The resonant boundary $Q$-curvature problem and boundary-weighted barycenters

Mohameden AHMEDOU, Sadok KALLEL, Cheikh Birahim NDIAYE

In fond memory of Abbas Bahri

Abstract

Given a compact four-dimensional Riemannian manifold $(M, g)$ with boundary, we study the problem of existence of Riemannian metrics on $M$ conformal to $g$ with prescribed $Q$-curvature in the interior $M$ of $M$, and zero $T$-curvature and mean curvature on the boundary $\partial M$ of $M$. This geometric problem is equivalent to solving a fourth-order elliptic boundary value problem (BVP) involving the Paneitz operator with boundary conditions of Chang-Qing and Neumann operators. The corresponding BVP has a variational formulation but the corresponding variational problem, in the case under study, is not compact. To overcome such a difficulty we perform a systematic study, à la Bahri, of the so called critical points at infinity, compute their Morse indices, determine their contribution to the difference of topology between the sublevel sets of associated Euler-Lagrange functional and hence extend the full Morse Theory to this noncompact variational problem. To establish Morse inequalities we were led to investigate from the topological viewpoint the space of boundary-weighted barycenters of the underlying manifold, which arise in the description of the topology of very negative sublevel sets of the related functional. As an application of our approach we derive various existence results and provide a Poincaré-Hopf type criterium for the prescribed $Q$-curvature problem on compact four dimensional Riemannian manifolds with boundary.

Key Words: Blow-up analysis, Critical points at infinity, $Q$-curvature, $T$-curvature, Morse theory, Spectral sequences, Boundary-weighted barycenters, Algebraic topological methods.

AMS subject classification: 53C21, 35C60, 58J60, 55R80.

Contents

1 Introduction and statement of the main results 2
2 Notation and preliminaries 9
3 Blow up analysis and deformation lemma 16
   3.1 Blow up points are isolated and interior ones are far from the boundary 17
   3.2 Harnack-type inequality around blow-up points 18
   3.3 Refined estimate around blow-up points 20
4 A Morse lemma at infinity 22
   4.1 Finite-dimensional reduction near infinity 23
   4.2 Construction of a pseudogradient near infinity 25
5 The boundary-weighted barycenters
5.1 The Euler characteristic of $B^p_q(M)$ 
5.2 The Homology of $B^p_q(M)$ 
5.2.1 The spaces $B^p_q := B^p_q / (B^p_{q-1} \cup B^p_{q+1})$ 
5.3 Barycenter spaces of disconnected spaces

6 Proof of the main results
6.1 Proof of Theorem 1.1 to Theorem 1.8 
6.2 Proof of Theorem 1.11 

7 Appendix
7.1 Bubble estimates 
7.2 Gradient and energy estimates 
7.2.1 The spaces $B^p_q$ 
7.2.2 Expansion of the Euler-Lagrange functional near Infinity 
7.2.2.2 Expansion of the gradient near infinity

1 Introduction and statement of the main results

On a four dimensional Riemannian manifold $(M, g)$, Paneitz[60] discovered in 1983, a conformally covariant fourth order differential operator denoted by $P^4_g$ and called the Paneitz operator. Brenson and Oersted[13] associated to this operator a natural concept of curvature called $Q$-curvature and denoted by $Q_g$. Both the Paneitz operator and the $Q$-curvature are defined in terms of the Ricci tensor $\text{Ric}_g$ and the scalar curvature $R_g$ of the Riemannian manifold $(M, g)$ as follows:

\[ P^4_g = \Delta^2_g + \text{div}_g \left( \frac{2}{3} R_g g - 2 \text{Ric}_g \nabla g \right), \quad Q_g = -\frac{1}{12} (\Delta_g R_g - R^g_3 + 3 |\text{Ric}_g|^2), \]

where $\nabla g$ is the covariant derivative with respect to $g$.

Likewise Chang-Qing[23] have discovered an operator $P^3_g$ which is associated to the boundary of a compact four-dimensional Riemannian manifold $(M, g)$ with boundary and a third-order curvature $T_g$ naturally associated to $P^3_g$. They are defined as follows

\[ P^3_g = \frac{1}{2} \partial \Delta_g + \Delta_g \frac{\partial}{\partial n_g} - 2 H_g \Delta_g + L_g (\nabla \tilde{g}, \nabla \tilde{g}) + \nabla \tilde{g} H_g \nabla \tilde{g} + (F_g - \frac{R_g}{3}) \frac{\partial}{\partial n_g}, \]

\[ T_g = -\frac{1}{12} \partial R_g + \frac{1}{2} R_g H_g - <G_g, L_g> + 3 H^3 - \frac{1}{3} \text{tr}_g (L^3) - \Delta_g H_g, \]

where $\tilde{g}$ is the metric induced by $g$ on $\partial M$, $\frac{\partial}{\partial n_g}$ is the inward Neumann operator on $\partial M$ with respect to $g$, $L_g = (L_{g, ab}) = \frac{1}{2} \partial g_{ab}$ is the second fundamental form of $\partial M$ with respect to $g$, $H_g = \frac{1}{3} \text{tr}_g (L_g) = \frac{1}{2} \tilde{g}^{ab} L_{g, ab}$ ($\tilde{g}^{ab}$ are the entries of the inverse $\tilde{g}^{-1}$ of the metric $\tilde{g}$) is the mean curvature of $\partial M$, $R^g_{ijkl}$ is the Riemann curvature tensor of $(M, g)$, $R_{g,ijkl} = g_{mi} R^m_{g,jkl}$ ($g_{ij}$ are the entries of the metric $g$), $F_g = R_{g, man}$ (with $n$ denoting the index corresponding to the normal direction in local coordinates) and $<G_g, L_g> = \tilde{g}^{ac} \tilde{g}^{bd} L_{g, ab} L_{g, cd}$. Moreover, the notation $L_g (\nabla \tilde{g}, \nabla \tilde{g})$, means $L_g (\nabla \tilde{g}, \nabla \tilde{g}) (u) = \nabla^a \tilde{g} (L_{g, ab} \nabla^b u)$. We point out that in all those notations above $i, j, k, l = 1, \cdots 4$ and $a, b, c, d = 1, \cdots 3$, and Einstein summation convention is used for repeated indices.

As the Laplace-Beltrami operator and the Neumann operator are conformally covariant, we have that $P^4_g$ is conformally covariant of bidegree $(0, 4)$ and $P^3_g$ of bidegree $(0, 3)$. Furthermore, as they govern the transformation laws of the Gauss curvature and the geodesic curvature on compact surfaces with boundary, the couple $(P^4_g, P^3_g)$ does the same for $(Q_g, T_g)$ on a compact four-dimensional Riemannian manifold with boundary $(\tilde{M}, g)$. In fact, under a conformal change of metric $g_u = e^{2u} g$, we have

\[(1) \quad \begin{cases} P^4_{g_u} = e^{-4u} P^4_g, & \text{in } \tilde{M}, \\ P^3_{g_u} = e^{-3u} P^3_g, & \text{on } \partial M. \end{cases} \quad \begin{cases} P^4_{g_u} + 2Q_{g_u} = 2Q_g e^{4u} & \text{in } \tilde{M}, \\ P^3_{g_u} + T_g = T_g e^{3u} & \text{on } \partial M. \end{cases} \]
Besides the above analogy, we have also an extension of the Gauss-Bonnet identity, known as the Gauss-Bonnet-Chern formula

\[ \int_M (Q_g + \frac{|W_g|^2}{8})dV_g + \oint_{\partial M} (T_g + Z_g)dS_g = 4\pi^2 \chi(M), \]

where \( W_g \) denote the Weyl tensor of \((M, g)\) and \( Z_g \) is given by the following formula

\[ Z_g = R_gH_g - 3H_gRic_{g,nn} + g^{ac}g^{bd}R_{g,anbn}L_{g,cd} - g^{ac}g^{bd}R_{g,acbc}L_{g,cd} + 6H_g^3 - 3H_g|L_g|^2 + tr_g(L_g^3), \]

with \( tr_g \) denoting the trace with respect to the metric induced on \( \partial M \) by \( g \) (namely \( \hat{g} \)) and \( \chi(M) \) the Euler characteristic of \( M \). Concerning the quantity \( Z_g \), we have that it vanishes when the boundary is totally geodesic and \( \int_{\partial M} Z_g dV_g \) is always conformally invariant, see [23]. Thus, setting

\[ \kappa_{(p^4,p^3)} := \kappa_{(p^4,p^3)}[g] := \int_M Q_g dV_g + \oint_{\partial M} T_g dS_g, \]

we have that thanks to (2), and to the fact that \( |W_g|^2 dV_g \) is pointwise conformally invariant, \( \kappa_{(p^4,p^3)} \) is a conformal invariant. We remark that \( 4\pi^2 \) is the total integral of the \((Q,T)\)-curvature of the standard four-dimensional hemisphere.

As was addressed in [1] by the first and third authors, a natural question is the following: given a compact four-dimensional Riemannian manifold with boundary \((M, g)\) and \( K : M \rightarrow \mathbb{R}_+ \) smooth, under which conditions on \( K \) does \( M \) carry a Riemannian metric conformal to \( g \) for which the corresponding \( Q \)-curvature is \( K \) and the corresponding \( T \)-curvature and mean curvature vanishes. Related questions have been studied for compact Riemannian surfaces with boundary regarding Gauss curvature and geodesic curvature, see for example [15], [16], [21], [27], and [61], and for compact Riemannian manifolds with boundary of dimension bigger than 2 regarding scalar curvature and mean curvature, see for example, [2], [17], [19], [27], [30], [32], [33], [40], [41] and [49] and the references therein. Thanks to (1), this problem is equivalent to finding a smooth solution to the following BVP:

\[
\begin{cases}
P_g^4 u + 2Q_g = 2Ke^4u & \text{in } \hat{M}, \\
P_g^3 u + T_g = 0 & \text{on } \partial M, \\
\frac{\partial u}{\partial n_g} - H_g u = 0 & \text{on } \partial M,
\end{cases}
\]

where \( \frac{\partial}{\partial n_g} \) is the inward normal derivative with respect to \( g \). Since the problem is conformally invariant, it is not restrictive to assume \( H_g = 0 \), since this can be always achieved through a conformal transformation of the background metric. Thus, from now on, we will assume that we are working with a background metric \( g \) satisfying \( H_g = 0 \) and hence BVP (4) becomes the following one with Neumann homogeneous boundary condition:

\[
\begin{cases}
P_g^4 u + 2Q_g = 2Ke^4u & \text{in } \hat{M}, \\
P_g^3 u + T_g = 0 & \text{on } \partial M, \\
\frac{\partial u}{\partial n_g} = 0 & \text{on } \partial M.
\end{cases}
\]

Defining \( \mathcal{H}_{2,2} \) as

\[ \mathcal{H}_{2,2} := \left\{ u \in W^{2,2}(M) : \frac{\partial u}{\partial n_g} = 0 \text{ on } \partial M \right\}, \]

where \( W^{2,2}(M) \) denotes the space of functions on \( M \) which are square integrable together with their first and second derivatives, and \( P_g^{4,3} \) as follows, for every \( u, v \in \mathcal{H}_{2,2} \)

\[
\langle P_g^{4,3} u, v \rangle_{L^2(M)} = \int_M \left( \Delta_g u \Delta_g v + \frac{2}{3} R_g \nabla_g u \cdot \nabla_g v \right) dV_g - 2 \int_M Ric_g(\nabla_g u, \nabla_g v) dV_g - 2 \int_{\partial M} L_g(\nabla_g u, \nabla_g v) dS_g.
\]
It follows from standard elliptic regularity theory that smooth solutions to (5) can be found by looking at critical points of the geometric functional

$$II(u) = \langle P^{4,3} u, u \rangle_{L^2(M)} + 4 \int_M Q_g udV_g + 4 \int_{\partial M} T_g u dS_g - \kappa_{(P^{4,3})} \ln \int_M e^{\kappa_g} dV_g, \quad u \in \mathcal{H}_{\bar{g}}.$$  

Similarly to the case of closed four-dimensional Riemannian manifolds, the spectral properties of $P^{4,3}_g$ and the sign of $\kappa_{(P^{4,3})}$ are strongly related. In fact, Catino-Ndiaye[20] proved that if $\partial M$ is umbilic in $(M, g)$, the Yamabe invariant $Y(M, \partial M, [g]) := \inf \{ \int_M R_g dV_g + \frac{1}{2} \int_{\partial M} H_u dS_u, g_u = e^{2u} g, \int_M e^{4u} dV_g = 1 \}$ is positive and $\kappa_{(P^{4,3})} + \frac{1}{6} Y(M, \partial M, [g])^2 > 0$, then $\ker P^{4,3}_g \simeq \mathbb{R}$ and $P^{4,3}_g$ is also nonnegative. They also observed that $\kappa_{(P^{4,3})}$ satisfies a rigidity type result, namely that if $\partial M$ is umbilic in $(M, g)$ and $Y(M, \partial M, [g]) \geq 0$, then $\kappa_{(P^{4,3})} \leq 4\pi^2$ with the equality holding if and only if $(M, g)$ is conformally equivalent to the standard four-dimensional hemisphere $S^4_+$. We point out that, the latter rigidity result has been noticed by Chen[26].

As in the case of closed four-dimensional Riemannian manifolds, here also, the analytic features of $II$ depend strongly on the values taken by $\kappa_{(P^{4,3})}$. Indeed depending on whether $\kappa_{(P^{4,3})}$ is a multiple of $4\pi^2$ or not, the way of finding critical points of $II$ changes drastically. To the best of our knowledge the first existence result for problem (5) has been obtained by Chang and Qing, see [24], under the assumptions that $P^{4,3}_g$ is nonnegative, $\ker P^{4,3}_g \simeq \mathbb{R}$ and $\kappa_{(P^{4,3})} < 4\pi^2$. An alternative proof using geometric flows method has been given in [54]. As already mentioned, a first sufficient condition to ensure those hypotheses (in the umbilic case) was given by Catino-Ndiaye[20]. Later, Ndiaye[52] developed a variant of the min-max scheme of Djadli-Malchiodi[29] to extend the result of Chang-Qing[24]. Precisely, he showed that problem (5) is solvable provided that $\ker P^{4,3}_g \simeq \mathbb{R}$ and $\kappa_{(P^{4,3})}$ is not a positive integer multiple of $4\pi^2$.

As in the case of closed four-dimensional Riemannian manifolds, here also, the assumptions $\ker P^{4,3}_g \simeq \mathbb{R}$ and $\kappa_{(P^{4,3})} \notin 4\pi^2\mathbb{N}^*$ will be referred to as nonresonant case. This terminology is motivated by the fact that in that situation the set of solutions to some appropriate perturbations of BVP (5) (including it) is compact, see [52]. Naturally, we call resonant case when $\ker P^{4,3}_g \simeq \mathbb{R}$ and $\kappa_{(P^{4,3})} \in 4\pi^2\mathbb{N}^*$. We divide the resonant case in two subcases. Precisely, we call the situation where $\kappa_{(P^{4,3})} = 4\pi^2$ the critical case and when $\kappa_{(P^{4,3})} \in 4\pi^2(\mathbb{N}^* \setminus \{1\})$ the supercritical one. With these terminologies, we have that the works of Chang-Qing[24] and Ndiaye[52] answer affirmatively the question raised above in the non resonant case. Unlike the non resonant case, up to the knowledge of the authors, there are no existence results in the resonant one...

To give some motivations of the study of the $(Q, T)$-curvature, we discuss some geometric applications of it. We have two results proven by Chen[26], and Catino-Ndiaye[20]. The first one follows from the works of Chen[26] and Catino-Ndiaye[20] and says that if the Yamabe invariant $Y(M, \partial M, [g])$ and $\kappa_{(P^{4,3})}$ are both positive and $(M, g)$ has umbilic boundary, then $M$ carries a conformal metric with positive Ricci curvature. Hence $M$ has a finite fundamental group. The second one due to Catino-Ndiaye[20] says that if $(M, g)$ has umbilic boundary, $Y(M, \partial M, [g]) > 0$, and $\kappa_{(P^{4,3})} > \frac{3}{4} \int_M |W_g|^2 dV_g$, then $M$ admits a Riemannian metric $\bar{g}$ such that $(M, \bar{g})$ has constant positive sectional curvature and $\partial M$ is totally geodesic in $(M, \bar{g})$. We remark also that the pair Paneitz, Chang-Qing operator, and the $(Q, T)$-curvature appear in the study of log-determinant formulas, Gauss-Bonnet type formulas, and the compactification of some locally conformally flat four-dimensional manifolds, see [22], [25], [23], [24].

In this paper, we are interested in the resonant case, namely when $\ker P^{4,3}_g \simeq \mathbb{R}$ and $\kappa_{(P^{4,3})} = 4k\pi^2$ with $k \in \mathbb{N}^*$. Namely we first completely identify the critical points at infinity of $II$, compute their Morse indices and determine their topological contribution to the difference of topology between the sublevel sets. Next, we combine the variational contribution of the critical points at infinity of $II$ with classical tools of Morse theory and precise knowledge of the topology of the boundary-weighted barycenters $B^2(M)$ (whose relevance in the problem under study was first discovered by Ndiaye[52]) to prove strong Morse inequalities and provide various type of existence results.

To state our main existence results, we need to fix some notation and make some definitions. For every...
\((p, q) \in (\mathbb{N}^*)^2\), we define \(F^M_p : (\hat{M}^p)^* \to \mathbb{R}\) as follows

\[
F^M_p(a_1, \ldots, a_p) := \sum_{i=1}^{p} \left( H(a_i, a_i) + \sum_{j \neq i} G(a_i, a_j) + \frac{1}{2} \ln(K(a_i)) \right),
\]

and \(F^\partial_q : (\partial M^q)^* \to \mathbb{R}\) as follows

\[
F^\partial_q(a_1, \ldots, a_q) := \sum_{i=1}^{q} \left( H(a_i, a_i) + \sum_{j \neq i} G(a_i, a_j) + \ln(K(a_i)) \right),
\]

where

\[
(\hat{M}^p)^* := M^p \setminus F(M^p), \quad ((\partial M)^q)^* := (\partial M)^q \setminus F((\partial M)^q)
\]

with \(F(M^p)\) and \(F((\partial M)^q)\) denoting respectively the fat diagonal of \((\hat{M}^p)\) and \((\partial M)^q\), \(G\) is the Green’s function of \((P^A_q) + \frac{1}{4} Q_q + \frac{1}{2} T_q\) under homogeneous Neumann condition with respect to \(g\) and satisfying the normalization \(\int_{\partial M} Q_q(x) G(\cdot, x) dV_g(x) + \int_{\partial M} T_q(x) G(\cdot, x) dS_g(x) = 0\), and \(H\) is its regular part, see Section 2 for more information.

On the other hand, for \((p, q) \in \mathbb{N}^2\) such that \(2p + q = k\), we define \(F^M_{p, q} : (\hat{M}^p)^* \times (\partial M^q)^* \to \mathbb{R}\) as follows

\[
F^M_{p, q}(a_1, \ldots, a_{p+q}) := F^M_p(a_1, \ldots, a_p) + \frac{1}{2} \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} G(a_i, a_j),
\]

and \(F^\partial_{p, q} : (\hat{M}^p)^* \times (\partial M^q)^* \to \mathbb{R}\) as follows

\[
F^\partial_{p, q}(a_1, \ldots, a_{p+q}) := F^\partial_q(a_{p+1}, \ldots, a_{p+q}) + 2 \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} G(a_i, a_j).
\]

Moreover, we set

\[
F_{p, q}(A) := 2F^M_{p, q}(A) + \frac{1}{2} F^\partial_{p, q}(A) = 2F^M_p(A_p) + \frac{1}{2} F^\partial_q(A_q) + 2 \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} G(a_i, a_j),
\]

with \(A = (a_1, \ldots, a_{p+q})\), \(A_p := (a_1, \ldots, a_p)\), \(A_q := (a_{p+1}, \ldots, a_{p+q})\), and define

\[
\text{Crit}(F_{p, q}) := \{ A \in ((\hat{M}^p)^* \times ((\partial M)^q)^* : A \text{ critical point of } F_{p, q}\}.
\]

Furthermore, for \(A := (a_1, \ldots, a_{p+q}) \in (\hat{M}^p)^* \times (\partial M^q)^*\), we set

\[
F^A_i(x) := e^{\frac{1}{2} H(a_i, x) + \frac{1}{4} \sum_{p=1, p \neq i} G(a_i, x) + \frac{1}{2} \ln(K(x)) + \frac{1}{2} \sum_{j=p+1}^{p+q} G(a_j, x)}, \quad i = 1, \ldots, p,
\]

and

\[
F^A_i(x) := e^{\frac{1}{2} H(a_i, x) + \frac{1}{4} \sum_{j=p+1, j \neq i} G(a_i, x) + \frac{1}{2} \ln(K(x)) + \frac{1}{2} \sum_{j=i+1}^{p} G(a_j, x)}, \quad i = p+1, \ldots, p+q.
\]

Moreover, we set

\[
L_K(A) := \left\{
\begin{array}{ll}
\sum_{i=1}^{p} (F^A_i) \hat{L}_g((F^A_i) \hat{L}_g)(a_i), & \text{if } q = 0, \\
\sum_{i=p+1}^{p+q} (F^A_i) \hat{L}_g((F^A_i) \hat{L}_g)(a_i) \frac{\partial \ln K}{\partial a_i}(a_i), & \text{if } q \neq 0,
\end{array}
\right.
\]

5
and

\[(15)\quad \iota_\infty(A) := 5p + 4q - 1 - \text{Morse}(\mathcal{F}_{p,q}, A),\]

where \(\text{Morse}(\mathcal{F}_{p,q}, A)\) denotes the Morse index of \(\mathcal{F}_{p,q}\) at \(A\). We set also

\[(16)\quad \mathcal{F}_{\infty}^{p,q} := \{ A \in \text{Crit}(\mathcal{F}_{p,q}) : \mathcal{L}_K(A) < 0 \},\]

and

\[(17)\quad \mathcal{F}_\infty := \bigcup_{(p,q) \in \mathbb{N}^2: 2p + q = k} \mathcal{F}_{\infty}^{p,q}.\]

Furthermore, we define

\[(18)\quad m_i^{p,q} := \text{card}\{ A \in \mathcal{F}_{\infty}^{p,q} : \iota_\infty(A) = i \}, \quad i = p + q - 1, \ldots, 5p + 4q - 1,\]

and

\[(19)\quad m_i^k := \sum_{(p,q) \in \mathbb{N}^2: 2p + q = k, p + q - 1 \leq i \leq 5p + 4q - 1} m_i^{p,q}, \quad i = 0, \ldots, 4k - 1.\]

For \(l \in \mathbb{N}^*\), we use the notation \(B_l^g(M)\) to denote the following set of barycentric type:

\[(20)\quad B_l^g(M) := \bigcup_{(p,q) \in \mathbb{N}^2, 0 < 2p + q \leq l} B_q^g(M, \partial M),\]

where

\[(21)\quad B_q^g(M, \partial M) := \left\{ \sum_{i=1}^{p+q} \alpha_i \delta_{a_i}, \quad a_i \in \hat{M}, \ p = 1, \ldots, p, \ a_i \in \partial M \ i = p + 1, \ldots, p + q, \right.\
\left. \alpha_i \geq 0 \ i = 1, \ldots, p + q, \quad \text{and} \ \sum_{i=1}^{p+q} \alpha_i = k \right\}.\]

Furthermore, we define

\[(22)\quad c_n^{k-1} = \dim H_n(B_{k-1}^g(M)), \quad n = 1, \ldots, 4k - 5,\]

where \(H_n(B_{k-1}^g(M))\) denotes the \(n\)-th homology group of \(B_{k-1}^g(M)\) with \(\mathbb{Z}_2\) coefficients and \(\dim H_n(B_{k-1}^g(M))\) is its dimension. For \((p,q) \in \mathbb{N}^2\) such that \(2p + q = k\), we say

\[(23)\quad (ND)_{p,q} \quad \text{holds if} \ \mathcal{F}_{p,q} \ \text{is a Morse function and for every} \ A \in \text{Crit}(\mathcal{F}_{p,q}), \ \mathcal{L}_K(A) \neq 0.\]

Finally, we say

\[(24)\quad (ND) \quad \text{holds if} \ (ND)_{p,q} \ \text{holds for every} \ (p,q) \in \mathbb{N}^2 : 2p + q = k.\]

Now, we are ready to state our main results, starting from those which follow from our strong Morse type inequalities. To do so, we start with the critical case, namely when \(k = 1\).

**Theorem 1.1** Let \((M, g)\) be a compact four-dimensional Riemannian manifold with boundary such that \(\text{Ker}P_{g}^{4,3} \simeq \mathbb{R}, \ \text{H}_g = 0\) and \(\kappa_{(p^2 + p^2)} = 4\pi^2\). Assuming that \(K\) is a smooth positive function on \(M\) such that \((ND)\) holds and the following system of non negative integers \((n_i)\)

\[
\begin{align*}
m_0^q &= 1 + n_0, \\
m_i^1 &= n_i + n_{i-1}, \quad i = 1, \ldots, 3, \\
0 &= n_4 \\
n_i &\geq 0, \quad i = 0, \ldots, 3
\end{align*}
\]

has no solutions, then \(K\) is the \(Q\)-curvature of a Riemannian metric conformal to \(g\) with zero \(T\)-curvature and vanishing mean curvature on \(\partial M\).
As a corollary of the above theorem, we derive the following Hopf-Poincaré index type criterion for existence.

**Corollary 1.2** Let \((M, g)\) be a compact four-dimensional Riemannian manifold with boundary such that \(\text{Ker} P^4_g \simeq \mathbb{R},\) \(H_g = 0,\) and \(\kappa(P^4_g, P^3_g) = 4\pi^2.\) Assuming that \(K\) is a smooth positive function on \(M\) such that \((ND)\) holds and

\[
\sum_{A \in F_\infty} (-1)^{i_\infty(A)} \neq 1,
\]

then \(K\) is the \(Q\)-curvature of a Riemannian metric conformal to \(g\) with zero \(T\)-curvature and vanishing mean curvature on \(\partial M.\)

As a by product of our analysis, we extend the above Hopf-Poincaré index criterium to include the case where the total sum equals 1 but a partial one is not, provided that there is a jump in the Morse indices of the function \(F_0.\) Namely we have the following criterium:

**Theorem 1.3** Let \((M, g)\) be a compact four-dimensional Riemannian manifold with boundary such that \(\text{Ker} P^4_g \simeq \mathbb{R},\) \(H_g = 0,\) and \(\kappa(P^4_g) = 4\pi^2.\) Assuming that \(K\) is a smooth positive function on \(M\) such that \((ND)\) holds and there exists a positive integer \(1 \leq l \leq 3\) and \(A_l \in F_\infty\) such that

\[
\sum_{A \in F_\infty, i_\infty(A) \leq l-1} (-1)^{i_\infty(A)} \neq 1
\]

and

\[
\forall A \in F_\infty, \quad i_\infty(A) \neq l,
\]

then \(K\) is the \(Q\)-curvature of a Riemannian metric conformal to \(g\) with zero \(T\)-curvature and vanishing mean curvature on \(\partial M.\)

Next, we state our main result in the supercritical case, namely when \(k \geq 2.\) It read as follows:

**Theorem 1.4** Let \((M, g)\) be a compact four-dimensional Riemannian manifold with boundary such that \(\text{Ker} P^4_g \simeq \mathbb{R},\) \(H_g = 0,\) and \(\kappa(P^4_g) = 4k\pi^2\) and \(k \in \mathbb{N}\) with \(k \geq 2.\) Assuming that \(K\) is a smooth positive function on \(M\) such that \((ND)\) holds and the following system of non negative integers \((n_i)\)

\[
\begin{align*}
m_0^k &= n_0, \\
m_1^k &= n_0 + n_1, \\
m_i^k &= c_{i-1}^{k-1} + n_i + n_{i-1}, & i = 2, \ldots, 4k - 4, \\
m_i^k &= n_i + n_{i-1}, & i = 4k - 3, \ldots, 4k - 1, \\
0 &= n_{4k-1}, \\
n_i &\geq 0, & i = 0, \ldots, 4k - 1,
\end{align*}
\]

has no solutions, then \(K\) is the \(Q\)-curvature of a Riemannian metric conformal to \(g\) with zero \(T\)-curvature and vanishing mean curvature on \(\partial M.\)

**Remark 1.5** The dimension of the homology groups of the boundary-weighted barycenters \(B_\alpha^\partial(M),\) namely

\[
c_n^\partial := \dim H_n(B_\alpha^\partial(M))
\]

are computed in Theorem 5.12 of Section 5 of this paper.

