On global bifurcation for the nonlinear Steklov problems

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Abstract

For $p \in (1, \infty)$, for an integer $N \geq 2$ and for a bounded Lipschitz domain $\Omega$, we consider the following nonlinear Steklov bifurcation problem

\[-\Delta_p \phi = 0 \text{ in } \Omega, \quad |\nabla \phi|^{p-2} \frac{\partial \phi}{\partial \nu} = \lambda \left( g|\phi|^{p-2} \phi + fr(\phi) \right) \text{ on } \partial \Omega,\]

where $\Delta_p$ is the $p$-Laplace operator, $g, f \in L^1(\partial \Omega)$ are indefinite weight functions and $r \in C(\mathbb{R})$ satisfies $r(0) = 0$ and certain growth conditions near zero and at infinity. For $f, g$ in some appropriate Lorentz-Zygmund spaces, we establish the existence of a continuum that bifurcates from $(\lambda_1, 0)$, where $\lambda_1$ is the first eigenvalue of the following nonlinear Steklov eigenvalue problem

\[-\Delta_p \phi = 0 \text{ in } \Omega, \quad |\nabla \phi|^{p-2} \frac{\partial \phi}{\partial \nu} = \lambda g|\phi|^{p-2} \phi \text{ on } \partial \Omega.\]

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1 Introduction

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N (N \geq 2)$ with the boundary $\partial \Omega$. For $p \in (1, \infty)$, we consider the following nonlinear Steklov bifurcation problem:

\[-\Delta_p \phi = 0 \text{ in } \Omega, \quad |\nabla \phi|^{p-2} \frac{\partial \phi}{\partial \nu} = \lambda \left( g|\phi|^{p-2} \phi + fr(\phi) \right) \text{ on } \partial \Omega,\]  \hfill (1.1)

where $\Delta_p$ is the $p$-Laplace operator defined as $\Delta_p(\phi) = \text{div}(|\nabla \phi|^{p-2} \nabla \phi)$, $f, g \in L^1(\partial \Omega)$ are indefinite weights functions and $r \in C(\mathbb{R})$ satisfying $r(0) = 0$. A function $\phi \in W^{1,p}(\Omega)$ is said to be a solution of (1.1), if

\[\int_{\Omega} |\nabla \phi|^{p-2} \nabla v \cdot v \, dx = \lambda \int_{\partial \Omega} (g|\phi|^{p-2} \phi v + fr(\phi)v) \, ds, \quad \forall v \in W^{1,p}(\Omega). \]

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Since \( r(0) = 0 \), the set \( \{ (\lambda, 0) : \lambda \in \mathbb{R} \} \) is always a trivial branch of solutions of (1.1). We say a real number \( \lambda \) is a bifurcation point of (1.1), if there exists a sequence \( \{ (\lambda_n, \phi_n) \} \) of nontrivial weak solutions of (1.1) such that \( \lambda_n \to \lambda \) and \( \phi_n \to 0 \) in \( W^{1,p}(\Omega) \) as \( n \to \infty \).

The bifurcation problem arises in numerous contexts in mathematical and engineering applications. For example, in reaction diffusion [30], elasticity theory [9, 47], population genetics [13], water wave theory [33], stability problems in engineering [49, 50]. Many authors considered the following nonlinear bifurcation problem with different boundary conditions

\[
-\Delta_p \phi = \lambda g|\phi|^{p-2}\phi + h(\lambda, x, \phi) \quad \text{in } \Omega, 
\]

where \( h \) is assumed to be a Carathéodory function satisfying \( h(\lambda, x, 0) = 0 \). There are various sufficient conditions available on \( g \) for the existence of a bifurcation point of (1.3). For Dirichlet boundary condition, \( g = 1 \) [19, 28, 37], \( g \in L^r(\Omega) \) with \( r > \frac{N}{2} \) [8], \( g \in L^\infty(\mathbb{R}^N) \) [20]. There are few works that deal with \( h \) of the form \( \lambda f(x)r(\phi) \) with continuous \( r \) satisfying \( r(0) = 0 \) and certain growth condition at zero and at infinity, see for \( g, f \) in Hölder continuous spaces [43], in certain Lebesgue spaces [27], in Lorentz spaces [7, 36]. The bifurcation problem (1.3) with Neumann boundary condition is considered for \( g = 1 \) in [19], for smooth \( f, g \) in [12].

For \( p = 2 \), (1.1) is considered in [15, 16, 46] for \( g = 1 \) and continuous \( f \), and in [40] for \( f, g \in L^\infty(\partial \Omega) \). Indeed, there are many singular weights (not belonging to any of the Lebesgue spaces) that appear in problems in quantum mechanics, molecular physics, see [24, 25, 26]. In this article, we enlarge the class of weight functions beyond the classical Lebesgue spaces. More precisely, we consider \( f, g \) in certain Lorentz-Zygmund spaces, and study the existence of bifurcation point for (1.1).

Using the weak formulation, it is easy to see that (1.3) is equivalent to the following operator equation:

\[
A(\phi) = \lambda G(\phi) + H(\lambda, \phi), \quad \phi \in X, 
\]

where \( X \) is the Banach space \( W^{1,p}(\Omega) \) or \( W^{1,p}_0(\Omega) \) depending on the boundary conditions, \( A, G, H(\lambda, \cdot) : X \to X' \) defined as \( \langle A(\phi), v \rangle = \int_\Omega \nabla \phi |\phi|^{p-2} \nabla \phi \nabla v \, dx; \langle G(\phi), v \rangle = \int_\Omega g|\phi|^{p-2}\phi v \, dx; \langle H(\lambda, \phi), v \rangle = \int_\Omega h(\lambda, x, \phi) v \, dx \). For \( p = 2 \), \( A \) is an invertible map. Using the Leray-Schauder degree [32], Krasnosel’skii in [31] gave sufficient conditions on \( L = A^{-1}G, K = A^{-1}H \) so that, for any eigenvalue \( \mu = \lambda^{-1} \) of \( L \) with odd multiplicity, \( \lambda, 0 \) is a bifurcation point of (1.4). Later, Rabinowitz [41, Theorem 1.3], extended this result by exhibiting a continuum of nontrivial solutions of (1.4) bifurcating from \( (\lambda, 0) \) which is either unbounded in \( \mathbb{R} \times X \) or meets at \( (\lambda^*, 0) \), where \( \mu = \lambda^{*-1} \) is an eigenvalue of \( L \). Further, if \( \mu \) has multiplicity one, then this continuum decompose into two subcontinua of nontrivial solutions of (1.4), see [3, 17, 18, 41, 42]. For \( p = 2 \), the Leray-Schauder degree is extended for certain maps between \( X \) to \( X' \) [11, 44] and then an analogue of Rabinowitz result is proved for the first eigenvalue of \( A = \lambda G \), see [19, 20, 28, 37].

To study the bifurcation problem (1.1), we consider the following nonlinear eigenvalue prob-
lem: $$-\Delta_p \phi = 0 \text{ in } \Omega,$$

$$|\nabla \phi|^{p-2} \frac{\partial \phi}{\partial \nu} = \lambda g |\phi|^{p-2} \phi \text{ on } \partial \Omega.$$ (1.5)

For $$N = 2, p = 2$$ and $$g = 1$$, the problem (1.5) is first considered by Steklov in [45]. A real number $$\lambda$$ is said to be an eigenvalue of (1.5), if there exists $$\phi \in W^{1,p}(\Omega) \setminus \{0\}$$ satisfying the following weak formulation

$$\int_\Omega |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla v \, dx = \lambda \int_{\partial \Omega} g |\phi|^{p-2} \phi v \, d\sigma, \quad \forall v \in W^{1,p}(\Omega).$$ (1.6)

For $$N > p$$, the classical trace embeddings ([38, Theorem 4.2 and Theorem 6.2]) gives

$$W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega), \text{ where } q \in \left[1, \frac{p(N-1)}{N-p}\right],$$

and for $$q \leq \frac{p(N-1)}{N-p}$$ the above embedding is compact. Thus, by the Hölder inequality the right hand side of (1.6) is finite for $$g \in L^r(\partial \Omega)$$ with $$r \in \left[\frac{N-1}{p-1}, \infty\right]$$ and for any $$\phi, v \in W^{1,p}(\Omega)$$. We say an eigenvalue $$\lambda$$ is principal, if there exists an eigenfunction of (1.5) corresponding to $$\lambda$$ that does not change it’s sign in $$\Omega$$. Notice that, zero is always a principal eigenvalue of (1.5) and if $$\int_{\partial \Omega} g \geq 0$$, then zero is the only principal eigenvalue. Thus for the existence of a positive principal eigenvalue of (1.5), it is necessary to have a $$g$$ satisfying $$\int_{\partial \Omega} g < 0$$ and the $$(N - 1)$$-dimensional Hausdorff measure of $$\text{supp}(g^+)$$ is nonzero. In [48], for $$g \in L^r(\partial \Omega)$$ with $$r \in \left(\frac{N-1}{p-1}, \infty\right]$$ satisfying the above necessary conditions, with the help of the above compact embedding, the authors proved the existence of a positive principal eigenvalue of (1.5). For $$N = p$$, $$W^{1,p}(\Omega)$$ is embedded compactly in $$L^q(\partial \Omega)$$ for $$q \in [1, \infty)$$. Thus for $$g \in L^r(\partial \Omega)$$ with $$r \in (1, \infty]$$ satisfying the above necessary condition, (1.5) admits a positive principal eigenvalue, as obtained in [48].

In order to enlarge the class of weight functions beyond $$L^r$$, we use the trace embeddings due to Cianchi-Kerman-Pick. In [14], the authors improved the classical trace embeddings by providing finer trace embeddings as below:

(i) For $$N > p$$: $$W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\partial \Omega) \subseteq L^{\frac{p(N-1)}{N-p}}(\partial \Omega).$$

(ii) For $$N = p$$: $$W^{1,p}(\Omega) \hookrightarrow L^{\infty,N-1}(\partial \Omega) \subseteq L^q(\partial \Omega), \quad \forall q \in [1, \infty).$$

Nevertheless, none of these embeddings are compact. In this article, we use the above trace embeddings and prove the existence of a positive principal eigenvalue of (1.5) for weight functions in certain Lorentz-Zygmund spaces. More precisely, for $$1 \leq d < \infty$$, we consider the following closed subspaces:

$$\mathcal{F}_d := \text{closure of } C^1(\partial \Omega) \text{ in the Lorentz space } L^{d,\infty}(\partial \Omega),$$

$$\mathcal{G}_d := \text{closure of } C^1(\partial \Omega) \text{ in the Lorentz-Zygmund space } L^{d,\infty;N}(\partial \Omega).$$

**Theorem 1.1.** Let $$p \in (1, \infty)$$ and $$N \geq p$$. Let $$g^+ \neq 0$$, $$\int_{\partial \Omega} g < 0$$ and

$$g \in \begin{cases} \mathcal{F}^{N-1}_{p-1} & \text{for } N > p, \\ \mathcal{G}_1 & \text{for } N = p. \end{cases}$$


Then
\[ \lambda_1 = \inf \left\{ \int_\Omega |\nabla \phi|^p : \phi \in W^{1,p}(\Omega), \int_{\partial \Omega} g |\phi|^p = 1 \right\} \]
is the unique positive principal eigenvalue of (1.5). Furthermore, \( \lambda_1 \) is simple and isolated.

Indeed, \( L^{N-1}(\partial \Omega) \) is contained in \( \mathcal{F}_{N-1}^{-1} \) (for \( N > p \)) and \( L^q(\partial \Omega) \) (for \( q > 1 \)) is contained in \( \mathcal{G}_1 \) (for \( N = p \)) (see Remark 4.1). Thus the above theorem extends the result of [48].

Having obtained the right candidate for bifurcation point, we can study (1.1) for weights in appropriate Lorentz-Zygmund spaces. For this, let us consider the following set:
\[ S = \{ (\lambda, \phi) \in \mathbb{R} \times W^{1,p}(\Omega) : (\lambda, \phi) \text{ is a solution of (1.1) and } \phi \not\equiv 0 \} \]
We say \( \mathcal{C} \subset S \) is a continuum of nontrivial solutions of (1.1) if it is connected in \( \mathbb{R} \times W^{1,p}(\Omega) \).

In this article, we prove the existence of a continuum \( \mathcal{C} \) of nontrivial solutions of (1.1) that bifurcates from \( (\lambda_1, 0) \).

For \( p \in (1, \infty) \) and \( g \) as in Theorem 1.1, depending on the dimension we make the following assumptions on \( r \) and \( f \):

\[
\begin{align*}
(\text{H1}) & \quad \left\{ \begin{array}{ll}
(a) & \lim_{|s| \to 0} \frac{|r(s)|}{|s|^{p-1}} = 0 \quad \text{and} \quad |r(s)| \leq C|s|^\gamma - 1 \text{ for some } \gamma \in \left( 1, \frac{p(N-1)}{N-p} \right), \\
(b) & \quad g \in \mathcal{F}_{N-1}^{\gamma-1}, \quad f \in \left\{ \begin{array}{ll}
\mathcal{F}_{\frac{N}{p-1}}, & \text{if } \gamma \geq p, \text{ where } \frac{1}{p} + \gamma(N-p) = 1; \\
\mathcal{F}_{p-1}^{N-1}, & \text{if } \gamma < p.
\end{array} \right.
\end{array} \right.
\end{align*}
\]

\[
(\text{H2}) \quad \left\{ \begin{array}{ll}
(a) & \lim_{|s| \to 0} \frac{|r(s)|}{|s|^{N-1}} = 0 \quad \text{and} \quad |r(s)| \leq C|s|^\gamma - 1 \text{ for some } \gamma \in (1, \infty), \\
(b) & \quad g \in \mathcal{G}_1, \quad f \in \mathcal{G}_d \text{ with } d > 1.
\end{array} \right.
\]

Theorem 1.2. Let \( p \in (1, \infty) \). Assume that \( r, g \) and \( f \) satisfy (H1) for \( N > p \) and satisfy (H2) for \( N = p \). Then \( \lambda_1 \) is a bifurcation point of (1.1). Moreover, there exists a continuum of nontrivial solutions \( \mathcal{C} \) of (1.1) such that \( (\lambda_1, 0) \in \mathcal{C} \) and either

(i) \( \mathcal{C} \) is unbounded, or

(ii) \( \mathcal{C} \) contains the point \( (\lambda, 0) \), where \( \lambda \) is an eigenvalue of (1.5) and \( \lambda \neq \lambda_1 \).

