Quantum Networks on Cubelike Graphs

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Cubelike graphs are the Cayley graphs of the elementary abelian group $\mathbb{Z}_2^n$ (e.g., the hypercube is a cubelike graph). We study perfect state transfer between two particles in quantum networks modeled by a large class of cubelike graphs. This generalizes results of Christandl et al. [Phys. Rev. Lett. 92, 187902 (2004)] and Facser et al. [Phys. Rev. A 92, 187902 (2008)].

I. INTRODUCTION

In view of applications like the distribution of cryptographic keys [2, 11], or the communication between registers in quantum devices [6, 20], the study of natural evolution of permanently coupled spin networks has become increasingly important. A special case of interest consists of homogenous networks of particles coupled by constant and fixed (nearest-neighbour) interactions.

An important feature of these networks is the possibility of faithfully transferring a qubit between specific particles without tuning the couplings or altering the network topology. This phenomenon is usually called perfect state transfer (PST). Since quantum networks (and communication networks in general) are naturally associated with directed graphs, there is a growing amount of literature on the relation between graph-theoretic properties and properties that allow PST (see [13, 15, 17, 18, 28]).

In the present paper we will give necessary and sufficient conditions for PST in quantum networks modeled by a large class of cubelike graphs. The vertices of a cubelike graph are the binary $n$-vectors; two vertices $u$ and $v$ are adjacent if and only if their symmetric difference belongs to a chosen set. Equivalently, cubelike graphs on $2^n$ vertices are the Cayley graphs of the elementary abelian group of order $2^n$ [10, 24].

Among cubelike graphs, the hypercube is arguably the most famous one, having many applications ranging from switching theory to computer architecture, etc. (see, e.g., [16, 22]). There are various and diverse results about quantum dynamics on hypercubes. These are essentially embraced by two areas: continuous-time quantum walks [19]; quantum communication in spin networks [7]. The common ingredient is the use of a Hamiltonian representing the adjacency structure of the graph.

Concerning state transfer, Christandl et al. [9] have shown that networks modeled by hypercubes are capable of transporting qubits between pairs of antipodal nodes, perfectly (i.e., with maximum fidelity) and in constant time. Facser et al. [12] generalized this observation, by considering a family of cubelike graphs whose members have the hypercube as a spanning subgraph (for this reason, these authors coined the term dressed hypercubes). Other questions related to quantum dynamics on hypercubes have been addressed in [2, 8, 21, 23, 26].

The paper is organized as follows: Section 2 contains the necessary definitions and the statements of our results. A proof will be given in Section 3. This is obtained by diagonalizing the Hamiltonians with simple tools from Fourier analysis on $\mathbb{Z}_2^n$.

II. SET UP AND RESULTS

Let $\mathbb{Z}_2^n$ be the additive abelian group $(\mathbb{Z}_2)^n$. Each element of $\mathbb{Z}_2^n$ is represented as a binary vector of length $n$. The zero vector $\mathbf{0}$ is made up of all 0’s. Let $f : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$ be a Boolean function on $n$ variables and let $\Omega_f = \{ w \in \mathbb{Z}_2^n \mid f(w) = 1 \}$. Let $d = |\Omega_f|$ be the number of vectors $w \in \mathbb{Z}_2^n$ such that $f(w) = 1$. Finally, if $w$ and $v$ are two binary vectors of the same length, then $w \oplus v$ denotes the vector obtained by computing their elementwise addition modulo 2, and $w^tv$ their scalar product.

The Cayley graph $X(\Gamma, T)$ of a group $\Gamma$ w.r.t. the set $T \subseteq \Gamma$ ($T = T^{-1}$) is the graph with vertex-set $V(X) = \{ \Gamma \}$ and an edge $\{g, h\} \in E(X)$, if there is $s \in T$ such that $gs = h$. The set $T$ is also called Cayley set. The Cayley graphs of the form $X(\mathbb{Z}_2^n, \Omega_f)$ are called cubelike graphs. Some cubelike graphs are illustrated in Figure 1 below.

