First-principle calculation of the $\eta_c \to 2\gamma$ decay width from lattice QCD

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We perform a lattice QCD calculation of the $\eta_c \to 2\gamma$ decay width using a model-independent method that requires no momentum extrapolation of the off-shell form factors. This method also provides a straightforward and simple way to examine the finite-volume effects. The calculation is accomplished using $N_f = 2$ twisted mass fermion ensembles. The statistically significant excited-state effects are observed and eliminated using a multi-state fit. The impact of fine-tuning the charm quark mass is also examined and confirmed to be well-controlled. Finally, using three lattice spacings for the continuum extrapolation, we obtain the decay width $\Gamma_{\eta_c\gamma\gamma} = 6.67(16)_{\text{stat}}(6)_{\text{syst}}$ keV, which differs significantly from the Particle Data Group’s reported value of $\Gamma_{\eta_c\gamma\gamma} = 5.4(4)$ keV (2.9 $\sigma$ tension). We provide insight into the comparison between our findings, previous theoretical predictions, and experimental measurements.

Keywords: Two-photon decay, Charmonium, Lattice QCD, Form factor, Finite-volume effects

INTRODUCTION

As a multi-scale system that can probe various regimes of quantum chromodynamics (QCD), heavy quarkonium presents an ideal laboratory for testing the interplay between perturbative and nonperturbative QCD [1]. In quarkonium physics, the two-photon decay widths of quarkonium play an important role in connecting QCD from perturbative to nonperturbative regime. These quantities are traditionally expressed as the product of short-distance quark-antiquark annihilation decay rates and the squared bound-state wave function at the origin [2]. Phenomenologically, the latter provides an essential, universal input for calculating the decay and production cross sections for the quarkonium states [3]. Therefore, quarkonium physics relies heavily on the accurate determination of these decay widths.

In this study, we focus on examining the two-photon decay of the lowest charmonium state, $\eta_c \to 2\gamma$, a topic that has attracted extensive attention from both experimental [4,21] and theoretical studies [2,22,30]. On the experimental side, the low statistics for direct measurements make the accurate determination of the two-photon decay extremely difficult. Despite decades of effort, direct measurements of decay width still have uncertainties ranging from 20% to 100%. The decay width $\Gamma_{\eta_c\gamma\gamma} = 5.4(4)$ keV favored by the Particle Data Group (PDG) is compiled using a combined fit with other decay channels, resulting in a branching ratio of $\text{Br}(\eta_c \to 2\gamma) = (1.68\pm0.12) \times 10^{-4}$, denoted here as the “PDG-fit” value. However, if one examines the PDG list in detail, there is another value of $\text{Br}(\eta_c \to 2\gamma) = 2.2^{+0.9}_{-0.6} \times 10^{-4}$ compiled based on the average of BESIII [20] and CLEO [21] measurements, with $\text{Br}(\eta_c \to 2\gamma) = (2.7 \pm 0.8 \pm 0.6) \times 10^{-4}$ and $\text{Br}(\eta_c \to 2\gamma) = (0.7^{+1.6}_{-0.2} \pm 0.2) \times 10^{-4}$, respectively.

These results are rather far apart from each other, but still consistent due to large errors. Such large errors propagate into the PDG-aver value and result in a 28% uncertainty. We denote this value as the “PDG-aver” value. Notably, PDG-fit value has a 7 times smaller uncertainty and a 24% lower central value compared to the PDG-aver one. As for the PDG-fit value, other decay channels are also taken into account. The constraints from different $\eta_c$ decays result in a much smaller error and a downward shift of the central value. The significantly suppressed uncertainty in the PDG-fit value indicates that the direct experimental measurements of the $\eta_c \to 2\gamma$ decay have little impact here.

On the theoretical side, a recent calculation based on Dyson-Schwinger equation [26] suggests a two-photon decay width $\Gamma_{\eta_c\gamma\gamma} = 6.32-6.39$ keV (with a branching ratio of $\text{Br}(\eta_c \to 2\gamma) = (1.98-2.00) \times 10^{-4}$), consistent with the PDG-aver value but much larger than the PDG-fit one. Meanwhile, a study from nonrelativistic QCD, including the next-to-next-to-leading-order perturbative corrections [27], gives the branching ratio $\text{Br}(\eta_c \to 2\gamma) = (3.1-3.3) \times 10^{-4}$, which is larger than other theoretical predictions and the experimental measurements. To clarify the discrepancies, a first-principle calculation of $\eta_c \to 2\gamma$ decay width from lattice QCD is crucial.

