Inference for First-Price Auctions with Guerre, Perrigne, and Vuong’s Estimator

Jun Ma† Vadim Marmer‡ Artyom Shneyerov§

Abstract

We consider inference on the probability density of valuations in the first-price sealed-bid auctions model within the independent private value paradigm. We show the asymptotic normality of the two-step nonparametric estimator of Guerre, Perrigne, and Vuong (2000) (GPV), and propose an easily implementable and consistent estimator of the asymptotic variance. We prove the validity of the pointwise percentile bootstrap confidence intervals based on the GPV estimator. Lastly, we use the intermediate Gaussian approximation approach to construct bootstrap-based asymptotically valid uniform confidence bands for the density of the valuations.

Keywords: Asymptotic Normality, Bootstrap, First-Price Auctions, Gaussian Approximation, Independent Private Values, Two-Step Nonparametric Estimators, Uniform Confidence Bands

JEL classification: C14, C57

©2019. This manuscript version is made available under the Creative Commons CC-BY-NC-ND 4.0 license http://creativecommons.org/licenses/by-nc-nd/4.0/. First version: March 3, 2016, This version: March 18, 2019.

*We thank the editor, Han Hong, the associate editor and two anonymous referees, whose comments have greatly improved the paper. We also thank Emmanuel Guerre, Jinyong Hahn, Nianqing Liu, Ryo Okui, Joris Pinkse, Christoph Rothe, and Quang Vuong for their helpful comments. Jun Ma’s research is supported by the Fundamental Research Funds for the Central Universities, and the Research Funds of Renmin University of China, #15XNF014 and fund for building world-class universities (disciplines) of Renmin University of China. Vadim Marmer gratefully acknowledges the financial support of the Social Sciences and Humanities Research Council of Canada under grants 435-2013-0331 and 435-2017-0329.

†School of Economics, Renmin University of China, 59 Zhongguancun Street, Haidian District, Beijing, China. Email: jun.ma@ruc.edu.cn. Tel.: 8610 62511102.

‡Corresponding author. Vancouver School of Economics, University of British Columbia, 6000 Iona Drive, Vancouver, BC, V6T 1L4, Canada. Email: vadim.marmer@ubc.ca. Tel.: 1 (604) 822 8217.

§Department of Economics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec, H3G 1M8, Canada. Deceased 24 October 2017.
1 Introduction

The structural estimation of auctions is an important and rapidly growing subfield at the junction of econometrics and industrial organization. Since the seminal work of Guerre, Perrigne, and Vuong (2000, GPV hereafter), much of theoretical and applied work has focused on nonparametric estimation of first-price, sealed-bid auctions. The object of interest is the probability density function (PDF) of latent valuations, which can then be used for a variety of policy counterfactuals such as the optimal reserve price (Paarsch, 1997 and Li et al., 2003).

The focus on nonparametric estimation is due to several reasons. First, in empirical applications, large auction datasets are often available. Second, nonparametric methods are flexible since no functional form assumptions are needed. Third, in auctions, nonparametric estimators are built directly from the identification arguments, and are often easy to implement.

GPV proposed a two-step nonparametric estimator of the PDF of valuations in first-price auctions, and showed that it is uniformly consistent and attains the minimax optimal uniform convergence rate. However, it has been an open question whether this estimator also converges in a distributional sense, which would allow empirical researchers to perform inferences, thereby increasing the scope of applications.

Recently, Marmer and Shneyerov (2012) developed an alternative quantile-based estimator of the PDF of valuations and showed its asymptotic normality. However, the GPV estimator is well established in the literature and is used in all the empirical applications we are aware of. Moreover, our results imply that the GPV estimator has a smaller asymptotic variance than that of the quantile-based estimator, as long as the two estimators use the same second-order kernel.

Inference and the closely related problem of nonparametric testing in structural auction models are important and have been receiving increasing attention in the literature. Beginning with the fundamental Haile et al. (2003)’s test for common values, recent contributions include testing for the monotonicity of bidding strategies (Liu and Vuong, 2013), endogenous entry (Li and Zheng, 2009 and Marmer et al., 2013), common versus private values (Hill and Shneyerov, 2013), the affiliation of bidder valuations (Jun et al., 2010, Li and Zhang, 2010 and de Castro and Paarsch, 2010), and inferences on bidder risk attitudes (Fang and Tang, 2014). In the
absence of the asymptotic distribution framework for the GPV estimator, these papers have adopted problem-specific approaches in each case.

The first main result we show in this paper is that the GPV estimator is asymptotically normal. The key difficulty is the presence of the nonparametric first step, which provides nonparametric estimates of the valuations in each auction. In the second step, the kernel density estimator is applied to those estimates, rather than the true valuations. This creates a unique challenge, to our knowledge not previously addressed in the econometrics literature. Our main insight is that the leading term in an asymptotic expansion of the estimator can be viewed as a V-statistic with a kernel that depends on the bandwidth. A projection argument shows that the distribution of this V-statistic is asymptotically normal. Using maximal inequalities for empirical processes and U-processes developed in recent literature, we show that the remainder term is uniformly negligible. The proof is rather long due to an intricate nature of the estimator, and involves some delicate steps.

Note that a working paper version of GPV (Guerre et al., 1995) also has an asymptotic normality result for the GPV estimator. However, the result therein is of a limited nature as it relies on a particular choice of tuning parameters, which insures that only the second stage of the estimating procedure contributes to the asymptotic variance. Thus, in their approach, the uncertainty due to the estimation of valuations in the first stage can be ignored asymptotically, which is achieved by applying different rates of smoothing of auction-specific covariates at both stages. The approach is restrictive in two respects: (i) While GPV’s smoothing strategy makes first-stage estimation errors negligible asymptotically, in finite samples their contribution to the variance may still be significant. Our approach takes into account the contribution of both stages and, as a result, is more accurate in finite samples. (ii) Equally importantly, their approach cannot be applied in cases with no auction-specific covariates or when covariates are modeled semi-parametrically as in Haile et al. (2003). Note that treating auction-specific characteristics semi-parametrically is particularly appealing to practitioners.

One unusual feature of our asymptotic normality result concerns the form of the asymptotic variance of the GPV estimator. Typically, the asymptotic variances of kernel density estimators depend on the integral of the squared kernel, which is a known constant that can be easily computed analytically or numerically. However, in the case of the GPV estimator, this constant is replaced by a convoluted integral transformation that involves the kernel function, its derivative, and the derivatives of the bidding strategy. This is a consequence of the two-step nature of the GPV estimator and happens due to the impact of the estimation errors from the first stage of the procedure on the distribution of the estimator. Since the bidding strategy is unknown, this fact complicates estimation of the asymptotic variance of the GPV estimator.

---

6See, e.g., Li and Racine (2007).
Our second contribution is to propose a consistent estimator of the asymptotic variance that avoids estimation of the bidding strategy and its derivative. Its uniform rate of convergence is established using the maximal inequalities. Our third contribution is to show the validity of the percentile bootstrap for the GPV estimator, which allows constructing confidence intervals without estimation of the asymptotic variance.

Our pointwise asymptotic normality results can be used for inference on the optimal reserve price, as the latter is determined by a nonlinear equation in the PDF of valuations (see Haile and Tamer, 2003). In our fourth contribution, however, we extend the pointwise results and develop valid uniform confidence bands for the PDF. The uniform confidence bands can be used, e.g., for specification of valuations' density. The extension utilizes the uniform rates of convergence of the remainder terms in our V-statistic approximation of the GPV estimator and its Hoeffding decomposition; it also relies on Gaussian anti-concentration inequalities and Gaussian coupling theorems developed in recent literature. This approach, referred to as the Intermediate Gaussian Approximation (IGA, hereafter) in the literature, is based on the seminal work of Chernozhukov et al. (2014a, b, 2016). Chernozhukov et al. (2014b) showed that although a random function based on nonparametric estimation errors does not typically weakly converge to any tight Gaussian random element, under certain conditions the supremum of its studentized version can be often approximated by the supremum of a tight Gaussian random element, the distribution of which changes with the sample size. Chernozhukov et al. (2016, 2014a) showed that under certain conditions the distribution of the Gaussian supremum can be approximated by bootstrapping, and the bootstrap consistency can be shown by applying the coupling theorems and the Gaussian anti-concentration inequality developed in these papers. Our paper is one of the first applications of these results. Our Monte Carlo simulation results show that the IGA approach produces confidence bands with excellent finite-sample coverage properties.

Our paper is also related to the recent literature on nonparametrically generated regressors in nonparametric regression. See, e.g., Rilstone (1996), Pinkse (2001), and Mammen et al. (2012). Note, however, that while that literature is concerned with nonparametrically estimated exogenous covariates, we deal with kernel estimation of the density of a nonparametrically generated “dependent” variable, potentially in presence of observable conditioning variables.

The rest of the paper proceeds as follows. Section 2 introduces the data-generating process (DGP) and describes the GPV estimator in detail. Due to complexity of the estimator, in Section 3 we show the asymptotic normality of the GPV estimator in a simplified model that has a constant number of bidders across auctions and no auction-specific heterogeneity.

---

7See, e.g., Kato and Sasaki (2017, 2018) for recent applications of these theorems for constructing confidence bands for different nonparametric curves.
Such a simplification allows us to present the main ideas in a more transparent fashion. In Section 4, we derive an estimator for the asymptotic variance and establish its uniform rate of convergence. We also show consistency of the percentile bootstrap confidence intervals. Section 5 provides results on constructing valid confidence bands within the same simplified framework. Proofs of the results in Sections 3–5 are given in the Appendix. Section 6 provides corresponding theorems in the general model with a random number of bidders and auction-specific heterogeneity. The proofs of these results can be found in the Supplement (included). Section 7 discusses how our approach can be extended to auctions with binding reserve prices. We report the results from our Monte Carlo study in Section 8. Section 9 concludes.

Notation. “a := b” is understood as “a is defined by b”. “a =: b” is understood as “b is defined by a”. \( \mathbb{1}(\cdot) \) denotes the indicator function, and we also denote \( \mathbb{1}_A := \mathbb{1}(\cdot \in A) \). Let \( \mathbb{1}([c,r]) \) denote a closed hypercube centered at some real vector \( c \) with edge \( r \). Let \( c^T \) denote the transpose of \( c \). “\( \sim \)” means “equal in distribution”. Let \( \ell^\infty(A) \) be the class of bounded functions defined on \( A \). For any \( f \in \ell^\infty(A) \), let \( \|f\|_A := \sup_{x \in A} |f(x)| \) be the sup-norm.

2 Data-Generating Process (DGP) and the GPV Estimator

The econometrician observes data from \( L \) auctions. Let \( X_l \) denote the \( d \)-dimensional relevant characteristics for the object in the \( l \)-th auction. Let \( N_l \) denote the number of bidders in the \( l \)-th auction. Let \( B_{il} \) denote the bid submitted by the \( i \)-th bidder in the \( l \)-th auction. The data observed by the econometrician is given by \( \{(B_{il}, X_l, N_l) : i = 1, ..., N_l, l = 1, ..., L\} \). Unobserved bidders’ valuations of the \( l \)-th auctioned object are denoted by \( \{V_{il} : i = 1, ..., N_l, l = 1, ..., L\} \). The following assumption describes the DGP.

Assumption 1 (DGP). (a) \( \{(X_l, N_l) : l = 1, ..., L\} \) are i.i.d.

(b) The marginal PDF of \( X_1 \), denoted by \( \varphi(\cdot) \), is strictly positive and continuous on its support \( \mathcal{X} := [x, \overline{x}]^d \) for some \( x < \overline{x} \) assumed to be known and admits up to \( R + 1 \) \( (R \geq 2) \) continuous partial derivatives.

(c) The conditional probability mass function of \( N_1 \) given \( X_1 = x \), denoted by \( \pi(\cdot|x) \), has a known support \( \mathcal{N} := \{\underline{n}, ..., \overline{n}\} \) for all \( x \in \mathcal{X}, \underline{n} \geq 2 \).

(d) For all \( n \in \mathcal{N}, \pi(n|\cdot) \) is strictly positive and admits up to \( R + 1 \) continuous partial derivatives.

(e) \( (n, x) \mapsto \pi(n|x)\varphi(x) \) is bounded above and away from zero on its support \( \mathcal{N} \times \mathcal{X} \).

Assumption 1 is similar to Assumptions A1 and A2 of GPV and Marmer and Shneyerov (2012, Assumption 1). Part (f) imposes the condition that the valuations and the random number of bidders are independent conditionally on the characteristics. See Footnote 14 of GPV.
(f) For each \( l = 1, ..., L \), given \( X_l = x \) and \( N_l = n \), \( \{V_{il} : i = 1, ..., n\} \) are i.i.d. with conditional PDF \( f(\cdot|x) \) and conditional cumulative distributional function (CDF) \( F(\cdot|x) \).

(g) For each \( n \in \mathcal{N} \), the support of \( (V_{11}, X_1) \) is \( S_{V,X} := \{ (v, x) : x \in \mathcal{X}, v \in [\underline{v}(x), \overline{v}(x)] \} \), with some positive boundary functions \( \underline{v}(\cdot) \) and \( \overline{v}(\cdot) \).

(h) \( f(\cdot|\cdot) \) is strictly positive and bounded away from zero and admits up to \( R \) continuous partial derivatives on \( S_{V,X} \).

Bidders’ valuations are not directly observable. Following GPV, we assume that \( B_{il} \) is the equilibrium bid of risk-neutral bidder \( i \) submitted in the \( l \)-th auction. Therefore the valuations are linked to the observed bids through the Bayesian Nash equilibrium (BNE) bidding strategy:

\[
B_{il} = s(V_{il}, X_l, N_l) := V_{il} - \frac{1}{(F(V_{il}|X_l))^{N_l-1}} \int_{\underline{v}(X_l)}^{V_{il}} (F(u|X_l))^{N_l-1} du. \tag{2.1}
\]

Under Assumptions 1(a) and 1(f), \( \{B_{il} : i = 1, ..., N_l\} \) are conditionally i.i.d. draws given \( X_l \) and \( N_l \). Let \( \overline{\nu}(x, n) := s(\overline{\nu}(x), x, n) \) and \( \underline{\nu}(x) := \underline{\nu}(x) \). Proposition 1(i) of GPV shows that the support of \( (B_{il}, X_l, N_l) \) is \( \{(b, x, n) : n \in \mathcal{N}, (b, x) \in S_{B,X}^n\} \), where \( S_{B,X}^n := \{ (b, x) : x \in \mathcal{X}, b \in [\underline{b}(x), \overline{b}(x, n)] \} \).

Let \( G(\cdot|x, n) \) denote the conditional CDF of \( B_{il} \) given \( X_l = x \) and \( N_l = n \). Let \( g(\cdot|x, n) \) be the corresponding conditional PDF. GPV established identification of the inverse bidding strategy:

\[
V_{il} = \xi(B_{il}, X_l, N_l) := B_{il} + \frac{1}{N_l-1} \frac{G(B_{il}|X_l, N_l)}{g(B_{il}|X_l, N_l)}. \tag{2.2}
\]

By replacing \( G(\cdot, \cdot, \cdot) \) and \( g(\cdot, \cdot, \cdot) \) in (2.2) with their nonparametric estimators, GPV proposed an estimator of \( \xi(\cdot, \cdot, \cdot) \), denoted by \( \hat{\xi}(\cdot, \cdot, \cdot) \). The GPV estimator of \( f(v|x) \) is the kernel density estimator that, in place of the true valuations, uses the so-called pseudo valuations \( \{ \hat{V}_{il} := \hat{\xi}(B_{il}, X_l, N_l) : i = 1, ..., N_l, l = 1, ..., L \} \).

Below we provide the details of GPV’s estimation procedure. Let \( K_0 \) and \( K_1 \) be univariate kernel functions of different orders satisfying the following assumption:

**Assumption 2 (Kernel).** (a) \( K_0 \) and \( K_1 \) are symmetric, compactly supported on \([-1,1]\) and twice continuously differentiable on \( \mathbb{R} \) with Lipschitz derivatives.

(b) \( K_0 \) is of order \( R \), and \( K_1 \) is of order \( 1 + R \): \( \int K_0(u)du = 1 \) and \( \int u^k K_0(u)du = 0 \) for \( k = 1, ..., R - 1 \); \( \int K_1(u)du = 1 \) and \( \int u^k K_1(u)du = 0 \) for \( k = 1, ..., R \).

\(^{9}\)See Equations (1) and (8) of GPV.
With \( K_g := K_1 \) and the multi-dimensional product kernels

\[
K_f (v, x) := K_0 (v) \cdot \prod_{k=1}^{d} K_0 (x_k) \quad \text{and} \quad K_X (x) := \prod_{k=1}^{d} K_1 (x_k), \quad \text{for} \ v \in \mathbb{R}, \ x = (x_1, ..., x_d) \in \mathbb{R}^d,
\]
define the following nonparametric estimators:

\[
\hat{\varphi} (x) := \frac{1}{L} \sum_{l=1}^{L} \frac{1}{h^d} K_X \left( \frac{X_l - x}{h} \right) \quad \text{and} \quad \hat{\pi} (n|x) := \frac{1}{\hat{\varphi} (x) L} \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} \mathbb{1} (N_l = n) \frac{1}{h^d} K_X \left( \frac{X_l - x}{h} \right),
\]

where \( \hat{\varphi} (\cdot) \) is the kernel density estimator of \( \varphi \) and \( \hat{\pi} (\cdot|\cdot) \) is the Nadaraya-Watson estimator of the conditional probability mass function \( \pi (\cdot|\cdot) \). Based on these, we define below the nonparametric estimators of the conditional CDF and PDF of the bids:

\[
\hat{G} (b|x, n) := \frac{1}{\hat{\pi} (n|x) \hat{\varphi} (x) L} \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} \mathbb{1} (N_l = n) \frac{1}{h^d} K_X \left( \frac{X_l - x}{h} \right),
\]

\[
\hat{g} (b|x, n) := \frac{1}{\hat{\pi} (n|x) \hat{\varphi} (x) L} \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} \frac{1}{h^{1+d}} K_g \left( \frac{B_{il} - b}{h} \right) K_X \left( \frac{X_l - x}{h} \right).
\]

Consider a partition of \( \mathbb{R}^d \) with generic half-open hypercubes of side \( h_\theta > 0 \):

\[
\Pi_{k_1, ..., k_d} := [k_1 h_\theta, (k_1 + 1) h_\theta) \times \cdots \times [k_d h_\theta, (k_d + 1) h_\theta),
\]

where \( (k_1, ..., k_d) \) runs over \( \mathbb{Z}^d \). Let \( \Pi_{h_\theta} (x) \) denote the hypercube that contains \( x \) in this partition. Define

\[
\hat{b} (x, n) := \max \{ B_{pl} : p = 1, ..., N_l, X_l \in \Pi_{h_\theta} (x), N_l = n, l = 1, ..., L \},
\]

\[
\hat{h} (x) := \min \{ B_{pl} : p = 1, ..., N_l, X_l \in \Pi_{h_\theta} (x), l = 1, ..., L \} \quad (2.3)
\]
to be the estimators of the boundaries of the support. Note that the estimators of the boundaries are super consistent. Let \( \hat{S}_{B,X}^n := \{ (b, x) : x \in \mathcal{X}, b \in [\hat{h} (x), \hat{b} (x, n)] \} \). The support of \( (B_{il}, X_l, N_l) \) then can be estimated by \( \{ (b, x, n) : n \in \mathcal{N}, (b, x) \in \hat{S}_{B,X}^n \} \).

The kernel density estimator \( \hat{g} (b|x, n) \) is asymptotically biased when \( (b, x) \) is near the boundaries of the support. GPV suggested that trimming should be applied to the observations near the estimated boundaries using the trimming factor \( T_{il} := 1 \left[ \mathbb{H} ((B_{il}, X_l), 2h) \in \hat{S}_{B,X}^n \right] \).

The two-step nonparametric estimator of \( f (v|x) \) developed by GPV is

\[
\hat{f}_{GPV} (v|x) := \frac{1}{\hat{\varphi} (x) L} \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} T_{il} \frac{1}{h^{1+d}} K_f \left( \frac{\hat{V}_{il} - v}{h}, \frac{X_l - x}{h} \right), \quad (2.4)
\]
For deriving the asymptotic properties of the GPV estimator, we make the following assumption on the bandwidths $h$ and $h_\partial$.\textsuperscript{10}

**Assumption 3** (Bandwidth).  (a) The bandwidth $h$ is of the form $h = \lambda_1 L^{-\gamma_1}$, for some strictly positive constants $\lambda_1$ and $\gamma_1$ satisfying $\frac{1}{(2R+3+d)} \leq \gamma_1 < \frac{1}{(3+d)}$.

(b) When $d > 0$, the “boundary” bandwidth is of the form $h_\partial = \lambda_\partial \left( \frac{\log(L)}{L} \right)^{\frac{1}{(1+d)}}$, where $\lambda_\partial$ is a strictly positive constant.

GPV showed that the optimal uniform convergence rate of their estimator is attained when the bandwidth $h$ is of order $O \left( \left( \frac{\log(L)}{L} \right)^{\frac{1}{(2R+3+d)}} \right)$. Note that the bandwidth in Assumption 3 is of smaller order. Under-smoothing imposed in Assumption 3 is needed to control the asymptotic bias of the GPV estimator, which is important for the validity of inference.

## 3 Asymptotic Normality of the GPV Estimator

For clarity of the presentation of the main ideas and results, in this section we first establish pointwise asymptotic normality of the GPV estimator in a simplified version of the model that has a fixed number of bidders and no auction-specific heterogeneity. When there are covariates capturing auction-specific heterogeneity present, these results can be used by treating the covariates additively semi-parametrically as in Haile et al. (2003, Section 6). In that case, there is no kernel smoothing over the covariates, as the GPV procedure would be applied to the “homogenized” bids, which are constructed as residuals from the parametric regression of the bids against the covariates.

In the simplified model, the econometrician observes data on bids in $L$ identical auctions, with a fixed number of bidders $N$ in each auction: $\{B_{il} : i = 1, \ldots, N; l = 1, \ldots, L\}$. Under Assumption 1, the valuations $\{V_{il} : i = 1, \ldots, N; l = 1, \ldots, L\}$ are i.i.d. with a compact support $[\underline{v}, \overline{v}] \subseteq \mathbb{R}_+$, PDF $f$ and CDF $F$. The object of interest is the PDF of the valuation at interior points of $[\underline{v}, \overline{v}]$. Suppose that $v_l > \underline{v}, v_u < \overline{v}$ and $I := [v_l, v_u]$ is an inner closed sub-interval of $[\underline{v}, \overline{v}]$. Fix

$$\overline{\delta} := \min \left\{ \frac{(\overline{v} - v_u)}{2}, \frac{(v_l - \underline{v})}{2} \right\}.$$

Under Assumption 1, $f$ is strictly positive and bounded away from zero on its support and admits at least $R$ continuous derivatives. Lemma A1 of GPV showed that under Assumption 1, the BNE bidding strategy is strictly increasing and $R + 1$ times continuously differentiable.

\textsuperscript{10}Assumption 3(a) is the same as the assumption on the rate of bandwidth for Marmer and Shneyerov (2012)’s quantile-based estimator. See Assumption 3 therein.
In this simplified framework, the inverse of the BNE bidding strategy is

\[
\xi (b) := b + \frac{1}{N - 1} \frac{G (b)}{g (b)},
\]

where \( G \) and \( g \) are the CDF and PDF of bids respectively. Denote \( \overline{b} := s (\overline{\tau}) \) and \( \underline{b} := s (\underline{\tau}) \).

Proposition 1(ii) of GPV shows that under Assumption 1, \( g \) is also bounded away from zero on its support \([\underline{b}, \overline{b}]\):

\[
C_g := \inf_{b \in [\underline{b}, \overline{b}]} g (b) > 0.
\]

The inverse bidding strategy (3.1) can be estimated by

\[
\hat{\xi} (b) := b + \frac{1}{N - 1} \frac{\hat{G} (b)}{\hat{g} (b)},
\]

where we use the usual nonparametric estimators of \( G \) and \( g \):

\[
\hat{G} (b) := \frac{1}{N \cdot L} \sum_{i,l} \mathbb{1} (B_{il} \leq b) \quad \text{and} \quad \hat{g} (b) := \frac{1}{N \cdot L} \sum_{i,l} \frac{1}{h} K_g \left( \frac{B_{il} - b}{h} \right),
\]

where \( \sum_{i,l} \) is understood as \( \sum_{l=1}^{L} \sum_{i=1}^{N} \).

Let \( \overline{\overline{b}} := \max \{ B_{il} : i = 1, \ldots, N, l = 1, \ldots, L \} \), and \( \underline{\underline{b}} := \min \{ B_{il} : i = 1, \ldots, N, l = 1, \ldots, L \} \). The trimming factor is now simply \( T_{il} := \mathbb{1} \left( \underline{\underline{b}} + h \leq B_{il} \leq \overline{\overline{b}} - h \right) \). The GPV estimator of \( f (v) \) is now given by

\[
\hat{f}_{GPV} (v) = \frac{1}{N \cdot L} \sum_{i,l} T_{il} \frac{1}{h} K_f \left( \frac{\overline{V}_{il} - v}{h} \right),
\]

where \( K_f = K_0 \) in this simplified framework.

We derive the following stochastic expansion of \( \hat{f}_{GPV} (v) \) around \( f (v) \):

\[
\hat{f}_{GPV} (v) - f (v) = \frac{1}{(N \cdot L)^2} \sum_{i,l} \overline{T}_{il} \frac{1}{h^2} K'_f \left( \frac{\overline{V}_{il} - v}{h} \right) \left( \overline{\overline{V}_{il}} - V_{il} \right) + o_p \left( (Lh^3)^{-1/2} \right),
\]

where \( \overline{T}_{il} := \mathbb{1} (|V_{il} - v| \leq \overline{b}) \) is an infeasible trimming factor and the remainder term is uniform in \( v \in I \). In the above expression, the derivative \( K'_f \) of the kernel function appears due to the linearization of \( K_f \left( (\overline{V}_{il} - v)/h \right) \) around \( K_f \left( (V_{il} - v)/h \right) \). The result in (3.3) shows that the distribution of the GPV estimator depends not only on the variation in \( V_{il} \)'s, but also on the estimation errors of pseudo valuations. In other words, the errors from estimation of the inverse bidding strategy affect the asymptotic distribution of the GPV estimator.

Since \( \hat{G} \) has a faster rate of convergence than \( \hat{g} \), the discrepancy between \( V_{il} \) and \( \overline{V}_{il} \) depends
on that between the true PDF $g (B_{ij})$ and the estimated PDF $\tilde{g}(B_{d})$, which in turn depends on the averaged discrepancy between $K_g ((B_{ijm} - B_{ij})/h)$ and $g(B_{ij})$, where the averaging is across $B_{ijm}$'s. Lemma A.1 establishes a further asymptotic expansion for the GPV estimator:

$$\tilde{f}_{GPV} (v) - f (v) = \frac{1}{(N - 1)(N \cdot L)} \sum_{i,l} \sum_{j,k} \mathcal{M} (B_{il}, B_{jk}; v) + o_p \left( (L h^3)^{-1/2} \right) ,$$

where the remainder term is uniform in $v \in I$, and

$$\mathcal{M} (b, b'; v) := - \frac{1}{h^2} K'_f \left( \frac{\xi (b) - v}{h} \right) \frac{G (b)}{g (b)^2} \left( \frac{1}{h} K_g \left( \frac{b' - b}{h} \right) - g (b) \right) .$$

For any fixed $v$, the leading term in (3.4) is a V-statistic with a kernel that depends on the bandwidth $h$. We now apply Hoeffding decomposition to this leading term. Define

$$\mathcal{M}_1 (b; v) := \int \mathcal{M} (b, b'; v) dG (b')$$

$$= - \frac{1}{h^2} K'_f \left( \frac{\xi (b) - v}{h} \right) \frac{G (b)}{g (b)^2} \left( E \left[ \frac{1}{h} K_g \left( \frac{B_{11} - b}{h} \right) \right] - g (b) \right) ,$$

and further,

$$\mathcal{M}_2 (b; v) := \int \mathcal{M} (b', b; v) dG (b')$$

Note that $\mu_M (v) = E [\mathcal{M}_1 (B_{11}; v)] = E [\mathcal{M}_2 (B_{11}; v)]$. The Hoeffding decomposition yields

$$\frac{1}{(N \cdot L)^2} \sum_{i,l} \sum_{j,k} \mathcal{M} (B_{il}, B_{jk}; v)$$

$$= \mu_M (v) + \left\{ \frac{1}{N \cdot L} \sum_{i,l} \mathcal{M}_1 (B_{il}; v) - \mu_M (v) \right\} + \left\{ \frac{1}{N \cdot L} \sum_{i,l} \mathcal{M}_2 (B_{il}; v) - \mu_M (v) \right\}$$

$$+ \frac{1}{(N \cdot L)(N \cdot L - 1)} \sum_{(i,l) \neq (j,k)} \left\{ \mathcal{M} (B_{il}, B_{jk}; v) - \mathcal{M}_1 (B_{il}; v) - \mathcal{M}_2 (B_{jk}; v) + \mu_M (v) \right\}$$

$$+ \frac{1}{(N \cdot L)^2} \sum_{i,l} \mathcal{M} (B_{il}, B_{il}; v) - \frac{1}{(N \cdot L)^2 (N \cdot L - 1)} \sum_{(i,l) \neq (j,k)} \mathcal{M} (B_{il}, B_{jk}; v) .$$

In the proof of Theorem 3.1 below, we use results for empirical processes and U-processes to show that the terms in the third and fourth lines of (3.7) and $(N \cdot L)^{-1} \sum_{i,l} \mathcal{M}_1 (B_{il}; v)$ are asymptotically negligible uniformly in $v \in I$. As is apparent from the definition of $\mathcal{M}_1$ in (3.6), the contribution of the $\mathcal{M}_1 (b; v)$ terms is negligible because they depend on the difference between the expectation $E [ h^{-1} K_g ((B_{ij} - b)/h)]$ and $g (b)$, i.e., the bias of the kernel.
density estimator, which is of order $O(h^{1+R})$. Thus, the asymptotic distribution of the GPV estimator is driven solely by

$$
\frac{1}{(N-1)^2} \frac{1}{N \cdot L} \sum_{i,l} \left( \mathcal{M}_2 (B_{il}; v) - \mu_M (v) \right),
$$

(3.8)

where $\mathcal{M}_2 (B_{il}; v) - \mu_M (v), i = 1, \ldots, N, l = 1, \ldots, L$ are independent, zero-mean and depend on the bandwidth.

We also show in the proof of Theorem 3.1 that the rescaled variance of (3.8) satisfies

$$
\begin{align*}
\mathbb{E} \left[ \frac{Lh^3}{(N-1)^2} \left( \frac{1}{N \cdot L} \sum_{i,l} \left( \mathcal{M}_2 (B_{il}; v) - \mu_M (v) \right) \right)^2 \right] \\
= \frac{1}{N (N-1)^2} \frac{h^3}{h^3} \int \left\{ \int K_f ' \left( \frac{\xi (b') - v}{h} \right) \frac{G (b')}{g (b')} K_g \left( \frac{b - b'}{h} \right) dG (b') \right\}^2 dG (b) + O (h^3) \\
= V_M (v) + O (h^3),
\end{align*}
$$

(3.9)

where the remainder term is uniform in $v \in I$. Thus, the asymptotic variance of the GPV estimator is the limit of the leading term in (3.9) as $h \downarrow 0$. Note that

$$
\begin{align*}
\lim_{h \downarrow 0} \frac{1}{h^3} \int \left\{ \int K_f ' \left( \frac{\xi (b') - v}{h} \right) \frac{G (b')}{g (b')} K_g \left( \frac{b - b'}{h} \right) dG (b') \right\}^2 dG (b) \\
= \frac{G(s(v))^2 (s'(v))^2}{g(s(v))^2} \int \left\{ \int K_f ' (u) K_g (w - s' (v) u) du \right\}^2 dw.
\end{align*}
$$

(3.10)

We have the following result.

**Theorem 3.1 (Asymptotic Normality).** Suppose Assumptions 1 - 3 hold. Then for any interior point $v \in (\underline{v}, \overline{v})$, $(Lh^{3/2} (\hat{f}_{GPV} (v) - f (v))) \rightarrow_d N (0, V_{GPV} (v))$, where

$$
V_{GPV} (v) := \frac{1}{N (N-1)^2} \frac{F (v)^2 f (v)^2}{g (s(v))^2} \int \left\{ \int K_f ' (u) K_g (w - s' (v) u) du \right\}^2 dw.
$$

(3.11)

**Remark 3.1.** The asymptotic variance $V_{GPV} (v)$ has an interesting and non-standard feature. Typically, the asymptotic variance of a kernel estimator involves a constant which is a simple functional of the kernel and does not involve any elements of the DGP. However, in the case of the GPV estimator, the constant has a convolution form. Moreover, this constant also involves the derivative of the unknown bidding function. As can be seen from (3.3), the convolution form is due to the fact that the variance of the GPV estimator is determined not only by the variation of $V_{it}$, but also by the estimation errors $\hat{V}_{it} - V_{it}$. The $s'(v)$ term appears due to averaging of $\xi (b)$’s in a small neighborhood of $v$, as one can see from (3.9).
The presence of the \( s'(v) \) term inside the integral in (3.11) can cause additional complications in estimation of the asymptotic variance \( V_{GPV}(v) \), as the econometrician now also has to estimate the functional of the kernel function. Potentially, one could estimate \( s'(v) \) and then estimate the convolution by the plug-in approach. However, we show in the next section that the asymptotic variance can be estimated directly without separate estimation of \( s'(v) \) and the convolution by using the sample analogue of the expression in the second line of (3.9).

**Remark 3.2 (Extensions).** Theorem 3.1 also implies asymptotic normality of some modified GPV estimators. Henderson et al. (2012) proposed to modify the standard GPV approach by estimating \( \xi \) under a monotonicity constraint implied by the structural model. However, if the auction model is correctly specified, the unconstrained estimator of \( \xi \) will be monotone with probability approaching one. Hence, we expect the standard GPV estimator and the estimator proposed in Henderson et al. (2012) to be first-order asymptotically equivalent.

Hickman and Hubbard (2014) proposed another modified GPV estimator by replacing sample trimming used in the standard GPV procedure with a boundary correction. Their method uses a boundary-bias-corrected kernel estimator in estimation of the density of bids. The corrected estimator is uniformly consistent over the entire support of the distribution of bids. When estimating the PDF of valuations away from the boundaries, our proof of (3.4) can be adapted to show that the same V-statistic approximation holds for the estimator in Hickman and Hubbard (2014). Hence, their estimator is first-order asymptotically equivalent to the standard GPV estimator, when \( v \) is chosen away from the boundaries.

**Remark 3.3 (Asymptotic Bias).** In the proof of Theorem 3.1, we incorporate the bias term:

\[
(Lh^3)^{1/2} \left( \hat{f}_{GPV}(v) - f(v) - \frac{1}{R!} f^{(R)}(v) \left( \int K_f(u) u^R du \right) h^R + o(h^R) \right) \rightarrow_d N(0, V_{GPV}(v)).
\]  

The leading bias term of the GPV estimator is the same as that of the infeasible estimator constructed using the unobserved true valuations. This is due to the following feature of the first-price auction model: at interior points, the bid density has \( 1 + R \) continuous derivatives rather than \( R \). It is natural to incorporate this structural feature in the estimation procedure by using a higher-order kernel in the first stage.

**Remark 3.4 (Comparison with the Quantile-Based Estimator).** The quantile-based (QB) estimator of Marmer and Shneyerov (2012) does not require estimation of latent valuations, and instead relies on a direct representation of the PDF of valuations using the distribution functions of observable bids. While the two estimators have the same rate of
convergence, the limiting distribution of the GPV estimator shows that it indeed improves on the QB estimator in the following sense. One can show that the GPV estimator has a smaller asymptotic variance than that of the quantile-based estimator, as long as the two estimators use the same second-order kernel functions. Suppose now $K_f = K_g = K$, for some second-order kernel function $K$. Let $V_{QB}(v)$ be the asymptotic variance of the QB estimator in the simplest setting without auction-specific heterogeneity.\textsuperscript{11} By Jensen’s inequality and standard calculus techniques, it can be easily shown that $V_{QB}(v)/V_{GPV}(v) \geq 1$. The details of the proof can be found in the Supplement.

Consider the following family of PDF’s: $f_\theta(v) = \theta v^{\theta-1} \cdot 1 \ (0 \leq v \leq 1)$ for some $\theta > 0$. The corresponding BNE bidding strategy is $s(v) = \left(1 - (\theta(N-1) + 1)^{-1}\right)v$. Note that in this case $s'(v)$ is constant, and we can compute the ratio $V_{QB}(v)/V_{GPV}(v)$ analytically. In the case of the triweight kernel $K$, ratio is found to be, e.g., 1.3259 when $(\theta, N) = (1, 2)$ and 2.3038 when $(\theta, N) = (2, 7)$. Thus, depending on the model, the GPV estimator could be substantially more precise than the QB estimator.\textsuperscript{12}

4 Pointwise Confidence Intervals

If the asymptotic variance $V_{GPV}(v)$ can be consistently estimated by some estimator $\hat{V}_{GPV}(v)$, one can construct an asymptotically valid pointwise confidence interval for $f(v)$ as

$$
CI^\dagger(v) := \left[\hat{f}_{GPV}(v) - z_{1-\alpha/2}\sqrt{\frac{\hat{V}_{GPV}(v)}{Lh^3}}, \hat{f}_{GPV}(v) + z_{1-\alpha/2}\sqrt{\frac{\hat{V}_{GPV}(v)}{Lh^3}}\right],
$$

where $z_{1-\alpha/2}$ denotes the $1 - \alpha/2$ quantile of the standard normal distribution.

While the formula for the asymptotic variance in (3.11) can be used for plug-in estimation of $V_{GPV}(v)$, such an estimator would be difficult to implement in practice. Firstly, it would require estimating the bidding strategy and its derivative. Secondly, even with an estimate of $s'(v)$, it is not always easy to compute analytically the double integral in the definition of $V_{GPV}(v)$. This issue becomes even more severe when there is auction-specific heterogeneity, as we discuss in Section 6. In that case, one would need to evaluate a multidimensional integral.

To avoid those issues, we propose an alternative approach to estimation of the asymptotic variance. As we discuss in the previous section, the asymptotic variance of the GPV estimator is the limit of the expression in (3.9). The leading term on the right-hand side of (3.9) can be

\textsuperscript{11}See Marmer and Shneyerov (2012, Theorem 2) for the expression of the asymptotic variance of the quantile-based estimator.

\textsuperscript{12}We believe that this finding is of interest in a more general context outside of the empirical auctions literature. It illustrates that two-step nonparametric estimators can outperform more direct estimators that avoid first-stage estimation of latent variables.
estimated using a U-type-statistic, while replacing the unknown $G$, $g$, and $\xi$ with $\hat{G}$, $\hat{g}$, and $\hat{\xi}$ respectively. The resulting estimator is given by

$$\hat{V}_{GPV}(v) := \frac{1}{N(N-1)^2 h^3 (N \cdot L) (N \cdot L - 1) (N \cdot L - 2)} \times \sum_{i,l} \sum_{(j,k) \neq (i,l)} \sum_{(j',k') \neq (i,l), (j',k') \neq (j,k)} \eta_{il,jk}(v) \eta_{il,j'k'}(v),$$

where

$$\eta_{il,jk}(v) := T_{jk} K_f \left( \frac{\hat{V}_{jk} - v}{h} \right) \hat{G}(B_{jk}) \frac{B_{il} - B_{jk}}{h} h K_g \left( \frac{B_{il} - B_{jk}}{h} \right).$$ (4.2)

The estimator avoids estimation of the bidding strategy and its derivative and evaluation of multidimensional integrals. It is very easily implementable in practice since it depends only on the bids, $\hat{G}$, $\hat{g}$, and the pseudo valuations. The next theorem shows consistency of the proposed estimator and provides an estimate of its uniform convergence rate.

**Theorem 4.1 (Variance Estimation).** Suppose Assumptions 1 - 3 hold. Then,

$$\sup_{v \in I} \left| \hat{V}_{GPV}(v) - V_M(v) \right| = O_p \left( \left( \frac{\log(L)}{L h^3} \right)^{1/2} + h^R \right).$$

An alternative to the confidence interval (4.1) is the bootstrap. We show below that the bootstrap approximation to the distribution of $S(v) := (L h^3)^{1/2} \left( \hat{f}_{GPV}(v) - f(v) \right)$ is asymptotically valid. Our focus is on the percentile bootstrap as it does not require estimation of the asymptotic variance, which makes it fairly popular among practitioners.

Let $\{B_{il}^*: i = 1, \ldots, N, l = 1, \ldots, L\}$ denote the bootstrap sample, i.e., a set of independent random variables drawn from the distribution $\hat{G}$ conditionally on the original sample of bids. Let $\hat{G}^*$ and $\hat{g}^*$ denote the bootstrap analogues of $\hat{G}$ and $\hat{g}$ respectively: they are constructed by following exactly the same procedure as that for constructing $\hat{G}$ and $\hat{g}$, however using the (empirical) bootstrap sample instead of the original sample. Let $\hat{\xi}^*$ be the bootstrap analogue of $\hat{\xi}$ defined using $\hat{G}^*$ and $\hat{g}^*$ in place of $\hat{G}$ and $\hat{g}$. We generate bootstrap samples of pseudo values as $\hat{V}_{il}^* := \hat{\xi}^*(B_{il}^*)$. Lastly, we construct a bootstrap analogue of $\hat{f}_{GPV}(v)$:

$$\hat{f}_{GPV}^*(v) := \frac{1}{N \cdot L} \sum_{i,l} \frac{1}{T_{il}^*} K_f \left( \frac{\hat{V}_{il}^* - v}{h} \right),$$

where $T_{il}^* := 1 \left( \hat{b} + h \leq B_{il}^* \leq \hat{b} - h \right)$.

Let $q^*_\tau(v)$ be the $\tau$-th quantile of the conditional distribution of $\hat{f}_{GPV}^*(v)$ given the original
The percentile bootstrap confidence interval is

\[ CI^\star (v) := \left[ q_{\alpha/2}(v), q_{1-\alpha/2}(v) \right] = \left[ \hat{f}_{GPV}(v) + \frac{s^\star_{\alpha/2}(v)}{\sqrt{Lh^3}}, \hat{f}_{GPV}(v) + \frac{s^\star_{1-\alpha/2}(v)}{\sqrt{Lh^3}} \right], \]

where \( s^\star_\tau(v) \) is the \( \tau \)-th quantile of the conditional distribution of \( S^\star(v) := (Lh^3)^{1/2} \left( \hat{f}^\star_{GPV}(v) - \hat{f}_{GPV}(v) \right) \) given the original sample. The conditional distributions of the bootstrap statistics \( \hat{f}^\star_{GPV}(v) \) and \( S^\star(v) \) given the original sample can be easily approximated by Monte Carlo methods. We show below that the bootstrap estimator of the finite-sample distribution of \( S(v) \) is consistent.

**Theorem 4.2 (Bootstrap Consistency).** Suppose Assumptions 1 - 3 hold. Then for any interior point \( v \in (\underline{v}, \overline{v}) \),

\[ \sup_{z \in \mathbb{R}} |P^* [S^\star(v) \leq z] - P [S(v) \leq z]| \to_p 0, \text{ as } L \uparrow \infty. \]

**Remark 4.1.** Theorems 3.1, 4.2 and Pólya’s theorem yield

\[ \sup_{z \in \mathbb{R}} |P^* [S^\star(v) \leq z] - P [N(0, V_{GPV}(v)) \leq z]| \to_p 0, \text{ as } L \uparrow \infty, \]

for each \( v \in (\underline{v}, \overline{v}) \). The above result and standard arguments (see, e.g., van der Vaart, 2000, Lemma 23.3) yield the asymptotic validity (consistency) of the percentile bootstrap confidence interval \( CI^\star(v) \), i.e., \( P [f(v) \in CI^\star(v)] \to 1 - \alpha \) as \( L \uparrow \infty \).

**Remark 4.2.** One can also studentize \( S(v) \) using our estimator of the asymptotic variance:

\[ Z(v) := \frac{\hat{f}_{GPV}(v) - f(v)}{(Lh^3)^{-1/2} \hat{V}_{GPV}(v)^{1/2}}. \]

(4.3)

Since \( \hat{V}_{GPV}(v) \) is consistent for the asymptotic variance, \( Z(v) \) is asymptotically distributed as a standard normal random variable. Let \( \hat{V}^*_{GPV}(v) \) be the bootstrap analogue of \( \hat{V}_{GPV}(v) \). The bootstrap analogue of \( Z(v) \) is \( Z^\star(v) := S^\star(v)/\sqrt{\hat{V}^*_{GPV}(v)} \). Let \( z^\star_\tau(v) \) be the \( \tau \)-th quantile of the conditional distribution of \( Z^\star(v) \) given the original sample. The “bootstrap-t” (or studentized bootstrap) confidence intervals can be obtained by replacing the critical value \( z_{1-\alpha/2} \) in \( CI^\dagger(v) \) with their bootstrap counterpart \( z^\star_\tau(v) \). The asymptotic validity of this alternative bootstrap confidence interval easily follows as a corollary to Theorem 4.2.
5 Uniform Confidence Bands

Consider the stochastic process
\[ \Gamma (v) := \frac{1}{N^{1/2}} \left( \frac{1}{N - 1} \right)^{1/2} \sum_{i,l} \frac{M_2(B_{il}; v) - \mu_M(v)}{\text{Var} \left[ N^{-1/2} (N - 1)^{-1} M_2(B_{11}; v) \right]}^{1/2}, \ v \in I. \]

(5.1)

Note that \( \mathbb{E} [\Gamma (v)] = 0 \) and \( \mathbb{E} \left[ \Gamma (v)^2 \right] = 1 \) for all \( v \in I \).

The following theorem shows that (a version of) the centered Gaussian process with index set \( I \) and covariance function \( \mathbb{E} [\Gamma (v) \Gamma (v')] \), for \( (v, v') \in I^2 \), is a tight random element in \( \ell^\infty (I) \). This Gaussian process, denoted by \( \{ \Gamma_G (v) : v \in I \} \), is the intermediate Gaussian process. The tightness of \( \Gamma_G \) as a random element in \( \ell^\infty (I) \) can be established using standard results (see, e.g., Chernozhukov et al., 2014b, Lemma 2.1). The following theorem also shows that one can approximate the distribution of the sup-norm \( \|Z\|_I = \sup_{v \in I} |Z(v)| \) with that of \( \Gamma_G \).

The result follows from uniform approximations of (3.8) and \( \hat{V}_{GPV} (v) \) and uses the coupling theorem for suprema of empirical processes of Chernozhukov et al. (2014b) and the Gaussian anti-concentration inequality of Chernozhukov et al. (2014a).

**Theorem 5.1.** Suppose Assumptions 1 - 3 hold. Then there exists a tight Gaussian random element \( \Gamma_G \) in \( \ell^\infty (I) \) that has mean zero and the same covariance structure as that of \( \Gamma \). Moreover,

\[ \sup_{z \in \mathbb{R}} |\mathbb{P} \left[ \|Z\|_I \leq z \right] - \mathbb{P} \left[ \|\Gamma_G\|_I \leq z \right]| \to 0, \text{ as } L \uparrow \infty. \]

The next result shows that the distribution of the sup-norm of the (empirical) bootstrap process
\[ Z^* (v) := \frac{\hat{f}_{GPV} (v) - \hat{f}_{GPV} (v)}{(Lh^3)^{-1/2}} \hat{V}_{GPV} (v)^{1/2}, \ v \in I \]

(5.2)
can be similarly approximated by that of \( \Gamma_G \).

**Theorem 5.2.** Suppose Assumptions 1 - 3 hold. Then,

\[ \sup_{z \in \mathbb{R}} |\mathbb{P}^* [\|Z^*\|_I \leq z] - \mathbb{P} [\|\Gamma_G\|_I \leq z]| \to_p 0, \text{ as } L \uparrow \infty. \]

Since the distributions of the suprema of (the absolute values of) \( \{Z(v) : v \in I\} \) and that of \( \{Z^*(v) : v \in I\} \) are both well approximated by that of \( \{\Gamma_G (v) : v \in I\} \), one can use the bootstrap critical values based on \( \|Z^*\|_I \) for construction of uniform confidence bands. Let

\[ \zeta^*_{L, \alpha} := \inf \{z \in \mathbb{R} : \mathbb{P}^* [\|Z^*\|_I \leq z] \geq 1 - \alpha\} \]

(5.3)
be the \((1 - \alpha)\)-quantile of the conditional distribution of \(\|Z^*\|_I\) given the original sample. The uniform confidence band is given by

\[
CB^* (v) := \left[ \hat{f}_{GPV} (v) - \zeta^*_{L,\alpha} \sqrt{\frac{\hat{V}_{GPV} (v)}{Lh^3}}, \hat{f}_{GPV} (v) + \zeta^*_{L,\alpha} \sqrt{\frac{\hat{V}_{GPV} (v)}{Lh^3}} \right], \text{ for } v \in I.
\]

The following corollary establishes its asymptotic validity and provides an estimate of the order of the bootstrap critical value \(\zeta^*_{L,\alpha}\).

**Corollary 5.1 (Validity of Bootstrap Confidence Band).** Suppose Assumptions 1 - 3 hold. Then, \(P \left[ f (v) \in CB^* (v), \text{ for all } v \in I \right] \to 1 - \alpha \text{ as } L \uparrow \infty.\) Moreover, \(\zeta^*_{L,\alpha} = O_p \left( \log (h^{-1} )^{1/2} \right) \).

**Remark 5.1 (Limiting Distribution of Uniform Error).** Since the seminal work of Bickel and Rosenblatt (1973), it has been found that in many cases a suitable normalization of the supremum of a studentized absolute difference between a nonparametric curve and its kernel-based estimator converges in distribution to standard Gumbel distribution. Asymptotically valid uniform confidence bands can be based on the Gumbel approximation. We do not pursue such an approach in this paper, since it is known that the accuracy of such approximation is poor. See Giné and Nickl (2015, Section 2.7) and Chernozhukov et al. (2014a) for discussion. On the other hand, for the auction model one can show that (a suitable normalization of) \(\|Z\|_I\) converges in distribution to standard Gumbel distribution when the true bidding strategy is linear. Derivation of the limiting distribution for the general case is interesting but beyond the scope of this paper. See the Supplement for more discussion.

**Remark 5.2.** Taking the IGA approach, we establish consistency of bootstrap uniform confidence bands by showing that the distributions of both \(\|Z\|_I\) and its empirical bootstrap counterpart can be approximated by the distribution of \(\|\hat{f}_G\|_I\). The proof hinges on using the coupling theorems of Chernozhukov et al. (2014b, 2016) and the Gaussian anti-concentration inequality of Chernozhukov et al. (2014a). In the literature, the multiplier bootstrap is used when the IGA approach is taken to construct confidence bands for nonparametric curves. See, e.g., Chernozhukov et al. (2014a) and Kato and Sasaki (2017, 2018). Here, we chose the empirical bootstrap since it is practically convenient given the two-step nature of the GPV estimator.

\[\text{13} \text{It also implies that the (supremum) width of the band } CB^* \text{ is of order } O_p \left( \log (h^{-1} )^{1/2} (Lh^3)^{-1/2} \right).\]
6 Auction-Specific Heterogeneity

6.1 Asymptotic Normality and Estimation of the Asymptotic Variance

We now turn to the general model with auction-specific heterogeneity and a random number of bidders. Firstly, we establish the asymptotic normality of the GPV estimator by following the same approach and steps as in the case of the simplified model in Section 3. While handling the general case is complicated by much heavier notations, all the results provided in this section can be viewed as straightforward generalizations of the results in Sections 3-5. The proofs of the results for the general case can be found in the Supplement.

In comparison with the simplified model, one of the main differences is in the form of the asymptotic variance. Recall that in the simplified case, the asymptotic variance of the GPV estimator depends on the derivative of the bidding strategy. As we show below, when there is auction-specific heterogeneity, the asymptotic variance also involves the partial derivatives of the bidding strategy with respect to the auction-specific characteristics. This is in addition to the partial derivative with respect to the valuation.

For some fixed \( x \) which is an interior point of \( \mathcal{X} \), let \( I(x) := [v_l(x), v_u(x)] \) be an inner closed sub-interval of \([v_l, v_u]\). The fact that the conditional density of the valuations given \( X = x \) and \( N = n \) is \( f(\cdot | x) \) under Assumption 1 motivates the following two-step estimator of \( f(v|X) \):

\[
\hat{f}_{GPV}(v|x) = \frac{1}{\hat{\pi}(n|x)}\hat{\varphi}(x)L \sum_{l=1}^{L} \mathbf{1}(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} T_{il} \frac{1}{h^{1+d}} K_f \left( \frac{\hat{V}_{il} - v}{h}, \frac{X_l - x}{h} \right).
\]

Note that the above estimator only uses data from auctions with \( N_l = n \). Since the PDF of valuations does not depend on the number of bidders, an estimator for \( f(v|x) \) can be constructed as a weighted average of \( \{ \hat{f}_{GPV}(v|x, n) : n \in \mathcal{N} \} \). E.g., GPV suggested using estimates of the conditional probabilities of drawing \( N_l = n \) as the weights:

\[
\hat{f}_{GPV}(v|x) = \sum_{n \in \mathcal{N}} \hat{\pi}(n|x) \hat{f}_{GPV}(v|x, n).
\]  

(6.1)

Note that this gives an expression that is the same as the right hand side of (2.4).

By repeating the steps from Section 3, one can show that the following analogue of the
linearization results in (3.4) holds for the general model:

\[
\hat{f}_{GPV}(v|x, n) - f(v|x) = \frac{1}{n|x} \sum_{l=1}^{L} \sum_{m=1}^{L} \mathcal{M}^n((\mathbf{B}_l, \mathbf{X}_l, N_l), (\mathbf{B}_m, \mathbf{X}_m, N_m); v) + o_p\left(Lh^{3+d}\right)^{-1/2},
\]

where \( B_1 := (B_{11}, ..., B_{N_1}) \), and the remainder term is uniform in \( v \in I(x) \). For \( b := (b_1, ..., b_m) \), the kernel function \( \mathcal{M}^n \) is given by:

\[
\mathcal{M}^n((b, z, m), (b', z', m'); v) := -1 (m = n) \frac{1}{m} \sum_{i=1}^{m} \frac{1}{h^{2+d}} K_f^\prime \left( \frac{\xi(b_i, z, m) - v, z - x}{h} \right) G(b, z, m) (m-1) g(b, z, m)^2 \times \left( 1 (m' = m) \frac{1}{m'} \sum_{j=1}^{m'} \frac{1}{h^{1+d}} K_g \left( \frac{b'_j - b_i}{h} \right) K_X \left( \frac{z' - z}{h} \right) - g(b, z, m) \right),
\]

where \( K_f^\prime (\cdot, \cdot) \) denotes the partial derivative function of \( K_f \) with respect to its first argument,

\[
G(b, z, m) := G(b|z, m) \pi(m|z) \varphi(z) \quad \text{and} \quad g(b, z, m) := g(b|z, m) \pi(m|z) \varphi(z).
\]

Note that the leading term on the right-hand side of (6.2) involves a V-statistic (with a kernel that depends on the bandwidth) and, therefore, can be analyzed using the Hoeffding decomposition. Thus, (3.7) can be generalized as

\[
\frac{1}{L^2} \sum_{l=1}^{L} \sum_{m=1}^{L} \mathcal{M}^n((\mathbf{B}_l, \mathbf{X}_l, N_l), (\mathbf{B}_m, \mathbf{X}_m, N_m); v) = \mu_{\mathcal{M}^n}(v) + \left\{ \frac{1}{L} \sum_{l=1}^{L} (\mathcal{M}^n(\mathbf{B}_l, \mathbf{X}_l, N_l; v) - \mu_{\mathcal{M}^n}(v)) \right\} + \left\{ \frac{1}{L} \sum_{l=1}^{L} (\mathcal{M}^n(\mathbf{B}_l, \mathbf{X}_l, N_l; v) - \mu_{\mathcal{M}^n}(v)) \right\} + o_p\left(Lh^{3+d}\right)^{-1/2},
\]

where the remainder term is uniform in \( v \in I(x) \),

\[
\mathcal{M}^n_b(b, z, m; v) := E[\mathcal{M}^n((b, z, m), (\mathbf{B}_1, \mathbf{X}_1, N_1); v)], \quad \mathcal{M}^n(\mathbf{B}_1, \mathbf{X}_1, N_1) \quad \text{and} \quad \mu_{\mathcal{M}^n}(v) := E[\mathcal{M}^n((\mathbf{B}_1, \mathbf{X}_1, N_1), (\mathbf{B}_2, \mathbf{X}_2, N_2); v)].
\]

The projection term \( \mathcal{M}^n_1 \) is the expectation of the kernel \( \mathcal{M}^n((b, z, m), (\mathbf{B}_1, \mathbf{X}_1, N_1); v) \)
with the first argument fixed at \((b_*, z, m)\). The expression for \(\mathcal{M}_1^0\) is:

\[
\mathcal{M}_1^0 (b_*, z, m; v) = -1 \{ m = n \} \frac{1}{m} \sum_{i=1}^{m} \frac{1}{h^{2+d}} K'_{f} \left( \frac{\xi(b_*, z, m) - v, z - x}{h} \right) \frac{G(b_*, z, m)}{g(b_*, z, m)^2}
\]

\[
\times \int \sum_{m' \in \mathcal{N}} \prod_{j=1}^{m'} \left( 1 \{ m' = n \} \frac{1}{m} \sum_{i=1}^{m'} \frac{1}{h^{1+d}} K_g \left( \frac{b'_j - b_i}{h} \right) K_X \left( \frac{z' - z}{h} \right) - g(b_*, z, m) \right)
\]

\[
\times \left( \prod_{j=1}^{m'} g \left( b'_j | z', m' \right) \pi \left( m' | z' \right) \varphi \left( z' \right) \right) \, db'_1 \cdots db'_m \, dz'.
\]

As in the case of the simplified model, the contribution of \(\mathcal{M}_1^0\) is asymptotically negligible. This happens for the same reason as in Section 3: \(\mathcal{M}_1^0\) depends on the difference between the expectation of the kernel function and the true density. Hence, the asymptotic distribution of the GPV estimator is driven solely by the \(\mathcal{M}_1^0\) term, which is the expectation of the kernel \(\mathcal{M}^n ((B_1, X_1, N_1), (b_*, z, m); v)\) with the second argument fixed at \((b_*, z, m)\).

We show in the supplement that a generalized version of (3.9) holds:

\[
E \left[ L h^{3+d} \left\{ \frac{1}{n} \sum_{l=1}^{L} \left( \mathcal{M}_2^0 (B_l, X_l, N_l; v) - \mu_{\mathcal{M}^n} (v) \right) \right\}^2 \right] = \frac{1}{n(n-1)^2} \frac{1}{h^{3(1+d)}} \int \int \left( \int \int \mathcal{N}(z', n) K'_{f} \left( \frac{\xi(b', z', n) - v, z' - x}{h} \right) \frac{G(b', z', n)}{g(b', z', n)^2} \right)
\]

\[
\times K_g \left( \frac{b - b'}{h} \right) K_X \left( \frac{z - z'}{h} \right) \, db' \, dz' \, g(b, z, n) \, dbd z + O(h^3)
\]

\[
= : V_{\mathcal{M}} (v | x, n) + O(h^3), \quad (6.4)
\]

where the remainder term is uniform in \(v \in I (x)\). Moreover, similarly to (3.10),

\[
\lim_{h \to 0} \frac{1}{h^{3(1+d)}} \int \int \left( \int \int \mathcal{N}(z', n) K'_{f} \left( \frac{\xi(b', z', n) - v, z' - x}{h} \right) \frac{G(b', z', n)}{g(b', z', n)^2} \right)
\]

\[
\times K_g \left( \frac{b - b'}{h} \right) K_X \left( \frac{z - z'}{h} \right) \, db' \, dz' \, g(b, z, n) \, dbd z
\]

\[
= \frac{G(s(v, x, n), x, n)^2 s_v(v, x, n)^2}{g(s(v, x, n), x, n)} \times \int \int \left\{ \int \int K'_{f} (w, y) K_X (y - z) K_g (u - s_v w - s_x y) \, dwd y \right\}^2 \, dud z, \quad (6.5)
\]
where \( s_v \) and \( s_x \) denote the partial derivatives of the bidding function:

\[
\begin{align*}
  s_v &:= \frac{\partial s (u, z, n)}{\partial u} \bigg|_{(u, z) = (v, x)} \\
  s_x &:= \frac{\partial s (u, z, n)}{\partial z} \bigg|_{(u, z) = (v, x)}.
\end{align*}
\]  

(6.6)

The asymptotic variance of the GPV estimator is the limit of \((\pi (n|x) \varphi (x))^{-2} \mathcal{V}_M (v|x, n)\). After using a change of variable argument and (6.5), the variance is shown to be

\[
\begin{align*}
  \mathcal{V}_{GPV} (v|x, n) :=& \frac{1}{n(n-1)^2} \frac{F (v|x)^2 f (v|x)^2}{\pi (n|x) \varphi (x) g (s (v, x, n) | x, n)^3} \\
  \times & \int \int \left\{ \int \int K'_f (w, y) K_X (y - z) K_g (u - s_v w - s_x^T y) \text{d} w \text{d} y \right\}^2 \text{d} u \text{d} z.
\end{align*}
\]  

(6.7)

The following theorem is a generalization of Theorem 3.1. The proof of the theorem as well as the proofs of all other results provided in Section 6 are in the Supplement.

**Theorem 6.1.** Suppose Assumptions 1 - 3 hold. Then, for any interior point \((v, x) \in S_{V, X}\),

\[
\left( L h^{3+d} \right)^{1/2} \left( \hat{f}_{GPV} (v|x, n) - f (v|x) \right) \to_d \mathcal{N} (0, \mathcal{V}_{GPV} (v|x, n)).
\]

Moreover, \(\left\{ \hat{f}_{GPV} (v|x, n) : n \in \mathcal{N} \right\}\) are asymptotically independent.

**Remark 6.1.** As in the simplified model, the asymptotic variance of the GPV estimator depends on a convoluted integral transformation involving the kernel function, its derivative and the derivatives of the bidding strategy. Auction-specific heterogeneity complicates the expression in two ways. Firstly, the dimension of the integral depends on the number of auction characteristics (covariates), and analytical calculation of the integral becomes cumbersome when there are many covariates. Secondly, the expression now contains the derivatives of the bidding strategy with respect to the covariates. The reason for that is apparent from the expression for the second moment of \(\mathcal{M}_n^d (\mathbf{B}_1, \mathbf{X}_1, n_1; v)\) in (6.4) as it involves averaging of the inverse bidding function \(\xi (u, z', n)\) over the auction characteristics \(z'\) in a shrinking neighborhood of \(x\).

While the GPV estimator and the quantile-based estimator have the same rate of convergence (see Marmer and Shneyerov, 2012, Theorem 2 for the rate of convergence and the expression of the asymptotic variance \(\mathcal{V}_{QB} (v|x, n)\) of the quantile-based estimator in the general model), one can show that the GPV estimator has a smaller asymptotic variance than that of the quantile-based estimator, as long as the two estimators use the same second-order kernel function. It can be shown that the ratio \(\mathcal{V}_{GPV} (v|x, n)/\mathcal{V}_{QB} (v|x, n) \leq 1\) by applying Jensen’s inequality and standard techniques for multi-dimensional integration. A detailed proof can be found in the Supplement.
Remark 6.2 (Optimal Weights). Since the distribution of valuations does not depend on the number of bidders, i.e., \( f(v|x,n) = f(v|x) \) for all \((v,x,n) \in S_{V,X} \times \mathcal{N}\), one can average the estimators \( \{\hat{f}_{GPV}(v|x,n) : n \in \mathcal{N}\} \) to obtain a weighted estimator of the density \( f(v|x) \):

\[
\hat{f}_{GPV}^w(v|x) := \sum_{n \in \mathcal{N}} \hat{w}(n,v,x) \hat{f}_{GPV}(v|x,n),
\]

where the weights \( \hat{w}(n,v,x), n \in \mathcal{N} \) should satisfy \( \hat{w}(n,v,x) \to_p w(n,v,x), n \in \mathcal{N} \) and \( \sum_{n \in \mathcal{N}} w(n,v,x) = 1 \). As in Marmer and Shneyerov (2012), the optimal weights that minimize the asymptotic variance of the resulting weighted estimator are inversely related to \( V_{GPV}(v|x,n) \) and given by

\[
w^{opt}(n,v,x) := \frac{1}{V_{GPV}(v|x,n)} \sum_{n \in \mathcal{N}} \frac{1}{V_{GPV}(v|x,n)}.
\]

These weights can be consistently estimated by the plug-in principle using an estimator of \( V_{GPV}(v|x,n) \). The original GPV estimator uses the weights \( \hat{w}(v,x,n) = \hat{\pi}_n(x), n \in \mathcal{N} \). See (2.4) and (6.1). Note, however, that such weights would be sub-optimal from the point of view of minimizing the asymptotic variance. We provide the asymptotic normality of \( \hat{f}_{GPV}(v|x) \) in the corollary below.

Corollary 6.1. Suppose Assumptions 1 - 3 hold. Then, for any interior point \((v,x) \in S_{V,X}\),

\[
\left(Lh^{3+d}\right)^{1/2} \left( \hat{f}_{GPV}(v|x) - f(v|x) \right) \to_d N(0,V_{GPV}(v|x)),
\]

where \( V_{GPV}(v|x) := \sum_{n \in \mathcal{N}} \pi(n|x)^2 V_{GPV}(v|x,n) \).

