ISOPARAMETRIC FUNCTIONS ON EXOTIC SPHERES

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Abstract. This paper extends widely the work in [GT12]. Existence or non-existence results of isoparametric functions on exotic spheres and Eells-Kuiper projective planes are established. In particular, every homotopy \( n \)-sphere \( (n > 4) \) is proved to carry an isoparametric function (with certain metric) with 2 points as the focal set. As an application, we improve a beautiful result of Bérard-Bergery [BB77] (see also pp.234-235 of [Be78]).

1. Introduction

Let \( N \) be a connected complete Riemannian manifold. A non-constant smooth function \( f \) on \( N \) is called transnormal, if there exists a smooth function \( b : \mathbb{R} \to \mathbb{R} \) such that \( |\nabla f|^2 = b(f) \), where \( \nabla f \) is the gradient of \( f \). If in addition, there exists a continuous function \( a : \mathbb{R} \to \mathbb{R} \) so that \( \Delta f = a(f) \), where \( \Delta f \) is the Laplacian of \( f \), then \( f \) is called isoparametric. Each regular level hypersurface is called an isoparametric hypersurface and the singular level set is called the focal set.

It was E. Cartan who first took up a systematical study of hypersurfaces with constant principal curvatures in space forms, especially in unit spheres. Many years later, a far-reaching generalization of Cartan’s work on those hypersurfaces in unit spheres was produced by Münzner in [Mü80]. They together found that every hypersurface with constant principal curvatures in a unit sphere determines an isoparametric function on the unit sphere, and vice versa (see also [CR85], chapter 3). Thus, the set of level hypersurfaces of an isoparametric function on a unit sphere consists of a family of parallel hypersurfaces with constant principal curvatures, the so called isoparametric foliation. Owing to Cartan and Münzner, the classification of isoparametric hypersurfaces in a unit sphere has been one of the most challenging problems in submanifold geometry. Up to now, the classification has almost been completed. We refer to [CR85] for the development of this subject. For the most recent progress and applications, see for example [CCJ07], [Miy12] and [TY13].

In general Riemannian manifolds, a series of beautiful results, similar to the case in a unit sphere has been proved or claimed by Wang in [Wa87]. Based on Wang’s work, Ge

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and Tang [GT12] made new contributions to the subject of isoparametric functions on general Riemannian manifolds, especially on exotic spheres. In a word, the results of [Wa87] and [GT12] show that the existence of an isoparametric function (even a transnormal function) on a Riemannian manifold $N$ may restrict strongly the geometry and topology of $N$. However, whether a given manifold admits a metric and an isoparametric function is still a problem. Actually, as posed in [GT12], it was not known that if any exotic sphere with dimension more than 4 admits a metric and an isoparametric function with 2 points as the focal set (one of the main results in [GT12] asserts that an exotic 4-sphere (if exists) admits no properly transnormal functions). Moreover, a transnormal function $f$ on Gromoll-Meyer 7-sphere with 2 points as the focal set was constructed in [GT12], however $f$ is not isoparametric.

One of our main aims is to develop a general way to construct metrics and isoparametric functions on a given manifold. According to [Wa87], one can see that a transnormal function on a complete Riemannian manifold is necessarily a Morse-Bott function. As is well known that a Morse-Bott function is a generalization of a Morse function, and it admits critical submanifolds satisfying certain non-degenerate condition. As one of our prime results we give a fundamental construction:

**Theorem 1.1.** (A fundamental construction) Let $N$ be a closed connected smooth manifold and $f$ a Morse-Bott function on $N$ with the critical set $C(f) = M_+ \sqcup M_-$, where $M_+$ and $M_-$ are both closed connected submanifolds of codimensions more than 1. Then there exists a metric $g$ on $N$ so that $f$ is an isoparametric function. Moreover, the metric can be chosen so that the critical submanifolds $M_+$ and $M_-$ are both totally geodesic.

Recall that a closed smooth $n$-manifold is called a homotopy $n$-sphere, if it has the homotopy type of the unit sphere $S^n$. By using a theorem of S. Smale, we have the following result as a consequence.

**Corollary 1.1.** Every homotopy $n$-sphere with $n > 4$ admits a metric and an isoparametric function with 2 points as the focal set.

**Remark 1.1.** Corollary 1.1 answers partially Problem 4.1 in [GT12].

Moreover, we can also construct metrics and isoparametric functions on homotopy spheres and the Eells-Kuiper projective planes so that at least one component of the critical set is not a single point.

Next, we intend to ask if our proof of Theorem 1.1 can be modified so that each regular level hypersurface has more geometric properties by using a more sagacious constructing method of the metric. As a partial negative solution to this problem, we prove the following non-existence result:

**Proposition 1.1.** Let $\Sigma^n$ be a homotopy sphere which admits a metric $g$ and an isoparametric function $f$ with 2 points as the focal set. Suppose that the level hypersurfaces are all totally umbilic. Then $\Sigma^n$ is diffeomorphic to $S^n$. 
By combining Proposition 1.1 with a topological argument, we obtain

**Theorem 1.2.** Every odd dimensional exotic sphere (a homotopy sphere not diffeomorphic to the unit sphere) admits no totally isoparametric functions with 2 points as the focal set.

Recall that a *totally isoparametric* function is an isoparametric function so that each regular level hypersurface has constant principal curvatures, as defined in [GTY11]. As mentioned before, an isoparametric function on a unit sphere must be totally isoparametric.

Last but not least, we have both existence and non-existence results of isoparametric functions on some homotopy spheres which also have $S^p$-property (see [Be78] for definition).

The paper is organized as follows. In Section 2, we review some basic definitions and give a proof to Theorem 1.1. Then we apply the construction in Theorem 1.1 to homotopy spheres and the Eells-Kuiper projective planes in Section 3. And in Section 4, we show Proposition 1.1 and Theorem 1.2. The last section will be concerned with the existence and non-existence of isoparametric functions on exotic spheres with $S^p$-property. For instance, Theorem 5.1 improves a beautiful result of Bérard-Bergery [BB77].

2. A FUNDAMENTAL CONSTRUCTION

This section is devoted to a complete proof of Theorem 1.1. Let’s start by recalling the definition of a Morse-Bott function. A smooth function $f$ on a smooth manifold $N$ is called a *Morse-Bott function* or a generalized Morse function if: (i) the critical set defined by equation $\text{d}f = 0$ consists of a family of smooth submanifolds, the so called critical submanifolds; (ii) the Hessian $H_f$ is a non-degenerate quadratic form in the normal direction of each critical submanifold to $N$.

In differential topology, it is well known that the topology of a manifold can be analyzed by studying Morse-Bott functions on it. The following result is essentially due to Wang [Wa87].

**Proposition 2.1.** A transnormal function $f$ on a complete connected Riemannian manifold $N$ is a Morse-Bott function. Moreover, the critical set $C(f)$ of $f$ coincides exactly with the focal set of $f$.

**Proof.** Combine Lemma 4 with Lemma 6 in [Wa87].

Owning to Proposition 2.1 it is natural to propose the following converse problem.

**Problem 2.1.** Let $f$ be a Morse-Bott function on a smooth manifold $N$. When does $N$ carry a Riemannian metric $g$ so that $f$ is transnormal with respect to this $g$?

