Regional controllability of anomalous diffusion generated by the
time fractional diffusion equations

Fudong Ge\textsuperscript{a,b}, YangQuan Chen\textsuperscript{b,*}, Chunhai Kou\textsuperscript{c}

\textsuperscript{a}College of Information Science and Technology, Donghua University, Shanghai 201620, PR China
\textsuperscript{b}Mechatronics, Embedded Systems and Automation Lab, University of California, Merced, CA 95343, USA
\textsuperscript{c}Department of Applied Mathematics, Donghua University, Shanghai 201620, PR China

Abstract

This paper is concerned with the investigation of the regional controllability of the time fractional diffusion equations. First, some preliminaries and definitions of regional controllability of the system under consideration are introduced, which promote the existence contributions on controllability analysis. Then we analyze the regional controllability with minimum energy of the time fractional diffusion equations on two cases: $B \in L(R^m, L^2(\Omega))$ and $B \notin L(R^m, L^2(\Omega))$. In the end, two applications are given to illustrate our obtained results.

Keywords: regional controllability; minimum energy; anomalous diffusion process; fractional order

1. Introduction

Recently fractional diffusion equations (FDEs) have attracted increasing interests and a great deal of contributions have been given to both in time and space variables \cite{1,2,3,4,5}. And it is confirmed that the fractional approach to anomalous diffusion models is appealing compared to other approaches. For instance, due to the nonlocal and hereditary properties of fractional operators, the anomalous diffusion models generated by FDEs are developed effectively to describe transport process in complex dynamic system.

As we all know, the anomalous diffusion processes in real world are essentially distributed and the continuous time random walk (CTRW), governed by the waiting time probability density function (PDF) and the jump length PDF, is a useful tool to describe this phenomenon \cite{6,7,8,9}. In addition, when the waiting time PDF and jump length PDF are power-law and independent, the anomalous transport process can be derived by the fractional partial differential equations, namely fractional Fokker-Planck and Klein-Kramers equations.

\textsuperscript{*}Corresponding author

Email address: ychen53@ucmerced.edu (YangQuan Chen)
And the time fractional diffusion equation models anomalous sub-diffusion and its solutions are transition densities of a stable Lévy motion, representing the accumulation of power law jumps 

Moreover, it can be easily seen that the control of anomalous diffusion problem generated by FDEs can be reformulated as a problem of analysis of infinite-dimensional control system. However, in the case of diffusion systems, it should be pointed out that, in general, not all the states can be reached. So in this paper, we first introduce some notations on the regional controllability of FDEs, i.e., the system under consideration is only exactly (or approximately) controllable on a subset of the state space, which can be regarded as a generalization of integer-order diffusion systems. Based on the semigroup theory, the regional controllability with minimum energy of time FDEs of two different kinds of cases: $B \in L(R^m, L^2(\Omega))$ and $B \notin L(R^m, L^2(\Omega))$ are discussed. More precisely, when $B \in L(R^m, L^2(\Omega))$, our main result is derived by utilizing the Balder’s theorem. And when $B \notin L(R^m, L^2(\Omega))$, the Hilbert Uniqueness Methods (HUMs), which were first introduced by Lions, are used to obtain the regional controllability with minimum energy of the system under consideration.

The rest of this paper is organized as follows: some concepts on regional controllability are presented in the next section. In Section 3, our main results on the regional controllability analysis of time FDEs are given. Two applications are worked out in the last section.

2. Preliminaries

Let $\Omega$ be an open bounded subset of $R^n$ with smooth boundary $\partial \Omega$, $Q = \Omega \times [0, T]$, $\Sigma = \partial \Omega \times [0, T]$. Let $L^p(0, T; \Omega)$ $(p \geq 1)$ be the space of Bochner integrable functions on $[0, T]$ with the norm $\|x\|_{L^p(\Omega)} = (\int_0^T \|x(s)\|^p_{R^n} ds)^{1/p}$ and consider the following abstract fractional state-space system

$$C_0 \mathcal{D}_t^\alpha z(t) = A z(t) + Bu(t), \quad z(0) = z_0 \in D(A),$$

where $t \in [0, T]$, $0 < \alpha < 1$, $z \in L^2(0, T; \Omega)$, $C_0 \mathcal{D}_t^\alpha$ is the Caputo fractional derivative, $D(A)$ holds for the domain of the operator $A$ and $A$ generates a strongly continuous semigroup $\{\Phi(t)\}_{t \geq 0}$ on the Hilbert state space $L^2(\Omega)$. In addition, $z_0 \in L^2(\Omega)$, $u \in L^2(0, T; R^m)$ and $B : R^m \to L^2(\Omega)$ is a linear operator to be specified later.

Next, we will introduce some definitions and lemmas to be used in the sequel.

**Definition 2.1.** [1, 2] The Caputo fractional derivative of order $\alpha > 0$ of a function $z$ is given by

$$C_0 \mathcal{D}_t^\alpha z(t) = \left\{ \begin{array}{ll} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{\partial^n}{\partial s^n} z(s) ds, & \alpha < n, \\
\frac{\partial^n}{\partial t^n} z(t), & \alpha = n, \end{array} \right. \quad (2.2)$$
where \( t \geq 0, n - 1 < \alpha \leq n, n \in \mathbb{N} \), provided that the right side of (2) is pointwise defined.

