A discussion of vacuum polarization correction to the cross section of
\[ e^+e^- \rightarrow \gamma^*/\psi \rightarrow \mu^+\mu^- \]

Jin Hong-Dou\(^1\,2\)) Zhou Li-Peng\(^1,2,1\)) Zhang Bing-Xin\(^2\) Hu Hai-Ming\(^2,1,2\))

\(^1\) University of Chinese Academy of Science, Beijing, 100049, China
\(^2\) Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China

Abstract: The vacuum polarization is a constituent of the initial state radiation correction for the cross section of the \( e^+e^- \) annihilation processes. In the energy region in vicinity of the narrow resonances \( J/\psi \) and \( \psi(3686) \) the vacuum polarization contribution from the resonant component has a significant effect to the line-shape of lepton pair production cross section. This paper discusses some basic concepts and gives an analytical calculation to the cross section of the \( e^+e^- \rightarrow \gamma^*/\psi \rightarrow \mu^+\mu^- \) considering the single and double vacuum polarization effect of the virtual photon propagator, and presents some numerical comparisons with the traditional treatments.

Key words: electron-positron annihilation, resonance, vacuum polarization, cross section.

PACS: 78.70.Bj, 13.20.Gd, 12.15.Lk

1 Introduction

The calculations of the cross sections of \( e^+e^- \rightarrow e^+e^- \) and \( \mu^+\mu^- \) to the first order \( \alpha^3 \) \((O(\alpha) \sim 1\%)\) corrections have been studied decades ago\(^4\). Usually, the contributions from the high level Feynman diagrams are called the radiative corrections, which include: vertex correction, vacuum polarization, electron self-energy and bremsstrahlung, et al.

For the theoretical calculation up to \( \alpha^3 \), the radiation correction terms are the interference between tree level and higher level one-loop Feynman diagrams. In the references mentioned above, all of the correction terms are treated as small quantities compared to the tree level term. Such approximations for the QED correction and nonresonant QCD hadronic correction are reasonable. But in the energy regions around narrow resonances, such as the charmonium \( J/\psi \) and \( \psi(3686) \), the contribution of hadronic resonant component to the vacuum polarization (VP) correction is no longer a small quantity and nor smooth function about the energy, the energy dependence of VP correction factor has a significant influence on the line shape of the total cross section of processes \( e^+e^- \rightarrow e^+e^- \) and \( \mu^+\mu^- \). Therefore, the VP correction has to be treated properly.

For the QED process \( e^+e^- \rightarrow \mu^+\mu^- \), the radiation corrections include the initial state and final state corrections. The final state radiation (FSR) correction is usually smaller than that of the initial state radiation (ISR) correction due to the mass relation \( m_e \ll m_\mu \), if one dose not seek very high accuracy, FSR correction can be neglected. The contributions of the two-photon-exchange diagrams and the asymmetry about \( e^\pm \) and \( \mu^\pm \) are less important. In this paper, only ISR correction of the process \( e^+e^- \rightarrow \mu^+\mu^- \) is considered in order to let the discussion succinct, and the discussions only concentrate on the effect of VP correction, other correction terms cite the expression given in the related references.

In general, the total cross section of a process in \( e^+e^- \) annihilation is measured with experimental data:

\[ \sigma_{ex}^{tot}(s) = \frac{N_{sig}}{L\epsilon}, \]

where \( N_{sig} \) is the number of signal events, \( L \) the integrated luminosity of data samples, and \( \epsilon \) the detection efficiency for \( e^+e^- \rightarrow \gamma\text{+signal} \) (the signal is \( \mu^+\mu^- \) in this paper). But the quantity interested in physics is the Born cross section \( \sigma_{ex}^{0} \), which relates to the \( \sigma_{ex}^{tot} \) by the ISR correction factor \((1+\delta)\) in following way:

\[ \sigma_{ex}^{0}(s) = \frac{\sigma_{ex}^{tot}(s)}{1+\delta(s)}. \]
In fact, the factor $1+\delta(s)$ indicates the fraction all of the higher order Feynman diagrams by the definition:

$$1+\delta(s) = \frac{\sigma_{\text{tot}}(s)}{\sigma_{\text{th}}(s)},$$

(3)

where $\sigma_{\text{th}}(s)$ and $\sigma_{\text{tot}}(s)$ are the theoretical Born cross section and total cross sections. It is clear that the correct theoretical calculation of $1+\delta(s)$ is a key factor for obtaining the accurate value of $\sigma_{\text{ex}}(s)$ from the direct measurement of $\sigma_{\text{ex}}(s)$. In order to get the consistent results of $\sigma_{\text{ex}}(s)$ and $\sigma_{\text{th}}(s)$, the theoretical calculation and experimental measurement is an iterative procedure. The following discussions concentrate on the application, not the pure theoretical study.

The outline of this paper is as follows. In section 2, the related Born cross sections are presented. In section 3, the general VP corrections to the virtual photon propagator described in the text books and references is reviewed. In section 4, the effective experiment leptonic widths with different definitions are reviewed. In section 5, the properties of the VP modified Born cross sections are given and the line-shape are shown numerically. In section 6, the analytical expressions of the total cross section of $e^+e^-\rightarrow\mu^+\mu^-$ with single and double VP corrections are deduced, and numerical results are shown. The section 7 gives some discussions and comments.

2 Born cross section

In the energy region of vicinity of the resonance $\psi$ with $J^{PC}=1^{--}$, the $\mu^+\mu^-$ pair can production via two channels in the $e^+e^-$ annihilation:

$$e^+e^- \Rightarrow \begin{cases} \gamma^* \\ \psi \end{cases} \Rightarrow \mu^+\mu^-,$$

where the mode via virtual photon $\gamma^*$ is the direct electro-magnetic production, and another is the electro-magnetic decay of intermediate resonant state $\psi$. The tree level Feynman diagram for this process is the coherent summation of following two diagrams:

![Fig. 1. The tree level Feynman diagram for $e^+e^-\rightarrow\mu^+\mu^-$.](image)

The Born cross section for $\gamma^*$ channel $e^+e^-\rightarrow\gamma^*\rightarrow\mu^+\mu^-$ corresponding to the left diagram of Fig. 1 is

$$\sigma_{\gamma^*}^0(s) = \frac{4\pi\alpha^2}{3s},$$

(4)

where $s$ is the square of center-of-mass energy of the initial state $e^+e^-$, $\alpha$ the fine structure constant. And for the channel via the intermediate resonance $\psi$ decay, the cross section of $e^+e^-\rightarrow\psi\rightarrow\mu^+\mu^-$ corresponding to the right diagram of Fig. 1 is then

$$\sigma_{\psi}^0(s) = \frac{4\pi\alpha^2}{3s}|A_{BW}|^2, \quad A_{BW} = \frac{Fr}{\Delta+iFr},$$

(5)

the following notations are used in the Breit-Wigner amplitude $A_{BW}$:

$$\Delta = \frac{s-M^2}{M^2} = t-1, \quad t = \frac{s}{M^2};$$

$$r = \frac{\Gamma}{M};$$

$$F = \frac{3\sqrt{\alpha}M\Gamma_f}{\alpha M} = \frac{3}{\alpha}\sqrt{tB_tB_f}.$$
where $M$ is the nominal mass (peak position of the Breit-Wigner function) of the resonance $\psi$, $\Gamma$ the total width, $\Gamma^0$ the theoretical (bare) electronic width, $\Gamma_f$ the final state width, $B_e$ and $B_f$ the branch fractions of $\psi$ decay to $e^+e^-$ and $f$ respectively. For the final state $\mu^+\mu^-$, $\Gamma_f = \Gamma_{\mu^0}$, and lepton universality means $\Gamma_{\mu^0} = \Gamma^0$ under the limit $m^2 \leq \Gamma \rightarrow 0$. The combination parameter $F$ is such taken that Eq. (5) gives correct Breit-Wigner cross section.

