DETERMINANTAL MODULES OVER PREPROJECTIVE ALGEBRAS
AND REPRESENTATIONS OF DYNKIN QUIVERS

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Abstract. In this paper, we study extension groups of modules over a preprojective algebra via the Auslander-Reiten translation of the quiver associated with it. More precisely, based on the recent work given by Lapid and Minguez, we give a description of extension groups of a preprojective algebra in terms of the AR translation and nilpotent matrices. As a result, we calculate the extension group of a sort of so-called determinantal modules, which is an analog of quantum minors in quantum coordinate rings. In particular, we give an equivalent combinatorial condition when the product of two quantum minors (up to $q$-power rescaling) belongs to the dual canonical basis of quantum coordinate rings in the Dynkin case.

1. Introduction

The present paper attempts to establish a bridge between the representation theory of a Dynkin quiver $Q$ and the modules over the corresponding preprojective algebra $\Lambda_Q$ using Auslander-Reiten theory. The motivation of the paper stems from the cluster structure on quantum coordinate rings $A_q(n(w))$ for an element $w \in W$ in the Weyl group (see [10] for more details). C.Geiß, B.Leclerc and J.Schröer in [10] gave this quantum cluster structure via a module category $C_w$ on the preprojective algebra $\Lambda_Q$. The key notion in their works is a sort of so-called determinantal modules over $\Lambda_Q$, this notion is an analogue of quantum minors in $A_q(n(w))$. Recall that Quantum minors in $A_q(n(w))$ belong in the dual canonical basis $B^*(w)$ of $A_q(n(w))$.

For two dual canonical bases $\mathbf{b}_1$ and $\mathbf{b}_2$ in $B^*(w)$, there are a lot of conjectures about their product $\mathbf{b}_1 \mathbf{b}_2$: For instance, Leclerc proposed [17, Conjecture 1 and Conjecture 2] or Berenstein and Zelevinsky proposed [4, Conjecture 1.7]. In this paper we focus on the following problem:

Question 1.1. For a Dynkin quiver $Q$, let $w_0$ be the unique Weyl group element with maximal length, and express $w_0 = C_Q^g$ as a $g$ power of the Coxeter element $C_Q$ adapted with $Q$. Let $\mathbf{b}_1$ and $\mathbf{b}_2$ be two dual canonical bases in $B^*$ of quantum coordinate ring $A_q(n(w_0)) = A_q(n)$. We write $\lambda_1$ and $\lambda_2$ for the indexing of $\mathbf{b}_1$ and $\mathbf{b}_2$ respectively. (In this paper, we just consider the Konstant partitions as the indexing set of dual canonical bases).

Describe the condition $\mathbf{b}_1 \mathbf{b}_2 \in q^\mathbb{Z}B^*$ in terms of the combinatorial conditions of $(\lambda_1, \lambda_2)$.

Recall that [4, Conjecture 1.7] said that $\mathbf{b}_1 \mathbf{b}_2 \in q^\mathbb{Z}B^*$ if and only if $\mathbf{b}_1 \mathbf{b}_2 = q^n \mathbf{b}_2 \mathbf{b}_1$ for some integer $n$. However, Leclerc gave a counter example to this conjecture, and then proposed
the notion of real dual canonical bases, e.t., $b^2 \in qZ^+$ and [17, Conjecture 1 and Conjecture 2].

Classification of the real dual canonical bases is a difficult open problem until now. The well-known example is a sort of so-called quantum minors, which play an important role in the cluster structure on $A_q(n(w))$. Recently, Lapid and Minguez [14] classify in type $A$ the real dual canonical bases satisfying a so-called regular condition. Even if we assume that $b_1, b_2$ are real dual canonical bases, this question is still difficult to answer (see [17]). In [11] Kang, Kashiwara, Kim and Oh introduce the $R$-matrix on the category of modules over quiver Hecke algebras and then describe the condition $b_1 b_2 \in qZ^+$ in terms of the denominators of $R$-matrices. However, finding out the meaning of this denominator is also a difficult problem (see [7]).

In this paper, we consider the case when $b_1$ and $b_2$ are quantum minors in $A_q(n)$ and then give an equivalent combinatorial conditions of the constant partitions $(\lambda_1, \lambda_2)$. We remark that the quantum minors are dependent on the choice of the reduced expression of $w_0$. In our concerning case, this reduced expression is given by the Coxeter element $C_Q$.

Let us explain this more explicitly. The quantum minors correspond to the so-called determinantal modules over the preprojective algebra $\Lambda_Q$ (see [10]). The Konstant partitions corresponding to determinantal modules are a sort of special partition: They can be regarded as a sequence of Auslander-Reiten translation $\tau$ orbit of an indecomposable representation of $Q$ in our concerning case. Surprisingly, they are similar with the objects in the cluster category given by derived category of representations of $Q$ (see [3]).

Therefore, the above question (1.1) turns out to be a problem when $\text{Ext}_{\Lambda_Q}^1(\widetilde{M}, \widetilde{N}) = 0$ for two determinantal modules $\widetilde{M}$ and $\widetilde{N}$ over $\Lambda_Q$. Bases on Ringel’s work [21] and Lapid and Minguez’s work [1], we calculate the $\text{Ext}_{\Lambda_Q}^1(\widetilde{M}, \widetilde{N})$ for any two determinantal modules $\widetilde{M}, \widetilde{N}$. More precisely, in [21], Ringel described a module $\widetilde{M}$ over $\Lambda_Q$ as $(M, a)$ where $M$ is the representation of $Q$ and $a \in \text{Hom}_Q(M, \tau M)$ where $\tau M$ refers to the Auslander-Reiten translation of $M$. In [1] Aizenbud and Lapid study $\text{Ext}_{\Lambda_Q}^1(\widetilde{M}, \widetilde{N})$ of representations $\widetilde{M} = (M, a)$ and $\widetilde{N} = (N, b)$ using the following map

$$r_{a,b}: \text{Hom}_Q(M, N) \rightarrow \text{Hom}_Q(M, \tau N)$$

$$f \mapsto \tau(f)a - bf$$

Proposition 1.2 ([1, Proposition 9.2]). For any two modules $\widetilde{M} = (M, a)$ and $\widetilde{N} = (N, b)$ of $\Lambda_Q$, we have

$$0 \rightarrow \text{Coker} r_{a,b} \rightarrow \text{Ext}_{\Lambda_Q}^1(\widetilde{M}, \widetilde{N}) \rightarrow (\text{Coker} r_{b,a})^* \rightarrow 0$$

As a corollary, $\text{Ext}_{\Lambda_Q}^1(\widetilde{M}, \widetilde{N}) = 0$ coincides with $r_{a,b}$ and $r_{b,a}$ are surjective. However, it is too tough to calculate the map $r_{a,b}$ for all $\widetilde{M}$ and $\widetilde{N}$. In [15, Section 2.4], Lapid and Minguez give several approaches to calculate extension group $\text{Ext}_{\Lambda_Q}^1(\widetilde{M}, \widetilde{N})$, such as randomized algorithm, and they calculate extension groups of some special modules.
Unlike their approaches, we interpret \( a \) as a nilpotent \( r \times r \) matrix where \( r \) is the number of indecomposable direct summands of \( M \). Namely, \( M = \bigoplus_{k=1}^{r} M_k \) where each \( M_k \) is indecomposable. This allows us to consider the map \( r_{a,b} \) as a Lie bracket of \( a, b \) on the matrix space \( M_{r \times s} \) where \( s \) is the number of indecomposable direct summands of \( N \). This approach makes the map \( r_{a,b} \) more easier to calculate. Because we just consider the decomposition of representations of \( Q \) to calculate the extension group \( \text{Ext}^1_{\Lambda}(M, \overline{N}) \).

Since any indecomposable representation \( M_i \) corresponds to a positive root \( \lambda_i \in R^+ \) by Gabriel’s Theorem. Hence, any representation \( M \) gives rise to a root sequence \( (\lambda_1, \lambda_2, \cdots, \lambda_r) \) such that \( \lambda_i \geq \lambda_j \) for any \( i < j \), and \( M = \bigoplus \lambda^\bullet M_{\lambda_i} \). We call this sort root sequences for Konstant Partitions and denote by \( K^\bullet \Lambda(\alpha) \) the set of Konstant partitions for \( \alpha \). Therefore, the map \( r_{a,b} \) is uniquely determined by the Konstant partitions \( \lambda, \kappa \) associated with modules \( \overline{M}, \overline{N} \) respectively. That means we should find a combinatorial condition on the pair \( (\lambda, \kappa) \) to describe the condition \( \text{Ext}^1_{\Lambda}(M, \overline{N}) = 0 \) and then the condition \( b_1 b_2 \in B^* \).

As we pointed out before, The Konstant partitions associated with quantum minors in our concerning case are of the following form
\[
(1.2) \quad (\tau^u \beta, \tau^{u-1} \beta, \cdots, \tau \beta, \beta)
\]
Where \( \tau = C_Q \) is the Coxeter element that adapted with the orientation of \( Q \). Namely, \( \tau M_\beta = M_\beta \), where \( M_\beta \) refers to the indcomposable representation corresponding to the root \( \beta \).

Let \( \alpha \) and \( \beta \) be two roots, set
\[
R(\alpha, \beta) = \{ r \in \mathbb{Z} \mid \text{Hom}_{\Lambda}(\tau^r M_\alpha, M_\beta) \neq 0 \}
\]
We show that \( R(\alpha, \beta) \) is of a segment of integers. For simplicity, we denote by \([a, b]\) the integers between \( a \) and \( b \), by \( l_u[a, b] \) the number of the set \( [0, u] \cap [a, b] \), and by \( l^c_u[a, b] \) the number of the complement of the set \( [0, u] \cap [a, b] \) in \([0, u]\). In what follows, we write \( M_\lambda \) for the representation associated with the Konstant partition \( \lambda \) and \( \overline{M}_\lambda \) for the module \((M_\lambda, a)\) such that \( a \) is the maximal in the sense of (2.7).

**Theorem 1.3.** Let \( \lambda = (\tau^u \beta_a, \cdots, \tau \beta_a, \beta_a) \) and \( \kappa = (\tau^v \beta_b, \cdots, \tau \beta_b, \beta_b) \). Let
\[
(1.3) \quad R(\beta_a, \beta_b) = [s, t] \quad \text{and} \quad d_{\lambda,\kappa} = u - v - 1
\]
we have
\[
[M_\lambda, M_\kappa] - [M_\lambda, \tau M_\kappa] = l_u[u - t, u - s] \cap l^c_u[d_{\lambda,\kappa} - t, d_{\lambda,\kappa} - s] - l^c_u[u - t, u - s] \cap l_u[d_{\lambda,\kappa} - t, d_{\lambda,\kappa} - s]
\]
and
\[
\dim \ker r_{a,b} = \begin{cases} 
  l_v[s - d_{\lambda,\kappa} - 1, t - d_{\lambda,\kappa} - 1] & \text{if } u \geq v \\
  l_v[s - d_{\lambda,\kappa} - 1, t - d_{\lambda,\kappa} - 1] \cap [v - u, v] & \text{if } u < v
\end{cases}
\]
In particular, \( r_{a,b} \) is surjective if and only if
\[
(1) \quad \text{if } u \geq v \\
\quad l_v[s - d_{\lambda,\kappa} - 1, t - d_{\lambda,\kappa} - 1] = l_u[u - t, u - s] \cap l^c_u[d_{\lambda,\kappa} - t, d_{\lambda,\kappa} - s] - l^c_u[u - t, u - s] \cap l_u[d_{\lambda,\kappa} - t, d_{\lambda,\kappa} - s]
\]
(2) if \( u < v \)
\[
l[u \mid s - d_{\lambda, \kappa} - 1, t - d_{\lambda, \kappa} - 1] \cap [v - u, v]
= l[u \mid u - t, u - s] \cap l[u \mid d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] - l[u \mid u - t, u - s] \cap l[u \mid d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s]
\]

This theorem seems strange for most readers. This precisely shows that the map \( r_{a, b} \) is too difficult to calculate in general. Otherwise, Lapid and Minguez don’t need a so-called randomized algorithm. Here are some corollary to interpret some well known results.

**Corollary 1.4 ([9]).** For any \( \tau \)-orbit Konstant partition \( \lambda \) we have that \( \widetilde{M}_{\lambda} \) is a rigid module over \( \Lambda_Q \). Namely, \( \text{Ext}^1_{\Lambda}(\widetilde{M}_{\lambda}, \widetilde{M}_{\lambda}) = 0 \).

**Corollary 1.5.** Given two \( \tau \)-orbit Konstant partition \( \lambda = (\tau^u \beta, \cdots, \beta) \) and \( \kappa = (\tau^v \alpha, \cdots, \alpha) \) such that \( \beta = \tau^k \theta_a \) and \( \alpha = \tau^k \theta_b \) for some integer \( k \) and two different roots \( \theta_a, \theta_b \). We have
\[
(1.4) \quad \text{Ext}^1_{\Lambda}(\widetilde{M}_{\lambda}, \widetilde{M}_{\kappa}) = 0
\]
Here \( \theta_a \) and \( \theta_b \) refers to \( (2.5) \).

**Corollary 1.6 ([9]).** For any \( \tau \)-orbit Konstant partition \( \lambda \) we have that
\[
\text{dim Ext}^1_{\Lambda}(\widetilde{M}_{\lambda}, \widetilde{M}_{\tau \lambda}) = 1
\]
Here \( \tau \lambda = (\tau^{u+1} \beta, \tau^u \beta, \cdots, \tau \beta) \)

**Remark 1.7.** If we consider \( \lambda = (\tau^u \beta, \cdots, \beta) \) as the Konstant partition associated with quantum minor \( D(a, b) \) such that \( \beta = \beta_a \) and \( \tau^u \beta = \beta_b \). The first corollary means \( D(a, b)^2 \in q^Z B^* \); the second corollary means the formula [10, Lemma 5.2]
\[
D(\mu \omega_{i_a}, \nu \omega_{i_a})D(\mu \omega_{i_b}, \nu \omega_{i_b}) = q^n D(\mu \omega_{i_a}, \nu \omega_{i_a})D(\mu \omega_{i_a}, \nu \omega_{i_a})
\]
for some integer \( n \). Here \( \nu \) refers to \( \beta = \nu(\alpha_{i_a}) \) and \( \mu = \tau^u \nu \). The condition \( \beta = \tau^k \theta_a \) and \( \alpha = \tau^k \theta_b \) for some integer \( k \) means \( \nu \omega_{i_b} = \alpha \).

The third corollary is an interpretation of the following formula. [10, Proposition 5.5]
\[
(1.6) \quad q^B D(a, b^-) D(a^+, b) = q^4 D(a, b) D(a^-, b^-) - q^C \prod_{j \neq i} D(b^-(j), d^-(j))
\]

The third corollary just shows that \( D(a, b^-) D(a^+, b) \notin q^Z B^* \).

For a Dynkin quiver, one defines a Coxeter element \( C_Q = s_{i_1} s_{i_2} \cdots s_{i_n} \) adapted with \( Q \).
Since Dynkin quivers are bipartite orientations, we have that \( w_0 = c_{\frac{h}{2}} \) if the Coxeter element \( h \) is even or \( w_0 = c_{\frac{h - 1}{2}} \) where \( c_1 \) is the sink part of \( c_Q \) if \( h \) is odd. It gives rise to a word of \( w_0 \) as
\[
(1.7) \quad \mathbf{i} = [i_1 i_2 \cdots i_n i_1 \cdots i_n i_1 \cdots i_n \cdots i_n]
\]
We remark that \( \mathbf{i} \) is adapted to \( Q \). We always write \( \tau \) for the Coxeter element \( C_Q \).
Here is our main theorem.

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Theorem 1.8. Let $D(b, d)$ and $D(k, l)$ be two quantum minors induced from the above word $i$, let us write $\lambda(b, d) = (\tau^b \beta_b = \beta_d, \cdots, \beta_b)$ $\lambda(k, l) = (\tau^v \beta_k = \beta_l, \cdots, \beta_k)$. Set

$$R(\beta_b, \beta_k) = [s, t], R(\beta_k, \beta_b) = [s', t'], \text{ and } d_{(b, d), (k, l)} = u - v - 1$$

we have $D(b, d)D(k, l) \in q^{\mathbb{Z}}B^*$ if and only if they satisfy the following condition

1. if $u = v$

$$l_u[s, t] = l_u[u - t, u - s] \cap l_u^c[-1 - t, -1 - s] - l_u^c[u - t, u - s] \cap l_u[-1 - t, -1 - s]$$

and

$$l_u[s', t'] = l_u[u - t', u - s'] \cap l_u^c[-1 - t', -1 - s'] - l_u^c[u - t', u - s'] \cap l_u[-1 - t', -1 - s']$$

2. if $u \neq v$ and we assume that $u > v$,

$$l_v[u - t, u - s] \cap l_v^c[d_{(b, d), (k, l)} - t, d_{(b, d), (k, l)} - s] - l_v^c[u - t, u - s] \cap l_v[d_{(b, d), (k, l)} - t, d_{(b, d), (k, l)} - s]$$

and

$$l_u[s' - d_{(b, d), (k, l)} - 1, t' - d_{(b, d), (k, l)} - 1] \cap [u - v, u]$$

$$= l_u[v - t', v - s'] \cap l_v^c[d_{(b, d), (k, l)} - t', d_{(b, d), (k, l)} - s'] - l_v^c[v - t', v - s'] \cap l_v[d_{(b, d), (k, l)} - t', d_{(b, d), (k, l)} - s']$$

This theorem looks like quite strange. We remark that the right side of these formulas is of the symmetric form. The left side of these formulas represents the replacement of $(D(b, d), D(k, l))$. This phenomena is similar with the $R$-matrices $R_{D(b, d), D(k, l)}$ in the sense of [11]. The advantage of this theorem is to make it feasible to calculate all quantum minor pairs $(D(b, d), D(k, l))$.

As an application, we give another combinatorial condition when $\text{Ext}^1_{(M_{\lambda}, \tilde{M}_{\kappa})} = 0$ for any two ladder multisegments $\lambda, \kappa$. In [13, Lemma 6.21], Lapid and Minguez give a combinatorial condition when $\text{Ext}^1_{(M_{\lambda}, \tilde{M}_{\kappa})} = 0$ for any two ladder multisegments $\lambda, \kappa$. Although our condition seems like more difficult, our combinatorial condition make it feasible to calculate $\text{Ext}^1_{(M_{\lambda}, \tilde{M}_{\kappa})} = 0$ for any two ladder multisegments $\lambda, \kappa$ (see Section (4.6)).