Let \( \omega \subseteq \Omega \) be a given region of positive Lebesgue measure and \( z_T \in L^2(\omega) \) (the target function) be a given element of the state space.

2.1. The case of \( B \in L(L^m(R^m, L^2(\Omega))) \)

If \( B \in L(L^m(R^m, L^2(\Omega))) \), i.e., \( B \) is a bounded continuous operator from \( R^m \) to \( L^2(\Omega) \) and there exists a constant \( M_B \) such that \( \|B\| \leq M_B \).

Based on the argument from the contribution [4], we get that

\[ \text{Definition 2.2. [4]} \]

For any given \( u \in L^2(0, T; R^m) \), a function \( z \in L^2(0, T; \Omega) \) is said to be a mild solution of the system (2.1), denoted by \( z(\cdot, u) \), if it satisfies

\[ z(t, u) = S_\alpha(t)z_0 + \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s)Bu(s)ds, \] (2.3)

where

\[ S_\alpha(t) = \int_0^\infty \phi_\alpha(\theta)\Phi(t^\alpha \theta)d\theta \] (2.4)

and

\[ K_\alpha(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta)\Phi(t^\alpha \theta)d\theta. \] (2.5)

Here \( \{\Phi(t)\}_{t \geq 0} \) is the strongly continuous semigroup generated by \( A \), \( \phi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \psi_\alpha(\theta^{-\frac{1}{\alpha}}) \) and \( \psi_\alpha \) is a probability density function defined by

\[ \psi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \theta \in (0, \infty) \]

satisfying the following properties [19]

\[ \int_0^\infty e^{-\lambda \theta} \psi_\alpha(\theta)d\theta = e^{-\lambda^\alpha}, \int_0^\infty \psi_\alpha(\theta)d\theta = 1, \alpha \in (0, 1) \] (2.6)

and

\[ \int_0^\infty \theta^\nu \phi_\alpha(\theta)d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha \nu)}, \nu \geq 0. \] (2.7)

In order to prove our main results, the following hypotheses are needed.

\( (S_1) \) The semigroup \( \{\Phi(t)\}_{t \geq 0} \) generated by operator \( A \) is uniformly bounded on \( L^2(\Omega) \), i.e., there exists a constant \( M > 0 \) such that

\[ \sup_{t \geq 0} \|\Phi(t)\| \leq M. \] (2.8)

\( (S_2) \) For any \( t > 0 \), \( \Phi(t) \) is a compact operator.
Lemma 2.1. [4]

(i) For any \( t \geq 0 \) the operator \( S_\alpha(t) \) and \( K_\alpha(t) \) are linear bounded operators, i.e., for any \( x \in L^2(\Omega) \),

\[
\|S_\alpha(t)x\|_{L^2(\Omega)} \leq M\|x\|_{L^2(\Omega)} \tag{2.9}
\]

and

\[
\|K_\alpha(t)x\|_{L^2(\Omega)} \leq \frac{\alpha M}{\Gamma(1 + \alpha)}\|x\|_{L^2(\Omega)}, \tag{2.10}
\]

where \( M \) is defined in the inequality (8).

(ii) Operators \( \{S_\alpha(t)\}_{t \geq 0} \) and \( \{K_\alpha(t)\}_{t \geq 0} \) are strongly continuous, this is, for \( \forall x \in L^2(\Omega) \) and \( 0 \leq t_1 \leq t_2 \leq T \), we have

\[
\|S_\alpha(t_1)x - S_\alpha(t_2)x\|_{L^2(\Omega)} \to 0 \tag{2.11}
\]

and

\[
\|K_\alpha(t_1)x - K_\alpha(t_2)x\|_{L^2(\Omega)} \to 0 \text{ as } t_1 \to t_2. \tag{2.12}
\]

(iii) For \( t > 0 \), \( S_\alpha(t) \) and \( K_\alpha(t) \) are all compact operators if \( \Phi(t) \) is compact.

Definition 2.3.

(a_1) The system (2.1) is said to be regionally exactly controllable(or \( \omega \)–exactly controllable) if for any \( z_T \in L^2(\omega) \), there exists a control \( u \in L^2(0,T;\mathbb{R}^m) \) such that

\[
p_\omega z(T,u) = z_T. \tag{2.13}
\]

(a_2) The system (2.1) is said to be regionally approximately controllable(or \( \omega \)–approximately controllable) if for all \( z_T \in L^2(\omega) \), given \( \varepsilon > 0 \), there exists a control \( u \in L^2(0,T;\mathbb{R}^m) \) such that

\[
\|p_\omega z(T,u) - z_T\|_{L^2(\omega)} \leq \varepsilon, \tag{2.14}
\]

where \( p_\omega : L^2(\Omega) \to L^2(\omega) \), defined by \( p_\omega z = z|_\omega \), is a projection operator.

