DEFORMATION INVARIANCE OF RATIONAL PAIRS

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Abstract. Rational pairs, recently introduced by Kollár and Kovács, generalize rational singularities to pairs \((X, D)\). Here \(X\) is a normal variety and \(D\) is a reduced divisor on \(X\). Integral to the definition of a rational pair is the notion of a thrifty resolution, also defined by Kollár and Kovács, and in order to work with rational pairs it is often necessary to know whether a given resolution is thrifty. In this paper we present several foundational results that are helpful for identifying thrifty resolutions and analyzing their behavior. In 1978, Elkik proved that rational singularities are deformation invariant. Our main result is an analogue of this theorem for rational pairs: given a flat family \(X \rightarrow S\) and a Cartier divisor \(D\) on \(X\), if the fibers over a smooth point \(s \in S\) form a rational pair, then \((X, D)\) is also rational near the fiber \(X_s\).

1. Introduction

We present a proof that rational pairs are deformation invariant. The notion of a rational pair, recently introduced in [Kol13, Section 2.5], is a generalization of rational singularities on a variety \(X\) to reduced pairs \((X, D)\). A reduced pair is a normal variety \(X\) together with a Weil divisor \(D\), all of whose coefficients are 1. Such a \(D\) is a reduced divisor. The main results in this paper include the assumption that the underlying field \(k\) has characteristic 0.

The analogue of a smooth variety in the pairs setting is an snc pair. A pair \((X, D)\) has simple normal crossings, or is snc, if \(X\) is smooth, every \(D_i\) is smooth, and all the intersections of the components \(D_i\) are transverse. If \((X, D)\) is not snc, we can still refer to the snc locus \(\text{snc}(X, D)\), which is the largest open set \(U \subset X\) so that \((U, D \cap U)\) is snc.

If \(f: Y \rightarrow X\) is birational, \(X\) is normal, and \(D \subset X\) is a divisor, then the birational transform of \(D\), denoted \(f^{-1}_* D\), is defined in the following way: let \(U\) be the open set where \(f\) is an isomorphism, map \(D \cap U\) into \(Y\) by the morphism \(f^{-1}\), and then take the closure in \(Y\) of the image. A resolution of pairs is a resolution of singularities \(f: Y \rightarrow X\), such that the pair \((Y, B := f^{-1}_* D)\) has simple normal crossings, and it is written using the notation \(f: (Y, B) \rightarrow (X, D)\).

If \((X, D = \bigcup D_i)\) is an snc pair, then an irreducible component of any intersection of the \(D_i\) is called a stratum. If \((X, D)\) is not snc, then we may consider the strata of its snc locus. In order to determine whether a pair is rational, we’ll need to examine the strata of its snc locus and how they behave under a resolution of pairs. A general reference for these definitions is [Kol13].

This new theory of rational pairs requires us to restrict our attention to certain resolutions, called thrifty resolutions, which were also introduced and developed in [Kol13]. These resolutions of pairs have the nice properties that are required to
generalize theorems about rational singularities of varieties to pairs. Specifically, when we restrict to resolutions of pairs that are also thrifty, we have an analogue of Grauert-Riemenschneider vanishing in characteristic 0 (see \( \text{(3.6)} \) below), and we know that if a pair has one thrifty rational resolution, then every other thrifty resolution is also rational.

A thrifty resolution \( f : (Y, B) \to (X, D) \), which we will discuss in much more detail below, is one that satisfies two conditions:

1. (Condition 1) \( f \) is an isomorphism over the generic point of any stratum of \( \text{snc}(X, D) \)
2. (Condition 2) \( f \) is an isomorphism at the generic point of any stratum of \( (Y, B) \)

In this paper we present several foundational results that greatly simplify the task of verifying that a given resolution is thrifty. The first is a simple criterion for Condition 2 in the definition, which is usually the more difficult of the two conditions to check.

**Proposition 1.1** (see \( \text{(3.1)} \)). If \( f : (Y, B) \to (X, D) \) is a log resolution, then \( f \) satisfies Condition 2.

We will also show that if a birational morphism \( f : Y \to X \) is an isomorphism over every stratum of \( \text{snc}(X, D) \), then it can be dominated by a thrifty log resolution of \( (X, D) \).

**Theorem 1.2** (see \( \text{(3.2)} \)). Suppose \( f : Y \to X \) is a proper birational morphism between normal varieties, and \( D \subset X \) is a reduced divisor. If \( f \) is an isomorphism over every stratum of \( \text{snc}(X, D) \), then there is a thrifty log resolution of \( (X, D) \) factoring through \( f \).

It is not known whether a rational resolution (defined below in \( \text{(2.1)} \)) is necessarily thrifty, but we give a significant partial result in that direction: it is true for log resolutions. Indeed, every resolution can be dominated by a log resolution, so our result suffices for many applications.

**Proposition 1.3** (see \( \text{(3.4)} \)). If a log resolution of a pair is rational, then it is thrifty.

The main result in this paper is on deformation invariance of rational pairs in a flat family. In [Elk78], Elkik proved that rational singularities are deformation invariant: given a variety \( X \) with rational singularities and a flat morphism \( X \to S \), then if \( s \) is a smooth point and the fiber \( X_s \) over \( s \) has rational singularities, so does \( X \) in a neighborhood of \( X_s \).

In this paper we show that rational pairs \( (X, D) \), with \( D \) a Cartier divisor in \( X \), are also deformation invariant.

**Theorem 1.4** (see \( \text{(5.2)} \)). Let \( (X, D) \) be a pair, with \( D \) Cartier. Suppose \( X \to S \) is a flat morphism, and \( s \in S \) is a smooth point so that the fibers \( (X_s, D_s) \) form a reduced pair. If \( (X_s, D_s) \) is a rational pair, then \( (X, D) \) is a rational pair in a neighborhood of \( (X_s, D_s) \).

That is, if \( (X_s, D_s) \) is rational at \( x \), then \( (X, D) \) is also rational at \( x \).
2. Rational resolutions of pairs

Recall that a resolution $f: Y \rightarrow X$ of a variety is rational if the natural morphism $\mathcal{O}_X \rightarrow Rf_*\mathcal{O}_Y$ is a quasi-isomorphism and the higher direct images $R^i f_*\omega_Y$ vanish for $i > 0$. (This second part holds automatically in characteristic 0 by the Grauert-Riemenschneider vanishing theorem ([GR70]).)

