Integers without divisors
from a fixed arithmetic progression

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Abstract

Let $a$ be an integer and $q$ a prime number. In this paper we find an
asymptotic formula for the number of positive integers $n \leq x$ with the
property that no divisor $d > 1$ of $n$ lies in the arithmetic progression
$a$ modulo $q$.

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1 Introduction

We consider the frequency of natural numbers which do not have any divisor from a given arithmetic progression. More precisely, for integers $0 \leq a < m$ and a real number $x \geq 1$, we define:

$$N(m, a) = \{n \geq 1 : d \not\equiv a \pmod{m} \text{ for all } d \mid n, \ d > 1\}$$

and denote by $N(x; m, a)$ the number of positive integers $n \leq x$ in $N(m, a)$.

Our goal is to determine an explicit asymptotic formula for $N(x; m, a)$. We exclude the divisor $d = 1$ in the above definition since including it would make the result trivial for one residue class while not affecting the result for any of the others. To avoid increasing the technical complications we give detailed consideration to the special case that $m = q$ is a prime number. In the final section we give some remarks about the case of general modulus and about the still more complicated problem of counting those integers whose divisors avoid a subset of the residue classes.

When $a = 0$, it is clear that $n \in N(q, 0)$ if and only if $q$ does not divide $n$, and in this case it follows that

$$N(x; q, 0) = (1 - q^{-1}) x + O(q).$$

Thus, we can assume that $a \geq 1$ in what follows.

If $a = 1$ and $q = 2$, it is also clear that $n$ is in $N(2, 1)$ if and only if $n$ is a power of two, and therefore,

$$N(x; 2, 1) = \frac{\log x}{\log 2} + O(1).$$

Hence, we can further assume that $q \geq 3$ throughout the sequel. The case $a = 1$ is essentially different from (and quite a bit easier than) the others. The result obtained is the following.

**Theorem 1.** For every fixed odd prime $q$ we have

$$N(x; q, 1) = (1 + o(1)) \frac{\varphi(q - 1) q}{(q - 1)^{q-1} (q - 3)!} \frac{x \log \log x}{\log x},$$

where $\varphi$ is the Euler function.
In view of Theorem 1, which is proved in Section 3.1 below, it remains only to consider the case that $1 < a < q$. In order to state this result we introduce three constants $P_{a,q}$, $V_{a,q}$, and $W_{a,q}$, as follows. First, let

$$q - 1 = \prod_{j=1}^{k} p_j^{\alpha_j}$$

and

$$\text{ord}_q(a) = \prod_{j=1}^{s} p_j^{\beta_j}$$

be the prime factorizations of $q - 1$ and $\text{ord}_q(a)$ (the multiplicative order of $a$ modulo $q$), respectively. Here, $p_1, \ldots, p_k$ are distinct primes, $s \leq k$, and the integers $\alpha_j$ and $\beta_j$ are positive. Using these data, we define

$$P_{a,q} = \min_{1 \leq j \leq s} \{ p_j^{\alpha_j - \beta_j + 1} \}. \quad (1)$$

Next, recalling that every subgroup of a cyclic group is determined uniquely by its cardinality, let $H(a)$ be the unique subgroup in $(\mathbb{Z}/q\mathbb{Z})^*$ of cardinality $|H(a)| = (q - 1)/P_{a,q}$, and put

$$V_{a,q} = \lim_{y \to \infty} \left\{ (\log y)^{-1/P_{a,q}} \prod_{p \leq y \text{ (mod } q \text{) } \in H(a)} \left( 1 - \frac{1}{p} \right)^{-1} \right\}. \quad (2)$$

Thanks to the work of Williams [15], one knows that the limit exists and $0 < V_{a,q} < \infty$. Finally, suppose that $P_{a,q}$ is the prime power $p^r$, and put

$$W_{a,q} = e^{-\gamma/P_{a,q}} (1 - q^{-1})^{-1+1/P_{a,q}} \Gamma(1/P_{a,q}) P_{a,q}^{\alpha_q-2} (P_{a,q} - 2)! \sum_{1 \leq j \leq p/2} P_{a,q} \left( p^r - 2 \right) \left( p^r - 1 \right)^{j-1}, \quad (3)$$

where $\gamma$ is the Euler-Mascheroni constant, and $\Gamma(s)$ is the usual gamma function.

**Theorem 2.** For every fixed odd prime $q$ and integer $a$ with $1 < a < q$, we have

$$N(x; q, a) = (1 + o(1)) V_{a,q} W_{a,q} \frac{x(\log \log x)^{P_{a,q}-2}}{(\log x)^{1-1/P_{a,q}}}. \quad (4)$$

We deal throughout with a fixed arithmetic progression and do not consider the question of uniformity of the estimations in the modulus $q$, although it is clear from the methods employed that some (probably not very large) range of uniformity could be obtained.
The question of counting the number of integers up to $x$ with no prime divisor in a given residue class is more familiar and has a simpler answer; see for example the theorem of Wirsing given below in Lemma 10. Our proofs use this result and similar analytic methods but are complicated by other considerations which are mostly of a combinatorial nature and with a bit of group theory.

As we shall see in Lemma 3, the group $H(a)$ is the subgroup of $(\mathbb{Z}/q\mathbb{Z})^*$ having the largest order amongst those which do not contain (the class of) $a$, and this suggests its relevance to our problem. The fact that this subgroup is not unique in general, when the group is not cyclic, is the main thing which complicates the case of arbitrary modulus. These facts also lead, in our case of prime modulus, to the following easy corollaries.

**Corollary 1.** In case $H(a) = H(b)$ we have

$$N(x; q, a) \sim N(x; q, b).$$

Special cases of this give the following two results.

**Corollary 2.** If $\overline{a}$ satisfies $a\overline{a} \equiv 1 \pmod{q}$ then

$$N(x; q, \overline{a}) \sim N(x; q, a).$$

**Corollary 3.** If $a$ and $b$ are both quadratic non-residues modulo $q$ then

$$N(x; q, a) \sim N(x; q, b).$$

*Proof.* In this case $H(a)$ and $H(b)$ are each the subgroup of quadratic residues. \hfill $\square$

Finally we have

**Corollary 4.** If $a$ is a quadratic residue modulo $q$ and $b$ is a quadratic non-residue then

$$N(x; q, a) = o(N(x; q, b)).$$

*Proof.* In this case, either $a = 1$ and the result follows on comparing the estimates of the two theorems or, if $a > 1$, then $H(a)$ is a subgroup of index greater than two and the result follows from the second theorem. \hfill $\square$
Although there seem to be no earlier results that consider the above asymptotic formulae in this rather basic question, there is a long history of work on closely related problems. Erdős [3] showed that, if $m \leq (\log x)^{\log 2 - \delta}$ where $\delta > 0$ is fixed, then almost all positive integers $n \leq x$ have a divisor $d$ in each one of the residue classes $a \pmod{m}$, with $\gcd(a, m) = 1$. The value log 2 is optimal. Indeed, if $n$ satisfies the above condition then $\tau(n) \geq \phi(m)$ and, since $\tau(n) = (\log x)^{\log 2 + o(1)}$ holds for almost all $n \leq x$, we find that $m \leq (\log x)^{\log 2 + o(1)}$. Since the appearance of [3], the distribution of integers having a divisor in a specific residue class has been studied by several authors. For example, in answer to a question of Erdős from [4], Hall [7] showed that, for any $\varepsilon > 0$ and natural number $N$, there exists $\eta_N$ with $\eta_N \to 0$ when $N \to \infty$ such that, if $m \geq (\log N)^{\log 2 + z_N \sqrt{\log \log N}}$ then the number of positive integers $n \leq x$ having a divisor $d$ in the interval $m \leq d \leq N$ with $d \equiv 1 \pmod{m}$ is $< \eta_N x$ provided $N \leq x$. Extending prior results of Hall [7] and Erdős and Tenenbaum [6], de la Bretèche [1] proved that, if $N$ is any positive integer and $z_N$ is defined implicitly by the relation $m = (\log N)^{\log 2 - z_N \sqrt{\log \log N}}$, then there exists $\eta_N \to 0$ when $N \to \infty$ such that, for any $a$ coprime to $m$, we have

$$\left| \{n \leq x : d \equiv a \pmod{m} \text{ for some } d \mid n, \ m \leq d \leq N \} \right| = \Phi(z_N) + O(\eta_N x)$$

for all $3 \leq N \leq x$, where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\tau^2/2} d\tau,$$

which in turn answered a conjecture of Hall from [8].

Throughout the paper, $x$ denotes a large positive real number. We use the Landau symbols $O$ and $o$, as well as the Vinogradov symbol $\ll$, with their usual meanings. As we do not consider the question of obtaining bounds which are uniform in the modulus of the arithmetic progression we allow the implied constants in many places to depend on various parameters, such as the modulus, without explicit mention.

For a positive integer $\ell$, we write $\log_\ell x$ for the function defined inductively by $\log_1 x = \max\{\log x, 1\}$ and $\log_\ell x = \log_{\ell-1}(\log x)$ for $\ell \geq 2$, where log denotes the natural logarithm function. In the case $\ell = 1$, we omit the subscript to simplify the notation; however, it should be understood that all the logarithms that appear are $\geq 1$. 

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We use various other standard notations, including those for basic arithmetic functions such as Euler’s \( \varphi \)–function. We use \(|A|\) to denote the number of elements in \( A \) when \( A \) is a finite group, or set, or multiset. Given a set \( S \) of positive integers, whether finite or infinite, we frequently denote by \( S(x) \) the number of integers \( n \leq x \) in \( S \).

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2 Preliminary results

2.1 Combinatorial results

Recall that a multiset is a list \( \langle a_1, \ldots, a_k \rangle \) of elements from a set in which the same element can occur more than once, but the order is unimportant. For example, \( \langle 1, 1, 2, 3 \rangle \) and \( \langle 3, 1, 2, 1 \rangle \) are the same multisets in \( \mathbb{Z} \), whereas \( \langle 1, 1, 2, 3 \rangle \) and \( \langle 1, 2, 3 \rangle \) are different.

Let \( G \) be an arbitrary finite abelian group, written additively. If \( G = \{0\} \), put \( \kappa(G) = 0 \); otherwise, let \( \kappa(G) \) be the largest integer \( k \) for which there exists a multiset \( \langle a_1, \ldots, a_k \rangle \) of elements of \( G \) with the property:

\[
\sum_{j \in S} a_j \neq 0 \quad \text{for every nonempty subset } S \subseteq \{1, 2, \ldots, k\}. \quad (4)
\]

Since \( |G| < \infty \), it is easy to see that \( \kappa(G) < \infty \). In the special case that \( G = \mathbb{Z}/m\mathbb{Z} \), we have the following result:

Lemma 1. Let \( G = \mathbb{Z}/m\mathbb{Z} \), where \( m \geq 1 \). Then \( \kappa(G) = m - 1 \). Moreover, if \( m \geq 2 \), then the multiset \( \langle a_1, \ldots, a_{m-1} \rangle \) has the property \( (4) \) if and only if \( a_1 = \cdots = a_{m-1} = a \) for some \( a \in G \) that is coprime to \( m \).
Proof. We can assume that $m \geq 2$ since the result is trivial for $m = 1$.

