Singular function emerging from one-dimensional
elementary cellular automaton Rule 150

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Abstract

In this paper, we give a singular function on a unit interval derived from the dynamic of the one-dimensional elementary cellular automaton Rule 150. We describe properties of the resulting function, that is strictly increasing, uniformly continuous, and differentiable almost everywhere, and we show that it is not differentiable at dyadic rational points. We also give functional equations that the function satisfies, and show that the function is the only solution of the functional ones.

1 Introduction

There exist many pathological functions. The Weierstrass function and the Takagi function, for example, are real-valued functions that are continuous everywhere but nowhere differentiable [1, 2]. Generalized results of the Takagi function were given in [3]. Okamoto’s function is a one-parameter family of self-affine functions whose differentiability is determined by the parameter; it is differentiable almost everywhere, non-differentiable almost everywhere, or nowhere differentiable [4, 5, 6]. A singular function is defined by monotonically increasing (or decreasing), continuous everywhere, and has zero derivative almost everywhere. The Cantor function is an example of a singular function [7], that is also referred to as the Devil’s staircase, and there are infinite number of steps in \([0, 1]\) while it is constant most of them. Salem’s function is a self-affine function, that is another example of a singular function [8, 9, 10, 11]. There are several works discussed the relationship between the function and cellular automata. For the one-dimensional elementary cellular automaton Rule 90 the limit set is characterized by Salem’s function [12], and for a two-dimensional automaton that is a mathematical model of a crystalline growth the limit set is also characterized by Salem’s one (numerical result was given in [13], proofs were given in [14, 15]).

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In this paper, we provide a new singular function by the elementary cellular automaton Rule 150. Figure 1 shows the spatio-temporal pattern of Rule 150 from time step 0 to 31, and Figure 2 shows its limit set. The authors have calculated the number of nonzero states in the spatial and spatio-temporal pattern of Rule 150 [16]. By normalizing the dynamic of the numbers, we provide a function. We write it down by an infinite sum of the numbers, and show that the resulting function is a singular function. We also show that the function is not differentiable at dyadic rational points, and give functional equations that the function satisfies.

The remainder of the paper is organized as follows. Section 2 describes the preliminaries concerning the cellular automaton Rule 150 and the number of nonzero states in its spatial and spatio-temporal patterns. In Section 3, we provide a definition of the given function and describe properties of it. We show that the function is a singular function, and give functional equations that the function satisfies. Lastly, Section 4 discusses the findings of this paper.

2 Preliminaries

In this section, we present some definitions and notations for elementary cellular automata and their limit sets. We also provide an overview of previous results about the number of nonzero states in spatial or spatio-temporal patterns of cellular automata.

2.1 One-dimensional elementary cellular automaton Rule 150 and its limit set

Let \( \{0, 1\} \) be a binary state set and \( \{0, 1\}^\mathbb{Z} \) be the one-dimensional configuration space. Suppose that \( (\{0, 1\}^\mathbb{Z}, T) \) is a discrete dynamical system consisting of a space \( \{0, 1\}^\mathbb{Z} \) and a transformation \( T \) on \( \{0, 1\}^\mathbb{Z} \). The \( n \)-th iteration of \( T \) is denoted by \( T^n \). Thus, \( T^0 \) is the identity map.

Definition 1. A one-dimensional elementary cellular automaton \( (\{0, 1\}^\mathbb{Z}, T) \) is given by

\[
(Tx)_i = f(x_{i-1}, x_i, x_{i+1})
\]

for \( i \in \mathbb{Z} \) and \( x = \{x_i\}_{i \in \mathbb{Z}} \in \{0, 1\}^\mathbb{Z} \), where \( f : \{0, 1\}^3 \to \{0, 1\} \) is a map depending on the nearest three states. We call \( f \) a local rule of \( T \).

This is the simplest nontrivial cellular automaton. This class includes 256 automata, referred to by the Wolfram code from Rule 0 to Rule 255. For each state \( x_i \) (\( i \in \mathbb{Z} \)), the next state \( (Tx)_i \) is determined by the nearest three states \( (x_{i-1}, x_i, x_{i+1}) \). In the case of Table 1, the Wolfram code is Rule 150, because \( 1 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 150 \). The local rule of Rule 150 \( (\{0, 1\}^\mathbb{Z}, T_{150}) \) is also given by

\[
(T_{150}x)_i = x_{i-1} + x_i + x_{i+1} \quad \text{(mod 2)},
\]
Table 1: Local rule of Rule 150

| $x_{i-1}x_ix_{i+1}$ | 111 | 110 | 101 | 100 | 011 | 010 | 001 | 000 |
|---------------------|-----|-----|-----|-----|-----|-----|-----|-----|
| $(T_{150}x)_i$     | 1   | 0   | 0   | 1   | 1   | 1   | 0   |

for $x \in \{0, 1\}^Z$. The local rules given in Table 1 and Equation 2 are mathematically equal.

The configuration $x_o \in \{0, 1\}^Z$ is called the single site seed, wherein

$$(x_o)_i = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \in Z \backslash \{0\}. \end{cases}$$

(3)

Figure 1 shows the orbit of Rule 150 from the single site seed $x_o$ as an initial configuration until time step $2^5 - 1$.

