Non-Schwarzschild black-hole metric in four dimensional higher derivative gravity: analytical approximation

K. Kokkotas,1 R. A. Konoplya,1,2 and A. Zhidenko3

1Theoretical Astrophysics, Eberhard-Karls University of Tübingen, Tübingen 72076, Germany
2Institute of Physics and Research Centre of Theoretical Physics and Astrophysics, Faculty of Philosophy and Science, Silesian University in Opava, Opava, Czech Republic
3Centro de Matemática, Computação e Cognição (CMCC), Universidade Federal do ABC (UFABC), Rua Abolição, CEP: 09210-180, Santo André, SP, Brazil

Higher derivative extensions of Einstein gravity are important within the string theory approach to gravity and as alternative and effective theories of gravity. H. Lü, A. Perkins, C. Pope, K. Stelle [Phys. Rev. Lett. 114 (2015), 171601] found a numerical solution describing a spherically symmetric non-Schwarzschild asymptotically flat black hole in the Einstein gravity with added higher derivative terms. Using the general and quickly convergent parametrization in terms of the continued fractions, we represent this numerical solution in the analytical form, which is accurate not only near the event horizon or far from black hole, but in the whole space. Thereby, the obtained analytical form of the metric allows one to study easily all the further properties of the black hole, such as thermodynamics, Hawking radiation, particle motion, accretion, perturbations, stability, quasinormal spectrum, etc. Thus, the found analytical approximate representation can serve in the same way as an exact solution.

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I. INTRODUCTION

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The paper is organized as follows. Sec. II briefly discuss the theory under consideration and shows that without loss of generality it can be reduced to the Einstein-Weyl theory when considering spherically symmetric solutions. Sec. III relates the deduction of the analytical expressions for the metric functions with the help of the continued
fraction parametrization. Sec. IV is devoted to testing the accuracy of the obtained analytical metric through calculation of observable characteristics: rotational frequency on the innermost stable circular orbit and eikonal quasinormal modes. Finally, in Sec. V we discuss the obtained results and spotlight the main potentially interesting applications that can be done based on the obtained here analytical form of the metric.

II. STATIC SOLUTIONS IN THE EINSTEIN

GRAVITY WITH ADDITIONAL QUADRATIC TERMS

A. Analytical approximation

One of the coupling constants can be fixed when choosing the system of units, so we take \( \gamma = 1 \). Then, the equations of motion take the form

\[
R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} - 4 \alpha B_{\mu \nu} + 2 \beta R \left( R_{\mu \nu} - \frac{1}{4} R g_{\mu \nu} \right) + 2 \beta \left( \nabla_\mu R \nabla_\nu R - \nabla_\mu \nabla_\nu R \right) = 0,
\]

where

\[
B_{\mu \nu} = \left( \nabla^\rho \nabla^\sigma + \frac{1}{2} R^\rho \sigma \right) C_{\mu \rho \nu \sigma}
\]

is the tracefree Bach tensor. It is the only conformally invariant tensor that is algebraically independent of the Weyl tensor.

One can write a static metric as follows

\[
ds^2 = -\lambda^2 dt^2 + h_{ij} dx^i dx^j,
\]

where \( \lambda \) and \( h_{ij} \) are functions of the spatial coordinates \( x^i \). In [2] it was shown that, taking the trace of the field equations [2] and integrating the equations of motion over the spatial domain from the event horizon to infinity, one can find that

\[
\int \sqrt{h} d^3 x \left[ D^i (\lambda R D_i R) - \lambda (D_i R)^2 - m_0^2 \lambda R^2 \right] = 0,
\]

where \( D_i \) is the covariant derivative with respect to the spatial 3-metric \( h_{ij} \).

By definition, \( \lambda \) vanishes on the event horizon, so that if \( D_i R \) goes to zero sufficiently rapidly at spatial infinity, then the total derivative term can be discarded and any static black-hole solution of [1] must have vanishing Ricci scalar \( R = 0 \). The latter means that, without loss of generality, we can be constrained by the Einstein-Weyl gravity (\( \beta = 0 \)). Then, since \( B_{\mu \nu} \) is tracefree, the trace of [2] implies the vanishing Ricci scalar \( R = 0 \). Therefore, the Schwarzschild solution is also a solution for the Einstein-Weyl gravity.

Summarizing, when considering static solutions in the most general Einstein gravity with quadratic in curvature corrections given by [1], one can take \( \gamma = 1 \) and \( \beta = 0 \) without loss of generality.

