1. Introduction and statement of results

Let $f(z) = \sum_{n=1}^{\infty} a_f(n) n^{-\frac{k}{2} + 2 \pi i n z} \in S^\text{new}_k(\Gamma_0(q))$ be a normalized cusp form (so that $a_f(1) = 1$) with trivial nebentypus that is an eigenform of all the Hecke operators and all of the Atkin–Lehner involutions $|_k W(q)$ and $|_k W(Q_p)$ for each prime $p | q$. We call such a cusp form a newform (see [22] Section 2.5)). Throughout, we assume that $f$ is non-CM, so there is no imaginary quadratic field $K$ such that for $p \nmid q$, $p$ is inert in $K$ if and only if $a_f(p) = 0$. Deligne’s proof of the Weil conjectures implies that for each prime $p$ there exists an angle $\theta_p \in [0, \pi]$ such that $a_f(p) = 2 \cos \theta_p$. Serre’s extension of the Sato–Tate conjecture [38], originally proposed for $f$ attached to a non-CM elliptic curve by modularity, asserts that if $f$ in non-CM, then the sequence $\{\theta_p\}$ is equidistributed in the interval $[0, \pi]$ with respect to the measure $d\mu_{\text{ST}} := (2/\pi) \sin^2 \theta d\theta$. Equivalently, one has

\begin{equation}
\pi_{f,I}(x) := \# \{p \leq x: \theta_p \in I, \ p \nmid q \} \sim \mu_{\text{ST}}(I) \pi(x) \text{ as } x \to \infty,
\end{equation}

where $\pi(x) = \# \{p \leq x \}$ and $I = [\alpha, \beta] \subseteq [0, \pi]$. Barnet-Lamb, Geraghty, Harris, and Taylor [41] proved Serre’s extension of the Sato–Tate conjecture. See [25] for an excellent overview.

In [40], the author bounded the error term in (1.1) assuming that the $m$-th symmetric power lift $\text{Sym}^m f$ corresponds with a cuspidal automorphic representation of $\text{GL}_{m+1}(\mathbb{A})$ for each $m \geq 1$, where $\mathbb{A}$ denotes the ring of adeles over $\mathbb{Q}$. This implies (among other things) that the symmetric power $L$-functions $L(s, \text{Sym}^m f)$ have an analytic continuation and functional equation of the expected type for all $m \geq 1$. With this hypothesis, the author proved for fixed $f$ and $I$ that for all $\varepsilon > 0$, there exist constants $c_{\varepsilon,f}, C_{\varepsilon,f} > 0$ such that

\begin{equation}
|\pi_{f,I}(x) - \mu_{\text{ST}}(I) \pi(x)| \leq c_{\varepsilon,f} \pi(x)(\log x)^{-\frac{1}{8} + \varepsilon}, \quad x > C_{\varepsilon,f}.
\end{equation}

Until recently, it was known that $\text{Sym}^m f$ corresponds with a cuspidal automorphic representation of $\text{GL}_{m+1}(\mathbb{A})$ only for $m \leq 8$ [8] [15] [19] [20]. Recently, Newton and Thorne [31] proved that if $q$ is squarefree, then $\text{Sym}^m f$ corresponds with a cuspidal automorphic representation of $\text{GL}_{m+1}(\mathbb{A})$ for all $m \geq 1$. This inspired the author to improve the quality of (1.2) in the $x$-aspect and specify the uniformity with respect to $f$ and $I$. In what follows, $c_1, c_2, c_3, \ldots$ denotes a sequence of positive, absolute, and effectively computable constants.

**Theorem 1.1.** Let $q$ be squarefree, let $f \in S^\text{new}_k(\Gamma_0(q))$ be a non-CM newform as above, and let $I = [\alpha, \beta] \subseteq [0, \pi]$. There exists a constant $c_1$ such that

\begin{equation}
|\pi_{f,I}(x) - \mu_{\text{ST}}(I) \pi(x)| \leq c_1 \pi(x) \frac{\log(kq \log x)}{\sqrt{\log x}}, \quad x \geq 3.
\end{equation}
Even though the error term in Theorem 1.1 saves less than a full power of \( \log x \) over \( \pi(x) \), some arithmetically significant consequences still follow from Theorem 1.1. For example, Luca, Radziwiłł, and Shparlinski [24] proved that the inequality \( |a_f(n)| \leq (\log n)^{-\frac{1}{2}+o(1)} \) (where \( o(1) \) denotes a quantity, possibly depending on \( f \), which tends to zero as \( n \to \infty \)) holds for a density one subset of integers \( n \). This improves on a standard argument which achieves the same bound with \( \log 2 \) replacing \( \frac{1}{2} \). One might ask whether the exponent \( -\frac{1}{2} \) might be lowered any further for a density one subset of \( n \). In the same paper, Luca, Radziwiłł, and Shparlinski [24] proved that Theorem 1.1 (and even [12]) suffices to show that if \( v \in \mathbb{R} \) is fixed, then

\[
\lim_{x \to \infty} \frac{\# \{ n \leq x: a_f(n) \neq 0, \frac{\log |a_f(n)| + \frac{1}{2} \log \log n}{\sqrt{(\frac{1}{2} + \frac{\pi^2}{12}) \log \log n}} \geq v \}}{\# \{ n \leq x: a_f(n) \neq 0 \}} = \frac{1}{\sqrt{2\pi}} \int_{v}^{\infty} e^{-u^2/2} du.
\]

This central limit theorem shows that the exponent \(-\frac{1}{2}\) cannot be lowered any further for a density one subset of \( n \).

Theorem 1.1 produces a rather large upper bound on the least prime \( p \nmid q \) such that \( \theta_p \in I \). Lemke Oliver and the author [23] noticeably improved this bound under some more hypotheses on \( I \) which facilitated proofs relying on automorphy results which already existed at the time [8]. The work in [31] enables us to improve [23] Theorem 1.6] for all \( I \).

**Theorem 1.2.** Let \( q \) be squarefree. Let \( f \) and \( I \) be as in Theorem 1.1 and let \( \mu = \mu_{ST}(I) \). There exist a constant \( c_2 \) and a prime \( p \nmid q \) such that \( \theta_p \in I \) and \( p \leq (kq/\mu)c_2 \log(e/\mu)/\mu^2 \).

If one assumes the generalized Riemann hypothesis (GRH) each symmetric power \( L \)-function \( L(s, \text{Sym}^m f) \), then the bounds in Theorems 1.1 and 1.2 improve as follows.

**Theorem 1.3.** Let \( q \) be squarefree. Let \( f \) and \( I \) be as in Theorem 1.1 and let \( \mu = \mu_{ST}(I) \). If \( L(s, \text{Sym}^m f) \) satisfies GRH for all \( m \geq 0 \), then there exists a constant \( c_3 \) such that

\[
|\pi_{f,I}(x) - \mu_{ST}(I) \pi(x)| \leq c_3 x^{3} \log(kqx) \log x, \quad x \geq 3.
\]

Also, there exist a constant \( c_4 \) and a prime \( p \nmid q \) which satisfies \( \theta_p \in I \) and \( p \leq c_4 \frac{1}{\mu^2} (\log \frac{kq}{\mu})^2 \).

**Proof.** Rouse and the author [36] proved the first result. Chen, Park, and Swaminathan [7] proved the second result when \( f \) corresponds with an elliptic curve by modularity, in which case \( k = 2 \). Their proof extends to other \( f \) with little additional effort. The work in [7, 36] assumed the automorphy of the symmetric powers of \( f \), but that is now known unconditionally [31].

**Remark.** The authors in [7, 36] assumed a squarefree level in order to produce strong explicit bounds on \( c_3 \) and \( c_4 \). The orders of magnitude do not change when \( q \) is not squarefree, but the task of obtaining strong bounds on the implied constants seems difficult.

**Remark.** An error term of size \( c_{f,\varepsilon} x^{\frac{1}{2}+\varepsilon} \) is expected for all fixed \( \varepsilon > 0 \). See [3] Theorem 1.4] for some compelling on-average results in this direction.

For \( i = 1, 2 \), let \( f_i \in S_{n_i}^{\text{new}}(q_i) \) be a non-CM newform, let \( \{ \theta_p^{(i)} \} \) be the sequence of angles in the Sato–Tate conjecture for \( f_i \), and let \( \mathbf{1}_{I_i} \) be the indicator function of the interval \( I_i = [\alpha_i, \beta_i] \subseteq [0, \pi] \). Suppose that the only primitive character \( \chi \) satisfying the property
that \( f_1 = f_2 \otimes \chi \) is the trivial character, in which case \( f_1 \) and \( f_2 \) are not twist-equivalent. We denote this by \( f_1 \not\sim f_2 \). If \( f_1 \not\sim f_2 \), then it is natural to ask whether the Sato–Tate distributions for \( f_1 \) and \( f_2 \) are independent, as quantified by the existence of the asymptotic estimate

\[
\pi_{f_1, f_2, t_1, t_2}(x) := \sum_{\substack{p \leq x \\ p \not\mid q_1 q_2}} 1_{I_1}(\theta_p^{(1)}) 1_{I_2}(\theta_p^{(2)}) \sim \mu_{\text{ST}}(I_1) \mu_{\text{ST}}(I_2) \pi(x) \quad \text{as } x \to \infty,
\]

where \( I_1, I_2 \subseteq [0, \pi] \) are two intervals. This question was posed independently by Katz and Mazur for \( f_i \) corresponding to elliptic curves by modularity. In this setting, it was answered affirmatively by Harris \cite{Harris}. See Wong \cite{Wong} for a generalization to the setting considered here. The work of Newton and Thorne \cite{NewtonThorne} and the ideas leading to Theorem 1.1 permit us to effectively quantify this independence without recourse to unproven hypotheses.