The rest of the article is organized as follows. In Section 2, we give the definition and list some properties of symmetrization and Lorentz-Zygmund spaces. We also state the classical trace embedding theorems and their refinements. The definition and some of the properties of degree of a certain class of nonlinear maps between \( W^{1,p}(\Omega) \) and \( (W^{1,p}(\Omega))^\prime \) are also given in this section. In Section 3, we develop a functional framework associated with our problem and prove many results that we needed to prove our main theorems. Section 4 contains the proofs of Theorem 1.1 and Theorem 1.2.
2 Preliminaries

In this section, we briefly describe the one-dimensional decreasing rearrangement with respect to \((N-1)\)-dimensional Hausdorff measure. Using this, we define Lorentz-Zygmund spaces over the boundary and give examples of functions in these spaces. Further, we state the classical trace embeddings of \(W^{1,p}(\Omega)\), and its refinements due to Cianchi et al. We also define the degree for a certain class of nonlinear maps and list some of the results that we use in this article.

2.1 Symmetrization

Let \(\Omega \subset \mathbb{R}^N\) be a bounded Lipschitz domain. Let \(\mathcal{M}(\partial \Omega)\) be the collection of all real valued \((N-1)\)-dimensional Hausdorff measurable functions defined on \(\partial \Omega\). Given a function \(f \in \mathcal{M}(\partial \Omega)\), and for \(s > 0\), we define \(E_f(s) = \{x \in \partial \Omega : |f(x)| > s\}\). The distribution function \(\alpha_f\) of \(f\) is defined as \(\alpha_f(s) = \mathcal{H}^{N-1}(E_f(s))\) for \(s > 0\). We define the \(one\ dimensional\ decreasing\ rearrangement\ \(f^*\) of \(f\) as

\[
f^*(t) = \inf \{s > 0 : \alpha_f(s) < t\}, \quad \text{for } t > 0.
\]

The map \(f \mapsto f^*\) is not sub-additive. However, we obtain a sub-additive function from \(f^*\), namely the maximal function \(f^{**}\) of \(f^*\), defined by

\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(\tau) \, d\tau, \quad t > 0.
\]

Next we state one important inequality concerning the symmetrization [22, Theorem 3.2.10].

**Proposition 2.1.** \((Hardy-Littlewood\ inequality)\) Let \(N \geq 2\) and let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^N\). Let \(f\) and \(g\) be nonnegative measurable functions defined on \(\partial \Omega\). Then

\[
\int_{\partial \Omega} fg \, d\sigma \leq \int_0^{\mathcal{H}^{N-1}(\partial \Omega)} f^*(t)g^*(t) \, dt.
\]

2.2 Lorentz-Zygmund space

The Lorentz-Zygmund spaces are three parameter family of function spaces that refine the classical Lebesgue spaces. For more details on Lorentz-Zygmund spaces, we refer to [10, 23]. Here we consider the Lorentz-Zygmund spaces over \(\partial \Omega\) of a bounded domain \(\Omega\).

Let \(\Omega \subset \mathbb{R}^N\) be a bounded Lipschitz domain. Let \(f \in \mathcal{M}(\partial \Omega)\) and let \(l_1(t) = 1 + |\log(t)|\). For \((p, q, \alpha) \in [1, \infty] \times [1, \infty] \times \mathbb{R}\), consider the following quantity:

\[
|f|_{(p,q,\alpha)} := \left\| \frac{1}{t^\frac{1}{q}} l_1(t)^\alpha f^*(t) \right\|_{L^p((0,\mathcal{H}^{N-1}(\partial \Omega)))}^{\frac{1}{q}}.
\]

\[
= \begin{cases} \left( \int_0^{\mathcal{H}^{N-1}(\partial \Omega)} \left[\frac{1}{t^\frac{1}{q}} l_1(t)^\alpha f^*(t) \right]^q \frac{dt}{t} \right)^\frac{1}{q}, & 1 \leq q < \infty; \\
\sup_{0<t<\mathcal{H}^{N-1}(\partial \Omega)} t^{\frac{1}{q}} l_1(t)^\alpha f^*(t), & q = \infty. \end{cases}
\]
The Lorentz-Zygmund space \( L^{p,q,\alpha}(\partial \Omega) \) is defined as
\[
L^{p,q,\alpha}(\partial \Omega) := \{ f \in \mathcal{M}(\partial \Omega) : |f|_{(p,q,\alpha)} < \infty \},
\]
where \(|f|_{(p,q,\alpha)}\) is a complete quasi norm on \( L^{p,q,\alpha}(\partial \Omega) \). For \( p > 1 \),
\[
\|f\|_{(p,q,\alpha)} = \left\| \frac{1}{|t|^{\frac{1}{p}} - \frac{1}{q} \mu_1(t)} f_\ast(t) \right\|_{L^p(\mathcal{H}_{10,1}(\partial \Omega))}
\]
is a norm in \( L^{p,q,\alpha}(\partial \Omega) \) which is equivalent to \(|f|_{(p,q,\alpha)} \) \cite{10, Corollary 8.2}. In particular, \( L^{p,q,\alpha}(\partial \Omega) \) coincides with the Lorentz space \( L^{p,q}(\partial \Omega) \) introduced by Lorentz in \cite{35}. In the following proposition we discuss some important properties of the Lorentz-Zygmund spaces that we will use in this article.

**Proposition 2.2.** Let \( p, q, r, s \in [1, \infty] \) and \( \alpha, \beta \in (-\infty, \infty) \).

(i) Let \( p \in (1, \infty) \). If \( f \in L^{\infty,p,-1}(\partial \Omega) \), then \(|f|^p \in L^{\infty,1;p}(\partial \Omega)\). Moreover, there exists \( C > 0 \) such that
\[
\|f|^p\|_{(\infty,1;p)} \leq C\|f\|_{(\infty,p,-1)}^p.
\]

(ii) Let \( p \in (1, \infty) \). Then the space \( L^{1,\infty;p}(\partial \Omega) \) is contained in the dual space of \( L^{\infty,1;p}(\partial \Omega) \).

(iii) If \( r > p \), then \( L^{r,s;\beta}(\partial \Omega) \rightarrow L^{p,q,\alpha}(\partial \Omega) \), i.e., there exists a constant \( C > 0 \) such that
\[
\|f\|_{(p,q,\alpha)} \leq C\|f\|_{(r,s,\beta)}, \quad \forall f \in L^{r,s;\beta}(\partial \Omega).
\]

(iv) If either \( q \leq s \) and \( \alpha \geq \beta \) or, \( q > s \) and \( \alpha + \frac{1}{q} > \beta + \frac{1}{s} \), then \( L^{p,q,\alpha}(\partial \Omega) \hookrightarrow L^{p,s;\beta}(\partial \Omega) \), i.e., there exists \( C > 0 \) such that
\[
\|f\|_{(p,s;\beta)} \leq C\|f\|_{(p,q,\alpha)}, \quad \forall f \in L^{p,q,\alpha}(\partial \Omega).
\]

(v) For \( p \in (1, \infty) \), \( L^p(\partial \Omega) \hookrightarrow L^{1,\infty;\alpha}(\partial \Omega) \).

**Proof.** (i) If \( f \in L^{\infty,p,-1}(\partial \Omega) \), then \(|f|_{(\infty,p,-1)} < \infty \). Hence using \((|f|^p)^* = (f^*)^p\), we get
\[
\|f|^p\|_{(\infty,1;p)} = \int_0^{H^{1,-1}(\partial \Omega)} \left( \frac{\|f|^p\|}{\mu_1(t)} \right)^\frac{1}{p} \frac{dt}{t} = \left( \frac{1}{\int_0^{H^{1,-1}(\partial \Omega)} \frac{f^*(t)}{\mu_1(t)} \frac{dt}{t}} \right) \cdot \int_0^{H^{1,-1}(\partial \Omega)} \left( \frac{f^*(t)}{\mu_1(t)} \right)^\frac{1}{s} \frac{dt}{t} \cdot \|f|^p\|_{(\infty,1;p)}^p = \|f\|_{(\infty,p,-1)}^p.
\]

Therefore, \(|f|^p \in L^{\infty,1;p}(\partial \Omega)\). Now by the equivalence of norms, there exists \( C_1, C_2 > 0 \) such that
\[
\|f|^p\|_{(\infty,1;p)} \leq C_1\|f\|_{(\infty,p,-1)}^p \leq C_1 C_2\|f\|_{(\infty,p,-1)}^p.
\]

Thus there exists \( C > 0 \) such that \( \|f|^p\|_{(\infty,1;p)} \leq C\|f\|_{(\infty,p,-1)}^p \).

(ii) Let \( f \in L^{\infty,1;p}(\partial \Omega) \) and \( g \in L^{1,\infty;p}(\partial \Omega) \). Then using the Hardy-Littlewood inequality (Proposition 2.1),
\[
\int_{\partial \Omega} fg \, d\sigma \leq \int_0^{H^{1,-1}(\partial \Omega)} f^*(t)g^*(t) \, dt
\]
\[ \sup_{0 \leq t < H^{N-1}(\partial \Omega)} t g^*(t)(l_1(t))^p \left( \int_0^{H^{N-1}(\partial \Omega)} \frac{f^*(t)}{l_1(t))^p}{dt} \right) \]
\[ = \|g\|_{(1,\infty;p)} \|f\|_{(\infty;1,-p)} \].

Thus \( f \) is in the dual space of \( L^{1,\infty;p}(\partial \Omega) \).

(iii) Follows from [10, Theorem 9.1]. (iv) Follows from [10, Theorem 9.3].

(v) Let \( f \in L^p(\partial \Omega) \). Since \( p > 1 \), using (2.1) there exists \( C > 0 \) such that
\[ \|f\|_{(1,\infty;\alpha)} \leq C \|f\|_{L^p(\partial \Omega)}. \]

Therefore, \( L^p(\partial \Omega) \) is continuously embedded into \( L^{1,\infty;\alpha}(\partial \Omega) \). \qed

The following characterization of the function space \( G_d \) follows by similar arguments as in the proof of [5, Theroem 16].

**Proposition 2.3.** Let \( N \geq 2 \) and \( d \in [1, \infty) \). Then \( f \in G_d \) if and only if
\[ \lim_{t \to 0} t^{\frac{1}{d} f^*(l_1(t))^N} = 0. \]

Next we list some properties of the Lorentz spaces. For more details on Lorentz spaces, we refer to [1, 22, 29].

**Proposition 2.4.** Let \( p, q, r \in [1, \infty] \).

(i) Generalized Hölder inequality: Let \( f \in L^{p_1, q_1}(\partial \Omega) \) and \( g \in L^{p_2, q_2}(\partial \Omega) \), where \( (p_i, q_i) \in (1, \infty) \times [1, \infty] \) for \( i = 1, 2 \). If \( (p, q) \) be such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \), then
\[ \|fg\|_{(p,q)} \leq C \|f\|_{(p_1,q_1)} \|g\|_{(p_2,q_2)}, \]

where \( C = C(p) > 0 \) is a constant such that \( C = 1 \), if \( p = 1 \) and \( C = p' \), if \( p > 1 \).

(ii) For \( r > 0 \), \( \|\|f\|^r\|_{(\frac{r}{1-r}, \frac{r}{1-r})} = \|f\|_{(p,q)} \).

Proof. Proof of (i) follows from [29, Theorem 4.5]. For \( \alpha = 0 \), proof of (ii) directly follows from the definition of the Lorentz-Zygmund space. \qed

In the following we list some properties of the function space \( F_d \).

**Proposition 2.5.** Let \( d, q \in (1, \infty) \). Then

(i) \( L^{d,q}(\partial \Omega) \subset F_d \).

(ii) Let \( h \in L^{d,\infty}(\partial \Omega) \) and \( h > 0 \). Let \( f \in L^1(\partial \Omega) \). If \( \int_{\partial \Omega} h^{d-q}|f|^q < \infty \) for \( q \geq d \), then \( f \in L^{d,q}(\partial \Omega) \) and hence \( f \in F_d \).

(iii) \( f \in F_d \) if and only if
\[ \lim_{t \to 0} t^{\frac{1}{d} f^*(t)} = 0 = \lim_{t \to H^{N-1}(\partial \Omega)} t^{\frac{1}{d} f^*(t)}. \]
Proof. (i) Using (2.2) for \( \alpha = \beta = 0 \) and by the density arguments, we get \( L^{d,q}(\partial \Omega) \subset F_d \).

(ii) The result is obvious for \( q = d \). For \( q > d \), set \( g = h^{\frac{d}{q} - 1} |f| \). Then \( g \in L^q(\partial \Omega) \). Using Proposition 2.4, \( h^{\frac{d}{q}} \in L^{\frac{dq}{d+q}}(\partial \Omega) \). Therefore, by the generalized Hölder inequality (Proposition 2.4), \( f \in L^{d,q}(\partial \Omega) \).

(iii) Follows by the similar arguments as in [7, Theorem 3.3].

2.3 Examples

Now we give some examples of functions in the Lorentz-Zygmund spaces that are defined on \( \partial \Omega \) of a Lipschitz bounded domain \( \Omega \).