As a corollary of Theorem 1.4, we derive the following Hopf-Poincaré index type criterion for existence.
Corollary 1.6 Let \((M, g)\) be a compact four-dimensional Riemannian manifold with boundary such that \(\text{Ker}P_{g}^{4,3} \simeq \mathbb{R}, \ H_{g} = 0, \) and \(\kappa_{P} = 4k\pi^{2}\) and \(k \in \mathbb{N} \) with \(k \geq 2\). Assume that \(K\) is a smooth positive function on \(M\) such that \((ND)\) holds and

\[
\sum_{A \in \mathcal{F}_{\infty}} (-1)^{i_{\infty}(A)} \neq 1 - \chi(B_{k-1}^{0}(M)),
\]

where \(\chi(B_{k-1}^{0}(M))\) stands for the Euler Characteristic of the space of boundary-barycenters of order \(k-1\). Then \(K\) is the \(Q\)-curvature of a Riemannian metric conformal to \(g\) with zero \(T\)-curvature and vanishing mean curvature on \(\partial M\).

Remark 1.7 \(\chi(B_{k-1}^{0}(M))\) the Euler Characteristic of the space of boundary weighted-barycenters of order \(k-1\) of \(M\) is computed in Theorem 5.1 of Section 5 of this paper.

Just as in the one mass case, we generalize the above criterium to the case where there is a jump in the indices of the functions \(F_{p,q}\) for all \((p, q) \in \mathbb{N}^{2}\) such that \(2p + q = k\). Namely we prove the following:

Theorem 1.8 Let \((M, g)\) be a compact four-dimensional Riemannian manifold with boundary such that \(\text{Ker}P_{g}^{4,3} \simeq \mathbb{R}, \ H_{g} = 0, \) and \(\kappa_{P} = 4k\pi^{2}\) and \(k \in \mathbb{N} \) with \(k \geq 2\). Assuming that \(K\) is a smooth positive function on \(M\) satisfying the non degeneracy condition \((ND)\) and there exists a positive integer \(1 \leq l \leq 4k-1\) and \(A^{l} \in \mathcal{F}_{\infty}\) with \(i_{\infty}(A^{l}) \leq l-1\) such that

\[
\sum_{A \in \mathcal{F}_{\infty}, \ i_{\infty}(A) \leq l-1} (-1)^{i_{\infty}(A)} \neq 1 - \chi(B_{k-1}^{0}(M)),
\]

and

\[
\forall A \in \mathcal{F}_{\infty}, \quad i_{\infty}(A) \neq l,
\]

then \(K\) is the \(Q\)-curvature of a Riemannian metric conformal to \(g\) with zero \(T\)-curvature and vanishing mean curvature on \(\partial M\).

Taking advantage from the precise knowledge of the location of critical points at infinity, we put condition on the function \(K\) to insure that some subcritical approximations of the critical case do not blow up and hence leading to the following existence result:

Theorem 1.9 Let \((M, g)\) be a compact four-dimensional Riemannian manifold with boundary such that \(\text{Ker}P_{g}^{4,3} \simeq \mathbb{R}, \ H_{g} = 0, \) and \(\kappa_{P} = 4k\pi^{2}\) and \(k \in \mathbb{N} \) with \(k \geq 2\). Assuming that \(K\) is a smooth positive function on \(M\) such that for every maximum point \(a\) of \(\mathcal{F}_{0,1}\), we have \(\frac{\partial K}{\partial n_{g}}(a) > 0\), then \(K\) is the \(Q\)-curvature of a Riemannian metric conformal to \(g\) with zero \(T\)-curvature, vanishing mean curvature on \(\partial M\), and conformal factor minimizing the Paneitz functional \(II\).

In the special case of the half-sphere, the above statement reads as follows:

Corollary 1.10 Let \((S_{+}^{4}, g_{S_{+}^{4}})\) be the standard four-dimensional hemisphere, \(K : S_{+}^{4} \longrightarrow \mathbb{R}_{+}\) a smooth positive function and \(\hat{K} := K|_{S_{+}^{4}} : S_{+}^{3} \longrightarrow \mathbb{R}_{+}\) its restriction on \(S_{+}^{3}\). Assuming that \(\frac{\partial \hat{K}}{\partial n_{g_{S_{+}^{4}}}}(a) > 0\) for every maximum point \(a\) of \(\hat{K}\), then \(\hat{K}\) is the \(Q\)-curvature of a Riemannian metric conformal to \(g_{S_{+}^{4}}\) with zero \(T\)-curvature, vanishing mean curvature on \(S_{+}^{3}\), and conformal factor minimizing the Paneitz functional \(II\).

Next, we present a new type of existence results based on the use of spectral information to rule out the blow up of some supercritical approximations... It read as follows:
Theorem 1.11 Let $(M,g)$ be a compact four-dimensional Riemannian manifold with boundary such hat $\text{Ker}P_g^{4,3}\simeq \mathbb{R}$, $H_g = 0$, $\kappa(p^4,\rho^3) = 4k\pi^2$, and $k \in \mathbb{N}$ with $k \geq 2$. Assuming that $K$ is a smooth positive function on $M$ satisfying the non degeneracy (ND) such that at every local minimum of $A$ of $\mathcal{F}_{0,k}$ we have that $\mathcal{L}_K(A) < 0$, then $K$ is the $Q$-curvature of a Riemannian metric conformal to $g$ with zero $T$-curvature and vanishing mean curvature on $\partial M$.

Remark 1.12 The above theorem has a counterpart in the closed case which improves Corollary 1.4 of [55] by dropping the assumption on critical points of index one. However the proof, which is rather analytic, differs drastically from the algebraic topological argument of [55].

The remainder of this paper is organized as follows. In section 2 we fix the notation used in the paper, give some useful preliminary results. Throughout

The first and the third author (M.A & C-B.N) have been supported by the DFG project "Fourth-order uniformization type theorems for 4-dimensional Riemannian manifolds".

2 Notation and preliminaries

In this section, we fix our notation and give some useful preliminary results. Throughout $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{N}^*$ stands for the set of positive integers, and for $n \in \mathbb{N}^*$, $\mathbb{R}^n$ the standard $n$-dimensional Euclidean space, $\mathbb{R}_{+}^n$ the open positive half-space of $\mathbb{R}^n$, and $\mathbb{R}_{+}^n$ its closure.

For $n \in \mathbb{N}^*$ and $r > 0$, $B_{0}^n(r)$ denotes the open ball of $\mathbb{R}^n$ of center 0 and radius $r$, $\bar{B}_{0}^n(r)$ for its closure, $B_{0}^{n+1}(r) := B_{0}^n(r) \cap \mathbb{R}_{+}^{n+1}$, and $B_{0}^{n+1}(r) := \bar{B}_{0}^n(r) \cap \mathbb{R}_{+}^{n+1}$. $S^n$ denotes the unit sphere of $\mathbb{R}^{n+1}$, and $S_{+}^n$ the positive spherical cap, namely $S_{+}^n := S^n \cap \mathbb{R}_{+}^{n+1}$. For a Riemannian metric $\bar{g}$ on $M$ and $\bar{g}$ its induced metric on $\partial M$, the ball $B_{\bar{g}}(r)$ (respectively $B_{\bar{g}}(r)$) is with respect to the normal geodesic (respectively Fermi) coordinates and similarly $d_{\bar{g}}(x,y)$ denotes the geodesic distance if $x \in M$ (respectively the distance inherited from the Fermi coordinates). Moreover $inj_{\bar{g}}(M)$ (respectively $inj_{\bar{g}}(M)$) stands for the injectivity radius. $dV_{\bar{g}}$ denotes the Riemannian measure associated to the metric $\bar{g}$, and $dS_{\bar{g}}$ the volume form on $\partial M$ with respect to the metric induced by $\bar{g}$ on $\partial M$.

We will write $\text{Diag}(M)$ the diagonal subspace of $M^2 = M \times M$.

In this paper, $(M,g)$ always refers to a given underlying compact four-dimensional Riemannian manifold with boundary $\partial M$ and interior $M$, and $K : M \rightarrow \mathbb{R}$ a smooth function. Furthermore we assume that:

\[
\ker P_g^{4,3} \simeq \mathbb{R}, \quad \kappa(p^4,\rho^3) = 4k\pi^2 \text{ for some } k \in \mathbb{N}^*, \text{ and } K > 0 \text{ on } M.
\]

Now, we recall $G$ the Green’s function of the operator $(P_g^4(\cdot) + \frac{4}{k}Q_g, P_g^3(\cdot) + \frac{2}{k}T_g)$ with homogeneous Neumann boundary condition satisfying the normalization

\[
\int_M G(x,y)Q_g(y)dV_g(y) + \int_{\partial M} G(x,y)T_g(y)dS_g(y) = 0, \quad \forall x \in M.
\]
For $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, $\beta \in [0,1]$, $L^p(M)$ and $L^p(\partial M)$, $W^{k,p}(M)$, $C^k(M)$, and $C^{k,\beta}(M)$ stand respectively for the standard $p$-Lebesgue space on $M$ and $\partial M$, $(k, p)$-Sobolev space, $k$-continuously differentiable space and $k$-continuously differential space of Hölder exponent $\beta$, all with respect to $g$ (if the definition needs a metric structure) and for precise definitions and properties, see for example [3] or [34]. Given a function $u : M \to \mathbb{R}$ such that $u \in L^1(M)$ and $u \in L^1(\partial M)$, we define $\overline{\pi}_{(Q,T)}$ as follows

$$\overline{\pi}_{(Q,T)} := \frac{1}{4k\pi^2} \left( \int_M u Q_g dV_g + \int_{\partial M} u T_g dS_g \right).$$

Given a generic Riemannian metric $\tilde{g}$ on $M$ and a function $F(x, y)$ defined on an open subset of $M^2$ which is symmetric and with $F(\cdot, \cdot) \in C^2$ with respect to $\tilde{g}$, we define

$$\frac{\partial F(a, a)}{\partial a} := \frac{\partial F(x, a)}{\partial x} \big|_{x=a} = \frac{\partial F(a, y)}{\partial y} \big|_{y=a},$$

and $\Delta_{\tilde{g}} F(a_1, a_2) := \Delta_{\tilde{g},x} F(x, a_2) |_{x=a_1} = \Delta_{\tilde{g},y} F(a_2, y) |_{y=a_1}$. Similarly, for a function $F(x, y)$ defined on an open subset of $(\partial M)^2$ which is symmetric and with $F(\cdot, \cdot) \in C^1$ with respect to $\tilde{g}$, we define

$$\frac{\partial F(a, a)}{\partial a} := \frac{\partial F(x, a)}{\partial x} \big|_{x=a} = \frac{\partial F(a, y)}{\partial y} \big|_{y=a}.$$ 

For $l \in \mathbb{N}^*$ and $a \in M$, $O_{l}(1)$ stands for quantities bounded uniformly in $a$, $O_{l}(1)$ stands for quantities bounded uniformly in $l$ and $O_{l}(1)$ stands for quantities which tends to 0 as $l \to +\infty$. For $\epsilon$ positive and small, $a \in M$ and $\lambda \in \mathbb{R}$, large $\lambda \geq \frac{1}{2}$, $O_{A,\lambda}(1)$ stands for quantities bounded uniformly in $a$, $\lambda$, and $\epsilon$. Similarly for $\epsilon$ positive and small, $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $2p + q = k$, $\lambda := (\lambda_1, \ldots, \lambda_{p+q}) \in (\mathbb{R}^+)^{p+q}$, $\lambda_i \geq \frac{1}{2}$ for $i = 1, \ldots, p + q$, and $A := (a_1, \ldots, a_{p+q}) \in M^p \times (\partial M)^q$ (where $(\mathbb{R}^+)^q$ denotes the cartesian product of $p + q$ copies of $\mathbb{R}_+$, and the convention that $M^p \times (\partial M)^q := M^p$ and $M^0 \times (\partial M)^q = (\partial M)^q$ is used), $O_{\lambda, A, \epsilon}(1)$ stands for quantities bounded uniformly in $A$, $\lambda$, and $\epsilon$. Similarly for $\epsilon$ positive and small, $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $2p + q = k$, $\lambda := (\lambda_1, \ldots, \lambda_{p+q}) \in (\mathbb{R}^+)^{p+q}$, $\lambda_i \geq \frac{1}{2}$ for $i = 1, \ldots, p + q$, $\alpha := (\alpha_1, \ldots, \alpha_{p+q}) \in \mathbb{R}^{p+q}$, $\alpha_i$ close to 1 for $i = 1, \ldots, p + q$, and $\tilde{A} := (\tilde{a}_1, \ldots, \tilde{a}_{p+q}) \in M^p \times (\partial M)^q$ with still the same convention as above (where $\mathbb{R}^{p+q}$ denotes the cartesian product of $p + q$ copies of $\mathbb{R}$), $O_{\lambda, \tilde{A}, \epsilon}(1)$ will mean quantities bounded from above and below independent of $\lambda$, $\tilde{A}$, $\lambda$, and $\epsilon$. For $x \in \mathbb{R}$, we will use the notation $O(x)$ to mean $|x|O(1)$ where $O(1)$ will be specified in all the contexts where it is used. Large positive constants are usually denoted by $O$ and the value of $C$ is allowed to vary from formula to formula and also within the same line. Similarly small positive constants are also denoted by $o$ and their value may varies from formula to formula and also within the same line.

Now, for $(X, A)$ a pair of topological spaces and $q \in \mathbb{N}$, we denote by $H_q(X, A)$ its $q$-th relative homology group with $\mathbb{Z}_2$ coefficients. Here $H_q(X) = H_q(X, \emptyset)$. We use the notation $b_q(X)$ and $b_q(X, A)$ to denote the $q$-th betti number of $X$ and $(X, A)$ respectively. We will write $\chi(X)$ and $\chi(X, A)$ for the respective Euler characteristics of $X$ and $(X, A)$.

We call $k$ the number of negative eigenvalues (counted with multiplicity) of $P^3_{g}$. We point out that $k$ can be zero, but is always finite. If $k \geq 1$, then we will denote by $E \subset \mathcal{H}_{\overline{\pi}}$ the direct sum of the eigenspaces corresponding to the negative eigenvalues of $P^3_{g}$. The dimension of $E$ is of course $k$. On the other hand, we have the existence of an $L^2$-orthonormal basis of eigenfunctions $v_1, \ldots, v_k$ of $E$ satisfying

$$P^3_{g} v_i = \mu_i v_i \quad \forall \; i = 1 \cdots k,$$

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k < 0 < \mu_{k+1} \leq \cdots,$$

where $\mu_i$'s are the eigenvalues of $P^3_{g}$ counted with multiplicity. From the fact that $P^3_{g}$ is self-adjoint and annihilates constants, we have $E \subset \{ u \in \mathcal{H}_{\overline{\pi}} : \overline{\pi}_{(Q,T)} = 0 \}$. We define also the positive definite (on $\{ u \in \mathcal{H}_{\overline{\pi}} : \overline{\pi}_{(Q,T)} = 0 \}$) pseudo-differential operator $P^{3,+,\overline{\pi}}_{g}$ as follows

$$P^{3,+,\overline{\pi}}_{g} u = P^3_{g} u - 2 \sum_{i=1}^{k} \mu_i \left( \int_M u v_i dV_g \right) v_i.$$ 

Basically $P^{3,+,\overline{\pi}}_{g}$ is obtained from $P^3_{g}$ by reversing the sign of the negative eigenvalues and we extend the latter definition to $\overline{m} = 0$ for uniformity in the analysis and recall that in that case $P^{3,+,\overline{\pi}}_{g} = P^3_{g}$. 

10
Using $P_{g}^{4,3,+}$, we set for $t > 0$

\[(30)\]

\[I_{t}(u) := \langle P_{g}^{4,3,+}u, u \rangle > 2t \sum_{r=1}^{m} \mu_{r}(u^{r})^{2} + 4t \int_{M} Q_{y}udV_{g} + 4t \oint_{\partial M} T_{y}udS_{g} - 4\pi^{2}tk \ln \int_{M} Ke^{4u}dV_{g}, \]

\[u \in \mathcal{H}_{\frac{\alpha}{m}}, \]

with

\[(31)\]

\[u^{r} := \int_{M} uv_{r}dV_{g}, \quad r = 1, \cdots, \bar{k}, \quad u \in \mathcal{H}_{\frac{\alpha}{m}}. \]

Now, using (29) and (30), we obtain

\[(32)\]

\[I_{t}(u) := \langle P_{g}^{4,3}u, u \rangle > 2(t - 1) \sum_{r=1}^{m} \mu_{r}(u^{r})^{2} + 4t \int_{M} Q_{y}udV_{g} + 4t \oint_{\partial M} T_{y}udS_{g} - 4\pi^{2}tk \ln \int_{M} Ke^{4u}dV_{g}, \]

\[u \in \mathcal{H}_{\frac{\alpha}{m}}, \]

and hence $I = I_{1}$. Furthermore, using (31), we define

\[(33)\]

\[u^{-} = \sum_{r=1}^{m} u^{r}v_{r}. \]

We will use the notation $\langle \cdot, \cdot \rangle$ to denote the $L^{2}$ scalar product. On the other hand, it is easy to see that

\[(34)\]

\[\langle u, v \rangle_{P^{4,3}} := \langle P_{g}^{4,3}u, v \rangle, \quad u, v \in \{w \in \mathcal{H}_{\frac{\alpha}{m}} : \mathfrak{u}(Q,T) = 0\} \]

defines an inner product on $\{u \in \mathcal{H}_{\frac{\alpha}{m}} : \mathfrak{u}(Q,T) = 0\}$ which induces a norm equivalent to $W^{2,2}$-norm (on $\{u \in \mathcal{H}_{\frac{\alpha}{m}} : \mathfrak{u}(Q,T) = 0\}$) and denoted by

\[(35)\]

\[\|u\| := \sqrt{\langle u, u \rangle_{P^{4,3}}} \quad u \in \{w \in \mathcal{H}_{\frac{\alpha}{m}} : \mathfrak{u}(Q,T) = 0\}. \]

As above, in the general case, namely $\bar{k} \geq 0$, for $\epsilon$ small and positive, $\bar{\beta} := (\beta_{1}, \cdots, \beta_{\bar{m}}) \in \mathbb{R}^{m}$ with $\beta_{i}$ close to 0, $i = 1, \cdots, \bar{k}$ (where $\mathbb{R}^{\bar{k}}$ is the empty set when $\bar{k} = 0$), $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $2p + q = k$, $\lambda := (\lambda_{1}, \cdots, \lambda_{p+q}) \in (\mathbb{R}_{+})^{p+q}$, $\lambda_{i} \geq \frac{1}{4}$ for $i = 1, \cdots, p + q$, $\alpha := (\alpha_{1}, \cdots, \alpha_{p+q}) \in \mathbb{R}^{p+q}$, $\alpha_{i}$ close to 1 for $i = 1, \cdots, p + q$, and $A := (a_{1}, \cdots, a_{p+q}) \in (M)^{p} \times (\partial M)^{q}$, $w \in \mathcal{H}_{\frac{\alpha}{m}}$ with $||w||$ small, $O_{\alpha,A,\lambda,\beta,\epsilon}(1)$ will stand for quantities bounded independent of $\alpha, A, \lambda, \beta, \epsilon$, and $O_{\alpha,A,\lambda,\beta,\epsilon}(1)$ will stand for quantities bounded independent of $\alpha, A, \lambda, \beta, w$ and $\epsilon$.

In the sequel also, $II^c$ with $c \in \mathbb{R}$ will stand for $II^c := \{u \in \mathcal{H}_{\frac{\alpha}{m}} : II(u) \leq c\}$. Similarly also, given $c \in \mathbb{R}$, $II^c_\ell$ stands for $II^c_\ell := \{u \in \mathcal{H}_{\frac{\alpha}{m}} : II_\ell(u) \leq c\}$. We would like to emphasize that in some places in the literature, a different notation is used for the sublevel, precisely with the $c$ as subscript. Here we adopt the notation of P. Rabinowitz.

Given a point $b \in \mathbb{R}^4$ and a positive real number, we define $\delta_{b,\lambda}$ to be the standard bubble, namely

\[(36)\]

\[\delta_{b,\lambda}(y) := \ln \left( \frac{2\lambda}{1 + \lambda^{2}|y - b|^{2}} \right), \quad y \in \mathbb{R}^4. \]

The functions $\delta_{b,\lambda}$ verify the following equation

\[(37)\]

\[\Delta^{2}\delta_{b,\lambda} = 6e^{4\delta_{b,\lambda}} \quad \text{in} \quad \mathbb{R}^4. \]
Geometrically, equation (37) means that the metric \( g = e^{2b_\lambda} dx^2 \) (after pull-back by the stereographic projection) has constant \( Q \)-curvature equal to 3 (with \( dx^2 \) denoting the standard metric on \( \mathbb{R}^4 \)). Furthermore, if \( b \in \mathbb{R}^3 = \partial \mathbb{R}^4 \), then \( \delta b_\lambda \) satisfies

\[
\begin{cases}
\Delta^2 \delta b_\lambda = 6e^{4b_\lambda} & \text{in } \mathbb{R}^4_+,
\partial x_4 \Delta \delta b_\lambda = 0 & \text{on } \mathbb{R}^3,
\partial x_4 \delta b_\lambda = 0 & \text{on } \mathbb{R}^3,
\end{cases}
\]

with \( \mathbb{R}^4_+ = \mathbb{R}^3 \times \mathbb{R}_+ \) and a point \( x \in \mathbb{R}^4_+ \) has the following representation \( x = (x_1, \cdots, x_4) \). As above, equation (38) has also a geometric interpretation. Indeed, it is equivalent to the fact that the metric \( g = e^{2b_\lambda} dx^2 \) (after pull-back by the stereographic projection) has constant \( Q \)-curvature equal to 3, zero \( T \)-curvature and vanishing mean curvature (with \( dx^2 \) denoting the standard metric on \( \mathbb{R}^4_+ \)). Using the existence of conformal normal coordinates (see [43], [35], and [49]) and recalling that \( H_g = 0 \), then for every \( m \) large positive integer, we have that for \( a \in M \), there exists a function \( u_a \in C^\infty(M) \) such that the metric \( g_a = e^{2u_a} g \) verifies

\[
det g_a(x) = 1 + O_{a,x}((d g_a(x, a))^m) \quad \text{for } x \in B^g_a(g_a).
\]

with \( O_{a,x}(1) \) meaning bounded by a constant independent of \( a \) and \( x \), \( 0 < \varrho_a < \max(\frac{\ln j_a(M)}{10}, \frac{\ln j_a(\partial M)}{10}) \).

Moreover, we can take the family of functions \( g_a \) and \( g_a \) such that

\[
u_a(1) \text{ are } C^1 \text{ and } g_a \geq \varphi_0 > 0,
\]

for some small positive \( \varphi_0 \) satisfying \( \varphi_0 < \max(\frac{\ln j_a(M)}{10}, \frac{\ln j_a(\partial M)}{10}) \) and

\[
\|u_a\|_{C^4(M)} = O_a(1),
\quad \frac{1}{C} g \leq g_a \leq C g, \quad a \in M,
\]

\[
u_a(x) = O_a(d g_a(a, x) = O_a(d g(a, x)) \quad \text{for } x \in B^g_a(\varphi_0) \supset B_a(\varphi_0), \quad a \in M,
\]

\[
u_a(a) = 0, \quad a \in M \quad R_{g_a}(a) = 0, \quad a \in M \quad \text{and} \quad H_{g_a}(a) = 0 \quad a \in \partial M,
\]

for some large positive constant \( C \) independent of \( a \). For the meaning of \( O_a(1) \) in (41), see section 2. Furthermore, for \( a \in M \), using the scalar curvature equation satisfied by \( e^{-u_a} \), namely \( -\Delta_{g_a}(e^{-u_a}) + \frac{1}{6} R_{g_a} e^{-u_a} = \frac{1}{6} R_g(a) e^{-3u_a} \) in \( M \), and (39)-(41), it is easy to see that the following holds

\[
\Delta_{g_a}(e^{-u_a})(a) = -\frac{1}{6} R_g(a).
\]

Similarly, for \( a \in \partial M \), using the mean curvature equation satisfied by \( e^{-u_a} \), namely \( -\frac{\partial}{\partial n_{g_a}}(e^{-u_a}) + H_{g_a} e^{-u_a} = H_g(a) e^{-2u_a} \) on \( \partial M \), and (39)-(41), or just (41), it is easy to see that the following holds

\[
\frac{\partial}{\partial n_{g_a}}(e^{-u_a})(a) = 0.
\]

For \( a \in M \), and \( r > 0 \), we set

\[
\begin{align*}
\exp^n_a := \exp^{g_a} \quad \text{and} \quad B^n_a(r) := B^{g_a}_a(r).
\end{align*}
\]

On the other hand, using the properties of \( g_a \) (see (39)-(41))), it is easy to check that for every \( u \in C^2(B^n_a(r)) \) with \( 0 < \varrho < \frac{\varphi_0}{2} \) there holds

\[
\begin{align*}
\nabla_{g_a} u(a) = \nabla_{g} u(a) = \nabla_4 \tilde{u}(0), \quad \Delta_{g_a} u(a) = \Delta_4 \tilde{u}(0), \quad \text{if } d_{g_a}(a, \partial M) \geq 4 \varrho,
\quad \nabla_{g_a} u(a) = \nabla_{g} u(a) = \nabla_3 \tilde{u}(0), \quad \frac{\partial}{\partial n_{g_a}} u(a) = \frac{\partial}{\partial n_{g}} u(a) = \partial_{x_4} \tilde{u}(0), \quad \text{if } a \in \partial M,
\end{align*}
\]

12
\[ (46) \quad \hat{u}(y) = u(\exp^a(y)), \; y \in B^4_0(\theta) \text{ if } d_{g_a}(a, \partial M) \geq 4\theta \text{ and } y \in B^4_\theta(\theta) \text{ if } a \in \partial M, \]

with
\[ \hat{g}_a := g_a|_{\partial M} \quad \text{and} \quad \hat{g} := g|_{\partial M}. \]

Now for \( 0 < \theta < \frac{\theta_0}{4} \), we define the cut-off function \( \chi_\theta : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying the following properties:
\[ (47) \quad \left\{ \begin{array}{ll}
\chi_\theta(t) = t & \text{for } t \in [0, \theta], \\
\chi_\theta(t) = 2\theta & \text{for } t \geq 2\theta, \\
\chi_\theta(t) \in [\theta, 2\theta] & \text{for } t \in [\theta, 2\theta]. 
\end{array} \right. \]

Using the cut-off function \( \chi_\theta \), we define for \( a \in M \) and \( \lambda \in \mathbb{R}_+ \) the function \( \hat{\delta}_{a, \lambda} \) as follows
\[ (48) \quad \hat{\delta}_{a, \lambda}(x) := \ln \left( \frac{2\lambda}{1 + \lambda^2 \chi_\theta^3(d_{g_a}(x, a))} \right). \]

Next, using the pull-back standard bubbles, namely (48), for \( a \in M, \; d_{g_a}(a, \partial M) \geq 4C\theta \), and \( \lambda \in \mathbb{R}_+ \), we define \( \varphi_{a, \lambda} \) to be the solution of the following projected boundary value problem
\[ (49) \quad \left\{ \begin{array}{ll}
P^4_g \varphi_{a, \lambda} + \frac{4}{k} Q_g = 16\pi^2 \frac{e^4(\hat{\delta}_{a, \lambda} + u_a)}{\int_M e^4(\hat{\delta}_{a, \lambda} + u_a) dV_g} & \text{in } \hat{M}, \\
P^3_g \varphi_{a, \lambda} + \frac{2}{k} T_g = 0 & \text{on } \partial M, \\
\frac{\partial \varphi_{a, \lambda}}{\partial n_g} = 0 & \text{on } \partial M, \\
\varphi_{a, \lambda}(Q, T) = 0 = 0. 
\end{array} \right. \]

Similarly, for \( a \in \partial M, \; \lambda \in \mathbb{R}_+ \), we define \( \varphi_{a, \lambda} \) to be the solution of the following projected boundary value problem
\[ (50) \quad \left\{ \begin{array}{ll}
P^4_g \varphi_{a, \lambda} + \frac{2}{k} Q_g = 8\pi^2 \frac{e^4(\hat{\delta}_{a, \lambda} + u_a)}{\int_M e^4(\hat{\delta}_{a, \lambda} + u_a) dV_g} & \text{in } \hat{M}, \\
P^3_g \varphi_{a, \lambda} + \frac{1}{k} T_g = 0 & \text{on } \partial M, \\
\frac{\partial \varphi_{a, \lambda}}{\partial n_g} = 0 & \text{on } \partial M, \\
\varphi_{a, \lambda}(Q, T) = 0 = 0. 
\end{array} \right. \]

