HONEYCOMBS AND SUMS OF HERMITIAN MATRICES

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ABSTRACT. Horn’s conjecture [Ho], which given the spectra of two Hermitian matrices describes the possible spectra of the sum, was recently settled in the affirmative. We discuss one of the many steps in this, which required us to introduce a combinatorial gadget called a honeycomb; the question is then reformulable as about the existence of honeycombs with certain boundary conditions. Another important tool is the connection to the representation theory of the group $U(n)$, by “classical vs. quantum” analogies.

If $H$ is an $n \times n$ Hermitian matrix, then the spectrum $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ can be written as a weakly decreasing sequence of $n$ real numbers. Conversely, for every spectrum $\lambda$ we can form the set $O_\lambda$ of Hermitian matrices with spectrum $\lambda$; this set is known as a co-adjoint orbit of $U(n)$.

If $\lambda, \mu, \nu$ are three spectra, we define the relation

\[ \lambda \boxplus \mu \sim_c \nu \]

if there exist Hermitian matrices $H_\lambda \in O_\lambda, H_\mu \in O_\mu, H_\nu \in O_\nu$ such that $H_\lambda + H_\mu = H_\nu$.

The “c” in $\sim_c$ stands for “classical”; we will define a quantum analogue $\sim_q$ later on.) For instance, one can easily verify that

\[ (3) \boxplus (4) \sim_c (7), \quad (3,0) \boxplus (4,0) \sim_c (7,0), \quad (3,0) \boxplus (4,0) \sim_c (4,3), \quad (2,0) \boxplus (2,0) \sim_c (3,1) \]

but that

\[ (3) \boxplus (4) \not\sim_c (5), \quad (3,0) \boxplus (4,0) \not\sim_c (8,-1) \]

In 1912, Hermann Weyl [W] posed the problem of determining the set of triples $\lambda, \mu, \nu$ for which (1) held. Or more informally: given the eigenvalues of two Hermitian matrices $H_\lambda$ and $H_\mu$, what are all the possible eigenvalues of the sum $H_\lambda + H_\mu$? The purpose of this article is to describe the successful resolution to this problem, based on recent breakthroughs [Kl, HR, KT, KTW].

It is fairly easy to obtain necessary conditions in order for (1) to hold. For instance, from the simple observation that the trace of $H_\lambda + H_\mu$ must equal the sum of the traces of $H_\lambda$ and $H_\mu$, we obtain the condition

\[ \nu_1 + \ldots + \nu_n = \lambda_1 + \ldots + \lambda_n + \mu_1 + \ldots + \mu_n. \]

Another immediate constraint is that

\[ \nu_1 \leq \lambda_1 + \mu_1, \]

since the largest eigenvalue of $H_\lambda + H_\mu$ is at most the sum of $H_\lambda$ and $H_\mu$’s individual largest eigenvalues. (Exercise for the reader: show equality occurs exactly when the same vector
is a principal eigenvector for both matrices.) Weyl found a number of similar necessary conditions, such as the statement \( \nu_{i+j+1} \leq \lambda_{i+1} + \mu_{j+1} \) whenever \( 0 \leq i, j, i + j < n \), and many more necessary conditions were found by later authors. All of these conditions took the form of homogeneous linear inequalities (e.g. \( \nu_1 + \nu_2 \leq \lambda_1 + \lambda_2 + \mu_1 + \mu_2 \)) This phenomenon was then explained in the work of Alfred Horn [Ho]. Based on extensive computations for small \( n \), Horn then described a recursive algorithm for generating a list of such linear inequalities, and conjectured that this list, together with the trace identity (2), gave a necessary and sufficient set of criteria for (1). His conjecture turned out to be correct, though it waited 36 years for resolution.

We approached this problem by first observing that the relation (1) could be converted into a statement about honeycombs, which we introduced (for this purpose) in [KT]. These are a family of planar arrangements of edges labeled with multiplicities (some examples are in figure 1.) We give the precise definition in the next section.

![Figure 1. Two honeycombs. The left one is more typical in having only Y vertices, as will be explained in theorem 3. All edges are multiplicity 1, except for the edge labeled 2 in the right-hand honeycomb.](image)

The relevance of honeycombs to sums of Hermitian matrices is the following theorem, which we explain in more detail later:

**Theorem 1.** The relationship (1) holds if and only if there exists a honeycomb with boundary values \((\lambda, \mu, -\nu)\).

We now briefly summarize the sequence of events leading to the full proof of Horn’s conjecture. It has been known for some time that the “classical” problem (1) of summing Hermitian matrices is closely related to the “quantum” problem

\[
\lambda \boxplus \mu \sim_q \nu
\]

of tensoring \( U(n) \) representations, which we shall define later. For instance, it was well-known that (4) implies (1). The converse implication (for integral \( \lambda, \mu, \nu \)) was known as the saturation conjecture. Using some formidable algebraic and geometric machinery, Klyachko [Kl] was able to demonstrate a further non-trivial recursive relationship between the classical and quantum problems, and noted that this, combined with the saturation conjecture, would imply Horn’s conjecture. In [KT], Theorem 1 (and the quantum analogue of this theorem) were used to convert the saturation conjecture into a statement
about honeycombs, and then proved this statement by combinatorial methods, thus proving the saturation and Horn conjectures. (Recently it has been shown \[KTW\] that one can derive Horn’s conjecture directly from the saturation conjecture by purely combinatorial techniques, bypassing the machinery of \[Kl\]. Also, a very different proof of saturation, based on the representation theory of quivers, has since been given in \[DW\]. Finally, a short rendition of \[KT\] can be found in \[Bu\].)

