ON THE LOCAL BEHAVIOUR OF SPECIALIZATIONS OF
FUNCTION FIELD EXTENSIONS

JOACHIM KÖNIG, FRANÇOIS LEGRAND, AND DANNY NEFTIN

Abstract. Given a field $k$ of characteristic zero and an indeterminate $T$ over $k$, we investigate the local behaviour at primes of $k$ of finite Galois extensions of $k$ arising as specializations of finite Galois extensions $E/k(T)$ (with $E/k$ regular) at points $t_0 \in \mathbb{P}^1(k)$. We provide a general result about decomposition groups at primes of $k$ in specializations, extending a fundamental result of Beckmann concerning inertia groups. We then apply our result to study crossed products, the Hilbert–Grunwald property, and finite parametric sets.

1. Introduction

1.1. Grunwald problems. Given a number field $k$, Grunwald problems concern the structure of completions $L_p/k_p$ of $G$-extensions $L/k$ at any finitely many given primes $p$ of $k$. Here, a $G$-extension $L/k$ is a finite Galois extension of $k$ with Galois group $G$, and $k_p$ (resp., $L_p$) denotes the completion of $k$ (resp., of $L$) at $p$ (resp., at a prime $\mathfrak{P}$ of $L$ lying over $p$). The original motivation for such problems arose from their key role in the structure theory of finite dimensional division algebras over number fields. Since then, they have received much attention, especially due to connections with the regular inverse Galois problem; see [DG11, DG12], and with weak approximation; see, e.g., [Har07, Dem10, LA14, DLAN17].

More precisely, given a finite set $S$ of primes of $k$, and given finite Galois extensions $L^{(p)}/k_p$, $p \in S$, with Galois groups embedding into $G$, the Grunwald problem $(G, (L^{(p)}/k_p)_{p \in S})$ asks whether there is a $G$-extension $L/k$ whose completion $L_p$ at $p$ is $k_p$-isomorphic to $L^{(p)}$, $p \in S$. If such an extension $L/k$ exists, it is called a solution to the Grunwald problem $(G, (L^{(p)}/k_p)_{p \in S})$. We note that, instead of prescribing the local extensions $L_p/k_p$, $p \in S$, weaker versions ask for the existence of a $G$-extension $L/k$ with prescribed local degrees $[L_p : k_p]$, $p \in S$, or with prescribed local Galois groups $\text{Gal}(L_p/k_p)$, $p \in S$. These weaker versions usually suffice for applications to classical problems, see, e.g., [Wan50, Corollary 2] and [Sch68].

Date: October 25, 2018.

Key words and phrases. Galois theory, local behaviour, specializations, Grunwald problems, crossed products, parametric sets.

1Recall that the latter is independent of the choice of $\mathfrak{P}$ (up to $k_p$-isomorphism).
Examples of Grunwald problems \((G, (L^{(p)}/k^p)_{p \in S})\) with no solution \(L/k\) occur already for cyclic groups \(G\), when \(S\) contains a prime of \(k\) lying over 2 [Wan48]. However, it is expected [Har07, §1] that, for solvable groups \(G\), every Grunwald problem \((G, (L^{(p)}/k^p)_{p \in S})\) has a solution, provided \(S\) is disjoint from some finite set \(S_{\text{exc}}\) of “exceptional” primes of \(k\), depending only on \(G\) and \(k\). This is known when (1) \(G\) is abelian, and \(S_{\text{exc}}\) is the set of primes of \(k\) dividing 2 [NSW08 (9.2.8)]; (2) \(G\) is an iterated semidirect product \(A_1 \rtimes (A_2 \rtimes \cdots \rtimes A_n)\) of finite abelian groups, and \(S_{\text{exc}}\) is the set of primes of \(k\) dividing \(|G|\); see [Har07, Théorème 1] and [DLAN17, Theorem 1.1]; (3) \(G\) is solvable of order prime to the number of roots of unity in \(k\), and \(S_{\text{exc}} = \emptyset\) [NSW08 (9.5.5)]; and (4) there exists a generic extension for \(G\) over \(k\), and \(S_{\text{exc}} = \emptyset\) [Sal82, Theorem 5.9]. Among the above, the latter is the only method which applies to non-solvable groups. However, the family of non-solvable groups for which a generic extension is known is quite restricted, e.g., it is unknown whether the alternating group \(A_n\) has a generic extension for \(n \geq 6\). See [JLy02] for an overview on generic extensions.

The main source of realizations of non-solvable groups \(G\) over \(k\) is via \(k\)-regular \(G\)-extensions, that is, via \(G\)-extensions \(E/k(T)\), where \(T\) is an indeterminate over \(k\) and \(E\) is algebraically closed in \(E\). Indeed, by Hilbert’s irreducibility theorem, every non-trivial \(k\)-regular \(G\)-extension \(E/k(T)\) has infinitely many linearly disjoint specializations \(E_{t_0}/k\), \(t_0 \in \mathbb{P}^1(k)\), with Galois group \(G\). Many groups have been realized by this method; see, e.g., [MM99], and references within, as well as [Zyw14] for more recent examples.

This specialization process provides a natural way to attack Grunwald problems for \(k\)-regular Galois groups, that is, for finite groups \(G\) admitting a \(k\)-regular \(G\)-extension of \(k(T)\). Namely, given such an extension \(E/k(T)\), it is natural to ask for the local behaviour of specializations \(E_{t_0}/k\), \(t_0 \in \mathbb{P}^1(k)\). That is, which local extensions \(L^{(p)}/k_p\), which local Galois groups \(\text{Gal}(L^{(p)}/k_p)\), and which local degrees \([L^{(p)}/k_p]: k_p]\) arise by completing the specialization \(E_{t_0}/k\) at primes \(p\) of \(k\), when \(t_0\) runs over \(\mathbb{P}^1(k)\)? For points \(t_0 \in \mathbb{P}^1(k)\) which are \(p\)-adically far from branch points of \(E/k(T)\), this approach was deeply investigated by Dèbes and Ghazi [DG11, DG12] and applies only to unramified local extensions. Namely, given a \(k\)-regular \(G\)-extension \(E/k(T)\), a finite set \(S\) of primes of \(k\), disjoint from some finite set \(S_{\text{exc}} := \text{exc}(E/k(T))\) (depending only on \(E/k(T)\)), and unramified extensions \(L^{(p)}/k_p\) with Galois group embedding into \(G\), \(p \in S\), [DG12] provides \(t_0 \in k\) such that \(E_{t_0}/k\) is a solution to the Grunwald problem \((G, (L^{(p)}/k_p)_{p \in S})\).

1.2. Main results. The goal of this paper is to study the local behaviour at a prime \(p\) of \(k\) of specializations \(E_{t_0}/k\), when \(t_0\) is \(p\)-adically close to a branch point of \(E/k(T)\).

1.2.1. Background. A first related conclusion can be derived from the algebraic cover theory of Grothendieck. Namely, if \(p\) is not in some finite set \(S_{\text{exc}} := \text{exc}(E/k(T))\) of primes of \(k\) and if \(p\) ramifies in the specialization \(E_{t_0}/k\), then, \(t_0\) and some branch
point $t_i$ of $E/k(T)$ meet modulo $p$. In the special case where $t_i$ is $k$-rational and $v_p(t_0)$ is non-negative, this means that $a_p := v_p(t_0 - t_i)$ is positive, where $v_p$ denotes the normalized $p$-adic valuation. A fundamental theorem by Beckmann then asserts that the inertia group $I_{t_0,p}$ of $E_{t_0}/k$ at $p$ is determined by $a_p$ and the inertia group $I_{t_i}$ at a fixed prime of $E$ lying over the prime $P_i$ of $k[T - t_i]$ generated by $T - t_i$. Namely, $I_{t_0,p}$ is conjugate to $I_{t_i}^{p}$. We refer to [3] for more details.

1.2.2. Decomposition groups of specializations. Given $t_0 \in \mathbb{P}^1(k)$, assumed to meet the branch point $t_i$ modulo $p$, we show that the entire decomposition group at $p$ of the specialization $E_{t_0}/k$ is determined by the local behaviour at $t_i$, thus extending Beckmann’s theorem. Namely, suppose for simplicity that $t_i$ is $k$-rational, and let $D_{t_i}$ denote the geometric decomposition group at a fixed prime of $E$ lying over the geometric prime $P_i$. Recall that the specialization $E_{t_i}/k$ has Galois group $D_{t_i}/I_{t_i}$; let $\varphi_i : D_{t_i} \to D_{t_i}/I_{t_i}$ be the natural projection. Let $D_{t_i,p}$ be the decomposition group at a prime $\mathfrak{p}$ of $E_{t_i}$ lying over $p$; this is a subgroup of $D_{t_i}/I_{t_i}$. Note that, up to conjugation, the subgroup $\varphi_i^{-1}(D_{t_i,p})$ of $G$ is independent of the choice of an arithmetic prime $\mathfrak{P}$ lying over $p$ and a geometric prime lying over $P_i$.

Theorem 1.1. There exists a finite set $\mathcal{S}_{\text{exc}} := \mathcal{S}_{\text{exc}}(E/k(T))$ of primes of $k$ that satisfies the following property. Suppose that the given prime $p$ of $k$ is outside $\mathcal{S}_{\text{exc}}$. Moreover, suppose that the given branch point $t_i$ is $k$-rational, that $t_0$ and $t_i$ meet modulo $p$, and that the exponent $a_p$ is coprime to $|I_{t_i}|$. Then, the decomposition group of $E_{t_0}/k$ at $p$ is conjugate by an element of $G$ to $\varphi_i^{-1}(D_{t_i,p})$.

Note that $\varphi_i^{-1}(D_{t_i,p})$ is the group generated by $I_{t_i}$ and a lift of the Frobenius of $E_{t_i}/k$ at $p$. We refer to Theorem 4.1 for a more general version of Theorem 1.1, where we relax the assumptions on $t_i$ and $a_p$, and which is stated over more general base fields.

1.2.3. Solving Grunwald problems. Theorem 1.1 shows that the decomposition group of $E_{t_0}/k$ at $p$ is determined by the local data $(\varphi_i, D_{t_i,p})$ at $t_i$, when $t_0$ and $t_i$ meet modulo $p$. Theorem 1.2 below shows that this is the only constraint on completions of $E_{t_0}/k$ at $p$ for such specialization points $t_0$.

Theorem 1.2. Suppose that $\mathcal{S}$ is a finite set of primes of $k$ disjoint from some finite set of primes $\mathcal{S}'_{\text{exc}} := \mathcal{S}'_{\text{exc}}(E/k(T))$. For each $p \in \mathcal{S}$, fix a $k$-rational branch point $t_i(p)$ of $E/k(T)$ and a finite Galois extension $L^p/k_p$ with Galois group (resp., inertia group) $\varphi_i^{-1}(D_{t_i(p),p})$ (resp., $I_{t_i(p)}$). Then, there exists $t_0 \in k$ such that $E_{t_0}/k$ is a solution to the Grunwald problem $(G, (L^p/k_p)_{p \in \mathcal{S}})$.

See Theorem 4.4 for a more general version of Theorem 1.2, where the assumption on the branch points is relaxed, and where $k$ is not necessarily a number field.

\footnote{Note that the set $\mathcal{S}_{\text{exc}}$ is chosen such that every $t_0$ meets at most one branch point modulo $p$.}
Given a single homomorphism $\varphi_i: D_{t_i} \to D_{t_i}/I_{t_i}$ at a $(k$-rational) branch point $t_i$ of $E/k(T)$, Theorem 1.2 provides a solution to all Grunwald problems $(G, (L^{(p)}/k_p)_{p \in S})$, where, for each $p$ in $S$, the local extension $L^{(p)}/k_p$ has inertia group $I_{t_i}$ and decomposition group $\varphi_i^{-1}(D_{t_i,p})$. Such results are especially applicable to problems where the primes are allowed to vary and the decomposition groups are fixed; see, in particular, §1.3.1.

1.2.4. About the proof. Let $R$ be the ring of integers of $k$. Our approach to Theorems 1.1 and 1.2 considers the fraction field $F$ of the two-dimensional domain $R[[T-t_i]]$, which is also the completion of $R[T]$ at the ideal generated by $p$ and $T-t_i$. We use a theorem of Eisenstein, recalled as Theorem 2.4, to show that, for all but finitely many primes $p$ of $k$, the group $\text{Gal}(E \cdot F/F)$ is determined by the local data $(\varphi_i, D_{t_i,p})$ at $t_i$. On the other hand, we show that, by reducing the extension $E \cdot F/F$ at a place extending the place $(T-t_0)$ of $k(T)$, the resulting extension of residue fields is the completion of $E_{t_0}/k$ at $p$, giving the desired connection between $D_{t_0,p}$ and the local data at $t_i$. See §4.2. The above method is considerably different from that of Dèbes and Ghazi, which, to our knowledge, does not apply to ramified extensions; see [DG12, §1.6], and which is based on specializations of extensions of $k_p(T)$.

We use the theory of Brauer embedding problems to determine the extensions $M/k_p$ which satisfy the constraints imposed by Theorem 1.1 on completions of specializations $(E_{t_0})_p$, and the above reduction process to show that such fields $M$ are obtained as specializations $(E_{t_0})_p$.

1.3. Applications. Theorem 1.1 is then used to study crossed product division algebras, the Hilbert–Grunwald property, and finite parametric sets.

