A type of localized quadripartite entanglement

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(Dated: May 7, 2014)

In this paper, we show that the average three-tangle of the reduced tripartite density matrix for some quadripartite pure states can be increased by some potential measurements on the fourth subsystem, which means this type of quadripartite entanglement can be localized. In particular, we prove that the maximal increment with all potential measurements taken into account is a quadripartite entanglement monotone, so it quantifies this localized quadripartite entanglement. By analyzing quadripartite pure states based on the previous classification, we find that this quadripartite entanglement monotone is not only present in the standard GHZ state and absent in the standard W states. In addition, based on the proposed entanglement measure, we construct a new monogamy relation.

PACS numbers: 03.67.Mn, 03.65.Ud

I. INTRODUCTION

Quantum entanglement is an important physical resource in quantum information processing [1]. Quantification of entanglement is one of the most important tasks in the quantum information theory. However, although it has been more than ten years since the first several remarkable works on the bipartite entanglement [2–5], we can safely say that the good understanding of quantum entanglement is still restricted to bipartite low-dimensional systems (see the Ref. [1] and their references). The quantification of the multipartite entanglement remains an open question [6].

One of the important difficulties in the quantification of multipartite entanglement is that multipartite quantum entanglement can be classified into different inequivalent classes [7, 8], which implies that 1) in general a single quantity is not enough to characterize the multipartite quantum entanglement; 2) there exist different approaches to classifying the multipartite entanglement. However, there isn’t to say that a single quantify is useless, because we can always use a single quantity to quantify the entanglement of some given types. The most obvious example is that, although tripartite quantum pure states of qubits can be entangled in two different ways, 3-tangle is a quite successful measure of GHZ type entanglement [9]. In addition, a good entanglement measure cannot only distinguish the given entangled state from others (in usual, the separable states), but also should be an entanglement monotone which is not increased under stochastic local operations and classical communications (SLOCC) [10]. Therefore, although many authors have classified the multipartite quantum states into various inequivalent classes, they can not provide a good and direct entanglement measure for each class of quantum states [11–20]. In particular, in order to measure the GHZ type entanglement of multipartite quantum states, some authors defined an n-tangle for multipartite quantum pure states to generalize the 3-tangle [21]. However, it is regretful that for quadripartite quantum states of qubits, the so-called n-tangle has value one for the product of two Bell states, even though the n-tangle is an entanglement monotone.

In usual, the classification of multipartite entanglement is based on the SLOCC. In this sense, tripartite pure states of qubits can be classified into GHZ state and W state [7] and quadripartite pure states of qubits can be classified into nine families [8]. That means that a general quantum quadripartite pure state can be converted into the standard quantum state of the particular family by SLOCC. However, a particular characteristic of the tripartite GHZ states and the tripartite W states is that some measurements on one qubit of GHZ state could increase the average entanglement of the residual two qubits. But one cannot find any measurements on one qubit of W state such that the average entanglement of the residual two qubits is increased. In other words, the tripartite entanglement in GHZ state can be localized to the bipartite entanglement if the SLOCC is available, but the tripartite entanglement in W state has no such property. Can we consider the quadripartite entanglement in the same manner?

In this paper, we consider the above question by studying the quadripartite quantum pure states of qubits. Based on whether the quadripartite entanglement can be localized under SLOCC, i.e. whether some measurements can increase the average GHZ type entanglement of the residual three qubits, we propose a quadripartite entanglement monotone for (2 ⊗ 2 ⊗ 2 ⊗ n)-dimensional quantum pure states. Comparing to the conventional classification of quadripartite quantum pure states of qubits, we find that not only the standard quadripartite GHZ state can be localized and the W state is not the unique quadripartite entangled state which can not be localized. In addition, based on the quadripartite entanglement measure, we find an interesting monogamy relation. This paper
is organized as follows. In Sec. II, we briefly introduce the fundamental definitions of various entanglement that will be used in the paper. In Sec. III, we give our entanglement measure and prove that it is an entanglement monotone. In Sec. IV, compared with the previous classification, we analyze our entanglement measure on each family. In Sec V, we give a new monogamy relationship for quadripartite pure states. Finally, the conclusion is drawn.

