Communication-efficient distributed SGD with Sketching

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Abstract

Large-scale distributed training of neural networks is often limited by network bandwidth, wherein the communication time overwhelms the local computation time. Motivated by the success of sketching methods in sub-linear/streaming algorithms, we propose a sketching-based approach to minimize the communication costs between nodes without losing accuracy. In our proposed method, workers in a distributed, synchronous training setting send sketches of their gradient vectors to the parameter server instead of the full gradient vector. Leveraging the theoretical properties of sketches, we show that this method recovers the favorable convergence guarantees of single-machine top-$k$ SGD. Furthermore, when applied to a model with $d$ dimensions on $W$ workers, our method requires only $\Theta(kW)$ bytes of communication, compared to $\Omega(dW)$ for vanilla distributed SGD. To validate our method, we run experiments using a residual network trained on the CIFAR-10 dataset. We achieve no drop in validation accuracy with a compression ratio of 4, or about 1 percentage point drop with a compression ratio of 8. We also demonstrate that our method scales to many workers.

1 Introduction

Since 2012, the amount of computation required to train large models has been doubling every several months – far faster than the scaling of Moore’s law (Amodei and Hernandez, 2018). Furthermore, with the end of Dennard scaling, DRAM capacity per module has not kept pace with the rapid growth in the size of modern deep learning models (Johnsson and Netzer, 2016). Together, these trends have motivated the development of distributed training techniques, which distribute the computational cost of training a model over many compute nodes. As distributed training rapidly becomes the only feasible way to train state-of-the-art models, the need for improvements in distributed training algorithms has become increasingly urgent. The simplest and most popular distributed training algorithm is distributed synchronous stochastic gradient descent (DS-SGD), in which each batch of data is partitioned and distributed among $W$ workers. Each worker computes a gradient locally, and then these gradients are aggregated to yield the gradient on the entire batch.

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This aggregation step is a key challenge facing DS-SGD, since it incurs a significant network communication overhead. In the simplest case, each worker sends its gradient to all other workers, and the communication cost grows as $W^2d$, where $d$ is the dimensionality of the gradient. Using a single “parameter server,” which gathers and averages gradients from all the workers and broadcasts back the resulting parameter update, is more communication efficient. However, even in this case, the communication cost still grows as $Wd$. This communication cost can easily become a bottleneck for training in applications with large models, such as recurrent neural networks (RNN) (Vaswani et al., 2017), or in settings with a large number of workers, such as in Federated Learning.

It was recently shown that SGD with a parameter step towards only the top-$k$ largest coordinates of the gradient at each iteration provably converges in the vanilla non-distributed setting (Stich et al., 2018; Alistarh et al., 2018). In this paper we refer to this approach as “top-$k$ SGD.” Extending top-$k$ SGD to the distributed setting would appear to be a natural way to tackle the communication bottleneck facing DS-SGD. One simple way to extend the top-$k$ algorithm to distributed training is for each worker to send only the top-$k$ components of its locally computed gradient to the parameter server, and for the parameter server to produce the weight update for that iteration by summing these local top-$k$ vectors. This is a popular approach, and ideally it reduces the communication cost from $dW$ to $kW$. However, top-$k$ SGD only provably converges when the parameter update at each iteration is in the direction of the top-$k$ coordinates of the gradient on the entire batch. This is generally not the case for the vector formed by summing the local top-$k$ from each worker, unless the gradients on each worker are disjoint, or if operating under other simplifying assumptions (Alistarh et al., 2018, Assumption 1). Moreover, it is possible to show that the local top-$k$ elements of each worker can be inconsistent with the top-$k$ of their averaged gradients. That is, a top-$k$ element of the average gradient is not necessarily among the top-$k$ elements of any worker. As a result, to compute the top-$k$ of the average of the local gradients, each worker must transmit its entire gradient to the parameter server, for a total communication cost of $\Omega(dW)$ (i.e. no savings).

In this paper we address the following question: is it possible to compute top-$k$ SGD with total communication $O(kW)$? We answer this question affirmatively. Our main method employs powerful sketching techniques, which compress gradient vectors before communicating them between compute nodes. In most cases, sketching is a linear operator, and we use this critical fact to reduce the total communication of the system. Indeed, one can construct, using shared randomness, a linear operator $A : \mathbb{R}^d \mapsto \mathbb{R}^l$ where $l = O(k)$ and distribute $A$ among all workers before the training. At each step, the workers compute sketched gradients $s_i = Ag_i$ and communicate them to the parameter server, which sums them to form a global sketch $S = \sum_{i=1}^{W} s_i = \sum_{i=1}^{W} Ag_i = A(\sum_{i=1}^{W} g_i)$, where the last equality is due to the linearity of sketches. Observe that the communication cost is $O(kW)$ as requested. Moreover, sketches often can be used to extract approximate top-$k$ elements, i.e., elements whose magnitude is at least $f_k(1-\alpha)$, where $f_k$ is the value of the $k^{th}$ largest element. Therefore, it is possible to extract approximate top-$k$ elements from $S$ using, for example, a popular Count Sketch method. By sketching the gradients as described, our method approximates the single-node top-$k$ SGD algorithm. Leveraging the results of top-$k$ SGD, in particular (Stich et al., 2018), we provably establish its convergence while requiring only $O(kW)$ network communication.

### 1.1 Contributions

1. We use sketching techniques, in particular Count Sketch, to provably extend top-$k$ SGD to the distributed setting. We show that, for $W$ workers, our method converges at a rate $\min \{ 1/WT, 1/T^2 \}$ for smooth strongly convex functions. Moreover, we can parallelize the local computations of the workers.
2. Our method reduces the total communication from $O(Wd)$ to $O(Wk)$ per update. In contrast, methods that sum the local top-$k$ elements of each worker’s gradient have a worst-case total communication cost of $O(W^2k)$ (Lin et al., 2017; Alistarh et al., 2018).

3. We show experimentally that, when training a state-of-the-art model on CIFAR-10, our method can reduce total gradient communication by a factor of $\sim 4$ without losing model accuracy.

2 Related Work

Recently, there has been a surge of work seeking to improve the overall run time of distributed training of neural networks by alleviating the gradient communication bottleneck. We limit our discussion to synchronous communications. Broadly, most of the techniques that have been developed can be divided into two categories: gradient sparsification, and quantization. These techniques are largely independent, and can often be applied together to the same problem.

In gradient sparsification, only the top-$k$ coordinates, in magnitude, are communicated. Several authors propose this method as a heuristic (Dryden et al., 2016; Strom, 2015), and Aji and Heafield (2017) obtain a speedup of 22% on 4 GPUs for Neural Machine Translation without sacrificing translation quality. Stich et al. (2018) combine gradient sparsification with error correction, wherein each worker stores in memory the smaller gradient elements that were not sent in the current iteration. In the next iteration, these accumulated errors are added to the next gradient estimate. This method has a convergence rate on par with that of SGD. Similarly, Alistarh et al. (2018) study a top-$k$ sparsification algorithm, in which the parameter server aggregates the local top-$k$ gradient elements. They provide convergence guarantees under the assumption that the total top-$k$ gradient elements are close to the global top-$k$ elements in $\ell_2$ norm. Lin et al. (2017) propose a method, called “deep gradient compression,” to train neural networks using SGD with momentum. Although their method doesn’t have theoretical guarantees, they are able to compress the gradient vectors by 270-600x over a range of networks and tasks, without losing accuracy. Deep gradient compression uses gradient sparsification along with other techniques like momentum correction, gradient clipping, and momentum factor masking. However, it is not clear if this is the reduction in total communication, as typically the number of iterations to get to a particular accuracy increases.