1.3.1. Prescribing decomposition groups and crossed products. Recall that the Galois group $D$ of a tamely ramified extension of $k_p$ is metacyclic, that is, $D$ admits a cyclic normal subgroup $I$ such that $D/I$ is cyclic. The following application of Theorem 1.1 ensures the appearance of certain metacyclic subgroups of $G$ as decomposition groups of specializations of the $k$-regular $G$-extension $E/k(T)$.

**Theorem 1.3.** Given a branch point $t_i$ of $E/k(T)$, let $D_{t_i}$ (resp., $I_{t_i}$) denote the decomposition (resp., inertia) group at $t_i$, and let $\tau$ be an element of $D_{t_i}$. Then, there exist infinitely many primes $p$ of $k$, and, for each such prime $p$, infinitely many $t_0 \in k$ such that $E_{t_0}/k$ is a $G$-extension with decomposition group $\langle I_{t_i}, \tau \rangle$ at $p$.

We refer to Theorem 4.6 for a generalization of Theorem 1.3 where we consider finitely many primes of $k$ at the same time.

We then apply this result to prove the existence of $G$-crossed product division algebras over number fields for various finite groups $G$. Recall that a finite dimensional division algebra over its center $k$ is a $G$-crossed product if it admits a maximal subfield $L$ that
is finite and Galois over $k$, and such that $\text{Gal}(L/k) = G$. A $G$-crossed product division algebra is equipped with an explicit structure, which plays a key role in the theory of central simple algebras. By Schacher’s result [Sch68, Theorem 4.1], the existence of a $G$-crossed product division algebra with center $\mathbb{Q}$ implies that $G$ has metacyclic Sylow subgroups. The converse is a long standing open conjecture, originating in [Sch68]: for every finite group $G$ with metacyclic Sylow subgroups, there exists a $G$-crossed product division algebra with center $\mathbb{Q}$. Although this conjecture has been extensively studied, cf. [ABGV11, Section 11.A], the only finite non-abelian simple groups for which the conjecture is known to hold are $A_5 \cong \text{PSL}(2, \mathbb{F}_4)$ [GS79, Theorem 1], $A_6 \cong \text{PSL}(2, \mathbb{F}_9)$ [FS91, Theorem 6], $A_7$ [FS91, Theorem 6], $\text{PSL}(2, \mathbb{F}_7)$ [AS01, Proposition 5], $\text{PSL}(2, \mathbb{F}_{11})$ [AS01, Theorem 2], and the Mathieu group $M_{11}$ [KN17].

Given a $k$-regular $G$-extension $E/k(T)$ with suitable properties, we use Theorem 1.3 to establish a general criterion for the existence of specialization points $t_0 \in k$ such that $E_{t_0}$ is a maximal subfield of a $G$-crossed product. We refer to Theorem 5.2 for our precise result. We then use this criterion to derive the first infinite family of finite non-abelian simple groups $G$ with a $G$-crossed product division algebra over $\mathbb{Q}$.

**Theorem 1.4.** Let $p$ be a prime number such that either $p \equiv 3 \pmod{8}$ or $p \equiv 5 \pmod{8}$. Then, there exists a $\text{PSL}(2, \mathbb{F}_p)$-crossed product division algebra with center $\mathbb{Q}$.

See Theorem 5.3 where we show more generally that Theorem 1.4 holds over an arbitrary number field $k$ (instead of $\mathbb{Q}$), provided primitive 4-th roots of unity are not in $k$. See also Theorem 5.4 where a similar conclusion is shown to hold for prime numbers lying in other arithmetic progressions.

### 1.3.2. On the Hilbert-Grunwald property.

Following a terminology due to Dèbes and Ghazi, and motivated by the partial positive answer provided by Theorem 1.2, one may ask whether a given $k$-regular $G$-extension $E/k(T)$ has the Hilbert–Grunwald property, that is, whether there exists a finite set $S_{\text{exc}}$ of primes of $k$ such that every Grunwald problem $(G, (L^p/kp)_{p \in S})$, with $S$ disjoint from $S_{\text{exc}}$, has a solution inside the set of specializations of $E/k(T)$. Recall that this property is always fulfilled, provided we restrict to unramified Grunwald problems [DG11, DG12].

In contrast, our results on the local behaviour of (ramified) specializations show in particular that, for many finite groups $G$, no $k$-regular $G$-extension of $k(T)$ has the Hilbert-Grunwald property.

**Theorem 1.5.** Assume that $G$ has a non-cyclic abelian subgroup. Then, no $k$-regular $G$-extension of $k(T)$ has the Hilbert–Grunwald property.

We refer to Theorem 6.2 for a more general version with finitely many $k$-regular $G$-extensions at the same time. Groups with a non-cyclic abelian subgroup have been classified in the literature relatively explicitly; see, e.g., the classical papers [Zas35] and
Examples of such groups contain, in addition to the obvious $S_n$ for $n \geq 4$ and $D_n$ with $n$ even, all non-abelian simple groups; see the proof of Corollary 6.5. In particular, these groups satisfy the conclusion of Theorem 1.5. Recall that there are many non-abelian simple groups $G$ such that the only known realizations of $G$ over $k$ are specializations of $k$-regular $G$-extensions of $k(T)$, and only finitely many $k$-regular $G$-extensions of $k(T)$ are known. For such finite groups $G$, Theorem 1.5 implies that one cannot solve all Grunwald problems for the group $G$ over the number field $k$ by using only the currently known realizations of $G$ over $k$.

1.3.3. Non-existence of finite parametric sets. As in [KL18], given a finite group $G$, we call a set $S$ of $k$-regular $G$-extensions of $k(T)$ parametric if every $G$-extension of $k$ occurs as a specialization of some extension $E/k(T)$ in $S$; see Definition 7.1. If $S$ consists of a single extension $E/k(T)$, the extension $E/k(T)$ is called parametric. This is a generalization of the Beckmann-Black problem (over $k$), which, with our phrasing, asks whether every finite group $G$ has a parametric set over $k$.[4] Here, we ask whether a given finite group $G$ has a finite parametric set over $k$.

On the one hand, a given finite group $G$ has a parametric extension over $k$, provided $G$ has a one parameter generic polynomial over $k$. However, this latter condition is very restrictive. For example, in the case $k = \mathbb{Q}$, only the subgroups of the symmetric group $S_3$ have a one parameter generic polynomial; see [JLY02, §2.1 & §8.2]. On the other hand, no finite group $G$ was known to have no finite parametric set over the given number field $k$, until a recent joint work by the first two authors [KL18]. However, the “global” strategy developed in that paper requires $G$ to have a non-trivial proper normal subgroup which satisfies some further properties. In particular, this cannot be used for finite simple groups.

As a further application of our “local” results, we obtain the first examples of finite non-abelian simple groups without finite parametric sets:

**Theorem 1.6.** Let $n \geq 4$ be an integer. Then, the alternating group $A_n$ has no finite parametric set over $k$.

We refer to Theorem 7.2 where we show more generally that a given finite group $G$ has no finite parametric set over the number field $k$, provided $G$ has a non-cyclic abelian subgroup and some very weak Grunwald property holds. This last property holds in particular if $G = A_n$, by Mestre [Mes90]; see Corollary 7.3.

---

[4] Of course, up to obvious manipulations such as changes of variable, or translates by rational extensions $k(s)/k(t)$, both of which do not yield new specializations.

[5] See, e.g., the survey paper [Deb01] for more on the Beckmann-Black problem.

[6] Note that no finite group $G$ with a finite parametric set over $k$, but with no one parameter generic polynomial over $k$, is available in the literature.
Acknowledgments. This work is partially supported by the Israel Science Foundation (grant No. 577/15), and the Technion startup grant.

2. Notation and preliminaries

The section is organized as follows. §2.1 is devoted to standard background on Dedekind domains, while we recall classical material on function field extensions in §2.2.

2.1. Basics on Dedekind domains. For more on below, we refer to [Ser79].

2.1.1. Residue, localization, and completion. Let $R$ be a domain of characteristic zero, and $k$ its fraction field. Given a prime $\mathfrak{p}$ of $k$, i.e., given a non-zero prime ideal $\mathfrak{p}$ of $R$, the residue field $\overline{k}_\mathfrak{p}$ of $R$ at $\mathfrak{p}$ is the fraction field $\text{Frac}(R/\mathfrak{p})$ of $R/\mathfrak{p}$. We shall assume that $\overline{k}_\mathfrak{p}$ is perfect. The local ring $R_{(\mathfrak{p})} := \{x/y : (x, y) \in R^2 \text{ and } y \notin \mathfrak{p}\}$ is the localization of $R$ at $\mathfrak{p}$. The unique maximal ideal of $R_{(\mathfrak{p})}$ is $\mathfrak{p}R_{(\mathfrak{p})}$, and the corresponding residue field $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ is canonically isomorphic to $\overline{k}_\mathfrak{p}$. If $\mathfrak{p}R_{(\mathfrak{p})}$ is principal, there is a discrete valuation $v_\mathfrak{p}$ on $k$ whose valuation ring is $R_{(\mathfrak{p})}$. Let $k_\mathfrak{p}$ be the completion of $k$ with respect to $v_\mathfrak{p}$. There is a unique valuation on $k_\mathfrak{p}$ extending $v_\mathfrak{p}$, the residue field of which coincides with $\overline{k}_\mathfrak{p}$ (up to canonical isomorphism).

2.1.2. Dedekind domains. From now on, we assume that $R$ is a Dedekind domain.

Let $L/k$ be a finite extension. The integral closure $S$ of $R$ in $L$ is also a Dedekind domain. Let $\mathfrak{P}$ be a prime of $L$ lying over $\mathfrak{p}$. The residue field $\overline{k}_\mathfrak{P} := S/\mathfrak{P}$ is a finite extension of $\overline{k}_\mathfrak{p}$. The residue degree of $L/k$ at $\mathfrak{P}$ is the degree $f_{\mathfrak{P}/\mathfrak{p}} := [L_\mathfrak{P} : \overline{k}_\mathfrak{P}]$ of this finite extension. The maximal positive integer $e := e_{\mathfrak{P}/\mathfrak{p}}$ for which $\mathfrak{P}^e$ is contained in $\mathfrak{p}S$ is the ramification index of $\mathfrak{P}$ in $L/k$. The prime $\mathfrak{p}$ is ramified in $L/k$ if there exists a prime $\mathfrak{P}$ lying over it with ramification index $e_{\mathfrak{P}/\mathfrak{p}} > 1$, and unramified otherwise. It is totally ramified if $e_{\mathfrak{P}/\mathfrak{p}} = [L : k]$ for the unique prime $\mathfrak{P}$ lying over $\mathfrak{p}$, and totally split in $L/k$ if $\mathfrak{p}$ is unramified in $L/k$, and if $f_{\mathfrak{P}/\mathfrak{p}} = 1$ for every prime $\mathfrak{P}$ of $L$ lying over $\mathfrak{p}$.

From now on, we assume that the extension $L/k$ is Galois with Galois group $G$. The subgroup of $G$ which consists of all elements $\sigma$ such that $\sigma(\mathfrak{P}) = \mathfrak{P}$ is the decomposition group $D_{\mathfrak{P}}$ of $L/k$ at $\mathfrak{P}$. The residue extension $\overline{k}_\mathfrak{P}/\overline{k}_\mathfrak{p}$ is Galois, and the restriction map $D_{\mathfrak{P}} \to \text{Gal}(\overline{k}_\mathfrak{P}/\overline{k}_\mathfrak{p})$ is an epimorphism, whose kernel is the inertia group $I_{\mathfrak{P}}$ of $L/k$ at $\mathfrak{P}$. The decomposition groups $D_{\mathfrak{P}_1}$ and $D_{\mathfrak{P}_2}$ (resp., the inertia groups $I_{\mathfrak{P}_1}$ and $I_{\mathfrak{P}_2}$) of two primes $\mathfrak{P}_1$ and $\mathfrak{P}_2$ lying over $\mathfrak{p}$ in $L/k$ are conjugate in $G$. When a prime $\mathfrak{P}$ lying over $\mathfrak{p}$ is fixed, we set $D_{\mathfrak{P}} := D_{\mathfrak{P}}$ and $I_{\mathfrak{P}} := I_{\mathfrak{P}}$. One has $I_{\mathfrak{P}} \leq D_{\mathfrak{P}}$, the cardinalities $|I_{\mathfrak{P}}|$ and $|D_{\mathfrak{P}}|$ are independent of the choice of the prime $\mathfrak{P}$ lying over $\mathfrak{p}$, and are equal to $e_{\mathfrak{P}/\mathfrak{p}}$ and $e_{\mathfrak{P}/\mathfrak{p}}f_{\mathfrak{P}/\mathfrak{p}}$, respectively. Similarly, the residue extension $\overline{k}_{\mathfrak{P}}/\overline{k}_{\mathfrak{p}}$ of $L/k$ at $\mathfrak{P}$ does not depend on the choice of $\mathfrak{P}$ (up to $\overline{k}_{\mathfrak{p}}$-isomorphism); we therefore denote it by $\overline{k}_{\mathfrak{P}}/\overline{k}_{\mathfrak{p}}$. The following basic lemma describes explicitly the residue field $\overline{k}_{\mathfrak{p}}$. 

