REMARKS ABOUT BUBBLES

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Abstract: We make some remarks about bubbling on, not necessarily proper, champs de Deligne-Mumford, i.e. compactification of the space of mappings from a given (wholly scheme like) curve, so, in particular, on quasi-projective projective varieties. Under hypothesis on both the interior and the boundary such as Remark 1.1 below, this implies an optimal logarithmic variant of Mori’s Bend-and-Break. The main technical remark is 4.7, while our final remark, the cone theorem 1.2, is a variant.

1. Introduction

Gromov convergence in the setting of, say, a compact Kähler manifold, $X$, with boundary $\Delta = \sum_i \Delta_i$ is amusing. The basic point is a lemma of Mark Green, [G], that if $f_n \to f$ are maps from a Riemann surface $\Sigma$ converging uniformly on compact sets and $D$ a divisor on $X$ such that $f_n^{-1}(D) = \emptyset$, then this can only fail for $f$ if it factors through $D$. Trivially, [M1, Fact 2.2.4], this implies that if $f_n : \Sigma \to X \backslash \Delta$ converges to a disc with (non-trivial) bubbles then there must be a map from $A^1$ to $X \backslash \Delta$ or some other stratum, $\cap_{i \in I} \Delta_i \cup \cup_{j \notin I} \Delta_j$, where for notational convenience we write the latter as $X_I \backslash \Delta_J$, which includes the former for $I = \emptyset$.

Once the local algebra in the purely meromorphic context is identified, 3.9 & 4.3, this combines with Mori’s bend & break technique to yield some further amusement, e.g.,

Remark 1.1. Let $V$ be a smooth quasi-projective variety, over an algebraically closed field of any characteristic, and suppose it admits a compactification $V \hookrightarrow X$ such that $X$ is projective with at worst quotient singularities and for $\Delta_i$ the irreducible components of the boundary $\Delta = X \backslash V$, no stratum $X_I \backslash \Delta_J$, $I \neq \emptyset$, admits a non-trivial map from $A^1$, then for every map from a curve $f : C \to X$ such that $(K_X + \Delta) \cdot f C < 0$ to every $x \in f(C) \cap V$ there is a rational curve $L_x \ni x$ whose intersection with $D$ is supported in at most 1 point and which satisfies,

$$M \cdot L_x \leq 2 \cdot \dim(X) \frac{M \cdot f C}{(K_X + \Delta) \cdot f C}$$

for every nef divisor $M$ on $X$. In particular by [BDPP], if $K_X + \Delta$ is not pseudo effective, then $V$ is covered by $A^1$’s.

This follows immediately from [4.7] which we shall not repeat in this introduction since it is designed to deal simultaneously with a situation such as the above where one has optimal control on the bubbling in the boundary, but otherwise it may be supposed to be any old rubbish, i.e. no klt, dlt, or whatever is required, and, as we shall see, hypothesis of much greater boundary regularity. Nevertheless, [4.7] is not a catch all. For example the same statement is true if we suppose that $V$ has quotient singularities with the same proof, but the nuisance is that it does not yield maps from $A^1$ to the Vistoli covering champ $\mathcal{V} \to V$, i.e. the smallest smooth champ with moduli $V$, $\overline{V}$, and, just as bad, even in the hypothesis of [1.1] with $\mathcal{X} \to X$ the Vistoli covering champ one cannot (without more thought) replace the condition of no $A^1$’s in the higher
strata $X_I \setminus \Delta_J$ by no $\mathbb{A}^1$’s in $\mathcal{X}_I \setminus \Delta_J$. The phenomenon which is occasioning this is explained in 4.6 while some more trivial obstructions are recorded in 4.1 & 4.2. Nevertheless, modulo the usual technical problem about smoothness assumptions in bend & break, one does produce $\mathbb{A}^1$’s in the full generality of champs de Deligne-Mumford with quasi-projective moduli. The misfortune is that one cannot necessarily guarantee without conditions such as 1.1, and even if there were no boundary, that these curves pass through a specified point. Consequently, even for klt. surfaces, there are things that are being missed. For example, if $S$ has no boundary, and an ample anti-canonical divisor, 4.7 might only yield a single map from $\mathbb{P}^1 \to \mathcal{S}$ to the Vistoli covering champ, even though its known, [KM], that $\mathcal{S}$ is ruled. Similarly, [M3, 2.1.3], bubbles (strong Gromov sense) cannot form in the boundary of minimal quasi-projective surfaces, but, as we’ve said, by 4.6, this is necessarily a global phenomenon, and not something local that can be attributed to the absence of parabolic champs in the strata of the boundary. On the other hand, these aren’t obstructions to proving a cone theorem, which, 5.3, we duly do, to wit:

Remark 1.2. Let $X$ be a projective algebraic variety over an algebraically closed field of any characteristic and let $\mathcal{X} \to X$ be a smooth tame champ de Deligne-Mumford over it (i.e. has the same moduli), with $\Delta = \sum_i \Delta_i$ a snc divisor on $\mathcal{X}$, then there exists a countable family $\{L_k \subset X\}$ of curves whose induced champs $\mathcal{L}_k \to L_k$ are parabolic (i.e. admit maps from some $\mathbb{A}^1$) in some stratum $\mathcal{X}_k \setminus \Delta_k$, cf. 2.2, satisfying,

$$0 < -(K_{\mathcal{X}} + \Delta) \cdot \mathcal{L}_k \leq 2 \cdot \dim(X),$$

such that

$$\text{NE}(\mathcal{X}) = \text{NE}(K_{\mathcal{X}} + \Delta)_{\geq 0} + \sum_k \mathbb{R}^+ [\mathcal{L}_k]$$

and the parabolic rays $\mathbb{R}^+ [\mathcal{L}_k] = \mathbb{R}^+ [L_k]$ are locally discrete in $N_1(\mathcal{X})_{(K_{\mathcal{X}} + \Delta) < 0}$.

