BSDEs driven by $G$-Brownian motion with uniformly continuous generators

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Abstract

The present paper is devoted to investigating the existence and uniqueness of solutions to a class of non-Lipschitz scalar valued backward stochastic differential equations driven by $G$-Brownian motion ($G$-BSDEs). In fact, when the generators are Lipschitz continuous in $y$ and uniformly continuous in $z$, we construct the unique solution to such equations by monotone convergence argument. The comparison theorem and related Feynman-Kac formula are stated as well.

Keywords: $G$-Brownian motion, BSDE, uniformly continuous generators.

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1 Introduction

Given a Wiener space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, P_0)$, under which the canonical process $W_t$ constitutes a Brownian motion. A typical nonlinear Lipschitz backward stochastic differential equations (BSDEs), which is formulated in Pardoux and Peng [18], takes the form,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$ 

The authors found a unique pair of adapted processes $(Y_s, Z_s)_{0\leq s\leq T}$ that satisfy the above equation for given squarely integrable terminal value $\xi$ and Lipschitz generator $f$.

From then on, extensive efforts have been made towards relaxing the Lipschitz assumptions on the generator $f$. To mention just a few, for the scalar case, i.e., when $Y$ is 1-dimensional, Lepeltier and Martin [16] confirmed the existence of solutions to BSDEs with continuous generator that is of linear growth. Kobylanski [14] developed the existence for BSDEs with continuous generator that has a quadratic growth in $z$ when the terminal value $\xi$ is bounded. Also for the quadratic cases, Briand and Hu [1, 2] successively obtained the existence of solution for unbounded $\xi$. More imaginative

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works on generalizing the classical BSDEs theory from different points of view are emerging, and it would be too ambitious for us to give an overview of all variants. In this paper, we focus on the study of BSDEs under $G$-expectation framework.

The $G$-expectation theory was put forth by Peng [19, 20, 21], which provides a unified tool for stochastic analysis problems that involve non-dominated family of probability measures. In particular, the $G$-Brownian motion process is constructed with uncertain quadratic variation process, a feature that is helpful in capturing the volatility fluctuations of financial market. However there are many challenges in the research of $G$-expectation due to the uncertainty, for instance, the general dominated convergence theorem is not available, see Example 11 in [5]. Furthermore, there exist non-increasing and continuous $G$-martingales called non-symmetrical $G$-martingales, which makes the $G$-martingale representation theorem more difficult, see [21, 22, 24].

Recently, Hu, Ji, Peng and Song [6] considered the well-posedness problem of BSDEs driven by $G$-Brownian motion $B$:

$$
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g^{ij}(s, Y_s, Z_s)d(B^i, B^j)_s - \int_t^T Z_sdB_s - (K_T - K_t),
$$

where $\langle B^i, B^j \rangle$ denotes the quadratic (co)variation process and the generators $f, g$ are Lipschitz continuous in $(y, z)$. The solution of $G$-BSDE (1.1) consists of a triple of processes $(Y, Z, K)$, where $K$ is a non-symmetrical $G$-martingale. Note that the classical Banach contraction mapping principle cannot be applied directly to this equation due to the existence of $K$. The authors use PDE techniques and an approximation of Galerkin type to obtain the existence and uniqueness result of $G$-BSDE (1.1). In an accompany paper [7], they established the comparison theorem, Girsanov theorem and the relevant nonlinear Feynman-Kac formula.

Note that there are at least two characteristics that make the study on $G$-BSDEs meaningful, for one thing, we could establish a connection with fully nonlinear parabolic partial differential equations (PDEs) using $G$-BSDEs, for the other, since there exists a family of non-dominated, mutually singular martingale measures underlying the $G$-Brownian motion, one can solve simultaneously a family of classical BSDEs driven by mutually singular continuous martingales through dealing with only one aggregated $G$-BSDE. Moreover, Song [25] obtained gradient estimates for certain nonlinear partial differential equations (PDEs) by combining $G$-expectation theory with coupling methods. A close approach to $G$-BSDEs is the so-called second order BSDEs framework proposed independently by Soner, Touzi and Zhang [23].

Still there are further research papers on getting rid of the Lipschitz assumptions, and extensions of $G$-BSDEs from different aspects. For instance, Hu, Lin and Soumana Hima [11] studied the $G$-BSDEs under quadratic assumptions on coefficients and Li, Peng and Soumana Hima [15] considered $G$-BSDE with reflection, for which situation the solution is forced to lie above a prescribed continuous process. This paper is devoted to the research of the existence of solution to equation (1.1) when $f, g$ are Lipschitz continuous in $y$ and uniformly continuous in $z$, yet with a linear growth in both arguments.

In classical situation, the term $B$ in (1.1) boils down to the standard Wiener process, $d\langle B^i, B^j \rangle_t = (dt)\mathbb{1}_{i=j}$, and the term $K$ vanishes. It was due to Lepeltier and Martin [16] that confirmed these equations allow for solutions by monotone convergence argument, indeed their results hold for BSDEs that have continuous coefficients with linear growth, and Jia [12, 13] supplemented with proofs on uniqueness of solution. However Lepeltier and Martin’s arguments cannot be applied directly to investigate the existence of solution to equation (1.1), because the monotone convergence theorem of $G$-expectation is hardly at hand and the convergence of approximating sequences of $G$-BSDEs is not obvious.
Our observation is, this obstacle can be overcome with the help of a uniform estimate for approximating sequences of \( G \)-BSDEs, see Lemma 3.4. Indeed, the uniformly continuous generators can be approximated uniformly by a sequence of Lipschitz generators (see \([13]\)), from which we could prove the convergence of approximating sequences of \( G \)-BSDEs based on the linearization method of \([7]\) and \([9]\). Then we obtain the existence and uniqueness of the solution to \( G \)-BSDE \((1.1)\) by \( G \)-stochastic analysis technique. Since our work relies heavily on the comparison theorem of Lipschitz \( G \)-PDEs are discussed, thanks to the stability of viscosity solution, we show the solution to Markovian \( G \)-BSDE \((1.1)\) defines the unique solution to the related PDE, in the spirit of Feynman-Kac formula.

The paper is organized as follows. In the Section 2, we provide with preliminary notions on \( G \)-expectation and Lipschitz \( G \)-BSDEs. In Section 3, we state and prove our main theorem and the comparison theorem of our version. As an application, a slightly more general form of nonlinear Feynman-Kac formula is obtained in Section 4.

2 Preliminaries

To begin with, we shall recall some ingredients of \( G \)-expectation theory mainly from the seminal work of Peng \([21]\), and then of \( G \)-BSDEs results from \([6, 7]\).

2.1 \( G \)-expectation

Consider the canonical path space \( \Omega = C_0([0, \infty), \mathbb{R}^d) \), all continuous paths \( \omega \) vanishing at zero, i.e., \( \omega_0 = 0 \). \((\Omega, \rho)\) is readily seen to be a complete separable metric space, where \( \rho \) is given by,

\[
\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} \left(\max_{i \in [0, t]} |\omega^1_t - \omega^2_t|\right) \wedge 1.
\]

In the sequel, we will make use of these notations,

- \( B \) denotes the \( d \)-dimensional canonical process, i.e. \( B_t(\omega) = \omega_t \), for any \( \omega \in \Omega \).
- \( \mathcal{B}(\Omega) \) denotes the Borel \( \sigma \)-algebra of \( \Omega \), similarly we have \( \mathcal{B}(\Omega_t) \) with \( \Omega_t := \{ \omega_{\cdot \wedge t} : \omega \in \Omega \} \).
- \( L_{ip}(\Omega) := \{ \varphi(B_{t_1}, \ldots, B_{t_k}) : k \in \mathbb{N}, t_1, \ldots, t_k \in [0, \infty), \varphi \in C_{b,Lip}(\mathbb{R}^{k \times d}) \} \), where \( C_{b,Lip}(\mathbb{R}^{k \times d}) \) collects all bounded Lipschitz functions on \( \mathbb{R}^{k \times d} \). \( L_{ip}(\Omega_t) \) denotes all \( \mathcal{B}(\Omega_t) \)-measurable elements in \( L_{ip}(\Omega) \).
- \( S_d \) denote all symmetric matrices of size \( d \).

