On superintegrable systems closed to geodesic motion.

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Abstract

In this work we consider superintegrable systems in the classical $r$-matrix method. By using other automorphisms of the loop algebras we construct new superintegrable systems with rational potentials from geodesic motion on $\mathbb{R}^{2n}$.

1 Introduction

We shall consider classical integrable hamiltonian systems on the coadjoint orbits of finite-dimensional Lie algebras according to [1]. The dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ is equipped with the natural Lie-Poisson brackets specified by the condition that the Poisson bracket of two linear functions on $\mathfrak{g}^*$ coincides with their Lie bracket in $\mathfrak{g}$. Let $H$ be a function on $\mathfrak{g}^*$, $\nabla H \in \mathfrak{g}$ the gradient of $H$. In the space $C^\infty(\mathfrak{g}^*)$ of smooth function $H$ determines the evolution with the associated hamiltonian equation

$$\dot{x} = -(\text{ad}^*_{\nabla H}) \cdot x, \quad x \in \mathfrak{g}^*. \quad (1.1)$$

If $\mathfrak{g}$ is self-dual, i.e. has a nondegenerate inner product which allows to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and $\text{ad}^*$ with $\text{ad}$, then (1.1) takes on the usual form

$$\dot{x} = \{H, x\} = -(\text{ad}_{\nabla H}) \cdot x, \quad x \in \mathfrak{g}. \quad (1.2)$$

Henceforth we shall always to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and $\text{ad}^*$ with $\text{ad}$.

Function $I \in C^\infty(\mathfrak{g})$ is called an integral of evolution with a hamiltonian $H$ if

$$\{H, I\} = 0. \quad (1.3)$$

The evolution on a $2n$-dimensional symplectic manifold $M$ with the hamiltonian $H$ is called completely integrable if there exists $n$ functions $I_1, \ldots, I_n$, which are independent integrals in the involution for the hamiltonian $H$

$$\{I_i, I_j\} = 0, \quad i, j \leq n. \quad (1.3)$$

The functions $I_1, \ldots, I_n$ are independent, if forms $dI_1, \ldots, dI_n$ are linearly independent on the common level surfaces of these functions.

The evolution on a manifold $M$, $\dim M = 2n$ with the hamiltonian $H$ is called superintegrable or degenerate, if there exists more than $n$ independent integrals of motion $\{I_j\}_{j=1}^k$, $k > n$ and $n$ of which are in the involution (1.3) [1, 18, 25]. For the superintegrable systems all the integrals $\{I_j\}_{j=1}^k$, $k > n$ are generators of the polynomial associative algebra, whose defining relations are polynomials of certain order in generators (see [1, 5] for a collection of
The main example of the superintegrable systems is a free motion with the following Hamiltonian and equations of motion

\[ H = \sum_{j=1}^{n} p_j^2, \quad \dot{q}_j = p_j, \quad \dot{p}_j = 0, \]  

(1.4)

here \( p_j, q_j \) are canonical variables on \( M = \mathbb{R}^{2n} \). Integrals of motion in the involution and additional integrals of motion may be defined as

\[ I_k = f(p_1, \ldots, p_n), \quad I_{jk} = p_j q_k - q_j p_k. \]  

(1.5)

Other known classical superintegrable systems with an arbitrary number of degrees of freedom are the harmonic oscillator, the Kepler problem and the Calogero system \([21]\). Notice, that the Kepler problem and the Calogero model may be obtained from the geodesic motion (1.4) on spaces of constant curvature \([21]\). Another examples of superintegrable systems can be constructed by using either purely algebraic techniques \([1, 4, 5, 18]\) or separation of variables method at \( n = 2, 3 \) \([23, 4, 8]\). Some individual examples of superintegrable systems are listed in \([21]\) with the corresponding references.

Our aim is to show how superintegrable systems fit into a general pattern based on the notion of the classical \( r \)-matrix \([11, 22]\). The main advantage of such embedding is that important structure elements for superintegrable systems, such as a Lax representation, separation of variables \([3]\) and spectrum-generating algebra (dynamical algebra) \([4, 19]\) can be systematically derived from the underlying standard \( r \)-matrix formalism, which is a prerequisite for the study of the quantum case.

We propose a dressing procedure allowing to construct the new superintegrable systems starting from known ones. As a natural initial point we shall select a geodesic motion (1.4) on the Riemannian spaces of constant curvature. The Lax representations for these superintegrable geodesic motion are known \([16]\). To construct the new Lax equations associated to a potential superintegrable motion we apply the outer automorphism of the corresponding loop algebras \([26]\) directly to the Lax equations associated to a geodesic motion.

The paper is organized as follows. In Section 2 we briefly recall notion of classical \( r \)-matrix method. By use a triangular decomposition of semi-simple Lie algebras, in Section 3, the algebraic approach to superintegrable systems is proposed and this scheme is applied to several examples. In Section 4 superintegrable systems are constructed in \( r \)-matrix formalism, while Section 5 contains some examples.

## 2 Method of the classical \( r \)-matrix

A systematic way for realizing integrable Hamiltonian system on coadjoint orbits of the Lie algebras is provided by the \( r \)-matrix method \([11, 22]\).

Recall that the classical \( r \)-matrix on a Lie algebra \( \mathfrak{g} \) is a linear operator \( R \in \text{End}(\mathfrak{g}) \) such that the bracket on \( \mathfrak{g} \)

\[ [X, Y]_R = \frac{1}{2} ([RX, Y] + [X, RY]), \quad X, Y \in \mathfrak{g}, \]  

(2.1)

satisfies the Jacobi identity \([22, 23]\). In this case there are two structures of a Lie algebra on the linear space \( \mathfrak{g} \) given by original Lie bracket and by the \( r \)-bracket (2.1), respectively. The Casimir functions \( \tau_j \) on \( \mathfrak{g}^* \) invariant with respect to the original Lie structure are in the
involution with respect to the $r$-bracket. If $\tau$ is an invariant function on $\mathfrak{g}^*$, the associated hamiltonian equation (1.1) on $\mathfrak{g}^*$ is equal to

$$\frac{dL}{dt} = -\text{ad}^*_A \cdot L, \quad A = \frac{1}{2} R(d\tau(L)), \quad L \in \mathfrak{g}^*. \quad (2.2)$$

If $\mathfrak{g}$ is self-dual, then (2.2) takes on the usual Lax form [22].

It is obvious, that for any $r$-matrix in (2.2) all the Casimir functions give rise to integrals of motion in the involution [11, 22]. We have to find the origin of an appearance of the special superintegrable hamiltonians and their additional integrals of motion. Application of the ad-invariant functions $\tau_j$ is a basic tool in the $r$-matrix method [11, 22] and, therefore, we consider these functions in greater detail to assume the standard identification of the dual spaces.

To begin with let us recall some necessary facts from the notion of a universal enveloping algebra [7]. Let $\mathfrak{g}$ be a Lie algebra and $T(\mathfrak{g})$ be the tensor algebra of the vector space $T = T^0 \oplus T^1 \oplus T^2 \ldots$, $T^m = \mathfrak{g} \otimes \mathfrak{g} \otimes \ldots \otimes \mathfrak{g}$ $n$ times. (2.3)

If $J$ be the two-sided ideal of $T$ generated by the tensors $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathfrak{g}$, then the associative algebra $T/J$ is termed the universal enveloping algebra, which is usual denoted by $U(\mathfrak{g})$.

Let $m \geq 0$ be an integer. The vector subspace of $U(\mathfrak{g})$ generated by the products $x_1 x_2 \cdots x_j$, where $x_1, x_2, \ldots, x_j \in \mathfrak{g}$ and $j \leq m$ is denoted by $U_m(\mathfrak{g})$. We have

$$U_0(\mathfrak{g}) = \mathbb{C} \cdot 1, \quad U_1(\mathfrak{g}) = \mathbb{C} \cdot 1 \oplus \mathfrak{g}, \quad U_i(\mathfrak{g})U_j(\mathfrak{g}) \subset U_{i+j}(\mathfrak{g}).$$

This sequence is termed the canonical filtration of $U(\mathfrak{g})$.

According by the Birkhoff-Witt theorem $T(\mathfrak{g}) = J \oplus S(\mathfrak{g})$ and algebra $U(\mathfrak{g})$ is isomorphic to the symmetric algebra $S(\mathfrak{g})$ as a vector space. If $x_1, x_2, \ldots, x_m \in \mathfrak{g}$, then

$$w(x_1 x_2 \cdots x_m) = \frac{1}{m!} \sum_{\pi} P_{\pi} x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(m)}, \quad (2.4)$$

here $P_{\pi}$ means the permutation operator corresponding to a certain Young diagram $\pi$ [7]. The map $w$ (2.4) is a bijection of $S(\mathfrak{g})$ onto $U(\mathfrak{g})$, which is called the symmetrization.

The Casimir functions or the ad-invariant functions on $\mathfrak{g}$ form a center $I(\mathfrak{g})$ of $U(\mathfrak{g})$. The symmetrization mapping (2.4) allows us to construct $I(\mathfrak{g})$ by using the ad-invariants of commutative algebra $S(\mathfrak{g})$ [7].

The most interesting class of examples in the classical $r$-matrix method is provided by loop algebra $\mathcal{L}(\mathfrak{g}, \lambda)$ [22]. Algebra $\mathcal{L}(\mathfrak{g}, \lambda)$ can be realized as an algebra of the Laurent polynomials with coefficients in $\mathfrak{g}$

$$\mathcal{L}(\mathfrak{g}, \lambda) = \mathfrak{g}[\lambda, \lambda^{-1}] = \left\{ x(\lambda) = \sum_i x^i \lambda^i, \quad x \in \mathfrak{g} \right\}$$

and with the commutator $[x \lambda^i, y \lambda^j] = [x, y] \lambda^{i+j}$.