The bound $2 \cdot \dim(X)$ is, of course, not optimal, and while it’s easy enough to get $\dim(X) + 1$ instead, one has to vary the construction, albeit even with the construction as is one quickly gets better than $\dim(X) + 1$ if the situation is very non-schematic, cf. 5.2. A final technical remark is that we never do any deformation theory which is more complicated than that from an honest curve to a champ. This eliminates the habitual difficulties such as the graph of a map may not be an embedding, and one doesn’t even need to know that there is such a thing as a Hilbert champ of a champ, even though, [HR], it’s re-assuring that there is. Nevertheless one does need to know that every smooth tame champ de Deligne-Mumford $\mathcal{C} \to C$ over a curve admits a map from a curve. One could have gone through the same hoops as [SGA1, Exposé X] to deduce this in all characteristics from [BN], but it seems easier to just prove it directly, 2.4, and, of course, one gets a description of the tame fundamental group of $\mathcal{C}$ en passant.

2. Preliminaries

2.1. Set-up. We work over an algebraically closed filed $k$ of any characteristic. For brevity we make,

Definition 2.1. By a champ $\mathcal{X}$ is to be understood a tame champ de Deligne-Mumford over $k$, with quasi-projective moduli $\pi : \mathcal{X} \to X$. As such $\mathcal{X}/k$ is separated, and it is proper iff $X$ is projective. Thus, consistent with the ideas in [SGA1 exposé VI], the above data may also, as convenience of exposition dictates, be referred to as a champ over $X$, and the mis-translation stack will be eschewed.

Let $\mathcal{X}$ be a champ. A Cartier divisor $\Delta = \sum_i \Delta_i$ on $\mathcal{X}$ is said to be snc, if $\mathcal{X}$ is smooth at every point of the support, every irreducible component $\Delta_i$ of $\Delta$ is smooth, and the components through any geometric point form a system of parameters. For any effective Weil divisor with
$\Delta = \sum_i \Delta_i$, its expression as a sum of irreducible components, and $I$ a set of irreducible components of $\Delta$ we define the $I$-th stratum:

\[(2.1) \quad \mathcal{A}_I := \bigcap_{i \in I} \Delta_i.\]

When it has sense, e.g. $\mathcal{X} \rightarrow \text{CM}$, each $\Delta_i$ Cartier with local equations forming a regular sequence, and $\mathcal{A}_I$ reduced, and only when it has sense as a $\mathbb{Q}$-Cartier divisor, its canonical divisor is:

\[(2.2) \quad K_{\mathcal{A}_I} := (K_{\mathcal{X}} + \sum_{i \in I} \Delta_i)|_{\mathcal{A}_I}.\]

and we continue to understand $K_{\mathcal{A}_I}$ by this formula, and say $K_{\mathcal{A}_I}$ is $\mathbb{Q}$-Cartier even if this only has the sense that the above right hand side is the restriction of a $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$.

**Convention 2.2.** On writing $f : C \rightarrow \mathcal{X}_I$, or $\mathcal{C} \subset \mathcal{X}_I$, unless otherwise specified, we suppose that $I$ is the maximal set of components of $\Delta$ containing $f(C)$, respectively $\mathcal{C}$. In the implicit presence of such an $I$, we denote by $J$ the set of components complementary to $I$ and should every $\Delta_j$, $j \in J$, be $\mathbb{Q}$-Cartier we put:

\[\Delta_J := \sum_{j \in J} \Delta_j|_{\mathcal{X}_I}.\]

### 2.2. Curves and Uniformisation.

**Definition 2.3.** Let $\mathcal{C}_0 \rightarrow C$ be a (smooth connected) champ over a curve, without generic stabiliser, then we extend the topological Euler characteristic $C \rightarrow \chi(C)$ by way of,

\[\chi(\mathcal{C}_0) := \chi(C) + \sum_{c \in \mathcal{C}} \left( \frac{1}{n_c} - 1 \right)\]

where $n_c$ the order of the local monodromy. For a general (smooth connected) champ $\mathcal{C} \rightarrow C$, with generic stabiliser $G$, there is a fibration $\mathcal{C} \rightarrow \mathcal{C}_0$, $\mathcal{C}_0$ as above, with fibre $B_C$ and:

\[|G|\chi(\mathcal{C}) := \chi(\mathcal{C}_0)\]

Should the champ be proper, this is equally the negative of the degree of the canonical bundle, and should $\chi(\mathcal{C}) > 0$ we will say that the champ is parabolic.

The presence of stabilisers, particularly generic ones, is a recipe for technical problems, e.g. graphs of mappings may fail to be embeddings, which we eliminate by way of,

**Proposition 2.4.** Let $\pi : \mathcal{C} \rightarrow C$ be a (smooth connected) champ over a curve, then there is a finite proper map from a curve $B$ to $\mathcal{C}$. Better still,

\[\begin{align*}
(a) \quad & \chi(B) < 0, \text{ respectively, } = 0, \text{ or } > 0, \text{ whenever the same is true of } \chi(\mathcal{C}). \\
(b) \quad & \text{If } \chi(\mathcal{C}) \leq 0, \text{ then we may take } B \rightarrow \mathcal{C} \text{ to be \'{e}tale, and even realise } \mathcal{C} \text{ as } [B/E] \text{ for } E \text{ an extension,}
\end{align*}\]

\[\begin{array}{cccc}
1 & \rightarrow & G & \rightarrow \quad E & \rightarrow \quad H & \rightarrow & 1 \\
& \quad & \quad & \quad & \quad & \quad & \quad \quad & \quad \\
& \quad & \quad & \quad & \quad & \quad & \quad & \quad \quad \\
acting \ via \ \rho \ & \text{for } \ H \ & \text{a sub-group of } \text{Aut}(B), \ & \text{and } \ G \ & \text{the monodromy around a generic point of } \mathcal{C}.\end{array}\]

\[(c) \quad \text{Otherwise } B = \mathbb{P}^1 \text{ or } \mathbb{A}^1.\]

**Proof.** The assertion is well known if $G$ is trivial, e.g. [K1] Lemma 6.5. In particular by way of a base change, $F \rightarrow \mathcal{C}_0$, \'{e}tale should (b) apply, with $\mathcal{C}_0$ as per 2.3 and $F$ a curve we may suppose that $C = \mathcal{C}_0$. Now write $\mathcal{C} = [U/R]$ for some \'{e}tale cover $U \rightarrow \mathcal{C}$. By hypothesis $(s, t) : R \rightarrow U \times U$ is proper, and taking $U$ sufficiently fine, we may suppose that its image is $U \times_C U$. In addition, the stabiliser, $S := (s, t)^{-1}(\Delta)$, $\Delta \rightarrow U \times_C U$ the diagonal, admits a, non-canonical, isomorphism $\varphi : S \rightarrow U \times \mathcal{C}$, while an arbitrary fibre of $R \rightarrow U \times_C U$ may
be identified with the stabiliser of its source, so \( R \to U \times_C U \) is étale, whence refining \( U \) as necessary we may suppose that it admits a section \( \alpha \). Conjugation by \( \alpha \) combined with \( \varphi \), yields a map \( \Sigma : U \times_C U \to \text{Aut}(G) \), whose image in \( \text{Out}(G) \) is a co-cycle, thus affording,

\[
\bar{\Sigma} \in H^2_{\text{et}}(C, \text{Out}(G))
\]

