Distributed Deterministic Broadcasting
in Wireless Networks of Weak Devices under the SINR Model

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Abstract

The Signal-to-Interference-and-Noise-Ratio model (SINR) is currently the most popular model for analyzing communication in wireless networks. Roughly speaking, it allows receiving a message if the strength of the signal carrying the message dominates over the combined strength of the remaining signals and the background noise at the receiver. There is a large volume of analysis done under the SINR model in the centralized setting, when both network topology and communication tasks are provided as a part of the common input, but surprisingly not much is known in the ad hoc setting, when nodes have very limited knowledge about the network topology. In particular, there is no theoretical study of deterministic solutions to multi-hop communication tasks, i.e., tasks in which packets often have to be relayed in order to reach their destinations. These kinds of problems, including broadcasting, routing, group communication, leader election, and many others, are important from perspective of development of future multi-hop wireless and mobile technologies, such as MANET, VANET, Internet of Things.

In this paper we initiate a study of distributed deterministic broadcasting in ad-hoc wireless networks with uniform transmission powers under the SINR model. We design algorithms in two settings: with and without local knowledge about immediate neighborhood. In the former setting, our solution has almost optimal $O(D \log^2 n)$ time cost, where $n$ is the size of a network, $D$ is the eccentricity of the network and $\{1, \ldots, N\}$ is the set of possible node IDs. In the latter case, we prove an $\Omega(n \log N)$ lower bound and develop an algorithm matching this formula, where $n$ is the number of network nodes. As one of the conclusions, we derive that the inherited cost of broadcasting techniques in wireless networks is much smaller, by factor around $\min\{n/D, \Delta\}$, than the cost of learning the immediate neighborhood. Finally, we develop a $O(D \Delta \log^2 N)$ algorithm for the setting without local knowledge, where $\Delta$ is the upper bound on the degree of the communication graph of a network. This algorithm is close to a lower bound $\Omega(D \Delta)$.

In the model without local knowledge, we take advantage of the fact that efficient deterministic distributed communication is possible (in the SINR model) between stations which are very close, despite large amount of interferences caused by other transmitters. This feature somehow compensates inconveniences caused by distant interferences and makes it possible to obtain a broadcasting algorithm with efficiency similar to that obtained for UDG radio networks. However, unlike in the UDG radio networks model, the (lower) bounds apply also for randomized solutions. In other words, randomization does not substantially help in ad hoc distributed broadcasting in a large class of networks.

Keywords: Ad Hoc wireless networks, Signal-to-Interference-and-Noise-Ratio model (SINR) model, Broadcasting, Distributed algorithms, Deterministic algorithms, Local knowledge.

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1 Introduction

In this work we consider a broadcasting problem in ad-hoc wireless networks under the Signal-to-Interference-and-Noise-Ratio model (SINR). Wireless network consists of $n$ stations, also called nodes, with unique integer IDs in the range $\{1, \ldots, N\}$ and uniform transmission powers, deployed in the two-dimensional space with Euclidean metric. Each station initially knows only its own ID and location, parameters $n$ and $N$. A communication (or reachability) graph of the network is the graph defined on network nodes and containing links $(v, w)$ such that if $v$ is the only transmitter in the network then $w$ receives the message transmitted by $v$. We consider two settings: one with local knowledge, in which each station knows also its neighbors (i.e., stations reachable by a direct transmission), and the other when no extra knowledge is assumed.

In the broadcasting problem, there is one designated node, called the source, which has a piece of information (called a source message or a broadcast message) that must be delivered to all other accessible nodes by using wireless communication. In the beginning, only the source is active from perspective of the broadcast task, and other nodes join the execution after receiving the broadcast message for the first time. The goal is to minimize the worst-case time for accomplishing the broadcasting task.

1.1 Previous and Related Results

Recent development of deterministic protocols for wireless communication, e.g., CDMA-based technologies, and rapidly growing scale of ad hoc wireless networks, poses new challenges for design of efficient deterministic distributed protocols. In this work, we study the problem of distributed deterministic broadcasting in ad hoc wireless networks, which, to the best of our knowledge, has not been theoretically studied under the SINR model, from perspective of worst-case complexity. SINR model is currently considered the most adequate among the models of wireless networks. Furthermore, no other communication task involving multi-hop message propagation has been theoretically studied from perspective of distributed deterministic solutions in the SINR setting. In what follows, we list most relevant results in the SINR model, and the state of the art obtained in the older Radio Network model.

**SINR model.** In the SINR model in ad hoc setting, deterministic local broadcasting, in which nodes have to inform only their neighbors in the corresponding reachability graph, was studied in [27]. The considered setting allowed power control by algorithms, in which, in order to avoid collisions, stations could transmit with any power smaller than the maximal one. Randomized solutions for contention resolution [19] and local broadcasting [16] were also obtained.

There is a vast amount of work on centralized algorithms under the SINR model. The most studied problems include connectivity, capacity maximization, link scheduling types of problems (e.g., [10, 18, 2]). For recent results and references we refer the reader to the survey [17]. Multiple Access Channel properties were also recently studied under the SINR model, c.f., [24].

**Radio network model.** There are several papers analyzing deterministic broadcasting in the radio model of wireless networks, under which a message is successfully heard if there are no other simultaneous transmissions from the neighbors of the receiver in the communication graph. This model does not take into account the real strength of the received signals, and also the signals from outside of some close proximity. In the geometric ad hoc setting, Dessmark and Pelc [7] were the first who studied this problem. They analyzed the impact of local knowledge, defined as a range within which stations can discover the nearby stations. Unlike most research on broadcasting problem and the assumptions of this paper, Dessmark et al. [7] assume spontaneous wake-up of stations. That is, stations are allowed to do some pre-processing (including sending/receiving messages) prior receiving
the broadcast message for the first time. Moreover it is assumed in [7] that IDs are from \( \{1, \ldots, n\} \), which makes the setting even less comparable with the one considered in this work. Emek et al. [8] designed a broadcast algorithm working in time \( O(Dg) \) in UDG radio networks with eccentricity \( D \) and granularity \( g \), where eccentricity was defined as the minimum number of hops to propagate the broadcast message throughout the whole network and granularity was defined as the inverse of the minimum distance between any two stations. Later, Emek et al. [9] developed a matching lower bound \( \Omega(Dg) \). There were several works analyzing deterministic broadcasting in geometric graphs in the centralized radio setting, c.f., [14, 15, 25].

The problem of broadcasting is well-studied in the setting of graph radio model, in which stations are not necessarily deployed in a metric space; here we restrict to only the most relevant results. In deterministic ad hoc setting with no local knowledge, the fastest \( O(n \log (n/D)) \)-time algorithm in symmetric networks was developed by Kowalski [20], and almost matching lower bound was given by Kowalski and Pelc [22]. For recent results and references in less related settings we refer the reader to [6, 23, 5, 3, 13].

There is vast literature on randomized algorithms for broadcasting in graph radio model. Since they are quite efficient, there are very few studies of the problem restricted to geometric setting. However, when mobility of stations is assumed, location and movement of stations on the plane is natural. Such settings were studied e.g., in [11, 12].

1.2 Our Results

In this paper we present the first study on deterministic broadcasting in wireless connected networks deployed in two dimensional Euclidean space under the SINR model. We distinguish between the two settings: with and without local knowledge about neighbors in the communication graph. In the former model, we developed a broadcasting algorithm with time complexity \( O(n \log N) \), which matches the lower bound (Section 4). Then, an algorithm finishing broadcasting in time \( O(D \Delta \log^2 N) \) is presented, where \( \Delta \) is the largest degree of a vertex in the reachability graph (Section 5). This algorithm is close to the lower bound \( \Omega(D \Delta) \) – see Section 6. Our solution for networks with local knowledge works in time \( O(D \log^2 n) \), which provides \( O(\log^2 n) \) overhead over the straightforward \( \Omega(D) \) lower bound, and is faster than the algorithms for anonymous networks in every network with eccentricity \( D = o(n/\log N) \) or maximal degree \( \Delta = \omega(1) \). It also implies that the cost of learning neighborhoods by stations in wireless network is much higher, by factor around \( n/D \) or \( \Delta \), than the cost of broadcast itself (performed when such neighborhoods are provided). Importantly, the algorithm for networks with local knowledge works for any path loss parameter \( \alpha \geq 2 \) (though additional multiplicative \( \log^2 N \) factor appears in complexities of algorithms for \( \alpha = 2 \)), while the algorithms without local knowledge are applicable only when \( \alpha > 2 \).

Our results rely on novel techniques which simultaneously exploit specific properties of conflict resolution in the SINR model (see e.g. [1]) and algorithmic techniques developed for radio networks model. In particular, in the model with local knowledge, we show how to efficiently combine a novel SINR-based leader election technique, ensuring several parallel communications inside range area of one station (which is unfeasible to achieve in radio networks model), with the approach simulating collision detection in radio networks (c.f. [21]). As a result, we develop a general transformation of algorithms relying on the knowledge of network granularity (Section 3.2) into algorithm of asymptotically similar performance that do not require such knowledge.

In the model without local knowledge, we take advantage of the fact that efficient deterministic distributed communication is possible (in the SINR model) between stations which are very close, despite large amount of interferences caused by other transmitters. This feature somehow compensates inconveniences caused by distant interferences and makes possible to achieve broadcasting
algorithm with efficiency similar to that obtained for UDG radio networks. However, unlike in the UDG radio networks model, the (lower) bounds apply also for randomized solutions. In other words, randomization does not substantially help in ad hoc distributed broadcasting in a large class of networks.

2 Model, Notation and Technical Preliminaries

Throughout the paper, \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{N}_+ \) denotes the set \( \mathbb{N} \setminus \{0\} \), and \( \mathbb{Z} \) denotes the set of integers. For \( i, j \in \mathbb{Z} \), we use the notation \([i, j] = \{ k \in \mathbb{N} \mid i \leq k \leq j \}\) and \([i] = [1, i]\).

We consider a wireless network consisting of \( n \) stations, also called nodes, deployed into a two dimensional Euclidean space and communicating by a wireless medium. All stations have unique integer IDs in set \([\mathbb{N}]\). Stations of a network are denoted by letters \( u, v, w \), which simultaneously denote their IDs. Stations are located on the plane with Euclidean metric \( \text{dist}(\cdot, \cdot) \), and each station knows its coordinates. Each station \( v \) has its transmission power \( P_v \), which is a positive real number. There are three fixed model parameters: path loss \( \alpha \geq 2 \), threshold \( \beta \geq 1 \), and ambient noise \( \mathcal{N} \geq 1 \).

The \( \text{SINR}(v, u, \mathcal{T}) \) ratio, for given stations \( u, v \) and a set of (transmitting) stations \( \mathcal{T} \), is defined as follows:

\[
\text{SINR}(v, u, \mathcal{T}) = \frac{P_v \text{dist}(v, u)^{-\alpha}}{\mathcal{N} + \sum_{w \in \mathcal{T} \setminus \{v\}} P_w \text{dist}(w, u)^{-\alpha}}
\]  

(1)

In the Signal-to-Interference-and-Noise-Ratio model (SINR) considered in this work, station \( u \) successfully receives a message from station \( v \) in a round if \( v \in \mathcal{T}, u \notin \mathcal{T} \), and:

- \( \text{SINR}(v, u, \mathcal{T}) \geq \beta \), where \( \mathcal{T} \) is the set of stations transmitting at that time, and
- \( P_v \text{dist}^{-\alpha}(v, u) \geq (1 + \varepsilon)\beta\mathcal{N} \),

where \( \varepsilon > 0 \) is a fixed sensitivity parameter of the model. The above definition is common in the literature, c.f., \cite{ref1}.

In the paper, we assume for the sake of clarity of presentation that \( \beta = 1 \) and \( \mathcal{N} = 1 \). These assumptions can be dropped without harming the asymptotic performances of the presented algorithms and lower bounds formulas.

Ranges and uniformity. The communication range \( r_v \) of a station \( v \) is the radius of the circle in which a message transmitted by the station is heard, provided no other station transmits at the same time. A network is uniform, when ranges (and thus transmission powers) of all stations are equal, or nonuniform otherwise. In this paper, only uniform networks are considered. For clarity of presentation we make the assumption that all powers are equal to 1, i.e., \( P_v = 1 \) for each \( v \). The assumption that the values of \( P_v \) are 1 can be dropped without changing asymptotic formulas for presented algorithms and lower bounds. Under these assumptions, \( r_v = r = (1 + \varepsilon)^{-1/\alpha} \) for each station \( v \). The range area of a station with range \( r \) located at the point \((x, y)\) is defined as the circle with radius \( r \).

Communication graph and graph notation. The communication graph \( G(V, E) \), also called the reachability graph, of a given network consists of all network nodes and edges \((v, u)\) such that \( u \) is in the range area of \( v \). Note that the communication graph is symmetric for uniform networks, which are considered in this paper. By a neighborhood of a node \( u \) we mean the set (and positions) of all neighbors of \( u \), i.e., the set \( \{ w \mid (w, u) \in E \} \) in the communication graph \( G(V, E) \) of the underlying network. The graph distance from \( v \) to \( w \) is equal to the length of a shortest path from \( v \) to \( w \).

\footnote{The first condition is a straightforward application of the SINR ratio, comparing strength of one of the received signals with the remainder. The second condition enforces the signal to be sufficiently strong in order to be distinguished from the background noise, and thus to be decoded. Moreover, this condition ensures that all transmission powers are high enough so that some interference can be tolerated.}
in the communication graph, where the length of a path is equal to the number of its edges. The 
 eccentricity of a node is the maximum graph distance from this node to all other nodes (note that 
 the eccentricity is of the order of the diameter if the communication graph is symmetric — this is 
 also the case in this work).

We say that a station $v$ transmits $c$-successfully in a round $t$ if $v$ transmits a message in round $t$ 
 and this message is heard by each station $u$ in distance smaller or equal to $c$ from $v$. We say that a 
 station $v$ transmits successfully in round $t$ if it transmits $r$-successfully, i.e., each of its neighbors in 
 the communication graph can hear its message. Finally, $v$ transmits successfully to $u$ in round $t$ if $v$ 
 transmits a message in round $t$ and $u$ receives this message.

**Synchronization.** It is assumed that algorithms work synchronously in rounds, each station can 
 either act as a sender or as a receiver during a round. We do not assume global clock ticking – as it 
 can be coordinated by updating round counter and passing it along the network with messages.

