EXACT BOUNDARY OBSERVABILITY AND CONTROLLABILITY OF THE WAVE EQUATION IN AN INTERVAL WITH TWO MOVING ENDPOINTS

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Abstract. We study the wave equation in an interval with two linearly moving endpoints. We give the exact solution by a series formula, then we show that the energy of the solution decays at the rate $1/t$. We also establish observability results, at one or at both endpoints, in a sharp time. Moreover, using the Hilbert uniqueness method, we derive exact boundary controllability results.

1. Introduction and main results.

1.1. Introduction. The wave equation is a simple mathematical model describing the transverse small vibrations of a homogeneous string under tension and constrained to move in a plane. Let $T > 0$. When the string endpoints are clamped and its length $L_0 > 0$ is invariant with time, this model can be stated as

$$
\begin{align*}
& w_{tt} - w_{xx} = 0, & \text{for } (x,t) \in (0,L_0) \times (0,T), \\
& w(0,t) = w(L_0,t) = 0, & \text{for } t \in (0,T), \\
& w(x,0) = w^0, \quad w_t(x,0) = w^1, & \text{for } x \in (0,L_0).
\end{align*}
$$

The function $x \to w(x,t), x \in (0,L_0)$, describes the shape of the string at time $t$. It is well known that for initial data satisfying $w^0 \in H_0^1(0,L_0)$ and $w^1 \in L^2(0,L_0)$, the above problem has a unique solution satisfying

$$
\phi \in C \left( [0,T]; H_0^1(0,L_0) \right) \quad \text{and} \quad \phi_t \in C \left( [0,T]; L^2(0,L_0) \right).
$$

In particular, this solution enjoys the following properties (see [11]):

- The “energy” of the solution, defined as

$$
E(t) = \frac{1}{2} \int_0^{L_0} w_x^2(x,t) + w_t^2(x,t) \, dx, \quad t \in (0,T),
$$

is a conserved quantity in time, i.e. $E(t) = \text{Constant}, \forall t \geq 0$.

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Due to the finite speed of propagation, taken here equal to 1, the boundary observability at one endpoint (resp. at the two endpoints) holds iff the length of the time interval satisfies $T \geq 2L_0$ (resp. $T \geq L_0$).

Using a control function acting at one of the endpoints, the wave equation is controllable iff $T \geq L_0$. If we use two controls, i.e. a control function at each endpoint, the controllability holds iff $T \geq L_0$.

In the paper at hand, we wonder if the solution of the wave equation has some analogue properties when the length of the string varies in time. Such situations, where the spacial domain is time-dependent, appear in many different areas of physics, from optics, electromagnetism, fluid dynamics to quantum mechanics. See for instance [6, 13, 15, 21] and the survey paper [7].

1.2. Assumptions and main results. To be more precise, we consider the wave equation in an interval with two linearly moving endpoints. We suppose that the left endpoint moves to the left with a constant speed $v_1$ and the other endpoint moves to the right at a constant speed $v_2$. To simplify the computations, we take as an initial time

$$t_0 = \frac{L_0}{v_1 + v_2}$$

where $L_0$ denotes now the initial length of the string. Then, we consider the following interval with moving ends

$$I_t := (-v_1 t, v_2 t), \quad t \geq t_0.$$

In the $xt$–plan, we have a noncylindrical domain $Q_{t_0+T}$, and its lateral boundary $\Sigma_{t_0+T}$, defined as

$$Q_{t_0+T} := \{ (x, t) \in \mathbb{R}^2 \mid x \in I_t, \text{ for } t \in (t_0, t_0 + T) \},$$

$$\Sigma_{t_0+T} := \bigcup_{t_0 < t < t_0 + T} \{ (-v_1 t, t), (v_2 t, t) \}.$$

We assume that the speeds $v_1$ and $v_2$ satisfy

$$v_1 + v_2 > 0 \quad \text{and} \quad 0 \leq v_1, v_2 < 1. \quad (3)$$

The first one, i.e. $v_1 + v_2 > 0$, ensures that at least one of the endpoints is moving and thus the length of the string is increasing with time. The assumption $0 \leq v_1, v_2 < 1$ in (3) ensures that $\Sigma_{t_0+T}$ satisfies the so-called timelike condition. This later one means that the speed of each moving endpoint is inferior to the speed of propagation of the wave (here equal to 1).

