Non-Linear Electrodynamics in Blandford–Znajek Energy Extraction

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1. Introduction

Maxwell’s electromagnetic theory (MED) is a widely used fundamental theory in both quantum physics and the context of cosmology. It is a well-known and recognized theory. In 1933 and 1934 Born and Infeld made the first attempts to change equations of MED \cite{1,2} and tried to eliminate the divergence of the electron’s self-energy in classical electrodynamics. The Born–Infeld electrodynamics model does not contain any singularities because its electric field starts at its highest value at the center (which is equal to the nonlinearity parameter $b$), and then gradually decreases until it behaves like the electric field of Maxwell at longer distances. This model also ensures that the energy of a single point charge is limited. The parameter $b$ has a connection to the tension of strings in the theory\cite{3,4}, and there have been studies done to determine potential constraints for the value of $b$ in refs. \cite{5–11}.

In contrast to the Euler–Heisenberg electrodynamics\cite{12}, the Born–Infeld model does not show vacuum birefringence when subjected to an external electric field. The Born–Infeld theory maintains both causality and unitarity principles. The Born–Infeld electrodynamics has served as inspiration for other models that are free of singularities and possess similar properties. For instance, various models presented by Kruglov in refs. \cite{13–24}. Fang and Wang have presented a fruitful method for finding black hole solutions that have either electric or magnetic charges, in a theory that combines general relativity with a non-linear electrodynamics.\cite{25} Since then, numerous models have been advocated, and the effects of these theories—known as non linear electrodynamics (NLED)—have been investigated in a wide range of contexts, not just those related to cosmology and astrophysics,\cite{26–44} but also in non-linear optics,\cite{45} high power laser technologies and plasma physics,\cite{46,47} nuclear physics,\cite{48,49} and superconductors.\cite{50} Many gravitational non-linear electrodynamics (G-NED), extensions of the Reissner–Nordstrom (RN) solutions of the Einstein–Maxwell field equations have gained a lot of attention (see refs. \cite{50–54} and references therein). Additionally, Stuchlik and Schee have demonstrated that models that produce the weak-field limit of Maxwell are considered relevant, as opposed to those that do not provide the correct enlargement of black hole shadows in the absence of charges.\cite{39} In
particular, the existence of axially symmetric non-linear charged black holes (at least transiently) has been studied,[59] indicating neutrinos as good probes thanks to their bountiful production in any astrophysical context. As a consequence, it would be interesting, in principle, to investigate the nature of electromagnetism (linear or not), due to different signatures in certain neutrino phenomena, such as neutrino oscillations, spin-flip, and r-processes. The effect of non-linear phenomena on the BH shadow, BH thermodynamics, deflection angle of light and also wormholes have been investigated too.[56–69] In the context of primordial physics, instead, NLED, when coupled to a gravitational field, can give the necessary negative pressure and enhance cosmic inflation[70] and some models also prevent cosmic singularity at the big bang[71–75] and ensure matter–antimatter asymmetry.[76] The reason to consider NLED in the primordial Universe comes from the assumption that electromagnetic and gravitational fields were very strong during the evolution of the early universe, thereby leading to quantum correction and giving birth to NLED.[77,78] Recently, the non-linear electrodynamics has also been invoked as an available framework for generating the primordial magnetic fields (PMFs) in the Universe.[79,80] The latter, indeed, is a still open problem of the modern cosmology, and although many mechanisms have been proposed, this issue is far to be solved. Seed of magnetic fields may arise in different contexts, for example, string cosmology,[83] inflationary models of the Universe,[82,83] non-minimal electromagnetic-gravitational coupling,[84,85] gauge invariance breakdown,[86,87] density perturbations,[88] gravitational waves in the early Universe,[89] Lorentz violation,[89] cosmological defects,[90] electroweak anomaly,[91] temporary electric charge non-conservation,[92] trace anomaly,[93] parity violation of the weak interactions.[94] The current state of art points to an unexplained physical mechanism that creates large-scale magnetic fields and seems to be present in all astrophysical contexts. They might be remnants of the early Universe that were amplified later in a galactic period, according to one idea. To create such large-scale fields, super-horizon correlations can only still be created during inflationary epochs. However, it is still unclear how the electromagnetic conformal symmetry is broken. Different theoretical techniques have been taken into consideration for this, most notably non-minimal coupling with gravity, which by its very nature broke conformal symmetry ([95] and references therein). In a minimal scenario, electromagnetic conformal invariance can also be overcome. In this instance, the major goal is to modify the electromagnetic Lagrangian to a non-linear function of $F = (1/4) F_{\mu \nu} F^{\mu \nu}$, as done in refs.[79, 80, 96].

