Semiclassical degeneracies and ordering for highly frustrated magnets in a field

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We discuss ground state selection by quantum fluctuations in frustrated magnets in a strong magnetic field. We show that there exist dynamical symmetries – one a generalisation of Henley’s gauge-like symmetry for collinear spins, the other the quantum relict of non-collinear weathervane modes – which ensure a partial survival of the classical degeneracies. We illustrate these for the case of the kagome magnet, where we find zero-point energy differences to be rather small everywhere except near the collinear ‘up-up-down’ configurations, where there is rotational but not translational symmetry breaking. In the effective Hamiltonian, we demonstrate the presence of a term sensitive to a topological ‘flux’. We discuss the connection of such problems to gauge theories by casting the frustrated lattices as medial lattices of appropriately chosen simplex lattices, and in particular we show how the magnetic field can be used to tune the physical sector of the resulting gauge theories.

I. INTRODUCTION

The rich behaviour presented by geometrically frustrated magnets has made them a popular subject of study.$^1$ Particular attention has been devoted to systems in which the spins reside on corner-sharing units: the kagome lattice in $d=2$ dimensions, consisting of corner-sharing triangles, and the $d=3$ pyrochlore lattice, made up of corner-sharing tetrahedra, whose sparse connectivity makes them particularly strongly frustrated.$^2$

Classical ground state degeneracies can be considered to be the hallmark of such geometrical frustration. One natural question is what happens to these ground state degeneracies in the presence of quantum fluctuations. Generically, one would expect the degeneracies to be lifted, so that they would show up in a quantum model as a large density of states at low energies, on a scale set by the strength of the quantum fluctuations. Indeed, such a large density of low-energy excitations is perhaps the most striking feature of the kagome spin $S=1/2$ Heisenberg magnet$^3$ although as yet there exists no agreement on its origin.

Given the difficulties in studying the limit of strong quantum fluctuations, small $S$, it has proven to be fruitful to consider instead the semiclassical limit$^4 S \to \infty$, where quantum fluctuations endow the classical ground states with an effective energy functional via the zero-point energy of the spin excitations,$^5 \sum \hbar \tilde{\omega}(\{S_i^{(0)}\})/2$, where $\tilde{\omega}$ denote the frequencies of the excitations around a ground state configuration $\{S_i^{(0)}\}$. (In the following, we set $\hbar=1$.) Perhaps the most striking result was that a sizeable, albeit reduced, degeneracy can survive in the semiclassical limit. The most prominent example of this was again provided by the kagome lattice, where the equations of motion for excitations around any coplanar ground state can be written in a similar form, so that all these states have the same zero-point energy.$^2$ The next important step was Henley’s observation$^6$ sketched below, that degeneracies for collinear ground states could be understood in terms of a gauge-like symmetry in the equations of motion.

In a related development, a great deal of attention has recently been focused on the behaviour of frustrated magnets in an external field, in particular on magnetisation plateaux and magnetocalorics.$^6,7,8,9,10,11,12,13$

In this paper, we follow up on those two strands of work by considering the properties of frustrated magnets on corner-sharing geometries in an external field in the semiclassical limit. We start from the observation that such corner-sharing lattices can be thought of as medial lattices of a ‘simplex lattice’; the spins are bond variables on the simplex lattice, whereas the corner-sharing units reside on its sites.

The ground state condition imposed by classical Hamiltonian then takes the form of a local constraint, which can conveniently encoded in a gauge theory, the ‘physical’ sector of which can be changed by the application of a magnetic field, so that one can tune between, e.g., loop and dimer models in this fashion.

We then consider the harmonic theory of excitations in such magnets by generalising the equations of motion derived in Ref. 2 to include a magnetic field. This will allow us to show that Henley’s gauge-like symmetry for collinear spins can be generalised to this case, but in addition that a different dynamical symmetry exists for non-collinear spin configurations.

We illustrate these ideas with the example of a Heisenberg magnet on the kagome lattice in an external field. This choice allows us to build on a substantial body of previous work (on classical equations of motion, order by disorder and magnetisation plateaux$^7,12,13,14,15,16,17$). In keeping with received wisdom, the semiclassical order by disorder we find favours coplanar or collinear states. In different field ranges, both states derived from $Q=0$ and $Q=\sqrt{3}\times\sqrt{3}$ configurations occur; we find $Z_3$ and $Z_3 \times U(1)$ order parameters, some of which will only be expressed at minuscule temperatures. For the value of the magnetic field which permits a collinear configuration, we find that the semiclassical ordering pattern differs from that proposed for $S=1/2$. 


We contrast this behaviour to the kagome lattice thought of as a network of corner-sharing hexagons, where the effective semiclassical model does not permit such simple solutions. Finally, we comment on experiment, and close with some more general remarks about gauge theories for this problem.