Let us now consider the wave equation, with homogeneous Dirichlet boundary conditions,

$$\begin{cases}
\phi_{tt} - \phi_{xx} = 0, & \text{in } Q_{t_0+T}, \\
\phi (-v_1 t, t) = \phi (v_2 t, t) = 0, & \text{for } t \in (t_0, t_0 + T), \\
\phi (x, t_0) = \phi^0 (x), & \phi_t (x, t_0) = \phi^1 (x), & \text{for } x \in I_{t_0}.
\end{cases} \quad (WP)$$

Under the assumption (3) and for every initial data

$$\phi^0 \in H_0^1 (I_{t_0}), \quad \phi^1 \in L^2 (I_{t_0}) \quad (4)$$

there exists a unique solution to Problem (WP) such that

$$\phi \in C ([t_0, t_0 + T]; H_0^1 (I)), \quad \phi_t \in C ([t_0, t_0 + T]; L^2 (I)). \quad (5)$$

This can be shown, for instance, by a penalisation method [10, p. 413], see also [2, 4].
Moreover, the exact solution of \((WP)\) can be given by a series formula. To do so, we need to introduce the notation of some constants that will frequently appear in the sequel. We set

\[
v_{\min} := \min \{v_1, v_2\}, \quad v_{\max} := \max \{v_1, v_2\},
\]

\[
\alpha_v := \frac{1 + v_1}{1 - v_2}, \quad \beta_v := \frac{1 + v_2}{1 - v_1}, \quad \kappa_v := \frac{2}{\log (\alpha_v \beta_v)},
\]

\[
V_1 := (1 - v_2) \alpha_v \beta_v - 1, \quad V_2 := (1 - v_1) \alpha_v \beta_v - 1.
\]

Here and in the sequel, the subscript \(v\) is used to emphasize the dependence on the speeds \(v_1\) and \(v_2\). Taking into account Assumption (3), we can easily check that

\[
\alpha_v > 1, \quad \beta_v > 1, \quad 0 < \kappa_v < +\infty, \quad -V_1 < - (v_1 + v_2) \leq -v_1 < v_2 \leq v_1 + v_2 < V_2.
\]

Then, the exact solution of \((WP)\) is given by the formula

\[
\phi(x, t) = \sum_{n \in \mathbb{Z}^*} c_n \left( e^{i \pi \kappa_v \log(t+x)} - e^{i \pi \kappa_v \log\left(\frac{1+v_1}{1+v_2}\right)} e^{i \pi \kappa_v \log(t-x)} \right)
\]

where \(x \in (-v_1 t, v_2 t), t \geq t_0\) and the coefficients \(c_n\) are complex numbers independent of \(t\).

This series formula was obtained by Balazs [1]. He managed to calculate the coefficients \(c_n\), in function of the initial data \(\phi^0\) and \(\phi^1\), when only one endpoint is moving, e.g. \(v_1 = 0\). If the two endpoints are moving, i.e. \(v_1 > 0\) and \(v_2 > 0\), the question remained open as far as we know. The determination of \(c_n, n \in \mathbb{Z}^*\) turns out to be a bit tricky, as we shall see in the proof of our first main result.

**Theorem 1.1.** Under the assumptions (3) and (4), the solution of Problem \((WP)\) is given by the series (9). The coefficients \(c_n\) are given by any of the two following formula

\[
c_n = \frac{1}{4 \pi n i} \int_{-V_1 t_0}^{V_2 t_0} \left( \tilde{\phi}_x^0 + \tilde{\phi}_x^1 \right) e^{-i \pi \kappa_v \log(t_0 + x)} dx,
\]

\[
e^{i \pi \kappa_v \log\left(\frac{1+v_1}{1+v_2}\right)} = \frac{1}{4 \pi n i} \int_{-V_1 t_0}^{V_2 t_0} \left( \tilde{\phi}^0_x - \tilde{\phi}^1_x \right) e^{-i \pi \kappa_v \log(t_0 - x)} dx, \quad \text{for } n \in \mathbb{Z}^*,
\]

where \(\tilde{\phi}_x^0\) and \(\tilde{\phi}_x^1\) are extensions of the initial data \(\phi^0\) and \(\phi^1\) on the interval \((-V_1 t_0, V_2 t_0)\) given below by (31) and (32) respectively.

