ACTIONS OF THE DERIVED GROUP OF A MAXIMAL UNIPOTENT
SUBGROUP ON G-VARIETIES

DMITRI I. PANYUSHEV

INTRODUCTION

The ground field \( k \) is algebraically closed and of characteristic zero. Let \( G \) be a semisimple simply-connected algebraic group over \( k \) and \( U \) a maximal unipotent subgroup of \( G \). One of the fundamental invariant-theoretic facts, which goes back to Hadžiev [9], is that \( k[G/U] \) is a finitely generated \( k \)-algebra and regarded as \( G \)-module it contains every finite-dimensional simple \( G \)-module exactly once. From this, one readily deduces that the algebra of \( U \)-invariants, \( k[G/U]^U \), is polynomial. More precisely, choose a maximal torus \( T \subset \text{Norm}_G(U) \). Let \( r \) be the rank of \( G \), \( \varpi_1, \ldots, \varpi_r \) the fundamental weights of \( T \) corresponding to \( U \), and \( \alpha_1, \ldots, \alpha_r \) the respective simple roots. Set \( \mathfrak{X}_+ = \sum_{i=1}^r \mathbb{N} \varpi_i \), and let \( R(\lambda) \) denote the simple \( G \)-module with highest weight \( \lambda \in \mathfrak{X}_+ \). Then

\[
k[G/U] \cong \bigoplus_{\lambda \in \mathfrak{X}_+} R(\lambda).
\]

Let \( f_i \) be a non-zero element of one-dimensional space \( R(\varpi_i)^U \subset k[G/U]^U \). Then \( k[G/U]^U \) is freely generated by \( f_1, \ldots, f_r \).

For an affine \( G \)-variety \( X \), the algebra of \( U \)-invariants, \( k[X]^U \), is multigraded (by \( T \)-weights). If \( X = V \) is a \( G \)-module, then there is an integral formula for the corresponding Poincaré series [4, Theorem 1]. Using that formula, M. Brion discovered useful “symmetries” of the Poincaré series and applied them (in case \( G \) is simple) to obtaining the classification of simple \( G \)-modules with polynomial algebras \( k[V]^U \) [4, Ch. III]. Afterwards, I proved that similar “symmetries” of Poincaré series occur for conical factorial \( G \)-varieties with only rational singularities [16], [17, Ch.5]. Since there is no integral formula for Poincaré series in general, another technique was employed. Namely, I used the transfer principle for \( U \), “symmetries” of the Poincaré series of \( k[G/U] \), and results of F. Knop relating the canonical module of an algebra and a subalgebra of invariants [13].

Our objective is to extend these results to the derived group \( U' = (U,U) \). In Section 1, we prove that \( R(\lambda)^U \) is a cyclic \( U/U' \)-module for any \( \lambda \in \mathfrak{X}_+ \) and \( \dim R(\lambda)^U = \prod_{i=1}^r ((\lambda, \alpha_i^\vee) + 1) \), where \( \alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i) \), see Theorem 1.6. From these properties, we deduce that \( k[G/U]^U \) is a polynomial algebra of Krull dimension \( 2r \). More precisely, we have \( \dim R(\varpi_i)^U = 2 \) for each \( i \), and if \( (f_i, \tilde{f}_i) \) is a basis in \( R(\varpi_i)^U \), then \( \{ f_i, \tilde{f}_i \mid i = 1, \ldots, r \} \)
freely generate \( \mathbb{k}[G/U]^{U'} \) (see Theorem 1.8). This fact seems to have remained unnoticed before. As a by-product, we show that the subgroup \( TU' \subset G \) is epimorphic (i.e., \( \mathbb{k}[G]^{TU'} = \mathbb{k} \)) if and only if \( G \neq SL_2, SL_3 \).

Section 2 is devoted to general properties of \( U' \)-actions on affine \( G \)-varieties. We show that \( \mathbb{k}[G/U] \) is generated by fundamental \( G \)-modules sitting in it, and using this fact we explicitly construct an equivariant affine embedding of \( G/U' \) with the boundary of codimension \( \geq 2 \) (Theorem 2.2). Since \( \mathbb{k}[G/U'] \) is finitely generated, \( \mathbb{k}[X]^{U'} \) is finitely generated for any affine \( G \)-variety \( X \) [8]. Furthermore, \( \text{Spec}(\mathbb{k}[X]^{U'}) \) inherits some other good properties of \( X \) (factoriality, rationality of singularities) (Theorem 2.3). We also give an algorithm for constructing a finite generating system of \( \mathbb{k}[X]^{U'} \), if generators of \( \mathbb{k}[X]^{U} \) are already known (Theorem 2.4). This appears to be very helpful in classifying simple \( G \)-modules with polynomial algebras of \( U' \)-invariants (for \( G \) simple).

In Section 3, we study the Poincaré series of multigraded algebras \( \mathbb{k}[X]^{U'} \), where \( X \) is factorial affine \( G \)-variety with only rational singularities (e.g. \( X \) can be a \( G \)-module). Assuming that \( G \neq SL_2, SL_3 \), we obtain analogues of our results for Poincaré series of \( \mathbb{k}[X]^{U} \). One of the practical outcomes concerns the case in which \( V \) is a \( G \)-module and \( \mathbb{k}[V]^{U'} \) is polynomial. If \( d_1, \ldots, d_m \) (resp. \( \mu_1, \ldots, \mu_m \)) are the degrees (resp. \( T \)-weights) of basic \( U' \)-invariants, then \( \sum_i d_i \leq \dim V \) and \( \sum_i \mu_i \leq 2\rho - \sum_{j=1}^r \alpha_j \), where \( \rho = \sum_{j=1}^r \alpha_j \). The second inequality requires some explanations, though. Unlike the case of \( U \)-invariants, there is no natural free monoid containing the \( T \)-weights of all \( U' \)-invariants. But for \( G \neq SL_2, SL_3 \), these \( T \)-weights generate a convex cone. Therefore, such a free monoid does exist, and the above inequality for \( \sum_i \mu_i \) is understood as componentwise inequality with respect to any such monoid and its basis. Moreover, \( \sum_i d_i = \dim V \) if and only if \( \sum_i \mu_i = 2\rho - \sum_{j=1}^r \alpha_j \). Again, these relations are to be useful for our classification of polynomial algebras \( \mathbb{k}[V]^{U'} \), which is obtained in Section 5. Note that \( 2\rho - \sum_{j=1}^r \alpha_j \) is the sum of all positive non-simple roots, i.e., the roots of \( U' \).

Section 4 is a kind of combinatorial digression. Let \( \mathcal{C} \) be the cone generated by all \( T \)-weights occurring in \( \mathbb{k}[G/U]^{U'} \). Our description of generators shows that \( \mathcal{C} \) is actually generated by \( \omega_i, \omega_i - \alpha_i \) (\( i = 1, \ldots, r \)). We prove that the dual cone of \( \mathcal{C} \) is generated by the non-simple positive roots (Theorem 4.2). We also obtain a partition of \( \mathcal{C} \) in simplicial cones, which is parametrised by the disjoint subsets on the Dynkin diagram of \( G \).

My motivation to consider \( U' \)-invariants arose from attempts to understand the structure of centralisers of certain nilpotent elements in simple Lie algebras. For applications to centralisers one needs Theorem 1.6 in case of \( SL_3 \), and this was the result initially proved. This application will be the subject of a subsequent article.

**Notation.** If an algebraic group \( Q \) acts on an irreducible affine variety \( X \), then

- \( Q_x = \{ q \in Q \mid q \cdot x = x \} \) is the stabiliser of \( x \in X \);
\[ \mathfrak{k}[X]^{Q} \text{ is the algebra of } Q\text{-invariant polynomial functions on } X. \text{ If } \mathfrak{k}[X]^{Q} \text{ is finitely generated, then } X/\!\!/Q := \text{Spec}(\mathfrak{k}[X]^{Q}), \text{ and the quotient morphism } \pi_{X, Q} : X \rightarrow X/\!\!/Q \text{ is the mapping associated with the embedding } \mathfrak{k}[X]^{Q} \hookrightarrow \mathfrak{k}[X]. \]

- \( \mathfrak{k}(X)^{Q} \) is the field of \( Q\)-invariant rational functions;

Throughout, \( G \) is a semisimple simply-connected algebraic group and \( r = rk\ G. \)

- \( \Delta \) is the root system of \((G, T), \Pi = \{\alpha_{1}, \ldots, \alpha_{r}\} \) are the simple roots corresponding to \( U, \) and \( \varpi_{1}, \ldots, \varpi_{r} \), are the corresponding fundamental weights.

- The character group of \( T \) is denoted by \( \mathfrak{x}. \) All roots and weights are regarded as elements of the \( r\)-dimensional vector space \( \mathfrak{x} \otimes \mathbb{Q} =: \mathfrak{x}_{Q}. \) For any \( \lambda \in \mathfrak{x}_{+}, \lambda^{\ast} \) is the highest weight of the dual \( G\)-module. The \( \mu\)-weight space of \( \mathcal{R}(\lambda) \) is denoted by \( \mathcal{R}(\lambda)_{\mu}. \)

**Acknowledgements.** This work was done during my stay at the Max-Planck-Institut für Mathematik (Bonn). I am grateful to this institution for the warm hospitality and support.

### 1. The Algebra of \( U^{\prime}\)-Invariants on \( G/U \)

For any \( \lambda \in \mathfrak{x}_{+}, \) we wish to study the subspace \( \mathcal{R}(\lambda)^{U^{\prime}}. \) First of all, we notice that \( B \subset \text{Norm}_{G}(U^{\prime}) \) (actually, they are equal if \( G \) has no simple factors \( SL_{2} \)) and therefore \( \mathcal{R}(\lambda)^{U^{\prime}} \) is a \( B/U^{\prime}\)-module. In particular, \( T \) normalises \( U^{\prime} \) and hence \( \mathcal{R}(\lambda)^{U^{\prime}} \) is a direct sum of its own weight spaces. Let \( \mathcal{P}(\lambda) \) be the set of weights of \( \mathcal{R}(\lambda). \) It is a poset with respect to the root order. This means that \( \mu \) covers \( \nu \) if \( \mu - \nu \in \Pi. \) Then \( \lambda \) is the unique maximal element of \( \mathcal{P}(\lambda). \) Let \( e_{i} \in \mathfrak{g} = \text{Lie}(G) \) be a root vector corresponding to \( \alpha_{i} \in \Pi. \)

Given a nonzero \( x \in \mathcal{R}(\lambda)^{U^{\prime}}, \) consider

\[
M_{x} = \{(n_{1}, \ldots, n_{r}) \in \mathbb{N}^{r} \mid e_{1}^{n_{1}} \cdots e_{r}^{n_{r}}(x) \neq 0\}.
\]

We also write \( n = (n_{1}, \ldots, n_{r}) \) and \( e^{n} = e_{1}^{n_{1}} \cdots e_{r}^{n_{r}}. \) Notice that \( e^{n}(x) \) does not depend on the ordering of \( e_{i}^{s} \) since \([e_{i}, e_{j}] \in \text{Lie}(U^{\prime})\) for all \( i, j \) and \( \mathcal{R}(\lambda)^{U^{\prime}} \) is an \( U/U^{\prime}\)-module. We regard \( M_{x} \) as poset with respect to the componentwise inequalities, i.e., \( n \succ n^{\prime} \) if and only if \( n_{i} \geq n_{i}^{\prime} \) for all \( i. \) Clearly, \( M_{x} \) is finite and \((0, \ldots, 0)\) is the unique minimal element of it.

