H-theorem for classical matter around a black hole

Piero Nicolini\textsuperscript{1,2,3} and Massimo Tessarotto\textsuperscript{1,4}

\textsuperscript{1}Dipartimento di Matematica e Informatica, Università di Trieste, Italy
\textsuperscript{2}Dipartimento di Matematica, Politecnico di Torino, Turin, Italy
\textsuperscript{3}Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy
\textsuperscript{4}Consorzio di Magnetofluidodinamica, Trieste, Italy

(Dated: March 1, 2022)

We propose a classical solution for the kinetic description of matter falling into a black hole, which permits to evaluate both the kinetic entropy and the entropy production rate of classical infalling matter at the event horizon. The formulation is based on a relativistic kinetic description for classical particles in the presence of an event horizon. An H-theorem is established which holds for arbitrary models of black holes and is valid also in the presence of contracting event horizons.

PACS: 65.40.Gr 04.70.Bw 04.70.Dy.

PACS numbers:

The remarkable mathematical analogy between the laws of thermodynamics and black hole (BH) physics following from classical general relativity still escapes a complete and satisfactory interpretation. In particular it is not yet clear whether this analogy is merely formal or leads to an actual identification of physical quantities belonging to apparently unrelated framework. The analogous quantities are $E \leftrightarrow M$, $T \leftrightarrow \alpha \kappa$ and $S \leftrightarrow (1/8\pi\alpha)A$, where $A$ and $\kappa$ are the area and the surface gravity of the BH, while $\alpha$ is a constant. A immediate hint to believe in the thermodynamical nature of BH comes from the first analogy which actually regards a unique physical quantity: the total energy. However, at the classical level there are obstacles to interpret the surface gravity as the BH temperature since a perfectly absorbing medium, discrete or continuum, which is by definition unable to emit anything, cannot have a temperature different from absolute zero. An reconciliation was partially achieved by in 1975 by Hawking \cite{1}, who showed, in terms of quantum particle pairs nucleation, the existence of a thermal flux of radiation emitted from the BH with a black body spectrum at temperature $T = \hbar \kappa / 2\pi k_B$ (Hawking BH radiation model). The last analogy results the most intriguing, since the area $A$ should essentially be the logarithm of the number of microscopic states compatible with the observed macroscopic state of the BH, if we identify it with the Boltzmann definition. In such a context, a further complication arise when one strictly refers to the internal microstates of the BH, since for the infinite red shift they are inaccessible to an external observer. An additional difficulty with the identification $S \leftrightarrow (1/8\pi\alpha)A$, however, follows from the BH radiation model, since it predicts the existence of contracting BH for which the radius of the BH may actually decrease. To resolve this difficulty a modified constitutive equation for the entropy was postulated\textsuperscript{2,3}, in order to include the contribution of the matter in the BH exterior, by setting

$$S' = S + \frac{1}{4} k \frac{c^3 A}{G \hbar},$$

($S'$ denoting the so-called Bekenstein entropy) where $S$ is the entropy carried by the matter outside the BH and $S_{bh} = \frac{1}{4} k \frac{c^3 A}{G \hbar}$ identifies the contribution of the BH. As a consequence a generalized second law

$$\delta S' \geq 0$$

was proposed\textsuperscript{2,3} which can be viewed as nothing more than the ordinary second law of thermodynamics applied to a system containing a BH. From this point of view one notices that, by assumption and in contrast to the first term $S$, $S_{bh}$ cannot be interpreted, in a proper sense, as a physical entropy of the BH, since, as indicated above, it may decrease in time. However, the precise definition and underlying statistical basis both for $S$ and $S_{bh}$ remain obscure. Thus a fundamental problem still appears their precise estimates based on suitable microscopic models. Since the evaluation of $S_{bh}$ requires the knowledge of the internal structure of the event horizon (excluding for causality the BH interior), the issue can be resolved only in the context of a consistent formulation of quantum theory of gravitation\textsuperscript{4,5}. This can be based, for example, on string theory\textsuperscript{6} and can be conveniently tested in the framework of semiclassical gravity\textsuperscript{7,8}. Regarding, instead the entropy produced by external matter $S$, its evaluation depends on the nature, not yet known, of the BH. However, even if one regards the BH as a purely classical object surrounded by a suitably large number of classical particles its estimate should be achievable in the context of classical statistical mechanics.