The definition of a rational resolution of a pair $(X, D)$ is formally very similar to that of a rational resolution of a variety $X$. Again, throughout this paper we assume all varieties are defined over a field of characteristic 0. The definitions from [Kol13] also make sense in positive characteristic, but the proofs of the main results do rely on the characteristic-0 assumption.

Definition 2.1 ([Kol13, 2.78]). Let $(X, D)$ be a reduced pair. A resolution $f: (Y, B) \rightarrow (X, D)$ is rational if

1. The natural map $\mathcal{O}_X(-D) \rightarrow Rf_*\mathcal{O}_Y(-B)$ is a quasi-isomorphism, and
2. The higher direct images $R^i f_*\omega_Y(B)$ vanish for $i > 0$.

If $D$ is a Cartier divisor in $X$, then there is a characterization of rational resolutions $f: (Y, B) \rightarrow (X, D)$ that looks very much like Kempf’s well-known criterion for rational resolutions of varieties $f: Y \rightarrow X$ (see [Kol13, 2.77]).

Theorem 2.2 ([Kol13, 2.84]). Let $(X, D)$ be a reduced pair, with $D$ Cartier, and let $f: (Y, B) \rightarrow (X, D)$ be a resolution. Then $f$ is rational if and only if two conditions are satisfied:

1. $X$ is CM, and
2. The natural map $Rf_*\omega_Y(B) \rightarrow \omega_X(D)$ is a quasi-isomorphism.

It is well known that if one resolution of a variety $X$ is rational, then every other resolution of $X$ is also rational. In this case we say that $X$ has rational singularities. The analogous statement does not hold for rational resolutions of pairs as defined in [24]: there are many pairs $(X, D)$ that have both rational and non-rational resolutions. For example, an snc pair $(X, D)$ that has a stratum of at least codimension 2 will have both types of resolutions; see [Kol13, p. 94].

The resolutions that cause this statement to fail are all of a certain type. If we exclude them, then the analogous claim for pairs is true. So we will restrict our attention to a certain type of resolution. The resolutions that we’ll consider are called thrifty.

Definition 2.3 ([Kol13, 2.79]). Let $(X, D)$ be a reduced pair, with $f: (Y, B) \rightarrow (X, D)$ a resolution. Then $f$ is a thrifty resolution if two conditions hold:

1. $f$ is an isomorphism over the generic point of every stratum of snc$(X, D)$; equivalently, $f(\text{Ex}(f))$ does not contain any stratum of snc$(X, D)$
2. $f$ is an isomorphism at the generic point of every stratum of $(Y, B)$; equivalently, the exceptional locus $\text{Ex}(f)$ does not contain any stratum of $(Y, B)$.

We’ll refer to these as Conditions 1 and 2 from now on.

It is shown in [Kol13, 2.86] that in characteristic 0, if a pair $(X, D)$ has a thrifty rational resolution, then every other thrifty resolution of $(X, D)$ is also rational. As long as we restrict our attention to thrifty resolutions, then, the situation for pairs is similar to the situation for varieties.

Now we have enough to define rational pairs.
Definition 2.4 ([Kol13 2.80]). If $X$ is a normal variety, with $D$ a reduced divisor on $X$, then the pair $(X, D)$ is rational if it has a thrifty rational resolution. (Equivalently, if every thrifty resolution of $(X, D)$ is rational.)

It is an open question whether a rational resolution is necessarily thrifty. For dlt pairs this is true; see [Kol13 2.87]. In (3.4) below, we give another partial answer: given a resolution of pairs $f: (Y, B) \to (X, D)$, if the entire preimage of $D$ is an snc divisor in $Y$ (that is, if $f$ is a log resolution), then $f$ is indeed thrifty.

3. Preliminary results: Thrifty resolutions

In order to work with rational pairs, it will be essential to be able to tell whether a given resolution is thrifty—that is, whether it satisfies Conditions 1 and 2 from (2.3). Condition 1 of thriftiness is a property of snc$(X, D)$: to see whether it holds, it is only necessary to check the snc locus, where $X$, the $D_i$, and all the intersections of the $D_i$ are smooth. Condition 2, on the other hand, is not a property of snc$(X, D)$, so we must examine points that map to the non-snc locus of $(X, D)$. Points outside of snc$(X, D)$ are trickier to deal with, so alternative ways to check for Condition 2 would be welcome.

Condition 2 is automatic if $f$ is a log resolution, as we’ll show below. Such an $f$ is then thrifty if and only if it satisfies Condition 1. Recall that a log resolution $(Y, B) \to (X, D)$ is a resolution with two additional conditions: the exceptional locus $\text{Ex}(f)$ is a divisor in $Y$, and the pair $(Y, B + \text{Ex}(f))$ is snc. This is a much stronger requirement than that $(Y, B)$ be snc, as in the definition of a resolution of pairs: for $f$ to be a log resolution, the components of $\text{Ex}(f)$ must be smooth divisors and must intersect each other and the components of $B$ transversally.

Proposition 3.1. If $f: (Y, B) \to (X, D)$ is a log resolution, then $f$ satisfies Condition 2.

Proof. Suppose $f: (Y, B) \to (X, D)$ is a log resolution. Let $E$ be the reduced divisor supported on $\text{Ex}(f)$, so that the pair $(Y, B + E)$ is snc and $B + E$ is reduced. Condition 2 fails exactly when a stratum of $(Y, B)$ is contained in $E$. Let $Z$ be a stratum of $(Y, B)$, so that $Z$ is a component of some intersection $\bigcap_{i \in I} D_i$, where $I = i_1, \ldots, i_s$. Now we appeal to [Kol13 4.16.2], originally stated and proved in [Fuj07 3.9.2], which says that an intersection of $s$ components of the divisor in a reduced dlt pair has pure codimension $s$. This theorem applies here because our $(Y, B)$ is snc and therefore dlt, so $Z$ has codimension $s$.