Since all NLED models significantly depend on scale factors (dimensionless or not), which may cause overlaps with other physics observables, it is obvious that determining the presence of non-linear phenomena is not free of uncertainty. Energy extraction from black holes, which is connected to various significant astrophysical events, including black hole jets and therefore Gamma-ray bursts (GRBs), is one area where NLED effects have not yet been properly studied.[97] GRBs are flashes of non-thermal bursts of low energy ($\approx 100 \text{ keV}–1 \text{ MeV}$) photons and release an energy of the order $10^{51}–10^{53}$ erg in a few seconds.[98] They are classified in two classes: the short-hard bursts (with duration $\approx 2$ s) and long soft bursts (with duration $\gtrsim 2$ s). Long duration bursts are produced by the core collapse of massive stars, while the origin of short-duration bursts is induced by the coalescing of compact binaries. The Blandford–Znajek (BZ) process[99–104] and the (very recent) magnetic reconnection mechanism[105,106] are the two different energy extraction techniques used today, along with a revised version of the original Penrose process[107] called magnetic Penrose process.[108–111] Among them, the BZ mechanism is still the most widely accepted theory to explain high energy phenomena[112,113] (even if there are still open questions in certain models or combinations[114–116]). It involves a magnetic field generated by the accretion disk, whose field lines are accumulated during the accretion process and twisted inside the rotating ergosphere. Charged particles within the cylinder of twisted lines can be accelerated away from the black hole, composing the jets. A characteristic feature of this mechanism is that the energy loss rate decays exponentially. This has been confirmed in a good fraction of observations (X-ray light curves) of GRBs.[117] Furthermore, black holes with brighter accretion disks have more powerful jets implying a correlation between the two. Even if accretion onto a black hole is the most efficient process for emitting energy from matter it is not able to reach the energy rate of the GRBs, while other energy extraction ways such as the Hawking radiation give predictions on temperature, time-scale, and energy rate highly in conflict with the observations.[118] Numerical models of black hole accretion systems have significantly progressed our understanding of relativistic jets indicating two types of jets, one associated with the disc that is mass-loaded by disc material and the other associated directly with the black hole.[119] In the first case, however, jets with high Lorentz factors are not supported. The BZ process, which produces highly relativistic jets by electromagnetically extracting black hole spin energy, remains the most astrophysically plausible mechanism to do so and is in good agreement with direct observations.[120] In this sense, understanding the general relativistic magnetohydrodynamic (GRMHD) model of the bulk flow dynamics near the black hole (where relativistic jets are formed) is essential to study the central engine.

In this paper, in order to determine if non-linear effects may change the rate of energy extraction and the magnetic field configuration surrounding a (non-charged) black hole encircled by its magnetosphere, we will investigate the Blandford–Znajek mechanism in the context of the NLED framework.

The layout of the paper is as follows: In Section 2 we derive, for the first time, the general version of energy flux up to second order in the spin parameter. Section 3 is devoted to computing and solving the magnetohydrodynamic problem, searching, in particular, for separated (monopole and paraboloid) solutions. In Section 4 we give some estimates of the energy extraction with respect to standard BZ mechanism. We study primordial magnetic fields from (minimally coupled) NLED for different non-linear models in Section 5, while discussion and conclusions are drawn in the Section 6. In this work, we adopt natural units $G = c = 1$ and for simplicity set $M = 1$ in order to handle adimensional quantities ($r, a, ...$). The negative metric signature $(+, −, −, −)$ is also adopted.

2. Non-Linear Magnetohydrodynamics

In this section, following refs. [99] and [121] we derive the energy extraction rate for a spinning, non-charged black hole in presence of stationary, axisymmetric, force-free, magnetized plasma,
and an externally sourced magnetic field. In the Boyer–Lindquist horizon penetrating coordinates, the axially symmetric spacetime line element is

\[
d s^2 = \left(1 - \frac{2r}{\Sigma}\right) dt^2 - \left(\frac{4r}{\Sigma}\right) dr^2 - \left(1 + \frac{2r}{\Sigma}\right) d\theta^2 - \sin^2 \theta \left[\Sigma + a^2 \left(1 + \frac{2r}{\Sigma}\right)\right] d\phi^2 + \left(\frac{4arsin^2 \theta}{\Sigma}\right) dt d\theta \\
+ 2a \left(1 + \frac{2r}{\Sigma}\right) \sin^2 \theta d\phi dr
\]

where \( \Sigma := r^2 + a^2 \cos^2 \theta, \Delta = r^2 - 2r + a^2. \) The metric determinant is \( g := |\det(g_{\mu \nu})| = -\Sigma \sin^2 \theta. \) We consider now a general electromagnetic Lagrangian governing the surrounding plasma and call it \( L_{\text{NLED}}^{\text{EM}}; \) it is generally a function of the two invariants \( X := (1/4)F_{\mu \nu} F^{\mu \nu} \) and \( G := (1/4)F_{\mu \nu} F^{\mu \nu}, \) where, called \( A_\mu = (\Phi, -A) \) the four-potential, \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field strength tensor and \( F^{\mu \nu} = \frac{1}{2} F_{\mu \lambda} e^{\lambda \mu \nu} \) is its dual \( (\epsilon \text{ is the anti-symmetric Levi–Civita tensor}). \) Clearly, Maxwell theory is recovered when \( L_{\text{NLED}}^{\text{EM}} = -X. \) The energy–momentum tensor, in absence of magnetic charges, is, which automatically satisfies Equation (4). The radial energy and angular momentum flux, as measured by a stationary long-distance observer, are given by

\[
F_{E}^{(\mu)} := T_{\nu}^{\mu}, \quad F_{\phi}^{(\mu)} := -T_{\nu}^{\mu}
\]

Therefore

\[
F_{E}^{(\mu)} = -L_X (F_{\mu \alpha} F_{\alpha \beta} + F_{\mu \alpha} F_{\beta \alpha^\prime} - F_{\alpha \beta^\prime} F_{\alpha^\prime \beta}) g^{\alpha \beta}
\]

and hence

\[
F_{E}^{(\mu)} = L_X \left[2 B_r^2 w_r \left(w - \frac{\alpha}{2r} \right) + w B_r \Delta \right] \sin^2 \theta
\]

while the angular momentum flux is \( F_{\phi}^{(\mu)} = \frac{F_{\phi}}{w}. \) On the horizon, \( r_+ := 1 + \sqrt{1 - a^2}, \) Equation (10) reads as

\[
F_{\phi}^{(\mu)} := -2 F_{\phi} \left| \frac{L_X}{r_+} \right| \sin^2 \theta
\]