In section 3, we use the formula (9) and the explicit formula of the coefficients \(c_n\) to study the asymptotic behaviour of the energy of the solution of Problem \((WP)\), defined as

\[
E_v(t) = \frac{1}{2} \int_{-v_1 t}^{v_2 t} \phi_x^2(x, t) + \phi_t^2(x, t) \, dx, \quad \text{for } t \geq t_0.
\]

In contrast with the conservation of the energy \(E(t)\), in the case of an interval with a fixed length, the quantity \(E_v(t)\) is decaying in time. The next theorem gives the precise decay rate.
Theorem 1.2. Under the assumptions (3) and (4), the solution of Problem (WP) satisfies
\[ tE(t) + \int_{-v_1t}^{v_2t} x\phi_x \phi_t \, dx = S_v, \quad \text{for } t \geq t_0, \quad (13) \]
where \( S_v := 2\pi^2 \kappa_v \sum_{n \in \mathbb{Z}} |nc_n|^2 \) is finite and independent of \( t \). Moreover, it holds that
\[ \frac{S_v}{(1 + v_{\max})t} \leq E_v(t) \leq \frac{S_v}{(1 - v_{\max})t}, \quad \text{for } t \geq t_0. \quad (14) \]

In section 5, we consider in one hand the observability problem for (WP) at the endpoint \( x = v_2t \). This problem can be stated as follows: To give sufficient conditions on the length of the time interval, denoted by \( T_v \), such that there exists a constant \( c_2(T_v) > 0 \) for which the observability inequality
\[ E_v(t_0) \leq c_2(T_v) \int_{t_0}^{t_0 + T_v} \phi_x^2 (v_2t, t) + \phi_t^2 (v_2t, t) \, dt \]
holds for all the solutions of (WP). This inequality is also called the inverse inequality.

Noting that \( \phi(v_2t, t) = 0 \), for \( t \geq t_0 \), implies that
\[ (\phi(v_2t, t))_t = v_2 \phi_x (v_2t, t) + \phi_t (v_2t, t) = 0, \]
then \( \phi_t^2 (v_2t, t) = v_2^2 \phi_x^2 (v_2t, t) \), for \( t \geq t_0 \). Denoting \( C_2(T_v) = c_2(T_v) \left( 1 + v_2^2 \right) \), the precedent inequality can be rewritten as
\[ E_v(t_0) \leq C_2(T_v) \int_{t_0}^{t_0 + T_v} \phi_x^2 (v_2t, t) \, dt. \quad (15) \]

The minimal value of \( T_v \), for which Inequality (15) holds, is called the time of observability. Due to the finite speed of propagation, one expects that \( T_v > 0 \) depends on the initial length \( L_0 \) and also on the two speeds of expansion \( v_1 \) and \( v_2 \). Indeed, the next theorem shows that the sharp time of observability is given by
\[ T_v := \frac{2L_0}{(1 - v_1)(1 - v_2)}. \quad (16) \]

Theorem 1.3. Under the assumptions (3) and (4), we have:
- For every \( T \geq 0 \), the solution of (WP) satisfies the direct inequality
  \[ \int_{t_0}^{t_0 + T} \phi_x^2 (v_2t, t) \, dt \leq K_v(T) E_v(t_0) \quad (17) \]
  with a constant \( K_v(T) \) depending only on \( v_1, v_2 \) and \( T \).
- If \( T \geq T_v \), Problem (WP) is observable at \( x = v_2t \) and it holds that
  \[ E_v(t_0) \leq \frac{\alpha_v \beta_v}{4} \left( 1 - v_2^2 \right)^2 \int_{t_0}^{t_0 + T} \phi_x^2 (v_2t, t) \, dt. \quad (18) \]
Conversely, if \( T < T_v \), (18) does not hold.