Lattice QCD calculations of $\eta_c \to 2\gamma$ decay have been conducted in the past, but the systematic effects in these computations are still not entirely under control. Apart from the earlier quenched studies [23], later unquenched lattice studies have utilized only one or two lattice ensembles [25,29]. In particular, substantial lattice spacing errors have been observed in Ref. [29]. Therefore, it is crucial to use at least three lattice ensembles to allow for a controlled continuum extrapolation. This work aims...
to systematically improve upon previous lattice studies of this radiative decay. Several improvements are made to obtain a more accurate result: (1) We adopt a novel method to extract the on-shell form factor directly, which avoids conventional model-dependent errors caused by the momentum extrapolation of the off-shell form factors. (2) We perform a spatial volume integral to obtain the form factor, with a truncation range introduced to monitor the finite-volume effects. (3) We remove the excited-state contaminations, which are found to be quite significant in this study. (4) We confirm that the systematic effects due to fine-tuning the valence charm quark mass are smaller than the statistical errors. (5) Lastly, lattice computations are done using three ensembles with three different lattice spacings and find that the lattice results are well described by a form linear in \( a^2 \), which is suggested by the automatic \( O(a) \) improvement of the ensembles. These efforts finally allow us to obtain the decay width with a precision of about 2.6%.

**METHODOLOGY**

We start the discussion of \( H \rightarrow \gamma \gamma \) in an infinite-volume continuum Euclidean space, where \( H \) indicates a hadron with mass \( m_H \). The relevant hadronic matrix element \( F_{\mu \nu}(p) \) for the two-photon decay process is,

\[
F_{\mu \nu}(p) = \int dt e^{im_H t} \int d^3 \vec{x} e^{-i\vec{p} \cdot \vec{x}} H_{\mu \nu}(t, \vec{x}),
\]

where the hadronic function \( H_{\mu \nu}(t, \vec{x}) \) is defined as

\[
H_{\mu \nu}(t, \vec{x}) = \langle 0| T[J^m_{\mu}(x) J^m_{\nu}(0)]|H(k)\rangle,
\]

with the initial state \( |H(k)\rangle \) carrying the four-momentum \( k = (im_H, \vec{0}) \). The four-momentum assigned to the electromagnetic current \( J^m_{\mu} = \sum_q e_q \bar{q} \gamma_\mu q \) \((e_q = 2/3, -1/3, -1/3, 2/3\) for \( q = u, d, s, c \)) takes the form \( p = (im_H/2, \vec{p}) \) with \( |\vec{p}| = m_H/2 \), so that it satisfies the on-shell condition for the photon.

We then assume that the hadron in the initial state is a pseudo-scalar particle. According to its negative parity, the hadronic tensor \( F_{\mu \nu}(p) \) can be parameterized as

\[
F_{\mu \nu}(p) = \epsilon_{\mu \nu \alpha \beta} p_\alpha k_\beta F_{H \gamma \gamma}.
\]

By multiplying \( \epsilon_{\mu \nu \alpha \beta} p_\alpha k_\beta \) to both sides, the form factor at on-shell momentum is extracted through

\[
F_{H \gamma \gamma} = -\frac{1}{2m_H|\vec{p}|^2} \int d^4x e^{-ipx} \epsilon_{\mu \nu \alpha 0} \frac{\partial H_{\mu \nu}(x)}{\partial x_\alpha}.
\]

After averaging over the spatial direction for \( \vec{p} \), \( F_{H \gamma \gamma} \) would be obtained through

\[
F_{H \gamma \gamma} = -\frac{1}{2m_H} \int d^4x e^{im_H t} \frac{1}{|\vec{p}|^2} \epsilon_{\mu \nu \alpha 0} x_\alpha H_{\mu \nu}(x),
\]

where \( j_n(x) \) are the spherical Bessel functions. The decay width is then given by

\[
\Gamma_{H \gamma \gamma} = \frac{\pi}{2} \cdot m_H^2 F^2_{H \gamma \gamma}.
\]

