STABILITY OF 2D STEADY EULER FLOWS WITH CONCENTRATED VORTICITY

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ABSTRACT. In this paper, we study the stability two-dimensional (2D) steady Euler flows with sharply concentrated vorticity in a simply-connected bounded domain. These flows are obtained as maximizers of the kinetic energy subject to the constraint that the vorticity is compactly supported in a finite number of disjoint regions of small diameter. We prove the nonlinear stability of these flows when the vorticity is concentrated in one small region, or in two small regions with opposite signs. The proof is achieved by showing that these flows constitute a compact and isolated set of local maximizers of the kinetic energy on an isovertical surface. The separation property of the stream function plays a crucial role in validating the isolatedness.

1. Introduction and main results

1.1. 2D Euler equation and its vorticity form. Let $D \subset \mathbb{R}^2$ be a smooth bounded domain. Consider the following Euler system governing the motion of an incompressible inviscid fluid of unit density in $D$

\[
\begin{aligned}
\partial_t v + (v \cdot \nabla)v &= -\nabla P, \quad x = (x^1, x^2) \in D, \quad t > 0, \\
\nabla \cdot v &= 0,
\end{aligned}
\]

(1.1)

where $v = (v^1, v^2)$ is the velocity field and $P$ is the pressure. The scalar vorticity of the fluid is defined as

\[\omega = \partial_{x^1} v^2 - \partial_{x^2} v^1.\]

From the momentum equation (i.e., the first equation of (1.1)), we can easily get the following evolution equation for $\omega$

\[\partial_t \omega + v \cdot \nabla \omega = 0.\]

(1.2)

Under suitable assumptions $v$ can be recovered from $\omega$ via the Biot-Savart law. For example, if $D$ is additionally simply-connected and the following impermeability boundary condition holds

\[v \cdot n = 0, \quad x \in \partial D,\]

(1.3)

where $n$ is the outward unit normal to $\partial D$, then the Biot-Savart law can be expressed as

\[v = \nabla^\perp G_\omega.\]

(1.4)
Here $G$ is the Green operator corresponding to $-\Delta$ in $D$ with zero boundary condition, or equivalently,

$$
\begin{align*}
-\Delta G\omega &= \omega, \quad x \in D, \\
G\omega &= 0, \quad x \in \partial D,
\end{align*}
$$

(1.5)

and $\nabla^\perp G\omega$ is the clockwise rotation through $\pi/2$ of $\nabla G\omega$, that is,

$$
\nabla^\perp G\omega = (\partial_{x_2} G\omega, -\partial_{x_1} G\omega).
$$

The function $G\omega$ is called the stream function related to $\omega$. If $D$ is not simply-connected, or $v$ is not tangential to the boundary, then the Biot-Savart law is no longer of the form (1.4) and has a more complicated expression. For simplicity, throughout this paper we assume that $D$ is simply-connected and (1.3) holds. Hence the evolution equation of the vorticity is

$$
\partial_t \omega + \nabla^\perp G\omega \cdot \nabla \omega = 0,
$$

(1.6)

which is usually called the vorticity equation. In the rest of this paper we mainly focus our attention on (1.6).

In the literature, there are a lot of existence results on the initial-value problem of (1.6) with initial vorticity in various function spaces, including those by Hölder \[19\] and Wolibner \[12\] in Hölder spaces, Yudovich \[13\] in $L^\infty$, DiPerna and Majda \[16\] in $L^p$ with $1 < p < +\infty$, and Delort \[15\] in the space of nonnegative Radon measures in $H^{-1}$. In this paper we only work in Yudovich’s setting, that is, the vorticity belongs to $L^\infty$. The reasons are threefold. First, it contains enough solutions that are physically interesting, such as flows with discontinuous vorticity. Second, uniqueness holds in this case, which makes many statements concise. Third, 2D Euler flows with bounded vorticity possess some good conservative properties that are crucial in proving stability.

Before stating Yudovich’s result, we list some notations that will be used throughout this paper.

- $\mathcal{L}$: the planar Lebesgue measure;
- $\mathcal{R}_f$: the rearrangement class of some function $f \in L^1_{\text{loc}}(D)$, that is,
  $$
  \mathcal{R}_f = \{ g \in L^1_{\text{loc}}(D) \mid \mathcal{L}(\{ x \in D \mid g(x) > s \}) = \mathcal{L}(\{ x \in D \mid f(x) > s \}), \forall s \in \mathbb{R} \};
  $$
- $\text{supp}(f)$: the essential support of some measurable function $f$ (see §1.5, \[21\] for the precise definition);
- $L^\infty_c(\mathbb{R}^2)$: the set of essentially bounded functions in $\mathbb{R}^2$ with compact support;
- $\text{sgn}(a)$: the sign of some real number $a$, that is,
  $$
  \text{sgn}(a) = \begin{cases} 
  -1, & \text{if } a < 0, \\
  0, & \text{if } a = 0, \\
  1, & \text{if } a > 0;
  \end{cases}
  $$
- $1_A$: characteristic function of some set $A$;
- $0$: the origin in $\mathbb{R}^2$;
- $B_r(x)$: the disk with radius $r$ and center $x$;
- $f_+(f_-)$: the positive (negative) part of $f$, that is, $f_+ = \max\{f, 0\}$ ($f_- = \max\{-f, 0\}$);
• \( G \): the Green function of \(-\Delta\) in \( D \) with zero boundary condition;

• \( h \): the regular part of the Green function \( G \), that is,

\[
h(x, y) = -\frac{1}{2\pi} \ln |x - y| - G(x, y), \quad x, y \in D;
\]

• \( H \): the Robin function of the domain \( D \), that is,

\[
H(x) = h(x, x), \quad x \in D.
\]

Yudovich’s result can be stated as follows.

**Yudovich’s Theorem.** Let \( \omega_0 \in L^\infty(D) \). Then there exists a unique weak solution \( \omega \in L^\infty(D \times (0, +\infty)) \) to the vorticity equation (1.6) in the following sense

\[
\int_D \omega_0(x)\phi(x, 0)dx + \int_0^{+\infty} \int_D \omega \partial_t \phi + \omega \nabla \phi \cdot \nabla^\perp G \omega dxdt = 0, \quad \forall \phi \in C_c^\infty(D \times \mathbb{R}).
\]

Moreover, \( \omega \) satisfies

(i) \( \omega(\cdot, t) \in \mathcal{R}_{\omega_0} \) for any \( t \geq 0 \);

(ii) \( \omega \in C([0, +\infty); L^p(D)) \) for any \( p \in [1, +\infty) \);

(iii) \( E(\omega(\cdot, t)) = E(\omega_0) \) for any \( t \in [0, +\infty) \), where

\[
E(\omega(\cdot, t)) = \frac{1}{2} \int_D |v|^2 dx = \frac{1}{2} \int_D |\nabla^\perp G \omega|^2 dx = \frac{1}{2} \int_D \omega G \omega dx
\]

is the kinetic energy of the fluid.

See Majda-Bertozzi [25] or Marchioro-Pulvirenti [30] for the detailed proof of Yudovich’s Theorem.

**Remark 1.1.** In Yudovich’s Theorem, for fixed \( t \geq 0 \), since \( \omega(\cdot, t) \in L^\infty(D) \), we can deduce from standard elliptic estimates that \( \nabla \omega(\cdot, t) \in C^1(\bar{D}) \), therefore the integrals in (1.7) and (1.8) make sense.

**Remark 1.2.** In Yudovich’s Theorem, by (1.7) and (ii) it is easy to check that

\[
\lim_{t \to 0^+} \|\omega(\cdot, t) - \omega_0\|_{L^p(D)} = 0
\]

for any \( p \in [1, +\infty) \).

By Yudovich’s Theorem, an ideal flow evolves on some “isovortical” surface with the kinetic energy unchanged. Here by an isovortical surface we mean a set of all instantaneous flows whose vorticities constitute the rearrangement class of a given measurable function. As we will see in Section 3, this fact plays an important role in the study of stability for 2D steady Euler flows.
1.2. **Steady solution and stability.** A steady solution to the vorticity equation is a solution that does not depend on the time variable. Hence a steady solution $\omega$ satisfies

$$\nabla \cdot \mathcal{G} \omega \cdot \nabla = 0, \ x \in D. \quad (1.9)$$

For $\omega$ only in $L^\infty(D)$, (1.9) should be interpreted in the following weak sense.

**Definition 1.3.** Let $\omega \in L^\infty(D)$. We call $\omega$ a steady weak solution to the vorticity equation if it satisfies

$$\int_D \omega \nabla \cdot \mathcal{G} \omega \cdot \nabla \phi dx = 0, \ \forall \phi \in C_0^\infty(D). \quad (1.10)$$

It is easy to check that Definition 1.3 is consistent with (1.7).

Without any restriction there are infinitely many steady weak solutions. For instance, any $\omega \in C^1(\bar{D})$ satisfying $\omega = f(\mathcal{G} \omega)$ in $D$, where $f \in C^1(\mathbb{R})$, must satisfy (1.10). A more general criterion is proved in [13].

**Lemma A** ([13], Theorem 1.2). Let $k$ be a positive integer. Suppose $\omega \in L^\infty(D)$ satisfies

$$\omega = \sum_{i=1}^{k} \omega_i, \ \min_{1 \leq i < j \leq k} \{\text{dist(supp}(\omega_i),\text{supp}(\omega_j))\} > 0, \ \omega_i = f_i(\mathcal{G} \omega) \ \text{a.e. in supp}(\omega_i)_\delta \quad (1.11)$$

for some $\delta > 0$, where supp$(\omega_i)_\delta$ is the $\delta$-neighborhood of supp$(\omega_i)$ in the Euclidean norm, that is,

$$\text{supp}(\omega_i)_\delta = \{x \in D | \text{dist}(x,\text{supp}(\omega_i)) < \delta\},$$

and each $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is either monotone or locally Lipschitz continuous. Then $\omega$ is a steady weak solution to the vorticity equation.

Lemma A will be used in the proof of Theorem 1.5 in Section 6.

Given a steady solution, an interesting and important problem to consider is its stability. In this paper, we only consider stability of Lyapunov type, also called nonlinear stability. To make it general, we give the definition of stability for a set of steady weak solutions.

**Definition 1.4.** Let $\mathcal{M} \subset L^\infty(D)$ be a nonempty set of steady weak solutions to the vorticity equation, $\| \cdot \|$ be a norm on $L^\infty(D)$, $\mathcal{P} \subset L^\infty(D)$ be nonempty. If for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $\omega_0 \in \mathcal{P}$ satisfying

$$\inf_{w \in \mathcal{M}} \|w - \omega_0\| < \delta,$$

it holds that

$$\inf_{w \in \mathcal{M}} \|w - \omega(\cdot, t)\| < \varepsilon, \ \forall t \geq 0,$$

where $\omega$ is the weak solution to the vorticity equation with initial vorticity $\omega_0$, then $\mathcal{M}$ is said to be stable in the norm $\| \cdot \|$ with respect to initial perturbations in $\mathcal{P}$.
Sometimes the stability in Definition 1.4 is also called orbital stability. When $M$ contains only one element $\bar{\omega}$ and is stable, we say $\bar{\omega}$ is stable.

Commonly used norms include the $L^p$ norm of the vorticity $\|\omega\|_{L^p(D)}$, $1 \leq p < +\infty$, and the energy norm $\|\omega\|_E = (2E(\omega))^{1/2}$.

1.3. Steady Euler flows with concentrated vorticity. In many natural phenomena such as tornados, the vorticity is sharply concentrated in a finite number of small regions and vanishes elsewhere. Mathematically, the vorticity should be assumed to possess the following form

$$\omega^\varepsilon = \sum_{i=1}^{k} \omega_i^\varepsilon, \quad \text{supp}(\omega_i^\varepsilon) \subset B_{o(1)}(x_i), \quad \int_D \omega_i^\varepsilon \, dx = \kappa_i + o(1), \quad i = 1, \ldots, k, \quad (1.12)$$

where $\varepsilon$ is a small positive parameter, $k$ is a positive integer, $x_1, \ldots, x_k$ are $k$ different points in $D$, $\kappa_1, \ldots, \kappa_k$ are $k$ nonzero real numbers, and $o(1) \to 0$ as $\varepsilon \to 0$. The study for such kind of flows is rather difficult in both theoretical and numerical analysis because of the presence of strong singularity. To simplify the problem, a related finite dimensional model, called the point vortex model, has been introduced. See [22] for example. In the point vortex model, each $\omega_i^\varepsilon$ is replaced by a Dirac measure located at $x_i$, and the evolution of each $x_i$ is described by the following system of ordinary differential equations

$$\kappa_i \frac{dx_i}{dt} = -\nabla_{x_i} W(x_1, \ldots, x_k), \quad i = 1, \ldots, k, \quad (1.13)$$

where $W$ is the Kirchhoff-Routh function related to $\kappa_1, \ldots, \kappa_k$, this is,

$$W(x_1, \ldots, x_k) = -\sum_{1 \leq m < n \leq k} \kappa_m \kappa_n G(x_m, x_n) + \frac{1}{2} \sum_{n=1}^{k} \kappa_n^2 h(x_n, x_n), \quad (1.14)$$

where

$$(x_1, \ldots, x_k) \in D \times \cdots \times D \setminus \{(x_1, \ldots, x_k) \mid x_m \in D, x_m = x_n \text{ for some } m \neq n\}.$$

The point vortex model is only an approximate model, and rigorous analysis on its connection with the vorticity equation with concentrated vorticity is an interesting research topic. We refer the interested readers to [26, 27, 28, 29, 30, 31, 35] for some deep results in this respect.

In this paper, we are mainly concerned with steady flows with concentrated vorticity. From the point vortex model, the locations of concentrated blobs of vorticity should constitute a critical point of $W$. This has been proved rigorously in [39] by using the anti-symmetry of the Biot-Savart kernel. The inverse problem is: given a critical point $(x_1, \ldots, x_k)$ of $W$, can we construct a family of steady Euler flows such that the vorticities “shrink” to these $k$ points? This problem has been studied by many authors in the past several decades and lots of solutions of the form (1.12) have been obtained via various methods. See [7, 8, 10, 17, 18, 32, 34] and the references therein.
Roughly speaking, these existence results can be divided into two types. For the first type, the rearrangement of the vorticity is prescribed, and the vorticity has the form (1.11); however, the profile function \( f \) is unknown. See [17, 18] for example. We will discuss this type of solutions in detail below. For the second type, the vorticity has the form (1.11) with each \( f \) prescribed, but the rearrangement of the vorticity is unknown. See [7, 9, 10, 32] for example. Of course, for some special kind of solutions such as vortex patches, both the rearrangement of the vorticity and the profile functions are known. See [8, 34]. Our stability result (i.e., Theorem 1.6 below) is about solutions of the first type; however, we will show in the last section that our method can also be used to handle the stability of the second type of flows under certain circumstance.

Below we give the precise statement for the existence of concentrated steady vortex flows of the first type. The basic idea comes from [18] and [34], but the presentation here is more general.

Let \( k \) be a positive integer, \( \vec{\kappa} = (\kappa_1, \cdots, \kappa_k) \), where each \( \kappa_i \) is a nonzero real number. Let \( (\bar{x}_1, \cdots, \bar{x}_k) \) be an isolated local minimum point of the Kirchhoff-Routh function \( W \) related to \( \vec{\kappa} \), that is,

\[
W(x_1, \cdots, x_k) = - \sum_{1 \leq i < j \leq k} \kappa_i \kappa_j G(x_i, x_j) + \frac{1}{2} \sum_{i=1}^{k} \kappa_i^2 h(x_i, x_i).
\]

Choose \( \bar{r} > 0 \) sufficiently small such that

(i) \( B_{\bar{r}}(\bar{x}_i) \subset D \) for any \( 1 \leq i \leq k \), where \( B_{\bar{r}}(\bar{x}_i) \) is the closure of \( B_{\bar{r}}(\bar{x}_i) \) in the Euclidean topology;

(ii) \( B_{\bar{r}}(\bar{x}_i) \cap B_{\bar{r}}(\bar{x}_j) = \emptyset \) for any \( 1 \leq i < j \leq k \);

(iii) \( (\bar{x}_1, \cdots, \bar{x}_k) \) is the unique minimum point of \( W \) in \( B_{\bar{r}}(\bar{x}_1) \times \cdots \times B_{\bar{r}}(\bar{x}_k) \).