Gradient quantization is another popular approach to reduce the communication cost in distributed model training (Seide et al., 2014; Strom, 2015). In essence, gradient quantization reduces communication cost by lowering the floating point precision of the encoded information to be communicated. Alistarh et al. (2017) introduce Q-SGD, which employs randomized rounding and an efficient encoding of gradients called 
Elias integer encoding. This method, when applied to ResNet-152, trains on ImageNet to full accuracy 1.8 times faster the the vanilla full-precision variant. Bernstein et al. (2018a) propose a method called signSGD, in which only the signs of the stochastic gradient estimates are communicated. Despite the reduced precision, signSGD converges at the same rate as SGD. Moreover, Bernstein et al. (2018b) show that, if the gradient signs from each worker are aggregated using a majority vote algorithm, signSGD is Byzantine fault tolerant. Wen et al. (2017) propose a similar method called TernGrad, where gradients are quantized to three values: 1, −1 or 0. With layerwise ternarizing and gradient clipping, TernGrad converges on AlexNet to the same accuracy at full-precision SGD.

Several researchers have proposed applying sketching to address the communication bottleneck in distributed and Federated Learning (Konečný et al., 2016; Jiang et al., 2018). However, these methods either do not have provable guarantees, or they apply sketches only to portions of the data, failing to alleviate the $\Omega(Wd)$ communication overhead. In particular, Konečný et al. (2016)
propose “sketched updates” in Federated Learning for structured problems, and Jiang et al. (2018) introduce a range of hashing and quantization techniques to improve the constant in $O(Wd)$.

3 Preliminaries

Notation. We use capital roman letters $X$ to denote matrices, lower-case roman letters $x$ to denote vectors, and normal and Greek letters $x, X, \xi$ to denote scalars. $[n]$ denotes the set of natural number from 1 to $n$.

Problem setup. Let $w \in \mathbb{R}^d$ be the parameters of the model to be trained. Let $f_i(w)$ be the loss incurred by $w$ at the $i^{th}$ data point. For example, in classification problems, the $i^{th}$ data point is the pair $(x_i, y_i)$ where $x_i$ is the feature vector and $y_i$ the label; in an unsupervised setting, we only have features $x_i$ as the $i^{th}$ data point. We use $f_i$ to generalize the discussion. The goal in machine learning is to minimize the generalization error $f(w) = \mathbb{E}_{(x_i,y_i) \sim D} [f_i(w)]$. We now introduce a class of functions called smooth strongly convex function. Our theoretical guarantees are limited to this class.

**Definition 1** (Smooth strongly convex function). $f : \mathbb{R}^d \to \mathbb{R}$ is a $L$-smooth $\mu$-strongly convex function if the following holds $\forall w_1, w_2 \in \mathbb{R}^d$,

1. $\|\nabla f(w_2) - \nabla f(w_1)\| \leq L \|w_2 - w_1\|$ (Smoothness)
2. $f(w_2) \geq f(w_1) + \langle \nabla f(w_1), w_2 - w_1 \rangle + \frac{\mu}{2} \|w_2 - w_1\|^2$ (Strong convexity)

3.1 Stochastic Gradient Descent

Stochastic Gradient Descent (SGD) is one of the most popular algorithms in machine learning. It is a first-order optimization method wherein at time $t$, we construct unbiased estimates $g_t$ of the true gradient $\nabla f(w_t)$ with bounded variance. The SGD update step is $w_{t+1} = w_t - \eta_t g_t$ where $\eta_t$ is the step size at the $t^{th}$ iteration. A simple way to construct gradient estimates is to sample uniformly a point or a mini-batch from the dataset and compute the gradient on it. We get $\mathbb{E} \left[ g_t \mid \{w_i\}_{i=0}^{t-1} \right] = \nabla f(w_{t-1})$, where $w_i$ is the $i^{th}$ iterate of the algorithm. Note that the expectation is taken only with respect to the randomness in the $t^{th}$ step and therefore we condition on all the other randomness.

As is standard, we assume that $\mathbb{E} \left[ \|g_t\|^2 \mid \{w_i\}_{i=0}^{t-1} \right] \leq G^2$ and $\mathbb{E} \left[ \|g_t - \nabla f(w_t)\|^2 \mid \{w_i\}_{i=0}^{t-1} \right] \leq \sigma^2$ for some constants $G$ and $\sigma$. For smooth strongly convex functions, SGD converges at a rate of $O \left( L G^2 / \mu^2 T \right)$ (Rakhlin et al., 2012).

3.2 Count Sketch

Sketching gained its fame in the streaming model (Muthukrishnan et al., 2005), where an algorithm is restricted to only one pass over the dataset and the memory is at most sublinear in the size and/or dimensionality of the dataset. A seminal paper by Alon et al. (1999) formalizes the model and delivers a series of important results, among which is the $l_2$-norm sketch (later referred to as the AMS sketch). Given a stream of updates $(a_i, w_i)$ to the $d$ dimensional vector $f$ (i.e. the $i$-th update is $f_{a_i} := w_i$), the AMS sketch initializes a vector of random signs: $s = (s_j)_{j=1}^{d}, s_j = \pm 1$. On each update $(a_i, w_i)$, it maintains the running sum $S := s_n w_n$, and at the end it reports $S^2$. Note that, if $s_j$ are at least 2-wise independent, then $E(S^2) = E(\sum_i f_i s_i)^2 = \sum_i f_i^2 = \|f\|_2^2$. Similarly, the
authors show that 4-wise independence is enough to bound the variance by \(4\|f\|_2^2\). Averaging over independent repetitions running in parallel provides control over the variance, while the median filter (i.e. the majority vote) controls the probability of failure. Formally, the result can be summarized as follows: AMS sketch, with a large constant probability, finds \(\hat{\ell}_2 = \|f\|_2 \pm \varepsilon\|f\|_2\) using only \(O\left(\frac{1}{\varepsilon^2}\right)\) space. Note that one does not need to explicitly store the entire vector \(s\), as its values can be generated on the fly using 4-wise independent hashing.

**Definition 2.** Let \(f \in \mathbb{R}^d\). The \(i\)-th coordinate \(f_i\) of \(f\) is an \((\alpha_1, \ell_2)\)-heavy hitter if \(|f_i| \geq \alpha_1 \|f\|_2\). \(f_i\) is an \((\alpha_2, \ell_2^2)\)-heavy hitter if \(f_i^2 \geq \alpha_2 \|f\|_2^2\).

The AMS sketch was later extended by Charikar et al. (2002) to detect heavy coordinates of the vector (see Definition 2). The resulting Count Sketch algorithm hashes the coordinates into \(b\) buckets, and sketches the \(\ell_2\) norm of each bucket. Assuming the histogram of the vector values is skewed, only a small number of buckets will have relatively large \(\ell_2\) norm. Intuitively, those buckets contain the heavy coordinates and therefore all coordinates hashed to other buckets can be discarded. Repeat the same routine independently and in parallel \(O\left(\log b \cdot d\right)\) times, and all items except the heavy ones will be excluded. Details on how to combine proposed hashing and \(\ell_2\) sketching efficiently are presented in Figure 1 and Algorithm 1.