Let us briefly explain the structure of this paper. In Section (2), we give the definition of dual representations of $Q$. Although this notion is known for related experts, we give a precise interpretation of this notion by the automorphism groups action on the extension groups. In Section (3), we calculate the map $r_{a, b}$ using the results in Section (2). In Section (4), we recall the cluster category over preprojective algebras and prove some well-known results about determinantal modules using our approach in Section (3). In Section (5), we recall the cluster structure on the quantum coordinate ring and show an equivalent combinatorial condition when the product of any two quantum minors is in the dual canonical bases (up to $q$-power rescaling).
Notions and conventions. In this paper, a quiver \( Q = (I, \Omega) \) consists of a set of vertices \( I \) and a set of arrows \( \Omega \). Denote by \( n \) the number of vertices if there is no danger of confusion. For an arrow \( h \) one denotes by \( s(h) \) and \( t(h) \) its source and target, respectively. For our purposes, we assume that \( Q \) is of Dynkin type. It follows that \( I \) endows with an order such that if there exists an arrow from \( i_k \) to \( i_l \) we have \( k < l \). \( \Omega \) gives rise to a symmetric matrix \( C_Q \), simply write \( C \), as follows.

\[
\begin{cases}
2 & \text{if } i = j \\
-\frac{\omega}{\pi} \{h : i \rightarrow j\} - \frac{\omega}{\pi} \{h : j \rightarrow i\} & \text{otherwise}
\end{cases}
\]

We write \( \alpha_i \) for its simple root. Let \( P = \mathbb{Z}[\alpha_i]_{i \in I} \) be its root lattice. The matrix \( C \) induces a bilinear form on \( P \), we write \((-,-)\) for it. We denote \( Q^+ \subset P \) as the positive lattice. Set a bilinear form on \( Q^+ \) by

\[
\langle \alpha, \beta \rangle = \sum_{i \in I} \alpha_i \beta_i - \sum_{h \in \Omega} \alpha_{s(h)} \beta_{t(h)}
\]

It is well known that \( \langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle \). We write \( \omega_i \) for the fundamental weights and let \( \Lambda^+ = \mathbb{N}[\omega_i]_{i \in I} \) be dominant weights. We write \( W \) for its Weyl group generated by simple reflections \( s_i \) associated with simple roots \( \alpha_i \) for any \( i \in I \). Denote \( R^+ \) by the set of positive roots.

Let us fix an orientation of \( Q \) such that the indexing set of \( I \) satisfying

\[
i_k \rightarrow i_j \text{ then } k < j
\]

The ground field is the complex number field \( \mathbb{C} \), we sometimes write \( k \) for simplicity. Let \( \alpha \in Q^+ \) and \( V \) be a \( I \)-graded space such that \( \text{dim} V = \alpha \), we denote by \( G_\alpha \) the group \( \prod_{i \in I} \text{GL}(V_i) \) and denote by \( E_\alpha \) the representation space \( \bigoplus_{h \in \Omega} \text{Hom}_k(V_{s(h)}, V_{t(h)}) \) endowed with a \( G_\alpha \) action given by \( g \cdot x = (g_{t(h)} x_h g_{s(h)}^{-1})_{h \in \Omega} \). For simplicity, we denote by \( \text{Hom}_I(V, W) \) the space \( \bigoplus_{i \in I} \text{Hom}_k(V_i, W_i) \) and by \( \text{Hom}_\Omega(V, W) \) the space \( \bigoplus_{h \in \Omega} \text{Hom}_k(V_{s(h)}, W_{t(h)}) \).

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2. Representation theory of Dynkin quivers

In this section we briefly review some basic facts on the representation theory of Dynkin quivers. We will construct some maps related to representation space \( E_\alpha \), and give the definition of dual representations of \( Q \) in the sense of Lusztig’s nilpotent varieties.

Fix a Dynkin quiver \( Q \). Let \( M = (V, x) \) be a representation of \( Q \) where \( V \) is a \( I \)-graded vector space and \( x = (x_i)_{i \in I} \) is a tuple of matrices \( x_i : V_{s(h)} \rightarrow V_{t(h)} \). We sometimes write the matrices \( x \) for the representation \( M \) if there is no danger of confusion. For a representation
Let $M = (V, x)$, we write $\dim M$ for its dimension $\sum_{i \in I} \dim(V_i)\alpha_i \in Q^+$. Given two representations $M, N$ of $Q$, we denote by $\text{Hom}_Q(M,N)$ the vector space of $Q$-morphisms between $M$ and $N$ and write $[M,N]$ (resp: $[M,N]^1$) for $\dim \text{Hom}_Q(M,N)$ (resp: $\dim \text{Ext}_Q^1(M,N)$). One has

$$\langle \dim M, \dim N \rangle = [M,N] - [M,N]^1$$

For any vertex $i \in I$, we associate it with the indecomposable projective (resp: injective) representation $P_i$ (resp: $I_i$), and the head (resp: socle) of $P_i$ (resp: $I_i$) is the simple module $S_i$.

### 2.1. Representations of Dynkin quivers.

Let us focus on the indecomposable representations of $Q$. Let $w \in W$ be element of the Weyl group $W$. A reduced expression of $w = s_{i_m} \cdots s_{i_2}s_{i_1}$ gives rise to a word $i = [i_1 i_2 \cdots i_m]$. This yields a set of roots as follows:

$$\alpha_{i_1} \cap s_{i_2} \cap \cdots s_{i_{k-1}} \cap (\alpha_{i_k}) \cap \cdots s_{i_2} \cap s_{i_1}$$

Let us denote them by $\beta_k = s_{i_1} \cdots s_{i_k} \cap (\alpha_{i_k})$ for $1 \leq k \leq m$ and give them an ordering so that $\beta_l < \beta_k$ if $l < k$. We write $w = w_0$ for the unique element in $W$ with maximal length and $C = s_{i_1} \cdots s_{i_n}$ for the Coxeter element of $W$. Denote by $h$ the Coxeter number.

Fix a word of reduced expression of $w$. For any $k \in [1,m]$, let us consider $\beta_k$. Define the following subset of $R^+$:

$$\tau^1 \beta_k = \{ \beta_j \mid i_j = i_k \} \quad \tau^1(k) = \{ i_j \mid i_j = i_k \} \quad k^+ = \min \{ j > k \mid i_j \in \tau^1(k) \} \quad k^- = \max \{ j < k \mid i_j \in \tau^1(k) \}$$

$$\beta_{[j,k]} = (\beta_j, \tau_j, \cdots, \tau^{-1}k, \beta_k) \quad i_j = i_k \quad j \leq k$$

From an orientation of $Q$ given by (1.11), one defines a Coxeter element $C_Q = s_{i_1} \cdots s_{i_n}$. It is easy to see that $i_1$ is a source and $i_n$ is a sink. Since Dynkin quivers are bipartite orientations, we have that $w_0 = c_Q^{h/2}$ if $h$ is even or $w_0 = c_1 c_Q^{-h+1}$ where $c_1$ is the sink part of $C_Q$ if $h$ is odd. It gives rise to a word of $w_0$ as

$$i = [i_1 i_2 \cdots i_n i_1 \cdots i_n \cdots i_1 \cdots i_n]$$

It is easy to see $i$ is adapted to $Q$.

Denote by $\theta_k$ the root $s_{i_1} \cdots s_{i_k-1} \cap (\alpha_{i_k})$ for any $i_k \in I$. It is easy to see that

$$\{ \theta_k \}_{k \in [1,n]} = \{ \alpha \in R^+ \mid C_Q^{-1}(\alpha) \in R^- \}$$

From the above word (2.4) we have that any roots $\beta$ with the form $w(\alpha_{i_k})$ for some $w \in W$ are identified with $C_Q^{-r}(\theta_k)$ for some $r \in \mathbb{N}$. 
Gabriel’s Theorem. Let \( i \) be a sink of \( Q \), one can define Reflection functor \( \Phi_i \) on the representations of \( Q \): For a representation \( M = (V, x) \), let us consider the following map

\[
0 \rightarrow W_i \rightarrow \bigoplus_{h; t(h) = i} V_{s(h)} \xrightarrow{\sum_h x_h} V_i
\]

where \( W_i \) is the kernel of the above map. Define \( \Phi_i(M) = (V', x') \) where \( V'_k = V_k \) for \( k \neq i \) and \( V'_i = W_i \), and where \( x'_h = x_h \) for \( t(h) \neq i \) and \( x'_h \) is the above induced map \( W_i \rightarrow V_{s(h)} \) for \( t(h) = i \). The reflection functors give rise to the following theorem.

**Theorem 2.1.** [Gabriel’s Theorem] If \( Q \) is a Dynkin quiver, then the map \( M \mapsto \dim M \) gives a bijection between the set of isomorphic class of indecomposable representations and the set of positive roots \( R^+ \).

We denote by \( M_\beta \) the module corresponding to the root \( \beta \). If the order of positive roots is given as before, one has

\[
\text{Hom}_Q(M_{\beta_k}, M_{\beta_l}) = \text{Ext}_Q(M_{\beta_k}, M_{\beta_l}) = 0 \quad \text{for} \quad \beta_k < \beta_l
\]

We sometimes write \([\beta_k, \beta_l]\) (resp: \([\beta_k, \beta_l]^1\)) for \( \dim \text{Hom}(M_{\beta_k}, M_{\beta_l}) \) (resp: \( \dim \text{Ext}(M_{\beta_k}, M_{\beta_l}) \)) for simplicity.

Since any representation \( M \) with \( \dim M = \alpha \) can be decomposed as a direct sum of indecomposable representations, we associate each \( M \) with a sequence of roots \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) such that \( M \cong \bigoplus_{k=1}^r M_{\lambda_k} \) and for any \( k \in [1, s] \) such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \). Therefore, one can define the Konstant partition \( \lambda \) by

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)
\]

where \( \lambda_k \in R^+ \) such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \). Denote by \( \text{KP}(\alpha) \) the set of Konstant partitions satisfying \( \sum_{i=1}^s \lambda_i = \alpha \). It is straight to see that the set of \( G_\alpha \)-orbits in \( E_\alpha \) coincides with \( \text{KP}(\alpha) \).

There is an ordering on representations with dimension \( \alpha \) (up to isomorphism) given by \( M \leq N \) if and only if \( \mathcal{O}_N \subset \overline{\mathcal{O}_M} \) where \( M, N \in E_\alpha \), \( \mathcal{O}_N \) is the \( G_\alpha \)-orbit of \( N \), and \( \overline{\mathcal{O}_M} \) is the Zariski closure of orbit \( \mathcal{O}_M \). Therefore, we define an ordering on \( \text{KP}(\alpha) \) so that \( \lambda' \leq \lambda \) if \( \mathcal{O}_\lambda \subset \overline{\mathcal{O}_\lambda} \).

### 2.2. An example of \( A_n \)

Let us consider the type \( A_n \). We denote by \([1, n]\) the set of integers \( k \) such that \( 1 \leq k \leq n \). We fix an orientation of \( A_n \) as follows:

\[
1 \rightarrow 2 \rightarrow \cdots \rightarrow n
\]

The Coxeter element \( C_Q \) is given by \( C_Q = s_{i_1}s_{i_2} \cdots s_{i_n} \). It induce a reduced expression of \( w_0 \) as follows.

\[
w_0 = s_1s_2s_3 \cdots s_n s_1s_2 \cdots s_n s_1 \cdots s_n \cdots
\]

Thus, we obtain a word

\[
\{1, 2, \cdots, n, 1, 2, \cdots, n, \cdots, 1, 2, \cdots, n, \cdots\}
\]
Let \( \alpha_i \) be the simple root associated with each point \( i \in [1, n] \), and let \( Q^+ = \mathbb{N}[\alpha_i]_{i \in [1, n]} \) be the positive root lattice. The positive root set \( R_i^+ \) is isomorphic to the set of segments \([a, b]\) such that \( a \leq b \). It is easy to check
\[
\tau(M[k, l]) = M[k + 1, l + 1] \text{ if } l \leq n - 1
\]
where \( \tau(M) \) refers to the Auslander-Retie translation of \( M \). We will show that

**Lemma 2.2.** The order of segments is given by
\[
[i, j] > [k, l] \text{ if } i > k \text{ or } i = k \text{ and } j > l
\]

**Proof.** Since the Coxeter element \( C = s_{i_1}s_{i_2}\cdots s_{i_n} \). By the definition \((2.5)\), we obtain
\[
\theta_k = [1, k] \quad \text{for } k \in [1, n]
\]
and for any \( i \), \( \tau^i \theta_k = [i, k + i] \) if \( k + i \leq n \). Since \( \tau^i \theta_k < \tau^j \theta_l \) if and only if \( i < j \) or \( i = j \) and \( k < l \). Namely, \([i, i + k] < [j, j + l]\) if and only if \( i < j \) or \( i = j \) and \( i + k < j + l \). \( \square \)

It is well known that any indecomposable representations of \( A_n \) is of the form \( M[a, b] \) such that \( \dim M[a, b] = \sum_{a \leq i \leq b} \alpha_i \). We list some properties of indecomposable representations.

\begin{align*}
[M[i, j], M[k, l]] & = 1 \text{ if and only if } k \leq i \leq l \leq j \\
[M[k, l], M[i, j]]^1 & = 1 \text{ if and only if } k + 1 \leq i \leq l + 1 \leq j.
\end{align*}

Moreover, for each \([M[k, l], M[i, j]]^1 = 1\), we have the following short exact sequence
\[
0 \rightarrow M[i, j] \rightarrow M[i, l] \oplus M[k, j] \rightarrow M[k, l] \rightarrow 0
\]
where we formally set \( U_{i, j} = 0 \) if \( i < 1 \) or \( j > n \) or \( j < i \). For each representation \( M \) of \( Q \), one can decompose every representation \( M \cong M[a_1, b_1] \oplus M[a_2, b_2] \oplus M[a_m, b_m] \) for some \( m \geq 1 \). We arrange this tuple \(([a_1, b_1], [a_2, b_2], \cdots [a_m, b_m])\) such that \([a_k, b_k] \geq [a_{k+1}, b_{k+1}]\) for any \( k \in [1, n - 1] \), which is called multisegment. It follows that the Grothendieck group of \( A_n\text{-mod} K_0(A_n - \text{mod}) \) is isomorphic to the set of multisegments. Therefore, the Konstant partitions are of the form multisegments.

We define a shift on multisegments as \( \lambda[1] = ([i_1 + 1, j_1 + 1], [i_2 + 1, j_2 + 1], \cdots [i_r + 1, j_r + 1]) \).

It is straightforward to see any two roots \( \alpha, \beta \) of the same length have the relation \( \alpha = \beta[k] \) for some integer \( k \). It implies that \( \beta[j, k] = \{\beta_k, \beta_k[-1], \cdots, \beta_j[1], \beta_j\} \).

2.3. **Extension spaces of representations.** Given two representations \( M = (V, x) \) and \( N = (W, y) \), one has an exact sequence
\[
(2.10) \quad 0 \rightarrow \text{Hom}_Q(M, N) \rightarrow \bigoplus_{i \in I} \text{Hom}_k(V_i, W_i) \overset{p_{x,y}}{\rightarrow} \bigoplus_{h \in \Omega} \text{Hom}_k(V_{s(h)}, W_{t(h)}) \overset{e_{x,y}}{\rightarrow} \text{Ext}_Q^1(M, N) \rightarrow 0
\]
where \( p_{x,y}(f) = (f_{t(h)}x_h - y_hf_{s(h)})_{h \in \Omega} \). Recall that for any \( \eta \in \text{Ext}_Q^1(M, N) \) there is a short exact sequence
\[
\eta: 0 \rightarrow N \rightarrow E_{\eta} \rightarrow M \rightarrow 0
\]
where $E_\eta$ refers to the block as \[
\begin{pmatrix}
y & \eta \\
0 & x
\end{pmatrix}\]
where $\eta$ is an element of $e_{x,y}^{-1}(\overline{\eta})$. Note that $E_\eta$ is independent of the choice of $\eta \in e_{x,y}^{-1}(\overline{\eta})$ up to isomorphism. This follows from the fact that the subgroup of $G_{\dim M \oplus N}$ consisting of the blocks as \[
\begin{pmatrix}
E_N & f \\
0 & E_M
\end{pmatrix}
\] where $f \in \text{Hom}_I(V,W)$ gives rise to an element of $\eta + p_{x,y}(f)$ via the $G_{\dim M \oplus N}$-action on $E_\eta$.

Let us consider the End$_Q(M)$-action on $\text{Hom}_\Omega(V,W)$ by $\eta \cdot f = (\eta h f_s(h))$. This action gives rise to an End$_Q(M)$-action on Ext$_{\Omega}^1(M,N)$. In order to show that it is a well-defined action, it is enough to check that for any $\eta \in \text{Im} p_{x,y}$ and $f \in \text{End}_Q(M)$ we have $\eta \cdot f \in \text{Im} p_{x,y}$. Considering $\eta = (g_t(h)x_h - y_h g_s(h))_{h \in \Omega}$ and the following equation, we obtain our conclusion.

\[
\eta \cdot f = (g_t(h)x_h f_s(h) - y_h g_s(h) f_s(h))_{h \in \Omega} = (g_t(h)f_t(h)x_h - y_h g_s(h) f_s(h))_{h \in \Omega} = p_{x,y}(gf)
\]

It follows that Ext$_\Omega^1(M,N)$ is a module of End$_Q(M)^{op}$. Similarly, Ext$_\Omega^1(M,N)$ is a module of End$_Q(N)$. Applying functor $D := \text{Hom}_k(-, k)$ to exact sequence (2.10), we obtain (2.11)

\[
0 \rightarrow D \text{Ext}_\Omega^1(M,N) \xrightarrow{\epsilon_{x,y}^\eta} \bigoplus_{h \in \Omega} \text{Hom}_k(W_{t(h)}, V_{s(h)}) \xrightarrow{p_{x,y}^*} \bigoplus_{i \in I} \text{Hom}_k(W_i, V_i) \rightarrow D \text{Hom}_Q(M,N) \rightarrow 0
\]

where $p_{x,y}^*$ is the map $p_{x,y}^*(z) = (\sum_{t(h)=i} x_h z_h - \sum_{s(h)=i} z_h y_h)_{i \in I}$. Because we can consider $p_{x,y}$ as a Lie bracket operation on the space $\text{Hom}_k(V,W)$ after forgetting the $I$-graded structure and $\text{Hom}_\Omega(V,W)$, $\text{Hom}_I(V,W)$ are the subspace of $\text{Hom}_k(V,W)$, the formula of $p_{x,y}^*$ follows from the standard fact. We denote by $\eta^\prime \in D \text{Ext}_\Omega^1(M,N)$ the dual object of $\eta \in \text{Ext}_\Omega^1(M,N)$. In fact, $\eta^\prime$ is the transport of matrices of $\eta$ under a given base.