Now if $Z \subset E$, then $Z$ is contained in some component $E_j$ of $E$. But then $Z \subset E_j \cap \left( \bigcap_{i \in I} D_i' \right)$. Now $(Y, B + E)$ is also snc and hence dlt, so this intersection has codimension $s + 1$ by another application of [Kol13 4.16.2]. But then $Z$, which has codimension $s$, cannot be a subset of this intersection, so this situation is impossible. Thus Condition 2 always holds for a log resolution. $\square$

The next result will be very useful in the proof of the main theorem (5.2). In that argument, we analyze a pair $(X, D)$ and a certain subpair $(X_t, D_t)$, where $X_t$ is a Cartier divisor in $X$, and $D_t$ in $D$. We’ll appeal to the following theorem to show that the restriction of a thrifty resolution of $(X, D)$ to $X_t$ is dominated by a thrifty resolution of $(X_t, D_t)$. 
Theorem 3.2. Suppose \( f : Y \to X \) is a proper birational morphism between normal varieties, and \( D \subset X \) is a reduced divisor. If \( f \) is an isomorphism over every stratum of snc\((X, D)\), then there is a thrifty log resolution of \((X, D)\) factoring through \( f \).

**Proof.** Let \( B = f_*^{-1}D \) be the birational transform of \( D \) in \( Y \). We'll construct a thrifty log resolution \( g \) of \((X, D)\) starting with the pair \((Y, B)\).

First, blow up the exceptional components of codimension at least 2 in \( Y \) of the birational morphism \( f : Y \to X \). From this we obtain a new proper birational morphism \( f' : Y' \to X \), and a new birational transform \( B' \) of \( D \) in \( Y' \). Now the exceptional locus of \( Y' \to X \) is a divisor in \( Y' \), and we'll call it \( E' \).

Let \( h \) be a log resolution of the pair \((Y', B' + E')\) that is an isomorphism over \( \text{snc}(Y', B' + E') \). To do this, we appeal to Szabó’s theorem from [Sza94], which says that choosing such a resolution is possible; see [Kol13, 10.45.2].

Now write \( g = f \circ f' \circ h \). It is a log resolution of \( X \) and the birational transform of \( D \) by \( g \) is an snc divisor. We'll show now that \( g \) is also an isomorphism over the generic point of every stratum of snc\((X, D)\).

Let \( U \subset X \) be the open set over which \( f \) is an isomorphism. Let \( x \) be the generic point of any stratum of snc\((X, D)\). Then by assumption \( x \in U \), so \( x \in U \cap \text{snc}(X, D) \).

The pair \((Y, B)\) is snc at the preimage of \( x \) in \( Y \), because \( f \) is an isomorphism there. Similarly, \((Y', B')\) is snc at the preimage of \( x \) in \( Y' \); again, the blowups \( f' : Y' \to Y \) do not affect points outside the exceptional locus of \( f \), so the entire preimage of \( U \cap \text{snc}(X, D) \) in \( Y' \) is inside the locus where \( f' \) is an isomorphism. The exceptional locus \( E' \) is disjoint from the preimage of \( x \) in \( Y' \), so the pair \((Y', B' + E')\) is also snc there. We chose the resolution \( h \) of \((Y', B' + E')\) to be an isomorphism over \( \text{snc}(Y', B' + E') \), so the composition \( g = f \circ f' \circ h \) is an isomorphism over \( x \).

In other words, \( g \) satisfies Condition 1 of thriftiness from (2.3). Since \( g \) is a log resolution of \((X, D)\), it follows from (3.1) that \( g \) is thrifty. So there is a thrifty log resolution of \((X, D)\) factoring through \( f \).

\[ \square \]

Corollary 3.3. Every thrifty resolution is dominated by a thrifty log resolution.

**Proposition 3.4.** If a log resolution of a pair is rational, then it is thrifty.

**Proof.** Let \( f : (Y, B) \to (X, D) \) be a rational log resolution. We'll verify that \( f \) satisfies Condition 1 and Condition 2. Since \( f \) is log, Condition 2 is automatic by (3.1). Condition 1, on the other hand, is a property of snc\((X, D)\). Rational resolutions are defined in terms of sheaves, and \( U = \text{snc}(X, D) \) is an open set, so the restriction of \( f \) to \( f^{-1}(U) \) is still a rational resolution of \((U, D \cap U)\).

Every snc pair is dlt, so by [Kol13, 2.87], which says that a resolution of a dlt pair is rational if and only if it is thrifty, we conclude that \( f \) is thrifty over snc\((X, D)\), and hence satisfies Condition 1 there. But it is sufficient to check the snc locus of \((X, D)\) to verify Condition 1, so \( f \) is thrifty.

\[ \square \]

Next we'll show that thrifty resolutions satisfy an analogue of the Grauert-Riemenschneider vanishing theorem (see [GR70]). Thus the second condition in (2.1) is automatically true, at least for resolutions that are known to be thrifty.

To prove this, we'll start with a recent result from the literature. The assumption that our varieties are defined over a field of characteristic 0 is necessary here.
Proposition 3.5 (Special case of [Kol13 10.34]). Let $(Y, B)$ be snc, and $f : Y \to X$ a projective morphism of varieties over a field of characteristic $0$. For any lc center $Z$ of $(Y, B)$, write $F_Z \subset Z$ for the generic fiber of $f|_Z : Z \to f(Z)$. Set
\[ c = \max\{\dim F_Z : Z \text{ is an lc center}\}. \]

Then $R^i f_* \omega_Y(B) = 0$ for $i > c$.

The statement of [Kol13 10.34] is more general, but we only need this version here.

Proposition 3.6 (GR-type vanishing for thrifty resolutions). If $f : (Y, B) \to (X, D)$ is a thrifty resolution, then $R^i f_* \omega_Y(B) = 0$ for all $i > 0$.

Proof. We’ll appeal to (3.5), since $(Y, B)$ is an snc pair and the resolution $f : Y \to X$ is projective. Let $Z$ be a stratum of $(Y, B)$, and let $F_Z$ be the generic fiber of the map $f|_Z : Z \to f(Z)$. The image $f(Z)$ is closed, since $f$ is projective, and it is also irreducible because $Z$ is, so it has a generic point.