Figure 1: Spatio-temporal pattern of Rule 150, $\{T_{150}x_o\}_{n=0}^{31}$

Figure 2: Limit set of Rule 150 from the single site seed $x_o$

Suppose that $\{T^nx_o\}$ is a dynamic of a cellular automaton from the single site seed, and a subset of a two-dimensional Euclidean space $S(n)$ is given by

$$S(n) = \{(i, t) \in Z^2 \mid (T^tx_o)_i > 0, 0 \leq t \leq n\},$$

(4) that consists of nonzero states from time step 0 until $n$. A limit set of a cellular automaton is defined by $\lim_{n \to \infty} S(n)/n$, if it exists, where $S(n)/n$ is a contracted set of $S(n)$ with a contraction rate of $1/n$. Before evaluating the limit, $S(n)/n$ for finite $n$ is called a prefractal set if the limit set exists. For limit sets of linear cellular automata the following two theorems have been shown.

**Theorem 1** ([17]). Consider a $p^m$-state linear cellular automaton ($p$ is a prime number, $m \in Z_{>0}$). If $p^m-1$ divides time step $n$, then $(T^mx_o)_p = (T^nx_o)_i$. If $p^m$ divides $n$ and at least one of the elements of $i$ is indivisible by $p$, then $(T^nx_o)_i$ equals 0.

**Theorem 2** ([17]). For a $p^m$-state linear cellular automaton ($p$ is a prime number, $m \in Z_{>0}$) its limit set $\lim_{k \to \infty} S(p^k-1)/p^k$ exists.
Based on Theorems 1 and 2, we obtain the following corollary, because Rule 150 is a two-state linear cellular automaton.

**Corollary 1.** A subset of a two-dimensional Euclidean space $S_{150}(2^k - 1)$ is given by

$$S_{150}(2^k - 1) = \{(i, t) \in \mathbb{Z}^2 \mid (T^t_{150}x_o)_i > 0, 0 \leq t \leq 2^k - 1\}.$$ (5)

The limit set of the orbit of Rule 150 from the single site seed $x_o$, $\lim_{k \to \infty} (S_{150}(2^k - 1)/2^k)$, is a fractal whose Hausdorff dimension is $\log(1 + \sqrt{5})/\log 2$.

Figure 2 shows the limit set of Rule 150 with time steps $n = 2^k - 1$ as $k$ tends to infinity.

### 2.2 Numbers of nonzero states for Rule 150

Let $num_T(n)$ be the number of nonzero states in a spatial pattern $T^n x_o$ for time step $n$, and $cum_T(n)$ be the cumulative sum of the number of nonzero states in a spatial pattern $T^m x_o$ from time step $m = 0$ to $n$ for a cellular automaton. Thus,

$$num_T(n) = \sum_{i \in \mathbb{Z}} (T^n x_o)_i, \quad cum_T(n) = \sum_{m=0}^{n} \sum_{i \in \mathbb{Z}} (T^m x_o)_i.$$ (6)

In the case of Rule 150, the numbers are denoted by $num_{150}(n)$ and $cum_{150}(n)$, respectively. Figure 3 shows the dynamic of $cum_{150}(n)$ for $0 \leq n < 256$. We introduce the previous results about the number of nonzero states in the spatio-temporal and spatial patterns according to self-similar structures.

![Figure 3: $\{cum_{150}(n)\}$ of Rule 150 for $0 \leq n < 256$](image)

**Proposition 1 ([16, 17, 18]).** We introduce the transition matrix $M$ and the vector $v_0$ as the initial values;

$$M = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}, \quad v_0 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$ (7)
The cumulative sum of the number of nonzero states from time step 0 to $2^k - 1$ for Rule 150, $\operatorname{cum}_{150}(2^k - 1)$, is given by $aM^{k-1}v_0$ for a vector $a = (1 0)$ and $k \geq 1$. Hence,

$$\operatorname{cum}_{150}(2^k - 1) = \frac{\sqrt{5}}{20}(1 + \sqrt{5})^{k+2} - \frac{\sqrt{5}}{20}(1 - \sqrt{5})^{k+2}. \quad (8)$$

**Proposition 2 ([16]).** We introduce the transition matrices and the vector

$$M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad u_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (9)$$

Assuming that the binary expansion of $n$ is $n_{l-1}n_{l-2}\cdots n_1n_0$ ($n_i \in \{0, 1\}$, $i = 0, 1, \ldots, l-1$), let $p_r$ be the number of clusters consisting of continuous $r$ 1s in the binary number. Thus,

$$\operatorname{num}_{150}(n) = aM_{n_{l-1}}M_{n_{l-2}}\cdots M_{n_0}u_0 = \prod_{r=0}^{l} (aM^r u_0)^{p_r} = \prod_{r=0}^{l} \left(\frac{2^{r+2} + (-1)^{r+1}}{3}\right)^{p_r}. \quad (10)$$

**Remark 1.** In this paper we set $\operatorname{num}_{150}(-1) = \operatorname{cum}_{150}(-1) = 0$ for $n = -1$, due to a technical reason.

### 3 Main results

In this section, we give a function on a unit interval by the cumulative sum of the number of nonzero states of Rule 150. For $x = \sum_{i=1}^{\infty} (x_i/2^i) \in [0, 1]$ and $k \in \mathbb{Z}_{>0}$, $F_k(x)$ is given by $\operatorname{cum}_{150}((\sum_{i=1}^{k} x_i2^{k-i}) - 1)/\operatorname{cum}_{150}(2^k - 1)$, that is a normalized sum of the number by $\operatorname{cum}_{150}(2^k - 1)$. Considering its limit, we name it $F$. Thus,

$$F(x) := \lim_{k \to \infty} F_k(x) = \lim_{k \to \infty} \frac{\operatorname{cum}_{150}\left((\sum_{i=1}^{k} x_i2^{k-i}) - 1\right)}{\operatorname{cum}_{150}(2^k - 1)}. \quad (11)$$

Figure 4 shows the graph of $F(x)$ for $x \in [0, 1]$ and the limit set of Rule 150.
Although we have obtained $\text{cum}_{150}(n)$ and $\text{num}_{150}(n)$ by Propositions 1 and 2, it is difficult to write down $F$ directly, because $\text{cum}_{150}(n)$ is obtained only for time step $n = 2^k - 1$, and it is not easy to calculate the sum of $\text{num}_{150}(n)$. In the next section, we construct the function $F$ focusing on self-similar structures of the spatio-temporal pattern $\{T_{150,x}\}$. Section 3.1 provides a definition of the function $F$, and Section 3.2 describes properties of $F$. In Section 3.3 we give functional equations that $F$ satisfies, and we prove that $F$ is the only solution of them.