Suppose that $\kappa(G) \geq m$. Then, for some $k \geq m$, there exists a multiset $\langle a_1, \ldots, a_k \rangle$ in $G$ with the property $\mathfrak{H}$. Since the elements $b_j = \sum_{i=1}^{j} a_i$, $j = 1, \ldots, k$, are all nonzero, and $G$ has only $m - 1$ nonzero elements, two of the elements $b_j$ must be equal by the pigeonhole principle; that is, $b_{j_1} = b_{j_2}$ for some $j_1 < j_2$. But this implies that $\sum_{j_1 < i \leq j_2} a_i = 0$, which contradicts $\mathfrak{H}$. Therefore, $\kappa(G) \leq m - 1$.

Next, suppose that $a_1 = \cdots = a_{m-1} = a$ for some $a \in G$ that is coprime to $m$. Then, for every nonempty subset $S \subseteq \{1, 2, \ldots, m-1\}$, one has $\sum_{j \in S} a_j = a|S|$. Since $\gcd(a, m) = 1$ and $m \nmid |S|$, it cannot be true that $a|S| \equiv 0 \pmod{m}$; therefore, the multiset $\langle a_1, \ldots, a_{m-1} \rangle$ has the property $\mathfrak{H}$ which shows that $\kappa(G) \geq m - 1$.

Finally, suppose that the multiset $\langle a_1, \ldots, a_{m-1} \rangle$ has the property $\mathfrak{H}$. As before, let $b_j = \sum_{i=1}^{j} a_i$, $j = 1, \ldots, m - 1$. Then the elements $b_1, \ldots, b_{m-1}$ are distinct and nonzero, and since $G$ has precisely $m - 1$ nonzero elements, it follows that $\{b_1, \ldots, b_{m-1}\} = G \setminus \{0\}$. Using $\mathfrak{H}$, we see that $a_2 \neq 0$, and $a_2 \neq b_j$ for $j = 2, \ldots, m-1$; therefore, $a_2 = b_1 = a_1$. By a similar argument, it follows that $a_j = a_1$ for $j = 2, \ldots, m-1$; in other words, $a_1 = \cdots = a_{m-1} = a$ holds for some $a \in G$. Thus, we have $b_j = ja$ for $j = 1, \ldots, m - 1$ and, since $b_j \equiv 1 \pmod{m}$ for some value of $j$, it follows that $\gcd(a, m) = 1$. \[ \square \]

Now, let $G$ be a nontrivial finite abelian group, written additively. If $G = \mathbb{Z}/2\mathbb{Z}$, put $\kappa(G, 1) = 0$; otherwise, for every $a \in G \setminus \{0\}$, let $\kappa(G, a)$ be the largest integer $k$ for which there exists a multiset $\langle a_1, \ldots, a_k \rangle$ of elements of $G \setminus \{0\}$ with the property:

$$\sum_{j \in S} a_j \neq a \text{ for every subset } S \subseteq \{1, 2, \ldots, k\}. \quad (5)$$

In general, $\kappa(G, a)$ need not be finite (e.g., if $p$ is prime, $G = \mathbb{Z}/p^2\mathbb{Z}$, and $a = 1$, then $\mathfrak{E}$ holds for the multiset $\langle a_1, \ldots, a_k \rangle$, where $a_1 = \cdots = a_k = p$, for every natural number $k$). However, we do have the following finiteness result, which suffices for our applications:

**Lemma 2.** Let $p$ be a fixed prime, and let $G_r = \mathbb{Z}/p^r\mathbb{Z}$ for every natural number $r$. If $a \in G_r \setminus \{0\}$ and $p^{r-1} \mid a$, then $\kappa(G_r, a) = p^r - 2$.

**Proof.** First, we argue by induction on $r$ that for every $a \in G_r \setminus \{0\}$ with $p^{r-1} \mid a$ and every multiset $\langle a_1, \ldots, a_k \rangle$ in $G_r \setminus \{0\}$ with the property $\mathfrak{E}$, the
following inequality holds:
\[
\left| \left\{ g \in G_r : g = \sum_{j \in S} a_j \text{ for some subset } S \subseteq \{1, 2, \ldots, k\} \right\} \right| \geq k + 1. \tag{6}
\]

Since the left side of (6) cannot exceed \( |G_r \setminus \{a\}| = p^r - 1 \), it follows that \( \kappa(G_r, a) \leq p^r - 2 \).

Suppose first that \( r = 1 \), and put \( G = G_1 = \mathbb{Z}/p\mathbb{Z} \). Let \( a \in G \setminus \{0\} \) be fixed, and suppose that \( \langle a_1, \ldots, a_k \rangle \) is a multiset in \( G \setminus \{0\} \) with the property (3). Let \( b_1, \ldots, b_s \) be the distinct values taken by \( a_i \) for \( i = 1, \ldots, k \), and let \( m_1, \ldots, m_s \) be the respective multiplicities; then \( \sum_{j=1}^s m_j = k \). Put
\[
A_j = \{ ub_j : u = 0, 1, \ldots, m_j \} \quad (1 \leq j \leq s).
\]

Since each \( m_j < p - 1 \) (otherwise, \( a \in A_j \) and (3) fails), \( A_j \) is a subset of \( G \) of cardinality \( m_j + 1 \). Let \( \sum_{j=1}^s A_j \) be the set of elements \( g \in G \) of the form \( g = \sum_{j=1}^s c_j \), where \( c_j \in A_j \) for \( j = 1, \ldots, s \). A corollary/generalization of the Cauchy-Davenport theorem (see for example [12, Theorem 2.3]) states that
\[
\left| \sum_{j=1}^s A_j \right| \geq \min \{ p, \sum_{j=1}^s |A_j| - s + 1 \},
\]
and in our situation,
\[
\sum_{j=1}^s |A_j| - s + 1 = \sum_{j=1}^s (m_j + 1) - s + 1 = k + 1.
\]

Since \( \sum_{j=1}^s A_j \) is the set of elements \( g \in G \) that can be written as \( \sum_{j \in S} a_j \) for some subset \( S \subseteq \{1, 2, \ldots, k\} \), we also have by (3):
\[
\left| \sum_{j=1}^s A_j \right| \leq |G \setminus \{a\}| = p - 1.
\]

Therefore, \( k + 1 \leq p - 1 \), and we obtain the inequality (6) when \( r = 1 \).

To complete the induction, we show that (6) holds for the integer \( r \geq 2 \) assuming that the corresponding inequality is true for \( r - 1 \).

Let \( a \in G_r \setminus \{0\} \) with \( p^{r-1} \mid a \), and suppose that \( \langle a_1, \ldots, a_k \rangle \) is a multiset in \( G_r \setminus \{0\} \) satisfying (3). Without loss of generality, we can assume that \( a_1, \ldots, a_{\ell} \in G_r \setminus G'_r \) and \( a_{\ell+1}, \ldots, a_k \in G'_r \setminus \{0\} \), where \( G'_r \) is the subgroup of \( G_r \) consisting of those elements divisible by \( p \).
Let $\bar{a}_j = a_{\ell+j}/p$ for $j = 1, \ldots, k-\ell$, and put $\bar{a} = a/p$. Then $\bar{a} \in G_{r-1} \setminus \{0\}$ with $p^{r-2} \mid \bar{a}$, and $\langle \bar{a}_1, \ldots, \bar{a}_{k-\ell} \rangle$ is a multiset in $G_{r-1} \setminus \{0\}$ that satisfies the analogous statement of (5) obtained after replacing $a$ by $\bar{a}$, each $a_j$ by $\bar{a}_j$, and $k$ by $k - \ell$, since the condition $\sum_{j \in S} \bar{a}_j \not\equiv \bar{a}$ in $G_{r-1}$ is equivalent to $\sum_{j \in S} a_{\ell+j} \not\equiv a$ in $G_r$ for every subset $S \subseteq \{1, 2, \ldots, k - \ell\}$. Applying the inductive hypothesis with the element $\bar{a}$ and the multiset $\langle \bar{a}_1, \ldots, \bar{a}_{k-\ell} \rangle$ in $G_{r-1}$, and considering its implication for the element $a$ and the multiset $\langle a_{\ell+1}, \ldots, a_k \rangle$ in $G_r$, one sees that if $B$ denotes the set of elements $g \in G_r$ equal to $\sum_{j \in S} a_j$ for some subset $S \subseteq \{\ell + 1, \ldots, k\}$, then $|B| \geq k - \ell + 1$.

Let $b_1, \ldots, b_s \in G_r \setminus G_r'$ be the distinct values taken by $a_i$ for $i = 1, \ldots, \ell$, and let $m_1, \ldots, m_s$ be the respective multiplicities; then $\sum_{j=1}^s m_j = \ell$. Let $A_j = \{0, b_j\}$, and put

$$m_j A_j = \underbrace{A_j + \cdots + A_j}_{m_j \text{ copies}} \quad (1 \leq j \leq s).$$

Since each $b_j$ is coprime to $p$, a theorem of I. Chowla (see [12, Theorem 2.1]) yields the inequality

$$|B + \sum_{j=1}^s m_j A_j| \geq \min \left\{ p^r, |B| + \sum_{j=1}^s m_j |A_j| - \sum_{j=1}^s (m_j - 1) - s \right\}.$$ 

Since $|A_j| = 2$ for $j = 1, \ldots, s$, we have

$$|B| + \sum_{j=1}^s m_j |A_j| - \sum_{j=1}^s (m_j - 1) - s \geq k - \ell + 1 + \sum_{j=1}^s m_j = k + 1.$$

As $B + \sum_{j=1}^s m_j A_j$ is the set of elements $g \in G_r$ that are equal to $\sum_{j \in S} a_j$ for some subset $S \subseteq \{1, 2, \ldots, k\}$, we also have by (5):

$$|B + \sum_{j=1}^s m_j A_j| \leq |G_r \setminus \{a\}| = p^r - 1.$$

Therefore, $k + 1 \leq p^r - 1$, and we obtain the inequality (6), which completes the induction.

As mentioned earlier, the inequality (6) implies that $\kappa(G_r, a) \leq p^r - 2$ for all $a \in G_r \setminus \{0\}$ with $p^{r-1} \mid a$. On the other hand, the lower bound $\kappa(G_r, a) \geq p^r - 2$ is an immediate consequence of the next lemma. \hfill \square
Lemma 3. Suppose that $p$, $r$, and $a$ satisfy the conditions of Lemma 2, and put $k = p^r - 2$. For every $b \in G_r$ such that $p \nmid b$, let $n$ be the least nonnegative integer for which the congruence $n \equiv ab^{-1} - 1 \pmod{p^r}$ holds, and let $M_{p,r,a}(b) = \langle a_1, \ldots, a_k \rangle$ be the multiset in $G_r \setminus \{0\}$ defined by

$$a_j = \begin{cases} b & \text{if } j \leq n; \\ -b & \text{if } j \geq n + 1. \end{cases}$$

Then $M_{p,r,a}(b)$ has the property (5).

Proof. For every subset $S \subseteq \{1, \ldots, k\}$, we have $\sum_{j \in S} a_j = mb$ for some integer $m$ in the range $-(k-n) \leq m \leq n$. Hence, $m \not\equiv (n+1) \pmod{p^r}$, and therefore $mb \not\equiv (n+1)b \equiv a \pmod{p^r}$.

The next lemma shows that the multisets $M_{p,r,a}(b)$ defined in Lemma 3 are the only critical multisets that arise under the conditions of Lemma 2.

Lemma 4. Suppose that $p$, $r$, and $a$ satisfy the conditions of Lemma 2, and put $k = p^r - 2$. If $\langle a_1, \ldots, a_k \rangle$ is a multiset in $G_r \setminus \{0\}$ with the property (5), then $\langle a_1, \ldots, a_k \rangle = M_{p,r,a}(b)$ for some choice of $b \in G_r$.