2.2. The case of \( B \notin L(\mathbb{R}^m, L^2(\Omega)) \)

If \( B \notin L(\mathbb{R}^m, L^2(\Omega)) \), similar to the argument in [10, 11, 12], the extension definitions on regional controllability are introduced.

Take into account that the system (1) is line, without loss of generality, we suppose that \( z_0 = 0 \) in the following discussion. Let \( H : L^2(0,T;\mathbb{R}^m) \to L^2(\Omega) \) be

\[
Hu = \int_0^T (T - s)^{\alpha - 1}K_\alpha(T - s)Bu(s)ds, \forall u \in L^2(0,T;\mathbb{R}^m). \tag{2.15}
\]
It follows from Definition 2.3 that the system (1) is regionally approximately (exactly) controllable on $\omega$ if and only if

$$\text{imp}_\omega H = L^2(\omega)$$  \hspace{1cm} (2.16)

Definition 2.4.

(b1) The system (2.1) is regionally exactly controllable if and only if

$$\ker p_\omega + \text{im} H = L^2(\Omega).$$  \hspace{1cm} (2.17)

(b2) The system (2.1) is said to be regionally approximately controllable if and only if

$$\ker p_\omega + \overline{\text{im} H} = L^2(\Omega).$$  \hspace{1cm} (2.18)

Suppose that $\{\Phi^*(t)\}_{t \geq 0}$, generated by the adjoint operator of $A$, is also a $C_0$ semigroup on the Hilbert state space $L^2(\Omega)$. Then for any $v \in L^2(\Omega)$, it follows from $\langle Hu, v \rangle = \langle u, H^*v \rangle$ that

$$H^*v = B^*(T - s)^{\alpha - 1} K_\alpha^*(T - s)v,$$  \hspace{1cm} (2.19)

where $\langle \cdot, \cdot \rangle$ is the duality pairing of space $L^2(\omega)$, $B^*$ is the adjoint operator of $B$ and $K_\alpha^*(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta) \Phi^*(t^\alpha \theta) d\theta$. Then we have $\overline{\text{im} p_\omega H} = L^2(\omega)$ is equivalent to

$$\ker H^* \cap \text{im} p_\omega^* = \{0\},$$  \hspace{1cm} (2.20)

where $p_\omega^*: L^2(\omega) \to L^2(\Omega)$, the adjoint operator of $p_\omega$, is

$$p_\omega^*z(x) := \begin{cases} z(x), & x \in \omega, \\ 0, & x \in \Omega \setminus \omega. \end{cases}$$  \hspace{1cm} (2.21)

3. Regional controllability analysis of the time FDEs

In this section, we will explore the possibility of finding a minimum energy control which steer the time FDEs (2.1) from the initial state $z_0$ to a target function $z_T$ on the region $\omega$.

Let $U_T = \{u \in L^2(0, T; \mathbb{R}^m) : p_\omega z(T, u) = z_T\}$. Consider the following minimum problem

$$\inf_u J(u) = \inf_u \left\{ \int_0^T \|u(t)\|_{\mathbb{R}^m}^2 dt : u \in U_T \right\}.$$  \hspace{1cm} (3.22)
3.1. The case of $B \in \mathbf{L}(\mathbb{R}^m, L^2(\Omega))$

**Theorem 3.1.** Suppose that $B \in \mathbf{L}(\mathbb{R}^m, L^2(\Omega))$ and the assumptions $(S_1), (S_2)$ hold, then the minimum problem (3.22) admits at least one optimal solution provided that the system $[2.1]$ is $\omega-$approximately controllable.

**Proof.** It is easy to see that $U_T$ is a closed and convex set. We first prove that the operator $H$ is strongly continuous (see p.597, [14]), which admits the existence of optimal control to the minimum problem (22). Moreover, according to the argument in [15], since the operator $H$ is linear and continuous, we only need to show that the operator is precompact.

For any $t \in [0, T], z_0 \in L^2(\Omega)$, it follows from Lemma 2.4 that the term $S_{\alpha}(t)z_0$ in Eq. (2.3) is strongly continuous. Let $N : L^2(\mathbb{R}^m) \rightarrow L^2(\Omega)$ be

\[ Nu(t) := \int_0^t (t-s)^{\alpha-1} K_{\alpha}(t-s) Bu(s)ds, \quad t \in [0, T]. \quad (3.23) \]

and we next show that $N$ is a relatively compact operator.