Define

\[
\Xi = \{ \xi \in L^\infty_c(\mathbb{R}^2) | \xi \geq 0 \text{ a.e., } \xi \text{ is radially symmetric and nonincreasing} \}. \tag{1.15}
\]

Let \( M \) be a fixed positive number, \( \Pi^1, \cdots, \Pi^k : (0, \bar{r}) \to \Xi \) be \( k \) maps satisfying

\[
\text{supp}(\Pi^i_\varepsilon) \subset \overline{B_{\varepsilon}(0)} \quad \text{for any } \varepsilon \in (0, \bar{r}),
\]

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \Pi^i_\varepsilon(x) dx = |\kappa_i|, \quad \tag{1.16}
\]

\[
||\Pi^i_\varepsilon||_{L^\infty(\mathbb{R}^2)} \leq M \varepsilon^{-2},
\]

where \( i = 1, \cdots, k \). Define

\[
\mathcal{K}_{\varepsilon} = \left\{ w = \sum_{i=1}^{k} w_i \Big| w_i \text{ is a rearrangement of } \text{sgn}(\kappa_i)\Pi^i_\varepsilon, \text{ supp}(w_i) \subset \overline{B_{\varepsilon}(\bar{x}_i)} \right\}.
\]

Denote \( \mathcal{M}_{\varepsilon} \) the set of maximizers of \( E \) over \( \mathcal{K}_{\varepsilon} \). It is easy to see that \( E \) attains its maximum over \( \mathcal{K}_{\varepsilon} \), the weak closure of \( \mathcal{K}_{\varepsilon} \) in \( L^2(D) \). We will see in Section 6 that any maximizer of \( E \) over \( \mathcal{K}_{\varepsilon} \) must belong to \( \mathcal{K}_{\varepsilon} \). Consequently \( \mathcal{M}_{\varepsilon} \) must be nonempty.
Theorem 1.5 (Existence of concentrated vortex flows). For any \( \varepsilon \in (0, \bar{r}) \), \( \mathcal{M}_\varepsilon \) is not empty. Moreover, for any \( \delta \in (0, \bar{r}) \), there exists \( \varepsilon_0 > 0 \), such that for any \( \varepsilon \in (0, \varepsilon_0) \), we have

(i) \( \operatorname{supp}(\omega^1_{B_{\varepsilon}(\bar{x}_i)}) \subset \overline{B_\delta(\bar{x}_i)} \) for any \( \omega \in \mathcal{M}_\varepsilon \), \( i = 1, \ldots, k \);

(ii) any \( \omega \in \mathcal{M}_\varepsilon \) is a steady weak solution to the vorticity equation.

The proof of Theorem 1.5 will be given in Section 6.

Roughly speaking, the conclusion (i) in Theorem 1.5 means that for each \( i \in \{1, \ldots, k\} \), \( \operatorname{supp}(\omega^1_{B_{\varepsilon}(\bar{x}_i)}) \) “shrinks” to \( \bar{x}_i \) uniformly as \( \varepsilon \to 0 \).

1.4. Stability result. Now we are ready to state our stability result.

Theorem 1.6 (Stability). In the setting of Theorem 1.5, suppose \( k = 1 \) or \( k = 2 \) with \( \kappa_1 \kappa_2 < 0 \). Then there exists \( \varepsilon_1 > 0 \), such that for any \( \varepsilon \in (0, \varepsilon_1) \), \( \mathcal{M}_\varepsilon \) is stable in the \( L^p \) norm of the vorticity with initial perturbations in \( L^\infty(D) \).

If we choose \( \Pi^i \) as follows

\[
\Pi^i_\varepsilon = \frac{\kappa_i}{\pi \varepsilon^2} 1_{B_{\varepsilon}(0)}, \quad i = 1, \ldots, k, \; \varepsilon \in (0, \bar{r}),
\]

then \( \mathcal{M}_\varepsilon \) is a set of steady vortex patches, and the corresponding stability results have been obtained in [12] for \( k = 1 \) and [11] for \( k = 2, \kappa_1 \kappa_2 < 0 \). However, the methods in [11] [12] rely heavily on the facts that the boundary of a concentrated steady vortex patch is \( C^1 \) and the stream function is nondegenerate on it. For general \( \Pi^i \), there is little hope to apply the methods in [11] [12] to obtain stability.

To prove Theorem 1.6, we introduce some new ideas. First, we prove a general stability criterion, i.e., Theorem 3.1 in Section 3, showing that an isolated and compact set of local maximizers of the kinetic on some isovortical surface must be stable. This generalizes Burton’s stability criterion in [6] for a single isolated local maximizer. From this criterion, we need only to verify the compactness and isolatedness of \( \mathcal{M}_\varepsilon \). Compactness is easy, and can be obtained in view of the variational nature of \( \mathcal{M}_\varepsilon \). The trouble is isolatedness. Our second innovation is that we develop a new technique to prove the isolatedness of \( \mathcal{M}_\varepsilon \). We discover that the stream function has a very good property, which we call the separation property, when the vorticity is sufficiently concentrated. With the separation property, together with the symmetry and positivity of the Green operator, we can employ a lemma (i.e., Lemma 2.3 in Section 2) from measure theory to obtain isolatedness.

It is worth mentioning that our method can also be used to handle the stability of some other concentrated vortex flows. An example is given in Section 7.

Before ending this section, we recall some related works on the stability of 2D steady Euler flows and make some comparisons. The study for the stability of steady Euler flows in two dimensions has a long history, dating back to the works of Kelvin [20] and Love [21]. The first systematic and general method of proving nonlinear stability was established by Arnol’d in 1960s. Arnol’d in [11] [2] proposed the well-known energy-Casimir (EC) functional method and used it to prove what is now usually called Arnol’d’s first and second stability theorems. In 1990s, Wolansky and Ghil [40] [41] developed the EC functional method by introducing the idea of supporting functionals and obtained some extensions of Arnol’d’s
second stability theorem. See also \cite{23, 38}. Although Arnol’d’s stability theorems and their extensions are powerful tools in studying the nonlinear stability of steady Euler flows, they have strong limitations as well. First, they can only deal with flows with vorticity satisfying
\[
\omega = f(G\omega) \text{ a.e. in } D,
\]
where the profile function \(f\) is given. However, in Section 6 we will see that the profile functions related to the flows in Theorem 1.6 are unknown. Second, they usually require \(f\) to be \(C^1\), which excludes the possibility of application to many interesting cases such as vortex patches. Third, these criteria have strong requirements on the energy of solutions. For example, Arnol’d’s second stability requires the operator \(-\Delta - f'(G\omega)\) in \(L^2(D)\) to be positive, which does not hold or is hard to verify in most cases. For these reasons, it is almost impossible to prove Theorem 1.6 by simply applying these general stability criteria.

Except for Arnol’d’s stability theorems and their extensions, there are also some stability results about special steady Euler flows in the literature. See \cite{11, 12, 33, 36} about steady vortex patches, \cite{3} about steady flows related to solutions of the mean field equation, and \cite{37} about steady flows related to least energy solutions of the Lane-Emden equation. However, the methods used therein are more or less based on some particularities of the flows, and seems not helpful with our problem.

A remaining open question is the stability of steady Euler flows with concentrated vorticity distributed in more than two small regions. In this case, there are at least two blobs of vorticity with the same sign, and therefore the separation property for the stream function does not hold anymore.

This paper is organized as follows. In Section 2, we present several useful lemmas that will be frequently used in subsequent sections. In Section 3, we prove a general stability criterion. In Sections 4 and 5, we give the proof of Theorem 1.6 for \(k = 1\) and \(k = 2\) respectively. In Section 6, we give the proof of Theorem 1.5 for completeness. In Section 7, we briefly discuss the stability of concentrated vortex flows with prescribed profile functions.

2. Preliminaries

The aim of this section is to present several useful lemmas, most of which are from Burton’s papers. To make it concise, we only give their simple versions, which are enough for our later use.

**Lemma 2.1.** Let \(1 \leq p < +\infty\), \(\Omega \subset \mathbb{R}^2\) be a Lebesgue measurable set such that \(\mathcal{L}(\Omega) < +\infty\), \(f \in L^p(\Omega)\), and \(\bar{\mathcal{R}}_f\) be the weak closure of \(\mathcal{R}_f\) in \(L^p(\Omega)\). Then \(\bar{\mathcal{R}}_f\) convex.

*Proof.* See Theorem 6 in \cite{4}. \(\square\)

**Lemma 2.2.** Let \(V, W \subset \mathbb{R}^2\) be two Lebesgue measurable sets such that \(\mathcal{L}(V) = \mathcal{L}(W) < +\infty\). Then for any \(w \in L^1(W)\), there exists some \(v \in L^1(V)\) such that
\[
\mathcal{L}\left(\{x \in W \mid w(x) > s\}\right) = \mathcal{L}\left(\{x \in V \mid v(x) > s\}\right), \quad \forall s \in \mathbb{R}.
\]

*Proof.* This lemma is an easy consequence of Lemma 4, (ii) in \cite{4}. \(\square\)
Lemma 2.3. Let $1 \leq p \leq +\infty$, $\Omega \subset \mathbb{R}^2$ be a Lebesgue measurable set such that $\mathcal{L}(\Omega) < +\infty$. Let $f \in L^p(\Omega), g \in L^{p'}(\Omega)$ be fixed, where $p'$ is the Hölder conjugate exponent of $p$. Then there exists $\tilde{f} \in \mathcal{R}_f$, such that

$$\int_{\Omega} \tilde{f} g dx = \sup_{u \in \mathcal{R}_f, v \in \mathcal{R}_g} \int_{\Omega} uv dx.$$ 

Proof. See Theorem 1 and Theorem 4 in [4]. □

Lemma 2.4. Let $p \in [1, +\infty)$ be fixed, $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $f_0 \in L^p(\Omega)$ be nonnegative, and $g \in H^2(\Omega)$. Suppose $\tilde{f} \in \mathcal{R}_{f_0}$ satisfying

(i) $\int_{\Omega} \tilde{f} g dx \geq \int_{\Omega} f g dx$ for all $f \in \mathcal{R}_{f_0}$;
(ii) $-\Delta g \geq \tilde{f}$ a.e. $x \in \Omega$.

Then $\tilde{f} \in \mathcal{R}_{f_0}$. Moreover, there exists some nondecreasing function $\phi : \mathbb{R} \to [-\infty, +\infty]$ such that $\tilde{f}(x) = \phi(g(x))$ a.e. $x \in \Omega$.

Proof. See Lemma 2.15 in [5]. □

Lemma 2.5. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $\omega \in L^\infty((0, +\infty) \times \Omega)$, $\zeta_0 \in L^\infty(\Omega)$. Then there exists a unique $\zeta \in L^\infty((0, +\infty) \times \Omega)$ satisfying the following conditions:

(i) $\zeta$ solves $\partial_t \zeta + \nabla^\perp G_\omega \cdot \nabla \zeta = 0$ in the sense of distributions, i.e.,

$$\int_0^\infty \int_{\Omega} \zeta \partial_t \phi + \zeta \nabla^\perp G_\omega \cdot \nabla \phi dx dt = 0, \quad \forall \phi \in C_c^\infty((0, +\infty) \times \Omega);$$

(ii) $\zeta \in C([0, +\infty); L^p(\Omega))$ for any $p \in [1, +\infty)$;
(iii) $\zeta(0, \cdot) = \zeta_0$;
(iv) $\zeta(t, \cdot) \in \mathcal{R}_{\zeta_0}$ for any $t \in [0, +\infty)$.

Proof. See Lemma 11 and Lemma 12 in [6]. □

The following two rearrangement inequalities can be found in Lieb and Loss’s book [21].

Lemma 2.6. Let $f, g$ be nonnegative Lebesgue measurable functions on $\mathbb{R}^2$, and $f^*, g^*$ be their symmetric-decreasing rearrangements. Then

$$\int_{\mathbb{R}^2} fg dx \leq \int_{\mathbb{R}^2} f^* g^* dx.$$ 

Proof. See §3.4 in [21]. □

Lemma 2.7. Let $f, g, h$ be nonnegative Lebesgue measurable functions on $\mathbb{R}^2$, and $f^*, g^*, h^*$ be their symmetric-decreasing rearrangements. Then

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x)g(x-y)h(y) dxdy \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f^*(x)g^*(x-y)h^*(y) dxdy.$$ 

Proof. See §3.7 in [21]. □
3. A General Stability Criterion

Throughout this section let \( p \in (1, +\infty) \) be fixed, and \( \mathcal{R} \) be the rearrangement class of some fixed function in \( L^\infty(D) \). Denote \( \mathcal{R} \) the weak closure of \( \mathcal{R} \) in \( L^p(D) \). For any nonempty set \( A \subset L^p(D) \), denote \( A_\delta \) the \( \delta \)-neighborhood of \( A \) in \( L^p(D) \), that is,

\[
A_\delta = \{ w \in L^p(D) \mid \text{dist}(w, A) < \delta \}, \quad \text{where dist}(w, A) = \inf_{v \in A} \| w - v \|_{L^p(D)}.
\]

Obviously \( A_\delta \) is an open set in \( L^p(D) \).

The purpose of this section is to prove the following stability criterion.

**Theorem 3.1.** Suppose \( \mathcal{M} \subset \mathcal{R} \) is nonempty and satisfies

(i) \( \mathcal{M} \) is compact in \( L^p(D) \);

(ii) \( \mathcal{M} \) is an isolated set of local maximizers of \( E \) over \( \mathcal{R} \), that is, there exists some \( \delta > 0 \) such that

\[
\mathcal{M} = \{ w \in \mathcal{M}_\delta \cap \mathcal{R} \mid E(w) = E_0 \}, \quad \text{where } E_0 = \sup_{w \in \mathcal{M}_\delta \cap \mathcal{R}} E(w). \tag{3.1}
\]

Then any \( \omega \in \mathcal{M} \) is a steady weak solution to the vorticity equation, and \( \mathcal{M} \) is stable in the \( L^p \) norm of the vorticity with respect to initial perturbations in \( L^\infty(D) \).

**Remark 3.2.** The assumption (ii) in Theorem 3.1 holds if and only the following two items are satisfied

1. \( E \) is a constant \( E_0 \) on \( \mathcal{M} \);
2. for any \( v \in (\mathcal{M})_\delta \cap \mathcal{R} \) satisfying \( E(v) \geq E_0 \), it holds that \( v \in \mathcal{M} \).

**Remark 3.3.** If \( \mathcal{M} \) is a singleton, then Theorem 3.1 is exactly Theorem 5 in [6].

**Remark 3.4.** Whether the compactness assumption (i) in Theorem 3.1 can be removed is unknown. There are two special cases in which compactness holds automatically:

1. \( \mathcal{M} \) contains only a finite number of elements;
2. \( E_0 = \sup_{w \in \mathcal{R}} E(w) \).