Example 2.6. For \( \Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \} \), we consider

\[
g_1(x, y) = |y|^{-\frac{1}{2}}, \quad \forall (x, y) \in \partial \Omega.
\]

For \( s > 0 \), we can compute

\[
\alpha_{g_1}(s) = \begin{cases} 
2\pi, & \text{for } 0 < s < 1, \\
4 \sin^{-1}(\frac{1}{s^2}), & \text{for } s \geq 1.
\end{cases}
\]

Thus \( g_1^*(t) = (\csc(\frac{t}{4}))^\frac{1}{4} \). Therefore,

\[
\sup_{0 < t < 2\pi} t^\frac{1}{2} \left( \csc \left( \frac{t}{4} \right) \right)^\frac{1}{4} < \infty; \quad \sup_{0 < t < 2\pi} t(l_1(t))^2 \left( \csc \left( \frac{t}{4} \right) \right)^\frac{1}{4} < \infty.
\]

Hence \( g_1 \in L^{2,\infty}(\partial \Omega) \) and \( g_1 \in L^{1,\infty;2}(\partial \Omega) \). Furthermore,

\[
\lim_{t \to 0} t^\frac{1}{2} \left( \csc \left( \frac{t}{4} \right) \right)^\frac{1}{4} > 0; \quad \lim_{t \to 0} t(l_1(t))^2 \left( \csc \left( \frac{t}{4} \right) \right)^\frac{1}{4} > 0.
\]

Hence \( g_1 \notin F_2 \) (by Proposition 2.5) and \( g_1 \notin G_1 \) (by Proposition 2.3).

Example 2.7. Let \( p \in (1, \infty) \) and \( N > p \). For \( 0 < R < \frac{1}{2} \), let

\[
\Omega = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : |x_i| < R \text{ (for } i = 1, \ldots, N - 1), 0 < x_N < 2R \}
\]

and \( A = \{(x_1, x_2, \ldots, x_{N-1}, 0) : |x_i| < R \} \). Now consider

\[
g_2(x) = \begin{cases} 
|x_1 \log(|x_1|)|^{-\frac{N-1}{N-1}}, & \text{for } x \in A, \\
0, & \text{for } x \in \partial \Omega \setminus A.
\end{cases}
\]

Clearly \( g_2 \in L^1(\partial \Omega) \) and \( g_2 \notin L^{r}(\partial \Omega) \) for \( r \in \left[ \frac{N-1}{p-1}, \infty \right) \). Let

\[
h(x) = \begin{cases} 
|x_1|^{-\frac{N-1}{N-1}}, & \text{for } x \in A, \\
0, & \text{for } x \in \partial \Omega \setminus A.
\end{cases}
\]
We calculate $\alpha_h(s) = 2^{N-1}R^{N-2}s^{\frac{N-1}{p-1}}$ and $h^*(t) = (2^{N-1}R^{N-2})^{\frac{N-1}{p-1}}t^{-\frac{N-1}{p-1}}$. Therefore, $h \in L^{\frac{N-1}{p-1}}(\partial \Omega)$. For $q = \frac{N}{p-1}$,

$$h^{\frac{N-1}{p-1}-q}(x) = \begin{cases} \frac{|x_1|^{\frac{N-1}{p-1}}}{N-1}, & \text{for } x \in A, \\ 0, & \text{for } x \in \partial \Omega \setminus A. \end{cases}$$

Further,

$$\int_{\partial \Omega} h^{\frac{N-1}{p-1}-q}g_2^q \, d\sigma = 2^{N-1}R^{N-2} \int_0^R t^{-\frac{1}{q}} \frac{1}{N-1} \log(t)^{-\frac{N}{p-1}} \, dt < \infty.$$ 

Therefore, by Proposition 2.5, $g_2 \in L^{\frac{N-1}{p-1}-q}(\partial \Omega)$ and hence $g_2 \in \mathcal{F}_{\frac{N-1}{p-1}}$.

**Example 2.8.** For $0 < R < 1$, let $\Omega$ and $A$ be given as in the above example. For $q \in (1, \infty)$, we consider

$$g_3(x) = \begin{cases} \frac{|x_1|^{\frac{N-1}{4}}}{N-1}, & \text{for } x \in A, \\ 0, & \text{for } x \in \partial \Omega \setminus A. \end{cases}$$

Clearly $g_3 \notin L^q(\partial \Omega)$ for $q \in (1, \infty)$. Further, we calculate $\alpha_{g_3}(s) = 2^{N-1}R^{N-2}s^{-q}$ and $g_3^*(t) = (2^{N-1}R^{N-2})^{\frac{1}{q}} t^{-\frac{q}{4}}$. Moreover,

$$\lim_{t \to 0} t^{\frac{q-1}{q}} (1 + |\log(t)|)^N = 0$$

and hence $g_3 \in \mathcal{G}_{1}$ (by Proposition 2.3).

### 2.4 Trace embeddings

Now we state the trace embeddings that play a vital role in this article. First, we state the classical trace embeddings to the Lebesgue spaces [38, Theorem 4.2, Theorem 4.6, Theorem 6.2].

**Proposition 2.9** (Classical trace embeddings). Let $N \geq 2$ and let $\Omega$ be a Lipschitz bounded domain in $\mathbb{R}^N$. Let $p \in (1, \infty)$. Then the following embeddings hold:

(i) If $N > p$ and $q \in \left[1, \frac{p(N-1)}{N-p}\right]$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega)$, i.e., there exists $C = C(N,p) > 0$ satisfying

$$\|\phi\|_{L^q(\partial \Omega)} \leq C\|\phi\|_{W^{1,p}(\Omega)}, \quad \forall \phi \in W^{1,p}(\Omega).$$

If $q \neq \frac{p(N-1)}{N-p}$, then the above embedding is compact.

(ii) If $N = p$ and $q \in [1, \infty)$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega)$, i.e., there exists $C = C(N) > 0$ satisfying

$$\|\phi\|_{L^q(\partial \Omega)} \leq C\|\phi\|_{W^{1,p}(\Omega)}, \quad \forall \phi \in W^{1,p}(\Omega),$$

and the above embedding is compact.
The following embeddings are due to Cianchi et al. [14, Theorem 1.3] that extends the classical trace embeddings to the Lebesgue spaces with the finer embeddings to the Lorentz-Zygmund spaces.

**Proposition 2.10** (Finer trace embeddings). Let $N \geq 2$ and let $\Omega$ be a Lipschitz bounded domain in $\mathbb{R}^N$. Let $p \in (1, \infty)$. Then the following embeddings hold:

(i) If $N > p$, then $W^{1,p}(\Omega) \hookrightarrow L^{\frac{p(N-1)}{N-p}}(\partial \Omega)$, i.e., there exists $C = C(N, p) > 0$ such that

$$\|\phi\|_{L^{\frac{p(N-1)}{N-p}}(\partial \Omega)} \leq C\|\phi\|_{W^{1,p}(\Omega)}, \quad \forall \phi \in W^{1,p}(\Omega).$$

(ii) Let $N = p$ and $g \in L^{\frac{N-1}{N-1}}(\partial \Omega)$. Then there exists a constant $C = C(N) > 0$ such that

$$\|\phi\|_{(\infty, N; 1)} \leq C\|\phi\|_{W^{1,p}(\Omega)}, \quad \forall \phi \in W^{1,p}(\Omega).$$

The above finer trace embeddings help us to get the weighted trace inequality for a class of weight functions defined on the boundary.

**Proposition 2.11.** (i) Let $N > p$ and $g \in L^{\frac{N-1}{N-1}}(\partial \Omega)$. Then there exists a constant $C = C(N, p) > 0$ satisfying

$$\int_{\partial \Omega} |g|\|\phi\|^p \leq C\|g\|_{(\frac{N-1}{N-1}, \infty)}\|\phi\|^p_{W^{1,p}(\Omega)}, \quad \forall \phi \in W^{1,p}(\Omega). \quad (2.3)$$

(ii) Let $N = p$ and $g \in L^{1,\infty}(\partial \Omega)$. Then there exists a constant $C = C(N) > 0$ satisfying

$$\int_{\partial \Omega} |g|\|\phi\|^p \leq C\|g\|_{(1,\infty; N)}\|\phi\|^p_{W^{1,p}(\Omega)}, \quad \forall \phi \in W^{1,p}(\Omega). \quad (2.4)$$

**Proof.** (i) For $\phi \in W^{1,p}(\Omega)$, by the generalized Hölder inequality (Proposition 2.4) and Proposition 2.4, we obtain

$$\int_{\partial \Omega} |g|\|\phi\|^p \leq \|g\|_{(\frac{N-1}{N-1}, \infty)}\|\phi\|^p_{W^{1,p}(\Omega)} = \|g\|_{(\frac{N-1}{N-1}, \infty)}\|\phi\|^p_{(\frac{p(N-1)}{N-p}, p)}.$$ 

Now using the finer trace embeddings (Proposition 2.10), we get

$$\int_{\partial \Omega} |g|\|\phi\|^p \leq C\|g\|_{(\frac{N-1}{N-1}, \infty)}\|\phi\|^p_{W^{1,p}(\Omega)}, \quad \forall \phi \in W^{1,p}(\Omega),$$

where $C = C(N, p)$ is the embedding constant.

(ii) For $\phi \in W^{1,N}(\Omega)$, using Proposition 2.2, we obtain

$$\int_{\partial \Omega} |g|\|\phi\|^N \leq \|g\|_{(1,\infty; N)}\|\phi\|^N_{(\infty, 1; -N)} \leq C\|g\|_{(1,\infty; N)}\|\phi\|^N_{(\infty, N; -1)}.$$ 

Again using the finer trace embeddings,

$$\int_{\partial \Omega} |g|\|\phi\|^N \leq C\|g\|_{(1,\infty; N)}\|\phi\|^N_{W^{1,N}(\Omega)}, \quad \forall \phi \in W^{1,N}(\Omega),$$

where $C = C(N) > 0$ is the embedding constant given in Proposition 2.10. \qed
2.5 Degree

We define the degree for certain class of maps from $W^{1,p} (\Omega)$ to it’s dual $(W^{1,p} (\Omega))'$. For more details on this topic, we refer to [11, 44].

Definition 2.12. Let $D \subset W^{1,p} (\Omega)$ be a set and let $F : D \to (W^{1,p} (\Omega))'$ be a map.

(i) **Demicontinuous**: $F$ is said to be demicontinuous on $D$, if for any sequence $(\phi_n) \subset D$ such that $\phi_n \to \phi_0$, then $\lim_{n \to \infty} \langle F(\phi_n), v \rangle = \langle F(\phi_0), v \rangle$, $\forall v \in W^{1,p} (\Omega)$.

(ii) **Class $\alpha (D)$**: $F$ is said to be in class $\alpha (D)$, if every sequence $(\phi_n)$ in $D$ satisfying $\phi_n \to \phi_0$ and $\lim_{n \to \infty} \langle F(\phi_n), \phi_n - \phi_0 \rangle \leq 0$, converges to some $\phi_0$ in $D$.

(iii) For $F \subset D$, $A (D, F)$ denotes the set of all bounded, demicontinuous map defined on $D$ that satisfies the class $\alpha (F)$.

(iv) **Isolated zero**: A point $\phi_0 \in D$ is called an isolated zero of $F$, if $F(\phi_0) = 0$ and there exists $r > 0$ such that the ball $B_r(\phi_0)$ (where $B_r(\phi_0) \subset D$) does not contain any other zeros of $F$.

(v) **Degree**: Let $F \in A (D, \partial D)$ satisfying $F(\phi) \neq 0$ for every $\phi \in \partial D$. Let $(v_i)$ be a Schauder basis for $W^{1,p} (\Omega)$ and let $V_n = \text{span} \{v_1, ..., v_n\}$. A finite-dimensional approximation $F_n$ of $F$ with respect to $V_n$ is defined as:

$$F_n(\phi) = \sum_{i=1}^{n} \langle F(\phi), v_i \rangle v_i$$

where $D_n = D \cap V_n$.

From [44, Theorem 2.1], $F_n(\phi) \neq 0$ for every $\phi \in \partial D_n$, the degree $\text{deg}(F_n, \overline{D_n}, 0)$ of $F_n$ with respect to $0 \in V_n$ is well defined and independent of $n$. Further from [44, Theorem 2.2], $\lim_{n \to \infty} \text{deg}(F_n, \overline{D_n}, 0)$ is independent of basis $(v_i)$. Now the degree of $F$ with respect to $0 \in (W^{1,p} (\Omega))'$ is defined as

$$\text{deg}(F, \overline{D}, 0) = \lim_{n \to \infty} \text{deg}(F_n, \overline{D_n}, 0).$$

(vi) **Homotopy**: Let $F, G \in A (D, \partial D)$ satisfying $F(\phi), G(\phi) \neq 0$ for every $\phi \in \partial D$. The mapping $F$ and $G$ is said to be homotopic on $\overline{D}$, if there exists a sequence of one parameter family $H_t : \overline{D} \to (W^{1,p} (\Omega))'$, $t \in [0, 1]$ such that $H_0 = F$ and $H_1 = G$ and $H_t$ satisfies the following:

(a) For $t \in [0, 1]$, $H_t \in A (D, \partial D)$ and $H_t(\phi) \neq 0$ for every $\phi \in \partial D$.

(b) For a sequence $t_n \in [0, 1]$ satisfying $t_n \to t$ and a sequence $\phi_n \in \overline{D}$ satisfying $\phi_n \to \phi_0$, $H_{t_n} \phi_n \to H_t \phi_0$ as $n \to \infty$.

(vii) **Index**: Let $F \in A (D, \overline{D})$ and let $\phi_0$ be an isolated zero of $F$. Then the index of a map $F$ is defined as $\text{ind}(F, \phi_0) = \lim_{r \to 0} \text{deg}(F, \overline{B_r(\phi_0)}, 0)$.

(viii) **Potential operator**: A map $F \in A (D, (W^{1,p} (\Omega))')$ is called a potential operator, if there exists a functional $f : W^{1,p} (\Omega) \to \mathbb{R}$ such that $f'(\phi) = F(\phi)$, for all $\phi \in W^{1,p} (\Omega)$.
The following Proposition is proved in [44] (Theorem 4.1, Theorem 4.4, Theorem 5.1, and Theorem 6.1).

**Proposition 2.13.** (i) Let \( F, G \in A(D, \partial D) \) satisfying \( F(\phi), G(\phi) \neq 0 \) for every \( \phi \in \partial D \). If \( F \) and \( G \) are homotopic in \( \overline{D} \), then \( \deg(H_t, \overline{D}, 0) = C, \forall t \in [0,1] \). In particular, \( \deg(F, \overline{D}, 0) = \deg(G, \overline{D}, 0) \).