Let $\rho_r = \{ u \in L^2(0, T; \mathbb{R}^m) : \|u\|_{L^2(0, T; \mathbb{R}^m)} \leq r \}$. For any fixed $t \in (0, T], \varepsilon, \delta \in (0, t)$, $u \in \rho_r$, let

\[ \tilde{N}_{(\varepsilon, \delta)}(t) = \alpha \int_0^{t-\varepsilon} \int_0^\infty (t-s)^{\alpha-1} \theta \phi(\theta) \Phi((t-s)^{\alpha} \theta) Bu(s)d\theta ds. \]

Since $\Phi(\varepsilon^\alpha) \delta$ is compact and

\[ \tilde{N}_{(\varepsilon, \delta)}(t) = \Phi(\varepsilon^\alpha) \delta \int_0^{t-\varepsilon} \int_0^\infty (t-s)^{\alpha-1} \theta \phi(\theta) \Phi((t-s)^{\alpha} \theta - \varepsilon^\alpha \delta) Bu(s)d\theta ds. \]

we get that $\tilde{N}_{(\varepsilon, \delta)}$ is relatively compact. Together with $\|Bu(\cdot)\| \leq M_B r < \infty$, by (i) in Lemma 2.1, for any $t \in [0, T]$, we have

\[
\| \tilde{N}u(t) - \tilde{N}_{(\varepsilon, \delta)}u(t) \| = \alpha \int_0^t \int_0^\delta (t-s)^{\alpha-1} \theta \phi(\theta) \Phi((t-s)^{\alpha} \theta) Bu(s)d\theta ds \\
+ \int_0^t \int_0^\delta (t-s)^{\alpha-1} \theta \phi(\theta) \Phi((t-s)^{\alpha} \theta) Bu(s)d\theta ds \\
- \int_0^{t-\varepsilon} \int_0^\delta (t-s)^{\alpha-1} \theta \phi(\theta) \Phi((t-s)^{\alpha} \theta) Bu(s)d\theta ds \|
\leq \alpha \int_0^t \int_0^\delta (t-s)^{\alpha-1} \theta \phi(\theta) \Phi((t-s)^{\alpha} \theta) Bu(s)d\theta ds \\
+ \alpha \int_0^{t-\varepsilon} \int_0^\delta (t-s)^{\alpha-1} \theta \phi(\theta) \Phi((t-s)^{\alpha} \theta) Bu(s)d\theta ds \\
\leq MM_B r \Gamma^{\alpha} \int_0^\delta \theta \phi(\theta) d\theta + \frac{MM_B r \varepsilon^\alpha}{\Gamma(1+\alpha)} \rightarrow 0
\]

as $\varepsilon, \delta \rightarrow 0$, where $M$ is defined in Eq. (8). Then we conclude that $N_{\rho_r}$ is a relatively compact set in $L^2(\Omega)$.
Next, we shall prove that $Nu$ is equicontinuous on $[0, T]$. For any $u \in \mathcal{U}$, $0 \leq \sigma_1 < \sigma_2 \leq T$,

$$\|Nu(\sigma_2) - Nu(\sigma_1)\| \leq \left\| \int_0^{\sigma_1} [(\sigma_2 - s)^{\alpha - 1} - (\sigma_1 - s)^{\alpha - 1}]K_\alpha(\sigma_2 - s)Bu(s)ds \right\|
$$

$$+ \left\| \int_{\sigma_1}^{\sigma_2} (\sigma_1 - s)^{\alpha - 1}[K_\alpha(\sigma_2 - s) - K_\alpha(\sigma_1 - s)]Bu(s)ds \right\|
$$

$$+ \left\| \int_{\sigma_1}^{\sigma_2} (\sigma_2 - s)^{\alpha - 1}K_\alpha(\sigma_2 - s)Bu(s)ds \right\|
$$

$$\leq \frac{MM_Br}{\Gamma(1 + \alpha)}(\sigma_2^{\alpha} - \sigma_1^{\alpha} + (\sigma_2 - \sigma_1)^{\alpha}) + A + \frac{MM_Br}{\Gamma(1 + \alpha)}(\sigma_2 - \sigma_1)^{\alpha}
$$

where $A = \left\| \int_0^{\sigma_1} (\sigma_1 - s)^{\alpha - 1}[K_\alpha(\sigma_2 - s) - K_\alpha(\sigma_1 - s)]Bu(s)ds \right\|$. Since $\varepsilon > 0$ small enough, we have

$$A \leq \int_0^{\sigma_1 - \varepsilon} (\sigma_1 - s)^{\alpha - 1}\|K_\alpha(\sigma_2 - s) - K_\alpha(\sigma_1 - s)\|\|Bu(s)\|ds
$$

$$+ \int_{\sigma_1 - \varepsilon}^{\sigma_1} (\sigma_1 - s)^{\alpha - 1}\|K_\alpha(\sigma_2 - s) - K_\alpha(\sigma_1 - s)\|\|Bu(s)\|ds
$$

$$\leq \left[ \frac{MM_Br}{\alpha} (\sigma_1^q - \varepsilon^q) \right] \sup_{s \in [\sigma_1 - \varepsilon, \sigma_1]} \|K_\alpha(\sigma_2 - s) - K_\alpha(\sigma_1 - s)\|
$$

$$+ \frac{2MM_Br}{\Gamma(1 + \alpha)}\varepsilon^q \to 0
$$

as $\sigma_2 \to \sigma_1$ due to the continuity of $K_\alpha(t)(t > 0)$ in the uniform operator topology. It follows from the Arzela-Ascoli theorem [16] that the operator $N$ is precompact. Thus, $H$ is strongly continuous, which guarantees the existence of optimal control to the minimum problem (3.22) under the fact that $U_T$ is a closed and convex set.

