FRAMIZATION OF THE TEMPERLEY–LIEB ALGEBRA

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Abstract. In this paper we propose a framization of the Temperley–Lieb algebra. The framization is a procedure that can briefly be described as the adding of framing to a known knot algebra in a way that is both algebraically consistent and topologically meaningful. Here, our framization is defined as a quotient of the Yokonuma–Hecke algebra. The main theorem provides necessary and sufficient conditions for the Markov trace defined on the Yokonuma–Hecke algebra to pass through to our framization.

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1. INTRODUCTION

Since the original construction of the Jones polynomial the Temperley–Lieb algebra has become a cornerstone of a fruitful interaction between Knot theory and Representation theory. The Temperley–Lieb algebra was introduced by Temperley and Lieb [23] and was rediscovered by Jones [8] as a knot algebra [9].

A knot algebra is an algebra that is used in the construction of invariants of classical links. Our main interest lies in those knot algebras that are involved in the construction of invariants through Jones’ formula [9]. More precisely, a knot algebra A is a triplet (A, π, τ), where π is an appropriate representation of the braid group in A and τ is a Markov trace function defined on A. The Temperley–Lieb algebra, the Iwahori–Hecke algebra and the BMW algebra are the most important examples of knot algebras.

The ‘framization’ is a mechanism designed by the second and fourth authors that consists of a generalization of a knot algebra via the addition of framing generators. In this way we obtain a new algebra which is related to framed braids and framed knots. More precisely, the framization...
procedure can roughly be described as the procedure of adding framing generators to the generating set of a knot algebra, of defining interacting relations between the framing generators and the original generators of the algebra and of applying framing on the original defining relations of the algebra. The resulting framed relations should be topologically consistent. The most difficult problem in this procedure is to apply the framization on the relations of polynomial type.

The basic example of framization is the Yokonuma–Hecke algebra, which can be regarded as a framization of the Iwahori–Hecke algebra [13, 16]. This framization of the Iwahori–Hecke algebra gives the recipe of how to apply the framization technique on the quadratic Iwahori–Hecke relation. Having in mind this example, the second and fourth authors proposed framizations of several knot algebras [17, 18]. Further, the second author constructed a unique Markov trace function, tr, on the Yokonuma–Hecke algebras $Y_{d,n}(u)$ with parameters $z, x_1, \ldots, x_{d-1}$ [11]. Consequently, invariants for framed, classical and singular oriented links have been constructed [16, 15, 14] by applying the ‘E–condition’ on the parameters $x_1, \ldots, x_{d-1}$ so that $tr$ re–scales and respects the negative stabilization move between framed braids [16]. The derived invariants for classical links do not coincide with the 2–variable Jones or HOMFLYPT polynomial (here denoted Homflypt) [1] except in few trivial cases, yet they are conjectured to be topologically equivalent to the Homflypt polynomial [5].

The Temperley–Lieb algebra can be regarded as a quotient of the Iwahori–Hecke algebra, therefore it is natural to look for a quotient of the Yokonuma–Hecke algebra that can be considered as a framization of the Temperley–Lieb algebra. In this paper we propose a framization of the Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra over an appropriate two–sided ideal. Although such an ideal is not unique, it will become clear that our choice for the ideal that leads to the framization of the Temperley–Lieb algebra is the one that reflects the construction of a ‘framed Jones Polynomial’ in the most natural way.

The outline of the paper is as follows. Section 2 is dedicated to providing necessary definitions and results, including: the Iwahori–Hecke algebra, the Ocneanu trace, and the Yokonuma–Hecke algebra $Y_{d,n}(u)$. In Section 3 we recall some basic tools from harmonic analysis of finite groups such as the convolution product, the product by coordinates and the Fourier transform, necessary for exploring the ‘E–system’.

In Section 4, where we present two natural quotients of $Y_{d,n}(u)$ that could possibly lead to a framization of the Temperley–Lieb algebra, the Yokonuma–Temperley–Lieb algebra, $YTL_{d,n}(u)$ (introduced and studied in [6]) and the Complex Reflection Temperley–Lieb algebra, $CTL_{d,n}(u)$. After deducing that these two quotient algebras are not suitable for our purpose, we introduce a third quotient of $Y_{d,n}(u)$, the Framization of the Temperley–Lieb algebra, $FTL_{d,n}(u)$. The connection between all possible quotients of $Y_{d,n}(u)$ is then analyzed and we prove that the defining ideals of $FTL_{d,n}(u)$ and $CTL_{d,n}(u)$ are principal. Furthermore, we provide presentations with non–invertible generators for the quotient algebras $FTL_{d,n}(u)$ and $CTL_{d,n}(u)$. We conclude this section with the formula for the dimension of $FTL_{d,n}(u)$ by Chlouveraki and Pouchin [2] and we provide a linear basis for the case $d = 2, n = 3$.

The main theorems are given in Section 5. They provide the necessary and sufficient conditions for the Markov trace $tr$ [11] on the Yokonuma–Hecke algebra to pass through to the quotient algebras $FTL_{d,n}(u)$ and $CTL_{d,n}(u)$ respectively. The corresponding results for the algebra $YTL_{d,n}(u)$ are given in [6]. More precisely, we first find the necessary and sufficient conditions for the case of $FTL_{d,3}(u)$ using tools from harmonic analysis on finite groups (Lemma 8) and then we generalize using induction on $n$ (Theorem 5). Using the same ideas we prove the analogous theorem for $CTL_{d,n}(u)$ (Theorem 6). In Section 6 we discuss the connection between the necessary and
sufficient conditions such that tr passes to all three quotients algebras $\text{CTL}_{d,n}(u)$, $\text{FTL}_{d,n}(u)$ and $\text{YTL}_{d,n}(u)$.

Finally, in Section 7 we define invariants for framed and classical links through the quotient algebras $\text{FTL}_{d,n}(u)$ and $\text{CTL}_{d,n}(u)$, and we deduce that the invariants for framed links that are derived from $\text{FTL}_{d,n}(u)$ provide a framed analogue of the Jones polynomial, while the invariants from the algebras $\text{CTL}_{d,n}(u)$ coincide with those from $\text{FTL}_{d,n}(u)$. We note that the invariants for classical links from the algebras $\text{YTL}_{d,n}(u)$ recover the Jones polynomial [6].

2. Preliminaries

2.1. Notations. Throughout the paper by the term algebra we mean an associative unital (with unity 1) algebra over $\mathbb{C}$. Thus we can regard $\mathbb{C}$ as a subalgebra of the center of the algebra. We will also fix two positive integers, $d$ and $n$.

As usual we denote $\mathbb{Z}/d\mathbb{Z}$ the group of integers modulo $d$, $\mathbb{Z}/d\mathbb{Z} = \{0, 1, \ldots, d-1\}$.

We denote $S_n$ the symmetric group on $n$ symbols. Let $s_i$ be the elementary transposition $(i, i + 1)$ and let $\langle s_i, s_j \rangle$ denote the the subgroup generated by $s_i$ and $s_j$. We also denote by $l$ the length function on $S_n$ with respect to the $s_i$'s.

Denote $\mathcal{C}_d = \langle t \mid t^d = 1 \rangle$ the cyclic group of order $d$. Let $t_i := (1, 1, \ldots, 1, t, 1, \ldots, 1) \in \mathcal{C}_d^n$, where $t$ is in the $i$–th position. We then have:

$$\mathcal{C}_d^n = \langle t_1, \ldots, t_n \mid t_i t_j = t_j t_i, t_i^d = 1 \rangle.$$

Define $C_{d,n} := \mathcal{C}_d^n \rtimes S_n$, where the action is defined by permutation on the indices of the $t_i$'s, namely: $s_i t_j = t_{s_i(j)} s_i$. Notice that $C_{d,n}$ is isomorphic to the complex reflection group $G(d, 1, n)$.

Denote also $B_n$ the braid group of type $A$, that is, the group generated by the elementary braidings $\sigma_1, \ldots, \sigma_{n-1}$, subject to the following relations: $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ for $|i - j| = 1$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$. We will also use the $d$–modular framed braid group $\mathcal{F}_{d,n} := \mathcal{C}_d^n \rtimes B_n$, where the action of $B_n$ on $\mathcal{C}_d^n$ is defined by the induced permutation on the indices of the $t_i$'s and $C$ is the infinite cyclic group, which surjects naturally on $\mathcal{F}_{d,n}$. We will also refer to the framed braid group $\mathcal{F}_n := \mathcal{C}^n \rtimes B_n$. Of course,

$$\mathcal{F}_n \cong \mathbb{Z} \rtimes B_n \quad \text{and} \quad \mathcal{F}_{d,n} \cong (\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n.$$

Remark 1. We would like to point out that $C_{d,n}$ and $\mathcal{F}_{d,n}$ appear in the theory of “fields with one element”. This is a theory dreamt by J. Tits in his study of algebraic groups. According to the seminal article of Kapranov and Smirnov [19], $\text{GL}_n(\mathbb{F}_1) = S_n$, $\text{GL}_n(\mathbb{F}_1[t]) = B_n$, $\text{GL}_n(\mathbb{F}_1^n) = C_{d,n}$ and $\text{GL}_n(\mathbb{F}_1^n[t]) = \mathcal{F}_{d,n}$, where $\text{GL}_n(\mathbb{F}_1^n)$ (resp. $\text{GL}_n(\mathbb{F}_1^n[t])$) is in “some sense” the limit case $q \to 1$ of $\text{GL}_n(\mathbb{F}_q)$ (resp. $\text{GL}_n(\mathbb{F}_q[t])$).

2.2. Background material. From now on we fix a non-zero complex number $u$. We denote by $H_n(u)$ the Iwahori–Hecke algebra associated to $S_n$, that is, the $\mathbb{C}$–algebra with linear basis $\{h_w \mid w \in S_n\}$ and the following rules of multiplication:

$$h_s h_w = \begin{cases} h_{s_i w} & \text{for } l(s_i w) > l(w) \\ u h_{s_i w} + (u - 1) h_w & \text{for } l(s_i w) < l(w) \end{cases} \quad (2.1)$$

Set $h_i := h_{s_i}$. Then $H_n(u)$ is presented by $h_1, \ldots, h_{n-1}$ subject to the following relations:

$$h_i h_j = h_j h_i \quad \text{for all } |i - j| > 1$$

$$h_i h_j h_i = h_j h_i h_j \quad \text{for all } |i - j| = 1$$

$$h_i^2 = u + (u - 1) h_i$$
Definition 1. The Temperley–Lieb algebra $\text{TL}_n(u)$ can be defined as the quotient of the algebra $H_n(u)$ over the two–sided ideal generated by the Steinberg elements $h_{i,j}$:

$$h_{i,j} := \sum_{w \in \langle s_i, s_j \rangle} h_w, \quad \text{for all } |i - j| = 1. \quad (2.2)$$

Equivalently, $\text{TL}_n(u)$ can be presented by $h_1, \ldots, h_{n-1}$ subject to the following relations:

$$h_i h_j = h_j h_i \quad \text{for all } |i - j| > 1$$

$$h_i h_j h_i = h_j h_i h_j \quad \text{for all } |i - j| = 1$$

$$h_i^2 = u + (u - 1)h_i$$

$$1 + h_i + h_j + h_i h_j + h_j h_i = 0 \quad \text{for all } |i - j| = 1$$

Furthermore, using the transformation:

$$f_i := \frac{1}{u+1}(h_i + 1) \quad (2.3)$$

the algebra $\text{TL}_n(u)$ can be presented by the non–invertible generators $f_1, \ldots, f_{n-1}$ subject to the following relations:

$$f_i^2 = f_i, \quad f_i f_j f_i = \delta f_i, \quad \text{for all } |i - j| = 1$$

$$f_i f_j = f_j f_i, \quad \text{for all } |i - j| > 1$$

where $\delta^{-1} = 2 + u + u^{-1}$ [9].

