REDUCIBILITY OF INDUCED DISCRETE SERIES REPRESENTATIONS FOR AFFINE HECKE ALGEBRAS OF TYPE $B$

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Abstract. Recently Delorme and Opdam have generalized the theory of $R$-groups towards affine Hecke algebras with unequal labels. We apply their results in the case where the affine Hecke algebra is of type $B$, for an induced discrete series representation with real central character. We calculate the $R$-group of such an induced representation, and show that it decomposes multiplicity free into $2^d$ irreducible summands. The power $d$ can be calculated combinatorially.

Contents

1. Introduction
2. The affine Hecke algebra
   2.1. Definition
2.2. Irreducible tempered representations of $\mathbb{H}$
2.3. Hecke algebras and reductive $p$-adic groups
2.4. The graded Hecke algebra
3. The $R$-group
4. Conjecture on $\hat{\mathbb{H}}$ for $R_0$ of type $B$
   4.1. Residual points
   4.2. Symbols and Springer correspondents
   4.3. Conjecture on $\hat{\mathbb{H}}^0$
   4.4. Reducibility of induced discrete series
5. Explicit calculation of the $R$-group
   5.1. Calculation of $R_0(\xi)$
   5.2. Calculation of $R(\xi)$
   5.3. Counting irreducible components
References

1. Introduction

Let $\mathbb{H}$ be an affine Hecke algebra of type $B$ or $C$ with possibly unequal labels (see 2.1 for its precise definition). In this article we study the reducibility of the induction of a discrete series representations of a parabolic subalgebra of $\mathbb{H}$. We restrict ourselves to the case where the irreducible components of the induced representation have real central character. In particular, we may assume that the
representation which is being induced has real central character. Under this assumption the results for type $C$ can be derived from those of type $B$ (see Remark 2.1), so we only compute the type $B$ case.

We study this reducibility by computing the relevant $R$-group. This is possible thanks to the recent extension of the $R$-group theory by Delorme and Opdam to the case of affine Hecke algebras. In the case of real reductive groups, this theory was developed by Knapp, Stein and Harish-Chandra. The extension to $p$-adic groups was done by Silberger. In each of the cases, one constructs, starting from a subgroup of the Weyl group which is computed using the Plancherel measure of the group (or the algebra), intertwining operators for a certain parabolically induced representation, and prove that these operators span its centralizer algebra. The $R$-group consists of those elements which yield nontrivial intertwiners. In general, the map $r \mapsto E(r)$ from the $R$-group to the intertwiners yields a projective representation.

We prove that in our situation, the $R$-group is always isomorphic to a certain $\mathbb{Z}^d$, where we can calculate $d$ explicitly using combinatorics involving the central character of the induced representation and the structure of the parabolic root subsystem which defines the Hecke subalgebra from which we induce. This allows us to prove that the induced representation under consideration always decomposes into $2^d$ irreducible and pairwise inequivalent irreducible components. Moreover, we also prove that the a priori projective representation $r \mapsto E(r)$ is linear in our setting.

Suppose that $R_0$ is of type $B_n$ and let $q_1$, resp. $q_2$ be the parameters of $\mathcal{H}$ corresponding to a simple reflection in a long, resp. short root (see 2.1 and (4.1)). Since $q_1, q_2 \in \mathbb{R}_{>0}$, there exists $m$ such that $q_2 = q_1^m$. The representation theory of $\mathcal{H}$ is largely determined by $m$ in the sense that raising $q_1, q_2$ to a common power $q_1^\epsilon, q_2^\epsilon$ does not change the parametrization of the irreducible modules of the corresponding Hecke algebras, even though the algebras themselves are not isomorphic.

For all $m$ such that reducibility of induced discrete series representation in our setting turns out to occur, the Hecke algebra is isomorphic to the centralizer algebra of a certain parahorically induced representation of a reductive $p$-adic group $\mathcal{G}$ (see below). Thus, our results fit naturally into the theory of $R$-groups for reductive $p$-adic groups of classical type. Goldberg has given an explicit description of the $R$-groups of the groups of classical Lie type $SO(n, F)$ and $Sp(2n, F)$ in [7]. He has proven that for these groups, the $R$-group is isomorphic to $\mathbb{Z}^d$ for some $d$. However, $d$ cannot be calculated explicitly, other than by assuming knowledge of the irreducibility of other parabolically induced discrete series representations (those induced from a so-called basic parabolic subgroup).

The outline of this paper is as follows. In section 2 we review some necessary material on affine and graded Hecke algebras of arbitrary type. We review the relation between the affine Hecke algebra and the representation theory of reductive $p$-adic groups, explaining how the results in this paper connect to the ones of Goldberg. In section 3 we review the portion of the theory of the $R$-group developed by Delorme and Opdam which is relevant for our purposes. Since our result fits naturally into a combinatorial framework called generalized Springer correspondence which we have introduced in [20], we quickly review some of the involved constructions.
in section 4. The result of this paper gives a partial affirmative answer to a conjecture in [20] which we also mention here. In section 5 we compute the $R$-group, making use of these combinatorics, and prove the results mentioned above. For the Iwahori-Hecke case, we also give an interpretation of the $R$-group in terms of the component groups which come up in the Springer correspondence. This is done using the combinatorics recalled in section 4.

2. The affine Hecke algebra

2.1. Definition. Let $R = (R_0, X, \tilde{R}_0, Y, \Pi_0)$ be a root datum. By this we mean that $X$ and $Y$ are free abelian groups with a perfect pairing $\langle \cdot, \cdot \rangle$ over $\mathbb{Z}$, that $R_0 \subset X$ is a reduced root system, that $\tilde{R}_0$ is the dual root system of coroots of $R_0$, and that $\Pi_0 = \{\alpha_1, \ldots, \alpha_n\}$ is a choice of simple roots of $R_0$. We denote the corresponding set of positive roots by $R_0^+$. Define $Q = \mathbb{Z}R_0 \subset X$, the root lattice. Let $W_0$ be the Weyl group of $R_0$, that is, the finite reflection group generated by the reflections $s_i : X \to X : x \to x - \langle x, \alpha_i \rangle \alpha_i$. Given $\Pi_L \subset \Pi_0$, we denote the corresponding sub root system of $R_0$ by $R_L$, and $W_L = W_0(R_L) \subset W_0$.

The extended affine Weyl group is defined as $W = W_0 \ltimes X$. It admits a decomposition $W = W^a \ltimes \Omega$, with $\Omega \cong X/Q$ and $W^a = W_0 \ltimes Q$. This group $W^a$ is called the affine Weyl group and is, like $W_0$, a Coxeter group. Let $S^m$ be the set of maximal coroots in $R_0^\vee$ (with respect to the dominance ordering on $Y$). Then a set of Coxeter generators for $W^a$ is given by $S = \{s_1, \ldots, s_n\} \cup \{s_\theta \mid \theta^\vee \in S^m\}$ where $s_1, \ldots, s_n$ are as above and $s_\theta(x) = x - (-\langle x, \theta \rangle + 1)(-\theta)$. If $R_0$ is irreducible then $S^m$ consists of a single element and we write $s_\theta = s_\theta$ for the corresponding reflection. Being a Coxeter group, $W^a$ has a length function $l$. We can uniquely extend it to a function on $W$, by defining $l(\omega) = 0$ for $\omega \in \Omega$.

Choose a function $q : W \to \mathbb{R}_{>0}$ such that $q(ww') = q(w)q(w')$ whenever $l(ww') = l(w) + l(w')$. Such a function is determined by its values on the simple reflections. Equivalently, it can be characterized by a set of “root labels” $q_\alpha \in \mathbb{R}_{>0}$, for $\alpha \in R_{ww'} = R_0 \cup \{2\alpha \mid \alpha \in 2Y \cap \tilde{R}_0\}$, with the condition that $q_\alpha$ depends only on the $W_0$-orbit of $\alpha$. For $\alpha$ such that $2\alpha \notin R_{ww'}$, we formally define $q_{\alpha/2} = 1$. For an irreducible root system, we define on simple affine roots $q(s_i) = q_\alpha q_{\alpha_i/2}$ ($i = 1, \ldots, n$) and $q(s_0) = q_\theta$.

The affine Hecke algebra $\mathcal{H} = \mathcal{H}(R, q)$ is then the unique associative, unital complex algebra with $\mathbb{C}$-basis $T_w, w \in W$ such that the multiplication satisfies

$$T_wT_{w'} = T_{ww'} \text{ if } l(ww') = l(w) + l(w');$$

$$(T_{s_i} + 1)(T_{s_i} - q(s_i)) = 0 \text{ for } i = 0, 1, \ldots, n.$$

2.2. Irreducible tempered representations of $\mathcal{H}$.

2.2.1. Bernstein-Zelevinsky presentation. Let $X^+ = \{x \in X \mid \langle x, \alpha \rangle \geq 0 \text{ for all } \alpha \in R_0^+\}$. For $x \in X^+$, let $t_x$ denote the corresponding element of $W$ and define $t_x^\alpha = q(t_x)^{-1/2}t_{\alpha x}$. If $z \in X$, write it as $z = x - y$ with $x, y \in X^+$, and put $\theta_z = \theta_x\theta_y^{-1}$ (notice that all $T_w$ are invertible). Let $\mathcal{A}$ be the subalgebra of $\mathcal{H}$ which has the $\theta_z, x \in X$ as $\mathbb{C}$-basis. The action of $W_0$ on $X$ also provides an action on $\mathcal{A}$. Let $\mathcal{H}_0$ be the subalgebra of $\mathcal{H}$ with basis $T_w, w \in W_0$. According to what is known as the Bernstein-Zelevinsky presentation, one has $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{A} = \mathcal{A} \otimes \mathcal{H}_0$. 


Moreover Bernstein (unpublished, see [13]), showed that the center $Z$ of $\mathcal{H}$ is equal to $Z = A^{w_0}$.

2.2.2. Central character. It is well known that all irreducible representations of $\mathcal{H}$ are of finite dimension. This follows from the fact that $\mathcal{H}$ is finitely generated over $Z$, and that by Dixmier’s version of Schur’s Lemma, $Z$ acts by scalars in an irreducible representation $(\pi, V_\pi)$. Let $T = \text{Hom}_Z(X, \mathbb{C}^\times) = \text{Spec}(A)$. In an irreducible representation $(\pi, V_{\pi})$ of $\mathcal{H}$, $Z$ acts by $W_0t \in W_0 \setminus T \simeq \text{Spec}(Z)$. We call $W_0t$ (or, by abuse of terminology, any of its elements) the central character of $(\pi, V_\pi)$. Let $T$ have polar decomposition $T = T_sT_r = \text{Hom}(X, S^1)\text{Hom}(X, \mathbb{R}_{>0})$. If $W_0t \subset T_r$, then we say that $(\pi, V_\pi)$ has real central character.

2.2.3. Tempered; discrete series. Let $(\pi, V_\pi)$ be an irreducible representation of $\mathcal{H}$. If, for $t \in T$, we have $V_\pi(t) = \{v \in V_\pi \mid (\pi(a) - t(a))v = 0 \text{ for all } a \in A\} \neq 0$, then we say that $t$ is an $A$-weight of $(\pi, V_\pi)$. If every $A$-weight $t$ of $(\pi, V_\pi)$ satisfies $|t(x)| \leq 1$ for all $x \in X^+$, then we say that $(\pi, V_\pi)$ is a tempered representation. If moreover $|t(x)| < 1$ for all $0 \neq x \in X^+$, then we say that $(\pi, V_\pi)$ is a discrete series representation. Let $\mathcal{H}^t$ be the set of equivalence classes of irreducible tempered representations of $\mathcal{H}$. The theory of the Plancherel formula for the affine Hecke algebra of [16] implies that the representations in $\mathcal{H}^t$ are precisely those which occur in the spectral decomposition of the natural trace of $\mathcal{H}$ (cf. [16] §). We will be concerned with $\mathcal{H}^t_{rs}$, the representations in $\mathcal{H}^t$ with real central character. We denote by $\mathcal{H}^t_{rs} \subset \mathcal{H}^t$ the subset of discrete series representations.

2.2.4. Residual cosets. We need some more terminology from [16]. Let $L = rLT_L$ be a coset of a subtorus $T_L$ of $T$. Then we define a parabolic root system $R_L = \{\alpha \in R_0 \mid \alpha(T_L) = 1\}$. Let $R_L^0 = \{\alpha \in R_L \mid \alpha(r_L) = q_0^{\frac{1}{2}}\text{ or } \alpha(r_L) = -q_0^{\frac{1}{2}}\}$ and $R_L^\pm = \{\alpha \in R_L \mid \alpha(r_L) = \pm 1\}$. Then we put $i_L = |R_L^0| - |R_L^\pm|$, and we say that $L = rLT_L$ is a residual coset if $i_L = \text{codim}(T_L)$. Clearly the notion of residual coset is $W_0$-invariant. Usually we will choose $L$ in its orbit such that $R_L$ is a standard parabolic with simple roots $\Pi_L \subset R_L$.

If moreover $r_L$ can be chosen in $T_r$, then we say that $L$ is a real residual coset. A residual coset $L = \{r_L\}$ (i.e., $T_L = \{1\}$) is called a residual point. As described in [16] Ch. 7, the classification of residual points for graded Hecke algebras in [8] leads to a classification of all residual cosets of $\mathcal{H}(R, q)$.

2.2.5. c-function. For later reference, we also recall the following definition. Let $\alpha \in R_0$, then define $c_\alpha$ in the fraction field of $A$ as

\begin{equation}
(2.1) \quad c_\alpha = \frac{(1 + q_\alpha^{-1} \theta_\alpha)(1 - q_\alpha^{-1} \theta_\alpha)}{(1 + \theta_\alpha)(1 - \theta_\alpha)}.
\end{equation}

We view $c_\alpha$ as a rational function on $T$. Notice that if $\alpha \notin 2Y$, then by definition $q_\alpha^{-1} = 1$, so $c_\alpha$ simplifies to $(1 - q_\alpha^{-1} \theta_\alpha)/(1 - \theta_\alpha)$ in this case.

2.2.6. Parabolic induction. Let $\Pi_L \subset \Pi_0$, then $R_L = (R_L, X, \tilde{R}_L, Y, \Pi_L)$ is a root datum. Let $X_L = X/(X \cap (R_0^\vee)^\mathbb{Z})$ and $Y_L = Y \cap Q R_L^\vee$. Then $R_L = (R_L, X_L, \tilde{R}_L, Y_L, \Pi_L)$ is also a root datum. Restriction of $\{q_\alpha \mid \alpha \in R_{mr}\}$ to $R_{sL, r} \subset R_{mr}$ yields $q_L$ on $W_L \ltimes X$ and $q_L$ on $W_L \ltimes X_L$. Let $\mathcal{H}_L = \mathcal{H}(R_L, q_L)$ and $\mathcal{H}_L = \mathcal{H}(R_L, q_L)$ be the associated affine Hecke algebras. Let $T_L = \text{Hom}(X_L, \mathbb{C}^\times)$. Define also the lattice $X_L = X/(R L \cap X)$ and the torus $T_L = \text{Hom}(X_L, \mathbb{C}^\times)$. It
follows from the results in [10] that the representations in \( \hat{H}^t \) all arise as irreducible components of

\[
\pi(\Pi_L, \delta, t^L) = \text{Ind}_{H_L}^G(\delta \circ \phi_L),
\]

where \( \delta \in \hat{H}_{L,R}^{\text{red}} \), \( t^L \in T^L \), and \( \phi_L \) denotes the map \( H^L \to H_L \) which takes \( T_w \to T_w \) for \( w \in W_L \), and sends \( \theta_x \to t^L(x)\theta_{\tilde{x}} \) with \( x \to \tilde{x} \) the projection \( X \to X_L \). Moreover, a representation in \( \hat{H}_L^t \) arises as an irreducible summand in (2.2) for a unique \( (\Pi_L, \delta, 1) \) if we consider the tripels \( (\Pi_L, \delta, 1) \) up to a suitable equivalence relation (which in the case of real central character and a root system \( R_0 \) of type \( B_n \) is just Weyl group conjugacy of \( L \), cf. Lemma. 3.1 below).