We shall give a rather ahistorical (and pro-honeycomb) survey of this circle of ideas, starting with honeycombs (which were actually the last piece of the puzzle to be discovered), then discussing the connections between the classical and quantum problems, followed by a sketch of the honeycomb-based proof of the saturation conjecture. Then we state Horn’s conjecture, and sketch the honeycomb-based proof of this from saturation.

There are many other closely related and interesting mathematical questions that we will not address, and we shall refer to the reader to the excellent survey article \[F2\].

1. Honeycombs

We shall shortly give the formal definition of a honeycomb as introduced in \[KT\], but first we shall rephrase Weyl’s problem in a more symmetric form. We say that the relation

\[
\lambda \boxplus \mu \boxplus \nu \sim_0 c
\]

holds if there exist matrices \(H_\lambda \in \mathcal{O}_\lambda, H_\mu \in \mathcal{O}_\mu, H_\nu \in \mathcal{O}_\nu\) such that \(H_\lambda + H_\mu + H_\nu = 0\). Clearly we have

\[
\lambda \boxplus \mu \sim_0 \nu \iff \lambda \boxplus \mu \boxplus (-\nu) \sim_0 0
\]

where \(-\nu := (-\nu_n, \ldots, -\nu_1)\) is the negation of \(\nu\). Thus to solve Weyl’s problem it suffices to determine the set of triples \(\lambda, \mu, \nu\) which obey \((5)\). This formulation has the advantage of \(S_3\) symmetry in \((\lambda, \mu, \nu)\), as opposed to mere \(S_2\) symmetry in \((\lambda, \mu)\).

The trace identity \((2)\) in this new formulation becomes

\[
\lambda_1 + \ldots + \lambda_n + \mu_1 + \ldots + \mu_n + \nu_1 + \ldots + \nu_n = 0,
\]

while \((3)\) becomes

\[
\lambda_1 + \mu_1 + \nu_n \geq 0.
\]

Based on these relations, it is natural to introduce the plane

\[
\mathbb{R}^3_{\sum = 0} := \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}.
\]

We shall always depict this plane with the six “cardinal directions” \((0, 1, -1), (-1, 1, 0), (-1, 0, 1), (0, -1, 1), (1, -1, 0)\) and \((1, 0, -1)\) drawn Northwest, North, Northeast, South, and Southwest respectively. (Of course, “Northwest” makes a 60 degree angle with North rather than a 45 degree angle, and similarly for the other diagonal cardinal directions.)

A diagram will be defined as a configuration of (possibly half-infinite) line segments in \(\mathbb{R}^3_{\sum = 0}\), with each edge parallel to one of the cardinal directions (North-South, Northeast-Southwest, Northwest-Southeast), and labeled with a positive integer which we refer to as the “multiplicity” or “tension”. To every diagram we can associate a measure on \(\mathbb{R}^3_{\sum = 0}\) defined as the sum of Lebesgue measure on each line segment, weighted by the multiplicity. We say that two diagrams \(d, d'\) are equivalent if their associated measures are equal.
If \( h \) is a diagram and \( v \) is a point in \( \mathbb{R}^3 \) such that the sum of the co-ordinate vectors of rays emanating from \( v \), multiplied by their tensions, equals zero, we say that \( v \) is a zero-tension point of \( h \).

The two possibilities that will interest us most are a point on a line segment, in which case the zero-tension condition says that the two rays must have the same multiplicity, and a point at the center of a \( Y \) with again three equal-multiplicity rays. There are several more complicated possibilities, as shown in figure 2.

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**Figure 2.** Zero-tension points, with the rays labeled by their multiplicities. All but the first type are called 'vertices'.

Define a honeycomb \( h \) as a diagram (or more precisely, an equivalence class of diagrams) such that

1. every point in \( \mathbb{R}^3 \) is a zero-tension point
2. there are only finitely many “vertices,” i.e. points with more than two rays emanating
3. the semiinfinite lines go only in the Northeast, Northwest, and South directions (i.e. no Southeast, Southwest, or North rays)

The lines mentioned in part 3 are called the boundary edges of the honeycomb. Two examples of honeycombs appear in figure 1.

It is a pleasant exercise to show that the number of boundary edges (with multiplicity) pointing in one cardinal direction is the same as the number in each of the other two directions. (This is basically because the net tension of the honeycomb must be zero.) We will call a honeycomb with \( n \) boundary edges in each direction an \( n \)-honeycomb and denote the space of such by \( \text{HONEY}_n \).

Since every edge in a honeycomb is parallel to a cardinal direction, each of which has one of its three coordinates being zero, every honeycomb edge has a constant coordinate (common to every point along that edge.) In particular we can read off the constant coordinates of boundary edges and call them

\[
(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n) = (\lambda, \mu, \nu)
\]

as in figure 3.

We can now phrase Theorem 1 in this symmetrized setting:

**Theorem 2.** The relationship \( \lambda \boxplus \mu \boxplus \nu \sim_c 0 \) holds if and only if there exists a honeycomb with boundary values \( (\lambda, \mu, \nu) \).