In [7, 9] Ocneanu constructed a unique Markov trace on $H_n(u)$. More precisely, we have the following theorem.

Theorem 1 (Ocneanu). For any $\zeta \in \mathbb{C}^\times$ there exists a linear trace $\tau$ on $\bigcup_{n=1}^{\infty} H_n(u)$ uniquely defined by the inductive rules:

1. $\tau(ab) = \tau(ba), \quad a, b \in H_n(u)$
2. $\tau(1) = 1$
3. $\tau(ah_n) = \zeta \tau(a), \quad a \in H_n(u)$ \quad (Markov property).

The Ocneanu trace $\tau$ passes through to $\text{TL}_n(u)$. Indeed, as it turned out [9], to factorize $\tau$ to the Temperley–Lieb algebra, we only need the fact that $\tau$ kills the expression of Eq. 2.2. So, in [9] it is proved that $\tau$ passes to the Temperley–Lieb algebra if and only if:

$$\zeta = -\frac{1}{u+1} \quad \text{or} \quad \zeta = -1. \quad (2.4)$$

2.3. The Yokonuma–Hecke algebra. The Yokonuma–Hecke algebra of type $A$, denoted $Y_{d,n}(u)$ [25], can be defined by generators and relations [11] and can be regarded as a quotient of $\mathbb{C}F_{d,n}$ over the two–sided ideal that is generated by the elements:

$$\sigma_i^2 - (u - 1)e_i - (u - 1)e_i \sigma_i - 1,$$

where $e_i$ is the idempotent defined by:

$$e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}, \quad i = 1, \ldots, n-1. \quad (2.5)$$

Equivalently, one can define $Y_{d,u}(u)$ as follows:
Definition 2. The Yokonuma–Hecke algebra $Y_{d,n}(u)$ is the algebra presented by generators $g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$ subject to the following relations:

$$g_ig_j = g_jg_i \quad \text{for all } |i - j| > 1$$

$$g_{i+1}g_{i+1} = g_{i+1}g_{i+1}$$

$$t_it_j = t_jt_i \quad \text{for all } i, j$$

$$t_i^2 = 1 \quad \text{for all } i$$

$$g_it_i = t_i+1g_i$$

$$g_it_{i+1} = t_ig_i$$

$$g jt_j = t_jg_i \quad \text{for } j \neq i, i + 1$$

$$g_i^2 = 1 + (u - 1)e_i + (u - 1)e_ig_i$$

Note that for $d = 1$ the quadratic relation (2.13) becomes:

$$g_i^2 = (u - 1)g_i + u$$

So the Yokonuma–Hecke $Y_{1,n}(u)$ coincides with the Iwahori–Hecke algebra.

The algebra $Y_{d,n}(u)$ can also be regarded as a $u$–deformation of the group algebra $\mathbb{C}C_{d,n}$. Indeed, if $w \in S_n$ is a reduced word in $S_n$ with $w = s_{i_1} \ldots s_{i_k}$ then the expression $g_w = g_{s_{i_1}} \ldots g_{s_{i_k}} \in Y_{d,n}(u)$ is well–defined since the generators $g_i := g_{s_i}$ satisfy the same braiding relations as the generators of $S_n$ [20]. We have the following multiplication rule in $Y_{d,n}(u)$ (see Proposition 2.4[10]):

$$g_{s_i}g_w = \begin{cases} 
g_{s_{i}w} & \text{for } l(s_{i}w) > l(w) 
g_{s_{i}w} + (u - 1)e_i g_{s_{i}w} + (u - 1)e_i g_w & \text{for } l(s_{i}w) < l(w).
\end{cases}$$

Note also that the generators $g_{s_i}$ correspond to $t_i$ and so we have that: $g_{t_iw} = g_{t_i}g_w = t_ig_w$.

The definition of the idempotents $e_i$ can be generalized in the following way. For any indices $i, j$ we define the following elements in $Y_{d,n}(u)$:

$$e_{i,j} := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_j^{d-s}.$$  

We also define, for any $0 \leq m \leq d - 1$, the shift of $e_i$ by $m$:

$$e_i^{(m)} := \frac{1}{d} \sum_{s=0}^{d-1} t_i^m t_j^{d-s}.$$  

Notice that $e_i = e_{i,i+1} = e_i^{(0)}$. Notice also that $e_i^{(m)} = t_i^m e_i = t_i^m e_{i,i+1}$. Then one deduces easily that:

$$t_i^a t_j^b t_k^c e_{i,j} = e_i^{(a+b+c)} e_{i,j}$$

for all $0 \leq m, a, b, c \leq d - 1$.

The following lemma collects some of the relations among the $e_i$’s, the $t_j$’s and the $g_i$’s. These relations will be used in the paper.
**Lemma 1** ([6] Lemma 1). For the idempotents $e_i$ and for $1 \leq i, j \leq n - 1$ the following relations hold:

\[
\begin{align*}
t_j e_i &= e_i t_j \\
e_{i+1} g_i &= g_i e_{i,i+2} \\
e_{i} g_j &= g_j e_i, \quad \text{for } j \neq i - 1, i + 1 \\
e_j g_i g_j &= g_j g_i e_i \quad \text{for } |i - j| = 1 \\
e_i e_{i+1} &= e_i e_{i,i+2} \\
e_i e_{i+1} &= e_{i,i+2} e_{i+1}.
\end{align*}
\]

Using the multiplication formulas (2.14), the second author proved in [11] that $Y_{d,n}(u)$ has the following standard linear basis:

\[
\{t_1^{a_1} \ldots t_n^{a_n} g_w \mid a_i \in Z/dZ, w \in S_n\}.
\]

(2.18)

Notice now that using relations (2.10) and (2.11) one can write any monomial $m$ in $Y_{d,n}(u)$ in the following form:

\[
m = t_1^{a_1} \ldots t_n^{a_n} m'
\]

where $m' = g_{i_1} \ldots g_{i_n}$. We say that every monomial in $Y_{d,n}(u)$ has the splitting property, which is in fact inherited from the framed braid group $F_n$. That is, one can separate the framing part from the braiding part.

Further, we have an inductive basis of the Yokonuma–Hecke algebra, which is used in the proof of the main theorem, Theorem 5. More precisely:

**Proposition 1** ([11] Proposition 8). Every element in $Y_{d,n+1}(u)$ is a unique linear combination of words, each of one of the following types:

\[
m_n g_n g_{n-1} \ldots g_i t_i^k \quad \text{or} \quad m_n t_{n+1}^k
\]

where $0 \leq k \leq d - 1$ and $m_n$ is a word in the inductive basis of $Y_{d,n}(u)$.

2.4. Using the above basis, the second author proved that $Y_{d,n}(u)$ supports a unique Markov trace. We have the following theorem:

**Theorem 2** ([11] Theorem 12). For indeterminates $z, x_1, \ldots, x_{d-1}$ there exists a unique linear Markov trace $\text{tr}$:

\[
\text{tr} : \cup_{n=1}^{\infty} Y_{d,n}(u) \longrightarrow \mathbb{C}[z, x_1, \ldots, x_{d-1}]
\]

defined inductively on $n$ by the following rules:

\[
\begin{align*}
\text{tr}(ab) &= \text{tr}(ba) \\
\text{tr}(1) &= 1 \\
\text{tr}(ag_n) &= z \text{tr}(a) \quad (\text{Markov property}) \\
\text{tr}(at_{s+1}^n) &= x_s \text{tr}(a) \quad (s = 1, \ldots, d - 1)
\end{align*}
\]

where $a, b \in Y_{d,n}(u)$.

Using the trace rules of Theorem 2 and including $x_0 := 1$, we deduce that $\text{tr}(e_i)$ takes the same value for all $i$, and this value is denoted by $E$:

\[
E := \text{tr}(e_i) = \frac{1}{d} \sum_{s=0}^{d-1} x_s x_{d-s}.
\]

Moreover, we also define the shift by $m$ of $E$, where $0 \leq m \leq d - 1$, by:

\[
E^{(m)} := \text{tr}(e_i^{(m)}) = \frac{1}{d} \sum_{s=0}^{d-1} x_{m+s} x_{d-s}.
\]
Notice that $E = E^{(0)}$.

3. Fourier transform and the $E$–system

An important tool in the proof of the main theorem are some classical identities of harmonic analysis on the group of integers modulo $d$. More precisely, we will use identities linking the convolution product and the product by coordinates through the Fourier transform. These tools were also used in solving the so–called $E$–system, see Appendix [16]. Thus, in this section we shall give some notations and recall some well–known and useful facts of the Fourier transform along with some facts for the $E$–system.

3.1. In our setting it is convenient to see the group of integers modulo $d$ as the group $C_d$. Hence, the product by coordinates in $C_d$ is defined by the formula

$$(d-1) \sum_{r=0}^{d-1} a_r t^r \cdot (d-1) \sum_{s=0}^{d-1} b_s t^s = \sum_{i=0}^{d-1} a_ib_it^i$$

and the convolution product is defined by the formula:

$$(d-1) \sum_{r=0}^{d-1} a_r t^r \ast (d-1) \sum_{s=0}^{d-1} b_s t^s = \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \sum_{t=0}^{d-1} (d-1) a_ib_{r-s} t^r$$

In order to define the Fourier transform on $C_d$ we need to introduce the following elements:

$$i_a := \sum_{s=0}^{d-1} \chi_a(s) t^s \quad (a \in \mathbb{Z}/d\mathbb{Z}).$$

where the $\chi_k$'s denote the characters of the group $\mathbb{Z}/d\mathbb{Z}$, namely:

$$\chi_k(m) = \cos \frac{2\pi km}{d} + i \sin \frac{2\pi km}{d} \quad (k, m \in \mathbb{Z}/d\mathbb{Z})$$

One can verify that:

$$i_a \ast i_b = \begin{cases} d_i_a & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

On the other hand, we shall denote by $\delta_a$ the element of the canonical linear basis of $C_d$, that is, $\delta_a := t^a$. It is clear that:

$$\delta_a \cdot \delta_b = \begin{cases} \delta_a & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

The Fourier transform is the linear automorphism on $C_d$, defined as:

$$y := \sum_{r=0}^{d-1} a_r t^r \mapsto \hat{y} := \sum_{s=0}^{d-1} (y \ast i_s)(0) t^s$$

where $(y \ast i_s)(0)$ denote the coefficient of $\delta_0$ in the convolution $y \ast i_s$.

The next proposition gathers the most important properties of the Fourier transform used in the paper.

**Proposition 2 ([22]).** For any $y$ and $y'$ in $C_d$, we have:

1. $\hat{y \ast y'} = \hat{y} \cdot \hat{y}'$
2. $\hat{y \cdot y'} = d^{-1} \hat{y} \ast \hat{y}'$
3. $\delta_a = i_{-a}$
If \( y = \sum_{r=0}^{d-1} a_r t^r \), then

\[
 \hat{y} = \sum_{r=0}^{d-1} a_{-r} t^r
\]

Finally, we note that the elements in the group algebra \( \mathbb{C} C_d \) can also be identified to the set of functions \( f : \mathbb{Z}/d\mathbb{Z} \to \mathbb{C} \), where the identification is as follows:

\[
 (f : \mathbb{Z}/d\mathbb{Z} \to \mathbb{C}) \mapsto \sum_{k=0}^{d-1} f(k) t^k \in \mathbb{C} C_d.
\]

Some times we shall use this identification, since it makes some computations more comfortable.