Proof. We proceed by induction on $r$, following the proof of Lemma 2.

First, let $r = 1$. Suppose there exist integers $b, c$ with $b \not\equiv \pm c \pmod{p}$ and indices $i, j$ such that $a_i \equiv b \pmod{p}$ and $a_j \equiv c \pmod{p}$. Reordering the elements $a_1, \ldots, a_k$ if necessary, we can assume that $i = 1$ and $j = 2$. Let $A = \{0, a_1\} + \{0, a_2\}$; clearly, $|A| = 4$. Let $A_j = \{0, a_j\}$ for $j = 1, \ldots, k$. By the Cauchy-Davenport theorem, we have

$$p - 1 \geq \left| A + \sum_{j=3}^{k} A_j \right| \geq \min \left\{ p, |A| + \sum_{j=3}^{k} |A_j| - (k - 1) + 1 \right\} = p,$$

which is impossible. Thus, there exists an integer $b$ such that $a_j \in \{b, -b\}$ for $j = 1, \ldots, k$. After reordering the elements $a_1, \ldots, a_k$, we can assume that $a_j = b$ if $j \leq m$ and $a_j = -b$ if $j \geq m + 1$, for some $0 \leq m \leq k$.

Now, let $n$ be the least positive integer for which $n \equiv ab^{-1} \pmod{p}$ holds. If $n \leq m$, then $a_1 + \cdots + a_n = nb \equiv a \pmod{p}$, which contradicts (5). On the other hand, if $n \geq m + 2$, then $p - n \leq p - 2 - m = k - m$, thus $a_{m+1} + \cdots + a_{m+n} = (p-n)(-b) \equiv a \pmod{p}$, which again contradicts (5). Therefore, $n = m + 1$, and the result follows for $r = 1$. 

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Now suppose that the result has been proved for all cyclic $p$-groups of order less than $p^r$; we need to prove it for $G_r = \mathbb{Z}/p^r\mathbb{Z}$.

To do this, let us first show that $p \nmid a_j$ for each $j = 1, \ldots, k$. Indeed, suppose this is not the case. If $p \mid a_j$ for all $j = 1, \ldots, k$, then writing $\tilde{a}_j = a_j/p$, we see that the multiset $\langle \tilde{a}_1, \ldots, \tilde{a}_k \rangle$ has the property \((\mathbf{3})\) with $a$ replaced by $\tilde{a} = a/p$. Since the elements $\tilde{a}_1, \ldots, \tilde{a}_k$ can be viewed as elements of the cyclic group with $p^{r-1}$ elements, the induction hypothesis implies that $p^r - 2 = k \leq p^{r-1} - 2$, which is impossible. This argument shows that there exists at least one element $a_j$ such that $p \nmid a_j$. Now, using the notation of the proof of Lemma 2, we can assume that $p \nmid a_j$ for $j = 1, \ldots, \ell$ and $p \nmid a_j$ for $j = \ell + 1, \ldots, k$, where $1 \leq \ell < k$. Let $B$ denote the set of elements $g \in G_r$ that are equal to $\sum_{j \in S} a_j$ for some subset $S \subseteq \{\ell + 1, \ldots, k\}$. As in the proof of Lemma 2 we have $|B| \geq k - \ell + 1$. Since $p \mid a_1$, it follows that $|B + \{0, a_1\}| = 2|B| > |B| + 1$.

Since the last inequality is strict, the argument based on the Chowla Theorem (see the proof of Lemma 2) implies that $|B + \sum_{j=1}^s m_j A_j| > \min\{p^r, k + 2\}$.

Taking into account that $B + \sum_{j=1}^s m_j A_j$ has at most $p^r - 1$ elements (since this set does not contain $a$), we see that $k \leq p^r - 3$, which is impossible. Thus, we have established our assertion that $p \nmid a_j$ for $j = 1, \ldots, k$.

To complete the proof of the lemma for the group $G_r$, we can use an argument identical to the one given above for the case $r = 1$, except that the Cauchy-Davenport Theorem is now replaced by the Chowla Theorem, which is applicable since $\gcd(a_j, p) = 1$ for $j = 1, \ldots, k$. 

The next lemma provides a complete list of the distinct multisets $\mathfrak{M}_{p,r,a}(b)$ which arise for various choices of $p$ and $r$ in the special case that $a = p^{r-1}$.

**Lemma 5.** Let $p^r$ be a prime power, and let $a = p^{r-1}$. For integers $\eta$ and $c$ let $\mathfrak{N}_{p,r}(\eta, c)$ be the multiset in $\mathbb{Z}$ defined by

\[
\mathfrak{N}_{p,r}(\eta, c) = \langle \underbrace{c, c, \ldots, c}_{\eta \text{ copies}}, \underbrace{-c, -c, \ldots, -c}_{p^r - 2 - \eta \text{ copies}} \rangle.
\]
For an integer \( \lambda \) not divisible by \( p \) let \( \lambda \) be the least positive integer such that \( \lambda \equiv 1 \pmod{p} \). Then, there is a one to one correspondence, given by the congruence modulo \( p^r \), between pairs of multisets \( M_{p, r, a}(\pm b) \) in \( G_r \setminus \{0\} \) and the family \( N_{p, r}(\eta, c) \) where, in case \( p \) is odd, \( \eta \) runs through the integers
\[
\eta \in \{ p^{r-1}\lambda - 1 : 1 \leq \lambda \leq (p - 1)/2 \}
\]
and \( c \) runs through the integers
\[
c \in \{ \lambda + p\mu : 0 \leq \mu \leq p^{r-1} - 1 \}
\]
while, in case \( p = 2 \), we have \( \eta = 1 \) and, in the range for \( c \), we must replace \( p^{r-1} - 1 \) by \( 2^{r-2} - 1 \).

Proof. Let \( M_{p, r, a}(b) \) be a multiset in \( G_r \setminus \{0\} \) of the type constructed in Lemma 3. We claim that \( M_{p, r, a}(b) = M_{p, r, a}(-b) \). Indeed, let \( n \) be the least nonnegative integer for which the congruence
\[
n \equiv p^{r-1}b - 1 \equiv p^{r-1}\lambda - 1 \pmod{p^r}
\]
holds. Clearly, \( n \neq p^r - 1 \), hence it follows that \( m = p^r - 2 - n \) is the least nonnegative integer for which the congruence \( m \equiv p^{r-1}(-b) - 1 \pmod{p^r} \) holds, and this implies the claim.

For a given multiset \( M_{p, r, a}(b) \), let \( d \) be the least positive integer congruent to \( b \) modulo \( p^r \), and let \( \mathcal{M} \) be the multiset in \( \mathbb{Z} \) defined by
\[
\mathcal{M} = \langle d, d, \ldots, d, -d, -d, \ldots, -d \rangle_{n \text{ copies}}, \langle -d, -d, \ldots, -d \rangle_{p^r - 2 - n \text{ copies}}.
\]
Then \( \mathcal{M} \) and \( M_{p, r, a}(b) \) are congruent modulo \( p^r \).

Suppose first that \( p = 2 \). Since \( M_{p, r, a}(b) = M_{p, r, a}(-b) \), then replacing \( b \) by \( -b \) if necessary, we can assume that \( d \leq 2^r - d \). Hence, \( d \) is a positive odd integer with \( d \leq 2^r - 1 \). Also,
\[
n \equiv 2^{r-1}b - 1 \equiv 2^{r-1} - 1 \pmod{2^r},
\]
where the second congruence follows from the fact that \( b \) is odd; in view of the minimality condition on \( n \), it follows that \( n = 2^{r-1} - 1 \). Therefore, \( \mathcal{M} = N_{p, r}(\eta, c) \) with \( \eta = n \) and \( c = d \).

Now suppose that \( p > 2 \). Since \( M_{p, r, a}(b) = M_{p, r, a}(-b) \), then replacing \( b \) by \( -b \) if necessary, we can assume that \( n \leq p^r - 2 - n \). Let \( \lambda \) be the least positive integer such that \( \lambda \equiv b^{-1} \pmod{p} \); then
\[
n \equiv p^{r-1}b^{-1} - 1 \equiv p^{r-1}\lambda - 1 \pmod{p^r}.
\]
In view of the minimality condition on $n$ and the fact that $n \leq (p^r - 2)/2$, it follows that $n \in \{p^{r-1}\lambda - 1 : 1 \leq \lambda \leq (p - 1)/2\}$. Also, defining $\lambda$ as in the statement of the lemma, we have

$$d \equiv b \equiv \lambda \pmod{p}. $$

Since $1 \leq d \leq p^r - 1$, it follows that $d \in \{\lambda + p\mu : 0 \leq \mu \leq p^{r-1} - 1\}$. Therefore, $\mathcal{M} = \mathcal{N}_{p,r}(\eta, c)$ with $\eta = n$ and $c = d$.

To prove the uniqueness assertion, we must show that the multisets $\mathcal{N}_{p,r}(\eta_1, c_1)$ defined in the statement of the lemma are all distinct modulo $p^r$. If $p^r = 2$, then $\eta_1 = \eta_2 = 2^{r-1} - 1$. Also, since $c_j < 2^r - c_j$ for $j = 1, 2$ (note that the inequalities are strict since $2^r \geq 4$), the congruence (7) implies that $c_1 \equiv c_2 \pmod{2^r}$; as $c_1, c_2 \in \{1, 3, \ldots, 2^{r-1} - 1\}$, this is possible only if $c_1 = c_2$. If $p > 2$, then the inequalities $\eta_j < p^r - 2 - \eta_j$, $j = 1, 2$, and the congruence (7) together imply that $\eta_1 = \eta_2$ and $c_1 \equiv c_2 \pmod{p^r}$. Since $1 \leq c_j \leq p^r - 1$ for $j = 1, 2$, it follows that $c_1 = c_2$. This completes the proof. \hfill \Box

### 2.2 Algebraic results

Let $G$ be a fixed nontrivial cyclic group, and let $a$ be an element of $G$ other than the identity. Among the subgroups $H < G$ that do not contain $a$, let $H(a)$ denote that subgroup $H$ which has the greatest cardinality; note that $H(a)$ is well-defined since every subgroup of a finite cyclic group is determined uniquely by its cardinality. Let

$$|G| = \prod_{j=1}^{k} p_j^{\alpha_j} \quad \text{and} \quad \text{ord}_G(a) = \prod_{j=1}^{s} p_j^{\beta_j}$$

(8)

be the prime factorizations of $|G|$ and $\text{ord}_G(a)$ (the order of $a$ in $G$). Here, $p_1, \ldots, p_k$ are distinct primes, $s \leq k$, and the integers $\alpha_j$ and $\beta_j$ are positive. Using these data, we define:

$$\mathcal{P}(G, a) = \min_{1 \leq j \leq s} \left\{ p_j^{\alpha_j - \beta_j + 1} \right\}. $$

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Note that the constant $P_{a,q}$ defined by (1) is equal to $P(G,a)$ in the case that $G$ is the cyclic group $(\mathbb{Z}/q\mathbb{Z})^*$.

**Lemma 6.** Let $G$ be a nontrivial cyclic group, and let $a$ be an element of $G$ other than the identity. Then $P(G,a)|H(a)| = |G|$.

Thus, the definition of $H(a)$ given here is consistent with the definition given in the introduction.