The remainder of this paper is organised as follows. In Sect. II we introduce notation by writing down the Hamiltonian in a field for a simplex lattice and deriving (Sect. III) the equations of motion. Sects. IV and V deal with the dynamical symmetries and the properties of effective Hamiltonians. Sect. VI explores in detail ground state selection in the kagome magnet, and addresses issues arising for the kagome-hexagon model. We conclude with some more general remarks about effective Hamiltonians. Sect. VII states selection in the kagome magnet, and addresses different forms of order by disorder.

II. MEDIAL LATTICE CONSTRUCTION

A frustrated lattice made up of corner-sharing units (also known as simplices), can be obtained by considering any lattice Λ, which we will refer to as the simplex lattice. The midpoints of the bonds of the simplex lattice define any lattice Λ, which we will refer to as the simplex lattice. (also known as simplices), can be obtained by considering

A magnet in a field on the simplex lattice Υ is thus equivalent to the non-degenerate saturated state with all spins aligned, and all links empty. Effectively collinear Ising spin configurations can also arise in highly frustrated continuous spin models, via different forms of order by disorder.2,9,10,21,22,23

If z is even (odd), so are the possible \( L_α \).

If we denote spins pointing up by a link occupied by a dimer, and spins pointing down by an empty link, one observes that \( L = z \) corresponds to the non-degenerate saturated state with all spins aligned, and all links empty. As has been noted in many contexts, \( L = z - 2 \) is equivalent to one link emanating from each site being occupied. A magnet in a field on the simplex lattice Υ is thus equivalent to the hardcore dimer models on its simplex lattice Λ. Similarly, \( L = z - 4 \) corresponds to loop models, which are fully packed in the sense that each site is part of a loop. A familiar example of this is the pyrochlore magnet in zero field.

For sufficiently large \( z \), this continues for further \( L \geq 0 \). We have illustrated these cases in Fig. 2 where \( L = 0 \) corresponds to the easy-axis kagome model of Ref. 24 which from the current perspective is a three dimer model on the triangular lattice.

For Ising spins, at the transition field where spin \( L = z - 2n \) becomes degenerate with \( L = z - 2n + 2 \), the allowed

\[
H = H_J + H_Z = \frac{J}{2} \sum_\alpha (\tilde{L}_\alpha^2 - \frac{g_\mu_B}{J} \hat{B} \cdot \tilde{L}_\alpha)
\]

where \( \tilde{L}_\alpha = \sum_{j \in v(\alpha)} \tilde{S}_j \).

The ground state condition \( L_α = 0 \) ∀\( \alpha \) represents such a local constraint. An external magnetic field provides a handle for tuning this constraint. Adding a Zeeman term (where \( g \) is the g-factor, \( \mu_B \) the Bohr magneton and \( \hat{B} \) the magnetic field),

\[
H_Z = - \sum_i g_\mu_B \tilde{S}_i \cdot \hat{B}
\]

yields the Hamiltonian,
vertex configurations are those of both cases. This is in particular the case for odd $z$ in zero field, and it goes along with a spike in the zero temperature entropy at the field strength in question.

III. EQUATIONS OF MOTION

The simple form of the Hamiltonian in terms of the $L$ suggests that these variables should be useful quantities to use when constructing effective theories. Indeed, the thermodynamics of the pyrochlore Heisenberg antiferromagnet is well captured by formulating a theory in terms of the $L$ of a single tetrahedron.$\Theta$

Similarly, the $L$ are useful quantities when discussing the spectrum of excitations around a classical ground state. In zero field, changes of the spin orientation that keep $L \equiv 0$ cost zero energy. Hence, all finite frequency modes involve $L \neq 0$. Indeed, the zero field equations of motion for Heisenberg spins around a ground state configuration $\{S^{(0)}\}$ is

$$\dot{L}_\alpha = JS \sum_{\beta,\alpha} S_{\alpha\beta}^{(0)} \times L_\beta \ .$$

Here and in the following, the symbol $\times$ denotes a vector product, and the sum $\sum_{\beta,\alpha}$ runs over all $S_{\beta\alpha} \in v(\alpha)$. In the following, we drop the superscript 0 as, in our harmonic theory, the spins $S_{\alpha\beta}$ will always refer to a ground state configuration.

By the same token, the simple form of Eq. (3.3) suggests formulating the equations of motion in terms of

$$l_\alpha = L_\alpha - h/2 \ .$$

The resulting form of the equations of motion have a similar structure (see App. A):

$$\frac{d}{dt} \left( -\frac{1}{2} h \times l_\alpha \right) = JS \sum_{\beta,\alpha} S_{\alpha\beta} \times l_\beta \ .$$

IV. DYNAMICAL SYMMETRIES

A. Collinear ground states

For collinear spins in zero field, we have $L = (L^1, L^2, 0)$ and $S_\alpha = (0, 0, S^3)$. It was pointed out by Henley that the form of the equations of motion remains unchanged when making the Ising ($\zeta = \pm 1$) gauge-like variable transformation

$$L_\alpha \to \zeta_\alpha L_\alpha \quad (4.1)$$

together with

$$S_{\alpha\beta} \to \zeta_\alpha S_{\alpha\beta} \zeta_\beta \ , \quad (4.2)$$

for any set of $\{\zeta\}$.