The local behaviour of specializations 7
Lemma 2.1. Let \( f(X) \in R[X] \) be a monic separable polynomial of positive degree \( n \) and splitting field \( L \) over \( k \). Denote the roots of \( f(X) \) by \( y_1, \ldots, y_n \). Let \( p \) be a prime of \( k \) such that the discriminant of \( f \) is not contained in \( p \). Then, one has \( \mathcal{L} = \mathcal{P}(y_1, \ldots, y_n) \), where \( \mathcal{P} \) denotes the reduction of \( x \in L \) modulo a prime of \( L \) lying over \( p \).

The completion \( L\mathcal{P} \) of \( L \) at \( \mathcal{P} \) is Galois over \( k\mathcal{P} \), with Galois group canonically isomorphic to \( D \) (via restriction from \( L\mathcal{P} \) to \( L \)). One has \( L\mathcal{P} = L \cdot k\mathcal{P} \), where \( L \cdot k\mathcal{P} \) denotes the compositum of \( L \) and \( k\mathcal{P} \) inside a given algebraic closure \( k\mathcal{P} \) of \( k\mathcal{P} \). Since, up to \( k\mathcal{P} \)-isomorphism, this does not depend on the choice of \( \mathcal{P} \), one can speak of the completion of \( L/k \) at \( p \), and denote it by \( L_p/k_p \). The maximal subextension of \( L_p/k_p \) which is unramified at \( p \) equals \( L^p/k^p \), where \( L_k^p \) denotes the fixed field of \( I_p \) in \( L_p \); we denote it by \( L^p_{ur}/k_p \), and call it the unramified part of \( L_p/k_p \). Furthermore, fixing a \( k \)-embedding \( \sigma \) of \( L \) into \( \mathcal{P} \) induces a choice of a prime \( \mathcal{P} \) lying over \( p \) in \( L/k \), via \( \mathcal{P} := \{ x \in L : \nu_p(\sigma(x)) > 0 \} \), where \( \nu_p \) is the \( p \)-adic valuation on \( k_p \), extended to \( k\mathcal{P} \).

2.2. Extensions of function fields. Let \( \widetilde{k} \) denote a fixed algebraic closure of a given field \( k \) of characteristic zero, and let \( T \) be an indeterminate over \( k \).

2.2.1. Classical background. Let \( E/k(T) \) be a finite Galois extension with Galois group \( G \), and such that \( E/k \) is regular, that is, such that \( E \cap \widetilde{k} = k \). We then say that \( E/k(T) \) is a \( k \)-regular \( G \)-extension.

A point \( t_0 \in \mathbb{P}^1(\widetilde{k}) \) is a branch point of \( E/k(T) \) if the prime ideal \( (T - t_0) \widetilde{k}[T - t_0] \) of \( \widetilde{k}[T - t_0] \) is ramified in \( E \cdot \widetilde{k}/k(T) \). The extension \( E/k(T) \) has finitely many branch points, denoted by \( t_1, \ldots, t_r \) (one has \( r = 0 \) if and only if \( G \) is trivial). For each \( i \in \{1, \ldots, r\} \), denote the decomposition group (resp., the inertia group) of \( E(t_i)/k(t_i)(T) \) at (a fixed prime of \( E(t_i) \) lying over) \( (T - t_i) k(t_i)[T - t_i] \) by \( D_{t_i} \) (resp., by \( I_{t_i} \)).

Given \( t_0 \in \mathbb{P}^1(k) \), the residue extension of \( E/k(T) \) at \( (T - t_0) k[T - t_0] \) is denoted by \( E_{t_0}/k \), and called the specialization of \( E/k(T) \) at \( t_0 \). It is a finite Galois extension whose Galois group is canonically isomorphic to \( D_{t_0}/I_{t_0} \), where \( D_{t_0} \) (resp., \( I_{t_0} \)) denotes the decomposition group (resp., the inertia group) of \( E/k(T) \) at \( (T - t_0) k[T - t_0] \).

If \( k \) is the fraction field of a Dedekind domain \( R \), we denote the decomposition group (resp., the inertia group) of the specialization \( E_{t_0}/k \) at a given non-zero prime ideal \( p \) of \( R \) such that the residue field \( \mathcal{L}_p \) is perfect by \( D_{t_0,p} \) (resp., by \( I_{t_0,p} \)). Note that \( D_{t_0,p} \) is canonically isomorphic to a subgroup of \( D_{t_0}/I_{t_0} \).

We recall the following well-known lemma, cf. [PV05], which is a standard consequence of Krasner’s lemma and the compatibility between the Hilbert specialization property and the weak approximation property of \( \mathbb{P}^1 \).

---

\( \footnote{Replace \( T - t_0 \) by \( 1/T \) if \( t_0 = \infty \). The compositum \( E \cdot \widetilde{k} \) of \( E \) and \( \widetilde{k}(T) \) is taken in a fixed algebraic closure of \( k(T) \) containing \( \widetilde{k} \).} \)
Lemma 2.2. Assume $k$ is a number field. Let $S$ be a finite set of primes of $k$, and $P(T, Y) \in k[T][Y]$ a monic, separable polynomial with splitting field $E$ over $k(T)$. For each $p \in S$, choose $t_p$ in $k$ such that $P(t_p, Y)$ is separable, and let $(E_{t_p})_p/k_p$ be the completion of $E_{t_p}/k$ at $p$. Then, there exist infinitely many $t_0 \in k$ such that $E_{t_0}/k$ has Galois group $G$ and its completion at $p$ is $(E_{t_0})_p/k_p$ for each $p \in S$. Moreover, one may require these specializations $E_{t_0}/k$ to be pairwise linearly disjoint.

2.2.2. Laurent series fields and their algebraic extensions. Given $t_0 \in \mathbb{P}^1(k)$, set $p := (T-t_0) k[T-t_0]$. The unramified part $E^\ur_p/k((T-t_0))$ of the completion $E_p/k((T-t_0))$ of $E/k(T)$ at $p$ is of the form $E_{t_0}((T-t_0))/k((T-t_0))$; see [Ser79, page 55]. Furthermore, one has $E_p = E^\ur_p(\sqrt{\pi})$ for some element $\pi \in E^\ur_p$ of $p$-adic valuation 1. Since $\pi = \alpha(T-t_0)(1+\beta)$ for some $\alpha \in E_{t_0}$ and $\beta \in pE^\ur_p$, and since $1+\beta$ is an $e$-th power in $E^\ur_p$, we may replace $\pi$ by $\alpha(T-t_0)$, so that $E_p = E^\ur_p(\sqrt{\alpha(T-t_0)})$. Moreover, since $E_p$ is the splitting field of $X^e - \alpha(T-t_0)$, the $e$-th roots of unity are contained in $E_p$, and then even in $E_p \cap \tilde{k} = E_{t_0}$, giving the following lemma.

Lemma 2.3. Let $t_i$ be a branch point of $E/k(T)$ of ramification index $e_i$. Then, the specialization $E(t_i)_{t_i}$ contains all $e_i$-th roots of unity.

Setting $k' := k(t_i)$, $E' := E(t_i)$, and $p_i := (T-t_i) k'[T-t_i]$, the completion $E'_{p_i}$ is then a solution to the embedding problem $D_{t_i} \rightarrow \varphi \text{Gal}(E^\ur_{p_i}/k'((T-t_i)))$, where the image is identified with $D_{t_i}/I_{t_i}$ and $\varphi$ is the natural projection. Since $E'_{p_i}$ is a Kummer extension of $E'_{t_i}((T-t_i))$, the conjugation action of $D_{t_i}$ on $I_{t_i}$ is isomorphic to the action on the group of $e_i$-th roots of unity in $E'_{t_i}$. Such an embedding problem is then called a Brauer embedding problem. See, e.g., [MM99, Chapter IV, §7] for more details.

Finally, we will make use of a theorem about the structure of algebraic power series, which was first stated over the integers by Eisenstein. For our purposes, let $R$ be an integral domain with fraction field $k$. The general version below can be found, e.g., in [BBC12, Lemma 2.1].

Theorem 2.4. Let $\alpha = \sum_{n=0}^{\infty} \alpha_n T^n \in k[[T]]$ be algebraic over $k(T)$. Then, there exist $r$ and $s$ in $R$ such that $r \cdot (\alpha_n s^n) \in R$ for each $n \geq 0$. In particular, if $R$ is a Dedekind domain, there exist only finitely many prime ideals $p$ of $R$ such that $\alpha$ is not in $R_p[[T]]$.

3. Ramification in specializations

In this section, we recall some classical properties on ramification in specializations of function field extensions.

Let $k$ be the fraction field of a Dedekind domain $R$ of characteristic zero, and let $p$ be a non-zero prime ideal of $R$ such that $k_p$ is perfect. Let $v_p$ denote the discrete valuation on $k$ with valuation ring $R_p$. Let $T$ be an indeterminate over $k$ and $G$ a finite group.

First, we recall the definition of meeting modulo $p$: 


Definition 3.1. (1) Let $F/k$ be a finite extension, $R_F$ the integral closure of $R$ in $F$, and $p_F$ a non-zero prime ideal of $R_F$. We say that two distinct points $t_0$ and $t_1$ in $\mathbb{P}^1(F)$ meet modulo $p_F$ if either $v_{p_F}(t_0) \geq 0$, $v_{p_F}(t_1) \geq 0$, and $v_{p_F}(t_0 - t_1) > 0$, or $v_{p_F}(t_0) \leq 0$, $v_{p_F}(t_1) \leq 0$, and $v_{p_F}((1/t_0) - (1/t_1)) > 0$ (where $v_{p_F}$ denotes the discrete valuation on $F$ associated with $p_F$). \hfill [1]

(2) We say that two distinct points $t_0$ and $t_1$ in $\mathbb{P}^1(\overline{k})$ meet modulo $p$ if $t_0$ and $t_1$ meet modulo a prime ideal of the integral closure of $R$ in $k(t_0, t_1)$ lying over $p$.

Note that, if $t_0$ and $t_1$ meet modulo $p$ and $F/k$ is a finite extension containing $k(t_0, t_1)$, then, $t_0$ and $t_1$ meet modulo a prime ideal of the integral closure of $R$ in $F$ lying over $p$.

In the case where $t_0$ is $k$-rational and meets $t_1$ modulo $p$, the following lemma asserts the existence of a unique degree 1 prime lying over $p$ at which $t_0$ and $t_1$ meet.

Lemma 3.2. For every $t_1 \in \mathbb{P}^1(\overline{k})$, there exists a finite set $S_1$ of primes of $k$, depending only on $t_1$, which satisfies the following property. Suppose $p \not\in S_1$ and let $t_0 \in \mathbb{P}^1(k)\setminus\{t_1\}$ be such that $t_0$ and $t_1$ meet modulo $p$. Then, there exists a unique prime $p' := p'(t_0, t_1, p)$ lying over $p$ in $k(t_1)/k$ with residue degree $f_{p'} = 1$ at which $t_0$ and $t_1$ meet.

Remark 3.3. (1) If $t_1$ is $k$-rational, then, one has $p'(t_0, t_1, p) = p$.

(2) For each prime ideal $p'$ lying over $p \not\in S_1$ in $k(t_1)/k$ with residue degree $f_{p'} = 1$, the weak approximation property of $\mathbb{P}^1$ provides infinitely many $t_0 \in k$ such that $p'(t_0, t_1, p) = p'$.

Proof. By part (1) of Remark 3.3, we may assume $t_1 \not= \infty$. We require the set $S_1$ to contain the (finite) set of primes $q$ of $k$ such that the minimal polynomial $m_{t_1}(T)$ of $t_1$ over $k$ is either not integral at $q$ or not separable modulo $q$. In particular, for every prime $p'$ of $k(t_1)$ lying over $p$, we may assume $v_p(t_1) \geq 0$ and $\overline{k(t_1)}_{p'}$ is generated over $\overline{k}_p$ by the reduction modulo $p'$ of $t_1$, by Lemma 2.1.

For the existence part, note that, since $t_0$ and $t_1$ meet modulo $p$, the above provides a prime ideal $p'$ lying over $p$ in $k(t_1)/k$ such that $v_{p'}(t_0) \geq 0$, $v_{p'}(t_1) \geq 0$, and $v_{p'}(t_0 - t_1) > 0$. Since $p$ is not in $S_1$, and since the reduction modulo $p'$ of $t_1$ equals the reduction modulo $t_0$ of $t_0$, we have $\overline{k(t_1)}_{p'} = \overline{k_p(t_0)} = \overline{k_p}$, that is, $f_{p'} = 1$.

