AFFINE MATSUKI CORRESPONDENCE FOR SHEAVES

TSAO-HSIEN CHEN AND DAVID NADLER

Abstract. We lift the affine Matsuki correspondence between real and symmetric loop group orbits in affine Grassmannians to an equivalence of derived categories of sheaves. In analogy with the finite-dimensional setting, our arguments depend upon the Morse theory of energy functions obtained from symmetrizations of coadjoint orbits. The additional fusion structures of the affine setting lead to further equivalences with Schubert constructible derived categories of sheaves on real affine Grassmannians.

Contents

1. Introduction 2
1.1. Matsuki correspondence for sheaves 3
1.2. Affine Matsuki correspondence for sheaves 4
1.3. Relation to Schubert geometry 7
1.4. Further directions 8
1.5. Organization 10
1.6. Acknowledgements 11

2. $K(\mathbb{K})$ and $LG_\mathbb{R}$-orbits on $Gr$ 11
2.1. Loop groups 11
2.2. The based loop spaces $\Omega X$ 12
2.3. Real affine Grassmannians 12
2.4. The energy flow on $\Omega G_c$ 12
2.5. Component groups of $Gr_{\mathbb{R}}$ 13
2.6. Parametrization of $K(\mathbb{K})$ and $LG_\mathbb{R}$-orbits 13
2.7. Geometry of $K(\mathbb{K})$ and $LG_\mathbb{R}$-orbits 14
2.8. The components $Gr^G_0$ 15

3. The Matsuki flow 16
3.1. The Matsuki flow on $Gr$ 16

4. Real Beilinson-Drinfeld Grassmannians 19
4.1. Beilinson-Drinfeld Grassmannians 19
4.2. Real forms 20

5. Uniformizations of real bundles 21
5.1. Stack of real bundles 21
1. Introduction

This is the first of several papers devoted to the geometry and representation theory of real loop groups $LG_{\mathbb{R}}$, i.e., groups of maps from the circle into a real reductive (not necessarily compact) Lie group $G_{\mathbb{R}}$. Some of our primary motivations established here or in the sequels include the following:
(1) A lift of the affine Matsuki correspondence [N1] between real and symmetric loop group orbits in affine Grassmannians to an equivalence of derived categories of sheaves.

(2) A lift of the Kostant-Sekiguchi correspondence [S] between real and symmetric nilpotent orbits to an equivariant stratified homeomorphism (see [CN2]).

(3) The development of a representation theory of real loop groups from the well-known setting of compact groups [PS] to general reductive groups.

Of the preceding goals, the current paper establishes (1) which in turn provides the geometry underlying our approach to (2). It also introduces and establishes basic properties of the moduli of quasi-maps that play a fundamental role in (3).

In what immediately follows, we describe the main results of (1) in more detail, including the remarkable relation of real and symmetric loop group orbits in affine flag varieties to real affine Schubert geometry. We then sketch some of the applications to (2), (3) and other topics to be established in sequel papers.

1.1. Matsuki correspondence for sheaves. We begin by recalling the Matsuki correspondence for sheaves [MUV]. It intertwines the Beilinson-Bernstein localization [BB] of Harish Chandra (g, K)-modules with the Kashiwara-Schmid localization [KS] of (infinitesimal classes of) admissible representations of GR.

Let GR be a connected real reductive algebraic group, and G = GR ⊗R C its complexification. From this starting point, one constructs the following diagram of Lie groups

\[
\begin{array}{ccc}
G & & Gc \\
\downarrow & & \downarrow \\
K & & G_R \\
\uparrow & & \uparrow \\
Kc & & Kc \\
\end{array}
\]

Here G = G(C) and G_R = G_R(R) are the Lie groups of complex and real points respectively, Kc is a maximal compact subgroup of G_R, with complexification K, and Gc is the maximal compact subgroup of G containing Kc.

Let B \simeq G/B be the flag manifold of Borel subgroups of G. The groups K and G_R act on B with finitely many orbits and the classical Matsuki correspondence [M] provides an anti-isomorphism of orbit posets

\[
|K\backslash B| \longleftrightarrow |G_R\backslash B|
\]

between the sets of K-orbits and G_R-orbits on B, each ordered with respect to orbit closures. The correspondence matches a K-orbit O^+ with the unique G_R-orbit O^- such that the intersection O = O^+ \cap O^- is a non-empty Kc-orbit.

The Matsuki correspondence for sheaves [MUV], as conjectured by Kashiwara, lifts this anti-isomorphism of posets to an equivalence

\[
D_c(K\backslash B) \simeq D_c(G_R\backslash B)
\]
between the bounded constructible $K$-equivariant and $G_\mathbb{R}$-equivariant derived categories of $\mathcal{B}$. The main ingredient of the proof is a Morse-theoretic interpretation and refinement of the Matsuki correspondence due to Uzawa.

1.2. Affine Matsuki correspondence for sheaves. Now let us turn to the affine setting.

Let $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of formal power series, and $\mathcal{X} = \mathbb{C}((t))$ the field of formal Laurent series. Let $D = \text{Spec}\mathcal{O}$ be the formal disk, and $D^\times = \text{Spec}\mathcal{X}$ the formal punctured disk. Let $\mathbb{C}(t, t^{-1})$ be the ring of Laurent polynomials so that $\mathbb{G}_m = \text{Spec}\mathbb{C}[t, t^{-1}]$.

In place of diagram (1.1), we take the diagram of loop groups

\[
\xymatrix{ & G(\mathcal{K}) \ar[dr] & \\
K(\mathcal{K}) \ar[ur] & L G_\mathbb{R} \ar[ur] & LG_c \\
& L K_\mathbb{c} \ar[ur] & }
\]

Here $G(\mathcal{K})$ and $K(\mathcal{K})$ are the formal loop groups of maps $D^\times \to G$ and $D^\times \to K$ respectively, $L G_\mathbb{R}, LG_c$, and $LK_\mathbb{c}$ are the subgroups of the polynomial loop group $LG = G(\mathbb{C}[t, t^{-1}])$ of those maps that take the unit circle $S^1$ into $K_\mathbb{c}, G_c$, and $G_\mathbb{R}$ respectively.

In this paper, the role of the flag manifold $\mathcal{B} \simeq G/B$ will be played by the affine Grassmannian $\text{Gr} = G(\mathcal{K})/G(0)$ \footnote{In a sequel paper [CN1], we will extend many of our results to the affine flag manifold $F_\ell = G(\mathcal{K})/I$, where $I \subset G(0)$ is an Iwahori subgroup. Our focus in this paper is the remarkable connection between the Matsuki correspondence for the affine Grassmannian and real Schubert geometry as highlighted in Sect. 1.3 below.)}

The paper [N1] establishes a Matsuki correspondence for the affine Grassmannian: there is an anti-isomorphism of orbit posets

\[
|K(\mathcal{K})\backslash\text{Gr}| \longleftrightarrow |LG_\mathbb{R}\backslash\text{Gr}|
\]

between the sets of $K(\mathcal{K})$-orbits and $LG_\mathbb{R}$-orbits on Gr, each ordered with respect to orbit closures. The correspondence matches a $K(\mathcal{K})$-orbit $\mathcal{O}_K$ with the unique $LG_\mathbb{R}$-orbit $\mathcal{O}_R$ such that the intersection $\mathcal{O}_c = \mathcal{O}_K \cap \mathcal{O}_R$ is a non-empty $LK_\mathbb{c}$-orbit.

Furthermore, the paper [N1] provides an explicit parametrization of the orbit posets (see Sect. 2 for a review).

The main result of this paper is the following Morse-theoretic interpretation and refinement of the Matsuki correspondence for the affine Grassmannian:

**Theorem 1.1** (Theorem 3.4 below). There is a $LK_\mathbb{c}$-invariant function $E : \text{Gr} \to \mathbb{R}$ and $LG_c$-invariant metric on Gr such that the associated gradient $\nabla E$ and gradient-flow $\phi_t$ satisfy the following:

1. The critical locus $\nabla E = 0$ is a disjoint union of $LK_\mathbb{c}$-orbits.
Thus the real analytic stack $LG\in \mathcal{O}_c$ union of those components consisting of $O(1.6)$ category of sheaves on it.

we give moduli interpretations of the quotient stacks $G$ we denote by $\text{Bun}$ bundles on the projective line $P$. On the real projective line $P$ finite-dimensional nor finite-codimensional (unlike the all complexes that are extensions by zero off of finite type substac ks.

make sense of the bounded constructible $K$ correspondence for sheaves on the affine Grassmannian in analogy with (1.3). In order to $(2)$ codimensional). Thus there is not a naive approach to sheaves on $K$ finite-

Next, introduce the ind-stack of quasi-maps $QM(2)(\mathbb{P}^1, G, K)$ classifying $(z_1, z_2, \varepsilon, \sigma)$ where $(z_1, z_2)$ is a point of $(\mathbb{P}^1)^2$, $\varepsilon$ is a $G$-bundle on $\mathbb{P}^1$, and $\sigma$ is a section $\mathbb{P}^1 \setminus \{z_1, z_2\} \rightarrow \varepsilon \times^G G/K$. The given conjugations on $(\mathbb{P}^1)^2$, $G, K$ induce a conjugation on $QM(2)(\mathbb{P}^1, G, K)$, and we denote by $QM(2)(\mathbb{P}^1, G, K)_R$ the real analytic ind-stack of its real points. There

Using the above refinement of the affine Matsuki correspondence $L(1.5)$, we prove a Matsuki correspondence for sheaves on the affine Grassmannian in analogy with (1.3). In order to $K$ and we denote by $\text{Bun}$ $G$ and equip $(\mathbb{P}^1)^2$ with the conjugation $(z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1)$. Its real points $(\mathbb{P}^1)^2_R$ form a real analytic space isomorphic to $\mathbb{P}^1(\mathbb{C})$ via the projection $(z_1, z_2) \mapsto z_1$.

In Proposition $5.7$, we prove the following:

$$\begin{align*}
(1.7) & \quad \text{The quotient } LG_R \setminus \text{Gr is a real analytic stack isomorphic to } \text{Bun}_G(\mathbb{P}^1)_{R, \alpha_0}.
\end{align*}$$

Thus the real analytic stack $LG_R \setminus \text{Gr}$ is locally of finite type and we have a well-defined category of sheaves on it.

**Definition 1.2.** Let $D_c(LG_R \setminus \text{Gr})$ be the bounded constructible derived category of sheaves on $LG_R \setminus \text{Gr}$. We set $D_i(LG_R \setminus \text{Gr})$ to be the full subcategory of $D_c(LG_R \setminus \text{Gr})$ consisting of all complexes that are extensions by zero off of finite type substacks.

Next let us discuss the quotient $K(\mathcal{X}) \setminus \text{Gr}$. In general, the $K(\mathcal{X})$-orbits on Gr are neither finite-dimensional nor finite-codimensional (unlike the $LG_R$-orbits on Gr which are finite-codimensional). Thus there is not a naive approach to sheaves on $K(\mathcal{X}) \setminus \text{Gr}$ with traditional methods. To overcome this, we use the observation that the quotient $LK_c \setminus \text{Gr}$ is a real analytic ind-stack of ind-finite type, i.e., an inductive limit of real analytic stacks of finite type. We will take a certain subcategory of sheaves on $LK_c \setminus \text{Gr}$ as a replacement for $K(\mathcal{X})$-equivariant sheaves on Gr.

To give more details, denote by $z \mapsto \bar{z}$ the standard conjugation of $\mathbb{P}^1$ with real form $\mathbb{P}^1$, and equip $(\mathbb{P}^1)^2$ with the conjugation $(z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1)$. Its real points $(\mathbb{P}^1)^2_R$ form a real analytic space isomorphic to $\mathbb{P}^1(\mathbb{C})$ via the projection $(z_1, z_2) \mapsto z_1$.

Next, introduce the ind-stack of quasi-maps $QM(2)(\mathbb{P}^1, G, K)$ classifying $(z_1, z_2, \varepsilon, \sigma)$ where $(z_1, z_2)$ is a point of $(\mathbb{P}^1)^2$, $\varepsilon$ is a $G$-bundle on $\mathbb{P}^1$, and $\sigma$ is a section $\mathbb{P}^1 \setminus \{z_1, z_2\} \rightarrow \varepsilon \times^G G/K$. The given conjugations on $(\mathbb{P}^1)^2$, $G, K$ induce a conjugation on $QM(2)(\mathbb{P}^1, G, K)$, and we denote by $QM(2)(\mathbb{P}^1, G, K)_R$ the real analytic ind-stack of its real points. There

(2) The gradient-flow $\phi_t$ preserves the $K(\mathcal{X})$-and $LG_R$-orbits.

(3) The limits $\lim_{t \to \pm \infty} \phi_t(\gamma)$ of the gradient-flow exist for any $\gamma \in \text{Gr}$. For each $LK_c$-orbit $O_c$ in the critical locus, the stable and unstable sets

\begin{align*}
\mathcal{O}_K &= \{ \gamma \in \text{Gr} | \lim_{t \to \infty} \phi_t(\gamma) \in O_c \} \\
\mathcal{O}_R &= \{ \gamma \in \text{Gr} | \lim_{t \to -\infty} \phi_t(\gamma) \in O_c \}
\end{align*}

are a single $K(\mathcal{X})$-orbit and $LG_R$-orbit respectively.

(4) The correspondence between orbits $O_K \leftrightarrow O_R$ defined by $(1.3)$ recovers the affine Matsuki correspondence $(1.5)$. 

In Proposition $5.7$, we prove the following:

(1.7) The quotient $LG_R \setminus \text{Gr}$ is a real analytic stack isomorphic to $\text{Bun}_G(\mathbb{P}^1)_{R, \alpha_0}$.

Thus the real analytic stack $LG_R \setminus \text{Gr}$ is locally of finite type and we have a well-defined category of sheaves on it.

**Definition 1.2.** Let $D_c(LG_R \setminus \text{Gr})$ be the bounded constructible derived category of sheaves on $LG_R \setminus \text{Gr}$. We set $D_i(LG_R \setminus \text{Gr})$ to be the full subcategory of $D_c(LG_R \setminus \text{Gr})$ consisting of all complexes that are extensions by zero off of finite type substacks.
is a natural projection $QM^{(2)}(\mathbb{P}^1, G, K)_{\mathbb{R}} \to \text{Bun}_G(\mathbb{P}^1)_\mathbb{R}$, $(z_1, z_2, \mathcal{E}, \sigma) \mapsto \mathcal{E}$, and we write $QM^{(2)}(\mathbb{P}^1, G, K)_{\mathbb{R}, \alpha_0}$ for the pre-image of the components $\text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}, \alpha_0}$.

We also have the natural projection $QM^{(2)}(\mathbb{P}^1, G, K)_{\mathbb{R}} \to (\mathbb{P}^1)^2_{\mathbb{R}} \simeq \mathbb{P}^1(\mathbb{C})$, $(z_1, z_2, \mathcal{E}, \sigma) \mapsto z_1$. For $z \in \mathbb{P}^1(\mathbb{C})$, denote by $QM^{(2)}(\mathbb{P}^1, z, G, K)_{\mathbb{R}}$ the fiber of $QM^{(2)}(\mathbb{P}^1, G, K)_{\mathbb{R}}$ over $z$, and by $QM^{(2)}(\mathbb{P}^1, z, G, K)_{\mathbb{R}, \alpha_0}$ the intersection of the fiber with $QM^{(2)}(\mathbb{P}^1, G, K)_{\mathbb{R}, \alpha_0}$. In particular, for the (non-real) point $i \in \mathbb{P}^1(\mathbb{C})$, we have the fiber $QM^{(2)}(\mathbb{P}^1, i, G, K)_{\mathbb{R}}$, and the intersection $QM^{(2)}(\mathbb{P}^1, i, G, K)_{\mathbb{R}, \alpha_0}$.

In section 6, we prove the following:

(1.8) The quotient $LK_c\backslash \text{Gr}$ is a real analytic ind-stack isomorphic to $QM^{(2)}(\mathbb{P}^1, i, G, K)_{\mathbb{R}, \alpha_0}$.

Thus the real analytic ind-stack $LK_c\backslash \text{Gr}$ is of ind-finite type and we have a well-defined category of sheaves on it.

Finally, denote by $S$ the stratification of $LK_c\backslash \text{Gr}$ with strata the $LK_c$-quotients of $K(\mathcal{K})$-orbits. By Theorem [1.1] each $K(\mathcal{K})$-orbit $O_K$ deformation retracts to an $LK_c$-orbit $O_c$. This suggests the following definition.

**Definition 1.3.** Let $D_c(LK_c\backslash \text{Gr})$ be the bounded constructible derived category of sheaves on $LK_c\backslash \text{Gr}$. We set $D_c(K(\mathcal{K})\backslash \text{Gr})$ to be the full subcategory of $D_c(LK_c\backslash \text{Gr})$ of complexes constructible with respect to the stratification $S$.

We are now ready to state our second main result, the affine Matsuki correspondence for sheaves.

**Theorem 1.4** (Theorem [7.1] below). There is an equivalence of categories

$$\Upsilon : D_c(K(\mathcal{K})\backslash \text{Gr}) \xrightarrow{\sim} D_c(LG_R \backslash \text{Gr})$$

**Remark 1.5.** The category $D_c(K(\mathcal{K})\backslash \text{Gr})$, respectively $D_c(LG_R \backslash \text{Gr})$, is generated by standard, respectively costandard, objects, and the equivalence $\Upsilon$ maps standard objects to costandard objects.

**Remark 1.6.** Our first two main results, Theorems [1.1] and [1.4], admit natural generalizations from the affine Grassmannian to any (partial) affine flag manifold. (Beyond the affine Grassmannian, we do not know whether the orbit posets have as simple a parameterization as recounted in Sect. [2].) The addition of Iwahori and other level structures offers further interesting geometry, especially in families as we vary their place along the curve, and we postpone details to the sequel paper [CNT].

**Remark 1.7.** In place of the standard conjugation $z \mapsto \bar{z}$ of the projective line $\mathbb{P}^1$, we could take the “antipodal” conjugation $z \mapsto -\bar{z}^{-1}$ whose real points are empty. In place of
**Diagram (1.4)**, we would find the diagram of “twisted” loop groups

\[
\begin{array}{ccc}
G(\mathcal{K}) & \rightarrow & G_{\theta}(\mathcal{K}) \\
\downarrow & & \downarrow \\
L_{\eta} G & \rightarrow & L_{\eta_{c}} G \\
\downarrow & & \downarrow \\
L_{\eta_{c}} K & \rightarrow & \end{array}
\]

Here \(G_{\theta}(\mathcal{K})\) is the subgroup of maps \(\gamma : D^{\times} \rightarrow G\) such that \(\gamma(-z) = \theta(\gamma(z))\) where \(\theta : G \rightarrow G\) is the involution that cuts out \(K \subset G\). Similarly, \(L_{\eta} G\), respectively \(L_{\eta_{c}} G\), is the subgroup of maps \(\gamma : \mathbb{C}^{\times} \rightarrow G\) such that \(\gamma(-z^{-1}) = \eta(\gamma(z))\), respectively \(\gamma(-z^{-1}) = \eta_{c}(\gamma(z))\), where \(\eta : G \rightarrow G\), respectively \(\eta_{c} : G \rightarrow G\), is the conjugation that cuts out \(G_{\mathbb{R}} \subset G\), respectively \(G_{c} \subset G\). Lastly, \(L_{\eta_{c}} K\) is the subgroup of maps \(\gamma : \mathbb{C}^{\times} \rightarrow K\) such that \(\gamma(-z^{-1}) = \eta_{c}(\gamma(z))\), and is the intersection of any two of the above three subgroups.

We expect statements analogous to our first two main results, Theorems 1.1 and 1.4, along with the moduli interpretations that underlie them, to hold with this setup as well.

**1.3. Relation to Schubert geometry.** In this section, we state our third main result, a remarkable connection between the Matsuki correspondence for the affine Grassmannian and real Schubert geometry.

Let \(\mathcal{O}_{\mathbb{R}} = \mathbb{R}[[t]]\), and \(\mathcal{X}_{\mathbb{R}} = \mathbb{R}((t))\). Consider the real affine Grassmannian \(\text{Gr}_{\mathbb{R}} = G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})/G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})\). The group \(G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})\), respectively \(G_{\mathbb{R}}(\mathbb{R}[t^{-1}])\), acts on \(\text{Gr}_{\mathbb{R}}\) with finite-dimensional, respectively finite-codimensional, orbits.

Recall the uniformization of real analytic stacks of (1.7):

\[
L_{G_{\mathbb{R}}} \backslash \text{Gr} \simeq \text{Bun}_{G}(\mathbb{P}^{1})_{\mathbb{R},0,0}
\]

In its construction, we view \(\text{Gr}\) as based at the (non-real) point \(i \in \mathbb{P}^{1}(\mathbb{C})\). When we instead focus on the (real) point \(0 \in \mathbb{P}^{1}(\mathbb{R})\), we obtain an alternative uniformization:

\[
G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_{\mathbb{R}} \simeq \text{Bun}_{G}(\mathbb{P}^{1})_{\mathbb{R},0,0}
\]

Let \(D_{c}(G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}}) \backslash \text{Gr}_{\mathbb{R}})\) be the bounded constructible \(G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}})\)-equivariant derived category of sheaves on the ind-scheme \(\text{Gr}_{\mathbb{R}}\). Let \(D_{c}(G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_{\mathbb{R}})\) be the bounded constructible derived category of sheaves on the stack \(G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_{\mathbb{R}}\), and set \(D_{c}(G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_{\mathbb{R}})\) to be the full subcategory of complexes that are extensions by zero off of finite type substacks.

Recall the affine Matsuki correspondence for sheaves of Theorem 1.4:

\[
\Upsilon : D_{c}(K(\mathcal{K}) \backslash \text{Gr}) \overset{\sim}{\longrightarrow} D_{c}(L_{G_{\mathbb{R}}} \backslash \text{Gr})
\]

In Proposition 8.8 we show the Radon transform provides an analogous equivalence:

\[
\Upsilon_{\mathbb{R}} : D_{c}(G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}}) \backslash \text{Gr}_{\mathbb{R}}) \overset{\sim}{\longrightarrow} D_{c}(G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_{\mathbb{R}})
\]

The main ingredient in the proof of the equivalence \(\Upsilon_{\mathbb{R}}\) is the natural \(\mathbb{R}_{>0}\)-action on \(\text{Gr}_{\mathbb{R}}\) (as Morse theory is the main ingredient in the proof of Theorem 1.4).
Our third main result is a nearby cycles equivalence intertwining the Matsuki correspondence of (1.12) and the Radon transform of (1.13). To state it, recall the quasi-map family $QM^{(2)}(\mathbb{P}^1, G, K)_\mathbb{R} \to \mathbb{P}^1(\mathbb{C})$. Recall as well for the (non-real) point $i \in \mathbb{P}^1(\mathbb{C})$, the identification of the fiber:

\[(1.14) \quad LK_c \backslash Gr \simeq QM^{(2)}(\mathbb{P}^1, i, G, K)_{\mathbb{R}, \alpha_0}.\]

For the (real) point $0 \in \mathbb{P}^1(\mathbb{C})$, we have an analogous identification of the fiber:

\[(1.15) \quad K_c \backslash Gr_{\mathbb{R}} \simeq QM^{(2)}(\mathbb{P}^1, 0, G, K)_{\mathbb{R}, \alpha_0}.\]

Taking nearby cycles in the family descends to a functor

\[(1.16) \quad \Psi : D_c(K(\mathbb{K}) \backslash Gr) \to D_c(G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}}) \backslash Gr_{\mathbb{R}}).\]

**Theorem 1.8** (Theorem 8.1 and 8.2 below). There is a canonical commutative square of equivalences

\[(1.17) \quad \begin{array}{ccc}
D_c(K(\mathbb{K}) \backslash Gr) & \xrightarrow{\Psi} & D_c(G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}}) \backslash Gr_{\mathbb{R}}) \\
\gamma & & \gamma_{\mathbb{R}} \\
D_c(LG_{\mathbb{R}} \backslash Gr) & \xrightarrow{\Psi_{\mathbb{R}}} & D_c(G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash Gr_{\mathbb{R}})
\end{array}\]

where the equivalence $\Psi_{\mathbb{R}}$ is given by transport along the uniformization isomorphisms

\[(1.18) \quad LG_{\mathbb{R}} \backslash Gr \simeq Bun_G(\mathbb{P}^1, \mathbb{R}, 0) \simeq G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash Gr_{\mathbb{R}}.\]

1.4. **Further directions.** In this final section of the introduction, we discuss results to appear in sequel papers that build on those of the current paper.

1.4.1. **Kostant-Sekiguchi correspondence.** One of our primary motivations for studying the affine Matsuki correspondence is its application to the Kostant-Sekiguchi correspondence. This will be the subject of the sequel paper [CN2] which we briefly survey here.

Let $\mathfrak{g}, \mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{k}$ be the Lie algebras of $G, G_{\mathbb{R}}$ and $K$ respectively, and introduce the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone, and introduce the real nilpotent cone $\mathcal{N}_{\mathbb{R}} = \mathcal{N} \cap \mathfrak{g}_{\mathbb{R}}$, and the $\mathfrak{p}$-nilpotent cone $\mathcal{N}_p = \mathcal{N} \cap \mathfrak{p}$. The adjoint actions of $G, G_{\mathbb{R}}$ and $K$ preserve $\mathcal{N}, \mathcal{N}_{\mathbb{R}}$ and $\mathcal{N}_p$ respectively and have finitely many orbits.

The celebrated Kostant-Sekiguchi correspondence [S] is a poset isomorphism

\[(1.19) \quad |K \backslash \mathcal{N}_p| \leftrightarrow |G_{\mathbb{R}} \backslash \mathcal{N}_{\mathbb{R}}|\]

between sets of $K$-orbits and $G_{\mathbb{R}}$-orbits, each ordered with respect to orbit closure.

Let $\mathcal{O} \subset \mathcal{N}_p$ be a $K$-orbit and $\mathcal{O}' \subset \mathcal{N}_{\mathbb{R}}$ the corresponding $G_{\mathbb{R}}$-orbit under the Kostant-Sekiguchi correspondence. The papers [SV, V] establish the following remarkable result:

\[(1.20) \quad \text{There is a real analytic } K_c\text{-equivariant isomorphism } \mathcal{O} \simeq \mathcal{O}'.\]

Now recall from Theorem 1.8 the equivalence

\[(1.21) \quad \Psi : D_c(K(\mathbb{K}) \backslash Gr) \xrightarrow{\sim} D_c(G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}}) \backslash Gr_{\mathbb{R}}).\]
given by nearby cycles in the quasi-map family $QM^{(2)}(\mathbb{P}^1, G, K)_R \to \mathbb{P}^1(\mathbb{C})$.