By (3.5), $R^i f_* \omega_Y(B) = 0$ for all $i > c$, where
\[ c = \max\{\dim F_Z : Z \text{ is a stratum}\}. \]

The lc centers of an snc pair are exactly its strata: see [Kol13 4.15]. Since $f$ is thrifty, it is birational on every stratum of $(Y, B)$. In particular, it is dominant when restricted to each $Z$, so the dimension of each generic fiber is 0: see [Eis95 p. 290]. Thus $c = 0$, so $R^i f_* \omega_Y(B) = 0$ for all $i > 0$. \hfill \Box

4. Preliminary results: base change by the local scheme near a point

We are now ready to develop the main theorem: that rationality of pairs is preserved by deforming a flat family defined over a base scheme $S$. First we’ll reduce to an especially simple case, where $S$ is the spectrum of a regular local ring, and then we’ll prove the claim in that situation with an induction argument. In order to reduce to the case where the base $S$ is Spec $R$ for a regular local ring $R$, we need several preliminary results about how rational pairs and thrifty resolutions behave with respect to the base change by a morphism Spec $\mathcal{O}_{S,s} \to S$, where $s \in S$ is a smooth point.

The results that follow are all closely related, and they share a common notation. For convenience, we collect all the notation here and will refer back to it throughout the rest of the paper.

Notation 4.1. Let $f : X \to S$ be a morphism, and $s \in S$ a point. Let $(X, D)$ be a reduced pair such that the fibers $(X_s, D_s)$ form a reduced pair. Also, let $f : (Y, B) \to (X, D)$ be a resolution. Let Spec $\mathcal{O}_{S,s} \to S$ be the inclusion of the local scheme near $s$, and base change by this morphism: let $X' = X \times_S \text{Spec } \mathcal{O}_{S,s}$, and similarly for $Y', Y', B'$. Write $\pi : X' \to X$ for the natural projection onto the first factor.
As we will see, all the salient aspects of this situation are preserved by the base change by \(\text{Spec} \mathcal{O}_{S,s} \to S\). The next few results are standard commutative algebra, but for lack of a reference we include the proofs here.

**Lemma 4.2.** With the notation of (4.1), let \(x' \in X'\). If \(\pi(x') = x\), and \(\mathcal{F}\) is a sheaf of \(\mathcal{O}_X\)-modules on \(X\), then \(\mathcal{F}_x \simeq (\pi^* \mathcal{F})_{x'}\). That is, stalks of \(\mathcal{O}_X\)-modules are preserved by the base change. In particular, \(\mathcal{O}_{X,x} \simeq \mathcal{O}_{X',x'}\).

**Proof.** We may assume the schemes are affine, so the base change corresponds to a pushout diagram of rings:

\[
\begin{array}{ccc}
B & \longrightarrow & B_p \\
\uparrow & & \uparrow \\
A & \longrightarrow & A_p
\end{array}
\]

Here \(A\) and \(B\) are rings, \(\phi: A \to B\) is the homomorphism corresponding to the affine morphism \(\text{Spec} B \to \text{Spec} A\), and \(p \triangleleft A\) is a prime ideal. Then the \(\mathcal{O}_X\)-module \(\mathcal{F}\) has the form \(\mathcal{F} = \mathcal{M}\) for a \(B\)-module \(\mathcal{M}\), and \(\pi^* \mathcal{F} = (\mathcal{M} \otimes_B B_p)^\sim \simeq (\mathcal{M} \otimes_A A_p)^\sim\). Let \(q'\) be a prime ideal in the ring \(B_p\). It corresponds to an ideal \(q\) in \(B\) that is disjoint from \(f(A - p)\). Here \(q'\) corresponds to \(x' \in X'\); \(q\) to \(x \in X\). It suffices to verify that \((\mathcal{M} \otimes_A A_p)_{q'} \simeq M_q\). By basic properties of localization (see [Sta14, Tag 02C7]) and the fact that \(\phi(A - p) \subset B - q\), this is true.

Note in particular the case where \(\mathcal{F} \simeq \mathcal{O}_X\): if \(\pi(x') = x\), then \(\mathcal{O}_{X,x} \simeq \mathcal{O}_{X',x'}\). □

**Corollary 4.3.** With the notation of (4.1), let \(P\) be a property of \(X\) that may be checked locally (i.e., at stalks). Then \(P\) holds on the image of the projection \(\pi(X') \subset X\) if and only if \(P\) holds on \(X'\).

**Example 4.4.** We continue to use the notation of (4.1). Before moving on, we’ll note a few specific applications of (4.3) to the varieties \(X\) and \(Y\), and their counterparts \(X'\) and \(Y'\) under the base change. Here are a few useful choices for the property \(P\) in (4.3). We’ll refer back to these in the rest of the paper.

1. **Nonsingularity.** From (4.3) we immediately see that \(Y'\) is nonsingular (because \(Y\) is), and each component of \(B' \subset Y'\) is nonsingular (because each component of \(B\) is).

2. **Codimension.** if \(x' \in X'\) maps to \(x \in X\), then \(x'\) and \(x\) have the same codimension. This is immediately clear from (4.2), because \(\text{codim}(x, X) = \dim \mathcal{O}_{X,x}\) and similarly for \(\text{codim}(x', X')\).

3. **Reduced divisors.** Moreover, since we are assuming that \(D \subset X\) is a reduced divisor, the base change \(D'\) is also reduced. The coefficient of a component of a divisor is checked at the stalk at the generic point of each component, and we have just seen that codimension-1 points in \(X'\) map to codimension-1 points in \(X\). Note that the function field also is involved in checking the coefficient of a divisor. The function field is also preserved—it is the stalk of the structure sheaf of any generic point of the base change—so reducedness is preserved.
(4) **The snc locus.** The property of being in the snc locus of a pair is checked using stalks of the structure sheaf and of quotients of the structure sheaf. Nonsingularity of the variety and of each component of the divisor is checked on stalks of the structure sheaf. The other requirement for a point to be in the snc locus of a pair is that any components of the divisor that pass through the point must meet transversally. That is, the local equations that cut out the components form part of a regular sequence in the stalk of the structure sheaf at each point of the variety (see [SGA77, 3.1.5]). Now the stalks of the structure sheaf are preserved by the base change, and a component of $D'$ passes through $x'$ in $X'$ if and only if its image passes through $x = \pi(x')$ in $X$. So $x'$ is in snc$(X', D')$ if and only if $x = \pi(x')$ is in snc$(X, D)$. Since $(X, B)$ is assumed to be snc in (4.1), we also have that $(Y', B')$ is snc.