Interestingly, almost all the proofs we know of this Theorem proceed by first proving a quantized version of this Theorem, which we define in the next section. We shall therefore not discuss the proof of this Theorem here, and content ourselves instead with producing evidence which strongly suggests that the Theorem is plausible.
We first consider the $n = 1$ case. In this case $\lambda = (\lambda_1)$, $\mu = (\mu_1)$, $\nu = (\nu_1)$, and it is clear that (5) holds if and only if $\lambda_1 + \mu_1 + \nu_1 = 0$. (In other words, the trace condition is already necessary and sufficient.) On the honeycomb side, this claim can be easily seen, if one accepts the fact (which is actually a little tricky to prove) that 1-honeycombs must have the shape of a “Y”. See Figure 4.

More generally, it is a pleasant exercise to show that the boundary values of any $n$-honeycomb must satisfy (6), basically because the three co-ordinates around every vertex sum to zero (by virtue of lying in $\mathbb{R}^3$).

Now consider the $n = 2$ case, so that $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2)$. In this case there can be at most one 2-honeycomb with the specified boundary values (Figure 5). The lengths of the three line segments in the honeycomb can be computed as $\lambda_2 + \mu_1 + \nu_1$, $\lambda_1 + \mu_2 + \nu_1$, $\lambda_1 + \mu_1 + \nu_2$. Since these line segments need to have non-negative length, we obtain the necessary conditions

$$\lambda_2 + \mu_1 + \nu_1, \lambda_1 + \mu_2 + \nu_1, \lambda_1 + \mu_1 + \nu_2 \geq 0.$$
These inequalities can be rephrased (6) as the statement that the quantities $\lambda_1 - \lambda_2$, $\mu_1 - \mu_2$, $\nu_1 - \nu_2$ form the lengths of a triangle. The reader may verify from some linear algebra that these conditions are indeed necessary and sufficient for (5).

\[
\begin{align*}
\lambda_1 - \lambda_2 &> 0, \\
\mu_1 - \mu_2 &> 0, \\
\nu_1 - \nu_2 &> 0,
\end{align*}
\]

**Figure 5.** A 2-honeycomb is uniquely determined by its boundary values. The boundary values must satisfy (6), and the edge lengths must be non-negative. The edges are labeled by their constant co-ordinates. The edge lengths can be computed by subtracting the constant co-ordinates of two parallel adjacent edges; for instance, the lower left edge has length $\lambda_1 - (-\mu_2 - \nu_1)$.

In the $n > 2$ case things become more complicated, because the boundary values no longer uniquely determine the honeycomb. In fact, every hexagon which is present in a honeycomb provides a degree of freedom; the hexagon can be “breathed” inwards or outwards (see Figure 6).

\[
\begin{align*}
\lambda_1 + \mu_2 + \nu_{n-1} &\geq 0,
\end{align*}
\]

**Figure 6.** A hexagon in a honeycomb, with a dotted line indicating a place to which one might dilate it.

However, it is still possible to demonstrate that inequalities such as (7) must hold for $n$-honeycombs. Indeed, one can simply extend the $\mu_1$ ray northward until it intersects the $\lambda_1$ ray. This intersection point must be Northwest of the intersection of $\nu_n$ and $\lambda_1$, which gives (7). The reader may be amused by locating proofs of similar inequalities such as $\lambda_1 + \mu_2 + \nu_{n-1} \geq 0$.

On the positive side, it is easy to see that (5) will hold if there exist permutations $\alpha, \beta \in S_n$ such that $\lambda_{\alpha(i)} + \mu_{\beta(i)} + \nu_i = 0$ for all $1 \leq i \leq n$. The honeycomb analogue of this is depicted in Figure 7; one can obtain a (rather degenerate) $n$-honeycomb by overlaying $n$ separate 1-honeycombs on top of one another.
Figure 7. An $n$-honeycomb can be obtained by overlaying $n$ 1-honeycombs on top of one another. Here $\alpha$ and $\beta$ map $1,2,3,4$ to $4,3,1,2$ and $4,3,2,1$ respectively.

More generally, there is a notion of overlaying an $n$-honeycomb $h$ and an $m$-honeycomb $h'$ to form an $n + m$-honeycomb $h \oplus h'$. To be precise, $h \oplus h'$ is the honeycomb whose associated measure is the sum of the measures associated to $h$ and $h'$. This operation corresponds to the direct sum operation on Hermitian matrices (which takes an $n \times n$ matrix and an $m \times m$ matrix and forms an $(n + m) \times (n + m)$ block-diagonal matrix), or of the concatenation operation on spectra (which takes a set of $n$ eigenvalues and a set of $m$ eigenvalues and forms the (sorted) set of $n + m$ eigenvalues.) Intuitively, an overlay can be demonstrated by drawing two honeycombs on transparencies and stacking both transparencies on the same projector; see figure 8. We shall have more to say about overlays later in this article.

Figure 8. Two honeycombs overlaid to produce a third. The origin $(0,0,0)$ is marked in each picture with a black dot.