In the sequel paper [CN2], we show the quasi-map family in fact admits a topological trivialization providing a $K_c$-equivariant homeomorphism

\begin{equation}
\Omega K_c \backslash \text{Gr} \sim \text{Gr}_R
\end{equation}

We also introduce another quasi-map family $QM^{(2)}(\mathcal{X}, G, K)_R \to A^1(\mathbb{C})$ induced by a degeneration $\mathcal{X} \to A^1$ of the projective line $\mathbb{P}^1$ to a nodal curve $\mathbb{P}^1 \vee \mathbb{P}^1$. Its restriction to an open subspace can be modeled by the flat family of quotients

\begin{equation}
\Omega K_c \backslash \text{Gr} \sim \text{Gr}_R \to K(\mathbb{C}[t^{-1}])_1 \backslash \text{Gr}
\end{equation}

where $K(\mathbb{C}[t^{-1}])_1 \subset K(\mathbb{C}[t^{-1}])$ is the kernel of evaluation at $\infty$.

Now we arrive at our intended application to the Kostant-Sekiguchi correspondence. It is well-known [L] that when $G$ is of type $A$, the nilpotent cone $N$ embeds in the affine Grassmannian $\text{Gr}$. Furthermore, this induces embeddings of the real and $p$-nilpotent cones:

\begin{equation}
N_R \subset \text{Gr}_R \quad N_p \subset K(\mathbb{C}[t^{-1}])_1 \backslash \text{Gr}
\end{equation}

Applying the geometry of (1.22) and (1.23), we obtain in type A a lift of the Kostant-Sekiguchi correspondence:

\begin{equation}
\text{There is a } K_c\text{-equivariant orbit-preserving homeomorphism } N_p \simeq N_R.
\end{equation}

Thanks to the compatibility of our constructions with inner automorphisms and Cartan involutions, we are in fact able to deduce (1.25) for all classical types from the case of type $A$.

1.4.2. **Comparison of dual groups.** The paper [N1] associates to each real form $G_R \subset G$ a reductive subgroup $H_{\text{real}}^\vee \subset G^\vee$ of the dual group.\(^2\) The construction of $H_{\text{real}}^\vee$ is via Tannakian formalism: its tensor category of finite-dimensional representations $\text{Rep}(H_{\text{real}}^\vee)$ is realized as a certain full subcategory $Q_R \subset D_c(G_R(O_R) \backslash \text{Gr}_R)$ of perverse sheaves on the real affine Grassmannian $\text{Gr}_R$.

On the other hand, the papers [GN1] [GN2] associate to every spherical subgroup $K \subset G$ a reductive subgroup $H_{\text{sph}}^\vee \subset G^\vee$ of the dual group. Again, the construction of $H_{\text{sph}}^\vee$ is via Tannakian formalism: its tensor category of finite-dimensional representations $\text{Rep}(H_{\text{sph}}^\vee)$ can be realized as a certain full subcategory $Q_K \subset D_c(K(G) \backslash \text{Gr})$ of perverse sheaves where as usual we understand $D_c(K(G) \backslash \text{Gr})$ as complexes on a quasi-map space with target $G/K$.

When $K \subset G$ is the symmetric subgroup of a real form $G_R \subset G$, we may ask whether the above two subgroups $H_{\text{real}}^\vee, H_{\text{sph}}^\vee \subset G^\vee$ coincide. Their inclusions into $G^\vee$ are determined under Tannakian formalism by the respective tensor functors of restriction $\text{Rep}(G^\vee) \to \text{Rep}(H_{\text{real}}^\vee), \text{Rep}(G^\vee) \to \text{Rep}(H_{\text{sph}}^\vee)$. By construction, these functors correspond to functors from the Satake category $\text{Sat}_G \subset D_c(G(O) \backslash \text{Gr})$ to the respective categories $Q_R, Q_K$.

\(^2\)While the notation suggests regarding $H_{\text{real}}^\vee$ itself as a dual group, we do not know of a concrete role for its dual group.
Details of the following compatibility will be given in [CNI]. Recall from Theorem 1.8 the nearby cycles equivalence

\[
\Psi : D_c(K(K)\Gr) \sim \rightarrow D_c(G_\mathbb{R}(\mathcal{O}_\mathbb{R})\Gr_\mathbb{R})
\]
given by nearby cycles in the quasi-map family \(QM^{(2)}(\mathbb{P}^1, G, \mathcal{K})_\mathbb{R} \rightarrow \mathbb{P}^1(\mathbb{C})\)

**Theorem 1.9.** The functor \(\Psi\) restricts to the horizontal tensor equivalence in a commutative diagram of tensor functors

\[
\begin{array}{ccc}
Sat_G & \sim & Q_\mathbb{R} \\
\downarrow & & \downarrow \\
Q_K & \sim & Q_\mathbb{R}
\end{array}
\]

1.4.3. **Representation theory of real loop groups.** Let us briefly sketch here another motivation for the results of this paper.

Recall the usual Matsuki correspondence for sheaves [MUV] intertwines the Beilinson-Bernstein localization [BB] of Harish Chandra \((\mathfrak{g}, K)\)-modules with the Kashiwara-Schmid localization [KS] of (infinitesimal classes of) admissible representations of \(G_\mathbb{R}\). We seek an analogous geometric approach to the representation theory of real loop groups.

For a compact real form \(G_c \subset G\), the positive energy representation theory of the real loop group \(LG_c\) offers analogues of the many beautiful geometric and combinatorial aspects of the representation theory of \(G_c\) itself. For example, there is a Borel-Weil-Bott construction of irreducibles, a Weyl-Kac character formula via localization, BGG resolutions via Schubert geometry, among other now standard results [PS]. Furthermore, there is the celebrated fusion structure on level \(k\) representations as organized by rational conformal field theory.

In comparison, for a non-compact real form \(G_\mathbb{R} \subset G\), relatively little representation theory of the real loop group \(LG_\mathbb{R}\) has been developed. This is so even though there are longstanding motivations coming from Chern-Simons theory for non-compact gauge group. With the results of this paper in hand, one might hope to engineer a representation theory of \(LG_\mathbb{R}\) by suitably “globalizing” \(K(\mathcal{K})\)-equivariant and \(LG_\mathbb{R}\)-equivariant sheaves on affine flag varieties. Unfortunately, traditional global sections constructions, following Beilinson-Bernstein or Kashiwara-Schmid, appear either to produce no new representations or to lead to semi-infinite pathologies.

We expect a theory of admissible representations of \(LG_\mathbb{R}\) to fit within the framework of representations on pro-vector spaces. More specifically, we conjecture the derived categories of equivariant sheaves in the affine Matsuki correspondence are equivalent to categories of \(LG_\mathbb{R}\)-representations that are admissible in the sense that they are pro-objects in the positive energy representations of \(LK_c\). We plan to approach this in future work.

1.5. **Organization.** In Section 2, we recall the parametrization of \(K(\mathcal{K})\)-orbits and \(LG_\mathbb{R}\)-orbits on the affine Grassmannian and the statement of the affine Matsuki correspondence. We also establish some geometric properties for those orbits. In Section 3 we construct the Matsuki flow on the affine Grassmannian and we give a Morse-theoretic interpretation and refinement of the Matsuki correspondence for the affine Grassmannian. In Section 4...
we study real forms of Beilinson-Drinfeld Grassmannians. In Sections 5 and 6 we study moduli stacks of real bundles on \( \mathbb{P}^1 \) and quasi-maps. We study uniformizations for those moduli stacks and use them to provide moduli interpretations for various quotients of the affine Grassmannian by subgroups of the loop group. In Section 7 we prove the affine Matsuki correspondence for sheaves (Theorem 1.4). In Section 8 we prove the nearby cycles equivalences and the Radon transform equivalence (Theorem 1.8). In Section 9 we study the compatibility of Hecke actions. In Appendix A we discuss real analytic stacks and categories of sheaves on real analytic stacks.

1.6. Acknowledgements. T.H. Chen would like to thank the Max Planck Institute for Mathematics for support, hospitality, and a nice research environment. D. Nadler would like to thank the Miller Institute for its inspiring environment. The research of T.H. Chen is supported by NSF grant DMS-1702337 and that of D. Nadler by NSF grant DMS-1502178.

2. \( K(\mathcal{X}) \) and \( LG_{\mathbb{R}} \)-orbits on Gr

In this section we study \( K(\mathcal{X}) \) and \( LG_{\mathbb{R}} \)-orbits on the affine Grassmannian \( Gr \).

2.1. Loop groups. The real forms \( G_{\mathbb{R}} \) and \( G_c \) of \( G \) correspond to anti-holomorphic involutions \( \eta \) and \( \eta_c \). The involutions \( \eta \) and \( \eta_c \) commutes with each other and \( \theta := \eta_c \circ \eta = \eta \circ \eta_c \) is an involution of \( G \). We have \( K = G^\theta \), \( G_{\mathbb{R}} = G^\eta \), and \( G_c = G^{\eta_c} \). We fix a maximal split tours \( S_{\mathbb{R}} \subset G_{\mathbb{R}} \) and a maximal torus \( T_{\mathbb{R}} \) such that \( S_{\mathbb{R}} \subset T_{\mathbb{R}} \). We write \( S \) and \( T \) for the complexification of \( S_{\mathbb{R}} \) and \( T_{\mathbb{R}} \). We denote by \( \Lambda_T \) the lattice of coweights of \( T \) and \( \Lambda_S \) the lattice of real coweights. We write \( \Lambda^+_T \) the set of dominant coweight with respect to the Borel subgroup \( B \) and define \( \Lambda^+_S := \Lambda_S \cap \Lambda^+_T \). For any \( \lambda \in \Lambda_T \) we define \( \eta(\lambda) \in \Lambda_T \) as

\[
\eta(\lambda) : \mathbb{C}^x \xrightarrow{\tau} \mathbb{C}^x \xrightarrow{\eta} T \xrightarrow{\eta_c} T,
\]

where \( \tau \) is the complex conjugation of \( \mathbb{C}^x \) with respect to \( \mathbb{R}^x \). The assignment \( \lambda \mapsto \eta(\lambda) \) defines an involution on \( \Lambda_T \), which we denote by \( \eta \), and \( \Lambda_S \) is the fixed points of \( \eta \).

Let \( LG := G(\mathbb{C}[t, t^{-1}]) \) be the (polynomial) loop group associated to \( G \). We define the following involutions on \( LG \): for any \( (\gamma : \mathbb{C}^x \to G) \in LG \) we set

\[
\eta^r(\gamma) : \mathbb{C}^x \xrightarrow{\tau} \mathbb{C}^x \xrightarrow{\gamma} \mathbb{C}^x \xrightarrow{\eta} G \xrightarrow{\eta} G,
\]

\[
\eta^c(\gamma) : \mathbb{C}^x \xrightarrow{\tau} \mathbb{C}^x \xrightarrow{\gamma} \mathbb{C}^x \xrightarrow{\eta_c} G \xrightarrow{\eta_c} G.
\]

Here \( \tau(x) = x^{-1} \) is the the inverse map. Denote by \( \mathcal{X} = \mathbb{C}((t)) \) and \( \Theta = \mathbb{C}[[t]] \). We have the following diagram
Here $LG_{\mathbb{R}}$ and $LG_{\mathbb{C}}$ are the fixed points subgroups of the involutions $\eta^r$ and $\eta^c$ on $LG$ respectively. Equivalently, $LG_{\mathbb{R}}$ (resp. $LG_{\mathbb{C}}$) is the subgroup of $LG$ consisting of maps that take the unit circle $S^1 \subset \mathbb{C}$ to $G_{\mathbb{R}}$ (resp. $G_{\mathbb{C}}$). We define the based loop group $\Omega G_{\mathbb{C}}$ to be the subgroup of $LG_{\mathbb{C}}$ consisting of maps that take $1 \in S^1$ to $e \in G_{\mathbb{C}}$.

2.2. The based loop spaces $\Omega X_c$. We define $X \subset G$ (resp. $X_c \subset G_c$) to be the identity component of the fixed point subspace of the involution $\tilde{\theta} = \theta^{-1}$ on $G$ (resp. $G_c$). The map $\pi : G \to X, \pi(g) = \tilde{\theta}(g)g$ induces a $G$-equivariant isomorphism $K \backslash G \simeq X$ (resp. $G_c$-equivariant isomorphism $K_c \backslash G_c \simeq X_c$). We define the loop space $LX_c$ to be the subspace of $LG_{\mathbb{C}}$ consisting of maps that take $S^1$ into $X_c$. We define the based loop space $\Omega X_c$ to be the subspace of $LX_c$ consisting of maps that take $1 \in S^1$ to $e \in X_c$.

2.3. Real affine Grassmannians. We recall results from [N1] about the real affine Grassmannian. Let $Gr := G(\mathbb{K})/G(\theta)$ be the affine Grassmannian for $G$ and $Gr_{\mathbb{R}} := G(\mathbb{K}_{\mathbb{R}})/G(\theta_{\mathbb{R}})$ be the real affine Grassmannian. For any $\lambda \in \Lambda^+_T$ we denote by $S^\lambda$ and $T^\lambda$ the $G(\theta)$ and $G(\mathbb{C}[t^{-1}])$-orbit of $t^\lambda \in Gr$. The orbits $S^\lambda$ and $T^\mu$ on $Gr$ are transversal and the intersection $C^\lambda = S^\lambda \cap T^\lambda$ is isomorphic to the flag manifold $G/P^\lambda$ where the parabolic subgroup $P^\lambda$ is the stabilizer of $\lambda$. The affine Grassmannian $Gr$ is the disjoint union of the orbits $S^\lambda$ (resp. $T^\lambda$) for $\lambda \in \Lambda^+_T$.

$Gr = \bigsqcup_{\lambda \in \Lambda^+_T} S^\lambda$ (resp. $Gr = \bigsqcup_{\lambda \in \Lambda^+_T} T^\lambda$)

and we have

$\overline{S}^\lambda = \bigsqcup_{\mu \leq \lambda} S^\mu$ (resp. $\overline{T}^\lambda = \bigsqcup_{\lambda \leq \mu} T^\mu$).

The intersection of $S^\lambda$ (resp. $T^\lambda$) with $Gr_{\mathbb{R}}$ is nonempty if and only if $\lambda \in \Lambda^+_S$ and we write $S^\lambda_S$ (resp. $T^\lambda_S$), $\lambda \in \Lambda^+_S$ for the intersection. We define $C^\lambda_S$ to be the intersection of $S^\lambda_S$ and $T^\lambda_S$. $S^\lambda$ (resp. $T^\lambda$) is equal to the $G(\mathbb{R}(\theta_{\mathbb{R}})$-orbit (resp. $G_{\mathbb{R}}(\mathbb{R}[t^{-1}])$-orbit) of $t^\lambda$ and $C^\lambda_S$ is isomorphic to the real flag manifold $G_{\mathbb{R}}/P^\lambda_{\mathbb{R}}$ where the parabolic subgroup $P^\lambda_{\mathbb{R}} \subset G_{\mathbb{R}}$ is the stabilizer of $\lambda$. The real affine Grassmannian $Gr$ is the disjoint union of the orbits $S^\lambda_{\mathbb{R}}$ (resp. $T^\lambda_{\mathbb{R}}$) for $\lambda \in \Lambda^+_T$.

$Gr_{\mathbb{R}} = \bigsqcup_{\lambda \in \Lambda^+_S} S^\lambda_S$ (resp. $Gr_{\mathbb{R}} = \bigsqcup_{\lambda \in \Lambda^+_S} T^\lambda_S$)

and we have

$\overline{S}^\lambda_{\mathbb{R}} = \bigsqcup_{\mu \leq \lambda} S^\mu_{\mathbb{R}}$ (resp. $\overline{T}^\lambda_{\mathbb{R}} = \bigsqcup_{\lambda \leq \mu} T^\mu_{\mathbb{R}}$).

2.4. The energy flow on $\Omega G_c$. We recall the construction of energy flow on $\Omega G_c$ following [PS, Section 8.9]. For any $\gamma \in LG_{\mathbb{C}}$ and $v \in T_\gamma LG_{\mathbb{C}}$ we denote by $\gamma^{-1}v \in LG_{\mathbb{C}}$ (resp. $v\gamma^{-1} \in LG_{\mathbb{C}}$) the image of $v \in T_\gamma LG_{\mathbb{C}}$ under the isomorphism $T_\gamma LG_{\mathbb{C}} \simeq T_\gamma LG_{\mathbb{C}} \simeq LG_{\mathbb{C}}$ induced by the left action (resp. right action).

Fix a $G_{\mathbb{C}}$-invariant metric $\langle \cdot, \cdot \rangle$ on $g_{\mathbb{C}}$. Observe that the formula

$\omega(v, w) := \int_{S^1} \langle (\gamma^{-1}v)', \gamma^{-1}w \rangle d\theta$

12
defines a left invariant symplectic form on $T_\gamma \Omega G_c$. According to [PS, Theorem 8.6.2], the composition $\Omega G_c \to G(\mathcal{K}) \to \text{Gr}$ defines a diffeomorphism

$$\Omega G_c \simeq \text{Gr}.$$ 

Let $J_\gamma$ be the automorphism of $T_\gamma \Omega G_c$ which corresponds to multiplication by $i$ in terms of the complex structure on $\text{Gr}$. The formula $g(v, w) = \omega(v, J_\gamma w)$ defines a positive inner product on $T_\gamma \Omega G_c$ and the Kähler form on $T_\gamma \Omega G_c$ is given by $g(v, w) + i\omega(v, w)$. Finally, for any smooth function $F : \Omega G_c \to \mathbb{R}$ there corresponds so-called Hamiltonian vector field $R(\gamma)$ and gradient vector field $\nabla F(\gamma)$ on $\Omega G_c$ characterized by

$$\omega(R(\gamma), v) = dF(\gamma)(u), \quad g(\nabla F(\gamma), u) = dF(\gamma)(u).$$

Consider the energy function on $\Omega G_c$:

$$E : \Omega G_c \to \mathbb{R}, \quad \gamma \to (\gamma', \gamma')_\gamma = \int_{S^1} \langle \gamma^{-1}\gamma', \gamma^{-1}\gamma' \rangle d\theta.$$

We have the following well-known facts.

**Proposition 2.1.** ([P, PS]

1. The Hamiltonian vector field of $E$ is equal to the vector field induced by the rotation flow $\gamma_a(t) = \gamma(t + a)\gamma(a)^{-1}$ and is given by $\gamma \to R(\gamma) = \gamma' - \gamma\gamma'(0)$. The gradient vector field of $E$ is equal to $\nabla E = -J \circ R$.

2. The critical locus $\nabla E = 0$ is the disjoint union $\bigsqcup_{\lambda \in \Lambda_T^+} C^\lambda$ of $G_c$-orbits of $\lambda \in \Omega G_c$.

3. The gradient flow $\psi_t$ of $\nabla E$ preserves the orbits $S^\lambda$ and $T^\lambda$. For each critical orbit $C^\lambda$, we have

$$S^\lambda = \{ \gamma \in \Omega G_c | \lim_{t \to \infty} \psi_t(\gamma) \in C^\lambda \} \quad T^\lambda = \{ \gamma \in \Omega G_c | \lim_{t \to -\infty} \psi_t(\gamma) \in C^\lambda \}.$$ 

That is $S^\lambda$ and $T^\lambda$ are the stable and unstable manifold of $C^\lambda$.

2.5. **Component groups of $\text{Gr}_R$.** The diffeomorphism $\Omega G_c \simeq \text{Gr}$ induces a diffeomorphism on the $g$-fixed points $(\Omega G_c)^g \simeq \text{Gr}_R$. Let $\Omega_{\text{top}}G_c$ and $\Omega_{\text{top}}X_c$ be the (topological) based loop spaces of $G_c$ and $X_c$. Note that for any $\gamma \in (\Omega_{\text{top}}G_c)^g$ we have $\gamma(-1) \in K_c$ and the map $e^{i\theta} \to \gamma'(e^{i\theta}) := \pi \circ \gamma(e^{i\theta}/2)$ defines a map $\gamma' : S^1 \to X_c$, that is, $\gamma' \in \Omega_{\text{top}}X_c$. According to [M] the composition

$$q : (\Omega G_c)^g \to (\Omega_{\text{top}}G_c)^g \to \Omega_{\text{top}}X_c$$

is a homotopic equivalence where the first map is the natural inclusion and the second map is given by $\gamma \to \gamma'$. Since $X_c$ is a deformation retract of $X$, the map $q$ induces an isomorphism

$$\pi_0(\text{Gr}_R) \simeq \pi_0((\Omega G_c)^g) \simeq \pi_0(\Omega_{\text{top}}X_c) \simeq \pi_1(X_c) \simeq \pi_1(X).$$

2.6. **Parametrization of $K(\mathcal{K})$ and $L\text{Gr}_R$-orbits.** We recall results from [N2] about the parametrization of $K(\mathcal{K})$ and $L\text{Gr}_R$-orbits on $\text{Gr}$. Consider the following diagram

$$\pi_1(G) \xrightarrow{\pi} \pi_1(X) \xleftarrow{[\_]} \Lambda_g,$$

where the first map is that induced by the map $\pi : G \to X$ and the second map $[\_]$ assigns to a loop its homotopy class.
**Definition 2.2.** We define $\mathcal{L} \subset \Lambda_+^r$ to be the inverse image of $\pi_*(\pi_1(G))$ along the map $[-]$.

**Remark 2.3.** If $K$ is connected, then we have $\mathcal{L} = \Lambda_+^r$.

**Proposition 2.4 (N1).** We have the following.

1. There is a bijection
   \[ |K(\mathcal{X}) \setminus \text{Gr}| \longleftrightarrow \mathcal{L} \]
   between $K(\mathcal{X})$-orbits on $\text{Gr}$ and $\mathcal{L}$ characterized by the following properties: Let $\mathcal{O}^\lambda_K$ be the $K(\mathcal{X})$-orbits corresponding to $\lambda \in \mathcal{L}$. Then for any $\gamma \in \mathcal{O}^\lambda_K$, thought of as an element in $\Omega G_c$, satisfies $\tilde{\theta}(\gamma) \gamma \in G(\mathbb{C}[t])t^\lambda G(\mathbb{C}[t])$. In addition, we have $\mathcal{O}^\lambda_K = \bigcup_{\mu \leq \lambda} \mathcal{O}^\mu_K$.

2. There is a bijection
   \[ |LG_R \setminus \text{Gr}| \longleftrightarrow \mathcal{L} \]
   between $LG_R$-orbits on $\text{Gr}$ and $\mathcal{L}$ characterized by the following property: Let $\mathcal{O}^\lambda_R$ be the $LG_R$-orbits corresponding to $\lambda \in \mathcal{L}$. Then for any $\gamma \in \mathcal{O}^\lambda_R$, thought of as an element in $\Omega G_c$, satisfies $\tilde{\eta}^\gamma(\gamma) \gamma \in G(\mathbb{C}[t^{-1}])t^\lambda G(\mathbb{C}[t])$. In addition, we have $\mathcal{O}^\lambda_R = \bigcup_{\mu \leq \lambda} \mathcal{O}^\mu_R$.

3. The correspondence
   \[ (2.3) \quad |K(\mathcal{X}) \setminus \text{Gr}| \longleftrightarrow |LG_R \setminus \text{Gr}|, \quad \mathcal{O}^\lambda_K \longleftrightarrow \mathcal{O}^\lambda_R \]
   provides an order-reversing isomorphism from the poset $|K(\mathcal{X}) \setminus \text{Gr}|$ to the poset $|LG_R \setminus \text{Gr}|$ (with respect to the closure ordering). In addition, for each $K(\mathcal{X})$-orbit $\mathcal{O}^\lambda_K$, $\mathcal{O}^\lambda_R$ is the unique $LG_R$-orbit such that $\mathcal{O}^\lambda_c := \mathcal{O}^\lambda_K \cap \mathcal{O}^\lambda_R$ is a single $LK_c$-orbit.

We will call (2.3) the Affine Matsuki correspondence.

**Corollary 2.5.** The $K(\mathcal{X})$-orbits and $LG_R$-orbits are stable under the rotation flow $\gamma_a(t)$ (see Proposition 2.4).

**Proof.** We give a proof for the case of $K(\mathcal{X})$-orbits. The proof for the $LG_R$-orbits is similar. Let $\mathcal{O}^\lambda_K$ be a $K(\mathcal{X})$-orbit and let $\gamma = \gamma(t) \in \mathcal{O}^\lambda_K$. By Proposition 2.4, we need to show that $\tilde{\theta}(\gamma(t)) \gamma(t) \in G(\mathbb{C}[t])t^\lambda G(\mathbb{C}[t])$. A direct computation shows that $\tilde{\theta}(\gamma(t)) \gamma(t) = \tilde{\theta}(\gamma(1)) \tilde{\theta}(\gamma(t + a)) \gamma(t + a) \gamma(t + a)^{-1}$. Note that $\tilde{\theta}(\gamma(t + a)) \gamma(t + a) \in G(\mathbb{C}[t])t^\lambda G(\mathbb{C}[t])$ as $\gamma(t) \in \mathcal{O}^\lambda_K$, the desired claim follows.

\[ \square \]

2.7. **Geometry of $K(\mathcal{X})$ and $LG_R$-orbits.** For $\lambda \in \Lambda_+^r$, we define $P^\lambda \subset \Omega X_c$ to be the intersection of $\Omega X_c$ with the orbit $S^\lambda \subset \Omega G_c \simeq \text{Gr}$, and we define $Q^\lambda \subset \Omega X_c$ to be the intersection of $\Omega X_c$ with the orbit $T^\lambda \subset \Omega G_c \simeq \text{Gr}$. We define $B^\lambda$ to be the intersection of $\Omega X_c$ with $C^\lambda \subset \Omega G_c \simeq \text{Gr}$. The projection map $\pi : G \to X, g \to \tilde{\theta}(g)g$ induces a projection $\pi : \Omega G_c \to \Omega X_c$.

**Lemma 2.6.** $P^\lambda$ is a vector bundle over $B^\lambda$. 

14
Proof. By [\text{N1}, Proposition 6.3], the restriction of the energy function $E$ to $P^\lambda$ is Bott-Morse and $B^\lambda$ is the only critical manifold. The lemma follows.