On the other hand, we consider the following boundary controllability problem: Given
\[ (u^0, u^1) \in L^2(I_{t_0}) \times H^{-1}(I_{t_0}), \quad (19) \]
\[ (u^0_T, u^1_T) \in L^2(I_{t_0 + T}) \times H^{-1}(I_{t_0 + T}), \quad (20) \]
find a control function \( f \in L^2(t_0, t_0 + T) \), acting at one of the endpoints, say \( x = v_2 t \), such that the solution of the problem
\[
\begin{aligned}
&u_{tt} - u_{xx} = 0, \\
&u(-v_1 t, t) = 0, \quad u(v_2 t, t) = f(t), \\
&u(x, t_0) = u^0(x), \quad u_t(x, t_0) = u^1(x),
\end{aligned}
\]
in \( Q_{t_0 + T} \), satisfies also
\[
u(x, t_0 + T) = u^0_T(x), \quad u_t(x, t_0 + T) = u^1_T(x), \quad \text{for} \ x \in I_{t_0} + T.
\]

Note that Problem \((CW P)\) admits a unique solution
\[
u \in C([t_0, t_0 + T]; L^2(I_t)) \cap C^1([t_0, t_0 + T]; H^{-1}(I_t))
\]
in the transposition sense, see [14].

Once the observability is established at the endpoint \( x = v_2 t \), we use the Hilbert uniqueness method (due to J.-L. Lions [11]) to show that the controllability of \((CW P)\) holds iff \( T \geq T_v \), see Theorem 4.2 in page 16.

In the last section, we establish the observability of the wave equation \((WP)\) by tow observers placed at the two endpoints \( x = -v_1 t \) and \( x = v_2 t \). The sharp time of observability, in this case, is given by
\[
\hat{T}_v := L_0 / (1 - v_{\text{max}}).
\]

More precisely, we have the following result.

**Theorem 1.4.** Under the assumptions \((3)\) and \((4)\), we have:
- For every \( T \geq 0 \), the solution of \((WP)\) satisfies the direct inequality
  \[
  \int_{t_0}^{t_0 + T} \phi_x^2(-v_1 t, t) + \phi_x^2(v_2 t, t)dt \leq \hat{K}_v(T) E_v(t_0)
  \]
  with a constant \( \hat{K}_v(T) \) depending only on \( v_1, v_2 \) and \( T \).
- If \( T \geq \hat{T}_v \), Problem \((WP)\) is observable at the two endpoints \( x = -v_1 t, x = v_2 t \), and it holds that
  \[
  E_v(t_0) \leq \frac{(1 - v_{\text{min}}^2)^2}{4(1 - v_{\text{max}})} \max(\alpha_v, \beta_v) \int_{t_0}^{t_0 + T} \phi_x^2(-v_1 t, t) + \phi_x^2(v_2 t, t)dt.
  \]

Conversely, if \( T < \hat{T}_v \), \((23)\) does not hold.

Finally, we consider the controllability problem at both endpoints. That is to say, for any
\[
(y^0, y^1) \in L^2(I_{t_0}) \times H^{-1}(I_{t_0}), \tag{24}
(y^0_T, y^1_T) \in L^2(I_{t_0 + T}) \times H^{-1}(I_{t_0 + T}), \tag{25}
\]
find two control functions \( f_1, f_2 \in L^2(t_0, t_0 + T) \) such that the solution of
\[
\begin{aligned}
y_{tt} - y_{xx} &= 0, \quad \text{in} \ Q_{t_0 + T}, \\
y(-v_1 t, t) &= f_1(t), \quad y(v_2 t, t) = f_2(t), \quad \text{for} \ t \in (t_0, t_0 + T), \tag{CW P2}
y(x, t_0) = y^0(x), \quad y_t(x, t_0) = y^1(x), \quad \text{for} \ x \in I_{t_0},
\end{aligned}
\]
satisfies the final conditions
\[
y(x, t_0 + T) = y^0_T(x), \quad y_t(x, t_0 + T) = y^1_T(x), \quad \text{for} \ x \in I_{t_0 + T}.
\]

We show that the controllability of \((CW P2)\) holds iff \( T \geq \hat{T}_v \), see Theorem 5.2 in page 23.
1.3. Some previous works and used techniques. Observability and controllability of the wave equation in noncylindrical domains were considered by several authors. Bardos and Chen [2] obtained the interior exact controllability by a “controllability via stabilisation” argument. Miranda [14] used a change of variable to transform the noncylindrical problem to a cylindrical one, shows the exact boundary controllability by HUM. Then going back to the noncylindrical problem, he obtains the desired results. In recent years, there is a renewed interest in the observability and controllability of such problems, see for instance [5, 12, 20].