The failure of \( \Sigma \) to be a co-cycle in \( \text{Aut}G \) is rather precise, \textit{viz:} applying the Cech co-boundary operator at the level of arrows yields a map, \( g : V \times_C V \times_C V \to G \) such that the co-boundary of \( \Sigma \) is the inner automorphism associated to \( g \). The naturality of \( g \) implies, in the usual Cech notation, that:

\[
g^{-1} \alpha \beta \gamma \delta g_{\alpha \beta \gamma} \delta g_{\alpha \beta \gamma} = \Sigma_{\alpha \beta}(g^{\beta \gamma \delta}).
\]

At this point we must distinguish the case (b) and (c) of the proposition. In the former \( C \) is a (tame) étale \( K(\pi, 1) \), and the above condition on \( g \) says exactly that there is a group extension,

\[
1 \to G \to E \to \pi \to 1
\]

with action of \( \pi \) by outer automorphisms defined via \( \Sigma \in \text{Hom}(\pi, \text{Out}(G)) \), and implied 2 co-cycle defined by \( g \) viewed as a map from \( \pi \times \pi \) to \( G \). Here \( \pi \) is the (prime to the characteristic) pro-fundamental group, so this proves the assertion (b) when \( C = \mathbb{C}_0 \) which is all that we’ll need in the sequel, while in general \( \mathbb{C}_0 \) is also a (tame) étale \( K(\pi, 1) \), and the same argument works at the price of some notational complication occasioned on replacing \( U \times_C U \) by the arrows \( R_0 \) defining \( \mathbb{C}_0 \), and so forth.

As to case (c) of the proposition, we may suppose that \( \mathbb{C}_0 = C \) is either \( \mathbb{P}^1 \) or \( \mathbb{A}^1 \), so that in either case \( \Sigma \) is homologous to zero. Adjusting both \( \varphi \) and \( \sigma \) accordingly, we may suppose that \( \Sigma \) is trivial, and \( g \) takes values in the centre \( Z \) of \( G \). In particular by the above formula it defines a class,

\[
\bar{g} \in H^2_{\text{et}}(C, Z)
\]

necessarily null if \( C = \mathbb{A}^1 \), and otherwise null after a base change of the form,

\[
\mathbb{P}^1 \to \mathbb{P}^1 : x \mapsto x^l
\]

for some sufficiently large \( l \). Consequently after a base change, and an adjustment of \( \sigma \) by central elements, we obtain a 1-isomorphism, given on arrows by,

\[
R \to C \times_C G : f \mapsto (s(f), f \sigma(s \times t(f))^{-1})
\]

for \( G \) acting trivially on \( C \), i.e. \( \mathcal{E} = C \times B_G \), and we can take \( B = C \) mapping to \( \mathcal{E} \) via the natural projection \( \text{pt} \to B_G \).

To which let us add,

**Remark 2.5.** Of course \( \mathbb{C}_0 \to C \) is parabolic only if the number of non-schematic points is at most 3, and should it be 3 then the universal cover of \( \mathbb{C}_0 \) is \( \mathbb{P}^1 \), and one may enumerate the possibilities where this occurs. In the general case, however, there is no such simple enumeration, so that being more precise than \( \text{[2.3]}(c) \) is pointless. Nevertheless it gives an equivalent criteria for parabolicity, \textit{viz:} A champ is parabolic iff it is dominated by \( \mathbb{A}^1 \), or \( \mathbb{P}^1 \) should it be proper.

The above suggests that a useful way to think of a quasi-projective variety is as a champs with infinite monodromy around the boundary. Unfortunately, this is, a priori, technically vacuous, and so we make,

**Definition 2.6.** A smooth quasi projective curve is a smooth projective curve \( C \) together with a reduced divisor \( D \). Exactly as per \[2.3\] the topological Euler characteristic of \( C \setminus D \) is equally the negative of the log-canonical degree \( K_C + D \). For \( \Delta = \sum \Delta_i \), an effective Weil divisor on a champ \( \mathcal{X} \), and \( f : C \to \mathcal{X}_1 \) respecting the convention \([2.3]\) the induced quasi-projective curve is \( C \) together with the unique reduced divisor whose support coincides with \( f^* \Delta_j \).
2.3. Dimension counts. We put ourselves in the situation of \[2.2\] including, by the way, a possibly empty set of indices \( I \). We wish to study \( \text{Mor}(C, \mathcal{X}_I) \). As it happens, the Hilbert champ of a champ is known to exist, \([HR]\), so the aforesaid space of morphisms may be viewed as a sous-champ of a connected component of the graph. This is not, however, in any way necessary since \( f^* \mathcal{O}_{\mathcal{X}_I} \) is a sheaf of admissible \( \mathcal{O}_C \) algebras, \([M2]\), whose formal spectrum is the completion of \( C \times \mathcal{X}_I \) in the graph. As such the local theory is wholly (formal) scheme like, and one could just as well define \( \text{Mor}(C, \mathcal{X}_I) \) as an open subscheme of the Chow scheme of the moduli. In any case, this object is known to exist, and thinking of a quasi-projective curve as a champ with infinite monodromy on the boundary, one realises that the relevant object of study is,

\[
\text{Definition 2.7. Let things be as above, then the sous-champ,}
\]

\[ \text{Mor}(C, \mathcal{X}_I, \Delta_J) \subset \text{Mor}(C, \mathcal{X}_I) \]

is the parameter space of morphisms \( h: C \to \mathcal{X}_I \) such that \( h^*(\Delta_J) = f^*(\Delta_J) \). In addition, bearing in mind both equation \[2.2\] and the convention \[2.2\] we say, cf. \([K1, \text{Definition 3.1}]\],

(i) There are enough deformations of \( f \) in \( \mathcal{X}_I \) if,

\[ \dim_{[f]} \text{Mor}(C, \mathcal{X}_I) \geq -K_{\mathcal{X}_I} \cdot f + (1 - g(C)) \dim(\mathcal{X}_I). \]

and there are enough deformations in \( \mathcal{X}_I \) if this holds for all \( f: C \to \mathcal{X}_I \).