Further, if the system (2.1) is $\omega-$approximately controllable, for any $z_T \in \omega$, suppose that $J(u^*) = \inf_u J(u) = \varepsilon < \infty$, by the definition of infimum, we can deduce that there exists a sequence $\{u_i\}_{i=1, 2, \ldots}$ such that $p_\omega z(T, u_i) = z_T$, $u_i \in U_T \subseteq L^2(0, T; \mathbb{R}^m)(i = 1, 2, 3, \ldots)$ and $J(u_i) \to J(u^*)$. Then we have $u_i \to u^*$ in $L^2(0, T, \mathbb{R}^m)$.

For any $t \in [0, T]$, by Definition 2.2 and Lemma 2.1, we get that

$$\|p_\omega z(t, u^*) - p_\omega z(t, u_i)\|_{L^2(\Omega)} = \left\| p_\omega \int_0^t (t - s)^{\alpha - 1}K_\alpha(t - s)B(u^*(s) - u_i(s))ds \right\|
$$

$$\leq \left\| \int_0^t (t - s)^{\alpha - 1}K_\alpha(t - s)B(u^*(s) - u_i(s))ds \right\|
$$

$$\leq \frac{\alpha MM_B}{\Gamma(1 + \alpha)} \int_0^t (t - s)^{\alpha - 1}\|u^*(s) - u_i(s)\|_{L^2(\mathbb{R}^m)}ds,$$
which yields that 
\[ p_\omega z(t, u_i) \to p_\omega z(t, u^*) \text{ in } C(0, T, \omega) \text{ as } i \to \infty. \]

And since \( U_T \) is closed and convex, from Marzur Lemma \[16\] we see that \( u^* \in U_T \). Thus it follows from the Balder’s theorem in \[17\] that
\[ \varepsilon = J(u^*) = \lim_{i \to \infty} J(u_i) \geq J(u^*) \geq \varepsilon, \]
which means that \( u^* \) is the optimal solution of the minimum problem \[3.22\]. This completes the proof.

3.2. The case of \( B \notin L(\mathbb{R}^m, L^2(\Omega)) \)

If \( B \notin L(\mathbb{R}^m, L^2(\Omega)) \), for example, when the control is pointwise or boundary control. The operator \( N \) defined in Eq. (23) is unbounded and then \( N \) is not relatively compact and new methods should be introduced.

Here we will introduce the Hilbert uniqueness methods (HUMs), which is first introduced by Lions in \[18\] to study the controllability problems of a linear distributed parameter systems. Further, we note that this method is also available when \( B \) is a bounded continuous operator.

Let \( Z = \text{imp}_\omega H \subseteq L^2(\omega) \), by duality \( Z \subseteq L^2(\omega) \subseteq Z^* \) and for any \( f \in Z^* \), define
\[ \| f \|_{Z^*} := \int_0^T \| B^* (T - s)^{\alpha - 1} K_\omega^* (T - s) p_\omega^* f \|^2 ds, \]
where \( p_\omega^* \) is defined in Eq. (2.21).

**Lemma 3.1.** \( \| \cdot \|_{Z^*} \) is a norm of space \( Z^* \) provided that the system \( (2.1) \) is \( \omega\)–approximately controllable.

**Proof.** If the system \( (2.1) \) is \( \omega\)–approximately controllable, we get that \( \ker H^* p_\omega^* = \{0\} \), i.e.,
\[ B^* (T - s)^{\alpha - 1} K_\omega^* (T - s) p_\omega^* f = 0 \Rightarrow f = 0. \]
Hence, for any \( f \in Z^* \), it follows from
\[ \| f \|_{Z^*} = \int_0^T \| B^* (T - s)^{\alpha - 1} K_\omega^* (T - s) p_\omega^* f \|^2 ds = 0 \Leftrightarrow B^* (T - s)^{\alpha - 1} K_\omega^* (T - s) p_\omega^* f = 0 \]
that \( \| \cdot \|_{Z^*} \) is a norm of space \( Z^* \) and the proof is complete.

Denote the completion of the set \( Z^* \) with the norm \( \| \cdot \|_{Z^*} \) again by \( Z^* \). For each \( f \in Z^* \), since \( f \) is a linear bounded functional on \( Z \), by the Riesz representation theorem, there exists a unique element in \( Z \), denoted by \( Pf \), such that
\[ f(y) = (Pf, y) \quad \text{for all } y \in Z, \]
where $(\cdot, \cdot)$ is the inner product in space $Z$. Then we get that $P : Z^* \rightarrow Z$ is a linear operator and the following lemma holds.