**Corollary 4.5.** Using (4.1), let $\mathcal{F} \to \mathcal{G}$ be a morphism of coherent sheaves of $\mathcal{O}_X$-modules on $X$, and assume that the induced morphism $\pi^* \mathcal{F} \to \pi^* \mathcal{G}$ on $X'$ is an isomorphism. Then $\mathcal{F} \to \mathcal{G}$ is an isomorphism in a neighborhood of the fiber $X_s$.

**Proof.** First, note that the natural inclusion morphism $X_s \to X$ factors through the projection $\pi: X' \to X$. This is just because the local ring $\mathcal{O}_{S,s}$ maps to the residue field $\mathcal{O}_{S,s}/m_s = k(s)$, so the fiber $X_s = X \times_S \text{Spec} k(s)$ maps to $X' = X \times_S \text{Spec} \mathcal{O}_{S,s}$. So the fiber $X_s$ is contained in the image of $\pi$. By (4.2), for any $x' \in X'$ with $\pi(x') = x$, there are isomorphisms

$$\mathcal{F}_x \simeq (\pi^* \mathcal{F})_{x'}, \quad \mathcal{G}_x \simeq (\pi^* \mathcal{G})_{x'}.$$ 

So $\mathcal{F} \to \mathcal{G}_x$ is an isomorphism for all $x \in X_s$. Now the stalk of the kernel sheaf is the kernel of the stalk morphism, and there is a natural isomorphism between the stalk of the cokernel sheaf and the cokernel of the morphism on stalks ([Vak13 2.5.A–B]). So $\ker(\mathcal{F} \to \mathcal{G})_x = 0$ and $\operatorname{coker}(\mathcal{F} \to \mathcal{G})_x = 0$ for all $x \in X_s$.

Now $\ker(\mathcal{F} \to \mathcal{G})$ and $\operatorname{coker}(\mathcal{F} \to \mathcal{G})$ are coherent because $\mathcal{F}$ and $\mathcal{G}$ are. For any coherent sheaf $\mathcal{H}$, if $\mathcal{H}_x = 0$, then $\mathcal{H}_U = 0$ for some neighborhood $U$ of $x$. So the sheaves $\ker(\mathcal{F} \to \mathcal{G})$ and $\operatorname{coker}(\mathcal{F} \to \mathcal{G})$ are zero in a neighborhood of $x$. Since this is true for all $x \in X_s$, it follows that $\mathcal{F} \to \mathcal{G}$ is an isomorphism in a neighborhood of the fiber $X_s$. \qed

**Lemma 4.6.** With the notation of (4.1), the morphism $f': Y' \to X'$ is a resolution of singularities. Moreover, $B'$ is the birational transform of $D'$ and $(Y', B')$ is snc; thus, $(Y', B') \to (X', D')$ is a resolution of pairs. If $f$ is a log resolution, then so is $f'$, and if $f$ is thrifty, so is $f'$.

**Proof.** First of all, $f'$ is proper and $Y'$ is smooth: properness is always preserved by base change, and $Y'$ is smooth by (4.1).

We’ll now verify that $f'$ is birational. Let $\eta'$ be some codimension-0 point in $X'$. Irreducibility is not necessarily preserved by the base change, so there might be multiple such points; we may choose any one. This $\eta'$ maps to some point $\eta$ in $X$, which also has codimension 0 by (4.3). Now this $\eta$ is the generic point of $X$, because $X$ is irreducible. Because $Y \to X$ is surjective, this point $\eta$ has a preimage...
\( \xi \) in \( Y \), and this \( \xi \) is the generic point of \( Y \). Then there is a point in \( Y' \simeq Y \times_X X' \) mapping to \( \xi \); call this point \( \xi' \). By (4.4), \( \xi' \) has codimension 0, and the map on stalks \( \mathcal{O}_{X',y'} \rightarrow \mathcal{O}_{Y',\xi'} \) is exactly the same map as \( \mathcal{O}_{X,y} \rightarrow \mathcal{O}_{Y,\xi} \), which is an isomorphism because \( Y \rightarrow X \) is birational. Then \( f' \) is an isomorphism over every generic point of \( X' \). So it is birational over each component of \( X' \).

Now \( (Y', B') \) is snc, by (4.4). The same argument as above shows that \( B' \rightarrow D' \) is birational over every component of \( D' \), so \( (Y', B') \rightarrow (X', D') \) is a resolution of pairs.

We’ll show next that the exceptional locus of \( f \) in \( X \) base changes to the exceptional locus of \( f' \) in \( Y' \), and the image of the exceptional locus of \( f \) in \( X \) base changes to the image of the exceptional locus of \( f' \) in \( X' \).

First, taking the base change of all the exceptional components in \( X \) gives us exactly the exceptional components in \( Y' \). If a point on \( X \) is in the image of \( \text{Ex}(f) \) and also of the projection \( X' \rightarrow X \), then its preimages in \( X' \) are in the image of \( \text{Ex}(f') \). Indeed, given an exceptional component of the resolution \( Y \rightarrow X \), the map \( \mathcal{O}_{X', f'(y')} \rightarrow \mathcal{O}_{Y', y'} \) at its generic point \( y' \) is not an isomorphism, and this persists in the base change to the resolution \( Y' \rightarrow X' \) by (4.2); it is precisely the same map as \( \mathcal{O}_{X, f(y)} \rightarrow \mathcal{O}_{Y, y} \), if \( y = y' \). From this the converse follows too: an exceptional component in \( Y' \) maps to an exceptional component in \( Y \), and a point in \( f'(\text{Ex}(f)) \subset X' \) maps to a point in \( f(\text{Ex}(f)) \subset X \). So the base change of the exceptional locus is the exceptional locus, and the base change of the image of the exceptional locus is the image of the exceptional locus.