The authors, in these cited works, relay on the multiplier method (see [8]) to establish the energy estimates and inequalities necessary to derive the observability and controllability results. In this work, we present a different approach. The key idea is that we analyse the series representation of solution and use it to derive the desired energy and observability estimates. This is made possible by considering (9) as the difference of two (generalised) Fourier series in weighted $L^2$-spaces.

Although Fourier series approach in control theory of problems in cylindrical domains is by now classic, see [9, 17, 22], the adaptation of this approach to problems in noncylindrical domains seems to be recent. In [18, 19], we have successfully used this approach for the wave equation, with Dirichlet or mixed boundary conditions, in an interval with one moving endpoint, e.g. $(0, v_2 t), 0 < v_2 < 1$. We showed that the boundary observability and controllability at one endpoint, whether it is the fixed or the moving one, holds in a sharp time $T_0 = 2L_0/(1 - v_2)$.

Of course, the considered problems in this paper are more challenging since the interval have two moving endpoints. The obtained results are new to our knowledge. In particular, the observability at the two endpoints and the controllability of Problem ($CWP2$) seems to be not considered before.

1.4. Organization of the paper. For the convenience of the reader, we recall in the next section some definitions and establish some lemmas that are necessary for the sequel. Then, we show how to calculate the coefficients of the series formula (9). In Section 3, we derive a sharp estimate for the energy of the solution of ($WP$). Then, the boundary observability and controllability at one and at both endpoints are considered in the fourth and fifth sections.

2. Exact solution.

2.1. Preliminaries. Let $a, b \in \mathbb{R}, b > a$, and consider a positive (weight) function $\rho : (a, b) \rightarrow \mathbb{R}$. In the sequel, we denote by $L^2(a, b, \rho ds)$ the weighted Hilbert space of measurable complex-valued functions on $\mathbb{R}$, endowed by the scalar product

$$\int_a^b h(s) g(s) \rho(s) \, ds$$

and its associated norm. As usual, we drop $\rho ds$ in the $L^2$ space notation if $\rho \equiv 1$.

If the set of functions $\{\varphi_n\}_{n \in \mathbb{Z}}$ is a complete orthonormal basis of $L^2(a, b, \rho ds)$, then every function $h \in L^2(a, b, \rho ds)$ can be written as

$$h(s) = \sum_{n \in \mathbb{Z}} a_n \varphi_n(s), \quad \text{where} \quad a_n := \int_a^b h(s) \varphi_n(s) \rho(s) \, ds.$$  

(26)
In particular, the following Parseval equality holds
\[
\int_a^b |h(s)|^2 \rho(s) ds = \sum_{n \in \mathbb{Z}} |a_n|^2,
\] (27)
see for instance [3, 16].

Let us check the completeness of some sets of functions used in the next sections.

**Lemma 2.1.** Let \(a, b \in \mathbb{R}^+\) satisfying \(b = \alpha_v \beta_v\). Then, the set of functions
\[
\left\{ \sqrt{\kappa_v/2} e^{i\pi \kappa_v \log z} \right\}_{n \in \mathbb{Z}}
\] (28)
is complete and orthonormal in the space \(L^2(a, b, dz/z)\).

The proof can be found in [19, p. 4].

**Remark 1.** Let \(m \in \mathbb{Z}\). Taking \(a = (\alpha_v \beta_v)^m t_0\) and \(b = (\alpha_v \beta_v)^{m+1} t_0\), then the set (28) is a complete orthonormal set in the space \(L^2((\alpha_v \beta_v)^m t_0, (\alpha_v \beta_v)^{m+1} t_0, dt/t)\).

**Remark 2.** Assume that \(b > a > 0\), then
\[
\frac{1}{b} \| \cdot \|_{L^2(a, b)} \leq \| \cdot \|_{L^2(a, b, dz/z)} \leq \frac{1}{a} \| \cdot \|_{L^2(a, b)}, \text{ for } z \in (a, b).
\]

This means that a function \(h \in L^2(a, b)\) iff \(h \in L^2(a, b, dz/z)\).