**Lemma 1.1.** Let \( x \in \mathcal{R}(\lambda)^{U^{\prime}} \) be a weight vector. The poset \( M_{x} \) contains a unique maximal element, say \( m = (m_{1}, \ldots, m_{r}). \) Furthermore, \( e^{m}(x) \) is a highest vector of \( \mathcal{R}(\lambda). \)

**Proof.** If \( n \in M_{x} \) is maximal, then \( e_{i}(e^{n}(x)) = 0 \) for each \( i. \) Hence \( e^{n}(x) \) is a highest vector of \( \mathcal{R}(\lambda). \) Next,

\[
\text{the weight of } e^{n}(x) = (\text{the weight of } x) + \sum_{i=1}^{r} n_{i}\alpha_{i}.
\]

Hence all nonzero vectors of the form \( e^{n}(x) \) are linearly independent. This yields the uniqueness of a maximal element. \( \square \)
Corollary 1.2. $M_x$ is a multi-dimensional array, i.e., $M_x = \{(n_1, \ldots, n_r) \mid 0 \leq n_i \leq m_i \forall i\}$.

Let $I_\lambda$ denote the set of $T$-weights in $R(\lambda)^{U'}$. It is a subset of $\mathcal{P}(\lambda)$.

Proposition 1.3. For any $\lambda \in \mathcal{X}_+$, $R(\lambda)^{U'}$ is a multiplicity free $T$-module. More precisely,

$$R(\lambda)^{U'} = \bigoplus_{\mu \in I_\lambda} R(\lambda)^{U_\mu},$$

where $\dim R(\lambda)^{U_\mu} = 1$ for each $\mu$ and $I_\lambda \subset \{\lambda - \sum_i a_i \alpha_i \mid 0 \leq a_i \leq (\lambda, \alpha_i^\vee)\}$.

Proof. If $x \in R(\lambda)^{U_\mu}$ and $(m_1, \ldots, m_r)$ is the maximal element of $M_x$, then $\mu + \sum_i m_i \alpha_i = \lambda$ and $\mu + \sum_i n_i \alpha_i \in \mathcal{P}(\lambda)$ for any $(n_1, \ldots, n_r) \in M_x$. In particular, $\lambda - m_i \alpha_i \in \mathcal{P}(\lambda)$. Whence $m_i \leq (\lambda, \alpha_i^\vee)$ and $I_\lambda \subset \{\lambda - \sum_i a_i \alpha_i \mid 0 \leq a_i \leq (\lambda, \alpha_i^\vee)\}$.

Assume that $x, y \in R(\lambda)^{U_\mu}$ are linearly independent. It follows from Lemma 1.1 that $M_x = M_y$. Since $e^m(x), e^m(y) \in R(\lambda)^{U}$, we have $e^m(xy) = 0$ for some $c \in k^\times$. This means that $M_{x-cy} \neq M_x$, a contradiction! Thus, each $R(\lambda)^{U_\mu}$ is one-dimensional. \qed

Lemma 1.4. $I_\lambda$ is a connected subset in the Hasse diagram of $\mathcal{P}(\lambda)$ that contains $\lambda$.

Proof. Indeed, suppose $0 \neq v \in R(\lambda)^{U_\mu}$. If $e_{\alpha_i}v = 0$ for all $i$, then $v$ is a $U$-invariant and hence $\mu = \lambda$. Otherwise, we have $e_{\alpha_i}v \neq 0$ for some $i$ and therefore $\mu + \alpha_i$ is also a weight of $R(\lambda)^{U'}$. Then we argue by induction. \qed

Proposition 1.5. For any fundamental weight $\varpi_i$, we have $R(\varpi_i)^{U'} = R(\varpi_i)_{\varpi_i} \oplus R(\varpi_i)_{\varpi_i - \alpha_i}$. In particular, $I_{\varpi_i} = \{\varpi_i, \varpi_i - \alpha_i\}$ and $\dim R(\varpi_i)^{U'} = 2$.

Proof. Note that $\varpi_i - \alpha_i \in \mathcal{P}(\varpi_i)$ and $\dim R(\varpi_i)_{\varpi_i - \alpha_i} = 1$, while $\varpi_i - 2\alpha_i \notin \mathcal{P}(\varpi_i)$. We obviously have $R(\varpi_i)^{U'} \supset R(\varpi_i)_{\varpi_i} \oplus R(\varpi_i)_{\varpi_i - \alpha_i}$. Any weight of $R(\varpi_i)$ covered by $\varpi_i - \alpha_i$ is of the form $\varpi_i - \alpha_i - \alpha_j$, where $\alpha_j$ is a simple root adjacent to $\alpha_i$ in the Dynkin diagram of $G$. Since $\varpi_i - \alpha_j \notin \mathcal{P}(\varpi_i)$, Kostant’s weight multiplicity formula shows that $\dim R(\varpi_i)_{\varpi_i - \alpha_i - \alpha_j} = 1$. Since $\alpha_i + \alpha_j$ is a root of $U'$, we have $R(\varpi_i)_{\varpi_i - \alpha_i - \alpha_j} \subset R(\varpi_i)^{U'}$ and it follows from Lemma 1.4 that there cannot be anything else in $R(\varpi_i)^{U'}$. \qed

Set $\tilde{X} = \text{Spec}(k[G/U])$. It is an affine $G$-variety containing $G/U$ as a dense open subset. Recall that $\tilde{X}$ has the following explicit model, see [25]. Let $v_{-\varpi_i}$ be a lowest weight vector in $R(\varpi_i)^*$. Then the stabiliser of $(v_{-\varpi_1}, \ldots, v_{-\varpi_r}) \in R(\varpi_1)^* \oplus \cdots \oplus R(\varpi_r)^*$ is the maximal unipotent subgroup that is opposite to $U$ and

$$\tilde{X} \simeq \frac{G \cdot (v_{-\varpi_1}, \ldots, v_{-\varpi_r})}{R(\varpi_1)^* \oplus \cdots \oplus R(\varpi_r)^*}.$$

Let $p_i : \tilde{X} \to R(\varpi_i)^*$ be the projection to the $i$-th component. Then the pull-back of the linear functions on $R(\varpi_i)^*$ yields the unique copy of the $G$-module $R(\varpi_i)$ in $k[\tilde{X}]$. The additive decomposition $k[\tilde{X}] = \bigoplus_{\lambda \in \mathcal{X}_+} R(\lambda)$ is a polygrading; i.e., if $f_i \in R(\lambda_i) \subset k[\tilde{X}]$, $i = 1, 2$, then $f_1f_2 \in R(\lambda_1 + \lambda_2)$.
Definition 1. Let $Q$ be an algebraic group with Lie algebra $\mathfrak{q}$. A $Q$-module $V$ is said to be cyclic if there is $v \in V$ such that $\mathcal{U}(\mathfrak{q}) \cdot v = V$, where $\mathcal{U}(\mathfrak{q})$ is the enveloping algebra of $\mathfrak{q}$. Such $v$ is called a cyclic vector.

Theorem 1.6. For any $\lambda \in \mathfrak{x}_+$, we have

\begin{enumerate}[(i)]
\item $I_\lambda = \{ \lambda - \sum_{i=1}^r a_i \alpha_i \mid 0 \leq a_i \leq (\lambda, \alpha_i^\vee) \}$;
\item $R(\lambda)^{U'}$ is a cyclic $U/U'$-module of dimension $\prod_{i=1}^r ((\lambda, \alpha_i^\vee) + 1)$. Up to a scalar multiple, there is a unique cyclic vector that is a $T$-eigenvector.
\end{enumerate}

Proof. In view of Lemma 1.1 and Proposition 1.3, it suffices to prove that $R(\lambda)^{U'}$ contains a vector of weight $\lambda - \sum_{i=1}^r (\lambda, \alpha_i^\vee) \alpha_i$. This vector have to be cyclic, because applying the $e_i$’s to it we obtain weight vectors with all weights from $\{ \lambda - \sum_{i=1}^r a_i \alpha_i \mid 0 \leq a_i \leq (\lambda, \alpha_i^\vee) \}$, hence the whole of $R(\lambda)^{U'}$.

Let $\hat{f}_i$ be a nonzero vector in one-dimensional space $R(\mathfrak{w}_i)_{\alpha_i - \alpha_i}$. Using the unique copy of $R(\mathfrak{w}_i)$ inside $\mathbb{k}[\hat{X}]$, we regard $\hat{f}_i$ as $U'$-invariant polynomial function on $\hat{X}$. Take the product (monomial) $F := \prod_{i=1}^r \hat{f}_i^{(\lambda, \alpha_i^\vee)} \in \mathbb{k}[\hat{X}]$. Since $\mathbb{k}[\hat{X}]$ is a domain, $F \neq 0$. The multiplicative structure of $\mathbb{k}[\hat{X}]$ shows that $F \in R(\lambda)^{U'}$ and the weight of $F$ equals $\sum_{i=1}^r (\lambda, \alpha_i^\vee)(\mathfrak{w}_i - \alpha_i) = \lambda - \sum_{i=1}^r (\lambda, \alpha_i^\vee) \alpha_i$. □

Remark 1.7. For the group $TU' \subset B$, we have $\dim TU' = \dim U$. It is well known that $TU'$ is a spherical subgroup of $G$ (e.g. apply [5, Prop. 1.1]). The sphericity also follows from the fact $R(\lambda)^{U'}$ is a multiplicity free $T$-module (Proposition 1.3). That $R(\lambda)^{U'}$ is a multiplicity free $T$-module follows also from [10, Corollary 8]. However, we obtain the explicit description of the corresponding weights and the $U/U'$-module structure of $R(\lambda)^{U'}$.

Theorem 1.8. Let $f_i$ (resp. $\hat{f}_i$) be a nonzero vector in one-dimensional space $R(\mathfrak{w}_i)_{\alpha_i}$ (resp. $R(\mathfrak{w}_i)_{\alpha_i - \alpha_i}$). Then the algebra of $U'$-invariants, $\mathbb{k}[G/U]^{U'}$, is freely generated by $f_1, \hat{f}_1, \ldots, f_r, \hat{f}_r$.

Proof. It follows from (the proof of) Theorem 1.6 that the monomials $\prod_{i=1}^r f_i^{c_i} \hat{f}_i^{(\lambda, \alpha_i^\vee) - c_i}$, $0 \leq c_i \leq (\lambda, \alpha_i^\vee)$, form a basis for $\dim R(\lambda)^{U'}$ for each $\lambda \in \mathfrak{x}_+$. Hence $\mathbb{k}[G/U]^{U'}$ is generated by $f_1, \hat{f}_1, \ldots, f_r, \hat{f}_r$. Since $U'$ is unipotent and $\dim (G/U) - \dim U' = 2r$, the Krull dimension of $\mathbb{k}[G/U]^{U'}$ is at least $2r$. Hence there is no relations between the above generators. □

Recall that a closed subgroup $H \subset G$ is said to be epimorphic if $\mathbb{k}[G/H] = \mathbb{k}$ or, equivalently, $R(\lambda)^H = \{0\}$ unless $\lambda = 0$, see e.g. [8, § 23B].

Proposition 1.9. Suppose $G$ is simple. The subgroup $TU'$ is epimorphic if and only if $G \neq SL_2$ or $SL_3$. 


Proof. The case of $SL_2$ is obvious, so we assume that $r \geq 2$. In view of Theorem 1.8, we have to check that neither of the monomials $\prod_{i=1}^{r} (f_i, j_i)^{\langle \lambda, \alpha_i \rangle}$, $0 \leq c_i \leq \langle \lambda, \alpha_i \rangle$, has zero weight if $G \neq SL_3$. The weight in question equals
\[
\mu := \sum_{i=1}^{r} (\lambda, \alpha_i \gamma) \omega_i - \sum_{i=1}^{r} c_i \alpha_i.
\]
Set $\rho^\gamma = \frac{1}{2} \sum_{\gamma \in \Delta^+} \gamma^\gamma$. Then $(\mu, \rho^\gamma) = \sum_{i=1}^{r} (\lambda, \alpha_i \gamma)(\omega_i, \rho^\gamma) - \sum_{i=1}^{r} c_i$. Notice that
\[
2(\omega_i, \rho^\gamma) = \sum_{\gamma \in \Delta^+} (\omega_i, \gamma^\gamma) \geq \# \{ \gamma \in \Delta^+ \mid (\gamma, \omega_i) > 0 \}.
\]
That is, $2(\omega_i, \rho^\gamma)$ is at least the dimension of the nilpotent radical of the maximal parabolic subalgebra corresponding to $\omega_i$. This readily implies that $(\omega_i, \rho^\gamma) > 1$ for all $i$ whenever $g \neq sl_3$. Whence $(\mu, \rho^\gamma)$ is positive.