(ii) There are enough deformations of \( f \) in \( \mathcal{X}_I \setminus \Delta_J \) if,

\[ \dim_{[f]} \text{Mor}(C, \mathcal{X}_I, \Delta_J) \geq -(K_{\mathcal{X}_I} + \Delta) \cdot f + (1 - g(C)) \dim(\mathcal{X}_I). \]

and there are enough deformations in \( \mathcal{X}_I \setminus \Delta_J \) if this holds for all \( f: C \to \mathcal{X}_I \).

As in the compact case, we will need to consider curves passing through a fixed point.

\[
\text{Definition 2.8. Conventions enforce, let } f: C \to \mathcal{X}_I \text{ be a curve and } \Gamma \subset C \text{ a finite subset such that } \Gamma \cap f^*\Delta_J = \emptyset. \text{ The sous-champ } \text{Mor}(C, \mathcal{X}_I, \Delta_J, f|_{\Gamma}) \subset \text{Mor}(C, \mathcal{X}_I, \Delta_J) \text{ is the parameter space of morphisms } h: C \to \mathcal{X}_I \text{ which furthermore verify } h|_{\Gamma} = f|_{\Gamma}. \text{ Manifestly if } f(\Gamma) \text{ is contained in the smooth locus of } \mathcal{X}_I \text{ then,}
\]

\[ (2.3) \quad \dim_{[f]} \text{Mor}(C, \mathcal{X}_I, \Delta_J, f|_{\Gamma}) \geq \dim_{[f]} \text{Mor}(C, \mathcal{X}_I, \Delta_J) - |\Gamma| \dim(\mathcal{X}_I) \]

To which, we have the following minor variant of Mori’s estimate,

\[
\text{Lemma 2.9. Let things be as above then,}
\]

(i) If \( \mathcal{X}_I \) is LCI, and \( f \) meets the smooth locus of the same, then there are enough deformations of \( f \) in \( \mathcal{X}_I \).

(ii) If there are enough deformations in \( \mathcal{X}_I \) and \( \Delta_J \) is Cartier where it meets \( f \), then there are enough deformations in \( \mathcal{X}_I \setminus \Delta_J \).

In particular if \( \mathcal{X} \) is smooth, then there are always enough deformations in \( \mathcal{X} \setminus \Delta \), and if \( (\mathcal{X}, \Delta) \) is smooth with snc boundary then there are even enough deformations in \( \mathcal{X}_I \setminus \Delta_J \) for every stratum.

\[
\text{Proof. Case (i) follows for the same reason as [M, Proposition 3], i.e. the functoriality of the obstruction group [SGA1, exposé III, 5.1], and the same calculation as [K2, II.1.3]. Case (ii) follows from case (i), since fixing the intersection with the boundary is at most } \Delta_J \cdot f \text{ conditions.} \]

\[ \square \]
3. Looking for Rational Curves

3.1. An instructive case. As ever we place ourselves in the setup \[2.1\] with \(F_b : C \to \mathcal{X}_I\) a one dimensional family of morphisms respecting the convention \[2.2\] i.e. we have a morphism, \(F : C \times B \to \mathcal{X}_I\)

where \(B\) is a (not necessarily complete) smooth curve. Suppose further that every \(\Delta_j, j \in J,\) is \(\mathbb{Q}\)-Cartier, and for some \(c \in C,\)

(i) \(F(c \times B)\) is a point and \(c \not\in F_b^*(\Delta_j);\)
(ii) for any \(b \in B\) the pull-back of the boundary \(F_b^*(\Delta_j)\) is constant on \(C = C \times b;\)
(iii) \(F\) is generically finite.

Let \(\overline{B}\) be a smooth compactification of \(B\). We denote by \(f\) the composition \(\pi F\), as well as the induced rational map:

\[
f : C \times \overline{B} \dashrightarrow X_I.
\]

By (i) this cannot be defined on all of \(c \times \overline{B}\). Let \(X\) be projective and \(S\) a smooth surface obtained by a sequence of blow ups in closed points such that the rational map \(\overline{f} : S \to X_I\) is a morphism, and:

(iv) \(\overline{f}\) is a smooth and minimal, i.e. it does not contract any \((-1)\)-curve.

Let \(E = \sum_{a} c_a E_a\) be the exceptional divisor in \(S\) over some indeterminacy in \(c \times \overline{B}\). Notice that by (ii) the divisor \(\overline{f}^* \Delta_j\) is supported either on \((\overline{f}_0^* \Delta_j) \times \overline{B}\), some fixed \(0 \in B\), or on the components of the exceptional divisor. Therefore any component \(E_a\) of \(E\) such that \(\overline{f}(E_a) \not\subset \Delta_j\) can only meet \(\Delta_j\) in another component \(E_b\) such that \(\overline{f}(E_b) \subset \Delta\) or along the proper transform of the fibre, \(C \times \{b\}\), through an indeterminacy should this be contained in \(\overline{f}^* \Delta\). Summarising for strata, yields:

**Lemma 3.1.** Let everything be as above, with \(E_a\) a component of the exceptional divisor of \(\overline{f}\). Suppose \(\overline{f}(E_a) \subset X_{I'},\) with \(I' \supset I\) maximal. Then \(E_a\) can only meet \(\Delta_j, j \not\in I',\) in another component \(E_b\) or along the proper transform \(\widetilde{C}_b\) of a fibre, \(C \times \{b\}\), through an indeterminacy, and this only if \(\overline{f}(E_b) \subset \Delta_{I'}\) or \(\overline{f}(\widetilde{C}_b) \subset \Delta_{I'}\).

We will consider the dual graph associated to \(E\) together with the proper transform \(\widetilde{C}\) of \(C \times b\), where the graph shall be rooted, and we endow it with the metric given by the distance from the same. Observe that a vertex at maximal distance from the root has valency one.