For the uniqueness part, assume that $t_0$ and $t_1$ meet modulo two distinct primes $p'_1$ and $p'_2$ of $k(t_1)$ lying over $p$, both with residue degree 1. Since $m_{t_1}(T)$ is separable modulo $p$, the primes $p'_1$ and $p'_2$ correspond to distinct linear factors $T - a_1$ and $T - a_2$ of the reduction of $m_{t_1}(T)$ in $\overline{k_p}[T]$, so that $t_1 \equiv a_j$ modulo $p'_j$ for each $j \in \{1, 2\}$. As $t_0$ and $t_1$ meet modulo $p'_j$, the difference $t_0 - t_1$ is in $p'_j$. Hence, $t_0 - a_j$ is in $p'_j \cap k = p$. We then get $a_1 \equiv a_2$ modulo $p$, which cannot happen. \hfill \Box

\footnote{We set $1/\infty = 0$, $1/0 = \infty$, $v_p(\infty) = -\infty$, and $v_p(0) = \infty$.}
Now, we recall the definition of intersection multiplicity at \( p \). Below, the minimal polynomial over \( k \) of any given element \( t_1 \in \mathbb{P}^1(\kbar) \) is denoted by \( m_{t_1}(T) \) (we set \( m_{t_1}(T) = 1 \) if \( t_1 = \infty \)). Denote the constant coefficient of \( m_{t_1}(T) \) by \( a_{t_1} \). Then, the minimal polynomial of \( 1/t_1 \) over \( k \) is

\[
\begin{align*}
&\bullet m_{1/t_1}(T) = (1/a_{t_1}) T^{\deg(m_{t_1}(T))} m_{t_1}(1/T) \text{ if } t_1 \in \kbar \setminus \{0\}, \\
&\bullet m_{1/t_1}(T) = 1 \text{ if } t_1 = 0, \\
&\bullet m_{1/t_1}(T) = T \text{ if } t_1 = \infty.
\end{align*}
\]

Let \( t_1 \) be in \( \mathbb{P}^1(\kbar) \) and \( t_0 \) in \( \mathbb{P}^1(k) \). Assume \( v_p(a_{t_1}) = 0 \) if \( t_1 \neq 0 \) to make the intersection multiplicity well-defined in Definition 3.4 below.

**Definition 3.4.** The intersection multiplicity \( I_p(t_0, t_1) \) of \( t_0 \) and \( t_1 \) at \( p \) is

\[
I_p(t_0, t_1) = \begin{cases} 
    v_p(m_{t_1}(t_0)) & \text{if } v_p(t_0) \geq 0, \\
    v_p(m_{1/t_1}(1/t_0)) & \text{if } v_p(t_0) \leq 0.
\end{cases}
\]

Lemma 3.5 below, which is \cite[Lemma 2.5]{Leg16b}, connects Definitions 3.1 and 3.4.

**Lemma 3.5.** Let \( t_1 \) be in \( \mathbb{P}^1(\kbar) \) and \( t_0 \) in \( \mathbb{P}^1(k) \). Assume \( v_p(a_{t_1}) = 0 \) if \( t_1 \neq 0 \).

1. If \( I_p(t_0, t_1) > 0 \), then, \( t_0 \) and \( t_1 \) meet modulo \( p \).
2. The converse in (1) holds, provided \( m_{t_1}(T) \) is in \( R_p[T] \).

Finally, we recall the following classical result on ramification in specializations; see, e.g., \cite[Proposition 4.2]{Bec91}, \cite{Con00}, and \cite[§2.2.3]{Leg16b}.

Let \( E/k(T) \) be a \( k \)-regular \( G \)-extension with branch points \( t_1, \ldots, t_r \). For every \( i \in \{1, \ldots, r\} \), denote the inertia group of \( E(t_i)/k(t_i)(T) \) at \( (T - t_i) k(t_i)[T - t_i] \) by \( I_{t_i} \).

**Specialization Inertia Theorem.** Assume that \( p \) is not in some finite set \( S_{bad} := \mathcal{S}_{bad}(E/k(T)) \) of prime ideals of \( R \). Let \( t_0 \in \mathbb{P}^1(k) \setminus \{t_1, \ldots, t_r\} \).

1. If \( p \) ramifies in \( E_{t_0}/k \), then, \( t_0 \) meets some branch point \( t_i \) of \( E/k(T) \) modulo \( p \).
2. Suppose that \( t_0 \) and \( t_i \) meet modulo \( p \). Then, \( I_{t_0,p} \) is conjugate to \( I_{t_i}^{I_p(t_0,t_i)} \).

### 4. Main Results

In this section, we state and prove Theorems 4.1, 4.4, and 4.6, which are the most general versions of Theorems 1.1, 1.2, and 1.3 respectively. Throughout this section, we use the notation of §2.3. Let \( k \) be the fraction field of a Dedekind domain \( R \) of characteristic zero such that, for every prime ideal \( p \) of \( R \), the residue field \( \kbar \) is perfect. Given an indeterminate \( T \) over \( k \) and a finite group \( G \), let \( E/k(T) \) be a \( k \)-regular \( G \)-extension with branch points \( t_1, \ldots, t_r \). Fix \( i \in \{1, \ldots, r\} \). Recall that \( D_{t_i} \) (resp., \( I_{t_i} \)) is the decomposition group (resp., the inertia group) of \( E(t_i)/k(t_i)(T) \) at \( (T - t_i) k[T - t_i] \), and that \( D_{t_i,p} \) is the decomposition group of \( (E(t_i))_{t_i}/k(t_i) \) at a prime \( p' \) of \( k(t_i) \), so

---

9See, e.g., \cite[§2.2.3]{Leg16b} for an explicit description of these “bad” primes.
that $D_{t_i, p'}$ is canonically identified with a subgroup of $D_{t_i}/I_{t_i}$. Set $e_i := |I_{t_i}|$, and let $\varphi : D_{t_i} \to D_{t_i}/I_{t_i}$ be the natural projection.

4.1. Statement of Theorem 4.1.

Theorem 4.1. Assume that $t_0 \in \mathbb{P}^1(k) \setminus \{t_1, \ldots, t_r\}$ and $t_i$ meet modulo a prime $p$ of $k$ avoiding a finite set $\mathcal{S}_{\text{exc}} := \mathcal{S}_{\text{exc}}(E/k(T))$ of primes of $k$. Let $p' = p'(t_0, t_i, p)$ be the unique prime ideal lying over $p$ in $k(t_i)/k$ provided by Lemma 3.3.

1. The decomposition group $D_{t_0, p}$ is conjugate in $G$ to a subgroup $U$ of $D_{t_i}$ such that $\varphi(U) = D_{t_i, p'}$. Moreover, if $I_{p}(t_0, t_i)$ is coprime to $e_i$, one has $U = \varphi^{-1}(D_{t_i, p'})$.

2. The unramified part of the completion $E_{t_0, k_p}$ of $E_{t_0}$ at $p$ contains $(E(t_i))_{t_i} \cdot k(t_i)_{p'}$, with equality if $I_{p}(t_0, t_i)$ is coprime to $e_i$.

As the specialization $E_{t_0}/k$ is Galois, the compositum $E_{t_0} \cdot k_p$ is independent of the embedding of $E_{t_0}$ into a given algebraic closure of $k_p$. Similarly, $(E(t_i))_{t_i} \cdot k(t_i)_{p'}$ is independent of the embedding of $(E(t_i))_{t_i}$ into a given algebraic closure of $k(t_i)_{p'}$.

Remark 4.2. (1) Let $p$ be a prime of $k$, not in $\mathcal{S}_{\text{exc}}$, and let $p'$ be a prime lying over $p$ in $k(t_i)/k$ with residue degree $f_p|p = 1$. As in part (2) of Remark 3.3, there exist infinitely many $t_0 \in k$ such that $t_0$ and $t_i$ meet modulo $p'$, and such that $I_{p}(t_0, t_i)$ is coprime to $e_i$. Thus, as a consequence of part (1) of Theorem 4.1 and of the Specialization Inertia Theorem, there exist infinitely many $t_0 \in k \setminus \{t_1, \ldots, t_r\}$ such that $I_{t_0, p}$ is conjugate to $I_{t_i}$, and $D_{t_0, p}$ is conjugate to $\varphi^{-1}(D_{t_i, p'})$.

2. Part (2) of Theorem 4.1 implies that the residue degree of $(E(t_i))_{t_i}/k(t_i)$ at $p'$ divides that of $E_{t_0}/k$ at $p$, with equality if $I_{p}(t_0, t_i)$ is coprime to $e_i$.

3. The assumption that $I_{p}(t_0, t_i)$ is coprime to $e_i$ is necessary in general for the more precise conclusions in parts (1) and (2) of Theorem 4.1 as the following easy example shows. Set $k := \mathbb{Q}$, $E := \mathbb{Q}(\sqrt{T})$, $t_i := 0$, and let $p$ be a prime number. Since $E/\mathbb{Q}(T)$ is totally ramified at $t_i$, one has $E_{t_i} = \mathbb{Q}$, and $|I_{t_i}| = |D_{t_i}| = 2$. Set $t_0 := \alpha \cdot p^2$, where $\alpha$ denotes a rational number of $p$-adic valuation 0. Then, one has $(E_{t_0})_p = \mathbb{Q}_p$ if $\alpha$ is a square modulo $p$ (and hence $1 = |D_{t_0, p}| < |\varphi^{-1}(D_{t_i, p})| = 2$), whereas $(E_{t_0})_p/\mathbb{Q}_p$ has degree 2 if $\alpha$ is not a square modulo $p$, and therefore $E_{t_i} \cdot \mathbb{Q}_p \subseteq E_{t_0} \cdot \mathbb{Q}_p$ in this case.

4.2. Proof of Theorem 4.1. By possibly changing the variable $T$, we may assume that $t_i$ is not equal to $\infty$, and that $t_i$ is integral over $R$. For simplicity, set $k' := k(t_i)$, $k'_{p'} := k(t_i)_{p'}$, $E' := E(t_i)$, $E'_i := (E(t_i))_{t_i}$, $S := T - t_i$, and let $R'$ be the integral closure of $R$ in $k'$. From now on, we fix an embedding of $E$ into a given algebraic closure $k'_{p'}((S))$ of $k'_{p'}((S))$. Every compositum of fields below has to be understood inside $k'_{p'}((S))$.

We break the proof into four steps.

10To make this well-defined, we assume in particular that $\mathcal{S}_{\text{exc}}$ contains the set $\mathcal{S}_1$ from Lemma 3.2.
4.2.1. Step I. Here, we identify the Galois group of $E' \cdot k'(S)/k'(S))$ with $D_{t}$ and that of $E' \cdot k'_{p'}((S))/k'_{p'}((S))$ with a subgroup of $D_{t}$, namely, with $\varphi^{-1}(D_{t, p'})$.

First, consider the compositum of $E'$ and $k'(S))$. The restriction of the Galois group $\text{Gal}(E' \cdot k'(S)/k'(S))$ to $E'/k'(S)$ preserves a prime $\mathfrak{Q}$ of $E'$ lying over the prime generated by $S$. Hence, we identify $\text{Gal}(E' \cdot k'(S)/k'(S))$ with the decomposition group $D_{t}$ of $E'/k'(S)$ at $\mathfrak{Q}$. Moreover, $E'_{t}/k'$ is the residue extension of $E'/k'(S)$ at $\mathfrak{Q}$, and the unramified part of $E' \cdot k'(S)/k'(S))$ is $E'_{t} \cdot k'(S)/k'(S))$; see [2.2.2]. Thus, $E' \cdot k'(S)/E'_{t} \cdot k'(S))$ is totally ramified at the prime generated by $S$, and its Galois group is identified with the inertia group $I_{t}$ of $E'/k'(S)$ at $\mathfrak{Q}$.

Now, consider the compositum $E'_{t} \cdot k'_{p'}$. This defines an embedding of $E'_{t}$ into the algebraic closure of $k'_{p'}$ that is contained in $k'_{p'}((S))$, and hence a prime $\mathfrak{Q}'$ of $E'_{t}$ lying over $p'$ such that the restriction of $\text{Gal}(E'_{t} \cdot k'_{p'}/k'_{p'})$ to $E'_{t}/k'$ preserves $\mathfrak{Q}'$. Therefore, we identify $\text{Gal}(E'_{t} \cdot k'_{p'}/k'_{p'})$ with the decomposition group $D_{t, p'}$ of $E'_{t}/k'$ at $\mathfrak{Q}'$.

Next, consider the compositum $E' \cdot k'_{p'}((S))$. Its Galois group $V$ over $k'_{p'}((S))$ is identified with a subgroup of $D_{t} = \text{Gal}(E' \cdot k'(S)/k'(S))$ via restriction. Note that, as $E' \cdot k'(S)/E'_{t} \cdot k'(S))$ is totally ramified, the fields $E' \cdot k'(S))$ and $E'_{t} \cdot k'_{p'}((S))$ are linearly disjoint over $E'_{t} \cdot k'(S))$. Hence, $\text{Gal}(E' \cdot k'_{p'}((S))/E'_{t} \cdot k'_{p'}((S)))$ is identified with $I_{t} = \text{Gal}(E' \cdot k'(S)/E'_{t} \cdot k'(S))$, so that $\varphi(V) = D_{t, p'}$.

We then obtain the following diagram of inclusions and Galois groups:

\[
\begin{array}{ccc}
E' & \rightarrow & E' \cdot k'(S)) \\
\rightarrow & & \uparrow \\
D_{t} (E'_{t} \cdot k'(S)) & \rightarrow & E'_{t} \cdot k'_{p'}((S)) \\
\rightarrow & & \uparrow \\
k'(S) & \rightarrow & k'_{p'}((S))
\end{array}
\]

\[
(4.1)
\]

4.2.2. Step II. Our reduction process modulo $(T - t_{0})$ will occur in the domain $R'_{p'}[[S]]$ see Step III [4.2.3]. Let $F$ be the fraction field of $R'_{p'}[[S]]$; note that $F \subseteq k'_{p'}((S))$. Here, we show that $E'_{t} \cdot F$ is contained in $E' \cdot F$, and identify the Galois groups of $E'_{t} \cdot F/F$ and $E' \cdot F/F$ with $D_{t, p'}$ and $\varphi^{-1}(D_{t, p'})$, respectively.