We define $\Omega X^0_c$ be the union of components of $\Omega X_c$ in $\pi_*(\pi_1(G)) \subset \pi_1(X) = \pi_0(\Omega X_c)$.

**Lemma 2.7.** We have $\Omega X^0_c = \bigcup_{\lambda \in \mathcal{L}} P^\lambda$.

**Proof.** Let $\lambda \in \Lambda^S_\mathcal{L}$. It suffices to show that $P^\lambda \subset \Omega X^0_c$ if and only if $\lambda \in \mathcal{L}$. We have $t^\lambda \in B^\lambda$ and it follows from the definition of the map $[-] : \Lambda^S_\mathcal{L} \to \pi_1(X) = \pi_0(\Omega X_c)$ that $t^\lambda$ lies in the component of $\Omega X^0_c$ corresponding to $[\lambda] \in \pi_0(\Omega X_c)$ (here $[\lambda]$ is the image of $\lambda$ under $[-]$). It implies $t^\lambda \in \Omega X^0_c$ if and only if $\lambda \in \mathcal{L}$. Since $B^\lambda = K_c \cdot t^\lambda$ and $\pi : G \to X$ is $K$-equivariant, it implies $B^\lambda \subset \Omega X^0_c$ if and only if $\lambda \in \mathcal{L}$. Finally, since $P^\lambda$ is a vector bundle over $B^\lambda$ we conclude that $P^\lambda \subset \Omega X^0_c$ if and only if $\lambda \in \mathcal{L}$. The lemma follows.

**Proposition 2.8.** We have the following.

1. The projection $\pi : \Omega G_c \to \Omega X_c$ maps $\mathcal{O}_K^\lambda$ into $P^\lambda$ and the resulting map $\mathcal{O}_K^\lambda \to P^\lambda$ is a principal $\Omega K_c$-bundle over $P^\lambda$.
2. The projection $\pi : \Omega G_c \to \Omega X_c$ maps $\mathcal{O}_\mathbb{R}^\lambda$ into $Q^\lambda$ and the resulting map $\mathcal{O}_\mathbb{R}^\lambda \to Q^\lambda$ is a principal $\Omega K_c$-bundle over $Q^\lambda$.
3. We have $\pi(\Omega G_c) = \Omega X^0_c$ and the resulting map $\pi : \Omega G_c \to \Omega X^0_c$ is a principal $\Omega K_c$-bundle over $\Omega X^0_c$.

**Proof.** Fix $\lambda \in \mathcal{L}$. Proposition 2.3 together with the fact that $\tilde{\theta} = \tilde{\eta}^\tau$ on $\Omega G_c$ imply $\pi(\mathcal{O}_K^\lambda) \subset P^\lambda$ and $\pi(\mathcal{O}_\mathbb{R}^\lambda) \subset Q^\lambda$. Note that $P^\lambda$ is a vector bundle over $B^\lambda$ and $\pi(C^\lambda) = B^\lambda$ as $\pi$ is $LK_c$-equivariant and $LK_c$ (resp. $K_c$) acts transitively on $C^\lambda$ (resp. $B^\lambda$). Thus the image $\pi(\mathcal{O}_K^\lambda)$ meets every connected component of $P^\lambda$ and, by [\text{N2}, Proposition 6.4], we have $P^\lambda \subset \pi(\Omega G_c)$. It implies $\pi(\mathcal{O}_K^\lambda) = P^\lambda$ and part (1) follows. For part (2) we observe that $Q^\lambda = \bigcup_{\lambda \leq \mu, \mu \in \Lambda^S_\mathcal{L}} Q^\lambda \cap P^\mu$. Since $B^\lambda = Q^\lambda \cap P^\lambda$ is in the closure of $Q^\lambda \cap P^\mu$, Lemma 2.7 implies $Q^\lambda = \bigcup_{\lambda \leq \mu, \mu \in \mathcal{L}} Q^\lambda \cap P^\mu$ and part (1) implies $Q^\lambda \subset \pi(\Omega G_c)$, hence $Q^\lambda = \pi(\mathcal{O}_\mathbb{R}^\lambda)$. Part (2) follow. Part (3) follows from part (1) and Lemma 2.7.

**Corollary 2.9.** $K(\mathcal{X})$ and $LG_\mathbb{R}$-orbits on $Gr$ are transversal.

**Proof.** By Proposition 2.8 it suffices to show that the strata $P^\lambda$ and $Q^\mu$ in $\Omega X_c$ are transversal. This follows from the fact that the orbits $S^\lambda$ and $T^\lambda$ on $\Omega G_c$ are transversal and both $S^\lambda, T^\lambda$ are invariant under the involution $\bar{\theta}$ on $\Omega G_c$ as $\tilde{\bar{\theta}} = \tilde{\eta}^\tau$ on $\Omega G_c$ and $S^\lambda$ (resp. $T^\lambda$) is $\tilde{\bar{\theta}}$-invariant (resp. $\tilde{\eta}^\tau$-invariant).

2.8. The components $Gr^0_\mathbb{R}$. We define $Gr^0_\mathbb{R}$ be the union of the components of $Gr_\mathbb{R}$ in the image $\pi_*(\pi_1(G)) \subset \pi_1(X) = \pi_0(Gr_\mathbb{R})$.

**Lemma 2.10.** We have $Gr^0_\mathbb{R} = \bigcup_{\lambda \in \mathcal{L}} S^\lambda_\mathbb{R}$.
Proof. Let $\lambda \in \Lambda^+_\mathbb{R}$. It suffices to show that $S^\lambda \mathbb{R}_\mathbb{R} \subset \text{Gr}^0 \mathbb{R}_\mathbb{R}$ if and only if $\lambda \in \mathcal{L}$. We have $t^\lambda \in S^\lambda \mathbb{R}_\mathbb{R}$ and it follows from (2.2) that $t^\lambda$ lies in the component of $\text{Gr} \mathbb{R}_\mathbb{R}$ corresponding to $[\lambda] \in \pi_0(\text{Gr} \mathbb{R}_\mathbb{R}) = \pi_1(X)$. It implies $t^\lambda \in \text{Gr}^0 \mathbb{R}_\mathbb{R}$ if and only if $\lambda \in \mathcal{L}$. Since $G_\mathbb{R}/P^\lambda_\mathbb{R} = K_c \cdot t^\lambda$ and $\pi : G \to X$ is $K$-equivariant, it implies $G_\mathbb{R}/P^\lambda_\mathbb{R} \subset \text{Gr}^0 \mathbb{R}_\mathbb{R}$ if and only if $\lambda \in \mathcal{L}$. Finally, since $S^\lambda \mathbb{R}$ is a vector bundle over $G_\mathbb{R}/P^\lambda_\mathbb{R}$ we conclude that $S^\lambda \mathbb{R}_\mathbb{R} \subset \text{Gr}^0 \mathbb{R}_\mathbb{R}$ if and only if $\lambda \in \mathcal{L}$. The lemma follows. \hfill \Box

**Definition 2.11.** We define $D_c(G_\mathbb{R}/\text{Gr} \mathbb{R}_\mathbb{R})$ to be the bounded constructible derived categories of sheaves on $G_\mathbb{R}/\text{Gr} \mathbb{R}_\mathbb{R}$. We set $D_c(G_\mathbb{R}(\mathcal{O}_\mathbb{R})/\text{Gr} \mathbb{R}_\mathbb{R})$ to be the full subcategory of $D_c(G_\mathbb{R}/\text{Gr})$ of complexes constructible with respect to the $G_\mathbb{R}(\mathcal{O}_\mathbb{R})$-orbits stratification. We set $D_c(G_\mathbb{R}(\mathcal{O}_\mathbb{R})/\text{Gr}^0 \mathbb{R}_\mathbb{R})$ be the full subcategory of $D_c(G_\mathbb{R}(\mathcal{O}_\mathbb{R})/\text{Gr} \mathbb{R}_\mathbb{R})$ of complexes supported on the components $\text{Gr}^0 \mathbb{R}_\mathbb{R}$.

### 3. The Matsuki flow

In this section we construct a Morse flow on the affine Grassmannian, called the Matsuki flow, and we use it to give a Morse-theoretic interpretation and refinement of the affine Matsuki correspondence.

**3.1. The Matsuki flow on Gr.** The Cartan decomposition $\mathfrak{g}_\mathbb{R} = \mathfrak{t}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}$ induces a decomposition of $\mathfrak{g}_c = \mathfrak{t}_c \oplus i\mathfrak{p}_\mathbb{R}$, $\mathfrak{g}_\mathbb{R} = \mathfrak{t}_c \oplus \mathfrak{p}_\mathbb{R}$ and the corresponding loop algebra $L\mathfrak{g} = Lt \oplus L\mathfrak{p}$, $L\mathfrak{g}_c = Lt_c \oplus L(i\mathfrak{p}_\mathbb{R})$, $L\mathfrak{g}_\mathbb{R} = Lt_c \oplus L\mathfrak{p}_\mathbb{R}$.

Recall the non-degenerate bilinear form $(\cdot, \cdot)_\gamma$ on $T_\gamma LG_c$

\[(v_1, v_2)_\gamma := \int_{S^1} \langle \gamma^{-1}v_1, \gamma^{-1}v_2 \rangle d\theta.\]

Let $\gamma \in LG_c$ and $T_\gamma(LK_c \cdot \gamma) \subset T_\gamma LG_c$ be the tangent space of the $LK_c$-orbit $LK_c \cdot \gamma$ through $\gamma$. The bilinear form above induces an orthogonal decomposition

\[T_\gamma LG_c = T_\gamma LK_c \cdot \gamma \oplus (T_\gamma LK_c \cdot \gamma)^\perp\]

and for any vector $v \in T_\gamma LG_c$ we write $v = v_0 + v_1$ where $v_0 \in T_\gamma LK_c \cdot \gamma$, $v_1 \in (T_\gamma LK_c \cdot \gamma)^\perp$. Note that we have

\[(3.1) \quad \gamma^{-1}v_0 \in \text{Ad}_{\gamma^{-1}} L\mathfrak{t}_c, \quad \gamma^{-1}v_1 \in \text{Ad}_{\gamma^{-1}} L(i\mathfrak{p}_\mathbb{R}).\]

Recall that the loop group $\Omega G_c$ can be identified with a “co-adjoint” orbit in $LG_c$ via the embedding

\[\Omega G_c \hookrightarrow L\mathfrak{g}_c, \quad \gamma \mapsto \gamma^{-1}\gamma'.\]

Consider the following functions on $\Omega G_c$

\[E : \Omega G_c \to \mathbb{R}, \quad \gamma \mapsto (\gamma', \gamma')_\gamma = \int_{S^1} \langle \gamma^{-1}\gamma', \gamma^{-1}\gamma' \rangle d\theta,\]

\[E_0 : \Omega G_c \to \mathbb{R}, \quad \gamma \mapsto (\gamma_0', \gamma_0')_\gamma = \int_{S^1} \langle \gamma^{-1}\gamma_0', \gamma^{-1}\gamma_0' \rangle d\theta,\]

\[E_1 : \Omega G_c \to \mathbb{R}, \quad \gamma \mapsto (\gamma_1', \gamma_1')_\gamma = \int_{S^1} \langle \gamma^{-1}\gamma_1', \gamma^{-1}\gamma_1' \rangle d\theta.\]
Note that $E$ is the energy function in (2.1).

**Lemma 3.1.** Recall the map $\pi : \Omega G_c \to \Omega G_c, \gamma \to \theta(\gamma)^{-1}\gamma$. We have
\[
4E_1 = E \circ \pi : \Omega G_c \to \mathbb{R}.
\]
In particular, the function $E_1$ is $LK_c$-invariant.

**Proof.** Write $||v|| = \langle v, v \rangle$ for $v \in \mathfrak{g}_c$. For any $\gamma \in \Omega G_c$ we have
\[
E \circ \pi(\gamma) = \int_{S^1} \|\pi(\gamma)^{-1}\pi(\gamma)'\|d\theta = \int_{S^1} \|\gamma^{-1}\gamma' - \gamma^{-1}\eta(\gamma)\eta(\gamma)^{-1}\gamma\|d\theta.
\]
Note that $\gamma^{-1}\gamma' - \gamma^{-1}\theta(\gamma)\theta(\gamma)^{-1}\gamma = 2\gamma^{-1}\gamma_1'$, hence we have $\|\gamma^{-1}\gamma - \gamma^{-1}\theta(\gamma)\theta(\gamma)^{-1}\gamma\| = 4\|\gamma^{-1}\gamma_1'\|$. The lemma follows.

**Lemma 3.2.** The Hamiltonian vector field on $\Omega G_c$ which correspond to $E_1$ (resp. $E_0$) is given by
\[
\gamma \to R_1(\gamma) = \gamma_1' - \gamma_1'(0) \quad (\text{resp. } \gamma \to R_0 = \gamma_0' - \gamma_0'(0)).
\]
In particular, we have
\[
\gamma^{-1}R_1(\gamma) \in \text{Ad}_{\gamma^{-1}} L\mathfrak{p}_R + i\mathfrak{p}_R \quad (\text{resp. } \gamma^{-1}R_0(\gamma) \in \text{Ad}_{\gamma^{-1}} L\mathfrak{k}_R + \mathfrak{t}_R).
\]

**Proof.** Since $R_0(\gamma) + R_1(\gamma) = R(\gamma) = \gamma' - \gamma\gamma(0)'$, it is enough to show that $R_1(\gamma) = \gamma_1' - \gamma_1'(0)$. Let $\gamma \in \Omega G_c, x = \pi(\gamma) = \theta(\gamma)^{-1}\gamma$, and $u \in T_{\gamma}\Omega G_c$. According to Proposition 2.1 and Lemma 3.1 we have
\[
4dE_1(\gamma)(u) = \pi^*dE(\gamma)(u) = dE(x)(\pi_*u) = \omega(x', \pi_*u) = \omega(x^{-1}x', x^{-1}\pi_*u).
\]
Using the equalities $x^{-1}x' = 2\gamma^{-1}\gamma_1', x^{-1}\pi_*u = 2\gamma^{-1}u_1$, and the fact that $\langle \gamma^{-1}\gamma_1', (\gamma^{-1}u_0)' \rangle = 0$, we get
\[
4dE_1(\gamma)(u) = 4\omega(\gamma^{-1}\gamma_1', \gamma^{-1}u_1) = 4\int_{S^1} \langle \gamma^{-1}\gamma_1', (\gamma^{-1}u_1)' \rangle d\theta = 4\int_{S^1} \langle \gamma^{-1}\gamma_1', (\gamma^{-1}u)' \rangle d\theta = 4\omega(R_1(\gamma), u).
\]
The lemma follows.

Let $\Omega G_c = \bigcup_{\lambda \in \mathcal{E}} \mathfrak{O}^\lambda_K$ and $\Omega G_c = \bigcup_{\lambda \in \mathcal{E}} \mathfrak{O}^\lambda_R$ be the $K(\mathcal{K})$-orbits and $LG_R$-orbits stratifications of $\Omega G_c$. Let $\mathfrak{O}^\lambda_c = \mathfrak{O}^\lambda_K \cap \mathfrak{O}^\lambda_R$ which is a single $LK_c$-orbit.

**Proposition 3.3.** Let $E_1 : \Omega G_c \to \mathbb{R}$ be the function above and $\nabla E_1$ be the corresponding gradient vector field.

(1) $\nabla E_1$ is tangential to both $\mathfrak{O}^\lambda_K$ and $\mathfrak{O}^\lambda_R$.

(2) The union $\bigcup_{\lambda \in \mathcal{E}} \mathfrak{O}^\lambda_c$ is the critical manifold of $\nabla E_1$.

(3) For any $\gamma \in \mathfrak{O}^\lambda_c$, let $T_{\gamma}\Omega G_c = T^+ \oplus T^0 \oplus T^-$ be the orthogonal direct sum decomposition into the positive, zero, and negative eigenspaces of the Hessian $d^2E_1$. We have
\[
\gamma T_{\gamma}\mathfrak{O}^\lambda_K = T^+ \oplus T^0, \quad \gamma T_{\gamma}\mathfrak{O}^\lambda_R = T^- \oplus T^0.
\]
Proof. Proof of (1). We first show that $\nabla E_1$ is tangential to $\mathcal{O}^\lambda_{\mathbb{R}} = \Omega G_c \cap LG_{\mathbb{R}} t^\lambda G(\mathbb{C}[t])$. Since the tangent space $T_\gamma \mathcal{O}^\lambda_{\mathbb{R}}$ at $\gamma \in \mathcal{O}^\lambda_{\mathbb{R}}$ is identified, by left translation, with the space

$$\Omega g_c \cap (\text{Ad}_{\gamma^{-1}} Lg_{\mathbb{R}} + g(\mathbb{C}[t])) \subset \Omega g_c,$$

it suffices to show that $\gamma^{-1}\nabla E_1(\gamma) \in \text{Ad}_{\gamma^{-1}} Lg_{\mathbb{R}} + g(\mathbb{C}[t])$. Recall that, by Proposition 2.1, we have $\gamma^{-1}\nabla E_1(\gamma) = J(\gamma^{-1} R_1(\gamma))$. Note that $J(v) + iv \in g(\mathbb{C}[t])$ for $v \in Lg$ and by Lemma 3.2 we have

$$i\gamma^{-1} R_1(\gamma) = i(\gamma^{-1}(\gamma'_1 - \gamma\gamma'_1(0)) \in \text{Ad}_{\gamma^{-1}} Lp_{\mathbb{R}} + p_{\mathbb{R}}.$$

All together, we get

$$\gamma^{-1}\nabla E_1(\gamma) = -i\gamma^{-1} R_1(\gamma) + (J(\gamma^{-1} R_1(\gamma)) + i\gamma^{-1} R_1(\gamma)) \in \text{Ad}_{\gamma^{-1}} Lp_{\mathbb{R}} + g(\mathbb{C}[t])$$

which is contained in $\text{Ad}_{\gamma^{-1}} Lg_{\mathbb{R}} + g(\mathbb{C}[t])$. We are done. The same argument as above, replacing $LG_{\mathbb{R}}$ by $K(\mathcal{X})$, shows that the gradient field $\nabla E_0$ of $E_0$ is tangential to $\mathcal{O}^\lambda_{\mathbb{K}}$. Since, by Corollary 2.5, the orbit $\mathcal{O}^\lambda_{\mathbb{K}}$ is a complex submanifold of $\Omega G_c = \text{Gr}$ invariant under the rotation flow $\gamma_0(t)$, it follows from Proposition 2.1 that $\nabla E$ is tangential to $\mathcal{O}^\lambda_{\mathbb{K}}$. Since $\nabla E_1 = \nabla E - \nabla E_0$, we conclude that $\nabla E_1$ is also tangential to $\mathcal{O}^\lambda_{\mathbb{K}}$. This finishes the proof of (1).

Proof of (2) and (3). Let $\Omega X^0_c$ be the components of $\Omega X_c$ in lemma 2.7. By proposition 2.8 and lemma 3.1, the function $E_1$ factors as

$$E_1 : \Omega G_c \rightarrow \Omega X^0_c \subset \Omega G_c \rightarrow \mathbb{R}.$$

Thus to prove (2) and (3), it is enough to prove following:

(i) The union $\bigsqcup_\lambda B^\lambda$ is the critical manifold of the restriction $E$ to $\Omega X^0_c$,

(ii) For $\gamma \in B^\lambda$ we have $T_\gamma P^\lambda = W^+ \oplus W^0$, $T_\gamma Q^\lambda = W^- \oplus W^0$, where $T_\gamma \Omega X^0_c = W^+ \oplus W^0 \oplus W^-$ is the orthogonal direct sum decomposition into the positive, zero, and negative eigenspaces of the Hessian $E|_{\Omega X^0_c}$.

By Proposition 2.1 we have $T_\gamma S^\lambda = U^+ \oplus U^0$, $T_\gamma T^\lambda = U^- \oplus U^0$, where $T_\gamma \Omega G_c = U^+ \oplus U^0 \oplus U^-$ is the orthogonal direct sum decomposition into the positive, zero, and negative eigenspaces of the Hessian $E$. Note that $\bar{\theta}$ induces a linear map on $T_\gamma \Omega G_c$, which we still denote by $\bar{\theta}$, and we have $T_\gamma \Omega X_c = (T_\gamma \Omega G_c)\bar{\theta}$ is the fixed point subspace. So to prove (i) and (ii) it suffices to show that the subspaces $T_\gamma S^\lambda$ and $T_\gamma T^\lambda$ are $\bar{\theta}$-invariant. It is true, since $\bar{\theta} = \bar{\eta}^*$ on $\Omega G_c$ and $S^\lambda$ (resp. $T^\lambda$) is $\bar{\eta}^*$-invariant (resp. $\bar{\eta}^*$-invariant). This finished the proof of (2) and (3).

Theorem 3.4. The gradient $\nabla E_1$ and gradient-flow $\phi_t$ associated to the $LK_c$-invariant function $E_1 : \text{Gr} \rightarrow \mathbb{R}$ and the $LG_{\mathbb{R}}$-invariant metric $g(\cdot)$ satisfy the following:

(1) The critical locus $\nabla E_1 = 0$ is the disjoint union of $LK_c$-orbits $\bigsqcup_{\lambda \in \mathcal{L}} \mathcal{O}^\lambda_c$

(2) The gradient-flow $\phi_t$ preserves the $K(\mathcal{X})$-and $LG_{\mathbb{R}}$-orbits.
The limits \( \lim_{t \to \pm \infty} \phi_t(\gamma) \) of the gradient-flow exist for any \( \gamma \in \Gr. \) For each \( LK_c \)-orbit \( \mathcal{O}_c^\lambda \) in the critical locus, the stable and unstable sets

\[
(3.3) \quad \mathcal{O}_c^\lambda = \{ \gamma \in \Gr | \lim_{t \to \infty} \phi_t(\gamma) \in \mathcal{O}_c^\lambda \} \quad \mathcal{O}_c^\mu = \{ \gamma \in \Gr | \lim_{t \to -\infty} \phi_t(\gamma) \in \mathcal{O}_c^\mu \}
\]

are a single \( K(\mathcal{X}) \)-orbit and \( LG_\mathbb{R} \)-orbit respectively.

(4) The correspondence between orbits \( \mathcal{O}_c^\lambda \leftrightarrow \mathcal{O}_c^\mu \) defined by (3.3) recovers the affine Matsuki correspondence (2.3).

Proof. Part (1) and (2) follows from Proposition 3.3. The \( LK_c \)-invariant function \( E_1 \), respectivley the \( LG_c \)-invariant metric \( g(\cdot, \cdot) \), and the flow \( \phi_t \), descends to a \( K_c \)-invariant Morse-Bott function \( \hat{E}_1 : \Omega K_c \setminus \Gr \to \mathbb{R} \), respectivley a \( K_c \)-invariant metric \( g(\cdot, \cdot) \) on \( \Omega K_c \setminus \Gr \), and a flow \( \hat{\phi}_t \). Since the function \( E_1 \) is bounded below and the quotient \( \Omega K_c \setminus \mathcal{O}_c^\lambda \) is finite dimensional with \( \Omega K_c \setminus \mathcal{O}_c^\lambda = \bigcup_{\mu \leq \lambda} \Omega K_c \setminus \mathcal{O}_c^\mu \), Proposition 3.3 and standard results for gradient flows (see, e.g., [AB] Proposition 1.19 or [P] Theorem 1) imply that the limit \( \lim_{t \to \pm \infty} \phi_t(\gamma) \) exists for any \( \gamma \in \Omega K_c \setminus \Gr \) and \( \Omega K_c \setminus \mathcal{O}_c^\lambda \) is the stable manifold for \( \Omega K_c \setminus \mathcal{O}_c^\lambda \) and \( \Omega K_c \setminus \mathcal{O}_c^\mu \) is the unstable manifold for \( \Omega K_c \setminus \mathcal{O}_c^\lambda \). Part (3) and (4) follows. \( \square \)

We will call the gradient flow \( \phi_t : \Gr \to \Gr \) the Matsuki flow on \( \Gr \).

4. Real Beilinson-Drinfeld Grassmannians

In this section we recall some basic facts about Real Beilinson-Drinfeld Grassmannians. The main reference is [N2].

4.1. Beilinson-Drinfeld Grassmannians. Let \( \Sigma \) be a smooth curve over \( \mathbb{C} \). Consider the functor \( G(\mathcal{O})_{\Sigma^n} \) from the category of affine schemes to sets

\[
S \to G(\mathcal{O})_{\Sigma^n}(S) := \{(x, \phi)| x \in \Sigma^n(S), \phi \in G(\hat{\Gamma}_x)\}.
\]

Here \( \hat{\Gamma}_x \) is the formal completion of the graphs \( \Gamma_x \) of \( x \) in \( \Sigma \times S \). Similarly, we define \( G(\mathcal{K})_{\Sigma^n} \) to be the functor from the category of affine schemes to sets

\[
S \to G(\mathcal{K})_{\Sigma^n}(S) := \{(x, \phi)| x \in \Sigma^n(S), \phi \in G(\hat{\Gamma}_x^0)\}.
\]

Here \( \hat{\Gamma}_x^0 := \hat{\Gamma}_x - \Gamma_x \) and \( \hat{\Gamma}_x = \text{Spec}(A_x) \) is the spectrum of ring of functions \( A_x \) of \( \hat{\Gamma}_x \). \( G(\mathcal{O})_{\Sigma^n} \) is represented by a formally smooth group scheme over \( \Sigma^n \) and \( G(\mathcal{K})_{\Sigma^n} \) is represented by a formally smooth group ind-scheme over \( \Sigma^n \).

Consider the functor \( LG_{\Sigma^n} \) that assigns to an affine scheme \( S \) the set of sections

\[
S \to LG_{\Sigma^n}(S) = \{(x, \gamma)| x \in \Sigma^n(S), \gamma \in G(\Sigma \times S - \Gamma_x)\}.
\]

There is a natural map \( LG_{\Sigma^n} \to G(\mathcal{K})_{\Sigma^n} \) sending \( (x, \gamma) \) to \( (x, \phi = \gamma|_{\Sigma^n}) \), where \( \gamma|_{\Sigma^n} \) is the restriction of the section \( \gamma : \Sigma \times S - \Gamma_x \to \hat{\Gamma}_x^0 \).