In statistical mechanics the “disorder” characterizing a physical system, classical or quantal, endowed by a large number of permissible microstates, is sometimes conventionally measured in terms of the so-called Boltzmann entropy $S_B = K \ln W$. Here $K$ is the Boltzmann constant while $W$ is a suitable real number to be identified with the total number of microscopic complexes compatible with the macroscopic state of the system, a number which generally depends on the specific micromodel of the system. Therefore, paradoxically, the concept of
Boltzmann entropy does not rely on a true statistical description of physical systems, but only on the classification of the internal microstates (quantal or classical). As is well known, $S_B$ can be axiomatically defined, demanding (i) that it results a monotonic increasing function of $W$ and (ii) that it satisfies the entropy additivity law $S_B(W_1W_2) = S_B(W_1)+S_B(W_2)$. Boltzmann entropy plays a crucial role in thermodynamics where (i) and (ii) have their corresponding laws in the entropy nondecreasing monotonicity and additivity. Since in statistical mechanics of finite system it is impossible to satisfy both laws exactly, the definition of $S_B$ is actually conditioned by the requirement of considering systems with $W >> 1$ (large physical systems).

An alternate definition of entropy in statistical mechanics is the one given by the Gibbs entropy, in turn related to the concept of Shannon information entropy. In contrast to the Boltzmann entropy, this is based on a statistical description of physical systems and is defined in terms of the probability distribution of the observable microstates of the system. In many cases it is sufficient for this purpose to formulate a kinetic description, and a corresponding kinetic entropy, both based on the one-particle kinetic distribution function. In particular, this is the case of classical many-particle systems, consisting of weakly interacting ultra relativistic point particles, such as those which may characterize the distribution of matter in the immediate vicinity of the BH exterior.

The goal of the paper is to provide an explicit expression for the contribution $S$, which characterizes Bekenstein law, to be evaluated in terms of a suitable kinetic entropy, and to estimate the corresponding entropy production rate due to infalling matter at the BH event horizon. In addition we intend to establish an H-theorem for the kinetic entropy which holds, in principle, for a classical BH characterized by event horizons of arbitrary shape and size and even in the presence of BH implosions or slow contractions. This is obtained in the framework where the classical description of outside matter and space is a good approximation to the underlying physics. The basic assumption is that the matter falling into the BH is formed by a system of $N \gg 1$ classical point particles moving in a classical spacetime. We adopt a covariant kinetic formalism taking into account the presence of an event horizon and assuming Hamiltonian dynamics for the point particles. The evolution of such a system is well known and results uniquely determined by the classical equations of motion, defined with respect to an arbitrary observer $O$. To this purpose let us choose $O$, without loss of generality, in a region where space time is (asymptotically) flat, endowing with the proper time $s$, with $\tau$ assumed to span the set $I \subset \mathbb{R}$ (observer’s time axis). Each point particle is described by the canonical state $x$ spanning the 8-dimensional phase space $\Gamma$, where $x = (r^{\mu}, p_{\mu})$. Moreover, its evolution is prescribed in terms of a suitable relativistic Hamiltonian $H = H(x)$, so that the canonical state $x = (r^{\mu}, p_{\mu})$ results parameterized in terms of the world line arc length $s$ (see [10]). As a consequence, requiring that $s = s(\tau)$ results a strictly monotonic function it follows that, the particle state can be also parameterized in terms of the observer’s time $\tau$. To obtain a kinetic description for a relativistic classical system of $N$ point particles we introduce the kinetic distribution function for the observer $O$, $\rho_K(x)$, defined as follows

$$\rho_K(x) \equiv \rho(x)\delta(s-s(\tau))\delta(\sqrt{u^{\mu}u_{\mu}}-1)$$

where $\rho(x)$ is the conventional kinetic distribution function in the 8-dimensional phase space. Notice that the Dirac deltas here introduced must be intended as physical realizability equations. In particular the condition placed on the arc length $s$ implies that the particle of the system is parameterized with respect to $s(\tau)$, i.e., it results functionally dependent on the proper time of the observer; instead the constraints placed on 4-velocity imply that $u^{\mu}$ is a tangent vector to a timelike geodesic. The event horizon of a classical BH is defined by the hypersurface $r_H$ specified by the equation