**Lemma 3.2.** Let \(v\) be a vertex having valency one, and denote by \(E_v\) the corresponding component of the exceptional divisor. Suppose that

\[
E_v \text{ is not contracted by } \overline{f} \text{ to a point.}
\]

Let \(I'\) be the maximal set such that \(\overline{f}(E_v) \subset X_{I'}\). Then \(\overline{f}(E_v)\) yields a map from \(\mathbb{A}^1\) to \(X_{I'} \setminus \Delta_{I'}\).

**Proof.** Since the vertex has valency one, by Lemma \[3.1\] the component \(E_v\) can meet \((\overline{f})^{-1} \Delta_{I'}\) at most at the point corresponding to the edge which connects it to the rest of the tree. Therefore it yields at worst a map from \(\mathbb{A}^1\) to \(X_{I'} \setminus \Delta_{I'}\). \(\square\)

In light of Lemma \[3.2\] the natural path to follow is to contract the vertices of valency one whose image in \(X\) is a point and analyse what happens on the new (singular) surface. The morphism \(F : C \times B \to \mathcal{X}_I\) will then be used to obtain a parabolic champ inside \(\mathcal{X}_I\). This will be undertaken in the following subsection and lead to Proposition \[3.9\]
3.2. The key algorithms. We keep the notation of the previous section §3.1. We set $S_0 := S$. When no confusion is possible, by abuse of language, we will often identify a vertex to the corresponding irreducible component and speak, for instance, of contraction or image of a vertex.

We construct inductively two families of surfaces according to the following algorithms:

**Algorithm 3.3** (Strong rule). The surface $S_{n+1}$ is constructed by contracting all the vertices of valency one in $S_n$ whose proper transform in $S$ is not a $(-1)$-curve.

**Algorithm 3.4** (Weak rule). The surface $W_{n+1}$, $W_0 = S_0$, is constructed by contracting all the vertices of valency one in $W_n$ whose image in $X$ is a point.

**Remark 3.5.** By the minimality of the resolution no vertex of the graph whose corresponding component is contracted by $f$ can be a $(-1)$-curve. 

**Proposition 3.6.** Let everything be as in §3.1. Consider the dual graph associated to the exceptional divisor of $S \to C \times \overline{B}$, union proper transform $\tilde{C}$ of $C \times b$, rooted at $\tilde{C}$, and endowed with the metric given by the distance from the root. Let $S_n$ be a surface obtained via the algorithm 3.3. Then

(i) the induced graph on $S_n$ is a tree;
(ii) any one-dimensional fibre of $S \to S_n$ is a chain;
(iii) any vertex in $S_n$ viewed as a vertex in the graph on $S$ meets at most one contracted connected component.
(iv) $S_n$ has at worst quotient singularities.

**Proof of Proposition 3.6.** Items (i) & (iv) follows from item (ii) and from the type of algorithm we are using. Item (ii) follows from item (iii).

We are then reduced to prove item (iii). Let $v$ be a vertex, meeting $s$ contracted connected component. In particular viewed as a vertex in $S$ it has valency at least $s + 1$, i.e. $s$ edges for the contracted components, and one giving the unique path to the root. Now consider undoing the procedure whereby $S$ was obtained from $C \times \overline{B}$, i.e. contract $-1$ curves in the reverse order. By construction this never destroys any of the above edges in the induced graph, but our vertex of interest eventually becomes a $-1$-curve, so, $s + 1 \leq 2$. 

**Definition 3.7.** By Remark 3.5 the previous proposition also holds for $W_n$, and we let $\tilde{W}$ be the conclusion of the algorithm.

Now denote by $\Gamma$ the graph of $F$ in $C \times \overline{B} \times \mathcal{X}_I$, and $|\Gamma|$ its moduli, equivalently the graph of $f$, and consider the following diagram,

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{f} & \mathcal{X}_I \\
\downarrow & & \downarrow \\
W & \xrightarrow{g} & |\Gamma|
\end{array}
\]

where $\mathcal{W}$ is the normalisation of the dominant component. It is tame because $\mathcal{X}_I$ is, and an isomorphism over $C \times B$. Better still:

**Lemma 3.8.** Let $\mathcal{V} \to W$ be the Vistoli covering champ of $W$, then there is a smooth champ $\mathcal{W} \to \mathcal{V}$ over $W$ with trivial generic stabiliser such that: $\mathcal{W} \to \mathcal{V}$ is the extraction of roots of components (irrespective of the order) of a (possibly empty) snc divisor in $\mathcal{V}$, i.e. everywhere étale locally $\mathcal{O}_{\mathcal{V}} = \mathcal{O}_{\mathcal{W}}[z]/(z^l = x)$, or $\mathcal{O}_{\mathcal{V}} = \mathcal{O}_{\mathcal{W}}[z, w]/(z^l = x, w^m = y)$ for $x = 0$, respectively $xy = 0$, an equation of the reduced fibre over $B$, and $l, m \in \mathbb{N}$ prime to the characteristic and depending on the component.
Proof. By 3.6 (iv), the Vistoli covering champ $\mathcal{V} \to W$ exists, and the pre-image of any fibre over $B$ is a snc divisor. Both $\mathcal{V}$ and $W$ contain $C \times B$ as an open set, so letting $\mathcal{V}'$ be the normalisation of the dominant component of $W \times_W \mathcal{V}$, the champ $\mathcal{V}'$ has pure ramification over a simple normal crossing divisor. The tame fundamental group of a simple normal crossing divisor is what one expects, $[\text{SGA1}]$ Exp. XIII, Cor. 5.3, so, in particular away from a crossing of components the extraction of a root as above for some $l_k$ depending on the component $k$, and taking $l_k$th roots in each component individually, irrespective of any order, yields $\mathcal{W}'$. □

The promised generalisation of Lemma 3.2 may now be given.

**Proposition 3.9.** Every irreducible component of the exceptional divisor in $\mathcal{W}'$ corresponding to the vertices of valency one in $W$ yields a parabolic champ in $\mathcal{X}_{I'}$, for some $I' \supseteq I$.