The first case is clear. For the second case, we give a brief explanation as follows. First, from the weak sequential continuity of \( E \) in \( L^p(D) \) we have

\[
E_0 = \sup_{w \in \mathcal{R}} E(w) = \sup_{w \in \mathcal{R}} E(w). \tag{3.2}
\]

Fix a sequence \( \{ w_n \}_{n=1}^{+\infty} \subset \mathcal{M} \). Then there exist a subsequence \( \{ w_{n_j} \}_{j=1}^{+\infty} \) and some \( \eta \in \mathcal{R} \) such that \( w_{n_j} \) converges to \( \eta \) weakly in \( L^p(D) \) as \( j \to +\infty \). The weakly sequential continuity of \( E \) in \( L^p(D) \) yields \( E(\eta) = E_0 \). Since \( \mathcal{R} \) is convex by Lemma 2.1 for any \( v \in \mathcal{R} \) and \( s \in [0, 1] \) we have \( sv + (1-s)\eta \in \mathcal{R} \). Hence

\[
\frac{d}{ds} E(sv + (1-s)\eta) \bigg|_{s=0^+} \leq 0,
\]

which gives

\[
\int_D v\mathcal{G}\eta dx \leq \int_D \eta\mathcal{G}\eta dx. \tag{3.3}
\]
Since in (3.3) the choice of $v$ in $\tilde{\mathcal{R}}$ is arbitrary, Lemma 2.4 yields $\eta \in \mathcal{R}$. By the uniform convexity of $L^p(D)$, we see that $w_n$ converges to $\eta$ strongly in $L^p(D)$ as $j \to +\infty$, and consequently $\eta \in \mathcal{M}_\delta \cap \mathcal{R}$. Taking into account (3.1) we get $\eta \in \mathcal{M}$.

To prove Theorem 3.1, we need several auxiliary lemmas.

**Lemma 3.5.** The strong and weak topologies of $L^p(D)$ restricted on $\mathcal{R}$ are the same.

**Proof.** It suffices to show that any strongly closed set $F \subset \mathcal{R}$ must be weakly closed (both in the sense of subspace topology of $\mathcal{R}$). Since $\mathcal{R}$ is bounded in $L^p(D)$ and $p \in (1, +\infty)$, the weak topology restricted on $\mathcal{R}$ is metrizable. Hence it suffices to prove that $F$ is weakly sequentially closed on $\mathcal{R}$. Suppose $\{w_n\}^{+\infty}_{n=1} \subset F$ and $w_n$ converges weakly to some $\eta \in \mathcal{R}$. Then $\|\eta\|_{L^p(D)} = \lim_{n \to +\infty} \|w_n\|_{L^p(D)}$. By the fact that $L^p(D)$ is uniformly convex, we immediately see that $w_n$ converges strongly to $\eta$, which yields $\eta \in F$. \hfill \Box

**Remark 3.6.** From the proof of Lemma 3.5 it is easy to check that for any $\mathcal{N} \subset \mathcal{R}$ the following three items are equivalent:

(1) $\mathcal{N}$ is compact;
(2) $\mathcal{N}$ is weakly closed;
(3) $\mathcal{N}$ is weakly sequentially closed.

**Lemma 3.7.** There exists a weakly open set $U$ such that $U \cap \mathcal{R} = \mathcal{M}_\delta \cap \mathcal{R}$.

**Proof.** Observe that $\mathcal{M}_\delta \cap \mathcal{R}$ is strongly open on $\mathcal{R}$ (in the subspace topology). By Lemma 3.5 $\mathcal{M}_\delta \cap \mathcal{R}$ is also weakly open on $\mathcal{R}$. Hence the existence of the desired $U$ follows immediately. \hfill \Box

**Lemma 3.8.** There exists $\tau \in (0, \delta)$ such that $\mathcal{M}_\tau \subset U$.

**Proof.** Suppose by contradiction that there exists a sequence $\{w_n\}^{+\infty}_{n=1} \subset U^c$ such that

$$\text{dist}(w_n, \mathcal{M}) < \frac{1}{n}.$$

Here $U^c$ is the complement of $U$. Since $U$ is weakly open, $U$ must be strongly open, which means $U^c$ is strongly closed. Choose a sequence $\{v_n\}^{+\infty}_{n=1} \subset \mathcal{M}$ such that

$$\|w_n - v_n\|_{L^p(D)} < \frac{1}{n}. \quad (3.4)$$

Since $\mathcal{M}$ is compact, there exist a subsequence $\{v_{n_j}\}^{+\infty}_{j=1}$ and some $v \in \mathcal{M}$ such that

$$\lim_{j \to +\infty} \|v_{n_j} - v\|_{L^p(D)} = 0,$$

which together with (3.4) yields

$$\lim_{j \to +\infty} \|w_{n_j} - v\|_{L^p(D)} = 0.$$

Taking into account the fact that $U^c$ is strongly closed, we get $v \in U^c$. This contradicts the fact that $v \in \mathcal{M} \subset U$. \hfill \Box
Lemma 3.9. Let $\tau$ be determined by Lemma 3.8, then
\[
E_0 = \sup_{w \in \mathcal{M}_\tau \cap \bar{\mathcal{R}}} E(w). 
\] (3.5)

Proof. First by (3.1) we have $E(w) = E_0$ for any $w \in \mathcal{M}$, thus
\[
E_0 \leq \sup_{w \in \mathcal{M}_\tau \cap \bar{\mathcal{R}}} E(w). 
\]
So it suffices to prove the inverse inequality, i.e.,
\[
E(w) \leq E_0, \quad \forall \ w \in \mathcal{M}_\tau \cap \bar{\mathcal{R}}. 
\] (3.6)

Fix $w \in \mathcal{M}_\tau \cap \bar{\mathcal{R}}$. Since $\bar{\mathcal{R}}$ is the weak closure of $\mathcal{R}$, there exists a sequence $\{w_n\}_{n=1}^{+\infty} \subset \mathcal{R}$ such that $w_n$ converges weakly to $w$ as $n \to +\infty$. On the other hand, since $w \in \mathcal{M}_\tau \subset U$ (by Lemma 3.8) and $U$ is weakly open, we obtain $w_n \in U, \ \forall n \geq N_0$ for some sufficiently large $N_0$. Therefore
\[
w_n \in U \cap \mathcal{R} \subset \mathcal{M}_\delta \cap \mathcal{R}, \ \forall n \geq N_0.
\]
Taking into account the definition of $E_0$ (see (3.1)), we have $E(w_n) \leq E_0$ for all $n \geq N_0$. Hence
\[
E(w) = \lim_{n \to +\infty} E(w_n) \leq E_0. 
\]
\[\square\]

Lemma 3.10. Let $\tau$ be determined by Lemma 3.8, then
\[
\mathcal{M} = \{w \in \mathcal{M}_\tau \cap \bar{\mathcal{R}} \mid E(w) = E_0\}. 
\] (3.7)

Proof. It suffices to prove
\[
\{w \in \mathcal{M}_\tau \cap \bar{\mathcal{R}} \mid E(w) = E_0\} \subset \mathcal{M}. 
\]
To this end, fix $w \in \mathcal{M}_\tau \cap \bar{\mathcal{R}}$ such that $E(w) = E_0$. By (3.1), we need only to show that $w \in \mathcal{R}$.

The argument is similar to that in Remark 3.4. Since $\bar{\mathcal{R}}$ is a convex set by Lemma 2.1, for any $v \in \mathcal{R}$ and $s \in [0, 1]$ we have $sv + (1-s)w \in \mathcal{R}$. On the other hand, it is clear that $sv + (1-s)w \in \mathcal{M}_\tau$, and thus $sv + (1-s)w \in \mathcal{M}_\tau \cap \bar{\mathcal{R}}$, provided that $s \geq 0$ is sufficiently small. In view of Lemma 3.9 we obtain
\[
\frac{d}{ds} E(sv + (1-s)w) \bigg|_{s=0^+} \leq 0, 
\]
which gives
\[
\int_D v Gwdx \leq \int_D w Gwdx. 
\] (3.8)
Since the function $v$ in (3.8) is arbitrarily chosen in $\bar{\mathcal{R}}$, we can apply Lemma 2.4 to obtain $w \in \mathcal{R}$. This finishes the proof.
\[\square\]
Lemma 3.11. Let \( \tau \) be determined by Lemma 3.8. Then for any \( \{w_n\}_{n=1}^{+\infty} \subset M_\tau \cap \bar{R} \) satisfying \( \lim_{n \to +\infty} E(w_n) = M \), there exist a subsequence \( \{w_{n_j}\}_{j=1}^{+\infty} \) and some \( \eta \in M \) such that \( w_{n_j} \) converges to \( \eta \) strongly in \( L^p(D) \) as \( j \to +\infty \).

Proof. Since \( \bar{R} \) is weakly sequentially closed in \( L^p(D) \), there exist a subsequence \( \{w_{n_j}\}_{j=1}^{+\infty} \) and some \( \eta \in \bar{R} \) such that \( w_{n_j} \) converges to \( \eta \) weakly in \( L^p(D) \) as \( j \to +\infty \). To finish the proof, it suffices to show that \( \eta \in M \) (this can ensure the strong convergence of \( w_{n_j} \) to \( \eta \) since \( M \subset \bar{R} \)).

First, it is clear from the weak sequential continuity of \( E \) in \( L^p(D) \) that

\[
E(\eta) = E_0. \tag{3.9}
\]

Second, we can show that \( \eta \in M_\tau \cap \bar{R} \). In fact, since \( \{w_{n_j}\}_{j=1}^{+\infty} \subset M_\tau \), there exists a sequence \( \{v_j\}_{j=1}^{+\infty} \subset M \) such that

\[
\|w_{n_j} - v_j\|_{L^p(D)} < \frac{\tau}{2}.
\]

Since \( M \) is compact, there exist a subsequence \( \{v_{j_k}\}_{k=1}^{+\infty} \) and some \( v \in M \) such that \( v_{j_k} \) converges to \( v \) strongly in \( L^p(D) \) as \( k \to +\infty \). Hence

\[
\|\eta - v\|_{L^p(D)} \leq \liminf_{k \to +\infty} \|w_{n_{j_k}} - v_{j_k}\|_{L^p(D)} \leq \frac{\tau}{2},
\]

which implies \( \eta \in M_\tau \). Therefore

\[
\eta \in M_\tau \cap \bar{R}. \tag{3.10}
\]

Now (3.9), (3.10) and Lemma 3.10 together yield \( \eta \in M \). □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. First we show that any \( \omega \in M \) is a steady weak solution to the vorticity equation. For any \( \phi \in C_c^\infty(D) \), define a family of smooth area-preserving transformations \( \Phi_t : D \to D \) as follows

\[
\begin{aligned}
\frac{d\Phi_t(x)}{dt} &= \nabla^\perp \phi(x), \quad t \in \mathbb{R}, \\
\Phi_0(x) &= x, \quad x \in D.
\end{aligned}
\]

It is clear that \( \omega(\Phi_{-t}(\cdot)) \in \mathcal{R}_\omega \) for all \( t \in \mathbb{R} \). Taking into account (ii) in Theorem 3.1, we see \( E(\omega(\Phi_{-t}(\cdot))) \) attains its local maximum at \( t = 0 \), which implies

\[
\left. \frac{d}{dt} E(\omega(\Phi_{-t}(\cdot))) \right|_{t=0} = 0. \tag{3.11}
\]

By a simple calculation we get from (3.11) that

\[
\int_D \omega \nabla^\perp G \omega \cdot \nabla \phi \, dx = 0.
\]

Since \( \phi \in C_c^\infty(D) \) is arbitrary, \( \omega \) is a steady weak solution to the vorticity equation in the sense of Definition 1.3.
Now we prove the stability of $\mathcal{M}$. Suppose by contradiction that $\mathcal{M}$ is not stable in the $L^p$ norm of the vorticity with initial perturbations in $L^\infty(D)$. Then there exist some $\epsilon_0 > 0$, a sequence $\{\omega_0^n\}_{n=1}^{+\infty} \subset L^\infty(D)$, and a sequence $\{t_n\}_{n=1}^{+\infty} \subset \mathbb{R}_+$, such that
\[ \text{dist}(\omega_0^n, \mathcal{M}) < \frac{1}{n}, \tag{3.12} \]
and
\[ \text{dist}(\omega_{t_n}^n, \mathcal{M}) \geq \epsilon_0, \]
where $\omega_{t_n}^n$ is the unique weak solution to the vorticity equation at time $t_n$ with initial vorticity $\omega_0^n$. Here $\text{dist}(. , .)$ denotes the distance between two sets in $L^p(D)$. Without loss of generality, we assume that
\[ 0 < \epsilon_0 < \frac{\tau}{3}, \tag{3.13} \]
where $\tau$ is the positive number determined in Lemma 3.8. Moreover, since any weak solution to the vorticity is continuous with respect to the time variable in $L^p(D)$, we can assume that
\[ \text{dist}(\omega_{t_n}^n, \mathcal{M}) = \epsilon_0 \quad \text{for each} \quad n. \tag{3.14} \]
From (3.13) and (3.14) we immediately get
\[ \omega_{t_n}^n \in \mathcal{M}_{\frac{\tau}{3}} \quad \text{for each} \quad n. \tag{3.15} \]

By (3.12), we can choose a sequence $\{w_n\}_{n=1}^{+\infty} \subset \mathcal{M}$ such that
\[ \|\omega_0^n - w_n\|_{L^p(D)} < \frac{1}{n}. \tag{3.16} \]
Since $\mathcal{M}$ is compact in $L^p(D)$, there exist a subsequence, still denoted by $\{w_n\}_{n=1}^{+\infty}$, and some $w_0 \in \mathcal{M}$, such that $w_n$ converges strongly to $w_n$ as $n \to +\infty$. Combining (3.16) we get
\[ \lim_{n \to +\infty} \|\omega_0^n - w_0\|_{L^p(D)} = 0. \tag{3.17} \]
Taking into account energy conservation we get
\[ \lim_{n \to +\infty} E(\omega_{t_n}^n) = \lim_{n \to +\infty} E(\omega_0^n) = E(w_0) = E_0. \tag{3.18} \]

Now for each fixed $n$ let $w^n$ be the weak solution to the following linear transport equation with initial value $w_0$ (in the sense of Lemma 2.5)
\[ \partial w^n + \nabla \perp G^{\omega^n} \cdot \nabla w^n = 0. \]
Then for each $n$ it holds that
\[ \omega_{t_n}^n \in \mathcal{R}_{w_0} = \mathcal{R}, \tag{3.19} \]
\[ w_{t_n}^n - \omega_{t_n}^n \in \mathcal{R}_{w_0 - \omega_0^n}. \tag{3.20} \]
By (3.20) we get
\[ \|w_{t_n}^n - \omega_{t_n}^n\|_{L^p(D)} = \|w_0 - \omega_0^n\|_{L^p(D)} \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.21} \]
This together with (3.15) yields
\[ w_{t_n}^n \in \mathcal{M}_{\frac{\tau}{2}} \quad \text{if} \quad n \quad \text{is sufficiently large}. \tag{3.22} \]
From (3.19) and (3.22) we obtain
\[ w^n_t \in \mathcal{R} \cap M \] if n is sufficiently large. (3.23)
On the other hand, by (3.18) and (3.21) we have
\[ \lim_{n \to +\infty} E(w^n_t) = E_0. \] (3.24)
To summarize, we have obtained a sequence \( \{w^n_t\}_{n=1}^{+\infty} \) satisfying (3.23) and (3.24). By Lemma 3.11, there exist a subsequence, still denoted by \( \{w^n_t\}_{n=1}^{+\infty} \), and some \( \eta \in M \) such that \( w^n_t \) converges strongly to \( \eta \) as \( n \to +\infty \). Taking into account (3.21) we see that \( \omega^n_t \) converges strongly to \( \eta \) as \( n \to +\infty \), which implies
\[ \lim_{n \to +\infty} \text{dist} (\omega^n_t, M) = 0. \] (3.25)
This is an obvious contradiction to (3.14).

\[ \square \]

4. Proof of Theorem 1.6: \( k = 1 \)

In this section we give the proof of Theorem 1.6 for \( k = 1 \).
Throughout this section, let \( 1 < p < +\infty, M > 0, \kappa > 0 \) be fixed, and \( \bar{x} \) be an isolated local minimum point of the Robin function \( H \). Fix a small positive number \( \bar{r} \) such that
(i) \( \overline{B_\bar{r}(\bar{x})} \subset D \);
(ii) \( \bar{x} \) is the unique minimum point of \( H \) in \( \overline{B_\bar{r}(\bar{x})} \).
Let \( \Xi \) be defined by (1.15), and \( \Pi : (0, \bar{r}) \to \Xi \) be a map satisfying
\[ \text{supp}(\Pi_\varepsilon) \subset \overline{B_\varepsilon(0)} \text{ for any } \varepsilon \in (0, \bar{r}), \]
\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \Pi_\varepsilon(x)dx = \kappa, \] (4.1)
\[ \|\Pi_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leq M\varepsilon^{-2}. \]
Below for convenience we denote
\[ \kappa_\varepsilon = \int_{\mathbb{R}^2} \Pi_\varepsilon(x)dx. \] (4.2)
Define
\[ \mathcal{R}_\varepsilon = \{ w \in L^\infty(D) \mid w \text{ is a rearrangement of } \Pi_\varepsilon, \} \]
\[ \mathcal{K}_\varepsilon = \{ w \in \mathcal{R}_\varepsilon \mid \text{supp}(w) \subset \overline{B_\varepsilon(\bar{x})} \}. \]
Denote \( \mathcal{M}_\varepsilon \) the set of maximizers of \( E \) over \( \mathcal{K}_\varepsilon \).
Now Theorem 1.5 can be stated as follows.