$$R(x) = r_H$$

where $x$ denotes a point of the space time manifold, while $R(x)$ reduces to the radial coordinate in the spherically symmetric case. According to a classical point of view, let us now assume that the particles are "captured" by the BH (i.e., for example, they effectively disappear for the observer since their signals are red shifted in such a way that they cannot be anymore detected) when they reach $\gamma$ of equation

$$R_{\gamma}(x) = r_{\gamma}.$$  

Here $r_{\gamma} = (1+\epsilon)r_H$, while $\epsilon$ depends on the detector and the 4-momentum of the particle. The presence of the BH event horizon is taken into by defining suitable boundary conditions for the kinetic distribution function on the hypersurface $\gamma$. For this purpose we distinguish between incoming and outgoing distributions on $\gamma$, $\rho_0^-(x)$ and $\rho_0^+(x)$ corresponding respectively to $n_{\alpha}u^\alpha \leq 0$ and $n_{\alpha}u^\alpha > 0$, where $n_{\alpha}$ is a locally radial outward 4-vector. Therefore the boundary conditions on $\gamma$ are specified as follows

$$\rho_0^+(x) = \rho_0^-(x)\delta(s-s(\tau))\delta(\sqrt{u^{\mu}u_{\mu}}-1)$$

$$\rho_0(x) = 0$$

It follows that it is possible to represent the kinetic distribution function in the whole space time manifold in the form

$$\rho_K(x) = \rho_0^-(x) + \rho_0^+(x)$$

where

$$\rho_0^\pm(x) \equiv \rho(x)\delta(s-s(\tau))\delta(\sqrt{u^{\mu}u_{\mu}}-1) \times \Theta^{\pm}(R_{\gamma}(x) - r_{\gamma}(s(\tau)))$$
with $\Theta^\pm$ respectively denoting the strong and the weak Heaviside functions

$$\Theta^+(a) = \begin{cases} 1 & \text{for } a \geq 0 \\ 0 & \text{for } a < 0. \end{cases} \quad (10)$$

and

$$\Theta^-(a) = \begin{cases} 1 & \text{for } a > 0 \\ 0 & \text{for } a \leq 0. \end{cases} \quad (11)$$

We stress that in the above boundary conditions no detailed physical model is actually introduced for the particle loss mechanism, since all particles are assumed to be captured on the same hypersurface $\gamma$, independent of their mass, charge and state. This provides a classical loss model for the BH event horizon.

Let us now consider the evolution of the kinetic distribution function $\rho G(x)$ in external domains, i.e. outside the event horizon. Assuming that binary collisions are negligible, or can be described by means of a mean field, and provided that the phase space volume element is conserved, it follows the collisionless Boltzmann equation, or the Vlasov equation in the case of charged particles [19],

$$\frac{d}{d\tau} \left( \frac{dV}{ds} \frac{\partial \hat{\rho}(x)}{\partial V} + \frac{dp}{ds} \frac{\partial \hat{\rho}(x)}{\partial p} \right) = 0 \quad (12)$$

with summation understood over repeated indexes, while $\hat{\rho}(x)$ denotes $\rho G(x)$ evaluated at $r^0 = r^0(s(\tau))$ and $p_0 = m \left| \frac{\partial r_0}{\partial s} \right|_{s=s(\tau)}$. This equation resumes the conservation of the probability in the relativistic phase space in the domain external to the event horizon. Invoking the Hamiltonian dynamics for the system of point particles, the kinetic equation takes the conservative form

$$\frac{d}{d\tau} [\hat{\rho}(x), H]_x = 0. \quad (13)$$

Let us now introduce the appropriate definition of kinetic entropy $S(\rho)$ in the context of relativistic kinetic theory. We intend to prove that in the presence of the BH event horizon it satisfies an $H$ theorem. The concept of entropy in relativistic kinetic theory can be formulated by direct extension of customary definition given in nonrelativistic setting [12, 13, 14, 15]. For this purpose we introduce the notion of kinetic entropy measured with respect to an observer endowed with proper time $\tau$ as follows