2.2.7. Classification of central characters of \( \hat{H}_L^t \). In [10], Opdam obtains a classification of the central characters of the irreducible tempered representations of \( H \), hence in particular of the representations in \( \hat{H}_L^t \). The central characters of \( \hat{H}_L^t \) are precisely the \( W_0 \)-orbits of the centers of the real residual cosets. This bijection is such that the central character of \( (\pi, V_\pi) \in \hat{H}_L^t \) which occurs in the induced representation \( \pi(\Pi_L, \delta, 1_L) \) is the center of a residual coset \( M \) with \( R_L = R_M \). Thus, the central character of a discrete series representation is a residual point, and conversely.

2.2.8. The \( R \)-group. Let \( L = r_L T^L \) be a real residual coset. In general, the description of \( \Delta_{W_0 \cap L} \subset \hat{H}_L^t \) of irreducible representations with central character \( W_0 \cap L \) is unknown, apart from the fact that it is a non-empty and finite set. For a Hecke algebra \( H \) of type \( A \), it is known that \( \hat{H}_{L,R}^{\text{red}} \) consists of a single one-dimensional representation. If \( q_0 > 1 \), the corresponding representation of the symmetric group is the sign representation. The discrete series representation itself is referred to as the Steinberg character. For arbitrary root systems, to classify \( \Delta_{W_0 \cap L} \) for all central characters of \( \hat{H}_L^t \), one needs to classify all \( \hat{H}_{L,R}^{\text{red}} \), and understand the induction (2.2). This latter problem is addressed in recent work of Delorme and Opdam, [11], where they give a description of a group which controls the decomposition into irreducible components of (2.2). They call this group the \( R \)-group, in analogy with the corresponding group for real or \( p \)-adic groups (see e.g. [11]).

2.3. Hecke algebras and reductive \( p \)-adic groups. Let \( G \) be a connected split reductive group defined over a non-archimedean non-discrete local field \( F \) of characteristic zero, with ring of integers \( O \) and residue field \( F_q \). Let \( G \) be the group of \( F \)-rational points of \( G \).

2.3.1. Parabolic induction in \( G \). Let \( P = MN \) be a parabolic subgroup with Levi subgroup \( M \) and unipotent radical \( N \). Let \( \rho \) be an irreducible, admissible representation of \( M \). Then we can extend \( \rho \) across \( N \) to become a representation of \( P \), and then induce to obtain \( I(P, \rho) = \text{Ind}_{M}^{G}(\rho) \). The compactness of \( G/P \) implies the admissibility of \( I(P, \rho) \). It is known that for each irreducible admissible representation \( \pi \) of \( G \), there exists a parabolic subgroup \( P = MN \) and an irreducible supercuspidal representation \( \rho \) of \( M \) such that \( \pi \) is a subrepresentation of \( I(P, \rho) \). Similarly, the \( I(P, \rho) \) where \( \rho \) is a discrete series representation, yield the tempered spectrum of \( G \).
2.3.2. Unipotent representations; the Hecke algebra. Let $\mathcal{P} \subset G$ be a parahoric subgroup, with pro-unipotent radical $U$. Then the quotient $M = \mathcal{P}/U$ is isomorphic to a Levi subgroup of $G(F_q)$, hence in particular it is a finite reductive group of Lie type. Let $\sigma$ be a cuspidal unipotent representation of $M$, and let $\sigma^\vee$ be its contragredient. We inflate $\sigma^\vee$ across $U$ to become a representation of $\mathcal{P}$, and then form the (compactly) induced representation $c - \text{Ind}_G^G(\sigma^\vee)$. Let $\mathcal{H}(G, \mathcal{P}, \sigma) = \text{End}_G(c - \text{Ind}_G^G(\sigma^\vee))$ be its centralizer algebra (the notation is such as to obtain the equivalence of categories below). By theorems of Morris, [10] and Lusztig, [13], $\mathcal{H}(G, \mathcal{P}, \sigma)$ is isomorphic to an (extended) affine Hecke algebra associated to a root system whose rank is the difference between the split ranks of $G$ and $M$, with explicitly determined possibly unequal labels.

Let $\mathfrak{S}\mathfrak{R}_\sigma(G)$ be the category of smooth representations of $G$ which are generated by their $\sigma$-isotypic component. Then by [2] Theorem 4.3, there is an equivalence of categories between $\mathfrak{S}\mathfrak{R}_\sigma(G)$ and the category $\mathcal{H}(G, \sigma) - \text{Mod}$ of modules of $\mathcal{H}(G, \mathcal{P}, \sigma)$.

The prototype is when one takes $\mathcal{P} = \mathcal{I}$, an Iwahori subgroup of $G$ and $\sigma = 1$. Suppose $G$ has root datum $(R_0^\vee, Y, R_0, X, \Pi_0^\vee)$. Then $\mathcal{H}(G, \mathcal{I}, 1)$ is the Iwahori-Hecke algebra with root datum $(R_0, X, R_0^\vee, Y, \Pi_0)$ and labels $q(s) = q$, the cardinality of the residue field of $F$. The equivalence of categories $\mathfrak{S}\mathfrak{R}_\sigma(G) \rightarrow \mathcal{H}(G, \mathcal{I}, 1) - \text{Mod}$ is given by taking Iwahori-invariant vectors, that is by the map $V \mapsto V^\mathcal{I}$, which is naturally a $\mathcal{H} = \mathcal{H}(G, \mathcal{I}, 1)$-module.

2.3.3. Relation between parabolic inductions in $\mathcal{H}$ and $G$. For the general case of an affine Hecke algebra $\mathcal{H}(G, \mathcal{P}, \sigma)$, it does not seem to be written up in the literature what the precise relation is between parabolic induction in $\mathcal{H}(G, \mathcal{P}, \sigma)$ and in $G$.

However in the Iwahori-spherical case where $\mathcal{P} = \mathcal{I}$ and $\sigma = 1$, the following is known from the work of Jantzen, cf. [9]. Let $P = MN$ be a parabolic subgroup with standard Levi subgroup $M$. Let $R_M$ be the root system of $M$. Let $\rho$ be a representation of $M$ which is generated by its $\mathcal{I}_M := \mathcal{I} \cap M$-fixed vectors (note that $\mathcal{I}_M$ is an Iwahori subgroup of $M$). Let $w_0$ (resp. $w_M$) be the longest element of $W_0$ (resp. of $W_0(R_M)$) and let $w^M = w_0 w_M$. Let $M' = w_0 M w_0 = w^M M (w^M)^{-1}$. Let $\mathcal{H}^{M'}$ be the subalgebra of $\mathcal{H}$ associated to $R_{M'}$ as in [2]. Then, according to [9] Prop. 2.1.2 (the case where $M$ is a maximal split torus goes back to [14]), we have

\begin{equation}
(\text{Ind}_{G}^{\mathcal{P}}(\rho))^\mathcal{I} \simeq \mathcal{H} \otimes_{\mathcal{H}^{M'}} (w^M \cdot \rho^{M'}).
\end{equation}

This means that the results which we will describe, relevant to the right hand side, can be interpreted as results on the reducibility of the induced representation on the left hand side, and thus refine the work of Goldberg in these cases.

2.3.4. Goldberg's results. For the induced representation $I(P, \rho)$, the decomposition into irreducibles of $\text{Ind}_P^G(\rho)$ is determined by a finite group, referred to as the $R$-group. Roughly speaking, the dual of this group parametrizes the irreducible constituents. For $G = GL(n, F)$, it is known that all $R$-groups are trivial. For the split groups of classical type $SO(n, F), Sp(2n, F)$, the $R$-group has been computed by Goldberg in [4]. He shows that if $\rho$ is a discrete series representation, then its $R$-group $R(\rho) \simeq \mathbb{Z}_d^2$ for some $d$. We discuss this in some more detail for the case where $G = SO(2n + 1, F)$, the case where $G = Sp(2n, F)$ being analogous (we are not concerned with the type $D$ case where $G = SO(2n, F)$ in this paper). Let $M$
be a Levi subgroup of $G$, then $M$ can be written as
\begin{equation}
M \simeq GL(m_1)^{n_1} \times \cdots \times GL(m_r)^{n_r} \times SO(2m+1),
\end{equation}
where $\sum m_i n_i + m = n$ and the $m_i$ are all different. If $r = 1$ then $M$ is called a basic Levi subgroup, and $P = MN$ a basic parabolic. Given $M$ as in (2.4) and a discrete series representation $\rho$ of $M$, then $\rho = \rho_1 \otimes \cdots \otimes \rho_t \otimes \rho_B$, where $\rho_t$ is a discrete series representation of $GL(m_t)^{n_t}$ and $\rho_B$ is a discrete series representation of $SO(2m+1)$. Let $R_i$ be the $R$-group of the representation $\rho_t \otimes \rho_B$, w.r.t. to the induction to $SO(2(m_i n_i + m) + 1)$. Then Goldberg has shown that
\begin{equation}
R(\rho) \simeq R_1 \times \cdots \times R_t.
\end{equation}
Every $R_i$ is of the form $R_i \simeq \mathbb{Z}^{m_i}_{2}$. Let $\rho_i = \rho_{i,1} \otimes \cdots \otimes \rho_{i,n_i}$, where each $\rho_{i,j}$ is a discrete series representation of $GL(m_i)$. Then, if $P'_i = M'_i N'_i$ is the parabolic subgroup of $G_i \simeq SO(2(m_i + m) + 1)$ such that $M'_i \simeq GL(m_i) \times SO(2m + 1)$, we have
\begin{equation}
d_i = \sharp\{[\rho_{i,j}] \mid \text{Ind}^{G'_i}_{P'_i}(\rho_{i,j} \otimes \rho_B) \text{ is reducible}\}.
\end{equation}
Here, $[\rho_{i,j}]$ denotes the equivalence class of $\rho_{i,j}$. In [3], Mœglin and Tadić compute the numbers $d_{i,j}$ in terms of so-called Jordan pairs. However, these pairs seem not to be easy to compute.

For affine Hecke algebras, Delorme and Opdam have shown in [4] that if one considers unitary parabolic induction of discrete series representations of a parabolic quotient algebra, one can also construct the $R$-group, and that it has the same significance. In this article, we explicitly compute this $R$-group in the case where the induced representation has real central character. We will show that in this special case the analogues of (2.4) and (2.5) hold. However for the analogue of (2.6), the assumption of real central character implies that each $\rho_{i,j}$ is the Steinberg representation, so we are in the simpler situation where $d_i \in \{0,1\}$. We compute $d_i$ combinatorially, using a certain Young tableau which is defined in terms of the parameters of the Hecke algebra, the type of the parabolic and the central character of the induced discrete series representation.

2.4. The graded Hecke algebra. In the proof of Theorem 5.10 which describes the decomposition into irreducibles of the representations $\pi(\Pi_L, \delta, 1_L)$ for $\delta \in \mathcal{H}^{ds}_{L, B}$, we will need the graded Hecke algebra and its relation to the affine Hecke algebra, as described by Lusztig in [13].

Let $\alpha \in R_0$. Then we put
\begin{equation}
k_\alpha = \log(q_\alpha q_{\alpha/2}^{-1/2}).
\end{equation}
Notice that if $\alpha \notin 2Y$, we simply have $k_\alpha = \log(q_\alpha)$. Let $\mathfrak{a}^* = X \otimes \mathbb{Z} \mathbb{R}$, and define $\mathcal{R}^{deg} = (R_0, \mathfrak{a}^*, R_0^\vee, \mathfrak{a}, \Pi_0)$ (we call this the degenerate root datum associated to $\mathcal{R}$). Then the graded Hecke algebra associated to $\mathcal{R}^{deg}$ and the label function $k : \alpha \mapsto k_\alpha$ is defined to be
\[H = H(\mathcal{R}^{deg}, k) = \mathbb{C}[W_0] \otimes S(\mathfrak{a}_C^\alpha),\]
where we have the relations
\[x \cdot s_\alpha - s_\alpha \cdot s_\alpha(x) = k_\alpha(x, \alpha),\]
for $x \in \mathfrak{a}_C^\alpha, \alpha$ simple. As reviewed in [20], it follows from theorems of Lusztig in [13] that there exists a natural bijection between $\hat{H}^l_{B}$ and $\widetilde{H}^l_{B}$ (defined analogously).
Remark 2.1. Suppose that \( \mathcal{H} = \mathcal{H}(\mathcal{R}', q') \) where \( \mathcal{R} = (R'_0, X', R'_0, Y', \Pi'_0) \) and \( R'_0 \) has type \( C_n \). Then, by the above mentioned theorems of Lusztig, there is a natural bijection \( \mathcal{H}' \leftrightarrow \mathcal{H} \), where \( \mathcal{H}' \) is the graded Hecke algebra associated to \( \mathcal{R}_{deg} \) and \( k' \) depending on \( q' \) as in \( \mathcal{H} \). Let \( k'_1 \) (resp. \( k'_2 \)) be the labels of the roots of type \( \pm e_i \pm e_j \) (resp. \( \pm 2e_i \)), expressed in standard basis vectors of \( X' \otimes \mathbb{Z} \). Let \( R_0 \) be the root system of type \( B \) obtained from \( R'_0 \) by replacing the \( \pm 2e_i \) with \( \pm e_i \). Let \( k_1 = k_{\pm e_i \pm e_j} = k'_1 \), and \( k_2 = k_{\pm e_i} = k'_2 / 2 \). It is not hard to see that \( \mathcal{H}' \simeq \mathcal{H} \), where \( \mathcal{H} \) is attached to the data \( \mathcal{R}_{deg} \), obtained from \( \mathcal{R}_{deg} \) by replacing \( R'_0 \) with \( R_0 \), and label function \( k \). Thus, the problem of reducibility of an induced representation with real central character for an affine Hecke algebra of type \( C \) is equivalent to a corresponding problem for an affine Hecke algebra of type \( B \). Therefore we will only treat the type-\( B \) case.

3. The \( R \)-group

We now review the definition and properties of the \( R \)-group, following \[4\]. Let \( \Xi \) be the (cf. \[10\]) set of objects of the “groupoid of standard induction data”, that is, an element of \( \Xi \) is a triple \( \xi = (\Pi_L, \delta, t^L) \), where \( \Pi_L \subset \Pi_0 \) defines a standard parabolic, \( \delta \in \mathcal{H}^{ds}_L \), and \( t^L \in T^L_L \). If moreover \( t^L \in T^L_L \), then we write \( \xi \in \Xi_w \).