This problem motivates us to establish Theorem 2.2. To complete a proof to Theorem 2.2 we still has a long way. Let’s first investigate the topology of a Morse-Bott function whose critical set has only two components.
**Proposition 2.2.** Let $N$ be a connected closed manifold and $f : N \to \mathbb{R}$ a Morse-Bott function so that $C(f)$ has two connected closed components, say $M_+$ and $M_-$. Then $N$ is diffeomorphic to the glued manifold

$$D(M_-) \sqcup M \sqcup \varphi D(M_+).$$

for some diffeomorphism $\varphi : \partial D(M_-) \to \partial D(M_+)$, where $M = \partial D(M_-) \times [0, 1]$. $D(M_+)$ and $D(M_-)$ are respectively the normal disc bundles over $M_+$ and $M_-$. 

**Proof.** By the generalized Morse lemma and regular interval theorem (see [Hi76], pp.149 and pp.153).

**Remark 2.1.** Duan and Rees [DR92] also considered the situation where the critical set of the smooth map $f : N \to \mathbb{R}$ has only two components, both of which are smooth submanifolds of $N$. Actually, they did not assume that the critical set of $f$ are non-degenerate in any sense.

Next, we set about constructing the desired metric. The construction is a modified version of A. Weinstein’s construction (see, for example, [Be78], pp.231-233).

**Theorem 2.2.** Let $N$ be a closed smooth manifold, and $f : N \to \mathbb{R}$ a Morse-Bott function with $C(f) = M_+ \sqcup M_-$, where $M_+$ and $M_-$ are both closed connected submanifolds of codimensions more than 1. Then there exists a Riemannian metric on $N$ so that $f$ is transnormal. In fact, the metric can be chosen so that $M_+$ and $M_-$ are both totally geodesic.

**Proof.** Without loss of generality, we can assume $\text{Im} f = [\alpha, \beta]$ and $M_+ = f^{-1}(\beta)$, $M_- = f^{-1}(\alpha)$. Our construction of the metric is divided into three steps.

**Step 1:**

At first, by the generalized Morse lemma, there exist a vector bundle (a tubular neighborhood) over $M_-$ with some metric, say $\xi^- = (\pi_-, E_-, M_-)$ and an embedding $\sigma_- : \mathcal{U} \to N$, where $\mathcal{U} \subset E_-$ is an open neighborhood of the zero section, such that

1. $\sigma_-|_{M_-} = \text{id}$, where $M_-$ is identified with the zero section of $\xi^-$ and $\sigma_-(\mathcal{U})$ is an open neighborhood of $M_-$ in $N$. Notice here rank $\xi^- = \dim N - \dim M_- \geq 2$.

2. The composition $\mathcal{U} \xrightarrow{\sigma_-} N \xrightarrow{f} \mathbb{R}$ is given by

$$f(\sigma_-(p, v)) = |v|^2 + \alpha$$

for any $(p, v) \in \pi^-(p) \cap \mathcal{U}$ and $p \in M_-$. 

Now, since $M_-$ is compact, we can choose $\epsilon > 0$ so small that

$$M_-(\epsilon) := \{x \in N | f(x) \leq \alpha + \epsilon^2\} \subset \sigma_-(\mathcal{U}).$$

Thus, we have a disc bundle, $D_\xi^-(-\epsilon) := \{(p, v) \in E_- | p \in M_-, |v| \leq \epsilon\}$ with $S_\xi^-(-\epsilon) := \partial D_\xi^-(-\epsilon)$ so that $\sigma_-(D_\xi^-(-\epsilon)) = M_-(\epsilon)$.
Next, for each dimension $d$, let us choose a smooth Riemannian metric $g^d$ on the disc $D^d(\epsilon) := \{ x \in \mathbb{R}^d | |x| \leq \epsilon \}$ as $g^d = F(r)^2 dr^2 + G(r)^2 ds_{d-1}^2$, where $ds_{d-1}^2$ is the canonical metric on $S^{d-1}(1)$, and $F, G$ are smooth functions on $[0, \epsilon]$ with

$$F(r) = \begin{cases} 1, & \text{for } r \leq \frac{\epsilon}{2}; \\ 2r, & \text{for } r \geq \frac{\epsilon}{4}, \end{cases}$$

and

$$G(r) = \begin{cases} r, & \text{for } r \leq \frac{\epsilon}{2}; \\ 1, & \text{for } r \geq \frac{\epsilon}{4}. \end{cases}$$

The choice of the metric ensures the following two properties:

1. $g^d$ is $O(d)$-invariant under the usual action of the orthogonal group $O(d)$ with the origin $o \in \mathbb{R}^d$ fixed;
2. $g^d$ is product near the boundary $\partial D^d(\epsilon) = S^{d-1}(\epsilon)$.

Moreover, $|\frac{\partial}{\partial r}| = 2r$ near the boundary, where $r = |x|$, and $\frac{\partial}{\partial r}$ is the radius direction.

With this metric, each radius has the same length, say $\delta$. Furthermore, it is not difficult to show each radius is a geodesic with arc length as parameter in $D^d(\epsilon)$.

Let $h_-$ be any Riemannian metric on $M_-$. By a construction due to J. Vilms [Vi70], there exists a Riemannian metric $h'_-$ on $D_{\xi^-}(\epsilon)$, such that the fibration $\pi : (D_{\xi^-}(\epsilon), h'_-) \to (M_-, h_-)$ is a Riemannian submersion with totally geodesic fibers isometric to $(D^d(\epsilon), g^d)$ with $d_- = \text{rank} \xi^-$. In the case at hand, this construction can be described as follows. Note that we have chosen the Euclidean metric on $(\pi_-, E_-, M_-)$, and let us choose a metric linear connection on $(\pi_-, E_-, M_-)$. This connection produces a distribution $\mathcal{H}_-$ on the tangent bundle $TE_-$ of $E_-$, complementary to the vertical distribution $\mathcal{V}_-$ (kernel of $d\pi_-$), and the associated parallel transport is by linear isometries. Then the desired metric $h'_-$ on $D_{\xi^-}(\epsilon)$ can be constructed in the following way: one takes the vertical and horizontal distributions to be orthogonal, lifts $h_-$ from $M_-$ to the horizontal distribution and then takes the metric $g_{\xi^-}$ on the fiber $D^d(\epsilon)$. Clearly parallel transport is still an isometry for these metrics on the fibers, so that the fibers are totally geodesic.

Notice that the metric $h'_-$ is still product near the boundary $S_{\xi^-}(\epsilon)$ of $D_{\xi^-}(\epsilon)$ and the zero section is totally geodesic in $D_{\xi^-}(\epsilon)$. Moreover, any geodesic issuing from a point in $S_{\xi^-}(\epsilon)$ and orthogonal to $S_{\xi^-}(\epsilon)$ in $D_{\xi^-}(\epsilon)$ stays in the same fiber of the fibration. In particular, all the geodesics orthogonal to the boundary $S_{\xi^-}(\epsilon)$ in $D_{\xi^-}(\epsilon)$ have the same length $2\delta$.

Now, by the diffeomorphism $\sigma_- : D_{\xi^-}(\epsilon) \to M_-(\epsilon)$, we obtain an induced metric on $M_-(\epsilon)$, say $h_-(\epsilon)$. On the other side, we can also construct an analogous Riemannian metric, say $h_+(\epsilon)$, on $M_+(\epsilon) := \{ x \in N | f(x) \geq \beta - \epsilon^2 \}$ which is diffeomorphic to a disc bundle $D_{\xi^+}(\epsilon)$ over $M_+$, where $\epsilon$ is small enough, the same as before.