For $SL_3$, the monomial $(f_1 f_2)^\alpha$ has zero weight. That is, $R(a(\omega_1 + \omega_2))^T U' \neq \{0\}$. \hfill $\square$

Remark 1.10. If $G = SL_3$, then $TU'$ is a Borel subgroup of a reductive subgroup $GL_2 \subset SL_3$. Proposition 1.9 can also be deduced from a result of Pomerening [18, Korollar 3.6].

Example 1.11. Let $U_n$ be a maximal unipotent subgroup of $G = SL_n$ and let $U_{n-1}$ be a maximal unipotent subgroup of a standardly embedded group $SL_{n-1} \subset SL_n$. It is well known that $\mathbb{k}[SL_n/U_n]^{U_{n-1}}$ is a polynomial algebra of Krull dimension $2(n - 1)$ and its generators have a simple description, see e.g. [1, Sect. 3]. The reason is that $SL_n/U_n$ is a spherical $SL_{n-1}$-variety and the branching rule $SL_n \downarrow SL_{n-1}$ is rather simple. That is, $\mathbb{k}[SL_n/U_n]^{U_{n-1}}$ and $\mathbb{k}[SL_n/U_n]^{U_n}$ are polynomial rings of the same dimension, and also $\dim U_{n-1} = \dim U_n$. However, the subgroups $U_n', U_{n-1} \subset SL_n$ are essentially different unless $n = 2, 3$.

2. SOME PROPERTIES OF ALGEBRAS OF $U'$-INVARIENTS

The main result of Section 1 says that $\mathbb{k}[G/U]'$ is a polynomial algebra of Krull dimension $2r$. This can also be understood in the other way around, since $\mathbb{k}[G/U]'$ and $\mathbb{k}[G/U]'$ are canonically isomorphic. Indeed, for any closed subgroup $H \subset G$, we regard $\mathbb{k}[G/H]$ as subalgebra of $\mathbb{k}[G]$:
\[
\mathbb{k}[G/H] = \{ f \in \mathbb{k}[G] \mid f(gh) = f(g) \text{ for any } g \in G, h \in H \}.
\]
Any subgroup of $G$ acts on $G/H$ by left translations. Therefore
\[
\mathbb{k}[G/U]' \simeq \{ f \in \mathbb{k}[G] \mid f(u_1 gu_2) = f(g) \text{ for any } g \in G, u_1 \in U', u_2 \in U \},
\]
\[
\mathbb{k}[G/U]' \simeq \{ f \in \mathbb{k}[G] \mid f(u_2 gu_1) = f(g) \text{ for any } g \in G, u_1 \in U', u_2 \in U \}.
\]
The involutory mapping $(f \in \mathbb{k}[G]) \mapsto \hat{f}$, where $\hat{f}(g) = f(g^{-1})$, takes $\mathbb{k}[G/U]'$ to $\mathbb{k}[G/U]'$, and vice versa.
One can deduce some properties of $k[G/U']$ using the known structure of $k[G/U']^U$. Set $A = k[G/U']$. It is a rational $G$-algebra, which can be decomposed as $G$-module:

$$A = \bigoplus_{\lambda \in X_+} m_{\lambda, A} R(\lambda).$$

By Frobenius reciprocity, the multiplicity $m_{\lambda, A}$ is equal to $\dim R(\lambda^*)^{U'}$. Therefore, it is finite. In our situation,

$$\dim R(\lambda^*)^{U'} = \dim R(\lambda)^{U'} = \prod_{i=1}^r ((\lambda, \alpha_i^\vee) + 1).$$

In particular, $m_{\varpi_i, A} = 2$ for any $i$. One can also argue as follows.

The group $G \times G$ acts on $G$ by left and right translations and the decomposition of $k[G]$ as $G \times G$-module is of the form:

$$k[G] = \bigoplus_{\lambda \in X_+} R(\lambda^*) \otimes R(\lambda),$$

where the first (resp. second) copy of $G$ in $G \times G$ acts on the first (resp. second) factor of tensor product in each summand [14, Ch. 2, § 3, Theorem 3]. Then

$$(2.1) \quad A = k[G/U'] = \bigoplus_{\lambda \in X_+} R(\lambda^*) \otimes R(\lambda)^{U'},$$

$$(2.2) \quad A^U = \bigoplus_{\lambda \in X_+} R(\lambda^*)^U \otimes R(\lambda)^{U'}.$$
Recall that \( f_i \) and \( \bar{f}_i \) are nonzero weight vectors in \( R(\varpi_i)_{\varpi_i} \) and \( R(\varpi_i)_{\varpi_i - \alpha_i} \), respectively.

**Theorem 2.2.** Let \( p = (f_1, \bar{f}_1, \ldots, f_r, \bar{f}_r) \in 2R(\varpi_1) \oplus \cdots \oplus 2R(\varpi_r) \). Then

(i) \( G_p = U' \);
(ii) \( \mathbb{k}[\overline{G:p}] = \mathbb{k}[G/U'] \) and \( \overline{G:p} \simeq \text{Spec}(A) \) is normal;
(iii) \( \text{codim}(\overline{G:p} \setminus G:p) \geq 2 \).

**Proof.** Part (i) is obvious. Then \( G:p \simeq G/U' \) and hence \( B := \mathbb{k}[\overline{G:p}] \) is a subalgebra of \( A \). By the very construction, \( m_{\varpi, B} \geq 2 \). (Consider different non-trivial projections \( \overline{G:p} \to R(\varpi_i) \) for all \( i \).) Since \( m_{\varpi, B} \leq m_{\varpi, A} = 2 \) and \( A \) is generated by the fundamental \( G \)-modules, we must have \( B = A \). This yields the rest. \( \square \)

Let \( X \) be an algebraic variety equipped with a regular action of \( G \). Then \( X \) is said to be a \( G \)-variety. The “transfer principle” ([3, Ch. 1], [20, §3], [8, §9]) asserts that

\[
\mathbb{k}[X]^H \simeq (\mathbb{k}[X] \otimes \mathbb{k}[G/H])^G
\]

for any affine \( G \)-variety \( X \) and any subgroup \( H \subset G \). In particular, if \( \mathbb{k}[G/H] \) is finitely generated, then so is \( \mathbb{k}[X]^H \). In view of Lemma 2.1, this applies to \( H = U' \), hence \( \mathbb{k}[X]^{U'} \) is always finitely-generated. Moreover, the polynomiality of \( \mathbb{k}[G/U']^U \) implies that \( \mathbb{k}[X]^{U'} \) inherits a number of other good properties from \( \mathbb{k}[X] \). Recall that \( \text{Spec}(\mathbb{k}[X]^{U'}) \) is denoted by \( X//U' \); hence \( \mathbb{k}[X//U'] \) and \( \mathbb{k}[X]^{U'} \) are the same objects.

We often use below the notion of a variety with rational singularities. Let us provide some relevant information for the affine case.

a) If \( \phi : \bar{X} \to X \) is a resolution of singularities, then \( X \) is said to have rational singularities if \( H^0(\bar{X}, \mathcal{O}_{\bar{X}}) = \mathbb{k}[X] \) and \( H^i(\bar{X}, \mathcal{O}_{\bar{X}}) = 0 \) for \( i \geq 1 \). In particular, \( X \) is necessarily normal.

b) If \( X \) has only rational singularities and \( G \) is a reductive group acting on \( X \), then \( X//G \) has only rational singularities (Boutot [2]).

c) If \( X \) has only rational singularities, then \( X \) is Cohen-Macaulay (Kempf [12]). It follows that if \( X \) is factorial and has rational singularities, then \( X \) is Gorenstein.

**Theorem 2.3.** Let \( X \) be an irreducible affine \( G \)-variety. If \( X \) has only rational singularities, then so has \( X//U' \). Furthermore, if \( X \) is factorial, then \( X//U' \) is factorial, too.

**Proof.** This is a straightforward consequence of known technique. Since \( \mathbb{k}[G/U']^U \) is a polynomial algebra, \( G//U' \) has rational singularities by Kraft’s theorem [3, Theorem 1.6], [20]. By the transfer principle for \( H = U' \), we have \( X//U' \simeq (X \times (G//U'))//G \). Applying Boutot’s theorem [2] to the right-hand side, we conclude that \( X//U' \) has rational singularities. The second assertion stems from the fact that \( U' \) has no non-trivial rational characters. \( \square \)
We have \( k[X]^U \subset k[X]^U' \), and both algebras are finitely generated. Assuming that generators of \( k[X]^U \) are known, we obtain a finite set of generators for \( k[X]^U' \), as follows.

**Theorem 2.4.** Suppose that \( f_1, \ldots, f_m \) is a set of \( T \)-homogeneous generators of \( k[X]^U \) and the weight of \( f_i \) is \( \lambda_i \). (That is, there is a \( G \)-submodule \( V_i \subset k[X] \) such that \( V_i \cong R(\lambda_i) \) and \( f \in (V_i)^U \).) Then the union of bases of the spaces \( (V_i)^U' \), \( i = 1, \ldots, m \), generate \( k[X]^U' \). In particular, \( k[X]^U' \) is generated by at most \( \sum_{i=1}^m \prod_{j=1}^r ((\lambda_i, \alpha_j^\gamma) + 1) \) functions.

**Proof.** Let \( B \) be the algebra generated by the spaces \( (V_i)^U' \). Clearly, \( B \) is \( B/U' \)-stable and contains \( k[X]^U \). Hence it meets every simple \( G \)-submodule of \( k[X] \). Therefore, it is sufficient to prove that \( B \) contains \( U/U' \)-cyclic vectors of all simple \( G \)-submodules.

We argue by induction on the root order \( \prec \) on the set of dominant weights. Let \( c_i \in (V_i)^U' \) be the unique \( U/U' \)-cyclic weight vector. By definition, \( c_i \in B \). We normalise \( f_i \) and \( c_i \) such that \( E(\lambda_i)(c_i) = f_i \), where the operator \( E(\lambda) \), \( \lambda \in \mathfrak{X}_+ \), is defined by \( E(\lambda) := \prod_{i=1}^r e_i^{(\lambda, \alpha_i^\gamma)} \). Assume that for any simple \( G \)-module \( W \) of type \( R(\mu) \) occurring in \( k[X] \), with \( \mu \prec \lambda \), the cyclic vector of \( W \) belong to \( B \). Consider an arbitrary simple submodule \( V \subset k[X] \) of type \( R(\lambda) \). Take a polynomial \( P \) in \( m \) variables such that \( f=P(f_1, \ldots, f_m) \) is a highest vector of \( V \). Without loss of generality, we may assume that every monomial of \( P \) is of weight \( \lambda \). We claim that \( P(c_1, \ldots, c_m) \neq 0 \). Indeed, it is easily seen that \( E(\lambda)P(c_1, \ldots, c_m) = P(E(\lambda_1)(c_1), \ldots, E(\lambda_m)(c_m)) = f \). The last equality does not guarantee us that \( P(c_1, \ldots, c_m) \in V \). However, this means that the projection of this element to \( V \) is well-defined and it must be a \( U/U' \)-cyclic vector of \( V \), say \( c \). More precisely, \( P(c_1, \ldots, c_m) = c + \tilde{c} \), where \( \tilde{c} \) belong to a sum of simple submodules of types \( R(\nu_i) \) with \( \nu_i \prec \lambda \). If \( P \) is a monomial, then this follows from the uniqueness of the Cartan component in tensor products. In our case, the Cartan component of the tensor product associated with every monomial of \( P \) is \( R(\lambda) \), which easily yields the general assertion. By definition, \( P(c_1, \ldots, c_m) \in B \), and by the induction assumption, \( \tilde{c} \in B \). Thus, \( c \in B \). \( \square \)

This theorem provides a good upper bound on the number of generators of \( k[X]^U' \). However, it is not always the case that a minimal generating system of \( k[X]^U \) is a part of a minimal generating system of \( k[X]^U' \). (See examples in Section 5.)