**Proof.** We begin by factoring $|G|$ and $\text{ord}_G(a)$ as in (3) above. By the Chinese Remainder Theorem, we have

$$G \cong \mathbb{Z}/|G|\mathbb{Z} \cong \prod_{j=1}^{k} \mathbb{Z}/p_j^{\alpha_j}\mathbb{Z}.$$ 

Under this isomorphism, the element $a \in G$ can be identified with an ordered $k$-tuple:

$$(p_1^{\alpha_1-\beta_1}a_1, \ldots, p_k^{\alpha_k-\beta_k}a_k) \in \prod_{j=1}^{k} \mathbb{Z}/p_j^{\alpha_j}\mathbb{Z},$$

where each $a_j$ is an integer in the range $1 \leq a_j \leq p_j^{\beta_j}$, and $p_j \nmid a_j$. Replacing $a$ by one of its automorphic images $\tilde{a} \in G$, if necessary, we can assume without loss of generality that $a_j = 1$ for $j = 1, \ldots, k$. Indeed, if a subset $S \subset G$ is characteristic (that is, invariant under all automorphisms of $G$), then $S$ does not contain $a$ if and only if $S$ does contain any automorphic image of $a$. Since $H(a)$ is characteristic, it follows that $H(a) = H(\tilde{a})$ for every automorphic image $\tilde{a}$ of $a$.

Now let $K$ be a subgroup of $G$ that does not contain $a$, and suppose that $|K| = \prod_{j=1}^{k} p_j^{\gamma_j}$ for some nonnegative integers $\gamma_j$. Then,

$$K \cong \prod_{j=1}^{k} p_j^{\alpha_j-\gamma_j} \mathbb{Z}/p_j^{\alpha_j}\mathbb{Z} \subseteq \prod_{j=1}^{k} \mathbb{Z}/p_j^{\alpha_j}\mathbb{Z}.$$ 

The condition that $a \not\in K$ is equivalent to the existence of an index $j$ such that $\alpha_j - \gamma_j > \alpha_j - \beta_j$; that is, $\gamma_j < \beta_j$. In particular, $\beta_j > 0$, and therefore $1 \leq j \leq s$. If $K$ is maximal among the subgroups of $G$ which do not contain $a$, it must be the case that $\gamma_j = \beta_j - 1$ and $\gamma_i = \alpha_i$ for all $i \neq j$; consequently, $|K| = |G|/p_j^{\alpha_j-\beta_j+1}$. Finally, since $H(a)$ has the largest cardinality of all
such subgroups $K$, it is clear that $|H(a)| = |G|/p_j^{\alpha_j - \beta_j + 1}$, where $j$ is the only index for which

$$p_j^{\alpha_j - \beta_j + 1} = \min_{1 \leq i \leq s} \{p_i^{\alpha_i - \beta_i + 1}\} = \mathcal{P}(G, a),$$

and this completes the proof.

**Lemma 7.** Let $a$ be a fixed element of $G = (\mathbb{Z}/q\mathbb{Z})^*$ other than the identity, and suppose that $\mathcal{P}_{a,q}$ is the prime power $p^r$. Put $G_r = \mathbb{Z}/p^r\mathbb{Z}$. Then there exists a generator $g$ of the cyclic group $G$ such that the map $n \mapsto g^n$ defines a group isomorphism $\phi_g : G_r \rightarrow G/H(a)$ which maps the congruence class $p^{r-1} \pmod{p^r}$ to the coset $aH(a)$.

**Proof.** First, let $g$ be an arbitrary generator of $G$. Since every subgroup of $G$ is determined uniquely by its cardinality, it follows from Lemma 6 that $H(a)$ is the subgroup of $G$ generated by $g^{\mathcal{P}_{a,q}} = g^{p^r}$. Then, it is easy to see that the map $n \mapsto g^n$ defines a group isomorphism $\phi_g : G_r \rightarrow G/H(a)$.

Let $\psi_g : G \rightarrow G_r$ be the homomorphism defined via the composition:

$$G \rightarrow G/H(a) \xrightarrow{\phi_g^{-1}} G_r.$$

Since $a \notin H(a)$, the element $\overline{a} = \psi_g(a)$ is not the identity in $G_r$. On the other hand, $\overline{a}$ is contained in every subgroup $K$ of $G_r$, for otherwise the preimage $\psi_g^{-1}(K)$ would be a subgroup of $G$ which properly contains $H(a)$ and such that $a \notin \psi_g^{-1}(K)$, contradicting the maximality of $H(a)$. In particular, $\overline{a}$ lies in the subgroup $K$ generated in $G_r$ by the congruence class $p^{r-1} \pmod{p^r}$. Thus, $\psi_g(a) = bp^{r-1} \pmod{p^r}$ for some integer $b$ with $p \nmid b$. Replacing $g$ by the generator $g^b$, the result follows immediately. \[\square\]

The following technical lemma, used in the proof of Theorem 2 below, combines the preceding two lemmas with the combinatorial results of the previous section.

**Lemma 8.** Let $a$ be a fixed element of $G = (\mathbb{Z}/q\mathbb{Z})^*$ other than the identity. Write $\mathcal{P}_{a,q} = p^r$, and put $G_r = \mathbb{Z}/p^r\mathbb{Z}$. Let $g$ be a generator of $G$ with the property described in Lemma 4 and let $\psi_g : G \rightarrow G_r$ be the homomorphism defined in the proof of that lemma.

Suppose that $\mathcal{M} = \langle a_1, \ldots, a_k \rangle$ is a multiset in $G$ with the property:

$$\prod_{j \in S} a_j \neq a \quad \text{for every subset } S \subseteq \{1, 2, \ldots, k\}. \quad (9)$$
Let $\mathcal{S}$ be the multiset consisting of the elements $a_j \in \mathcal{M}$ that occur with multiplicity at least $q - 2$, and let $H$ be the subgroup of $G$ generated by the elements of $\mathcal{S}$. Finally, let $\mathcal{R}$ be the multiset consisting of those elements of $\mathcal{M}$ which do not lie in $H$. Then:

$(i)$ $|\mathcal{R}| \leq (q - 1)(q - 3)$;

$(ii)$ $|H| \leq |H(a)|$, and equality holds if and only if $H = H(a)$;

$(iii)$ If $H = H(a)$, then $|\mathcal{R}| \leq \mathcal{P}_{a,q} - 2$;

$(iv)$ Suppose that $H = H(a)$ and $|\mathcal{R}| = \mathcal{P}_{a,q} - 2$. Then $\psi_g(\mathcal{R})$ is a multiset $\mathcal{N}_{p,r}(\eta, c)$ of the type considered in Lemma 5.

**Proof.** The assertion $(i)$ is trivial, since $|G| = q - 1$ and every element of $\mathcal{R}$ occurs with multiplicity at most $q - 3$.

Let $b_1, \ldots, b_s$ be the distinct elements that occur in the multiset $\mathcal{S}$, and let $m_1, \ldots, m_s$ be the respective multiplicities. Since every element of $H$ can be expressed as a product $\prod_{i=1}^{s} b_i^{\nu_i}$, where $0 \leq \nu_i \leq q - 2 \leq m_i$ for $i = 1, \ldots, s$, it follows that every element of $H$ is a product of the form $\prod_{j \in S} a_j$ for some subset $S \subseteq \{1, \ldots, k\}$. Using (9), we see that $a \notin H$, hence $(ii)$ follows immediately from the definition of $H(a)$ and the fact that every subgroup of $G$ is determined uniquely by its cardinality.

From now on, we assume $H = H(a)$. Write $\mathcal{R} = \langle k_1, \ldots, k_t \rangle$, and observe that

$$\prod_{i \in T} k_i \notin aH(a) \quad \text{for every subset } T \subseteq \{1, 2, \ldots, t\}. \quad (10)$$

Indeed, assuming that $\prod_{i \in T} k_i = ah^{-1}$ for some $h \in H(a)$, the argument above shows that $h = \prod_{j \in S} a_j$ for some subset $S \subseteq \{1, \ldots, k\}$, and as $\mathcal{R} \subseteq \mathcal{M}$, it follows that $\prod_{i \in T} k_i = \prod_{j \in R} a_j$ for another subset $R \subseteq \{1, \ldots, k\}$. Clearly, $R \cap S = \emptyset$ since $\mathcal{R} \cap \mathcal{S} = \emptyset$; therefore, $\prod_{j \in R \cup S} a_j = a$, which contradicts (10).

Let $\overline{\mathcal{R}} = \langle \overline{k}_1, \ldots, \overline{k}_t \rangle$ be the image of $\mathcal{R}$ under the map $\psi_g$, that is $\mathcal{R} = \psi_g(\mathcal{R})$, and put $\overline{a} = \psi_g(a)$. Using (10), we deduce that

$$\prod_{i \in T} \overline{k}_i \neq \overline{a} \quad \text{for every subset } T \subseteq \{1, 2, \ldots, t\}.$$  

Therefore, Lemma 2 immediately implies that

$$|\mathcal{R}| = |\overline{\mathcal{R}}| = t \leq \kappa(G_r, \overline{a}) = p^r - 2 = \mathcal{P}_{a,q} - 2,$$  

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which proves \((iii)\). In the case that \(|\mathcal{R}| = \mathcal{P}_{a,q} - 2\), we can apply Lemmas 4 and 5 to conclude that \(\overline{\mathcal{R}} = \mathcal{N}_{p,r}(\eta, c)\) for a unique choice of \(\eta\) and \(c\), which proves \((iv)\).

\[\blacklozenge\]

### 2.3 Analytic results

For the proofs of Theorems 1 and 2, we need a variant of the classical result of Landau \[11\]:

\[
\left|\left\{ n \leq x : \Omega(n) = k \right\}\right| \sim \left|\left\{ n \leq x : \omega(n) = k \right\}\right| \sim \frac{x(\log_2 x)^{k-1}}{(k-1)! \log x}, \tag{11}
\]

where \(k \geq 1\) is a fixed integer, and \(\Omega(n)\) and \(\omega(n)\) denote the total number of prime factors of \(n\) counted with and without multiplicity, respectively. Specifically, we need an estimate for number of positive integers \(n \leq x\) with \(\Omega(n) = k\) and such that every prime factor of \(n\) lies in a prescribed subset of the residue classes modulo \(m\).

In this section the implied constants, frequently without explicit mention, may depend not only on \(m\) but on \(k\) and on various other parameters; virtually everything but \(x\) is fair game.

For given \(m\) let \(\mathcal{A}\) be a nonempty subset of \((\mathbb{Z}/m\mathbb{Z})^*\). Define

\[
\mathcal{Q}(m, \mathcal{A}) = \{ n \geq 1 : p \mid n \Rightarrow p \equiv a \pmod{m} \text{ for some } a \in \mathcal{A} \}.
\]

For each \(k\) define \(\mathcal{Q}_k(m, \mathcal{A})\) to be the set of positive integers \(n\) in \(\mathcal{Q}(m, \mathcal{A})\) for which \(\Omega(n) = k\).