Proof. Let $v$ be a valency one vertex in $W$, $\mathcal{E}_v$ the corresponding irreducible component of the exceptional divisor in $\mathcal{W}'$, with $F$ the morphism from $\mathcal{W}'$ to $\mathcal{X}$, and $I' \supseteq I$ maximal amongst sets such that $F(\mathcal{E}_v) \subset \mathcal{X}_{I'}$. By Proposition 3.6, the component $\mathcal{E}_v$ meets the rest, $\mathcal{R}$, of the fibre of $\mathcal{W}$ over $B$ in at most one point. By 3.8 the picture is as follows:

Or, more precisely, in the notation of the proof of 3.8, $\mathcal{W}' \to \mathcal{V}$ restricted to $\mathcal{E}_v \setminus \mathcal{R}$ is an étale covering of degree $l^{-1}$. Therefore:

$$\chi(\mathcal{E}_v \setminus \mathcal{R}) = \frac{1}{l} \cdot (2 - 1 - (1 - \frac{1}{n})) = \frac{1}{ln} > 0,$$

where $l$ is the order of the stabiliser of the generic point $\mathcal{E}_v$, and $n$ is the order of the stabiliser at the non-scheme like point on the image in $\mathcal{V}$ should a contracted chain occur- otherwise $\chi = 1/l$, and the proposition is proved. □

4. **Log Bend-and-Break**

Let us first consider some ways in which Abhyankar’s theorem: any positive dimensional component of a fibre of a birational map to a smooth variety is ruled, fails for champs, i.e.,

**Example 4.1.** Let $X$ be a smooth surface. Let $X_p$ be the blow-up of a point $p \in X$ and $E$ the exceptional divisor. Let $Y$ be the blow-up of three points $q_1, q_2, q_3$ on $E$, $F_1, F_2, F_3$ the corresponding exceptional divisors, $\bar{E}$ the proper transform of $E$ in $Y$ and $f : Y \to X$ the
composite morphism. Taking $m_i \geq 3$th roots in the $F_i$ yields a champ $\rho : \mathcal{W} \to Y$ over $Y$, proper and birational to $X$, but $\rho^{-1}(\tilde{E})$ is not parabolic.

Similarly, we have the following example of Campana, in which we find further obstructions to Bend-and-break in the presence of a boundary.

**Example 4.2** (Campana, [C]. Example 9.19). Take an isotrivial family of smooth plane cubics $C_t$ passing through a point $o$ degenerating to a union of 3 lines $L_1, L_2, L_3$, only one of which, say $L_1$, passes through $o$. For instance one can take $C_t = x^3 + y^3 + t z^3 = 0$ and $o = [1 : -1 : 0]$. Let $X$ be the blow-up of two general points on $L_1$ distinct from $o$ and let $E_1, E_2$ be the corresponding exceptional divisors. Set $\Delta := E_1 + E_2$. Let $\delta \in X$ be the point over $o$. The proper transform $\tilde{L}_1$ is the only rational curve in the family through $\delta$, but it does not yield an $\mathbb{A}^1$ in $X \setminus \Delta$.

These and other difficulties limit how much one can improve 3.9, viz:

**Proposition 4.3.** Let everything be as in 3.1 and suppose further that the family fixes not just a point but a finite set, $\emptyset \neq \Gamma \subset C$ complimentary to $f^* \text{Supp}(\Delta_J)$, then for $H$ a nef divisor on $X$ there is a parabolic champ $\mathcal{L}$ in some possibly different stratum $\mathcal{X}_I \setminus \Delta_J$, $I' \supseteq I$, such that:

$$H \cdot \mathcal{L} \leq \frac{2(H \cdot f^* C)}{|\Gamma|}.$$  

If in addition,

(a) $\mathcal{X}_I \setminus \Delta_J \to X_I \setminus |\Delta_J|$ is étale.

(b) For any $I' \supseteq I$, no proper sub-stratum $X_{I'} \setminus |\Delta_{I'}|$ of the moduli contains a parabolic curve. Then the parabolic champ $\mathcal{L}$ of 4.1 may be taken to meet $f(\Gamma)$ and lie in the stratum $\mathcal{X}_I \setminus \Delta_J$.

**Remark 4.4.** Both hypotheses in Proposition 4.3 are necessary. For example, if we remove 4.3 (b), then 4.2 applies. The necessity of 4.3 (a) and the further impossibility of replacing $X_{I'} \setminus |\Delta_{I'}|$ by $\mathcal{X}_{I'} \setminus |\Delta_{I'}|$ will be discussed in Remark 4.6.

**Proof of Proposition 4.3.** Put $\Gamma = \{c_1, \ldots, c_\delta\}$. As a first step in the resolution of an indeterminacy at a point on a section $c_j \times B$, one blows up successively in the point where the proper transform crosses the exceptional divisor until the map is well defined at the proper transform of the section. This connects the proper transform of the fibre to that of the section by way of a chain of rational curves, with $E$, say, the unique curve meeting the section. The degree bound 4.1 comes from [MM, Theorem 4.1], and applies to the total transform of such an $E$ at some such indeterminacy of $f$. Choose this indeterminacy, and form a graph $G$ whose vertices are the curves in the total transform of $E$ together with one other $o$ for the rest of the curves in the fibre. The latter intersect the total transform in a single point on a single curve in the said transform, and between the corresponding vertices we add an edge, together with edges for all intersections between curves in the total transform. Finally, we root the whole thing in $o$, so 3.9 or more correctly the proof adapted to the above graph, yields 4.1. Furthermore:

**Claim 4.5.** Suppose 4.3 (a) $\not\in$ (b), and let $v \neq o$ be a vertex in $G$ (or the same at any other such indeterminacy, albeit we may not have a degree bound there) such that the irreducible component $E_v$ is not contracted to a point in $X$, then $\mathcal{X}(E_v) \not\subseteq |\Delta_J|$. Better still each $E_v$ yields a parabolic champ in $\mathcal{X}_I \setminus \Delta_J$.

**Proof of Claim 4.5.** By decreasing induction on the distance from the root in the image, $K$, of the graph (more correctly dual graph of the image) in $W$ for $W$ as per 3.7, albeit with no contractions being performed whenever these occur in $o$. Let $v_{\max}$ be a vertex at maximal distance which is not contracted in $\mathcal{X}$. Since it has valency one, we may apply Proposition 4.3 (b) to deduce that $v_{\max}$ yields a parabolic champ in $\mathcal{X}_{I'} \setminus |\Delta_{I'}|$, for some $I' \supset I$. By hypothesis 4.3 (b) we must have $I' = I$ and we are done. Let now $v_n$ be a vertex, corresponding to an
irreducible component of the exceptional divisor, which is not contracted in $\mathcal{X}$ and has distance $n$ from the root, as for example in the following picture.