Suppose that \( \delta \) has central character \( W_{0}^r \mathcal{T}_{L} \subset T_L \), then we will say that \( \xi \) has central character \( W_{0}^r \mathcal{T}_{L} \). Let

\[
\pi(\xi) = \pi(\Pi_L, \delta, t^L) = \text{Ind}_{W_{0}^r \mathcal{T}_{L}}^{\mathcal{H}_{0}^L}(\delta \circ \phi_L).
\]

The central character of the irreducible components of \( \pi(\xi) \) is \( W_{0}^r \mathcal{T}_{L} \subset T \). Define

\[
W_{L,L} = \{ k \times w \in (T^L_L \cap T_L) \times W_0 \mid w(L) = L \},
\]

and (cf. \[4\] p. 34)

\[
W_{\xi, \xi} = \{ g = k \times w \in W_{L,L} \mid g \cdot \xi = (\Pi_L, \Psi_g(\delta), kw(t^L)) = \xi \}.
\]

(3.1)

This means the following, cf. \[16\]. If \( w(L) = L \), then \( w \) induces an automorphism of affine Hecke algebras \( \psi_w : \mathcal{H}_L \rightarrow \mathcal{H}_L \). If \( k \in K_L = T^L_L \cap T_L \) then we also have an automorphism \( \psi_k : \mathcal{H}_L \rightarrow \mathcal{H}_L \): \( \theta_{L}N_w \mapsto k(\bar{x})\theta_{L}N_w \) for \( \bar{x} \in X_L \) and \( w \in W_{L,L} \). Let \( g = k \times w \), and define \( \psi_g : \mathcal{H}_L \rightarrow \mathcal{H}_L \) to be the composition \( \psi_g = \psi_k \circ \psi_w \). This leads to a bijection \( \Psi_g : \Delta_{W_{L,L}} \rightarrow \Delta_{W_{L,L}w((L))} \) given by \( \Psi_g(\delta) = \delta \circ \psi_g^{-1} \).

Lemma 3.1. If \( R_0 \) is of type \( B_n \) and \( \xi = (\Pi_L, \delta, 1) \) has real central character, then \( \Psi_g \) is the identity map for all \( g \in \mathcal{W}_{\xi, \xi} \).

Proof: The fact that \( t^L = 1 \) implies \( k = 1 \) for all \( g = k \times w \in \mathcal{W}_{\xi, \xi} \). Suppose that the root system \( R_L \) is of type \( \prod L_i A_{\lambda_i} \times B_l \) for some \( \lambda_i, l \). Then \( w(L) = L \) implies that \( w \) acts as the identity on the part of type \( B_l \). On the parts of type \( A_l \), it acts as a combination of diagram automorphisms and permutations of \( A \)-factors of equal
rank. Since $W_L r_L$ is the central character of a discrete series representation of $\mathcal{H}_L$, it is the product of the central characters of discrete series representations of affine Hecke algebras of type $A_L$, and a central character of discrete series representations of an affine Hecke algebra of type $B_L$. An affine Hecke algebra of type $A$ which occurs as a factor in $\mathcal{H}_L$ has only one discrete series representation with real central character (the Steinberg representation). Therefore $\Psi_g$ acts trivially on $\Delta_{W_L r_L}$. □

It follows that, if $R_0$ has type $B_n$ and for $\xi$ with real central character, we have

$$W_{\xi,\xi} = \{(w, 1) \mid w(L) = L\}.$$  

In particular, $W_{\xi,\xi}$ depends only on $L$ and not on $\delta$.

Let $t^* = X \otimes \mathbb{C} \supset R_0$. For $L \subset I$, define

$$t^{L,\ast} = \{\lambda \in t^* \mid \lambda(\alpha) = 0 \text{ for all } \alpha \in L\}.$$  

**Definition 3.2.** (cf. [4] Definition 5.1) Let $\alpha \in R_0 \setminus R_L$, then we denote by $[\alpha]^L \in t^{L,\ast}$ the restriction of $\alpha \in t^*$ to $t^{L,\ast}$. Let $\beta = [\alpha]^L$ for $\alpha \in R_0 \setminus R_L$. Let $\xi = (\Pi_\xi, \delta, t^L) \in \Xi$ where $\delta$ has central character $W_L r_L$. Then we define

$$c_\beta(\xi) = \prod_{\{\alpha \mid [\alpha]^L \in \mathbb{R}_{>0}\}} c_\alpha(r_L t^L).$$  

**Lemma 3.3.** With the above notation, $c_\beta(\xi)$ does not depend on the choice of $r_L$ in its $W_L$-orbit.

**Proof:** Let $w \in W_L$. Then, for any root $\alpha$ we have $\alpha = \alpha|_{t_L} + \alpha|_{t_L}$. Since $R_L \subset t_L$, we have $w(\alpha) = \alpha|_{t_L} + w(\alpha)|_{t_L}$, hence $[w(\alpha)]^L = [\alpha]^L$. The lemma follows. □

**Definition 3.4.** Let $\beta = [\alpha]^L$ for $\alpha \in R_0 - R_L$. Then we say that $\beta$ is primitive, if for every $\alpha' \in R_0 - R_L$ we have $[\alpha']^L \in \mathbb{R}_{>0} \Rightarrow [\alpha'] \in \mathbb{Z}_{>0}$.

**Theorem 3.5.** ([4] Prop. 6.5) Let $\xi \in \Xi_u$. Let $\beta = [\alpha]^L$ be primitive. Then the pole order of $\xi$ is at most one. Moreover, the primitive $\beta = [\alpha]^L$ for which (3.3) has a pole form an integral root system in $t^{L,\ast}$. We will denote this root system by $R_0(\xi)$.

A Weyl group element $w \in W_0$ for which $w(L) = L$ acts on the restricted roots by $w(([\alpha]^L) = [w(\alpha)]^L$. Therefore one can define (cf. [4] Def. 6.8)

$$R(\xi) = \{g \in W_{\xi,\xi} \mid g(R_{0,+}(\xi)) = R_0(\xi)\}.$$  

Suppose that $\xi \in \Xi_u$. Then, according to [4] Prop. 6.7:

$$W_{\xi,\xi} \cong W_0(\xi) \rtimes R(\xi),$$

where $W_0(\xi)$ denotes the Weyl group of $R_0(\xi)$.

For every $w \in W_{\xi,\xi}$, Opdam has defined (cf. [10] 4.4.4) an intertwining operator $E(w, \xi)$ of $\pi(\xi)$.

**Theorem 3.6.** ([4] Theorem 6.3) Let $\xi \in \Xi_u$. If $w \in W_0(\xi)$ then $E(w, \xi) = \lambda d_{\pi(\xi)}$.

Moreover, the elements of $R(\xi)$ lead to the nontrivial $\mathcal{H}$-endomorphisms of $\pi(\xi)$ and determine its decomposition into irreducibles:

**Theorem 3.7.** ([4] Theorem 7.4) For all $\xi \in \Xi_u$, one has

- $\text{dim}(\text{End}_{\mathcal{H}}(\pi(\xi))) = \sum_{r \in R(\xi)} \text{dim}(E(\xi, r));$
- $\text{dim}(\text{End}_{\mathcal{H}}(\pi(\xi))) = |R(\xi)|.$
\[ E(r, \xi)E(r', \xi) = \eta_\xi(r, r')E(rr', \xi) \text{ for all } r, r' \in R(\xi), \text{ where } \eta_\xi \text{ is a 2-cocycle}. \]

The first property says that the intertwining operators \( E(r, \xi) \) with \( r \in R(\xi) \) span the space of intertwiners of \( \pi(\xi) \), and by the second property they are linearly independent, hence form a basis of this space.

If \( \eta_\xi \) splits, i.e., satisfies
\[
\eta_\xi(r, r') = \sigma_\xi(rr') \sigma_\xi(r)^{-1} \sigma_\xi(r')^{-1}
\]
for some \( \sigma_\xi : R(\xi) \to \mathbb{C}^* \), then by standard arguments (see [11]) one obtains from Theorem 3.7 a bijection between the irreducible representations of \( R(\xi) \) and the irreducible components of \( \pi(\xi) \). We will see below that if \( R_0 \) is of type \( B_n \) and \( \xi \in \Xi_n \) has real central character, then we are in this situation.

4. Conjecture on \( \hat{\mathcal{H}}_R^t \) for \( R_0 \) of type \( B \)

Given an orthonormal basis \( e_1, \ldots, e_n \) of \( Q \otimes_{\mathbb{Z}} \mathbb{R} \), we let \( R_0 = \{ \pm e_i \pm e_j \mid 1 \leq i, j \leq n \} \cup \{ \pm e_i \mid 1 \leq i \leq n \} \) and we choose as simple roots \( \alpha_1, \ldots, \alpha_n \) where \( \alpha_i = e_i - e_{i+1} \) for \( 1 \leq i < n \) and \( \alpha_n = e_n \). One has \( \theta = e_1 \). Define (cf. [20], noticing that \( \hat{\alpha}_1 \not\in 2Y \) for any choice of root datum),
\[
q_1 = q_{\hat{\alpha}_1}, q_2 = q_{\hat{\alpha}_n}q_{\hat{\alpha}_n/2}^{1/2}.
\]
We will always assume that \( q_1 \neq 1 \). Furthermore we assume that \( \text{rank}(X) = \text{rank}(Q) \), a necessary condition in order to have discrete series representations. Thus, \( X \) is either the root lattice \( Q \) or the weight lattice. In the latter case \( Y = \hat{Q} \) is the coroot lattice, hence \( R_{nr} = R_0 \) and all \( q_{\hat{\alpha}_i/2} = 1 \).

4.1. Residual points. First we recall some facts from [8] and [20] (see also [19]). Define (since we assume that \( q_1 \neq 1 \) \( m \) by \( q_2 = q_1^m \). We say that the parameters \( q_1, q_2 \) are \textit{generic} if \( m \not\in \pm \{0, 1/2, 1, 3/2, \ldots, n-1\} \). Otherwise, we call them \textit{special}. According to [8], the real residual points for \( \mathcal{H} \) are, for generic parameters, indexed by the partitions of \( n \). Given \( \lambda \vdash n \), let \( Y(\lambda) \) be its Young tableau. For a square \( \square \in Y(\lambda) \), let \( c(\square) \) be its content (its column coordinate minus its row coordinate).

Then we define \( c(\lambda; q_1, q_2) \) to be the point with coordinates \( q_1^{c(\square)} q_2 \). We do not specify an order on the coordinates, since we are interested in \( W_0 \)-orbits only.

For \( m \in \mathbb{R} \), we define for a partition \( \lambda \) its \( m \)-tableau \( T_m(\lambda) \) to be its Young tableau where box \( \square \) is filled with entry \( e_m(\square) = |c(\square) + m| \). An example for \( m = 2 \) is given in Figure 1. Then we recall from [20] the splitting map \( \mathcal{S}_m \):

\[
\begin{array}{cccccc}
2 & 3 & 4 & 5 & 6 & \\
1 & 2 & 3 & 4 & & \\
0 & 1 & 2 & 3 & & \\
1 & & & & & \\
\end{array}
\]

\textbf{Figure 1.} \( T_2(\lambda) \) for \( \lambda = (1445) \).
**Definition 4.1.** Let $\lambda \vdash n$. The splitting procedure $S_m$ divides $T_m(\lambda)$ into horizontal and vertical blocks, by subsequently enclosing the biggest remaining entry in the thus far unselected part into a horizontal or vertical box, in such a way that at every moment the thus far created boxes form the diagram of a partition.

Figure 2 contains, for $m = 1$, two examples of $\lambda$ such that $S_m$ is well-defined on $Y(\lambda)$.

![Figure 2](image)

**Figure 2. Examples of the splitting map**

The map $S_m$ is not well-defined for all $m$ and all $\lambda$: it may happen that the maximal remaining entry occurs twice. But we have:

**Proposition 4.2.** Let $q_1 > 0, q_2 > 0$ and $q_2 = q_1^n$. Then $c(\lambda; q_1, q_2)$ is a residual point if and only if $S_m$ is well-defined on $T_m(\lambda)$.

Let $P(n, 2)$ be the set of bipartitions $(\xi, \eta)$ of total weight $n$. If $S_m$ is well-defined on $\lambda$, then we define $S_m(\lambda) = (\xi, \eta) \in P(n, 2)$, where $\xi$ consists of the lengths of the horizontal blocks, and $\eta$ of the lengths of the vertical blocks in the split tableau. A $1 \times 1$-block $\Box$ is considered to be horizontal, resp. vertical, if $c(\Box) > -m$, resp. $c(\Box) < -m$ (that is, if $\Box$ lies above, resp. below, the zero-diagonal).

If $m \not\in \pm\{0, 1/2, 1, \ldots, n - 1\}$ then $S_m$ is well-defined for all $\lambda$. For special parameters, one may have $W_0c(\lambda; q_1, q_2) = W_0c(\mu; q_1, q_2)$ even if $\lambda \neq \mu$ and this point need not be residual.

### 4.2. Symbols and Springer correspondents

Conjecture 4.3 below is formulated using a generalization of the symbols which were introduced by Lusztig in [12] to describe the Springer correspondence. These symbols are defined as follows. Here and in the sequel, it is our convention to write the parts of a partition in increasing order: $\lambda = (0 \leq \lambda_1 \leq \lambda_2 \leq \ldots) = (0^r 1^s 2^t \ldots)$.

Let $m \in \mathbb{Z}$. Then the $m$-symbol $\Lambda^m(\xi, \eta)$ of $(\xi, \eta) \in P(n, 2)$ is the two-lined array whose top line contains $\xi_1, \xi_2 + 2, \xi_3 + 4, \ldots$ and whose bottom line contains $\eta_1, \eta_2 + 2, \eta_3 + 4, \ldots$, where we assume that $l(\xi) = l(\eta) + m$ by adding zeroes to $\xi$ or $\eta$, but such that $\eta_1$ or $\xi_1$ is non-zero. In the symbol we write the entry of $\xi_i$ to the left of the corresponding entry of $\eta_i$ if $m > 0$, and to the right of the corresponding entry of $\eta_i$ if $m < 0$. For $m = 0$, we have two symbols, the $+0$-symbol and the $-0$-symbol. For example, $\Lambda^2(12, 3), \Lambda^{+0}(12, 3), \Lambda^{-0}(12, 3)$ and $\Lambda^{-2}(12, 3)$ are, respectively

\[
\begin{pmatrix}
0 & 1 & 3 & 6 \\
1 & 0 & 4 & 5
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 4 & 5 \\
0 & 1 & 5 & 4
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 2 & 4 & 9 \\
0 & 1 & 2 & 4 & 9
\end{pmatrix}.
\]
If $m \notin \mathbb{Z}, m \in \frac{1}{2}\mathbb{Z}$, then the $m$-symbol $\Lambda^m(\xi, \eta)$ is defined similarly. We ensure that $l(\xi) = l(\eta) + sgn(m)(|m| + 1)/2$. The symbol $\Lambda^m(\xi, \eta)$ has entries $\xi_1, \xi_2 + 2, \xi_3 + 4, \ldots$ in the top row and $\eta_1 + 1, \eta_2 + 3, \eta_3 + 5, \ldots$ in the bottom row.