### 3.2. The \( E \)-system

The \( E \)-system is a non-linear system of equations that was introduced in order to find the necessary and sufficient conditions that need to be applied on the parameters \( x_i \) of \( \text{tr} \) so that the definition of link invariants from the Yokonuma–Hecke algebra would be possible [16].

In [16, Appendix] the full set of solutions of the \( E \)-system is given. To do that, Gérardin interpreted the solutions of the \( E \)-system as certain complex functions \( x \) over \( \mathbb{Z}/d\mathbb{Z} \). More precisely, the function \( x \) is defined by mapping 0 to \( x_0 := 1 \) and \( k \) is mapped in the parameters \( x_k \) of \( \text{tr} \). Thus, the function \( x \) is identified as:

\[
 x = \sum_{k=0}^{d-1} x_k t^k
\]

Notice that the coefficient of \( t^k \) in the convolution \( x \ast x \) (respectively \( x \ast x \ast x \)) is:

\[
 E^{(k)} \quad \text{(respectively } \text{tr}(e_1^{(k)} e_2))\]

(see Lemma 2 [6]). Equivalently, we have

\[
 x \ast x : k \mapsto E^{(k)} \quad \text{and} \quad x \ast x \ast x : k \mapsto \text{tr}(e_1^{(k)} e_2)
\]

(3.3)

**Definition 3.** We say that the \((d-1)\)-tuple of complex numbers \((x_1, \ldots, x_{d-1})\) satisfies the \( E \)-condition if \( x_1, \ldots, x_{d-1} \) satisfy the following system of non-linear equations in \( \mathbb{C} \), the \( E \)-system:

\[
 E^{(m)} = x_m E \quad (1 \leq m \leq d-1).
\]

(3.4)

More precisely, in [16, Appendix], Gérardin achieved to solve the \( E \)-system by interpreting each equation of (3.4) as the value at \( m \) of the convolution product of the functional \( x : s \mapsto x_s \) by itself in the complex algebra \( \mathbb{C} C_d \) and then using some tools of harmonic analysis on finite groups. It turns out that the solutions of the \( E \)-system are parametrized by the non-empty subsets of the cyclic group \( \mathbb{Z}/d\mathbb{Z} \), that is, given a non-empty subset \( D \) of \( \mathbb{Z}/d\mathbb{Z} \), the corresponding solution is:

\[
 x_D = \frac{1}{|D|} \sum_{k \in D} x_k
\]

(3.5)

**Remark 2.** It is worth noting that the formula for the solutions of the \( E \)-system can be interpreted as a generalization of the Ramanujan’s sum. Indeed, by taking the subset \( P \) of \( \mathbb{Z}/d\mathbb{Z} \) consisting of the numbers coprime to \( d \), then the solution parametrized by \( P \) is, up to the factor \( |P| \), the Ramanujan’s sum \( c_d(k) \) (see [21]).
Notice now, that the characters of $C_d$ are given by $t^a \mapsto \chi_k(a)$, where $k$ runs in $\mathbb{Z}/d\mathbb{Z}$; thus, we shall denote also this character as $\chi_k$. We note also that the solutions $x_D$ can be seen as elements in $\mathbb{C}C_d$, namely:

$$x_D = \sum_{j=0}^{d-1} x_j t^j$$

where $x_j = \frac{1}{|D|} \sum_{m \in D} \chi_m(j)$.

We finish this section with a theorem which yields the main connection among the solutions of the $E$–system and the trace $\text{tr}$.

**Theorem 3** ([13] Theorem 7). If the trace parameters $(x_1, \ldots, x_{d-1})$ satisfy the $E$–condition, then

$$\text{tr}(ae_n) = \text{tr}(a)\text{tr}(e_n) \quad (a \in Y_{d,n}(u)).$$

4. A Framization of the Temperley–Lieb Algebra

As discussed during the Introduction, the Yokonuma–Hecke algebra can be interpreted as the framization of the Iwahori–Hecke algebra, which is a knot algebra. Thus a natural question arises, the definition of a framization for the knot algebra Temperley–Lieb. Considering the fact that the Temperley–Lieb algebra can be defined as a quotient of the Iwahori–Hecke algebra, it is natural to try and define a framization of the Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra. Recall now that the defining ideal of the Temperley–Lieb algebra (Definition 1) is generated by the Steinberg elements which are related to the subgroups $\langle s_i, s_{i+1} \rangle$ of $S_n$, for all $i$. These subgroups can be also regarded as subgroups of $C_{d,n}$. Therefore, using the multiplication rule of Eq. 2.14 we are able to define the analogous Steinberg elements $g_{i,i+1}$ in $Y_{d,n}(u)$,

$$g_{i,i+1} := \sum_{w \in \langle s_i, s_{i+1} \rangle} g_w \quad \text{for all } i$$

In [6, Definition 5] we defined a potential candidate for the framization of the Temperley–Lieb algebra, the Yokonuma–Temperley–Lieb algebra, denoted $YTL_n(u)$, which is the quotient of $Y_{d,n}(u)$ over the 2–sided ideal generated by all the $g_{i,i+1}$’s for all $i$. It is not difficult to show that this ideal is in fact principal and it is generated by the element $g_{1,2}$. Moreover, the necessary and sufficient conditions for the trace $\text{tr}$ to pass through to $YTL_n(u)$ were studied [6, Theorem 5]. Unfortunately, these conditions turn out to be too strong. Namely, the trace parameters $x_i$ must be $d^{th}$ roots of unity, furnishing obvious, special solutions of the $E$–system and resulting in loss of the framing information on the level of the invariant. Thus, the framed analogues of the Jones polynomial obtained from the algebras $YTL_{d,n}(u)$ turn out to be of no interest. For this reason, $YTL_{d,n}(u)$ is discarded as framization of $TL_n(u)$. However, using the results of [1], the classical Jones polynomial can be still recovered through these algebras by representing the Artin braid group $B_n$ in $Y_{d,n}(u)$, considering the $t_i$’s as formal generators, and then taking the quotient over the ideal that is generated by the $g_{i,i+1}$’s [6, Corollary 2]. Finally, we note that the representation theory of this algebra has been studied extensively in [2].

Given the fact that $Y_{d,n}(u)$ can be considered as a $u$–deformation of $\mathbb{C}C_{d,n}$ (recall the discussion in Section 2.3), it is natural to consider subgroups of $C_{d,n}$ that involve in their generating set the framing generators of the $i$–th and $j$–th strands along with $\langle s_i, s_j \rangle$. Therefore, we have considered the following subgroups of $C_{d,n}$:

$$C_{i,i+1} := \langle t_i, t_{i+1}, t_{i+2} \rangle \times \langle s_i, s_{i+1} \rangle \quad \text{for all } i.$$
Further, notice that these subgroups are isomorphic to $C_{d,3}$ and thus analogous to the classical case. We define now the elements $c_{i,i+1}$ in $Y_{d,n}(u)$ as follows:

$$c_{i,i+1} = \sum_{c \in C_{i,i+1}} gc$$

We then have the following definition:

**Definition 4.** For $n \geq 3$, we define the algebra $\text{CTL}_{d,n}(u)$ as the quotient of the algebra $Y_{d,n}(u)$ by the two-sided ideal of generated by the $c_{i,i+1}$'s, for all $i$. We shall call $\text{CTL}_{d,n}(u)$ the Complex Reflection Temperley–Lieb algebra.

As it will be shown in Theorem 6 the necessary and sufficient conditions such that $\text{tr}$ passes to $\text{CTL}_{d,n}(u)$ are, contrary to the case of $\text{YTL}_{d,n}(u)$, too relaxed. In particular, the E-condition does not appear on the trace parameters $x_i$, which is necessary for the definition of link invariants through $\text{tr}$ (see [16, Section 4] for details). Thus, in order to define link invariants from the algebras $\text{CTL}_{d,n}(u)$, the E-condition must be imposed on the $x_i$'s resulting in invariants that are naturally obtained from a different construction (see below). Therefore, both of the natural definitions for the framed analogue of the Temperley–Lieb algebra that are derived from $Y_{d,n}(u)$ should be discarded as possible framizations of the Temperley–Lieb algebra.

Since the conditions such that the trace $\text{tr}$ passes to the quotient algebras are either too strong or too relaxed, it indicates that the desired framization of the Temperley–Lieb algebra should be an intermediate algebra between $\text{YTL}_{d,n}(u)$ and $\text{CTL}_{d,n}(u)$. Regarding the definitions of these two algebras, it is reasonable to construct the framization in an analogous way using an intermediate subgroup between $\langle s_i, s_{i+1} \rangle$ and $C_{i,i+1}$. Thus, we consider the following subgroups of $C_{d,n}$:

$$H_{i,i+1} := \langle t_i^{-1}t_{i+1}^{-1}, t_{i+1}^{-1}t_{i+2}^{-1} \rangle \times \langle s_i, s_{i+1} \rangle \quad \text{for all } i.$$  

We now introduce the following elements:

$$r_{i,i+1} := \sum_{x \in H_{i,i+1}} gx \quad \text{for all } i.$$

**Definition 5.** For $n \geq 3$, the Framization of the Temperley–Lieb algebra, denoted $\text{FTL}_{d,n}(u)$, is defined as the quotient $Y_{d,n}(u)$ over the two-sided ideal generated by the elements $r_{i,i+1}$, for all $i$.

The reason of considering $\text{FTL}_{d,n}(u)$ as a framization of the Temperley–Lieb algebra is due to the fact that through this algebra the definition of a non-trivial framization of the Jones polynomial is possible. Therefore, the necessary and sufficient conditions for the trace $\text{tr}$ to pass to $\text{FTL}_{d,n}(u)$ are determined in Theorem 6, while in Section 7 we discuss the invariants derived from this algebra.

**Remark 3.** Notice that when $d = 1$, the Yokonuma–Hecke algebra collapses to the Iwahori–Hecke algebra, hence it follows that $Y_{1,n}(u)$ also collapses to $TL_n(u)$. Moreover, in this case the subgroups $H_{1,i+1}$ and $C_{i,i+1}$ also collapse to $\langle s_i, s_{i+1} \rangle$. Hence, $\text{FTL}_{1,n}(u)$ and $\text{CTL}_{1,n}(u)$ collapse to $TL_n(u)$ too.

4.1. We shall now show how the algebras defined above are related. We start noting that

$$C_{i,i+1} = H_{i,i+1} \times C_d$$

Indeed, we have a homomorphism $\phi$ from $C_{i,i+1}$ onto $C_d$ defined by $\phi(x) = t_i^{a+b+c}$, where $C_d$ is regarded as the cyclic group generated by $t_i$ and $x$ is written uniquely as $x = t_i^{a+b+c} t_{i+1}^{c} w$, with
It is known that the defining ideal of the Temperley–Lieb algebra is principal. We are going now to prove that the defining ideals of $\text{FTL}_{d,n}(u)$ and $\text{CTL}_{d,n}(u)$ respectively are principal ideals too. The method used in the proof is standard and we start with two technical lemmas.