**Step 2:**
The task we face is to extend smoothly the metrics $h_-(e)$ and $h_+(e)$ on $M_-(e)$ and $M_+(e)$ to the whole manifold $N$ with appropriate properties. We are left to work on $M := \{ x \in N | \alpha + \epsilon^2 \leq f(x) \leq \beta - \epsilon^2 \}$, since $N = M_-(e) \cup M \cup M_+(e)$.

To do it, by regular interval theorem (see [Hi76]), $\partial M_-(e)$ is diffeomorphic to $\partial M_+(e)$. More precisely, there is a diffeomorphism $\tau : M \rightarrow S_{\xi^-}(e) \times [\alpha + \epsilon^2, \beta - \epsilon^2]$ satisfying

$$\tau(f^{-1}(a)) = S_{\xi^-}(e) \times \{a\}, \forall a \in [\alpha + \epsilon^2, \beta - \epsilon^2].$$

Let us now choose a family $g_t$ of Riemannian metrics on $S_{\xi^-}(e)$, which are smooth in the parameter $t$ for $t \in [\alpha + \epsilon^2, \beta - \epsilon^2]$ and constant near $\alpha + \epsilon^2$ and $\beta - \epsilon^2$, respectively. Moreover, we ask that $g_{\alpha + \epsilon^2}$ is the metric induced from $h_-(e)$, and $g_{\beta - \epsilon^2}$ is induced from $h_+(e)$ by the diffeomorphism $\tau$.

Then a metric $h_0$ on $S_{\xi^-}(e) \times [\alpha + \epsilon^2, \beta - \epsilon^2]$ can be constructed in such a way that the direct decomposition given by the projections onto the two factors is orthogonal, the metric induced on each $S_{\xi^-}(e) \times \{t\}$ is $g_t$, and the metric induced on $\{p\} \times [\alpha + \epsilon^2, \beta - \epsilon^2]$ is the canonical metric with length $\beta - \alpha - 2\epsilon^2$ on $[\alpha + \epsilon^2, \beta - \epsilon^2]$ for each point $p \in S_{\xi^-}(e)$. It is not difficult to show that for any given point $p \in S_{\xi^-}(e)$, the curve

$$\gamma : [\alpha + \epsilon^2, \beta - \epsilon^2] \rightarrow S_{\xi^-}(e) \times [\alpha + \epsilon^2, \beta - \epsilon^2], \ t \mapsto (p, t),$$

is a geodesic in $S_{\xi^-}(e) \times [\alpha + \epsilon^2, \beta - \epsilon^2]$.

**Step 3:**

At last, these metrics $h_-(e)$, $\tau^* h_0$ and $h_+(e)$ on $M_-(e)$, $M$ and $M_+(e)$ respectively can be glued together giving a desired metric $g_N$ on $N$. Observe that $|\nabla f| \equiv 1$ on $M$. On the other hand, near the boundary of $M_-(e)$, one has $|\nabla f| = |2r \nabla r| = |\frac{1}{2} \frac{\partial f}{\partial r}| = 1$, which also holds near the boundary of $M_+(e)$. These properties guarantee that the glued metric $g_N$ on $N$ is smooth. From this construction, it is obvious that every geodesic starting orthogonally from any point in $M_-(e)$ must arrive at $M_+(e)$ orthogonally with length $2\delta + \beta - \alpha - 2\epsilon^2$.

By a direct computation, we see

$$|\nabla f|^2 = \begin{cases} \frac{4(f-a)}{F^2(\sqrt{f-a})} & \alpha \leq f \leq \alpha + \epsilon^2 \\ 0 & \alpha + \epsilon^2 \leq f \leq \beta - \epsilon^2 \\ \frac{4(\beta-f)}{F^2(\sqrt{\beta-f})} & \beta - \epsilon^2 \leq f \leq \beta, \end{cases}$$

which means that $f$ is transnormal. Now, the proof is complete. \(\square\)

**Remark 2.2.** In general, $M_+$ and $M_-$ of a transnormal function $f$ are not necessarily totally geodesic.

According to the proof of Theorem 2.2, we have the following enjoyable observation.

**Proposition 2.3.** The function $f$ is isoparametric on the domain $N - M$ with respect to the metric constructed in Theorem 2.2.
Proof. We only need to verify $f$ on $M_\ast(e) = \sigma_\ast(D_{\xi}(e))$. Since we are working on a Riemannian submersion with totally geodesic fibers, it follows that $\triangle$ on $M_\ast(e)$ has a decomposition $\triangle = \triangle^H + \triangle^V$, where $\triangle^H$ and $\triangle^V$ are the horizontal and vertical Laplacians on $M_\ast(e)$ with respect to the Riemannian submersion $\pi \circ \sigma^{-1} : M_\ast(e) \to M_\ast$. In our case, it is not difficult to see that $\triangle^H f = 0$. Thus, by a direct computation,

$$\triangle f = \triangle^V(r^2) = \frac{2}{F^2} - \frac{2rF}{F^3} + \frac{2(\dim N - \dim M_\ast - 1)rG}{F^2G},$$

that is, $\triangle f$ is a continuous function of $f$ on $M_\ast(e)$. The proof then follows. \qed

Up to now, we have constructed a metric $g_N$ on $N$ such that $f$ is transnormal on whole $N$, and isoparametric near $M_\ast$ and $M_s$. In order to prove Theorem 1.1 we have to deform the metric $g_N$ on $N$ so that $f$ is isoparametric. For this purpose, we prepare three lemmas.

**Lemma 2.1** (Local version of Moser’s theorem). Let $Q^n = (0, 1)^n$ and $\tilde{Q}^n = [0, 1]^n$ be the open and closed unit cubes in Euclidean space, respectively. Let $f, g$ be two positive smooth functions on $\tilde{Q}^n$ such that $f = g$ near $\partial \tilde{Q}^n$ and $\int_{\tilde{Q}^n} (f - g) dx = 0$. Then there exists a diffeomorphism $\psi : Q^n \to \tilde{Q}^n$ satisfying

1. $g \circ \psi \det \nabla \psi = f$ in $Q^n$;
2. $\psi$ is id near $\partial \tilde{Q}^n$;
3. $\psi$ is isotopic to id.

In fact, the diffeomorphism $\psi$ can be decomposed as $\psi = \psi_n \circ \psi_{n-1} \circ \ldots \circ \psi_1 : Q^n \to \tilde{Q}^n$, and for any $1 \leq s \leq n$,

$$\psi_s : Q^n \to Q^n, (x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{s-1}, v_s(x_1, x_2, \ldots, x_n), x_{s+1}, \ldots, x_n)$$

is a diffeomorphism such that $\frac{\partial v_s}{\partial x_i} > 0$, $\psi_s = \text{id}$ near $\partial \tilde{Q}^n$, and $\psi_s$ preserves the line segments \{$(x_1, x_2, \ldots, x_n) \in Q^n | x_i = c_i, 1 \leq i \leq n, i \neq s$\}.

**Lemma 2.2** (Global version of Moser’s theorem). Let $\tau$ and $\sigma$ be two volume elements on a compact connected manifold $M$ with

$$\int_M \tau = \int_M \sigma.$$

Then there exists a diffeomorphism $\psi : M \to M$ so that $\psi$ is isotopic to id : $M \to M$ and satisfies $\psi^\ast \tau = \sigma$.

**Remark 2.3.** For the proofs of Lemma 2.1 and Lemma 2.2 we refer to [Mo65] and [DM90]. However, we should indicate that Lemma 2 was not proved correctly in [Mo65], and it was rectified in [DM90]. As Calabi pointed out that the global version of Moser’s theorem holds even for non-orientable manifolds if one uses the concept of “odd” forms (see [Mo65]).