where \( L_X^{(\phi)} := L_X (r_+, \theta) \) and \( \Omega_{\phi} := a/(2r_+) \) is the angular velocity of the horizon. Apart from the factor \( L_X, \) these relations are equal to the linear (Maxwell) case. However, although the change is minimal, the physical consequences could be decisive. Indeed, \( F_{\phi}^{(\mu)} > 0 \) not only if \( 0 < w < \Omega_{\phi}, \) but also if \( L_X < 0 \) at the horizon. Moreover, since \( L_X \) is a function of \( X, \) and [121]

\[
X = \frac{1}{2} \left[R_r^2 (1 - w^2) + B_r^2 (1 - w^2) + B_\phi^2 \right]
\]

the energy flux will depend not only on the radial magnetic field \( B_r, \) but in general also on the other two components, namely \( B_\theta \) and \( B_\phi. \) The power extracted (energy rate) is

\[
p^{\text{NLED}} := \int \int d\phi d\theta \sqrt{-g} F_{\phi}^{(\mu)} = 4\pi \int_{0}^{\epsilon \phi} d\theta \sqrt{-g} F_{\phi}^{(\mu)}
\]

In order to evaluate \( p^{\text{NLED}}, \) we need to solve MHD equations and find the expressions for \( B_r, B_\theta, \) and \( B_\phi. \) This is not an easy task, being quite laborious already in the standard Maxwell theory. As a first approach, we can certainly proceed with a perturbative series expansion in powers of \( a, \) as originally done in ref. [99]. Since typically one assumes \( w = \Omega/2, \) then \( F_{\phi} \propto a^2 \) so a Schwarzschild solution (i.e., \( a = 0 \)) is fine to obtain an expression for \( p^{\text{NLED}} \) good up to second order in the spin parameter. It is clear that such a relation would be accurate only in the regime \( a \ll 1. \)

Since we want to completely solve the magnetohydrodynamic equations, instead of Equation (3), we use the (equivalent) set of equations \( F_{\mu \nu} J^\mu = 0, \) coming from free-force approximation. Only two equations are independent, and they give

\[
J_{\phi} = -\mu(r, \theta) B_r, \quad J_{\phi} = -\mu(r, \theta) B_\theta, \quad J_{\phi} = -\mu(r, \theta) B_\phi + J w
\]

where we defined \( \mu := J_{\phi}/B_\phi = -J_{\phi}/B_\phi + J w. \) The above equations are formally equivalent to those of ref. [99] and seem not to depend a priori on the specific NLED model. However, when coupled to Maxwell equations, difference with the linear theory appears clear. Indeed, in order to find the explicit expression for \( \mu \) and \( J \), from Equation (4), we get the following set of equations:
\[ \frac{\partial}{\partial t} \left[ \sin^2 \theta L_{\chi} B_\theta \left( \Delta w Y + 4r^2 w - 2ra \right) \right] + \partial_\theta \left[ \sin^2 \theta L_{\chi} \left( 2r B_\phi - w B_\theta \right) \right] = -J_\Sigma \sin \theta \]

\[ A_{\chi} \left[ \sin^2 \theta L_{\chi} B_\theta \left( \Delta w Y + 4r^2 w - 2ra \right) \right] + \partial_\theta \left[ \sin^2 \theta L_{\chi} \left( 2r B_\phi - w B_\theta \right) \right] = -J_\Sigma \sin \theta \]

Together with Equations (14) and in a very similar way to ref. [99], they lead to\[99,123,124\]

where the explicit dependence on \( L_{\chi} \) is shown. We will call \( B_\theta \) the expression in square brackets by analogy with ref. [99], even if, in our notation and coordinates, it will not be properly the toroidal field.

By putting \( \partial_\theta \) from Equation (15) into Equation (14) and by using Equation (16), the important differential equation for \( A_{\phi} \) is found

\[ \frac{d B_\phi}{d A_{\phi}} = \frac{\mu}{\Sigma \sin \theta} \left[ \sin^2 \theta L_{\chi} \left( \Delta w Y + 4r^2 w - 2ra \right) \right] + \partial_\theta \left[ \sin^2 \theta L_{\chi} \left( 2r B_\phi - w B_\theta \right) \right] = -J_\Sigma \sin \theta \]

Notice that \( w, B_\phi, \) and \( B_\theta \) are functions (only) of \( A_\phi \) by definition of \( F_{\mu\nu} \), hence Equation (16) implies \( B_\theta \) is only a function of \( A_\phi \). In summary, our first unknowns \( (B_\phi, B_\theta, B_\phi, w) \), after using the ideal approximation in Equation (6), Maxwell equations (Equation (4)) and the free-force approximation, have been reduced to one, namely \( A_\phi \). Equation (17), for \( L_{\chi} = -1 \), is also known as “stream equation,” and its solution \( A_\phi \) is called “stream function.”\[125\]

### 3. Separated Solutions

In this section, we solve Equation (17) in the static limit \( a = 0 \). This will be sufficient to have an expression for the extracted power up to second order in \( a \).