Thus, provided the new state, with the $S_{\alpha\beta}$ changed in sign, is again a ground state of the Hamiltonian $\Theta$, the spectrum of its excitation energies is going to be identical to that of the starting state.

It can be seen that this continues to be the case for the equations of motion for the $\{l\}$ in the presence of a field, which can be seen by writing them out explicitly:

$$<\zeta_\alpha l^1_\alpha > + \frac{h}{2} <\zeta_\beta l^2_\beta > = -(\zeta_\alpha S^3_{\alpha\beta} \zeta_\beta) (\zeta_\beta l^2_\beta)$$

$$<\zeta_\alpha l^2_\alpha > - \frac{h}{2} <\zeta_\alpha l^1_\alpha > = (\zeta_\alpha S^3_{\alpha\beta} \zeta_\beta) (\zeta_\beta l^1_\beta) \quad (4.3)$$

$$l^3_\alpha = 0 \ ,$$

where we take the field along the 3-direction.

For a non-collinear state, this transformation does not work as the bottom member of Eq. (4.3) does not hold in that case. This implies that all spins need to point along the field axis, as in the ground state $L_\alpha = h/2$ implies that not all $S^3 \in v(\alpha)$ can be zero.

Further down, we show how the kagome magnet at 1/3 of its saturation field provides an example of such a symmetry, including a subextensive degeneracy as well as a resulting $Z_3$ order parameter.

B. Non-collinear ground states

There exists a different dynamical symmetry for the case of non-collinear spins. This can be seen by writing the equations of motion in component form:

$$l^1_\alpha + \frac{h}{2} l^2_\alpha = (\eta S^2_{\alpha\beta})(\eta l^2_\beta) - S^3_{\alpha\beta} l^2_\beta$$

$$l^2_\alpha - \frac{h}{2} l^1_\alpha = S^3_{\alpha\beta} l^1_\beta - (\eta S^1_{\alpha\beta})(\eta l^1_\beta) \quad (4.4)$$

$$\eta l^3_\alpha = (\eta S^1_{\alpha\beta} l^2_\beta) - (\eta S^2_{\alpha\beta} l^1_\beta) \ .$$

Eq. (3.3) corresponds to $\eta = 1$.

For $\eta = -1$, these equations retain their form under the transformation

$$l^3_\alpha \to \eta l^3_\alpha \ ; \ S^1_{\alpha\beta} \to \eta S^1_{\alpha\beta} \ . \quad (4.5)$$
Notice that this transformation is distinct from the collinear gauge transformation: for collinear states, \( l^3 \equiv S_1^3 \equiv S_2^3 \equiv 0 \), and the transformation does not generate a new spin state.

This transformation needs to be applied consistently: from the bottom member of Eq. 4.4 it follows that a change (4.5) involving the plaquette \( \beta \) requires a concomitant change on those neighbouring \( \alpha \) plaquettes for which \( |S_{\alpha\beta}^3| \neq 1 \). If all \( |S_{\alpha\beta}^3| \neq 1 \), one therefore ends up transforming all the spins in the system, which amounts to a global symmetry operation.

However, if one can find a set of spins the neighbours of which all point in the 3-direction, the transformation is non-trivial. This is the same condition as that for the weathervane modes of the classical kagome magnet in zero field. The local dynamical \( Z_2 \) symmetry in the semiclassical problem is therefore the vestige of the ‘accidental’ local \( U(1) \) symmetry present in the classical case, and it persists to the case of nonzero fields.

Here, we have demonstrated that it holds not only for the zero-field case, in which the angle between any two neighbouring spins is uniform. Below we provide an example of this for the kagome magnet in a field, and discuss the order parameter resulting from the ensemble of semiclassically degenerate states.

V. EFFECTIVE HAMILTONIANS

The basic idea of effective Hamiltonians is to discard as much microscopic information as possible, essentially by integrating out fluctuations, in the hope that this procedure will give rise to an expression which is simple (and, ideally, local).

A. Selection of collinear configurations

It has been suggested, first in the seminal work by Shender on quantum order by disorder, that quantum fluctuations favour collinear spin configurations, integrating them out in a semiclassical framework leads, crudely, to an effective biquadratic Hamiltonian of the form

\[
H_4 = J_4 S \sum_{\langle ij \rangle} (S_i \cdot S_j)^2 .
\] (5.1)

We have tested this assertion in the present context by considering the \( Q = 0 \) configurations at a magnetic field where a collinear state is possible: at \( h = 2 \), the up-up-down state is possible, in which the spins on two sublattices of the kagome lattice point up, and down on the third (Fig. 3).