**Lemma 9.** Let \(m\) and \(k\) be fixed positive integers and \(\mathcal{A}\) a nonempty subset of \((\mathbb{Z}/m\mathbb{Z})^*\). For real \(x \geq 1\), let \(\mathcal{Q}_k(x; m, \mathcal{A})\) be the number of positive integers \(n \leq x\) in the set \(\mathcal{Q}_k(m, \mathcal{A})\). Then,

\[
\mathcal{Q}_k(x; m, \mathcal{A}) = (1 + o(1)) \left( \frac{|\mathcal{A}|}{\varphi(m)} \right)^k \frac{x(\log_2 x)^{k-1}}{(k-1)! \log x}.
\]

**Proof.** For the proof we may follow an argument given in Section 9.4 in the book by Nathanson \[13\].

Let \(\mathcal{P}\) be the set of primes \(p\) such that \(p \equiv a \pmod{m}\) for some \(a \in \mathcal{A}\), let \(\mathcal{P}^k\) be the set of ordered \(k\)-tuples of primes in \(\mathcal{P}\), and for every positive integer \(n\), let

\[
r_k(n) = \left| \{(p_1, \ldots, p_k) \in \mathcal{P}^k : p_1 \cdots p_k = n \} \right|.
\]
For any real number \( x \geq 1 \), put
\[
f_k(x) = \sum_{n \leq x} r_k(n) = \sum_{p_1 \cdots p_k \leq x, (p_1, \ldots, p_k) \in P^k} 1,
\]
\[
g_k(x) = \sum_{n \leq x} \frac{r_k(n)}{n} = \sum_{p_1 \cdots p_k \leq x, (p_1, \ldots, p_k) \in P^k} \frac{1}{p_1 \cdots p_k},
\]
\[
h_k(x) = \sum_{n \leq x} r_k(n) \log n = \sum_{p_1 \cdots p_k \leq x, (p_1, \ldots, p_k) \in P^k} \log(p_1 \cdots p_k).
\]
Note that, for every \( k \geq 1 \), the relations
\[
g_{k+1}(x) = \sum_{p \leq x, p \in A} \frac{g_k(x/p)}{p}, \tag{12}
\]
and
\[
k \cdot h_{k+1}(x) = (k + 1) \sum_{p \leq x, p \in A} h_k(x/p) \tag{13}
\]
follow easily from the above definitions. Finally, let \( Q_k^b(m, A) \) denote the set of all squarefree elements of \( Q_k(m, A) \) and let \( Q_k^b(x; m, A) \) count the number of these up to \( x \). For these, we of course have \( \Omega(n) = \omega(n) = k \).

The following properties of \( r_k(n) \) are immediate:

- \( 0 \leq r_k(n) \leq k! \) for all \( n \geq 1 \);
- \( r_k(n) > 0 \iff n \in Q_k(m, A) \);
- \( r_k(n) = k! \iff n \in Q_k^b(m, A) \).

Consequently,
\[
f_k(x) = \sum_{n \leq x} r_k(n) \leq k! \sum_{n \leq x, r_k(n) > 0} 1 = k! \cdot Q_k(x; m, A), \tag{14}
\]
and
\[
f_k(x) = \sum_{n \leq x} r_k(n) \geq k! \sum_{n \leq x, r_k(n) = k!} 1 = k! \cdot Q_k^b(x; m, A). \tag{15}
\]

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If \( n \in Q_k(m, A) \setminus Q_k^\flat(m, A) \), then \( \omega(n) < \Omega(n) = k \), and therefore,

\[
Q_k(x; m, A) - Q_k^\flat(x; m, A) \leq \left| \{ n \leq x : \omega(n) < k \} \right| \ll \frac{x (\log_2 x)^{k-2}}{\log x},
\]

where we have used (11) in the last step. Hence, from (14) and (15) we deduce that

\[
Q_k(x; m, A) = \frac{f_k(x) + O}{k!} + O \left( \frac{x (\log_2 x)^{k-2}}{\log x} \right).
\]

To prove the theorem, it therefore suffices to establish the estimate:

\[
f_k(x) = (1 + o(1)) \frac{C k x (\log_2 x)^{k-1}}{\log x}, \tag{16}
\]

where, for brevity, we have put \( C = |A|/\varphi(m) \). As it is clear that \( f_k(x) = O(x) \), by partial summation we have

\[
h_k(x) = \sum_{n \leq x} r_k(n) \log n = f_k(x) \log x - \int_1^x \frac{f_k(t)}{t} \, dt = f_k(x) \log x + O(x),
\]

and thus (16) follows immediately from the estimate:

\[
h_k(x) = (1 + o(1)) C k x (\log_2 x)^{k-1}. \tag{17}
\]

Using the prime number theorem for arithmetic progressions we have

\[
h_1(x) = \sum_{p \leq x, \ p \in \mathcal{P}} \log p = (1 + o(1)) C x. \tag{18}
\]

In particular, this yields (17) in the special case \( k = 1 \). By the analogue for arithmetic progressions of the theorem of Mertens, or by partial summation from the previous formula, we also have

\[
g_1(x) = \sum_{p \leq x, \ p \in \mathcal{P}} \frac{1}{p} = (1 + o(1)) C \log_2 x.
\]

The latter estimate implies that

\[
g_1(x^{1/k}) = (1 + o(1)) C \log_2(x^{1/k}) = (1 + o(1)) C \log_2 x.
\]

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Thus, from the trivial inequalities
\[ g_1(x^{1/k})^k \leq g_k(x) \leq g_1(x)^k, \]
we see that
\[ g_k(x) = (1 + o(1)) C^k (\log x)^k. \] 
(19)

Now, for \( k \geq 1 \), define
\[ F_k(x) = h_k(x) - Ckxg_{k-1}(x), \]
where we have put \( g_0(x) = 1 \) for all \( x \geq 1 \). We claim that the bound
\[ F_k(x) = o(x(\log x)^{k-1}) \] 
(20)
holds for each fixed \( k \geq 1 \). Observe that this statement implies the desired result; indeed, if (20) holds for some integer \( k \geq 2 \), then in view of (19), we have
\[ h_k(x) = Ckxg_{k-1}(x) + F_k(x) = C^k kx(\log x)^{k-1} + o(x(\log x)^{k-1}), \]
which gives (17).

To prove (20), we use induction on the parameter \( k \). The initial case \( k = 1 \) follows immediately from (18) and the fact that \( g_0(x) = 1 \). Now suppose that (20) holds for some integer \( k \geq 1 \). By relations (12) and (13), we have
\[
k F_{k+1}(x) = k h_{k+1}(x) - Ck(k+1) x g_k(x) \\
= (k + 1) \sum_{p \leq x, \ p \in A} h_k(x/p) + Ck + 1 x \sum_{p \leq x, \ p \in A} g_{k-1}(x/p) \\
= (k + 1) \sum_{p \leq x, \ p \in A} \left( h_k(x/p) + Ck(x/p) g_{k-1}(x/p) \right) \\
= (k + 1) \sum_{p \leq x, \ p \in A} F_k(x/p). \]  
(21)

For fixed \( \varepsilon > 0 \), there is a constant \( x_0 = x_0(\varepsilon) \) such that
\[ F_k(x/p) \leq \frac{\varepsilon x(\log (x/p))^{k-1}}{p} \leq \frac{\varepsilon x(\log x)^{k-1}}{p}, \]
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whenever \( x/p \geq x_0 \). On the other hand, \( F_k(x/p) = O_\varepsilon(1) \) if \( x/p < x_0 \). Consequently,

\[
\sum_{p \leq x, \ p \in A} F_k(x/p) = \sum_{p \leq x/x_0, \ p \in A} F_k(x/p) + \sum_{x/x_0 < p \leq x, \ p \in A} F_k(x/p)
\]

\[
\leq \varepsilon x(\log_2 x)^{k-1} \sum_{p \leq x/x_0} \frac{1}{p} + O_\varepsilon \left( \sum_{x/x_0 < p \leq x} \frac{1}{p} \right)
\]

\[
= (\varepsilon + o(1))x(\log_2 x)^k + O_\varepsilon(x/\log x).
\]

Combining this estimate with (21), it follows that for every \( \varepsilon > 0 \), there is a constant \( x_1 = x_1(\varepsilon) \) such that the inequality

\[
F_{k+1}(x) \leq 2\varepsilon(1 + 1/k)x(\log_2 x)^k
\]

holds whenever \( x \geq x_1 \); it follows that (20) holds with \( k \) replaced by \( k + 1 \). This completes the induction and finishes the proof of the lemma.

We need to count, in addition to the integers in \( Q_k(m, A) \), the same integers without the restriction on \( \Omega(n) \), that is those in the set \( Q(m, A) \). For this we shall use the following result of Wirsing [16]:

**Lemma 10.** Suppose that \( f \) is a fixed real-valued multiplicative function with the following properties:

(i) For every natural number \( n \), \( f(n) \geq 0 \);

(ii) For some constants \( c_1, c_2 \) with \( c_2 < 2 \), the bound \( f(p^\nu) \leq c_1 c_2^\nu \) holds for all primes \( p \) and integers \( \nu \geq 2 \);

(iii) There exists a constant \( \tau > 0 \) such that

\[
\sum_{p \leq x} f(p) = (\tau + o(1)) \frac{x}{\log x}.
\]

Then,

\[
\sum_{n \leq x} f(n) = \left( \frac{1}{e^{\tau} \Gamma(\tau)} + o(1) \right) \frac{x}{\log x} \prod_{p \leq x} \sum_{\nu = 0}^{\infty} \frac{f(p^\nu)}{p^\nu},
\]

where \( \gamma \) is the Euler-Mascheroni constant and \( \Gamma(s) \) is the gamma function.
The classical result of Mertens that
\[ \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) = e^{-\gamma} (\log x)^{-1} + O\left( (\log x)^{-2} \right), \]
has been generalized in the paper of Williams [15] (see also [14]), which gives a similar estimate when the product above is restricted to primes lying in a fixed arithmetic progression. To state this result we first recall some notation from [15]. Let \( m \) be a positive integer and let \( \chi \) be a non-principal Dirichlet character modulo \( m \). Let \( L(s, \chi) \) be the corresponding \( L \)-function and define the Dirichlet series
\[ K(s, \chi) = \sum_{n=1}^{\infty} \frac{k_{\chi}(n)}{n^s} = \prod_{p} \left( 1 - \frac{k_{\chi}(p)}{p^s} \right)^{-1}, \]
where \( k_{\chi}(n) \) is the completely multiplicative function whose value at the prime \( p \) is given by
\[ k_{\chi}(p) = p \left( 1 - \left( 1 - \frac{\chi(p)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-\chi(p)} \right). \]

The main result of [15] is the following:

**Lemma 11.** Let \( a \) and \( m \geq 1 \) be coprime integers. Then,
\[ \prod_{\substack{p \leq x \quad (\text{mod} \ m) \quad p \equiv a}} \left( 1 - \frac{1}{p} \right) = \frac{\varpi(a, m)}{(\log x)^{1/\varphi(m)}} + O\left( \frac{1}{(\log x)^{1+1/\varphi(m)}} \right), \quad (22) \]

where
\[ \varpi(a, m) = e^{-\gamma} \frac{m}{\varphi(m)} \prod_{\chi \neq \chi_0} \left( \frac{K(1, \chi)}{L(1, \chi)} \right)^{\chi(a)} \left( \frac{1}{\varphi(m)} \right)^{1/\varphi(m)}. \quad (23) \]

We are now ready to count the integers in \( Q(m, A) \). Recall that these are just the integers all of whose prime factors lie in the set \( A \). For real \( x \geq 1 \) let \( Q(x; m, A) \) denote the number of such integers \( n \leq x \).