**Collision detection.** We consider the model without collision detection, that is, if a station $u$ does 
 not receive a message in a round $t$, it has no information whether any other station was transmitting 
 in that round and about the value of $SINR(v,u,T)$, for any station $u$, where $T$ is the set of 
 transmitting stations in round $t$.

**Broadcasting problem and complexity parameters.** In the broadcasting problem studied 
 in this work, there is one distinguished node, called the source, which initially holds a piece of 
 information (also called a source message or a broadcast message). The goal is to disseminate this 
 message to all other nodes by sending messages along the network. The complexity measure is the 
 worst-case time to accomplish the broadcast task, taken over all connected networks with specified 
 parameters. Time, also called the round complexity, denotes here the number of communication 
 rounds in the execution of a protocol: from the round when the source is activated with its broadcast 
 message till the broadcast task is accomplished (and each station is aware of this fact). For the sake 
 of complexity formulas, we consider the following parameters: $n$, $N$, $D$, and $g$, where: $n$ is the 
 number of nodes, $[N]$ is the range of IDs, $D$ is the eccentricity of the source, and $g$ is the granularity 
 of the network, defined as $r$ times the inverse of the minimum distance between any two stations 
 (c.f., [8]) divided by $r$.

**Messages and initialization of stations other than source.** We assume that a single message 
 sent in the execution of any algorithm can carry the broadcast message and at most polynomial, in 
 the size of the network, number of control bits in the size of the network. For simplicity of analysis, 
 we assume that every message sent during the execution of our broadcast protocols contains the 
 broadcast message; in practice, further optimization of a message content could be done in order to 
 reduce the total number of transmitted bits in real executions. A station other than the source starts 
 executing the broadcasting protocol after the first successful receipt of the broadcast message; we 
 call it a non-spontaneous wake-up model, to distinguish from other possible settings, not considered 
 in this work, where stations could be allowed to do some pre-processing (including sending/receiving 
 messages) prior receiving the broadcast message for the first time. We say that a station that received 
 the broadcast message is informed.

**Knowledge of stations.** Each station knows its own ID, location, and parameters $n$, $N$. Some 
 subroutines use the granularity $g$ as a parameter, though our main algorithms can use these subrou-
 tines without being aware of the actual granularity of the input network. We distinguish between 
 ad hoc networks, where stations do not know anything about the topology of the network at the 
 beginning of the execution of an algorithm, and networks with local knowledge, in which each station 
 knows locations and IDs of its neighbors in the communication graph.
2.1 Grids and Schedules

Given a parameter $c > 0$, we define a partition of the 2-dimensional space into square boxes of size $c \times c$ by the grid $G_c$, in such a way that: all boxes are aligned with the coordinate axes, point $(0,0)$ is a grid point, each box includes its left side without the top endpoint and its bottom side without the right endpoint and does not include its right and top sides. We say that $(i,j)$ are the \( \text{coordinates} \) of the box with its bottom left corner located at \((c \cdot i, c \cdot j)\), for \(i,j \in \mathbb{Z}\). A box with \( \text{coordinates} \) $(i,j) \in \mathbb{Z}^2$ is denoted $C(i,j)$. As observed in [7,8], the \( \text{grid} \) $G_{r/\sqrt{2}}$ is very useful in design of algorithms for geometric radio networks, provided $r$ is equal to the range of each station. This follows from the fact that $r/\sqrt{2}$ is the largest parameter of a grid such that each station in a box is in the range of every other station in that box. In the following, we fix $\gamma = r/\sqrt{2}$, where $r = (1 + \varepsilon)^{-1/\alpha}$, and call $G_{\gamma}$ the pivotal \( \text{grid} \).

Two boxes $C,C'$ are \( \text{neighbors} \) in a network if there are stations $v \in C$ and $v' \in C'$ such that edge $(v,v')$ belongs to the communication graph of the network. Boxes $C(i,j)$ and $C'(i',j')$ are \( \text{adjacent} \) if $|i-i'| \leq 1$ and $|j-j'| \leq 1$ (see Figure 1). For a station $v$ located in position $(x,y)$ on the plane we define its \( \text{grid coordinates} \) with respect to the grid $G_c$ as the pair of integers $(i,j)$ such that the point $(x,y)$ is located in the box $C(i,j)$ of the grid $G_c$ (i.e., $ic \leq x < (i+1)c$ and $jc \leq y < (j+1)c$).

A (general) \( \text{broadcast schedule} \) $S$ of length $T$ wrt $N \in \mathbb{N}$ is a mapping from $[N]$ to binary sequences of length $T$. A station with identifier $v \in [N]$ \( \text{follows} \) the \( \text{schedule} \) $S$ of length $T$ in a fixed period of time consisting of $T$ rounds, when $v$ transmits a message in round $t$ of that period iff the position $t$ mod $T$ of $S(v)$ is equal to 1.

A \( \text{geometric broadcast schedule} \) $S$ of length $T$ with parameters $N, \delta \in \mathbb{N}$, $(N, \delta)$-gbs for short, is a mapping from $[N] \times [0, \delta - 1]^2$ to binary sequences of length $T$. Let $v \in [N]$ be a station whose grid coordinates with respect to the grid $G_c$ are equal to $(i,j)$. We say that $v$ \( \text{follows} \) $(N, \delta)$-gbs $S$ for the grid $G_c$ in a fixed period of time, when $v$ transmits a message in round $t$ of that period iff the $t$th position of $S(v,i \mod \delta,j \mod \delta)$ is equal to 1. A set of stations $A$ on the plane is \( \delta\text{-diluted} \) wrt $G_c$, for $\delta \in \mathbb{N} \setminus \{0\}$, if for any two stations $v_1,v_2 \in A$ with grid coordinates $(i_1,j_1)$ and $(i_2,j_2)$, respectively, the relationships $(|i_1 - i_2| \mod \delta) = 0$ and $(|j_1 - j_2| \mod \delta) = 0$ hold.

Let $S$ be a general broadcast schedule wrt $N$ of length $T$, let $c > 0$ and $\delta > 0$, $\delta \in \mathbb{N}$. A $\delta$-dilution of a $S$ is defined as a $(N, \delta)$-gbs $S'$ such that the bit $(t - 1)\delta^2 + a\delta + b$ of $S'(v,a,b)$ is equal to 1 iff the bit $t$ of $S(v)$ is equal to 1. That is, each round $t$ of $S$ is partitioned into $\delta^2$ rounds of $S'$, indexed by pairs $(a,b) \in [0, \delta - 1]^2$, such that a station with grid coordinates $(i,j)$ in $G_c$ is allowed to send messages only in rounds with index $(i \mod \delta,j \mod \delta)$, provided schedule $S$ admits a transmission in its (original) round $t$. Since we will usually apply dilution to the pivotal grid, it is assumed that all references to a dilution concern that grid, unless stated otherwise.

Observe that, since ranges of stations are equal to the length of diagonal of boxes of the pivotal grid, a box $C(i,j)$ can have at most 20 \( \text{neighbors} \) (see Figure 1). We define the set DIR $\subset [-2,2]^2$ such that $(d_1,d_2) \in$ DIR if it is possible that boxes with \( \text{coordinates} \) $(i,j)$ and $(i+d_1,j+d_2)$ can be neighbors. Given $(i,j) \in \mathbb{Z}^2$ and $(d_1,d_2) \in$ DIR, we say that the box $C(i+d_1,j+d_2)$ is \( \text{located in direction} \) $(d_1,d_2)$ from the box $C(i,j)$.

3 Algorithms for Networks with Local Knowledge

In this section we describe our broadcasting algorithms for networks with local knowledge, i.e., under the assumption that each stations knows (IDs and locations) of all stations in its range area. Recall that we also assume that stations know $n$, the size of the network and $N$, the range of identifiers.
Figure 1: If \( v, w, z \) are in the range area of \( u \), then boxes containing \( v, w, \) and \( z \) are neighbors of \( C \). The first figure contains all 20 boxes which can be neighbors of \( C \). The boxes \( C_1, \ldots, C_8 \) are adjacent to \( C \).

We start with presenting a generic algorithmic scheme and tools for analysis. Next, we describe an algorithm for networks with additionally known granularity bound \( g \), i.e., parameters \( n, N \) and \( g \) are known to the stations in the beginning of the execution. Complexity of this algorithm is expressed in terms of \( D \) and \( g \); note however that stations do not need any information about \( D \) in order to execute our algorithms. Finally, using this algorithm as a subroutine, we provide a solution for the general setting when only \( n \) and \( N \) are known.

### 3.1 Generic Algorithmic Scheme

In the first step of each broadcasting algorithm, the source sends the broadcast message. Then, our broadcasting algorithms repeat several times the procedure Inter-Box-Broadcast, whose \( i \)th repetition is aimed to transmit the broadcast message from boxes of the pivotal grid containing at least one station that has received the broadcast message in the previous execution of Inter-Box-Broadcast (or from the source) to boxes which are their neighbors.

Each station \( v \) of the network is in state \( s(v) \), which may be equal to one of the following three values: asleep, active, or idle. At the beginning of execution of each of our broadcasting algorithms, the source sends the broadcast message and all stations in its box of the pivotal grid set their states to active, while all the remaining stations are in the asleep state. The states of stations change only at the end of Inter-Box-Broadcast, according to the following rules:

- All stations in state active change their state to idle.

- A station \( u \) changes its state from asleep to active if it has received the broadcast message from a station \( v \) in the current execution of Inter-Box-Broadcast such that either \( v \) was in state active (at the beginning of the current execution of Inter-Box-Broadcast) or \( v \) belongs to the same box of the pivotal grid as \( u \). That is, let \( C \) be a box of the pivotal grid, let \( u \in C \) be in state asleep at the beginning of Inter-Box-Broadcast. The only possibility that \( u \) receives a message and it does not change its state from asleep to active at the end of Inter-Box-Broadcast is that each message received by \( u \) is sent by a station \( v \) which is in state asleep when it sends the message and \( v \not\in C \).

Our goal is to preserve the following invariant during the execution of our algorithms:

(I) For each box \( C \) of the pivotal grid, states of all stations located inside \( C \) are equal.

The intended property of an execution of Inter-Box-Broadcast is:
(P) The broadcast message is (successfully) sent from each box $C$ containing stations in state \textit{active} to all stations located in boxes which are neighbors of $C$. (Recall that a box $C'$ is a neighbor of a box $C$ if there are stations $v \in C$ and $v' \in C'$ such that edge $(v,v')$ belongs to the communication graph.)

Note that, since stations move to the state \textit{active} only after receiving the broadcast message, the following fact holds.

**Proposition 1.** If (I) and (P) are satisfied, the source message is transmitted to the whole network in time $O(D \cdot T(n))$, where $T(n)$ is time complexity of one execution of Inter-Box-Broadcast.

In what follows, we give a specification of Inter-Box-Broadcast first under the assumption of known granularity $g$, and later we remove that assumption.

### 3.2 A Granularity-Dependent Algorithm

In this section we describe a broadcasting algorithm whose complexity depends on granularity. We assume that granularity $g$ is known to all stations of the network. First, we present a general leader election algorithm, which, given a set of stations $V$ with granularity $g$, elects a leader in each box of the pivotal grid containing at least one element of $V$, in time $O(\log g)$. Then, using this algorithm, we describe how to implement Inter-Box-Broadcast in time $O(\log g)$ in such a way that (I) and (P) are preserved.

#### 3.2.1 Leader Election

Let $I_1 = [i_1,j_1)$, $I_2 = [i_2,j_2)$ be segments on a line, whose endpoints belong to the grid $G_x$. The box-distance between $I_1$ and $I_2$ with respect to $G_x$ is zero when $I_1 \cap I_2 \neq \emptyset$, and it is equal to $\min(|i_1 - j_2|/x,|i_2 - j_1|/x)$ otherwise. Given two rectangles $R_1$, $R_2$, whose vertices belong to $G_x$, the box-distance $\text{dist}_M(R_1,R_2)$ between $R_1$ and $R_2$ is equal to the maximum of the box-distances between projections of $R_1$ and $R_2$ on the axes defining the first and the second dimension in the Euclidean space.

We say that a function $d_\alpha : \mathbb{N} \to \mathbb{N}$ is flat for $\alpha \geq 2$ if

$$d_\alpha(n) = \begin{cases} O(1) & \text{for } \alpha > 2 \\ O(\log n) & \text{for } \alpha = 2 \end{cases}$$

**Lemma 1.** Given a set of stations $V$ with granularity $g$, one can choose the leader in each box of the pivotal grid containing at least one element of $V$ in $O(d_\alpha^2(n) \log g)$ rounds, where $d_\alpha(n)$ is a flat function.

Moreover, if polynomial size of messages is allowed, each station can learn positions of all (active) stations located in its box in $O(\log g)$ rounds.

The remaining part of this section is devoted to the proof of Lemma 1.

**Proposition 2.** For each $\alpha \geq 2$ and $\varepsilon > 0$, there exists a flat function $d_\alpha(n)$ such that the following properties hold. Assume that a set of $n$ stations $A$ is $d$-diluted wrt the grid $G_x$, where $x = \gamma/c$, $c \in \mathbb{N}$, $c > 1$ and $d \geq d_\alpha(n)$. Moreover, at most one station from $A$ is located in each box of $G_x$. Then, if all stations from $A$ transmit simultaneously, each of them is $\frac{2}{\alpha}$-successful. Thus, in particular, each station from a box $C$ of $G_x$ can transmit its message to all its neighbors located in $C$ and in boxes $C'$ of $G_x$ which are adjacent to $C$. 

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Figure 2: Boxes in distance \( id \) from \( C \) form a frame partitioned into four rectangles of size \( x \times (2id + 2)x \). Each of these rectangles contain at most \( i + 1 \) boxes such that any two of them are in box-distance at least \( d \).