Since \( U' \) has no rational characters, \( \dim X/U' = \trdeg k(X)^{U'} = \dim X - \dim U' + \min_{x \in X} \dim (U')_x \). To compute the last quantity, we use the existence of a generic stabiliser for \( U \)-actions on irreducible \( G \)-varieties [6, Thm. 1.6].

**Lemma 2.5.** Let \( U_* \) be a generic stabiliser for \( (U : X) \). Then \( \min_{x \in X} \dim (U')_x = \dim (U_* \cap U') \).

**Proof.** Let \( \Psi \subset X \) be a dense open subset of generic points, i.e., \( U_x \) is \( U \)-conjugate to \( U_* \) for any \( x \in \Psi \). Since \( U' \) is a normal subgroup, \( U_x \cap U' \) is also \( U \)-conjugate to \( U_* \cap U' \). Thus, all \( U' \)-orbits in \( \Psi \) are of dimension \( \dim U' - \dim (U_* \cap U') \). \( \square \)
Remark 2.6. 1) If $X$ is (quasi)affine, then one can choose $U_*$ in a canonical way. Let $\mathcal{M}(X)$ be the monoid of highest weight of all simple $G$-modules occurring in $k[X]$. Then $U_*$ is the product of all root unipotent subgroup $U^\mu$ ($\mu \in \Delta^+$) such that $(\mu, \mathcal{M}(X)) = 0$ [17, Ch. 1, §3]. Equivalently, $U_*$ is generated by the simple root unipotent subgroups $U^{\alpha_i}$ such that $(\alpha_i, \mathcal{M}(X)) = 0$. It follows that $U_* \cap U' = (U_*, U_*)$. This also means that if $\mathcal{M}(X)$ is known, then $\min_{x \in X} \dim(U'_x)$ can effectively be computed.

2) The group $U_*$ is a maximal unipotent subgroup of a generic stabiliser for the diagonal $G$-action on $X \times X^*$ [17, Theorem 1.2.2]. Here $X^*$ is the so-called dual $G$-variety. It coincides with the dual $G$-module, if $X$ is a $G$-module. Using tables of generic stabilisers for representations of $G$, one can again compute $U_*$ and $(U_*, U_*)$.

3. POINCARÉ SERIES OF MULTIGRADED ALGEBRAS OF $U'$-INVARIA NTS

Let $X$ be an irreducible affine $G$-variety. (Eventually, we impose other constraints on $X$.) Since $T$ normalises $U'$, it acts on $X//U'$ and the algebra $k[X]/U'$ acquires a multigrading (by $T$-weights). Our objective is to describe some properties of the corresponding Poincaré series. Before we stick to considering $U'$-invariants, let us give a brief outline of notation and results to be used below.

Let $\mathcal{R}$ be a finitely generated $\mathbb{N}^m$-graded $k$-algebra such that $k[\mathcal{R}]_0 = 0$. Set $X = \text{Spec}(\mathcal{R})$.

- The Poincare series of $\mathcal{R}$ is (the Taylor expansion of) a rational function in $t_1, \ldots, t_m$:
  $$F(\mathcal{R}; \underline{t}) = P(\underline{t})/Q(\underline{t})$$
  for some polynomials $P, Q$.
- If $\mathcal{R}$ is Cohen-Macaulay, then $\Omega_{\mathcal{R}}$ (or $\Omega_X$) stands for the canonical module of $\mathcal{R}$; $\Omega_{\mathcal{R}}$ is naturally $\mathbb{Z}^m$-graded such that the Poincaré series of $\Omega_{\mathcal{R}}$ is
  $$F(\Omega_{\mathcal{R}}; \underline{t}) = (-1)^{\dim X} F(\mathcal{R}; \underline{t}^{-1}) .$$
- If $\mathcal{R}$ is Gorenstein, then the rational function $F(\mathcal{R}; \underline{t})$ satisfies the equality
  $$F(\mathcal{R}; \underline{t}^{-1}) = (-1)^{\dim X} q(X) F(\mathcal{R}; \underline{t}),$$
  for some $q(X) = (q_1(X), \ldots, q_m(X)) \in \mathbb{Z}^m$, and the degree of a homogeneous generator $\omega_{\mathcal{R}}$ of $\Omega_{\mathcal{R}}$ is $\deg(\omega_{\mathcal{R}}) = q(X)$ [22, Theorem 6.1], [23, 1.12].
- If $X$ has only rational singularities, then $q_i(X) \geq 0$ and $q(X) \neq (0, \ldots, 0)$ [3, Proposition 4.3]
- Let $G$ be a semisimple group acting on $X$ (of course, it is assumed that $G$ preserves the $\mathbb{N}^m$-grading of $\mathcal{R}$). Then there is a relationship between $\Omega_{\mathcal{R}}$ and $\Omega_{\mathcal{R}^G}$ [13] and hence between $q(X)$ and $q(X//G)$, see below.
We begin with the case of $X = G$, where $G$ is regarded as $G$-variety with respect to right translations. That is, we are going to study the graded structure of $A = \mathbb{k}[G/U']$. Since $G$ is simply-connected, it is a factorial variety. Therefore, $\text{Spec}(A) = G//U'$ is factorial (and has only rational singularities). In particular, $G//U'$ is Cohen-Macaulay (= CM). There is the direct sum decomposition

$$A = \bigoplus_{\gamma \in \mathcal{X}} A_{\gamma},$$

where $A_{\gamma} = \{ f \in A \mid f(gt) = \gamma(t)f(g) \text{ for any } g \in G, t \in T \}$. The weights $\gamma$ such that $A_{\gamma} \neq 0$ form a finitely generated monoid, which is denoted by $\Gamma$. Since $R(\lambda)^{U'}$ is a multiplicity free $T$-module, it follows from Eq. (2.1) that, for any $\lambda \in \mathcal{X}_+$, different copies of $R(\lambda^\vee)$ lie in the different weight spaces $A_{\gamma}$. More precisely, the corresponding set of weights is $I_\lambda$ (see Section 1). In particular, two copies of $R(\omega_i^\vee)$ belong to $A_{\omega_i}$ and $A_{\omega_i - \alpha_i}$. Therefore, $\Gamma$ is generated by the weights $\omega_i, \omega_i - \alpha_i, i = 1, \ldots, r$. Note that the group generated by $\Gamma$ coincides with $\mathcal{X}$, since $\Gamma$ contains all fundamental weights.

**Lemma 3.1.** If $G$ has no simple factors $SL_2$ or $SL_3$, then $\Gamma \setminus \{0\}$ lies in an open half-space of $\mathcal{X}_Q$, $A_0 = \mathbb{k}$, and $\dim A_{\gamma} < \infty$ for all $\gamma \in \Gamma$.

**Proof.** It is shown in the proof of Proposition 1.9 that $(\rho^\vee, \omega_i - \alpha_i) > 0$ for all $i$. Hence the half-space determined by $\rho^\vee$ will do. We have $A_0 = \mathbb{k}[G/TU'] = \mathbb{k}$, since $TU'$ is epimorphic. This also implies the last claim, because $A$ is finitely generated. □

The algebra $A$ is $\Gamma$-graded, and we are going to study the corresponding Poincaré series. Unfortunately, $\Gamma$ is not always a free monoid. Therefore we want to embed $\Gamma$ into a free monoid $\mathbb{N}^r$. This is always possible, if $\Gamma$ generates a convex cone in $\mathcal{X}_Q$, see e.g. [15, Corollary 7.23]. For this reason, we assume below that $G$ has no simple factors $SL_2$ or $SL_3$, and choose an embedding $\Gamma \hookrightarrow \mathbb{N}^r$. In other words, we find $v_1, \ldots, v_r \in \mathcal{X}$ such that $\mathcal{X} = \bigoplus_{i=1}^r \mathbb{Z}v_i$ and $\Gamma \subset \bigoplus_{i=1}^r \mathbb{N}v_i$. Furthermore, one can achieve that $(v_i, \rho^\vee) > 0$ for all $i$. Then $(v_1, \ldots, v_r)$ is said to be a $\Gamma$-adapted basis for $\mathcal{X}$. Thus, every $\gamma \in \Gamma$ gains a unique expression of the form $\gamma = \sum_i k_i(\gamma)v_i, k_i(\gamma) \in \mathbb{N}$.

Now, we define the multigraded Poincaré series of $A$ as the power series

$$\mathcal{F}(A; t_1, \ldots, t_r) = \mathcal{F}(A; {\underline{t}}) = \sum_{\gamma \in \Gamma} (\dim A_{\gamma}) {\underline{t}}^{\gamma},$$

where $t_i = t_i^{k_i(\gamma)} \ldots t_i^{k_r(\gamma)}$. As is well-known, $\mathcal{F}(A; {\underline{t}})$ is a rational function. Since $A$ is a factorial CM domain, it is Gorenstein. Therefore, there exists $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ such that

$$\mathcal{F}(A; {\underline{t}}^{-1}) = (-1)^{\dim G/U'} {\underline{a}} \mathcal{F}(A; {\underline{t}}),$$

where $\underline{t}^{-1} = (t_1^{-1}, \ldots, t_r^{-1})$ [22, § 6]. Moreover, since $G\!/U'$ has only rational singularities, all $a_i$ are actually non-negative, and $\underline{a} \neq (0, \ldots, 0)$ [3, Proposition 4.3].
Set \( b(A) := \sum_{i=1}^{r} a_i v_i \in \mathcal{X} \). A priori, this element might depend on the choice of an embedding \( \Gamma \hookrightarrow \mathbb{N}^r \). Fortunately, it doesn’t. Roughly speaking, this can be explained via properties of the canonical module \( \Omega_A \), which is a free \( A \)-module of rank one. However, even if we accurately accomplish this program, then we still do not find the very element \( b(A) \in \mathcal{X} \). Therefore, we choose another path. Our plan consists of the following steps:

1. \( A^U \) is a polynomial algebra and its Poincaré series can be written down explicitly;
2. Using the formula for this Poincaré series, we determine \( b(A^U) \in \mathcal{X} \);
3. Using results of \([17, 5.4]\), we prove that \( b(A) = b(A^U) \).

The algebra \( A^U \) is acted upon by \( T \times T \). Two copies of \( T \) acts on \( A^U \subset \mathbb{k}[G] \) via left and right translations. For the presentation of Eq. (2.2), the first (resp. second) copy of \( T \) acts on the first (resp. second) factor in tensor products. Then

\[
A^U = \bigoplus_{\lambda \in \mathcal{X}, \gamma \in \Gamma} A_{\lambda, \gamma}^U,
\]

where \( A_{\lambda, \gamma}^U = \{ f \in A^U \subset \mathbb{k}[G] \mid f(tg t') = \lambda(t)^{-1} \gamma(t') f(g) \text{ for all } t, t' \in T \} \), and we set

\[
\mathcal{F}(A^U; \underline{s}, \underline{t}) = \sum_{\lambda, \gamma}(\dim A_{\lambda, \gamma}^U) \underline{s}^\lambda \underline{t}^\gamma.
\]

Here \( \underline{s} = (s_1, \ldots, s_r) \) and \( \underline{s}^\lambda = s_1^{\alpha_1} \cdots s_r^{\alpha_r} \) if \( \lambda = \sum_i n_i \omega_i \).