Suppose that $\Lambda^m(\xi, \eta)$ and $\Lambda^m(\alpha, \beta)$ have the same entries with the same multiplicities. Then we say that these symbols are similar, which we denote by $(\xi, \eta) \sim_m (\alpha, \beta)$. The similarity class in $\mathcal{P}(n, 2)$ under $\sim_m$ of $(\xi, \eta)$ is denoted by $[(\xi, \eta)]_m$.

4.2.1. Truncated induction. Let $(\xi, \eta) \in \mathcal{P}(n, 2)$. Then the $m$-symbol $\Lambda^m(\xi, \eta)$ can be written as $\Lambda^m(\xi, \eta) = (\xi, \eta) + \Lambda^m(0^{l(\xi)}, 0^{l(\eta)})$. We define

$$a_m(\xi, \eta) = \sum_{x, y \in \Lambda^m(\xi, \eta)} \min(x, y) - \sum_{x, y \in \Lambda^m(0^{l(\xi)}, 0^{l(\eta)})} \min(x, y).$$

In this summation, we sum over every pair of entries of the symbols. This function $a_m$ gives rise to truncated induction: if $W'$ is a subgroup of $W_0$ and $\chi'$ is a character of $W'$, let

$$\text{Ind}_{W'}^{W_0}(\chi') = \sum_{\chi \in W_0} n_{\chi', \chi} \chi.$$  

Suppose that $\chi_0 \in W_0$ with $n_{\chi', \chi_0} > 0$ is such that $n_{\chi', \chi} > 0$ implies $a_m(\chi) \leq a_m(\chi_0)$. Then we define

$$\text{tr}_{m} - \text{Ind}_{W'}^{W_0}(\chi') = \sum_{\chi : a_m(\chi) = a_m(\chi_0)} n_{\chi', \chi} \chi.$$  

In other words, we only keep the representations whose $a_m$-value is maximal among the representations which occur in the induction.

4.2.2. Definition of $\Sigma_m(W_0 r)$. Let $W_0 r$ be the central character of an element of $\mathcal{H}_G^L$. We recall the definition of Springer correspondents of $W_0 r$, as in [20]. Let $q_2 = q^m$. If $W_0 r$ is a residual point, then we define its Springer correspondents to be $\Sigma_m(W_0 r) = [\Sigma_m(\lambda)]_m$, where $W_0 r = W_0 c(\lambda; k, mk)$). It is shown in [20] that $\Sigma_m(W_0 r)$ is well-defined.

Otherwise, we can choose $r$ in its orbit $W_0 r$ such that $r = r_L$ is the center of a real residual coset $L$ for which $R_L$ is a standard parabolic root subsystem of $R_0$. Then $R_L$ is of type $A_{\kappa} \times B_l := A_{\kappa-1} \times A_{\kappa-1} \times \ldots A_{\kappa-1} \times B_l$ for some partition $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_r)$ of $n - l$. It is known that an affine Hecke algebra of type $A_{p-1}$ and parameter $q$ has one Weyl group orbit of residual points (assuming that $\text{rank}(X) = \text{rank}(Q)$), which is $W_0 r_p$ with

$$r_p(q) = (q^{-(p-1)/2}, q^{-(p-3)/2}, \ldots, q^{(p-3)/2}, q^{(p-1)/2}).$$

The associated discrete series representation of $\mathcal{H}(A_{p-1}, q)$ is one-dimensional, and its restriction to $\mathcal{H}_0(A_{p-1}, q)$ is, for $q > 1$, the sign representation. Therefore, we associate to $r_p$ a strip $S(p)$ with $\kappa$ boxes, which have entries $-((p-1)/2, -(p-3)/2, \ldots, (p-3)/2, (p-1)/2)$ (the exponents of $q$ in (44)). We will call such a strip an $A$-strip. Notice that its entries do not depend on $q$. We will also need the strip with the absolute values of the same entries, we denote it by $|S(p)|$ and also call it a (positive) $A$-strip.

The residual point $W_L r_L$ can be written as

$$W_L r_L(q_1, q_2) = r_{\kappa_1}(q_1) c(\mu; q_1, q_2) = r_{\kappa_1}(q_1) r_{\kappa_2}(q_1) \ldots r_{\kappa_r}(q_1) c(\mu; q_1, q_2),$$
where \( \mu \vdash l \) is such that \( c(\mu; q, q^m) \) is residual. By this notation, we mean that the first \( \kappa_1 \) coordinates of \( r_L \) are as in \((4.4)\) with \( p = \kappa_1 \), the next \( \kappa_2 \) coordinates are as in \((4.4)\) with \( p = \kappa_2 \), etc., until finally the last \( l \) coordinates are those of \( c(\mu; q_1, q_2) \).

Then we define, letting \((\kappa_1)\) denote the trivial representation of \( W_0(A_{\kappa_1}, -1) \):

\[
\Sigma_m(W_L r_L) = (\kappa_1) \otimes (\kappa_2) \otimes \cdots (\kappa_r) \otimes [\text{triv}_\mu]_m = \text{triv}_\mu \otimes [\text{triv}_\mu]_m,
\]

that is, we tensor the Springer correspondents of \( W_0(B_l) c(\mu; q_1, q_2) \) with the Springer correspondents of \( W_0(A_{\kappa_1}) r_{\kappa_1}(q_1) \). Finally, we let

\[
(4.6) \quad \Sigma_m(W_0 r_L) = \{ (\xi, \eta) \mid (\xi, \eta) \text{ occurs in } \text{tr}_{m} - \text{Ind}_{W_L}^{W_0}(\Sigma_m(W_L r_L)) \}.
\]

In other words, we truncatedly induce the Springer correspondents of \( W_L r_L \) to obtain those of \( W_0 r_L \).

It is shown in \([20]\) that (under the identification of bipartitions and irreducible characters of \( W_0 \)) we have, for all \( m \), a disjoint union

\[
(4.7) \quad \bigcup_{L \text{ real residual coset}} \Sigma_m(W_0 r_L) = \hat{W}_0.
\]

It is not hard to show that all \( \Sigma_m(W_0 r_L) \) are singletons unless \( q_1, q_2 \) are special parameters.

### 4.3. Conjecture on \( \hat{H}_L^t \). We can now formulate our conjecture on \( \hat{H}_L^t \). If \( M \) is a \( \hat{H}_0 \)-module, we denote by \( R(M) \) the corresponding \( W_0 \)-module which we obtain in the limit \( q_0^{1/2} \to 1 \).

**Conjecture 4.3.** ([20]) Let \( \mathcal{H} \) be the affine Hecke algebra with labels \( q_2 = q_1^m \) obtained from the \( q_0 \) as in \((4.4)\) and \( q_1 > 1, q_2 \geq 1 \). Then we have the following description of \( \hat{H}_L^t \). Let \( W_0 r \) be the \( W_0 \)-orbit of the center of a real residual coset of \( \mathcal{H} \).

(i) \( \Delta_{W_0 r} \) is indexed by \( \Sigma_m(W_0 r) \).

(ii) The modules in \( \Delta_{W_0 r} \) are naturally graded for the action of \( \hat{H}_0 \). The top non-zero degree of these modules is \( a_m(\Sigma_m(W_0 r)) \), and the \( \hat{H}_0 \)-representations in this degree are irreducible. The corresponding \( W_0 \)-representations are \( \Sigma_m(W_0 r) \otimes \epsilon \).

(iii) Let \( M_\chi \in \hat{H}_L^t \) be the \( \mathcal{H} \)-module labelled by \( \chi \in \hat{W}_0 \) using (ii) and \((4.7)\). That is, its top degree is \( R(M^{\text{top}}_\chi) \cong \chi \otimes \epsilon \). Let \( \succ_m \) be an ordering on \( W_0 \) which is a refinement of the pre-order given by \( \psi \succ \chi \Rightarrow a_m(\psi) \leq a_m(\chi) \), in which similarity classes form intervals. Then the matrix \( (R(M_\chi), \psi \psi, \chi)_{\psi, \chi} \in W_0 \) is block-lower triangular where the blocks are given by the similarity classes.

(iv) The \( \hat{H}_0 \)-structure of \( M_\chi \) can be computed with the generalized Green functions of Shoji (see \([18]\)).

The details of (iv), as well as the corresponding statements for arbitrary \( q_1, q_2 \) can be found in \([20]\). We remark that the conjecture is known to hold for \( m = 1 \), by the theory of Kazhdan-Lusztig (cf. \([11]\)).

### 4.4. Reducibility of induced discrete series. In this article we will show that \((4.3)\) for arbitrary central character follows from \((4.3)\) for residual points. Indeed, Conjecture \((4.3)\) implies in particular the number of irreducible components in an induced representation \( \pi(\xi) = \pi(\Pi_L, \delta, 1) \) for \( \delta \in \hat{H}_L^t \) as follows. Indeed, consider a residual coset \( L \) such that \( R_L \) has simple roots \( \Pi_L \). Let its center \( r_L \) be as in...
According to (4.31), the cardinality of $\Delta_{W_Lr_L}$ is equal to the cardinality of $\Sigma_m(W_Lr_L)$, and the cardinality of $\Delta_{W_0r_L}$ is equal to the cardinality of $\Sigma_m(W_0r_L)$. It is shown in [20] that we have

$$|\Sigma_m(W_0r_L)| = 2^d|\Sigma_m(W_Lr_L)|,$$

where $d$ is the number of $A$-strips of length $k_i$ ($i = 1, 2, \ldots, r$) which can be glued to $T_m(\mu)$ while maintaining the $m$-tableau of a partition. This number $d$ is independent of $\mu'$ in the set $\{\mu' : l \mid S_m(\mu') \sim_m S_m(\mu)\}$.

As a consequence, we expect that $\pi(\xi)$ has $2^d$ irreducible components. We give an example in Figure 3 for $m = 3$, $\kappa = (3, 4, 7, 11)$ and $\mu = (1, 1, 2, 3, 4)$. The upper part depicts the $A$-strips $|S(3)|, |S(4)|, |S(7)|, |S(11)|$ and the 3-tableau $T_3(\mu)$. As shown in the lower part of the Figure, two of the strips (those of $A_6$ and $A_{10}$) can be glued to $T_3(\mu)$, while this can not be done for the strips of $A_2$ and $A_3$ (the dashed lines). Therefore, $d = 2$ and we expect $\pi(\xi)$ to have four irreducible components.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{m = 3, $\kappa = (3, 4, 7, 11)$, $\mu = (1, 1, 2, 3, 4)$}
\end{figure}

5. Explicit calculation of the $R$-group

Let $R_L \subset R_0$ be a standard parabolic root system, then $R_L$ has type $A_\kappa \times B_l = A_{\kappa_1-1} \times A_{\kappa_2-1} \times \cdots \times A_{\kappa_r-1} \times B_l$, for some $l \geq 0$ and $\kappa = (\kappa_1, \ldots, \kappa_r) \vdash n - l$. The central character of a discrete series representation $\delta$ of $H_L$ is then given by $W_Lr_L$, with $r_L$ as in (4.5). Let $\kappa_0 = 0$, $\kappa_{r+1} = l$, and put $K_i = \kappa_0 + \kappa_1 + \cdots + \kappa_i$ for $i = 0, 1, \ldots, r + 1$. Define $L_i = \{\alpha_j \mid j = K_i - 1, K_i - 1 + 1, \ldots, K_i\}$ ($i = 1, \ldots, r$), $L_{r+1} = \{\alpha_j \mid j = K_r + 1, \ldots, n\}$ and $I_i = \{K_i - 1, \ldots, K_i\}$. Then $I = \{1, \ldots, n\} = \cup_i I_i$ and $L = \cup_i L_i$. Define $E_i = [\alpha_{K_i-1+1}]^L$ for $i = 1, 2, \ldots, r$, then $\{E_1, \ldots, E_r\}$ forms a basis of $t^L$. We begin by calculating the root system $R_0(\xi)$.

5.1. Calculation of $R_0(\xi)$. One checks that we have the following relations between the projections of $R_0 \setminus R_L$ onto $t^L$:
(i-a) Let $\alpha = e_{i_1} + e_{j_1}$ with $i_1 \in I_{p_1}, j_1 \in I_{p_2}, p_1 \neq r + 1, p_2 \neq r + 1, p_1 \neq p_2$. Then $[\alpha]^L = [e_{i_1} + e_{j_1}]^L$ for all $i_2 \in I_{p_1}, j_2 \in I_{p_2}$.

(i-b) Let $\alpha = e_{i_1} - e_{j_1}$ with $i_1 \in I_{p_1}, j_1 \in I_{p_2}, p_1 \neq r + 1, p_2 \neq r + 1, p_1 \neq p_2$. Then $[\alpha]^L = [e_{i_1} - e_{j_1}]^L$ for all $i_2 \in I_{p_1}, j_2 \in I_{p_2}$.

(ii) Let $\alpha = e_{i_1} + e_{j_1}$ with $i_1, j_1 \in I_p$ with $p \neq r + 1$ (so that $\alpha \notin R_L$). Then $[\alpha]^L = [e_{i_1} + e_{j_1}]^L = 2[e_{i_1}]^L = 2[e_{i_2} + e_{j_1}]^L$ for all $i_2, j_2 \in I_p, j_3 \in I_{r+1}$.

Clearly, $[e_{i_2}]^L$ is the primitive one.

To calculate the pole order of $c_\beta(r_L)$ for each primitive $\beta$, we note that

$$e_{K_{i-1}+d}(r_L) = e_d(r_{K_i})(q_1) = q_1^{-(\kappa_i-1)/2+(d-1)}, \quad i = 1, \ldots, r; \quad d = 1, \ldots, \kappa_i$$

and

$$e_{K_{i-1}+d}(r_L) = e_i(c(\mu); q_1, q_2) = q_1^{c(\square)}q_2; \quad i = 1, \ldots, l.$$ 

for some numbering $\square, 1, \ldots, \square$ of the boxes of $Y(\mu)$, and $c(\square)$ the content of $\square$.

We are going to compute the pole order of $c_\beta(r_L)$ for every primitive $\beta$. Notice that, since the central character $W_Lr_L$ is real, this amounts to calculating the pole order of (see 2.1)

$$C_\beta = \prod_{(\alpha)[\alpha] \in \mathbb{Z}_{>,0}\beta} \frac{1 - q_{\alpha}^{-1/2}q_\alpha^{-1}\theta_{-\alpha}}{1 - \theta_{-\alpha}},$$

when evaluated in $r_L$. We calculate $C_\beta(r_L)$ for each case above separately.