Proof. Recall that \( r = (1 + \varepsilon)^{-1/\alpha} \) and \( \gamma = r/\sqrt{2} \). First, assume that \( \alpha > 2 \). Consider any station \( u \) in distance smaller or equal to \( \frac{2r}{c} \leq 2\sqrt{2}x < 3x \) to a station \( v \in A \). Then, the signal from \( v \) received by \( u \) is at least

\[
\frac{1}{(\frac{2r}{c})^\alpha} = \left( \frac{c}{2r} \right)^\alpha.
\]

Now, we would like to derive an upper bound on interferences caused by stations in \( A \setminus \{v\} \) at \( u \). Let \( C \) be a box of \( G_x \) which contains \( v \). The fact that \( A \) is \( d \)-diluted wrt \( G_x \) implies that the number of boxes containing elements of \( A \) which are in box-distance \( id \) from \( C \) is at most \( 8(i + 1) \) (see Figure 2). Moreover, no box in distance \( j \) from \( C \) such that \( (j \mod d \neq 0) \) contains elements of \( A \). Finally, for a station \( v \in C \) and a station \( w \in C' \) such that \( \text{distM}(C, C') = j \), the inequality \( \text{dist}(v, u) \geq jx \) is satisfied. Note that our goal is not to evaluate interferences at \( v \in C \), but at any station \( u \) such that \( \text{dist}(u, v) \leq \frac{2r}{c} < 3x \). Therefore, \( u \in C' \) such that \( \text{distM}(C, C') < 3 \), where \( C' \) is a box of \( G_x \). For a fixed \( d > 3 \), the total noise and interferences \( I \) caused by all elements of \( A \setminus \{v\} \) at \( u \) is at most

\[
\mathcal{N} + \sum_{i=1}^{n} 8(i + 1) \cdot \frac{1}{(idx)^\alpha}
\]

where \( d \geq \bar{d} \geq d - 3 \), since there are at most \( 8(i + 1) \) nonempty boxes in box-distance \( i \cdot d \) from the box \( C \) in \( d \)-diluted instance and the box-distance between \( C \) and the box \( C' \) containing \( u \) is at most 2. Furthermore,

\[
I \leq 1 + 8 \cdot \left( \frac{1}{dx} \right)^\alpha \cdot \sum_{i=0}^{n} (i + 1)^{1-\alpha} \leq 1 + 8 \left( \frac{c\sqrt{2}}{rd} \right)^\alpha \sum_{i=1}^{n} i^{1-\alpha} = 1 + 8d_\alpha(n) \left( \frac{\sqrt{2}c}{rd} \right)^\alpha
\]

where \( d_\alpha(n) = \sum_{i=1}^{n} i^{1-\alpha} = 1 + \zeta(\alpha - 1) \), \( \zeta \) is the Riemann zeta function and \( \mathcal{N} = 1 \). So, the signal from \( v \) is received at \( u \) if the following inequality is satisfied

\[
1 + 8d_\alpha(n) \left( \frac{\sqrt{2}c}{rd} \right)^\alpha \leq \left( \frac{c}{2r} \right)^\alpha
\]
which is equivalent to
\[
\bar{d} \geq 2\sqrt{2} \left( \frac{8d_\alpha(n)}{1 - (2r/c)^\alpha} \right)^{1/\alpha}.
\]
Assuming that \( c \geq 2 \), we have \( 1 - (\frac{2r}{c})^\alpha \geq 1 - r^\alpha \) and therefore (3) is satisfied for each \( \bar{d} \geq 2\sqrt{2} \left( \frac{8}{1 - r^\alpha} \right)^{1/\alpha} d_\alpha(n) \) or \( d \geq 3 + 2\sqrt{2} \left( \frac{8}{1 - r^\alpha} \right)^{1/\alpha} d_\alpha(n) \).

**Note on dependence on \( \varepsilon \):** by substituting \( r := (1 + \varepsilon)^{-1/\alpha} \), one can check that \( d = O((1/\varepsilon)^{1/\alpha}) \) for \( \alpha > 2 \).

The following corollary is a straightforward application of Proposition 2 for \( c = 2 \).

**Corollary 1.** For each \( \alpha \geq 2 \) there exists a flat function \( d_\alpha : \mathbb{N} \rightarrow \mathbb{N} \) such that the following property is satisfied:

Let \( A \) be a set of \( O(n) \) stations on the plane which is \( \delta \)-diluted wrt the pivotal grid \( G_\gamma \), where \( d \geq d_\alpha(n) \) and each box contains at most one element of \( A \). Then, if all elements of \( A \) transmit messages simultaneously in the same round \( t \) and no other station is transmitting a message in \( t \), each of them transmits successfully.

We say that a box \( C \) of the grid \( G_x \) has the leader from set \( A \) if there is one station \( v \in A \) located in \( C \) with status leader and all stations from \( A \) located in \( C \) know which station it is.

**Proposition 3.** Assume that \( A \) is a set of leaders in some boxes of the grid \( G_x \), \( x \leq \frac{\gamma}{2} \), and each station knows whether it belongs to \( A \). Then, it is possible to choose the leader of each box of \( G_{2x} \) containing at least one element of \( A \) in \( O(d_\alpha(n)) \) rounds.

**Proof.** Note that each cell of \( G_{2x} \) consists of four boxes of \( G_x \). Let us fix some labeling of this four boxes by the numbers \( \{1, 2, 3, 4\} \), the same in each box of \( G_{2x} \). Now, assign to each station from \( A \) the label \( l \in [1, 4] \) corresponding to its position in the box of \( G_{2x} \) containing it. We “elect” leaders in \( G_{2x} \) in four phases \( F_1, \ldots, F_4 \). Phase \( F_i \) is just the application of Proposition 2 for \( A \) equal to the set of leaders with label \( i \). That is, we first have a general broadcast schedule \( S \) of length 4 such that position \( i \) of \( S(v) \) is equal to 1 iff label of \( v \) is 1. Then, \( S \) is \( d \)-diluted wrt \((N, x)\), where \( d \leftarrow d_\alpha(n) \) and \( d_\alpha \) is the constant from Proposition 2. Therefore, each leader from \( A \) can hear messages of all other (at most) three leaders located in the same box of \( G_{2x} \). Then, for a box \( C \) of \( G_{2x} \), the leader with the smallest label (if any) among leaders of the four sub-boxes of \( C \) becomes the leader of \( C \). \( \square \)

Assume that granularity of a network is equal to \( g \). Let \( h = \min_{i \in \mathbb{N}} (2^i | 2^i \geq g) \). Since \( h \geq g \), each box of \( G_{\gamma/h} \) is occupied by at most one station – its leader. We choose the leader of each box of the pivotal grid by the algorithm **GranLeaderElection** (Algorithm 1), which starts from assuming that all (active) stations are leaders of respective boxes of \( G_{\gamma/h} \) (note that there is at most one station in each box of this grid). Then, it repeatedly applies the technique from Proposition 3 in order to gradually obtain leaders of larger boxes.

**Algorithm 1** **GranLeaderElection**\((V, g)\)

1. \( h \leftarrow \min_{i \in \mathbb{N}} (2^i | 2^i \geq g) \)
2. \( x \leftarrow r/h \)
3. Each station \( v \in V \) gets status leader of the appropriate box of \( G_x \).
4. for \( i = 1, 2, \ldots, \log h \) do
5. \( \quad \) Choose leaders of boxes of \( G_{2x} \) from leaders of \( G_x \), using Proposition 3
6. \( x \leftarrow 2 \cdot x \)
Finally, we summarize properties of Algorithm GranLeaderElection in the following proposition.

**Proposition 4.** Algorithm GranLeaderElection chooses the leader in each box of the pivotal grid containing at least one element of \( V \) in time \( O(\log gd_\alpha^2(n)) \), where \( d_\alpha \) is a flat function, provided granularity of \( V \) is not larger than \( g \).

### 3.2.2 Broadcasting Algorithm

Given the algorithm electing the leaders in boxes of the pivotal grid, we describe implementation of procedure Inter-Box-Broadcast, called here Gran-Inter-Box-Broadcast. In this way we obtain algorithms GRANUBr, which repeats Gran-Inter-Box-Broadcast several times.

We say that a station \( u \) is \((d_1, d_2)\)-connected, for \((d_1, d_2) \in \text{DIR} \) iff \( v \in C(i, j) \) for a box \( C(i, j) \) of the pivotal grid and \( v \) has a neighbor in the box \( C(i + d_1, j + d_2) \) of the pivotal grid. Below, we formally describe Inter-Box-Broadcast procedure, which applies the leader election procedure in order to transmit a message from each box containing stations in state active to its neighbors. More precisely, for each direction \((d_1, d_2) \in \text{DIR} \), the application of leader election chooses one station \( v \) in \( C \) which has a neighbor in the box \( C' \) located in the direction \((d_1, d_2) \) from \( C \) (if there is such a station in \( C \)) and that station transmits successfully. Then, the neighbor \( u \in C' \) of \( v \) with the smallest ID is chosen to broadcast the message to all stations from \( C \). In order to formalize this idea, assume that \( u, v \) are such stations that \( u \in C' \) for a box \( C' \) of the pivotal grid and \( u \) is in the range area of \( v \). We say that \( u \) dominates box \( C' \) with respect to \( v \) if \( u = \min\{w \mid w \in C' \text{ and } w \text{ is in the range area of } v\} \).

**Algorithm 2** Gran-Inter-Box-Broadcast\((g)\)

1. **for** \((d_1, d_2) \in \text{DIR} \) **do**
2. \( V(d_1, d_2) \leftarrow \{v \mid s(v) = \text{active} \text{ and } v \text{ is } (d_1, d_2)-\text{connected}\} \)
3. \( \text{GranLeaderElection}(V(d_1, d_2), g) \quad \triangleright \text{leader of } (d_1, d_2)-\text{connected stations} \)
4. \( d \leftarrow d_\alpha(n), \quad \triangleright d_\alpha \text{ is a flat function from Corollary} \)
5. **for** \((j, k) \in [0, d - 1]^2 \) **do**
6. **Round 1**: a station \( v \) transmits if it is elected the leader of its box (of the pivotal grid)
7. in step 3 during \( \text{GranLeaderElection}(V(d_1, d_2), g) \) and \( v \in C(j', k') \) such that
8. \((j' \mod d, k' \mod d) = (j, k)\).
9. **Round 2**: station \( u \) transmits if: \( s(u) = \text{asleep} \), \( u \) could hear \( v \) in Round 1, \( u \in C(j', k') \)
10. such that \(((j' - d_1) \mod d, (k' - d_2) \mod d) = (j, k)\),
11. and \( u \) dominates its box wrt \( v \).
12. For each \( v \in V \) such that \( s(v) = \text{active} \): \( s(v) \leftarrow \text{idle} \).

**Proposition 5.** Algorithm Gran-Inter-Box-Broadcast works in time \( O(d_\alpha^2(n) \log g) \) for a flat function \( d_\alpha : \mathbb{N} \rightarrow \mathbb{N} \) and it preserves properties (I) and (P).

**Proof.** Time complexity bound follows directly from Proposition \( \) and Corollary \( \).

In order to prove (I), it is sufficient to show that in each box \( C \) of the pivotal grid and each execution of Gran-Inter-Box-Broadcast, either all stations in \( C \) move from the state asleep to active, or none station in \( C \) changes its state from asleep to active during that execution of Gran-Inter-Box-Broadcast. Here we benefit from the fact that stations know their neighborhood. If a station \( u \) from a box \( C \) and in state asleep receives a message from a station \( v \) in state idle, and \( u \) knows that \( v \) transmits successfully, then \( u \) is also able to determine which other stations in box \( C \) receive the same message in the current round (since it knows positions of these stations and \( v \) sends its position inside a message). In this way, the unique station \( u \) (with smallest ID) among stations from box \( C \) that have received the message from \( v \) can be determined, and this station transmits a message.
This message is successfully heard by all other stations in C in the appropriate Round 2 (see line 9 of the algorithm), since the set of stations sending messages in Round 2 is \( d \)-diluted. Assuming that all stations located in C are in the state asleep at the beginning of Inter-Box-Broadcast, they change their states to active at the end of this execution of Inter-Box-Broadcast.

As for (P), we make use of the fact that (I) is satisfied at the beginning of each Inter-Box-Broadcast. Thus, either all stations in a box \( C \) are in state active at the beginning of Inter-Box-Broadcast or none is. In the former case, the correctness of GranLeaderElection (see Proposition \[4\]) guarantees that if \( C' \) is a neighbor of \( C \) in direction \((d_1, d_2)\), then a unique station \( v \) from \( C \) is chosen in line 3, which has a neighbor in \( C' \) and then \( v \) transmits successfully in line 6 (i.e., in Round 1, see Corollary \[1\] for justification).

Finally, we obtain the following result.

**Theorem 1.** Algorithm GranUBR performs broadcasting in a \( n \)-node network of diameter \( D \) with granularity \( g \) in time \( O(D d^2_o(n) \log g) \), where \( d_o \) is a flat function.

### 3.3 General Algorithm

In order to deal with networks with unlimited granularity, we propose a method of “decreasing” granularity to the level of \( 2^{O(\log^2 n)} \) in time \( O(\log^2 n) \). When granularity is decreased, we apply protocols designed for networks with bounded granularity.

Our method of decreasing granularity applies a technique of simulating collision detection in radio networks without collision detection, called Echo, c.f., \[21\]. Using a modified Echo procedure, we can choose “representatives” of dense areas of a (box of a) network, which will work “on behalf” of whole such areas. In this way we decrease granularity of the network. Importantly, this procedure does not harm connectivity of the network nor changes its eccentricity more than by a constant multiplicative factor. We describe this technique in Section 3.3.2.

The above mentioned method of choosing representatives (of “dense” areas) works correctly when applied to one set of stations such that each of them is in the range area of each other. However, when one tries to apply it simultaneously to several remote groups of stations, interferences incurred in the SINR model can disrupt these executions. Therefore, before applying the above method of decreasing granularity, we first design an offline procedure — based on the local views of stations — that partitions the set of stations in a box of the pivotal grid into log \( n \) families of sets. (Note that each station knows all elements of its box of the pivotal grid, since these stations are in its range area.)

The key property of this partition is that the sets in one family \( F \) (called color) are located in such a way that one can execute the leader election procedure (i.e., the choice of representatives) based on Echo simultaneously on all sets from \( F \). Since each set in each family covers a square with side’s length at least \( r/2^{O(\log^2 n)} \), the leaders (representatives) elected in separated sets form subnetworks with granularity \( 2^{O(\log^2 n)} \). This local pre-processing procedure is described in Section 3.3.1.

Finally, in Section 3.3.3 we provide algorithm DiamUBR. This algorithm follows the generic scheme described in Section 3.1 with additional local pre-processing (c.f., Section 3.3.1) and with specific implementations of Election and Inter-Box-Broadcast based on the method of decreasing granularity described in Section 3.3.2.