According to an algebra homomorphism [13]

$$U(\mathcal{L}(\mathfrak{g}, \lambda)) \to \mathbb{C}[\lambda, \lambda^{-1}] \otimes U(\mathfrak{g})$$
the Casimir functions on the loop algebras can be recovered by the ad-invariants \( \tau_j(x) \) in \( I(\mathfrak{g}) \)

\[
\tau_{j,\phi} = \text{Res}_{\lambda=0} \phi(\lambda) \cdot \tau_j(x(\lambda)), \quad \tau_j(x) \in I(\mathfrak{g}), \quad \phi(\lambda) \in \mathbb{C}[\lambda, \lambda^{-1}],
\]

(2.5)

where \( \phi(\lambda) \) is some rational function on spectral parameter \( \lambda \) with numerical values \( \mathbb{Z}^2 \).

Studying the superintegrable systems we have to introduce the usual tensor algebra

\[
T(\mathfrak{g}, \lambda, \mu, \ldots) = T^0 \oplus T^1 \oplus T^2 \ldots,
\]

\[
T^m(\mathfrak{g}, \lambda, \mu, \ldots, \nu) = \mathcal{L}(\mathfrak{g}, \lambda) \otimes \mathcal{L}(\mathfrak{g}, \mu) \otimes \ldots \otimes \mathcal{L}(\mathfrak{g}, \nu) - m - \text{times},
\]

(2.6)

and canonical filtration of the corresponding enveloping algebra \( U(\mathfrak{g}, \lambda, \mu, \ldots) \) generated by subspaces \( U_m(\mathfrak{g}, \lambda, \mu, \ldots, \nu) \). These vector subspaces are produced by subspaces \( U_m(\mathfrak{g}) \)

\[
x_{i_1i_2\ldots i_k}(\lambda, \mu, \ldots, \nu) = \sum_{j_1j_2\ldots j_k} x_{i_1i_2\ldots i_k} \lambda^{j_1} \mu^{j_2} \ldots \nu^{j_k}, \quad k \leq m, \quad x_{i_1i_2\ldots i_k} \in U_m(\mathfrak{g}).
\]

In just the same way as for a Lie algebra \( \mathfrak{g} \) [7], we can define the canonical mapping of the loop algebra \( \mathcal{L}(\mathfrak{g}, \lambda) \) into \( U(\mathfrak{g}, \lambda, \mu, \ldots) \). Any element \( L(\lambda) \) of \( \mathcal{L}(\mathfrak{g}, \lambda) \) can be embedded into \( U_m(\mathfrak{g}, \lambda, \mu, \ldots, \nu) \)

\[
L_j(\lambda_j) = id_1 \otimes \cdots \otimes id_{j-1} \otimes L(\lambda_j) \otimes id_{j+1} \otimes \cdots \otimes id_m \in U_m(\mathfrak{g}, \lambda, \mu, \ldots, \nu).
\]

(2.7)

and

\[
L^{(k)}_{j_1j_2\ldots j_k}(\lambda_1, \lambda_2, \ldots, \lambda_k) = \prod_{n=1}^{k} L_{j_n}(\lambda_{j_n}), \quad 1 \leq k \leq m
\]

(2.8)

\[
L^{(m)}(\lambda, \mu, \ldots, \nu) = L(\lambda) \otimes L(\mu) \otimes \cdots \otimes L(\nu), \quad - m - \text{times}.
\]

(2.9)

here \( \lambda_1 = \lambda, \lambda_2 = \mu, \ldots \lambda_m = \nu \).

If \( \mathfrak{g} \) is identified with its dual, then \( r \)-bracket [2.1] can be rewritten in the tensor form [11, 22, 23]. Let \( L(\lambda) \in \mathcal{L}(\mathfrak{g}, \lambda) \) be a generic point in the loop algebra, which is regarded as a Lax matrix. The corresponding \( r \)-bracket is given by

\[
\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] - [r_{21}(\lambda, \mu), L_2(\mu)],
\]

(2.10)

\[
r_{21}(\lambda, \mu) = P_{12} r_{12}(\lambda, \mu) P_{12},
\]

where \( P_{12} \) is a permutation operator in \( \mathcal{L}(\mathfrak{g}, \lambda) \otimes \mathcal{L}(\mathfrak{g}, \mu) \) and \( r_{ij}(\lambda, \mu) \) are kernels of the corresponding operators \( R \) and \( R^* \) in [2.1, 2]. Notice, that the \( r \)-matrix scheme is extended easily to the twisted subalgebras of loop algebra \( \mathcal{L}(\mathfrak{g}, \lambda) \) and the corresponding matrices \( r_{12} \)

have rational, trigonometric and elliptic dependence on spectral parameter.

The Poisson brackets between the elements \( L^{(k)}_{j_1j_2\ldots j_k}(\lambda, \mu, \ldots, \nu) \) (2.8) can be written in the "generalized" \( r \)-matrix form. For instance

\[
\{L_{12}(\lambda, \mu), L_3(\nu)\} = [r_{13}(\lambda, \nu) + r_{23}(\mu, \nu), L_{12}(\lambda, \mu)]
\]

(2.11)

\[
- \quad [r_{31}(\lambda, \nu), L_{23}(\mu, \nu)] - [r_{23}(\mu, \nu), L_{13}(\lambda, \nu)],
\]

where \( r_{ij}(\lambda_i, \lambda_j) \) are \( r \)-matrices, which act nontrivially in the corresponding subspaces of \( T_m(\mathfrak{g}, \lambda, \mu, \ldots, \nu) \) [2, 6] and

\[
\{L_{12}(\lambda, \mu), L_{34}(\nu, \eta)\} = [r^{(1)}(\lambda, \mu, \nu, \eta), L_1(\lambda)] + [r^{(2)}(\lambda, \mu, \nu, \eta), L_2(\mu)]
\]

(2.12)

\[
- \quad [r^{(3)}(\lambda, \mu, \nu, \eta), L_3(\nu)] - [r^{(4)}(\lambda, \mu, \nu, \eta), L_4(\eta)],
\]
where

\[
\begin{align*}
    r^{(1)}(\lambda, \mu, \nu, \eta) &= \tilde{r}(\lambda, \mu, \nu, \eta) + P_{34}\tilde{r}(\lambda, \mu, \eta, \nu)P_{34}, \\
    r^{(2)}(\lambda, \mu, \nu, \eta) &= P_{12}r^{(1)}(\mu, \lambda, \nu, \eta)P_{12}, \\
    r^{(3)}(\lambda, \mu, \nu, \eta) &= P_{13}\tilde{r}(\lambda, \mu, \nu, \eta) + P_{12}\tilde{r}(\mu, \lambda, \nu, \eta)P_{12}P_{13}, \\
    r^{(4)}(\lambda, \mu, \nu, \eta) &= P_{34}r^{(3)}(\lambda, \mu, \eta, \nu)P_{34}, \\
    \tilde{r}(\lambda, \mu, \nu, \eta) &= r^{13}(\lambda, \nu)L_{24}(\mu, \eta).
\end{align*}
\]

Here \(P_{ij}\) are operators of pairwise permutations in the tensor algebra \(T_m(\mathfrak{g}, \lambda, \mu, \ldots, \nu)\) (2.6).

Integrals of motion in the involution are completely defined by the ad-invariants (2.5) of \(L(\mathfrak{g}, \lambda)\) and by the linear \(r\)-bracket (2.10) [11, 22]. For description of the superintegrable systems we have to consider ad-invariants of \(U(\mathfrak{g}, \lambda, \mu, \ldots)\) and more complicated embedding \(L^{(k)}_{\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_k}}(\lambda, \mu, \ldots, \nu)\) (2.8–2.9). In the next section we consider several examples, that allows us to understand the origin of the appearance of additional integrals of motion in the classical \(r\)-matrix method.

### 3 One class of superintegrable systems

Let \(\mathfrak{g}\) be a simple Lie algebra with a Cartan subalgebra \(\mathfrak{b}\) and a system of simple roots \(\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}\). Let

\[
\begin{align*}
    e_i &= e_{\alpha_i}, \quad h_i = h_{\alpha_i}, \quad f_i = e_{-\alpha_i}, \quad i = 1 \ldots n,
\end{align*}
\]

be the Chevalley generators and \(\{h_i, e_i, f_i\}\) be a Cartan-Weil basis \([6]\), normalized by \((e_i, f_i) = 1:

\[
\begin{align*}
    [h_i, h_j] &= 0, \quad [e_i, f_j] = \delta_{ij}h_i, \\
    [h_i, e_j] &= c_{ij}e_j, \quad [h_i, f_j] = -c_{ij}f_j, \\
    (\text{ad} e_i)^{-c_{ij}+1} \cdot e_j &= 0, \quad (\text{ad} f_i)^{-c_{ij}+1} \cdot f_j = 0.
\end{align*}
\]

Here \(c_{ij}\) are entries of the Cartan matrix, which are integers in this normalization.

Let \(\mathfrak{n}_+ (\mathfrak{n}_-)\) be the linear span of root vectors \(e_j (f_j)\) such that

\[
\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b} \oplus \mathfrak{n}_+.
\]

This decomposition is termed the triangular decomposition of \(\mathfrak{g}\).

Every basis in the commutative subalgebra \(U_m(\mathfrak{b}) \subset U_m(\mathfrak{g})\) could be associated to the family of functionally independent integrals in the involution. Let us define a hamiltonian \(H\) as a function on alone generator \(h_i\) of the Cartan subalgebra \(\mathfrak{b}\)

\[
H = H(h_i), \quad h_i \in \mathfrak{b}, \quad H \in U_m(\mathfrak{b}).
\]

It is immediately seen that the \(n - 1\) functionally independent integrals in the involution may be constructed for the hamiltonian \(H\) (3.3) in \(U_m(\mathfrak{b})\)

\[
I_j = I_j(h_1, \ldots, h_n), \quad h_i \in \mathfrak{b}, \quad I_j \in U_m(\mathfrak{b}),
\]

here \(n = \dim \mathfrak{b} = \text{rank} \mathfrak{g}\).
Moreover, we can introduce another $n(n - 1)$ integrals of evolution in $U(n_\pm \oplus n_+)$

$$I_{jk}^l = e_j^l \cdot J_k^m \in U_p(n_\pm \oplus n_+), \quad \frac{l}{m} = \frac{c_{ik}}{c_{ij}}, \quad p = \max \{l, m\}, \quad (3.5)$$

$$\{H(h_i), I_{jk}^l \} = (lc_{ij} - mc_{ik}) \frac{dH}{dh_i} \cdot I_{jk}^l = 0,$$

where $l \geq 1$ and $m \geq 1$ in monomials $I_{jk}^l$ are positive integers in normalization $\{3.1\}$.