So differentiating with respect to \( \lambda \) and \( a \) (respectively) the relation \( \varphi_{a, \lambda}(Q, T) = 0 \), we get (respectively)
\[ (51) \quad \frac{\partial \varphi_{a, \lambda}(x)}{\partial \lambda} \bigg|_{(Q, T)} = 0, \]

and
\[ (52) \quad \frac{\partial \varphi_{a, \lambda}(x)}{\partial a} \bigg|_{(Q, T)} = 0. \]
Now, we recall that \( G \) is the unique solution of the following BVP

\[
\begin{align*}
P^A_3 G(a,\cdot) + \frac{4}{k} Q_g &= 16\pi^2 \delta_a(\cdot) \quad \text{in } \hat{M}, \\
\left(\frac{2}{k} - 1\right) G(a,\cdot) &= 0 \quad \text{on } \partial M, \\
\frac{\partial G(a,\cdot)}{\partial n_g} &= 0 \quad \text{on } \partial M, \\
G(a,\cdot)_{(Q,T)} &= 0.
\end{align*}
\]

Using (53), it is easy to see that the following integral representation formula holds

\[
u(x) - \hat{\nu}(Q,T) = \frac{1}{16\pi^2} \left( \int_{\hat{M}} G(x,y) P_g u(y) dV_g(y) + 2 \int_{\partial M} G(x,y) P_g^3 u(y) dS_g(y) \right), \quad u \in C^4(M), \quad x \in M,
\]

where \( \hat{\nu}(Q,T) \) is defined as in Section 2. It is a well-known fact that \( G \) has a logarithmic singularity. In fact \( G \) decomposes as follows

\[
G(a,x) = S(a,x) + H(a,x),
\]

where

\[
S(x,y) = \begin{cases} 
\ln \left( \frac{1}{\lambda^2(d_{ya}(a,x))} \right) & \text{if } B^\circ_2(a) \cap \partial M = \emptyset, \\
\ln \left( \frac{1}{\lambda^2(d_{ya}(a,x))} \right) + \ln \left( \frac{1}{\lambda^2(d_{ya}(a,x))} \right) & \text{otherwise},
\end{cases}
\]

with \( \bar{x} \) denoting the normal reflection of \( x \) through \( \partial M \) with respect to \( g_a \) and

\[
G \in C^\infty(M^2 \setminus \text{Diag}(M)), \quad \text{and } H \text{ extend to a } C^{3,\beta}(M^2) \text{ function, } \forall \beta \in (0,1).
\]

Now, using (6), (7), and (12), (13) combined with the symmetry of \( H \), it is easy to see that for every \( (p,q) \in \mathbb{N}^2 \) such that \( 2p + q = k \), there holds

\[
\frac{\partial F_{p,q}(a_1,\cdots,a_{p+q})}{\partial a_i} = \frac{\nabla_g F_{i}^A(a_i)}{F_{i}^A(a_i)} \quad i = 1,\cdots,p.
\]

and

\[
\frac{\partial F_{p,q}(a_1,\cdots,a_{p+q})}{\partial a_i} = \frac{\nabla_g F_{i}^A(a_i)}{2F_{i}^A(a_i)} \quad i = p+1,\cdots,p+q.
\]

Next, for \( (p,q) \in \mathbb{N}^2 \) such that \( 2p + q = k \) and \( A \in (\hat{M}^p)^* \times ((\partial M)^q)^* \), we set

\[
l_K(A) := \begin{cases} 
\sum_{i=1}^{p} \left( \frac{\Delta_{g_a} F_{i}^A(a_i)}{2F_{i}^A(a_i)} - \frac{2}{3} R_g(a_i) \frac{F_{i}^A(a_i)}{4F_{i}^A(a_i)} \right), & \text{if } q = 0, \\
\sum_{i=p+1}^{p+q} \frac{1}{4(F_{i}^A)^*(a_i)} \frac{\partial F_{i}^A}{\partial g_a}(a_i), & \text{if } q \neq 0,
\end{cases}
\]

and use the properties of the metrics \( g_a, i = 1,\cdots,p \), the transformation rule of the conformal Laplacian under conformal change of metrics and direct calculations, to get

\[
l_K(A) = 4\mathcal{L}_K(A), \quad \forall A \in \text{Crit}(F_{p,q}).
\]

For \( k \geq 2 \) and \( k \geq 1 \), we define \( A^{\theta}_{k-1,k} \) to be the colimit of the diagram

14
There exists a blowing up sequence of solutions to the following type:

\[
\begin{align*}
 P_g^4 u_l + 2t_l Q_g &= 2t_l Ke^{4u_l} \quad \text{in } \bar{M}, \\
P_g^3 u_l + t_l T_g &= 0 \quad \text{on } \partial M, \\
\frac{\partial u_l}{\partial n_g} &= 0 \quad \text{on } \partial M,
\end{align*}
\]

(62) where

(63) \( t_l \to 1 \) as \( l \to +\infty \),

the following lemma holds.

**Lemma 2.1** Assuming that \((u_l)\) is a sequence of solutions of (62) with \( t_l \) satisfying (63), then there exists \((p, q) \in \mathbb{N}^2 \) with \( p + q \in \mathbb{N}^+ \) such that up to a subsequence, there exists \( p + q \) converging sequences of points \((x_{i,l})_{i \in \mathbb{N}}\) with limits \( x_i, i = 1, \cdots, p + q, x_i \in M \) for \( i = 1, \cdots, p, x_i \in \partial M \) for \( i = p + 1, \cdots, p + q, \) \( p + q \) sequences \((\mu_{i,l})_{i \in \mathbb{N}}\) \( i = 1, \cdots, p + q \) of positive real numbers converging to 0 such that the following hold:

a) \( \frac{d_g(x_{i,l}, x_{j,l})}{\mu_{i,l}} \to +\infty \) as \( l \to +\infty \) \( i \neq j, i, j = 1, \cdots, p + q, \)

(64)

\[
t_l K(x_i) \mu_{i,l}^4 e^{4u_l(x_i)} e^{-4\ln 2} = 3, \quad i = 1, \cdots, p + q.
\]

b)

For \( i = 1, \cdots, p, \)

(65) \[ v_{i,l}(x) = u_l(\exp x, (\mu_{i,l} x)) - u_l(x_i) + \ln 2 \to V_0(x) := \ln \left( \frac{2}{1 + |x|^2} \right) \quad \text{in } C^4_{\text{loc}}(\mathbb{R}^4) \text{ as } l \to +\infty, \]

and for \( i = p + 1, \cdots, p + q, \)

(65) \[ v_{i,l}(x) = u_l(\exp x, (\mu_{i,l} x)) - u_l(x_i) + \ln 2 \to V_0(x) := \ln \left( \frac{2}{1 + |x|^2} \right) \quad \text{in } C^4_{\text{loc}}(\mathbb{R}^4) \text{ as } l \to +\infty. \]

c)

There exists \( C > 0 \) such that

(66) \[ \inf_{i=1,\ldots,p+q} d_g(x_{i,l}, x)^4 e^{4u_l(x)} \leq C \quad \forall x \in M, \quad \forall l \in \mathbb{N}. \]
\[ d) \]
\[
t_{l}Ke^{4u_{l}}dV_{g} \rightarrow 8\pi^{2} \sum_{i=1}^{p} \delta_{x_{i}} + 4\pi^{2} \sum_{i=p+1}^{p+q} \delta_{x_{i}} \quad \text{in the sense of measure as } l \rightarrow +\infty,
\]
\[
(67) \quad \lim_{l \rightarrow +\infty} \int_{M} t_{l}Ke^{4u_{l}}dV_{g} = 4(2p+q)\pi^{2}.
\]
\[ e) \]
\[
u_{l} - \overline{\nu}_{l}(Q,T) \rightarrow \sum_{i=1}^{p} G(x_{i},\cdot) + \frac{1}{2} \sum_{i=p+1}^{p+q} G(x_{i},\cdot) \quad \text{in } C_{\text{loc}}^{1}(M \setminus \{x_{1}, \ldots, x_{p+q}\}),
\]
\[
(68) \quad \overline{\nu}_{l}(Q,T) \rightarrow -\infty \quad \text{as } l \rightarrow +\infty.
\]

### 3 Blow up analysis and deformation lemma

In this section, we derive a Bahri-Lucia type deformation lemma which is a refined version of the classical Bahri-Lucia deformation Lemma. Indeed from the works of Bahri[7], [8] and Lucia[45], we have the following deformation Lemma which will improve using refined blow-up analysis.

**Lemma 3.1** Assuming that \(a, b \in \mathbb{R}\) such that \(a < b\) and there is no critical values of II in \([a, b]\), then there are two possibilities

1) Either

\[ II^{a} \text{ is a deformation retract of } II^{b}. \]

2) Or there exists a sequence \(t_{l} \rightarrow 1\) as \(l \rightarrow +\infty\) and a sequence of critical point \(u_{l}\) of \(II_{t_{l}}\) verifying

\[ a \leq II(u_{l}) \leq b \quad \text{for all } l \in \mathbb{N}. \]

Actually the proof of above Lemma provides a pseudogradient whose the noncompact \(\omega-\text{limit set}\) (i.e. the endpoints of noncompact orbits) is in one to one correspondence to the blow up set of some approximation of type (62). Hence for a better understanding of such noncompact orbits, one has to describe the behavior of the blowing up solutions near their blow up point. To that aim we prove the following Lemma:

**Proposition 3.2** Assuming that \((u_{l})_{l \in \mathbb{N}}\) is a bubbling sequence of solutions to (62) with \(t_{l}\) satisfying (63), then up to a subsequence and keeping the notation in Lemma 2.1, we have that the points \(x_{i,l}\) are uniformly isolated, and the scaling parameters \(\lambda_{i,l} := \mu_{i,l}^{-1}\) are comparable, namely there exists \(0 < \eta_{k} < \frac{90}{10}\) and \(0 < \varrho_{k} < \frac{90}{10}\) two positive and small real numbers, where \(g_{0}\) is as in (40), and a large positive constant \(\Lambda_{k}\) such that for \(l\) large enough, there holds

\[
d_{g}(x_{i,l}, x_{j,l}) \geq 4\overline{C}\eta_{k}, \quad \forall \; i \neq j = 1, \ldots, p + q, \quad \text{and} \quad \Lambda_{k}^{-1} \lambda_{j,l} \leq \lambda_{i,l} \leq \Lambda_{k} \lambda_{j,l}, \quad \forall \; i, j = 1, \ldots, p + q.
\]

(69)

Furthermore, the interior concentrations (if any) are uniformly far from \(\partial M\), namely if \(p > 0\), then for \(l\) large enough, there holds

\[
d_{g}(x_{i,l}, \partial M) \geq 4\overline{C}\varrho_{k}, \quad \forall \; i = 1, \ldots, p.
\]

(70)

Moreover, we have that the following estimate around the blow-up points holds (for \(l\) large enough)

\[
u_{l}(x) + \frac{1}{4} \ln \frac{t_{l}K(x_{i})}{3} = \ln \frac{2\lambda_{i,l}}{1 + \lambda_{i,l}^{2}(d_{g_{\varrho_{k}}}(x, x_{i}))^{2}} + O((d_{g_{\varrho_{k}}}(x, x_{i}))), \quad \forall \; x \in B_{\varrho_{k}}^{\ast}(\eta_{k}), \; i = 1, \ldots, p + q,
\]

(71)

where \(g_{\varrho_{k}}, \varrho, \text{ and } \overline{C}\) are as in Section 2, see (39) - (41).
Now, before proving Proposition 3.2, we would like first to show how it provides the refined Bahri-Lucia type deformation Lemma that we mentioned above. To do so, we start by defining the neighborhood of potential critical points at infinity of $II$ and for that we first fix $\Lambda > \Lambda_k$ to be a large positive constant. Next, for $(p,q) \in \mathbb{N}^2$ with $2p + q = k$, $\epsilon$, $\varrho$ and $\eta$ small positive real numbers with $0 < \varrho < \varrho_k$, and $0 < \eta < \eta_k$, we define the $(p,q,\epsilon,\varrho,\eta)$-neighborhood of potential critical points at infinity of $II$ as follows

$$V(p,q,\epsilon,\varrho,\eta) := \{ u \in H_{\text{loc}}(\mathbb{R}^n) : a_1, \ldots, a_p \in M, \quad a_{p+1}, \ldots, a_{p+q} \in \partial M, \quad \lambda_1, \ldots, \lambda_{p+q} > 0, \quad \| u - \overline{u}_{(Q,T)} - \sum_{i=1}^{p+q} \varphi_{a_i, \lambda_i} \| + \| \nabla^{p+3} II(u - \overline{u}_{(Q,T)}) \| = O\left( \sum_{i=1}^{p+q} \frac{1}{\lambda_i} \right) \}$$

(72)

where $\overline{C}$ is as in (41), $\Lambda_k$, $\varrho_k$, and $\eta_k$ are given by Proposition 3.2. Let $u_1$ be a sequence of blow-up critical point of $II_l$ with $(\overline{u}_l)_l = 0$, $l \in \mathbb{N}$ and $l_t \rightarrow 1$ as $l \rightarrow +\infty$, then there exists $(p,q) \in \mathbb{N}^2$ with $2p + q = k$ and $l_{\epsilon,\varrho,\eta}$ a large positive integer such that for every $l \geq l_{\epsilon,\varrho,\eta}$, we have $u_l \in V(p,q,\epsilon,\varrho,\eta)$.

On the other hand, as in [1] and by the same arguments, we have that Lemma 3.1 and Lemma 3.3 implies the following refined version of the classical Bahri-Lucia deformation lemma.

Lemma 3.4 Assuming that $\epsilon, \varrho, \eta$ are small positive real numbers with $0 < \varrho < \varrho_k$, $0 < \eta < \eta_k$, where $\varrho_k$ and $\eta_k$ are given by Proposition 3.2, then for $a,b \in \mathbb{R}$ such that $a < b$, we have that if there is no critical values of $II$ in $[a,b]$, then there are two possibilities:

1) Either

$$II^a$$

is a deformation retract of $II^b$.

2) Or there exists a sequence $t_l \rightarrow 1$ as $l \rightarrow +\infty$ and a sequence of critical point $u_l$ of $II_l$ verifying $a \leq II(u_l) \leq b$ for all $l \in \mathbb{N}^*$, $(p,q) \in \mathbb{N}^2$ with $2p + q = k$, and $l_{\epsilon,\varrho,\eta}$ a large positive integer such that $u_l \in V(p,q,\epsilon,\varrho,\eta)$ for all $l \geq l_{\epsilon,\varrho,\eta}$.

Next we come back to the proof of Proposition 3.2 and for that we are going to divide the remainder of this section into three subsections. In the first one, we show that the blow-up points are uniformly isolated and the interior concentration points (if any) are uniformly far from $\partial M$. In the second one, we prove that the Proposition 3.2 holds with the classical form of the sup+inf estimate (71), precisely (for non experts) with $O(\|d_{a_i}(x,x_i)\|)$ replaced just by $O(1)$. The last subsection deals with the full version of sup+inf estimate (71).

### 3.1 Blow up points are isolated and interior ones are far from the boundary

As already mentioned above, in this subsection, we show that the blow up points are uniformly isolated and that the interior blow-up points (if any) are uniformly far from $\partial M$. Precisely, we prove the following Lemma:

Lemma 3.5 Assuming that $(u_l)_l \in \mathbb{N}$ is a bubbling sequence of solutions to BVP (62) with $t_l$ satisfying (63), then keeping the notations in Lemma 2.1, we have that the points $x_{i,l}$ are uniformly isolated, namely there exists $0 < \eta_k < \frac{\varrho}{\eta}$ (where $\varrho_0$ is as in (40)) such that for $l$ large enough, there holds

$$d_{a}(x_{i,j},x_{j,j}) \geq 4\overline{C}\eta_k, \quad \forall i \neq j = 1, \ldots, p + q.$$
Furthermore, there exists $0 < \rho_k < \frac{\mu}{4}$ (where $\rho_0$ is as in \eqref{eq:40}) such that if $p > 0$, then for $l$ large enough, there holds
\begin{equation}
    d_g(x_{i,l}, \partial M) \geq 4C\rho_k, \quad \forall i = 1, \cdots, p.
\end{equation}

**Proof.** We are going to use the method of \cite{56} to prove Lemma and hence we will be sketchy in many arguments. As in \cite{56}, we first fix $1 < \nu < 2$, and for $i = 1, \cdots, p + q$, we set
\begin{equation}
    \bar{u}_{i,l}(r) = \text{Vol}_g(\partial B_{x_i}(r))^{-1} \int_{\partial B_{x_i}(r)} u_l(x) d\sigma_g(x), \quad \forall 0 \leq r < \text{inj}_g(M),
\end{equation}
and
\begin{equation}
    \psi_{i,l}(r) = r^{4\nu} \exp(4\bar{u}_{i,l}(r)), \quad \forall 0 \leq r < \text{inj}_g(M).
\end{equation}
Furthermore, as in \cite{56}, we define $r_{i,l}$ as follows
\begin{equation}
    r_{i,l} := \sup\left\{ R_{\nu,\mu_i,l} \leq r \leq \frac{R_{i,l}}{2} \text{ such that } \psi'_{i,l}(r) < 0 \text{ in } [R_{\nu,\mu_i,l}, r]\right\};
\end{equation}
where $R_{i,l} := \min_{j \neq i} d_g(x_{i,l}, x_{j,l})$. Thus, by continuity and the definition of $r_{i,l}$, we have that
\begin{equation}
    \psi'_{i,l}(r_{i,l}) = 0.
\end{equation}
Now, as in \cite{56}, to prove \eqref{eq:73}, it suffices to show that $r_{i,l}$ is bounded below by a positive constant in dependent of $l$. Thus, we assume by contradiction that (up to a subsequence) $r_{i,l} \rightarrow 0$ as $l \rightarrow +\infty$ and look for a contradiction. In order to do that, we use the integral representation formula for $(P^4_g, P^3_g)$ under homogeneous Neumann boundary condition and the integral method of \cite{54}, to derive the following estimate
\begin{equation}
    \psi'_{i,l}(r_{i,l}) \leq (r_{i,l})^{4\nu-1} \exp(\bar{u}_{i,l}(r_{i,l})) (4\nu - 8C + o_l(1) + O_l(r_{i,l})),
\end{equation}
with $C > 1$. So from $1 < \nu < 2$, $C > 1$ and $r_{i,l} \rightarrow 0$ as $l \rightarrow +\infty$, we deduce that for $l$ large enough, there holds
\begin{equation}
    \psi'_{i,l}(r_{i,l}) < 0.
\end{equation}
Thus, \eqref{eq:76} and \eqref{eq:77} lead to a contradiction, thereby concluding the proof of \eqref{eq:73}. Hence, the proof of the lemma is complete, since clary \eqref{eq:73} implies \eqref{eq:74}.

### 3.2 Harnack-type inequality around blow-up points

In this subsection, we present the weak form of Proposition 3.2 that we mentioned above, namely we show that the difference of a bubbling sequence of solutions to BVP \eqref{eq:62} with $t_l$ satisfying \eqref{eq:63} and the pull back bubble around a blow-up point is a $O(1)$. Indeed, we will prove the following Lemma:

**Lemma 3.6** Assuming that $(u_l)_{l \in \mathbb{N}}$ is a bubbling sequence of solutions to BVP \eqref{eq:62} with $t_l$ satisfying \eqref{eq:63}, then keeping the notations in Lemma 2.1 and Lemma 3.5, we have that for $l$ large enough, there holds
\begin{equation}
    u_l(x) + \frac{1}{4} \ln \frac{t_l K(x_i)}{3} = \ln \left(1 + \frac{2\lambda_{i,l}}{1 + \lambda_{i,l}^2 (d_{g_{x_i}}(x, x_i))^2}\right) + O(1), \quad \forall x \in B_{x_i}(\eta_k),
\end{equation}
up to choosing $\eta_k$ smaller than in Lemma 3.5.

**Remark 3.7** We point out that the comparability of the scaling parameters $\lambda_{i,l}$’s follows directly from Lemma 3.6.
On the other hand, using the conformal covariance properties of the Paneitz operator and of the Chang-Qing one, see (1), we have that $\hat{u}_l := u_l - \hat{w}$ satisfies
\begin{align*}
P_g^4\hat{u}_l + 2\hat{Q}_l = 2t_l Ke^{4\hat{u}_l} & \quad \text{in } \tilde{M}, \\
P_g^3\hat{u}_l + \hat{T}_l = 0 & \quad \text{on } \partial M, \\
\frac{\partial \hat{u}_l}{\partial n_{\tilde{g}}} = 0 & \quad \text{on } \partial M.
\end{align*}
(79)

Using standard elliptic regularity theory and (41), we derive
\begin{equation}
\hat{w}(y) = O(d_g(y,x)) \quad \text{in } B_{\tilde{g}}^3(2\eta_1).
\end{equation}
(80)

On the other hand, using the conformal covariance properties of the Paneitz operator and of the Chang-Qing one, we get, by
\begin{equation}
\frac{\partial \hat{w}}{\partial n_{\tilde{g}}} = 0 \quad \text{on } \partial M,
\end{equation}
(81)

where $\hat{Q}_l = t_l e^{-4\hat{u}}Q_g + \frac{1}{2}P_g^4\hat{w}$ and $\hat{T}_l = t_l e^{-3\hat{u}}T_g + P_g^3\hat{w}$.

Next, as in [56], we are going to establish the classical sup-inf-estimate for $\hat{u}_l$ (and even the full version which will be done in the next Lemma), since thanks (80) all terms coming from $\hat{w}$ can be absorbed on the right hand side of (78). Now, we are going to rescale the functions $\hat{u}_l$ around the points $x_\ldots$. In order to do that, we define $\varphi_l : B_{0,\eta_1}^{\mu_1} \to B_{\tilde{g}}^3(2\eta_1)$ if $x \in M$, and $\varphi_l : B_{0}^{\eta_1} \to B_{\tilde{g}}^3(2\eta_1)$ if $x \in \partial M$ by the formula $\varphi_l(z) := \mu_1 z$ and $\mu_1$ is the corresponding scaling parameter given by Lemma 2.1. Furthermore, as in [56], we define the following rescaling of $\hat{u}_l$
\begin{equation}
v_l := \hat{u}_l \circ \varphi_l + \ln \mu_1 + \frac{1}{4} \ln \frac{t_l K(x)}{3}.
\end{equation}
(82)

Using the Green’s representation formula for $(P_g^4, P_g^3)$ under homogenous Neumann boundary condition with respect to $\tilde{g}$, the method of [54] and standard doubling argument to deal with the situation $x \in \partial M$, we get, by
\begin{equation}
v_l(z) + 2 \ln |z| = O(1), \quad \text{for } z \in \tilde{B}_{0}^{\eta_1} \setminus B_{0}^{\eta_1}(-\ln \mu_1),
\end{equation}
(83)
in case of interior blow-up, and in case of boundary blow-up, we obtain
\begin{equation}
v_l(z) + 2 \ln |z| = O(1), \quad \text{for } z \in \tilde{B}_{0}^{\eta_1} \setminus B_{0}^{\eta_1}(-\ln \mu_1).
\end{equation}
(84)

Now, we are going to show that the estimate (81) holds also in $B_{0}^{\eta_1}(-\ln \mu_1)$ (in case of interior blow-up), and that the estimate (82) holds in $B_{0}^{\eta_1}(-\ln \mu_1)$ as well (in case of boundary blow-up). To do so, we use Lemma 2.1, the same arguments as in [56], and standard doubling argument (when $x \in \partial M$), to get
\begin{equation}
v_l(z) + 2 \ln |z| = O(1), \quad \text{for } z \in B_{0}^{\eta_1}(-\ln \mu_1), \quad \text{if } x \in M.
\end{equation}
(85)
and

\( v_l(z) + 2 \ln |z| = O(1), \) for \( z \in \bar{B}_0^{\mathbb{R}^4}(-\ln \mu_l), \) if \( x \in \partial M. \)

Now, combining (81) and (83) when \( x \in M, \) and (82) and (84) when \( x \in \partial M, \) we obtain

\( v_l(z) + 2 \ln |z| = O(1), \) for \( z \in \bar{B}_0^{\mathbb{R}^4}(\frac{\eta}{\mu_l}), \) if \( x \in \hat{M}. \)

\( v_l(z) + 2 \ln |z| = O(1), \) for \( z \in \bar{B}_0^{\mathbb{R}^4}(\frac{\mu}{\mu_l}), \) if \( x \in \partial M. \)

Thus scaling back, namely using \( y = \mu z \) and the definition of \( v_l, \) we obtain the desired \( O(1)-\)estimate. Hence the proof of the Lemma is complete.

### 3.3 Refined estimate around blow-up points

As already mentioned above, in this subsection, we show formula (71). Precisely, we prove the following Lemma:

**Lemma 3.8** Assuming that \((u_l)_{l \in \mathbb{N}}\) is a bubbling sequence of solutions to BVP (62) with \( t_i \) satisfying (63), then keeping the notations in Lemma 2.1, Lemma 3.5, and Lemma 3.6, we have that the following estimate holds (l large enough)

\[
u_l(x) + \frac{1}{4} \ln \frac{t_i}{3} = \ln \frac{2\lambda_{i,t}}{1 + \lambda_{i,t}^2(d_{g_{x_i}}(x,x_i))^2} + O(d_{g_{x_i}}(x,x_i)), \quad \forall x \in B_{\eta i}(x_i).
\]

**Proof.** We are going to use the method of [56], hence we will be sketchy in many arguments. Now, let \( V_0 \) be the unique solution of the following conformally invariant integral equation

\[V_0(z) = \frac{3}{4\pi^2} \int_{\mathbb{R}^4} \ln \frac{|y|}{|z - y|} e^{V_0(y)} dy + \ln 2, \quad V_0(0) = \ln 2, \quad \nabla V_0(0) = 0.\]

Next, we set \( w_l(z) = v_l(z) - V_0(z) \) for \( z \in B_0^{\mathbb{R}^4}(\eta_1 \mu_l^{-1}) \) when \( x \in M, \) and \( w_l(z) = v_l(z) - V_0(z) \) for \( z \in \bar{B}_0^{\mathbb{R}^4}(\eta_1 \mu_l^{-1}) \) when \( x \in \partial M, \) and use Lemma 3.6 to infer that

\( |w_l| \leq C \) in \( B_0^{\mathbb{R}^4}(\eta_1 \mu_l^{-1}) \) if \( x \in \hat{M}. \)

and

\( |w_l| \leq C \) in \( B_0^{\mathbb{R}^4}(\eta_1 \mu_l^{-1}) \) if \( x \in \partial M. \)

On the other hand, it is easy to see that to achieve our goal, it is sufficient to show

\( |w_l| \leq C \mu_l |z| \) in \( B_0^{\mathbb{R}^4}(\eta_1 \mu_l^{-1}) \), if \( x \in \hat{M}. \)

and

\( |w_l| \leq C \mu_l |z| \) in \( B_0^{\mathbb{R}^4}(\eta_1 \mu_l^{-1}) \), if \( x \in \partial M. \)

To show (89) and (90), we first set

\[A_l := \max_{z \in \Omega_l} \frac{|w_l(z)|}{\mu_l(1 + |z|)}\]

with

\[
\Omega_l = B_0^{\mathbb{R}^4}(\eta_1 \mu_l^{-1}), \quad \text{if} \quad x \in \hat{M}.
\]
and

\[ \Omega_l = \overline{B}^m_{\eta l}(\eta_l \mu_l^{-1}), \quad \text{if } x \in \partial M. \]

We remark that to show (89) and (90), it is equivalent to prove that \( \Lambda_l \) is bounded. Now, let us suppose that \( \Lambda_l \to +\infty \) as \( l \to +\infty \), and look for a contradiction. To do so, we will use the method of [54] combined with a standard doubling argument. For this, we first choose a sequence of points \( z_l \in \Omega_l \) such that \( \Lambda_l = \frac{|w_l(z_l)|}{\mu_l(1+|z_l|)} \). Next, up to a subsequence, we have that either \( z_l \to z^* \) as \( l \to +\infty \) (with \( z^* \in \mathbb{R}^4 \)) or \( |z_l| \to +\infty \) as \( l \to +\infty \). Now, we make the following definition

\[ \bar{w}_l(z) := \frac{w_l(z)}{\Lambda_l \mu_l(1+|z_l|)}, \]

and have

\[ |\bar{w}_l(z)| \leq \left( \frac{1+|z|}{1+|z_l|} \right), \]

and

\[ |\bar{w}_l(z_l)| = 1. \]

Now, we consider the case where the points \( z_l \) escape to infinity.

**Case 1: \( |z_l| \to +\infty \)**

In this case, using (87), (88), the Green’s representation formula for \( (P_\hat{g}, P_\hat{g}) \) under homogeneous Neumann boundary condition with respect to \( \hat{g} \), and the method of [56], we obtain

\[ \bar{w}_l(z_l) = \frac{3}{4\pi^2} \int_{\Omega_l} \ln \left( \frac{|\xi|}{|z_l - \xi|} \right) \left( \frac{O(1)(1+|\xi|)^{-7}}{(1+|z_l|)} + \frac{O(1)(1+|\xi|)^{-7}}{\Lambda_l(1+|z_l|)} \right) d\xi + o(1). \]

Now, using the fact that \( |z_l| \to +\infty \) as \( l \to +\infty \), one can easily check that

\[ \bar{w}_l(z_l) = \frac{3}{4\pi^2} \int_{\Omega_l} \ln \left( \frac{|\xi|}{|z_l - \xi|} \right) \left( \frac{O(1)(1+|\xi|)^{-7}}{(1+|z_l|)} + \frac{O(1)(1+|\xi|)^{-7}}{\Lambda_l(1+|z_l|)} \right) d\xi + o(1). \]

Hence, we reach a contradiction to (92).