#### 3.3.1 Partition into collision avoiding families

In the following, a square in the grid \( G_a \) is a square whose vertices belong to \( G_a \) (thus the length of the side of each such square is a multiplicity of \( a \)). We associate such squares with stations of a network located in them in the following way:
(a) a square (box) $R$ of size $a \times a$ is associated with all stations located in it;

(b) any larger square $R$ contains some subset of stations of the network located inside $R$; however, for each square $R'$ of size $a \times a$ included in $R$, either $R$ contains all stations of $R'$ or none of them.

Let $S$ be a set of squares in a grid $G_a$, each $R \in S$ has associated a set of stations $V_R$ located inside $R$. We say that $S$ is collision avoiding if for each $R \in S$ and each $v \in V_R$, the following condition is satisfied:

if the set of transmitting stations in a round is equal to $\{v\} \cup \bigcup_{R' \in S \setminus \{R\}} V_{R'}$

then the message of $v$ is received by each station from $V_R \setminus \{v\}$.

In other words, transmissions in squares different from $R$ cannot disrupt communication in $R$ (even if all elements of other squares are transmitting simultaneously), provided exactly one station from $R$ is transmitting.

Assume that there are given an upper bound $d \cdot a$ on the length of the side of a square and an upper bound $y$ on the number of stations associated with a square. As we show in the following proposition, in order a set $S$ of squares satisfying these bounds be collisions avoiding, it is sufficient that the box-distance between each two elements of $S$ is at least $d_a(n)dy$, where $d_a$ is a flat function.

**Proposition 6.** For each $\alpha \geq 2$, there exists a flat function $d_\alpha$ satisfying the following property. Let $S$ be a set of squares in a grid $G_a$, where $a = \gamma/c$ for some $c \in \mathbb{N}$, such that

- each square $R \in S$ has associated at most $y$ stations located inside $R$,
- the length of the side of each $R \in S$ is at most $d \cdot a$,
- for each $R_1, R_2 \in S$, the box-distance between $R_1$ and $R_2$ is not smaller than $x \cdot a$,
- the number of stations associated to all squares is equal to $n$,

for some $y, d, x \in \mathbb{N}_+$ such that $c > 2d$. If $x \geq d_a(n)dy$ then $S$ is collision avoiding.

**Proof.** Let $c \in \mathbb{N}$, $d, x, y \in \mathbb{N}_+$ be such that $c > 2d$ and $x \geq d$ (note that the proposition concerns $x \geq d_a(n)dy$ only). Recall that $r = (1 + \varepsilon)^{-1/\alpha}$, $\gamma = r/\sqrt{2}$, $a = \gamma/c$.

Let $R \in S$ and $v \in R$. Since the side of $R$ is at most $d a = d \gamma/c$, the distance from $v$ to any other station $w \in R$ is at most $\sqrt{2} \gamma/c = dr/c$. Therefore the power of signal from $v$ received by $w$ is at least

$$\frac{1}{(dr/c)^\alpha} = \left(\frac{c}{rd}\right)^\alpha.$$

On the other hand, $I$, the total noise plus interference received by $w$ and caused by all elements of $\bigcup_{R' \in S \setminus \{R\}} V_{R'} \setminus V_R$ is at most

$$\mathcal{N} + \sum_{j=1}^{n} 4 \cdot 5 j \cdot \frac{y}{(j \cdot xa)^\alpha} \leq 1 + d'_a(n) \cdot \frac{y}{xa} \cdot \left(\frac{c}{r}\right)^\alpha,$$

where $d'_a = \max(1, 20 \cdot 2^{\alpha/2} \cdot \zeta(\alpha - 1))$, $\zeta$ is the Riemann zeta function and $\mathcal{N} = 1$. The above formula follows from the fact that there are at most $20 j$ squares such that the box-distance of each of them to $R$ is in the interval $[j \cdot xa, (j + 1) \cdot xa)$ and the box-distance between each two of them is not smaller than $x \cdot a$ (see Figure 3). Therefore,
Each square whose distance to $R$ is in $[j \cdot xa, (j + 1) \cdot xa)$ has a nonempty intersection with the gray frame. Moreover, the box-distance between any two such squares is at least $xa$, the “width” of the frame.

$$\left( \frac{c}{rd} \right)^{\alpha} \geq 1 + c' \cdot \frac{y}{xa} \cdot \left( \frac{c}{r} \right)^{\alpha},$$

then the message from $v$ is received by $w$ if This implies that the constraint

$$x'^{\alpha} \geq \frac{d'_a(n) \cdot y}{1/d^{\alpha} - \left( \frac{r}{c} \right)^{\alpha}}$$

guarantees that $w$ receives a message from $v$. By the assumption $c > 2d$ and $r < 1$, we see that

$$\frac{1}{d^{\alpha}} - \left( \frac{r}{c} \right)^{\alpha} > \frac{2^{\alpha} - r^{\alpha}}{2^{\alpha} d^{\alpha}} > \frac{1}{2^{\alpha} d^{\alpha}};$$

and therefore

$$\frac{d'_a(n) \cdot y}{d^{\alpha} - \left( \frac{r}{c} \right)^{\alpha}} < d'_a(n) \cdot 2^{\alpha} \cdot d^{\alpha} \cdot y.$$ 

Thus, if $x \geq d_a(n)dy$ where $d_a(n) = 2 \cdot (d'_a(n))^{1/\alpha}$, then the condition \( \square \) for collision avoidance is satisfied.

Below, we present algorithm NoGran which splits a set of stations in $O(\log n)$ collision avoiding families of squares. More precisely, for each box $C$ of the pivotal grid, the algorithm builds $\log n$ collision avoiding families of squares in $C$, such that each station from $C$ belongs to some square in those families.

Let $C$ be a box of the pivotal grid. We start with the set of squares of size $a \times a$ of the grid $G_a$ included in $C$ and containing at least one station, for some sufficiently small $a$ (line 2). The goal is to build such a set of squares in each box of the pivotal grid that subset of squares with similar — up to the multiplicative factor 2 — number of associated stations is collision avoiding. In stages $i \in [0, \log n]$, we consider squares with the number of associated stations in the interval $[2^{i-1}, 2^i]$ (see line 6) and we keep an upper bound $d_a$ on the length of the side of (so far unconsidered) squares. In each stage, we choose greedily as large as possible subsets of squares such that each two squares of a subset are in large distance (to avoid interferences), see lines 8-10 (c.f., Proposition \( \square \)). These squares form the $i$th family of squares (color $i$). The remaining squares are combined into larger
squares containing more than $2^i$ elements each (see lines 7, and 11-13). As we show, it is possible to ensure that the upper bound on the lengths of the side of a square increases sufficiently slow to guarantee that eventually each station belongs to some square and the set of squares is split into \( \log n \) collision avoiding families, assuming \( a = \gamma/2^{O(\log^2 n)} \) (or \( c = O(\log n) \)).

The key issue is that our ultimate goal is to guarantee that the set of squares with a fixed color in all boxes (not only in one fixed box) are collision avoiding, since the algorithm has to perform further computation in various boxes simultaneously. (By the way, if we restrict to one box of the pivotal grid, it is sufficient to associate the same color to all stations. On the other hand, NoGran is executed locally (in one box) since stations should be able to perform this procedure without communication, on the basis of their knowledge about neighborhood. One cannot exclude that squares \( R_1, R_2 \) with the same color which belong to two adjacent boxes of the pivotal grid are very close to each other. Therefore, we refine our coloring in order to avoid the situation that two squares from adjacent boxes have the same color (line 9).

**Algorithm 3 NoGran**

\[
1: \quad a \leftarrow \gamma/c = r/((\sqrt{2}c)) \\
2: \quad S \leftarrow \text{all nonempty boxes of } G_a \text{ inside the box } C \text{ of the pivotal grid} \\
3: \quad \text{for each } R \in S: \quad V_R \leftarrow \text{all stations located in } R; \\
4: \quad d_0 \leftarrow 1 \\
5: \quad \text{for } i = 0, 1, \ldots, \log n \text{ do} \quad \triangleright \text{Iteration of phases} \\
6: \quad x_i \leftarrow c_0 d_i 2^i \\
7: \quad W_i \leftarrow \{ R \in S \mid 2^{i-1} < |V_R| \leq 2^i \} \\
8: \quad E_i \leftarrow \{ (R_1, R_2) \mid R_1, R_2 \in W_i, \text{distM}(R_1, R_2) \leq x_i \cdot a \} \\
9: \quad \text{for each separated vertex } R \text{ of the graph } G_i(W_i, E_i) \text{ do} \\
10: \quad \quad \text{color}(R) \leftarrow (i, j \mod 2, k \mod 2) \\
11: \quad \quad \text{delete } R \text{ from } W_i \\
12: \quad \quad \text{for each connected component } W' \subseteq W_i \text{ do} \\
13: \quad \quad \quad \text{Form a smallest square } R' \text{ containing all elements of } W', \text{ and add } R' \text{ to } S \\
14: \quad \quad \text{Remove all elements of } W' \text{ from } S \\
15: \quad d_{i+1} \leftarrow 4(x_i + d_i)
\]

Now, we formally analyze algorithm NoGran. Let phase \( i \) denote the execution of the body of the main loop, i.e., lines 5-14, of the algorithm NoGran for the corresponding \( i \). Let \( \text{side}(R) \), for a square \( R \), denote the length of the side of \( R \). We will show that the following invariants are satisfied at the beginning of the phase \( i \), for every \( i \geq 0 \):

(A1) Each square \( R \in S \) has more than \( 2^{i-1} \) stations (i.e., \( |V_R| > 2^{i-1} \));

(A2) For each \( R \in S \), the length of the side of \( R \) is not larger than \( \frac{|V_R|d_i}{2^{i-1}} \cdot a \).

**Proposition 7.** The algorithm NoGran satisfies the invariants (A1) and (A2) at the beginning of each phase.

**Proof.** The proof goes by induction. One can easily verify that the invariants are satisfied at the beginning of phase 0. Next, assuming that the invariants are satisfied at the beginning of phase \( i \), we show that they are satisfied at the beginning of phase \( i + 1 \) as well.

As for the invariant (A1), observe that each square having at most \( 2^i \) elements is removed from \( S \) during phase \( i \) (in line 10 or 13). Moreover, each new square added to \( S \) during phase \( i \) contains stations of at least two removed squares (see line 12 and the fact that each separated vertex/square
R is deleted in line 10). Since (A1) is satisfied at the beginning of phase i, the number of station in such a new square is larger than $2^{i-1} + 2^{i-1} = 2^i$.

Concerning (A2), observe that a square that is in $S$ at the beginning of phase $i$ and is not removed from $S$ during phase $i$ satisfies the condition

$$\text{side}(R) \leq \frac{|V_R|d_i}{2^i} \leq \frac{|V_R|d_{i+1}}{2^{i+1}}$$

at the beginning of phase $i + 1$, because $d_i < d_{i+1}/2$ (line 14.). Now, consider a square $R'$ added to $S$ during phase $i$. Let $W'$ be the connected component of $W_i$ whose elements form $R'$. Let $x_1, x_2$ ($y_1, y_2$, respectively) be the smallest and largest values of the first (second, respectively) coordinate of vertices of squares from $W_i$. W.l.o.g. assume that $x_2 - x_1 \geq y_2 - y_1$. Thus, $\text{side}(R') = x_2 - x_1$. Then, there exists a path $(R_1, \ldots, R_p)$ in $W'$ such that $x_1$ is the first coordinate of some vertex of $R_1$, $x_2$ is the first coordinate of some vertex of $R_p$. Our inductive assumptions imply that:

- $2^i \geq |V_{R_i}| > 2^{i-1}$ for each $j \in [p]$;
- $\text{side}(R_j) \leq \frac{|V_{R_i}|d_i}{2^i} \cdot a \leq d_i \cdot a$ for each $i \in [p]$;
- $\text{distM}(R_j, R_{j+1}) \leq x_j \cdot a$ for each $j \in [p-1]$;
- $\sum_{j=1}^{p} \text{side}(R_j) + \sum_{j=1}^{p-1} \text{distM}(R_j, R_{j+1}) \geq x_2 - x_1 = \text{side}(R')$.

Thus,

$$\text{side}(R') \leq (pd_i + (p - 1)x_i) \cdot a \leq p(x_i + d_i) a = pd_{i+1} a/4$$

and

$$|V_{R'}| \leq p \cdot 2^{i-1}.$$ 

Therefore, $\text{side}(R') \leq pd_{i+1} a/4 < \frac{|V_{R'}|d_{i+1}}{2^{i+1}} \cdot a = \frac{|V_{R'}|d_{i+1}}{2^{i+1}} \cdot a$, which confirms that the invariant (A2) is satisfied at the beginning of phase $i + 1$. □

**Proposition 8.** There exists a constant $c_1$, which depends only on $\alpha$, such that: if $c \geq 2^{c_1 \log^2 n}$ then the set of stations with assigned color $i$ by NOGRAN($i, c$) is collision avoiding, for each $i \in [\log n]$.

**Proof.** First, assume that all stations are located in one box of the pivotal grid. The choice of $x_i$ in Algorithm NOGRAN (line 5) guarantees that the set of squares with color $(i, j, k)$ is collision avoiding due to Proposition [3] provided $c > 2d_i$. Since $d_0 = 1$, $d_{i+1} = 4(x_i + d_i)$ and $x_i = c_\alpha d_i 2^i$, the relationship

$$d_{i+1} = 4(d_i + x_i) = 4d_i(1 + c_\alpha 2^i) \leq 8c_\alpha 2^i \cdot d_i$$

holds for $i \geq 0$, where the last inequality follows from the fact that $c_\alpha \geq 1$ (see Proposition [4]). Thus,

$$d_{i+1} \leq (8c_\alpha)^{i+1} \prod_{j=0}^{i} 2^j = 2^{i(i+1)/2+(i+1)\log(8c_\alpha)}.$$ 

Therefore $d_{\log n} = 2^{O(\log^2 n)}$ and the appropriate choice of $c_1$ guarantees that $d_{\log n} < 2^{c_1 \log^2 n}/2$. So, the proposition holds for $c = 2^{c_1 \log^2 n}$, since $c > 2d_i$ for each $i \in [\log n]$ and squares with each color are collision avoiding by Proposition [5].

