Resonance and decay phenomena lead to quantum mechanical time asymmetry

A Bohm and H V Bui
Department of Physics, University of Texas at Austin, Austin TX 78712, USA
E-mail: bohm@physics.utexas.edu, viethai@physics.utexas.edu

Abstract. The states (Schrödinger picture) and observables (Heisenberg picture) in the standard quantum theory evolve symmetrically in time, given by the unitary group with time extending over \(-\infty < t < +\infty\). This time evolution is a mathematical consequence of the Hilbert space boundary condition for the dynamical differential equations. However, this unitary group evolution violates causality. Moreover, it does not solve an old puzzle of Wigner: How does one describe excited states of atoms which decay exponentially, and how is their lifetime \(\tau\) related to the Lorentzian width \(\Gamma\)? These questions can be answered if one replaces the Hilbert space boundary condition by new, Hardy space boundary conditions. These Hardy space boundary conditions allow for a distinction between states (prepared by a preparation apparatus) and observables (detected by a registration apparatus). The new Hardy space quantum theory is time asymmetric, i.e., the time evolution is given by the semigroup with \(t_0 \leq t < +\infty\), which predicts a finite “beginning of time” \(t_0\), where \(t_0\) is the ensemble of time at which each individual system has been prepared. The Hardy space axiom also leads to the new prediction: the width \(\Gamma\) and the lifetime \(\tau\) are exactly related by \(\tau = h/\Gamma\).

1. Introduction
Most physicists believe that quantum theory, non-relativistic and relativistic, is time-symmetric which means that time \(t\) extends over \(-\infty < t < +\infty\). But asymmetric time evolution with \(t_0 < t < +\infty\), \(t_0\) finite, has been discussed in a variety of forms, e.g., [1] [2] and references thereof. Here we want to discuss the quantum mechanical time asymmetry due to the boundary condition for the dynamical equations of quantum mechanics.

The main reason for the belief in time symmetric quantum theory with \(-\infty < t < +\infty\) is probably of mathematical origin: To solve the dynamical differential equations (Schrödinger equation for states, or Heisenberg equation for observables in the Heisenberg picture) one has to use boundary conditions. Von Neumann’s choice for these boundary conditions was the Hilbert space (i.e., Lebesgue-square integrable wave functions). From a mathematical theorem (Stone-von Neumann theorem [3]) then follows the (unitary) group evolution with \(-\infty < t < +\infty\), for the states (in the Schrödinger picture) or for the observables (in the Heisenberg picture). A similar theorem, not as well known holds if one uses the Schwartz space boundary condition of the Dirac ket formalism [4].

But Hilbert space and Schwartz space are not the only choice for boundary conditions of the Schrödinger and Heisenberg equation. In Lax-Phillips scattering theory [5], Hardy spaces were used in applications to scattering of classical waves. Hardy spaces have also been suggested more
recently in the quantum theory of scattering. They are needed to obtain a quantum theory of resonance phenomena and decaying states [6, 7, 8, 9, 10, 11, 12, 13].

That scattering resonances and decaying states may be connected with each other has long been suspected, but an exact theory that unifies these phenomena was not possible within the conventional mathematical frame of (non-relativistic and relativistic) quantum theory provided by the Hilbert space, or by the Schwartz space triplet for Dirac kets. Hardy spaces are perhaps the only mathematical tools that allow an unification of resonance scattering and of decay phenomena.

2. Decaying States and Resonances
Many people have been thinking about the nature of the relationship between resonance and decay phenomena. According to an anecdote we heard from Larry Schulman, Wigner was going around Princeton in the 1960s asking people: “Tell me, what is the state vector of the decay phenomena.” According to an anecdote we heard from Larry Schulman, Wigner was going around Princeton in the 1960s asking people: “Tell me, what is the state vector of the decay phenomena?”

According to an anecdote we heard from Larry Schulman, Wigner was going around Princeton in the 1960s asking people: “Tell me, what is the state vector of the decay phenomena?”

The conclusion from these and other results is:
1. Using strictly Hilbert space methods one cannot have a state vector which has exact exponential decay.
2. Starting with Breit-Wigner energy wave function (3), there will be deviations from
exponential decay law (4). A state, with the Breit-Wigner energy wave function restricted
to the positive real axis $0 \leq E \leq \infty$ as in (3), cannot have purely exponential time evolution.

In spite of these mathematical results, many people think that a Breit-Wigner resonance
state with something like the energy distribution (3), and a decaying Gamow state, with the
time evolution (1) or (2), are the same physical entities:

A Breit-Wigner resonance is an exponentially decaying state.

This is in particular the prevailing opinion in non-relativistic quantum mechanics, and for
relativistic resonances one still speaks of energy dependent width. Therefore, one may ask the
questions: Can one construct a mathematical theory in which a resonance of width $\Gamma$ and an
exponentially decaying state with exponential lifetime $\tau$,

$$\tau = \frac{\hbar}{\Gamma}, \quad (5)$$

are just two different observations of the same quasi-stable physical state? Can one find a
generalized state vector, which is described by something like a Dirac ket, or by something
which is a little more “generalized” than the Dirac ket?

In standard Hilbert space quantum mechanics, there is no vector like a Dirac ket or something
a little “more” generalized “than a Dirac ket” and “... there does not exit (in 1959 [19])... a
rigorous theory to which these various [Weisskopf-Wigner] methods can be considered as
approximations”.

The task, therefore is: to find a mathematical theory which contains new generalized vectors,
or kets, and other mathematical entities which

1. describe Breit-Wigner lineshape, i.e, something like (3), and
2. also describe exact exponential time evolution, like the Gamow vector (1), and for which in
   addition the relation (5) holds.
If such a theory can be found, does such a theory have some surprising properties?

3. Unifying Exponential Time Evolution and Breit-Wigner Energy Lineshape

The mathematical ingredients of quantum theory are: linear superposition, scalar products and
operators in linear spaces, dynamical equations ( Schrödinger for states and Heisenberg for
observables), (and symmetry transformations of non-relativistic (Galileo group) and relativistic
space-time (Poincare group) which will not be discussed here.)

Dynamical differential equations need to be solved under boundary conditions. The standard
boundary condition for the dynamical differential equations in quantum mechanics is the Hilbert
space boundary condition. Does one really have to use the Hilbert space axiom as the boundary
conditions for the dynamical (Schrödinger or Heisenberg) equations?

The Hilbert space of von Neumann requires Lebesgue integrals in order that every Cauchy
sequence has a limit element in the Hilbert space $\mathcal{H}$ (completeness). But physicists do not
use Lebesgue integrals; the energy wave functions are not given by a class of Lebesgue square
integrable function $f(E)$ ($0 \leq E < \infty$), but by one Schwartz space function $\varphi(E') = \langle E' | \varphi \rangle$
which is smooth, infinitely differentiable, rapidly decreasing function. The Dirac kets $|E\rangle$ are
continuous (in Schwartz space convergence) antilinear functionals $F_E(E')$:

$$F_E(E') = \langle E' | E \rangle = \delta(E' - E) \text{ for } 0 \leq E < \infty \quad \text{(or also for } -\infty \leq E < \infty),$$

not vectors.
For the Schwartz space $\Phi$, one uses Riemann integrals, rather than Lebesgue integrals; however Dirac kets $|E\rangle$ are in the space of antilinear continuous functionals, $F_E = |E\rangle$, $|E\rangle \in \Phi^\times$, and one has a triplet of spaces:

$$\Phi \subset \mathcal{H} = \mathcal{H}^\times \subset \Phi^\times.$$  \tag{7}

The Schwartz space and the Dirac kets $|E\rangle \in \Phi^\times$, have been well accepted by the majority of physicists. But even the Dirac kets in $\Phi^\times$ are still insufficient to describe exponentially decaying states with complex energy $E_R - i\hbar/2\tau$, like the Gamow vector $|E - i\hbar/2\tau\rangle$. There is no such vector in the dual space $\Phi^\times$ of the Schwartz space. But one can modify the mathematical axioms of quantum physics further, such that the Gamow state vectors $|G\rangle$ with the property (1), and with the Breit-Wigner energy distribution like (3), are mathematically well defined, and that in addition $\tau = \frac{\hbar}{\Gamma}$ holds as exact relation. In this theory, a resonance is an exponentially decaying state. Furthermore, both have a common mathematical origin, the pole of the $S$-matrix.

Physicists use only Riemann integrals. But they also use Dirac kets $|E\rangle$. Furthermore, they associate resonances with poles of the $S$-matrix at $z_R = E_R - i\Gamma/2$. For Gamow vectors, one can use mathematically defined kets with complex eigenvalues $|E - i\Gamma/2\rangle$. The Lippmann-Schwinger kets [23] also suggest analytic continuation into the complex energy plane. If all of these properties are to be put together into a mathematical theory, one needs energy wave functions, which are not only smooth and rapidly decreasing, like the (Schwartz functions), but which can also be analytically continued into the complex energy plane.

Before looking for a theory with complex energy variable that unifies the Breit-Wigner resonances and exponentially decaying states, one needs to look at the experimental situation:

In contrast to the discussions of the Weisskopf-Wigner methods and the consequences of the mathematical theorems for Hilbert space vectors, the analysis of the experimental data is usually done in terms of exponential time evolution and Breit-Wigner energy distribution.

A compilation of the experimental data, for a large number of spontaneously decaying quantum systems in different areas of physics, shows good agreement with the exponential law $P(t) \sim e^{-t/\tau}$ [20]. In addition, the relation $\tau = \frac{\hbar}{\Gamma}$ has been tested more recently for the $\beta$ decay of Na to very high accuracy. Both line-width $\Gamma$ [21] and lifetime $\tau$ [22] have been measured with sufficiently high accuracy:

The Breit-Wigner line width measurement gives:

$$\Gamma = (40.537 \pm 0.087) \times 10^{-9} \text{ eV}, \quad \frac{\hbar}{\Gamma} = (16.237 \pm 0.035) \text{ ns}$$

and the exponential lifetime measurement gives $\tau = (16.254 \pm 0.022) \text{ ns}$.

On the basis of these and other experimental data, it is not unjustified to make the hypothesis that:

A scattering resonance $(E_R, \Gamma)$ (but without the non-resonant background $B_j(E)$ which is also present in the data of scattering experiments), and an exponentially decaying state $(E_D, \tau)$ are the same physical entities. The center of the Breit-Wigner resonance energy $E_R$ usually incorporate corresponds to the energy of the decaying state $E_D$ and the width $\Gamma$ of the Breit-Wigner lineshape corresponds to the inverse lifetime $\tau$ of the decaying state:

$$(E_R, \Gamma) = \left( E_D, \frac{\hbar}{\tau} \right).$$  \tag{8}

The standard mathematical theory with Hilbert space axiom as well as the Schwartz space theory with Dirac kets do not contain the mathematical means to incorporate a state with the property (8). To incorporate these new “state” vectors $|E_R - i\Gamma/2\rangle$, one needs to solve the dynamical equations of quantum mechanics, (9) and (10) below, under new boundary conditions.
4. Causality and Various Solutions of the Dynamical Differential Equations of Quantum Mechanics

The dynamical evolution of quantum systems can be described as the evolution of the observable, in which the states do not change in time (Heisenberg picture), or as the evolution of the state with the observable kept fixed (Schrödinger picture).

The Heisenberg equation for the observable $A(t)$ or in the special case $A(t) = |\psi^{-}(t)\rangle\langle\psi^{-}(t)|$ for the observable vector $\psi^{-}(t)$ are:

$$i\hbar \frac{\partial A(t)}{\partial t} = -[H, A(t)] \quad (9a)$$

$$i\hbar \frac{\partial}{\partial t} \psi^{-}(t) = -H \psi^{-}(t) \quad (9b)$$

The Schrödinger equation for the state operator $\rho(t)$ (“density matrix”) or for the state vector $\phi^{+}(t)$ are:

$$i\hbar \frac{\partial \rho(t)}{\partial t} = [H, \rho(t)] \quad (10a)$$

$$i\hbar \frac{\partial}{\partial t} \phi^{+}(t) = H \phi^{+}(t) \quad (10b)$$

We use the label $-$ for the observables and $+$ for the state in order to emphasize that $H$ is the exact Hamiltonian which includes the interaction and shall not use the interaction-free state vectors usually denoted $\phi = \phi^{in}$ and correspondingly $\psi = \psi^{out}$ which evolve with a hypothetical free Hamiltonian called $K = H_{0} = H - V$, where $V$ is an interaction Hamiltonian. The label $+$ and $-$ at the in-state $\phi^{+}$ and the out-observable $\psi^{-}$ have been taken to conform with the Lippmann-Schwinger equation (22) below.

The predictions of the theory that are compared with the experimental data (measured as ratios of detector count $N(t)/N$) are the Born probabilities:

$$\text{Tr} (A(t) \rho) = \text{Tr} (A \rho(t)) \sim \frac{N(t)}{N} \quad (11)$$

For the special case of the observable $|\psi^{-}(t)\rangle\langle\psi^{-}(t)|$ in the state $\phi^{+}$, these probabilities are calculated as:

$$P_{\phi^{+}} (|\psi^{-}(t)\rangle\langle\psi^{-}(t)|) = \text{Tr} (|\psi^{-}(t)\rangle\langle\psi^{-}(t)| \phi^{+}) \langle\phi^{+}|) \quad (12)$$

In order to calculate these probabilities, one has to solve the dynamical equation (9) or (10). To solve differential equations requires the choice of a boundary condition, i.e. a statement what kind of the solutions should be admitted as possible solutions of (9a) (9b) or of (11) (12). In standard quantum mechanics, boundary conditions of the dynamical equations are given by the Hilbert space axiom:

for the admitted set of state vectors $\{\phi^{+}\}$ choose the Hilbert space $\mathcal{H}$ (Schrödinger picture),

or

for the set of vectors representing observables $\{\psi^{-}\}$ choose the Hilbert space $\mathcal{H}$ (Heisenberg picture).
In place of the Hilbert space axiom, one can also choose other possible boundary conditions. For instance, one can choose the Schwartz space boundary condition (i.e. using the mathematical version of Dirac’s ket formalism):

for the set of admitted state vectors \( \{ \phi^+ \} \) choose the Schwartz space \( \Phi \),
\[ \text{or} \]
for set of observable vectors \( \{ \psi^- \} \) choose the Schwartz space \( \Phi \).

The solutions of the Heisenberg equation (9b) for \( \psi(t) \) as well as the solutions of the Schrödinger equation (12) for \( \phi^+(t) \) are according to a mathematical theorem, (the Stone-von Neumann theorem [3] for the Hilbert space \( \mathcal{H} \) (i.e., for Lebesgue integrable \( |\phi^+(E)|^2 \)) given by the unitary group evolution. Thus, as a consequence of this Stone-von Neumann theorem, one obtains from the Hilbert space axiom of mathematics:

\[
\psi^-(t) = U(t) \psi^- = e^{iHt/\hbar} \psi^-, \quad -\infty < t < +\infty \quad \text{in the Heisenberg picture},
\]

or

\[
\phi^+(t) = U^\dagger(t) \phi^+(t) = e^{-iHt/\hbar} \phi^+, \quad -\infty < t < +\infty \quad \text{in the Schrödinger picture}. \tag{15}
\]

Consequently: Using the Hilbert space boundary conditions (13), the probability for the observable \( |\psi^-(t)\rangle\langle\psi^-(t)| \) in the state \( \phi^+ \) is predicted for “all time” \( -\infty \leq t \leq +\infty \):

\[
P_{\phi^+}(|\psi^-(t)\rangle\langle\psi^-(t)|) = |\langle\psi^-(t)|\phi^+\rangle|^2 \quad \text{for } \psi^-, \phi^+ \in \mathcal{H} \text{ and for } -\infty \leq t \leq +\infty. \tag{16}
\]

By a similar theorem [4], one obtains under the Schwartz space boundary condition (14) the same prediction:

\[
P_{\phi^+}(|\psi^-(t)\rangle\langle\psi^-(t)|) = |\langle\psi^-(t)|\phi^+\rangle|^2 \quad \text{for } \psi^-, \phi^+ \in \Phi \text{ and for } -\infty \leq t \leq +\infty. \tag{17}
\]

Summarizing, if one solves the dynamical equations of quantum mechanics (9b) or (12) under the Hilbert space axiom or under the Schwartz space boundary condition, one predicts from (15) the probability \( |\langle\psi^-(t)|\phi^+\rangle|^2 \) of the observable \( |\psi^-(t)\rangle\langle\psi^-(t)| \) (or \( A(t) \)) in the state \( \phi^+ \) for the time: \( -\infty \leq t \leq +\infty \).

What is the physical evidence that we should have a time evolution group (15) or (16) i.e that \( U(t) = e^{iHt/\hbar} \) extends over \( -\infty < t < +\infty \)?

The prediction (15) or (16) would mean that the probability \( |\langle\psi^-(t)|\phi^+\rangle|^2 \) for the observable \( |\psi^-(t)\rangle\langle\psi^-(t)| \) in the state \( \phi^+ \) is predicted for all time \( -\infty \leq t \leq +\infty \), even for \( t \rightarrow -\infty \). But it is obvious that an observable \( |\psi^-(t)\rangle\langle\psi^-(t)| \) can not be measured in a state \( \phi^+ \) at the time \( t \leq t_0 \), where \( t_0 \) is the finite time at which the state \( \phi^+ \) has been prepared, since a state \( \phi^+ \) must be prepared first before an observable \( |\psi^-(t)\rangle\langle\psi^-(t)| \) can be measured in this state (causality). E.g. a detector for \( |\psi^-(t)\rangle\langle\psi^-(t)| \) cannot count the decay products of a decaying state \( \phi^G(t_0) \) before the decaying state \( \phi^G(t_0) \) has been prepared at a time \( t_0 < t \), but only for \( t > t_0 \). Experiments indicate that there is such a finite time \( t_0 \) for a decaying quantum system, which of course will be measured as an ensemble of finite times \( t_0^{(i)} \)[23]. The onset times of the dark periods in quantum jump experiments might provide an experimental conformation of this ensemble of times [24].

This means the Born probability to measure the observable \( |\psi^-(t)\rangle\langle\psi^-(t)| \) or \( A(t) \) in the state \( \phi^+ \):

\[
P_{\phi^+}(A(t)) = \text{Tr}(A(t) |\phi^+(t_0)\rangle\langle\phi^+(t_0)|)
\]

\[
P_{\phi^+}(\psi^-(t)) = |\langle\psi^-(t)|\phi^+|\rangle|^2 = |\langle e^{iHt}\psi^- |\phi^+\rangle|^2 = |\langle\psi^- | e^{-iHt}\phi^+\rangle|^2 = |\langle\psi^- |\phi^+(t)|\rangle|^2 \tag{19}
\]

exists (experimentally) only for \( t \geq t_0(=0) \).
Here \( t_0 \) is the time at which the state \( \phi^+ \) has been prepared, thus the detection time of the decay products \( t \) must be later than \( t_0 \). Causality provides the intuitive justification that \( t \geq t_0 \); the evolution in the real world is time-asymmetric.

This causality should be reflected in the theory. Therefore, it would be good to find a new theory (precisely *new boundary conditions for the dynamical equations* (9) or (10)), replacing the Hilbert space axiom (13); and similarly, the Schwartz space axiom (14).

The new boundary conditions for the dynamical equation need to be chosen such that the solutions of the Schrödinger equation (12) for the state \( \phi^+(t) \) will be given by the *semigroup*:

\[
U^x(t) = e^{-iHt/\hbar} \quad 0 \leq t < \infty \quad \text{for state } \phi^+.
\]

Similarly using the Heisenberg picture, the solutions of the Heisenberg equation (9b) for the observable, \( \psi^-(t) \), must be given by the *semigroup*:

\[
U(t) = e^{+iHt/\hbar} \quad 0 \leq t < \infty \quad \text{for observable } \psi^-.
\]

The dynamical equations (9a), (9b), or (11), (12) remain the same, but the boundary conditions (13) and (14) need to be revised.

### 5. What the \( i\epsilon \) in the Lippmann-Schwinger Equation Suggests

To conjecture such a new “causal” theory in which the time evolution will be given by (20) or (21), we start with the phenomenological theory of scattering and decay.

One uses in- and out-plane wave “state” \( |E^+\rangle \) and \( |E^-\rangle \) which fulfill the Lippmann-Schwinger equation [25]:

\[
|E^\pm\rangle = |E \pm i\epsilon\rangle = |E\rangle + \frac{1}{E - H \pm i\epsilon}V|E\rangle = \Omega^\pm|E\rangle, \quad \epsilon \to +0
\]

One speaks of complex energies (as dictated by the \( \pm i\epsilon \) in (22):

- for the analytic \( S \) matrix [26] \( S_j(E) \rightarrow S_j(z) \)
- for the Gamow states [14] \( \phi^G \) \( z_R = E_R - i\Gamma/2 \)
- or uses the \( \pm i\epsilon \) in the propagator of field theory [27] \( z = E \pm i\epsilon, \epsilon \text{ infinitesimal} \)

The \( |E^+\rangle \) of (22) are taken as basis system for the Dirac basis vector expansion of in-state vectors:

\[
\phi^+ = \sum_{j,j_3,\eta} \int_0^\infty dE|E,j,j_3,\eta^+\rangle\langle E,j,j_3,\eta^+|\phi^+\rangle = \int_0^\infty dE|E^+\rangle\langle E^+|\phi^+\rangle.
\]

The \( |E^-\rangle \) of (22) are taken as basis system of vectors \( \psi^- \)

\[
\psi^- = \sum_{j,j_3,\eta} \int_0^\infty dE|E,j,j_3,\eta^-\rangle\langle E,j,j_3,\eta^-|\psi^-\rangle = \int_0^\infty dE|E^-\rangle\langle E^-|\psi^-\rangle,
\]

which are now postulated to represent *observables* and therefore fulfill the Heisenberg equation.

The measured quantities are the Born probabilities

\[
|\langle \psi^-, \phi^+ \rangle |^2 = \text{Tr}(|\psi^-\rangle\langle \psi^-| \phi^+\rangle \phi^+ \rangle),
\]

for the observable \( |\psi^-\rangle\langle \psi^-| \) in the state \( \phi^+ \). These vectors \( \psi^- \) and the observable \( A \) in general represent the registration apparatus (e.g. a detector); note that \( |\psi^-\rangle \) of (24), will not be
interpreted as an “out-state” (which makes no physical sense), but as an observable representing the detector.

Causality requires that the observable $|\psi^-\rangle\langle\psi^-|$ cannot be measured in the state $\phi^+$ before the state $\phi^+$ has been prepared at $t_0$ (which in (20) and (21) has been chosen as $t_0 = 0$, but which could be any finite time). ($t_0 = 0$ in (20) and (21) represents an ensemble of times $t_0^{(1)}, t_0^{(2)}, \ldots$ for an ensemble of quantum particles, cf. [23, 24].)

The Dirac basis vector expansions (23), (24) use two different kind of kets:

$$|E_{jj\eta\eta^\mp}\rangle \in \Phi^\pm$$

suggested by the Lippmann-Schwinger “out-plane waves” $|E^-\rangle$ and in-plane waves $|E^+\rangle$, respectively.

Because of the $+i\epsilon$ in the Lippmann-Schwinger equation (22), the energy wave function of the prepared in-state $\phi^+$,

$$\phi^+(E) = \langle^+ E_{jj\eta}\rangle|\phi^+\rangle = \langle \phi^+ | E_{jj\eta} \rangle^+,$$

is the boundary value of an analytic function in the lower complex energy semi-plane (for complex energy $z = E + i\epsilon = E - i\epsilon$ immediately below the cut along the real axis on the second sheet of the $S$-matrix $S_j(z)$, cf. Fig 1 below.)

Similarly, the energy wave function of the observable $|\psi^-\rangle\langle\psi^-|$, $\psi^-(E) \equiv \langle - E_{jj\eta}\rangle|\psi^-\rangle$, (27a)

needs to be a function that can be extended to an analytic function in the upper complex energy semi-plane so that its complex conjugate

$$\bar{\psi}^-(E) = \langle \psi^- | E j j_3 \eta^- \rangle,$$ (27b)

can be extended to an analytic function in the lower complex semi-plane.
6. Calculating the Born Probabilities

In order to calculate the Born probability amplitude \((\psi^-|\phi^+\rangle)\) for the observable \(\psi^-\) in the state \(\phi^+\), one uses basis vector expansions (23), (24). In this amplitude \((\psi^-|\phi^+\rangle)\), the \(S\)-matrix element \((-E', j', j_3', \eta'|E, j, j_3, \eta^+)\) will appear, which is the bra-ket of the basis system \(|E, j, j_3, \eta^+\rangle\) of the interacting in-state vector \(\phi^+\) in (23) and the basis system \(|E, j, j_3, \eta^-\rangle\) of the interacting out-observable vector \(\psi^-\) in (24). The modules squared: \(|(\psi^-, \phi^+)^2|\), represents the Born probabilities (11), (12), and (25) for the observable \(\psi^-\) in the state \(\phi^+\).

This matrix element can be simplified using energy- and angular momentum conservation:

\[
(-E', j', j_3', \eta'|E, j, j_3, \eta^+) = \delta(E' - E)\delta_{j,j_3}\delta_{j',j}\mathcal{S}^{\eta\eta}(E) .
\]  

(28)

Thus, the matrix element has been expressed in terms of a “reduced” \(S\)-matrix elements \(S^{\eta\eta}(E)\), where \(\eta', \eta\) are the channel quantum numbers or particle species quantum numbers.

Consider the simplest case that there is just one first order pole of \(S_j(E)\) at \(z = z_R\) and there are no other singularities. The reduced \(S\)-matrix \(S^{\eta\eta}(E)\) encapsulates the properties of the specific interaction between target and projectile, which show up as such specific features like Breit-Wigner bumps (3) in the cross section, and are due to a pole of the \(S\)-matrix at \(z_R = E_R - i\Gamma/2\):

\[
S^{\eta\eta}(E) = 2i\alpha_j^{\eta\eta}(E) + \delta^{\eta\eta} = 2i\frac{R^{\eta\eta}}{E - z_R} + \delta^{\eta\eta} + B^{\eta\eta}(E) .
\]  

(29)

\(B^{\eta\eta}(E)\) denotes some slowly varying background.

From the analyticity property of the energy wave functions (26) (27b) follows that \((\psi^-|E^-\rangle\langle E^+|\phi^+\rangle)S_j(E)\) can be analytically continued into the lower complex plane (second sheet), except for singularities of \(S_j(E) = S^{\eta\eta}(E)\). A singularity can be a first order pole at \(z = z_R = E_R - i\Gamma/2\) indicating that there is a resonance in the \(S\)-matrix element \(S_j(E) = 2i\alpha_j(E) + 1\), at the position \(z_R\) as shown in Fig 1.

The contour of integration for the \(S\)-matrix element can be deformed from the cut along the positive real axis as shown in Fig 1. Using (23), (24) and (28), the Born probability amplitude \((\psi^-, \phi^+)\) for the observable \(\psi^-\) in the state \(\phi^+\) is then written as

\[
(\psi^\text{out}, S\phi^\text{in}) \equiv (\psi^-, \phi^+) = \sum_j \int_{E_0 = 0}^{\infty} dE \sum_{j_3} \sum_{\eta, \eta'} \langle \psi^-|E, j, j_3, \eta^-\rangle S_j^{\eta\eta}(E) \langle E^+|E, j, j_3, \eta^+|\phi^+\rangle .
\]  

(30)

For the present purpose, \((\psi^\text{out}, S\phi^\text{in})\) is just another way of writing \((\psi^-, \phi^+)\). In simplified notation, this equation (30) is also written as:

\[
(\psi^-, \phi^+) = \int_{E_0 = 0}^{\infty} dE \langle \psi^-|E^-\rangle S_j(E) \langle E^+|\phi^+\rangle = \int_{\text{Cut}} \overline{\psi^-}(E) S_j(E) \phi^+ (E) .
\]  

(31)

With the right properties of the energy wave functions \((\psi^-|E^-\rangle\langle E^+|\phi^+\rangle)\), the contour of integration in (31) can be deformed from the cut along the positive real axis first sheet, (which is also the positive real axis second sheet) into the contour in the lower complex energy plane on second sheet given by \(C_-\), \(C_\infty\) and along the circle \(C_1\) around the pole at \(z_R\) on the second sheet of the Riemann surface, (where the resonance pole of the \(S\)-matrix element or of the scattering amplitude \(a_j(E)\)) is located).

Here we want to consider the integral along the contour \(C_1\) around the pole term \(\frac{R^{\eta\eta}}{z - z_R}\) of the \(S\)-matrix (29):

\[
\oint_{C_1} dz \langle \psi^-|z^-\rangle \frac{R^{\eta\eta}}{z - z_R} \langle z^+|\phi^+\rangle .
\]  

(32a)
The contour of this integral can be deformed into the integral:

$$\int_{-\infty I}^{+\infty I} dE \langle \psi^- | E^- \rangle \frac{R^n}{E-z_R} \langle \phi^+ | E \rangle.$$  (32b)

The program is now to determine the property of the spaces $\Phi_-$ (of $\{\phi^+\}$ representing states) and $\Phi_+$ (of $\{\psi^-\}$ representing observables), that means to determine the mathematical properties of the energy wave functions $\psi^-(E) \equiv \langle -E | \psi^- \rangle$ and $\phi^+(E) = \langle E | \phi^+ \rangle$ of (23) and (24).

We start from the integral along $C_1$ in (32b). The mathematical properties of $\langle \psi^- | E^- \rangle$ and $\langle \phi^+ | E \rangle$ are then determined, i.e. conjectured, in such a way that a scattering resonance with Breit-Wigner line shape at $z = z_R$ in (29) and a decaying state $|z_R^-\rangle$ with exponential time evolution,

$$e^{-iHt/\hbar} | z_R^- \rangle = e^{-iE_Rt/\hbar} e^{-\eta (t/2)} | z_R^- \rangle,$$  (33)

are both derived from the $S$-matrix pole at $z_R$. If that is possible, it would mean that the $S$-matrix pole represents the exponential decaying state $|z_R^-\rangle$, as well as the Breit-Wigner resonance with scattering amplitude $a_j^{BW} = \frac{R}{E-z_R}$.

### 7. Conjecturing the Hardy space axiom from the Pole Term of the $S$-Matrix

We focus our attention on the $S$-matrix pole term and the contour of integration $C_1$ around the pole at $z_R$, and take the $S$-matrix pole at $z_R = E_R - i\Gamma/2$ as the definition for a resonance. The scattering amplitude of a resonance at $z_R$, is then $a_j^{BW} = \frac{1}{\pi} S_j^{BW} (E) = \frac{R}{E-z_R}$. The integral along the infinite semi-circle $C_\infty$ on the second sheet of the $S$-matrix needs to vanish (this is part of the conditions which the energy wave function, whose properties we wish to conjecture by this process, needs to fulfill in order to obtain the new axiom -new boundary conditions for solutions of the dynamical equations, which will replace the Hilbert space boundary condition (13) and Schwartz space boundary condition (14)). The integral around the first order pole $z_R$ of the $S$-matrix is deformed into an integral along $-\infty I$ to 0 and an integral along the cut from 0 to $+\infty$, i.e. into an integral along the negative real axis $-\infty I < E < 0$ and along the cut $0 < E < +\infty$ as indicated in (32b).

The Breit-Wigner resonance amplitude is thus related to the integral from $-\infty I$ to 0 on the second sheet and on to $+\infty I$ along the cut: $-\infty I < E < +\infty I$.

In this way, at the $S$-matrix pole $z_R = E_R - i\Gamma/2$:

The Breit-Wigner will be a Gamow vector $|z_R, j, j_3, \eta^-\rangle$ with resonance amplitude: related to Breit-Wigner energy wave function:

$$\langle \psi^- | \phi_j^G \rangle = \langle \psi^- | z_R, j, j_3, \eta^- \rangle \sqrt{2\pi \Gamma} =$$
$$= \int_{-\infty I}^{+\infty I} dE \langle \psi^- | E, j, j_3, \eta^- \rangle \frac{i \sqrt{\pi \Gamma}}{E-z_R}$$  (34a)

Thus the Breit-Wigner: will be the ket, i.e the functional $|\psi^-\rangle \equiv \Phi_+$ related to the space of observables $\{\psi^-\}$ \equiv $\Phi_+$

$$a_j^{BW} (E) = \frac{R}{E-z_R} \iff \phi_j^G = |z_R, j, j_3, \eta^- \rangle \sqrt{2\pi \Gamma} =$$
$$= \int_{-\infty I}^{+\infty I} dE \langle E, j, j_3, \eta^- \rangle \frac{i \sqrt{\pi \Gamma}}{E-z_R}$$  (34b)
This vector $|E_R-i\Gamma/2, j, j_3, \eta^-\rangle = |z_R, j, j_3, \eta^-\rangle = |z_R\rangle = |E_R-i\Gamma/2\rangle$ can then be shown to:

1. be an eigenket of $H$ with the discrete complex eigenvalue $(E_R-i\Gamma/2)$ (as Gamow had wanted):

$$\langle H\psi^-|z_R\rangle = (\psi^-|H^\times|z_R\rangle = z_R\langle \psi^-|z_R\rangle \quad \text{for all } \psi^- \in \Phi_+,$$

or in ket notation

$$H^\times|E_R-i\Gamma/2\rangle = (E_R-i\Gamma/2)|E_R-i\Gamma/2\rangle.$$

2. have the property:

$$\langle e^{iHt/\hbar}\psi^-|E_R-i\Gamma/2\rangle \equiv \langle \psi^-|e^{-iH^\times t/\hbar}|E_R-i\Gamma/2\rangle = e^{-iE_Rt/\hbar}e^{-(\Gamma/2)t/\hbar}\langle \psi^-|E_R-i\Gamma/2\rangle.$$

This means that the generalized eigenvector $|z_R, j, j_3, \eta^-\rangle$ on the space of $\{\psi^-\} \equiv \Phi_+$ represents a scattering resonance with resonance energy $E_R$ and Breit-Wigner width $\Gamma$, as shown in (34b) and it is also an exponentially decaying state of lifetime $\tau$ as shown in (36). Thus

$$\text{a resonance of width } \Gamma \text{ in (34b) } \equiv \text{a decaying state with lifetime } \tau = \hbar \Gamma \text{ in (36).} \quad (37)$$

Therewith we would have a theory that unifies the Breit-Wigner resonances with the exponentially decaying Gamow vectors.

In order to be able to make the contour deformations described above, and to derive the results (34b), (35), and (36), new mathematical properties have to be assumed for the energy functions $\langle +E|\phi^+\rangle$ and $\langle -E|\psi^-\rangle$. These new mathematical properties turn out to agree exactly with mathematical conditions that define the smooth Hardy functions. This is an example of what Wigner calls the “unreasonable effectiveness of mathematics in physical sciences”.

Summarizing: In order that a Breit-Wigner resonance in (29) can be associated to a Gamow state with lifetime $\tau$ in (36).

In (38) (39), the $\mathcal{H}_\pm^2$ denote the Hardy class spaces and $\mathcal{S}$ denotes the space of Schwartz functions.

It is not clear that the smooth Hardy functions (38) (39) are the only functions that lead to the relation (34b). If one insists on Lebesgue integrals, one can probably obtain something similar using Hardy classes $\mathcal{H}_\pm^2$ in place of smooth Hardy functions $(\mathcal{H}_\pm^2 \cap \mathcal{S})$. But for quantum physics, the beautiful Dirac kets of Schwartz space $\mathcal{S}$ should be retained. Therefore, (38) (39) is probably the best choice for the energy wave functions. This is expressed by the Hardy space axiom (40), (41) for the space of in-states and out-observables.

Therewith we have arrived at a new axiom, the Hardy space axiom:
The set of prepared (in-) states \( \{ \phi^+ \} \) defined by the preparation apparatus (e.g. accelerator etc.) is:
\[
\{ \phi^+ \} = \Phi_- \subset \mathcal{H} \subset \Phi_-^\times, \quad |E, j, j_3, \eta^+ \rangle \in \Phi_-^\times.
\] (40)

The set of (out-) observables \( \{ \psi^- \} \) defined by the registration apparatus (e.g. detector) is
\[
\{ \psi^- \} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times, \quad |E, j, j_3, \eta^- \rangle \in \Phi_+^\times.
\] (41)

Here \( \Phi_\mp \) denotes the Hardy spaces of the complex semiplanes \( \mathbb{C}_\mp \) whose energy wave functions are given by (38) (39), and \( \Phi_\mp^\times \) are their duals (spaces of continuous antilinear functionals).

The discrepancy in the label \( \mp \) for the kets \( |E^\pm \rangle \) in (22) and the label \( \mp \) for the Hardy spaces \( \Phi_\mp \) in (40) (41) and the Hardy classes \( \mathcal{H}^2_\mp \) in (38) (39) has its origin in the different conventions in physics for the Lippmann-Schwinger equation of (22) and the convention in mathematics for the Hardy spaces, \( \Phi_\mp \) and \( \mathcal{H}^2_\mp \). The kets \( |z^\mp \rangle \), \( |E^\pm \rangle \), \( |E_R - i\Gamma/2^\mp \rangle \) are now defined as continuous antilinear functionals on the Hardy spaces \( \Phi_\mp \): \( |E^\pm \rangle \in \Phi_\mp^\times \). One has finally a consistent mathematical theory which unifies resonance and decay phenomena (with \( \Gamma = \hbar/\tau \) as an exact equality).

With the mathematical theory defined by the Hardy space axiom (40) (41), one can use mathematical theorems: (similar to the Stone-von Neumann theorem which applied to the Hilbert space boundary condition (13) of the dynamical equation, and which led to the unitary group evolution \( -\infty < t < +\infty \)) Solving the same dynamical equations (9b) and (12) (Heisenberg or Schrödinger) now under the Hardy space boundary conditions (40) (41) leads, (by the Paley-Wiener theorem [28]), to a \textit{semigroup evolution}:

For the observable \( \psi^- \) obeying the Heisenberg equation (9b), one obtains:
\[
\psi^-(t) = e^{iH(t-t_0)}\psi^-(t_0) \quad t_0 \leq t < \infty.
\] (42)

For the prepared state obeying the Schrödinger equation (12), one obtains:
\[
\phi^+(t) = e^{-iH(t-t_0)}\phi^+(t_0) \quad t_0 \leq t < \infty.
\] (43)

This means that the Hardy space boundary condition, which was needed in order to establish \( \tau = \hbar/\Gamma \) as an exact relation, predicts a time \textit{asymmetric} dynamical evolution (42) or (43) and therewith a beginning of time \( t_0 = \text{finite} \), for quantum systems.

A straightforward consequence of the semigroup solutions (42) (43) is that it predicts the Born probabilities
\[
P_{\phi^+}(\psi^-(t)) = |\langle \psi^-(t) | \phi^+(t_0) \rangle|^2 \quad \text{only for} \quad t \geq t_0 (= 0).
\] (44)

This solves the causality problem (it also avoids the exponential catastrophe for Gamow states): The observable \( |\psi^-(t)\rangle \langle \psi^-(t) | \) or any other observable \( A(t) \) (solution of the Heisenberg equation) is predicted \textit{only} for time \( t \) after the preparation time \( t_0 \) of the state \( \phi^+ \).

The finite quantum mechanical beginning of time \( t_0 \) is probably observed as an ensemble of finite times [23, 24]. Some people may want to consider the big bang time as an example of a quantum mechanical beginning of time, then one can speculate about many more bangs. A more important consequence of the beginning of time is probably that it provides a mathematical theory which unifies the resonance and decay phenomena of quantum physics into an exact mathematical theory, with lifetime \( \tau = \text{inverse width} \ h/\Gamma \).

**Acknowledgments**

The kind hospitality at Tsukub University is gratefully acknowledged.
References

[1] Halliwell J J, Pérez-Mercader J and Zurek W H 1994 Physical Origins of Time Asymmetry (Cambridge University Press)
[2] Bohm A, Doebner H D and Kielanowski P 1998 Irreversibility and Causality Lecture Notes in Physics Vol 504 (Berlin: Springer-Verlag)
[3] Stone M H 1932 Ann. Math. 33 No. 3 643; Neumann V J 1932 Ann. Math. 33 No. 3 567
[4] Bohm A and Gadella M 1989 Dirac Kets, Gamow Vectors and Gelfand Triplet Lecture Note in Physics Vol 348 (New York and Berlin: Springer-Verlag) Proposition II p 82
[5] Lax P D and Phillips R S 1967 Scattering Theory 1st Edition (New York: Academic Press)
[6] Bohm A 1978 Proc. 7th Int. Group Theory Coll. Lecture Note in Physics Vol 94, ed Bohm A et al p 245
[7] Bohm A 1979 Quantum Mechanics: Foundations and Applications 1st Edition (New York: Springer)
[8] Strauss Y, Horwitz L P and Eisenberg E 2000 J. Math. Phys. 41 No. 12 8050
[9] Strauss Y 2003 Int. J. Theor. Phys. 42 2285
[10] Strauss Y, Horwitz L P and Volovick A 2006 J. Math. Phys. 47, 123505
[11] Baumgärtel H 2007 Int. J. Theor. Phys. 46 1959
[12] Baumgärtel H 2010 J. Math. Phys. 51 113508
[13] Baumgärtel H 2006 Rev. Math. Phys. 18 61
[14] Gamow G 1928 Z. Phys. 51 204-12
[15] Weisskopf V and Wigner E P 1930 Z. Phys. 65 18
[16] Breit G and Wigner E P 1936 Phys. Rev. 49 519
[17] Goldberger M and Watson K 1964 Collision Theory (New York: Wiley) chapter 8
[18] Khalifin L A 1958 Sov. Phys. JETP 6 1053-63
[19] Levy M 1959 Nuovo Cimento 13 115
[20] Norman E B 1988 Phys. Rev. Lett. 60 2246
[21] Oates C W, Vogel K R and Hall J L 1996 Phys. Rev. Lett. 76 2866
[22] Volz U, Majerus M, Liebel H, Schmitt A and Schmoranzer A 1996 Phys. Rev. Lett. 76 2862
[23] Bohm A 1999 Phys. Rev. A 60 861
[24] Bohm A, Kielanowski P and Wickramasekara S 2006 Ann. Phys. 321 2299
[25] Lippmann B A and Schwinger J 1950 Phys. Rev. 79 469480; Gell-Mann M and Goldberger M L 1953 Phys. Rev. 91 398
[26] Eden R J, Landshoff P V, Olive V P and Polkinghorne J C 1962 The Analytic S-Matrix (New York: Cambridge University Press); Taylor J R 1972 Scattering Theory (John Wiley & Sons, Inc)
[27] Weinberg S 1995 The Quantum Theory of Field Vol 1 (Cambridge University Press)
[28] Paley R and Wiener N 1934 Fourier Transform in the Complex Domain (New York: American Mathematical Society)