Starting with the Van Royen-Weisskopf formula, the $\Gamma^0_e$ can be expressed by following formula \[6\]:

$$
\Gamma^0_e = \frac{16}{3} \pi \alpha^2 e^2 N_c \frac{|R(0)|^2}{M^2} (1 - \frac{16 \alpha^4}{3 \pi}),
$$

(9)

where $e_c$ is the charge of charm quark in the unit of the electron charge, $N_c$ the number of colors, $\alpha_s$ the strong coupling constant evaluated at $s = M^2$, and $R(0)$ the radial wave function of $\psi$ at origin. Some phenomenological models can give rough estimation for the value of $R(0)$, but its accurate value has to be measured by experiment.

The total production amplitude of $\mu^+\mu^-$ via above two channels should be coherently summed,

$$
A_{eff} = 1 + \frac{F \Re e^i\delta}{\Delta + i\Gamma},
$$

(10)

where $\delta$ is the relative phase angle between the two channels. The total Born cross section can be written as:

$$
\sigma^0(s) = \frac{4 \pi \alpha^2}{3s} |A_{eff}|^2.
$$

(11)

### 3 Vacuum polarization correction

In the point of view of quantum field theory, two charged particle interact via exchanging the quanta of electromagnetic field, which corresponds to the virtual photon propagator between two charges in the presentation of the Feynman diagram. In other word, a factor $e$ lies at each end of the photon propagator. The VP corrections modify the photon propagator; this effect is equivalent to the coupling strength between charges is changed. In the 1-particle-irreducible (1PI) chain approximation, the infinite series of 1PI diagrams are summed, and then the photon propagator is modified by VP correction in following way \[6\]:

$$
\frac{-ig_{\mu \nu}}{q^2} \rightarrow \frac{-ig_{\mu \nu}}{q^2[1 - \Pi(q^2)]},
$$

(12)

where $g_{\mu \nu}$ is the metric tensor, the $\Pi(q^2)$ the VP function. For the $e^+e^-$ annihilation process, the transferred squared 4-momentum $q^2$ equals to the square of the center-of-mass energy, $q^2 = s$. After the charge renormalization, the effect of the VP correction can be explained as the energy-independent electro-magnetic fine structure constant $\alpha$ is replaced by the effective energy-dependent running coupling factor:

$$
\alpha \rightarrow \alpha(s) = \frac{\alpha}{[1 - \Pi(s)]}.
$$

(13)

In general, the Born cross section of QED processes are proportional to $\alpha^2$, the running coupling $\alpha(s)$ leads to a new energy-dependence of the cross section.

The VP function $\Pi(s)$ can be conveniently written as the following sum according to the origin of VP \[10, 12\]:

$$
\Pi(s) = \hat{\Pi}_{QED}(s) + \hat{\Pi}_{QCD}(s) + \hat{\Pi}_{res}(s),
$$

(14)

where $\hat{\Pi}_{QED}(s)$ includes the contributions of $e^+e^-$, $\mu^+\mu^-$ and $\tau^+\tau^-$ loops \[13\], and $\hat{\Pi}_{QCD}$ the hadronic nonresonant contribution. The term $\hat{\Pi}_{QCD}$ can be expressed by the dispersion relation \[14, 15\], or by the equivalent analytic expression of quasi-fermion-pairs $q\bar{q}$ with a QCD modification factor \[16\]. The resonant part of the VP function has following form \[12\]:

$$
\hat{\Pi}_{res}(s) = \frac{3s\Gamma^0_e}{\alpha M^2} \frac{1}{s - M^2 + i\Gamma} = \frac{h}{\Delta + i\Gamma},
$$

(15)

where the constants

$$
h = \frac{3s\Gamma^0_e}{\alpha M^2} = \frac{3}{\alpha} \Gamma^0, \quad r_e = \frac{\Gamma^0_e}{M},
$$

(16)
For the narrow intermediate resonant states $J/\psi$ and $\psi(3686)$, $\Gamma \ll M$, numerical calculations show that the value of $\hat{\Pi}_{res}(s)$ is very sensitive to that of $\Gamma_0$ and $\Gamma$, which means that $\Gamma_0$ and $\Gamma$ will influence the cross section of $e^+e^- \rightarrow \mu^+\mu^-$ significantly, and determining the values of these two parameters with high accuracy is very important.