Let $p \geq 1$ be a largest power of monomials $I_{jk}^l$, $j, k = 1, \ldots, n$. Taking $\{3.1\}$ into account we observe that other monomials in $U_p(\mathfrak{g})$

$$T_{j_1j_2j_3}^{k_1k_2k_3} = h_{k_1}^{j_1} e_{k_2}^{j_2} f_{k_3}^{j_3}, \quad j_1 + j_2 + j_3 \leq p, \quad (3.6)$$

belong to the generalized dynamical algebra $[4, 5]$ defined as

$$\{H(h_i), T_{j_1j_2j_3}^{k_1k_2k_3} \} = \left( j_2c_{ik_2} - j_3c_{ik_3} \right) \frac{dH}{dh_i} \cdot T_{j_1j_2j_3}^{k_1k_2k_3}. \quad (3.7)$$

In this case the generalized dynamical algebra $[4, 5]$ of $H$ $\{3.3\}$ generated by elements $\{3.7\}$ coincides with the polynomial subalgebra $U_p(\mathfrak{g})$. Operators

$$Q_{j_1j_2j_3}^{k_1k_2k_3} = \left( \frac{dH}{dh_i} \right)^{-1} T_{j_1j_2j_3}^{k_1k_2k_3}, \quad (3.8)$$

are counterparts of the ladder operators of hamiltonian $H$ $\{3.3\}$ in quantum mechanics. If the hamiltonian $H$ is a linear function on generators, the spectrum-generating algebra $[4, 19]$ generated by elements $\{3.8\}$ coincides with the polynomial subalgebra $U_p(\mathfrak{g})$.

The realization of a proposed scheme for a given hamiltonian is a quite difficult task, which is not always dictated by an a priori obvious procedure of the searching of a necessary algebra $\mathfrak{g}$. Moreover, the different algebras $\mathfrak{g}$ could be associated to the single superintegrable system.

On the other hand, taking some fixed algebra $\mathfrak{g}$, we have to introduce a suitable representation for a realization of superintegrable system in canonical variables on it’s coadjoint orbits.

Let us take the $n$ three-dimensional Heisenberg algebras $\mathfrak{h}_j$ ($\mathfrak{g} = \oplus \mathfrak{h}_j$) having basis $(h_j, e_j, f_j)$ such that $[e_j, f_j] = h_j$ and $[h_j, e_j] = [h_j, f_j] = 0$. The center of nilpotent algebra $\mathfrak{h}_j$ is $\mathbb{C} \cdot h_j$. The corresponding universal enveloping algebra $U(\mathfrak{g} = \oplus \mathfrak{h}_j)$ is a Weil algebra $W_n$. Polynomial subalgebra $U_2(\mathfrak{g})$ of $W_n$ is well-known in mathematics $[4, 8]$ and physics $[4, 9, 19]$. It consists of quadratic generators $T_{ij}$, $P_{ij}$, $S_{ij}$ and linear generators $(h_j, e_j, f_j)$

$$T_{ij} = e_i f_j, \quad S_{ij} = e_i e_j, \quad P_{ij} = f_i f_j, \quad (3.9)$$

where we do not consider polynomials in elements of the center of $\mathfrak{h}_j$.

The non-vanishing polynomial Poisson brackets in $U_2(\mathfrak{g})$ are

$$\{e_i, T_{jm} \} = h_i \delta_{im} e_j, \quad \{e_i, P_{jm} \} = h_i (\delta_{ij} f_m + \delta_{im} f_j),$$
$$\{f_i, T_{jm} \} = -h_i \delta_{ij} f_m, \quad \{f_i, S_{jm} \} = -h_i (\delta_{ij} e_m + \delta_{im} e_j),$$
$$\{T_{ij}, T_{lm} \} = (\delta_{lm} h_i T_{ij} - \delta_{il} h_j T_{im}), \quad (3.10)$$
$$\{T_{ij}, S_{lm} \} = -h_j (\delta_{jm} S_{il} + \delta_{il} S_{jm}), \quad \{T_{ij}, P_{lm} \} = h_i (\delta_{im} P_{jl} + \delta_{il} P_{jm}),$$
$$\{S_{ij}, P_{lm} \} = h_j (\delta_{jm} T_{il} + \delta_{il} T_{jm}) + h_i (\delta_{im} T_{jl} + \delta_{il} T_{jm}),$$
The coadjoint orbits of the Heisenberg algebras are planes $h_j = \alpha_j \in \mathbb{C}$. On the fixed orbit of the algebra $\mathfrak{g} = \oplus \mathfrak{h}_j$ we can reinterpret the brackets (3.10) not as polynomial brackets in generators $\mathfrak{g}$, but rather as linear brackets between generators $T_{ij}$, $P_{ij}$, $S_{ij}$ and $e_j$, $f_j$ with the structure constants $h_j = \alpha_j$. It allows us to identify representations of $U_2(\mathfrak{g})$ with representation of the known Lie algebras. If we consider $n$ equivalent orbits of nilpotent algebra $\mathfrak{h}$ ($\alpha_i = \alpha_j$), the complete algebra $U_2(\mathfrak{g})$ is isomorphic to the semi-direct product $W_n \otimes sp(2n)$ [1, 2]. Subalgebra of generators $T_{ij}$ is isomorphic to the algebra $u(n)$ or, after separation of Casimir operator $\sum T_{kk}$, to the algebra $su(n)$.

The $n$-dimensional commutative ideal in $U_2(\mathfrak{g})$ (the Cartan subalgebra) is generated by elements $\tilde{h}_j = e_j f_j = T_{jj}$. If we introduce the hamiltonian $H$ by the rule (3.3) on this $n$-dimensional ideal, then another generators of $U_2(\mathfrak{g})$ have the following brackets with it

$$H = \sum_{i=1}^{n} \tilde{h}_j = \sum_{j=1}^{n} T_{jj}, \quad \{H, T_{ij}\} = (h_i - h_j)T_{ij},$$
$$\{H, e_j\} = -h_j e_j, \quad \{H, f_j\} = h_j f_j, \quad (3.11)$$
$$\{H, S_{ij}\} = -(h_i + h_j)S_{ij}, \quad \{H, P_{ij}\} = (h_i + h_j)P_{ij}.$$  

The hamiltonian $H$ is a superintegrable hamiltonian in $U_2(\mathfrak{g})$ on the following coadjoint orbit of $\mathfrak{g} = \oplus \mathfrak{h}_j$

$$h_i = h_j, \quad i, j = 1, \ldots, n.$$  

(3.12)

Note, that we can select different $n$-dimensional commutative ideals in the various subalgebras $U_m(\mathfrak{g})$ to construct superintegrable systems (3.3-3.5) with some fixed hamiltonian (3.3).

The $n$-dimensional harmonic oscillator in the euclidean coordinates is described by the hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{n} \left( p_j^2 + \alpha_j^2 q_j^2 \right).$$  

(3.13)

The phase space of oscillator (3.13) can be identified with the coadjoint orbits of Heisenberg algebras according to

$$e_j = a_j = \frac{1}{\sqrt{2}} (p_j - i\alpha_j q_j),$$
$$f_j = a_j^+ = \frac{1}{\sqrt{2}} (p_j + i\alpha_j q_j),$$
$$h_j = i\alpha_j,$$  

(3.14)

where $a_j, \quad a_j^+$ are the standard creation and annihilation operators. In this realization, on the orbit (3.12) hamiltonian $H$ (3.13) of isotropic oscillator is superintegrable in $U_2(\mathfrak{g})$ [1].

However, it will be useful to identify the phase space of oscillator with the coadjoint orbits of another algebras. For instance, if the phase space of oscillator be identified with the coadjoint orbits in $sl(2)^*$, we can introduce integrable systems isomorphic to oscillator

$$H' = \frac{1}{2} \sum_{j=1}^{n} \left( p_j^2 + \alpha_j^2 q_j^2 \right) + \sum_{j=1}^{n} \beta_j q_j^2, \quad \beta_j \in \mathbb{R}$$  

(3.15)

which is known as the Smorodinsky-Winternitz system at $n = 2, 3$ [23].

Let us prove it. Consider an infinite-dimensional representation $W$ of the Lie algebra $sl(2)$ in linear space $V$ defined in the Cartan-Weil basis $\{h, e, f\}$ in End(V) equipped with the natural bracket

$$[h, e] = e, \quad [h, f] = -f, \quad [e, f] = 2h, \quad \Delta = h^2 + \frac{1}{2}(ef + fe).$$  

(3.16)
If operator $e$ is invertible in $\text{End}(V)$, then the mapping
\[
\begin{align*}
h &\rightarrow h' = h, \quad e \rightarrow e' = e, \\
f &\rightarrow f' = f + \beta e^{-1}, \quad \beta \in \mathbb{C}
\end{align*}
\] (3.17)
is an outer automorphism of the space of infinite-dimensional representations of $sl(2)$ in $V^{[2]}$. The mapping (3.17) shifts spectrum of $\Delta$ on the parameter $\beta$
\[
\Delta \rightarrow \Delta' = \Delta + \beta.
\]
We can suppose that automorphisms (3.17) defines one-parametric realization $W(\beta)$ of $sl(2)$. For instance, realization of $sl(2)$ with one free parameter $\beta$ in the classical mechanics is given by
\[
\begin{align*}
h &= \frac{qp}{2}, \quad e = \frac{q^2}{2}, \quad f = -\frac{p^2}{2} + \frac{\beta}{q^2}, \quad \Delta' = \beta,
\end{align*}
\]
(3.18)
where $(q, p)$ is a pair of canonical coordinate and momenta with the classical Poisson bracket $\{p, q\} = 1$. Motivated by realization (3.18) we present one application of the automorphism (3.17) in the theory of integrable systems. Consider a classical hamiltonian system completely integrable on the $\mathbb{R}^{2n}$ with the natural hamiltonian
\[
H = \sum_{j=1}^{n} p_j^2 + V(q_1, \ldots, q_n).
\]
Let the phase space be identified completely or partially with the $m$ coadjoint orbits in $sl(2)^*$ as (3.18). Then the mapping
\[
H \rightarrow H' = H + \sum_{j=1}^{m} \frac{\beta_j}{q_j^2}, \quad \beta_j \in \mathbb{R}
\] (3.19)
preserves the properties of integrability and separability. The list of such systems can be found in [21].