Notice that this definition embraces every possible set $\Omega_f$. When $f$ is the characteristic function of the standard generating set of $\mathbb{Z}_2^n$, the graph $X(\mathbb{Z}_2^n, \Omega_f)$ is called hypercube. (For example, the leftmost graph in Figure 1). The adjacency matrix of $X(\mathbb{Z}_2^n, \Omega_f)$ is the $2^n \times 2^n$ matrix

$$A_f = \sum_{w \in \Omega_f} \rho_{reg}(w),$$

where $\rho_{reg}(x)$ is the regular (permutation) representation.
of $w \in \Omega_f$. In particular, if $w = w_1 w_2 \cdots w_n$ then

$$
\rho_{\text{reg}}(w) = \bigotimes_{i=1}^{n} \sigma_x^{w_i},
$$

where $\sigma_x$ is a Pauli matrix. It is clear that $A_f$ commutes with the adjacency matrix of any other cubelike graph, given that the group $\mathbb{Z}_2^n$ is abelian.

Now, let us choose a bijection between vertices of $X(\mathbb{Z}_2^n, \Omega_f)$ and the elements of the standard basis $\{|1\rangle, |2\rangle, \ldots, |N\rangle\}$ of an Hilbert space $\mathcal{H} \cong \mathbb{C}^{N}$, where $N = 2^n$. This is the usual space of $n$ qubits. If we look at the single excitation case in the XY model, the evolution of a network of spin 1/2 quantum mechanical particles on the vertices of $X(\mathbb{Z}_2^n, \Omega_f)$ can be seen as induced by the adjacency matrix $A_f$, which then plays the role of an Hamiltonian (for details see [12] or [9]). On the light of this observation, given two vectors $a, b \in \mathbb{Z}_2^n$, the (unnormalized) transition amplitude between $a$ and $b$ induced by $A_f$ is the expression

$$
T(a, b) = \langle b| e^{-iA_f t}|a\rangle = \sum_{w \in \mathbb{Z}_2^n} (-1)^{\sigma_x^w} e^{-i\lambda_w t} (-1)^{b}^{\sigma_x^w} = \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)^w} w e^{-i\lambda_w t},
$$

where $t \in \mathbb{R}^+$. The fidelity of state transfer between $a$ and $b$ is then $F(a, b) = \frac{1}{2^{n}} |T(a, b)|$. By definition, the evolution under $A_f$ is periodic if there is $t \in \mathbb{R}^+$ such that $F(a, a) = 1$ for every $a \in \mathbb{Z}_2^n$. Graph-theoretic properties responsible of periodic evolution (sometime also called perfect revival) have been considered in the literature (14, 15, 27, 28, 29).

With the next proposition, we show that every network modeled by a cubelike graph has a periodic evolution. The period is $\pi$ and it does not depend on the number of vertices of the graph. Equivalently, it does not depend on the dimension of $\mathcal{H}$.

**Theorem 1** Let $X(\mathbb{Z}_2^n, \Omega_f)$ be a cubelike graph and let $a, b \in \mathbb{Z}_2^n$.

1. For $t = \pi$, we have $F(a, b) = 1$ if and only if $a = b$.

2. For $t = \pi/2$, we have $F(a, b) = 1$ if $a \oplus b = u$ and $u = \bigoplus_{w \in \Omega_f} w \neq 0$.

As a simple consequence of this statement, we have various ways to route information between any two nodes of a network whose vertices correspond to the elements of $\mathbb{Z}_2^n$. Let $f$ be a Boolean function such that $\Omega_f = \{w_1, \ldots, w_r\}$ is a generating set of $\mathbb{Z}_2^n$. Let $\bigoplus_{w \in \Omega_f} w \neq 0$. Let us define $C = \{w, w_1, \ldots, w_r\}$ and $C_1 = C \setminus \{w\}$. Since the sum of the elements of $C_1$ is nonzero, the Cayley graph $X(\mathbb{Z}_2^n, C_1)$ has PST between $a$ and $b$ such that $a \oplus b = w_i$ at time $\pi/2$. The simplest case arises when we chose $r = n$ and take the vectors $w_1, \ldots, w_r$ to be the standard basis of $\mathbb{Z}_2^n$. Then $X(\mathbb{Z}_2^n, C)$ is the folded d-cube and, by using a suitable sequence of the graphs $X(\mathbb{Z}_2^n, C_1)$, we can arrange PST from the zero vector to any desired element of $\mathbb{Z}_2^n$.

For example, consider the case $d = 3$. We write $w_1 = (100), w_2 = (010)$ and $w_3 = (001)$. Then $w = (111)$. The graph $X(\mathbb{Z}_2^3, C)$ is illustrated in Figure 1 – left. Since $w_1 = w_2 \oplus w_3 \oplus w$, there is PST between 000 and 111 at time $\pi/2$, given that 000 $\oplus$ 100 = 100 (see Figure 2 – right). Also, there is PST for the pairs $\{010, 110\}, \{001, 101\}$ and $\{011, 111\}$. Notice that $X(\mathbb{Z}_2^3, C_1)$ is isomorphic to the 3-dimensional hypercube.