**Lemma 12.** Let \( m \) be a fixed positive integer and \( A \) a nonempty subset of \((\mathbb{Z}/m\mathbb{Z})^*\). Then,
\[ Q(x; m, A) = (1 + o(1)) \vartheta(m, A) \frac{x}{(\log x)^{1-|A|/\varphi(m)}}, \]

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where
\[
\vartheta(m, \mathcal{A}) = e^{-\gamma |\mathcal{A}|/\varphi(m)} \frac{1}{\Gamma(|\mathcal{A}|/\varphi(m))} \prod_{a \in \mathcal{A}} \varpi(a, m)^{-1},
\] (24)

with the constants \(\varpi(a, m)\) defined as in Lemma 11.

**Proof.** This follows immediately by applying Lemma 10 to the multiplicative function \(f\) which is defined on prime powers by
\[
f(p^\nu) = \begin{cases} 1 & \text{if } p \equiv a \pmod{m} \text{ for some } a \in \mathcal{A}; \\ 0 & \text{otherwise}; \end{cases}
\]
making use of the estimates of Lemma 9 (with \(k = 1\)) and of Lemma 11.

The next lemma evaluates \(\vartheta(m, \mathcal{A})\) in the special case \(m = q, \mathcal{A} = H(a)\).

**Lemma 13.** We have
\[
\vartheta(q, H(a)) = e^{-\gamma/\mathcal{P}_{a,q}} \frac{(1 - q^{-1})^{1/\mathcal{P}_{a,q}}}{\Gamma(1/\mathcal{P}_{a,q})} \mathcal{V}_{a,q},
\]
where \(\mathcal{P}_{a,q}\) and \(\mathcal{V}_{a,q}\) are given by (23) and (24), respectively.

**Proof.** By the definitions (23) and (24), we have
\[
\vartheta(q, H(a)) = e^{-\gamma/\mathcal{P}_{a,q}} \frac{1}{\Gamma(1/\mathcal{P}_{a,q})} \prod_{h \in H(a)} \varpi(h, q)^{-1},
\]
where
\[
\varpi(h, q) = \left( e^{-\gamma - \frac{q}{q-1}} \prod_{\chi \neq \chi_0} \frac{K(1, \chi)}{L(1, \chi)} \right)^{1/(q-1)}.
\]
From the orthogonality relation
\[
\sum_{h \in H(a)} \overline{\chi(h)} = \begin{cases} |H(a)| & \text{if } \chi|_{H(a)} = 1, \\ 0 & \text{otherwise}, \end{cases}
\]
it follows that
\[
\vartheta(q, H(a)) = \frac{(1 - q^{-1})^{1/\mathcal{P}_{a,q}}}{\Gamma(1/\mathcal{P}_{a,q})} \prod_{\chi \neq \chi_0, \chi|H(a) = 1} \left( \frac{K(1, \chi)}{L(1, \chi)} \right)^{-1/\mathcal{P}_{a,q}}.
\]
By (3.2) of [15], we have
\[ K(1, \chi) = \frac{L(1, \chi)}{L(1, \chi)} = \lim_{y \to \infty} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right)^{\chi(p)} \text{.} \]

Therefore, in view of the relation
\[ \sum_{\chi \neq \chi_0} \chi(p) = \begin{cases} \mathcal{P}_{a,q} - 1 & \text{if } p \pmod{q} \in H(a), \\ -1 & \text{otherwise,} \end{cases} \]

and the Mertens’ formula
\[ \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) = \left( 1 + o(1) \right) \frac{e^{-\gamma}}{\log y}, \]

we derive that
\[ \vartheta(q, H(a)) = \frac{(1 - q^{-1})^{1/\mathcal{P}_{a,q}}}{\Gamma(1/\mathcal{P}_{a,q})} \left( \lim_{y \to \infty} \prod_{p \leq y} \prod_{\chi \neq \chi_0} \chi_{H(a)} = 1 \left( 1 - \frac{1}{p} \right)^{\chi(p)} \right) -1/\mathcal{P}_{a,q} \]
\[ = \frac{(1 - q^{-1})^{1/\mathcal{P}_{a,q}}}{\Gamma(1/\mathcal{P}_{a,q})} \lim_{y \to \infty} \left( \prod_{p \leq y} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right)^{\mathcal{P}_{a,q}} \right) -1/\mathcal{P}_{a,q} \]
\[ = \frac{e^{-\gamma/\mathcal{P}_{a,q}} (1 - q^{-1})^{1/\mathcal{P}_{a,q}}}{\Gamma(1/\mathcal{P}_{a,q})} \lim_{y \to \infty} \left( (\log y)^{-1/\mathcal{P}_{a,q}} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right)^{-1} \right) \text{.} \]

Inserting the definition (2), we finish the proof. \( \square \)

**Lemma 14.** Let \( a \) be a nonnegative integer, \( b \) a real number in the half-open interval \((0, 1]\), \( c \) a nonnegative real number, and \( K \) a positive real number. Let \( S \) be a set of positive integers, and for \( x \geq 1 \) put
\[ S(x) = \left| \{ n \leq x : n \in S \} \right| \text{.} \]

Finally, suppose that the following estimate holds as \( x \to \infty \):
\[ S(x) = (1 + o(1)) \frac{K x (\log x)^a}{(\log x)^b} \text{.} \]
Then,
\[
\sum_{h \in S \atop h \leq x^{1/2}} \frac{1}{h (\log(x/h))^c} = \begin{cases} 
O\left( \frac{(\log_2 x)^a}{(\log x)^{b+c-1}} \right) & \text{if } b \in (0, 1); \\
(1 + o(1)) \frac{K}{a+1} \frac{(\log_2 x)^{a+1}}{(\log x)^c} & \text{if } b = 1.
\end{cases}
\]

Proof. Since
\[
\sum_{h \leq \log_2 x} \frac{1}{h (\log(x/h))^c} = (1 + o(1)) \frac{\log_3 x}{(\log x)^c},
\]
we have
\[
\sum_{h \in S \atop h \leq x^{1/2}} \frac{1}{h (\log(x/h))^c} = \sum_{h \in S \atop \log_2 x < h \leq x^{1/2}} \frac{1}{h (\log(x/h))^c} + O\left( \frac{\log_3 x}{(\log x)^c} \right).
\]
Since the estimate
\[
S(t) = (K + o(1)) \frac{t(\log_2 t)^a}{(\log t)^b}
\]
holds uniformly for all \( t \geq \log_2 x \), by partial summation we deduce that
\[
\sum_{h \in S \atop \log_2 x < h \leq x^{1/2}} \frac{1}{h (\log(x/h))^c} = \int_{\log_2 x}^{x^{1/2}} \frac{dS(t)}{t (\log(x/t))^c} = (K + o(1)) (J_1 + J_2 - J_3),
\]
where
\[
J_1 = \left[ \frac{(\log_2 t)^a}{(\log t)^b(\log(x/t))^c} \right]_{\log_2 x}^{x^{1/2}},
\]
\[
J_2 = \int_{\log_2 x}^{x^{1/2}} \frac{(\log_2 t)^a}{(\log t)^b(\log(x/t))^c} \frac{dt}{t},
\]
\[
J_3 = c \int_{\log_2 x}^{x^{1/2}} \frac{(\log_2 t)^a}{t (\log t)^b(\log(x/t))^{c+1}} \frac{dt}{t}.
\]
Clearly,
\[
J_1 \ll \frac{(\log_4 x)^a}{(\log_3 x)^b(\log x)^c} \quad \text{and} \quad J_3 \ll \frac{J_2}{\log x}.
\]
Making the change of variables $t = x^s$ in the integral $J_2$, it follows that

$$J_2 = \frac{1}{(\log x)^{b+c-1}} \int_{u_0}^{1/2} \frac{(\log x + \log s)^a}{s^b (1 - s)^c} \, ds,$$

where $u_0 = (\log_3 x)/\log x$. To complete the proof, it suffices to show that

$$\int_{u_0}^{1/2} \frac{(\log x + \log s)^a}{s^b (1 - s)^c} \, ds = \begin{cases} O((\log_2 x)^a) & \text{if } b \in (0, 1); \\ (1 + o(1)) \frac{(\log x)^{a+1}}{a+1} & \text{if } b = 1. \end{cases}$$

First, we discuss the integral over $u_0 \leq s \leq u_1$, where $u_1 = (\log_3 x)^{-1}$. If $b \in (0, 1)$, then

$$\int_{u_0}^{u_1} \frac{(\log x + \log s)^a}{s^b (1 - s)^c} \, ds = (1 + o(1)) \int_{u_0}^{u_1} \frac{(\log x + \log s)^a}{s^b} \, ds$$

$$= (1 + o(1)) \left[ \frac{s^{1-b} \sum_{j=0}^{a} \frac{a!}{(b-1)^j (a-j)!} (\log x + \log s)^{a-j}}{1-b} \right]_{u_0}^{u_1}$$

$$= (1 + o(1)) \frac{(\log x)^a}{(1-b)(\log_3 x)^{1-b}},$$

and for $b = 1$, we have

$$\int_{u_0}^{u_1} \frac{(\log x + \log s)^a}{s^b (1 - s)^c} \, ds = (1 + o(1)) \int_{u_0}^{u_1} \frac{(\log x + \log s)^a}{s} \, ds$$

$$= (1 + o(1)) \left[ \frac{(\log x + \log s)^{a+1}}{a+1} \right]_{u_0}^{u_1} = (1 + o(1)) \frac{(\log x)^{a+1}}{a+1}.$$

Next, we consider the integral over $u_1 \leq s \leq 1/2$. If $b \in (0, 1)$, then

$$\int_{u_1}^{1/2} \frac{(\log x + \log s)^a}{s^b (1 - s)^c} \, ds = (1 + o(1))(\log x)^a \int_{u_1}^{1/2} \frac{ds}{s^b (1 - s)^c}$$

$$= (1 + o(1))(\log x)^a,$$

and for $b = 1$, we have

$$\int_{u_1}^{1/2} \frac{(\log x + \log s)^a}{s^b (1 - s)^c} \, ds = (1 + o(1))(\log x)^a \int_{u_1}^{1/2} \frac{ds}{s (1 - s)^c}$$

$$= (1 + o(1))(\log x)^a \log_3 x.$$

Combining the preceding results, we obtain the stated estimates. \qed
Lemma 15. For $j = 1, 2$, let $S_j$ be a set of positive integers and, for $x \geq 1$, put

$$S_j(x) = |\{n \leq x : n \in S_j\}|.$$  

Suppose that, for $j = 1, 2$,

$$S_j(x) = (1 + o(1)) \frac{K_j x(\log_2 x)^{a_j}}{(\log x)^{b_j}}$$

where $a_1, a_2$ are nonnegative integers, $K_1, K_2 > 0$, and $0 < b_1 < 1$, $b_2 = 1$. Let $S(x)$ be the number of ordered pairs $(h_1, h_2) \in S_1 \times S_2$ such that $h_1 h_2 \leq x$. Then the following estimate holds:

$$S(x) = (1 + o(1)) \frac{K_1 K_2}{a_2 + 1} x(\log_2 x)^{a_1 + a_2 + 1}$$

$(\log x)^{b_1}$.