**Lemma 2.2.** For every \(t \geq t_0\), the set of functions
\[
\left\{ \sqrt{\kappa_v/2} e^{i\pi \kappa_v \log(t+x)} \right\}_{n \in \mathbb{Z}}, \quad \text{resp.} \quad \left\{ \sqrt{\kappa_v/2} e^{i\pi \kappa_v \log(t-x)} \right\}_{n \in \mathbb{Z}}
\] (29)
is complete and orthonormal in the space \(L^2(-v_1 t, V_2 t, \frac{dx}{t+x})\), resp. \(L^2(-V_1 t, v_2 t, \frac{dx}{t-x})\).

**Proof.** Let \(t \geq t_0\). We use the change of variables
\[
s = \kappa_v \log \frac{t+x}{(1-v_1) t}, \quad x \in (-v_1 t, V_2 t),
\]
\[
\text{resp. } s = \kappa_v \log \frac{\alpha_v \beta_v (1-v_2) t}{t-x}, \quad x \in (-V_1 t, v_2 t),
\]
to obtain \(s \in (0, 2)\). The result follows for by arguing as in [19].

**Remark 3.** For \(t \geq t_0\), the time-like condition in (3) ensures that the weight functions \(1/(t \pm x)\) are positives, hence
\[
\frac{1}{t(1 + V_2)} \| \cdot \|_{L^2(-v_1 t, V_2 t)} \leq \| \cdot \|_{L^2(-v_1 t, V_2 t, \frac{dx}{t+x})} \leq \frac{1}{t(1 - v_1)} \| \cdot \|_{L^2(-v_1 t, V_2 t)},
\]
\[
\frac{1}{t(1 + V_1)} \| \cdot \|_{L^2(-V_1 t, v_2 t)} \leq \| \cdot \|_{L^2(-V_1 t, v_2 t, \frac{dx}{t-x})} \leq \frac{1}{t(1 - v_2)} \| \cdot \|_{L^2(-V_1 t, v_2 t)},
\]
and therefore a function \(h\) belongs \(L^2(-v_1 t, V_2 t, \frac{dx}{t+x})\) or \(L^2(-V_1 t, v_2 t, \frac{dx}{t-x})\) iff \(h\) belongs to \(L^2(-v_1 t, V_2 t)\) or \(L^2(-V_1 t, v_2 t)\) respectively.
2.2. Extensions of the initial data. To determine the coefficients $c_n$ of the series formula (9), we first note that the two orthogonal sets considered in Lemma 2.2 appear in the series (9). To use their orthogonality properties, we need to extend the function $\phi$, defined only on $(-v_1 t, v_2 t)$, to the intervals $(-V_1 t, v_2 t)$ and $(-v_1 t, V_2 t)$, (recall that $-V_1 < -v_1 < v_2 < V_2$).

This extension is realised as follows

$$
\tilde{\phi}(x, t) = \begin{cases} 
-\phi \left(-t + \frac{1-v_1}{1+v_1} (t-x), t\right), & x \in (-V_1 t, -v_1 t), \\
\phi (x, t), & x \in (-v_1 t, v_2 t), \\
-\phi \left(t - \frac{1-v_2}{1+v_2} (t+x), t\right), & x \in (v_2 t, V_2 t).
\end{cases}
$$

(30)

The obtained function is well defined since the first variable of $\phi$ remains in the interval $(-v_1 t, v_2 t)$. In particular, $\tilde{\phi}(-v_1 t, t) = \tilde{\phi}(v_2 t, t) = 0$, hence the homogeneous boundary conditions at $x = -v_1 t$ and $x = v_2 t$ remain satisfied, for every $t \geq t_0$. See Figure 1 where the extension of an initial data $\phi^0$ is represented.