**Lemma 3.2.** The operator $P : Z^* \rightarrow Z$ is isometry.

**Proof.** For any $f \in Z^*$, it follows from (3.26) that

$$
\| Pf \|_Z = \sup_{\| y \|_Z = 1} (Pf, y) = \sup_{\| y \|_Z = 1} \| f(y) \| = \| f \|_{Z^*}.
$$

Then $\mathcal{R}(P) \subseteq Z$ is a closed subspace. To complete the proof, we should only show that $\mathcal{R}(P) = Z$. If not so, then there exists a $y_0 \in Z$, $y_0 \neq 0$ such that $(Px, y_0) = 0$. By (3.26), we have

$$
f(y_0) = 0 \quad \text{for all } f \in Z^*,
$$

which implies that $y_0 = 0$, a contradiction. Then we see that $\mathcal{R}(P) = Z$ and the proof is complete.

Further, let $\wedge : Z^* \rightarrow Z$ be

$$
\wedge f = p_\omega \varphi_1(T),
$$

where $\varphi_1(t)$ is defined by

$$
\begin{cases}
C_0 D_\alpha \varphi_1(t) = A \varphi_1(t) + BB^*(T-t)^{\alpha-1}K_\alpha^*(T-t)f, \\
\varphi_1(0) = 0.
\end{cases}
$$

(3.28)

Since for any $f \in Z^*$, $y \in Z$, by Hölder’s inequality, we have

$$
(\wedge f, y) = \int_\Omega p_\omega \int_0^T (T-s)^{\alpha-1}K_\alpha(T-s)B \times B^*(T-s)^{\alpha-1}K_\alpha^*(T-s)p_\omega^* f(x)dsy(x)dx
$$

$$
\leq \int_0^T \| B^*(T-s)^{\alpha-1}K_\alpha^*(T-s)p_\omega^* f \|^2 ds \| y \|_Z
$$

$$
\leq \| f \|_{Z^*} \| y \|_Z
$$

and $\| \wedge f \|_Z \leq \| f \|_{Z^*}$. Further, for any $f \in Z^*$, we have

$$
(\wedge f, f) = \int_\Omega p_\omega \int_0^T (T-s)^{\alpha-1}K_\alpha(T-s)B \times B^*(T-s)^{\alpha-1}K_\alpha^*(T-s)p_\omega^* f(x)dsf(x)dx
$$

$$
= \int_0^T \int_\Omega \left[ B^*(T-s)^{\alpha-1}K_\alpha^*(T-s)p_\omega^* f(x) \right]^2 dx ds.
$$

Then if the system (2.1) is $\omega-$approximately controllable on $[0, T]$, we get that $f = 0$. Thus it follows from the uniqueness of $P$ that $\wedge$ is an isomorphism from $Z^*$ to $Z$. 

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Next, suppose that \( \varphi_0(t) \) satisfies

\[
\begin{align*}
\frac{C}{\alpha} D_t^\alpha \varphi_0(t) &= A \varphi_0(t), \\
\varphi_0(0) &= z_0 \in D(A),
\end{align*}
\]

for all \( z_T \in L^2(\omega) \), we have \( z_T = p_\omega [\varphi_1(T) + \varphi_0(T)] \). Further, let \( f \) be the solution of the following equation

\[
\wedge f := z_T - p_\omega \varphi_0(T). 
\]

Then we are ready to state the following theorem.

**Theorem 3.2.** If the system (2.1) is \( \omega \)–approximately controllable, then for any \( z_T \in L^2(\omega) \), (3.30) has a unique solution \( f \in Z^* \) and the control

\[
u^* = B^*(T - \cdot)^{\alpha-1} K_\alpha^*(T - \cdot)p_\omega^* f
\]

steers the system to \( z_T \) at time \( T \) in \( \omega \). Moreover, \( u^* \) is the solution of the minimum problem (3.22).

**Proof.** By Lemma 3.1, we get that if the system (2.1) is \( \omega \)–approximately controllable, then \( \| \cdot \|_{Z^*} \) is a norm of space \( Z^* \). Let the completion of \( Z^* \) with respect to the norm \( \| \cdot \|_{Z^*} \) again by \( Z^* \). Then next we show that the equation (3.30) has a unique solution in \( Z^* \).

For any \( f \in Z^* \), by the definition of operator \( \wedge \) in (3.27), we get that

\[
< f, \wedge f > = < f, p_\omega \varphi_1(T) > \\
= < f, p_\omega \int_0^T (T - s)^{\alpha-1} K_\alpha(T - s) Bu^*(s) ds > \\
= \int_0^T < f, p_\omega(T - s)^{\alpha-1} K_\alpha(T - s) Bu^*(s) > ds \\
= \int_0^T \| B^*(T - s)^{\alpha-1} K_\alpha^*(T - s)p_\omega^* f \|_{Z^*}^2 ds \\
= \| f \|_{Z^*}^2.
\]

Hence, it follows from Lemma 3.2 and the Theorem 2.1 in [20] that the equation (3.30) admits a unique solution in \( Z^* \). Further, let \( u = u^* \) in problem (2.1), we see that \( p_\omega z(T, u^*) = z_T \).