Proof. Observe that

$$S(x) = \sum_{h_1 \in S_1 \atop h_1 \leq x^{1/2}} S_2(x/h_1) + \sum_{h_2 \in S_2 \atop h_2 \leq x^{1/2}} S_1(x/h_2) - S_1(x^{1/2}) S_2(x^{1/2}).$$

Uniformly for $h_1 \leq x^{1/2}$, we have

$$S_2(x/h_1) = (1 + o(1)) \frac{K_2 x(\log_2(x/h_1))^{a_2}}{h_1 \log(x/h_1)};$$

thus Lemma 14 implies that

$$\sum_{h_1 \in S_1 \atop h_1 \leq x^{1/2}} S_2(x/h_1) \ll x^{a_2} \sum_{h_1 \in S_1(x^{1/2})} \frac{1}{h_1 \log(x/h_1)} \ll \frac{x(\log_2 x)^{a_1 + a_2}}{(\log x)^{b_1}}.$$  

Similarly,

$$\sum_{h_2 \in S_2 \atop h_2 \leq x^{1/2}} S_1(x/h_2) = (1 + o(1)) K_1 x(\log_2 x)^{a_1} \sum_{h_2 \in S_2(x^{1/2})} \frac{1}{h_2 (\log(x/h_2))^{b_1}}$$

$$= (1 + o(1)) \frac{K_1 K_2}{a_2 + 1} x(\log_2 x)^{a_1 + a_2 + 1}$$

$(\log x)^{b_1}$,

where we have again used Lemma 14. Since

$$S_1(x^{1/2}) S_2(x^{1/2}) \ll \frac{x(\log_2 x)^{a_1 + a_2}}{(\log x)^{b_1+1}},$$

the result follows. $\square$
3  Proofs of the theorems

In this section we frequently use the notation \( S(x) \) for the number of positive integers \( n \leq x \) in the set \( S \).

3.1 Proof of Theorem 1

Fix the prime \( q \geq 3 \), and write \( N \) and \( N(x) \) respectively for \( N_1(q, 1) \), and \( N(x; q, 1) \). Let \( N^* \) be the set of integers \( n \in N \) that are not divisible by \( q \). Then \( N^* \) can be expressed as a disjoint union \( N_1 \cup N_2 \), where \( N_1 \) is the set of integers \( n \in N^* \) with \( \Omega(n) \leq q - 3 \), and \( N_2 = N \setminus N_1 \).

Since \( N_1 \) is contained in the set of all integers with \( \Omega(n) \leq q - 3 \), it follows from (11) that the number of such integers \( n \leq x \) satisfies

\[
N_1(x) \ll \frac{x \log x}{\log x}. 
\]  

(25)

Next, let \( n \in N_2 \), and factor \( n = p_1 p_2 \ldots p_k \), where \( p_1 \leq p_2 \leq \cdots \leq p_k \) are primes, none of which is equal to \( q \); note that \( k \geq q - 2 \). Let \( a_j \) denote the residue class of \( p_j \) modulo \( q \) for \( j = 1, \ldots, k \). For any nonempty subset \( S \subseteq \{1, \ldots, k\} \), \( \prod_{j \in S} a_j \) is the residue class of the divisor \( d_S = \prod_{j \in S} p_j \) of \( n \). Since \( d_S \equiv 1 \pmod{q} \), it follows that \( k \leq \kappa(G) \), where \( G \) is the abelian group \( (\mathbb{Z}/q\mathbb{Z})^* \cong \mathbb{Z}/(q-1)\mathbb{Z} \). Hence, by Lemma 11 we have \( k \leq q - 2 \). Since \( k \geq q - 2 \) for each \( n \in N_2 \), it follows that \( k = q - 2 \), and Lemma 11 further shows that \( a_1 \equiv \cdots \equiv a_k \equiv a \pmod{q} \) for some primitive root \( a \) modulo \( q \). Therefore, denoting by \( U(q) \) the set of primitive roots modulo \( q \), we have

\[
N_2(x) = \sum_{a \in U(q)} Q_{q-2}(x; q, \{a\}).
\]

Since \( |U(q)| = \varphi(q - 1) \), from Lemma 9 we deduce that

\[
N_2(x) = (1 + o(1)) \frac{\varphi(q - 1) x \log x}{(q - 1)^{q-2} (q - 3)! \log x}.
\]  

(26)

Combining the estimates (25) and (26), we have

\[
N^*(x) = (1 + o(1)) \frac{\varphi(q - 1) x \log x}{(q - 1)^{q-2} (q - 3)! \log x}.
\]  

(27)
In view of the obvious relation
\[ N(x) = \sum_{\nu \geq 0} N^*(x/q^\nu), \]
we see that
\[ N(x) = (1 + o(1))(1 - q^{-1})^{-1} N^*(x), \]
which, together with (27) yields the stated estimate of Theorem 1.

3.2 Proof of Theorem 2

Fix the prime \( q \geq 3 \) and the integer \( 2 \leq a < q \), write \( N \) for \( N(q, a) \), and let \( N^* \) be the set of integers \( n \in N \) that are not divisible by \( q \).

Throughout the proof, we fix a generator \( g \) of the group \( G_r = \mathbb{Z}/p^r\mathbb{Z} \) with the property stated in Lemma 7. Here, \( p^r = \mathcal{P}_{a,q} \) as usual. We also denote by \( \phi_g : G_r \to G/H(a) \) and \( \psi_g : G \to G_r \) the maps defined in the statement and proof of Lemma 7. Here, \( G = (\mathbb{Z}/q\mathbb{Z})^* \) as before.

For each \( n \in N^* \), let \( n = p_1 \cdots p_k \) be a factorization of \( n \) as a product of primes, where \( k = \Omega(n) \), and let \( \mathcal{M}_n = \langle a_1, \ldots, a_k \rangle \) be the multiset in \( G \) whose elements are the congruence classes \( p_j \pmod{q} \) for \( j = 1, \ldots, k \). As in the statement of Lemma 8, we associate to \( \mathcal{M}_n \) a subgroup \( H_n \) of \( G \) and a multiset \( \mathcal{N}_n \subseteq \mathcal{M}_n \).

For every subgroup \( H \) of \( G \) with \( a \notin H(a) \), and every multiset \( \mathcal{N} \) in \( G \), let \( N_{H,\mathcal{N}} \) denote the set of integers \( n \in N^* \), \( n \leq x \) such that \( H_n = H \) and \( \mathcal{N}_n = \mathcal{N} \). Our goal is to estimate the number \( N_{H,\mathcal{N}}(x) \) of these, for every pair \((H, \mathcal{N})\).

First, suppose that \( H \neq H(a) \), and let \( H \) and \( \mathcal{N} \) be fixed. Put \( y = \exp((\log x \log_3 x)/\log_2 x) \), and let
\[ N_1 = \{ n \in N_{H,\mathcal{N}} : P(n) \leq y \}, \]
where \( P(n) \) denotes the largest prime factor of \( n \). Using a well–known result on smooth numbers; i.e., positive integers \( n \) whose largest prime factor is small with respect to \( n \) (see for example [2] or [10]), we have
\[ N_1(x) \leq x \exp(-(1 + o(1))u \log u) \]
\[ = \frac{x}{(\log x)^{1+o(1)}} = O\left( \frac{x}{(\log x)^{1-1/\mathcal{P}_{a,q}}} \right), \quad (28) \]
where \( u = (\log x) / \log y = (\log_2 x) / \log_3 x \).

Now let \( \mathcal{N}_2 = \mathcal{N}_{H,R} \setminus \mathcal{N}_1 \). For every integer \( n \) in \( \mathcal{N}_2 \), let \( n = p_1 \cdots p_k \) be a factorization of \( n \) such that \( p_k = P(n) \), and put \( m = p_1 \cdots p_{k-1} \). For any fixed value of \( m \) obtained in this way, \( p_k \) is a prime that satisfies the inequalities
\[
\frac{x}{m} \geq p_k > y = \exp \left( \frac{\log x \log_3 x}{\log_2 x} \right);
\]
therefore, the number of possibilities for \( p_k \) is at most
\[
\pi(x/m) = \frac{x}{m \log(x/m)} \leq \frac{x \log_2 x}{m \log x \log_3 x}.
\]

Note that \( m = h_0 k_0 \), with
\[
h_0 = \prod_{a_j \in H} p_j \quad \text{and} \quad k_0 = \prod_{a_j \in K} p_j,
\]
where each element \( a_j \in G \) corresponds to the congruence class \( p_j \) (mod \( q \)) as before. Then \( h_0 \in \mathcal{Q}(q, H) \) in the notation of Lemma 12, and we have \( \Omega(k_0) \leq |\mathcal{R}| \leq L = (q-1)(q-3) \) by Lemma 8(i). Thus, summing over the possible choices of \( h_0 \) and \( k_0 \), we see that
\[
\mathcal{N}_2(x) \ll \frac{x \log_2 x}{\log x \log_3 x} \left( \sum_{h_0 \in \mathcal{Q}(q, H)} \frac{1}{h_0^{\Omega(k_0)}} \right) \left( \sum_{k_0 \leq x \Omega(k_0) \leq L} \frac{1}{k_0} \right).
\]
(29)

Using Lemma 12 and partial summation, we derive the bound
\[
\sum_{h_0 \in \mathcal{Q}(q, H), h_0 \leq x} \frac{1}{h_0} \ll (\log x)^{|H|/(q-1)}.
\]
(30)

On the other hand, we have
\[
\sum_{k_0 \leq x \Omega(k_0) \leq L} \frac{1}{k_0} \ll \sum_{j \leq L} \frac{1}{j!} \left( \sum_{p \leq x} \frac{1}{p^\nu} \right)^j \ll \sum_{j \leq L} \frac{1}{j!} (\log_2 x + O(1))^j \ll (\log_2 x)^L.
\]
(31)
Inserting the estimates (30) and (31) into (29), it follows that

\[ N_2(x) \ll \frac{x \log x}{(\log x)^{L+1} x^{1-|H|/(q-1)} \log x}. \]  

Finally, by Lemma 8 (ii), we have \(|H| < |H(a)|\) since \(H \neq H(a)\) (and the group \(G = (\mathbb{Z}/q\mathbb{Z})^*\) is cyclic). As \(|H(a)|/(q-1) = 1/\mathcal{P}_{a,q}\) by Lemma 3 the estimates (28) and (32) together imply that

\[ N_{H,\mathcal{R}}(x) = o\left(\frac{x}{(\log x)^{1-1/\mathcal{P}_{a,q}}}\right). \]  

Recall that the number of such pairs \((H, \mathcal{R})\) is bounded in terms of \(q\) so the above estimate is sufficient to easily absorb this case into the error term.

It remains to consider the pairs with \(H = H(a)\) and we turn our attention to the problem of estimating \(N_{H(a),\mathcal{R}}(x)\) for a fixed multiset \(\mathcal{R}\). In the case that \(\mathcal{R} = \emptyset\), it is easy to see that

\[ N_{H(a),\emptyset}(x) = Q(x; q, H(a)). \]

Hence, by Lemma 12 we have

\[ N_{H(a),\emptyset}(x) = (1 + o(1)) \vartheta(q, H(a)) \frac{x}{(\log x)^{1-1/\mathcal{P}_{a,q}}}. \]  

From now on, we assume that \(\mathcal{R} \neq \emptyset\). We recall that the inequality \(|\mathcal{R}| \leq \mathcal{P}_{a,q} - 2\) holds by Lemma 8 (iii); in particular, \(\mathcal{P}_{a,q} \geq 3\) if \(\mathcal{R} \neq \emptyset\).

