, v^2, v^3\bar{\beta}(u, v^2)).
\]

Then by a straightforward calculation, the unit normal vector field \( \nu \) of \( f \) is obtained as

\[
\nu := \frac{1}{\Delta} \left( -v^3\bar{\beta}_u, -\frac{3}{2}v\bar{\beta} - v^3\bar{\beta}_v, 1 \right),
\]

\[
\Delta = \left[ 1 + v^2 \left( \left( \frac{3}{2}\bar{\beta} + v^2\bar{\beta}_v \right)^2 + v^4(\bar{\beta}_u)^2 \right) \right]^{1/2}.
\]

Since \( \nu_u(0, 0) = (0, -3\bar{\beta}(u, 0)/2, 0) \), we have

\[
\tilde{\psi}(u) = \det(f_u, \nu_u, \nu) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{2}\bar{\beta}(u, 0) & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\frac{3}{2}\bar{\beta}(u, 0).
\]

Thus (3) of Theorem 1.4 holds if and only if

\[
(1.4) \quad \bar{\beta}(0, 0) = 0, \quad \bar{\beta}_u(0, 0) \neq 0.
\]

Then by the implicit function theorem, there exists a \( C^\infty \)-function \( \delta(u, v) \) such that \( \delta(0, 0) = 0 \), and

\[
(1.5) \quad \bar{\beta}(\delta(u, v), v) = u
\]

holds. Using this, we have a local diffeomorphism on \( \mathbb{R}^2 \) as \( \varphi : (u, v) \mapsto (\delta(u, v^2), v) \), and

\[
f \circ \varphi(u, v) = (\delta(u, v^2), v^3, uv^3).
\]

Since \( \delta_u \neq 0 \) by (1.5), \( \Phi_2 : (x, y, z) \mapsto (\delta(x, y), y, z) \) gives a local diffeomorphism on \( \mathbb{R}^3 \), and

\[
\Phi_2^{-1} \circ f \circ \varphi = (u, v^2, uv^3)
\]

gives the standard cuspidal cross cap \( f_{\text{CCR}} \) mentioned in the introduction. \( \square \)

In \( \text{SUY} \), the last three authors introduced the notion of singular curvature of cuspidal edges, and studied the behavior of the Gaussian curvature near a cuspidal edge:

**Fact 1.11.** Let \( f : U \rightarrow \mathbb{R}^3 \) be a front, \( p \in U \) a cuspidal edge, and \( \gamma(t) (|t| < \varepsilon) \) a singular curve consisting of non-degenerate singular points with \( \gamma(0) = p \). Then the Gaussian curvature \( K \) is bounded on a sufficiently small neighborhood of \( J := \gamma((-\varepsilon, \varepsilon)) \) if and only if the second fundamental form vanishes on \( J \). Moreover, if the Gaussian curvature \( K \) is non-negative on \( U \setminus J \) for a neighborhood of \( U \) of \( p \), then the singular curvature is non-positive.

The singular curvature at a cuspidal cross cap is also defined in a similar way to the cuspidal edge case. Since the unit normal vector field \( \nu \) is well-defined at a cuspidal cross caps, the second fundamental form is well-defined. Since singular points sufficiently close to a cuspidal cross cap are cuspidal edges, the following assertion immediately follows from the above fact.

**Proposition 1.12.** Let \( f : U \rightarrow \mathbb{R}^3 \) be a frontal, \( p \in U \) a cuspidal cross cap, and \( \gamma(t) (|t| < \varepsilon) \) a singular curve consisting of non-degenerate singular points with \( \gamma(0) = p \). Then the Gaussian curvature \( K \) is bounded on a sufficiently small neighborhood of \( J := \gamma((-\varepsilon, \varepsilon)) \) if and only if the second fundamental form vanishes
on $J$. Moreover, if the Gaussian curvature $K$ is non-negative on $U \setminus J$ for a neighborhood of $U$ of $p$, then the singular curvature is non-positive.

Now, we give an example of a surface with umbilic points accumulating at a cuspidal cross cap point. For a space curve $\gamma(t)$ with arc-length parameter, we take $\{\xi_1(t), \xi_2(t), \xi_3(t)\}$, $\kappa(t) > 0$ and $\tau(t)$ as the Frenet frame, the curvature and the torsion functions of $\gamma$. We consider a tangent developable surface $f(t,u) = \gamma(t) + u\xi_1(t)$ of $\gamma$. The set of singular points of $f$ is $\{(t,0)\}$.

We remark that this surface is frontal, since $\nu(t,u) = \xi_3(t)$ gives the unit normal vector. By a direct calculation, the first fundamental form $ds^2$ and the second fundamental form $II$ are written as

$$ds^2 = \left(1 + u^2(\kappa(t))^2\right) dt^2 + 2 dt du + du^2, \quad II = u\kappa(t)\tau(t) dt^2,$$

and the Gaussian curvature $K$ and the mean curvature are

$$K = 0, \quad H = \frac{\tau(t)}{2u\kappa(t)}.$$ 

So a regular point $(t,u)$ is an umbilic point if and only if $\tau(t) = 0$. On the other hand, it is easy to show that $f$ is a front at $(t,0)$ if and only if $\tau(t) \neq 0$. Moreover, Cleave showed that a tangent developable surface $f$ at $(t,0)$ is $A$-equivalent to a cuspidal cross cap if and only if $\tau(t) = 0$ and $\tau'(t) \neq 0$, which also follows from our criterion directly. Hence we consider a tangent developable surface with space curve $\gamma(t)$ with $\tau(t) = 0$ and $\tau'(t) \neq 0$, and then we have the desired example.