(ii) Let \( F \in A(D, \partial D) \). Suppose that \( 0 \in \overline{D} \setminus \partial D \) and \( \langle F(\phi), \phi \rangle \geq 0, F(\phi) \neq 0 \) for \( \phi \in \partial D \). Then \( \deg(F, \overline{D}, 0) = 1 \).

(iii) Let \( F \in A(D, \overline{D}) \) satisfying \( F(\phi) \neq 0 \), for every \( \phi \in \partial D \). If \( F \) has only finite number of isolated zeros in \( \overline{D} \), then
\[
\deg(F, \overline{D}, 0) = \sum_{i=1}^{n} \text{ind}(F, \phi_i),
\]
where \( \phi_i (i = 1, \ldots, n) \) are all zeros of \( F \) in \( D \).

(iv) Let \( F \in A(D, (W^{1,p}(\Omega))') \) be a potential operator. Suppose that the point \( \phi_0 \) is a local minimum of \( f \) and it is an isolated zero of \( F \). Then \( \text{ind}(F, \phi_0) = 1 \).

### 3 Functional framework

In this section, we set up a suitable functional framework for our problem. We consider the following functional on \( W^{1,p}(\Omega) \):
\[
G(\phi) = \int_{\partial\Omega} |g|\phi|^p, \quad \forall \phi \in W^{1,p}(\Omega).
\]

For \( g \in L^{\frac{N-p}{p-1}}(\partial\Omega) \) (if \( N > p \)) and \( g \in L^{1,\infty:N}(\partial\Omega) \) (if \( N = p \)), Proposition 2.11 ensures that \( G \) is well defined. Now we study the continuity, compactness and differentiability of \( G \).

**Proposition 3.1.** Let
\[
g \in \begin{cases} 
L^{\frac{N-p}{p-1}}(\partial\Omega) & \text{for } N > p, \\
L^{1,\infty:N}(\partial\Omega) & \text{for } N = p.
\end{cases}
\]

Then \( G \) is continuous.

**Proof.** We only consider the case \( N > p \). For \( N = p \), the proof will follow using similar arguments. Let \( \phi_n \to \phi \) in \( W^{1,p}(\Omega) \) and let \( \epsilon > 0 \) be given. Clearly,
\[
|G(\phi_n) - G(\phi)| \leq \int_{\partial\Omega} |g||(|\phi_n|^p - |\phi|^p)|.
\]

Using the inequality due to Lieb and Loss [34, Page 22], there exists \( C = C(\epsilon, p) > 0 \) such that
\[
(|\phi_n|^p - |\phi|^p) \leq \epsilon |\phi|^p + C|\phi_n - \phi|^p \quad \text{a.e. on } \partial\Omega.
\]
Hence
\[
\int_{\partial \Omega} |g||(|\phi_n|^p - |\phi|^p)| \leq \epsilon \int_{\partial \Omega} |g||\phi|^p + C \int_{\partial \Omega} |g||\phi_n - \phi|^p. \tag{3.1}
\]

Now using (2.3), we obtain
\[
\int_{\partial \Omega} |g||\phi_n - \phi|^p \leq C||g|| \left( \frac{\lambda}{p-1} \right) ||\phi_n - \phi||_{W^{1,p}(\Omega)}^p, \tag{3.2}
\]

where \( C = C(N, p) > 0 \) is the embedding constant and \( p' \) is the conjugate exponent of \( p \). Now from (3.1) and (3.2), we easily conclude that \( G(\phi_n) \to G(\phi) \) as \( n \to \infty \).

**Proposition 3.2.** Let
\[
g \in \begin{cases} 
F_{N, \frac{1}{p-1}} & \text{for } N > p, \\
G_1 & \text{for } N = p. 
\end{cases}
\]

Then \( G \) is compact.

**Proof.** As before, we only consider the case \( N > p \). Let \( \phi_n \to \phi \) in \( W^{1,p}(\Omega) \) and let \( \epsilon > 0 \) be given. Set \( L = \sup \{||\phi_n||_{W^{1,p}(\Omega)} + ||\phi||_{W^{1,p}(\Omega)} \} \). For \( g \in F_{N, \frac{1}{p-1}} \), we split \( g = g_\epsilon + (g - g_\epsilon) \) where \( g_\epsilon \in C^1(\partial \Omega) \) such that \( \|g - g_\epsilon\|_{(\frac{N}{p-1}, \infty)} < \frac{\epsilon}{2} \). Then
\[
\int_{\partial \Omega} |g||(|\phi_n|^p - |\phi|^p)| \leq \int_{\partial \Omega} |g_\epsilon||(|\phi_n|^p - |\phi|^p)| + \int_{\partial \Omega} |g - g_\epsilon||(|\phi_n|^p - |\phi|^p)|. \tag{3.3}
\]

We estimate the second integral of (3.3) using (2.3) as,
\[
\int_{\partial \Omega} |g - g_\epsilon||(|\phi_n|^p - |\phi|^p)| \leq C\|g - g_\epsilon\|_{(\frac{N}{p-1}, \infty)} \left( \|\phi_n\|_{W^{1,p}(\Omega)}^p + \|\phi\|_{W^{1,p}(\Omega)}^p \right). \tag{3.4}
\]

Since \( W^{1,p}(\Omega) \) is compactly embedded into \( L^p(\partial \Omega) \) (Proposition 2.9), there exists \( n_1 \in \mathbb{N} \) such that \( \int_{\partial \Omega} \|g_\epsilon||(|\phi_n|^p - |\phi|^p)| < \epsilon, \; \forall n \geq n_1 \). Now from (3.3) and (3.4), we obtain
\[
\int_{\partial \Omega} |g||(|\phi_n|^p - |\phi|^p)| < (C + 1)\epsilon, \quad \forall n \geq n_1.
\]

Thus \( G(\phi_n) \) converges to \( G(\phi) \) as \( n \to \infty \).

**Proposition 3.3.** Let \( p \in (1, \infty) \). Let \( N, g \) be given as in Proposition 3.2. Then \( G \) is differentiable at every \( \phi \in W^{1,p}(\Omega) \) and
\[
\langle G'(\phi), v \rangle = p \int_{\partial \Omega} g|\phi|^{p-2}\phi v, \quad \forall v \in W^{1,p}(\Omega).
\]

Moreover, the map \( G' \) is compact.

**Proof.** For \( \phi, v \in W^{1,p}(\Omega) \), let \( f : \partial \Omega \times [-1, 1] \to \mathbb{R} \) defined by \( f(y, t) = g(y)|(\phi + tv)(y)|^p \). Then \( \frac{\partial f}{\partial t}(\cdot, t) = pg|\phi + tv|^{p-2}(\phi + tv)v \) and
\[
\frac{\partial f}{\partial t}(\cdot, t) \leq p2^{p-1}|g|||(|\phi|^{p-1} + |v|^{p-1})|v|.
\]
Set \( h = p^{2p-1} |g| (|\phi|^{p-1} + |v|^{p-1}) |v| \) and for each \( n \in \mathbb{N} \), set
\[
h_n(y) = n \left( f(y, \frac{1}{n}) - f(y, 0) \right).
\]
Clearly, \( h_n(y) \to \frac{\partial f}{\partial t}(y, 0) \) a.e. on \( \partial \Omega \) and by mean value theorem, we also have
\[
|h_n(y)| \leq \sup_{t \in [-1,1]} \left| \frac{\partial f}{\partial t}(y, t) \right| \leq h(y).
\]
Furthermore, using a similar set of arguments as given in the proof of Proposition 2.11, one can show that \( h_n, h \in L^1(\partial \Omega) \), for each \( n \in \mathbb{N} \). Therefore, by the dominated convergence theorem,
\[
\lim_{n \to \infty} \int_{\partial \Omega} n \left( f(y, \frac{1}{n}) - f(y, 0) \right) \, dy = \int_{\partial \Omega} \frac{\partial f}{\partial t}(y, 0) \, dy = p \int_{\partial \Omega} g|\phi|^{p-2} \phi v.
\]
Thus
\[
\langle G'(\phi), v \rangle = \frac{d}{dt} G(\phi + tv) \bigg|_{t=0} = p \int_{\partial \Omega} g|\phi|^{p-2} \phi v.
\]
The proof of compactness is quite similar to that of Proposition 3.2. \( \Box \)

For \( p \in (1, \infty) \), consider the following functional
\[
J(\phi) = \int_{\Omega} |\nabla \phi|^p, \quad \forall \phi \in W^{1,p}(\Omega).
\]
Then \( J \) is differentiable on \( W^{1,p}(\Omega) \), and the derivative is given by
\[
\langle J'(\phi), u \rangle = p \int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla u, \quad \forall u \in W^{1,p}(\Omega).
\]

**Proposition 3.4.** Let \( p \in (1, \infty) \). Then

(i) \( J' \) is continuous.

(ii) \( J' \) is of class \( \alpha(W^{1,p}(\Omega)) \).

**Proof.** (i) Let \( \phi_n \to \phi \) in \( W^{1,p}(\Omega) \). For \( v \in W^{1,p}(\Omega) \),
\[
|\langle J'(\phi_n) - J'(\phi), v \rangle | \leq \int_{\Omega} |(|\nabla \phi_n|^{p-2} \nabla \phi_n - |\nabla \phi|^{p-2} \nabla \phi)| |\nabla v|
\leq \left( \int_{\Omega} \left( |(|\nabla \phi_n|^{p-2} \nabla \phi_n - |\nabla \phi|^{p-2} \nabla \phi)|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \left( \int_{\Omega} |\nabla v|^p \right)^{\frac{1}{p}}.
\]
Therefore,
\[
\|J'(\phi_n) - J'(\phi)\| \leq \left( \int_{\Omega} \left( |(|\nabla \phi_n|^{p-2} \nabla \phi_n - |\nabla \phi|^{p-2} \nabla \phi)|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}.
\]
Now consider the map $J_1$ defined as $J_1(\phi) = |\nabla \phi|^{p-2}\nabla \phi$. Clearly $J_1$ maps $W^{1,p}(\Omega)$ into $L^p(\Omega)$ and $J_1$ is continuous. Hence we conclude $\|J'(\phi_n) - J'(\phi)\| \to 0$ as $n \to \infty$.

(ii) Let $\phi_n \to \phi$ in $W^{1,p}(\Omega)$ and let $\lim_{n \to \infty} \langle J'(\phi_n), \phi_n - \phi \rangle = 0$. Then

$$\lim_{n \to \infty} \langle J'(\phi_n) - J'(\phi), \phi_n - \phi \rangle = \lim_{n \to \infty} \langle J'(\phi_n), \phi_n - \phi \rangle - \lim_{n \to \infty} \langle J'(\phi), \phi_n - \phi \rangle \leq 0. \quad (3.5)$$

Now for each $n \in \mathbb{N}$,

$$\langle J'(\phi_n) - J'(\phi), \phi_n - \phi \rangle \geq p (\|\nabla \phi_n\|^{p-1} - \|\nabla \phi\|^{p-1}) (\|\nabla \phi_n\| - \|\nabla \phi\|) \geq 0.$$

Hence from (3.5), we get

$$\lim_{n \to \infty} \langle J'(\phi_n) - J'(\phi), \phi_n - \phi \rangle = 0.$$

Therefore, $\|\nabla \phi_n\|_p \to \|\nabla \phi\|_p$ as $n \to \infty$. Hence by uniform convexity of $(L^p(\Omega))^N$, we obtain $\nabla \phi_n \to \nabla \phi$ in $(L^p(\Omega))^N$. Further, since $W^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$, we get $\phi_n \to \phi$ in $L^p(\Omega)$. Therefore, $\phi_n \to \phi$ in $W^{1,p}(\Omega)$. Thus the map $J$ is of class $\alpha(W^{1,p}(\Omega))$. \qed

**Proposition 3.5.** Let $p \in (1, \infty)$ and let $N$, $r$ and $f$ satisfy (H1) or (H2). Then the map $F$ defined by

$$\langle F(\phi), v \rangle = \int_{\Omega} f r(\phi)v$$

is a well-defined map from $W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$. Moreover, $F$ is continuous and compact.

**Proof.** First, we assume that $N, r$ and $f$ satisfy (H1). In this case $\gamma \in (1, \frac{\gamma^*}{N-1})$ and we use different arguments for $\gamma \in (1, p)$ and $\gamma \in [\frac{\gamma^*}{N-1}, \frac{\gamma^*}{N-p}]$. For $\gamma \in (1, p)$, there exists $C > 0$ such that $|r(s)| \leq C|s|^{p-1}$ for $s \in \mathbb{R}$. Therefore, using the finer trace embeddings (Proposition 2.10), for $\phi, v \in W^{1,p}(\Omega)$, clearly we have

$$|\langle F(\phi), v \rangle| \leq C\|f\|_{(\frac{\gamma^*}{N-1})} \|\phi\|_{W^{1,p}(\Omega)}^{p-1} \|v\|_{W^{1,p}(\Omega)} \quad (3.6)$$

For $\gamma \in [p, \frac{\gamma^*}{N-p})$, using Proposition 2.4 and the finer trace embeddings (Proposition 2.10), we have

$$W^{1,p}(\Omega) \hookrightarrow L^{\frac{\gamma^*}{N-p}}(\partial \Omega). \quad (3.7)$$

Since $\frac{1}{p} + \frac{(\gamma-1)(N-p)}{p(N-1)} + \frac{N-p}{p(N-1)} = 1$, for $\phi, v \in W^{1,p}(\Omega)$, using the generalized Hölder inequality (Proposition 2.4), we obtain

$$\int_{\partial \Omega} |f| |r(\phi)v| \leq C\bar{p} \|f\|_{(\bar{\gamma}, \infty)} \|\phi\|_{W^{1,p}(\Omega)}^{\gamma-1} \|v\|_{L^{\frac{\gamma^*}{N-p}}(\partial \Omega)} \gamma.$$

Therefore, from (3.7),

$$|\langle F(\phi), v \rangle| \leq C\|f\|_{(\bar{\gamma}, \infty)} \|\phi\|_{W^{1,p}(\Omega)}^{\gamma-1} \|v\|_{W^{1,p}(\Omega)}, \quad \forall \phi, v \in W^{1,p}(\Omega), \quad (3.8)$$
where \( C = C(N, p) > 0 \).