II. DEFINITIONS

The three-tangle $\mu_3(\psi)$ for a tripartite pure state of qubits $|\psi\rangle = \sum_{i,j,k=0}^1 a_{ijk} |ijk\rangle$ can be given by Eq. (2)

$$\mu_3 = 4|d_1 - 2d_2 + 4d_3|,$$

with

\begin{align*}
  d_1 &= a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2, \\
  d_2 &= a_{000}a_{111}a_{001}a_{110} + a_{001}a_{110}a_{010}a_{101} + a_{010}a_{101}a_{000}a_{111} + a_{100}a_{011}a_{110}a_{001}, \\
  d_3 &= a_{000}a_{110}a_{011}a_{100} + a_{011}a_{100}a_{010}a_{110} + a_{100}a_{011}a_{110}a_{001} + a_{011}a_{110}a_{100}a_{001}.
\end{align*}

It is easy to find that $\tau_3 = \sqrt{\mu_3}$ is also a good entanglement measure. Thus for a tripartite mixed state $\rho$, the 3-tangle can be defined, based on the convex-roof-construction, as

$$\tau_3(\rho) = \min_{\{p_i, \pi_i\}} \sum_i p_i \tau_3(\pi_i),$$

with $\{p_i, \pi_i\}$ is a possible realization of $\rho$. In particular, if for a quadripartite $(2 \otimes 2 \otimes 2 \otimes n)$-dimensional quantum pure state $|\psi\rangle_{ABCS}$, $\rho = Tr_S|\psi\rangle_{ABCS}\langle\psi|$, we can define the 3-tangle of assistance as

$$\tau_a(\rho) = \max_{\{p_i, \pi_i\}} \sum_i p_i \tau_3(\pi_i).$$

Since for a tripartite quantum pure state of qubits $|\phi\rangle$ has

$$\tau_3(A \otimes B \otimes C |\phi\rangle) = |\det (A)| |\det (B)| |\det (C)| \tau_3(|\phi\rangle),$$

holds for any local operations $A$, $B$ and $C$, a similar proof as Ref. [22] can show that the three-tangle of assistance defined in Eq. (4) is a quadripartite entanglement monotone.

III. QUADRIPARTITE ENTANGLEMENT MONOTONE

Let $|\psi\rangle_{ABCS}$ be a quadripartite $(2 \otimes 2 \otimes 2 \otimes n)$-dimensional quantum pure state, $\rho_{ABC} = Tr_S|\psi\rangle_{ABCS}\langle\psi|$, based on the definitions we can easily find that $\tau_a(\rho_{ABC}) \geq \tau_3(\rho_{ABC})$. Define

$$\tau_4(|\psi\rangle_{ABCS}) = \tau_4(\rho_{ABC}) = \sqrt{\tau_3^2(\rho_{ABC}) - \tau_3^2(\rho_{ABC})}$$

we can find that $\tau_4(|\psi\rangle_{ABCS})$ has the following properties.

i) $\tau_4(|\psi\rangle_{ABCS})$ is invariant under determinant-one operations on subsystems $A$, $B$ and $C$.

ii) $\tau_4(|\psi\rangle_{ABCS})$ is a concave function on $\rho_{ABC}$.

One can find that the property i) is very obvious based on Eq. (4). Now we will prove the property ii). Let $\lambda \in [0, 1]$ and $\rho_1, \rho_2$ be the tripartite quantum states of qubits [23], then

$$\lambda \tau_4(\rho_1) + (1 - \lambda) \tau_4(\rho_2) \geq \sqrt{\lambda \tau_4(\rho_1)^2 + (1 - \lambda)^2 \tau_4(\rho_2)^2},$$

where the first inequality holds based on Cauchy-Schwarz inequality, and the second inequality holds based on the definition of $\tau_a$ and $\tau_3$.