At \( h = 2 \), there is in fact a two-parameter family of possible \( Q = 0 \) states, as depicted in Fig. 3. In Fig. 4, we have evaluated the zero-point energy as a function of \( \theta \), that is to say for coplanar spin configurations. Here, \( \theta = 0 \) corresponds to the collinear state.

From this picture, two conclusions can be drawn. Firstly, the biquadratic form broadly has the correct shape, with the minimum and maximum in the right places, and a monotonic behaviour in between. Secondly, the shape of the curve is in quantitative disagreement with the simple biquadratic term. Therefore, when trying to extract the effective \( J_4 \), one should proceed in two different ways, depending on what information one is interested in.

If the question one is asking concerns the overall energy scale at which quantum fluctuations start to assert themselves, one is basically interested in the difference between the highest and the lowest point on that curve. By contrast, if one starts from the lowest temperatures and asks about the energy cost of small deviations from collinearity, one needs to fit the curvature at low \( \theta \). (In fact, fitting to a power law gives an exponent deviating substantially from 2 already for small \( \theta \); however, this exponent does seem to extrapolate to 2 within the limits of our numerical accuracy).

Note that the values for \( J_4 \) thus obtained differ by an
order of magnitude when the low-$\theta$ fit is done to be accurate around $\theta = 5^\circ$ (inset of Fig. 4). This discrepancy continues to grow as the range of $\theta$ is lowered.

Is the form of the effective biquadratic interaction more accurate in the absence of a field? For the kagome magnet, there are no collinear states available in zero field. However, for the checkerboard (planar pyrochlore) magnet, such configurations do exist, and we have checked that the corresponding $E_0(\theta)$ has the same behaviour in that case.

**B. Gauge form of Hamiltonian**

By defining $l^\pm = l^i \pm il^j$, one can rewrite the equations of motion for collinear spins as

$$\pm \omega l^\pm_\alpha = \left( \frac{h}{2} \delta_{\alpha\beta} + S_{\alpha\beta} \right) l^\pm_\beta .$$  

(5.2)

To obtain $E_0 = JS(\langle |\omega| \rangle)/2$, this form allows us to make use of Henley’s trick for calculating the zero point energy by writing

$$\text{Tr}|\omega| = \text{Tr}\sqrt{\omega^2} = \text{Tr}\sqrt{\left( \frac{h^2}{4} + z \right) \delta_{\alpha\beta} + hS_{\alpha\beta} + M_{\alpha\beta}} ,$$  

and then Taylor expanding the square root to evaluate the trace term by term. Here $z$ is the coordination of the simplex lattice, and $M_{\alpha\beta} = S_{\alpha\gamma}S_{\gamma\beta} - z\delta_{\alpha\beta}$, so that its diagonal entries vanish. Thus, formally (and up to a constant), for $N_{\alpha\beta} = -(hS_{\alpha\beta} + M_{\alpha\beta})/\sqrt{h^2/4 + z}$,

$$\text{Tr}|\omega| = \sqrt{\frac{h^2}{4} + z} \text{Tr}\sqrt{\delta_{\alpha\beta} - N_{\alpha\beta}}$$

$$= -\frac{h^2}{4} + z \sum_{m=2}^{\infty} \frac{(2m - 3)!!}{2^m m!} \text{Tr}N^m .$$  

(5.4)

$\text{Tr}N^m$, as in zero field, only contains terms of the type $S_{\alpha\beta}S_{\gamma\delta}...S_{\omega\alpha}$, which can be identified with closed loops $\alpha \rightarrow \beta \rightarrow \gamma \rightarrow \cdots \rightarrow \omega \rightarrow \alpha$ on the lattice $\Lambda$ as each site of $\Lambda$ occurs an even number of times.

Besides altering the coefficients of different loops, the inclusion of the magnetic field has changed the structure of this expression by including a term linear in $S_{\alpha\beta}$ in the Taylor expansion, thus allowing for the possibility that $\text{Tr}N^m$ contains terms with an odd number of factors of $S_{\alpha\beta}$. For bipartite lattices $\Lambda$, such loops are not possible, but they do exist for nonbipartite ones, where the structure of the expansion for $h \neq 0$ thus contains terms which are absent in zero field.

In any case, these loops have the form of fluxes $\Phi$ – the factors

$$\Phi = S_{\alpha\beta}S_{\gamma\delta}...S_{\omega\alpha}$$  

(5.5)

have the appearance of line integrals of the Ising ‘gauge field’ $S$. Knowing the form of the terms appearing in the Hamiltonian, one still needs to fix their coefficients. This problem has recently been addressed in some detail by Hizi and Henley.