At last, we need a lemma for deforming metrics.
Lemma 2.3. Let $S^{n-1}$ be an $(n-1)$-dimensional smooth manifold and $M^n = S^{n-1} \times [\alpha_0, \beta_0]$. Let $h$ be a metric on $M$ defined by a smooth one-parameter family of metrics $\{h_t, t \in [\alpha_0, \beta_0]\}$ on $S$, i.e. $h_t(x, y) = h(x) + dt^2$ for $(x, t) \in S^{n-1} \times [\alpha_0, \beta_0]$. Define a new metric $\tilde{h}$ on $M$ given by $\{\tilde{h}_t = e^{2u(h_t)}, t \in [\alpha_0, \beta_0]\}$ depending only on a smooth function $u$ on $M$. Considering the smooth projection function $f : S^{n-1} \times [\alpha_0, \beta_0] \to \mathbb{R}$, $(x, t) \mapsto t$, we have

$$\tilde{\Delta} f = \Delta f + (n-1) \frac{\partial u}{\partial t},$$

where $\tilde{\Delta}$ and $\Delta$ are the Laplacians with respect to $\tilde{h}$ and $h$ respectively.

Proof. By a straightforward computation.

Proof of Theorem 1.1: Assume the same notations as in Theorem 2.2. By the proof of Lemma 2.3, let $\phi : S_{\xi}(\epsilon) = \partial D_\xi(\epsilon) \to S_{\xi}(\epsilon) = \partial D_\xi(\epsilon)$ such that $N$ is diffeomorphic to the glued manifold $D_\xi(\epsilon) \sqcup M' \sqcup_\phi D_\xi(\epsilon)$, where $M' = S_{\xi}(\epsilon) \times [\alpha + \epsilon^2, \beta - \epsilon^2]$. Additionally, we have constructed metrics $h', h_\epsilon$ and $h_0$ on $D_\xi(\epsilon)$, $D_\xi(\epsilon)$ and $M'$ respectively. As we asserted before, $f$ is isoparametric on $D_\xi(\epsilon)$ and $D_\xi(\epsilon)$.

Let $\omega_+$ and $\omega_\epsilon$ be the volume elements of $(S_{\xi}(\epsilon), h', |_{S_{\xi}(\epsilon)})$ and $(S_{\xi}(\epsilon), h_\epsilon |_{S_{\xi}(\epsilon)})$ respectively. Then $(S_{\xi}(\epsilon), g_{\beta-\epsilon}) = (S_{\xi}(\epsilon), \psi^* (h'_{\epsilon} |_{S_{\xi}(\epsilon)}))$ has the volume element $\psi^* \omega_+$. By Lemma 2.2 (Global version of Moser’s theorem), there exists a diffeomorphism $\psi : S_{\xi}(\epsilon) \to S_{\xi}(\epsilon)$, which is isotopic to $id$ and satisfies

$$\psi^* (\psi^* \omega_+) = \lambda \omega_+,$$

where $\lambda = \int \psi^* \omega_+$. Defining $\bar{\psi} := \psi \circ \psi$, we see that $\bar{\psi}$ is isotopic to $\psi$ and

$$\tilde{\omega} = (\psi \circ \psi)^* \omega_+ = \psi^* (\psi^* \omega_+) = \lambda \omega_+,$$

where $\tilde{\omega}$ is the volume element of $(S_{\xi}(\epsilon), \psi^* g_{\beta-\epsilon})$. Since $\bar{\psi}$ is isotopic to $\psi$, it follows that $D_\xi(\epsilon) \sqcup M' \sqcup D_\xi(\epsilon)$ is diffeomorphic to $D_\xi(\epsilon) \sqcup M' \sqcup D_\xi(\epsilon)$.

Consider now $N := D_\xi(\epsilon) \sqcup M' \sqcup D_\xi(\epsilon)$. Let $\tilde{f} : N \to \mathbb{R}$ be a function such that $\tilde{f}|_{D_{\xi\epsilon}^+} = f \circ \tau_\epsilon|_{D_{\xi\epsilon}^+}$ and $\tilde{f}|_{M'} = f \circ \tau^{-1}|_{M'}$. It is obvious that $N$ is diffeomorphic to $N$ and $\tilde{f}$ is a Morse-Bott function with the critical submanifolds $C(\tilde{f}) = M_+ \sqcup M_-, \text{ where } M_+ \text{ and } M_- \text{ are diffeomorphic to } M_+ \text{ and } M_-$. With respect to the metrics constructed in Theorem 2.2, $\tilde{f}$ is isoparametric on $(D_\xi(\epsilon), h')$ and $(D_\xi(\epsilon), h'_\epsilon)$. On $M'$, choose a metric $g$ defined by a smooth family of metrics $\{g_t^\epsilon, t \in [\alpha + \epsilon^2, \beta - \epsilon^2]\}$ on $S_{\xi}(\epsilon)$ such that

$$g_{\alpha+\epsilon^2}^\epsilon = g_{\alpha+\epsilon^2}^\epsilon, g_{\beta-\epsilon^2}^\epsilon = \psi^* g_{\beta-\epsilon^2}^\epsilon$$

and $g_\epsilon$ is constant near $\alpha + \epsilon^2$ and $\beta - \epsilon^2$. By lemma 2.3, let $u : M' \to \mathbb{R}$ be a smooth function to be determined and $\bar{g}$ a new metric on $M'$ determined by the new family of metrics $\{\bar{g}_t^\epsilon, \bar{g}_t^\epsilon = e^{2u} g_t^\epsilon, t \in [\alpha + \epsilon^2, \beta - \epsilon^2]\}$. Then

$$\tilde{\Delta} \tilde{f} = \Delta \tilde{f} + (n-1) \frac{\partial u}{\partial \alpha},$$
where \( n = \dim N \), \( \tilde{\Delta} \) and \( \Delta \) are Laplacians with respect to \( \tilde{g} \) and \( g \), respectively.

Next let \( h : [\alpha + \epsilon^2, \beta - \epsilon^2] \to \mathbb{R} \) be a smooth function with \( h(t) = \Delta \tilde{f}|_{S_{\epsilon^2}(e) \times \{t\}} \) for \( t \) near \( \alpha + \epsilon^2 \) and \( \beta - \epsilon^2 \). In order to find such an \( u \) that \( \tilde{f} \) is isoparametric on \((M', \tilde{g})\), it is sufficient to construct \( u \) as the solution of the following equation

\[
\tilde{\Delta} \tilde{f} = h
\]

or equivalently

\[
h(t) = \Delta \tilde{f} + (n - 1) \frac{\partial u}{\partial t}
\]

which implies that for any \((x, t) \in M'\)

\[
(n - 1)(u(x, t) - u(x, \alpha + \epsilon^2)) = \int_{\alpha + \epsilon^2}^{t} [h(s) - \Delta \tilde{f}(x, s)] ds.
\]

On the other hand, by a direct computation, we have

\[
\int_{\alpha + \epsilon^2}^{\beta - \epsilon^2} \Delta \tilde{f}(x, s) ds = \log \left( \frac{\det(g_{ij}(x, \beta - \epsilon^2))}{\det(g_{ij}(x, \alpha + \epsilon^2))} \right)
\]

which is just equal to \( \log \lambda \), independent of the choice of \( x \in S_{\epsilon^2}(e) \). Hence, an appropriate choice of the function \( h \) satisfies the equation \( \int_{\alpha + \epsilon^2}^{\beta - \epsilon^2} [h(s) - \Delta \tilde{f}(\cdot, s)] ds \equiv 0 \). At last we get a solution

\[
u(x, t) = \frac{1}{n - 1} \int_{\alpha + \epsilon^2}^{t} [h(s) - \Delta \tilde{f}(x, s)] ds,
\]

for \((x, t) \in M'\), such that \( u|_{S_{\epsilon^2}(e) \times \{0\}} \equiv 0 \) for \( t \) near \( \alpha + \epsilon^2 \) and \( \beta - \epsilon^2 \). The argument above implies that \( \tilde{\alpha} \) still has the product metric near the boundary of \( M' \) as before. Therefore the three parts \((D_{\epsilon^2}(e), h'), (D_{\epsilon^2}(e), h'')\) and \((M', \tilde{g})\) can be glued to gain a smooth metric \( \tilde{g}_N \) on \( N \) so that \( \tilde{f} \) is isoparametric.