For practical purposes, it is important to have a consistent estimator of the asymptotic variance \( V_{GPV}(v|x) \) that avoids estimation of the bidding strategy \( s(\cdot,\cdot,n) \) and its derivatives. It is also highly desirable to avoid analytical or numerical evaluation of a multidimensional integral in the definition of the asymptotic variance. Following the same approach we used in the case of the simplified model (see (4.2)), we rely on the sample analogue of (6.4):

\[
\hat{V}_{GPV}(v|x,n) := \frac{1}{n(n-1)^2 \hat{\pi}(n|x)^2 \hat{\varphi}(x)^2 h^{3(1+d)}} \frac{1}{L(L-1)(L-2)} \sum_{l=1}^{L} \sum_{k \neq l} \sum_{k' \neq k, k' \neq l} 1 (N_l = n, N_k = n, N_{k'} = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \eta_{l,k}(v,x)\eta_{l,k'}(v,x), \quad (6.8)
\]
where

\[
\eta_{il,k}(v,x) := \frac{1}{N_k} \times \sum_{j=1}^{N_k} \mathbb{T}_{jk} \mathbb{K}_f' \left( \frac{V_{jk} - v}{h}, \frac{X_k - x}{h} \right) \mathbb{G} \left( B_{jk}, X_k, N_k \right) \mathbb{K}_g \left( B_{il,jk} - B_{jk} \right) K_X \left( \frac{X_l - X_k}{h} \right).
\]

The following result is a generalization of Theorem 4.1.

**Theorem 6.2.** Suppose Assumptions 1 - 3 hold. Then, for any interior point \( x \),

\[
\sup_{v \in I(x)} \left| \mathbb{V}_{GPV}(v|x,n) - \left( \pi(n|x) \varphi(x) \right)^{-2} \mathbb{V}_M(v|x,n) \right| = O_p \left( \left( \frac{\log(L)}{Lh^{3+d}} \right)^{1/2} + h^R \right).
\]

**Remark 6.3.** An estimator for the asymptotic variance \( \mathbb{V}_{GPV}(v|x) \) of the estimator \( \hat{f}_{GPV}(v|x) \) in Corollary 6.1 can be constructed using the plug-in approach:

\[
\mathbb{V}_{GPV}(v|x) := \sum_{n \in \mathbb{N}} \hat{\pi}(n|x)^2 \mathbb{V}_{GPV}(v|x,n).
\]

Its rate of convergence is the same as in Theorem 6.2.

### 6.2 Bootstrap-based Inference and Uniform Confidence Bands

To generate bootstrap samples, we apply the same resampling procedure as that proposed in Marmer and Shneyerov (2012, Section 4). First, we randomly draw \( L \) observations from \( \{(X_l, N_l) : l = 1, \ldots, L\} \) (i.e., the auction-specific characteristics) with replacement. Next, we randomly draw bids with replacement from the bids corresponding to each selected auction. Given \( (X_l^*, N_l^*) = (X_{l'}, N_{l'}) \) in the first step, in the second step \( \{B_{il}^* : i = 1, \ldots, N_{l'}^*\} \) is generated as an empirical bootstrap sample drawn from \( \{B_{il'} : i = 1, \ldots, N_{l'}\} \). Let \( \tilde{\xi}^* (\cdot, \cdot, \cdot) \) and \( \tilde{\varphi}^* (\cdot) \) be the bootstrap analogues of \( \hat{\xi} (\cdot, \cdot, \cdot) \) and \( \hat{\varphi} (\cdot) \) respectively. Let \( \hat{f}_{GPV}(v|x) \) denote the bootstrap version of the GPV estimator:

\[
\hat{f}_{GPV}^*(v|x) := \frac{1}{\varphi^*(x)L} \sum_{l=1}^{L} \frac{1}{N_l^*} \sum_{i=1}^{N_l^*} \mathbb{T}_{il}^* \frac{1}{h^{1+d}} K_f \left( \hat{V}_{il}^* - v, \frac{X_l^* - x}{h} \right),
\]

where \( \hat{V}_{il}^* := \hat{\xi}^* (B_{il}^*, X_l^*, N_l^*) \) and the bootstrap version of the trimming factor is given by

\[
\mathbb{T}_{il}^* := 1 \left( \mathbb{H}((B_{il}^*, X_l^*), 2h) \subseteq \mathbb{S}^{N_l}_{B_l^*, X_l} \right).
\]

Consider the scaled deviation of the GPV estimator from the true PDF, and its bootstrap
analogue:

\[ S(v|x) := \left(Lh^{3+d}\right)^{1/2} \left(\hat{f}_{GPV}(v|x) - f(v|x)\right) \] and
\[ S^*(v|x) := \left(Lh^{3+d}\right)^{1/2} \left(\hat{f}^*_c GPV(v|x) - \hat{f}_{GPV}(v|x)\right) . \]

The following result, which is a generalization of Theorem 4.2, establishes the validity of the percentile bootstrap for \( f(v|x) \).

**Theorem 6.3.** Suppose Assumptions 1 - 3 hold. Then, for any interior point \((v, x) \in S_{V,X}\),

\[ \sup_{z \in \mathbb{R}} |P^*\left[S^*(v|x) \leq z\right] - P\left[S(v|x) \leq z\right]| \rightarrow_p 0, \text{ as } L \uparrow \infty. \]

We now turn to construction of uniform confidence bands for \( \{f(v|x): v \in I(x)\} \) given a fixed interior point \( x \). Consider the following processes:

\[ Z(v|x) := \frac{\hat{f}_{GPV}(v|x) - f(v|x)}{\left(Lh^{3+d}\right)^{-1/2} \hat{V}_{GPV}(v|x)^{1/2}} \] and \( Z^*(v|x) := \frac{\hat{f}^*_c GPV(v|x) - \hat{f}_{GPV}(v|x)}{\left(Lh^{3+d}\right)^{-1/2} \hat{V}_{GPV}(v|x)^{1/2}}, \ v \in I(x) . \)

Similarly to the simplified model, the distribution of \( \|Z(\cdot|x)\|_{I(x)} \) can be approximated by the conditional distribution of \( \|Z^*(\cdot|x)\|_{I(x)} \). Let \( \zeta^*_{L,a} \) be the \((1 - \alpha)\)-quantile of the conditional distribution of \( \|Z^*(\cdot|x)\|_{I(x)} \) given the original sample. Consider the following confidence band: for \( v \in I(x) \),

\[ CB^*(v|x) := \left[ \hat{f}_{GPV}(v|x) - \zeta^*_{L,a} \sqrt{\frac{\hat{V}_{GPV}(v|x)}{Lh^{3+d}}}, \hat{f}_{GPV}(v|x) + \zeta^*_{L,a} \sqrt{\frac{\hat{V}_{GPV}(v|x)}{Lh^{3+d}}} \right] . \]

The following result, which is a generalization of Corollary 5.1, establishes the validity of \( CB^*(\cdot|x) \).

**Theorem 6.4.** Suppose Assumptions 1 - 3 hold. Then,

\[ P\left[f(v|x) \in CB^*(v|x), \text{ for all } v \in I(x)\right] \rightarrow 1 - \alpha, \text{ as } L \uparrow \infty. \]

7 Binding Reserve Price

Section 4 of GPV shows how to modify their identification and estimation strategy when there is a binding reserve price. Here, we discuss how our approach can be applied in that case.

When there is a binding reserve price, it is assumed that only bidders with valuations exceeding the reserve price submit bids. Thus, one has to distinguish between the numbers
of potential and actual (active) bidders. Let \( N \) denote the number of potential bidders, which is assumed to be known to players. The bidding strategy depends on the number of potential bidders instead of the number of active bidders. As discussed in GPV, \( N \) can be estimated by taking the maximum of the observed numbers of actual bidders across the auctions: \( \hat{N} := \max \{ N_l : l = 1, \ldots, L \} \), where \( N_l \) is the number of actual bidders in auction \( l \).

GPV assume that the reserve price in auction \( l \), denoted \( P_{0l} \), is some unknown deterministic function of the auction characteristics \( X_l \): \( P_{0l} = p_0 (X_l) \). The probability of drawing a valuation below the reserve price is given by \( \Phi (X_l) := F (P_{0l} | X_l) \). The conditional CDF and PDF of the distribution of valuations given participation (submitting a bid) are

\[
F^\ast (v|x) := \frac{F(v|x) - \Phi (x)}{1 - \Phi (x)} \quad \text{and} \quad f^\ast (v|x) := \frac{f(v|x)}{1 - \Phi (x)} \tag{7.1}
\]

respectively. The third displayed equation on page 550 of GPV shows that \( \Phi (\cdot) \) can be estimated using a nonparametric regression of the number of actual bidders:

\[
\hat{\Phi} (x) := 1 - \frac{1}{\hat{N} \hat{\varphi} (x)} \sum_{l=1}^{L} \frac{1}{h^d} N_l K_X \left( \frac{X_l - x}{h} \right).
\]

GPV point out that the density of bids is unbounded at the reserve price \( p_0 (x) \) and, in its neighborhood, behaves as \( 1/\sqrt{b - p_0(x)} \). To avoid technical problems due to the unbounded density, they propose to transform the bids as

\[
B_{il} = (B_{il} - P_{0l})^{1/2}.
\]

The support of \( (B_{i11}, X_1) \) is given by \( S_{B_{i1}, X} := \{ (b, x) : x \in X, b \in [0, \bar{b}_1 (x)] \} \), where \( \bar{b}_1 (z) := (b_1 (z) - p_0 (z))^{1/2} \). The support can be estimated by \( \hat{S}_{B_{i1}, X} := \{ (b, x) : z \in X, b \in [0, \bar{b}_1 (x)] \} \), where \( \bar{b}_1 (x) := \max \{ B_{ipl} : p = 1, \ldots, N_l, X_l \in \Pi_{b_0} (x), l = 1, \ldots, L \} \), see page 550 in GPV.

Let \( G_1 (\cdot | \cdot) \) and \( g_t (\cdot | \cdot) \) denote respectively the conditional CDF and PDF of the transformed bids \( B_{i11} \) given \( X_1 \). Let \( G_1 (b_1, z) := G_1 (b_1 | z) \varphi (z) \) and \( g_t (b_1, z) := g_t (b_1 | z) \varphi (z) \). GPV show that \( g_t (\cdot, \cdot) \) is bounded on its support, and that latent valuations can be recovered using

\[
V_{il} = \xi_1 (B_{il}, X_l) := P_{0l} + B_{il}^2 \left( 1 - \Phi (X_l) \right) \frac{G_1 (B_{il}, X_l) + \Phi (X_l)}{(1 - \Phi (X_l)) g_t (B_{il}, X_l)}.
\]

In the modified GPV procedure, one first estimates \( \xi_1 (\cdot, \cdot) \) by replacing \( N, \Phi (\cdot), G_1 (\cdot, \cdot), g_t (\cdot, \cdot) \) and \( \varphi (\cdot) \) with their estimators. \( G_1 (\cdot, \cdot) \) and \( g_t (\cdot, \cdot) \) can be estimated using the trans-
formed bids $B_{it}$:

$$
\hat{G}_t(b_{it}, x) := \frac{1}{L} \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} \mathbb{1}(B_{it} \leq b_{it}) \frac{1}{h^d} K_X \left( \frac{X_l - x}{h} \right),
$$

$$
\hat{g}_t(b_{it}, x) := \frac{1}{L} \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} \frac{1}{h^{1+d}} g \left( \frac{B_{it} - b_{it}}{h} \right) K_X \left( \frac{X_l - x}{h} \right).
$$

In the second step of the modified GPV procedure, one uses the pseudo valuations

$$
\left\{ \hat{V}_{it} := \hat{\xi}_t(B_{it}, X_l) : i = 1, \ldots, N_l, l = 1, \ldots, L \right\},
$$

where $\hat{\xi}_t(\cdot, \cdot)$ is the estimated version of $\xi_t(\cdot, \cdot)$, in place of latent valuations to construct a kernel density estimator of $f^*(v|\mathbf{x})$:

$$
\hat{f}_{GPV}^*(v|\mathbf{x}) := \frac{1}{\hat{\varphi}(\mathbf{x}) L} \sum_{l=1}^{L} \frac{1}{N_l} \sum_{i=1}^{N_l} T_{il} \frac{1}{h^{1+d}} K_f \left( \frac{\hat{V}_{it} - v}{h}, \frac{X_l - x}{h} \right),
$$

where the trimming factors $T_{il}$ can be defined analogously to the case with no binding reserve price: $T_{il} := 1 \left( \mathcal{H}( (B_{it}, X_l), 2h) \subseteq \hat{S}_{B_{it}, X_l} \right)$.

Our approach can be used to obtain the asymptotic distribution of the modified GPV estimator as follows. In view of the definitions of $\xi_t(\cdot, \cdot)$ and its estimator, the $\hat{V}_{it} - V_{it}$ term in the analogue of (3.3) can be expanded as

$$
\hat{V}_{it} - V_{it} = 2B_{it} \frac{(1 - \Phi(X_l)) G_t(B_{it}, X_l) + \Phi(X_l) \varphi(X_l)}{(1 - \Phi(X_l))^2} (\hat{g}_t(B_{it}, X_l) - g_t(B_{it}, X_l)) + o.s.o.,
$$

where "s.o." stands for smaller order terms. Hence, the GPV estimator of $f^*(v|\mathbf{x})$ still has a representation of the same form as in (6.2):

$$
\hat{f}_{GPV}^*(v|\mathbf{x}) - f^*(v|\mathbf{x}) = \frac{1}{\hat{\varphi}(\mathbf{x}) L^2} \sum_{l=1}^{L} \sum_{k=1}^{L} \mathcal{M}_t((B_{it}, X_l, N_l), (B_{it+k}, X_k, N_k); v) + o_p \left( (Lh^{3+d})^{-1/2} \right),
$$

27
where $B_{1:t} := (B_{1:1}, ..., B_{1:N_t})$, and

$$
\mathcal{M}_1 ((b, z, m), (b', z', m'); v)
:= -\frac{1}{m} \sum_{i=1}^{m} \frac{1}{h^{2+d}} K_f \left( \frac{\xi_i (b, z) - v}{h}, \frac{z - x}{h} \right) \frac{2b_i ((1 - \Phi (z)) G_i (b_i, z) + \Phi (z) \varphi (z))}{(N - 1) (1 - \Phi (z)) g_i (b_i, z)^2}
\times \left( \frac{1}{m'} \sum_{j=1}^{m'} \frac{1}{h^{1+d}} K_g \left( \frac{b'_j - b_i}{h} \right) K_X \left( \frac{z' - z}{h} \right) - g_t (b_i, z) \right).
$$

Similarly to Theorem 6.1, the expression in (6.5), and for any interior point $v \in (p_0 (x), \bar{v} (x))$, the GPV estimator of $f^* (v \mid x)$ is asymptotically normal with the asymptotic variance given by

$$
V_{GPV} (v, x)
:= \frac{\pi (x)}{\varphi (x)^2} \frac{(2s_t (v, x) s_{tv} (v, x) ((1 - \Phi (x)) G_t (s_t (v, x), x) + \Phi (x) \varphi (x)))^2}{(N - 1)^2 (1 - \Phi (x))^2 g_t (s_t (v, x), x)}
\times \int \int \left\{ \int K_f (w, y) K_X (y - z) K_g (u - s_{tv} w - s_{tx} y) dwdy \right\}^2 du dz,
$$

as $h \downarrow 0$, where $\pi (x) := E \left[ N^{-1}_1 \mid X_1 = x \right]$. Note that conditionally on $X_1$, the number of active bidders $N_1$ has a binomial distribution with parameters $N$ and $1 - \Phi (X_1)$. Lastly, similarly to Theorem 6.1, the expression in (6.5), and for any interior point $v \in (p_0 (x), \bar{v} (x))$, the GPV estimator of $f^* (v \mid x)$ is asymptotically normal with the asymptotic variance given by
where \( s_v(v, x) := \xi(\cdot, x) \), and the partial derivatives \( s^*_v \) and \( s^*_x \) are defined similarly to \( s_v \) and \( s_x \) in (6.6). Similarly to Corollary 6.1, one can show:

\[
\left( Lh^{3+d} \right)^{1/2} \left( \hat{f}^*(v \mid x) - f^*(v \mid x) \right) \rightarrow_d N \left( 0, \nu_{\theta GPV}(v, x) \right).
\]

To estimate the asymptotic variance \( \nu_{\theta GPV}(v, x) \), one can use the sample analogue of (7.2) in the same way as that used to construct the estimator of \( \nu_{GPV}(v, x) \) defined by (6.8) from (6.4). As before, the approach does not require estimation of the bidding strategy \( s^*(v, x) \) or its derivatives. The analogue estimator is given by

\[
\hat{\nu}_{\theta GPV}(v, x) := \frac{1}{\hat{\pi}(x)^2} \frac{1}{(N - 1)^2 h^{3(1+d)}} \frac{1}{L(L - 1)(L - 2)} \times \sum_{l=1}^{L} \sum_{k \neq l} \sum_{k' \neq k, k' \neq l} \frac{1}{N_l} \sum_{i=1}^{N_l} \eta_{il,k}(v, x) \eta_{il,k'}(v, x),
\]

where \( \hat{\pi}(x) \) is the Nadaraya-Watson estimator of \( \pi(x) \), and

\[
\eta_{il,k}(v, x) := \frac{1}{N_k} \sum_{j=1}^{N_k} T_{jk} K' \left( \frac{\hat{V}_{jk} - v}{h}, \frac{X_k - x}{h} \right) K_g \left( \frac{B_{il} - B_{jk}}{h} \right) K_X \left( \frac{X_l - X_k}{h} \right) \times \frac{2B_{jk} \left( (1 - \hat{\Phi}(X_k)) \hat{G}_t (B_{jk}, X_k) + \hat{\Phi}(X_k) \hat{\varphi}(X_k) \right)}{1 - \hat{\Phi}(X_k) \hat{g}_t (B_{jk}, X_k)^2}.
\]

Suppose \( I(x) \) is an inner closed sub-interval of \([p_0(x), \bar{v}(x)]\). The uniform convergence rate of \( \hat{\nu}_{\theta GPV}(v, x) \) to (7.2) can be shown to be the same as that in the statement of Theorem 6.2. In view of the definitions in (7.1), the nonparametric estimator for \( f(v \mid x) \) is \( \left( 1 - \hat{\Phi}(x) \right) \hat{f}^*(v \mid x) \). Since \( \hat{\Phi}(x) \) converges at a faster rate than the PDF estimator \( \hat{f}^*(v \mid x) \), one can see that

\[
\left( Lh^{3+d} \right)^{1/2} \left( \left( 1 - \hat{\Phi}(x) \right) \hat{f}^*(v \mid x) - f(v \mid x) \right) \rightarrow_d N \left( 0, (1 - \Phi(x))^2 \nu_{\theta GPV}(v, x) \right).
\]

A valid uniform confidence band of \( \{ f(v \mid x) : v \in I(x) \} \) can be constructed by adapting the methods described in Section 6.
Table 1: Coverage probabilities of the uniform confidence band $CB^*$ for the number of bidders $N = 3, 5, 7$, the distribution parameter $\theta = 1, 2$, different ranges of valuations $v$, and the nominal coverage probability $= 0.90, 0.95, 0.99$. The number of auctions $L$ is determined by $N \cdot L = 2100$.

| $N$ | $\theta = 1, \ v \in [0.3, 0.7]$ | $\theta = 1, \ v \in [0.2, 0.8]$ | $\theta = 2, \ v \in [0.3, 0.7]$ | $\theta = 2, \ v \in [0.2, 0.8]$ |
|-----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 3   | 0.880 0.924 0.988               | 0.880 0.930 0.982               | 0.912 0.954 0.994               | 0.900 0.944 0.988               |
| 5   | 0.922 0.962 0.998               | 0.918 0.962 0.994               | 0.882 0.938 0.990               | 0.872 0.932 0.990               |
| 7   | 0.888 0.946 0.990               | 0.898 0.948 0.984               | 0.916 0.950 0.986               | 0.914 0.964 0.994               |

8 Monte Carlo Simulations

In this section, we assess the finite-sample coverage accuracy of the uniform confidence bands. Our simulation design follows Marmer and Shneyerov (2012), and the DGP is described in Remark 3.4. We consider $\theta \in \{1, 2\}$ and draw valuations from $f_\theta$. We choose the triweight kernel when implementing the two-step estimator. We used the second-order triweight kernel in the second step and used the fourth-order triweight kernel in the first step.

We need to choose the bandwidths in the first step when we construct the pseudo valuations and in the second step when we implement kernel density estimation using the pseudo valuations. We follow GPV (see Section 2.4) and use $h_g = 3.72 \cdot \widehat{\sigma}_b \cdot (N \cdot L)^{-1/5}$ as the first-step bandwidth, where $\widehat{\sigma}_b$ is the estimated standard deviation of the observed bids. We use $h_f = 3.15 \cdot \widehat{\sigma}_v \cdot ((N \cdot L)_T)^{-1/5}$ as the second-step bandwidth, where $\widehat{\sigma}_v$ is the estimated standard deviation of the trimmed pseudo valuations and $(N \cdot L)_T$ is the number of bids remaining after the trimming. The constants 3.72 and 3.15 are Silverman’s rule-of-thumb constants corresponding to fourth-order and second-order triweight kernels.\(^{14}\) We consider different numbers of bidders $N \in \{3, 5, 7\}$, and also the density function over different ranges: $v \in [0.2, 0.8]$ and $v \in [0.3, 0.7]$.\(^{15}\) The number of auctions $L$ is chosen so the total number of observations is fixed as $N \cdot L = 2100$.

\(^{14}\)See Li and Racine (2007) for a description of the Silverman approach; see also Li et al. (2003, Section 3.2).

\(^{15}\)We use grid maximization, where the grid is chosen as $[v_l : 0.001 : v_u]$. We have also tried a finer grid $[v_l : 0.0001 : v_u]$, which produced similar results.
In Table 1, we report our simulation results for the bootstrap-based IGA uniform confidence band $CB^*$. We find that the IGA bootstrap approach provides accurate coverage probabilities. Additional simulation results are reported in the Supplement.

9 Conclusion

The GPV estimator has proven to be the essential input in virtually all nonparametric structural auction models. By proving the asymptotic normality and the first-order validity of the bootstrap uniform confidence bands, this paper completes the econometric theory of the GPV estimator and opens way to new applications.

Our pointwise asymptotic normality results can be used for inference on an important policy variable: the optimal reserve price. As discussed, e.g., in Haile and Tamer (2003), the optimal reserve price $r(x)$ in auctions with $X_l = x$ satisfies the following equation:

$$r(x) - \frac{1 - F(r(x)|x)}{f(r(x)|x)} = c(x),$$

where $c(x)$ is the seller’s own valuation. Suppose that the estimator $\hat{r}(x)$ is constructed by solving an estimated version of the above equation with $f(\cdot|x)$ replaced by its GPV estimator $\hat{f}_{GPV}(\cdot|x)$. In that case, our pointwise normality results imply the asymptotic normality of the estimated optimal reserve price:

$$\left( Lh^{3+d} \right)^{1/2} (\hat{r}(x) - r(x)) \to_d N \left( 0, \left( \frac{1 - F(r(x)|x)}{2f(r(x)|x)^2 + f'(r(x)|x)(1 - F(r(x)|x))} \right)^2 V_{GPV}(r(x)|x) \right).$$

Our results for the validity of the percentile bootstrap of the GPV estimator naturally carry over to the above estimator of the optimal reserve price. Thus in practice, one can use the percentile bootstrap for inference on the optimal reserve price.

Our uniform confidence bands can be used for specification of the density of valuations.

In future research, our results could be extended in several directions. Below, we briefly describe some of potentially interesting extensions. These extensions would address the limitations of the independent private values model that underlies the GPV estimator.

First, in GPV the bidders are treated symmetrically. Empirically this is not always the case. See, e.g., Flambard and Perrigne (2006) for an application to snow removal contracts. Second, we abstract from the empirically relevant issue of unobserved heterogeneity, as in Krasnokutskaya (2011), Hu et al. (2013) and Roberts (2013). Third, correlated values may also...
be important empirically. Li et al. (2002) and Hubbard et al. (2012) extend the GPV estimator to the affiliated value environment. Fourth, Guerre et al. (2009) and Zincenko (2018) provide extensions to an environment with risk-averse bidders. Fifth, the literature on endogenous entry in auctions has developed rapidly. See, e.g., Li and Zheng (2009), Krasnokutskaya and Seim (2011), Marmer et al. (2013), Roberts and Sweeting (2013) and Gentry and Li (2014).

In the above models, the basic GPV estimator is often adapted to suit the needs of a particular application. But most of these estimators share the underlying two-step structure of the GPV estimator. We conjecture that our main results will also prove useful for establishing the asymptotic normality and validity of certain uniform confidence bands for these GPV-like estimators.

A remaining unresolved important practical issue is bandwidth selection for the GPV estimator. It is possible that the recent advances in that area (e.g., Calonico et al., 2014 and Armstrong and Kolesár, 2016) can be adapted to the framework of GPV.

References

Armstrong, T. B., Kolesár, M., 2016. Simple and honest confidence intervals in nonparametric regression, working paper, Yale University.

Athey, S., Haile, P. A., 2002. Identification of standard auction models. Econometrica 70 (6), 2107–2140.

Athey, S., Haile, P. A., 2007. Nonparametric approaches to auctions. Handbook of Econometrics 6, 3847–3965.

Augenblick, N., 2015. The sunk-cost fallacy in penny auctions. Review of Economic Studies 83 (1), 58–86.

Bickel, P. J., Rosenblatt, M., 1973. On some global measures of the deviations of density function estimates. Annals of Statistics 1 (6), 1071–1095.

Calonico, S., Cattaneo, M. D., Titunik, R., 2014. Robust nonparametric confidence intervals for regression-discontinuity designs. Econometrica 82 (6), 2295–2326.

Chen, X., Kato, K., 2017. Jackknife multiplier bootstrap: Finite sample approximations to the U-process supremum with applications, working paper, University of Illinois at Urbana-Champaign.

Chernozhukov, V., Chetverikov, D., Kato, K., 2014a. Anti-concentration and honest, adaptive confidence bands. Annals of Statistics 42 (5), 1787–1818.
Chernozhukov, V., Chetverikov, D., Kato, K., 2014b. Gaussian approximation of suprema of empirical processes. Annals of Statistics 42 (4), 1564–1597.

Chernozhukov, V., Chetverikov, D., Kato, K., 2016. Empirical and multiplier bootstraps for suprema of empirical processes of increasing complexity, and related Gaussian couplings. Stochastic Processes and their Applications 126 (12), 3632–3651.

de Castro, L. I., Paarsch, H. J., 2010. Testing affiliation in private-values models of first-price auctions using grid distributions. Annals of Applied Statistics, 2073–2098.

Dudley, R. M., 2002. Real Analysis and Probability. Cambridge University Press.

Fang, H., Tang, X., 2014. Inference of bidders’ risk attitudes in ascending auctions with endogenous entry. Journal of Econometrics 180 (2), 198–216.

Flambard, V., Perrigne, I., 2006. Asymmetry in procurement auctions: evidence from snow removal contracts. Economic Journal 116 (514), 1014–1036.

Folland, G. B., 1999. Real Analysis: Modern Techniques and Their Applications, 2nd Edition. John Wiley & Sons.

Gentry, M., Li, T., 2014. Identification in auctions with selective entry. Econometrica 82 (1), 315–344.

Ghosal, S., Sen, A., van der Vaart, A. W., 2000. Testing monotonicity of regression. Annals of statistics 28 (4), 1054–1082.

Gimenes, N., 2017. Econometrics of ascending auctions by quantile regression. Review of Economics and Statistics 99 (5), 944–953.

Gimenes, N., Guerre, E., 2016. Quantile methods for first-price auction: A signal approach, working paper, PUC-Rio.

Giné, E., Nickl, R., 2015. Mathematical Foundations of Infinite-dimensional Statistical Models. Cambridge University Press.

Guerre, E., Perrigne, I., Vuong, Q., 1995. Nonparametric estimation of first-price auctions, working paper.

Guerre, E., Perrigne, I., Vuong, Q., 2000. Optimal nonparametric estimation of first-price auctions. Econometrica 68 (3), 525–574.

Guerre, E., Perrigne, I., Vuong, Q., 2009. Nonparametric identification of risk aversion in first-price auctions under exclusion restrictions. Econometrica 77 (4), 1193–1227.
Haile, P. A., Hong, H., Shum, M., 2003. Nonparametric tests for common values at first-price sealed-bid auctions, NBER Working paper No.10105.

Haile, P. A., Tamer, E., 2003. Inference with an incomplete model of english auctions. Journal of Political Economy 111 (1), 1–51.

Henderson, D. J., List, J. A., Millimet, D. L., Parmeter, C. F., Price, M. K., 2012. Empirical implementation of nonparametric first-price auction models. Journal of Econometrics 168 (1), 17–28.

Hendricks, K., Porter, R. H., 2007. An empirical perspective on auctions. Handbook of Industrial Organization 3, 2073–2143.

Hickman, B. R., Hubbard, T. P., 2014. Replacing sample trimming with boundary correction in nonparametric estimation of first-price auctions. Journal of Applied Econometrics 30, 739–762.

Hickman, B. R., Hubbard, T. P., Sağlam, Y., 2012. Structural econometric methods in auctions: A guide to the literature. Journal of Econometric Methods 1 (1), 67–106.

Hill, J. B., Shneyerov, A., 2013. Are there common values in first-price auctions? a tail-index nonparametric test. Journal of Econometrics 174 (2), 144–164.

Hu, Y., McAdams, D., Shum, M., 2013. Identification of first-price auctions with non-separable unobserved heterogeneity. Journal of Econometrics 174 (2), 186–193.

Hubbard, T. P., Li, T., Paarsch, H. J., 2012. Semiparametric estimation in models of first-price, sealed-bid auctions with affiliation. Journal of Econometrics 168 (1), 4–16.

Jun, S. J., Pinkse, J., Wan, Y., 2010. A consistent nonparametric test of affiliation in auction models. Journal of Econometrics 159 (1), 46–54.

Kato, K., Sasaki, Y., 2017. Uniform confidence bands for nonparametric errors-in-variables regression, working paper, University of Tokyo.

Kato, K., Sasaki, Y., 2018. Uniform confidence bands in deconvolution with unknown error distribution. Journal of Econometrics 207 (1), 129–169.

Kawai, K., Nakabayashi, J., 2015. Detecting large-scale collusion in procurement auctions, working paper, University of California, Berkeley.

Kosorok, M. R., 2007. Introduction to Empirical Processes and Semiparametric Inference. Springer.
Krasnokutskaya, E., 2011. Identification and estimation of auction models with unobserved heterogeneity. Review of Economic Studies 78 (1), 293–327.

Krasnokutskaya, E., Seim, K., 2011. Bid preference programs and participation in highway procurement auctions. American Economic Review 101 (6), 2653–2686.

Lahiri, S. N., 2013. Resampling Methods for Dependent Data. Springer.

Li, Q., Racine, J. S., 2007. Nonparametric Econometrics: Theory and Practice. Princeton University Press, Princeton, New Jersey.

Li, T., Perrigne, I., Vuong, Q., 2002. Structural estimation of the affiliated private value auction model. RAND Journal of Economics, 171–193.

Li, T., Perrigne, I., Vuong, Q., 2003. Semiparametric estimation of the optimal reserve price in first-price auctions. Journal of Business & Economic Statistics 21 (1), 53–64.

Li, T., Zhang, B., 2010. Testing for affiliation in first-price auctions using entry behavior. International Economic Review 51 (3), 837–850.

Li, T., Zheng, X., 2009. Entry and competition effects in first-price auctions: theory and evidence from procurement auctions. Review of Economic Studies 76 (4), 1397–1429.

Liu, N., Luo, Y., 2017. A nonparametric test for comparing valuation distributions in first-price auctions. International Economic Review 58 (3), 857–888.

Liu, N., Vuong, Q., 2013. Nonparametric test of monotonicity of bidding strategy in first price auctions, working paper, Shanghai University of Finance and Economics.

Luo, Y., Wan, Y., 2018. Integrated-quantile-based estimation for first-price auction models. Journal of Business & Economic Statistics 36 (1), 173–180.

Mammen, E., 1992. Bootstrap, wild bootstrap, and asymptotic normality. Probability Theory and Related Fields 93 (4), 439–455.

Mammen, E., Rothe, C., Schienle, M., et al., 2012. Nonparametric regression with nonparametrically generated covariates. Annals of Statistics 40 (2), 1132–1170.

Marmer, V., Shneyerov, A., 2012. Quantile-based nonparametric inference for first-price auctions. Journal of Econometrics 167 (2), 345–357.

Marmer, V., Shneyerov, A., Xu, P., 2013. What model for entry in first-price auctions? A nonparametric approach. Journal of Econometrics 176 (1), 46–58.
Newey, W. K., 1994. Kernel estimation of partial means and a general variance estimator. Econometric Theory 10 (2), 1–21.

Nolan, D., Pollard, D., 1987. U-processes: rates of convergence. Annals of Statistics 15 (2), 780–799.

Paarsch, H. J., 1997. Deriving an estimate of the optimal reserve price: an application to British Columbian timber sales. Journal of Econometrics 78 (2), 333–357.

Pinkse, J., 2001. Nonparametric regression estimation using weak separability. Unpublished manuscript.

Rilstone, P., 1996. Nonparametric estimation of models with generated regressors. International Economic Review 37 (2), 299–313.

Roberts, J. W., 2013. Unobserved heterogeneity and reserve prices in auctions. RAND Journal of Economics 44 (4), 712–732.

Roberts, J. W., Sweeting, A., 2013. When should sellers use auctions? American Economic Review 103 (5), 1830–1861.

Serfling, R. J., 2009. Approximation Theorems of Mathematical Statistics. John Wiley & Sons.

van der Vaart, A. W., 2000. Asymptotic Statistics. Cambridge University Press.

van der Vaart, A. W., Wellner, J. A., 1996. Weak Convergence and Empirical Processes with Applications to Statistics. Springer.

Zincenko, F., 2018. Nonparametric estimation of first-price auctions with risk-averse bidders. Journal of Econometrics 205 (2), 303–335.
Appendix

Let $\leq$ denote an inequality up to a universal constant that does not depend on the sample size $L$. Denote $\|f\|_{Q,2} := \left( \int |f|^2 \, dQ \right)^{1/2}$. For a sequence of classes of functions $\mathcal{F}_L$ (that may depend on the sample size) defined on $[\underline{b}, \overline{b}]^d$, let $N\left( \epsilon, \mathcal{F}_L, \|\cdot\|_{Q,2} \right)$ denote the $\epsilon$—covering number, i.e., the smallest integer $m$ such that there are $m$ balls of radius $\epsilon$ (with respect to the metric induced by the norm $\|\cdot\|_{Q,2}$) centered at points in $\mathcal{F}_L$. A function $F_L : [\underline{b}, \overline{b}]^d \rightarrow \mathbb{R}_+$ is an envelope of $\mathcal{F}_L$ if $F_L \geq \sup_{f \in \mathcal{F}_L} |f|$. We say that $\mathcal{F}_L$ is a (uniform) Vapnik-Chervonenkis-type (VC-type) class with respect to the envelope $\mathcal{F}_L$ if there exist some positive constants $A$ and $V$ such that

$$N\left( \epsilon, F_L, \mathcal{F}_L, \|\cdot\|_{Q,2} \right) \leq \left( \frac{A}{\epsilon} \right)^V,$$

for all finitely discrete probability measure $Q$ on $[\underline{b}, \overline{b}]^d$. Note that the all function classes that appear later are dependent on $L$. We suppress the dependence for notational simplicity.

Denote $\overline{C}_{K_f} := \sup_{u \in \mathbb{R}} |K_f(u)|$ and $\overline{C}_{K_g} := \sup_{u \in \mathbb{R}} |K_g(u)|$. Since $\sup_{u \in \mathbb{R}} K''_f(u) < \infty$, the restriction of $K'_f$ on $[-1, 1]$ is of bounded variation. We can decompose $K'_f = D_1 - D_2$, where $D_1$ and $D_2$ are non-decreasing and bounded. Similarly, denote $\overline{C}_{D_s} := \sup_{u \in \mathbb{R}} |D_s(u)|$, for $s \in \{1, 2\}$. Let $\overline{C}_{s'} := \sup_{u \in [\underline{b}, \overline{b}]} s'(u)$. Let $C_k, k = 1, 2, \ldots$ denote positive universal constants that are independent of the sample size and whose values may change in different places. $\delta_{k,L}, k = 1, 2, 3, \ldots$ denote positive null sequences of real numbers (i.e., $\delta_{k,L} \downarrow 0$, as $L \uparrow \infty$).

Let $E^* \left[ \cdot \right]$ denote the expectation under $P^*$, the conditional probability distribution given the original sample. We use the following notation for bootstrap asymptotics. We say that $\xi_L = o^*_p (\lambda_L)$ if for all $\epsilon > 0$, $P^* \left[ |\xi_L/\lambda_L| > \epsilon \right] \rightarrow 0$ as $L \uparrow \infty$. We say that $\xi_L = O^*_p (\lambda_L)$ if for all $\epsilon > 0$, there is $M_\epsilon > 0$ and some $L_\epsilon \in \mathbb{N}$ such that $P \left[ P^* \left[ |\xi_L/\lambda_L| \geq M_\epsilon \right] > \epsilon \right] < \epsilon$ for all $L \geq L_\epsilon$. Properties of $O^*_p$ and $o^*_p$ notations carry over to $O^*_p$ and $o^*_p$ naturally.

“With probability approaching 1” is abbreviated as “w.p.a.1”. For notational simplicity, in the proofs, $\max_{i,l}$ is understood as $\max_{(i,l) \in \{1, \ldots, N\} \times \{1, \ldots, L\}}$. $\sum$ is understood as $\sum_{(i,l) \in \{1, \ldots, N\} \times \{1, \ldots, L\}}$. $\sum$ is understood as

$$\sum_{(i,l) \in \{1, \ldots, N\} \times \{1, \ldots, L\}} \sum_{(j,k) \neq (i,l)} \sum_{(j',k') \neq (j,k)} ,$$

i.e., summing over all distinct indices. We also adopt the following notation: $(N \cdot L)_2$ is understood as $(N \cdot L) (N \cdot L - 1)$ and $(N \cdot L)_3$ is understood as $(N \cdot L) (N \cdot L - 1) (N \cdot L - 2)$. 

37
In the proofs, we will often invoke maximal inequalities for empirical processes. A sharper one is Chernozhukov et al. (2014b, Corollary 5.1). It is more convenient to apply a modified version of this inequality, i.e., Chen and Kato (2017, Corollary 5.5) (with \( r = k = 1 \)). To avoid ambiguity, we will still refer to this inequality as the “CCK inequality” in the proofs. In some applications, a less sharp but simpler maximal inequality for empirical processes suffices. See, e.g., van der Vaart and Wellner (1996, Theorem 2.14.1). We will refer to this inequality as the “VW inequality”. We will also often invoke a (less sharp) maximal inequality for U-processes, i.e., Chen and Kato (2017, Corollary 5.6). We will refer to this inequality as the “CK inequality”. The following lemmas provide asymptotic expansions that are crucial for proving the distributional results provided in Theorems 3.1 and 5.1 and also the bootstrap consistency results. Their proofs are relegated to the Supplement.

### A Proofs for Sections 3–5

**Lemma A.1.** Suppose Assumptions 1 - 3 hold. Then,

\[
\hat{f}_{GPV} (v) - f (v) = \frac{1}{(N - 1)(N \cdot L)^2} \sum_{i,l} \sum_{j,k} \mathcal{M} (B_{il}, B_{jk}; v)
\]

\[
+ \frac{1}{R!} f^{(R)} (v) \left( \int K_f (u) u^R du \right) h^R + O_p \left( \left( \frac{\log (L)}{Lh} \right)^{1/2} + \frac{\log (L)}{Lh^3} \right) + o (h^R),
\]

where the remainder term is uniform in \( v \in I \) and \( \mathcal{M} \) is defined by (3.5).

**Lemma A.2.** Suppose Assumptions 1 - 3 hold. Then,

\[
\hat{f}_{GPV} (v) - f (v) = \frac{1}{N - 1} \frac{1}{N \cdot L} \sum_{i,l} \left( \mathcal{M}_2 (B_{il}; v) - \mu_M (v) \right)
\]

\[
+ \frac{1}{R!} f^{(R)} (v) \left( \int K_f (u) u^R du \right) h^R + O_p \left( \left( \frac{\log (L)}{Lh} \right)^{1/2} + \frac{\log (L)}{Lh^3} \right) + o (h^R),
\]

where the remainder term is uniform in \( v \in I \).

**Proof of Theorem 3.1.** It follows from Lemma A.2 that

\[
(Lh^3)^{1/2} \left( \hat{f}_{GPV} (v) - f (v) \right) = \frac{1}{N^{1/2} (N - 1)} \frac{1}{(N \cdot L)^{1/2}} \sum_{i,l} h^{3/2} \left( \mathcal{M}_2 (B_{il}; v) - \mu_M (v) \right) + o_p (1).
\]

Next, we show that a central limit theorem for triangular arrays can be applied to the leading term on the right-hand side.
Let

\[ M_\ast^\dagger (b, b'; v) := -\frac{1}{h^3} K' f \left( \frac{\xi (b) - v}{h} \right) \frac{G (b)}{g (b)^2} K g \left( \frac{b' - b}{h} \right), \]

\[ M_\ast^\dagger (v) := \int \int M_\ast^\dagger (b, b'; v) \, dG (b') \, dG (b) = \int M_\ast^\dagger (b, v) \, dG (b). \]

It is easy to see that

\[ M_2 (B_{il}; v) - \mu_M (v) = M_2^\dagger (B_{il}; v) - \mu_M^\dagger (v), \] for all \( i = 1, ..., N \) and \( l = 1, ..., L. \) (A.1)

Define

\[ U_{il} (v) := \frac{1}{N^{1/2} (N - 1)} \frac{1}{(N \cdot L)^{1/2}} \frac{h^{3/2}}{2} \left( M_2^\dagger (B_{il}; v) - \mu_M^\dagger (v) \right) \]

and

\[ \sigma (v) := \left( \sum_{i,l} \mathbb{E} \left[ U_{il} (v)^2 \right] \right)^{1/2} = \left( \frac{1}{N (N - 1)^2} h^3 \mathbb{E} \left[ \left( M_2^\dagger (B_{11}; v) - \mu_M^\dagger (v) \right)^2 \right] \right)^{1/2}. \] (A.2)

By the definition of \( U_{il} (v), \)

\[ (L h^{3/2}) \left( \tilde{f}_{GPV} (v) - f (v) \right) = \sum_{i,l} U_{il} (v) + o_p (1). \] (A.3)

Next, we show that

\[ \frac{1}{N (N - 1)^2} h^3 \mathbb{E} \left[ \left( M_2^\dagger (B_{11}; v) - \mu_M^\dagger (v) \right)^2 \right] - \mathcal{V}_M (v) = O (h^3) \] (A.4)

uniformly in \( v \in I. \) It is easy to check that

\[ \mu_M (v) = \mu_M^\dagger (v) + \left( \int_2^\beta - \frac{1}{h^2} K' f \left( \frac{\xi (b) - v}{h} \right) G (b) \, db \right). \]
By change of variable and a mean value expansion,

\[
\int_{b}^{b} K_f' \left( \frac{\xi(b) - v}{h} \right) G(b) \, db = h \int_{\frac{v}{h}}^{\frac{v}{h}} K_f' (u) G(s(hu + v)) \, s'(hu + v) \, du
\]

\[
= h \int_{\frac{v}{h}}^{\frac{v}{h}} K_f'(u) \left\{ G(s(v)) s'(v) + \left( g(s(\hat{v})) s'(\hat{v})^2 + G(s(\hat{v})) s''(\hat{v}) \right) hu \right\} \, du \quad \text{(A.5)}
\]

for some mean value \( \hat{v} \) (depending on \( u \)) such that \( |\hat{v} - v| \leq h |u| \). Since \( \int K_f'(u) \, du = 0 \) and \( K_f' \) is supported on \([-1, 1] \), we have

\[
\sup_{v \in I} \left| \int_{\frac{v}{h}}^{\frac{v}{h}} K_f'(u) G(s(hu + v)) \, s'(hu + v) \, du \right| \leq h \left( \int |K_f'(u) u| \, du \right) \left( \sup_{u \in [1, 1]} |g(s(u)) s'(u)^2 + G(s(u)) s''(u)| \right).
\]

By the above result, (A.5), the continuity of \( s' \) and \( s'' \) and the continuity of \( g \) and \( G \), we have

\[
\sup_{v \in I} \left| \int_{\frac{v}{h}}^{\frac{v}{h}} K_f' \left( \frac{\xi(b) - v}{h} \right) G(b) \, db \right| = O(h^2) \quad \text{(A.6)}
\]

It is shown in the proof of Lemma A.2 that \( \sup_{v \in I} |\mu_M(v)| = o(h^R) \). Therefore we have

\[
\sup_{v \in I} |\mu_M^R(v)| = O(1). \quad \text{(A.7)}
\]

It is clear that

\[
E \left[ h^3 \mathcal{M}_2^R(B_{11}; v)^2 \right] = h^{-3} \int \left\{ \int_{\frac{v}{h}}^{\frac{v}{h}} K_f' \left( \frac{\xi(b) - v}{h} \right) \frac{G(b)}{g(b)} K_g \left( \frac{b' - b}{h} \right) \, db \right\}^2 \, dG(b')
\]

\[
= N (N - 1)^2 V_M(v).
\]

Now (A.4) follows from the above result and (A.7).

By change of variables \( u = (\xi(b) - v)/h \) and \( w = (b - s(v))/h \),

\[
E \left[ h^3 \mathcal{M}_2^L(B_{11}; v)^2 \right] = \int_{\frac{v}{h}}^{\frac{v}{h}} \left\{ \int_{\frac{v}{h}}^{\frac{v}{h}} \rho(u, w; v) \, du \right\}^2 g(hw + s(v)) \, dw, \quad \text{(A.8)}
\]

\[\text{See Proposition 1 and Lemma A1 of GPV.}\]
where $\psi(z) := G(s(z))s'(z)/g(s(z))$ and

$$\rho(u, w; v) := K_f'(u) \psi(hu + v) K_g \left( w - \frac{s(hu + v) - s(v)}{h} \right).$$

Denote

$$\overline{\rho}(w; v) := \psi(v) \left\{ \int K_f'(u) K_g \left( w - s'(v) u \right) du \right\}. $$

Next, we show that

$$E \left[ h^3 M_2^4 \left( B_{11}; v \right)^2 \right] = g(s(v)) \int \overline{\rho}(w; v)^2 \, dw + o(1)$$

$$= \frac{F(v)^2 f(v)^2}{g(s(v))^3} \int \left\{ \int K_f'(u) K_g \left( w - s'(v) u \right) du \right\}^2 \, dw + o(1), \quad \text{(A.9)}$$

where the remainder term is uniform in $v \in I$ and we applied the equality $f(v) = g(s(v)) s'(v)$ to obtain the second equality. Since $s'$ is continuous and $K_g$ and $K_f'$ are supported on $[-1, 1]$ and bounded, it follows from (S.1.5) and the reverse triangle inequality that

$$\sup_{u \in I} \left\{ 1 \left( u \in \left[ \frac{u - v}{h}, \frac{\overline{v} - v}{h} \right] \right) \left| \rho(u, w; v) \right| \right\} \lesssim 1 \left( |u| \leq 1 \right) 1 \left( \left| w - \frac{s(hu + v) - s(v)}{h} \right| \leq 1 \right)$$

$$\lesssim 1 \left( |u| \leq 1 \right) 1 \left( |w| \leq 1 + \overline{C}s' \right), \quad \text{(A.10)}$$

for all $(u, w) \in \mathbb{R}^2$. Similarly,

$$\sup_{v \in I} \left| \overline{\rho}(w; v) \right| \lesssim 1 \left( |w| \leq 1 + \overline{C}s' \right). \quad \text{(A.11)}$$

Next, by the triangle inequality,

$$\left| \int_{b-s(v)/h}^{b-s(g(v))/h} \left\{ \int_{b-s(v)/h}^{b-s(g(v))/h} \rho(u, w; v) \, du \right\}^2 g(hw + s(v)) \, dw - g(s(v)) \int \overline{\rho}(w; v)^2 \, dw \right|$$

$$\leq \left| \int_{b-s(v)/h}^{b-s(g(v))/h} \left\{ \int_{b-s(v)/h}^{b-s(g(v))/h} \rho(u, w; v) \, du \right\}^2 \{g(hw + s(v)) - g(s(v))\} \, dw \right|$$

$$+ \left| \int_{b-s(v)/h}^{b-s(g(v))/h} \left( \left\{ \int_{b-s(v)/h}^{b-s(g(v))/h} \rho(u, w; v) \, du \right\}^2 - \overline{\rho}(w; v)^2 \right) g(s(v)) \, dw \right|$$

$$+ g(s(v)) \left| \int_{b-s(v)/h}^{b-s(g(v))/h} \overline{\rho}(w; v)^2 \, dw - \int \overline{\rho}(w; v)^2 \, dw \right|. \quad \text{(A.12)}$$
Equation (A.11) implies that the last term of the right-hand side of (A.12) is zero for all \( v \in I \), when \( h \) is sufficiently small. Now (A.10) implies

\[
\sup_{v \in I} \left| \int_{\frac{\pi - s(v)}{h}}^{\pi + s(v)} \left\{ \int_{\frac{\pi - v}{h}}^{\pi + v} \rho(u, w; v) \, du \right\}^2 \{ g(hw + s(v)) - g(s(v)) \} \, dw \right|
\]

\[
\leq \sup_{v \in I} \int_{1} \left( |w| \leq 1 + \overline{C}_s \right) |g(hw + s(v)) - g(s(v))| \, dw
\]

\[
\leq \sup \{ |g(b') - g(b)| : b \in [s(v_l), s(v_u)], |b' - b| \leq (1 + \overline{C}_s') h \}
\]

\[= o(1), \quad (A.13)\]

where the inequalities hold when \( h \) is sufficiently small and the equality follows from the fact that \( g \) is uniformly continuous on any inner closed subinterval of \([b, b']\).

By (A.10) and (A.11),

\[
\sup_{v \in I} \left| \int_{\frac{\pi - s(v)}{h}}^{\pi + s(v)} \left\{ \int_{\frac{\pi - v}{h}}^{\pi + v} \rho(u, w; v) \, du \right\}^2 - \overline{\rho}(w; v)^2 \right| \, dw
\]

\[
\leq \sup_{v \in I} \int_{1} \left( |w| \leq 1 + \overline{C}_s \right) \int_{\frac{\pi - v}{h}}^{\pi + v} \rho(u, w; v) \, du - \overline{\rho}(w; v) \, dw. \quad (A.14)\]

It follows from the uniform continuity of \( \psi \), which is implied by GPV Proposition 1 and Lemma A1, that

\[
\sup_{u \in \mathbb{R}} \sup_{v \in I} \left| \int_{\frac{\pi - v}{h}}^{\pi + v} \rho(u, w; v) \, du - \psi(v) \int_{\frac{\pi - v}{h}}^{\pi + v} K_j'(u) K_g \left( w - \frac{s(hu + v) - s(v)}{h} \right) \, du \right| = o(1). \]

By the mean value theorem, since \( K_j' \) and \( K_g \) are both supported on \([-1, 1]\) and bounded,

\[
\sup_{u \in \mathbb{R}} \sup_{v \in I} \left| \psi(v) \int_{\frac{\pi - v}{h}}^{\pi + v} K_j'(u) K_g \left( w - \frac{s(hu + v) - s(v)}{h} \right) \, du - \overline{\rho}(w; v) \right|
\]

\[
\leq \sup_{v \in I} \left| K_j'(u) \right| \frac{s(hu + v) - s(v)}{hu} - s'(v) \, du
\]

\[
\leq \sup \{ |s'(v') - s'(v)| : v \in I, |v - v'| \leq h \}
\]

\[= o(1), \quad (A.15)\]

where the inequalities hold when \( h \) is sufficiently small and the equality holds since \( s' \) is
uniformly continuous. Now it follows that
\[
\sup_{w \in \mathbb{R}} \sup_{v \in I} \left| \int_{\frac{\pi - v}{h}}^{\pi - s(v)/h} \rho(u, w; v) \, du - \bar{\rho}(w; v) \right| = o(1).
\]

It follows from the above result and (A.14) that
\[
\sup_{v \in I} \left| \int_{\frac{\pi - s(v)}{h}}^{\pi - a(v)/h} \left( \int_{\frac{\pi - u}{h}}^{\pi - s(v)/h} \rho(u, w; v) \, du \right)^2 - \bar{\rho}(w; v)^2 \right| \, dw = o(1).
\]

Then (A.9) follows from the above result, (A.12) and (A.13).

We have
\[
\begin{aligned}
E\left[ h^3 \left( \mathcal{M}_2^f(B_{11}; v) - \mu_{\mathcal{M}_2^f}(v) \right)^2 \right] \\
= E\left[ h^3 \mathcal{M}_2^f(B_{11}; v)^2 \right] - h^3 \mu_{\mathcal{M}_2^f}(v)^2 \\
= \frac{F(v)^2 f(v)^2}{g(s(v))^3} \int \left\{ \int K_f'(u) K_g(w - s'(v)u) \, du \right\}^2 \, dw + o(1),
\end{aligned}
\]

(A.16)

where the remainder term is uniform in \( v \in I \).

Lastly, we verify Lyapunov’s condition for each fixed \( v \in I \). It is clear from the definition of \( \sigma(v) \) (see (S.3.5)) and (A.16) that
\[
\sigma(v) = \frac{1}{N^{1/2}(N - 1)} \left\{ \frac{F(v)^2 f(v)^2}{g(s(v))^3} \int \left\{ \int K_f'(u) K_g(w - s'(v)u) \, du \right\}^2 \, dw \right\}^{1/2} + o(1).
\]

(A.17)

By LoÅšve’s \( c_r \) inequality,
\[
\begin{aligned}
\sum_{i,l} E \left[ \left| \frac{U_{il}(v)}{\sigma(v)} \right|^3 \right] &= \sigma(v)^{-3} (N - 1)^{-3}(N \cdot L)^{-1/2} E \left[ h^{3/2} \left( \mathcal{M}_2^f(B_{11}; v) - \mu_{\mathcal{M}_2^f}(v) \right)^3 \right] \\
&\leq \sigma(v)^{-3} (N \cdot L)^{-1/2} \left( h^{3/2} E \left[ \mathcal{M}_2^f(B_{11}; v) \right]^3 \right) + h^{3/2} \mu_{\mathcal{M}_2^f}(v)^3.
\end{aligned}
\]

(A.18)

It follows from the \( c_r \) inequality and change of variables that
\[
E \left[ h^{3/2} \left| \mathcal{M}_2^f(B_{11}; v) \right|^3 \right] \leq h^{-1/2} \int_{\frac{\pi - s(v)}{h}}^{\pi - a(v)/h} \int_{\frac{\pi - u}{h}}^{\pi - s(v)/h} \rho(u, w; v) \, du \left| g(hw + s(v)) \right| \, dw.
\]

43
Now it is clear from (A.10) that
\[
\sup_{v \in I} \mathbb{E} \left[ h^{3/2} \left| M_2^v (B_{11}; v) \right|^3 \right] = \mathcal{O} \left( h^{-1/2} \right).
\]
(A.19)

It now follows from the above result, (A.7), (A.17) and (A.18) that
\[
\sum_{i,l} \mathbb{E} \left[ \left| \frac{U_{il} (v)}{\sigma (v)} \right|^3 \right] \downarrow 0, \text{ as } L \uparrow \infty.
\]
(A.20)

Hence, by Lyapunov’s central limit theorem,
\[
\sum_{i,l} \frac{U_{il} (v)}{\sigma (v)} \rightarrow_d N (0, 1), \text{ as } L \uparrow \infty.
\]

The conclusion follows from the above result, (A.3), (A.17) and Slutsky’s lemma. ■

**Proof of Theorem 4.1.** Write
\[
N (N - 1)^2 \hat{V}_{GPV} (v) = \Delta_1^v (v) + \Delta_2^v (v) + \Delta_3^v (v),
\]
where
\[
\Delta_1^v (v) := \frac{1}{(N \cdot L)_3} \sum_{(3)} h^3 T_{jk} K_f' \left( \frac{\hat{V}_{jk} - v}{h} \right) \frac{G (B_{jk})}{g (B_{jk})^2} K_g \left( \frac{B_{il} - B_{jk}}{h} \right)
\times T_{j'k'} K_f' \left( \frac{\hat{V}_{j'k'} - v}{h} \right) \frac{G (B_{j'k'})}{g (B_{j'k'})^2} K_g \left( \frac{B_{il} - B_{j'k'}}{h} \right),
\]
\[
\Delta_2^v (v) := \frac{2}{(N \cdot L)_3} \sum_{(3)} h^3 T_{jk} K_f' \left( \frac{\hat{V}_{jk} - v}{h} \right) \frac{G (B_{jk})}{g (B_{jk})^2} K_g \left( \frac{B_{il} - B_{jk}}{h} \right)
\times T_{j'k'} K_f' \left( \frac{\hat{V}_{j'k'} - v}{h} \right) \left( \frac{\hat{G} (B_{j'k'})}{\hat{g} (B_{j'k'})^2} - \frac{G (B_{j'k'})}{g (B_{j'k'})^2} \right) K_g \left( \frac{B_{il} - B_{j'k'}}{h} \right)
\]
and
\[
\Delta_3^v (v) := \frac{1}{(N \cdot L)_3} \sum_{(3)} h^3 T_{jk} K_f' \left( \frac{\hat{V}_{jk} - v}{h} \right) \left( \frac{\hat{G} (B_{jk})}{\hat{g} (B_{jk})^2} - \frac{G (B_{jk})}{g (B_{jk})^2} \right) K_g \left( \frac{B_{il} - B_{jk}}{h} \right)
\times T_{j'k'} K_f' \left( \frac{\hat{V}_{j'k'} - v}{h} \right) \left( \frac{\hat{G} (B_{j'k'})}{\hat{g} (B_{j'k'})^2} - \frac{G (B_{j'k'})}{g (B_{j'k'})^2} \right) K_g \left( \frac{B_{il} - B_{j'k'}}{h} \right).
\]

44
By standard arguments (see, e.g., Marmer and Shneyerov, 2012, Lemma 1),

\[
\sup_{b \in [b-h,b+h]} \left| \tilde{G}(b) - G(b) \right| = O_p \left( \frac{\log (L)}{L}^{1/2} + h^{1+R} \right).
\]

Then by the triangle inequality, (S.1.5), (S.1.1) and the identity

\[
a/b = a/c - \frac{a(b-c)}{c^2} + \frac{a(b-c)^2}{bc^2}.
\]

\[
\left| \Delta^1_2(v) \right| \leq \max_{j',k'} \mathbb{T}_{j'k'} \left| \tilde{G} \left( B_{j'k'} \right) - G \left( B_{j'k'} \right) \right|^2 \left| K_f' \left( \frac{\tilde{V}_{jk} - v}{h} \right) \right|
\times |K_g \left( \frac{B_{il} - B_{jk}}{h} \right)| \mathbb{T}_{j'k'} \left| K_f' \left( \frac{\tilde{V}_{j'k'} - v}{h} \right) \right| |\mathbb{I} \left( \frac{\tilde{V}_{jk} - v}{h} \leq h \right) |
\times |K_g \left( \frac{B_{il} - B_{jk}}{h} \right)| \mathbb{T}_{j'k'} \left| \tilde{V}_{j'k'} - v \right| \left( \mathbb{I} \left( \frac{\tilde{V}_{jk} - v}{h} \leq h \right) \right) \left| K_g \left( \frac{B_{il} - B_{jk}}{h} \right) \right| \left( \mathbb{I} \left( \frac{\tilde{V}_{j'k'} - v}{h} \leq h \right) \right) \right).
\]

(A.22)

Since it is shown in the proof of Lemma A.1 that

\[
\max_{i,l} \mathbb{T}_{il} \left| \tilde{V}_{il} - V_{il} \right| = O_p \left( \frac{\log (L)}{Lh}^{1/2} + h^{1+R} \right) = o_p(h),
\]

(A.23)

by the triangle inequality,

\[
\frac{1}{(N \cdot L)_3} \sum_{(3)} h^{-3} \mathbb{T}_{jk} \mathbb{I} \left( \frac{\tilde{V}_{jk} - v}{h} \leq h \right) \left| K_g \left( \frac{B_{il} - B_{jk}}{h} \right) \right| \mathbb{T}_{j'k'} \mathbb{I} \left( \frac{\tilde{V}_{j'k'} - v}{h} \leq h \right) \left| K_g \left( \frac{B_{il} - B_{jk}}{h} \right) \right| \left( \mathbb{I} \left( \frac{\tilde{V}_{jk} - v}{h} \leq h \right) \right) \left( \mathbb{I} \left( \frac{\tilde{V}_{j'k'} - v}{h} \leq h \right) \right) \\
\leq \frac{1}{(N \cdot L)_3} \sum_{(3)} h^{-3} \mathbb{I} \left( |\xi(B_{jk}) - v| \leq 2h \right) \left| K_g \left( \frac{B_{il} - B_{jk}}{h} \right) \right| \left( |\xi(B_{j'k'}) - v| \leq 2h \right) \left| K_g \left( \frac{B_{il} - B_{jk}}{h} \right) \right| \mathcal{K} \left( B_{il}, B_{jk}, B_{j'k'}; v \right),
\]

(A.24)
where the inequality holds w.p.a.1.