Theorem 1. $\tau_4(|\psi\rangle_{ABCS})$ is a quadripartite entanglement monotone. It can measure one type of localized quadripartite entanglement.

Proof. We will first prove $\tau_4(|\psi\rangle_{ABCS})$ does not increase under SLOCC operations. We suppose the pure state is shared by $A$, $B$, $C$ and $S$. Let us consider the following four-way SLOCC. Firstly, $S$ performs a measurement denoted by the Kraus operators $M_l$ on the qudit $S$ and send the result $i$ to others; Secondly, based on the result of $S$, $A$ performs a measurement $A_{ij}$ on qubit $A$ and send his result $j$ to others; Thirdly, based on the previous results of $A$ and $S$, $B$ performs a measurement $B_{ik}$ on qubit $B$ and send the result $k$ to others, and similarly, based on the results of $A$, $B$ and $S$, $C$ performs a measurement $C_{ij}$ on qudit $C$ and send the result $l$ to others; Finally, based on all the previous results, $S$ performs a measurement $F_{ijkl}$ on qudit $S$ and send his result $m$ to others. After the SLOCC operations, the initial state $|\psi\rangle_{ABCS}$ will become a mixed state as $|\psi\rangle_{ijkl}$, where

$$|\psi\rangle_{ijkl} = |\psi\rangle_{ABCS} \otimes A_{ij}^{\dagger} \otimes B_{ik}^{\dagger} \otimes C_{ij}^{\dagger} \otimes F_{ijkl} M_l^{\dagger} \|\psi\rangle_{ABCS},$$

and $N_{ijkl}$ is the corresponding probability. Therefore,
we can obtain the average $\tau_4$ as

$$\sum_{ijklm} N_{ijklm} \tau_4 (|\psi\rangle_{ABCD}) = \sum_{ijklm} D_{ijklm} \tau_4 \left( F_{ijklm} M_i |\psi\rangle_{ABCS} \right)$$

$$= \sum_{ijklm} D_{ijklm} \left[ T_{RS} F_{ijklm} M_i |\psi\rangle_{ABCS} \langle \psi | M_i^\dagger (F_{ijklm})^\dagger \right],$$

(9)

with $D_{ijklm} = \det(A_i^j) \det(B_k^j) \det(C_l^j)$. Let $\tilde{\rho}_i = T_{RS} M_i |\psi\rangle_{ABCS} \langle \psi | M_i^\dagger$. Since the reduced density matrix seen from A, B and C cannot be changed by the local operations $F_{ijklm}$ alone, we have $\tilde{\rho}_i = \sum_m \sigma_m$, with $\sigma_m = T_{RS} F_{ijklm} M_i |\psi\rangle_{ABCS} \langle \psi | M_i^\dagger (F_{ijklm})^\dagger$. Based on the property ii), one can easily find that the right hand side (r.h.s.) of Eq. (9) can lead to the following inequality:

$$r.h.s. \leq \sum_{ijklm} D_{ijklm} \left[ T_{RS} M_i |\psi\rangle_{ABCS} \langle \psi | M_i^\dagger \right],$$

(10)

According to the geometric-arithmetic inequality: $\sum_{j} \left| \det A_j^i \right| \leq \frac{1}{2} \sum_{j} \text{Tr} A_j^i A_j^i = 1$ and the similar inequalities for $B_k^j$ and $C_l^j$, we can get

$$r.h.s. \leq \sum_i \tau_4 \left[ T_{RS} M_i |\psi\rangle_{ABCS} \langle \psi | M_i^\dagger \right].$$

(11)

where the second inequality holds similar to the Eq. (9). Eq. (11) shows that $\tau_4$ is a quadripartite entanglement monotone (similar as Ref. [22]).