**Theorem 4.1.** For any \( \varepsilon \in (0, \bar{r}) \), \( \mathcal{M}_{\Pi, \varepsilon} \) is nonempty. Moreover, for any \( \delta > 0 \), there exists some \( \varepsilon_0 > 0 \), such that for any \( \varepsilon \in (0, \varepsilon_0) \), we have
(i) \( \text{supp}(\omega) \subset \overline{B_\delta(\bar{x})}, \forall \omega \in \mathcal{M}_\varepsilon; \)
(ii) any \( \omega \in \mathcal{M}_\varepsilon \) is a weak solution to the steady vorticity equation.
By Theorem 3.1, in order to prove the stability of $M_\varepsilon$, it suffices to show that $M_\varepsilon$ is compact and isolated in the sense of Theorem 3.1.

First we prove compactness, which relies on Lemma 2.4.

**Proposition 4.2 (Compactness).** For any $\varepsilon \in (0, \bar{r})$, $M_\varepsilon$ is compact in $L^p(D)$.

**Proof.** Fix $\varepsilon \in (0, \bar{r})$. Let $\{\omega_n\}_{n=1}^{+\infty}$ be an arbitrary sequence in $M_\varepsilon$. We will show that it contains a subsequence that converges to some element of $M_\varepsilon$ in $L^p(D)$.

Obviously $\{\omega_n\}_{n=1}^{+\infty}$ is bounded in $L^p(D)$, thus there exist a subsequence, still denoted by $\{\omega_n\}_{n=1}^{+\infty}$, and some $\eta \in \overline{K}_\varepsilon$, such that $\omega_n$ converges weakly to $\eta$ in $L^p(D)$ as $n \to +\infty$. Here $\overline{K}_\varepsilon$ denotes the weak closure of $K_\varepsilon$ in $L^p(D)$, which is also equal to the weak closure of $K_\varepsilon$ in $L^p(\overline{B}_\bar{r}(\bar{x}))$. It is clear that

$$E(\eta) = \lim_{n \to +\infty} E(\omega_n) = \sup_{\omega \in K_\varepsilon} E(\omega) = \sup_{\omega \in \overline{K}_\varepsilon} E(\omega).$$

(4.3)

To finish the proof, it is sufficient to show that $\eta \in K_\varepsilon$ (obviously this implies $\eta \in M_\varepsilon$ and $\omega_n$ converges strongly to $\eta$ as $n \to +\infty$). Since $\overline{K}_\varepsilon$ is convex by Lemma 2.1, for any $v \in \overline{K}_\varepsilon$ and $s \in [0, 1]$ we have $sv + (1 - s)\eta \in \overline{K}_\varepsilon$. Hence

$$\left. \frac{d}{ds} E(sv + (1 - s)\eta) \right|_{s=0^+} \leq 0,$$

which yields

$$\int_{\overline{B}_\bar{r}(\bar{x})} v\mathcal{G}\eta \, dx \leq \int_{\overline{B}_\bar{r}(\bar{x})} \eta \mathcal{G}\eta \, dx.$$

Since $v \in \overline{K}_\varepsilon$ is arbitrary, we can take $\Omega = B_{\bar{r}}(\bar{x})$ and $\mathcal{R}_{f_0} = K_\varepsilon$ in Lemma 2.4 to get $\eta \in K_\varepsilon$.

□

Now we turn to the proof of isolatedness, which is a little complicated and only holds when $\varepsilon$ is sufficiently small. To begin with, we prove the following separation property for the stream function.

**Lemma 4.3 (Separation property).** Define

$$r_\varepsilon = \inf \{r > 0 \mid \text{for any } \omega \in M_\varepsilon, \omega = 0 \text{ a.e. in } D \setminus B_r(\bar{x})\}$$

Then there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, it holds that

$$\sup_{D \setminus B_r(\bar{x})} \mathcal{G}\omega < \inf_{B_{r_\varepsilon}(\bar{x})} \mathcal{G}\omega, \forall \omega \in M_\varepsilon.$$  \hspace{1cm} (4.4)

**Proof.** Obviously $\text{supp}(\omega) \subset B_{r_\varepsilon}(\bar{x})$ for any $\omega \in M_\varepsilon$. This implies $r_\varepsilon > 0$ for any $\varepsilon \in (0, \bar{r})$. Moreover, by (i) in Theorem 4.1 we see that $r_\varepsilon \to 0$ as $\varepsilon \to 0$. 

□
Fix \( \omega \in \mathcal{M}_\varepsilon \). For any \( x \in D \setminus B_r(\bar{x}) \), we estimate \( G_\omega(x) \) as follows

\[
G_\omega(x) = -\frac{1}{2\pi} \int_D \ln |x - y| \omega(y)dy - \int_D h(x, y)\omega(y)dy
\]

\[
= -\frac{1}{2\pi} \int_{B_{r_\varepsilon}(\bar{x})} \ln |x - y| \omega(y)dy - \int_D h(x, y)\omega(y)dy
\]

\[
\leq -\frac{1}{2\pi} \ln(\bar{r} - r_\varepsilon) \int_D \omega(y)dy + \|h\|_{L^\infty(B_r(\bar{x}) \times B_r(\bar{x}))} \int_D \omega(y)dy
\]

\[
= -\frac{k_\varepsilon}{2\pi} \ln(\bar{r} - r_\varepsilon) + \|h\|_{L^\infty(B_r(\bar{x}) \times B_r(\bar{x}))} K_\varepsilon
\]

\[
\to -\frac{k}{2\pi} \ln \bar{r} + \|h\|_{L^\infty(B_r(\bar{x}) \times B_r(\bar{x}))} K
\]

as \( \varepsilon \to 0 \). Here we used the fact that \( \omega = 0 \) a.e. in \( D \setminus B_{r_\varepsilon}(\bar{x}) \). This can be easily verified from the definition of \( r_\varepsilon \).

On the other hand, for any \( x \in B_{r_\varepsilon}(\bar{x}) \), we have

\[
G_\omega(x) = -\frac{1}{2\pi} \int_D \ln |x - y| \omega(y)dy - \int_D h(x, y)\omega(y)dy
\]

\[
= -\frac{1}{2\pi} \int_{B_{r_\varepsilon}(\bar{x})} \ln |x - y| \omega(y)dy - \int_D h(x, y)\omega(y)dy
\]

\[
\geq -\frac{1}{2\pi} \ln(2r_\varepsilon) \int_D \omega(y)dy - \|h\|_{L^\infty(B_r(\bar{x}) \times B_r(\bar{x}))} \int_D \omega(y)dy
\]

\[
= -\frac{k_\varepsilon}{2\pi} \ln(2r_\varepsilon) - \|h\|_{L^\infty(B_r(\bar{x}) \times B_r(\bar{x}))} K_\varepsilon
\]

\[
\to +\infty
\]

as \( \varepsilon \to 0 \).

Combining (4.5) and (4.6), we deduce that

\[
\sup_{D \setminus B_r(\bar{x})} G_\omega < \inf_{B_{r_\varepsilon}(\bar{x})} G_\omega, \ \forall \omega \in \mathcal{M}_\varepsilon
\]

if \( \varepsilon \) is sufficiently small.

The following lemma will also be used in obtaining isolatedness.

**Lemma 4.4.** Let \( U \subset \mathbb{R}^2 \) be a bounded domain, \( B_1, B_2 \) be open balls such that \( B_1 \subset B_2 \subset U \), and \( \psi \in C(\bar{U}) \) satisfies

\[
\inf_{B_1} \psi > \sup_{U \setminus B_2} \psi. \tag{4.7}
\]

Let \( \mathcal{R} \) be the rearrangement class of some nonnegative function \( f_0 \in L^1(U) \) satisfying

\[
\mathcal{L}(\{x \in U \mid f_0(x) > 0\}) \leq \mathcal{L}(B_1). \tag{4.8}
\]
Suppose $u \in \mathcal{R}$ satisfies
\[
\int_U w \psi \, dx \geq \int_U u \psi \, dx \quad \forall w \in \mathcal{R},
\] (4.9)
then $\text{supp}(u) \subset \bar{B}_2$.

**Proof.** Suppose by contradiction that $\text{supp}(u) \not\subset \bar{B}_2$. Define
\[W := \{ x \in U \setminus B_2 \mid u(x) > 0 \}.\]
Then it is obvious that
\[\mathcal{L}(W) > 0, \quad \int_W u \, dx > 0.\]
Now we claim that
\[\mathcal{L}(\{ x \in B_1 \mid u(x) = 0 \}) \geq \mathcal{L}(W).\] (4.10)
In fact,
\[
\begin{align*}
\mathcal{L}(\{ x \in B_1 \mid u(x) = 0 \}) & = \mathcal{L}(B_1) - \mathcal{L}(\{ x \in B_1 \mid u(x) > 0 \}) \\
& \geq \mathcal{L}(B_1) - \mathcal{L}(\{ x \in B_2 \mid u(x) > 0 \}) \\
& = \mathcal{L}(B_1) - \mathcal{L}(\{ x \in U \mid u(x) > 0 \}) + \mathcal{L}(\{ x \in U \setminus B_2 \mid u(x) > 0 \}) \\
& = \mathcal{L}(B_1) - \mathcal{L}(\{ x \in U \mid f_0(x) > 0 \}) + \mathcal{L}(W) \\
& \geq \mathcal{L}(W).
\end{align*}
\]
Note that in the last inequality we used (4.8). By (4.10), there exists some Lebesgue measurable set $V \subset \{ x \in B_1 \mid u(x) = 0 \}$ such that
\[\mathcal{L}(V) = \mathcal{L}(W).\]
Applying Lemma 2.2, we can choose some Lebesgue measurable function $v$ defined on $V$, such that $v$ is a rearrangement of $u1_W$. Extend $v$ such that $v(x) = 0$ a.e. $x \in U \setminus V$. Define
\[\hat{u} = u - u1_W + v.\]
Then it is clear that $\hat{u} \in \mathcal{R}$. Now we calculate as follows
\[
\int_U w \psi \, dx - \int_U \hat{u} \psi \, dx = \int_U (u1_W - v) \psi \, dx
\]
\[
= \int_W w \psi \, dx - \int_V v \psi \, dx
\]
\[
\leq \sup_{U \setminus B_2} \psi \int_W u \, dx - \inf_{B_1} \psi \int_V v \, dx
\]
\[
= \left( \sup_{U \setminus B_2} \psi - \inf_{B_1} \psi \right) \int_W u \, dx
\]
\[
< 0,
\]
which is a contradiction to (4.9).
Now we are ready to prove the isolatedness of $\mathcal{M}_\varepsilon$ for small $\varepsilon$.

**Proposition 4.5** (Isolatedness). Let $\varepsilon_0 > 0$ be determined by Lemma 4.3. Then for any $\varepsilon \in (0, \varepsilon_0)$, there exists some $\delta > 0$ such that

$$\mathcal{M}_\varepsilon = \left\{ w \in (\mathcal{M}_\varepsilon)_{\delta} \cap \mathcal{R}_\varepsilon \mid E(w) = \sup_{w \in (\mathcal{M}_\varepsilon)_{\delta} \cap \mathcal{R}_\varepsilon} E(w) \right\},$$

(4.11)

where $(\mathcal{M}_\varepsilon)_{\delta}$ is the $\delta$-neighborhood of $\mathcal{M}_\varepsilon$ in $L^p(D)$.

**Proof.** Let $\varepsilon \in (0, \varepsilon_0)$ be fixed. Then (4.4) holds by Lemma 4.3.

Denote

$$M_\varepsilon = \sup_{w \in \mathcal{K}_\varepsilon} E(w).$$

(4.12)

Obviously for any $w \in \mathcal{K}_\varepsilon$, $w \in \mathcal{M}_\varepsilon$ if and only if $E(w) = M_\varepsilon$.

To prove (4.11), it suffices to show that there exists some $\delta > 0$ such that for any $w \in (\mathcal{M}_\varepsilon)_{\delta} \cap \mathcal{R}_\varepsilon$ satisfying $E(w) \geq M_\varepsilon$, it holds that $w \in \mathcal{M}_\varepsilon$. Suppose by contradiction that this is false. Then for any positive integer $n$, there exists some $w_n$ such that

$$w_n \in (\mathcal{M}_\varepsilon)_{\frac{1}{n}} \cap \mathcal{R}_\varepsilon,$$

(4.13)

$$E(w_n) \geq M_\varepsilon,$$

(4.14)

$$w_n \notin \mathcal{M}_\varepsilon.$$  

(4.15)

By (4.14), we can choose some sequence $\{v_n\}_{n=1}^{\infty} \subset \mathcal{M}_\varepsilon$ such that

$$\|v_n - w_n\|_{L^p(D)} < \frac{1}{n}.$$  

(4.16)

On the other hand, by the compactness of $\mathcal{M}_\varepsilon$ we can choose a subsequence of $\{v_n\}_{n=1}^{\infty}$, still denoted by $\{v_n\}_{n=1}^{\infty}$, and some $\eta \in \mathcal{M}_\varepsilon$, such that

$$\lim_{n \to +\infty} \|v_n - \eta\|_{L^p(D)} = 0.$$  

(4.17)

Combining (4.16) and (4.17) we obtain

$$\lim_{n \to +\infty} \|w_n - \eta\|_{L^p(D)} = 0.$$  

(4.18)

By standard elliptic estimate, we get from (4.19) that

$$\lim_{n \to +\infty} \|Gw_n - G\eta\|_{L^\infty(D)} = 0.$$  

(4.19)

On the other hand, since $\eta \in \mathcal{M}_\varepsilon$, we get from (4.14) that

$$\sup_{D \setminus B_{r}(\bar{x})} G\eta < \inf_{B_{r}(\bar{x})} G\eta.$$  

(4.20)

Now (4.19) and (4.20) together yield

$$\sup_{D \setminus B_{r}(\bar{x})} Gw_n < \inf_{B_{r}(\bar{x})} Gw_n.$$  

(4.21)
if \( n \) is sufficiently large. Below let \( n \) be fixed and large enough such that (4.21) holds.