Now suppose \( f \) is a log resolution. We’ll show the same is true of \( f' \). We already know that the base change of a divisor is a divisor, because codimension of points is preserved. Consider an irreducible component \( F \) of the exceptional locus in \( Y' \). We’ll show \( F \) has codimension 1 in \( Y' \). Assume the opposite: that \( F \) has codimension \( k \geq 2 \) in \( Y' \). If \( \nu \) is the generic point of \( F \), then the map on stalks

\[
\mathcal{O}_{X', f'((\nu))} \rightarrow \mathcal{O}_{Y', \nu}
\]

is not an isomorphism, because \( Z \) is exceptional, but there is a closed, codimension-1 subset \( V \) of \( Y' \) containing \( F \) so that the map on stalks at the generic point of \( V \) is an isomorphism. Then the map on stalks at the corresponding codimension-\( k \) point in \( Y \) is not an isomorphism, but at the codimension-1 point it is. In other words, the resolution \( Y \rightarrow X \) also has an exceptional component that is not a divisor—but this is impossible, because we assume \( Y \rightarrow X \) is a log resolution, so its exceptional components must be divisors.

So the exceptional locus is a divisor in \( Y' \). Taking the snc locus commutes with this base change by (4.3), so \( f' \) is a log resolution.

Finally we show that if \( f \) is thrifty, then so is \( f' \). To verify Conditions 1 and 2 of (2.3), we must examine the resolution \( f' \) at generic points of strata of \( \text{snc}(X', D') \) and of \( (Y', B') \).

We’ll start with Condition 1. Let \( Z \) be a stratum of \( \text{snc}(X', D') \), so that \( Z \) is an irreducible component of some intersection \( D'_i \cap \cdots \cap D'_r \). Then, with \( \pi : X' \rightarrow X \) the projection, \( \pi(Z) \) is irreducible and contained in a component of the intersection \( D_i \cap \cdots \cap D_r \) in \( X \). Now, by applying (4.4), we see that \( \pi(Z) \) has the same codimension in \( X \) as \( Z \) does in \( X' \), and \( \pi(Z) \subset \text{snc}(X, D) \). By [Kol13, 4.16.2], which says that the intersection of \( r \) components of a dlt divisor has pure codimension \( r \), \( \pi(Z) \) is actually equal to a stratum of \( \text{snc}(X, D) \) (and is not merely contained in one). Now the assumption that \( f \) is thrifty means that \( \pi(Z) \) is not contained
in \(f(\text{Ex}(f))\). By the argument above, which showed that taking the image of the exceptional locus commutes with the base change \(X' \to X\), \(Z\) is not in \(f'(\text{Ex}(f'))\), and so \(f'\) satisfies Condition 1.

Condition 2 is almost the same. If a stratum of \((Y', B')\) lies in \(\text{Ex}(f')\), then its image is a stratum of \((Y, B)\) and lies in \(\text{Ex}(f)\), but this is impossible because \(f\) is thrifty. So \(f'\) also satisfies Condition 2, and hence is thrifty. \(\square\)

5. Deformation invariance of rational pairs

In this section we prove the main theorem: given a pair \((X, D)\) with \(D\) Cartier and a flat morphism \(X \to S\), if the fibers \((X_s, D_s)\) over a smooth point \(s \in S\) are a rational (reduced) pair, then \((X, D)\) is also rational near \(X_s\).

Because we assume \(D\) is Cartier, we may use (2.2), the analogue of Kempf’s criterion for rational pairs, to conclude that \((X, D)\) is rational near \(X_s\). In order to check the second part of (2.2), we will need to exhibit a thrifty resolution \(f\) of \(X\) so that that \(f^* \omega_Y(B) \cong \omega_X(D)\), at least near \(X_s\). The next lemma shows that actually we only need to verify that \(f^* \omega_Y(B) \to \omega_X(D)\) is surjective: injectivity is automatic.

**Lemma 5.1.** Suppose \(f: Y \to X\) is a birational and proper morphism between normal varieties, \(D\) is a divisor in \(X\), \(B = f^{-1}_* D\) is the birational transform of \(D\) in \(Y\), and \(\omega_Y\) is torsion free. (For example, let \(f: (Y, B) \to (X, D)\) be a resolution, so that \(\omega_Y\) is invertible.) Then there is a logarithmic trace morphism \(f^* \omega_Y(B) \to \omega_X(D)\), and it is injective.

**Proof.** Let \(U \subset X\) be the largest open set over which \(f\) is an isomorphism. The complement of \(U\) has codimension at least 2, because \(X\) is normal. Let \(i: U \hookrightarrow X\) be the inclusion. On \(U\), we have

\[
f^* \omega_Y(B)|_U \cong \omega_X(D)|_U,
\]

because the restricted map \(f: f^{-1}(U) \to U\) is an isomorphism. Now \(i^*\) and \(i_*\) are adjoint functors, so there is a natural morphism

\[
f^* \omega_Y(B) \to i_* i^* f^* \omega_Y(B),
\]

and the sheaf on the right can also be written as \(i_*(f^* \omega_Y(B)|_U)\). Putting these maps together, we obtain a composition

\[
f^* \omega_Y(B) \to i_*(f^* \omega_Y(B)|_U) \to i_*(\omega_X(D)|_U).
\]

On the open set \(U\), \(\omega_X(D)\) and \(i_*(\omega_X(D)|_U)\) are equal. The complement \(X \setminus U\) has codimension at least 2 and the sheaves are reflexive, so they are equal on \(X\) by \([\text{Har}80, 1.6]\). From this we have the desired map \(f^* \omega_Y(B) \to \omega_X(D)\).