Following\[99\] we assume that for \( a \ll 1 \)

\[ A_{\phi} = A_{\phi}^{(0)} + a^2 A_{\phi}^{(2)} + O(a^4) \]

\[ B_\phi = a B_\phi^{(1)} + O(a^2) \]

\[ w = aw^{(1)} + O(a^3) \]

while \( B_\phi = w = 0 \) when \( a = 0 \). The functions \( B_\phi^{(1)}, w^{(1)}, \) and \( A_{\phi}^{(2)} \) are unknowns, while \( A_{\phi}^{(0)} \) is just the solution for Schwarzschild case. The other components of \( B \)

\[ B_\phi = -\frac{1}{\sqrt{-g}} \left( \partial_\phi A_{\phi}^{(0)} + a^2 \partial_\phi A_{\phi}^{(2)} \right) \]

\[ B_\theta = \frac{1}{\sqrt{-g}} \left( \partial_\theta A_{\phi}^{(0)} + a^2 \partial_\theta A_{\phi}^{(2)} \right) \]

It is clear that, in the static limit, the only unknown function is \( A_{\phi}^{(0)} \). Indeed, at zero order in \( a \approx O(1) \), Equation (17) becomes

\[ L A_{\phi}^{(0)} = 0 \]

where

\[ L := \frac{1}{\sin \theta \partial r} \left( 1 - \frac{2}{r} \right) \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \]

where \( L_{\chi}^{(0)} \) is \( L_{\chi} \) in the Schwarzschild limit.\[126\] For a power-law model \( L_{\chi}^{(0)} = -C X - \gamma X^2 \), it would be\[127\]

\[ L_{\chi}^{(0)} = -C - \gamma \delta \left( \frac{-1}{2 r^2 \sin \theta} \right)^{\delta-1} \]

We now consider separated solutions for \( A_{\phi} \) and also assume a similar form for \( L_{\chi} \), that is,

\[ A_{\phi}^{(0)} = R(r) \cdot U(\theta) \]

\[ L_{\chi}^{(0)} = f(r) \cdot g(\theta) \]

With this ansatz, Equation (23) reads as

\[ \frac{\partial}{\partial r} \left[ \frac{g(\theta) \partial U}{\sin \theta \partial \theta} \right] = -K \frac{g(\theta) U}{\sin \theta} \]

\[ \frac{\partial}{\partial r} \left[ f(r) \left( 1 - \frac{2}{r} \right) \frac{\partial R(r)}{\partial r} \right] = K \frac{R(r)}{r^2} \]

where \( K \) is a separation constant. We will choose \( K = 0 \) so as to obtain the simplest solution (the lowest order)\[125,128\] one.

From here on, the specific NLED model must be chosen. Assuming a power-law model\[14,126\] and hence Equation (25), we have to set \( C = 0 \), unless one assumes \( L_{\chi}^{(0)} \) is a function of just one variable, that is \( f(r) = 1 \) or \( g(\theta) = 1 \), but this would exclude most of NLED models. As a check, when \( \delta = 1 \), we obtain the known solution as given in refs. [100, 125]. For \( \delta = 2 \), the latitudinal part does not change, that is,

\[ U(\theta) = a \cos \theta + \beta \]

while the radial part strongly changes

\[ R(r) = c (6r^3 + 15r^4 + 40r^5 + 120r^6 + 480r^7 + 960 \ln(r - 2) - 2192 + d)^{1/3} \]
where $\alpha$, $\beta$, $c$, and $d$ are constants. Following [125], we note that it is impossible to have a monopole solution [130] by default, as there are no combinations of constants to eliminate the radial dependence in $A_{\psi}^{(0)}$ without canceling all $R(r)$; it follows from Equation (22) that $B_{\psi} \neq 0$. A separable paraboloidal solution $(\alpha + \beta = 0 = d)$ is instead possible:

$$A_{\psi}^{(0)} \approx (\cos \theta - 1)(6r^5 + 15r^4 + 40r^3 + 120r^2 + 480r + 960\ln(r - 2) - 2192)^{1/3}$$

(31)

For $\delta = 3$ and higher values, the angular part will be equal to Equation (29), while the radial one will be consistent only if $r < 2$, that is, beyond the event horizon, so we discard them. Same epilogue if one chooses negative powers ($\delta < 0$): no monopole solution would exist and paraboloidal one would be valid only for $r < r_+$. This could be an interesting point: monopole solutions are actually not physical, while paraboloidal magnetic configurations can explain the collimation of the jets [131,132]. It must be emphasized that the geometry of the magnetic lines depends on the distance and thickness of the accretion disk, the only structure capable of generating a magnetic field. Therefore, exact solutions would require boundary conditions (see ref. [125] and references therein) and therefore specific astrophysical scenarios. Moreover, also numerical simulations could come to our aid as done in refs. [102, 116, 119, 133]. An interesting point of difference of Equation (31) with respect to the analogous Maxwell solution is the forward displacement of the flow inversion point ($r \approx 2.35$ vs $r \approx 2.31$), that is the point in which $A_{\psi}^{(0)}$ change sign (and hence $R(r) = 0$). However, as shown in Figure 1, the main difference with linear theory is the asymptotic behavior ($r \gg 1$) of the solution, being $A_{\psi}^{(0)} \sim r'^3(1 - \cos \theta)$ with $s > 1$ in the non-linear case ($s = 1$ in linear theory). This stronger "verticality" could favor these kind of solutions in the formation of jets.

### 4. Some Estimates

In this section, starting from the result of the previous section, we find an estimate of the extracted power comparing it with the linear theory (Maxwell) case. Here, we propose two different ways.

Given the presence of the singularity at $r = 2$ we have to discard this point. In order to use Equation (13), which is evaluated on the horizon, we assume the condition $B_{\psi} \gg B_{\phi}$, which is often used in simulations [131,134]. From Equation (13), we find for the power extracted in the Maxwell case [125,135] $P$, at the second order

$$P \approx \frac{4\pi}{3r} \left[ r + 2\ln(r - 2) \right]^2 \Omega_{\psi}^2$$

(32)

where $r > 2$. On the other hand, in power-law model (31), similar computations lead to

$$P_{\text{NLED}} \approx \frac{4\pi}{3r^5} (6r^5 + 15r^4 + 40r^3 + 120r^2 + 480r + 960\ln(r - 2) - 2192)^{1/3} \Omega_{\psi}^2$$

(33)
where we used

\[ L_X^p = -\frac{1}{r^2} (6r^5 + 15r^4 + 40r^3 + 120r^2 + 480r) + 960 \ln(r - 2) - 2192 r^{2/3} \]  

(34)

instead of \( L_X^{\text{BZ}} \). Apart from the radial field approximation \( B_r \gg B_\theta \), the rate is quite accurate;\(^{[136]} \) it has been plotted as a function of \( r \) in Figure 2. From the latter, it is clear that in principle such a NLED model could really extract more energy than in the conventional case. However, it would not have a Maxwellian limit because we had to impose \( C = 0 \) to achieve the analytical solution to Equation (31).