**Example 1.13.** Let $\gamma(t) = (t,t^2,t^4)$ and consider a tangent developable surface $f$ of $\gamma$. Since $\tau(0) = 0$ and $\tau'(0) = 12 \neq 0$, all points on the ruling passing through $\gamma(0)$ are umbilic points and $f$ at $(0,0)$ is a cuspidal cross cap (see Figure 2).

**2. Singularities of maximal surfaces**

A holomorphic map $F = (F^1, F^2, F^3) : M^2 \to C^3$ of a Riemann surface $M^2$ into the complex space form $C^3$ is called null if $\sum_{j=1}^3 F^j \cdot F^j$ vanishes, where $F^j := dF^j/dz$. We consider two projections, the former is the projection into the Euclidean 3-space

$$p_E : C^3 \ni (\zeta^1, \zeta^2, \zeta^3) \mapsto \text{Re}(\zeta^1, \zeta^2, \zeta^3) \in R^3,$$

and the latter one is the projection into Lorentz-Minkowski 3-space

$$p_L : C^3 \ni (\zeta^1, \zeta^2, \zeta^3) \mapsto \text{Re}(-\sqrt{-1}\zeta^3, \zeta^1, \zeta^2) \in L^3,$$

where $L^3$ is the Lorentz-Minkowski space-time of dimension 3 with signature $(-,+,+)$.

It is well-known that the projection of null holomorphic immersions into $R^3$ by $p_E$ gives conformal minimal immersions. Moreover, conformal minimal immersions are always given locally in such a manner.
On the other hand, the projection of null holomorphic immersions into \( L^3 \) by \( p_L \) gives spacelike maximal surfaces with singularities, called maxfaces (see \( \text{[UY2]} \) for details). Moreover, \( \text{[UY2]} \) proves that maxfaces are all frontal and gives a necessary and sufficient condition for their singular points to be cuspidal edges and swallowtails. In this section, we shall give a necessary and sufficient condition for their singular points to be cuspidal cross caps and will show that generic singular points of maxfaces consist of cuspidal edges, swallowtails and cuspidal cross caps (see Theorem \( \text{[A]} \) in the introduction).

The following fact is known (see \( \text{[UY2]} \)):

**Fact 2.1.** Let \( U \subset \mathbb{C} \) be a simply connected domain containing a base point \( z_0 \), and \( (g, \omega) \) a pair of a meromorphic function and a holomorphic 1-form on \( U \) such that

\[
(1 + |g|^2)^2|\omega|^2 \neq 0
\]

on \( U \). Then

\[
f(z) := \text{Re} \int_{z_0}^z (-2g, 1 + g^2, \sqrt{-1}(1 - g^2)) \omega
\]

gives a maxface in \( L^3 \). Moreover, any maxfaces are locally obtained in this manner.

The first fundamental form (that is, the induced metric) of \( f \) in (2.2) is given by

\[
ds^2 = (1 - |g|^2)^2|\omega|^2.
\]

In particular, \( z \in U \) is a singular point of \( f \) if and only if \( |g(z)| = 1 \), and at \( f : U \setminus \{|g| = 1\} \to L^3 \) is a spacelike maximal (that is, vanishing mean curvature) immersion. The meromorphic function \( g \) can be identified with the Lorentzian Gauss map. We call the pair \((g, \omega)\) the Weierstrass data of \( f \). In \( \text{[UY2]} \), the last two authors proved that \( f \) is a front on a neighborhood of a given singular point \( z = p \) if and only if \( \text{Re}(dg/(g^2\omega)) \neq 0 \). Moreover, the following assertions are proved in \( \text{[UY2]} \):

**Fact 2.2** (\( \text{[UY2]} \) Theorem 3.1). Let \( U \) be a domain of the complex plane \((\mathbb{C}, z)\) and \( f : U \to L^3 \) a maxface constructed from the Weierstrass data \((g, \omega = \omega dz)\), where \( \omega \) is a holomorphic function on \( U \). Then \( f \) is a frontal into \( L^3 \) (which is identified with \( \mathbb{R}^3 \)). Take an arbitrary point \( p \in U \). Then \( p \) is a singular point of \( f \) if and only if \( |g(p)| = 1 \), and \( f \) is a front at a singular point \( p \) if and only if \( \text{Re}(g'/g^2\omega) \neq 0 \) holds at \( p \), where \( \omega' = d\omega/dz \).

Suppose now \( \text{Re}(g'/g^2\omega) \neq 0 \) at a singular point \( p \). Then

1. \( f \) is \( \mathcal{A} \)-equivalent to a cuspidal edge at \( p \) if and only if \( \text{Im}(g'/g^2\omega) \neq 0 \), and
2. \( f \) is \( \mathcal{A} \)-equivalent to a swallowtail at \( p \) if and only if

\[
\frac{g'}{g^2\omega} \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad \text{Re} \left[ \frac{g}{g'} \left( \frac{g'}{g^2\omega} \right) \right] \neq 0.
\]

**Remark 2.3.** In \( \text{[UY2]} \) Lemma 3.3, the third and fourth authors proved that \( \text{Re}(g'/g^2\omega) \neq 0 \) at a singular point if and only if \( f \) is a front at \( p \) and \( p \) is non-degenerate singular point. More precisely, the condition \( \text{Re}(g'/g^2\omega) \neq 0 \) is equivalent to that \( f \) is a front, and if this is the case, \( p \) is automatically a non-degenerate singular point because \( g' \neq 0 \). See page 25 in \( \text{[UY2]} \).