### 3.1 Constructing the function $F$

The function $F : [0, 1] \rightarrow [0, 1]$ is given by the following definition.

**Definition 2.** Let

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad u_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{12}$$

For $x = \sum_{i=1}^{\infty} (x_i/2^i) \in [0, 1]$ the function $F : [0, 1] \rightarrow [0, 1]$ is given by

$$F(x) = \sum_{i=1}^{\infty} x_i r(x)_i \alpha^i, \tag{13}$$

where $r(x)_i = a M_{x_{i-1}} M_{x_{i-2}} \cdots M_{x_2} M_{x_1} M_{x_0} u_0$ and $\alpha = (\sqrt{5} - 1)/4$.

From here we explain how we obtain Definition 2. Assuming $S$ is the set of nonzero points in the limit set of Rule 150 $\lim_{k \to \infty} (S_{150}(2^k - 1)/2^k)$, we find that $S$ consists of self-similar sets of $S$. Let $S_0 := S$ and $S_1$ be a subset of $S$ on $x \in [0, 1/2)$, that is self-similar to $S$ (see Figure 5(a1) and (b1)). Hence, comparing $S_1$ and $S$, we obtain

$$F\left(\frac{1}{2}\right) = \lim_{k \to \infty} \frac{\text{cum}_{150}(2^k - 1)}{\text{cum}_{150}(2^{k+1} - 1)} = \frac{\sqrt{5} - 1}{4} =: \alpha. \tag{14}$$
Figure 5: \( S_i, f_i(x) \) and \( \sum_{j=0}^{i} f_j(x) \) (\( i = 0, 1, 2, 3 \))
Let $S_i$ be a self-similar subset of $S$ on $x \in [0,1/2^i)$ (see Figure 5 (a1), (b1), (c1), and (d1)). For $i \geq 0$ comparing $S_i$ and $S$, we also obtain

$$F \left( \frac{1}{2^i} \right) = \lim_{k \rightarrow \infty} \frac{\text{num}_{150}(2^{k+1})}{\text{num}_{150}(2^{k+1} - 1)} = \left( \frac{\sqrt{5} - 1}{4} \right)^i. \quad (15)$$

![Figure 6: The set $S(11/16)$ consists of one $S_1$, three $S_3$s, and nine $S_4$s.](image)

For any $x \in [0,1]$ $S(x)$ is denoted by a subset of $S$ on $[0,x)$. We can represent $S(x)$ by a union of $S_i$s. Considering the binary expansion of $x$, we know the components $S_i$s of $S(x)$. When $x = 1/2^i$, $S(x) = S_i$, and the set $S(x)$ is equal to one $S_i$. As another example, when $x = 11/16 = 1/2 + 1/2^2 + 1/2^4 \in [0,1]$, the set $S(11/16)$ consists of one $S_1$, three $S_3$s, and nine $S_4$s (see Figure 6). The coefficients 1, 3, and 9 are given by $x,r(x)$, for $i = 1,3,4$, respectively.

In Equation (13) $x,r(x)$ shows that how many $S_i$s with the size of $\alpha^i \cdot S(x)$ includes. If $x_i = 0$, there are no $S_i$s, and if $x_i = 1$, there are some $S_i$s. The value $r(x_i)$ shows the number of $S_i$s, that is given by $\text{num}_{150}(n)$ where $n = \sum_{j=0}^{i-1} x_j \cdot 2^j$. Based on Theorem 1 for any $m \in \mathbb{Z}_{\geq 0}$ we have

$$\text{num}_{150} \left( \sum_{j=0}^{i-1} x_j \cdot 2^j \right) = \text{num}_{150} \left( \sum_{j=0}^{i-1} x_j \cdot 2^{j+m} \right). \quad (16)$$

Considering the spatio-temporal pattern of Rule 150, we find that nonzero states of $T_{150}^n x_o$ for $n = \sum_{j=0}^{i-1} x_j \cdot 2^{j+m}$ ($\forall m \in \mathbb{Z}_{\geq 0}$) are the apexes of $S_i$s. Lines in Figure 7 (a), (b), and (c), for example, show $T_{150}^{2^m+3+2^m+1+2^m} x_o$ for $m = 0, 1,$ and 4, respectively. All of the numbers are the same, $\text{num}_{150}(2^3 + 2^1 + 2^4) = \text{num}_{150}(2^4 + 2^2 + 2^4) = \text{num}_{150}(2^7 + 2^5 + 2^4) = 15$. Hence, we find that there exist fifteen $S_i$s on the interval $[(2^m+3 + 2^m+1 + 2^m)/2^m+4, (2^m+3 + 2^m+1 + 2^m)/2^m+4 + 1/2^4]$. By Proposition 2 we can represent $r(x_i)$ by $\text{num}_{150}(n) = a M_{x,-1} M_{x,-2} \cdots M_{x_i} M_{x_0} u_0$. 