Now, consider the case when stations are located in various boxes of the pivotal grid. The choice of colors guarantees that $\text{distM}(R_1, R_2) \geq \epsilon r$ for any two squares $R_1, R_2$ with color $(i, j, k)$ such that $R_1 \in C, R_2 \not\in C$, where $C$ is a box of the pivotal grid (the method of assigning $j, k$ guarantees that $R_1$ and $R_2$ are not in adjacent boxes). In order to guarantee the correctness of the proposition, it is sufficient that $\text{distM}(R_1, R_2) \geq c_\alpha d_{\log n}^{\log n}$ (see Proposition [6]). Since $\text{distM}(R_1, R_2) \geq \epsilon r = (1 + \epsilon)^{-1/\alpha}$, it is enough to assure that $c_\alpha d_{\log n}^{\log n} < r$, which can also be guaranteed for $c = 2^{O(\log^2 n)}$. □
Finally, we can state the key property of the algorithm NoGran.

**Lemma 2.** Algorithm NoGran forms the set of $O(\log n)$ collision avoiding families of squares such that each station belongs to (exactly) one square in these families.

**Proof.** Since there are $n$ stations overall, $|V_R| \leq n$ and therefore each station is assigned to a square. Proposition 8 implies that those families are collision avoiding. Finally, it follows directly from the algorithm that each station belongs to exactly one square from those families. 

### 3.3.2 Election by Echo

Before we specify exactly how our application of the procedure Echo [21] works, let us explain what is the task we would like to solve by using this technique. During Inter-Box-Broadcast, if stations in a box $C$ of the pivotal grid are in state active, the goal is to send a message to at least one station in each box $C'$ of $C$ which is a neighbor of $C$. To inform a station in $C'$, it is sufficient that exactly one station from $C$ that has a neighbor in $C'$ is transmitting in some step successfully. We are going to assure this property by guaranteeing that exactly one station is transmitting among stations having neighbors in $C'$. However, although each station from $C$ knows whether it has a neighbor in $C'$, it does not necessarily know which other stations from $C$ have also neighbors in $C'$.

The goal of the algorithm ChooseRepByEcho is as follows. We are given a set $V_1$ of stations such that $(v, w)$ is an edge in the communication graph, for each $v, w \in V_1$ and the set $V_1$ is known to each $v \in V_1$. Moreover, $V_2 \subseteq V_1$ is defined such that each $v \in V_1$ knows whether it belongs to $V_2$ (i.e., whether $v \in V_2$), but it may not have a knowledge which of the remaining elements of $V_1$ belong to $V_2$. As a result, a unique representative $w$ of $V_2$ should be chosen and all elements of $V_1$ should be aware of $w$; in case of $V_2 = \emptyset$, all elements of $V_1$ should be aware of that fact.

**Algorithm 4 ChooseRepByEcho**

```
1: $\psi \leftarrow \min_{v \in V_1} (v)$
2: $\psi$ transmits a message with information whether $\psi \in V_2$ 
3: if $\psi \in V_2$ then return $\psi$ and finish 
4: Let $\varphi \in V_1$ be $\min_{w \in V_1 \setminus \{\psi\}} \{w \mid \text{dist}(\psi, v) \leq \text{dist}(\psi, w)\}$ every $v \in V_1$ 
5: $\varphi$ transmits a message with information whether $\varphi \in V_2$ 
6: if $\varphi \in V_2$ then return $\varphi$ and finish 
7: Assign unique temporary IDs (TIDs) in $|V_1|$ to all elements of $V_1$: $\text{TID}(v) \leftarrow |\{u \in V_1 \mid u \leq v\}|$ 
8: $bot \leftarrow 1$; $top \leftarrow |V_1|$ 
9: while $bot \leq top$ do 
10: mid $\leftarrow [(bot + top)/2]$ 
11: $T \leftarrow \{v \in V_2 \mid bot \leq \text{TID}(v) \leq mid\}$ 
12: **Round R1:** each $v \in T$ transmits the message $m_v$ encoding $v$ 
13: **Round R2:** each $v \in T \cup \{\varphi\}$ transmits the message $m_v$ encoding $v$ 
14: **Round R3:** 
15: if $\psi$ can hear $m_v$ in $R_1$ for $v \in V_1$ then $\psi$ transmits $m_v$ 
16: else if $\psi$ can hear $m_\varphi$ in $R_2$ then $\psi$ transmits $m_\varphi$ 
17: if $m_\varphi$ is heard in $R_3$ for $v \in V_1 \setminus \{\varphi\}$ then return $v$ and finish the algorithm’s execution 
18: if $m_\varphi$ is heard in $R_3$ then $bot \leftarrow mid + 1$ 
19: else $top \leftarrow mid$
```

Just for further consideration we would like to point out that $\text{dist}(\psi, \varphi)$ is the largest among distances between between elements of $V_1$. This implies that $\psi$ can hear $\varphi$ only when no other
Proposition 9. Assume that the algorithm ChooseRepByEcho is executed in parallel on a family of collision avoiding squares. Then, each execution of ChooseRepByEcho($V_1, V_2$) finishes in $O(\log n)$ rounds and it gives the following result:

- if $V_2 = \emptyset$: each station of $V_1$ knows that $V_2$ is empty;
- otherwise, each $v \in V_1$ knows a fixed station $w \in V_2$ called a representative of $V_2$.

Proof. As for time complexity, note that top − bot becomes roughly twice smaller in each execution of the loop 9-19 (see lines 18-19).

The assumption that the algorithm is executed on collision avoiding squares implies that we can assume that each execution of ChooseRepByEcho($V_1, V_2$) satisfies the following condition: if exactly one element of $V_1$ (different from $\varphi$) transmits a message in a round, then this message is received by all elements of $V_1$. Moreover, since $\text{dist}(\psi, \varphi) \geq \text{dist}(\psi, v)$ for each $v \in V_1$, $\psi$ cannot receive a message from $\varphi$ if any element of $V_1 \setminus \{\varphi\}$ transmits a message at the same round. These observations imply that, after round $R_3$, all stations from $V_1$ can determine whether the subset of $V_2$, which consists of stations with TIDs in the range $[\text{bot}, \text{mid}]$, contains 0, 1, or more than one element. Thanks to this fact, an execution of lines 10-19 gives each element of $V_1$ information whether $X = V_2 \cap \{v \mid TID(v) \in [\text{bot}, \text{mid}]\}$ is empty. Using this property, the while-loop 9-19 applies binary search in order to choose a representative of $V_2$, if $V_2 \neq \emptyset$. More precisely, if $m_{\varphi}$ is heard in $R_3$ then $X$ is empty (and searching is restricted to the range $[\text{mid}+1, \text{top}]$), and it is not empty otherwise. \[\square\]

3.3.3 Broadcasting Algorithm

Finally, we define a broadcasting algorithm DiamUBr, which repeats several times the algorithm Gen-Inter-Box-Broadcast given below. Algorithm Gen-Inter-Box-Broadcast resembles the algorithm Gran-Inter-Box-Broadcast from Section 3.2. However Gen-Inter-Box-Broadcast, first applies the technique of “decreasing” granularity introduced in Sections 3.3.1 and 3.3.2.

Recall the following definitions. A station $v$ is $(d_1, d_2)$-connected for $d_1, d_2 \in \{0, 1, 2\}$ iff $v \in C(i, j)$ for a box $C(i, j)$ of the pivotal grid and $v$ has a neighbor in the box $C'(i+d_1, j+d_2)$ of the pivotal grid. Let $u, v$ be such stations that $u \in C$, for a box $C$ of the pivotal grid, and $u$ is in the range area of $v$. We say that $u$ dominates $C$ with respect to $v$ if $u = \min\{w \mid w \in C \text{ and } w \text{ is in the range area of } v\}$. We also set $g_{\alpha} = 2^{c_1 \log^2 n}$, where $c_1$ is the constant from Proposition 8.
Algorithm 5 Gen-Inter-Box-Broadcast

1: Each station $v$ in state active executes $\text{NoGran}(C, c_α)$, where $C$ is the box of the pivotal grid containing $v$, $c_α \leftarrow d_α(n)$ and $d_α$ is a flat function satisfying properties stated in Proposition 6.

2: for each $(d_1, d_2) \in \text{DIR}$ do
3:  for $(i, j, k) \in [\log n] \times \{0, 1\} \times \{0, 1\}$ do
4:    for each square $R$ of color $(i, j, k)$ in-parallel do
5:      $\text{ChooseReprByEcho}(V_R, V_R \cap \{v \mid v \text{ is } (d_1, d_2)-\text{connected}\})$
6:    $d \leftarrow$ parameter from Corollary 1 applied for the set of leaders of boxes of $G_γ$.
7:  for $(j, k) \in [0, d - 1]^2$ do
8:    Round 1: A station $v$ transmits if:
9:      $v$ is elected the leader of its box of the pivotal grid in line 6 during $\text{GranLeaderElection}$,
10:     and $v \in C(j', k')$ such that $(j' \mod d, k' \mod d) = (j, k)$
11:    Round 2: A station $u$ transmits if:
12:      $s(u) = \text{asleep}$,
13:      $u$ heard $v$ in Round 1,
14:     $u \in C(j', k')$ such that $((j' - d_1) \mod d, (k' - d_2) \mod d) = (j, k)$,
15:     and $u$ dominates its box wrt $v$.

Proposition 10. Algorithm Gen-Inter-Box-Broadcast works in time $O(d_α^2(n) \log^2 n)$ for a flat function $d_α$ and it preserves the properties (I) and (P) from page 4.

Proof. As for time complexity, the execution of $\text{ChooseReprByEcho}$ in line 5 requires $O(\log n)$ rounds, and the execution of $\text{GranLeaderElection}$ in line 6 requires $O(\log^2 n)$ rounds. Since $d$ and the size of DIR are constant, Gen-Inter-Box-Broadcast works in time $O(\log^2 n)$.

As algorithm Gen-Inter-Box-Broadcast follows the structure of Gran-Inter-Box-Broadcast, the fact that it preserves (I) and (P) can be proved similarly as Proposition 5. In fact, it is sufficient to prove that if there is $v \in C$ in state active for a box $C$ which is $(d_1, d_2)$-connected, then $C$ has the leader after step 6. This claim is a consequence of the following facts:

- $\text{NoGran}(C, g_α)$ in line 1 guarantees that each active station which is $(d_1, d_2)$-connected is associated with some square which has assigned a color in $[\log n] \times \{0, 1\}^2$; moreover, squares with the same color are collision avoiding (Proposition 5);
- $\text{ChooseReprByEcho}$ (line 5) chooses a representative of $V_R \cap \{v \mid v \text{ is } (d_1, d_2)\text{-connected}\}$ for each square $R$, provided $V_R \cap \{v \mid v \text{ is } (d_1, d_2)-\text{connected}\} \neq \emptyset$ thanks to the fact that squares with a fixed color are collision avoiding (Proposition 9);
- Granularity of the set of representatives in line 6 of the algorithm is at most $g_α$ by Proposition 9 and item (b) on page 12 defining restrictions on associations of squares with stations. Therefore, $\text{GranLeaderElection}$ in line 6 chooses the leader in the box $C$, if the set of station from $\{v \mid v \text{ is a representative chosen in line 5}\}$ located in $C$ is nonempty (Proposition 4).

Below, we state a theorem which follows directly from the specification of Algorithm $\text{DiamUBr}$ (i.e., repeating algorithm Gen-Inter-Box-Broadcast) and from Proposition 10.

Theorem 2. Algorithm $\text{DiamUBr}$ performs broadcasting in a $n$-node network of diameter $D$ in time $O(Dd_α^2(n) \log^2 n)$, where $d_α$ is a flat function.
4 Size Dependent Algorithm for Anonymous Networks

In this section we consider fully anonymous ad hoc networks in which, at the beginning of a protocol, execution each station knows only $n$, $N$, its own ID and its position in the Euclidean space (i.e., its coordinates). We develop a deterministic broadcasting algorithm $\text{SizeUBr}$, which matches the lower bound $\Omega(n \log N)$ (see Theorem 6).

4.1 High-Level Idea of Algorithm $\text{SizeUBr}$

Our algorithm executes repeatedly two threads.

The first thread keeps combining stations into groups in such a way that eventually, for any box $C$ of the pivotal grid, all stations located in $C$ form one group. Moreover, each group should have the leader, and each station should be aware of (i) which group it belongs to, (ii) which station is the leader of that group, and (iii) which stations belong to that group (i.e., a station should know the set of IDs and positions of all stations in the group). These properties are achieved as follows.

Upon waking up, each station forms a group with a single element (itself), and then the groups increase gradually by merging. The merging process builds upon the following observation. Let $\sigma$ be the smallest distance between two stations taking part in the first thread, and let $u, v$ be two closest stations. Thus, there is at most one transmitting station in each box of the grid $G_{\sigma/\sqrt{2}}$. Then, if $u$ (resp., $v$) transmits a message and no other station in distance $d \cdot \sigma$, for some constant $d$, transmits at the same time, then $v$ (resp., $u$) can hear that message (see Proposition 6). Using combinatorial structure called strongly-selective family (ssf) as a broadcast schedule, one can assure that a round satisfying these properties occurs in $O(\log n)$ rounds. If $u$ can hear $v$ and $v$ can hear $u$ during such a schedule, the groups of $u$ and $v$ can be merged into one larger group.

The second thread, on the other hand, is supposed to guarantee that in each round $t$ of the algorithm and for each group of stations $H$, exactly one station from $H$ is transmitting a message in round $t$. This property will be satisfied provided each station knows its group, so it can determine its temporary ID (TID) as the rank of its ID in the sequence of IDs of stations from the group, taken in a nondecreasing order. Using these TIDs, the stations of the group apply round-robin strategy. Thus, if each group corresponds to all stations in the appropriate box, transmissions in the second thread are successful (see Corollary 1, Proposition 6 for $y = 1$, $a = 1$ and $d = 1$), and therefore they guarantee that all neighbors of the box will have informed stations, provided the second thread is executed for sufficiently long time.

In order to apply the above described ideas for global broadcasting, it is necessary to repeat Threads 1 and 2 several times. The main problem with implementation and its analysis is that there is no simple way to determine whether group(s) already covers the whole box of the pivotal grid. Moreover, as long as there are many groups inside a box, transmissions in the second thread may cause unwanted interferences. Another problem is that the set of stations attending the protocol changes gradually, when new stations become informed and can join the execution of the protocol. Therefore we modify the above described ideas in the following way:

- The two threads — one forming groups and the other transmitting in a round-robin fashion — are interleaved such that one round of the former is followed by one round of the latter. This will be conceptually implemented in a form of two parallel threads.