**Lemma 2.** The following hold in $Y_{d,n}(u)$ for all $i = 1, \ldots, n - 1$ and $j = 1, \ldots, n$:

1. $t_j = (g_1 \cdots g_{n-1})^{j-1} t_1 (g_1 \cdots g_{n-1})^{-(j-1)}$
2. $g_i = (g_1 \cdots g_{n-1})^{i-1} g_1 (g_1 \cdots g_{n-1})^{-(i-1)}$

**Proof.** The statement (1) is true for $j = 2$. Indeed:

\[
(g_1 \cdots g_{n-1}) \cdot t_1 (g_1 \cdots g_{n-1}) = t_2 (g_1 \cdots g_{n-1})(g_1 \cdots g_{n-1})^{-1} = t_2.
\]
Suppose that the statement is true for $j = k$. We will show that the statement holds for $j = k + 1$. We have:

\[
(g_1 \ldots g_{n-1})^k t_1 (g_1 \ldots g_{n-1})^{-k} = (g_1 \ldots g_{n-1})(g_1 \ldots g_{n-1})^{k-1} t_1 (g_1 \ldots g_{n-1})^{-(k-1)} (g_1 \ldots g_{n-1})^{-1} \\
= (g_1 \ldots g_{n-1}) t_k (g_1 \ldots g_{n-1})^{-1} \\
= g_1 \ldots g_{k-1} g_k t_{k+1} \ldots g_{n-1} (g_1 \ldots g_{n-1})^{-1} \\
= t_{k+1} (g_1 \ldots g_{n-1}) (g_1 \ldots g_{n-1})^{-1} \\
= t_{k+1}.
\]

The second statement of the Lemma has been proved in [6, (2) Lemma 5].

\[\Box\]

**Lemma 3.** The following hold in $Y_{d,n}(u)$ for all $i = 1, \ldots, n - 2$ and $0 \leq a, b, c \leq d - 1$:

1. \[t_i^{ab} t_{i+1} c^{i+2} = (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-1} (g_1 \ldots g_{n-1})^{-i} \]

2. \[t_i^{ab} t_{i+1} c^{i+2} g_i = (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-1} (g_1 \ldots g_{n-1})^{-i} \]

3. \[t_i^{ab} t_{i+1} c^{i+2} g_i + 1 = (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-1} (g_1 \ldots g_{n-1})^{-i} \]

4. \[t_i^{ab} t_{i+1} c^{i+2} g_i + 1 = (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-1} (g_1 \ldots g_{n-1})^{-i} \]

5. \[t_i^{ab} t_{i+1} c^{i+2} g_i + 1 = (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-1} (g_1 \ldots g_{n-1})^{-i} \]

6. \[t_i^{ab} t_{i+1} c^{i+2} g_i + 1 = (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-1} (g_1 \ldots g_{n-1})^{-i} \]

**Proof.** We will demonstrate the proof the statement (1) and (6). The other statements are proved in an analogous manner. For statement (1) we have from Lemma 2:

\[
t_i^{ab} t_{i+1} c^{i+2} = (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-i} \]

\[
\cdot (g_1 \ldots g_{n-1})^{i+1} t_i c^{i} (g_1 \ldots g_{n-1})^{-(i+1)} \\
= (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-i} \]

\[
\cdot (g_1 \ldots g_{n-1})^{i-1} (g_1 \ldots g_{n-1})^{2} t_i c^{i} (g_1 \ldots g_{n-1})^{-2} (g_1 \ldots g_{n-1})^{-(i-1)} \\
= (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)}. 
\]

For statement (6) we have from Lemma 2:

\[
t_i^{ab} t_{i+1} c^{i+2} g_i + 1 = (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-i} \]

\[
\cdot (g_1 \ldots g_{n-1})^{i+1} t_i c^{i} (g_1 \ldots g_{n-1})^{-(i+1)} \\
= (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-i} \]

\[
\cdot (g_1 \ldots g_{n-1})^{i-1} (g_1 \ldots g_{n-1})^{2} t_i c^{i} (g_1 \ldots g_{n-1})^{-2} (g_1 \ldots g_{n-1})^{-(i-1)} \\
= (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-i} \\
\cdot (g_1 \ldots g_{n-1})^{i-1} (g_1 \ldots g_{n-1})^{-1} (g_1 \ldots g_{n-1})^{-1} (g_1 \ldots g_{n-1})^{-i} \\
= (g_1 \ldots g_{n-1})^{i-1} a_1 b_2 c^{i+2} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{-i}. 
\]

\[\Box\]

**Theorem 4.** The defining ideal of $\text{FTL}_{d,n}(u)$ is generated by any single element $r_{i,i+1}$. 

Proof. It is enough to prove that \( r_{i,i+1} = (g_1 \ldots g_{n-1})^{(i)} r_{1,2}(g_1 \ldots g_{n-1})^{-(i-1)} \). Expanding \( r_{1,2} \), we have

\[
(g_1 \ldots g_{n-1})^{i-1} r_{1,2}(g_1 \ldots g_{n-1})^{-(i-1)} = \sum_{a+b+c=0} w \in S_3 (g_1 \ldots g_{n-1})^{i-1} t_1^a t_2^b t_3^c g_w (g_1 \ldots g_{n-1})^{-(i-1)}
\]

Applying now Lemma 3 in each factor of the summation, we arrive to

\[
(g_1 \ldots g_{n-1})^{i-1} r_{1,2}(g_1 \ldots g_{n-1})^{-(i-1)} = r_{i,i+1}
\]

Therefore the proof is concluded. □

From Theorem 4 we have the following corollaries:

**Corollary 1.** \( \text{FTL}_{d,n}(u) \) is the algebra generated by \( t_1, \ldots, t_n, g_1, \ldots, g_{n-1} \) which are subject to the defining relations of \( \text{Y}_{d,n}(u) \) and the relation:

\[
r_{1,2} = 0.
\]

**Corollary 2.** The defining ideal of \( \text{CTL}_{d,n}(u) \) is generated by any single element \( c_{i,i+1} \). Hence \( \text{CTL}_{d,n}(u) \) can be presented by \( t_1, \ldots, t_n, g_1, \ldots, g_{n-1} \) together with the defining relations of \( \text{Y}_{d,n}(u) \) and the relation:

\[
c_{1,2} = 0.
\]

Proof. As in the proof Theorem 4, it is enough to prove that \( c_{i,i+1} = \gamma c_{1,2} \gamma^{-1} \) where \( \gamma := (g_1 \ldots g_{n-1})^{i-1} \). From Eq. 4.1, we have

\[
\gamma c_{1,2} \gamma^{-1} = \left( \sum_{0 \leq k \leq d-1} \gamma t_1^k \gamma^{-1} \right) \gamma r_{1,2} \gamma^{-1}
\]

By using now (1) of Lemma 2 and Eq. 4.3, it follows that \( \gamma c_{1,2} \gamma^{-1} = c_{i,i+1} \). The rest of the statement is now clear. □

4.3. By using the analogous transformation to Eq. 2.3, we obtain presentations for \( \text{FTL}_{d,n}(u) \) and \( \text{CTL}_{d,n}(u) \) through non–invertible generators. More precisely, set

\[
\ell_i := \frac{1}{u+1}(g_i + 1).
\]

Then, by a direct computation, which is not necessary to reproduce here, we obtain the following:

**Proposition 3.** The algebra \( \text{FTL}_{d,n}(u) \) can be presented with generators:

\[
\ell_1, \ldots, \ell_{n-1}, t_1, \ldots, t_n
\]
subject to the following relations:

\[ t_i^d = 1, \quad t_it_j = t_jt_i \]
\[ \ell_it_j = t_j\ell_i, \quad \ell_it_j = t_j\ell_i, \quad \text{for } |i - j| > 1 \]
\[ \ell_i\ell_j = \ell_j\ell_i, \quad \text{for } |i - j| > 1 \]
\[ \ell_it_i = t_{i+1}\ell_i + \frac{1}{u+1}(t_i - t_{i+1}) \]
\[ \ell_it_{i+1} = t_{i+1}\ell_i + \frac{1}{u+1}(t_{i+1} - t_i) \]
\[ \ell_i^2 = \frac{(u-1)e_i + 2}{u+1} \ell_i, \]
\[ \ell_i\ell_{i+1}\ell_i - \frac{(u-1)e_i + 1}{(u+1)^2} \ell_i = \ell_{i+1}\ell_i\ell_{i+1} - \frac{(u-1)e_{i+1} + 1}{(u+1)^2} \ell_{i+1} \]
\[ e_i^{(k)}e_{i+1}\ell_i\ell_{i+1}\ell_i = \frac{u}{(u+1)^2} e_i^{(k)}e_{i+1}\ell_i \]

**Proposition 4.** The algebra CTL\(_{d,n}(u)\) can be presented with generators:

\[ \ell_1, \ldots, \ell_{n-1}, t_1, \ldots, t_n \]

subject to the following relations:

\[ t_i^d = 1, \quad t_it_j = t_jt_i \]
\[ \ell_it_j = t_j\ell_i, \quad \ell_it_j = t_j\ell_i, \quad \text{for } |i - j| > 1 \]
\[ \ell_i\ell_j = \ell_j\ell_i, \quad \text{for } |i - j| > 1 \]
\[ \ell_it_i = t_{i+1}\ell_i + \frac{1}{u+1}(t_i - t_{i+1}) \]
\[ \ell_it_{i+1} = t_{i+1}\ell_i + \frac{1}{u+1}(t_{i+1} - t_i) \]
\[ \ell_i^2 = \frac{(u-1)e_i + 2}{u+1} \ell_i, \]
\[ \ell_i\ell_{i+1}\ell_i - \frac{(u-1)e_i + 1}{(u+1)^2} \ell_i = \ell_{i+1}\ell_i\ell_{i+1} - \frac{(u-1)e_{i+1} + 1}{(u+1)^2} \ell_{i+1} \]
\[ \sum_{k=0}^{d-1} e_i^{(k)}e_{i+1}\ell_i\ell_{i+1}\ell_i = \sum_{k=0}^{d-1} e_i^{(k)}e_{i+1}\ell_i \]

**Remark 4.** We know that a linear basis of the Temperley–Lieb algebra can be constructed from the interpretation of the generators \( f_i \) as diagrams. In virtue of Remark 3, then it is desirable to construct a basis of FTL\(_{d,n}(u)\) from the presentation given in Proposition 3. Unfortunately, we do not have a diagrammatic interpretation for the generators \( l_i \) yet. Also, an explicit linear basis for both FTL\(_{d,n}(u)\) and CTL\(_{d,n}(u)\) is still an open problem. Indeed, the construction of such a basis for FTL\(_{d,n}(u)\) seems like a non–trivial task. However, in [3], by using the representation theory of Yokonuma–Hecke algebra (see [4]), Chlouveraki and Pouchin have found a formula for the dimension of FTL\(_{d,n}(u)\). More precisely, they proved:

\[ \dim\text{FTL}_{d,n}(u) = \sum_{|k_1| + |k_2| + \ldots + |k_d| = n} \left( \frac{n!}{|k_1|! \ldots |k_d|!} \right)^2 c_{|k_1| \ldots c_{|k_d|}} \]  \( (4.6) \)
where \( n = (k_1, \ldots, k_d) \) is a \( d \)-partition of \( n \), \( k_i \) is a usual partition whose Young diagram has at most two columns and \( c_k \) is the \( k \)-th Catalan number. We shall show an explicit basis for the case \( n = 3 \) and \( d = 2 \). To do that, we shall use the Corollary 1 to describe \( \text{FTL}_{d,n}(u) \). Therefore, we have that the inductive basis of \( Y_{2,3}(u) \) described in Proposition 1 spans \( \text{FTL}_{2,3}(u) \). From Eq. 4.6 it follows that the dimension of \( \text{FTL}_{2,3}(u) \) is 46. Thus, a linear basis of \( \text{FTL}_{2,3}(u) \) can be obtained by omitting 2 linearly dependent elements, from the 48 elements of the inductive basis of \( Y_{2,3}(u) \). Now Eq. 4.2, implies that \( t_m e_1 e_2 g_{1,2} = 0 \), for \( m = 0, 1 \). Therefore, it follows that the elements 1 and \( t_1 \) can be expressed, in \( \text{FTL}_{2,3}(u) \), as a linear combination of the standard basis of \( Y_{2,3}(u) \). Thus, we have a basis of \( \text{FTL}_{2,3}(u) \) obtained from the inductive basis \( \text{FTL}_{2,3}(u) \) by omitting the monomials 1 and \( t_1 \).