III. BLACK HOLES IN HIGHER-DERIVATIVE GRAVITY

Here, first, we shall find a general analytical form of the non-Schwarzschild metric and then expand it in terms of the small deviation \( k \) from the Schwarzschild branch.

A. Analytical approximation

The line element of a black hole is given by

\[
ds^2 = -h(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where the functions \( h(r) \) and \( f(r) \) satisfy the Einstein-Weyl equations of motion, which have the following form:

\[
h''(r) = \frac{4 h(r)^2 (1 - r f'(r) - f(r)) - r h(r) (r f'(r) + 4 f(r)) h'(r) + r^2 f(r) h'(r)^2}{2 r^2 f(r) h(r)},
\]

\[
f''(r) = \frac{h(r) - f(r) h(r) - r f(r) h'(r)}{\alpha f(r) (2 h(r) - r h'(r))} - \frac{h(r) (3 r^2 f'(r)^2 + 12 r f(r) f'(r) - 4 r f(r) + 12 f(r)^2 - 8 f(r))}{2 r^2 f(r) (r h'(r) - 2 h(r))} + \frac{f(r)}{2 r^2 h(r)^2} h'(r)^2 - r h(r) f'(r) (r h'(r) + 4 h(r)).
\]

The metric functions \( f(r) \) and \( h(r) \) can be expanded into the Taylor series near the horizon \( r_0 \):

\[
h(r) = c \left[ (r - r_0) + h_2 (r - r_0)^2 + \ldots \right]
\]

\[
f(r) = f_1 (r - r_0) + f_2 (r - r_0)^2 + \ldots,
\]

where \( f_1 \) is the shooting parameter, which we choose in such a way that the solution is asymptotically flat, and \( c \) is the arbitrary scaling factor, which we choose such that \( t \) is the time coordinate of a remote observer, i.e.,

\[
\lim_{r \to \infty} h(r) = 1.
\]

Substituting [9] into [7] and [8], one can express all the coefficients in terms of \( c \) and \( f_1 \), which we find using numerical integration as prescribed in [4].
It is useful to introduce the dimensionless parameter, which parameterizes the solutions up to the rescaling
\[ p = \frac{r_0}{\sqrt{2a}}. \]  
(10)

Notice that for all \( p \) the Schwarzschild metric is the exact solution of the Einstein-Weyl equations as well, but at some minimal nonzero \( p_{\text{min}} \), in addition to the Schwarzschild solution, there appears the non-Schwarzschild branch (found numerically in [4]) which describes the asymptotically flat black hole, whose mass is decreasing, when \( p \) grows, and vanishing at some \( p_{\text{max}} \).

The approximate maximal and minimal values of \( p \) are:
\[ p_{\text{min}} \approx 1054/1203 \approx 0.876, \quad p_{\text{max}} \approx 1.14 \]  
(11)

Following the parametrization procedure given in [6] we define the functions \( A \) and \( B \) through the following relations:
\[ h(r) \equiv xA(x), \]  
\[ \frac{h(r)}{f(r)} \equiv B(x)^2, \]  
where \( x \) denotes the dimensionless compact coordinate
\[ x \equiv 1 - \frac{r_0}{r}. \]  
(14)

We represent the above two functions as follows:
\[ A(x) = 1 - \epsilon(1 - x) + (a_0 - \epsilon)(1 - x)^2 + \hat{A}(x)(1 - x)^3, \]
\[ B(x) = 1 + b_0(1 - x) + \hat{B}(x)(1 - x)^2, \]  
(15)

where \( \hat{A}(x) \) and \( \hat{B}(x) \) are introduced in terms of the continued fractions, in order to describe the metric near the event horizon \( x = 0 \):
\[ \hat{A}(x) = \frac{a_1}{1 + \frac{a_2x}{1 + \frac{a_3x}{1 + \cdots}}}, \]
\[ \hat{B}(x) = \frac{b_1}{1 + \frac{b_2x}{1 + \frac{b_3x}{1 + \cdots}}}. \]  
(16)

At the event horizon one has: \( \hat{A}(0) = a_1, \quad \hat{B}(0) = b_1 \).

We notice that \([7] \) and \([8] \) imply that \( a_0 = b_0 = 0 \), i.e. the post-Newtonian parameters for the non-Schwarzschild solution coincide with those in General Relativity. We fix the asymptotic parameter \( \epsilon \) as

\[ \epsilon = -\left(1 - \frac{2\tilde{M}}{r_0}\right), \]  
(17)

using the value of the asymptotic mass which can be found by numerical fitting of the asymptotic behavior of the metric functions.