The proof is now complete.

\[ \square \]

3. Applications and examples

In this section, the general construction in Theorem 1.1 is applied to the existence of isoparametric functions on various exotic spheres and Eells-Kuiper projective planes.

First, we give a short

**Proof of Corollary 1.1:**

*Proof.* A well known result of S. Smale ( ref. [Mi07], pp.128 ) states that there exists a Morse function with 2 critical points on each homotopy sphere with dimension more than 4. Thus, it follows from Theorem 1.1 that each homotopy sphere with dimension more than 4 admits a metric and an isoparametric function with 2 points as the focal set.

The proof is now complete. \[ \square \]
More generally, we also investigate the existence of isoparametric functions on exotic spheres such that at least one component of the critical set is not a single point.

Following J. Milnor, let \( g : S^m \times S^n \to S^m \times S^n \) be an orientation-preserving diffeomorphism. Then a smooth manifold \( M^{m+n+1}(g) \) depending on \( g \) is obtained from disjoint spaces \( \mathbb{R}^{m+1} \times S^n \) and \( S^m \times \mathbb{R}^{n+1} \) by matching \((tx,y)\) and \((x',t'y')\), where \((x',y') = g(x,y)\) and \( t' = \frac{1}{t} \) for any \( x \in S^m, y \in S^n \) and \( t \in (0, +\infty) \). In light of this construction, we have the following observation.

**Proposition 3.1.** Define a function \( f \) on \( M^{m+n+1}(g) \) by

\[
f : \mathbb{R}^{m+1} \times S^n \to \mathbb{R}, (X, y) \mapsto \frac{|X|^2}{1 + |X|^2},
\]
and

\[
f : S^m \times \mathbb{R}^{n+1} \to \mathbb{R}, (x, Y) \mapsto \frac{1}{1 + |Y|^2},
\]

\( f \) is a Morse-Bott function with \( C(f) = \{0\} \times S^n \sqcup S^m \times \{0\} \). Moreover, \( M^{m+n+1}(g) \) admits an isoparametric function with focal set diffeomorphic to \( S^m \sqcup S^n \) under some metric.

**Proof.** It is obvious that \( f \) is a well defined smooth function. Moreover, \( f \) is a Morse-Bott function with critical set \( \{0\} \times S^n \sqcup S^m \times \{0\} \). Therefore, \( M^{m+n+1}(g) \) admits a metric and an isoparametric function with focal set diffeomorphic to \( S^m \sqcup S^n \) under some metric, by virtue of Theorem [11].

In order to construct concrete examples, we have the following method by J. Milnor [Mil59]. Starting with two smooth maps \( f_1 : S^m \to SO(n+1) \) and \( f_2 : S^n \to SO(m+1) \), a diffeomorphism \( g : S^m \times S^n \to S^m \times S^n \) is defined by

\[
g(x,y) = (f_2^{-1}(f_1(x) \cdot y) \cdot x, f_1(x) \cdot y)
\]
for \( x \in S^m \) and \( y \in S^n \). Denote \( M(f_1, f_2) := M(g) \). There is also another way to construct manifolds, the so called Milnor pairing. The Milnor pairing is a bilinear pairing given by

\[
\beta_{m,n} : \pi_m SO(n) \otimes \pi_n SO(m) \to \pi_0 \text{Diff}^+ S^{m+n} \to \Gamma_{m+n+1} = \Theta_{m+n+1},
\]

where \( \text{Diff}^+ S^{m+n} \) is the group of all orientation preserving diffeomorphisms from \( S^{m+n} \) onto itself; \( \Gamma_l \) denotes the group of oriented diffeomorphism classes of twisted \( l \)-spheres, \( \Theta_l \) denotes the group of oriented homotopy \( l \)-spheres up to relation \( h \)-cobordant, and in fact they are isomorphic for each \( l \) ( [Mil07], pp.5 ). We remark that two homotopy \( l \)-spheres, \( l > 4 \), are \( h \)-cobordant if and only if they are diffeomorphic (ref. [KM63]). The following theorem of J. Milnor tells us how the two constructing methods above are related.

**Theorem 3.1** ([Mil07], pp. 218). Given two smooth maps \( f_1 : S^m \to SO(n) \hookrightarrow SO(n+1) \) and \( f_2 : S^n \to SO(m) \hookrightarrow SO(m+1) \), the resulting manifold \( M(f_1, f_2) \) is diffeomorphic to \( \beta_{m,n}(f_1, f_2) \).
Applying Proposition 3.1 and Theorem 3.1 will lead to the following examples.

**Example 3.1.** Each homotopy $7$-sphere admits a metric and an isoparametric function with focal set diffeomorphic to $S^3 \sqcup S^3$, since $\Theta_7 = \text{Im} \beta_{3,3}$ (see [Mil07], pp. 187).

**Remark 3.1.** This example generalizes Proposition 4.1 in [GT12], where only Milnor $7$-spheres (those can be represented as $S^3$ bundles over $S^4$) are considered. It is well known that not every homotopy $7$-sphere is a Milnor sphere.

Recall that the group of homotopy spheres $\Theta_n$ has an important subgroup $bP_{n+1}$ defined by Kervaire and Milnor in [KM63]. A homotopy $n$-sphere $\Sigma^n$ represents an element of $bP_{n+1}$ if and only if $\Sigma^n$ bounds a parallelizable manifold. To determine the group $\Theta_n$, they did a systematic research of such subgroups. In particular, they showed that $\Theta_7 = bP_8$ and $bP_{4m+2} = \mathbb{Z}_2$ or $0$. As in the literature, a Kervaire sphere $\Sigma_{4m+1}$ is the boundary of the pluming of two copies of the unit disc bundle $D(TS^{2m+1})$ of the tangent bundle of $S^{2m+1}$ and consequently it is a generator of $bP_{4m+2}$ (see [KM63], [HH67] and [MM80]). For such homotopy spheres, we have the following result.

**Example 3.2.** For $m > 0$, the Kervaire sphere $\Sigma_{4m+1}$ admits a metric and an isoparametric function with focal set diffeomorphic to $S^1 \sqcup S^3$. The reason is that one can identify $\Sigma_{4m+1}$ with $M(f, f)$, where $f : S^{2m} \to SO(2m) \leftrightarrow SO(2m + 1)$ is the characteristic map for the tangent bundle of $S^{2m+1}$ (see [HH67] and [MM80]). Additionally, more examples concerning even dimensional homotopy sphere, as well as the odd dimensional homotopy sphere which is not the boundary of any parallelizable manifold are given.