Let us take the $n$ algebras $sl(2)$ ($\mathfrak{g} = \oplus sl(2)$) having basis $(h_j, e_j, f_j)$ with the brackets
\[
\begin{align*}
[h_j, e_j] &= \alpha_j e_j, \quad [h_j, f_j] = -\alpha_j f_j, \quad [e_j, f_j] = 2\alpha_j h_j,
\end{align*}
\]
(3.20)
where $\{\alpha_j\}_{j=1}^{n} \in \mathbb{C}$ is a set of arbitrary constants.

As usual, we can define the polynomial subalgebra $U_2(\mathfrak{g})$ of the corresponding universal enveloping algebra. Here we consider a special subalgebra of $U_2(\mathfrak{g})$ consisting of quadratic operators $T_{ij} \in U_2(\mathfrak{n}_- \oplus \mathfrak{n}_+), \quad i \neq j$ and linear operators $T_{ii} \in U_1(\mathfrak{b})$
\[
T_{ij} = e_i f_j, \quad i \neq j, \quad T_{ii} = h_i.
\] (3.21)
These operators are closed under the Poisson bracket
\[
\begin{align*}
\{T_{jk}, T_{lm}\} &= 2 \left( \alpha_j \delta_{jm} - \alpha_k \delta_{lk} \right) T_{jm} T_{lk}, \quad j \neq k, l \neq m, \\
\{T_{jj}, T_{ba}\} &= \alpha_j \left( \delta_{jl} - \delta_{jm} \right) T_{lm}, \quad l \neq m, \\
\{T_{jj}, T_{kk}\} &= 0,
\end{align*}
\]
(3.22)
or, in the unit polylinear form,
\[
\begin{align*}
\{T_{jk}, T_{lm}\} &= \left[ \alpha_j \delta_{jk} \left( \delta_{jl} - \delta_{jm} \right) T_{lm} + \alpha_l \delta_{ml} \left( \delta_{kl} - \delta_{jl} \right) T_{jk} + 2 \left( 1 - \delta_{jk} \right) \left( 1 - \delta_{lm} \right) \left( \alpha_j \delta_{jm} - \alpha_k \delta_{lk} \right) T_{jm} T_{lk} \right],
\end{align*}
\]
(3.23)
As before, we introduce linear hamiltonian $H = \sum h_j = \sum T_{ij}$ in the Cartan subalgebra of $\mathfrak{g} = \oplus sl(2)$. Elements $T_{ij}$ of $U_2(n_- \oplus n_+)$ have the following brackets with the hamiltonian

$$\{H, T_{ij}\} = (\alpha_j - \alpha_i) T_{ij}.$$ 

The coadjoint orbits in $sl(2)$ algebras (3.21) are fixed by the value of the Casimir operator $\Delta$ (3.16): $\Delta_j = \alpha_j^2 \in \mathbb{C}$. On the special coadjoint orbit of $\mathfrak{g} = \oplus sl(2)$ determined by $\alpha_i = \alpha_j$, $i, j = 1, \ldots, n$ the linear hamiltonian $H$ (3.3) has additional integrals of evolution $T_{ij}$ (3.21).

The phase space of oscillator can be identified with the coadjoint orbit of $\mathfrak{g}$ by using the representation (3.18) and the similar transformation

$$\left( \begin{array}{cc} h_j & e_j \\ f_j & -h_j \end{array} \right) = (\alpha_j U_j)^{-1} \left( \begin{array}{cc} h_j' & e_j' \\ f_j' & -h_j' \end{array} \right) U_j, \quad U_j = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} i\alpha_j & -1 \\ \alpha_j & -i \end{array} \right).$$

Matrix elements $\{h_j', e_j', f_j'\}$ in (3.23) are generators of $sl(2)$ in representation (3.13). More explicitly

$$
\begin{align*}
  e_j &= {1 \over 4}(p_j^2 - \alpha_j^2 q_j^2 + 2i\alpha_j p_j q_j) = \frac{1}{2}(a_j^+)^2, \\
  f_j &= {1 \over 4}(p_j^2 - \alpha_j^2 q_j^2 - 2i\alpha_j p_j q_j) = \frac{1}{2}a_j^2, \\
  h_j &= -{i \over 4}(p_j^2 + \alpha_j^2 q_j^2) = -{i \over 2}N_j,
\end{align*}
$$

where $\{a_j, a_j^+, N_j\}_{j=1}^n$ are the standard operators (3.14). Hamiltonian $H = 2i \sum T_{jj}$ of oscillator (3.13) or of the Smorodinsky-Winternitz system (3.15) and integrals of motion $T_{ij}$ (3.21) are at most second order polynomials in generators of $sl(2)^*$. If the ratio of frequencies $\alpha_j/\alpha_k = \lambda/m$ is rational, then the hamiltonian of oscillator (3.13) has additional integrals of motion in $U_p(\mathfrak{g})$ (3.5) [1]. The corresponding dynamical algebra coincides with $U_p(\mathfrak{g})$. For instance, the anisotropic oscillator in two dimensions with a $2:1$ ratio of the frequencies and with a third order polynomial algebra $U_3(\mathfrak{g})$ has been considered in [5].

So, we can see that for the superintegrable hamiltonian $H$ (3.3) in $U_m(\mathfrak{b})$ the additional integrals of motion belong to $U_m(n_- \oplus n_+)$.

In the next section, motivated by the presented examples, we consider superintegrable systems in the classical $r$-matrix method.

### 4 Superintegrable systems on $sl(n)$

In application to integrable systems every element $L(\lambda)$ of the loop algebra $L(\mathfrak{g}, \lambda)$ be a matrix in some fixed matrix representation of the algebra $\mathfrak{g}$ and representation space be an auxiliary space [1]. Let us consider an algebra $\mathfrak{g} = sl(n, \mathbb{C})$ in the fundamental representation and begin with the standard rational $r$-matrix [1, 22]. In this case element $L(\lambda)$ of $L(\mathfrak{g}, \lambda)$ be a $n \times n$ matrix in the auxiliary space $\mathbb{C}^n$ and the matrix elements of $L(\lambda)$ are some rational functions of spectral parameter $\lambda$.

Basis of invariant functions in $I(sl(n, \mathbb{C}))$ could be selected as

$$\tau_k(L(\lambda)) = {1 \over k} \text{tr} L^k(\lambda), \quad k \leq n.$$  

The family of the integrals of motion in the involution is generated by $\tau_k(L(\lambda))$ [1, 22]

$$I_{i,k}(L) = \Phi^{(i)}(\tau_k(\lambda)), \quad (4.2)$$

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where \( \Phi^{(i)}_\lambda \) are various linear functionals defining a set of the functionally independent integrals of evolution. For instance,

\[
\Phi^{(i)}_\lambda(z) = \text{Res}_{\lambda=0} \left( \phi_i(\lambda) \cdot z \right), \quad \phi(\lambda) \in \mathbb{C}[\lambda, \lambda^{-1}],
\]

(4.3)

here \( \phi_i(\lambda) \) are some functions of spectral parameter with numerical values. In this case integrals \( I_{i,k} \) are at most \( k \)-order polynomials in generators \( \mathfrak{g}, I_{i,k} \in U_k(\mathfrak{g}) \).

The Lax equation (2.2) associated with the hamiltonian \( I_{i,k} \) (4.2) is equal to

\[
\frac{dL(\mu)}{dt} = \{ I_{i,k}, L(\mu) \} = [L(\mu), A_{i,k}(\mu)] ,
\]

(4.4)

where the second Lax matrix \( A_{i,k} \) has the form

\[
A_{i,k}(\mu) = \Phi^{(i)}_\lambda \text{tr}_1 \left( v_{21}(\lambda, \mu) L_1^{k-1}(\lambda) \right),
\]

(4.5)

here \( \text{tr}_1 \) is taken over the first auxiliary space.

The representation space of the subalgebra \( U_m(\mathfrak{g}, \lambda, \mu, \ldots, \nu) \) be an extended auxiliary space

\[
V^{(m)} = \otimes_{i=1}^m V_i = V_1 \otimes V_2 \otimes \cdots \otimes V_m, \quad V_i \simeq \mathbb{C}^n.
\]

(4.6)

The elements \( L_j(\lambda_j) \) (2.7) and \( L^{(k)}_{j_1j_2\cdots j_k}(\lambda, \mu, \ldots, \nu) \) (2.8) belonging to the algebra \( U_m(\mathfrak{g}, \lambda, \mu, \ldots, \nu) \) are the \( n^m \times n^m \) matrices in \( V^{(m)} \)

\[
L_j(\lambda_j) = I_1 \otimes \cdots \otimes I_{j-1} \otimes L(\lambda_j) \otimes I_{j+1} \otimes \cdots \otimes I_m, \quad \lambda_1 = \lambda, \quad \lambda_2 = \mu, \ldots, \lambda_m = \nu,
\]

\[
L^{(m)}(\lambda, \mu, \ldots, \nu) = \prod_{j=1}^m L_j(\lambda_j),
\]

(4.7)

here \( I \) means a \( n \times n \) unit matrix and subscript \( j \) shows in which of the spaces \( V_j \) in the whole space \( V^{(m)} \) the matrix \( L(\lambda) \) acts nontrivially.

The equation of evolution for the matrix \( L^{(m)} \) has a commutator Lax form

\[
\frac{d}{dt} L^{(m)}(\lambda, \mu, \ldots, \nu) = \left[ L^{(m)}(\lambda, \mu, \ldots, \nu), A^{(m)}(\lambda, \mu, \ldots, \nu) \right],
\]

(4.8)

with the following second matrix

\[
A^{(m)}(\lambda, \mu, \ldots, \nu) = \sum_{j=1}^m A_j(\lambda_j), \quad \lambda_1 = \lambda, \quad \lambda_2 = \mu, \ldots, \lambda_m = \nu
\]

(4.9)

which is a sum of the matrices \( A_j(\lambda_j) \) of the type (4.7) acting in the whole spaces \( V^{(m)} \).