8
\[ S \in x^x = 1. \] The graphs of \( x \) if \( S \) no

Similarly to the above, we obtain the graphs of \( f \) and (a3). If \( k \) for

\[ T_{150} \]

Figure 7: Lines on the spatio-temporal patterns of Rule 150 show \( T_{150}^{m+3+2m+1+2^m} x \) for \( m = 0, 1, \) and 4, respectively. The numbers of nonzero states on these lines are the same, i.e.,

\[ \text{num}_{150}(2^4 + 2^1 + 2^0) = \text{num}_{150}(2^4 + 2^2 + 2^1) = \text{num}_{150}(2^7 + 2^5 + 2^4) = 15. \]

Next, we consider a sequential function \( \{ f_i \} (i \in \mathbb{Z}_{\geq 0}) \). For each \( i \) \( f_i : [0, 1] \to [0, 1] \) is given by, if \( i = 0 \)

\[ f_0(x) := x \alpha^0 = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2^0}, \\ 1 & \text{if } x = \frac{1}{2^0}, \end{cases} \] (17)

and if \( i > 0 \)

\[ f_i(x) := x_i \alpha^i = \begin{cases} 0 & \text{if } \frac{2i}{2^i} \leq x < \frac{2k + 1}{2^i}, \ x = 1, \\ r(x) \alpha^i & \text{if } \frac{2k + 1}{2^i} \leq x < \frac{2k + 2}{2^i}, \end{cases} \] (18)

for \( k \in \mathbb{Z}_{>0} \) and \( k \leq 2^{i-1} - 1 \). Figure 5 shows \( S_i, f_i(x) \) and \( \sum_{j=0}^{i-1} f_j(x) \) \( (i = 0, 1, 2, 3) \). For \( x \in [0, 1] \) first we consider \( S(x) \). Each function \( f_i(x) \) represents how many \( S_i \)s are included in \( S(x) \) for \( x \in [0, 1] \). The graph of \( f_0 = \sum_{j=0}^{0} f_j \) is shown in Figure 5 (a2) and (a3). If \( x \in [0, 1] \) \( f_0(x) = 0 \), because there are no \( S_0 \)s in \( S(x) \) for \( x \in [0, 1] \), and if \( x = 1 \) \( f_0(x) = 1 \), because \( S(1) \) is equal to \( S_0 \) itself. Thus, \( f_0(x) > 0 \) only when \( x = 1 \). The graphs of \( f_1 \) and \( \sum_{j=0}^{i} f_j \) are shown in Figure 5 (b2) and (b3). When \( x \in [0, 1/2] \) \( S(x) \) does not include any \( S_i \)s, and \( f_1(x) = 0 \). When \( x = 1 \) \( S(1) \) has already covered by \( S_0 \), and also \( f_1(1) = 0 \). When \( x \in [1/2, 1] \) \( S(x) \) includes one \( S_1 \), and then \( f_1(x) = \alpha \), where \( \alpha \) is the size of \( S_1 \) and the coefficient 1 is given by \( r(x) \). Similarly to the above, we obtain the graphs of \( f_i \) and \( \sum_{j=0}^{i} f_j \) for \( i \geq 0 \). If there are no \( S_i \)s in \( S(x), f_i(x) = 0 \), and if \( S(x) \) includes \( r(x) \), \( S_i \), \( f_i(x) = r(x) \alpha^i \).

Lastly, we check that the infinite series in Equation (13) is convergent. We have \(|x, r(x), \alpha^i| \leq 3^{-1} \alpha^i \), and \( \sum_{i=1}^{\infty} 3^{-1} \alpha^i = \alpha/(1 - 3\alpha) \) < \( +\infty \). The series converges because it converges absolutely. Therefore, we can consider the infinite sum of \( f_i(x) \) from \( i = 0 \) to infinity, that is exactly \( F(x) \), and we have obtained Definition 2.

**Corollary 2.** By Proposition 3 for \( x = \sum_{i=1}^{\infty} (x_i/2^i) \in [0, 1] \) we have \( N_i \) and \( k_j \) \((j = 1, \ldots, N_i) \) such that \( a M_{x_{i-1}} M_{x_{i-2}} \cdots M_{x_j} M_{x_0} u_0 = \prod_{j=1}^{N_i} (a M_1^{k_j} u_0) \). Because
Based on the result, Equation (24) is obtained by

\[
F(x) = \sum_{i=1}^{\infty} x_i \alpha^i \prod_{j=1}^{N_i} \left( \frac{2^{k_j+2} + (-1)^{k_j+1}}{3} \right),
\]

(19)

**Remark 2.** The function \( F \) given in Definition A is equal to \( F \) in Equation (11). Based on the result, \( \text{cum}_{150}((\sum_{i=1}^{\infty} x_i 2^{k_i-1}) - 1) = \sum_{i=1}^{\infty} x_i r(x_i) \text{cum}_{150}(2^{k_i-1} - 1) \), we have

\[
F_k(x) = \frac{\text{cum}_{150} \left( (\sum_{i=1}^{k} x_i 2^{k_i-1} - 1 \right)}{\text{cum}_{150}(2^k - 1)}
\]

(20)

\[
= \frac{\text{cum}_{150} \left( (\sum_{i=1}^{\infty} x_i 2^{k_i-1} - 1 \right) - \text{cum}_{150} \left( (\sum_{i=k+1}^{\infty} x_i 2^{k_i-1} - 1 \right)}{\text{cum}_{150}(2^k - 1)}
\]