- In order to tackle the lack of knowledge about the progress in computation, each station participates in the protocol for $T(n)$ rounds, where $T(n)$ is the upper bound on the round complexity of accomplishing our broadcasting algorithm derived in the analysis.

\footnote{It is sufficient that $O(\log^2 n)$ bits of coordinates of stations are stored.}
Finally, our proof of complexity bound is based on measuring the progress of computation at round \( t \) by using amortized analysis, in a way reflecting the advancement of the process of merging groups and receiving the broadcast message by consecutive stations.

### 4.2 Formal Implementation of Algorithm \( \text{SIZEUBR} \)

Each station \( v \) keeps in its local memory a boolean variable \( L(v) \) indicating whether \( v \) has the status of the leader of its group, and local variables \( M(v) \in V \) and \( G(v) \subseteq V \). Let us think of a directed graph defined by edges \((v, M(v))\). Our goal is to preserve the invariant that the graph is a forest and each edge \((v, M(v))\) is directed from a child to its parent in the appropriate tree of \( F \). Provided this invariant is preserved, we define \( \text{master}(v) \) as the transitive closure of \( M(v) \), i.e., \( \text{master}(v) = v \) if \( M(v) = v \) and \( \text{master}(v) = \text{master}(M(v)) \) otherwise. Moreover, \( \text{group}(v) = G(\text{master}(v)) \). The fact that pointers \( M(v) \) define a forest gives a partition of the set of stations in the following way:

- each tree of this graph forms one group;
- each group has the leader which is equal to the root of the appropriate tree; that is, the leader of the group to which \( v \) belongs is equal to \( \text{master}(v) \).

We say that a station \( v \) is consistent if \( M(v) = \text{master}(v) \) and \( G(v) = \text{group}(v) \). Initial values of the local variables of stations are as follows: \( L(v) \leftarrow \text{true} \), \( M(v) \leftarrow v \), \( G(v) \leftarrow \{v\} \). Thus, all stations are consistent at the beginning. A leader is each station \( v \) such that \( L(v) = \text{true} \).

We say that a network satisfies integrity at time \( t \) iff

(a) groups \( G(v) \) known by leaders at the end of round \( t \) form a partition of the set of all stations \( V \) (i.e., \( V = \bigcup_{v \mid L(v)} G(v) \) and \( G(v) \cap G(u) = \emptyset \) for each \( v \neq u \) such that \( L(v) = L(u) = \text{true} \));

(b) \( G(v) \subseteq G(M(v)) \) for each station \( v \);

(c) \( M(v) \in \text{box}(v) \) and \( G(v) \) contains only stations located in \( \text{box}(v) \).

One of invariants which we are going to be preserved along executions of \( \text{SIZEUBR} \) is that all leaders are consistent, and the network satisfies integrity. Ideally, we would also like to achieve consistency of stations which are not leaders — unfortunately this property will not be guaranteed by our solution, however our algorithm will be able to achieve it at some crucial stages of the broadcasting task.

The algorithm proceeds in two parallel threads: Thread 1 and Thread 2. We assume that Thread 1 is executed in odd rounds (i.e., in rounds \( t \) such that \( t \mod 2 = 0 \)) and Thread 2 in even rounds. In order to simplify presentation, we assume that rounds of Thread 1/Thread 2 have consecutive numbers \( 1, 2, 3, \ldots \). Below, we describe both threads in more detail.

**Thread 1.** The main goal of Thread 1 is to merge groups such that consistency of leaders and integrity of network are preserved. The following technical proposition is the key for guaranteeing process of merging groups is fast enough.

**Proposition 11.** For each \( \alpha > 2 \), there exists a constant \( d \), which depends only on the parameters \( \varepsilon, \beta \) and \( \alpha \) of the model, satisfying the following property. Let \( W \) be a set of stations such that there is at most one station from \( W \) in each box of the grid \( G_x \), for some \( x \leq \gamma \), and \( \min_{u,v \in W} \{ \text{dist}(u,v) \} = x \cdot \sqrt{2} \). If station \( u \in C \) for a box \( C \) of \( G_x \) is transmitting in a round \( t \) and no other station in any box \( C' \) of \( G_x \) in the box-distance at most \( d \) from \( C \) is transmitting at that round, then \( v \) can hear the message from \( u \) at round \( t \).
Proof. Let \( u, v \) satisfy properties stated in the proposition. If \( u \) is transmitting in round \( t \) then the power of the signal of \( u \) arriving at \( v \) is

\[
\frac{1}{(\sqrt{2}x)^\alpha} \geq (1 + \varepsilon)N,
\]

where the inequality follows from the fact that \( \sqrt{2}x \leq r = (1 + \varepsilon)^{-1/\alpha} \) (recall that we assume \( \beta = 1 \)). Observe that, under the assumptions of the proposition, the number of stations whose distance to \( v \) is in the interval \([ix, (i + 1)x)\) is not larger than the number of boxes of \( G_x \) in box-distance \( i \) from the box containing \( v \), which in turn is equal to \( 8(i + 1) \). Assuming that no station in any box \( C' \) in the box-distance at most \( d \) from \( C \) is transmitting, the amount of interference and noise at \( v \) is smaller than

\[
N + \sum_{i=d}^{\infty} 8(i + 1) \cdot \frac{1}{(ix)^\alpha} = N + \frac{8}{x^\alpha} \cdot c_d,
\]

where \( c_d = \sum_{i=d+1}^{\infty} i^{1-\alpha} \). Thus, by (5) it is sufficient to show that there exists \( d \) which guarantees that

\[
N + \frac{8}{x^\alpha} c_d \leq (1 + \varepsilon)N \quad \text{or} \quad N + \frac{8}{x^\alpha} c_d \leq \frac{1}{2^{1/2} x^\alpha}
\]

for each \( x > 0 \), which is equivalent to:

\[
c_d \leq \frac{1 - N (\sqrt{2}x)^\alpha}{8 \cdot 2^{1/2}} \quad \text{or} \quad c_d \leq \frac{\varepsilon N x^\alpha}{8}.
\]

Consider two cases:

Case A: \( N (\sqrt{2}x)^\alpha \leq \frac{1}{2} \)

This case reduces the first inequality of (6) to \( c_d \leq \frac{1}{16 \cdot 2^{1/2}} \) which is satisfied for sufficiently large \( d \), due to convergence of \( \sum_i i^{1-\alpha} \).

Case B: \( N (\sqrt{2}x)^\alpha > \frac{1}{2} \)

In this case, the second inequality of (6) reduces to \( c_d \leq \frac{\varepsilon}{16 \cdot 2^{1/2}} \) which is also satisfied for sufficiently large \( d \), due to convergence of \( \sum_i i^{1-\alpha} \).

A family \( S = (S_0, \ldots, S_{s-1}) \) of subsets of \([N]\) is a \((N, k)\)-ssf (strongly-selective family) of length \( s \) if, for every non empty subset \( Z \) of \([N]\) such that \(|Z| \leq k\) and for every element \( z \in Z \), there is a set \( S_i \) in \( S \) such that \( S_i \cap Z = \{z\} \). It is known that there exists \((N, k)\)-ssf of size \( O(k^2 \log N) \) for every \( k \leq N \), c.f., [4]. Let \( k = (2d + 1)^2 \), let \( S \) be a \((N, k)\)-ssf, and let \( s = |S| = O(\log N) \). The sets \( S_0, \ldots, S_{s-1} \) of the family \( S \) define a broadcast schedule in such a way that station \( v \) transmits in round \( t \) iff \( v \in S_t \mod s \) (formally, the bit \( t \) of \( S(v) \) is equal to \( 1 \) iff \( v \in S_t \)).

**Corollary 2.** For each \( \alpha > 2 \), there exists a constant \( d \), which depends only on the parameters \( \varepsilon, \beta \) and \( \alpha \) of the model, satisfying the following property. Let \( W \) be a set of stations such that \( \min_{u,v \in W, \text{box}(u)=\text{box}(v)} \{\text{dist}(u, v)\} = x \) and let \( \text{dist}(u, v) = x \) for some \( u, v \in W \) such that \( \text{box}(u) = \text{box}(v) \) and \( W \) is \( d \)-diluted for \( d \geq 2 \). Then, \( v \) can hear the message from \( u \) during an execution of a \((N, k)\)-ssf on \( W \).

Now, we are ready to describe Thread 1 in detail. Given a \((N, k)\)-ssf \( S \) of length \( s \), Thread 1 consists of blocks of \( 2s \) rounds, each block split in two stages of length \( s \). Importantly, a station which becomes informed during a block, starts participating in the execution of the protocol in the next block of Thread 1. Algorithm [1] describes behavior of a station \( v \) in step \( t \). Note that the initial
value of $X_v$ is equal to the empty set for each $v$ at the beginning of a block (see Algorithm 7) and then, it is equal to the set of station which transmitted successfully to $v$ during the block.

### Algorithm 6 Thread1($v, t$)

1: $t' \leftarrow t \mod 2$
2: if $v$ informed before step $t - t'$ then \(\triangleright v\) informed before the current block
3: \hspace{1em} if $t' < t$ then \(\triangleright\) (Stage 1 of a block)
4: \hspace{2em} if $L(v)$ and $v \in S_{t \mod s}$ then
5: \hspace{3em} $v$ transmits a message including $v$ and $G(v)$
6: \hspace{2em} else
7: \hspace{3em} if $L(v)$ then
8: \hspace{4em} if $v$ can hear $u$ then $X_v \leftarrow X_v \cup \{u\}$
9: \hspace{3em} else
10: \hspace{4em} if $v$ can hear $u$ such that $G(v) \subset G(u)$ then $M(v) \leftarrow u; G(v) \leftarrow G(u)$
11: \hspace{2em} else \(\triangleright\) (Stage 2 of a block)
12: \hspace{1em} if $L(v)$ and $v \in S_{t \mod s}$ then
13: \hspace{2em} $v$ transmits a message including $v$ and $X_v$
14: \hspace{1em} Modify($v, t$)

In a single block of Thread 1, the $(N,k)$-ssf $S$ is executed twice: once in Stage 1 and once in Stage 2. At the end of the block, the procedure Modify is executed, whose goal is to merge groups using information gathered in Stages 1 and 2 of the current block. In Stage 1, each station $v$ determines $X_v$, the set of stations $u$ such that $v$ can hear $u$ during the execution of $S$ (on the set of stations active at the beginning of Stage 1 of the block). In Stage 2, each station $v$ sends $X_v$, and in this way, at the end of Stage 2, it also collects information about $X_u$ for each $u \in X_v$.

For a fixed block of computation, let $G'(V, E')$ be a symmetric graph which consists of such edges $(u, v)$ that $u$ and $v$ have the status of leaders, $u$ can hear $v$ and $v$ can hear $u$ during the block of computation. Note that $(u, v) \in E'$ iff $v \in X_u$ and $u \in X_v$. Thus, each station can determine its neighbors in $G'$ at the end of each block (since $v$ knows $X_v$ after Stage 1, and it learns $X_u$, for each $u \in X_v$, during Stage 2).

At the end of each block of Thread 1, each station modifies its local variables appropriately, by executing procedure Modify, c.f., the pseudo-code of Algorithm 6. The goal is to make at least one merge of two groups. In order to achieve this goal, we implement an algorithm which builds (in distributed way) a matching in $G'$ such that the matching is nonempty iff the set of edges of $G'$ is nonempty as well. (Actually, our algorithm builds such a matching that each station $v$ satisfying the following properties chooses its “partner” in the matching: $v$ can hear another station during a block and $v$ is smaller than IDs of stations which transmitted successfully a message to $v$ in the block.) Then, the groups of the pairs of stations in the matching are merged.
Algorithm 7 Modify($v, t$)

1: if $t \mod 2s = 0$ then
   \hspace{1em} $\triangleright$ Execute at the end of round $t$ such that $t \mod 2s = 0$
2: \hspace{2em} \text{match}(v) \leftarrow \text{nil}$
3: \hspace{2em} if $L(v)$ and $X_v \neq \emptyset$ then
4: \hspace{3em} $u \leftarrow \min(X_v)$
5: \hspace{3em} if $v = \min(X_u)$ then
6: \hspace{4em} \text{match}(v) \leftarrow u$
7: \hspace{3em} if $v > u$ then
8: \hspace{4em} $M(v) \leftarrow u; L(v) \leftarrow false$
9: \hspace{3em} $G(v) \leftarrow G(v) \cup G(u)$
10: $X_v \leftarrow \emptyset$

Thread 2. In Thread 2, each station applies round-robin algorithm inside its group. This is done successfully provided the stations possess up to date information about their groups — which is the goal of the previously described Thread 1.

Algorithm 8 Thread2($v, t$)

1: $\Delta \leftarrow \abs{G(v)}$
2: $TID(v) \leftarrow \{u \mid u \in G(v) \text{ and } u < v\}$
3: if $t \mod \Delta = TID(v)$: $v$ transmits a message.

4.3 Analysis

Recall that we make a simplifying assumption that, if at most one station from each box of the pivotal grid transmits in a round $t$, then each such transmission is successful. Due to Corollary 1, one can achieve this property using dilution with constant parameter $d$ (provided $\alpha > 2$), which does not change the asymptotic complexity of our algorithm.

First, we prove some basic properties of Thread 1.

Proposition 12. Thread 1 preserves consistency of leaders and integrity of network at any round.

Proof. Assume that consistency of leaders and integrity of network are satisfied at the beginning of a block of Thread 1. Since variables determining integrity of the network and consistency of stations change only at the end of blocks (i.e., during the execution of algorithm Modify), let us consider round $t$ at the end of a block. Note that $u = \text{match}(v)$ iff $v = \text{match}(u)$ at the end of Modify($v, t$). Moreover, if $u = \text{match}(v)$ and $v = \text{match}(u)$, then exactly one of $u, v$ becomes non-leader and one of them remains the leader. Thus, as a result, the groups $G(v), G(u)$ are replaced by $G(v) \cup G(u)$ after step $t$, which proves integrity. Since the group of the station $v$ changes only in case $u = \text{match}(v)$, $v = \text{match}(u)$ and $L(v) = L(u) = true$ for some $u$, it preserves consistency thanks to the fact that such $u$ and $v$ exchange messages with $u$ during the analyzed block of Thread 1.