This description rests on the following lemma:

**Lemma 4.3.** For every finite extension $M/k'(S)$, there exists a finite set $S_{M}$ of primes of $k'$ (depending only on $M/k'(S)$) such that, if $p'$ is not in $S_{M}$, then, the fields $M \cdot F$ and $k'_{p'}((S))$ are linearly disjoint over $F$.

\[\text{Eis93 Exercise 7.11}.\]
Proof. Up to replacing $M/k'(S)$ by its Galois closure, we may assume that $M/k'(S)$ is Galois. Given an intermediate field $k'(S) \subseteq M_0 \subseteq M$, let $\alpha := \alpha(M_0)$ be a primitive element of $M_0/k'(S)$. We have

$$\alpha = \sum_{i \geq -n_0} \alpha_i \cdot S^{i/e}$$

for some positive integer $e$, some non-negative integer $n_0$, and coefficients $\alpha_i$, $i \geq -n_0$, which lie in a finite extension $K/k'$. By possibly replacing $\alpha$ by $S^{n_0} \alpha$, we may assume $n_0 = 0$. As $\alpha$ is algebraic over $k'(S)$, Theorem 2.4 asserts that the set $\mathcal{T}_{M_0}$ of all primes $p'$ of $k'$ such that $v_{\mathcal{T}}(\alpha_j) < 0$ for some $j \geq 0$ and some prime $\Omega$ of $K$ lying over $p'$ is finite, and depends only on $\alpha$, that is, only on $M_0$.

Set $\mathcal{S}_M := \cup_{M_0} \mathcal{T}_{M_0}$, where $M_0$ runs over all intermediate fields in $M/k'(S)$. Suppose that $p'$ is not in $\mathcal{S}_M$, and set $F_1 := (M \cdot F) \cap k'_p((S))$. To prove the lemma, it suffices to show that $F_1 = F$. Since $F \subseteq F_1 \subseteq M \cdot F$, and since the latter is Galois over $F$, the field $M_1 := F_1 \cap M$ satisfies $M_1 \cdot F = F_1$. Set $\alpha := \alpha(M_1)$. As $\alpha$ is an element of $k'_p((S))$, one has $e = 1$ (with the notation of (4.2)), so that $\alpha$ is an element of $R_p([S])$ (as $p'$ is not in $\mathcal{S}_M$). Thus, $\alpha$, and hence $M_1$, are contained in $F$. Hence, $F_1$, which is equal to $M_1 \cdot F$, is equal to $F$, thus ending the proof of the lemma.

Apply Lemma 4.3 with $M = E' \cdot E'_{t_i}$ to get a finite set $\mathcal{S}_M$ of primes of $k'$, depending only on $E/k(T)$, such that $(E' \cdot E'_{t_i}) \cdot F$ and $k'_p((S))$ are linearly disjoint over $F$, provided $p'$ is not in $\mathcal{S}_M$. Set $L := (E'_{t_i} \cdot k'_p((S))) \cap (E' \cdot F)$. Since $(E'_{t_i} \cdot F) \cdot k'_p((S)) \subseteq (E' \cdot F) \cdot k'_p((S))$, the previous linear disjointness provides $E'_{t_i} \cdot F \subseteq E' \cdot F$. Hence, $E'_{t_i} \cdot F \subseteq L$. Conversely, $L$ contains $E'_{t_i} \cdot F$ by its definition and the inclusion $E'_{t_i} \cdot F \subseteq E' \cdot F$. Letting $\mathcal{S}_2$ denote the finite set of primes of $k$ obtained by restricting a prime in $\mathcal{S}_M$, the equality $L = E'_{t_i} \cdot F$ holds, provided $p \notin \mathcal{S}_2$. Set $d_{i,p'} := [E'_{t_i} \cdot k'_p : k'_p]$. As $F$ contains $k'_p = \text{Frac}(R'_p)$,

$$d_{i,p'} = [E'_{t_i} \cdot k'_p : k'_p] \geq [E'_{t_i} \cdot F : F] \geq [E'_{t_i} \cdot k'_p((S)) : k'_p((S))] = d_{i,p'}'$$

the last equality following from $E'_{t_i} \cdot k'_p((S))/k'_p((S))$ being unramified. Hence, $[E'_{t_i} \cdot F : F] = d_{i,p'}$. Moreover, as $E' \cdot F$ and $k'_p((S))$ are linearly disjoint over $F$, the Galois group $\text{Gal}(E' \cdot F/F)$ is identified with $V := \text{Gal}(E' \cdot k'_p((S))/k'_p((S)))$; see (4.11) via restriction. This identification gives $\text{Gal}(E' \cdot F/E'_{t_i} \cdot F) = I_{t_i}$, and $\text{Gal}(E'_{t_i} \cdot F/F) = D_{t_i,p'}$. 


We then obtain the following diagram of inclusions and Galois groups:

\[
\begin{array}{c}
E' \\
\frac{E'_i \cdot k'_p}{L = E'_i' \cdot F} \\
\frac{k'_p}{F}
\end{array}
\begin{array}{c}
E' \\
\frac{E'_i \cdot k'_p((S))}{D_i,p'} \\
\frac{k'_p((S))}{F}
\end{array}
\]

4.2.3. Step III. Here, we reduce the extension $E' \cdot F/F$ modulo $(T - t_0)$, and identify the reduction with $E_{t_0} k_p / k_p$. More precisely, we show that there exists a place $\mathfrak{M}$ of $E' \cdot F$ whose restriction to $E$ has residue field $E_{t_0}$ and whose restriction to $F$ has residue field $k_p$. We then show that the residue field at $\mathfrak{M}$ is the compositum $E_{t_0} k_p$.

As $t_0$ and $t_i$ meet modulo $\mathfrak{p}'$ by the definition of $\mathfrak{p}'$, and as $t_i$ is integral over $R$, one has $v_{\mathfrak{p}'}(t_0 - t_i) > 0$. Hence, $T - t_0 = S - (t_0 - t_i)$ is in $R_{\mathfrak{p}'}[[S]]$. Moreover, as $((t_0 - t_i)^m)_{m \geq 1}$ converges to 0 in $R_{\mathfrak{p}'}$, the specialization map $R_{\mathfrak{p}'}[[S]] \to R_{\mathfrak{p}'}$, which sends $S$ to $t_0 - t_i$, is well-defined. As it is onto, there is a canonical isomorphism $R_{\mathfrak{p}'}[[S]] / (T - t_0) R_{\mathfrak{p}'}[[S]] \cong R_{\mathfrak{p}'}$. In particular, $(T - t_0) R_{\mathfrak{p}'}[[S]]$ is a prime ideal of $R_{\mathfrak{p}'}[[S]]$. Let $\mathfrak{A}$ be the localization of $R_{\mathfrak{p}'}[[S]]$ at $(T - t_0) R_{\mathfrak{p}'}[[S]]$. The previous isomorphism shows that the residue field of $\mathfrak{A}$ at $(T - t_0) \mathfrak{A}$ is canonically isomorphic to Frac($R_{\mathfrak{p}'}$) = $k_{\mathfrak{p}'}$.

Let $S_3$ be the finite set of primes of $k'$ which ramify in $k'/k$. Assuming $\mathfrak{p} \not\in S_3$, and using that $f_{\mathfrak{p}'}|_{\mathfrak{p}} = 1$ by the definition of $\mathfrak{p}'$, we get $k_{\mathfrak{p}'} = k_p$. Hence, $\mathfrak{A}/(T - t_0) \mathfrak{A}$ is canonically isomorphic to $k_p$. We use this canonical isomorphism to identify the two fields.

The integral closure $\mathfrak{G}$ of $\mathfrak{A}$ in $E' \cdot F$ is a Dedekind domain containing a prime ideal $\mathfrak{M}$ lying over $(T - t_0) \mathfrak{A}$. In particular, $\mathfrak{M} \cap E$ is a prime of $E$ lying over $(T - t_0) k[T]$. Let $\mathfrak{A}_{t_0}$ (resp., $\mathfrak{M}_{t_0}$) be the restriction to $E_{t_0} \cdot F$ of $\mathfrak{G}$ (resp., of $\mathfrak{M}$). Since $E_{t_0}$ and $k_{\mathfrak{p}'}$ are contained in the residue field $\mathfrak{A}_{t_0}/\mathfrak{M}_{t_0}$, and since $[E_{t_0} \cdot k_{\mathfrak{p}'} : k_{\mathfrak{p}'}] = d_{\mathfrak{p},\mathfrak{p}'} = [E_{t_0} \cdot F : F]$, we have $\mathfrak{A}_{t_0}/\mathfrak{M}_{t_0} = E_{t_0} \cdot k_{\mathfrak{p}'}$. Thus, we have the following diagram of inclusions:

\[
\begin{array}{c}
E_{t_0} = \overline{E}_{\mathfrak{M} \cap E} \\
\frac{\mathfrak{G}/\mathfrak{M}}{\mathfrak{A}_{t_0}/\mathfrak{M}_{t_0} = E_{t_0} \cdot k_{\mathfrak{p}'}} \\
k = \mathfrak{A}/(T - t_0) \mathfrak{A} = k_p = k_{\mathfrak{p}'}.
\end{array}
\]
Now, we claim that $\mathcal{G}/\mathfrak{M}$ is equal to the compositum of $k_\mathbf{p}$ and $E_{t_0}$, provided $\mathbf{p}$ is not in some finite set of primes $\mathbf{p}$ of $k$ defined below. Indeed, let $P(T, Y) \in R[T][Y]$ be the minimal polynomial over $k(T)$ of some primitive element of $E/k(T)$, assumed to be integral over $R[T]$. If $P(t_0, Y)$ is not separable, then, $t_0$ belongs to the finite set $D$ of all roots in $k$ of the discriminant $\Delta(T) \in R[T]$ of $P(T, Y)$ which are not branch points of $E/k(T)$. As $t_0$ and $t_i$ meet modulo $\mathbf{p}$, the inseparability of $P(t_0, Y)$ implies that $\mathbf{p}$ is in the finite set $\mathcal{S}_4$ of primes $\mathbf{p}$ of $k$ such that $v_{\mathbf{p}}(d - t_i) > 0$ for some $d \in D$ and some prime $q^\prime$ of $k^\prime$ lying over $\mathbf{p}$. Henceforth, we shall assume that $\mathbf{p}$ is not in $\mathcal{S}_1$, and then that $P(t_0, Y)$ is separable. Denote the roots of $P(T, Y)$ in $E$ by $y_1, \ldots, y_n$, and their reductions modulo $\mathfrak{M}$ and $E$ by $\overline{y}_1, \ldots, \overline{y}_n$, respectively. As $\Delta(t_0) \neq 0$, the discriminant $\Delta(T)$ is not in $(T - t_0)\mathcal{R}$. Since $E'/F$ is generated by $y_1, \ldots, y_n$ over $F$, the residue field $\mathcal{G}/\mathfrak{M}$ is generated by $\overline{y}_1, \ldots, \overline{y}_n$ over $\mathcal{R}/(T - t_0)\mathcal{R} = k_\mathbf{p}$ by Lemma 2.1. Thus, one has $\mathcal{G}/\mathfrak{M} = E_{t_0} \cdot k_\mathbf{p}$, proving the claim. The general containment in part (2) of Theorem 4.4 follows from (4.4), provided $\mathbf{p}$ is not in $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$.

4.2.4. Step IV. Here, we use Step III to identify the Galois group of $E_{t_0}k_\mathbf{p}/k_\mathbf{p}$ with a subgroup of $\text{Gal}(E'/F/F) = \varphi^{-1}(D_{t_i, p'})$ whose projection under $\varphi$ equals $D_{t_i, p'}$.

Assume that $\mathbf{p} \not\in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$, so that we are in the situation of (4.3) and (4.4). Up to replacing $D_{t_0}$ by a conjugate of it, we may assume that $D_{t_0}$ is the decomposition group of $E/k(T)$ at $\mathfrak{M} \cap E$. Let $U$ be the decomposition group of $E'/F/F$ at $\mathfrak{M}$, so that it identifies via restriction with a subgroup of $D_{t_0}$. As the prime of $k[T]$ generated by $T - t_0$ is unramified in $E/k(T)$, it is also unramified in $E'/F/F$. Thus, $U$ (resp., $D_{t_0}$) is also the Galois group of $E_{t_0} \cdot k_\mathbf{p}/k_\mathbf{p}$ (resp., of $E_{t_0}/k$). Hence, the restriction of $U$ to $E_{t_0}/k$ is the decomposition group of some prime of $E_{t_0}$ lying over $\mathbf{p}$. Thus, we may identify $U$ with $D_{t_0, \mathbf{p}}$.

On the other hand, $U$ is a subgroup of $\text{Gal}(E'/F/F)$. The latter is identified with $V$ in Step II (4.2.2). Hence, $U$ is a subgroup of $D_{t_i}$ via the identification in Step I (4.2.3). Moreover, $\varphi(U)$ is the decomposition group of $E_{t_i}' \cdot F/F$ at $\mathfrak{M}_{t_i}$. As shown in Step III (4.2.3), one has $\text{Gal}(E_{t_i}' \cdot F/F) = \text{Gal}(E_{t_i}' \cdot k_\mathbf{p}/k_\mathbf{p})$. But the latter is $D_{t_i, p'}$ by Step II. Hence, $\varphi(U) = D_{t_i, p'}$, thus proving the general case of part (1) of Theorem 4.1.