**Algorithm 1 Count Sketch (Charikar et al., 2002)**

1: function init\((r, c)\):
2:    \(\text{init sign hashes } \{s_j\}_{j=1}^r\) and \(\text{bucket hashes } \{h_j\}_{j=1}^r\)
3:    \(\text{init } r \times c \text{ table of counters } S\)
4: function update\((i, f_i)\):
5:    \(\text{for } j \in 1 \ldots r:\)
6:        \(S[j, h_j(i)] += s_j(i) f_i\)
7: function estimate\((i)\):
8:    \(\text{init length } r \text{ array estimates}\)
9:    \(\text{for } j \in 1, \ldots, r:\)
10:       \(\text{estimates}[r] = s_j(i) S[j, h_j(i)]\)
11: return median\(\text{of estimates}\)

Charikar et al. (2002) define the following approximation scheme for finding the list \(T\) of the top-\(k\) coordinates: \(\forall i \in [d]: i \in T \Rightarrow f_i \geq (1 - \varepsilon)\theta\) and \(f_i \geq (1 + \varepsilon)\theta \Rightarrow i \in T\), where \(\theta\) is chosen to be the \(k\)-th largest value of \(f\).

![Figure 1: Low level intuition behind the update step of the Count Sketch.](image-url)
Theorem 1 (Charikar et al., 2002). The Count Sketch algorithm finds approximate top-k coordinates with probability at least $1 - \delta$, in space $O\left(\log \frac{d}{\delta} \left( k + \frac{\|f^\text{tail}\|_2^2}{(\alpha)^2} \right) \right)$, where $\|f^\text{tail}\|_2^2 = \sum_{i \notin \text{top } k} f_i^2$ and $\theta$ is the $k$-th largest coordinate.

Note that, if $\theta = \alpha \|f\|_2$, the Count Sketch finds all $(\alpha, \ell_2)$-heavy coordinates and approximates their values with error $\pm \varepsilon \|f\|_2$. It does so with a memory footprint of $O\left(\frac{1}{\sqrt{\alpha}} \log d \right)$. We are more interested in finding $(\alpha, \ell_2^2)$-heavy hitters, which, by an adjustment to Theorem 1, the Count Sketch can approximately find with a space complexity of $O\left(\frac{1}{\alpha} \log d \right)$, or $O\left(k \log d \right)$ if we choose $\alpha = O\left(\frac{1}{k} \right)$.

Both the Count Sketch and the Count-Min Sketch, which is a similar algorithm presented by Cormode and Muthukrishnan (2005) that achieves a $\pm \varepsilon \ell_1$ guarantee, gained popularity in distributed systems primarily due to the mergeability property formally defined by Agarwal et al. (2013): given a sketch $S(f)$ computed on the input vector $f$ and a sketch $S(g)$ computed on input $g$, there exists a function $F$, s.t. $F(S(f), S(g))$ has the same approximation guarantees and the same memory footprint as $S(f + g)$. Note that sketching the entire vector can be rewritten as a linear operation $S(f) = Af$, and therefore $S(f + g) = S(f) + S(g)$. We take advantage of this crucial property in our sketched SGD algorithm, since, on the parameter server, the sum of the workers’ sketches is identical to the sketch that would have been produced with only a single worker operating on the entire batch.

Besides having sublinear memory footprint and mergeability, the Count Sketch is simple to implement and straightforward to parallelize, facilitating GPU acceleration (Ivkin et al., 2018).

4 Our Approach

Finding frequent items is a classical problem in streaming algorithms. It has been extensively studied, and algorithms have been designed with optimal space and time complexity (Misra and Gries, 1982; Cormode and Muthukrishnan, 2005; Charikar et al., 2002; Braverman et al., 2017). We formulate the problem of finding top gradient coordinates as that of finding approximate top-$k$ values, or $(\alpha, \ell_2^2)$-heavy hitters, in the gradient vector. However, our setting has significant differences from the streaming model of computation. In particular, we are only limited by communication bandwidth, and can afford more memory and computation than is typical for a streaming algorithm.
Our algorithm builds on sparsified SGD with error accumulation (Stich et al., 2018; Alistarh et al., 2018). The idea is to use only the top \( k \) elements from the stochastic gradient to make the update, while maintaining a local error vector to accumulate gradient elements that were too small to communicate in the current round. We describe the method which we called HeavyMix below.

**HeavyMix.** Based on the intuition described in the previous paragraph, our HeavyMix subroutine (see Algorithm 2) returns all \((1/k, \ell_2^2)\)-heavy hitters. First, it queries estimations \( \hat{g}_i \) for all coordinates of the gradient \( g \) (line 2), and then it filters out all non-heavy items using simple rule \( \hat{g}_i \geq \ell_2^2 / k \) (line 3). To ensure the algorithm converges even when there are no heavy hitters (as in Stich et al. (2018)), we also sample up to \( k \) non-heavy items at random (line 4). Before returning the resulting set, HeavyMix carries out a second round of communication to retrieve the exact values of the gradient.

**Sketched-SGD.** We first discuss the sparsified SGD algorithm in the single-machine setting (Stich et al., 2018). The accumulated gradient at time \( t \) is \( \bar{g}_t = \eta_t g_t + a_{t-1} \), where \( a \) is the accumulated error. The algorithm computes \( \tilde{g}_t = \text{Top}_k(\bar{g}_t) \), updates the model, and stores the new accumulated error \( a_t = \bar{g}_t - \tilde{g}_t \). Stich et al. (2018) show that, for smooth strongly convex functions, this method converges at the same rate as SGD. Our method differs in that, instead of directly taking the top-\( k \) elements of \( \bar{g}_t \), we sketch \( \bar{g}_t \) using a Count Sketch to get \( S_t \). We then call HeavyMix to extract \( k \) elements from \( S_t \) as \( \tilde{g}_t \), and we use \( \tilde{g}_t \) to perform the gradient descent update. Finally, we accumulate the error by subtracting these \( \tilde{g}_t \) from the gradient. This method is presented as Algorithm 3.

**Sketched-D-SGD.** We now discuss the algorithm in the distributed setting, presented in full in Algorithm 4. At time \( t \), the \( i \)-th worker (\( i = 1 \ldots W \)) sketches its gradient vector as \( S^i_t \) and sends it to the parameter server. The parameter server merges all worker sketches, computes \( \tilde{g}_t \) as the top \( k \) elements extracted from the merged sketch using HeavyMix, and updates the model using \( \tilde{g}_t \) as the gradient descent step. Then, the top \( k \) elements are send back to the workers, each of which computes and stores its locally accumulated error \( a^i_t \).

**Sketched-D-SGD with momentum.** In practice, vanilla SGD is not usually sufficient to produce state-of-the-art results on standard datasets. In particular, momentum (heavy ball and Nesterov) are widely used. We extend this to the distributed setting with sketching in Algorithm 5. Moreover, following Lin et al. (2017), we incorporate the momentum correction technique to keep the correct momentum accumulated locally. Our method works well in practice, as reported in Section 7. We leaved establishing theoretical guarantees for Sketched-D-SGD with momentum as an open problem for future work.

## 5 Theoretical Results

We first restate the result from Stich et al. (2018). To do so, we first need to define a \( \tau \)-contraction.