Now, we are going to show that, when the points \( z_l \to z^* \) as \( l \to +\infty \), we reach a contradiction as well.

**Case 2: \( z_l \to z^* \)**

In this case, using the assumption \( z_l \to z^* \), the Green’s representation formula for \( (P_\hat{g}, P_\hat{g}) \) under homogeneous Neumann boundary condition with respect to \( \hat{g} \), and the method of [56], we obtain that up to a subsequence

\[ \bar{w}_l \to w \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^4) \quad \text{as } l \to +\infty, \quad \text{if } x \in \hat{M}, \]

(93)

\[ \bar{w}_l \to w \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^4) \quad \text{as } l \to +\infty, \quad \partial_4 w = 0 \quad \text{on } \mathbb{R}^3, \quad \text{if } x \in \partial M, \]

(94)

and

\[ \bar{w}_l(z) = \frac{3}{\pi^2} \int_{\Omega_l} \ln \left( \frac{|\xi|}{|z - \xi|} \right) \frac{K \circ \varphi_l(\xi)}{K \circ \varphi_l(0)} e^{A_4h_l(\xi)} \bar{w}(\xi) d\xi + \frac{3}{\Lambda_l \mu_l(1+|z_l|) \pi^2} \int_{\Omega_l} \ln \left( \frac{|\xi|}{|z - \xi|} \right) O(\mu_l(1+|\xi|)^{-7}) d\xi \]

\[ + \frac{O(1)+O(|z|)}{\Lambda_l(1+|z|)}. \]

(95)
where $e^{4\theta} := \int_0^1 e^{4(su + (1-s)v_0)} ds$. Thus, appealing to (93), (94), and (95), we infer that $w$ satisfies
\begin{equation}
(96) \quad w(z) = \frac{3}{\pi^2} \int_{\mathbb{R}^4} \ln \frac{|\xi|}{|z - \xi|} e^{4V_0(\xi)} w(\xi) d\xi, \quad \text{if} \quad x \in \partial M,
\end{equation}
and
\begin{equation}
(97) \quad w(z) = \frac{3}{\pi^2} \int_{\mathbb{R}^4} \ln \frac{|\xi|}{|z - \xi|} e^{4V_0(\xi)} w(\xi) d\xi, \quad \partial_x w = 0 \text{ on } \mathbb{R}^3 \quad \text{if} \quad x \in \partial M.
\end{equation}

Now, using (91), we have that $w$ satisfies the following asymptotic
\begin{equation}
(98) \quad |w(z)| \leq C(1 + |z|).
\end{equation}

On the other hand, from the definition of $v_l$, it is easy to see that
\begin{equation}
(99) \quad w(0) = 0, \quad \text{and} \quad \nabla w(0) = 0.
\end{equation}

So, using (96)-(99), Lemma 3.7 in [56], and a standard doubling argument, we obtain
\[ w = 0. \]

However, from (92), we infer that $w$ satisfies also
\begin{equation}
(100) \quad |w(z^*)| = 1.
\end{equation}

So we reach a contradiction in the second case also. Hence the proof of the Lemma is complete. ■

**Proof of Proposition 3.2**

Proposition 3.2 follows directly from Lemma 3.5, Lemma 3.6, Remark 3.7, and Lemma 3.8. ■

### 4 A Morse lemma at infinity

In this section, we characterize the critical points at infinity of $II$ and establish a Morse type Lemma for them. To do so, we will divide this section into two subsections. In the first one, we perform a finite-dimensional Lyapunov-Schmidt type reduction. In the second one, we combine the latter finite-dimensional reduction and the construction of a suitable pseudogradient at infinity to achieve our goal. For all we will parameterize the neighborhood of potential critical points at infinity. Namely following the ideas of Bahri-Coron [11], and using Lemma 7.1 and Lemma 7.2, we have that for every $\varrho$ and $\eta$ small positive real numbers with $0 < \varrho < \varrho_k$, and $0 < \eta < \eta_k$ where $\varrho_k$ and $\eta_k$ are given by Proposition 3.2, there exists $\epsilon_k = \epsilon_k(\varrho, \eta) > 0$ such that for every $(p, q) \in \mathbb{N}^2$ with $2p + q = k$, there holds
\begin{equation}
\forall \ 0 < \epsilon \leq \epsilon_k, \ \forall u \in V(p, q, \epsilon, \varrho, \eta), \text{the minimization problem}
\end{equation}
\begin{equation}
(101) \quad \min_{B^{p,q}_{\epsilon,\varrho,\eta}} \|u - \bar{u}_{Q,T} - \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} - \sum_{r=1}^{\bar{k}} \beta_r (v_r - [v_r])_{(Q,T)}\|
\end{equation}
has a unique solution, up to permutations, where $B^{p,q}_{\epsilon,\varrho,\eta}$ is defined as follows
\begin{equation}
(102) \quad B^{p,q}_{\epsilon,\varrho,\eta} := \{(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in \mathbb{R}^{p+q} \times \mathbb{M}^p \times (\partial M)^q \times (0, +\infty)^{p+q} \times \mathbb{R}^{\bar{k}} : |\alpha_i - 1| \leq \epsilon, \lambda_i \geq \frac{1}{\epsilon}, i = 1, \cdots, p + q, \ d_g(a_i, a_j) \geq 4\mathcal{C}_{\bar{\eta}}, i \neq j = 1, \cdots, p + q, \ d_g(a_i, \partial M) \geq 4\mathcal{C}_{\bar{\varrho}}, \ |\beta_r| \leq R, r = 1, \cdots, \bar{k}\}.
\end{equation}
Moreover, using the solution of (101), we have that every \( u \in V(p, q, \epsilon, \varrho, \eta) \) can be written as

\[
(103) \quad u - \overline{u}_{(Q, T)} = \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^k \beta_r (v_r - \overline{v_r}(Q, T)) + w,
\]

where \( w \) verifies the following orthogonality conditions

\[
(104) \quad \overline{u}_{(Q, T)} = \varphi_{a_i, \lambda_i}, w > p+3 = < \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_i}, w > p+3 = < \frac{\partial \varphi_{a_i, \lambda_i}}{\partial a_i}, w > p+3 = < v_r, w > = 0, i = 1, \ldots, p + q, r = 1, \ldots, k
\]

and the estimate

\[
(105) \quad ||w|| = O \left( \frac{1}{\sum_{i=1}^{p+q} \frac{1}{\lambda_i}} \right),
\]

where here \( O(1) := O_{\alpha, A, \lambda, \beta, w, \epsilon}(1) \), and for the meaning of \( O_{\alpha, A, \lambda, \beta, w, \epsilon}(1) \), see Section 2. Furthermore, the concentration points \( a_i \), the masses \( \alpha_i \), the concentrating parameters \( \lambda_i \) and the negativity parameter \( \beta_r \) in (103) verify also

\[
(106) \quad d_g(a_i, a_j) \geq 4C\eta \text{ for } i \neq j = 1, \ldots, p + q, d_g(a_i, \partial M) \geq 4C\eta \text{ for } i = p + 1, \ldots, p + q,
\]

\[
\frac{1}{\overline{\lambda}} \leq \frac{\lambda_i}{\lambda_j} \leq \overline{\lambda} \text{ for } i, j = 1, \ldots, p + q, \lambda_i \geq \frac{1}{\epsilon} \text{ for } i = 1, \ldots, p + q,
\]

\[
\text{and } \sum_{r=1}^k |\beta_r| + \sum_{i=1}^{p+q} |\alpha_i| - 1 \cdot \sqrt{\ln \lambda_i} = O \left( \sum_{i=1}^{p+q} \frac{1}{\lambda_i} \right),
\]

with still \( O(1) \) as in (105).

### 4.1 Finite-dimensional reduction near infinity

In this sub-section, we perform a finite-dimensional Lyapunov-Schmidt type reduction by exploiting the stability property of the standard bubbles as in [1]. Indeed, in doing so, we first derive the following proposition...

**Proposition 4.1** Assuming that \((p, q) \in \mathbb{N}^2\) such that \(2p + q = k\), \(0 < \varrho < \varrho_k\), \(0 < \eta < \eta_k\), and \(0 < \epsilon \leq \epsilon_k\), where \(\varrho_k\) and \(\eta_k\) are given by Proposition 3.2, and \(\epsilon_k\) is given by (101), and \(u = \overline{u}_{(Q, T)} + \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^k \beta_r (v_r - \overline{v_r}(Q, T)) + w \in V(p, q, \epsilon, \varrho, \eta)\) with \(w\), the concentration points \(a_i\), the masses \(\alpha_i\), the concentrating parameters \(\lambda_i\) (\(i = 1, \ldots, p + q\)), and the negativity parameters \(\beta_r\) (\(r = 1, \ldots, k\)) verifying (104)-(106), then we have

\[
(107) \quad H(u) = H \left( \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^k \beta_r (v_r - \overline{v_r}(Q, T)) - f(w) + Q(w) + o(||w||^2) \right),
\]

where

\[
(108) \quad f(w) := 16k \pi^2 \frac{\int_M K e^4 \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + 4 \sum_{r=1}^k \beta_r v_r \omega dV_g}{\int_M K e^4 \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + 4 \sum_{r=1}^k \beta_r v_r dV_g},
\]

and

\[
(109) \quad Q(w) := ||w||^2 - 32k \pi^2 \frac{\int_M K e^4 \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + 4 \sum_{r=1}^k \beta_r v_r w^2 dV_g}{\int_M K e^4 \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + 4 \sum_{r=1}^k \beta_r v_r dV_g}.
\]
Moreover, setting

$$E_{a_i, \lambda_i} := \{ w \in H^1_0 : < \varphi_{a_i, \lambda_i}, w >_{p+2} = \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_i}, \lambda_i >_{p+2} = \frac{\partial \varphi_{a_i, \lambda_i}}{\partial a_i}, \lambda_i >_{p+2} = 0, \}$$

where

$$\overline{\nu}_{(Q,T)} = < v_k, w > = 0, \quad k = 1, \ldots, \bar{k} \quad \text{and} \quad ||w|| = O \left( \sum_{i=1}^{p+1} \frac{1}{\lambda_i} \right),$$

and

$$A := (a_1, \ldots, a_{p+q}), \quad \bar{\lambda} = (\lambda_1, \ldots, \lambda_{p+q}), \quad E_{A, \bar{\lambda}} := \cap_{i=1}^{p+q} \overline{E}_{a_i, \lambda_i},$$

we have that, the quadratic form $Q$ is positive definite in $E_{A, \bar{\lambda}}$. Furthermore, the linear part $f$ verifies that, for every $w \in E_{A, \bar{\lambda}}$, there holds

$$f(w) = O \left( ||w|| \left( \sum_{i=1}^{p+q} \frac{1}{\lambda_i^2} \right) + \sum_{i=p+1}^{p+q} \frac{1}{\lambda_i} + \sum_{i=1}^{p+q} \frac{1}{\lambda_i} \right).$$

where for $i = 1, \ldots, p+q$, $F_i^A$ is as in Lemma 7.11, $o(1) = o_{A, \bar{\lambda}, \lambda, w, e}(1), O(1) := O_{A, \bar{\lambda}, \lambda, w, e}(1)$, and for the meaning of $o_{A, \bar{\lambda}, \lambda, w, e}(1)$ and $O_{A, \bar{\lambda}, \lambda, w, e}(1)$, see Section 2.

**Proof.** It follows from Lemma 7.12, Lemma 7.1, Lemma 7.9, Lemma 7.10, and the same strategy as in the proof of Proposition 6.1 in [1].

As in [1], we have that Proposition 4.1 implies the following direct corollaries.

**Corollary 4.2** Assuming that $(p, q) \in \mathbb{N}^2$ such that $2p + q = k$, $0 < q < q_k$, $0 < \eta < \eta_k$, and $0 < \epsilon \leq \epsilon_k$, where $q_k$ and $\eta_k$ are given by Proposition 3.2, and $\epsilon_k$ is given by (101), and $u := \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{v_r} \beta_r (v_r - (v_r)_{(Q,T)})$ with the concentration points $a_i, \alpha_{i}, \beta_i$, the masses $\alpha_{i}^0$, the concentrating parameters $\lambda_i$, $i = 1, \ldots, p+q$ and the negativity parameters $\beta_r (r = 1, \ldots, \bar{k})$ satisfying (106), then there exists a unique $\bar{w}(\bar{a}, A, \bar{\lambda}, \bar{\beta}) \in E_{A, \bar{\lambda}}$ such that

$$II(u + \bar{w}(\bar{a}, A, \bar{\lambda}, \bar{\beta})) = \min_{w \in E_{A, \bar{\lambda}}, \mu + w \in V(p, q, e, \eta)} II(u + w),$$

where $\bar{a} := (a_1, \ldots, a_{p+q}), A := (a_1, \ldots, a_{p+q}), \bar{\lambda} := (\lambda_1, \ldots, \lambda_{p+q})$ and $\bar{\beta} := (\beta_1, \ldots, \beta_k)$. Furthermore, $(\bar{a}, A, \bar{\lambda}, \bar{\beta}) \rightarrow \bar{w}(\bar{a}, A, \bar{\lambda}, \bar{\beta})$ is $C^1$ and satisfies the following estimate

$$\frac{1}{C} \left| \bar{w}(\bar{a}, A, \bar{\lambda}, \bar{\beta}) \right|^2 \leq \left| f(\bar{w}(\bar{a}, A, \bar{\lambda}, \bar{\beta})) \right| \leq C \left| \bar{w}(\bar{a}, A, \bar{\lambda}, \bar{\beta}) \right|^2,$$

for some large positive constant $C$ independent of $\bar{a}$, $A$, $\bar{\lambda}$, $\bar{\beta}$, and $\epsilon$, hence

$$||\bar{w}(\bar{a}, A, \bar{\lambda}, \bar{\beta})|| = O \left( \sum_{i=1}^{p+q} \frac{1}{\lambda_i^2} + \sum_{r=1}^{v_r} \frac{1}{\lambda_i} + \sum_{i=p+1}^{p+q} \frac{1}{\lambda_i^2} + \sum_{r=1}^{v_r} \frac{1}{\lambda_i} \right),$$

where $O(1) := O_{A, \bar{\lambda}, \lambda, w, e}(1)$ and for its meaning see Section 2.

**Corollary 4.3** Assuming that $(p, q) \in \mathbb{N}^2$ such that $2p + q = k$, $0 < q < q_k$, $0 < \eta < \eta_k$, and $0 < \epsilon \leq \epsilon_k$, where $q_k$ and $\eta_k$ are given by Proposition 3.2, and $\epsilon_k$ is given by (101), and $u_0 := \sum_{i=1}^{p+q} \alpha_{i}^0 \varphi_{a_i, \lambda_i}^0 + \sum_{r=1}^{v_r} \beta_r^0 (v_r - (v_r)_{(Q,T)})$ with the concentration points $a_i^0, \alpha_{i}^0, \beta_r^0 (r = 1, \ldots, \bar{k})$ satisfying (106), then there exists an open neighborhood $U$ of $(\bar{a}^0, A^0, \bar{\lambda}^0, \delta^0)$ (with $\bar{a}^0 := (a_1^0, \ldots, a_{p+q}^0), A^0 := (a_1^0, \ldots, a_{p+q}^0), \bar{\lambda} := \bar{\lambda}^0$
Proposition 4.4 Assuing that $(p, q) \in \mathbb{N}^2$ such that $2p + q = k$, $0 < q < q_k$, $0 < \eta < \eta_k$, and $0 < \epsilon \leq \epsilon_k$, where $q_k$ and $\eta_k$ are given by Proposition 3.2, and $\epsilon_k$ is given by (7.11), then there exists a pseudogradient $W$ of $II(\bar{\alpha}, \lambda, \bar{\beta})$ such that

1) For every $u := \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{k} \beta_r (v_r - (v_r)_{(Q,T)}) \in V(p, q, \epsilon, \eta, \eta) + w \in V(p, q, \epsilon, \eta, \eta)$, we have that if $q = 0$, then there holds

\[
< -\nabla II(u), W > \geq c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i=1}^{p} \frac{|\nabla v F^A(a_i)|}{\lambda_i} + \sum_{i=1}^{p} |\alpha_i - 1| + \sum_{i=1}^{p} |\tau_i| + \sum_{r=1}^{k} |\beta_r| \right),
\]

Thus, as in [1], in $V(p, q, \epsilon, \eta, \eta)$ we have a splitting of the variables $(\bar{\alpha}, \lambda, \bar{\beta})$ and $V$ and one can decrease the functional $II$ in the variable $V$ without touching the variable $(\bar{\alpha}, \lambda, \bar{\beta})$ by considering just the flow

\[
\frac{dV}{dt} = -V.
\]

So, since $II$ is invariant by translations by constants, then the variational study of $II$ in $V(p, q, \epsilon, \eta, \eta)$ is equivalent to the one of the following finite-dimensional functional

\[
II(\bar{\alpha}, \lambda, \bar{\beta}) := II(\sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{k} \beta_r (v_r - (v_r)_{(Q,T)}) + \tilde{w}(\bar{\alpha}, \lambda, \bar{\beta})),
\]

where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{p+q})$, $\lambda = (\lambda_1, \ldots, \lambda_{p+q})$ and $\bar{\beta} = (\beta_1, \ldots, \beta_k)$ with the concentration points $a_i$, the masses $\alpha_i$, the concentrating parameters $\lambda_i$ $(i = 1, \ldots, p + q)$ and the negativity parameters $\beta_r$ $(r = 1, \ldots, k)$ satisfying (106), and $\tilde{w}(\bar{\alpha}, \lambda, \bar{\beta})$ is as in Corollary 4.2. Hence the goal of this subsection is achieved.

4.2 Construction of a pseudogradient near infinity

In this subsection, we construct a pseudogradient for the finite-dimensional functional $II(\bar{\alpha}, \lambda, \bar{\beta})$ given by (119), and use it to characterize the critical points at infinity of $II$ and establish a Morse type Lemma for them. Indeed, we have:

Proposition 4.4 Assuming that $(p, q) \in \mathbb{N}^2$ such that $2p + q = k$, $0 < q < q_k$, $0 < \eta < \eta_k$, and $0 < \epsilon \leq \epsilon_k$, where $q_k$ and $\eta_k$ are given by Proposition 3.2, and $\epsilon_k$ is given by (7.11), then there exists a pseudogradient $W$ of $II(\bar{\alpha}, \lambda, \bar{\beta})$ such that

1) For every $u := \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{k} \beta_r (v_r - (v_r)_{(Q,T)}) \in V(p, q, \epsilon, \eta, \eta)$ with the concentration points $a_i$, the masses $\alpha_i$, the concentrating parameters $\lambda_i$ $(i = 1, \ldots, p + q)$ and the negativity parameters $\beta_r$ $(r = 1, \ldots, k)$ satisfying (106), we have that if $q = 0$, then there holds

\[
< -\nabla II(u), W > \geq c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i=1}^{p} \frac{|\nabla v F^A(a_i)|}{\lambda_i} + \sum_{i=1}^{p} |\alpha_i - 1| + \sum_{i=1}^{p} |\tau_i| + \sum_{r=1}^{k} |\beta_r| \right),
\]

25
To move the concentration points we make use of the vector field:

\[ W = \sum a \cdot \nabla F_i(a) \]

We will provide the proof for the case \( k = 1 \), \( \bar{\alpha} = (\alpha_1, \cdots, \alpha_p) \), \( \bar{\lambda} = (\lambda_1, \cdots, \lambda_p) \), \( \bar{\beta} = (\beta_1, \cdots, \beta_k) \) and \( \epsilon \).

2) Furthermore, for every \( u := \sum \alpha_i \phi \alpha_i + \sum \beta_i (v_r - \nabla ) \) satisfies \( \mathcal{V}(p,q,e,q,\eta) \) with the concentration points \( \alpha_i \), the masses \( \alpha_i \), the concentration parameters \( \lambda_i \) \( \beta_i \) \( c_i \) satisfying \( (106) \), and \( \bar{\omega}(\bar{\alpha}, \bar{\lambda}, \bar{\beta}) \) is as in \( (113) \), we have that if \( q = 0 \), then there holds

\[ < -\nabla II(u), W > + \frac{\partial \bar{\omega}(W)}{\partial (\bar{\alpha}, \bar{\lambda}, \bar{\beta})} \geq c \left( \sum_{i=1}^{p+q} \frac{1}{\lambda_i} + \sum_{i=1}^{p} \frac{\nabla \phi \alpha_i}{\lambda_i} + \sum_{i=p+1}^{p+q} \frac{\nabla \phi \alpha_i}{\lambda_i} \right) \]

and if \( q \neq 0 \), then there holds

\[ < -\nabla II(u), W > + \frac{\partial \bar{\omega}(W)}{\partial (\bar{\alpha}, \bar{\lambda}, \bar{\beta})} \geq c \left( \sum_{i=1}^{p+q} \frac{1}{\lambda_i} + \sum_{i=1}^{p} \frac{\nabla \phi \alpha_i}{\lambda_i} + \sum_{i=p+1}^{p+q} \frac{\nabla \phi \alpha_i}{\lambda_i} \right) \]

where \( c \) is a small positive constant independent of \( A := (a_1, \cdots, a_{p+q}) \), \( \bar{\alpha} = (\alpha_1, \cdots, \alpha_{p+q}) \), \( \bar{\lambda} = (\lambda_1, \cdots, \lambda_{p+q}) \), \( \bar{\beta} = (\beta_1, \cdots, \beta_k) \) and \( \epsilon \).

3) Moreover, the pseudogradient \( W \) is a bounded vector field and the region where the concentration rates \( \lambda_i \)'s are not bounded along the flow lines of \( W \) are the one where \( A := (a_1, \cdots, a_{p+q}) \) converges along the flow lines of \( W \) to a critical point \( B \) of \( \mathcal{F}_{p,q} \) satisfying \( l_k(B) < 0 \).

Proof.

Case: \( q = 0 \)

In this case, it follows from \( (57) \), \( (59) \), \( (60) \), Lemma 7.11, Lemma 7.12, Corollary 7.13, Proposition 4.4 and the arguments to derive it go along with the ones in the proof of Proposition 8.1 in [1].

Case: \( q \neq 0 \)

We will provide the proof for the case \( k \geq 2 \). The case \( k = 1 \) can be treated similarly.

Let \( \zeta \) be a cut off function satisfying

\[ 0 \leq \zeta \leq 1, \quad \zeta(s) = 0 \text{ if } |s| \leq 1/2, \quad \zeta(s) = 1 \text{ if } |s| \geq 1. \]

The claimed global pseudogradient will be constructed as a convex combination of local ones. To derive these local pseudogadients, we make use of the following vector fields:

To move the concentration rates we will use of the vector fields

\[ W_{\lambda_i} := \frac{1}{\lambda_i} \frac{\partial \phi \alpha_i}{\partial \lambda_i}, \quad W_{\tau_i} := -\frac{\tau_i}{|\tau_i|} \zeta \left( \frac{\lambda_i}{\ln \lambda_i} |\tau_i| \right) \frac{1}{\lambda_i} \frac{\partial \phi \alpha_i}{\partial \lambda_i}. \]

To move the concentration points we make use of the vector field:

For \( i = 1, \cdots, p \)

\[ W_{\alpha_i} := \frac{\nabla \phi \alpha_i}{|\nabla \phi \alpha_i|} \zeta \left( \frac{\lambda_i}{\ln \lambda_i} |\nabla \phi \alpha_i| \right) \frac{1}{\lambda_i} \frac{\partial \phi \alpha_i}{\partial \alpha_i}. \]
and for \( i = p+1, \ldots, p+q \) we define

\[
W_{a_i} := \frac{\nabla_{\hat{g}} F_i^A(a_i)}{\left|\nabla_{\hat{g}} F_i^A(a_i)\right|} \left( \frac{\lambda_i}{\ln \lambda_i} \nabla_{\hat{g}} F_i^A(a_i) \right) \left( \frac{1}{\lambda_i} \frac{\partial \phi_i}{\partial a_i} - \frac{3}{8} \lambda_i \frac{\partial \phi_i}{\partial a_i} \right)
\]

We divide the set \( V(p,q,\varepsilon,\eta) \) into subsets. To that aim we define for \( C > 0 \) large to be chosen later the set

\[
V_1(p,q,\varepsilon) := \left\{ u \in V(p,q,\varepsilon,\eta), \text{ s.t. } \exists i \in \{1, \ldots, p+q\}; |\tau_i| \geq \frac{B}{2C} \lambda_i \right\}
\]

and setting \( F := \{ i \in \{1, \ldots, p+q\} \} \) such that \( |\tau_i| \geq \frac{B}{2C} \lambda_i \), we define in \( V_1(p,q,\varepsilon) \) the vector field

\[
W_1 := \sum_{i \in F} W_{\tau_i}.
\]

Using (7.12) and taking \( C \) large, we derive that for some positive constant \( c_1 \) there holds:

\[
< -\nabla I(\overline{\tau}), W_1 > \geq c_1 \left( \sum_{i \in F} \frac{1}{\lambda_i} \right),
\]

hence the claimed estimate holds in this set since all \( \lambda_i \) are comparable and according to Lemma (7.8) we have that \( \sum_i \tau_i = O\left( \sum_i \frac{1}{\lambda_i} \right) \).

Now for \( \delta \) a small positive number, we define the following subset

\[
V_2(p,q,\varepsilon) := \left\{ u \in V(p,q,\varepsilon,\eta), \text{ s.t. } \forall i \in \{1, \ldots, p+q\}; |\tau_i| \leq \frac{2C}{\lambda_i} \text{ and } \exists i \in \{1, \ldots, p\} \text{ such that } |\nabla_{\hat{g}} F_i^A(a_i)| \geq \delta \right\}
\]

Setting

\[
W_2 := \sum_{i=1}^{p+q} W_{a_i},
\]

we derive using (7.14) that for some positive constant \( c_2 \) we have that:

\[
< -\nabla I(\overline{\tau}), W_2 > \geq c_2 \left\{ \sum_{i=1}^{p} \frac{|\nabla_{\hat{g}} F_i^A(a_i)|}{\lambda_i} + \sum_{i=p+1}^{p+q} \frac{|\nabla_{\hat{g}} F_i^A(a_i)|}{\lambda_i} \right\},
\]

hence the claimed estimate holds also in this region.

Now we define the following region

\[
V_3(p,q,\varepsilon) := \left\{ u \in V(p,q,\varepsilon,\eta), \text{ s.t. } \forall i \in \{1, \ldots, p+q\}; |\tau_i| \leq \frac{2C}{\lambda_i} \text{ and } \forall i \in \{1, \ldots, p\} \text{ such that } |\nabla_{\hat{g}} F_i^A(a_i)| < 2\delta \right\}
\]

We observe that in this region the concentration points are in an arbitrarily small \( \delta \)-neighborhood of some critical point \( B \) of \( F_{p,q} \) and we subdivide it in two subsets

\[
V_3^{-}(p,q,\varepsilon) = \{ u \in V_3(p,q,\varepsilon) \text{ such that } \mathcal{L}_K(B) < 0, \}
\]

and

\[
V_3^{+}(p,q,\varepsilon) = \{ u \in V_3(p,q,\varepsilon) \text{ such that } \mathcal{L}_K(B) > 0, \}
\]

In the set \( V_3(p,q,\varepsilon) \) we define the vector field

\[
W_3 := -\text{sign}(\mathcal{L}_K(B)) \sum_{i=1}^{p+q} W_{\lambda_i}.
\]
Using (7.13) we derive that for some positive constants $c_3$ there holds
\[ < -\nabla II(\bar{u}, W_3) > \geq e \sum_{i=1}^{p+q} \frac{1}{\lambda_i}, \]
hence the claimed estimate holds also in this region and the global pseudogradient $W$ is a convex combination of $W_i$, $i = 1, 2, 3$.

The proof of the second claim (123) follows from (121) using the fact that
\[ \|\bar{w}\|^2 = o \left( \sum_{i=1}^{p+q} \frac{1}{\lambda_i} + \sum_{i=p+1}^{p+q} \frac{\|\nabla g F_i(a_i)\|}{\lambda_i} + \sum_{i=p+1}^{p+q} (\alpha_i - 1) + \sum_{i=1}^{p+q} |\tau_i| + \sum_{r=1}^{k} |\beta_r| \right) \]

Now we observe that the only region where the $\lambda_i$’s are not bounded along the flow lines of $W$ is the region $\mathcal{V}_3^-$ where the critical points converge to some critical point $B$ of $F_{p,q}$ such that $\mathcal{L}_K(B) < 0$.