Define

\[
\begin{align*}
K_1^{(1)} (b; v) &:= \int \int K(b, b', b''; v) \, dG(b') \, dG(b''), \\
K_2^{(1)} (b; v) &:= \int \int K(b', b, b''; v) \, dG(b') \, dG(b''), \\
K_3^{(1)} (b; v) &:= \int \int K(b', b'', b; v) \, dG(b') \, dG(b'') , \tag{A.25}
\end{align*}
\]

\[
\begin{align*}
K_1^{(2)} (b, b'; v) &:= \int K(b, b', b''; v) \, dG(b''), \\
K_2^{(2)} (b, b'; v) &:= \int K(b, b'', b'; v) \, dG(b''), \\
K_3^{(2)} (b, b'; v) &:= \int K(b'', b, b'; v) \, dG(b'') , \tag{A.26}
\end{align*}
\]

and

\[
\mu_K (v) := \int \int \int K(b, b', b''; v) \, dG(b) \, dG(b') \, dG(b''). \tag{A.27}
\]

The Hoeffding decomposition yields

\[
\frac{1}{(N \cdot L)^3} \sum_{(3)} K(B_{il}, B_{jk}, B_{ij'k''}; v)
= \mu_K (v) + \frac{1}{N \cdot L} \sum_{i,l} \left( K_1^{(1)} (B_{il}; v) - \mu_K (v) \right) + \frac{1}{N \cdot L} \sum_{i,l} \left( K_2^{(1)} (B_{il}; v) - \mu_K (v) \right) \\
+ \frac{1}{N \cdot L} \sum_{i,l} \left( K_3^{(1)} (B_{il}; v) - \mu_K (v) \right) + \Upsilon_1^K (v) + \Upsilon_2^K (v) + \Upsilon_3^K (v) + \Psi_K (v) , \tag{A.28}
\]

where \( \Upsilon_1^K (v) \), \( \Upsilon_2^K (v) \) and \( \Upsilon_3^K (v) \) are degenerate U-statistics of order two and \( \Psi_K (v) \) is a degenerate U-statistic of order three:

\[
\begin{align*}
\Upsilon_1^K (v) &:= \frac{1}{(N \cdot L)_2} \sum_{(2)} \left\{ K_1^{(2)} (B_{il}, B_{jk}; v) - K_1^{(1)} (B_{il}; v) - K_2^{(1)} (B_{jk}; v) + \mu_K (v) \right\} , \\
\Upsilon_2^K (v) &:= \frac{1}{(N \cdot L)_2} \sum_{(2)} \left\{ K_2^{(2)} (B_{il}, B_{jk}; v) - K_1^{(1)} (B_{il}; v) - K_3^{(1)} (B_{jk}; v) + \mu_K (v) \right\} , \\
\Upsilon_3^K (v) &:= \frac{1}{(N \cdot L)_2} \sum_{(2)} \left\{ K_3^{(2)} (B_{il}, B_{jk}; v) - K_2^{(1)} (B_{il}; v) - K_3^{(1)} (B_{jk}; v) + \mu_K (v) \right\} , \tag{A.29}
\end{align*}
\]
and
\[
\Psi_K(v) := \frac{1}{(N \cdot L)_3} \sum_{(3)} \left\{ K(B_{il}, B_{jk}, B_{j'k'}; v) - K_1^{(2)}(B_{il}, B_{jk}; v) - K_2^{(2)}(B_{il}, B_{j'k'}; v) - K_3^{(2)}(B_{jk}, B_{j'k'}; v) + K_1^{(1)}(B_{il}; v) + K_2^{(1)}(B_{jk}; v) + K_3^{(1)}(B_{j'k'}; v) - \mu_K(v) \right\}. \tag{A.30}
\]

By change of variables, the expression for \( \mu_K(v) \) is given by
\[
\int_b^{-s(v)} h_b^{-s(v)} \left\{ \int_v^{-v} h_v^{-v} \left( |w| \leq 2 \right) K_g \left( u - s(hw + v) - s(v) \right) \frac{d}{d\omega} (hw + v) g(s(hw + v)) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \
envelope $h^{-3}C_{K_2}$. The CK inequality yields
\[
\mathbb{E} \left[ \sup_{v \in I} \left| Y_{k}^2 (v) \right| \right] \leq (Lh^3)^{-1}, \text{ for } k = 1, 2, 3 \text{ and } \mathbb{E} \left[ \sup_{v \in I} |\Psi_K (v)| \right] \leq L^{-3/2}h^{-3}. \quad (A.32)
\]

Since $\mathcal{X}$ is VC-type, it follows from Chen and Kato (2017, Lemma 5.4) that the function classes $\left\{ K_k (\cdot ; v) : v \in I \right\}$, for $k = 1, 2, 3$ are all VC-type with respect to the constant envelope $h^{-3}C_{K_2}$. By Jensen’s inequality,
\[
\sup_{v \in I} \mathbb{E} \left[ K_k^{(1)} (B_{11} ; v)^2 \right] \leq \sup_{v \in I} \int \int \int K (b, b', b'' ; v)^2 \, dG (b) \, dG (b') \, dG (b''), \text{ for } k = 1, 2, 3.
\]

By change of variables,
\[
\sup_{v \in I} \int \int \int K (b, b', b'' ; v)^2 \, dG (b) \, dG (b') \, dG (b'') \\
\leq h^{-3}C_{K_2} \left\{ \sup_{v \in I} \frac{\mathbb{E} (s(v))}{h} \{ \int \frac{s(v)}{h} \, 1 (|w| \leq 2) 1 (|u| \leq 1 + 2C_{s'}) \, g (s (hw + v)) \, d w \}^2 \right. \\
\left. \times g (hu + s (v)) \, d u \right\}
\]
and hence,
\[
\sup_{v \in I} \mathbb{E} \left[ K_k^{(1)} (B_{11} ; v)^2 \right] \leq h^{-3}, \text{ for } k = 1, 2, 3. \quad (A.33)
\]

Now we apply the CCK inequality with $\sigma^2$ being the left-hand side of (A.33) and $F$ being $h^{-3}C_{K_2}$. It follows that
\[
\mathbb{E} \left[ \sup_{v \in I} \frac{1}{N \cdot L} \sum_{i,l} \left( K_k^{(1)} (B_{il} ; v) - \mu_K (v) \right) \right] \\
\leq C_1 \left\{ (Lh^3)^{-1/2} \log (C_2 L)^{1/2} + (Lh^3)^{-1} \log (C_2 L) \right\} \\
= O \left( \left( \frac{\log (L)}{Lh^3} \right)^{1/2} \right), \text{ for } k = 1, 2, 3. \quad (A.34)
\]

Now,
\[
\sup_{v \in I} \frac{1}{N \cdot L} \sum_{i,l} K (B_{il}, B_{jkl}, B_{jkl} ; v) = O_p (1) \quad (A.35)
\]
follows from (A.28), (A.31), (A.34) and (A.32) and
\[
\sup_{v \in I} \left| \Delta_2^2 (v) \right| = O_p \left( \left( \frac{\log (L)}{Lh} \right)^{1/2} + h^{1+R} \right)
\]
follows from (A.35) and (A.22). Similarly, we can show

$$\sup_{v \in I} |\Delta_3^I(v)| = O_p \left( \frac{\log (L)}{Lh} + h^{2+2\epsilon} \right).$$

Using mean-value expansion arguments,

$$\Delta_1^I(v) = \frac{1}{(N \cdot L)^3} \sum_{j,k}^3 h^{-3} \mathbb{T}_{jk} K_f' \left( \frac{V_{jk} - v}{h} \right) \frac{G(B_{jk})}{g(B_{jk})^2} K_g \left( \frac{B_{il} - B_{jk}}{h} \right)$$

$$+ \mathbb{T}_{j'k'} K_f' \left( \frac{V_{j'k'} - v}{h} \right) \frac{G(B_{j'k'})}{g(B_{j'k'})^2} K_g \left( \frac{B_{il} - B_{j'k'}}{h} \right)$$

$$+ \frac{2}{(N \cdot L)^3} \sum_{j,k}^3 h^{-4} \mathbb{T}_{jk} K_f'' \left( \frac{\hat{V}_{jk} - v}{h} \right) \left( \hat{V}_{jk} - V_{jk} \right)$$

$$+ \mathbb{T}_{j'k'} K_f'' \left( \frac{\hat{V}_{j'k'} - v}{h} \right) \left( \hat{V}_{j'k'} - V_{j'k'} \right) \frac{G(B_{j'k'})}{g(B_{j'k'})^2} K_g \left( \frac{B_{il} - B_{j'k'}}{h} \right)$$

$$=: \Delta_4^I(v) + 2 \cdot \Delta_5^I(v) + \Delta_6^I(v),$$

where $\hat{V}_{jk}$ ($\hat{V}_{j'k'}$) is the mean value that lies between $V_{jk}$ ($V_{j'k'}$) and $\hat{V}_{jk}$ ($\hat{V}_{j'k'}$). Now by (S.1.5), the triangle inequality, the fact $\max_{i,l} T_{il} \left| V_{il} - V_{il} \right| = O_p(h)$ and the fact that $K_f'$ and $K_f''$ are both compactly supported on $[-1, 1]$,

$$|\Delta_5^I(v)| \lesssim h^{-1} \left\{ \max_{j,k} \mathbb{T}_{jk} \left| \hat{V}_{jk} - V_{jk} \right| \right\} \left( \frac{1}{(N \cdot L)^3} \sum_{j,k}^3 \mathbb{K}(B_{il}, B_{jk}, B_{j'k'}; v) \right)^{1/2}$$

$$= O_p \left( \frac{\log (L)}{Lh^3} + h^{2R} \right),$$

where the inequality holds w.p.a.1 and the equality is uniform in $v \in I$, and also

$$|\Delta_6^I(v)| \lesssim h^{-2} \left\{ \max_{j,k} \mathbb{T}_{jk} \left| \hat{V}_{jk} - V_{jk} \right| \right\} \left( \frac{1}{(N \cdot L)^3} \sum_{j,k}^3 \mathbb{K}(B_{il}, B_{jk}, B_{j'k'}; v) \right)^2$$

$$= O_p \left( \frac{\log (L)}{Lh^3} + h^{2R} \right),$$

49
where the inequality holds w.p.a.1 and the equality is uniform in \( v \in I \).

Denote
\[
\mathcal{H}(b, b', b''; v) := \frac{1}{h^3} K_f' \left( \frac{\xi(b') - v}{h} \right) K_g \left( \frac{b - b'}{h} \right) \frac{G(b')}{(b')^2} K_f' \left( \frac{\xi(b'') - v}{h} \right) K_g \left( \frac{b - b''}{h} \right) \frac{G(b'')}{(b'')^2}.
\]

Since \( K_f' \) is compactly supported on \([-1, 1]\), the trimming is asymptotically negligible:
\[
\Delta_4^\dagger(v) = \frac{1}{(N \cdot L)^3} \sum_{(3)} \mathcal{H}(B_{ild}, B_{ijk}, B_{jk'}; v), \text{ for all } v \in I,
\]
w.p.a.1.\(^{17}\) The Hoeffding decomposition yields
\[
\frac{1}{(N \cdot L)^3} \sum_{(3)} \mathcal{H}(B_{ild}, B_{ijk}, B_{jk'}; v) = \mu_{\mathcal{H}}(v) + \frac{1}{N \cdot L} \sum_{i,l} \left( \mathcal{H}^{(1)}_1(B_{ild}; v) - \mu_{\mathcal{H}}(v) \right) + \frac{1}{N \cdot L} \sum_{i,l} \left( \mathcal{H}^{(1)}_2(B_{ild}; v) - \mu_{\mathcal{H}}(v) \right)
+ \frac{1}{N \cdot L} \sum_{i,l} \left( \mathcal{H}^{(1)}_3(B_{ild}; v) - \mu_{\mathcal{H}}(v) \right) + \Gamma_{\mathcal{H}}^1(v) + \Gamma_{\mathcal{H}}^2(v) + \Gamma_{\mathcal{H}}^3(v) + \Psi_{\mathcal{H}}(v), \quad (A.36)
\]
where the terms in the decomposition are defined by (S.1.55) to (S.1.60) with \( \mathcal{K} \) replaced by \( \mathcal{H} \). Note that we have \( \mu_{\mathcal{H}}(v) = N(N - 1)^2 V_M(v) \). Also define \( \mathcal{H} := \{ \mathcal{H}(\cdot, \cdot, \cdot; v) : v \in I \} \).

By the arguments used to show that \( \mathcal{K} \) is (uniformly) VC-type, we can show that \( \mathcal{H} \) is (uniformly) VC-type with respect to the constant envelope
\[
h^{-3} \left( C_{D1} + C_{D2} \right)^2 C_{Kg}^2 C_{g}^{-4}. \quad (A.37)
\]
Since \( \mathcal{H} \) is VC-type, it again follows from Chen and Kato (2017, Lemma 5.4) that the function classes \( \{ \mathcal{H}^{(1)}_k(\cdot; v) : v \in I \} \), for \( k = 1, 2, 3 \) are all VC-type with respect to the constant envelope \( (A.37) \).

\(^{17}\)Formally, we can show that the supremum (over \( v \in I \)) of the difference between \( \Delta_4^\dagger(v) \) and the same term with \( \tilde{T}_{jk} \) \((T_{jk'}; v)\) replaced by \( \tilde{T}_{jk} \) \((\tilde{T}_{jk'}; v)\) is equal to zero w.p.a.1. Then since \( K_f' \) is compactly supported on \([-1, 1]\), it is straightforward to see that the contribution of the trimming factors \( \tilde{T}_{jk} \) and \( \tilde{T}_{jk'} \) is negligible.
Similarly, by Jensen’s inequality and change of variables, for \(k = 1, 2, 3\),

\[
E \left[ \mathcal{H}_k^{(1)} (B_{11}; v)^2 \right] \leq \int \int \int \mathcal{H} (b, b', b''; v)^2 \, dG (b) \, dG (b') \, dG (b'')
\]

\[
= h^{-6} \int \left\{ \int_{b} K_f' \left( \frac{\xi (b') - v}{h} \right)^2 K_g \left( \frac{b - b'}{h} \right)^2 \, \frac{G (b')^2 \, db'}{g (b')^3} \right\}^2 \, g (b) \, db
\]

\[
\leq h^{-3} \int \left\{ \int \mathbb{1} (|u| \leq 1) \, \mathbb{1} (|w| \leq 1 + C_s') \, du \right\}^2 \, dw,
\]

for all \(v \in I\). Therefore,

\[
\sup_{v \in I} E \left[ \mathcal{H}_k^{(1)} (B_{11}; v)^2 \right] \leq h^{-3}, \ k = 1, 2, 3. \quad \text{(A.38)}
\]

We apply the CCK inequality again with \(\sigma^2\) being the left-hand side of (A.38) and \(F\) being (A.37) and also the CK inequality. It now follows that the bounds (A.32) and (A.34) also hold as we replace \(K\) by \(H\). \(\blacksquare\)

Let \(\widetilde{f}\) denote the infeasible estimator that uses the unobserved true valuations:

\[
\widetilde{f} (v) = \frac{1}{N \cdot L} \sum_{i,l} \frac{1}{h} K_f \left( \frac{V_{il} - v}{h} \right).
\]

Let

\[
\widetilde{f}^* (v) := \frac{1}{N \cdot L} \sum_{i,l} \frac{1}{h} K_f \left( \frac{V_{il}^* - v}{h} \right)
\]

with \(V_{il}^* := \xi (B_{il}^*)\) be the empirical bootstrap analogue of \(\widetilde{f} (v)\). The following lemma is used in the the proofs of bootstrap consistency results (Theorems 4.2 and 5.2). Its proof can be found in the Supplement.

**Lemma A.3.** Suppose Assumptions 1 - 3 hold. Then,

\[
\sup_{v \in I} \left| \widetilde{f}^* (v) - \widetilde{f} (v) \right| = O_p \left( \left( \log \left( \frac{L}{L} \right) \right)^{1/2} \right).
\]

The proofs of the bootstrap consistency results also hinge on an asymptotic expansion for \(\hat{f}_{GPV}^* (v) - \hat{f}_{GPV} (v)\), which is the empirical bootstrap analogue of \(\hat{f}_{GPV} (v) - f (v)\). The following lemma provides a crucial asymptotic expansion result that is invoked in the proof of bootstrap consistency. Its proof is relegated to the Supplement.
Lemma A.4. Suppose that Assumptions 1 - 3 hold. Then

\[ \hat{f}_{\text{GPV}}(v) - \hat{f}^*(v) = \frac{1}{(N - 1)(N \cdot L)^2} \sum_{i,l} \sum_{j,k} \mathcal{M}(B_{il}, B_{jk}; v) \]

\[ + \frac{1}{N - 1} \left\{ \frac{1}{N \cdot L} \sum_{i,l} \mathcal{M}_2(B_{il}^*; v) - \frac{1}{N \cdot L} \sum_{i,l} \mathcal{M}_2(B_{il}; v) \right\} \]

\[ + O_p^* \left( \left( \frac{\log(L)}{Lh} \right)^{1/2} + \frac{\log(L)}{Lh^3} + h^R \right), \]

where the remainder term is uniform in \( v \in I \).

Proof of Theorem 4.2. Write

\[ \hat{f}_{\text{GPV}}^*(v) - \hat{f}_{\text{GPV}}^*(v) = \left( \hat{f}_{\text{GPV}}^*(v) - \hat{f}^*(v) \right) + \left( \hat{f}^*(v) - \tilde{f}(v) \right) - \left( \hat{f}_{\text{GPV}}^*(v) - \tilde{f}(v) \right). \] (A.39)

It follows from Lemma A.1 and the fact \( \tilde{f}(v) - f(v) = O_p \left( \left( \frac{\log(L)}{Lh} \right)^{1/2} + h^R \right) \) that

\[ \hat{f}_{\text{GPV}}^*(v) - \tilde{f}(v) = \frac{1}{(N - 1)(N \cdot L)^2} \sum_{i,l} \sum_{j,k} \mathcal{M}(B_{il}, B_{jk}; v) \]

\[ + O_p \left( \left( \frac{\log(L)}{Lh} \right)^{1/2} + \frac{\log(L)}{Lh^3} + h^R \right), \]

where the remainder term is uniform in \( v \in I \). It follows from the above result, Lemmas A.3, A.4 and Marmer and Shneyerov (2012, online supplement, Lemma S.1) that for any fixed \( v \in (v, \pi) \),

\[ (Lh^3)^{1/2} \left( \hat{f}_{\text{GPV}}^*(v) - \hat{f}_{\text{GPV}}^*(v) \right) \]

\[ = \frac{1}{N^{1/2} (N \cdot L)^{1/2}} \sum_{i,l} h^{3/2} \left( \mathcal{M}_2(B_{il}^*; v) - \frac{1}{NL} \sum_{i,l} \mathcal{M}_2(B_{il}; v) \right) + o_p^*(1), \]

where the leading term of the right-hand side is an empirical bootstrap analogue of

\[ \frac{1}{N^{1/2} (N \cdot L)^{1/2}} \sum_{i,l} h^{3/2} \left( \mathcal{M}_2(B_{il}; v) - \mu_{\mathcal{M}}(v) \right), \]

which was shown to converge in distribution to a normal random variable in the proof of Theorem 3.1. The conclusion now follows from Theorem 1 of Mammen (1992), Pólya’s theorem and the conditional Slutsky’s lemma (see, e.g., Lahiri, 2013, Lemma 4.1). ■
The following lemma is invoked in the proof of Theorem 5.1. It essentially follows from Lemma A.2 and Theorem 4.1. Its proof is relegated to the Supplement. It is easy to check that

$$\Gamma(v) = \frac{1}{(N \cdot L)^{1/2}} \sum_{i,t} \frac{\mathcal{M}_2^i (B_{it}; v) - \mu_{\mathcal{M}_2} (v)}{\text{Var} \left[ \mathcal{M}_2^i (B_{11}; v) \right]^{1/2}}, \text{ for all } v \in I.$$  

**Lemma A.5.** Suppose Assumptions 1 - 3 hold. Then,

$$\sup_{v \in I} |Z(v) - \Gamma(v)| = O_p \left( \log (L)^{3/2} h + \frac{\log (L)}{(Lh^3)^{1/2}} + L^{1/2} h^{3/2 + R} \right),$$

where \( \{Z(v) : v \in I\} \) was defined by (4.3) and \( \{\Gamma(v) : v \in I\} \) was defined by (5.1).

The following result is useful (see Chernozhukov et al., 2016, Lemma 2.1).

**Lemma A.6.** Let \( V \) and \( W \) be random variables such that \( P[|V - W| > r_1] \leq r_2, \) for some positive constants \( r_1 \) and \( r_2. \) Then,

$$|P[V \leq t] - P[W \leq t]| \leq P[|W - t| \leq r_1] + r_2, \text{ for all } t \in \mathbb{R}.$$  

**Proof of Theorem 5.1.** Recall that \( \text{Var} \left[ h^{3/2} \mathcal{M}_2^i (B_{11}; v) \right] \) converges to \( N (N - 1)^2 V_{GPV} (v) \) uniformly in \( v \in I \) as \( h \downarrow 0, \) see the proof of Theorem 3.1. Therefore, when \( h \) is sufficiently small,

$$\inf_{v \in I} \text{Var} \left[ h^{3/2} \mathcal{M}_2^i (B_{11}; v) \right] > C_1 > 0.$$  

By standard arguments (see the proof of Lemma A.1), we can easily verify that \( \mathcal{M}^\dagger := \{\mathcal{M}^\dagger (\cdot, \cdot; v) : v \in I\} \) is (uniformly) VC-type with respect to the constant envelope

$$h^{-3} (C_{D_1} + C_{D_1'}) C_s^{-2} C_{K_g}. \quad (A.40)$$

It follows from the fact that \( \mathcal{M}^\dagger \) is a VC-type class with respect to the constant envelope (S.1.69) and Chen and Kato (2017, Lemma 5.4) that \( \{\mathcal{M}_2^i (\cdot; v) : v \in I\} \) is also VC-type with respect to the constant envelope (S.1.69). It follows from this result and Chernozhukov et al. (2014b, Corollary A.1) that the function class

$$\mathcal{S} := \left\{ \frac{h^{3/2} (\mathcal{M}_2 (\cdot; v) - \mu_{\mathcal{M}_2} (v))}{\text{Var} \left[ h^{3/2} \mathcal{M}_2 (B_{11}; v) \right]^{1/2}} : v \in I \right\} = \left\{ \frac{h^{3/2} \left( \mathcal{M}_2^i (\cdot; v) - \mu_{\mathcal{M}_2} (v) \right)}{\text{Var} \left[ h^{3/2} \mathcal{M}_2^i (B_{11}; v) \right]^{1/2}} : v \in I \right\}$$

is (uniformly) VC-type with respect to a constant envelope that is a multiple of \( h^{-3/2} \) when \( h \) is sufficiently small.
Note that \( \{ \Gamma (v) : v \in I \} \) is an empirical process indexed by \( \mathcal{S} \). Chernozhukov et al. (2014b, Lemma 2.1) implies the existence of a tight Gaussian element in \( \ell^\infty (\mathcal{S}) \), denoted by \( S_G \) with mean zero and covariance function

\[
E [S_G (f) S_G (g)] = E [f (B_{11} g) (B_{11})], \text{ for all } (f, g) \in \mathcal{S}^2.
\]

Define another mean-zero Gaussian process

\[
\Gamma_G (v) := S_G \left( \frac{h^{3/2} \left( \mathcal{M}_2 \left( ; v \right) - \mu_{\mathcal{M}_2} (v) \right)}{\text{Var} \left[ h^{3/2} \mathcal{M}_2 (B_{11}; v) \right]^{1/2}} \right), \quad v \in I.
\]

It is easy to check that the process \( \{ \Gamma_G (v) : v \in I \} \) is a tight Borel measurable random element in \( \ell^\infty (I) \) (for any fixed \( L \)) by referring to the definitions. See, e.g., Kosorok (2007, Page 105). Note that \( \{ \Gamma_G (v) : v \in I \} \) has the same covariance structure as \( \{ \Gamma (v) : v \in I \} \). Kosorok (2007, Lemma 7.2 and Lemma 7.4) yield that \( I \) is totally bounded if endowed with the intrinsic pseudo metric \((v, v') \mapsto \left[ E \left( (\Gamma_G (v) - \Gamma_G (v'))^2 \right) \right]^{1/2} \) and \( \{ \Gamma_G (v) : v \in I \} \) is separable as a stochastic process. Application of Chernozhukov et al. (2014b, Corollary 2.2) with \( q = \infty, b \approx h^{-3/2}, \gamma = \log (L)^{-1} \) and \( \sigma = 1 \) yields that there exists a sequence of random variables \( W_L \) with

\[
W_L \overset{d}{=} \| S_G \|_{\mathcal{S}} = \| \Gamma_G \|_I
\]

satisfying

\[
\| \| \Gamma \|_I - W_L \| = O_p \left( \frac{\log (L)}{(Lh^{3/2})^{1/6}} \right). \quad (A.41)
\]

Note that the distribution of \( W_L \) (also that of \( \| \Gamma_G \|_I \)) changes with \( L \). Since \( E \left[ \Gamma_G (v)^2 \right] = 1 \) for all \( v \in I \), the diameter of \( I \) under the intrinsic metric is less than or equal to 2. By the calculations used in the proof of Chernozhukov et al. (2014b, Corollary 5.1) and approximation based on the strong law of large numbers (see van der Vaart and Wellner, 1996, Problem 2.5.1), we have

\[
\int_0^1 \sqrt{\log \left( 2N \left( \epsilon, \mathcal{S}, \| \|_{G,2} \right) \right)} d\epsilon \leq \int_0^1 \sqrt{1 + \log \left( \frac{h^{-3/2}}{\epsilon} \right)} d\epsilon \leq \log \left( h^{-1} \right)^{1/2}.
\]

Then in view of the fact that \( E \left[ \Gamma_G (v)^2 \right] = 1 \) for all \( v \in I \), Dudley’s bound (see, e.g., Giné and Nickl, 2015, Theorem 2.3.7) yields

\[
E \left[ \| \Gamma_G \|_I \right] = E \left[ \| S_G \|_{\mathcal{S}} \right] = O \left( \log \left( h^{-1} \right)^{1/2} \right). \quad (A.42)
\]

Since \( \{ \Gamma_G (v) : v \in I \} \) is a centered separable Gaussian process with \( E \left[ \Gamma_G (v)^2 \right] = 1 \) for
all $v \in I$, the Gaussian anti-concentration inequality Chernozhukov et al. (2014a, Corollary 2.1) yields
\[
\sup_{z \in \mathbb{R}} P \left[ \| \Gamma_G \|_I - |z| \leq \epsilon \right] \leq 4 \epsilon (E \| \Gamma_G \|_I + 1), \text{ for all } \epsilon \geq 0. \tag{A.43}
\]

Note that Lemma A.5 and (A.41) yield $\| Z \|_I - W_L = o_p \left( \log \left( h^{-1} \right)^{-1/2} \right)$. This result and Dudley (2002, Theorem 9.2.2) imply that there exists some null sequence $\delta_{1,L} \downarrow 0$,
\[
P \left[ \| Z \|_I - W_L \right] \geq \log \left( h^{-1} \right)^{-1/2} \delta_{1,L} \] < \delta_{1,L}.

By the above result, Lemma A.6, the fact $W_L \overset{d}{=} \| \Gamma_G \|_I$, (A.42) and (A.43),
\[
\sup_{z \in \mathbb{R}} \left| P \left[ \| Z \|_I \leq z \right] - P \left[ \| \Gamma_G \|_I \leq z \right] \right| \leq \sup_{z \in \mathbb{R}} \left[ \| \Gamma_G \|_I - |z| \leq \log \left( h^{-1} \right)^{-1/2} \delta_{1,L} \right] + \delta_{1,L} = O \left( \delta_{1,L} \right).
\]

Denote $\mu_{\mathcal{M}^2_2}(v) := (N \cdot L)^{-1} \sum_{i,l} \mathcal{M}^1_2(B_{il}; v)$ and consider the following bootstrap analogue of $\Gamma$:
\[
\Gamma^*(v) := \frac{1}{(N \cdot L)^{1/2}} \sum_{i,l} \frac{\mathcal{M}^1_2(B_{il}; v) - \hat{\mu}_{\mathcal{M}^2_2}(v)}{\text{Var} \left[ \mathcal{M}^1_2(B_{11}; v) \right]^{1/2}}, \text{ for } v \in I. \tag{A.44}
\]

The following lemma is invoked in the proof of Theorem 5.2. Its proof is similar to that of Lemma A.5. It essentially follows from Lemmas A.1, A.3, A.4 and Theorem 4.1. We relegate its proof to the Supplement.

**Lemma A.7.** Suppose that Assumptions 1 - 3 hold. Then
\[
\sup_{v \in I} |Z^*(v) - \Gamma^*(v)| = O_p^* \left( \log (L)^{1/2} h + \frac{\log (L)}{(Lh^3)^{1/2}} + L^{1/2}h^{3/2+R} \right),
\]
where $\{ Z^*(v) : v \in I \}$ is defined by (5.2) and $\{ \Gamma^*(v) : v \in I \}$ is defined by (A.44).

We also need the following technical lemma in the proof of Theorem 5.2. Its proof can be found in the Supplement.

**Lemma A.8.** Let $V^*_L$ and $W^*_L$ be statistics computed using the bootstrap sample with $|V^*_L - W^*_L| = O_p^* (\lambda L)$ for some $\lambda L \downarrow 0$. Suppose for any fixed $C_1 > 0$,
\[
\sup_{z \in \mathbb{R}} P^* \left[ |W^*_L - z| \leq C_1 \lambda L \right] \to_p 0, \text{ as } L \uparrow \infty.
\]
Then,
\[
\sup_{z \in \mathbb{R}} |\mathbb{P}^* \left[ V^*_L \leq z \right] - \mathbb{P}^* \left[ W^*_L \leq z \right]| \to_p 0, \text{ as } L \uparrow \infty.
\]

**Proof of Theorem 5.2.** Application of Chernozhukov et al. (2016, Theorem 2.3) with \( B(f) = 0, q = \infty, b \leq h^{-3/2}, \gamma = \log(L)^{-1} \) and \( \sigma = 1 \) yields that there exists a sequence of random variables, \( W^*_L \), with the property that the conditional distribution of \( V^*_L \) given the original sample is the same as the (marginal) distribution of \( \| S_G \|_I = \| I_G \|_I \) almost surely, and
\[
\| \Gamma^*_I - W^*_L \| = O_p \left( \frac{\log(L)}{(L^3)^{1/6}} \right).
\]

This result, Lemma A.7 and Markov’s inequality yield
\[
\| Z^*_I - W^*_L \| = O_p \left( \lambda^*_L \right),
\]
where
\[
\lambda^*_L := \log(L)^{1/2} h + \log(L) \left( L^{-1} h^{-3} \right)^{1/6} + L^{1/2} h^{-3/2}.\]

Since the conditional distribution of \( W^*_L \) under \( \mathbb{P}^* \) is the same as the distribution of \( \| S_G \|_I = \| I_G \|_I \), (A.42) and the anti-concentration bound (A.43) now yield
\[
\sup_{z \in \mathbb{R}} \mathbb{P}^* \left[ |W^*_L - z| \leq C_1 \lambda^*_L \right] \leq 4C_1 \lambda^*_L \left( \mathbb{E}[\| I_G \|_I] + 1 \right) = o(1),
\]
for any \( C_1 > 0 \). The conclusion now follows from the fact that \( \mathbb{P}^* \left[ W^*_L \leq z \right] = \mathbb{P} \left[ \| I_G \|_I \leq z \right] \) for all \( z \in \mathbb{R} \) almost surely, and Lemma A.8 (with \( V^*_L = \| Z^*_I \|_I \)).

**Proof of Corollary 5.1.** Note that (A.43) implies that the CDF of \( \| I_G \|_I \) is Lipschitz continuous. Let \( \zeta(1-\tau) \) denote the \((1-\tau)\)-th quantile of \( \| I_G \|_I \). Now Theorem 5.2 and Dudley (2002, Theorem 9.2.2) imply that there exists some null sequence \( \delta_{2,L} \downarrow 0 \) such that
\[
\mathbb{P} \left[ \mathbb{P}^* \left[ \| Z^*_I \|_I \leq \zeta(1-\alpha + \delta_{2,L}) \right] - \mathbb{P} \left[ \| I_G \|_I \leq \zeta(1-\alpha + \delta_{2,L}) \right] > \delta_{2,L} \right] \leq \delta_{2,L},
\]
which clearly implies \( \mathbb{P} \left[ \mathbb{P}^* \left[ \| Z^*_I \|_I \leq \zeta(1-\alpha + \delta_{2,L}) \right] < 1-\alpha \right] < \delta_{2,L} \). By this result and the definition of \( \zeta^*_{L,\alpha} \) (see (5.3)), we have
\[
\mathbb{P} \left[ \zeta^*_{L,\alpha} > \zeta(1-\alpha + \delta_{2,L}) \right] = \delta_{2,L}. \quad (A.45)
\]
This further implies
\[
\Pr \left[ \|Z\| \leq \zeta_{L,\alpha} \right] \leq \Pr \left[ \|Z\| \leq \zeta(1 - \alpha + \delta_{2,L}) + \delta_{2,L} \right] \\
\leq \Pr \left[ \|\Gamma_G\| \leq \zeta(1 - \alpha + \delta_{2,L}) + \delta_{2,L} + \delta_{3,L} \right] \\
= (1 - \alpha) + 2\delta_{2,L} + \delta_{3,L},
\]
for some null sequence $\delta_{3,L} \downarrow 0$, where the second inequality follows from Theorem 5.1 and the equality follows from van der Vaart (2000, Lemma 21.1(ii)) and the continuity of the CDF of $\|\Gamma_G\|$. Similarly, by using
\[
\Pr \left[ \|Z\| \leq \zeta(1 - \alpha + \delta_{2,L}) \right] - \Pr \left[ \|\Gamma_G\| \leq \zeta(1 - \alpha + \delta_{2,L}) \right] > \delta_{2,L} \leq \delta_{2,L}
\]
which implies $\Pr \left[ \zeta_{L,\alpha} \leq \zeta(1 - \alpha + \delta_{2,L}) \right] < \delta_{2,L}$, we can show that a lower bound holds:
\[
\Pr \left[ \|Z\| \leq \zeta_{L,\alpha} \right] \geq (1 - \alpha) - (2\delta_{2,L} + \delta_{3,L}).
\]
Now we have
\[
\Pr \left[ \|Z\| \leq \zeta_{L,\alpha} \right] \to 1 - \alpha, \quad \text{as } L \uparrow \infty,
\]
which is exactly the first conclusion.

By the definition of $\zeta(\cdot)$ and the Borell-Sudakov-Tsirelson inequality (see Giné and Nickl, 2015, Theorems 2.2.7 and 2.5.8), we have
\[
\zeta(1 - \alpha + \delta_{2,L}) \leq \mathbb{E} [\|\Gamma_G\|] + \left(2\log \left( \frac{1}{\alpha - \delta_{2,L}} \right) \right)^{1/2} = O \left( \log (h^{-1})^{1/2} \right).
\]
By (A.45), $\zeta_{L,\alpha} \leq \zeta(1 - \alpha + \delta_{2,L})$ w.p.a.1, and the second conclusion follows. \qed

57
Supplement

S.1 Proofs of the Lemmas in Appendix A

To prove Lemmas A.1 and A.2, we derive the following intermediate asymptotic expansion first.

**Lemma S.1.1.** Suppose Assumptions 1 - 3 hold. Let \( \tilde{T}_{il} := \mathbb{1} \left( |V_{il} - v| \leq \tilde{\delta} \right) \) be an infeasible trimming factor. Then,

\[
\tilde{f}_{GPV}(v) - f(v) = \frac{1}{N \cdot L} \sum_{i,l} \tilde{T}_{il} \frac{1}{h^2} K_i^T \left( \frac{V_{il} - u}{h} \right) \left( \tilde{V}_{il} - V_{il} \right) + \frac{1}{R!} \tilde{f}^{(R)}(v) \left( \int K_i u^R du \right) h^R + O_p \left( \left( \frac{\log (L)}{Lh} \right)^{1/2} + \frac{\log (L)}{Lh^3} \right) + o \left( h^R \right),
\]

where the remainder term is uniform in \( v \in I \).

**Proof of Lemma S.1.1.** By standard arguments (see, e.g., Marmer and Shneyerov, 2012, Lemma 1),

\[
\sup_{b \in [\delta, \delta]} |\hat{G}(b) - G(b)| = O_p \left( \frac{\log (L)}{L} \right)^{1/2} \quad \text{and} \quad \sup_{b \in [\delta + h, \delta - h]} |\hat{g}(b) - g(b)| = O_p \left( \frac{\log (L)}{Lh^3} \right) + h^{1 + R}.
\]

(S.1.1)

It follows from the definitions of \( \tilde{V}_{il} \) and \( V_{il} \) and the identity

\[
a b = \frac{a}{c} - \frac{a (b - c)}{c^2} + \frac{a (b - c)^2}{bc^2}
\]

(S.1.2)

that

\[
\tilde{V}_{il} - V_{il} = \frac{1}{N - 1} \left\{ \hat{G}(B_{il}) - G(B_{il}) \right\} g(B_{il}) - \hat{G}(B_{il}) \hat{g}(B_{il}) - g(B_{il})^2 + \hat{G}(B_{il}) \hat{g}(B_{il}) - g(B_{il})^2 \}
\]

(S.1.3)

Then by the triangle inequality,

\[
\max_{i,l} T_{il} \left| \tilde{V}_{il} - V_{il} \right| \leq \max_{i,l} T_{il} \left| \hat{G}(B_{il}) - G(B_{il}) \right| g(B_{il}) + \max_{i,l} T_{il} \left| \hat{g}(B_{il}) - g(B_{il}) \right| g(B_{il})^2 + \max_{i,l} T_{il} \left| \hat{G}(B_{il}) - g(B_{il}) \right| g(B_{il})^2.
\]

(S.1.4)

The order of magnitude of the first and second terms of the right hand side of (S.1.4) is easily obtained by using

\[
C_{il} := \inf_{b \in [\delta, \delta]} g(b) > 0,
\]

(S.1.5)

and the fact that \( \hat{b} \geq b \) and \( \hat{b} \leq b \).

For the third term, since \( \max_{i,l} T_{il} \left| \hat{g}(B_{il}) - g(B_{il}) \right| = o_p \left( 1 \right) \), we have \( \max_{i,l} T_{il} \left| \hat{g}(B_{il}) - g(B_{il}) \right|^{-1} \leq (C_{il}/2)^{-1} \) w.p.a.1 and consequently,

\[
\max_{i,l} T_{il} \left| \frac{\hat{G}(B_{il}) - g(B_{il})}{\hat{g}(B_{il})} \right| g(B_{il})^2 \leq 2C_{il} \left\{ \max_{i,l} T_{il} \left| \hat{g}(B_{il}) - g(B_{il}) \right|^2 \right\} = O_p \left( \frac{\log (L)}{Lh} + h^{2 + 2R} \right),
\]

(S.1.6)

where the inequality holds w.p.a.1.

Now it follows that

\[
\max_{i,l} T_{il} \left| \tilde{V}_{il} - V_{il} \right| = O_p \left( \frac{\log (L)}{Lh} + h^{1 + R} \right).
\]

(S.1.7)

Similarly, we can also obtain

\[
\sup_{v \in I} \max_{i,l} T_{il} \left| \tilde{V}_{il} - V_{il} \right| = O_p \left( \frac{\log (L)}{Lh} + h^{1 + R} \right).
\]

(S.1.8)

by observing that \( \tilde{T}_{il} = \mathbb{1} \left( B_{il} \in [s (v - \delta), s (v + \delta)] \right) \) and using (S.1.1) and (S.1.4) (with \( T_{il} \) replaced by \( \tilde{T}_{il} \)).
Write
\[
\hat{f}_{GPV}(v) = \frac{1}{N \cdot L} \sum_{i,l} \left( \frac{\bar{T}_{il}}{h} K_f \left( \frac{\tilde{V}_{il} - v}{h} \right) + T_{il} \left( 1 - \bar{T}_{il} \right) \frac{1}{h} K_f \left( \frac{\tilde{V}_{il} - v}{h} \right) + (T_{il} - 1) \bar{T}_{il} \frac{1}{h} K_f \left( \frac{\tilde{V}_{il} - v}{h} \right) \right)
\]
\[= \frac{1}{N \cdot L} \sum_{i,l} \frac{\bar{T}_{il}}{h} K_f \left( \frac{\tilde{V}_{il} - v}{h} \right) + k_1^f(v) + k_2^f(v).
\]

Since \(K_f\) is compactly supported on \([-1,1]\), \(K_f((\tilde{V}_{il} - v)/h)\) is zero if \(\tilde{V}_{il}\) is outside of an \(h\)--neighborhood of \(v\). By the triangle inequality, we have
\[
\left| k_1^f(v) \right| \leq \frac{1}{N \cdot L} \sum_{i,l} h^{-1} T_{il} \left( 1 - \bar{T}_{il} \right) \left( \frac{\left| \tilde{V}_{il} - v \right|}{h} \leq 1 \right)
\]
\[\leq \frac{1}{N \cdot L} \sum_{i,l} h^{-1} T_{il} \left( 1 - \bar{T}_{il} \right) \left( \left| V_{il} - v \right| \leq h + \max_{j,k} T_{jk} \left| \tilde{V}_{jk} - V_{jk} \right| \right).
\]

Therefore,
\[
P \left[ \sup_{v \in I} \left| k_1^f(v) \right| = 0 \right] \geq P \left[ \max_{i,l} T_{il} \left| \tilde{V}_{il} - V_{il} \right| < \frac{\bar{T}}{2} \right],
\]
when \(h\) is sufficiently small. The right hand side of the above inequality tends to one as \(L \uparrow \infty\) (see (S.1.7)). Therefore, we have \(\sup_{v \in I} \left| k_1^f(v) \right| = 0\) w.p.a.1.

Similarly, we have
\[
\left| k_2^f(v) \right| \leq \frac{1}{N \cdot L} \sum_{i,l} h^{-1} (1 - T_{il}) \bar{T}_{il}
\]
\[\leq \frac{1}{N \cdot L} \sum_{i,l} h^{-1} T_{il} \left( \bar{b} - \tilde{b} + b + h > B_{il} \right) \bar{T}_{il} + \frac{1}{N \cdot L} \sum_{i,l} h^{-1} T_{il} \left( b - \bar{b} - \tilde{b} - h \right) \bar{T}_{il}
\]
\[\leq h^{-1} T_{il} \left( \bar{b} - \tilde{b} + b + h > s \left( v_l - \tilde{\sigma} \right) \right) + h^{-1} T_{il} \left( b - \bar{b} - \tilde{b} - h < s \left( v_u + \tilde{\sigma} \right) \right)
\]
and hence
\[
P \left[ \sup_{v \in I} \left| k_2^f(v) \right| > 0 \right] \leq P \left[ \bar{b} - \tilde{b} + b + h > s \left( v_l - \tilde{\sigma} \right) \right] + P \left[ b - \bar{b} - \tilde{b} - h < s \left( v_u + \tilde{\sigma} \right) \right].
\]
The right hand side of the above inequality tends to zero as \(L \uparrow \infty\) since by the Borel-Cantelli lemma (see also GPV Proposition 2) and (S.1.5), we have
\[
\left| \tilde{\sigma} - \sigma \right| = O_p \left( \frac{\log (L)}{L} \right), \quad \left| \tilde{b} - b \right| = O_p \left( \frac{\log (L)}{L} \right).
\]
Therefore now we have
\[
\hat{f}_{GPV}(v) = \frac{1}{N \cdot L} \sum_{i,l} \bar{T}_{il} \frac{1}{h} K_f \left( \frac{\tilde{V}_{il} - v}{h} \right), \quad \text{for all } v \in I, \text{ w.p.a.1.}
\]

Since \(K_f\) is compactly supported on \([-1,1]\),
\[
\sup_{v \in I} \max_{i,l} \left| \left( \frac{\bar{T}_{il}}{h} - 1 \right) \frac{1}{h} K_f \left( \frac{\tilde{V}_{il} - v}{h} \right) \right| = 0, \quad \text{when } h < \bar{T}.
\]
(S.1.9)

It now follows that
\[
\hat{f}_{GPV}(v) - \bar{f}(v) = \frac{1}{N \cdot L} \sum_{i,l} \bar{T}_{il} \frac{1}{h} K_f \left( \frac{\tilde{V}_{il} - v}{h} \right) - K_f \left( \frac{\tilde{V}_{il} - v}{h} \right), \quad \text{for all } v \in I, \text{ w.p.a.1.}
\]

By a second-order Taylor expansion of the right-hand side of the above equality,
\[
\hat{f}_{GPV}(v) - \bar{f}(v) = \frac{1}{N \cdot L} \sum_{i,l} \bar{T}_{il} \frac{1}{h^2} K_f' \left( \frac{\tilde{V}_{il} - v}{h} \right) \left( \tilde{V}_{il} - V_{il} \right) + \frac{1}{2} \frac{1}{N \cdot L} \sum_{i,l} \bar{T}_{il} \frac{1}{h^3} K_f'' \left( \frac{\tilde{V}_{il} - v}{h} \right) \left( \tilde{V}_{il} - V_{il} \right)^2,
\]
(S.1.10)
for some mean value \(\tilde{V}_{il}\) that lies on the line joining \(\tilde{V}_{il}\) and \(V_{il}\).
Since \( K'' \) is compactly supported on \([-1, 1]\) and bounded, by the triangle inequality, we have

\[
\begin{align*}
\sup_{v \in I} \frac{1}{N \cdot L} \sum_{i, l} \tilde{T}_{il} \frac{1}{h^2} K''_f \left( \tilde{V}_{il} - v \right) \left( \tilde{V}_{il} - V_{il} \right)^2
\leq \sup_{v \in I} \frac{1}{N \cdot L} \sum_{i, l} \tilde{T}_{il} h^{-3} \mathbb{E} \left[ \left| \tilde{V}_{il} - v \right| \leq h \right] \mathbb{E} \left[ \sup_{v \in I} \tilde{T}_{il} \left( \tilde{V}_{il} - V_{il} \right)^2 \right] \\
\leq \left\{ \sup_{v \in I} \frac{1}{N \cdot L} \sum_{i, l} \tilde{T}_{il} h^{-3} \mathbb{E} \left[ \left| V_{il} - v \right| \leq 2h \right] \mathbb{E} \left[ \sup_{v \in I} \tilde{T}_{il} \left( \tilde{V}_{il} - V_{il} \right)^2 \right] \right\} .
\end{align*}
\] (S.1.11)

where the last inequality holds w.p.a.1, since \( \sup_{v \in I} \tilde{T}_{il} \left| V_{il} - v \right| = o_p (h) \) (see (S.1.8)).

Denote \( \tilde{T}_{il} := \mathbb{I} \left( \left| V_{il} - v \right| \leq 2h \right) \). The CCK inequality and Markov’s inequality yield

\[
\sup_{v \in I} \frac{1}{N \cdot L} \sum_{i, l} h^{-1} \tilde{T}_{il} = O_p \left( \frac{\log (L)}{Lh} \right)^{1/2} .
\] (S.1.12)

Since \( \sup_{v \in I} \mathbb{E} \left[ h^{-1} \tilde{T}_{il} \right] \leq 4 \sup_{u \in [1, 2]} f (u) \), it follows that

\[
\sup_{v \in I} \frac{1}{N \cdot L} \sum_{i, l} h^{-1} \tilde{T}_{il} = O_p (1) .
\] (S.1.13)

It follows from the above result, (S.1.8), (S.1.11) and (S.1.12) that

\[
\begin{align*}
\sup_{v \in I} \frac{1}{N \cdot L} \sum_{i, l} \tilde{T}_{il} \frac{1}{h^2} K''_f \left( \tilde{V}_{il} - v \right) \left( \tilde{V}_{il} - V_{il} \right)^2 = O_p \left( \frac{\log (L)}{Lh^{3/2}} + h^{2R} \right) .
\end{align*}
\] (S.1.14)

By standard arguments for kernel density estimation (see, e.g., Newey, 1994),

\[
\tilde{f}_k = f - \mathbb{E} \left[ \tilde{f}_k \right] = \frac{1}{R!} f^{(R)} (\tilde{v}) \left( \int K_f (u) u^R du \right) h^{R} + o (h^{R}) ,
\] (S.1.15)

where the remainder terms are uniform in \( v \in I \). The conclusion now follows from (S.1.10), (S.1.14) and (S.1.15).

**Proof of Lemma A.1.** By using (S.1.3),

\[
\frac{1}{N \cdot L} \sum_{i, l} \tilde{T}_{il} \frac{1}{h^2} K''_f \left( \tilde{V}_{il} - v \right) \left( \tilde{V}_{il} - V_{il} \right)
\]

\[
= - \frac{1}{(N - 1) \cdot N \cdot L} \sum_{i, l} \tilde{T}_{il} \frac{1}{h^2} K''_f \left( \tilde{V}_{il} - v \right) \frac{G (B_{il})}{g (B_{il})} \left( \tilde{g} (B_{il}) - g (B_{il}) \right) + \Delta^1_1 (v) + \Delta^2_1 (v) + \Delta^3_1 (v) ,
\] (S.1.16)

where

\[
\Delta^1_1 (v) := \frac{1}{(N - 1) \cdot N \cdot L} \sum_{i, l} \tilde{T}_{il} \frac{1}{h^2} K''_f \left( \tilde{V}_{il} - v \right) \frac{G (B_{il}) - G (B_{il})}{g (B_{il})} ,
\]

\[
\Delta^2_1 (v) := - \frac{1}{(N - 1) \cdot N \cdot L} \sum_{i, l} \tilde{T}_{il} \frac{1}{h^2} K''_f \left( \tilde{V}_{il} - v \right) \frac{G (B_{il}) - G (B_{il})}{g (B_{il})^2} \left( \tilde{g} (B_{il}) - g (B_{il}) \right)
\]

\[
\Delta^3_1 (v) := - \frac{1}{(N - 1) \cdot N \cdot L} \sum_{i, l} \tilde{T}_{il} \frac{1}{h^2} K''_f \left( \tilde{V}_{il} - v \right) \frac{G (B_{il})}{g (B_{il})} \left( \tilde{g} (B_{il}) - g (B_{il}) \right)^2 .
\]
We have
\[
\sup_{v \in I} \left| \Delta_2^1 (v) \right| = O_p \left( \frac{\log (L)}{L h^{\alpha/2}} + h R \left( \frac{\log (L)}{L} \right)^{1/2} \right)
\]
(S.1.17)
by using the triangle inequality, (S.1.5), (S.1.1) and the fact that
\[
\sup_{v \in I} \frac{1}{N \cdot L} \sum_{i,l} \left| \frac{1}{h} K'_{ij} \left( \frac{V_{il} - v}{h} \right) \right| \leq \sup_{v \in I} \frac{1}{N \cdot L} \sum_{i,l} h^{-1} |\bar{T}_{il}| = O_p (1).
\]
It also follows from the above result, the triangle inequality and (S.1.6) that
\[
\sup_{v \in I} \left| \Delta_2^1 (v) \right| = O_p \left( \frac{\log (L)}{L h^{\alpha/2}} + h^{2R+1} \right).
\]
(S.1.18)

Next, we apply the maximal inequalities for empirical processes and degenerate U-processes to obtain the order bound for \( \sup_{v \in I} \left| \Delta_2^1 (v) \right| \). Since \( K'_{ij} \) is compactly supported on \([-1,1] \), the contribution of the trimmed observations is asymptotically negligible:
\[
\sup_{v \in I} \max_{i,l} \left| (\bar{T}_{il} - 1) K'_{ij} \left( \frac{V_{il} - v}{h} \right) \frac{(\hat{G}(B_{il}) - G(B_{il}))}{g(B_{il})} \right| = 0, \text{ when } h \leq \delta.
\]
(S.1.19)
Define
\[
G (b, b'; v) := \frac{1}{h^2} K'_{ij} \left( \frac{\xi (b) - v}{h} \right) \frac{(1 (b' \leq b) - G (b))}{g (b)}.
\]
By the definition of \( \hat{G} \) and (S.1.19),
\[
\Delta_2^1 (v) = \frac{1}{(N - 1)} \frac{1}{(N - L)^2} \sum_{(2)} G (B_{il}, B_{jk}; v) + \frac{1}{(N - 1)} \frac{1}{(N - L)^2} \sum_{i,l} G (B_{il}, B_{il}; v), \text{ for all } v \in I,
\]
(S.1.20)
when \( h \) is sufficiently small. The kernel \( G \) satisfies
\[
G_1 (b; v) := \int G (b, b'; v) dG (b') = 0 \text{ and } \mu_G (v) := \int \int G (b, b'; v) dG (b') dG (b) = 0.
\]
Also define
\[
G_2 (b; v) := \int G (b', b; v) dG (b').
\]
The Hoeffding decomposition yields
\[
\frac{1}{(N - L)^2} \sum_{(2)} G (B_{il}, B_{jk}; v) = \frac{1}{N \cdot L} \sum_{i,l} G_2 (B_{il}; v) + \frac{1}{(N - L)^2} \sum_{(2)} \left\{ G (B_{il}, B_{jk}; v) - G_2 (B_{il}; v) \right\}.
\]
(S.1.21)

By van der Vaart and Wellner (1996, Lemma 2.6.16), for any positive \( h \), the class \( \left\{ K'_{ij} \left( \frac{\xi (v)}{h} \right) : v \in I \right\} \) is the (pointwise) difference of two Vapnik-Chervonenkis (VC) subgraph classes of functions on \( [v, \xi] \): \( \left\{ D_s \left( \frac{\xi (v)}{h} \right) : v \in I \right\}, s \in \{1, 2\} \). Each of these classes has VC index less than or equal to two. Let
\[
\mathcal{D}_s := \left\{ D_s \left( \frac{\xi (v)}{h} \right) : v \in I \right\}, s \in \{1, 2\}.
\]
By Kosorok (2007, Lemma 9.9(ii)), each of \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) has VC index less than or equal to two. Let \( \tilde{g} (b, b') := h^{-2} g (b)^{-1} (1 (b' \leq b) - G (b)) \). It then follows from Kosorok (2007, Lemma 9.9 (vi)) that for any positive \( h \), each of the classes \( \mathcal{D}_s \cdot \tilde{g} \ (s \in \{1, 2\}) \) is VC-subgraph class with VC index less than or equal to three. They have constant envelopes \( h^{-2} \mathcal{D}_s \mathcal{C}_{-1} \) and \( h^{-2} \mathcal{D}_s \mathcal{C}_{-1} \) respectively. Giné and Nickl (2015, Theorem 3.6.9) (see also Kosorok, 2007, Theorem 9.3) yields the following non-asymptotic bound:
\[
N \left( \epsilon \left( h^{-2} \mathcal{D}_s \mathcal{C}_{-1} \right), \mathcal{D}_s \cdot \tilde{g}, \|Q, 2\| \right) \leq \left( \frac{A \epsilon}{\epsilon} \right)^V, \text{ for } \epsilon \in (0, 1],
\]
(S.1.22)
for any (not necessarily discrete) probability measure \( Q \), where \( A > 1 \) and \( V > 1 \) are universal constants that are independent of \( L \). Now
\( \mathcal{G} = \{ \mathcal{G}(\cdot; v) : v \in I \} \) is a subset of the pointwise difference of \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \). It follows from (S.1.22) and Nolan and Pollard (1987, Lemma 16) that \( \mathcal{G} \) is a (uniform) VC-type class with respect to the constant envelope

\[
\begin{equation}
       h^{-22} (\overline{\tau}_{\mathcal{D}_1} + \overline{\tau}_{\mathcal{D}_2}) \leq
\end{equation}
\]

For the higher-order term in the Hoeffding decomposition (S.1.21), the CK inequality suffices to yield

\[
\begin{equation}
E \left[ \sup_{v \in I} \left| \frac{1}{(N \cdot L)^2} \sum_{\{2\}} \mathcal{G}(B_{il}, B_{ih}; v) - \mathcal{G}_2(B_{il}; v) \right| \right] \leq (Lh^2)^{-1}.
\end{equation}
\]

To obtain a bound for the order of the supremum of the first term in the Hoeffding decomposition (S.1.21), a sharper maximal inequality for empirical processes is needed.

Compute

\[
\begin{equation}
E \left[ \mathcal{G}_2(B_{il}; v)^2 \right] = \int \left( \frac{h^2}{h} K_f' \left( \frac{\xi(b) - v}{h} \right) \left( \frac{b'}{b} - G(b) \right) db \right)^2 g(b') db'
\end{equation}
\]

where we applied the Fubini-Tonelli theorem to obtain the second equality. Applying the Fubini-Tonelli theorem again, we have

\[
\begin{equation}
\int \left( \frac{h^2}{h} K_f' \left( \frac{\xi(b) - v}{h} \right) \left( \frac{b'}{b} - G(b) \right) db \right)^2 g(b') db' = \int \left( \frac{h^2}{h} K_f' \left( \frac{\xi(b) - v}{h} \right) \right) K_f' \left( \frac{\xi(b) - v}{h} \right) \left( \text{min} \{b', b\} \right) db' db.
\end{equation}
\]

By change of variables \( u = (\ell(b') - v)/h \) and \( w = (\ell(b) - v)/h \), and since \( K_f' \) is supported on \([-1, 1] \), we have

\[
\begin{align*}
       & \int \left( \frac{h^2}{h} K_f' \left( \frac{\xi(b') - v}{h} \right) \right) K_f' \left( \frac{\xi(b) - v}{h} \right) \left( \text{min} \{b', b\} \right) db' db \\
     = & h^2 \int_0^w \int_{-w}^w K_f' (u) K_f' (w) G \left( \text{min} \{s (h u + v), s (h w + v)\} \right) s' (h u + v) s' (h w + v) du dw \\
     = & 2h^2 \int K_f' (w) s' (h w + v) \left( \int_{-w}^w K_f' (u) G (s (h u + v) s' (h u + v) du \right) dw,
\end{align*}
\]

when \( h \) is sufficiently small, where the last equality follows from symmetry. By a mean value expansion,

\[
\begin{align*}
       & \int_{-w}^w K_f' (u) G (s (h u + v)) s' (h u + v) du = \int_{-w}^w K_f' (u) \left\{ G (s (v)) s' (v) + \left( g (s (v)) s'' (v) + G (s (v)) s'' (v) \right) u \right\} du
\end{align*}
\]

for some mean value \( \hat{v} \) (depending on \( u \)) such that \( |v - \hat{v}| \leq h |u| \). Now it follows that

\[
\begin{align*}
       & \int \left( \frac{h^2}{h} K_f' \left( \frac{\xi(b) - v}{h} \right) \left( \frac{b'}{b} - G(b) \right) db \right)^2 g(b') db' \\
     = & 2G (s (v)) s' (v) h^2 \left( \int K_f' (w) s' (h w + v) \int_{-w}^w K_f' (u) du \right) \\
     & + 2h^3 \left( \int K_f' (w) s' (h w + v) \int_{-w}^w K_f' (u) \left( g (s (\hat{v})) s'' (\hat{v}) + G (s (\hat{v})) s'' (\hat{v}) \right) du \right).
\end{align*}
\]

Another mean value expansion with some mean value \( \bar{v} \) (depending on \( u \)) such that \( |\bar{v} - v| \leq h |u| \) yields

\[
\begin{align*}
       & \sup_{v \in I} \left| \int K_f' (w) s' (h w + v) \int_{-w}^w K_f' (u) du \right| = \sup_{v \in I} \left| \int K_f' (w) \left( s' (v) + s'' (\bar{v}) h w \right) \int_{-w}^w K_f' (u) du \right| \\
     \leq & h \left( \sup_{u \in \mathbb{R}} \left| s'' (u) \right| \right) \left( \int |K_f' (w) K_f (w) | dw \right).
\end{align*}
\]
where the inequality holds when \( h \) is sufficiently small and we used the fact
\[
\int K'_f (w) \int_{-\infty}^w K'_f (u) \, du \, dw = \int K'_f (w) \, K_f (w) \, dw = 0
\]
which holds under our assumption imposed on the kernel functions. We also have
\[
\sup_{v \in \mathcal{I}} \left| \int K'_f (w) s' (hw + v) \int_{-\infty}^w K'_f (u) \left( g (s (\hat{v})) s' (\hat{v})^2 + G (s (\hat{v})) s'' (\hat{v}) \right) \, du \, dv \right|
\leq \left( \int \left| K'_f (w) \right| \, dw \right) \left( \int K'_f (u) \, du \right) \sum_{v \in \mathcal{I}} \left| g (s (u)) s' (u)^2 + G (s (u)) s'' (u) \right|
\]
when \( h \) is sufficiently small, by the definition of \( \hat{v} \) and the fact that \( K'_f \) is supported on \([-1, 1]\). It follows from the above result, (S.1.27), (S.1.28), our assumptions imposed on the kernel functions, the continuity of \( s \), \( s' \) and \( s'' \) and the continuity of \( g \) and \( G \) that
\[
\sup_{v \in \mathcal{I}} \int \left( \int K'_f \left( \frac{\xi (b) - v}{h} \right) \right) \left( \frac{b'}{b} \right) \, db' \leq h^3,
\quad (S.1.29)
\]
when \( h \) is sufficiently small. The above result and (S.1.25) imply
\[
\sup_{v \in \mathcal{I}} E \left[ \mathcal{G}_2 (B_{11}; v)^2 \right] \leq h^{-1},
\quad (S.1.30)
\]
when \( h \) is sufficiently small.

It follows from the fact that \( \mathcal{G} \) is a VC-type class with respect to the constant envelope (S.1.23) and Chen and Kato (2017, Lemma 5.4) that \( \{ \mathcal{G} (\cdot; v) : v \in \mathcal{I} \} \) is also VC-type with respect to the constant envelope (S.1.23). Now an application of the CCK inequality with \( \sigma^2 \) being the left-hand side of (S.1.30) and \( F \) being (S.1.23) yields
\[
E \left[ \sup_{\mathcal{I}} \left| \frac{1}{N \cdot L} \sum_{i,l} \mathcal{G} (B_{il}; v) \right| \right] \leq C_1 \left( Lh \right)^{-1/2} \log \left( C_2 L \right)^{1/2} + \left( Lh^2 \right)^{-1} \log \left( C_2 L \right) = O \left( \frac{\log (L)}{Lh} \right)^{1/2},
\]
where the inequality is non-asymptotic and holds when \( h \) is sufficiently small. Now the above result, (S.1.21), (S.1.24) and Markov’s inequality yield
\[
\sup_{\mathcal{I}} \left| \frac{1}{(N \cdot L)^2} \sum_{i,l} \mathcal{G} (B_{il}; B_{jk}; v) \right| \leq O_p \left( \frac{\log (L)}{Lh^2} + \frac{\log (L)}{Lh} \right)^{1/2},
\]
Since \( \mathcal{G} \) is uniformly bounded by (S.1.23),
\[
\sup_{\mathcal{I}} \left| \frac{1}{(N \cdot L)^2} \sum_{i,l} \mathcal{G} (B_{il}; B_{11}; v) \right| \leq \left( Lh^2 \right)^{-1}.
\]
Now (S.1.20) and these results yield
\[
\sup_{\mathcal{I}} \left| \mathcal{G}_1 (v) \right| = O_p \left( \frac{\log (L)}{Lh^2} + \frac{\log (L)}{Lh} \right)^{1/2},
\quad (S.3.11)
\]

Since \( K'_f \) is compactly supported on \([-1, 1]\),
\[
\sup_{\mathcal{I}} \max_i \left| \frac{\mathcal{G}_1 (B_{il}) - 1}{K'_f \left( \frac{\xi (B_{il}) - v}{h} \right)} \right| G (B_{il}) \left( \hat{g} (B_{il}) - g (B_{il}) \right) = 0, \text{ when } h < \bar{r}.
\]
By the above result, Lemma S.1.1, (S.1.16), (S.1.17), (S.1.18) and (S.1.31), we have
\[
\hat{f} (v) = f (v) = \frac{1}{N - 1} \frac{1}{N \cdot L} \sum_{i,l} \frac{1}{h^2 K'_f \left( \frac{\xi (B_{il}) - v}{h} \right)} G (B_{il}) \left( \hat{g} (B_{il}) - g (B_{il}) \right) + O_p \left( \frac{\log (L)}{Lh} \right)^{1/2} + h^R + \frac{\log (L)}{Lh^2},
\]
S.6
where the remainder term is uniform in \( v \in I \). The conclusion now follows from the definition of \( \hat{g} \).

**Proof of Lemma A.2.** Again we observe that the maximal inequalities provided in Chernozhukov et al. (2014b), Chen and Kato (2017, Section 5) and van der Vaart and Wellner (1996) yield non-asymptotic bounds for the suprema of the absolute values of the terms in the Hoeffding decomposition

\[
\frac{1}{(N \cdot L)^2} \sum \sum_{j,k} \mathcal{M} (B_i; B_j; v) = \mu_\mathcal{M} (v) + \left\{ \frac{1}{N \cdot L} \sum_{i,j} \mathcal{M}_1 (B_i; v) - \mu_\mathcal{M} (v) \right\} + \left\{ \frac{1}{N \cdot L} \sum_{i,j} \mathcal{M}_2 (B_i; v) - \mu_\mathcal{M} (v) \right\}
\]

\[
+ \frac{1}{N \cdot L} \sum_{(i,j) \neq (j,k)} \{ \mathcal{M} (B_i; B_j; v) - \mathcal{M}_1 (B_i; v) - \mathcal{M}_2 (B_j; v) + \mu_\mathcal{M} (v) \}
\]

\[
+ \frac{1}{(N \cdot L)^2} \sum_{i,j} \mathcal{M} (B_i; B_j; v) - \frac{1}{(N \cdot L)^2} \sum_{(i,j) \neq (j,k)} \mathcal{M} (B_i; B_j; v).
\]

We use the same arguments as those showing that \( \mathcal{F} \) is (uniformly) VC-type to show that the class \( \mathcal{M} := \{ \mathcal{M} (\cdot, \cdot; v) : v \in I \} \) is (uniformly) VC-type with respect to the constant envelope

\[
h^{-3} (\mathcal{C}_{D_1} + \mathcal{C}_{D_2}) \mathcal{C}_\mathcal{M} + h^{-2} (\mathcal{C}_{D_1} + \mathcal{C}_{D_2}) \mathcal{C}_\mathcal{M}^{-1}.
\]  

(S.1.32)

Then the CK inequality yields

\[
E \left[ \sup_{v \in I} \left| \frac{1}{(N \cdot L)^2} \sum_{(2)} \mathcal{M} (B_i; B_j; v) - \mathcal{M}_1 (B_i; v) - \mathcal{M}_2 (B_j; v) + \mu_\mathcal{M} (v) \right| \right] \leq (Lh^3)^{-1}.
\]  

(S.1.33)

Since \( \mathcal{M} \) is uniformly bounded by (S.1.32), we have

\[
\sup_{v \in I} \left| \frac{1}{(N \cdot L)^2} \sum_{i,j} \mathcal{M} (B_i; B_j; v) \right| = O \left( (Lh^3)^{-1} \right)
\]  

(S.1.34)

and

\[
\sup_{v \in I} \left| \frac{1}{(N \cdot L)^2} \sum_{(2)} \mathcal{M} (B_i; B_j; v) \right| = O \left( (Lh^3)^{-1} \right).
\]  

(S.1.35)

By the definition, \( \mu_\mathcal{M} (v) \) is given by

\[
\mu_\mathcal{M} (v) := \int \int \mathcal{M} (b, b'; v) dG (b) dG (b') = \int \int \frac{1}{h^2} K'_g \left( \frac{\xi (b) - v}{h} \right) g (b) \beta (b) db'
\]

where

\[
\beta (b) := \int \left( \frac{1}{h} K_g \left( \frac{b' - b}{h} \right) - g (b) \right) g (b') db'
\]  

(S.1.36)

denotes the bias of the kernel density estimator for \( g (b) \). Since we assume that \( K_g \) is supported on \([-1, 1]\) and differentiable everywhere on \( \mathbb{R} \), it is straightforward to verify that \( \beta \) is differentiable on \( \left[ a, v_a - \delta \right] \cup \left[ v_a + \delta, b \right] \)

and

\[
\beta' (b) = \int \left( - \frac{1}{h^2} K'_g \left( \frac{b' - b}{h} \right) - g' (b) \right) g (b') db'
\]

which is the bias of the kernel estimator for the density derivative \( g' (b) \). By Proposition 1(iv) of GPV, \( g \) has \( 1 + R \) continuous derivatives instead of \( R \). By a standard argument for the bias of kernel estimators for the density (see, e.g., Newey, 1994), since \( K_g \) is supported on
\[-1,1], \text{ for each } b \in \left[ s \left( v_i - \delta \right), s \left( v_u + \delta \right) \right],
|\beta'(b)| \leq \frac{h^{1+R}}{(1+R) |\psi'|_{\psi \in [b-h, b+h]}} \sup_{b \in [b-h, b+h]} \left| \psi^{(1+R)} (b') \right| \int \left| u^{1+R} K_g (u) \right| du,
\tag{S.37.37}
\end{align}

when \( h \) is sufficiently small (so that \([b-h, b+h]\) is an inner closed subset of \([\delta, \delta]\)). By change of variable and Taylor expansion,
\begin{align}
\sup_{b \in [s(v_i - \delta), s(v_u + \delta)]} |\beta'(b)| &= \sup_{b \in [s(v_i - \delta), s(v_u + \delta)]} \left| \frac{\int 1 h^R K_g \left( \frac{b' - b}{h} \right) g'(b') \, db'}{\psi'} \right| \\
&\leq \sup_{b \in [s(v_i - \delta), s(v_u + \delta)]} \frac{h^R}{R^!} \left| K_g (u) u^R \left( g^{(1+R)} \left( \frac{b}{h} \right) - g^{(1+R)} (b) \right) \right| du,
\tag{S.37.38}
\end{align}

when \( h \) is sufficiently small, where \( \hat{b} \) is the mean value depending on \( u \) with \( |\hat{b} - b| \leq h |u| \). Since \( g^{(1+R)} \) is uniformly continuous on any inner closed subset of \([\delta, \delta]\), the assumption that \( K_g \) is supported on \([-1,1]\) and (S.37.38) imply that \( \beta'(b) = o(h^R) \) uniformly in \( b \in [s \left( v_i - \delta \right), s \left( v_u + \delta \right)] \).