Now assume that \( N, r \) and \( f \) satisfy (H2). For \( d \in (1, \infty) \), choose \( a_i, b_i \in (1, \infty) \) (for \( i = 1, 2 \)) such that

\[
a_1, b_1 > \frac{1}{\gamma - 1}, \quad \frac{1}{d} + \frac{1}{a_1} + \frac{1}{a_2} = 1 = \frac{1}{N} + \frac{1}{b_1} + \frac{1}{b_2}.
\]

For \( \phi, v \in W^{1,p}(\Omega) \), using the generalized Hölder inequality (Proposition 2.4), we obtain

\[
\int_{\partial\Omega} |f||r(\phi)v| \leq Cd||f||_{(d,N)}\|\phi\|_{(a_1(\gamma-1), b_1(\gamma-1))}^\gamma \|v\|_{(a_2, b_2)}.
\]

Now by Proposition 2.2 and using the trace embeddings (Proposition 2.9 and Proposition 2.10), we have

\[
L^{d,\infty;N}(\partial\Omega) \hookrightarrow L^{d,N}(\partial\Omega),
\]

\[
W^{1,N}(\Omega) \hookrightarrow L^{\infty,N;1}(\partial\Omega) \hookrightarrow L^{a_1(\gamma-1),b_1(\gamma-1)}(\partial\Omega),
\]

\[
W^{1,N}(\Omega) \hookrightarrow L^q(\partial\Omega) \hookrightarrow L^{a_2,b_2}(\partial\Omega), \quad q > a_2.
\]

Therefore, from (3.9) we get

\[
|\langle F(\phi), v \rangle| \leq C\|f\|_{(d,\infty;N)}\|\phi\|_{W^{1,N}(\Omega)}^\gamma \|v\|_{W^{1,N}(\Omega)}, \quad \forall \phi, v \in W^{1,N}(\Omega),
\]

where \( C = C(N) \) > 0. Thus the map \( F \) is well defined in both the cases. The continuity and the compactness of \( F \) will follow from the similar set of arguments as given in the proof of Proposition 3.2. So we omit the proof.

**Proposition 3.6.** Let \( p \in (1, \infty) \). Let \( N, r \) and \( f \) be given as in Proposition 3.5. Then

\[
\frac{\|F(\phi)\|_{(W^{1,p}(\Omega))^\prime}}{\|\phi\|_{W^{1,p}(\Omega)^\prime}} \to 0, \quad \text{as} \quad \|\phi\|_{W^{1,p}(\Omega)} \to 0.
\]

**Proof.** Let \( \epsilon > 0 \) be given. We only prove the case when \( N, r \) and \( f \) satisfy (H1). For (H2), the proof is similar. For \( \gamma \in [p, \frac{p(N-1)}{N-p}] \), using (3.8) we have,

\[
\|F(\phi)\| \leq C\|f\|_{(\tilde{p}, \infty)}\|\phi\|_{W^{1,p}(\Omega)}^\gamma, \quad \forall \phi \in W^{1,p}(\Omega).
\]

Therefore,

\[
\frac{\|F(\phi)\|_{(W^{1,p}(\Omega))^\prime}}{\|\phi\|_{W^{1,p}(\Omega)^\prime}} \leq C\|f\|_{(\tilde{p}, \infty)}\|\phi\|_{W^{1,p}(\Omega)}^\gamma.
\]

If \( \gamma \in (1, p) \), then from (H1) there exists \( s_0 > 0 \) and \( C = C(s_0) > 0 \) such that

\[
|r(s)| < \frac{\epsilon}{\|f\|_{\left(\frac{N-1}{p-1}, \infty\right)}} |s|^{p-1}, \quad \text{for} \quad |s| < s_0,
\]

\[
|r(s)| \leq C|s|^{p-1} \quad \text{and} \quad |r(s)| \leq C|s|^\frac{p(N-1)}{N-p-1}, \quad \text{for} \quad |s| \geq s_0.
\]
For \( \phi \in W^{1,p}(\Omega) \), set \( A = \{ y \in \partial \Omega : |\phi(y)| < s_0 \} \) and \( B = \partial \Omega \setminus A \). For \( v \in W^{1,p}(\Omega) \), using (3.10) and (3.6), we get

\[
\int_A |f||r(\phi)||v| < \frac{\epsilon}{\|f\|(\frac{N-1}{p-1},\infty)} \int_A |f||\phi|^{p-1}|v| \leq C\epsilon\|\phi\|_{W^{1,p}(\Omega)}^{p-1}\|v\|_{W^{1,p}(\Omega)}. \tag{3.11}
\]

To estimate the above integral on \( B \), we split \( f = f_\epsilon + (f - f_\epsilon) \) where \( f_\epsilon \in C^1(\partial \Omega) \) with \( \|f - f_\epsilon\|(\frac{N-1}{p-1},\infty) < \epsilon \). Now (3.10) and (3.6) yield

\[
\int_B |f - f_\epsilon||r(\phi)||v| \leq C \int_B |f - f_\epsilon||\phi|^{p-1}|v| < C\epsilon\|\phi\|_{W^{1,p}(\Omega)}^{p-1}\|v\|_{W^{1,p}(\Omega)}, \tag{3.12}
\]

where \( C = C(s_0, N, p) > 0 \). On the other hand using (3.10), Hölder inequality (Proposition 2.2) and the classical trace embeddings (Proposition 2.9), we obtain

\[
\int_B |f_\epsilon||r(\phi)||v| \leq C \int_B |f_\epsilon||\phi|^{\frac{N(p-1)}{N-p}}|v| \\
\leq C\|f_\epsilon\|_{L^\infty(\partial \Omega)}\|\phi\|_{L^{\frac{N(p-1)}{N-p}}(\partial \Omega)} \|v\|_{L^{\frac{N(p-1)}{N-p}}(\partial \Omega)},
\]

where \( C = C(N, p) > 0 \). Now using (3.12) we conclude

\[
\int_B |f||r(\phi)||v| \leq C \left( \epsilon\|\phi\|_{W^{1,p}(\Omega)}^{p-1} + \|f_\epsilon\|_{L^\infty(\partial \Omega)}\|\phi\|_{W^{1,p}(\Omega)}^{\frac{N(p-1)}{N-p}} \right) \|v\|_{W^{1,p}(\Omega)},
\]

where \( C = C(s_0, N, p) > 0 \). Thus (3.11) and the above inequality yield:

\[
\|F(\phi)\|_{(W^{1,p}(\Omega))'} \leq C \left( \epsilon\|\phi\|_{W^{1,p}(\Omega)}^{p-1} + \|f_\epsilon\|_{L^\infty(\partial \Omega)}\|\phi\|_{W^{1,p}(\Omega)}^{\frac{N(p-1)}{N-p}} \right).
\]

Therefore,

\[
\frac{\|F(\phi)\|_{(W^{1,p}(\Omega))'}}{\|\phi\|_{W^{1,p}(\Omega)}^{p-1}} \leq C \left( \epsilon + \|f_\epsilon\|_{L^\infty(\partial \Omega)}\|\phi\|_{W^{1,p}(\Omega)}^{\frac{N(p-1)}{N-p}} \right) \to 0.
\]

as \( \|\phi\|_{W^{1,p}(\Omega)} \to 0 \). \( \square \)

For \( g \) as given in Theorem 1.1, we consider the set

\[
M_g = \left\{ \phi \in W^{1,p}(\Omega) : \int_{\partial \Omega} |\phi|^p > 0 \right\}.
\]

Since \( g^+ \neq 0 \), we can show that the set \( M_g \) is nonempty. The functional \( J \) is not coercive on \( W^{1,p}(\Omega) \). However, using a Poincaré type inequality on \( M_g \) we show that \( J \) is coercive on \( M_g \).

**Lemma 3.7.** Let \( g^+ \neq 0 \), \( \int_{\partial \Omega} g < 0 \), and

\[
g \in \begin{cases} F_{\frac{N}{p-1}} & \text{for } N > p, \\ G_1 & \text{for } N = p. \end{cases}
\]

Then there exists \( m \in (0, 1) \) such that

\[
\int_{\Omega} |\nabla \phi|^p \geq m \int_{\Omega} |\phi|^p, \quad \forall \phi \in M_g. \tag{3.13}
\]
Proof. On the contrary, assume that (3.13) does not hold for any \( m \in (0,1) \). Thus for each \( n \in \mathbb{N} \), there exists \( \phi_n \in M_g \) such that

\[
\int_{\Omega} |\nabla \phi_n|^p < \frac{1}{n} \int_{\Omega} |\phi_n|^p.
\]

If we set \( w_n = \|\phi_n\|^{-1} \phi_n \), then \( \|w_n\|_p = 1 \) and \( \int_{\Omega} |\nabla w_n|^p < \frac{1}{n} \). Thus \( (w_n) \) is bounded and hence there exists a subsequence \( (w_{n_k}) \) of \( (w_n) \) such that \( w_{n_k} \rightharpoonup w \) in \( W^{1,p}(\Omega) \). By weak lowersemicontinuity of \( \|\nabla \|_p \) we have \( \|\nabla w\|_p = 0 \). Hence the connectedness yields \( w = c \) a.e. in \( \Omega \). By the compactness of the embedding of \( W^{1,p}(\Omega) \) into \( L^p(\Omega) \), we get \( \|w\|_p = 1 \) and hence \( |c||\Omega|^{\frac{1}{p}} = 1 \). Therefore, \( \int_{\partial \Omega} |g|^p = \frac{1}{|\Omega|} \int_{\partial \Omega} g < 0 \). On the other hand, \( \int_{\partial \Omega} |w_{n_k}|^p = \|\phi_{n_k}\|_p^p \int_{\partial \Omega} g|\phi_{n_k}|^p > 0 \). Thus by the compactness of \( G \) (Proposition 3.2), we get \( \int_{\partial \Omega} |g|^p = \lim_{k \to \infty} \int_{\partial \Omega} g|w_{n_k}|^p \geq 0 \), a contradiction. Thus there must exists \( m \in (0,1) \) satisfying (3.13). \( \square \)

**Remark 3.8.** For \( g \) as given in Lemma 3.7, consider the set

\[
N_g = \left\{ \phi \in W^{1,p}(\Omega) : \int_{\partial \Omega} |g|^p = 1 \right\} = G^{-}(1).
\]

For \( \phi \in N_g \), \( \langle G'(\phi), \phi \rangle \neq 0 \). Thus 1 is a regular point of \( G \) and \( N_g \) is a \( C^1 \) manifold. Moreover (see [21, Proposition 6.4.35]),

\[
\|dJ(\phi)\| = \min_{\lambda \in \mathbb{R}} \| (J' - \lambda G') (\phi) \|, \quad \forall \phi \in N_g.
\]

**Definition 3.9.** A map \( f \in C^1(Y, \mathbb{R}) \) is said to satisfy **Palais-Smale (P. S.)** condition on a \( C^1 \) manifold \( M \subset Y \), if \( (\phi_n) \) is a sequence in \( M \) such that \( f(\phi_n) \to c \in \mathbb{R} \) and \( \|df(\phi_n)\| \to 0 \), then \( (\phi_n) \) has a subsequence that converges in \( M \).

**Lemma 3.10.** Let \( g \) be as given in Lemma 3.7. Then \( J \) satisfies the P. S. condition on \( N_g \).

*Proof.* Let \( (\phi_n) \) be a sequence in \( N_g \) and \( \lambda \in \mathbb{R} \) such that \( J(\phi_n) \to \lambda \) and \( \|dJ(\phi_n)\| \to 0 \). By Remark 3.8, there exists a sequence \( (\lambda_n) \) such that \( (J' - \lambda_n G')(\phi_n) \to 0 \) as \( n \to \infty \). By Lemma 3.7, the sequence \( (\phi_n) \) is also bounded in \( W^{1,p}(\Omega) \). Now using the reflexivity of \( W^{1,p}(\Omega) \), we get a subsequence \( (\phi_{n_k}) \) such that \( \phi_{n_k} \rightharpoonup \phi \) in \( W^{1,p}(\Omega) \). Since \( N_g \) is weakly closed, \( \phi \in N_g \). Also \( \lambda_{n_k} \to \lambda \) as \( k \to \infty \), since

\[
\langle (J' - \lambda_{n_k} G')(\phi_{n_k}), \phi_{n_k} \rangle = p(J(\phi_{n_k}) - \lambda_{n_k}).
\]

Furthermore,

\[
\langle J'(\phi_{n_k}), \phi_{n_k} - \phi \rangle = \langle (J' - \lambda_{n_k} G')(\phi_{n_k}), \phi_{n_k} - \phi \rangle + \lambda_{n_k} \langle G'(\phi_{n_k}), \phi_{n_k} - \phi \rangle.
\]

Now using the compactness of \( G' \), we get \( \langle J'(\phi_{n_k}), \phi_{n_k} - \phi \rangle \to 0 \). Moreover, as \( J' \) is of class \( \alpha(W^{1,p}(\Omega)) \) (Proposition 3.4), the sequence \( (\phi_{n_k}) \) converges to \( \phi \) in \( W^{1,p}(\Omega) \). Therefore, \( J \) satisfies the P. S. condition on \( N_g \). \( \square \)
4 Proof of main theorems

In this section, we prove all our main theorems.

4.1 The existence and some of the properties of the first eigenvalue

Proof of Theorem 1.1:

First, recall that

\[ \lambda_1 = \inf_{\phi \in N_g} \int_{\Omega} |\nabla \phi|^p. \]

From Lemma 3.7, we clearly have \( \lambda_1 > 0 \). Since the functional \( J \) is coercive on \( N_g \), a sequence that minimizes \( J \) over \( N_g \) will be bounded and hence admits a weakly convergent subsequence that converges to say \( \phi_1 \). As \( N_g \) is weakly closed, \( \phi_1 \in N_g \) and \( J(\phi_1) = \lambda_1 \). Thus \( \lambda_1 \) is the minimum of \( J \) on \( N_g \) and hence \( \|dJ(\phi_1)\| = 0 \).