For any given monotonic sublinear continuous function \( G : S_d \to \mathbb{R} \), Peng \([21]\) associated it with a nonlinear \( G \)-expectation \( \hat{\mathbb{E}}[\cdot] \) using a nonlinear parabolic PDE, which in turn makes the canonical processes \( B \) a \( d \)-dimensional \( G \)-Brownian motion, ending up with the so-called \( G \)-expectation space \((\Omega, L_{ip}(\Omega), \hat{\mathbb{E}}[\cdot], (\hat{\mathbb{E}}_t[\cdot])_{t \geq 0})\). The readers are referred to \([19, 20, 21]\) for detailed construction and so forth.

For each \( p \geq 1 \), the completion of \( L_{ip}(\Omega) \) under the norm \( \|X\|_{L^p_{ip}} := (\hat{\mathbb{E}}[|X|^p])^{1/p} \) is denoted by \( L^p_{G}(\Omega) \). Similarly, we can define \( L^p_{G}(\Omega_t) \) for each \( t > 0 \). The \( G \)-expectation \( \hat{\mathbb{E}}[\cdot] \) and conditional \( G \)-expectation can be extended continuously to the completion \( L^p_{G}(\Omega) \). And the \( G \)-expectation can be regarded as a upper expectation.
Theorem 2.1 ([4, 8]) There exists a weakly compact set \( \mathcal{P} \) of probability measures on \((\Omega, \mathcal{B}(\Omega))\), such that

\[
\hat{E}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi], \text{ for all } \xi \in L^1_G(\Omega).
\]

For this \( \mathcal{P} \), we define a capacity

\[
c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).
\]

A set \( A \in \mathcal{B}(\Omega) \) is polar if \( c(A) = 0 \). A property holds “quasi-surely” (q.s.) if it holds except for a polar set. In what follows, we do not distinguish two random variables between \( X \) and \( Y \), if \( X = Y \) q.s.

Now we state the nonlinear monotone convergence theorem, which is different from the linear case.

Proposition 2.2 ([4]) Suppose \( X_n, n \geq 1 \) and \( X \) are \( \mathcal{B}(\Omega) \)-measurable. If \( \{X_n\}_{n=1}^\infty \) in \( L^2_G(\Omega) \) satisfies that \( X_n \downarrow X \), q.s., then \( \hat{E}[X_n] \downarrow \hat{E}[X] \).

Peng also introduced the stochastic integral with respect to \( G \)-Brownian motion, which led to a symmetric \( G \)-martingale. Given a fixed constant \( T > 0 \), the following spaces of stochastic processes will be useful,

- \( M^0_G(0,T) := \{ \eta : \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t), \ \xi_i \in L_{lip}(\Omega_{t_i}) \text{ for some partition } t_0 \leq t_1 \leq \ldots \leq t_N \text{ of } [0,T] \} \).
- \( M^2_G(0,T) \) is the completion of \( M^0_G(0,T) \) under norm \( \|\eta\|_{M^2_G} = \{\hat{E}[\int_0^T |\eta_s|^2 ds]\}^{1/2} \).
- \( S^0_G(0,T) = \{h(t,B_{t_1 \wedge t}, \ldots, B_{t_n \wedge t}) : t_1, \ldots, t_n \in [0,T], h \in C_b(Lip(\mathbb{R}^{n+1}))\} \).
- \( S^2_G(0,T) \) is the completion of \( S^0_G(0,T) \) under norm \( \|\eta\|_{S^2_G} = \{\hat{E}[\sup_{t \in [0,T]} |\eta_t|^2]\}^{1/2} \).

For each \( 1 \leq i, j \leq d \), we denote by \( \langle B^i, B^j \rangle \) the mutual variation process. Then for two processes \( \eta \in M^2_G(0,T) \) and \( \xi \in M^1_G(0,T) \), the \( G \)-Itô integrals \( \int_0^t \eta_s dB^i_s \) and \( \int_0^t \xi_s d\langle B^i, B^j \rangle_s \) are well defined, see Peng [21]. Moreover, the corresponding \( G \)-Itô formula were established.

2.2 \( G \)-BSDEs with Lipschitz assumptions

From now on, we always assume that the function \( G \) is non-degenerate throughout our paper, i.e. there are two constants \( 0 < \underline{\sigma}^2 \leq \sigma^2 < \infty \) such that

\[
\frac{1}{2} \sigma^2 \text{tr}[A - A'] \leq G(A) - G(A') \leq \frac{1}{2} \underline{\sigma}^2 \text{tr}[A - A'], \text{ for } A \geq A'.
\]

Then there exists a bounded and closed subset \( \Gamma \subset \mathbb{S}^+(d) \) such that

\[
G(A) = \frac{1}{2} \sup_{Q \in \Gamma} \text{tr}[AQ],
\]

where \( \mathbb{S}^+(d) \) denotes the space of all \( d \times d \) symmetric positive-definite matrices.
Consider the following $G$-BSDEs (recall that we use Einstein summation convention):

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g^{ij}(s, Y_s, Z_s)d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),
\]

(2.1)
in which the generators

\[
f(t, \omega, y, z), g^{ij}(t, \omega, y, z): [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}
\]

and the terminal value $\xi$ fulfill these assumptions,

(A1) For some $\beta > 2, \xi \in L^\beta_G(\Omega_T)$; for any $(y, z), f(\cdot, y, z) \in M^\beta_G(0, T), g^{ij}(\cdot, y, z) \in M^\beta_G(0, T)$.  

(A2) There is a Lipschitz constant $L_0 > 0$, so that

\[
|f(t, y, z) - f(t, y', z')| + |g^{ij}(t, y, z) - g^{ij}(t, y', z')| \leq L_0|y - y'| + L_0|z - z'|.
\]

For simplicity, we denote by $\mathcal{S}^2_G(0, T)$ the collection of process $(Y, Z, K)$ such that $Y \in S^2_G(0, T), Z \in M^2_G(0, T; \mathbb{R}^d)$, $K$ is a non-increasing $G$-martingale with $K_0 = 0$ and $K_T \in L^2_G(\Omega_T)$. Hu et al. [6, 7] firstly obtained the existence and uniqueness result on Lipschitz $G$-BSDEs (2.1), and the comparison principle.

**Theorem 2.3** ([6]) Assume the conditions (A1) and (A2) hold. Then the equation (2.1) admits a unique solution $(Y, Z, K) \in \mathcal{S}^2_G(0, T)$.

**Theorem 2.4** ([7]) Assume $(\xi^\nu, f^\nu, g^\nu_{ij})$ satisfy assumption (A1) for $\nu = 1, 2$. Moreover, one of them satisfies assumption (A2). Suppose $(Y^{\nu}, Z^{\nu}, K^{\nu})$ is a $\mathcal{S}^2_G(0, T)$-solution to the $G$-BSDE (2.1) with data $(\xi^{\nu}, f^{\nu}, g^{\nu}_{ij})$. If $\xi^1 \leq \xi^2, f^2 \leq f^1$ and the matrix $(g^2_{ij})_{i,j=1} \leq (g^1_{ij})_{i,j=1}$, then we have $Y^2_t \leq Y^1_t$ for all $t \in [0, T]$.

The linear $G$-BSDEs will be repeatedly used in our paper, so we sketch the idea on how to construct the solution. Consider linear $G$-BSDE of the form,

\[
Y_t = \xi + \int_t^T [a_s Y_s + b_s Z_s + m_s]ds + \int_t^T [c^{ij}_s Y_s + d^{ij}_s Z_s + n^{ij}_s]d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),
\]

(2.2)

where $(a_s)_{s \in [0, T]}, (c^{ij}_s)_{s \in [0, T]} \in M^2_G(0, T), (b_s)_{s \in [0, T]}, (d^{ij}_s)_{s \in [0, T]} \in M^2_G(0, T; \mathbb{R}^d)$ are bounded processes and $\xi \in L^2_G(\Omega_T), (m_s)_{s \in [0, T]}, (n^{ij}_s)_{s \in [0, T]} \in M^2_G(0, T)$.  