$$S(\rho) = -P \int_{\Gamma} d\mathbf{x}(s) \delta(s - s(\tau)) \delta(\sqrt{u_{\mu}u^{\mu}} - 1) \rho(x) \ln \rho(x), \quad (14)$$

where $\rho(x)$ is strictly positive and, in the 8-dimensional integral, the state vector $x$ is parameterized with respect to $s$, with $s$ denoting an arbitrary arc length. Here $P$ is the principal value of the integral introduce in order to exclude from the integration domain the subset in which the distribution function vanishes. Hence $S(\rho)$ can also be written:

$$S(\rho) = -P \int_{\Gamma} d\mathbf{x}(s) \delta(s - s(\tau)) \rho_1(x) \ln \rho(x), \quad (15)$$

where $\rho_1(x)$ now reads

$$\rho_1(x) = \Theta(r(s) - r_e(s)) \delta(\sqrt{u_{\mu}u^{\mu}} - 1) \rho(x). \quad (16)$$

Differentiating with respect to $\tau$ and introducing the invariant volume element $d^3r d^3p$, the entropy production rate results manifestly proportional to the area $A$ of the event horizon

$$\frac{dS(\rho)}{d\tau} = -P \int_{\Gamma} d^3r d^3p F_{rr} \left[ \delta(r - r_e) \hat{\rho} \ln \hat{\rho} \right]. \quad (17)$$

Indeed, the r.h.s represents the entropy flux across the event horizon. Moreover here, $\Gamma^{-}$ is the subdomain of phase space corresponding to the particle falling into the BH and $F_{rr}$ is the characteristic integrating factor

$$F_{rr} = \frac{ds(\tau)}{d\tau} \left( \frac{dr}{ds} - \frac{dr_e}{ds} \right). \quad (18)$$

We can write the above expression in terms of the kinetic probability density evaluated at the hypersurface $\sqrt{u_{\mu}u^{\mu}} = \Theta$, defined as $\hat{f}(\chi) \equiv \hat{\rho}/N$. It follows

$$\hat{\rho} \ln \hat{\rho} \equiv N \hat{f} \ln N \hat{f}. \quad (19)$$

At this point we adopt a customary procedure in statistical mechanics [12] invoking the inequality

$$N \hat{f} \ln N \hat{f} \geq N \hat{f} - 1 \quad (20)$$

and notice that in the subdomain of phase space $\Gamma^{-}$ in which $F_{rr} \geq 0$ there results by definition $\hat{\rho} = 0$. Hence it follows that in $\Gamma^{-}$

$$F_{rr}, < 0 \quad (21)$$

where by construction $\frac{ds(\tau)}{d\tau} > 0$. This result holds independent of the value of $\frac{dR}{d\tau}$. Next let us introduce the bounded subset $\Omega \subset \Gamma^{-}$ such that $N \hat{f}$ results infinitesimal (of order $\delta$) in the complementary set $\Gamma^{-} \setminus \Omega$ and moreover the ordering estimate

$$P \int_{\Gamma^{-} \setminus \Omega} d^3r d^3p \left| F_{rr} \right| \delta(r - r_e) \left[ N \hat{f} \ln N \hat{f} \right] \sim O (\delta \ln \delta)$$

is required to hold. Manifestly this domain includes the set of improper points of $\Gamma^{-}$. In the set $\Omega$, $N \hat{f}$ is by assumption positive and such that $N \hat{f} > \delta$. Therefore there results

$$0 < P \int_{\Omega} d^3r d^3p \left| F_{rr} \right| \delta(r - r_e) \equiv M_{\delta} \quad (22)$$
where $M_\delta$ is a suitable finite constant. Thus one obtains

$$\frac{dS(\rho)}{d\tau} \geq P \int_\Omega d^3r d^3p \left| F_{rr} \right| \delta (r - r_e) \left[N f - 1 + O(\delta \ln \delta) \right]$$