Now from Remark 3.8, we obtain

\[ \int_{\Omega} |\nabla \phi_1|^{p-2} \nabla \phi_1 \cdot \nabla v \, dx = \lambda_1 \int_{\partial \Omega} g|\phi_1|^{p-2} \phi_1 v \, d\sigma, \quad \forall v \in W^{1,p}(\Omega). \]  (4.1)

\( \lambda_1 \) is a principal eigenvalue: Clearly \( |\phi_1| \) is also an eigenfunction of (1.5) corresponding to \( \lambda_1 \). Moreover, as \( |\phi_1| \) is \( p \)-harmonic, \( |\phi_1| \in C^{1,\alpha}(\Omega) \). Since \( |\phi_1| \geq 0 \), by the maximum principle in [51, Theorem 5], \( |\phi_1| > 0 \) in \( \Omega \). Without loss of generality we may assume \( \phi_1 > 0 \) in \( \Omega \). We show that \( \phi_1 \) is positive also on \( \partial \Omega \).

For \( \epsilon > 0 \), consider the function \( \frac{\phi_1}{\phi_1 + \epsilon} + \epsilon \). It is easy to verify that \( \frac{\phi_1}{\phi_1 + \epsilon} + \epsilon \in W^{1,p}(\Omega) \) and \( \frac{\phi_1}{\phi_1 + \epsilon} + \epsilon \to 1 \) in \( L^p(\Omega) \). We show that \( \frac{\phi_1}{\phi_1 + \epsilon} + \epsilon \to 1 \) in \( W^{1,p}(\Omega) \) as well. This together with trace embedding will ensure that \( \phi_1 > 0 \) in \( \Omega \). Thus it is enough to prove \( \nabla \frac{\phi_1}{\phi_1 + \epsilon} + \epsilon \to 0 \) in \( L^p(\Omega) \) as \( \epsilon \to 0 \). Notice that,

\[ \left| \nabla \frac{\phi_1}{\phi_1 + \epsilon} \right|^p = \left( \frac{\epsilon}{\phi_1 + \epsilon} \right)^p \frac{|\nabla \phi_1|^p}{(\phi_1 + \epsilon)^p} \leq \frac{|\nabla \phi_1|^p}{\phi_1^p}. \]  (4.2)

Furthermore, by taking \( \frac{1}{(\phi_1 + \epsilon)^{p-1}} \in W^{1,p}(\Omega) \) as a test function in (4.1), we obtain

\[ (p-1) \int_{\Omega \setminus \{0\}} \frac{|\nabla \phi_1|^p}{(\phi_1 + \epsilon)^p} = \lambda_1 \int_{\partial \Omega} g \left( \frac{\phi_1}{\phi_1 + \epsilon} \right)^{p-1} \leq \lambda_1 \int_{\partial \Omega} |g|. \]

We apply Fatou’s lemma and let \( \epsilon \to 0 \) in the above inequality to get

\[ (p-1) \int_{\Omega} \frac{|\nabla \phi_1|^p}{\phi_1^p} \leq \lambda_1 \int_{\partial \Omega} |g|. \]

Now (4.2) together with the dominated convergence theorem ensures that \( \nabla \frac{\phi_1}{\phi_1 + \epsilon} \to 0 \) in \( L^p(\Omega) \).

The uniqueness and the simplicity: The usual arguments (for example, see [48, Lemma 3.1] for a proof) using the Picone’s identity [2, Theorem 1.1] gives the uniqueness of the positive principal eigenvalue and the simplicity of \( \lambda_1 \).
\[ \lambda_1 \text{ is an isolated eigenvalue: } \text{We adapt the proof of [6, Proposition 2.12]. On the contrary, we suppose that there exists a sequence \((\lambda_n)\) of eigenvalues of (1.5) converging to } \lambda_1. \text{ For each } n \in \mathbb{N}, \text{ let } \psi_n \in N_g \text{ be an eigenfunction corresponding to } \lambda_n. \text{ Then } J(\psi_n) = \lambda_n \to \lambda_1 \text{ and} \]

\[ \langle (J' - \lambda_n G')(\psi_n), \psi_n \rangle = (J - \lambda_n G)(\psi_n) = 0, \]

i.e., \(\|dJ(\psi_n)\| = 0\). Hence using Lemma 3.10 and the continuity of \(J'\) and \(G'\), we get \(\psi_n \to \psi\), an eigenfunction corresponding to \(\lambda_1\). Since \(\lambda_1\) is simple, \(\psi = \pm \phi_1\), where \(\phi_1\) is a first eigenfunction such that \(\phi_1 > 0\) on \(\Omega\). If we let \(\psi = \phi_1\), then by Egorov’s theorem there exists \(E \subset \Omega\) and \(n_1 \in \mathbb{N}\) such that \(|E| < \epsilon\) and \(\psi_n = 0\) a.e. in \(E^c\) for \(n \geq n_1\). Also from (1.5) we have

\[ \int_{\Omega} |\nabla \psi_n|^p = \lambda_n \int_{\partial \Omega} g|\psi_n|^p. \]

Notice that \(\int_{\Omega} |\nabla \psi_n|^p \neq 0\), since \(\psi_n\) changes sign on \(\Omega\). Now by setting \(v_n = (\int_{\partial \Omega} g|\psi_n|^p)^{-\frac{1}{p}} \psi_n\), we have \(v_n \in N_g\) and \(\int_{\Omega} |\nabla v_n|^p = \lambda_n \to \lambda_1\). Therefore, \(v_n\) must converge to \(\phi_1\), a contradiction as \(v_n = 0\) a.e. in \(E^c\) for \(n \geq n_1\). Thus \(\lambda_1\) must be an isolated eigenvalue. \(\square\)

**Remark 4.1.** (a) Let

\[ g \in \begin{cases} L^{\frac{N-1}{p-1}}(\partial \Omega) & \text{for } N > p, \\ L^{1,\infty;N}(\partial \Omega) & \text{for } N = p. \end{cases} \]

Then \(\frac{1}{\lambda_1}\) is the best constant in the following weighted trace inequality:

\[ \int_{\partial \Omega} |g||\phi|^p \leq C \int_{\Omega} |\nabla \phi|^p, \quad \forall \phi \in W^{1,p}(\Omega). \]

In addition, if \(g\) satisfy all the assumptions of Theorem 1.1, then this best constant is also attained.

(b) Since \(\Omega\) is bounded, we have

\[ L^q(\partial \Omega) \subset L^{\frac{N-1}{p-1}}(\partial \Omega), \forall q > \frac{N-1}{p-1}, \quad \text{and} \quad L^q(\partial \Omega) \subset G_1, \forall q \in (1, \infty). \]

Thus, Theorem 1.2 of [48] follows from Theorem 1.1. Furthermore, Example 2.7 and Example 2.8 give examples of weight functions for which Theorem 1.2 of [48] is not applicable, however admits a positive principal eigenvalue by our Theorem 1.1.

**Remark 4.2.** For \(g\) as given in Theorem 1.1, the functional \(J\) and the set \(N_g\) satisfy all the properties of [39, Theorem 5.3]. Therefore, by [39, Theorem 5.3], there exists a sequence of eigenvalues \((\lambda_n)\) of (1.5) and the sequence \((\lambda_n)\) is unbounded.

### 4.2 Bifurcation

For proving Theorem 1.2, we adapt the degree theory arguments given in [20], also see [4]. We split our proof into several lemmas and propositions.
Lemma 4.3. Let \( g^+ \neq 0 \), \( \int_{\partial \Omega} g < 0 \), and
\[
g \in \begin{cases} \mathcal{F}_{N-1}^\frac{2}{N-1} & \text{for } N > p, \\ G_1 & \text{for } N = p. \end{cases}
\]

Let \((\phi_n)\) be a sequence in \( W^{1,p}(\Omega) \) such that
\[
\int_{\Omega} |\nabla \phi_n|^p - \lambda \int_{\partial \Omega} g|\phi_n|^p < C
\]
for some \( C > 0 \) and \( \lambda > 0 \). If \((||\nabla \phi_n||_p)\) is bounded, then \((||\phi_n||_p)\) is bounded.

**Proof.** Our proof is by method of contradiction. Suppose that the sequence \((||\nabla \phi_n||_p)\) is bounded and \(||\phi_n||_p \to \infty\) as \( n \to \infty \). By setting \( w_n = \frac{1}{n} \phi_n \), we obtain \( ||w_n||_p = 1 \) and \( ||\nabla w_n||_p \to 0 \) as \( n \to \infty \). Thus there exists a subsequence \((w_{n_k})\) of \((w_n)\) such that \( w_{n_k} \to w \) in \( W^{1,p}(\Omega) \). Now the weak lowersemicontinuity of \( ||\nabla||_p \) gives \( ||\nabla w||_p = 0 \). Since \( \Omega \) is connected, we get \( w = c \) a.e. in \( \Omega \) and from the compactness of the embedding of \( W^{1,p}(\Omega) \) into \( L^p(\Omega) \), \( ||\psi||_p = 1 \). Thus \( \int_{\partial \Omega} g|w|^p = \frac{1}{p} \int_{\Omega} g < 0 \). On the other hand from (4.3) we also have
\[
\int_{\Omega} |\nabla w_{n_k}|^p - \lambda \int_{\partial \Omega} g|w_{n_k}|^p \leq C ||\phi_{n_k}||_p^p.
\]
Now we let \( k \to \infty \) so that the compactness of \( G \) gives \( -\lambda \int_{\partial \Omega} g|w|^p \leq 0 \). A contradiction to \( \int_{\partial \Omega} g|w|^p < 0 \).

In the next proposition, for \( \lambda \in (0, \lambda_1 + \delta) \), we find a lower estimate of the functional \( J - \lambda G \).

**Proposition 4.4.** Let \( \delta > 0 \) and let \( \lambda \in (0, \lambda_1 + \delta) \setminus \lambda_1 \). Then for \( \phi \in W^{1,p}(\Omega) \setminus \{0\}, \)
\[
J(\phi) - \lambda G(\phi) > \begin{cases} 0, & \text{if } \lambda \in (0, \lambda_1); \\ \frac{\delta}{\lambda_1} J(\phi), & \text{if } \lambda \in (\lambda_1, \lambda_1 + \delta). \end{cases}
\]

**Proof.** Firstly, for any \( \lambda > 0 \) and \( \phi \in W^{1,p}(\Omega) \setminus \{0\} \), we consider the following cases:

(i) \( G(\phi) \leq 0 \) and \( J(\phi) > 0 \) : clearly \( J(\phi) - \lambda G(\phi) > 0 \).

(ii) \( G(\phi) = 0 \) and \( J(\phi) = 0 \) : using the connectedness of \( \Omega \) and the fact that \( \int_{\partial \Omega} g < 0 \), we get \( \phi = 0 \). So this case does not arise, since \( \phi \neq 0 \).

(iii) \( G(\phi) > 0 \) : in this case \( \lambda_1 \leq \frac{J(\phi)}{G(\phi)} \). Thus for \( \lambda \in (0, \lambda_1) \), we get \( J(\phi) - \lambda G(\phi) > 0 \).

Secondly, for \( \lambda \in (\lambda_1, \lambda_1 + \delta) \) and \( \phi \in W^{1,p}(\Omega) \), we have
\[
J(\phi) - \lambda G(\phi) = J(\phi) - \lambda_1 G(\phi) + (\lambda_1 - \lambda) G(\phi) \\
\geq (\lambda_1 - \lambda) G(\phi) > \frac{\lambda_1 - \lambda}{\lambda_1} J(\phi) > -\frac{\delta}{\lambda_1} J(\phi),
\]
where the inequalities follow from the facts \( J(\phi) - \lambda_1 G(\phi) \geq 0 \) and \( \lambda \in (\lambda_1, \lambda_1 + \delta) \).
For $\lambda \in (\lambda_1, \lambda_1 + \delta)$, we consider a differentiable function $\eta(t)$ such that

$$
\eta(t) = \begin{cases} 
0, & 0 \leq t \leq 1, \\
\text{strictly convex}, & 1 < t < 2, \\
\frac{2\delta}{\lambda_1}(t - 1), & t \geq 2.
\end{cases}
$$

Therefore,

$$
\eta'(t) = \begin{cases} 
0, & 0 \leq t < 1; \\
\frac{2\delta}{\lambda_1}, & t \geq 2,
\end{cases}
$$

and $\eta'(t) \geq 0$, $1 \leq t \leq 2$.

(4.7)

A similar function (with an additional parameter $k$) is considered in the proof of [20, Theorem 4.1]. We would like to point out that, their proof also works by fixing a value for $k$. Since, the functional $J - \lambda G$ is not bounded below for $\lambda \in (\lambda_1, \lambda_1 + \delta)$, we add a non-negative term to it. The following result is proved as a part of the proof of [20, Theorem 4.1].

**Lemma 4.5.** Let $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and let $\eta$ be given as above. Then the functional $\eta_\lambda(\phi) = J(\phi) - \lambda G(\phi) + \eta(J(\phi))$ satisfies the following:

(a) $\eta_\lambda$ is weakly lower semicontinuous.

(b) $\eta_\lambda$ is coercive.

(c) $\eta_\lambda$ is bounded below.

(d) there exists $R_0 > 0$ such that the map $\eta'_\lambda : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ does not vanish on $\partial B_R(0)$ for all $R \geq R_0$.

**Proof.** (a) Let $\phi_n \rightharpoonup \phi$ in $W^{1,p}(\Omega)$. Since $J$ is weakly lower semicontinuous, $G$ is compact and $\eta$ is increasing and continuous, we get

$$
\lim_{n \to \infty} \eta_\lambda(\phi_n) = \lim_{n \to \infty} J(\phi_n) - \lambda \lim_{n \to \infty} G(\phi_n) + \eta(\lim_{n \to \infty} (J(\phi_n)))
\geq J(\phi) - \lambda G(\phi) + \eta(J(\phi)) = \eta_\lambda(\phi).
$$

Therefore, $\eta_\lambda$ is weakly lower semicontinuous.