**Theorem 1.4.** For \( i = 1, 2 \), let \( q_i \) be squarefree, let \( f_i \) be a newform as in Theorem 1.1, and let \( I_i = [\alpha_i, \beta_i] \subseteq [0, \pi] \). If \( f_1 \not\sim f_2 \), then there exists a constant \( c_5 \) such that

\[
|\pi_{f_1, f_2, t_1, t_2}(x) - \mu_{\text{ST}}(I_1) \mu_{\text{ST}}(I_2) \pi(x)| \leq c_5 \pi(x) \frac{\log(k_1 q_1 k_2 q_2 \log \log x)}{\sqrt{\log \log x}}, \quad x \geq 16.
\]

**Remark.** It is likely that one could establish a much stronger ineffective error term, where the ineffectivity arises from the possible existence of a Landau–Siegel zero (see Sections \ref{sec:LandauSiegel} and \ref{sec:effectiveness}). See Molteni \cite{Molteni} (and also his PhD thesis) for a discussion on how to bound such zeros in our setting. As one sees in Section \ref{sec:ineffectiveness}, Landau–Siegel zeros do not plague the error term in Theorem 1.4.

If one assumes GRH for the Rankin–Selberg \( L \)-functions associated to the tensor products of Sym\(^m_1 f_1 \) and Sym\(^m_2 f_2 \) for all \( m_1, m_2 \geq 0 \), then Theorem 1.4 improves as follows.

**Theorem 1.5.** For \( i = 1, 2 \), let \( q_i \) be squarefree, and let \( f_i \in S_{k_i}^{\text{new}}(\Gamma_0(q_i)) \) and \( I_i \) be as in Theorem 1.4. If \( f_1 \not\sim f_2 \) and the Rankin–Selberg \( L \)-functions \( L(s, \text{Sym}^m f_1 \times \text{Sym}^m f_2) \) satisfy GRH for all integers \( m_1, m_2 \geq 0 \), then there exists a constant \( c_6 \) such that

\[
|\pi_{f_1, f_2, t_1, t_2}(x) - \mu_{\text{ST}}(I_1) \mu_{\text{ST}}(I_2) \pi(x)| \leq c_6 x^{\frac{2}{3}} \log(k_1 k_2 q_1 q_2)^{\frac{1}{3}}, \quad x \geq 3.
\]

**Proof.** Bucur and Kedlaya \cite{BucurKedlaya} proved this when \( f_1 \) and \( f_2 \) correspond with elliptic curves by modularity. One can extend their proof to other \( f_1 \) and \( f_2 \) with little additional effort. \( \square \)

We now describe an application of Theorem 1.4. An open conjecture of Lang and Trotter \cite{LangTrotter} asserts that if \( f \) is the newform associated to a non-CM elliptic curve over \( \mathbb{Q} \) and \( t \in \mathbb{Z} \) is fixed, then there exists a constant \( c_{f, t} \geq 0 \) such that

\[
\# \{ p \leq x : a_f(p) \sqrt{p} = t \} \sim c_{f, t} \sqrt{x} / \log x \quad \text{as } x \to \infty.
\]

We interpret \( c_{f, t} = 0 \) to mean that finitely many primes \( p \) satisfy \( a_f(p) \sqrt{p} = t \). There is a long history of bounding (1.4) under GRH \cite{SatoTate2, IwaniecKowalski, XuZhang, Xiao} and unconditionally \cite{Buchholz2, Xu, Xiao}, \cite{SteinWong, LiWong}. Given integers \( t_1 \) and \( t_2 \), one might ask a similar question about the distribution of \( p \) such that \( a_{f_1}(p) \sqrt{p} = t_1 \) and \( a_{f_2}(p) \sqrt{p} = t_2 \), where \( f_1 \) and \( f_2 \) correspond with non-isogenous non-CM elliptic curves. For such \( f_1 \) and \( f_2 \), it is conjectured that

\[
\# \{ p \leq x : a_{f_1}(p) \sqrt{p} = t_1, \ a_{f_2}(p) \sqrt{p} = t_2 \} \sim c_{f_1, f_2, t_1, t_2} \log \log x \quad \text{as } x \to \infty.
\]

This joint distribution is known on average; see \cite{Xing, ShinYang}. Using Theorem 1.4, we give the first unconditional pointwise bounds in the direction of (1.5).
Corollary 1.6. For $i = 1,2$, let $q_i$ be squarefree, let $t_i \in \mathbb{R}$, and let $f_i \in S^{\text{new}}_{k_i}(\Gamma_0(q_i))$ be as in Theorem 1.4. If $f_1 \not\sim f_2$, then there exists a constant $c_7$ such that
\[
\#\{p \leq x : a_{f_1}(p)p^{k_1-1} = t_1, a_{f_2}(p)p^{k_2-1} = t_2\} \leq c_8(x) \frac{\log(k_1q_1k_2q_2 \log \log x)}{\sqrt{\log \log x}}, \quad x \geq 16.
\]

We have stated Theorems 1.1–1.4 for squarefree levels since this is setting where Newton and Thorne prove automorphy for all symmetric powers of $f$. Apart from the remark which follows the proof of Theorem 6.1 below, our proofs will hold for all levels once the automorphy of the pertinent symmetric powers is established. With a few additional alterations, one could replace the newforms in our theorems with Hilbert modular forms over totally real fields once a suitable analogue of the work of Newton and Thorne is proved.

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2. Proof of Theorems 1.1, 1.2, and 1.4 and Corollary 1.6

In this section, we reduce the proofs of Theorems 1.1, 1.2, and 1.4 to the task of proving a uniform version of the prime number theorem for certain classes of $L$-functions. We use the Vinogradov notation $F \ll G$ to denote the existence of an absolute and effectively computable constant $c > 0$ (not necessarily the same in each occurrence) such that $|F| \leq c|G|$ in the range indicated. We write $F = G + O(H)$ to denote that $|F - G| \ll H$.

2.1. Proof of Theorems 1.1 and 1.2. Let $I = [\alpha, \beta] \subseteq [0, \pi]$ be an interval. Let $1_I$ be the indicator function of the interval $I$, and define $\pi_{f,I}(x)$ as in (1.1). Let
\[
U_m(\cos \theta_p) = \frac{\sin((m+1)\theta_p)}{\sin \theta_p} = \sum_{j=0}^{m} e^{(2j-m)\theta_p}
\]
be the $m$-th Chebyshev polynomial of the second type. These polynomials form an orthonormal basis for $L^2([0, \pi], \mu_{\text{ST}})$ with respect to the usual inner product $\langle f, g \rangle = \int_0^\pi f(\theta)g(\theta)\,d\mu_{\text{ST}}$.

Montgomery [27, Chapter 1] used the work of Beurling and Selberg to efficiently majorize and minorize the indicator function of a subinterval of $[0,1]$ using carefully constructed trigonometric polynomials. One performs a suitable change of variables to handle subintervals of $[0, \pi]$ and a change of basis to express the trigonometric polynomials in terms of the Chebyshev polynomials $U_m(\cos \theta)$ (see [36, Section 3]). For any integer $M \geq 3$, we find that
\[
(2.1) \quad |\pi_{f,I}(x) - \mu_{\text{ST}}(I)\pi(x)| \ll \frac{\pi(x)}{M} + \sum_{m=1}^{M} \frac{1}{m} \left| \sum_{p \leq x \atop p \not| q} U_m(\cos \theta_p) \right|, \quad x \geq 3.
\]
Note that $\pi(x) \sim x/\log x$ by the prime number theorem.

By partial summation, we have
\[
(2.2) \quad \sum_{p \leq x \atop p \not| q} U_m(\cos \theta_p) = \frac{\theta_{f,m}(x)}{\log x} - \int_2^x \frac{\theta_{f,m}(t)}{t(\log t)^2} \,dt, \quad \theta_{f,m}(x) = \sum_{p \leq x \atop p \not| q} U_m(\cos \theta_p) \log p.
\]
We will estimate $\theta_{f,m}(x)$ as follows.
Proposition 2.1. Let $f$ be as in Theorem 1.4. There exist constants $c_9$ (suitably large) and $c_{10}$ and $c_{11}$ (suitably small) such that if $1 \leq m \leq c_{11} \sqrt{\log x / \log(kq \log x)}$, then

$$|\theta_{f,m}(x)| \ll m^2 x^{\frac{1}{c_9m}} + m^2 x \left( \exp \left[-c_{10} \frac{\log x}{m^2 \log(kqm)} \right] + \exp \left[-c_{10} \frac{\sqrt{\log x}}{\sqrt{m}} \right] \right).$$

By (2.1), (2.2), and Proposition 2.1, if $3 \leq M \leq c_{11} \sqrt{\log x / \log(kq \log x)}$, then

$$|\pi_{f,I}(x) - \mu_{ST}(I)\pi(x)| \ll \pi(x) \left( \frac{1}{M} + \sum_{m=1}^{M} m \left( x^{-\frac{1}{c_9m}} + \exp \left[-c_{10} \frac{\log x}{m^2 \log(kqm)} \right] + \exp \left[-\frac{\sqrt{\log x}}{\sqrt{m}} \right] \right) \right).$$

Proof of Theorem 1.4 and Corollary 1.6. This follows from (2.3) by choosing $c_{13} > 0$ to be a sufficiently small constant and $M = [c_{13} \sqrt{\log x / \log(kq \log x)}] \geq 3$. For all remaining values of $x$, the theorem is trivial.