**Definition 3 (\( \tau \)-contraction (Stich et al., 2018)).** A \( \tau \)-contraction operator is a possibly randomized operator \( \text{comp} : \mathbb{R}^d \rightarrow \mathbb{R}^d \) that satisfies the contraction property:

\[
\forall x \in \mathbb{R}^d : \mathbb{E} \left[ \|x - \text{comp}(x)\|^2 \right] \leq (1 - \tau) \|x\|^2
\]
Algorithm 2 HeavyMix

**Input:** $S$ - sketch of gradient $g$, $k$ - parameter
1. Query $\tilde{g}_t^2 = (1 \pm 0.5)\|g\|_2^2$ from sketch $S$
2. $\forall i$ query $\tilde{g}_t^2 = g_t^2 \pm \frac{1}{2k}\|g_t\|_2^2$ from sketch $S$
3. $H \leftarrow \{i | \tilde{g}_t^2 \geq \tilde{g}_t^2/k\}$ and $NH \leftarrow \{i | \tilde{g}_t^2 < \tilde{g}_t^2/k\}$
4. $\text{Top}_k = H \cup \text{rand}_1(NH)$, where $l = k - |H|$
5. second round of communication to get exact values of $\text{Top}_k$

**Output:** $\breve{g}$: $\forall i \in \text{Top}_k : \breve{g}_t = g_t$ and $\forall i \notin \text{Top}_k : \breve{g}_t = 0$

Algorithm 3 SKETCHED-SGD

**Input:** $k, \xi, \delta$
1. $\eta_t \leftarrow \frac{1}{\xi (\xi+t)}, q_t \leftarrow (\xi + t)^2$, $Q_T = \sum_{t=1}^{T} q_t, a_0 = 0$
2. for $t = 1, 2, \ldots, T$ do
3. Compute stochastic gradient $g_t$
4. Error correction: $\breve{g}_t = \eta_t g_t + a_{t-1}$
5. $\breve{g}_t = \text{HeavyMix}(S_t, k)$, where $S_t$ is a sketch of $\breve{g}_t$
6. Error accumulation: $a_t = \breve{g}_t - \check{g}_t$
7. Update $w_{t+1} = w_t - \breve{g}_t$
8. end for

**Output:** $\hat{w}_T = \frac{1}{Q_T} \sum_{t=1}^{T} q_t w_t$

Given this, and assuming that the stochastic gradients $g$ are unbiased and bounded in norm, i.e. $\mathbb{E} [\|g\|^2] \leq G^2$, we have the following result.

**Theorem 2 (Stich et al., 2018).** Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a $\mu$-smooth $\mu$-strongly convex function. Given $T > 0$ and $0 < k \leq d$, sparsified SGD with step size $\eta_t = \frac{1}{\xi (\xi+\ell)}$, with $\xi > 1 + \frac{d(1+\beta)}{\ell (1+\rho)}$, with $\beta > 4$ and $\rho = \frac{4\beta}{(\beta-4)(\beta+1)^2}$, after $T$ steps outputs $\hat{w}_T$:

$$\mathbb{E} [f(\hat{w}_T)] - f(w^*) \leq \mathcal{O} \left( \frac{G^2}{\mu T} + \frac{d^2G^2L}{k^2\mu^2 T^2} + \frac{d^3G^3}{k^3\mu T^3} \right).$$

**One Machine.** We first discuss how to reproduce this result using sketching on a single machine. It suffices to show that our compression scheme, HeavyMix, is a $k/d$-contraction.

**Lemma 1.** HeavyMix, with sketch size $\Theta(k \log(d/\delta))$ is a $k/d$-contraction with probability at least $1 - \delta$.

**Proof.** Given $g \in \mathbb{R}$, the HeavyMix algorithm extracts all $(1/k, \ell_2^2)$-heavy elements from a Count Sketch $S$ of $g$. Let $\check{g}$ be the values of all elements recovered from its sketch. For a fixed $k$, we create two sets $H$ (heavy), and $NH$ (not-heavy). All coordinates of $\check{g}$ with values at least $\frac{1}{k} \ell_2^2$ are put in $H$, and all others in $NH$, where $\ell_2^2$ is the estimate of $\|g\|_2$ from the Count Sketch. Note that the number of elements in $H$ can be at most $k$. Then, we sample uniformly at random $l = k - |H|$ elements from $NH$, and finally output its union with $H$. We then do a second round of communication to get exact values of these $k$ elements.

Note that, because of the second round of communication in HeavyMix and the properties of the Count Sketch, with probability at least $1 - \delta$ we get the exact values of all elements in $H$. Call this the “heavy hitters recovery” event. Let $g_H$ be a vector equal to $g$ at the coordinates in $H$, and
Algorithm 4 SKETCHED-D-SGD

Input: $k, \xi, \delta, W$

1: $\eta_t \leftarrow \frac{1}{T} \xi, q_t \leftarrow (\xi + t)^2, Q_T = \sum_{t=1}^{T} q_t, a_0 = 0$

2: for $t = 1, 2, \ldots, T$ do

3: Compute stochastic gradient $g_t^i$

4: Error correction: $g_t^i = \eta_t g_t^i + a_{t-1}$

5: Compute sketches $S_t^i$ of $g_t^i$ and send to Master

6: Aggregate sketches $S_t = \frac{1}{W} \sum_{i=1}^{W} S_t^i$

7: $\tilde{g}_t = HEAVYMIX(S_t, k)$

8: Update $w_{t+1} = w_t - \tilde{g}_t$ and send it to Workers

9: Error accumulation: $a_{t+1} = b_{t+1} - \tilde{g}_t$

10: end for

Output: $\tilde{w}_T = \frac{1}{Q_T} \sum_{t=1}^{T} q_t w_t$

zero otherwise. Define $g_{NH}$ analogously. Conditioning on the heavy hitters recovery event, and taking expectation over the random sampling, we have

$$E \left[ \|g - \tilde{g}\|^2 \right] = \|g_H - \tilde{g}_H\|^2 + E \left[ \|g_{NH} - \text{rand}_i(g_{NH})\|^2 \right]$$

$$\leq \left( 1 - \frac{k - |H|}{d - |H|} \right) \|g_{NH}\|^2 \leq \left( 1 - \frac{k - |H|}{d - |H|} \right) \left( 1 - \frac{|H|}{2k} \right) \|g\|^2$$

Note that, because we condition on the heavy hitter recovery event, $\tilde{g}_H = g_H$ due to the second round communication (line 5 of Algorithm 2). The first inequality follows using Lemma 1 from Stich et al. (2018). The second inequality follows from the fact that the heavy elements have values at least $\frac{1}{k^2} \geq \frac{1}{2k} \|g\|^2$, and therefore $\|g_{NH}\|^2 = \|g\|^2 - \|g_H\|^2 \leq \left( 1 - \frac{|H|}{2k} \right) \|g\|^2$.

Simplifying the expression, we get

$$E \left[ \|g - \tilde{g}\|^2 \right] \leq \left( \frac{2k - |H|}{2k} \right) \left( \frac{d - k}{d - |H|} \right) \|g\|^2 = \left( \frac{2k - |H|}{2k} \right) \left( \frac{d}{d - |H|} \right) \left( 1 - \frac{k}{d} \right) \|g\|^2.$$

Note that the first two terms can be bounded as follows:

$$\left( \frac{2k - |H|}{2k} \right) \left( \frac{d}{d - |H|} \right) \leq 1 \iff kd - |H| \leq kd - 2k |H| \iff |H| (d - 2k) \geq 0$$

which holds when $k \leq d / 2$ thereby completing the proof.

We can now appeal to Stich et al. (2018) to get a convergence guarantee directly. Note that we can distribute the failure probabilities among the number of iterations $T$, and do a union bound. The sketch size in that case becomes $\Theta(k \log(Td/\delta))$. This gives us a convergence rate of $O(G^2 / T)$, as in the complete Theorem 5 stated in the appendix. However, we make minor modifications here so as to reap improvements in distributed setting. In particular, we will assume that the stochastic oracle gives us unbiased gradients with bounded variance along with bounded norms. This is because, as discussed, without the variance bound condition, appealing to Stich et al. (2018) yields a rate of $O(G^2 / T)$. However, using the variance bound, with minor modifications to the analysis of Stich et al. (2018), we get a rate of $O(\sigma^2 / T)$. In the distributed setting, we will employ variance reduction to get an $O(\sigma^2 / WT)$ rate. We defer the modified analysis of the one machine setting to the appendix. Summarizing, we assume that the stochastic gradients $g$ are unbiased,
bounded in norm and variance as \( \|g\|^2 \leq G^2 \|g - E[g]\|^2 \leq \sigma^2 \) respectively. These conditions yield the following result.