By Lemma 2.3 there exists some \( u_n \in \mathcal{R}_\varepsilon \) such that

\[
\int_D u_n \mathcal{G} w_n dx \geq \int_D w \mathcal{G} w_n dx \quad \forall \ w \in \mathcal{R}_\varepsilon.
\]  (4.22)

Taking into account (4.21), (4.22), and choosing \( B_2 = B_r(\bar{x}), B_1 = B_{r_\varepsilon}(\bar{x}) \), \( u = u_n \) in Lemma 4.4, we get

\[
\text{supp}(u_n) \subset B_{r_\varepsilon}(\bar{x}).
\]  (4.23)

Note that (4.8) is satisfied since \( \text{supp}(\omega) \subset B_{r_\varepsilon}(\bar{x}) \) for any \( \omega \in \mathcal{M}_\varepsilon \) by the definition of \( r_\varepsilon \).

Hence we obtain

\[
u_n \in \mathcal{K}_\varepsilon.
\]  (4.24)

To get a contradiction, we prove the following statement:

\[
E(u_n) \geq E(w_n), \text{ and the equality holds if and only if } u_n = w_n.
\]  (4.25)

In fact, since \( w_n \in \mathcal{R}_\varepsilon \) (see (4.13)), we get from (4.22) that

\[
\int_D u_n \mathcal{G} w_n dx \geq \int_D w_n \mathcal{G} w_n dx.
\]  (4.26)

Now we calculate \( E(u_n) - E(w_n) \) as follows

\[
E(u_n) - E(w_n) = \frac{1}{2} \int_D u_n \mathcal{G} u_n - w_n \mathcal{G} w_n dx
\]  (4.27)

\[
= \frac{1}{2} \int_D (u_n - w_n) \mathcal{G} (u_n + w_n) dx
\]  (4.28)

\[
= \frac{1}{2} \int_D (u_n - w_n) \mathcal{G} (u_n - w_n) dx + \int_D (u_n - w_n) \mathcal{G} w_n dx
\]  (4.29)

\[
\geq \int_D (u_n - w_n) \mathcal{G} w_n dx
\]  (4.30)

\[
\geq 0.
\]  (4.31)

Note that in (4.27) we used the symmetry of the Green operator \( \mathcal{G} \), in (4.30) we used the positivity of \( \mathcal{G} \), and in (4.31) we used (4.26). Moreover, it is easy to see that (4.31) is an equality if and only if \( u_n = w_n \). Therefore \( E(u_n) = E(w_n) \) if and only if \( u_n = w_n \). This proves (4.25).

Based on (4.25), we can easily get a contradiction. In fact, since \( u_n \in \mathcal{K}_\varepsilon \), we get from (4.12) that \( E(u_n) \leq M_\varepsilon \). Taking into account (4.14) we obtain \( E(w_n) \geq E(u_n) \). This together with (4.25) gives \( E(w_n) = E(u_n) \), and thus \( w_n = u_n \), which contradicts (4.15).

Remark 4.6. Repeating the argument in Proposition 4.5, we can in fact prove the following conclusion. Let \( \mathcal{R} \) be the rearrangement class of some nonnegative function \( f_0 \in L^\infty(D) \) and \( \mathcal{M} \subset \mathcal{R} \) be nonempty. Suppose there exist two open ball \( B_1, B_2 \) with \( B_1 \subset \subset B_2 \subset \subset D \) such that

(i) \( \mathcal{M} \) is compact in \( L^p(D) \), where \( 1 < p < +\infty \) is fixed;
(ii) \( \mathcal{M} \subset \mathcal{K}_1 \), where \( \mathcal{K}_1 = \{ w \in \mathcal{R} \mid \text{supp}(w) \subset \bar{B}_1 \} \);
(iii) \( \mathcal{M} = \{ w \in \mathcal{K}_2 \mid E(w) = \sup_{\mathcal{K}_2} E \} \), where \( \mathcal{K}_2 = \{ w \in \mathcal{R} \mid \text{supp}(w) \subset \bar{B}_2 \} \);
(iv) for any \( \omega \in \mathcal{M} \), it holds that
\[
\sup_{D \setminus B_2} G_\omega < \inf_{B_1} G_\omega.
\]
Then \( \mathcal{M} \) must be an isolated set of local maximizers of \( E \) over \( \mathcal{R} \).

5. PROOF OF THEOREM 1.6 \( k = 2 \)

In this section, we prove Theorem 1.6 for \( k = 2 \) and \( \kappa_1\kappa_2 < 0 \). The process is parallel to that in Section 4, therefore we may omit some similar arguments to avoid verbosity.

Throughout this section let \( 1 < p < +\infty \), \( M > 0 \), \( \kappa_1 > 0 \), \( \kappa_2 < 0 \) be fixed, and \( (\bar{x}_1, \bar{x}_2) \) be a given isolated local minimum point of the corresponding Kirchhoff-Routh function
\[
W(x_1, x_2) = -2\kappa_1\kappa_2 G(x_1, x_2) + \kappa_1^2 h(x_1, x_1) + \kappa_2^2 h(x_2, x_2). \tag{5.1}
\]
Fix a small positive number \( \bar{r} \) such that

(i) \( B_r(\bar{x}_i) \subset D, \ i = 1, 2 \);
(ii) \( B_r(\bar{x}_1) \cap \overline{B_r(\bar{x}_2)} = \emptyset \);
(iii) \( (\bar{x}_1, \bar{x}_2) \) is the unique minimum point of \( W \) in \( \overline{B_r(\bar{x}_1) \times B_r(\bar{x}_2)} \).

Let \( \Pi^1, \Pi^2 : (0, \bar{r}) \to \Xi \) be two maps satisfying
\[
\text{lim}_{\varepsilon \to 0} \int_{\mathbb{R}^2} \Pi^i_\varepsilon(x) dx = |\kappa_i|, \quad \text{supp}((\Pi^i_\varepsilon)_\varepsilon) \subset \overline{B_\varepsilon(0)} \quad \text{for any } \varepsilon \in (0, \bar{r}), \quad ||\Pi^i_\varepsilon||_{L^\infty(\mathbb{R}^2)} \leq M \varepsilon^{-2}, \tag{5.2}
\]
where \( i = 1, 2 \). For simplicity, denote
\[
\kappa_{1,\varepsilon} = \int_{\mathbb{R}^2} \Pi^1_\varepsilon(x) dx, \quad \kappa_{2,\varepsilon} = \int_{\mathbb{R}^2} \Pi^2_\varepsilon(x) dx.
\]

Define
\[
\mathcal{K}_\varepsilon = \left\{ w = \sum_{i=1}^{2} w_i \mid \text{supp}(w_i) \subset \overline{B_r(\bar{x}_i)}, \ w_i \text{ is a rearrangement of } \text{sgn}(\kappa_i)\Pi^i_\varepsilon, \ i = 1, 2 \right\},
\]
\[
\mathcal{R}_\varepsilon = \{ w \in L^\infty(D) \mid w \text{ is the rearrangement of some } v \in \mathcal{K}_\varepsilon \}.
\]
Note that the above definition of \( \mathcal{R}_\varepsilon \) is reasonable since any two elements in \( \mathcal{K}_\varepsilon \) have the same distribution function.

Denote \( \mathcal{M}_\varepsilon \) the set of maximizers of \( E \) over \( \mathcal{K}_\varepsilon \).

Theorem 1.5 now can be stated as follows.

**Theorem 5.1.** For any \( \varepsilon \in (0, \bar{r}) \), \( \mathcal{M}_\varepsilon \) is nonempty. Moreover, for any \( \delta > 0 \), there exists some \( \varepsilon_0 > 0 \), such that for any \( \varepsilon \in (0, \varepsilon_0) \), we have

(i) \( \text{supp}(\omega 1_{B_r(\bar{x}_i)}) \subset \overline{B_\delta(\bar{x}_i)}, \ \forall \omega \in \mathcal{M}_\varepsilon, \ i = 1, 2; \)
(ii) any \( \omega \in \mathcal{M}_\varepsilon \) is a steady weak solution to the vorticity equation.
Below we show that $\mathcal{M}_\varepsilon$ is compact and isolated. Compactness still relies on Lemma 2.4.

**Proposition 5.2** (Compactness). For any $\varepsilon \in (0, \bar{r})$, $\mathcal{M}_\varepsilon$ is compact in $L^p(D)$.

**Proof.** Fix some sequence $\{\omega_n\}_{n=1}^{+\infty} \subset \mathcal{M}_\varepsilon$. Denote $\omega_{n,i} = \omega_n 1_{B_r(\bar{x}_i)}$, $i = 1, 2$.

Below let $i \in \{1, 2\}$ be fixed. For $\{\omega_{n,i}\}_{n=1}^{+\infty}$, there exist a subsequence, still denoted by $\{\omega_{n,i}\}_{n=1}^{+\infty}$, and some $\eta_i \in L^p(D)$, such that $\omega_{n,i}$ converges strongly to $\eta_i$ as $n \to +\infty$. It is easy to check that $\eta_i \in \mathcal{K}_{i,\varepsilon}$, the weak closure of $\mathcal{K}_{i,\varepsilon}$ in $L^p(B_r(\bar{x}_i))$ (or $L^p(D)$), where

$$\mathcal{K}_{i,\varepsilon} = \{w \in L^\infty(D) \mid \text{supp}(w) \subset B_r(\bar{x}_i), \text{ } w \text{ is the rearrangement of } \text{sgn}(\kappa_i)\Pi^i\}.$$

A similar argument as in Proposition 4.2 gives

$$\int_{B_r(\bar{x}_i)} \eta_i \mathcal{G}\eta_i dx \geq \int_{B_r(\bar{x}_i)} v \mathcal{G}\eta_i dx \quad \forall \ v \in \mathcal{K}_{i,\varepsilon}.$$

By Lemma 2.4, we get $\eta \in \mathcal{K}_{i,\varepsilon}$, and thus $\omega_{n,i}$ converges strongly to $\eta_i$ as $n \to +\infty$.

Denote $\eta = \eta_1 + \eta_2$, then $\eta \in \mathcal{K}_\varepsilon$ and $\omega_n$ converges strongly to $\eta$ as $n \to +\infty$. Hence the compactness of $\mathcal{M}_\varepsilon$ is proved.

\[\square\]

**Lemma 5.3** (Separation property). Define

$$r_\varepsilon = \inf \left\{r > 0 \mid \omega 1_{B_r(\bar{x}_i)} = 0 \text{ a.e. in } D \setminus B_r(\bar{x}_i) \text{ for any } \omega \in \mathcal{M}_\varepsilon \text{ and } i = 1, 2 \right\}.$$

Then there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, it holds that

$$\sup_{D \setminus B_{r_\varepsilon}(\bar{x}_1)} \mathcal{G}\omega < \inf_{B_{r_\varepsilon}(\bar{x}_1)} \mathcal{G}\omega, \quad \inf_{D \setminus B_{r_\varepsilon}(\bar{x}_2)} \mathcal{G}\omega > \sup_{B_{r_\varepsilon}(\bar{x}_2)} \mathcal{G}\omega, \quad \forall \omega \in \mathcal{M}_\varepsilon.$$

**Proof.** It is clear that $r_\varepsilon > 0$ for any $\varepsilon \in (0, \varepsilon_0)$. Moreover, $r_\varepsilon \to 0$ as $\varepsilon \to 0$ by (i) in Theorem 5.1.

Below let $\omega \in \mathcal{M}_\varepsilon$ be fixed. For any $x \in D \setminus B_{r_\varepsilon}(\bar{x}_1)$, we estimate $\mathcal{G}\omega(x)$ as follows

$$\mathcal{G}\omega(x) = \int_{B_{r_\varepsilon}(\bar{x}_1)} G(x, y)\omega(y)dy + \int_{B_{r_\varepsilon}(\bar{x}_2)} G(x, y)\omega(y)dy \quad (5.3)$$

$$\leq \int_{B_{r_\varepsilon}(\bar{x}_1)} G(x, y)\omega(y)dy + \|G\|_{L^\infty(B_{r_\varepsilon}(\bar{x}_1) \times B_{r_\varepsilon}(\bar{x}_2))} |\kappa_{2,\varepsilon}|. \quad (5.4)$$

In (5.3) we used the fact $\omega = \omega 1_{B_{r_\varepsilon}(\bar{x}_1)} + \omega 1_{B_{r_\varepsilon}(\bar{x}_2)}$. Since $\overline{B_{r_\varepsilon}(\bar{x}_1)} \cap \overline{B_{r_\varepsilon}(\bar{x}_2)} = \emptyset$, we get

$$\|G\|_{L^\infty(B_{r_\varepsilon}(\bar{x}_1) \times B_{r_\varepsilon}(\bar{x}_2))} < +\infty.$$
The first term in (5.4) can be estimated as follows

\[
\int_{B_\epsilon(x_1)} G(x, y) \omega(y) dy = -\frac{1}{2\pi} \int_{B_\epsilon(x_1)} \ln |x - y| \omega(y) dy - \int_{B_\epsilon(x_1)} h(x, y) \omega(y) dy
\]

\[
= -\frac{1}{2\pi} \int_{B_\epsilon(x_1)} \ln |x - y| \omega(y) dy - \int_{B_\epsilon(x_1)} h(x, y) \omega(y) dy
\]

\[
\leq -\frac{1}{2\pi} \int_{B_\epsilon(x_1)} \ln(\rho - r_\epsilon) \omega(y) dy + \|h\|_{L^\infty(B_\epsilon(x_1) \times B_\epsilon(x_1))} |\kappa_{1, \epsilon}|
\]

(5.5)

\[
= -\frac{\kappa_{1, \epsilon}}{2\pi} \ln(\rho - r_\epsilon) + \|h\|_{L^\infty(B_\epsilon(x_1) \times B_\epsilon(x_1))}\kappa_{1, \epsilon}
\]

\[
\to -\frac{\kappa_{1, \epsilon}}{2\pi} \ln(\rho) + \|h\|_{L^\infty(B_\epsilon(x_1) \times B_\epsilon(x_2))}\kappa_1
\]

as \(\epsilon \to 0\). On the other hand, for any \(x \in \overline{B_\epsilon(x_1)}\), we have

\[
\mathcal{G}_\omega(x) = \int_{B_\epsilon(x_2)} G(x, y) \omega(y) dy + \int_{B_\epsilon(x_1)} G(x, y) \omega(y) dy
\]

(5.6)

\[
\geq \int_{B_\epsilon(x_1)} G(x, y) \omega(y) dy - \|G\|_{L^\infty(B_\epsilon(x_1) \times B_\epsilon(x_2))}\kappa_2\epsilon.
\]

(5.7)

For first term in (5.7),

\[
\int_{B_\epsilon(x_1)} G(x, y) \omega(y) dy = -\frac{1}{2\pi} \int_{B_\epsilon(x_1)} \ln |x - y| \omega(y) dy - \int_{B_\epsilon(x_1)} h(x, y) \omega(y) dy
\]

\[
= -\frac{1}{2\pi} \int_{B_\epsilon(x_1)} \ln |x - y| \omega(y) dy - \int_{B_\epsilon(x_1)} h(x, y) \omega(y) dy
\]

(5.8)

\[
\geq -\frac{\kappa_{1, \epsilon}}{2\pi} \ln(2\epsilon) - \|h\|_{L^\infty(B_\epsilon(x_1) \times B_\epsilon(x_1))}\kappa_{1, \epsilon}
\]

\[
\to +\infty
\]

as \(\epsilon \to 0\). Combining (5.4), (5.5), (5.7) and (5.8), we see that

\[
\sup_{D \setminus \overline{B_\epsilon(x_1)}} \mathcal{G}_\omega < \inf_{B_\epsilon(x_1)} \mathcal{G}_\omega, \forall \omega \in \mathcal{M}_\epsilon
\]

provided that \(\epsilon\) is sufficiently small. Similarly, for small \(\epsilon\) we have

\[
\inf_{D \setminus \overline{B_\epsilon(x_2)}} \mathcal{G}_\omega > \sup_{B_\epsilon(x_2)} \mathcal{G}_\omega, \forall \omega \in \mathcal{M}_\epsilon.
\]

Hence the proof is finished.

\[\square\]

**Lemma 5.4.** Let \(U \subset \mathbb{R}^2\) be a bounded domain, \(B_1, B_2, B_3, B_4\) be open balls such that \(B_1 \subset B_2 \subset \overline{U}, B_3 \subset B_4 \subset U\), and \(B_2 \cap \overline{B_4} = \emptyset\). Let \(f_1, f_2 \in L^1(U)\) such that \(f_1 \geq 0, f_2 \leq 0\) a.e. in \(U\), \(\text{supp}(f_1) \subset B_2, \text{supp}(f_2) \subset B_4\), and

\[
\mathcal{L}(\{x \in U \mid f_1(x) > 0\}) \leq \mathcal{L}(B_1), \quad \mathcal{L}(\{x \in U \mid f_2(x) < 0\}) \leq \mathcal{L}(B_3).
\]

(5.9)
Let $\mathcal{R}$ be the rearrangement class of $f_1 + f_2$. Let $\psi \in C(\bar{U})$ satisfy
\begin{equation}
\inf_{B_1} \psi > \sup_{U \setminus B_2} \psi, \quad \sup_{B_3} \psi < \inf_{U \setminus B_4} \psi.
\end{equation}
If $u \in \mathcal{R}$ satisfies
\begin{equation}
\int_{U} w \psi dx \geq \int_{U} w \psi dx, \quad \forall \ w \in \mathcal{R},
\end{equation}
then $\text{supp}(u_+) \subset \bar{B}_1$, $\text{supp}(u_-) \subset \bar{B}_3$.