Now we use the assumption that \(\omega_Y\) is torsion free. This guarantees that \(\omega_Y(B)\) is also torsion free, as is its pushforward \(f^* \omega_Y(B)\). Indeed, for any open set \(V \subset X\), the sections of \(f^* \omega_Y(B)\) on \(V\) are by definition the same as those of \(\omega_Y(B)\) on
f^{-1}(V). Now $f_*\omega_Y(B) \to \omega_X(D)$ is an isomorphism at the generic point of $X$, so the kernel of the morphism is a torsion sheaf. But $f_*\omega_Y(B)$ is torsion free, so the logarithmic trace map is injective. \hfill \square

**Theorem 5.2** (Deformation invariance for rational pairs). Let $(X, D)$ be a pair, with $D$ Cartier. Suppose $X \to S$ is a flat morphism, and $s \in S$ is a smooth point so that the fibers $(X_s, D_s)$ form a reduced pair. If $(X_s, D_s)$ is a rational pair, then $(X, D)$ is a rational pair in a neighborhood of $(X_s, D_s)$.

Proof. The first step is to show that we may assume the base $S$ is the spectrum of a regular local ring $R$. To see this, first base change the morphism $X \to S$ by the flat morphism $\text{Spec } \mathcal{O}_{S, s} \to S$, and let $X' = X \times_S \text{Spec } \mathcal{O}_{S, s}$ as in the notation of \ref{4.1}. Similarly, let $D' = D \times_S \text{Spec } \mathcal{O}_{S, s}$. Then $X' \to \text{Spec } \mathcal{O}_{S, s}$ is again flat.

We'll show that it suffices to prove the result for the new pair $(X', D')$: if $(X', D')$ is rational near $X_s$, then $(X, D)$ is also rational near $X_s$. Let $f: (Y, B) \to (X, D)$ be a thrifty log resolution, not necessarily rational. Then there is a Cartesian diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{p} & Y' \\
\downarrow{f} & & \downarrow{f'} \\
X & \xrightarrow{\pi} & X' \\
\downarrow & & \downarrow \\
S & \xrightarrow{} & \text{Spec } \mathcal{O}_{S, s}
\end{array}
\]

By \ref{4.3}, if $X'$ is normal and CM, then so is $X$ at every point in the image of the projection $X' \to X$, and if $D'$ is reduced, then so is $D$ at every point in the image of $D' \to D$. Also, by \ref{1.6}, $f': (Y', B') \to (X', D')$ is a thrifty log resolution.

Suppose for now that we have shown $(X', D')$ is a rational pair in a neighborhood $U$ of $X_s$, so that every thrifty resolution of the pair is rational over $U$. Then $f'$ is a rational resolution; that is, on $U$ we have

\[\mathcal{R}(f'), \mathcal{O}_{Y'}(-B') \simeq \mathcal{O}_{X'}(-D').\]

The map $\text{Spec } \mathcal{O}_{S, s} \to S$ is flat, so by cohomology and base change for flat morphisms ([Har77 III.9.3]), there is an isomorphism on $U$:

\[\pi^*\mathcal{R}_f \mathcal{O}_Y(-B) \simeq \mathcal{O}_{X'}(-D').\]

To prove the original thrifty resolution $f$ is then rational in a neighborhood of $X_s$, we need to verify that $\mathcal{R}_f \mathcal{O}_Y(-B) \simeq \mathcal{O}_{X}(-D)$ near $X_s$. Now $\mathcal{O}_{X'}(-D') = \pi^*\mathcal{O}_X(-D)$, and the pushforwards $\mathcal{R}_f \mathcal{O}_Y(-B)$ are coherent. By \ref{1.6} it follows that $\mathcal{R}_f \mathcal{O}_Y(-B) \simeq \mathcal{O}_{X}(-D)$ in a neighborhood of $X_s$.

So it suffices to prove the result in the case where the base is $\text{Spec } \mathcal{O}_{S, s}$. We may then assume that $S = \text{Spec } R$, where $R$ is a regular local ring of dimension $n$. 


We’ll prove that $X$ is normal in a neighborhood of the fiber $X_s$. Then, by (2.2) and (3.0), we will only need to prove that $X$ is CM near $X_s$ and for some thrifty resolution $f: (Y, B) \to (X, D)$, the logarithmic trace $f_*\omega_Y(B) \to \omega_X(D)$ is an isomorphism in a neighborhood of $X_s$. By (5.1) the logarithmic trace is injective, so we will show that $X$ is normal and CM in a neighborhood of $X_s$.

Following the idea of the proof of [Eli78 Théorème 2], we’ll prove this using induction on $n$. Our base case is $n = 0$. If $n = 0$, then $S = \text{Spec } K = \{s\}$ for some field $K$. In this case $(X, D) = (X_s, D_s)$, so the conclusion is trivially true.

Now let $\dim R = n$. For the inductive hypothesis, assume the result is true for $\dim R < n$; if the base scheme is the spectrum of a regular local ring of dimension less than $n$, and if $(X_s, D_s)$ is rational, then $(X, D)$ is rational in a neighborhood of $X_s$.

Let $t$ be a regular parameter in $R$. Then let $X_t = X \times_S \text{Spec}(R/tR)$, so that $X_t$ is the pullback of the divisor defined by $t$ in $S = \text{Spec } R$, and define $D_t$ similarly. Then $X_t$ is Cartier in $X$, and $D_t$ in $D$. Note that $(X_s, D_s)$ is a subpair of $(X_t, D_t)$: $X_s \subset X_t$ and $D_s \subset D_t$.

Now the regular local ring $R/tR$ has dimension $n - 1$, so $(X_t, D_t)$ is rational near $X_s$ by the inductive hypothesis. In particular, $X_t$ is normal and CM. Then $X$ is also normal and CM in a neighborhood of $X_t$: these are properties that pass from a Cartier divisor to an open set in the whole space. Any neighborhood of $X_t$ is also a neighborhood of $X_s$, so $X$ is CM and normal near $X_s$. Working in this neighborhood, normality allows us to use (2.2), and since we also have in this neighborhood that $X$ is CM, $f$ is thrifty, and $f_*\omega_Y(B) \to \omega_X(D)$ is injective, it just remains to show that $f_*\omega_Y(B) \to \omega_X(D)$ is surjective near $X_s$.