**Example 3.3.** According to D. L. Frank [Fr68], it is known that $\Theta_8 = \text{Im} \beta_{3,4}$, $2\Theta_{10} = \text{Im} \beta_{3,6}$. Hence, every homotopy $8$-sphere admits a metric and an isoparametric function with focal set diffeomorphic to $S^3 \sqcup S^4$; every element in $2\Theta_{10}$ admits a metric and an isoparametric function with focal set diffeomorphic to $S^3 \sqcup S^6$.

It is also proved by D. L. Frank in [Fr68] that there exists a homotopy $15$-sphere $\Sigma^{15}$ which is contained in $\text{Im} \beta_{11,3}$ but not in $bP_{16}$. Therefore, this $\Sigma^{15}$ admits an isoparametric function with focal set $S^{11} \sqcup S^3$ for certain metric, and it is not the boundary of any parallelizable manifold.

In fact, Proposition 3.1 gives us more examples than we show here (see [Mil59] and related references).

At last, we will investigate the existence of isoparametric functions on Eells-Kuiper projective planes. Recall that a closed manifold is called an Eells-Kuiper projective plane if it admits a Morse function with three critical points. Eells-Kuiper [EK1] obtained many remarkable results on such manifolds. For instance, they showed that the integral cohomology ring...
of an Eells-Kuiper projective plane $M$ is isomorphic to that of the real, complex, quaternionic or Cayley projective plane. Consequently, the dimension of $M$ must be equal to 2, 4, 8, or 16. Moreover, for $m \in \{1, 4, 8\}$, an Eells-Kuiper projective plane $M^{2m}$ is diffeomorphic to $D(\xi) \sqcup \varphi D^{2m}$ for some diffeomorphism $\varphi : S(\xi) = \partial D(\xi) \to S^{2m-1}$, where $D(\xi)$ and $S(\xi)$ are the associated disc bundle and sphere bundle for certain vector bundle of rank $m$ over $S^m$. It follows that on every Eells-Kuiper projective plane we can construct a Morse-Bott function whose critical set has two components, a point and $S^m$. Therefore, by Theorem 1.1, we get the following result.

**Proposition 3.2.** For $m = 4$ or 8, each Eells-Kuiper projective plane $M^{2m}$ admits a metric and an isoparametric function so that, one component of the focal set is a single point and the other is diffeomorphic to $S^m$.

**Remark 3.2.** On the projective planes $CP^2$, $HP^2$ and $CaP^2$ with Fubini-Study metrics, there are isoparametric functions so that the focal sets have two components, a single point, and $S^2$, $S^4$ and $S^8$, respectively. In fact, these isoparametric foliations are all homogeneous. For $CP^2$ and $HP^2$, one uses the Hopf fibrations to derive the desired conclusion. As for $CaP^2$, one uses the cohomogeneity one action of Spin(9) on $F_4/\text{Spin}(9)$ which is isometric to $CaP^2$.

4. Non-existence results on exotic spheres

In this section, Proposition 1.1 and Theorem 1.2 are proved. Although any exotic $n$-sphere with $n > 4$ admits an isoparametric function with 2 points as the focal set by Theorem 1.1, Proposition 1.1 means that regular level hypersurfaces of such isoparametric functions on an exotic sphere can not have strong extrinsic geometric properties. This is a way to distinguish exotic spheres from standard spheres by geometry.

**Proof of Proposition 1.1.** Let $2\delta := d(m_+, m_-)$, where $d$ is the distance function with respect to the Riemannian metric $g$. The existence of thus isoparametric function $f$ implies $\Sigma^n$ can be decomposed as

$$\Sigma^n = \exp D_\delta(T_m \Sigma^n) \cup \exp D_\delta(T_m \Sigma^n),$$

where $D_\delta(T_m \Sigma^n)$ and $D_\delta(T_m \Sigma^n)$ are the discs of radius $\delta$ at $T_m \Sigma^n$ and $T_m \Sigma^n$ respectively. It follows that $\Sigma^n$ is diffeomorphic to $D^n(1) \sqcup \eta D^n(1)$, where $\eta : S^{n-1}(1) \to S^{n-1}(1)$ is determined by the following diagram

$$\begin{array}{c}
S^{n-1}(1) \hookrightarrow D^n(1) \xrightarrow{\rho^+_\delta} T_m \Sigma^n \xrightarrow{\exp_m} \Sigma^n \\
\downarrow \eta \\
S^{n-1}(1) \hookrightarrow D^n(1) \xrightarrow{\rho^-_\delta} T_m \Sigma^n \xrightarrow{\exp_m} \Sigma^n,
\end{array}$$

where $\rho^+_\delta$ and $\rho^-_\delta$ are the canonical linear diffeomorphisms.
Since the regular hypersurfaces of $f$ are all totally umbilical, we can assume the regular hypersurface $X_t := \{ x \in \Sigma^n | d(p, m_\pm) = t \}$ has constant principal curvature $\lambda(t)$ for any $t \in (0, 2\delta)$. According to Kowalski and Vanhecke ([KV86], Theorem 12), we have

$$\exp_m^* g = \frac{r}{H} \sum_{i=1}^n |dx_i|^2 + \frac{1}{H} \sum_{i=1}^n x_i dx_i^2,$$

where $H(r) = \exp[2 \int_0^r (\lambda(t) - \frac{1}{2})dt]$, $r^2 = \sum_{i=1}^n (x_i)^2 \in (0, \delta^2]$, and $(x_1, x_2, \ldots, x_n)$ is the normal coordinate around $m_-$. Similarly, near $m_+$,

$$\exp_m^* g = \frac{r}{H} \sum_{i=1}^n |dy_i|^2 + \frac{1}{H} \sum_{i=1}^n y_i dy_i^2,$$

where $H(r) = \exp[2 \int_0^r (\lambda(2\delta-r) - \frac{1}{2})dt]$, $r^2 = \sum_{i=1}^n (y_i)^2 \in (0, \delta^2]$, and $(y_1, y_2, \ldots, y_n)$ is the normal coordinate around $m_+$. By the diagram introducing the diffeomorphism $\eta$ and the formulae of the metric around $m_+$ and $m_-$, we conclude that $\eta : S^{n-1}(1) \to S^{n-1}(1)$ is an isometry. It follows at once that $\Sigma^n$ is diffeomorphic to $S^n(1)$, which completes the proof. 

At last, for odd dimensional homotopy spheres, we have Theorem 1.2 by a topological argument.

**Proof of Theorem 1.2**

*Proof.* Let $\Sigma^{2n+1}$ be a homotopy sphere and $f : \Sigma^{2n+1} \to \mathbb{R}$ a totally isoparametric function with respect to a metric $g$ on $\Sigma^{2n+1}$. Then for any regular value $t$, the regular hypersurface $M_t := f^{-1}(t)$ has constant principal curvatures in $(\Sigma^{2n+1}, g)$. Let $m(t)$ be the number of distinct principal curvatures of $M_t$ and assume $m(t) > 1$ for some $t$. Then the tangent bundle of $M_t$ has the decomposition as follows,

$$TM_t = T_1 \oplus T_2 \oplus \cdots \oplus T_{m(t)},$$

where $T_1, T_2, \ldots, T_{m(t)}$ are the principal distributions. On the other hand, we observe that each regular hypersurface $M_t$ is diffeomorphic to $S^{2n}$, since $M_t$ is a geodesic hypersphere of each of two focal points. Hence, the Euler characteristic classes satisfy

$$0 \neq e(TM_t) = e(T_1)e(T_2) \cdots e(T_{m(t)}).$$

However, the right side of the equality is equal to 0, since $\operatorname{rank} T_k < 2n$, each class $e(T_k) \in H^{\operatorname{rank} T_k}(M_t) = H^{\operatorname{rank} T_k}(S^{2n})$ vanishes, for $k \in \{ 1, 2, \ldots, m(t) \}$. That is a contradiction, which implies that $m(t) = 1$ and the regular hypersurface $M_t$ is totally umbilical for any regular value $t$. The conclusion we hoped follows immediately from Proposition 1.1.