In the lattice calculation, we adopt the infinite-volume reconstruction method proposed in Ref. [31]. This method has been successfully applied to various processes [32–41] to reconstruct the infinite-volume hadronic function using the finite-volume ones. In this work, we use it for the lattice calculation of the \( \eta_c \rightarrow 2\gamma \) decay. It is natural to introduce an integral truncation \( t_s \) and write the contribution as \( F_{\eta_c \gamma \gamma}(t_s) \). The parameter \( t_s \) is chosen sufficiently large to guarantee that the time dependence of \( H_{\mu \nu}(t, \vec{x}) \) for \( t > t_s \) is dominated by the ground state, which is the \( J/\psi \) state when neglecting the disconnected diagrams. Then the residual integral from \( t_s \) to \( \infty \) can be calculated as

\[
\delta F_{\eta_c \gamma \gamma}(t_s) = -\frac{1}{2m_{\eta_c}} \frac{e^{i|\vec{p}|t_s}}{\sqrt{m_{\eta_c}^2 + |\vec{p}|^2 - |\vec{p}|}} \times \int d^3 \vec{x} j_1(|\vec{p}| \cdot \vec{x}) \epsilon_{\mu \nu \alpha 0} x_\alpha H_{\mu \nu}(t_s, \vec{x}).
\]

The total contribution of \( F_{\eta_c \gamma \gamma} \) is given by

\[
F_{\eta_c \gamma \gamma} = F_{\eta_c \gamma \gamma}(t_s) + \delta F_{\eta_c \gamma \gamma}(t_s),
\]

where the \( t_s \) dependence cancels if the ground state is saturated. We then use the hadronic function \( H_{\mu \nu}^L(t, \vec{x}) \) calculated on a finite-volume lattice to replace the infinite-volume \( H_{\mu \nu}(t, \vec{x}) \) for \( t \leq t_s \). Such replacement only amounts for exponentially suppressed finite-volume effects as the hadronic function \( H_{\mu \nu}(t, \vec{x}) \) itself is suppressed exponentially when \( |\vec{x}| \) becomes large. One can introduce a spatial integral truncation \( R \) and examine at large \( R \) whether the finite-volume effects are well under control or not.

The method used in our calculation is generically different from the conventional approach where the photon momenta are assigned by discrete Fourier transformation and the off-shell form factors [23] or amplitude squares [29] with nonzero photon virtualities. In those cases, the physical result can only be obtained after a momentum extrapolation to the on-shell limit. The situation becomes much easier here as the approach presented above allows us to extract the on-shell form factor directly. Therefore, the systematic uncertainties arising from the model-dependent extrapolation are avoided and the computational cost is also significantly reduced.

**NUMERICAL SETUP**

The calculation is performed using three \( N_f = 2 \) flavor twisted mass gauge field ensembles generated by the...
Extended Twisted Mass Collaboration (ETMC) \cite{42,43} with lattice spacing $a \simeq 0.0667, 0.085, 0.098$ fm. We call these ensembles $a67$, $a85$, and $a98$, respectively. More parameters of these ensembles are listed in Table I. The valence charm quark mass is tuned by requiring the lattice result of charmonium masses to coincide with that of 1) $\eta_c$ or 2) $J/\psi$. These two choices will make a difference on our physical quantities. For simplicity, we add the suffix “I” or “II” to the ensemble name to specify the case of 1) or 2) mentioned above. For detailed information on the tuning, we refer to the supplemental material.

In this work, we calculate the three-point correlation function $C_{\mu\nu}^{(3)}(x, y, t_i) \equiv \langle J_{\mu}^{(c)}(x)J_{\nu}^{(c)}(y)O_{\eta_c}^\dagger(t_i) \rangle$ with $t_i = \min\{t_x, t_y\} - \Delta t$. The $Z_4$-stochastic wall source propagator is placed at time $t_i$ so that the $O_{\eta_c}^\dagger$ operator carries the zero momentum. It is found in our study that the uncertainty is reduced by nearly a factor of 2 by using a stochastic propagator compared to that using the point source propagator. We also find that the excited-state contamination associated with the $O_{\eta_c}^\dagger$ operator is significant. We thus apply the APE \cite{44} and Gaussian smearing \cite{45} to the $\eta_c$ field and it efficiently reduces the excited-state effects. Nevertheless, when the precision reaches 1-3% in our calculation the excited-state effects are statistically significant unless $\Delta t \gtrsim 1.6$ fm. Such systematic effects affect both two-point correlation function $C^{(2)}(t) = \langle O_{\eta_c}(t)O_{\eta_c}^\dagger(0) \rangle$ and three-point function $C_{\mu\nu}^{(3)}$. Using a two-state fit form, we write the time dependence for $C^{(2)}(t)$ as