Let $(Y, B) \to (X, D)$ be a thrifty log resolution. There is then a proper morphism $Y_t \to X_t$ and a component $Y_1$ of $Y_t$ mapping birationally to $X_t$. Write $f_1 = f|_{Y_1}: Y_1 \to X_t$, and let $B_1 = (f_1)^{-1}(D_t)$ be the birational transform of $D_t$. Now $(Y_1, B_1)$ is not snc—$Y_1$ need not even be smooth—but the morphism $(Y_1, B_1) \to (X_t, D_t)$ is birational and satisfies Condition 1 from (2.3). By (3.2), there is a thrifty log resolution $f_2: (Y_2, B_2) \to (X_t, D_t)$ factoring through $(Y_1, B_1)$. Write $\tilde{f}$ for the intermediate birational morphism $Y_2 \to Y_1$.

\[
\begin{array}{ccc}
(Y_2, B_2) & \xrightarrow{f} & (Y_1, B_1) \\
\downarrow{f_2} & & \downarrow{f_1} \\
(X_t, D_t) & \xrightarrow{\tilde{f}} & (X_t, D_t)
\end{array}
\]

By assumption, $(X_t, D_t)$ is a rational pair, so $f_2$ is a rational resolution. Now $D_t$ is Cartier in $X_t$, so by (2.2), the logarithmic trace map is an isomorphism:

\[(f_2)_*\omega_{Y_1}(B_1) \simeq \omega_{X_t}(D_t).\]

Because $t$ is a regular parameter, we have an exact sequence

\[
0 \longrightarrow \omega_Y \xrightarrow{\omega_Y} \omega_Y \xrightarrow{t} \omega_{Y_t} \longrightarrow 0
\]
and similarly for $\omega_X, \omega_{X_1}$. Twist the sequences by $B$ and $D$, respectively. Both operations are exact because $B, D$ are Cartier. Push forward the sequence on $Y$ by $f$. The result is a commutative diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & f_*\omega_Y(B) & \rightarrow & f_*\omega_Y(B) & \rightarrow & f_*\omega_Y(B_1) & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \omega_X(D) & \rightarrow & \omega_X(D) & \rightarrow & \omega_X(D_1) & \rightarrow & 0 
\end{array}
$$

Both rows are exact: the top is exact by (3.6), the analogue of the Grauert-Riemenschneider vanishing theorem for thrifty resolutions.

By Grothendieck duality, $\omega_Y(B_1)$ is a subsheaf of $\omega_Y(B_1)$. Moreover, by (5.1), the logarithmic trace map $f_*\omega_Y(B_2) \rightarrow \omega_Y(B_1)$ is injective. Composing these injective maps, we get an injection $f_*\omega_Y(B_2) \rightarrow \omega_Y(B_1)$. Pushing forward by $f$ and using that $f_2 = f \circ f$, we have an injective map

$$f_*\tilde{f}_*\omega_Y(B_2) = (f_2)_*\omega_Y(B_2) \hookrightarrow f_*\omega_Y(B_1).$$

The isomorphism $(f_2)_*\omega_Y(B_2) \rightarrow \omega_X(D_1)$ then factors through $f_*\omega_Y(B_1)$, so the composition

$$f_*\omega_Y(B_2) \rightarrow f_*\omega_Y(B_1) \rightarrow \omega_X(D_1)$$

is injective and surjective. Thus the right vertical arrow in the diagram is surjective.

To prove our desired result—that $f$ is a rational resolution—we just need to verify that the middle vertical arrow is also surjective in a neighborhood of $X$. Let $x \in X_s$ be any point, not necessarily closed. Then $x$ maps to the single closed point—the maximal ideal $m$—of $\text{Spec } R$. This is clear from the definition of $R$: it is $O_{S,s}$ for a smooth point $s \in S$, and $X_s$ is just the fiber over $s$.

Working in an affine neighborhood of our point $x$, we may assume $X = \text{Spec } A, Y = \text{Spec } C$, and $\omega_X(D), \omega_Y(B)$ correspond to finite modules $M, N$ over $A, C$ respectively. We have ring maps $R \rightarrow A \rightarrow C$. Considering $N$ as an $A$-module via the map $A \rightarrow C$, and then thinking of $N \rightarrow M$ as a morphism of $A$-modules, we have the local version of the morphism $f_*\omega_Y(B) \rightarrow \omega_X(D)$.

The logarithmic trace map $f_*\omega_Y(B) \rightarrow \omega_X(D)$ is injective by (5.1), so $N \rightarrow M$ is injective and $N$ is a submodule of $M$. The morphism of sheaves $f_*\omega_Y(B_1) \rightarrow \omega_X(D_1)$ corresponds locally to the map of modules $N/tN \rightarrow M/tM$. Let $p$ be the ideal in $A$ corresponding to the point $x$. Since $t \in m$ and $p$ pulls back to $m$, $t$ is also in $p$ when we view $t$ as an element of $A$.

Next we localize the entire diagram of $A$-modules at $p$, to get a diagram of maps between $A_p$-modules:
Localization is exact, so the rows of the diagram are short exact, the first two arrows are injective, and the right arrow is surjective. Also, localization commutes with taking quotients, so $(M/tM)_p \simeq M_p/tM_p$, $(N/tN)_p \simeq N_p/tN_p$, and $(M/N)_p \simeq M_p/N_p$. By the snake lemma, there is a short exact sequence:

$$0 \to K \to (M/N)_p \overset{t}{\to} (M/N)_p \to 0$$

In particular, notice that the map $(M/N)_p \to (M/N)_p$ given by multiplication by $t$ is surjective, so $(M/N)_p = t(M/N)_p$.

All the modules we had before localizing were finite over $A$, so all the localized modules in the new diagram are finite over $A_p$. Now $t$ is in the single maximal ideal of $A_p$ and $(M/N)_p = t(M/N)_p$, so $(M/N)_p = 0$ by Nakayama’s lemma.

This argument holds for every point $x \in X_s$, so the cokernel sheaf $\omega_X(D)/f_*\omega_Y(B)$ is zero at every $x \in X_s$. Then, because the cokernel sheaf is coherent, it is actually zero on an open set of $X$ containing $X_s$.

Now we’ve shown that $\omega_X(D) \simeq f_*\omega_Y(B)$ in a neighborhood of the fiber $X_s$. By the reductions above, the induction is complete and $(X, D)$ is rational in a neighborhood of $X_s$. □

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