By change of variable,
\begin{align}
\int_{\hat{b}}^b K_f \left( \frac{\xi (b) - v}{h} \right) G(b) \beta (b) \, db &= h \int_{s(v_i - \delta)}^{s(v_u + \delta)} K_f (u) \int_{\hat{b}}^h G(s(hu + v)) \beta (s(hu + v)) s'(hu + v) \, du,
\end{align}

Let \( \psi (z) := G(s(z)) s'(z)/g(s(z)) \). By Lemma A1 and Proposition 1 of GPV, both \( \psi \) and \( \psi' \) are uniformly continuous on \([\delta, \delta]\). By a mean value expansion,
\begin{align}
\int_{s(v_i - \delta)}^{s(v_u + \delta)} K_f (u) \psi (hu + v) \beta (s(hu + v)) \, du &= \int_{s(v_i - \delta)}^{s(v_u + \delta)} K_f (u) \psi (v) \beta (s(v)) \, du \\
&\quad + \int_{s(v_i - \delta)}^{s(v_u + \delta)} K_f (u) \left\{ \psi' (v) \beta (s(v)) + \psi (v) \beta' (s(v)) s'(v) \right\} hu \, du,
\tag{S.37.39}
\end{align}

where \( \hat{v} \) is the mean value depending on \( u \) with \( |\hat{v} - v| \leq h |u| \). Since \( \int K_f (u) \, du = 0 \) by symmetry of the kernel and \( K_f \) is compactly supported on \([-1,1]\), the first term of the right-hand side of (S.37.39) vanishes when \( h \) is sufficiently small and for the second term, we have
\begin{align}
\left| \int_{s(v_i - \delta)}^{s(v_u + \delta)} K_f (u) \left\{ \psi' (v) \beta (s(v)) + \psi (v) \beta' (s(v)) s'(v) \right\} hu \, du \right| \\
\leq \left( \int \left| K_f (u) u \right| \, du \right) \sup_{u \in [s(v_i - \delta), s(v_u + \delta)]} \left| \psi' (u) \beta (s(u)) + \psi (u) \beta' (s(u)) s'(u) \right| \, du,
\tag{S.37.40}
\end{align}

when \( h \) is sufficiently small.

Since \( \beta (b) \) and \( \beta' (b) \) are \( o(h^R) \) uniformly in \( b \in [s \left( v_i - \delta \right), s \left( v_u + \delta \right)] \). It follows from (S.37.39) and (S.37.40) that
\begin{align}
\sup_{v \in I} |\mu_{\mathcal{M}} (v)| &= \sup_{v \in I} \left| \int_{\hat{b}}^b K_f' \left( \frac{\xi (b) - v}{h} \right) \frac{G(b) \beta (b)}{g(b)} \, db \right| = o \left( h^R \right).
\tag{S.37.41}
\end{align}

It is clear from the definition of \( \mathcal{M} \) and the arguments used in the proof of Lemma A.1 that for any positive \( h \) the class \( \{ \mathcal{M}_1 (\cdot; v) : v \in I \} \) is a subset of the difference of two VC-subgraph classes, each of which has VC index less than or equal to three. Giné and Nickl (2015, Theorem 3.6.9) and Nolan and Pollard (1987, Lemma 16) imply that \( \{ \mathcal{M}_1 (\cdot; v) : v \in I \} \) is (uniformly) VC-type with respect to the constant envelope:
\begin{align}
\left\{ \sup_{b \in [s(v_i - \delta), s(v_u + \delta)]} \left| \beta (b) \right| \right\} = O \left( h^{R-1} \right),
\tag{S.37.42}
\end{align}

S.8
when $h$ is sufficiently small. The VW inequality suffices to yield
\[
E \left[ \sup_{v \in I} \frac{1}{N \cdot L} \sum_{i,t} M_1(B_{il}; v) - \mu_M(v) \right] \leq (Lh^2)^{-1/2} h^R,
\]
when $h$ is sufficiently small. The conclusion follows from Lemma A.1, (S.1.33), (S.1.34), (S.1.35), (S.1.41), the above inequality and Markov's inequality. 

The following lemma is essentially the same as Marmer and Shneyerov (2012, online supplement, Lemma S.2). It is clear from its proof that it suffices to bound the first (absolute) moment. The proof of the following lemma is almost the same as that of Marmer and Shneyerov (2012, Lemma S.2) and hence omitted.

**Lemma S.1.2.** Let $\hat{\sigma}^*$ be a statistic computed using the bootstrap sample satisfying $E^* \left[ |\hat{\sigma}^*| \right] = O_p(\epsilon_L)$ for some null sequence $\epsilon_L \downarrow 0$. Then $\hat{\sigma}^* = O_p(\epsilon_L)$.

**Proof of Lemma A.3.** Note that we have $\hat{f}(v) = E^* \left[ \hat{f}^*(v) \right]$ and also the function class $\{h^{-1}K_f(\frac{\cdot}{h}) : v \in I\}$ is (uniformly) VC-type with respect to the constant envelope $h^{-1}C_{K_f}$. Note that
\[
\tilde{\sigma}_V^2 := \sup_{v \in I} E^* \left[ \frac{h}{2} K_f \left( \frac{\hat{V}_v - v}{h} \right)^2 \right] = \sup_{v \in I} \frac{1}{N \cdot L} \sum_{i,t} \frac{1}{h^2} K_f \left( \frac{V_{il} - v}{h} \right)^2 \leq \sup_{v \in I} \frac{1}{N \cdot L} \sum_{i,t} h^{-2} T_{il}.
\]
Now it is clear that $\tilde{\sigma}_V = O_p(h^{-1/2})$ follows from (S.1.13).

Next we apply the CCK inequality with $\sigma = \tilde{\sigma}_V$ and $F$ being the constant envelope $h^{-1}C_{K_f}$. Observing the non-asymptotic nature of the CCK inequality and applying it, we have
\[
E^* \left[ \sup_{v \in I} |\tilde{f}^*(v) - \tilde{f}(v)| \right] \leq C_1 \left( L^{-1/2} \tilde{\sigma}_V \log(C_2L)^{1/2} + (Lh)^{-1} \log(L) \right) = O_p \left( \left( \frac{\log(L)}{Lh} \right)^{1/2} \right),
\]
where the inequality is non-asymptotic. The conclusion follows from the above result and Lemma S.1.2.

To prove Lemma A.4, we derive intermediate asymptotic expansions that are empirical bootstrap analogues of those provided in Lemmas S.1.1 and A.1 first. These expansions are given in the following two lemmas.

**Lemma S.1.3.** Suppose that Assumptions 1 - 3 hold. Let $\hat{G}_{il}^* := \mathbb{I} \left( |V_{il}^* - v| \leq \delta^{(S.1.44)} \right)$. Then
\[
\hat{f}_{G_{PV}}^*(v) - \tilde{f}^*(v) = \frac{1}{N \cdot L} \sum_{i,t} \hat{\sigma}_V^2 \frac{1}{h^2} K_f \left( \frac{V_{il} - v}{h} \right) \left( \hat{V}_{il}^* - V_{il}^* \right) + O_p \left( \left( \frac{\log(L)}{Lh^3} + h^R \right) \right),
\]
where the remainder term is uniform in $v \in I$.

**Proof of Lemma S.1.3.** By Marmer and Shneyerov (2012, Lemmas 1, S.1 and S.4), we have
\[
\sup_{b \in \mathbb{B}_L} |\hat{G}^*(b) - G(b)| = O_p \left( \left( \frac{\log(L)}{L} \right)^{1/2} \right) \quad \text{and} \quad \sup_{b \in \mathbb{B}_{h}, \|v\|_\mathbb{R}} \left| \hat{g}^*(b) - g(b) \right| = O_p \left( \left( \frac{\log(L)}{L} \right)^{1/2} + h^{1+R} \right). \tag{S.1.43}
\]
Note that it is straightforward to verify that (S.1.5) and (S.1.43) imply
\[
P^* \left[ \max_{i,l} T_{il}^* \left| \hat{g}^*(B_{il}^*)^{-1} \right| \leq \left( \frac{\epsilon_L}{2} \right)^{-1} \right] \rightarrow_p 1, \text{ as } L \uparrow \infty,
\]
which further implies $\max_{i,l} T_{il}^* \left| \hat{g}^*(B_{il}^*)^{-1} \right| = O_p(1)$. Now bootstrap analogues of (S.1.7) and (S.1.8) can be easily obtained by using (S.1.43) and this result. We have
\[
\max_{i,l} T_{il}^* \left| \hat{V}_{il}^* - V_{il}^* \right| = O_p \left( \left( \frac{\log(L)}{Lh} \right)^{1/2} + h^{1+R} \right) \tag{S.1.44}
\]
S.9
and

\[
\sup_{v \in I} \max_{i,l} \left| \hat{\nabla}_i^h V_{il} - V_{il}^h \right| = O_p \left( \left( \frac{\log (L)}{L} \right)^{5/2} \right) + h^{1+R} .
\]  

(S.1.45)

Write

\[
\hat{f}_{GPV} (v) = \frac{1}{N \cdot L} \sum_{i,l} \left\{ \frac{\hat{\tau}_i^h K_f \left( \hat{\nabla}_i^h v - h \right)}{h} + \tau_i^h \left( 1 - \frac{\hat{\tau}_i^h}{h} \right) \frac{1}{h} K_f \left( \hat{\nabla}_i^h v - h \right) + (\tau_i^h - 1) \frac{\tau_i^h}{h} K_f \left( \hat{\nabla}_i^h \frac{v}{h} \right) \right\} 
= \frac{1}{N \cdot L} \sum_{i,l} \frac{\tau_i^h}{h} K_f \left( \hat{\nabla}_i^h v - h \right) + \kappa_1^*(v) + \kappa_2^*(v).
\]

By the arguments used in the proof of Lemma S.1.1,

\[
P^* \left[ \sup_{v \in I} \left| \kappa_1^*(v) \right| > 0 \right] \leq P^* \left[ \max_{i,l} \tau_i^h \left| \hat{\nabla}_i^h V_{il} - V_{il}^h \right| \geq \frac{3}{2} \right] = o_p (1),
\]

where the inequality holds when \( h \) is sufficiently small.

Since for all \( v \in I, \)

\[
|\kappa_2^*(v)| \leq \sup_{v \in I} \frac{1}{N \cdot L} \sum_{i,l} h^{-1} (1 - \tau_i^h) \hat{\tau}_i^h \\
\leq h^{-1} \left( \left( \hat{\beta} - \beta \right) + \hat{\beta} + h > s (v - \beta) \right) + h^{-1} \left( \beta - \hat{\beta} - \beta - h < s (v + \beta) \right)
\]

by Markov’s inequality,

\[
P^* \left( \sup_{v \in I} |\kappa_2^*(v)| > 0 \right) \leq 1 \left( \left( \hat{\beta} - \beta \right) + \hat{\beta} + h > s (v - \beta) \right) + 1 \left( \beta - \hat{\beta} - \beta - h < s (v + \beta) \right) = o_p (1).
\]

Thus we have

\[
\hat{f}_{GPV} (v) = \frac{1}{N \cdot L} \sum_{i,l} \frac{\tau_i^h}{h} K_f \left( \hat{\nabla}_i^h \frac{v}{h} \right) + o_p (\epsilon_L),
\]

where the remainder term is uniform in \( v \in I, \) for any null sequence \( \epsilon_L \downarrow 0, \) and hence is negligible.

Then by a Taylor expansion and (S.1.9) with \( \hat{\nabla}_i (V_{il}^h) \) replaced by their bootstrap counterparts \( \hat{\nabla}_i^h (V_{il}^h), \)

\[
\hat{f}_{GPV} (v) - \hat{f}^* (v) = \frac{1}{N \cdot L} \sum_{i,l} \frac{\tau_i^h}{h^2} K_f^* \left( \hat{\nabla}_i^h \frac{v}{h} \right) \left( \hat{\nabla}_i^h V_{il} - V_{il}^h \right) + \frac{1}{2} \frac{1}{N \cdot L} \sum_{i,l} \frac{\tau_i^h}{h^2} K_f^* \left( \hat{\nabla}_i^h \frac{v}{h} \right) \left( \hat{\nabla}_i^h V_{il} - V_{il}^h \right)^2
\]

(S.1.46)

for some mean value \( \hat{V}_{il}^h \) that lies on the line joining \( \hat{V}_{il}^h \) and \( V_{il}^h, \) with some remainder error term that is \( o_p (\epsilon_L) \) for any null sequence \( \epsilon_L \downarrow 0. \)

Since \( K_f^* \) is supported on \([-1, 1], \) by the triangle inequality,

\[
\sup_{v \in I} \left| \frac{1}{N \cdot L} \sum_{i,l} \frac{\tau_i^h}{h^2} K_f^* \left( \hat{\nabla}_i^h \frac{v}{h} \right) \left( \hat{\nabla}_i^h V_{il} - V_{il}^h \right)^2 \right| \\
\leq \left\{ \sup_{v \in I} \frac{1}{N \cdot L} \sum_{i,l} h^{-3} \tau_i^h \right\} \left\{ \sup_{v \in I} \max_{i,l} \tau_i^h \left( \hat{\nabla}_i^h V_{il} - V_{il}^h \right)^2 \right\},
\]

(S.1.47)

where \( \tau_i^h := \mathbb{I} \left( |V_{il}^h - v| \leq 2h \right), \) and

\[
\kappa_3^*(v) := \frac{1}{N \cdot L} \sum_{i,l} \tau_i^h h^{-3} \mathbb{I} (|V_{il}^h - v| > 2h) \mathbb{I} \left( V_{il}^h - v \leq h + \max_{i,l} \tau_i^h \right| \hat{\nabla}_i^h V_{il} - V_{il}^h \right) .
\]

S.10
Then it is clear that
\[ P^* \left( \sup_{v \in I} |\sigma_3^* (v)| > 0 \right) \leq P^* \left[ \sup_{v \in I} \max_{i,l} \left| \tilde{T}_{i,l}^* - V_{i,l}^* \right| > h \right] = O_p (1), \]
where the equality follows from (S.1.45). Therefore, sup_{v \in I} |\sigma_3^* (v)| = O_p (s_L), for any null sequence s_L \downarrow 0. A application of the CCK inequality and Lemma S.1.2 yields
\[ \sup_{v \in I} \left| \frac{1}{N - L} \sum_{i,l} h^{-1} \tilde{T}_{i,l}^* - E^* [h^{-1} \tilde{T}_{i,l}^*] \right| = O_p \left( \frac{\log (L)}{Lh} \right)^{1/2}. \] (S.1.48)
It is argued in the proof of Lemma S.1.1 that sup_{v \in I} E^* [h^{-1} \tilde{T}_{i,l}^*] = O_p (1). It then follows from this result, (S.1.45), (S.1.47), (S.1.48) and Marmer and Shneyerov (2012, Lemma S.1) that
\[ \sup_{v \in I} \left| \frac{1}{N - L} \sum_{i,l} h^{-1} \tilde{T}_{i,l}^* \right| = O_p \left( \frac{\log (L)}{Lh^3} + h^2 R \right). \]
The conclusion follows from the above result and (S.1.46).

**Lemma S.1.4.** Suppose that Assumptions 1 - 3 hold. Then
\[ \tilde{G}_{PV} (v) - \hat{G}^* (v) = \frac{1}{(N - 1)} \frac{1}{(N - L)} \sum_{i,k} \sum_{i,j} \left( B_{i,l}^*, B_{j,k}^*; v \right) + O_p \left( \frac{\log (L)}{Lh^3} + h^2 R \right), \]
where the remainder term is uniform in v \in I.

**Proof of Lemma S.1.4.** By using (S.1.3) with all objects replaced by their bootstrap counterparts, we have
\[ \frac{1}{N - L} \sum_{i,l} \tilde{T}_{i,l}^* \frac{1}{h} K_f^* \left( \frac{V_{i,l}^* - v}{h} \right) \left( \tilde{V}_{i,l}^* - V_{i,l}^* \right) = - \frac{1}{N - 1} \frac{1}{N - L} \sum_{i,l} \tilde{T}_{i,l}^* \frac{1}{h^2} K_f^* \left( \frac{V_{i,l}^* - v}{h} \right) \frac{G (B_{i,l}^*)}{g (B_{i,l}^*)} \left( \tilde{G}^* (B_{i,l}^*) - g (B_{i,l}^*) \right) + \Delta_1 (v) + \Delta_2 (v) + \Delta_3 (v), \]
where
\[ \Delta_1 (v) := \frac{1}{(N - 1)} \frac{1}{N - L} \sum_{i,l} \tilde{T}_{i,l}^* \frac{1}{h^2} K_f^* \left( \frac{V_{i,l}^* - v}{h} \right) \frac{\tilde{G}^* (B_{i,l}^*) - G (B_{i,l}^*)}{g (B_{i,l}^*)}, \]
and the order bounds
\[ \sup_{v \in I} |\Delta_2 (v)| = O_p \left( \frac{\log (L)}{Lh^{3/2}} + h R \left( \frac{\log (L)}{L} \right)^{1/2} \right) \quad \text{and} \quad \sup_{v \in I} |\Delta_3 (v)| = O_p \left( \frac{\log (L)}{Lh^2} + h^{2 R + 1} \right) \]
can be easily obtained by using (S.1.43) and the fact
\[ \sup_{v \in I} \frac{1}{N - L} \sum_{i,l} h^2 K_f^* \left( \frac{V_{i,l}^* - v}{h} \right) \leq \sup_{v \in I} \frac{1}{N - L} \sum_{i,l} h^{-1} \tilde{T}_{i,l}^* = O_p (1). \]

Since K_f^* is supported on [-1, 1], by (S.1.19) with all objects replaced by their bootstrap counterparts, we have
\[ \Delta_1 (v) = \frac{1}{(N - 1)} \frac{1}{(N - L)^2} \sum_{i,l} \sum_{i,k} G \left( B_{i,l}^*, B_{j,k}^*; v \right), \] for all v \in I,
when h is sufficiently small. Note that the conditional distribution of each of \{ B_{i,l}^* : i = 1, \ldots, N, l = 1, \ldots, L \} is \tilde{G}. Let
\[ \tilde{\mu} : (b, b') \rightarrow \int \int G (b, b'; v) d\tilde{G} (b') d\tilde{G} (b') = \Delta_1 (v) = O_p \left( \frac{\log (L)}{Lh} + \frac{\log (L)}{Lh} \right)^{1/2} \], (S.1.50)
where the last equality is uniform in \( v \in I \) and shown in the proof of Lemma A.1. Let

\[ \tilde{G}_1 (b; v) := \int \mathcal{G} (b, b'; v) \, d\tilde{G} (b') = \frac{1}{h^2} K_f \left( \frac{\xi (b) - v}{h} \right) \tilde{G} (b) - G (b) \]

and

\[ \tilde{G}_2 (b; v) = \frac{1}{N \cdot L} \sum_{j,k} \frac{1}{K_f} \left( \frac{\xi (B_{jk}) - v}{h} \right) \mathbb{I} \left( b \leq B_{jk} \right) - G (B_{jk}) \cdot \]

The Hoeffding decomposition gives

\[
\frac{1}{(N \cdot L)^2} \sum_{i,l} \sum_{j,k} \mathcal{G} \left( B_{i1}^*, B_{jk}^*; v \right) = \tilde{\mu} (v) + \left\{ \frac{1}{N \cdot L} \sum_{i,l} \tilde{G}_1 (B_{i1}^*; v) - \tilde{\mu} (v) \right\} + \left\{ \frac{1}{N \cdot L} \sum_{i,l} \tilde{G}_2 (B_{i1}^*; v) - \tilde{\mu} (v) \right\} + \frac{1}{(N \cdot L)^2} \sum_{(2)} \mathcal{G} \left( B_{i1}^*, B_{jk}^*; v \right) - \tilde{G}_1 (B_{i1}^*; v) - \tilde{G}_2 (B_{jk}^*; v) + \tilde{\mu} (v) \right\} + \frac{1}{(N \cdot L)^2} \sum_{(2)} \mathcal{G} \left( B_{i1}^*, B_{jk}^*; v \right).
\] (S.1.51)

We argued that \( \mathcal{G} \) is a (uniform) VC-type class with respect to the constant envelope (S.1.23). Due to the non-asymptotic nature of the maximal inequalities we invoked in the proofs, we can apply the same inequalities in the “bootstrap world” combined with Lemma S.1.2 to obtain the desired result. The CK inequality yields the following non-asymptotic bound:

\[
E^* \left[ \sup_{v \in I} \left\{ \frac{1}{(N \cdot L)^2} \sum_{(2)} \mathcal{G} \left( B_{i1}^*, B_{jk}^*; v \right) - \tilde{G}_1 (B_{i1}^*; v) - \tilde{G}_2 (B_{jk}^*; v) + \tilde{\mu} (v) \right\} \right] \leq (Lh^2)^{-1}.
\] (S.1.52)

It is clear from the definition of \( \tilde{G}_1 \) and the standard arguments that (conditionally on the original sample, in the bootstrap world) for any positive \( h \) the (non-random) class \( \left\{ \tilde{G}_1 (\cdot; v) : v \in I \right\} \) is a subset of the difference of two VC-subgraph classes, each of which has VC index less than or equal to three. Giné and Nickl (2015, Theorem 3.6.9) and Nolan and Pollard (1987, Lemma 16) imply that \( \left\{ \tilde{G}_1 (\cdot; v) : v \in I \right\} \) is (uniformly) VC-type with respect to the (conditionally) constant envelope:

\[
h^{-2} \left( C_{D_1} + C_{D_2} \right) \leq \left\{ \sup_{b \in \mathbb{R}} \left| \tilde{G} (b) - G (b) \right| \right\}.
\]

Now the VW inequality yields:

\[
E^* \left[ \sup_{v \in I} \left\{ \frac{1}{N \cdot L} \sum_{i,l} \tilde{G}_1 (B_{i1}^*; v) - \tilde{\mu} (v) \right\} \right] \leq \frac{1}{L^{1/2} h^2} \left\{ \sup_{b \in \mathbb{R}} \left| \tilde{G} (b) - G (b) \right| \right\} = O_p \left( \log \left( \frac{L}{h^2} \right) \right),
\] (S.1.53)

where the inequality is non-asymptotic.

Note that

\[
\frac{1}{N \cdot L} \sum_{i,l} \tilde{G}_2 (B_{i1}^*; v) - \tilde{\mu} (v) = \frac{1}{N \cdot L} \sum_{i,l} \tilde{G}_2^1 (B_{i1}^*; v) - E^* \left[ \tilde{G}_2^1 (B_{i1}^*; v) \right],
\]

where

\[
\tilde{G}_2^1 (b; v) := \frac{1}{N \cdot L} \sum_{j,k} \frac{1}{h^2} K_f \left( \frac{\xi (B_{jk}) - v}{h} \right) \frac{1}{g (B_{jk})} (b \leq B_{jk}),
\]

since

\[
\tilde{G}_2^1 (B_{i1}^*; v) - \tilde{\mu} (v) = \tilde{G}_2^1 (B_{i1}^*; v) - E^* \left[ \tilde{G}_2^1 (B_{i1}^*; v) \right], \quad \text{for all } i = 1, \ldots, N \text{ and } l = 1, \ldots, L.
\]

It is easy to check that the class \( \left\{ \tilde{G}_2^1 (\cdot; v) : v \in I \right\} \) is non-random conditionally on the original sample and VC-type with respect to the constant envelope (S.1.23) by Chen and Kato (2017, Lemma 5.4). Compute

\[
E^* \left[ \tilde{G}_2^1 (B_{i1}^*; v) \right]^2 = \frac{1}{(N \cdot L)^2} \sum_{i,l} \sum_{j' \neq j} \frac{1}{h^2} K_f \left( \frac{\xi (B_{jk}) - v}{h} \right) \frac{1}{g (B_{jk})} (B_{i1} \leq B_{jk}) K_f \left( \frac{\xi (B_{j'k'}) - v}{h} \right) \frac{1}{g (B_{j'k'})} (B_{i1} \leq B_{j'k'})
\]
Now by observing that $\mathcal{J}$ is symmetric with respect to the second the third arguments and the V-statistic decomposition argument of Serfling (2009, 5.7.3),

$$
\frac{1}{(N \cdot L)^3} \sum_{i,l} \sum_{j,k,j',k'} \mathcal{J} (B_{il}, B_{jk}, B_{j'k'}; v)
$$

$$
= \frac{1}{(N \cdot L)^3} \sum_{i,l} \mathcal{J} (B_{il}, B_{jk}, B_{j'k'}; v)
$$

$$
+ \frac{O (L^{-1})}{3(N \cdot L)^2 - 2(N \cdot L)} \left\{ \sum_{i,l} (2 \mathcal{J} (B_{il}, B_{il}, B_{j'k}; v) + \mathcal{J} (B_{jk}, B_{il}, B_{il}; v)) + \sum_{i,l} \mathcal{J} (B_{il}, B_{il}, B_{il}; v) \right\}. \quad (S.1.54)
$$

Define

$$
\mathcal{J}_1^{(1)} (b; v) := \int \int \mathcal{J} (b, b', b''; v) \, dG (b') \, dG (b''),
$$

$$
\mathcal{J}_2^{(1)} (b; v) := \int \int \mathcal{J} (b', b, b''; v) \, dG (b') \, dG (b''),
$$

$$
\mathcal{J}_3^{(1)} (b; v) := \int \int \mathcal{J} (b', b', b; v) \, dG (b') \, dG (b'),
$$

(S.1.55)

$$
\mathcal{J}_1^{(2)} (b, b'; v) := \int \mathcal{J} (b, b', b''; v) \, dG (b''),
$$

$$
\mathcal{J}_2^{(2)} (b, b'; v) := \int \mathcal{J} (b', b, b'; v) \, dG (b'),
$$

$$
\mathcal{J}_3^{(2)} (b, b'; v) := \int \mathcal{J} (b'', b, b'; v) \, dG (b'')
$$

(S.1.56)

and

$$
\mu_{\mathcal{J}} (v) := \int \int \mathcal{J} (b, b', b''; v) \, dG (b) \, dG (b') \, dG (b'').
$$

(S.1.57)

The Hoeffding decomposition yields

$$
\frac{1}{(N \cdot L)^3} \sum_{i,l} \mathcal{J} (B_{il}, B_{jk}, B_{j'k'}; v) = \mu_{\mathcal{J}} (v) + \frac{1}{N \cdot L} \sum_{i,l} \left( \mathcal{J}_1^{(1)} (B_{il}; v) - \mu_{\mathcal{J}} (v) \right) + \frac{1}{N \cdot L} \sum_{i,l} \left( \mathcal{J}_2^{(1)} (B_{il}; v) - \mu_{\mathcal{J}} (v) \right)
$$

$$
+ \frac{1}{N \cdot L} \sum_{i,l} \left( \mathcal{J}_3^{(1)} (B_{il}; v) - \mu_{\mathcal{J}} (v) \right) + \Upsilon_{\mathcal{J}}^{(1)} (v) + \Upsilon_{\mathcal{J}}^{(2)} (v) + \Upsilon_{\mathcal{J}}^{(3)} (v) + \Psi_{\mathcal{J}} (v), \quad (S.1.58)
$$

where $\Upsilon_{\mathcal{J}}^{(1)} (v)$, $\Upsilon_{\mathcal{J}}^{(2)} (v)$ and $\Upsilon_{\mathcal{J}}^{(3)} (v)$ are degenerate U-statistics of order two and $\Psi_{\mathcal{J}} (v)$ is a degenerate U-statistic of order three:

$$
\Upsilon_{\mathcal{J}}^{(1)} (v) := \frac{1}{(N \cdot L)^2} \sum_{(2)} \left\{ \mathcal{J}_1^{(2)} (B_{il}, B_{jk}; v) - \mathcal{J}_1^{(1)} (B_{il}; v) - \mathcal{J}_2^{(1)} (B_{jk}; v) + \mu_{\mathcal{J}} (v) \right\},
$$

$$
\Upsilon_{\mathcal{J}}^{(2)} (v) := \frac{1}{(N \cdot L)^2} \sum_{(2)} \left\{ \mathcal{J}_2^{(2)} (B_{il}, B_{jk}; v) - \mathcal{J}_1^{(1)} (B_{il}; v) - \mathcal{J}_3^{(1)} (B_{jk}; v) + \mu_{\mathcal{J}} (v) \right\},
$$

$$
\Upsilon_{\mathcal{J}}^{(3)} (v) := \frac{1}{(N \cdot L)^2} \sum_{(2)} \left\{ \mathcal{J}_3^{(2)} (B_{il}, B_{jk}; v) - \mathcal{J}_1^{(1)} (B_{il}; v) - \mathcal{J}_3^{(1)} (B_{jk}; v) + \mu_{\mathcal{J}} (v) \right\}
$$

(S.1.59)

and

$$
\Psi_{\mathcal{J}} (v) := \frac{1}{(N \cdot L)^3} \sum_{(3)} \{ \mathcal{J} (B_{il}, B_{jk}, B_{j'k'}; v) - \mathcal{J}_1^{(2)} (B_{il}, B_{jk}; v) - \mathcal{J}_2^{(2)} (B_{il}, B_{j'k'}; v) - \mathcal{J}_3^{(2)} (B_{jk}, B_{j'k'}; v)
$$

$$
+ \mathcal{J}_1^{(1)} (B_{il}; v) + \mathcal{J}_2^{(1)} (B_{jk}; v) + \mathcal{J}_3^{(1)} (B_{j'k'}; v) - \mu_{\mathcal{J}} (v) \}.
$$

(S.1.60)
It is easy to check that
\[ \mu_J(v) = \int_{b}^{\infty} \int_{b'}^{\infty} \frac{1}{h^2} K'_f \left( \frac{\xi(b') - v}{h} \right) K'_f \left( \frac{\xi(b) - v}{h} \right) G \left( \min \{b, b'\} \right) \, db \, db'. \]

It is shown in the proof of Lemma A.1 that
\[ \sup_{v \in I} |\mu_J(v)| = O \left( h^{-1} \right), \tag{S.1.61} \]
see (S.1.26) and (S.1.29).

Next, we obtain the order bounds for the suprema of the absolute values of the remainder terms in the Hoeffding decomposition (S.1.58). Firstly, we observe that \( J : \{J(v), v \in I\} \) is (uniformly) VC-type with respect to the constant envelope \( h^{-4}C_\sigma^{-2} (C_{D_1} + C_{D_2})^2 \) by using the same arguments applied to show that \( \mathcal{X} \) is VC-type in the proof of Theorem 4.1. The CK inequality yields
\[ \mathbf{E} \left[ \sup_{v \in I} |\Psi(v)| \right] \leq (Lh^4)^{-1}, \quad \text{for } k = 1, 2, 3 \text{ and } \mathbf{E} \left[ \sup_{v \in I} |\Psi(v)| \right] \leq L^{-3/2}h^{-4}. \tag{S.1.62} \]

Since \( J \) is VC-type with respect to the constant envelope \( h^{-4}C_\sigma^{-2} (C_{D_1} + C_{D_2})^2 \), the fact that \( \left\{ J_k^{(1)} (v) : v \in I \right\}, k = 1, 2, 3 \) are all VC-type with respect to the same constant envelope follows from \( \text{Chen and Kato} (2017, \text{Lemma} 5.4) \). Now by change of variable and using the fact that \( K'_f \) is compactly supported on \([-1, 1], \]
\[ \sup_{v \in I} E \left[ J_1^{(1)} (B_{11}; v)^2 \right] = \sup_{v \in I} \int \left\{ \int_{b}^{\infty} \frac{1}{h^2} K'_f \left( \frac{\xi(b') - v}{h} \right) \, db' \right\}^4 \, db \]
\[ \leq \sup_{v \in I} \left\{ \int_{b}^{\infty} \frac{1}{h^2} K'_f \left( \frac{\xi(b') - v}{h} \right) \, db' \right\}^4 \]
\[ \leq h^{-4}, \]

where the last inequality holds when \( h \) is sufficiently small. Now the CCK inequality with \( \sigma^2 \) being
\[ \sup_{v \in I} E \left[ J_1^{(1)} (B_{11}; v)^2 \right] \text{ or } \sup_{v \in I} E \left[ J_2^{(1)} (B_{11}; v)^2 \right] \]
and \( F \) being \( h^{-4}C_\sigma^{-2} (C_{D_1} + C_{D_2})^2 \) yield
\[ E \left[ \sup_{v \in I} \left\{ \frac{1}{N \cdot L} \sum_{i,j} \left( J_1^{(1)} (B_{i1}; v) - \mu_J(v) \right) \right\} \right] \leq C_1 \left\{ (Lh^4)^{-1/2} \log (C_2 L)^{1/2} + (Lh^4)^{-1} \log (C_2 L) \right\} \]
\[ \text{and} \]
\[ E \left[ \sup_{v \in I} \left\{ \frac{1}{N \cdot L} \sum_{i,j} \left( J_2^{(1)} (B_{i1}; v) - \mu_J(v) \right) \right\} \right] \leq C_1 \left\{ (Lh^5)^{-1/2} \log (C_2 L)^{1/2} + (Lh^4)^{-1} \log (C_2 L) \right\}, \]
when \( h \) is sufficiently small. Note that the second inequality holds if \( J_2^{(1)} \) is replaced by \( J_3^{(1)} \), since \( J(v) \) is symmetric with respect to the second and the third arguments. Now
\[ \sup_{v \in I} \left| \frac{1}{(N \cdot L)^3} \sum_{i,j,k,k'} J (B_{i1}, B_{j1}, B_{j,k,k'}; v) \right| = O_p \left( h^{-1 + \left( \frac{\log (L)}{Lh^3} \right)^{1/2} + \frac{\log (L)}{Lh^4}} \right) \tag{S.1.63} \]
follows from these inequalities, Markov’s inequality, (S.1.58), (S.1.61) and (S.1.62).
and

\[ \sup_{\nu \in I} \left| \frac{1}{N \cdot L} \sum_{i,l} J(B_{il}, B_{il}; \nu) \right| \leq h^{-4} \]

follow from the fact that \( J \) is uniformly bounded by \( h^{-4} \left( \overline{C}_{D_1} + \overline{C}_{D_2} \right)^2 \). Now it is clear from these inequalities, (S.1.54) and (S.1.63) that

\[ \hat{\sigma}_{B_2}^2 := \sup_{\nu \in I} E^* \left[ \hat{G}_{B_2}^2 (B_{11}; \nu)^2 \right] = O_p \left( h^{-1} + \left( \frac{\log (L)}{L h^2} \right)^{1/2} + \frac{\log (L)}{L h^2} \right). \]

The CCK inequality with \( \sigma = \hat{\sigma}_{B_2} \) and \( F \) being the constant envelope (S.1.23) yields

\[
E^* \left[ \sup_{\nu \in I} \left| \frac{1}{N \cdot L} \sum_{i,l} \hat{G}_{B_2}^2 (B_{il}; \nu) - E^* \left[ \hat{G}_{B_2}^2 (B_{11}; \nu) \right] \right| \right] \leq C_1 \left\{ L^{-1/2} \hat{\sigma}_{B_2} \log (C_2 L)^{1/2} + (Lh^2)^{-1} \log (L) \right\} = O_p \left( \left( \frac{\log (L)}{L h} \right)^{1/2} + \frac{\log (L)}{L h^2} \right),
\]

where the inequality is non-asymptotic. Now

\[
\sup_{\nu \in I} | \Delta_1^*(\nu) | = O_p \left( \frac{\log (L)}{L h} \right)^{1/2} \frac{\log (L)}{L h^2} \]

follows from the above result, Lemma S.1.2, (S.1.50), (S.1.51), (S.1.52), (S.1.53), Marmer and Shneyerov (2012, Lemma S.1) and uniform boundedness of \( \mathcal{G} \) which implies that the last two terms in the decomposition (S.1.51) are both \( O \left( \left( L h^2 \right)^{-1} \right) \) uniformly in \( \nu \in I \). The conclusion follows from Lemma S.1.3, the order bounds for the suprema of \( \Delta_1^*, k = 1, 2, 3 \) and also the definition of \( \hat{\sigma}_*^2 \).

**Proof of Lemma A.4.** Define

\[
\hat{M}_1 (b; \nu) := \int \mathcal{M} (b, b'; \nu) \, d\hat{G} (b'), \quad \hat{M}_2 (b; \nu) := \int \mathcal{M} (b', b; \nu) \, d\hat{G} (b') \quad \text{and} \quad \tilde{\mu}_\mathcal{M} (\nu) := \int \int \mathcal{M} (b, b'; v) \, d\hat{G} (b) \, d\hat{G} (b').
\]

Note that we have

\[
\tilde{\mu}_\mathcal{M} (\nu) = \frac{1}{(N \cdot L)^2} \sum_{i,l} \sum_{j,k} \mathcal{M} (B_{il}, B_{jk}; \nu)
\]

by definition. The Hoeffding decomposition yields

\[
\frac{1}{(N \cdot L)^2} \sum_{i,l} \sum_{j,k} \mathcal{M} (B_{il}^*, B_{jk}^*; \nu) = \tilde{\mu}_\mathcal{M} (\nu) + \left\{ \frac{1}{(N \cdot L)^2} \sum_{i,l} \hat{M}_1 (B_{il}^*; \nu) - \tilde{\mu}_\mathcal{M} (\nu) \right\} + \left\{ \frac{1}{(N \cdot L)^2} \sum_{i,l} \hat{M}_2 (B_{il}^*; \nu) - \tilde{\mu}_\mathcal{M} (\nu) \right\}
\]

\[
+ \frac{1}{(N \cdot L)^2} \sum_{i,l} \mathcal{M} (B_{il}^*, B_{jk}^*; \nu) - \hat{M}_1 (B_{il}^*; \nu) - \hat{M}_2 (B_{jk}^*; \nu) + \tilde{\mu}_\mathcal{M} (\nu)
\]

\[
+ \frac{1}{(N \cdot L)^2} \sum_{i,l} \mathcal{M} (B_{il}^*, B_{jk}^*; \nu) - \frac{1}{(N \cdot L)^2} \left\{ \sum_{i,l} \mathcal{M} (B_{il}^*, B_{jk}^*; \nu) \right\}.
\]

(S.1.64)

It is argued in the proof of Lemma A.2 that \( \mathcal{M} \) is VC-type with respect to the constant envelope (S.1.32). Now the CK inequality yields

\[
E^* \left[ \sup_{\nu \in I} \left| \frac{1}{(N \cdot L)^2} \sum_{i,l} \mathcal{M} (B_{il}^*, B_{jk}^*; \nu) - \hat{M}_1 (B_{il}^*; \nu) - \hat{M}_2 (B_{jk}^*; \nu) + \tilde{\mu}_\mathcal{M} (\nu) \right| \right] \leq (Lh^3)^{-1} + (Lh^2)^{-1}.
\]

(S.1.65)

Note that by definition,

\[
\hat{M}_1 (b; \nu) = - \frac{1}{h^2} K_{\nu} \left( \frac{\beta (b) - \nu}{h} \right) \frac{G (b)}{g (b)} \{ \hat{g} (b) - g (b) \}.
\]

By the arguments used to show that \( \{ \hat{G}_1 (; \nu) : \nu \in I \} \) is VC-type, we can show that \( \{ \hat{M}_1 (; \nu) : \nu \in I \} \) is (non-random) VC-type (condi-
tionally on the original sample) with respect to the (conditionally) constant envelope:

\[ h^{-2} (\overline{C}_{D_1} + \overline{C}_{D_2}) \sum_{i,l} \sup_{b \in [(v_i-h), (v_i+h)]} |\bar{g}(b) - g(b)|, \]

where \( h \) is sufficiently small. The VW inequality yields:

\[ E^* \left[ \sup_{v \in I} \left| \frac{1}{N \cdot L} \sum_{i,l} \hat{M}_1 (B_{il}'; v) - \hat{\mu}_M (v) \right| \right] \leq L^{-1/2} h^{-2} \sup_{b \in [(v_i-h), (v_i+h)]} |\bar{g}(b) - g(b)| = O_p \left( \frac{\log(L)^{1/2}}{Lh^{1/2}} + \frac{hR^{-1}}{L^{1/2}} \right), \]  

(S.1.66)

where the inequality is non-asymptotic.

Consider the following process:

\[ \Delta^* (v) := \frac{1}{N \cdot L} \sum_{i,l} \left\{ \left( \hat{M}_2 (B_{il}'; v) - \hat{\mu}_M (v) \right) - \left( M_2 (B_{il}'; v) - \frac{1}{N \cdot L} \sum_{j,k} M_2 (B_{jk}; v) \right) \right\}, \quad v \in I. \]

Let

\[ \hat{M}_2 (b; v) := \int \mathcal{M}^i (b'; b, v) \, d\hat{G} (b') \quad \text{and} \quad \hat{\mu}_M (v) := \int \int \mathcal{M}^i (b', b; v) \, d\hat{G} (b') \, d\hat{G} (b). \]

It follows from

\[ M_2 (B_{il}'; v) - \frac{1}{N \cdot L} \sum_{j,k} M_2 (B_{jk}; v) = M_2^i (B_{il}'; v) - \frac{1}{N \cdot L} \sum_{j,k} M_2^i (B_{jk}; v), \quad \text{for all } i = 1, ..., N \text{ and } l = 1, ..., L \]

and

\[ \hat{M}_2 (B_{il}'; v) - \hat{\mu}_M (v) = \hat{M}_2^i (B_{il}'; v) - \hat{\mu}_M^i (v), \quad \text{for all } i = 1, ..., N \text{ and } l = 1, ..., L \]

that

\[ \Delta^* (v) = \frac{1}{N \cdot L} \sum_{i,l} \left\{ \left( \hat{M}_2^i (B_{il}'; v) - \hat{\mu}_M^i (v) \right) - \left( M_2^i (B_{il}'; v) - \frac{1}{N \cdot L} \sum_{j,k} M_2^i (B_{jk}; v) \right) \right\}, \quad \text{for all } v \in I. \]

Simple algebra yields

\[ E^* \left[ \left( \hat{M}_2^i (B_{1l}'; v) - M_2^i (B_{1l}'; v) \right)^2 \right] = \frac{1}{(N \cdot L)^3} \sum_{i,l} \sum_{j,k} \sum_{j',k'} \mathcal{L} (B_{il}, B_{jk}, B_{j'k'}; v), \]

(S.1.67)

where

\[ \mathcal{L} (b, b', b''; v) := \left( \mathcal{M}^i (b', b; v) - \mathcal{M}^i_2 (b; v) \right) \left( \mathcal{M}^i (b', b; v) - \mathcal{M}^i_2 (b; v) \right). \]

By the V-statistic decomposition argument of Serfling (2009, 5.7.3), since the kernel \( \mathcal{L} \) is symmetric with respect to the second and the third arguments, we have

\[ \frac{1}{(N \cdot L)^3} \sum_{i,l} \sum_{j,k} \sum_{j',k'} \mathcal{L} (B_{il}, B_{jk}, B_{j'k'}; v) \]

\[ = \frac{1}{(N \cdot L)^3} \sum_{(3)} \mathcal{L} (B_{il}, B_{jk}, B_{j'k'}; v) \]

\[ + \frac{O(L^{-1})}{3(N \cdot L)^2 - 2(N \cdot L)} \sum_{(2)} \left( 2 \mathcal{L} (B_{il}, B_{jk}; v) + \mathcal{L} (B_{jk}, B_{il}; v) \right) + \sum_{i,l} \mathcal{L} (B_{il}, B_{il}, B_{il}; v) \]  

(S.1.68)

Since we argued that \( \mathcal{M}^i \) and \( \left\{ \mathcal{M}^i_2 (\cdot; v) : v \in I \right\} \) are both VC-type with respect to the constant envelope

\[ h^{-3} (\overline{C}_{D_1} + \overline{C}_{D_2}) \sum_{i=1}^{n} \mathcal{K}_{\mathcal{L}} \]

(S.1.69)

it follows from Nolan and Pollard (1987, Lemma 16) and Chernozhukov et al. (2014b, Lemma A.1) that \( \mathcal{L} := \{ \mathcal{L} (\cdot, \cdot, \cdot; v) : v \in I \} \) is VC-type with respect to the constant envelope

\[ 4 \left\{ h^{-3} (\overline{C}_{D_1} + \overline{C}_{D_2}) \sum_{i=1}^{n} \mathcal{K}_{\mathcal{L}} \right\}^2. \]

(S.1.70)
Note that the leading U-process is first-order degenerate:

\[ \mu_L(v) = L_1^{(1)}(b; v) = L_2^{(1)}(b; v) = L_3^{(1)}(b; v) = 0, \text{ for all } (b, v) \in \left[ \frac{1}{b}, b \right] \times I \]

and it is also clear that

\[ L_1^{(2)}(b, b'; v) = L_2^{(2)}(b, b'; v) = 0, \text{ for all } (b, b', v) \in \left[ \frac{1}{b}, b \right]^2 \times I, \]

where these terms in the Hoeffding decomposition are defined by (S.1.55) to (S.1.57) with \( K \) replaced by \( L \). Now the Hoeffding decomposition is simply

\[
\frac{1}{(N - L)^3} \sum_{(3)} L(B_{it}, B_{jk}, B_{j'k'}; v) = \frac{1}{(N - L)^2} \sum_{(2)} L_3^{(2)}(B_{it}, B_{jk}; v) + \frac{1}{(N - L)^3} \sum_{(3)} \left\{ L(B_{it}, B_{jk}, B_{j'k'}; v) - L_3^{(2)}(B_{jk}, B_{j'k'}; v) \right\}.
\]

The CCK inequality yields

\[
E \left[ \sup_{v \in I} \left| \frac{1}{(N - L)^2} \sum_{(2)} L_3^{(2)}(B_{it}, B_{jk}; v) \right| \right] = O \left( (Lh^6)^{-1} \right)
\]

and

\[
E \left[ \sup_{v \in I} \left| \frac{1}{(N - L)^3} \sum_{(3)} \left\{ L(B_{it}, B_{jk}, B_{j'k'}; v) - L_3^{(2)}(B_{jk}, B_{j'k'}; v) \right\} \right| \right] = O \left( L^{-3/2}h^{-6} \right).
\]

It follows from these inequalities, (S.1.68), the fact that \( \mathcal{L} \) is uniformly bounded by (S.1.70) and Markov’s inequality that

\[
\sup_{v \in I} \left| \frac{1}{(N - L)^3} \sum_{i, j, j', k, k'} L(B_{it}, B_{jk}, B_{j'k'}; v) \right| = O_p \left( (Lh^6)^{-1} \right).
\]

It then follows from the above result and (S.1.67) that

\[
\tilde{\sigma}_D^2 := \sup_{v \in I} E^* \left[ \left( \delta_2 B_{i1}^{*2}(v) - \delta_2 B_{i1}^{*2}(v) \right)^2 \right] = O_p \left( (Lh^6)^{-1} \right).
\]

Since \( \mathcal{M}^{(i)} \) is VC-type with respect to the constant envelope (S.1.69), the fact that both \( \left\{ \mathcal{M}^{(i)}(\cdot; v) : v \in I \right\} \) and \( \left\{ \tilde{\mathcal{M}}^{(i)}(\cdot; v) : v \in I \right\} \) are VC-type classes (conditionally on the original sample) with respect to the same constant envelope follows from Chen and Kato (2017, Lemma 5.4). Now Nolan and Pollard (1987, Lemma 16) implies that the class \( \left\{ \tilde{\mathcal{M}}^{(i)}(\cdot; v) - \mathcal{M}^{(i)}(\cdot; v) : v \in I \right\} \) is also VC-type (conditionally on the original sample) with respect to a constant envelope that is twice of (S.1.69). The CCK inequality with \( \sigma = \tilde{\sigma}_D \) and \( F \) being this constant envelope yields:

\[
E^* \left[ \sup_{v \in I} |\Delta^*(v)| \right] \leq C_1 \left\{ L^{-1/2} \tilde{\sigma}_D \log (C_2 L)^{1/2} + L^{-1} \log^3 (C_2 L) \right\} = O_p \left( \frac{\log(L)}{Lh^3} \right),
\]

where the inequality is non-asymptotic. The conclusion follows from the above result, Lemma S.1.4, (S.1.64), (S.1.65), (S.1.66), Lemma S.1.2 and the fact that \( \mathcal{M} \) is uniformly bounded by (S.1.32).

**Proof of Lemma A.5.** We first show

\[
\sup_{v \in I} \left| \frac{\hat{f}_{GPV}(v) - f(v)}{(Lh^3)^{-1/2} \sqrt{M(v) / v}} - \Gamma(v) \right| = O_p \left( \left( \frac{\log(L)}{Lh^3} \right)^{1/2} + \frac{\log(L)}{Lh^3} + \frac{\log(L)}{Lh^3} + h \right),
\]

(S.1.71)

Lemma A.2 showed that

\[
\varphi_1 := \sup_{v \in I} \left| \frac{\hat{f}_{GPV}(v) - f(v)}{(N - 1) \sqrt{M_2(B_{it}; v)} - \mu_M(v)} \right| = O_p \left( \left( \frac{\log(L)}{Lh} \right)^{1/2} + \frac{\log(L)}{Lh^3} + h \right),
\]

where the remainder is uniform in \( v \in I \).
Since $V_M(v)$ uniformly converges to $V_{GPV}(v)$ as $h \downarrow 0$, we have
\[\mathbb{V}_M := \inf_{v \in I} V_M(v) > C_1 > 0,\] (S.1.72)

when $h$ is sufficiently small. It is shown in the proof of Theorem 3.1 (see (A.8) and (A.10)) that
\[\sup_{v \in I} E \left[ \mathcal{M}_2^2(B_{11}; v)^2 \right] \leq h^{-3}.\] (S.1.73)

An application of the CCK inequality with $\sigma^2$ being the left-hand side of (S.1.73) and $F$ being (A.38) gives
\[E \left[ \sup_{v \in I} \frac{1}{N - 1} \frac{1}{N - L} \sum_{i,t} \left( \mathcal{M}_2(B_{il}; v) - \mu_M(v) \right) \right] \leq C_1 \left( (Lh^3)^{-1/2} \log(C_2L)^{1/2} + (Lh^3)^{-1} \log(C_2L) \right),\] (S.1.74)

where the inequality is non-asymptotic. Then Markov’s inequality yields
\[\vartheta_2 := \sup_{v \in I} \left| \frac{1}{N - 1} \frac{1}{N - L} \sum_{i,t} \left( \mathcal{M}_2(B_{il}; v) - \mu_M(v) \right) \right| = O_p \left( \frac{(\log(L))^{1/2}}{Lh^3} \right).\] (S.1.75)

In the proof of Theorem 3.1, we showed
\[\vartheta_3 := \sup_{v \in I} \left| V_M(v)^{1/2} - \text{Var} \left[ \left( N^{-1/2}(N - 1)^{-1} h^{3/2} \mathcal{M}_2(B_{11}; v) \right)^{1/2} \right] \right| = O(h^3).\]

Then by the triangle inequality and
\[a \leq \frac{a}{c} - \frac{a(b - c)}{\sqrt{2}} \leq \frac{a(b - c)^2}{bc^2},\] (S.1.76)

we have
\[\sup_{v \in I} \left| \left( \frac{\hat{V}_{GPV}(v) - f(v)}{(Lh^3)^{-1/2} V_M(v)^{1/2}} - \hat{\vartheta}_3 \right) \right| \leq (Lh^3)^{-1/2} \left( \sum_{i,t} \vartheta_1 + \sum_{i,t} \vartheta_2 \vartheta_3 + \text{Var} \left[ \left( N^{-1/2}(N - 1)^{-1} h^{3/2} \mathcal{M}_2(B_{11}; v) \right)^{1/2} \right] \right).\]

Now (S.1.71) follows from the above result and the the rates of convergence of $\vartheta_1$, $\vartheta_2$ and $\vartheta_3$.

By Theorem 4.1 and (S.1.72), we have
\[\sup_{v \in I} \left| \hat{V}_{GPV}(v)^{1/2} - V_M(v)^{1/2} \right| \leq \frac{1}{2} \sup_{v \in I} \left( \max \left\{ \hat{V}_{GPV}(v)^{-1}, V_M(v)^{-1} \right\} \right)^{1/2} \left| \hat{V}_{GPV}(v) - V_M(v) \right| = O_p \left( \frac{(\log(L))^{1/2}}{Lh^3} + \vartheta_R \right).\] (S.1.77)

It then follows from the above result, (S.1.76), (S.1.72) and the rates of convergence of $\vartheta_1$ and $\vartheta_2$ that
\[\sup_{v \in I} \left| Z(v) - \frac{\hat{f}_{GPV}(v) - f(v)}{(Lh^3)^{-1/2} V_M(v)^{1/2}} \right| = O_p \left( \frac{(\log(L))^{1/2}}{(Lh^3)^{1/2}} + (\log(L))^{1/2} \vartheta_R \right).\]

The conclusion follows from the above result and (S.1.71).

**Proof of Lemma A.7.** By decomposition (see (A.37)), Lemmas A.1, A.3, A.4 and Marmer and Shneyerov (2012, online supplement, Lemma S.1), we have
\[\vartheta_1^* := \sup_{v \in I} \left| \hat{f}_{GPV}(v) - \hat{f}_{GPV}(v) - \left\{ \frac{1}{N - 1} \frac{1}{N - L} \sum_{i,t} \left( \mathcal{M}_2(B_{il}; v) - \frac{1}{N} \sum_{j,k} \mathcal{M}_2(B_{jk}; v) \right) \right\} \right|\]
Now by the above result, Lemma \ref{lem:cck} follows from the CCK inequality that
\[
\frac{\sigma^2}{\tilde{M}_2} = \frac{1}{N^2 \cdot L} \sum_{i,j} \left( M_2(B^*_i;v) - \frac{1}{N} \sum_{j,k} M_2(B_{j,k};v) \right),
\]
where the second equality follows since it is easy to check that
\[
M_2(B^*_i;v) - \frac{1}{N} \sum_{j,k} M_2(B_{j,k};v) = M_2^1(B^*_i;v) - \frac{1}{N} \sum_{j,k} M_2^1(B_{j,k};v), \quad \text{for all } i = 1, \ldots, N \text{ and } t = 1, \ldots, L.
\]
(S.1.78)

Now let
\[
\tilde{\vartheta}_2^2 := \sup_{v \in I} \mathbb{E}^{*} \left[ \tilde{M}_2^1 (B_{11};v)^2 \right]
\]
\[
\leq h^{-3} \left\{ \sup_{v \in I} \mathbb{E} \left[ h^3 \tilde{M}_2^1 (B_{11};v)^2 \right] + \sup_{v \in I} \frac{1}{N \cdot L} \sum_{i,j} h^3 \tilde{M}_2^1 (B_{ij};v)^2 - \mathbb{E} \left[ h^3 \tilde{M}_2^1 (B_{11};v)^2 \right] \right\}.
\]
(S.1.79)

Since it was shown that \( \mathbb{E} \left[ h^3 \tilde{M}_2^1 (B_{11};v)^2 \right] \) converges to \( N (N - 1)^2 V_{GPV}(v) \) uniformly in \( v \in I \), we have
\[
\sup_{v \in I} \mathbb{E} \left[ h^3 \tilde{M}_2^1 (B_{11};v)^2 \right] = O(1).
\]
(S.1.80)

It follows from Chernozhukov et al. (2014b, Corollary A.1) that \( \left\{ h^3 \tilde{M}_2^1 (\cdot;v)^2 : v \in I \right\} \) is VC-type with respect to the constant envelope \( h^{-3} \left( C_{D_1} + C_{D_1}^2 \right) \). Now by change of variables we have
\[
\sup_{v \in I} \mathbb{E} \left[ h^6 \tilde{M}_2^1 (B_{11};v)^4 \right] = h^{-1} \sup_{v \in I} \int \frac{T_{(v)}}{\lambda(v)} \left( \int \frac{\tau_{(v)}}{\lambda(v)} \rho(u,w;v) du \right)^4 g(hw + s(v))dw \leq h^{-1}.
\]
(S.1.81)

Applying the CCK inequality with \( \sigma^2 \) being the left-hand side of the inequality of (S.1.81) and \( F \) being this constant envelope yields
\[
\mathbb{E} \left[ \sup_{v \in I} \frac{1}{N \cdot L} \sum_{i,j} h^3 \tilde{M}_2^1 (B_{ij};v)^2 - \mathbb{E} \left[ h^3 \tilde{M}_2^1 (B_{11};v)^2 \right] \right] \leq C_1 \left\{ (Lh)^{-1/2} \log (2L)^{-1/2} + (Lh^3)^{-1} \log (2L) \right\}
\]
\[
= O \left( \left( \frac{\log (L)}{Lh} \right)^{1/2} + \frac{\log (L)}{Lh^3} \right),
\]
where the inequality is non-asymptotic. It follows from the above result, Markov’s inequality, (S.1.79) and (S.1.80) that \( \mathbb{E}^{*} \left[ \tilde{\vartheta}_2^2 \right] = O_P(h^{-3}) \). It now follows from the CCK inequality that
\[
\mathbb{E}^{*} \left[ \tilde{\vartheta}_2^2 \right] \leq C_1 \left\{ L^{-1/2} \sigma^2 \tilde{M}_2^1 \log (2L)^{1/2} + (Lh^3)^{-1} \log (2L) \right\} = O_P \left( \left( \frac{\log (L)}{Lh^3} \right)^{1/2} \right).
\]

Now by the above result, Lemma \ref{lem:s12}, the triangle inequality, (S.1.76) and the rates of convergence of \( \vartheta_2^2 \) and \( \vartheta_3 \), we have
\[
\sup_{v \in I} \left| \tilde{I}_{GPV}(v) - I^{**}(v) \right| \leq (Lh^3)^{1/2} \left\{ \sum_{\tilde{M}}^{-1/2} \vartheta_2^2 \tilde{\vartheta}_3 + \sum_{\tilde{M}}^{-1} \vartheta_2^2 \tilde{\vartheta}_3 + \operatorname{Var} \left[ \bar{a}(N-1)^{-1} \right] \right\}^{1/2} \left( \sum_{\tilde{M}}^{-1} \vartheta_2^2 \tilde{\vartheta}_3 \right)^{1/2}
\]
S.19
\[= O_p \left( \log(L)^{1/2} h + \frac{\log(L)}{(Lh)^{1/2}} + L^{1/2} h^{3/2 + R} \right).\]

The conclusion follows from the above result, (S.1.77), the rates of convergence of \(\vartheta_1^\ast\) and \(\vartheta_2^\ast\).

**Proof of Lemma A.8.** By definition, fix any \(\epsilon > 0\), there exists some \(M_\epsilon > 0\) and also some \(L_\epsilon \in \mathbb{N}\), such that

\[P \left[ \sup_{z \in \mathbb{R}} P^\ast \left[ |V^\ast_L - W^\ast_L| > \lambda_L M_\epsilon \right] \geq \frac{\epsilon}{2} \right] < \frac{\epsilon}{2},\]

when \(L \geq L_\epsilon\). By Lemma A.6,

\[\sup_{z \in \mathbb{R}} |P^\ast [V^\ast_L \leq z] - P^\ast [W^\ast_L \leq z]| \leq \sup_{z \in \mathbb{R}} P^\ast \left[ |W^\ast_L - z| \leq \lambda_L M_\epsilon \right] + \frac{\epsilon}{2},\]

when \(P^\ast \left[ |V^\ast_L - W^\ast_L| > \lambda_L M_\epsilon \right] \leq \epsilon/2\).

Since

\[\sup_{z \in \mathbb{R}} P^\ast \left[ |W^\ast_L - z| \leq C_1 \lambda_L \right] \to 0,\]

as \(L \uparrow \infty\), there exists some \(L_\epsilon' \in \mathbb{N}\) such that

\[P \left[ \sup_{z \in \mathbb{R}} P^\ast \left[ |W^\ast_L - z| \leq \lambda_L M_\epsilon \right] \geq \frac{\epsilon}{2} \right] < \frac{\epsilon}{2},\]

when \(L \geq L_\epsilon'\). Since the three events satisfy

\[\left\{ P^\ast \left[ |V^\ast_L - W^\ast_L| > \lambda_L M_\epsilon \right] \leq \frac{\epsilon}{2} \right\} \cap \left\{ \sup_{z \in \mathbb{R}} P^\ast \left[ |W^\ast_L - z| \leq \lambda_L M_\epsilon \right] < \frac{\epsilon}{2} \right\} \subseteq \left\{ \sup_{z \in \mathbb{R}} P^\ast \left[ |V^\ast_L - z| - P^\ast |W^\ast_L - z| \right] < \epsilon \right\},\]

it is clear that

\[P \left[ \sup_{z \in \mathbb{R}} P^\ast \left[ |V^\ast_L - z| - P^\ast |W^\ast_L - z| \right] \geq \epsilon \right] \leq P \left[ P^\ast \left[ |V^\ast_L - W^\ast_L| > \lambda_L M_\epsilon \right] > \frac{\epsilon}{2} \right] + P \left[ \sup_{z \in \mathbb{R}} P^\ast \left[ |W^\ast_L - z| \leq \lambda_L M_\epsilon \right] \geq \frac{\epsilon}{2} \right] < \epsilon,\]

when \(L \geq \max \{L_\epsilon, L_\epsilon'\}\). The conclusion now follows.

**S.2 Limiting Distribution of the Uniform Error**

Denote

\[\hat{M}^\ast_1 (b; v) := \frac{G(bv) s'(v)}{g(s(v))} \int \frac{\partial}{\partial u} K'_f (u) K_g \left( \frac{b - s(v)}{h} - s'(v) u \right) du \]

\[\sqrt{\frac{g(b)}{2\pi}} \int K'_f (u) K_g \left( w - s'(v) u \right) du \] \[d w,\]

and

\[\hat{\mu}_{\hat{M}^\ast_1} (v) : = \int \hat{M}^\ast_1 (b; v) dG (b).\]

Since \(K_g\) is of bounded variation, it follows from Nolan and Pollard (1987, Lemma 22(ii)) that the function class

\[\left\{ (b, u) \mapsto K_g \left( \frac{b - s(v)}{h} - s'(v) u \right) : v \in I \right\}\]

is uniformly VC-type with respect to some constant envelope. Since \(K'_f\) is supported on \([-1, 1]\), it follows from Ghosal et al. (2000, Lemma A.2) the function class

\[\left\{ b \mapsto \int K'_f (u) K_g \left( \frac{b - s(v)}{h} - s'(v) u \right) du : v \in I \right\}\]

is uniformly VC-type with respect to some constant envelope. Then by standard arguments, we can verify that the function class \(\hat{M}^\ast_1 (\cdot; v) : v \in I\) is uniformly VC-type with respect to some constant envelope that is a multiple of \(h^{-1/2}\).
Consider the following empirical processes:

\[
\tilde{F}(v) := \frac{1}{(N \cdot L)^{1/2}} \sum_{i,t} \left( \tilde{M}^2_{ij} (B_{ij}; v) - \mu_{\tilde{M}^2} (v) \right)
\]

and

\[
\tilde{\Delta}(v) := \frac{1}{(N \cdot L)^{1/2}} \sum_{i,j} \left( \tilde{M}^2_{ij} (B_{ij}; v) - \frac{\tilde{M}^2_{ij} (B_{ij}; v)}{\text{Var} \left[ \tilde{M}^2_{ij} (B_{ij}; v) \right]} - \mu_{\tilde{M}^2} (v) + \frac{\mu_{\tilde{M}^2} (v)}{\text{Var} \left[ \tilde{M}^2_{ij} (B_{ij}; v) \right]} \right),
\]

(S.2.1)

for \( v \in I \). It follows from Nolan and Pollard (1987, Lemma 16) that

\[
\left\{ \tilde{M}^2_{ij} (;v) - \frac{\tilde{M}^2_{ij} (;v)}{\text{Var} \left[ \tilde{M}^2_{ij} (B_{ij}; v) \right]} : v \in I \right\}
\]

is uniformly VC-type with respect to some constant envelope that is a multiple of \( h^{-3/2} + h^{-1/2} \), when \( h \) is sufficiently small.