$$C^{(2)}(t) = V \sum_{i=0.1} Z_i^2 e^{-E_i t} e^{-E_i (T-t)}$$

with $V$ the spatial-volume factor, $E_0 = m_{\eta_c}$ the ground-state energy and $E_1$ the energy of the first excited state. $Z_i = \langle i | O_{\eta_c}^\dagger(0) | i \rangle$ are the overlap amplitudes for the ground and the first excited state. We then use $Z_0$ and $m_{\eta_c}$ as the inputs to determine the hadronic function $H_{\mu\nu}$ through $H_{\mu\nu}(t_x, t_y, \vec{x} - \vec{y}) = C_{\mu\nu}^{(3)}(x, y, t_i)/[Z_0/(2m_{\eta_c})] e^{-m_{\eta_c}(t_x - t_i)}].$ The excited-state effects carried by $H_{\mu\nu}$ propagate into the form factor $F_{\eta_c \gamma \gamma}$ and can be parameterized using a relatively simple two-state form

$$F_{\eta_c \gamma \gamma}(\Delta t) = F_{\eta_c \gamma \gamma} + \xi e^{-(E_1 - E_0)\Delta t},$$

with two unknown parameters $F_{\eta_c \gamma \gamma}$ and $\xi$. To fully control the systematic effects, we use various $\Delta t$ from the range of 0.7 - 1.6 fm and fit the lattice data to the form \cite{10}. The range of $\Delta t$ in lattice units is also tabulated in the last column of Table I. To compute the correlation function for the whole set of $\vec{x} - \vec{y}$, we place the point source propagator on one vector current and treat the other one as the sink. Both the point and the stochastic wall source propagators are placed on all time slices and thus the average based on time translation invariance can be performed to increase the statistics.

In our calculation, the electromagnetic current is replaced by a local charm quark current as $J_{\mu}^{em}(x) = Z_V c_v J_{\mu}^{(c)}(x)$ with $J_{\mu}^{(c)}$ defined as $J_{\mu}^{(c)} \equiv \bar{c}(x) \gamma_{\mu} c(x)$. Here $Z_V$ is a renormalization factor, which converts the local vector current to the conserved one at the cost of at most of $O(a^2)$. For the details of the determination of $Z_V$, we refer to the supplemental material.

LATTICE RESULTS

The lattice results of $F_{\eta_c \gamma \gamma}$ as a function of the truncation time $t_s$ with different separations $\Delta t$ are shown in the top panel in Fig. 1. Here we take the ensemble with the finest lattice spacing, namely $a67$ as an example. The results are shown with the charm quark mass tuned to the physical point $m_c \simeq m_{\eta_c}^{phys}$. The integral in Eq. 10 is performed with $\vec{x}$ summed over the whole spatial volume. We find that for all the separation $\Delta t$ and all ensembles used in this work, a temporal truncation $t_s \simeq 1$ fm is a conservative choice for the ground-state saturation. With this choice, the results for $F_{\eta_c \gamma \gamma}$ as a function of $\Delta t$ are shown on the bottom panel in Fig. 1. It shows that $F_{\eta_c \gamma \gamma}$ has an obvious $\Delta t$ dependence, indicating sizable excited-state effects associated with $O_{\eta_c}$ operator as we have pointed out before. Using a two-state fit described by Eq. 10 we can extract the ground-state contribution to the form factor at $\Delta t \rightarrow \infty$.

To examine the finite-volume effects, we introduce a spatial integral truncation parameter $R$ in both Eqs. 3 and 4. As the hadronic function $H_{\mu\nu}(x)$ is dominated by the $J/\psi$ state at large $|\vec{x}|$, the size of the integrand is exponentially suppressed when $|\vec{x}|$ becomes large. In Fig. 2 the form factor $F_{\eta_c \gamma \gamma}$ is shown as a function of $R$. For $R \gtrsim 0.8$ fm, there exists a plateau, indicating that the hadronic function $H_{\mu\nu}(x)$ at $|\vec{x}| \gtrsim 0.8$ fm has negligible contribution to $F_{\eta_c \gamma \gamma}$. For each of the three ensembles used in this work, the lattice size satisfies $L > 2$ fm which is sufficiently large to accommodate the hadron. We thus

Table I. Parameters of gauge ensembles used in this work. From left to right, we list the ensemble name, the lattice spacing $a$ (with errors taken from Ref. \cite{42}), the spatial and temporal lattice size $L \times T$, the number of the configurations $N_{\text{conf}}$, the light quark mass $a\mu$, the corresponding pion mass $m_\pi$, and the range of the time separation between the hadron and the nearest current $\Delta t$, see the discussion after Eq. \cite{10}. Here, $L$, $T$, and $\Delta t$ are given in lattice units. For all ensembles, $\Delta t$ takes a consistent range of 0.7-1.6 fm.
conclude that finite-volume effects are well under control in our calculation. Here the parameter $R$ is simply introduced for the examination of the size of the finite-volume effects. The lattice results reported throughout the paper are obtained based on the whole spatial-volume summation.