Now we define the notion of critical points at infinity for $II$

**Definition 4.5** A critical point at infinity for $II$, with respect to the pseudogradient $W$ is an accumulation of some non compact orbits of $W$ such that the flow lines enter and remain for ever in some $V(p,q, \varepsilon_k, g, \eta)$ for $\varepsilon_k \to 0$.

As a corollary of Proposition 4.4, we derive the following characterization of the critical points at infinity of $II$.

**Corollary 4.6** 1) The critical points at infinity of $II$ are uniquely described by a number of interior masses $p, q \in \mathbb{N}$ and boundary masses $q \in \mathbb{N}$ with $2p + q = k$ and with respect to which they correspond to the "configurations" $\alpha_i = 1$, $\lambda_i = +\infty$, $\tau_i = 0$ $i = 1, \ldots, p + q$, $\beta_i = 0$, $r = 1, \ldots, k$, $A$ is a critical point of $F_{p,q}$ such that $\mathcal{L}_K(A) < 0$ and $V = 0$ and we denote them by $z^\infty$ with $z$ being the corresponding critical point of $F_{p,q}$.

2) The $II$-energy of a critical point at infinity $z^\infty$ denoted by $\mathcal{E}_{II}(z^\infty)$ is given by
\[ \mathcal{E}_{II}(z^\infty) = -\frac{20}{3} k \pi^2 - 4k \pi^2 \ln \left( \frac{k \pi^2}{6} \right) - 8 \pi^2 F_{p,q}(z_1, \ldots, z_{p+q}) \]
where $z = (z_1, \ldots, z_{p+q})$.

Furthermore, using Lemma 7.11, Proposition 4.4, (115), Corollary 4.3, and classical Morse lemma, we derive the following Morse type reduction near a critical point at infinity of $II$.

**Lemma 4.7** (Morse type reduction near infinity) Assuming that $(p, q) \in \mathbb{N}^2$ such that $2p + q = k$, $0 < \rho < \rho_k$, $0 < \eta < \eta_k$, and $0 < \epsilon \leq \epsilon_k$, where $\rho_k$ and $\eta_k$ are given by Proposition 3.2, and $\epsilon_k$ is given by (101), and $u_0 := \sum_{i=1}^{p+q} \alpha_i \varphi_{\alpha_i, \lambda_i} + \sum_{r=1}^{k} \beta_r (v_r - (v_r)(Q,T)) + \tilde{w}(\tilde{\alpha}^0, A^0, \bar{\lambda}^0, \beta^0) \in \mathcal{V}_3(p, q, \epsilon, \rho, \eta)$ (where $\tilde{\alpha}^0 := (\alpha_1, \ldots, \alpha_{p+q})$, $A^0 := (a_1, \ldots, a_{p+q})$, $\bar{\lambda} := (\lambda_1, \ldots, \lambda_{p+q})$ and $\beta^0 := (\beta_1, \ldots, \beta_k)$) with the concentration points $\alpha_i^0$, the masses $\alpha_i^0$, the concentrating parameters $\lambda_i^0$ ($i = 1, \ldots, p + q$) and the negativity parameters $\beta_i^0$ ($r = 1, \ldots, k$) satisfying (106) and furthermore $A^0 \in \text{Crit}(F_{p,q})$, then there exists an open neighborhood $U$ of $(\tilde{\alpha}^0, A^0, \bar{\lambda}^0, \beta^0)$ such that for every $(\tilde{\alpha}, A, \bar{\lambda}, \beta) \in U$ with $\tilde{\alpha} := (\alpha_1, \ldots, \alpha_{p+q})$, $A := (a_1, \ldots, a_{p+q})$, $\bar{\lambda} := (\lambda_1, \ldots, \lambda_{p+q})$, $\beta := (\beta_1, \ldots, \beta_k)$, and the $a_i$, the $\alpha_i$, the $\lambda_i$ ($i = 1, \ldots, p + q$) and the $\beta_r$ ($r = 1, \ldots, k$) satisfying (106), and $w$ satisfying (106) with $\sum_{i=1}^{p+q} \alpha_i \varphi_{\alpha_i, \lambda_i} + \sum_{r=1}^{k} \beta_r (v_r - (v_r)(Q,T)) + w \in \mathcal{V}_3(p, q, \epsilon, \rho, \eta)$, we have the
In this section will often drop writing $\mathbf{B}$ the usual barycenter spaces 
Before going into details in the investigation of the space of weighted -barycenters we point that, unlike stratum
We observe that it is a stratified subspace of the barycenter space
Furthermore we also define for $p,q,\epsilon$
This section is devoted to the investigation from algebraic topological viewpoint of the boundary-weighted barycenters

5 The boundary-weighted barycenters

Throughout section $M$ denotes a compact Riemannian manifold of dimension $m \geq 2$ with Boundary $\partial M$ and interior $M$.

5.1 The Euler characteristic of $B^\partial_p(M)$

Throughout this section and for the sake of simplicity, we normalize the sum weights to be 1 instead $k$ in the rest of the paper. For the sake of clarity we rewrite the definitions of the basic spaces under this new normalization. That is we define

$$B_{p,q}(M,\partial M) := \left\{ \sum_{i=1}^{p} \alpha_i a_i + \sum_{j=1}^{q} \beta_j b_j, \alpha_i, \beta_j \in [0,1], \ a_i \in \tilde{M}, b_j \in \partial M, \text{ and } \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j = 1 \right\}. $$

and observe that the set $B_{p,q}(M,\partial M)$ is a subspace of the space $B_{p+q}(M)$ of weighted barycenters of order $p+q$ and is endowed with the induced topology.

Furthermore we also define for $l \in \mathbb{N}^*$ the space of weighted-barycenters of order $l$ as

(129) $B^\partial_l(M) := \bigcup_{2p+q \leq l} B_{p,q}(M,\partial M).$

We observe that it is a stratified subspace of the barycenter space $B_l(M)$. Moreover the closure of each stratum $B_{p,q}(M,\partial M)$ in $B_{p+q}(M)$ is contained in $\bigcup B_{p-i,q+i}(M,\partial M)$ and we have inclusions $B^\partial_{l-1}(M) \subset B^\partial_l(M).$

In this section will often drop writing $M$ from the notation for simplicity.
Before going into details in the investigation of the space of weighted-barycenters we point that, unlike the usual barycenter spaces $B_l(M)$, the subspaces $B^\partial_l(M)$ are not homotopy invariant. In particular if $M$
is contractible, it is not the case in general that \( B_t^0(M) \) is also contractible. We will illustrate this with a simple non-trivial example which is the closed disk \( D^2 \) with connected boundary \( S^1 \). The space \( B_t^2(M) \) is made out of two points on boundary or a single point in interior. This has the topology of the union \( B_2(\partial M) \cup B_1(M) \), with \( B_2(\partial M) \cap B_1(M) = B_1(\partial M) = \partial M \) (see (133) for a schematic description). When \( M \) is a disk, \( B_t^2(D) \) is obtained by gluing a disk \( D = B_1(D) \) to the circle \( S^1 = B_1(\partial D) \) sitting inside \( B_2(S^1) \). It is well-known that \( B_2(S^1) \cong S^3 \) ([42], Corollary 1.4 (b)). Thus \( B_t^2(D) \) is obtained up to homotopy from \( S^3 \) by collapsing out a circle \( S^1 \). To know what this quotient \( S^3/S^1 \) is, we need understand the nature of the inclusion \( B_1(\partial M) = S^1 \hookrightarrow S^3 = B_2(\partial M) \) (this is by definition a knot). We don’t know the precise nature of this knot (we suspect it is the trefoil knot, see [50]) but since all we need is the homology, we can use Lefshetz duality

\[
\tilde{H}^*(S^3/S^1) \cong H_{3-*}(S^3 \setminus S^1)
\]

The homology of the complement of (any) knot is independent of the embedding and is given by

\[
H_1(S^3 \setminus S^1) \cong \mathbb{Z}, \quad H_i(S^3 \setminus S^1) = 0, \quad i > 1
\]

This calculation is obtained by analyzing the Mayer-Vietoris sequence of the union \( S^3 = (S^3 \setminus S^1) \cup T \) where \( T \) is a tubular neighborhood of \( S^1 \) homeomorphic to a torus. The upshot is that

\[
(130) \quad \tilde{H}^*(S^3/S^1) = \tilde{H}^*(B_t^2(D)) \cong \begin{cases} \mathbb{Z}, & * = 3 \\ \mathbb{Z}, & * = 2 \end{cases}
\]

and is zero in all other degrees. We will later check our main theorem against this calculation. The objective of this section is to compute the Euler characteristic of the space \( \chi(B_t^0) \) for \( l \in N^* \).

**Theorem 5.1** Suppose \( M \) is a compact even dimensional manifold with boundary \( \partial M \). Then

\[
\chi(B_{2l-1}^2) = \chi(B_{l-1}(M)) \quad \text{and} \quad \chi(B_{2l}^2) = \chi(B_l(M)).
\]

In particular

\[
\chi(B_0^1(M)) = 0 \quad \text{if} \quad \chi(M) = 0.
\]

We list some useful known facts about the Euler characteristic of familiar constructions:

- The Euler characteristic \( \chi \) satisfies the inclusion-exclusion principle on closed subsets; i.e. \( \chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \). This has a generalization. We say \( Y \) is the colimit of the diagram

\[
B \longrightarrow X \longrightarrow A
\]

if the diagram maps to \( Y \) and if \( Z \) is any space to which this diagram maps to, there must be an arrow \( Y \rightarrow Z \) factoring this diagram as in the figure

\[
\begin{array}{c}
X \swarrow A \\
\downarrow \downarrow \\
B \searrow \downarrow Y \\
\downarrow \downarrow \\
Z
\end{array}
\]

i.e. \( Y \) is the “smallest” space to which the diagram can be mapped. If \( Y \) is such a colimit, then

\[
(131) \quad \chi(Y) = \chi(A) + \chi(B) - \chi(X)
\]

This equality is true because there is a Mayer-Vietoris sequence associated to this diagram. In particular if all maps in the diagram are inclusions, then \( X = A \cap B, Y = A \cup B \) and we recover the inclusion-exclusion principle for \( \chi \).
Before giving a proof of Theorem 5.1, we look closely at the first few cases. When to prove Lemma 5.3 i.e. \( \chi_l \) the case

\[
\chi(B_4(M)) = 1 - \frac{1}{4}(1 - \chi) \cdots (l - \chi)
\]

where \( \chi := \chi(M) \). In particular, observe that when \( M \) is closed odd dimensional, \( \chi(M) = 0 \) and hence necessarily \( \chi(B_k(M)) = 0 \). This fact will be used throughout.

- If \( X * Y \) is the join of \( X \) and \( Y \), then

\[
\chi(X * Y) = \chi(X) + \chi(Y) - \chi(X)\chi(Y)
\]

A cute way to see this is to notice that \( X * Y \) is the colimit of the obvious projection maps in the diagram \( CX \times Y \leftarrow X \times Y \rightarrow X \times CY \) where \( CX \) is the cone on \( X \) (a contractible space). Now apply (131).

- Let \( \vee \) denote the one point union; i.e. \( X \vee Y = X \sqcup Y / x_0 = y_0 \). Then * is (up to homotopy) distributive with respect to \( \vee \); i.e. \( X * (Y \vee Z) \simeq (X * Y) \vee (X * Z) \).

We need a definition

**Definition 5.2** Define \( B^p_q(M) \) the subspace of \( B_{p+q}(M) \) given by

\[
B^p_q(M) := \{ \sum t_i x_i \in B_{p+q}(M) \text{ with at most } p \text{ of the } x_i \text{'s in the interior of } M \}
\]

This is a closed subset of \( B_{p+q}(M) \) and since points in the interior are allowed to move into the boundary, we have inclusions \( B^p_{q+i} \subseteq B^p_q \). Naturally \( B^p_q \) is a subspace of \( B^q_1 \) if \( 2p + q \leq l \). For example \( B^1_2 \subseteq B^4_4 \) but \( B^2_2 \) is not a subspace of \( B^4_4 \). Note also that

**Lemma 5.3** \( B^p_q \) is the closure of \( B_{p,q} \) in \( B_{p+q}(M) \).

Before giving a proof of Theorem 5.1, we look closely at the first few cases. When \( k = 1 \), there isn’t much to prove

\[
B^0_1(M) = B_{0,1}(M, \partial M) = B_1(\partial M) = \partial M
\]

If \( M \) is even dimensional, then \( \partial M \) is closed odd dimensional and its Euler characteristic must vanish; i.e. \( \chi(B^0_1(M)) = 0 \).

The case \( l = 2 \) was stated in (130). We indicated that \( B^0_2(X) \) is the colimit of

\[
B_1(M) \longrightarrow \partial M \longrightarrow B_2(\partial M)
\]

so that \( \chi(B^0_2) = \chi(B_2(\partial M)) + \chi(M) - \chi(\partial M) \). If \( M \) is even dimensional, \( \chi(\partial M) \) is trivial and so is \( \chi(B^0_2(M)) \) according to (132). Thus \( \chi(B^0_2(M)) = \chi(M) \) in accordance with Theorem 5.1.

**Definition 5.4** Let \( P \) be a poset which we view as a category with morphisms pointing upward. Here \( p < q \) in \( P \) means \( p \rightarrow q \). We will assume that \( P \) is a lower semilattice meaning that for any \( p, q \in P \) there is a greatest lower bound. A diagram of spaces over \( P \) is a functor from \( P \) into the category of topological spaces. Given a diagram of spaces, we define the colimit of this diagram to be the “smallest” space to which the diagram can map to (we refer to [62] for the precise definitions). When all maps in the diagram are inclusions (this is our case), suffices to say that the colimit is constructed by taking the union of all spaces in the diagram glued over common intersections.

It turns out that \( B^p_q \) is the colimit of a poset diagram. Let’s consider the first few cases. When \( l = 2 \), this is the poset with two edges given in (133) which takes the form

\[
\begin{array}{c}
\text{\( B^0_2 \)} \\
\partial M
\end{array}
\begin{array}{c}
\text{\( B^1_1 \)} \\
\text{\( B^2_0 \)}
\end{array}
\]

31
For $l = 3$, $B^0_3$ consists of those barycenters with only one point in interior and one point on the boundary, or with 3 points on boundary and no points in the interior. This is the union $B_3(\partial M) \cup B^1_1$ over the intersection $B_2(\partial M)$, thus it is the colimit of

$$B^0_3 \to B^0_2 = B_2(\partial M) \to B^1_1$$

Similarly $B^0_4$ is the colimit of the following diagram

$$\begin{array}{ccc}
B_4(\partial M) & \to & B^2_3 \\
\downarrow & & \downarrow \\
B_3(\partial M) & \to & B^1_1 \\
\downarrow & & \downarrow \\
B_2(\partial M) & \to & B^1_1
\end{array}$$

This is again interpreted the following way: $B^0_4$ is the union $B_4(\partial M) \cup B^1_1 \cup B_2(\partial M)$ with all three subspaces overlapping according to the arrows in the diagram; i.e. $B_4(\partial M) \cap B^1_2 = B_3(\partial M)$, etc.

In general we have

**Proposition 5.5** The space $B^0_{2l}(M)$ is the colimit of the following diagram of spaces

$$\begin{array}{ccc}
B^0_{2l} = B_{2l}(\partial M) & \to & B^1_{2l-2} \\
\downarrow & & \downarrow \\
B^0_{2l-1} & \to & \cdots \\
\downarrow & & \downarrow \\
\vdots & & \downarrow \\
B^0_{l+1} & \to & B^0_l \\
\downarrow & & \downarrow \\
B^0_l & \to & B^0_l
\end{array}$$

Similarly, the diagram whose colimit is $B^0_{2l-1}(M)$ is obtained by truncating the top row.

The proof is self-evident. Since all vertical maps are inclusions, the colimit of our diagram is obtained by taking the union of all spaces in the top row whose pairwise intersections are given by the row underneath....

**Remark 5.6** We can similarly define the boundary weighted barycenter spaces $B^0_l(M,r)$ consisting of points in the interior with integer weight $r \geq 1$, not just $r = 2$. This is a modification of (129) where now $rp + q \leq l$. Here too we can give a complete description of the space as in Proposition 5.5.

**Lemma 5.7** $\chi(B^0_{2l}) = \sum_{i=0}^l \chi(B^1_{2l-2i}) - \sum_{i=0}^{l-1} \chi(B^1_{2l-2-2i})$.

**Proof.** Look at the first top two rows of the colimit diagram in Proposition 5.5. By the inclusion-exclusion principle $\chi(B^0_{2l} \cup B^1_{2l-2}) = \chi(B^0_{2l}) + \chi(B^1_{2l-2}) - \chi(B^0_{2l-1})$. The next space in the top row $B^0_{2l-4}$ intersects with this union along the subspace $B^1_{2l-3}$ (that’s the point) so that

$$\chi(B^0_{2l} \cup B^1_{2l-2} \cup B^2_{2l-4}) = \chi(B^0_{2l} \cup B^1_{2l-2}) + \chi(B^1_{2l-4}) - \chi(B^0_{2l-3}) = \chi(B^0_{2l}) + \chi(B^1_{2l-2}) + \chi(B^1_{2l-4}) - \chi(B^0_{2l-3}) - \chi(B^0_{2l-1})$$

32
The lemma follows by induction. ■

We therefore need to compute \( \chi(B^p_q) \) for various \( p, q \). When \( p = 0 \) or \( q = 0 \), the computation is obvious: \( \chi(B^0_q) = 0 \) if \( M \) is even dimensional while \( \chi(B^0_0) = \chi(B_p(M)) \). It remains to determine \( \chi(B^p_q) \) for non-zero \( p, q \). This is done by induction. The inductive hypothesis is ensured by the following Lemma.

**Lemma 5.8** For \( q \geq 1 \), \( B^p_q \) is the colimit of the diagram

\[
B_p(M) \ast B_q(\partial M) \longrightarrow B^{p-1}_1 \ast B_q(\partial M) \longrightarrow B^{p-1}_{q+1}
\]

In particular when \( M \) is even-dimensional with boundary then \( \chi(B^p_q) = \chi(B_p) \).

**Proof.** Note that in the diagram above, only the lefthanded map \( B^{p-1}_1 \ast B_q(\partial M) \longrightarrow B_p(M) \ast B_q(\partial M) \) is an inclusion. An element in \( B^p_q \) consists again of barycenters with at least \( q \)-points in \( \partial M \). Let’s now look at a typical element in \( B^p_q \) which we write as \( \sum t_i x_i \) with at most \( p \) of the \( x_i \)'s in the interior of \( M \). When exactly \( p \) such points are in the interior, this is an element of the join \( B_p(M) \ast B_q(\partial M) \) since it can be written as

\[
t\left( \sum_{i=1}^{p} \frac{t_i}{t} a_i \right) + (1-t) \left( \sum_{j=1}^{q} s_j b_j \right), \quad t = \sum t_i = 1 - \sum s_j, \quad a_i \in \bar{M}, b_j \in \partial M
\]

On \( B^p_q(M) - B^{p-1}_{q+1} \), the map \( B_p(M) \ast B_q(\partial M) \longrightarrow B^p_q(M) \) is one-to-one. When in \( \sum t_i x_i \) one of the \( x_i \)'s goes to the boundary, \( \sum t_i x_i \) approaches an element in \( B^{p-1}_{q+1} \). This means that the subspace \( B^p_q(M) \ast B_q(\partial M) \) maps to a quotient. Rephrasing this in terms of colimits, we obtain the diagram of the proposition.

Using the formula for the Euler characteristic of a colimit (131) and the formula for \( \chi(X \ast Y) \), we obtain readily that

\[
\chi(B^p_q) = \chi(B_p) + \chi(B^{p-1}_{q+1}) - \chi(B^p_q - B^{p-1}_{q+1})
\]

(we have applied here repeatedly that \( \chi(B_q(\partial M)) = 0 \) if \( \chi(\partial M) = 0 \)). We can then proceed by induction. The computation \( \chi(B^p_q) = \chi(B_p) \) for all \( q \geq 1 \) will be immediate if we establish that \( \chi(B^3_3) = \chi(M) \) for all \( q \geq 1 \). But \( B^3_3 \) is the colimit of the diagram

\[
B^{q+1}_q(\partial M) \longrightarrow \partial M \ast B_q(\partial M) \longrightarrow M \ast B_q(\partial M)
\]

from which we get that \( \chi(B^3_q) = \chi(M) \) as desired. ■

**Proof of Theorem 5.1:**

This is a direct consequence of Lemmas 5.7 and 5.8.

### 5.2 The Homology of \( B^p_q(M) \)

Throughout \( \tilde{H}(X) \) means reduced homology; that is for \( X \) connected, \( \tilde{H}_*(X) \) (any coefficients) is \( H_*(X) \) if \( * > 0 \) while \( \tilde{H}_0(X) = 0 \). We also make the convention that \( H_*(X) = 0 \) if \( * < 0 \). We have the easy observation

**Lemma 5.9** For homological degree \( * > 0 \) and field coefficients

\[
H_*(B^p_l/B^p_{l-1}) \cong H_{*+1}(B^0_{l-1}) \oplus H_*(B^0_l)
\]

**Proof.** The subspace \( B^0_{l-1} \) is contractible in \( B^l_0 \). Indeed choose \( x_0 \) some chosen basepoint in \( \partial M \) (call it the “conepoint”) and define

\[
W_l = \left\{ \sum_i \alpha_i x_i \in B^0_l \setminus B^0_{l-1} \mid x_i = x_0 \text{ for some } i \right\} \bigcup B^0_{l-1}
\]

(134)
Then $W_l$ is contractible into itself via the contraction $(t, \sum_1^l \alpha_i x_i) \mapsto tx_0 + \sum (1-t) \alpha_i x_i$. This says that $B^0_{l-1}$ is contractible in $B^0_l$ and the homology long exact sequence for the pair $(B^0_l, B^0_{l-1})$ breaks down into short exact sequences and so the claim is immediate with field coefficients.

We start our homology computation of $H_*(B^0_l(M))$ by looking back at the diagram in Proposition 5.5. This diagram is organized in rows and we let $T_i$ the colimit of the first $i$ rows, $0 \leq i \leq l$. We have $T_0 = B^0_l$, $T_1 = B^0_{l+1} \cup B^1_{l-1}$, and more generally

$$T_i = \left\{ \sum t_j x_j \in B^0_{l+t}, \text{ where at most } l \text{ of the } x_j \text{'s in interior of } M \right\}.$$

Obviously $T_k = B^0_{2k}$ and $T_{l-1} = B^0_{2l-1}$. It is not true however that $T_{l-2}$ is $B^0_{2l-2}$.

As for the case of $B^0_{l-1}$ inside $B^0_l$, the subspace $T_i$ is contractible in $T_{i+1}$ so that (with field coefficients)

$$H_*(T^0_i/T^0_{i-1}) \cong H_{i-1}(T^0_{i-1}) \oplus H_*(T^0_i).$$

Consider the corresponding rows in $i$ and $i-1$

$$\begin{array}{cccc}
B^0_{l+i} & B^1_{l+i-2} & \cdots & B^i_{l-1} \\
B^0_{l+i-1} & B^1_{l+i-3} & \cdots & B^i_{l-i+1} \\
\end{array}$$

The consecutive quotients $\overline{T}_i := T_i/T_{i-1}$ for $0 \leq i \leq l$ are given according to

$$\overline{T}_i = \frac{B^0_{l+i}}{B^0_{l+i-1}} \lor \bigvee_{0 \leq j \leq i-1} \frac{B^1_{l+i-2j}}{(B^1_{l+i-2j+1} \cup B^1_{l+i-2j-1})} \lor B^1_{l-i+1}$$

5.2.1 The spaces $\overline{B^0_q} := B^0_p/(B^p_{q-1} \cup B^p_{q+1})$

We can think of this quotient as the space of formal barycenters $\sum_{i=0}^{p+q} t_i x_i$ with precisely $p$ points in the interior and $q$ points on the boundary, with the topology that points on the boundary never leave that boundary and if either $t_i \to 0$ or if a point from the interior approaches the boundary, the whole configuration approaches the basepoint.

To describe $\overline{B^0_q}$, the following preliminary lemma is needed. We make use of some notation:

- $X \ast \emptyset = X$. An element of $X \ast Y$ is a segment starting at $X$ when $t = 0$ and ending at $Y$ when $t = 1$. We view $X$ (resp. $Y$) as a subspace of $X \ast Y$ corresponding to when $t = 0$ (resp. $t = 1$).

- The “half smash” product of two based spaces is one of

$$X \times Y := \frac{X \times Y}{x_0 \times Y}, \quad X \times Y := \frac{X \times Y}{X \times y_0}$$

- The “unreduced suspension” of a space $\Sigma X$, is the join $X \ast S^0$ where $S^0 = \{1, -1\}$. Equivalently $\Sigma X = [0, 1] \times X/(0, x) \sim (0, x'), (1, x) \sim (1, x')$. An element of the suspension is written as an equivalence class $[t, x]$.

Lemma 5.10 Let $(X, A)$ and $(Y, B)$ be two connected CW pairs. Then $X \ast Y/A \ast Y \simeq X/A \ast Y$ and

$$(X \ast Y)/(X \ast B \cup A \ast Y) \simeq \begin{cases} X/A \ast Y/B, & A \neq \emptyset, B \neq \emptyset; \\
X/A \ast \Sigma Y, & A \neq \emptyset, B = \emptyset; \\
\Sigma(X \times Y), & A = \emptyset, B = \emptyset. \end{cases}$$
PROOF. Here \( X \ast Y/A \ast Y \) is obtained from \( X/A \ast Y \) by collapsing \( x_0 \ast Y \), where \( x_0 \) is the natural basepoint of quotient \( X/A \). Since \( x_0 \ast Y \) is a cone hence contractible, collapsing out \( x_0 \ast Y \) from \( (X/A) \ast Y \) doesn’t change homotopy type.

On the other hand, let \( (X, x_0) \) and \( (Y, y_0) \) be pointed spaces. Consider the subspace \( X \ast y_0 \cup x_0 \ast Y \) of the join. This is the union of two cones along the segment \( x_0 \ast y_0 \). This space is 1-connected (Van-Kampen) and has trivial homology in positive degrees (Mayer-Vietoris). This means (by Whitehead theorem and since we are working with CW complexes) that this subspace is contractible. We thus have the equivalence

\[
(X \ast Y)(X \ast B \cup A \ast Y) = [(X/A) \ast (Y/B)] \cup x_0 \times (Y/B) \cup (X/A) \ast y_0 \simeq (X/A) \ast (Y/B)
\]

Consider next the case \( A = \emptyset, B = \emptyset \) (third case). Elements of the join are classes \([x, t, y]\) with identifications at \( t = 0 \) and \( t = 1 \). When we pass to the quotient \( X \ast Y/X \cup Y \), we obtain the identification space consisting of \([x, t, y]\) with everything collapsed out to \( x_0 \) (resp. \( y_0 \)) when \( t = 0 \) (resp. \( t = 1 \)). This is by definition the unreduced suspension \( \Sigma(X \times Y) \). This can be seen directly from the depiction of the join construction in the figure below. The second case is proven similarly but with a little trick. Let’s look at the figure again and notice that \( X \ast Y/Y \) is \( X \times \Sigma Y \) with a copy of \( X \) (in the right of the middle diagram) collapsed out. This is of the homotopy type of \( X \ltimes \Sigma Y \); that is

\[
X \ast Y/Y \simeq X \ltimes \Sigma Y
\]

from which we deduce the series of equivalences \( X \ast Y/X \cup A \ast Y \simeq (X/A) \ast Y/(X/A) \simeq (X/A) \ltimes \Sigma Y \) and the claim follows. ■

We turn back to \( \overline{B}^p_q \). According to Lemma 5.8 (or by inspection) we have the homeomorphism

\[
B^p_q \overline{B}^p_{q+1} = B^p_q(M) \ast B^p_{q}(\partial M),
\]

and similarly

\[
\overline{B}^p_q = \frac{B^p_q}{B^p_q \cup B^p_{q+1}} = \frac{B^p_q(M) \ast B^p_q(\partial M)}{B^p_1(\partial M) \cup B^p_1(M) \ast B^p_1(\partial M)},
\]

\[
\simeq \frac{B^p_q(M)}{B^p_1(\partial M) \cup B^p_1(M) \ast B^p_1(\partial M)},
\]

\[
\simeq \frac{B^p_q(M)}{B^p_1(\partial M) \cup \Sigma B^p_1(\partial M)}.
\]

where for the last equivalence we have used the standard fact that if \( A \) is contractible in \( X \), then \( X/A \simeq X \vee \Sigma A \) (and \( B^p_{q-1}(\partial M) \) is contractible in \( B^p_1(\partial M) \) as already mentioned). By the distributivity property \( X \ast (Y \vee Z) \simeq (X \ast Y) \vee (X \ast Z) \), we can write further

\[
\overline{B}^p_q \simeq \left( \frac{B^p_q(M)}{B^p_1(\partial M)} \vee \frac{B^p_q(M)}{B^p_1(\partial M)} \right) \vee \left( \frac{B^p_q(M)}{B^p_1(\partial M)} \vee \Sigma B^p_1(\partial M) \right).
\]

35
We must therefore understand the quotient \( B_p(M)/B^p_{l-1} \).

**Lemma 5.11** There is a homotopy equivalence

\[
\frac{B_p(M)}{B^p_{l-1}} \simeq B_p(M/\partial M).
\]

**Proof.** When \( p = 1 \), \( B^0_{l-1} = \partial M \) and \( B_p(M)/B^p_{l-1} = M/\partial M \). When \( p = 2 \), \( B_2(M) \) is the symmetric join \( \text{Sym}^2(M) := M* M/\Sigma_2 \) and \( B^2_1(M) = (\partial M * M) \cup (M * \partial M)/\Sigma_2 \). The quotient is \( B_2(M)/B^2_1(M) = \text{Sym}^2(M/\partial M)/\tilde{A} \) where \( \tilde{A} \) is the subspace of barycenters \( tx + (1-t)y \) with \( x = * \) the basepoint; that is \( A \) is up to homotopy the cone on \( (M/\partial M) \) which is contractible. We’ve just shown that \( B_2(M)/B^2_1(M) \simeq \text{Sym}^2(M/\partial M) \). The general case is similar. Note that \( M/\partial M \) has a preferred basepoint \( x_0 \). It is now clear that

\[
\frac{B_p(M)}{B^p_{l-1}} = \frac{B_p(M/\partial M)}{W_p}
\]

where \( W_p \) is as defined in (134) with “cone point” \( x_0 \). This subspace is contractible so that up to homotopy \( B_p(M/\partial M)/W_p \simeq B_p(M) \) and the claim follows. \( \square \)

We can now put everything previous together to get our main calculation. Let \( \sigma \) be the suspension operator acting on homology so that if \( x \) is a homology class, then \( \deg(\sigma(x)) = \deg(x) + 1 \). Recall in our notation that \( B_0(Y) = 0 \) and that \( \emptyset * X = X \).