**Proposition 3.2.** We have

\[
\mathcal{F}(A^U; \underline{s}, \underline{t}) = \prod_{i=1}^{r} \frac{1}{(1 - s_i \ell_1^{\omega_i}) (1 - s_i \ell_1^{\omega_i - \alpha_i})},
\]

where \( i^* \) is defined by \( (\omega_i)^* = \omega_i^* \).

**Proof.** This follows from the fact that \( A^U \) is freely generated by the space \( R = \bigoplus_{i=1}^{r} R(\omega_i^*)^U \otimes R(\omega_i)^U \), and the \((T \times T)\)-weights of a bi-homogeneous basis of \( R \) are \((\omega_i^*, \omega_i), (\omega_i^*, \omega_i - \alpha_i), i = 1, \ldots, r \). \( \square \)

Of course, \( \ell_1^{\omega_i} \) should be understood as \( t_1^{k_1(\omega_i)} \cdots t_r^{k_r(\omega_i)} \), and likewise for \( \omega_i - \alpha_i \). Since \( \sum_i (\omega_i + \omega_i - \alpha_i) = 2 \rho - |\Pi| = |\Delta^+ \setminus \Pi| \), we readily obtain

**Corollary 3.3.** \( \mathcal{F}(A^U; \underline{s}^{-1}, \underline{t}^{-1}) = (s_1 \cdots s_r)^{2\rho - |\Pi|} \mathcal{F}(A^U; \underline{s}, \underline{t}) \).

One can disregard (for a while) the \( \mathcal{X}_+ \)-grading of \( A^U \) and consider only the \( \Gamma \)-grading induced from \( A \). This amount to letting \( s_i = 1 \) for all \( i \). Then we obtain \( b(A^U) = 2 \rho - |\Pi| \), and, surely, this does not depend on the choice of \( \Gamma \hookrightarrow \mathbb{N}^r \). Thus, we have completed steps (1) and (2) of the above plan.

Now, we recall a relationship between the multigraded Poincaré series of algebras \( \mathbb{k}[X] \) and \( \mathbb{k}[X]^U \). For \( G \)-modules, these results are due to M. Brion \([3, \text{Ch. IV}], [4, \text{Theorem 2}]\).
A general version is found in [16], [17, Ch. 5]. We will consider two types of conditions imposed on $G$-varieties $X$:

1. $(\mathcal{C}_1) \quad \begin{cases} X \text{ is an irreducible factorial } G\text{-variety with only rational singularities and } \\ \mathbb{k}[X]^G = \mathbb{k}. \end{cases}$

2. $(\mathcal{C}_2) \quad \begin{cases} X \text{ is an irreducible factorial } G\text{-variety with only rational singularities; } \mathbb{k}[X] \text{ is } \\ \mathbb{N}^m\text{-graded, } \mathbb{k}[X] = \bigoplus_{n \in \mathbb{N}^m} \mathbb{k}[X]_n, \text{ and } \mathbb{k}[X]_0 = \mathbb{k}. \end{cases}$

In particular, $X$ is Gorenstein in both cases. Suppose $X$ satisfies $(\mathcal{C}_2)$. The Poincaré series of the Gorenstein algebra $\mathbb{k}[X]$ satisfies an equality of the form

$$(3.2) \quad F(\mathbb{k}[X]; t^{-1}) = (-1)^{\dim X} \frac{q(X)}{t} F(\mathbb{k}[X]; t),$$

where $t = (t_1, \ldots, t_m)$ and $q(X) = (q_1(X), \ldots, q_m(X))$. The affine variety $X//U$ inherits all good properties of $X$, i.e., it is irreducible, factorial, etc. Furthermore, $\mathbb{k}[X]^U$ is naturally $X_+ \times \mathbb{N}^m$-graded, and one defines the Poincaré series

$$F(\mathbb{k}[X]^U; s, t) = \sum_{\lambda \in X_+, n \in \mathbb{N}^m} (\dim \mathbb{k}[X]_{\lambda,n}) s^\lambda t^n.$$

Since $X//U$ is again Gorenstein, this series satisfies an equality of the form

$$F(\mathbb{k}[X]^U; s^{-1}, t^{-1}) = (-1)^{\dim X//U} \frac{q(X//U)}{s} F(\mathbb{k}[X]^U; s, t)$$

for some $b = b(X//U) = (b_1, \ldots, b_r)$ and $q(X//U) = (q_1(X//U), \ldots, q_m(X//U))$.

**Theorem 3.4** (see [17, Theorem 5.4.26]). *Suppose that $X$ satisfies condition $(\mathcal{C}_2)$. Then*

1. $0 \leq b_i \leq 2$;
2. $0 \leq q_i(X//U) \leq q_i(X)$ for all $i$;
3. the following conditions are equivalent:
   - $b = (2, \ldots, 2)$;
   - For $D = \{z \in X \mid \dim U_z > 0\}$, we have $\text{codim}_X D \geq 2$;
   - $q(X//U) = q(X)$;

Let us apply this theorem to the $G$-variety $\text{Spec}(\mathcal{A}) = G//U'$. The algebra $\mathcal{A}$ is $\Gamma$-graded and hence suitably $\mathbb{N}^r$-graded, as explained before. Note that $\text{Spec}(\mathcal{A})$ satisfies both conditions $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$. At the moment, we consider $X = \text{Spec}(\mathcal{A})$ as variety satisfying condition $(\mathcal{C}_2)$, with $m = r$. Comparing Eq. (3.1) and (3.2), we see that $a = q(X)$. Proposition 3.2 and Corollary 3.3 show that here $b(X//U) = (2, \ldots, 2)$ and $q(X//U)$ corresponds to $b(\mathcal{A}^U) = 2\rho - |\Pi|$. Now, Theorem 3.4(3) guarantee us that $q(X) = q(X//U)$, i.e.,

$$(3.3) \quad b(\mathcal{A}) = b(\mathcal{A}^U) = 2\rho - |\Pi|.$$ 

This completes our computation of $b(\mathcal{A})$. Note that we computed $b(\mathcal{A})$ without knowing an explicit formula of the Poincaré series $F(\mathcal{A}; t)$. 

Our next goal is to obtain analogues of results of [17, 5.4], where $U$ is replaced with $U'$, i.e., results on Poincaré series of algebras $k[X]^{U'}$.

Suppose $X$ satisfies $(\mathfrak{C}_1)$. The algebra $k[X]^{U'}$ is $\Gamma$-graded, and we consider the Poincaré series

$$\mathcal{F}(k[X]^{U'}; t) = \sum_{\gamma \in \Gamma} \dim k[X]^{U'}_\gamma t^\gamma,$$

where $k[X]^{U'}_\gamma = \{ f \in k[X]^{U'} \mid f(t,z) = \gamma(t)^{-1} f(z) \}$ and, as above, $t^\gamma$ is determined via the choice of a $\Gamma$-adapted basis $(v_1, \ldots, v_r)$ for $\mathfrak{F}$. The assumption $k[X]^G = k$ and the convexity of the cone generated by $\Gamma$ guarantee us that $k[X]_0^{U'} = k$ and all spaces $k[X]^{U'}_\gamma$ are finite-dimensional. Since $X//U'$ is again factorial, with only rational singularities (Theorem 2.3), it is Gorenstein and hence

$$\mathcal{F}(k[X]^{U'}; t^{-1}) = (-1)^{\dim X//U'} t a \mathcal{F}(k[X]^{U'}; t)$$

for some $a = (a_1, \ldots, a_r) \in \mathbb{N}^r$. Using the basis $(v_1, \ldots, v_r)$, we set $b(X//U') = \sum_{i=1}^r a_i v_i \in \mathfrak{F}$.

**Theorem 3.5.** Suppose that $X$ satisfies $(\mathfrak{C}_1)$. Then

1. $0 \leq b(X//U') \leq b(\mathcal{A}) = 2\rho - |\Pi|$
   (componentwise, with respect to any $\Gamma$-adapted basis $v_1, \ldots, v_r$);
2. the following conditions are equivalent:
   a) $b(X//U') = 2\rho - |\Pi|$;
   b) For $D = \{ x \in X \mid \dim(U')_x > 0 \}$, we have $\codim X D \geq 2$;

**Proof.** Using our results on $\mathcal{A}$ and $\mathcal{A}^U$ obtained above, one can easily adapt the proof of [17, Theorem 5.4.21]. For the reader’s convenience, we recall the argument.

1. We have $0 \leq b(X//U')$, since $X//U'$ has rational singularities.

Set $Z = X \times (G//U')$. It is a factorial $G$-variety with only rational singularities and $k[Z] = k[X] \otimes \mathcal{A}$. Define the $\Gamma$-grading of $k[Z]$ by $k[Z]_\beta = k[X] \otimes \mathcal{A}_\beta$, $\beta \in \Gamma$. By the transfer principle, $k[Z]^G \simeq k[X]^{U'}$ and the $\Gamma$-grading of $k[X]^{U'}$ corresponds under this isomorphism to the $\Gamma$-grading of $k[Z]^G$ as subalgebra of $k[Z]$.

In this situation (a semisimple group $G$ acting on a factorial variety $Z$ with only rational singularities), one can apply results of Knop to the quotient morphism $\pi_G : Z \to Z//G$. Set $m = \max_{z \in Z} \dim G.z$. Recall that $\Omega_X$ is the canonical module of $k[X]$. By Theorems 1,2 in [13], there is an injective $G$-equivariant homomorphism of degree 0 of graded $k[Z]$-modules

$$\gamma : \Omega_Z \to \wedge^m \mathfrak{g}^* \otimes \pi_G^*(\Omega_{Z//G}).$$
Here $\Omega_Z = \Omega_X \otimes \Omega_{G/U'}$ and grading of $\Omega_Z$ comes from the grading of $\Omega_{G/U'}$. The injectivity of $\bar{\gamma}$ implies that

$$b(X//U') = \begin{cases} \text{degree of a homogeneous} \\ \text{generator of } \Omega_{X/U'} \simeq \Omega_{Z/G} \end{cases} \leq \begin{cases} \text{degree of a homogeneous} \\ \text{generator of } \Omega_{G/U'} \end{cases} = b(A).$$

This yields the rest of part (1).

(2) To prove the equivalence of a) and b), we replace each of them with an equivalent condition stated in terms of $Z$:

a') $\deg(\omega_{Z//G}) = \deg(\omega_Z)$;

b') $\text{codim}_Z \tilde{D} \geq 2$, where $\tilde{D} = \{z \in Z \mid \dim G_z > 0\}$.

The argument in part (1) shows that a) and a') are equivalent. The equivalence of b) and b') follows from the fact that $G/U'$ is dense in $G//U'$ and the complement is of codimension $\geq 2$, see Theorem 2.2.

The injectivity and $G$-equivariance of $\bar{\gamma}$ means that there is $c \in (\wedge^m g^* \otimes k[Z])^G$ such that $\bar{\gamma}(\omega_Z) = c \cdot \omega_{Z//G}$. We can regard $c$ as $G$-equivariant morphism $c' : Z \to \wedge^m g^*$. It is shown in [13] that if $\dim G_z = m$ and $z \in Z_{\text{reg}}$, then $c'(z)$ is nonzero and it yields (normalised) Plücker coordinates of the $m$-dimensional space $g^*_z \subset g^*$.