(i-a) Let $\alpha = e_{i_1} + e_{j_1}$ with $i \in I_{p_1}, j \in I_{p_2}, p_1 \neq p_2, p_1 \neq r + 1, p_2 \neq r + 1$. Then

$$C_\beta(r_L) = \prod_{d_1, d_2 = 1}^{\kappa_{p_1}, \kappa_{p_2}} \frac{1 - q_1^{-1}q_1^{-(\kappa_{p_1} - \kappa_{p_2})/2+(d_1-1)-\kappa_{p_2}/2+(d_2-1)}}{1 - q_1^{-(\kappa_{p_1} - \kappa_{p_2})/2+(d_1-1)-\kappa_{p_2}/2+(d_2-1)}}.$$

We claim that (5.2) has a pole only if $\kappa_{p_1} = \kappa_{p_2}$. Indeed, (5.2) has a pole for every $(d_1, d_2)$ such that $d_1 + d_2 = (\kappa_{p_1} + \kappa_{p_2})/2 + 1$ and a zero for every $(d_1, d_2)$ such that $d_1 + d_2 = (\kappa_{p_1} + \kappa_{p_2})/2 + 2$. Hence, if $\kappa_{p_1} + \kappa_{p_2}$ is odd then (5.2) does not have a pole. If $\kappa_{p_1} + \kappa_{p_2}$ is even, then write $\kappa_{p_2} = \kappa_{p_1} + 2a$. The poles in (5.2) arise for $\{1, \kappa_{p_1} + a\}, (2, \kappa_{p_1} + a - 1), \ldots, (\kappa_{p_1}, a + 1)\}$ and the zeroes at $\{1, \kappa_{p_1} + a + 1\}, (2, \kappa_{p_1} + a), \ldots, (\kappa_{p_1} + 2a, a + 1)\}$. 

(ii-b) Suppose $\beta = [e_{i_1} - e_{j_2}]$ and $i_1 \in I_{p_1}, j_1 \in I_{p_2}$, with $p_1 \neq p_2, p_1 \neq r + 1, p_2 \neq r + 1$. Then

$$C_\beta(r_L) = \prod_{d_1, d_2 = 1}^{p_{11}, p_{22}} \frac{1 - q_1^{-1}q_1^{-(\kappa_{p_1} - \kappa_{p_2})/2-(d_1-d_2)}}{1 - q_1^{-(\kappa_{p_1} - \kappa_{p_2})/2-(d_1-d_2)}}.$$

We claim that this expression has a pole only for $\kappa_{p_1} = \kappa_{p_2}$. The proof is identical to the one above: if $\kappa_{p_1} + \kappa_{p_2}$ is odd then there are no poles at all, hence suppose that $\kappa_{p_2} \geq \kappa_{p_1}$, say $\kappa_{p_2} = \kappa_{p_1} + 2a$ for some $a \in \mathbb{Z}_{>0}$. Then the poles are indexed by $(d_1, d_2) \in \{(a + 1, 1), (a + 2, 2), \ldots, (a + \kappa_{p_1}, \kappa_{p_2})\}$ and the zeroes by $(d_1, d_2) \in \{(a, 1), (a + 1, 1), \ldots, (a + \kappa_{p_1} - 1, \kappa_{p_1})\}$, except when $a = 0$ (i.e., when $\kappa_{p_1} = \kappa_{p_2}$), in which case $(a, 1)$ does not occur.

(ii) Let $\alpha = e_i$ for $i \in I_p, p \neq r + 1$ and let $\beta = [\alpha]^L$. Then

$$C_\beta(r_L) = \prod_{d = 1}^{\kappa_p} \frac{1 - q_1^{-1}q_1^{-(\kappa_p - 1)/2+(d-1)}}{1 - q_1^{-(\kappa_p - 1)/2+(d-1)}} \prod_{1 \leq d_1 < d_2 \leq \kappa_p} \frac{1 - q_1^{-1}q_1^{-(\kappa_p + d_1 + d_2 - 1)}}{1 - q_1^{-(\kappa_p + d_1 + d_2 - 1)}} \times$$

$$\prod_{d_1, d_2 = 1}^{d_1 + d_2, \kappa_p} \frac{1 - q_1^{-1}q_1^{-(\kappa_p - 1)/2-(d_1-d_2)}}{1 - q_1^{-(\kappa_p - 1)/2-(d_1-d_2)}}.$$
We write this expression as

\[(5.4)\]

\[C_\beta(r_L) = C_\beta(\kappa_p)C_\beta(\kappa_p, T_m(\mu)),\]

where \(C_\beta(\kappa_p)\) denotes the top line of \ref{5.3} and \(C_\beta(\kappa_p, T_m(\mu))\) the bottom line. Thus, \(C_\beta(\kappa_p)\) denotes the part of \ref{5.3} which corresponds to the factor \(A_{\kappa_p,-1}\), and \(C_\beta(\kappa_p, T_m(\mu))\) denotes the part of \ref{5.3} which calculates the “interaction” between \(A_{\kappa_p,-1}\) and \(T_m(\mu)\).

(ii-a) First we calculate the pole order of \(C_\beta(\kappa_p)\). In the left hand product of \(C_\beta(\kappa_p)\), we find a pole if and only if \(\kappa_p\) is odd. In the right hand side product, we have a pole for \((d_1, d_2) \in \{(1, \kappa_p), (2, \kappa_p - 1), \ldots, ([\frac{\kappa_p}{2}], [\frac{\kappa_p}{2}] + 1)\}\). The total number of poles in \(C_\beta(\kappa_p)\) is therefore \([\frac{\kappa_p}{2}]\).

Next, we count the number of zeroes in \(C_\beta(\kappa_p)\). In the left hand product, we find one zero factor if \(q_2 = q_1^m\) for \(m \in \{(\kappa_p - 1)/2, (\kappa_p - 3)/2, \ldots\}\), and none otherwise. In the right hand side we have, independently of the relation between \(q_1\) and \(q_2\), a zero for every \((d_1, d_2) \in \{(2, \kappa_p), (3, \kappa_p - 1), \ldots, ([\frac{\kappa_p}{2}], [\frac{\kappa_p}{2}])\}\), i.e., a total on the right hand side of \([\frac{\kappa_p}{2}] - 1 = [\frac{\kappa_p}{2}] - 1\) zeroes.

Thus,

\[(5.5)\]

\[C_\beta(\kappa_p)\text{ has a pole of order } \begin{cases} 0 & \text{if } q_2 = q_1^m, m \in \{(\kappa_p - 1)/2, (\kappa_p - 3)/2, \ldots\}, \\ 1 & \text{otherwise}. \end{cases}\]

(ii-b) Now we calculate the pole order of \(C_\beta(\kappa_p, T_m(\mu))\). Suppose that \(q_2 = q_1^m\) with \(m \in \frac{1}{2}\mathbb{Z}\) (otherwise the parameters are generic and \(C_\beta(\kappa_p, T_m(\mu))\) does not have a pole). Then \(C_\beta(\kappa_p, T_m(\mu)) = \)

\[(5.6)\]

\[
\prod_{d=1}^{\kappa_p} \prod_{i=1}^l \frac{1 - q_1^{-1}q_1^{-((\kappa_p - 1)/2 + (d - 1))c(\square)_i + m}}{1 - q_1^{-1}q_1^{-((\kappa_p - 1)/2 + (d - 1))c(\square)_i + m}} \frac{1 - q_1^{-1}q_1^{-((\kappa_p - 1)/2 + (d - 1)) - c(\square)_i + m}}{1 - q_1^{-1}q_1^{-((\kappa_p - 1)/2 + (d - 1)) - c(\square)_i + m}}.
\]

Notice that the \([c(\square)] + m\) are the entries of the \(m\)-tableau \(T_m(\mu)\).

Let \(S(\kappa_p)\) be the A-strip with entries \((-((\kappa_p - 1)/2), -((\kappa_p - 3)/2), \ldots, (\kappa_p - 3)/2, (\kappa_p - 1)/2) = (-z, \ldots, z)\). For \(\square \in S(\kappa_p)\) a square of the A-strip, let \(c(\square)\) be its entry. So, \(c(\square) = c_{K_p - 1 + i}(r_L) = q_1^{-2i+1}(i = 1, \ldots, \kappa_p)\). For a square \(\square\) of \(Y(\lambda)\), let \(e_m(\square)\) be its entry in \(T_m(\lambda)\). With these notations,

\[(5.7)\]

\[
C_\beta(\kappa_p, T_m(\mu)) = \prod_{\square \in S(\kappa_p)} \prod_{\square' \in T_m(\mu)} \frac{1 - q_1^{-1}q_1^{-c(\square) + e_m(\square') + e_m(\square)}}{1 - q_1^{-1}q_1^{-c(\square) + e_m(\square') + e_m(\square)}} \frac{1 - q_1^{-1}q_1^{-c(\square) - e_m(\square')} - e_m(\square')}}{1 - q_1^{-1}q_1^{-c(\square) - e_m(\square') - e_m(\square')}}.
\]

Thus, \(C_\beta(\kappa_p, T_m(\mu))\) has a pole for every \(\square \in S(\kappa_p)\), \(\square' \in T_m(\mu)\) such that \(c(\square) = \pm e_m(\square')\) (where, if \(e_m(\square') = 0\), we count this equality twice); and a zero for every \(\square \in S(\kappa_p)\), \(\square' \in T_m(\mu)\) such that \(c(\square) \pm e_m(\square') = 1\) (where again, we count twice if \(e_m(\square') = 0\)). Now recall the splitting map \(S_m\) which divides \(T_m(\mu)\) into horizontal and vertical blocks. We write \(B \in S_m(\mu)\) to denote that \(B\) is one of the blocks into
which $T_m(\mu)$ is partitioned by $S_m$. We can thus write

$$C_\beta(\kappa_p, T_m(\mu)) = \prod_{\square \in S(\kappa_p)} \prod_{B \in S_m(\mu)} \prod_{\square' \in B} \frac{1 - q_1^{-1}q_1^{-e(\square) + e_m(\square')}}{1 - q_1^{-e(\square) + e_m(\square')}} \frac{1 - q_1^{-1}q_1^{-e(\square') - e_m(\square')}}{1 - q_1^{-e(\square') - e_m(\square')}}$$

$$= \prod_{B \in S_m(\mu)} C_\beta(\kappa_p, B).$$

We can thus calculate the pole-order of $C_\beta(r_L)$ as the sum of the pole orders of $C_\beta(\kappa_p)$ and those of the $C_\beta(\kappa_p, B)$. We recall from [20] that we may choose $\mu$ such that the splitting procedure $S_m$ first selects $m$ horizontal blocks, and then alternatingly a vertical and a horizontal block. This means that we can suppose that a block $B$ contains entries $(x, x+1, \ldots, y)$ with $0 \leq x \leq y$. However, $x = y = 0$ does not occur since $T_m(\mu)$ does not contain a block of length one containing a zero.

Denote by $B$ one of these blocks. Then the pole order of $C_\beta(\kappa_p, B)$ is given by

$$|P(\kappa_p, B)| - |Z(\kappa_p, B)|,$$

where

$$P(\kappa_p, B) = \{(e(\square), e_m(\square'), \varepsilon) \mid \square \in S(\kappa_p), \square' \in B, \varepsilon \in \{+,-\}, e(\square) + e_m(\square') = 0\},$$

$$Z(\kappa_p, B) = \{(e(\square), e_m(\square'), \varepsilon) \mid \square \in S(\kappa_p), \square' \in B, \varepsilon \in \{+,-\}, e(\square) + e_m(\square') = 1\}.$$

We calculate $P(\kappa_p, B)$ and $Z(\kappa_p, B)$ by a case by case analysis. Recall $z = (\kappa_p - 1)/2$. Clearly, if $x - z \notin \mathbb{Z}$ then $P(\kappa_p, B) = Z(\kappa_p, B) = \emptyset$ since $e(\square) \pm e_m(\square') \notin \mathbb{Z}$. Otherwise,

- If $z < x - 1$ then $P(\kappa_p, B) = Z(\kappa_p, B) = \emptyset$.
- If $z = x - 1$ then $P(\kappa_p, B) = \{(-z, x, +)\}$ and $P(\kappa_p, B) = \emptyset$.
- If $x \leq z < y$ then $P(\kappa_p, B) = \{(\pm z, z, \mp), (\pm z-1, z-1, \mp), \ldots, (\pm x, x, \mp)\}$ and $Z(\kappa_p, B) = \{(-z, z+1, +), (-z+1, z, +), \ldots, (-x+1, x, +)\}$.

We find a pole of order one.

- If $z > y$, then $P(\kappa_p, B) = \{(\pm y, y, \mp), (\pm y-1, y-1, \mp), \ldots, (\pm x, x, \mp)\}$ and $Z(\kappa_p, B) = \{(y-2, y-1, +), (y-1, y, +), \ldots, (x-1, x, +)\}$.

We find a pole of order one.

Summarizing, we have:

**Lemma 5.1.** Let $z = (\kappa_p - 1)/2$. Let $B \in S_m(\mu)$ be a block with entries $(x, x+1, \ldots, y)$ with $0 \leq x \leq y$. Then $C_\beta(\kappa_p, B)$ has pole order

$$\begin{align*}
-1 & \text{ if } z = x - 1 \\
0 & \text{ if } z \notin \{x - 1, y\} \\
1 & \text{ if } z = y.
\end{align*}$$

This enables us to show

**Proposition 5.2.** Let $(\Pi_L, \delta, 1) \in \Xi$ have real central character $W_{L, \ell L}$ with $r_L$ as in [3]. Consider the primitive vector $\beta = E_p = [e_{K_{p-1}+1}] \in \ell_t^{T_m}$. Then $C_\beta$ has a pole in $r_L$ unless a (positive) $A$-strip $|S(\kappa_p)|$ of length $\kappa_p$ can be glued to $T_m(\mu)$. 

**Proof:** Let $z = (\kappa_p - 1)/2$. If $m - z \notin \mathbb{Z}$, then $S(\kappa_p)$ can certainly not be glued to $T_m(\lambda)$. Recall that we decompose $C_\beta(r_L) = C_\beta(\kappa_p)C_\beta(\kappa_p, T_m(\mu))$. By [16],
$C_\beta(\kappa_p)$ has a pole whereas by Lemma 5.1, $C_\beta(\kappa_p, T_m(\mu))$ has pole order zero. Thus $C_\beta(r_L)$ has a pole of order one as required.

We therefore assume that $m - z \in \mathbb{Z}$. If $l = 0$, then $T_m(\mu)$ is the empty tableau, so $C_\beta(r_L) = C_\beta(\kappa_p)$. One can place a strip of length $\kappa_p$ on the empty tableau if and only if $(\kappa_p - 1)/2 \geq m$. By (5.4), this can be done if and only if $C_\beta(\kappa_p)$ does not have a pole.

Now suppose $l > 0$.

(i) Suppose that $|S(\kappa_p)|$ can not be glued to $T_m(\lambda)$. Then there are two possibilities.

(i-a) We have $z < m - l(\mu)$. In particular, $l(\mu) < m$ and thus $T_m(\mu)$ contains only horizontal blocks: we have $S_m(\mu) = (\mu, -)$. These blocks have initial entries $m, m - 1, \ldots, m - l(\mu) + 1$. Since $z < m$, $C_\beta(\kappa_p)$ has a pole by (5.3). By Lemma 5.1, $C_\beta(\kappa_p, T_m(\mu))$ has pole order zero, so $C_\beta$ has a pole of order one in $r_L$.

(i-b) $T_m(\mu)$ contains a (unique) block $B$ whose last entry is $z$. In this case, $C_\beta(\kappa_p, B)$ has a pole of order one. If $m > z$ then $C_\beta(\kappa_p)$ has a pole, but $m > z$ implies that there exists a block $B'$ of $T_m(\mu)$ which starts on $z + 1$. If $z > 0$ then this block is unique and the total pole order of $C_\beta(r)$ is one. If $z = 0$, then $B$ has last entry zero, which is impossible. If $m \leq z$ then $C_\beta(\kappa_p)$ does not have a pole. But $m \leq z$ implies that no block starts on $z + 1$, so again $C_\beta(r)$ has a pole of order one.