In addition, we claim that our entanglement monotone is a quadripartite entanglement measure, we will prove that $\tau_4$ vanishes for any separable state. A quadripartite separable quantum pure state can be given as $|\varphi_1\rangle_{ABCD} = |\chi_1\rangle_A \otimes |\gamma_1\rangle_{BCD}$, $|\varphi_2\rangle_{ABCD} = |\chi_2\rangle_{AB} \otimes |\gamma_2\rangle_{BCD}$ and $|\varphi_3\rangle_{ABCD} = |\chi_3\rangle_{ABC} \otimes |\gamma_3\rangle_D$, where the state $|\chi_i\rangle$ with multiparticle subscripts out of the ket denotes a general multipartite quantum pure state, so a completely separable state is also included. For $|\varphi_1\rangle_{ABCD}$, if we trace out the subsystem $D$, the final reduced density can be given by $\sigma_{1BCD} = |\chi_1\rangle_A \langle \chi_1 | \otimes \sigma_{BCD}$. Hence, we can find $\tau_4 (|\varphi_1\rangle_{ABCD}) = 0$, which shows $\tau_4 (|\varphi_1\rangle_{ABCD}) = 0$. Similarly, one can find that $\tau_4 (|\varphi_2\rangle_{ABCD}) = 0$. For $|\varphi_3\rangle_{ABCD}$, when we trace out the subsystem D, we can obtain the reduced density matrix as $\sigma_{ABC} = |\chi_3\rangle_{ABC} \langle \chi_3 |$. So it is easy to find that $\tau_4 (|\varphi_3\rangle_{ABCD}) = 0$, which also means that $\tau_4 (|\varphi_3\rangle_{ABCD}) = 0$.

One can directly find that our Eq. (6) actually implies a relationship of monogamy, which shows that $\tau_4$ can lead to the increasing of the average 3-tangle (the maximum is the 3-tangle of assistance). The definition of $\tau_4$ obviously shows that the entanglement quantified by $\tau_4$ can be localized.

IV. ANALYZE THE QUANTUM STATES OF DIFFERENT FAMILIES USING THE ENTANGLEMENT MEASURE

We have shown that $\tau_4$ can quantify one type of localized quadripartite entanglement which is the generalization of some property of the tripartite GHZ state. After all, there are not only the GHZ state for quadripartite states. So in this section, we would like to discuss which states of four qubits hold this type of so-called localized entanglement. As we know, quadripartite quantum pure state of qubits has nine families up to the permutation of the qubits in the sense of SLOCC. The standard state of each family can be given as follows [8].

$$G_{abcd} = \frac{a + d}{2} (|0000\rangle + |1111\rangle) + \frac{a - d}{2} (|0011\rangle + |1100\rangle),$$

$$+ b + c (|0101\rangle + |1010\rangle) + \frac{b - c}{2} (|1001\rangle + |0110\rangle),$$

$$L_{abc} = \frac{a + b}{2} (|0000\rangle + |1111\rangle) + \frac{a - b}{2} (|0011\rangle + |1100\rangle) + c (|0101\rangle + |1010\rangle) + |0110\rangle + |0011\rangle,$$

$$L_{ab} = \frac{a + b}{2} (|0000\rangle + |1111\rangle) + \frac{a - b}{2} (|0011\rangle + |1100\rangle) + \frac{i}{2} (|0001\rangle + |0100\rangle + |1011\rangle + |1101\rangle),$$

$$L_{a} = \frac{a + d}{2} (|0000\rangle + |1111\rangle) + \frac{a - d}{2} (|0101\rangle + |1010\rangle + |1101\rangle + |0110\rangle),$$

$$L_{0000} = |0000\rangle + |1111\rangle + |0011\rangle + |1100\rangle + |1010\rangle + |0110\rangle,$$

$$L_{0011} = |0000\rangle + |1100\rangle + |1010\rangle + |0110\rangle.$$
in this case, $L_{a_4}$ is not separable, but $L_{a_2 | a_3}$ is separable. In addition, we can find that $L_{a_2 | a_3}$ and $L_{a_2 | a_3, a_3}$ have no this type of localized entanglement, even though $L_{a_2 | a_3}$ is not separable. On the contrary, $L_{a_2 | a_3}$ always has nonzero the localized quadripartite entanglement.