By tedious and lengthy calculations, we have

\[
E \left[ \left( \tilde{M}^2_{ij} (B_{ij}; v) - \frac{\tilde{M}^2_{ij} (B_{ij}; v)}{\text{Var} \left[ \tilde{M}^2_{ij} (B_{ij}; v) \right]} \right)^2 \right]
\]

\[
= \int \int K^2 (u, v) \left\{ \frac{G(u) G(v)}{g(u) g(v)} \int \frac{1}{h^2} K' (u) K_g \left( \frac{b-v}{h} \right) \right\} du \left\{ \frac{G(u) G(v)}{g(u) g(v)} \int \frac{1}{h^2} K' (v) K_g \left( \frac{b-v}{h} \right) \right\} dv
\]

\[
= O \left( h^2 \right),
\]

uniformly in \( v \in I \). Then by the CCK inequality and Markov’s inequality,

\[
\sup_{v \in I} \tilde{\Delta}(v) = O_p \left( \log (L)^{1/2} h + \log (L) (Lh^3)^{-1/2} \right).
\]

(S.2.2)

Again, by tedious and lengthy calculations, it can be shown that

\[
\sup_{v \in I} \left| \mu_{\tilde{M}^2} (v) \right| = O \left( h^{3/2} \right).
\]

(S.2.3)

By the arguments used in the proof of Theorem 5.1, we can easily verify that there exists a centered Gaussian process \( \tilde{F}_G (v) : v \in I \) which is a tight random element in \( \ell^\infty (I) \) and has the following covariance function:

\[
E \left[ \tilde{F}_G (v) \tilde{F}_G (v') \right] = E \left[ \left( \tilde{M}^2_{ij} (B_{ij}; v) - \mu_{\tilde{M}^2} (v) \right) \left( \tilde{M}^2_{ij} (B_{ij}; v') - \mu_{\tilde{M}^2} (v') \right) \right], \text{ for all } (v, v') \in I^2.
\]

Application of the coupling theorem Chernozhukov et al. (2014b, Corollary 2.2) with \( q = \infty, b \leq h^{-1/2}, \gamma = \log (L)^{-1} \) and \( \sigma = 1 \) yields that there exists a sequence of random variables \( \tilde{W}_L \) with \( \tilde{W}_L \overset{d}{=} \left\| \tilde{F}_G \right\|_I \) satisfying

\[
\left\| \left\| \tilde{F}_G \right\|_I - \tilde{W}_L \right\| = O_p \left( \log (L) (Lh^2) \right). \]

(S.2.4)

It follows from Lemma A.5, (S.2.1), (S.2.2) and (S.2.4) that

\[
\left\| \left\| Z \right\|_I - \tilde{W}_L \right\| = o_p \left( \log (L)^{-1} \right).
\]

(S.2.5)
Let $Z$ be a standard normal random variable that is independent of $\left\{ \tilde{\Gamma}_G(v) : v \in I \right\}$. Define

$$\tilde{\Gamma}_{GR}^R(v) := \tilde{\Gamma}_G(v) + Z \cdot \mu_{M^2_2}(v), \quad v \in I.$$ 

Note that $\left\{ \tilde{\Gamma}_{GR}^R(v) : v \in I \right\}$ is a centered Gaussian process with covariance function

$$E \left[ \tilde{\Gamma}_{GR}^R(v) \tilde{\Gamma}_{GR}^R(v') \right] = E \left[ \tilde{\Gamma}_G(v) \tilde{\Gamma}_G(v') \right] + O_p \left( h^{3/2} \right).$$

(S.2.3)

$\left\{ \tilde{\Gamma}_{GR}^R(v) : v \in I \right\}$ is a Bickel-Rosenblatt-type Gaussian approximation.

Suppose that $s$ is a linear function so that $s'(v) = \bar{\tau}$ for all $v \in I$, for some positive constant $\bar{\tau}$. It is easy to see that in this special case, the covariance function is

$$E \left[ \tilde{\Gamma}_{GR}^R(v) \tilde{\Gamma}_{GR}^R(v') \right] = \frac{\int \int K'_f(u) K_g(y - \bar{\tau} u) du \int K'_f(u') K_g(y - \bar{\tau} u' + \bar{\tau} (v - v')) du' dy}{\int \left\{ \int K'_f(u) K_g(w - \bar{\tau} u) du \right\}^2 dw},$$

which is a function of $(v - v')$. So the process $\left\{ \tilde{\Gamma}_{GR}^R(v) : v \in I \right\}$ is stationary.

Denote

$$\rho(t) = \frac{\int \int K'_f(u) K_g(y - \bar{\tau} u) du \int K'_f(u') K_g(y - \bar{\tau} u' + \bar{\tau} t) du' dy}{\int \left\{ \int K'_f(u) K_g(w - \bar{\tau} u) du \right\}^2 dw}.$$ 

It is easy to check that

$$\rho(0) = 1, \quad \rho'(0) = \frac{\tau \int \int K'_f(u) K_g(y - \bar{\tau} u) du \int K'_f(u') K_g'(y - \bar{\tau} u') du' dy}{\int \left\{ \int K'_f(u) K_g(w - \bar{\tau} u) du \right\}^2 dw} = 0,$$

and

$$\rho''(0) = \frac{-\tau^2 \int \int K'_f(u) K_g(y - \bar{\tau} u) du \int K'_f(u') K''_g(y - \bar{\tau} u') du' dy}{\int \left\{ \int K'_f(u) K_g(w - \bar{\tau} u) du \right\}^2 dw} = \frac{\tau^2}{\int \left\{ \int K'_f(u) K_g(w - \bar{\tau} u) du \right\}^2 dw} \frac{\int \int K'_f(u) K'_g(y - \bar{\tau} u) dy}{\int \left\{ \int K'_f(u) K_g(w - \bar{\tau} u) du \right\}^2 dw},$$

where the second equality follows from integration by parts.

Define $\tilde{\Gamma}_{GR}^R(y) := \tilde{\Gamma}_{GR}^R(v + h \cdot y)$, for $y \in \left[ 0, \frac{\alpha - \alpha'}{h} \right]$. It is easy to see that $E \left[ \tilde{\Gamma}_{GR}^R(y) \tilde{\Gamma}_{GR}^R(y') \right] = \rho(y - y')$ and

$$\left\| \tilde{\Gamma}_{GR}^R \right\|_I = \sup_{y \in \left[ 0, \frac{\alpha - \alpha'}{h} \right]} \left| \tilde{\Gamma}_{GR}^R(y) \right|.$$
Note that $\mathbb{E} \left[ \mathcal{F} \mathcal{B} \mathcal{R}_G (y) \right] = 0$ and $\mathbb{E} \left[ \mathcal{F} \mathcal{B} \mathcal{R}_G (y)^2 \right] = 1$. \{\mathcal{F} \mathcal{B} \mathcal{R}_G (y) : y \in \mathbb{R}_+ \} is a centered, normalized and stationary Gaussian process. It is easy to check that since $K'_y$ and $K_y$ are both supported on $[-1, 1]$, $\rho(t) = 0$ when $|t| > 2$. Since $\rho(0) = 1$ and $\rho'_0 (0) = 0$, we also have $\rho(t) = 1 - \lambda t^2 + o(t^2)$, as $t \downarrow 0$, where $\lambda = -\rho''(0)$. Therefore the conditions in the statement of Giné and Nickl (2015, Theorem 2.7.9) are all satisfied. Let

$$a_L := \left( 2 \cdot \log \left( \frac{v_u - v_l}{h} \right) \right)^{1/2},$$

$$b_L := \left( 2 \cdot \log \left( \frac{v_u - v_l}{h} \right) \right)^{1/2} + \frac{\log \left( \frac{\lambda^{1/2}}{2\pi} \right)}{\left( 2 \cdot \log \left( \frac{v_u - v_l}{h} \right) \right)^{1/2}}.$$

By Giné and Nickl (2015, Theorem 2.7.9),

$$\lim_{L \uparrow \infty} \mathbb{P} \left[ a_L \left( \mathcal{F} \mathcal{B} \mathcal{R}_G \| I \| - b_L \right) \leq x \right] = \exp (-\exp (-x)), \text{ for all } x \in \mathbb{R},$$

where $x \mapsto \exp (-\exp (-x))$ is the standard Gumbel CDF. We note that $a_L = O \left( \log (L)^{1/2} \right)$. It then follows from (S.2.6) and Slutsky’s lemma that

$$\lim_{L \uparrow \infty} \mathbb{P} \left[ a_L \left( \mathcal{F} \mathcal{B} \mathcal{R}_G \| I \| - b_L \right) \leq x \right] = \exp (-\exp (-x)), \text{ for all } x \in \mathbb{R}.$$

Since $\mathcal{W}_L \overset{d}{=} \mathcal{F} \mathcal{B} \mathcal{R}_G \| I \|$, we also have

$$\lim_{L \uparrow \infty} \mathbb{P} \left[ a_L \left( \mathcal{W}_L - b_L \right) \leq x \right] = \exp (-\exp (-x)), \text{ for all } x \in \mathbb{R},$$

and therefore by (S.2.5) and Slutsky’s lemma,

$$\lim_{L \uparrow \infty} \mathbb{P} \left[ a_L \left( \| Z \| - b_L \right) \leq x \right] = \exp (-\exp (-x)), \text{ for all } x \in \mathbb{R}.$$

Note that unlike many other asymptotic results for limiting distributions of uniform errors in the literature (see, e.g., Bickel and Rosenblatt, 1973 and Ghosal et al., 2000), the normalizing constants for $\| Z \|_I$ depend on the unknown slope $\overline{\gamma}$.

For the general case, the difficulty is that the approximating Gaussian process $\{ \mathcal{F} \mathcal{B} \mathcal{R}_G (v) : v \in I \}$ is non-stationary. Deriving the limiting distribution of $\mathcal{F} \mathcal{B} \mathcal{R}_G \| I \|$ (and $\| Z \|_I$) requires non-standard techniques and is beyond the scope of this paper.

### S.3 Proofs of the Results in Section 6

#### S.3.1 Preliminaries and Notation

For fixed $\mathbf{x} \in \text{int} (\mathcal{X})$, let $I (\mathbf{x}) = [v_l (\mathbf{x}), v_u (\mathbf{x})]$ be an inner closed subset of $[\mathbf{\underline{y}} (\mathcal{X}), \mathbf{\overline{y}} (\mathcal{X})]$. Fix

$$\delta_0 := \min \left\{ (\overline{\gamma} (\mathbf{x}) - v_u (\mathbf{x})) / 2, (v_l (\mathbf{x}) - \underline{\gamma} (\mathbf{x})) / 2 \right\}.$$

Then for any $n' \in \mathcal{N}$, by the strict monotonicity of $s (\cdot, \mathbf{x}, n')$ we have

$$s (v_l (\mathbf{x}) - \delta_0, \mathbf{x}, n') > s (\mathbf{\underline{y}} (\mathbf{x}), \mathbf{x}, n') = \mathbf{\underline{h}} (\mathbf{x}).$$

Since $\mathbf{x}$ is an interior point, by the continuity of $s (\cdot, \cdot, n')$ and $\mathbf{\underline{h}} (\cdot)$, there exists a neighborhood $\mathcal{H} (\mathbf{x}; \delta_{0'}, n')$ for some $\delta_{0'} > 0$ and some $\underline{\delta}_{0'} > 0$ such that $\mathcal{H} (\mathbf{x}; \underline{\delta}_{0'}, n') \subseteq \text{int} (\mathcal{X})$ and

$$v_l (\mathbf{x}) - \delta_0 > \underline{\gamma} (\mathbf{x}'), \text{ for all } \mathbf{x}' \in \mathcal{H} (\mathbf{x}; \underline{\delta}_{0'}, n') \quad \text{and}$$

$$\inf_{\mathbf{x}' \in \mathcal{H} (\mathbf{x}; \underline{\delta}_{0'}, n')} s (v_l (\mathbf{x}) - \delta_0, \mathbf{x}', n') > \sup_{\mathbf{x}'' \in \mathcal{H} (\mathbf{x}; \underline{\delta}_{0'}, n')} \mathbf{\underline{h}} (\mathbf{x}'') + \underline{\delta}_{0'}.$$
Similarly, we can find some $\delta_{\alpha'} > 0$ and some $\overline{\delta}_{\alpha'} > 0$ such that $H_\alpha(x, \overline{\delta}_{\alpha'}) \subseteq \text{int} (\mathcal{X})$ and

$$v_u(\alpha) + \delta_0 < v(\alpha'), \text{ for all } \alpha' \in \mathcal{X}(x, \overline{\delta}_{\alpha'})$$

and

$$\sup_{\alpha' \in \mathcal{X}(x, \overline{\delta}_{\alpha'})} s(v_u(\alpha) + \delta_0, \alpha', n') < \inf_{\alpha'' \in \mathcal{X}(x, \overline{\delta}_{\alpha'}, n')} \overline{\delta}(n', n').$$

Let

$$\overline{\overline{\delta}} := \min \{ \delta_0, \delta_\alpha/2, ..., \delta_\alpha/2, \overline{\delta}_{\alpha'}/2, ..., \overline{\delta}_{\alpha'}/2 \}.$$

Note that now

$$C_{V, X} := \left[ v(\alpha) - \overline{\overline{\delta}}, v_u(\alpha) + \overline{\overline{\delta}} \times H_\alpha(x, \overline{\overline{\delta}})$$

is a inner closed subset of $S_{V, X}$, when $x$ is an interior point of $X$. Denote

$$C_{B, V, X}^{n'} := \left\{(s, (v', x', n')) : (v', x') \in C_{V, X}, \alpha' \in \mathcal{X}(x, \overline{\overline{\delta}}) \right\}$$

(S.3.1)

for each $n' \in \mathcal{N}$. By the continuity of $s(\cdot', n'(\cdot'))$ (see Lemma A2 of GPV), $C_{B, V, X}^{n'}$ is a compact inner subset of $S_{B, V, X}^{n'}$.

Proposition 1(ii) of GPV gives that

$$\min_{n' \in \mathcal{N}(b, \alpha') \subseteq S_{B, V, X}^{n'}} g(b, x', n') > C_\overline{\overline{\overline{\delta}}} > 0$$

(S.3.2)

for some constant $C_\overline{\overline{\overline{\delta}}}$. Also denote

$$\overline{\overline{\overline{\delta}}} := \sup_{x \in \mathcal{X}} \overline{\overline{\overline{\delta}}}(x).$$

Denote

$$G(b, x', n') := G(b|x', n') \pi(n'|x') \varphi(x'), \quad g(b, x', n') := g(b|x', n') \pi(n'|x') \varphi(x')$$

and

$$\hat{G}(b, x', n') := \hat{G}(b|x', n') \hat{\pi}(x'), \quad \hat{g}(b, x', n') := \hat{g}(b|x', n') \hat{\pi}(n'|x') \hat{\varphi}(x').$$

Denote

$$I'(v|x) := \frac{1}{L^{1/2}} \sum_{l=1}^L \sum_{n \in \mathcal{N}} \left\{ M_2^3 (B_l, X_l, N_l^2; v) \right\} - \frac{1}{L^{1/2}} \sum_{l=1}^L \sum_{n \in \mathcal{N}} \left\{ M_2^3 (B_l, X_l, N_l^2; v) \right\}$$

and its bootstrap analogue

$$I''(v|x) := \frac{1}{L^{1/2}} \sum_{l=1}^L \sum_{n \in \mathcal{N}} \left\{ M_2^3 (B_l, X_l, N_l^2; v) \right\} - \frac{1}{L^{1/2}} \sum_{l=1}^L \sum_{n \in \mathcal{N}} \left\{ M_2^3 (B_l, X_l, N_l^2; v) \right\},$$

where

$$\hat{M}_2^3 (v) := \mathbb{E}^* \left[ M_2^3 (B_1^*, X_1^*, N_1^2; v) \right]$$

$$= \mathbb{E}^* \left[ E^* \left[ M_2^3 (B_1^*, X_1^*, N_1^2; v) | X_1^* \right] \right]$$

$$= \frac{1}{L} \sum_{l=1}^L M_2^3 (B_l, X_l, N_l; v),$$

where the second equality follows from LIE and the third equality can be verified using the fact that the bids in the bootstrap sample are conditionally i.i.d.

Denote

$$\hat{f}(v, x, n) := \frac{1}{L} \sum_{l=1}^L \mathbb{I} (N_l = n) \frac{1}{N_l} \sum_{i=1}^N \frac{1}{h^{1+\delta}} K_f \left( \frac{V_i - v}{h}, \frac{X_i - x}{h} \right),$$

$$\hat{f}(v|x) := \frac{1}{\hat{\varphi}(x)} \sum_{n \in \mathcal{N}} \hat{f}(v, x, n)$$

S.24
and

\[
\hat{f}_{GPV}(v, x, n) := \hat{f}_{GPV}(v|x, n) \tilde{\varphi}(n|x) \hat{\varphi}(x) = \frac{1}{L} \sum_{l=1}^{L} \sum_{N_l = n} \frac{N_l}{N_l} \sum_{i=1}^{N_l} T_{il} \frac{1}{h^{1+d}} K_f \left( \frac{\tilde{V}_{il} - v}{h}, \frac{X_i - x}{h} \right).
\]

Note that now we have

\[
\hat{f}_{GPV}(v|x) = \frac{1}{\hat{\varphi}(x)} \sum_{n \in N} \hat{f}_{GPV}(v, x, n).
\]

Let

\[
\hat{G}^* (b', x', n') := \frac{1}{L} \sum_{l=1}^{L} \sum_{N_l = n'} \frac{N_l}{N_l} \sum_{i=1}^{N_l} \sum_{j=1}^{N_l} (B_{il} \leq b') \frac{1}{h^{1+d}} K_f \left( \frac{B_{il} - b'}{h} \right) K_K \left( \frac{X_i^{*} - x'}{h} \right)
\]

and

\[
\tilde{g}^* (b', x', n') := \frac{1}{L} \sum_{l=1}^{L} \sum_{N_l = n'} \frac{N_l}{N_l} \sum_{i=1}^{N_l} \sum_{j=1}^{N_l} (B_{il} \leq b') \frac{1}{h^{1+d}} K_f \left( \frac{B_{il} - b'}{h} \right) K_K \left( \frac{X_i^{*} - x'}{h} \right).
\]

Note that now we have

\[
\hat{V}_{il} = B_{il} + \frac{1}{N_l - 1} \hat{g}^* (B_{il}^{*}, X_i^{*}, N_l^{*}).
\]

Let

\[
\hat{\tilde{f}}_{GPV}(v|x, n) := \frac{1}{\pi^* (n|x) \tilde{\varphi}^*(x)} L \sum_{l=1}^{L} \sum_{N_l = n} \frac{N_l}{N_l} \sum_{i=1}^{N_l} T_{il} \frac{1}{h^{1+d}} K_f \left( \frac{\tilde{V}_{il} - v}{h}, \frac{X_i^{*} - x}{h} \right)
\]

and

\[
\hat{\tilde{f}}_{GPV}(v, x, n) := \hat{\tilde{f}}_{GPV}(v|x, n) \tilde{\varphi}^*(n|x) \tilde{\varphi}^*(x).
\]

Note that we have

\[
\hat{f}_{GPV}(v|x) = \frac{1}{\tilde{\varphi}^*(x)} \sum_{n \in N} \hat{f}_{GPV}(v, x, n).
\]

Denote

\[
\tilde{f}^*(v, x, n) := \frac{1}{L} \sum_{l=1}^{L} \sum_{N_l = n} \frac{N_l}{N_l} \sum_{i=1}^{N_l} \frac{1}{h^{1+d}} K_f \left( \frac{V_{il} - v}{h}, \frac{X_i^{*} - x}{h} \right),
\]

where \(V_{il} := \xi (B_{il}^{*}, X_i^{*}, N_l^{*})\).

For a function \(\phi : \mathbb{R}^{1+d} \rightarrow \mathbb{R}\), denote

\[
D_{z_j}^d \phi (x_1, ..., x_{1+d}) := \left. \frac{\partial^d \phi (z_1, ..., z_{1+d})}{\partial z_j^d} \right|_{(z_1, ..., z_{1+d}) = (x_1, ..., x_{1+d})},
\]

for \(\alpha \in \mathbb{Z}_+\) and \(j = 1, ..., 1 + d\). It is also convenient to denote

\[
\phi' (x_1, ..., x_{1+d}) := \frac{\partial \phi (z_1, ..., z_{1+d})}{\partial z_1} \bigg|_{(z_1, ..., z_{1+d}) = (x_1, ..., x_{1+d})} \quad \text{and} \quad \phi'' (x_1, ..., x_{1+d}) := \frac{\partial \phi (z_1, ..., z_{1+d})}{\partial z_1^2} \bigg|_{(z_1, ..., z_{1+d}) = (x_1, ..., x_{1+d})}
\]

to be the partial derivatives with respect to the first argument.

Let \(\|\cdot\|_1\) denote the \(L_1\) norm on \(\mathbb{R}^k\): \(\|(z_1, ..., z_k)\|_1 = |z_1| + \cdots + |z_k|\). Also denote

\[
K_X^0 (x) := \prod_{k=1}^{d} K_0 (x_k), \quad \text{for } x = (x_1, ..., x_d) \in \mathbb{R}^d
\]

and

\[
S_R := \left\{ (\alpha_1, ..., \alpha_{1+d}) \in \mathbb{Z}_{1+d}^{1+d} : \sum_{j=1}^{1+d} \alpha_j = R \right\}.
\]
Law of iterated expectations is abbreviated as “LIE”. max is understood as \( \max_{l \in \{1, \ldots, L\}} \max_{i \in \{1, \ldots, n_l\}} \). \( \sum \) is understood as \( \sum_{k \neq l} \) and \( \sum \) is understood as \( \sum_{j \in \{1, \ldots, j\}} \sum_{k \neq l, l' \neq l} \) \( k' \). i.e., summing over all distinct indices. \( (L)_{2} \) is understood as \( (L)_{L} \) and \( (L)_{3} \) is understood as \( (L-1)_{L} \). 

S.3.2 Proofs of the Results in Section 6

Proof of Theorem 6.1. Now it follows from Lemma S.3.3 that

\[
\left( Lb^{3+d}\right)^{1/2} \left( \frac{f_{G_{PN}(v|x, n)} - f(v|x)}{\hat{\varphi}(x)} \right) := \frac{1}{\pi} \frac{1}{L^{1/2}} \sum_{i=1}^{L} h^{(3+d)/2} (M_{2} (B_{i}, X_{i}, N_{i}; v) - M_{2} (v)) + o_p(1) \quad \text{(S.3.3)}
\]

We now show that a central limit theorem for triangular arrays can be applied to the leading term of the right hand side of (S.3.3).

Let

\[
M_{2}^{l,n} (b, z ; m ; v) := E \left[ M_{2}^{l,n} (B, X, N_{i}) (b, z ; m ; v) \right]
\]

and

\[
\mu_{M_{2}^{l,n}} (v) := E \left[ M_{2}^{l,n} (B, X, N_{i}) (b, z ; m ; v) \right].
\]

By the LIE, we have

\[
M_{2}^{l,n} (b_{m}, z, m ; v) = E \left[ M_{2}^{l,n} (B_{1}, X_{1}, N_{1}) (b_{m}, z, m ; v) \right]
\]

and

\[
M_{2}^{l,n} (b_{m}, z, m ; v) = E \left[ M_{2}^{l,n} (B_{2}, X_{2}, N_{2}) (b_{m}, z, m ; v) \right].
\]

It is straightforward to check

\[
M_{2}^{l,n} (B_{l}, X_{l}, N_{l}; v) - \mu_{M_{l,n}} (v) = M_{2}^{l,n} (B_{l}, X_{l}, N_{l}; v) - \mu_{M_{l,n}} (v) \quad \text{for all } l = 1, \ldots, L.
\]

Denote

\[
U_{l}^{n} (v) := L^{-1/2} h^{(3+d)/2} \left( M_{2}^{l,n} (B_{l}, X_{l}, N_{l}; v) - \mu_{M_{l,n}} (v) \right)
\]

S.26
and
\[
\sigma^n (v) := \left( \sum_{l=1}^{L} E \left[ U^n_l (v)^2 \right] \right)^{1/2} = \left( E \left[ h^{d+1} \left( M_{2,n} \left( B_1, X_1, N_1; v \right) - \mu_{M_{2,n}} (v) \right)^2 \right] \right)^{1/2}.
\] (S.3.5)

By the definition of \( U^n_l (v) \)'s and (S.3.3),
\[
\left( Lh^{d+1} \right)^{1/2} \left( \int_{G^PV} (v|x, n) - f(v|x) \right) = \frac{1}{\pi (n|x)} \sum_{l=1}^{L} U^n_l (v) + o_P (1).
\] (S.3.6)

It is easy to check that
\[
\mu_{M_{2,n}} (v) = \mu_M (v) + \left( - \frac{1}{n-1} \int \xi (z', n) \frac{1}{h^{d+1}} K_f^0 \left( \frac{z' - z_h}{h} \right) G (h, z', n) \, db \, dz' \right).
\]

By change of variables and a mean value expansion, we have
\[
\int \frac{1}{h^{d+1}} K_f^0 \left( \frac{z' - z_h}{h} \right) G \left( s \left( v(z', n), z_h \right) \right) u \left( \frac{\sigma (z', n)}{h} \right) K_0^0 \left( \frac{\sigma (z', n)}{h} \right) \, du \, dz' \leq \int \frac{1}{h^{d+1}} K_f^0 \left( \frac{z' - z_h}{h} \right) \left( \frac{\sigma (z', n)}{h} \right) \, du \, dz'
\]
\[
\leq \left( \int \frac{1}{h^{d+1}} K_f^0 \left( \frac{z' - z_h}{h} \right) \, du \, dz' \right) \left( \sup \left( u(z') \right) \right) G \left( s \left( v(z', n), z_h \right) \right) \left( \frac{\sigma (z', n)}{h} \right) \, du \, dz' \leq O (1),
\]
where the inequality holds when \( h \) is sufficiently small and the equality follows from the fact that \( C_{V,X} \) is an inner closed subset of \( S_{V,X} \), the continuity of \( g(\cdot, n) \), \( G(\cdot, n) \), \( s' (\cdot, n) \) and \( s'' (\cdot, n) \) and the fact
\[
\int \frac{1}{h^{d+1}} K_f^0 \left( \frac{z' - z_h}{h} \right) \, dz' = O (1),
\] (S.3.8)
which follows from change of variables. Now it follows that
\[
\sup_{v \in I (x)} \left| \int \frac{1}{h^{d+1}} K_f^0 \left( \frac{z' - z_h}{h} \right) G (h, z', n) \, db \, dz' \right| = O (1).
\] (S.3.9)

Therefore it follows from the fact \( \sup_{v \in I (x)} |\mu_{M_{2,n}} (v)| = O (h^R) \) which is shown in the proof of Lemma S.3.3 and (S.3.9) that
\[
\sup_{v \in I (x)} |\mu_{M_{2,n}} (v)| = O (1), \text{ for all } n \in N.
\] (S.3.10)

S.27
By the LIE, we have

$$E \left[ h^{3+d}M_{t}^{X}(B, X, N; v)^{2} \right]$$

$$= E \left[ \frac{1}{h^{1+d}} \left\{ 1 \right\} \frac{1}{N_{1}} \sum_{n=1}^{N_{1}} \int_{X} \int_{h(z')} \frac{1}{h^{1+d}K_{g}'} \left( \frac{\xi(b', z', n) - v}{h} \right) \right.$$ 

$$\times \frac{G(b', z', n)}{g(b', z', n)} K_{g} \left( \frac{X_{1} - z'}{h} \right) \text{d}b' \text{d}z' \right]$$

$$= \frac{1}{n} \frac{1}{h^{1+d}} \int_{X} \int_{h(z')} \left\{ \int_{X} \int_{h(z')} \frac{1}{h^{1+d}K_{g}'} \left( \frac{\xi(b', z', n) - v}{h} \right) \right.$$ 

$$\times \frac{G(b', z', n)}{g(b', z', n)} K_{g} \left( \frac{b - b'}{h} \right) K_{X} \left( \frac{z - z'}{h} \right) \text{d}b' \text{d}z' \right]$$

$$+ \frac{1}{n} \frac{1}{h^{1+d}} \int_{X} \int_{h(z')} \left\{ \int_{X} \int_{h(z')} \frac{1}{h^{1+d}K_{g}'} \left( \frac{\xi(b', z', n) - v}{h} \right) \right.$$ 

$$\times \frac{G(b', z', n)}{g(b', z', n)} K_{g} \left( \frac{b - b'}{h} \right) K_{X} \left( \frac{z - z'}{h} \right) \text{d}b' \text{d}z' \right]$$

By change of variables, we have

$$= \frac{1}{n} \frac{1}{h^{1+d}} \int_{X} \int_{h(z')} \left\{ \int_{X} \int_{h(z')} \frac{1}{h^{1+d}K_{g}'} \left( \frac{\xi(b', z', n) - v}{h} \right) \right.$$ 

$$\times \frac{G(b', z', n)}{g(b', z', n)} K_{g} \left( \frac{b - b'}{h} \right) K_{X} \left( \frac{z - z'}{h} \right) \text{d}b' \text{d}z' \right]$$

$$= \frac{1}{n} \frac{1}{h^{1+d}} \int_{X} \int_{h(z')} \left\{ \int_{X} \int_{h(z')} \frac{1}{h^{1+d}K_{g}'} \left( \frac{\xi(b', z', n) - v}{h} \right) \right.$$ 

$$\times \frac{G(b', z', n)}{g(b', z', n)} K_{g} \left( \frac{b - b'}{h} \right) K_{X} \left( \frac{z - z'}{h} \right) \text{d}b' \text{d}z' \right] \right] \pi \left( n | z \right) \psi \left( z \right) \text{d}z.$$

(S.3.11)

By change of variables, we have

$$= \frac{1}{h} \frac{1}{n} \frac{1}{h^{1+d}} \int_{Y} \left\{ \int_{Y} \int_{h(y')} \frac{1}{h^{1+d}K_{g}'} \left( \frac{\xi(b', z', n) - v}{h} \right) \right.$$ 

$$\times \frac{G(b', z', n)}{g(b', z', n)} K_{g} \left( \frac{b - b'}{h} \right) K_{X} \left( \frac{z - z'}{h} \right) \text{d}b' \text{d}z' \right] \pi \left( n | y + x \right) \varphi \left( y + x \right) \text{d}y$$

$$= \frac{1}{h} \frac{1}{n} \frac{1}{h^{1+d}} \int_{Y} \left\{ \int_{Y} \int_{h(y')} \frac{1}{h^{1+d}K_{g}'} \left( \frac{\xi(b', z', n) - v}{h} \right) \right.$$ 

$$\times \frac{G(b', z', n)}{g(b', z', n)} K_{g} \left( \frac{b - b'}{h} \right) K_{X} \left( \frac{z - z'}{h} \right) \text{d}b' \text{d}z' \right] \pi \left( n | y + x \right) \varphi \left( y + x \right) \text{d}y.$$

where

$$Y := \left[ \frac{x_{1} - x_{1}}{h}, \frac{x_{d} - x_{d}}{h} \right] \times \cdots \times \left[ \frac{x_{d} - x_{d}}{h}, \frac{x_{d} - x_{d}}{h} \right].$$

By the c_{e} inequality, we have

$$\leq h \int_{Y} \left\{ \int_{Y} \int_{h(y')} \frac{1}{h^{1+d}K_{g}'} \left( \frac{\xi(b', z', n) - v}{h} \right) \right.$$ 

$$\times \frac{G(b', z', n)}{g(b', z', n)} K_{g} \left( \frac{b - b'}{h} \right) K_{X} \left( \frac{z - z'}{h} \right) \text{d}b' \text{d}z' \right] \pi \left( n | y + x \right) \varphi \left( y + x \right) \text{d}y$$

$$+ h \int_{Y} \left\{ \int_{Y} \int_{h(y')} \frac{1}{h^{1+d}K_{g}'} \left( \frac{\xi(b', z', n) - v}{h} \right) \right.$$ 

$$\times \frac{G(b', z', n)}{g(b', z', n)} K_{g} \left( \frac{b - b'}{h} \right) K_{X} \left( \frac{z - z'}{h} \right) \text{d}b' \text{d}z' \right] \pi \left( n | y + x \right) \varphi \left( y + x \right) \text{d}y.$$
When $h$ is sufficiently small ($h \leq \min \left\{ \frac{\delta_1}{\mu_1}, \ldots, \frac{\delta_k}{\mu_k}, \frac{\delta_{k+1}}{\mu_{k+1}}, \ldots, \frac{\delta_m}{\mu_m} \right\}$),
\[
\int_{\mathcal{H}(0,1)}^\infty \int_{\mathcal{H}(0,1)}^\infty K_g(w) \, dw = 1, \text{ for all } y' \in \mathcal{H}(0,1), \, |u| \leq 1, \, y \in \mathcal{H}(0,2) \text{ and } v \in I(x),
\] (S.3.12)

By a mean value expansion, for any $y \in \mathcal{Y}$,
\[
\int_{\mathcal{Y}} \int_{\mathcal{H}(0,1)}^\infty K_f(u, y') \psi(hu + v, hy' + x, n) \, dz \, du \psi(y - y' \psi(v, hy' + x, n) g(s(hu + v, hy' + x, n)|hy + x, n) \, du dudy' \]
\[
= \int_{\mathcal{Y}} K_X(y - y') \int_{\mathcal{H}(0,1)}^\infty K_f(u, y') \psi(v, hy' + x, n) g(s(v, hy' + x, n)|hy + x, n)
\]
\[
+ hu(\psi(v, hy' + x, n) g(s(v, hy' + x, n)|hy + x, n) + \psi(hu + v, hy' + x, n) g(s(hu + v, hy' + x, n)|hy + x, n) s'(v, hy' + x, n)) \, du dudy' \]
\[
= \int_{\mathcal{Y}} K_X(y - y') \psi(v, hy' + x, n) g(s(v, hy' + x, n)|hy + x, n) K_X^0(y') \left( \int_{\mathcal{H}(0,1)}^\infty K_0(u) \, du \right) dudy' \]
\[
+ h \int_{\mathcal{Y}} K_X(y - y') \int_{\mathcal{H}(0,1)}^\infty K_f(u, y') \psi(hu + v, hy' + x, n) g(s(hu + v, hy' + x, n)|hy + x, n)
\]
\[
+ \psi(hu + v, hy' + x, n) g(s(hu + v, hy' + x, n)|hy + x, n) s'(h, hy' + x, n) \, du dudy',
\] (S.3.13)

where $\tilde{v}$ is the mean value with $|\tilde{v} - v| \leq h|u|$. Since $K_0^0$ is odd,
\[
\int_{\mathcal{H}(0,1)}^\infty K_0^0(u) \, du = 0, \text{ for all } y' \in \mathcal{H}(0,1) \text{ and } v \in I(x),
\]
when $h$ is sufficiently small. It follows from this fact, (S.3.12), (S.3.13), continuity of $\psi(\cdot, \cdot, n)$, $\psi'(\cdot, \cdot, n)$, $s(\cdot, \cdot, n)$, $s'(\cdot, \cdot, n)$, $g(\cdot, \cdot, n)$ and $g'(\cdot, \cdot, n)$ that
\[
\sup_{v \in I(x)} \int_{\mathcal{Y}} \left\{ \int_{\mathcal{H}(0,1)}^\infty K_f(u, y') \psi(hu + v, hy' + x, n) \, du \int_{\mathcal{H}(0,1)}^\infty K_g(w) \, dw \right\}^2 \pi(n|hy + x) \varphi(hy + x) \, dy
\]
\[
= O(h^3).
\]

By a second-order Taylor expansion and the fact
\[
\int_{\mathcal{H}(0,1)}^\infty K_g(w) \, dw = 0, \text{ for all } y' \in \mathcal{H}(0,1), \, y \in \mathcal{H}(0,2), \, |u| \leq 1 \text{ and } v \in I(x),
\]
when $h$ is sufficiently small, we have
\[
\sup_{v \in I(x)} \int_{\mathcal{Y}} \left\{ \int_{\mathcal{H}(0,1)}^\infty K_f(u, y') \psi(hu + v, hy' + x, n) \, du \int_{\mathcal{H}(0,1)}^\infty K_g(w) \, dw \right\}^2 \pi(n|hy + x) \varphi(hy + x) \, dy
\]
\[
= O(h^5).
\]
Now it follows that
\[
\sup_{v \in I(x)} \frac{1}{h^{n+1}} \frac{1}{(n-1)!} \int_X \int_{\mathbb{R}^d} \sum_{\mathcal{H}} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(b, z, n) \, db \, dz \right) \frac{1}{h^{n+1}} K_f \left( \frac{\xi (b', z', n) - v}{h}, \frac{z' - x}{h} \right) G \left( b', z', n \right) K_y \left( \frac{b - b'}{h} \right) K_X \left( \frac{z - z'}{h} \right) \times g(b, z, n) \, db \, dz
\]
\[= O \left( h^n \right). \]

Now by this result, (S.3.11) and change of variables, we have
\[
E \left[ h^{3+d} \mathcal{M}_{21}^{1,n} (B_1, X_1, N_1; v) \right] = \frac{1}{n(n-1)^2} \int_Y \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_f (u, y') \psi \left( hu + v, hy' + x, n \right) K_g \left( w - \frac{s(hv + x, y', n) - s(v, x, n)}{h} \right) K_X (y - y') \, du \, dv \, dx \, dy' \left( \text{S.3.14} \right)
\]
\[
\quad \times g(hw + s(v, x, n), hy + x, n) \, du \, dv \, dx + O \left( h^3 \right).
\]

where the remainder term is uniform in \( v \in I(x) \).

Denote
\[
\rho^n (u, y', w, v; y) := K_f (u, y') \psi \left( hu + v, hy' + x, n \right) K_g \left( w - \frac{s(hv + x, y', n) - s(v, x, n)}{h} \right) K_X (y - y')
\]
and
\[
\mathcal{P}^n (w, y; v) := \psi \left( v, x, n \right) \int K_f (u, y') K_g \left( w - s(v, x, n) - \frac{s_y x}{n} y' \right) K_X (y - y') \, du \, dv \, dx.
\]

Now we have
\[
E \left[ h^{3+d} \mathcal{M}_{21}^{1,n} (B_1, X_1, N_1; v) \right] = \frac{1}{n(n-1)^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_f (u, y') K_g \left( w - s(v, x, n) - \frac{s_y x}{n} y' \right) K_X (y - y') \, du \, dv \, dx \left( \text{S.3.15} \right)
\]
\[
\times g(hw + s(v, x, n), hy + x, n) \, du \, dv \, dx + O \left( h^3 \right).
\]

Let
\[
\mathcal{P} := \sup_{(u, x) \in \mathcal{C} \cap X, j = 1} \sum_{j=1}^{1+d} D_j s(u, x, n).
\]

When \( h \) is sufficiently small,
\[
\sup_{v \in I(x)} \left| \rho^n (u, y', w, v; y) \right| \leq 1 \left( y' \in \mathbb{R} \setminus (0, 1) \right) 1 \left( y \in \mathbb{R} \setminus (0, 2) \right) 1 \left( |w| \leq 1 + \mathcal{P} \right) 1 \left( |u| \leq 1 \right) \left( \text{S.3.15} \right)
\]
and
\[
\sup_{v \in I(x)} \left| \mathcal{P}^n (w, y; v) \right| \leq 1 \left( y \in \mathbb{R} \setminus (0, 2) \right) 1 \left( |w| \leq 1 + \mathcal{P} \right). \left( \text{S.3.16} \right)
\]

By the triangle inequality, we have
\[
\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho^n (u, y', w, v; y) \, du \, dv \, dx \, dy' \right|^2 \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho^n (u, y', w, v; y) \, du \, dv \, dx \left( \text{S.3.16} \right).
\]

S.30
where the remainder terms are uniform in $v$. It is clear that when $h$ is sufficiently small,

\[
g(\mathbf{s}(v, x, n), x, n) \left| \int_y \int_{\frac{\mathbf{s}(h y + z, n) - s(v, x, n)}{h}} \mathbf{p}^n(w, y; v)^2 \, d\mathbf{w} \right| = 0, \text{ for all } v \in I(x). \tag{S.3.18}
\]

And

\[
\sup_{v \in I(x)} \left| \int_y \int_{\frac{\mathbf{s}(h y + z, n) - s(v, x, n)}{h}} \mathbf{p}^n(w, y; v)^2 \, d\mathbf{w} \right| = o(1) \tag{S.3.19}
\]

follows from uniform continuity of $g(\cdot, \cdot, n)$ and $s(\cdot, \cdot, n)$. It is also clear that

\[
\sup_{v \in I(x)[w, y] \in \mathbb{R}^{d+1}} \left| \psi(v, x, n) \int_y \int_{\frac{\mathbf{s}(h y + z, n) - s(v, x, n)}{h}} K_{h}^f(u, y) K_g\left( w - \frac{s(h u + v, h y + x, n) - s(v, x, n)}{h} \right) K_X(y - y') \, d\mathbf{w} \right| = o(1),
\]

where the equality follows from uniform continuity of the partial derivatives of $s(\cdot, \cdot, n)$ and now

\[
\sup_{v \in I(x)} \left| \int_y \int_{\frac{\mathbf{s}(h y + z, n) - s(v, x, n)}{h}} \rho^n(u, y', w, y; v) \, d\mathbf{w} \right|^2 - \mathbf{p}^n(w, y; v)^2 \, d\mathbf{w} = o(1) \tag{S.3.20}
\]

follows from this result, continuity of $\psi(\cdot, \cdot, n)$, (S.3.15) and (S.3.16).

Now by (S.3.10), (S.3.14), (S.3.17), (S.3.18), (S.3.19) and (S.3.20), we have

\[
E \left[ h^{3+d} \left( M_{2}^{n}(B_1, X_1, N_1; v) - \mu_{M_{1,n}}(v) \right)^2 \right] = E \left[ h^{3+d} M_{2}^{n}(B_1, X_1, N_1; v)^2 \right] - h^{3+d} \mu_{M_{1,n}}(v)^2 = \frac{1}{n(n-1)^2} \left( \varphi(x) g(s(v, x, n) | x, n) \right)^2 \int \left\{ \int \int K_{h}^f(u, y) K_g\left( w - s_0 u - s_2 y \right) K_X(z - y) \, d\mathbf{w} \right\}^2 \, d\mathbf{w} + o(1)
\]

and also

\[
\sigma^n(v) = \frac{1}{n^{1/2}} \left( \frac{F(v|x)^2}{\varphi(x) g(s(v, x, n) | x, n)} \right)^{1/2} \int \left\{ \int \int K_{h}^f(u, y) K_g\left( w - s_0 u - s_2 y \right) K_X(z - y) \, d\mathbf{w} \right\}^2 \, d\mathbf{w}^{1/2} + o(1), \tag{S.3.21}
\]

where the remainder terms are uniform in $v \in I(x)$. 

S.31
By the $c_r$ inequality, we have

$$
\sum_{l=1}^{L} E \left[ \left( \frac{U^n_l(v)}{\sigma^n_l(v)} \right)^3 \right] = \sigma^n(v)^{-3} L^{-1/2} E \left[ h^{3(3+d)/2} \left| M_{2,1}^{n,l}(B_1, X_1, N_1; v) - \mu_{M_{1,1} l,n} (v) \right|^{3} \right] \\
\leq \sigma^n(v)^{-3} L^{-1/2} \left( E \left[ h^{3(3+d)/2} \left| M_{2,1}^{n,l}(B_1, X_1, N_1; v) \right|^{3} \right] + h^{3(3+d)/2} |\mu_{M_{1,1} l,n} (v)|^{3} \right). \quad (S.3.22)
$$

By the $c_r$ inequality, Jensen’s inequality and change of variables, we have

$$
E \left[ h^{3(3+d)/2} \left| M_{2,1}^{n,l}(B_1, X_1, N_1; v) \right|^{3} \right] \\
\leq E \left[ h^{3(3+d)/2} \left( 1 (N_1 = n) \right) \frac{1}{N_1} \sum_{j=1}^{N_1} \int_{\mathcal{C}(v')} K_{j'} \left( \frac{z'-x}{h} \right) G (b', z', n) \right. \\
\times \left. K_{g} \left( \frac{b'-b_{1,j}}{h} \right) K_{X} \left( \frac{z'-X_1}{h} \right) \right) \right] \\
\leq h^{-(1+d)/2} \int_{\mathcal{C}(v')} \int_{\mathcal{C}(v')} \int_{\mathcal{C}(v')} \int_{\mathcal{C}(v')} \rho^n \left( u, y', w, y, v \right) du dy' \\
\times \int_{\mathcal{C}(v')} \int_{\mathcal{C}(v')} \rho^n \left( h w + s (v, x, n), h y + x, n \right) du dy.
$$

Now it follows from this result and (S.3.15) that

$$
\sup_{v \in I(\mathcal{R})} E \left[ h^{3(3+d)/2} \left| M_{2,1}^{n,l}(B_1, X_1, N_1; v) \right|^{3} \right] = O \left( h^{-(1+d)/2} \right). \quad (S.3.23)
$$

It now follows from this result, (S.3.10), (S.3.21) and (S.3.22) that

$$
\sum_{l=1}^{L} E \left[ \left( \frac{U^n_l(v)}{\sigma^n_l(v)} \right)^3 \right] \downarrow 0, \text{ as } L \uparrow \infty. \quad (S.3.24)
$$

By the Lyapunov’s central limit theorem, we have

$$
\sum_{l=1}^{L} \frac{U^n_l(v)}{\sigma^n_l(v)} \rightarrow_{d} N (0, 1), \text{ as } L \uparrow \infty. \quad (S.3.25)
$$

Standard arguments for kernel-smoothing based nonparametric estimation shows that \( \hat{\pi} (n|x) \hat{\varphi} (x) \) is consistent for \( \pi (n|x) \varphi (x) \). Now the asymptotic normality follows from this result, (S.3.6), (S.3.21), (S.3.25), the above displayed result and Slutsky’s lemma.

Fixing any \( n_1, n_2 \in N \) with \( n_1 \neq n_2 \), we have

$$
\left(L h^{3+d} \right)^{1/2} \left( \tilde{f}_{GPV} (v|x, n_1) - f (v|x) \right) = \frac{1}{\widehat{\pi} (n_1|x) \hat{\varphi} (x) L^{1/2}} \sum_{l=1}^{L} h^{3(3+d)/2} \left( M_{2,1}^{n_1,l} (B_1, X_1, N_1; v) - \mu_{M_{1,1} n_1} (v) \right) + o_p (1)
$$

$$
\left(L h^{3+d} \right)^{1/2} \left( \tilde{f}_{GPV} (v|x, n_2) - f (v|x) \right) = \frac{1}{\widehat{\pi} (n_2|x) \hat{\varphi} (x) L^{1/2}} \sum_{l=1}^{L} h^{3(3+d)/2} \left( M_{2,1}^{n_2,l} (B_1, X_1, N_1; v) - \mu_{M_{1,1} n_2} (v) \right) + o_p (1) \quad (S.3.26)
$$

Denote

$$
Y_i (v) := \left( L^{-1/2} h^{3(3+d)/2} \left( M_{2,1}^{i,n_1} (B_1, X_1, N_1; v) - \mu_{M_{1,1} n_1} (v) \right), L^{-1/2} h^{3(3+d)/2} \left( M_{2,1}^{i,n_2} (B_1, X_1, N_1; v) - \mu_{M_{1,1} n_2} (v) \right) \right)^T.
$$

It follows from (S.3.10) that

$$
E \left[ k^{3+d} \left( M_{2,1}^{n_1,l} (B_1, X_1, N_1; v) - \mu_{M_{1,1} n_1} (v) \right) \left( M_{2,1}^{n_2,l} (B_1, X_1, N_1; v) - \mu_{M_{1,1} n_2} (v) \right) \right] = O \left( h^{3+d} \right),
$$

S.32
uniformly in \( v \in I(x) \). It follows from the above result and (S.3.21) that

\[
L \sum_{i=1}^{L} \text{Var}[Y_i(v)] = \begin{bmatrix}
V_{GPV}(v, x, n_1) & 0 \\
0 & V_{GPV}(v, x, n_2)
\end{bmatrix}, \text{ as } L \uparrow \infty.
\]

It is straightforward to verify that given the above result,

\[
L \sum_{i=1}^{L} E[\|Y_i(v)\|^2] \downarrow 0, \text{ as } L \uparrow \infty \tag{S.3.27}
\]

is sufficient for the Lyapunov condition for the the multi-dimensional Lyapunov central limit theorem which can be established as a consequence of the Cramer-Wold device and the one-dimensional central limit theorem for triangular arrays. (S.3.27) can be established by using the \( c_r \) inequality and (S.3.23). The joint asymptotic normality of \( \sum_{i=1}^{L} Y_i(v) \) and asymptotic independence of

\[
\left( (Lh^{3+q})^{1/2} \left( \hat{f}_{GPV}(v|x, n_1) - f(v|x) \right) , (Lh^{3+q})^{1/2} \left( \hat{f}_{GPV}(v|x, n_2) - f(v|x) \right) \right)^T
\]

follows from (S.3.26), (S.3.27), the consistency of \( \hat{\pi}(n_1|x) \hat{\varphi}(x) \) and \( \hat{\pi}(n_2|x) \hat{\varphi}(x) \) and Slutsky’s lemma. \[\blacksquare\]

**Proof of Corollary 6.1.** The conclusion of this corollary follows from Theorem 6.1, consistency of \( \hat{\pi}(n|x) \), \( n \in N \) and the continuous mapping theorem. \[\blacksquare\]

Note that the estimator for the asymptotic variance is numerically equivalent to

\[
\hat{V}_{GPV}(v|x, n) = \sum_{n \in N} \hat{\pi}(n|x)^2 \hat{V}_{GPV}(v|x, n)
\]

\[
= \hat{\varphi}(x)^2 \binom{h^{3+q}}{L} \sum_{(M)} \frac{1}{N_1} \sum_{l=1}^{N_l} \frac{1}{N_k} \sum_{j=1}^{N_k} \frac{1}{N_l} - 1 \text{ T}_{jk} K_f \left( \frac{\hat{V}_{jk} - v}{h}, \frac{X_k - x}{h} \right) \frac{\hat{G}(B_{jk}, X_k, N_k)}{g(B_{jk}, X_k, N_k)^2}
\]

\[
\times \mathbb{2} (N_l = N_k) K_g \left( B_{il} - B_{jk}, h \right) K_X \left( X_l - X_k \right) \frac{1}{N_k} \sum_{j'=1}^{N_k} \frac{1}{N_k} - 1 \text{ T}_{j'k} K_f \left( \frac{\hat{V}_{j'k} - v}{h}, \frac{X_{k'} - x}{h} \right)
\]

\[
\times \frac{\hat{G}(B_{jk'}, X_{k'}, N_{k'})}{g(B_{jk'}, X_{k'}, N_{k'})} \mathbb{2} (N_l = N_{k'}) K_g \left( B_{il} - B_{jk'}, h \right) K_X \left( X_l - X_{k'} \right) \frac{1}{h} .
\]

**Proof of Theorem 6.2.** Write

\[
\hat{\pi}(n|x)^2 \hat{\varphi}(x)^2 \hat{V}_{GPV}(v|x, n) = \Delta_1^1(v) + \Delta_1^1(v) + \Delta_1^1(v),
\]

where

\[
\Delta_1^1(v) := \frac{1}{h^{3+q}} \sum_{(M)} \frac{1}{N_1} \sum_{l=1}^{N_l} \frac{1}{N_k} \sum_{j=1}^{N_k} \frac{1}{N_l} - 1 \text{ T}_{jk} K_f \left( \frac{\hat{V}_{jk} - v}{h}, \frac{X_k - x}{h} \right) \frac{\hat{G}(B_{jk}, X_k, N_k)}{g(B_{jk}, X_k, N_k)^2}
\]

\[
\times \mathbb{2} (N_l = N_k) K_g \left( B_{il} - B_{jk}, h \right) K_X \left( X_l - X_k \right) \frac{1}{N_k} \sum_{j'=1}^{N_k} \frac{1}{N_k} - 1 \text{ T}_{j'k} K_f \left( \frac{\hat{V}_{j'k} - v}{h}, \frac{X_{k'} - x}{h} \right)
\]

\[
\times \frac{\hat{G}(B_{jk'}, X_{k'}, N_{k'})}{g(B_{jk'}, X_{k'}, N_{k'})} \mathbb{2} (N_l = N_{k'}) K_g \left( B_{il} - B_{jk'}, h \right) K_X \left( X_l - X_{k'} \right) \frac{1}{h}.
\]
where the second inequality holds w.p.a.1. Let

\[ K ([b, z, m], (b', z', m'), (b'', z'', m''); v) \]

\[ := h^{-3(1+\delta)} \mathbb{I} (m = n) \frac{1}{m} \sum_{i=1}^{m} \frac{1}{m'} \sum_{j=1}^{m'} \frac{1}{m'' - 1} \mathbb{I} ([\xi (b', z', m') - v] \leq 2h) \left| K_{[b, z, m]}^{0}(\frac{z' - x}{h}) \right| \]

\[ \times \mathbb{I} (m = m') \left| K_{b, z, m}^{0}(\frac{b - b'}{h}) \right| \left| K_{x, -x, h}^{0}(\frac{x - x'}{h}) \right| \sum_{j=1}^{m''} \frac{1}{m'' - 1} \mathbb{I} ([\xi (b'', z'', m'') - v] \leq 2h) \right| \]
\[ \times \left| K_X \left( \frac{z'' - z}{h} \right) \right| | 1 (m = m') | \left| K \left( \frac{b_i - b_i'}{h} \right) \right| \left| K_X \left( \frac{z - z''}{h} \right) \right|. \]

(S.3.28) can be written as

\[ \left| \Delta_2 (v) \right| \leq \left\{ \max_{j', j''} \mathbb{T}_{j', j''} \left| \frac{G (B_{j'; k}, X_{k'}, N_{k'})}{g (B_{j'; k}, X_{k}, N_{k})^2} - \frac{G (B_{j''; k}, X_{k'}, N_{k'})}{g (B_{j''; k}, X_{k}, N_{k})^2} \right| \right\} \times \left\{ \frac{1}{(L)_{3}} \sum_{(3)} K \left( \left| (B_{1}, X_{1}, N_{1}) \right|, (B_{k}, X_{k}, N_{k}), (B_{k'}, X_{k'}, N_{k'}) \right); v) \right\}. \]

It can be verified that \( \{ K (\cdot, \cdot, v); v \in I (\{z\}) \} \) is uniformly VC-type with respect to the envelope

\[ F_K (z, z', z'') := \frac{C_{K}^2}{n (n - 1)^{2}} \left| K_X \left( \frac{z' - z}{h} \right) \right| \left| K \left( \frac{z - z''}{h} \right) \right| \left| K_{X} \left( \frac{z'' - x}{h} \right) \right|. \]  

(S.3.29)

Define

\[ \mu_K (v) := E \left[ \frac{K ((B_{1}, X_{1}), (B_{2}, X_{2}), (B_{3}, X_{3}), (B_{4}, X_{4}))}{v} \right], \]  

(S.3.30)

\[ K_{K_{1}} (b, z, m; v) := E \left[ \frac{K ((b, z, m); (B_{1}, X_{1}), (B_{2}, X_{2}), (B_{3}, X_{3}, v))}{v} \right], \]  

\[ K_{K_{2}} (b, z, m; v) := E \left[ \frac{K ((b, z, m); (B_{1}, X_{1}), (B_{2}, X_{2}), (B_{3}, X_{3}, v))}{v} \right], \]  

\[ K_{K_{3}} (b, z, m; v) := E \left[ \frac{K ((b, z, m); (B_{1}, X_{1}), (B_{2}, X_{2}), (B_{3}, X_{3}, v))}{v} \right], \]  

(S.3.31)

and

\[ K_{K_{1}} (b, z, m; b', z', m'; v) := E \left[ \frac{K ((b, z, m); (b', z', m'); (B_{1}, X_{1}, v))}{v} \right], \]  

\[ K_{K_{2}} (b, z, m; b', z', m'; v) := E \left[ \frac{K ((b, z, m); (b', z', m'); (B_{1}, X_{1}, v))}{v} \right], \]  

\[ K_{K_{3}} (b, z, m; b', z', m'; v) := E \left[ \frac{K ((b, z, m); (b', z', m'); (B_{1}, X_{1}, v))}{v} \right]. \]  

(S.3.32)

The Hoeffding decomposition yields

\[ \frac{1}{(L)_{3}} \sum_{(3)} K \left( (B_{1}, X_{1}, N_{1}), (B_{k}, X_{k}, N_{k}), (B_{k'}, X_{k'}, N_{k'}) ; v \right) \]

\[ = \mu_K (v) + \frac{1}{L} \sum_{l=1}^{L} \left( K_{K_{1}} (B_{1}, X_{1}, N_{l}; v) - \mu_K (v) \right) + \frac{1}{L} \sum_{l=1}^{L} \left( K_{K_{2}} (B_{1}, X_{l}, N_{l}; v) - \mu_K (v) \right) + \frac{1}{L} \sum_{l=1}^{L} \left( K_{K_{3}} (B_{1}, X_{l}, N_{l}; v) - \mu_K (v) \right) \]

\[ + \Upsilon_{K} (v) + \Psi_{K} (v), \]

where \( \Upsilon_{K} (v) \), \( \Upsilon_{K} (v) \) and \( \Upsilon_{K} (v) \) are degenerate U-statistics of order two and \( \Psi_{K} (v) \) is a degenerate U-statistic of order three:

\[ \Upsilon_{K} (v) := \frac{1}{(L)_{2}} \sum_{(2)} \left\{ K_{K_{1}} (B_{1}, X_{1}, N_{l}; v) - K_{K_{1}} (B_{1}, X_{l}, N_{l}; v) - K_{K_{1}} (B_{1}, X_{l}, N_{l}; v) + \mu_K (v) \right\}, \]

(S.3.33)

\[ \Upsilon_{K} (v) := \frac{1}{(L)_{2}} \sum_{(2)} \left\{ K_{K_{2}} (B_{1}, X_{1}, N_{l}; v) - K_{K_{1}} (B_{1}, X_{l}, N_{l}; v) - K_{K_{1}} (B_{k}, X_{k}, N_{k}; v) + \mu_K (v) \right\}, \]

\[ \Upsilon_{K} (v) := \frac{1}{(L)_{2}} \sum_{(2)} \left\{ K_{K_{3}} (B_{1}, X_{1}, N_{l}; v) - K_{K_{1}} (B_{1}, X_{l}, N_{l}; v) - K_{K_{1}} (B_{k}, X_{k}, N_{k}; v) + \mu_K (v) \right\}, \]

\[ \Psi_{K} (v) := \frac{1}{(L)_{3}} \sum_{(3)} K \left( (B_{1}, X_{1}, N_{1}), (B_{k}, X_{k}, N_{k}), (B_{k'}, X_{k'}, N_{k'}) ; v \right) \]

\[ - K_{K_{1}} ((B_{1}, X_{1}, N_{l}), (B_{k}, X_{k}, N_{k}), (B_{k'}, X_{k'}, N_{k'}) ; v)) - K_{K_{2}} ((B_{1}, X_{l}, N_{l}), (B_{k}, X_{k}, N_{k}) ; v)) \]

\[ - K_{K_{3}} ((B_{1}, X_{l}, N_{l}), (B_{k}, X_{k}, N_{k}) ; v)) \]

\[ S.35 \]
Then, by change of variables, we have

\[\mu_K(v) = \int_{\mathbb{R}} \sum_{m \in \mathbb{N}} \int_{\mathbb{R}} \sum_{m' \in \mathbb{N}} \int_{\mathbb{R}} \sum_{m'' \in \mathbb{N}} \int_{\mathbb{R}} \frac{1}{h^{3(1+d)}} \left(\begin{array}{c}
m = n \\frac{1}{m} \frac{1}{m' - 1}
\end{array}\right) \times \left(\begin{array}{c}
\left| (\xi (b', z', m') - v) \right| \leq 2h
\end{array}\right) \left| K_X \left(\frac{z' - x}{h}\right) \right| \left| K_g \left(\frac{b - b'}{h}\right) \right| \times \left(\begin{array}{c}
\left| K_X \left(\frac{z'' - z}{h}\right) \right| g(b, z, m) g(b', z', m') g(b'', z'', m'') db db' db'' dz dz' dz''
\end{array}\right) \right)
\]

Then, by change of variables, we have

\[\mu_K(v) \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{h^{3(1+d)}} \left(\begin{array}{c}
\left| (\xi (b', z') - v) \right| \leq 2h
\end{array}\right) \left| K_X \left(\frac{z' - x}{h}\right) \right| \left| K_g \left(\frac{b - b'}{h}\right) \right| \times \left(\begin{array}{c}
\left| K_X \left(\frac{z'' - z}{h}\right) \right| g(b, z, m) \left(\begin{array}{c}
g(b', z', m') g(b'', z'', m'') db db' db'' dz dz' dz''
\end{array}\right)
\end{array}\right) \right)
\]

It is now clear that

\[\mu_K(v) \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{h^{3(1+d)}} \left(\begin{array}{c}
\left| (\xi (b', z') - v) \right| \leq 2h
\end{array}\right) \left| K_X \left(\frac{z' - x}{h}\right) \right| \left| K_g \left(\frac{b - b'}{h}\right) \right| \times \left(\begin{array}{c}
\left| K_X \left(\frac{z'' - z}{h}\right) \right| g(b, z, m) \left(\begin{array}{c}
g(b', z', m') g(b'', z'', m'') db db' db'' dz dz' dz''
\end{array}\right)
\end{array}\right) \right)
\]

when \(h\) is sufficiently small. Now it is clear that \(\sup_{v \in (\mathbb{R})} \mu_K(v) = O(1)\).

By the LIE, we have

\[K_1^{(1)} (b, z, m; v) = \frac{1}{m!} \frac{1}{m'} \frac{1}{m''} \sum_{m, m', m'' \in \mathbb{N}} \left(\begin{array}{c}
(m = n) \\frac{1}{m} \frac{1}{m' - 1}
\end{array}\right) \times \left(\begin{array}{c}
\left| (\xi (b', z', m') - v) \right| \leq 2h
\end{array}\right) \left| K_X \left(\frac{z' - x}{h}\right) \right| \left| K_g \left(\frac{b - b'}{h}\right) \right| \times \left(\begin{array}{c}
\left| K_X \left(\frac{z'' - z}{h}\right) \right| g(b', z', m') g(b'', z'', m'') db db' db'' dz dz' dz''
\end{array}\right) \right)
\]

and

\[K_2^{(1)} (b, z, m; v) = \frac{1}{n!} \frac{1}{n'} \frac{1}{n''} \sum_{n, n', n'' \in \mathbb{N}} \left(\begin{array}{c}
(n = n) \\frac{1}{n} \frac{1}{n' - 1}
\end{array}\right) \times \left(\begin{array}{c}
\left| (\xi (b', z', m') - v) \right| \leq 2h
\end{array}\right) \left| K_X \left(\frac{z' - x}{h}\right) \right| \left| K_g \left(\frac{b - b'}{h}\right) \right| \times \left(\begin{array}{c}
\left| K_X \left(\frac{z'' - z}{h}\right) \right| g(b', z', m') g(b'', z'', m'') db db' db'' dz dz' dz''
\end{array}\right) \right)
\]

\(S.36\)
By Jensen’s inequality, the LIE and change of variables, we have

\[ E \left[ K_1^{(1)} (B_1, X_1, N_1; v) \right] \]

\[ \leq E \left[ \frac{1}{N_1^2 (N_1 - 1)^{3}} \sum_{N_1 - 1} \int_{\mathcal{X}} \int_{\mathcal{X}} \int_{\mathcal{X}} \int_{\mathcal{Z}} \frac{1}{h^{3(1+d)}} \mathbb{1} \left( \| (B', z', N_1) - v \| \leq 2h \right) \right] \]

\[ \times K_2 \left( \left( k' \left( \frac{B - B'}{h} \right) \right) \left| K_2 \left( \frac{X - X_1}{h} \right) \right| \left| \xi \left( \left( b'' + y'' + x, n \right), \left( b'' + x, n \right) \right) \right| \right] \]

\[ \times g \left( b', z', N_1 \right) g \left( b'', z'', N_1 \right) \mathbb{d}b' \mathbb{d}z' \mathbb{d}b'' \mathbb{d}z'' \]

and

\[ E \left[ K_2^{(1)} (B_1, X_1, N_1; v) \right] \]

\[ \leq E \left[ \frac{1}{N_1^2 (N_1 - 1)^{3}} \sum_{N_1 - 1} \int_{\mathcal{X}} \int_{\mathcal{X}} \int_{\mathcal{X}} \int_{\mathcal{Z}} \frac{1}{h^{3(1+d)}} \mathbb{1} \left( \| (B_1, X_1, N_1) - v \| \leq 2h \right) \right] \]

\[ \times K_2 \left( \left( k' \left( \frac{B_1 - B'}{h} \right) \right) \left| K_2 \left( \frac{X_1 - X_1}{h} \right) \right| \left| \xi \left( \left( b'' + y'' + x, n \right), \left( b'' + x, n \right) \right) \right| \right] \]

\[ \times g \left( b', z', N_1 \right) g \left( b'', z'', N_1 \right) \mathbb{d}b' \mathbb{d}z' \mathbb{d}b'' \mathbb{d}z'' \]

Now it is clear that

\[ \sigma_{K_1}^2 \quad := \quad \sup_{v \in \mathcal{I}(x)} E \left[ K_1^{(1)} (B_1, X_1, N_1; v) \right] \quad = \quad O \left( h^{-(1+d)} \right). \]  

(S.335)

Since \{ \mathcal{K} (\cdot ; \cdot ; v) : v \in \mathcal{I}(x) \} is (uniformly) VC-type with respect to the envelope (S.329), it follows from Chen and Kato (2017, Lemma 5.4) that the class \{ K_1^{(1)} (\cdot ; v) : v \in \mathcal{I}(x) \} is VC-type with respect to the envelope

\[ F_{K_1^{(1)}} (z) \quad := \quad \overline{C}_{K_2} \frac{1}{n (n - 1)^d} \int_{\mathcal{X}} \left[ K_2 \left( \frac{z - \bar{x}}{h} \right) \right] \left| \mathbb{E} \left[ K_2 \left( \frac{z - \bar{x}}{h} \right) \right] \right| \left( \mathbb{d}z' \right)^2 \]

\[ \leq h^{-(3+d)} \int_{\mathcal{X}} \left[ K_2 \left( \frac{z - \bar{x}}{h} \right) \right] \left( \mathbb{d}z' \right)^2 \]

\[ \quad = \quad O \left( h^{-(3+d)} \right). \]
Now applying the CCK inequality with $\sigma = \sigma_{K_1}$ and $F = F_{K_1}$, we have

$$E \left[ \sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l=1}^{L} K_{1}^{(1)} (B_1, X_1, N_l; v) - \mu_K (v) \right| \right] \leq C_1 \left\{ L^{-1/2} \sigma_{K_1} \log (C_2 L)^{1/2} + L^{-1} \left\| F_{K_1} \right\| \right\| \log (C_2 L) \right\}$$

$$= O \left( \frac{\log (L)}{L h^{3+d}} \right)^{1/2} + \frac{\log (L)}{L h^{3+d}} .$$

(S.3.36)

Similarly,

$$E \left[ \sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l=1}^{L} K_{2}^{(1)} (B_1, X_1, N_l; v) - \mu_K (v) \right| \right] = O \left( \frac{\log (L)}{L h^{3+d}} \right)^{1/2} + \frac{\log (L)}{L h^{3+d}} .$$

(S.3.37)

follows from

$$\sup_{v \in I(x)} E \left[ K_{2}^{(1)} (B_1, X_1, N_l; v)^2 \right] = O \left( h^{-(1+d)} \right).$$

We also have

$$E \left[ \sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l=1}^{L} K_{3}^{(1)} (B_1, X_1, N_l; v) - \mu_K (v) \right| \right] = O \left( \frac{\log (L)}{L h^{1+d}} \right)^{1/2} + \frac{\log (L)}{L h^{1+d}} .$$

(S.3.38)

since $K$ is symmetric with respect to the second and the third arguments.