The above estimate necessarily requires the stream function, that is, a solution of the (very involved) stream equation. Moreover, it required to force \( C = 0 \) for the power-law model. We can overcome these issues in the following way. As before, let us assume a radial field in the form \( B_r = B_0 \sin \zeta \), where \( B_0 \approx \sqrt{\sigma_0} \) is the magnetic strength as given by plasma magnetization \( \sigma_0 \) (\( \zeta \) is the angle between \( B \) and \( \phi \) at the equator). Unlike before, let us evaluate Equation (13) on the horizon \( r = r_* \). Just by assuming \( B_\theta \) negligible, it is straightforward to obtain an expression for \( p^{\text{NLED}} \) without solving the stream equation and accurate up to second order in the spin parameter. This means that such an estimate would be suitable also for non-separable NLED model, like the Kruglov one \( L_{\text{NLED}} = -X \times (1 + \beta X)^{-1}. \)\(^{[144]} \) Since in this framework \( B_\theta = B_0 \cos \zeta \), the rate with respect to Maxwell case simply is

\[ \frac{p^{\text{NLED}}}{p} = \frac{1}{\left( \frac{\varphi}{2} \frac{B^2}{B_0^2} + 1 \right)^2} \]  

(35)

A similar computation was done for \( L_{\text{NLED}} = -X - \gamma X^2 \) and a comparison between these two different NLED models has been reported in Figure 3. It is evident the advantage of power-law model with positive exponent.\(^{[137,138]} \) In general, we have

\[ \frac{p^{\text{NLED}}}{p} = -L_X(X_0) \]  

(36)

where we defined \( X_0 := B_0^2/2 \). The strong dependence on the specific NLED model is clearly explicit: the ratio of energy power in presence of a non-linear electrodynamic model \( p^{\text{NLED}} \) to linear (Maxwell) case \( p \) is simply given by the opposite of the derivative of the Lagrangian with respect to \( X \) evaluated at \( X_0 := B_0^2/2 \). Notice that the two methods hold true in different regimes. While in the first way an assumption of type \( B_r \gg B_\theta \) must be made, in the second estimate one needs \( B_\theta \ll 1 \). One can use one or the other depending on the specific context. From an astrophysical point of view, both observations and simulations suggest that the magnetic field around massive black holes has a poloidal configuration, that is, the field lines lie in planes containing the axis of rotation.\(^{[139–141]} \) Therefore, the assumption \( B_r \gg B_\theta \) is physically achievable. About the second assumption, since a purely radial solution (monopole) is not likely, we generally expect \( B_\theta \neq 0 \). However, except in extreme paraboloidal cases, the polar component of the magnetic field is negligible for distances well beyond the event horizon (see figure 1 in ref. [100]).

5. Primordial Magnetic Field from NLED

According to general relativity (GR) primordial fields decayed adiabatically due to conservation of the flux that is, \( a^2 B \approx \text{const} \), and hence \( B \approx 1/a^2 \). Consequently, the magnetic energy density \( \rho_\phi = |B|^2/(8\pi) \) should have scaled as \( 1/a^4 \), where \( a \) is the scale factor of the (flat) Friedman–Robertson–Walker (FRW) metric. Since the scale factor diverges during inflation, this type of decay implies very faint magnetic fields at the end of the inflation period. This scaling is valid for every cosmic energy density present in the Universe, and then also for the cosmic microwave background (CBM), whose energy density (assumed almost constant during inflationary era) is given by \( \rho = \pi^2 T^4/25, \) or, in function of \( a \), by \( \rho \sim 1/a^4 \) (the extra factor \( 1/a \) with respect to matter, which decays as \( \approx 1/a^3 \), comes from energy redshift). Therefore, the ratio \( r = \rho_B/\rho_\phi \) remained constant until today, with a current value of \( r \approx 1 \) and this constraint is a good tool to study primordial fields. The origin of large-scale magnetic fields has been studied not only in the context of GR but also in extended or alternative theories of gravity.\(^{[142–144]} \) The main idea behind such works is to assume the non-conservation of the flux, breaking the conformal invariance of the electromagnetic sector and hence making possible a different trend from the adiabatic one for \( B \).

In this section, following refs. [79, 95, 96] and in the context of GR, we try to find constrains on some NLED models, exploiting existence and survival of PMFs. We start from the action

\[ S = \int d^4x \sqrt{-g} \frac{R}{2k^2} + L_{\text{NLED}} \]  

(37)

where \( \chi^2 = 8\pi \) and \( L_{\text{NLED}}(X,G) \) encodes a general electromagnetic theory (see Section 2). It is clear that Maxwell theory is
Figure 3. Estimation of the ratio between the power extracted from a black hole (through BZ mechanism) in presence of non-linear electrodynamics \( P^{\text{NLED}} \) and the same quantity in Maxwell theory \( P \), as a function of the magnetic field strength (see Equation (36)). Several NLED models have been taken into account: power-law \( (C = 1, \gamma = 0.01, \delta = 2) \), Kruglov \( (\beta = 0.01) \), Ovgun–Benaoum \( (a = \beta = 0.01) \), and Ovgun exponential \( (a = 0.001, \beta = 0.99) \). The ratio is accurate up to second order in the spin parameter for a BH with \( M = 1 \). Only one approximation has been used, namely \( B \approx 0 \).