The quotient ind-scheme

\[
\text{Gr}_{\Sigma^n} := G(\mathcal{K})_{\Sigma^n}/G(\mathcal{O})_{\Sigma^n}.
\]
is called the Beilinson-Drinfeld Grassmannian. We have
\[ \text{Gr}_{\Sigma^n}(S) = \{(x, \xi, \phi) | x \in \Sigma^n(S), \xi \text{ a } G\text{-torsor on } \Sigma \times S, \phi \text{ a trivialization of } \xi \text{ on } \Sigma \times S - \Gamma_x \}. \]

4.2. **Real forms.** From now we assume \( \Sigma = \mathbb{P}^1 = \mathbb{C} \cup \mathbb{\infty} \). We write \( \text{Gr}(n) = \text{Gr}_{\Sigma^n}, G(\mathcal{K})^{(n)} = G(\mathcal{K})_{\Sigma^n}, \) etc. Let \( c : \mathbb{P}^1 \to \mathbb{P}^1 \) be the complex conjugation. Consider the following anti-holomorphic involution \( c^{(2)} : (\mathbb{P}^1)^2 \to (\mathbb{P}^1)^2, c^{(2)}(a, b) = (c(b), c(a)). \) The involution \( c^{(2)} \) together with the involution \( \eta \) on \( G \) defines anti-holomorphic involutions on \( G(\mathcal{O})^{(2)}, G(\mathcal{K})^{(2)}, \) and \( LG^{(2)} \) and we write \( G(\mathcal{O})^{(2)}, G(\mathcal{K})^{(2)}, \) and \( LG^{(2)} \) for the corresponding real analytic spaces of real points. We define \( \text{Gr}^{(2)}_\mathbb{R} = G(\mathcal{O})^{(2)} \backslash G(\mathcal{K})^{(2)} \) a real form of \( \text{Gr}^{(2)}. \)

**Lemma 4.1.** We have the following:

1. There are canonical isomorphisms
   \[ LG^{(2)}_\mathbb{R}|_0 \simeq G_{\mathbb{R}}(\mathbb{R}[t^{-1}]_1), \quad LG^{(2)}_\mathbb{R}|_{i\mathbb{R}^\times} \simeq LG_\mathbb{R} \times i\mathbb{R}^\times; \]

2. There are canonical isomorphisms
   \[ \text{Gr}^{(2)}_\mathbb{R}|_0 \simeq \text{Gr}_\mathbb{R}, \quad \text{Gr}^{(2)}_\mathbb{R}|_{i\mathbb{R}^\times} \simeq \text{Gr} \times i\mathbb{R}^\times \]
   compatible with the natural action of \( LG^{(2)}_\mathbb{R} \) on \( \text{Gr}^{(2)}_\mathbb{R}. \)

**Proof.** The isomorphism in (1) is the restriction of the natural isomorphisms \( LG^{(2)}_\mathbb{R}|_{i\mathbb{R}^\times} \simeq LG^{(2)}_\mathbb{R}|_1 \times i\mathbb{R}^\times \simeq LG \times i\mathbb{R}^\times \) and \( LG^{(2)}_\mathbb{R}|_0 \simeq G(\mathbb{C}[t^{-1}]) \). Here we regard \( i\mathbb{R} \subset \mathbb{C} \times \mathbb{C}, z \to (z, -z) \) and the isomorphism \( LG^{(2)}_\mathbb{R}|_i \simeq LG, \gamma(z) \to \gamma(t) \) is induced by the change of coordinate \( t = \frac{z-i}{z+i} \) of \( \mathbb{P}^1 \) sending \( i \) to \( 0, -i \) to \( \infty \), and \( \infty \) to \( 1 \). The isomorphism in (2) is the restriction of the factorization isomorphism \( \text{Gr}^{(2)}_\mathbb{R}|_{i\mathbb{R}^\times} \simeq \text{Gr}^{(2)}_\mathbb{R}|_1 \times i\mathbb{R}^\times \simeq \text{Gr} \times \text{Gr} \times i\mathbb{R}^\times \) and \( \text{Gr}^{(2)}_\mathbb{R}|_0 \simeq \text{Gr} \). Here the isomorphism \( \text{Gr}^{(2)}_\mathbb{R}|_i \times i\mathbb{R}^\times \simeq \text{Gr} \times \text{Gr} \times i\mathbb{R}^\times \) is induced by the above coordinate \( t = \frac{z-i}{z+i}. \)

**Lemma 4.2.** Assume \( G_{\mathbb{R}} \) is compact. We have \( G_{\mathbb{R}}(\mathbb{R}[t^{-1}]_1) = G_{\mathbb{R}} \) and \( G_{\mathbb{R}}(\mathcal{K}_{\mathbb{R}}) = G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}}) \).

**Proof.** Note that the real affine Grassmannian \( \text{Gr}_{\mathbb{R}} \) for a compact group \( G_{\mathbb{R}} \) is equal to a point and \( G_{\mathbb{R}}(\mathbb{R}[t^{-1}]_1) = \{ \gamma \in G_{\mathbb{R}}(\mathbb{R}[t^{-1}]_1) | \gamma(\infty) = e \} \) is an open \( G_{\mathbb{R}}(\mathbb{R}[t^{-1}]_1) \)-orbit in \( \text{Gr}_{\mathbb{R}} \) (see Section 2.3). Hence \( G_{\mathbb{R}}(\mathcal{K}_{\mathbb{R}}) = G_{\mathbb{R}}(\mathcal{O}_{\mathbb{R}}) \) and \( G_{\mathbb{R}}(\mathbb{R}[t^{-1}]_1) = e \). The lemma follows.

Consider the group ind-scheme \( \Omega K^{(2)} \subset LK^{(2)} \) that assigns to each affine scheme \( S \) the set
\[ \Omega K^{(2)}(S) = \{(x, \phi) | x \in \mathbb{C}(S), \phi \in K(\mathbb{P}^1 \times S - \Gamma_{xU-x}), \phi(\{\infty\} \times S) = e \}. \]

The involution on \( LK^{(2)} \) restricts to an involution on \( \Omega K^{(2)} \) and we write \( \Omega K^{(2)}_{\mathbb{R}} \) for the corresponding real analytic ind-space of real points.

**Lemma 4.3.** We have the following:

1. There are canonical isomorphisms
   \[ LK^{(2)}_{\mathbb{R}}|_0 \simeq K_c, \quad LK^{(2)}_{\mathbb{R}}|_{i\mathbb{R}^\times} \simeq LK_c \times i\mathbb{R}^\times. \]
(2) There are canonical isomorphisms
\[ \Omega K^{(2)}_R|_0 \simeq e, \quad \Omega K^{(2)}_R|_{i\mathbb{R}^\times} \simeq \Omega K_c \times i\mathbb{R}^\times. \]

Proof. It follows directly from Lemma 4.1 and Lemma 4.2. □

5. Uniformizations of real bundles

In this section we study uniformizations of the stack of real bundles on \( \mathbb{P}^1 \) and use it to provide a moduli interpretation for the quotient \( LG_\mathbb{R}\backslash \text{Gr} \).

In the rest of the paper, all the (ind-)stacks are of Bernstein-Lunts type, that is, they are unions of open substacks \( X_i \), each \( X_i \) being a quotient stack \( G\backslash X \) of finite type and the bounded derived category of \( \mathbb{C} \)-constructible sheaves on \( D_c(X) \) is the limit of \( D_c(G\backslash X) \), where each \( D_c(G\backslash X) \) can be defined as an equivariant derived category in the sense of Bernstein-Lunts (see Appendix A).

5.1. Stack of real bundles. Let \( \text{Bun}_G(\mathbb{P}^1) \) be the moduli stack of \( G \)-bundles on the complex projective line \( \mathbb{P}^1 \). The standard complex conjugation \( z \rightarrow \bar{z} \) on \( \mathbb{P}^1 \) together with the involution \( \eta \) of \( G \) defines a real structure \( c : \text{Bun}_G(\mathbb{P}^1) \rightarrow \text{Bun}_G(\mathbb{P}^1) \) with real form \( \text{Bun}_{cG}(\mathbb{P}^1_R) \), the real algebraic stack of \( G_\mathbb{R} \)-bundles on the projective real line \( \mathbb{P}^1_\mathbb{R} \). We write \( \text{Bun}_G(\mathbb{P}^1)_R \) for the real analytic stack of real points of \( \text{Bun}_{cG}(\mathbb{P}^1_R) \). By definition, we have \( \text{Bun}_G(\mathbb{P}^1)_R \simeq \Gamma(\mathbb{R}) \backslash \Gamma_\mathbb{R} \), where \( Y \rightarrow \text{Bun}_{cG}(\mathbb{P}^1_R) \) is a \( \mathbb{R} \)-surjective presentation of the real algebraic stack \( \text{Bun}_{cG}(\mathbb{P}^1_R) = Y \times_{\text{Bun}_{cG}(\mathbb{P}^1_R)} Y \) is the corresponding groupoid, and \( X_\mathbb{R}, \Gamma_\mathbb{R} \) are the real analytic spaces of real points of \( X, \Gamma \) (see Appendix A).

A point of \( \text{Bun}_G(\mathbb{P}^1)_R \) is a \( G_\mathbb{R} \)-bundle \( \mathcal{E}_\mathbb{R} \) on \( \mathbb{P}^1_\mathbb{R} \) and, by descent, corresponds to a pair \((\mathcal{E}, \gamma)\) where \( \mathcal{E} \) is a \( G \)-bundle on \( \mathbb{P}^1 \) and \( \gamma : \mathcal{E} \simeq c(\mathcal{E}) \) is an isomorphism such that the induced composition is the identity
\[ \mathcal{E} \xrightarrow{\gamma} c(\mathcal{E}) \xrightarrow{c(\gamma)} c(c(\mathcal{E})) = \mathcal{E}. \]

We call such pair \((\mathcal{E}, \gamma)\) a real bundle on \( \mathbb{P}^1 \) and \( \text{Bun}_G(\mathbb{P}^1)_R \) the stack of real bundles on \( \mathbb{P}^1 \).

For any \( G_\mathbb{R} \)-bundle \( \mathcal{E}_\mathbb{R} \), the restriction of \( \mathcal{E}_\mathbb{R} \) to the (real) point \( \infty \) is a \( G_\mathbb{R} \)-bundle on \( \text{Spec}(\mathbb{R}) \) and the assignment \( \mathcal{E}_\mathbb{R} \rightarrow \mathcal{E}_\mathbb{R}|_{\infty} \) defines a morphism
\[ \text{Bun}_{cG}(\mathbb{P}^1)_R \rightarrow \mathbb{B}G_\mathbb{R}. \]

For each \( \alpha \in H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), G) \), let \( T_\alpha \) be a \( G_\mathbb{R} \)-torsor on \( \text{Spec}(\mathbb{R}) \) in the isomorphism class of \( \alpha \) and we define \( G_{\mathbb{R}, \alpha} = \text{Aut}_{cG}(T_\alpha) \). The collection \( \{ G_{\mathbb{R}, \alpha} : \alpha \in H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), G) \} \) is the set of pure inner forms of \( G_\mathbb{R} \). Let \( G_{\mathbb{R}, \alpha} = G_{\mathbb{R}, \alpha}(\mathbb{R}) \) be the real analytic group associated to \( G_{\mathbb{R}, \alpha} \). We denote by \( \alpha_0 \) the isomorphism class of trivial \( G_{\mathbb{R}, 0} \)-torsor. By Example 4.3, the morphism above induces a morphism
\[ c_\infty : \text{Bun}_G(\mathbb{P}^1)_R \rightarrow \bigsqcup_{\alpha \in H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), G)} \mathbb{B}G_{\mathbb{R}, \alpha} \]
on the corresponding real analytic stacks. Define
\[ \text{Bun}_G(\mathbb{P}^1)_{R, \alpha} := (c_\infty)^{-1}(\mathbb{B}G_{\mathbb{R}, \alpha}) \]

\(^3\)A presentation of a real algebraic stack is \( \mathbb{R} \)-surjective if it induces a surjective map on the isomorphism classes of \( \mathbb{R} \)-points.
for the inverse image of $BG_{\mathbb{R},\alpha}$ under $cl_{\infty}$. Note that each $\text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},\alpha}$ is an union of connected components of $\text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}}$ and we obtain the following decomposition of the stack of real bundles

$$\text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}} = \bigsqcup_{\alpha \in H^1(Gal(\mathbb{C}/\mathbb{R}), G)} \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},\alpha}.$$  

We will call $\text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},\alpha}$ the stack of real bundles of class $\alpha$.

**Example 5.1.** Consider $G = \mathbb{C}^\times$. In the case $\eta$ is the split conjugation, the cohomology group $H^1(Gal(\mathbb{C}/\mathbb{R}), G)$ is trivial and we have

$$\text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}} \simeq \mathbb{Z} \times B\mathbb{R}^\times.$$  

In the case $\eta = \eta_c$ is the compact conjugation, we have $H^1(Gal(\mathbb{C}/\mathbb{R}), G) = \{\alpha_0, \alpha_1\} \simeq \mathbb{Z}/2\mathbb{Z}$ and

$$\text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}} \simeq \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},\alpha_0} \cup \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},\alpha_1},$$  

where $\text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},\alpha_i} \simeq B\mathbb{S}^1$.

5.2. **Uniformizations of real bundles.** We shall introduce and study two kinds of uniformization of real bundles: one uses a real point of $\mathbb{P}^1$ called the real uniformization the other uses a complex point of $\mathbb{P}^1$ called the complex uniformization.

5.2.1. **Real uniformizations.** The uniformization morphism

$$u : \text{Gr} \to \text{Bun}_G(\mathbb{P}^1)$$  

for $\text{Bun}_G(\mathbb{P}^1)$ exhibits $\text{Gr}$ as a $G(\mathbb{C}[t^{-1}])$-torsor over $\text{Bun}_G(\mathbb{P}^1)$, in particular, we have an isomorphism

$$(5.2) \quad G(\mathbb{C}[t^{-1}]) \backslash \text{Gr} \simeq \text{Bun}_G(\mathbb{P}^1).$$  

The map $u$ is compatible with the real structures on $\text{Gr}$ and $\text{Bun}_G(\mathbb{P}^1)$ and we denote by

$$(5.3) \quad u_{\mathbb{R}} : \text{Gr}_{\mathbb{R}} \to \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}}$$  

the associated map between the corresponding real analytic stacks of real points. We call the morphism $u_{\mathbb{R}}$ the real uniformization. It follows from (5.2) that $u_{\mathbb{R}}$ factors through an embedding

$$(5.4) \quad G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_{\mathbb{R}} \to \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}}.$$  

We shall describe the image of $u_{\mathbb{R}}$.

**Proposition 5.2.** The map $u_{\mathbb{R}}$ factors through

$$u_{\mathbb{R}} : \text{Gr}_{\mathbb{R}} \to \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},\alpha_0} \subset \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}}$$  

and induces an isomorphism of real analytic stacks

$$G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_{\mathbb{R}} \xrightarrow{\sim} \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},\alpha_0}.$$  

22
Proof. Since every $G_\mathbb{R}$-bundle $E_\mathbb{R}$ in the image of $u_\mathbb{R}$ is trivial over $\mathbb{P}_\mathbb{R}^1 - \{0\}$, in particular at $\infty$, we have $E_\mathbb{R} \in \text{Bun}_G(\mathbb{P}_\mathbb{R}^1)_{\mathbb{R},0}$. Thus the map $u_\mathbb{R}$ factors through $\text{Bun}_G(\mathbb{P}_\mathbb{R}^1)_{\mathbb{R},0}$. We show that the resulting morphism $u_\mathbb{R} : G_\mathbb{R} \to \text{Bun}_G(\mathbb{P}_\mathbb{R}^1)_{\mathbb{R},0}$ is surjective. Let $f : S \to \text{Bun}_G(\mathbb{P}_\mathbb{R}^1)_{\mathbb{R},0}$ be a smooth presentation (note that $S$ is smooth as $\text{Bun}_G(\mathbb{P}_\mathbb{R}^1)_{\mathbb{R},0}$ is smooth). It suffices to show that, étale locally on $S$, $f$ admits a lifting to $G_\mathbb{R}$. Consider the fiber product $Y := S \times_{\text{Bun}_G(\mathbb{P}_\mathbb{R}^1)_{\mathbb{R}}} G_\mathbb{R}$ and we denote by $h : Y \to S$ the natural projection map. It suffices to show that $h$ is surjective and admit a section étale locally on $S$. By Theorem 1.1 in [MS], every $G_\mathbb{R}$-bundle $E_\mathbb{R}$ on $\mathbb{P}_\mathbb{R}^1$ which is trivial at $\infty$ admits a trivialization on $\mathbb{P}_\mathbb{R}^1 - \{0\}$. It implies $h$ is surjective. To show that $h$ admits a section, we observe that $Y$ is a real analytic ind-space smooth over $G_\mathbb{R}$ and, as $u_\mathbb{R}$ is formally smooth, for any $y \in Y$ and $s = h(y) \in S$, the tangent map $dh_y : T_y Y \to T_s S$ is surjective. Choose a finite dimensional subspace $W \subset T_y Y$ such that $dh_y(W) = T_y S$. We claim that there exists a smooth real analytic space $U \subset Y$ such that $y \in U$ and $T_y U = W$. This implies $h|_U : U \to S$ is smooth around $y$, thus $f$ admits a section étale locally around $s = h(y)$. Finally, by (5.4), we obtain an isomorphism $G_\mathbb{R}(\mathbb{R}[t^{-1}]) \backslash G_\mathbb{R} \simeq \text{Bun}_G(\mathbb{P}_\mathbb{R}^1)_{\mathbb{R},0}$.

To prove the claim, we observe that $Y$ is locally isomorphic to $G_\mathbb{R}$ times a smooth real analytic space. So it suffices to show for any finite dimensional subspace $W \subset T_y G_\mathbb{R}$, there exists a smooth real analytic space $U$ such that $T_y U = W$. This follows from the fact that the exponential map $\exp : T_y G_\mathbb{R} \to G_\mathbb{R}$ associated to the metric $g(\cdot)|_{G_\mathbb{R}}$ (here $g(\cdot)$ is the metric on $G$ in Section 2.4) is a local diffeomorphism.

□

5.2.2. Generalization to other components $\text{Bun}_G(\mathbb{P}_\mathbb{R}^1)_{\mathbb{R},\alpha}$. In this section we briefly discuss generalization of Proposition 5.2 to the component $\text{Bun}_G(\mathbb{P}_\mathbb{R}^1)_{\mathbb{R},\alpha} \in H^1(\text{Gal}(\mathbb{G}/\mathbb{R}), G)$. Recall the $G_\mathbb{R}$-torsor $T_\alpha$ and the corresponding pure inner form $G_{\mathbb{R},\alpha}$. Note that for each $G_\mathbb{R}$-bundle $E_\mathbb{R}$ the $T_\alpha$-twist $F_\mathbb{R} := E_\mathbb{R} \times_{G_\mathbb{R}} T_\alpha$ is a $G_{\mathbb{R},\alpha}$-torsor and the assignment $E_\mathbb{R} \to F_\mathbb{R}$ defines an isomorphism

$$\text{Bun}_{G_{\mathbb{R}}}(\mathbb{P}_\mathbb{R}^1) \simeq \text{Bun}_{G_{\mathbb{R},\alpha}}(\mathbb{P}_\mathbb{R}^1)$$

of real algebraic stacks. Let $G_{\mathbb{R},\alpha}$ be the affine Grassmannian for $G_{\mathbb{R},\alpha}$. Consider the uniformization map

$$u_\alpha : G_{\mathbb{R},\alpha} \to \text{Bun}_{G_{\mathbb{R},\alpha}}(\mathbb{P}_\mathbb{R}^1) \simeq \text{Bun}_{G_{\mathbb{R}}}(\mathbb{P}_\mathbb{R}^1).$$

Let $G_{\mathbb{R},\alpha} := G_{\mathbb{R},\alpha}(\mathbb{R})$ and we denote by

$$u_{\alpha,\mathbb{R}} : G_{\mathbb{R},\alpha} \to \text{Bun}_G(\mathbb{P}_\mathbb{R}^1)$$

the map associated to $u_\alpha$. Let $|u_{\alpha,\mathbb{R}}| : G_{\mathbb{R},\alpha} \to |\text{Bun}_G(\mathbb{P}_\mathbb{R}^1)|$ the associated map on the isomorphism classes of points.

Lemma 5.3. We have $|u_{\alpha,\mathbb{R}}|(G_{\mathbb{R},\alpha}) = |\text{Bun}_G(\mathbb{P}_\mathbb{R}^1)\rangle$.

Proof. It suffices to show that every $G_\mathbb{R}$-bundle $E_\mathbb{R}$ on $\mathbb{P}_\mathbb{R}^1$ such that $E_\mathbb{R}|_\infty \simeq T_\alpha$ is in the image $u_{\alpha,\mathbb{R}}(G_{\mathbb{R},\alpha})$. By Theorem 1.1 in [MS], for any such bundle $E_\mathbb{R}$ the restriction of $E_\mathbb{R}$ to $U_\infty = \mathbb{P}_\mathbb{R}^1 - \{\infty\}$ (resp. $U_0 = \mathbb{P}_\mathbb{R}^1 - \{0\}$) is isomorphic to $T_\alpha \times U_\infty$ (resp. $T_\alpha \times U_0$). Since the image $u_{\alpha,\mathbb{R}}(G_{\mathbb{R},\alpha})$ consists of real bundles which can be obtained from glueing of $T_\alpha \times U_\infty$}
and $T_\alpha \times U_0$ along the open subset $U_\infty \cap U_0 = \mathbb{P}_R^1 - \{0, \infty\}$. It implies $\mathcal{E}_R \in u_{0,\alpha,R}(\text{Gr}_R,\alpha)$ and the proof is complete.

The lemma above implies that the morphism $u_{\alpha,R}$ factors through

$$u_{\alpha,R} : \text{Gr}_R,\alpha \to \text{Bun}_G(\mathbb{P}_R^1),\alpha \subset \text{Bun}_G(\mathbb{P}_R^1)$$

and the same argument as in the proof of Proposition 5.2 shows that

**Proposition 5.4.** The map $u_{\alpha,R}$ induces an isomorphism

$$G_{R,\alpha}(\mathbb{R}[t^{-1}])\backslash \text{Gr}_R,\alpha \sim \text{Bun}_G(\mathbb{P}_R^1),\alpha$$

of real analytic stacks.

5.2.3. **Complex uniformizations.** We now discuss complex uniformizations. The natural map

$$u^{(2)} : \text{Gr}^{(2)} \to \text{Bun}_G(\mathbb{P}_R^1) \times (\mathbb{P}_R^1)^2$$

exhibits $\text{Gr}^{(2)}$ as a $L\text{Gr}^{(2)}$-torsor over $\text{Bun}_G(\mathbb{P}_R^1) \times (\mathbb{P}_R^1)^2$, that is, we have an isomorphism

$$L\text{Gr}^{(2)} \sim \text{Bun}_G(\mathbb{P}_R^1) \times (\mathbb{P}_R^1)^2$$

The morphism $u^{(2)}$ is compatible with the complex conjugations on $\text{Gr}^{(2)}$ and $\text{Bun}_G(\mathbb{P}_R^1) \times (\mathbb{P}_R^1)^2$ and we denote by

(5.5) $$u^{(2)}_R : \text{Gr}^{(2)}_R \longrightarrow \text{Bun}_G(\mathbb{P}_R^1) \times \mathbb{P}_R^1$$

the map between the corresponding real analytic stacks. Note that the map above factors through an imbedding

(5.6) $$L\text{Gr}^{(2)}_R \sim \text{Bun}_G(\mathbb{P}_R^1) \times \mathbb{P}_R^1.$$ 

Recall that, by Proposition 4.1, we have isomorphisms

$$\text{Gr}_R^{(2)}|_0 \simeq \text{Gr}_R, \quad L\text{Gr}_R^{(2)}|_0 \simeq \text{Gr}_R(\mathbb{R}[t^{-1}])$$

$$\text{Gr}^{(2)}_R|_{i\mathbb{R}^\times} \simeq \text{Gr} \times i\mathbb{R}^\times, \quad L\text{Gr}^{(2)}_R|_{i\mathbb{R}^\times} \simeq L\text{Gr}_R \times i\mathbb{R}^\times.$$ 

The restriction

$$u_{0,R} := u^{(2)}_R|_0 : \text{Gr}_R \simeq \text{Gr}^{(2)}_R|_0 \to \text{Bun}_G(\mathbb{P}_R^1)_R$$

of (5.5) to the real point $0 \in i\mathbb{R}$ is isomorphic to the real uniformization map in (5.3). Consider the case when $x \in i\mathbb{R}^\times$. It follows from the isomorphism above that there is a unique map

(5.7) $$u_{x,C} : \text{Gr} \longrightarrow \text{Bun}_G(\mathbb{P}_R^1)_R$$

making the following diagram commutative

$$\begin{array}{ccc}
\text{Gr}^{(2)}_R|x & \sim & \text{Gr} \\
\downarrow u^{(2)}_R|x & & \downarrow u_{x,C} \\
\text{Bun}_G(\mathbb{P}_R^1)_R \times \{x\} & \sim & \text{Bun}_G(\mathbb{P}_R^1)_R
\end{array}$$
We call the map (5.7) the complex uniformization associated to \( x \). Note that, by (5.6), the map \( u_{x,\mathbb{C}} \) induces an embedding

\[
LG_R \backslash \text{Gr} \rightarrow \text{Bun}_G(\mathbb{P}^1)_R.
\]

We shall give a description of \( u_{x,\mathbb{C}} \). Let \((\mathcal{E}, \phi) \in \text{Gr}\) where \( \mathcal{E} \) is a \( G \)-bundle on \( \mathbb{P}^1 \) and \( \phi : \mathcal{E}_{|\mathbb{P}^1\setminus\{0\}} \simeq G \times (\mathbb{P}^1 - \{0\}) \) is a trivialization of \( \mathcal{E} \) over \( \mathbb{P}^1 - \{0\} \). Let \((\mathcal{E}_x, \phi_x)\) be the pull back of \((\mathcal{E}, \phi)\) along the isomorphism \( \mathbb{P}^1 \simeq \mathbb{P}^1, t \rightarrow z = \frac{t-x}{t+x} \). So \( \mathcal{E}_x \) is a \( G \)-bundle on \( \mathbb{P}^1 \) and \( \phi_x \) is a trivialization of \( \mathcal{E}_x \) on \( \mathbb{P}^1 - \{x\} \). Let \( c(\mathcal{E}_x) \) be complex conjugation of \( \mathcal{E}_x \) (see Sect. 5.1) and let \( \mathcal{F} \) be the \( G \)-bundle on \( \mathbb{P}^1 \) obtained from gluing of \( \mathcal{E}_{|\mathbb{P}^1\setminus\{x\}} \) and \( c(\mathcal{E})_{|\mathbb{P}^1\setminus\{x\}} \) using the isomorphism \( c(\phi_x)^{-1} \circ \phi_x : \mathcal{E}_{|\mathbb{P}^1\setminus\{x\}} \simeq c(\mathcal{E})_{|\mathbb{P}^1\setminus\{x\}} \). By construction, there is a canonical isomorphism \( \gamma : \mathcal{F} \simeq c(\mathcal{F}) \) and the resulting real bundle \((\mathcal{F}, \gamma) \in \text{Bun}_G(\mathbb{P}^1)_R\) is the image \( u_{x,\mathbb{C}}((\mathcal{E}, \phi)) \). Note that the cohomology class in \( H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), G) \) given by the restriction of the real bundle \( \mathcal{F} \) to \( \infty \) is represented by the co-boundary \( c(\phi_x(v))^{-1}(\phi_x)(v) \) (here \( v \in \mathcal{E}_x|_\infty \)), hence is trivial. Thus the complex uniformization \( u_{x,\mathbb{C}} \) factors as

\[
u_{x,\mathbb{C}} : \text{Gr} \rightarrow \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}_{00}}.
\]

We shall describe the image of \( u_{x,\mathbb{C}} \). For each \( z \in \mathbb{C}^\times \) let \( a_z : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) be the multiplication map by \( z \). Consider the flows on \( \text{Gr}^{(2)} \) and \( \text{Bun}_G(\mathbb{P}^1) \):

\[
\psi_z : \text{Gr}^{(2)} \rightarrow \text{Gr}^{(2)}, \quad (x, \mathcal{E}, \phi) \rightarrow (a_z(x), (a_z^{-1})^* \mathcal{E}, (a_z^{-1})^* \phi).
\]

\[
\psi_z : \text{Bun}_G(\mathbb{P}^1) \rightarrow \text{Bun}_G(\mathbb{P}^1), \quad \mathcal{E} \rightarrow (a_z^{-1})^* \mathcal{E}
\]

For \( z \in \mathbb{R}_{>0} \) the flows above restrict to flows

\[
\psi^1_z : \text{Gr}^{(2)}_{R} \rightarrow \text{Gr}^{(2)}_{R}, \quad \psi^2_z : \text{Bun}_G(\mathbb{P}^1)_R \rightarrow \text{Bun}_G(\mathbb{P}^1)_R
\]

and we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Gr}^{(2)}_{R} & \xrightarrow{\psi^1_z} & \text{Gr}^{(2)}_{R} \\
\downarrow{q} & & \downarrow{q} \\
\text{Bun}_G(\mathbb{P}^1)_{R,00} & \xrightarrow{\psi^2_z} & \text{Bun}_G(\mathbb{P}^1)_{R,00}
\end{array}
\]

Here \( q \) is the natural projection map.