We say that station $u$ joins the group of station $v$ during the block of Thread 1 if $L(u) = L(v) = true$ at the beginning of the block, while $L(u) = false$, $L(v) = true$, and $M(u) = v$ at the end of that block.

Lemma 3. Assume that the set $W$ of leaders at the beginning of a block of Thread 1 contains at least two elements, which are located in the same box of the pivotal grid. Then, there exist $u, v \in W$ such that $u$ joins the group of $v$ during the block.
Proof. Let $y$ be equal to the smallest distance between a pair of stations $u, v \in W$ such that $u$ and $v$ belong to the same box of the pivotal grid. Let $u, v$ be the elements of $W$ such that $\text{dist}(u, v) = y$ and $\text{box}(u) = \text{box}(v)$. Let $x = y/\sqrt{2}$. Let $u \in C$ for a box $C$ of the grid $G_x$ and let $A$ be the set of elements of $W$ located in boxes of $G_x$ which are in box-distance at most $d$ from $C$, where $d$ is the constant from Proposition $[11]$. The set $A$ contains at most $(2d + 1)^2$ elements, since each box of $G_x$ contains at most one element of $W$. Therefore, there exists a round $t \leq s$ in the ssf $S$ such that $v$ is transmitting a message at round $t$ and no other element of $A$ is transmitting at that round. Proposition $[11]$ implies that $u$ can hear $v$ in such a round. Similarly, $v$ can hear $u$ during an execution of $S$. Therefore, there exists at least one pair $(u, v)$ such that $u \in X_v$ and $v \in X_u$ at round $2s$ of the block, which is equivalent to the fact that $E' = \{(u, v) | u \in X_v \text{ and } v \in X_u\}$, the set of edges of a graph $G'(V, E')$, is nonempty. Now, let $u$ be the smallest ID of a node whose degree in $G'$ is larger than zero. Let $v$ be its neighbor in $G'$ with the smallest ID. It is clear from the construction that $v$ joins the group of $u$ in such case (see algorithm Modify($v, t^*$), for $t^*$ being the last round of the block).

In general, it might happen that a station which is not a leader is not consistent. Such a situation occurs, for example, when $u$ joins the group of $v$ and then $v$ joins the group of $w$. Simultaneously, while $v$ can hear $w$ when it joins the group of $w$, it is possible that $u$ cannot hear $w$. The following lemma states that eventually, when there is at most one leader in each box at the beginning of a block of Thread 1, then for each leader, all stations in its box correctly update the information about their masters and groups during the considered block and become consistent.

**Lemma 4.** Assume that there is at most one leader in each box of the pivotal grid containing active stations, at the beginning of a block of Thread 1. Then, for each box $C$ containing a leader and each $v \in C$ that is informed at the beginning of the block, $v$ is consistent at the end of the block.

**Proof.** Let $v \in C$ be informed and let $u \in C$ be the only leader in $C$ at the beginning of a block. Integrity of the network and consistency of leaders (Proposition $[12]$) guarantee that $u = \text{master}(v)$. The station $v$ can hear $u$ during the block, which follows from the fact that each leader broadcasts successfully during the block (due to our simplifying assumption concerning situation that at most one station in each box of the pivotal grid is transmitting). Thus, since $v$ receives a message from $u = \text{master}(v)$, it updates its local variables in line 10 of pseudo-code of Thread 1 and becomes consistent.

We say that a block $j$ of Thread 2 is partially stable if the following conditions are satisfied:

- each box of the pivotal grid contains at most one leader;
- at least one informed station is not consistent;

at the beginning of the block $j$. Formally, we define progress of algorithm SIZEUBr at the end of block $j$ as $\pi(j)$, equal to the sum of the following four components:

(a) the number of informed stations;

(b) $n$ minus the number of groups;

(c) the number of tuples $(v, d_1, d_2)$ such that $v$ is an informed station, $d_1, d_2 \in \text{DIR}$, $v$ belongs to $C(i, l)$ for some $i, l \in \mathbb{Z}$, and there is an informed station in the box $C' = C(i + d_1, l + d_2)$, where $C, C'$ are boxes of the pivotal grid;

(d) the number of partially stable blocks of Thread 1 up to round $t$. 

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It is clear that the expressions described in the above items (a)–(c) have always values in $O(n)$. We show that (d) is also in $O(n)$, which directly implies that $\pi(j) = O(n)$ for every $j$.

**Proposition 13.** For each network with $n$ stations, the number of partially stable blocks of Thread 1 is smaller than $n$.

**Proof.** Consider two consecutive blocks $j_1 < j_2$ of Thread 1 satisfying (i) and (ii). Lemma 4 implies that all stations informed at the beginning of block $j_1$ are consistent at the beginning of block $j_2$. Note that an informed station located in a box $C$ of the pivotal grid with one leader may lose its consistency only in the case when a new station from box $C$ becomes informed. Since there is an informed station that is not consistent at the beginning of block $j_2$ (c.f., (iii)), the number of informed stations at the beginning of block $j_2$ is larger than the number of informed stations at the beginning of block $j_1$. Therefore the number of blocks of Thread 1 satisfying (i) and (ii) is smaller than $n$. \hfill $\square$

Now, we show that the amortized increase of cost $\pi$ during each block of Thread 1 — defined as the time period including block of Thread 1 and rounds of Thread 2 interleaved with the block of Thread 1 — is at least one.

In the following, we analyze progress of computation during blocks of Thread 1, however we take into account also rounds of Thread 2 occurring during the time span of the analyzed block of Thread 1 (recall that the executions of the two threads are interleaved).

**Lemma 5.** Assume that some stations are not yet informed at the beginning of some block $j$ of Thread 1. Then, there exists a block $k \geq j$ such that the total increase of progress function in blocks $j, j+1, \ldots, k$ is at least $k - j + 1$.

**Proof.** If there are two informed stations $u, v \in C$, for a box $C$ of the pivotal grid, such that $L(u) = L(v) = \text{true}$ (i.e., $u, v$ are leaders) at the beginning of block $j$, progress increase is guaranteed in block $j$ by Lemma 5 since at least one merge of two groups takes place.

If there is at most one leader in each box at the beginning of block $j$, then we consider two cases:

**Case 1.** All informed stations are consistent at the beginning of block $j$.

In this case all transmissions in both Threads are successful, as long as the number of informed stations does not change. Therefore, each informed station can transmit successfully. And, since not all stations in the network are informed and the network is connected, a new station becomes informed eventually. Let $k \geq j$ be the smallest number of a block in which a new station $v$ becomes informed.

If this station $v$ belongs to a box which has an informed leader at the beginning of block $j$, then $v$ becomes informed in block $j$ and the progress increase is 1 in round $j$, which certifies the claimed result for $k = j$.

If the box $C'$ containing station $v$ does not have an informed leader at the beginning of block $j$, then $C'$ does not have any informed station at the beginning of block $j$ either (due to integrity of the network). Let $u \in C$ be a station that informed $v$ and $k \geq j$ be the number of the block in which $v$ becomes informed. Since each transmission of Thread 2 is successful in this case, and Thread 2 applies a round-robin protocol on stations from $C$, $u$ does not transmit in blocks $j, j+1, \ldots, k-1$ implies that the number of stations in box $C$ is at least $(k - j) + 1$ (since at least one station from box $C$ transmits during the time span of one block in Thread 2). Moreover, there are $k - j$ various stations in $C$ such that each of them transmits successfully in blocks $j, \ldots, k-1$. Let $C = C(i, j), C' = C(i + d_1, j + d_2)$. Therefore, the number of tuples $(v, d_1, d_2)$ such that $v$ is an informed station and belongs to $C(i, j)$ for some $i, j \in \mathbb{Z}$, $d_1, d_2 \in \text{DIR}$ and there is an informed station in the box $C'(i + d_1, j + d_2)$, increases by at least $k - j + 1$ throughout blocks $j, \ldots, k$. Therefore, the progress $\pi$ increases by at least $k - j + 1$. 

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Case 2. There is a station which is not consistent at the beginning of block \( j \).

Then, the part (d) of the potential function \( \pi \) increases until the end of block \( j \), according to Lemma 4.

Finally, we obtain the following theorem as a direct consequence of Lemma 5.

Theorem 3. Algorithm SIZEUBR performs broadcasting in each \( n \)-node network in time \( O(n \log N) \).

5 Degree Dependent Algorithm for Anonymous Networks

In this section we present a broadcasting algorithm which achieves complexity \( O(D\Delta \log^2 N) \) in anonymous networks, i.e., when neighborhood is not known.

The core of the algorithm is a leader election procedure which, given a set of stations \( V \), chooses exactly one station (the leader) in each box \( C \) of the pivotal grid which contains at least one element of \( V \). This procedure works in \( O(\log n \cdot \log N) \) rounds and it is executed several times. The set of stations attending a particular leader election execution consists of all stations which received the broadcast message and have not been chosen leaders of their boxes in previous executions of the leader election procedure. Moreover, at the end of each execution of the leader election procedure, each leader chosen in that execution transmits a message successfully (see Corollary 1). In this way, each station receives the broadcast message after \( O(D\Delta \log^2 N) \) rounds.

5.1 Leader Election

In the following, we describe the leader election algorithm. We are given a set of stations \( V \) of size at most \( n \). The set \( V \) is not known to stations, each station knows merely whether it belongs to \( V \) or it does not belong to \( V \). In the algorithm, we use \( (N,d)\)-ssf \( S \) of size \( s = O(\log N) \), where \( d \) is the constant from Proposition 11. As before, \( X_v \) for a given execution of \( S \) is defined as the set of stations which belong to \( box(v) \) and \( v \) can hear them during that execution. The key observation for our construction is in fact a consequence of Corollary 2.

Proposition 14. For each \( \alpha > 2 \), there exists a constant \( k \), which depends only on the parameters \( \varepsilon, \beta \) and \( \alpha \) of the model, satisfying the following property. Let \( W \) be a 3-diluted (wrt the pivotal grid) set of stations and let \( C \) be a box of the pivotal grid. If \( \min_{u,v \in C \cap W} x = \leq 1/n \) and \( \text{dist}(u,v) = x \) for some \( u,v \in W \) such that \( box(u) = box(v) = C \), then \( v \) can hear the message from \( u \) during an execution of a \( (N,k)\)-ssf on \( W \).

Proof. Let \( u,v \) and \( x \) be as specified in the proposition and let \( C = box(u) = box(v) \). Let \( S \) be a \( (N,k)\)-ssf. If all stations from \( W \) are located in \( C \), then the claim follows directly from Corollary 2. So, let \( W' \) be the set of all elements of \( W \) which are not located in \( C \). Let us (conceptually) “move” all stations from \( W' \) to boxes adjacent to \( C \), preserving the invariant that \( \min_{u,v \in W, box(u) = box(v) = C} \{\text{dist}(u,v)\} = x \). Note that such a movement is possible, since there are at most \( n \) stations in \( W' \) and the side of a box of the pivotal grid is larger than \( 1/2 \). Since \( W \) is 3-diluted, the distance from \( w \in C \) to any station \( w' \in W' \) before movement of \( w' \) is larger than the distance from \( w \) to \( w' \) after movement. Let \( W'' \) define \( W \) with new locations of stations (after movements). Therefore, if \( u \) can hear \( v \) in the execution of \( S \) on \( W'' \) (i.e., after movements of stations), it can hear \( v \) in the execution of \( S \) on \( W \) (i.e., with original placements of stations). However, the fact that \( u \) can hear \( v \) on \( W'' \) follows directly from the fact that \( \min_{u,v \in W''} \{\text{dist}(u,v)\} = x \) by Corollary 2.

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The leader election algorithm consists of two stages. The first stage gradually eliminates elements from the set of candidates for the leader in consecutive executions of a selector $S$ in the first for loop. Therefore, we call this stage Elimination. Let block $l$ of Elimination stage denote the executions of $S$ for $i = l$. Each station $v$ “eliminated” in block $l$ has assigned the value $ph(v) = l$. Let $V(l) = \{v | ph(v) > l\}$ and $V_C(l) = \{v | ph(v) > l \text{ and box}(v) = C\}$ for $l \in \mathbb{N}$ and $C$ which is a box of the pivotal grid. The key property of sets $V_C(l)$ is that $|V_C(l+1)| \leq |V_C(l)|/2$ and the granularity of $V_C(l)\{l\}$ is smaller than $n$ for each box $C$ and $l \in \mathbb{N}$, where $l_C$ is the largest $l \in \mathbb{N}$ such that $V_C(l)$ is not empty. Therefore, we can choose the leader of each box $C$ applying (simultaneously in each box) the granularity dependent leader election algorithm on $V_C(l_C)$. It is done by the second stage, which applies the granularity dependent leader election on $V_C(\log n)$, $V_C(\log n - 1)$, $V_C(\log n - 2)$ and so on, until the leader of $C$ is chosen. After it is done all stations in $C$ become silent. This idea is implemented in the second part of the algorithm, called Selection. Now, we provide the pseudo-code of the leader election algorithm and then its correctness and complexity are formally analyzed.