Using the form factors as inputs, the dimensionless quantity $\Gamma_{\eta_c\gamma\gamma}/m_{\eta_c}$ can be evaluated and the results are listed in Table I. The lattice results for $\Gamma_{\eta_c\gamma\gamma}/m_{\eta_c}$ at different lattice spacings are shown in Fig. 3 together with an extrapolation that is linear in $a^2$. This is an expected behavior for the twisted mass ensembles which have the automatic $O(a)$ improvement at maximal twist. It is evident that the fitting curves describe the lattice data well. However, there is a possibility that ensemble a98 may not be optimally tuned, leading to residual $O(a)$ discretization errors [13]. To further explore this possibility, we perform the continuum extrapolations both with and without the coarsest lattice, a98. These results are consistent within statistical errors. After the continuous extrapolations of dimensionless $\Gamma_{\eta_c\gamma\gamma}/m_{\eta_c}$, we obtain

$$
\Gamma_{\eta_c\gamma\gamma}/m_{\eta_c} = \begin{cases} 
2.204(61) \times 10^{-6} & \text{I, with a98} \\
2.234(52) \times 10^{-6} & \text{II, with a98} \\
2.211(100) \times 10^{-6} & \text{I, without a98} \\
2.227(85) \times 10^{-6} & \text{II, without a98} 
\end{cases}
$$

To have a direct comparison with the experimental results of the decay width, we rescale the dimensionless $\Gamma_{\eta_c\gamma\gamma}/m_{\eta_c}$ to physical values by multiplying the experimental mass of $\eta_c$, $m_{\eta_c}^{\text{exp}} = 2.9839$ GeV. The following results for $\Gamma_{\eta_c\gamma\gamma}$ are then obtained:

$$
\Gamma_{\eta_c\gamma\gamma} = \begin{cases} 
6.61(18) \text{ keV} & \text{I, with a98} \\
6.67(16) \text{ keV} & \text{II, with a98} \\
6.60(30) \text{ keV} & \text{I, without a98} \\
6.64(25) \text{ keV} & \text{II, without a98} 
\end{cases}
$$

The results of $\Gamma_{\eta_c\gamma\gamma}$ show good consistency across variations in the charm quark mass and with or with-
out ensemble a98. The latter suggests that no indication of $O(a)$ effects are observed in ensemble a98, which is consistent with findings from other lattice studies [42, 43, 46, 47]. Therefore, we have included the results from ensemble a98 in our report. Specifically, we use the central value and statistical error from the second line in Eq. (12) - which uses $J/\psi$ for the charm quark mass setting - as the central value and statistical error for our final result. To estimate the systematic error, we consider the deviation of the central values between the first two lines in Eq. (12). Our final result for the decay width is

$$\Gamma_{\eta_c \to \gamma\gamma} = 6.67(16)_{\text{stat}}(6)_{\text{syst}} \text{ keV.}$$

This result is larger than the PDG-fit value by 24% with a 2.9 $\sigma$ tension, but compatible with PDG-aver value as it carries a much larger uncertainty. Our lattice results using two different valence charm quark mass setting procedures, together with the two PDG values are illustrated in Fig. 3 for comparison.

In this work, we present a lattice calculation of $\eta_c \to \gamma\gamma$ with various systematic effects under well control using a novel method. The systematic error budget in our current lattice calculation is relatively complete. The remaining systematic effects requiring further investigation are the neglected disconnected diagrams, the quenching of strange quark, and the use of up and down quarks heavier than their physical values in our calculation.

DISCUSSION

The PDG-fit relies on the assumption that all individual measurements are correctly statistically distributed and the correlations among the different decay modes are explicitly known. This is relatively difficult and the assumption may not be valid for all the measurements. According to the table of the correlation coefficients used in the PDG fit, there exists a large correlation between the $\eta_c \to 2\gamma$ decay and other decay modes, indicating that the PDG-fit result is easily affected by the parametrization in the constrained fit. Thus a direct and precise experimental measurement of $\eta_c \to 2\gamma$ is essential for our better understanding of the charm quark physics.