2.2. Proof of Theorem 1.4 and Corollary 1.6

For $i = 1, 2$, let $\mu_i = \mu_{ST}(I_i)$ and let $f_i$ be as in Theorem 1.4 with Sato–Tate angles $\{\theta_p^{(i)}\}$. Let $\pi_{f_1,f_2,I_1,I_2}(x)$ be as in (1.3). Cochrane [4] carried out a version of Montgomery’s analysis which constructs trigonometric polynomials which efficiently majorize and minorize the indicator function of $R_1 \times R_2$, where $R_i = [a_i, b_i] \subseteq [0, 1]$. Upon performing a change of variables and a change of basis similar to [36] Section 3 to express these polynomials in terms of Chebyshev polynomials, we find for any integer $M \geq 3$ that

$$|\pi_{f_1,f_2,I_1,I_2}(x) - \mu_{ST}(I_1)\mu_{ST}(I_2)\pi(x)| \ll \frac{\pi(x)}{M} + \sum_{0 \leq m_1, m_2 \leq M \atop m_1m_2 \neq 0} \frac{1}{(m_1 + 1)(m_2 + 1)} \sum_{p \leq x \atop p \nmid q_1 q_2} U_{m_1}(\cos \theta_p^{(1)})U_{m_2}(\cos \theta_p^{(2)}), \quad x \geq 3.$$

By partial summation, we have

$$\sum_{p \leq x \atop p \nmid q_1 q_2} U_{m_1}(\cos \theta_p^{(1)})U_{m_2}(\cos \theta_p^{(2)}) = \frac{\theta_{f_1,f_2,m_1,m_2}(x)}{\log x} - \int_2^x \frac{\theta_{f_1,f_2,m_1,m_2}(t)}{t (\log t)^2} \, dt,$$

where

$$\theta_{f_1,f_2,m_1,m_2}(x) = \sum_{p \leq x \atop p \nmid q_1 q_2} U_{m_1}(\cos \theta_p^{(1)})U_{m_2}(\cos \theta_p^{(2)}) \log p.$$

We will estimate $\theta_{f_1,f_2,m_1,m_2}(x)$ as follows.

Proposition 2.2. For $i = 1, 2$, let $f_i$ be as in Theorem 1.4. There exist constants $c_{15}$ (suitably large) and $c_{16}$, $c_{17}$, and $c_{18}$ (suitably small) such that if

$$1 \leq m_1, m_2 \leq M \leq c_{17} \sqrt{\log \log x / \log(k_1 q_1 k_2 q_2 \log \log x)},$$
then
\[ |\theta_{f_1,f_2,m_1,m_2}(x)| \ll x^{1-\frac{c_{18}}{(k_1 q_1 k_2 q_2 M)^4 M^2}} + (m_1 m_2)^2 x^{1-\frac{1}{c_{15} M^2}} + (m_1 m_2)^2 x \left( \exp \left[ -c_{16} \frac{\log x}{M^2 \log(k_1 q_1 k_2 q_2 M)} \right] + \exp \left[ -c_{16} \sqrt{\frac{\log x}{M}} \right] \right). \]

Note that \( \theta_{f_1,f_2,0,m_3}(x) = \theta_{f_2,m_2}(x) \) and \( \theta_{f_1,f_2,m_1,0}(x) = \theta_{f_1,m_1}(x) \). Therefore, by Proposition 2.1, (2.4), (2.5), (2.6), and Proposition 2.2, we find that if (2.7) holds, then
\[
\begin{align*}
|\pi_{f_1,f_2,t_1,t_2}(x) - \mu_{ST}(I_1)\mu_{ST}(I_2)\pi(x)| \\
\ll \pi(x) \left\{ \frac{1}{M} + M^2 x^{-\frac{c_{18}}{(k_1 q_1 k_2 q_2 M)^4 M^2}} + M^4 x^{-\frac{1}{c_{15} M^2}} \right. \\
+ M^4 \left( \exp \left[ -c_{16} \frac{\log x}{M^2 \log(k_1 q_1 k_2 q_2 M)} \right] + \exp \left[ -c_{16} \sqrt{\frac{\log x}{M}} \right] \right) \right\}.
\end{align*}
\]

Proof of Theorem 1.4. Write \( \mu = \mu_{ST}(I_1)\mu_{ST}(I_2) \). There exists a suitably large constant \( c_{19} \) such that if \( x \geq \exp(\exp(c_{19} \frac{1}{\mu} \log \frac{k_1 q_1 k_2 q_2 M}{\mu})) \), then the theorem follows from (2.3) by choosing \( c_{20} > 0 \) to be a sufficiently small constant and \( M = \left\lceil \frac{\sqrt{\log \log x}}{c_{20} \log(k_1 q_1 k_2 q_2 \log \log x)} \right\rceil \geq 3 \). For all remaining values of \( x \), the theorem is trivial. \( \square \)

Proof of Corollary 1.6. The result is trivial for small \( x \), so it suffices to let \( x \) be large with respect to \( k_i, q_i \), and \( t_i \). We apply Theorem 1.4 with \( I_1 = I_2 = \left\lceil \frac{x}{2} - \frac{\pi}{4 R}, \frac{x}{2} + \frac{\pi}{4 R} \right\rceil \), where
\[ R = \sqrt{\log(k_1 q_1 k_2 q_2 \log \log x)/(\log \log x)^{\frac{1}{4}}} \]
so that \( \mu_{ST}(I_1)\mu_{ST}(I_2) \leq R^{-2} \). If \( p \) is sufficiently large with respect to \( t_i \) and \( k_i \), then it is straightforward to check that if \( a_{f_i}(p)p^{\frac{k_i-1}{x}} = t_i \), then \( \cos \theta_p^{(i)} \in I_i \). Therefore, we have
\[ \# \{ p \leq x : a_{f_1}(p)\sqrt{p} = t_1, \ a_{f_2}(p)\sqrt{p} = t_2 \} \ll \pi_{f_1,f_2,t_1,t_2}(x), \]
and the result follows immediately from Theorem 1.4. \( \square \)

2.3. Outline for proofs of Propositions 2.1 and 2.2. Propositions 2.1 and 2.2 will follow from a sufficiently uniform unconditional prime number theorem for the \( L \)-functions associated to symmetric powers of the cuspidal automorphic representations associated to newforms and the Rankin–Selberg convolutions of said symmetric powers. To prove such prime number theorems, we first review well-known properties of \( L \)-functions in Section 3. We prove zero-free regions and log-free zero density estimates for \( L \)-functions which satisfy the generalized Ramanujan conjecture in Section 3. In Section 5 we use the results in Section 4 to prove a highly uniform prime number theorem for \( L \)-functions satisfying the generalized Ramanujan conjecture. Finally, in Section 6 we use the aforementioned work of Newton and Thorne 31 to show how \( L \)-functions of symmetric powers and their convolutions fit into the framework of Section 5. We then prove Propositions 2.1 and 2.2.

Our work requires careful attention to the dependence of various estimates on the degrees of the \( L \)-functions under consideration. Other problems in analytic number theory typically do not require such care, but in our setting, the degree aspect of our estimates is the aspect that matters most since it directly determines the quality of our results. In order to make the degree dependencies in the necessary estimates as strong as we can, we will refine the work in [17, Appendix] and [39] to prove the necessary zero-free regions and log-free zero density estimates.
3. Properties of $L$-functions

3.1. Standard $L$-functions. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$, and $\mathcal{F}_m$ be the set of cuspidal automorphic representations of $GL_m(\mathbb{A})$ with unitary central character, which assume to be normalized so that it is trivial on the diagonally embedded copy of the positive real numbers. Given $\pi \in \mathcal{F}_m$, let $\tilde{\pi}$ be the representation which is contragredient to $\pi$, and let $q_\pi \geq 1$ be the conductor of $\pi$. Write the finite part of $\pi$ as a tensor product $\otimes_p \bar{\pi}_p$ of local representations over primes $p$. For each $p$, there exist Satake parameters $\alpha_{1,\pi}(p), \ldots, \alpha_{m,\pi}(p) \in \mathbb{C}$ such that the local $L$-function $L(s, \pi_p)$ is given by

$$L(s, \pi) = \prod_{j=1}^m \left(1 - \frac{\alpha_{j,\pi}(p)}{p^s}\right)^{-1} = 1 + \sum_{j=1}^{\infty} \frac{a_\pi(p^j)}{p^js}.$$

We have $\alpha_{j,\pi}(p) \neq 0$ for all $j$ when $p \nmid q_\pi$, and some of the $\alpha_{j,\pi}(p)$ might equal zero when $p|q_\pi$. The standard $L$-function $L(s, \pi)$ attached to $\pi$ is

$$L(s, \pi) = \prod_{p} L(s, \pi_p) = \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s},$$

which converges absolutely for $\text{Re}(s) > 1$.

The gamma factor corresponding to the infinite place of $\mathbb{Q}$ is given by

$$L(s, \pi_\infty) = q^{s/2} \prod_{j=1}^{m} \Gamma_\mathbb{R}(s + \mu_\pi(j)), \quad \Gamma_\mathbb{R}(s) = \pi^{-s/2} \Gamma(s/2),$$

where $\mu_\pi(1), \ldots, \mu_\pi(n) \in \mathbb{C}$ are the Langlands parameters. The bounds

$$|\alpha_{j,\pi}(p)| \leq \theta_m, \quad \text{Re}(\mu_\pi(j)) \geq -\theta_m$$

hold for some $0 \leq \theta_m \leq \frac{1}{2} - \frac{1}{m+1}$. The generalized Ramanujan conjecture and generalized Selberg eigenvalue conjectures assert that the above inequalities hold with $\theta_m = 0$.