To find the closed-form solution to equation (2.2), a standard method is to introduce a dual process. However for the $G$-expectation case, unless the $G$-Brownian motion degenerates to the standard Wiener process, the measures $ds$ and $d\langle B \rangle_s$ are mutually singular, therefore to cancel terms involving $ds$ and $d\langle B^i, B^j \rangle_s$ is even harder. To adapt the classical dual method, Hu et al. [7] came up with a strategy of enlarging the original $G$-expectation space to $G$-expectation space $(\tilde{\Omega}, L^1_G(\tilde{\Omega}), \tilde{G})$ with $\tilde{\Omega} = C_0([0, \infty), \mathbb{R}^{2d})$ and

\[
\tilde{G}(A) = \frac{1}{2} \sup_{Q \in \Gamma} \text{tr} \left[ A \begin{bmatrix} Q & I_d \\ I_d & Q^{-1} \end{bmatrix} \right], A \in \mathbb{S}_{2d}.
\]

(2.3)

Let $(B_t, \tilde{B}_t)_{t \geq 0}$ be the canonical process in the extended space. Then
Lemma 2.5 ([7]) In the extended $\tilde{G}$-expectation space, the solution of the linear $G$-BSDE (2.2) can be represented as

$$Y_t = \mathbb{E}^\tilde{G}_t[\tilde{Y}_T] + \int_t^T m_s \tilde{Y}_s ds + \int_t^T n^i_s \tilde{Y}_s d(B^i, B^j)_s,$$

where $\{\tilde{Y}_s\}_{s \in [t, T]}$ is the solution of the following $\tilde{G}$-SDE:

$$\tilde{Y}_s = 1 + \int_t^s a_r \tilde{Y}_r dr + \int_t^s c^i_r \tilde{Y}_r d(B^i, B^j)_r + \int_t^s d^i_r \tilde{Y}_r dB_r + \int_t^s b_r \tilde{Y}_r dB_r. \quad (2.4)$$

Moreover,

$$\tilde{E}^\tilde{G}_t[\tilde{Y}_T - K_T] - \int_t^T a_s K_s \tilde{Y}_s ds - \int_t^T c^i_s K_s \tilde{Y}_s d(B^i, B^j)_s] = K_t. \quad (2.5)$$

3 $G$-BSDEs with uniformly continuous generators

In this section, we shall investigate the well-posedness problem of the subsequent $G$-BSDEs

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g^{ij}(s, Y_s, Z_s) d(B^i, B^j)_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad (3.1)$$

where the generators

$$f(t, \omega, y, z), \ g^{ij}(t, \omega, y, z) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R},$$

satisfy the following assumptions:

(H1) There exists a constant $\beta > 2$ such that $f(\cdot, \cdot, y, z), g(\cdot, \cdot, y, z) \in M^\beta_G(0, T)$ for any $y, z$.

(H2) $f$ and $g$ are Lipschitz continuous in $y$, are of linear growth and uniformly continuous in $z$, i.e. there is a constant $L$ and a continuous function $\phi$, both independent of $(t, \omega)$, such that

$$|f(t, \omega, y, z) - f(t, \omega, y', z')| + |g^{ij}(t, \omega, y, z) - g^{ij}(t, \omega, y', z')| \leq L|y - y'| + \phi(|z - z'|),$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and sub-additive, with $\phi(0) = 0$ as well as $\phi(z) \leq L(1 + |z|)$.

(H3) $g^{ij} \equiv 0$ whenever $i \neq j$.

Remark 3.1 Note that assumption (H3) is necessary to construct a sequence of $G$-BSDEs monotonically converges to $Y$, see (i) of Lemma 3.4.

According to Lemma 1 in Lepeltier and Martin [16] or Lemma 2 in Jia [12], there exists a sequence of Lipschitz functions that nicely approximates $f$ and $g_{ij}$ respectively. Indeed, for any $(t, y, z), n \in \mathbb{N}$ and for every $\omega$, denote

$$\varphi_n(t, y, z) := \inf_{q \in \mathbb{Q}} \{\varphi(t, y, q) + n|z - q|\} - \varphi_0(t), \ \bar{\varphi}_n(t, y, z) := \sup_{q \in \mathbb{Q}} \{\varphi(t, y, q) - n|z - q|\} - \varphi_0(t),$$

where $\varphi = f, g^{ij}$ and $\varphi_0(t) = \varphi(t, 0, 0)$. Their main technical lemma can be summarized as,

Lemma 3.2 Assume (H1)-(H2) hold. Then for each $n > L$, the following properties hold
(i) both $\varphi$ and $\varphi_n$ are of linear growth, moreover, for all $(t, y, z)$,
\[-L(1 + |y| + |z|) \leq \varphi_n(t, y, z) \leq \varphi(t, y, z) - \varphi_0(t) \leq \varphi_n(t, y, z) \leq L(1 + |y| + |z|);
\]
(ii) for all $(t, y, z)$, $\varphi(t, y, z)$ is non-decreasing and $\varphi(t, y, z)$ is non-increasing;
(iii) $\varphi_n(t, y, \cdot)$ and $\varphi_n(t, \cdot, z)$ are Lipschitz functions with constant $n$, $\varphi_n(t, \cdot, \cdot)$ and $\varphi_n(\cdot, \cdot, z)$ are Lipschitz functions with Lipschitz constant $L$;
(iv) if $(y_n, z_n) \rightarrow (y, z)$, then $\varphi_n(t, y_n, z_n) \rightarrow \varphi(t, y, z) - \varphi_0(t)$ and $\varphi_n(t, y_n, z_n) \rightarrow \varphi(t, y, z) - \varphi_0(t)$;
(v) for all $(t, \omega, y, z)$,
\[0 \leq \varphi(t, y, z) - \varphi_0(t) - \varphi(t, y, z) \leq \phi\left(\frac{2L}{n-L}\right), 0 \leq \varphi_n(t, y, z) + \varphi(t, y, z) \leq \phi\left(\frac{2L}{n-L}\right).
\]

Based on the above approximation results, we construct two sequences of $G$-BSDEs corresponding respectively to $(f_n, g_n^i)$ and $(f_n, g_n^{ij})$, i.e.,
\[
Y^n_t = \xi + \int_t^T f_n(s, Y^n_s, Z^n_s) + g_n^i(s)dB^i_s + \int_t^T g_n^{ij}(s, Y^n_s, Z^n_s)dB^j_s - \int_t^T Z^n_s dB_s - (K^n_t - K^n_0),
\]
\[
\bar{Y}^n_t = \xi + \int_t^T f_n(s, \bar{Y}^n_s, \bar{Z}^n_s) + g_n^i(s)dB^i_s + \int_t^T g_n^{ij}(s, \bar{Y}^n_s, \bar{Z}^n_s)dB^j_s - \int_t^T \bar{Z}^n_s dB_s - (K^n_T - K^n_0). \tag{3.2}
\]

We need an additional assumption to ensure the existence of $Y^n$ and $\bar{Y}^n$:

(H4) For each $n$ and for any $(y, z)$, $\varphi_n(t, y, z)$ and $\bar{\varphi}_n(t, y, z)$ all belong to $M^\beta_G(0, T)$, with $\varphi = f, g^{ij}$.