**Theorem 5.12** Let \( M \) be a compact manifold with boundary \( \partial M \). The reduced homology of \( B^0_l(M) \) with field coefficients is given by

\[
\hat{H}_*(B^0_{2l}(M)) \cong \hat{H}_*(B_{2l}(\partial M)) \oplus \bigoplus_{0 < i \leq l} \hat{H}_*(B_i(M/\partial M) * B_{2l-2i}(\partial M))
\]

\[
\cong \hat{H}_*(B_{2l}(\partial M)) \oplus \hat{H}_*(B_l(M/\partial M)) \oplus \sigma \bigoplus_{i=1}^{l-1} \hat{H}_*(B_i(M/\partial M)) \otimes \hat{H}_*(B_{2l-2i}(\partial M))
\]

where \( \sigma \) is the suspension operator. Similarly

\[
\hat{H}_*(B^0_{2l-1}(M)) \cong \hat{H}_*(B_{2l-1}(\partial M)) \oplus \bigoplus_{0 < i \leq l-1} \hat{H}_*(B_i(M/\partial M) * B_{2l-2i-1}(\partial M))
\]

\[
\cong H_*(B_{2l-1}(\partial M)) \oplus \sigma \bigoplus_{i=1}^{l-1} H_*(B_i(M/\partial M)) \otimes H_*(B_{2l-2i-1}(\partial M))
\]

The formula is valid for \( l = 1 \) by setting \( \bigoplus^0 = 0 \).

**Proof.** We go back to \( T_i \) the colimit of the bottom \( i \) rows of the colimit diagram in Proposition 5.5, for \( 0 \leq i \leq k \). Since \( T_{i-1} \) contractible in \( T_i \), we have

\[
T_i := T_i/T_{i-1} = T_i \vee \Sigma T_{i-1},
\]

where \( \Sigma T_{i-1} \) is the suspension of \( T_{i-1} \). It is known that \( H_*(\Sigma X) = \sigma H_*(X) \). On the other hand there is the wedge decomposition (135)

\[
T_i = \frac{B^{0}_{l+i}}{B^{0}_{l+i-1}} \vee \bigcup_{0 < j < i} \frac{B^{j}_{l+i-2j}}{B^{j-1}_{l+i-2j+1} \cup B^{j-1}_{l+i-2j-1}} \vee \frac{B^{i-1}_{l-i}}{B^{i-1}_{l-i+1}}.
\]

36
For the same reason as earlier indicated, \( \frac{B^0_{i+i}}{B^0_{i+i-1}} \simeq B_{i+i} (\partial M) \lor \Sigma B_{i+i-1} (\partial M) \), while

\[
\frac{B^1_{i+i}}{B^1_{i+i-1}} \simeq \frac{B_1 (M)}{B^1_{i+i-1}} \ast B_{i-1} (\partial M) \simeq B_* (M/\partial M) \ast B_{i-1} (\partial M)
\]

according to (136), Lemma 5.10 and Lemma 5.11.... Putting everything together gets us

\[
\overline{T}_i = T_i \lor \Sigma T_{i-1} \simeq B_{i+i} (\partial M) \lor \Sigma B_{i+i-1} (\partial M) \lor B_i (M/\partial M) \ast B_{i-1} (\partial M)
\]

Using induction starting with \( B^0_1 (M) = \partial M \) we can infer from the decomposition of \( \overline{T}_i \) above that the integral homology (not just with field coefficients) is given by

\[
\tilde{H}_* (T_i) \cong \tilde{H}_* \left( B_{i+i} (\partial M) \lor \bigvee_{0 \leq j < i} (B_j (M/\partial M) \ast B_{i+i-2j} (\partial M)) \right)
\]

(139)

\[
\cong \tilde{H}_* (B_{i+i} (\partial M)) \oplus \bigoplus_{j=1}^i \tilde{H}_* (B_j (M/\partial M) \ast B_{i+i-2j} (\partial M))
\]

(140)

The argument behind this deduction is to first apply homology to both sides of (138) and to use the cancelation property for finite abelian groups\(^1\)We pointed out that for connected based spaces \( X \ast Y \simeq \Sigma X \land Y \) so that (with field coefficients now) \( \tilde{H}_* (X \ast Y) \cong \sigma \tilde{H}_* (X) \otimes \tilde{H}_* (Y) \). The theorem is immediate replacing \( T_i \) by \( B^0_{2i} \) and \( T_{i-1} \) by \( B^0_{2i-1} \).

**Remark 5.13** The case \( l = 1 \) in the theorem gives

\[
\tilde{H}_* (B^0_2 (M)) \cong \tilde{H}_* (B_2 (\partial M)) \oplus \tilde{H}_* (M/\partial M)
\]

When \( M = D \) is the disk with boundary \( S^1 \),

\[
\tilde{H}_* (B^0_2 (D)) \cong \tilde{H}_* (B_2 (S^1)) \oplus \tilde{H}_* (B_1 (S^2)) \cong \tilde{H}_* (S^3) \oplus \tilde{H}_* (S^2) \cong \begin{cases} \mathbb{Z}, \ast = 3 \\ \mathbb{Z}, \ast = 2 \end{cases}
\]

and this is precisely the computation in (130).

One first consequence of Theorem 5.12 is that we can compute the connectivity and homological dimension of our spaces \( B^p_q \). A space \( X \) is said to be \( r \)-connected for some \( r \geq 0 \) if all the homotopy groups of \( X \) vanish up to \( r \). A space is 0-connected if it is connected and it is 1-connected if it is simply connected.

**Lemma 5.14** Suppose \( \partial M \) is 1-connected, then \( B^p_q (M) \) is also 1-connected

**Proof.** Given a diagram as in Proposition 5.5, it is well-known that if all spaces \( B^p_q \) making up the diagram are 1-connected, then the colimit is 1-connected (this is a consequence of the Van-Kampen theorem). When \( p = 0 \), \( B^0_q = B_q (\partial M) \) and this is at least \( 2q - 1 \) connected according to [42] so in particular 1-connected. We can now use induction on \( p \) and the colimit diagram in Lemma 5.8 to complete the proof. More precisely recall that if \( X \) and \( Y \) are connected, then \( X \ast Y \) is simply connected. Since by induction \( B^p_{q-1} \) is connected, it follows that all constituent subspaces in the colimit diagram in Lemma 5.8 are 1-connected and hence so is \( B^p_q \). The proof is complete. \( \blacksquare \)

\(^1\) For finite complexes, the homology groups are finitely generated abelian groups. For those groups, if \( A \times G \cong B \times G \), then \( A \cong B \) (cancellation)
Lemma 5.15 Suppose $M$ is a manifold of dim $M = m$. Then $H_*(B^q_0) = 0$ for $* \geq lm$. If in addition both $M$ and $\partial M$ are r-connected with $r \geq 1$, then $B^q_0(M)$ is $l + r - 2$-connected if $l$ is even and $\min\{l + 2r - 2, 2l + r - 2\}$-connected if $l \geq 3$ is odd.

Proof. The first claim is by inspection knowing that the homological dimension of $B_n(X)$ is at most $nh(X) + n - 1$, where $h(X)$ is the homological dimension of $X$. On the other hand it is shown in [42] that if $M$ is r-connected with $r \geq 1$, then $B_n(M)$ is $2n + r - 2$ connected. If both $M$ and $\partial M$ are r-connected\(^2\), then so is $M/\partial M$. This means that $\tilde{H}_i(B_n(\partial M)) = \tilde{H}_i(B_i(M/\partial M)) = 0$ if $i \leq 2l + r - 2$. We obtain our connectivity bound by computing the homological connectivity of $B^q_0(M)$ from Theorem 5.12 and then invoking Lemma 5.14 (3). Similar calculation when $l$ is odd.

A second main consequence of Theorem 5.12 is that we can recover the Euler characteristic computation of $B^q_0(M)$ when $M$ even dimensional.

**Alternative Proof of Theorem 5.1**

From (139)

$$\chi(T_i - 1) = \chi(B_{t+i}(\partial M)) - 1 + \sum_{j=1}^{i} (\chi(B_j(M/\partial M) \ast B_{t+i-2j}(\partial M)) - 1),$$

$$= \chi(B_{t+i}(\partial M)) - 1,$$

$$+ \sum_{j=1}^{i} [\chi(B_j(M/\partial M)) + \chi(B_{t+i-2j}(\partial M)) - \chi(B_j(M/\partial M))\chi(B_{t+i-2j}(\partial M)) - 1].$$

Suppose $M$ is even-dimensional compact with connected boundary. We can use the fact that $\chi(B_l(\partial M)) = 0$ since $\chi(\partial M) = 0$. From the previous formula

$$\chi(T_i) = -i + \sum_{j=1}^{i} \chi(B_j(M/\partial M))$$

From which we get

$$\chi(B^q_{2l}) = \chi(T_k) = -l + \sum_{j=1}^{l} \chi(B_j(M/\partial M)),$$

$$\chi(B^q_{2l-1}) = \chi(T_{l-1}) = -l - 1 + \sum_{j=1}^{l-1} \chi(B_j(M/\partial M)).$$

According to (131) again, $\chi(M/\partial M) = \chi(M) - \chi(\partial M) + 1 = \chi(M) + 1$. Set $\chi := \chi(M)$. Combining with (132) yields

$$\chi(B_j(M/\partial M)) = 1 - \frac{1}{j!}(-\chi)(1 - \chi) \cdots (j - 1 - \chi),$$

$$= 1 - \frac{1}{j!}(1 - \chi)(1 - \chi) \cdots (j - \chi) + \frac{1}{(j-1)!}(1 - \chi) \cdots (j - 1 - \chi),$$

$$= \chi(B_j(M)) - \chi(B_{j-1}(M)) + 1.$$ 

Adding those up for $0 < j \leq l$ and plugging into (141) yields immediately our formula $\chi(B^q_{2l}) = \chi(B_l(M))$ and that $\chi(B^q_{2l-1}) = \chi(B_{l-1}(M))$. This is precisely the content of Theorem 5.1.

\(^2\)There is in general no relationship between the connectivity of $M$ and $\partial M$.

\(^3\)Here we recall that if $X$ is simply connected with $H_i(X) = 0$ for $i \leq r$, then $\pi_i(X) = 0$ for $i \leq r$.
5.3 Barycenter spaces of disconnected spaces

This section which is of independent interest discusses $B_t(X)$ when $X = A \sqcup B$ is a disjoint union. This case is not treated in [42] and it is needed in case our manifold $M$ has disconnected boundary. Obviously $B_t(A \sqcup B) = A \sqcup B$ is disconnected. For $n \geq 2$, $B_n(A \sqcup B)$ is however connected. By convention set $B_0(Y) = \emptyset$ as before. An element of the join $X \ast Y$ is written $[x, t, y]$. As for Lemma 5.10, there are inclusions $X \subset X \ast Y$ and $Y \subset X \ast Y$ where $X = \{[x, 0, y]\}$ and $Y = \{[x, 1, y]\}$. We will write these subspaces as $\emptyset \ast Y$ and $X \ast \emptyset$.

**Proposition 5.16** $B_n(A \sqcup B)$ is the colimit of the lattice diagram

\[
\begin{array}{cccccc}
B_t(A) & B_{t-1}(A) \ast B_1(B) & \cdots & B_1(B) \\
B_{t-1}(A) & B_{t-2}(A) \ast B_1(B) & \cdots & B_{t-1}(B) \\
& \vdots & \ddots & \ddots & \ddots \\
& & & B_1(A) & B_1(B)
\end{array}
\]

All arrows are inclusions.

This diagram expresses the fact that points can be either in $A$ or $B$. Points in $A$ alone or $B$ alone "interact" with each other, while points in $A$ do not interact with points in $B$. Proposition 5.16 gives a homotopy decomposition which we will use to get to the homology of $B_n(A \sqcup B)$.\(^4\)

Next and throughout, $\overline{B_t}(X) := B_t(X)/B_{t-1}(X)$. Note that because $B_{t-1}(X)$ is contractible in $B_t(X)$, $\overline{B_t}(X) \simeq B_t(X) \vee \Sigma B_{t-1}(X)$. This is a fact we will use routinely below.

**Lemma 5.17** Let $X = A \sqcup B$ and $l \geq 3$. Then

\[
\overline{B}_l(A \sqcup B) \simeq \overline{B}_l(A) \lor \overline{B}_{l-1}(A) \times \Sigma B \lor \overline{B}_{l-2}(A) \ast \overline{B}_2(B) \lor \overline{B}_2(A) \ast \overline{B}_{l-2}(B) \lor \Sigma A \times \overline{B}_{l-1}B \lor \overline{B}_l(B).
\]

**Proof.** In the diagram of Proposition 5.16, the colimit of the first $l$ rows is $B_l(A \sqcup B)$ while the colimit of the first $l-1$ rows is $B_{l-1}(A \sqcup B)$. By collapsing out this latter subspace we obtain the wedge

\[
\overline{B}_l(A \sqcup B) = \overline{B}_l(A) \lor \bigvee_{1 \leq i \leq j \leq l-1} \frac{B_i(A) \ast B_j(B)}{B_{i-1}(A) \ast B_j(B) \lor B_i(A) \ast B_{j-1}(B) \lor \overline{B}_l(B)},
\]

where we convene that $B_0 = \emptyset$. But

\[
\frac{B_i(A) \ast B_j(B)}{B_{i-1}(A) \ast B_j(B) \lor B_i(A) \ast B_{j-1}(B)} \simeq \overline{B}_i(A) \ast \overline{B}_j(B) \quad \text{(Lemma 5.10, } i, j \geq 1),
\]

while for $j = 1$ (similarly if $i = 1$) we have

\[
\frac{B_{l-1}(A) \ast B}{B_{l-1}(A) \lor B_{l-2}(A) \ast B} \simeq \overline{B}_{l-1} \times \Sigma B.
\]

Combining with (142) gets us the answer. ■

All homology below is reduced and is taken with field coefficients.

\(^4\)In the algebraic topological jargon, a homotopical decomposition of a space $X$ is a diagram of spaces whose "homotopy colimit" is weakly equivalent to $X$. When the diagram is indexed over a poset and all maps are "nice inclusions" (i.e. cofibrations), the homotopy colimit and the colimit are equivalent. This is our case.
**Lemma 5.18** In the case $l=2$, $B_2(A \sqcup B)$ has the homology of

$$B_2(A) \vee \Sigma(A \times B) \vee B_2(B).$$

**Proof.** The corresponding colimit diagram for $B_2(A \sqcup B)$ has just two rows

\[
\begin{array}{ccc}
B_2(A) & \rightarrow & B_3(B) \\
\downarrow & & \downarrow \\
B_1(A) & \rightarrow & B_1(B)
\end{array}
\]

so that as we’ve already seen

$$
\overline{B}_2(A \sqcup B) \cong \overline{B}_2(A) \vee \Sigma(A \times B) \vee \overline{B}_2(B) \\
\cong B_2(A) \vee \Sigma A \vee \Sigma(A \times B) \vee B_2(B) \vee \Sigma B.
$$

But the left hand side has another description. Since $A$ and $B$ are disjoint in $B_2(A \sqcup B)$, take a path $ta + (1-t)b, t \in I$ with $a \in A$ and $b \in B$. The union $I \cup A \cup B$ is connected and contractible in $B_2(A \sqcup B)$. It follows that $\overline{B}_2(A \sqcup B) \cong B_2(A \sqcup B) \vee \Sigma(A \vee B)$. Comparing with the above, we have the equivalence

$$B_2(A \sqcup B) \vee \Sigma A \vee \Sigma B \cong B_2(A) \vee \Sigma(A \times B) \vee B_2(B) \vee \Sigma A \vee \Sigma B.$$

In homotopy theory, cancelation of wedge summands doesn’t hold in general. However passing to homology with field coefficients, we will obtain direct summands that we can cancel. This is saying precisely that $H_\ast(B_2(A \sqcup B)) \cong H_\ast(B_2(A) \vee \Sigma(A \times B) \vee B_2(B))$ as desired. □

We are now in a position to state the main result of this section.

**Theorem 5.19** For $A, B$ two disjoint connected spaces and $l \geq 2$, $B_k(A \sqcup B)$ has the same reduced homology as

$$B_l(A) \vee \Sigma B_{l-1}(A) \vee B_l(B) \vee \Sigma B_{l-1}B$$

$$\vee \bigvee_{i=1}^{l-1} B_{l-1}(A) \ast B_i(B) \vee \bigvee_{i=2}^{l-1} \Sigma B_{l-1}(A) \ast B_{i-1}(B)$$

(if the lower index of the wedge is greater than the upper index, which happens when $l=2, 3$, then the corresponding wedge term doesn’t exist).

**Proof.** This is an inductive argument based on the decomposition of Lemma 5.17 and the fact that $\overline{B}_l(X) \cong B_l(X) \vee \Sigma B_{l-1}(X)$. Note that the formula agrees with the decomposition in Lemma 5.18 since $\Sigma(X \times Y) \cong X \times Y \vee \Sigma X \vee \Sigma Y$ so that

$$H_\ast(B_2(A \sqcup B)) \cong H_\ast(B_2(A)) \oplus H_\ast(B_2(B)) \oplus H_\ast(\Sigma A) \oplus H_\ast(\Sigma B) \oplus H_\ast(A \ast B)$$

which is precisely the statement of the theorem when $l=2$.

To proceed further we use the same idea as in Lemma 5.18. We need use here that

$$H_\ast(X \times \Sigma Y) \cong H_\ast((X \times \Sigma Y) \vee \Sigma Y) \cong H_\ast(X \times Y) \oplus H_\ast(\Sigma Y)$$

According to the decomposition in Lemma 5.17, $H_\ast(\overline{B}_l(A \sqcup B))$ is therefore given by

$$H_\ast(\overline{B}_l(A \sqcup B)) \cong H_\ast(\overline{B}_l(A)) \oplus H_\ast(\overline{B}_{l-1}(A) \ast B) \oplus H_\ast(\Sigma B) \oplus$$

$$\bigoplus_{i=2}^{l-2} H_\ast(\overline{B}_{l-i}(A) \ast \overline{B}_i(B)) \oplus H_\ast(A \ast \overline{B}_{l-1}B) \oplus H_\ast(\Sigma A) \oplus H_\ast(\overline{B}_i(B))$$

$$H_\ast(\overline{B}_l(A)) \oplus H_\ast(\overline{B}_{l-1}(A)) \oplus \bigoplus_{i=1}^{l-1} H_\ast(\overline{B}_{l-i}(A) \ast \overline{B}_i(B)) \oplus H_\ast(\overline{B}_{l-1}(B)) \oplus H_\ast(\overline{B}_l(B))$$

(143)
Here we should recall that
\[
\overline{\mathcal{B}}_i(A) \ast \overline{\mathcal{B}}_j(B) \simeq (B_i(A) \vee \Sigma B_{i-1}(A)) \ast (B_j(B) \ast \Sigma B_{j-1}(B)) \\
\simeq B_i(A) \ast B_j(B) \vee B_i(A) \ast \Sigma B_{j-1}(B) \vee \Sigma B_{i-1}(A) \ast B_j(B) \\
\vee \Sigma B_{i-1}(A) \ast \Sigma B_{j-1}(B) \quad \text{(distributivity property)}
\]

We can combine this with (143) to obtain (using reduced homology)
\[
\bigoplus_{i=1}^{l-1} H_*(\overline{\mathcal{B}}_{l-i}(A) \ast \overline{\mathcal{B}}_i(B)) \simeq H_*(B_{l-1}(A) \ast B) \oplus H_*(\Sigma B_{l-2}(A) \ast B)
\]
\[
\bigoplus_{i=2}^{l-2} H_*(B_{l-i}(A) \ast B_i(B)) \oplus H_*(\Sigma B_{l-i-1}(A) \ast B_i(B))
\]
\[
\bigoplus_{i=2}^{l-2} H_*(\Sigma B_{l-i}(A) \ast B_{i-1}(B)) \oplus H_*(\Sigma^2 B_{l-i-1}(A) \ast B_{i-1}(B))
\]
\[
\oplus H_*(A \ast B_{l-1}(B)) \oplus H_*(\Sigma A \ast B_{l-2}(B))
\]

If we now suppose that \(H_*(B_{l-1}(A \cup B))\) has the homology stated in the statement of the theorem, then by identifying the expression for \(H_*(\overline{\mathcal{B}}_1(A \cup B))\) in (143) with \(H_*(B_1(A \cup B)) \oplus H_*(\Sigma B_{l-1}(A \cup B))\) and using induction, we end up with the expression of the homology \(H_*(B_1(A \cup B))\) stated in the theorem. ■

We look at particular cases below.

**Remark 5.20** Let \(A = S^1 = B\) so that \(X\) is the disjoint union of two circles. Then \(B_3(S^1) \simeq S^5\), \(B_2(S^1) \simeq S^3\) and \(B_3(S^1 \sqcup S^1)\) has the homology of
\[
B_3(A) \vee \Sigma B_2(A) \vee B_2(A) \ast B \vee \Sigma A \ast B \vee A \ast B_2(B) \vee B_3(B) \vee \Sigma B_2(B)
\]
\[
\simeq 4 \vee S^5 \vee 3 \vee S^4
\]

In general if \(X\) is a disjoint union of circles, then \(B_n(X)\) will be a wedge of spheres, the top dimensional spheres being of dimension \(2n - 1\).

## 6 Proof of the main results

In this section, we present the proof of our main results. The first type of these are based on our strong Morse inequalities and the related Hopf-Poincaré index type formula, while second type of results takes advantage for the precise knowledge of lack of compactness, to derive that, under appropriate conditions of the function \(K\) blow up of subcritical respectively supercritical approximation can rule out.

### 6.1 Proof of Theorem 1.1 to Theorem 1.8

The basic idea of the proof is to extend the classical Morse theory to our problem. To do so we first characterize the topology of the very high and the very negative ones of \(II\). We treat first the very high sublevels of \(II\) and for that we start with the following lemma.

**Lemma 6.1** Assuming that \((p, q) \in \mathbb{N}^2\) such that \(2p + q = k\), \(0 < q < q_k\), \(0 < \eta < \eta_k\), and \(0 < \epsilon \leq \epsilon_k\), where \(q_k\) and \(\eta_k\) are given by Proposition 3.2, and \(\epsilon_k\) is given by (101), then there exists \(C_0^k := C_0^k(q, \eta)\) such that for every \(0 < \epsilon \leq \epsilon_k\) where \(\epsilon_k\) is as in (101), there holds
\[
V(p, q, \epsilon, q, \eta) \subset II^{C_0^k} \setminus II^{-C_0^k}.
\]
Proof. It follows directly from (103)-(106), Lemma 7.11, Lemma 7.12, and Proposition 4.1. ■

Next, combining Proposition 3.3 and the latter Lemma, we have the following Corollary.

**Corollary 6.2** There exists a large positive constant $C^k_1$ such that

$$\text{Crit}(II) \subset II^{-C^k_1} \setminus II^{-C^k_1}.$$

Proof. It follows, via a contradiction argument, from the fact that $II$ is invariant by translation by constants, Proposition 3.3, and Lemma 6.1. ■

Next, we are ready to characterize the topology of very high sublevels of $II$. Indeed, as in [1] and for the same reasons, we have that Lemma 3.4, Lemma 6.1 and Corollary 6.2 imply the following Lemma which describes the topology of very high sublevels of the Euler-Lagrange functional $II$.

**Lemma 6.3** Assuming that $(p, q) \in \mathbb{N}^2$ such that $2p + q = k$, $0 < q < q_k$, and $0 < \eta < \eta_k$, where $q_k$ and $\eta_k$ are given by Proposition 3.2, there exists a large positive constant $L^k := L^k(p, \eta)$ such that for every $L \geq L^k$, we have that $II^k$ is a deformation retract of $\mathcal{H}_{II}$, and hence it has the homology of a point, where $C^k_0$ is as in Lemma 6.1 and $C^k_1$ as in Lemma 6.2.

Next, we turn to the study of the topology of very negative sublevels of $II$ when $k \geq 2$ or $\bar{k} \geq 1$. Indeed, as in [1] and for the same reasons, we have that the well-know topology of very negative sublevels in the nonresont case (see [52]), Lemma 3.3, Lemma 6.1 and Corollary 6.2 imply the following Lemma which gives the homotopy type of the very negative sublevels of the Euler-Lagrange functional $II$.

**Lemma 6.4** Assuming that $k \geq 2$ or $\bar{k} \geq 1$, $(p, q) \in \mathbb{N}^2$ such that $2p + q = k$, $0 < q < q_k$, and $0 < \eta < \eta_k$, where $q_k$ and $\eta_k$ are given by Proposition 3.2, then there exists a large positive constant $L_{k, \bar{k}} := L_{k, \bar{k}}(p, \eta)$ with $L_{k, \bar{k}} > 2 \max\{C^k_0, C^k_1\}$ such that for every $L \geq L_{k, \bar{k}}$, we have that $II^{-L}$ has the same homotopy type as $B_{k-1}^k(M)$ if $k \geq 2$ and $\bar{k} = 0$, as $A_{k-1, \bar{k}}^k$ if $k \geq 2$ and $\bar{k} \geq 1$ and as $S^{k-1}$ if $k = 1$ and $\bar{k} \geq 1$, where $C^k_0$ is as in Lemma 6.1 and $C^k_1$ as in Lemma 6.2.

Now, we present the proof of Theorem 1.1-Theorem 1.8.

**Proof of Theorem 1.1-Theorem 1.8**

We will provide only the proof of Theorem 1.4, its corollary and Theorem 1.8. The proofs of the remaining statements concerning the critical case $\kappa = 4\pi^2$ are similar and even simpler since they involve only single boundary blow up points.

For the sake of simplicity we assume that the Paneitz operator $P^4_{\eta}$is non negative. The argument extends virtually to the general case, see [1].

Arguing by contradiction we assume that the the functional $II$ does not admit a critical point. Thanks to Lemma (6.1), we may choose $L$ large enough such that all critical point at Infinity are contained in the strip $(II^L, II^{-L})$.

Now we define

$$M(t) := \sum_{i=1}^{4k-1} m^k_i t^i, \quad P(t) := \sum_{i=0}^{\infty} b_i(II^L, II^{-L}) t^i,$$

where

$$b_i := \text{rank} H_i(II^L, II^{-L}).$$

We notice that it follows from the exact sequence of the pair $(II^L, II^{-L})$ that

$$H_0(II^L, II^{-L}) \simeq H_1(II^L, II^{-L}) \text{ and } H_i(II^L, II^{-L}) \simeq H_{i-2}(II^{-L}), \forall i \geq 2.$$

Hence it follows that Lemma 6.4 that

$$P(t) = \sum_{i=2}^{4k-5} c_{i-1}^{k-1} t^i.$$
Moreover it follows from our Morse Lemma 4.7 that strong Morse inequalities hold. Namely we have that:

\[
M(t) - P(t) = (1 + t)R(t),
\]

where \( R(t) := \sum_{i \geq 1} n_it^i \) is a polynomial in \( t \) with non negative integer coefficients. Equating the coefficients of \( t \) in the polynomials on the left and right hand side of (144) we obtain a solution of the system (27) hence contradicting the assumption of Theorem 1.4.