For any \( u_1, u_2 \in L^2(0, T, R^m) \) with \( p_\omega z(T, u_1) = z_T \) and \( p_\omega z(T, u_2) = z_T \), we obtain that \( p_\omega [z(T, u_1) - z(T, u_2)] = 0 \). And for any \( f \in Z^* \), we have

\[
< f, p_\omega [z(T, u_1) - z(T, u_2)] >= 0.
\]

It follows from

\[
< p_\omega H(u_1 - u_2), f >=< u_1 - u_2, H^* p_\omega^* f >
\]
that
\[ \int_0^T < u_1(s) - u_2(s), B^*(T-s)^{\alpha-1}K_\alpha^*(T-s)p_\omega^*f > ds = 0. \]
Then by
\[
J'(u_1)(u_1 - u_2) = 2 \int_0^T < u_1(s), u_1(s) - u_2(s) > ds \\
= 2 \int_0^T < u^*(s), u_1(s) - u_2(s) > ds \\
= 0,
\]
we obtain that \( u^* \) is the solution of the minimum problem (3.22). This completes the proof.

4. Example

In this section, we will introduce two examples which is reachable on \( \omega \) but not reachable on the whole domain.

Example 4.1. Let us consider the following one dimension time FDEs with \( Bu = p_{[a_1,a_2]}u, 0 \leq a_1 \leq a_2 \leq 1 \)
\[
\begin{aligned}
\frac{C}{6}D_t^{0.7}z(x, t) &= \frac{\partial^2}{\partial x^2}z(x, t) + p_{[a_1,a_2]}u(t), \quad [0, 1] \times [0, T] \\
z(x, 0) &= z_0, \quad [0, 1] \\
z(0, t) &= z(1, t) = 0. \quad [0, T]
\end{aligned}
\]
Corresponding to system (2.11), we have \( A = \frac{\partial^2}{\partial x^2} \) and
\[
\Phi(t)z(x) = \sum_{i=1}^{\infty} \exp(\lambda_1^i t)(z, \xi_i)_{L^2([0,1])}\xi_i(x), \quad x \in [0, 1],
\]
where
\[
\lambda_i = -i^2\pi^2 \quad \text{and} \quad \xi_i(x) = \sqrt{2}\sin(i\pi x), \quad x \in [0, 1].
\]
Then we get that the hypotheses (S1) and (S2) hold with \( M = 1 \). Further, we have
\[
K_{0.7}(t)z(x) = 0.7 \int_0^\infty \theta \phi_{0.7}(\theta)\Phi(t^{0.7}\theta)z d\theta \\
= 0.7 \int_0^\infty \theta \phi_{0.7}(\theta) \sum_{i=1}^{\infty} \exp(\lambda_i^i t^{0.7}\theta)(z, \xi_i)_{L^2([0,1])}\xi_i(x) d\theta \\
= 0.7 \sum_{i=1}^{\infty} (z, \xi_i)_{L^2([0,1])}\xi_i(x) \int_0^\infty \theta \phi_{0.7}(\theta) \exp(\lambda_i^i t^{0.7}\theta) d\theta.
\]
It follows from (2.7) and the Taylor expansion of exponential function that
\[
K_{0.7}(t)z(x) \\
= 0.7 \sum_{i=1}^{\infty} (z, \xi_i)_{L^2([0,1])}\xi_i(x) \sum_{j=0}^{\infty} \frac{(\lambda_i^i t^{0.7})^j}{j!}d\theta \sum_{j=0}^{\infty} \frac{(\lambda_i^i t^{0.7})^j}{(1+j\theta_{0.7}+0.7)} \\
= \sum_{i=1}^{\infty} E_{0.7,0.7}(\lambda_i^i t^{0.7})(z, \xi_i)_{L^2([0,1])}\xi_i(x),
\]
where $E_{\alpha,\beta}(z) := \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha+i\beta)}$, $\Re \alpha > 0$, $\beta, z \in \mathbb{C}$ is known as the generalized Mittag-Leffler function. Similarly, we have

$$S_{0,7}(t)z(x) = \int_0^t \phi_{0,7}(\theta)\Phi(t^{0.7}\theta)d\theta$$

(4.35)

$$= \sum_{i=1}^{\infty} (z, \xi_i)_{L^2(0,1)} E_{0,7,1}(\lambda_i t^{0.7})\xi_i(x).$$

(4.36)

What’s more, since $A = \frac{\partial^2}{\partial x^2}$ generates a compact, analytic, self-adjoint $C_0$ semigroup, we have

$$(H^*z)(t) = B^*(T-t)^{-0.3}K_{0,7}^*(T-t)z(t)$$

$$= B^*(T-t)^{-0.3} \sum_{i=1}^{\infty} E_{0,7,0.7}(\lambda_i(T-t)^{0.7})(z, \xi_i)_{L^2(0,1)}\xi_i(x)$$

$$= (T-t)^{-0.3} \sum_{i=1}^{\infty} E_{0,7,0.7}(\lambda_i(T-t)^{0.7})(z, \xi_i)_{L^2(0,1)} \int_{a_1}^{a_2} \xi_i(x)dx.$$ 

Then it follows from $\int_{a_1}^{a_2} \xi_i(x)dx = \frac{2}{\sqrt{7}} \sin \frac{i\pi(a_1+a_2)}{2} \sin \frac{i\pi(a_1-a_2)}{2}$ that $\ker H^* \neq \{0\}$ ($\text{im} H \neq L^2(\omega)$) when $a_2 - a_1 \in Q$, i.e., the system (4.31) is not weakly controllable when $a_2 - a_1 \in Q$.