In Fig. 4 we summarize the experimental measurements of $\Gamma_{\eta_c \to \gamma\gamma}$ from 1986 to 2013 [4-21]. These results are very consistent with our lattice calculation but carry quite large errors, which range from 20% to 100%. It is still challenging to reduce the uncertainty to the level of a few percent. Regarding this situation, for a certain period, a first-principle determination of $\Gamma_{\eta_c \to \gamma\gamma}$ from lattice QCD will play an irreplaceable role for a better understanding of the QCD dynamics in charm quark physics.
first effect is Okubo-Zweig-Iizuka (OZI) suppressed and believed to only give a small contribution in the charmonium system [50,53]. For the other two, previous lattice calculations [54] indicate that they will also result in only small effects. Nevertheless, these further improvements become more straightforward for future lattice calculations using the gauge ensembles with physical pion mass, heavy sea quarks, and more different lattice spacings. The calculation including the disconnected diagrams will also provide a direct estimation of the OZI suppressed contribution.

**CONCLUSION**

In this work, we propose a new method to compute the decay width of \( \eta_c \rightarrow 2 \gamma \), where the on-shell form factor is obtained by combining the hadronic function calculated from lattice QCD and an appropriate weight function known analytically. As this method requires no model-dependent extrapolation of the off-shell form factors, it provides a theoretically clean framework to determine the radiative decay width. Such a method can also be applied to other processes which involve the leptonic or radiative decay width. It is believed to only give a small contribution in the charmonium states. Our lattice result for the decay width is consistent with the PDG-aver value, but notably different from the previous experimental measurements and also the PDG-aver value, by about 2.9 standard deviations. We suspect that the error of the PDG-fit value might be underestimated due to the limited knowledge of the direct experimental measurements of \( \eta_c \rightarrow 2 \gamma \) and strong correlation from other decay channels. It is therefore crucial for the forthcoming experiments, e.g. BESIII, to further reduce the experimental uncertainties. The cross-check of our results by other lattice QCD calculations are also very helpful to clarify the discrepancy between theory and experiments.

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Supplementary Material

In this section, we expand on a selection of technical details and add results to facilitate cross-checks of different calculations of the \( \eta_c \rightarrow 2\gamma \) decay width.

Derivation of Eq. (5)

For convenience, we introduce a scalar function \( \mathcal{I} \) by multiplying \( \epsilon_{\mu\nu\alpha\beta}p_\alpha k_\beta \) to both sides of the Eq. (5). For the right side, it has

\[
\mathcal{I} = \epsilon_{\mu\nu\alpha\beta}p_\alpha k_\beta \epsilon_{\mu\nu\alpha'}p_{\alpha'} k_{\beta'} F_{H\gamma}\gamma = 2F_{H\gamma}\gamma [p^2 k^2 - (p \cdot k)^2] = -2F_{H\gamma\gamma} m_0^2 |\vec{p}|^2
\]

(14)

where \( |\vec{p}| = m_H/2 \) is considered. For the left side, it has

\[
\mathcal{I} = i m_H \epsilon_{\mu\nu\alpha\beta} \int d^3 \vec{x} e^{-i\vec{p}\cdot\vec{x}} \mu_{\mu\nu\alpha\beta} \partial_{x_\alpha} \mathcal{H}_{\mu\nu}(t, \vec{x})
\]

(15)

Combining these two equations, we obtain

\[
F_{H\gamma\gamma} = -\frac{1}{2m_H |\vec{p}|^2} \int d^3 \vec{x} e^{-i\vec{p}\cdot\vec{x}} \epsilon_{\mu\nu\alpha\beta} \partial_{x_\alpha} \mathcal{H}_{\mu\nu}(x).
\]

(16)

Averaging the direction of \( \vec{p} \), then the factor \( e^{-i\vec{p}\cdot\vec{x}} \) can be replaced by

\[
\frac{1}{4\pi} \int d^3 \vec{x} e^{-i\vec{p}\cdot\vec{x}} = \frac{1}{2} \int_0^1 d \cos \theta \epsilon_{\mu\nu\alpha\beta} \frac{\partial \mathcal{H}_{\mu\nu}(x)}{\partial x_\alpha} = \sin(\vec{p} |\vec{x}|) \equiv J_0(\vec{p} |\vec{x}|)
\]

(17)

Therefore,

\[
F_{H\gamma\gamma} = -\frac{1}{2m_H |\vec{p}|^2} \int d^3 \vec{x} e^{-i\vec{p}\cdot\vec{x}} \epsilon_{\mu\nu\alpha\beta} \frac{\partial \mathcal{H}_{\mu\nu}(x)}{\partial x_\alpha} \times J_0(\vec{p} |\vec{x}|) = \frac{1}{2m_H |\vec{p}|^2} \int d^3 \vec{x} e^{-i\vec{p}\cdot\vec{x}} \epsilon_{\mu\nu\alpha\beta} \frac{\partial J_0(\vec{p} |\vec{x}|)}{\partial x_\alpha} \mathcal{H}_{\mu\nu}(x)
\]

\[
= -\frac{1}{2m_H} \int d^3 \vec{x} e^{-i\vec{p}\cdot\vec{x}} J_1(\vec{p} |\vec{x}|) \epsilon_{\mu\nu\alpha\beta} \mathcal{H}_{\mu\nu}(x),
\]

(18)

Both in the penultimate lines of Eq. (15) and Eq. (18), the space integrals for the total differentiation are omitted.