The statements of Theorem 3.1 of \( \text{[UY2]} \) are criteria to be locally diffeomorphic to a cuspidal edge or a swallowtail. However, in this case, local diffeomorphicty implies \( \mathcal{A} \)-equivalency. See the appendix of \( \text{[KRSUY]} \). We shall prove the following:
Theorem 2.4. Let $U$ be a domain of the complex plane $(C, z)$ and $f: U \to L^3$ a maxface constructed from the Weierstrass data $(g, \omega = \bar{\omega}dz)$, where $\omega$ is a holomorphic function on $U$. Take an arbitrary singular point $p \in U$. Then $f$ is $A$-equivalent to a cuspidal cross cap at $p$ if and only if

$$\frac{g'}{g^2}\omega \in \sqrt{-1}R \setminus \{0\} \quad \text{and} \quad \text{Im} \left[ \frac{g'}{g^2}\omega \right] \neq 0,$$

where $' = d/dz$.

Proof. We identify $L^3$ with the Euclidean 3-space $R^3$. Let $f$ be a maxface as in (2.2). Then

$$\nu := \frac{1}{\sqrt{1 + |g|^2 + 4|g|^2}}(1 + |g|^2, 2 \text{Re} g, 2 \text{Im} g)$$

is the unit normal vector field of $f$ with respect to the Euclidean metric of $R^3$. Let $p \in U$ be a singular point of $f$, that is, $|g(p)| = 1$ holds. Then by (2.4), $\omega$ does not vanish at $p$. Here,

$$\lambda = \det(f_u, f_v, \nu) = (|g|^2 - 1)|\omega|^2 \sqrt{1 + |g|^2 + 4|g|^2},$$

under the complex coordinate $z = u + \sqrt{-1}v$ on $U$. Then the singular point $p$ is non-degenerate if and only if $dg \neq 0$.

The singular direction $\xi$ and the null direction $\eta$ are given by $\xi = \sqrt{-1}(g'/g)$, and $\eta = \sqrt{-1}((g/\omega))$, respectively. Thus, we can parametrize the singular curve $\gamma(t)$ as

$$\dot{\gamma}(t) = \sqrt{-1}\left(\frac{g'}{g}\right)(\gamma(t)) \quad \left(\dot{\gamma} = \frac{d}{dt}\right)$$

(see the proof of Theorem 3.1 in [UY2]). Here, we identify $T_pU$ with $R^2$ and $C$ with

$$\zeta = a + \sqrt{-1}b \in C \leftrightarrow (a, b) \in R^2 \leftrightarrow a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} = \zeta\frac{\partial}{\partial z} + \bar{\zeta}\frac{\partial}{\partial \bar{z}},$$

where $z = u + \sqrt{-1}v$. Then $\dot{\gamma}$ and $\eta$ are transversal if and only if

$$\det(\xi, \eta) = \text{Im} \xi \bar{\eta} = \text{Im} \left(\frac{g'}{g^2\omega}\right) \neq 0.$$

On the other hand, one can compute $\psi$ as in Theorem 1.5 as

$$\psi = \det(f_u \dot{\gamma}, dv(\eta), \nu) = \text{Re} \left(\frac{g'}{g^2\omega}\right) \cdot \psi_0,$$

where $\psi_0$ is a smooth function on a neighborhood of $p$ such that $\psi_0(p) \neq 0$. Then the second condition of Theorem 1.5 is written as

$$\text{Re} \left(\frac{g'}{g^2\omega}\right) = 0 \quad \text{and} \quad \text{Im} \left[ \left(\frac{g'}{g^2\omega}\right) \left(\frac{g'}{g}\right) \right] \neq 0.$$

Here, we used the relation $d/dt = \sqrt{-1}(g'/g)(\partial/\partial z) - (g'/g)(\partial/\partial \bar{z})$, which comes from (2.3). Using the relation $(g'/g) = (g/g')$ (real valued function), we have the conclusion. \hfill $\square$

Let $f: U \to L^3$ be a maxface with a Weierstrass data $(g, \omega)$. Then the associated maxface $\bar{f}: U \to L^3$ with the Weierstrass data $(g, \sqrt{-1}\omega)$ is called the conjugate maxface, which has the same first fundamental form and principal curvatures as $f$.

The following assertion follows immediately:

...
**Corollary 2.5** (A duality between swallowtails and cuspidal cross caps). A maxface \( f : U \to \mathcal{L}^3 \) is \( \mathcal{A} \)-equivalent to a swallowtail (resp. a cuspidal cross cap) at \( p \in \overline{U} \) if and only if its conjugate maxface \( \tilde{f} \) is \( \mathcal{A} \)-equivalent to a cuspidal cross cap (resp. a swallowtail) at \( p \in \overline{U} \).

**Remark.** The same assertion holds for CMC-1 faces in de Sitter 3-space, see Corollary 2.5.

**Example 2.6.** The Lorentzian Enneper surface is a maxface \( f : \mathcal{C} \to \mathcal{L}^3 \) with the Weierstrass data \((g, \omega) = (z, dz)\) (see [UY2 Example 5.2]), whose set of singularities is \( \{ z ; |z| = 1 \} \). As pointed out in [UY2, Fact 2.2] implies that the points of the set

\[
\{ z ; |z| = 1 \} \setminus \{ \pm 1, \pm \sqrt{-1}, \pm e^{\pm \pi i/4} \}
\]

are cuspidal edges and the points \( \pm 1, \pm \sqrt{-1} \) are swallowtails. Moreover, using Theorem 2.4 we deduce that the four points \( \pm e^{\pm \pi i/4} \) are cuspidal cross caps.