**Remark 3.3** As can be easily seen, assumption (H4) is imposed mainly to keep all processes under investigation lying in space $M^\beta_G(0, T)$. This condition can be verified for lots of situations. For instance, assume (H1) hold. Suppose for $\varphi = f, g$ that $\varphi(\cdot, \cdot, y, z)$ is uniformly continuous in $(t, \omega)$ with the modulus of continuity independent of $(y, z)$,
\[
|\varphi(t, \omega, y, z) - \varphi(t', \omega', y, z)| \leq \phi(|t - t'| + \sup_{s \in [0, t]} |\omega(s) - \omega'(s)|).
\]

Then it is straightforward to observe that $\varphi_n(\cdot, \cdot, y, z)$ and $\bar{\varphi}_n(\cdot, \cdot, y, z)$ are uniformly continuous in $(t, \omega)$. Recalling the property (i) from Lemma 3.2, we have $\varphi_n(\cdot, \cdot, y, z)$ and $\bar{\varphi}_n(\cdot, \cdot, y, z)$ are bounded and then
\[
\lim_{N \to \infty} \hat{\mathbb{E}} \int_0^T |\varphi_n(t, y, z)|^2 1_{\{|\varphi_n(t, y, z)| \geq N\}} dt = 0, \quad \lim_{N \to \infty} \hat{\mathbb{E}} \int_0^T |\bar{\varphi}_n(t, y, z)|^2 1_{\{|\bar{\varphi}_n(t, y, z)| \geq N\}} dt = 0.
\]
Thus by Theorem 4.16 in [10], we know $\varphi$ satisfies assumption (H4).

The following lemma is important in our future discussion.

**Lemma 3.4** Let $\xi$ be in $L^\beta_G(\Omega, T)$ and the assumptions (H1)-(H4) hold. Then the G-BSDE (3.2) has a unique $\mathcal{G}^2(0, T)$-solution. Moreover, we have
(i) for any $n,m \in \mathbb{N}$, the comparisons $Y^n \leq Y^{n+1} \leq Y^m$ hold;
(ii) both $Y^n$ and $\bar{Y}^n$ are uniformly bounded in $S^2_{\bar{G}}(0,T)$;
(iii) for each $n > L$, the differences between $Y^n$ and $\bar{Y}^n$ can be uniformly controlled, that is,
\[ |Y^n_t - \bar{Y}^n_t| \leq C_G \phi\left( \frac{2L}{n-L} \right), \forall t \in [0,T], \]
where $C_G$ is a constant depending on $G, L$, and $T$.

**Proof.** The proof is built on the conclusions of lemma 3.2. By assumption (H3), we have $g_{ij} = \bar{g}_{ij} = 0$ whenever $i \neq j$. Thus $(\bar{g}_{ij} - g_{ij})_{i,j=1}^d$ is a nonnegative definite matrix. Then from Theorem 2.3 and the comparison theorem 2.4, it is trivial to verify (i) in view of assertions (i)-(ii) from lemma 3.2.

In order to prove (ii), setting $w(y,z) = L(1+|y|+|z|)$, consider the following G-BSDEs
\[ U_t = \xi + \int_t^T [w(U_s, V_s) + f_0(s)] ds + \int_t^T [w(U_s, V_s) + g_0(s)] dB(s), \]
\[ U_t' = \xi + \int_t^T [-w(U'_s, V'_s) + f_0(s)] ds + \int_t^T [-w(U'_s, V'_s) + g_0(s)] dB(s), \]
for each $\bar{U}_t$ and ($i$)
\[ \|\phi\|_{\mathbb{F}^d} \text{ is uniformly bounded}, \]
where $\phi = f, g$. It follows from Theorem 2.3 that the above G-BSDE admits a unique $\bar{S}^2_{\bar{G}}(0,T)$-solution $(U, V, R)$ and $(U', V', R')$, respectively. Then by (i) of lemma 3.2 and the comparison theorem 2.4, it holds that for any $n \in \mathbb{N}$
\[ U'_t \leq Y^n_t \leq \bar{Y}^n_t \leq U_t, \forall t \in [0,T], \]
which implies the desired result.

Finally, we proceed to verify the third assertion (iii). Without loss of generality, assume that $d = 1$. Set $(\bar{Y}, \bar{Z}) = (Y^n - \bar{Y}^n, \bar{Z}^n - \bar{Z}^n)$. Then for each $t \in [0,T]$, we have
\[ \bar{Y}_t = K^n_T + \int_t^T \hat{f}_s ds + \int_t^T \hat{g}_s dB_s - \int_t^T \hat{Z}_s d\bar{B}_s - (K^n_T - K^n_t), \]
where $\hat{\varphi}_s = \varphi_n(s, \bar{Y}_s, \bar{Z}_s) - \bar{\varphi}_n(s, Y^n_s, Z^n_s)$ for $\varphi = f, g$.

By Lemma 3.5 in [9], for each $\varepsilon > 0$, there exist four bounded processes $a^\varepsilon, b^\varepsilon, c^\varepsilon, d^\varepsilon \in \mathbb{M}^2_d(0,T)$ such that for all $s \in [0,T]$, $\hat{f}_s = a^\varepsilon_s \bar{Y}_s + b^\varepsilon_s \bar{Z}_s + m_s - c^\varepsilon_s, \hat{g}_s = c^\varepsilon_s \bar{Y}_s + d^\varepsilon_s \bar{Z}_s + n_s - n^\varepsilon_s$, and $|a^\varepsilon_s| \leq L, |b^\varepsilon_s| \leq L, |c^\varepsilon_s| \leq n, |d^\varepsilon_s| \leq n, |m_s| \leq 2(L+n)\varepsilon, |n^\varepsilon_s| \leq 2(L+n)\varepsilon, m_s = \hat{f}_n(s, Y^n_s, Z^n_s) - \hat{f}_s(s, Y^n_s, Z^n_s), n_s = \hat{g}_n(s, Y^n_s, Z^n_s) - \hat{g}_s(s, Y^n_s, Z^n_s)$.

In order to estimate the solution to the above linearized equation (3.3), as in [7], we shift from the underlying $G$-expectation space to an auxiliary extended $\tilde{G}$-expectation space $(\tilde{\Omega}, L^1_{\tilde{G}}(\tilde{\Omega}), \tilde{E}^{\tilde{G}})$ with $\tilde{\Omega} = C_0([0,\infty), \mathbb{R}^2)$, where $\tilde{G}$ is given by equation (2.3), within which, $(B_t, \tilde{B}_t)_{t \geq 0}$ denotes the corresponding canonical process.

Applying Lemma 2.5 yields that
\[ \tilde{Y}_t + K^{\tilde{G}}_t = \tilde{E}^{\tilde{G}} T \int_t^T (m_s + 2G(n_s) - m^\varepsilon_s - a^\varepsilon_t K^n_s) \tilde{Y}^t_{\varepsilon} d\tilde{s}_s + \int_t^T (m_s + 2G(n_s) - m^\varepsilon_s - a^\varepsilon_t K^n_s) \tilde{Y}^t_{\varepsilon} d\tilde{s}_s - \int_t^T 2G(n_s) \tilde{Y}^t_{\varepsilon} d\tilde{s}_s, \]
Then by equation (2.4). From G-Itô’s formula, we conclude that
\[ \hat{\Gamma}_s^{t,\varepsilon} = \exp\left(\int_t^s \left(a_r^\varepsilon - b_r^\varepsilon d\xi_r\right) dr + \int_t^s c_r^\varepsilon d(B)_r\right) \mathcal{E}_s^B \mathcal{E}_t^B. \]
Here \( \mathcal{E}_s^B = \exp\left(\int_t^s b_r^\varepsilon d(B)_r - \frac{1}{2} \int_t^s \vert b_r^\varepsilon \vert^2 d(B)_r\right) \) and \( \mathcal{E}_t^B = \exp\left(\int_t^\infty b_r^\varepsilon d(B)_r - \frac{1}{2} \int_t^\infty \vert b_r^\varepsilon \vert^2 d(B)_r\right) \). Therefore using equations (3.3) and G-Itô’s formula we get that
\[ \hat{Y}_t + \mathcal{K}_n \leq \mathcal{E}_t^\beta \left[ \int_t^T (m_s + 2G(n_s)) \hat{\Gamma}_s^{t,\varepsilon} ds - \int_t^T m_s \hat{\Gamma}_s^{t,\varepsilon} ds - \int_t^T n_s \hat{\Gamma}_s^{t,\varepsilon} d(B)_s \right] + \mathcal{K}_n, \text{ q.s.} \quad (3.4) \]
By (v) of Lemma 3.2, we get
\[ 0 \leq m_s + 2G(n_s) \leq 2(1 + \sigma^2) \phi\left(\frac{2L}{n - L}\right). \]
Note that for each \( s \geq t \), \( \hat{\Gamma}_s^{t,\varepsilon} \leq \exp(L(1 + \sigma^2)(s - t)) \Gamma_s^{t,\varepsilon} \), where \( \Gamma_s^{t,\varepsilon} = 1 + \int_t^s d\xi_r \Gamma_r^{t,\varepsilon} d(B)_r + \int_t^s b_r^\varepsilon \Gamma_r^{t,\varepsilon} d\tilde{B}_r \). Then by equation (3.4), we derive that
\[ \hat{Y}_t \leq [2(1 + \sigma^2) \phi\left(\frac{2L}{n - L}\right) + 2(1 + \sigma^2)\epsilon] \mathcal{E}_t^\beta \left[ \int_t^T \exp(L(1 + \sigma^2)(s - t)) \Gamma_s^{t,\varepsilon} ds \right] \]
\[ \leq \frac{\exp(L(1 + \sigma^2)(T - t))}{L} \left[ 2\phi\left(\frac{2L}{n - L}\right) + 2(L + \epsilon) \right]. \]
Sending \( \epsilon \to 0 \), we have
\[ \hat{Y}_t \leq \frac{2 \exp(L(1 + \sigma^2)(T - t))}{L} \phi\left(\frac{2L}{n - L}\right), \]
which completes the proof. \( \Box \)