(21)

\[
= \frac{\sum_{i=1}^{\infty} x_i r(x_i) \text{cum}_{150}(2^{k_i-1} - 1) - \sum_{i=k+1}^{\infty} x_i r(x_i) \text{cum}_{150}(2^{k_i-1} - 1}}{\text{cum}_{150}(2^k - 1)}
\]

(22)

\[
= \frac{\sum_{i=1}^{\infty} x_i r(x_i) \alpha^i}{1 - (-4\alpha^2)^{k+2}} - \frac{\sum_{i=1}^{k} x_i r(x_i) (-4\alpha)^{-i}}{(-4\alpha^{2k})^{-k-2} - 1} - \frac{\sum_{i=1}^{\infty} x_i r(x_i) \alpha^i}{\alpha^{-k} - \alpha^2 (-4\alpha)^{k+2}}
\]

(23)

\[
\rightarrow \sum_{i=1}^{\infty} x_i r(x_i) \alpha^i - 0 - 0 \quad (k \to \infty)
\]

(24)

\[
= F(x).
\]

(25)

Equation (24) is obtained by \(|-4\alpha^2| < 1, |1/(-4\alpha)| < 1, \) and \( r(x_{i+k} < 3^{i+k-1} \).

**Remark 3.** We verify that \( F(0) = 0 \) and \( F(1) = 1 \). By the definition of \( F \)

\[
F(0) = F \left( \sum_{i=1}^{\infty} \frac{0}{2^i} \right) = 0,
\]

(26)

\[
F(1) = F \left( \sum_{i=1}^{\infty} \frac{1}{2^i} \right) = \sum_{i=1}^{\infty} r(x_i) \alpha^i = \sum_{i=1}^{\infty} b_i \alpha^i = 1,
\]

(27)

where \( b_i := a M_i^{-1} u_0 = (2^{i+1} + (-1)^i)/3 \).

**Remark 4.** The binary expansion of \( x \) is unique except for dyadic rationals \( x = m/2^j \), which have two possible expansions. We check that the definition of \( F \) is consistent for the numbers having two binary expansions. Let \( x = \sum_{i=1}^{k} (x_i/2^i) + 1/2^{k+1} + \sum_{i=k+2}^{\infty} (1/2^i) \) for \( x_i \in \{0, 1\} \) and \( k \in \mathbb{Z}_{\geq 0} \). Thus, \( x = y \). Here we confirm that \( F(x) = F(y) \). By the definition of \( F \)

\[
F(x) = F \left( \sum_{i=1}^{k} \frac{x_i}{2^i} + \frac{1}{2^{k+1}} + \sum_{i=k+2}^{\infty} \frac{1}{2^i} \right) = \left( \sum_{i=1}^{k} x_i r(x_i) \alpha^i \right) + r(x)_{k+1} \alpha^{k+1},
\]

(28)

\[
F(y) = F \left( \sum_{i=1}^{k} \frac{x_i}{2^i} + \sum_{i=k+2}^{\infty} \frac{1}{2^i} \right) = \left( \sum_{i=1}^{k} x_i r(x_i) \alpha^i \right) + \sum_{i=k+2}^{\infty} r(y_i) \alpha^i.
\]

(29)
Set $\tilde{M}_k = M_{x_k}M_{x_{k-1}}\cdots M_{x_1}M_0$. We have

$$F(x) - F(y) = r(x)_{k+1}a^{k+1} - \sum_{i=k+2}^{\infty} r(y)_i a^i \quad (30)$$

$$= (a\tilde{M}_{x_k}u_0)\alpha^{k+1} - \sum_{i=k+2}^{\infty} (a\tilde{M}_1^{-k-2}M_0\tilde{M}_{x_k}u_0)\alpha^i \quad (31)$$

$$= (a\tilde{M}_{x_k}u_0)\alpha^{k+1} - \sum_{i=k+2}^{\infty} (a\tilde{M}_1^{-k-2}u_0)(a\tilde{M}_{x_k}u_0)\alpha^i \quad (32)$$

$$= (a\tilde{M}_{x_k}u_0)\alpha^{k+1} \left( 1 - \sum_{i=1}^{\infty} (a\tilde{M}_1^{-i-1}u_0)\alpha^i \right) \quad (33)$$

$$= (a\tilde{M}_{x_k}u_0)\alpha^{k+1} \left( 1 - \sum_{i=1}^{\infty} b_i \alpha^i \right) = 0. \quad (34)$$

### 3.2 Properties of the function $F$

In this section we describe properties of the function $F$ given in Definition 2.

**Theorem 3.** The function $F$ on $[0, 1]$ holds the following properties.

1. $F$ is strictly increasing,
2. $F$ is uniformly continuous,
3. $F$ is differentiable with derivative zero almost everywhere, and
4. $F$ is a singular function.

**Proof of Theorem 3 (i).** Assuming that $y > x$, we have $k \in \mathbb{Z}_{\geq 0}$ such that $y_i = x_i$ for $\forall i \leq k$, $y_{k+1} = 1$, and $x_{k+1} = 0$.

We consider the following three cases (excepting the case of $x = \sum_{i=1}^{k}(x_i/2^i) + \sum_{i=k+2}^{\infty}(1/2^i)$ and $y = \sum_{i=1}^{k}(x_i/2^i) + 1/2^{k+1}$, because it means $x = y$).

(a) Suppose that $x = \sum_{i=1}^{k}(x_i/2^i) + \sum_{i=k+2}^{\infty}(1/2^i)$ and $y = (\sum_{i=1}^{k}(x_i/2^i)) + 1/2^{k+1} + (\sum_{i=k+2}^{\infty}(y_i/2^i))$, where $\sum_{i=k+2}^{\infty} y_i > 0$.