(ii) Now suppose that $|S(\kappa_p)|$ can be glued to $T_m(\lambda)$. Then no block in $T_m(\mu)$ ends on $z$. If $m \leq z$, then by (5.2), $C_\beta(\kappa_p)$ has pole order zero. There is no block $B'$ in $T_m(\mu)$ which starts on $z + 1 \geq m + 1$, hence $C_\beta(\kappa_p, B')$ has pole order zero for all $B'$. Thus, $C_\beta(r_L)$ does not have a pole.

If $m > z$, then $C_\beta(\kappa_p)$ has a pole. But $m > z$ implies that $T_m(\mu)$ contains a block $B'$ which starts on $z + 1 \leq m$, and so $C_\beta(\kappa_p, B')$ cancels the pole of $C_\beta(\kappa_p)$. Hence, $C_\beta(r_L)$ does not have a pole. \[\square\]

5.1.1. Description of $R_0(\xi)$. We can now describe the root system $R_0(\xi)$. We have seen that $E_i \pm E_j \in R_0(\xi)$ if and only if $\kappa_i = \kappa_j$, and $E_i \in R_0(\xi)$ unless an $A$-strip $|S(\kappa_i)|$ can be glued to $T_m(\mu)$. Therefore, we have

**Proposition 5.3.** Write

\[\sum_{\nu \in \Delta} \langle \kappa, \nu \rangle = (l_1^\nu, \ldots, l_s^\nu)\]

by grouping the equal $\kappa_i$ together. Then the root system $R_0(\xi)$ is of type

\[R_0(\xi) \cong R_0(X_{r_1}) \times R_0(X_{r_2}) \times \cdots \times R_0(X_{r_s}),\]

where

- $X_{r_i} = \begin{cases} D_{r_i} & \text{if a strip of length } l_i \text{ fits into } T_m(\mu), \\ B_{r_i} & \text{if not}. \end{cases}$

In this formula, we use the convention that $R_0(D_1)$ is the empty root system, while for $n \geq 2$, the root system of type $D_n$ on an orthogonal basis $\{e_1, \ldots, e_n\}$ is given by the vectors $\{ \pm e_i \pm e_j \mid i \neq j \}$.\[\]

5.2. Calculation of $R(\xi)$. We are going to compute

\[R(\xi) = \{ w \in W_0 \mid w(L) = L, w(R_0^+(\xi)) = R_0^+(\xi) \}.\]
Lemma 5.4. Let \( w \in R(\xi) \) be such that \( w(R_0^+(X_{r_j})) = R_0^+(X_{r_k}) \) where \( X \in \{B, D\} \). Then \( j = k \).

Proof: Suppose \( j \neq k \). By construction, \( l_j \neq l_k \). Let \( r_j = r_k = p \). Then we may suppose that the root systems \( R_0(X_p) \) are realised on basis vectors \( E_1, \ldots, E_p \) (for \( \kappa_1 = \cdots = \kappa_p = l_j \)) and \( E_{p+1}, \ldots, E_{2p} \) (for \( \kappa_{p+1} = \cdots = \kappa_{2p} = l_k \)).

If \( X = D \), then \( w(\{E_i \pm E_j \mid 1 \leq i < j \leq p\}) = \{E_i \pm E_j \mid p + 1 \leq i < j \leq 2p\} \). Moreover, \( w \in W_0 \) so \( w(E_i) \in \{E_1, \ldots, E_{2p}\} \) for all \( 1 \leq i \leq 2p \). This means that we have \( w(E_i) = E_{p+i} \) for all \( 1 \leq i < p \) and \( w(E_p) = \pm E_{2p} \).

Thus, \( [w(e_1)]^L = E_{p+1} \), so \( w(e_1) = e_t \) for \( t \geq pl_j + 1 \). But \( w(e_1) = w(\alpha_1) + w(e_2) \), and \( w(\alpha_1) = \alpha_s \) with \( s < pl_j \). Thus, \( w(e_2) = e_4 - \alpha_s \notin R_0 \), which is a contradiction.

If \( X = B \), then similarly, we find that \( w(E_i) = E_{p+i} \) for all \( 1 \leq i \leq p \) which also leads to a contradiction. Thus, \( j = k \). \( \square \)

It follows that if \( w \in R(\xi) \), then \( w(L_j) = L_j \) for all \( j \) (i.e., the parts of equal length in \( L \) are not interchanged). In particular, if \( w \in R(\xi) \) then \( w(E_i) = \pm E_i \) for all \( i = 1, 2, \ldots, r \).

Proposition 5.5. If \( R_0(\xi) \) contains only root systems of type \( B \), then \( R(\xi) = \{1\} \).

Let \( w \in R(\xi) \). We want to show that \( w = 1 \), and therefore it suffices to show that \( w = 1 \) on all \( L_i \) separately, which we prove using (downward) induction on \( i \).

(i) Suppose \( i = r+1 \). Then \( R_{L_{r+1}} \) is a root system of type \( B_1 \), hence \( w(L_{r+1}) = L_{r+1} \) implies \( w = 1 \) on \( R_{L_{r+1}} \).

(ii) Now suppose that \( w = 1 \) on \( \cup_{j>i+1}I_j \). Then we consider \( e_{K,i+1} = \alpha_{K,i+1} + \cdots + \alpha_n \). One has \( w(e_{K,i+1}) = w(\alpha_{K,i+1} + \alpha_{K,i+2} + \cdots + \alpha_{K,i+1-1} + \alpha_{K,i+1} + \alpha_{K,i+1+1} + \cdots + \alpha_n) = \alpha_{K,i+1} + \alpha_{K,i+2} + \cdots + \alpha_{K,i+1-1} + w(\alpha_{K,i+1}) + \alpha_{K,i+1+1} + \cdots + \alpha_n \). On the other hand, since \( w(E_i) = E_i \), we have \( w(e_{K,i+1}) = e_j \) for some \( j \in K_i + 1, \ldots, K_{i+1} \). Now, \( w(\alpha_{K,i+1}) = w(e_{K,i+1}) - (\alpha_{K,i+1} + \cdots + \alpha_{K,i+1-1}) - (\alpha_{K,i+1} + \cdots + \alpha_n) = e_j - (e_{K,i+1} - e_{K,i+1} ) - e_{K,i+1} \) must be a root. Therefore, \( j = K_i + 1 \), \( w(e_{K,i+1}) = e_{K,i+1} \) and \( w(\alpha_{K,i+1}) = \alpha_{K,i+1} \). Since we already knew that \( w(L_{i+1}) = L_{i+1} \), it follows that \( w(I_{i+1}) = I_{i+1} \). But then \( w(\alpha_{K,i+1}) = \alpha_{K,i+1} \) implies that \( w = 1 \) on \( I_{i+1} \). This proves the induction step and therefore the claim. \( \square \)

It remains to calculate \( R(\xi) \) for \( R_0(\xi) \) which contains a root system of type \( D \). The idea is that \( w \) in \( R(\xi) \) can interchange the order of the simple roots on the last part of \( L \) which gives a basis vector for the type \( D \) factor in \( \xi \). For example, if \( L \) is

\[
\begin{array}{ccccccccc}
\bullet & - & - & - & o & - & o & - & \circ & - & o & - & \circ & - & o & = & \circ
\end{array}
\]

then \( t^L \) has basis vectors \( E_1, E_2, E_3 \). If \( q_2 = q_1 \), then \( R_0^+(\xi) = \{E_1 \pm E_2, E_1 \pm E_3, E_2 \pm E_3\} \) (cf. \( 5.4 \)). As seen in the proof of Lemma 5.4 an element \( w \in R(\xi) \) must fix \( E_1 \) and \( E_2 \) and may send \( E_3 \) to \( \pm E_3 \). Together with \( w(L) = L \), we will see that this implies that if \( w(E_3) = E_3 \) then \( w = 1 \), and if \( w(E_3) = -E_3 \) then \( w \) interchanges the last two simple roots in \( L \) (\( \alpha_7 \) and \( \alpha_8 \)).

Theorem 5.6. Let \( \xi = (\Pi_L, \delta, 1) \in \Xi \) have real central character \( W_L r_L \) as in \( 5.5 \). Suppose \( R_L \) is of type \( A_r \times B_1 \). Then

\[ R(\xi) \simeq \mathbb{Z}_2 \]
where \( d \) is the number of A-strips \(|S(\kappa_i)|\) which can be glued to \( T_m(\lambda) \).

**Proof:** By conjugating, we may assume that the root systems of type \( D \) are in the beginning of the Dynkin diagram (i.e., arise on \( \kappa_1, \kappa_2, \ldots \)), so by Proposition 5.4 we may assume that \( w(\alpha_i) = \alpha_i \) for all \( \alpha_i \) not in these \( D \)-factors. Otherwise stated, we can suppose for the calculation that \( R_0(\xi) \) only contains \( D \)-factors.

By Lemma 5.4, if \( w \in R(\xi) \), then \( w \) fixes the positive roots in each irreducible factor of \( R_0(\xi) \) separately.

(1) We first consider the case where all \( A \)-factors are of equal rank, i.e., \( \kappa = (l')^2 \) and \( R_0(\xi) = R_0(L_r) \). Recall the basis vectors \( E_1, \ldots, E_r \) of \( L^r \). Suppose \( w \in R(\xi) \). Then, as seen in (the proof of) Lemma 5.4, we have \( w(E_i) = E_i \) for all \( i < r \) and \( w(E_r) = \pm E_r \). If \( w(E_r) = E_r \), we have seen in Proposition 5.5 that \( w = 1 \). We thus assume that \( w(E_r) = -E_r \) and show that this uniquely determines \( 1 \neq w \in R(\xi) \). We proceed in several steps.

(1a) Let \( \beta_r = e_{(r-1)l+1} \), that is, \( \beta_r \) is the first coordinate vector of the last copy of \( A_{l-1} \) in \( R_L \). Then \( [\beta_r]^L = E_r \), hence \( w(\beta_r)^L = -[\beta_r]^L \). This means that \( w(\beta_r) = -e_i \) for some \((r-1)l + 1 \leq i \leq rl \). We have, since \( w(L_i) = L_i \) for all \( L_i \):

\[
w(\beta_r) = w(\alpha_{(r-1)l+1} + \cdots + \alpha_{rl-1} + \alpha_{rl} + e_{rl+1}) = \alpha_{(r-1)l+1} + \cdots + \alpha_{rl-1} + w(\alpha_{rl}) + e_{rl+1} = \beta_r + w(\alpha_{rl}) - \alpha_{rl}.
\]

Thus, \( R_0 \ni w(\alpha_{rl}) = w(\beta_r) - \beta_r + \alpha_{rl} = -e_i - e_{(r-1)l+1} + e_{rl} - e_{rl+1} \), which implies that \( i = rl \). Thus, \( w(\beta_r) = -e_i \) and \( w(\alpha_{rl}) = -e_{(r-1)l+1} - e_{rl+1} \). This implies the values of \( w \) on \( L_r \), since \( w(\alpha_{rl-1} + \alpha_{rl}) = w(\alpha_{rl-1}) + w(\alpha_{rl}) = e_i - e_{i+1} - e_{(r-1)l+1} - e_{rl+1} = 0 \) for all \( i \in \{ (r-1)l + 1, rl - 1 \} \). It follows that \( w(\alpha_{rl-1}) = \alpha_{(r-1)l+1} \), and analogously that \( w(\alpha_{rl-j}) = \alpha_{(r-1)l+j} \) for all \( j \). Thus, \( w \) acts on \( L_r \) by flipping it to its mirror image.

(1b) Let \( \beta_{r-1} = e_{(r-2)l+1} \), that is, \( \beta_{r-1} \) is the first coordinate vector of the last-but-one copy of \( A_{l-1} \) in \( R_L \). Then \( [\beta_{r-1}]^L = E_{r-1} \), hence \( w(\beta_{r-1})^L = [\beta_{r-1}]^L \). This means that \( w(\beta_{r-1}) = e_i \) for some \((r-2)l + 1 \leq i \leq (r-1)l \). We have, since \( w(L_i) = L_i \) for all \( L_i \):

\[
w(\beta_{r-1}) = w(\alpha_{(r-2)l+1} + \cdots + \alpha_{(r-1)l+1} + \alpha_{(r-1)l} + \beta_r) = \alpha_{(r-2)l+1} + \cdots + \alpha_{(r-1)l+1} + w(\alpha_{(r-1)l}) - e_{rl}.
\]

Thus, \( R_0 \ni w(\alpha_{(r-1)l}) = w(\beta_{r-1}) - e_{(r-2)l+1} + e_{(r-1)l} + e_{rl} \), which implies that \( i = (r-2)l + 1 \). Thus, \( w(\beta_{r-1}) = \beta_r - e_{rl} \). This implies the values of \( w \) on \( L_{r-1} \), since \( w(\alpha_{(r-1)l-1} + \alpha_{(r-1)l}) = w(\alpha_{(r-1)l-1}) + w(\alpha_{(r-1)l}) = e_i - e_{i+1} + e_{(r-1)l} + e_{rl} = 0 \) for all \( i \in \{ (r-2)l + 1, (r-1)l - 1 \} \). It follows that \( w(\alpha_{(r-1)l-1}) = \alpha_{(r-1)l-1} \), and analogously that \( w(\alpha_{(r-1)l-j}) = \alpha_{(r-1)l-j} \) for all \( j \). Thus, \( w \) acts on \( L_{r-1} \) as the identity.

(1c) Let \( \beta_{r-2} = e_{(r-3)l+1} \). Then \( [\beta_{r-2}]^L = E_{r-2} \), hence \( w(\beta_{r-2})^L = [\beta_{r-2}]^L \). This means that \( w(\beta_{r-2}) = e_i \) for some \((r-3)l + 1 \leq i \leq (r-2)l \). We have, since \( w(L_i) = L_i \) for all \( L_i \):

\[
w(\beta_{r-2}) = w(\alpha_{(r-3)l+1} + \cdots + \alpha_{(r-2)l-1} + \alpha_{(r-2)l} + \beta_r) = \alpha_{(r-3)l+1} + \cdots + \alpha_{(r-2)l-1} + w(\alpha_{(r-2)l}) + e_{(r-2)l+1}.
\]

Thus, \( R_0 \ni w(\alpha_{(r-2)l}) = w(\beta_{r-2}) - e_{(r-3)l+1} + e_{(r-2)l} - e_{(r-2)l+1} \), which implies that \( i = (r-3)l + 1 \). Thus, \( w(\alpha_{(r-2)l}) = e_{(r-2)l} + e_{(r-2)l+1} \) and \( w(\beta_{r-2}) = \beta_r - e_{rl} \).
This implies the values of \( w \) on \( L_{r-2} \), as before. We find that \( w \) acts on \( L_{r-1} \) as the identity.