By (2.4), \( \omega \) does not vanish at a singular point \( p \). Hence there exists a complex coordinate system \( z \) such that \( \omega = dz \). On the other hand, \( g \neq 0 \) at the singular point \( p \). Hence there exists a holomorphic function \( h \) in \( z \) such that \( g = e^h \). We denote by \( f_h \) the maxface defined by the Weierstrass data \((g, \omega) = (e^h, dz)\). Let \( \mathcal{O}(U) \) be the set of holomorphic functions defined on \( U \), which is endowed with the compact open \( C^\infty \)-topology. Then we have the induced topology on the set of maxfaces \( \{ f_h \}_{h \in \mathcal{O}(U)} \). We shall prove Theorem A in the introduction. To prove the theorem, we rewrite the criteria in Fact 2.2 and Theorem 2.4 in terms of \( h \).

**Lemma 2.7.** Let \( h \in \mathcal{O}(U) \) and set

\[
\alpha_h := e^{-h}h', \quad \beta_h := e^{-2h}(h'' - (h')^2),
\]

where \( \prime = d/dz \). Then

1. A point \( p \in U \) is a singular point of \( f_h \) if and only if \( \text{Re} h = 0 \),
2. A singular point \( p \) is non-degenerate if and only if \( \alpha_h \neq 0 \),
3. A singular point \( p \) is a cuspidal edge if and only if \( \text{Re} \alpha_h \neq 0 \) and \( \text{Im} \alpha_h \neq 0 \),
4. A singular point \( p \) is a swallowtail if and only if \( \text{Re} \alpha_h = 0 \), \( \text{Im} \alpha_h \neq 0 \), and \( \text{Re} \beta_h \neq 0 \),
5. A singular point \( p \) is a cuspidal cross cap if and only if \( \text{Re} \alpha_h = 0 \), \( \text{Im} \alpha_h \neq 0 \), and \( \text{Re} \beta_h \neq 0 \).

**Proof.** Since \( g = e^h \), (1) is obvious. Moreover, a singular point \( p \) is non-degenerate if and only if \( g' = e^{2h} \alpha_h \) does not vanish. Hence we have (2). Since \( g'/(g^2 \omega) = \alpha_h \), the criterion for a front (Fact 2.2) is \( \text{Re} \alpha_h \neq 0 \). Then by Fact 2.2 we have (3).

Here,

\[
\left( \frac{g'}{g^2 \omega} \right)' = e^{-h}h'' - (h')^2.
\]

Then, if \( \text{Im} \alpha_h = 0 \) and \( \alpha_h \neq 0 \),

\[
\text{Re} \left[ \frac{g'}{g^2 \omega} \right]' = \text{Re} \left[ e^{-2h}(h'' - (h')^2) \right] \frac{1}{\alpha_h} = \frac{1}{\alpha_h} \text{Re} \beta_h.
\]

Then by Fact 2.2 we have (4). On the other hand, if \( \text{Re} \alpha_h = 0 \) and \( \alpha_h \neq 0 \),

\[
\text{Im} \left[ \frac{g'}{g^2 \omega} \right]' = \text{Im} \left[ e^{-2h}(h'' - (h')^2) \right] = -\sqrt{-1} \frac{1}{\alpha_h} \text{Re} \beta_h.
\]

Thus we have (5). \( \square \)
Let $J^2_0(U)$ be the space of 2-jets of holomorphic functions on $U$, which is identified with an 8-dimensional manifold

$$J^2_0(U) = U \times C \times C \times C = U \times F \times F_1 \times F_2,$$

where $F$, $F_1$ and $F_2$ are 1-dimensional complex vector spaces corresponding to the jets $h(p)$, $h'(p)$ and $h''(p)$ for $h \in O(U)$ at $p$, respectively. Then, the canonical map $j^2_h : U \to J^2_0(U)$ is given by $j^2_h(p) = (p, h(p), h'(p), h''(p))$. The point $P \in J^2_0(U)$ is expressed as

$$(2.5) \quad P = (p, \hat{h}, \hat{h}_1, \hat{h}_2) = (p, \hat{u}, \hat{v}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2),$$

where $\hat{h} = \hat{u} + \sqrt{-1} \hat{v}$, $\hat{h}_j = \hat{u}_j + \sqrt{-1} \hat{v}_j$ ($j = 1, 2$). We set

$$A := \{ P \in J^2_0(U) ; \Re \hat{h} = 0, \hat{\alpha} = 0 \},$$

$$B := \{ P \in J^2_0(U) ; \Re \hat{h} = 0, \Im \hat{\alpha} = 0, \Re \hat{\beta} = 0 \},$$

$$C := \{ P \in J^2_0(U) ; \Re \hat{h} = 0, \Re \hat{\alpha} = 0, \Re \hat{\beta} = 0 \},$$

where

$$\hat{\alpha} = e^{-\hat{h}} \hat{u}_1, \quad \hat{\beta} = e^{-2\hat{h}} (\hat{h}_2 - (\hat{h}_1)^2).$$

**Lemma 2.8.** Let $S = A \cup B \cup C$ and

$$G := \{ h \in O(U) ; j^2_h(U) \cap S = \emptyset \}.$$

Then all singular points of $f_h$ are cuspidal edges, swallowtails or cuspidal cross caps if $h \in G$.