The relation (5) is clearly $S_3$ symmetric, however the statement that $(\lambda, \mu, \nu)$ admits a honeycomb with these boundary values only appears to be symmetric under cyclic permutations and the inversion $(\lambda, \mu, \nu) \mapsto (-\nu, -\mu, -\lambda)$. Nevertheless, it is possible to show that honeycombs do indeed obey the full $S_3$ symmetry; an elegant proof of this based on scattering arguments is given in [Wo]. The same argument also gives the associativity property

$$\exists \nu : \lambda \oplus \mu \sim_c \nu; \nu \oplus \rho \sim_c \sigma) \iff (\exists \nu' : \mu \oplus \rho \sim_c \nu'; \lambda \oplus \nu \sim_c \sigma);$$

this property, combined with a “Pieri rule” to handle the generating cases when $\lambda$ or $\mu$ is equal to $(\epsilon, 0, \ldots, 0)$ for some small $\epsilon$, can be used to give an inductive proof of Theorem 1.

An interesting degenerate case occurs when $\nu$ is kept fixed, while the spacings between eigenvalues of $\lambda$ and $\mu$ are allowed to become very large. In this case the honeycomb
degenerates into a pattern known as a Gelfand-Cetlin pyramid, while Weyl’s problem degenerates to Schur’s problem of determining which \( n \)-tuples can be the diagonal entries of a Hermitian matrix with specified eigenvalues. (In fact, we discovered honeycombs by extrapolating from this degenerate case.)

We encourage the reader to try out the honeycomb Java applet at

\[ \text{http://www.math.ucla.edu/~tao/java/honeycombs.html} \]

2. ORGANIZING HONEYCOMBS INTO A POLYHEDRAL CONE

The space \( \text{HONEY}_n \) of all \( n \)-honeycombs has been defined as an abstract set, but one can in fact give this space the structure of a polyhedral cone inside some finite dimensional vector space.

Call a honeycomb nondegenerate if

1. all its edges are multiplicity 1, and
2. all its vertices are either right-side-up or upside-down Y’s.

It is straightforward to prove that all nondegenerate \( n \)-honeycombs have the same topological shape, namely that of \( \mathbb{P} \) in particular there is a natural way to correspond the edges in one \( n \)-honeycomb with those in any other.

\[ \text{Figure 9. Three nondegenerate 3-honeycombs. Note that there is a natural way to correspond the edges in one with the edges in any other.} \]

This gives us a way of making the space of nondegenerate \( n \)-honeycombs into an open polyhedral cone in a real vector space, with defining hyperplanes given by rational coefficients. The coordinates are given by the constant coordinates of the edges, linear equalities must be imposed saying that the constant coordinates of three edges meeting at a vertex add to zero, and linear inequalities which say that every edge has strictly positive length.

**Theorem 3.** [KT] The identification in the above paragraph, between nondegenerate \( n \)-honeycombs and points in a certain rational polyhedral open cone, extends to an identification of all \( \text{HONEY}_n \) with the closure of this cone. In particular, nondegenerate \( n \)-honeycombs form a dense open set in \( \text{HONEY}_n \).

This theorem is surprisingly annoying to prove, and takes pp. 1067-1074 of [KT]. Its virtue is in enabling us to use the theory of such cones to define the special honeycombs that we will deform to, as explained in the introduction.
For a honeycomb \(h\), let \(\partial h \in (\mathbb{R}^n)^3\) denote the list \((\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n)\) of constant coordinates on the boundary edges of \(h\), and \(\text{BDRY}_n \subseteq (\mathbb{R}^n)^3\) be the image of this map \(\partial : \text{HONEY}_n \to (\mathbb{R}^n)^3\). Then we can think of the main question as being to list the inequalities determining \(\text{BDRY}_n\).

It is not hard to show directly that this map is proper, so each fiber is a compact, convex polyhedron.

As one application of this formalism, we can easily show that for any \(\lambda, \mu, \nu\), the truth or falsity of (5) or (1) can be determined in polynomial time with respect to the dimension \(n\); this fact appears to be previously unknown. Indeed, the problem is equivalent to determining whether the polytope \(\partial^{-1}(\lambda, \mu, \nu)\) is non-empty. Since the cone \(\text{HONEY}_n\) has about \(O(n^2)\) faces, this can be achieved in polynomial time by standard algorithms (e.g. the simplex method). [The authors thank Peter Shor for pointing out this fact].

One can also use this formalism to create a more quantitative version of Theorems 1 and 2. Let \(H_\lambda\) be the random variable with the uniform distribution on \(O_\lambda\) (where “uniform” can be defined using induced Lebesgue measure, or the \(U(n)\) action). In other words, \(H_\lambda\) is a random matrix with spectrum \(\lambda\). One can then define \(P(\lambda \boxplus \mu \sim_c \nu)\) to be the probability density of the spectrum of the sum \(H_\lambda + H_\mu\) of two independent random matrices evaluated at \(\nu\). Similarly define \(P(\lambda \boxplus \mu \boxplus \nu \sim_c 0)\).

**Theorem 4.** Up to inessential factors (constants and Vandermonde determinants), \(P(\lambda \boxplus \mu \sim_c 0)\) is essentially the volume of the symplectic reduction of the manifold \(O_\lambda \times O_\mu\) (with the diagonal \(U(n)\) action) at the point \(-\nu\), and similarly for \(P(\lambda \boxplus \mu \boxplus \nu \sim_c 0)\). We shall have more to say about this later on.