4.4. We finish this section with two technical lemmas which will be used in the proof of Theorems 5 and 6.

**Lemma 4.** For the element \( g_{1,2} \) we have in \( Y_{d,n}(u) \) the following:

\[
\begin{align*}
(1) \quad g_{1,1} &= [1 + (u - 1)e_1]g_{1,1} \\
(2) \quad g_{2,1} &= [1 + (u - 1)e_2]g_{1,1} \\
(3) \quad g_{1,2} &= [1 + (u - 1)e_1 + (u - 1)e_{1,3} + (u - 1)^2 e_1 e_2]g_{1,1} \\
(4) \quad g_{2,2} &= [1 + (u - 1)e_2 + (u - 1)e_{1,3} + (u - 1)^2 e_1 e_2]g_{1,1} \\
(5) \quad g_{1,2} &= [1 + (u - 1)(e_1 + e_2 + e_{1,3}) + (u - 1)^2 (u + 2) e_1 e_2]g_{1,1}.
\end{align*}
\]

**Proof.** See [6, Lemma 5]. Cf. [12, Lemma 7.5].

**Lemma 5.** For the element \( r_{1,2} \) we have in \( Y_{d,n}(u) \):

\[
\begin{align*}
(1) \quad g_{1} &= [1 + (u - 1)e_1]r_{1,2} \\
(2) \quad g_{2} &= [1 + (u - 1)e_2]r_{1,2} \\
(3) \quad g_{1,2} &= [1 + (u - 1)e_1 + (u - 1)e_{1,3} + (u - 1)^2 e_1 e_2]r_{1,2} \\
(4) \quad g_{2,2} &= [1 + (u - 1)e_2 + (u - 1)e_{1,3} + (u - 1)^2 e_1 e_2]r_{1,2} \\
(5) \quad g_{1,2} &= [1 + (u - 1)(e_1 + e_2 + e_{1,3}) + (u - 1)^2 (u + 2) e_1 e_2]r_{1,2}.
\end{align*}
\]

**Proof.** In order to prove this lemma we will make extensive use of Lemmas 4 and 1. For statement (1) we have:

\[
g_{1} = g_{1} e_1 e_2 g_{1,1} = e_1 e_3 g_{1,1} \]

\[
= e_1 e_2 [1 + (u - 1)e_1]g_{1,1}
\]

\[
= [1 + (u - 1)e_1]e_1 e_2 g_{1,1}
\]

\[
= [1 + (u - 1)e_1]r_{1,2}
\]

In an analogous way we prove statement (2). For statement (3) we have that:

\[
g_{1,2} = g_{1,2} e_1 e_2 g_{1,2} = e_2 e_3 g_{1,2} \]

\[
= e_1 e_2 [1 + (u - 1)e_1 + (u - 1)e_{1,3} + (u - 1)^2 e_1 e_2]g_{1,1}
\]

\[
= [1 + (u - 1)e_1 + (u - 1)e_{1,3} + (u - 1)^2 e_1 e_2]g_{1,1}
\]

\[
= [1 + (u - 1)e_1 + (u - 1)e_{1,3} + (u - 1)^2 e_1 e_2]r_{1,2}
\]
In an analogous way we prove statement (4). Finally, we have for statement (5):

\[
g_{12}g_1 r_{1,2} = g_{12}g_1 e_1 e_2 g_{1,2} \\
= e_1 e_2 g_{12} g_1 g_{1,2} \\
= e_1 e_2 [1 + (u - 1) (e_1 + e_2 + e_{1,3}) + (u - 1)^2 (u + 2) e_1 e_2] g_{1,2} \\
= [1 + (u - 1) (e_1 + e_2 + e_{1,3}) + (u - 1)^2 (u + 2) e_1 e_2] e_1 e_2 g_{1,2} \\
= [1 + (u - 1) (e_1 + e_2 + e_{1,3}) + (u - 1)^2 (u + 2) e_1 e_2] r_{1,2}. \\
\]

□

Lemma 6. For the element \( c_{1,2} \) we have in \( Y_{d,n}(u) \):

\[
(1) \quad g_1 c_{1,2} = [1 + (u - 1) e_1] c_{1,2} \\
(2) \quad g_2 c_{1,2} = [1 + (u - 1) e_2] c_{1,2} \\
(3) \quad g_{12} c_{1,2} = [1 + (u - 1) e_1 + (u - 1) e_{1,3} + (u - 1)^2 e_1 e_2] c_{1,2} \\
(4) \quad g_2 g_1 c_{1,2} = [1 + (u - 1) e_2 + (u - 1) e_{1,3} + (u - 1)^2 e_1 e_2] c_{1,2} \\
(5) \quad g_{12} g_1 c_{1,2} = [1 + (u - 1) (e_1 + e_2 + e_{1,3}) + (u - 1)^2 (u + 2) e_1 e_2] c_{1,2} \\
\]

Proof. The proof is completely analogous to the proof of Lemma 5. □

5. Markov traces

The main purpose of this section is to find the necessary and sufficient conditions in order that the trace \( tr \) defined on \( Y_{d,n}(u) \) \([11]\) passes to the quotient algebras \( FTL_{d,n}(u) \) and \( CTL_{d,n}(u) \). Since the defining ideal of \( FTL_{d,n}(u) \) (respectively of \( CTL_{d,n}(u) \)) is principal, by the linearity of \( tr \), we have that \( tr \) passes to \( FTL_{d,n}(u) \) (respectively to \( CTL_{d,n}(u) \)) if and only if we have:

\[
tr(m r_{1,2}) = 0 \quad (\text{respectively} \quad tr(m c_{1,2}) = 0) \tag{5.1}
\]

for all monomials \( m \) in the inductive basis of \( Y_{d,n}(u) \). So, we seek necessary and sufficient conditions for (5.1) to hold. The strategy is to find such conditions first for \( n = 3 \) and then generalise using induction.

5.1. Recall that elements in the inductive basis of \( Y_{d,3}(u) \) are of the following forms:

\[
l_{1}^{a}t_{1}^{b}t_{2}^{c}, \quad l_{1}^{a}g_{1}t_{1}^{b}t_{3}^{c}, \quad l_{1}^{a}g_{1}t_{2}^{b}g_{1}t_{1}^{c}, \quad l_{1}^{a}g_{1}t_{2}^{b}g_{1}t_{2}^{c}, \quad l_{1}^{a}g_{1}t_{1}^{b}g_{2}g_{1}t_{2}^{c}, \quad t_{1}^{a}g_{1}t_{1}^{b}g_{2}g_{1}t_{1}^{c}, \tag{5.2}
\]

where \( 0 \leq a, b, c \leq d - 1 \) (see Proposition 1). We need now to compute the trace of the elements \( m r_{1,2} \), where \( m \) runs the monomials listed in (5.2). To do these computations we will use the following lemma and proposition.

Lemma 7. For all \( 0 \leq m \leq d - 1 \), we have:

\[
tr\left(e_{1}^{(m)} e_{2} g_{1,2}\right) = (u + 1) z^{2} x_{m} + (u + 2) z E^{(m)} + tr(e_{1}^{(m)} e_{2})
\]
Proof. By direct computation we have:
\[
\text{tr}(e_1^{(m)} e_{2,g_{1,2}}) = \text{tr}(e_1^{(m)} e_{2,g_1}) + \text{tr}(e_1^{(m)} e_{2,g_2}) + \text{tr}(e_1^{(m)} e_{2,g_1g_2}) \\
+ \text{tr}(e_1^{(m)} e_{2,g_2g_1}) + \text{tr}(e_1^{(m)} e_{2,g_1g_2}) + \text{tr}(e_1^{(m)} e_2)
\]
\[
= \frac{1}{d^2} \sum_{s=0}^{d-1} (t_1^{m+s} t_2^{-s} t_3^{-k} g_1) + \frac{1}{d^2} \sum_{s=0}^{d-1} (t_1^{m+s} t_2^{-s} t_3^{-k} g_2)
\]
\[
+ \frac{1}{d^2} \sum_{s=0}^{d-1} (t_1^{m+s} t_2^{-s} t_3^{-k} g_1 g_2) + \frac{1}{d^2} \sum_{s=0}^{d-1} (t_1^{m+s} t_2^{-s} t_3^{-k} g_2 g_1)
\]
\[
+ \frac{1}{d^2} \sum_{s=0}^{d-1} (t_1^{m+s} t_2^{-s} t_3^{-k} g_1 g_2 g_1) + \text{tr}(e_1^{(m)} e_2)
\]
\[
= 2 z E^{(m)} z^2 x_m + z E^{(m)} + (u-1) z E^{(m)} + (u-1) z^2 x_m \\
= (u+1) z^2 x_m + (u+2) z E^{(m)} + \text{tr}(e_1^{(m)} e_2).
\]

\[\square\]

Proposition 5. For all 0 ≤ a, b, c ≤ d − 1, we have:

(1) If \( m = t_1^a t_2^b t_3^c \),
\[
\text{tr}(mr_{1,2}) = (u+1) z^2 x_{a+b+c} + (u+2) E^{(a+b+c)} z + \text{tr}(e_1^{(a+b+c)} e_2)
\]

(2) If \( m = t_1^a g_1 t_1^b t_3^c \) and \( m = t_1^a b^2 g_2 t_3^c \),
\[
\text{tr}(mr_{1,2}) = u \left[ (u+1) z^2 x_{a+b+c} + (u+2) E^{(a+b+c)} z + \text{tr}(e_1^{(a+b+c)} e_2) \right]
\]

(3) If \( m = t_1^a b^2 g_2 t_1^c t_3^c \) and \( m = t_1^a g_1 t_1^b t_3^c \),
\[
\text{tr}(mr_{1,2}) = u^2 \left[ (u+1) z^2 x_{a+b+c} + (u+2) E^{(a+b+c)} z + \text{tr}(e_1^{(a+b+c)} e_2) \right]
\]

(4) If \( m = t_1^a g_1 t_1^b t_2^c t_3^c \),
\[
\text{tr}(mr_{1,2}) = u^3 \left[ (u+1) z^2 x_{a+b+c} + (u+2) E^{(a+b+c)} z + \text{tr}(e_1^{(a+b+c)} e_2) \right].
\]

Proof. We will prove claim (1). According to Eq. 4.2 we have: \( mr_{1,2} = t_1^{a,b,c} e_{r_{1,2}} = t_1^{a,b,c} e_{1 e_{2,g_{1,2}}} \).

But \( t_1^{a,b,c} e_{1 e_{2,g_{1,2}}} = e_{1^{(a+b+c)} e_2} \), hence
\[
m_{r_{1,2}} = e_{1^{(a+b+c)} e_{2,g_{1,2}}}
\]

Thus, claim (1) follows by applying Lemma 7.

The proofs for the rest claims use Lemmas 5 and 7 and follow the same argument, so we finish the proof of the proposition by proving only one more representative case. We shall prove claim (3) for \( m = t_1^a g_1 t_1^b t_2^c t_3^c \). This monomial can be rewritten as \( t_1^{a,b,c} e_{r_{1,2}} g_{1,2} \). Now, by using Lemma 5 on \( g_1 g_{r_{1,2}} \), we obtain:
\[
m_{r_{1,2}} = t_1^{a,b,c} e_{r_{1,2}} g_{1,2} = t_1^{a,b,c} \left[ 1 + (u-1) e_1 + (u-1) e_1 e_2 + (u-1)^2 e_1 e_2 \right] r_{1,2}
\]
then using now Eq. 4.2 and the fact the \( e_i \)'s are idempotent, follows that:
\[
m_{r_{1,2}} = t_1^{a,b,c} \left[ e_1 e_2 + (u-1) e_1 e_2 + (u-1) e_1 e_2 + (u-1)^2 e_1 e_2 \right] g_{1,2}
\]
\[
= u^2 t_1^{a,b,c} e_{1 e_{2,g_{1,2}}}
\]
Then, applying Eq. 2.17 we have:
\[
m_{r,1,2} = u^2 t_1 t_2 t_3 e_1 e_2 g_{1,2} = u^2 e_1^{(a+b+c)} e_2 g_{1,2}.
\]
Therefore, by using Lemma 7, we obtain the desired expression for \( \text{tr}(m_{r,1,2}) \).