**Proof.** We set

$$S_A := \{ h \in O(U) ; j^2_h(U) \cap A \neq \emptyset \},$$

$$S_B := \{ h \in O(U) ; j^2_h(U) \cap B \neq \emptyset \},$$

$$S_C := \{ h \in O(U) ; j^2_h(U) \cap C \neq \emptyset \}.$$

Then we have $G = (S_A)^c \cap (S_B)^c \cap (S_C)^c$. Let $h \in G$, and let $p \in U$ be a singular point of $f_h$. Since $h \notin S_A$, $p$ is a non-degenerate singular point. If $f_h$ is a front at $p$, then $\Re \alpha_h = 0$. Since $h \notin S_C$, this implies that $\Re \beta_h \neq 0$, and hence $p$ is a cuspidal cross cap. If $f_h$ is a front at $p$ and not a cuspidal edge, $p$ is a swallowtail since $h \notin S_B$. \hfill $\square$

By this lemma, a singular point $p \in U$ is neither a cuspidal edge, a swallowtail nor a cuspidal cross cap if and only if $\hat{j}^2_h(p) \in S$. Thus if $S$ is a union of a finite number of submanifolds in $J^2_0(U)$ of codimension 3, then we can say generic singular points of $h$ consist of cuspidal edges, swallowtails and cuspidal cross caps. In fact, Theorem [A] can be proved in a similar way to Theorem 3.4 of [KRSUY] using the following lemma.

**Lemma 2.9.** $S = A \cup B \cup C$ is the union of a finite number of submanifolds in $J^2_0(U)$ of codimension 3.

**Proof.** Using parameters in (2.5), we can write

$$A = \{ \hat{u} = \hat{u}_1 = \hat{v}_1 = 0 \},$$

which is a codimension 3 submanifold in $J^2_0(U)$. Moreover, one can write

$$B = \{ \zeta_1 = 0, \zeta_2 = 0, \zeta_3 = 0 \}, \text{ where }$$

$$\zeta_1 = \hat{u},$$

$$\zeta_2 = e^{-\hat{u}} (\hat{v}_1 \cos \hat{v} - \hat{u}_1 \sin \hat{v}),$$

$$\zeta_3 = e^{-2\hat{u}} \left[ (\hat{v}_2 - (\hat{v}_1)^2)^2 \cos 2\hat{v} + (\hat{v}_2 - 2\hat{u}_1 \hat{v}_1) \sin 2\hat{v} \right].$$
Here, we can compute that

\begin{equation}
\frac{\partial (\xi_1, \xi_2, \xi_3)}{\partial (\bar{u}, \bar{u}_1, \bar{v}_1)} = 2e^{-3\hat{u}}(\bar{u}_1 \cos \hat{v} + \bar{v}_1 \sin \hat{v}).
\end{equation}

Since \((\bar{u}_1, \bar{v}_1) \neq (0, 0)\) and \( \hat{v}_1 \cos \hat{v} - \bar{u}_1 \sin \hat{v} = 0 \) hold on \(B \setminus A\), (2.6) does not vanish on \(B \setminus A\). Hence by the implicit function theorem, \(B \setminus A\) is a submanifold of codimension 3.

Similarly, \(C\) is written as

\[C = \{\xi_1 = 0, \xi_2 = 0, \xi_3 = 0\},\]

where

\[\xi_1 = \hat{u}, \quad \xi_2 = e^{-\hat{u}}(\bar{u}_1 \cos \hat{v} + \bar{v}_1 \sin \hat{v}), \]

\[\xi_3 = e^{-2\hat{u}}\left(\bar{u}_2 - (\bar{u}_1)^2 + (\bar{v}_1)^2\right) \cos 2\hat{v} + (\hat{v}_2 - 2\bar{u}_1 \bar{v}_1) \sin 2\hat{v} \right).\]

Then we have

\[\frac{\partial (\xi_1, \xi_2, \xi_3)}{\partial (\bar{u}, \bar{u}_1, \bar{v}_1)} = 2e^{-3\hat{u}}(\bar{v}_1 \cos \hat{v} - \bar{u}_1 \sin \hat{v}).\]

Thus, \(C \setminus A\) is a submanifold of codimension 3.

Hence \(S = A \cup (B \setminus A) \cup (C \setminus A)\) is a union of submanifolds of codimension 3. \(\square\)

3. Singularities of CMC-1 surfaces in de Sitter 3-space

It is well-known that spacelike CMC-1 (constant mean curvature one) surfaces in de Sitter 3-space \(S^3_1\) have similar properties to spacelike maximal surfaces in \(L^3\). In this section, we shall give an analogue of Theorem 1.4 for such surfaces. Though the assertion is the same, the method is not parallel: For maxfaces, one can easily write down the Euclidean normal vector explicitly, as well as the Lorentzian normal, in terms of the Weierstrass data. However, the case of CMC-1 surfaces in \(S^3_1\) is different, as it is difficult to express the Euclidean normal vector, and we apply Theorem 1.4 instead of Corollary 1.5 since Theorem 1.4 is independent of the metric of the ambient space.

A holomorphic map \(F: M^2 \to \text{SL}(2, C)\) of a Riemann surface \(M^2\) into the complex Lie group \(\text{SL}(2, C)\) is called null if \(\det F_z = 0\) holds on \(M^2\), where \(z\) is a local complex coordinate of \(M^2\). This condition does not depend on the choice of complex coordinates. We consider two projections of holomorphic null immersions, the former is the projection into the hyperbolic 3-space

\[p_E: \text{SL}(2, C) \ni F \mapsto FF^* \in H^3 = \text{SL}(2, C)/\text{SU}(2)\]

and the latter one is the projection into de Sitter 3-space \(S^3_1\)

\[p_L: \text{SL}(2, C) \ni F \mapsto F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F^* \in S^3_1 = \text{SL}(2, C)/\text{SU}(1, 1),\]

where \(F^* = F^*\). It is well-known that the projection of null holomorphic immersions into \(H^3\) by \(p_E\) gives conformal CMC-1 immersions (see [4] and [5]). Moreover, conformal CMC-1 immersions are always given locally in such a manner.