2.4. The action of automorphism groups on extension spaces. Let us fix a representation $M = (V, x)$ with dimension $\alpha \in Q^+$. We denote by Aut$_Q(M)$ the isomorphism group of $M$ in End$_Q(M)$. We define an Aut$_Q(M)$ action on Ext$_\Omega^1(M,M)$ by $g \ast \eta = (g_t(h) \eta_h g_s^{-1}(h))_{h \in \Omega}$. If we consider Aut$_Q(M)$ as a subset of $G_\alpha$, as we discussed before, this action is compatible with the $G_\alpha$ on $E_\alpha$ when we consider $\overline{\eta} \in \text{Ext}_\Omega^1(M,M)$ as an element in $E_\alpha$ by the formula (2.10). Denote by $O_\overline{\eta}$ the Aut$_Q(M)$-orbit of $\overline{\eta} \in \text{Ext}_\Omega^1(M,M)$.

Considering the formula (2.11) and replacing $N$ by $M$, we obtain a representation $\eta^\prime \in E_\alpha^*$ for any $\eta \in E_\alpha$. The $G_\alpha$-action is compatible with respect to the transport of matrices. It follows that $\eta \leq \eta^\prime$ if and only if $\eta^\prime \leq \eta^\prime$.

**Lemma 2.3.** Let $\eta, \eta^\prime \in E_\alpha$ and suppose $\eta \leq \eta^\prime$. (2.10) implies that they induce elements in Ext$_\Omega^1(M,M)$. Consider their Aut$_Q(M)$-orbits in Ext$_\Omega^1(M,M)$ via (2.10). We have

\[
O_{\eta^\prime} \subset O_{g \ast \eta}
\]

for some $g \in G_\alpha$. 
For an element \( \eta \in D\text{Ext}_Q^1(M, M) \), set
\[
N_{G_\alpha}(\eta) = \{ g \in G_\alpha \mid g \cdot \eta \in D\text{Ext}_Q^1(M, M) \}
\]
It is easy to see that \( \text{Aut}_Q(M) \) is a subgroup of \( N_{G_\alpha}(\eta) \). The conjugate class of \( \text{Aut}_Q(M) \) in \( N_{G_\alpha}(\eta) \) is classified by quotient variety \( N_{G_\alpha}(\eta)/\text{Aut}_Q(M) \), denoted by \( B \). It follows that
\[
(2.12) \quad \mathcal{O}_{\eta} \cap D\text{Ext}_Q^1(M, M) = \bigcup_{g \in B} \mathcal{O}_{g \cdot \eta}
\]
and each orbit of the same dimension.

**Proposition 2.4.** Under the above assumption, \( \text{Ext}_Q^1(M, M) \) admits an open \( \text{Aut}_Q(M) \)-orbit if and only if there exists an element \( \eta \in D\text{Ext}_Q^1(M, M) \) with the minimal dimension \([\eta, \eta]^1\) and \( N_{G_\alpha}(\eta) = \text{Aut}_Q(M) \).

**Proof.** If \( \text{Ext}_Q^1(M, M) \) admits an open \( \text{Aut}_Q(M) \)-orbit \( \mathcal{O}_{\eta} \). Since there is only \( \mathcal{O}_{\eta} \) with the maximal dimension, it follows by (2.12) that \( B = e \) and then \( N_{G_\alpha}(\eta) = \text{Aut}_Q(M) \). Because \( \mathcal{O}_{\eta} \cap D\text{Ext}_Q^1(M, M) \) is an open dense subset of \( D\text{Ext}_Q^1(M, M) \), we get that \( D\text{Ext}_Q^1(M, M) \subset \mathcal{O}_{\eta} \), this means for any element \( \eta' \in D\text{Ext}_Q^1(M, M) \), we have \( \mathcal{O}_{\eta'} \subset \mathcal{O}_{\eta} \). Hence, we obtain \( \eta \leq \eta' \) for any \( \eta \in D\text{Ext}_Q^1(M, M) \). In other words, \([\eta, \eta]^1 \leq [\eta', \eta'^1] \) for any \( \eta' \in D\text{Ext}_Q^1(M, M) \).

On the other hand, if \([\eta, \eta]^1 \) is of the minimal with respect to any \( \eta' \in D\text{Ext}_Q^1(M, M) \), then \( \mathcal{O}_{\eta} \) is of maximal dimension for all \( \eta' \in D\text{Ext}_Q^1(M, M) \). Since \( \text{Aut}_Q(M) = N_{G_\alpha}(\eta) \) and (2.12), \( \mathcal{O}_{\eta} \cap D\text{Ext}_Q^1(M, M) = \mathcal{O}_{\eta} \) is an open dense subvariety of \( D\text{Ext}_Q^1(M, M) \). \( \square \)

**Remark 2.5.** This proposition is useful, as we always find the element \( \eta \in D\text{Ext}_Q^1(M, M) \) with minimal \([\eta, \eta]^1\) for any representation \( M \) of \( Q \). The problem of the open \( \text{Aut}_Q(M) \)-orbit in \( D\text{Ext}_Q^1(M, M) \) turns out to be calculating the group \( N_{G_\alpha}(\eta) \).

**Proposition 2.6.** \( \text{Ext}_Q^1(M, M) \) admits an open \( \text{Aut}_Q(M) \)-orbit if and only if there is an element \( \eta \in \text{Ext}_Q^1(M, M) \) such that \([\eta, \text{End}_Q(M)] = \text{Ext}_Q^1(M, M) \). Moreover, the dimension of center of \( \eta \) in \( \text{End}_Q(M) \), which is denoted by \( \text{dim} \text{Z}(\eta) \), is equal to \( \langle \alpha, \alpha \rangle \).

**Proof.** Following from Lie(\( \text{Aut}_Q(M) \)) = \( \text{End}_Q(M) \) and (2.1).

We consider \( \text{End}_Q(M) \) as a Lie subalgebra of \( \prod_{i \in I} \mathfrak{gl}(V_i) \). The above proposition shows that \( \text{Hom}_Q(M, \tau M) \) as its cyclic module.

**Definition 2.7.** Let \( M = (V, x) \) be a representation of \( Q \). We define the dual representation of \( M \) by
\[
M^* = \{ \eta \in D\text{Ext}_Q^1(M, M) \mid [\eta, \eta]^1 \leq [\eta', \eta]^1 \quad \forall \eta' \in D\text{Ext}_Q^1(M, M) \}
\]
By Lemma (2.3), we have that \( M^* \) is unique up to isomorphism.
3. Auslander-Reiten Theory

In this section we will recall the Auslander-Reiten theory on Dynkin quivers. As we discussed before, the \( \text{Aut}_Q(M) \) action on the extension space \( \text{Ext}_Q^1(M, N) \) turns out to be the \( \text{Aut}_Q(M) \) action on the space \( D \text{ Ext}_Q^1(M, N) \). The advantage of the space \( D \text{ Ext}_Q^1(M, N) \) is that we can consider it as a subspace of \( \text{Hom}_\Omega(V, W) \), which is easier than the quotient space \( \text{Ext}_Q^1(M, N) \). In order to describe the space \( D \text{ Ext}_Q^1(M, N) \), we introduce the notion of Auslander-Reiten translation on Dynkin quivers.

3.1. Notions of AR translation. For any representation \( M \), let us consider its minimal projective resolution

\[
0 \rightarrow P_1 \xrightarrow{p_1} P_0 \rightarrow M \rightarrow 0
\]

Applying the functor \((-)^t := \text{Hom}_Q(-, kQ)\), we obtain

\[
(3.1) \quad 0 \rightarrow M^t \xrightarrow{p_1^t} P_0^t \rightarrow \text{Coker} p_1^t \rightarrow 0
\]

then applying \( D = \text{Hom}_k(-, k) \) to the above exact sequence, we obtain

\[
0 \rightarrow D \text{ Coker} p_1^t \rightarrow D P_1^t \rightarrow D P_0^t \rightarrow D M^t \rightarrow 0
\]

We call \( D \text{ Coker} p_1^t \) the Auslander-Reiten translation of \( M \), which is denoted by \( \tau M = (\tau V, \tau x) \). It is easy to see that \( \tau M = D \text{ Ext}_Q^1(M, kQ) \). We list some well-known properties of Auslander-Reiten translation as follows.

**Proposition 3.1.** The following propositions are not only true for the Dynkin cases.

1. In acyclic cases, we have \( \tau M = \Phi C_Q M \), where \( \Phi C_Q \) refers the reflection functor corresponding to the Coxeter element \( C_Q \) (see Section (2.1)). In particular, \( \dim \tau V = C_Q(\dim V) \) as an element in \( Q^+ \).

2. In acyclic cases, we have

\[
(3.2) \quad \text{Hom}_Q(M, N) = D \text{ Ext}_Q^1(N, \tau M) = D \text{ Ext}_Q^1(\tau^{-1} N, M)
\]

\[
\text{Hom}_Q(M, \tau N) = D \text{ Ext}_Q^1(N, M) = D \text{ Ext}_Q^1(\tau N, \tau M)
\]

3. If \( M, N \) have no direct summand of projective representations, we have

\[
\text{Hom}_Q(M, N) \cong \text{Hom}_Q(\tau M, \tau N); \quad \text{Ext}_Q^1(M, N) \cong \text{Ext}_Q^1(\tau M, \tau N)
\]

Suppose \( Q \) be a Dynkin quiver. For any two non-projective indecomposable representations \( \beta_k \) and \( \beta_l \), we have \( \beta_k < C \beta_k \leq \beta_l \) if \( [\beta_k, \beta_l] \) is equal to 1.

4. In [21] Ringel inductively constructed the maps \( u_h \) as \( u_h : (\tau V)_{t(h)} \rightarrow V_{s(h)} \) such that for any \( i \in I \).

\[
(3.3) \quad 0 \rightarrow (\tau V)_i \xrightarrow{u_h, t(h) = i} \bigoplus_{t(h) = i} V_{s(h)} \oplus \bigoplus_{s(h') = i} (\tau V)_{t(h')} \xrightarrow{(x_{h'}, u_{h'})} V_i
\]

and for any element \( \eta \in \text{Ker} p_{x, x} \), there exists an unique element \( g \) in \( \text{Hom}_Q(M, \tau M) \) such that \( u g = \eta \). For any \( f \in \text{Hom}_Q(M, M) \), we have

\[
(3.4) \quad u_h(\tau f)_{t(h)} = f_{s(h)} u_h, \text{ for any arrow } h
\]
Proof. We just prove (3). As $[\beta_k, \beta_l] = [\beta_l, C_\beta_k] = 1$, and formula (2.8), we obtain $\beta_k < \beta_l$ and $\beta_l < C_\beta_k \leq \beta_l$. \hfill \Box

Let us consider the formula (2.11). Applying $D$ on that exact sequence, we obtain

**Lemma 3.2.**

\[(3.5)\]
\[0 \rightarrow \text{Hom}_Q(N, \tau M) \xrightarrow{e_{x,y}} \bigoplus_{h \in \Omega} \text{Hom}_k(W_{t(h)}, V_{s(h)}) \xrightarrow{p_{x,y}} \bigoplus_{i \in I} \text{Hom}_k(W_i, V_i) \rightarrow D \text{Hom}_Q(M, N) \rightarrow 0\]

where $e_{x,y}^*$ is given by $e_{x,y}^*(f) = (u_h f_{t(h)})_{h \in \Omega}$.

**Proof.** In order to get the formula, we first show that $(u_h f_{t(h)})_{h \in \Omega} \in \text{Ker} p_{x,y}^*$. Recall $p_{x,y}^* (z) = (\sum_{t(h)=i} x_h z_h - \sum_{s(h)=i} z_h y_h)_{i \in I}$. It implies that

\[
p_{x,y}^* ((u_h f_{t(h)})_{h \in \Omega})
\]
\[= (\sum_{t(h)=i} x_h u_h f_i - \sum_{s(h)=i} u_h f_{t(h)} y_h)_{i \in I} \quad \text{by (3.4)}
\]
\[= (\sum_{t(h)=i} x_h u_h f_i - \sum_{s(h)=i} u_h \tau x_h f_i)_{i \in I} \quad \text{by (3.3)}
\]
\[= (0 \cdot f_i)_{i \in I} = 0
\]

If $u_h f_{t(h)} = 0$ for any $h \in \Omega$, we will prove $f_i = 0$ by induction. If $i$ is a sink, then $u_h$ is injective for any $t(h) = i$, it implies $f_i = 0$. Let us assume that for any $j > i$ we have $f_j = 0$. Considering the exact sequence (3.3) and $u_h f_i = 0$, we have $(u_h f_i, \tau x_h f_i) = (0, f_j \tau x_h)$ for some $j > i$. As $f_j = 0$, it follows by our assumption that $(u_h f_i, \tau x_h f_i) = 0$. Thus we get $f_i = 0$ for any vertex $i \in I$.

The fact $\text{Im} e_{x,y}^* \subset \text{ker} p_{x,y}^*$ and $\dim \text{Hom}_Q(N, \tau M) = \dim \text{Ext}_Q^1(M, N)$ implies that $\text{Hom}_Q(M, \tau N)$ is the kernel of $p_{x,y}^*$ via the map $e_{x,y}^*$. \hfill \Box

Let us consider the representations of the opposite quiver $Q^{op}$. For any two $I$-graded vector spaces $V, W$, one can consider $\text{Hom}_Q(V, W)$ as the dual space of $\text{Hom}_Q(W, V)$. It follows that one can consider $E_{\alpha}^*$ as the representation space $E_{\alpha}(Q^{op})$ for any $\alpha \in Q^+$. Let $M' = (V, a)$ and $N' = (W, b)$ be two representations of $Q^{op}$ such that $a \in \text{Hom}_Q(M, \tau M) \subset \text{Hom}_Q(V, V)$ and $b \in \text{Hom}_Q(N, \tau N) \subset \text{Hom}_Q(W, W)$ via (3.5). By (2.10) we have

\[(3.6)\]
\[0 \rightarrow \text{Hom}_{Q^{op}}(N', M') \rightarrow \bigoplus_{i \in I} \text{Hom}_k(W_i, V_i) \xrightarrow{p_{a,b}} \bigoplus_{h \in \Omega} \text{Hom}_k(W_{t(h)}, V_{s(h)}) \rightarrow \text{Ext}_{Q^{op}}(N', M') \rightarrow 0
\]

Denote by $i_{x,y}$ the injective map $\text{Hom}_Q(N, M) \xrightarrow{i_{x,y}} \bigoplus_{i \in I} \text{Hom}_k(W_i, V_i)$.

**Lemma 3.3.** The restriction of $p_{a,b}$ to $\text{Hom}_Q(N, M)$ becomes the following map.

\[r_{a,b} : \text{Hom}_Q(N, M) \rightarrow \text{Hom}_Q(N, \tau M)
\]
\[f \mapsto af - \tau(f)b
\]
Proof. We just show the following commutative diagram.

\[
\begin{array}{ccc}
\text{Hom}_Q(N, M) & \xrightarrow{r_{a,b}} & \text{Hom}_Q(N, \tau M) \\
\downarrow i_{x,y} & & \downarrow \epsilon_{x,y} \\
\bigoplus_{i \in I} \text{Hom}_k(W_i, V_i) & \xrightarrow{p_{a,b}} & \bigoplus_{h \in \Omega} \text{Hom}_k(W_{i(h)}, V_{s(h)})
\end{array}
\]

Consider

\[
e_{x,y}^* r_{a,b}(f) = e_{x,y}^* (af - \tau(f)b) = (u_h^V af - u_h^V \tau(f)b)_{h \in \Omega} = (e_{x,x}^* (a)f - f e_{x,y}^*(b))_{h \in \Omega} = p_{a,b} i_{x,y}(f)
\]

\[\square\]

3.2. Representations and root system. In this section, we will give an interpretation of the condition \([\beta, \alpha] = 1\) in terms of the relation between roots \(\beta\) and \(\alpha\). This seems like an interesting link. This relation become more interesting when we consider the AR translation \(\tau\). Due to Proposition (3.1), in what follows, we denote by \(\tau\) the Coxeter element \(C_Q\) in Section (2.1).

Lemma 3.4. Suppose that \(\alpha < \beta \in R^+\). If \([\beta, \alpha] = 0\), then we have \(\alpha + \beta \in R^+\) or \((\alpha, \beta) = 0\). In fact, we have \(0 \rightarrow M_\beta \rightarrow M_{\alpha + \beta} \rightarrow M_\alpha \rightarrow 0\) or \((\alpha, \beta) = 0\).

Proof. If \([\beta, \alpha] = 0\), it implies that \(\langle \beta, \alpha \rangle = 0\) by formula (2.6). On the other hand, we have \(\langle \alpha, \beta \rangle = [\alpha, \beta] - [\alpha, \beta]^1 = -[\alpha, \beta]^1\). If \(-[\alpha, \beta]^1 = 0\), it implies that \((\alpha, \beta) = 0\). If \(-[\alpha, \beta]^1 = -1\), we obtain \((\alpha, \beta) = -1\). It follows that \(s_\beta(\alpha) = \alpha + \beta \in R^+\).

\[\square\]

Proposition 3.5. Let \(\alpha < \beta \in R^+\). There are four cases as follows:

1. \([\beta, \alpha] = [\beta, \tau \alpha] = 1\);
2. \(\alpha + \beta \in R^+\) if and only if \([\beta, \alpha] = 0\) and \([\beta, \tau \alpha] = 1\);
3. \(\beta - \alpha \in R^+\) if and only if \([\beta, \alpha] = 1\) and \([\beta, \tau \alpha] = 0\);
4. \([\beta, \alpha] = [\beta, \tau \alpha] = 0\).