![Figure 1. Extension of an initial data $\phi^0$ when $v_1 < v_2$.](image)

On one hand, taking the derivative of (30) with respect to $x$, we obtain for $t = t_0$,

$$
\tilde{\phi}_x^0 (x) = \begin{cases} 
\frac{1-v_1}{1+v_1} \phi_x^0 \left(-t_0 + \frac{1-v_1}{1+v_1} (t_0-x)\right), & x \in (-V_1 t_0, -v_1 t_0), \\
\phi_x^0 (x, t_0), & x \in (-v_1 t_0, v_2 t_0), \\
\frac{1-v_2}{1+v_2} \phi_x^0 \left(t_0 - \frac{1-v_2}{1+v_2} (t_0+x)\right), & x \in (v_2 t_0, V_2 t_0).
\end{cases}
$$

(31)

On the other hand, $\tilde{\phi}^1$ is extended as follows

$$
\tilde{\phi}^1 (x) = \begin{cases} 
-\frac{1-v_1}{1+v_1} \phi^1 \left(-t_0 + \frac{1-v_1}{1+v_1} (t_0-x)\right), & x \in (-V_1 t_0, -v_1 t_0), \\
\phi^1 (x), & x \in (-v_1 t_0, v_2 t_0), \\
-\frac{1-v_2}{1+v_2} \phi^1 \left(t_0 - \frac{1-v_2}{1+v_2} (t_0+x)\right), & x \in (v_2 t_0, V_2 t_0).
\end{cases}
$$

(32)

The extension $\tilde{\phi}_x^0$ (resp. $\tilde{\phi}^1$) satisfies some even-like (resp. odd-like) symmetry in the variable $x$ with respect to the point $x = -v_1 t_0$ on the interval $(-V_1 t_0, v_2 t_0)$ and with respect to the point $x = v_2 t_0$ on the interval $(-v_1 t_0, V_2 t_0)$.

Remark 4. Going back to (7) and (8), we see that $V_1 = v_2$ if $v_1 = 0$. Then, $\tilde{\phi}_x^0, \tilde{\phi}^1$ are odd functions and $\tilde{\phi}_x^0$ is an even function (in the usual sense) on the interval
solution of Problem (WP). Similarly, if \( v_2 = 0 \), then \( V_2 = v_1 \) and \( \tilde{\phi}_t^0, \tilde{\phi}_t^1 \) are odd function and \( \phi_x^0 \) is an even function in the interval \((-v_1 t_0, v_1 t_0)\). This justifies the terminology “odd-like” and “even-like” used above.

**Remark 5.** Figure 1 shows why the constants \( \alpha_v, \beta_v, V_1, V_2 \) are related to the solution of Problem (WP). For instance,

i) the forward characteristic line starting from the endpoint \( x = -v_1 t_0 \) (resp. \( x = v_2 t_0 \)), hits the boundary \( x = v_2 t \) (resp. \( x = -v_1 t \)) at time \( t = \alpha_v t_0 \) (resp. \( t = \beta_v t_0 \)).

ii) consider \((x_1, t_1)\) as the intersection of these two characteristic lines after one reflection on the boundary, see Figure 1. We can check that, the two backward characteristic lines from this point intersect the \( x \)-axis precisely at \( x = -V_1 t_0 \) and \( x = V_2 t_0 \).

### 2.3. Proof of Theorem 1.1

Now we are ready to give the proof of the first main theorem.

**Proof of Theorem 1.1.** Thanks to Assumption (5) and Remark 3, we can derive term by term the series (9), it comes that

\[
\begin{align*}
\phi_x(x, t) &= i\pi \kappa_v \sum_{n \in \mathbb{Z}}^{} n \left( c_n \frac{e^{i\pi \kappa_v \log(t+x)}}{t+x} + C_n \frac{e^{i\pi \kappa_v \log(t-x)}}{t-x} \right), \\
\phi_t(x, t) &= i\pi \kappa_v \sum_{n \in \mathbb{Z}}^{} n \left( c_n \frac{e^{i\pi \kappa_v \log(t+x)}}{t+x} - C_n \frac{e^{i\pi \kappa_v \log(t-x)}}{t-x} \right),
\end{align*}
\]

where \( t \geq t_0, x \in (-v_1 t, v_2 t) \) and

\[
C_n := c_n e^{i\pi \kappa_v \log\left(\frac{1+v_2}{1-v_2}\right)} \quad \text{for } n \in \mathbb{Z}^*.
\]