Fig.2 shows that around $J/\psi$ and $\psi(3686)$, the running coupling constant $\alpha(s)$ is a sensitive function about $s$. The minimum and maximum values of $\alpha(s)$ are not at the position the nominal mass $M$ of the resonance. The VP correction or $\alpha(s)$ will deform the intrinsic line-shape of Breit-Winger function.

It should be stressed that the partition of Eq.(14) is only a convenient algorithm, not real physical implication. Writing the total VP correction in the form of the summation of the three terms dose not mean these terms correspond real observable states in experiment. In fact, VP means virtual fluctuations of quantum vacuum, and the expression Eq.(14) only sum up the contributions of the chain approximation graphs, and there are any number of bubbles graphs are neglected. These complex graphs originally should be taken into account in the VP corrections, but they are not be included in Eq.(14) [13]. So, the resonant term in VP function dose not equal to the real resonant states measured in experiments, the QED and QCD terms are similar too. Making clear this concept is important for the analysis and calculation.

4 Effective leptonic width

In all most of the related references, the value of the electronic width in the Breit-Wigner function takes the experimental partial width, i.e. the VP effect is absorbed into the theoretical electronic width, but there are two different conventions. In the reference [16], the experimental electronic width is defined as:

$$\Gamma^\text{ex}_e = \frac{\Gamma_0^e}{|1-\hat{\Pi}(M^2)|^2}, \quad (17)$$

which means that for every points, no matter the true energies are at peak or off-peak of the resonance, the value of the VP function take only its peak value. In reference [12], the following definition is adopted:

$$\Gamma^\text{ex}_e = \frac{\Gamma^0_e}{|1-\hat{\Pi}_0(M^2)|^2}, \quad \hat{\Pi}_0(s) = \hat{\Pi}_\text{QED}(s) + \hat{\Pi}_\text{QCD}(s), \quad (18)$$

this convention means that the electronic width absorbs only the contribution of nonresonant parts, the resonant component of VP correction is absorbed into $M$ and $\Gamma$. The conventions Eq.(17) and Eq.(18) are equivalent for calculation results.

Considering the application in scan experiment, this work will discuss a natural definition:

$$\Gamma^\text{ex}_e = \frac{\Gamma^0_e}{|1-\hat{\Pi}(s)|^2}, \quad (19)$$

which is similar to Eq.(17) in form, but the energy scale of VP factor uses the actual value $s$, not fixed at peak $M^2$. For the line-shape scan experiment, some energy points are not at $\psi$ position, adopting definition Eq.(19) will be natural and convenient. Following sections will discuss the energy dependence of the cross section under the definition Eq.(19).
5 VP modified Born cross section

From the point of view of Feynman diagram, the VP correction modifies photon propagator. This effect can be understood from another side, i.e. VP effect modifies the fine structure constant $\alpha$ to the running coupling constant $\alpha(s)$. In this section, the single and double VP effects will be discussed, and their differences will be compared numerically.

5.1 VP modified cross section of $\gamma^*$ channel

The Born cross section $\sigma^0_\gamma(s)$ of $\gamma^*$ channel expressed by Eq.(4) is a smooth function about energy. When the VP correction is applied on it in the following way:

$$\tilde{\sigma}^0_\gamma(s) = \sigma^0_\gamma(s) \left| \frac{1}{1 - \Pi(s)} \right|^{-2},$$

the line-shape of $\tilde{\sigma}^0_\gamma(s)$ appears obvious resonant structure. The Fig.3 shows the line-shape of $\sigma^0_\gamma(s)$ by Eq.(4) and $\tilde{\sigma}^0_\gamma(s)$ by Eq.(20). Obviously, the resonant structure of $\tilde{\sigma}^0_\gamma(s)$ is due to the VP effect. In other word, the resonant shape of $\gamma^*$ channel cross section does not mean real resonant state $J/\psi$ or $\psi(3686)$ produced.