**Remark 4.1.** According to [HH67], there exists an exotic Kervaire sphere $\Sigma^{4m+1}$ which has a cohomogeneity one action. Consequently, by a pretty observation in [GT12], $\Sigma^{4m+1}$ admits a totally isoparametric function $f$ under an invariant metric. However, each component of the focal set of $f$ is not just a point, but a smooth submanifold. Hence, the assumption on the focal set in Theorem 1.2 is essential.
Due to Theorem 1.2, it is reasonable to ask

**Problem 4.1.** Does there exist an even dimensional exotic sphere $\Sigma^{2n}(n > 2)$ which admits a metric and a totally isoparametric function with 2 points as the focal set?

## 5. Isoparametric functions and $SC^p$-property

In this section, both isoparametric functions and $SC^p$-structure are investigated simultaneously. First, we recall the definition. Let $(M, g)$ be a Riemannian manifold and $p$ be a point in $M$. If all the geodesics issued from $p$ are simply closed geodesics with the same length, we say that $(M, g)$ satisfies the $SC^p$-property at $p \in M$.

For background knowledge and a systematical research, we refer to the classic book [Be78].

Next, we give the following existence theorem which improves a beautiful result of Bérard-Bergery [BB77] (see also pp.159, pp.234-235 in [Be78]).

**Theorem 5.1.** On any homotopy sphere $\Sigma^{2n}$ in $2\Theta_{2n}(n \geq 3)$, there exists a Riemannian metric so that it possesses $SC^p$-property at two points, say $m_+$ and $m_-$. Furthermore under the same metric, there exists an isoparametric function $f$ on $\Sigma^{2n}$ with focal set $C(f) = \{m_+, m_\}$. The later property means that $\Sigma^{2n}$ is locally harmonic at both points $m_+$ and $m_-$. 

**(Proof.** We first observe that, given any homotopy sphere $\Sigma^{2n} \in 2\Theta_{2n}$, there exists an orientation-preserving diffeomorphism $\eta : S^{2n-1} \to S^{2n-1}$ such that

$$
\Sigma^{2n} = D^{2n} \cup_{\eta \circ \tau_0} D^{2n}.
$$

Let $(S^{2n-1}, g_0)$ be the standard unit sphere with volume element $\omega_0$ and the standard antipodal map $\tau_0$. By the constructions in Theorem 2.2 and Theorem 1.1, we know that $\Sigma^{2n}$ can be represented as a union of two $D^{2n}$ spheres. The orientations of $\eta$ and $\tau_0$ are fixed so that $\eta \circ \tau_0$ and $\tau_0 \circ \eta$ are both orientation-preserving.

The conditions (1) and (2) imply the existence of a metric $g$ and an isoparametric function $f$ with $C(f) = \{m_+, m_\}$, and condition (3) guarantees that the metric $g$ satisfies the $SC^p$-property at $m_+$ and $m_-$. 

To construct the desired $\varphi$, let $S_+ := \{x = (x_1, x_2, \ldots, x_{2n}) \in S^{2n-1} | x_1 \geq 0\}$, and $S_- := \{x = (x_1, x_2, \ldots, x_{2n}) \in S^{2n-1} | x_1 \leq 0\}$. First, choose a diffeomorphism $\zeta : S^{2n-1} \to S^{2n-1}$, which is the restriction of an orientation-preserving diffeomorphism from $D^{2n}$ to $D^{2n}$, such that the composition $\eta' := \eta \circ \zeta$ satisfies $\eta'|_{S_+} = \text{id}$ and $\eta'$ is isotopic to $\eta$. Thus we get two forms $(\eta')^* \omega_0$ and $\omega_0$ on $S^{2n-1}$, which are equal on the domain $S_+$ and have equal integral over the domain $S_-$. Applying Lemma 2.1 to the two forms, we have a diffeomorphism $u : S^{2n-1} \to S^{2n-1}$ such
that \( u \) is isotopic to id, \( u|_{S^+} = \text{id} \), and \( u'((\eta')^*\omega_0) = \omega_0 \). Define a diffeomorphism \( \tilde{\eta} := \eta' \circ u \). It is clear that \( \tilde{\eta}|_{S^+} = \text{id} \), \( (\tilde{\eta})^*\omega_0 = \omega_0 \).

Now, the diffeomorphism \( \varphi \) is defined by the compositions \( \varphi := \tau_0 \circ \tilde{\eta} \circ \tau_0 \circ \tilde{\eta} \). We are left to verify that \( \varphi \) possesses the required properties. For condition (1), we note that \( \tau_0 \) is isotopic to id on the odd dimensional sphere \( S^{2n-1} \). While condition (2) is evident to verify. As for condition (3), since \( \tau_0 \circ \tilde{\eta} \circ \tau_0 \) is the identity on \( S_- \), it follows that \( \tau_0 \circ \tilde{\eta} \circ \tau_0 \) commutes with \( \tilde{\eta} \), and thus \( \varphi \circ \tau_0 = \tau_0 \circ \varphi \). It completes the proof.

**Remark 5.1.** The first half conclusion of Theorem 5.1 was proved by Bérard-Bergery [BB77] (pp.237. see also pp.4, pp.234, pp.235 of [Be78]). The latter half means that every geodesic hypersphere centered at \( m_+ \) or \( m_- \) is of constant mean curvature. In particular, there exists at least one exotic sphere of dimension 10 which possesses these properties.

As for odd dimensional homotopy spheres, we have the following non-existence result.

**Theorem 5.2.** Let \( \Sigma^7 \) be a 7-dimensional homotopy sphere and \( f : \Sigma^7 \to \mathbb{R} \) a transnormal function with the focal set \( C(f) = \{ m_+, m_- \} \) under some metric \( g \). Suppose that \( (\Sigma^7, g) \) satisfies the \( SCP \)-property at either \( m_+ \) or \( m_- \). Then \( \Sigma^7 \) is diffeomorphic to \( S^7 \) or the element [14] in \( \Theta_7 = \mathbb{Z}/28 \).

**Proof.** As in the proof of Proposition [14], let \( 2\delta := d(m_+, m_-) \), where \( d \) is the distance function with respect to the Riemannian metric \( g \). Since \( f \) is a transnormal function with \( C(f) = \{ m_+, m_- \} \), \( \Sigma^7 \) has the following decomposition,

\[
\Sigma^7 = \exp D_\delta(T_{m_+}, \Sigma^7) \cup \exp D_\delta(T_{m_-}, \Sigma^7),
\]

where \( D_\delta(T_{m_+}, \Sigma^7) \) and \( D_\delta(T_{m_-}, \Sigma^7) \) are the discs of radius \( \delta \) at \( T_{m_+} \Sigma^7 \) and \( T_{m_-} \Sigma^7 \) respectively. What’s more, \( \Sigma^7 \) is diffeomorphic to \( D^7(1) \sqcup_{\sigma} D^7(1) \), where \( \sigma : S^6(1) \to S^6(1) \) is determined by the following diagram,

\[
\begin{array}{c}
S^6(1) \leftarrow D^7(1) \xrightarrow{\rho^+} T_{m_+} \Sigma^7 \xrightarrow{\exp_{m_+}} \Sigma^7 \\
\downarrow \sigma \hspace{1cm} \| \\
S^6(1) \leftarrow D^7(1) \xrightarrow{\rho^-} T_{m_-} \Sigma^7 \xrightarrow{\exp_{m_-}} \Sigma^7,
\end{array}
\]

where \( \rho^+ \) and \( \rho^- \) are the canonical linear diffeomorphisms.