Proof. The proof is very similar to that of Lemma 4.4, we omit it therefore. \hfill \Box

Proposition 5.5 (Isolatedness). Let $\varepsilon_0 > 0$ be determined in Lemma 5.3. Then for any $\varepsilon \in (0, \varepsilon_0)$, there exists some $\delta > 0$ such that
\begin{equation}
M_{\varepsilon} = \sup_{w \in (M_{\varepsilon})_{\delta}} E(w) = \sup_{w \in (M_{\varepsilon})_{\delta} \cap \mathcal{R}_{\varepsilon}} E(w),
\end{equation}
where $(M_{\varepsilon})_{\delta}$ is the $\delta$-neighborhood of $M_{\varepsilon}$ in $L^p(D)$.

Proof. Let $r_\varepsilon$ be defined in Lemma 5.3. Then for any $\varepsilon \in (0, \varepsilon_0)$, it holds that
\begin{equation}
\sup_{D \setminus B_r(\bar{x}_1)} \mathcal{G} \omega < \inf_{B_{r_\varepsilon}(\bar{x}_1)} \mathcal{G} \omega, \quad \inf_{D \setminus B_r(\bar{x}_2)} \mathcal{G} \omega > \sup_{B_{r_\varepsilon}(\bar{x}_2)} \mathcal{G} \omega, \quad \forall \omega \in M_{\varepsilon}.
\end{equation}

Denote
\begin{equation}
M_{\varepsilon} = \sup_{w \in K_{\varepsilon}} E(w).
\end{equation}
Below we show that there exists some $\delta > 0$, such that for any $w \in (M_{\varepsilon})_{\delta} \cap \mathcal{R}_{\varepsilon}$ satisfying $E(w) \geq M_{\varepsilon}$, it holds that $w \in M_{\varepsilon}$. Suppose that this statement is not true, then for any positive integer $n$ there exists some $w_n$ such that
\begin{equation}
w_n \in (M_{\varepsilon})_{\delta} \cap \mathcal{R}_{\varepsilon},
\end{equation}
\begin{equation}
E(w_n) \geq M_{\varepsilon},
\end{equation}
\begin{equation}
w_n \notin M_{\varepsilon}.
\end{equation}
Below we deduce a contradiction from (5.14), (5.15) and (5.16).

Since $M_{\varepsilon}$ is compact by Lemma 5.2, there exists some $\eta \in M_{\varepsilon}$ such that up to a subsequence
\begin{equation}
\lim_{n \to +\infty} \|w_n - \eta\|_{L^p(D)} = 0.
\end{equation}
Consequently
\begin{equation}
\lim_{n \to +\infty} \|\mathcal{G} w_n - \mathcal{G} \eta\|_{L^\infty(D)} = 0.
\end{equation}
Since $\eta \in M_{\varepsilon}$, taking into account (5.12), we deduce that
\begin{equation}
\sup_{D \setminus B_r(\bar{x}_1)} \mathcal{G} w_n < \inf_{B_{r_\varepsilon}(\bar{x}_1)} \mathcal{G} w_n, \quad \inf_{D \setminus B_r(\bar{x}_2)} \mathcal{G} w_n > \sup_{B_{r_\varepsilon}(\bar{x}_2)} \mathcal{G} w_n, \quad \forall \omega \in M_{\varepsilon},
\end{equation}
provided that $n$ is large enough.
Below fix a large $n$ such that (5.19) holds. By Lemma 2.3, there exists some $u_n \in \mathcal{R}_\varepsilon$ such that
\[ \int_D u_n G w_n \, dx \geq \int_D w G w_n \, dx \quad \forall w \in \mathcal{R}_\varepsilon. \] (5.20)
From (5.19) and (5.20), we can apply Lemma 5.4 to get
\[ \text{supp}(u_n 1_{\{u_n > 0\}}) \subset B_{r_\varepsilon}(\bar{x}_1), \quad \text{supp}(u_n 1_{\{u_n < 0\}}) \subset B_{r_\varepsilon}(\bar{x}_2) \] (5.21)
which implies
\[ u_n \in \mathcal{K}_\varepsilon. \] (5.22)
From (5.13), (5.15) and (5.22) we see that
\[ E(u_n) \leq M_\varepsilon \leq E(w_n). \] (5.23)
On the other hand, by (5.20), using symmetry and positivity of the Green operator, we can prove the following assertion
\[ E(u_n) \geq E(w_n), \text{ and the equality holds if and only if } u_n = w_n. \] (5.24)
By (5.23), (5.24) we get $w_n = u_n$, and thus $E(w_n) = M_\varepsilon$. This means $w_n = u_n \in \mathcal{M}_\varepsilon$, a contradiction to (5.16).

\[ \square \]

Remark 5.6. From the proof of Proposition 5.5, we in fact obtain the following conclusion. Let $B_1, B_2, B_3, B_4$ be open balls such that $B_1 \subset B_2 \subset \subset D$, $B_3 \subset B_4 \subset \subset D$, and $\overline{B_2} \cap \overline{B_4} = \emptyset$. Let $f, g \in L^\infty(D)$ such that $f \geq 0, g \leq 0 \text{ a.e. in } D$, supp($f$) $\subset B_1$, supp($g$) $\subset B_3$. Let $\mathcal{R}$ be the rearrangement class of $f + g$ in $D$ and $\mathcal{M} \subset \mathcal{R}$ be nonempty. Suppose that $\mathcal{M}$ satisfies the following conditions:

(i) $\mathcal{M}$ is compact in $L^p(D)$, where $1 < p < +\infty$ is fixed;
(ii) $\mathcal{M} \subset \mathcal{K}_1$, where
\[ \mathcal{K}_1 = \{w = w_1 + w_2 \mid w_1 \in \mathcal{R}_f, \text{supp}(w_1) \subset \overline{B}_1, w_2 \in \mathcal{R}_g, \text{supp}(w_2) \subset \overline{B}_3\}; \]
(iii) $\mathcal{M} = \{w \in \mathcal{K}_2 \mid E(w) = \sup_{\mathcal{K}_2} E\}$, where
\[ \mathcal{K}_2 = \{w = w_1 + w_2 \mid w_1 \in \mathcal{R}_f, \text{supp}(w_1) \subset \overline{B}_2, w_2 \in \mathcal{R}_g, \text{supp}(w_2) \subset \overline{B}_4\}; \]
(iv) for any $\omega \in \mathcal{M}$, it holds that
\[ \sup_{D \setminus B_2} G \omega < \inf_{B_1} G \omega, \quad \inf_{D \setminus B_4} G \omega > \sup_{B_3} G \omega. \]
Then $\mathcal{M}$ must be an isolated set of local maximizers of $E$ over $\mathcal{R}$.

6. Construction of 2D steady Euler flows with concentrated vorticity

In this section we prove Theorem 1.5. For clarity, we only provide the proof for $k = 1$. For general $k$, there is no essential difficulty.
6.1. Existence of a maximizer and its profile.

**Proposition 6.1.** For any $\varepsilon \in (0, \bar{r})$, $\mathcal{M}_\varepsilon$ is nonempty. Moreover, for any $\omega \in \mathcal{M}_\varepsilon$, there exists some nondecreasing function $\phi : \mathbb{R} \to \mathbb{R}$, depending on $\omega$, such that $\omega = \phi(G\omega)$ a.e. in $B_{\bar{r}}(\bar{x})$.

**Proof.** For fixed $\varepsilon \in (0, \bar{r})$, since $\mathcal{K}_\varepsilon$ is bounded in $L^2(D)$, we can easily get

$$M_\varepsilon = \sup_{w \in \mathcal{K}_\varepsilon} E(w) < +\infty.$$  

Choose $\{w_n\}_{n=1}^{+\infty}$ such that $E(w_n) \to M_\varepsilon$ as $n \to +\infty$. Up to a subsequence, we can assume that $w_n$ converges weakly to some $\omega \in \bar{\mathcal{K}}_\varepsilon$ in $L^2(D)$. Here $\bar{\mathcal{K}}_\varepsilon$ denotes the weak closure of $\mathcal{K}_\varepsilon$ in $L^2(D)$. By the weak sequential continuity of $E$ in $L^2(D)$, we have $E(\omega) = M_\varepsilon$, that is, $\omega$ is a maximizer of $E$ over $\bar{\mathcal{K}}_\varepsilon$.

Now we show that $\omega \in \mathcal{K}_\varepsilon$. Since $\bar{\mathcal{K}}_\varepsilon$ is convex by Lemma 2.1 for any $v \in \bar{\mathcal{K}}_\varepsilon$ and $s \in [0, 1]$ we have $sv + (1 - s)\omega \in \bar{\mathcal{K}}_\varepsilon$. Therefore

$$\frac{d}{ds} E(sv + (1 - s)\omega) \bigg|_{s=0^+} \leq 0,$$

which yields

$$\int_D \omega G\omega dx \geq \int_D v G\omega dx.$$  

By taking $\Omega = B_{\bar{r}}(\bar{x})$, $\mathcal{R}_{f_0} = \mathcal{K}_\varepsilon$, $\tilde{f} = \omega$ and $g = G\omega$ in Lemma 2.1, we get

(i) $\omega \in \mathcal{K}_\varepsilon$;
(ii) $\omega = \phi(G\omega)$ a.e. in $B_{\bar{r}}(\bar{x})$ for some nondecreasing function $\phi : \mathcal{R} \to [-\infty, +\infty]$.

Since $\omega \in L^\infty(D)$, we can redefine $\phi$ such that it is real-valued. Thus the proof is completed.

For each $\omega \in \mathcal{M}_\varepsilon$, let $\phi : \mathcal{R} \to \mathcal{R}$ be the nondecreasing function in Proposition 6.1. Define the Lagrangian multiplier related to $\omega$ as follows

$$\mu_\omega = \inf \{s \in \mathcal{R} \mid \phi(s) > 0\}.$$  

It is clear from the definition of $\mu_\omega$ that

$$\begin{aligned}
\omega > 0 & \quad \text{a.e. on } \{x \in B_{\bar{r}}(\bar{x}) \mid G\omega(x) > \mu_\omega\}; \\
\omega = 0 & \quad \text{a.e. on } \{x \in B_{\bar{r}}(\bar{x}) \mid G\omega(x) < \mu_\omega\}.
\end{aligned} \quad (6.1)$$

On $\{x \in B_{\bar{r}}(\bar{x}) \mid G\omega(x) = \mu_\omega\}$, by the property of Sobolev functions we have $\omega = -\Delta G\omega = 0$ a.e.

Note that $\mu_\omega$ is always positive. In fact, if $\mu_\omega \leq 0$, then $G\omega \geq 0 \geq \mu_\omega$ in $D$ by maximum principle, hence the above discussion yields $\omega > 0$ a.e. in $B_{\bar{r}}(\bar{x})$, which contradicts the following fact

$$\mathcal{L}(\{x \in B_{\bar{r}}(\bar{x}) \mid \omega(x) > 0\}) = \mathcal{L}(\{x \in \mathbb{R}^2 \mid \Pi_\varepsilon(x) > 0\}) \leq \pi \varepsilon^2 < \pi \bar{r}^2 \leq \mathcal{L}(B_{\bar{r}}(\bar{x})).$$

For the convenience of later use, we define the vortex core related to $\omega \in \mathcal{M}_\varepsilon$ as follows

$$A_\omega = \{x \in B_{\bar{r}}(\bar{x}) \mid G\omega(x) > \mu_\omega\}.$$
It is clear that

$$L(A_\omega) \leq \pi \varepsilon^2, \quad \forall \omega \in M_\varepsilon. \quad (6.2)$$

**Remark 6.2.** It is not known whether $\text{dist}(A_\omega, \partial B_\varepsilon(\bar{x})) > 0$, so we are not sure whether $\omega$ is a steady weak solution to the vorticity equation. However, we will prove in the sequel that this is true if $\varepsilon$ is small enough.

### 6.2. Size of the vortex core.

**Proposition 6.3.** There exists some $R_0 > 0$, not depending on $\varepsilon$, such that for any $\varepsilon \in (0, \bar{r})$, it holds that

$$\text{diam}(\text{supp}(\omega)) \leq R_0 \varepsilon, \quad \forall \omega \in M_\varepsilon.$$  \hspace{1cm} (6.3)

Before giving the proof of Proposition 6.3, we present some necessary asymptotic estimates for the maximizers as $\varepsilon \to 0$.

Recalling (4.2), we have

$$\kappa_\varepsilon = \int_D w(x) dx, \quad \forall w \in K_\varepsilon.$$  \hspace{1cm} (6.4)

Also recall that $\kappa_\varepsilon \to \kappa$ as $\varepsilon \to 0$.

Below we use $C$ to denote various positive numbers that do not depend on $\varepsilon$, but possibly depend on $D, \bar{x}, \kappa, M$ and $\Pi$.  