In the vicinity of narrow resonance, both full Born cross section $\sigma^0(s)$ expressed by Eq.(11) and VP function $\Pi(s)$ are the sensitive function of energy. So, the energy dependence of the effective cross section $\tilde{\sigma}^0(s)$ is not only determined by $\sigma^0(s)$, but also by $\Pi_{\text{res}}(s)$. Therefore, in the calculation of the line-shape of cross section around $J/\psi$ and $\psi(3686)$ the VP correction of the electronic width should not take their values at peak like did in [16] or [12], nor expand Eq.(15) into series and only take the first two terms like did in [10, 11].

5.2 VP modified cross section of $\psi$ channel

Usually, the cross section of a resonance is expressed by the Breit-Wigner form, see Eq.(5). If the value of leptonic width still takes $\Gamma^0_e$, the effective Breit-Wigner cross section is modified by the VP correction. The reference [16] used the convention for VP correction described in section 4, the VP modified Breit-Wigner form is,

$$\tilde{\sigma}^0_\psi(s) = \frac{\sigma^0_\psi(s)}{|1 - \Pi(M^2)|^2}.\quad(21)$$

It is clear that the numerator and denominator use different energy scales. In fact, a natural VP correction for Breit-Wigner cross section should be the original form,

$$\tilde{\sigma}^0_\psi(s) = \frac{\sigma^0_\psi(s)}{|1 - \Pi(s)|^2}.\quad(22)$$

Fig.4 shows the line-shape of $\sigma^0_\psi(s)$ defined by Eq.(5) and $\tilde{\sigma}^0_\psi(s)$ defined by Eq.(21) and Eq.(22) for $J/\psi$ and $\psi(3686)$ respectively. In the calculations, the values of $M$ and $\Gamma$ adopt the PDG values, and the $\Gamma^0_e$ adopt the theoretical values $\Gamma^0_e = 4.8$ keV for $J/\psi$ and $\Gamma^0_e = 2.1$ keV for $\psi(3686)$ [17]. The peak positions of $\sigma^0_\psi(s)$ and $\tilde{\sigma}^0_\psi(s)$ defined by Eq.(21) are the same, and the relative difference of their peak values are about 6% for both $J/\psi$ and $\psi(3686)$. The shift of peak positions between $\sigma^0_\psi(s)$ and $\tilde{\sigma}^0_\psi(s)$ defined by Eq.(22) are about 1.0 Mev and 0.4 Mev,
and the relative difference of their peak values are about 31% and 3% for J/ψ and ψ(3686) respectively. The J/ψ is narrower than ψ(3686), the shift of in the vicinity of J/ψ is much larger than that of ψ(3686). The line-shapes of the VP modified Breit-Wigner cross section for adopting Eq. (21) and Eq. (22) are different. It is clear that adopting Eq. (22) is reasonable, and it is consistent with the VP correction to γ∗ channel, see Eq. (20).

![Fig. 4. The comparisons of the line-shapes of e^+e^− → J/ψ → µ^+µ^- and e^+e^- → ψ(3686) → µ^+µ^- between the original σ_ψ^0(s) (left peak) and the VP modified ˜σ_ψ^0(s) (right peak).]

It should be emphasized that the VP effect never be equivalent to the resonances J/ψ and ψ(3686) measured in experiments. The former is only the virtual vacuum fluctuation without definite quantum number, and the later is a real bound state of q̅q with J^PC = 1−− and limited life time.