Assume a'), i.e., $\deg(\omega_{Z//G}) = \deg(\omega_Z)$. Then $\deg c = 0$, i.e.,

$$c \in (\wedge^m g^* \otimes k[Z]_0)^G = (\wedge^m g^* \otimes k[X])^G.$$

This means that $c'$ can be pushed through the projection to $X$:

$$Z = X \times (G//U') \to X \to \wedge^m g^*.$$

Let $z = (x, v) \in X \times (G//U')$ be a generic point, i.e., $x \in X_{\text{reg}}$, $v \in G/U'$, and $\dim G_z = m$. Since $c'(z)$ depends only on $x$, we see that $g_z$ does not depend on $v$. But this is only possible if $\dim g_z = 0$, that is, $m = \dim G$. This already proves that $\text{codim}_Z \tilde{D} \geq 1$. If $\text{codim}_Z \tilde{D} = 1$, then formulae (6), (7), (12) in [13] show that $\tilde{D} = \{z \in Z \mid c'(z) = 0\}$. However, $\wedge^m g^*$ is the trivial 1-dimensional $G$-module, hence $c \in k[X]^G = k$. That is, $c'$ is a constant (nonzero) mapping. This contradiction shows that $\text{codim}_Z \tilde{D} \geq 2$.

Conversely, if b') holds, then $\tilde{D}$ is a proper subvariety of $Z$, i.e., $m = \dim G$ and $c \in (\wedge^{\dim G} g^* \otimes k[Z])^G = k[Z]^G$. Furthermore, since $\text{codim}_Z \tilde{D} \geq 2$, $c$ has no zeros on $Z$ (because $Z$ is normal and $c'(z) = c(z) \neq 0$ for any $z \in Z_{\text{reg}} \setminus \tilde{D}$). It follows that $c$ is constant, $\deg c = 0$ and hence $\deg(\omega_{Z//G}) = \deg(\omega_Z)$. \hfill \Box

If $X$ satisfies ($\mathcal{C}_2$), then the algebra $k[X]^{U'}$ is naturally $\Gamma \times \mathbb{N}^m$-graded, and we consider the Poincaré series

$$\mathcal{F}(k[X]^{U'}; s, t) = \sum_{n \in \mathbb{N}^m, \gamma \in \Gamma} \dim k[X]^{U'}_{n, \gamma} s^{n} t^{\gamma},$$
where $k[X]_{n,\gamma}^{U'} = \{ f \in k[X]_{n}^{U'} \mid f(tz) = \gamma(t)^{-1}f(z) \}$ and $t^\gamma$ is as above. Since $X//U'$ is again Gorenstein, we have

$$\mathcal{F}(k[X]^{U'}; \underline{\gamma}^{-1}, \underline{t}^{-1}) = (-1)^{\dim X//U'} a \sum q(X//U') \mathcal{F}(k[X]^{U'}; \underline{\gamma}, \underline{t})$$

for some $a = (a_1, \ldots, a_r) \in \mathbb{N}^r$ and $q(X//U') \in \mathbb{N}^m$. Using the basis $(v_1, \ldots, v_r)$, we set $b(X//U') = \sum_{i=1}^r a_iv_i \in \mathcal{X}$. The following is a $U'$-analogue of Theorem 3.4.

**Theorem 3.6.** Suppose that $X$ satisfies $(\mathcal{E}_2)$. Then

1. $0 \leq b(X//U') \leq b(\mathcal{A}) = 2\rho - |\Pi|$ (componentwise, with respect to any $\Gamma$-adapted basis $v_1, \ldots, v_r$);
2. $0 \leq q_i(X//U') \leq q_i(X)$ for all $i$;
3. the following conditions are equivalent:
   i. $b(X//U') = 2\rho - |\Pi|$;
   ii. For $D = \{ x \in X \mid \dim(U')_x > 0 \}$, we have $\operatorname{codim}_X D \geq 2$;
   iii. $q(X//U') = q(X)$.

We leave it to the reader to adapt the proof of Theorem 5.4.26 in [17] to the $U'$-setting.

These results may (and will) be applied to describing $G$-varieties $X$ with polynomial algebras $k[X]^{U'}$. Suppose for simplicity that $k[X]$ is $\mathbb{N}$-graded (i.e., $m = 1$). If $f_1, \ldots, f_s$ are algebraically independent homogeneous generators of $k[X]^{U'}$, then $\sum \deg f_i = q(X//U') \leq q(X)$. In particular, if $X$ is a $G$-module with the usual $\mathbb{N}$-grading of $k[X]$, then $\sum \deg f_i \leq \dim X$. Similarly, if $\omega_i$ is the $T$-weight of $f_i$, then $\sum_{i=1}^{s} \omega_i \leq 2\rho - |\Pi|$. The idea to use an a priori information on the Poincaré series for classifying group actions with polynomial algebras of invariants is not new. It goes back to T.A. Springer [21]. Since then it was applied many times to various group actions.

4. SOME COMBINATORICS RELATED TO $U'$-INVARIANTS

In previous sections, we have encountered some interesting objects in $\mathcal{X}$ related to the study of $U'$-invariants. These are $b(\mathcal{A}) = 2\rho - |\Pi|$, the set of $T$-weights in $R(\lambda)^{U'}$ (denoted $I_{\lambda}$), and the monoid $\Gamma$ generated by $\omega_i, \omega_i - \alpha_i$ for all $i \in \{1, \ldots, r\} =: [r].$

**Proposition 4.1.**

i. If $G$ has no simple ideals $SL_2$, then $2\rho - |\Pi|$ is a strictly dominant weight;
ii. For any $\lambda \in \mathcal{X}_+$, the weight $|I_{\lambda}|$ is dominant. Furthermore, $(|I_{\lambda}|, \alpha_i) > 0$ if and only if there is $j$ such that $(\lambda, \alpha_j^\vee) > 0$ and $(\alpha_i, \alpha_j) > 0$.

**Proof.** (i) is obvious.
(ii) Recall that \( I_\lambda = \{ \lambda - \sum_{i=1}^r c_i \alpha_i \mid 0 \leq c_i \leq (\lambda, \alpha_i^\vee), \ i = 1, \ldots, r \} \). Choose \( i \in [r] \) and slice \( I_\lambda \) into the layers, where all coordinates \( c_j \) with \( j \neq i \) are fixed, i.e., consider

\[
I_\lambda(c_1, \ldots, c_i, \ldots, c_r) = \{ \lambda - \sum_{j \neq i} c_j \alpha_j - c_i \alpha_i \mid 0 \leq c_i \leq (\lambda, \alpha_i^\vee) \}.
\]

Then one easily verifies that \((|I_\lambda(c_1, \ldots, c_i, \ldots, c_r)|, \alpha_i^\vee) = ((\lambda, \alpha_i^\vee) + 1)(-\sum_{j \neq i} c_j \alpha_j, \alpha_i^\vee) \geq 0 \). Hence \((|I_\lambda|, \alpha_i^\vee) \geq 0 \), and the condition of positivity is also inferred. \( \square \)

Let \( \mathcal{C} \) be the cone in \( \mathfrak{X}_\mathbb{Q} \) generated \( \Gamma \), i.e., by all weights \( \varpi_i, \varpi_i - \alpha_i \). Consider the dual cone \( \check{\mathcal{C}} := \{ \eta \in \mathfrak{X}_\mathbb{Q} \mid (\eta, \varpi_i) \geq 0 \ \&\ \ (\eta, \varpi_i - \alpha_i) \geq 0 \ \text{for all} \ i \} \).

**Theorem 4.2.** The cone \( \check{\mathcal{C}} \) is generated by the non-simple positive roots.

**Proof.** 1) Let \( \mathcal{K} \) denote the cone in \( \mathfrak{X}_\mathbb{Q} \) generated by \( \Delta^+ \setminus \Pi \). It is easily seen that \( \mathcal{K} \subset \check{\mathcal{C}} \).

Indeed, let \( \delta \in \Delta^+ \setminus \Pi \). Then \( (\varpi_i, \delta) \geq 0 \). If \( s_i \in W \) is the reflection corresponding to \( \alpha_i \in \Pi \), then \( s_i(\varpi_i) = \varpi_i - \alpha_i \) and \( s_i(\delta) \in \Delta^+ \). Hence \( (\varpi_i - \alpha_i, \delta) = (\varpi_i, s_i(\delta)) \geq 0 \).

2) Conversely, we prove that \( \check{\mathcal{K}} \subset \mathcal{C} \). We construct a partition of \( \check{\mathcal{K}} \) into finitely many simplicial cones, and show that each cone belong in \( \mathcal{C} \).

Suppose that \( \mu \in \mathfrak{X} \) and \( (\mu, \delta) \geq 0 \) for all \( \delta \in \Delta^+ \setminus \Pi \). Set \( J = J(\mu) = \{ j \in [r] \mid (\mu, \alpha_j) < 0 \} \).

We identify the elements of \( [r] \) with the corresponding nodes of the Dynkin diagram of \( G \). The obvious but crucial observation is that the nodes in \( J \) are disjoint on the Dynkin diagram. (Such subsets \( J \) are said to be disjoint.)

**Claim.** The \( r \) vectors \( \varpi_i \) \((i \notin J)\), \( \varpi_j - \alpha_j \) \((j \in J)\) form a basis for \( \mathfrak{X}_\mathbb{Q} \).

**Proof.** Since \( J \) is disjoint, \( \prod_{j \in J} s_j \in W \) takes these \( r \) vectors to \( \varpi_1, \ldots, \varpi_r \).

Thus, we can uniquely write

\[
\mu = \sum_{i \notin J} b_i \varpi_i + \sum_{j \in J} a_j (\varpi_j - \alpha_j), \quad b_i, a_j \in \mathbb{Q}.
\]

By the assumption, \((\mu, \alpha_i) \geq 0 \) if and only if \( i \notin J \). For \( j \in J \), we have \((\mu, \alpha_j^\vee) = -a_j < 0 \), i.e., \( a_j > 0 \). It is therefore suffices to prove that all \( b_i \) are nonnegative. Choose any \( i \notin J \). Let \( J[i] \) denote the set of all nodes in \( J \) that are adjacent to \( i \). Set \( w_i = \prod_{j \in J[i]} s_j \in W \). (If \( J[i] = \emptyset \), then \( w = 1 \).) Then \( w_i(\alpha_i) \) is either \( \alpha_i \) or a non-simple positive root. In both cases, we know that \((\mu, w_i(\alpha_i)) \geq 0 \). On the other hand, this scalar product is equal to \((w_i(\mu), \alpha_i) = b_i (\varpi_i, \alpha_i) \). Thus, each \( b_i \) is nonnegative and \( \mu \in \mathcal{C} \). \( \square \)

**Remark 4.3.** The argument in the second part of proof shows that \( \mathcal{C} \) is the union of simplicial cones parametrised by the disjoint subset of the Dynkin diagram. For any such set \( J \subset [r] \), let \( \mathcal{C}_J \) denote the simplicial cone generated by \( \varpi_i \) \((i \notin J)\), \( \varpi_j - \alpha_j \) \((j \in J)\). Then

\[
\mathcal{C} = \bigcup_{J \ \text{disjoint}} \mathcal{C}_J.
\]
Here $C_{\emptyset}$ is the dominant Weyl chamber and $C_J = (\prod_{j \in J} s_j)C_{\emptyset}$. Furthermore, if $C_J = \{ \sum_{i \in J} b_i \varpi_i + \sum_{j \in J} a_j (\varpi_j - \alpha_j) \mid a_j > 0, b_i \geq 0 \}$, then
\[ C = \bigsqcup_{J \text{ disjoint}} C_J. \]

**Remark 4.4.** It is a natural problem to determine the edges (one-dimensional faces) of the cone $\tilde{C}$. We can prove that, for $A_r$ and $C_r$, the edges are precisely the roots of height 2 and 3. However, this is no longer true in the other cases, because a root of height 4 is needed.

### 5. Irreducible Representations of Simple Lie Algebras with Polynomial Algebras of $U'$-Invariants

In this section, we obtain the list of all irreducible representations of simple Lie algebras with polynomial algebras of $U'$-invariants. If $G = SL_2$, then $U'$ is trivial and so is the classification problem. Therefore we assume that $rk G \geq 2$.