**Remark 3.5** Note that from (i) and (ii) of Lemma 3.4, in general we cannot conclude that \( Y^n \) (or \( \hat{Y}^n \)) is a Cauchy sequence in \( M^2_\mathcal{G}(0, T) \) according to Proposition 2.2, which is different from the classical case.

Now we are ready to state the main result of this section.

**Theorem 3.6** Given assumptions (H1)-(H4) and \( \xi \in L^2_\mathcal{G}(\Omega_T) \), the G-BSDE (3.1) admits a unique solution \( (Y, Z, K) \in \mathcal{S}^2_\mathcal{G}(0, T) \).

**Proof.** We shall deal with the existence and uniqueness of solution to G-BSDE (3.1) separately. For the uniqueness, suppose that both of \( (Y^i, Z^i, K^i) \), \( i = 1, 2 \) are \( \mathcal{S}^2_\mathcal{G}(0, T) \)-solution to G-BSDE (3.1), by comparison theorem 2.4, we obtain that for each \( n \)
\[ Y^n \leq Y^1 \leq \hat{Y}^n, \forall t \in [0, T], \]
which, together with Lemma 3.4, implies
\[ |Y^1_t - Y^2_t| \leq |Y^n_t - \hat{Y}^n_t| \leq C_G \phi\left(\frac{2L}{n - L}\right), \forall n > L. \]
Note that \( Y^1 \) is a continuous process. Sending \( n \to \infty \), we deduce that \( Y^1 = Y^2 \) q.s.. Then applying G-Itô’s formula upon \( |Y^1_t - Y^2_t|^2 \) on \([0, T]\), we have \( Z^1 = Z^2 \) and hence \( K^1 = K^2 \), which shows that G-BSDE (3.1) allows for at most one \( \mathcal{S}^2_\mathcal{G}(0, T) \)-solution.
The rest of the proof is devoted to studying the existence, which will be divided into three steps. Without loss of generality, we assume \(d = 1 \) and \(g \equiv 0\).

1 The uniform estimates. Let \(C(\alpha)\) denote a constant depending on parameter \(\alpha\) that may change from line to line. From (ii) of Lemma 3.4, we have for all \(n\)

\[
\|\bar{Y}^n\|_{\mathcal{S}^2_G} \leq C(L, \bar{\sigma}, \varrho, \bar{T}).
\]

Calculating by Itô’s formula upon \(|\bar{Y}^n|^2\), we have for any \(t \in [0, T]\),

\[
|\bar{Y}^n_t|^2 + \int_0^T |\bar{Z}^n_s|^2 d(B)_s = |\xi|^2 + 2 \int_0^T \bar{Y}^n_s (f_n(s, \bar{Y}^n_s, \bar{Z}^n_s) + f_0(s)) ds - 2 \int_0^T \bar{Y}^n_s \bar{Z}^n_s dB_s - \int_0^T 2 \bar{Y}^n_s d\bar{K}^n_s.
\]

(3.5)

Since

\[
|\bar{f}^n(t, y, z)| \leq L(1 + |y| + |z|),
\]

we get that

\[
2 \bar{Y}^n_s f_n(s, \bar{Y}^n_s, \bar{Z}^n_s) \leq 2L(|\bar{Y}^n|^2 + |\bar{Y}^n|^2) + \frac{4L^2}{\varrho^2} |\bar{Y}^n|^2 + \frac{\sigma^2}{4} |\bar{Z}^n|^2.
\]

Using BDG inequality and Hölder’s inequality, we derive that

\[
\mathbb{E}[\int_0^T |\bar{Y}^n_s \bar{Z}^n_s dB_s|^2] \leq C(\bar{\sigma}) \mathbb{E}[\int_0^T |\bar{Y}^n_s \bar{Z}^n_s|^2 ds|^\bar{T}] \leq C(\bar{\sigma}) \|\bar{Y}^n\|_{\mathcal{S}_G^2} \|\bar{Z}^n\|_{\mathcal{M}^2_G}
\]

\[
\leq C(\bar{\sigma}, \varrho) \|\bar{Y}^n\|_{\mathcal{G}^2}^2 + \frac{\sigma^2}{8} \|\bar{Z}^n\|_{\mathcal{M}^2_G}.
\]

Thus, in view of equation (3.5) we have

\[
\mathbb{E}[\int_0^T |\bar{Z}^n_s|^2 d(B)_s] \leq C(L, \bar{\sigma}, \varrho, \bar{T}) + \frac{\sigma^2}{2} \|\bar{Z}^n\|_{\mathcal{M}^2_G} + 2 \mathbb{E}[\sup_{s \in [0, T]} |\bar{Y}^n_s| |\bar{K}^n_T|].
\]

(3.6)

Recalling that

\[
\bar{K}^n_T = \xi - \bar{Y}^n_0 + \int_0^T [f_n(s, \bar{Y}^n_s, \bar{Z}^n_s) + f_0(s)] ds - \int_0^T \bar{Z}^n_s dB_s.
\]

By a similar analysis as above, we obtain

\[
\mathbb{E}[\sup_{s \in [0, T]} |\bar{Y}^n_s| |\bar{K}^n_T|] \leq C(L, \bar{\sigma}, \varrho, \bar{T}) + \frac{\sigma^2}{8} \|\bar{Z}^n\|_{\mathcal{M}^2_G},
\]

putting together equation (3.6) with the fact that \(\varrho^2 \|\bar{Z}^n\|_{\mathcal{M}^2_G(0, T)}^2 \leq \mathbb{E}[\int_0^T |\bar{Z}^n_s|^2 d(B)_s]\) indicates that

\[
\|\bar{Z}^n\|_{\mathcal{M}^2_G(0, T)} + \|\bar{K}^n_T\|_{\mathcal{L}^2_G} \leq C(L, \bar{\sigma}, \varrho, \bar{T}), \quad \forall n \in \mathbb{N}.
\]

2 The convergence. From assertions (i) and (iii) of Lemma 3.4, we get that for each \(n, m > L\)

\[
\|\bar{Y}^n - \bar{Y}^m\|_{\mathcal{S}^2_G} \leq \|\bar{Y}^{n \wedge m} - \bar{Y}^{n \wedge m}\|_{\mathcal{S}^2_G} \leq C_G \phi\left(\frac{2L}{n \wedge m - L}\right).
\]

from which we conclude that \(\{\bar{Y}^n\}_{n \in \mathbb{N}}\) is a Cauchy sequence in \(\mathcal{S}^2_G(0, T)\). Then there is a process \(\bar{Y} \in \mathcal{S}^2_G(0, T)\) such that \(\bar{Y}^n\) converges to \(\bar{Y}\) in \(\mathcal{S}^2_G(0, T)\).