As it is clear, the behavior strongly depends on the specific NLED model; however, a power law model with a positive exponent could in principle extract more energy (while at low magnetic field it behaves like linear electromagnetism), although some regimes could be excluded in order not to exceed the Eddington limit.

obtained when \( L_{\text{NLED}} = -X \). Varying the action with respect to the electromagnetic field \( A_{\mu} \), the field equations are

\[
\partial_{\mu} \left[ \sqrt{-g} \left( L_{\text{X}} F^{\mu \nu} + L_{C} F^{\mu \nu} \right) \right] = 0
\]

\[
\partial_{\mu} F^{\mu \nu} = 0
\]

which are the source free of zero density version of Equations (4) and (5). We consider here a conformally flat FRW metric

\[
d^2 s = a^2(\eta) \left( d\eta^2 - dx^2 \right) = dt^2 - a(t) dx^2
\]

where \( \eta = \int_0^t a^{-1}(t) dt \) is the conformal time and \( a(t) \) is a dimensionless scale factor. In this metric and with our signature, \( F_{\mu \nu} \) can be written as

\[
F_{\mu \nu} = a^2(\eta) \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}
\]

in order to separate highlight the electric and magnetic fields as measured by a comoving (inertial) observer. With this ansatz and assuming the non-existence of magnetic charge, Equation (38) becomes

\[
A_{\mu}^\prime + \frac{L_{\text{X}}}{L_{\text{X}}^\prime} A_{\mu} - \partial_{\nu} \frac{L_{\text{X}}}{L_{\text{X}}^\prime} (\partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu}) - \Delta A_{\mu} = 0
\]

where \( j = 1, 2, 3 \), \( \Delta = \delta_{ij} \partial_{j} \partial_{i} \) and a prime denotes derivative w.r.t. conformal time. The above equation can be also written in terms of the electric \( E_{j} \) and magnetic \( B_{j} \) fields as

\[
\partial_{\mu} \left( a^2 L_{\text{X}} E_{\mu} \right) - a^2 \nabla \left( L_{\text{X}} B_{\mu} \right) = 0
\]

The zero component of Equation (38) reads as

\[
\nabla \left( L_{\text{X}} E \right) = 0
\]

while the Bianchi identity gives

\[
\partial_{\mu} \left( a^2 B_{\mu} \right) + a^2 \nabla \times E = 0
\]

as well as the usual constrain \( \nabla \cdot B = 0 \). Combining Equations (43) and (45) one obtains

\[
L_{\text{X}} F_{\mu \nu} + \left( \partial_{\nu} L_{\text{X}} \right) F_{\mu} = 0
\]

where we defined \( F := a^2 B \) with \( B := |B| \) and assumed the long-wavelength approximation \( \frac{a(t)}{a(t)} \) (i.e., disregarding spatial derivatives). Notice that in this approximation \( F = F(\eta) \). By choosing a power-law model \( L_{\text{NLED}} = -\alpha X - \gamma X^2 \), we recover the same results of ref. [96]. Here, we focus on other non-linear Lagrangian. As a first model, we consider \( L_{\text{NLED}}(X) = -\alpha X - \beta X^\gamma \)

\[
L_{\text{NLED}}(X) = \alpha + \beta X^\gamma
\]

where \( \alpha \) and \( \beta \) are two real parameters with \( \beta \) controlling the non-linearity contributions. A suitable condition for obtaining an
analytical solution is the strong regime \((2a^4/F^2) \ll \beta\), that is, \(B \gg B_0\) with \(B_0 := \sqrt{2}/\rho_{\text{min}}\), it follows from Equation (46)

\[
\frac{d^2 F}{da^2} + \left[ 1 - \frac{4}{a} (a + 1) \right] \frac{d F}{da} + \frac{8 (a + 1)}{a^2} F^2 = 0
\]

(48)

where \(s = -1, 2, 1\) depending on which primordial phase we are considering, that is, inflation, reheating, radiation [95,345]. Assuming a power-law solution for \(F, F \approx a^\alpha\), the inflationary exponent is

\[
p_\alpha = \frac{1}{2} \left[ 3 + 4\alpha \pm \sqrt{\alpha} \right]
\]

(49)

where we defined \(\Sigma := 16\alpha^2 - 8\alpha - 23\). This solution clearly constrains the parameter \(\alpha\) to be either \(\alpha > \frac{1}{4}(1 + 2\sqrt{6})\) or \(\alpha < \frac{1}{4}(1 - 2\sqrt{6})\). When \(\alpha = \frac{1}{4}(1 \pm 2\sqrt{6})\) a pseudo-power-law solution is possible, namely

\[
F(a) = c_1 a^\alpha - c_2 a^\alpha \ln(3 + 4a)
\]

(50)

with \(p = 3/2 + 2\alpha\).