**Theorem 3.** Let \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) be a \( L \)-smooth \( \mu \)-strongly convex function. Given \( T > 0 \) and \( 0 < k \leq d, 0 < \delta < 1 \), and a \( \tau_k \)-contraction, Algorithm 3 SKETCHED-SGD with sketch size \( \mathcal{O}(k \log(dT/\delta)) \) and step size \( \eta_t = \frac{1}{T+\xi} \), with \( \xi = 1 + \frac{1+\beta}{\tau_k(1+\rho)} \), with \( \beta > 4 \) and \( \rho = \frac{4\beta}{(\beta-4)(\beta+1)^2} \), after \( T \) steps outputs \( \hat{w}_T \) such that with probability at least \( 1 - \delta \)

\[
E[f(\hat{w}_T)] - f(w^*) \leq \mathcal{O}
\left( \frac{\sigma^2}{\mu T} + \frac{G^2 L}{\tau_k^3 \mu^2 T^2} + \frac{G^3}{\tau_k^3 \mu T^3} \right)
\]

**Corollary 1.** With the same conditions as Theorem 3, using HeavyMix for compression yields

\[
E[f(\hat{w}_T)] - f(w^*) \leq \mathcal{O}
\left( \frac{\sigma^2}{\mu T} + \frac{d^2 G^2 L}{k^2 \mu^2 T^2} + \frac{d^3 G^3}{k^3 \mu T^3} \right)
\]

**Distributed Setting.** In the case of \( W \) workers, there are two important changes. The first, as discussed above, is that we assume the stochastic oracle gives us unbiased gradients with bounded variance along with bounded norms, so as to get variance reduction. The second change is that we will maintain an approximation of the squared \( \ell_2 \) norm of the sum of gradients from each worker. This is needed to show that the compression in the distributed setting is a \( k/d \)-contraction, and can be done using the Count Sketch, since it approximates \( \ell_2 \) norms. We already proved Lemma 1 in the general case when norms are approximated up to a constant factor. We now present the main result.

**Theorem 4 (Main Theorem).** Let \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) be a \( L \)-smooth \( \mu \)-strongly convex function, and let the data be shared among \( W \) workers. Given \( 0 < k \leq d, 0 < \alpha, \delta < 1 \) Algorithm 4 SKETCHED-D-SGD run with sketch size \( = \mathcal{O}(k \log(dT/\delta)) \), step size \( \eta_t = \frac{1}{T+\xi} \), with \( \xi = 1 + \frac{d(1+\beta)}{k(1+\rho)} \), with \( \beta > 4 \) and \( \rho = \frac{4\beta}{(\beta-4)(\beta+1)^2} \) after \( T \) steps outputs \( \hat{w}_T \) such that the following holds,

1. With probability at least \( 1 - \delta \),

\[
E[f(\hat{w}_T)] - f(w^*) \leq \mathcal{O}
\left( \frac{\sigma^2}{\mu WT} + \frac{d^2 G^2 L}{k^2 \mu^2 T^2} + \frac{d^3 G^3}{k^3 \mu T^3} \right)
\]

2. The total communication per update is \( \Theta(k \log(dT/\delta)W) \) messages.

**Rate and dimensionality dependence.** We first note that the convergence rate is \( \min \{1/WT, 1/T^2 \} \).

In regimes with \( W = o(T) \), we get a \( 1/WT \) rate, however as soon as \( W = \omega(T) \), the rate becomes \( 1/T^2 \). As in Stich et al. (2018), the dimensionality dependence is only in the higher order terms: when \( T = \omega(\delta) \), we get a dimensionality-independent rate of convergence.

**\( \epsilon \)-sub-optimality.** It is often more insightful to interpret various aspects of the result in terms of \( \epsilon \)-sub-optimality of generalization; i.e \( f(\hat{w}_T) - f(w^*) \leq \epsilon \). For simplicity, we look at the \( W = o(T) \) regime. Let \( q = \Theta(k \log(dT/\delta))W \) be the number of messages communicated per update. For an \( \epsilon \)-sub-optimal solution, we get,

- Sample complexity = \( \Theta(1/\epsilon) \).
- Number of rounds \( T = \Theta(1/W\epsilon) \).
- Communication complexity = \( Tq = \Theta((k/\epsilon) \log(d/\delta\epsilon)) \)
Computational Complexity = $T \cdot W \cdot \text{computation per worker} = \Theta((\delta k / \epsilon) \log(d / \delta W \epsilon))$

Space Complexity = size of accumulation + size of sketch = $\Theta(d + k \log(d / \delta W \epsilon))$

Proof of Theorem 4. First note that, from linearity of sketches (see section 3.2), the top-$k$ (or heavy) elements from the merged sketch $S_t = \sum_{i=1}^{W} S_i$ are the top-$k$ of the sum of vectors that were sketched. We have already shown in Lemma 1 that that extracting the top-$k$ elements from $S - T$ using HeavyMix gives us a $k$-contraction on the sum of gradients. Moreover since the guarantee is relative and norms are positive homogeneous, the same holds for the average, i.e. when dividing by $W$. Now since the average of stochastic gradients is still an unbiased estimate, this reduces to SKETCHED-SGD on one machine, and the convergence therefore follows from Theorem 5. Moreover, by taking an average of independent stochastic gradients, the variance is reduced by a factor of $W$; hence the convergence is accelerated by a factor of $W$. The communication cost follows since we only need to communicate the sketch, which finishes the proof.

6 Implementation

6.1 Training Procedure

We train the model from the winning entry in the DAWNBench 2017 competition under the “training time” category (Coleman et al., 2017; Page, 2018). The model is a custom ResNet9, with $\sim 6.6M$ parameters. We train on CIFAR-10, a dataset of 60,000 32x32 pixel RGB images drawn from 10 categories (Krizhevsky and Hinton, 2009). See §B of the appendix for additional details.

We modify the training procedure by splitting each batch into $W$ sub-batches, each of which is processed by a separate worker. Each worker accumulates its gradient locally, zeroing out at each iteration any element that the parameter server determined to be in the global top-$k$. One change from the theoretical method proposed is that in round one of communication, we retrieve the top $P_k$ elements, for some fixed $P$, and then query for the exact values of these $P_k$ coordinates. From these, we take the top $k$ elements. This ensures robust estimation of top $k$ elements, while only increasing the communication by a small constant factor $P$.

We also apply momentum correction and momentum factor masking, as in Lin et al. (2017). In momentum correction, each worker accumulates gradients, with momentum, in a velocity vector $u$. At each iteration, $u$ is accumulated into an error accumulation vector $v$, which is the vector that is compressed and sent to the parameter server. Elements of $v$ that the parameter server determined to be in the top-$k$ are zeroed out in each worker at the end of the iteration. In momentum factor masking, the corresponding elements of $u$ are also zeroed out, with the intuition that the momentum from delayed updates is stale, and should be canceled to prevent the stale momentum from moving the optimizer in the wrong direction.

Our training procedure is summarized in Algorithm 5.