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We continue to show the convergence of $\bar{Z}^n$ in $M^2_G(0, T)$. For each $n, m > L$, applying Itô’s formula to $|Y^n - Y^m|^2$ yields that
\[
\mathbb{E}^2 \mathbb{E} \left[ \int_0^T |\bar{Z}^n_s - \bar{Z}^m_s|^2 ds \right] \leq \mathbb{E} \left[ \int_0^T |\bar{Z}^n_s - \bar{Z}^m_s|^2 d(B)_s \right]
\leq 2\mathbb{E} \left[ \int_0^T (\bar{Y}^n_s - \bar{Y}^m_s)(\bar{f}_n(s, \bar{Y}^n_s, \bar{Z}^n_s) - \bar{f}_m(s, \bar{Y}^m_s, \bar{Z}^m_s))ds \right] - \int_0^T (\bar{Y}^n_s - \bar{Y}^m_s)d(\bar{K}^n_s - \bar{K}^m_s)]
\leq 2\mathbb{E} \left[ \sup_{s \in [0, T]} |\bar{Y}^n_s - \bar{Y}^m_s| \right] \cdot \left[ L \int_0^T (2 + |\bar{Y}^n_s| + |\bar{Y}^m_s| + |\bar{Z}^n_s| + |\bar{Z}^m_s|)ds + |\bar{K}^n_T| + |\bar{K}^m_T|) \right]
\leq C(L, \sigma, \sigma, T)\|\bar{Y}^n - \bar{Y}^m\|_{S^2_G},
\]
where we have used the estimates of step 1 and Hölder’s inequality in the last inequality. Consequently, we can find some process $Z \in M^2_G(0, T)$ so that $\bar{Z}^n$ converges to $Z$ in $M^2_G(0, T)$.

Denote
\[ K_t := Y_t - Y_0 + \int_0^t f(s, Y_s, Z_s)ds - \int_0^t Z_s dB_s, \]
we claim that
\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T |\bar{f}_n(s, \bar{Y}^n_s, \bar{Z}^n_s) + f_0(s) - f(s, Y_s, Z_s)|^2 ds \right] = 0, \tag{3.7}
\]
whose proof will be given in step 3. Thus it is easy to check that for each $t \in [0, T]$
\[
\lim_{n \to \infty} \mathbb{E} |K_t - \bar{K}^n_t|^2 = 0,
\]
which implies that $K$ is a non-increasing $G$-martingale and then $(Y, Z, K) \in \mathcal{G}_2^2(0, T)$ is the solution to $G$-BSDE (3.1).

3 The proof of equation (3.7). For each $n > L$, applying lemma 3.2, we get that
\[
\mathbb{E} \left[ \int_0^T |\bar{f}_n(s, \bar{Y}^n_s, \bar{Z}^n_s) + f_0(s) - f(s, Y_s, Z_s)|^2 ds \right]
\leq 3\mathbb{E} \left[ \int_0^T |\bar{f}_n(s, \bar{Y}^n_s, \bar{Z}^n_s) + f_0(s) - f(s, \bar{Y}^n_s, \bar{Z}^n_s)|^2 ds \right] + \int_0^T |f(s, \bar{Y}^n_s, \bar{Z}^n_s) - f(s, Y_s, Z_s)|^2 ds
+ \int_0^T |f(s, Y_s, \bar{Z}^n_s) - f(s, Y_s, Z_s)|^2 ds
\leq 3T\phi(\frac{2L}{n - L}) + 3TL^2 \|\bar{Y}^n - Y\|_{S^2_G}^2 + 3\mathbb{E} \left[ \int_0^T |f(s, Y_s, \bar{Z}^n_s) - f(s, Y_s, Z_s)|^2 ds \right].
\]

By the uniform continuity of $f$ in $z$, for any fixed $\varepsilon > 0$, there exists a $\delta$, so that $|f(\cdot, \cdot, x) - f(\cdot, \cdot, y)| < \varepsilon$ whenever $|z - q| \leq \delta$. Then for each $N > 0$, we obtain that
\[
\mathbb{E} \left[ \int_0^T |f(s, Y_s, \bar{Z}^n_s) - f(s, Y_s, Z_s)|^2 ds \right]
\leq 2\mathbb{E} \left[ \int_0^T |f(s, Y_s, \bar{Z}^n_s) - f(s, Y_s, Z_s)|^2 1_{|\bar{Z}^n_s - Z_s| \leq \varepsilon} ds \right] + 2\mathbb{E} \left[ \int_0^T |f(s, Y_s, \bar{Z}^n_s) - f(s, Y_s, Z_s)|^2 1_{|\bar{Z}^n_s - Z_s| > \delta} ds \right]
\leq 2T\varepsilon^2 + 2\mathbb{E} \left[ \int_0^T (|f_0(s)| + 2L|Y_s| + L|\bar{Z}^n_s| + L|Z_s|)^2 1_{|\bar{Z}^n_s - Z_s| > \delta} ds \right].
\]
Since $Z^n$ converges to $Z$ in $M^2_G(0, T)$, it is easy to check that $\mathbb{E}\left[\int_0^T 1_{|Z^n_t - Z_t| > \delta}\right]$ is vanishing as $n \to \infty$. Note that $|f_0(s)| + 2L|Y_s| + L|Z^n_s| + L|Z_s| \in M^2_G(0, T)$. Thus with the help of Theorem 4.7 in [10], we get that

$$\lim_{n \to \infty} \mathbb{E}\left[\int_0^T (|f_0(s)| + 2L|Y_s| + L|Z^n_s| + L|Z_s|)^2 1_{|Z^n_t - Z_t| > \delta}ds\right] = 0.$$ 

Consequently, putting together the above two inequalities we deduce that

$$\limsup_{n \to \infty} \mathbb{E}\left[\int_0^T |f_0(s) - f(s, Y_s, Z_s)|^2 ds\right] \leq 2T\varepsilon^2.$$ 

Letting $\varepsilon \to 0$, we get the desired result. $\square$

**Example 3.7** For a 1-dimensional G-Brownian motion $B$ with $\mathbf{g}^2 := -\mathbb{E}[|B_1|^2]$, consider the following G-BSDE:

$$Y_t = \frac{1}{6}|B_T|^6 - \frac{5}{2}\mathbf{g}^2 \int_t^T |Z_s|^2 ds - \int_t^T Z_s dB_s - (K_T - K_t).$$

Note that $f(z) = -\frac{5}{2}\mathbf{g}^2 |z|^2$ is a uniformly continuous function. Then by G-Itô’s formula and Theorem 3.6, it is easy to check that $(\frac{1}{6}|B_t|^6, (B_t)^5, \frac{5}{2}\mathbf{g}^2 \int_0^t |B_s|^4 ds - \frac{5}{2} \int_0^t |B_s|^4 dB(s))$ is the unique $\mathcal{G}_T^2(0, T)$-solution.

**Theorem 3.8 (Comparison Theorem)** Suppose $\xi^\nu \in L^\beta_G(\Omega_T)$, $\nu = 1, 2$ and $f^\nu, g^{\nu, ij}$ satisfy assumption (H1)-(H4). Let $(Y^\nu, Z^\nu, K^\nu)$ be the $\mathcal{G}_T^2(0, T)$-solution of G-BSDE (3.1) with data $(\xi^\nu, f^\nu, g^{\nu, ij})$. If $\xi^1 \leq \xi^2$, $f^1(t, y, z) \leq f^2(t, y, z)$ and $g^{1, ij}(t, y, z) \leq g^{2, ij}(t, y, z)$ for any $(t, \omega, y, z)$, then $Y^1_t \leq Y^2_t$ for each $t$.