### 3. Quantum Analogues

We now describe the quantum analogue of the classical relation. Roughly speaking, is to the representation theory of \(U(n)\) as \(\text{U(1)}\) is to the symplectic geometry of \(U(n)\).

Recall that the irreducible unitary representations of \(U(1)\) are all one-dimensional. In fact, for each integer \(\lambda\) we can define the irreducible representation \(V_\lambda\) as a one-dimensional vector space, with the action of \(e^{i\theta}\) given by multiplication by \(e^{i\lambda\theta}\) on \(V_\lambda\).

More generally, for any weakly decreasing sequence \(\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)\) of integers we can define an irreducible unitary representation \(V_\lambda\) of \(U(n)\) by standard constructions (see e.g. [F]). The \(n\)-tuple \(\lambda\) is referred to as the weight of \(V_\lambda\). For instance, if \(\lambda\) consists of \(k\) 1’s and \(n-k\) 0’s then \(V_\lambda\) is the space of \(k\)-forms \(\bigwedge^k \mathbb{C}^n\) with the standard \(U(n)\) action. More generally, if \(\lambda_n \geq 0\) we define \(V_\lambda\) to be the highest weight irreducible representation in

\[
\bigotimes_{i=1}^n \text{Sym}^{\lambda_i - \lambda_{i+1}} \bigwedge^k \mathbb{C}^n
\]

with the convention \(\lambda_{n+1} = 0\), and the \(\lambda_n < 0\) representations can be defined dually.

Given two irreducible representations \(V_\lambda, V_\mu\) of \(U(n)\), the tensor product \(V_\lambda \otimes V_\mu\) is another representation of \(U(n)\). In the \(n = 1\) case the tensor product is again an irreducible
representation: \( V_\lambda \otimes V_\mu \equiv V_{\lambda+\mu} \). However, in general the tensor product is not irreducible, and splits up as a direct sum of many smaller irreducible representations \( V_\nu \). We can now define the relation (4) as the statement that a copy of \( V_\nu \) appears at least once in the tensor product \( V_\lambda \otimes V_\mu \). Note that the quantum relation is only defined for \( \text{integral } \lambda, \mu, \nu \), whereas the classical relation (1) is defined for \( \text{real } \lambda, \mu, \nu \).

There is a close parallel between (1) and (4). For instance, one can obtain the trace identity (2) as a necessary condition for (4) by considering the action of the center \( U(1) \) of \( U(n) \). One can similarly obtain the necessary condition (3) by considering the highest weights of the action of a maximal torus \( U(1) \times \ldots \times U(1) \) in \( U(n) \). From a more physical viewpoint, one can view the classical problem as a problem of describing how the moments of inertia of bodies in \( \mathbb{C}^n \) behave under superposition, while the quantum problem is the problem of describing how the spin states of particles in \( \mathbb{C}^n \) behave under superposition. (The \( n = 2 \) case is especially interesting to physicists, since \( U(2) \) is closely related to \( O(3) \). In this case every representation \( V_\nu \) appears at most once in \( V_\lambda \otimes V_\mu \) (this corresponds to the fact that 2-honeycombs are determined by their boundary values), and one can parameterize the decomposition explicitly using the Clebsch-Gordan coefficients.)

This connection between the classical and quantum problems seems to have been noted first in [L] (and in a more general context in [He], both in 1982), and appears in detail in [Kl]; the most natural framework for such results is exposed in [Kn]. Explicitly, the connection is given by

**Theorem 5.** Let \( \lambda, \mu, \nu \) be weakly decreasing sequences of \( n \) integers.

1. (Quantum implies classical.) If (4) holds, then (1) holds.
2. (Classical implies asymptotic quantum.) Conversely, if (1) holds, then there exists an integer \( N > 0 \) such that \( N\lambda + N\mu \sim_q N\nu \). (Here \( N\lambda \) is the sequence \( (N\lambda_1, \ldots, N\lambda_n) \).)

From this theorem it is natural to phrase

**Conjecture 1** (Saturation conjecture). One can take \( N = 1 \) in the above theorem. In other words, (1) and (4) are equivalent for integer \( \lambda, \mu, \nu \).

This conjecture seems to be special to \( U(n) \); the naïve analogue of this conjecture for other Lie groups can be easily shown to be false. The saturation conjecture is so named because it is equivalent to the set of triples \( (\lambda, \mu, \nu) \) obeying (4) being a saturated submonoid of \( \mathbb{Z}^m \).

To attack the saturation conjecture using honeycombs we need a quantum analogue of Theorems 1, 2. We first phrase a symmetric form of (4). We say that

\[
\lambda \boxplus \mu \boxplus \nu \sim_q 0
\]

holds if \( V_\lambda \otimes V_\mu \otimes V_\nu \) contains a non-trivial \( U(n) \)-invariant vector. It is easy to show that (9) is equivalent to \( \lambda \boxplus \mu \sim_q -\nu \).

A honeycomb is said to be integral if its vertices lie on \( \mathbb{Z}^m_{\sum = 0} := \mathbb{R}^m_{\sum = 0} \cap \mathbb{Z}^m \). Note that the boundary values of an integral honeycomb are necessarily integers.

**Theorem 6.** The relationship (9) holds if and only if there exists an integral honeycomb with boundary values \( (\lambda, \mu, \nu) \). As a corollary, the relationship (4) holds if and only if there exists an integer honeycomb with boundary values \( (\lambda, \mu, -\nu) \).
Note that Theorems 6 and 5 imply Theorems 1 and 2.