Finally, assume that $I_\mathbf{p}(t_0, t_i)$ is coprime to $e_i$. Let $I_{t_0}$ denote the inertia group of $E'/F/F$ at $\mathfrak{M}_{t_i}$, and $E_{t_i}' \cdot F/F$ is unramified at $\mathfrak{M}_{t_i}$, one has $I_{t_0} \subseteq \text{Gal}(E'/F/E_{t_i}' \cdot F)$ (= $I_{t_i}$; see 4.2.2). Set $S_{\text{exc}} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup S_{\text{bad}}$, where $S_{\text{bad}}$ is the finite set of primes of $k$ from the Specialization Inertia Theorem (3), and assume that $\mathbf{p}$ is not in $S_{\text{exc}}$. As $I_\mathbf{p}(t_0, t_i)$ is coprime to $e_i$, one has $|I_{t_0, \mathbf{p}}| = e_i$ by the Specialization Inertia Theorem. Since the conjugation of $U$ to $D_{t_0, \mathbf{p}}$ sends $I_{t_0}$ to $I_{t_0, \mathbf{p}}$, we also have $|I_{t_0}| = e_i$. As $I_{t_0}$ is a subgroup of $I_{t_i}$, and since $|I_{t_i}| = e_i$, we get the equality $I_{t_0} = I_{t_i}$. Thus, $I_{t_i}$ is contained in $U$, and $\mathfrak{M}_{t_i}$ is totally ramified in $E'/F/E_{t_i}' \cdot F$. Hence, $U = \varphi^{-1}(D_{t_i, p'})$, and $(E_{t_0} \cdot k_\mathbf{p})^\text{ur} = \mathfrak{R}_{t_i}/\mathfrak{M}_{t_i} = E_{t_i}' \cdot k_\mathbf{p}'$, completing the proof of Theorem 4.1.
4.3. On specifying completions in specializations. Theorem 4.4 restricts the structure of the completion \((E_{t_0})_p\) at \(p\) of \(E_{t_0}\). Namely, it implies that \((E_{t_0})_p\) contains the field \(N^{(p)} := \langle E(t_i) \rangle_{t_i} \cdot k(t_i)_{p'}\) whose Galois group over \(k(t_i)_{p'}\) is \(D_{t_i,p'}\), and that the Galois group \(\text{Gal}(E_{t_0} / k_p)\) is a subgroup of \(\varphi^{-1}(D_{t_i,p'})\), where \(\varphi : D_{t_i} \to D_{t_i} / I_{t_i}\) is the natural projection. The following theorem shows that this is the only restriction for extensions of \(k_p\) with Galois group \(\varphi^{-1}(D_{t_i,p'})\):

Theorem 4.4. Let \(S\) be a finite set of primes of \(k\), disjoint from some finite set \(S'_{\text{exc}} = S'_{\text{exc}}(E/k(T))\). For each \(p \in S\), assume that there exists \(i := i(p) \in \{1, \ldots, r\}\) and a prime \(p'\) lying over \(p\) in \(k(t_i) / k\) with residue degree 1, and let \(L^{(p)}/k_p\) be a finite Galois extension containing \(N^{(p)} := \langle E(t_i) \rangle_{t_i} \cdot k(t_i)_{p'}\) such that there exists an isomorphism \(\psi\) from \(\text{Gal}(L^{(p)}/k_p)\) to \(\varphi^{-1}(D_{t_i,p'})\) which maps \(\text{Gal}(L^{(p)}/N^{(p)})\) onto \(I_{t_i}\). Then, there exist infinitely many \(t_0 \in k\) such that the completion of \(E_{t_0}/k\) at \(p\) equals \(L^{(p)}/k_p\) for each \(p \in S\). Moreover, if \(k\) is a number field, we may assume that these specializations of \(E/k(T)\) have Galois group \(G\).

Remark 4.5. (1) If \(L^{(p)}/N^{(p)}\) is totally ramified, \(p \in S\), and if \(k\) is a number field, then, we are in the settings of Theorem 4.4. In this case, by the Specialization Inertia Theorem, the intersection multiplicity of \(t_0\) and \(t_i\) at \(p\) is coprime to \(|I_{t_i}|\), \(p \in S\).

(2) By combining Theorem 4.4 [DG12, Theorem 1.2], and the Chinese remainder theorem, one can more generally require the extensions \(L^{(p)}/k_p\), \(p \in S\), to be either of the above form or unramified of degree \(d\), where \(d\) is an arbitrary positive integer such that \(G\) contains at least one element of order \(d\).

Proof. Fix a prime \(p \in S\). As in (4.1) set \(k' := k(t_i), k'_{p'} := k(t_i)_{p'}, E' := E(t_i), E'_{t_i} := \langle E(t_i) \rangle_{t_i}\), and \(S := T - t_i\). Moreover, set \(N := N^{(p)}\), and \(\Gamma := \varphi^{-1}(D_{t_i,p'})\). As shown in Step I of the proof of Theorem 4.1 (4.2.1), one has \(\text{Gal}(N/k_p) = D_{t_i,p'}\). Then, all fields \(L^{(p)}\) with the properties mentioned in Theorem 4.4 are solution fields to the embedding problem \(\Gamma \rightarrow \text{Gal}(N/k_p)\). Since \(I_{t_i}\) (resp., \(\Gamma\)) is the inertia group (resp., the Galois group) of \(E' \cdot k'_p((S)) / k'_p((S))\) (as shown in Step I of the proof of Theorem 4.1 (4.2.1)), the action of \(\Gamma\) on the cyclic kernel \(I_{t_i}\) is isomorphic to the action on \(c_{t_i}\)-th roots of unity in \(N\) (see 2.2.2). The above embedding problem is then a Brauer embedding problem. By [MM99, Chapter IV, Theorem 7.2], if \(x \in N\) is chosen such that \(N(\sqrt{\alpha})\) is a solution field to this embedding problem, then, all the solutions fields are of the form \(N(\sqrt{\beta}x)\) with \(\beta \in k_p^{	imes}\). Furthermore, upon multiplying \(\beta\) by a suitable \(c_{t_i}\)-th power, we can require \(\beta x\) to be of positive \(p\)-adic valuation.

The field \(E' \cdot k'((S))\) is generated over \(E'_{t_i} \cdot k'((S))\) by \(\sqrt{\alpha}S\) for some \(\alpha \in E'_{t_i}\); see 2.2.2. Up to enlarging the set \(S'_{\text{exc}}\), we may assume that \(\alpha\) is of \(p\)-adic valuation 0. Set \(M := (E'_{t_i} \cdot k'((S)))\langle \sqrt{\alpha}S\rangle\), and consider the field \(F\) from Step II of the proof of Theorem 4.1 (4.2.2). Assume that \(S'_{\text{exc}}\) contains the set \(S'_{\text{exc}}\) from Theorem 4.1 Then,
(E' · M) · F and \(k'_p((S))\) are linearly disjoint over \(F\) by Lemma 4.3. But one also has \((M · F) · k'_p((S)) = E' · k'_p((S))\). Hence, \(M · F = E' · F\).

Thus, we have \(E' · F = (E'_t · F)(\sqrt[\beta]{\alpha S})\). Next, we choose \(t_0 \in k_p\) with \(I_0(t_0, t_i) > 0\) and specialize \(E' · F/F\) at \(T - t_0\) as in Step III of the proof of Theorem 4.1 (\(\S 4.2.3\)). Since \(E'_t · F\) specializes to \(E'_t · k'_p\) and \(\sqrt[\beta]{\alpha S}\) specializes to \(\sqrt[\beta]{\alpha(t_0 - t_i)}\), Lemma 2.1 yields that all fields of the form \((E'_t · k'_p)(\sqrt[\beta]{\alpha(t_0 - t_i)})\), with \(\pi\) an element of \(k_p = k'_p\) of positive \(p\)-adic valuation, can be reached via specializing \(T\) to \(t_0 \in k_p\). Hence, if \(N(\sqrt[\beta]{x})\) is one of the fields obtained in this way, then, we reach all fields of the form \(N(\sqrt[\beta]{\beta x})\), where \(\beta\) is any element of \(k^\times\) such that \(\beta x\) is of positive \(p\)-adic valuation.

As shown above, this covers in particular the extension \(L\langle \psi \rangle/k_p\). Krasner’s lemma shows that \(L\langle \psi \rangle/k_p\) can even be obtained by restricting to specialization values \(t_p \in k\). The extension \(L\langle \psi \rangle/k_p\) therefore equals \((E_{t_p})_p/k_p\), the latter being the specialization of \(E'_t · F/F\) at \(T - t_p\).

Finally, choose \(t_0 \in k\) sufficiently close \(p\)-adically to every \(t_p \in S\). This yields the assertion in the general case. In the number field case, Lemma 2.2 provides the conclusion on the Galois group of the produced specializations.

4.4. On specifying inertia and decomposition groups of specializations. Our next result is devoted to a re-occurrence property for subgroups of \(G\) appearing as decomposition groups of specializations, in the case where \(k\) is a number field.

Theorem 4.6. Assume that \(k\) is a number field, and let \(s\) be a positive integer. Given \(j \in \{1, \ldots, s\}\), choose a branch point \(t_{i(j)}\) of \(E/k(T)\), and an element \(\tau_{i(j)}\) of \(D_{t_{i(j)}}\). Then, there exist \(s\) infinite sets \(S_1, \ldots, S_s\) of primes of \(k\) which satisfy the following property. For every tuple \((p_1, \ldots, p_s) \in S_1 × \cdots × S_s\) of distinct primes, there exist infinitely many \(t_0 \in k\) for which \(E_{t_0}/k\) is a \(G\)-extension whose decomposition group (resp., inertia group) at \(p_j\) is \(\langle I_{t_{i(j)}}, \tau_{i(j)} \rangle\) (resp., \(I_{t_{i(j)}}\)) for \(j = 1, \ldots, s\).

Proof. We may assume without loss that \(s = 1\). Indeed, Lemma 2.2 yields that the assertion for one prime \(p_i\) at a time \((i = 1, \ldots, s)\) implies the assertion for all primes \(p_1, \ldots, p_s\) simultaneously. Also, by that lemma, we do need to prove that \(E_{t_0}/k\) has Galois group \(G\). As before, set \(i := i(1), t_i := t_{i(1)}, \tau := \tau_{i(1)}, k' := k(t_i),\) and \(E'_t := (E(t_i))_t\). Let \(\tau\) be the image of \(\tau\) under the natural projection \(D_{t_i} \to D_{t_i}/I_{t_i}\), and let \(C\) be the conjugacy class in \(D_{t_i}/I_{t_i}\) of \(\tau\). By part (1) of Theorem 4.1 and part (1) of Remark 4.2, it suffices to show that there is an infinite set \(S\) of primes of \(k\) such that, for every \(p \in S\), there is a prime \(p'\) lying over \(p\) in \(k'/k\), with residue degree \(f_{p'/p} = 1\), and such that the Frobenius \(Frob_{p'}(E'_t/k')\) lies in \(C\). By the Chebotarev density theorem, the natural density of the set \(S'\) of all primes \(p'\) of \(k'\) such that \(Frob_{p'}(E'_t/k')\) lies in \(C\) equals \(|C| · |I_{t_i}|/|D_{t_i}|\). This remains true for the set \(S'' = \{p' \in S' : f_{p'/p'\cap k} = 1\}\), since the set of residue degree 1 primes \(p'\) of \(k'\) is of natural density 1, by the prime
number theorem for number fields. In particular, the set of all primes $p$ of $k$ which are contained in a prime $p' \in S'$ is infinite, completing the proof. 

5. APPLICATION TO $G$-CROSSED PRODUCTS AND ADMISSIBILITY

This section is devoted to our application to $G$-crossed products over number fields. We state and prove our admissibility criterion, Theorem 5.2 in §5.2, and apply it to obtain explicit families of examples in §5.3. For this section, let $k$ be a number field, $R$ its ring of integers, $T$ an indeterminate over $k$, and $G$ a finite group.

5.1. Background. Recall that $G$ is called $k$-admissible if there exists a $G$-crossed product division algebra with center $k$. A $G$-extension $L/k$ such that $L$ is a maximal subfield of a $G$-crossed product division algebra is called $k$-adequate. To prove Theorem 5.2, we need Schacher’s admissibility criterion [Sch68, §2] over number fields:

**Theorem 5.1.** Let $L/k$ be a $G$-extension. Then, $L$ is $k$-adequate if and only if, for each prime number $p$ dividing $|G|$, there exist two distinct prime ideals $p_1$ and $p_2$ of $R$ such that the decomposition group of $L/k$ at $p_i$ contains a $p$-Sylow subgroup of $G$ for $i = 1, 2$.

5.2. A new general admissibility criterion.

**Theorem 5.2.** Assume that $G$ has a $k$-regular realization $E/k(T)$ such that, for every prime number $p$ such that $G$ has a non-cyclic $p$-Sylow subgroup, the following holds:

(H) for some branch point $t_i$ of $E/k(T)$, the decomposition group $D_{t_i}$ contains a $p$-Sylow subgroup $P$ of $G$ such that $PI_{t_i}/I_{t_i}$ is cyclic.

Then, there exist infinitely many pairwise linearly disjoint $k$-adequate $G$-extensions of $k$, arising as specializations of $E/k(T)$. In particular, $G$ is $k$-admissible.