Here, we make some further remarks on the structure of the relevant solutions for the case that the dominant terms in the effective Hamiltonian are those involving the elementary plaquettes of the simplex lattice $\Lambda$. By elementary plaquettes we mean the plaquettes the tiling of which makes up $\Lambda$, for instance a square in the case of the square lattice.

For a two-dimensional lattice $\Lambda$ which is a tiling of such elementary plaquettes, fixing the flux through the elementary plaquettes determines the fluxes on all contractible loops of the system. If the pattern of elementary fluxes is the same in two different states, this means that the contribution to the energy is the same in both states even for longer loops.

This does not mean that the energy is uniquely determined by specifying the elementary fluxes: the fluxes through non-contractible loops such as a loop winding around a cylinder in the case of periodic boundary conditions are not thus determined. Below, we present a numerical determination, for the case of the kagome lattice, that two states having locally the same flux but differing in the flux through a non-contractible loop have a zero-point energy difference which approaches a constant, $JS$, in the thermodynamic limit.

Therefore, even though the energy measured locally in the flux expansion is identical everywhere, there is a nonvanishing nonlocal contribution to the energy.

**VI. GROUND STATE SELECTION IN THE KAGOME MAGNET**

In this section, we give examples of the dynamical symmetries discussed above, and discuss the effective models and order parameters which arise.

**A. Collinear states**

We start with the case of collinear spins at a magnetic field $h/2 = 1$, where $L \equiv 1$ allows for states in which two spins in each triangle are aligned with the field, and one is antialigned. As mentioned above, these states map onto dimer model on honeycomb lattice, by placing a dimer on links where $S_{\alpha\beta}$ points down (Fig. 5).

There are exponentially many such dimer coverings, leading to an entropy of $S = 0.1077$ per spin once the collinear configurations have been selected. The zero-point fluctuations will lift the remaining degeneracy as different collinear configurations have different ‘fluxes’ through their elementary plaquettes.

Two particularly simple representatives are the $Q = 0$ configuration (all dimers residing on vertical links, Fig. 5), and the $Q = \sqrt{3} \times \sqrt{3}$ configuration, which involves a tripling of the unit cell. The former has uni-
formly zero flux in each plaquette, whereas the latter has a flux of $\pi$ in two plaquettes out of three.

We have evaluated the zero-point energy for these states for a large system (and for all collinear configurations of a finite system, see below); as we disagree in part with several of the published results on this, we have given details of the calculation in App. B. The fact that the $Q = 0$ state has a lower zero would imply that fluxless configurations are preferred.

It turns out to be easy to characterise these configurations completely. Only the states in which exactly two dimers occur on each hexagon are allowed, for periodic boundary conditions. For the readers familiar with height models, these are the maximally tilted states, or equivalently the non-flippable ones.

For a system of linear length $l$, with $3l^2$ spins and periodic boundary conditions, we find the number $N_0$ of zero flux states to be

$$ N_0 = 3(2^l - 1) . \quad (6.1) $$

However, not all of these can be related to one another via the transformations described above.

To see this, let us consider in more detail what kind of transformations are generated by (4.1, 4.2). Note that, applied to a single triangle $\alpha$, the total spins of triangle $\alpha$ and the neighbouring $\beta$’s change, so that these violate the ground-state condition. To undo this change, one needs to apply the transformation to one-dimensional sets of triangles, as depicted in Fig. 5. This transformation is analogous to the one discussed by Tchernyshyov et al. for the planar pyrochlore lattice; the zero-field case discussed there exhibits the additional feature of independent transformations on different sublattices.26

The dimers of the resulting configuration, superimposed on the initial configuration, form two ‘strings’ (lines of alternating new and old dimers). Even though a single string does not change the local fluxes, such strings can only be created in pairs by the gauge transformation (and they can be separated from one another by further transformations).

One would therefore in general expect the configurations differing by an odd number of strings to differ in energy; we have checked that this is indeed the case, see Fig. 5 the energy cost for an odd number of strings remarkably equals, to within our numerical accuracy, precisely $JS$, which suggests that this is a robust result. As the fluxes through any contractible loop are the same independent of the parity of the number of strings, this energy difference in the effective flux Hamiltonian is due to a topological flux.

To find the number of ground states, we thus need to enumerate the number of even, $N_0^g$, and odd, $N_0^u$, configurations separately. For odd $l$, Eq. (6.1) gives the correct number of ground states as the flux around a non-contractible loop $\prod_\alpha S_{\alpha\beta}$ always equals $-1$ in one direction. For $l$ even, we find

$$ N_0^g = 3(2^l - 1) , \quad N_0^u = 3(2^{l-1}) . \quad (6.2) $$

We have checked these results against numerics: we have computed the zero-point energy for all collinear configurations for $l = 4, 5$ and 6, the total number of which is 417, 7623 and 263640, respectively. The number of ground states is given by 21, 93 and 93, respectively; these include the $Q = 0$ states. For the even case, the odd first excited states number 24 and 96, in accordance with the above expressions.