**Proof.** For each $n \in \mathbb{N}$, let $(\bar{Y}^{2,n}, \bar{Z}^{2,n}, \bar{K}^{2,n})$ be the $\mathcal{G}_T^2(0, T)$-solution of G-BSDE (3.2) corresponding to $(f^2, g^{2, ij})$. It is obvious that $\bar{f}^2_n(t, y, z) + \bar{f}^2_n(t) \geq f^1(t, y, z)$ and $\bar{g}^{2, ij}_n(t, y, z) + \bar{g}^{2, ij}_n(t) \geq g^{1, ij}(t, y, z)$. Note that $g^{2, ij} = g^{1, ij}$ and $\bar{g}^{2, ij}_n(t) = 0$ whenever $i \neq j$. Then using Theorem 2.4, we get that $Y^1_t \leq Y^{2,n}_t$ for each $t$. Note that $\bar{Y}^{2,n}$ converges to $Y^2$ in $\mathcal{S}_T^2(0, T)$. Sending $n \to \infty$, we derive that $Y^1_t \leq Y^2_t$. The proof is complete. $\square$

### 4 Nonlinear Feynman-Kac formula

In this section, we shall utilize Theorem 3.6 to establish a nonlinear Feynman-Kac formula that slightly generalizes the corresponding result of [7, 21]. Retaining the notations in previous sections, for each $(t, x) \in [0, T] \times \mathbb{R}^m$, let’s consider G-BSE

$$\begin{align*}
\int_t^T - f(s, X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)ds - g^i(s, X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)d(B^i, B^i)_s + Z^{t,x}_s dB_s + dK^{t,x}_s, \quad s \in [t, T]\nY^{t,x}_t = \Phi(X^{t,x}_t),
\end{align*}$$

where $X^{t,x}$ is defined through a forward G-SDE on the interval $[t, T]$.

$$\begin{align*}
\int_t^T \int_t^s b(s, X^{t,x}_s)ds + \int_t^T h^i(s, X^{t,x}_s)d(B^i, B^i)_s + \sigma(s, X^{t,x}_s)dB_s, \quad X^{t,x}_t = x.
\end{align*}$$

In the sequel, we use these running assumptions abbreviated as (H5):
(i) $b, h^{ij} : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m; \sigma : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m \times d}; f, g^i : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}; \Phi : \mathbb{R}^m \to \mathbb{R}$, are all deterministic continuous functions.

(ii) There exist two positive integers $q, L$ and a modulus of continuity $\phi$ such that

$$|b(t, x) - b(t, x')| + \sum_{i,j=1}^{d} |h^{ij}(t, x) - h^{ij}(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq L|x - x'|,$$

$$|\Phi(x) - \Phi(x')| + |f(t, x, y, z) - f(t, x', y', z')| + \sum_{i=1}^{d} |g^i(t, x, y, z) - g^i(t, x', y', z')| \leq L(1 + |x|^q + |x'|^q)|x - x'| + L|y - y'| + \phi(|z - z'|).$$

To link the above $G$-BSDE system with PDE, we need several estimates from [7, 21],

**Lemma 4.1** Assuming (H5), for any $\delta \in [0, T - t]$, there exists a constant $C$ depending on $L', G, p, n, T$ such that

$$\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,x}|^p] \leq C(1 + |x|^p),$$

$$\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,x} - X_{t+\delta}^{t,x'}|^p] \leq C|x - x'|^p,$$

$$\hat{\mathbb{E}}_t\left[\sup_{x\in[t,t+\delta]} |X_{t+\delta}^{t,x} - x|^p\right] \leq C(1 + |x|^p)\delta^{p/2}.$$

**Theorem 4.2** Suppose (H5) hold. Then $G$-BSDE (4.1) has a unique solution triplet $(Y^{t,x}, Z^{t,x}, K^{t,x}) \in \mathfrak{S}_G^2(t, T)$.

**Proof.** By Lemma 4.1 and assumption (H5), for each $p \geq 1$, it is easy to get that $\Phi(X_{T}^{t,x}) \in L^p_G(\Omega_T)$.

The facts $f(\cdot, X_{t+\delta}^{t,x}, y, z), g^i(\cdot, X_{t+\delta}^{t,x}, y, z) \in M^p_G(t, T)$ follow from Theorem 4.16 in [10]. Therefore it suffices to verify conditions (H4), before applying Theorem 3.6 to complete the proof.

For any $(t, x, y, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d$ and $n \in \mathbb{N}$, set

$$\varphi_n(t, x, y, z) = \inf_{q \in \mathbb{Q}} \{\varphi(t, x, y, q) + n|z - q|\} - \varphi_0(t, x),$$

$$\bar{\varphi}_n(t, x, y, z) = \sup_{q \in \mathbb{Q}} \{\varphi(t, x, y, q) - n|z - q|\} - \varphi_0(t, x),$$

for $\varphi = f, g^i$ and $\varphi_0(t, x) = \varphi(t, x, 0, 0)$. By property (ii) of (H5), we derive that

$$|\varphi_n(t, x, y, z) - \varphi_n(t, x', y, z)| + |\bar{\varphi}_n(t, x, y, z) - \bar{\varphi}_n(t, x', y, z)| \leq 4L(1 + |x|^q + |x'|^q)|x - x'|,$$

which, together with Theorem 4.16 in [10] and Lemma 4.1, implies that both $\varphi_n(\cdot, X_{t+\delta}^{t,x}, y, z)$ and $\bar{\varphi}_n(\cdot, X_{t+\delta}^{t,x}, y, z)$ belong to $M^p_G(t, T)$ for each $p \geq 1$. □

Using the same notations appearing in the above argument, for each $(t, x) \in [0, T] \times \mathbb{R}^m$ and $n \in \mathbb{N}$, we consider a sequence of approximating $G$-BSDEs corresponding respectively to generators
\( (f_n, g_n^i) \) and \((f_n, g_n^{ii})\) on \([t, T]\),

\[
\begin{align*}
Y_{n,t,x} &= \Phi(X_{T,n}^{t,x}) + \int_t^T [f_n(r, X_{r}^{t,x}, Y_{r}^{n,t,x}, Z_{r}^{n,t,x}) + f_0(r, X_{r}^{t,x})]dr - \int_t^T Z_{r}^{n,t,x} dB_r \\
&\quad + \int_t^T [g_n^i(r, X_{r}^{t,x}, Y_{r}^{n,t,x}, Z_{r}^{n,t,x}) + g_0^i(r, X_{r}^{t,x})]d\langle B^i, B^i \rangle_r - (K_{n,t,x}^{r,t,x} - K_{s,t,x}^{r,t,x}), \\
\bar{Y}_{n,t,x} &= \Phi(X_{T,n}^{t,x}) + \int_t^T [f_n(r, X_{r}^{t,x}, \bar{Y}_{r}^{n,t,x}, \bar{Z}_{r}^{n,t,x}) + f_0(r, X_{r}^{t,x})]dr - \int_t^T \bar{Z}_{r}^{n,t,x} dB_r \\
&\quad + \int_t^T [g_n^i(r, X_{r}^{t,x}, \bar{Y}_{r}^{n,t,x}, \bar{Z}_{r}^{n,t,x}) + g_0^i(r, X_{r}^{t,x})]d\langle B^i, B^i \rangle_r - (\bar{K}_{n,t,x}^{r,t,x} - \bar{K}_{s,t,x}^{r,t,x}).
\end{align*}
\]

(4.3)

If we denote \( \tilde{u}^n(t, x) := Y_{n,t,x}^{t,x}, \bar{u}^n(t, x) := \bar{Y}_{n,t,x}^{t,x}, (t, x) \in [0, T] \times \mathbb{R}^m \).