**Lemma 5.5.** We have the following properties of the flows:

1. The critical manifold of the flow \( \psi^1_z \) are the cores \( C^\lambda_R \subset \text{Gr}_R \simeq \text{Gr}^{(2)}_{R}|_0 \) and the stable manifold for \( C^\lambda_R \) is the strata \( S^\lambda_R \subset \text{Gr}_R \).

2. For each \( \lambda \in \Lambda^+_S \), we denote by

\[
\tilde{T}^\lambda_R = \{ \gamma \in \text{Gr}^{(2)}_R \mid \lim_{z \rightarrow 0} \psi^1_z(\gamma) \in C^\lambda_R \}
\]

the corresponding unstable manifold. We have \( \tilde{T}^\lambda_R|_0 \simeq T^\lambda_R \subset \text{Gr}_R \) for \( \lambda \in \Lambda^+_S \). The isomorphism \( \text{Gr}^{(2)}_R|_x \simeq \text{Gr}, \ x \in \mathbb{R}_{>0} \), restricts to an isomorphism

\[
\tilde{T}^\lambda_R|_x \simeq \mathcal{O}^\lambda_R
\]

for \( \lambda \in \mathcal{L} \) and \( \tilde{T}^\lambda_R|_x \) is empty for \( \lambda \in \Lambda^+_S - \mathcal{L} \).
Lemma 5.6. \( (1) \) For any \( \gamma \in \text{Gr}_R^{(2)} \), the action map \( \mathbb{R}_0 \to \text{Gr}_R^{(2)} \), \( z \to \psi^1_z(\gamma) \) given by the flow \( \psi^1_z \) extends to a map \( a_\gamma : \mathbb{R}_0 \to \text{Gr}_R^{(2)} \) such that \( a_\gamma(0) = \lim_{z \to 0} \psi^1_z(\gamma) \).

\( (2) \) For any \( E \in \text{Bun}_G(\mathbb{P}^1)_{R,0} \), the action map \( \mathbb{R}_0 \to \text{Bun}_G(\mathbb{P}^1)_{R,0} \), \( z \to \psi^2_z(E) \) given by the flow \( \psi^2_z \) extends to a map

\[
(5.12) \quad a_E : \mathbb{R}_0 \to \text{Bun}_G(\mathbb{P}^1)_{R,0}.
\]

Moreover, we have \( a_E(z) \simeq E \) for all \( z \in \mathbb{R}_0 \), and for any \( \gamma \in \text{Gr}_R^{(2)} \), there is a commutative diagram

\[
(5.13) \quad \begin{array}{ccc}
\mathbb{R}_0 & \xrightarrow{a_\gamma} & \text{Gr}_R^{(2)} \\
& a_E \downarrow & \downarrow q \\
& \text{Bun}_G(\mathbb{P}^1)_{R,0} & \\
\end{array}
\]

where \( E = q(\gamma) \in \text{Bun}_G(\mathbb{P}^1)_R \).

Proof. Part (1) follows from Lemma 5.5 (2). Proof of part (2). Let \( \gamma \in \text{Gr}_R \) and let \( E = q(\gamma) \in \text{Bun}_G(\mathbb{P}^1)_R \). Consider the the composed map

\[
a_E : \mathbb{R}_0 \to \text{Gr}_R \to \text{Gr}_R(\mathbb{R}[t^{-1}]) \setminus \text{Gr}_R \simeq \text{Bun}_G(\mathbb{P}^1)_{R,0}
\]

where \( a_\gamma \) is the map in part (1) and the last isomorphism is the real uniformization (see Prop.5.2). It is elementary to check that the map \( a_E \) only depends on \( E \) and \( a_E(z) = \psi^2_z(E) \) for \( z \in \mathbb{R}_0 \), hence defines the desired map in (5.12). Moreover, since \( G_R(\mathbb{R}[t^{-1}]) \)-orbits \( T^\lambda_R \) on \( \text{Gr}_R \) are unstable manifolds for the flow \( \psi^1_z \), we have \( a_\gamma(\mathbb{R}_0) \subset T^\lambda_R \) if \( \gamma \in T^\lambda_R \), and it implies \( a_E(z) \simeq E \) for all \( a \in \mathbb{R}_0 \). The commutativity of diagram (5.13) follows from the construction of \( a_E \).

Recall the components \( \text{Gr}_R^0 = \bigcup_{E \in S_R^\lambda} \) in Section 2. We define

\[
\text{Bun}_G(\mathbb{P}^1)_{R,0}
\]

be the image of \( \text{Gr}_R^0 \) under the real uniformization \( u_R : \text{Gr}_R \to \text{Bun}_G(\mathbb{P}^1)_{R,0} \). Note that

\[
(5.14) \quad \text{Bun}_G(\mathbb{P}^1)_{R,0} \simeq G_R(\mathbb{R}[t^{-1}]) \setminus \text{Gr}_R^0 \subset \text{Bun}_G(\mathbb{P}^1)_{R,0} \simeq G_R(\mathbb{R}[t^{-1}]) \setminus \text{Gr}_R
\]

is a union of components of \( \text{Bun}_G(\mathbb{P}^1)_{R,0} \).

Proposition 5.7. The complex uniformization \( u_{x,C} : \text{Gr} \to \text{Bun}_G(\mathbb{P}^1)_{R,0} \) factors as

\[
u_{x,C} : \text{Gr} \to \text{Bun}_G(\mathbb{P}^1)_{R,0}
\]

and induces an isomorphism

\[
LG_R \setminus \text{Gr} \sim \to \text{Bun}_G(\mathbb{P}^1)_{R,0}
\]

of real analytic stacks.
Proof. Let \( \gamma \in \text{Gr} \) and \( \gamma_x \in \text{Gr}_{\mathbb{R}}^{(2)} \) be the image of \( \gamma \) under the isomorphism \( \text{Gr} \simeq \text{Gr}_{\mathbb{R}}^{(2)} \). Let \( \mathcal{E} = u_{x,C}(\gamma) = q(\gamma_x) \in \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},a_0} \) be the image of the complex uniformization map. By Lemma 5.6(2) we have
\[
|\mathcal{E}| = |a_\mathcal{E}(0)| = |q(a_{\gamma_x}(0))| = |q(\lim_{z \to 0} \psi_z^{(1)}(\gamma_x))|,
\]
and the propositions above, the components \( \text{Bun}_{\mathbb{R}}^{(2)} \) imply that \( u_{x,C} \) factors through \( u_{x,C} : \text{Gr} \to \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} \) and induces a surjection between the sets of isomorphism classes of objects. Now a similar argument as in the proof Proposition 5.2 shows that \( u_{x,C} : \text{Gr} \to \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} \) is surjective and, by (5.8), we obtain an isomorphism \( LG_{\mathbb{R}} \backslash \text{Gr} \simeq \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} \). \( \square \)

Remark 5.8. We have \( \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} = \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},a_0} \) if and only if \( K \) is connected. So in the case when \( K \) is disconnected, the map \( u_{x,C} : \text{Gr} \to \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},a_0} \) is not surjective, that is, not every real bundle of class \( a_0 \) admits a complex uniformization.

Example 5.9. In the case \( G = \mathbb{C}^* \) with split conjugation, we have
\[
\text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} \simeq 2\mathbb{Z} \times B\mathbb{R}^\times \subset \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}} \simeq \mathbb{Z} \times B\mathbb{R}^\times
\]
and the complex uniformization is given by
\[
u_{x,C} : \text{Gr} \simeq \mathbb{Z} \times \{ \text{pt} \} \xrightarrow{\times 2,p} \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} \simeq 2\mathbb{Z} \times B\mathbb{R}^\times.
\]
Here \( p : \{ \text{pt} \} \to B\mathbb{R}^\times \) is the quotient map.

5.3. Categories of sheaves on \( LG_{\mathbb{R}} \backslash \text{Gr} \). Since \( \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}} \) is a real analytic stack of finite type, by the propositions above, the components \( \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},a}, \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} \), and the quotient stacks \( LG_{\mathbb{R}} \backslash \text{Gr}, G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_{\mathbb{R}}, \text{and } G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_{\mathbb{R}}^0 \) are also of finite type and there are well-defined categories of sheaves on them.

Definition 5.10. We define \( D_c(LG_{\mathbb{R}} \backslash \text{Gr}) \) to be the bounded derived category of \( \mathbb{C} \)-constructible sheaves on \( LG_{\mathbb{R}} \backslash \text{Gr} \). We define \( D_c(LG_{\mathbb{R}} \backslash \text{Gr}) \) to be the full subcategory of \( D_c(LG_{\mathbb{R}} \backslash \text{Gr}) \) consisting of all constructible complexes that are extensions by zero off of finite type substacks of \( LG_{\mathbb{R}} \backslash \text{Gr} \). We denote by \( D_c(G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_{\mathbb{R}}), D_c(G_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_{\mathbb{R}}^0) \), etc, for the similar defined categories.

5.4. Uniformizations in family. Consider the open subset \( \text{Gr}_R^{(2),0} \subset \text{Gr}_R^{(2)} \) such that \( \text{Gr}_R^{(2),0} |_x = \text{Gr}_R^{(2)} |_x \) for \( x \neq 0 \) and \( \text{Gr}_R^{(2),0} |_0 \simeq \text{Gr}_R^{(2)} |_0 \simeq \text{Gr}_R^{(2)} \). Let
\[
u_{x,R}^{(2),0} : \text{Gr}_R^{(2),0} \to \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}} \times i\mathbb{R}
\]
be the restriction of (5.3) to \( \text{Gr}_R^{(2),0} \).

Proposition 5.11. The map \( \nu_{x,R}^{(2),0} \) factors through
\[
u_{x,R}^{(2),0} : \text{Gr}_R^{(2),0} \to \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} \times i\mathbb{R}
\]
and induces an isomorphism
\[ LG_\mathbb{R}^{(2)} \backslash \text{Gr}_\mathbb{R}^{(2),0} \simto \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} \times i\mathbb{R}. \]
of real analytic stacks.

Proof. This follows from Proposition 5.2 and 5.7.

6. Quasi-maps

6.1. Definition of quasi-maps. Let \( \Sigma \) be a smooth complex projective curve. For \( n \geq 0 \), define the stack of quasi-maps with poles \( QM^{(n)}(\Sigma, G, K) \) to classify triples \((x, E, \sigma)\) comprising a point \( x = (x_1, \ldots, x_n) \in \Sigma^n \), a \( G \)-torsor \( E \) over \( \Sigma \), and a section \( \sigma : \Sigma \setminus |x| \to E \times^G G/K \) where we write \( |x| = \bigcup_{i=1}^n x_i \subset \Sigma \). According to [GN1], \( QM^{(n)}(\Sigma, G, K) \) is an ind-stack of ind-finite type. Note the natural maps \( QM^{(n)}(\Sigma, G, K) \to \Sigma^n \) and \( QM^{(n)}(\Sigma, G, K) \to \text{Bun}_G(\Sigma) \). For any \( x \in \Sigma^n \), we will write \( QM^{(n)}_G(\Sigma, x, G, K) \) for the fiber \( QM^{(n)}(\Sigma, G, K) \times_{\Sigma^n} \{x\} \).

6.2. Real forms of quasi-maps. Now specialize to \( \Sigma = \mathbb{P}^1 \) and \( n = 2 \). The standard conjugation of \( \mathbb{P}^1 \), denoted by \( x \mapsto \bar{x} \), induces a twisted conjugation on \( (\mathbb{P}^1)^2 \), defined by \( c(x_1, x_2) = (\bar{x}_2, \bar{x}_1) \) with real points isomorphic to \( (\mathbb{P}^1)_\mathbb{R}^2 \cong \mathbb{P}^1 \) regarded as a real variety. Let us fix the isomorphism given by the choice of \( x_1 \). Together with the conjugation of \( G \) preserving \( K \), the twisted conjugation of \( (\mathbb{P}^1)^2 \) induces a conjugation of \( QM^{(2)}(\mathbb{P}^1, G, K) \). Let us denote its real points by \( QM^{(2)}_G(\mathbb{P}^1, G, K)_\mathbb{R} \). Note there are natural maps
\[ QM^{(2)}(\mathbb{P}^1, G, K)_\mathbb{R} \to (\mathbb{P}^1)_\mathbb{R}^2 \cong \mathbb{P}^1(\mathbb{C}), \quad QM^{(2)}(\mathbb{P}^1, G, K)_\mathbb{R} \to \text{Bun}_G(\mathbb{P}^1)_\mathbb{R}. \]

Define \( QM^{(2)}(\mathbb{P}^1, G, K)_{\mathbb{R},0} \) (resp. \( QM^{(2)}_G(\mathbb{P}^1, G, K)_{\mathbb{R},0} \)) be the pre-image of \( \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} \) (resp. \( \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} \)) under the morphism \( QM^{(2)}(\mathbb{P}^1, G, K)_\mathbb{R} \to \text{Bun}_G(\mathbb{P}^1)_\mathbb{R} \). For any \( x \in \mathbb{P}^1(\mathbb{C}) \) we have the fiber \( QM^{(2)}(\mathbb{P}^1, x, G, K)_\mathbb{R} \) and the intersections \( QM^{(2)}(\mathbb{P}^1, x, G, K)_{\mathbb{R},0} \) and \( QM^{(2)}(\mathbb{P}^1, x, G, X)_{\mathbb{R},0}. \)

6.3. Uniformizations of quasi-maps. We have a natural uniformization map
\[ \text{Gr}^{(2)} \to QM^{(2)}(\mathbb{P}^1, G, K) \]
equivribbons.png

exhibits \( \text{Gr}^{(2)} \) as a \( LK^{(2)} \)-torsor on \( QM^{(2)}(\mathbb{P}^1, G, K) \). In particular, there is a canonical isomorphism of ind-stacks
\[ q^{(2)} : LK^{(2)} \backslash \text{Gr}^{(2)} \simto QM^{(2)}(\mathbb{P}^1, G, K). \]
The morphism in (6.1) is compatible with the real structures and we denote by
\[ \text{Gr}^{(2)}_\mathbb{R} \longrightarrow QM^{(2)}(\mathbb{P}^1, G, K)_\mathbb{R} \]
the associated map on the corresponding real algebraic stacks of real points. It follows from (6.2) that the map above factors through an embedding
\[ q^{(2)}_\mathbb{R} : LK^{(2)}_\mathbb{R} \backslash \text{Gr}^{(2)}_\mathbb{R} \longrightarrow QM^{(2)}(\mathbb{P}^1, G, K)_\mathbb{R}. \]
By Lemma 4.3, there are natural isomorphisms

\[ LK_R^{(2)} \setminus \text{Gr}_R^{(2)} |_x \simeq LK_c \setminus \text{Gr}, \quad x \in i\mathbb{R}^\times, \quad LK_R^{(2)} \setminus \text{Gr}_R^{(2)} |_0 \simeq K_c \setminus \text{Gr}_R \]

and the map \( q_x^{(2)} \) gives rise to maps

\[ q_x : LK_c \setminus \text{Gr} \longrightarrow QM^{(2)}(\mathbb{P}^1, x, G, K)_R, \quad x \in i\mathbb{R}^\times \]
\[ q_0 : K_c \setminus \text{Gr}_R \longrightarrow QM^{(2)}(\mathbb{P}^1, 0, G, K)_R. \]

**Lemma 6.1.** We have the following:

1. The map \( q_x \) induces an isomorphism \( q_x : LK_c \setminus \text{Gr} \sim \longrightarrow QM^{(2)}(\mathbb{P}^1, x, G, K)_{R,0}. \)

2. The map \( q_0 \) induces an isomorphism \( q_0 : K_c \setminus \text{Gr}_R \sim \longrightarrow QM^{(2)}(\mathbb{P}^1, 0, G, K)_{R,0}. \)

**Proof.** This follows from Proposition 5.7 (resp. Proposition 5.2) that every real bundle \( E \) in \( \text{Bun}_G(\mathbb{P}^1)_{R,0} \) (resp. \( \text{Bun}_G(\mathbb{P}^1)_{R,0} \)) admits a complex uniformization (resp. a real uniformization).

Recall the open family \( \text{Gr}_R^{(2),0} \to i\mathbb{R} \) and the family of uniformizations \( LG^{(2)}_R \setminus \text{Gr}_R^{(2),0} \simeq \text{Bun}_G(\mathbb{P}^1)_{R,0} \times i\mathbb{R} \) in Proposition 5.11. The above lemma implies the following.

**Proposition 6.2.** The natural map \( \text{Gr}_R^{(2),0} \to QM^{(2)}(\mathbb{P}^1, G, K)_R \) induces an isomorphism

\[ LK_R^{(2)} \setminus \text{Gr}_R^{(2),0} \simeq QM^{(2)}(\mathbb{P}^1, G, K)_{R,0}|_{i\mathbb{R}} \]

and we have the following commutative diagram

\[
\begin{array}{ccc}
LK_R^{(2)} \setminus \text{Gr}_R^{(2),0} & \longrightarrow & QM^{(2)}(\mathbb{P}^1, G, K)_{R,0}|_{i\mathbb{R}} \\
\downarrow & & \downarrow \\
LG_R^{(2)} \setminus \text{Gr}_R^{(2),0} & \longrightarrow & \text{Bun}_G(\mathbb{P}^1)_{R,0} \times i\mathbb{R}
\end{array}
\]

where the vertical maps are the natural quotient and projection maps. In addition, there are canonical isomorphisms

\[ LK_c \setminus \text{Gr} \times i\mathbb{R}^\times \simeq QM^{(2)}(\mathbb{P}^1, G, K)_{R,0}|_{i\mathbb{R}^\times} \]
\[ K_c \setminus \text{Gr}_R^0 \simeq QM^{(2)}(\mathbb{P}^1, 0, G, K)_{R,0} \]

6.4. **Categories of sheaves on** \( LK_c \setminus \text{Gr} \). By Lemma 6.1, the real analytic ind-stack \( LK_c \setminus \text{Gr} \) is of ind-finite type and we have a well-defined category of sheaves on it. Introducing the stratification \( S \) of \( LK_c \setminus \text{Gr} \) with strata the \( LK_c \)-quotients of \( K(\mathcal{X}) \)-orbits.

**Definition 6.3.** Let \( D_c(LK_c \setminus \text{Gr}) \) be the bounded constructible derived category of sheaves on \( LK_c \setminus \text{Gr} \). We set \( D_c(K(\mathcal{X}) \setminus \text{Gr}) \) to be the full subcategory of \( D_c(LK_c \setminus \text{Gr}) \) of complexes constructible with respect to the stratification \( S \).
6.5. Rigidified quasi-maps. Let $QM^{(n)}(\mathbb{P}^1, G, K, \infty)$ be the ind-scheme classifies quadruple $(x, \mathcal{E}, \phi, \iota)$ where $x \in \mathbb{C}^n$, $\mathcal{E}$ is a $G$-bundle on $\mathbb{P}^1$, $\phi : \mathbb{P}^1 - |x| \to \mathcal{E} \times^G X$, and $\iota : \mathcal{E}_K|_{\infty} \simeq K$, here $\mathcal{E}_K$ is the $K$-reduction of $\mathcal{E}$ on $\mathbb{P}^1 - |x|$ given by $\phi$. We have a natural map $QM^{(n)}(\mathbb{P}^1, G, K, \infty) \to \mathbb{C}^n$. The ind-scheme $QM^{(n)}(\mathbb{P}^1, G, K, \infty)$ is called rigidified quasi-maps. Note that we have natural map

$$QM^{(n)}(\mathbb{P}^1, G, K, \infty) \to QM^{(n)}(\mathbb{P}^1, G, K)$$

sending $(x, \mathcal{E}, \phi, \iota)$ to $(x, \mathcal{E}, \phi)$ and it induces an isomorphism

$$K\backslash QM^{(n)}(\mathbb{P}^1, G, K, \infty) \simeq QM^{(n)}(\mathbb{P}^1, G, K)|_{\mathbb{C}^n}$$

where the twisted $K$ acts on $QM^{(n)}(\mathbb{P}^1, G, K, \infty)$ by changing the trivialization $\iota$.

The twisted conjugation on $(x_1, x_2) \to (\bar{x}_2, \bar{x}_1)$ together with the involution $\eta$ on $G$ defines a real form $QM^{(2)}(\mathbb{P}^1, G, K, \infty)_R$ of $QM^{(2)}(\mathbb{P}^1, G, K, \infty)$. We have a natural map $QM^{(2)}(\mathbb{P}^1, G, K, \infty)_R \to Bun_G(\mathbb{P}^1)_R$ and we denote by $QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{R,0}$ the pre-image of the components $Bun_G(\mathbb{P}^1)_{R,0}$. The isomorphism (6.3) induces an embedding

$$K_c\backslash QM^{(2)}(\mathbb{P}^1, G, K, \infty)_R \to QM^{(2)}(\mathbb{P}^1, G, K)_{R,0}|_C.$$ 

It follows from Proposition 6.2 that the above embedding restricts to an isomorphism

$$K_c\backslash QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{R,0} \simeq QM^{(2)}(\mathbb{P}^1, G, K)_{R,0}|_C$$

and there are canonical isomorphisms

$$\Omega K^{(2)}_R \backslash Gr^{(2),0} \simeq QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{R,0}|_R$$

$$\Omega K_c \backslash Gr \times i\mathbb{R}^\times \simeq QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{R,0}|_{i\mathbb{R}^\times}$$

$$Gr^0_\mathbb{R} \simeq QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{R,0}|_0$$

Consider the stratifications $S_1 = \{\Omega K_c \backslash O^1_\mathbb{R}\}_{\lambda \in \mathcal{L}}$ of $\Omega K_c \backslash Gr^0$ and $S_2 = \{S^1_\lambda\}_{\lambda \in \mathcal{L}}$ of $Gr^0_\mathbb{R}$. By (6.5), the union $S^{(2)} = S_1 \times i\mathbb{R}^\times \cup S_2 \cup \{0\}$ forms a stratification of $QM^{(2)}(\mathbb{P}^1, G, X, \infty)_{R,0}|_i\mathbb{R}$. In section 8 we will need following technical lemma, which is proved in [CN2 Proposition 6.7].

**Lemma 6.4.** The stratification $S^{(2)}$ above is Whitney and the natural map

$$QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{R,0}|_i\mathbb{R} \longrightarrow i\mathbb{R}$$

is a Thom stratified map. Here $i\mathbb{R}$ is equipped with the stratification $i\mathbb{R} = i\mathbb{R}^\times \cup \{0\}$.