By the definition $x$ is represented by $\sum_{i=1}^{k}(x_i/2^i) + 1/2^{k+1}$. Let $l = \max\{i \mid x_j = y_j \text{ for } \forall j \leq i\}$. The number $l$ always exists, because $y \neq x$, and the following inequality is obtained.

$$F(y) - F(x) = \sum_{i=l+1}^{\infty} y_i r(y)_i a^i \geq r(y)_{l+1} a^{l+1} > 0, \quad (35)$$

since $\sum_{i=k+2}^{\infty} y_i > 0$. 

11
(b) Suppose that \( x = \sum_{i=1}^{k} (x_i/2^i) + \sum_{i=k+2}^{\infty} (x_i/2^i) \) and \( y = \sum_{i=1}^{k} (x_i/2^i) + 1/2^{k+1} \), where \( \prod_{i=k+2}^{\infty} x_i = 0 \). 

By the definition \( y \) is represented by \( \sum_{i=1}^{k} (x_i/2^i) + \sum_{i=k+2}^{\infty} (1/2^i) \). Since \( \prod_{i=k+2}^{\infty} x_i = 0 \), we have the following inequality.

\[
F(y) - F(x) = \left( \sum_{i=k+2}^{\infty} r(y)_i \alpha^i \right) - \left( \sum_{i=k+2}^{\infty} x_i r(x)_i \alpha^i \right)
\]

\[
= \left( \sum_{i=1}^{\infty} r(y)_{i+k+1} \alpha^{i+k+1} \right) - \left( \sum_{i=1}^{\infty} x_{i+k+1} r(x)_{i+k+1} \alpha^{i+k+1} \right)
\]

\[
= r(x)_{k+1} \alpha^{k+1} \left( \sum_{i=1}^{\infty} x_{i+k+1} r(x)_i \alpha^i \right) > 0. 
\]

(c) Suppose that \( x = \sum_{i=1}^{k} (x_i/2^i) + \sum_{i=k+2}^{\infty} (x_i/2^i) \) and \( y = \sum_{i=1}^{k} (x_i/2^i) + 1/2^{k+1} + (\sum_{i=k+2}^{\infty} y_i/2^i) \), where \( \prod_{i=k+2}^{\infty} x_i = 0 \) and \( \sum_{i=k+2}^{\infty} y_i > 0 \).

Let \( m = \min \{i | y_i = 1, i > k + 1\} \). Thus, we have

\[
F(y) - F(x) = \left( \sum_{i=k+1}^{\infty} y_i r(y)_i \alpha^i \right) - \left( \sum_{i=k+2}^{\infty} x_i r(x)_i \alpha^i \right)
\]

\[
= r(x)_{k+1} \alpha^{k+1} + \sum_{i=k+2}^{\infty} (y_i r(y)_i - x_i r(x)_i) \alpha^i
\]

\[
> r(x)_{k+1} \alpha^{k+1} + r(y)_m \alpha^m - \sum_{i=k+2}^{\infty} r(x)_i \alpha^i
\]

\[
= r(x)_{k+1} \alpha^{k+1} + r(y)_m \alpha^m - r(x)_{k+1} \alpha^{k+1} \sum_{i=1}^{\infty} b_i \alpha^i
\]

\[
= r(y)_m \alpha^m > 0. 
\]

The inequality \((42)\) is satisfied, because \( \prod_{i=k+2}^{\infty} x_i = 0 \).

Therefore, if \( y > x \), then \( F(y) > F(x) \).

Proof of Theorem 3 (ii). Let \( x = \sum_{i=1}^{\infty} (x_i/2^i) \in [0, 1] \), \( y = \sum_{i=1}^{\infty} (y_i/2^i) \in [0, 1] \) \((x_i, y_i \in \{0, 1\})\), and \( x \neq y \). We have \( k \in \mathbb{Z}_{\geq 0} \) such that \( x_i = y_i \) for \( \forall i \leq k \). Hence,

\[
|y - x| \leq \frac{1}{2^k} < (3\alpha)^k := \frac{1 - 3\alpha}{\alpha}. 
\]
Without loss of generality, we assume that \( y > x \).

\[
|F(y) - F(x)| = \left| \sum_{n=k+1}^{\infty} (y_n r(y_n) - x_n r(x_n)) \alpha^n \right| \quad (46)
\]

\[
\leq \sum_{n=k+1}^{\infty} r(y_n) \alpha^n \quad (47)
\]

\[
< \sum_{n=k+1}^{\infty} 3^{n-1} \alpha^n \quad (48)
\]

\[
= \alpha \frac{1}{1 - 3\alpha} (3\alpha)^k = \epsilon. \quad (49)
\]

Since \( F \) is a function on a finite bounded section \([0, 1]\), \( F \) is uniformly continuous.

Proof of Theorem 3 (iii). The function \( F \) is bounded variation, because \( F \) is strictly increasing by Theorem 3 (i). Hence, \( F \) is differentiable almost everywhere on \([0, 1]\) (e.g., [19, Theorem 6.3.3]).

Suppose that \( x \in [0, 1] \) is a differentiable point. For any \( k > 1 \) we have \((y_1, \ldots, y_{k-1})\) such that \( \sum_{i=1}^{k-1} y_i/2^i \leq x \leq (\sum_{i=1}^{k-1} y_i/2^i) + 1/2^k \), and

\[
\frac{F \left( \left( \sum_{i=1}^{k-1} (y_i/2^i) \right) + (1/2^k) \right) - F \left( \sum_{i=1}^{k-1} (y_i/2^i) \right)}{2^{-k}} = 2^k r(x) \alpha^k. \quad (50)
\]

Assuming that the derivative at \( x \) is not zero, the derivative is finite and positive because \( F \) is strictly increasing.