The problem (4) has had a long history, and a solution is given by the famous Littlewood-Richardson rule. This rule has been formulated in many different ways, most if which involve Young tableaux; a variant due to Berenstein and Zelevinsky can be easily adapted to give Theorem 6 (Fulton has also shown that this theorem can be proven directly from the Littlewood-Richardson rule.) Other proofs are known; for instance, one can combine the quantum version of (3) with Pieri’s rule for tensoring a $U(n)$ representation with the tautological $C^n$ representation to give an inductive proof of Theorem 6.

A quantum analogue of Theorem 4 is also known:

**Theorem 7.** The number of times $V_\nu$ appears in the tensor product of $V_\lambda \otimes V_\mu$ is equal to the number of integral honeycombs with boundary values $(\lambda, \mu, -\nu)$. Equivalently, the dimension of the $U(n)$-invariant subspace of $V_\lambda \otimes V_\mu \otimes V_\nu$ is equal to the number of integral honeycombs with boundary values $(\lambda, \mu, \nu)$.

All the proofs of Theorem 6 mentioned above can also be used to prove Theorem 7. Theorem 4 can be viewed as a crude asymptotic version of Theorem 7. Variants of this theorem appear in [J, BZ], and particularly in [GP], though honeycombs are not explicitly used in these papers. We remark that the representation theoretic quantities in Theorem 7 can also be calculated by the Steinberg product rule (for instance), though we do not know a proof of this Theorem that goes via this rule.

Readers who are familiar with the representation theory of $SL_2$ (or $SU(2)$) may verify that the honeycomb rule given in Theorem 6 corresponds to the usual triangle inequalities for the weights. The fact that 2-honeycombs are uniquely determined by their boundary values corresponds to the fact that each irreducible representation of $SL_2$ appears exactly once in a tensor product of irreducibles.

As mentioned in the introduction, the saturation conjecture would give a complete solution to the (now equivalent) problems (1), (4), given by Horn’s conjecture. We shall keep the reader in suspense as to the actual statement of Horn’s conjecture for a little while longer, and instead turn to the honeycomb-based proof of saturation.

### 4. Proof of the Saturation Conjecture

In light of the theorems of the previous section, the saturation conjecture can be reduced to the following purely honeycomb-theoretic problem:

**Theorem 8.** Let $h$ be a (real-valued) honeycomb with integer boundary values. Then there exists an integer honeycomb $h'$ with the same boundary values as $h$.

Or in other words: if $\lambda, \mu, \nu$ are integers and the polytope $\partial^{-1}(\lambda, \mu, \nu)$ is non-empty, then $\partial^{-1}(\lambda, \mu, \nu)$ must contain at least one integer point.

The most obvious thing to do is to look for a vertex of $\partial^{-1}(\lambda, \mu, \nu)$; however one can give examples of vertices which are non-integral even when $\lambda, \mu, \nu$ are integers. Thus we have to be a little more careful as to how to locate our integer honeycomb.

We call a functional $f : \text{HONEY}_n \rightarrow \mathbb{R}$ superharmonic if it increases when we dilate a hexagon (which one can do to any hexagon in any nondegenerate honeycomb, as in figure 6).
Fix a generic superharmonic functional \( f \), and define the **largest lift** of a triple \((\lambda, \mu, \nu)\) as the honeycomb \( h \) that maximizes \( f(h) \) subject to \( \partial h = (\lambda, \mu, \nu) \). It is straightforward to prove that the largest-lift map \( \text{BDRY}_n \to \text{HONEY}_n \) is uniquely defined (for a given generic \( f \)), continuous, and piecewise-linear. To show Theorem 8 it then suffices to show that the largest map takes integer boundary values to integer honeycombs.

We need some more notation. Say that a honeycomb \( h \) has **only simple degeneracies** if all its edges are multiplicity 1, and its vertices are either Ys (possibly upside down) or crossings of two straight lines. In this case, define the **underlying graph** of \( h \) as the graph whose vertices consist of the (possibly upside down) Ys but not the crossings; the crossings we instead interpret as two edges missing one another. (In the examples in figure 1 every vertex except the bottom right vertex of the right-hand honeycomb are only simple degeneracies.)

With this in mind, we can talk about **loops** in a simply degenerate honeycomb (meaning, in the underlying graph), or call the honeycomb **acyclic** if there are none. For example, in the left honeycomb in figure 10 there is a loop, whereas the honeycomb on the right is acyclic.

![Two simply degenerate honeycombs](image)

**Figure 10.** Two simply degenerate honeycombs, with black dots on the vertices of the their underlying graphs. The left one has a loop that can breathe in and out (to, say, the dotted-line position), but the right has none.

The importance of loops in simply degenerate honeycombs is that they can be breathed in and out, as in figure 10 generalizing the case of dilating a single hexagon.

Call a largest lift **regular** if the boundary spectra \( \lambda, \mu, \nu \) each contain no repeated eigen-values.

The main technical part of [KT] is to prove

**Theorem 9.** [KT] Regular largest lifts can only have simple degeneracies.

In particular, regular largest lifts come with underlying graphs. Roughly speaking, this theorem is proven by showing that every non-simple degeneracy can be “blown up” in a way which increases the superharmonic functional.