5.3 Single VP correction case

The Feynman diagram with single VP correction can be shown in Fig. 5. The bubble represents the VP correction in 1-particle-irreducible (1PI) approximation. The coherent amplitude is given by sum of two diagrams,

\[ \tilde{A}_{eff} \sim \frac{1}{1 - \Pi(s)} \left( 1 + \frac{F r e^{i\delta}}{\Delta + i\Gamma} \right), \]

which is same as in other references, only the photon propagator connecting to the initial state e^+e^- is modified by VP correction, and the photon propagator connecting to the final state µ^+µ^- keeps no VP correction.

![Fig. 5. The Feynman diagram with single VP correction.]

Considering VP effect, and if the electro-magnetic coupling is still expressed by α, the Born cross section is modified by the following expression:

\[ \sigma^0(s) \rightarrow \tilde{\sigma}^0(s) = \frac{4\pi\alpha^2}{3s} |\tilde{A}_{eff}|^2 = \frac{\sigma^0(s)}{|1 - \Pi(s)|^2}, \]

where Born cross section σ^0(s) is expressed by Eq. (11), and VP function ˆΠ(s)^2 by Eq. (14). The energy dependence of σ^0(s) and ˜σ^0(s) in vicinity of J/ψ and ψ(3686) is shown in Fig. 6. It is clear that the VP correction or equivalent α(s) distorts the line-shape of the original resonant structure.
5.4 Double VP correction case

In quantum field theory, the processes \(e^+e^- \rightarrow \mu^+\mu^-\) are time reversal invariant under \(T \rightleftharpoons -T\). Looking at the right Feynman diagram for the channel \(e^+e^- \rightarrow \psi \rightarrow \mu^+\mu^-\) in Fig. 5, the three-line vertex of \(e^+\gamma^*\) and \(\mu^+\mu^-\gamma^*\) are unsymmetrical. This issue corresponds to the electro-magnetic coupling is \(\alpha(s)\) for vertex \(e^+\gamma^*\), and the coupling is fixed at \(\alpha(0)\) for vertex \(\mu^+\mu^-\gamma^*\), which means they are not invariant \(T \rightleftharpoons -T\) inversion. This problem can be simply solved by considering the double VP correction.

According to the Feynman rule and ISR correction principle, each virtual photon propagator will be modified by one VP correction factor. The Feynman diagram with time reversal symmetry can be plotted as the Fig. 7. There are two photon propagators for the channel \(e^+e^- \rightarrow \psi \rightarrow \mu^+\mu^-\), and every photon propagator is modified by one VP factor independently, and the two VP factors cannot be combined into one. But the channel \(e^+e^- \rightarrow \gamma^* \rightarrow \mu^+\mu^-\) still has one VP factor.

\[
\hat{A}_{eff} \sim 1 - \frac{1}{1 - \Pi(s)} + \frac{1}{1 - \Pi(s)} F r e^{is} \frac{1}{\Delta + i\rho} 1 - \Pi(s),
\]

and the corresponding cross section can be written as:

\[
\hat{\sigma}^0(s) = \frac{4\pi\alpha^2}{3s}|\hat{A}_{eff}|^2.
\]

In order to give an intuitionistic comparison, Fig. 8 shows the line-shape of the \(\sigma^0(s)\) expressed by Eq. (11), and the \(\hat{\sigma}^0(s)\) expressed by Eq. (25) and Eq. (26).
Comparing Fig. 6 and Fig. 8, the single and double VP correction lead different line-shapes for the cross section. This issue will lead different results for the extracting resonant parameters from experimental data.

6 Total cross section

The tree level cross section reflects the basic property of a process, which is interesting in physics. But in experiments, the measured is the total cross section. In this section, the general form of the total cross section for $e^+e^- \rightarrow \mu^+\mu^-$ is given first, then the analytical expressions are deduced for the cases of single and double VP corrections, and they are compared numerically.