The CK inequality yields

$$E \left[ \sup_{v \in I(x)} \left| \Psi_{K_1} (v) \right| \right] \leq L^{-1} \left\{ \int_{X} \int_{X} \left( \int_{X} \frac{1}{h^{3+d}} K_{0}^{(1)} \left( \frac{z'-z}{h} \right) \right) \left| K_{X} \left( \frac{z-z'}{h} \right) \right| \left| K_{0}^{(2)} \left( \frac{z''-z}{h} \right) \right| \left| K_{X} \left( \frac{z-z''}{h} \right) \right| \varphi (z') dz' \right)^{2}$$

$$\times \varphi (z) \varphi (z'') dz dz' dz''$$

$$= \left( \frac{\log (L)}{L h^{3+d}} \right)^{-1} \left\{ \int_{Y} \int_{Y} \left( \int_{Y} K_{0}^{(2)} (y') \right) \left| K_{X} (y-y') \right| \left| K_{0}^{(2)} (y'') \right| \left| K_{X} (y-y'') \right| \left| K_{X} (y-y''') \right| \varphi (hy+y) dy'' \right)^{2}$$

$$\times \varphi (hy+x) \varphi (hy''+x) dy dy'' \right)^{1/2}$$

$$= O \left( \left( \frac{\log (L)}{L h^{3+d}} \right)^{-1} \right),$$

(S.3.39)

and

$$E \left[ \sup_{v \in I(x)} \left| \Psi_{K_1} (v) \right| \right] \leq L^{-3/2} \left\{ \int_{X} \int_{X} \left( \int_{X} \frac{1}{h^{3/2}} K_{0}^{(1)} \left( \frac{z'-z}{h} \right)^2 \right) K_{X} \left( \frac{z-z'}{h} \right)^2 \left| K_{0}^{(2)} \left( \frac{z''-z}{h} \right)^2 \right| \left| K_{X} \left( \frac{z-z''}{h} \right)^2 \right| \varphi (z) \right)^{2}$$

$$\times \varphi (z') \varphi (z'') dz dz' dz''$$

$$= L^{-3/2} \left( \frac{\log (L)}{L h^{3/2}} \right)^{-1} \left\{ \int_{Y} \int_{Y} K_{0}^{(2)} (y')^2 K_{X} (y-y')^2 K_{0}^{(2)} (y'')^2 K_{X} (y-y'')^2 \varphi (hy+x) \right)^{2}$$

$$\times \varphi (hy+x) \varphi (hy''+x) dy dy'' \right)^{1/2}$$

$$= O \left( \left( \frac{\log (L)}{L h^{3/2}} \right)^{-1} \right).$$

(S.3.41)
Since $\mathcal{K}$ is symmetric with respect to the second and the third arguments, we have

$$
E \left[ \sup_{v \in I(\mathbf{z})} |T_{\mathcal{K}}^2(v)| \right] = O \left( \left( Lh^{3+d} \right)^{-1} \right),
$$

(S.3.42)

Write

$$
\Delta_1^i(v) = \Delta_1^i(v) + 2\Delta_2^i(v) + \Delta_3^i(v),
$$

where

$$
\Delta_1^i(v) := \frac{1}{h^{3+1-d}(L)^3} \sum_{i=1}^{(3)} \left( N_i = n \right) N_i^2 \sum_{i=1}^{N_i} \left( \mathbb{1} \right) K_x \left( \frac{V_{jk} - v}{h}, \frac{X_k - x}{h} \right) G(B_{jk}, \mathbf{X}_k, N_k) \left( \frac{V_{jk} - v}{h}, \frac{X_k - x}{h} \right)
\times \left( \frac{B_{il} - B_{jk}}{h} \right) K_x \left( \frac{X_l - X_k}{h} \right) \left( \frac{1}{N_k} \sum_{j'=1}^{N_k} \left( \frac{1}{N_{k'}} \sum_{j''=1}^{N_{k'}} T_{jk} \right) K_x \left( \frac{V_{jk} - v}{h}, \frac{X_k - x}{h} \right) \left( \frac{B_{il} - B_{jk'}}{h} \right) K_x \left( \frac{X_l - X_{k'}}{h} \right) \right)
\times \left( \frac{g(B_{jk}, \mathbf{X}_k, N_k)}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) \left( \frac{N_i = N_{k'}}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) K_x \left( \frac{X_l - X_{k'}}{h} \right)
\times \left( \frac{g(B_{jk}, \mathbf{X}_k, N_k)}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) \left( \frac{N_i = N_{k'}}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) K_x \left( \frac{X_l - X_{k'}}{h} \right)
\times \left( \frac{g(B_{jk}, \mathbf{X}_k, N_k)}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) \left( \frac{N_i = N_{k'}}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) K_x \left( \frac{X_l - X_{k'}}{h} \right)
$$

and

$$
\Delta_3^i(v) := \frac{1}{h^{3+1-d}(L)^3} \sum_{i=1}^{(3)} \left( N_i = n \right) N_i^2 \sum_{i=1}^{N_i} \left( \mathbb{1} \right) K_x \left( \frac{V_{jk} - v}{h}, \frac{X_k - x}{h} \right) G(B_{jk}, \mathbf{X}_k, N_k) \left( \frac{V_{jk} - v}{h}, \frac{X_k - x}{h} \right)
\times \left( \frac{B_{il} - B_{jk}}{h} \right) K_x \left( \frac{X_l - X_k}{h} \right) \left( \frac{1}{N_k} \sum_{j'=1}^{N_k} \left( \frac{1}{N_{k'}} \sum_{j''=1}^{N_{k'}} T_{jk} \right) K_x \left( \frac{V_{jk} - v}{h}, \frac{X_k - x}{h} \right) \left( \frac{B_{il} - B_{jk'}}{h} \right) K_x \left( \frac{X_l - X_{k'}}{h} \right) \right)
\times \left( \frac{g(B_{jk}, \mathbf{X}_k, N_k)}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) \left( \frac{N_i = N_{k'}}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) K_x \left( \frac{X_l - X_{k'}}{h} \right)
\times \left( \frac{g(B_{jk}, \mathbf{X}_k, N_k)}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) \left( \frac{N_i = N_{k'}}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) K_x \left( \frac{X_l - X_{k'}}{h} \right)
\times \left( \frac{g(B_{jk}, \mathbf{X}_k, N_k)}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) \left( \frac{N_i = N_{k'}}{g(B_{jk}, \mathbf{X}_k, N_k)^2} \right) K_x \left( \frac{X_l - X_{k'}}{h} \right)
$$

where $\hat{V}_{jk}$ ($\hat{V}_{jk'}$) is the mean value that lies between $V_{jk}$ ($V_{jk'}$) and $\tilde{V}_{jk}$ ($\tilde{V}_{jk'}$).

Now by the fact $\max_{i,j} \mathbb{E}[|\tilde{V}_{ij} - V_{ij}|] = o_p(h)$ and the fact that $K_0'$ and $K_0''$ are both compactly supported on $[-1, 1]$, we have

$$
|\Delta_1^i(v)| \lesssim h^{-1} \left\{ \max_{j,k} T_{jk} \left| \hat{V}_{jk} - V_{jk} \right| \right\} \left\{ \frac{1}{(L)^3} \sum_{i=1}^{(3)} \mathbb{K}(B_{i,j}, X_l, N_l), (B_{k,l}, X_k, N_k), (B_{k',l'}, X_k', N_k') ; v) \right\} = O_p \left( \left( \frac{\log(L)}{Lh^{3+\frac{d}{2}}} \right)^{1/2} + h^R \right),
$$

where equality holds w.p.a.1. and the equality is uniform in $v \in I(\mathbf{z})$ and also

$$
|\Delta_3^i(v)| \lesssim h^{-2} \left\{ \max_{j,k} T_{jk} \left| \tilde{V}_{jk} - V_{jk} \right| \right\} \left\{ \frac{1}{(L)^3} \sum_{i=1}^{(3)} \mathbb{K}(B_{i,j}, X_l, N_l), (B_{k,l}, X_k, N_k), (B_{k',l'}, X_k', N_k') ; v) \right\} = O_p \left( \frac{\log(L)}{Lh^{3+\frac{d}{2}}} + h^R \right)
$$

where the equality holds w.p.a.1 and the equality is uniform in $v \in I(\mathbf{z})$.

Since $K_0'$ is compactly supported on $[-1, 1]$, the trimming is asymptotically negligible:

$$
\Delta_1^i(v) = \frac{1}{(L)^3} \sum_{i=1}^{(3)} \mathbb{K}(B_{i,j}, X_l, N_l), (B_{k,l}, X_k, N_k), (B_{k',l'}, X_k', N_k') ; v), \text{ for all } v \in I(\mathbf{z}), \text{ w.p.a.1},
$$

S.39
where

\[ \mathcal{H}((b, z, m), (b', z', m'); v) \]

\[ := h^{-3(1+d)} \| (m = n) \sum_{i=1}^{m} \frac{1}{m'} \sum_{j=1}^{m'} \frac{1}{m''-1} K_j^\prime \left( \frac{\xi (b'_j, z', m') - v, z' - x}{h} \right) \]

\[ \times g \left( \frac{b'_j, z', m'}{h} \right) 1 (m = m') K_g \left( \frac{b_i - b'_j}{h} \right) K_X \left( \frac{z - z'}{h} \right) \]

\[ \times K_j^\prime \left( \frac{\xi (b'_j, z', m'') - v, z'' - x}{h} \right) \]

\[ g \left( \frac{b'_j, z', m''}{h} \right) 1 (m = m'') K_g \left( \frac{b_i - b'_j}{h} \right) K_X \left( \frac{z - z''}{h} \right) . \]

The Hoeffding decomposition yields

\[ \frac{1}{(L_h)^{\ell}} \sum_{i=1}^{\ell} \mathcal{H}((B_i, X_i, N_i), (B_k, X_k, N_k), (B_{k'}, X_{k'}, N_{k'}); v) \]

\[ = \mu_{\mathcal{H}} (v) + \frac{1}{L} \sum_{i=1}^{L} \left( H_{11} (B_i, X_i, N_i; v) - \mu_{\mathcal{K}} (v) \right) + \frac{1}{L} \sum_{i=1}^{L} \left( H_{12} (B_i, X_i, N_i; v) - \mu_{\mathcal{K}} (v) \right) \]

\[ + \frac{1}{L} \sum_{i=1}^{L} \left( H_{3} (B_i, X_i, N_i; v) - \mu_{\mathcal{K}} (v) \right) + T_{\mathcal{H}} (v) + T_{\mathcal{K}} (v) + \Psi_{\mathcal{H}} (v), \]

where the terms in the decomposition are defined by (S.3.30) to (S.3.34) with \( \mathcal{K} \) replaced by \( \mathcal{H} \). By the LIE, we can easily check that

\[ \mu_{\mathcal{H}} (v) = V_{\mathcal{M}} (v|x, n), \]

\[ H_{11}^{(1)} (b, z, m; v) := \frac{1}{(n - 1)^2} \frac{1}{h^{1(1+d)}} \| (m = n) \frac{1}{m^2} \sum_{i=1}^{m} \left( \int_{X} \int_{(z')^2} K_j^\prime \left( \frac{\xi (b'_j, z', m) - v, z' - x}{h} \right) \right) \]

\[ g \left( \frac{b'_j, z', m}{h} \right) K_X \left( \frac{z' - z}{h} \right) dB' dz' \]

and

\[ H_{12}^{(1)} (b, z, m; v) := \frac{1}{m} \sum_{i=1}^{m} \frac{1}{h^{1(1+d)}} \frac{1}{m - 1} K_j^\prime \left( \frac{\xi (b_i, z, m) - v, z - x}{h} \right) g \left( \xi (b'_j, z', m) \right) \int_{X} \int_{(z')^2} K_j^\prime \left( \frac{\xi (b'_j, z', m) - v, z' - x}{h} \right) \]

\[ g \left( \frac{b'_j, z', m}{h} \right) K_X \left( \frac{z' - z}{h} \right) \times \frac{1}{n - 1} K_j^\prime \left( \frac{\xi (b_i, z, m) - v, z - x}{h} \right) . \]

We have \( H_3^{(1)} = H_2^{(1)} \) since \( \mathcal{H} \) is symmetric with respect to the second and the third arguments.

Then it can be easily verified that

\[ \sup_{v \in \mathcal{I}(x)} E \left[ H_{11}^{(1)} (B_1, X_1, N_1; v)^2 \right] = O \left( h^{-(1+d)} \right) \]

and

\[ \sup_{v \in \mathcal{I}(x)} E \left[ H_{12}^{(1)} (B_1, X_1, N_1; v)^2 \right] = O \left( h^{-(1+d)} \right) . \]

The bounds in (S.3.36) to (S.3.42) also hold if \( \mathcal{K} \) is replaced by \( \mathcal{H} \).

Now we have shown that

\[ \sup_{v \in \mathcal{I}(x)} \left( \hat{\mathcal{R}} (n|x)^2 \mathcal{R} (x)^2 \mathcal{V} GPV (v|x, n) - V_{\mathcal{M}} (v|x, n) \right) = O_p \left( \left( \frac{\log (L)}{Lh^{3+d}} \right)^{1/2} + h^R \right) . \]
The first conclusion of the theorem follows from the above fact,
\[
\left| \hat{V}_{GPV}(v|x, n) - (\pi(n|x)\varphi(x))^{-2} V_M(v|x, n) \right| \leq \left| \frac{1}{\pi(n|x)^2 \hat{\varphi}(x)^2} - \frac{1}{\pi(n|x)^2 \varphi(x)^2} \right| \left| \hat{\pi}(n|x) \hat{\varphi}(x)^2 \hat{V}_{GPV}(v|x, n) \right| + \left| \hat{\pi}(n|x)^2 \hat{\varphi}(x)^2 \hat{V}_{GPV}(v|x, n) - V_M(v|x, n) \right| \nonumber
\]
and the fact
\[
\hat{\pi}(n|x) - \pi(n|x) = O_p \left( \left( \frac{\log(L)}{Lh^d} \right)^{1/2} + h^{1+R} \right) \quad \text{and} \quad \hat{\varphi}(x) - \varphi(x) = O_p \left( \left( \frac{\log(L)}{Lh^d} \right)^{1/2} + h^{1+R} \right). \quad (S.3.43)
\]

The second conclusion of the theorem follows similarly.

**Proof of Theorem 6.3.**

Note that by Marmer and Shneyerov (2012, Lemma S.1) that
\[
\sum_{n \in N} \hat{f}_{GPV}(v, x, n) - \sum_{n \in N} \tilde{f}_{GPV}(v, x, n) = \left( \sum_{n \in N} \hat{f}_{GPV}(v, x, n) - \sum_{n \in N} \tilde{f}_{GPV}(v, x, n) \right) + \left( \sum_{n \in N} \tilde{f}_{GPV}(v, x, n) - \sum_{n \in N} \tilde{f}(v, x, n) \right) - \left( \sum_{n \in N} \tilde{f}_{GPV}(v, x, n) - \sum_{n \in N} \tilde{f}(v, x, n) \right). \quad (S.3.44)
\]

It follows from this result, (S.3.44), Lemmas S.3.5 and S.3.9 and Marmer and Shneyerov (2012, Lemma S.1) that
\[
\sum_{n \in N} \hat{f}_{GPV}(v, x, n) - \sum_{n \in N} \tilde{f}_{GPV}(v, x, n) = \frac{1}{L^2} \sum_{l=1}^{L} \sum_{k=1}^{L} M^n(\{B_l, X_l, N_l\}, \{B_k, X_k, N_k\}; v) - \frac{1}{L} \sum_{k=1}^{L} M^2(\{B_k, X_k, N_k\}; v) + O_p \left( \left( \frac{\log(L)}{Lh^{1+d}} \right)^{1/2} + \frac{\log(L)}{Lh^{1+d}} + h^R \right).
\]

It is also shown in the proof of Lemma S.3.10 that
\[
\sup_{v \in I(x)} \left| \frac{1}{L^2} \sum_{l=1}^{L} \sum_{k=1}^{L} \left( M^n(\{B_l, X_l, N_l\}; v) - \frac{1}{L} \sum_{k=1}^{L} M^2(\{B_k, X_k, N_k\}; v) \right) \right| = O_p \left( \left( \frac{\log(L)}{Lh^{1+d}} \right)^{1/2} \right).
\]

Note that by Marmer and Shneyerov (2012, Lemmas S.1 and S.4), we have
\[
\tilde{\varphi}^+(x) - \varphi(x) = O_p \left( \left( \frac{\log(L)}{Lh^d} \right)^{1/2} + h^{1+R} \right).
\]

Now using these results, (S.1.76), (S.3.43) and the fact
\[
\sum_{n \in N} \hat{f}_{GPV}(v, x, n) \rightarrow_p f(v|x) \varphi(x), \quad \text{uniformly in } v \in I(x),
\]
we can obtain
\[
\hat{f}_{GPV}(v|x) - \tilde{f}_{GPV}(v|x) = \frac{1}{\tilde{\varphi}^+(x)} \sum_{n \in N} \hat{f}_{GPV}(v, x, n) - \frac{1}{\varphi(x)} \sum_{n \in N} \tilde{f}_{GPV}(v, x, n)
\]
\[
= \frac{1}{L} \sum_{l=1}^{L} \frac{1}{\varphi(x)} \sum_{n \in N} \left( M^n(\{B_l, X_l, N_l\}; v) - \frac{1}{L} \sum_{k=1}^{L} M^2(\{B_k, X_k, N_k\}; v) \right)
\]
\[
= \frac{1}{L} \sum_{l=1}^{L} \frac{1}{\varphi(x)} \sum_{n \in N} \left( M^n(\{B_l, X_l, N_l\}; v) - \frac{1}{L} \sum_{k=1}^{L} M^2(\{B_k, X_k, N_k\}; v) \right)
\]
\[
S.41
Similarly, we have
\[ \hat{f}_{GPV}(v|x) - f(v|x) = \frac{1}{L} \sum_{i=1}^{L} \frac{1}{\varphi(x)} \sum_{n \in N} \{ M_{2}^{n}(B_i, X_i; N_i; v) - \mu_{M^{n}}(v) \} + o_{p}\left( \frac{\log(L)}{(Lh)^{3/4}} + \frac{\log(L)}{Lh^{3/4}} + h^{R} \right). \]

Now it follows that
\[ \left( Lh^{3/4} \right)^{1/2} \left( \hat{f}_{GPV}(v|x) - f(v|x) \right) = \frac{1}{L^{1/2}} \sum_{i=1}^{L} \frac{1}{\varphi(x)} \sum_{n \in N} h^{(3/4)d/2} \left\{ M_{2}^{n}(B_i, X_i; N_i; v) - \frac{1}{L} \sum_{k=1}^{L} M_{2}^{n}(B_k, X_k, N_k; v) \right\} + o_{p}(1), \]
where the leading term converges in distribution to \( N(0, V_{GPV}(v|x)) \) and
\[ \left( Lh^{3/4} \right)^{1/2} \left( f_{GPV}(v|x) - \hat{f}_{GPV}(v|x) \right) = \frac{1}{L^{1/2}} \sum_{i=1}^{L} \frac{1}{\varphi(x)} \sum_{n \in N} h^{(3/4)d/2} \left\{ M_{2}^{n}(B_i, X_i; N_i; v) - \frac{1}{L} \sum_{k=1}^{L} M_{2}^{n}(B_k, X_k, N_k; v) \right\} + o_{p}(1), \]
where the leading term is the bootstrap analogue of the leading term on the right hand side of (S.3.45). The desired result follows from this observation and the arguments used in the proof of Theorem 4.2.

**Proof of Theorem 6.4.** It is shown in the proof of Theorem 6.1 that when \( h \) is sufficiently small,
\[ \inf_{v \in I(x)} \text{Var} \left[ L^{(3/4)d/2} \sum_{n \in N} M_{2}^{n}(B_1, X_1; N_1; v) \right] > C_1 > 0. \] (S.3.46)

Since it is argued in the proof of Lemma S.3.4 that \( \{ M_{2}^{1, n}(.; v) : v \in I(x) \} \) is uniformly VC-type with respect to the envelope (S.3.47) which satisfies \( \| F_{M_{2}^{1, n}} \|_{X} = O(h^{-3/4+d}) \), it follows from Nolan and Pollard (1987, Lemma 16) and (S.3.46) that the class
\[ \left\{ \frac{h^{(3/4)d/2} \sum_{n \in N} \left( M_{2}^{1, n}(.; v) - \mu_{M^{1, n}}(v) \right)}{\text{Var} \left[ h^{(3/4)d/2} \sum_{n \in N} M_{2}^{1, n}(B_1, X_1; N_1; v) \right]^{1/2}} : v \in I(x) \right\}, \]
is uniformly VC-type with respect to an envelope that is a multiple of \( h^{(3/4)d/2} \sum_{n \in N} F_{M_{2}^{1, n}} \), which satisfies
\[ \left\| h^{(3/4)d/2} \sum_{n \in N} F_{M_{2}^{1, n}} \right\|_{X} = O(h^{-3/4+d}), \]
when \( h \) is sufficiently small.

By the arguments used in the proof of Theorem 5.1, we can show the existence of a tight Gaussian random element in \( \ell^{\infty}(I(x)) \), denoted by \( \{ \Gamma(v|x) : v \in I(x) \} \) that has the same covariance structure as the empirical process \( \{ \Gamma(v|x) : v \in I(x) \} \). Application of Chernozhukov et al. (2014b, Corollary 2.2) with \( q = \infty, b \leq h^{-3/4+d/2}, \gamma = \log(L)^{-1} \) and \( \sigma = 1 \) yields the existence of a coupling \( W_{L} \) with \( W_{L} \overset{d}{=} \| \Gamma(v|x) \|_{I(x)} \) satisfying
\[ \| \Gamma(v|x) \|_{I(x)} - W_{L} = O_{p}\left( \frac{\log(L)}{(Lh)^{3/4}} \right). \]
This result and Lemma S.3.4 yield
\[ \| Z(v|x) \|_{I(x)} - W_{L} = O_{p}(\lambda_{L}), \] (S.3.47)
where
\[ \lambda_{L} := \log(L)^{1/2} h + \frac{\log(L)}{(Lh)^{3/4}} + L^{1/2} h^{(3/4+d)/2+R}. \]
Then by applying Lemma A.6, (S.3.47) and the Gaussian anti-concentration inequality of Chernozhukov et al. (2014a), we have

$$\sup_{z \in \mathbb{R}} \left| P \left[ \| Z (\cdot | x) \|_{I(x)} \leq z \right] - P \left[ \| I_G (\cdot | x) \|_{I(x)} \leq z \right] \right| = o(1).$$  \hspace{1cm} (S.3.48)

Application of Chernozhukov et al. (2016, Theorem 2.3) with $B(f) = 0$, $q = \infty$, $b \leq h^{-(3+d)/2}$, $\gamma = \log (L)^{-1}$ and $\sigma = 1$ yields that there exists a coupling $W^*_L$, with the property that the conditional distribution of $W^*_L$ given the original sample is the same as the marginal distribution of $\| I_G (\cdot | x) \|_{I(x)}$ almost surely, and

$$\left| \| I^* (\cdot | x) \|_{I(x)} - W^*_L \right| = O_p \left( \frac{\log (L)}{Lh^{3+d}} 1_{\gamma} \right).$$

This result, Lemma S.3.10 and Markov’s inequality yield

$$\left| \| Z^* (\cdot | x) \|_{I(x)} - W^*_L \right| = O_p (\lambda^*_L).$$

Next we apply the arguments used in the proof of Theorem 5.2. It follows from the above displayed result, the Gaussian anti-concentration inequality of Chernozhukov et al. (2014a) and Lemma A.8 that

$$\sup_{z \in \mathbb{R}} \left| P^* \left[ \| Z^* (\cdot | x) \|_{I(x)} \leq z \right] - P \left[ \| I_G (\cdot | x) \|_{I(x)} \leq z \right] \right| = o_p (1).$$  \hspace{1cm} (S.3.49)

The conclusion follows from (S.3.48), (S.3.49) and the arguments used in the proof of Corollary 5.1.

\section*{S.3.3 Lemmas}

\textbf{Lemma S.3.1.} Suppose that Assumptions 1 - 3 hold. Let $x$ be an interior point of $\mathcal{X}$ and $n \in \mathcal{N}$ be fixed. Let

$$\bar{\tau}_{il} := 1 \left( (V_{il}, X_i) \in \| v, x, \bar{d} \| \right).$$

Then

$$\tilde{f}_{GPV} (v, x, n) - f (v|x) \varphi (x) \pi (v|x) = \frac{1}{2} \sum_{l=1}^{L} 1 (N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \bar{\tau}_{il} \frac{1}{h^{3+d}} K^L_x \left( \frac{V_{il} - v}{h} - \frac{X_i - x}{h} \right) \left( \tilde{V}_{il} - V_{il} \right) + O_p \left( \frac{\log (L)}{Lh^{1+d}} 1/2 + \frac{\log (L)}{Lh^{3+d}} + h^R \right),$$

where the remainder term is uniform in $v \in I (x)$.

\textbf{Proof of Lemma S.3.1.} It is clear from the proof of Lemma B2 of GPV that

$$\max_{n^1 \in \mathcal{N}_{\| ((b', x'), h) \|} \subseteq \mathcal{N}'_{B, x}} \sup_{\xi (b^1, X^1, N_1) \subseteq \mathcal{S}'_{B, x}} \left| \tilde{G} (b', x', n') - G (b', x', n') \right| = O_p \left( \frac{\log (L)}{Lh^d} 1/2 + h^{1+R} \right),$$

$$\text{max}_{n^2 \in \mathcal{N}_{\| (b', x', h) \|} \subseteq \mathcal{N}'_{B, x}} \sup_{\xi (b^2, X^2, N_2) \subseteq \mathcal{S}'_{B, x}} \left| \tilde{g} (b', x', n') - g (b', x', n') \right| = O_p \left( \frac{\log (L)}{Lh^d} 1/2 + h^{1+R} \right).$$  \hspace{1cm} (S.3.50)

It follows from

$$V_{il} = \xi (B_{il}, X_i, N_i) := B_{il} + \frac{1}{N_i - 1} G (B_{il}, X_i, N_i)$$

$$\tilde{V}_{il} := \xi (\tilde{B}_{il}, X_i, N_i) = B_{il} + \frac{1}{N_i - 1} \tilde{G} (B_{il}, X_i, N_i)$$

and

$$\tilde{V}_{il} := \xi (\tilde{B}_{il}, X_i, N_i)$$

S.43
(S.1.76) that
\[
\tilde{V}_{il} - V_{il} = \frac{1}{N_l - 1} \left( \tilde{G}(B_{il}, X_{il}, N_l) - G(B_{il}, X_{il}, N_l) \right) - \frac{\tilde{G}(B_{il}, X_{il}, N_l) (\tilde{g}(B_{il}, X_{il}, N_l) - g(B_{il}, X_{il}, N_l))}{g(B_{il}, X_{il}, N_l)^2} + \frac{\tilde{G}(B_{il}, X_{il}, N_l) (\tilde{g}(B_{il}, X_{il}, N_l) - g(B_{il}, X_{il}, N_l))^2}{g(B_{il}, X_{il}, N_l)^4} \right).
\]

(S.3.51)

It then follows from the triangle inequality that
\[
\max_{i,l} T_{il} \left| \tilde{V}_{il} - V_{il} \right| \leq \max_{i,l} T_{il} \left| \tilde{G}(B_{il}, X_{il}, N_l) - G(B_{il}, X_{il}, N_l) \right| + \max_{i,l} T_{il} \left| \tilde{G}(B_{il}, X_{il}, N_l) (\tilde{g}(B_{il}, X_{il}, N_l) - g(B_{il}, X_{il}, N_l)) \right| + \max_{i,l} T_{il} \left| \frac{\tilde{G}(B_{il}, X_{il}, N_l) (\tilde{g}(B_{il}, X_{il}, N_l) - g(B_{il}, X_{il}, N_l))^2}{g(B_{il}, X_{il}, N_l)^4} \right|.
\]

(S.3.52)

Denote
\[
T_{il} := 1 \left( \mathbb{P}((B_{il}, X_{il}), h) \subseteq S_{B_{il}, X}^{N_l} \right).
\]

For the first term of the right hand side of (S.3.52), we have
\[
\max_{i,l} T_{il} \tilde{V}_{il} \leq \max_{i,l} T_{il} \left| \tilde{G}(B_{il}, X_{il}, N_l) - G(B_{il}, X_{il}, N_l) \right| + \max_{i,l} T_{il} \left| \tilde{G}(B_{il}, X_{il}, N_l) (\tilde{g}(B_{il}, X_{il}, N_l) - g(B_{il}, X_{il}, N_l)) \right| + \max_{i,l} T_{il} \left| \frac{\tilde{G}(B_{il}, X_{il}, N_l) (\tilde{g}(B_{il}, X_{il}, N_l) - g(B_{il}, X_{il}, N_l))^2}{g(B_{il}, X_{il}, N_l)^4} \right|.
\]

(S.3.53)

and by (S.3.2) and (S.3.50), we have
\[
\max_{i,l} T_{il} \tilde{V}_{il} \leq \max_{i,l} T_{il} \left| \tilde{G}(B_{il}, X_{il}, N_l) - G(B_{il}, X_{il}, N_l) \right| \leq O_p \left( \left( \frac{\log(L)}{Lh^d} \right)^{1/2} + h^{1+R} \right).
\]

For the second term of the right hand side of (S.3.53), we first note that \( \mathbb{H}((B_{il}, X_{il}), 2h) \subseteq \tilde{S}_{B_{il}, X}^{N_l} \) if and only if \( \mathbb{H}(X_{il}, 2h) \subseteq X \) and for all \( x' \in H(X_{il}, 2h) \), \( B_{il} + 2h \leq \tilde{b}(x', N_l) \) and \( B_{il} - 2h \geq \tilde{b}(x') \). Proposition 2 of GPV gives that
\[
\sup_{(x', n') \in X \times \mathcal{N}} \left| \tilde{b}(x', n') - \tilde{b}(x', n) \right| = o_p(h) \quad \text{and} \quad \sup_{x' \in X} \left| \tilde{b}(x') - b(x') \right| = o_p(h).
\]

(S.3.54)

For any \((i, l)\), if \( \mathbb{H}((B_{il}, X_{il}), 2h) \subseteq \tilde{S}_{B_{il}, X}^{N_l} \), we have for all \( x' \in H(X_{il}, h) \),
\[
B_{il} + h \leq \tilde{b}(x', N_l) + \left( \sup_{(z, n') \in X \times \mathcal{N}} \left| \tilde{b}(z, n') - b(z, n') \right| - h \right) \quad \text{and} \quad B_{il} - h \geq \tilde{b}(x') - \left( \sup_{x' \in X} \left| \tilde{b}(x') - b(x') \right| - h \right).
\]

Therefore, if \( \sup_{(z, n') \in X \times \mathcal{N}} \left| \tilde{b}(z, n') - b(z, n') \right| \leq h \), \( \sup_{z \in X} \left| \tilde{b}(z) - b(z) \right| \leq h \) and \( \mathbb{H}((B_{il}, X_{il}), 2h) \subseteq \tilde{S}_{B_{il}, X}^{N_l} \), we must have \( \mathbb{H}((B_{il}, X_{il}), h) \subseteq \tilde{S}_{B_{il}, X}^{N_l} \). Now it is clear that for sufficiently small \( h \),
\[
P \left[ \max_{i,l} T_{il} (1 - \pi_{il}) = 0 \right] \geq P \left[ \sup_{(z, n') \in X \times \mathcal{N}} \left| \tilde{b}(z, n') - \tilde{b}(z, n) \right| \leq h \right] \quad \text{and} \quad \sup_{z \in X} \left| \tilde{b}(z) - b(z) \right| \leq h.
\]

By using (S.3.54), we have \( \max_{i,l} T_{il} (1 - \pi_{il}) = 0 \), w.p.a.1. Therefore we have
\[
\max_{i,l} T_{il} \left| \tilde{G}(B_{il}, X_{il}, N_l) - G(B_{il}, X_{il}, N_l) \right| = O_p \left( \left( \frac{\log(L)}{Lh^d} \right)^{1/2} + h^{1+R} \right).
\]

(S.3.55)
Similarly, by using (S.3.50) and the triangle inequality, we have
\[
\max_{i,l} T_{il} \left| \frac{\hat{G}(B_{il}, X_l, N_l) - g(B_{il}, X_l, N_l)}{g(B_{il}, X_l, N_l)^2} \right| \\
\leq \max_{i,l} T_{il} \left| \frac{\hat{G}(B_{il}, X_l, N_l) - G(B_{il}, X_l, N_l)}{g(B_{il}, X_l, N_l)^2} \right| \left( \frac{\log(L)}{Lh^{1+\sigma}} \right)^{1/2} + h^{1+R}.
\]
\[
= O_p \left( \frac{\log(L)}{Lh^{1+\sigma}} + h^{1+2R} \right), \tag{S.3.56}
\]
where the inequality holds w.p.a.1.

For the third term of the right hand side of (S.3.52), it follows from
\[
\max_{i,l} T_{il} |\hat{g}(B_{il}, X_l, N_l) - g(B_{il}, X_l, N_l)| = o_p(1)
\]
and (S.3.2) that
\[
\max_{i,l} T_{il} \hat{g}(B_{il}, X_l, N_l)^{-1} \leq \left( \frac{C}{2} \right)^{-1}
\]
and therefore we have
\[
\max_{i,l} T_{il} \left| \frac{\hat{G}(B_{il}, X_l, N_l) - g(B_{il}, X_l, N_l)}{g(B_{il}, X_l, N_l)^2} \right| \\
\leq \max_{i,l} T_{il} \left| \frac{\hat{G}(B_{il}, X_l, N_l) - G(B_{il}, X_l, N_l)}{g(B_{il}, X_l, N_l)^2} \right| \left( \frac{\log(L)}{Lh^{1+\sigma}} \right)^{1/2} + h^{1+2R}.
\]
\[
= O_p \left( \frac{\log(L)}{Lh^{1+\sigma}} + h^{1+2R} \right),
\]
where the inequality holds w.p.a.1. Therefore it follows from (S.3.52), (S.3.55), (S.3.56) and the above result that
\[
\max_{i,l} T_{il} |\tilde{V}_{il} - V_{il}| = O_p \left( \frac{\log(L)}{Lh^{1+\sigma}} + h^{1+R} \right). \tag{S.3.57}
\]

Note that $H((B_{il}, X_l), h) \subseteq S_{B_{il}, X_l}$ if and only if $H(X_l, h) \subseteq X'$ and for all $x' \in H(X_l, h)$, $B_{il} + h \leq \tilde{h}(x', N_l)$ and $B_{il} - h > \tilde{h}(x')$. Now we can show that when $h$ is sufficiently small, for every $v \in I(x)$, $(V_{il}, X_l) \in H((v, x), \tilde{h})$ implies $H((B_{il}, X_l), h) \subseteq S_{B_{il}, X_l}$. When $h < \tilde{h}$ and $(V_{il}, X_l) \in H((v, x), \tilde{h})$, clearly, for any $n' \in N'$ we have $H(X_l, h) \subseteq H(x, \tilde{h}, n') \subseteq X'$ since for any $x' \in H(X_l, h)$ we have $X_l \in H(x, \tilde{h}, n')$ by assumption and thus $x' \in H(x, \tilde{h}, n')$ by the triangle inequality. By the definition of $\tilde{h}(x')$, since $|V_{il} - v| \leq \tilde{h}$ and $v \in I(x)$, we have
\[
\tilde{h}(x') + \tilde{h}(x) < s(V_{il}, X_l, n'),
\]
for all $x' \in H(X_l, h)$. Since this holds for all $n' \in N'$, we have $B_{il} > \tilde{h}(x') + h$, for all $x' \in H(X_l, h)$, when $h < \min \left\{ \frac{\tilde{h}}{2}, \ldots, \frac{\tilde{h}}{2} \right\}$. Similarly, we have $B_{il} + h < \tilde{h}(x', N_l)$ when $h$ is sufficiently small. Therefore we have
\[
\sup_{v \in I(x)} \max_{i,l} T_{il} |\tilde{V}_{il} - V_{il}| = O_p \left( \frac{\log(L)}{Lh^{1+\sigma}} + h^{1+R} \right). \tag{S.3.58}
\]

Write
\[
\frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} T_{il} \frac{1}{h^{1+\sigma}} K_f \left( \frac{\tilde{V}_{il} - v}{h}, \frac{X_l - x}{h} \right) \\
= \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} T_{il} \frac{1}{h^{1+\sigma}} K_f \left( \frac{\tilde{V}_{il} - v}{h}, \frac{X_l - x}{h} \right) + k_1(v) + k_2(v)
\]
where

\[ \kappa_1^+(v) := \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} T_{il} (1 - \tilde{T}_{il}) \frac{1}{h^{1+d}} K_f \left( \frac{\tilde{V}_{il} - v}{h}, \frac{X_i - x}{h} \right) \]

\[ \kappa_2^+(v) := \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \tilde{T}_{il} (\tilde{T}_{il} - 1) \frac{1}{h^{1+d}} K_f \left( \frac{\tilde{V}_{il} - v}{h}, \frac{X_i - x}{h} \right). \]

Since \( K_0 \) is supported on \([-1, 1] \), \( K_0 ((\tilde{V}_{il} - v)/h) \) is zero if \( \tilde{V}_{il} \) is outside of a \( h \)-neighborhood of \( v \). By the triangle inequality, we have

\[ |\kappa_1^+(v)| \leq \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} (1 - \tilde{T}_{il}) h^{-(1+d)} \| (\tilde{V}_{il} - v) \| 1(X_i \in H(x, h)) \]

\[ \leq \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} (1 - \tilde{T}_{il}) h^{-(1+d)} \| (V_{il} - v) \| \leq h + \max_{j,k} \tilde{T}_{jk} |\tilde{V}_{jk} - V_{jk}| 1(X_i \in H(x, h)). \]

Therefore it is clear that

\[ P \left[ \sup_{v \in I(x)} |\kappa_1^+(v)| = 0 \right] \geq P \left[ \max_{j,k} \tilde{T}_{jk} |\tilde{V}_{jk} - V_{jk}| < \frac{\pi}{2} \right], \]

when \( h \) is sufficiently small. Therefore, we have \( \sup_{v \in I(x)} |\kappa_1^+(v)| = 0 \), w.p.a.1.

It is clear that

\[ \sup_{v \in I(x)} \| \kappa_2^+(v) \| \leq \sup_{v \in I(x)} \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} h^{-(1+d)} \tilde{T}_{il} 1 \left( B_{il} + 2h > \inf_{x' \in \tilde{H}(X_i, 2h) \cap X} \tilde{f}(x', n) \right) 1(X_i \in H(x, h)) \]

\[ + \sup_{v \in I(x)} \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} h^{-(1+d)} \tilde{T}_{il} 1 \left( B_{il} - 2h < \sup_{x' \in \tilde{H}(X_i, 3h) \cap X} \tilde{f}(x', n) \right) \]

\[ \leq h^{-(1+d)} 1 \left( \sup_{x' \in \tilde{H}(X, 3h)} s \left( v_u (x) + \xi, x', n \right) + 2h > \inf_{x' \in \tilde{H}(X, 3h)} \tilde{f}(x', n) \right) \]

\[ + h^{-(1+d)} 1 \left( \inf_{x' \in \tilde{H}(X, 3h)} s \left( v_l (x) - \xi, x', n \right) - 2h < \sup_{x' \in \tilde{H}(X, 3h)} \tilde{f}(x', n) \right) \]

\[ \leq h^{-(1+d)} 1 \left( \sup_{x' \in \tilde{H}(X, 3h)} s \left( v_u (x) + \xi, x', n \right) + 2h > \inf_{x' \in \tilde{H}(X, 3h)} \tilde{f}(x', n) - \sup_{x' \in X} \tilde{f}(x', n) \right) \]

\[ + h^{-(1+d)} 1 \left( \inf_{x' \in \tilde{H}(X, 3h)} s \left( v_l (x) - \xi, x', n \right) - 2h < \sup_{x' \in \tilde{H}(X, 3h)} \tilde{f}(x', n) + \sup_{x' \in X} \tilde{f}(x', n) \right), \]

where the second inequality holds when \( h \) is sufficiently small and follows from the definition of \( \tilde{T}_{il} \). Now it follows that

\[ P \left[ \sup_{v \in I(x)} |\kappa_2^+(v)| > 0 \right] \leq P \left[ \sup_{x' \in \tilde{H}(X, 3h)} s \left( v_u (x) + \xi, x', n \right) + 2h > \inf_{x' \in \tilde{H}(X, 3h)} \tilde{f}(x', n) - \sup_{x' \in X} \tilde{f}(x', n) \right] \]

\[ + P \left[ \inf_{x' \in \tilde{H}(X, 3h)} s \left( v_l (x) - \xi, x', n \right) - 2h < \sup_{x' \in \tilde{H}(X, 3h)} \tilde{f}(x', n) + \sup_{x' \in X} \tilde{f}(x', n) \right] \]

when \( h \) is sufficiently small. It follows from (S.3.54) that the right-hand side of the inequality tends to zero as \( L \uparrow \infty \).

Now we have

\[ \tilde{f}_{G P V}(v, x, n) - \tilde{f}(v, x, n) \]

\[ = \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \tilde{T}_{il} \frac{1}{h^{1+d}} \left( K_f \left( \frac{\tilde{V}_{il} - v}{h}, \frac{X_i - x}{h} \right) - K_f \left( \frac{\tilde{V}_{il} - v}{h}, \frac{X_i - x}{h} \right) \right) \] (S.3.59)

S.46
for all \( v \in I(x) \), w.p.a.1. Then a second-order Taylor expansion of the right-hand side of (S.3.59) gives

\[
\hat{f}_{GPV}(v, x, n) - \hat{f}(v, x, n) = \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \frac{1}{h^{2+\alpha}} K_f^{(0)} \left( \frac{V_l - v}{h}, \frac{X_l - x}{h} \right) \left( \hat{V}_{il} - V_{il} \right) + \frac{1}{2} \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \frac{1}{h^{3+\alpha}} K_f^{(2)} \left( \frac{V_l - v}{h}, \frac{X_l - x}{h} \right) \left( \hat{V}_{il} - V_{il} \right)^2, \tag{S.3.60}
\]

for some mean value \( \hat{V}_{il} \) that lies on the line joining \( \hat{V}_{il} \) and \( V_{il} \). It follows from triangle inequality and the Lipschitz condition imposed on the kernel that

\[
\left| \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \frac{1}{h^{1+\alpha}} \left| K_f^{(0)} \left( \frac{X_l - x}{h} \right) \right| \right| \left( |V_l - v| \leq h \right) \left( \max_{i,j,k} \left| \hat{V}_{il} \right| \right),
\]

\[
\left( \left| V_{il} - v \right| \leq h \right) \left( \max_{i,j,k} \left| \hat{V}_{il} \right| \right).
\]

By the triangle inequality, we have

\[
\frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \frac{1}{h^{1+\alpha}} \left| K_f^{(0)} \left( \frac{X_l - x}{h} \right) \right| \left( \left| V_{il} - v \right| \leq h \right) \leq \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \frac{1}{h^{1+\alpha}} \left| K_f^{(0)} \left( \frac{X_l - x}{h} \right) \right| \left( \left| V_{il} - v \right| \leq h + \max_{j,k} \left| \hat{V}_{ij} - V_{ij} \right| \right)
\]

\[
\leq \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \frac{1}{h^{1+\alpha}} \left| K_f^{(0)} \left( \frac{X_l - x}{h} \right) \right| \left( \left| V_{il} - v \right| \leq h + \max_{j,k} \left| \hat{V}_{ij} - V_{ij} \right| \right).
\]

where

\[
\tilde{\tau}_{il} := \mathbb{I} \left( \left| V_{il} - v \right| \leq 2h \right)
\]

and

\[
\kappa^0(v) := \frac{1}{L} \sum_{l=1}^{L} 1(N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \tilde{\tau}_{il} \mathbb{I} \left( \left| V_{il} - v \right| > 2h \right) \frac{1}{h^{1+\alpha}} \left| K_f^{(0)} \left( \frac{X_l - x}{h} \right) \right| \left( \left| V_{il} - v \right| \leq h + \max_{j,k} \left| \hat{V}_{ij} - V_{ij} \right| \right).
\]

Clearly we have

\[
P \left[ \sup_{v \in I(x)} |\kappa^0(v)| = 0 \right] \geq P \left[ \sup_{v \in I(x)} \max_{j,k} \left| \hat{V}_{ij} - V_{ij} \right| \leq h \right]
\]

where the right hand side tends to 1 as \( L \to \infty \) since \( \sup_{v \in I(x)} \max_{j,k} \left| \hat{V}_{ij} - V_{ij} \right| = o_p(h) \) (see (S.3.58)). Therefore, we have \( \sup_{v \in I(x)} |\kappa^0(v)| = 0 \) w.p.a.1.

By the LIE and change of variables, we have

\[
\sup_{v \in I(x)} E \left[ \mathbb{I} \left( N_1 = n \right) \frac{1}{N_1} \sum_{i=1}^{N_1} \frac{1}{h^{1+\alpha}} \left| K_f^{(0)} \left( \frac{X_1 - x}{h} \right) \right| \right]
\]

\[
= \sup_{v \in I(x)} E \left[ \mathbb{I} \left( N_1 = n \right) \frac{1}{h^{1+\alpha}} \left| K_f^{(0)} \left( \frac{X_1 - x}{h} \right) \right| \mathbb{E} \left[ \tilde{\tau}_{11} | X_1, N_1 \right] \right]
\]

\[
= \sup_{v \in I(x)} \int_{\mathbb{H} \times [0, \infty)} h^{-1} \mathbb{I} \left( \left| w - v \right| \leq 2h \right) \left| K_f^{(0)}(z) \right| f(w|hz + \varphi) \pi(nhz + \varphi) \varphi(hz + \varphi) dw dz
\]

\[
\leq \left( \sup_{(w, \varphi) \in C_{\mathbb{H}, X}} f(w|z, n) \right) \left( \sup_{z \in \mathbb{H}[\varphi]} \pi(n|z) \varphi(z) \right), \tag{S.3.63}
\]

where the inequality holds when \( h \) is sufficiently small, since \( K_0 \) is supported on \([-1, 1] \).
By Jensen’s inequality and LIE, we have

\[
\sigma_{\mathcal{F}_n}^2 \geq \sup_{v \in I(\mathbf{a})} E \left[ \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l = n) \left\{ \frac{1}{N_l} \sum_{i=1}^{N_l} \tilde{V}_i \right\} \right] \leq \sup_{v \in I(\mathbf{a})} E \left[ \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l = n) \frac{1}{h^{1+d}} K_X \left( \frac{X_1 - x}{h} \right) \right] \leq h^{-(1+d)},
\]

(S.3.64)

where the second inequality holds when \( h \) is sufficiently small. Let \( u := (u_1, ..., u_m) \) and

\[
\mathcal{F}_n^u (u, x, m, v) := \mathbb{I} (m = n) \frac{1}{m} \sum_{i=1}^{m} \mathbb{I} (|u_i - v| \leq 2h) \frac{1}{h^{1+d}} K_X \left( \frac{z - x}{h} \right).
\]

By standard arguments, we can verify that the class \( \{ \mathcal{F}_n (v; v) : v \in I(\mathbf{a}) \} \), which implicitly depends on \( L \), is uniformly VC-type with respect to the envelope

\[
F_{\mathcal{F}_n} (z) := \frac{1}{h^{1+d}} K_X \left( \frac{z - x}{h} \right).
\]

The CCK inequality yields

\[
E \left[ \sup_{v \in I(\mathbf{a})} \left| \frac{1}{L} \sum_{l=1}^{L} \mathcal{F}_n (V_l, X_l, N_l; v) - E [\mathcal{F}_n (V_1, X_1, N_1; v)] \right| \right] \leq C_1 \left\{ L^{-1/2} \sigma_{\mathcal{F}_n} \log (C_2 L)^{1/2} + L^{-1} \| F_{\mathcal{F}_n} \|_\infty \log (C_2 L) \right\} = O \left( \frac{\log (L)}{Lh^{1+d}} \right)^{1/2}
\]

and

\[
\sup_{v \in I(\mathbf{a})} \left| \frac{1}{L} \sum_{l=1}^{L} \mathcal{F}_n (V_l, X_l, N_l; v) - E [\mathcal{F}_n (V_1, X_1, N_1; v)] \right| = O_p \left( \frac{\log (L)}{Lh^{1+d}} \right)^{1/2}
\]

(S.3.65)

follows from Markov’s inequality. It now follows that

\[
\sup_{v \in I(\mathbf{a})} \frac{1}{L} \sum_{l=1}^{L} \mathcal{F}_n (V_l, X_l, N_l; v) = O_p (1).
\]

(S.3.66)

It follows from the above result, (S.3.58), (S.3.61), (S.3.62) and (S.3.63) that

\[
\sup_{v \in I(\mathbf{a})} \left| \frac{1}{L} \sum_{l=1}^{L} \mathbb{I} (N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \tilde{V}_i \right| \frac{1}{h^{1+d}} K_X \left( \frac{X_l - x}{h} \right) K'' \left( \frac{\tilde{V}_i - v}{h} \right) \left( \tilde{V}_i - V_i \right)^2 = O_p \left( \frac{\log (L)}{Lh^{3+d}} + h^2 R \right).
\]

It follows from the above result, (S.3.60) and (S.3.61) that

\[
\hat{f}_{\text{G}P} (v, x, n) - \hat{f} (v, x, n) = \frac{1}{L} \sum_{l=1}^{L} \mathbb{I} (N_l = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \tilde{V}_i \frac{1}{h^{1+d}} K'' \left( \frac{\tilde{V}_i - v}{h} \right) \left( \tilde{V}_i - V_i \right)^2 = O_p \left( \frac{\log (L)}{Lh^{3+d}} + \frac{\log (L)}{Lh^{1+d}} \right) + O_p \left( \frac{\log (L)}{Lh^{1+d}} \right)^{1/2} + h^2 R.
\]

(S.3.67)

Standard arguments for the kernel density estimation for conditional densities gives

\[
\hat{f} (v, x, n) = f (v|x) \varphi (x) \pi (n|x) + O_p \left( \frac{\log (L)}{Lh^{1+d}} \right)^{1/2} + h^2 R,
\]

where the remainder term is uniform in \( I(\mathbf{a}) \). See, e.g., Marmer and Shneyerson (2012, Lemma 1(f)). Then the conclusion follows from (S.3.67) and the above result.
Lemma S.3.2. Suppose that Assumptions 1 - 3 hold. Let \( x \) be an interior point of \( X \) and \( n \in N \) be fixed. Then

\[
\hat{g}_{GPV}(v, x, n) - f(v|x) \varphi(n|x) = \frac{1}{L} \sum_{l=1}^L \sum_{i=1}^n M^n((B_l, X_l, N_l), (B_k, X_k, N_k); v) + \mathcal{O}_p \left( \left( \frac{\log(L)}{Lh^{2+d}} \right)^{1/2} + \frac{\log(L)}{Lh^{2+d}} + h^R \right),
\]

where the remainder term is uniform in \( v \in I(x) \).

Proof of Lemma S.3.2. Using (S.3.51), we have

\[
\frac{1}{L} \sum_{l=1}^L \sum_{i=1}^n \frac{N_l}{N_i} \sum_{i=1}^n \mathbb{E}_i \left[ K'_f \left( \frac{V_{il} - v}{h}, \frac{X_l - x}{h} \right) \right] (\hat{V}_{il} - V_{il})
\]

where

\[
\Delta^1_v(v) = \frac{1}{L} \sum_{l=1}^L \sum_{i=1}^n \frac{N_l}{N_i} \sum_{i=1}^n \mathbb{E}_i \left[ K'_f \left( \frac{V_{il} - v}{h}, \frac{X_l - x}{h} \right) \right] \left[ \frac{G(B_{il}, X_l, N_l) - \hat{g}(B_{il}, X_l, N_l)}{g(B_{il}, X_l, N_l)} \right],
\]

\[
\Delta^2_v(v) = -\frac{1}{L} \sum_{l=1}^L \sum_{i=1}^n \frac{N_l}{N_i} \sum_{i=1}^n \mathbb{E}_i \left[ K'_f \left( \frac{V_{il} - v}{h}, \frac{X_l - x}{h} \right) \right] \left[ \frac{\hat{g}(B_{il}, X_l, N_l) - g(B_{il}, X_l, N_l)}{g(B_{il}, X_l, N_l)} \right],
\]

By using the triangle inequality, (S.3.2), (S.50) and

\[
\sup_{v \in I(x)} \left| \Delta^2_v(v) \right| = \mathcal{O}_p \left( \frac{\log(L)}{Lh^{2+d}} + h^{2R+1} \right).
\]

It follows from the triangle inequality, (S.3.2) and the inequality \( G(b, \alpha', n') \leq \pi(n'|\alpha') \varphi(\alpha') \) that

\[
\left| \Delta^3_v(v) \right| \leq \left( \min_{i,l} \mathbb{E}_i \hat{g}(B_{il}, X_l, N_l) \right)^{-1} \left\{ \frac{1}{L} \sum_{l=1}^L \sum_{i=1}^n \frac{N_l}{N_i} \sum_{i=1}^n \mathbb{E}_i \left[ K'_f \left( \frac{V_{il} - v}{h}, \frac{X_l - x}{h} \right) \right] \left[ \frac{\hat{g}(B_{il}, X_l, N_l) - g(B_{il}, X_l, N_l)}{g(B_{il}, X_l, N_l)} \right] \right\}.
\]

It follows from (S.3.50), (S.3.69) and the fact

\[
\left( \min_{i,l} \mathbb{E}_i \hat{g}(B_{il}, X_l, N_l) \right)^{-1} < \left( \frac{C}{2} \right)^{-1}, \text{ w.p.a.1}
\]

that

\[
\sup_{v \in I(x)} \left| \Delta^4_v(v) \right| = \mathcal{O}_p \left( \frac{\log(L)}{Lh^{2+d}} + h^{2R-1} \right).
\]

For \( \Delta^4_v(v) \), firstly, the contribution of the trimmed observations is asymptotically negligible, since \( K'_f \) has a bounded support. When \( h \)
is sufficiently small,
\[
\sum_{l=1}^L \mathbb{1} \left( N_l = n \right) \frac{1}{N_l} \sum_{i=1}^{N_l} \left( \hat{G}_{i,l} - 1 \right) K_f^\prime \left( \frac{V_{i,l} - v}{h}, \frac{X_i - x}{h} \right) \frac{G \left( B_{i,l}, X_l, N_l \right) - \hat{G} \left( B_{i,l}, X_l, N_l \right)}{(N_l - 1) g \left( B_{i,l}, X_l, N_l \right)} = 0. \tag{S.3.70}
\]

Since
\[
\Delta_1^L (v) = - \frac{1}{L} \sum_{l=1}^L \mathbb{1} \left( N_l = n \right) \frac{1}{N_l} \sum_{i=1}^{N_l} \frac{1}{h^{2 + d}} K_f^\prime \left( \frac{V_{i,l} - v}{h}, \frac{X_i - x}{h} \right) \frac{1}{(N_l - 1) g \left( B_{i,l}, X_l, N_l \right)} \times \left\{ \mathbb{1} \left( N_k = N_l \right) \frac{1}{N_k} \sum_{j=1}^{N_k} \mathbb{1} \left( B_{j,k} \leq B_{i,l} \right) \frac{1}{h^d} K_f \left( \frac{X_{k,l} - X_i}{h} \right) - G \left( B_{i,l}, X_l, N_l \right) \right\},
\]

when \( h \) is sufficiently small, it is clear that we can write
\[
\Delta_1^L (v) = \frac{1}{L^2} \sum_{l=1}^L \sum_{k=1}^L G^n \left( (B_{i,l}, X_l, N_l), (B_{k,l}, X_k, N_k); v \right), \tag{S.3.71}
\]
where
\[
G^n \left( (b, z, m), (b', z', m'); v \right) := - \mathbb{1} \left( m = n \right) \frac{1}{m} \sum_{l=1}^m \frac{1}{h^{2 + d}} K_f^\prime \left( \frac{\xi (b_l, z, m) - v}{h}, \frac{z - x}{h} \right) \frac{1}{(m - 1) g \left( b_l, z, m \right)} \times \left\{ \mathbb{1} \left( m' = m \right) \frac{1}{m'} \sum_{j=1}^{m'} \mathbb{1} \left( b'_j \leq b_l \right) \frac{1}{h^d} K_f \left( \frac{z'_j - z}{h} \right) - G \left( b_l, z, m \right) \right\}. \tag{S.3.72}
\]

Let
\[
\mu_{G^n} \left( v \right) := \int \sum_{m \in \mathbb{N'}} \int \cdots \int \sum_{m^* \in \mathbb{N'}} \int \cdots \int G_1^n \left( b, z, m; v \right) \prod_{j=1}^m g \left( b_j, z, m \right) \varphi \left( m \right) \varphi \left( z \right) db_1 \cdots db_m dz
\]
and
\[
G_1^n \left( b, z, m; v \right) := E \left[ G^n \left( (b, z, m), (B_1, X_1, N_1); v \right) \right] \quad \text{and} \quad G_2^n \left( b, z, m; v \right) := E \left[ G^n \left( (B_1, X_1, N_1), (b, z, m); v \right) \right].
\]

The Hoeffding decomposition yields
\[
\Delta_1^L (v) = \mu_{G^n} \left( v \right) + \left\{ \frac{1}{L} \sum_{l=1}^L G_1^n \left( B_{i,l}, X_l, N_l; v \right) - \mu_{G^n} \left( v \right) \right\} + \left\{ \frac{1}{L} \sum_{l=1}^L G_2^n \left( B_{i,l}, X_l, N_l; v \right) - \mu_{G^n} \left( v \right) \right\}
\]
\[
+ \frac{1}{L (L - 1)} \sum_{l \neq k} G^n \left( (B_{i,l}, X_l, N_l), (B_{k,l}, X_k, N_k); v \right) - G_1^n \left( B_{i,l}, X_l, N_l; v \right) - G_2^n \left( B_{k,l}, X_k, N_k; v \right) + \mu_{G^n} \left( v \right) \}
\]
\[
+ \frac{1}{L^2} \sum_{l=1}^L G^n \left( (B_{i,l}, X_l, N_l), (B_{i,l}, X_l, N_l); v \right) - \frac{1}{L^2 (L - 1)} \sum_{l \neq k} G^n \left( (B_{i,l}, X_l, N_l), (B_{k,l}, X_k, N_k); v \right) \].
\]

By the LIE, we have
\[
G_1^n \left( b, z, m; v \right) = - \mathbb{1} \left( m = n \right) \frac{1}{m} \sum_{l=1}^m \frac{1}{h^{2 + d}} K_f^\prime \left( \frac{\xi (b_l, z, m) - v}{h}, \frac{z - x}{h} \right) \frac{1}{(m - 1) g \left( b_l, z, m \right)} \times \left\{ E \left[ \mathbb{1} \left( N_1 = m \right) G \left( b_l, X_1, N_1 \right) \right] \frac{1}{h^n} K_f \left( \frac{X_1 - z}{h} \right) - G \left( b_l, z, m \right) \right\}
\]

S.50
and

\[ G_n^2 (b, z, m; v) = \mathbb{E} \left[ -1 (N_1 = n) \frac{1}{h^{2+2d}} K_f' \left( \frac{\xi (B_{11}, X_1, N_1) - v}{h}, \frac{X_1 - z}{h} \right) \right. \]
\[ \times \left. \left\{ \mathbb{E} \left( \frac{1}{m} \sum_{j=1}^m (b_j \leq B_{11}) \frac{1}{h^d} K_X \left( \frac{z - X_1}{h} \right) - G (B_{11}, X_1, N_1) \right) \right\} \right] \]
\[ = - m \sum_{j=1}^m \mathbb{E} \left( \frac{1}{m} \sum_{j=1}^m (b_j \leq B_{11}) \frac{1}{h^d} K_X \left( \frac{z - X_1}{h} \right) \right) \frac{1}{n-1} (m = n) \frac{1}{m} \sum_{j=1}^m (b_j \leq b') \frac{1}{h^d} K_X \left( \frac{z - z'}{h} \right) \, db' \, dz' \]
\[ + \int_X \int_{\mathbb{R}^d} \mathbb{E} \left( \frac{1}{m} \sum_{j=1}^m (b_j \leq B_{11}) \frac{1}{h^d} K_X \left( \frac{z - X_1}{h} \right) \right) \frac{1}{n-1} (m = n) \frac{1}{m} \sum_{j=1}^m (b_j \leq b') \frac{1}{h^d} K_X \left( \frac{z - z'}{h} \right) \, db' \, dz'. \]

Let

\[ \beta_n (b, z) := \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^n (m' = n) \frac{1}{h^d} K_X \left( \frac{z - Z_j}{h} \right) \right] - G (b, z, n) \]
\[ = \int_X G (b, z', n) \frac{1}{h^d} K_X \left( \frac{z' - z}{h} \right) \, dz' - G (b, z, n) \]

be the bias of the kernel estimator of \( G (b, z, n) \). It is clear that

\[ \mu g_n (v) = - \frac{1}{n-1} \int_X \int_{\mathbb{R}^d} \frac{1}{h^{1+d}} K_f' \left( \frac{\xi (b, z, n) - v}{h}, \frac{z - x}{h} \right) \beta_n (b, z) \, db \, dx. \]
\[ = - \frac{1}{n-1} \int_X \int_{\mathbb{R}^d} \frac{1}{h^{1+d}} K_f' \left( \frac{\xi (u, y, z)}{h}, \frac{z - x}{h} \right) \beta_n (s (hu + v, hy + x, n)) \, db \, dx \]

where the second equality follows from change of variables.

Since we assume that \( K_1 \) is supported on \([-1, 1]\) and differentiable on \( \mathbb{R} \), it is straightforward to verify that for all \( z \in \mathbb{H} (b, n) \), \( \beta (\cdot, z) \) satisfies the assumptions of Theorem 2.27 of Folland (1999) and therefore \( \beta_n (\cdot, z) \) is differentiable on \( [s (v_1 (x) - h, z, n), s (v_u (x) + h, z, n)] \), which is well-defined when \( h \) is sufficiently small, and

\[ \beta_n' (b, z) = \int_X \frac{1}{h^d} K_X \left( \frac{z' - z}{h} \right) g (b, z', n) \, dz' - g (b, z, n) \]

for all \( b \in [s (v_1 (x) - h, z, n), s (v_u (x) + h, z, n)] \).

By the usual argument for the bias of kernel estimators (see, e.g., Newey (1994)) and the assumption that \( K_0 \) is supported on \([-1, 1]\), for each \( (b, z) \in C_{B, X}^n \),

\[ |\beta_n (b, z)| \leq \frac{h^{R+1}}{(R+1)!} \left( \sup_{(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^{R+1}_+} \sup_{\|z\|_1 \leq 1} \left| D_2^{\alpha_1} \cdots D_{R+1}^{\alpha_d} G (b, z', n) \right| \right) \left( \int \|z\|_1^{R+1} |K_X (z')| \, dz' \right) \]

(S.3.73)

and

\[ |\beta_n' (b, z)| \leq \frac{h^{R+1}}{(R+1)!} \left( \sup_{(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^{R+1}_+} \sup_{\|z\|_1 \leq 1} \left| D_2^{\alpha_1} \cdots D_{R+1}^{\alpha_d} g (b, z', n) \right| \right) \left( \int \|z\|_1^{R+1} |K_X (z')| \, dz' \right), \]

(S.3.74)

when \( h \) is sufficiently small. Since \( C_{B, X}^n \) is a compact inner closed subset of \( S_{B, X} \), it is clear that \( \beta_n (b, z) \) and \( \beta_n' (b, z) \) are \( O (h^{R+1}) \) uniformly in \( (b, z) \) in \( C_{B, X}^n \).