The following lemma is our main result for \( n = 3 \).

**Lemma 8.** The trace \( \text{tr} \) passes to \( \text{FTL}_{d,3} \) if and only if the parameters of the trace \( \text{tr} \) satisfy:

\[
x_k = -z \left( \sum_{m \in \text{Sup}_1} \chi(km) + (u + 1) \sum_{m \in \text{Sup}_2} \chi(km) \right)
\]
and
\[
z = -\frac{1}{|\text{Sup}_1| + (u + 1)|\text{Sup}_2|},
\]
where \( \text{Sup}_1 \cup \text{Sup}_2 \) (disjoint union) is the support of the Fourier transform of \( x \), and \( x \) is the complex function on \( \mathbb{Z}/d\mathbb{Z} \), that maps 0 to 1 and \( k \) to the trace parameter \( x_k \) (cf. Subsection 3.2).

**Proof.** Recall that the trace \( \text{tr} \) passes to \( \text{FTL}_{d,3} \) if and only if the Eqs. 5.1 hold, for all \( m \) in the inductive basis of \( Y_{d,3} \). By using Proposition 5 follows that the trace \( \text{tr} \) passes to the quotient algebra \( \text{FTL}_{d,n}(u) \) if and only if the trace parameters \( x_1, \ldots, x_{d-1} \) satisfy the following system of equations:

\[
E_0 = E_1 = \cdots = E_{d-1} = 0
\]
where
\[
E_m := (u + 1)z^2 x_m + (u + 2)E^{(m)} z + \text{tr}(e_1^{(m)} e_2) = 0, \quad 0 \leq m \leq d - 1
\]
We note now that this system of equations above is equivalent to the system:

\[
\begin{align*}
E_0 &= 0 \\
E_m - x_mE_0 &= 0 & \text{where} & & 1 \leq m \leq d - 1
\end{align*}
\]
(5.3)

We will solve this system of equations, obtaining thus the proof of the lemma.

Recall that \( x_0 := 1 \), \( E^{(0)} = E \) and \( e_1^{(0)} = e_1 \), hence \( E_0 = (u + 1)z^2 + (u + 2)Ez + \text{tr}(e_1 e_2) \). Then the \((d - 1)\) equations \( E_m - x_mE_0 = 0 \) of Eq. 5.3 become:

\[
z(u + 2)(E^{(m)} - x_mE) = -\left( \text{tr}(e_1^{(m)} e_2) - x_m \text{tr}(e_1 e_2) \right), \quad 1 \leq m \leq d - 1.
\]
(5.4)

Interpreting now the above equation in the functional notation of Section 3 and having in mind Eq. 3.3, it follows that Eq. 5.4 can be rewritten as:

\[
(u + 2)z \left( \frac{1}{d} x * x - Ex \right) = -\left( \frac{1}{d^2} x * x * x - \text{tr}(e_1 e_2) x \right).
\]

Applying now the Fourier transform on the above functional equality and using Proposition 2, we obtain:

\[
(u + 2)z \left( \frac{x^2}{d} - E\tilde{x} \right) = -\left( \frac{x^3}{d^2} - \text{tr}(e_1 e_2)\tilde{x} \right)
\]
(5.5)

Let now \( \tilde{x} = \sum_{m=0}^{d-1} y_m t_m \). Then Eq. 5.5 becomes:

\[
(u + 2)z \left( \frac{y_m^2}{d} - Ey_m \right) = -\left( \frac{y_m^3}{d^2} - \text{tr}(e_1 e_2)y_m \right)
\]

Hence

\[
y_m \left( \frac{y_m^2}{d^2} + (u + 2)z \frac{y_m}{d} - (u + 2)zE - \text{tr}(e_1 e_2) \right) = 0
\]
(5.6)
Now, from equation $E_0 = 0$, we have that $-(u + 2)zE = (u + 1)z^2 + \text{tr}(e_1e_2)$. Replacing this expression of $-(u + 2)zE$ in Eq. 5.6 we have that:

$$y_m \left( \frac{y_m^2}{d^2} + (u + 2)z \frac{y_m}{d} + (u + 1)z^2 \right) = 0$$

or equivalently:

$$y_m (y_m + dz)(y_m + dz(u + 1)) = 0 \quad (5.7)$$

Denote $\text{Sup}_1 \cup \text{Sup}_2$ the support of $\hat{x}$, where

$\text{Sup}_1 := \{m \in \mathbb{Z}/d\mathbb{Z}; \; y_m = -dz\}$ and $\text{Sup}_2 := \{m \in \mathbb{Z}/d\mathbb{Z}; \; y_m = -dz(u + 1)\}$

hence

$$\hat{x} = \sum_{m \in \text{Sup}_1} -dz t^m + \sum_{m \in \text{Sup}_2} -dz(u + 1)t^m$$

Then

$$\hat{x} = -dz \sum_{m \in \text{Sup}_1} \hat{\delta}_m - dz(u + 1) \sum_{m \in \text{Sup}_2} \hat{\delta}_m$$

thus from argument (4) of Proposition 2 we have:

$$\hat{x} = -z \left( \sum_{m \in \text{Sup}_1} i_{-m} + (u + 1) \sum_{m \in \text{Sup}_2} i_{-m} \right)$$

Therefore, having in mind now (5) of Proposition 2, we deduce that:

$$x_k = -z \left( \sum_{m \in \text{Sup}_1} \chi(km) + (u + 1) \sum_{m \in \text{Sup}_2} \chi(km) \right) \quad (5.8)$$

Having in mind that $x_0 = 1$, one can determine the values of $z$. Indeed, from Eq. 5.8, we have that:

$$1 = x_0 = -z(|\text{Sup}_1| + (u + 1)|\text{Sup}_2|)$$

or equivalently:

$$z = -\frac{1}{|\text{Sup}_1| + (u + 1)|\text{Sup}_2|}. \quad (5.9)$$

Keeping the same notation with the above lemma, we have:

**Theorem 5.** The trace $\text{tr}$ defined on $Y_{d,n}(u)$ passes to the quotient algebra $\text{FTL}_{d,n}(u)$ if and only if the trace parameters $z, x_1, \ldots, x_{d-1}$ satisfy the conditions of Lemma 8, i.e. Eqs. 5.8 and 5.9.

**Proof.** The proof is by induction on $n$. The case $n = 3$ is the lemma above. Assume now that the statement holds for all $\text{FTL}_{d,k}(u)$, where $k \leq n$, that is:

$$\text{tr}(a_k r_{1,2}) = 0$$

for all $a_k \in Y_{d,k}(u)$, $k \leq n$. We will show the statement for $k = n + 1$. It suffices to prove that the trace vanishes on any element of the form $a_{n+1}r_{1,2}$, where $a_{n+1}$ belongs to the inductive basis of $Y_{d,n+1}(u)$ (recall Eq. 1), given the conditions of the theorem. Namely:

$$\text{tr}(a_{n+1} r_{1,2}) = 0.$$
Since \( a_{n+1} \) is in the inductive basis of \( Y_{d,n+1}(u) \), it is of one of the following forms:

\[
a_{n+1} = a_n g_n \ldots g_i t_i^k \quad \text{or} \quad a_{n+1} = a_{n+1}^k
\]

where \( a_n \) is in the inductive basis of \( Y_{d,n}(u) \). For the first case we have:

\[
\text{tr}(a_{n+1} r_{1,2}) = \text{tr}(a_n g_n \ldots g_i t_i^k r_{1,2}) = z \text{tr}(a_n g_n^{-1} \ldots g_i t_i^k r_{1,2}) = z \text{tr}(a r_{1,2}),
\]

where \( a := a_n g_n^{-1} \ldots g_i t_i^k \). Notice now that \( a \) is a word in \( Y_{d,n}(u) \) and so, by the linearity of the trace, we have that \( \text{tr}(a r_{1,2}) \) is a linear combination of traces of the form \( \text{tr}(a_n r_{1,2}) \), where \( a_n \) is in the inductive basis of \( Y_{d,n}(u) \). Therefore, by the induction hypothesis, we deduce that:

\[
\text{tr}(a r_{1,2}) = 0
\]

if and only if the conditions of the Theorem are satisfied. Therefore the statement is proved. The second case is proved similarly. Hence, the proof is concluded.

**Corollary 3.** In the case where one of the sets \( \text{Sup}_1 \) or \( \text{Sup}_2 \) is the empty set, the values of the \( x_k \)'s are solutions of the \( E \)-system. More precisely, if \( \text{Sup}_1 \) is the empty set, the \( x_k \)'s are the solutions of the \( E \)-system parametrized by \( \text{Sup}_2 \) and \( z = -1/(u+1)|\text{Sup}_2| \). If \( \text{Sup}_2 \) is the empty set, then \( x_k \)'s are the solutions of the \( E \)-system parametrized by \( \text{Sup}_1 \) and \( z = -1/|\text{Sup}_1| \).

**Proof.** The proof follows from Eq. 3.5 and the expression given in theorem above for the \( x_k \)'s. \( \square \)

5.2. The method of finding the necessary and sufficient conditions for \( \text{tr} \) to pass to the quotient algebra \( \text{CTL}_{d,n}(u) \) is completely analogous to that of the previous subsection. Thus, we will need the following analogue of Proposition 5.

**Proposition 6.** Define \( G \), as follows:

\[
G = (u + 1)z^2 \sum_{k=0}^{d-1} x_k + (u + 2)z \sum_{k=0}^{d-1} E^{(k)} + \sum_{k=0}^{d-1} \text{tr}(e_1^{(k)} e_2)
\]

Then for all \( 0 \leq a, b, c \leq d - 1 \), we have:

1. \( \text{tr}(mc_{1,2}) = G \) for \( m = t_1^a t_2^b t_3^c \)
2. \( \text{tr}(mc_{1,2}) = uG \) for \( m = t_1^a g_1 t_2^b t_3^c \) and \( m = t_1^a b_2 g_2 t_3^c \),
3. \( \text{tr}(mc_{1,2}) = u^2 G \) for \( m = t_1^a b_2 g_2 t_1^c \) and \( m = t_1^a g_1 t_2^b t_2^c \),
4. \( \text{tr}(mc_{1,2}) = u^3 G \) for \( m = t_1^a g_1 t_1^b g_2 t_3^c \).

Following now the analogous reasoning that was used to prove Theorem 5 and having in mind Eq. 5.1, Corollary 2, Lemma 6 and Proposition 6, we obtain the following theorem.

**Theorem 6.** The trace \( \text{tr} \) passes to the quotient if and only if the parameter \( z \) and the \( x_i \)'s are related through the equation:

\[
(u + 1)z^2 \sum_{k \in \mathbb{Z}/d \mathbb{Z}} x_k + (u + 2)z \sum_{k \in \mathbb{Z}/d \mathbb{Z}} E^{(k)} + \sum_{k \in \mathbb{Z}} \text{tr}(e_1^{(k)} e_2) = 0. \quad (5.10)
\]

6. COMPARISON OF THE TRACE CONDITIONS

In this section we will compare the conditions that need to be applied to the trace parameters \( z \) and \( x_i, i = 1, \ldots, d - 1 \) so that \( \text{tr} \) passes to each of the quotient algebras.