6.2 Sketching Implementation

We implement a parallelized Count Sketch with PyTorch (Paszke et al., 2017). The Count Sketch data structure supports a query method, which returns a provable $\pm \|f\|_2$ approximation to each coordinate value. However, efficiently extracting which coordinates are heavy hitters is not always straightforward. In applications of Count Sketches to streaming data problems, there are two common efficient techniques to find heavy coordinates: 1) maintaining a heap of the largest coordinates seen so far (Charikar et al., 2002); 2) maintaining a hierarchy of sketches with exponentially reducing cardinality of embedded vectors (i.e. create a search tree) (Agarwal et al., 2013). However, both techniques break when the sketched vector is allowed to have negative inputs.
Algorithm 5 EMPIRICAL TRAINING

Input: $k, \eta_t, m, T$

1: $\forall i: u^i, v^i \leftarrow 0$
2: for $t = 1, 2, \ldots, T$ do
3: Compute stochastic gradient $g^i_t$ Worker $i$
4: Momentum: $u^i_t \leftarrow m u^i_t + \eta_t g^i_t$ Worker $i$
5: Error accumulation: $v^i_t \leftarrow v^i_t + u^i_t$ Worker $i$
6: Compute sketch $S^i_t$ of $v^i_t$ and send to Master Worker $i$
7: Aggregate sketches $S_t = \frac{1}{W} \sum_{i=1}^{W} S^i_t$ Master
8: $\tilde{g}_t = \text{HEAVY MIX}(S_t, k)$ Master
9: Update $w_{t+1} = w_t - \tilde{g}_t$ and send it to Workers Master
10: $u^i, v^i \leftarrow 0$, for all $i$ s.t. $\tilde{w}_t^i \neq 0$ Worker $i$
11: end for

![Learning Curves](image)

Figure 3: Learning curves for vanilla distributed SGD, global top-k SGD, and sketched SGD.

(as is the case when sketching gradients). To the best of our knowledge, there is no efficient way to find heavy coordinates in the presence of negative inputs. Fortunately, in our application, the dimensionality of the underlying vector being sketched is not very large, and most of the SGD computations are performed on the GPUs. Therefore, we can query the Count Sketch for the value of every gradient coordinate without undue computational cost.

7 Empirical Results

We compare our method to results achieved when carrying out vanilla distributed SGD, with no gradient compression. We also compare to the top-k SGD algorithm, where the true, global top-k of the workers’ gradients are computed at each iteration. Sketching approximates this baseline, so it serves as a reasonable upper bound on the sketching performance. Figure 3 shows sample learning curves for these three algorithms.

To choose $k$, we investigate how performance of the top-k baseline varies with $k$. Figure 4 shows maximum validation accuracy as a function of $k$, when trained for 50 epochs. For very small $k$, the model is unable to train in the limited number of epochs. And for very large $k$, the momentum factor masking zeros out momentum terms even before they become stale, hindering convergence.
In the sketched SGD algorithm, each worker must send $P k$ gradient elements to the parameter server, so it is necessary to choose a small $k$ in order to achieve a high compression ratio. Based on these results, we therefore choose $k = 10,000$ for further experiments.

To explore the tradeoff between trained model accuracy and compression, we vary the size of the sketch and the constant $P$ from Algorithm 5. In the no-compression baseline, the amount of communication required per worker to compute the next weight update is simply the number of parameters in the model, $d$; the total communication required throughout training to send gradients is $d W T_{baseline}$. In the sketched method, the workers must transmit a sketch, followed by $P k$ uncompressed values on line 8 of Algorithm 5. (For simplicity, we ignore the communication required to request a particular $P k$ elements from the workers, since the required sparse binary vector is highly compressible.) The total communication cost throughout training for the workers to send gradients is therefore $(s + P k) W T_{sketch}$. In practice, we run the baseline for 24 epochs and the sketched version for 50 epochs, so the compression ratio is $\frac{d}{s + P k} \left( \frac{24}{50} \right)$.

Note that above, we consider only the communication cost to aggregate the gradients computed on each worker. However, the parameter server must also distribute new parameter values once it computes the weight update, incurring additional communication costs. In particular, the parameter server in a pure gradient sparsification method must send at worst $k W$ new parameter values to each worker after every iteration, if none of the top-$k$ elements from any of the workers happen to overlap. In contrast, the parameter server in sketched SGD only needs to send $k$ new parameter values, since only $k$ parameters are updated at each iteration.

Figure 5 shows the tradeoff between achieving high validation accuracy and high compression ratio. In all runs (sketched SGD and top-$k$ baseline) $k = 10,000$, and we train for 50 epochs. The baseline without compression only needs to train for 24 epochs.

As shown in the figure, sketched SGD can deliver a compression ratio close to 4 with no loss in validation accuracy, or about 8 with 1 percentage point drop in validation accuracy. The overall compression ratio, including the communication needed to send updated parameter vectors from the parameter server to each worker, will be much higher, and in particular will scale linearly with

![Global Top-k Baseline](image-url)
Figure 5: Tradeoff between achieving high validation accuracy and high compression ratio. The baseline model (orange line, no compression) achieves about 94% accuracy after 24 epochs. True top-k SGD (dashed blue line) achieves a slightly higher accuracy after 50 epochs, and serves as an upper bound for the sketched SGD algorithm (blue circles). We vary the sketch size and $P$, keeping $k$ constant at 10,000.

To confirm that sketched SGD is suitable for a large number of workers, we investigate how performance varies with the number of workers used. Sketches are linear operators, so in principle the number of workers has no effect on which elements the parameter server determines to be in the top-k. To confirm this, Figure 6 plots the maximum validation accuracy as a function of the number of workers, where all other parameters are kept constant. As expected, performance does not appear to correlate with the number of workers; we attribute the fluctuations in performance to the inherent stochasticity of floating point computations on the GPU.

8 Discussion

This paper demonstrates the theoretical soundness and practical validity of sketching methods when applied to distributed SGD. We demonstrate that sketched top-k distributed SGD converges at a rate $O(1/T)$ and can reduce communication, in the worst case, from $O(Wd)$ to $O(Wk)$ where $W$ is the number of workers, $d$ is the dimension of the gradient.

The Count Sketch, as an $\ell_2$-heavy hitter (HH) algorithm, provides asymptotically better approximation compared to $\ell_1$-HH algorithms, such as Count-Min Sketch (Charikar et al., 2002; Misra and Gries, 1982). However, trying different sketches and extensive parameter tuning could potentially bring better compression rates in practice.

Recall that the sketch size grows as $O(k \log d)$; hence for a larger $d$ (i.e. larger networks) we expect to observe higher compression rates, as $\frac{k \log d}{d}$ is decreasing rapidly with $d$. Theoretical guarantees hold the same for any number of workers, and in practice we observed similar performance with increase in the number of workers. Thus we believe our method can be extended to larger networks and to the Federated Learning setting (McMahan et al., 2016).
Figure 6: Maximum validation accuracy achieved as a function of the number of workers, with all other parameters held constant. These models were trained for 50 epochs, using a sketch with 29 rows and 10,000 columns, with $k=10,000$ and $P=20$.

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A Proofs

Proof of Theorem 3. The proof, as in Stich et al. (2018), just follows using convexity and Lemmas 3, 2 and fact 3. The lemmas which are exactly same as Stich et al. (2018), are stated as facts. However, the proofs of lemmas, which change are stated in full for completeness, with the changes highlighted.