By change of variable, the fact that \( s \) is twice continuously differentiable (see Lemma A1 of GPV) and a mean value expansion, we have

\[ \mu g_n (v) = - \frac{1}{n-1} \int_X \frac{1}{h^{1+d}} K_0' \left( \frac{z - x}{h} \right) \left( \int \frac{\xi (u, x, n)}{h} \beta_n (s (hu + v, n), z) s' (hu + v, z, n) \, du \right) \, dz. \]

S.51
\[
\begin{align*}
\frac{1}{n-1} \int_X \frac{1}{h^{1+d}} K_X^h \left( \frac{z - x}{h} \right) \left\{ \int_{v(x)}^{\frac{z - x}{h}} K_0^h(u) \left( \beta_{\bar{g}^n} (s(v, z, n), z) s'(v, z, n) \right. \right. \\
+ \left. \left. \left( \beta_{\bar{g}^n} (s(v, z, n), z) s''(v, z, n)^2 + \beta_{\bar{g}^n} (s(v, z, n), z) s''(v, z, n) \right) hu \right\} du \right\} dz,
\end{align*}
\]
where \( \dot{v} \) is the mean value (depending on \( u \) and \( z \)) with \( |\dot{v} - v| \leq h |u| \). By the triangle inequality, we have
\[
|\mu_{\bar{g}^n}(v)| \leq \int_X \frac{1}{h^{1+d}} K_X^h \left( \frac{z - x}{h} \right) \left| \beta_{\bar{g}^n} (s(v, z, n), z) s'(v, z, n) \int_{v(x)}^{\frac{z - x}{h}} K_0^h(u) du \right| dz
+ \int_X \frac{1}{h^{1+d}} K_X^h \left( \frac{z - x}{h} \right) \left| \int_{u(x)}^{\frac{z - x}{h}} K_0^h(u) \right| \left| \beta_{\bar{g}^n} (s(v, z, n), z) s''(v, z, n)^2 + \beta_{\bar{g}^n} (s(v, z, n), z) s''(v, z, n) \right| du \right| \right||dz.
\]
(S.3.75)

Since \( K_0 \) is assumed to be supported on \([-1, 1]\), \( K_X(x) ((s - x)/h) \) is zero for all \( z \notin \mathbb{H}(x, \delta) \) when \( h \) is sufficiently small \( (h < \delta) \). But if \( h < \delta \) and \( z \in \mathbb{H}(x, \delta) \), \( K_X^h(u)du = 0 \) for all \( v \in I(x) \) since \( C_{V,X} \) is an inner closed subset of \( S_{V,X} \) (i.e., for all \( z \in \mathbb{H}(x, \delta) \), \( |v(x) - \bar{v}, v_u(x) + \bar{v}| \subseteq (\mathbb{H}(x), \bar{v}(z)) \)). Therefore the first term on the right hand side of (S.3.75) vanishes when \( h \) is sufficiently small. Therefore,
\[
\sup_{v \in I(x)} |\mu_{\bar{g}^n}(v)| \leq \left( \int_X \frac{1}{h^{1+d}} K_X^h \left( \frac{z - x}{h} \right) \right) \left\{ \sup_{(u,x) \in C_{V,X}} \left| \beta_{\bar{g}^n} (s(u, z, n), z) s'(u, z, n)^2 + \beta_{\bar{g}^n} (s(u, z, n), z) s''(u, z, n) \right| \right\}.
\]
(S.3.76)

It follows from the fact that \( \beta_{\bar{g}^n} (b, z) \) and \( \beta'_{\bar{g}^n} (b, z) \) are \( O(h^{R+1}) \) uniformly in \( (b, z) \in \mathcal{C}^n_{\beta;X} \) and (S.3.8) that
\[
\sup_{v \in I(x)} |\mu_{\bar{g}^n}(v)| = O(h^R).
\]
(S.3.77)

Standard arguments can be applied to verify that class \( \{\bar{g}^n(\cdot, \cdot, \cdot ; v) : v \in I(x)\} \) is (uniformly) VC-type with respect to the envelope
\[
F_{\bar{g}^n}(z, z') := \frac{1}{(n-1)C_{n+1}h^{2+d}} \left| K_X^h \left( \frac{z - x}{h} \right) \right| \left| K_X^h \left( \frac{z' - x}{h} \right) \right| + C_{n+1}^{-1} h^{2+d} \left| K_X^h \left( \frac{z - y}{h} \right) \right|.
\]
(S.3.77)

The CK inequality yields
\[
E \left[ \sup_{v \in I(x)} \left| \frac{1}{(L)^2} \sum_{(2)}^{(G^n) \left( (B_1, X_1, N_1), (B_k, X_k, N_k) : v \right) - \bar{G}^n_1 \left( B_1, X_1, N_1 ; v \right) - \bar{G}_2 \left( B_k, X_k, N_k ; v \right) + \mu_{\bar{g}^n}(v) \right| \right] \leq L^{-1} \left( E \left[ F_{\bar{g}^n} (X_1, X_2)^2 \right] \right)^{1/2}
= O \left( \left( L^2 h^{2+d} \right)^{-1} \right).
\]

It is clear from the definition that when \( h \) is sufficiently small, the class \( \{\bar{g}^n(\cdot ; v) : v \in I(x)\} \) is uniformly VC-type with respect to the envelope
\[
F_{\bar{g}^n}(z) := \frac{1}{h^{2+d}} \left| K_X^h \left( \frac{z - x}{h} \right) \right| \left\{ \sup_{(b', z') \in \mathcal{C}^n_{\beta;X}, \bar{g}^n} \left| \beta_{\bar{g}^n} (b', z') \right| \right\}.
\]

The VW inequality yields
\[
E \left[ \sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l=1}^{L} \bar{g}^n_1 \left( B_1, X_1, N_l ; v \right) - \mu_{\bar{g}^n}(v) \right| \right] \leq L^{-1/2} E \left[ F_{\bar{g}^n} (X_1)^2 \right]^{1/2} = O \left( \frac{h^{R-1}}{(L h^{2+d})^{1/2}} \right),
\]
where the inequality holds when \( h \) is sufficiently small and the equality follows from a standard change of variable argument and (S.3.73).
Let
\[ G^{1,n}_2 (b, z, m; (b', z', m'); v) = -1 (m = n) \frac{1}{m} \sum_{j=1}^{m} \frac{1}{h^{2+2d}} K_j' \left( \frac{\xi (b_j, z, m) - v}{h}, \frac{z' - x}{h} \right) \frac{1}{(m-1)} g (b, z, m) \]
\times 1 (m' = m) \frac{1}{m'} \sum_{j'=1}^{m'} 1 (b'_j \leq b_t) K_X \left( \frac{z' - z}{h} \right),
where the second equality follows from LIE and
\[ \mu_{G^{1,n}_2} (v) := E \left[ G^{1,n}_2 ((B_1, X_1, N_1), (b, z, m); v) \right] \]
It follows from symmetry that

\[
E \left[ \sigma_2^1 (B_1, X_1, N_1; v)^2 \right] = 2 \int_Y \int_Y \frac{\sigma(hy' + x_v)}{h} \int_Y \frac{\sigma(hy'' + x_v)}{h} \frac{1}{(n - 1)^2 h^{2+d}} K' (u, y') K_X (y' - y) K' (w, y'') K_X (y'' - y) \\
\times s' (h + v, hy' + x_v) G (s' (h + v, hy'' + x_v), h + x_v) s' (hy' + x_v) \mathrm{d}y' \mathrm{d}y'' \mathrm{d}y.
\]

By a mean value expansion, we have

\[
E \left[ \sigma_2^1 (B_1, X_1, N_1; v)^2 \right] = \varphi_1 (v) + \varphi_2 (v)
\]

where

\[
\varphi_1 (v) = 2 \int_Y \int_Y \frac{\sigma(hy' + x_v)}{h} \int_Y \frac{\sigma(hy'' + x_v)}{h} \frac{1}{(n - 1)^2 h^{2+d}} K' (u, y') K_X (y' - y) K' (w, y'') K_X (y'' - y) \\
\times \mathbb{1} \left( (h + v, hy' + x_v) \leq (h + v, hy'' + x_v) \right) G (s' (h + v, hy'' + x_v), h + x_v) s' (hy' + x_v) \mathrm{d}y' \mathrm{d}y'' \mathrm{d}y
\]

and

\[
\varphi_2 (v) = 2 \int_Y \int_Y \frac{\sigma(hy' + x_v)}{h} \int_Y \frac{\sigma(hy'' + x_v)}{h} \frac{1}{(n - 1)^2 h^{2+d}} K' (u, y') K_X (y' - y) K' (w, y'') K_X (y'' - y) \\
\times \mathbb{1} \left( (h + v, hy' + x_v) \leq (h + v, hy'' + x_v) \right) u \left( \sigma' (h', hy'' + x_v), h' + x_v \right) s' (h + v, hy'' + x_v) \mathrm{d}y' \mathrm{d}y'' \mathrm{d}y
\]

for some mean value \( \hat{v} \) with \( |\hat{v} - v| \leq h |v| \). It is clear that when \( h \) is sufficiently small,

\[
\sup_{v \in I (x)} |\varphi_2 (v)| \leq h^{-(1+d)} \left\{ \sup_{u \in I (x)} \sup_{(w, z', n) \in \mathbb{H} ((v, x), \sigma')} s' (u, z', n) g (s (w, z', n), z, n) s' (w, z'', n) \right\} + \sup_{u \in I (x)} \sup_{(w, z', z'') \in \mathbb{H} ((v, x), \sigma')} s' (u, z', n) G (s (w, z'', n), z, n) s'' (w, z'', n)
\]

(S.3.78)

and therefore \( \sup_{v \in I (x)} |\varphi_2 (v)| = O (h^{-(1+d)}) \).

When \( h \) is sufficiently small,

\[
K_0 \left( s (h + v, hy' + x_v) \leq s (h + v, hy'' + x_v) \right) du = K_0 \left( \frac{\xi (s (h + v, hy' + x_v), hy' + x_v) - v}{h} \right)
\]

for all \( y', y'' \in \mathcal{H} (0, 1), |w| \leq 1 \) and \( v \in I (x) \) and thus

\[
\varphi_1 (v) = 2 \int_Y \int_Y \frac{\sigma(hw + x_v)}{h} \int_Y \frac{\sigma(hw + x_v)}{h} \frac{1}{(n - 1)^2 h^{2+d}} K_0 \left( \frac{\xi (s (h + v, hy' + x_v), hy'' + x_v) - v}{h} \right) K' (y') K_X (y' - y) \\
\times K' (w, y'') K_X (y'' - y) G (s (v, hy'' + x_v), h + x_v) s' (v, hy'' + x_v) s' (hy' + x_v) \mathrm{d}y' \mathrm{d}y'' \mathrm{d}y
\]

for all \( v \in I (x) \).
Let\[
\xi_x := \frac{\partial \xi(u, x, n)}{\partial z}(u, x, n)\]and\[
\tau_1(u, x, n) := \xi(s(u, x, n), x, n) = \xi(s(u, x, n), x, n).
\]

For all \(y' \in H(0, 1), y \in H(0, 2)\) and \(|w| \leq 1\),\[
\sup_{v \in I(x)} \left| K_0 \left( \frac{\xi \left( s(hw + v, hy' + x, n), hy' + x, n) - v \right)}{h} \right| - K_0 \left( \frac{w + \xi' \left( s(v, x, n), x, n \right) s_y y' + \xi_y y'}{h} \right) \right| \leq h \left( \sup_{v \in I(x)} \left| D_1 \cdots D_1^{1+2d} \tau_1(u, x, n) \right| \right)
\]

Let\[
\tau_2(u, x, n) := \tau'(s(u, x, n), x, n) \tau'(v, x, n).
\]

For all \(y', y'' \in H(0, 1), y \in H(0, 2)\) and \(|w| \leq 1\),\[
\sup_{v \in I(x)} \left| s'(hw + v, hy' + x, n) G(s(v, x, n), x, n) s'(v, x, n) - s'(v, x, n) G(s(v, x, n), x, n) \right| \leq h \left( \sup_{v \in I(x)} \left| D_{1+3d} \tau_2(u, x, n) \right| \right),
\]

when \(h\) is sufficiently small. Therefore by the change of variable argument, we have\[
\vartheta_1(v) = \frac{2s'(v, x, n)^2 G(s(v, x, n), x, n)}{h^{2+2d}(n - 1)^2} \int \int \int K_X(y' - y) K'_y(w, y'') K_X(y'' - y) \times K'_{x}(w + \xi'(s(v, x, n), x, n) s_y y' + \xi_y y') \, dw \, dy \, dy + O \left( h^{-(1+d)} \right),
\]

where the remainder term is uniform in \(v \in I(x)\), when \(h\) is sufficiently small. We note that the leading term on the right hand side of the above displayed equation vanishes since the integrand is an odd function. Therefore we have\[
\vartheta_2 = \sup_{v \in I(x)} \left| \vartheta_1(v) \right| + \sup_{v \in I(x)} \left| \vartheta_2(v) \right| = O \left( h^{-(1+d)} \right).
\]

Since standard arguments can be applied to verify that the class \(\mathcal{G}^{1, n}(\cdot, \cdot; v) : v \in I(x)\) is (uniformly) VC-type with respect to the envelope\[
F_{\mathcal{G}^{1, n}}(z, z') := \frac{1}{(n - 1)c_0 h^{2+2d}} \left| K_X' \left( \frac{z - x}{h} \right) \right| K_X \left( \frac{z' - x}{h} \right),
\]

it follows from Chen and Kato (2017, Lemma 5.4) that the class \(\mathcal{G}^{1, n}(\cdot; v) : v \in I(x)\) is uniformly VC-type with respect to the envelope\[
F_{\mathcal{G}^{1, n}}(z) := \int F_{\mathcal{G}^{1, n}}(z', z) \varphi(z') \, dz' = \frac{1}{(n - 1)c_0} \int \frac{1}{h^{2+2d}} \left| K_X' \left( \frac{z' - x}{h} \right) \right| K_X \left( \frac{z' - x}{h} \right) \varphi(z') \, dz'.
\]

The CCK inequality yields\[
E \left[ \sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l=1}^{L} \mathcal{G}^{1, n} \left( B_l, X_l, N_l; v \right) - \mu_{\mathcal{G}^{1, n}}(v) \right| \right] \leq C_1 \left( L^{-1/2} \sigma_{\mathcal{G}^{1, n}} \log(C_2 L)^{1/2} + L^{-1} \left\| \mathcal{G}^{1, n} \right\|_X \log(C_2 L) \right) = O \left( \left( \log(L) \right)^{1/2} + \frac{\log(L)}{L h^{1+2d}} \right),
\]

where the inequality is non-asymptotic and the equality follows from (S.3.79) and \(\left\| \mathcal{G}^{1, n} \right\|_X = O \left( h^{-(2+d)} \right)\) (which follows from change of variables).
It is easy to check

$$
sup_{v \in I(x)} \left| \frac{1}{L^2} \sum_{l=1}^{L} \mathcal{G}^{n}_{v}((B_{i}, X_{i}, N_{i}), (B_{k}, X_{k}, N_{k}); v) \right| \leq \frac{1}{L^2} \sum_{l=1}^{L} \left\{ \frac{1}{h^{2+d}} \left| K_{X}^{0} \left( \frac{X_{i} - x}{h} \right) \right| \left| K_{X} \left( \frac{X_{k} - X_{i}}{h} \right) \right| + \frac{1}{h^{2+d}} \left| K_{X}^{0} \left( \frac{X_{k} - x}{h} \right) \right| \right\} = O_{p} \left( \left( Lh^{2} \right)^{-1} \right),
$$

where the equality follows from change of variables and Markov’s inequality and

$$
sup_{v \in I(x)} \left| \frac{1}{L^2} \sum_{l=1}^{L} \mathcal{G}^{n}_{v}((B_{i}, X_{i}, N_{i}), (B_{i}, X_{i}, N_{i}); v) \right| \leq \frac{1}{L^2} \sum_{l=1}^{L} \left\{ \frac{1}{h^{2+d}} \left| K_{X}^{0} \left( \frac{X_{i} - x}{h} \right) \right| + \frac{1}{h^{2+d}} \left| K_{X}^{0} \left( \frac{X_{i} - x}{h} \right) \right| \right\} = O_{p} \left( \left( Lh^{2} \right)^{-1} \right),
$$

where the equality follows from change of variables and Markov’s inequality. Now it follows that

$$
sup_{v \in I(x)} \left| \Delta_{v} \left( v \right) \right| = O_{p} \left( h^{R} + \frac{\log(L)}{Lh^{1+a}} \right)^{1/2} + \frac{\log(L)}{Lh^{a} + h^{R}}
$$

and the conclusion follows.

**Lemma S.3.3.** Suppose that Assumptions 1 - 3 hold. Let \( x \) be an interior point of \( X \) and \( n \in \mathcal{N} \) be fixed. Then

$$
\tilde{G}_{GPv} (v, x, n) - f (v | x) \varphi (x) \pi (n | x) = \frac{1}{L} \sum_{l=1}^{L} \{ \mathcal{M}^{n}_{v} ((B_{i}, X_{i}, N_{i}), (B_{i}, X_{i}, N_{i}); v) - \mu_{M^{n}} (v) \} + O_{p} \left( \frac{\log(L)}{Lh^{1+a}} \right)^{1/2} + \frac{\log(L)}{Lh^{a} + h^{R}},
$$

where the remainder term is uniform in \( v \in I (x) \).

**Proof of Lemma S.3.3.** The Hoeffding decomposition yields

$$
\frac{1}{L^2} \sum_{l=1}^{L} \sum_{k \neq l}^{L} \mathcal{M}^{n} ((B_{i}, X_{i}, N_{i}), (B_{k}, X_{k}, N_{k}); v) = \mu_{M^{n}} (v) + \frac{1}{L} \sum_{l=1}^{L} \{ \mathcal{M}^{n}_{v} ((B_{i}, X_{i}, N_{i}), (B_{i}, X_{i}, N_{i}); v) - \mu_{M^{n}} (v) \} + \frac{1}{L} \sum_{l \neq k}^{L} \{ \mathcal{M}^{n}_{v} ((B_{i}, X_{i}, N_{i}), (B_{k}, X_{k}, N_{k}); v) - \mu_{M^{n}} (v) \}
$$

$$
+ \frac{1}{L(L-1)} \sum_{l \neq k}^{L} \{ \mathcal{M}^{n}_{v} ((B_{i}, X_{i}, N_{i}), (B_{k}, X_{k}, N_{k}); v) - \mu_{M^{n}} (v) \}
$$

$$
+ \frac{1}{L^2} \sum_{l=1}^{L} \mathcal{M}^{n} ((B_{i}, X_{i}, N_{i}), (B_{j}, X_{j}, N_{j}); v) - \frac{1}{L^2} \sum_{l \neq k}^{L} \mathcal{M}^{n} ((B_{i}, X_{i}, N_{i}), (B_{k}, X_{k}, N_{k}); v).
$$

Let

$$
\beta_{M^{n}} (b, z) := E \left[ (N_{1} = n) \frac{1}{N_{2}} \sum_{j=1}^{N_{2}} \frac{1}{h^{1+a}} K_{y} \left( \frac{B_{j} - b}{h} \right) K_{X} \left( \frac{X_{1} - z}{h} \right) \right] - g (b, z, n)
$$

$$
= \int_{X} \int_{Z} \frac{1}{h^{1+a}} K_{y} \left( \frac{b' - b}{h} \right) K_{X} \left( \frac{z' - z}{h} \right) g (b', z', n) \, db' \, dz' - g (b, z, n)
$$

where the second equality follows from LIE.

By the definition of \( \mathcal{M}^{n} (\cdot, \cdot), \mu_{M^{n}} (v) \) is given by

$$
\mu_{M^{n}} (v) = - \int_{X} \sum_{m \in \mathcal{N}} \sum_{l=1}^{m} \frac{1}{m} \sum_{l=1}^{m} \frac{1}{h^{1+a}} K_{y} \left( \frac{\xi (b_{i}, z, n) - v}{h} \right) g (b_{i}, z, n) \frac{\beta_{M^{n}} (b_{i}, z)}{(m-1) g (b_{i}, z, n)^{2}}.
$$
A mean value expansion gives

\[
S.3.82
\]

It is clear that \( \beta_{M^n} (b, z) \) is the bias of the kernel estimator for \( g(b, z, n) \). Since we assume that \( K_0 \) is supported on \([-1, 1]\) and differentiable everywhere on \( \mathbb{R} \), it is straightforward to verify that for \( \beta_{M^n} (b, z) \), the assumptions of Theorem 2.27 of Folland (1999) are satisfied. Therefore \( \beta_{M^n} (b, z) \) is differentiable on \([s(v_1(x) - h, z, n), s(v_0(x) + h, z, n)]\) for all \( z \in H(x, h) \) when \( h \) is sufficiently small and

\[
\beta_{M^n} (b, z) = \int_X \int_{\mathbb{R}^d} K_g \left( \frac{b' - b}{h} \right) K_X \left( \frac{z' - z}{h} \right) g(b', z', n) \, db' \, dz' - g'(b, z, n),
\]

which is the bias of the kernel estimator for the partial derivative \( g'(b, z, n) \). By the usual argument for the bias of kernel estimators for the density (see, e.g., Newey (1994)) and the assumption that \( K_0 \) is supported on \([-1, 1]\), for each \((b, z) \in C^n_{b,X}\),

\[
\begin{align*}
| \beta_{M^n} (b, z) | & \leq \frac{h^{R+1}}{(R + 1)!} \left\{ \sup_{(a_1, \ldots, a_{R+1})} \sup_{(v, z') \in H(b, z', h)} \left| D^{a_1} \cdots D^{a_{R+1}} g(b', z', n) \right| \right\} \\
& \times \left\{ \int \int \| (b', z') \|_1^R \left| K_g (b') K_X (z') \right| \, db' \, dz' \right\},
\end{align*}
\]

(S.3.81)

when \( h \) is sufficiently small. It follows from Proposition 1(iv) of GPV that \( g(\cdot, v, n) \) admits \( R + 1 \) continuous partial derivatives on the interior of \( S^n_{b,X} \) for each \( n \in \mathbb{N} \). By using the standard argument for the bias of kernel estimators for the density derivatives (see, e.g., Newey (1994)), for each \((b, z) \in C^n_{b,X}\),

\[
\begin{align*}
| \beta_{M^n} (b, z) | & \leq \frac{h^{R}}{R!} \left\{ \sup_{(a_1, \ldots, a_{R})} \sup_{(v, z') \in H(b, z', h)} \left| D^{a_1} \cdots D^{a_{R}} g'(b', z', n) \right| \right\} \\
& \times \left\{ \int \int \| (b', z') \|_1^R \left| K_g (b') K_X (z') \right| \, db' \, dz' \right\},
\end{align*}
\]

(S.3.82)

when \( h \) is sufficiently small. Since \( C^n_{b,X} \) is an inner closed subset of \( S^n_{b,X} \). (S.3.81) and (S.3.82) imply that \( \beta_{M^n} (b, z) \) and \( \beta_{M^n}' (b, z) \) are also \( O(h^{R}) \) uniformly in \((b, z) \in C^n_{b,X}\).

By change of variables, we have

\[
\mu_{M^n} (v) = -\frac{1}{n - 1} \int_X \int_{\mathbb{R}^d} K_f \left( \frac{u - x}{h} \right) \left| G \left( s(hu + v, z, n) \right) \right| g \left( s(hu + v, z, n) \right) \beta_{M^n} \left( s(hu + v, z, n) \right) \, du \, dz.
\]

A mean value expansion gives

\[
\begin{align*}
\mu_{M^n} (v) & = -\frac{1}{n - 1} \int_X \int_{\mathbb{R}^d} K_f \left( \frac{u - x}{h} \right) \psi \left( hu + v, z, n \right) \beta_{M^n} \left( s(hu + v, z, n) \right) \, du \, dz \\
& = -\frac{1}{n - 1} \int_X \int_{\mathbb{R}^d} K_f \left( \frac{u - x}{h} \right) \left\{ \psi (v, z, n) \beta_{M^n} (s(v, z, n), z) + \psi (v, z, n) \beta_{M^n}' (s(v, z, n), z) s'(v, z, n) hu \right\} \, du \, dz,
\end{align*}
\]

where \( \hat{v} \) is the mean value (depending on \( u \) and \( z \)) with \( |\hat{v} - v| \leq h|u| \). By the triangle inequality, we have

\[
\begin{align*}
| \mu_{M^n} (v) | & \leq \int_X \frac{1}{h^{R+1}} K_f \left( \frac{z - x}{h} \right) \left| \psi \left( v, z, n \right) \beta_{M^n} \left( s(v, z, n) \right) \right| K_0 \left( u \right) \, du \, dz \\
& + \int_X \frac{1}{h^{R+1}} K_f \left( \frac{z - x}{h} \right) \left| \int \frac{\psi (v, z, n) \beta_{M^n}' (s(v, z, n), z) s'(v, z, n) hu}{h^{R+1}} \right| K_0 \left( u \right) \, du \, dz.
\end{align*}
\]

(S.3.83)

By the argument used in the proof of Lemma S.3.2, the first term on the right hand side of (S.3.83) vanishes for all sufficiently small \( h \).
Therefore now we have

\[
\sup_{v \in I(\mathbf{x})} |\mu_{M^n}(v)| \leq \left\{ \int \frac{1}{h^d} \left| K^n_X \left( \frac{z-x}{h} \right) \right| dz \right\} \times \left\{ \sup_{(u,z) \in C^n_{B,X}} |\psi'(u,z) \beta_{M^n}(s(u,z,n),z) + \psi(u,z) \beta'_{M^n}(s(u,z,n),z) s'(u,z,n)| \right\},
\]

(S.3.84)

when \( h \) is sufficiently small. It follows from this result, the fact that \( \beta_{M^n}(b,z) \) and \( \beta'_{M^n}(b,z) \) are both \( O(h^R) \) uniformly in \( (b,z) \in C^n_B \)

and (S.3.87) that \( \sup_{v \in I(\mathbf{x})} |\mu_{M^n}(v)| = O(h^R) \).

Standard arguments can be applied to verify that class \( \{M^n (\cdot; v) : v \in I(\mathbf{x})\} \) is (uniformly) VC-type with respect to the envelope

\[
F_{M^n}(z,z') := \frac{\overline{\mathcal{C}}}{(n-1) C_{h^{d+2}}^2} \left| K^n_X \left( \frac{z-x}{h} \right) \right| + \frac{\overline{\mathcal{C}}^0}{(n-1) C_{h^{2+d}}^2} K^n_X \left( \frac{z-x}{h} \right).
\]

(S.3.85)

The CK inequality gives

\[
E \left[ \sup_{v \in I(\mathbf{x})} \left| \frac{1}{L^2} \sum_{i=1}^L \{M^n((B_i,B_1,1;N_i),(B_k,K_k,N_k);v) - M^n(B_i,B_1,1;N_i);v - M^n(B_k,K_k,N_k;v) + \mu_{M^n}(v) \} \right| \right] \leq L^{-1} \left( E \left( F_{M^n}(X_1,X_2)^2 \right) \right)^{1/2} = O \left( \left( Lh^{3+d} \right)^{-1} \right)
\]

where the equality follows from change of variables.

Next, we have

\[
M^n_b(b,z,m;v) = E[M^n((b,z,m),(B_1,B_1,1;N_1);v)]
\]

\[
= -1(m = n) \frac{1}{m} \sum_{i=1}^m \frac{1}{h^{2+d}} K^n_i \left( \frac{z-x}{h}, \frac{z-x}{h} \right) \times \left\{ 1(N_1 = m) \right\} \frac{1}{N_1} \sum_{j=1}^{N_1} h^{1+d} K_g \left( \frac{B_j - b_i}{h} \right) K^n_i \left( \frac{X_1 - z}{h} \right)
\]

\[
-1(m = n) \frac{1}{m} \sum_{i=1}^m \frac{1}{h^{2+d}} K^n_i \left( \frac{z-x}{h}, \frac{z-x}{h} \right) \times \left\{ 1(N_1 = m) \right\} \frac{1}{N_1} \sum_{j=1}^{N_1} h^{1+d} K_g \left( \frac{B_j - b_i}{h} \right) K^n_i \left( \frac{X_1 - z}{h} \right)
\]

It is clear from the definition that when \( h \) is sufficiently small, the class \( \{M^n_b(\cdot; v) : v \in I(\mathbf{x})\} \) is uniformly VC-type with respect to the envelope

\[
F_{M^n}(z) := \frac{\overline{\mathcal{C}}_3 + \overline{\mathcal{C}}_2}{(n-1) C_{h^{2+d}}^2} \left| K^n_X \left( \frac{z-x}{h} \right) \right| \sup_{(b',z) \in C^n_{B,X}} |\beta_{M^n}(b',z')|
\]

The VW inequality yields

\[
E \left[ \sup_{v \in I(\mathbf{x})} \left| \frac{1}{L} \sum_{i=1}^L M^n_b(B_i,B_1,1;N_i;v) - \mu_{M^n}(v) \right| \right] \leq L^{-1/2} E \left( F_{M^n}(X_1)^2 \right)^{1/2} = O \left( \frac{h^{R-1}}{(Lh^{1+d})^{1/2}} \right),
\]

where the inequality holds when \( h \) is sufficiently small and the equality follows from a standard change of variable argument and (S.3.81).

It follows from Markov’s inequality and change of variables that

\[
\sup_{v \in I(\mathbf{x})} \left| \frac{1}{L^2} \sum_{i=1}^L M^n((B_i,B_1,1;N_i),(B_1,B_1,1);v) \right| \leq \frac{1}{L^2} \sum_{i=1}^L \frac{1}{h^{2+d}} K^n_X \left( \frac{X_1 - x}{h} \right) + \frac{1}{L^2} \sum_{i=1}^L \frac{1}{h^{2+d}} K^n_X \left( \frac{X_i - x}{h} \right)
\]

\[
= O_p \left( \left( Lh^{3+d} \right)^{-1} \right)
\]

S.58
and similarly, we have
\[
\sup_{v \in I(x)} \left| \frac{1}{L^2 (L - 1)} \sum_{i \neq k} M^n ((B_i, X_i, N_i), (B_k, X_k, N_k); v) \right| \\
\leq \frac{1}{L^2 (L - 1)} \sum_{i \neq k} \frac{1}{h^{3+2d}} \left| K^0 X \left( \frac{X_i - x}{h} \right) \right|, \\
\quad \left| K^0 X \left( \frac{X_k - X_i}{h} \right) \right| + \frac{1}{L^2} \sum_{i = 1}^L \frac{1}{h^{3+2d}} \left| K^0 X \left( \frac{X_i - x}{h} \right) \right| \\
= O_p \left( \frac{L h^3}{L h^3} \right).
\]

The conclusion follows from these bounds. 

\textbf{Lemma S.3.4.} Suppose Assumptions 1 - 3 hold. Then
\[
\sup_{v \in I(x)} \left| Z (v \mid x) - \Gamma (v \mid x) \right| = O_p \left( \log (L)^{1/2} h + \frac{\log (L)}{(L h^3)^{1/2} + L^{1/2} h^{(3+d)/2 + R}} \right).
\]

\textbf{Proof of Lemma S.3.4.} It is straightforward to check that
\[
\Gamma (v \mid x) = \frac{1}{L^{1/2}} \sum_{l = 1}^L \sum_{n \in N} \left\{ M_2^n (B_l, X_l, N_l; v) - \mu_{M_2^n} (v) \right\} \\
\quad \text{Var} \left[ \sum_{n \in N} M_2^n (B, X, N_l) \right]^{1/2}
\]
and
\[
\frac{1}{L} \sum_{l = 1}^L M_2^n (B_l, X_l, N_l; v) - \mu_{M_2^n} (v) = \frac{1}{L} \sum_{l = 1}^L M_2^n (B_l, X_l, N_l; v) - \mu_{M_2^n} (v)
\]
for all $v \in I(x)$, since
\[
M_2^n (B_l, X_l, N_l) - \mu_{M_2^n} (v) = M_2^n (B_l, X_l, N_l) - \mu_{M_2^n} (v), \text{ for all } l = 1, ..., L.
\]

It follows from standard arguments that $\{ M_1^n (:, :; v) : v \in I(x) \}$ is uniformly VC-type with respect to the envelope
\[
F_{M_1^n} (z, z') := \frac{\bar{C} (C_1 + C_2) \bar{C}_{K_y}}{(n - 1) C_p} \left| K^0 X \left( \frac{z - x}{h} \right) \right|, \\
\left| K^0 X \left( \frac{z' - x}{h} \right) \right| \varphi (z') \, dz'.
\]

Then it follows from Chen and Kato (2017, Lemma 5.4) that $\{ M_2^n (:, :; v) : v \in I(x) \}$ is uniformly VC-type with respect to the envelope
\[
F_{M_2^n} (z) := \frac{\bar{C} (C_1 + C_2) \bar{C}_{K_y}}{(n - 1) C_p} \int \frac{1}{h^{3+2d}} \left| K^0 X \left( \frac{z' - x}{h} \right) \right|, \\
\left| K^0 X \left( \frac{z' - x}{h} \right) \right| \varphi (z') \, dz'.
\]

Let
\[
\sigma_{M_2^n}^2 := \sup_{v \in I(x)} E \left[ M_2^n (B_1, X_1, N_1; v)^2 \right] = O \left( h^{-(3+d)} \right).
\]

The CCK inequality yields
\[
E \left[ \sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l = 1}^L M_2^n (B_l, X_l, N_l; v) - \mu_{M_2^n} (v) \right| \right] \\
\leq C_1 \left\{ L^{-1/2} \sigma_{M_2^n} \log (C_2 L)^{1/2} + L^{-1/2} \left\| F_{M_2^n} \right\|_{\text{lin}} \log (C_2 L) \right\}, \\
= O \left( \frac{\log (L)}{(L h^3)^{1/2}} \right),
\]
where the equality follows from (S.3.88) and $\left\| F_{M_2^n} \right\|_{\text{lin}} = O \left( h^{-(3+d)} \right)$ (which follows from change of variables). By Markov’s inequality, we have
\[
\sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l = 1}^L M_2^n (B_l, X_l, N_l; v) - \mu_{M_2^n} (v) \right| = \sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l = 1}^L M_2^n (B_l, X_l, N_l; v) - \mu_{M_2^n} (v) \right| = O_p \left( \frac{\log (L)}{(L h^3)^{1/2}} \right).
\]
By the above result, Lemma S.3.3, (S.3.43) and (S.1.76), we have

\[
\hat{f}_{GPV}(v|x) - f(v|x) = \frac{1}{\varphi(x)} \sum_{l=1}^{L} \sum_{n \in \mathbb{N}} \left\{ M_{2}^{n}(B_{l}, X_{l}, N_{l}; v) - \mu_{M_{l}^{n}}(v) \right\} + O_{p}\left( \frac{\log(L)}{L^{1/2}} \right)^{1/2} + \frac{\log(L)}{L^{1/2} + h^{1/2}} + h^{R}. \tag{S.3.90}
\]

We showed that

\[
\sup_{v \in l(x)} \left| V_{M}(v|x, n) - \text{Var} \left[ h^{(3+d)/2} M_{2}^{1,n}(B_{1}, X_{1}, N_{1}; v) \right] \right| = O \left( h^{3} \right)
\]

and

\[
\sup_{v \in l(x)} \left| \text{Var} \left[ \sum_{n \in \mathbb{N}} h^{(3+d)/2} M_{2}^{1,n}(B_{1}, X_{1}, N_{1}; v) \right] - \sum_{n \in \mathbb{N}} V_{M}(v|x, n) \right| = O \left( h^{3} \right)
\]

in the proof of Theorem 6.1. Therefore we have

\[
\sup_{v \in l(x)} \left| \text{Var} \left[ \sum_{n \in \mathbb{N}} h^{(3+d)/2} M_{2}^{1,n}(B_{1}, X_{1}, N_{1}; v) \right] - \sum_{n \in \mathbb{N}} V_{M}(v|x, n) \right| = O \left( h^{3} \right). \tag{S.3.91}
\]

Next, by (S.1.76), we have

\[
\frac{\hat{f}_{GPV}(v|x) - f(v|x)}{(Lh^{3+d})^{-1/2} \left\{ \sum_{n \in \mathbb{N}} \varphi(x)^{-2} V_{M}(v|x, n) \right\}^{1/2}} = \frac{\left\{ \sum_{n \in \mathbb{N}} \varphi(x)^{-2} V_{M}(v|x, n) \right\}^{1/2}}{L^{1/2} h^{(3+d)/2} + L^{1/2} h^{(3+d)/2 + R} + O_{p}\left( \frac{\log(L)}{L^{1/2} h^{(3+d)/2}} \right)}
\]

follows from the above decomposition, (S.3.89), (S.3.90) and (S.3.91).

Next, we have

\[
\frac{\hat{f}_{GPV}(v|x) - f(v|x)}{(Lh^{3+d})^{-1/2} \sum_{n \in \mathbb{N}} \varphi(x)^{-2} V_{M}(v|x, n)}^{1/2} - \frac{\hat{f}_{GPV}(v|x) - f(v|x)}{(Lh^{3+d})^{-1/2} \left\{ \sum_{n \in \mathbb{N}} \varphi(x)^{-2} V_{M}(v|x, n) \right\}^{1/2}} = \frac{\left\{ \sum_{n \in \mathbb{N}} \varphi(x)^{-2} V_{M}(v|x, n) \right\}^{1/2}}{L^{1/2} h^{(3+d)/2} + L^{1/2} h^{(3+d)/2 + R}}
\]

where the second equality follows from Theorem 6.2, (S.3.89) and Lemma S.3.3. The conclusion follows.
Lemma S.3.5. Suppose Assumptions 1 - 3 hold. Then

\[
\sup_{v \in I(\mathbf{x})} \left| \tilde{f}^*(v, \mathbf{x}, n) - \tilde{f}(v, \mathbf{x}, n) \right| = \mathcal{O}_p \left( \frac{\log(L)}{Lh^{1+d}} \right). 
\]

Proof of Lemma S.3.5. By the fact that the bids in the bootstrap sample are conditionally i.i.d. and LIE, we have

\[
E^* \left[ \tilde{f}^*(v, \mathbf{x}, n) \right] = E^* \left[ 1 (N_1^* = n) \frac{1}{N_1^*} \sum_{i=1}^{N_1^*} 1_{h^1+d} K_f \left( \frac{V_{\hat{i}} - v}{h}, \frac{X_{\hat{i}} - \mathbf{x}}{h} \right) \right] 
\]

By Jensen’s inequality and LIE, we have

\[
E^* \left[ \tilde{f}^*(v, \mathbf{x}, n) \right] \leq E^* \left[ 1 (N_1^* = n) \frac{1}{N_1^*} \sum_{i=1}^{N_1^*} 1_{h^1+d} K_f \left( \frac{V_{\hat{i}} - v}{h}, \frac{X_{\hat{i}} - \mathbf{x}}{h} \right)^2 \right] 
\]

By applying the arguments used to show (S.3.66), we have

\[
\tilde{\sigma}_n^2 := \sup_{v \in I(\mathbf{x})} E^* \left[ 1 (N_1^* = n) \frac{1}{N_1^*} \sum_{i=1}^{N_1^*} 1_{h^1+d} K_f \left( \frac{V_{\hat{i}} - v}{h}, \frac{X_{\hat{i}} - \mathbf{x}}{h} \right)^2 \right] = \mathcal{O}_p \left( h^{-(1+d)} \right). 
\]

Let

\[
\tilde{F}_{*;n}(\mathbf{u}, \mathbf{z}, m; v) := \mathbb{I} (m = n) \frac{1}{m} \sum_{i=1}^{m} 1_{h^1+d} K_f \left( \frac{u_i - v}{h}, \frac{z - \mathbf{x}}{h} \right). 
\]

Standard arguments can be applied to verify that the class \( \{ \tilde{F}_{*;n} (\cdot; v) : v \in I(\mathbf{x}) \} \) is uniformly VC-type with respect to the envelope

\[
F_{*;n}(z) := (C_{D_1} + C_{D_2}) \frac{1}{h^1+d} K^0_X \left( \frac{z - \mathbf{x}}{h} \right). 
\]

Now the CCK inequality yields

\[
E^* \left[ \sup_{v \in I(\mathbf{x})} \left| \tilde{f}^*(v, \mathbf{x}, n) - \tilde{f}(v, \mathbf{x}, n) \right| \right] \leq C_1 \left\{ L^{-1/2} \tilde{\sigma} \log(C_2 L)^{1/2} + L^{-1} \| F_{*;n} \|_X \log(C_2 L) \right\}. 
\]

The conclusion follows from this result, (S.3.92) and Lemma S.1.2.

Lemma S.3.6. Suppose Assumptions 1 - 3 hold. Then

\[
\max_{n' \in N_{\hat{L}}((b', \mathbf{x}'), h)} \sup_{(b', \mathbf{x}', n') \in S_{B, X}^{n'}} \left| \tilde{G}^*(b', \mathbf{x}', n') - \tilde{G}(b', \mathbf{x}', n') \right| = \mathcal{O}_p \left( \frac{\log(L)}{Lh^{1+d}} \right)^{1/2} 
\]

and

\[
\max_{n' \in N_{\hat{L}}((b', \mathbf{x}'), h)} \sup_{(b', \mathbf{x}', n') \in S_{B, X}^{n'}} \left| \tilde{g}^*(b', \mathbf{x}', n') - \tilde{g}(b', \mathbf{x}', n') \right| = \mathcal{O}_p \left( \frac{\log(L)}{Lh^{1+d}} \right)^{1/2}. 
\]

S.61
Proof of Lemma S.3.6. Fix $n' \in \mathcal{N}$. Again by the fact that the bids in the bootstrap sample are conditionally i.i.d. and LIE, we have

\[
\mathbb{E}^* \left[ \tilde{G}^* (b', x', n') \right] = \mathbb{E}^* \left[ \mathbb{I} \left( N_1^* = n' \right) \frac{1}{N_1^*} \sum_{i=1}^{N_1^*} 1 \left( B_{i1} \leq b' \right) \frac{1}{h^d} K_{X} \left( \frac{X_i^* - x'}{h} \right) \right]
\]

By Jensen’s inequality and LIE, we have

\[
\mathbb{E}^* \left[ \mathbb{I} \left( N_1^* = n' \right) \frac{1}{N_1^*} \sum_{i=1}^{N_1^*} 1 \left( B_{i1} \leq b' \right) \frac{1}{h^d} K_{X} \left( \frac{X_i^* - x'}{h} \right) \right] \leq \mathbb{E}^* \left[ \mathbb{I} \left( N_1^* = n' \right) \frac{1}{N_1^*} \sum_{i=1}^{N_1^*} 1 \left( B_{i1} \leq b' \right) \frac{1}{h^d} K_{X} \left( \frac{X_i^* - x'}{h} \right)^2 \right]
\]

\[
= \frac{1}{L} \sum_{l=1}^{L} \mathbb{I} \left( N_l = n' \right) \frac{1}{N_l} \sum_{i=1}^{N_l} 1 \left( B_{il} \leq b' \right) \frac{1}{h^d} K_{X} \left( \frac{X_i^* - x'}{h} \right)^2 .
\]

By arguments used to show (S.3.50), we have

\[
\sup_{\mathbb{H}((b', x'), h) \subseteq \mathcal{S}^n_{b', X}} \left| \frac{1}{L} \sum_{l=1}^{L} \mathbb{I} \left( N_l = n' \right) \frac{1}{N_l} \sum_{i=1}^{N_l} 1 \left( B_{il} \leq b' \right) \frac{1}{h^d} K_{X} \left( \frac{X_i^* - x'}{h} \right)^2 - \mathbb{E} \left[ \mathbb{I} \left( N_l = n' \right) \frac{1}{N_l} \sum_{i=1}^{N_l} 1 \left( B_{il} \leq b' \right) \frac{1}{h^d} K_{X} \left( \frac{X_i - x'}{h} \right)^2 \right] \right|
\]

\[
= \mathcal{O}_p \left( \left( \log \frac{L}{h^{1+\epsilon}} \right)^{1/2} \right)
\]

Now by the LIE and change of variables, we have

\[
\mathbb{E} \left[ \mathbb{I} \left( N_1 = n' \right) \frac{1}{N_1} \sum_{i=1}^{N_1} 1 \left( B_{i1} \leq b' \right) \frac{1}{h^d} K_{X} \left( \frac{X_i^* - x'}{h} \right)^2 \right] = \int_{x} G \left( b', h y + x', n' \right) K_{X} \left( y \right)^2 dy.
\]

Now it is clear that since $K_1$ is compactly supported on $[-1, 1]$,

\[
\sup_{\mathbb{H}((b', x'), h) \subseteq \mathcal{S}^n_{b', X}} \mathbb{E}^* \left[ \mathbb{I} \left( N_1 = n' \right) \frac{1}{N_1} \sum_{i=1}^{N_1} 1 \left( B_{i1} \leq b' \right) \frac{1}{h^d} K_{X} \left( \frac{X_i^* - x'}{h} \right)^2 \right] = O(1).
\]

Now it easily follows that

\[
\hat{\sigma}^2_B := \sup_{\mathbb{H}((b', x'), h) \subseteq \mathcal{S}^n_{b', X}} \mathbb{E}^* \left[ \mathbb{I} \left( N_1 = n' \right) \frac{1}{N_1} \sum_{i=1}^{N_1} 1 \left( B_{i1} \leq b' \right) \frac{1}{h^d} K_{X} \left( \frac{X_i^* - x'}{h} \right)^2 \right] = \mathcal{O}_p \left( h^{-d} \right).
\]

Let

\[
\mathcal{G}^{*, n'} \left( u, z, m ; b', x' \right) := \mathbb{I} \left( m = n' \right) \frac{1}{m} \sum_{i=1}^{m} \frac{1}{h^d} 1 \left( u_i \leq b' \right) K_X \left( \frac{z - x'}{h} \right).
\]

Standard arguments can be applied to verify that the class $\left\{ \mathcal{G}^{*, n'} \left( ; b', x' \right) : \mathbb{H}((b', x'), h) \subseteq \mathcal{S}^n_{b', X} \right\}$ is uniformly VC-type with respect to a constant envelope $h^{-d} \left( \mathcal{C}_{D_1} + \mathcal{C}_{D_2} \right) \sup_{x' \in \mathbb{R}^d} \left[ K_X \left( x' \right) \right]$. Then the CCK inequality yields

\[
\mathbb{E}^* \left[ \sup_{\mathbb{H}((b', x'), h) \subseteq \mathcal{S}^n_{b', X}} \left| \tilde{G}^* \left( b', x', n' \right) - \tilde{G} \left( b', x', n' \right) \right| \right] \leq C_1 \left( L^{-1/2} \hat{\sigma}^2 B \log \left( C_2 L \right)^{1/2} + (L h^d)^{-1} \log \left( C_2 L \right) \right)
\]

\[
= \mathcal{O}_p \left( \left( \log \frac{L}{h^{1+\epsilon}} \right)^{1/2} \right).
\]
The first conclusion follows from Lemma S.1.2 and also the assumption that $\mathcal{N}$ is a bounded set of positive integers. The second conclusion follows from similar arguments.

Lemma S.3.7. Suppose that Assumptions 1 - 3 hold. Let $x$ be an interior point of $\mathcal{X}$ and $n \in \mathcal{N}$ be fixed. Let

$$
\tilde{T}_{il}^* := \mathbb{I} \left( (V_{il}^*, X_i^*) \in \mathbb{H} \left( (v, x), \delta \right) \right).
$$

Then

$$
\hat{r}_{PV} (v, x, n) - \tilde{r}^* (v, x, n) = \frac{1}{L} \sum_{l=1}^{L} \frac{1}{h^{2+d}} \nu_f \left( \frac{\tilde{V}_{il}^* - v}{h}, \frac{X_i^* - x}{h} \right) \left( \tilde{V}_{il}^* - V_{il}^* \right) + O_p \left( \frac{\log (L)}{L h^{1+\delta}} + h^R \right),
$$

where the remainder term is uniform in $v \in I (x)$.

Proof of Lemma S.3.7. It follows from Lemma S.3.6, (S.3.50) and Marmer and Shneyerov (2012, Lemma S1) that

$$
\max_{n' \in \mathcal{N}_i} \mathbb{E} \left( \sup_{(b', \mathcal{X})} \left| \tilde{G}^* (b', x', n') - G (b', x', n') \right| \right) = O_p \left( \frac{\log (L)}{L h^d} \right)^{1/2} + h^{1+R},
$$

(S.3.93)

We apply again the arguments used in the proof of Lemma S.3.1. First, we have

$$
\max_{i,l} T_{il}^* \left| \tilde{V}_{il}^* - V_{il}^* \right| \leq \max_{i,l} T_{il} \left| \frac{\tilde{G}^* (B_{il}^*, X_i^*, N_i^*) - g (B_{il}^*, X_i^*, N_i^*)}{g (B_{il}^*, X_i^*, N_i^*)} \right| + \max_{i,l} T_{il} \left| \frac{\tilde{G}^* (B_{il}^*, X_i^*, N_i^*) (\tilde{g}^* (B_{il}^*, X_i^*, N_i^*) - g (B_{il}^*, X_i^*, N_i^*))}{g (B_{il}^*, X_i^*, N_i^*)^2} \right|(S.3.94)
$$

Denote

$$
\overline{T}_{il}^* := \mathbb{I} \left( ((B_{il}^*, X_i^*), h) \subseteq \mathcal{S}_{B', \mathcal{X}}^{N_i} \right).
$$

For the first term of the right hand side of (S.3.94), we have

$$
\max_{i,l} \overline{T}_{il} \left| \frac{\tilde{G}^* (B_{il}^*, X_i^*, N_i^*) - g (B_{il}^*, X_i^*, N_i^*)}{g (B_{il}^*, X_i^*, N_i^*)} \right| = \max_{i,l} \overline{T}_{il} \left| \frac{\tilde{G}^* (B_{il}^*, X_i^*, N_i^*) (\tilde{g}^* (B_{il}^*, X_i^*, N_i^*) - g (B_{il}^*, X_i^*, N_i^*))}{g (B_{il}^*, X_i^*, N_i^*)^2} \right| \leq \max_{i,l} \overline{T}_{il} \left| \frac{\tilde{G}^* (B_{il}^*, X_i^*, N_i^*) - g (B_{il}^*, X_i^*, N_i^*)}{g (B_{il}^*, X_i^*, N_i^*)} \right| + \max_{i,l} \overline{T}_{il} \left| \frac{\tilde{G}^* (B_{il}^*, X_i^*, N_i^*) - g (B_{il}^*, X_i^*, N_i^*)}{g (B_{il}^*, X_i^*, N_i^*)} \right| \leq O_p \left( \frac{\log (L)}{L h^d} \right)^{1/2} + h^{1+R}.\tag{S.3.95}
$$

and by (S.3.2) and (S.3.93),

$$
\max_{i,l} \overline{T}_{il} \left| \frac{\tilde{G}^* (B_{il}^*, X_i^*, N_i^*) - g (B_{il}^*, X_i^*, N_i^*)}{g (B_{il}^*, X_i^*, N_i^*)} \right| \leq \max_{i,l} \overline{T}_{il} \left| \tilde{g}^* (B_{il}^*, X_i^*, N_i^*) - g (B_{il}^*, X_i^*, N_i^*) \right| = O_p \left( \frac{\log (L)}{L h^d} \right)^{1/2} + h^{1+R}.\tag{S.3.96}
$$

It is argued in the proof of Lemma S.3.1 that if

$$
\sup_{(x', n') \in \mathcal{X} \times \mathcal{N}} \left| \tilde{b} (x', n') - \tilde{b} (x', n') \right| \leq h, \sup_{x' \in \mathcal{X}} \left| \tilde{b} (x') - \tilde{b} (x') \right| \leq h \quad \text{and} \quad \mathbb{E} ( (B_{il}^*, X_i) , 2h) \subseteq \mathcal{S}_{B', \mathcal{X}}^{N_i},
$$

we must have \( \mathbb{E} ((B_{il}^*, X_i) , 2h) \subseteq \mathcal{S}_{B', \mathcal{X}}^{N_i} \). Now it is clear that when $h$ is sufficiently small,

$$
P \left[ \max_{i,l} \overline{T}_{il} \left( 1 - \overline{T}_{il} \right) > 0 \right] \leq \mathbb{I} \left( \max_{i,l} \overline{T}_{il} \left| \tilde{b} (x', n') - \tilde{b} (x', n') \right| \leq h \right) + \mathbb{I} \left( \sup_{x' \in \mathcal{X}} \left| \tilde{b} (x') - \tilde{b} (x') \right| \leq h \right) = o_p (1),
$$

S.63
Similarly, we have for any null sequence $\epsilon \downarrow 0$. By the above result, (S.3.95) and (S.3.96), we have

$$
\max_{i,t} T^*_{il} \left| \tilde{g}^* (B^*_{il}, X^*_i, N^*_i) - g (B^*_{il}, X^*_i, N^*_i) \right| = O_p^* \left( \left( \frac{\log (L)}{Lh^d} \right)^{1/2} + h^{1+R} \right).
$$

(S.3.97)

Similarly, we have

$$
\max_{i,t} T^*_{il} \left| \tilde{g}^* (B^*_{il}, X^*_i, N^*_i) \right| = O_p^* \left( \left( \frac{\log (L)}{Lh^d} \right)^{1/2} + h^{1+R} \right).
$$

(S.3.98)

It follows from (S.3.93) that

$$
\max_{i,l} T^*_{il} \left| \tilde{g}^* (B^*_{il}, X^*_i, N^*_i) - g (B^*_{il}, X^*_i, N^*_i) \right| = O_p^* (1).
$$

It follows from the above result and (S.3.2) that

$$
P^* \left[ \max_{i,t} T^*_{il} \left| \tilde{g}^* (B^*_{il}, X^*_i, N^*_i)^{-1} \right| \leq \left( \frac{C}{p} \right)^{-1} \right] \rightarrow_p 1, \quad \text{as } L \uparrow \infty,
$$

which further implies $\max_{i,l} T^*_{il} \left| \tilde{g}^* (B^*_{il}, X^*_i, N^*_i)^{-1} \right| = O_p^* (1)$. Therefore we have

$$
\max_{i,t} T^*_{il} \left| \tilde{g}^* (B^*_{il}, X^*_i, N^*_i) \right|^2 = O_p^* \left( \left( \frac{\log (L)}{Lh^d} \right)^{1/2} + h^{2+2R} \right).
$$

(S.3.99)

It follows from the above result, (S.3.94), (S.3.97) and (S.3.98) that

$$
\max_{i,t} T^*_{il} \left| \tilde{V}^*_{il} - V^*_{il} \right| = O_p^* \left( \left( \frac{\log (L)}{Lh^d} \right)^{1/2} + h^{1+R} \right).
$$

We showed in the proof of Lemma S.3.1 that when $h$ is sufficiently small, for every $v \in I (x)$, $(V^*_i, X^*_i) \in \mathbb{E} ((v, x), \delta)$ implies $\mathbb{E} ((B^*_{il}, X^*_i), h) \subseteq S_{B, X}^N$. Therefore we have

$$
\sup_{v \in I (x)} \max_{i,t} T^*_{il} \left| \tilde{V}^*_{il} - V^*_{il} \right| = O_p^* \left( \left( \frac{\log (L)}{Lh^d} \right)^{1/2} + h^{1+R} \right).
$$

(S.3.100)

Write

$$
\tilde{f}^{GPV} (v, x, n) = \frac{1}{L} \sum_{i=1}^L 1 (N^*_i = n) \frac{1}{N^*_i} \sum_{i=1}^N \sum_{t=1}^{N^*_i} T^*_{il} \left( 1 - T^*_{il} \right) \frac{1}{h^{1+d}} \log \left( \frac{1}{h^{1+d}} \log \frac{1}{h} \right)
$$

where

$$
\kappa^*_1 (v) := \frac{1}{L} \sum_{i=1}^L 1 (N^*_i = n) \frac{1}{N^*_i} \sum_{i=1}^N \sum_{t=1}^{N^*_i} T^*_{il} \left( 1 - T^*_{il} \right) \frac{1}{h^{1+d}} \log \left( \frac{1}{h^{1+d}} \log \frac{1}{h} \right) \left( \frac{V^*_{il} - v}{h} - X^*_i - x \right)
$$

$$
\kappa^*_2 (v) := \frac{1}{L} \sum_{i=1}^L 1 (N^*_i = n) \frac{1}{N^*_i} \sum_{i=1}^N \sum_{t=1}^{N^*_i} T^*_{il} \left( 1 - T^*_{il} \right) \frac{1}{h^{1+d}} \log \left( \frac{1}{h^{1+d}} \log \frac{1}{h} \right) \left( \frac{V^*_{il} - v}{h} - X^*_i - x \right).
$$

Since $K_0$ is supported on $[-1, 1]$, $K_0 ((\tilde{V}^*_{il} - v)/h)$ is zero if $\tilde{V}^*_{il}$ is outside of a $h$–neighborhood of $v$. Then by the triangle inequality, we
have

\[|\kappa_1^*(v)| \leq \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l^* = n) \frac{1}{N_l^*} \sum_{i=1}^{N_l^*} \mathbb{T}_{il} \left( 1 - \tilde{w}_{il} \right) h^{-5(1+d)} \mathbb{1} \left( |\hat{V}_{il}^* - v| \leq h \right) \mathbb{1} \left( X^*_i \in \mathbb{H}(x, h) \right) \]

\[\leq \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l^* = n) \frac{1}{N_l^*} \sum_{i=1}^{N_l^*} \mathbb{T}_{il} \left( 1 - \tilde{w}_{il} \right) h^{-5(1+d)} \mathbb{1} \left( |V_{il}^* - v| \leq h + \max_{l,i} \mathbb{T}_{il} \right) \mathbb{1} \left( X^*_i \in \mathbb{H}(x, h) \right).\]

Therefore it is clear that

\[P^* \left[ \sup_{v \in I(x)} |\kappa_1^*(v)| > 0 \right] \leq P^* \left[ \max_{i,l} \mathbb{T}_{il} \left| \hat{V}_{il}^* - V_l^* \right| > \frac{\sigma}{2} \right] = o_p(1),\]

where the inequality holds when \( h \) is sufficiently small. Therefore, \( \sup_{v \in I(x)} |\kappa_1^*(v)| = o_p^*(\epsilon_L) \) for any null sequence \( \epsilon_L \downarrow 0. \)

As argued in the proof of Lemma \( \text{S.3.1} \), when \( h \) is sufficiently small,

\[\sup_{v \in I(x)} |\kappa_1^*(v)| \leq h^{-5(1+d)} \mathbb{1} \left( \sup_{\mathbb{X}} \left( s \left( v_u(x) + \delta, \mathbb{X}, n \right) + 2h > \inf_{\mathbb{X}} \tilde{b}(x') \right) \right) \]

\[+ h^{-5(1+d)} \mathbb{1} \left( \inf_{\mathbb{X}} \tilde{b}(x') - h(x') \right) \]

Now it follows that

\[P^* \left[ \sup_{v \in I(x)} |\kappa_1^*(v)| > 0 \right] \leq 1 \left( \sup_{\mathbb{X}} \left( s \left( v_u(x) + \delta, \mathbb{X}, n \right) + 2h > \inf_{\mathbb{X}} \tilde{b}(x') \right) \right) \]

\[+ \mathbb{1} \left( \inf_{\mathbb{X}} \tilde{b}(x') - h(x') \right) \]

\[= o_p(1),\]

where the equality follows from \( \text{S.3.54} \) and Markov’s inequality. Therefore, \( \sup_{v \in I(x)} |\kappa_1^*(v)| = o_p^*(\epsilon_L) \) for any null sequence \( \epsilon_L \downarrow 0. \)

A second-order Taylor expansion gives

\[\tilde{f}_{GPV}^*(v, x, n) - \tilde{f}^*(v, x, n) = \quad \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l^* = n) \frac{1}{N_l^*} \sum_{i=1}^{N_l^*} \mathbb{T}_{il} \frac{1}{h^{5+d}} K_f^l \left( \frac{V_{il}^* - v}{h}, \frac{X^*_i - x}{h} \right) \left( \hat{V}_{il}^* - V_{il}^* \right)
\]

\[+ \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l^* = n) \frac{1}{N_l^*} \sum_{i=1}^{N_l^*} \mathbb{T}_{il} \frac{1}{h^{5+d}} K_f^l \left( \frac{X^*_i - x}{h} \right) \left( \hat{V}_{il}^* - V_{il}^* \right)^2 \quad (S.3.101)\]

for some mean value \( \hat{V}_{il}^* \) that lies on the line joining \( \hat{V}_{il}^* \) and \( V_{il}^* \), with some remainder term that is \( o_p^*(\epsilon_L) \) for any null sequence \( \epsilon_L \downarrow 0. \) It follows from the triangle inequality and the Lipschitz condition imposed on the kernel that

\[\left| \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l^* = n) \frac{1}{N_l^*} \sum_{i=1}^{N_l^*} \mathbb{T}_{il} \frac{1}{h^{5+d}} K_f^l \left( \frac{X^*_i - x}{h} \right) \left( \hat{V}_{il}^* - V_{il}^* \right)^2 \right|
\]

\[\leq \left( \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l^* = n) \frac{1}{N_l^*} \sum_{i=1}^{N_l^*} \mathbb{T}_{il} \frac{1}{h^{5+d}} \right) \left| K_f^l \left( \frac{X^*_i - x}{h} \right) \right| \left( \max_{l,i} \mathbb{T}_{il} \right) \left( \hat{V}_{il}^* - V_{il}^* \right)^2 \quad (S.3.102)\]

By the triangle inequality, we have

\[\frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l^* = n) \frac{1}{N_l^*} \sum_{i=1}^{N_l^*} \mathbb{T}_{il} \frac{1}{h^{5+d}} \left| K_f^l \left( \frac{X^*_i - x}{h} \right) \right| \left( \hat{V}_{il}^* - V_{il}^* \right) \leq \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l^* = n) \frac{1}{N_l^*} \sum_{i=1}^{N_l^*} \mathbb{T}_{il} \left( 1 - \tilde{w}_{il} + \tilde{w}_{il} \right) \frac{1}{h^{5+d}} \left| K_f^l \left( \frac{X^*_i - x}{h} \right) \right| \left( \hat{V}_{il}^* - V_{il}^* \right) \]

\[\leq \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l^* = n) \frac{1}{N_l^*} \sum_{i=1}^{N_l^*} \mathbb{T}_{il} \left( 1 - \tilde{w}_{il} + \tilde{w}_{il} \right) \frac{1}{h^{5+d}} \left| K_f^l \left( \frac{X^*_i - x}{h} \right) \right| \left( \hat{V}_{il}^* - V_{il}^* \right) \leq \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}(N_l^* = n) \frac{1}{N_l^*} \sum_{i=1}^{N_l^*} \mathbb{T}_{il} \left( 1 - \tilde{w}_{il} + \tilde{w}_{il} \right) \frac{1}{h^{5+d}} \left| K_f^l \left( \frac{X^*_i - x}{h} \right) \right| \left( \hat{V}_{il}^* - V_{il}^* \right)
\]
\[ \sum_{i=1}^{N_l} \left( \frac{\tau_{il}^1}{h} \right) K_X \left( \frac{X_i^l - x}{h} \right) + \kappa^*_3 (v), \] (S.3.103)

where

\[ \tau_{il}^1 := 1 \left( |V_{il}^1 - v| \leq 2h \right) \]

and

\[ \kappa^*_3 (v) := \sum_{i=1}^{N_l} \left( \frac{\tau_{il}^1}{h} \right) \left( |V_{il}^1 - v| > 2h \right) \frac{1}{h^{3+\delta}} K_X \left( \frac{X_i^l - x}{h} \right) \left( |V_{il}^1 - v| \leq h + \max_{j,k} \tau_{jkl}^1 \left| \tilde{V}_{jkl} - V_{jkl} \right| \right). \]

Clearly we have

\[ P^* \left[ \sup_{v \in I(x)} \left| \kappa^*_3 (v) \right| > 0 \right] \leq P^* \left[ \sup_{v \in I(x)} \max_{j,k} \tau_{jkl}^1 \left| \tilde{V}_{jkl} - V_{jkl} \right| > h \right] = o_p (1), \]

where the equality follows from (S.3.100). Therefore, \( \sup_{v \in I(x)} \left| \kappa^*_3 (v) \right| = o_p (\epsilon_L), \) for any null sequence \( \epsilon_L \downarrow 0. \)

By arguments used in the proof of Lemma S.3.5, we can easily show

\[ \sup_{v \in I(x)} \left| \frac{1}{L} \sum_{i=1}^{N_l} 1 (N_l^* = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \tau_{il}^1 \frac{1}{h^{3+\delta}} K_X \left( \frac{X_i^l - x}{h} \right) - E^* \left[ \frac{1}{L} \sum_{i=1}^{N_l} \tau_{il}^1 \frac{1}{h^{3+\delta}} K_X \left( \frac{X_i^l - x}{h} \right) \right] \right| = O_p \left( \left( \frac{\log (L)}{Lh^{3+\delta}} \right)^{1/2} \right). \]

We also have

\[ \sup_{v \in I(x)} \left[ \frac{1}{L} \sum_{i=1}^{N_l} 1 (N_l^* = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \tau_{il}^1 \frac{1}{h^{3+\delta}} K_X \left( \frac{X_i^l - x}{h} \right) \right] = \sup_{v \in I(x)} \frac{1}{L} \sum_{i=1}^{L} f_n^e (V_{il}, X_i, N_l; v) = O_p (1), \]

where the first equality follows from LIE and the fact that the bids in the bootstrap sample are conditionally i.i.d. and the second equality was shown in the proof of Lemma S.3.1. See (S.3.66). Now it follows that

\[ \frac{1}{L} \sum_{i=1}^{L} 1 (N_l^* = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \tau_{il}^1 \frac{1}{h^{3+\delta}} K_X \left( \frac{X_i^l - x}{h} \right) = O_p \left( h^{-2} \right), \]

uniformly in \( v \in I (x). \) Then it follows from the above result that

\[ \left( \frac{1}{L} \sum_{i=1}^{L} \tau_{il}^1 \frac{1}{h^{3+\delta}} K_X \left( \frac{X_i^l - x}{h} \right) \left( |V_{il}^1 - v| \leq h \right) \right) \left( \max_{i,l} \tau_{i,l}^1 \left| \tilde{V}_{i,l}^1 - V_{i,l}^1 \right|^2 \right) = O_p \left( \frac{\log (L)}{Lh^{3+\delta}} + h^{2R} \right), \]

uniformly in \( v \in I (x). \) The conclusion follows.