Let $r_\pi \in \{0, 1\}$ be the order of the pole of $L(s, \pi)$ at $s = 1$, where $r_\pi = 1$ if and only if $\pi$ is the trivial representation $1$ of $GL_1(\mathbb{A})$ whose $L$-function is the Riemann zeta function $\zeta(s)$. The function $\Lambda(s, \pi) = (s(s-1))^{r_\pi} L(s, \pi) L(s, \pi_\infty)$ is entire of order one. There exists a complex number $W(\pi)$ of modulus one such that $\Lambda(s, \pi) = W(\pi) \Lambda(1-s, \bar{\pi})$, where $\bar{\pi} \in \mathcal{F}_m$ is the contragredient representation. We have the equalities of sets

$$\{\alpha_{j,\pi}(p)\} = \overline{\{\alpha_{j,\pi}(p)\}}, \quad \{\mu_\pi(j)\} = \overline{\{\mu_\pi(j)\}},$$

and $q_\pi = q_{\bar{\pi}}$. We define the analytic conductor $C(\pi)$ to be

$$C(\pi, t) = q^n \prod_{j=1}^{n} (1 + |\mu_\pi(j) + it|), \quad C(\pi) = C(\pi, 0).$$

Define the numbers $\Lambda_\pi(n)$ by the Dirichlet series identity

$$\sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)}{n^s} = - \frac{L'}{L}(s, \pi) = \sum_{p} \sum_{\ell=1}^{\infty} \frac{\sum_{j=1}^{m} \alpha_{j,\pi}(p)^\ell \log p}{p^{\ell s}} \left[ \text{Re}(s) > 1 \right].$$

It was proved in the discussion following [39, Lemma 2.3] that for all $\eta > 0$, we have

$$\sum_{n=1}^{\infty} \frac{|\Lambda_\pi(n)|}{n^{1+\eta}} \leq \frac{1}{\eta} + m \log C(\pi) + O(m^2).$$
3.2. Rankin–Selberg $L$-functions. Let $\pi \in \mathfrak{F}_m$ and $\pi' \in \mathfrak{F}_{m'}$. For each prime $p$, we let

$$L(s, \pi_p \times \pi'_p) = \prod_{j=1}^{m} \prod_{j'=1}^{m'} \left(1 - \frac{\alpha_{j,j',\pi \times \pi'}(p)}{p^s}\right)^{^{-1}} = 1 + \sum_{j=1}^{\infty} \frac{a_{\pi \times \pi'}(p^j)}{p^{js}}$$

for suitable complex numbers $\alpha_{j,j',\pi \times \pi'}(p)$. If $p \nmid q_\pi q_{\pi'}$, then we have the equality of sets $\{\alpha_{j,j',\pi \times \pi'}(p)\} = \{\alpha_{j,\pi}(p)\alpha_{j',\pi'}(p)\}$. A complete description of $\alpha_{j,j',\pi \times \pi'}(p)$ is given in \cite{39} Appendix. From these Satake parameters, one defines the Rankin–Selberg $L$-function

$$L(s, \pi \times \pi') = \prod_{p} L(s, \pi_p \times \pi'_p) = \sum_{n=1}^{\infty} \frac{a_{\pi \times \pi'}(n)}{n^s}$$

associated to the tensor product $\pi \otimes \pi'$, which converges absolutely for $\Re(s) > 1$. We write $q_{\pi \times \pi'}$ for the conductor of $\pi \otimes \pi'$. Bushnell and Henniart \cite{6} proved that $q_{\pi \times \pi'} q_{\pi} q_{\pi'}^n$.

The gamma factor corresponding to the infinite place of $\mathbb{Q}$ is given by

$$L(s, \pi_\infty \times \pi'_\infty) = q_{\pi \times \pi'}^{s/2} \prod_{j=1}^{m} \prod_{j'=1}^{m'} \Gamma_{\mathbb{R}}(s + \mu_{\pi \times \pi'}(j, j'))$$

for suitable complex numbers $\mu_{\pi \times \pi'}(j, j')$. If both $\pi$ and $\pi'$ are unramified at the infinite place of $\mathbb{Q}$, then we have the equality of sets $\{\mu_{\pi \times \pi'}(j, j')\} = \{\mu_{\pi}(j) + \mu_{\pi'}(j')\}$. A complete description of the numbers can be found in \cite{39} Proof of Lemma 2.1. From the explicit descriptions of the numbers $\alpha_{j,j',\pi \times \pi'}(p)$ and $\mu_{\pi \times \pi'}(j, j')$, we find that

$$|\alpha_{j,j',\pi \times \pi'}(p)| \leq \theta_{m} + \theta_{m'}, \quad \Re(\mu_{\pi \times \pi'}(j, j')) \geq -\theta_{m} - \theta_{m'}.$$

Let $r_{\pi \times \pi'} \in \{0, 1\}$ be the order of the pole of $L(s, \pi \times \pi')$ at $s = 1$. We have that $r_{\pi \times \pi'} = 1$ if and only if $\pi' = \overline{\pi}$. The function $\Lambda(s, \pi \times \pi') = (s(s - 1))^{r_{\pi \times \pi'}} L(s, \pi \times \pi') L(s, \pi_\infty \times \pi'_\infty)$ is entire of order one. There exists a complex number $W(\pi \times \pi')$ of modulus one such that $\Lambda(s, \pi \times \pi') = W(\pi \times \pi') \Lambda(1 - s, \overline{\pi} \times \overline{\pi'})$. We define the analytic conductor $C(\pi \times \pi')$ to be

$$C(\pi \times \pi', t) = q_{\pi \times \pi'} \prod_{j=1}^{m} \prod_{j'=1}^{m'} (1 + |\mu_{\pi \times \pi'}(j, j') + it|), \quad C(\pi \times \pi') = C(\pi \times \pi', 0).$$

The combined work of Bushnell and Henniart \cite{6} and Brumley \cite{17} Appendix] proves that

$$C(\pi \times \pi', t) \leq C(\pi \times \pi')(1 + |t|)^m, \quad C(\pi \times \pi') \leq e^{O(m'm)} C(\pi)^{m'} C(\pi')^m.$$

Define the numbers $\Lambda_{\pi \times \pi'}(n)$ by the Dirichlet series identity

$$\sum_{n=1}^{\infty} \frac{\Lambda_{\pi \times \pi'}(n)}{n^s} = -\frac{L'}{L}(s, \pi \times \pi') = \sum_{p} \sum_{\ell=1}^{\infty} \sum_{j=1}^{m} \sum_{j'=1}^{m'} \frac{\alpha_{j,j',\pi \times \pi'}(p)\ell \log p}{p^{j+s}}, \quad \Re(s) > 1.$$

It was proved in the discussion following \cite{39} Lemma 2.3] that for all $\eta > 0$, we have

$$\sum_{n=1}^{\infty} \frac{|\Lambda_{\pi \times \pi'}(n)|}{n^{1+\eta}} \leq \frac{1}{\eta} + m'm \log C(\pi \times \pi') + O((m'm)^2).$$
3.3. Isobaric automorphic representations. Let \( d \geq 1 \) be an integer; let \( m_1, \ldots, m_d \) be positive integers; let \( t_1, \ldots, t_d \in \mathbb{R} \); and let \( \pi_i \in \mathfrak{F}_{m_i} \) with \( 1 \leq i \leq d \). Consider the isobaric automorphic representation \( \Pi \) of \( \text{GL}_r(\mathbb{A}) \) given by the isobaric sum

\[
\Pi = \pi_1 \otimes \det |t_1| \cdots \otimes \det |t_d|.
\]

The \( L \)-function \( L(s, \Pi) \) factors as the product \( \prod_{j=1}^d L(s + it_j, \pi_j) \). Its analytic conductor is \( C(\Pi, t) = \prod_{j=1}^d C(\pi_j, t + t_j) \).

Let \( d' \geq 1 \) be an integer; let \( m'_1, \ldots, m'_d \) be positive integers; let \( t'_1, \ldots, t'_d \in \mathbb{R} \); and let \( \pi'_{i'} \in \mathfrak{F}_{m'_{i'}} \) with \( 1 \leq i' \leq d' \). Consider the isobaric automorphic representation \( \Pi' \) of \( \text{GL}_{r'}(\mathbb{A}) \) given by the isobaric sum \( \Pi' = \pi'_1 \otimes |t'_1| \cdots \otimes |t'_{d'}| \). We then define

\[
L(s, \Pi \times \Pi') = \prod_{j=1}^d \prod_{j'=1}^{d'} L(s + it_j + it'_{j'}, \pi_j \times \pi'_{j'}). \]

We define \( C(\Pi \times \Pi', t) = \prod_{j=1}^d \prod_{j'=1}^{d'} C(\pi_j \times \pi'_{j'}, t + t_j + t'_{j'}) \) and \( C(\Pi \times \Pi') = C(\Pi \times \Pi', 0) \).

4. Zeros of \( L \)-functions

We require two results on the distribution of zeros of standard \( L \)-functions and Rankin–Selberg \( L \)-functions. First, we require a standard zero-free region. We present a modification to the work of Brumley [17] Appendix which will improve the degree dependence. Second, we require a log-free zero density estimate. We will use the aforementioned zero-free region to improve the degree dependence in the work of Soundararajan and the author [39].

4.1. Zero-free regions. A proof of the following proposition is sketched in [23, Lemma 3.1]. We give a complete proof here.