**Lemma 6.4 (Lower bound of the kinetic energy).** For any $\varepsilon \in (0, \bar{r})$ and $\omega \in M_\varepsilon$, it holds that

$$E(\omega) \geq -\frac{\kappa_\varepsilon^2}{4\pi} \ln \varepsilon - C \kappa_\varepsilon^2. \quad (6.3)$$

**Proof.** Let $\omega \in M_\varepsilon$ be fixed. Define $w(x) = \Pi_\varepsilon(x - \bar{x})$. Then it is clear that $w \in K_\varepsilon$, thus $E(\omega) \geq E(w)$. We can estimate the lower bound of $E(w)$ as follows

$$E(w) = \frac{1}{2} \int_{B_\varepsilon(x)} \int_{B_\varepsilon(y)} G(x,y)w(x)w(y)dxdy$$

$$= -\frac{1}{4\pi} \int_{B_\varepsilon(x)} \int_{B_\varepsilon(y)} \ln |x - y|w(x)w(y)dxdy - \frac{1}{2} \int_{B_\varepsilon(x)} \int_{B_\varepsilon(y)} h(x,y)w(x)w(y)dxdy$$

$$= -\frac{1}{4\pi} \int_{B_\varepsilon(0)} \int_{B_\varepsilon(0)} \ln |x - y|\Pi_\varepsilon(x)\Pi_\varepsilon(y)dxdy - \frac{1}{2} \int_{B_\varepsilon(x)} \int_{B_\varepsilon(x)} h(x,y)w(x)w(y)dxdy$$

$$\geq -\frac{1}{4\pi} \ln(2\varepsilon)\kappa_\varepsilon^2 - \frac{1}{2} \|h\|_{L^\infty(B_\varepsilon(x) \times B_\varepsilon(x))}\kappa_\varepsilon^2.$$  

Notice that $\|h\|_{L^\infty(B_\varepsilon(x) \times B_\varepsilon(x))}$ is a positive number depending only on $D$ and $\bar{x}$. Therefore (6.3) is proved. \hfill \Box

For any $\omega \in M_\varepsilon$, define

$$T_\omega = \frac{1}{2} \int_D \omega(\mathcal{G}\omega - \mu_\omega) dx, \quad (6.4)$$

which represents the kinetic energy on the vortex core related to the flow with vorticity $\omega$. 
Lemma 6.5 (Uniform upper bound of $T_\omega$). For any $\omega \in \mathcal{M}_\varepsilon$, it holds that

$$T_\omega \leq C.$$  

Proof. Let $\omega \in \mathcal{M}_\varepsilon$ be fixed. For convenience denote $\zeta = G\omega - \mu_\omega$. Since $\mu_\omega > 0$ and $G\omega = 0$ on $\partial D$, we see that $\zeta_+ \in H^1_0(D)$. Moreover, by the fact that $\{x \in D \mid \omega(x) > 0\} = \{x \in D \mid G\omega \geq \mu_\omega\}$, we get

$$\omega \zeta = \omega \zeta_+ \text{ a.e. in } D, \quad \omega = -\Delta \zeta_+ \text{ a.e. in } D.$$  

Therefore we can apply the formula of integration by parts to obtain

$$T_\omega = \frac{1}{2} \int_D \omega \zeta dx = \frac{1}{2} \int_D \omega \zeta_+ dx = \frac{1}{2} \int_D |\nabla \zeta_+|^2 dx. \quad (6.5)$$

On the other hand,

$$T_\omega = \frac{1}{2} \int_{A_\omega} \omega \zeta_+ dx \leq \frac{1}{2} M\varepsilon^{-2} \int_{A_\omega} \zeta_+ dx \leq \frac{1}{2} M\varepsilon^{-2} L(A_\omega)^{1/2} \|\zeta_+\|_{L^2(B_r(\bar{x}))}$$

$$\leq \frac{1}{2} M\pi^{1/2}\varepsilon^{-1} \|\zeta_+\|_{L^2(B_r(\bar{x}))}. \quad (6.6)$$

Note that in the first inequality we used (6.1), and in the third inequality we used (6.2). Using Hölder’s inequality and the Sobolev embedding $W^{1,1}(B_r(\bar{x})) \hookrightarrow L^2(B_r(\bar{x}))$, we get

$$\|\zeta_+\|_{L^2(B_r(\bar{x}))} \leq C \left( \|\zeta_+\|_{L^1(B_r(\bar{x}))} + \|\nabla \zeta_+\|_{L^1(B_r(\bar{x}))} \right)$$

$$= C_1 \left( \|\zeta_+\|_{L^1(A_\omega)} + \|\nabla \zeta_+\|_{L^1(A_\omega)} \right)$$

$$\leq C_1 L(A_\omega)^{1/2} \left( \|\zeta_+\|_{L^2(B_r(\bar{x}))} + \|\nabla \zeta_+\|_{L^2(B_r(\bar{x}))} \right)$$

$$\leq C_1 \pi^{1/2}\varepsilon \left( \|\zeta_+\|_{L^2(B_r(\bar{x}))} + \|\nabla \zeta_+\|_{L^2(B_r(\bar{x}))} \right), \quad (6.7)$$

$$= C_1 \left( \|\zeta_+\|_{L^1(A_\omega)} + \|\nabla \zeta_+\|_{L^1(A_\omega)} \right)$$

$$\leq C_1 \pi^{1/2}\varepsilon \left( \|\zeta_+\|_{L^2(B_r(\bar{x}))} + \|\nabla \zeta_+\|_{L^2(B_r(\bar{x}))} \right), \quad (6.8)$$

$$\leq C_1 \pi^{1/2}\varepsilon \left( \|\zeta_+\|_{L^2(B_r(\bar{x}))} + \|\nabla \zeta_+\|_{L^2(B_r(\bar{x}))} \right), \quad (6.9)$$

where $C_1$ depends only on $\bar{r}$, thus depends only on $D$ and $\bar{x}$. If $C_1 \pi^{1/2}\varepsilon > 1/2$, then it is easy to verify that $T_\omega$ has a uniform upper bound depending only on $D, \bar{x}$ and $M$. Thus without loss of generality we assume that $C_1 \pi^{1/2}\varepsilon \leq 1/2$, then

$$\|\zeta_+\|_{L^2(B_r(\bar{x}))} \leq C_2 \varepsilon \|\nabla \zeta_+\|_{L^2(B_r(\bar{x}))} \quad (6.10)$$

for some $C_2$ depending only on $D$ and $\bar{x}$. Now (6.6) and (6.11) together yield

$$T_\omega \leq \frac{1}{2} C_2 M\pi^{1/2} \|\nabla \zeta_+\|_{L^2(B_r(\bar{x}))} \leq \frac{1}{2} C_2 M\pi^{1/2} \|\nabla \zeta_+\|_{L^2(D)}. \quad (6.12)$$

Combining (6.5) and (6.11) we get the desired result.

Lemma 6.6 (Lower bound of the Lagrangian multiplier). For any $\omega \in \mathcal{M}_\varepsilon$, it holds that

$$\mu_\omega \geq -\frac{\kappa_\varepsilon}{2\pi} \ln \varepsilon - C \left( \kappa_\varepsilon + \frac{1}{\kappa_\varepsilon} \right).$$
Proof. By the definition of $T_\omega$ (see (6.4)) we have the relation

$$T_\omega = E(\omega) - \frac{1}{2} \mu_\omega \kappa_\varepsilon.$$

Then the desired result follows from Lemma 6.4 and Lemma 6.5.

Now we are ready to prove Proposition 6.3.

Proof of Proposition 6.3. Fix $\omega \in M_\varepsilon$. For any $x \in \text{supp}(\omega)$, it holds that $G\omega(x) \geq \mu_\omega$. In view of Lemma 6.6, we get

$$G\omega(x) \geq -\frac{\kappa_\varepsilon}{2\pi} \ln \varepsilon - C \left( \kappa_\varepsilon + \frac{1}{\kappa_\varepsilon} \right),$$

or equivalently

$$-\frac{1}{2\pi} \int_{B_r(\bar{x})} \ln |x-y| \omega(y)dy - \int_{B_r(\bar{x})} h(x,y)\omega(y)dy \geq -\frac{\kappa_\varepsilon}{2\pi} \ln \varepsilon - C \left( \kappa_\varepsilon + \frac{1}{\kappa_\varepsilon} \right). \quad (6.13)$$

Since $h$ is bounded from below on $B_r(\bar{x}) \times B_r(\bar{x})$, we get from (6.13) that

$$-\frac{1}{2\pi} \int_{B_r(\bar{x})} \ln |x-y| \omega(y)dy \geq -\frac{\kappa_\varepsilon}{2\pi} \ln \varepsilon - C \left( \kappa_\varepsilon + \frac{1}{\kappa_\varepsilon} \right). \quad (6.14)$$

Notice that (6.14) can be written as

$$\int_{B_r(\bar{x})} \ln \frac{\varepsilon}{|x-y|} \omega(y)dy \geq -C \left( \kappa_\varepsilon + \frac{1}{\kappa_\varepsilon} \right). \quad (6.15)$$

Let $R > 1$ be a positive number to be determined. Divide the integral on the left side of (6.15) into two parts

$$I = \int_{B_r(\bar{x})} \ln \frac{\varepsilon}{|x-y|} \omega(y)dy = I + II,$$

where

$$I = \int_{D \cap B_R(\bar{x})} \ln \frac{\varepsilon}{|x-y|} \omega(y)dy, \quad II = \int_{D \setminus B_R(\bar{x})} \ln \frac{\varepsilon}{|x-y|} \omega(y)dy. \quad (6.17)$$

For $I$, using Lemma 2.6 we have

$$I \leq \int_{B_\varepsilon(0)} \ln \frac{\varepsilon}{|y|} \Pi_\varepsilon(y)dy \leq M \varepsilon^{-2} \int_{B_\varepsilon(0)} \ln \frac{\varepsilon}{|y|}dy \leq C. \quad (6.18)$$

Here we used (4.1). For $II$, we have

$$II = \int_{D \setminus B_R(\bar{x})} \ln \frac{\varepsilon}{|x-y|} \omega(y)dy \leq \ln \frac{1}{R} \int_{D \setminus B_R(\bar{x})} \omega(y)dy. \quad (6.19)$$

By (6.15), (6.18) and (6.19), we get

$$\int_{D \setminus B_R(\bar{x})} \omega(y)dy \leq \frac{C \left( \kappa_\varepsilon + \frac{1}{\kappa_\varepsilon} + 1 \right) \ln R}{\ln R} \quad (6.20)$$
In view of the fact
\[ \lim_{\varepsilon \to 0} \kappa_{\varepsilon} = \kappa, \]
we get from (6.20) that
\[ \int_{D \setminus B_{R_{\varepsilon}}(x)} \omega(y) dy \leq \frac{1}{3} \kappa_{\varepsilon} \]  
provided that \( R \) is sufficiently large, not depending on \( \varepsilon \). In view of
\[ \int_{D} \omega(y) dy = \kappa_{\varepsilon}, \]  
we get from (6.21) that
\[ \int_{B_{R_{\varepsilon}}(x)} \omega(y) dy \geq \frac{2}{3} \kappa_{\varepsilon}. \]  

Since \( x \in \text{supp}(\omega) \) is arbitrary, we must have
\[ \text{diam}(\text{supp}(\omega)) \leq 2R_{\varepsilon}. \]  
In fact, if (6.24) is false, then we can choose two points \( x_1, x_2 \in \text{supp}(\omega) \) such that
\[ |x_1 - x_2| > 2R_{\varepsilon}, \]  
a contradiction to (6.22). Choosing \( R_0 = 2R \) we complete the proof.

\[ \square \]

6.3. Limiting location of the vortex core.

Proposition 6.7. For any \( \delta > 0 \), there exists some \( \varepsilon_0 \in (0, \bar{r}) \), depending only on \( D, \bar{x}, M, \kappa, \Pi \) and \( \delta \), such that for any \( \varepsilon \in (0, \varepsilon_0) \), it holds that
\[ \text{supp}(\omega) \subset B_{\delta}(\bar{x}), \quad \forall \omega \in M_\varepsilon. \]

Proof. We prove this proposition by contradiction. Suppose there exist \( \delta_0 > 0 \), such that for any positive integer \( n \), there exist \( \varepsilon_n \in (0, 1/n) \), \( \omega_n \in M_{\varepsilon_n} \), and \( x_n \in \text{supp}(\omega_n) \) such that
\[ |x_n - \bar{x}| > \delta_0. \]  
Since \( \text{supp}(\omega_n) \subset B_{\bar{\varepsilon}}(\bar{x}) \), we have \( x_n \in B_{\bar{\varepsilon}}(\bar{x}) \). Up to a subsequence, we can assume that \( x_n \to \hat{x} \) for some \( \hat{x} \in B_{\bar{\varepsilon}}(\bar{x}) \) as \( n \to +\infty \). It is clear from (6.26) that
\[ |\hat{x} - \bar{x}| \geq \delta_0. \]  

Define \( w_n = \Pi_{\varepsilon_n}(\cdot - \bar{x}) \), then \( w_n \in K_{\varepsilon_n} \), and thus \( E(w_n) \leq E(\omega_n) \). This means
\[ - \frac{1}{2\pi} \int_{B_{\bar{\varepsilon}}(\hat{x})} \int_{B_{\bar{\varepsilon}}(\hat{x})} \ln |x - y| w_n(x) w_n(y) dx dy - \int_{B_{\bar{\varepsilon}}(\hat{x})} \int_{B_{\bar{\varepsilon}}(\hat{x})} h(x, y) w_n(x) w_n(y) dx dy \]
\[ \leq - \frac{1}{2\pi} \int_{B_{\bar{\varepsilon}}(\hat{x})} \int_{B_{\bar{\varepsilon}}(\hat{x})} \ln |x - y| \omega_n(x) \omega_n(y) dx dy - \int_{B_{\bar{\varepsilon}}(\hat{x})} \int_{B_{\bar{\varepsilon}}(\hat{x})} h(x, y) \omega_n(x) \omega_n(y) dx dy. \]
Observing that \( w_n \) is radially symmetric and nonincreasing with respect to \( \bar{x} \), we can apply Lemma 2.7 to obtain

\[
-\frac{1}{2\pi} \int_{B_r(\bar{x})} \int_{B_r(\bar{x})} \ln |x-y| w_n(x) w_n(y) dxdy \geq -\frac{1}{2\pi} \int_{B_r(\bar{x})} \int_{B_r(\bar{x})} \ln |x-y| \omega_n(x) \omega_n(y) dxdy.
\]

Therefore we get

\[
\int_{B_r(\bar{x})} \int_{B_r(\bar{x})} h(x, y) w_n(x) w_n(y) dxdy \geq \int_{B_r(\bar{x})} \int_{B_r(\bar{x})} h(x, y) \omega_n(x) \omega_n(y) dxdy. \tag{6.28}
\]

Obviously

\[
\lim_{n \rightarrow +\infty} \int_{B_r(\bar{x})} \int_{B_r(\bar{x})} h(x, y) w_n(x) w_n(y) dxdy = \kappa^2 h(\hat{x}, \hat{x}). \tag{6.29}
\]

On the other hand, by Proposition 6.3 and the fact that \( \lim_{n \rightarrow +\infty} x_n = \hat{x} \), we see that the support of \( \omega_n \) “shrinks” to \( \hat{x} \) as \( n \rightarrow +\infty \). Therefore

\[
\lim_{n \rightarrow +\infty} \int_{B_r(\bar{x})} \int_{B_r(\bar{x})} h(x, y) \omega_n(x) \omega_n(y) dxdy = \kappa^2 h(\hat{x}, \hat{x}). \tag{6.30}
\]

Combining (6.28), (6.29) and (6.30) we obtain

\[
H(\bar{x}) \geq H(\hat{x}).
\]

Since \( \bar{x} \) is the unique minimum point of \( H \) in \( \overline{B_r(\bar{x})} \), we deduce that \( \bar{x} = \hat{x} \). This obviously contradicts (6.27). Hence the proof is finished.

\( \square \)

6.4. **Proof of Theorem 1.5.** Now we can complete the proof of Theorem 1.5

**Proof of Theorem 1.5.** It suffices to show that \( \omega \in M_\varepsilon \) is a steady weak solution to the vorticity equation if \( \varepsilon \) is sufficiently small. By Proposition 6.7, we see that

\[
\text{dist}(\text{supp}(\omega, \partial B_r(\bar{x}))) > 0, \quad \forall \omega \in M_\varepsilon
\]

when \( \varepsilon \) is small. The desired result follows from Lemma A in Section 1.

\( \square \)

7. **Concentrated vortex flows with prescribed profile functions**

In this section, we discuss the stability of concentrated vortex flows with prescribed profile functions. To begin with, we state the following existence result proved in [10].