When \( y_k = 0 \),

\[
\frac{2^{k+1} r(x)_{k+1} \alpha^{k+1}}{2^k r(x)_{k} \alpha^k} = \frac{2^{k+1} r(x)_{k} \alpha^{k+1}}{2^k r(x)_{k} \alpha^k} = 2\alpha. \quad (51)
\]

When \( y_{k-1} = 0 \) and \( y_k = 1 \),

\[
\frac{2^{k+1} r(x)_{k+1} \alpha^{k+1}}{2^k r(x)_{k} \alpha^k} = \frac{2^{k+1} (3r(x)_{k}) \alpha^{k+1}}{2^k r(x)_{k} \alpha^k} = 6\alpha. \quad (52)
\]

When \( y_{k-1} = 1, y_k = 1, \) and \( l \geq 2 \), where \( l \) is the length of continuous 1s including \( y_k \),

\[
\frac{2^{k+1} r(x)_{k+1} \alpha^{k+1}}{2^k r(x)_{k} \alpha^k} = \frac{2\alpha(aM_1 u_0)}{aM_{l-1} u_0} = \frac{2\alpha(2^{l+2} + (-1)^{l+1})}{2^{l+1} + (-1)^l} =: D_l. \quad (53)
\]

By a simple calculation we have \( \min_{l \geq 2} D_l = D_2 = 10\alpha/3, \max_{l \geq 2} D_l = D_3 = 22\alpha/5. \)

Hence,

\[
\frac{10\alpha}{3} \leq \frac{2^{k+1} r(x)_{k+1} \alpha^{k+1}}{2^k r(x)_{k} \alpha^k} \leq \frac{22\alpha}{5}. \quad (54)
\]
On the other hand, because $F$ is differentiable at $x$ by Equation (50),
\[
\lim_{k \to \infty} \left( (2^{k+1}r(x)_{k+1}\alpha^{k+1}) - (2^kr(x)\alpha^k) \right) = \lim_{k \to \infty} (K_{y_{k-1},y_k} \alpha - 1)(2^kr(x)\alpha^k) = 0, 
\]
(55)
where $K_{y_{k-1},y_k}$ is 2, 6, or $10/3 \leq K_{y_{k-1},y_k} \leq 22/5$ by Equations (51), (52), and (54). For any $k$ $K_{y_{k-1},y_k}\alpha - 1 \neq 0$, and $\lim_{k \to \infty}(2^kr(x)\alpha^k)$ should be zero.

It contradicts the assumption. We conclude that the derivative at $x$ is zero when $F$ is differentiable at $x$.

**Proof of Theorem 3 (iv).** By properties of $F$ in Theorem 3 (i), (ii), and (iii) it means that the function $F$ is a singular function.

Next, we provide non-differentiable points for $F$. If $x \in (0, 1)$ is a dyadic rational $m/2^i$ for some $m, i \in \mathbb{Z}_{>0}$, $x$ is represented by a finite binary fraction.

**Proposition 3.** If $x \in (0, 1)$ is represented by a finite binary fraction $(\sum_{i=1}^{k-1}(x_i/2^i)) + 1/2^k$ for some $k \in \mathbb{Z}_{>0}$, then $F$ is not differentiable at $x$.

**Proof.** Let $y_m = \sum_{i=1}^{k-1}(x_i/2^i) + \sum_{i=k+1}^{m}(1/2^i)$ for $m > k$ and $x_i \in \{0, 1\}$.

\[
\frac{F(x) - F(y_m)}{x - y_m} = \frac{\sum_{i=m+1}^{\infty} r(x_i)\alpha^i}{\sum_{i=m+1}^{\infty} \alpha^i} 
\]
(56)
\[
= \frac{2^mr(x)\alpha^m}{3} \sum_{i=m+1}^{\infty} b_i\alpha^i 
\]
(57)
\[
= \frac{r(x)\alpha^m}{3} \left( \frac{2(4\alpha)^m}{(2\alpha)^{k+1}(1 - 2\alpha)} + \frac{(-2\alpha)^m}{(-\alpha)^{k-1}(1 + \alpha)} \right) 
\]
(58)
\[
\to +\infty \quad (m \to \infty). 
\]
(59)
Let $z_m = (\sum_{i=1}^{k-1}(x_i/2^i)) + 1/2^k + (\sum_{i=m}^{\infty}(1/2^i))$ for $m \geq k + 2$. On the other hand,

\[
\frac{F(z_m) - F(x)}{z_m - x} = \frac{2^{m-1}r(z_m)\alpha^{m-1}}{2^{m-1}} 
\]
(60)
\[
= \frac{r(x)\alpha^{m-1}}{2^{m-1}} \to 0 \quad (m \to \infty). 
\]
(61)
Hence, $F$ is not differentiable at $x \in (0, 1)$.

### 3.3 Functional equations for $F$

Lastly, we give functional equations that the singular function $F$ satisfies. Because of the self-similarity of the limit set of Rule 150, the function $F$ is self-affine satisfying $F(x) = \alpha F(2x)$ for $0 \leq x \leq 1/2$, and $F(x) = F(x/\alpha)$ for $0 \leq x \leq 1$. Including this equation, we obtain the following result.
Theorem 4.  (i) The singular function $F$ satisfies functional equations

$$F(x) = \begin{cases} 
\alpha F(2x) & \text{if } 0 \leq x < \frac{1}{2}, \\
3F \left( \frac{2x-1}{2} \right) + \alpha & \text{if } \frac{1}{2} \leq x < \frac{3}{4}, \\
F \left( \frac{2x-1}{2} \right) + 2F \left( \frac{4x-3}{4} \right) + \alpha + 2\alpha^2 & \text{if } \frac{3}{4} \leq x \leq 1.
\end{cases} \quad (62a)$$

(ii) The function $F$ is the unique continuous function on $[0, 1]$ that satisfies the upper functional equations, (62a), (62b), and (62c).