**Proposition 4.1.** Let \( \Pi \) be an isobaric automorphic representation of \( \text{GL}_r(\mathbb{A}) \). If \( L(s, \Pi \times \Pi) \) has a pole of order \( r_{\Pi \times \Pi} \geq 1 \) at \( s = 1 \), then \( L(1, \Pi \times \Pi) \neq 0 \), and there exists a constant \( 0 < c_{21} < 1 \) such that \( L(s, \Pi \times \Pi) \) has at most \( r_{\Pi \times \Pi} \) real zeros in the interval

\[
s \geq 1 - \frac{c_{21}}{(r_{\Pi \times \Pi} + 1) \log C(\Pi \times \Pi)}. \tag{4.1}
\]

**Proof.** By proceeding as in the proof of [18, Equations 5.28 and 5.37], we find that

\[
\sum_{1 - c_{21} / \log C(\Pi \times \Pi) < \beta \leq 1} \frac{1}{\sigma - \beta} < \frac{r_{\Pi \times \Pi}}{\sigma - 1} + \text{Re} \left( \frac{L'(s, \Pi \times \Pi)}{L(s, \Pi \times \Pi)} \right) + O(\log C(\Pi \times \Pi)),
\]

where \( s = \sigma \geq 1 \). Define \( \Lambda_{\Pi \times \Pi}(n) \) by the identity \( \sum_{n=1}^{\infty} \frac{\Lambda_{\Pi \times \Pi}(n)}{n^s} = -\frac{L'}{L}(s, \Pi \times \Pi) \). The work in [39, Section A.2] leading up to Equation A.9 shows that \( \Lambda_{\Pi \times \Pi}(n) \geq 0 \) for all \( n \geq 1 \) (even if \( n \) shares a prime factor with a conductor of one of the constituents of \( \Pi \)). Thus by nonnegativity, we find that

\[
\sum_{1 - c_{21} / \log C(\Pi \times \Pi) < \beta \leq 1} \frac{1}{\sigma - \beta} < \frac{r_{\Pi \times \Pi}}{\sigma - 1} + O(\log C(\Pi \times \Pi)). \tag{4.2}
\]
Let $N$ be the number of real zeros in the sum on the left hand side of (4.2). We choose 
\[ \sigma = 1 + 2c_{21}/\log C(\Pi \times \tilde{\Pi}) \] 
and conclude that 
\[ \frac{N \log C(\Pi \times \tilde{\Pi})}{2c_{21} + \frac{c_{21}}{r_{\Pi \times \tilde{\Pi}} + 1}} < \left( r_{\Pi \times \tilde{\Pi}} + O(1) \right) \log C(\Pi \times \tilde{\Pi}). \]

This implies that $N < r_{\Pi \times \tilde{\Pi}} + \frac{r_{\Pi \times \tilde{\Pi}}}{2(r_{\Pi \times \tilde{\Pi}} + 1)} + O(c_{21}),$ so $N \leq r_{\Pi \times \tilde{\Pi}}$ when $c_{21}$ is small enough.

The possibility that $L(1, \Pi \times \tilde{\Pi}) = 0$ is ruled out by [22, Theorem A.1]. □

Remark. Our proof removes the extraneous factor of $d$ in the denominator in [18, Lemma 5.9] when $f = \Pi \times \tilde{\Pi}$. While this may seem like a small improvement, the quality of our main theorems depends heavily on it.

**Corollary 4.2.** Let $\pi \in \mathfrak{F}_m$ and $\pi' \in \mathfrak{F}_{m'}$. Suppose that both $\pi$ and $\pi'$ are self-dual (that is, $\pi = \overline{\pi}$ and $\pi' = \overline{\pi'}$). There exists a constant $0 < c_{22} < 1$ such that the following results hold.

1. $L(s, \pi) \neq 0$ in the region
\[ \text{Re}(s) \geq 1 - \frac{c_{22}}{m \log(C(\pi)(1 + |\text{Im}(s)|))} \]
apart from at most one zero. If the exceptional zero exists, then it is real and simple.

2. $L(s, \pi \times \pi') \neq 0$ in the region
\[ \text{Re}(s) \geq 1 - \frac{c_{22}}{(m + m') \log(C(\pi)C(\pi')(1 + |\text{Im}(s)|)^{\min\{m,m'\}})} \]
apart from at most one zero. If the exceptional zero exists, then it is real and simple.

Remark. This improves the denominator $(m + m')^3 \log(C(\pi)C(\pi')(1 + |\text{Im}(s)|)^m)$ in [17, Theorem A.1] when $F = \mathbb{Q}$ and both $\pi$ and $\pi'$ are self-dual. A similar improvement holds when $\pi$ and $\pi'$ are defined over number fields.

**Proof.** We prove the second part; the proof of the first part follows by choosing $\pi' = 1$. Without loss, suppose that $m \leq m'$. First, let $\gamma \neq 0$, and suppose to the contrary that $\rho = \beta + i\gamma$ is a zero in the region (4.3). Inspired by [17, Appendix], we apply Proposition 4.1 to $\Pi = \pi' \otimes | \det |^{\gamma} \otimes | \det |^{-\overline{\gamma}} \otimes | \pi$, in which case (since $\pi$ and $\pi'$ are self-dual)
\[ L(s, \Pi \times \Pi) = L(s, \pi \times \overline{\pi})L(s, \pi' \times \overline{\pi'})L(s + it, \pi \times \pi')^2L(s - it, \pi \times \pi')^2 \]
\[ \times L(s + 2i\gamma, \pi' \times \overline{\pi'})L(s - 2i\gamma, \pi' \times \overline{\pi'}). \]

Since $\pi$ and $\pi'$ are self-dual, it follows that $\rho$ is a zero of $L(s, \pi \times \pi')$ if and only if $\overline{\rho}$ is. Thus if $\rho$ is a zero of $L(s, \pi)$, then $L(s, \Pi \times \Pi)$ has a real zero at $s = \beta$ of order 4 in the region (4.1). This contradicts Proposition 4.1 since $r_{\Pi \times \Pi} = 3$ when $\gamma \neq 0$. The desired result follows from the bound $\log C(\Pi \times \Pi) \ll (m + m') \log(C(\pi)C(\pi')(1 + |\gamma|)^{m'})$, which holds via (3.4).

Second, suppose that $\gamma = 0$. The same arguments as when $\gamma \neq 0$ hold, except that now $r_{\Pi \times \Pi} = 5$. Since the presence of a single zero of $L(s, \pi \times \pi')$ in the claimed region contributes a zero of multiplicity four to $L(s, \Pi \times \Pi)$, we must conclude that if a real zero of $L(s, \pi \times \pi')$ exists in the region (4.3) (with Im($s$) = 0), then such a zero must be simple. □

We also record a bound on the exceptional zero in Part (2) of Corollary 4.2 (if it exists).

**Lemma 4.3.** There exists a constant $0 < c_{23} < 1$ such that the exceptional zero in Part (2) of Corollary 4.2 when $\pi \neq \pi'$ is bounded by $1 - c_{23}(C(\pi)C(\pi'))^{-m-m'}$. 


4.2. Log-free zero density estimates. Our log-free zero density estimates are as follows.

Proposition 4.4. Let \( \pi \in \mathfrak{F}_m \) and \( \pi' \in \mathfrak{F}_{m'} \). There exists a constant \( c_{24} > 2 \) such that the following are true.

1. Suppose that \( \pi \) satisfies GRC at all primes \( p \nmid q_\pi \). If \( 0 \leq \sigma \leq 1 \) and \( T \geq 1 \), then
   \[
   N_{\pi}(\sigma, T) := \# \{ \rho = \beta + i\gamma : \beta \geq \sigma, \ |\gamma| \leq T, \ L(\rho, \pi) = 0 \} \ll m^2 (C(\pi)T)^{c_{24}(1-\sigma)}. \]

2. If \( 0 \leq \sigma \leq 1, \ T \geq 1, \ m, m' \leq M, \) and \( \pi \) and \( \pi' \) satisfy GRC at all \( p \nmid q_\pi q_{\pi'} \), then
   \[
   N_{\pi \times \pi'}(\sigma, T) := \# \{ \rho = \beta + i\gamma : \beta \geq \sigma, \ |\gamma| \leq T, \ L(\rho, \pi \times \pi') = 0 \} \ll M^4 ((C(\pi)C(\pi'))^M T)^{c_{24}M^2(1-\sigma)}. \]

For the sake of notational compactness, we will only prove Part (1). There are no structural differences in the proof of Part (2) except that \( m \) is replaced by \( m' \) and \( C(\pi) \) is replaced by \( C(\pi)^m C(\pi')^m \) (which is an upper bound for \( C(\pi \times \pi') \)). Our proof runs parallel to that of [39, Theorem 1.2], so we only point out the key differences. We begin with some adjustments to the lemmas in [39] which will improve the degree dependence.

Lemma 4.5. If \( t \in \mathbb{R} \) and \( 0 < \eta \leq 1 \), then
\[
\sum_{\rho} \frac{1 + \eta - \beta}{|1 + \eta + it - \rho|^2} \leq 2m \log C(\pi) + m \log(2 + |t|) + \frac{2}{\eta} + O(m^2)
\]
so that \( \# \{ \rho : |\rho - (1 + it)| \leq \eta \} \leq 10m\eta \log C(\pi) + 5m\eta \log(2 + |t|) + O(m^2\eta + 1) \).

Proof. The proof proceeds just as in [39, Lemma 3.1], but we use [22] instead of [39, Equation 1.9], and we use the fact that \( r_\pi \leq 1 \) in our case. \( \square \)

Lemma 4.6. Let \( T \geq 1 \), and let \( \tau \in \mathbb{R} \) satisfy \( 200\eta \leq |\tau| \leq T \). If \( c_{22}/(m \log(C(\pi)T)) \leq \eta \leq (200m)^{-1} \) and \( s = 1 + \eta + i\tau \), then
\[
\left( \frac{-1}{k!} \left( \frac{L'}{L} (s, \pi) \right) \right)^{(k)} = \sum_{\rho \in |s - \rho| \leq 200\eta} \frac{1}{(s - \rho)^{k+1}} + O \left( \frac{m \log(C(\pi)T)}{(200\eta)^k} \right).
\]

Proof. The proof is the same as that of [39, Equation 4.1] with three changes. First, we have that \( r_\pi \leq 1 \). Second, we widen the range of \( \eta \) all the way to the edge of the zero-free region in Corollary 4.2. Finally, we use Lemma 4.5 instead of [39, Lemma 3.1]. \( \square \)