First, suppose that \(|\mathcal{R}| < \mathcal{P}_{a,q} - 2\); note that this is possible only if \(\mathcal{P}_{a,q} \geq 4\). For each \(n \in N_{H(a),\mathcal{R}}\), write \(n = h_0 k_0\), where

\[ h_0 = \prod_{j=1}^{k} p_{a_j}^{j} \quad \text{and} \quad k_0 = \prod_{j=1}^{k} p_j^{a_j}. \]

Then \(h_0 \in S_1\) and \(k_0 \in S_2\), where

\[ S_1 = Q(q, H(a)) \quad \text{and} \quad S_2 = \{n : \Omega(n) \leq \mathcal{P}_{a,q} - 3\}, \]

and therefore,

\[ N_{H(a),\mathcal{R}}(x) \leq |\{(h_0, k_0) \in S_1 \times S_2 : h_0k_0 \leq x\}|. \]
Applying Lemma 15 and making use of the estimates provided by Lemma 12 and (11), we obtain the bound

\[ N_{H(a),\mathfrak{R}}(x) \ll \frac{x(\log_2 x)^{p_{a,q}-3}}{(\log x)^{1-1/p_{a,q}}}, \]  

which again is of smaller order of magnitude than the main term claimed by the theorem.

Now let \( \mathfrak{R} \) be a multiset with cardinality \(|\mathfrak{R}| = p_{a,q} - 2\). According to Lemma 8 (iv), \( \psi_g(\mathfrak{R}) \) is a multiset \( \mathfrak{N}_{p,r}(\eta, c) \) of the type considered in Lemma 5; in other words, \( \mathfrak{R} \equiv \phi_g(\mathfrak{N}_{p,r}(\eta, c)) \pmod{H(a)} \), or

\[ \mathfrak{R} = \langle g^c h_1, g^c h_2, \ldots, g^c h_{\eta+1}, g^{p_{a,q}-c} h_{\eta+2}, \ldots, g^{p_{a,q}-c} h_{p_{a,q}-2} \rangle \]

for some sequence \( h_1, \ldots, h_{p_{a,q}-2} \) in \( H(a) \).

For a fixed pair \((\eta, c)\), let \( \mathcal{N}_{\eta,c} \) be the disjoint union

\[ \mathcal{N}_{\eta,c} = \bigcup_{\psi_g(\mathfrak{R}) = \mathfrak{N}_{p,r}(\eta, c)} \mathcal{N}_{H(a),\mathfrak{R}}, \]

and define the following subsets of \( G \):

\[ G^+ = \{ g^c h : h \in H(a) \} \quad \text{and} \quad G^- = \{ g^{p_{a,q}-c} h : h \in H(a) \}. \]

For each \( n \in \mathcal{N}_{\eta,c} \), we can factor \( n = h_0 k_0 l_0 \), where

\[ h_0 = \prod_{j=1}^{k} p_{a_j H(a)}, \quad k_0 = \prod_{j=1}^{k} p_{a_j G^+}, \quad \text{and} \quad l_0 = \prod_{j=1}^{k} p_{a_j G^-}. \]

Then \( h_0 \in \mathcal{S}_1 \), \( k_0 \in \mathcal{S}_2 \), and \( l_0 \in \mathcal{S}_3 \), where

\[ \mathcal{S}_1 = \mathcal{Q}(q, H(a)), \]
\[ \mathcal{S}_2 = \mathcal{Q}_0(q, G^+), \]
\[ \mathcal{S}_3 = \mathcal{Q}_\xi(q, G^-), \]

with \( \xi = p_{a,q} - 2 - \eta \). Conversely, if \( h_0 \in \mathcal{S}_1 \), \( k_0 \in \mathcal{S}_2 \), and \( l_0 \in \mathcal{S}_3 \), and \( n = h_0 k_0 l_0 \leq x \), then \( n \in \mathcal{N}_{\eta,c} \). Let us also define

\[ \mathcal{V} = \{ n : n = h_0 l_0 \text{ for some } h_0 \in \mathcal{S}_1 \text{ and } l_0 \in \mathcal{S}_3 \} \]
and

\[ \mathcal{W} = \{ n : n = h_0 k_0 l_0 \text{ for some } h_0 \in S_1, k_0 \in S_2, \text{ and } l_0 \in S_3 \}. \]

Then, since the sets \( H(a), G^+ \) and \( G^- \) are pairwise disjoint, it is easy to see that the natural map \( S_1 \times S_3 \to \mathcal{V} \) given by \( (h_0, l_0) \mapsto h_0 l_0 \) is a bijection. Similarly, the natural map \( \mathcal{V} \times S_2 \to \mathcal{W} \) given by \( (h_0 l_0, k_0) \mapsto h_0 k_0 l_0 \) is also a bijection. To estimate \( N_{\eta,c}(x) \), we apply Lemma \ref{lemma15} twice: first to the pair of sets \( S_1 \) and \( S_3 \), then to the pair of sets \( \mathcal{V} \) and \( S_2 \).

By Lemma \ref{lemma12}, we have

\[ S_1(x) = Q(x; q, H(a)) = (1 + o(1)) \frac{\vartheta(q, H(a))}{(\log x)^{1 - 1/\mathfrak{P}_{a,q}}} \]

and by Lemma \ref{lemma9} we have

\[ S_3(x) = Q_{\xi}(x; q, G^-) = (1 + o(1)) \frac{1}{\mathfrak{P}_{a,q}^\xi (\xi - 1)!} \frac{x (\log_2 x)^{\xi - 1}}{\log x}, \]

where we have used the fact that \( |G^-| = |H(a)| \). Applying Lemma \ref{lemma15} to the pair of sets \( S_1 \) and \( S_3 \), and taking into account the bijection \( S_1 \times S_3 \to \mathcal{V} \) mentioned above, we get

\[ \mathcal{V}(x) = \left| \{(h_0, l_0) \in S_1 \times S_3 : h_0 l_0 \leq x \} \right| \]

\[ = (1 + o(1)) \frac{\vartheta(q, H(a))}{\mathfrak{P}_{a,q}^\xi (\xi - 1)!} \frac{x (\log_2 x)^\xi}{(\log x)^{1 - 1/\mathfrak{P}_{a,q}}}. \]

To complete the estimate of \( N_{\eta,c}(x) \), we must now consider separately the cases \( \eta = 0 \) and \( \eta \neq 0 \). Suppose first that \( \eta = 0 \) and \( \xi = \mathfrak{P}_{a,q} - 2 \) (which can occur only if \( \mathfrak{P}_{a,q} \) is an odd prime; see Lemma \ref{lemma5}). In this case, \( G^+ = \emptyset \), \( S_2 = \{1\} \), and \( \mathcal{W} = \mathcal{V} \); consequently,

\[ N_{\eta,c}(x) = \mathcal{W}(x) = (1 + o(1)) \frac{\vartheta(q, H(a))}{\mathfrak{P}_{a,q}^\xi (\xi - 2)! (\xi - 1)!} \frac{x (\log_2 x)^{\xi}}{(\log x)^{1 - 1/\mathfrak{P}_{a,q}}}. \]

Next, suppose that \( \eta \neq 0 \). By Lemma \ref{lemma9} we have

\[ S_2(x) = Q_{\eta}(x; q, G^+) = (1 + o(1)) \frac{1}{\mathfrak{P}_{a,q}^\eta (\eta - 1)!} \frac{x (\log_2 x)^{\eta - 1}}{\log x}. \]
Applying Lemma 15 to the pair of sets $\mathcal{V}$ and $\mathcal{S}_2$, and recalling the bijection $\mathcal{V} \times \mathcal{S}_2 \to \mathcal{W}$ described earlier, one has

$$N_{\eta,c}(x) = \mathcal{W}(x) = |\{(h_0 l_0, k_0) \in \mathcal{V} \times \mathcal{S}_2 : h_0 k_0 l_0 \leq x\}| = (1 + o(1)) \frac{\vartheta(q, H(a))}{\mathcal{P}_{a,q}^{\eta+\xi}} x (\log x)^{\eta+\xi}.$$

Therefore, for all choices of $\eta$ and $c$, we obtain the estimate

$$N_{\eta,c}(x) = (1 + o(1)) \left( \mathcal{P}_{a,q} - 2 \right) \frac{\vartheta(q, H(a))}{\mathcal{P}_{a,q}^{\mathcal{P}_{a,q}-2} (\mathcal{P}_{a,q} - 2)!} \frac{x (\log x)^{\mathcal{P}_{a,q}-2}}{(\log x)^{1-1/\mathcal{P}_{a,q}}}. \quad (36)$$

Taking into account the estimates (28), (33), (34), (35) and (36), we find

$$N^*(x) = \sum_{|\mathcal{R}|=\mathcal{P}_{a,q}-2} N_{H(a),\mathcal{R}}(x) + o \left( \frac{x (\log x)^{\mathcal{P}_{a,q}-2}}{(\log x)^{1-1/\mathcal{P}_{a,q}}} \right).$$

Thus, if $\mathcal{P}_{a,q} = 2$, then $N_{H(a),\mathcal{R}}(x)$ is the only term in the above sum and

$$N^*(x) = (1 + o(1)) \vartheta(q, H(a)) \frac{x}{(\log x)^{1-1/\mathcal{P}_{a,q}}}.$$

If, on the other hand, $\mathcal{P}_{a,q} \geq 3$, then

$$N^*(x) = (1 + o(1)) \sum_{\eta,c} \left( \mathcal{P}_{a,q} - 2 \right) \frac{\vartheta(q, H(a))}{\mathcal{P}_{a,q}^{\mathcal{P}_{a,q}-2} (\mathcal{P}_{a,q} - 2)!} \frac{x (\log x)^{\mathcal{P}_{a,q}-2}}{(\log x)^{1-1/\mathcal{P}_{a,q}}},$$

where the sum runs over the possible values of $\eta$ and $c$ corresponding to the prime power $p^r = \mathcal{P}_{a,q}$ (see Lemma 50). It is easy to see that

$$\sum_{\eta,c} \left( \mathcal{P}_{a,q} - 2 \right) = p^{r-1} \sum_{1 \leq j \leq p/2} \left( p^r - 2 \right)$$

holds for all possible values of $\mathcal{P}_{a,q}$ (and both sides are equal to 1 if $\mathcal{P}_{a,q} = 2$); therefore, making use of Lemma 13, the definition (3), and the relation

$$N(x) = (1 + o(1))(1 - q^{-1})^{-1} N^*(x),$$

we obtain the estimate stated in the theorem.
4 Concluding remarks

We touch very briefly on a number of directions in which this work might well be extended.

(1) Further development of the main term in the asymptotic formula: It is apparent that there are terms of only slightly lower order in the asymptotic formula, some stepping down by powers of \( \log_2 x \) and others by powers of \( \log x \). There seems no reason why these could not be further elucidated although a convenient description of the involved constants might be a lot to expect.

(2) Uniformity in the modulus: Certainly one can trace through the above arguments to obtain results of this type. If one wants however to obtain more than a very limited range of applicability one would need to get at least some useful bounds for the “constants” in the lower order main terms.

(3) Subset avoidance: Rather than ask for the number of integers whose divisors avoid a single residue class \( a \) it seems natural to ask for the number of those whose divisors avoid a subset \( A \) of the reduced residue classes. Here, two cases stand out as probably being quite similar to our existing results, in the case that \( A \) is a subgroup, to our first theorem, and in the case that \( A \) is a coset, to our second one.

(4) General modulus: Although it could be combined with any of the above, the removal of the restriction that the modulus be prime is probably the most natural next step. In this case it seems that little is needed beyond giving a count on the number of different groups avoiding \( a \) and having the same maximal order, and then multiplying the previous result by this number. It is clear that the contribution coming from integers which correspond to more than one of these groups will give a lower order of magnitude. From the fundamental theorem for finite abelian groups it is not hard to find a group-theoretic expression for the number of such subgroups but to give this answer as an explicit reasonable–looking function of the modulus may be a different story.

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