What is the order parameter describing this ensemble of configurations? In the presence of a field, the SU(2) symmetry is broken down to U(1), but this symmetry is unbroken for the collinear states. The ensemble, however, does break (real space) rotational symmetry. This happens because the strings defined above cannot cross, so that any configuration only includes dimers of at most
two different orientations. Also, shifting the dimers along
one string (e.g. by having a defect tunnel along the
string) connects only dimer configurations with the same
dimer orientation absent. The three pure $Q = 0$ (zero-
string) configurations are special in this respect as they
are the only ones to contain only one dimer orientation.

The phase at the collinear point therefore is described
by an order parameter with $Z_3$ symmetry. Note that for
$S = 1/2$, a $Q = \sqrt{3} \times \sqrt{3}$ ordering has been proposed so
that there should be a phase transition as $S$ is lowered.
This appears reasonable given the fact that at small $S$,
collective quantum tunnelling around hexagons can be
expected to play a role. These processes, which are ab-
sent in the $1/S$ expansion, will favour the $Q = \sqrt{3} \times \sqrt{3}$
structures.

\section{B. Coplanar states}

The analysis in this case is quite analogous to the pre-
ceding subsection. Briefly, we have checked explicitly
(in intervales of $h$ of 0.2) that, among all $Q = 0$ con-
figurations, the favoured ones are the same as found in
previous studies for triangle-based lattices; these are de-
picted in Fig. 7. The spin orientations for $0 \leq h \leq 2$
are given as

\begin{align}
S_1 &= (0, 0, -1) \\
S_2 &= (\sin \theta, 0, \cos \theta) \\
S_3 &= (-\sin \theta, 0, \cos \theta) \\
2 \cos \theta &= h/2 + 1.
\end{align}

For $2 \leq h \leq 6$, one has

\begin{align}
S_1 &= (\sin \beta, 0, \cos \beta) \\
S_2 = S_3 &= (\sin \alpha, 0, \cos \alpha) \\
2 \cos \alpha &= (3 + h^2/4)/h, \; 2 \cos \beta = h - 4 \cos \alpha.
\end{align}

Fig. 7 shows the zero-point energy of the $Q = 0$ con-
figuration as a function of $h$. As for the collinear case, we
compare the energies of $Q = 0$ and $Q = \sqrt{3} \times \sqrt{3}$; this
is shown in Fig. 8. The data in that plot has been ob-
tained by numerically integrating the zero point energy
using mathematica, and by extrapolation of the finite size
results.

We find that for $h < h_p$, the $Q = 0$ configuration is
always energetically favoured over the $Q = \sqrt{3} \times \sqrt{3}$. For
a region of finite width for $h > h_p$, the $Q = \sqrt{3} \times \sqrt{3}$ is in
fact the one with a lower energy. In any case, note that
the energy differences are very small – the $Q = \sqrt{3} \times \sqrt{3}$
is never lower in energy by more than a few $10^{-3} JS$
per spin. The location where the energy difference is
appreciable is around the collinear point, $h = 2$.

Let us now look at the order parameter involved which
exists in the ensemble of states connected by the gauge
transformations.

The vector normal to the plane of the spins defines an
azimuthal angle with respect to the field direction; this
reflects the residual $U(1)$ symmetry. This symmetry will
be broken in the coplanar states.

For $h > 2$, all spins have $S^3 \neq 0$ and hence our dynam-
ical symmetry $U(1)$ cannot be used to obtain a degener-
ate configuration. [There do exist degeneracies between
states at higher energies.] As two of the three spin di-
rections are the same, this will again mean that only the
position of one sublattice is special, so that the order
parameter will be $Z_3 \times U(1)$, on top of any translational
symmetry breaking.

By contrast, for $h < 2$, all the three spin directions are
different. However, the spins on one sublattice points
along the field direction, and hence applying the trans-
formation leads to other degenerate configurations.

The strings involved in the dynamical symmetry inter-
change the spins with $S^3 > 0$ and alternating $\pm S^1$, so
that in the overall ensemble, $S_{1,2}$ have no preferred sub-
lattice, whereas $S_3$ does. The resulting symmetry break-
ing is therefore restricted to the sublattice on which the
spin with $S^3 = -1$ resides. The order parameter is again
$Z_3 \times U(1)$.

\section{C. Kagome-hexagon model}

In the collinear examples so far, the effective gauge
Hamiltonian has been simple in that there have existed
configurations which minimise all the terms for the short-
est loop that arises. These effective classical Hamiltoni-
ans are therefore unfrustrated. Different scenarios, how-
ever, are possible. For instance, in the pyrochlore slice, the
actual effective Hamiltonian is not dominated by the
shortest loop.