Let us consider the surface $W'$ obtained by contracting the irreducible components of the exceptional divisor of $W$ which are contracted to a point in $\mathcal{X}$ and correspond to the vertices of the subgraph at distance $> n$ from the root. We denote by $f'$ the induced map from $W'$ into $X$. Let $K'$ be the corresponding graph in $W'$ (which is not necessarily a tree), again rooted at $o$ and endowed with the metric given by the distance from the root. Notice that since $K$ is a tree and we have not changed anything at distance $< n$ there is a unique vertex $v_{n-1}$ in $K'$ at distance $n - 1$ from the root which is connected to $v_n$. Observe also that $(f')^*|\Delta J|$ either contains $E_{v_n}$ or meets it in at most $E_{v_n} \cap E_{v_{n-1}}$. Indeed, by the principal ideal theorem, $(f')^*|\Delta J|$ is empty or pure co-dimension 1 while, by the inductive hypothesis, it is supported at most in vertices a distance $\leq n$ from the root. On the other hand, if $(f')^*|\Delta J|$ were supported on $E_{v_n}$, there would be a proper sub-stratum $X_I'$ containing $E_{v_n}$ such that $E_{v_n}$ meets $(f')^{-1}|\Delta J|$ in at most the point $E_{v_n} \cap E_{v_{n-1}}$, which contradicts 4.3 (b).

The induction concluded, we return to the graph $K$ in $W$, and colour $o$ and any vertices contracted in $X_I$ black. As such the sub-graph whose vertices are black has a connected component $O \ni o$. Any white vertex which meets $O$ in $K$ does so along a unique edge—otherwise it would admit two paths to $o$. As per the proof of 4.3 we pass to a singular surface $W''$, but now the contraction of all black vertices not in $O$, and again the principal ideal theorem yields that the pull-back of $\Delta J$ can have support at most in $O$. Consequently the whole of the proper transform of $E$ except $O$ is in $X_I \setminus |\Delta J|$, with the (non-empty set) of white vertices not meeting $O$ corresponding to $\mathbb{P}^1$'s in the same, and otherwise $\mathbb{A}^1$'s, and by 4.3 (a), this remains true in $\mathcal{X} \setminus \Delta J$. Finally introduce another graph by removing $o$ and its unique edge and replacing it by a vertex, $o'$, for the proper transform of the section, and an edge to the unique exceptional curve that it meets, then this is still a connected tree. Again we colour $o'$ black, and let $O'$ be its connected component in the black sub-graph. By definition there is a white vertex meeting $O'$, and every curve in $O'$ contracts to a point in $f(\Gamma)$, so we’re done.

Remark 4.6. In order to see the need whether for 4.3 (a) or the impossibility of replacing the moduli strata in 4.3 in (b) by champ strata, consider a situation as in the following picture.
This can be obtained, for example, by blowing up points on a surface with the vertex \( v \) of valency 5 being occasioned by the first blow up. As such the black vertices are -2 curves, the other white vertices, \( w_i \), -1, and the former may be blown down to non-scheme like points on a smooth champ. Equally, on this champ the picture, on adding the red vertex, can be obtained as a degeneracy of a family of curves blowing down a section to a point \( p \) which meets the picture in \( v \) alone. Each vertex of valency 1 is a parabolic champ, but the vertex which passes through \( p \) is not. This vertex could also be the boundary, so one cannot replace moduli by champ in 4.3 (b) either.

Thus without 4.3 (a) & (b) one adds nothing to 3.9 beyond quoting the degree bound 4.1 of Miyaoka-Mori. Nevertheless, we can continue the blatant plagerism of the same by way of,

**Proposition 4.7.** Let \( \mathcal{X} \to X \) be a champ over a projective variety, and \( \Delta = \sum_i \Delta_i \) the irreducible components of an effective Weil divisor on \( \mathcal{X} \), with \( H \) nef on \( X \). Suppose further that for some stratum \( \mathcal{X}_I \) each \( \Delta_j, j \in J \) is \( \mathbb{Q} \)-Cartier, and that there are enough deformations in \( \mathcal{X}_I \setminus \Delta \) at every curve whose image is some \( f : C \to \mathcal{X}_I \), then should \( (K_{\mathcal{X}} + \Delta) \cdot f C \) be negative there is a parabolic champ \( \mathcal{L} \) in some stratum \( \mathcal{X}_{I'} \setminus \Delta' \), \( I' \supseteq I \) such that,

\[
H \cdot \mathcal{L} \leq 2 \dim(\mathcal{X}_I) \left( \frac{(H \cdot f C)}{(K_{\mathcal{X}} + \Delta) \cdot f C} \right).
\]

If, in addition, 4.3 (a)& (b) hold, then for generic \( x \in C \) there is such a champ \( \mathcal{L} \ni x \) parabolic in \( \mathcal{X}_I \setminus \Delta \).

**Proof.** Whenever the normalisation \( \mathcal{C} \) of the image of \( f \) is not parabolic this follows from [MM, Theorem 5] with the same proof by 2.9 and 4.3. Otherwise, to make the trick work of replacing \( f \) by a composition with the geometric Frobenius \( C^{(1)} \to C \) modulo large primes, or a power of the same if \( k \) already has positive characteristic, we need that there are more deformations of \( C \to \mathcal{X}_I \) then there are of \( C \to \mathcal{C} \). This amounts to,

\[
(K_{\mathcal{X}} + \Delta) \cdot \mathcal{C} < -2
\]

and once one has this, one argues exactly as in the non-parabolic case, while if this fails the above bound is at least \( H \cdot \mathcal{C} \), and one takes \( \mathcal{L} = \mathcal{C} \). \( \square \)
5. The cone theorem

Cone theorems are weaker than \[ \text{I.7} \] so we need less, e.g. the following variant:

**Proposition 5.1.** Let \( \mathcal{X} \rightarrow X \) be a champ over a projective variety with snc boundary \( \Delta = \sum_i \Delta_i \) on \( \mathcal{X} \), and, bearing in mind \[ \text{II.4}, \mathcal{P} \) a parabolic champ in some \( \mathcal{X}_{I'} \setminus \Delta_{I'} \), then:

\[
\mathcal{P} \sim Z + \mathcal{L}
\]

where \( Z \) is a 1-cycle on \( \mathcal{X} \), \( \mathcal{L} \) is parabolic in \( \mathcal{X}_{I'} \setminus \Delta_{I'} \) for some \( I' \supset I \), and satisfies:

\[
-\mathcal{L} \cdot (K_{\mathcal{X}} + \Delta) \leq 2 \dim(X),
\]