Proof. we just prove (3). \((\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle = 1\) if and only if \([\beta, \alpha] = 1\) and \([\beta, \tau \alpha] = 0\), it is equivalent to \(s_\alpha(\beta) = \beta - (\alpha, \beta) \alpha = \beta - \alpha \in R\).

Let \(E\) be a non trivial extension in \(\text{Ext}_Q^1(M_\alpha, M_\beta)\). Namely, we have the following exact sequence

\[
0 \rightarrow M_\beta \rightarrow E \rightarrow M_\alpha \rightarrow 0
\]

This implies that \([E, E]^1 < [M_\beta \oplus M_\alpha, M_\beta \oplus M_\alpha]^1 = 1\). That is: \([E, E]^1 = 0\). As the rigid representation with dimension vector \(\alpha + \beta\) is unique up to isomorphism and any indecomposable representation \(M_{\alpha, \beta}\) is rigid, we get \(M_{\alpha, \beta} \cong E\). We proved our conclusion.

\[\square\]

Lemma 3.6. For two roots \(\theta_a > \theta_b\).
2.6

Proof. Recall that \( \theta_k = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k}) \) for \( k = a, b \). It is easy to see by the ordering on \( I \) that \( i_a \) and \( i_b \) are the sinks of the support of \( \theta_a \) and \( \theta_b \) respectively. The relation \( \theta_a > \theta_b \) implies \( a > b \). Let us put \( w = s_{i_b}\cdots s_{i_{a-1}} \) and write \( \theta_a = s_{i_1}s_{i_2}\cdots s_{i_{b-1}}w\alpha_{i_a} \).

If \( w(\alpha_{i_a}) = \sum_{k>b} n_k\alpha_{i_k} + n_b\alpha_{i_b} \) for \( n_b > 0 \), this means there exists a path \( \rho : i_b \to i_a \). Meanwhile, we see that

\[
w(\alpha_{i_a}) = s_{i_b}(\sum_{k>b} n_k\alpha_{i_k}) = \sum_{k>b} n_k\alpha_{i_k} - (\sum_{k>b} n_k\alpha_{i_k}, \alpha_{i_b})\alpha_{i_b} = \sum_{k>b} n_k\alpha_{i_k} + n_b\alpha_{i_b}
\]

It follows that \( \theta_a = n_b\theta_b + \beta \) for some \( \beta \in R^+ \). Let us consider

\[
(\theta_a, \theta_b) = (\sum_{k>b} n_k\alpha_{i_k} + n_b\alpha_{i_b}, \alpha_{i_a})
\]

\[
= (\sum_{k>b} n_k\alpha_{i_k}, \alpha_{i_b}) + (n_b\alpha_{i_b}, \alpha_{i_b})
\]

\[
= -n_b + 2n_b = n_b > 0
\]

It implies that \( \theta_a - \theta_b \in R^+ \).

If \( w(\alpha_{i_a}) = \sum_{k>b} n_k\alpha_{i_k} \), namely, there is no path from \( i_b \) to \( i_a \), then similarly we have \( (\theta_a, \theta_b) = 0 \). It is easy to see that \( \theta_a - \theta_b = \sum_{k>b} n_k\alpha_{i_k} - (\sum_{k>b} n_k\alpha_{i_k}, \alpha_{i_b}) = 0 \), as Gabriel’s functor preserve the dimension of \( \theta_a, \theta_b \) and the supports of the latter two roots have no common vertex. On the other hand, \( \tau \theta_b > \theta_a \), it follows by (2.6) that \( (\theta_a, \theta_b) = 0 \).

We will discuss the relation between Auslander-Reiten translations and the support of roots. For a root \( \beta = \sum_{k \in [1, s]} n_k\alpha_{i_k} \), we denote by \( \text{soc} \beta \) (resp: \( \text{hd} \beta \)) the set of source (resp: sink) vertices in the support of \( \beta \), such as \( s \) (resp: \( t \)). It is easy to see that the simple representation \( S_{i_s} \) (resp: \( S_{i_t} \)) is a socle (resp: head) of the corresponding representation \( M_{\beta} \) (recall that \( i_k \to i_l \) then \( k < l \)).

**Lemma 3.7.** Under the above assumption, for any \( k \) such that \( \tau^k\beta \in R^+ \), then we have

\[
(3.9) \quad \text{hd} \tau^k\beta \geq \text{hd} \beta \text{ and } \text{soc} \tau^k\beta \geq \text{soc} \beta
\]

Here we say that \( S \leq T \) if for any \( u \in S \) there exists an element \( v \in T \) such that \( u \leq v \).

**Proof.** For simplicity, It is enough to prove the case when \( k = 1 \). Let us first consider the simple representations \( S_{i_k} \) for all non-source vertices of \( Q \).

Recall that \( i_k \in \text{soc} \beta \) if and only if \( [S_{i_k}, M_\beta] \neq 0 \) From the AR-exact sequence for \( S_{i_k} \)

\[
0 \to S_{i_k} \to \oplus_{\beta} M_\beta \to \tau^{-1} S_{i_k} \to 0
\]

We see that \( \text{supp} \tau^{-1} S_{i_k} \subset \{i_c \mid c \leq k\} \).

Let us take \( i_k \in \text{hd} \tau^k \beta \) and consider

\[
(3.10) \quad [\tau M_\beta, S_{i_k}] = [M_\beta, \tau^{-1} S_{i_k}]
\]
Let \( \beta = \sum_{j \in [t, s]} n_j \alpha_j \). If \( k < t \), then \( \supp \tau^{-1} S_{ik} \subset [1, k] \). It follows by the fact \( [t, s] \cap [1, k] = \emptyset \) that \( [\tau M_\beta, S_{ik}] = 0 \). It leads to a contradiction. Therefore, we get \( \text{hd} \tau \beta \geq \text{hd} \beta \).

On the other hand, let us consider the AR-exact sequence for \( S_{ik} \)

\[
0 \to \tau S_{ik} \to \bigoplus \beta M_\beta \to S_{ik} \to 0
\]

We see that \( \supp \tau S_{ik} \subset \{ i_c \mid c \geq k \} \).

Let us take \( i_k \in \text{soc} \tau^{-1} \beta \) and consider

\[
[S_{ik}, \tau^{-1} M_\beta] = [\tau S_{ik}, M_\beta]
\]

Let \( \beta = \sum_{j \in [t, s]} n_j \alpha_j \). If \( k > s \), then \( \supp S_{ik} \subset [k, n] \). It follows by the fact \( [t, s] \cap [1, k] = \emptyset \) that \( [S_{ik}, \tau^{-1} M_\beta] = 0 \). This means \( \text{soc} \tau^{-1} \beta \leq \text{soc} \beta \). Namely, \( \text{soc} \beta \leq \text{soc} \tau \beta \).

Therefore, we obtain

\[
\text{hd} \tau \beta \geq \text{hd} \beta \quad \text{and} \quad \text{soc} \tau \beta \geq \text{soc} \beta
\]

We remark that if \( i_k \in \text{hd} \beta \) (resp: \( i_k \in \text{soc} \beta \)) such that \( i_k \) is a source (resp: sink) of \( Q \), this formula follows from the AR-sequence

\[
0 \to \tau M_\beta \to E \to M_\beta \to 0
\]

and \( \text{hd} \tau M_\beta \geq \text{hd} E \geq \text{hd} M_\beta \) and \( \text{soc} \tau M_\beta \geq \text{soc} E \geq \text{soc} M_\beta \)

\[\square\]

Next we will show that for any two roots \( \alpha, \beta \), we get

\[
R(\alpha, \beta) = \left\{ k \in \mathbb{Z}_{\geq 0} \mid [\tau^k \beta, \beta] = 1 \right\} = [a, b]
\]

Where \( \tau^{b+1} \alpha + \beta \in R^+ \). We prove this using the classification of the root pair \( (\alpha, \beta) \). Let us first consider the case \( \alpha = \beta \).

**Lemma 3.8.** Let \( M_\beta \) be the indecomposable representation of \( Q \) for the root \( \beta \). Let us set

\[
n_\beta = \max\{ k \in \mathbb{N} \mid [\tau^k \beta, \beta] = 1 \} = \min\{ k \in \mathbb{N} \mid [\tau^k \beta, \beta] = 0 \} - 1
\]

We have

\[
\tau^{n_\beta + 1} \beta + \beta \in R^+
\]

This formula means \( \left\{ k \in \mathbb{Z}_{\geq 0} \mid [\tau^k \beta, \beta] = 1 \right\} = [0, n_\beta] \)

**Proof.** For simplicity, we just consider the case \( \beta = \theta_a \) for all \( a \in I \), as \( [\tau^k \beta, \beta] = [\tau^{k-1} \beta, \beta] \) and any \( \beta = \tau^n \theta_a \) for a \( n \in \mathbb{N}, a \in I \).

We first show that

\[
\left\{ k \in \mathbb{Z}_{\geq 0} \mid [\tau^k \theta_a, \theta_a] = 1 \right\} = [0, n_a]
\]

This means there exists no \( n' \) such that \( [\tau^{n'} \theta_a, \theta_a] = 0 \) and \( [\tau^{n'+1} \theta_a, \theta_a] = 1 \). Otherwise, we take \( s < n_a \) such that \( [\tau^s \theta_a, \theta_a] = [\tau^{s+1} \theta_a, \theta_a] = 0 \) and \( [\tau^{s+1} \theta_a, \theta_a] = 1 \).

It follows by Proposition (3.5) that \( \alpha = \tau^{s+1} \theta_a - \theta_a \in R \). This is equivalent to \( (\tau^{s+1} \theta_a, \theta_a) > 0 \). Let \( w = s_{i_1} s_{i_2} \cdots s_{i_{a-1}} \) and \( \theta_a = w(\alpha_{i_a}) \). Recall that \( \tau = s_{i_1} s_{i_2} \cdots s_{i_{a-1}} \). We see that

\[
w^{-1} \tau w = s_{i_a} s_{i_{a-1}} \cdots s_{i_1} s_{i_n} \cdots s_{i_{a-1}}
\]
gives rise to a quiver $Q^a$ such that $i_a$ is a source of $Q^a$ and $w^{-1}\tau w$ is a coxeter element adapted with $Q^a$. Let us denote it by $\tau_a$.

Since

$$(\tau^{s+1}\theta_a, \theta_a) = (\tau^{s+1}w\alpha_i, w\alpha_i) = (\tau^{s+1}\alpha_i, \alpha_i) > 0$$

This means $\alpha := \tau^{s+1}\alpha_i - \alpha_i \in R$.

If $\alpha \in R^+$, by Lemma (3.4), we have

$$0 \to M_{\alpha} \to \tau^s S_{i_a} \to S_{i_a} \to 0$$

Therefore, we have $\text{hd} \tau^s S_{i_a} = \text{hd} S_{i_a}$. By Lemma (3.7), we have

$$i_a \leq \text{hd} \tau^s S_{i_a} \leq \text{hd} \tau^s S_{i_a} = i_a$$

This implies that there exists a surjective map: $\tau^s S_{i_a} \to S_{i_a} \to 0$. This means

$$[\tau^s S_{i_a}, S_{i_a}] = [\tau^s S_{i_a}, \tau S_{i_a}] = 1$$

Following Proposition (3.5), we get $\tau^{a+1} S_{i_a} - S_{i_a} \notin R$. This means

$$w(\tau^{a+1} S_{i_a} - S_{i_a}) = \tau^{a+1} \theta_a - \theta_a \notin R$$

This leads to a contradiction.

On the other hand, if $\tau^m \beta + \beta \in R^+$, it implies that $(\tau^m \beta, \beta) = -1$. Since

$$(\tau^m \beta, \beta) = (\tau^m \beta, \beta) + (\beta, \tau^m \beta)$$

$$=[\tau^m \beta, \beta] - [\beta, \tau^m \beta]$$

$$=[\tau^m \beta, \beta] - [\tau^m \beta, \tau \beta] = -1$$

It implies by Proposition (3.5) that $[\tau^m \beta, \beta] = 0$ and $[\tau^{m-1} \beta, \beta] = [\tau^m \beta, \tau \beta] = 1$. By equation (3.14), we have that $m = n_\beta + 1$. \qed

**Lemma 3.9.** Let $n > a > b$ and $r > 0$. Set

$$R(a, b) = \{r \in \mathbb{N} \mid [\tau^r \theta_a, \theta_b] = 1\}$$

Then we have $R(a, b) = \emptyset$ if $[\theta_a, \theta_b] = 0$; In the case when $[\theta_a, \theta_b] = 1$ we have $R(b, a) = [0, t]$ such that $\tau^{r+1} \theta_a + \theta_b \in R^+$. \ned

**Proof.** Let us consider the Coxeter element $w_b^{-1} C_Q w_b$, where $w_b = s_{i_b} s_{i_b+1} \cdots s_{i_a} s_{i_1} \cdots s_{i_b-1}$. It can be expressed as $s_{i_b} s_{i_b+1} \cdots s_{i_a} s_{i_1} \cdots s_{i_b-1}$. Denote by $Q^b$ the orientation induced from this Coxeter element. It is easy to see that $i_b$ is a source of $Q^b$. It implies that $S_{i_b}$ is an injective representation of $Q^b$. By the properties of Reflection functor, we have $[\tau^r \theta_a, \theta_b] = [\tau^r w \alpha_i, \alpha_{i_b}]$, where $\tau_b$ refers to the Coxeter element $C_{Q^b}$ for $Q^b$.

First, we consider the case $[\theta_a, \theta_b] = 0$. Set $u = s_{i_b} \cdots s_{i_a-1}$, we see that

$$\theta_a = w_b u \alpha_i$$

It follows by Lemma (3.6) that there is no path from $i_a$ to $i_b$. On the other hand, $\text{soc} \theta_a = i_a$ and $\text{soc} \theta_b = i_b$. Since $[\theta_a, \theta_b] = [u \alpha_i, \alpha_{i_b}] = 0$, we have $\text{supp} u \alpha_i \cap \{i_b\} = \emptyset$. (3.15) implies that $S_{i_b}$ is not a head of $M_{u \alpha_i}$ of $Q^b$. In particular, by the form of $u \alpha_i$, we see that
Let us denote $M_{\alpha_{i_a}}$, here we denote by $<_{\tau}$ the ordering of $I$ given by the Coxeter element $\tau$. It follows by Lemma (3.7) that $\tau_{b}^{k}u\alpha_{i_a} \geq_{\tau} \tau_{b}^{k}u\alpha_{i_b}$. This means $[\tau_{b}M_{\alpha_{i_a}},S_{i_b}] = 0$. It follows by (3.15) and $[\tau_{b}M_{\alpha_{i_a}},S_{i_b}] = [w_{b}\tau_{b}M_{\alpha_{i_a}},\theta_{b}]$ that
\[ [\tau^{k}\theta_{a},\theta_{b}] = 0 \text{ for all } k \in \mathbb{N} \]

Next, we suppose that there is a path from $i_a$ to $i_b$. Since $[\tau^{r}\theta_{a},\theta_{b}] = [\tau^{r}_{b}u\alpha_{i_a},\alpha_{i_b}]$, we get
\[ [\tau^{r}_{b}u\alpha_{i_a},\alpha_{i_b}] = 1 \text{ if and only if } i_b \in \text{hd} \tau_{b}^{r}\alpha_{i_a} \]

It implies by Lemma (3.7) that if $[\tau_{b}^{s+1}u\alpha_{i_a},\alpha_{i_b}] = 1$ then $[\tau_{b}^{k}u\alpha_{i_a},\alpha_{i_b}] = 1$ for all $k \in [0,s]$. It follows by the fact $[\tau_{b}^{k}u\alpha_{i_a},\alpha_{i_b}] = [\tau^{k}\theta_{a},\theta_{b}]$ that
\[ R(a,b) = [0,t] \]

Proposition (3.5) (2) yields that $\tau^{t+1}\theta_{a} + \theta_{b} \in R^{+}$.

\[ \square \]

**Lemma 3.10.** Let $\theta_{a} < \theta_{b}$. Set
\[ R(a,b) = \{ r \in \mathbb{N} | [\tau^{r}\theta_{a},\theta_{b}] = 1 \} \]

We have $R(a,b) = 0$ if $[\tau\theta_{a},\theta_{b}] = 0$. And $R(a,b) = [1,s]$ such that $\tau^{s+1}\theta_{a} + \theta_{b} \in R^{+}$.

**Proof.** Replacing $(\theta_{a}, \theta_{b})$ in the above lemma with $(\tau\theta_{a}, \theta_{b})$, we obtain this conclusion. \( \square \)

Therefore, we obtain the following proposition.

**Proposition 3.11.** Set
\[ R(a,b) = \{ r \in \mathbb{N} | \text{Hom}_{Q}(\tau^{r}M_{\theta_{a}},M_{\theta_{b}}) \neq 0 \} \]

Here $M_{\theta_{a}}$ and $M_{\theta_{b}}$ refers to the indecomposable representations for $\theta_{a}$ and $\theta_{b}$, respectively.

(1) If $a = b$, we have $R(a,a) = [0,s]$ such that $C^{s+1}_{Q}\theta_{a} + \theta_{a} \in R^{+}$.

(2) If $a > b$, we have $R(a,b) = \emptyset$ if there exists no path from $i_a$ to $i_b$. Otherwise, we have $R(a,b) = [0,s]$ such that $C^{s+1}_{Q}\theta_{a} + \theta_{b} \in R^{+}$.

(3) If $a < b$ and in the case $A_{n}$, we have $R(a,b) = \emptyset$ if there exists no path from $i_b$ to $i_a$. Otherwise, we have $R(a,b) = [1,s]$ such that $C^{s+1}_{Q}\theta_{a} + \theta_{b} \in R^{+}$.

Let us denote $n(a,b)$ by the maximal number $s$ in $R(a,b)$.

**Example 3.12.** Recall Section (2.2), let us discuss the $R(\alpha,\beta)$ for any two segments $\alpha, \beta$. We denote by $|\alpha|$ the length of $\alpha$. In other words, $|\alpha| = b - a$ if $\alpha = [a,b]$. Let $\alpha$ and $\beta$ be two roots such that $|\alpha| = |\beta| = m$, we have $[M(\alpha),M(\beta)]^{1} \neq 0$ if and only if $\alpha = \beta[-k]$ such that $1 \leq k \leq m$.

**Proof.** We write $\beta$ as $[i,i+m-1]$ and $\alpha$ as $[i-k,i+m-k-1]$ for some integer $k$. The relation $[M(\alpha),M(\beta)]^{1} \neq 0$ is equivalent to the relation $i-k+1 \leq i \leq i+m-k \leq i+m-1$. It implies that $1 \leq k \leq m$. \( \square \)
If $|\alpha| = m$ and $|\beta| = n$ such that $m > n$, let $\alpha = [a, b]$ and $\beta = [c, d]$, we have

$$R(\alpha, \beta) = \{k \in \mathbb{Z} \mid [[a + k, b + k], [c, d]] = 1\}$$

By (2.8), $[[a + k, b + k], [c, d]] = 1$ if and only if $c \leq a + k \leq d \leq b + k$. It is easy to see that $R(\alpha, \beta) = [c - a, d - a]$. On the other hand, $R(\beta, \alpha) = [d - b, d - a]$.

We remark that the relation $\Delta = [a, b] \prec \Delta' = [c, d]$ in the sense of [16] coincides with the condition $[M[a, b], M[c, d]]^1 = 1$. By Proposition (3.5), that means there are two cases for this relation: (a) $[M[a, b], M[c, d]] = 1$ and $[M[c, d], M[a + 1, b + 1]] = 1$; (b) $[a, b] \cup [c, d] = [a, d]$.

3.3. AR-orbits Konstant partitions. In this section, we will consider a special Konstant partitions: $\tau$-orbit Konstant partition. This notion is from the quantum minors $D(p, q)$ in the quantum nilpotent coordinate ring $A_q(n)$. In our cases, the Konstant partitions for these $D(p, q)$ are the sequences with the following form

$$(\tau^r \beta_p, \tau^{r-1} \beta_p, \ldots, \beta_p)$$

for some $r \in \mathbb{N}$ and $\beta_p \in R^+$. On the other hand, In [3] Buan, Marsh, Reineke, Reiten and Todorov constructed the cluster category using the $\tau$-quotient of the derived category of representations of $Q$. Therefore, It is important to consider $\tau$-orbit Konstant partitions. However, this notion is new as for as we know. We just mention some previous works related to this notion.

Example 3.13. The $\tau$-orbit Konstant partitions are of the form multisegments

$$\lambda = ([a + r, b + r], \ldots, [a + 1, b + 1], [a, b])$$

3.4. The map $r_{a,b}$. Recall the map $r_{a,b}$ in Lemma (3.3). This map plays an important role in the following section. This map is used to calculate the Ext$^1_{A}(\tilde{M}, \tilde{N})$ for modules $\tilde{M}, \tilde{N}$ over $\Lambda_Q$ (see [1, Proposition 9.1]). Here we just focus on the case of $\tau$-orbit’s Konstant partitions. In fact, it is impossible to describe the map $r_{a,b}$ in all cases, even in the case of $A_n$ with usual orientation (see [15, Section 2.4]).

Lemma 3.14. Let $a = (a_{i,j})_{i,j \in [1, r]}$ and $b = (b_{i,j})_{i,j \in [1, t]}$ be two nilpotent matrices such that $a_{i,i+1} = b_{i,j+1} = 1$ for all $i \in [1, r]$ and $j \in [1, s]$ and $a_{i,j} = b_{i,j} = 0$ if $i \geq j$. Consider the subspace $\tilde{Z}(a, b)$ of $Mat_{r \times s}$ consisting of elements $f = (f_{i,j})_{i \in [1, r], j \in [1, t]}$ satisfying the equation $af - fb = 0$. We have $\dim \tilde{Z}(a, b) = \min\{r, t\}$.

Proof. Let us define the map $r_{a,b} : Mat_{r \times s} \rightarrow Mat_{r \times s}$ by sending $f$ to $af - fb$. Let consider the paths from $k$ to $l$. There are two kinds of paths:

$$k \xrightarrow{a_{k,l}} d \xrightarrow{f_{d,l}} l$$

and

$$k \xrightarrow{f_{k,s}} s \xrightarrow{b_{s,l}} l$$

Hence, for a pair $(i, j)$, we have

$$r_{a,b}(f)_{i,j} = \sum_{d=1}^{r} a_{i,d} f_{d,j} - \sum_{s=1}^{t} f_{i,s} b_{s,j}$$
\[
= \sum_{d=i+1}^{r} a_{i,df_{d,j}} - \sum_{s=1}^{j-1} f_{i,s}b_{s,j}
\]

We remark that here \( f_{d,j} \) satisfies \( d \geq i + 1 \) and \( f_{i,s} \) satisfies \( s < j \). Let us first show \( f_{r,1} = 0 \).

Consider
\[
\begin{align*}
r_{a,b}(f)_{r-1,1} &= a_{r-1,r}f_{r,1} - \sum_{s=1}^{t} f_{i,s}b_{s,1} \\
&= a_{r-1,r}f_{r,1} = f_{r,1} = 0
\end{align*}
\]

We will show that \( f_{r,j-1} = 0 \) except \( f_{r,t} \) by induction on \( j \). Let us assume that \( f_{r,k} = 0 \) for any \( k < j - 1 \) and consider the pair \((r, j)\). The above equation becomes
\[
\begin{align*}
r_{a,b}(f)_{r,j} &= a_{r,r}f_{r,j} - \sum_{s=1}^{j-1} f_{r,s}b_{s,j} \\
&= -\sum_{s=1}^{j-1} f_{r,s}b_{s,j} - f_{r,j-1}b_{j-1,j} \\
&= -f_{r,j-1}b_{j-1,j} \text{ by induction } f_{r,s} = 0 \text{ for all } s < j - 1
\end{align*}
\]

We have \( f_{r,j-1} = 0 \), as \( b_{j-1,j} = 1 \).

Similarly, Let us consider the pair \((i, 1)\) for any \( 1 \leq i \leq r - 1 \), we will show that \( f_{i+1,1} = 0 \) except \( f_{1,1} \) by
\[
\begin{align*}
r_{a,b}(f)_{i,1} &= \sum_{d=i+1}^{r} a_{i,df_{d,1}} - 0 \\
&= \sum_{d=i+2}^{r} a_{i,df_{d,1}} + a_{i,i+1}f_{i+1,1} \\
&= f_{i+1,1} \text{ by induction } f_{d,1} = 0 \text{ for all } d > i + 1
\end{align*}
\]

we will show that \( f_{i,j} = 0 \) if \( i > j \) except \((r, t)\). We prove this by induction on \( d = i - j \).

As we discussed before, for \( d = r - 1 \) we have \( f_{r,1} = 0 \).

\[
\begin{align*}
r_{a,b}(f)_{i-1,j} &= \sum_{d=i}^{r} a_{i,df_{d,j}} - \sum_{s=1}^{j-1} f_{i-1,s}b_{s,j} \\
&= a_{i-1,i}f_{i,j} + \sum_{d=i+1}^{r} a_{i-1,df_{d,j}} - \sum_{s=1}^{j-1} f_{i,s}b_{s,j}
\end{align*}
\]

\( d - j \geq i + 1 - j > i - j \) and \( i - s \geq i - (j - 1) > i - j \). By induction we have \( r_{a,b}(f)_{i-1,j} = a_{i-1,i}f_{i,j} = 0 \). It implies that \( f_{i,j} = 0 \) for \( i > j \). Therefore, we get that
\[
\begin{align*}
r_{a,b}(f)_{i,j} &= \sum_{d=i+1}^{j} a_{i,df_{d,j}} - \sum_{s=i}^{j-1} f_{i,s}b_{s,j}
\end{align*}
\]
Next, we will show that $f_{i,i} = f_{i,j}$ for any $i, j \in \min\{s, t\}$.

$$r_{a,b}(f)_{i,j+1} = a_{i,i+1} f_{i+1,i+1} - f_{i,i} b_{i,i+1} = f_{i+1,i+1} - f_{i,i} = 0$$

Let us give a tuple $(f_{1,i})_{i\in[1,t]}$. We will show that this tuple gives rise to a unique solution of the equation $r_{a,b}(f) = 0$. As above, we see $f_{i,i} = f_{1,1}$ for all $i \leq \min\{r, t\}$. We will show that $f_{i,i+k}$ is determined by $f_{1,i}$ for $i \leq k \leq t - 1$ by induction.

$$r_{a,b}(f)_{i,i+k+1} = \sum_{d=i+1}^{i+k+1} a_{i,d} f_{d,i+k+1} - \sum_{s=i}^{i+k} f_{i,s} b_{s,i+k+1}$$

$$= \sum_{d>i+1}^{i+k+1} a_{i,d} f_{d,i+k+1} - \sum_{s=i}^{i+k-1} f_{i,s} b_{s,i+k+1} + (a_{i,i+1} f_{i+1,i+k+1} - f_{i,i+k} b_{i+k,i+k+1})$$

Since $i + k + 1 - d < k$ and $s - i < k$, $f_{d,i+k+1}, f_{i,s}$ is given by the tuple $(f_{1,j})$ for $j < k$. Thus, we obtain $f_{i+1,i+1+k} = f_{i+k} + b_{i}$ where $b_{i} \in \langle f_{1,1}, \cdots, f_{1,k-1} \rangle$. It implies that $f_{i,i+k} = f_{1,k} + c_{i}$ where $c_{i} \in \langle f_{1,1}, \cdots, f_{1,k-1} \rangle$.

If $r = t$, we have that $(f_{1,i})_{i\in[1,t]}$ is a basis of the equation $r_{a,b}(f) = 0$. If $r < t$, we have $f_{r,k} = 0$ for $k \in [r, t - 1]$. It leads to $f_{1,1} = 0$ and $f_{1,j} = 0$ for $j \in [1, t - r]$. Thus, $(f_{1,i})_{i\in[r-t+1,t]}$ is a basis of the equation $r_{a,b}(f) = 0$. If $r > t$, $(f_{1,i})_{i\in[1,t]}$ is a basis of the equation $r_{a,b}(f) = 0$. In a word, dim Ker $r_{a,b} = \min\{r, t\}$. 

**Theorem 3.15.** Let $\lambda = (\lambda_{1}, \lambda_{2}, \cdots, \lambda_{r}) \in \text{KP}(\alpha)$ and $\kappa = (\kappa_{1}, \kappa_{2}, \cdots, \kappa_{t}) \in \text{KP}(\beta)$ be two $\tau$-orbit Konstant partitions. Let $a \in \text{Hom}_{Q}(M_{\lambda}, \tau M_{\lambda})$ and $b \in \text{Hom}_{Q}(M_{\kappa}, \tau M_{\kappa})$ such that $a_{i,j} = b_{k,l} = 1$ if $[\lambda_{i}, \tau \lambda_{j}] = [\kappa_{k}, \tau \kappa_{l}] = 1$. Set $\{k \mid [\lambda_{1}, \kappa_{k}] = 1\} = [p, q]$. Let us consider the map

$$r_{a,b} : \text{Hom}_{Q}(M_{\lambda}, M_{\kappa}) \to \text{Hom}_{Q}(M_{\lambda}, \tau M_{\kappa})$$

We have

$$\dim \text{Ker} r_{a,b} = \begin{cases} 2[1, t] \cap [p, q] & \text{if } r \geq t \\ 2[t-r+1, t] \cap [p, q] & \text{if } r < t \end{cases}$$

In particular, $r_{a,b}$ is surjective if and only if

$$[M_{\lambda}, M_{\kappa}] - [M_{\lambda}, \tau M_{\kappa}] = \begin{cases} 2[1, r] \cap [p, q] & \text{if } r \geq t \\ 2[t-r+1, t] \cap [p, q] & \text{if } r < t \end{cases}$$

**Proof.** In the case $r \geq t$, from the above Lemma we have that $(f_{1,k})_{k\in[1,t]}$ is a basis for the equation $r_{a,b} = 0$. On the other hand, $f_{1,k} \neq 0$ if and only if $k \in [p, q]$. Hence, we obtain $\dim \text{Ker} r_{a,b} = 2[1, t] \cap [p, q]$. Similarly, in the case $r < t$, we have that $(f_{1,k})_{k\in[t-r+1,t]}$ is a basis for the equation $r_{a,b} = 0$, then $\dim \text{Ker} r_{a,b} = 2[t-r+1, t] \cap [p, q]$. Let $i \in [1, r]$ and set

$$\Gamma_{i} = \{j \in [1, t] \mid [\lambda_{i}, \kappa_{j}] = 1\}$$

and

$$\tau \Gamma_{i} = \{j \in [1, t] \mid [\lambda_{i}, \tau \kappa_{j}] = [\lambda_{i}, \kappa_{j-1}] = 1\}$$
It is easy to see that \( j \in \Gamma_i \) the \( j + 1 \in \Gamma_i \) for any \( j \neq t \). If \( t \in \Gamma_i \),
\[
\square
\]

For simplicity, we denote by \( l_u[a, b] \) the number of the set \([0, u] \cap [a, b]\) and by \( l_u^c[a, b] \) the number of the complement of the set \([0, u] \cap [a, b]\) in \([0, u]\). we always write \( l_u[a, b] \) for the set or the number of this set if there is no confusion.

**Theorem 3.16.** Let \( \lambda = (\tau^u \beta_a, \ldots, \tau \beta_a, \beta_a) \) and \( \kappa = (\tau^v \beta_b, \ldots, \tau \beta_b, \beta_b) \), and put
\[
(3.22) \quad R(a, b) = [s, t] \quad \text{and} \quad d_{\lambda, \kappa} = u - v - 1
\]

we have
\[
[M_\lambda, M_\kappa] - [M_\lambda, \tau M_\kappa] = l_u[u - t, u - s] \cap l_u^c[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] - l_u^c[u - t, u - s] \cap l_u[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s]
\]

In particular, \( r_{a, b} \) is surjective if and only if
(1) if \( u \geq v \)
\[
l_u[s - d_{\lambda, \kappa} - 1, t - d_{\lambda, \kappa} - 1] = l_u[u - t, u - s] \cap l_u^c[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] - l_u^c[u - t, u - s] \cap l_u[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s]
\]
(2) if \( u < v \)
\[
l_u[s - d_{\lambda, \kappa} - 1, t - d_{\lambda, \kappa} - 1] \cap [v - u, v] = l_u[u - t, u - s] \cap l_u^c[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] - l_u^c[u - t, u - s] \cap l_u[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s]
\]

Proof. For any \( i \in [0, u] \), let us set
\[
\Gamma_i = \{ j \in [0, v] \mid [\tau^{u - i} \beta_a, \tau^{v - j} \beta_b] = 1 \}
\]
and
\[
\tau \Gamma_i = \{ j \in [0, v] \mid [\tau^{u - i} \beta_a, \tau^{v - j + 1} \beta_b] = 1 \}
\]
It is easy to see that we have \( j + 1 \in \tau \Gamma_i \) if \( j \in \Gamma_i \) and \( j \neq v \). On the other hand, for any \( k \in \tau \Gamma_i \) then \( k - 1 \in \Gamma_i \) for any \( k \neq 0 \). Since \( v \in \Gamma_i \) is equivalent to the condition \( u - i \in [s, t] \).
In other words,
\[
(3.23) \quad v \in \Gamma_i \iff i \in l_u[u - t, u - s]
\]
Meanwhile, \( 0 \in \tau \Gamma_i \) is equivalent to
\[
(3.24) \quad 0 \in \tau \Gamma_i \iff i \in l_u[u - v - 1 - t, u - v - 1 - s]
\]
There are four cases:
(1) \( v \in \Gamma_i \) and \( 0 \in \tau \Gamma_i \); then \( \Gamma_i = \tau \Gamma_i = [0, v] \).
(2) \( v \in \Gamma_i \) and \( 0 \notin \tau \Gamma_i \); then \( \Gamma_i = [a, v] \) and \( \tau \Gamma_i = [a + 1, v] \).
(3) \( v \notin \Gamma_i \) and \( 0 \in \tau \Gamma_i \); then \( \Gamma_i = [0, b] \) and \( \tau \Gamma_i = [0, b + 1] \).
(4) \( v \notin \Gamma_i \) and \( 0 \notin \tau \Gamma_i \); then \( \Gamma_i = [a, b] \) and \( \tau \Gamma_i = [a + 1, b + 1] \).

Therefore, we have
\[
[M_\lambda, M_\kappa] - [M_\lambda, \tau M_\kappa]
\]
\[
= \# \{ i \in [0, u] \mid v \in \Gamma_i \text{ and } 0 \notin \tau \Gamma_i \} - \# \{ i \in [0, u] \mid v \notin \Gamma_i \text{ and } 0 \notin \tau \Gamma_i \}
\]
\[
= l_u[u - t, u - s] \cap l_u^c[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] - l_u^c[u - t, u - s] \cap l_u[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s]
\]
Let us consider the segment \([p, q]\) in Theorem (3.15).

\[
[p, q] = \{ j \in [0, v] \mid [\tau^a \beta_a, \tau^v \beta_b] = 1 \}
\]

Since

\[
[\tau^a \beta_a, \tau^v \beta_b] = [\tau^{a-v} \beta_a, \beta_b]
\]

This means \(j \in [p, q]\) then \(u - v + j \in [s, t]\). Hence, we obtain

\[
[p, q] = [s + v - u, t + v - u] \cap [0, v] = l_v[s - d_{\lambda, \kappa} - 1, t - d_{\lambda, \kappa} - 1]
\]

Therefore, \(r_{a, b}\) is surjective if and only if

(1) if \(u \geq v\)

\[
l_v[s - d_{\lambda, \kappa} - 1, t - d_{\lambda, \kappa} - 1]
= l_u[u - t, u - s] \cap l_u^e[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] - l_u^e[u - t, u - s] \cap l_u[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s]
\]

(2) if \(u < v\)

\[
l_v[s - d_{\lambda, \kappa} - 1, t - d_{\lambda, \kappa} - 1] \cap [v - u, v]
= l_u[u - t, u - s] \cap l_u^e[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] - l_u^e[u - t, u - s] \cap l_u[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s]
\]

\[
4. \textbf{Representation theory of preprojective algebras}
\]

In this section, we briefly recall some basic notions of preprojective algebras. The goal of this section is to interpret the properties of modules of preprojective algebras in terms of the results we obtained before. Thus, the conclusions in this section may be well known in other papers. We just try to give a new viewpoint to consider these problems.

Let us consider a quiver \(Q\) and its opposite quiver \(Q^{\text{op}}\). One defines an algebra \(\Lambda_Q\) as a quotient algebra of \(k(Q \cup Q^{\text{op}})\) by the ideal \(I\) which is generated by

\[
\sum_{s(h)=i,t(h')=i} \bar{h}h - h'\bar{h}'
\]

for any \(i \in I\). We simply denote by \(\Lambda\)

The modules \(M\) of \(\Lambda\) are of the form \((V, x, a)\) where \(x_h : V_{s(h)} \rightarrow V_{t(h)}\) and \(a_h : V_{t(h)} \rightarrow V_{s(h)}\) subject to the relation

\[
\sum_{s(h)=i,t(h')=i} a_{j} x_h - x_{h'} a_{h'} = 0 \text{ for any } i \in I.
\]

For any two modules \(\widetilde{M}, \widetilde{N}\) of \(\Lambda\), one has

\[
\text{D Ext}^1_{\Lambda}(\widetilde{M}, \widetilde{N}) = \text{Ext}^1_{\Lambda}(\widetilde{N}, \widetilde{M})
\]

\[
\dim \text{Ext}^1_{\Lambda}(\widetilde{M}, \widetilde{N}) = \dim \text{Hom}_{\Lambda}(\widetilde{M}, \widetilde{N}) + \dim \text{Hom}_{\Lambda}(\widetilde{N}, \widetilde{M}) - (\dim \widetilde{M} \cdot \dim \widetilde{N})
\]

For a module \(\widetilde{M}\), we denote by \(\text{soc}_{i_k}(\widetilde{M})\) the direct summands of \(\text{soc}(\widetilde{M})\) consisting of \(S_{i_k}\), where \(S_{i_k}\) is the simple module for the vertex \(i_k\). Denote by \(\widetilde{I}_{i_k}\) the projective (injective) module of \(\Lambda\) for the vertex \(i_k\).
4.1. Cluster category of preprojective algebras. In this section, we briefly review the cluster category of preprojective algebras (see [9] and [10]). Give an \( w \in W \) as in section (2.1), and fix a reduced expression \( w = s_{i_r} \cdots s_{i_2} s_{i_1} \) and \( i = [i_1 i_2 \cdots i_r] \). Denote by \( R(w) \) the set of roots associated with it. For any \( \beta_k \in R(w) \), we define a module \( \bar{V}_k \) associated with it by
\[
0 = M_0 \subset M_1 \subset \cdots \subset V_k \subset I_k
\]
such that \( \text{soc}_{\bar{V}_{k-p+1}}(I_{i_k}/M_{p-1}) = M_p/M_{p-1} \). Set
\[
\bar{V}_i = \bigoplus_{k=1}^{r} \bar{V}_k
\]
Define \( C_1 \) (or \( C_w \)) as the full subcategory of \( \Lambda - \text{mod} \) consisting of objects as quotient modules of some direct sum of a finite number of \( \bar{V}_i \). For \( j \in I \), let \( k_j := \max\{ k \in [1, r] | i_k = j \} \). Define \( I_j = \bar{V}_{k_j} \) and set \( I_i = \bigoplus_{j \in [1, r]} I_j \), which is an injective module in \( C_w \). Define
\[
\bar{V}[k, l] = \bar{V}_l / \bar{V}_k
\]
Next we will recall the notion of rigid modules of \( \Lambda \). We call the module \( T \) as a rigid module if \( \text{Ext}^1_{\Lambda}(T, T) = 0 \). One denotes by \( A_T \) its endomorphism ring \( \text{End}_\Lambda(T) \) and denote by \( \Gamma_T \) the Gabriel’s quiver of \( A_T \). More precisely, the set of vertices of \( \Gamma_T \) is \( [1, r] \), where \( r \) is the number of indecomposable direct summands of \( T \); the number of arrows between \( i \) and \( j \) is \( \dim \text{Ext}_{A_T}(S_i, S_j) \). One defines the antisymmetric matrix
\[
B_T = (b_{i,j})_{i \in [1, r], j \in [1, n-r]}
\]
by
\[
b_{i,j} = \sharp \{ \text{arrows } i \to j \text{ in } \Gamma_T \} - \sharp \{ \text{arrows } j \to i \text{ in } \Gamma_T \}
\]

**Definition 4.1.** Let \( T \) be a rigid module in \( C_w \). We call \( T \) a cluster tilting module if \( \text{Ext}^1_{\Lambda}(T, X) = 0 \) implies \( X \in \text{add}(T) \).

We give some properties of the category \( C_w \) as follows.

**Proposition 4.2.** [10] Let \( T \) be a basic cluster tilting module in \( C_w \), one has
1. \( \bar{V}_i \) is a cluster tilting object;
2. \( T \) has \( r \) pairwise non-isomorphic indecomposable direct summands;
3. (Mutations) There exists a unique indecomposable module \( T^*_k \cong T_k \) such that \( (T/T_k) \oplus T^*_k \) is \( C_w \)-maximal rigid. We call \( (T/T_k) \oplus T^*_k \) the mutation of \( T \) in direction \( k \), and denote it by \( \mu_k(T) \). We have \( \dim \text{Ext}^1_{\Lambda}(T_k, T^*_k) = 1 \). Let \( 0 \to T_k \to T_k' \to T_k^* \to 0 \) and \( 0 \to T_k' \to T_k'' \to T_k \to 0 \) be non-split short exact sequences. Then
\[
T_k' \cong \bigoplus_{b_{ik} < 0} T_j^{-b_{ik}}, \quad T_k'' \cong \bigoplus_{b_{ik} > 0} T_j^{b_{ik}}
\]

4.2. Ringel’s interpretation of \( C_w \). In what follows, we assume that \( w = w_0 \) and \( i \) is \( (2.4) \). In the Dynkin case, it is well known that
\[
C_{w_0} \cong \Lambda_Q - \text{mod}
\]
In [21] Ringel introduced a category \( D_Q \) as follows: its objects consist of a pair \((M, a)\) where \( M \) is a representation of \( Q \) and \( a \in \text{Hom}_Q(M, \tau M) \); the morphisms from \((M, a)\) to \((N, b)\) are the morphisms \( f \) in \( \text{Hom}_Q(M, N) \) such that \( b \tau = \tau(f) a \). By Lemma (3.3), we have \( \text{Hom}_{D_Q}((M, a), (N, b)) = \ker r_{a,b} = \text{Hom}_Q(M, N) \cap \text{Hom}_{Q^{op}}(a, b) \). There is a canonical functor \( \Psi : D_Q \to kQ - \text{mod} \) by sending \((M, a)\) to \( M \).
**Definition 4.3.** Let $M$ be a representation of $Q$. We define $M_{\beta}^r = \oplus_{k \in \mathbb{N}} \tau^k M_{\beta}$ for an indecomposable representation $M_{\beta}$ of $Q$. Similarly, we define $M_{\beta}^l = \oplus_{k \in \mathbb{N}} \tau^{-k} M_{\beta}$.

For any $k \in [1, m]$, we define $V_k = M_{\beta_k}^r$. Namely,
\[ V_k = \bigoplus_{i \in \tau^k(k), l \leq k} M_{\beta_i} \]
If $k^+ = \emptyset$, we have $V_k = \bigoplus_{i \in \tau^k(k)} M_{\beta_i}$. We define
\[ V[k, l] = \bigoplus_{\beta_j \in \beta[k, l]} M_{\beta_j} \text{ for } i_k = i_l \]
and set
\[ V_i = \bigoplus_{k \in [1, m]} V_k \]

Next, let us define the following category:

**Definition 4.4.** We define $\mathcal{D}_w$ by the category consisting of representations $M$ of $Q$ such that
\[ 0 \to \tau^l T \to T \to M \to 0 \]
where $T', T \in \text{add}(V_i)$. It is easy to see $\mathcal{D}_w = \text{Rep-Q}$.

Following [21], we have the following Theorem:

**Theorem 4.5.** There is a functor $\Psi_w : \mathcal{C}_w \to \mathcal{D}_w$ by sending $\widetilde{M} = (V, x, a)$ to $M = (V, x)$. Moreover, the following conditions hold:

1. $\Psi_w(V[k, l]) = V[k, l]$, and $\widetilde{V}[k, l] = (V[k, l], \eta)$ for $\eta = V[k, l]^*$ (see Definition (2.7));
2. For each rigid module $T$ in $\mathcal{C}_w$, we have $\overline{T} = (T, T^*)$ where $T \in \mathcal{D}_w$;
3. $\Psi_w$ is an exact functor.

*Proof.* Following [21, Theorem A].

4.3. **AR Quivers.** In this section, we will relate the cluster tilting $\overline{V}_i$ with the Auslander-Reiten quiver of $Q$. Let $I_i$ be indecomposable injective representations of $Q$ with dimension vector $\gamma_i$ for $i \in I$. The set of indecomposable representations equals $\{\tau^k I_i \mid C^k \gamma_i \in R^+\}$. The AR-sequence of indecomposable representations gives rise to a quiver structure $\Gamma_Q$ of $R^+$. In other words, $\tau^k I_i \to \tau^l I_j$ if there exists an irreducible morphism $f : \tau^k I_i \to \tau^l I_j$. The Auslander-Reiten sequences are given by
\[ 0 \to \tau^{k+1} I_i \to \bigoplus_{(j, l) \sim (i, k)} \tau^l I_j \to \tau^k I_i \to 0 \]
where $(j, l) \sim (i, k)$ refers to $\tau^l I_j$ such that there exist arrows from $\tau^l I_j$ to $\tau^k I_i$.

Recall that the section (2.1), a reduced expression $i = [i_1 i_2 \cdots i_m]$ of $w_0$, which is adapted to $Q$, gives rise to an ordering on $R^+$ by equation (2.2). In ([20]) Oh defines a quiver structure $\Gamma_i$ on $R^+$ as follows.

$\beta_k \to \beta_l$ if $k > l$ such that $i_k \sim i_l$ and there is no $k > j > l$ such that $i_j \in \{i_k, i_l\}$
By [2], we have \( Y_1 = \Gamma_Q \).

**Lemma 4.6.** Let \( \beta_k \in R^+ \) such that \( M_{\beta_k} \) is not a projective representation, we have \( \tau M_{\beta_k} = M_{\tau \beta_k} \).

*Proof.* Since the roots of projective representations are those \( i_k \) satisfying \( k^+ = \emptyset \). Considering \( i (2.4) \) we have that \( [i_{k+1} \cdots i_{k+1}] \) contains \( n - 1 \) different vertices except \( i_k \). Therefore, it implies that all arrows \( \beta_l \to \beta_k \) in \( Y_1 \) satisfy \( i_l \in [i_{k+1} \cdots i_{k+1}] \). On the other hands, we have all arrows \( \beta_k \to \beta_l \) from \( \beta_k \) in \( Y_1 \) satisfy \( i_l \in [i_{k+1} \cdots i_{k+1}] \). Since \( Y_1 = \Gamma_Q \), we obtain that \( M_{\beta_k} = \tau M_{\beta_k} = M_{\tau \beta_k} \). \( \square \)

We define a tuple of Konstant partitions

\[ \beta(i) = (\beta_{[1,1]}, \beta_{[2,2]}, \ldots, \beta_{[m,m]}) \]

We define a quiver \( Q_1 \) by for any \( k > l \) we have \( \beta_{[k,k]} \to \beta_{[l,l]} \) if and only if there is an arrow \( \beta_k \to \beta_l \) in the AR quiver \( \Gamma_Q \) or \( \beta_{[k,k]} \to \beta_{[l,l+1]} \).

Following \( [10, \text{section 9.4}] \) or \([5]\), we define a quiver \( \Gamma_1 \) as follows. The vertex set of \( \Gamma_1 \) is equal to \([1, m]\). For \( 1 \leq s < t \leq m \) such that \( i_s \neq i_t \), there is an arrow \( s \to t \) if \( i_s \sim i_t \) and \( s < t < s^+ \leq t^+ \). These are called ordinary arrows. Furthermore, for each \( s \in [1, m] \), there is an arrow \( s \to s^- \) if \( s^+ > 0 \). These are horizontal arrows.

**Proposition 4.7 (8, Theorem 1).** We have \( Q_1^{op} = \Gamma_1 \).

*Proof.* In [8], Geiß, Lecerc and Schröer showed this proposition. Here we will give another proof. Note that \( Q_1 \) is a quiver obtained from \( \Gamma_Q \) by adding arrows \( \beta_k \to \beta_{k+1} \). Recall the section (4.3), we have that \( \Gamma_1^{op} \) is a quiver obtained from \( Y_1 \) by adding arrows \( \beta_s \to \beta_{s+1} \). Since the vertices in \((k, k^+)\) are \( n - 1 \) different vertices except \( i_k \). In [2] we have \( Y_1 = \Gamma_1 \). It implies our conclusion. \( \square \)

**Remark 4.8.** The reason we here mention this quiver \( \Gamma_1 \) is that this data will give rise to a quantum cluster structure on \( A_q(n(w)) \) for \( w \in W \), (see Section (5.4) or [10] for more details).

### 4.4. Relations with representations of quivers.

Fix a \( I \)-graded vector space \( V \) such that \( \text{dim} V = \alpha \in Q^+ \), consider \( E_\alpha^* \) as the space \( \bigoplus_{h \in \Omega} \text{Hom}_k(V_{\ell(h)}, V_{\ell(h)}) \). Denote by \( x_h \) the elements in \( \text{Hom}_k(V_{\ell(h)}, V_{\ell(h)}) \). The space \( E_\alpha \oplus E_\alpha^* \) admits a symmetric structure \( \omega(-,-) \) given by

\[ \omega(z, w) = \text{Tr}_V \sum_{h \in \Omega} z_h w_h - w_h z_h \]

The \( G_\alpha \) action on \( E_\alpha \oplus E_\alpha^* \) induced by its action on \( E_\alpha \). It gives rise to the following moment map

\[ \mu : E_\alpha \oplus E_\alpha^* \to gl_{\alpha} \]

\[ (x, a) \mapsto \sum_{i \in I} \sum_{s(h) = i; \ell(h') = i} a_{h} x_h - x_{h'} a_{h'} \] (6.4)

Where \( gl_{\alpha} = \bigoplus_{i \in I} \text{Hom}_k(V_i, V_i) \). Recall formula ( (3.5)), we have \( \mu(x, -) = p^x_{x,x} \) and \( \mu(-, a) = p^a_{a,a} \). We define the nilpotent variety as follows.
**Definition 4.9.** We define the nilpotent variety \( \Lambda_\alpha \) by
\[
\Lambda_\alpha = \{(x, a) \in \mu^{-1}(0) \mid a \text{ is nilpotent}\}
\]
In acyclic cases, any element \( a \in E^*_\alpha \) is nilpotent. It follows that \( \Lambda_\alpha = \mu^{-1}(0) \).

We give another proof of main result in [21]

**Lemma 4.10.** [21] The fiber of the canonical map \( \pi : \Lambda_\alpha \to E_\alpha \) at \( M = (V, x) \in E_\alpha \) is equal to \( \text{Hom}_Q(M, \tau M) \). Similarly, the fiber of the canonical map \( \pi^* : \Lambda_\alpha \to E^*_\alpha \) at \( M' = (V, a) \in E_\alpha \) is equal to \( \text{Hom}_{Q^{op}}(M', \tau M') \).

**Proof.** By equation (3.5), we have \( \pi^{-1}(M) = \{a \in E_\alpha \mid \mu(x, a) = 0\} \) is equal to \( \text{Hom}_Q(M, \tau M) \).

For the case of \( \pi^* \), we get our conclusion similarly. \( \square \)

Let us consider the following sequence.

\[
0 \to \text{Hom}_\Lambda(\tilde{M}, \tilde{M}) \to \mathfrak{gl}_\alpha \overset{(p_{x, x}, p_{a, a})}{\longrightarrow} E_\alpha \oplus E^*_\alpha \overset{(p_{a, a}, p_{x, x})}{\longrightarrow} \mathfrak{gl}_\alpha \to D \text{Hom}_\Lambda(\tilde{M}, \tilde{M}) \to 0
\]

We denote \( H^0_{\tilde{M}}(E_\alpha \oplus E^*_\alpha) = \ker(p_{a, a}^{*}, p_{x, x}^{*})/\text{Im}(p_{x, x}, p_{a, a}). \) Note that other places in the sequence are exact.

**Theorem 4.11.** We have that
\[
\text{Ext}^1_\Lambda(\tilde{M}, \tilde{M}) = H^0_M(E_\alpha \oplus E^*_\alpha)
\]

**Proof.** First we show that \( H^0(E_\alpha \oplus E^*_\alpha) \subset \text{Ext}^1_\Lambda(\tilde{M}, \tilde{M}) \). Since any \((u, v) \in E_\alpha \oplus E^*_\alpha\) corresponds to a representation
\[
\bar{E}_{u,v} = \left\{ \begin{pmatrix} x & u \\ 0 & x \\ a & v \\ 0 & a \end{pmatrix} \right\}
\]
It is easy to see that \( \bar{E}_{u,v} = \bar{E}_{u',v'} \) if and only if \((u - u', v - v') \in \text{Im}(p_{x, x}, p_{a, a}). \) On the other hand, we have
\[
\dim H^0(E_\alpha \oplus E^*_\alpha) = 2 \dim \text{Hom}_\Lambda(\tilde{M}, \tilde{M}) - (\alpha, \alpha) = \dim \text{Ext}_\Lambda(\tilde{M}, \tilde{M})
\]
This implies that \( \text{Ext}^1_\Lambda(\tilde{M}, \tilde{M}) = H^0_M(E_\alpha \oplus E^*_\alpha). \) \( \square \)

**Corollary 4.12.** The representation \( \tilde{M} = (V, x, a) \) is rigid if and only if the sequence (4.7) is exact.

**Proof.** It is easy to see this conclusion. \( \square \)

**Proposition 4.13.** Let \( M = (V, x) \) be a representation of \( Q \). It admits an open orbit in \( \text{Hom}_Q(M, \tau M) \) if and only if its dual \( M^* = (V, x) \) admits an open orbit in \( \text{Hom}_{Q^{op}}(M^*, \tau M^*) \).

**Proof.** Since \( M \) admits an open orbit in \( \text{Hom}_Q(M, \tau M) \) if and only if \( \tilde{M} = (V, x, a) \) is a rigid module. Replacing \( M' \) by \( M^* \) in above diagram, we obtain the above conclusion. \( \square \)

**Proposition 4.14** ([1, Corollary 9.3]). For any two representations \( \tilde{M} = (V, x, a) \) and \( \tilde{N} = (W, y, b) \) of \( \Lambda_Q \), we have
\[
0 \to \text{Coker} r_{a,b} \to \text{Ext}^1_{\Lambda_Q}(\tilde{M}, \tilde{N}) \to \text{Ker} r_{b,a}^{*} \to 0
\]
4.5. Calculate the extension groups of determinantal modules. In what follows, we assume that the word \(i\) as (2.4). Following Section 4.1 and Section 4.2, we will calculate the extension groups of the determinantal modules \(\tilde{\mathcal{V}}[a, b]\) and \(\tilde{\mathcal{V}}[c, d]\). In Section 4.2, we see that the Konstant partitions corresponding to determinantal modules are of the form \(\tau\)-orbit Konstant partitions. Here we will give some well known results, (see [9]). For a \(\tau\)-orbit Konstant partition \(\lambda = (\tau^u \beta, \cdots, \beta)\), let us recall the notion of the dual module \(\tilde{M}_\lambda\) for \(\tilde{\lambda}\), it is an element of \(a \in \text{Hom}_Q(M_\lambda, \tau M_\lambda)\) such that

\[
\begin{align*}
0 & \to \text{Hom}_Q(M, \tilde{N}) \to \text{Hom}_Q(M, \tau N) \\
& \to \text{Ext}^1_{\text{A}_Q}(M, \tilde{N}) \to \text{Ext}^1_{\text{A}_Q}(M, \tau N) \\
& \to \text{Ext}^2_{\text{A}_Q}(M, \tilde{N}) \to \text{Ext}^2_{\text{A}_Q}(M, \tau N) \\
& \to \cdots \\
& \to \text{Ext}^r_{\text{A}_Q}(M, \tilde{N}) \to \text{Ext}^r_{\text{A}_Q}(M, \tau N) \\
& \to 0
\end{align*}
\]

(4.8)

where \(e_{a, b}(\eta) = \{ M \oplus N, \begin{pmatrix} a \\ \eta \\ b \end{pmatrix} \} \).

\[\square\]

In particular, we have \(a_{i, i+1} = 1\) for any \(i \in [0, u - 1]\). Applying Theorem (3.16), we see the following corollaries.

**Corollary 4.15** ([9]). For any \(\tau\)-orbit Konstant partition \(\lambda\) we have that \(\tilde{\lambda}\) is a rigid module over \(\text{A}_Q\). In other words, \(\text{Ext}^1_{\text{A}_Q}(\tilde{\lambda}, \tilde{\lambda}) = 0\)

\[\text{Proof.}\] Let us write \(\lambda\) for \((\tau^u \beta, \cdots, \beta)\). Following Theorem (3.16), we have

\[u = v, [s, t] = [0, n_\beta], \text{ and } l_u(d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s) = l_u[-1 - n_\beta, -1] = 0.\]

This implies \(l_u[1 - n_\beta, 1] = [0, u]\). It is enough to show that

\[
(4.10) \quad \mathfrak{z}[0, n_\beta] \cap [0, u] = \mathfrak{z}[u - n_\beta, u] \cap [0, u]
\]

it is straight to see this equation. \[\square\]

**Corollary 4.16** ([9]). Under the above assumption, for a \(\tau\)-orbit Konstant partition \(\lambda\) we have that

\[\dim \text{Ext}^1_{\text{A}_Q}(\tilde{\lambda}, \tilde{\tau\lambda}) = 1\]

Here \(\tau\lambda = (\tau^{u+1} \beta, \tau^u \beta, \cdots, \beta)\).

\[\text{Proof.}\] Let us write \(\lambda\) for \((\tau^u \beta, \cdots, \beta)\) and \(\tau\lambda = (\tau^{u+1} \beta, \cdots, \beta)\). Following Theorem (3.16), we have \(u = v, [s, t] = [1, n_\beta + 1]\), and \(d_{\lambda, \tau\lambda} = -1\). This means \(l_u(d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s) = l_u[-n_\beta - 2, -2] = 0\). Thus \(l_u[1 - n_\beta, 1] = [0, u]\) and

\[
\begin{align*}
l_u[u - t, u - s] & \cap l_u[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] = l_u[u - t, u - s] \cap l_u[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] \\
& = l_u[u - n_\beta - 1, u - 1] \cap [0, u]
\end{align*}
\]

If \(u - n_\beta - 1 \geq 0\), this term is equal to

\[u - n_\beta - 1, u - 1 = n_\beta = l_u[1, n_\beta + 1] = \ker r_{\lambda, \tau\lambda}\]

It implies that \(r_{\lambda, \tau\lambda}\) is surjective.
If \( u - n_\beta - 1 < 0 \), this term is equal to
\[
[0, u - 1] = u = l_u[1, n_\beta + 1] = [1, u]
\]
This follows that \( r_{\lambda, \tau \lambda} \) is surjective

Let us consider \( r_{M_\tau \lambda, M_\lambda} \). It is easy to see that \([s, t] = [-1, n_\beta - 1] \) and
\[
l_u[d_{\tau \lambda, \lambda} - t, d_{\tau \lambda, \lambda} - s] = l_u[-n_\beta, 0] = [0]
\]
It implies that \( l_u^n[-n_\beta, 0] = [1, u] \) and
\[
l_u[u - t, u - s] = l_u^n[d_{\lambda, \lambda} - t, d_{\lambda, \lambda} - s] - l_u^n[u - t, u - s] \cap l_u[u[d_{\lambda, \lambda} - t, d_{\lambda, \lambda} - s] = l_u[u - n_\beta + 1, u + 1] \cap [0]
\]
(4.11)

If \( u \geq n_\beta - 1 \), then \( l_u[u - n_\beta + 1, u + 1] = [u - n_\beta + 1, u] \), \( l_u^n[u - n_\beta + 1, u + 1] = [0, u - n_\beta] \), and then (4.11) is equal to
\[
\sharp[u - n_\beta + 1, u] \cap [1, u] = \begin{cases} 
\frac{u - 1}{n_\beta - 1} & u = n_\beta \\
1 & u \geq n_\beta + 1
\end{cases}
\]
On the other hand, \( l_u[s - d_{\lambda, \lambda} - 1, t - d_{\lambda, \lambda} - 1] = l_u[s, t] = l_u[-1, n_\beta - 1] = n_\beta \). This means
\[
\dim \ker r_{\tau \lambda, \lambda} = n_\beta > [M_{\tau \lambda, \lambda}, M_{\tau \lambda, \lambda}] - [M_{\tau \lambda, \lambda}, M_{\tau \lambda, \lambda}] = n_\beta - 1
\]
(4.12)

Let us consider the case \( u < n_\beta - 1 \). \( l_u[u - n_\beta + 1, u + 1] = [0, u] \), \( l_u^n[u - n_\beta + 1, u + 1] = \emptyset \), and then (4.11) is equal to
\[
\sharp[0, u] \cap [1, u] = [1, u]
\]
Thus we get
\[
\dim \ker r_{\tau \lambda, \lambda} = l_u[-1, n_\beta - 1] = [0, u] = u + 1 > u
\]
(4.13)
This implies that \( r_{\tau \lambda, \lambda} \) is not surjective.

Considering the above discussion, we see that \( r_{\lambda, \tau \lambda} \) is surjective and the dimension of cokernel of \( r_{\tau \lambda, \lambda} \) is equal to 1. It follows by Proposition (4.14) that
\[
\text{Ext}_A^1(\widetilde{M}_\lambda, \widetilde{M}_\lambda) = 1
\]
\[\square\]

Following [10, Lemma 5.2], we have

**Corollary 4.17.** Given two \( \tau \)-orbit Konstant partition \( \lambda = (\tau^u \beta, \cdots, \beta) \) and \( \kappa = (\tau^u \alpha, \cdots, \alpha) \) such that \( \beta = \tau^k \theta_a \) and \( \alpha = \tau^k \theta_b \) for some integer \( k \) and two different roots \( \theta_a, \theta_b \). We have
\[
\text{Ext}_A^1(\widetilde{M}_\lambda, \widetilde{M}_\kappa) = 0
\]

**Proof.** We assume that \( \beta > \alpha \). Let us consider the map \( r_{\lambda, \kappa} \). Since \( d_{\lambda, \kappa} = -1 \) and \([s, t] = [0, n_a, b] \), it is easy to see that \( l_u[s - d_{\lambda, \kappa} - 1, t - d_{\lambda, \kappa} - 1] = l_u[0, n_a, b] \).

Since \( l_u[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] = l_u[-1 - n_a, b, -1] = \emptyset \), we see that \( l_u^n[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] = [0, u] \).

On the other hand, \( l_u[u - t, u - s] = l_u[u - n_a, b, u] \) implies that
\[
l_u[u - t, u - s] \cap l_u^n[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s] = l_u^n[u - t, u - s] \cap l_u[d_{\lambda, \kappa} - t, d_{\lambda, \kappa} - s]
\]
It is easy to see that \([u - n_{a,b}, u] \cap [0, u] = [0, n_{a,b}] \cap [0, u]\). It follows by Theorem (3.16) that \(r_{\lambda, \kappa}\) is surjective.

Let us consider \(r_{\nu, \lambda}\), Since \(d_{\nu, \lambda} = -1\) and \([s, t] = [1, n_{b,a}]\), it is easy to see that \(l_u[s - d_{\nu, \lambda} - 1, t - d_{\nu, \lambda} - 1] = l_u[1, n_{a,b}].\) Since \(l_u[d_{\nu, \lambda} - t, d_{\nu, \lambda} - s] = l_u[-1 - n_{b,a}] = \emptyset\), we see that \(l_u^u[d_{\nu, \lambda} - t, d_{\nu, \lambda} - s] = [0, u]\). On the other hand, \(l_u[u - t, u - s] = l_u[u - n_{b,a}, u - 1] \implies\)

\[
l_u[u - t, u - s] \cap l_u^u[d_{\nu, \lambda} - t, d_{\nu, \lambda} - s] - \#l_u^u[u - t, u - s] \cap l_u[d_{\nu, \lambda} - t, d_{\nu, \lambda} - s] = l_u[u - n_{b,a}, u] \cap [1, u] = l_u^u[1, n_{b,a}] = \dim \ker r_{\nu, \lambda}.
\]

It leads to \(r_{\nu, \lambda}\) is surjective.

Therefore, we have shown that \(r_{\lambda, \kappa}\) and \(r_{\nu, \lambda}\) are both surjective in any case. By Proposition (4.14), we obtain \(\Ext^1_\lambda(\tilde{M}_\lambda, \tilde{M}_\kappa) = 0\). \(\square\)

4.6. Relation with Lapid and Minguez’s works. In this section, we will give an interpretation of some results in [13].

Following [13], we call \(\lambda = ([a_1, b_1], \cdots, [a_r, b_r])\) a (strict) ladder multisegments if

\[
[a_i, b_i] \succ [a_{i+1}, b_{i+1}] \quad \text{for all } i \in [1, r].
\]

See Example (3.12). This means

\[
[[a_{i+1}, b_{i+1}], [a_i, b_i]] = [[a_i, b_i], [a_{i+1} + 1, b_{i+1} + 1]] = 1
\]

It follows that the module \(\tilde{M}_\lambda = (M_\lambda, a)\) satisfying \(a_{i, i+1} = 1\) for any \(i \in [1, r - 1]\). Let \(\kappa = ([c_1, d_1], \cdots, [c_t, d_t])\) be another ladder multisegment and write \(\tilde{M}_\kappa = (M_\kappa, b)\) for the modules over \(\Lambda\) associated to \(\kappa\). We have

\[
[p, q] = \{k \in [1, t] \mid ([a_1, b_1], [c_k, d_k]) = 1\} = \{k \in [1, t] \mid [c_k \leq a_1 \leq d_k \leq b_1]\}
\]

It follows by Theorem (3.15) that

\[
dim \ker r_{\lambda, \kappa} = \begin{cases} \# [1, t] \cap [p, q] & \text{if } r \geq t \\ \# [t - r + 1, t] \cap [p, q] & \text{if } r < t \end{cases}
\]

Following [13, Section 6.3], let us set

\[
X_{\lambda, \kappa} = \{(i, j) \in [1, r] \times [1, t] \mid ([a_i, b_i], [c_j + 1, d_j + 1]) = 1\}
\]

\[
Y_{\lambda, \kappa} = \{(i, j) \in [1, r] \times [1, t] \mid ([a_i, b_i], [c_j, d_j]) = 1\}
\]

We remark that the notion of \(X_{\lambda, \kappa}\) and \(Y_{\lambda, \kappa}\) in [13, Section 6.3] uses the relation \([a_i, b_i] \prec [c_j, d_j]\). Therefore, \(r_{a, b}\) is surjective if and only if

\[
\#Y_{\lambda, \kappa} - \#X_{\lambda, \kappa} = \begin{cases} \# [1, r] \cap [p, q] & \text{if } r \geq t \\ \# [t - r + 1, t] \cap [p, q] & \text{if } r < t \end{cases}
\]
By Proposition (3.5), there are four cases for any pair \( (i, j) \):

\[
\begin{align*}
&[[a_i, b_j], [c_j, d_j]] = 1 \text{ and } [[a_i, b_j], [c_j + 1, d_j + 1]] = 1; \\
&[[a_i, b_j], [c_j, d_j]] = 0 \text{ and } [[a_i, b_j], [c_j + 1, d_j + 1]] = 0 \\
&[[a_i, b_j], [c_j, d_j]] = 0 \text{ and } [[a_i, b_j], [c_j + 1, d_j + 1]] = 1; \\
&[[a_i, b_j], [c_j, d_j]] = 1 \text{ and } [[a_i, b_j], [c_j + 1, d_j + 1]] = 0
\end{align*}
\]

We don’t need consider the first two cases, as they don’t change the number \( \sharp Y_{\lambda, \kappa} - \sharp X_{\lambda, \kappa} \).

Therefore, we just consider the last two case. Namely, \([a_i, b_j] \cup [c_j, d_j] = [a_i, d_j]\) and \([a_i, b_j] - [c_j, d_j] \in R\). The first relation means that \( b_i + 1 = d_j \); the second relation means \( a_i = c_j, b_i \geq d_j \) or \( a_i > c_j, b_i = d_j \).

Therefore, we obtain

\[
\sharp Y_{\lambda, \kappa} - \sharp X_{\lambda, \kappa} = \sharp \{(i, j) \mid a_i = c_j, b_i \geq d_j \text{ or } a_i > c_j, b_i = d_j\} - \sharp \{(i, j) \mid b_i + 1 = c_j\}
\]

Therefore, Theorem (3.15) implies that \( r_{a, b} \) is surjective if and only if

\[
\sharp \{(i, j) \mid a_i = c_j, b_i \geq d_j \text{ or } a_i > c_j, b_i = d_j\} - \sharp \{(i, j) \mid b_i + 1 = c_j\}
\]

(4.19)

\[
\sharp \{k \in [1, t] \mid c_k \leq a_1 \leq d_k \leq b_1\} \quad \text{if } r \geq t
\]

\[
\sharp \{k \in [t - r + 1, t] \mid c_k \leq a_1 \leq d_k \leq b_1\} \quad \text{if } r < t
\]

Since \( \text{Ext}_1^\Lambda(M_{\lambda}, \tilde{M}_\kappa) = 0 \) if and only if \( r_{a, b} \) and \( r_{b, a} \) are surjective, we give a combinatorial condition for any pair ladder multisegments.

Recall the combinatorial condition in [13, Lemma 6.21], they say \( NC(\lambda, \kappa) \) condition that there exist indices \( i \in [1, r] \) and \( j \in [1, t] \) and \( k \geq 0 \) such that

1. \((i + l, j + l) \in X_{\lambda, \kappa}\) for all \( l \in [0, k]\);  
2. \((i - 1, j), (i + k, j + k + 1) \notin Y_{\lambda, \kappa}\).

In [13, Lemma 6.21], Lapid and Minguez show that \( \text{Ext}_1^\Lambda(\tilde{M}_\lambda, \tilde{M}_\kappa) = 0 \) if and only if they satisfy \( NC(\lambda, \kappa) \) condition. But \( NC(\lambda, \kappa) \) condition is difficult to find the indices \( i, j, k \) for each ladder pair \( (\lambda, \kappa) \), our condition makes it feasible to calculate the \( \text{Ext}_1^\Lambda(\tilde{M}_\lambda, M_\kappa) = 0 \) for any ladder pair \( (\lambda, \kappa) \), although it looks like more difficult.

5. Quantum groups

In this section we will briefly recall some basic notions of quantum groups. First, for the Cartan matrix \( C_Q \) associated with a quiver \( Q \), (1), one defines a quantum group \( U_q(\mathfrak{g}) \) as an algebra over \( \mathbb{Q}(q) \) generated by \( e_i; f_i; q^h \) where \( i \in I \) and \( h \in P^\vee \) with the following condition.

(1) \( q^0 = 1; q^h q^{h'} = q^{h+h'} \); 
(2) \( q^h e_i q^{-h} = q^{(h, \alpha_i)} e_i, q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i \); 
(3) \( e_if_j - f_je_i = \delta_{ij} \frac{q^{h_i}-q^{-h_j}}{q-q^{-1}} \);
where \([n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \) \([n] = [n][n-1] \cdots [1],\) and \(e_i^{(k)} = \frac{e_i^k}{[k]}, f_i^{(k)} = \frac{f_i}{[k]}\) for all \(i \in I\) and \(k \in \mathbb{N}.\) 

\(U_q(\mathfrak{g})\) admits a \(Q\)-graded by defining 

\[U_q(\mathfrak{g})_\alpha = \left\{ x \in U_q(\mathfrak{g}) \mid q^h x q^{-h} = q^{(\alpha,h)} x \text{ for all } h \in P^+ \right\}\]

for any \(\alpha \in Q.\) It is easy to see that \(U_q(\mathfrak{g})_0\) is generated by \(q^h\) for all \(h \in P^+.\) We remark that here we use \(q\) for all \(i \in I\) rather than \(q_i,\) as in Dynkin cases \(q_i = q\) for all \(i \in I.\) We have the root decomposition as follows.

\[U_q(\mathfrak{g})^\pm = \bigoplus_{\alpha \in Q^\pm} U_q(\mathfrak{g})_\alpha\]

We call \(x\) as a homogeneous element if \(x \in U_q(\mathfrak{g})_\alpha\) for some \(\alpha \in Q\) and call \(\alpha = \text{wt}(x)\) its weight.

Denote by \(A\) the ring \(\mathbb{Z}[q, q^{-1}].\) Let \(U_q(\mathfrak{g})_A^+\) be a algebra generated by \(e_i^{(k)}\) for all \(i \in I\) and \(k \in \mathbb{N}.\) For simplicity, we denote it by \(U_q(\mathfrak{n}).\)

### 5.1. Notions in quantum groups.

Here is a list of operations on \(U_q(\mathfrak{n}):\)

1. (bar involution): \(\overline{e_i} = e_i; \overline{q} = q^{-1};\) (Anti-involution) \(e_i = e_i\) and \((xy)^* = y^* x^*.\)
2. Set the product on \(U_q(\mathfrak{n}) \otimes U_q(\mathfrak{n})\) by

\[(x_1 \otimes y_1)(x_2 \otimes y_2) = q^{-(\text{wt}(x_2),\text{wt}(y_1))} x_1 x_2 \otimes y_1 y_2\]

3. \(r : U_q(\mathfrak{n}) \to U_q(\mathfrak{n}) \otimes U_q(\mathfrak{n})\) is a algebraic map given by \(r(e_i) = 1 \otimes e_i + e_i \otimes 1.\) We call it twist coproduct.
4. (Lusztig’s bilinear form) Set \((-,-)_L : U_q(\mathfrak{n})_\alpha \times U_q(\mathfrak{n})_\alpha \to \mathbb{Q}(q)\) by

\[
\begin{align*}
(1,1)_L &= 1; (e_i; e_j)_L = \delta_{i,j} \frac{1}{1-q}; \\
(xy,z)_L &= (x \otimes y, r(z))_L; (z,xy)_L = (r(z), x \otimes y)_L.
\end{align*}
\]

Here the form on \(U_q(\mathfrak{n}) \otimes U_q(\mathfrak{n})\) is given by \((x \otimes y, x' \otimes y')_L = (x,x')_L(y,y')_L.\)
5. (Kashiwara’s bilinear form) For \(\alpha = \sum_{i \in I} n_i \alpha_i \in Q^-\) and any \(x, y \in U_q(\mathfrak{n})_\alpha,\) set

\[\prod_{i \in I} (1 - q^2)^{n_i} (x,y)_L
\]

6. (dual bar involution) Set \(\sigma : U_q(\mathfrak{n}) \to U_q(\mathfrak{n})\) such that

\[\sigma(x, y)_K = (x, \overline{y})_K
\]

For \(x \in U_q(\mathfrak{n})_\alpha\) one has \(\sigma(x) = q^{N(\alpha)}(\ast \circ -)(x)\) where \(N(\alpha) = \frac{1}{2}((\alpha, \alpha) + 2 | \alpha |,\) and \(\sigma(xy) = q^{(\text{wt}(x),\text{wt}(y))} \sigma(y) \sigma(x).\)
Canonical bases. For a $\alpha \in Q^+$ let $\mathcal{Q}_\alpha$ be the semisimple category generated by Lusztig’s sheaves $L_i$ for $i \in (I)_\alpha$. Denote by $\mathcal{P}_\alpha$ the subset of $\mathcal{Q}_\alpha$ consisting of simple perverse sheaves in $\mathcal{Q}_\alpha$. Here is a key theorem in quantum groups.

**Theorem 5.1.** ([19]; [22]) There is an isomorphism between $K_0(\mathcal{Q}_\alpha) \cong U_q(n)_{\alpha}$ for any $\alpha \in Q^+$. Under the Induction and Restriction functors on $K_0(\mathcal{Q}) \overset{\text{def}}{=} \oplus_{\alpha \in Q^+} K_0(\mathcal{Q}_\alpha)$, one obtains an algebraic isomorphism $\Phi : K_0(\mathcal{Q}) \cong U_q(n)$. We call $b \in U_q(n)_{\alpha}$ a canonical base if $b = \Phi(\text{IC})$ for some $\text{IC} \in \mathcal{P}_\alpha$. Denote by $\mathcal{B}$ the set of canonical bases.

**Remark 5.2.** In Dynkin case $\mathcal{P}_\alpha$ is exactly the set of simple perverse sheaves $\text{IC}(\lambda)$ where $\lambda$ run over all the Konstant partitions $\text{KP}(\alpha)$.

5.2. Quantum unipotent groups. Following [18] and [12], we define

**Definition 5.3.** Set

$$\mathcal{A}_q(n) = \{u \in U_q(n) \mid (u, v)_K \in \mathcal{A} \text{ for all } v \in U_q(n)\}$$

and

$$\mathcal{B}^* = \{b^* \in \mathcal{A}_q(n) \mid \exists b \in \mathcal{B} (b^*, b) = 1 \text{ and } (b^*, b') = 0 \text{ for any } b' \neq b \in \mathcal{B}\}$$

We call $\mathcal{B}^*$ the set of dual canonical bases. We remark that $\mathcal{A}_q(n)$ can be considered as a quantum coordinate ring of nilpotent subgroup $N$ with Lie algebra $n$, (see [10]).

Here is the list of properties of $\mathcal{A}_q(n)$ and $\mathcal{B}^*$.

1. There is an isomorphism as follows.

$$\Pi : U_q(n)_{\alpha} \xrightarrow{\sim} \mathcal{A}_q(n)_{\alpha}$$

$$u \mapsto (u, -)_K$$

2. $\sigma(b^*) = b^*$ for any $b^* \in \mathcal{B}^*$.

In what follows, we consider the quantum nilpotent groups, .

Give a Weyl element $w \in W$ and fix a reduced expression of $w$ as in Section (2.1), let us consider the roots $R(w)$ associated with $w$. For any $\beta \in R(w)$ one defines the root base $F(\beta) \subset U_q(n)_{\beta}$ as in [12, Section 4.3]. We remark that here we consider the positive part of $U_q(n)$ rather than the negative part as in this paper. For any Konstant partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \in \text{KP}(\alpha)_w$ we define $F(\lambda) = F(\lambda_1) \cdots F(\lambda_r)$.

**Definition 5.4.** [19, Proposition 40.2.1]; or[10] Define $U_q(n(w))$ by an algebra generated by root bases $F(\beta)$ for all $\beta \in R(w)$. The set of $F(\lambda)$ for all $\lambda \in \text{KP}(\alpha)$ form a basis of this algebra, which is called PBW basis. Define

$$\mathcal{A}_q(n(w)) = \Pi(U_q(n(w)))$$

(5.1)

Let us recall the notion of unipotent quantum minors $d_{u(\omega_i),v(\omega_i)}$ where $\omega_i$ refers to the basic weight for $i \in I$, and $u, v \in W$ (see [10, Section 6.5]). Let $i = [i_1 i_2 \cdots i_m]$ be a reduced expression word of $w \in W$. Following [10, Section 5.4], we set

$$D(b, d) \overset{\text{def}}{=} d_{s_{i_1} s_{i_2} \cdots s_{i_d}(\omega_{i_k}), s_{i_1} s_{i_2} \cdots s_{i_d}(\omega_{i_k})}$$

such that $i_d = i_d$

(5.2)

Here is a list of properties of $D(b, d)$.
(1) \(D(b, d) \in B^*\) for any \(b < d\) such that \(i_b = i_d\) (this condition is not necessary, as \(s_i(\omega_j) = \omega_j\) if \(i \neq j\)).

(2) For the word \((2.4)\), we have the Konstant partition \(\lambda(b, d)\) for \(D(b, d)\) is

\[
(\beta_d, \beta_{d-1}, \ldots, \beta_k, \ldots, \beta_b)
\]

where \(\beta_k\) run over the root of the form \(s_i s_i \cdots s_i (\alpha_{i_l})\) such that \(i_l = i_d\) and \(b \leq l \leq d\). In particular, when \(w = w_0\) and the reduced expression word is giveb by \((2.4)\), \(\lambda(b, d) = (\tau^u \beta_b = \beta_a, \ldots, \beta_b)\) for some integer \(u\), which is a \(\tau\)-orbit Konstant partition.

5.3. **Quantum cluster algebras.** In this section we will recall the notion of quantum cluster algebras (see [6] or [10, Section 8] for more details).

Let \(L = (\lambda_{i,j})\) be a skew-symmetric \(r \times r\) matrix with integer entries. The **basic quantum torus** \(A(L)\) is the \(\mathbb{Z}[q^{\pm \frac{1}{2}}]\)-algebra generated by \(X_i^{\pm}\) for \(i \in [1, r]\) with the condition

\[
X_i X_i^{-1} = 1; X_i X_j = q^{\lambda_{i,j}} X_j X_i, \quad (i, j \in [1, r])
\]

For \(a = (a_1, a_2, \ldots, a_r) \in \mathbb{Z}^r\), set

\[
X^a = q^{\frac{1}{2} \sum_{i>j} a_i a_j \lambda_{i,j}} X_1^{a_1} \cdots X_r^{a_r}
\]

Then \(\{X^a \mid a \in \mathbb{Z}^r\}\) is a \(\mathbb{Z}[q^{\pm \frac{1}{2}}]\)-base of \(A(L)\), and we have for any \(a, b \in \mathbb{Z}^r\)

\[
X^a X^b = q^{\frac{1}{2} \sum_{i>j} (a_i b_j - b_i a_j) \lambda_{i,j}} X^{a+b}
\]

Denote by \(F\) its skew field of fractions.

Fix a positive integer \(n < r\). Let \(\widetilde{B} = (b_{i,j})\) be a \(r \times (r - n)\)-matrix with integer coefficients. The submatrix \(B\) consisting of first \(r - n\) rows of \(\widetilde{B}\) is called the **principal part** of \(\widetilde{B}\). We will require \(B\) is skew-symmetric. We call \(B\) the **exchange matrix**. We say that the pair \((L, B)\) is **compatible** if we have

\[
(5.3) \quad \sum_{k=1}^r b_{kj} \lambda_{ki} = \delta_{ij} d \quad (j \in [1, r-n]; i \in [1, r])
\]

for some positive integer \(d\).

If \((L, B)\) is compatible, the datum \(T = ((X_1, X_2, \cdots, X_r), L, \widetilde{B})\) is called a **quantum seed** in \(F\). The set \((X_1, X_2, \cdots, X_r)\) is called the **cluster** of \(T\) and its elements the **cluster variables**. The cluster variables \(X_{r-n+1}, \ldots, X_r\) are called **frozen variables**, as they will not be affected by the mutation to be defined now. The elements \(X^a\) with \(a \in \mathbb{N}^r\) are called **quantum cluster monomials**.

For \(k \in [1, r-n]\) we define the mutation \(\mu_k(L, \widetilde{B})\) of a compatible pair \((L, \widetilde{B})\). Let \(E\) be the \(r \times r\)-matrix with entries

\[
(5.4) \quad e_{ij} = \begin{cases} 
\delta_{i,j} & \text{if } j \neq k \\
-1 & \text{if } i = j = k \\
\max(0, -b_{ik}) & \text{if } i \neq j = k
\end{cases}
\]
Let $F$ be the $(r - n) \times (r - n)$-matrix with entries

\[
\begin{pmatrix}
\delta_{ij} & \text{if } i \neq k \\
-1 & \text{if } i = j = k \\
\max(0, b_{kj}) & \text{if } j \neq i = k
\end{pmatrix}
\]

Then $\mu_k(L, \bar{B}) = (\mu_k(L), \mu_k(\bar{B}))$ where

\[
\mu_k(L) = E^t L E; \mu_k(\bar{B}) = E \bar{B} F.
\]

It is easy to check that $\mu_k(L, \bar{B})$ is also a compatible pair with the same $d$. Define $a' = (a'_1, a'_2, \ldots, a'_r)$ and $a'' = (a''_1, a''_2, \ldots, a''_r)$ by

\[
a'_i = \begin{cases}
-1 & \text{if } i = k \\
\max\{0, b_{ik}\} & \text{if } i \neq k
\end{cases}
\]

\[
a''_i = \begin{cases}
-1 & \text{if } i = k \\
\max\{0, -b_{ik}\} & \text{if } i \neq k
\end{cases}
\]

We then define

\[
\mu_k(X_i) = \begin{cases}
X^{a'_i} + X^{a''_i} & \text{if } i = k \\
X_i & \text{if } i \neq k
\end{cases}
\]

The elements $X'_i \overset{\text{def}}{=} \mu_k(X_i)$ satisfy

\[
X'_i X'_j = q^{\delta_{ij}} X'_i X'_j \quad (i, j \in [1, r])
\]

where $\mu_k(L) = L'$. Moreover,

\[
\mu_k(T) \overset{\text{def}}{=} ((X'_1, X'_2, \ldots, X'_r), \mu_k(L), \mu_k(\bar{B}))
\]

is a new quantum seed in $\mathcal{F}$, called the mutation of $T$ in direction $k$.

**Definition 5.5.** The quantum cluster algebra $\mathcal{A}_{q^{1/2}}(\mathcal{C}_w)$ is the $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of the skew field $\mathcal{F}$ generated by the union of clusters of all quantum seeds obtained from $\mathcal{T}$ by any sequence of mutations.

5.4. **Cluster structure on quantum coordinate rings.** Let us recall the cluster category $\mathcal{C}_w$ in section (4.1). In $\mathcal{C}_w$ $\mathcal{C}_w$ admits a structure of a quantum cluster algebra. This is given by the initial seed

\[
T_{\bar{V}_1} = ((X_{\bar{V}_1}, \ldots, X_{\bar{V}_r}), L_{\bar{V}_1}, B_{\bar{V}_1})
\]

where $L_{\bar{V}_1} = (\lambda_{ij})$ is given by

\[
\lambda_{ij} = \text{Hom}_\Lambda(\bar{V}_i, \bar{V}_j) - \text{Hom}_\Lambda(\bar{V}_j, \bar{V}_i), \quad (i, j \in [1, r])
\]

and $B_{\bar{V}_1}$ refers to the skew-matrix of $\Gamma_1$ in (4.3). The frozen variables are the injective modules $I_j$ for $j \in I$. $(L_{\bar{V}_1}, B_{\bar{V}_1})$ is compatible with $d = 2$ in equation (5.3). The mutations of $T_{\bar{V}_1}$ is given by the mutations in (3) of Proposition (4.2). Denote by $\mathcal{A}_{q^{1/2}}(\mathcal{C}_w)$ this quantum cluster algebra. For any rigid module $T \in \mathcal{C}_w$, we have $X_T$ is a quantum cluster monomial in $\mathcal{A}_{q^{1/2}}(\mathcal{C}_w)$. To modify $q^{1/2}$, we define the variables $Y_T$ for any rigid module $T \in \mathcal{C}_w$ by

\[
Y_T = q^{-[T,T]/2} X_T
\]
Because $Y_{R}Y_{S} = q^{[R,S]}Y_{R}Y_{S}$ and $Y_{T}Y_{T} = q^{[T,T]_k}(q^{-1}Y_{T}^{T} + Y_{T}^{T'})$ in Proposition (4.2). After rescaling $X_T$ by $Y_T$, we obtain a quantum cluster algebra $A_q(C_w)$ generated by $Y_T$ for all rigid modules $T \in C_w$. It is a $\mathbb{Z}[q^\pm]$-subalgebra of $A_q(C_w)$.

Let us set

$$\mathcal{A}_{Q(q)}(C_w) = \mathbb{Q}(q) \otimes_{\mathbb{Z}[q^\pm]} A_q(C_w)$$

**Theorem 5.6.** [10, Theorem 12.3] There is an isomorphism $\Upsilon : \mathcal{A}_{Q(q)}(C_w) \cong U_q(\mathfrak{n}(w))$* by sending $Y_{\tilde{V}_k}$ to $D(0, k)$ for any $k \in [1, r]$. Moreover, $\Upsilon(\tilde{V}[k^+, l]) = D(k,l)$.

Let us assume that $i$ is the word as in (2.4) and recall the notations in (2.3). We set $l(b,d) = \ell(\lambda(b,d))$. In other words, $l(b,d) = u+1$ if $\lambda(b,d) = (\tau^u\beta_b = \beta_d, \ldots, \beta_b)$. Following Theorem (3.16), we have

**Theorem 5.7.** Let $D(b,d)$ and $D(k,l)$ be two quantum minors induced from the word (2.4). Let us write $\lambda(b,d) = (\tau^u\beta_b = \beta_d, \ldots, \beta_b) \lambda(k,l) = (\tau^v\beta_k = \beta_l, \ldots, \beta_k)$. Set

$$R(\beta_b, \beta_k) = [s,t], R(\beta_b, \beta_d) = [s',t'],$$

and $d(b,d),(k,l) = u-v-1 = l(b,d) - l(k,l) - 1$ we have $D(b,d)D(k,l)$ is a dual canonical base in $A_q(\mathfrak{n}(w))$ if and only if they satisfy the following conditions:

1. if $u = v$
   
   $$l_u[s,t] = l_u[u-t, u-s] \cap l_u'[1-t, 1-s] - l_u'[u-t, u-s] \cap l_u[-1-t, -1-s]$$
   
   and
   
   $$l_u[s',t'] = l_u[u-t', u-s'] \cap l_u'[1-t', 1-s'] - l_u'[u-t', u-s'] \cap l_u[-1-t', -1-s']$$

2. if $u \neq v$ and we assume that $u > v$,

   $$l_v[s - d(b,d),(k,l)1, t - d(b,d),(k,l)1 - 1]$$

   $$= l_u[u-t, u-s] \cap l_u'[d(b,d),(k,l)1 - t, d(b,d),(k,l)1 - s] - l_u'[u-t, u-s] \cap l_u[d(b,d),(k,l)1 - t, d(b,d),(k,l)1 - s]$$

   and

   $$l_v[s' - d(b,d),(k,l)1 - 1, t' - d(b,d),(k,l)1 - 1] \cap [u-v, u]$$

   $$= l_u[v-t', v-s'] \cap l_u'[d(b,d),(k,l)1 - t', d(b,d),(k,l)1 - s'] - l_u'[v-t', v-s'] \cap l_u[d(b,d),(k,l)1 - t', d(b,d),(k,l)1 - s']$$

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