Here, we display a third way the effective Hamiltonian
can turn out. This occurs in kagome-hexagon model, in
which the kagome lattice arises from a triangular simplex lattice, that is to say, it is considered as a lattice of corner-sharing hexagons. The zero-field Ising case has been studied in Ref. 24. In the present situation, as discussed in Sect. V, the collinear manifold is selected dynamically, and the first sign of the presence of quantum fluctuations should be incipient nematic correlations (never long-ranged, by Mermin-Wagner) as the temperature is lowered.

For the collinear states with $L = 4$, which can be mapped onto the dimer coverings on the triangular lattice (Fig. 2), it is the external field which determines the direction of the collinear axis, and no spontaneous symmetry breaking is required.

The effective flux Hamiltonian for the resulting dimer model differs from the previous cases in several ways. Firstly, loops which appear in the expansion of Ref. 24 can have odd lengths, as the simplex lattice is non-bipartite. In fact, the shortest loop has length three.

Secondly, the term in the effective Hamiltonian corresponding to the shortest loop does not act to lift the degeneracy between the dimer states: the number of loops with an odd number of dimers equals twice the number of dimers; all the other loops contain no dimers. This follows from the collinear ground-state constraint (the hardcore dimer condition) and is independent of dimer configuration.

Thirdly, whereas this first term is trivial, the next term, corresponding to loops of length four, is frustrated: there is no state in which all loops of length four contain an even (or all an odd) number of dimers. The resulting effective Hamiltonian is therefore simple in that it is a purely classical one, but still complicated, as it is frustrated. We leave an analysis of this model as a subject for future work.

In closing this section, we note that, unlike in the conventional kagome model, the selection of collinear states does not take place classically, by the counting argument of Ref. 2. Rather, this is an example where there will be quantum, but not thermal, order by disorder.

VII. SUMMARY AND OUTLOOK

In this paper, we have considered quantum fluctuations of frustrated magnets in a magnetic field in a semiclassical framework, focusing on the issue of dynamical symmetries. In closing, we remark on some connections between this work and experiment, as well as other theories of frustrated magnets.

A. Energy scales and experiment

Even though all energy scales in the semiclassical treatment are set by $JS$, the prefactors which describe at what temperature different effects will become visible differ greatly. Although none of these prefactors are large, they differ considerably in their smallness.

The tendency towards collinear ordering (Fig. 4) – where possible – is the largest effect. With four bonds per unit cell having non-collinear spins, and three spins per unit cell, a zero-point energy per spin of 0.069 $JS \cos^2 \theta$ implies an effective coefficient of the biquadratic term of about $JS/20$. This might just be visible in the most frustrated magnets. If we take a jarosite with a spin as large as $S = 5/2$, there is a factor of 50 between $JS^2$ and $JS/20$; this is comparable to the suppression of the ordering temperature due to frustration, which is typically also lowered by a factor of about 50 from the Curie-Weiss temperature.

Selection among those collinear states is weaker – the zero-point energy per spin differs by at most $JS/50$. To observe this effect, one requires not only the absence of other terms in the Hamiltonian (beyond the classical nearest-neighbour exchange) on that energy scale. Rather, the system might not be able to equilibrate into the eventual ground state ensemble for reasons of ‘topological glassiness’ that is to say it might not be able on experimental timescales to tunnel through the collinear anisotropy barriers. From the data plotted in Fig. 5, it would nonetheless seem that the best place to look for such quantum order by disorder is near fields for which collinear configurations are available. Luckily, a breaking of a $Z_3$ symmetry goes along with a clear signature in the specific heat, as has been observed elsewhere for kagome magnets with a Potts order parameter.

B. (Effective) gauge theories

The general medial lattice construction $\Lambda \rightarrow \Upsilon$ makes the emergence of gauge structures in highly frustrated magnetism look rather natural – the spins for systems of...
corner-sharing simplices quite naturally reside on links of the simplex lattice, Λ. A particularly striking manifestation of the tendency of frustrated magnets to self-generate gauge descriptions is provided by dipolar spin ice.\footnote{\cite{31,32}}

The classical ground-state constraint is a simple local constraint on the sites of Λ. For the theories considered here, it is a U(1) constraint on the lattice level, which as we have shown can be tuned by the application of a magnetic field (the magnitude of the at each site of Λ is fixed by \( h \)). The effective long-wavelength gauge theories depend on further details of the lattice under consideration, such as bipartiteness.\footnote{\cite{33,34}} However, it is not so surprising from the current perspective that applying a field \( h \) by itself does not change the effective theory drastically.\footnote{\cite{35}}

It is this ‘dual’ role of the spins in the current context which leads to many of the unusual features of the theories in frustrated magnets exhibiting very unusual phase structures.\footnote{\cite{36,37}}

In the context of the gauge theories describing the correlations in such dimer models, it is natural to think of the spins as (gauge-invariant) electric fluxes residing on the links of Λ. However, in the gauge-like theories of the Henley type considered here, the collinear spins have a second, complementary role as an Ising gauge (rather than gauge-invariant electric) field, the magnetic fluxes \( \Phi \) of which (Eq. \ref{Phi}) determine the zero-point energies.