Lemma 4.7. Let \( T \geq 1, \ 200\eta \leq |\tau| \leq T, \) and \( c_{22}/(m \log(C(\pi)T)) < \eta \leq (200m)^{-1} \). Let \( K > \lceil 200m\eta \log(C(\pi)T) + O(m^2\eta + 1) \rceil \) and \( s = 1 + \eta + i\tau \). If \( L(s, \pi) \) has a zero \( \rho_0 \) satisfying \( |\rho_0 - (1 + i\tau)| \leq \eta \), then
\[
\left| \sum_{\rho \in |s - \rho| \leq 200\eta} \frac{1}{(s - \rho)^{k+1}} \right| \geq \left( \frac{1}{100\eta} \right)^{k+1}.
\]

Proof. The proof is the same as that of [39, Lemma 4.2], but we use Lemmas 4.5 and 4.6 instead of [39, Equations 3.6 and 4.1]. \( \square \)
Theorem 4.8. Let \( \eta \) and \( \tau \) be real numbers satisfying \( c_{22}/(m \log(C(\pi)T)) < \eta \leq (200m)^{-1} \) and \( 200\eta \leq |\tau| \leq T \). Let \( K \geq 1 \) be an integer, and let \( N_0 = \exp(K/(300\eta)) \) and \( N_1 = \exp((40K)/\eta) \). Let \( s = 1+\eta+i\tau \), If \( K \leq k \leq 2K \), then

\[
\left| \frac{\eta^{k+1}}{k!} \left( \frac{L'}{L}(s, \pi) \right)^{(k)} \right| \leq \eta^2 \int_{N_0}^{N_1} |\Lambda(n)| \frac{du}{u} + O\left( \frac{m\eta \log(C(\pi)T)}{(110)^k} \right).
\]

Proof. The proof is the same as that of \cite[Lemma 4.3]{39} except that it incorporates \cite[Equation 19]{32} instead of \cite[Equation 19]{39} as well as our wider range of \( \eta \).

Proof of Proposition 4.4. We reiterate that we have only given the details for \( L(s, \pi) \), but the details for \( L(s, \pi \times \pi') \) run parallel apart from bounding \( C(\pi \times \pi') \). Compare with \cite[Theorem 1.2]{39}, but we use Lemmas 4.7 and 4.8 instead of \cite[Lemmas 4.2 and 4.3]{39}. We conclude that

\[
\#\{\rho = \beta + i\gamma : \beta \geq 1 - \eta/2, |\gamma| \leq T\} \ll 102^{4K} \eta^2 m \log(C(\pi)T)T^2 \int_{N_0/e}^{N_1} \sum_{x < n \leq xe^{1/T}} |\Lambda(n)| \frac{2dx}{x^3} + \eta m \log(C(\pi)T)
\]

\[
\ll 102^{4K} K \eta T^2 \int_{N_0/e}^{N_1} \sum_{x < n \leq xe^{1/T}} |\Lambda(n)| \frac{2dx}{x^3} + K
\]

By hypothesis, we have the bounds \(|\alpha_{j,\pi}(p)| \leq 1\) for \( p \nmid q_\pi \) and \(|\alpha_{j,\pi}(p)| \leq p^{1-1/m} \) for \( p|q_\pi \).

Since \( q_\pi \) has \( O(\log q_\pi) \) distinct prime divisors, for \( x \) in the range \([N_0/e, N_1]\), we have

\[
\sum_{x < n \leq xe^{1/T}} |\Lambda(n)| \leq m \sum_{x < n \leq xe^{1/T}} \Lambda(n) + m \sum_{p|N_\pi} \log p \sum_{\log x < \log p \leq \log x + 1} p^{\ell(1-1/m)} \ll m \frac{x}{T} (1 + x^{-\frac{1}{2m} \log q_\pi}) \ll m \frac{x}{T},
\]

hence

\[
\#\{\rho = \beta + i\gamma : \beta \geq 1 - \eta/2, |\gamma| \leq T\} \ll m^2 102^{4K} K \eta T^2 \int_{N_0/e}^{N_1} \frac{x}{T} \frac{2dx}{x^3} + K \ll m^2 102^{4K} K \eta T^2 \cdot \frac{K}{\eta T^2} + K \ll m^2 105^{4K}.
\]

If we write \( \eta = 2(1-\sigma) \), then it follows that

\[
N_\pi(\sigma, T) \ll m^2 e^{O(m)} C(\pi)T \cdot 10^{10} m(1-\sigma), \quad 1 - \frac{1}{400m} \leq \sigma < 1 - \frac{c_{22}}{2m \log(C(\pi)T)}.
\]

If \( \sigma \geq 1 - c_{22}/(2m \log(C(\pi)T)) \), then \( N_\pi(\sigma, T) \leq 1 \) per Corollary 1.2. Our proposition is trivial for \( \sigma < 1 - 1/(400m) \) since there are \( \ll m T \log(C(\pi)T) \) zeros with \( 0 < \beta < 1 \) and \( |\gamma| \leq T \); see \cite[Theorem 5.8]{18}. To finish the proof, we apply the bound \( e^{O(m)} \leq C(\pi)^{O(1)} \). 

□
5. An uniform prime number theorem

We will prove a uniform version of the prime number theorem for the $L$-functions $L(s, \pi)$ and $L(s, \pi \times \pi')$ which satisfy the hypotheses of Corollary 4.2 and Proposition 4.4. The use of Proposition 4.4 helps us to improve the range of $x$ (which important for Theorem 1.2) and simplify certain aspects of the proof. We define the weighted prime counting functions

$$\theta_\pi(x) := \sum_{p \leq x \atop p \equiv \pi} a_\pi(p) \log p, \quad \theta_{\pi \times \pi'}(x) := \sum_{p \leq x \atop p \equiv \pi, p \equiv \pi'} a_{\pi \times \pi'}(p) \log p.$$ 

Proposition 5.1. Let $\pi \in \mathfrak{F}_m$ and $\pi' \in \mathfrak{F}_{m'}$. Suppose that $\pi$ and $\pi'$ satisfy the hypotheses of Corollary 4.2 and Proposition 4.4.

(1) Suppose that either $\mu_\pi(j) = 0$ or $\Re(\mu_\pi(j)) \geq \frac{1}{2}$ for each $j$. If $2 \leq C(\pi)^m \leq x^{1/(36c_{24})}$ and $32c_{24}m^{1/2} < \frac{1}{4}$, then

$$\left| \theta_\pi(x) - r_\pi x + \frac{x^{\beta_1}}{\beta_1} \right| \ll m^2 x^{1 - \frac{1}{36c_{24}m}} + m^2 x \left( \exp \left[ - \frac{c_{22} \log x}{2m \log C(\pi)} \right] + \exp \left[ - \frac{\sqrt{c_{22} \log x}}{2M} \right] \right).$$

We omit the $\beta_1$ term if the exceptional zero in Corollary 4.2 (Part 1) does not exist.

(2) Let $m, m' \leq M$. Suppose that either $\mu_{\pi \times \pi'}(j, j') = 0$ or $\Re(\mu_{\pi \times \pi'}(j, j')) \geq \frac{1}{2}$. If $2 \leq (C(\pi)C(\pi'))^M \leq x^{1/(36c_{24})}$ and $32c_{24}M^{1/2} < \frac{1}{4}$, then

$$\left| \theta_{\pi \times \pi'}(x) - r_{\pi \times \pi'} x + \frac{x^{\beta_1}}{\beta_1} \right| \ll (m'm)^2 x^{1 - \frac{1}{36c_{24}M^2}} + (m'm)^2 x \left( \exp \left[ - \frac{c_{22} \log x}{4M \log(C(\pi)C(\pi'))} \right] + \exp \left[ - \frac{\sqrt{c_{22} \log x}}{2M} \right] \right).$$

We omit the $\beta_1$ term if the exceptional zero in Corollary 4.2 (Part 2) does not exist.

As with our proof of Proposition 4.4 we will only prove Part (1) of Proposition 5.1 since the proof of Part (2) runs entirely parallel apart from an application of (3.4) to bound $C(\pi \times \pi')$. To begin our work toward Proposition 5.1 we introduce a carefully chosen smooth weight for sums over prime powers.

Lemma 5.2. Choose $x \geq 3$, $\varepsilon \in (0, 1/4)$, and an integer $\ell \geq 1$. Define $A = \varepsilon/(2\ell \log x)$. There exists a continuous function $\phi(t) = \phi(t; x, \ell, \varepsilon)$ which satisfies the following properties:

(1) $0 \leq \phi(t) \leq 1$ for all $t \in \mathbb{R}$, and $\phi(t) \equiv 1$ for $\frac{1}{2} \leq t \leq 1$.

(2) The support of $\phi$ is contained in the interval $\left[ \frac{1}{2} - \frac{\varepsilon}{\log x}, 1 + \frac{\varepsilon}{\log x} \right]$.

(3) Its Laplace transform $\Phi(z) = \int_\mathbb{R} \phi(t)e^{-zt}dt$ is entire and is given by

$$\Phi(z) = e^{-(1+2\ell A)z} \cdot \left( \frac{1 - e^{(1+2\ell A)z}}{-z} \right) \left( \frac{1 - e^{2A z}}{-2A z} \right)^\ell.$$

(4) Let $s = \sigma + it$, $\sigma > 0$, $t \in \mathbb{R}$ and $\alpha$ be any real number satisfying $0 \leq \alpha \leq \ell$. Then

$$|\Phi(-s \log x)| \leq \frac{e^{\sigma \log x}}{|s| \log x} \cdot (1 + x^{-\sigma/2}) \cdot \left( \frac{2\ell}{\varepsilon |s|} \right)^\alpha.$$ 

Moreover, $|\Phi(-s \log x)| \leq e^{\sigma \log x}$ and $\frac{1}{2} < \Phi(0) < \frac{3}{4}$. 

(5) If \( \frac{3}{4} < \sigma \leq 1 \), \( x \geq 10 \), and \( \delta_{\pi,1}, r_{\pi} \in \{0, 1\} \), then

\[
\psi_{\pi}(x) = \psi_{\pi}(x, \phi) = \sum_{n=1}^{\infty} \phi\left(\frac{\log n}{\log x}\right) \Lambda_{\pi}(n).
\]

The next lemma shows that \( \psi_{\pi}(x) \) closely approximates \( \theta_{\pi}(x) \).