The assumption that \( (\Sigma^7, g) \) satisfies the \( SCP \)-property at the point \( m_+ (m_-) \) implies that it satisfies the \( SCP \)-property at the point \( m_- (m_+) \). That is to say, the assumption can deduce that \( (\Sigma^7, g) \) satisfies the \( SCP \)-property at both of the 2 points \( m_+ \) and \( m_- \). Thus we obtain

\[
\sigma \circ \tau_0 = \tau_0 \circ \sigma.
\]

Define a map

\[
\mathcal{A} : \Theta_7 = \mathbb{Z}/28 \to \Theta_7 = \mathbb{Z}/28,
\]

\[
D^7 \sqcup \varphi D^7 \mapsto D^7 \sqcup_{\tau_0 \circ \varphi \circ \tau_0} D^7.
\]
It is obvious that $\mathcal{A}$ is a well-defined homomorphism. According to [ADPR], $\mathcal{A}$ is determined as follows,

$$\mathcal{A} : \Theta_7 = \mathbb{Z}_{28} \to \Theta_7 = \mathbb{Z}_{28},$$

$$[n] \mapsto [-n]$$

Now, we have the diffeomorphisms $\mathcal{A}(\Sigma^7) = \mathcal{A}(D^7 \sqcup_{\iota} D^7) = D^7 \sqcup_{\iota \circ \sigma_0} D^7 = D^7 \sqcup_{\tau} D^7 = \Sigma^7$.

Hence, it follows that $\Sigma^7 = [0]$ or $[14] \in \mathbb{Z}_{28} = \Theta_7$. □

**Remark 5.2.** Similarly, let $\Sigma^{15}$ be a 15-dimensional homotopy sphere which bounds a parallelizable manifold and $f : \Sigma^{15} \to \mathbb{R}$ a transnormal function with the focal set $C(f) = \{m_+, m_-\}$ under some metric $g$. Suppose in addition that $(\Sigma^{15}, g)$ satisfies the $SCP$-property at either the point $m_+$ or $m_-$. Then $\Sigma^{15}$ is diffeomorphic to $S^{15}$ or $\Sigma^{15} = [4064] \in bP_{16} = \mathbb{Z}_{8128}$.

**Remark 5.3.** Very recently, Tang and Zhang [TZ13] solved a problem of Béard-Bergery and Besse. That is, they showed that every Eells-Kuiper quaternionic projective plane carries a Riemannian metric with $SCP$ property for certain point $p$. We don’t know how to improve our Proposition 3.2. For instance, we don’t know whether there is a metric on every Eells-Kuiper quaternionic projective plane with not only the property in Proposition 3.2, but also the $SCP$-property.

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**References**

[ADPR] U. Abresch, C. Durán, T. Püttmann and A. Rigas, *Wiedersehen metrics and exotic involutions of Euclidean spheres*, J. für die reine und angew. Math., 605 (2007), 1–21.

[BB77] L. Béard-Bergery, *Quelques exemples de variétés riemanniennes où toutes les géodésiques issues dun point sont fermées et de même longueur suivis de quelques résultats sur leur topologie*, Ann. Inst. Fourier, 27 (1977), 231–249.

[Be78] A. L. Besse, *Manifolds all of whose geodesics are closed*, with appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger, and J. L. Kazdan, Ergeb. Math. Grenzgeb. Vol. 93, Springer, Berlin, 1978.

[CCJ07] T. E. Cecil, Q. S. Chi and G. R. Jensen, *Isoparametric hypersurfaces with four principal curvatures*, Ann. Math., 166 2007, no.1, 1–76.

[CR85] T. E. Cecil and P. T. Ryan, *Tight and taut immersions of manifolds*, Research Notes in Math. 107, Pitman, London, (1985).

[DM90] B. Dacorogna and J. Moser, *On a partial differential equation involving the Jacobian determinant*, Ann. de l’I. H. P., 7 (1990), 1–26.

[DR92] H. B. Duan and E. G. Rees, *Functions whose critical set consists of two connected manifolds*, Papers in honor of Jos Adem (Spanish). Bol. Soc. Mat. Mexicana (2), 37 (1992), no. 1-2, 139–149.

[EK1] J. Eells and N. H. Kuiper, *Manifolds which are like projective planes*, Pub. Math. IHES, 14 (1962), 5–46.

[Fr68] D. L. Frank, *An invariant for almost closed manifolds*, Bull. Amer. Math. Soc., 74 (1968), 562–567.

[GT12] J. Q. Ge and Z. Z. Tang, *Isoparametric functions and exotic spheres*, J. für die reine und angew. Math., DOI: 10.1515/crelle-2012-0005, March 2012.
[GTY11] J. Q. Ge, Z. Z. Tang and W. J. Yan, A filtration for isoparametric hypersurfaces in Riemannian manifolds, arXiv:1107.5234, 2011.

[Hi76] M. W. Hirsch, Differential Topology, Graduate Texts in Math., 33, Springer-Verlag, New York, 1976.

[HH67] W. C. Hsiang and W. Y. Hsiang, On compact subgroups of the diffeomorphism groups of Kervaire spheres, Ann. Math., 85 (1967), 359–369.

[KM63] M. A. Kervaire and J. Milnor, Group of homotopy spheres: I, Ann. Math., 77 (1963), 504–537.

[KV86] O. Kowalski and L. Vanhecke, A new formula for the shape operator of a geodesic sphere and its application, Math. Z., 192 (1986), 613–625.

[MM80] B. M. Mann and E. Y. Miller, The construction of the Kervaire sphere by means of an involution, Michigan Math. J., 27 (1980), 301–308.

[Mil56] J. Milnor, On manifolds homeomorphic to the 7-sphere, Ann. Math., 64 (1956), 399–405.

[Mil59] J. Milnor, Differential structures on spheres, Amer. J. Math., 81 (1959), 962–972.

[Mil07] J. Milnor, Collected papers of J. Milnor, III, Differential Topology, Amer. Math. Soc., (2007).

[Miy12] R. Miyaoka, Isoparametric hypersurfaces with \((g, m) = (6, 2)\), Ann. Math., 177 (2013), 53–110.

[Mü80] H. F. Münzner, Isoparametric hyperflächen in sphären, I and II, Math. Ann., 251 (1980), 57–71 and 256 (1981), 215–232.

[Mo65] J. Moser, On the volume elements on a manifold, Trans. Amer. Math. Soc., 120 (1965), 286–294.

[TY13] Z. Z. Tang and W. J. Yan, Isoparametric foliation and Yau conjecture on the first eigenvalue, to appear in J. of Diff. Geom.

[TZ13] Z. Z. Tang and W. P. Zhang, On a problem of Bérard-Bergery and Besse, arXiv:1302.2792.

[Vil70] J. Vilms, Totally geodesic maps, J. of Diff. Geom., 4 (1970), 73–79.

[Wa87] Q. M. Wang, Isoparametric functions on Riemannian manifolds. I, Math. Ann., 277 (1987), 639–646.