On the other hand, the projection of null holomorphic immersions into \(S^3_1\) by \(p_L\) gives spacelike CMC-1 surfaces with singularities, called CMC-1 faces (see [6] for details). In this section, we shall give a necessary and sufficient condition for their singular points to be cuspidal edges, swallowtails and cuspidal cross caps, and will show that CMC-1 faces admitting only these singular points are generic.

Recall that de Sitter 3-space is

\[S^3_1 = S^3_1(1) = \{ (x^0, x^1, x^2, x^3) \in L^4 : -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \},\]
with metric induced from $L^4$, which is a simply-connected 3-dimensional Lorentzian manifold with constant sectional curvature 1. We can consider $L^4$ to be the set of $2 \times 2$ Hermitian matrices $\text{Herm}(2)$ by the identification

$$L^4 \ni X = (x^0, x^1, x^2, x^3) \leftrightarrow X = \sum_{k=0}^{3} x^k e_k = \left( \begin{array}{cc} x^0 + x^3 & x^1 + \sqrt{-1} x^2 \\ x^1 - \sqrt{-1} x^2 & x^0 - x^3 \end{array} \right),$$

where

$$e_0 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad e_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad e_2 = \left( \begin{array}{cc} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{array} \right), \quad e_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$

Then $S^3_1$ is written as

$$S^3_1 = \{ X ; X^* = X, \det X = -1 \} = \{ F e_3 F^* ; F \in \text{SL}(2, \mathbb{C}) \}$$

with the metric

$$\langle X, Y \rangle = -\frac{1}{2} \text{trace} (X e_2(Y) e_2).$$

In particular, $\langle X, X \rangle = -\det X$.

The following fact is known (see [F]):

**Fact 3.1.** Let $U \subset \mathbb{C}$ be a simply connected domain containing a base point $z_0$, and $(g, \omega)$ a pair of a meromorphic function and a holomorphic 1-form on $U$ such that \[(2.1)\] holds on $U$. Then by solving the ordinary differential equation

$$F^{-1} dF = \left( \begin{array}{cc} g & -g^2 \\ 1 & -g \end{array} \right) \omega$$

with $F(z_0) = e_0$, where $z_0 \in U$ is the base point,

$$f := F e_3 F^*$$

gives a CMC-1 face in $S^3_1$. Moreover, any CMC-1 face is locally obtained in this manner.

The first fundamental form of $f$ in \[(3.2)\] is given by

$$ds^2 = (1 - |g|^2)^2 |\omega|^2.$$
Lemma 3.2. Any section $X$ of the limiting tangent bundle is parametrized as
\begin{equation}
X = F \left( \frac{\zeta g + \zeta \bar{g}}{\zeta (|g|^2 + 1)} \right) F^* \tag{3.4}
\end{equation}

for some $\zeta : U \to C$.

Proof. Let $p$ be an arbitrary point in $U$. Since $X_p \in \text{Herm}(2)$, $X_p \in T_pL^4$. Because $\langle f_p, X_p \rangle = 0$, $X_p \in T_pS_1^3$. Since $\langle \nu_p, X_p \rangle = 0$, and $\langle , \rangle$ is a non-degenerate inner product, we get the conclusion. \hfill $\square$

The above lemma will play a crucial role in giving a criterion for cuspidal cross caps in terms of the Weierstrass data.

Proposition 3.3. Let $U$ be a domain of the complex plane $(C, z)$. Let $f : U \to S_1^3$ be a CMC-1 face and $F$ a holomorphic null lift of $f$ with Weierstrass data $(g, \omega = \hat{\omega}dz)$, where $\hat{\omega}$ is a holomorphic function on $U$. Then a singular point $p \in U$ is non-degenerate if and only if $dg(p) \neq 0$.

Proof. Define $\xi \in T_{f(p)}L^4$ as
\begin{equation}
\xi := FF^* \tag{3.5}
\end{equation}
Then $\xi \in T_{f(p)}S_1^3$, because $\langle f, \xi \rangle = 0$. Define a 3-form $\Omega$ on $S_1^3$ as
\begin{equation}
\Omega(X_1, X_2, X_3) := \det(f, X_1, X_2, X_3) \tag{3.6}
\end{equation}
for arbitrary vector fields $X_1, X_2, X_3$ of $S_1^3$. Then $\Omega$ gives a volume element on $S_1^3$. Since
\[
\Omega(f_u, f_v, \xi) = \det(f, f_u, f_v, \xi)
\]
\[
= \det \begin{pmatrix} 0 & 2 \text{Re}(g\hat{\omega}) & -2 \text{Im}(g\hat{\omega}) & 1 \\ 0 & \text{Re}((1 + g^2)\hat{\omega}) & -\text{Im}((1 + g^2)\hat{\omega}) & 0 \\ 0 & -\text{Im}((1 - g^2)\hat{\omega}) & -\text{Re}((1 - g^2)\hat{\omega}) & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]
we see that
\[
d(\Omega(f_u, f_v, \nu)) = -\frac{1}{2} \left( d(g\hat{\omega})(1 + |g|^2)|\hat{\omega}|^2 - (1 - |g|^2)d((1 + |g|^2)|\hat{\omega}|^2) \right)
\]
\[
= -d(g\hat{\omega})|\hat{\omega}|^2
\]
at $p$, because $|g(p)| = 1$, proving the proposition by Proposition 1.2. \hfill $\square$