**Proof.** Let $p$ be a prime number dividing $|G|$, and let $P$ be a $p$-Sylow subgroup of $G$. First, assume that $P$ is cyclic. Let $t_p \in k$ be such that $E_{t_p}/k$ has Galois group $G$; such a $t_p$ exists by Hilbert’s irreducibility theorem. By the Chebotarev density theorem, there exist infinitely many prime ideals $p$ of $R$ such that $D_{t_p,p}$ contains $P$. Now, assume that $P$ is not cyclic. Let $t_i(p)$ be a branch point of $E/k(T)$ such that the decomposition group $D_{t_i(p)}$ contains $P$, and such that $PI_{t_i(p)}/I_{t_i(p)}$ is cyclic (condition (H)). Then, by Theorem 4.6, there exists an infinite set $S_p$ of prime ideals of $R$ such that, for each $p \in S_p$, there exist infinitely many $t_p \in k$ such that $D_{t_p,p} = PI_{t_i(p)}$. By applying Lemma 2.2, we obtain $t_0 \in k$ such that $E_{t_0}/k$ is a $G$-extension and, for each $p$ dividing $|G|$, one has $D_{t_0,p} = PI_{t_i(p)}$ for at least two distinct primes $p$ of $k$, verifying Theorem 5.1.

5.3. Examples. Roughly speaking, in order to apply Theorem 5.2, one has to produce $k$-regular realizations of $G$ with “sufficiently large” ramification indices and residue degrees at some branch points. Ramification indices can often be prescribed purely
group-theoretically (e.g., via the rigidity method). To ensure “large” residue degrees we use Lemma 2.3. This method yields a large class of new examples. Below, we derive the first \( \mathbb{Q} \)-admissibility results for infinite families of finite non-abelian simple groups. More precisely, Theorem 5.3 covers half of the groups \( \text{PSL}_2(\mathbb{F}_p) \) (\( p \) prime), whereas Theorem 5.4 covers a further 1/8, giving a total of 62.5% of the groups.

**Theorem 5.3.** Assume that \( k \) does not contain \( \sqrt{-1} \). Let \( p \) be a prime number such that either \( p \equiv 3 \pmod{8} \) or \( p \equiv 5 \pmod{8} \). Then, the groups \( \text{PSL}_2(\mathbb{F}_p) \) and \( \text{PGL}_2(\mathbb{F}_p) \) are \( k \)-admissible.

**Proof.** Recall that \( \text{PGL}_2(\mathbb{F}_p) \) has order \( (p-1)p(p+1) \). Then, by our assumption on \( p \), the 2-Sylow subgroups of \( \text{PGL}_2(\mathbb{F}_p) \) (resp., of \( \text{PSL}_2(\mathbb{F}_p) \)) have order 8 (resp., 4). By [MM99, Chapter I, Corollary 8.10], there exists a \( k \)-regular \( \text{PGL}_2(\mathbb{F}_p) \)-extension \( E/k(T) \) with \( k \)-rational branch points \( t_1 \), \( t_2 \), and \( t_3 \), and such that the corresponding inertia groups \( I_{t_1} \), \( I_{t_2} \), and \( I_{t_3} \) are generated by elements in the conjugacy classes \( 2B \), \( 4A \), and \( pA \) of \( \text{PGL}_2(\mathbb{F}_p) \), respectively (according to the ATLAS notation \( [C+85] \)). Consider the branch point \( t_2 \) of \( E/k(T) \) with ramification index 4. By Lemma 2.3, and as \( k \) does not contain \( \sqrt{-1} \), the residue degree at this branch point is divisible by 2. It then remains to combine Theorem 5.2 and the classical fact that the Sylow subgroups of \( \text{PGL}_2(\mathbb{F}_p) \) of odd order are cyclic to get that \( \text{PGL}_2(\mathbb{F}_p) \) is \( k \)-admissible.

Furthermore, the fixed field of \( \text{PSL}_2(\mathbb{F}_p) \) in \( E \) is a rational function field, say \( k(S) \), by, e.g., [Ser92, Lemma 4.5.1], that has degree 2 over \( k(T) \). Denote by \( s_2 \) the unique point lying over \( t_2 \) in \( k(S)/k(T) \). Then, \( s_2 \) is a branch point of \( E/k(S) \) with ramification index 2, and the residue degree at \( s_2 \) in \( E/k(S) \) is divisible by 2. As above, we conclude that \( \text{PSL}_2(\mathbb{F}_p) \) is \( k \)-admissible. \( \square \)

**Theorem 5.4.** Let \( p \) be a prime number such that the following two conditions hold:

1. either \( p \equiv 7 \pmod{16} \) or \( p \equiv 9 \pmod{16} \),
2. either \( p \equiv 2 \pmod{5} \) or \( p \equiv 3 \pmod{5} \).

Assume that \( k \) contains neither \( \sqrt{-1} \) nor \( \sqrt{p^*} \), where \( p^* = (-1)^{(p-1)/2}p \). Then, the group \( \text{PSL}_2(\mathbb{F}_p) \) is \( k \)-admissible.

**Proof.** By condition (1), the 2-Sylow subgroups of \( \text{PSL}_2(\mathbb{F}_p) \) have order 8. Moreover, 5 is not a square modulo \( p \) by condition (2). By [MM99, Chapter I, Theorem 7.10], one then has a \( \mathbb{Q} \)-regular \( \text{PSL}_2(\mathbb{F}_p) \)-extension of \( \mathbb{Q}(T) \) with 4 branch points, and the corresponding inertia groups are generated by elements in the conjugacy classes \( 4A \), \( 4A \), \( pA \), and \( pB \) of \( \text{PSL}_2(\mathbb{F}_p) \) (according to the ATLAS notation). The proof of that theorem also gives the position of the branch points of this \( \mathbb{Q} \)-regular extension. Namely, in the notation of that proof, the \( \mathbb{Q} \)-regular realization of \( \text{PSL}_2(\mathbb{F}_p) \) is \( E/\mathbb{Q}(v) \), and the branch points with ramification index 4 are the roots of the equation \( u^2 - 6u + 25 = 0 \), where \( u \) is related to \( v \) via \( v := \sqrt{p^*(u+5)/(u-5)} \). From the above data, one computes that the fields \( \mathbb{Q}(v_i) \), where \( v_i \) is a branch point of ramification index 4, do not contain
In fact, one has $v_i = \pm 2\sqrt{-p^*}$. Clearly, all of the above remains true for the extension $E \cdot k/k(v)$ instead of $E/\mathbb{Q}(v)$. In particular, as $k$ contains neither $\sqrt{-1}$ nor $\sqrt{p^*}$, one has $\sqrt{-1} \not\in k(v_i)$. By Lemma 2.3, the residue degree at a branch point of $E \cdot k/k(v)$ with ramification index 4 is divisible by 2. Hence, the decomposition group at such a branch point is of order divisible by 8. It then remains to apply Theorem 5.2 to conclude.

6. Application to the Hilbert-Grunwald property

The present section is devoted to our application to the Hilbert-Grunwald property. We state and prove Theorem 6.2 which is a more general version of Theorem 1.5, and show that it applies to any finite non-abelian simple group; see Corollary 6.5. For this section, let $k$ be a number field, $R$ its ring of integers, $T$ an indeterminate over $k$, and $G$ a finite group. Given finitely many $k$-regular $G$-extensions $E_1/k(T), \ldots, E_s/k(T)$, we are interested in the following question, for which the partial positive answer provided by Theorem 4.4 is a natural motivation:

**Question 6.1.** Does there exist a finite set $S_{\text{exc}}$ of non-zero prime ideals of $R$ such that every Grunwald problem $(G, (L^p/k_p)_{p \in S})$, with $S$ disjoint from $S_{\text{exc}}$, has a solution inside the union of the sets of specializations of $E_1/k(T), \ldots, E_s/k(T)$?

It follows from earlier works by the first two authors [Leg16a, Køn17b] that, for each non-trivial finite group $G$ for which there exists a $k$-regular $G$-extension of $k(T)$, there always is one of these realizations for which the answer to Question 6.1 is negative.

In Theorem 6.2 below, we show that the answer to Question 6.1 is in fact always negative for many finite groups:

**Theorem 6.2.** Assume that $G$ has a non-cyclic abelian subgroup. Then, the answer to Question 6.1 is negative for all finite sets of $k$-regular $G$-extensions of $k(T)$.

The proof of Theorem 6.2 requires the following auxiliary result, which strengthens the special case of Theorem 4.6 where $\tau_{i(j)}$ is chosen to be the identity element of $D_{t_i(j)}$.

**Proposition 6.3.** Let $E/k(T)$ be a $k$-regular $G$-extension, and $F$ the compositum of the residue fields $(E(t_1))_{t_1}, \ldots, (E(t_r))_{t_r}$ of $E/k(T)$ at the branch points $t_1, \ldots, t_r$. Moreover, let $p$ be a prime ideal of $R$ that is totally split in $F/k$ (avoiding a finite set of primes depending only on $E/k(T)$). Then, the decomposition group of $E_{t_0}/k$ at $p$ is cyclic for every $t_0 \in \mathbb{P}^1(k)$.

**Proof.** Let $t_0$ be in $\mathbb{P}^1(k)$. First, assume that $t_0$ is in $\{t_1, \ldots, t_r\}$. Then, by the definition of $F$, the decomposition group $D_{t_0,p}$ of $E_{t_0}/k$ at $p$ is trivial. Now, assume that $t_0$ is not in $\{t_1, \ldots, t_r\}$. If $t_0$ does not meet any branch point of $E/k(T)$ modulo $p$, then, by the Specialization Inertia Theorem (4.3), $p$ is unramified in $E_{t_0}/k$ (up to excluding finitely many primes of $k$). Hence, $D_{t_0,p}$ is cyclic. So we may assume that $t_0$ meets some branch
point $t_i$ modulo $p$. As $p$ is totally split in $F/k$, part (1) of Theorem 4.1 shows that $D_{t_i,p}$ is contained in the inertia group $I_{t_i}$ of $E(t_i)/k(t_i)(T)$ at $(T - t_i) k(t_i)[T - t_i]$ (up to excluding finitely many primes of $k$). As $I_{t_i}$ is cyclic, we are done.

Proof of Theorem 6.2. Given a positive integer $s$, let \{${E_1/k(T), \ldots, E_s/k(T)}$\} be a finite set of $k$-regular $G$-extensions. Let $F$ be the compositum of the residue extensions at branch points of $E_1/k(T), \ldots, E_s/k(T)$. Then, by Proposition 6.3 for every non-zero prime ideal $p$ of $R$ that is totally split in $F/k$ (outside some finite set $S_{\text{exc}}$ depending only on $E_1/k(T), \ldots, E_s/k(T)$), the completion at $p$ of every specialization of any of the extensions $E_1/k(T), \ldots, E_s/k(T)$ has cyclic Galois group. Let $H$ be a non-cyclic abelian subgroup of $G$. Without loss of generality, we may assume $H = \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ for some prime number $q$. Let $\zeta_q$ be a primitive $q$-th root of unity, and let $p$ be a non-zero prime ideal of $R$, not in $S_{\text{exc}}$, that is totally split in $F(\zeta_q)/k$. As $p$ is totally split in $k(\zeta_q)/k$, there exists a finite Galois extension $L(p)/k_p$ with Galois group $H$ [12]. In particular, $L(p)/k_p$ cannot occur as the completion at $p$ of a specialization of any of the extensions $E_1/k(T), \ldots, E_s/k(T)$, completing the proof.

Remark 6.4. Our method cannot provide more examples of finite groups such that the answer to Question 6.1 is negative. Indeed, it requires the following assumption on $G$:

(*) $G$ has a non-cyclic subgroup $H$ such that, for every number field $F$ containing $k$, there exist infinitely many non-zero prime ideals $p$ of $R$ such that $p$ is totally split in $F/k$, and such that $H$ occurs as a Galois group over $k_p$.

As shown in the proof of Theorem 6.2, condition (*) holds if $G$ has a non-cyclic abelian subgroup. However, the converse is true, as it is easy to see that any subgroup $H$ of $G$ as in condition (*) has to be abelian.

As an immediate consequence of Theorem 6.2, the answer to Question 6.1 is always negative for finite non-cyclic abelian groups, dihedral groups $D_n$ (of order $2n$) with $n$ even, symmetric groups $S_n$ with $n \geq 4$, alternating groups $A_n$ with $n \geq 4$. In Corollary 6.5 below, we show that the same is true for arbitrary finite non-abelian simple groups:

Corollary 6.5. Assume that $G$ is a finite non-abelian simple group. Then, the answer to Question 6.1 is negative for all finite sets of $k$-regular $G$-extensions of $k(T)$.

Proof. By Theorem 6.2 it suffices to show that $G$ has a non-cyclic abelian subgroup. Suppose by contradiction that $G$ does not. Then, by, e.g., [CE56, Chapter XI, Theorem 11.6], every Sylow subgroup of $G$ is either cyclic or a generalized quaternion group. If every 2-Sylow subgroup of $G$ was cyclic, then, every Sylow subgroup of $G$ would then be cyclic. As Hölder proved in 1895 (see, e.g., [Rot95, Theorem 7.53] for more details),

---

[12] Indeed, one can take $L(p)$ to be the compositum of the fields $L_1^{(p)}$ and $L_2^{(p)}$, where $L_1^{(p)}/k_p$ is the unique degree $q$ unramified extension of $k_p$, and $L_2^{(p)}/k_p$ is a finite Galois extension with Galois group $\mathbb{Z}/q\mathbb{Z}$ that is totally ramified (such an extension exists; see, e.g., [Ser79, Chapter IV]).
G would be solvable\[3, which cannot happen. One may then apply \cite{Suz55, Theorem C} to get that G has a normal subgroup H which satisfies the following two conditions:

1. \((G : H) \leq 2\),
2. \(H = H' \times SL_2(\mathbb{F}_p)\) for some prime number \(p\) and some subgroup \(H'\) of \(H\) whose Sylow subgroups are cyclic.