Algorithm 9 LeaderElection($V, n$)
1: For each $v \in V$: $\text{cand}(v) \leftarrow \text{true}$; 
2: for $i = 1, \ldots, \log n + 1$ do \hspace{1cm} \hspace{1cm} // Elimination 
3: \hspace{1cm} for $j, k \in [0, 2]$ do 
4: \hspace{2cm} Execute $S$ twice on the set: 
5: \hspace{2.5cm} $\{w \in V | \text{cand}(w) = \text{true} \text{ and } w \in C(j', k') \text{ such that } (j' \mod 2, k' \mod 2) = (j, k)\}$; 
6: \hspace{2cm} Each $w \in V$ determines and stores $X_w$ during the first execution of $S$ and 
7: \hspace{2cm} $X_w$ for each $v \in X_w$ during the second execution of $S$, 
8: \hspace{2cm} for each $v \in V$ do 
9: \hspace{3cm} $u \leftarrow \min(X_v)$ 
10: \hspace{3cm} if $X_v = \emptyset$ or $v > \min(X_v \cup \{u\})$ then 
11: \hspace{4cm} $\text{cand}(v) \leftarrow \text{false}; \text{ph}(v) \leftarrow i$ 
12: For each $v \in V$: $\text{state}(v) \leftarrow \text{active}$ \hspace{1cm} \hspace{1cm} // Selection 
13: for $i = \log n, (\log n) - 1, \ldots, 2, 1$ do 
14: \hspace{1cm} $V_i \leftarrow \text{GranLeaderElection}(\{v \in V | \text{ph}(v) = i, \text{state}(v) = \text{active}\}, 1/n)$ \hspace{1cm} \hspace{1cm} // $V_i$ - leaders 
15: Each element $v \in V_i$ sets $\text{state}(v) \leftarrow \text{leader}$ and transmits successfully 
16: using constant dilution (see Corollary[1]) 
17: Simultaneously, for each $v \in V$ which can hear $u \in \text{box}(v)$: $\text{state}(v) \leftarrow \text{passive}$

**Lemma 6.** Let $C$ be a box of the pivotal grid and $l \in \mathbb{N}$. Then,

1. $|V_C(l+1)| \leq |V_C(l)|/2$;

2. If $V_C(l+1)$ is empty, then the smallest distance between elements of $V_C(l)$ is at least $1/n$.

**Proof.** Similarly as in Section[1] our algorithm implicitly builds matchings in the graphs whose vertices are $V_C(l)$ and an edge connects such $u$ and $v$ that $u$ can hear $v$ and $v$ can hear $u$ during an execution of $S$. Note that the station $v \in V_C(l)$ belongs to $V_C(l+1)$ only if the following conditions are satisfied:

- $v = \min(X_u)$;
- $u = \min(X_v)$; $v < u$

for some $u \in V_C(l)$. That is, only elements of the matching belong to $V_C(l+1)$ and exactly one element from each matched pair belongs to $V_C(l+1)$. 

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Therefore, the inequality $|V_C(l + 1)| \leq |V_C(l)|$ holds. This gives item 1 of the lemma.

As for item 2, assume that $V_C(l)$ is not empty. Observe that $V_C(l + 1)$ is not empty if there exist $v, u \in V_C(l)$ such that $v$ can hear $u$ and $u$ can hear $v$. (Indeed, $v \in V_C(l + 1)$ for the smallest $v \in V_C(l)$ such that $v$ can hear $u$ and $u$ can hear $v$ for some $u \in V_C(l)$.) However, such $v$ and $u$ exist if the smallest distance between elements of $V_C(l)$ is at least $\frac{1}{n}$ by Proposition 14.

**Theorem 4.** Algorithm LeaderElection chooses the leader in each box of the pivotal grid containing at least one element of $V$ in $O(\log n \log N) = O(\log^2 N)$ rounds, provided $\alpha > 2$.

*Proof.* Time complexity $O(\log^2 n)$ follows immediately from the bounds on the size of selectors and complexity of GranLeaderElection.

Lemma 6.1 implies that $V_C(l) = \emptyset$ for each box $C$ and $l > \log n$. (In other words, $ph(v) \leq \log n$ for each $v \in V$.) Moreover, by Lemma 6.2, the smallest distance between stations of $V_C(l_0)$ is at least $1/n$, where $l_0 = \max_l \{V_C(l) \neq \emptyset\}$. In other words the smallest distance between stations of $\{v \in V \mid ph(v) = l_0, state(v) = active\}$ is $\geq 1/n$, where $l_0$ is the largest number $l$ such that $ph(v) = l$ for some $v \in V$.

Let us focus on a box $C$ which contains at least one station from $V$. Selection stage (the for-loop in lines 13-16) tries to choose the leader of $C$ among $V_C(\log n), V_C(\log n - 1), \ldots$. Moreover, when the leader is elected, all stations from $C$ are switched off (i.e., their state is set to passive which implies that they do not attend further GranLeaderElection executions). Since $l_0 = \max_l (V_C(l) \neq \emptyset) \leq \log n$ and the smallest distance between elements of $V_C(l_0)$ is $\geq 1/n$, each execution of GranLeaderElection is applied on a set of stations with the smallest distance between stations $\geq 1/n$, and therefore the leader in each box $C$ containing (at least one) element of $V$ is chosen by LeaderElection.

### 5.2 Broadcasting Protocol

Algorithm 10 implements our broadcasting algorithm which repeats leader election procedure several times and each station is “switched off” after it is elected a leader of its box (assuring that each leader $v$ transmits the broadcast message successfully to all station accessible from $v$).

**Algorithm 10 GeneralBroadcast($V, n$)**

1. The source transmits the broadcast message
2. $V_1 \leftarrow \{v \in V \mid v$ received the broadcast message$\}$
3. for $i = 1, 2, \ldots, D\Delta$ do
4. \hspace{1em} LeaderElection($V_i, n$)
5. \hspace{1em} $V_{i+1} \leftarrow \{v \in V \mid state(v) \neq leader, v$ received the broadcast message$\}$

**Theorem 5.** Algorithm GeneralBroadcast finishes broadcasting in $O(D\Delta \log^2 N)$ rounds in ad hoc networks, provided that $\alpha > 2$ and each station knows $N, D$ and $\Delta$.

*Proof.* Let $P$ be a shortest path in the network graph from the source to a station $v$. Then, the length of $P$ is at most $D$. Theorem 4 guarantees that each station $v$ is elected a leader of its box $C$ after at most $\Delta$ executions of LeaderElection following the execution in which $v$ receives the broadcast message. Moreover, a station elected the leader of its box successfully sends the broadcast message to all its neighbors in the network graph. Therefore, the broadcast message arrives to the last vertex of $P$ in $O(D\Delta \log^2 N)$ rounds.

In order to implement Algorithm GeneralBroadcast, the knowledge of $n$, $D$ and $\Delta$ is required. However, if $n$ is not known, one can implement LeaderElection in $O(\log^2 N)$ rounds using the bound $n \leq N$. Moreover, each station $v$ which is elected a leader of its box in GeneralBroadcast, does not
attend the protocol after the execution of LeaderElection in which it is chosen a leader. And, each station is eventually elected a leader. Therefore, instead of the for-loop repeated \( D \Delta \) times, it is sufficient that each station participates in the protocol until its state changes to the value \( \text{leader} \). This observation leads to the following corollary.

**Corollary 3.** One can build a protocol which finishes broadcasting in \( O(D\Delta \log^2 N) \) rounds in ad hoc networks, provided that \( \alpha > 2 \) and each station knows merely \( N \).

### 6 Lower Bounds

In this section we provide lower bounds which are close to the the upper bounds provided so far. (In fact, they leave the gap \( O(\log N) \) in most cases.)

For a network with distinguished source station \( s \), \( L_i \) denotes the set of nodes in distance \( i \) from \( s \) in the communication graph (thus, in particular, \( L_0 = \{s\} \) and \( L_1 \) is equal to the set of neighbors of \( s \)).

**Theorem 6.** There exists an infinite family of networks requiring \( \Omega(n \log N) \) rounds in order to accomplish deterministic ad hoc broadcasting in the SINR model without local knowledge.

*Proof.* First, we describe a family of networks \( F \) such that broadcasting in SINR requires time \( \Omega(D \log N) \).

Each element of \( F \) is formed as a sequential composition of \( D \) networks \( V_1, \ldots, V_D \) of eccentricity 3 each, such that:

- the source \( s \) is connected with two nodes \( v_1, v_2 \) in \( L_1 \) with arbitrary IDs;
- \( v_1, v_2 \) are connected with \( w \), the only element of \( L_2 \), and satisfy the condition:
  \[
  P \cdot \text{dist}(v_1, w)^{-\alpha} = P \cdot \text{dist}(v_2, w)^{-\alpha} - N/2.
  \] (7)

Moreover, we assume that \( \beta = 1 \). Finally, sequential composition of networks \( V_1, \ldots, V_D \) stands for identifying the element \( w \) of network component \( V_i \) with the source \( s \) of network component \( V_{i+1} \).

Note that if \( v_1 \) and \( v_2 \) transmit simultaneously in a network component \( V_i \), the message is not received by \( w \). Using simple counting argument, one can force such choice of IDs of \( v_1 \) and \( v_2 \) that \( \Omega(\log N) \) rounds are necessary until a round in which exactly one of \( v_1, v_2 \) transmits a message under the SINR model. Since \( D = \Theta(n) \) in the above construction, the bound \( \Omega(n \log N) \) holds. \( \square \)

**Theorem 7.** For any deterministic broadcasting algorithm \( A \) and for every \( D \geq 3 \) and \( \Delta \geq 4 \), there exists a network of at most \( D\Delta \) nodes with eccentricity \( D \) and maximal degree \( \Delta \) on which algorithm \( A \) completes broadcasting in \( \Omega(D\Delta) \) rounds.

*Proof.* Let \( \gamma = 1/\sqrt{2} \). Let \( F \) be a family of networks \( F_j \), for \( 1 \leq j \leq \Delta \), of eccentricity 3 which consist of three layers:

- the source \( s \), located in the origin point \((0, 0)\), is the only element of \( L_1 \);
- \( L_2 \) consists of \( \Delta \) nodes \( v_0, \ldots, v_{\Delta-1} \), where the position of \( v_i \) is \((\gamma \cdot \frac{i}{\Delta}, \gamma)\) for \( 0 \leq i \leq \Delta - 1 \);\n- \( L_3 \) contains only one node \( w_j \) with coordinates \((\gamma \cdot \frac{j}{\Delta}, \gamma + 1)\).

Thus, the family \( F \) consists of \( \Delta \) elements, each network \( F_j \in F \) is uniquely determined by the value \( j \) fixing the position of node \( w_j \in L_3 \).

In what follows, we assume that the ranges of \( s \) and \( v_0, \ldots, v_{\Delta-1} \) are equal to 1. Then,
• \(v_0, \ldots, v_{\Delta-1}\) are in the range area of \(s\);
• \(w_j\) is in the range of \(v_j\) and it is not in the range of any other station from \(L_1 \cup L_2\);
• if more than \(2^{\alpha/2}\) stations from \(L_2\) transmit in a round, node \(w_j\) cannot hear a message.

The first two bullets follow directly from the location of points and the value of range. The last bullet holds because the minimum (maximum) of the distances between \(v_i\) and \(w_j\) is larger than or equal to 1 (smaller than \(\sqrt{2}\)), which guarantees that \(SIR(v_i, w_j, T)\) is smaller than 1 for each of the transmitting stations \(v_i\) if \(|T \cap L_2| \geq 3\), where \(T\) is the set of transmitting stations.

Consider any broadcasting algorithm \(A\). We specify an adversary who simultaneously, round after round, decides what is heard by stations in \(L_1 \cup L_2\) in consecutive rounds of \(A\) and restricts the family of considered networks \(\mathcal{F}\) to the networks on which such answers are valid. The goal of the adversary is to prevent the arrival of a message to \(w_j\) as long as possible. Assume that the source sends the broadcast message to all nodes in \(L_1\) in round 0. The adversary determines the family \(\mathcal{F}_t\), for every \(t \leq \lfloor \Delta/2 \rfloor - 1\), in the following way:

1. \(\mathcal{F}_0 \leftarrow \mathcal{F}\)
2. \(c \leftarrow \lceil 2^{\alpha/2} \rceil\)
3. For \(t = 1, 2, \ldots, \lfloor \Delta/c \rfloor - 1\) do:
   a. if \(v_{i_1}, \ldots, v_{i_{\ell}}\) are the only stations from \(L_1\) that transmit a message in the \(t\)-th round of \(A\) on the networks from \(\mathcal{F}_{t-1}\), and \(c' \leq c\)
      then \(\mathcal{F}_t \leftarrow \mathcal{F}_{t-1} \setminus \{F_{i_1}, \ldots, F_{i_{\ell}}\}\);
   b. otherwise, \(\mathcal{F}_t \leftarrow \mathcal{F}_{t-1}\).

One can easily verify that, for each \(t \leq \lfloor \Delta/c \rfloor - 1\), the following conditions are satisfied:

• \(\mathcal{F}_t\) is not empty;
• the history of communication (i.e., messages/noise heard by all stations in consecutive rounds) is the same in each network from \(\mathcal{F}_t\) up to the round \(t\);
• \(w_j\) does not receive the broadcast message by round \(t\) in the execution of \(A\) on any network in \(\mathcal{F}_t\).

This provides the claimed lower bound for constant eccentricity \(D\). In order to generalize this bound for arbitrary \(D\), one can consider a family of networks which consists of \((D - 1)/2\) networks from \(\mathcal{F}\) shifted such that the source of the \(i\)th network is equal to the only element in layer \(L_3\) in the \((i - 1)\)st network, for \(2 \leq i \leq (D - 1)/2\). The above strategy of the adversary can be applied sequentially to every subsequently shifted network from \(\mathcal{F}\), to gain the multiplicative factor \(D\). Note also that the size of the obtained network is \(\frac{D-1}{2} \cdot (\Delta + 1) + 1 \leq D \Delta\), its maximum degree is \(\Delta\) and its eccentricity is \(\frac{D-1}{2} \cdot 2 + 1 = D\).

As we argue next, the complexity of broadcasting depends also on granularity of the network.

**Corollary 4.** For any deterministic broadcasting algorithm \(A\) in unknown uniform model, and for any each \(D \geq 3\) and \(g \geq 4\), there exists a network with eccentricity \(D\) and granularity \(g\) on which algorithm \(A\) completes broadcasting in \(\Omega(Dg)\) rounds.

**Proof.** Note that granularity of the family of networks considered in the proof of Theorem\(\square\)is \(\Omega(\Delta)\), which immediately gives the claimed result. \(\square\)

Finally, we make an observation that one can transform lower bounds from Theorems\(\square\)and\(\square\)to the case of randomized algorithms. We sketch an idea of these transformations by considering networks from the family \(\mathcal{F}\) described in Theorem\(\square\)Recall that each element of the layer \(L_2\) should transmit as the only element of \(L_2\) in order to guarantee that the only element of \(L_3\) is informed, regardless of its location. However, by simple counting arguments, the expectation of the number of steps after which some of elements of \(L_2\) transmit as the only ones is \(\Omega(\Delta)\).
7 Conclusions

In this work we provided several novel algorithmic techniques for broadcasting in ad hoc wireless networks with uniform power, supported by theoretical analysis. We also discovered that the lack of knowledge about stations on close proximity results in substantially higher performance cost for majority of network parameters $D, \Delta$, and even randomization does not help much. The main open problem is to extend this study to networks with non-uniform power and to other fundamental communication problems.

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