We shall now prove the following:

Theorem 3.4. Let $U$ be a domain of the complex plane $(C, z)$ and $f : U \to S_1^3$ a CMC-1 face constructed from the Weierstrass data $(g, \omega = \hat{\omega}dz)$, where $\hat{\omega}$ is a holomorphic function on $U$. Then:

1. A point $p \in U$ is a singular point if and only if $|g(p)| = 1$.
2. $f$ is $A$-equivalent to a cuspidal edge at a singular point $p$ if and only if $\text{Re} \left( \frac{g'}{g^2\hat{\omega}} \right) \neq 0$ and $\text{Im} \left( \frac{g'}{g^2\hat{\omega}} \right) \neq 0$ hold at $p$, where $' = d/dz$.
3. $f$ is $A$-equivalent to a swallowtail at a singular point $p$ if and only if
\[
\frac{g'}{g^2\hat{\omega}} \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad \text{Re} \left\{ \frac{g}{g'} \left( \frac{g'}{g^2\hat{\omega}} \right)' \right\} \neq 0
\]
hold at $p$. 

Holds at $p$. If this is the case, $p$ is a non-degenerate singular point.

Proof. Let $\nu$ be as in $(3.3)$. Then by the similar argument in the proof of Lemma 1.6, $f$ is a front at $p$ if and only if $D^{L^4}_\eta \nu$ is not proportional to $\nu$, where $\eta$ is the null direction which is given by

$$\eta = \frac{-1}{\sqrt{g\omega}} \partial_{z} - \frac{-1}{\sqrt{g\omega}} \partial_{\bar{z}}$$

at a singular point $p$, where $z$ is a complex coordinate and $\omega = \bar{\omega} dz$. Using $(3.4)$ and noticing $|g| = 1$ at $p$, we have

$$D^{L^4}_\eta \nu = \frac{-1}{\sqrt{g\omega}} \partial_{z} (F(\beta^2 F^*) - \frac{-1}{g\omega} \partial_{\bar{z}} (\beta^2 F^*))$$

$$= \frac{-1}{\sqrt{g\omega}} F (F^{-1} F_\beta \beta^2 + (\beta^2)_z F^*) - \frac{-1}{g\omega} F (\beta^2 (F^*)_z (F^*)^{-1} + (\beta^2)_z F^*)$$

$$= 2 F \left( \begin{array}{c} \text{Im} \ g\mu \\ \text{Im} \ g\bar{\mu} \end{array} \right) F^*, \quad \text{where} \quad \mu = \frac{g'}{g\omega}$$

at $p$, where $D^{L^4}$ is the canonical connection of $L^4$. On the other hand, we have

$$\langle D^{L^4}_\eta \nu, \nu \rangle = 0.$$

Here, $\langle \nu, \nu \rangle = 0$ at $p$ (that is, $\nu$ is a null vector in $L^4$). Then $D^{L^4}_\eta \nu$ is proportional to $\nu$ if and only if it is a null vector, which is equivalent to

$$0 = \det D^{L^4}_\eta \nu = 4 (\text{Im} \ g\mu)^2 - \mu \bar{\mu} = 4 (\text{Im} \ g\bar{\mu})^2 - g\bar{g} \mu \bar{\mu})$$

$$= -4 (\text{Re} \ g\bar{\mu})^2 = -4 (\text{Re} \ g\mu)^2 = -4 \left( \frac{\text{Re} L}{g} \right)^2 = -4 \left( \frac{\text{Re} g'}{g^2\omega} \right)^2,$$

because $g\bar{g} = 1$. Thus we have the conclusion. Moreover, if this is the case, $dg \neq 0$ at $p$. Then by Proposition $(3.5)$, $p$ is a non-degenerate singular point. $\square$

Lemma 3.6. Let $U$ be a domain of the complex plane $(C, z)$. Let $f : U \rightarrow S^3$ be a CMC-1 face and $F$ a holomorphic null lift of $f$ with Weierstrass data $(g, \omega = \bar{\omega})$, where $\omega$ is a holomorphic function on $U$. Let $X$ be a section of the limiting tangent bundle defined as in equation $(3.4)$. Take a singular point $p \in U$. Then

$$\bar{\psi} := \langle \nu, D^{L^4}_\eta X \rangle = 2 \text{Re} \left( \frac{g'}{g^2\omega} \right) \text{Im} (\bar{\xi} g)$$
Proof. We set

\[ T = \left( \frac{\zeta \bar{g} + \zeta g}{\zeta(g)^2 + 1}, \frac{\zeta(|g|^2 + 1)}{\zeta g + \zeta \bar{g}} \right). \]

Then \( X = FTF^* \). On the other hand, the null direction \( \eta \) is given as (3.8) at \( p \).