The first term of the r.h.s of (20) can be interpreted in terms of the effective radial velocity of incoming particles

$$V_r^{eff} \equiv \frac{1}{n_0} \int_\Omega d^3p \left| F_{rr} \right| \delta (r - r_e) \hat{f},$$

while $n_0$ is the surface number density of the incoming particle distribution function

$$\int_\Omega d^3r d^3p \hat{f}(x) \delta (r - r_0(s(t))) = An_o.$$  

Finally we invoke the majorization

$$\frac{dS(\rho)}{d\tau} \geq \hat{S} \equiv N \inf \left\{ \int_\Omega d^3r n_0 V_r^{eff} \right\} - M_\delta + O(\delta \ln \delta)$$

and impose that $N \gg 1$ be sufficiently large to satisfy the inequality $\hat{S} > 0$. We stress that $\inf \left\{ \int_\Omega d^3r n_0 V_r^{eff} \right\}$ can be assumed strictly positive for non isolated BHs surrounded by matter. This proves the relativistic H-theorem

$$\frac{dS(\rho)}{d\tau} > 0.$$  

Let us briefly analyze the basic implications of our results. First we notice that the H-theorem here obtained appears of general validity even if achieved in the classical framework and under the customary requirement $N \gg 1$ (large classical system) and in validity of the subsidiary condition $\hat{S} > 0$. Indeed the result applies to BH having, in principle, arbitrary shape of the event horizon. The description adopted is purely classical both for the falling particles (charged or neutral [17, 18, 19, 20]) and for the gravitational field and is based on the relativistic collisionless Boltzmann equation and/or the Vlasov equation respectively for neutral and charged particles. A key aspect of our formalism is the definition of suitable boundary conditions for the kinetic distribution function in order to take into account the presence of the event horizon. Second, the expressions for the entropy and entropy production rate, respectively Eqs. (14) and (17), can be used to determine the Bekenstein entropy for classical BH [1] and the related generalized second law [2], although we stress that the present results are independent of the assumptions involved in the definition of the Bekenstein entropy [17]. Finally interesting features of the derivation are that the entropy production rate results proportional to the area of the event horizon and that the formalism is independent of the detailed model adopted for the BH. In particular also the possible presence of an imploding star (contracting event horizon) is permitted.

Acknowledgments

P. N. is supported by the Ministero dell’Istruzione, dell’Università e della Ricerca (MIUR) via the Programma PRIN 2004: ‘Metodi matematici delle teorie cinetiche’.

[1] S.W. Hawking, Comm. Math. Phys. 43, 199 (1975).
[2] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).
[3] J. D. Bekenstein, Phys. Rev. D 9, 3292 (1974).
[4] J. D. Bekenstein, Phys. Rev. D 12, 3077 (1975).
[5] S. W. Hawking, Phys. Rev. D 13, 191 (1976).
[6] For reviews see. for instance J. M. Maldacena, PhD thesis, Princeton University, 1996 hep-th/9607235
A.W. Peet, Class. Quantum Grav. 15, 3291 (1998);
R. M. Wald, Living Rev. Rel. 4, 6 (2001).
[7] R. Balbinot, A. Fabbri, V. Frolov, P. Nicolini, P. J. Sutton, A. Zelnikov, Phys. Rev. D 63 084029 (2001).
[8] R. Balbinot, A. Fabbri, P. Nicolini, P. J. Sutton, Phys. Rev. D 66 024014 (2002).
[9] C. G. Chakrabarti and Kajal De, Internat. J. Math & Math. Sci. 23, 243 (2000).
[10] J. L. Synge, Relativity: the general theory, North Holland PC, Amsterdam 1960.
[11] R.M. Wald, General relativity, The University of Chicago Press, Chicago and London 1984.
[12] H. Grad, Handbuch der physik, Vol. XII, 205, Springer Verlag, Berlin 1956.
[13] W. Israel, J. Math. Phys. 4, 1163 (1963).
[14] C. Cercignani, Theory and applications of the Boltzmann equation, Scottish Academic Press, Edinburgh and London 1975.
[15] S. R. De Groot, Relativistic Kinetic Theory - Principles and Applications, North-Holland 1980.
[16] J. Yvon, Correlations and entropy in classical mechanics, Pergamon Press, Oxford 1969.
[17] M. Pozzo and M. Tessarotto, Phys. Plasmas, 5, 2232 (1998).
[18] A. Beklemishev and M. Tessarotto, Phys. Plasmas, 6, 4548 (1999).
[19] A. Beklemishev and M. Tessarotto, A. & A. 428, 1 (2004).
[20] A. Beklemishev, P. Nicolini and M. Tessarotto, AIP Conf. Proc. 762, 1283 (2005).