By Proposition 4.2 in [7], both \( \tilde{u}^n \) and \( \bar{u}^n \) are continuous functions. Similarly we can define

\[
u(t, x) := Y_{t,x}^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^m.
\]

Clearly \( \nu(t, x) \) is a well-defined deterministic function from the above theorem. And some regularity can be derived from that of \( \tilde{u}^n \), \( \bar{u}^n \), indeed we have

**Lemma 4.3** Given assumption \((H5)\), \(u\) is a continuous function of polynomial growth.

**Proof.** Without loss of generality, assume that \(|\phi(z)| \leq L(1 + |z|)\). Setting \(w(y, z) = L(1 + |y| + |z|)\), consider the following \(G\)-BSDEs on \([t, T]\)

\[
U_s = \Phi(X_{T}^{t,x}) + \int_s^T [w(U_r, V_r) + f_0(r, X_{r}^{t,x})]dr + \int_s^T [w(U_r, V_r) + g_0^i(r, X_{r}^{t,x})]d\langle B^i, B^i \rangle_r \\
- \int_s^T V_r dB_r - (R_T - R_s),
\]

\[
U'_s = \Phi(X_{T}^{t,x}) + \int_s^T [w(U'_r, V'_r) + f_0(r, X_{r}^{t,x})]dr + \int_s^T [w(U'_r, V'_r) + g_0^i(r, X_{r}^{t,x})]d\langle B^i, B^i \rangle_r \\
- \int_s^T V'_r dB_r - (R'_T - R'_s).
\]

By (i) of lemma 3.2 and the comparison theorem 2.4, we have for each fixed \((t, x)\)

\[
U'_s \leq Y_{s,t,x}^{n,t,x} \leq Y_{s,t,x}^{t,x} \leq \bar{Y}_{s,t,x}^{n,t,x} \leq U_s, \forall s \in (t, T],
\]

Recalling Proposition 4.2 in [7], we can find some constant \(\bar{C}\) depending on \(L, G, m\) and \(T\) so that

\[
|U'_s| + |U_s| \leq \bar{C}(1 + |x|^q + 1),
\]

which indicates that \(u\) is of polynomial growth.

Applying (iv) of lemma 3.2 yields that for each \((t, x) \in [0, T] \times \mathbb{R}^m\),

\[
|\tilde{u}^n(t, x) - u(t, x)| \leq CG\phi\left(\frac{2L}{n - L}\right), \quad n > L,
\]

i.e., \(\tilde{u}^n(t, x)\) converges to \(u(t, x)\) uniformly in \((t, x)\). Consequently, \(u\) is continuous in \((t, x)\), which ends the proof. \(\Box\)

The main result of this section is,
Theorem 4.4 Let assumption (H5) be given. Then u is the unique viscosity solution to the following PDE:

\[
\begin{cases}
\partial_t u + G(H(t, x, u, D_x u, D^2_x u)) + \langle b(t, x), D_x u \rangle + f(t, x, u, \sigma^\top(t, x) D_x u) = 0, \\
u(T, x) = \Phi(x), \quad x \in \mathbb{R}^m,
\end{cases}
\]

(4.4)

where

\[
H_{ij}(t, x, v, p, A) = (\sigma^\top(t, x) A \sigma(t, x))_{ij} + 2\langle p, \sigma_{ij}(t, x) \rangle + 2g^i(t, x, v, \sigma(t, x))\mathbf{1}_{i=j}
\]

for any \((t, x, v, p, A) \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \times S_m\).

For reader’s convenience, we provide with the definition of viscosity solution to equation (4.4), see [3]. For every \( v \in \mathcal{C}([0, T] \times \mathbb{R}^m) \), denote by \( \mathcal{P}^{2,+}v(t, x) \) the “parabolic superjet” of \( v \) at \((t, x)\), which refers to the set of triples \((a, p, X) \in \mathbb{R} \times \mathbb{R}^m \times S_m\) such that

\[
v(s, y) \leq v(t, x) + a(s-t) + \langle p, y-x \rangle + \frac{1}{2} \nabla^2 X(y-x, y-x) + o(|s-t|+|y-x|^2).
\]

Similarly the “parabolic subjet” of \( v \) at \((t, x)\) can be defined by \( \mathcal{P}^{2,-}v(t, x) := -\mathcal{P}^{2,+}(-v)(t, x) \).

Definition 4.5 (i) For \( v \in \mathcal{C}([0, T] \times \mathbb{R}^m) \), \( v \) is a viscosity subsolution of (4.4) on \([0, T] \times \mathbb{R}^m\), if \( v(T, x) \leq \Phi(x) \) and for all \((t, x) \in (0, T) \times \mathbb{R}^m\),

\[
a + G(H(t, x, v(t, x), p, X)) + \langle b(t, x), p \rangle + f(t, x, v(t, x), \sigma(t, x)) \geq 0, \quad \text{for} \quad (a, p, X) \in \mathcal{P}^{2,+}v(t, x).
\]

(ii) A viscosity supersolution of equation (4.4) on \([0, T] \times \mathbb{R}^m\) refers to function \( v \in \mathcal{C}([0, T] \times \mathbb{R}^m) \) with \( v(T, x) \geq \Phi(x) \) such that for each \((t, x) \in (0, T) \times \mathbb{R}^m\),

\[
a + G(H(t, x, v(t, x), p, X)) + \langle b(t, x), p \rangle + f(t, x, v(t, x), \sigma(t, x)) \leq 0, \quad \text{for} \quad (a, p, X) \in \mathcal{P}^{2,-}v(t, x).
\]

A function \( v \in \mathcal{C}([0, T] \times \mathbb{R}^m) \) is called a viscosity solution of equation (4.4) if it is simultaneously a viscosity subsolution and a viscosity supersolution of equation (4.4) on \([0, T] \times \mathbb{R}^m\).

The proof of Theorem 4.4. Since the uniqueness of viscosity solution to equation (4.4) is well established, c.f. [17, 21], by the symmetry of supersolution and subsolution, we only check that \( u \) is a viscosity subsolution.

Given \((t, x) \in (0, T) \times \mathbb{R}^n\) and \((a, p, X) \in \mathcal{P}^{2,+}u(t, x)\), since \( u^n \) converges to \( u \) uniformly in \((t, x)\), we get that

\[
\lim_{n \to \infty} |u^n(t_n, x_n) - u(t, x)| = 0,
\]

whenever \((t_n, x_n) \to (t, x)\). With the help of Proposition 4.3 in [3], there exist sequences

\[
n_k \to \infty, \quad (t_k, x_k) \to (t, x), \quad \text{and} \quad (a_k, p_k, X_k) \to (a, p, X),
\]

such that

\[
(a_k, p_k, X_k) \in \mathcal{P}^{2,+}u^{n_k}(t_k, x_k).
\]
From the Feynman-Kac formula in [7], we know $u^n(t, x)$ is the unique viscosity solution to
\[
\begin{align*}
\partial_t V + G(H^n(t, x, V, D_x V, D_x^2 V)) + \langle b(t, x), D_x V \rangle + \int_n(t, x, V, \sigma^\top(t, x) D_x V) + f_0(t, x) &= 0, \\
v(T, x) &= \Phi(x), \quad x \in \mathbb{R}^m,
\end{align*}
\]
with
\[
H^n_{ij}(t, x, v, p, A) = (\sigma^\top(t, x) A \sigma(t, x))_{ij} + 2\langle p, h_{ij}(t, x) \rangle + 2\tilde{g}_i(t, x, v, \sigma(t, x)) 1_{\{i=j\}} + 2\tilde{g}_0(t, x) 1_{\{i=j\}}.
\]
Thus by the definition of viscosity solution we derive that
\[
a_k + G(H^n(t, x, u^n_k(t, x_k), p_k, X_k)) + \langle b(t, x_k), p_k \rangle + f(t, x, u^n_k(t, x_k), p_k \sigma(t, x_k)) + f_0(t, x_k) \geq 0.
\]
Recalling (i) of lemma 3.2, we obtain that for $\varphi = f, g$,
\[
\varphi(t, x, y, z) \geq \varphi_0(t, x) + \varphi_0(t, x), \quad \text{for all } (t, x, y, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d,
\]
which implies that $H(t, x, v, p, A) \geq H^n(t, x, v, p, A)$. Then we derive that
\[
a_k + G(H(t, x, u^n_k(t, x_k), p_k, X_k)) + \langle b(t, x_k), p_k \rangle + f(t, x, u^n_k(t, x_k), p_k \sigma(t, x_k)) \geq 0.
\]
Sending $k \to \infty$, we get
\[
a + G(H(t, x, u(t, x), p, X)) + \langle b(t, x), p \rangle + f(t, x, u(t, x), p \sigma(t, x)) \geq 0,
\]
which is the desired result. \hfill $\square$

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