Thus, we can conclude that the system (4.31) is not weakly controllable on $[0, 1]$ but on some appropriately subregion $[a_1, a_2] \subseteq [0, 1]$ and according to Theorem 3.1, the minimum problem (3.22) admits at least one optimal solution.

**Example 4.2.** Consider the following time FDEs with pointwise control $Bu = u(t)\delta(x-b)$, $0 < b < 1$, i.e., $B \notin L^1(\mathbb{R}^m, L^2(\Omega))$

$$\begin{cases}
C D_t^{0.7} z(x, t) = \frac{\partial^2}{\partial x^2} z(x, t) + u(t)\delta(x-b), & [0, 1] \times [0, T] \\
z(x, 0) = 0, & [0, 1] \\
z(0, t) = z(1, t) = 0. & [0, T]
\end{cases}$$

(4.37)

Here $Z = L^2(0, 1)$, let $\omega = [\sigma_1, \sigma_2] \subseteq [0, 1]$ and if the system (4.37) is $\omega$—approximately controllable, since $A = \frac{\partial^2}{\partial x^2}$ generates a compact, analytic, self-adjoint $C_0$ semigroup, similarly to the argument above, we have

$$\lambda_i = -i^2\pi^2, \quad \xi_i(x) = \sqrt{2} \sin(i\pi x), \quad x \in [0, 1],$$

(4.38)

$$\Phi(t)z(x) = \sum_{i=1}^{\infty} \exp(\lambda_i t)(z, \xi_i)_{L^2(0,1)}\xi_i(x), \quad x \in [0, 1]$$

(4.39)

and

$$K_{0,7}(t)z(x) = \sum_{i=1}^{\infty} E_{0,7,0.7}(\lambda_i t^{0.7})(z, \xi_i)_{L^2(0,1)}\xi_i(x),$$

(4.40)
Moreover, by Lemma 3.1, we get that if the system (2.1) is \( \omega \)-approximately controllable,

\[
\begin{align*}
    f & \to \|f\|_{Z^*} \\
    & = \int_0^T \|(T - s)^{-0.3} K_\alpha^*(T - s)p_\omega^* f(b)\|^2 ds \\
    & = \int_0^T \left\| (T - t)^{-0.3} \sum_{i=1}^{\infty} E_{0.7,0.7}(\lambda_i(T - t)^{0.7})(z, \xi_i)_{L^2(0,1)} p_\omega^* f(b) \right\|^2 ds
\end{align*}
\]

defines a norm on \( Z^* \). It follows from Lemma 3.2 that

\[
\wedge f = p_\omega \varphi_1(T),
\]

is a isometry form \( Z^* \) to \( Z \), where \( \varphi_1(x, t) \) is the solution of the following equations

\[
\begin{align*}
    & C_0 D_t^{0.7} \varphi_1(x, t) = \frac{\partial^2}{\partial x^2} \varphi_1(x, t) + (T - t)^{\alpha - 1} K_\alpha^*(T - t)f(b), \\
    & \varphi_1(x, 0) = 0. \\
    & \varphi_1(0, t) = \varphi_1(1, t) = 0.
\end{align*}
\]

Then by Theorem 3.2, we see that the control

\[
u^*(t) = (T - t)^{-0.3} \sum_{i=1}^{\infty} E_{0.7,0.7}(\lambda_i(T - t)^{0.7})(z, \xi_i)_{L^2(0,1)} p_\omega^* f(b)
\]

steers the system to \( z_T \) at time \( T \) in \( \omega \), where \( f \) is the solution of equations

\[
\wedge f = z_T - p_\omega \varphi_0(\cdot, T),
\]

and \( \varphi_0(t) \) solves

\[
\begin{align*}
    & C_0 D_t^{0.7} \varphi_0(x, t) = \frac{\partial^2}{\partial x^2} \varphi_0(x, t), \\
    & \varphi_0(x, 0) = z_0(x) \in D(A), \\
    & \varphi_0(0, t) = \varphi_0(1, t) = 0.
\end{align*}
\]

Moreover, \( u^* \) is the solution of the minimum problem (3.22).

5. CONCLUSIONS

This paper is the first time to study the regional controllability analysis of the time fractional diffusion equations on two cases: \( B \in L(R^m, L^2(\Omega)) \) and \( B \notin L(R^m, L^2(\Omega)) \), which can be regarded as the extension of the existence contributions on controllability analysis of integer order \[10, 11, 12\]. The results we present here can also be extended to model real dynamic systems in complex dynamic system. For instance, the problem of regional observability of FDEs as well as the case of fractional super-diffusion equations with more complicated dynamics are of great interest.

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