For generic dimension, the hypercube is $X(\mathbb{Z}_2^n, E)$, where $E = \{(10\ldots0), (010\ldots0), \ldots, (0\ldots01)\}$ and $J_n = \bigoplus_{w \in E} w$, the all-ones vector of length $2^n$. Since the Hamming distance between two different elements $a, b \in \mathbb{Z}_2^n$ is exactly their distance in $X(\mathbb{Z}_2^n, E)$, we have PST between any two antipodal vertices of the hypercube, as it was already observed in [2] and [12]. Recall that the distance between two vertices in a graph is the length of the geodesic (equivalently, the shortest path) connecting the vertices. Two vertices are said to be antipodal if their
distance is the diameter of the graph, i.e., the longest among all the geodesics. Antipodal vertices in a Cayley graphs are connected via a sequence of all elements of the Cayley set. It remains as an open problem to verify that whenever there is PST between two vertices of a cubelike graph then the vertices are antipodal.

### III. PROOF OF THE THEOREM

The abstract Fourier transform of a Boolean function $f$ is the rational valued function $f^* : \mathbb{Z}_2^n \to \mathbb{Q}$ which defines the coefficients of $f$ with respect to the orthonormal basis of the functions

$$Q_w(x) = (-1)^{w^T x},$$

that is,

$$f^*(w) = 2^{-n} \sum_{x \in \mathbb{Z}_2^n} (-1)^{w^T x} f(x).$$

Then

$$f(x) = \sum_{w \in \mathbb{Z}_2^n} (-1)^{w^T x} f^*(w)$$

is the Fourier expansion of $f$. Note that the zero-order Fourier coefficient is equal to the probability that the function takes the value 1, i.e., $f^*(0) = \frac{1}{2^n}$, while the other Fourier coefficients measure the correlation between the function and the parity of subsets of its arguments.

Using a vector-representation for the functions $f$ and $f^*$, and considering the natural ordering of the binary vectors $w \in \mathbb{Z}_2^n$, one can derive a convenient matrix formulation for the transform pair: $f = H_n f^*$ and $f^* = \frac{1}{\sqrt{2^n}} H_n f$, where $H_n$ is the Hadamard transform matrix. Given a function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$, the set $\Omega_f$ defines the Cayley graph $X(\mathbb{Z}_2^n, \Omega_f)$, whose spectrum coincides, up to a factor $2^n$, with the Fourier spectrum of the function: $\frac{1}{\sqrt{2^n}} H_n A_f H_n = D_f$, where $A_f$ is the adjacency matrix of $X(\mathbb{Z}_2^n, \Omega_f)$ and $D_f$ is an $2^n \times 2^n$ diagonal matrix. In particular, for $w \in \mathbb{Z}_2^n$, we have $\lambda_w = 2^n f^*(w)$. Theorem 3 needs the following two technical lemmas.

**Lemma 2** Let $f$ be a Boolean function such that $\bigoplus_{w \in \Omega_f} w = 0$. Then, for all $v \in \mathbb{Z}_2^n$,

$$\lambda_v = d - 4k_v,$$

where $d = |\Omega_f|$ and $k_v \in \mathbb{N}$, $0 \leq k_v \leq |d/2|$.

**Proof.** Let $a \in \Omega_f$, $a \neq 0$, and let $\Omega'_f = \Omega \setminus \{a\}$. Since $\bigoplus_{w \in \Omega_f} w = 0$, we have that $a = \bigoplus_{w \in \Omega'_f} w$. Now observe that, for any $v \in \mathbb{Z}_2^n$,

$$\lambda_v = \sum_{w \in \mathbb{Z}_2^n} (-1)^{w^T v} f(w) = \sum_{w \in \Omega_f} (-1)^{w^T v}$$

$$= \sum_{w \in \Omega'_f} (-1)^{w^T v} + (-1)^{w^T v}$$

$$= \sum_{w \in \Omega'_f} (-1)^{w^T v} + (-1)^{\bigoplus_{w \in \Omega'_f} w}$$

$$= \sum_{w \in \Omega'_f} (-1)^{w^T v} + \prod_{w \in \Omega'_f} (-1)^{w^T v}.$$
Thus,

$$\lambda_v = \mu_v - (-1)^v = d + 1 - 4k_v - (-1)^v,$$

and the thesis immediately follows.