Now choosing \( t = -1 \) in equation (144) we derive that:

\[
\sum_{A \in \mathcal{F}_\infty} (-1)^{i_{\infty}(A)} = \chi(\mathcal{II}_L, \mathcal{II}_L - L) = 1 - \chi(B_{k-1}^0),
\]

which violates the condition of corollary (1.6).

The proof of Theorem 1.8 follows for similar arguments using the Morse subcomplex related to the critical points at Infinity whose Morse indices are less or equal some \( l \leq 4k - 1 \).

6.2 Proof of Theorem 1.11

Our next theorem is based on the construction of a solution of supercritical non resonant approximation. Such a solution is built using the top homology class of the boundary-weighted barycenters \( B^0_k(M) \). For such a solution we derive an accurate estimate of its Morse index. We then use such a spectral information to rule out its blow up, proving that it should converge to a solution of our equation.

**Proof of Theorem 1.11**

We consider the following superapproximation of the resonant case

\[
(P_\varepsilon) \begin{cases}
P^4_gu + 2Q_g = 2(\kappa + \varepsilon) \frac{k^2}{4} \sum_{i=1}^k \alpha_i \delta_{a_i} & \text{in } \mathcal{M}, \\
P^3_gu + T_g = 0 & \text{on } \partial \mathcal{M}, \\
\frac{\partial u}{\partial n_g} = 0 & \text{on } \partial \mathcal{M},
\end{cases}
\]

where \( \varepsilon \) is a small positive number and \( \kappa := \kappa(p^4, p^3) = 4\pi^2 k \), where \( k \in \mathbb{N}^* \). Regarding the problem \( (P_\varepsilon) \) we prove the following claim:

**Claim 1:** For a sequence of \( \varepsilon_k \to 0 \), the problem \( (P_\varepsilon) \) admits a solution \( u_\varepsilon \) whose Morse index \( \text{Morse}(u_\varepsilon) \) satisfies

\[
4k + \bar{k} \leq \text{Morse}(u_\varepsilon) \leq 4k + 1 + \bar{k}.
\]

For the sake of simplicity of notation we provide the proof only in the case \( \bar{k} = 0 \). The arguments extend virtually to the case \( \bar{k} \neq 0 \).

To prove the above claim we argue as follows: We embedded the \( B_k(\partial M) \) into the space of variation \( \mathcal{H}_{\bar{\partial}M} \) through the map:

\[
f_k(\lambda) : B_k(\partial M) \rightarrow \mathcal{H}_{\bar{\partial}M}
\]

as follows

\[
f_k(\lambda)(\sum_{i=1}^k \alpha_i \delta_{a_i}) := \sum_{i=1}^k \alpha_i \varphi_{a_i, \lambda},
\]

with the \( \varphi_{a_i, \lambda} \)'s defined by (49).

Now we notice that it follows from Lemma 7.5, Lemma 7.6 and Lemma 6.4 that \( f_k(\lambda) \) maps for \( \lambda \) and \( L \) large

\[
f_k(\lambda) : B_k(\partial M) \rightarrow II_{\varepsilon}^{L},
\]

where \( II_{\varepsilon} \) is the Euler-Lagrange functional associated to \( (P_\varepsilon) \). Hence \( M_k(\lambda) := f_k(\lambda)(B_k(\partial M)) \) is a stratified set of top dimension \( 4k - 1 \). Now observe that \( M_k(\lambda) \) is contractible in \( \mathcal{H}_{\bar{\partial}M} \) (by taking its
We now deform $U_k$ using the pseudogradient flow obtained as a convex combination of the Bahri-Lucia pseudogradient of Lemma 3.1 and the pseudogradient constructed in Proposition 4.4. By transversality arguments, we may assume that such a deformation avoids the stable manifolds of critical points of $\Pi_{\epsilon}$ whose indices are grater or equal $4k + 2$.

Now using the compactness of the variational problem in the non resonant case and a theorem of Bahri-Rabinowitz [12], we derive that $U_k(\lambda)$ retracts by deformation onto $\Pi_{\epsilon}^{-L} \cup \Sigma$, where $\Sigma$ is the union of the unstable manifolds of some critical points of $\Pi_{\epsilon}$ caught by the flow. Since $U_k(\lambda)$ is contractible whereas $\Pi_{\epsilon}^{-L}$ is not, $\Sigma$ is not empty. Moreover from the above transversality arguments, the Morse indices of its critical points are upper bounded by $4k + 1$. Hence

$$\Sigma = \bigcup(W_u(x); x \text{ is a critical point of } \Pi_{\epsilon} \text{ whose Morse index } m(x) \leq 4k + 1).$$

Now using the exact homology sequence of the pair $(\Pi_{\epsilon}^{-L} \cup \Sigma)$ we derive that

$$\cdots \longrightarrow H_{4k}(\Pi_{\epsilon}^{-L} \cup \Sigma) \longrightarrow H_{4k}(\Pi_{\epsilon}^{-L} \cup \Sigma, \Pi_{\epsilon}^{-L}) \longrightarrow H_{4k-1}(\Pi_{\epsilon}^{-L}) \longrightarrow \cdots$$

Since $\Pi_{\epsilon}^{-L} \cup \Sigma$ is retract by deformation of $U_k$ which is contractible, we derive that

$$H_{4k}(\Pi_{\epsilon}^{-L} \cup \Sigma, \Pi_{\epsilon}^{-L}) = H_{4k-1}(\Pi_{\epsilon}^{-L}) \neq 0.$$ 

It follows that $\Sigma$ contains at least a critical point of $\Pi_{\epsilon}$ whose Morse index is either $4k$ or $4k + 1$.

To conclude the proof of the theorem we prove the following claim:

**Claim 2:**

$$u_{\epsilon_k} \to u_{\infty} \text{ in } C^{4,\alpha}(M),$$

where $u_{\infty}$ is a solution of equation (5).

To prove the claim it is enough to rule out the blow up of $u_{\epsilon_k}$. Arguing by contradiction we assume that

$$\|u_{\epsilon_k} - (u_{\epsilon_k})_{Q,T}\| \to +\infty.$$

Arguing as in Lemma 3.3 we derive that

$$u_{\epsilon_k} \in V(p, q, \delta_k, \vartheta, \eta),$$

for some $\delta_k \to 0$.

Testing the equation $\mathcal{P}_\epsilon$ by $\sum_{i=1}^{p+q} \frac{\partial^2 \mathcal{F}_\epsilon}{\partial m_{\alpha_i}}(a_i)$, very much like in Corollary 7.13, we derive that:

$$\epsilon_k = 2\pi^2 \sum_{i=p+1}^{p+q} \frac{\partial^2 \mathcal{F}_\epsilon}{\partial m_{\alpha_i}}(a_i) + \frac{1}{\lambda_i} \frac{\partial^2 \mathcal{F}_\epsilon}{\partial m_{\alpha_i}}(a_i) + \frac{1}{\lambda_i} + |\alpha_i - 1|^2 + |\tau_i|^2.$$

Now expanding $\Pi_{\epsilon_k}(u_{\epsilon_k})$, just like in Lemma 7.11 we obtain:

$$\Pi_{\epsilon_k}(u_{\epsilon_k}) = \frac{20}{3} k^3 - 4k \ln(k \pi \partial m_{\alpha_i}) - 8 \pi^2(1 + \epsilon_k) \mathcal{F}_p, q(a_1, \ldots, a_{p+q})$$

$$\times \left( \sum_{i=1}^{p+q} (\alpha_i - 1)^2 \ln \lambda_i \right) + \sum_{i=p+1}^{p+q} (\alpha_i - 1)^2 \ln \lambda_i - 4 \pi^2 \sum_{i=1}^{p+q} \sigma_i^2 - \sum_{i=p+1}^{p+q} \sigma_i^2$$

$$- 2 \pi^2 \sum_{i=p+1}^{p+q} \frac{1}{\lambda_i} \frac{\partial^2 \mathcal{F}_\epsilon}{\partial m_{\alpha_i}}(a_i) + O \left( \epsilon_k^2 + \sum_{k=1}^{p+q} |\alpha_k - 1|^3 + |\sigma_k|^3 + \frac{1}{\lambda_k^2} \right).$$

Denoting by $A$ the critical point of $\mathcal{F}_p, q$ to which the concentration points converge, we derive from the above expansion that
\[
Morse(u_{εk}) = \begin{cases} 
5p + 4q - Morse(F_{p,q}, A) - 1 & \text{if } \mathcal{L}_K(A) < 0, \\
5p + 4q - Morse(F_{p,q}, A) & \text{if } \mathcal{L}_K(A) > 0. 
\end{cases}
\]

Therefore we see that the Morse index of such a blowing solution is superabound by 4k, where \( k = 2p + q \).

Taking into account the Morse estimates of the Claim 1, we derive that a necessary condition for \( u_{εk} \) to blow up is that:

\[
Morse(u_{εk}) = 4k, \quad p = 0 \quad q = k
\]

and \( A \) is local minimum of \( F_{0,k} \) satisfying that \( \mathcal{L}_K(A) > 0 \). Hence we reach a contradiction with the assumption of Theorem 1.11. Therefore the theorem is fully proven.

**Proof of Theorem 1.9**

We consider the following subcritical approximation

\[
(P^ε) \begin{cases} 
P_g^4 u + 2Q_g = 2(κ - ε) \frac{Kε^u}{\int_M Kε^u} & \text{in } M, \\
P_g^3 u + T_g = 0 & \text{on } ∂M, \\
\frac{∂u}{∂n_g} = 0 & \text{on } ∂M,
\end{cases}
\]

and denote by \( II^ε \) its associated Euler-Lagrange functional.

Thanks to Moser-Trudinger inequality the functional \( II^ε \) achieves its minimum, say \( u^ε \).

We claim that \( u^ε \) converges to \( u^0 \) a critical point of \( II \). Since otherwise it would blow up and hence due to Lemma 3.3 and corollary 4.6 that \( u^ε \) has to concentrate at a maximum point \( a ∈ ∂M \) of \( K \) such that

\[
\frac{∂K}{∂n_g}(a) < 0
\]

which contradicts the assumption of the theorem. Hence Theorem 1.9 is fully proven.

### 7 Appendix

In this section we collect various estimates of the projected bubble \( ϕ_{a,λ} \) and its derivative as well as useful estimates of the functional \( II \) and its gradient. These estimates are standard and use elementary arguments. Hence we to keep the paper at reasonable length we omit their proofs.

#### 7.1 Bubble estimates

Using the conformal invariance of the Paneitz operator, Chang-Qing operator and the conformal Neuman operator (recalling \( H_g = 0 \)), the properties of the metric \( g_a \) (see (39)-(41)), the BVP satisfied by the \( ϕ_{a,λ} \)'s (see (49) and (50)), and the Green’s representation formula (54), we derive the following two Lemmas.

**Lemma 7.1** Assuming that \( ε \) is positive and small, \( 0 < 0 < 0_k \) where \( 0_k \) is as in Proposition 3.2, and \( λ ≥ \frac{1}{2} \), then

1) If \( d_g(a, ∂M) ≥ 4C_0 \), then

\[
ϕ_{a,λ}(·) = \hat{δ}_{a,λ}(·) + \ln \frac{λ}{2} + H(a, ·) + \frac{1}{4λ^2} Δ_{g_a} H(a, ·) + O \left( \frac{1}{λ^3} \right),
\]

and if \( a ∈ ∂M \), then

\[
ϕ_{a,λ}(·) = \hat{δ}_{a,λ}(·) + \ln \frac{λ}{2} + \frac{1}{2} H(a, ·) + \frac{1}{8λ^2} Δ_{g_a} H(a, ·) + O \left( \frac{1}{λ^3} \right),
\]

45
where $O(1)$ means $O_{a,\lambda,\epsilon}(1)$ and for it meaning see Section 2.

2) If $d_g(a, \partial M) \geq 4C_\theta$, then
\[
\frac{\lambda}{\partial \lambda} \frac{\partial \varphi_{a,\lambda}(\cdot)}{\partial \lambda} = \frac{2}{1 + \lambda^2 \chi^2(d_g(a, \cdot))} - \frac{1}{2\lambda^2} \Delta_{g_a} G(a, \cdot) + O \left( \frac{1}{\lambda^3} \right),
\]
and if $a \in \partial M$, then
\[
\frac{1}{\lambda} \frac{\partial \varphi_{a,\lambda}(\cdot)}{\partial a} = \frac{\chi_g(d_g(a, \cdot))}{d_g(a, \cdot)} \frac{2\lambda \exp^{-1}(\cdot)}{1 + \lambda^2 \chi^2(d_g(a, \cdot))} + \frac{1}{2\lambda} \frac{\partial \varphi_{a,\lambda}(\cdot)}{\partial a} + O \left( \frac{1}{\lambda^3} \right),
\]
where $O(1)$ is as in point 1).

3) If $d_g(a, \partial M) \geq 4C_\theta$, then
\[
\frac{1}{\lambda} \frac{\partial \varphi_{a,\lambda}(\cdot)}{\partial a} = \chi_g(d_g(a, \cdot)) \frac{\chi_g'(d_g(a, \cdot))}{d_g(a, \cdot)} \frac{2\lambda \exp^{-1}(\cdot)}{1 + \lambda^2 \chi^2(d_g(a, \cdot))} + \frac{1}{2\lambda} \frac{\partial \varphi_{a,\lambda}(\cdot)}{\partial a} + O \left( \frac{1}{\lambda^3} \right),
\]
where $O(1)$ is as in point 1).

**Lemma 7.2** Assuming that $\epsilon$ is small and positive, $0 < \varrho < \varrho_k$ and $0 < \eta < \eta_k$ where $0 < \varrho < \varrho_k$ and $0 < \eta < \eta_k$ are as in Proposition 3.2, and $\lambda \geq \frac{1}{\lambda}$, then we have:

1) If $d_g(a, \partial M) \geq 4C_\theta$, then
\[
\varphi_{a,\lambda}(\cdot) = G(a, \cdot) + \frac{1}{4\lambda^2} \Delta_{g_a} G(a, \cdot) + O \left( \frac{1}{\lambda^3} \right) \quad \text{in} \quad M \setminus B^a_\varrho(\eta),
\]
and if $a \in \partial M$, then
\[
\varphi_{a,\lambda}(\cdot) = \frac{1}{2} G(a, \cdot) + \frac{1}{8\lambda^2} \Delta_{g_a} G(a, \cdot) + O \left( \frac{1}{\lambda^3} \right) \quad \text{in} \quad M \setminus B^a_\varrho(\eta),
\]
where $O(1)$ means $O_{a,\lambda,\epsilon}(1)$ and for it meaning see Section 2.

2) If $d_g(a, \partial M) \geq 4C_\theta$, then
\[
\frac{1}{\lambda} \frac{\partial \varphi_{a,\lambda}(\cdot)}{\partial \lambda} = -\frac{1}{2\lambda^2} \Delta_{g_a} G(a, \cdot) + O \left( \frac{1}{\lambda^3} \right) \quad \text{in} \quad M \setminus B^a_\varrho(\eta),
\]
and if $a \in \partial M$, then
\[
\frac{1}{\lambda} \frac{\partial \varphi_{a,\lambda}(\cdot)}{\partial \lambda} = -\frac{1}{4\lambda^2} \Delta_{g_a} G(a, \cdot) + O \left( \frac{1}{\lambda^3} \right) \quad \text{in} \quad M \setminus B^a_\varrho(\eta),
\]
where $O(1)$ is as in point 1).

3) If $d_g(a, \partial M) \geq 4C_\theta$, then
\[
\frac{1}{\lambda} \frac{\partial \varphi_{a,\lambda}(\cdot)}{\partial a} = \frac{1}{\lambda} \frac{\partial G(a, \cdot)}{\partial a} + O \left( \frac{1}{\lambda^3} \right) \quad \text{in} \quad M \setminus B^a_\varrho(\eta),
\]
and if \( a \in \partial M \), then
\[
\frac{1}{\lambda} \frac{\partial \varphi_{a,\lambda}(\cdot)}{\partial a} = \frac{1}{2\lambda} \frac{\partial G(a,\cdot)}{\partial a} + O \left( \frac{1}{\lambda^3} \right) \quad \text{in} \quad M \setminus B_\delta^a(\eta),
\]
where \( O(1) \) is as in point 1).

Next, using the above two Lemmas, we obtain the following three Lemmas.

**Lemma 7.3** Assuming that \( \epsilon \) is small and positive, \( 0 < \varrho < \varrho_k \) where \( \varrho_k \) is as in Proposition 3.2, and \( \lambda \geq \frac{1}{\epsilon} \), then we have:

1) If \( d_g(a, \partial M) \geq 4C \varrho \), then
\[
< P_g \varphi_{a,\lambda}, \varphi_{a,\lambda} > = 32\pi^2 \ln \lambda - \frac{40\pi^2}{3} + 16\pi^2 H(a,a) + \frac{8\pi^2}{\lambda^2} \Delta g_a H(a,a) + O \left( \frac{1}{\lambda^3} \right),
\]
and if \( a \in \partial M \), then
\[
< P_g \varphi_{a,\lambda}, \varphi_{a,\lambda} > = 16\pi^2 \ln \lambda - \frac{20\pi^2}{3} + 4\pi^2 H(a,a) + \frac{2\pi^2}{\lambda^2} \Delta g_a H(a,a) + O \left( \frac{1}{\lambda^3} \right),
\]
where \( O(1) \) means \( O(a,\lambda,\epsilon)(1) \) and for its meaning see Section 2.

2) If \( d_g(a, \partial M) \geq 4C \varrho \), then
\[
< P_g \varphi_{a,\lambda}, \varphi_{a,\lambda} > = 16\pi^2 \ln \lambda - \frac{20\pi^2}{3} + 4\pi^2 H(a,a) + \frac{2\pi^2}{\lambda^2} \Delta g_a H(a,a) + O \left( \frac{1}{\lambda^3} \right),
\]
and if \( a \in \partial M \), then
\[
< P_g \varphi_{a,\lambda}, \varphi_{a,\lambda} > = 8\pi^2 - \frac{2\pi^2}{\lambda^2} \Delta g_a H(a,a) + O \left( \frac{1}{\lambda^3} \right),
\]
where \( O(1) \) is as in point 1).

3) If \( d_g(a, \partial M) \geq 4C \varrho \), then
\[
< P_g \varphi_{a,\lambda}, \varphi_{a,\lambda} > = 16\pi^2 \ln \lambda - \frac{40\pi^2}{3} + 16\pi^2 H(a,a) + \frac{8\pi^2}{\lambda^2} \Delta g_a H(a,a) + O \left( \frac{1}{\lambda^3} \right),
\]
and if \( a \in \partial M \), then
\[
< P_g \varphi_{a,\lambda}, \varphi_{a,\lambda} > = 3\pi^2 + \frac{4\pi^2}{\lambda} \frac{\partial H(a,a)}{\partial a} + O \left( \frac{1}{\lambda^3} \right).
\]
where \( O(1) \) is as in point 1).

**Lemma 7.4** Assuming that \( \epsilon \) is small and positive, \( 0 < \varrho < \varrho_k \), \( 0 < \eta < \eta_k \), \( \Lambda > \Lambda_k \) where \( 0 < \varrho < \varrho_k \), \( 0 < \eta < \eta_k \), and \( \Lambda_k \) are as in Proposition 3.2, \( a_i, a_j \in M \), \( d_g(a_i, a_j) \geq 4C \eta \), \( \frac{1}{\Lambda} \leq \frac{\lambda_i}{\lambda_j} \leq \Lambda \), and \( \lambda_i, \lambda_j \geq \frac{1}{\epsilon} \) where \( C \) is as in (41), then we have:

1) \( (1)_{i,j} \) If \( d_g(a_i, \partial M) \geq 4C \varrho \) and \( d_g(a_j, \partial M) \geq 4C \varrho \), then
\[
< P_g \varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j} > = 16\pi^2 G(a_j, a_i) + \frac{4\pi^2}{\lambda_i^2} \Delta g_{a_i} G(a_i, a_j) + \frac{4\pi^2}{\lambda_j^2} \Delta g_{a_j} G(a_j, a_i) + O \left( \frac{1}{\lambda_i^2} + \frac{1}{\lambda_j^2} \right),
\]
where $O(1)$ means here $O_{A,\bar{\lambda},c}(1)$ with $A = (a_i,a_j)$ and $\bar{\lambda} = (\lambda_i,\lambda_j)$ and for the meaning of $O_{A,\bar{\lambda},c}(1)$, see Section 2.

2) \hspace{1cm} (1)_{i,j} \hspace{0.5cm} \text{If } d_g(a_i,\partial M) \geq 4\bar{C}_g \text{ and } d_g(a_j,\partial M) \geq 4\bar{C}_g, \text{ then}

\begin{equation}
<P_{g}\varphi_{a_i,\lambda_i},\varphi_{a_j,\lambda_j}> = 8\pi^2 G(a_j,a_i) + \frac{2\pi^2}{\lambda_i^2} \Delta_{g_{a_i}} G(a_i,a_j) + \frac{2\pi^2}{\lambda_j^2} \Delta_{g_{a_j}} G(a_j,a_i) + O\left(\frac{1}{\lambda_i^3} + \frac{1}{\lambda_j^3}\right),
\end{equation}

and

\begin{equation}
<\partial_{\lambda_i} P_{g}\varphi_{a_i,\lambda_i},\varphi_{a_j,\lambda_j}> = -\frac{8\pi^2}{\lambda_j^2} \Delta_{g_{a_j}} G(a_j,a_i) + O\left(\frac{1}{\lambda_j^3}\right),
\end{equation}

(2)_{i,j} \hspace{0.5cm} \text{if } d_g(a_i,\partial M) \geq 4\bar{C}_g \text{ and } a_j \in \partial M, \text{ then}

\begin{equation}
<\partial_{\lambda_j} P_{g}\varphi_{a_i,\lambda_i},\varphi_{a_j,\lambda_j}> = -\frac{4\pi^2}{\lambda_j^2} \Delta_{g_{a_j}} G(a_j,a_i) + O\left(\frac{1}{\lambda_j^3}\right),
\end{equation}

(2)'_{i,j} \hspace{0.5cm} \text{if } \lambda_i \in \partial M \text{ and } d_g(a_j,\partial M) \geq 4\bar{C}_g, \text{ then}

\begin{equation}
<\partial_{\lambda_j} P_{g}\varphi_{a_i,\lambda_i},\varphi_{a_j,\lambda_j}> = -\frac{4\pi^2}{\lambda_j^2} \Delta_{g_{a_j}} G(a_j,a_i) + O\left(\frac{1}{\lambda_j^3}\right),
\end{equation}

(3)_{i,j} \hspace{0.5cm} \text{if } a_i \in \partial M \text{ and } a_j \in \partial M, \text{ then}

\begin{equation}
<\partial_{\lambda_j} P_{g}\varphi_{a_i,\lambda_i},\varphi_{a_j,\lambda_j}> = -\frac{2\pi^2}{\lambda_j^2} \Delta_{g_{a_j}} G(a_j,a_i) + O\left(\frac{1}{\lambda_j^3}\right),
\end{equation}

where $O(1)$ is as in point 1).

3) \hspace{1cm} (1)_{i,j} \hspace{0.5cm} \text{If } d_g(a_i,\partial M) \geq 4\bar{C}_g \text{ and } d_g(a_j,\partial M) \geq 4\bar{C}_g, \text{ then}

\begin{equation}
<P_{g}\varphi_{a_i,\lambda_i},\frac{1}{\lambda_j} \partial_{a_j} \varphi_{a_j,\lambda_j}> = \frac{16\pi^2}{\lambda_j} \frac{\partial G(a_j,a_i)}{\partial a_j} + O\left(\frac{1}{\lambda_j}\right),
\end{equation}

(2)_{i,j} \hspace{0.5cm} \text{if } d_g(a_i,\partial M) \geq 4\bar{C}_g \text{ and } a_j \in \partial M,

\begin{equation}
<P_{g}\varphi_{a_i,\lambda_i},\frac{1}{\lambda_j} \partial_{a_j} \varphi_{a_j,\lambda_j}> = \frac{8\pi^2}{\lambda_j} \frac{\partial G(a_j,a_i)}{\partial a_j} + O\left(\frac{1}{\lambda_j}\right),
\end{equation}

(2)'_{i,j} \hspace{0.5cm} \text{if } a_i \in \partial M \text{ and } d_g(a_j,\partial M) \geq 4\bar{C}_g, \text{ then}

\begin{equation}
<P_{g}\varphi_{a_i,\lambda_i},\frac{1}{\lambda_j} \partial_{a_j} \varphi_{a_j,\lambda_j}> = \frac{8\pi^2}{\lambda_j} \frac{\partial G(a_j,a_i)}{\partial a_j} + O\left(\frac{1}{\lambda_j}\right).
\end{equation}
(3) If $a_i \in \partial M$ and $a_j \in \partial M$, then
\[
< P_g \varphi_{a_i, \lambda}, \frac{1}{\lambda_j} \frac{\partial \varphi_{a_i, \lambda}}{\partial a_j} > = \frac{4\pi^2}{\lambda_j} \frac{\partial G(a_j, a_i)}{\partial a_j} + O \left( \frac{1}{\lambda_j^3} \right).
\]

**Lemma 7.5**
1) If $\epsilon$ is small and positive, $a \in \partial M$, $q \in \mathbb{N}^+$, and $\lambda \geq \frac{1}{\epsilon}$, then there holds
\[
C^{-1} \lambda^{q-4} \leq \int_M e^{A\varphi_{a, \lambda}} dV_g \leq C \lambda^{q-4},
\]
where $C$ is independent of $a$, $\lambda$, and $\epsilon$.

2) If $\epsilon$ is positive and small, $a_i, a_j \in \partial M$, $\lambda \geq \frac{1}{\epsilon}$ and $\lambda d_g(a_i, a_j) \geq 4\overline{C} R$, then we have
\[
< P_g^{\lambda, 3} \varphi_{a_i, \lambda}, \varphi_{a_j, \lambda} > \leq 4\pi^2 G(a_i, a_j) + O(1),
\]
where $O(1)$ means here $O_{A, \lambda, \epsilon}(1)$ with $A = (a_i, a_j)$, and for the meaning of $O_{A, \lambda, \epsilon}(1)$, see section 2.

3) If $\epsilon$ is positive and small, $a_i, a_j \in \partial M$, $\lambda, \lambda_j \geq \frac{1}{\epsilon}$, $\frac{1}{4} \leq \frac{\alpha_j}{\lambda_j} \leq \Lambda$ and $\lambda d_g(a_i, a_j) \geq 4\overline{C} R$, then we have
\[
< P_g^{\lambda, 3} \varphi_{a_i, \lambda}, \varphi_{a_j, \lambda} > \leq 4\pi^2 G(a_i, a_j) + O(1),
\]
where $O(1)$ means here $O_{A, \lambda, \epsilon}(1)$ with $A = (a_i, a_j)$ and $\lambda = (\lambda_i, \lambda_j)$ and for the meaning of $O_{A, \lambda, \epsilon}(1)$, see Section 2.

**Lemma 7.6**
Let $q \in \mathbb{N}^+$, $R$ be a large positive constant, $\epsilon$ be a small positive number, $\alpha_i \geq 0$, $i = 1, \cdots, q$, $\sum_{i=1}^q \alpha_i = k$, $\lambda \geq \frac{1}{\epsilon}$ and $u = \sum_{i=1}^q \alpha_i \varphi_{a_i, \lambda}$ with $a_i \in \partial M$ for $i = 1, \cdots, q$. Assuming that there exist two positive integer $i, j \in \{1, \cdots, p\}$ with $i \neq j$ such that $\lambda d_g(a_i, a_j) \leq \frac{\overline{C} R}{\epsilon}$, where $\overline{C}$ is as in (41), then we have
\[
H(u) \leq H(v) + O(\ln \hat{R}),
\]
with
\[
v := \sum_{k \leq p, k \neq i, j} \alpha_k \varphi_{a_k, \lambda} + (\alpha_i + \alpha_j) \varphi_{a_i, \lambda},
\]
where here $O(1)$ stand for $O_{G, A, \lambda, \epsilon}(1)$, with $\hat{\alpha} = (\alpha_1, \cdots, \alpha_q)$ and $A = (a_1, \cdots, a_q)$, and for the meaning of $O_{G, A, \lambda, \epsilon}(1)$, we refer the reader to Section 2.