From convexity we have that

\[
f \left( \frac{1}{Q_T} \sum_{t=1}^{T} q_t \tilde{w}_t \right) - f(w^*) \leq \frac{1}{Q_T} \sum_{t=1}^{T} q_t f(w_t) - f(w^*) = \frac{1}{Q_T} \sum_{t=1}^{T} q_t (f(w_t) - f(w^*))
\]

Define \( \epsilon_t = f(w_t) - f(w^*) \), the excess error of iterate \( t \). From Lemma 2 we have

\[
\mathbb{E} \left[ \| \tilde{w}_{t+1} - w^* \|^2 \right] \leq \left( 1 - \frac{\eta \mu}{2} \right) \mathbb{E} \left[ \| \tilde{w}_t - w^* \|^2 \right] + \sigma^2 \eta_t^2 - \left( 1 - \frac{2}{\xi} \right) \epsilon_t \eta_t + (\mu + 2L) \mathbb{E} \left[ \| a_t \|^2 \right] \eta_t
\]

Bounding the last term using Lemma 3, with probability at least \( 1 - \delta \), we get,

\[
\mathbb{E} \left[ \| \tilde{w}_{t+1} - w^* \|^2 \right] \leq \left( 1 - \frac{\eta \mu}{2} \right) \mathbb{E} \left[ \| \tilde{w}_t - w^* \|^2 \right] + \sigma^2 \eta_t^2 - \left( 1 - \frac{2}{\xi} \right) \epsilon_t \eta_t + \frac{(\mu + 2L)4\beta G^2}{\tau_k^2(\beta - 4)} \eta_t^3
\]

where \( \tau_k \) is the contraction we get from HeavyMix. We have already show that \( \tau_k \leq \frac{5}{\beta} \).

Now using Lemma 3 and the fist equation, we get,

\[
f \left( \frac{1}{Q_T} \sum_{t=1}^{T} q_t \tilde{w}_t \right) - f(w^*) \leq \frac{\mu \xi^2 4 \mathbb{E} \left[ \| w_0 - w^* \|^2 \right]}{8(\xi - 2) Q_T} + \frac{4T(T + 2 \xi) \xi \sigma^2}{\mu(\xi - 2) Q_T} + \frac{256(\mu + 2L)4\beta G^2 T}{\mu^2(\beta - 4) \tau_k^2(\xi - 2) Q_T}
\]

Note that \( \xi > 2 + \frac{1+\beta}{\kappa(1+\rho)} \). Moreover \( Q_T = \sum_{t=1}^{T} q_t = \sum_{t=1}^{T} (\xi + t)^2 \geq \frac{1}{3} T^3 \) upon expanding and using the conditions on \( \xi \). Also \( \xi/(\xi - 2) = \mathcal{O}(1 + 1/\tau_k) \).

Finally using \( \sigma^2 \leq G^2 \) and Fact 1 to bound \( \mathbb{E} \left[ \| w_0 - w^* \|^2 \right] \leq 4G^2/\mu^2 \) completes the proof.

\[ \square \]

Lemma 2. Let \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) be a \( L \)-smooth \( \mu \)-strongly convex function, and \( w^* \) be its minima. Let \( \{w_t\} \) be a sequence of iterates generated by Algorithm 3.

Define error \( \epsilon_t := \mathbb{E} [ f(w_t) - f(w^*) ] \) and \( \tilde{w}_{t+1} = \tilde{w}_t - \eta_t g_t \) be a stochastic gradient update step at time \( t \), with \( \mathbb{E} \left[ \| g_t - \nabla f(w_t) \|^2 \right] \leq \sigma^2 \), \( \mathbb{E} \left[ \| g_t \|^2 \right] \leq G^2 \) and \( \eta_t = \frac{1}{\mu(1+\xi)} \xi > 2 \) then we have,

\[
\mathbb{E} \left[ \| \tilde{w}_{t+1} - w^* \|^2 \right] \leq \left( 1 - \frac{\eta \mu}{2} \right) \mathbb{E} \left[ \| \tilde{w}_t - w^* \|^2 \right] + \sigma^2 \eta_t^2 - \left( 1 - \frac{2}{\xi} \right) \epsilon_t \eta_t + (\mu + 2L) \mathbb{E} \left[ \| a_t \|^2 \right] \eta_t
\]

Proof. This is the first step of the perturbed iterate analysis framework Mania et al. (2015). We follow the steps as in Stich et al. (2018). The only change is that the proof of Stich et al. (2018) works with bounded gradients i.e. \( \mathbb{E} \left[ \| g_t \|^2 \right] \leq G^2 \). This assumption alone, doesn’t provide the variance reduction effect in the distributed setting. We therefore adapt the analysis with the variance bound \( \mathbb{E} \left[ \| g_t - \nabla f(w_t) \|^2 \right] \leq \sigma^2 \).

\[
\| \tilde{w}_{t+1} - w^* \|^2 = \| \tilde{w}_{t+1} - \tilde{w}_t + \tilde{w}_t - w^* \|^2 = \| \tilde{w}_{t+1} - \tilde{w}_t \|^2 + \| \tilde{w}_t - w^* \|^2 + 2 \langle \tilde{w}_{t+1} - \tilde{w}_t, \tilde{w}_t - w^* \rangle
\]

\[
= \eta_t^2 \| g_t \|^2 + \| \tilde{w}_t - w^* \|^2 + 2 \langle \tilde{w}_t - w^*, \tilde{w}_{t+1} - \tilde{w}_t \rangle
\]

\[
= \eta_t^2 \| g_t \|^2 + \| \tilde{w}_t - w^* \|^2 + 2 \| \tilde{w}_t - w^* \| \| \tilde{w}_{t+1} - \tilde{w}_t \| + \eta_t^2 \| \nabla f(w_t) \|^2 + \| \tilde{w}_t - w^* \|^2
\]

\[
+ 2 \eta_t \langle g_t, \nabla f(w_t) \rangle + 2 \eta_t \langle w^* - \tilde{w}_t, g_t \rangle
\]
We now claim that the last term $2L\epsilon$ where in the last step, we define $\epsilon$. The first term is bounded by $\mu$-strong convexity as,

$$ f(w^*) \geq f(w_t) + \langle \nabla f(w_t), w^* - w_t \rangle + \frac{\mu}{2} \|w_t - w^*\|^2 $$

$$ \iff \langle \nabla f(w_t), w^* - w_t \rangle \leq f(w^*) - f(w_t) - \frac{\mu}{2} \|w_t - w^*\|^2 $$

$$ \leq -\epsilon_t + \frac{\mu}{2} \|\hat{w}_t - w_t\| - \frac{\mu}{4} \|w^* - \hat{w}_t\| $$

where in the last step, we define $\epsilon_t := f(w_t) - f(w^*)$ and use $\|u + v\|^2 \leq 2(\|u\|^2 + \|v\|^2)$. The second term is bounded by using $2 \langle u, v \rangle \leq a \|u\|^2 + \frac{1}{a} \|v\|^2$ as follows,

$$ 2 \langle w_t - \hat{w}_t, \nabla f(w_t) \rangle \leq 2L \|w_t - \hat{w}_t\|^2 + \frac{1}{2L} \|\nabla f(w_t)\|^2 $$

Moreover, from Fact 2, we have $\|\nabla f(w_t)\|^2 \leq 2L\epsilon_t$. Taking expectation and putting everything together, we get,

$$ \mathbb{E} \left[ \|\tilde{w}_{t+1} - w^*\|^2 \right] \leq \left( 1 - \frac{\mu \eta_t}{2} \right) \mathbb{E} \left[ \|\tilde{w}_t - w^*\|^2 \right] + \eta_t^2 \sigma^2 $n\right) \mathbb{E} \left[ \|\tilde{w}_t - \hat{w}_t\|^2 \right] + (2L \eta_t^2 - \eta_t) \epsilon_t $$