Expanding \([12] \) and \([13] \) near the event horizon we find the parameters \( a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots \) as functions of \( \epsilon \) and \( f_1 \). In their turn, the values of \( \epsilon \) and \( f_1 \) can be found numerically for each value of the parameter \( p \). It appears that \( \epsilon, a_1, \) and \( b_1 \) approach zero for
\[ p \approx \frac{1054}{1203}, \]
where the numerical non-Schwarzschild solution coincides with the Schwarzschild one.

The fitting of numerical data for various values of \( p \) and \( r_0 \) shows that \( \epsilon, a_1, \) and \( b_1 \) can be approximated within the maximal error \( \lesssim 0.1\% \) by parabolas as follows:
\[ \epsilon \approx (1054 - 1203p) \left(\frac{3}{1271} + \frac{p}{1529}\right), \]  
(18)
\[ a_1 \approx (1054 - 1203p) \left(\frac{7}{1746} - \frac{5p}{2421}\right), \]  
(19)
\[ b_1 \approx (1054 - 1203p) \left(\frac{p}{1465} - \frac{2}{1585}\right). \]  
(20)

These fittings are almost linear in \( p \). Indeed, if one uses \( k = 1054 - 1203p \), then the above relations read
\[ \epsilon \approx \frac{6857795}{2337860877} k \left(1 - \frac{1271}{6857795} k\right) + \mathcal{O}(k^3), \]  
(21)
\[ a_1 \approx \frac{1242869}{565017822} k \left(1 + \frac{970}{1242869} k\right) + \mathcal{O}(k^3), \]  
(22)
\[ b_1 \approx -\frac{370840}{558679215} k \left(1 + \frac{317}{370840} k\right) + \mathcal{O}(k^3), \]  
(23)
where one can see that the coefficients of the quadratic form are quite small. Nevertheless they cannot be neglected if one aims at 0.1% accuracy.

With the same accuracy we are able to find \( a_2 \) and \( b_2 \) as
\[ a_2 \approx \frac{6p^2}{17} + \frac{5p}{6} - \frac{131}{102}, \]  
(24)
\[ b_2 \approx \frac{81p^2}{242} + \frac{109p}{118} - \frac{16}{89}, \]  
(25)
\[ a_3 \text{ and } a_4 \text{ diverge at} \]
\[ p \approx \frac{237}{223}. \]

Therefore we find out that these parameters can be well approximated as
\[
\begin{align*}
    a_3 &\approx \frac{9921p^2}{31} - \frac{385p + 4857}{237 - 223p}, \\
    a_4 &\approx \frac{9p^2}{14} + \frac{3149p - 2803}{237 - 223p}.
\end{align*}
\] (26, 27)

In this way, although each of the parameters \( a_3 \) and \( a_4 \) diverges at \( p \approx 237/223 \), its ratio is finite and thereby has finite contribution into the continued fraction (see fig. 1).

Finally, we observe that \( b_3 \) and \( b_4 \) are well approximated by the straight lines as
\[
\begin{align*}
    b_3 &\approx \frac{2p}{57} + \frac{29}{56}, \\
    b_4 &\approx \frac{13p}{95} - \frac{121}{98}.
\end{align*}
\] (28, 29)

Within the chosen accuracy of a fraction of 0.1\% for the metric functions \( f(r) \) and \( h(r) \) (see fig. 2) we can set \( a_5 = b_5 = 0 \) and, substituting the found above coefficients \( \epsilon, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \) into (15), obtain the final analytic expressions for the metric functions as the forth order continued fraction expansion (see appendix B).

If one is limited by a rougher approximation of the second order, all the parameters \( \epsilon, a_1, b_1, a_2, b_2 \) can also be well approximated by linear (in \( p \)) polynomials instead of the quadratic ones. Such an approximation would have much simpler analytical form (which will be discussed in subsection C), leading to the larger maximal error of about a few percents.