(2.) Consider the function $g$ such that $\Omega_g = \Omega_f \setminus \{u\}$. Let $\mu_v$ denote the eigenvalues of the Cayley graph associated to $g$. As $u = \Theta_{w \in \Omega_f} w = \Theta_{w \in \Omega_g} w$, we have $\Theta_{w \in \Omega_g} w = 0$. By applying Lemma 2, we obtain

$$\mu_v = |\Omega_g| - 4k_v = d - 1 - 4k_v,$$

with $0 \leq k_v \leq \frac{d}{4}$. Now, as in (1.) observe that

$$\lambda_v = \sum_{w \in \mathbb{Z}_2^n} (-1)^v f(w) = \sum_{w \in \Omega_f} (-1)^v$$

$$= \sum_{w \in \Omega_g} (-1)^v w + (-1)^v = \mu_v + (-1)^v.$$

So we get

$$\lambda_v = d - 1 - 4k_v + (-1)^v,$$

concluding the proof of the lemma. ■

Proof of Theorem 1. (1.) All eigenvalues $\lambda_w$ are integers with the same parity. In particular, they are all odd if $d = |\Omega_f|$ is odd, and all even, otherwise. Thus, by Eq. (1), for $t = \pi$ we have

$$T(a, b) = \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} e^{-i\lambda_w \pi}$$

$$= \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} \cos(\lambda_w \pi)$$

$$= \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} (-1)^{\lambda_w}$$

$$= (-1)^d \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w}.$$

Thus the statement follows since

$$\sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} = \begin{cases} 2^n, & a \oplus b = 0, \text{ i.e., } a = b; \\ 0, & \text{otherwise.} \end{cases}$$

(2.) First, let us suppose that $f$ is such that $\Theta_{w \in \Omega_f} w = 0$. Then, applying Lemma 2 we can see that

$$T(a, b) = \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} e^{-i\lambda_w \pi/2}$$

$$= \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} e^{-i(d - 4k_w)\pi/2}$$

$$= e^{-id\pi/2} \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} e^{i2k_w \pi}$$

$$= e^{-id\pi/2} \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w}.$$

The thesis is verified, since $\sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} = 2^n$ if and only if $a \oplus b = 0$, but here $a \neq b$. The remaining case is when $\Theta_{w \in \Omega_f} w = u \neq 0$. Applying Lemma 3 we can write

$$T(a, b) = \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} e^{-i(d - 4k_w)\pi/2}$$

$$+ \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} e^{-i(d - 4k_w)\pi/2}$$

$$= e^{-id\pi/2} \left( \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} e^{i2k_w \pi} \right)$$

$$= e^{-id\pi/2} \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b)T_w} (-1)^{T_w}$$

$$= e^{-id\pi/2} \sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b \oplus u)T_w}.$$

The statement holds since $\sum_{w \in \mathbb{Z}_2^n} (-1)^{(a \oplus b \oplus u)T_w} = 2^n$ if and only if $a \oplus b \oplus u = 0$, i.e., $a \oplus b = u = \Theta_{w \in \Omega_f} w$. ■

IV. CONCLUSION

We have given a necessary and sufficient condition for PST in quantum networks modeled by a large class of cubelike graphs.

A special case is left open: when $\Theta_{w \in \Omega_f} w = 0$. Numerical evidence suggests that cubelike graphs with this property do not allow PST.

An application of our result is a straightforward method to distinguish if, for a Boolean function $f$, we have $\Theta_{w \in \Omega_f} w \neq 0$ or $\Theta_{w \in \Omega_f} w = 0$, when this promise holds. By Theorem 3 we know that $|b| e^{-iA_f \pi/2} |a| = 1$ if $a \oplus b \neq 0$. If we let evolve the system under the Hamiltonian $A_f$ for a time $\pi/2$, we then obtain the state $|b|$ by performing a von Neumann measurement $w.r.t$ the standard basis of $\mathcal{H}$. When $n$ gets large, the probability of observing $|b|$ with distinct measurements tends to decrease exponentially if $\Theta_{w \in \Omega_f} w = 0$.

On the other side, this problem is easy without the use of any quantum technique. Beyond this trivial application, given the link between cubelike graphs and Boolean functions, it is natural to ask whether quantum dynamics on these graphs can help in getting useful information about the corresponding functions.

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