We now claim that the last term $2L\eta_t^2 - \eta_t \leq -\frac{\xi - 2}{\xi} \eta_t$ or equivalently $2L\eta_t^2 - \left( 1 - \frac{\xi - 2}{\xi} \right) \eta_t \leq 0$. Note that this is a quadratic in $\eta_t$ which is satisfied between its roots $0$ and $\frac{1}{2L}$. So it suffices to show is that our step sizes are in this range. In particular, the second root (which is positive by choice of $\xi$) should be no less than step size. We have $\eta_t = \frac{1}{\mu L \epsilon_t}$, $\eta_t \leq \frac{1}{\mu \xi} \forall t$, the second root $\frac{1}{L\xi} \geq \frac{1}{\mu \xi}$ because smoothness parameter $L \geq \mu$, the strong convexity parameter, or equivalently the condition number $\kappa := L/\mu \geq 1$. Combining the above with $a_t = w_t - \hat{w}_t$, we get,

$$ \mathbb{E} \left[ \|\tilde{w}_{t+1} - w^*\|^2 \right] \leq \left( 1 - \frac{\mu \eta_t}{2} \right) \mathbb{E} \left[ \|\tilde{w}_t - w^*\|^2 \right] + \eta_t^2 \sigma^2 $n\right) \mathbb{E} \left[ \|a_t\|^2 \right] - \left( 1 - \frac{2}{\xi} \right) \eta_t \epsilon_t $$

Fact 1. Rakhlin et al. (2012) Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a $\mu$-strongly convex function, and $w^*$ be its minima. Let $g$ be an unbiased stochastic gradient at point $w$ such that $\mathbb{E} \left[ \|g\|^2 \right] \leq G^2$, then

$$ \mathbb{E} \left[ \|w - w^*\|^2 \right] \leq \frac{4G^2}{\mu^2} $$

Fact 2. For $L$-smooth convex function $f$ with minima $w^*$, then the following holds for all points $w$,

$$ \|\nabla f(w) - \nabla f(w^*)\|^2 \leq 2L(f(w) - f(w^*)) $$

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We first claim that particular, the compression is provided by the recovery guarantees of Count Sketch, and we do a Proof of Lemma 3. The proof repeats the steps in Stich et al. (2018) with minor modifications. In for constants $A, B > 0, \mu \geq 0, \xi > 1$. Then,

$$1 \frac{\sum_{t=0}^{T-1} q_t \epsilon_t}{Q_T} \leq \frac{\mu \xi^3 b_0}{8 Q_T} + \frac{4 T(T + 2 \xi) A}{\mu Q_T} + \frac{64 T B}{\mu^2 Q_T}$$

for $\eta_t = \frac{8}{\mu(\xi + t)}, q_t = (\xi + t)^2, Q_T = \sum_{t=0}^{T-1} q_t \geq \frac{T^3}{3}$

**Fact 4. Stich et al. (2018)** Let $\{h_t\}_{t>0}$ be a sequence satisfying $h_0 = 0$ and

$$h_{t+1} \leq \min \left\{ (1 - \tau / 2) h_t + \frac{2}{\tau k} \eta_t^2 A, (t + 1) \sum_{i=0}^{t} \eta_i^2 A \right\}$$

for constant $A > 0$, then with $\eta_t = \frac{1}{\tau_k}$ with $\xi > 1 + \frac{4+\beta}{\eta_k(1+\rho)}$, with $\beta > 4$ and $\rho = \frac{4\beta}{(\beta-4)(\beta+1)}$, for $t \geq 0$ we get,

$$h_t \leq \frac{4\beta}{(\beta - 4)} \cdot \frac{\eta_t^2 A}{\tau_k}$$

**Lemma 3.** With probability at least $1 - \delta$

$$\mathbb{E} \left[ \|a_t\|^2 \right] \leq \frac{4\beta}{(\beta - 4)} \cdot \frac{\eta_t^2 G^2}{\tau_k}$$

**Proof of Lemma 3.** The proof repeats the steps in Stich et al. (2018) with minor modifications. In particular, the compression is provided by the recovery guarantees of Count Sketch, and we do a union bound over all its instances. We write the proof in full for the sake of completeness. Note that

$$a_t = a_{t-1} + \eta_{t-1} g_{t-1} - \tilde{g}_{t-1}$$

We first claim that $\mathbb{E} \left[ \|a_t\|^2 \right] \leq t \eta_t^2 G^2$. Since $a_0 = 0$, we have $a_t = \sum_{i=1}^{t} (a_i - a_{i-1}) = \sum_{i=0}^{t-1} (\eta_i g_i - \tilde{g}_i)$. Using $(\sum_{i=1}^{n} a_i)^2 \leq (n + 1) \sum_{i=1}^{n} a_i^2$ and taking expectation, we have

$$\mathbb{E} \left[ \|a_t\|^2 \right] \leq t \sum_{i=0}^{t-1} \mathbb{E} \left[ \|\eta_i g_i - \tilde{g}_i\|^2 \right] \leq t \sum_{i=0}^{t-1} \eta_i^2 G^2$$

Also, from the guarantee of Count Sketch, we have that, with probability at least $1 - \delta / T$, the following holds give that our compression is a $\tau_k$ contraction.

Therefore

$$\|a_{t+1}\|^2 \leq (1 - \tau_k) \|a_t + \eta_t g_t\|^2$$
Using inequality $(a + b)^2 \leq (1 + \gamma)a^2 + (1 + \gamma^{-1})b^2$, $\gamma > 0$ with $\gamma = \frac{2}{F}$, we get

$$
\|a_{t+1}\|^2 \leq \tau_k \left( (1 + \gamma) \|a_t\|^2 + (1 + \gamma^{-1}) \eta_t \|g_t\|^2 \right)
$$

$$
\leq \frac{(2 - \tau_k)}{2} \|a_{t-1}\|^2 + \frac{2}{\tau_k} \eta_t^2 \|g_t\|^2
$$

Taking expectation on the randomness of the stochastic gradient oracle, and using $E[\|g_t\|^2] \leq G^2$, we have,

$$
E[\|a_{t+1}\|^2] \leq \frac{(2 - \tau_k)}{2} E[\|a_t\|^2] + \frac{2}{\tau_k} \eta_t^2 G^2
$$

Note that for a fixed $t \leq T$ this recurrence holds with probability at least $1 - \delta/T$. Using a union bound, this holds for all $t \in [T]$ with probability at least $1 - \delta$. Conditioning on this and using Fact 4 completes the proof. \qed

### A.1 Without variance bound

We now state theorem which uses on the norm bound on stochastic gradients. It follows by directly plugging the fact the HeavyMix is a $k/d$-contraction in the result of Stich et al. (2018).

**Theorem 5.** Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a $L$-smooth $\mu$-strongly convex function. Given $T > 0$ and $0 < k \leq d, 0 < \delta < 1$, Algorithm 3 SKETCHED-SGD with access to stochastic gradients such that $E[\|g\|^2] \leq G^2$, with sketch size $O(k \log(dT/\delta))$ and step size $\eta_t = \frac{1}{T^k}$, with $\xi > 1 + \frac{d(1+\beta)}{k(1+\rho)}$, with $\beta > 4$ and $\rho = \frac{48}{(\beta - 4)(\beta + 1)}$, after $T$ steps outputs $\hat{w}_T$ such that with probability at least $1 - \delta$:

$$
E[f(\hat{w}_T)] - f(w^*) \leq O\left( \frac{G^2}{\mu T} + \frac{d^2G^2L}{k^2\mu^2T} + \frac{d^3G^3}{k^3\mu T^3} \right).
$$

### B Model Training Details

We train our models with a batch size of 512, a learning rate varying linearly at each iteration from 0 (beginning of training) to 0.4 (epoch 5) back to 0 (end of training). We augment the training data by padding images with a 4-pixel black border, then cropping randomly back to 32x32, making 8x8 random black cutouts, and randomly flipping images horizontally. We use a cross-entropy loss with L2 regularization of magnitude 0.0005.