### B. Expansion of the forth order approximation near the Schwarzschild solution

When one is interested in relatively small deviations from the Schwarzschild geometry, a more concise expressions can be obtained by using the expansion in terms of \( k \). In order to have a positive asymptotic mass, the values of \( k \) can vary from 0 until approximately -321.727. For example, the final formula for \( h(r) \) expanded up to the first order in \( k \) then reads
\[
h(r) = \left(1 - \frac{r_0}{r}\right) \left(1 - k \frac{r_0 h_1(r)}{r h_2(r)} + \mathcal{O}(k^2)\right),
\] (30)

where

\[ h_1(r) = 7094296364854698294656777815r^3 + 2700140790021572890363934045r^2 r_0 \\
\quad + 3285298486678922219083981378rr_0^2 - 4194480693404458083513273360r_0^3, \]
\[ h_2(r) = 6100180386561r (39646131244569649r^2 - 24556525364789942r_0 + 156809140779977329r_0^2). \] (31, 32)
This formula is considerably shorter than the full formula for \( h(r) \), what might be useful when one numerically models various process around the black hole. The ratio of both metric functions is

\[
\frac{f(r)}{h(r)} = 1 + \frac{37793605455056k_0^2(50038777r + 84360383r_0)}{278643r(626529735540821298r^2 - 3742896005107026923rr_0 + 11207698915983181988r_0^2)} + \mathcal{O}(k^2). \tag{33}
\]

When \(|k| \lesssim 100\), first order expansions in \( k \) for \( h(r) \) and \( f(r) \) stay within a few tenths of one percent from the full analytical metric, keeping thereby the same order of the general error. The \( k \)-expanded metric is also included into the Mathematica\textsuperscript{®} notebook we share.

C. Second order approximation: more compact, but robust analytical metric

In case one is interested in a much more robust, but compact expression for the metric, one can be limited by the second order in the expansion \([10]\), i.e. take \( a_3 = b_3 = 0 \). Then, it is sufficient to consider a linear fit for \( \epsilon, a_1, a_2, b_1, b_2 \) as follows

\[
\begin{align*}
\epsilon &\approx \frac{1054 - 1203p}{326}, \\
a_1 &\approx \frac{1054 - 1203p}{556}, \\
a_2 &\approx \frac{18 - 17p}{11}, \\
b_1 &\approx \frac{1054 - 1203p}{1881}, \\
b_2 &\approx \frac{2 + p}{4}.
\end{align*}
\tag{34-38}
\]

Thus, we obtain even simpler (than in the two previous subsections) form for the metric functions \( A(r) \) and \( B(r) \) in \([11]\).

\[
\begin{align*}
A(r) &= 1 - \frac{(1054 - 1203p)r_0^2}{2r^2} \times \left( r + r_0 + \frac{11r_0}{163r_0 + 2787r + 18r_0 - 17p(r - r_0)} \right), \\
B(r) &= 1 - \frac{4(1054 - 1203p)r_0^2}{1881r^2(2r + r_0) - p(r - r_0)}. \tag{39-40}
\end{align*}
\]

Yet, the obtained metric is considerably less accurate: for \( p < 0.97 \) the relative error stays within a fraction of one percent, but for near extremal values of \( p \) it may reach a few percents. The accuracy of the approximation for a given value of \( p \) of the second and forth order approximation can be learnt from the Mathematica notebook we share with readers.

IV. TESTING THE ACCURACY OF THE APPROXIMATION

The metric is not a gauge-invariant characteristic and, strictly speaking, comparing the metric functions has no direct physical interpretation. Therefore, the best way to test the accuracy of the analytical metric obtained in the previous section is to calculate basic observable quantities for the analytical metric and compare them with the accurate ones found for the numerical metric. Here we shall consider two kinds of such observable characteristics: the frequency of a massive particle on the innermost stable circular orbit (ISCO) and frequencies of the quasi-normal modes in the eikonal (short wavelength) regime.

A. Innermost stable circular orbit

First, we shall compute the radius of the smallest circular orbit of a massive test particle rotating around the black hole. The circular movement of a massive particle is described by the following potential

\[
V_m(r) = \frac{E^2}{h(r)} - \frac{L^2}{r^2} - 1,
\]

where \( E \) and \( L \) are, respectively, the energy and momentum per unit mass.

The innermost stable circular orbit corresponds to

\[
V_m(r) = V_m'(r) = V_m''(r) = 0,
\]

which is reduced to the following equation for the radial coordinate of the orbit

\[
rh(r)h''(r) - 2rh'(r)^2 + 3h(r)h'(r) = 0.
\]

We solve the above equation numerically with \( h(r) \) given in the Appendix A in the analytical form and in \([4]\) numerically. The corresponding orbital frequencies are given

\[
\Omega = \sqrt{\frac{h'(r)}{2r}}. \tag{41}
\]