(b) Let $(\phi_n)$ be a sequence in $W^{1,p}(\Omega)$ such that $\eta_\lambda(\phi_n) \leq C$, $\forall n \in \mathbb{N}$. We show that the sequence $(\phi_n)$ is bounded in $W^{1,p}(\Omega)$. From (4.4), we have

$$
C \geq \eta_\lambda(\phi_n) > -\frac{\delta}{\lambda_1} J(\phi_n) + \eta(J(\phi_n)), \quad \forall n \in \mathbb{N}.
$$

(4.8)

Thus, for $\phi_n$ with $J(\phi_n) \geq 2$, using the definition of $\eta$, we have

$$
C \geq \eta_\lambda(\phi_n) > -\frac{\delta}{\lambda_1} J(\phi_n) + \frac{2\delta}{\lambda_1}(J(\phi_n) - 1) = \frac{\delta}{\lambda_1} J(\phi_n) - \frac{2\delta}{\lambda_1}.
$$

Hence, $J(\phi_n) \leq \max\left\{2, \frac{\lambda_1 C}{\delta} + 2\right\}$. Now, we can use Lemma 4.3 to obtain $C_1 > 0$ so that $\|\phi_n\|_p \leq C_1$. Therefore, the sequence $(\phi_n)$ is bounded in $W^{1,p}(\Omega)$. 

22
(c) From (4.4), we have \( \eta_\lambda(\phi) > -\frac{\phi}{\lambda_1} J(\phi) + \eta(J(\phi)), \ \forall \phi \in W^{1,p}(\Omega) \). Therefore,
\[
\eta_\lambda(\phi) > \begin{cases} 
\frac{\delta}{\lambda_1} J(\phi) - \frac{2\delta}{\lambda_1} > 0, & \text{if } J(\phi) > 2; \\
-\frac{\phi}{\lambda_1} J(\phi) + \eta(J(\phi)) \geq -\frac{2\delta}{\lambda_1}, & \text{if } J(\phi) \leq 2.
\end{cases}
\]
Thus \( \eta_\lambda \) bounded below.

(d) By Lemma 3.7, there exists \( m > 0 \) such that
\[
J(\phi) \geq m\|\phi\|_p^p, \ \forall \phi \in W^{1,p}(\Omega) \text{ with } G(\phi) > 0.
\]
We choose \( R_0 = 2(1 + \frac{1}{m}) \). Thus, for \( \phi \in \partial B_R(0) \) with \( R > R_0 \), either \( J(\phi) > 2 \) or \( \|\phi\|_p > \frac{2}{m} \). Notice that,
\[
\langle \eta'_\lambda(\phi), \phi \rangle = p \left( J(\phi) - \lambda G(\phi) + \eta'(J(\phi))J(\phi) \right).
\]
Thus, using (4.4), we obtain
\[
\frac{1}{p} \langle \eta'_\lambda(\phi), \phi \rangle \geq -\frac{\delta}{\lambda_1} J(\phi) + \eta'(J(\phi))J(\phi).
\]
In particular, for \( J(\phi) > 2 \), we have
\[
\frac{1}{p} \langle \eta'_\lambda(\phi), \phi \rangle \geq \frac{\delta}{\lambda_1} J(\phi).
\]
On the other hand, for \( J(\phi) \leq 2 \), we have \( \|\phi\|_p > \frac{2}{m} \). Hence from (4.9), we conclude that \( G(\phi) \leq 0 \). Now from the part (i) and (ii) of proof of Proposition 4.4, we get \( \langle \eta'_\lambda(\phi), \phi \rangle > 0 \). Therefore, \( \eta'_\lambda(\phi) \neq 0 \) for \( \phi \in \partial B_R(0) \) for any \( R > R_0 \).  

Recall that a function \( \phi \in W^{1,p}(\Omega) \) is a weak solution of (1.1), if it satisfies the following weak formulation:
\[
\int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla v - \lambda \int_{\partial \Omega} (g|\phi|^{p-2}v + f(\phi)v) = 0, \ \forall v \in W^{1,p}(\Omega).
\]
Therefore, \( \phi \) is a solution of (1.1) if and only if
\[
\langle (J' - \lambda(G' + F))\phi, v \rangle = 0, \ \forall v \in W^{1,p}(\Omega).
\]

**Proposition 4.6.** The maps \( J' - \lambda(G' + F) \) and \( J' - \lambda G' \) are well-defined maps from \( W^{1,p}(\Omega) \) to its dual \( (W^{1,p}(\Omega))^\prime \). Moreover, these maps are bounded, demicontinuous and of class \( \alpha(W^{1,p}(\Omega)) \).

**Proof.** From Proposition 3.3, Proposition 3.4, and Proposition 3.5, we obtain \( J' - \lambda(G' + F) \) and \( J' - \lambda G' \) are well defined, bounded and demicontinuous. Since \( J' \) is of class \( \alpha(W^{1,p}(\Omega)) \) and \( G', F \) are compact, the maps \( J' - \lambda(G' + F) \) and \( J' - \lambda G' \) are of class \( \alpha(W^{1,p}(\Omega)) \).  

**Proposition 4.7.** Let \( q, \lambda_1 \) be as given in Theorem 1.1. Then there exists \( \delta > 0 \) such that for each \( \lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\}, \ \text{ind}(J' - \lambda G', 0) \) is well defined. Furthermore,
\[
(a) \ \text{ind}(J' - \lambda G', 0) = 1 \text{ for } \lambda \in (0, \lambda_1),
\]

(b) \( \text{ind}(J' - \lambda G', 0) = -1 \) for \( \lambda \in (\lambda_1, \lambda_1 + \delta) \).

**Proof.** Since \( \lambda_1 \) is an isolated eigenvalue of (1.5), there exists \( \delta > 0 \) such that \( \lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\} \) is not an eigenvalue of (1.5). Thus for \( \lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\} \), 0 is the only solution of \( J' - \lambda G' \) and hence \( \text{ind}(J' - \lambda G', 0) \) is well defined.

(a) For \( \lambda \in (0, \lambda_1) \), from (4.4), we have
\[
\langle (J' - \lambda G')(\phi), \phi \rangle = p(J(\phi) - \lambda G(\phi)) > 0, \quad \forall \phi \in W^{1,p}(\Omega) \setminus \{0\}.
\]
Therefore, by Proposition 2.13, \( \text{deg}(J' - \lambda G', B_r(0), 0) = 1 \) for every \( r > 0 \). Thus
\[
\text{ind}(J' - \lambda G', 0) = \lim_{r \to 0} \text{deg}(J' - \lambda G', B_r(0), 0) = 1.
\]

(b) In this case, we adapt a technique used in the proof of [20, Theorem 4.1]. First, we compute \( \text{ind}(\eta'_\lambda, 0) \). Clearly, 0 is a zero of \( \eta'_\lambda \). If \( \phi_0 \neq 0 \) is a zero of \( \eta'_\lambda \), then \( \lambda = \frac{\lambda_1}{1 + \eta'(\phi_0)} \) is an eigenvalue of (1.5) and \( \phi_0 \) is a corresponding eigenfunction. Since \( 0 < \frac{\lambda_1}{1 + \eta'(\phi_0)} < \lambda_1 + \delta \), we must have \( \lambda = \lambda_1 \) and \( \phi_0 = c\phi_1 \) for some \( c \in \mathbb{R} \), where \( \phi_1 \) is the first eigenfunction of (1.5) normalized as \( \int_{\partial \Omega} g\phi_1^2 = 1 \) and \( \phi_1 > 0 \) in \( \Omega \). Notice that,
\[
\eta'(J(\phi_0)) = \frac{\lambda}{\lambda_1} - 1 \in \left( 0, \frac{\delta}{\lambda_1} \right).
\]
Thus from (4.7), we assert that \( J(\phi_0) \in (1, 2) \). Moreover, since \( \eta' \) is strictly increasing in (1, 2) and the functional \( J \) is even, there exists a unique \( c > 0 \) such that \( \phi_0 = \pm c\phi_1 \). Conversely, if we choose \( c > 0 \) such that \( \eta'(J(c\phi_1)) = \frac{\lambda}{\lambda_1} - 1 \), then \( \pm c\phi_1 \) is a zero of \( \eta'_\lambda \). Therefore, the map \( \eta'_\lambda \) has precisely three zeros \( -c\phi_1, 0, c\phi_1 \). Now we will show that \( \text{ind}(\eta'_\lambda, \pm c\phi_1) = 1 \). It is enough to prove \( \pm c\phi_1 \) are the minimizers for \( \eta_\lambda \). From Lemma 4.5, the functional \( \eta_\lambda \) is coercive, weak lowersemicontinuous and bounded below. Thus \( \eta_\lambda \) admits a minimizer. Notice that, \( \eta_\lambda(t\phi_1) = (\lambda_1 - \lambda)t^p G(\phi_1) + \eta(p^t J(\phi_1)) \) and hence \( \eta_\lambda(t\phi_1) < 0 \) for sufficiently small \( t > 0 \). Thus 0 is not a minimizer and hence \( \pm c\phi_1 \) are the only minimizers of \( \eta_\lambda \). Therefore, by Proposition 2.13, we get
\[
\text{ind}(\eta'_\lambda, \pm c\phi_1) = 1. \tag{4.10}
\]
For \( R_0 \) as given in Lemma 4.5, we choose \( R > R_0 \), so that \( \pm c\phi_1 \in B_R(0) \) and \( \langle \eta'_\lambda(\phi), \phi \rangle > 0 \) for \( \phi \in \partial B_R(0) \). By Proposition 2.13, \( \text{deg}(\eta'_\lambda, \overline{B_R(0)}, 0) = 1. \) Thus by the additivity of degree (Proposition 2.13) and from (4.10), we obtain \( \text{deg}(\eta'_\lambda, \overline{B_r(0)}, 0) = -1 \) for sufficiently small \( r > 0 \). Since \( \eta'_\lambda = J' - \lambda G' \) on \( B_r(0) \) for \( r < 1 \), we conclude that \( \text{ind}(J' - \lambda G', 0) = -1 \). \( \square \)

**Lemma 4.8.** Let \( \lambda_1 \) be given as in Theorem 1.1. Then for \( \lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\} \), \( \text{ind}(J' - \lambda(G' + F), 0) = \text{ind}(J' - \lambda G', 0) \).

**Proof.** For \( \lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\} \), define \( H_\lambda : W^{1,p}(\Omega) \times [0, 1] \to (W^{1,p}(\Omega))' \) as
\[
H_\lambda(\phi, t) = J'(\phi) - \lambda G'(\phi) - \lambda t F(\phi).
\]
Clearly, $H_\lambda(.,0) = J' - \lambda G'$ and $H_\lambda(.,1) = J' - \lambda(G' + F)$. From Proposition 4.6, for each $t \in [0,1]$, $H_\lambda(.,t)$ is bounded, demicontinuous and of class $\alpha(W^{1,p}(\Omega))$. We prove the existence of a sufficiently small $r > 0$ such that for each $t \in [0,1]$, $H_\lambda(.,t)$ does not vanish in $\overline{B_r(0)} \setminus \{0\}$. On the contrary, assume that no such $r$ exists. Then for any $r > 0$, there exists $t_r \in [0,1]$ and $\phi_r \in W^{1,p}(\Omega) \setminus \{0\}$ such that $\|\phi_r\|_{W^{1,p}(\Omega)} \leq r$ and $H_\lambda(\phi_r, t_r) = 0$. In particular, for a sequence of positive numbers $(r_n)$ converging to 0, there exist a sequence $t_n \in [0,1]$ and a sequence $\phi_n \in W^{1,p}(\Omega) \setminus \{0\}$ such that $\|\phi_n\|_{W^{1,p}(\Omega)} \leq r_n$ and

$$J'(\phi_n) - \lambda G'(\phi_n) - \lambda t_n F(\phi_n) = 0. \quad (4.11)$$

If we set $v_n = \phi_n\|\phi_n\|_{W^{1,p}(\Omega)}^{-1}$, then $\|v_n\|_{W^{1,p}(\Omega)} = 1$ and hence admits a subsequence $(v_{n_k})$ such that $v_{n_k} \to v$ in $W^{1,p}(\Omega)$. From (4.11) we also have

$$\langle J'(v_{n_k}) - \lambda G'(v_{n_k}), v_{n_k} - v \rangle = \lambda t_{n_k} \left\langle \frac{F(\phi_{n_k})}{\|\phi_{n_k}\|_{W^{1,p}(\Omega)}^{p-1}}, v_{n_k} - v \right\rangle.$$

By Proposition 3.6, the right hand side of the above inequality goes to zero as $k \to \infty$. Therefore,

$$\lim_{k \to \infty} \langle J'(v_{n_k}) - \lambda G'(v_{n_k}), v_{n_k} - v \rangle = 0.$$

Now, since $J' - \lambda G'$ is of class $\alpha(W^{1,p}(\Omega))$ (Proposition 4.6), we get $v_{n_k} \to v$ as $k \to \infty$. Thus using (4.11), we deduce that $J'(v) - \lambda G'(v) = 0$ and $\|v\|_{W^{1,p}(\Omega)} = 1$. A contradiction, as $\lambda \in (0,\lambda_1 + \delta) \setminus \{\lambda_1\}$ is not an eigenvalue of (1.5). Therefore, there exists $R > 0$ such that $H_\lambda(.,t)$ does not vanish in $\overline{B_R(0)} \setminus \{0\}$. Thus 0 is an isolated zero of $H(.,t)$ for any $t \in [0,1]$. Hence by homotopy invariance of degree (Proposition 2.13), we obtain

$$\text{ind}(J' - \lambda(G' + F), 0) = \text{ind}(J' - \lambda G', 0) = \begin{cases} 1, & \text{for } \lambda \in (0,\lambda_1); \\ -1, & \text{for } \lambda \in (\lambda_1,\lambda_1 + \delta). \end{cases} \quad (4.12)$$

The following theorem gives a sufficient condition [44, Theorem 7.5, Page-61] under which $\lambda_1$ is a bifurcation point of (1.1).