6.1 General form

In the Feynman diagram (FD) scheme, the total cross section up to order $O(\alpha^3)$ can be written as:

$$\sigma_{\text{tot}}(s) = (1 - x_m^\beta + \delta_{\text{vert}})\tilde{\sigma}_0(s) + \beta \int_0^{x_m} dx H(x; s)\tilde{\sigma}_0(s'),$$

where $x \equiv E_\gamma/\sqrt{s}$ is the energy fraction carried by the bremsstrahlung photon, $x_m = 1 - 4m^2_\mu/s$ the maximum energy fraction of the radiative electron, $s' = (1 - x)s$ the effective square of center-of-mass energy of final $\mu^+\mu^-$ pair after radiation, $\delta_{\text{vert}}$ the vertex correction, and the function

$$H(x; s) = \beta \frac{x^\beta}{x} (1 - x + \frac{x^2}{2}),$$

$$\beta = \frac{2\alpha}{\pi} \ln \left(\frac{s}{m^2_e}\right) - 1,$$

is the radiator.

In principle, the integral Eq. (27) can be calculated using numerical method. But in the application for the narrow resonances $J/\psi$ and $\psi(3686)$ scan experiment, the $e^\pm$ beam energy spread effect must be considered. The theoretical total cross section which can match the experiment data is the following effective one,

$$\sigma_{\text{eff}}^{\text{tot}}(s_0) = \int dsG(s; s_0)\sigma_{\text{tot}}^{\text{tot}}(s),$$

where $G(s; s_0)$ is the Gaussian function representing the energy spread distribution of the initial $e^\pm$ beams, and $\sqrt{s_0}$ the nominal center-of-energy of $e^\pm$. Eq. (29) is a two-dimensional integral about variables $x$ and $s$. The integral Eq. (29) contains Eq. (27), the outer integral about $s$ has to be calculated numerically, but the inner integral Eq. (27) about $x$ can be calculated analytically. The analytical calculation of Eq. (27) can save much CPU time and arrives high numerical accuracy.

In the following sections, the analytical expression of Eq. (27) will be deduced for the two cases of single and double VP corrections, and the total cross sections are calculated using the analytical results respectively.

6.2 Analytical calculation for single VP case

If the initial $e^\pm$ radiates a photon with energy fraction $x$, then the notations in Eq. (6) and Eq. (15) are changed,

$$\Delta \Rightarrow \Delta(x) = (1 - x)t - 1,$$

$$\Pi_{\text{res}}(s) \Rightarrow \Pi_{\text{res}}(x; s) = h \frac{1 - x}{\Delta(x) + i\rho},$$

the VP corrected Born cross section can be written as:

$$\sigma^0(s) \Rightarrow \tilde{\sigma}^0(x; s) = \frac{4\pi\alpha^2}{3s} \frac{1}{1 - x} \frac{U(x)}{V(x)},$$

where the quadratic polynomials

$$U(x) = u_2x^2 + u_1x + u_0,$$
$$V(x) = v_2x^2 + v_1x + v_0,$$

are
and the coefficients \( u_i \) and \( v_i \) are the combinations of the known constants. The integrand in Eq. (27) has following polynomial form:

\[
H(x)\tilde{\sigma}^0(x; s) = \frac{4\pi\alpha^2}{3s} \beta^2 x^\beta \left[ \frac{1}{1-x} \sum_{n=0}^{4} w_n x^n + \frac{1}{V(x)} \sum_{n=0}^{5} d_n x^n \right],
\]

where the coefficients \( w_n \) and \( d_n \) are the combinations of known constants. The integral of Eq. (27) can be calculated analytically. The numerical results of the analytical integrals of total cross section are shown by the Fig. 9, the line-shape of Born cross section is also plotted in order to show the effect of initial state radiation. It is clear that the peak positions of \( \sigma^0(s) \) and \( \sigma^{\text{tot}}(s) \) are shifted.

Fig. 9. The line-shape of the \( \sigma^0(s) \) (dashed) and \( \sigma^{\text{tot}}(s) \) of single VP correction (real line) in vicinity of \( J/\psi \) (left) and \( \psi(3686) \) (right).