Let \( k \) be a positive integer, \( \kappa_1, \ldots, \kappa_k \) be \( k \) nonzero real numbers. Let \( (\bar{x}_1, \ldots, \bar{x}_k) \) be an isolated local minimum point of the Kirchhoff-Routh function \( W \) related to \( \vec{\kappa} \). Choose \( \bar{r} > 0 \) sufficiently small such that

(i) \( \overline{B_r(\bar{x}_i)} \subset D \) for any \( 1 \leq i \leq k \), where \( \overline{B_r(\bar{x}_i)} \) is the closure of \( B_r(\bar{x}_i) \) in the Euclidean topology;

(ii) \( \overline{B_r(\bar{x}_i)} \cap \overline{B_r(\bar{x}_j)} = \emptyset \) for any \( 1 \leq i < j \leq k \);

(iii) \( (\bar{x}_1, \ldots, \bar{x}_k) \) is the unique minimum point of \( W \) in \( \overline{B_r(\bar{x}_1)} \times \cdots \times \overline{B_r(\bar{x}_k)} \).
Let $\varepsilon, \Lambda$ be two positive parameters. Define
\[
A_{\varepsilon, \Lambda} = \left\{ w = \sum_{i=1}^{k} w_i \middle| w_i \in L^\infty(D), \text{supp}(w_i) \subset B_{\varepsilon}(\bar{x}_i), 0 \leq \text{sgn}(\kappa_i)w_i \leq \frac{\Lambda}{\varepsilon^2}, \int_D w_i dx = \kappa_i \right\}.
\]
It is easy to see that $A_{\varepsilon, \Lambda}$ is nonempty if
\[
\Lambda > \max\left\{ 1, \frac{\varepsilon^2|\kappa_1|}{\pi r^2}, \ldots, \frac{\varepsilon^2|\kappa_k|}{\pi r^2} \right\}. \tag{7.1}
\]
Below we always assume that (7.1) holds.

Let $\mu_0, \tau_0$ be fixed positive numbers. We will need the following conditions on the profile function $f$:

(C1) $f \in C(\mathbb{R})$, $f(s) = 0$ if $s \leq 0$, and $f$ is strictly increasing on $[0, +\infty)$;
(C2) there exists $\mu_0 \in (0, 1)$ such that
\[
\int_0^s f(r) dr \leq \mu_0 f(s) s, \quad \forall s \geq 0;
\]
(C3) for all $\tau_0 > 0$, it holds that
\[
\lim_{s \to +\infty} f(s) e^{-\tau_0 s} = 0.
\]

For $f$ satisfying (C1)(C2)(C3), we define a new function $H_f$ as follows
\[
H_f(s) = \int_0^s h(r) dr, \tag{7.2}
\]
where
\[
h(s) = \begin{cases} f^{-1}(s), & \text{if } s > 0; \\ 0, & \text{if } s \leq 0. \end{cases} \tag{7.3}
\]

Let $f_1, \ldots, f_k$ be $k$ real functions satisfying (C1)(C2)(C3). Define
\[
\mathcal{E}(\omega) = \frac{1}{2} \int_D \omega(x) G \omega(x) dx - \frac{1}{\varepsilon^2} \sum_{i=1}^{k} \int_D H_{f_i}(\varepsilon^2 \text{sgn}(\kappa_i) \omega) 1_{B_{\varepsilon}(\bar{x}_i)} dx.
\]
Denote $\mathcal{N}_{\varepsilon, \Lambda}$ the set of maximizers of $\mathcal{E}$ over $A_{\varepsilon, \Lambda}$.

The following theorem has been proved in [10].

**Theorem 7.1** ([10]). There exists $\varepsilon_0, \Lambda_0 > 0$, depending only on $D, \bar{x}_1, \ldots, \bar{x}_k, \kappa, \mu_0, \tau_0$, such that for any $0 < \varepsilon < \varepsilon_0$ and $\Lambda > \Lambda_0$, $\mathcal{N}_{\varepsilon, \Lambda}$ is nonempty, and any $\omega \in \mathcal{N}_{\varepsilon, \Lambda}$ satisfies
\[
\omega = \frac{1}{\varepsilon^2} \text{sgn}(\kappa_i) f_i (\text{sgn}(\kappa_i) G \omega - \mu_{\omega, i}) \text{ a.e. in } B_{\varepsilon}(\bar{x}_i), \quad i = 1, \ldots, k,
\]
where each $\mu_{\omega, i}$ is a real number depending on $\omega$. Moreover, for fixed $\Lambda > \Lambda_0$, we have

(i) $\mu_{\omega, i} \geq -\frac{|\kappa_i|}{2\pi} \ln \varepsilon - C$ for any $\omega \in \mathcal{N}_{\varepsilon, \Lambda}$ and $i = 1, \ldots, k$, where $C$ is a positive number depending only on $D, \bar{x}_1, \ldots, \bar{x}_k, \kappa, \mu_0, \tau_0, \Lambda$;

(ii) $\text{diam}(\text{supp}(\omega)) \leq R_0 \varepsilon$ for any $\omega \in \mathcal{N}_{\varepsilon, \Lambda}$, where $R_0$ is a positive number depending only on $D, \bar{x}_1, \ldots, \bar{x}_k, \kappa, \mu_0, \tau_0, \Lambda$;
(iii) for any $\delta > 0$, there exists some $\varepsilon_1 > 0$, depending only on $D, \bar{x}_1, \cdots, \bar{x}_k, \bar{r}, \mu_0, \tau_0, \Lambda, \delta$, such that for any $\varepsilon \in (0, \varepsilon_1)$, it holds that

$$\text{supp}(\omega 1_{B_{\bar{r}}(\bar{x}_i)}) \subset B_\delta(\bar{x}_i), \quad \forall \omega \in \mathcal{N}_{\varepsilon, \Lambda},$$

and consequently by Lemma A any $\omega \in \mathcal{N}_{\varepsilon, \Lambda}$ is a steady weak solution to the vorticity equation.

Unlike the variational problem with prescribed rearrangement of vorticity in Section 1, here we do not know whether the elements in $\mathcal{N}_{\varepsilon, \Lambda}$ have the same rearrangements, and it is also hard to prove the compactness of $\mathcal{N}_{\varepsilon, \Lambda}$. Hence we can not apply Theorem 3.1 directly to obtain stability.

However, the good thing here is that we know what the profile functions are, which means that the stream function satisfies a definite semilinear elliptic equation. This allows us to employ the methods from the field of elliptic equations to obtain fine estimates for the solutions.

From the viewpoint of the stream function, recently Cao-Yan-Yu-Zou [14] proved the following local uniqueness result when each $f_i$ is a power function.

**Theorem 7.2 (Local uniqueness).** Let $q \in (0, +\infty)$ be fixed. In the setting of Theorem 7.1 Suppose additionally

(i) $(\bar{x}_1, \cdots, \bar{x}_k)$ is a nondegenerate critical point of $D$;
(ii) $\kappa_i > 0$ for any $i = 1, \cdots, k$;
(iii) $f_i(s) = s^q$ for any $i = 1, \cdots, k$.

Then for fixed $\Lambda > \Lambda_0$, there exists some $\varepsilon_2 > 0$, depending only on $D, \bar{x}_1, \cdots, \bar{x}_k, \bar{r}, \Lambda$, such that for any $\varepsilon \in (0, \varepsilon_2), \mathcal{N}_{\varepsilon, \Lambda}$ is a singleton $\{\omega_{\varepsilon, \Lambda}\}$.

Based on Theorem 7.2, we can proved the following stability result for $k = 1$.

**Theorem 7.3 (Stability).** In the setting of Theorem 7.2 if $k = 1$ and $\bar{x} \in D$ is a non-degenerate local minimum point of $H$, then for fixed $\Lambda > \Lambda_0$, there exists some $\varepsilon_3 > 0$, depending only on $D, \bar{x}, \kappa, \Lambda$, such that for any $\varepsilon \in (0, \varepsilon_3), \omega_{\varepsilon, \Lambda}$ is stable in the $L^p$ norm of the vorticity with respect to initial perturbations in $L^\infty(D)$ for any $p \in (1, +\infty)$.

**Proof.** By Theorem 3.1 in Section 3 or Theorem 5 in [6], it suffices to show that $\omega_{\varepsilon, \Lambda}$ is an isolated maximizer of $E$ over $\mathcal{R}_{\omega_{\varepsilon, \Lambda}}$. By Theorem 7.2 it is easy to see that $\omega_{\varepsilon, \Lambda}$ is the unique maximizer of $E$ over

$$\left\{ w \in L^\infty(D) \mid w \in \mathcal{R}_{\omega_{\varepsilon, \Lambda}}, \text{supp}(w) \subset \overline{B_{\bar{r}}(\bar{x})} \right\}$$

if $\varepsilon$ is sufficiently small. Then the isolatedness follows from a similar argument as in Section 4. \qed

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References

[1] V. I. Arnol’d, Conditions for nonlinear stability plane curvilinear flow of an idea fluid, *Sov. Math. Dokl.*, 6(1965), 773–777.
[2] V. I. Arnol’d, On an a priori estimate in the theory of hydrodynamical stability, *Amer. Math. Soc. Transl.*, 79(1969), 267–269.
[3] C. Bardos, Y. Guo, W. Strauss, Stable and unstable ideal plane flows, *Chinese Ann. Math. Ser. B*, 23(2002), 149–164.
[4] G. R. Burton, Rearrangements of functions, maximization of convex functionals, and vortex rings, *Math. Ann.*, 276(1987), 225–253.
[5] G. R. Burton, Variational problems on classes of rearrangements and multiple configurations for steady vortices, *Ann. Inst. H. Poincaré. Anal. Non Linéaire.*, 6(1989), 295-319.
[6] G. R. Burton, Global nonlinear stability for steady ideal fluid flow in bounded planar domains, *Arch. Ration. Mech. Anal.*, 176(2005), 149-163.
[7] D. Cao, Z. Liu and J. Wei, Regularization of point vortices for the Euler equation in dimension two, *Arch. Ration. Mech. Anal.*, 212(2014), 179–217.
[8] D. Cao, S. Peng and S. Yan, Planar vortex patch problem in incompressible steady flow, *Adv. Math.*, 270(2015), 263–301.
[9] D. Cao, S. Peng and S. Yan, Regularization of planar vortices for the incompressible flow, *Acta Math. Sci. Ser. B(Engl. Ed.)*, 38(2018), 1443–1467.
[10] D. Cao, G. Wang and W. Zhan, Desingularization of vortices for two-dimensional steady Euler flows via the vorticity method, *SIAM J. Math. Anal.*, 52(2020), 5363–5388.
[11] D. Cao and G. Wang, Steady vortex patches with opposite rotation directions in a planar ideal fluid, *Calc. Var. Partial Differential Equations*, 58 (2019), 58–75.
[12] D. Cao and G. Wang, Nonlinear stability of planar vortex patches in an ideal fluid, *J. Math. Fluid Mech.*, 58(2021), https://doi.org/10.1007/s00021-021-00588-w.
[13] D. Cao and G. Wang, A note on steady vortex flows in two dimensions, *Proc. Amer. Math. Soc.*, 148(2020), 1153–1159.
[14] D. Cao, S. Yan, W. Yu and C. Zou, Local uniqueness of vortices for 2D steady Euler flow, arXiv:2004.13512v3.
[15] J. M. Delort, Existence de nappes de tourbillon en dimension deux, *J. Amer. Math. Soc.*, 4(1991), 553–586.
[16] R. DiPerna and A. Majda, Concentrations in regularizations for 2D incompressible flow, *Comm. Pure Appl. Math.*, 40(1987), 301–345.
[17] A. R. Elcrat and K. G. Miller, Rearrangements in steady vortex flows with circulation, *Proc. Amer. Math. Soc.*, 111(1991), 1051-1055.
[18] A. R. Elcrat and K. G. Miller, Rearrangements in steady multiple vortex flows, *Comm. Partial Differential Equations*, 20(1994), no.9-10, 1481–1490.
[19] E. Hölder, Über unbeschränkte Fortsetzbarkeit einer stetigen ebenen Bewegung in einer unbegrenzten inkompressiblen Flüssigkeit (German), *Math. Z.*, 37(1933), 727–738.
[20] Thomson, Sir W.(Lord Kelvin), Maximum and minimum energy in vortex motion, *Mathematical and Physical Papers.*, 4(1910), 172–183.
[21] E. H. Lieb and M. Loss, Analysis, Second edition, *Graduate Studies in Mathematics*, Vol. 14. American Mathematical Society, Providence, RI (2001).
[22] C. C. Lin, On the motion of vortices in two dimension – I. Existence of the Kirchhoff-Routh function, *Proc. Natl. Acad. Sci. USA*, 27(1941), 570–575.
[23] Z. Lin, Some stability and instability criteria for ideal plane flows, *Comm. Math. Phys.* 246(2004), 87–112.
[24] A. E. H. Love, On the stability of certain vortex motions, *Proc. Roy. Soc. London*, (1893), 18–42.
[25] A. J. Majda and A. L. Bertozzi, Vorticity and incompressible flow, *Cambridge Texts in Applied Mathematics*, Vol. 27. Cambridge University Press, 2002.

[26] C. Marchioro, On the localization of the vortices, *Bollettino U.M.I.*, Serie 8, Vol. 1-B(1998), 571–584.

[27] C. Marchioro and M. Pulvirenti, Euler evolution for singular data and vortex theory, *Comm. Math. Phys.*, 91(1983), 563–572.

[28] C. Marchioro, Euler evolution for singular initial data and vortex theory: a global solution, *Comm. Math. Phys.*, 116(1988), 45–55.

[29] C. Marchioro and M. Pulvirenti, Vortices and localization in Euler flows, *Comm. Math. Phys.*, 154(1993), 49–61.

[30] C. Marchioro and M. Pulvirenti, Mathematical theory of incompressible noviscous fluids, Springer-Verlag, 1994.

[31] C. Marchioro and E. Pagani, Evolution of two concentrated vortices in a two-dimensional bounded domain, *Math. Meth. Appl. Sci.*, 8(1986), 328–344.

[32] D. Smets and J. Van Schaftingen, Desingulariation of vortices for the Euler equation, *Arch. Ration. Mech. Anal.*, 198(2010), 869–925.

[33] Y. Tang, Nonlinear stability of vortex patches, *Trans. Amer. Math. Soc.*, 304(1987), 617–637.

[34] B. Turkington, On steady vortex flow in two dimensions. I, II, *Comm. Partial Differential Equations*, 8(1983), 999–1030, 1031–1071.

[35] B. Turkington, On the evolution of concentrated vortex in an idea fluid, *Arch. Ration. Mech. Anal.*, 97(1987), 75–87.

[36] Y.-H. Wan and M. Pulvirenti, Nonlinear stability of circular vortex patches, *Comm. Math. Phys.*, 99(1985), 435–450.

[37] G. Wang, Orbital stability of 2D steady Euler flows related to least energy solutions of the Lane-Emden equation, arXiv:2104.12406.

[38] G. Wang, Nonlinear stability of planar steady Euler flows associated with semistable solutions of elliptic problems, arXiv: 2107.00217.

[39] G. Wang and B. Zuo, Location of concentrated vortices in planar steady Euler flows, arXiv:2101.06620.

[40] G. Wolansky, M. Ghil, An extension of Arnol’d’s second stability theorem for the Euler equations, *Phys. D*, 94(1996), 161–167.

[41] G. Wolansky, M. Ghil, Nonlinear stability for saddle solutions of ideal flows and symmetry breaking. *Comm. Math. Phys.*, 193(1998), 713–736.

[42] W. Wölbner, Un théorème sur l’existence du mouvement plan dun fluide parfait, homogène, incompressible, pendant un temps infiniment long (French), *Math. Z.*, 37(1933), 698–726.

[43] V. I. Yudovich, Non-stationary flow of an ideal incompressible fluid, *USSR Comp. Math. & Math.Phys.*, 3(1963), 1407–1456 [English].