**Lemma 5.3.** If \( \ell \geq 2 \) is an integer, \( x \) satisfies the hypotheses of Proposition 2.1, and \( \varepsilon \in \left( -\frac{1}{2}, \frac{1}{4} \right) \), then \( \theta_{\pi}(x) = \psi_{\pi}(x) + O(mx^{1-\frac{1}{3m}} + \varepsilon mx) \).

**Proof.** By Lemma 5.2, we have

\[
\psi_{\pi}(x) = \sum_{\sqrt{x} < n \leq x} \Lambda_{\pi}(n) + O\left( \sum_{\sqrt{x} < n \leq x} |\Lambda_{\pi}(n)| + \sum_{x < n \leq xe^{\varepsilon}} |\Lambda_{\pi}(n)| \right).
\]

Note that by (3.1) and the Brun–Titchmarsh theorem, we have

\[
\sum_{\sqrt{x} < n \leq x} \Lambda_{\pi}(n) = \sum_{n \leq x} \Lambda_{\pi}(n) + \sum_{\sqrt{x} < n \leq x} \Lambda_{\pi}(n) - \sum_{n \leq x} \Lambda_{\pi}(n).
\]

By Mellin inversion, we have

\[
\psi_{\pi}(x) = \frac{\log x}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \pi) \Phi(-s \log x) ds.
\]
**Lemma 5.4.** If \( \text{Re}(s) = -\frac{1}{4} \), then \(-\frac{L'}{L}(s, \pi) \ll \log C(\pi) + m \log |\text{Im}(s)| + 3\).

**Proof.** With the hypotheses of Proposition 5.1, this follows from [18, Proposition 5.7]. □

We begin our proof of Proposition 5.1 by shifting the contour to the line \( \text{Re}(s) = -\frac{1}{4} \), accumulating contributions from the residues at the nontrivial zeros of \( L(s, \pi) \) and a trivial zero at \( s = 0 \) of order \( O(m) \) with residue \( O(1) \). We bound the shifted contour integral on the line \( \text{Re}(s) = -\frac{1}{4} \) using Lemmas 5.2 and 5.4 and conclude that

\[
\psi_\pi(x) = r_\pi \Phi(-\log x) \log x - \delta_{\pi,1} \Phi(-\beta_1 \log x) \log x - (\log x) \sum_{\rho \neq \beta_1} \Phi(-\rho \log x)
+ O\left(\frac{(2l/\varepsilon)\log C(\pi)}{x^{\frac{1}{4}}} + m\right)
\]

(5.2) \[
= r_\pi x - \delta_{\pi,1} \frac{x^{\beta_1}}{\beta_1} - (\log x) \sum_{\rho \neq \beta_1} \Phi(-\rho \log x) + O\left(\frac{(2l/\varepsilon)\log C(\pi)}{x^{\frac{1}{4}}} + m + \sqrt{x + \varepsilon x}\right).
\]

By [18, Proposition 5.7], there are \( \ll \log C(\pi) \) nontrivial zeros \( \rho \) of \( L(s, \pi) \) with \( |\rho| < \frac{1}{4} \). Thus it follows from Lemma 5.2 that

(5.3) \[
\sum_{|\rho| \leq \frac{1}{4}, \rho \neq \beta_1} |\Phi(-\rho \log x)| \ll \sum_{|\rho| \leq \frac{1}{4}, \rho \neq \beta_1} x^{\frac{1}{2}} \ll x^{\frac{1}{2}} \log C(\pi).
\]

**Lemma 5.5.** Let \( \phi \) be defined as in Lemma 5.2 with \( \varepsilon = 8x^{-\frac{1}{5\pi}} \) and \( \ell = 4c_{24}m \). If \( 2 \leq C(\pi)^m \leq x^{1/(8c_{24})} \) and \( \varepsilon < \frac{1}{15} \), then

\[
\log x \sum_{|\rho| \geq \frac{1}{4}, \rho \neq \beta_1} |\Phi(-\rho \log x)| \ll m^2 x e^{-\eta_\pi(x)/2}, \quad \eta_\pi(x) := \inf_{\ell \geq 3} \left( \frac{c_{22} \log x}{m \log (C(\pi)t)} + \log t \right).
\]

**Proof.** Let \( T_0 = 0 \) and \( T_j = 2^{j-1} \) for all \( j \geq 1 \). We consider the sums

\[
S_j := \log x \sum_{T_{j-1} \leq |\gamma| \leq T_j} |\Phi(-\rho \log x)|.
\]

We estimate \( |F(-\rho \log x)| \) for \( \rho \) in the sum \( S_j \) using Lemma 5.2 with \( \alpha = \ell (1 - \beta) \). Our choices of \( \varepsilon \) and \( \ell \) and our restriction \( 2 \leq C(\pi)^m \leq x^{1/(8c_{24})} \) imply that

\[
|\Phi(-\rho \log x)| \log x \ll \frac{x^\beta}{|\rho|^{\ell (1-\beta)}} \ll x T_j^{\frac{1}{2}} (|\gamma| + 3)^{-\frac{1}{2}} x^{-\frac{1-\beta}{2}} (x^\pi T_j^\ell)^{-(1-\beta)}
\]

\[
\ll x T_j^{\frac{1}{2}} (|\gamma| + 3)^{-\frac{1}{2}} x^{-\frac{1-\beta}{2}} (C(\pi)T)^{-2mc_{24}(1-\beta)}.
\]

By the definition of \( \eta_\pi(x) \) and Corollary 4.2, we have the bound

\[
(|\gamma| + 3)^{-\frac{1}{2}} x^{-\frac{1-\beta}{2}} = e^{-\frac{1}{2} \log(|\gamma| + 3) + (1-\beta) \log x} \leq e^{-\eta_\pi(x)/2}.
\]

Consequently, we have that

\[
S_j \ll x e^{-\eta_\pi(x)/2} T_j^{-\frac{1}{2}} \sum_{T_{j-1} \leq |\gamma| \leq T_j} (C(\pi)T)^{-2mc_{24}(1-\beta)}
\]
\[ \leq xe^{-\eta_x(x)/2}T_j^{-\frac{1}{2}} \int_0^1 (C(\pi)T)^{-2mc_{24}\sigma} dN_m(1 - \sigma, T_j). \]

The Stieltjes integral equals
\[ (C(\pi)T)^{-2mc_{24}m} N_\pi(0, T_j) + m \log(C(\pi)T) \int_0^1 (C(\pi)T)^{-2mc_{24}\sigma} N_\pi(1 - \sigma, T_j) d\sigma, \]
which we estimate using Proposition 4.4. We conclude that \( S_j \ll m^2 xe^{-\eta_x(x)/2}T_j^{-1/2} \), hence
\[ \log x \sum_{|\rho| \geq \frac{1}{2}} |\Phi(-\rho \log x)| \ll \sum_{j=1}^\infty S_j \ll m^2 xe^{-\eta_x(x)/2} \sum_{j=1}^\infty T_j^{-\frac{1}{2}} \ll m^2 xe^{-\eta_x(x)/2}, \]
as desired. \( \square \)

**Lemma 5.6.** If \( m \geq 1 \) and \( x \geq 2 \), then \( e^{-\eta_x(x)/2} \leq \exp[-\frac{c_2 \log x}{2m \log C(x)}] + \exp[-\frac{\sqrt{c_2 \log x}}{2\sqrt{m}}] \).

**Proof.** This is a straightforward optimization problem. \( \square \)

**Proof of Proposition 6.1.** Collect the estimates in Lemma 5.3, (5.2), (5.3), Lemma 5.5, and Lemma 5.6 and apply the prescribed choices for \( \ell \) and \( \varepsilon \) in our range of \( x \). \( \square \)

### 6. Proof of Propositions 6.1 and 6.2

Let \( f \in S_k^{\text{new}}(\Gamma_0(q)) \) be a newform of squarefree level \( q \) as in the statement of Theorem 6.1. For each prime \( p \), let \( \theta_p \in [0, \pi] \) be the unique angle such that \( a_f(p) = 2 \cos \theta_p \). The modular \( L \)-function \( L(s, f) \) associated to \( f \) has the Euler product representation
\[ L(s, f) = \prod_{p} \left( 1 - \frac{a_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}} \right)^{-1}, \quad \text{Re}(s) > 1, \]
where \( \chi_0 \) is the trivial Dirichlet character modulo \( q \). We rewrite the Euler product as
\[ L(s, f) = \prod_{p|q} \left( 1 - \frac{(-\lambda_p p^{-\frac{s}{2}})}{p^s} \right) \prod_{p|q} \prod_{j=0}^1 \left( 1 - e^{i(2j-1)\theta_p} \right)^{-1}, \quad \text{Re}(s) > 1, \]
where \( \lambda_p \in \{-1, 1\} \) is the eigenvalue of the Atkin–Lehner operator \( |k W(Q_p) \).