In [6] we found the necessary and sufficient conditions so that \( \text{tr} \) passes to \( \text{YTL}_{d,n}(u) \). Indeed, we have the following:

**Theorem 7.** The trace \( \text{tr} \) passes to the quotient \( \text{YTL}_{d,n}(u) \) if and only if the \( x_i \)'s are solutions of the \( E \)-system and one of the two cases holds:
(i) For some $0 \leq m_1 \leq d - 1$ the $x_\ell$'s are $d^{th}$ roots of unity and $z = -\frac{1}{u+1}$ or $z = -1$.

(ii) For some $0 \leq m_1, m_2 \leq d - 1$, $m_1 \neq m_2$, the $x_\ell$'s are expressed as:

$$x_\ell = \frac{1}{2}(\exp(m_1 \ell) + \exp(m_2 \ell)) \quad (0 \leq \ell \leq d - 1).$$

In this case we have $z = -\frac{1}{2}$.

The conditions for the $x_i$'s in this case are particular solutions of the E–system. Thus, the conditions such that tr passes to $YTL_{d,n}(u)$ are contained in those of Theorem 5.

Moreover, Theorem 5 can be rephrased in the following way:

**Theorem 8.** The trace tr passes to the quotient algebra $FTL_{d,n}(u)$ if and only if the parameter $z$ and the $x_i$'s are related through the equation:

$$(u + 1)z^2 x_k + (u + 2)z E^{(k)} + \text{tr}(e_1^{(k)} e_2) = 0, \quad \forall k \in \mathbb{Z}/d\mathbb{Z}$$

This implies that the conditions such that the trace passes to the quotient algebra $YTL_{d,n}(u)$ are contained in those of Theorem 6. All of the above can be summarised in the following table:

| $Y_{d,n}(u)$ | $CTL_{d,n}(u)$ | $FTL_{d,n}(u)$ | $YTL_{d,n}(u)$ |
|-------------|----------------|----------------|----------------|
| free        | free           | $\leftarrow$ Theorem 6 | $\leftarrow$ Theorem 8 |
| free        | free           | $\leftarrow$ Theorem 7 |

**Table 1.** Relations of algebras and trace conditions.

The first row includes the projections between the algebras while the second shows the inclusions of the trace conditions for each case.

### 7. Knot invariants

#### 7.1. The 2–variable Jones or Homflypt polynomial, $P(\lambda, u)$, can be defined through the Ocneanu trace $\tau$ on $H_n(u)$ [9]. Indeed, for any braid $\alpha \in \cup_{\infty} B_n$, we have:

$$P(\lambda, u)(\hat{\alpha}) = \left( -\frac{1 - \lambda u}{\sqrt{\lambda}(1 - u)} \right)^{n-1} (\sqrt{\lambda})^{\varepsilon(\alpha)} \tau(\pi(\alpha)),$$

where: $\lambda = \frac{1 - u + \zeta}{u \zeta}$, $\pi$ is the natural epimorphism of $\mathbb{C}B_n$ onto $H_n(u)$ that sends the braid generator $\sigma_i$ to $h_i$ and $\varepsilon(\alpha)$ is the algebraic sum of the exponents of the $\sigma_i$'s in $\alpha$. Further, the Jones polynomial, $V(u)$, related to the algebras $TL_n(u)$, can be redefined through the Homflypt polynomial by specializing $\zeta$ to $-\frac{1}{u+1}$ [9]. This is the non–trivial value for which the Ocneanu trace $\tau$ passes to the quotient algebra $TL_n(u)$. Namely:

$$V(q)(\hat{\alpha}) = \left( -\frac{1 + u}{\sqrt{u}} \right)^{n-1} (\sqrt{u})^{\varepsilon(\alpha)} \tau(\pi(\alpha)) = P(u, u)(\hat{\alpha}).$$

#### 7.2. In [16] it is proved that the trace $\text{tr}$ defined on $Y_{d,n}(u)$ can be re–scaled according to the braid equivalence corresponding to isotopic framed links if and only if the $x_i$’s furnish a solution of the E–system (recall discussion in Section 3). Let $X_D = (x_1, \ldots, x_{d-1})$ be a solution of the E–system parametrized by the non–empty set $D$ of $\mathbb{Z}/d\mathbb{Z}$. We have the following definition:

**Definition 6** (Definition 3 [1]). The trace map $\text{tr}_D$ defined as the trace $\text{tr}$ with the parameters $x_i$ specialized to the values $x_i$, shall be called the *specialized trace* with parameter $z$. 

By normalizing \( \text{tr}_D \), an invariant for framed links can be obtained [16]:

\[
\Gamma_D(w, u)(\hat{\alpha}) = \left( \frac{(1 - wu)|D|}{\sqrt{w(1 - u)}} \right)^{n-1} (\sqrt{w})^{\epsilon(\alpha)} \text{tr}_D(\gamma(\alpha)),
\]

where: \( D \) is a non–empty subset of \( \mathbb{Z}/d\mathbb{Z} \) which parametrizes a solution of the \( E \)-system, \( w = \frac{z + (1 - u)E}{uz} \), \( E = \frac{1}{|D|} \), \( \gamma \) the natural endomorphism of the framed braid group algebra \( \mathbb{C}F_n \) onto the algebra \( Y_{d,n}(u) \), and \( \alpha \in \cup_{\infty}F_n \).

Furthermore, in [15] Juyumaya and Lambropoulou represented the classical braid group algebra \( Y_{d,n}(u) \) by regarding the framing generators \( t_i \) as formal elements. This is equivalent to restricting \( \Gamma_D \) to classical links, seen as framed links with all framings zero. This gives rise to an invariant of classical oriented links, denoted \( \Delta_D(\hat{\alpha}) \), where \( \alpha \in \cup_{\infty}B_n \). Namely:

\[
\Delta_D(w, u)(\hat{\alpha}) = \left( \frac{(1 - wu)|D|}{\sqrt{w(1 - u)}} \right)^{n-1} (\sqrt{w})^{\epsilon(\alpha)} \text{tr}_D(\delta(\alpha)),
\]

where: \( D, w, E \) as above, \( \delta \) the natural homomorphism of the classical braid group algebra \( \mathbb{C}B_n \) to the algebra \( Y_{d,n}(u) \) and \( \alpha \in \cup_{\infty}B_n \). Further, in [14] the invariant \( \Delta_D(w, u) \) was extended to an invariant for singular links.

Note that for \( d = 1 \) the traces \( \text{tr} \) and \( \text{tr}_D \) coincide with the Ocneanu trace \( \tau \), so the invariants \( \Gamma_D(u, w) \) and \( \Delta_D(u, w) \) coincide with the Homflypt polynomial. Moreover, in [1] it is shown that for generic values of the parameters \( u, z \) the invariants \( \Delta_D(w, u) \) do not coincide with the Homflypt polynomial except in the trivial cases \( u = 1 \) and \( E = 1 \). Yet, computational data [5] indicate that these invariants may be topologically equivalent to the Homflypt polynomial.

In [6] the invariants that are defined through the Yokonuma–Temperley–Lieb were studied. More precisely, it was shown that in order that the trace \( \text{tr} \) passes to the quotient algebra \( YTL_{d,n}(u) \) it is necessary that the \( x_i \)'s are \( d^{th} \) roots of unity. These furnish a (trivial) solution of the \( E \)-system and in this case \( E = 1 \). So, by [1], the invariants we obtain from \( YTL_{d,n}(u) \) coincide with the Jones polynomial. This is the main reason that the algebras \( YTL_{d,n}(u) \) do not qualify for being the framization of the Temperley–Lieb algebra.

In this section we will define the invariants for framed and classical links that can be obtained from the algebras \( YTL_{d,n}(u) \) and \( CTL_{d,n}(u) \).

### 7.3. Invariants from \( YTL_{d,n}(u) \)

As it has already been stated, the trace parameters \( x_i \) should be solutions of the \( E \)-system so that a link invariant through \( \text{tr} \) is well–defined. Moreover, the conditions of Theorem 5 include these solutions for the \( x_i \)'s. In order to define a link invariant on the level of the quotient algebra \( YTL_{d,n}(u) \), we discard any value of the \( x_i \)'s that does not comprise a solution of the \( E \)-system. Using Corollary 3 we choose a solution of the \( E \)-system and denote with \( D \) the subset of \( \mathbb{Z}/d\mathbb{Z} \) that parametrizes the said solution. This leads to the following values for \( z \):

\[
z = -\frac{1}{(u + 1)|D|} \quad \text{or} \quad z = -\frac{1}{|D|}.
\]

Further, we do not take into consideration the case where \( z = -\frac{1}{|D|} \), since important topological information is lost. Indeed, the trace \( \text{tr} \) gives the same value for all even (resp. odd) powers of the \( g_i \)’s, for \( m \in \mathbb{Z}^{>0} \) [16]:

\[
\text{tr}(g^m_i) = \left( \frac{u^m - 1}{u + 1} \right) z + \left( \frac{u^m - 1}{u + 1} \right) \frac{1}{|D|} + 1 \quad \text{if } m \text{ is even}
\]

and

\[
\text{tr}(g^m_i) = \left( \frac{u^m - 1}{u + 1} \right) z + \left( \frac{u^m - 1}{u + 1} \right) \left| \frac{1}{|D|} - \frac{1}{|D|} \right| \quad \text{if } m \text{ is odd}.
\]
From the remaining case where the $x_i$'s are solutions of the E–system and $z = -\frac{1}{(u+1)|D|}$ we deduce that $w = u$ in Eq. 7.1. We then have the following definition:

**Definition 7.** Let $X_D$ be a solution of the E–system, parametrized by the non–empty subset $D$ of $\mathbb{Z}/d\mathbb{Z}$ and let $z = -\frac{1}{(u+1)|D|}$. We obtain from $\Gamma_D(w, u)$ the following invariant for $\alpha \in \cup_\infty \mathcal{F}_n$:

\[
(i) \quad \theta_D(u)(\tilde{\alpha}) = \left(-\frac{(1+u)|D|}{\sqrt{u}}\right)^{n-1} (\sqrt{u})^\epsilon(\alpha) \text{tr}_D (\gamma(\alpha)) = \Gamma_D(u, u)(\tilde{\alpha}),
\]

Further, from $\Delta_D$, we obtain the following invariant for $\alpha \in \cup_\infty \mathcal{B}_n$:

\[
(ii) \quad \theta_D(u)(\tilde{\alpha}) = \left(-\frac{(1+u)|D|}{\sqrt{u}}\right)^{n-1} (\sqrt{u})^\epsilon(\alpha) \text{tr}_D (\delta(\alpha)) = \Delta_D(u, u)(\tilde{\alpha}),
\]

The invariants $\theta_D(u)$ are analogues of the Jones polynomial in the framed category.

**Remark 5.** If the invariants $\Delta_D(w, u)$ on the level of the Yokonuma–Hecke algebras turn out to be topologically equivalent to the Homflypt polynomial [5] then the invariants $\theta_D(u)$ will be topologically equivalent to the Jones polynomial, and the invariants $\theta_D(u)$ framed analogues of the Jones polynomial.