From Fig. 8 we see that the frequency of ISCO decreases quite a few times, as the parameter \( p \) grows. This means that the ISCO moves outward the black hole at a great extent. The relative error stays within a few percents for the parametric region under consideration, being much less for the near Schwarzschild and near extremal (almost massless) cases. At the same time, for the intermediate values of \( p \), when the error of the analytical approximation is maximal (see Fig. 8), the effect of the deviation from the Schwarzschild metric on the orbital frequency is already much larger than the error. This means that the forth order approximation developed here is adequate.
where the equation can be reduced to the following form:

\[ \Omega \text{ defined as follows} \]

Implying that the general relativistic Klein-Gordon equation the same. Perturbations of a test scalar field obey the eikonal formulas for the test fields of other spin will be only for the test massless scalar field, while the resultant equation for the large value of the multipole number \( \ell \) takes the form

\[ V(r) = \left( \ell + \frac{1}{2} \right)^2 \left( \frac{h(r)}{r^2} + \mathcal{O} \left( \frac{1}{\ell^2} \right) \right) . \]  

From (30) we find that the effective potential \( \mathcal{O} \) has the maximum at

\[ r_m = \frac{3\sigma_0}{2} \left( 1 + 0.000393k \right) + \mathcal{O} \left( k^2, \ell^{-2} \right) . \]

As \( k \) is negative, then the peak of the effective potential is closer to the black hole horizon for non-Schwarzschild solution than for the Schwarzschild one.

At high \( \ell \), once the effective potential has the form of the potential barrier, falling off at the event horizon and spacial infinity, the WKB formula found in [11] (for improvements and extensions of this formula, see [12–13]) can be applied for finding quasinormal modes:

\[ \omega^2 = V(r_m) - i \left( \frac{1}{2} + 1 \right) \sqrt{-2 \frac{d^2V(r_m)}{dr_m^2}} , \]  

which depends on the value and the second derivative of the potential in its maximum \( r_m \). Using the above eikonal formula for the QNMs we find that

\[ \omega = \frac{2}{3\sqrt{3r_0}} \left[ \left( \ell + \frac{1}{2} \right)(1 - 0.001308k) - i \left( \frac{1}{2} + 1 \right)(1 - 0.002743k) \right] + \mathcal{O} \left( k^2, \ell^{-1} \right) , \]

implying higher real (photon circular orbit) frequency and faster decay of the oscillations for the non-Schwarzschild branch. When \( k = 0 \) the above formula is reduced to its Schwarzschild analogue.

We compare the above formulas with the precise values for the real and imaginary parts of the frequencies, obtained by substituting the numerical solutions into the eikonal WKB formula. From Fig. 4 we see that the first-order in \( k \) formula \( \mathcal{O} \) provides quite an accurate result for the real (fractions of a percent) and imaginary (a few percents) parts of the eikonal frequencies. Notice, that the obtained analytical form in terms of the deviation from the Schwarzschild solution \( k \), allowed us to find easily the concise analytical non-Schwarzschild generalization of the well-known eikonal formulas for quasinormal modes of the Schwarzschild black hole. Usually the obtained eikonal quasinormal modes of a test scalar field determine the parameters of the circular null geodesic: the real and imaginary parts of the quasinormal mode are multiples of the frequency and instability timescale of the circular null geodesics respectively. However, quasinormal modes of non-tested, e. g., gravitational, fields may not obey this rule \( \mathcal{O} \).

V. DISCUSSION

In the present paper we have obtained the approximate analytic expression for the black-hole solution of the non-Schwarzschild metric in the most general Einstein gravity with quadratic in curvature corrections. The

\[ \phi(t, r, \theta, \phi) = \frac{8}{3\sqrt{3}} \left[ \left( \ell + \frac{1}{2} \right)(1 - 0.001308k) - i \left( \frac{1}{2} + 1 \right)(1 - 0.002743k) \right] + \mathcal{O} \left( k^2, \ell^{-1} \right) . \]
FIG. 4. Relative error (in percents) of the first-order in \( k = 1054 - 1203p \) approximated eikonal formula (46) for the real (left panel) and imaginary (right panel) parts of the eikonal formula for the QNMs.

obtained analytical metric represents asymptotically flat black hole which has the same post-Newtonian behavior as in General Relativity, but is essentially different in the strong field regime. The metric is expressed in terms of event horizon radius \( r_0 \) and the dimensionless parameter \( p = r_0 / \sqrt{2\alpha} \), where \( \alpha \) is the coupling constant. The minimal value of \( p \approx 0.876 \) corresponds to the merger of the Schwarzschild and non-Schwarzschild solutions, while at \( p \approx 1.14 \) the black-hole mass approaches zero (\( \epsilon = -1 \)).