**Theorem 5.1.** Let $G$ be a connected simple algebraic group with $rk G \geq 2$ and $\mathbb{R}(\lambda)$ a simple $G$-module. The following conditions are equivalent:

(i) $\mathbb{k}[\mathbb{R}(\lambda)]^{U'}$ is generated by homogeneous algebraically independent polynomials;

(ii) Up to the symmetry of the Dynkin diagram of $G$, the weight $\lambda$ occurs in Table 1.

For each item in the table, the degrees and weights of homogeneous algebraically independent generators are indicated. We use the numbering of simple roots as in [24].

**Table 1:** The simple $G$-modules with polynomial algebras of $U'$-invariants

| $G$     | $\lambda$ | Degrees and weights of homogeneous generators of $\mathbb{k}[\mathbb{R}(\lambda)]^{U'}$ |
|---------|-----------|-------------------------------------------------------------------------------------|
| $A_r \ (r \geq 2)$ | $\varpi_r$ | $(1, \varpi_1), (1, \varpi_1 - \alpha_1)$ |
| $A_{2r-1} \ (r \geq 2)$ | $\varpi_2^r$ | $(1, \varpi_2), (2, \varpi_4), \ldots, (r-1, \varpi_{2r-2}), (r, 0)$, $(1, \varpi_2 - \alpha_2), (2, \varpi_4 - \alpha_4), \ldots, (r-1, \varpi_{2r-2} - \alpha_{2r-2})$ |
| $A_{2r} \ (r \geq 2)$ | $\varpi_2^r$ | $(1, \varpi_2), (2, \varpi_4), \ldots, (r-1, \varpi_{2r-2}), (r, \varpi_{2r})$, $(1, \varpi_2 - \alpha_2), (2, \varpi_4 - \alpha_4), \ldots, (r, \varpi_{2r} - \alpha_{2r})$ |
| $B_r$ | $\varpi_1$ | $(1, \varpi_1), (1, \varpi_1 - \alpha_1), (2, 0)$ |
| $B_3$ | $\varpi_3$ | $(1, \varpi_3), (1, \varpi_3 - \alpha_3), (2, 0)$ |
| $B_4$ | $\varpi_4$ | $(1, \varpi_4), (1, \varpi_4 - \alpha_4), (2, \varpi_1 - \alpha_1), (2, 0)$ |
| $B_5$ | $\varpi_5$ | $(1, \varpi_5), (1, \varpi_5 - \alpha_5), (2, \varpi_1), (2, \varpi_1 - \alpha_1), (2, \varpi_2), (2, \varpi_2 - \alpha_2), (3, \varpi_5), (3, \varpi_5 - \alpha_5), (4, \varpi_3 - \alpha_3), (4, \varpi_4), (4, \varpi_4 - \alpha_4), (4, 0)$ |
| $C_r$ | $\varpi_1$ | $(1, \varpi_1), (1, \varpi_1 - \alpha_1)$ |
| $D_r \ (r \geq 4)$ | $\varpi_1$ | $(1, \varpi_1), (1, \varpi_1 - \alpha_1), (2, 0)$ |
| $D_5$ | $\varpi_5$ | $(1, \varpi_4), (1, \varpi_4 - \alpha_4), (2, \varpi_1), (2, \varpi_1 - \alpha_1)$ |
| $G$ | $\lambda$ | Degrees and weights of homogeneous generators of $k[R(\lambda)]^{U'}$ |
|-----|-------|-------------------------------------------------------------|
| $D_6$ | $\varpi_6$ | $(1, \varpi_6), (1, \varpi_6-\alpha_6), (2, \varpi_2), (2, \varpi_2-\alpha_2), (3, \varpi_6), (3, \varpi_6-\alpha_6), (4, \varpi_4-\alpha_4), (4, 0)$ |
| $E_6$ | $\varpi_1$ | $(1, \varpi_5), (1, \varpi_5-\alpha_5), (2, \varpi_1), (2, \varpi_1-\alpha_1), (3, 0)$ |
| $E_7$ | $\varpi_1$ | $(1, \varpi_1), (1, \varpi_1-\alpha_1), (2, \varpi_6), (2, \varpi_6-\alpha_6), (3, \varpi_1), (3, \varpi_1-\alpha_1), (4, \varpi_2-\alpha_2), (4, 0)$ |
| $F_4$ | $\varpi_1$ | $(1, \varpi_1), (1, \varpi_1-\alpha_1), (2, \varpi_1), (2, \varpi_1-\alpha_1), (3, \varpi_2-\alpha_2), (2, 0), (3, 0)$ |
| $G_2$ | $\varpi_1$ | $(1, \varpi_1), (1, \varpi_1-\alpha_1), (2, 0)$ |

Before starting the proof, we develop some more tools. Let $V$ be a simple $G$-module. A posteriori, it appears to be true that if $\text{rk } G > 1$ and $k[V]^{U'}$ is polynomial, then so is $k[V]^G$. Therefore our list is contained in Brion’s list of representations with polynomial algebras $k[V]^U$ [4, p.13]. However, we could not find a conceptual proof. The following is a reasonable substitute:

**Proposition 5.2.** Suppose that $k[V]^{U'}$ is polynomial and $G \neq SL_3$. Then $k[V]^G$ is polynomial.

**Proof.** As in Section 3, consider the $\Gamma$-grading $k[V]^{U'} = \bigoplus_{\gamma \in \Gamma} k[V]^{U'}$. If $G \neq SL_3$, then $TU'$ is epimorphic and hence $k[V]^{U'}_0 = k[V]^G$. Furthermore, since $\Gamma$ generates a convex cone, $\bigoplus_{\gamma \neq 0} k[V]^{U'}_\gamma$ is a complementary ideal to $k[V]^G$. In this situation, a minimal system of homogeneous generators for $k[V]^G$ is a part of a minimal system of homogeneous generators for $k[V]^{U'}$. □

**Remark 5.3.** For $G = SL_3$, it is not hard to verify that the only representations with polynomial algebras of $U'$-invariants are $R(\varpi_1)$ and $R(\varpi_2)$. The reason is that $U'$ is the maximal unipotent subgroup of $SL_2 \subset SL_3$. Therefore, by classical Roberts’ theorem, we have $k[V]^{U'} \simeq k[V \oplus R_1]^{SL_2}$, where $V$ is regarded as $SL_2$-module and $R_1$ is the tautological $SL_2$-module. All $SL_2$-modules with polynomial algebras of invariants are known [19, Theorem 4], and the restriction of the simple $SL_3$-modules to $SL_2$ are easily computed.

Let $U'_*$ denote a $U'$-stabiliser of minimal dimension for points in $R(\lambda)$. Recall that Lemma 2.5 and Remark 2.6 provide effective tools for computing $U'_*$ and $\dim U'_*$. If a ring of invariants $\mathfrak{A}$ is polynomial, then elements of a minimal generating system of $\mathfrak{A}$ are said to be basic invariants.

**Proposition 5.4.** Suppose that $k[R(\lambda)]^{U'}$ is polynomial and $G \neq SL_3$. Then

$$\dim R(\lambda) \leq 2 \dim(U'/U'_*) + \prod_{i=1}^r((\lambda, \alpha_i^\vee) + 1).$$

In particular, $\dim R(\lambda) \leq 2 \dim U' + \prod_{i=1}^r((\lambda, \alpha_i^\vee) + 1)$. 
Proof. We consider \( \mathbb{k}[R(\lambda)] \) with the usual \( \mathbb{N} \)-grading by the total degree of polynomial. Then \( \mathbb{k}[R(\lambda)]^{U'} \) is \( \Gamma \times \mathbb{N} \)-graded, and it has a minimal generating system that consists of (multi)homogeneous polynomials. Let \( f_1, \ldots, f_s \) be such a system. By Theorem 3.6(ii), we have
\[
\sum \deg(f_i) = q(R(\lambda) \cap U') \leq q(R(\lambda)) = \dim R(\lambda).
\]
On the other hand, \( s = \dim R(\lambda) - \dim(U' / U'_*) \) and the number of basic invariants of degree 1 equals \( a(\lambda) := \prod_{i=1}^{r} ((\lambda, \alpha_i^\vee) + 1) \). All other basic invariants are of degree \( \geq 2 \), and we obtain
\[
a(\lambda) + 2(\dim R(\lambda) - \dim(U' / U'_*) - a(\lambda)) = a(\lambda) + 2(s - a(\lambda)) \leq q(R(\lambda) \cap U') \leq \dim R(\lambda).
\]
Hence \( \dim R(\lambda) \leq 2 \dim(U' / U'_*) + a(\lambda) \). \qed

Proof of Theorem 5.1.

(i) \( \Rightarrow \) (ii). The list of irreducible representations of simple Lie algebras with polynomial algebras \( \mathbb{k}[V]^G \) is obtained in [11]. By Proposition 5.2, it suffices to prove that the representations in [11, Theorem 1] that do not appear in Table 1 cannot have a polynomial algebra of \( U' \)-invariants. The list of representation in question is the following:

I) \((A_r, \varpi_3), r = 6, 7, 8; (A_7, \varpi_1); (A_2, 3\varpi_1); (B_r, 2\varpi_1), r \geq 2; (D_r, 2\varpi_1), r \geq 4; (B_6, \varpi_6); (D_8, \varpi_8); (C_r, \varpi_2), r \geq 4; (C_4, \varpi_4); \) the adjoint representations.

II) \((A_5, \varpi_3); (C_3, \varpi_2); (C_3, \varpi_3); (D_7, \varpi_7); (A_r, 2\varpi_r). \)

• For list I), a direct application of Proposition 5.4 yields the conclusion. For instance, consider \( R(\varpi_3) \) for \( A_r \), and \( r = 6, 7, 8 \). Here \( a(\varpi_3) = 2 \) and the second inequality in Proposition 5.4 becomes
\[(r + 1)r(r - 1)/6 \leq r(r - 1) + 2,
\]
which is wrong for \( r = 6, 7, 8 \). The same argument applies to all representations in I), except \((A_2, 3\varpi_1). \) (The \( SL_3 \)-case is explained in Remark 5.3.)

• For list II), the inequality of Proposition 5.4 is true, and more accurate estimates are needed.

Consider the case \((A_5, \varpi_3)\). Here \( \dim R(\varpi_3) = 20, \dim U' = 10 \) and \( U'_* = \{1\} \). Hence \( \dim R(\varpi_3) \cap U' = 10 \). Assume that \( R(\varpi_3) \cap U' \simeq \mathbb{A}^{10} \). The number of basic invariants of degree 1 equals \( a(\varpi_3) = 2 \). It is known that \( \mathbb{k}[R(\varpi_3)]^G \) is generated by a polynomial of degree 4. This is our third basic invariant. Since we must have \( \sum_{i=1}^{10} \deg f_i \leq \dim R(\varpi_3) = 20 \), the only possibility is that the other 7 basic invariants are of degree 2. However, \( S^2(R(\varpi_3)) = R(2\varpi_3) \oplus R(\varpi_1 + \varpi_5) \), which shows that the number of basic invariants of degree 2 is at most \( \dim R(\varpi_1 + \varpi_5) \cap U' = 4 \). This contradiction shows that \( \mathbb{k}[R(\varpi_3)]^{U'} \) cannot be polynomial. Such an argument also works for \((C_3, \varpi_2), (C_3, \varpi_3), \) and \((D_7, \varpi_7) \).