**Proof.** If \(-\mathcal{P} \cdot (K_{\mathcal{X}} + \Delta) \leq 2 \dim(\mathcal{X})\) we set \( Z = 0 \) and \( \mathcal{P} = \mathcal{L} \). So we assume \(-\mathcal{P} \cdot (K_{\mathcal{X}} + \Delta) > 2 \dim(\mathcal{X})\). Take a covering \( \mathbb{P}^1 \rightarrow \mathcal{P} \) of degree \( d \in \mathbb{N} \) over the moduli. By Lemma \[ \text{2.9} \] we get:

\[
\dim_{[f]} \text{Mor}(\mathbb{P}^1, \mathcal{X}_I, \Delta_j) > (1 + 2d) \cdot \dim(\mathcal{X}_I).
\]

We fix an ample divisor \( H \) on \( X \) and argue by induction on the degree \( |\mathcal{P}| \cdot H \in \mathbb{N} \) of the moduli. As in the proof of \[ \text{I.7} \] we need more deformations in \( \mathcal{X}_I \) than there are map from \( \mathbb{P}^1 \rightarrow |\mathcal{P}| \), i.e. \( 2d + 1 \). This we have, so fixing \( 2d + 1 \) points in a deformation, \[ \text{I.3} \] yields a parabolic champ \( \mathcal{L} \) in some \( \mathcal{X}_{I'} \setminus \Delta_{I'}, I' \supset J' \) satisfying,

\[
H \cdot |\mathcal{L}| \leq 2d \cdot \frac{H \cdot |\mathcal{P}|}{1 + 2d} < H \cdot |\mathcal{P}|
\]

If the degree of \( |\mathcal{P}| \) is 1, this is nonsense, and we have the base of the induction. Otherwise, \( |\mathcal{L}| \) has smaller degree, and we conclude. \( \square \)

Let us observe,

**Remark 5.2.** The bound \( 2 \dim(X) \) is far from optimal, and it’s fastidious rather than difficult to do better. In the wholly scheme like case the optimal bound \( \dim(X) + 1 \) is known to hold, \[ \text{K2} \], but one has to vary the construction from \[ \text{MM} \]. However, even without varying it, things actually get better rather than worse the less scheme like the situation becomes since, in general, there are much fewer maps from a \( \mathbb{P}^1 \) to a parabolic champ than there are to its moduli.

In any case the following deduction from \[ \text{I.1}, \text{and K2 Theorem III.1.2} \] is formal,

**Proposition 5.3.** Let \( \mathcal{X} \rightarrow X \) be a smooth champ over a projective variety, and \( \Delta = \sum_i \Delta_i \) a snc divisor on \( \mathcal{X} \), then there exists a countable family \( \{L_k \subset X\} \) of curves whose induced champs \( \mathcal{L}_k \rightarrow L_k \) are parabolic in some \( \mathcal{X}_{I_k} \setminus \Delta_{I_k} \), satisfying,

\[
0 < -(K_{\mathcal{X}} + \Delta) \cdot \mathcal{L}_k \leq 2 \cdot \dim(X),
\]

such that

\[
\text{NE}(\mathcal{X}) = \text{NE}(\mathcal{X})(K_{\mathcal{X}} + \Delta) \geq 0 + \sum_k \mathbb{R}_+[\mathcal{L}_k]
\]

and the parabolic rays \( \mathbb{R}_+[\mathcal{L}_k] = \mathbb{R}_+[L_k] \) are locally discrete in \( \text{N}_1(\mathcal{X})(K_{\mathcal{X}} + \Delta) < 0 \).

**Proof.** Since we have somewhat weaker theorems at our disposition, e.g. \[ \text{I.1} \] rather than \[ \text{K2 Theorem II.5.7} \] we’ll go through the proof of III.1.2 in op. cit. to check that it works.

The first stage is to note that we have countably many classes \( L_k \) such that,

(i) \( 0 < -(K_{\mathcal{X}} + \Delta) \cdot L_k \leq 2 \cdot \dim(\mathcal{X}); \)

(ii) We may identify \( L_k \) with a curve such that the champ \( \mathcal{L}_k \) over the same is parabolic in some \( \mathcal{X}_{I_k} \setminus \Delta_{I_k} \).
Next form the closed cone $W$ of the right hand side of (5.2) and suppose this is not the left hand side, then there is a linear functional $M$ non-negative on the left hand side, strictly positive on $W\setminus \{0\}$, but vanishing on some $0 \neq z \in \overline{\text{NE}}(X)$. In particular $-(K_X+\Delta) \cdot z > 0$.

Now, say, $C_i = \sum_j a_{ij} C_{ij}$ are effective cycles limiting on $z$, then for every $i$ sufficiently large there is a $j$ such that,

$$\frac{M \cdot C_{ij}}{-(K_X+\Delta) \cdot C_{ij}} \leq \frac{M \cdot C_i}{-(K_X+\Delta) \cdot C_i}$$

By (4.7) applied to the $C_{ij}$ and (5.1) applied to the resulting parabolic champs $\mathcal{P}_{ij}$, there are champs $L_{k(i)}$ parabolic in some $\mathcal{X}_{k(i)} \setminus \Delta_{k(i)}$ such that,

$$M \cdot |L_{k(i)}| \leq 2 \dim(X) \cdot \frac{M \cdot C_i}{-(K_X+\Delta) \cdot C_i}$$

and $-(K_X+\Delta) \cdot L_{k(i)} < 2 \cdot \dim(X)$. By definition, the classes $[L_{k(i)}]$ are integral and belong to $W$, so for $i$ sufficiently large the left hand side is bounded below independent of $i$, while the right hand side tends to zero. This is nonsense, so, indeed, $W = \text{NE}(X)$.

To conclude we need to know that the parabolic rays are locally discrete and the right hand side of (5.2) is a closed cone. These statements, however, now follow verbatim as in [K2, Theorem III.1.2] up to the simple expedient of replacing $K_X$ in op. cit. by $K_X + \Delta$.

\[\square\]

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