7.4. **Invariants from $\text{CTL}_{d,n}(u)$.** The conditions of Theorem 6 do not involve the solutions of the E–system at all, so in order to obtain a well–defined link invariant on the level of $\text{CTL}_{d,n}(u)$ we must impose this condition on the $x_i$’s. Recall that the solutions of the E–system can be expressed in the form:

\[
x_D = \frac{1}{|D|} \sum_{k \in D} i_k \in \mathbb{C}C_d
\]

where $i_k = \sum_{j=0}^{d-1} \chi_k(j)t^j$, $\chi_k$ is the character that sends $m \mapsto \cos \frac{2\pi km}{d} + i \sin \frac{2\pi km}{d}$ and $D$ is the subset of $\mathbb{Z}/d\mathbb{Z}$ that parametrizes a solution of the E–system. Let now $\varepsilon$ be the augmentation function of the group algebra $\mathbb{C}C_d$, sending $\sum_{j=0}^{d-1} x_j t^j$ to $\sum_{j=0}^{d-1} x_j$. We have that:

\[
\varepsilon(x_D) = \frac{1}{|D|} \sum_{k \in D} \varepsilon(i_k) = \frac{1}{|D|} \sum_{j=0}^{d-1} \sum_{k \in D} \chi_j(k) = \left\{ \begin{array}{ll} \frac{d}{|D|} & \text{if } 0 \in D \\ 0 & \text{if } 0 \notin D \end{array} \right. \quad (7.5)
\]

From this we deduce that:

\[
\sum_{j=0}^{d-1} E^{(j)} = \varepsilon \left( \frac{x \ast x}{d} \right) = \frac{1}{d|D|^2} \sum_{k \in D} \varepsilon(i_k \ast i_k) = \frac{1}{|D|^2} \sum_{k \in D} \varepsilon(i_k) = \left\{ \begin{array}{ll} \frac{d}{|D|^2} & \text{if } 0 \in D \\ 0 & \text{if } 0 \notin D \end{array} \right. \quad (7.6)
\]

and also that:

\[
\sum_{j=0}^{d-1} \text{tr}(e_1^{(j)} e_2) = \varepsilon \left( \frac{x \ast x \ast x}{d^2} \right) = \frac{1}{d^2|D|^3} \sum_{k \in D} \varepsilon(i_k \ast i_k \ast i_k) = \frac{1}{|D|^3} \sum_{k \in D} \varepsilon(i_k) = \left\{ \begin{array}{ll} \frac{d}{|D|^3} & \text{if } 0 \in D \\ 0 & \text{if } 0 \notin D \end{array} \right. \quad (7.7)
\]

Using now Eqs. 7.5, 7.6 and 7.7, Eq. 5.10 becomes for the case where $0 \in D$:

\[
\frac{d}{|D|} \left( (u+1)x^2 + \frac{(u+2)}{|D|} x + \frac{1}{|D|^2} \right) = 0.
\]

Therefore, the trace $\text{tr}$ passes to the quotient for the following values of $z$:

\[
z = -\frac{1}{(u+1)|D|} \quad \text{or} \quad z = -\frac{1}{|D|}.
\]
Much like the case of FTL_{d,n}(u), the value $z = -\frac{1}{u^{|D|}}$ is not taken into consideration, since from Eq. 5.10 we deduce that $E = \frac{1}{|D|}$ and therefore from Eqs. 7.3 and 7.4 the trace $\text{tr}$ gives the same value for all even (resp. odd) powers of the $g_i$'s.

We have the following definition:

**Definition 8.** Let $X_D$ be a solution of the E-system, parametrized by the non-empty subset $D$ of $\mathbb{Z}/d\mathbb{Z}$ and let $z = -\frac{1}{(u+1)|D|}$. We obtain from $\Gamma_D(w, u)$ the following invariant for $\alpha \in \cup_{\infty} F_n$:

\[(i) \quad \mathcal{W}_D(u)(\hat{\alpha}) = \left(\frac{(1+u)|D|}{\sqrt{n}}\right)^{n-1} (\sqrt{u})^{\varepsilon(\alpha)} \text{tr}_D(\gamma(\alpha)) = \Gamma_D(u, u)(\hat{\alpha}),\]

Further, from $\Delta_D$, we obtain the following invariant for $\alpha \in \cup_{\infty} B_n$:

\[(ii) \quad W_D(u)(\hat{\alpha}) = \left(\frac{(1+u)|D|}{\sqrt{n}}\right)^{n-1} (\sqrt{u})^{\varepsilon(\alpha)} \text{tr}_D(\delta(\alpha)) = \Delta_D(u, u)(\hat{\alpha}),\]

**Remark 6.** It should be clear that the invariants $\mathcal{W}_D$ and $W_D$ that are obtained from $\text{tr}$ on the level of the quotient algebra CTL_{d,n}(u) coincide with the invariants $\vartheta_D$ and $\theta_D$ on the level of FTL_{d,n}(u). More precisely, the conditions that are applied to the trace parameters are the same for both of the quotient algebras and, consequently, so are the related invariants.

Furthermore, the solutions of the E-system (which are the necessary and sufficient conditions so that topological invariants for framed links can be defined) are included in the conditions of Theorem 5, while for the case of CTL_{d,n}(u) we still have to impose them. These are the main reasons that lead us to consider the quotient algebra FTL_{d,n}(u) as the most natural non-trivial analogue of the Temperley–Lieb algebra in the context of framed links.

The following table give a full overview of the invariants for each quotient algebra:

| $d, D$ | $\mathcal{F}_{d,n}$ | $B_n$ | $u$ | $w$ |
|--------|----------------|-------|-----|-----|
| V_{d,n}(u) | $\Gamma_D$ | $\Delta_D$ | $u$ | $w$ |
| YTL_{d,n}(u) | $\mathcal{V}_D$ | $V_D$ | $u$ | $u$ |
| FTL_{d,n}(u) | $\vartheta_D$ | $\theta_D$ | $u$ | $u$ |
| CTL_{d,n}(u) | $\vartheta_D$ | $\theta_D$ | $u$ | $u$ |

| $d, |D| = 1$ | $\mathcal{F}_{d,n}$ | $B_n$ | $u$ | $w$ |
|--------|----------------|-------|-----|-----|
| V_{d,n}(u) | $\Gamma_D$ | $P$ | $u$ | $\lambda$ |
| YTL_{d,n}(u) | $\mathcal{V}_D$ | $V$ | $u$ | $u$ |
| FTL_{d,n}(u) | $\vartheta_D$ | $\theta_D$ | $u$ | $u$ |
| CTL_{d,n}(u) | $\vartheta_D$ | $\theta_D$ | $u$ | $u$ |

| $d, |D| > 1$ | $\mathcal{F}_{d,n}$ | $B_n$ | $u$ | $w$ |
|--------|----------------|-------|-----|-----|
| V_{d,n}(u) | $\Gamma_D$ | $P$ | $u$ | $\lambda$ |
| YTL_{d,n}(u) | $\mathcal{V}_D$ | $V$ | $u$ | $u$ |
| FTL_{d,n}(u) | $\vartheta_D$ | $\theta_D$ | $u$ | $u$ |
| CTL_{d,n}(u) | $\vartheta_D$ | $\theta_D$ | $u$ | $u$ |

**Table 2.** Overview of the invariants for each algebra.

7.5. **Concluding note.** The knot invariants from the algebras FTL_{d,n}(u) and CTL_{d,n}(u) still remain under investigation. If the invariants from the Yokonuma–Hecke algebras prove to be topologically equivalent to the Homflypt polynomial, then the invariants from FTL_{d,n}(u) and CTL_{d,n}(u) will be topologically equivalent to the Jones polynomial. If not, it would be then meaninglessfull to consider the corresponding 3–manifold invariants (as obtained from work of Wenzl [24]). In the case of the algebras YTL_{d,n}(u) the Witten invariants of 3–manifolds can be recovered, since the related knot invariants recover the Jones polynomial [6].
REFERENCES

[1] M. Chlouveraki, S. Lambropoulou, The Yokonuma–Hecke algebras and the Homflypt polynomial, J. Knot Theory and Its Ramifications 22, No. 14 (2013) 1350080 (35 pages), DOI: 10.1142/S0218216513500806. See also arXiv:1204.1871 [math.GT].
[2] M. Chlouveraki, G. Pouchin, Determination of the representations and a basis for the Yokonuma–Temperley–Lieb algebra, to appear in Algebras and Representation Theory. See also arXiv:1311.5626 [math.RT].
[3] M. Chlouveraki, G. Pouchin, Determination of the representations and a basis for the framization of the Temperley–Lieb algebra, work in progress (private communication).
[4] M. Chlouveraki, L. Poulain D’Andecy, Representation theory of the Yokonuma–Hecke algebra, Advances in Mathematics 259 (2014), 134–172.
[5] S. Chmutov, S. Jablan, J. Juyumaya, K. Karvounis, S. Lambropoulou, On the knot invariants from the Yokonuma–Hecke algebras, in preparation. See http://www.math.ntua.gr/~sofia/yokonuma/index.html.
[6] D. Goundaroulis, J. Juyumaya, A. Kontogeorgis, S. Lambropoulou, The Yokonuma–Temperley–Lieb algebras, arXiv:1012.1557v3 [math.GT].
[7] P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millett and A. Ocneanu, A new polynomial invariant of knots and links, Bull AMS 12 (1985), 183-312.
[8] V. F.R. Jones, Index for subfactors, Invent Math 72 (1983), 1–25.
[9] V. F.R. Jones, Hecke algebra representations of braid groups and link polynomials, Ann Math 126, 335–388 (1987).
[10] J. Juyumaya, Sur les nouveaux générateurs de l’algèbre de Hecke H(G, U, 1), J Algebra 204, 40–68 (1998).
[11] J. Juyumaya, Markov trace on the Yokonuma–Hecke algebra, J Knot Theo Ramif 13, 25–39 (2004).
[12] J. Juyumaya, A partition Temperley–Lieb algebra, arXiv:1304.5158 [math.QA].
[13] J. Juyumaya, S. Lambropoulou, p–adic framed braids, Topol Appl 154 (2007) 1804–1826.
[14] J. Juyumaya, S. Lambropoulou, An invariant for singular knots, J Knot Theo Ramif, 18, No. 6 (2009) 825–840.
[15] J. Juyumaya, S. Lambropoulou, An adelic extension of the Jones polynomial, The mathematics of knots, M. Banagl, D. Vogel (eds.), Contributions in the Mathematical and Computational Sciences, Vol. 1, Springer (2011).
[16] J. Juyumaya, S. Lambropoulou, p–adic framed braids II, Adv Math, 234 (2013), 149–191.
[17] J. Juyumaya, S. Lambropoulou, Modular framization of the BMW algebra, arXiv:1007.0092v1 [math.GT].
[18] J. Juyumaya, S. Lambropoulou, On the framization of knot algebras, to appear in New Ideas in Low–dimensional Topology, L. Kauffman, V. Manturov (eds.), Series on Knots and everything, World Scientific.
[19] M. Kapranov, A. Smirnov Cohomology determinants and reciprocity laws: Number field case, unpublished preprint (1996).
[20] H. Matsumoto, Générateurs et relations des groupes de Weyl généralisés, C. R. Acad. Sci. Paris 258, 3419–3422 (1964).
[21] S. Ramanujan, On Certain Trigonometric Sums and their Applications in the Theory of Numbers, Transactions of the Cambridge Philosophical Society, 22, No 13 (1918), 297–276.
[22] A. Terras, Fourier Analysis of Finite Groups and Applications, London Math. Soc. student text 43, 1999.
[23] H.N.V. Temperley, E. H. Lieb, Relations between the ‘percolation’ and ‘coulloring’ problem and other graph-theoretical problem associated with regular planar lattice: some exact results for the ‘percolations problems’, Proc. Roy. Soc. London Ser. A 322 (1971), 251–280.
[24] H. Wenzl, Braids and invariants of 3–manifolds, Invent math, 114, 235–275 (1993).
[25] T. Yokonuma, Sur la structure des anneaux de Hecke d’un group de Chevalley fin, C.R. Acad. Sc. Paris, 264 (1967), 344–347.
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