(1d) It is clear that by repetition of this argument, one finds that \( w \) is uniquely determined, and that \( w(L_i) = L_i \) pointwise on every \( i = 1, \ldots, r-1 \). From the description of \( w \) in (1a-c) one sees easily that \( w(e_i) = e_i \) except when \((r-1)l+1 \leq i \leq rl\), for which we have \( w(e_{r-1}(j+1)) = -e_{r-l-j} \) \((j = 1, \ldots, l) \). It follows that \( w \) is an involution, and so \( R(\xi) \simeq \mathbb{Z}_2 \).

(2) Now we consider the general case where \( \kappa = (l_1^r \cdots l_s^r) \). Let \( N_i = \sum_{j=1}^{i-1} l_is_i \) for \( 2 \leq i \leq s+1 \) and \( N_1 = 0 \). Then \( e_{N_i+1} \) corresponds to the first coordinate vector of the first factor of type \( A_{l_i-1} \). In part (1) we have seen that \( w \in R(\xi) \) satisfies \( w(e_{N_i+1}) = e_{N_i+1} \) (if \( r_s > 1 \)) or \( w(e_{N_i+1}) = -e_{N_i+1} \) (if \( r_s = 1 \)). It is easy to check that this implies that we can repeat the arguments (1a-d) on each root system of type \( D \) in \( R(\xi) \) separately, to obtain for every root system of type \( D_r \) in \( R_0(\xi) \) (including the empty ones, which correspond to \( r \)) and as the identity on the others. It is easy to see from the construction of the \( w_i \) that they commute. We have thus found, for every \( l_i \), such that a strip \( S(l_i) \) can be glued to \( T_{m}(\mu) \), an involution \( w_i \in R(\xi) \) and these commute with each other. We conclude that \( R(\xi) \simeq \mathbb{Z}_2^d \), where \( d \) is the number of type \( D \) root systems in \( R_0(\xi) \) (including the empty ones, see \( \text{[3.21]} \)), which is equal to the number of \( A \)-strips of length \( \kappa \), which can be glued to \( T_{m}(\mu) \).

\[ \square \]

**Remark 5.7.** Notice that if \( l_i = 1 \) (resp. \( l_i = 2 \)), then the corresponding \( L_{l_i} \) are empty (resp. consist of a single root). In these cases, \( w_i \in R(\xi) \) does not correspond to a diagram automorphism. For example, in the case of minimal principal series where \( \kappa = (1^s) \) we have \( R_0(\xi) = R_0 \), unless \( m = 0 \) in which case \( R_0(\xi) = D_n \) and hence \( R(\xi) \simeq \mathbb{Z}_2 \).

### 5.3. Counting irreducible components

In this section we will show that the induced representation \( \pi(\xi) \) as in the above theorem, decomposes into \( 2^d \) irreducible components.

Suppose that \( R(\xi) \simeq \mathbb{Z}_2^d \). We compute \( H^2(R(\xi), \mathbb{C}^*) \). It is well known that for a direct product of finite abelian groups \( G_1, G_2 \), one has

\[
H^2(G_1 \times G_2, \mathbb{C}^*) \simeq H^2(G_1, \mathbb{C}^*) \times H^2(G_2, \mathbb{C}^*) \times (G_1 \otimes G_2).
\]

Furthermore, for a cyclic group \( G \) it is known that \( H^2(G, \mathbb{C}^*) = 0 \). Since \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \simeq \mathbb{Z}_2 \), it is not hard to see that

\[
H^2(\mathbb{Z}_2^d, \mathbb{C}^*) \simeq \mathbb{Z}_2^{(d-1)/2}.
\]

Let \( \eta_\xi \) be the 2-cocycle of Theorem \( \text{[3.21]} \). We recall from \([1] \text{ p. } 87\) how to obtain from \( \eta_\xi \) a parametrization of the irreducible constituents of \( \pi(\xi) \). One considers a central extension \( R'(\xi) \) of \( R(\xi) \) such that \( \eta_\xi \) splits when pulled back to \( R'(\xi) \).

\[
1 \to H \to R'(\xi) \to R(\xi) \to 1.
\]

We may take \( H \) to be the cyclic group generated by \([\eta_\xi]\), the image of \( \eta_\xi \) in \( H^2(R(\xi), \mathbb{C}^*) \). Choose a function \( \sigma_\xi : R'(\xi) \to \mathbb{C}^* \) which splits \( \eta_\xi \) over \( R'(\xi) \), i.e.,

\[
\eta_\xi(r'_1, r'_2) = \sigma_\xi(r'_1 r'_2) \sigma_\xi(r'_1)^{-1} \sigma_\xi(r'_2)^{-1}; \ r'_1, r'_2 \in R'(\xi).
\]
Here we have written, by abuse of notation, \( \eta \) for the pull-back of \( \eta \) to \( R'(\xi) \times R'(:) \). It follows easily that \( \sigma \) satisfies \( \sigma(zr') = \chi(z)\sigma(r') \) for all \( z \in H, r' \in R'(\xi) \). The irreducible characters of \( R'(\chi) \) with central character \( \chi^{-1} \) on \( H \) are the ones which index the irreducible components of \( \pi(\xi) \), cf. [II p. 87].

Clearly, since \( [\eta] \in H^2(R(\xi), C^*) \simeq Z_2^d \), we either have \( H \) is trivial or \( \pi \simeq Z_2 \). If \( H \) is trivial, then we obtain a bijection between the irreducible components of \( \pi(\xi) \) and the irreducible characters of \( R(\xi) \), in which case we are done. Thus, let us consider the second case.

**Proposition 5.8.** Consider a central extension
\[
(5.12) \quad 1 \to H \to G \to Z_2^n \to 1.
\]
where \( H \simeq Z_2 \). Denote the elements of \( H \) by \( \pm 1 \), and identify them with the corresponding elements in \( G \). Then we have, for every irreducible character \( \chi \in \hat{G} \), \( \chi(-1) = \pm \chi(1) \) and
\[
(5.13) \quad |\{ \chi \in \hat{G} \mid \chi(1) = \chi(-1) \}| = 2^n.
\]
Moreover, all characters in the above set correspond to one-dimensional representations.

**Proof:** By Schur’s lemma, the elements of \( H \) must act by scalar multiplication. Therefore the first statement follows.

Since \( G \) is a finite group we have
\[
\sum_{\chi \in \hat{G}} \chi(1)\chi(g) = \chi_{\text{reg}}(g),
\]
where \( \chi_{\text{reg}} \) is the character of the regular representation of \( G \). Applied with \( g = 1 \) and \( g = -1 \) and summing up, we get
\[
\sum_{\chi \in \hat{G}} \chi(1)(\chi(1) + \chi(-1)) = |G| = 2^{n+1}.
\]
In view of \( \chi(1) = \pm \chi(-1) \), we get
\[
\sum_{\chi \in \hat{G} : \chi(-1) = \chi(1)} \chi(1)^2 = 2^n.
\]

By construction \( G/H \) is isomorphic to \( Z_2^n \). Thus, by pulling back the \( 2^n \) irreducible one-dimensional representations of \( G/H \) to \( H \), we obtain \( 2^n \) one-dimensional representations of \( G \) which are trivial on \( H \).

By the above formula, there can be no others and the proposition follows. □

**Corollary 5.9.** Consider the central extension \( 5.12 \). Let \( N_+(G) := |\{ \chi \in \hat{G} \mid \chi(-1) = \pm \chi(1) \}| \). Then \( N_+(G) = 2^n \) if and only if \( G \) is abelian. Let us therefore suppose that \( G \) is not abelian.

Since \( [G, G] \) is mapped into \( [Z_2^n, Z_2] = 1 \), we have \( [G, G] \subset \ker(\pi) = \text{im}(i) \) where \( \pi : G \to Z_2^n \) is the projection map and \( i : Z_2 \to G \) is the inclusion map. Thus \( [G, G] \subset Z_2 \) but since \( G \) is not abelian, \( [G, G] \simeq Z_2 \).

Therefore the exact sequence \( 5.12 \) reads \( 1 \to [G, G] \to G \to G/[G, G] \to 1 \) and so \( G/[G, G] \simeq Z_2^n \). The number of one-dimensional representations of a group \( G \) being equal to the cardinality of \( G/[G, G] \), we see that \( G \) has \( 2^n \) one-dimensional
representations and they are exactly those representations for which the restriction to $H$ is trivial.

We get that

$$2^{n+1} = \sum_{\chi(1) = \chi(-1)} \chi(1)^2 + \sum_{\chi(1) = -\chi(-1)} \chi(1)^2 = 2^n + \sum_{\chi(1) = -\chi(-1)} \chi(1)^2 \geq 2^n + 4N_-(G),$$

so indeed $N_-(G) \leq 2^{n-2}$ as claimed. \hfill \Box

**Theorem 5.10.** Consider $H(\mathcal{R}, q)$ of type $B_n$, and let $q_1, q_2$ be defined by \textcolor{red}{[11]}. Suppose that $q_2 = q_1^n, q_1 \neq 1$. Let $\Pi_L \subset \Pi_0$ such that $R_L$ is of type $A_n \times B_l$, and consider the induction datum $\xi = (\Pi_L, \delta, 1)$ for $\delta \in \mathcal{H}_{L, R}^{\mathcal{R}}$. Let $W_L r_L$ be the central character of $\delta$. Let $\mu \vdash l$ be such that $W_L r_L = W_L(r_{\kappa_1}(q_1) \times \cdots \times r_{\kappa_l}(q_1) \times c(\mu, q_1, q_2)).$

Suppose that we can glue $d$ strips of the form $|S(\kappa_i)|$ to $T_m(\mu)$. Then the induced representation $\pi(\xi)$ decomposes into a direct sum of $2^d$ inequivalent tempered representations.

**Proof:** We will prove the theorem by induction on $d$. If $d = 0$ then $R(\xi) = 1$ and thus $\pi(\xi)$ is irreducible. If $d > 0$ then suppose that $R_L$ has type $A_n \times B_l$. Write $\kappa = (l_1, \ldots, l_l)$ (as in \textcolor{red}{[11]}) and $\kappa' = (l_2', \ldots, l_m')$. We may assume for simplicity of notation that every $A$-strip $|S(l_i)|$ can be glued to the $m$-tableau $T_m(\mu)$ (i.e., we assume $s = d$).

Let $\kappa = (l_1, \ldots, l_m)$ and put $n' = l + |\kappa'|$. Consider the parabolic Hecke algebra (as in \textcolor{red}{[11]}) $H_{L'}$ where $R_{L'}$ has type $A_{m'} \times B_{n'}$, and the obvious choice of simple roots $\Pi_{L'}$. Then, by transitivity of induction we have

$$\pi(\xi) = \text{Ind}_{H_{L'}}(\delta) = \text{Ind}_{H_{L'}}(\text{Ind}_{H_{L'}}(\delta)) = \text{Ind}_{H_{L'}}(\pi'(\xi)).$$

Clearly the number of irreducible constituents of $\pi(\xi)$ is at least equal to the number of irreducible constituents of $\pi'(\xi)$. We want to apply the induction hypothesis to $\pi'(\xi)$, but we have to circumvent the technical complication that $H_{L'}$ is not of type $B$.

We therefore return to the setting of \textcolor{red}{[2.3]} Let $H_L$ (resp. $H_{L'}$) be the graded Hecke algebra associated to $H_L$ (resp. $H_{L'}$) as in \textcolor{red}{[2.3]}. Thus, $H_L$ is associated to the degenerate root datum $(R_L, a^*, R_L^\vee, a, \Pi_L)$ and labels $k_n$’s as in \textcolor{red}{[2.6]}; and $H_{L'}$ is associated to $(R_{L'}, a^*, R_{L'}^\vee, a, \Pi_{L'})$ and labels depending on those of $H_L$ as in \textcolor{red}{[2.7]}.

By Lusztig’s theorems of \textcolor{red}{[13]}, we have an equivalence of categories between $H_{W_{L, r_L}}$ (the category of irreducible representations of $H_L$ with central character $W_{L, r_L}$) and $H_{W_{L, \gamma_L}}$ (the category of irreducible representations of $H_L$ with central character $W_L \gamma_L$), where $\exp(\gamma_L) = r_L$. Likewise, we have a categorical equivalence between $H_{W_{L', r_{L'}}}$ and $H_{W_{L', \gamma_L}}$ (the central character of any irreducible constituent of $\pi'(\xi)$ is $W_{L', r_{L'}}$).

Let $V$ be the representation space of $\delta$, then $V$ is also a representation space for the $H_L$-representation corresponding to $\delta$ under the above equivalence. By Lusztig’s theorems, the number of irreducible constituents of $\pi'(\xi)$ is equal to the
number of irreducible constituents of the corresponding induced representation

$$\text{Ind}_{H}^{H'} (V) = H' \otimes_{H} V.$$  

Recall that $\mathbb{H}' = \mathbb{C}[W_0(R_{L'})] \otimes S(a_C)$ and $\mathbb{H} = \mathbb{C}[W_0(R_L)] \otimes S(a_C)$. Let $L = L_1 \cup L_2$ where $L_1$ consists of those simple roots in $L$ corresponding to the factors $A_i^{(1)}$ and $L_2$ to the others. Put $a_{L_1} = \{ x \in a | \langle x, R_{L_1}' \rangle = 0 \}$ and $a_{L_2} = \{ x \in a | \langle x, R_{L_2}' \rangle = 0 \}$. Then we obtain a decomposition $\mathbb{H}_L = \mathbb{H}_{L_1} \otimes \mathbb{H}_{L_2}$, where $\mathbb{H}_{L_i} (i = 1, 2)$ is associated to $(R_{L_i}, a_{L_1}, a_{L_2}, \Pi_{L_i})$ and $k_{L_i}$, the restriction of $k$ to $R_{L_i}$. Therefore, we have a decomposition $V = V_1 \otimes V_2$ where $V_i$ is a representation of $\mathbb{H}_{L_i}$. Likewise, we have a decomposition $\mathbb{H}' = \mathbb{H}_{L_1} \otimes \mathbb{H}'_{L_2}$, where $\mathbb{H}'_{L_1}$ is associated to $(R_0(B_{n'}), a_{L_1}^{*}, a_{L_2}, \Pi_{n'})$ and the restriction of $k$ to $R_0(B_{n'})$.

Therefore, we have

$$\text{Ind}_{H}^{H'} (V) = \mathbb{H}' \otimes_{H} V = (\mathbb{H}_1 \otimes \mathbb{H}'_{L_2}) \otimes_{H} (V_1 \otimes V_2) = V_1 \otimes (\mathbb{H}'_{L_2} \otimes_{H} V_2) = V_1 \otimes (\text{Ind}_{H_2}^{H'} (V_2)).$$

We now apply again Lusztig’s theorems, for the appropriately defined affine Hecke algebras corresponding to $\mathbb{H}_{L_2}$ and $\mathbb{H}_{n'}$. Since the latter is an affine Hecke algebra of type $B$ and the former a parabolic subalgebra, we can apply the induction hypothesis to the induced representation on the right hand side. It is easy to see that the central character of the $\mathbb{H}_{L_2}$-representation afforded by $V_2$ is equal to $r_{\kappa_1, \ldots, \kappa, (q_1) \times \ldots \times r_{\kappa_1}, q_1, q_2}$. Since by construction, we can glue $d - 1$ strips $|S(l)| (i = 2, \ldots, d)$ to $T_m (\mu)$, we may thus assume that $\pi (\xi)$ has $2^d - 1$ irreducible components.