If \((G : H) = 2\), then, \(H\) is trivial as \(G\) is simple. Hence, \(G\) has order 2, which cannot happen. Then, by (1), we get \(G = H' \times SL_2(\mathbb{F}_p)\), with \(H'\) and \(p\) as in (2). As \(G\) is simple and \(SL_2(\mathbb{F}_p) \neq \{1\}\), we get \(H' = \{1\}\), i.e., \(G = SL_2(\mathbb{F}_p)\). Hence, \(SL_2(\mathbb{F}_p)\) is simple, which cannot happen as \(SL_2(\mathbb{F}_2) \cong S_3\), and, for \(p \geq 3\), the center of \(SL_2(\mathbb{F}_p)\) has order 2.

Solvability of Grunwald problems via specialization is being investigated further by the first author, in a context of infinite families of regular extensions. See \cite{Kon17a}.

7. Application to finite parametric sets

This section is devoted to our application to the non-existence of finite parametric sets over number fields, as already mentioned in \S1.3.3. In \S7.1, we state and prove Theorem 7.2 which is our new general criterion to provide examples of finite groups with no finite parametric set over number fields. Explicit examples, including those given in Theorem 1.6 from the introduction (\S1.3.3), are then given in \S7.2.

For this section, let \(k\) be a number field, \(R\) its ring of integers, \(T\) an indeterminate over \(k\), and \(G\) a finite group.

Let us recall the following definition \cite{KL18}:

**Definition 7.1.** We say that a set \(S\) of \(k\)-regular \(G\)-extensions of \(k(T)\) is parametric if every \(G\)-extension of \(k\) occurs as a specialization of some extension \(E/k(T)\) in \(S\).

It is shown in \cite{KL18} that certain finite groups do not possess finite parametric sets over the given number field \(k\). However, the “global” strategy developed in that paper requires such finite groups to have a non-trivial proper normal subgroup which satisfies some further properties. In particular, this cannot be used for finite simple groups.

7.1. A new general criterion for non-parametricity. Here, we rather use a “local” approach. Namely, in Theorem 7.2 below, we provide a sufficient condition, based on the proof of Theorem 6.2 for the finite group \(G\) to have no finite parametric set over \(k\).

Given a prime number \(q\), let \(\zeta_q\) denote a primitive \(q\)-th root of unity.

**Theorem 7.2.** Suppose that the following condition holds:

\[\text{More precisely, any given finite group whose every Sylow subgroup is cyclic is metacyclic; see, e.g., \cite[page 290]{Rob96}.}\]
(**) there exists a prime number $q$ and a number field $F$ containing $k(ζ_q)$ such that, for all but finitely non-zero prime ideals $p$ of $R$ which are totally split in $F/k$, there exists a $G$-extension of $k$ whose completion at $p$ has Galois group $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$.

Then, given a finite set $S$ of $k$-regular $G$-extensions of $k(T)$, there exist infinitely many $G$-extensions of $k$ each of which is a specialization of no extension $E/k(T)$ in $S$. In particular, $G$ has no finite parametric set over $k$.

Proof. Given a positive integer $s$, let $E_1/k(T), \ldots, E_s/k(T)$ be $k$-regular $G$-extensions. Pick a prime number $q$ and a number field $F$ as in condition (**). In particular, $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ occurs as a subgroup of $G$. By Proposition 6.3, there exist infinitely many prime ideals $p$ of $R$ which are totally split in $F/k$, and such that the Galois group of the completion at $p$ of any specialization of $E_i/k(T)$ is cyclic $(i = 1, \ldots, s)$. Indeed, one may take all prime ideals $p$ which are totally split in the compositum of $F$ and all the residue extensions at branch points of $E_i/k(T)$, $i = 1, \ldots, s$ (up to finitely many exceptions depending only on $E_1/k(T), \ldots, E_s/k(T)$). In particular, for such a $p$, a $G$-extension of $k$ as in condition (**) is a specialization of $E_i/k(T)$ for no $i \in \{1, \ldots, s\}$, completing the proof.

7.2. Explicit examples. Corollary 7.3 below contains our main examples of finite groups with no finite parametric set over number fields:

**Corollary 7.3.** Assume that either one of the following two conditions holds:

1. $G$ has a non-cyclic abelian subgroup, and there exists a finite set $S_{\text{exc}}$ of non-zero prime ideals of $R$ such that every Grunwald problem $(G, (L^{(p)}/k_p)_{p \in S})$, with $S$ disjoint from $S_{\text{exc}}$, has a solution.
2. $G = A_n$ $(n \geq 4)$.

Then, $G$ has no finite parametric set over $k$.

Proof. In either case, it suffices to show that condition (**) of Theorem 7.2 holds. This is clear in case (1). Now, assume that we are in case (2). Then, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ occurs as a subgroup of $G$. It then suffices to show that, for all prime ideals $p$ of $R$, there exists a $G$-extension of $k$ whose completion at $p$ has Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $p$ be a prime ideal of $R$, and let $L^{(p)}/k_p$ be a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-extension. Pick a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-extension $L/k$ whose completion at $p$ is $L^{(p)}/k_p$. By a classical result of Mestre (see [Mes90] and [KM01, Theorem 3]), $L/k$ occurs as a specialization of some $k$-regular $A_n$-extension $E/k(T)$. More precisely, there is a polynomial $P(T, Y) \in k[T][Y]$ which is monic and separable in $Y$, with splitting field $E$ over $k(T)$, and such that $P(0, Y)$ is separable with splitting field $L$ over $k$. Lemma 2.2 now ensures the existence of $t_0 \in k$ such that the specialization of $E/k(T)$ at $t_0$ has Galois group $A_n$ and has the same completion at $p$ as $L/k$. This completes the proof. □
Remark 7.4. (1) Aside from alternating groups, we are not aware of any other infinite family of non-abelian simple groups which have been shown to satisfy condition (**) of Theorem 7.2. However, by using the same tools, the following variant is derived:

Let $G$ be a finite non-abelian simple group, and let $p$ be a prime number such that $H := \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ occurs as a subgroup of $G^{14}$. Then, given a finite set $S$ of $k$-regular $G$-extensions of $k(T)$, there exist infinitely many $H$-extensions of $k$ each of which is a specialization of $E/k(T)$ for no $E/k(T) \in S$.

(2) Under the expectation of [Har07, §1] mentioned in §1.1, condition (1) of Corollary 7.3 holds (and then the conclusion that $G$ has no finite parametric set over $k$ holds as well), provided $G$ is solvable and has a non-cyclic abelian subgroup.

References

[ABGV11] Asher Auel, Eric Brussel, Skip Garibaldi, and Uzi Vishne. Open problems on central simple algebras. *Transform. Groups*, 16(1):219–264, 2011.

[AS01] Elizabeth Allman and Murray Schacher. Division algebras with $\text{PSL}(2, q)$-Galois maximal subfields. *J. Algebra*, 240(2):808–821, 2001.

[BBC12] Jason Bell, Nils Bruin, and Michael Coons. Transcendence of generating functions whose coefficients are multiplicative. *Trans. Amer. Math. Soc.*, 364(2):933–959, 2012.

[Bec91] Sybilla Beckmann. On extensions of number fields obtained by specializing branched coverings. *J. Reine Angew. Math.*, 419:27–53, 1991.

[C+85] J. H. Conway et al. *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups*. With computational assistance from J. G. Thackray. Oxford University Press, Eynsham, 1985.

[CE56] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N.J., 1956. xv+390 pp.

[Con00] Brian Conrad. Inertia groups and fibers. *J. Reine. Angew. Math.*, 522:1–26, 2000.

[Déb01] Pierre Débes. Théorie de Galois et géométrie: une introduction. In *Arithmétique des revêtements algébriques (Saint-Étienne, 2000)*, volume 5 of *Sém. Congr.*, pages 1–26. Soc. Math. France, Paris, 2001.

[Dem10] Cyril Demarche. Groupe de Brauer non ramifié d’espaces homogènes à stabilisateurs finis. (French). *Math. Ann.*, 346(4):949–968, 2010.

[DG11] Pierre Débes and Nour Ghazi. Specializations of Galois covers of the line. In “Alexandru Myller” *Mathematical Seminar*, volume 1329 of *AIP Conf. Proc.*, pages 98–108. Amer. Inst. Phys., Melville, NY, 2011.

[DG12] Pierre Débes and Nour Ghazi. Galois covers and the Hilbert-Grunwald property. *Ann. Inst. Fourier (Grenoble)*, 62(3):989–1013, 2012.

[DLAN17] Cyril Demarche, Giancarlo Lucchini Arteche, and Danny Neftin. The Grunwald problem and approximation properties for homogeneous spaces. *Ann. Inst. Fourier (Grenoble)*, 67(3):1009–1033, 2017.

[Eis95] David Eisenbud. *Commutative algebra. With a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

\[14\]Such a prime number $p$ exists by the proof of Corollary 6.5.
[FS91] Burton Fein and Murray Schacher. \(\mathbb{Q}\)-admissibility questions for alternating groups. *J. Algebra*, 142(2):360–382, 1991.

[GS79] Basil Gordon and Murray Schacher. The admissibility of \(A_5\). *J. Number Theory*, 11(4):498–504, 1979.

[Har07] David Harari. Quelques propriétés d’approximation reliées à la cohomologie galoisienne d’un groupe algébrique fini. (French). *Bull. Soc. Math. France*, 135(4):549–564, 2007.

[JLY02] Christian U. Jensen, Arne Ledet, and Noriko Yui. Generic polynomials. *Constructive Aspects of the Inverse Galois Problem*. Cambridge University Press, 2002.

[KL18] Joachim König and François Legrand. Non-parametric sets of regular realizations over number fields. *J. Algebra*, 497:302–336, 2018.

[KM01] Jürgen Klüners and Gunter Malle. A database for field extensions of the rationals. *LMS J. Comput. Math.*, 4:182–196, 2001.

[KN17] Joachim König and Danny Neftin. The admissibility of \(M_{11}\) over number fields. 2017. To appear in Journal of Pure and Applied Algebra.

[Kö17a] Joachim König. The Grunwald problem and specialization of families of regular Galois extensions. *Manuscript*, 2017. arXiv 1710.05548.

[Kö17b] Joachim König. Non-parametricity of rational translates of regular Galois extensions. *Acta. Arith.*, 179(3):267–275, 2017.

[LA14] Giancarlo Lucchini Arteche. Approximation faible et principe de Hasse pour des espaces homogènes à stabilisateur fini résoluble. (French). *Math. Ann.*, 360(3-4):1021–1039, 2014.

[Leg16a] François Legrand. On parametric extensions over number fields. 2016. To appear in Annali della Scuola Normale Superiore di Pisa - Classe di Scienze. arXiv 1602.06706.

[Leg16b] François Legrand. Specialization results and ramification conditions. *Israel J. Math.*, 214(2):621–650, 2016.

[Mes90] Jean-François Mestre. Extensions régulières de \(\mathbb{Q}(T)\) de groupe de Galois \(\tilde{A}_n\). *J. Algebra*, 131(2):483–495, 1990.

[MM99] Gunter Malle and B. Heinrich Matzat. *Inverse Galois Theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of Number Fields*, volume 323 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, second edition, 2008.

[PV05] Bernat Plans and Núria Vila. Galois covers of \(\mathbb{P}^1\) over \(\mathbb{Q}\) with prescribed local or global behavior by specialization. *J. Théor. Nombres Bordeaux*, 17(1):271–282, 2005.

[Rob96] Derek J. S. Robinson. *A Course in the Theory of Groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996. xviii+499 pp.

[Rot95] Joseph J. Rotman. *An Introduction to the Theory of Groups*, volume 148 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, fourth edition, 1995. xvi+513 pp.

[Sal82] David J. Saltman. Generic Galois extensions and problems in field theory. *Adv. in Math.*, 43(3):250–283, 1982.

[Sch68] Murray M. Schacher. Subfields of division rings. I. *J. Algebra*, 9:451–477, 1968.

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg. viii+241 pp.

[Ser92] Jean-Pierre Serre. *Topics in Galois Theory*, volume 1 of *Research Notes in Mathematics*. Jones and Bartlett Publishers, Boston, MA, 1992.
Michio Suzuki. On finite groups with cyclic Sylow subgroups for all odd primes. *Amer. J. Math.*, 77:657–691, 1955.

Shianghaw Wang. A counter-example to Grunwald’s theorem. *Ann. of Math. (2)*, 49:1008–1009, 1948.

Shianghaw Wang. On Grunwald’s theorem. *Ann. of Math. (2)*, 51:471–484, 1950.

Hans Zassenhaus. Über endliche Fastkörper. (German). *Abh. Math. Sem. Univ. Hamburg*, 11(1):187–220, 1935.

David Zywina. The inverse Galois problem for orthogonal groups. *Manuscript*, 2014. arXiv 1409.1151.

E-mail address: koenig.joach@technion.ac.il

Department of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel

E-mail address: legrandfranc@technion.ac.il

Department of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel

E-mail address: dneftin@technion.ac.il

Department of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel