Off-Shell Covariantization of Algebroid Gauge Theories

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Abstract

We present a generalized method to construct field strengths and gauge symmetries, which yield a Yang-Mills type action with Lie n-algebroid gauge symmetry. The procedure makes use of off-shell covariantization in a supergeometric setting. We apply this method to the system of a 1-form gauge field and scalar fields with Lie n-algebroid gauge symmetry. We work out some characteristic examples.

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1 Introduction

Recently, many approaches for a generalization of gauge theories are being discussed. Among them, there are the so-called higher gauge theories [1], where in addition to the gauge potential higher rank forms are introduced. Such theories are expected to appear, for example, in the construction of the effective theory of multiple M5-branes where a 2-form gauge potential appears.

Another approach is the promotion of the gauge algebra to an algebroid structure. This can be thought of as a generalization of the gauged non-linear sigma model, where the structure constants of the Lie algebra become scalar field dependent [2, 3, 4].

A systematic way to construct higher gauge theories is to use an $L_\infty$-structure [5]. Any truncated $L_\infty$-algebra defines a gauge theory of higher form gauge fields and the corresponding gauge symmetries are generalized to Lie n-algebras [6]. As we shall see, both generalizations, i.e., to higher gauge theory and to algebroid Yang-Mills can be understood in a unified way using supergeometry. Actually, there is a common phenomenon in both approaches, i.e. the higher gauge theory using an $L_\infty$-structure and the approach via algebroid structure, when it comes to the formulation of the corresponding field theories. This phenomenon is the so-called fake curvature condition [7].

In a general higher gauge theory also lower form gauge fields exist. However, the field strength of the higher form gauge potential is only covariant under the condition that the field strengths associated to the lower form gauge fields vanish. This is called the fake curvature condition and results in a non-interactive theory. Therefore, it is desirable to deform the higher algebra structure to circumvent this obstruction. Such a deformation process, known as off-shell covariantization, has been analyzed in the higher gauge theory context in our previous paper [8]. There, we solved the fake curvature condition by reducing the symmetries to Lie n-subalgebras, while imposing proper conditions on the auxiliary gauge fields.

In this paper, we want to address the problem of off-shell covariantization in the context of algebroid gauge theories. We apply our method to systems consisting of a 1-form gauge field and a scalar field. We formulate the corresponding higher algebroid gauge symmetries and associated gauge invariant actions. To obtain off-shell Yang-Mills type actions, we consider deformations of gauge transformations and field strengths. Auxiliary gauge fields are projected
out and field strengths are deformed by terms proportional to the lower curvatures.

In order to obtain proper gauge symmetries of gauge fields and field strengths, we use the supermanifold method on a so-called QP-manifold [9, 10], which is a useful tool to generate a BRST-BV formalism of topological field theories [11]. Instead of starting from fields and an action, we start with a graded symplectic manifold and its Hamiltonian function corresponding to a BRST charge of the gauge algebra. Gauge fields, field strengths and their gauge transformations are induced from the QP-manifold structure. This idea is similar to the free differential algebra method [12, 13]. In our formalism, consistency is guaranteed by the underlying QP-manifold structure [11, 14, 15, 16].

The advantage of the supermanifold method is that the gauge transformations and field strengths can be derived in a systematic manner. The starting point of our analysis is a general theory unifying gauge theories with algebroid symmetry and those with Lie n-algebra symmetry. Examples are the Kotov-Strobl model [4] and the Ho-Matsuo model [17]. See also [18] for a gauge theory with a Lie 2-algebra symmetry.

The organization of this paper is as follows. In section 2, we briefly review QP-manifolds and explain the off-shell covariantization procedure used in this paper. In section 3, we discuss the construction of \((n + 1)\)-dimensional higher algebroid gauge theories based on general QP-manifold structures. In section 4, we construct and analyze 4-dimensional algebroid gauge theories. We derive the relations between the structure functions necessary for off-shell covariantization. Furthermore, we discuss examples including the Stückelberg formalism, non-abelian off-shell covariantization and an example from the Kotov-Strobl models. In section 5, we examine the closure of the gauge symmetry algebra. Section 6 is devoted to discussion.

2 QP-manifolds and off-shell covariantization

In this section, we briefly review how to construct gauge transformations and field strengths using QP-manifolds. Then, we shortly explain the off-shell covariantization procedure of field strengths. Please refer to [19, 20] for conventional details.

A QP-manifold \((\mathcal{M}, \omega, Q)\) of degree \(n\) consists of a nonnegatively graded manifold \(\mathcal{M}\), a symplectic structure \(\omega\) of degree \(n\) and a homological vector field \(Q\) of degree 1 on \(\mathcal{M}\) such that \(L_Q \omega = 0\). The requirement of \(Q\) to be homological is equivalent to saying \(Q\) is nilpotent,
\( Q^2 = 0 \). The graded symplectic structure induces a Poisson bracket \( \{ -, - \} \) of degree \( (-n) \).

For any QP-manifold one can find a function \( \Theta \in C^\infty(\mathcal{M}) \) of degree \( n + 1 \), such that
\[
Q = \{ \Theta, - \}.
\] (2.1)
The nilpotency of \( Q \) then translates to the classical master equation
\[
\{ \Theta, \Theta \} = 0.
\] (2.2)
A QP-manifold can also be called a symplectic NQ-manifold. The operator \( Q \) generates BRST transformations of the associated gauge theory.

Though the method can be used to construct general \( p \)-form gauge theories, in this paper we focus on a theory containing scalar fields \( X^i(\sigma) \) and a 1-form gauge field \( A^a = d\sigma^\mu A^a_\mu(\sigma) \).

Our set-up is as follows. We consider a QP-manifold of degree \( n \), where the graded manifold is given by \( \mathcal{M}_n = T^*[n]E[1] \), \( n \in \mathbb{N} \), and \( E \to M \) is a vector bundle. \( M \) is a smooth manifold. We take the following local coordinates: \( x^i \) of degree 0 on \( M \) and \( q^a \) of degree 1 on the fiber of the vector bundle. When we construct the associated field theory, the degree corresponds to the ghost degree. With respect to the graded cotangent bundle \( T^*[n] \), we take coordinates \( (\xi_i, p_a) \) of degree \( (n, n-1) \) conjugate to \( (x^i, q^a) \). To summarize, the local coordinates on \( \mathcal{M}_n \) are \( (x^i, q^a, \xi_i, p_a) \) of degree \( (0, 1, n, n-1) \).

The symplectic form on \( \mathcal{M}_n \) is defined by
\[
\omega = \delta x^i \wedge \delta \xi_i + (-1)^n \delta q^a \wedge \delta p_a.
\] (2.3)
This induces the following graded Poisson bracket,
\[
\{ f, g \} = \frac{\delta f}{\delta x^i} \frac{\delta g}{\delta \xi_i} - \frac{\delta f}{\delta \xi_i} \frac{\delta g}{\delta x^i} + \frac{\delta f}{\delta q^a} \frac{\delta g}{\delta p_a} + (-1)^n \frac{\delta f}{\delta p_a} \frac{\delta g}{\delta q^a},
\] (2.4)
where \( f, g \in C^\infty(\mathcal{M}_n) \).

By making use of the BV-ASKZ formalism [11, 19], a topological field theory in \( n + 1 \) dimensions can be constructed from a QP-manifold of degree \( n \). Let \( \Sigma \) be the worldvolume.

The starting point of the construction is the promotion of the worldvolume to a graded space

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6[1] and \( [n] \) denote shifting of degree by 1 and \( n \), respectively.

7The relation between right and left derivative is given by \( \frac{\delta f}{\delta x} = (-1)^{|x||f|-|x|}) \frac{\partial f}{\partial x} \).
space $T[1]\Sigma$. We denote the local coordinates of $\Sigma$ by $\sigma^\mu$, which are of degree 0, and those of the fiber by $\theta^\mu$, which are of degree 1.

Let $\mathcal{M}_n$ be our QP-manifold. Then we can define the map $a : T[1]\Sigma \to \mathcal{M}_n$, such that

$$Z(\sigma, \theta) \equiv a^*(z) = \sum_{j=0}^{n+1} Z^{(j)}(\sigma, \theta) = \sum_{j=0}^{n+1} \frac{1}{j!} \theta^{\mu_1} \cdots \theta^{\mu_j} Z_{\mu_1 \cdots \mu_j}^{(j)}(\sigma)$$

(2.5)

is a superfield. Here, $z$ is a coordinate of degree $k$ on $\mathcal{M}_n$. The map $a$ is degree-preserving so that $|Z| = k$. Since the resulting object is a superfield in the BV sense, it contains associated gauge fields, ghosts and antifields as component fields. The physical component is the ghost number 0 component. In general, the ghost number of a field $\Psi$ is defined by degree minus form degree, $\text{gh}(\Psi) = |\Psi| - \text{deg}(\Psi)$, where form degree $(0, 1)$ is assigned to $(\sigma^\mu, \theta^\mu)$.

By degree counting, $Z_{\mu_1 \cdots \mu_j}^{(j)}$ has ghost number $(k - j)$, $Z_{\mu_1 \cdots \mu_k}^{(k)}$ has ghost number 0 and $Z_{\mu_1 \cdots \mu_{k-1}}^{(k-1)}$ has ghost number 1. The ghost number 0 component is a physical $k$-form gauge field and the ghost number 1 component its FP ghost, i.e., the gauge parameter of the associated gauge transformation.

The super field strength and its physical component corresponding to a coordinate $z$ on $\mathcal{M}$ are defined by

$$F_Z = d \circ a^*(z) - a^* \circ Q(z),$$

(2.6)

$$F_z = (d \circ a^*(z) - a^* \circ Q(z))|_{|z|+1}.$$  

(2.7)

Here, $d = \theta^\mu \partial_\mu$ denotes the superderivative and $|z|+1$ denotes projection to the degree $|z| + 1$ part, while setting all antifield components to zero. We get the super Bianchi identity for free,

$$(d \circ a^* - a^* \circ Q)^2 = 0 \Rightarrow d F_z = -F \circ Q(z).$$

(2.8)

The associated gauge transformation is encoded in the super field strength as the degree $|z|$ part

$$\delta Z = (d \circ a^*(z) - a^* \circ Q(z))|_{|z|}.$$  

(2.9)

Again, while projecting, we set all antifields to zero.

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8This formula gives a BRST transformation.
To extract the physical field directly, we define the map $\tilde{a} : T[1]\Sigma \to \mathcal{M}_n$ by

$$\tilde{a}^*(z) = \frac{1}{k!} d\sigma^{\mu_1} \wedge \cdots \wedge d\sigma^{\mu_k} Z^{(k)}_{\mu_1 \cdots \mu_k}(\sigma),$$  \hspace{1cm} (2.10)$$

where $z$ is a coordinate of degree $k$ on $\mathcal{M}_n$. Note that we have identified $\theta^{\mu}$ with $d\sigma^{\mu}$. Using this map, we can rewrite the physical field strength by

$$F_z = F(z) = d \circ \tilde{a}^*(z) - \tilde{a}^* \circ Q(z).$$  \hspace{1cm} (2.11)$$

For the QP-manifold $\mathcal{M}_n$ under consideration, we get a scalar field associated to the degree 0 coordinate $x^i$ and a 1-form gauge field associated to the degree 1 coordinate $q^a$, and associated field strengths,

$$\tilde{a}^*(x^i) \equiv X^i(\sigma),$$  \hspace{1cm} (2.12)$$

$$\tilde{a}^*(q^a) \equiv A^a(\sigma) = A^a d\sigma^\mu,$$  \hspace{1cm} (2.13)$$

$$F^i_X = d\tilde{a}^*(x^i) - \tilde{a}^*(Q x^i),$$  \hspace{1cm} (2.14)$$

$$F^a_A = d\tilde{a}^*(q^a) - \tilde{a}^*(Q q^a).$$  \hspace{1cm} (2.15)$$

In addition to that, we find $(n-1)$- and $n$-form auxiliary gauge fields $C_a$ and $\Xi_i$ associated with the conjugate coordinates on our graded symplectic manifold,

$$\tilde{a}^*(x^i) = X^i, \quad \tilde{a}^*(q^a) = A^a,$$  \hspace{1cm} (2.16)$$

$$\tilde{a}^*(\xi_i) = \Xi_i, \quad \tilde{a}^*(p_a) = C_a.$$  \hspace{1cm} (2.17)$$

In this very scenario, we have gauge transformations with three independent gauge parameters corresponding to the fields $A^a$, $\Xi_i$ and $C_a$:

$$a^*(q^a)|_{\text{deg}=0} = e^a, \quad a^*(\xi_i)|_{\text{deg}=n-1} = \mu^i, \quad \tilde{a}^*(p_a)|_{\text{deg}=n-2} = \epsilon^a.$$  \hspace{1cm} (2.18)$$

However, in general, the field strength $F_z$ of a gauge field $\tilde{Z} = \tilde{a}^*(z)$ is transformed adjointly, $\delta F_z \sim F_z$, only on-shell since the above procedure is derived from the theory of AKSZ sigma models \[11]. An action of the AKSZ sigma models is a topological field theory of BF type and the equation of motion is $F_z = 0$. If $F_z$ transforms adjointly without use of the equations of motion, we call $F_z$ off-shell covariant. If $F_z$ is off-shell covariant, the construction of a gauge invariant Yang-Mills type action $S \sim F_z \wedge * F_z$ is possible.
The procedure to obtain off-shell covariant field strengths is as follows. First, we drop the auxiliary gauge fields \((\Xi_i, C_a)\) and extra gauge degrees of freedom \(\mu_i'\) and \(\epsilon_a'\) by the projection \(\Xi_i = C_a = 0\). Then, we deform the field strengths \(F_z\) and gauge symmetries \(\delta\) by adding deformation terms proportional to lower field strengths. Note that the algebra of the structure constants (or functions) is not deformed. Choosing proper coefficients for the deformation leads to off-shell covariantized field strengths without changing the original gauge symmetry algebra.

3 Hamiltonian functions

A Hamiltonian function \(\Theta\) on a general QP-manifold \(\mathcal{M}_n\) of degree \(n\) is of degree \(n + 1\). In this section, we examine the most general Hamiltonian function on \(\mathcal{M}_n\) by expanding it in conjugate coordinates \((\xi_i, p_a)\),

\[
\Theta = \sum_k \Theta^{(k)},
\]

where \(\Theta^{(k)}\) is a \(k\)-th order function in \((\xi_i, p_a)\).

The following cases occur.

A) \(n \geq 4\): Since the degrees of \((\xi_i, p_a)\) are \((n, n - 1)\), the degree of \(\Theta^{(k)}\) for \(k \geq 2\) is larger than \(2n - 2\). Therefore, if \(n \geq 4\), then \(\Theta^{(k)} = 0\) for \(k \geq 2\) by degree counting, i.e. the general form of the Hamiltonian function is

\[
\Theta = \Theta^{(0)} + \Theta^{(1)}.
\]

B) \(n = 3\): In this case, \(\Theta^{(k)} = 0\) for \(k \geq 3\) by degree counting. Therefore, the expansion stops at second order,

\[
\Theta = \Theta^{(0)} + \Theta^{(1)} + \Theta^{(2)}.
\]

this QP-manifold defines a Lie 3-algebroid structure on \(E\), see \([21]\). Only for \(n \leq 3\) the Hamiltonian \(\Theta\) provides freedom for deformations. We discuss case \(n = 3\) in detail in section 4.
C)  $n = 1, 2$: The Hamiltonian $\Theta$ contains more deformation terms. In the $n = 2$ case, since $(x^i, q^a, \xi_i, p_a)$ is of degree $(0, 1, 2, 1)$, the graded manifold is $\mathcal{M}_2 = T^*[2]E[1]$, and

$$\Theta = \Theta^{(0)} + \Theta^{(1)} + \Theta^{(2)} + \Theta^{(3)}.$$  

(3.22)

Then, this defines a Courant algebroid on $E$.  

For $n = 1$, $\Theta$ defines a Poisson structure on $E$.

### 3.1 Gauge fields and field strengths induced from Hamiltonian functions

First, the Hamiltonian function $\Theta^{(1)}$ reproduces a Lie algebroid for general $n$. It contains the following terms,

$$\Theta^{(1)} = \rho^i_a(x)\xi^a + \frac{1}{2} f^c_{ab}(x)q^aq^bp_c,$$  

(3.23)

where $\rho^i_a(x), f^c_{ab}(x)$ are structure functions depending on $x$.

Lie algebroid operations are given by the following derived brackets,

$$[e_1, e_2] = -\{\{e_1, \Theta^{(1)}\}, e_2\},$$

(3.24)

$$\rho(e)f = \{\{e, \Theta^{(1)}\}, f\},$$

(3.25)

where $e, e_1, e_2 \in \Gamma(E)$ are sections of a Lie algebroid which is locally expressed by $e = e^a(x)p_a$ and $f \in C^\infty(M)$.

For details and notation, see appendix A.

The classical master equation, $\{\Theta^{(1)}, \Theta^{(1)}\} = 0$, implies the following conditions on the structure constants,

$$\rho^j_a \frac{\partial \rho^i_b}{\partial x^j} - \rho^j_b \frac{\partial \rho^i_a}{\partial x^j} + \rho^i_c f^c_{ab} = 0,$$  

(3.26)

$$\rho^j_a \frac{\partial f^d_{bc}}{\partial x^j} + f^e_d \epsilon_{[a} f^e_{bc]} = 0.$$  

(3.27)

The pullback $a^*$ maps the four coordinates to superfields as follows,

$$X^i \equiv a^*(x^i), \quad A^a \equiv a^*(q^a),$$

(3.28)

$$\Xi_i \equiv a^*(\xi_i), \quad C_a \equiv a^*(p_a).$$

(3.29)
The super field strengths are given by $F_z = da^*(z) - a^*Q(z)$.

\[ F^i_X = dx^i - \rho^i_a(X)A^a, \]  
\[ F^a_A = dA^a + \frac{1}{2} f^a_{bc}(X) A^b A^c, \]  
\[ F^{(C)}_a = dC_a + f^{b}_{ac}(X) C_b A^c - \rho^i_a(X) \Xi_i, \]  
\[ F^{(\Xi)}_i = d\Xi_i - \frac{1}{2} \frac{\partial f^a_{bc}(X)}{\partial x^i} (X) C_a A^b A^c - \frac{\partial \rho^i_a(X)}{\partial x^i} (X) \Xi_j A^a, \]

where $F^{(C)}$ and $F^{(\Xi)}$ are the super field strengths of $C$ and $\Xi$, respectively. When we substitute the component expansions to (3.30)–(3.33), then the corresponding degree $|z| + 1$ parts are the field strengths:

\[ F^i_X = dx^i - \rho^i_a(X)A^a, \]  
\[ F^a_A = dA^a + \frac{1}{2} f^a_{bc}(X) A^b A^c, \]  
\[ F^{(C)}_a = dC_a + f^{b}_{ac}(X) C_b A^c - \rho^i_a(X) \Xi_i, \]  
\[ F^{(\Xi)}_i = d\Xi_i - \frac{1}{2} \frac{\partial f^a_{bc}(X)}{\partial x^i} (X) C_a A^b A^c - \frac{\partial \rho^i_a(X)}{\partial x^i} (X) \Xi_j A^a. \]

The degree $|z|$ parts of the component expansions of the super field strengths yield the gauge transformations,

\[ \delta X^i = -\rho^i_a(X)\epsilon^a, \]  
\[ \delta A^a = d\epsilon^a + f^a_{bc}(X) A^b \epsilon^c, \]  
\[ \delta C_a = d\epsilon'_a + f^{bc}_{ac}(X) (\epsilon'_b \wedge A^c + C_b \wedge \epsilon^c) - \rho^i_a(X) \mu'_i, \]  
\[ \delta \Xi_i = d\mu'_i - \frac{1}{2} \frac{\partial f^a_{bc}(X)}{\partial x^i} (X) (\epsilon'_a A^b A^c + 2 C_a A^b \epsilon^c) - \frac{\partial \rho^i_a(X)}{\partial x^i} (X) (\mu'_j A^a + \Xi_j \epsilon^a). \]

The gauge transformations of the gauge field strengths are

\[ \delta F^i_X = \frac{\partial \rho^i_a}{\partial X^j} F^j_X \epsilon^a, \]  
\[ \delta F^a_A = - f^a_{bc} F^b_A \epsilon^c - \frac{\partial f^a_{bc}}{\partial X^j} F^j_X A^b \epsilon^c. \]

In general, $F^a_A$ is on-shell covariant.
4 Off-shell covariantization of 4d algebroid 1-form gauge theories

In the previous sections, we discussed the structure of the Hamiltonian and canonical transformations for general $n$. To make the discussion concrete, we take a field theory for the specific case $n = 3$, i.e. $\mathcal{M}_3 = T^*[3]E[1]$. In this case, $\Theta^{(2)}$ can be included in the Hamiltonian function and we obtain interesting nontrivial examples.

First, we describe the structure of the Hamiltonians based on $\mathcal{M}_3$. Local coordinates are $(x^i, q^a, \xi_i, p_a)$ of degree $(0, 1, 3, 2)$, respectively. Since $\Theta$ is of degree 4, the Hamiltonian function is at most a second order function in $(\xi_i, p_a)$, by degree counting, and can be expanded as $\Theta = \Theta^{(0)} + \Theta^{(1)} + \Theta^{(2)}$. Therefore, the concrete expressions are

$$\Theta^{(0)} = \frac{1}{4!} h_{abcd}(x) q^a q^b q^c q^d,$$

$$\Theta^{(1)} = \frac{1}{2} f_{ab}^c(x) q^a q^b p_c + \rho^i a(x) \xi_i q^a,$$

$$\Theta^{(2)} = \frac{1}{2} k_{ab}(x) p_a p_b,$$

with additional structure functions $h_{abcd}(x)$, $f_{ab}^c(x)$, $\rho^i a(x)$ and $k_{ab}(x)$.

From the classical master equation, $\{\Theta, \Theta\} = 0$, we obtain the following identities,

$$\rho^i_b k^{ba} = 0,$$

$$\rho^k_c \frac{\partial k_{ab}}{\partial x^k} + k_{da} f_{c}^{ba} + k_{db} f_{a}^{cd} = 0,$$

$$\rho^k_b \frac{\partial \rho^i_a}{\partial x^k} - \rho^k_a \frac{\partial \rho^i_b}{\partial x^k} + \rho^i_c f_{c}^{ab} = 0,$$

$$2\rho^k_{[a} \frac{\partial f_{bc]}^{a}}{\partial x^k} + k^{ae} h_{be}^{cd} - 2 f_{e[b}^{a} f^{e} c d] = 0,$$

$$2\rho^k_{[a} \frac{\partial h_{bcde]}^{a}}{\partial x^k} + \rho^f_{[ab} h_{cde]} = 0,$$

which define a Lie 3-algebroid [21].

Based on the general theory that we explained in the beginning, we consider the restriction of the 4-dimensional theory. The pullback $a^*$ maps the four coordinates to superfields as follows,

$$X^i \equiv a^*(x^i), \quad A^a \equiv a^*(q^a),$$

$$\Xi_i \equiv a^*(\xi_i), \quad C_a \equiv a^*(p_a),$$

with additional structure functions $h_{abcd}(x)$, $f_{ab}^c(x)$, $\rho^i a(x)$ and $k_{ab}(x)$.
where \((x, A, \Xi, C)\) are of degree \((0, 1, 3, 2)\). The super field strengths are given by

\[
F^i_X = dx^i - \rho^i_a(X)A^a, \tag{4.54}
\]
\[
F^a_A = dA^a + \frac{1}{2}f^a_{bc}(X)A^bA^c + k^{ab}(X)C_b, \tag{4.55}
\]
\[
F^{(C)}_a = dC_a + f^{bc}_a(X)C_b \wedge A^c - \rho^i_a(X)\Xi_i + \frac{1}{3!}h_{abcd}(X)A^bA^cA^d, \tag{4.56}
\]
\[
F^{(\Xi)}_i = d\Xi_i - \frac{1}{2}\partial f^a_{bc}(X)C_aA^b \wedge A^c - \partial\rho^a_i(X)\Xi_i \wedge A^a - \frac{1}{2}\partial k^{ab}_i(X)C_a \wedge C_b + \frac{1}{4!}\partial h_{abcd}(X)A^a \wedge A^b \wedge A^c \wedge A^d. \tag{4.57}
\]

where \(F^{(C)}\) and \(F^{(\Xi)}\) are the super field strengths of \(C\) and \(\Xi\), respectively. When we substitute the component expansions to \(4.54\) - \(4.57\), then the corresponding degree \(|z| + 1\) parts are the field strengths:

\[
\delta X^i = -\rho^i_a(X)e^a, \tag{4.62}
\]
\[
\delta A^a = de^a + f^a_{bc}(X)A^b \wedge A^c + k^{ab}(X)e^b, \tag{4.63}
\]
\[
\delta C_a = d\epsilon^a_a + f^b_{ac}(X)e^b \wedge A^c + C_b \wedge e^c - \rho^i_a(X)\epsilon^i_a + \frac{1}{2}h_{abcd}(X)A^b \wedge A^c \wedge A^d, \tag{4.64}
\]
\[
\delta \Xi_i = d\mu^i_a + \frac{1}{2}\partial f^a_{bc}(X)(\epsilon^a_a \wedge A^b \wedge A^c + 2C_a \wedge A^b \wedge A^c - \partial\rho^a_i(X)(\mu^i_a \wedge A^a + \Xi_i \wedge e^a - \frac{1}{3!}\partial h_{abcd}(X)A^a \wedge A^b \wedge A^c \wedge A^d. \tag{4.65}
\]

The gauge transformations of the field strengths are

\[
\delta F^i_X = \frac{\partial \rho^i_a}{\partial X^j}F^j_X e^a, \tag{4.66}
\]
\[
\delta F^a_A = -f^a_{bc}F^b_X e^c - \frac{\partial k^{ab}}{\partial X^j}F^j_X \wedge e^a - \frac{\partial f^a_{bc}}{\partial X^j}F^j_X \wedge A^b e^c. \tag{4.67}
\]
One recognizes from (4.67), that $F_A^a$ does not transform off-shell covariantly unless $k^{ab}(X)$ and $f^a_{bc}(X)$ are constants.

We seek nontrivial deformations of gauge transformations and field strengths, that lead to off-shell covariant gauge structures. This is done by adding terms to the field strengths and gauge transformations using the fundamental fields and lower form field strengths. Before introducing deformation terms, the auxiliary gauge fields are projected out by imposing $\Xi_i = C_a = 0$.

By form degree counting, we assume the following structure of deformations of the field strengths in terms of $X^i$ and $A^a$,

$$\hat{F}_X^i = F_X^i = dX^i - \rho^i_a(X)A^a, \quad (4.68)$$

$$\hat{F}_A^a = F_A^a|_{C_a = 0} + K^a_{cj}(X)F_X^j \wedge A^c + L^a_{ij}(X)F_X^i \wedge F_X^j$$

$$= dA^a + \frac{1}{2} f^a_{bc}(X)A^b \wedge A^c + K^a_{cj}(X)F_X^j \wedge A^c + L^a_{ij}(X)F_X^i \wedge F_X^j, \quad (4.69)$$

where $K^a_{cj}(X)$ and $L^a_{ij}(X)$ are functions. The gauge transformations of $(X^i, A^a)$ should be of the following form,

$$\hat{\delta}X^i = \delta X^i = -\rho^i_a(X)\epsilon^a, \quad (4.70)$$

$$\hat{\delta}A^a = \delta A^a + N^a_{ci}(X)F_X^i \epsilon^c$$

$$= d\epsilon^a + f^a_{bc}(X)A^b \epsilon^c + N^a_{ci}(X)F_X^i \epsilon^c, \quad (4.71)$$

where $N^a_{ci}(X)$ is a function.

Let us compute the gauge transformations of $(4.68)$ and $(4.69)$ using $(4.70)$ and $(4.71)$. Employing the Bianchi identities derived from (2.8),

$$dF_X^i = \frac{\partial \rho^i_a}{\partial X^j} F_X^j A^a + \rho^i_a F_A^a, \quad (4.72)$$

we can compute $\hat{\delta}F_A^a$. The requirement that the coefficients of $F_X^i d\epsilon^b$, $F_X^i A^a \epsilon^b$, $F_X^i \wedge A^a \epsilon^b$ and $F_X^i \wedge F_X^j \epsilon^b$ in $\hat{\delta}F_A^a$ vanish gives relations among $K$, $L$ and $N$,

$$N_{bi}^a = K_{bi}^a, \quad (4.73)$$

$$\frac{\partial f^a_{bc}}{\partial x^a} + f^a_{db} K_{ci}^d + K_{di}^a f^d_{bc} - \frac{\partial N_{ci}^a}{\partial x^j} \rho^j_b \rho^i_c + \frac{\partial K_{bi}^a}{\partial x^j} \rho^j_c + K_{aj}^a \frac{\partial \rho^j_c}{\partial x^j} = f^a_{dc} K_{bi}^d; \quad (4.74)$$

$$\frac{1}{2} \frac{\partial N_{cj}^a}{\partial x^i} + \frac{1}{2} K_{bi}^a N_{cj}^b + L_{ki}^a \frac{\partial \rho^j_c}{\partial x^j} - (i \leftrightarrow j) = f^a_{bc} L_{ij}^b - \frac{\partial L_{ij}^a}{\partial x^k} \rho^k_c. \quad (4.75)$$
Under these conditions the field strength is off-shell covariant,

\[ \hat{\delta} \hat{F}_A^a = -(f_{bc}^a + N_{ci}^a \rho_b^i) \hat{F}_A^{b} \epsilon^c. \]  

(4.76)

4.1 Examples

4.1.1 Stückelberg formalism

First, we consider a trivial example to show that this formalism is a generalization of a known formalism. The starting point is the QP-manifold \( \mathcal{M}_3 = T^*[3]E[1] \), where \( E = TM \) is a tangent bundle. We take

\[
\begin{align*}
    f_{bc}^a &= k^{ab}(x) = h_{abcd} = 0, \\
    \rho^i_a &= m \delta^i_a = \text{constant},
\end{align*}
\]

where \( i \) and \( a \) run over the same index range. Then, the Hamiltonian function is

\[
\Theta = m \xi_a q^a, \tag{4.77}
\]

which trivially satisfies the classical master equation, \( \{\Theta, \Theta\} = 0 \). The resulting field strengths are

\[
\begin{align*}
    F_X^a &= dX^a - mA^a, \tag{4.78} \\
    F_A^a &= dA^a, \tag{4.79} \\
    F_{a}^{(C)} &= dC_a + m \Xi_a, \tag{4.80} \\
    F_{a}^{(\Xi)} &= d\Xi_a. \tag{4.81}
\end{align*}
\]

The gauge transformations of the gauge fields are

\[
\begin{align*}
    \delta X^a &= -m e^a, \tag{4.82} \\
    \delta A^a &= de^a, \tag{4.83} \\
    \delta C_a &= d\epsilon'_a + m \mu'_a, \tag{4.84} \\
    \delta \Xi_a &= d\mu'_a. \tag{4.85}
\end{align*}
\]
From these equations, the gauge transformations of the field strengths are trivially covariant,

\[ \delta F_X^a = 0, \]  
\[ \delta F_A^a = 0. \]  
\[ (4.86) \]
\[ (4.87) \]

The gauge invariant action,

\[ S = \int \text{tr}(F_A \wedge *F_A) + \text{tr}(F_X \wedge *F_X) \]
\[ = \int F_{A\mu\nu}F_A^{\mu\nu} + (\partial_\mu X^a - mA_\mu^a)(\partial^\mu X^a - mA^\mu_a). \]  
\[ (4.88) \]

is the so-called Stückelberg formalism of the massive vector field $A_\mu^a$. We conclude that our formalism provides a nonlinear generalization of the Stückelberg formalism.

4.1.2 Nonabelian gauged nonlinear sigma models

We list a simple but nontrivial example, taking again $\mathcal{M}_3$ as a starting point. Let the structure constants be

\[ f_{bc}^a = \text{constant}, \quad \rho_a^i = h_{abcd} = 0, \quad k_{ab}(x) = \text{arbitrary}. \]

Then, the Hamiltonian function is

\[ \Theta = \frac{1}{2} f_{bc}^a q^b q^c p_a + \frac{1}{2} k_{ab}(x) p_a p_b. \]  
\[ (4.89) \]

The resulting field strengths are

\[ F_X^i = dX^i, \]  
\[ (4.90) \]
\[ F_A^a = dA^a + \frac{1}{2} f_{bc}^a A_b^c + k_{ab} C_b, \]  
\[ (4.91) \]
\[ F_a^{(C)} = dC_a + f_{ac}^b A^c \wedge C_b, \]  
\[ (4.92) \]
\[ F_i^{(\Xi)} = d\Xi_i - \frac{1}{2} \frac{\partial k_{ab}}{\partial X^i}(X) C_a \wedge C_b. \]  
\[ (4.93) \]

The gauge transformations of the gauge fields are

\[ \delta X^i = 0, \]  
\[ (4.94) \]
\[ \delta A^a = d\epsilon^a + f_{bc}^a A^b \epsilon^c + k_{ab}(X) \epsilon'_b, \]  
\[ (4.95) \]
\[ \delta C_a = d\epsilon'_a + f_{ac}^b (A^c \wedge \epsilon'_b + \epsilon C_b), \]  
\[ (4.96) \]
\[ \delta \Xi_i = d\mu'_i - \frac{\partial k_{ab}}{\partial X^i}(X) C_a \wedge \epsilon'_b. \]  
\[ (4.97) \]
Using these equations, we compute the gauge transformations of the field strengths as

\[ \delta F^i_X = 0, \]  
\[ \delta F^a_A = - f^a_{bc} F^b_A \epsilon^c - \frac{\partial k_{ab}}{\partial X^i} F^i_X \wedge \epsilon^b, \]  

(4.98) \hspace{1cm} (4.99)

The gauge transformation of \( F^a_A \) is not off-shell covariant.

Let us apply our formalism to this system. A solution of (4.73)–(4.75) in this example is

\[ K^a_{bi} = N^a_{bi} = \delta_a^b \frac{\partial w}{\partial x^i}(x), \quad L^a_{ij} = 0, \]  

(4.100)

where \( w(x) \) is an arbitrary function. The covariantized field strengths and gauge transformations are computed as

\[ \hat{F}^i_X = F^i_X = dX^i, \]  
\[ \hat{F}^a_A = dA^a + \frac{1}{2} f^a_{bc} A^b A^c + \frac{\partial w}{\partial X^i} F^i_X \wedge A^a, \]  
\[ \hat{\delta} A^a = d\hat{\epsilon}^a + f^a_{bc} \hat{A}^b \hat{\epsilon}^c + \frac{\partial w}{\partial X^i} F^i_X \hat{\epsilon}^a, \]  
\[ \hat{\delta} X^i = 0. \]  

(4.101) \hspace{1cm} (4.102) \hspace{1cm} (4.103) \hspace{1cm} (4.104)

Finally, we obtain

\[ \hat{\delta} F^i_X = 0, \]  
\[ \hat{\delta} F^a_A = - f^a_{bc} \hat{F}^b_A \hat{\epsilon}^c. \]  

(4.105) \hspace{1cm} (4.106)

Assume that \( M \) is 1-dimensional. Then, we drop the index \( i \) and take

\[ K^a_b = N^a_b = \frac{\delta_a^b}{x}, \]  

(4.107)

which yields

\[ F_X = dX, \]  
\[ \hat{F}^a_A = dA^a + \frac{1}{2} f^a_{bc} A^b A^c + \frac{1}{X} F_X \wedge A^a, \]  
\[ \hat{\delta} A^a = d\hat{\epsilon}^a + f^a_{bc} \hat{A}^b \hat{\epsilon}^c + \frac{1}{X} F_X \hat{\epsilon}^a, \]  
\[ \hat{\delta} X = 0. \]  

(4.108) \hspace{1cm} (4.109) \hspace{1cm} (4.110) \hspace{1cm} (4.111)
By the redefinition of $x$ via

$$\varphi = \log |X|,$$

(4.112)

the equations can be rewritten in a nonsingular form,

$$F_x = e^\varphi d\varphi,$$

(4.113)

$$\hat{F}_A = dA^a + \frac{1}{2} f^a_{bc} A^b A^c + d\varphi \wedge A^a,$$

(4.114)

$$\delta A^a = d\epsilon^a + f^a_{bc} \hat{\epsilon}^c + d\varphi \epsilon^a,$$

(4.115)

$$\hat{\delta} \varphi = 0.$$

(4.116)

### 4.1.3 Kotov-Strobl model

As third example we formulate the model proposed in [4].

Here, we consider a QP-manifold of degree two, $\mathcal{M}_2 = T^*[2]E[1]$, in order to demonstrate the covariantization procedure for the Kotov-Strobl model. Note that the resulting gauge theory is not restricted to any dimension. The local coordinates of $\mathcal{M}_2$ are denoted by $(x^i, \xi_i, q^a)$ of degree $(0, 2, 1)$. The fiber coordinates of $E[1]$ and $E^*[1]$ are identified by introducing a fiber metric $\lambda_{ab}$. The graded symplectic form is defined by

$$\omega = \delta x^i \wedge \delta \xi_i + \frac{1}{2} \lambda_{ab}(x) \delta q^a \wedge \delta q^b.$$

(4.117)

The most general form of the Hamiltonian is given by

$$\Theta = \rho^i_a(x) \xi_i q^a + \frac{1}{3!} h_{abc}(x) q^a q^b q^c.$$

(4.118)

In order to construct the Kotov-Strobl model, we take $M$ be a 2-dimensional manifold and $E$ a vector bundle over $M$ with 1-dimensional fiber. Let us denote the local coordinates of $\mathcal{M}_2$ by $(x, y) := (x^1, x^2)$, $(\xi, \eta) := (\xi_1, \xi_2)$ and $q := q^1$ and take the following Hamiltonian function,

$$\Theta = -e^{-\frac{1}{2} x y} \eta q.$$

(4.119)

---

9We can use the fiber coordinates $(q^a, p_a)$ of $E[1]$ and $E^*[1]$. 
where $\lambda$ is a constant. That corresponds to choosing

$$h_{abc} = 0, \quad \lambda_{11} = 1,$$

$$\rho^1 = 0, \quad \rho^2 = e^{-\frac{1}{2}xy}.$$

The associated superfields are defined as

$$x \equiv a^*(x), \quad y \equiv a^*(y),$$

$$A \equiv a^*(q),$$

$$\Xi \equiv a^*(\xi), \quad H \equiv a^*(\eta).$$

Using the formulas (2.11) and (2.9), we obtain the following field strengths,

$$F_X = dX,$$  \hspace{1cm} (4.120)

$$F_Y = dY - e^{-\frac{1}{2}XY} A,$$  \hspace{1cm} (4.121)

$$F_A = dA + e^{-\frac{1}{2}XY} H,$$  \hspace{1cm} (4.122)

$$F_\Xi = d\Xi - \frac{\lambda}{2} Ye^{-\frac{1}{2}XY} HA,$$  \hspace{1cm} (4.123)

$$F_H = dH - \frac{\lambda}{2} X e^{-\frac{1}{2}XY} HA,$$  \hspace{1cm} (4.124)

and gauge transformations of the gauge fields,

$$\delta X = 0,$$  \hspace{1cm} (4.125)

$$\delta Y = -e^{-\frac{1}{2}XY} \epsilon,$$  \hspace{1cm} (4.126)

$$\delta A = d\epsilon + e^{-\frac{1}{2}XY} \mu'_2,$$  \hspace{1cm} (4.127)

$$\delta \Xi = d\mu'_1 - \frac{\lambda}{2} Ye^{-\frac{1}{2}XY} (\mu'_2 A + H \epsilon),$$  \hspace{1cm} (4.128)

$$\delta H = d\mu'_2 - \frac{\lambda}{2} X e^{-\frac{1}{2}XY} (\mu'_2 A + H \epsilon).$$  \hspace{1cm} (4.129)

Here, $\epsilon$ is the 0-form gauge parameter corresponding to $A$, and $\mu'_1$ and $\mu'_2$ are the 1-form gauge parameters corresponding to $\Xi$ and $H$, respectively. We are only interested in the gauge transformations and field strengths of the fields $(X, Y, A)$. The gauge transformations
of the field strengths are computed as
\[
\delta F_X = 0, \quad \delta F_Y = \frac{\lambda}{2} Ye^{-\frac{\lambda}{2}XY} F_X e - \frac{\lambda}{2} X e^{-\frac{\lambda}{2}XY} F_Y e - e^{-\lambda XY} \mu_2', \quad (4.131)
\]
\[
\delta F^a_A = \frac{\lambda}{2} Y e^{-\frac{\lambda}{2}XY} F_X \mu_2' + \frac{\lambda}{2} X e^{-\frac{\lambda}{2}XY} F_Y \mu_2'. \quad (4.132)
\]
The gauge transformations of \( F_Y \) and \( F_A \) are not off-shell covariant.

We apply the off-shell covariantization procedure to this theory. The possible deformations of the field strengths and gauge transformations are
\[
\hat{F}_A = F_A + J(X, Y) F_X \wedge A + K(X, Y) F_Y \wedge A + L(X, Y) F_X \wedge F_Y, \quad (4.133)
\]
\[
\hat{A} = d\hat{\epsilon} + M(X, Y) F_X \wedge \hat{\epsilon} + N(X, Y) F_Y \wedge \hat{\epsilon}, \quad (4.134)
\]
where we determine the functions \( J, K, L, M, N \) of the scalar fields \( X, Y \). Deformations of the other field strengths and gauge transformations need not to be considered.

One solution is \( M = -\frac{\lambda}{2} Y \) and \( N = 0 \). In this case, \( \hat{\delta} F_Y \) is covariantized as
\[
\hat{\delta} F_Y = -\frac{\lambda}{2} e^{-\frac{\lambda}{2}XY} X F_Y \hat{\epsilon}. \quad (4.135)
\]
In the next step, we require off-shell covariance of \( \hat{\delta} \hat{F}_A \). This determines \( J = -\frac{\lambda}{2} Y \), \( K = 0 \) and \( L = -\frac{\lambda}{2} Y e^{\frac{\lambda}{2}XY} \). The resulting field strengths and gauge transformations are
\[
\hat{F}_A = dA - \frac{\lambda}{2} Y F_X \wedge A - \frac{\lambda}{2} Y e^{\frac{\lambda}{2}XY} F_X \wedge F_Y
\]
\[
= dA - \frac{\lambda}{2} Y e^{\frac{\lambda}{2}XY} dX \wedge dY, \quad (4.136)
\]
\[
\hat{A} = d\hat{\epsilon} - \frac{\lambda}{2} Y dX \hat{\epsilon}. \quad (4.137)
\]
The gauge transformation of \( \hat{F}_A \) is computed as
\[
\hat{\delta} \hat{F}_A = 0, \quad (4.138)
\]
which is off-shell covariant.

**Invariant Action** Since the scalar field strength \( F_X^i = dX^i - \rho^i_a(X) A^a \) transforms off-shell covariantly,
\[
\hat{\delta} F_X^i = \frac{\partial \rho^i_a}{\partial X^j} F_X^j \hat{\epsilon}^a, \quad (4.139)
\]
the action

\[ \int g_{ij}(X) F^i_X \wedge \ast F^j_X \]  

(4.140)
is invariant if \( g_{ij}(X) \) satisfies

\[ \hat{\delta} g_{ij}(X) = - \left( g_{kj} \frac{\partial \rho^k_a}{\partial X^i} + g_{ik} \frac{\partial \rho^k_a}{\partial X^j} \right) \epsilon^a. \]  

(4.141)

In this example, the action is given by

\[ S = \int F_X \wedge \ast F_X + V(X) + e^{\lambda XY} F_Y \wedge \ast F_Y + \hat{F}_A \wedge \ast \hat{F}_A \]  

(4.142)

and is invariant under gauge transformations. The gauge transformation of the third term is given by

\[ \hat{\delta} (e^{\lambda XY} F_Y \wedge \ast F_Y) = 2e^{\lambda XY} (\hat{\delta} F_Y) \wedge \ast F_Y - (\hat{\delta} e^{\lambda XY}) F_Y \wedge \ast F_Y \]

\[ = e^{\lambda XY} \lambda X \epsilon F_Y \wedge \ast F_Y - e^{\lambda XY} \lambda X \epsilon F_Y \wedge \ast F_Y = 0. \]  

(4.143)

**Correspondence to the Kotov-Strobl model**  
By the off-shell covariantization procedure, we obtain the field strengths

\[ F_X = dX, \]  

(4.144)

\[ F_Y = dY - e^{-\frac{\lambda}{2} XY} A, \]  

(4.145)

\[ \hat{F}_A = dA + \frac{\lambda}{2} e^{\frac{\lambda}{2} XY} Y dY \wedge dX, \]  

(4.146)

and the gauge transformations of the gauge fields

\[ \hat{\delta} X = 0, \]  

(4.147)

\[ \hat{\delta} Y = -e^{-\frac{\lambda}{2} XY} \hat{\epsilon}, \]  

(4.148)

\[ \hat{\delta} A = d\hat{\epsilon} - \frac{\lambda}{2} dXY \hat{\epsilon}. \]  

(4.149)

From these equations, the gauge transformations of the field strengths are computed as

\[ \hat{\delta} F_X = 0, \]  

(4.150)

\[ \hat{\delta} F_Y = -\frac{\lambda}{2} e^{-\frac{\lambda}{2} XY} X F_Y \hat{\epsilon}, \]  

(4.151)

\[ \hat{\delta} F_A = 0. \]  

(4.152)
Redefining the gauge field $A$, the gauge parameter $\epsilon$ and the field strength $F_A$ as

\[ \tilde{\epsilon} \equiv e^{-\frac{1}{2}XY} \epsilon, \]  
\[ \tilde{A} \equiv e^{-\frac{1}{2}XY} A, \]  
\[ G_A \equiv e^{-\frac{1}{2}XY} F_A, \]  

we obtain the following field strengths from eqs. (4.144)–(4.146),

\[ F_X = dX, \]  
\[ F_Y = dY - \tilde{A}, \]  
\[ G_A = d\tilde{A} + \frac{\lambda}{2} (XdY \wedge \tilde{A} + YF_Y \wedge dX). \]  

These are the field strengths discussed in [4]. We can rewrite the gauge transformations of the gauge fields using $\tilde{\epsilon}$ by

\[ \tilde{\delta}X = 0, \]  
\[ \tilde{\delta}Y = -\tilde{\epsilon}, \]  
\[ \tilde{\delta}\tilde{A} = d\tilde{\epsilon} - \frac{\lambda}{2} XF_Y \tilde{\epsilon}. \]  

Then, the gauge transformations of the field strengths are given by

\[ \tilde{\delta}F_X = 0, \]  
\[ \tilde{\delta}F_Y = \frac{\lambda}{2} XF_Y \tilde{\epsilon}, \]  
\[ \tilde{\delta}G_A = \frac{\lambda}{2} XG_A \tilde{\epsilon} - \frac{\lambda}{2} XF_Y \wedge F_X \tilde{\epsilon} + \left( \frac{\lambda}{2} \right)^2 X(1 - Y) F_Y \wedge F_X \tilde{\epsilon}, \]  

which are the same expressions as in [4].

### 5 Gauge algebras

Finally, we discuss the closure of the gauge symmetry algebra. For this, we write the gauge transformations as

\[ \tilde{\delta}X^i = -\rho^i_a (X) \tilde{\epsilon}^a, \]  
\[ \tilde{\delta}A^a = d\tilde{\epsilon}^a + f^a_{bc} A^b \tilde{\epsilon}^c + N^a_{ci} F^{ci}_X \tilde{\epsilon}^c, \]  

\[ 19 \]
where the gauge parameter \( \tilde{\epsilon}^a \) is an ordinary function. We find, that two gauge transformations \( \tilde{\delta}_1 \) and \( \tilde{\delta}_2 \) close to \( \tilde{\delta}_3 \) by \( [\tilde{\delta}_1, \tilde{\delta}_2] = \tilde{\delta}_3 \) with \( \tilde{\epsilon}_3^a = f_{bc}^a \tilde{\epsilon}_1^b \tilde{\epsilon}_2^c \), where \( \tilde{\delta}_i \) denotes the gauge transformation with respective gauge parameters \( \epsilon_i \),

\[
[\tilde{\delta}_1, \tilde{\delta}_2]X^i = \tilde{\delta}_3X^i,
\]

\[
[\tilde{\delta}_1, \tilde{\delta}_2]A^a = \tilde{\delta}_3A^a + \Lambda_{abc}^a \tilde{\epsilon}_1^b \tilde{\epsilon}_2^c,
\]

where

\[
\Lambda_{abc}^a = -\frac{1}{2} N_{abc} f^d_{bc} + f^a_{dc} N_{bc}^d + \frac{\partial N_{bc}^d}{\partial x^j} \rho^j_c + N_{cij}^a \frac{\partial \rho^j_b}{\partial x^i} - (b \leftrightarrow c).
\]

The gauge transformation of \( x^i \) is off-shell closed. Off-shell closure of the gauge transformation of \( A^a \) requires

\[
\Lambda_{abc}^a = 0,
\]

which is satisfied in our examples.

6 Discussion

In this paper, we generalized the method to obtain off-shell covariant gauge transformations and field strengths of higher gauge theories in [8] and applied it to a system of algebroid gauge theory with 1-form gauge field and scalars. We demonstrated off-shell covariantization of a gauge theory based on a Lie 2-algebroid and a Lie 3-algebroid. Recall that the resulting gauge theory is not restricted to any dimension. For covariantization, we deform field strengths and gauge transformations. The starting point of this procedure is an on-shell (i.e. \( F_Z = 0 \)) covariant theory. Since the gauge transformations and field strengths are deformed proportional to the lower field strengths, they are consistent if the theory is kept on-shell.

There are several directions to develop the approach presented in this paper. The extension of the method to gauge theories with Lie n-algebroid gauge symmetry induced from a QP-manifold of degree \( n \) is straightforward. Similar conditions corresponding to (4.73)–(4.75) can be computed for arbitrary \( n \).

Here, we have formulated the Kotov-Strobl model using a QP-manifold of degree two. However, we can also construct the Kotov-Strobl model from a QP-manifold of degree three. For this, a further generalization of the procedure is necessary. Another possible application
of our method is to investigate multiple M5-brane systems [24, 25]. We can add scalar fields to the analysis conducted in [8]. The procedure in this paper can also be applied to investigate the properties of supergravity in connection with tensor hierarchy. Furthermore, gauge theoretical formulations of gravity such as the vielbein formalism or the gauge theory of the Poincaré group can also be treated in this formalism. It would also be interesting to compare the present formalism with the approach taken in [26]. We expect that our approach will shed new light on the analysis of such systems.

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References

[1] J. C. Baez, “Higher Yang-Mills theory”, hep-th/0206130.

[2] N. Ikeda, “Deformation of BF theories, topological open membrane and a generalization of the star deformation”, JHEP 0107 (2001) 037 [hep-th/0105286].

[3] T. Strobl, “Algebroid Yang-Mills theories”, Phys. Rev. Lett. 93 (2004) 211601 [hep-th/0406215].

[4] A. Kotov and T. Strobl, “Curving Yang-Mills-Higgs gauge theories”, Phys. Rev. D 92 (2015) no.8, 085032.

[5] T. Lada and J. Stasheff, “Introduction to SH Lie algebras for physicists”, Int. J. Theor. Phys. 32 (1993) 1087 doi:10.1007/BF00671791 [hep-th/9209099].

[6] J. C. Baez and A. S. Crans, “Higher-Dimensional Algebra VI: Lie 2-Algebras”, Theor. Appl. Categor. 12 (2004) 492 [math/0307263 [math.QA]].
[7] J. C. Baez and J. Huerta, “An Invitation to Higher Gauge Theory”, Gen. Rel. Grav. 43 (2011) 2335 [arXiv:1003.4485 [hep-th]].

[8] U. Carow-Watamura, M. A. Heller, N. Ikeda, Y. Kaneko and S. Watamura, “Higher Gauge Theories from Lie n-algebras and Off-Shell Covariantization”, JHEP 1607 (2016) 125 [arXiv:1606.03861 [hep-th]].

[9] A. Schwarz, “Geometry of Batalin-Vilkovisky quantization”, Commun. Math. Phys. 155 (1993) 249, [hep-th/9205088].

[10] A. S. Schwarz, “Semiclassical approximation in Batalin-Vilkovisky formalism”, Commun. Math. Phys. 158 (1993) 373 [arXiv:hep-th/9210115].

[11] M. Alexandrov, M. Kontsevich, A. Schwartz and O. Zaboronsky, “The Geometry of the master equation and topological quantum field theory”, Int. J. Mod. Phys. A 12 (1997) 1405 [arXiv:hep-th/9502010].

[12] D. Sullivan, "Infinitesimal computations in topology", Bull. de l’Institut des Hautes Etudes Scientifiques, Publ. Math. 47 (1977)

[13] R. D’Auria and P. Fre, “Geometric Supergravity in d = 11 and Its Hidden Supergroup”, Nucl. Phys. B 201 (1982) 101 Erratum: [Nucl. Phys. B 206 (1982) 496].

[14] D. Fiorenza, U. Schreiber and J. Stasheff, “Čech cocycles for differential characteristic classes: an ∞-Lie theoretic construction”, Adv. Theor. Math. Phys. 16 (2012) no.1, 149 [arXiv:1011.4735 [math.AT]].

[15] M. Grützmann and T. Strobl, “General Yang-Mills type gauge theories for p-form gauge fields: From physics-based ideas to a mathematical framework or From Bianchi identities to twisted Courant algebroids”, Int. J. Geom. Meth. Mod. Phys. 12 (2014) 1550009 [arXiv:1407.6759 [hep-th]].

[16] S. Lavau, H. Samtleben and T. Strobl, “Hidden Q-structure and Lie 3-algebra for non-abelian superconformal models in six dimensions”, J. Geom. Phys. 86 (2014) 497 [arXiv:1403.7114 [math-ph]].
[17] P. M. Ho and Y. Matsuo, “Note on non-Abelian two-form gauge fields”, JHEP 1209 (2012) 075 [arXiv:1206.5643 [hep-th]].

[18] T. Strobl, “Non-abelian Gerbes and Enhanced Leibniz Algebras”, Phys. Rev. D 94 (2016) no.2, 021702 [arXiv:1607.00060 [hep-th]].

[19] N. Ikeda, “Lectures on AKSZ Sigma Models for Physicists”, arXiv:1204.3714 [hep-th].

[20] T. Bessho, M. A. Heller, N. Ikeda and S. Watamura, “Topological Membranes, Current Algebras and H-flux - R-flux Duality based on Courant Algebroids”, JHEP 1604 (2016) 170 [arXiv:1511.03425 [hep-th]].

[21] N. Ikeda and K. Uchino, “QP-Structures of Degree 3 and 4D Topological Field Theory”, Commun. Math. Phys. 303 (2011) 317 [arXiv:1004.0601 [hep-th]].

[22] Z.-J. Liu, A. Weinstein and P. Xu, “Manin triples for Lie bialgebroids”, J. Diff. Geom. 45 (1997), 547-574.

[23] D. Roytenberg, ”Courant algebroids, derived brackets and even symplectic supermanifolds”, math.DG/9910078.

[24] E. Witten, “Some comments on string dynamics”, in ’Los Angeles 1995, Future perspectives in string theory’ 501-523 [hep-th/9507121].

[25] E. Witten, “Geometric Langlands From Six Dimensions”, arXiv:0905.2720 [hep-th].

[26] I. A. Batalin and P. M. Lavrov, “Representation of a gauge field via intrinsic ’BRST’ operator”, Phys. Lett. B750 (2015) 325-330 [arXiv:1507.02361 [hep-th]].