#### 6.1. Standard \( L \)-functions.

For each \( m \geq 1 \), we define the Euler product
\[ L(s, \text{Sym}^m f) = \prod_{n=1}^\infty a_{\text{Sym}^m f}(n) n^{-s} = \prod_{p|q} \left( 1 - \frac{(-\lambda_p p^{-\frac{s}{2}})^m}{p^s} \right)^{-1} \prod_{p|q} \prod_{j=0}^m \left( 1 - \frac{e^{i(2j-m)\theta_p}}{p^s} \right)^{-1}, \quad \text{for Re}(s) > 1. \]
for \( \text{Re}(s) > 1 \). We also define
\[ L(s, (\text{Sym}^m f)_\infty) = \begin{cases} q^{(m+1)/2} \prod_{j=1}^m \Gamma_C(s + (j - \frac{1}{2})(k - 1)) & \text{if } m \text{ is odd}, \\ q^m \Gamma_R(s + r) \prod_{j=1}^{m/2} \Gamma_C(s + j(k - 1)) & \text{if } m \text{ is even}, \end{cases} \]
where $\Gamma_C(s) = \Gamma_R(s)\Gamma_R(s + 1)$, $r = 0$ if $m \equiv 0 \pmod{4}$, and $r = 1$ if $m \equiv 2 \pmod{4}$. Note that $L(\Sym^1 f) = L(s, f)$ and $L(\Sym^0 f) = \zeta(s)$. One easily checks via (6.2) that

$$a_{\Sym^m f}(p) = U_m(\cos \theta_p), \quad p \nmid q.$$

**Theorem 6.1.** Let $q$ be squarefree, let $f \in S^\text{new}_{k}(\Gamma_0(q))$ be as in Theorem 1.7 and let $\pi_f \in \mathfrak{F}_2$ correspond with $f$. If $m \geq 1$, then $L(s, \Sym^m f)$ is the standard $L$-function associated to the representation $\Sym^m \pi_f \in \mathfrak{F}_{m+1}$, with $L(s, (\Sym^m \pi_f)_\infty)$ given by (6.3).

**Proof.** Let $\pi_f$ be the cuspidal automorphic representation of $\GL_2(\mathbb{A})$ with unitary central character which corresponds with $f$. Newton and Thorne [31 Theorem B] recently proved that if $m \geq 1$, then the $m$-th symmetric power lift $\Sym^m \pi_f$ is a self-dual automorphic representation of $\GL_{m+1}(\mathbb{A})$ with trivial central character whose standard $L$-function is given by (6.1). Ramakrishnan [34] Theorems A and A’ proved that the automorphy of $\Sym^m \pi_f$ for all $m \geq 1$ implies the cuspidality of $\Sym^m \pi_f$ for all $m \geq 1$. Hence $\Sym^m \pi_f \in \mathfrak{F}_{m+1}$ for all $m \geq 1$. Moreno and Shahidi [28] and Cogdell and Michel [10, Section 3] computed $L(s, (\Sym^m \pi_f)_\infty)$ under the assumption of cuspidality, which we now have.

**Remark.** When $q$ is not squarefree and $p^2 | q$, the Euler factor at $p$ in (6.2) becomes more complicated when $m \geq 2$ (it is the same when $m = 1$). For computations of these Euler factors when $k = 2$ and $f$ corresponds with an elliptic curve, and see [12]. See [30 Appendix] for more general computations. Also, if $q$ is not squarefree, then Rouse [35 Section 5], following a suggestion of Serre, proved that $\log q_{\Sym^m f} \ll m \log q$. It seems difficult to prove a good explicit inequality when $q$ is non-squarefree except for the case where $f$ is associated to a non-CM elliptic curve [11 Appendix].

For $m \geq 1$, a straightforward calculation using (6.3) and Stirling’s formula yields

$$C(\Sym^m f) := C(\Sym^m \pi_f) \ll (kqm)^m, \quad \log C(\Sym^m f) \asymp m \log (kqm).$$

**6.2. Rankin–Selberg $L$-functions.** Given $f$ as in Theorem 6.1 let $\pi_f$ be the cuspidal automorphic representation of $\GL_2(\mathbb{A})$ corresponding to $f$. Given an integer $m \geq 0$, let $\Sym^m \pi_f$ be the $m$-th symmetric power lift, which is shown in Theorem 6.1 to be a cuspidal automorphic representation of $\GL_{m+1}(\mathbb{A})$. (If $m = 0$, then $\Sym^m \pi_f = \mathbb{1}$.)

For $i = 1, 2$, let $f_i \in S^\text{new}_{k_i}(\Gamma_0(q_i))$ be a newform as in Theorem 6.1 and let $\{\theta_p^{(j)}\}$ be the sequence of Sato–Tate angles for $f_i$. Suppose that $\pi_{f_1} \neq \pi_{f_2}$. For integers $m_i \geq 0$, we consider the tensor product $\Sym^{m_1} \pi_{f_1} \otimes \Sym^{m_2} \pi_{f_2}$, whose Rankin–Selberg $L$-function is

$$L(s, \Sym^{m_1} f_1 \times \Sym^{m_2} f_2) = \prod_{p \nmid q_1 q_2} \prod_{j_1 = 0, j_2 = 0}^{m_1} \prod_{j_1 = 0, j_2 = 0}^{m_2} \left(1 - \frac{e^{i(2j_1 - m_1)\theta_p^{(1)}} e^{i(2j_2 - m_2)\theta_p^{(2)}}}{p^s}\right)^{-1},$$

in view of Theorem 6.1. We have suppressed the (more complicated) Euler factors at primes $p | q_1 q_2$. Instead of providing the unwieldy (and, for our purposes, unenlightening) explicit descriptions for the Euler factors at primes $p | q_1 q_2$ and the gamma factors, we observe that the bound (3.3) applied to the primes $p | q_1 q_2$, while probably very inefficient, is strong enough for us to prove Theorem 1.4 and we can estimate $C(\Sym^{m_1} f_1 \times \Sym^{m_2} f_2)$ using (3.4) and (6.5). A tedious calculation shows that the Langlands parameters of $L(s, \Sym^{m_1} f_1 \times \Sym^{m_2} f_2)$ satisfy the hypotheses of Proposition 5.1; we omit this calculation. One easily checks that

$$a_{\Sym^{m_1} f_1 \times \Sym^{m_2} f_2}(p) = U_{m_1}(\cos \theta_p^{(1)}) U_{m_2}(\cos \theta_p^{(2)}), \quad p \nmid q_1 q_2.$$
Lemma 6.2. If $m_1m_2 \neq 0$ and $f_1 \not\sim f_2$ are as in Theorem 6.1, then $L(s, \text{Sym}^{m_1} f_1 \times \text{Sym}^{m_2} f_2)$ extends to an entire function with no pole at $s = 1$.

Proof. This follows from work of Rajan [33, Theorem 1.1]. For related work in the context of potential automorphy, see Harris [16, Section 5]. □

Lemma 6.3. There exists a constant $0 < c_{25} < 1$ such that if $m \geq 1$, then $L(s, \text{Sym}^m f) \neq 0$ for

$$\text{Re}(s) \geq 1 - \frac{c_{25}}{m^2 \log(kqm(1 + |\text{Im}(s)|))}.$$ 

Proof. When $\text{Im}(s) \neq 0$, this follows from Corollary 4.2 and (6.5) (once $c_{25}$ is made suitably small). It remains to handle the case where $\text{Im}(s) = 0$. Consider the isobaric automorphic representation $\Pi_m = 1 \oplus \text{Sym}^2 \pi_f \oplus \text{Sym}^m \pi_f$. Using the identities

$$\text{Sym}^m \pi_f \otimes \text{Sym}^m \pi_f = 1 \oplus \left( \bigoplus_{j=1}^m \text{Sym}^{2j} \pi_f \right)$$

and

$$\text{Sym}^m \pi_f \otimes \text{Sym}^2 \pi_f = \begin{cases} \bigoplus_{j=0}^2 \text{Sym}^{m+2-2j} \pi_f & \text{if } m \geq 2, \\ \pi_f \oplus \text{Sym}^3 \pi_f & \text{if } m = 1, \end{cases}$$

we find for $m \geq 1$ that

$$L(s, \Pi_m \times \tilde{\Pi}_m) = \zeta(s)^3 L(s, \text{Sym}^m f)^4 \prod_{j=1}^m L(s, \text{Sym}^{2j} f)$$

$$\times L(s, \text{Sym}^2 f)^3 L(s, \text{Sym}^4 f) L(s, \text{Sym}^{m+2} f)^2 L(s, \text{Sym}^{m-2} f)^2$$

(with $L(s, \text{Sym}^{m-2} f)^2$ omitted when $m = 1$). The bound $\log C(\Pi_m \times \tilde{\Pi}_m) \ll m^2 \log(kqm)$ follows from (3.4) and (6.5). Note that $L(s, \Pi_m \times \tilde{\Pi}_m)$ always has a pole of order exactly 3, but a proposed zero of $L(s, \text{Sym}^m f)$ ensures that $L(s, \Pi_m \times \tilde{\Pi}_m)$ has a real zero of order at least 4 in the region (4.1). This contradicts Proposition 4.1, hence no such zero can exist (once $c_{25}$ is made suitably small). □

Lemma 6.4. Let $1 \leq m_1, m_2 \leq M$. There exists a constant $0 < c_{26} < 1$ such that The Rankin–Selberg $L$-function $L(s, \pi \times \pi') \neq 0$ for

$$\text{Re}(s) \geq 1 - \frac{c_{26}}{M^2 \log(k_1 q_1 k_2 q_2 M(1 + |\text{Im}(s)|))}$$

apart from at most one zero $\beta_{m_1, m_2}$. If $\beta_{m_1, m_2}$ exists, then it is real and simple, and there exists a constant $0 < c_{27} < 1$ such that $\beta_{m_1, m_2} \leq 1 - c_{27}(k_1 q_1 k_2 q_2 M)^{-4M^2}$.

Proof. This follows from Corollary 4.2, Lemma 4.3 and (6.5). □

6.3. Proofs of Propositions 2.1 and 2.2.

Proof of Proposition 2.1. This follows from Part (1) of Proposition 5.1, (6.4), Theorem 6.1 (6.5), and Lemma 6.3. The two conditions in Proposition 5.1 are satisfied when $m$ is in the claimed range. □

Proof of Proposition 2.2. This follows from Part (2) of Proposition 5.1, Theorem 6.1 (6.5), Lemma 6.2 (6.0), and Lemma 6.4. The two conditions in Proposition 5.1 are satisfied when $m_1$ and $m_2$ are in the claimed range. □
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