**Lemma 1.** The underlying graphs of regular largest lifts are acyclic.

**Sketch of proof.** If a simply degenerate honeycomb has a loop, we can breathe it in and out; one direction will increase the value of any (generic) superharmonic functional. A largest
From this lemma one can show that the co-ordinates of regular largest lifts are integral linear combinations of the boundary values. Those who wish to see the details should go to [KT], but the argument is intuitively clear. Given a honeycomb with some edges labeled by their constant coordinates and some still mysterious, look for vertices with two known constant coordinates. The remaining one is minus the sum of the other two. Label it such and repeat. The reader is invited to play this game on the honeycombs in figure 10 and see how in the left-hand honeycomb one gets stuck exactly because of the loop.

In figure 11 we have labeled the boundary edges “A”, the edges whose constant coordinates can be determined from those “B”, those at the next stage of this recursive algorithm “C”, and so on.

**Figure 11.** A honeycomb integrally determined by its boundary, in four stages A,B,C,D.

Since every largest lift can be obtained as a limit of regular largest lifts, we thus have that the co-ordinates of all largest lifts are integral linear combinations of their boundary values. In particular, if the boundary values are integral, then the largest lifts are also linear. This proves Theorem 8 which gives the saturation conjecture.

5. **Klyachko’s result and Horn’s conjecture**

In this section we state a version of Klyachko’s result (one direction of which was also proven by Helmke and Rosenthal) and Horn’s conjecture. (We give slightly revisionist versions in order to avoid introducing Schubert calculus on Grassmannians, which is one of the equivalent problems explained in [F2].)

Horn [Ho] showed that the solution set to (1) must be given by (2) and a finite number of inequalities of the form

\[
\lambda_{i_1} + \ldots + \lambda_{i_r+1} + \mu_{j_1} + \ldots + \mu_{j_r+1} \geq \nu_{k_1} + \ldots + \nu_{k_{r+1}}
\]

where \(1 \leq r < n\) and \(i = (i_1 \geq \ldots \geq i_r)\), \(j = (j_1 \geq \ldots \geq j_r)\), and \(k = (k_1 \geq \ldots \geq k_r)\) are weakly decreasing sequences of integers between 0 and \(n - r\) inclusive. Let us call triples \((i, j, k)\) of this form *admissible*.

As an example, (3) is (10) for the admissible triple \(((0), (0), (0))\), while Weyl’s inequalities correspond to admissible triples of the form \(((i), (j), (i+j))\). The inequality \(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 \geq \nu_1 + \nu_2\) corresponds to \(((0, 0), (0, 0), (0, 0))\) and so forth.
After many computations of such inequalities, Horn made the remarkable

**Conjecture 2** (Horn’s conjecture). Let $\lambda, \mu, \nu$ be weakly decreasing sequences of reals. Then (1) holds if and only if (2) holds, and (10) holds whenever $i, j, k$ are admissible and $i \oplus j \sim_c k$.

Note that $i, j, k$ have length $r$, which is strictly less than $n$, and that the set of all admissible triples $(i, j, k)$ is finite for each fixed $n$. Thus, if Horn’s conjecture is true, the solution set to (1) can in principle be determined by a recursive algorithm. (In fact, Horn’s original formulation of this conjecture was phrased in terms of this algorithm.)

Helmke, Rosenthal, and Klyachko showed that Horn’s conjecture was true provided that the $\sim_c$ relation on $(i, j, k)$ was replaced by the quantum counterpart $\sim_q$:

**Theorem 10.** Let $\lambda, \mu, \nu$ be weakly decreasing sequences of reals obeying (2).

1. [HR, Kl] If (1) holds, then (10) holds whenever $(i, j, k)$ are admissible triples obeying $i \oplus j \sim_q k$.
2. [Kl] Conversely, if (10) holds whenever $(i, j, k)$ are admissible triples obeying $i \oplus j \sim_q k$, then (1) holds.

So in a sense, solvability of the “classical problem in dimension $n$” (about summing $n \times n$ Hermitian matrices) is determined by the solvability of the “quantum problem in dimension $m < n$” (about tensoring representations of $U(m)$.) Given the saturation theorem proven in the last section, which says that each such quantum problem is solvable exactly if the corresponding classical problem (in the same dimension) is solvable, this gives a recursive way to answer the problem.

Theorem 10 connects the classical and quantum problems in a way markedly different to the standard classical/quantum analogy as codified by Theorem 5. The proofs of this theorem are highly non-trivial, and first proceed by showing (4) is equivalent to a certain intersection problem in the Schubert calculus of Grassmanians. We do not discuss this further here, but refer the interested reader to [F2]. More recently, a purely honeycomb-theoretic proof of Theorem 10 has been obtained, which we discuss briefly in the next section.

### 6. Other Consequences

We close with mention of a few other applications of honeycombs and their properties proven above.

Horn’s proof that the solution set of (1) is determined by (2) and a finite number of inequalities of the form (10) is based on the following stronger fact: if (10) held with equality and $\lambda, \mu, \nu$ are regular, then the associated triple of matrices $(H_\lambda, H_\mu, H_\nu)$ is necessarily block diagonalizable. Put another way, the Hermitian triple is the direct sum of two smaller Hermitian triples.