In the reheating epoch, the power-law solution \(F \approx a^\alpha\) reads as

\[
q_\alpha = \frac{1}{2} \left[ \frac{1}{2} + 4(1 + \alpha) \pm \sqrt{\alpha} \right]
\]

(51)

with \(\Sigma := 16\alpha^2 + 4\alpha - 47/4\) and is valid when \(\alpha < \frac{1}{8}(-1 - 4\sqrt{3})\) or \(\alpha > \frac{1}{8}(-1 + 4\sqrt{3})\). When a pseudo-power-law solution for the remaining cases has \(q = 1/4 + 2(1 + \alpha)\). With these solutions, it is possible to express the strong regime assumption in terms of conformal time, that is, \(\eta \gg \eta^*\) with \(\eta^*\)

\[
\eta^* := \left[ \frac{\sqrt{2}}{c^2 + \gamma} \right]^{1/\lambda - 1/2} F
\]

(52)

where \(\lambda = [p, p\pm, q, q\pm]\) and \(c\) is the Hubble constant. Following ref. [96] we can try now to constrain the only present parameter \(\alpha\) exploiting the astrophysical observation \(r \approx 1\). Specifically, we found that this is possible only with the combination \((p, q, q\pm)\), as shown, for different primordial conditions, in Table 1. Using these estimations, one can obtain a complete solution for the remaining radiation epoch. For example, taking \(\alpha \approx 15\), the power-law solution for the radiation era goes like \(F \approx a^\alpha\) with \(\nu \approx [2, 63]\).

Similarly to what happens for the Born–Infeld model [1,2,96,146] also the exponential model [73]

\[
L_{\text{NLED}} = -\frac{X e^{-\alpha x}}{\alpha X + \beta}
\]

(53)

does not allow power-law solutions for \(F\). Indeed, in this case, Equation (46) becomes \((\alpha X \gg \beta)\)

\[
\frac{d^2 F}{da^2} + \left[ \frac{s - 1}{a} - 2 \frac{a F^2}{a^2} \right] \frac{d F}{da} + \frac{4 a F^3}{a^6} = 0
\]

(54)

Alternatively, after expanding, the above model reduces to a power-law one with \(\delta = 2\) \((C = 1/\beta\) and \(\gamma = -\alpha/\beta)\) as long as \(\beta \leq 1\) and \(B \ll \sqrt{2}/a\). As for the Born–Infeld model and based on the results of ref. [96], such a model is not compatible with the observation \(r \approx 1\).

Concluding, while for the model in Equation (47) an analytical solution is available and an estimate of the parameter \(\alpha\) is possible, the exponential one (Equation (53)), at least in the regimes studied to have an analytical solution, is excluded from the observations, in a context of primordial magnetic fields.

6. Conclusions

In the literature, there are no known exact solutions for the magnetosphere of a spinning BH. In this paper, we analytically solved the magnetohydrodynamic problem in the context of nonlinear electrodynamics (NLED) models, trying to keep an approach as much model-independent as possible. We explored the Blandford–Znajek mechanism in the light of this framework in order to establish if nonlinear effects can modify the energy extraction rate and the magnetic field configuration around a (non-charged) black hole surrounded by its magnetosphere. This attempt goes in the opposite direction to what was done in ref. [147], where the electromagnetic sector was unaffected, but a deviation from the familiar Kerr metric was switched on. Unlike ref. [147], in our work the black hole spin frequency \(\Omega_s\) does not undergo changes, but, on the other hand, the formula for the extracted power deviates (not only numerically but also formally) from conventional one, that is with Kerr geometry and Maxwell theory. In particular, we found that the energy flux will depend not only on the radial magnetic field \(B_r\) but in general also on the other two components, namely \(B_\theta\) and \(B_\phi\) and that, after perturbative expansion in powers of \(a\) (as done in ref. [99]), no monopole solutions exist for power-law models \(L_{\text{NLED}} = -CX - R X^2\) and that only the case \(\delta = 2\) is significant. Paraboloidal solutions, instead, seems to be possible, and they strongly change if compared to linear theory case, especially in the radial part, as shown in Figure 1. An interesting point of difference is the forward displacement of the flow inversion point \((r \approx 2.35\) vs \(r \approx 2.31)\), that is, the point in which \(A_{\phi}^0\) change sign (and hence \(R(r) = 0)\).

However, as shown in Figure 1, the main difference with linear theory is the asymptotic behavior \((r \gg 1)\) of the solution, being \(A_{\phi}^0 \approx r^2(1 - \cos \theta)\) with \(s > 1\) in the non-linear case \((s = 1\) in linear theory). The more pronounced “verticality” could favor these kinds of solutions in the formation of jets, even if we do not explored the astrophysical consequences.
We also tried to derive several estimates for the extracted power. We used two different approaches valid in different magnetic regimes \((B_\ast \gg B_p \text{ or } B_p \ll 1)\): one requiring the solution of the stream equation and one assuming the magnetic field strength as independent variable (see Figures 2 and 3, respectively). In both cases, it appears evident that NLED power-law models with positive power can in principle extract more energy with respect to classical Maxwell theory, and that, on the other hand, models like the Kruglov’s, for example, perform no better than the standard EM theory already does, making them rather unlikely. Even if separated solutions are not the only option, separable paraboloidal solutions seem to be in good agreement with numerical simulations even for rotating black holes, as rotation does not dramatically affect the magnetosphere configuration in this case.\cite{125} We notice that solutions have been found in the static limit \((a = 0)\) and that the expression for the extracted power are up to second order in \(a\). Moreover, it would be interesting to investigate on higher order solutions \((K \neq 0)\) as done in section 3 of ref. \cite{125}, as well as on non-separable ones. Notice also that exact solutions would require boundary conditions (see ref. \cite{125} and references therein), therefore only numerical simulations (as done in refs. \cite{102,116,119,133}) could be astrophysically meaningful. All these observations could be starting points for other works.