Proof of Theorem 4 (i). Let $x = \sum_{i=1}^{\infty} (x_i/2^i)$. If $0 \leq x < 1/2$, then $x_1 = 0$, and we have $2x = \sum_{i=1}^{\infty} (x_i/2^i)$.

$$\alpha F(2x) = \alpha \left( \sum_{i=1}^{\infty} x_{i+1} r(2x) \alpha^i \right) = \sum_{i=1}^{\infty} x_{i+1} (aM_{x_1} M_{x_{i-1}} \cdots M_{x_2} u_0) \alpha^{i+1} \quad (63)$$

$$= \sum_{i=2}^{\infty} x_i (aM_{x_{i-1}} M_{x_{i-2}} \cdots M_{x_2} u_0) \alpha^i = F(x). \quad (64)$$

If $1/2 \leq x < 3/4$, then $x_1 = 1$ and $x_2 = 0$. Since $(2x-1)/2 = \sum_{i=3}^{\infty} (x_i/2^i)$, we have $F((2x-1)/2) = \sum_{i=3}^{\infty} x_i r((2x-1)/2) \alpha^i$. Thus,

$$F(x) - F \left( \frac{2x-1}{2} \right) = \left( \alpha + \sum_{i=3}^{\infty} x_i r(x) \alpha^i \right) - \sum_{i=3}^{\infty} x_i r \left( \frac{2x-1}{2} \right) \alpha^i \quad (65)$$

$$= \alpha + \sum_{i=3}^{\infty} x_i (aM_{x_{i-1}} \cdots M_{x_2} (M_{x_1} - M_0) u_0) \alpha^i \quad (66)$$

$$= \alpha + 2 \sum_{i=3}^{\infty} x_i (aM_{x_{i-1}} \cdots M_{x_2} u_0) \alpha^i \quad (67)$$

$$= \alpha + 2F \left( \frac{2x-1}{2} \right). \quad (68)$$

Equation (68) follows directly from $x_2 = 0$. Hence, we have $F(x) = 3F((2x-1)/2) + \alpha$.

If $3/4 \leq x \leq 1$, then $x_1 = 1$ and $x_2 = 1$. Since $(2x-1)/2 = \sum_{i=2}^{\infty} (x_i/2^i)$, we have $F((2x-1)/2) = \sum_{i=2}^{\infty} x_i r((2x-1)/2) \alpha^i$, and since $(4x-3)/4 = \sum_{i=2}^{\infty} (x_i/2^i)$, we
have $F((4x - 3)/4) = \sum_{i=3}^{\infty} x_i r((4x - 3)/4) \alpha^i$.

$$F(x) - \alpha - 2\alpha^2 = \left( \alpha + 3\alpha^2 + \sum_{i=3}^{\infty} x_i r(x) \alpha^i \right) - \alpha - 2\alpha^2$$  \hfill (70)

$$= \alpha^2 + \sum_{i=3}^{\infty} x_i (aM_{x_{i-1}} \cdots M_{x_3} M_{x_2} M_{x_1} u_0) \alpha^i$$  \hfill (71)

$$= \alpha^2 + \sum_{i=3}^{\infty} x_i (aM_{x_{i-1}} \cdots M_{x_3} (M_1 + 2I_2) u_0) \alpha^i$$  \hfill (72)

$$= \sum_{i=2}^{\infty} x_i (aM_{x_{i-1}} \cdots M_{x_2} u_0) \alpha^i + 2 \sum_{i=3}^{\infty} x_i (aM_{x_{i-1}} \cdots M_{x_3} u_0) \alpha^i$$  \hfill (73)

$$= F\left(\frac{2x - 1}{2}\right) + 2F\left(\frac{4x - 3}{4}\right),$$  \hfill (74)

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof of Theorem 4 (ii). The uniqueness is obtained by showing that the functional equations determine the value for each dyadic rational on $[0, 1]$.

We first obtain $F(0) = 0$ by Equation (62a), and $F(1/2) = \alpha$ by Equation (62b). Thus $F(1/2^i) = \alpha^i$ for $i \in \mathbb{Z}_{\geq 0}$ by Equation (62a), and $F(1) = F(1/2^0) = 1$ by Equation (62c). Calculating $F(3/4)$, we obtain $F(3/2^i)$ for $i \geq 2$, and calculating $F(5/8)$ and $F(7/8)$, we obtain $F(5/2^i)$ and $F(7/2^i)$ for $i \geq 3$. Accordingly, iterating the same procedure for $i \geq 4$, we determine $F(m/2^i)$ for each dyadic rational point $m/2^i$ on $[0, 1]$.

Since the dyadic rationals on $[0, 1]$ are dense, there is a unique continuous function.

4 Conclusions

In this paper, a function on a unit interval has given by the dynamic of the one-dimensional elementary cellular automaton Rule 150. Since the limit set of Rule 150 holds a self-similarity, the resulting function is self-affine. We have shown that the function is strictly increasing, uniformly continuous, and differentiable almost everywhere, that means it is a singular function. We also have given the functional equations that the singular function satisfies, and we have proven that the function is the only solution of the functional ones.

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