**Remark 6.5.** In fact, in loc. cit., we show that the quasi-maps family above admits a $K_c$-equivariant topological trivialization.
6.6. Flows on quasi-maps. For each $z \in \mathbb{C}^\times$ we have the following flow (6.6)

$$\psi_z : QM^{(2)}(\mathbb{P}^1, G, K, \infty) \rightarrow QM^{(2)}(\mathbb{P}^1, G, K, \infty), \ (x, \mathcal{E}, \psi, \iota) \rightarrow (a_z(x), (a_{z-1})^* \mathcal{E}, (a_{z-1})^* \psi, \iota).$$

For $z \in \mathbb{R}_{>0}$ the flow $\phi_z$ restricts to a flow

$$\psi_z^3 : QM^{(2)}_{\mathbb{R}}(\mathbb{P}^1, G, K, \infty) \rightarrow QM^{(2)}_{\mathbb{R}}(\mathbb{P}^1, G, K, \infty),$$

and we have the following commutative diagrams

$$\begin{array}{ccc}
QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{\mathbb{R}} & \xrightarrow{\psi_z^3} & QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{\mathbb{R}} \\
\mathbb{P}^1 & \xrightarrow{a_z} & \mathbb{P}^1
\end{array}$$

Lemma 6.6. We have the following properties of the flows:

1. The flow $\psi_z$ on $QM^{(2)}_{\mathbb{R}}(\mathbb{P}^1, G, K, \infty)$ is $K_c$-equivariant.
2. Recall the flow $\psi_1^3$ on $Gr^{(2)}_{\mathbb{R}}$ (5.10). We have the following commutative diagram

$$\begin{array}{ccc}
Gr^{(2)}_{\mathbb{R}} |_C & \xrightarrow{\psi_1^3} & Gr^{(2)}_{\mathbb{R}} |_C \\
QM^{(2)}_{\mathbb{R}}(\mathbb{P}^1, G, K, \infty) & \xrightarrow{\psi_3^3} & QM^{(2)}_{\mathbb{R}}(\mathbb{P}^1, G, K, \infty)
\end{array}$$

3. For each $\lambda \in \Lambda^+_S$, the core $C^\lambda_{R} \subset Gr_{\mathbb{R}} \subset QM^{(2)}_{\mathbb{R}}(\mathbb{P}^1, G, K, \infty)$ is a union of components of the critical manifold of the flow $\psi_z^3$ on $QM^{(2)}_{\mathbb{R}}(\mathbb{P}^1, G, K, \infty)$ and the stable manifold for $C^\lambda_{R}$ is the strata $S^\lambda_{R} \subset Gr_{\mathbb{R}}$.
4. For each $\lambda \in \Lambda^+_S$, we denote by

$$\tilde{T}_R^\lambda = \{ x \in QM^{(2)}_{\mathbb{R}}(\mathbb{P}^1, G, K, \infty) |_\mathbb{R} | \lim_{z \rightarrow 0} \psi_z^3(x) \in C^\lambda_{R} \}$$

the corresponding unstable manifold. We have $\tilde{T}_R^\lambda |_0 \simeq T_R^\lambda \subset Gr_{\mathbb{R}}$ for $\lambda \in \Lambda^+_S$. The open embedding $\Omega K_c \setminus Gr \rightarrow QM^{(2)}_{\mathbb{R}}(\mathbb{P}^1, G, K, \infty) |_\mathbb{R} |_i$ restricts to an isomorphism

$$\Omega K_c \setminus \mathcal{O}_R^\lambda \simeq \tilde{T}_R^\lambda |_i$$

for $\lambda \in \mathcal{L}$.

Proof. Part (1) and (2) follows from the construction of the flows. Part (3) and (4) follows from Lemma 5.5 and diagram (6.7).

7. Affine Matsuki correspondence for sheaves

In this section we prove the affine Matsuki correspondence for sheaves.
7.1. The functor $\Upsilon$. Let $u : LK_c \backslash Gr \to LG_r \backslash Gr$ be the quotient map. Define

$$\Upsilon : D_c(K(\mathcal{X}) \backslash Gr) \to D_c(LG_r \backslash Gr)$$

to be the restriction of $u : D_c(LK_r \backslash Gr) \to D_c(LG_r \backslash Gr)$ to $D_c(K(\mathcal{X}) \backslash Gr) \subset D_c(LK_r \backslash Gr)$.

**Theorem 7.1** (Affine Matsuki correspondence for sheaves). The functor $\Upsilon$ defines an equivalence of categories

$$\Upsilon : D_c(K(\mathcal{X}) \backslash Gr) \xrightarrow{\sim} D_c(LG_r \backslash Gr).$$

The rest of the section is devoted to the proof of Theorem 7.1.

7.2. Bijection between local systems. Write $[\mathcal{O}^\lambda_K] = LK_c \backslash \mathcal{O}^\lambda_K$, $[\mathcal{O}^\lambda_r] = LK_c \backslash \mathcal{O}^\lambda_r$, $[\mathcal{O}^\lambda_c] = LK_c \backslash \mathcal{O}^\lambda_c$, and $[\mathcal{E}^\lambda] = LG_r \backslash \mathcal{O}^\lambda_r \in LG_r \backslash Gr$. Recall the Matsuki flow $\phi_t : Gr \to Gr$ in Theorem 3.4. As $\phi_t$ is $LK_c$-equivariant, it descends to a flow $\tilde{\phi}_t : LK_c \backslash Gr \to LK_c \backslash Gr$ and we define

$$\phi_\pm : LK_c \backslash Gr \to \bigsqcup_{\lambda \in \mathcal{C}} [\mathcal{O}^\lambda_c] \subset LK_c \backslash Gr, \quad \gamma \to \lim_{t \to \pm \infty} \tilde{\phi}_t(\gamma).$$

Consider the following Cartesian diagrams:

$$
\begin{array}{ccc}
[\mathcal{O}^\lambda_c] & \xrightarrow{i^\lambda_c} & LK_c \backslash Gr \\
\downarrow \phi_+ & & \downarrow \phi_+ \\
[\mathcal{O}^\lambda_r] & \xrightarrow{j^\lambda_r} & \bigsqcup_{\lambda \in \mathcal{C}} [\mathcal{O}^\lambda_c] \\
\downarrow \phi_- & & \downarrow \phi_- \\
[\mathcal{O}^\lambda_r] & \xrightarrow{j^\lambda_r} & LG_r \backslash Gr \\
\downarrow \phi_- & & \downarrow \phi_- \\
[\mathcal{E}^\lambda] & \xrightarrow{j^\lambda} & LG_r \backslash Gr \\
\downarrow u^\lambda & & \downarrow u \\
[\mathcal{O}^\lambda_r] & \xrightarrow{p^\lambda} & [\mathcal{E}^\lambda] \\
\end{array}
$$

Here $i^\lambda_c$ and $j^\lambda_r$ are the natural embeddings and $\phi_+^\lambda$ (resp. $u^\lambda$) is the restriction of $\phi_+$ (resp. $u$) along $j^\lambda_r$ (resp. $j^\lambda_r$).

**Lemma 7.2.** We have the following:

1. There is a bijection between isomorphism classes of local systems $\tau^+$ on $[\mathcal{O}^\lambda_c]$, local systems $\tau^-$ on $[\mathcal{O}^\lambda_r]$, local systems $\tau$ on $[\mathcal{O}^\lambda_c]$, and local systems $\tau^\mathcal{R}$ on $[\mathcal{E}^\lambda] = [LG_r \backslash \mathcal{O}^\lambda_r]$, characterizing by the property that $\tau^\pm \simeq (\phi_\pm^\lambda)^* \tau$ and $\tau^- \simeq (u^\lambda)^* \tau^\mathcal{R}$.

2. The map $u^\lambda$ factors as

$$u^\lambda : [\mathcal{O}^\lambda_r] \xrightarrow{\phi^\lambda} [\mathcal{O}^\lambda_c] \xrightarrow{p^\lambda} [\mathcal{E}^\lambda]$$

where $p^\lambda$ is smooth of relative dimension $\dim[\mathcal{E}^\lambda] - \dim[\mathcal{O}^\lambda_c]$. Moreover, we have $(p^\lambda)^* \tau^\mathcal{R} \simeq \tau$.

**Proof.** Since the fibers of $\phi_\pm$ are contractible, pull-back along $\phi_+^\lambda$ (resp. $\phi_-^\lambda$) defines an equivalence between $LK_c$-equivariant local systems on $\mathcal{O}^\lambda_c$ and $LK_c$-equivariant local systems on $\mathcal{O}^\lambda_K$ (resp. $\mathcal{O}^\lambda_r$). We show that the fiber of $u^\lambda$ is contractible, hence pull back along $u^\lambda$ defines an equivalence between local systems on $[\mathcal{E}^\lambda]$ and $LK_c$-equivariant local systems on $\mathcal{O}^\lambda_r$. Pick $y \in \mathcal{O}^\lambda_c$ and let $LK_c(y)$, $LG_r(y)$ be the stabilizers of $y$ in $LK_c$ and $LG_r$ respectively. The group $LK_c(y)$ acts on the fiber $l_y := (\phi_+^\lambda)^{-1}(y)$ and we have $\mathcal{O}^\lambda_r \simeq LK_c \times^{LK_c(y)} l_y$. 

32
Moreover, under the isomorphism \([\mathcal{O}_\mathbb{R}^\lambda] \simeq \text{LK}_c/\mathcal{O}_\mathbb{R}^\lambda \simeq \text{LK}_c(y)/\mathcal{O}_c^\lambda \simeq \text{LK}_c(y)/y\), and \([\mathcal{E}^\lambda] \simeq \text{LG}_{\mathbb{R}}(y)/y\), the map \(u^\lambda\) takes the form

\[ u^\lambda : [\mathcal{O}_\mathbb{R}^\lambda] \simeq \text{LK}_c(y)/y \xrightarrow{\phi^\lambda} [\mathcal{O}_c^\lambda] \simeq \text{LK}_c(y)/y \xrightarrow{p^\lambda} [\mathcal{E}^\lambda] \simeq \text{LG}_{\mathbb{R}}(y)/y, \]

where the first map is induced by the projection \(l_y \to y\) and the second map is induced by the inclusion \(\text{LK}_c(y) \to \text{LG}_{\mathbb{R}}(y)\). We claim that the quotient \(\text{LK}_c(y)/\text{LG}_{\mathbb{R}}(y)\) is contractible, hence \(u^\lambda\) has contractible fibers and \(p^\lambda\) is smooth of relative dimension \(\dim[\mathcal{E}^\lambda] - \dim[\mathcal{O}_c^\lambda]\). Part (1) and (2) follows.

Proof of the claim. Pick \(y' \in C_\mathbb{R}^\lambda \subset \text{Gr}_{\mathbb{R}}\) and let \(K_c(y')\) and \(G_{\mathbb{R}}(\mathbb{R}[t^{-1}](y'))\) be the stabilizers of \(y'\) in \(K_c\) and \(G_{\mathbb{R}}(\mathbb{R}[t^{-1}])\) respectively. The composition of the complex and real uniformizations of \(\text{Bun}_G(\mathbb{P}^1)_\mathbb{R},0\)

\[ \text{LG}_{\mathbb{R}}(\text{Gr}^\text{Prop.5.1}) \simeq \text{Bun}_G(\mathbb{P}^1)_\mathbb{R},0 \text{ Prop.5.2} \simeq \text{G}_{\mathbb{R}}(\mathbb{R}[t^{-1}]) \text{ Gr}_{\mathbb{R}}^0 \]

identifies

\[ \text{LG}_{\mathbb{R}}(y)/y \simeq [\mathcal{E}^\lambda] \simeq \text{G}_{\mathbb{R}}(\mathbb{R}[t^{-1}](y'))/y'. \]

Hence we obtain a natural isomorphism

\[ \text{LG}_{\mathbb{R}}(y)/y \simeq \text{Aut}([\mathcal{E}^\lambda]) \simeq \text{G}_{\mathbb{R}}(\mathbb{R}[t^{-1}](y'))/y' \]

sending \(\text{LK}_c(y) = K_c(y) \subset \text{LG}_{\mathbb{R}}(y)\) to \(K_c(y') \subset \text{G}_{\mathbb{R}}(\mathbb{R}[t^{-1}](y'))\). Thus we reduce to show that the quotient \(K_c(y')/G_{\mathbb{R}}(\mathbb{R}[t^{-1}](y'))\) is contractible. This follows from the fact that evaluation map \(K_c(y')/G_{\mathbb{R}}(\mathbb{R}[t^{-1}](y')) \to K_c(y')/G_{\mathbb{R}}(y'), \gamma(t^{-1}) \to \gamma(0)\) has contractible fibers and the quotient \(K_c(y')/G_{\mathbb{R}}(y')\) is contractible as \(K_c(y')\) is a maximal compact subgroup of the Levi subgroup of \(G_{\mathbb{R}}(y')\).

\[ \square \]

7.3. **Proof of Theorem 7.1.** For each \(\lambda \in \mathcal{L}\) and a local system \(\tau\) on \([\mathcal{O}_c^\lambda]\) one has the standard sheaves

\[ S^+(\lambda, \tau) := (i^\lambda)^* (\tau^+), \quad \text{and} \quad S^- (\lambda, \tau) := (j^\lambda)^* (\tau) \]

and co-standard sheaves

\[ S^+(\lambda, \tau) := (i^\lambda)_! (\tau^+), \quad \text{and} \quad S^- (\lambda, \tau) := (j^\lambda)_! (\tau). \]

Here \(\tau^+\) and \(\tau\) are local system on \([\mathcal{O}_K^\lambda]\) and \([\mathcal{E}^\lambda]\) corresponding to \(\tau\) as in Lemma 7.2. Let \(d_\lambda := \dim \text{Bun}_G(\mathbb{P}^1)_\mathbb{R} - \dim[\mathcal{O}_K^\lambda]\).

Write

\[ (\iota^\lambda)_!: [\mathcal{O}_c^\mu] \to [\mathcal{O}_K^\mu], \quad (\iota^\lambda)_!: [\mathcal{O}_c^\mu] \to [\mathcal{O}_K^\mu] \]

for the natural embeddings. We recall the following fact, see [MUV, Lemma 5.4].

**Lemma 7.3.** (1) Consider \([\mathcal{O}_c^\mu] \xrightarrow{\iota^\mu} [\mathcal{O}_K^\mu] \xrightarrow{\phi^\mu} [\mathcal{O}_c^\mu]\). Let \(\mathcal{F} \in D_c([\mathcal{O}_K^\mu])\). If \(\mathcal{F}\) is smooth (= locally constant) on the trajectories of the flow \(\phi_t\), then we have canonical isomorphisms \((\iota^\mu)_! \mathcal{F} \simeq (\phi_t^\mu)_! \mathcal{F}\) and \((\iota^\mu)_* \mathcal{F} \simeq (\phi_t^\mu)_* \mathcal{F}\).
(2) Consider \([\mathcal{O}_c] \xrightarrow{\iota^\mu} [\mathcal{O}_c^\mu] \xrightarrow{\phi^\mu} [\mathcal{O}_c]^\mu\) where \(\iota^\mu\) is the natural embedding. Let \(\mathcal{F} \in D_c([\mathcal{O}_c^\mu])\). If \(\mathcal{F}\) is smooth (= locally constant) on the trajectories of the flow \(\tilde{\phi}_t\) and is supported on a finite dimensional substack \(\mathfrak{V} \subset [\mathcal{O}_c^\mu]\), then we have canonical isomorphisms \((\iota^\mu)^*\mathcal{F} \simeq (\phi^\mu)_*\mathcal{F}\) and \((\iota^\mu)^*\mathcal{F} \simeq (\phi^\mu)_*\mathcal{F}\).

We shall show that the functor \(\Upsilon\) sends standard sheaves to co-standard sheaves. Introduce the following local system on \([\mathcal{O}_c^\lambda]\)

\[
(7.5) \quad \mathcal{L}_\lambda := (\iota^\lambda)^*L^\lambda \otimes \mathcal{L}_\lambda'' \otimes \text{or}_{\lambda}^\mu
\]

where

\[
(7.6) \quad (\mathcal{L}_\mu')^\vee := (i_+^\mu)^!(\mathcal{C})[\text{codim}[\mathcal{O}_c^\mu]] \quad \text{and} \quad \mathcal{L}_\mu := (i_+^\lambda)^!\mathcal{C}[\text{codim}[\mathcal{O}_c^\lambda][\mathcal{O}_c^\mu]]
\]

are local systems on \([\mathcal{O}_c^\lambda]\) and \([\mathcal{O}_c^\mu]\) respectively and \(\text{or}_{\lambda}^\mu := (\mathcal{F})^!\mathcal{C}[-\dim[\mathcal{E}_\lambda] + \dim[\mathcal{O}_c^\lambda]]\) is the orientation sheaf for the smooth map \(\mathcal{P}^\lambda : [\mathcal{O}_c^\lambda] \to [\mathcal{E}_\lambda]\) in \((7.1)\).

**Lemma 7.4.** For any local system \(\tau\) on \([\mathcal{O}_c^\lambda]\) we have

\[
\Upsilon(S^+_s(\lambda, \tau)) \simeq S^-_s(\lambda, \tau \otimes \mathcal{L}_\lambda)[d_\lambda].
\]

**Proof.** Let \(\lambda, \mu \in \mathcal{L}\). Consider the following diagram

\[
\begin{array}{ccc}
[\mathcal{O}_c^\lambda] \cap [\mathcal{O}_c^\mu] & \xrightarrow{u^\mu} & [\mathcal{E}^\mu] = LG_\mathbb{R}\setminus [\mathcal{O}_c^\mu] \\
\downarrow s & & \downarrow j^\mu \\
[\mathcal{O}_c^\lambda] & \xrightarrow{i_+^\mu} & LK_c\setminus \text{Gr} \xrightarrow{u} LG_\mathbb{R}\setminus \text{Gr}
\end{array}
\]

Let \(\mathfrak{G} = (j^\mu)^*\Upsilon(S_s(\lambda, \tau)) \simeq (j^\mu)^*(i_+^\lambda)*s(\tau^+) \simeq (u^\mu)_!(i_+^\lambda)_!(i_+^\mu)_!(\tau^+)\). It suffices to show that \(\mathfrak{G} \simeq 0\) if \(\lambda \neq \mu\) and \(\mathfrak{G} \simeq (\tau \otimes \mathcal{L}_\lambda)|_{\text{Gr}}\) if \(\lambda = \mu\).

By Corollary 2.3 the orbits \([\mathcal{O}_c^\mu]\) and \([\mathcal{O}_c^\lambda]\) are transversal to each other, hence we have

\[
(7.8) \quad (i_+^\mu)_!(\tau^+) \simeq (i_+^\lambda)_!(\tau^+) \otimes \mathcal{L}_\mu'[\text{codim}[\mathcal{O}_c^\mu]]
\]

where

\[
(7.9) \quad (\mathcal{L}_\mu')^\vee = (i_+^\mu)^!(\mathcal{C})[\text{codim}[\mathcal{O}_c^\mu]]
\]

is a local system on \([\mathcal{O}_c^\mu]\). Thus

\[
\mathfrak{G} \simeq u^\mu_!(i_+^\mu)_!(\tau^+) \simeq (u^\mu)_!(i_+^\mu)_!(\tau^+) \otimes \mathcal{L}_\mu'[\text{codim}[\mathcal{O}_c^\mu]] \simeq (u^\mu)_!(s_\ast i^\prime_\tau(\tau^+) \otimes \mathcal{L}_\mu')[\text{codim}[\mathcal{O}_c^\mu]].
\]

According to Lemma 7.3 the map \(u^\mu\) factors as

\[
u^\mu : [\mathcal{O}_c^\mu] \xrightarrow{p^\mu} [\mathcal{O}_c^\mu] \xrightarrow{\phi^\mu} [\mathcal{E}^\mu]
\]

where \(p^\mu\) is smooth of relative dimension \(\dim[\mathcal{E}^\mu] - \dim[\mathcal{O}_c^\mu]\). Since \(s_\ast i^\prime_\tau(\tau^+)\text{codim}[\mathcal{O}_c^\mu] \in D_c([\mathcal{O}_c^\mu])\) is smooth on the trajectories of the flow \(\tilde{\phi}_t\), by Lemma 7.3 we have

\[
(7.10) \quad \mathfrak{G} \simeq u^\mu_!(s_\ast i^\prime_\tau(\tau^+) \otimes \mathcal{L}_\mu')[\text{codim}[\mathcal{O}_c^\mu]] \simeq p^\mu_!(\phi^\mu)_!(s_\ast i^\prime_\tau(\tau^+) \otimes \mathcal{L}_\mu')[\text{codim}[\mathcal{O}_c^\mu]] \overset{\text{Lem 7.3}}{\simeq}
\]
Here $\iota^\mu : [\mathcal{O}_\mu^\mu] \to [\mathcal{O}_\mu^\mu]$ is the embedding.

If $\lambda \neq \mu$ then $[\mathcal{O}_\mu^\mu] \cap [\mathcal{O}_\lambda^\lambda]$ is empty, thus we have

$$(\iota^\mu)^! s_\ast \iota^!(\tau^+) \otimes \mathcal{L}_\lambda^\mu)[\text{codim}[\mathcal{O}_\mu^\mu]] = 0$$

and (7.10) implies $\mathcal{G} = 0$.

If $\lambda = \mu$, then $[\mathcal{O}_\mu^\mu] \cap [\mathcal{O}_\lambda^\lambda] = [\mathcal{O}_c^\lambda]$, $s = \iota^\mu$, $\iota = \iota^\lambda$ are closed embeddings and by Lemma 7.2 we have

$$(u^\lambda)! (s_\ast \iota^! (\tau^+) \otimes \mathcal{L}_\lambda^\mu)[\text{codim}[\mathcal{O}_\mu^\mu]] \simeq (u^\lambda)! (s_\ast (\iota \otimes \mathcal{L}_\lambda^\mu)^\mu)[\text{codim}[\mathcal{O}_\mu^\mu] - \text{codim}[\mathcal{O}_\mu^\mu][\mathcal{O}_c^\lambda]]$$

where $\mathcal{L}_\lambda^\mu := (\iota^\lambda)^! \mathcal{C}[\text{codim}[\mathcal{O}_\mu^\mu][\mathcal{O}_c^\lambda]]$

is a local system on $[\mathcal{O}_c^\lambda]$. Now an elementary calculation shows that

$\mathcal{G} \simeq (u^\lambda)! (s_\ast \iota^! (\tau^+) \otimes \mathcal{L}_\lambda^\mu)[\text{codim}[\mathcal{O}_\mu^\mu]] \simeq (u^\lambda)! (s_\ast (\tau \otimes \mathcal{L}_\lambda^\mu)^\mu)[\text{codim}[\mathcal{O}_\mu^\mu] - \text{codim}[\mathcal{O}_\mu^\mu][\mathcal{O}_c^\lambda]] \simeq$

$$(u^\lambda)! (\iota^\lambda)^* (\tau \otimes \mathcal{L}_\lambda^\mu \otimes (\mathcal{L}_\lambda^\mu)^\mu)[\text{codim}[\mathcal{O}_\mu^\mu] - \text{codim}[\mathcal{O}_\mu^\mu][\mathcal{O}_c^\lambda]] \simeq (\tau \otimes \mathcal{L}_\lambda^\mu)[\text{codim}[\mathcal{O}_\mu^\mu] - \text{codim}[\mathcal{O}_\mu^\mu][\mathcal{O}_c^\lambda]]$$

where

$$(\tau \otimes \mathcal{L}_\lambda^\mu)[\text{codim}[\mathcal{O}_\mu^\mu] - \text{codim}[\mathcal{O}_\mu^\mu][\mathcal{O}_c^\lambda]] \simeq \cdots$$

is a local system on $[\mathcal{O}_c^\lambda]$ and

$$d_\lambda = \text{codim}[\mathcal{O}_\mu^\mu] - \text{codim}[\mathcal{O}_\mu^\mu][\mathcal{O}_c^\lambda] + \text{dim}[\mathcal{E}_\lambda] - \text{dim}[\mathcal{O}_c^\lambda] = \dim \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R}} - \text{dim}[\mathcal{O}_K^\lambda]$$

The lemma follows.

We shall show that $\Upsilon$ is fully-faithful. Consider a diagram of closed substacks of $LK_c \setminus \text{Gr}$

$$U_0 \xrightarrow{j_0} U_1 \xrightarrow{j_1} U_2 \rightarrow \cdots \rightarrow U_k \rightarrow \cdots$$

such that

1. $\bigcup_i U_i = LK_c \setminus \text{Gr}$,
2. Each $U_i$ is a finite union of $[\mathcal{O}_K^\lambda]$,
3. Each $j_k$ is a closed embedding.

Let $f_i : U_i \to LK_c \setminus \text{Gr}$ be the natural embedding and we define

$s_i = u \circ f_i : U_i \to LG_{\mathbb{R}} \setminus \text{Gr}$.

Note that each $s_i$ is of finite type.

**Lemma 7.5.** For any $\mathcal{F}, \mathcal{F}' \in D_\text{c}(K(\mathcal{K}) \setminus \text{Gr})$ we have

$$\text{Hom}_{D_\text{c}(K(\mathcal{K}) \setminus \text{Gr})}(\mathcal{F}, \mathcal{F}') \simeq \text{Hom}_{D_\text{c}(LG_{\mathbb{R}} \setminus \text{Gr})}(\Upsilon(\mathcal{F}), \Upsilon(\mathcal{F}'))$$.
Proof. Choose $k$ such that $\mathcal{F} = (j_k)_{*} \mathcal{F}_k$ and $(j_k)_{*} \mathcal{F}_k'$ for $\mathcal{F}_k, \mathcal{F}_k' \in D_c(K(X) \setminus U_k)$. We have

$$\text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}, \mathcal{F}') \simeq \text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, \mathcal{F}_k')$$

and

$$\text{Hom}_{D_c(LG \setminus G \setminus U_k)}(\mathcal{F}, \mathcal{F}') \simeq \text{Hom}_{D_c(K(X) \setminus U_k)}((s_k)_{*}\mathcal{F}_k, (s_k)_{*}\mathcal{F}_k') \simeq \text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, (s_k)^{!}(s_k)_{!}\mathcal{F}_k').$$

We have to show that the map

$$(7.13) \quad \text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, \mathcal{F}_k') \to \text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, (s_k)^{!}(s_k)_{!}\mathcal{F}_k')$$

is an isomorphism. Since $D_c(K(X) \setminus U_k)$ is generated by $w_{*}(\tau^{+}_{\lambda})$ (resp. $w_{*}(\tau^{+}_{\mu})$) for $[\mathcal{O}^{\lambda}_{K}] \subset U_k$ (here $w_{*} : [\mathcal{O}^{\lambda}_{K}] \to U_k$ is natural inclusion), it suffices to verify $(7.13)$ for $\mathcal{F}_k = (w_{*})_{!}(\tau^{+}_{\lambda})$ and $\mathcal{F}_k' \simeq (w_{*})_{!}(\tau^{+}_{\mu})$.