Given that we have already drawn an analogy between direct sums of matrices and overlaying of honeycombs, there should be a corresponding statement, stating that the faces of $\text{BDRY}_n$ correspond to honeycombs which are overlays.

Not every direct sum of Hermitian matrices (or an overlay of two honeycombs) corresponds to an inequality (10) in Horn’s list. However, we have
Theorem 11. \[\text{[KTW]}\] Let \( h \) be a generic point in \( \text{HONEY}_n \). Then \( \partial(h) \) is on a facet of \( \text{BDRY}_n \) if and only if \( h \) is an overlay of two smaller honeycombs \( A \) and \( B \), such that at each point of intersection some neighborhood of \( h \) looks like the left figure in figure 12 ("A turns clockwise to B").

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure12.png}
\caption{In the left figure, A turns clockwise to B, whereas in the right the reverse is true. Any transverse point of intersection of two overlaid honeycombs must look like exactly one of these.}
\end{figure}

In this case one can read off the inequality on \( \text{BDRY}_n \) directly from \( h \): it says that the sum of the boundary coordinates on \( A \) are nonnegative. Some examples are in figure 13.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13.png}
\caption{All the consistently clockwise overlays of size 3, up to rotation and deformation, and the associated inequalities on \( \text{BDRY}_3 \). In each one the solid honeycomb turns clockwise to the dashed honeycomb.}
\end{figure}

If \( A \) always turns clockwise to \( B \) at the intersections, we can construct a new \( m \)-honeycomb \( A' \) by shifting \( A \) so the bottom right vertex is at \((0,0,0)\), replacing each of \( A' \)'s edges by one whose length is the number of intersections with \( B \), and removing \( B \) altogether. The result is in fact a new honeycomb, integral and of size \( m \) (an example is in figure 14). This construction can be used to give a purely honeycomb-theoretic proof \[\text{[KTW]}\] of the results of Klyachko and Helmke-Rosenthal, and gives enough additional insight to cut down Horn’s overcomplete list of inequalities to the minimum possible.

6.1. An open question. The present proof of Theorem 4 is very unsatisfying; it comes as an asymptotic limit of Theorem 7, which itself is proved only indirectly.

Consider the horizontal projection of the 2-sphere of height \( 1/(2\pi) \) onto the diameter between the poles. Archimedes’ theorem states that the length of an interval in that diameter equals the area of the preimage on the sphere. Today we say that the horizontal projection is \textit{measure-preserving}, which at first seems marvelous since the interval has only half the dimension of the sphere. The question: is there a corresponding map from \( \mathcal{M}_{\lambda \mu \nu} \) to the polytope of honeycombs, giving a direct proof of Theorem 4 (and perhaps of Theorem 7)?
Figure 14. The honeycomb on the right comes from the solid honeycomb on the left, with each edge rescaled to the number of times it intersects the dashed honeycomb, and translated to put the bottom right vertex at the origin (marked with a dot).

References

[Bu] A. Buch, The saturation conjecture (after A. Knutson and T. Tao), to appear, L’enseignement Math.
[BZ] A. Berenstein, A. Zelevinsky, Triple multiplicities for $\mathfrak{sl}(r+1)$ and the spectrum of the exterior algebra of the adjoint representation, J. Alg. Comb., 1 (1992), 7-22.
[DW] H. Derksen and J. Weyman, Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients, to appear in J. Amer. Math. Soc.
[F] W. Fulton, Young Tableaux, Cambridge University Press, 1997.
[F2] W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, Bulletin of the AMS 37 (2000), 209-249.
[GP] O. Gleizer, A. Postnikov, Littlewood-Richardson coefficients via Yang-Baxter equation, [math.QA/9909124] to appear in Internat. Math. Res. Notices.
[He] G. J. Heckman, Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups, Invent. Math. 67 (1982), 333-356.
[HR] U. Helmke and J. Rosenthal, Eigenvalue inequalities and Schubert calculus, Math. Nachr. 171 (1995), 207-225.
[Ho] A. Horn, Eigenvalues of sums of Hermitian matrices, Pacific J. Math., 12 (1962), 225-241.
[J] S. Johnson, The Schubert calculus and eigenvalue inequalities for sums of Hermitian matrices, Ph.D. thesis, University of California. Santa Barbara, 1979.
[Kl] A.A. Klyachko, Stable vector bundles and Hermitian operators, Selecta Math. (N.S.) 4 (1998), 419-445.
[Kn] A. Knutson, The symplectic and algebraic geometry of Horn’s problem, math.LA/9911088 to appear in Linear Algebra and its Applications.
[KT] A. Knutson, T. Tao, The honeycomb model of $\text{GL}_n(\mathbb{C})$ tensor products I: proof of the saturation conjecture, Journal of the AMS 12 (1999), 1055-1090.
[KTW] A. Knutson, T. Tao, C. Woodward, The honeycomb model of $\text{GL}_n(\mathbb{C})$ tensor products II: facets of the Littlewood-Richardson cone, in preparation.
[L] B. V. Lidskii, Spectral polyhedron of the sum of two Hermitian matrices, Funct. Anal. Appl. 16 (1982), 139-140.
[W] H. Weyl, Das asymtotische Verteilungsgesetz der Eigenwerte lineare partieller Differentialgleichungen, Math. Ann. 71 (1912), 441-479.
[Wo] C. Woodward, Honeycomb Associativity via Scattering Theory, preprint.

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