Hadronic function \( \mathcal{H}_{\mu\nu}(t, \vec{x}) \)

The hadronic function \( \mathcal{H}_{\mu\nu}(t, \vec{x}) \) in Eq. (2) can be extracted from a three-point function \( C_{\mu\nu}(t, \vec{x}; \Delta t) \)

\[
C_{\mu\nu}(t, \vec{x}; \Delta t) = \begin{cases} 
\langle J_{\mu}^T(\vec{x}, t) J_{\nu}^T(\vec{0}, 0) \phi_{\eta_c}(\Delta t) \rangle, & t \geq 0 \\
\langle J_{\mu}^T(\vec{0}, 0) J_{\nu}^T(\vec{x}, t) \phi_{\eta_c}(t - \Delta t) \rangle, & t < 0 
\end{cases}
\]

(19)

with only the connected Wick contractions included

\[
\langle J_{\mu}^T(\vec{x}, t) J_{\nu}^T(\vec{0}, 0) \phi_{\eta_c}(\Delta t) \rangle = -Z_c^2 \epsilon_{\gamma\gamma}(\vec{c}\gamma c(\vec{x}, t)\bar{c}\gamma \bar{c}(\vec{0}, 0)\bar{c}\gamma \gamma c(-\Delta t))
\]

\[
= Z_c^2 \epsilon_{\gamma\gamma}(\bar{c}\gamma \bar{c}(\vec{0}, 0)\bar{c}\gamma \gamma c(-\Delta t)) \times \gamma_{\mu} S_{c}(\vec{0}, \vec{x}; t, \Delta t) + \gamma_{\nu} S_{c}(\vec{0}, \vec{x}; t, \Delta t)) \}
\]

(20)

and

\[
\langle J_{\mu}^T(\vec{0}, 0) J_{\nu}^T(\vec{x}, t) \phi_{\eta_c}(t - \Delta t) \rangle = -Z_c^2 \epsilon_{\gamma\gamma}(\bar{c}\gamma \bar{c}(\vec{0}, 0)\bar{c}\gamma \gamma c(t - \Delta t))
\]

\[
= Z_c^2 \epsilon_{\gamma\gamma}(\bar{c}\gamma \bar{c}(\vec{0}, 0)\bar{c}\gamma \gamma c(t - \Delta t)) \times \gamma_{\nu} S_{c}(\vec{0}, \vec{x}; t - \Delta t) + \gamma_{\nu} S_{c}(\vec{0}, \vec{x}; t - \Delta t)) \}
\]

(21)

Then, the hadronic function \( \mathcal{H}_{\mu\nu}(t, \vec{x}) \) is determined directly through

\[
\mathcal{H}_{\mu\nu}(t, \vec{x}) = \begin{cases} 
\frac{2E_0}{Z_0} e^{-E_0 t} C_{\mu\nu}(t, \vec{x}; \Delta t), & t \geq 0 \\
\frac{2E_0}{Z_0} e^{-E_0 (t-\Delta t)} C_{\mu\nu}(t, \vec{x}; \Delta t), & t < 0 
\end{cases}
\]

(22)

where \( E_0, Z_0 \) are extracted from the two point function as given in Eq. (9) and the renormalization constant \( Z_\gamma \) is calculated by Eq. (23).