Thus

\[ D^L_\eta X = \frac{\sqrt{-1}}{g^\omega} F(F^{-1}F_zT + T_z)F - \sqrt{-1} \frac{g^\omega}{F} F(F^{-1}F_zT)^* + T_z)F^*. \]

Since \( \bar{g} = g^{-1} \) at any singular point \( p \), and by (3.11), we see that

\[ \sqrt{-1} \frac{1}{g^\omega} F(F^{-1}F_zT)F^* = -\sqrt{-1} \frac{g^\omega}{g^\omega} F(F^{-1}F_zT)^* F^* = \sqrt{-1}(\zeta \bar{g} - \zeta g)F \left( \begin{array}{c} 1 \\ \bar{g} \\ \bar{g} \end{array} \right) F^*. \]

Thus

\[ \langle D^L_\eta X, \nu \rangle = \langle \sqrt{-1} \frac{1}{g^\omega} F \bar{T}_z F^* - \sqrt{-1} \frac{g^\omega}{F} F \bar{T}_z F^*, \nu \rangle. \]

Since

\[ \langle \sqrt{-1} \frac{1}{g^\omega} F \bar{T}_z F^*, \nu \rangle = -\frac{1}{2} \text{trace} \left[ \sqrt{-1} \frac{g'}{2g^\omega} \left( \begin{array}{c} \zeta \bar{g} - \zeta g \\ \zeta \bar{g} - \zeta g \end{array} \right) \left( \begin{array}{c} \bar{g} \\ -\bar{g} \\ 1 \end{array} \right) \right] \]

and

\[ \langle \sqrt{-1} \frac{g'}{2g^2\omega} \bar{T}_z F^*, \nu \rangle = -\frac{1}{2} \text{trace} \left[ \sqrt{-1} \frac{g'}{2g^\omega} \left( \begin{array}{c} \zeta \bar{g} - \zeta g \\ \zeta \bar{g} - \zeta g \end{array} \right) \left( \begin{array}{c} \bar{g} \\ -\bar{g} \\ 1 \end{array} \right) \right] \]

we have

\[ \tilde{\psi} = \langle D^L_\eta X, \nu \rangle = 2 \text{Re} \left( \frac{g'}{g^\omega} \frac{g'}{2g^2\omega} \right) \text{Im}(\zeta g), \]

proving the lemma.

\[ \square \]

Now assume that \( X \) defined as in (3.14) satisfies (2) in Theorem 1.3. Then by the definition of \( X \), \( \text{Im}(\zeta g) \) cannot be zero at a singular point.

Proof of Theorem 1.4. Since the criteria for cuspidal edges and swallowtails are described intrinsically, and the first fundamental form of \( f \) is the same as in the case of maxfaces, so the assertions (1) and (2) are parallel to the case of maxfaces in \( L^3 \). So it is sufficient to show the last assertion: Let \( \gamma \) be the singular curve with \( \gamma(0) = p \). Since the induced metric \( ds^2 \) is in the same form as for the maxface case, we can parametrize \( \gamma \) as (3.9):

\[ \dot{\gamma}(t) = \sqrt{-1} \left( \frac{g'}{g} \right)(\gamma(t)), \]

where \( \dot{\cdot} = \frac{d}{dt} \).

On the other hand, the null direction is given as in (3.8). Assume \( X \) satisfies (2) in Theorem 1.4. Then the necessary and sufficient condition for a cuspidal cross...
cap is $\bar{\psi} = 0$ and $d\bar{\psi}/dt \neq 0$, by Theorem 1.4. Thus, Lemma 3.6 implies the last assertion, since
\[
\left. \frac{d}{dt} \right|_{t=0} \langle DL^* X, \nu \rangle = 2 \operatorname{Im}(\bar{\zeta} g) \operatorname{Re} \left[ \left( \frac{g'}{(g^2 \omega)} \right)' \frac{d\gamma}{dt} \right] \\
= -2 \operatorname{Im}(\bar{\zeta} g) \operatorname{Im} \left[ \left( \frac{g'}{(g^2 \omega)} \right)' \frac{g'}{g} \right] \\
= -2 |g|^2 \operatorname{Im}(\bar{\zeta} g) \operatorname{Im} \left[ \left( \frac{g'}{(g^2 \omega)} \right)' \frac{g'}{g} \right].
\]

We take a holomorphic function $h$ defined on a simply connected domain $U \subset \mathbb{C}$. Then there is a CMC-1 face $f_h$ with Weierstrass data $(g, \omega) = (e^h, dz)$, where $z$ is a complex coordinate of $U$. Let $O(U)$ be the set of holomorphic functions on $U$, which is endowed with the compact open $C^\infty$-topology. Since the criteria for cuspidal edges, swallowtails and cuspidal cross caps in terms of $(g, \omega)$ are exactly the same as in the case of maxfaces, we have the following:

**Corollary 3.7.** Let $U \subset \mathbb{C}$ be a simply connected domain and $K$ an arbitrary compact set, and let $S(K)$ be the subset of $O(U)$ consisting of $h \in O(U)$ such that the singular points of $f_h$ are cuspidal edges, swallowtails or cuspidal cross caps. Then $S(K)$ is an open and dense subset of $O(U)$.

As the same as case of maxfaces, the conjugate CMC-1 face $\tilde{f}$ of a CMC-1 face $f$ is defined by the Weierstrass data $(g, \sqrt{-1} \omega)$, where $(g, \omega)$ is the Weierstrass data of $f$.

**Corollary 3.8** (A duality between swallowtails and cuspidal cross caps). A CMC-1 face $f : U \to S^3_1$ is $A$-equivalent to a swallowtail (resp. a cuspidal cross cap) at $p \in U$ if and only if its conjugate CMC-1 face $\tilde{f}$ is $A$-equivalent to a cuspidal cross cap (resp. a swallowtail) at $p \in U$.

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(Shoichi Fujimori) Department of Mathematics, Fukuoka University of Education, Munakata, Fukuoka 811-4192, Japan
E-mail address: fujimori@fukuoka-edu.ac.jp

(Kentaro Saji) Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: saji@math.sci.hokudai.ac.jp

(Masaaki Umehara) Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: umehara@math.sci.osaka-u.ac.jp

(Kotaro Yamada) Faculty of Mathematics, Kyushu University, Higashi-ku, Fukuoka 812-8581, Japan
E-mail address: kotaro@math.kyushu-u.ac.jp