**Lemma S.3.8.** Suppose that Assumptions 1 - 3 hold. Let \( x \) be an interior point of \( X \) and \( n \in N \) be fixed. Then

\[ \hat{f}_{G_{PV}} (v, x, n) - \hat{f}^* (v, x, n) = \frac{1}{L^2} \sum_{i=1}^{L} \sum_{k=1}^{L} M^n ((B_{i,k}^l, X_{i,k}^l, N_{i,k}^l), (B_{i,k}^l, X_{i,k}^l, N_{i,k}^l); v) + O_p \left( \left( \frac{\log (L)}{Lh^{3+\delta}} \right)^{1/2} + \frac{\log (L)}{Lh^{3+\delta}} + h^{3R} \right), \]

where the remainder term is uniform in \( v \in I (x). \)

**Proof of Lemma S.3.8.** We have

\[ \frac{1}{L} \sum_{i=1}^{L} 1 (N_l^* = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \tau_{il}^1 \frac{1}{h^{2+\delta}} R_{ij-l} \left( \frac{X_i^l - v}{h} \right) \left( \frac{X_j^l - x}{h} \right) \left( \tilde{V}_{il}^1 - V_{il}^1 \right) \]

\[ = - \frac{1}{L} \sum_{i=1}^{L} 1 (N_l^* = n) \frac{1}{N_l} \sum_{i=1}^{N_l} \tau_{il}^1 \frac{1}{h^{2+\delta}} R_{ij-l} \left( \frac{X_i^l - v}{h} \right) \left( \frac{X_j^l - x}{h} \right) \frac{1}{N_l^* - 1} G \left( B_{il}^l, X_{il}^l, N_{il}^l \right) \left( g \left( B_{il}^l, X_{il}^l, N_{il}^l \right) - g \left( B_{il}^l, X_{il}^l, N_{il}^l \right) \right) \]

\[ + \Delta^*_1 (v) + \Delta^*_2 (v) + \Delta^*_3 (v), \] (S.3.104)
where \( \Delta_1^*(v), \Delta_2^*(v) \) and \( \Delta_3^*(v) \) are bootstrap versions of \( \Delta_1^*(v), \Delta_2^*(v) \) and \( \Delta_3^*(v) \). Now

\[
\sup_{v \in I(x)} |\Delta_2^*(v)| = O_p\left(\frac{\log(L)}{Lh} + h^{2R+1}\right) \quad \text{and} \quad \sup_{v \in I(x)} |\Delta_3^*(v)| = O_p\left(\frac{\log(L)}{Lh} + h^{2R+1}\right)
\]

follow from

\[
\sup_{v \in I(x)} \frac{1}{L} \sum_{l=1}^{L} \mathbb{1}(N_l^* = n) \frac{1}{N_l^* \cdot 1 - h^{1+\alpha}} \left| K_f \left( \frac{V_{d,l} - v}{h} \cdot \frac{X_l^* - x}{h} \right) \right| \leq \sup_{v \in I(x)} \frac{1}{L} \sum_{l=1}^{L} \mathbb{1}(N_l^* = n) \frac{1}{N_l^* \cdot 1} \left| K_f \left( \frac{X_l^* - x}{h} \right) \right|
\]

(S.3.105)

where the right hand side was shown to be \( O_p(1) \) in the previous lemma and

\[
\sup_{v \in I(x)} \max_{i,l} \left| g^* (B_{il}^*, X_l^*, N_l^*)^{-1} \right| \leq \max_{i,l} \mathbb{E} \left| g^* (B_{il}^*, X_l^*, N_l^*)^{-1} \right| = O_p(1).
\]

Since \( K_f \) has a bounded support, the contribution of the trimmed observations is asymptotically negligible. When \( h \) is sufficiently small, we have

\[
\Delta_1^*(v) = \frac{1}{L^2} \sum_{l=1}^{L} \sum_{k=1}^{L} \mathbb{G}^n ((B_{l}^*, X_l^*, N_l^*), (B_{k}^*, X_k^*, N_k^*); v), \text{ for all } v \in I(x).
\]

(S.3.106)

Let

\[
\mu_{G^n}(v) := \mathbb{E}^* \left[ \mathbb{G}^n ((b, z, m), (B_{l}^*, X_l^*, N_l^*); v) \right] \quad \text{and} \quad \mu_{G^n}^*(v) := \mathbb{E}^* \left[ \mathbb{G}^n ((B_{l}^*, X_l^*, N_l^*); v) \right].
\]

The Hoeffding decomposition yields

\[
\frac{1}{L^2} \sum_{l=1}^{L} \sum_{k=1}^{L} \mathbb{G}^n ((B_{l}^*, X_l^*, N_l^*), (B_{k}^*, X_k^*, N_k^*); v) = \mu_{G^n}(v) + \frac{1}{L} \sum_{l=1}^{L} \mathbb{G}^n ((B_{l}^*, X_l^*, N_l^*); v) - \mu_{G^n}(v) + \frac{1}{L} \sum_{l=1}^{L} \mu_{G^n}(v)
\]

\[
+ \frac{1}{L (L-1)} \sum_{l \neq k} \mathbb{G}^n ((B_{l}^*, X_l^*, N_l^*), (B_{k}^*, X_k^*, N_k^*); v) - \mu_{G^n}^*(v) (B_{l}^*, X_l^*, N_l^*; v) - \mu_{G^n}^*(v) (B_{k}^*, X_k^*, N_k^*; v) + \mu_{G^n}(v)
\]

\[
+ \frac{1}{L^2} \sum_{l=1}^{L} \mathbb{G}^n ((B_{l}^*, X_l^*, N_l^*), (B_{l}^*, X_l^*, N_l^*); v) - \frac{1}{L^2 (L-1)} \sum_{l \neq k} \mathbb{G}^n ((B_{l}^*, X_l^*, N_l^*), (B_{k}^*, X_k^*, N_k^*); v).
\]

(S.3.107)

Since \( \mathbb{G}^n (\cdot, \cdot; v) : v \in I(x) \) is uniformly VC-type with respect to the envelope (S.3.77), the CK inequality yields

\[
E^* \left[ \sup_{v \in I(x)} \left| \frac{1}{(L^2)} \sum_{l=1}^{L} \mathbb{G}^n ((B_{l}^*, X_l^*, N_l^*), (B_{k}^*, X_k^*, N_k^*); v) - \mu_{G^n}(v) (B_{l}^*, X_l^*, N_l^*; v) - \mu_{G^n}^*(v) (B_{k}^*, X_k^*, N_k^*; v) + \mu_{G^n}(v) \right| \right]^2
\]

\[
\leq L^{-1} \left( E^* \left[ F_{G^n} (X_1^*, X_2^*)^2 \right] \right)^{1/2}.
\]

By change of variables, we have

\[
E \left[ \frac{1}{L^2} \sum_{l=1}^{L} \mathbb{G}^n (X_l, X_k) \right] = \frac{L - 1}{L} E \left[ F_{G^n} (X_1, X_2)^2 \right] + \frac{L}{L} E \left[ F_{G^n} (X_1, X_1)^2 \right]
\]

\[
= O \left( h^{-4+2\delta} \right).
\]
Then it follows that
\[ E^* \left[ F_{\hat{g}_n} (\mathbf{X}_1^*, \mathbf{X}_2^*)^2 \right] = \frac{1}{L^2} \sum_{l=1}^L \sum_{k=1}^L F_{\hat{g}_n} (\mathbf{X}_l, \mathbf{X}_k)^2 = O_p \left( n^{-4/2d} \right), \]
where the second equality follows from the previous result and Markov’s inequality. Now it follows that
\[
E^* \left[ \sup_{v \in I(\mathbf{x})} \left\{ \frac{1}{(L^2)} \sum_{(l,m) \in I(\mathbf{x})} G^n \left( (B^n_{l,m}, \mathbf{X}^*_l, N^*_l), (B^n_{m,l}, \mathbf{X}^*_m, N^*_m) ; v \right) - \hat{\varphi}_1^n (B^n_{l,m}, \mathbf{X}^*_l, N^*_l; v) - \hat{\varphi}_2^n (B^n_{m,l}, \mathbf{X}^*_m, N^*_m; v) + \tilde{\mu}_{g^n} (v) \right\} \right] = O_p \left( \left( Lh^{2+d} \right)^{-1} \right).
\]

Now by the LIE and the fact that the bids in the bootstrap sample are conditionally i.i.d., we have
\[
\hat{\varphi}^n_1 (b, z, m; v) = -1 (m = n) \frac{1}{m} \sum_{i=1}^m h^{2+d} K_f \left( \frac{\xi (b_i, z, m) - v}{h} \right) G \left( b_i, z, m \right) - G \left( 0, z, m \right).
\]
It follows from standard arguments that when \( h \) is sufficiently small, \( \left\{ \hat{\varphi}^n_1 (\cdot; v) : v \in I(\mathbf{x}) \right\} \) is VC-type with respect to the envelope
\[
F_{\hat{g}_n} (z) := \frac{C_{D_1} + C_{D_2}}{(n-1) C_g} \left| K_X \left( \frac{z - x}{h} \right) \right| \left\{ \sup_{(b', z') \in C_{D_1}^0} \left| \hat{G} (b', z', n) - G (b', z', n) \right| \right\},
\]
conditionally on the original sample. The VW inequality yields
\[
E^* \left[ \sup_{v \in I(\mathbf{x})} \frac{1}{L} \sum_{l=1}^L \hat{\varphi}^n_1 (B^n_{l}, \mathbf{X}^*_l, N^*_l; v) - \tilde{\mu}_{g^n} (v) \right] \leq L^{-1/2} \left( E^* \left[ F_{\hat{g}_n} (\mathbf{X})^2 \right] \right)^{1/2} \leq L^{-1/2} \left\{ \frac{1}{L} \sum_{l=1}^L \frac{1}{h^{2+d}} K_X \left( \frac{\mathbf{X}_l - x}{h} \right) \right\}^{1/2} \left\{ \sup_{(b', z') \in C_{D_1}^0} \left| \hat{G} (b', z', n) - G (b', z', n) \right| \right\} \leq O_p \left( \left( Lh^{2+d} \right)^{-1} \right) O_p \left( \frac{\log (L)}{L^{d+1}} \right)^{1/2} + h^{1+R},
\]
where the equality follows from change of variables, Markov’s inequality and (S.3.50).

Let
\[
\hat{\varphi}_1^n (B^n_{l}, \mathbf{X}^*_l, N^*_l; v) := E^* \left[ \hat{\varphi}^n_1^n (B^n_{l}, \mathbf{X}^*_l, N^*_l; v) \right] = \frac{1}{L} \sum_{l=1}^L \frac{1}{(N_k - 1)} \sum_{j=1}^{N_k} \frac{1}{h^{2+d}} K_f \left( \frac{\xi (B_{j,k}, \mathbf{X}_k, N_k) - v}{h} \right) \left| \mathbf{X}_k - \mathbf{X}_l \right| \frac{1}{N_k-1} \frac{1}{h^{2+d}} K_f \left( \frac{z - \mathbf{X}_k}{h} \right),
\]
where the second equality follows from LIE and the fact that the bids in the bootstrap sample are conditionally i.i.d.. It is straightforward to verify that
\[
\hat{\varphi}_2^n (B^n_{l}, \mathbf{X}^*_l, N^*_l; v) = \hat{\varphi}_2^n (B^n_{l}, \mathbf{X}^*_l, N^*_l; v) - E^* \left[ \hat{\varphi}_2^n (B^n_{l}, \mathbf{X}^*_l, N^*_l; v) \right], \text{ for all } l = 1, \ldots, L
\]
and thus
\[
\frac{1}{L} \sum_{l=1}^L \hat{\varphi}_2^n (B^n_{l}, \mathbf{X}^*_l, N^*_l; v) = \frac{1}{L} \sum_{l=1}^L \hat{\varphi}_2^n (B^n_{l}, \mathbf{X}^*_l, N^*_l; v) - E^* \left[ \hat{\varphi}_2^n (B^n_{l}, \mathbf{X}^*_l, N^*_l; v) \right].
\]

Since \( \left\{ \hat{\varphi}^n_1 (\cdot; v) : v \in I(\mathbf{x}) \right\} \) is uniformly VC-type with respect to the envelope (S.3.80) (see the proof of Lemma S.3.2), it follows from Chen and Kato (2017, Lemma 5.4) that the class \( \left\{ \hat{\varphi}_2^n (\cdot; v) : v \in I(\mathbf{x}) \right\} \) is uniformly VC-type with respect to the envelope
\[
F_{\hat{g}_2^n} (z) := \frac{C_{D_1} + C_{D_2}}{(n-1) C_g} \frac{1}{L} \sum_{l=1}^L \frac{1}{h^{2+d}} \left| K_X \left( \frac{\mathbf{X}_k - x}{h} \right) \right|^2 K_X \left( \frac{z - \mathbf{X}_k}{h} \right),
\]

S.68
conditionally on the original sample.

By Jensen’s inequality, LIE and the fact that the bids in the bootstrap sample are conditionally i.i.d., we have

$$E^* \left[ \frac{1}{N_1^n} \sum_{i=1}^{N_1^n} \frac{1}{L} \sum_{k=1}^{L} \mathbb{1} (N_k = n) \frac{1}{N_k} \sum_{j=1}^{N_k} \frac{1}{h^{2+2d}} K_j \left( \frac{(B_{jk}, X_k, N_k) - v}{h} \right) \mathbb{1} (N_1^* = N_k) \mathbb{1} (B_{1j}^* \leq B_{jk}) \mathbb{1} \left( \frac{X_1^* - X_k}{h} \right) \right]^2$$

$$= \frac{1}{L} \sum_{l=1}^{L} \sum_{k=1}^{L} \sum_{k'=1}^{L} J \left( (B_1, X_1, N_1), (B_k, X_k, N_k), (B_k', X_{k'}, N_{k'}) ; v \right),$$

where

$$J \left( (b, z, m), (b', z', m'), (b'', z'', m'') ; v \right) := \frac{1}{h^{4+4d}} \frac{1}{m} \sum_{j=1}^{m} \left( \frac{1}{m'} \sum_{j'=1}^{m'} K_j' \left( \frac{(b_j', z_j', m'_j) - v}{h} \right) \right) \frac{1}{m''} \sum_{j''=1}^{m''} K_j'' \left( \frac{(b_j'', z_j'', m''_j) - v}{h} \right) \mathbb{1} (m = m') \mathbb{1} (b_j \leq b_j') \mathbb{1} (m'' = n) \mathbb{1} (b_j \leq b_j'') \mathbb{1} \left( \frac{z - z'}{h} \right) \mathbb{1} \left( \frac{z'' - z'}{h} \right).$$

Standard arguments can be applied to verify that \( \{ J(\cdot, \cdot, \cdot, v) : v \in I(\cdot) \} \) is uniformly VC-type with respect to the envelope

$$F_J (z, z', z'') := \frac{(C_0 + C_2)^2}{(n-1)^2 C_0 h^{-2d}} \left| K_X \left( \frac{z' - x}{h} \right) \right| \left| K_X \left( \frac{z - z'}{h} \right) \right| \left| K_X \left( \frac{z'' - z'}{h} \right) \right|. \quad (S.3.108)$$

By observing that \( J \) is symmetric with respect to the second and the third arguments and the V-statistic decomposition argument of Serfling (2009, 5.7.3), we have

$$\frac{1}{L} \sum_{l=1}^{L} \sum_{k=1}^{L} \sum_{k'=1}^{L} J \left( (B_1, X_1, N_1), (B_k, X_k, N_k), (B_k', X_{k'}, N_{k'}) ; v \right)$$

$$= \frac{1}{L} \sum_{l=1}^{L} J \left( (B_1, X_1, N_1), (B_k, X_k, N_k), (B_k', X_{k'}, N_{k'}) ; v \right) + O \left( \frac{(L-1)}{3L^2 - 2L} \right) \left\{ \sum_{l=1}^{L} J \left( (B_1, X_1, N_1), (B_k, X_k, N_k), (B_l, X_l, N_l) ; v \right) + \sum_{l \neq k} 2J \left( (B_1, X_1, N_1), (B_l, X_l, N_l), (B_k, X_k, N_k) ; v \right) + J \left( (B_k, X_k, N_k), (B_l, X_l, N_l) ; v \right) \right\}.$$
\[
\begin{align*}
&= \frac{1}{m} \sum_{i=0}^{m} \frac{1}{h^{(1+d)}} \left\{ \int_{X} \int_{\mathbb{Z}^d} K_f' \left( \frac{\xi(b, z, n) - v}{h}, \frac{z - x}{h} \right) \left[ \frac{1}{n} (m - n) - 1 \right] (b_i \leq b') K_X \left( \frac{z - z'}{h} \right) db'dz' \right\}^2 \\
\text{and}

J_2^{(1)} (b, z, m; v) := E \left[ J \left( (B_1, X_1, N_1), (b, z, m), (B_2, X_2, N_2); v \right) \right] = \frac{1}{m} \sum_{i=0}^{m} \frac{1}{h^{(1+d)}} \left\{ \int_{X} \int_{\mathbb{Z}^d} K_f' \left( \frac{\xi(b, z, m) - v}{h}, \frac{z - x}{h} \right) \left[ \frac{1}{n} (m - n) - 1 \right] g(b, z, m) \right\}^2
\end{align*}
\]

We observe that

\[
\mu_J (v) = E \left[ J \left( (B_1, X_1, N_1), (B_2, X_2, N_2), (B_3, X_3, N_3); v \right) \right] = E \left[ J_2^{(1)} (B_1, X_1, N_1; v) \right]^2
\]

and thus

\[
\sup_{v \in I(x)} \mu_J (v) = O \left( h^{-(1+d)} \right), \quad \text{(S.3.110)}
\]

which was shown in the proof of Lemma S.3.2.

By Jensen's inequality, LIE and change of variables, we have

\[
E \left[ J_2^{(1)} (B_1, X_1, N_1; v) \right]^2 \leq E \left[ J \left( (B_1, X_1, N_1), (B_2, X_2, N_2); v \right) \right] = \frac{1}{m} \sum_{i=0}^{m} \frac{1}{h^{(1+d)}} \left\{ \int_{X} \int_{\mathbb{Z}^d} K_f' \left( \frac{\xi(b, z, m) - v}{h}, \frac{z - x}{h} \right) \left[ \frac{1}{n} (m - n) - 1 \right] g(b, z, m) \right\}^2
\]

and

\[
E \left[ J_2^{(1)} (B_1, X_1, N_1; v) \right]^2 \leq E \left[ J \left( (B_1, X_1, N_1), (B_2, X_2, N_2); v \right) \right] = \frac{1}{m} \sum_{i=0}^{m} \frac{1}{h^{(1+d)}} \left\{ \int_{X} \int_{\mathbb{Z}^d} K_f' \left( \frac{\xi(b, z, m) - v}{h}, \frac{z - x}{h} \right) \left[ \frac{1}{n} (m - n) - 1 \right] g(b, z, m) \right\}^2
\]
Now it is easy to observe
\[
\sigma^2_{\mathcal{J}_1^{(1)}} := \sup_{v \in \mathcal{I}(x)} E \left[ J_1^{(1)}(B_1, X_1, N_1; v)^2 \right] = O \left( h^{-4+3d} \right)
\]  
(S.3.111)

and
\[
\sigma^2_{\mathcal{J}_2^{(1)}} := \sup_{v \in \mathcal{I}(x)} E \left[ J_2^{(1)}(B_1, X_1, N_1; v)^2 \right] = O \left( h^{-5+3d} \right).
\]

Since \( \mathcal{J}(\cdot, \cdot; v) : v \in \mathcal{I}(x) \) is uniformly VC-type with respect to the envelope (S.3.108), it follows from Chen and Kato (2017, Lemma 5.4) that \( \{ J_1^{(1)}(\cdot; v) : v \in \mathcal{I}(x) \} \) is VC-type with respect to the envelope
\[
F_{\mathcal{J}_1^{(1)}}(z) := \frac{(C_{D_1} + C_{D_2})^2}{(n-1)^2 C_d^2} \int \int \frac{1}{h^{3(1+d)}} K_X^0 \left( \frac{z-x}{h} \right) \left| K_X \left( \frac{z-x}{h} \right) \right| K_X^0 \left( \frac{z''-x}{h} \right) \left| K_X \left( \frac{z''-x}{h} \right) \right| \varphi (z') \varphi (z'') dz' dz''.
\]
Then the CCK inequality yields
\[
E \left[ \sup_{v \in \mathcal{I}(x)} \left| \frac{1}{L} \sum_{l=1}^L J_1^{(1)}(B_l, X_l, N_l; v) - \mu_\mathcal{J}(v) \right| \right] \leq C_1 \left\{ L^{-\frac{3}{2}} \sigma_{\mathcal{J}_1^{(1)}} \log (C_2 L)^{\frac{1}{2}} + L^{-1} \left\| F_{\mathcal{J}_1^{(1)}} \right\|_\mathcal{X} \log (C_2 L) \right\} = O \left( \frac{\log (L)}{L h^{3+3d}} \right)^{\frac{1}{2}} + \frac{\log (L)}{L h^{3+2d}},
\]
where the equality follows from (S.3.111) and \( \left\| F_{\mathcal{J}_1^{(1)}} \right\|_\mathcal{X} = O \left( h^{-4+2d} \right) \) (which follows from change of variables).

Similarly, since \( \{ J_2^{(1)}(\cdot; v) : v \in \mathcal{I}(x) \} \) is VC-type with respect to the envelope
\[
F_{\mathcal{J}_2^{(1)}}(z') := \frac{(C_{D_1} + C_{D_2})^2}{(n-1)^2 C_d^2} \int \int \frac{1}{h^{3(1+d)}} K_X^0 \left( \frac{z'-z}{h} \right) \left| K_X \left( \frac{z'-z}{h} \right) \right| K_X^0 \left( \frac{z''-x}{h} \right) \left| K_X \left( \frac{z''-x}{h} \right) \right| \varphi (z') \varphi (z'') dz' dz'',
\]
the CCK inequality yields
\[
E \left[ \sup_{v \in \mathcal{I}(x)} \left| \frac{1}{L} \sum_{l=1}^L J_2^{(1)}(B_l, X_l, N_l; v) - \mu_\mathcal{J}(v) \right| \right] \leq C_1 \left\{ L^{-\frac{3}{2}} \sigma_{\mathcal{J}_2^{(1)}} \log (C_2 L)^{\frac{1}{2}} + L^{-1} \left\| F_{\mathcal{J}_2^{(1)}} \right\|_\mathcal{X} \log (C_2 L) \right\} = O \left( \frac{\log (L)}{L h^{5+3d}} \right)^{\frac{1}{2}} + \frac{\log (L)}{L h^{3+2d}}.
\]
Since \( \mathcal{J} \) is symmetric with respect to the second and the third arguments, we have
\[
E \left[ \sup_{v \in \mathcal{I}(x)} \left| \frac{1}{L} \sum_{l=1}^L J_3^{(1)}(B_l, X_l, N_l; v) - \mu_\mathcal{J}(v) \right| \right] = O \left( \frac{\log (L)}{L h^{5+3d}} \right)^{\frac{1}{2}} + \frac{\log (L)}{L h^{3+2d}}.
\]

Let
\[
F_{\mathcal{J}_2^{(2)}}(z, z') := \int F_\mathcal{J}(z, z', z'') \varphi (z'') dz'', F_{\mathcal{J}_1^{(2)}}(z, z') := \int F_\mathcal{J}(z, z'', z') \varphi (z'') dz'' and F_{\mathcal{J}_3^{(2)}}(z, z') := \int F_\mathcal{J}(z'', z', z') \varphi (z'') dz''.
\]

The CK inequality and change of variables yield
\[
E \left[ \sup_{v \in \mathcal{I}(x)} \left| \mathcal{Y}_1^{(2)}(v) \right| \right] \leq L^{-1} \left( E \left[ F_{\mathcal{J}_2^{(2)}}(X_1, X_2)^2 \right] \right)^{\frac{1}{2}} = O \left( \frac{L^2 h^{4+2d}}{L h^{4+2d}} \right)^{-1},
\]
\[
E \left[ \sup_{v \in \mathcal{I}(x)} \left| \mathcal{Y}_2^{(2)}(v) \right| \right] \leq L^{-1} \left( E \left[ F_{\mathcal{J}_1^{(2)}}(X_1, X_2)^2 \right] \right)^{\frac{1}{2}} = O \left( \frac{L^2 h^{4+2d}}{L h^{4+2d}} \right)^{-1},
\]
\[
E \left[ \sup_{v \in \mathcal{I}(x)} \left| \mathcal{Y}_3^{(2)}(v) \right| \right] \leq L^{-1} \left( E \left[ F_{\mathcal{J}_2^{(2)}}(X_1, X_2)^2 \right] \right)^{\frac{1}{2}} = O \left( \frac{L^2 h^{4+2d}}{L h^{4+2d}} \right)^{-1}.
\]
and

$$E \left[ \sup_{v \in I(x)} |\Psi_J(v)| \right] \leq L^{-3/2} \left( E \left[ F_J^2 (X_1, X_2, X_3) \right] \right)^{1/2} = O \left( L^{-3/2} k^{4.5d/2} \right).$$

It follows from these bounds for expectations of suprema of empirical and degenerate U-processes, (S.3.109), (S.3.110) and Markov’s inequality that

$$\sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l=1}^L J \left( (B_l, X_l, N_l), (B_k, X_k, N_k), (B_{k'}, X_{k'}, N_{k'}) ; v \right) \right| = O_p \left( h^{-1(1+d)} + \frac{\log(L)}{Lh^{4+3d}} \right)^{1/2} + \frac{\log(L)}{Lh^{4+2d}}. $$

It is also straightforward to check

$$\sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l=1}^L \frac{1}{L(L - 1)} \sum_{l \neq k} J \left( (B_l, X_l, N_l), (B_k, X_k, N_k), (B_{k'}, X_{k'}, N_{k'}) ; v \right) \right| \leq \frac{1}{L(L - 1)} \sum_{l \neq k} \frac{1}{L h^{4(1+d)}} K_2^0 \left( \frac{X_l - X}{h} \right)^2 K \left( \frac{X_l - X_k}{h} \right)^2 \leq O_p \left( h^{-4(1+d)} \right).$$

where the equalities follow from change of variables and Markov’s inequality. Then it follows that

$$\sigma_{\hat{g}^2, n}^2 := \sup_{v \in I(x)} E^* \left[ \hat{g}_{\hat{g}, n}^2 \left( B_1^*, X_1^*, N_1^* ; v \right) \right]^2 = O_p \left( h^{-1(1+d)} + \frac{\log(L)}{Lh^{4+3d}} \right)^{1/2} + \frac{\log(L)}{Lh^{4+2d}} \right.$$ (S.3.112)

and the CCK inequality yields

$$E^* \left[ \sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l=1}^L \hat{g}_{\hat{g}, n}^2 \left( B_1^*, X_1^*, N_1^* ; v \right) - E^* \left[ \hat{g}_{\hat{g}, n}^2 \left( B_1^*, X_1^*, N_1^* ; v \right) \right] \right| \right] \leq C_1 \left\{ L^{-1/2} \sigma_{\hat{g}^2, n} \log(C_2 L)^{1/2} + L^{-1} \left\| F_{\hat{g}^2, n} \right\|_X \log(C_2 L) \right\}$$

$$= O_p \left( \frac{1}{Lh^{1+d}} \right)^{1/2} + \frac{\log(L)}{Lh^{2+d}} \right)$$

where the equality follows from (S.3.112) and \( \left\| F_{\hat{g}^2, n} \right\|_X = O_p \left( h^{-(2+d)} \right) \) (which follows from change of variables and Markov’s inequality).

It is easy to check

$$\sup_{v \in I(x)} \left| \frac{1}{L^2} \left( \sum_{l \neq k} \hat{g}_{\hat{g}, n} \left( (B_1^*, X_1^*, N_1^*), (B_k^*, X_k^*, N_k^*) ; v \right) \right) \right| \leq \frac{1}{L^2} \left( \sum_{l \neq k} \frac{1}{h^{2+d}} K_2^0 \left( \frac{X_l^* - X}{h} \right)^2 K \left( \frac{X_l^* - X_k^*}{h} \right)^2 \right) = O_p \left( (Lh^2)^{-1} \right).$$

S.72
Lemma S.3.9. Suppose that Assumptions 1 - 3 hold. Let \( x \) be an interior point of \( \mathcal{X} \) and \( n \in \mathcal{N} \) be fixed. Then

\[
\widehat{f}_{GPV}^* (v, x, n) - \widehat{f}^* (v, x, n) = \frac{1}{L} \sum_{l=1}^{L} \sum_{k=1}^{L} \mathcal{M}^n ((B^*_k, X^*_k, N^*_k), (B^*_l, X^*_l, N^*_l); v)
\]

\[
= \frac{1}{L} \sum_{l=1}^{L} \sum_{k=1}^{L} \mathcal{M}^n ((B^*_k, X^*_k, N^*_k), (B^*_l, X^*_l, N^*_l); v) + \left\{ \frac{1}{L} \sum_{l=1}^{L} \widehat{\mathcal{M}}^n ((B^*_k, X^*_k, N^*_k), (B^*_l, X^*_l, N^*_l); v) - \widehat{\mu}_{M^n} (v) \right\}
+ \frac{1}{(L)^2} \sum_{(2)} \mathcal{M}^n ((B^*_1, X^*_1, N^*_1), (B^*_2, X^*_2, N^*_2); v) - \widehat{\mathcal{M}}^n ((B^*_1, X^*_1, N^*_1), (B^*_2, X^*_2, N^*_2); v) - \widehat{\mu}_{M^n} (v)
\]

where the remainder term is uniform in \( v \in I(x) \).

**Proof of Lemma S.3.9.** Let

\[
\widehat{\mu}_{M^n} (v) := \mathbb{E}^* [\mathcal{M}^n ((B^*_1, X^*_1, N^*_1), (B^*_2, X^*_2, N^*_2); v)]
\]

\[
\widehat{\mathcal{M}}^n_1 (b, z; m; v) := \mathbb{E}^* [\mathcal{M}^n ((b, z, m), (B^*_1, X^*_1, N^*_1); v)] \quad \text{and} \quad \widehat{\mathcal{M}}^n_2 (b, z, m; v) := \mathbb{E}^* [\mathcal{M}^n ((B^*_1, X^*_1, N^*_1), (b, z, m); v)].
\]

The Hoeffding decomposition yields

\[
\frac{1}{L} \sum_{l=1}^{L} \sum_{k=1}^{L} \mathcal{M}^n ((B^*_k, X^*_k, N^*_k), (B^*_l, X^*_l, N^*_l); v)
\]

\[
= \widehat{\mu}_{M^n} (v) + \left\{ \frac{1}{L} \sum_{l=1}^{L} \widehat{\mathcal{M}}^n ((B^*_k, X^*_k, N^*_k), (B^*_l, X^*_l, N^*_l); v) - \widehat{\mu}_{M^n} (v) \right\}
+ \frac{1}{(L)^2} \sum_{(2)} \mathcal{M}^n ((B^*_1, X^*_1, N^*_1), (B^*_2, X^*_2, N^*_2); v) - \widehat{\mathcal{M}}^n ((B^*_1, X^*_1, N^*_1), (B^*_2, X^*_2, N^*_2); v) - \widehat{\mu}_{M^n} (v)
\]

By the LIE and the fact that the bids in the bootstrap sample are conditionally i.i.d., we have

\[
\widehat{\mu}_{M^n} (v) = \frac{1}{L} \sum_{l=1}^{L} \sum_{k=1}^{L} \mathcal{M}^n ((B^*_k, X^*_k, N^*_k), (B^*_l, X^*_l, N^*_l); v).
\]

Since \( \{\mathcal{M}^n (\cdot, \cdot; v) : v \in I(x)\} \) is uniformly VC-type with respect to the envelope \( \mathcal{S} , \mathcal{G} \), the CK inequality yields

\[
\mathbb{E}^* \left[ \sup_{v \in I(x)} \left\{ \frac{1}{(L)^2} \sum_{(2)} \mathcal{M}^n ((B^*_1, X^*_1, N^*_1), (B^*_2, X^*_2, N^*_2); v) - \widehat{\mathcal{M}}^n ((B^*_1, X^*_1, N^*_1), (B^*_2, X^*_2, N^*_2); v) - \widehat{\mu}_{M^n} (v) \right\} \right] 
\]

\[
\leq L^{-1} \left( \mathbb{E}^* \left[ \mathcal{M}^n (X^*_1, X^*_2)^2 \right] \right)^{1/2} .
\]

S.73
Since

$$E\left[ \frac{1}{L^2} \sum_{l=1}^{L} \sum_{k=1}^{L} F_{\mathcal{M}^n} (X_l, X_k)^2 \right] = \frac{L-1}{L} E \left[ F_{\mathcal{M}^n} (X_1, X_2)^2 \right] + \frac{1}{L} E \left[ F_{\mathcal{M}^n} (X_1, X_1)^2 \right] = O\left( h^{-(6+2d)} \right),$$

where the second equality follows from change of variables, we have

$$E^* \left[ F_{\mathcal{M}^n} (X_1^*, X_2^*)^2 \right] = \frac{1}{L^2} \sum_{l=1}^{L} \sum_{k=1}^{L} F_{\mathcal{M}^n} (X_l^*, X_k^*)^2 = O_p \left( h^{-(6+2d)} \right),$$

where the second equality follows from Markov’s inequality. Therefore we have

$$E^* \left[ \sup_{v \in I(x)} \left| \frac{1}{(L_2)^2} \sum_{(l_1, l_2) \in \mathcal{C}_{m,n}} K_{l_2}^2 \left( \frac{X_{l_1} - x}{h} \right) \right| \right] = O_p \left( \left( L h^{d+4} \right)^{-1} \right). \quad (S.3.115)$$

By the LIF and the fact that the bids in the bootstrap sample are conditionally i.i.d., we have

$$\hat{\mathcal{M}}_m^a (b, z, m; v) = -1 (m = n) \frac{1}{m} \sum_{i=1}^{m} \frac{1}{h^{2+d}} K_i^a \left( \frac{\xi (b_i, z, m) - v}{h} \right) \frac{G (b_i, z, m)}{g(b_i, z, m)} \left\{ \hat{g} (b_i, z, m) - g (b_i, z, m) \right\}. \quad (S.3.134)$$

It is clear from the definition that when \( h \) is sufficiently small, \( \hat{\mathcal{M}}_m^a (\cdot; v) : v \in I(x) \) is uniformly VC-type with respect to the envelope

$$F_{\hat{\mathcal{M}}_m^a} (z) := \left\{ \sup_{(b', z') \in C_{n,b}^n} |\hat{g} (b', z', n) - g (b', z', n)| \right\}. \quad (S.3.135)$$

Then the VW inequality yields

$$E^* \left[ \sup_{v \in I(x)} \left| \frac{1}{L} \sum_{l=1}^{L} \hat{\mathcal{M}}_m^a (B_{l, 1}^*, X_{l, 1}^*, N_{l, 1}^*; v) - \hat{\mu}_{\mathcal{M}^n} (v) \right| \right] \leq L^{-1/2} \left( E^* \left| F_{\hat{\mathcal{M}}_m^a} (X_{1, 1}^*)^2 \right| \right)^{1/2} \leq L^{-1/2} \left\{ \frac{1}{L} \sum_{l=1}^{L} \frac{1}{h^{2+d}} K_i^a \left( \frac{X_{l, 1} - x}{h} \right) \right\}^{1/2} \left\{ \sup_{(b', z') \in C_{n,b}^n} |\hat{g} (b', z', n) - g (b', z', n)| \right\} \leq O_p \left( \left( L h^{d+4} \right)^{-1/2} \right) O_p \left( \left( \frac{\log (L)}{L h^{d+4}} \right)^{1/2} + h^{1+R} \right), \quad (S.3.136)$$

where the equality follows from change of variables, Markov’s inequality and (S.3.50).

Let

$$\hat{\mathcal{M}}_{m, 1}^a (b, z, m; v) := E^* \left[ \mathcal{M}_{m, 1}^a \left( (B_{1, 1}^*, X_{1, 1}^*, N_{1, 1}^*); (b, z, m); v \right) \right]$$

and

$$\hat{\mu}_{\mathcal{M}_{m, 1}^a} (v) := E^* \left[ \mathcal{M}_{m, 1}^a \left( (B_{1, 1}^*, X_{1, 1}^*, N_{1, 1}^*); (B_{2, 1}^*, X_{2, 1}^*, N_{2, 1}^*); v \right) \right]. \quad (S.3.137)$$

Consider

$$\Delta^* (v) := \frac{1}{L} \sum_{l=1}^{L} \left\{ \hat{\mathcal{M}}_m^a (B_{l, 1}^*, X_{l, 1}^*, N_{l, 1}^*; v) - \hat{\mu}_{\mathcal{M}^n} (v) \right\} - \left( \hat{\mathcal{M}}_m^a (B_{1, 1}^*, X_{1, 1}^*, N_{1, 1}^*; v) - \frac{1}{L} \sum_{k=1}^{L} \hat{\mathcal{M}}_m^a (B_{k, 1}^*, X_{k, 1}^*, N_{k, 1}^*; v) \right), \quad v \in I(x). \quad (S.3.138)$$
Since it is straightforward to check
\[ \tilde{\mathcal{M}}_t^n(B^*_i, X^*_i, N^*_i; v) - \bar{\mu}_{\mathcal{M}_t^n}(v) = \tilde{\mathcal{M}}_t^n(B^*_i, X^*_i, N^*_i; v) - \bar{\mu}_{\mathcal{M}_t^n}(v), \] for all \( l = 1, \ldots, L \)
and
\[ \mathcal{M}_l^n(B^*_i, X^*_i, N^*_i; v) - \frac{1}{L} \sum_{k=1}^L \mathcal{M}_l^n(B_k, X_k, N_k; v) = \mathcal{M}_l^n(B^*_i, X^*_i, N^*_i; v) - \frac{1}{L} \sum_{k=1}^L \mathcal{M}_l^n(B_k, X_k, N_k; v), \] for all \( l = 1, \ldots, L \),
we have
\[ \Delta^n(v) = \frac{1}{L} \sum_{l=1}^L \left\{ \left( \tilde{\mathcal{M}}_l^n(B^*_i, X^*_i, N^*_i; v) - \bar{\mu}_{\mathcal{M}_l^n}(v) \right) - \left( \mathcal{M}_l^n(B^*_i, X^*_i, N^*_i; v) - \frac{1}{L} \sum_{k=1}^L \mathcal{M}_l^n(B_k, X_k, N_k; v) \right) \right\}, \ v \in I(x). \]
Simple algebra yields
\[ E^* \left[ \left( \tilde{\mathcal{M}}_l^n(B^*_i, X^*_i, N^*_i; v) - \mathcal{M}_l^n(B^*_i, X^*_i, N^*_i; v) \right)^2 \right] = \frac{1}{L} \sum_{l=1}^L \sum_{k=1}^L \sum_{k'=1}^L \mathcal{L}((B_l, X_l, N_l), (B_k, X_k, N_k), (B_{k'}, X_{k'}, N_{k'}); v), \]
where
\[ \mathcal{L}((b, z, m), (b', z', m'), (b'', z'', m''): v) := \left( \mathcal{M}^{*,n}((b', z', m'), (b, z, m); v) - \mathcal{M}^{*,n}(b, z, m; v) \right) \left( \mathcal{M}^{*,n}((b'', z'', m''), (b, z, m); v) - \mathcal{M}^{*,n}(b, z, m; v) \right). \]
Standard arguments can be applied to verify that \( \{\mathcal{L}((\cdot, \cdot, \cdot); v) : v \in I(x)\} \) is VC-type with respect to the envelope
\[ F_{\mathcal{L}}(z, z', z'') := F_{\mathcal{M}^{*,n}}(z', z) F_{\mathcal{M}^{*,n}}(z'', z) + F_{\mathcal{M}^{*,n}}(z') F_{\mathcal{M}^{*,n}}(z'', z) + F_{\mathcal{M}^{*,n}}(z') F_{\mathcal{M}^{*,n}}(z', z) + F_{\mathcal{M}^{*,n}}(z)^2. \]

By observing that \( \mathcal{J} \) is symmetric with respect to the second and the third arguments and the V-statistic decomposition argument of Serfling (2009, 5.7.3), we have
\[ \frac{1}{L^3} \sum_{l=1}^L \sum_{k=1}^L \sum_{k'=1}^L \mathcal{L}((B_l, X_l, N_l), (B_k, X_k, N_k), (B_{k'}, X_{k'}, N_{k'}); v) = \frac{O(L^{-1})}{3L^2} \left\{ \sum_{l=1}^L \mathcal{L}((B_l, X_l, N_l), (B_l, X_l, N_l), (B_l, X_l, N_l); v) + \sum_{l=1}^L (2\mathcal{L}((B_l, X_l, N_l), (B_l, X_l, N_l), (B_k, X_k, N_k); v) + \mathcal{L}((B_k, X_k, N_k), (B_l, X_l, N_l), (B_l, X_l, N_l); v)) \right\}, \]
The Hoeffding decomposition yields
\[ \frac{1}{L^3} \sum_{l=1}^L \mathcal{L}((B_l, X_l, N_l), (B_k, X_k, N_k), (B_{k'}, X_{k'}, N_{k'}); v) \]
\[ = \frac{1}{L^2} \sum_{(l)} \mathcal{L}^{(2)}((B_l, X_l, N_l), (B_k, X_k, N_k); v) \]
\[ + \frac{1}{L^3} \sum_{(l)} \left\{ \mathcal{L}((B_l, X_l, N_l), (B_k, X_k, N_k), (B_{k'}, X_{k'}, N_{k'}); v) - \mathcal{L}^{(2)}((B_k, X_k, N_k), (B_{k'}, X_{k'}, N_{k'}); v) \right\}, \]
where the terms in the decomposition are defined by (S.330) to (S.342) with \( \mathcal{K} \) replaced by \( \mathcal{L} \). Note that it is easy to check that all other terms in the Hoeffding decomposition vanish.
Denote \( F_{\mathcal{L}^2} (z, z') := \int F_{\mathcal{L}} (z'', z'', z'') \, dz'' \).

The CK inequality yields

\[
E \left[ \sup_{v \in I(a)} \left| \frac{1}{(L)_{2}} \sum_{i=1}^{L} \mathcal{L}^2 ((B_i, X_i, N_i), (B_k, X_k, N_k); v) \right| \right] \leq L^{-1} \left( E \left[ F_{\mathcal{L}^2} (X_1, X_2)^2 \right] \right)^{1/2} = O \left( (Lh^{(6+2d)})^{-1} \right)
\]

and

\[
E \left[ \sup_{v \in I(a)} \left| \frac{1}{(L)_{2}} \sum_{i=1}^{L} \mathcal{L}^2 ((B_i, X_i, N_i), (B_k, X_k, N_k), (B_k', X_k', N_k'); v) - \mathcal{L}^2 ((B_k, X_k, N_k), (B_k', X_k', N_k'); v) \right| \right] \leq L^{-3/2} \left( E \left[ F_{\mathcal{L}} (X_1, X_2, X_3)^2 \right] \right)^{1/2} = L^{-3/2} h^{-(6+3d/2)},
\]

where the equalities follow from change of variables.

It is also straightforward to check that

\[
\sup_{v \in I(a)} \left| \frac{1}{(L)_{2}} \sum_{i=1}^{L} \mathcal{L} ((B_i, X_i, N_i), (B_i, X_i, N_i), (B_k, X_k, N_k), (B_k', X_k', N_k'); v) \right| \leq \frac{1}{(L)_{2}} \sum_{i=1}^{L} F_{\mathcal{M}^1, n} (X_i, X_i)^2 + \frac{2}{(L)_{2}} \sum_{i=1}^{L} F_{\mathcal{M}^2, n} (X_i) F_{\mathcal{M}^1, n} (X_i, X_i) + \frac{1}{(L)_{2}} \sum_{i=1}^{L} F_{\mathcal{M}^2, n} (X_i)^2 = O_p \left( h^{-(6+3d)} \right),
\]

and

\[
\sup_{v \in I(a)} \left| \frac{1}{(L)_{2}} \sum_{i=1}^{L} \mathcal{L} ((B_k, X_k, N_k), (B_i, X_i, N_i), (B_i, X_i, N_i); v) \right| \leq \frac{1}{(L)_{2}} \sum_{i=1}^{L} F_{\mathcal{M}^1, n} (X_k, X_i) F_{\mathcal{M}^1, n} (X_i, X_i) + \frac{1}{(L)_{2}} \sum_{i=1}^{L} F_{\mathcal{M}^2, n} (X_i) F_{\mathcal{M}^1, n} (X_k, X_i)
+ \frac{1}{(L)_{2}} \sum_{i=1}^{L} F_{\mathcal{M}^2, n} (X_i) F_{\mathcal{M}^1, n} (X_i, X_i) + \frac{1}{(L)_{2}} \sum_{i=1}^{L} F_{\mathcal{M}^2, n} (X_i)^2 = O_p \left( h^{-(6+2d)} \right),
\]

and

\[
\sup_{v \in I(a)} \left| \frac{1}{(L)_{2}} \sum_{i=1}^{L} \mathcal{L} ((B_k, X_k, N_k), (B_i, X_k, N_i), (B_i, X_i, N_i), (B_i, X_i, N_i); v) \right| \leq \frac{1}{(L)_{2}} \sum_{i=1}^{L} F_{\mathcal{M}^1, n} (X_k, X_k)^2 + \frac{2}{(L)_{2}} \sum_{i=1}^{L} F_{\mathcal{M}^2, n} (X_k) F_{\mathcal{M}^1, n} (X_k, X_k) + \frac{1}{(L)_{2}} \sum_{i=1}^{L} F_{\mathcal{M}^2, n} (X_k)^2 = O_p \left( h^{-(6+2d)} \right),
\]

where the equalities follow from change of variables and Markov’s inequality.

Now

\[
\vartheta_2^2 := \sup_{v \in I(a)} E^* \left[ \left\{ \hat{M}_{\mathcal{L}^2}^1 (B_i^*, X_i^*, N_i^*; v) - M_{\mathcal{L}^2}^1 (B_i^*, X_i^*, N_i^*; v) \right\}^2 \right] = O_p \left( (Lh^{6+2d})^{-1} \right) \quad (S.3.117)
\]

S.76
follows from these bounds and Markov’s inequality.

Since \( \{ \mathcal{M}_{1,n}^N (\cdot; v) : v \in I (\mathbf{x}) \} \) is uniformly VC-type with respect to the envelope (S.3.86), it follows from Chen and Kato (2017, Lemma 5.4) that \( \{ \tilde{\mathcal{M}}_{1,n}^N (\cdot; v) : v \in I (\mathbf{x}) \} \) is uniformly VC-type with respect to the envelope

\[
F_{\tilde{\mathcal{M}}_{1,n}}^N (\mathbf{z}) := \frac{(v \mathcal{U}_D + \mathcal{D}_1 + \mathcal{D}_2) C K_0}{(n - 1) C} \int \frac{1}{L} \sum_{k=1}^{L} \frac{1}{h^{3+2d}} \left| K_X \left( \frac{X_k - x}{h} \right) \right| \left| K_X \left( \frac{z - X_k}{h} \right) \right|, 
\]

conditionally on the original sample. Now \( \left\| F_{\tilde{\mathcal{M}}_{1,n}}^N (\cdot) \right\|_X = O_p (h^{-3-d}) \) follows from change of variables and Markov’s inequality. It follows from Nolan and Pollard (1987, Lemma 16) that \( \{ \tilde{\mathcal{M}}_{1,n}^N (\cdot; v) - \mathcal{M}_{1,n}^N (\cdot; v) : v \in I (\mathbf{x}) \} \) is uniformly VC-type with respect to \( F_{\tilde{\mathcal{M}}_{1,n}}^N + F_{\mathcal{M}_{1,n}}^N \), conditionally on the original sample. The CCR inequality yields

\[
E^* \left[ \sup_{v \in I (\mathbf{x})} | \Delta^*(v) | \right] \leq C_1 \left( L^{-1/2} \delta \log (C_2 L)^{1/2} + L^{-1} \left( \left\| F_{\tilde{\mathcal{M}}_{1,n}}^N \right\|_X + \left\| F_{\mathcal{M}_{1,n}}^N \right\|_X \right) \log (C_2 L) \right) 
= O_p \left( \log \left( \frac{L}{h^{3+d}} \right) \right),
\]

(S.3.118)

where the equality follows from (S.3.117), \( \left\| F_{\tilde{\mathcal{M}}_{1,n}}^N \right\|_X = O_p (h^{-3-d}) \) and \( \left\| F_{\mathcal{M}_{1,n}}^N \right\|_X = O (h^{-3-d}) \).

It is also straightforward to check that

\[
\sup_{v \in I (\mathbf{x})} \left| \frac{1}{L^2} \sum_{l=1}^{L} \mathcal{M}^N \left( (B_{l}^r, X_{l}^r, N_{l}^r), (B_{l}^r, X_{l}^r, N_{l}^r) ; v \right) \right| \leq \frac{1}{L^2} \sum_{l=1}^{L} \frac{1}{h^{3+2d}} \left| K_X \left( \frac{X_l^r - x}{h} \right) \right| + \frac{1}{L^2} \sum_{l=1}^{L} \frac{1}{h^{3+2d}} \left| K_K \left( \frac{X_l^r - x}{h} \right) \right|
= O_p \left( L h^{-3-d} \right)
\]

and

\[
\sup_{v \in I (\mathbf{x})} \left| \frac{1}{L^2} \sum_{l=1}^{L} \mathcal{M}^N \left( (B_{l}^r, X_{l}^r, N_{l}^r), (B_{l}^r, X_{l}^r, N_{l}^r) ; v \right) \right| \leq \frac{1}{L^2} \sum_{l=1}^{L} \frac{1}{h^{3+2d}} \left| K_X \left( \frac{X_l^r - x}{h} \right) \right| + \frac{1}{L^2} \sum_{l=1}^{L} \frac{1}{h^{3+2d}} \left| K_K \left( \frac{X_l^r - x}{h} \right) \right|
= O_p \left( L h^{-3-d} \right),
\]

where the equalities follow from change of variables, Markov’s inequality and Lemma S.1.2.

The conclusion follows from these results, (S.3.113), (S.3.114), (S.3.115), (S.3.116) and (S.3.118).

Lemma S.3.10. Suppose Assumptions 1 - 3 hold. Then

\[
\sup_{v \in I (\mathbf{x})} | Z^*_v (v|\mathbf{x}) - \Gamma^*_v (v|\mathbf{x}) | = O_p \left( \log (L)^{1/2} h + \frac{\log (L)}{(L h^{3+d})^{1/2}} + L^{1/2} h^{(3+d)/2 + 2} R \right).
\]

Proof of Lemma S.3.10. It is straightforward to verify that

\[
\mathcal{M}_2^N (B_{l}^r, X_{l}^r, N_{l}^r ; v) - \tilde{\mu}_{\mathcal{M}_2^N} (v) = \mathcal{M}_2^N (B_{l}^r, X_{l}^r, N_{l}^r ; v) - \mu_{\mathcal{M}_2^N} (v), \text{ for all } l = 1, \ldots, L.
\]

(S.3.119)

It follows from (S.3.119) that

\[
\frac{1}{L} \sum_{l=1}^{L} \mathcal{M}_2^N (B_{l}^r, X_{l}^r, N_{l}^r ; v) - \frac{1}{L} \sum_{l=1}^{L} \mathcal{M}_2^N (B_{l}^r, X_{l}, N_{l} ; v) = \frac{1}{L} \sum_{l=1}^{L} \mathcal{M}_2^N (B_{l}^r, X_{l}^r, N_{l}^r ; v) - \frac{1}{L} \sum_{l=1}^{L} \mathcal{M}_2^N (B_{l}^r, X_{l}, N_{l} ; v).
\]
Note that by the LIE and the fact that the bids in the bootstrap sample are conditionally i.i.d., we have

$$E^* \left[ M_2^{1,n} ( B_1^*, X_1^*, N_1^*; v) \right] = \frac{1}{L} \sum_{l=1}^{L} M_2^{1,n} ( B_l, X_1, N_l; v).$$

Let

$$\hat{\sigma}^2_{M_2^{1,n}} := \sup_{v \in I(x)} E^* \left[ M_2^{1,n} ( B_1^*, X_1^*, N_1^*; v)^2 \right] \leq \frac{\sigma^2_{M_2^{1,n}}}{h^{(3+d)}} \left( \sum_{l=1}^{L} h^{3+d} M_2^{1,n} ( B_l, X_1, N_l; v)^2 \right) + \frac{1}{L} \sum_{l=1}^{L} h^{3+d} M_2^{1,n} ( B_l, X_1, N_l; v)^2 - E \left[ h^{3+d} M_2^{1,n} ( B_1, X_1, N_1; v)^2 \right].$$

We have

$$\sup_{v \in I(x)} E \left[ h^{3+d} M_2^{1,n} ( B_1, X_1, N_1; v)^2 \right] = O \left( 1 \right)$$

since it was shown in the proof of Theorem 6.1 that

$$\sup_{v \in I(x)} \left[ E \left[ h^{3+d} M_2^{1,n} ( B_1, X_1, N_1; v)^2 \right] - \nu_{GPV} (v|x, n) \right] \rightarrow 0, \text{ as } L \uparrow \infty.$$

It follows from Chernozhukov et al. (2014b, Corollary A.1) that \( \{ h^{3+d} M_2^{1,n} ( v; v^2 ) : v \in I(x) \} \) is uniformly VC-type with respect to the envelope

$$\tilde{F}_{M_2^{1,n}} (z) := \left( \frac{\int (C^2 L) \tilde{K}_\phi \left( \frac{z'}{h} \right) }{(n-1) \int \tilde{K}_\phi \left( \frac{z'}{h} \right) } \right)^2 \frac{1}{h^{3+d}} \left( \sum_{l=1}^{L} h^{3+d} M_2^{1,n} ( B_1, X_1, N_1; v)^2 \right).$$

It follows from change of variables that

$$\sup_{v \in I(x)} \left\| \tilde{F}_{M_2^{1,n}} \right\|_x = O ( h^{-(3+d)} ),$$

Let

$$\hat{\sigma}^2_{M_2^{1,n}} := \sup_{v \in I(x)} E \left[ h^{3+d} M_2^{1,n} ( B_1, X_1, N_1; v)^2 \right] = O \left( h^{-(3+d)} \right),$$

where the second equality is shown in the proof of Theorem 6.4. Now the CCK inequality yields

$$E \left[ \sum_{l=1}^{L} h^{3+d} M_2^{1,n} ( B_1, X_1, N_1; v)^2 - E \left[ h^{3+d} M_2^{1,n} ( B_1, X_1, N_1; v)^2 \right] \right] \leq C_1 \left\{ L^{-1/2} \hat{\sigma}_{M_2^{1,n}} \log (C_2 L)^{1/2} + L^{-1} \left\| \tilde{F}_{M_2^{1,n}} \right\|_x \log (C_2 L) \right\} = O \left( \frac{\log (L)}{L h^{3+d}} \right) + \frac{\log (L)}{L h^{3+d}}.$$}

Now it follows that \( \hat{\sigma}^2_{M_2^{1,n}} = O_p \left( h^{-(3+d)} \right) \). Since \( \left\| M_2^{1,n} \right\|_x = O \left( h^{-(3+d)} \right) \), the CCK inequality yields

$$E^* \left[ \sum_{l=1}^{L} M_2^{1,n} ( B_1^*, X_1^*, N_1^*; v) - \frac{1}{L} \sum_{l=1}^{L} M_2^{1,n} ( B_l, X_1, N_l; v) \right] \leq C_1 \left\{ L^{-1/2} \hat{\sigma}_{M_2^{1,n}} \log (C_2 L)^{1/2} + L^{-1} \left\| M_2^{1,n} \right\|_x \log (C_2 L) \right\} = O_p \left( \frac{\log (L)}{L h^{3+d}} \right)^{1/2}.$$}

The conclusion follows from the above result, Lemmas S.3.2, S.3.5, S.3.9, Theorem 6.2, Lemma S.1.2, the decomposition (S.3.44) and Marmer and Shneyerov (2012, Lemma S.1).
S.4 Justification of Remark 3.4 and Remark 6.1

To prove the result stated in Remark 3.4, note that

\[
\frac{V_{GPV}(v)}{V_{QB}(v)} = \frac{\left\{ \int K'(u)K(w-s'(v)u)\,du \right\}^2\,dw}{\int K'(w)^2\,dw} \leq \frac{\int \left\{ \int K(u)K'(w-s'(v)u)\,du \right\}^2\,dudw}{\int K'(w)^2\,dw},
\]

where the second equality follows from integration by parts and the inequality follows from Jensen’s inequality. It follows easily from the Fubini-Tonelli theorem and the fact \( \int K(u)\,du = 1 \) that

\[
\int \int K(u)K'(w-s'(v)u)^2\,dudw = \int K(u)\left( \int K'(w-s'(v)u)^2\,dw \right)\,du = \int K'(w)^2\,dw.
\]

Now it follows that \( V_{GPV}(v) \leq V_{QB}(v) \).

Next, we proceed to the general case with a random number of bidders and auction-specific heterogeneity. We provide details about the proof of the result stated in Remark 6.1. Marmer and Shneyerov (2012) showed that the quantile-based estimator \( \hat{f}_{QB}(v|x,n) \) is asymptotically normal:

\[
\left( \frac{\hat{f}_{QB}(v|x,n) - f(v|x)}{V_{QB}(v|x,n)} \right) \rightarrow_d N(0, V_{QB}(v|x,n)),
\]

where

\[
V_{QB}(v|x,n) := \frac{F(v|x)^2}{n} \left( \int K(u)^2\,du \right)^d \int K'(u)^2\,du
\]

and \( K \) is a kernel function. See Marmer and Shneyerov (2012, Theorem 2).

Now instead of Assumption 2(b), we assume \( K_0 = K_1 = K \), for some common second-order kernel function \( K \) that is supported on \([-1,1]\) and used in both steps. Denote

\[
K_d(x) := \prod_{k=1}^d K(x_k), \text{ for } x = (x_1,...,x_d) \in \mathbb{R}^d.
\]

Now the ratio between \( V_{QB}(v|x,n) \) and \( V_{GPV}(v|x,n) \) is given by

\[
\frac{V_{QB}(v|x,n)}{V_{GPV}(v|x,n)} = \frac{\int (v|x)^2\,g(s(v,x,n)|x,n)\,du}{\int \left( \int K(u)^2\,du \right)^d \int K'(u)^2\,du} = \frac{s^2}{\int \left( \int K(u)^2\,du \right)^d \int K'(u)^2\,du} \left( \int K(u)^2\,du \right)^d \int K'(u)^2\,du.
\]

(S.4.1)

Since the kernel is supported on \([-1,1]\]. By integration by parts, we have

\[
\int_{-1}^1 K'(w)K\left(u-s_vw-s_x^Ty\right)\,dw + \int_{-1}^1 K(w)K'\left(u-s_vw-s_x^Ty\right)(-s_v)\,dw = 0
\]

for any \( w \in \mathbb{R} \). We now have

\[
\int \left( \int K_d(y)K'(w)K_d(y-z)K(u-s_vw-s_x^Ty)\,dwdy\right)^2 \,dz = s^2 \int \left( \int K_d(y)K(w)K'(u-s_vw-s_x^Ty)K_d(y-z)\,dudy\right)^2 \,dz \leq s^2 \int \left( \int K_d(y)K(w)K'(u-s_vw-s_x^Ty)^2K_d(y-z)^2\,dudydz\right),
\]

where the inequality follows from Jensen’s inequality under the assumption that \( K \) is second-order (i.e., \( K \) is a probability density function). It is clear that the integrand of the multiple integral on the third line of (S.4.2) is supported on a compact set. Since \( K \) and \( K' \) are continuous
In an auction with co-variates, we extend the design of Marmer and Shneyerov (2012). We consider a co-variates $X$, which is a truncated normal random variable truncated between 0 and 2, with the mean equal to 1, and some variance $\sigma^2$. The PDF of $X$ is given by:

$$\varphi(x) = \frac{\phi\left(\frac{x-1}{\sigma}\right)}{\sigma \left\{ \Phi\left(\frac{2-1}{\sigma}\right) - \Phi\left(\frac{0-1}{\sigma}\right) \right\}} \mathbb{1} (0 \leq x \leq 2),$$

where $\phi$ is the standard normal PDF, and $\Phi$ is the standard normal CDF.

Given $X$, the valuations are drawn from the CDF $F(\cdot|X)$:

$$F(v|X) := \begin{cases} 0, & v < 0 \\ v^X, & 0 \leq v \leq 1 \\ 1, & v \geq 1. \end{cases}$$

In an auction with $N$ bidders, the Bayesian Nash equilibrium bidding strategy is now given by

$$s(u|X) = \left(1 - \frac{1}{X(N-1)+1}\right) u.$$

We choose the triweight kernel when implementing the two-step estimator. We use the second-order triweight kernel in the second step and use the fourth-order triweight kernel in the first step.

We again follow Silverman’s rule-of-thumb approach when selecting the bandwidths. We use $h_y = 3.72 \cdot \hat{\sigma}_h \cdot (N \cdot L)^{-1/6}$ and $h_{X,1} = 3.72 \cdot \hat{\sigma}_X \cdot (N \cdot L)^{-1/6}$ as the first-step bandwidths, where $\hat{\sigma}_X$ is the estimated standard deviation of the observed co-variates. We use $h_f = 3.15 \cdot \hat{\sigma}_v \cdot (N \cdot L)^{-1/5}$ and $h_{X,2} = 3.15 \cdot \hat{\sigma}_X \cdot (N \cdot L)^{-1/6}$ as the second-step bandwidths. The rate is changed to $-1/5$ because now there is also smoothing over the values of the co-variates. For Nadaraya-Watson estimation of $\varphi(x)$ in the second step, we use the fourth-order triweight kernel and a rule-of-thumb bandwidth selection rule: $h_{X,3} = 3.72 \cdot \hat{\sigma}_X \cdot (N \cdot L)^{-1/5}$.

The results are reported in Table S.1. Under auction-specific heterogeneity, the coverage accuracy of the IGA uniform confidence bands is less precise than that in the case with no co-variates, especially for the wider range of values $v \in [0.2, 0.8]$. We believe this is due to smaller
Table S.1: Coverage probabilities of the Intermediate Gaussian Approximation uniform confidence band $CB^*$ in presence of auction-specific heterogeneity, for the number of bidders $N = 3, 5, 7$, the distribution parameter $\sigma = 0.8, 1, 1.2$, different ranges of values $v$, and the nominal coverage probability $= 0.90, 0.95, 0.99$. The number of auctions $L$ is determined by $N \cdot L = 2100$.

| Range of $v$ | $N$ | 0.90 | 0.95 | 0.99 |
|--------------|-----|------|------|------|
| [0.3, 0.7]   | 3   | 0.818| 0.888| 0.958|
|              | 5   | 0.804| 0.892| 0.962|
|              | 7   | 0.816| 0.892| 0.964|
| [0.2, 0.8]   | 3   | 0.642| 0.762| 0.924|
|              | 5   | 0.768| 0.864| 0.930|
|              | 7   | 0.798| 0.862| 0.952|

$\sigma = 1$

| Range of $v$ | $N$ | 0.90 | 0.95 | 0.99 |
|--------------|-----|------|------|------|
| [0.3, 0.7]   | 3   | 0.798| 0.892| 0.966|
|              | 5   | 0.810| 0.880| 0.948|
|              | 7   | 0.818| 0.898| 0.952|
| [0.2, 0.8]   | 3   | 0.694| 0.806| 0.928|
|              | 5   | 0.764| 0.860| 0.932|
|              | 7   | 0.832| 0.880| 0.922|

$\sigma = 1.2$

| Range of $v$ | $N$ | 0.90 | 0.95 | 0.99 |
|--------------|-----|------|------|------|
| [0.3, 0.7]   | 3   | 0.862| 0.916| 0.968|
|              | 5   | 0.826| 0.896| 0.942|
|              | 7   | 0.834| 0.888| 0.952|
| [0.2, 0.8]   | 3   | 0.700| 0.818| 0.936|
|              | 5   | 0.804| 0.878| 0.934|
|              | 7   | 0.812| 0.886| 0.942|

effective sample sizes, which arise because of smoothing over the covariate values. Our conjecture is that increasing the number of auctions $L$ would restore the accuracy.