Note that in this case the left hand side of $(7.13)$ becomes

$$(7.14) \quad \text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, \mathcal{F}_k') = 0 \quad \text{if} \quad \lambda \neq \mu$$

$$(7.15) \quad \text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, \mathcal{F}_k') \simeq \text{Hom}_{D_c([\mathcal{O}^{\lambda}_{K}])}(\tau^{+}_{\lambda}, \tau^{+}_{\lambda}) \quad \text{if} \quad \lambda = \mu.$$
Consider the case \( \lambda \neq \mu \). Then by Lemma 7.3 we have
\[
(\phi^\lambda_\ast)_\ast (\iota_\ast (u^\mu \circ s)^* \tau_{\mu, R} \otimes \mathcal{L})[d_\mu - d_\lambda]) \simeq (\phi^\lambda_\ast)_\ast (\iota_\ast ((u^\mu \circ s)^* \tau_{\mu, R} \otimes \iota^\ast \mathcal{L}))[d_\mu - d_\lambda]) \simeq
\]
\[
(\iota^\lambda_\ast)_\ast (\iota_\ast (u^\mu \circ s)^* \tau_{\mu, R} \otimes \iota^\ast \mathcal{L}))[d_\mu - d_\lambda]) = 0,
\]
here \( \iota^\lambda : [\mathcal{O}^\lambda_k] \to [\mathcal{O}^\lambda K] \), and it follows from (7.18) that
\[
\text{(7.19)} \quad \text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, (s_k)^\dagger(s_k); \mathcal{F}'_k) = 0.
\]
Hence we have
\[
\text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, \mathcal{F}'_k) \simeq \text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, (s_k)^\dagger(s_k); \mathcal{F}'_k) \simeq 0 \quad \text{if } \lambda \neq \mu.
\]

Consider the case \( \lambda = \mu \). We have
\[
(\phi^\lambda_\ast)_\ast (\iota_\ast (u^\lambda \circ s)^* \tau_{\lambda, R} \otimes \mathcal{L})[d_\lambda - d_\lambda]) \simeq (u^\lambda \circ s)^* \tau_{\lambda, R} \otimes \iota^\ast \mathcal{L}''' \simeq \tau_\lambda \otimes \mathcal{L}_\lambda \otimes \iota^\ast \mathcal{L}'''.
\]
We claim that \( \mathcal{L}_\lambda \otimes \iota^\ast \mathcal{L}''' \simeq \mathbb{C} \) is the trivial local system hence above isomorphism implies
\[
(\phi^\lambda_\ast)_\ast (\iota_\ast (u^\lambda \circ s)^* \tau_{\lambda, R} \otimes \mathcal{L})[d_\lambda - d_\lambda]) \simeq \tau_\lambda, \quad \text{if } \lambda = \mu,
\]
and by (7.18), we obtain
\[
\text{(7.20)} \quad \text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, (s_k)^\dagger(s_k); \mathcal{F}'_k) \simeq \text{Hom}_{D_c([\mathcal{O}^\lambda_k])}(\tau_\lambda, \tau_\lambda).
\]
By unwinding the definition of the map in (7.13), we obtain that (7.13) satisfies
\[
\text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, \mathcal{F}'_k) \xrightarrow{(7.13)} \text{Hom}_{D_c(K(X) \setminus U_k)}(\mathcal{F}_k, (s_k)^\dagger(s_k); \mathcal{F}'_k),
\]
\[
\sim \xrightarrow{(7.15)} \text{Hom}_{D_c([\mathcal{O}^\lambda_k])}(\tau_\lambda, \tau_\lambda) \xrightarrow{(7.20)} \text{Hom}_{D_c([\mathcal{O}^\lambda_k])}(\tau_\lambda, \tau_\lambda),
\]
hence is an isomorphism. The lemma follows.

To prove the claim, we observe that, up to cohomological shifts, we have
\[
\mathcal{L}_\lambda \simeq (s)^* \mathcal{L}_\lambda' \otimes \mathcal{L}_\lambda'' \otimes \text{or}_{p, \lambda} \simeq (p^\lambda)^*((j^\lambda)^\ast(\mathcal{C}^\vee) \otimes \text{or}_{p, \lambda} [-])
\]
\[
\iota^\ast \mathcal{L}''' \simeq \iota^\ast ((u \circ i^\lambda_\ast) \mathcal{C}) [-].
\]
Using the canonical isomorphisms \( \iota^\ast(...) \simeq \iota^\ast(...) \otimes \iota^\ast(...) \otimes \text{or}_{p, \lambda} [-] \), we see that
\[
\mathcal{L}_\lambda \otimes \iota^\ast \mathcal{L}''' \simeq (p^\lambda)^*((j^\lambda)^\ast(\mathcal{C}^\vee) \otimes \text{or}_{p, \lambda} \otimes \iota^\ast((u \circ i^\lambda_\ast) \mathcal{C}) [-]) \simeq
\]
\[
\simeq (p^\lambda)^*((j^\lambda)^\ast(\mathcal{C}^\vee)) \otimes \text{or}_{p, \lambda} \otimes (p^\lambda)^*((j^\lambda)^\ast(\mathcal{C})) [-] \simeq
\]
\[
\simeq (p^\lambda)^*((j^\lambda)^\ast(\mathcal{C}^\vee)) \otimes (p^\lambda)^*((j^\lambda)^\ast(\mathcal{C})) [-] \simeq \mathbb{C} [-].
\]
The claim follows. \( \square \)

It follows from Lemma 7.4 and Lemma 7.5 that the image of \( \Upsilon \) is equal to \( D_!(\text{LG}_{\mathbb{R}} \setminus \text{Gr}) \) and the resulting functor \( \Upsilon : D_c(K(X) \setminus \text{Gr}) \to D_!(\text{LG}_{\mathbb{R}} \setminus \text{Gr}) \) is fully-faithful, hence an equivalence. This finishes the proof of Theorem 7.1.
8. NEARBY CYCLES FUNCTORS AND THE RADON TRANSFORM

We study the nearby cycles functors associated to the quasi-maps in Section 6 and the Radon transform for the real affine Grassmannian.

8.1. A square of equivalences. Recall the quasi-map family $QM^{(2)}(\mathbb{P}^1, G, K)_{\mathbb{R},0} \to \mathbb{P}^1(\mathbb{C})$ in Section 6.2. By Proposition 6.2, we have the following cartesian diagram

\[
\begin{array}{ccc}
(LK_c \backslash \text{Gr}) \times i\mathbb{R}_{>0} & \xrightarrow{j} & QM^{(2)}(\mathbb{P}^1, G, K)_{\mathbb{R},0}|_{i\mathbb{R}_{>0}} \\
\downarrow f^0 & & \downarrow f \downarrow f_0 \\
LG_\mathbb{R} \backslash \text{Gr} \times i\mathbb{R}_{>0} & \xrightarrow{j} & \text{Bun}_G(\mathbb{P}^1)_{\mathbb{R},0} \times i\mathbb{R}_{>0} \\
\downarrow i\mathbb{R}_{>0} & & \downarrow i\mathbb{R}_{>0} \\
& & \{0\}
\end{array}
\]

Define the following nearby cycles functors

\[\Psi: D_c(LK_c \backslash \text{Gr}) \to D_c(K_c \backslash \text{Gr}_\mathbb{R}^0), \quad \mathcal{F} \to \Psi(\mathcal{F}) := i^! j_!(\mathcal{F} \boxtimes C_{i\mathbb{R}_{>0}}),\]

\[\Psi_\mathbb{R}: D_c(LG_\mathbb{R} \backslash \text{Gr}) \to D_c(G_\mathbb{R}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_\mathbb{R}^0), \quad \mathcal{F} \to \Psi_\mathbb{R}(\mathcal{F}) = (i^! j_!(\mathcal{F} \boxtimes C_{i\mathbb{R}_{>0}})).\]

We also have the Radon transform

\[\Upsilon_\mathbb{R}: D_c(G_\mathbb{R}(O_\mathbb{R}) \backslash \text{Gr}_\mathbb{R}) \to D_c(G_\mathbb{R}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_\mathbb{R}),\]

given by the restriction to $D(G_\mathbb{R}(O_\mathbb{R}) \backslash \text{Gr}_\mathbb{R}) \subset D_c(G_\mathbb{R} \backslash \text{Gr}_\mathbb{R})$ of the push-forward $p_! : D_c(G_\mathbb{R} \backslash \text{Gr}_\mathbb{R}) \to D_c(G_\mathbb{R}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_\mathbb{R})$ along the quotient map $p : G_\mathbb{R} \backslash \text{Gr} \to G_\mathbb{R}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_\mathbb{R}$.

Here are the main results of this section.

**Theorem 8.1.** The nearby cycles functors and the Radon transform induce equivalences of categories:

\[\Psi: D_c(K(\mathbb{K}) \backslash \text{Gr}) \xrightarrow{\sim} D_c(G_\mathbb{R}(O_\mathbb{R}) \backslash \text{Gr}_\mathbb{R}^0),\]

\[\Psi_\mathbb{R}: D_!(LG_\mathbb{R} \backslash \text{Gr}) \xrightarrow{\sim} D_!(G_\mathbb{R}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_\mathbb{R}^0),\]

\[\Upsilon_\mathbb{R}: D_c(G_\mathbb{R}(O_\mathbb{R}) \backslash \text{Gr}_\mathbb{R}) \xrightarrow{\sim} D_!(G_\mathbb{R}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_\mathbb{R}).\]

**Theorem 8.2.** We have a commutative square of equivalences

\[
\begin{array}{ccc}
D_c(K(\mathbb{K}) \backslash \text{Gr}) & \xrightarrow{\Psi} & D_c(G_\mathbb{R}(O_\mathbb{R}) \backslash \text{Gr}_\mathbb{R}^0) \\
\downarrow \Upsilon & & \downarrow \Upsilon_\mathbb{R} \\
D_!(LG_\mathbb{R} \backslash \text{Gr}) & \xrightarrow{\Psi_\mathbb{R}} & D_!(G_\mathbb{R}(\mathbb{R}[t^{-1}]) \backslash \text{Gr}_\mathbb{R}^0).
\end{array}
\]

Here $\Upsilon$ is the affine Matsuki correspondence for sheaves.

The rest of the section is devoted to the proof of Theorem 8.1 and Theorem 8.2.
8.2. Images of standard sheaves under $\Psi$. We begin with the following constructibility result for $\Psi$.

**Lemma 8.3.** We have $\Psi(\mathcal{F}) \in D_c(G_\mathbb{R}(\mathcal{O}_\mathbb{R}) \setminus \text{Gr}_\mathbb{R}^0)$ for any $\mathcal{F} \in D_c(K(K) \setminus \text{Gr})$.

**Proof.** Consider the stratification of $S_1$ of $\Omega K_c \setminus \text{Gr}$ with strata the $\Omega K_c$-quotients of $K(K)$-orbits. By Lemma 6.4, the pull-back of $S$ as stacks over $i\mathbb{R}_{\geq 0}$.

Furthermore, the map

$$QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{R,0}|_{i\mathbb{R} \geq 0} \to \Omega K_c \setminus \text{Gr}$$

together with the $G_\mathbb{R}(\mathcal{O}_\mathbb{R})$-orbits stratification of $\text{Gr}_{R}^0 \simeq QM^{(2)}(\mathbb{P}^1, G, K)_{R,0}|_{i\mathbb{R} \geq 0}$ forms a Whitney stratification of $QM^{(2)}(\mathbb{P}^1, G, K)_{R,0}|_{i\mathbb{R} \geq 0}$. Moreover, the map

$$QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{R,0}|_{i\mathbb{R} \geq 0} \to i\mathbb{R}_{\geq 0}$$

is Thom-stratified with respect to the above stratification of $QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{R,0}|_{i\mathbb{R} \geq 0}$ and the stratification $i\mathbb{R}_{\geq 0} = i\mathbb{R}_{\geq 0} \cup \{0\}$. Since

$$K_c \setminus QM^{(2)}(\mathbb{P}^1, G, K, \infty)_{R,0}|_{i\mathbb{R} \geq 0} \simeq QM^{(2)}(\mathbb{P}^1, G, K)_{R,0}|_{i\mathbb{R} \geq 0}$$

as stacks over $i\mathbb{R}_{\geq 0}$ (see §6.4), the nearby cycles functor $\Psi$ takes $\{LK_c \setminus \mathcal{O}_\mathbb{R}^\lambda\}_{\lambda \in \Lambda}$-constructible complexes on $LK_c \setminus \text{Gr}$ to $\{K_c \setminus S_{\mathbb{R}}^\lambda\}_{\lambda \in \Lambda}$-constructible complexes on $K_c \setminus \text{Gr}_{R}^0$. The lemma follows.

By the lemma above, the nearby cycles functor $\Psi$ restricts to a functor

$$\Psi : D_c(K(K) \setminus \text{Gr}) \to D_c(G_\mathbb{R}(\mathcal{O}_\mathbb{R}) \setminus \text{Gr}_\mathbb{R}^0).$$

We shall show that $\Psi$ sends standard sheaves to standard sheaves. Recall the flow

$$\psi : QM^{(2)}(\mathbb{P}^1, G, X, \infty)_R \to QM^{(2)}(\mathbb{P}^1, G, X, \infty)_R$$

in §6.6. For $\lambda \in \Lambda_{\mathbb{R}}^+$, we have the critical manifold $C_{\mathbb{R}}^\lambda$, the stable manifold $S_{\mathbb{R}}^\lambda$, and the unstable manifold $\bar{T}_{\mathbb{R}}^\lambda$. We write

$$s^+_\lambda : S_{\mathbb{R}}^\lambda \to QM^{(2)}(\mathbb{P}^1, G, K, \infty)_R, \quad \bar{i}^\lambda : \bar{T}_{\mathbb{R}}^\lambda \to QM^{(2)}(\mathbb{P}^1, G, K, \infty)_R$$

for the inclusion maps and we write

$$c^+_\lambda : S_{\mathbb{R}}^\lambda \to C_{\mathbb{R}}^\lambda, \quad \bar{d}^\lambda : \bar{T}_{\mathbb{R}}^\lambda \to C_{\mathbb{R}}^\lambda$$

for the contraction maps. Note that all the maps above are $K_c$-equivariant with respect to natural $K_c$-actions. The following lemma follows from a topological version of Braden’s theorem, see [N2, Theorem 9.2].

**Lemma 8.4.** For every $\mathcal{F} \in D_c(QM^{(2)}(\mathbb{P}^1, G, K, \infty)_R)$ which is $\mathbb{R}_{\geq 0}$-constructible with respect to the flow $\psi$, we have

$$(c^+_\lambda)_*(s^+_\lambda)^*\mathcal{F} \simeq (\bar{d}^\lambda)_!(\bar{i}^\lambda)^*\mathcal{F}.$$
Recall that, by Lemma 6.6, we have isomorphisms \( \tilde{t}^\lambda_{|i} \simeq \Omega K_c \backslash O^\lambda_R, \tilde{t}^\lambda_{|0} \simeq T^\lambda_R \), for \( \lambda \in \mathcal{L} \) and we write
\[
s^\lambda_+: T^\lambda_R \to QM^{(2)}(\mathbb{P}^1, G, K, \infty)_R, \quad t^\lambda_+: \Omega K_c \backslash O^\lambda_R \to QM^{(2)}(\mathbb{P}^1, G, K, \infty)_R
\]
for the restriction of \( \tilde{t}^\lambda \) and
\[
c^\lambda_-: T^\lambda_R \to C^\lambda_R, \quad d^\lambda_+: \Omega K_c \backslash O^\lambda_R \to C^\lambda_R
\]
for the restriction of the contractions \( \tilde{d}^\lambda \).

**Lemma 8.5.** For every \( \mathcal{F} \in D_c(\Omega K_c \backslash \text{Gr}) \) which is \( \mathbb{R}_{>0} \)-constructible with respect to the flow \( \psi_z \), we have
\[
(c^\lambda_+)_*(s^\lambda_+)^! \Psi(\mathcal{F}) \simeq (d^\lambda_+)_!(t^\lambda_+)^* \mathcal{F}, \quad \text{if } \lambda \in \mathcal{L}.
\]

**Proof.** Same argument as in \([N2, \text{Corollary 9.2}]\) \( \square \)

We write \( k^\lambda_+: \Omega K_c \backslash O^\lambda_c \to C^\lambda_R \) for the restriction of \( d^\lambda_+ \) and \( p^\lambda_+: T^\lambda_R \to G_R(\mathbb{R}[t^{-1}]) \backslash T^\lambda_R \) for the natural quotient map.

**Lemma 8.6.** The map \( k^\lambda_+: \Omega K_c \backslash O^\lambda_c \to C^\lambda_R \) is a \( K_c \)-equivariant isomorphism. There is a bijection between isomorphism classes of \( K_c \)-equivariant local systems \( \omega^+ \) on \( S^\lambda_R \), \( K_c \)-equivariant local systems \( \omega^- \) on \( T^\lambda_R \), \( K_c \)-equivariant local systems \( \omega \) on \( C^\lambda_R \), \( K_c \)-equivariant local systems \( \tau \) on \( \Omega K_c \backslash O^\lambda_c \), and local system \( \omega_R \) on \( G_R(\mathbb{R}[t^{-1}]) \backslash T^\lambda_R \), characterizing by the property that \( \omega^+ \simeq (c^\lambda_+)^* \omega, \tau \simeq (k^\lambda_+)^* \omega \), and \( (p^\lambda_+)^* \omega_R \simeq (c^\lambda_+)^* \omega \).

**Proof.** The first claim follows from the fact that \( \Omega K_c \backslash O^\lambda_c \simeq C^\lambda_R \simeq K_c(\lambda) \backslash K_c \), where \( K_c(\lambda) \) is the stabilizer of \( \lambda \) in \( K_c \), and the \( K_c \)-equivariant property of \( k^\lambda_+ \). The second claim follows from the facts that the contraction maps \( c^\lambda_+ \) are \( K_c \)-equivariant and the fibers of \( c^\lambda_+ \) and the quotient \( K_c \backslash G_R(\mathbb{R}[t^{-1}]) \) are contractible. \( \square \)

For any \( \lambda \in \mathcal{L} \) and a \( K_c \)-equivaraint local system \( \omega \) on \( C^\lambda_R \), one has the standard and co-standard sheaves
\[
\mathcal{F}^+_\tau(\lambda, \omega) := (s^\lambda_+)_!(\omega^+) \quad \text{and} \quad \mathcal{F}^+_\tau(\lambda, \omega) := (s^\lambda_+)_!(\omega^+)
\]
in \( D_c(G_R(\mathfrak{O}_R) \backslash \text{Gr}_R) \). Recall the standard sheaf \( S^\lambda_+(\lambda, \tau) \) in \( D_c(K(\mathcal{K}) \backslash \text{Gr}) \) (see \([7,2]\)).

**Proposition 8.7.** We have \( \Psi(S^\lambda_+(\lambda, \tau)) \simeq \mathcal{F}^+_\tau(\lambda, \omega) \)

**Proof.** It suffices to show that
\[
(a) \quad (s^\lambda_+)^! \Psi(S^\lambda_+(\lambda, \tau)) \simeq \omega^+ \quad \text{and} \quad (b) \quad (s^\lambda_+)^! \Psi(S^\lambda_+(\lambda, \tau)) = 0 \text{ for } \mu \neq \lambda.
\]

**Proof of (a).** By Lemma 8.6, it suffices to show that \( (c^\lambda_+)_*(s^\lambda_+)^! \Psi(S^\lambda_+(\lambda, \tau)) \simeq \omega \). But it follows from Lemma 8.5 and Lemma 7.2 that, indeed, we have
\[
(c^\lambda_+)_*(s^\lambda_+)^! \Psi(S^\lambda_+(\lambda, \tau)) \simeq (d^\lambda_+)_!(t^\lambda_+)^* S^\lambda_+(\lambda, \tau) \simeq (k^\lambda_+)_!(S^\lambda_+(\lambda, \tau)|_{\Omega K_c \backslash O^\lambda_c}) \simeq (k^\lambda_+)_! \tau \simeq -\omega.
\]

**Proof of (b).** It suffices to show that \( \mathcal{F}_\mu := (c^\lambda_+)_*(s^\lambda_+)^! \Psi(S^\lambda_+(\lambda, \tau)) = 0 \) for \( \mu \neq \lambda \). For this, it is enough to show that
\[
H^*_c(\mathcal{F}_\mu \otimes \mathcal{L}) = 0
\]
for any $K_c$-equivariant local system $\mathcal{L}$ on $C^\mu_R$. Recall the contraction map $\phi^\mu_\omega : \Omega K_c \setminus O^\mu_R \to \Omega K_c \setminus O^\mu_c$ coming from the Matsuki flow $\phi_\omega : \text{Gr} \to \text{Gr}$ in §3 By Lemma 7.2 and Lemma 8.6 for any such $\mathcal{L}$, there is a $K_c$-equivariant local system $\mathcal{L}'$ on $\Omega K_c \setminus O^\mu_c$ satisfying

\[(d_\mu)^*\mathcal{L} \simeq (\phi^\mu_\omega)^*\mathcal{L}' \quad (\text{as equivariant local systems on } \Omega K_c \setminus O^\mu_R).\]

Thus we have

\[
H^c_*(\mathcal{F}_\mu \otimes \mathcal{L}) \simeq H^c_*((c^1_\mu)_*(s_+^1)\Psi_K(S^+_\mu(\lambda, \tau) \otimes \mathcal{L})
\]

\[
\simeq H^c_*((d_\mu)_!(t_\mu)^*S^+_\mu(\lambda, \tau) \otimes \mathcal{L}) \quad (\text{by Lemma 8.5})
\]

\[
\simeq H^c_*(d_\mu)_!((t_\mu)^*S^+_\mu(\lambda, \tau) \otimes (d_\mu)^*\mathcal{L})) \quad (\text{by projection formula})
\]

\[
\simeq H^c_*((\phi^-_\mu)_!((t_\mu)^*S^+_\mu(\lambda, \tau) \otimes (\phi^-_\mu)^*\mathcal{L}')) \quad (\text{by (8.4)})
\]

\[
\simeq H^c_*((\phi^-_\mu)_!(t_\mu)^*S^+_\mu(\lambda, \tau) \otimes \mathcal{L}') \quad (\text{by projection formula})
\]

Note that $t_\mu = i^\mu_+ : \Omega K_c \setminus O^\mu_R \to \Omega K_c \setminus \text{Gr}$ is the embedding for the unstable manifold of the Morse flow $\phi_\omega$ on $\Omega K_c \setminus \text{Gr}$ and we have

\[(\phi^-_\mu)_!(t_\mu)^*S^+_\mu(\lambda, \tau) \simeq (\phi^-_\mu)_!(i^\mu_+)^*S^+_\mu(\lambda, \tau) \simeq (\phi^-_\mu)_!(i^\mu_+)^*1^\alphaS^+_\mu(\lambda, \tau),\]

here $i^\mu_+ : \Omega K_c \setminus O^\mu_R \to \Omega K_c \setminus \text{Gr}$ is the embedding for the stable manifold of the flow $\phi_\omega$ and $\phi^\mu_+ : \Omega K_c \setminus O^\mu_R \to \Omega K_c \setminus O^\mu_c$ is the contraction map. Since $\mu \neq \lambda$, we have $(i^\mu_+)^*S^+_\mu(\lambda, \tau) = 0$ and it implies

\[
H^c_*(\mathcal{F}_\mu \otimes \mathcal{L}) \simeq H^c_*((\phi^-_\mu)_!(t_\mu)^*S^+_\mu(\lambda, \tau) \otimes \mathcal{L}') \quad (\text{by Lemma 8.5})
\]

\[
\simeq H^c_*(((\phi^-_\mu)_!(i^\mu_+)^*1^\alphaS^+_\mu(\lambda, \tau) \otimes \mathcal{L}').
\]

Claim (b) follows and the proposition is proved.

\[
\square
\]

8.3. The Radon transform. Recall the flow $\psi^1_\omega : \text{Gr}^{(2)}_R \to \text{Gr}^{(2)}_R$ in (5.10). By Lemma 5.5 it restricts to a flow on the special fiber $\text{Gr}_R \simeq \text{Gr}^{(2)}_R|_0$ with critical manifolds $\bigcup_{\lambda \in \Lambda^+_\Lambda} C^\alpha_R$ and $S^\alpha_R$, respectively. $T^\lambda_R$ is the stable manifold, respectively unstable manifold, of $C^\alpha_R$. Let $T^\lambda_R : G_R(\mathbb{R}[t^{-1}]) \setminus T^\lambda_R \to G_R(\mathbb{R}[t^{-1}]) \setminus \text{Gr}_R$ be the natural inclusion map. According to Lemma 8.6 for any $K_c$-equivariant local system $\omega$ on $C^\alpha_R$, we have the standard and co-standard sheaves

\[
\mathcal{T}^-\lambda(\lambda, \omega) := (t^\lambda)^*(\omega_R) \quad \text{and} \quad \mathcal{T}^+\lambda(\lambda, \omega) := (t^\lambda)^*(\omega_R).
\]

Recall the Radon transform

\[
\Upsilon_R : D_c(G_R(\mathcal{O}_R) \setminus \text{Gr}_R) \to D_c(G_R(\mathbb{R}[t^{-1}] \setminus \text{Gr}_R)
\]

in (8.3). The same argument as in the proof of Theorem 7.1 replacing the Matsuki flow $\phi_t : \text{Gr} \to \text{Gr}$ by the $\mathbb{R}_{>0}$-flow $\psi^1_\omega : \text{Gr}_R \to \text{Gr}_R$ and Lemma 7.2 by Lemma 8.6 gives us:

**Proposition 8.8.** The Radon transform defines an equivalence of categories

\[
\Upsilon_R : D_c(G_R(\mathcal{O}_R) \setminus \text{Gr}_R) \to D_i(G_R(\mathbb{R}[t^{-1}] \setminus \text{Gr}_R).
\]

Moreover, for any $K_c$-equivariant local system $\omega$ on $C^\alpha_R$ we have

\[
\Upsilon_R(\mathcal{T}^+\lambda(\lambda, \omega)) \simeq \mathcal{T}^-\lambda(\lambda, \omega \otimes \mathcal{L}_\lambda)[d_\lambda].
\]
Here we regard the local system \( L_i \) in \( \text{7.12} \) as a local system on \( C_\mathbb{A}/\mathbb{C} \) via the isomorphism \( k_\lambda : \Omega G_\mathbb{A}/\mathbb{C} \cong C_\mathbb{A} \) in Lemma \( \text{8.6} \).

### 8.4. The functor \( \Psi_\mathbb{R} \)

Note that, by Proposition \( \text{5.11} \), the map \( LG_\mathbb{R}^{(2)}(\mathbb{P}^1) \cong \text{Bun}_G(\mathbb{P}^1,0) \times i\mathbb{R} \to i\mathbb{R} \) is isomorphic to a constant family. It implies

**Proposition 8.9.** The nearby cycles functor

\[
\Psi_\mathbb{R} : D_c(G[\mathbb{R}] \text{Gr}) \to D_c(G[\mathbb{R}] [\mathbb{R}] \text{Gr})
\]

is a t-exact equivalence (with respect to the natural \( t \)-structures) satisfying \( \Psi_\mathbb{R}(S_1^- (\lambda, \tau)) \cong \mathcal{T}_1^- (\lambda, \omega) \).