For \((A_r, 2\varpi_r), r \geq 2, \) we argue as follows. Here the algebra of \( U \)-invariants is polynomial, and the degrees and weights of basic \( U \)-invariants are \((1, 2\varpi_1), (2, 2\varpi_2), \ldots, (r, 2\varpi_r), \)
(r + 1, 0) \cite{4}. Using Theorem 2.4, we conclude that \( \mathbb{k}[R(2\varpi_r)]^{U'} \) can be generated by 3\( r + 1 \) polynomials whose degrees are 1, 1, 1; 2, 2, 2; \ldots; r, r, r; r + 1. This set of polynomials can be reduced somehow to a minimal generating system. Here \( \dim R(2\varpi_r)/U' = \dim R(2\varpi_r) - \dim U' = 2r + 1 \). Assume that \( R(2\varpi_r)/U' \sim A^{2r+1} \). Then we can remove \( r \) polynomials from the above (non-minimal) generating system such that the sum of degrees of the remaining polynomials is at most \( \dim R(2\varpi_r) = (r + 1)(r + 2)/2 \). This means that the sum of degrees of the \( r \) removed polynomials must be at least \( r(r + 1) \). Clearly, this is impossible.

(ii) \( \Rightarrow \) (i). All representations in Table 1 have a polynomial algebra of \( U \)-invariants whose structure is well-understood. Therefore, using Theorem 2.4 we obtain an upper bound on the number of generators of \( \mathbb{k}[R(\lambda)]^{U'} \). On the other hand, we can easily compute \( \dim R(\lambda)/U' \). In many cases, these two numbers coincide, which immediately proves that \( \mathbb{k}[R(\lambda)]^{U'} \) is polynomial. In the remaining cases, we use a simple procedure that allows us to reduce the non-minimal generating system provided by Theorem 2.4. This appears to be sufficient for our purposes.

- For \( G = D_5 \), the algebra \( \mathbb{k}[R(\varpi_5)]^{U} \) has two generators whose degrees and weights are \( (1, \varpi_4) \) and \( (2, \varpi_1) \). By Theorem 2.4, \( \mathbb{k}[R(\varpi_5)]^{U''} \) can be generated by polynomials of degrees and weights \( (1, \varpi_4), (1, \varpi_4 - \alpha_4), (2, \varpi_1), (2, \varpi_1 - \alpha_1) \). On the other hand, the monoid \( \mathcal{M}(R(\varpi_5)) \) is generated by \( \varpi_1, \varpi_4 \). Therefore, a generic stabiliser \( U_* \) is generated by the root unipotent subgroups \( U^{\alpha_2}, U^{\alpha_3}, \) and \( U^{\alpha_5} \) (see Remark 2.6). Hence \( \dim U_* = 6 \) and \( \dim U_*^{'} = 3 \). Thus \( \dim R(\varpi_5)/U' = 16 - 15 + 3 = 4 \) and the above four polynomials freely generate \( \mathbb{k}[R(\varpi_5)]^{U'} \).

The same method works for \( (A_r, \varpi_r); (A_r, \varpi_{r-1}); (B_r, \varpi_1); (C_r, \varpi_1); (D_r, \varpi_1); (B_r, \varpi_r), r = 3, 4; (E_6, \varpi_1) \).

There still remain four cases, where this method yields the number of generators that is one more than \( \dim R(\lambda)/U' \). Therefore, we have to prove that one of the functions provided by Theorem 2.4 can safely be removed. The idea is the following. Suppose that \( \mathbb{k}[R(\lambda)]{^U} \) contains two basic invariants of the same fundamental weight \( \varpi_i \), say \( p_1 \sim (d_1, \varpi_i), p_2 \sim (d_2, \varpi_i) \). Consider the corresponding \( U' \)-invariant functions \( p_1, q_1, p_2, q_2 \), where \( q_j \sim (d_j, \varpi_i - \alpha_i), j = 1, 2 \). Assuming that \( p_j, q_j \) are normalised such that \( e_i q_j = p_j \), the polynomial \( p_1 q_2 - p_2 q_1 \in \mathbb{k}[R(\lambda)] \) appears to be \( U \)-invariant, of degree \( d_1 + d_2 \) and weight \( 2\varpi_i - \alpha_i \). If we know somehow that there is a unique \( U \)-invariant of such degree and weight, then this \( U \)-invariant is not required for the minimal generating system of \( \mathbb{k}[R(\lambda)]^{U'} \). For instance, consider the case \( (F_4, \varpi_1) \). According to Brion \cite{4}, the free generators of \( \mathbb{k}[R(\varpi_1)]^{U(F_4)} \) are \( (1, \varpi_1), (2, \varpi_1), (3, \varpi_2), (2, \varpi_2), (3, \varpi_2), (3, \varpi_2 - \alpha_2), (2, \varpi_1), (3, \varpi_1) \). Theorem 2.4 provides a generating system for \( \mathbb{k}[R(\varpi_1)]^{U(F_4)} \) that consists of eight polynomials, namely:

\[
(1, \varpi_1), (1, \varpi_1 - \alpha_1), (2, \varpi_1), (2, \varpi_1 - \alpha_1), (3, \varpi_2), (3, \varpi_2 - \alpha_2), (2, \varpi_1), (3, \varpi_1).
\]
Here the weight \( \varpi_1 \) occurs twice and \( 2\varpi_1 - \alpha_1 = \varpi_2 \). Therefore the polynomial \((3, \varpi_2)\) can be removed from this set. Since \( \dim R(\varpi_1) = 26 \), \( \dim U' = 20 \), and \( \dim U'_* = 1 \), we have \( \dim R(\varpi_1) / U' = 7 \). The other three cases, where it works, are \((B_5, \varpi_5), (D_6, \varpi_6), (E_7, \varpi_1)\).

This completes the proof of Theorem 5.1. \( \square \)

**Remark 5.5.** For a \( G \)-module \( V \), let \( \text{ed}(k[V]^{U'}) \) denote the embedding dimension of \( k[V]^{U'} \), i.e., the minimal number of generators. Since \( k[V]^{U'} \) is Gorenstein, \( \text{ed}(k[V]^{U'}) - \dim V / U' = \text{hd}(k[V]^{U'}) \) is the homological dimension of \( k[V]^{U'} \) (see [19]). The same argument as in the proof of (ii) \( \Rightarrow \) (i) shows that for \((C_3, \varpi_2), (C_3, \varpi_3)\), and \((A_5, \varpi_3)\), we have \( \text{hd}(k[V]^{U'}) \leq 2 \).

Hence these Gorenstein algebras of \( U' \)-invariants are complete intersections. We can also prove that \( k[R(2\varpi_r)]^{U'(A_r)} \) is a complete intersection, of homological dimension \( r - 1 \). This means that a posteriori the following is true: If \( G \) is simple, \( V \) is irreducible, and \( k[V]^{U'} \) is polynomial, then \( k[V]^{U'} \) is a complete intersection. It would be interesting to realise whether it is true in a more general situation.

**Remark 5.6.** There is a unique item in Table 1, where the sum of degrees of the basic invariants equals \( \dim R(\lambda) \) or, equivalently, the sum of weights equals \( 2\rho - |\Pi| \). This is \((B_5, \varpi_5)\). By Theorem 3.6(iii), this is also the only case, where the set of points in \( R(\lambda) \) with non-trivial \( U' \)-stabiliser does not contain a divisor.

**References**

[1] D. Akhiezer and D. Panyushev. Multiplicities in the branching rules and the complexity of homogeneous spaces, *Moscow Math. J.* 2(2002), 17–33.

[2] J.-F. Boutot. Singularités rationnelles et quotients par les groupes réductifs, *Invent. Math.* 88, no. 1 (1987), 65–68.

[3] M. Brion. Sur la théorie des invariants, *Publ. Math. Univ. Pierre et Marie Curie*, no. 45 (1981), pp. 1–92.

[4] M. Brion. Invariants d’un sous-groupe unipotent maximal d’un groupe semi-simple. *Ann. Inst. Fourier* 33(1983), 1–27.

[5] M. Brion. Classification des espaces homogènes sphériques. *Compositio Math.* 63, no. 2 (1987), 189–208.

[6] M. Brion, D. Luna and Th. Vust. Espaces homogènes sphériques, *Invent. Math.* 84(1986), no. 3, 617–632.

[7] F. Grosshans. Hilbert’s fourteenth problem for nonreductive groups. *Math. Z.* 193 (1986), no. 1, 95–103.

[8] F. Grosshans. “Algebraic homogeneous spaces and Invariant Theory”, Lect. Notes Math. 1673, Berlin: Springer, 1997.

[9] Д. Хаджиеев. Некоторые вопросы теории векторных инвариантов, *Матем. Сборник* т.72, № 3 (1967), 420–435 (Russian). English translation: Dž. Hadžiev. Some questions in the theory of vector invariants, *Math. USSR-Sbornik*, 1(1967), 383–396.

[10] J. Horvath. Weight spaces of invariants of certain unipotent group actions, *J. Algebra* 126 (1989), 293–299.

[11] V.G. Kac, V.L. Popov and E.B. Vinberg. Sur les groupes linéaires algébriques dont l’algèbre des invariants est libre, *C. R. Acad. Sci. Paris*, Sér. A-B, 283, no. 12 (1976), A875–A878.
[12] G. KEMPFE. Cohomology and convexity, in “Toroidal Embeddings”, Lect. Notes Math. 339 (1973), 42–52.

[13] F. KNOP. Über die Glattheit von Quotientenabbildungen, Manuscripta Math. 56(1986), no. 4, 419–427.

[14] H. KRAFT. “Geometrische Methoden in der Invariantentheorie”, Aspekte der Mathematik D1, Braunschweig: Vieweg & Sohn, 1984.

[15] E. MILLER and B. STURMFELS. “Combinatorial commutative algebra”, Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005. xiv+417 pp.

[16] D. PANJSHYHEV. A restriction theorem and the Poincare series for $U$-invariants, Math. Annalen 301(1995), 655–675.

[17] D. PANJSHYHEV. Complexity and rank of actions in invariant theory, J. Math. Sci. (New York) 95(1999), 1925–1985.

[18] K. POMMERENING. Observable radizielle Untergruppen von halbeinfachen algebraischen Gruppen, Math. Z. 165(1979), 243–250.

[19] V.L. POPOV. Homological dimension of algebras of invariants, J. Reine Angew. Math., Bd. 341 (1983), 157–173.

[20] В.Л. Попов. Стягивания действий редуктивных алгебраических групп, Матем. Сборник т.130 (1986), 310–334 (Russian). English translation: V.L. POPOV. Contraction of actions of reductive algebraic groups, Math. USSR-Sbornik 58 (1987), 311–335.

[21] T.A. SPRINGER. On the invariant theory of $SU_2$, Indag. Math. 42(1980), no. 3, 339–345.

[22] R.P. STANLEY. Hilbert functions of graded algebras, Adv. Math., 28(1978), 57–83.

[23] R.P. STANLEY. Combinatorics and commutative algebra, 2nd ed., (Progress in Math. vol. 41). Basel: Birkhäuser, 1996.

[24] Э.Б. Винберг, В.В. Горбацевич, А.Л. Онищик. “Группы и алгебры Ли - 3”, Современные проблемы математики. Фундаментальные направления, т. 41. Москва: ВИНИТИ 1990 (Russian). English translation: V.V. GORBATSEVICH, A.L. ONISHCHIK and E.B. VINBERG. “Lie Groups and Lie Algebras” III (Encyclopaedia Math. Sci., vol. 41) Berlin Heidelberg New York: Springer 1994.

[25] Э.Б. Винберг, В.Л. Попов. Об одном классе аффинных квазиоднородных многообразий, Изв. АН СССР. Сер. Матем., 36(1972), 749–764 (Russian). English translation: E.B. VINBERG and V.L. POPOV. On a class of quasihomogeneous affine varieties, Math. USSR Izv., 6(1972), 743–758.

INDEPENDENT UNIVERSITY OF MOSCOW, BOL’SHOI VLASEVSKII PER. 11, 119002 MOSCOW, RUSSIA INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, B. KARETYI PER. 19, MOSCOW 127994
E-mail address: panyush@mccme.ru
URL: http://www.mccme.ru/~panyush