This is the ‘dual’ role of the spins in the current context which leads to many of the unusual features of the problems under consideration. This is yet another example of (otherwise familiar and well-understood) effective theories in frustrated magnets exhibiting very unusual supplementary structures.

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APPENDIX A: EQUATION OF MOTION IN A FIELD

A spin precesses in its effective (exchange+Zeeman) field:

\[
\dot{S}_i = i[H, S_i] = -\sum_{j:j\neq i} S_j \times S_i + h \times S_i
\]

Summing this over \( i \in \nu(\alpha) \) gives

\[
\dot{L}_\alpha = \sum_{i \in \nu(\alpha)} \dot{S}_i = -\sum_{j:j\neq \alpha} \sum_{i \in \nu(\alpha)} S_j \times S_i + h \times L_\alpha.
\]

Using \( \sum_{j:\alpha} S_{\alpha\beta} = L_\alpha + L_\beta - 2S_{\alpha\beta} \), this becomes

\[
\dot{L}_\alpha = -\sum_{\beta \neq \alpha} (L_\alpha + L_\beta) \times S_{\alpha\beta} + h \times L_\alpha.
\]

As \( \sum_{\beta \neq \alpha} S_{\alpha\beta} = L_\alpha \),

\[
\dot{L}_\alpha = -\sum_{\beta \neq \alpha} (L_\beta + \frac{h}{2}) \times S_{\alpha\beta} + h \times (L_\alpha + \frac{h}{2})
\]

\[
\implies \left( \frac{d}{dt} - \frac{1}{2} h \times \right) L_\alpha = \sum_{\beta \neq \alpha} S_{\alpha\beta} \times L_\beta.
\]

As the \( L \) are small deviations from the ground state, this expression can be linearised by replacing the spins \( \{ S_{\alpha\beta} \} \) by their ground state values \( \{ S_{\alpha\beta}^{(0)} \} \):

\[
\left( \frac{d}{dt} - \frac{1}{2} h \times \right) L_\alpha = \sum_{\beta \neq \alpha} S_{\alpha\beta}^{(0)} \times L_\beta.
\] (A1)

APPENDIX B: SPECTRUM FOR \( Q = 0 \) AND \( Q = \sqrt{3} \times \sqrt{3} \) PLATEAU CONFIGURATIONS

Iterating Eq. \ref{Phi} once to eliminate the degrees of freedom on one sublattice (corresponding to one sublattice of the bipartite honeycomb lattice Λ) gives

\[
(\omega - \frac{h}{2})^2 l_\alpha^+ = \sum_{\gamma: \beta \neq \alpha} S_{\alpha\beta} S_{\beta\gamma} l_\gamma^+.
\] (B1)

In the following, we define the unit vectors \( e \) as \( e_1 = (1, 0) \), \( e_2 = (-1/2, \sqrt{3}/2) \) and \( e_2 = (-1/2, -\sqrt{3}/2) \). We have chosen twice the nearest-neighbour distance of the kagome lattice as our unit of length.

\section{Q = 0}

The resulting spectrum is

\[
\omega = \pm \frac{h}{2} \pm \sqrt{3 + 2(\cos q_1 - \cos q_2 - \cos q_3)},
\] (B2)

where \( q_i = q^1 e_1^i + q^2 e_2^i \), and the two choices of \( \pm \) can be made independently.

The zero-point energy per spin is then the average \( E_0 = \langle |\omega| \rangle / 2 \). We find it to equal\footnote{\cite{38}}

\[
E_0 = 0.5484 JS.
\] (B3)

Note that the simple form \( \omega = \pm \frac{h}{2} \pm \sqrt{3} \) is deceptive, as the square root can be larger than \( h/2 \), so that \( |\omega_1| + |\omega_2| = h \neq |\omega_1| + |\omega_2| \).
\[ Q = \sqrt{3} \times \sqrt{3} \]

To compute the frequencies in this case requires solving the characteristic polynomial of a 3 × 3 matrix. Denoting \( \omega = \pm \frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \delta} \), this polynomial is given by

\[ -\lambda^3 + f\lambda + g = 0 , \quad (B4) \]

Here, \( f = 9 - 2c_- \), \( g = 4 + 2c_3 - 2c_- \), with \( q_i = q_1 e_1 + q_2 e_2 \) and

\[ c_- = \cos(q_1 - q_2) + \cos(q_2 - q_3) + \cos(q_3 - q_1) \]

The frequencies are cumbersome to write down, but a closed form expression can be obtained by substituting the above into the formula for solving the cubic equation Eq. (B4)

Finally, we find a zero-point energy of

\[ E_0 = 0.5643JS . \quad (B5) \]

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