The spectral invariants of the matrices \( L^{(m)} \) give rise to an involute family of the integrals of motion as before.

Taking into account the symmetrization mapping \( w \) (2.4) let us introduce matrices

\[
L^{(m,\pi)}(\lambda, \mu, \ldots, \nu) = P_\pi L^{(m)}(\lambda, \mu, \ldots, \nu) = P_\pi L_1(\lambda)L_2(\mu)\cdots L_m(\nu).
\]

(4.10)

Here permutation matrix \( P_\pi \) in \( V^{(m)} \) is determined by

\[
P_\pi (x_1 \otimes x_2 \otimes \cdots \otimes x_m) = x_{\pi(1)} \otimes x_{\pi(2)} \otimes \cdots \otimes x_{\pi(m)}, \quad P_\pi \cdot A_1 B_2 \cdots D_m = A_{\pi(1)} B_{\pi(2)} \cdots D_{\pi(m)} \cdot P_\pi, \quad P_\pi^2 = I,
\]

(4.11)

for any set of vectors \( x_j \) in \( \mathbb{C}^n \) or

\[
\text{for any } x_j \text{ in } \mathbb{C}^n \text{ or }
\]

(4.12)
for any $n \times n$ matrices $A, B, \ldots, D$ embedding in $V^{(m)}$ according to (4.7). The permutation of subscripts in (4.11) and (4.12) is defined by a certain Young diagram $\pi$ [15].

The equations of evolution for the matrices $L^{(m,\pi)}$ are equal to
\[
\frac{d}{dt} L^{(m,\pi)}(\lambda, \mu, \ldots, \nu) = L^{(m,\pi)}(\lambda, \mu, \ldots, \nu) - A^{(m,\pi)}(\lambda, \mu, \ldots, \nu) L^{(m,\pi)},
\]
where matrix $A^{(m)}$ is given by (4.9) and the second matrix $A^{(m,\pi)}$ differs from it by the permutation of spectral parameters in accordance with the Young diagram $\pi$

\[
A^{(m,\pi)}(\lambda, \mu, \ldots, \nu) = P_{\pi} A^{(m)}(\lambda, \mu, \ldots, \nu) P_{\pi}^{-1},
\]

\[
= \sum_{j=1}^{m} A_{j}(\lambda_{\pi(j)}), \quad \lambda_{1} = \lambda, \quad \lambda_{2} = \mu, \ldots, \lambda_{m} = \nu.
\]

The right-hand side of (4.13) is a matrix commutator if and only if
\[
A^{(m)}(\lambda, \mu, \ldots, \nu) = A^{(m,\pi)}(\lambda, \mu, \ldots, \nu),
\]
\[
\iff A(\lambda_{j}) = A(\lambda_{\pi(j)}).
\]

Assuming (4.14) holds we can define new multivariable generating functions of the integrals of motion
\[
s_{\pi}^{(m)}(\lambda, \mu, \ldots, \nu) = \frac{1}{m} \text{tr}_{(m)} L^{(m,\pi)}(\lambda, \mu, \ldots, \nu),
\]
\[
= \sum_{j=1}^{m} A_{j}(\lambda_{\pi(j)}), \quad \lambda_{1} = \lambda, \quad \lambda_{2} = \mu, \ldots, \lambda_{m} = \nu.
\]

The trace $\text{tr}_{(m)}$ in (4.15) is taken over the whole space $V^{(m)}$.

For any hamiltonians $I_{i,k}$ (4.2) condition (4.14) is always fulfilled for the equivalent spectral parameters
\[
\lambda = \lambda_{j} = \lambda_{\pi(j)}, \quad j = 1, \ldots, n, \quad \forall \pi
\]
that corresponds to a choice of another basis of ad-invariant functions in the center $I(g)$ [15] by using the outer powers of matrix $L(\lambda)$
\[
s_{m}(\lambda) = \frac{1}{m} \text{tr}_{(m)} L^{(m,\pi)}(\lambda, \lambda, \ldots, \lambda).
\]

Symmetric functions $s_{m}(\lambda)$ can be expressed in the symmetric functions $\tau_{m}(\lambda)$ (4.1) according to Newton’s formulas [15] and functions $s_{m}(\lambda)$ give rise to integrals in the involution as before.

In additional, condition (4.14) is fulfilled for the independent on spectral parameter matrix $A$ in (4.4)
\[
A_{i,m}(\mu) = \Phi_{\lambda}^{(i)} \text{tr}_{1} \left( r_{21}(\lambda, \mu) L_{1}^{m-1}(\lambda) \right) = \text{const} \neq 0.
\]

We observe that matrix $A_{i,m}(\mu)$ depends on spectral parameter $\mu$ via $r$-matrix only. Hence, the constant in spectral sense matrix $A(\mu)$ and the corresponding special hamiltonian are related to the singular points of the $r$-matrix.

The standard rational $r$-matrix for algebra $sl(n, \mathbb{C})$ is equal to
\[
r_{12}(\lambda, \mu) = \frac{P_{12}}{\lambda - \mu}.
\]

We can choose the special linear functional in (4.3) as residue at infinity
\[
\Phi_{\lambda}(z) = - \text{Res}_{\lambda=\infty} (\phi(\lambda) \cdot z),
\]
\[11\]
such that the second matrix

\[ A = \text{tr}_1 \left[ P \cdot \lim_{\lambda \to \infty} \phi(\lambda) L_1^{m-1}(\lambda) \right]. \tag{4.20} \]

is independent on spectral parameter. Moreover, if some invariant polynomial \( \tau_k(\lambda) \) \((4.1)\) has a nontrivial residue at \( \lambda = \infty \) on the phase space

\[ H = - \text{Res}_{\lambda=\infty} \phi(\lambda) \tau_k(\lambda), \tag{4.21} \]

which is chosen as a hamiltonian, the corresponding Lax matrix \( A \) \((4.20)\) do not equal to zero. In this case the singular point \( \lambda = \infty \) of \( r \)-matrix \((4.18)\) is associated with the superintegrable hamiltonian \( H \) \((4.21)\) and with the new multivariable generating functions of the integrals of motion \((4.15)\)

\[ s_m^\pi(\lambda, \mu, \ldots, \nu) = \frac{1}{m} \text{tr}(m) L^{(m,\pi)}(\lambda, \mu, \ldots, \nu), \tag{4.22} \]

\[ \{H, s_m^\pi(\lambda, \mu, \ldots, \nu)\} = 0. \]

We may to lose the property of involution for these new multivariable functions and the corresponding integrals of motion

\[ \{s_m^\pi(\lambda_1, \lambda_2, \ldots, \lambda_m), s_k^\pi(\mu_1, \mu_2, \ldots, \mu_k)\} \neq 0. \tag{4.23} \]

These brackets are completely recovered by the polynomial \( r \)-brackets for the matrices \( L^{(m,\pi)} \), as an example see \((2.1)\) \((2.12)\).

The coefficients of the spectral curve of \( L(\lambda) \) determined by the characteristic equation

\[ C(z, \lambda) = \det(zI + L(\lambda)) = 0. \tag{4.24} \]

give rise to integrals in the involution only. Complete set of integrals is determined by the generalized spectral surfaces

\[ C(z, \lambda, \mu \ldots \nu) = \det(zI + L^{(m,\pi)}(\lambda, \mu \ldots \nu)) = 0, \quad m \leq n, \quad \forall \pi. \tag{4.25} \]

As an example, let us consider the \( 2 \times 2 \) Lax matrix

\[ L(\lambda) = \begin{pmatrix} h & e \\ f & -h \end{pmatrix}(\lambda), \tag{4.26} \]

in the two-dimensional auxiliary space with a suitable basis \([11]\]

\[ P = \frac{1}{2} (I + \sum \sigma_j \otimes \sigma_j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

where Pauli matrices \( \sigma_j \) form an orthonormal basis of \( sl(2) \) in \( \mathbb{C}^2 \).

Matrix \( L(\lambda) \) has alone invariant polynomial \((4.17)\)

\[ s_2(\lambda) = \frac{1}{2} \text{tr}[PL(\lambda) \otimes L(\lambda)] = \det L(\lambda) = -\frac{1}{2} \text{tr}L^2(\lambda) = -\frac{1}{2} \tau_2(\lambda), \tag{4.27} \]

and one second order complementary polynomial \((4.22)\)

\[ s_2(\lambda, \mu) = \frac{1}{2} \text{tr}[PL(\lambda) \otimes L(\mu)] = h(\lambda)h(\mu) + \frac{e(\lambda)f(\mu) + f(\lambda)e(\mu)}{2}. \tag{4.28} \]
These polynomials may be the generating functions of the integrals of motion for some super-integrable hamiltonian (compare with (5.3)).

Next let us consider the Belavin \( r \)-matrices [3], which are meromorphic solutions of the classical Yang-Baxter equation such that

\[
\begin{align*}
    r_{12}(\lambda, \mu) &= r_{12}(\lambda - \mu) = -r_{21}(\mu - \lambda), & (4.29) \\
    r_{12}(\lambda - \mu) &= \frac{P_{12}}{\lambda - \mu} + O(1), & P_{12} x \otimes y = y \otimes x.
\end{align*}
\]

Here \( r_{ij} \) are rational, trigonometric or elliptic matrix-function on spectral parameters [4, 11, 22]. These matrices have a pole at \( \lambda = \mu \) with a permutation operator \( P_{12} \) (4.23) as residue. In additional the rational function \( r(\lambda - \mu) \) of two variables \( \lambda \) and \( \mu \) has the special point at \( \lambda = \infty \) in its domain of definition. In this point function \( r(\lambda - \mu) \) has a distinct from zero residue, which is independent from second spectral parameter \( \mu \). The generally accepted elliptic \( r \)-matrices [3, 11, 22] have not such special points in its domains. Nevertheless, the similar to the rational case construction can be proposed as well.

The Lax equation (2.22) and \( r \)-bracket (2.10) are covariant under the similar transformation

\[
L \rightarrow U^{-1} L U, \quad A \rightarrow U^{-1} A U, \quad r_{ij} \rightarrow U_{12}^{-1} U_{21} r_{ij} U_{12} U_{21}, \quad (4.30)
\]

where \( U \) is a constant matrix on a phase space. If the matrix \( U \) depends on spectral parameter, we can use such similar transformations to construct the additional poles of \( r \)-matrix. In this case the multivariable generating functions \( s_m^\pi(\lambda, \mu, \ldots, \nu) \) are changed

\[
s_m^\pi(\lambda, \mu, \ldots, \nu) \rightarrow s_m^\pi(\lambda, \mu, \ldots, \nu) = \frac{1}{m} \mbox{tr}_{(m)} [Z \pi L(\lambda) \otimes L(\mu) \ldots \otimes L(\nu)], \quad (4.31)
\]

here projector \( P_{\pi} \) in (4.22) is substituted by matrix

\[
Z_{\pi} = [U(\lambda) \otimes U(\mu) \ldots \otimes U(\nu)]^{-1} \cdot P_{\pi} \cdot U(\lambda) \otimes U(\mu) \ldots \otimes U(\nu). \quad (4.32)
\]

depending on spectral parameters. For the equivalent spectral parameters functions \( s_m(\lambda) \) are covariant under the similar transformations.

As an example, here we study an elliptic \( r \)-matrix on the twisted loop algebra \( \mathcal{L}(sl(2), \sigma) \). Let us consider the period lattice \( \Gamma = 2K\mathbb{Z} + 2iK'\mathbb{Z} \), where \( K \) and \( K' \) are the standard elliptic integrals of the module \( k \in [0, 1] \), and introduce the corresponding elliptic theta function \( \Theta_{ij}(\lambda, k) \) as in [24]. In these notations the standard elliptic \( r \)-matrix is

\[
r(\lambda - \mu) = \sum_{k=1}^{3} w_k(\lambda - \mu) \cdot \sigma_k \otimes \sigma_k, \quad (4.33)
\]

where \( \sigma_k \) are the Pauli matrices and

\[
\begin{align*}
    w_1(\lambda) &= \frac{\Theta_{11}(0, k) \Theta_{10}(\lambda, k)}{\Theta_{10}(0, k) \Theta_{11}(\lambda, k)}, \quad w_2(\lambda) = \frac{\Theta_{11}'(0, k) \Theta_{00}(\lambda, k)}{\Theta_{00}(0, k) \Theta_{11}(\lambda, k)}, \quad w_3(\lambda) = \frac{\Theta_{11}'(0, k) \Theta_{01}(\lambda, k)}{\Theta_{01}(0, k) \Theta_{11}(\lambda, k)}.
\end{align*}
\]

Function \( r(\lambda - \mu) \) (4.33) is meromorphic in \( \mathbb{C} \) and has simple poles at \( \lambda = \mu \mod \Gamma \).

According to [24] we introduce the similar \( r \)-matrix

\[
\begin{align*}
    \rho(\lambda, \mu) &= U_{12}^{-1}(\lambda, \mu, \lambda_{\infty}) r(\lambda - \mu) U_{12}(\lambda, \mu, \lambda_{\infty}), \\
    U_{12}(\lambda, \mu, \lambda_{\infty}) &= U_1(\lambda - \lambda_{\infty} + K) U_2(\mu - \lambda_{\infty} + K).
\end{align*}
\]

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with the following matrix $U(\lambda)$

$$
U(\lambda) = \begin{pmatrix} \Theta_{01} & \Theta_{00} \\ -\Theta_{00} & -\Theta_{01} \end{pmatrix} \left( \xi, \tilde{k} \right), \quad \lambda = \frac{2\xi}{1+k}, \quad \tilde{k} = \frac{2\sqrt{k}}{1+k},
$$

(4.34)

Here $\lambda_\infty$ is an arbitrary point. This $r$-matrix $\rho(\lambda, \mu)$ has been introduced for the purpose of separation variables in [24]. More explicitly

$$
\rho(\lambda, \mu) = \sum_{j=1}^{3} w_j (\lambda - \mu, k) \cdot \sigma_j \otimes \sigma_j + w_3 (\lambda - \lambda_\infty, k) \cdot \sigma_3 \otimes \sigma_3,
$$

(4.35)

$$
w(\lambda, k) \equiv w_1 (\lambda, k) = w_2 (\lambda, k) = \frac{\Theta_{11}'(\lambda, k)}{\Theta_{10}} \Theta_{11}(\lambda, k), \quad w_3 (\lambda) = \frac{\Theta_{11}'(\lambda, k)}{\Theta_{11}(\lambda, k)},
$$

Here $w_0$ is a some constant [24]. The poles of $\rho(\lambda, \mu)$ by the first spectral parameter $\lambda$ be at the points

$$
\lambda = \mu \mod \Gamma, \quad \text{Res}_{\lambda=\mu} \rho(\lambda, \mu) = P,
$$

$$
\lambda = \lambda_\infty \mod \Gamma, \quad \text{Res}_{\lambda=\lambda_\infty} \rho(\lambda, \mu) = Z = (\sigma_3 + \sigma_- + \sigma_+) \otimes \sigma_3.
$$

So, if $L(\lambda)$ is an orbit of the new $r$-matrix $\rho(\lambda, \mu)$, such that the second matrix

$$
A = \text{tr}_1 \left[ Z \cdot \lim_{\lambda \to \lambda_\infty} \phi(\lambda) L(\lambda) \right] \neq 0,
$$

is nontrivial constant in spectral sense, the special hamiltonian

$$
H = \text{Res}_{\lambda=\lambda_\infty} \text{tr} \left[ \phi(\lambda) L^2(\lambda) \right]
$$

is superintegrable.

In the next section we present some nontrivial examples of $r$-matrix orbits associated with superintegrable systems for the rational $r$-matrix.

## 5 Examples

The above construction of superintegrable systems can be applied to the Gaudin magnet, which was introduced in the quantum mechanics [12]. The classical version turned out to be useful example for developing a general group-theoretic approach to integrable system [11, 22].

We shall consider the rational Gaudin magnet related to $sl(N)$ algebra. The model in question is defined on the $M$ coadjoint orbits of $sl(N)^*$ in variables $X^{(m)}_{ij}$, $(m = 1, \ldots, M, i, j = 1, \ldots, N)$. The corresponding Lie-Poisson brackets are

$$
\{X^{(m)}_{ij}, X^{(n)}_{kl} \} = \delta_{mn} \left( X^{(m)}_{il} \delta_{jk} - X^{(m)}_{kj} \delta_{il} \right).
$$

(5.1)

The coadjoint orbits are fixed by values $t^{(m)}_k$ of the ad-invariant functions on $sl(N)$

$$
t^{(m)}_k = k^{-1} \text{tr}(X^{(m)}) k \in \mathbb{C}.
$$

(5.2)
The Poisson bracket (5.1) is nondegenerate on the manifold (5.2) having dimension \( n = MN(N - 1)/2 \) for the case of generic orbit (all \( t_i^{(m)} \) are distinct). In what follows we assume that the orbit is generic.

Fixing some element \( Z \in sl(N) \) as a residue at infinity we consider the special Lax matrix
\[
L(\lambda) = Z + \sum_{m=1}^{M} \frac{X^{(m)}}{\lambda - \delta_m},
\]
(5.3)
where \( \{\delta_m\} \) is a set of \( M \) arbitrary constants. Matrix \( L(\lambda) \) obeys the linear \( r \)-bracket (2.10) with the rational \( r \)-matrix (4.18).

The basis elements \( \tau_k \) (4.1) are meromorphic functions of \( \lambda \)
\[
\tau_k(\lambda) = \xi_k + \sum_{m=1}^{M} \sum_{j=1}^{k} I_{m,k}^{j} (\lambda - \delta_m)^j,
\]
here \( \xi_k = k^{-1} \text{tr} Z^k \) and \( I_{m,k}^{j} = t_i^{(m)} \) are fixed constants. Another residues \( I_{m,k}^{j} \) form a family of \( n = MN(N - 1)/2 \) independent integrals in the involution. It is immediately seen that the special hamiltonians in this family
\[
H^{(k)} = - \text{Res}_{\lambda=\infty} \tau_k(\lambda) = \sum_{m=1}^{M} \text{Res}_{\lambda=\delta_m} \tau_k(\lambda),
\]
are nondegenerate functions on the generic coadjoint orbits of \( sl(N)^* \) and they are corresponded to the constant in spectral sense second Lax matrix (4.13)
\[
A^{(k)} = Z^{k-1}.
\]
So, hamiltonians \( H^{(k)} \) are superintegrable hamiltonians and complete set of the integrals of motion can be generated by (4.22). As an example, additional integrals of evolution may be constructed from the quantities
\[
I_{m,k}^{\pi} = \text{Res} \ s^{\pi}_{m}(\lambda, \mu, \ldots, \nu).
\]
(5.4)
Here Res means residue of some fixed order \( k = (k_1, k_2, \ldots, k_m), k_j \leq K_j \) at the points
\[
\lambda = \delta_{j_1}, \quad \mu = \delta_{j_2}, \ldots, \quad \nu = \delta_{j_m},
\]
which belong to the divisor of the poles of multivariable function \( s^{\pi}_{m}(\lambda, \mu, \ldots, \nu) \)
\[
D = \{ (\delta_j, K_j), j = 1, \ldots, M, (\infty, 1) \}.
\]

As a second example, let us consider superintegrable natural systems on \( \mathbb{R}^{2n} \) with the following hamiltonian
\[
H = T + V = \sum a_{ij} p_i p_j + V(q_1, \ldots, q_n), \quad a_{ij} \in \mathbb{R},
\]
(5.5)
where \( \{p_j, q_j\}_{j=1}^{n} \) are canonical variables. By \( V = 0 \) in (5.3) this hamiltonian of geodesic motion is superintegrable (1.4). In the classical mechanics the geodesic motion on the Riemannian spaces of constant curvature is isomorphic to the rational Gaudin magnet on the algebra \( \mathfrak{g} = \oplus sl(2) \) [16]. Taking the known Lax matrix \( L(\lambda) \) in \( \mathcal{L}(sl(2), \lambda) \) associated to a free geodesic motion we can get an infinite set of the Lax matrices associated to the completely
integrable potential systems \[26\]. Construction of the new Lax equations consists of the application of the outer automorphism of infinite-dimensional representations of \(sl(2)\) \((3.17)\) to the loop algebras. As a second step, we have to apply the consequent projections of this mapping onto a suitable Poisson subspaces of \(r\)-bracket \((ad^*_R\)-invariant subspaces). The superintegrable systems are related to the special projections, which lead to a constant in spectral sense Lax matrix \(A(\lambda)\) \((2.2)\).

Let us consider the Lax matrix \(L(\lambda)\) in \(L(sl(2),\lambda)\) associated to geodesic motion. Hereafter we propose that this matrix \(L(\lambda)\) in the form \((4.26)\) obeys the \(r\)-matrix bracket \((2.10)\) with the rational \(r\)-matrix \((4.18)\). Following to the general procedure \([8,14,26]\) let us introduce new Lax pairs

\[
L'(\lambda) = L(\lambda) - \sigma_- \cdot \left[\phi(\lambda)e^{-1}(\lambda)\right]_{MN},
\]

\[
A'(\lambda) = A - \sigma_- \cdot \left[\phi(\lambda)e^{-2}(\lambda)\right]_{MN} = 
\begin{pmatrix}
0 & 1 \\
u_{MN}(\lambda) & 0
\end{pmatrix}
\]

Here \(\phi(\lambda)\) is a function on spectral parameter and \([z]_{MN}\) means restriction of \(z\) onto the \(ad^*_R\)-invariant Poisson subspace of the initial \(r\)-bracket \([14,26]\). For the rational \(r\)-matrix \((4.18)\) we can use the linear combinations of the following Laurent projections

\[
[z]_{MN} = \left[ \sum_{k=-\infty}^{+\infty} z_k \lambda^k \right]_{MN} \equiv \sum_{k=-M}^{N} z_k \lambda^k,
\]

or the Taylor projection by \(M = 0\).

The mappings \((5.6)\) play the role of a dressing procedure allowing to construct the Lax matrices \(L'_{MN}(\lambda)\) for an infinite set of new integrable systems starting from the single known Lax matrix \(L(\lambda)\) associated to one integrable model. The Lax matrix \(L'(\lambda)\) \((5.6)\) obeys the linear \(r\)-bracket \((2.10)\), where constant \(r_{ij}\)-matrices substituted by \(r'_{ij}\)-matrices depending on dynamical variables

\[
r_{12}(\lambda, \mu) \rightarrow r'_{12} = r_{12} - \left(\frac{[\phi(\lambda)e^{-2}(\lambda)]_{MN} - [\phi(\mu)e^{-2}(\mu)]_{MN}}{\lambda - \mu}\right) \cdot \sigma_- \otimes \sigma_-.
\]

The Poisson map \(\mathbb{R}^{2n} \rightarrow L(\mathfrak{g})\) may be determined by using a Newton form of the geodesic equations of motion \([14]\). Letting

\[
A = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} = \sigma_+,
\]

which is independent on spectral parameter \(\lambda\), we observe that the Lax matrix \(L(\lambda)\) \((4.26)\) is recovered by the alone entry \(e(\lambda)\) and by the hamiltonian \(H\) according to

\[
L(\lambda) = \begin{pmatrix}
-e_x/2 & e \\
-e_{xx}/2 & e_x/2
\end{pmatrix}(\lambda), \quad e_x = \{H,e\},
\]

In this case the generating functions \(s_2(\lambda)\) and \(s'_2(\lambda)\) obey the following equations of motion

\[
\frac{ds_2(\lambda)}{dt} = 0, \quad \Rightarrow \quad \partial_x^2 e(\lambda) = e_{xxx} = 0,
\]

\[
\frac{ds'_2(\lambda)}{dt} = 0, \quad \Rightarrow \quad \left[\frac{1}{4} \partial_x^4 + u_{MN}(\lambda) \partial_x + \frac{1}{2} u_{MN,x}(\lambda)\right] \cdot e(\lambda) = B_1[u_{MN}(\lambda)] \cdot e(\lambda) = 0.
\]

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Here $B_1[u_{MN}(\lambda)]$ is the Hamiltonian pencil operator for the coupled KdV equation \[14\]. The superintegrable systems correspond to the special solution of (5.11) by $\partial u_{MN}(\lambda)/\partial \lambda = 0$. The integrate form of the KdV recursion relations (5.11) are

$$s_2(\lambda) = \frac{e \cdot e_{xx}}{2} - \frac{e_x^2}{4},$$

$$s_2'(\lambda) = \frac{e \cdot e_{xx}}{2} - \frac{e_x^2}{4} + e^2 \cdot u_{MN}(\lambda).$$

A simple substitution for the entries of matrix $L(\lambda)$

$$e(\lambda) = B^2, \quad h(\lambda) = -e_x/2 = -BB_x,$$

$$f(\lambda) = -e_{xx}/2 = -B_x^2 - BB_{xx},$$

turns determinants (5.12) into the form

$$s_2(\lambda) = B^3 B_{xx}, \quad s_2'(\lambda) = B^3 B_{xx} + B^4 \left[ \frac{\phi(\lambda)}{B^4} \right]_{MN},$$

if we use an explicit formula for the potential $u_{MN}(\lambda)$. These equations have the form of Newton’s equations for the function $B$

$$B_{xx} = s_2(\lambda) B^{-3},$$

$$B_{xx} = s_2'(\lambda) B^{-3} - B \left[ \frac{f(\lambda)}{B^4} \right]_{MN},$$

If we assume that $B = \sum_{j=0}^{N} q_{N-j} \lambda^j$ is a polynomial, then its coefficients $q_j$ obey the Newton equation of motion (5.13) with $s_2(\lambda) = \sum I_k \lambda^k$, where $I_k$ are integrals of motion. Here we reinterpret the coefficients of $s_2(\lambda)$ and $s_2'(\lambda)$ in (5.13) not as functions on the phase space, but rather as integration constants. In variables $q_j$ mapping (5.6) affects only on the potential ($q$-dependent) part of the integrals of motion $I_k$. The kinetic (momentum dependent) part of $I_k$ remains unchanged. So, the dressing mapping (5.6) allows us to get over from a free motion on $\mathbb{R}^{2n}$ to a potential motion on $\mathbb{R}^{2n}$.

To construct the multipole Lax equation we introduce an appropriate completion $\mathcal{L}_D(sl(2))$ of the standard loop algebra associated to some fixed divisor of poles \[22\]

\[ D = \{ (\delta_j, l_j), \ j = 1, \ldots, M, (\infty, K) \}. \]

According by (5.10) the Poisson map $\mathbb{R}^{2n} \to \mathcal{L}_D(sl(2))$ is completely defined by the alone entry $e(\lambda)$ and by the hamiltonian $H$ related to the linear functional $\Phi_{\lambda}$. We put

$$e(\lambda) = \sum_{i=1}^K e_i \lambda^i + \sum_{j=1}^M \sum_{k=1}^{l_j} \frac{e_{jk}}{(\lambda - \delta_j)^k} = \frac{u_0 \prod (\lambda - u_j)}{\prod (\lambda - \delta_j)},$$

$$\Phi_{\lambda}(z) = \text{Res}_{\lambda=\infty} \left( \lambda^{-K} z \right),$$

that self-consistent with (5.9) and (5.13) \[14\]. Residues $e_i$ and $e_{jk}$ are the special functions of the canonical coordinates $\{q_j\}_{j=1}^n$, and variables $u_j$ are curvilinear separated coordinates \[14\]. By using function $B$ residues $e_i$ and $e_{jk}$ are easily restored in canonical variables $\{q_j\}_{j=1}^n$. 

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In the simple poles at \( \lambda = \delta j \) function \( e(\lambda) = B^2 \) \([5.16]\) has the following residues \( e_{j1} = q_j^2 \). Parametrization of residues in the higher order poles is discussed in \([14]\). As an example, we consider the polynomial part of \( e(\lambda) \) corresponding to a pole at infinity. Let

\[
B(\lambda) = \sum_{j=0}^{K} q_{K-j} \lambda^j, \quad (5.17)
\]

with \( q_0 = 1 \) and

\[
e(\lambda) = \sum_{j=0}^{K} e_j \lambda^j, \quad h(\lambda) = \sum_{j=0}^{K} h_j \lambda^j, \quad f(\lambda) = \sum_{j=0}^{K} f_j \lambda^j, \quad (5.18)
\]

with \( e_K = 1, h_K = 0 \) and \( f_K = 0 \) due to the definition of \( A \) \([5.9]\) and \( \Phi_\lambda \) \([5.16]\). Taking \([5.13]\) and \([5.16]\) into account we get

\[
e_j = \sum_{i=0}^{K-j} q_i q_{K-j-i}, \quad h_j = -\sum_{i=0}^{K-j} q_{i,x} q_{K-j-i}, \quad (5.19)
\]

\[f_j = -\sum_{i=0}^{K-j} q_{i,x} q_{K-j-i,x} - \sum_{i=0}^{K-j} q_{i,xx} q_{K-j-i},\]

where \( q_x = \{H, q\} \) and we used the Newton formulae for a product of two sets. Canonically conjugate to the coordinates \( q_j \) momenta \( p_j \) can be derived from the \( r \)-matrix algebra \([2.10]\) \([14]\). Some first polynomials are equal to

\[
K = 0, \quad e(\lambda) = 1, \quad h(\lambda) = 0,
\]

\[
K = 1, \quad e(\lambda) = \lambda + 2q_1, \quad -h(\lambda) = p_1,
\]

\[
K = 2, \quad e(\lambda) = \lambda^2 + 2\lambda q_1 + (2q_2 + q_1^2),
\]

\[-h(\lambda) = \lambda p_2 + (p_1 + p_2 q_1), \quad (5.20)\]

\[
K = 3, \quad e(\lambda) = \lambda^3 + 2\lambda^2 q_1 + \lambda(2q_2 + q_1^2) + 2(q_3 + q_1 q_2),
\]

\[-h(\lambda) = \lambda^2 p_3 + \lambda(p_2 + p_3 q_1) + (p_1 + p_2 q_1 + p_3 q_2). \]

Notice that the kinetic part of the hamiltonian \( H \) \([5.3]\) has a nondiagonal form in these variables

\[T = \sum_{j=1}^{K} p_j p_{K+1-j}. \quad (5.21)\]

Now we turn to the superintegrable systems. For the Taylor projections (\( M=0 \)) \([5.7]\) the mapping \([5.6]\) preserves the property of superintegrability if and only if \( N \leq K \) according to \([5.6]\) and \([5.16]\). Here \( K \) is a highest order of a pole of the entry \( e(\lambda) \) \([5.16]\) at infinity and \( N \) is a highest power in projection \([5.7]\). By \( N > K \) dynamical \( r \)-matrix \([5.8]\) has the higher poles at \( \lambda = \infty \) and the corresponding dynamical systems are no longer the superintegrable
All these superintegrable systems are related to the special point of \( r \)-matrix and the associated second Lax matrix \( A' \) (5.6) remains a constant in spectral sense under the mapping (5.6). All these superintegrable systems are related to the special stationary flows of the KdV hierarchy (5.11) and genus of the associated spectral curve of \( L'(\lambda) \) determined by the characteristic equation (4.24) is no more than number of the degrees of freedom.

As an example, we present here several superintegrable systems. Let the entry \( e(\lambda) \) has the simple poles at \( \lambda = \delta_j \) and the \( K \) poles at \( \lambda = \infty \)

\[
e(\lambda) = P_K(\lambda) + \sum_{j=K+1}^{N} \frac{q_j^2}{\lambda - \delta_j},
\]

where \( P_K(\lambda) \) are polynomials given by (5.20). For the some first values of \( K = N \) the corresponding superintegrable potentials in (5.5) are

\[
\begin{align*}
K &= 0, & V &= \sum_{j=1}^{n} \left( q_j^2 + \beta_j \frac{q_j^2}{q_j^2} \right), \\
K &= 1, & V &= 4q_1^2 + \sum_{j=2}^{n} \left( q_j^2 + \beta_j \frac{q_j^2}{q_j^2} \right), \\
K &= 2, & V &= 4q_1^3 - 8q_1q_2 + \sum_{j=3}^{n} \left( q_j^2 + \beta_j \frac{q_j^2}{q_j^2} \right), \\
K &= 3, & V &= -5q_1^4 + 12q_1^2q_2 - 4q_2^2 - 8q_1q_3 + \sum_{j=4}^{n} \left( q_j^2 + \beta_j \frac{q_j^2}{q_j^2} \right).
\end{align*}
\]

If \( N = K \), then the dressing mapping (5.6) has the \( N + 1 \) arbitrary parameters given by the function \( \phi(\lambda) = \sum_{j=0}^{N} \alpha_j \lambda^j \). These parameters are related to the canonical shifts of variables \( q_j \to q_j + \alpha_j \) and to the common rescaling \( V \to \alpha_N V \) in the presented potentials. Additional integrals of motion may be constructed by the rule (5.4) from the multivariable generating function \( s_2(\lambda, \mu) \) (4.28). Of course, for the oscillator all these integrals (5.4) coincide with known ones (3.21).

We may construct the similar superintegrable systems with rational potentials by using a general form of the entry \( e(\lambda) \) (5.16) with the higher order poles and applying more general Laurent projection (5.7). The presented method can be employed to construct superintegrable systems on the other Riemannian spaces of constant curvature [16]. The corresponding quantum systems may be obtained by canonical quantization [8, 26].

Taking into account the \( r \)-bracket (2.10) one can conclude that the entries \( e(\lambda) \) and \( f(\lambda) \) could play the roles similar to the standard creation and annihilation operators for harmonic oscillator [15, 10]. By using the similar transformation of matrices \( L(\lambda) \) or \( L'(\lambda) \) with matrix \( U \) (3.23) at \( \alpha_j = \alpha_k = 1 \) we can obtain the symmetric representation of these matrices

\[
e(\lambda) = e(\lambda, a_1, a_2, \ldots, a_n), \quad f(\lambda) = e^+(\lambda), \quad h(\lambda) = h^+(\lambda).
\]

In this symmetric representation of the Lax matrix the usual method of spectrum-generating algebras [4, 19] is a part of the standard Bethe ansatz [15, 10]. It should be emphasized, that the algebraic Bethe ansatz is a sufficiently universal procedure, which slightly depends from particular system in question. It allows us to interpret various concrete models as some representations of alone generalized model, which is defined by its \( r \)-matrix only.

As a third example, let us consider the Calogero-Moser systems. It is well known [21], that both the Toda models and the Calogero-Moser models are obtained by hamiltonian reduction
of the geodesic motion on the cotangent bundle $T^*G$ of a Lie group $G$. For the geodesic motion on symmetric spaces of zero curvature the canonical 2-form, the free hamiltonian and equations of motion are equal to

\[ w = \text{tr}(dy \wedge dx), \quad H = \frac{1}{2} \text{tr}(y^2), \quad (5.22) \]
\[ \dot{x} = y, \quad \dot{y} = 0. \]

For the geodesic motion on symmetric spaces of positive or negative curvature these quantities read

\[ w = \text{tr}(x^{-1}dy \wedge x^{-1}dx), \quad H = \frac{1}{2} \text{tr}(yx^{-1}), \quad (5.23) \]
\[ \dot{x} = y, \quad \dot{y} = yx^{-1}y. \]

The hamiltonians (5.22) and (5.23) have the following sets of integrals in the involution

\[ I_k = \text{tr}(y^k) \quad \text{and} \quad I_k = \text{tr}(yx^{-1})^k. \]

The additional integrals - "projections of angular momentum" (1.3) - are equal to

\[ I_{jk} = \text{tr}(qp^{j-1})\text{tr}(p^k) - \text{tr}(p^j)\text{tr}(qp^{k-1}). \quad (5.24) \]

Here $q = x$ and $p = y$ for the first equations of geodesic motion (5.22) and $q = \ln x$ or $q = \ln y$ with $p = yx^{-1}$ for the second equations of geodesic motion (5.23).

In the reduction process the Lax matrices of the reduced system are expressed in terms of $x$ by a formula of the type $L = zzx^{-1}$, where $z$ is some elements in $G$ [21]. For the geodesic motion (5.22) associated to the Calogero model with the rational potentials, the hamiltonian $H$ (5.22) remains superintegrable and images of integrals (5.24) are integrals of a reduced system [27]. For the second geodesic motion (5.23) associated to the Calogero-Moser model with the trigonometric potentials and to the Toda model, the reduced hamiltonian do not commute with the images of logarithmic additional integrals (5.24). In the quantum mechanics whole polynomial algebra of the integrals of motion for the Calogero model has been introduced in [17].

Let us show as the superintegrable hamiltonian (5.22) appears in the $r$-matrix formalism. For instance, consider the Euler-Calogero-Moser system [27]. Introduce a set of dynamical variables $\{(q_j, p_j)\}_{j=1}^N$ and $\{f_{ij}\}_{i,j=1}^N (f_{ij} = -f_{ji})$ together with the Poisson brackets

\[ \{p_j, q_k\} = \delta_{jk}, \quad (5.25) \]
\[ \{f_{ij}, f_{kl}\} = \frac{1}{2} (\delta_{il}f_{jk} + \delta_{ki}f_{lj} + \delta_{jk}f_{il} + \delta_{lj}f_{ki}). \quad (5.26) \]

In order to have a nondegenerate Poisson bracket it is assumed that the variables $f_{ij}$ are restricted to a symplectic submanifold of (5.26). The hamiltonian and the Lax matrix for the Euler-Calogero-Moser system [27] are given by

\[ H = \frac{1}{2} \sum_{j=1}^N p_j^2 + \frac{1}{2} \sum_{i,j=1}^N \frac{f_{ij}^2}{(q_i - q_j)^2}, \quad (5.27) \]
\[ L(\lambda) = \sum_{j=1}^N p_j e_{jj} + \sum_{i,j=1}^N \left( \frac{1}{q_i - q_j} + \frac{1}{\lambda} \right) f_{ij} e_{ij}. \]
with the corresponding $r$-matrix in the form \[ r_{12}(\lambda, \mu) = -\frac{\lambda}{\lambda^2 - \mu^2} \sum_{j=1}^{N} e_{jj} \otimes e_{jj} \]

\[ - \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^{N} \left( \frac{1}{q_i - q_j} + \frac{1}{\lambda + \mu} \right) e_{ij} \otimes e_{ij} \]

\[ - \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^{N} \left( \frac{1}{q_i - q_j} + \frac{1}{\lambda - \mu} \right) e_{ij} \otimes e_{ji} , \]

where $(e_{ij})_{kl} = \delta_{ik} \delta_{jl}$. In the reduction process this $r$-matrix inherits the singular point $\lambda = \infty$ from the initial rational $r$-matrix. The corresponding superintegrable hamiltonian (5.27) may be defined by (4.2) with $\phi = 1/2 \cdot \lambda^{-1}$

\[ H = \Phi_{\lambda} \left[ \text{tr} L^2(\lambda) \right] = \frac{1}{2} \text{Res}_{\lambda=\infty} \left[ \lambda^{-1} \cdot \text{tr} L^2(\lambda) \right] . \]

and the second Lax matrix is independent on spectral parameter

\[ A = \Phi_{\lambda} \text{tr}_1 \left[ r_{21}(\lambda, \mu) L_1(\lambda) \right] = \sum_{i,j=1 \atop i \neq j}^{N} \frac{f_{ij}}{(q_i - q_j)^2} e_{ij} . \]

The higher flows with $\phi(\lambda) = 1/k \cdot \lambda^{-k-1}$ in (1.2) are superintegrable as well [27].

6 Conclusions

We have seen that superintegrable systems closed to geodesic motion can be realizing as isospectral flows on coadjoint orbits of loop algebras in framework of the $r$-matrix formalism. All these systems with rational potentials are associated to the special singular point of $r$-matrix.

Another classical superintegrable systems with an arbitrary number of degrees of freedom are the Kepler problem [1, 21] and the rational Calogero-Moser systems [27]. In the proposed scheme we can consider a free geodesic motion on the momentum sphere and use the stereographic projection with appropriate change of the time variable to study the Kepler problem [20]. However, this transformation could violate the $r$-bracket (2.10) for the corresponding Lax matrix. It would be interesting to construct the $2 \times 2$ Lax matrix for the Kepler problem and for the Kepler-like superintegrable potentials listed in [1].

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