Thus we have divided the nine families into two parts based on our requirements. However, one could ask a natural question whether our measure depends on the permutation of qubits. It is unfortunate that our definition strongly depends on which qubit is traced out. A simple example is the state $L_{a_2 | a_3}$. One can find that $\tau_4$ for this state will vanish if we trace out the second, the third or the fourth qubit, even though we have shown it is not the case if we trace out the first qubit. Therefore, from the point of entanglement measure of view, we have to use a vector to completely characterize the type of localized entanglement of a quadripartite pure state. Here one can define an entanglement vector as $\tau_4 = [\tau_{ABCD}, \tau_{AB|CD}, \tau_{ABC|D}, \tau_{ABCD}]$, with bracket in subscripts denoting trace over the corresponding subsystem. Based on the entanglement vector, one can further consider what type the entanglement of a quadripartite pure state is.

V. A NEW MONOGAMY

Besides the definition which has presented a monogamy relation, we can find another new monogamy. For a pure state $|\phi\rangle_{ABCD}$, consider an optimal decomposition $\{p_i, |\psi_{BCD}\rangle\}$ of $\rho_{BCD} = Tr_A |\phi\rangle \langle \phi|$ such that $\tau_{a}(\rho_{BCD}) = \sum_i p_i \tau_3(|\psi_{BCD}\rangle_i)$, we will have

$$\tau_{(A)BCD} = \sqrt{\sum_i p_i \tau_3(|\psi_{BCD}\rangle_i)}^2 - \tau_3^2(\rho_{BCD}). \quad (12)$$

Since for each tripartite pure state $|\psi_{BCD}\rangle$, we have $\tau_3(|\psi_{BCD}\rangle) = \sqrt{C_3^2(\rho_{CD}) - C_2^2(\rho_{CD})}, \quad (13)$

where $\rho_{CD} = Tr_B |\psi_{BCD}\rangle \langle \psi_{BCD}|$ and $C(\rho_{CD})$ is the concurrence and $C_n(\rho_{CD})$ is the concurrence of assistance. Substitute Eq. (13) into Eq. (12), we will obtain

$$\tau_{(A)BCD}^2 + \tau_3^2(\rho_{BCD}) = \left[ \sum_i p_i \sqrt{C_3^2(\rho_{CD}) - C_2^2(\rho_{CD})} \right] \geq C_3^2(\rho_{CD}) - C_2(\rho_{CD}) \quad \geq \tau_3^2(|\phi\rangle_{AB|CD}) \quad (14)$$

where $\tau_3(|\phi\rangle_{AB|CD})$ denotes the 3-tangle by considering subsystems A and B as one party. In addition, the inequality holds based on Cauchy-Schwarz inequality. This inequality is not trivial, because an intuition observation can show that the inequality can be saturated if $|\phi\rangle_{AB|CD}$ is the standard quadripartite GHZ state $|\text{GHZ}_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$. Thus Eq. (14) provides us with a new inequality of monogamy. Analogously, we can also obtain other type inequalities of monogamy by considering the trace over different subsystems. These inequalities have the similar nature, so we omit them.

VI. CONCLUSIONS AND DISCUSSION

In this paper, we have found a quadripartite entanglement monotone which quantifies one type of localized quadripartite entanglement. That is, for one kind of quadripartite quantum pure state, the proposed localized quadripartite entanglement can converted to 3 subsystems by some measurements on the fourth subsystem. Along this line, we find that not only the standard GHZ state of four quits have nonzero localized entanglement and not only the standard W state has no localized entanglement, which is quite different from the case of tripartite pure states. However, it is actually consistent with that quadripartite quantum entanglement can be classified into more than 2 classes. In addition, a new monogamy besides the definition itself has also been proposed. At last, we would like to emphasize it is a quite interesting question what kind of other monogamy relations can be constructed based on the current localized entanglement measure or some others similar.

VII. ACKNOWLEDGEMENT

This work was supported by the National Natural Science Foundation of China, under Grant No. 10805007 and No. 10875020, and the Doctoral Startup Foundation of Liaoning Province.

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