### 8.5. Proof of Theorem 8.1 and Theorem 8.2

It remains to prove that \( \Psi : D_c(K(\mathcal{X}) \text{Gr}) \to D_c(K(\mathcal{X}) \text{Gr}) \) is an equivalence and Theorem 8.2. Note that for \( \mathcal{F} \in D_c(K(\mathcal{X}) \text{Gr}) \) there is a natural transformation (induced by the natural transformation \( f_0 i^! \to (i^! f_0)^! \))

\[
\Upsilon_\mathbb{R} \circ \Psi(\mathcal{F}) = (f_0) i^! j_! (\mathcal{F} \boxtimes C_{[\mathbb{R}] > 0}) \to (i^!) f_0 j_!(\mathcal{F} \boxtimes C_{[\mathbb{R}] > 0}) \cong (i^!) (\mathcal{F} \boxtimes C_{[\mathbb{R}] > 0}) = \Psi_\mathbb{R} \circ \Upsilon(\mathcal{F}).
\]

Moreover, it follows from Lemma \( \text{7.4} \), Proposition \( \text{8.7} \), Proposition \( \text{8.8} \), and Proposition \( \text{8.9} \) that \( \text{8.7} \) is an isomorphism for the standard sheaf \( S_1^- (\lambda, \tau) \). Since the category \( D_c(K(\mathcal{X}) \text{Gr}) \) is generated by \( S_1^- (\lambda, \tau) \), it implies \( \text{8.7} \) is an isomorphism. By Theorem \( \text{7.1} \), Proposition \( \text{8.8} \), and Proposition \( \text{8.9} \), the functors \( \Psi_\mathbb{R} \), \( \Upsilon_\mathbb{R} \), and \( \Upsilon_\mathbb{R} \) are equivalences and \( \text{8.7} \) implies \( \Psi : D_c(K(\mathcal{X}) \text{Gr}) \to D_c(G[\mathbb{R}] \text{Gr}) \) is an equivalence. This finishes the proof of Theorem 8.1 and Theorem 8.2.

### 9. Compatibility of Hecke actions

Recall the derived Satake category \( D_c(G(0) \text{Gr}) \) is naturally monoidal with respect to convolution. We will write \( \mathcal{T}_1 \star \mathcal{T}_2 \) for the convolution product of \( \mathcal{T}_1, \mathcal{T}_2 \in D_c(G(0) \text{Gr}) \).

Here we enhance the equivalences and commutative square of Theorems 8.1 and 8.2 to \( D_c(G(0) \text{Gr}) \)-modules. Roughly speaking, we will take advantage of the natural right actions on the categories involved, whereas the prior Radon transforms were performed on the left.

#### 9.1. Hecke actions

First, the affine Matsuki correspondence for sheaves

\[
\Upsilon : D_c(K(\mathcal{X}) \text{Gr}) \sim D_c(LG_\mathbb{R} \text{Gr})
\]

is naturally an equivalence of \( D_c(G(0) \text{Gr}) \)-modules by convolution on the right. To see this, recall \( \Upsilon \) is the restriction to \( D_c(K(\mathcal{X}) \text{Gr}) \subset D_c(LK_\mathbb{R} \text{Gr}) \) of the push-forward \( u : D_c(LK_\mathbb{R} \text{Gr}) \to D_c(LG_\mathbb{R} \text{Gr}) \) along the quotient map \( u : LK_\mathbb{C} \text{Gr} \to LG_\mathbb{R} \text{Gr} \). We can equip this construction with compatibility with convolution on the right by using the commutative action diagram

\[
\begin{array}{ccc}
LK_\mathbb{C} \times G(0) \text{Gr} & \xrightarrow{u \times \text{id}} & LG_\mathbb{R} \times G(0) \text{Gr} \\
\downarrow & & \downarrow \\
LK_\mathbb{C} \text{Gr} & \xrightarrow{u} & LG_\mathbb{R} \text{Gr}
\end{array}
\]
and its natural iterations.

Similarly, the Radon equivalence

\[ \Upsilon_\mathbb{R} : D_c(G_\mathbb{R}(O_\mathbb{R}) \backslash \text{Gr}_\mathbb{R}) \simto D_!(G_\mathbb{R}([t^{-1}]) \backslash \text{Gr}_\mathbb{R}) \]

is naturally an equivalence of \( D_c(G_\mathbb{R}(O_\mathbb{R}) \backslash \text{Gr}_\mathbb{R}) \)-modules by convolution on the right. To see this, recall \( \Upsilon_\mathbb{R} \) is the restriction to \( D_c(G_\mathbb{R}(O_\mathbb{R}) \backslash \text{Gr}_\mathbb{R}) \subset D_c(G_\mathbb{R} \backslash \text{Gr}_\mathbb{R}) \) of the push-forward \( p_! : D_c(G_\mathbb{R} \backslash \text{Gr}_\mathbb{R}) \to D_c(G_\mathbb{R}([t^{-1}]) \backslash \text{Gr}_\mathbb{R}) \) along the quotient map \( p : G_\mathbb{R} \backslash \text{Gr} \to G_\mathbb{R}([t^{-1}]) \backslash \text{Gr}_\mathbb{R} \).

We can equip this construction with compatibility with convolution on the right by using the commutative action diagram

\[
\begin{array}{ccc}
G_\mathbb{R} \backslash G(K_\mathbb{R}) \times^{G(0_k)} \text{Gr}_\mathbb{R} & \xrightarrow{p \times \text{id}} & G_\mathbb{R}([t^{-1}]) \backslash G(K_\mathbb{R}) \times^{G(0_k)} \text{Gr}_\mathbb{R} \\
\downarrow & & \downarrow \\
G_\mathbb{R} \backslash \text{Gr}_\mathbb{R} & \xrightarrow{p} & G_\mathbb{R}([t^{-1}]) \backslash \text{Gr}_\mathbb{R}
\end{array}
\]

and its natural iterations.

9.2. From complex to real kernels. Following [N1], nearby cycles in the real Beilinson-Drinfeld Grassmannian \( \text{Gr}^{(2)}_\mathbb{R} \) over \( i\mathbb{R} \geq 0 \) gives a functor

\[ m : D_c(G(0) \backslash \text{Gr}) \to D_c(G_\mathbb{R}(O_\mathbb{R}) \backslash \text{Gr}^{(2)}_\mathbb{R}) \subset D_c(G_\mathbb{R}(O_\mathbb{R}) \backslash \text{Gr}_\mathbb{R}) \]

Namely, by Lemma 4.1, there is a canonical diagram of \( G_\mathbb{R} \)-equivariant maps

\[ (9.1) \quad \text{Gr} \xleftarrow{\pi} \text{Gr} \times i\mathbb{R} \geq 0 \simeq \text{Gr}^{(2)}_\mathbb{R} \big|_{i\mathbb{R} \geq 0} \xrightarrow{j} \text{Gr}^{(2)}_\mathbb{R} \big|_{i\mathbb{R} \geq 0} \xrightarrow{i} \text{Gr}^{(2)}_\mathbb{R} \big|_0 \simeq \text{Gr}_\mathbb{R} \]

where we view \( G_\mathbb{R} \subset LG^{(2)}_\mathbb{R} \) as the constant group-scheme. One defines \( m = i^*j_* \pi^* f_\mathbb{R} \) where we write \( f_\mathbb{R} : D_c(G(0) \backslash \text{Gr}) \to D_c(G_\mathbb{R} \backslash \text{Gr}) \) for the forgetful functor.

Note the domain and codomain of \( m \) both have natural convolution monoidal structures. To equip \( m \) with a monoidal structure, we proceed as follows.

Let \( \text{Gr}^{(2)}_\mathbb{R} \times \text{Gr}^{(2)}_\mathbb{R} \) be the moduli of \( x_1, x_2 \in \mathbb{P}^1 \), \( E_1, E_2 \) \( G \)-torsors on \( \mathbb{P}^1 \), \( \phi \) a trivialization of \( E_1 \) over \( \mathbb{P}^1 \setminus \{x_1, x_2\} \), and \( \alpha \) an isomorphism from \( E_1 \) to \( E_2 \) over \( \mathbb{P}^1 \setminus \{x_1, x_2\} \). Let \( \text{Gr}^{(2)}_\mathbb{R} \times \text{Gr}^{(2)}_\mathbb{R} \) be the real form of \( \text{Gr}^{(2)}_\mathbb{R} \times \text{Gr}^{(2)}_\mathbb{R} \) with respect to the twisted conjugation that exchanges \( x_1 \) and \( x_2 \).

Then there is a canonical diagram of \( G_\mathbb{R} \)-equivariant maps

\[ (9.2) \quad G(K) \times^{G(0)} \text{Gr} \xrightarrow{\pi} G(K) \times^{G(0)} \text{Gr} \times i\mathbb{R} \geq 0 \simeq \text{Gr}^{(2)}_\mathbb{R} \times \text{Gr}^{(2)}_\mathbb{R} \big|_{i\mathbb{R} \geq 0} \xrightarrow{j} \]

\[ \text{Gr}^{(2)}_\mathbb{R} \times \text{Gr}^{(2)}_\mathbb{R} \big|_{i\mathbb{R} \geq 0} \xrightarrow{i} \text{Gr}^{(2)}_\mathbb{R} \times \text{Gr}^{(2)}_\mathbb{R} \big|_0 \simeq G_\mathbb{R}(K_\mathbb{R}) \times_{G(0_k)} \text{Gr}_\mathbb{R} \]

Moreover, the convolution maps on the end terms naturally extend to the entire diagram. By standard identities, we arrive at a canonical isomorphism \( m(F_1 \star F_2) \simeq m(F_1) \star m(F_2) \). By using iterated versions of the above moduli spaces, we may likewise equip \( m \) with the associativity constraints of a monoidal structure.
9.3. **Compatibility of actions.** Note we can view the Radon equivalence \( \Upsilon_R \) as an equivalence of \( D_c(G(\emptyset)\backslash Gr) \)-modules via the monoidal functor

\[
m : D_c(G(\emptyset)\backslash Gr) \longrightarrow D_c(G(\emptyset)(\emptyset_R)\backslash Gr_0(\emptyset_R))
\]

Now we have the following further compatibility of our constructions.

**Theorem 9.1.** Via the monoidal functor

\[
m : D_c(G(\emptyset)\backslash Gr) \longrightarrow D_c(G(\emptyset)(\emptyset_R)\backslash Gr_0(\emptyset_R))
\]

the equivalences

\[
\Psi : D_c(K(\emptyset)\backslash Gr) \longrightarrow D_c(G(\emptyset)(\emptyset_R)\backslash Gr_0(\emptyset_R)),
\]

\[
\Psi_R : D_c(LG_\emptyset(\emptyset)\backslash Gr) \longrightarrow D_c(G(\emptyset)(\emptyset_R[t^{-1}])\backslash Gr_0(\emptyset_R))
\]

of Theorem 8.2 and commutative square

\[
\begin{array}{ccc}
D_c(K(\emptyset)\backslash Gr) & \overset{\Psi}{\longrightarrow} & D_c(G(\emptyset)(\emptyset_R)\backslash Gr_0(\emptyset_R)) \\
\downarrow & & \downarrow \Upsilon_R \\
D_c(LG_\emptyset(\emptyset)\backslash Gr) & \overset{\Psi_R}{\longrightarrow} & D_c(G(\emptyset)(\emptyset_R[t^{-1}])\backslash Gr_0(\emptyset_R)).
\end{array}
\]

of Theorem 8.2 are naturally of \( D_c(G(\emptyset)\backslash Gr) \)-modules.

**Proof.** We will focus on the compatibility for the top row and indicate the moduli spaces needed. We leave it to the reader to pass to sheaves and apply standard identities. The compatibility for the bottom row and entire square can be argued similarly.

Let \( QM_2^2(\mathbb{P}^1, G, K)\times Gr_2 \) be the moduli of \( x_1, x_2 \in \mathbb{P}^1, \mathcal{E}_1, \mathcal{E}_2 \) \( G \)-torsors on \( \mathbb{P}^1, \sigma \) a section of \( \mathcal{E}_1 \times^G G/K \) over \( \mathbb{P}^1 \setminus \{x_1, x_2\} \), and \( \alpha \) an isomorphism from \( \mathcal{E}_1 \) to \( \mathcal{E}_2 \) over \( \mathbb{P}^1 \setminus \{x_1, x_2\} \). Let \( QM_2^2(\mathbb{P}^1, G, K)\times Gr_2 \) be the real form of \( QM_2^2(\mathbb{P}^1, G, K)\times Gr_2 \) with respect to the twisted conjugation that exchanges \( x_1 \) and \( x_2 \).

Then there is a canonical diagram of \( K_c \)-equivariant maps

\[
(9.3)
\]

\[
LK_c \backslash G(\emptyset) \times G(\emptyset) Gr \overset{\pi}{\longrightarrow} LK_c \backslash G(\emptyset) \times G(\emptyset) Gr \times i_{\mathbb{R}>0} \simeq QM_2^2(\mathbb{P}^1, G, K)\times Gr_2 |_{i_{\mathbb{R}>0}} \overset{j}{\longrightarrow} QM_2^2(\mathbb{P}^1, G, K)\times Gr_2 |_{i_{\mathbb{R}>0}} \simeq K_c \backslash G(\emptyset) Gr_0(\emptyset_R) Gr_\mathbb{R}
\]

Note we could equivalently obtain diagram (9.3) by taking diagram (9.2) and quotienting by the left action of the group-scheme \( LK_\mathbb{R}^2 \).

As with the convolution maps in diagram (9.2), the actions maps on the end terms of diagram (9.3) naturally extend to the entire diagram. By standard identities, we arrive at a canonical isomorphism \( \Psi(M \star \mathcal{F}) \simeq \Psi(M) \star m(\mathcal{F}) \). By using iterated versions of the above moduli spaces, we may likewise equip \( \Psi \) with the associativity constraints of a module map. \( \square \)
Appendix A. Real analytic stacks

A.1. Basic definitions. Let $\text{RSp}$ be the site of real analytic spaces where the coverings are étale (=locally biholomorphic) maps $\{S_i \to S\}_{i \in I}$ such that the map $\bigsqcup S_i \to S$ is surjective. A real analytic pre-stack is a functor $\mathcal{X} : \text{RSp} \to \text{Grpd}$ from $\text{RSp}$ to the category of groupoids $\text{Grpd}$ and a real analytic stack is a pre-stack which is a sheaf. Let $\Gamma \rightrightarrows X$ be a groupoid in real analytic spaces. We define $\Gamma \\backslash X$ be the stack associated to the pre-stack $S \to \{\Gamma(S) \rightrightarrows X(S)\}$. A morphism $\mathcal{X} \to \mathcal{Y}$ between real analytic stacks is called representable if for any morphism from a real analytic space $Y \to \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} Y$ is representable by a real analytic space. We say that a representable morphism $\mathcal{X} \to \mathcal{Y}$ has property P if it has property P after base change along any morphism from a real analytic space.

A.2. From real algebraic stacks to real analytic stacks. For any $\text{R}$-scheme $X$ locally of finite type, its $\text{R}$-points $X(\mathbb{R})$ is naturally a real analytic space, denoted by $X_\mathbb{R}$, and the assignment $X \to X_\mathbb{R}$ defines a functor from the category of $\text{R}$-scheme to the category of real analytic spaces. We are going to extend the above construction to real algebraic stacks. Let $X$ be a real algebraic stack. A presentation $f : X \to X$ of $X$ is called a $\text{R}$-surjective presentation if it induces a surjective map $X(\mathbb{R}) \to |X(\mathbb{R})|$ on the set of isomorphism classes of objects.

Lemma A.1. Let $f_1 : X_1 \to \mathcal{X}$ and $f_2 : X_2 \to \mathcal{X}$ be two $\text{R}$-surjective presentations of $\mathcal{X}$. Let $\Gamma_i = X_i \times_{\mathcal{X}} X_i \rightrightarrows X_i$ be the corresponding groupoid. Then there is a canonical isomorphism of real analytic stacks

$$\Gamma_1 \mathbb{R} \\backslash X_1 \mathbb{R} \cong \Gamma_2 \mathbb{R} \\backslash X_2 \mathbb{R}.$$

Proof. Let $Y = X_1 \times_{\mathcal{X}} X_2$ and $\Gamma = Y \times_{\mathcal{X}} Y$ be the corresponding groupoid. As $X_i$ is a presentation of $\mathcal{X}$ the natural map $Y \to X_i$ is smooth and one can check that the natural map $\Gamma \mathbb{R} \\backslash Y \mathbb{R} \to \Gamma_i \mathbb{R} \\backslash X_i \mathbb{R}$ is an isomorphism. The lemma follows.

Definition A.2. Given a real algebraic stack $\mathcal{X}$ which admits a $\text{R}$-surjective presentation, we define the associated real analytic stack to be

$$\mathcal{X}_\mathbb{R} := \Gamma \mathbb{R} \\backslash X \mathbb{R}$$

where $X \to \mathcal{X}$ is a $\text{R}$-presentation of $\mathcal{X}$.

By the lemma above $\mathcal{X}_\mathbb{R}$ is well-defined and the assignment $\mathcal{X} \to \mathcal{X}_\mathbb{R}$ defines a functor from the 2-category of real algebraic stacks which admit $\text{R}$-presentations to the 2-category of real analytic stacks.

Example A.3. Let $X$ be a $\text{R}$-scheme and $G$ be an algebraic group over $\text{R}$ acting on $X$. Consider the algebraic stack $\mathcal{X} = G \setminus X$. Let $T_1, ..., T_s$ be the isomorphism classes of $G$-torsors. Define $G_i := \text{Aut}_G(T_i)$ and the $\text{R}$-scheme $X_i := \text{Hom}_G(T_i, X)$. Note that $G_i$ acts on $X_i$ and the collection $\{G_1, ..., G_s\}$ gives all the pure-inner forms of $G$. Consider the
real algebraic stack $G_i \backslash X_i$. We have $G_i \backslash X_i \simeq \mathcal{X}$ and the map $\bigsqcup_{i=1}^s X_i \to \mathcal{X}$ is a $\mathbb{R}$-presentation. In addition, the $\mathbb{R}$-presentation above induces an isomorphism of real analytic stacks $\bigsqcup_{i=1}^s G_{i,\mathbb{R}} \backslash X_{i,\mathbb{R}} \simeq \mathcal{X}_{\mathbb{R}}$.

**Definition A.4.** Let $\mathcal{X}$ be a real analytic stack (resp. a real algebraic stack). The stack $\mathcal{X}$ is called of Bernstein-Lunts type (BL-type) if it is an union of open substacks $\mathcal{X} = \bigcup_i \mathcal{X}_i$, each $\mathcal{X}_i$ being a quotient $G \backslash X$ where $X$ is a real analytic space (resp. $\mathbb{R}$-scheme) and $G$ is a real analytic group (resp. affine algebraic group over $\mathbb{R}$) acting on $X$.

Note that, by the example above, each real algebraic stack $\mathcal{X}$ of BL-type admits a $\mathbb{R}$-surjective presentation and the corresponding real analytic stack $\mathcal{X}_{\mathbb{R}}$ is also of BL-type.

The discussion above can be generalized to real ind-schemes and real ind-stacks. Let $X_0 \to X_1 \to \cdots X_k \to \cdots$ be a diagram of closed embedding of $\mathbb{R}$-schemes. Let $X = \varinjlim X_i$ be the corresponding ind-scheme over $\mathbb{R}$. We define $X_{\mathbb{R}} = \varinjlim X_{i,\mathbb{R}}$ to be the corresponding real analytic space associated to the diagram $X_0 \to X_1 \to \cdots X_k \to \cdots$. Similarly, let $\mathcal{X} = \varinjlim \mathcal{X}_i$ be a real ind-stack associated to a diagram $\mathcal{X}_0 \to \mathcal{X}_1 \to \cdots \mathcal{X}_k \to \cdots$ of real algebraic stacks which admit $\mathbb{R}$-presentations. We define $\mathcal{X}_{\mathbb{R}} = \varinjlim \mathcal{X}_{i,\mathbb{R}}$. An ind-algebraic stack $\mathcal{X} = \varinjlim \mathcal{X}_i$ (resp. an real analytic ind-stack) is called of BL-type if each $\mathcal{X}_i$ is of BL-type.

Let $\mathcal{X} = \varinjlim \mathcal{X}_i$ be a in real algebraic ind-stack (resp. an real analytic ind-stack). By definition, a morphism $f : \mathcal{X} \to \mathcal{Y}$ from $\mathcal{X}$ to a real algebraic stack $\mathcal{Y}$ (resp. a real analytic stack) is the limit of morphism $f_i : \mathcal{X}_i \to \mathcal{Y}$. It is called representable if each $f_i$ is representable.

A.2.1. One can regard real algebraic stacks as complex algebraic stacks with real structures and the discussion above has an obvious generalization to this setting. Let $\mathcal{X}$ be a complex algebraic stack and let $\sigma$ be a real structure on $\mathcal{X}$, that is, a complex conjugation (or a semi-linear involution) $\sigma : \mathcal{X} \to \mathcal{X}$. Then a presentation $f : X \to \mathcal{X}$ of $\mathcal{X}$ is called a $\mathbb{R}$-surjective presentation if it satisfies the following properties. (1) There is a real structure $\sigma$ on $X$ such that $f$ is compatible with the real structures on $X$ and $\mathcal{X}$. (2) The map $f$ induces a surjective map $X(\mathbb{C})^\sigma \to |\mathcal{X}(\mathbb{C})^\sigma|$. One can check that Lemma A.1 still holds in this setting, thus for a pair $(\mathcal{X}^\sigma)$ as above which admits a $\mathbb{R}$-surjective presentation, there is a well-defined real analytic stack $\mathcal{X}_{\mathbb{R}}$ given by $\mathcal{X}_{\mathbb{R}} := \Gamma(\mathbb{C})^\sigma \backslash X(\mathbb{C})^\sigma$, where $X \to \mathcal{X}$ is a $\mathbb{R}$-surjective presentation, $\Gamma = X \times_\mathcal{X} X$ is the corresponding groupoid (Note that $\Gamma$ has a canonical real structure $\sigma$ coming from $X$ and $\mathcal{X}$). Finally, the previous discussions about stacks of BL-type and ind-stacks can be easily generalized to this new setting. The details are left to the reader.

A.3. **Sheaves on real analytic stacks.** Let $X$ be a real analytic space. We will denote by $D_c(X)$ the corresponding bounded derived category of $\mathbb{C}$-constructible sheaves. Let $\mathcal{X}$ be a real analytic stack of BL-type. We define $D_c(\mathcal{X}) = \varinjlim D_c(\mathcal{X}_i)$, where each $D_c(\mathcal{X}_i)$ is the bounded equivariant derived category in the sense of Bernstein-Lunts [BL]. Let $\mathcal{X} = \varinjlim \mathcal{X}_i$ be a real analytic ind-stack of BL-type. We define $D_c(\mathcal{X}) = \varinjlim D_c(\mathcal{X}_i)$. 

46
Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a representable morphism of real analytic stacks of BL-type. We will denote by $f_*, f^*, f_!, f!$ the corresponding functors between $D(\mathcal{X})$ and $D(\mathcal{Y})$ always understood in the derived sense. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a representable morphism from a real analytic ind-stack to a real analytic stack. Assume both $\mathcal{X}$ and $\mathcal{Y}$ are of BL-type, then the functors $f_*, f!$ are well-defined.

References

[AB] M.F. Atiyah, R.Bott. The Yang-Mills equations over Riemann surfaces. Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences Vol. 308, No. 1505, 1983, 523-615
[BB] A. Beilinson, J. Bernstein. Localisations de g-modules, C. R. Acad. Sci. Paris 292 (1982), 15-18.
[BL] J. Bernstein, V. Lunts. Equivariant derived categories, Lecture Notes in Mathematics 1578, Springer, 1994.
[CN1] T.H. Chen, D. Nadler. Affine Matsuki correspondence for sheaves II, in preparation.
[CN2] T.H. Chen, D. Nadler. Kostant-Sekiguchi homeomorphisms, preprint.
[GN1] D. Gaitsgory, D. Nadler. Spherical varieties and Langlands duality, Moscow Math. J. 10 (2010), no. 1, (Special Issue: In honor of Pierre Deligne), 65-137.
[GN2] D. Gaitsgory, D. Nadler. Hecke operators on quasimaps into horospherical varieties, Documenta Math. 14 (2009) 19-46.
[KS] M. Kashiwara and W. Schmid, Quasi-equivariant D-modules, equivariant derived category, and representations of reductive Lie groups, Lie theory and geometry, 457-488, Progress in Mathematics, Birkhäuser, 1994.
[L] G. Lusztig. Green polynomials and singularities of unipotent classes, Adv in Math. 42 (1981), 169-178.
[M] S. A. Mitchell. Quillen’s theorem on buildings and the loops on a symmetric space, Enseign. Math. 34 (1988), 123-166.
[MS] V.B. Mehta, S Subramanian. Principal bundles on the affine line, Proceedings of the Indian Academy of Sciences - Mathematical Sciences 1984, 137-145.
[MUV] I. Mirkovic, T. Uzawa, K. Vilonen. Matsuki correspondence for sheaves, Inventiones Math. 109 (1992), 231-245.
[N1] D. Nadler. Matsuki correspondence for the affine Grassmannian, Duke Math. 124 (2004), 421-457.
[N2] D. Nadler. Perverse sheaves on real loop Grassmannians, Inventiones Math. 159 (2005), 1-73.
[P] A. Pressley, The energy flow on the loop space of a compact Lie group, Journal of the London Mathematical Society, 2 (1982), 557-566.
[PS] A. Pressley, G. Segal. Loop groups. Oxford Univ. Press, New York, 1986.
[S] J. Sekiguchi. Remarks on real nilpotent orbits of a symmetric pair, J. Math. Soc. Japan 39 (1) (1987), 127-138.
[SV] W. Schmid, K. Vilonen, On the geometry of nilpotent orbits, Asian J. Math. 3 (1999), 233-274.
[V] M. Vergne, Instantons et correspondance de Kostant-Sekiguchi, C. R. Acad. Sci. Paris 320 (1995), 901-906.

Department of Mathematics, University of Chicago, Chicago, 60637, USA
Email address: chenth@math.uchicago.edu

Department of Mathematics, UC Berkeley, Evans Hall, Berkeley, CA 94720
Email address: nadler@math.berkeley.edu

47