**Lemma 7.7**
1) If $\epsilon$ is positive and small, $a_i, a_j \in \partial M$, $\lambda \geq \frac{1}{\epsilon}$ and $\lambda d_g(a_i, a_j) \geq 4\overline{C} R$, then
\[
\varphi_{a_j, \lambda}(\cdot) = \frac{1}{2} G(a_j, \cdot) + O(1) \text{ in } B_{\overline{C}}^{\alpha_i}(\frac{R}{\lambda_j}),
\]
where here $O(1)$ means here $O_{A, \lambda, \epsilon}(1)$, with $A = (a_i, a_j)$, and for the meaning of $O_{A, \lambda, \epsilon}(1)$, see section 2.

2) If $\epsilon$ is positive and small, $a_i, a_j \in \partial M$, $\lambda_i, \lambda_j \geq \frac{1}{\epsilon}$, $\frac{1}{4} \leq \frac{\alpha_j}{\lambda_j} \leq \Lambda$, and $\lambda_i d_g(a_i, a_j) \geq 4\overline{C} R$, then
\[
\varphi_{a_j, \lambda_j}(\cdot) = \frac{1}{2} G(a_j, \cdot) + O(1) \text{ in } B_{\overline{C}}^{\alpha_i}(\frac{R}{\lambda_j}),
\]
where here $O(1)$ means here $O_{A, \lambda, \epsilon}(1)$, with $A = (a_i, a_j)$, $\lambda = (\lambda_i, \lambda_j)$ and for the meaning of $O_{A, \lambda, \epsilon}(1)$, see Section 2.
Lemma 7.8 Let $\pi = \sum_{i=1}^{p} \alpha_i \phi_i + \sum_{i=p+1}^{p+q} \alpha_i \phi_i + \sum_{r=1}^{q} \beta_r v_r \in V(p,q,\varepsilon,\eta)$. Then there holds

$$
\int_M Ke^{\pi} dv_g = \Gamma \left[ 1 + \frac{1}{2} \sum_{i=p+1}^{p+q} \frac{\partial \varphi_i}{\partial x_j} (a_i) \Gamma_i + O(\frac{1}{\lambda^2}) \right]
$$

where

$$
\Gamma_i := \frac{\pi^2 \lambda^{8a_i - 4} \varphi_i(a_i) \chi_i(a_i)}{4(2a_i - 1)(4a_i - 1)}, \quad \Gamma := 2 \sum_{i=1}^{p} \Gamma_i + \sum_{i=p+1}^{p+q} \Gamma_i
$$

and $\chi_i(a_i)$ is defined in (157) and (158). Moreover setting

$$
\tau_i := 1 - \frac{k \Gamma_i}{\int_M Ke^{\pi}}
$$

there holds

$$
\sum_{i=1}^{p} 2 \tau_i + \sum_{i=p+1}^{p+q} \tau_i = \frac{1}{2} \sum_{i=p+1}^{p+q} \frac{\partial \varphi_i}{\partial x_j} (a_i) \frac{\lambda_i \varphi_i(a_i)}{\lambda_i \varphi_i(a_i)} + O(\sum_k |\tau_k|^2 + \frac{1}{\lambda^2})
$$

Next, using the relation between $(\mathbb{R}^4, g_{\mathbb{R}^4})$ and $(S^1, g_{S^1})$, the spectral property of the Paneitz operator $P_{g_{S^1}}$, a standard doubling argument to deal with boundary points, and a standard blow-up argument (as in Brendle [18]), we obtain the following last two lemmas of this subsection.

Lemma 7.9 Assuming that $0 < \varrho < \varrho_0$, then there exists $\bar{\Gamma}_0 := \Gamma_0(\varrho)$ and $\bar{\Lambda}_0 := \Lambda_0(\varrho)$ two large positive constant such that for every $a \in M$ such that either $d_g(a, \partial M) \geq \overline{C} \varrho$ or $a \in \partial M$, $\lambda \geq \Lambda_0$, and $w \in F_{a,\lambda} := \{w \in \mathcal{H}_{a,\lambda} : \|w\|_{(Q,T)} = <\varphi_{a,\lambda}, w > > p+3 = 0\}$, we have

$$
\int_M e^{\delta_{a,\lambda}} w^2 dv_{g_a} \leq \Gamma_0 \|w\|^2.
$$

Lemma 7.10 Assuming that $0 < \varrho < \varrho_k$, $0 \leq \eta < \eta_k$, then there exists a small positive constant $c_0 := c_0(\varrho, \eta)$ and a large positive constant $\Lambda_0 := \Lambda_0(\varrho, \eta)$ such that for every $(p,q) \in \mathbb{N}^2$ such that $2p + q = k$, for every $a_i \in M$ concentrations points $i = 1, \ldots, p + q$ such that $d_g(a_i, a_j) \geq \overline{C} \varrho$ for $i \neq j = 1, \ldots, p + q$, $d_g(a, \partial M) \geq \overline{C} \varrho$, $i = 1, \ldots, p$, where $\overline{C}$ is as in (41), and $a_i \in \partial M$, $i = p + 1, p + q$, for every $\lambda_i > 0$ concentrations parameters satisfying $\lambda_i \geq \Lambda_0$, with $i = 1, \ldots, p + q$, and for every $w \in E_{A,\lambda}^{a_i} = \bigcap_{i=1}^{p+q} E_{A,\lambda}^{a_i}$ with $A := (a_1, \ldots, a_{p+q})$, $\lambda := (\lambda_1, \ldots, \lambda_{p+q})$ and $E_{A,\lambda}^{a_i} = \{w \in \mathcal{H}_{a,\lambda} : <\varphi_{a_i,\lambda}, w > > p+3 \leq \frac{\partial \varphi_{a_i,\lambda}}{\partial x_i}, w > > p+3 \leq \frac{\partial \varphi_{a_i,\lambda}}{\partial x_i}, w > > p+3 = \|w\|_{(Q,T)} = 0\}$, there holds

$$
\|w\|^2 - 4 \sum_{i=1}^{p+q} \int_M e^{\delta_{a_i,\lambda}} w^2 dv_{g_a} \geq c_0 \|w\|^2.
$$

7.2 Gradient and energy estimates

This section is devoted to the expansion of the functional $II$ and its gradient in the neighborhood of potential critical points at infinity.
7.2.1 Expansion of the Euler-Lagrange functional near Infinity

In this subsection, we derive an expansion of the Euler-Lagrange functional II for the part at infinity which is characterized by a useful (topologically) piece of $B^p_q(M, \partial M)$ via the variational bubbles $\varphi_{a, \lambda}$ (where $B^p_q(M, \partial M)$ is as in (21)), namely for elements at infinity with zero $w$-part in the representation (103). Indeed, we have:

**Lemma 7.11** Assuming that $(p, q) \in \mathbb{N}^2$ such that $2p + q = k$, $0 < q < q_k$, $0 < \eta < \eta_k$, and $0 < \epsilon \leq \epsilon_k$, where $q_k$ and $\eta_k$ are given by Proposition 3.2, and the $\epsilon_k$ is given by (101), then for $a_i \in M$ concentration points, $\alpha_i$ masses, $\lambda_i$ concentration parameters ($i = 1, \cdots, p + q$), and $\beta_r$ negativity parameters ($r = 1, \cdots, \tilde{k}$) satisfying (106), we have

\begin{equation}
II \left( \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - (v_r)(Q,T)) \right) = -\frac{20}{3} k \pi^2 - 4k \pi^2 \ln \left( \frac{k \pi^2}{6} \right) - 8\pi^2 F_{p,q}(a_1, \ldots, a_{p+q})
\end{equation}

\begin{equation}
16\pi^2 \left( 2 \sum_{i=1}^{p} (\alpha_i - 1)^2 \ln \lambda_i + \sum_{i=p+1}^{p+q} (\alpha_i - 1)^2 \ln \lambda_i \right) + \sum_{r=1}^{\tilde{k}} \mu_k \beta_r^2 - 4\pi^2 \left[ \sum_{i=1}^{p} 2\sigma_i^2 + \sum_{i=p+1}^{p+q} \sigma_i^2 \right]
\end{equation}

\begin{equation}
-2\pi^2 \sum_{i=p+1}^{p+q} \frac{1}{\lambda_i} \frac{\partial F^A_i}{\partial a_{n_{a_i}}} (a_i) + O \left( \sum_{r=1}^{\tilde{k}} |\beta_r|^3 + \sum_{k=1}^{p+q} |\alpha_k - 1|^3 + |\sigma_k|^3 + \frac{1}{\lambda_k^2} \right),
\end{equation}

where $F_{p,q}$ is as in (8), $O(1)$ means here $O_{\alpha, A, \lambda, \beta, \epsilon}(1)$ with $\alpha = (\alpha_1, \cdots, \alpha_{p+q}), A := (a_1, \cdots, a_{p+q})$, $\bar{\alpha} := (\alpha_1, \cdots, \lambda_{p+q}), \beta := (\beta_1, \cdots, \beta_{\tilde{k}})$ and for $i = 1, \cdots, p + q$,

$$\tilde{\alpha}_i := 1 - \frac{\Gamma_i \alpha_i}{\Gamma}, \quad \Gamma := 2 \sum_{i=1}^{p} \gamma_i + \sum_{i=p+1}^{p+q} \gamma_i, \quad \Gamma_i := \frac{\pi^2 \lambda_i^{2\alpha_i - 4} F^A_i(a_i) G_i(a_i)}{4 (2\alpha_i - 1)(4\alpha_i - 1)},$$

with for $i = 1, \cdots, p$, $F^A_i$ is as in (12),

\begin{equation}
G_i(a_i) := e^{\frac{4}{p+q+1} \sum_{j=p+1}^{p+q} \alpha_j (\alpha_j - 1) G(a_j, a_i) + \frac{1}{p+q+1} \sum_{j=p+1}^{p+q} \beta_j \Delta_{\alpha_j} G(a_j, a_i)} e^{-\frac{1}{p+q+1} \sum_{j=p+1}^{p+q} \alpha_j \Delta_{\beta_j} G(a_j, a_i)} e^{\frac{1}{p+q+1} \sum_{j=p+1}^{p+q} \beta_j \Delta_{\beta_j} G(a_j, a_i)} e^{\frac{1}{p+q+1} \sum_{j=p+1}^{p+q} \beta_j \Delta_{\beta_j} G(a_j, a_i)},
\end{equation}

for $i = p+1, \cdots, p+q$, $F^A_i$ is as in (13),

\begin{equation}
G_i(a_i) := e^{\frac{4}{p+q+1} \sum_{j=p+1}^{p+q} \alpha_j (\alpha_j - 1) G(a_j, a_i) + \frac{1}{p+q+1} \sum_{j=p+1}^{p+q} \beta_j \Delta_{\alpha_j} G(a_j, a_i) + \frac{1}{p+q+1} \sum_{j=p+1}^{p+q} \alpha_j \Delta_{\beta_j} G(a_j, a_i) + \frac{1}{p+q+1} \sum_{j=p+1}^{p+q} \beta_j \Delta_{\beta_j} G(a_j, a_i)} e^{-\frac{1}{p+q+1} \sum_{j=p+1}^{p+q} \alpha_j \Delta_{\beta_j} G(a_j, a_i)} e^{\frac{1}{p+q+1} \sum_{j=p+1}^{p+q} \beta_j \Delta_{\beta_j} G(a_j, a_i)} e^{\frac{1}{p+q+1} \sum_{j=p+1}^{p+q} \beta_j \Delta_{\beta_j} G(a_j, a_i)},
\end{equation}

7.2.2 Expansion of the gradient near infinity

As mentioned above, in this subsection, we perform an expansion of $\nabla II$ on the same set as in the previous subsection. To do so, we start with the gradient of $II$ in the direction of $\lambda$. Precisely, we have:

**Lemma 7.12** Assuming that $(p, q) \in \mathbb{N}^2$ such that $2p + q = k$, $0 < q < q_k$, $0 < \eta < \eta_k$, and $0 < \epsilon \leq \epsilon_k$, where $q_k$ and $\eta_k$ are given by Proposition 3.2, and $\epsilon_k$ is given by (101), then for $a_i \in M$ concentration points, $\alpha_i$ masses, $\lambda_i$ concentration parameters ($i = 1, \cdots, p + q$), and $\beta_r$ negativity parameters ($r = 1, \cdots, \tilde{k}$) satisfying (106), we have for every $j = 1, \cdots, p$, there holds

\begin{equation}
< \nabla II \left( \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - (v_r)(Q,T)), \lambda \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_j} \right) > = 32\pi^2 \alpha_j \frac{\Gamma_j}{\lambda_j^2} \left( \frac{\Delta_{\alpha_j} F^A_j(a_j)}{F^A_j(a_j)} - \frac{2}{3} R_g(a_j) \right)
\end{equation}

\begin{equation}
+ O \left( \sum_{i=1}^{p+q} \alpha_i - 1 \right)^2 + \sum_{i=1}^{p+q} |\tau_i|^2 + \sum_{r=1}^{\tilde{k}} |\beta_r|^3 + \sum_{i=1}^{p+q} \frac{1}{\lambda_i^2} \right),
\end{equation}
and for every \( j = p + 1, \ldots, p + q \), there holds
\[
< \nabla II \left( \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - \overline{v_r}) (Q, T), \lambda \right), \lambda > = 16\pi^2 \alpha_j \tau_j - \frac{6\pi^2}{\lambda_j} \frac{1}{\mathcal{F}_j(a_j)} \frac{\partial \mathcal{F}_j^A(a_j)}{\partial \alpha_j}
\]
\[
+ O \left( \sum_{i=1}^{p+q} |\alpha_i - 1|^2 + \sum_{r=1}^{\tilde{k}} |\tau_r|^2 + \sum_{r=1}^{\tilde{k}} |\beta_r|^2 + \sum_{i=1}^{p+q} \frac{1}{\lambda^2_i} \right),
\]
where \( A := (a_1, \ldots, a_{p+q}) \), \( O(1) \) and for \( i = 1, \ldots, p + q \), \( \mathcal{F}_i^A \) is defined in (12) (resp. in (13)) and \( \tau_i := 1 - \frac{k\Gamma_i}{D} \), where \( \Gamma_i := \frac{\pi^2}{4\alpha_i(2\alpha_i - 1)(4\alpha_i - 1)} D := \int_M K(x) e^{4\sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r v_r(x)) dV_g(x) \).

Using Lemma 7.12, we have the following corollary:

**Corollary 7.13** Assuming that \( (p, q) \in \mathbb{N}^2 \) such that \( 2p + q = k \) and \( q \neq 0 \) and \( 0 < \varrho < \varrho_k \), \( 0 < \eta < \eta_k \), and \( 0 < \varepsilon \leq \varepsilon_k \), where \( \varrho_k \) and \( \eta_k \) are given by Proposition 3.2, and \( \varepsilon_k \) is given by (101), then for \( a_i \in M \) concentration points, \( \alpha_i \) masses, \( \lambda_i \) concentration parameters \( (i = 1, \ldots, p+q) \), and \( \beta_i \) negativity parameters \( (r = 1, \ldots, \tilde{k}) \) satisfying (106), we have that
\[
< \nabla II \left( \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - \overline{v_r}) (Q, T), \lambda \right), \lambda > = 2\pi^2 \sum_{i=p+1}^{p+q} \frac{1}{\lambda_i} \frac{\partial \mathcal{F}_i^A(a_i)}{\partial \alpha_i} \sum_{i=p+1}^{p+q} \frac{\partial \mathcal{F}_i^A(a_i)}{\partial \alpha_i}
\]
\[
+ O \left( \sum_{i=1}^{p+q} |\alpha_i - 1|^2 + \sum_{r=1}^{\tilde{k}} |\tau_r|^2 + \sum_{r=1}^{\tilde{k}} |\beta_r|^2 + \sum_{i=1}^{p+q} \frac{1}{\lambda^2_i} \right),
\]
where \( A := (a_1, \ldots, a_{p+q}) \), \( O(1) \) as in Lemma 7.11, and for \( i = 1, \ldots, p + q \), \( \mathcal{F}_i^A \) as in Lemma 7.11.

Next, we give the estimate of the gradient of the Euler-Lagrange functional II in the direction of A. Precisely, we have that:

**Lemma 7.14** Assuming that \( (p, q) \in \mathbb{N}^2 \) such that \( 2p + q = k \), \( 0 < \varrho < \varrho_k \), \( 0 < \eta < \eta_k \), and \( 0 < \varepsilon \leq \varepsilon_k \), where \( \varrho_k \) and \( \eta_k \) are given by Proposition 3.2, and \( \varepsilon_k \) is given by (101), then for \( a_i \in M \) concentration points, \( \alpha_i \) masses, \( \lambda_i \) concentration parameters \( (i = 1, \ldots, p+q) \), and \( \beta_i \) negativity parameters \( (r = 1, \ldots, \tilde{k}) \) satisfying (106), we have that for every \( j = p + 1, \ldots, p + q \), there holds
\[
< \nabla II \left( \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - \overline{v_r}) (Q, T), \lambda \right), \lambda > = -\frac{4\pi^2}{\lambda_j} \frac{\partial \mathcal{F}_j^A(a_j)}{\partial \alpha_j}
\]
\[
+ O \left( \sum_{i=1}^{p+q} |\alpha_i - 1|^2 + \sum_{i=1}^{p+q} \frac{1}{\lambda^2_i} + \sum_{r=1}^{\tilde{k}} |\beta_r|^2 + \sum_{i=1}^{p+q} \frac{1}{\lambda^2_i} \right),
\]
for every \( j = p + 1, \ldots, p + q \), there holds
\[
< \nabla II \left( \sum_{i=1}^{p+q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - \overline{v_r}) (Q, T), \lambda \right), \lambda > = 6\pi^2 \tau_j - \frac{4\pi^2}{\lambda_j} \frac{\partial \mathcal{F}_j^A(a_j)}{\partial \alpha_j}
\]
\[
+ O \left( \sum_{i=1}^{p+q} |\alpha_i - 1|^2 + \sum_{i=1}^{p+q} \frac{1}{\lambda^2_i} + \sum_{r=1}^{\tilde{k}} |\beta_r|^2 + \sum_{i=1}^{p+q} \frac{1}{\lambda^2_i} \right),
\]
where \( \hat{g} := g \triangledown M, A := (a_1, \ldots, a_{p+q}) \), \( O(1) \) is as in Lemma 7.11, and for \( i = 1, \ldots, p + q \), \( \mathcal{F}_i^A \) is as in Lemma 7.11 and \( \tau_i \) is as in Lemma 7.12.
References

[1] Ahmedou, M. and Ndiaye C. B., *Morse theory and the resonant Q-curvature problem*, preprint 2014 submitted.

[2] Ambrosetti, A., Li, Y.Y., Malchiodi, A., *On the Yamabe and the scalar curvature problems under boundary conditions*, Math. Ann. 322 (2002), 667–699.

[3] Aubin T., Some nonlinear problems in Riemannian geometry, Springer Monographs in Mathematics, Springer-Verlag, Berlin 1998.

[4] Aubin T., Bahri A., *Methods of algebraic topology for the problem of prescribed scalar curvature*. J. Math. Pures Appl. (9) 76 (1997), no. 6, 525-549.

[5] Aubin T., Bahri A., *A topological hypothesis for the problem of prescribed scalar curvature*. J. Math. Pures Appl. (9) 76 (1997), no. 10, 843-850.

[6] Bahri A., Critical points at infinity in some variational problems, Research Notes in Mathematics, 182, Longman-Pitman, London, 1989.

[7] Bahri A., *Un problème variationel sans compacité dans la géométrie de contact*, Comptes Rendus Mathématique Académie des Sciences, Paris, Série I 299 (1984) 754-760.

[8] Bahri A., *Pseudo-orbits of Contact Forms*, Pitman Research Notes in Mathematics Series, 173. Longman Scientific & Technical, Harlow, 1988.

[9] Bahri A., *An invariant for Yamabe-type flows with applications to scalar-curvature problems in high dimension*. A celebration of John F. Nash, Jr. Duke Math. J. 81 (1996), no. 2, 323-466.

[10] Bahri A., Brezis H., *Equations elliptiques non linéaires sur des variétés avec exposant de Sobolev critique*. C. R. Acad. Sci. Paris Sr. I Math. 307 (1988), no. 11, 573–576.

[11] Bahri A., Coron J.M., *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. 41-3 (1988), 253-294.

[12] Bahri A., and Rabinowitz P., *Periodic solutions of Hamiltonian systems of 3-body*, Ann. Inst. Poincaré, Anal. non linéaire 8(1991), 561-649.

[13] Branson T.P., Oersted., *Explicit functional determinants in four dimensions*, Proc. Amer. Math. Soc 113-3(1991), 669-682.

[14] Bredon G.E., *Topology and geometry*, Graduate Texts in Mathematics, 139, 1997. cand., 57-2 (1995), 293-345.

[15] Brendle S., *Curvature flows on surfaces with boundary*, Math. Ann. 324 (2002), no. 3, 491-519.

[16] Brendle S., *A family of curvature flows on surfaces with boundary*, Math. Z. 241, 829-869 (2002).

[17] Brendle S., *A generalization of the Yamabe flow for manifolds with boundary*. Asian J. Math. 6 (2002), no. 4, 625–644.

[18] Brendle S., *Convergence of the Yamabe flow for arbitrary initial energy*, J. Diff. Geom. 69 (2005),217-278.

[19] Brendle S., Chen S. Z., *An existence theorem for the Yamabe problem on manifolds with boundary*. J. Eur. Math. Soc. (JEMS) 16 (2014), no. 5, 991-1016.

[20] Catino G., Ndiaye C. B., *Integral pinching results for manifolds with boundary*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 9 (2010), 785-813.
[21] Chang, K.C., Liu, J.Q., A prescribing geodesic curvature problem, Math.Z. 213(1996), 343-365.
[22] Chang S.Y.A., Qing J., Yang P.C., Compactification of a class of conformally flat 4-manifold, Invent. Math. 142-1(2000), 65-93.
[23] Chang S.Y.A., Qing J., The Zeta Functional Determinants on manifolds with boundary I. The Formula, Journal of Functional Analysis 147, 327-362 (1997)
[24] Chang S.Y.A., Qing J., The Zeta Functional Determinants on manifolds with boundary II. Extremal Metrics and Compactness of Isospectral Set, Journal of Functional Analysis 147, 363-399 (1997)
[25] Chang S.Y.A., Qing J., Yang P.C., On the Chern-Gauss-Bonnet integral for conformal metrics on $\mathbb{R}^4$, Duke Math. J.03-3(2000),523-544.
[26] Chen, S.S., Conformal deformation on manifolds with boundary, Geom. Funct. Anal. 19 (2009), no. 4, 1029–1064.
[27] Cherrier, P., Problèmes de Neumann non linéaires sur les variétés riemanniennes, J. Funct. Anal. 57 (1984),154–206.
[28] Dold A., Algebraic topology, Lectures on algebraic topology. Reprint of the 1972 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. xii+377 pp. ISBN: 3-540-58660-1 55-02 (01A75).
[29] Djadli Z., Malchiodi A., Existence of conformal metrics with constant $Q$-curvature, Ann. of Math. (2) 168 (2008), no. 3, 813–858.
[30] Dajdli Z., Malchiodi, A., Ould Ahmedou, M. Prescribing scalar and boundary mean curvature on the three dimensional half sphere, J... Geom. Anal. 13 (2003), 255–289.
[31] Druet O., Robert F., Bubbling phenomena for fourth-order four-dimensional PDEs with exponential growth, Proc. Amer. Math. Soc 134(2006) no. 3, 897-908.
[32] Escobar J.F., The Yamabe problem on manifolds with boundary, Journal Differential Geometry 35 (1992) no.1, 21-84.
[33] Escobar J.F., Conformal deformation of a riemannian metric to a scalar flat with constant mean curvature on the boundary, Ann. of Math. 136(1992), 1–50.
[34] Gilbar D., Trudinger N., Elliptic Partial Differential Equations of Second Order, 2nd edition, Springer-Verlag, 1983.
[35] Günther M., Conformal normal coordinates, Ann. Global.... Anal. Geom. 11 (1993), 173-184.
[36] Gursky M., The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE, Comm. Math. Phys. 207-1 (1999),131-143.
[37] Hebey, E., Robert, F., Asymptotic analysis for fourth order Paneitz equations with critical growth, Adv. Calc. Var. 4 (2011), no. 3, 229275.
[38] Hebey, E., Robert, F., Wen, Y., Compactness and global estimates for a fourth order equation of critical Sobolev growth arising from conformal geometry, Commun. Contemp. Math. 8 (2006), 965.
[39] Druet, O., Hebey, E., Robert, F., Blow-up theory for elliptic PDEs in Riemannian geometry. Mathematical Notes, 45. Princeton University Press, Princeton, NJ, 2004.
[40] Han, Z.C., Li, Y.Y. The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature, Comm. Anal. Geom 8(2000), 809–869.
[41] Han, Z.C., Li, Y.Y. The Yamabe problem on manifolds with boundary: existence and compactness results, Duke Math. J. 99(1999), 489–542.
[42] Kallel S., Karoui R., *Symmetric joins and weighted barycenters*, Advanced Nonlinear Studies, 11(2011), 117–143.

[43] Lee J., Parker T., *The Yamabe problem*, Bull. A.M.S. 17 (1987), 37-81.

[44] Lin C. S., Wei J., *Sharp estimates for bubbling solutions of a fourth order mean field equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 6 (2007), no. 4, 599-630.

[45] Lucia M., *A deformation lemma with an application to a mean field equation*, Topol. Methods Nonlinear Anal. 30 (2007), no. 1, 113–138.

[46] Lucia M., Orák J., *A minimax theorem in the presence of unbounded Palais-Smale sequences*, Israel J. Math. 172 (2009).

[47] Malchiodi A., *Morse theory and a scalar field equation on compact surfaces*, Adv. Diff. Eq., 13 (2008), 1109–1129.

[48] Malchiodi A., *Compactness of solutions to some geometric fourth-order equations*, J. Reine Angew. Math. 594 (2006), 137–174.

[49] Marques F. C., *Existence results for the Yamabe problem on manifolds with boundary*, Indiana Univ. Math. J. (2005), 1599-1620.

[50] J. Mostovoy, *Lattices in \( \mathbb{C} \) and finite subsets of the circle*, Amer. Math. Monthly 111 (2004), no. 4, 357–360.

[51] Ndiaye C. B., *Constant \( Q \)-curvature metrics in arbitrary dimension*, J. Funct. Anal. 251 (2007), no. 1, 1–58.

[52] Ndiaye C.B., *Conformal metrics with constant \( Q \)-curvature for manifolds with boundary*, Comm. Anal. Geom. 16 (2008), no. 5, 1049–1124.

[53] Ndiaye C.B., *Constant \( T \)-curvature conformal metric on 4-manifolds with boundary*, Pacific J. Math. 240 (2009), no. 1, 151-184.

[54] Ndiaye C. B., *\( Q \)-curvature flow on manifolds with boundary*, Math. Z. 269 (2011), 83-114.

[55] Ndiaye C. B., *Algebraic topological methods for the supercritical \( Q \)-curvature problem*, Adv. Math. 277 (2015), 56–99.

[56] Ndiaye C. B., *Topological methods for the supercritical \( Q \)-curvature problem*, Preprint.

[57] Ndiaye C. B., *Blow-up phenomena for the boundary \( Q \)-curvature problem*, in preparation.

[58] Ndiaye C. B., *Sharp estimates for bubbling solutions to some fourth-order geometric equations*, to appear in IMRN.

[59] Ndiaye C. B., Xiao J, *An upper bound of the total \( Q \)-curvature and its isoperimetric deficit for higher-dimensional conformal Euclidean metrics*, Cal. Var, Partial Differential Equations 51 (2014), 291314

[60] Paneitz S., *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, preprint, 1983.

[61] Osgood B., Phillips R., Sarnak P., *Extremal of determinants of Laplacians*, J. Funct. Anal, 80, 148-211 (1988).

[62] J. Strom, Modern Classical Homotopy Theory, AMS Graduate Studies in Mathematics, vol. 127.

[63] Weinstein G., Zhang L., *The profile of bubbling solutions of a class of fourth order geometric equations on 4-manifolds*, J. Funct. Anal. 257 (2009), no. 12, 3895-3929
Mohameden Ahmedou
Mathematisches Institut der Justus-Liebig-Universität Giessen
Arndtsrasse 2, D-35392 Giessen
Germany
Mohameden.Ahmedou@math.uni-giessen.de

Sadok Kallel
American University of Sharjah (UAE)
and Laboratoire Painlevé, USTL(France)
sadok.kallel@math.univ-lille.fr,

Cheikh Birahim Ndiaye
Mathematisches Institut der Justus-Liebig-Universität Giessen
Arndtsrasse 2, D-35392 Giessen
Germany
and Tübingen University, Auf der Morgenstelle 10, D-72076 Tübingen
ndiaye@math.uni-tuebingen.de