Finally, following refs. \cite{79,95,96} we tried to find constraints on some NLED models, exploiting existence and survival of PMFs. We focused on two recent NLED models, namely Equations \((47)\) and \((53)\). After solving Maxwell equations, we found the constraint \(\alpha \approx 1\) for the first model, while the second model seems to be incompatible with the observation \(r \approx 1\) (see Section 1), at least in the regime we studied. In conclusion, our \textit{analytical} results emphasize that the existence and the behavior of non-linear electromagnetic phenomena strongly depend on the model and the physical context, and that power-law models \(L_{\text{NLED}} = -CX - \gamma X^6\) with \(\delta \leq 2\) should be further studied.

Appendix A

A1. Second Order Terms

Up to second order, the expression for \(X\) is

\[
X = \frac{1}{2r^2 \sin^2 \theta} \left[ \left( \partial_\phi A_\phi^{(0)} \right)^2 + \sigma^2 \left( \partial_\phi A_\phi^{(2)} \right)^2 + 2a^2 \partial_\phi A_\phi^{(0)} \partial_\phi A_\phi^{(2)} \right] \left( 1 - a^2 w^{(1)} \right) + \frac{\sigma^2 \left( \partial_\phi A_\phi^{(2)} \right)^2}{r^2} \left. \left( 1 - a^2 w^{(1)} \right) + a^2 B_\phi^{(1)} \right] \tag{A1}
\]

while Equation \((17)\) becomes

\[
L A_\phi^{(2)} = -r^2 \sin \theta S(r, \theta) \tag{A2}
\]

where \(L\) is given in Equation \((24)\) and

\[
S(r, \theta) := B_\phi^{(1)} \frac{d \bar{B}_\phi^{(1)}}{d \theta} - \frac{w^{(1)}}{r \sin \theta} \left[ \partial_\theta \left( \frac{L_x}{r^2} \frac{\sin \theta}{r} \frac{\partial_\theta A_\phi^{(0)}}{r^2 w^{(1)}} \right) - 2r + w^{(1)} \right] \left( r^2 - 4r^2 \right)
\]

\[
+ \partial_\theta \left( \frac{L_x^{(0)} \sin^2 \theta 2r B_\phi^{(1)}}{r^2} + \frac{L_x^{(0)} \sin \theta}{r^2} \frac{\partial_\theta A_\phi^{(0)} 2w^{(1)} - 1}{r^2} \right) \nonumber
\]

\[
+ \frac{1}{r^2 \sin^2 \theta} \partial_\theta \left( \frac{L_x^{(0)} \sin \theta}{r^2} \frac{\partial_\theta A_\phi^{(0)}}{r^2} \right) \nonumber
\]

\[
+ \frac{1}{r^2 \sin^2 \theta} \partial_\theta \left( \frac{L_x^{(0)} \sin \theta}{r^2} \frac{\partial_\theta A_\phi^{(0)}}{r^2} \right) \tag{A3}
\]

with

\[
\bar{B}_\phi^{(1)} := L_x^{(0)} \sin^2 \theta \left( r^2 - 2r \right) B_\phi^{(1)} - \frac{L_x^{(0)}}{r} \sin \theta 2w^{(1)} \partial_\theta A_\phi^{(0)} + \frac{L_x^{(0)} \sin \theta}{r^2} \partial_\theta A_\phi^{(0)} \tag{A4}
\]

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Keywords

black holes, Blandford–Znajek, energy extraction, non-linear electrodynamics
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Notice that here we use a different metric signature than ref. [121] and that in ref. [99] Kerr coordinates are used.

\[ X = \frac{1}{2}(B^2 - |E|^2) \]

Notice that our definition for \( B_g \) differs from that of ref. [99] by a factor \( \sqrt{g} \) as assumed in ref. [121].

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A solution for \( A^{(2)} \) requires a second order equation. See Appendix.

Generally, \( L_x \) is an even function of \( a \), that is, \( L_x = L_x^0 + a^2 L_x^2 + O(a^4) \). It is essential that \( L_x^0 \neq 0 \) in order to have a solution.

One in principle can generalize to higher orders as done, for example, in ref. [125].

The so-called Kruglov model,[14] for example, is not separable, while the Born–Infeld one reduces to a power-law.

The logarithmic singularity also present in the linear limit, simply means that solutions are valid in regions of space which exclude event horizon.

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A purely radial magnetic field (monopole), although not realistic, is still considered today being the simplest configuration to implement,[131] both numerically and analytically.

The (separable) paraboloidal Schwarzschild solution in linear theory goes like \( A^{(R)}_\phi \approx (\cos \theta - 1)(r + 2 \ln(r - 2)) \) as reported in ref. [125].

Expressions for \( P \) and \( p^{NLED} \) are at fault only for a constant depending on the field configuration (monopole, paraboloidal, etc.). We assume that they are of the same order in both cases, as it is plausible.

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A power-law electromagnetic model seems capable of extracting much more energy than models employing Kerr metric deformations. For example, in the case of a Johannsen metric, the extracted energy is no more than \( \approx 10 \) times larger.[131] In our framework, the ratio \( P^{NLED}/P \) can exceed 10\(^4\) (see Figure 3).

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We are assuming a scale factor of the type \( a(\eta) = c_\eta \eta^\gamma \), where \( c_\eta \) is the Hubble constant for the specific primordial era. Notice that the values of \( s \) are those of general relativity; changing the gravitational sector in the action (Equation (37)) leads to different \( s \) values. See for example.[85] This is the only point in which, in a minimal approach, gravity comes into play.

A Born–Infeld model,[2,1] does not allow either pseudo-power-law or power-law solutions. Expanding the Lagrangian in powers of \( X \), however, leads to a power-law model with \( \delta = 2 \) studied in ref. [96], and hence to the same solutions, which, however, seems to exclude power-law Lagrangian \( \mathcal{L}^{NLED} = -CX - \gamma X^4 \) with \( \delta > 2 \).

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