**Theorem 4.9.** Let $\lambda_1$ be given as in Theorem 1.1 and $g, r, f$ be given as in Theorem 1.2. Let

$$\bar{\lambda}_\pm = \lim_{\lambda \to \lambda_1 \pm 0} \text{ind}(J' - \lambda(G' + F), 0); \quad \hat{\lambda}_\pm = \lim_{\lambda \to \lambda_1 \pm 0} \text{ind}(J' - \lambda(G' + F), 0).$$

If at least two of the numbers $\bar{\lambda}_+, \hat{\lambda}_+, \bar{\lambda}_-, \hat{\lambda}_-$, $\text{ind}(J' - \lambda(G' + F), 0)$ are distinct, then $\lambda_1$ is a bifurcation point of (1.1).

**Theorem 4.10.** Let $\lambda_1$ be given as in Theorem 1.1 and $g, r, f$ be given as in Theorem 1.2. Then $\lambda_1$ is a bifurcation point of (1.1).

**Proof.** From Proposition 4.7 and Lemma 4.8, we have

$$\text{ind}(J' - \lambda(G' + F), 0) = \begin{cases} 1, & \text{for } \lambda \in (0,\lambda_1); \\ -1, & \text{for } \lambda \in (\lambda_1,\lambda_1 + \delta). \end{cases}$$
Thus, by Theorem 4.11, \( \lambda_1 \) is a bifurcation point of (1.1).

The following lemma is proved as a part of [41, Theorem 1.3].

**Lemma 4.11.** Let \( r, g \) and \( f \) be given as in Theorem 1.2. For \( \lambda \in \mathbb{R} \), define

\[
  r(\lambda) = \inf \left\{ \| \phi \|_{W^1,p(\Omega)} > 0 : (J' - \lambda(G' + F))(\phi) = 0 \right\}.
\]

Then \( r \) is lower semicontinuous. Further more, if \( \lambda \) is not an eigenvalue of (1.5), then \( r(\lambda) > 0 \).

**Proof.** \( r \) is lower semicontinuous: Let \( (\lambda_n) \) be a sequence in \( \mathbb{R}^+ \) such that \( \lambda_n \to \lambda \). Without loss of generality we assume that \( r(\lambda_n) \) is finite. Now by definition of \( r \), there exists \( \phi_n \in W^{1,p}(\Omega) \setminus \{0\} \) such that \( \|\phi_n\|_{W^1,p(\Omega)} < r(\lambda_n) + \frac{1}{n} \) and \( (J' - \lambda_n(G' + F))(\phi_n) = 0 \). Since \( (\phi_n) \) is bounded, up to a subsequence \( \phi_n \to \phi \) in \( W^{1,p}(\Omega) \). Now by writing

\[
  (J' - \lambda(G' + F))(\phi_n) = (J' - \lambda_n(G' + F))(\phi_n) + (\lambda_n - \lambda)(G' + F)(\phi_n),
\]

we observe that \( \lim_{n \to \infty} (J' - \lambda(G' + F))(\phi_n, \phi_n - \phi) = 0 \). As \( J' - \lambda(G' + F) \) is of class \( \alpha(W^{1,p}(\Omega)) \) (Proposition 4.6), we get \( \phi_n \to \phi \) in \( W^{1,p}(\Omega) \). Therefore,

\[
  (J' - \lambda(G' + F))(\phi) = 0 \tag{4.13}
\]

We claim that \( \phi \neq 0 \). If not, then \( \|\phi_n\|_{W^1,p(\Omega)} \to 0 \), as \( n \to \infty \). Set \( v_n = \phi_n ||\phi_n||_{W^1,p(\Omega)}^{-1} \). Then \( v_n \to v \) in \( W^{1,p}(\Omega) \) and (by the similar arguments as in the proof of Lemma 4.8) \( v \) must be an eigenfunction corresponding to \( \lambda \). A contradiction and hence \( \phi \neq 0 \). Thus,

\[
  r(\lambda) \leq \|\phi\|_{W^1,p(\Omega)} = \lim_{n \to \infty} \|\phi_n\|_{W^1,p(\Omega)} \leq \lim_{n \to \infty} \left( r(\lambda_n) + \frac{1}{n} \right) = \lim_{n \to \infty} r(\lambda_n).
\]

\( r \) is positive: Suppose \( r(\lambda) = 0 \) for some \( \lambda \). Then there exists a sequence \( (\phi_n) \in W^{1,p}(\Omega) \setminus \{0\} \) such that \( \|\phi_n\|_{W^1,p(\Omega)} < \frac{1}{n} \) and \( (J' - \lambda(G' + F))(\phi_n) = 0 \). Set \( v_n = \phi_n ||\phi_n||_{W^1,p(\Omega)}^{-1} \). Then \( \|v_n\|_{W^1,p(\Omega)} = 1 \) and \( v_n \to v \) in \( W^{1,p}(\Omega) \). Now using the similar arguments as in Lemma 4.8, we obtain

\[
  J'(v) - \lambda G'(v) = 0, \quad \text{where } \|v\|_{W^1,p(\Omega)} = 1.
\]

Thus \( \lambda \) must be an eigenvalue of (1.5). Therefore, \( r(\lambda) > 0 \), if \( \lambda \) is not an eigenvalue of (1.5). \( \square \)

**Remark 4.12.** If \( (\lambda, 0) \) is a bifurcation point of (1.1), then \( r(\lambda) = 0 \) and hence from Lemma 4.11, \( \lambda \) must be an eigenvalue of (1.5). Thus for the existence of a bifurcation point \( (\lambda, 0) \) of (1.1), it is necessary that \( \lambda \) is an eigenvalue of (1.5).

In the next proposition we prove a generalized homotopy invariance property for the maps \( J' - \lambda(G' + F) \). A similar result for Leray-Schauder degree is obtained in [32]. For a set \( U \) in \([a, b] \times W^{1,p}(\Omega)\), let \( U_\lambda = \{ \phi \in W^{1,p}(\Omega) : (\lambda, \phi) \in U \} \) and \( \partial U_\lambda = \{ \phi \in W^{1,p}(\Omega) : (\lambda, \phi) \in \partial U \} \).
Proposition 4.13. Let $U$ be a bounded open set in $[a, b] \times W^{1,p}(\Omega)$. If $(J' - \lambda(G' + F))(\phi) \neq 0$ for every $\phi \in \partial U_\lambda$, then $\deg(J' - \lambda(G' + F), U_\lambda, 0) = C$, $\forall \lambda \in [a, b]$.

Proof. It is enough to show that $\deg(J' - \lambda(G' + F), U_\lambda, 0)$ is locally constant on $[a, b]$. Then the proof will follow from the connectedness of $[a, b]$ and the continuity of the degree. For each $\lambda \in [a, b]$, consider the set $N_\lambda = \{\phi \in U_\lambda : (J' - \lambda(G' + F))(\phi) = 0\}$. For $\lambda_0 \in [a, b]$, let $I_0 \subset [a, b]$ be a neighbourhood of $\lambda_0$ and let $V_0$ be an open set such that $N_{\lambda_0} \subset V_0 \subset \overline{V_0} \subset U_{\lambda_0}$ and $I_0 \times V_0 \subset U$. We claim that there exists

$$I_1 \subset I_0 \text{ such that } \lambda_0 \in I_1 \text{ and } N_\lambda \subset V_0, \forall \lambda \in I_1.$$ 

If not, then there exists a sequence $(\lambda_n, \phi_n)$ in $U$ such that $\phi_n \in N_{\lambda_n} \setminus V_0$ and $\lambda_n \to \lambda_0$. As $(\phi_n)$ is bounded in $W^{1,p}(\Omega)$, $\phi_n \to \phi$ for some $\phi \in W^{1,p}(\Omega)$. Now following the steps that yield (4.13), we get $\phi_n \to \phi$ in $W^{1,p}(\Omega)$ and $(J' - \lambda_0(G' + F))(\phi) = 0$. Since $\phi \in \overline{U_{\lambda_0}}$ and $J' - \lambda_0(G' + F)$ is not vanishing on $\partial U_{\lambda_0}$, we conclude $\phi \in U_{\lambda_0}$. Thus $\phi \in N_{\lambda_0}$, a contradiction since $\phi \notin V_0$. Therefore, our claim must be true. Now consider the homotopy, $H : I_1 \times V_0 \to (W^{1,p}(\Omega))'$ defined as $H(\lambda, \phi) = (J' - \lambda(G' + F))(\phi)$. By construction, for every $\lambda \in I_1$, $H(\lambda, \cdot)$ does not vanish on $\partial V_0$. Thus by the classical homotopy invariance of degree (Proposition 2.13), $\deg(H(\lambda, \cdot), V_0, 0) = C$, $\forall \lambda \in I_1$. Since $H(\lambda, \cdot) \neq 0$ in $U_{\lambda_0} \setminus V_0$, by the additivity of degree, we obtain $\deg(H(\lambda, \cdot), U_{\lambda_0}, 0) = C$, $\forall \lambda \in I_1$. \hfill \square

Proof of Theorem 1.2: We adapt the technique used in the proof of [41, Theorem 1.3]. Recall that $S \subset \mathbb{R} \times W^{1,p}(\Omega)$ is the set of all nontrivial solutions of $(J' - \lambda(G' + F))(\phi) = 0$. Suppose there does not exist any continuum $C \subset S$ such that $(\lambda_1, 0) \in C$ and $C$ is either unbounded, or meets at $(\lambda, 0)$ where $\lambda$ is an eigenvalue of (1.5) and $\lambda \neq \lambda_1$. Then by [41, Lemma 1.2], there exists a bounded open set $U \subset \mathbb{R} \times W^{1,p}(\Omega)$ containing $(\lambda_1, 0)$ such that $\partial U \cap S = \emptyset$ and $\overline{U} \cap \mathbb{R} \times \{0\} = \overline{T} \times \{0\}$, where $I = (\lambda_1 - \delta, \lambda_1 + \delta)$ with $0 < \delta < \min\{\lambda_1, \lambda_2 - \lambda_1\}$. Thus $(\lambda \times \partial U_\lambda) \cap S = \emptyset$ for every $\lambda \in \mathbb{R}$ and $(\lambda, 0) \notin \partial U$ for $\lambda \in I$. In particular, $J' - \lambda(G' + F)$ does not vanish on $\partial U_\lambda$ for every $\lambda$ in $I$. Hence $\deg((J' - \lambda(G' + F), U_{\lambda}, 0)$ is well defined and by homotopy invariance of degree (Proposition 4.13), we have

$$\deg(J' - \lambda(G' + F), U_{\lambda}, 0) = C,$$ 

for $\lambda \in I$. \hfill (4.14)

Next we compute $\text{ind}(J' - \lambda(G' + F), 0)$ for $\lambda \in I$. Let

$$d := \text{dist}((-\infty, 0] \cup [\lambda_2, \infty), \overline{U}).$$

Since $\overline{U} \cap \mathbb{R} \times \{0\} = \overline{T} \times \{0\}$, we observe that $d > 0$. Now set

$$\rho(\lambda) = \begin{cases} \frac{d}{2}, & \text{for } \lambda \in (-\infty, 0] \cup [\lambda_2, \infty), \\ \min\{1, \frac{d}{2}\rho(\lambda), & \text{for } \lambda \in (0, \lambda_2) \setminus \{\lambda_1\}. \end{cases}$$

Thus using 4.11 we easily conclude that $\rho(\lambda) > 0$ for each $\lambda \neq \lambda_1$ and $\overline{B_{\rho(\lambda)}} \setminus \{0\}$ does not contain any solution of $J' - \lambda(G' + F)$. Let

$$I^* := \{\lambda : (\lambda, \phi) \in U \text{ for some } \phi\}, \quad \lambda^* := \sup\{\lambda : \lambda \in I^*\}, \quad \lambda_* := \inf\{\lambda : \lambda \in I^*\}$$

27
For $\lambda \in (\lambda_1, \lambda^*)$, let $\rho = \inf \{ \rho(\mu) : \mu \in [\lambda, \lambda^*] \}$. By Lemma (4.11), we have $\rho > 0$. Now consider the set $V = U \setminus [\lambda, \lambda^*] \times \overline{B_\rho}$. Observe that, $V$ is bounded and open in $[\lambda, \lambda^*] \times W^{1,p}(\Omega)$. Furthermore, for each $\mu \in [\lambda, \lambda^*]$, $V_\mu = U_\mu \setminus \overline{B_\rho}$ and $(J' - \mu(G' + F))$ does not vanish on $\partial V_\mu = \partial (U_\mu \setminus \overline{B_\rho})$. Therefore, by the homotopy invariance of degree (Proposition 4.13) and noting that $U_{\lambda^*} = \emptyset$, we get

$$\deg(J' - \lambda(G' + F), U_\lambda \setminus \overline{B_\rho}, 0) = \deg(J' - \lambda(G' + F), U_{\lambda^*} \setminus \overline{B_\rho}, 0) = 0.$$ 

Similarly, for $\lambda \in [\lambda_*, \lambda^*)$ we get $\deg(J' - \lambda(G' + F), U_\lambda \setminus \overline{B_\rho}, 0) = 0$. Since $(J' - \lambda(G' + F))(\phi) \neq 0$ for $\phi \in B_{\rho(\lambda)} \setminus \overline{B_\rho}$, by the additivity of the degree we get

$$\deg(J' - \lambda(G' + F), U_\lambda \setminus \overline{B_{\rho(\lambda)}}, 0) = 0, \quad \lambda \in [\lambda_*, \lambda^*) \setminus \{\lambda_1\}.$$ 

Again using the additivity of the degree, we conclude that

$$\deg(J' - \lambda(G' + F), U_\lambda, 0) = \deg(J' - \lambda(G' + F), B_{\rho(\lambda)}, 0), \quad \forall \lambda \in I \setminus \{\lambda_1\}.$$ 

Thus from (4.14) we obtain

$$\text{ind}(J' - \lambda(G' + F), 0) = C, \quad \text{for } \lambda \in I \setminus \{\lambda_1\}.$$ 

A contradiction to (4.12). Thus there must exist a continuous branch of non-trivial solutions from $(\lambda_1, 0)$ and is either unbounded, or meets at $(\lambda, 0)$ where $\lambda$ is an eigenvalue of (1.5).

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