As the metric is written in terms of the black-hole parameter and coupling constant and is accurate everywhere outside the event horizon, it can be used in the study of the basic properties of the black hole and the description of various phenomena in its vicinity in the same way as the exact analytical solution. Our main future aim is to generalize the obtained analytical metric to the case of rotating black holes [9]. At the same time a number of other appealing problems associated with the obtained metric could be solved:

- Perturbations and analysis of stability of the non-Schwarzschild black hole;
- Quasinormal modes of gravitational and test fields in its vicinity (this was partially done for the numerical solution in [17] and comparison between analytical and numerical metrics would be appealing). As higher curvature corrections frequently lead to a new branch of non-perturbative (in coupling constant) modes [18], it is interesting to check whether this phenomena takes place for the considered here quadratic gravity.
- Analysis of massless and massive particles’ motion, binding energy, innermost stable circular orbits, stability of orbits;
- Analysis of the accretion disks and the corresponding radiation in the electromagnetic spectra;
- Consideration of tidal and external magnetic fields in the vicinity of a black hole, etc.
- Hawking radiation in the semiclassical and beyond semiclassical regimes;
- Detailed study of the black-hole thermodynamics.

The obtained here analytical approximation for the metric (given in Appendix A) has two evident advantages over the numerical solution. First, it allows one to solve all the above enumerated problems in a much more economic and elegant way. Second, the analytical metric allows applications of a greater variety of methods for its analysis. For example, in order to get the full knowledge of the characteristic quasinormal spectrum of a black hole, one has to use the Leaver method [19], which simply cannot be applied to the numerical interpolation function and requires the analytically written metric. Applications of other methods, for example (used here for illustration) WKB method [11, 12] or the time-domain integration [20], are considerably constrained and give information only about the lowest modes.

In addition, we found the eikonal quasinormal frequencies of test fields and the frequency and positions of ISCO for the non-Schwarzschild black hole solution in the higher derivative gravity. Comparison of the data obtained for the analytical and numerical metrics allowed us to test the accuracy of our approximation. It is shown that the non-Schwarzschild black hole is characterized by a much further position of ISCO and much slower rotational frequency of a massive particle. The eikonal quasinormal modes of the non-Schwarzschild black hole have smaller real oscillation frequencies and damping rates.

Here we used the expansion up to the forth order and achieved accuracy in the metric functions with the maximal error of fractions of a percent. Once it is necessary, expansion to higher orders will produce much more accurate representation of the metric.
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Appendix A: Analytical form of the metric functions

Here we provide the obtained analytical form of the metric. In the attachment to this article we share with readers the Mathematica notebook, where the metric functions are explicitly written down.

The black-hole metric can be written in the form:

$$ds^2 = -\left(1 - \frac{r_0}{r}\right)A(r)dt^2 + \frac{B(r)^2dr^2}{\left(1 - \frac{r_0}{r}\right)A(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (A1)$$

Here, we present the metric functions $A(r)$ and $B(r)$ in terms of the parameters $b$ and $r_0$. Notice, that one can get the black hole radius $r_0 = 1$ (which leads to the re-definition of the black-hole mass) and express everything in terms of $r_0$.

$$A(r) = \left[152124199161\left(873828p^4 - 199143783p^3 + 806771764p^2 - 1202612078p + 604749333\right)r^4 + 78279\left(1336094371764p^6 - 300842119184823p^5 + 393815823540843p^4 + 2680050514097926p^3 - 950139215294689p^2 + 10978748485369369p - 4249747766121792\right)r^3r_0 - 70372821\left(1486200636p^6 + 180905642811p^5 + 417682197141p^4 - 1208134566031p^3 - 324990706209p^2 + 3382539200269p - 10458813132731415617\right)r^2r_0^2 - 23549620247668759617p^5 - 435688050031083222417p^4 + 2389090517292988952355p^3 - 3731827099716921879958p^2 + 2186684376605688462974p - 389142952738481370396\right)r r_0^3 + 31\left(3383106857977876p^6 + 410672271594465801p^5 - 14105000476530678231p^4 + 51431640078486304191p^3 - 71532183052581307042p^2 + 43250367615320791700p - 9476049523901501640\right)r_0^2 \right]^{1/2}, \quad (A2)$$
Here, the minimal value of $p \approx 1054/1203 \approx 0.876$ corresponds to the merger of the Schwarzschild and non-Schwarzschild solutions and at the maximal value of $p \approx 1.14$ the black-hole mass approaches zero ($\epsilon = -1$).