Now we consider the number of irreducible components of $\pi (\xi)$. We have $R(\xi) \simeq \mathbb{Z}_d$. If the cocycle $\eta_\xi$ of Theorem 5.7 is trivial in $H^2 (R(\xi), C^*)$ then as remarked there, it follows that $\pi (\xi)$ has $2^d$ irreducible components. If not, then (see [11, p. 86]) we have to consider a central extension $1 \rightarrow \mathbb{Z}_2 \rightarrow R'(\xi) \rightarrow \mathbb{Z}_d \rightarrow 1$. Then the irreducible components of $\pi (\xi)$ are in bijection with the irreducible representations of $G$ with a fixed restriction to $\mathbb{Z}_2$, as explained above. We denote this character of $\mathbb{Z}_2$ by $\chi$.

Recall that $N_+(R' (\xi)) = 2^d$ if and only if $N_-(R' (\xi)) \neq 2^d$ then $N_-(R' (\xi)) \leq 2^{d-2}$. Since we know that $\pi (\xi)$ has at least $2^{d-1}$ irreducible components, it follows in this case as well that $\pi (\xi)$ is a direct sum of $2^d$ irreducible components, which are indexed by the irreducible representations of $R(\xi)$.

These components are tempered by [10, Thm 4.23]. The fact that they are mutually inequivalent follows since the multiplicity of the constituent of $\pi (\xi)$ which is indexed by $\rho \in \hat{R} (\xi)$ is equal to $\dim (\rho) = 1$, see again [11, p. 87].

In turn, it follows that $\eta_\xi$ must split:

**Proposition 5.11.** The 2-cocycle $\eta_\xi$ has trivial image in $H^2 (R(\xi), C^*)$.

**Proof:** Suppose this is not true, then the group $H$ in (5.10) is isomorphic to $\mathbb{Z}_2$.

Suppose that $R'(\xi)$ is not abelian. Then, as we have seen in the proof of Theorem 5.10, $\chi_\xi$ must be the trivial character. But this means that $\sigma_\xi$, the function of (5.11) which splits $\eta_\xi$ on $R'(\xi)$, satisfies $\sigma_\xi (z r') = \sigma_\xi (r')$ for all $z \in H, r' \in R'(\xi)$. So $\sigma_\xi$ descends to $R(\xi)$ and splits $\eta_\xi$. In other words, the image of $\eta_\xi$ in $H^2 (R(\xi), C^*)$ is trivial, which is a contradiction.
On the other hand, suppose that $R^r(\xi)$ is abelian. The equivalence classes of extensions of the form \[ \mathcal{H}_L^R \to \mathcal{H}_L^R \to \mathcal{H}_L^R(\xi) \] are in a natural bijection with the elements of $H^2(\mathcal{H}_L^R(\xi), \mathbb{Z})$. Therefore, under the map $H^2(\mathcal{H}_L^R(\xi), \mathbb{Z}) \to H^2(\mathcal{H}_L^R, \mathbb{C}^*)$, the cohomology class of the central extension is mapped to $[\eta_\xi] \in \text{Ext}(\mathcal{H}_L^R, \mathbb{C}^*)$. Since $\mathbb{C}^*$ is a divisible abelian group, it is injective. Hence $\text{Ext}(\mathcal{H}_L^R, \mathbb{C}^*) = 1$ and in particular, $[\eta_\xi] = 1$. \hfill \Box

5.3.1. A combinatorial remark. By Theorem 5.10 the number of irreducible constituents of an induced discrete series representation with real central character is as predicted by Conjecture 4.3. Moreover, the parametrization of the constituents of $\pi(\xi)$ by the irreducible characters of $\mathcal{H}_L^R(\xi)$ can be transferred to a parametrization in terms of Young tableaux. Indeed, consider $\pi(\xi)$ where $\xi = (\Pi_L, \delta, 1) \in \Xi$ has real central character $W_L r_L$ with $r_L$ as in 4.5. Then the number of irreducible components in $\pi(\xi)$ is $2^d$ where $d$ is the number of strips of length $\kappa_i$ which can be glued to $T_m(\mu)$. For every $J \subset \{1, \ldots, d\}$, we obtain a partition $\mu_J$, consisting of $T_m(\mu)$ with the strips corresponding to $J$ glued to it. We obtain $2^d$ Young tableau associated to $\pi(\xi)$. Let $\chi_J$ be the character of $\mathcal{H}_L^R(\xi)$ which coincides with the non-trivial character of $\mathbb{Z}_2$ on the factors which correspond to $J$ and which coincides with the trivial representation on the other factors. Then we have a natural bijection $\chi_J \leftrightarrow \mu_J$. Since $\chi_J$ corresponds to an irreducible component of $\pi(\xi)$, we can also parametrize this component by $\mu_J$. In this way we obtain a parametrization of the irreducible constituents of $\pi(\xi)$ by the $m$-tableaux $T_m(\mu_J)$.

5.3.2. The Iwahori-Hecke cases. Let $q_0 = q$, the cardinality of the residue field of the $p$-adic field $F$. Then $\mathcal{H} \simeq \text{End}_G(\text{Ind}_L^G(1))$ for the group $G = G(F)$ with root datum $(R_0, Y, R_0, X, \Pi_0)$. It is known from the work of Kazhdan and Lusztig, 11, that the restrictions to $\mathcal{H}_0$ of the modules in $\mathcal{H}_0^L$ specialize, for $q_0^{1/2} = 1$, into the Springer modules for the Weyl group of the complex group $\mathcal{G}$ with root datum $(R_0, X, R_0, Y, \Pi_0)$. For example, if $R_0$ is of type $A_n$ (resp. $B_n$, resp. $C_n$) then $\mathcal{G} = SL_{n+1}(\mathbb{C})$ (resp. Spin$_{2n+1}(\mathbb{C})$, resp Sp$_{2n}(\mathbb{C})$). It is also known that we obtain a bijection between the central characters of the modules in $\mathcal{H}_0^L$ and the unipotent conjugacy classes of $\mathcal{G}$. Since there is a canonical bijection between the unipotent conjugacy classes of $\mathcal{G}$ and those of $\mathcal{G}/Z$ (where $Z$ denotes its center) we replace $\mathcal{G}$ with the corresponding group of adjoint type. We denote the bijection between the central characters of $\mathcal{H}_0^L$ and the unipotent classes of $\mathcal{G}$ by $W_{0r} \mapsto u_{W_{0r}}$ where $u_{W_{0r}}$ is a representative of the unipotent class associated to $W_{0r}$. Let, for $u \in \mathcal{G}$, $A(u)$ be the group of connected components of the centralizer $C_G(u)$.

In our situation where $R_0$ is of type $B_n$ we obtain two of these cases, cf. remark 2.1. If $m = 1$, then the modules of $\mathcal{H}_0^L$ specialize into the Springer modules of $\mathcal{G} = SO_{2n+1}(\mathbb{C})$. If $m = 1/2$, then the modules of $\mathcal{H}_0^L$ are in bijection with those of an affine Hecke algebra with root system of type $C_n$ and equal labels. In this case, the modules of $\mathcal{H}_0^L$, restricted to $\mathcal{H}_0$, specialize into the Springer modules of the adjoint group Spin$_{2n}(\mathbb{C})$.

In this context, the $R$-group admits the following characterization:

**Proposition 5.12.** Let $m = 1$ or $m = 1/2$. Let $\xi = (\Pi_L, \delta, 1) \in \Xi$. Let $W_L r_L$ be the central character of $\delta \in \mathcal{H}_L^{ds}$. Then

$$R(\xi) \simeq A(u_{W_{0r} r_L})/A(u_{W_{0} r_L}).$$
Proof: First we recall the relation between \(A(u_{W_0 r L})\) and the \(m\)-symbols of the Springer correspondents \(\Sigma_m(W_0 r L)\), for a central character \(W_0 r L\) of a representation in \(H_q^L\). Let \(S\) be the set of entries which occur exactly once in the symbol \(\Sigma_m(W_0 r L)\). Lusztig has defined in [22] (for an overview of these results, see [23] p. 419) an interval in \(S\) to be a subset of the form \(S = (i, i + 1, \ldots, j)\), such that \(0 \leq i \leq j\), \(i - 1 \notin S\), \(j + 1 \notin S\). If \(m = 1/2\), then we also require \(i \geq 1\). Consider the group \(A'(u) = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2\), one copy of \(\mathbb{Z}_2\) for every interval in \(S\). Let \(a_i\) generate the \(i\)-th copy of \(\mathbb{Z}_2\). Then, if \(m = 1\), \(A(u) \simeq \{\sum_i n_i a_i \mid \sum_i n_i \text{ even}\} \subset A'(u)\). If \(m = 1/2\) then we need to consider \(\lambda(u)\) such that \(u \in \text{Spin}_{2m}(\mathbb{C})\) has elementary divisors partition \(\lambda(u)\). Let \(\lambda(u) = (1^{r_1}2^{r_2} \cdots 2^{r_m})\). Then the number of intervals is equal to the number of \(r_i\) such that \(i\) is even and \(r_i > 0\). We associate to every interval a generator \(a_i\) of \(A'(u)\) and a \(r_i\), using the natural ordering. Then \(A(u) = A'(u)/\sum_i \text{even}, r_i \text{ odd } a_i\).

Now let \(W_{L r L}\) be as in the statement, and write it in the form \(A(\mu)\). Observe that \(A(u_{W_0 r L}) = A(u_{W_0 (B_i c) (\mu; q_1, q_2)})\), then \(A(u) = 1\) for all \(u\) in groups of type \(A\). Suppose that \(d\) strips \([S(\kappa_i)]\) can be glued to \(T_m(\mu)\). Then \(R(\xi) \simeq \mathbb{Z}_2^d\) and we have to show that the number of intervals in the \(m\)-symbols of \(\Sigma_m(W_0 r L)\) is exactly \(d\) greater than the number of intervals in the \(m\)-symbols of \(\Sigma_m(W_0 (B_i c) (\mu; q_1, q_2))\), where for \(m = 1/2\) we have to also check that \(\lambda(u_{W_0 r L})\) has even parts with odd multiplicity if and only if \(\lambda(u_{W_0 (B_i c) (\mu; q_1, q_2)})\) has even parts with odd multiplicity. But this is obvious since \(\lambda(u_{W_0 r L})\) is obtained from \(\lambda(u_{W_0 (B_i c) (\mu; q_1, q_2)})\) by adding the parts \((\kappa_1, \kappa_2, \kappa_2, \ldots, \kappa_r, \kappa_r)\) (this follows from the Bala–Carter classification, see e.g. [10] for a nice presentation).

Let \(W_{0 r}\) be a residual point. Then we have seen in [20] that the entries of \(\Sigma_m(W_0 r)\) all have a difference of at least two, hence every entry forms its own interval, except for the entry zero and \(m = 1/2\).

We then need to consider the symbols of the Springer correspondents \(\Sigma_m(W_0 r L)\). Since truncated induction is transitive \([20\ Cor. \ 4.35]\), we define \((\alpha^{(0)}, \beta^{(0)}) = \Sigma_m(\mu)\) and we choose \((\alpha^{(i)}, \beta^{(i)}) \in \text{tr}_m - \text{Ind}((\kappa_i) \otimes (\alpha^{(i-1)}, \beta^{(i-1)}))\) for \(i = 1, \ldots, r\). By [20] Prop. 4.37, we have \((\alpha^{(i)}, \beta^{(i)}) = (\alpha^{(i-1)}, \beta^{(i-1)}) \cup (a_i, b_i)\) where \(a_i + b_i = \kappa_i\). The similarity class of \(\Sigma_m((\alpha^{(i)}, \beta^{(i)}))\) is independent of this choice. Let \(S^{(i)}\) denote the number of intervals in the \(m\)-symbol of \((\alpha^{(i)}, \beta^{(i)}))\).

We recall the following from [20] Props. 4.41-4.43 and their proofs. If \([S(\kappa_i)]\) cannot be glued to \(T_m(\mu)\), then \((\alpha^{(i)}, \beta^{(i)})\) is uniquely determined by the choice of \((\alpha^{(i-1)}, \beta^{(i-1)}))\), and the entries of \(a_i, b_i\) in the \(m\)-symbol \(A^{m}(\alpha^{(i)}, \beta^{(i)})\) are either equal, or have a difference of one. Suppose that they are equal. Since no entry can occur more than twice, it follows that \([S^{(i)}] = [S^{(i-1)}]\). If the entries of \(a_i, b_i\) are not equal, then they have a difference of one, and moreover form an interval together with exactly one of the other already existing intervals. Thus, in this case \([S^{(i)}] = [S^{(i-1)}]\) as well. On the other hand, if one can glue a strip \([S(\kappa_i)]\) to \(T_m(\mu)\), then the entries of \(a_i, b_i\) in the symbol of \((\alpha^{(i)}, \beta^{(i)})\) have a difference of one and do not form an interval with the other entries. Thus, in this case we get \([S^{(i)}] = [S^{(i-1)}] + 1\).

In total, the number of intervals in the \(m\)-symbol of any element of \(\Sigma_m(W_0 r L)\) is therefore equal to \([S^{(i)}] = [S^{(0)}] + d\), where \(d\) is the number of strips \([S(\kappa_i)]\) which can be glued to \(T_m(\mu)\) and \([S^{(0)}]\) is the number of intervals in the \(m\)-symbol of \(\Sigma_m(W_0 (B_i c) (\mu; q_1, q_2))\). Therefore, the result follows. \(\square\)
Example 5.13. We continue the example of Figure 8 even though \( m > 1 \) there. However, even without the interpretation in terms of component groups, the statement on the number of intervals needed in the proof remains true, if we define for \( m \in \mathbb{Z} \) an interval to be a maximal set of consecutive entries in the set of entries which occur precisely once. In the example, we have \( S_3(\mu) = (234, 2) \) which has 3-symbol
\[
\begin{pmatrix}
0 & 2 & 4 & 7 & 14 \\
5 & 8 & 11 & 13 & 16 & 18 \\
1 & 4 & 6 & 10 & 15
\end{pmatrix}.
\]
As remarked in the proof of 5.12, this symbol has 5 intervals, one for every entry. We now perform the truncated induction \( T_3 - \text{Ind}(\langle 3 \rangle \otimes \langle 4 \rangle \otimes \langle 7 \rangle \otimes \langle 11 \rangle \otimes \langle 234, 2 \rangle) \). One computes, using the methods of [20], that the \( m \)-symbols of the 2-partitions which occur in this induction are all in the similarity class of
\[
\begin{pmatrix}
0 & 2 & 6 & 8 & 11 & 13 & 16 & 18 \\
1 & 4 & 6 & 10 & 15
\end{pmatrix}
\]
This symbol has 7 intervals, namely \((0, 1, 2), (4), (8), (10, 11), (13), (15, 16), (18)\). Indeed, this is two more than the number of intervals in the symbol of \( S_3(\mu) \), and one can glue two strips to \( T_3(\mu) \); those of length 7 and 11. These correspond to the intervals \((10, 11)\) and \((15, 16)\). The strips of length 3 and 4 correspond to the interval \((0, 1, 2)\) (the extension of an already existing one) and to the numbers 6, 6 which do not appear in any interval.

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