Tuning of the valence charm quark mass

The detailed information of the charm quark mass tuning is given in Table III. We attach the suffix “-I” and “-II” to distinguish the cases with \( m_{\eta_c} \approx m_{c^{\text{phys}}} \) and \( m_{J/\psi} \approx m_{c^{\text{phys}}} \). Together with the charmonium masses, we also list the values of the hyperfine splitting \( \delta m \equiv m_{J/\psi} - m_{\eta_c} \). Using three ensembles, an extrapolation that is linear in \( a^2 \) can be performed, see Fig. [1], and in the continuum limit we obtain \( \delta m = 123(1) \text{ MeV} \), which is 10 MeV larger than the PDG value. Similar increase of the hyperfine splitting has been observed by the HQQCD collaboration [24], where the discarded \( \eta_c \) annihilation effects are expected to cause a 7.3(1.2) MeV enhancement in \( \delta m \). In our calculation, a similar shift in \( \delta m \) could lead to a ~ 0.3% uncertainty in the \( \eta_c \) mass, which has little effect on our final result for the decay rate in Eq. (12).
Table III. Tuning of the bare charm quark mass $\mu_c$. The uncertainties of $m_{\eta_c}$ and $m_{J/\psi}$ are statistical only. The mass $\mu_c$ is tuned such that the mass of $\eta_c$ or $J/\psi$ approaches to its physical value, with the deviation controlled to be less than 0.2%. To distinguish the cases with $m_{\eta_c} \approx m_{\eta_c}^{\text{phys}}$ and $m_{J/\psi} \approx m_{J/\psi}^{\text{phys}}$, we attach the suffix “-I” and “-II” to the ensemble name. $\delta m \equiv m_{J/\psi} - m_{\eta_c}$ designates the hyperfine splitting of the charmonia.

| Ensemble | $\mu_c$ | $m_{\eta_c}$ (MeV) | $m_{J/\psi}$ (MeV) | $\frac{\delta m}{m_{\eta_c}} \times 10^2$ |
|----------|---------|---------------------|---------------------|----------------------------------|
| a98-I    | 0.2896  | 2984.9(4)           | 3064.8(5)           | 2.677(12)                        |
| a85-I    | 0.2563  | 2988.6(5)           | 3078.0(7)           | 2.991(18)                        |
| a67-I    | 0.2024  | 2987.6(5)           | 3090.4(7)           | 3.434(15)                        |
| Cont.Limt| —       | —                   | —                   | 4.11(11)                         |
| a98-II   | 0.2951  | 3018.9(4)           | 3097.2(5)           | 2.596(11)                        |
| a85-II   | 0.2586  | 3005.7(5)           | 3094.4(7)           | 2.953(18)                        |
| a67-II   | 0.2036  | 2999.5(5)           | 3101.7(7)           | 3.411(15)                        |
| Cont.Limt| —       | —                   | —                   | 4.13(12)                         |
| PDG      | —       | 2983.9(4)           | 3096.9(0)           | 3.79(1)                          |

Table IV. The vector-current renormalization constants $Z_V$ calculated using Eq. (23).

| Ensemble | a98-I | a85-I | a67-I |
|----------|-------|-------|-------|
| $Z_V$    | 0.6033(20) | 0.6255(22) | 0.6517(15) |

Figure S1. (Color online) The continuum extrapolation of hyperfine splitting. The errors of lattice spacing are presented by the horizontal error bars.

Determination of $Z_V$

In our calculation the electromagnetic current is replaced by a local charm quark current as $J_{\mu}^{em}(x) = Z_V e_c J_{\mu}^{(c)}(x)$ with $J_{\mu}^{(c)}$ defined as $\bar{c}_{\gamma\mu} c$, at the cost of introducing at most $O(a^2)$ errors, most of which are taken care of by the continuum extrapolation procedure described in main body of this paper. The factor $Z_V$ is a vector-current renormalization factor, which can be calculated by applying the condition of charge conservation and using a ratio between $C^{(2)}(t)$ and the three-point function $C^{(3)}(t) = \sum_{\vec{x}} \langle 0 | O(\eta_c(t), J_{0}^{(c)}(t/2, \vec{x}) O(0) \rangle$ with zero three-momentum inserted for both initial and final states. As the charge conservation holds for both ground and excited-states, we find that the excited-state effects in $C^{(3)}(t)$ and $C^{(2)}(t)$ cancel efficiently. The plateau of the ratio starts at $t \approx 1$ fm. The main systematic effect appears as the around-of-world effect in $C^{(2)}(t)$ at $t \approx T/2$. To account for this effect, we use the following ansatz,

$$Z_V = \frac{C^{(2)}(t)}{C^{(3)}(t)} \frac{1}{1 + e^{-m_{\eta_c}(T-2t)}}$$

for $t \lesssim T/2$, (23)

to extract $Z_V$ from the ratio. We find that the uncertainty in the determination of the charm quark mass makes a nearly negligible impact on $Z_V$, whose numerical values are listed in Table IV. We have also checked that the values for $Z_V$ presented here are consistent with those calculated by the RI-MOM scheme in Ref. [? ], where the results are given as 0.604(07), 0.624(04) and 0.659(04) for $a = 0.098, 0.085, 0.0667$ fm, respectively.