### 6.3 Analytical calculation for double VP case

Similarly, the integrand of Eq. (27) for the double VP correction can be expressed as the following elementary function:

\[
H(x)\tilde{\sigma}^0(x; s) = \frac{4\pi\alpha^2}{3s} \beta^2 x^\beta \left[ \frac{1}{1-x} \sum_{n=0}^{4} p_n x^n + \frac{1}{V(x)} \sum_{n=0}^{5} q_n x^n + \frac{1}{V^2(x)} \sum_{n=0}^{5} r_n x^n \right]
\]

where the coefficients \( p_n \), \( q_n \) and \( r_n \) are the combinations of the known constants. Input Eq. (36) into Eq. (27), and the integral Eq. (27) can be calculated analytically. The numerical results of the analytical integrals are shown by the Fig. 10.

Fig. 10. The line-shape of the \( \sigma^0(s) \) (dashed) and \( \sigma^{\text{tot}}(s) \) of double VP correction (real line) in vicinity of \( J/\psi \) (left) and \( \psi(3686) \) (right).

### 7 Discussions

This work discusses two problems: (1) treating the VP correction for \( \gamma^* \) channel and \( \psi \) channel by a natural and consistent scheme; (2) comparing the cross section of \( e^+e^- \rightarrow \mu^+\mu^- \) in single and double VP corrections schemes.

In the ISR calculations, treatment of electronic width adopts the form Eq. (19) instead of Eq. (17) or Eq. (18), and considering the double VP correction for \( \psi \) channel are new tries, and they don’t violate the basic principles of quantum field theory.
Fig. 11 puts together the comparisons of the original Born cross section $\sigma^0(s)$, the single and double VP modified Born cross sections $\tilde{\sigma}^0(s)$ in the vicinity of $J/\psi$ and $\psi(3686)$. The line-shapes of $\tilde{\sigma}^0(s)$ for single and double VP correction are different significantly.

The resonant parameters $(M, \Gamma, \Gamma_0^e, \delta)$ are the basic quantities in Breit-Wigner formula, which characterize the main properties of a resonance. The values of these parameters can be estimated by phenomenological (potential) models [17, 18]. But, the accurate values of these parameters must be measured by fitting the experimental data.

Usually, the cross section directly measured in experiments is the total cross section, which includes all of the radiative effects. In order to extract the resonant parameters from the measured cross section correctly, the proper treatment of the ISR correction (including VP) is crucial.

As seen in the previous sections, the value of the total cross section $\sigma_{\text{tot}}^{\text{th}}(s)$ depends on the VP correction scheme, and it is also the function of the resonant parameters. The ISR correction factor $1 + \delta$ is a theoretical calculation quantity defined by Eq. (3), it has influence of the Born cross section according to Eq. (2).

The values of the parameters of $J/\psi$ and $\psi(3686)$ can be extracted by fitting the measured cross section of the scan experiment based on the least square method:

$$\chi^2 = \sum_{i=1}^{n} \frac{[\sigma_{\text{tot}}^{\text{ex}}(s_i) - \sigma_{\text{tot}}^{\text{th}}(s_i)]^2}{\Delta_i^2}, \quad (37)$$

where $\Delta_i$ is the error of $\sigma_{\text{ex}}^{\text{tot}}(s_i)$ at energy point $s_i$. The optimized values of $(M, \Gamma, \Gamma_0^e, \delta)$ corresponding the minimum of $\chi^2$ are the measured values of the related parameters.

As long as the value of $\Gamma_0^e$ is extracted, one may obtain $\Gamma_{\text{ex}}^e$ by any reasonable convention. The $\Gamma_0^e$ connects the radial wave function of $c\bar{c}$ bound state $\psi$ at origin $R(0)$ according to Eq. (9), the correct value of $\Gamma_0^e$ can deduce the value of $R(0)$, and can test potential model.

The previous values of the resonant parameters are measured based on the VP treatment schemes described in [12, 19]. It is expected that if they are extracted using the scheme raised in this paper, the results will not be same as previous measurements, which one is reasonable should be judged by the experiments. This will be the work in the next step.
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