Combining this, with (31) and (32), for \( t = t_0 \), the extensions of the initial data are given by

\[
\tilde{\phi}_x^0(x) = \begin{cases} 
 i\pi \kappa_v \sum_{n \in \mathbb{Z}^*}^{} n \left( c_n \frac{e^{i\pi \kappa_v \log\left(\frac{1+v_1}{1-v_1}(t_0-x)\right)}}{t_0-x} \\
+ \frac{1-v_1}{1+v_1} C_n \frac{e^{i\pi \kappa_v \log(2t_0 - \frac{1-v_1}{1+v_1}(t_0-x))}}{2t_0 - \frac{1-v_1}{1+v_1}(t_0-x)} \right), & \text{if } x \in (-V_1 t_0, -v_1 t_0), \\
 i\pi \kappa_v \sum_{n \in \mathbb{Z}^*}^{} n \left( c_n \frac{e^{i\pi \kappa_v \log(t_0+x)}}{t_0+x} \\
+ C_n \frac{e^{i\pi \kappa_v \log(t_0-x)}}{t_0-x} \right), & \text{if } x \in (-v_1 t_0, v_2 t_0), \\
 i\pi \kappa_v \sum_{n \in \mathbb{Z}^*}^{} n \left( \frac{1-v_2}{1+v_2} C_n \frac{e^{i\pi \kappa_v \log\left(\frac{1-v_2}{1+v_2}(t_0+x)\right)}}{2t_0 - \frac{1-v_2}{1+v_2}(t_0+x)} \\
+ C_n \frac{e^{i\pi \kappa_v \log\left(\frac{1-v_1}{1+v_1}(t_0+x)\right)}}{t_0+x} \right), & \text{if } x \in (v_2 t_0, V_2 t_0). \end{cases}
\]

(36)
Due to (35), we can write

\[ \phi^1(x) = \begin{cases} 
  i\pi\kappa_v \sum_{n \in \mathbb{Z}^*} n \left( -c_n \frac{e^{in\pi\kappa_v \log\left(\frac{t_0-x}{1+v_1}\right)}}{t_0-x} 
  + \frac{1-v_1}{1+v_1} c_n \frac{e^{in\pi\kappa_v \log\left(2t_0 - \frac{1-v_1}{1+v_1}(t_0-x)\right)}}{2t_0 - \frac{1-v_1}{1+v_1}(t_0-x)} \right), & \text{if } x \in (-V_1t_0,-v_1t_0), \\
  i\pi\kappa_v \sum_{n \in \mathbb{Z}^*} n \left( c_n \frac{e^{in\pi\kappa_v \log(t_0+x)}}{t_0+x} 
  - c_n \frac{e^{in\pi\kappa_v \log(t_0-x)}}{t_0-x} \right), & \text{if } x \in (-v_1t_0,v_2t_0), \\
  i\pi\kappa_v \sum_{n \in \mathbb{Z}^*} n \left( -c_n \frac{e^{in\pi\kappa_v \log\left(\frac{1-v_2}{1+v_2}(t_0-x)\right)}}{1+v_2} \right), & \text{if } x \in (v_2t_0,V_2t_0). 
\end{cases} \]  

(37)

On one hand, taking the sum of (36) and (37), we get

\[ \tilde{\phi}_x^0 + \tilde{\phi}_x^1 = \begin{cases} 
  2\pi i\kappa_v \sum_{n \in \mathbb{Z}^*} nC_n \frac{e^{in\pi\kappa_v \log\left(\frac{t_0-x}{1+v_1}\right)}}{2t_0 - \frac{1-v_1}{1+v_1}(t_0-x)}, & \text{if } x \in (-V_1t_0,-v_1t_0), \\
  2\pi i\kappa_v \sum_{n \in \mathbb{Z}^*} nC_n \frac{e^{in\pi\kappa_v \log(t_0+x)}}{t_0+x}, & \text{if } x \in (-v_1t_0,v_2t_0), \\
  2\pi i\kappa_v \sum_{n \in \mathbb{Z}^*} nC_n \frac{e^{in\pi\kappa_v \log\left(\frac{1-v_2}{1+v_2}(t_0+x)\right)}}{t_0+x}, & \text{if } x \in (v_2t_0,V_2t_0). 
\end{cases} \]

In particular, on the interval \((-v_1t_0,V_2t_0)\) we have

\[ (t_0+x)\left(\tilde{\phi}_x^0 + \tilde{\phi}_x^1\right) = \begin{cases} 
  2\pi i\kappa_v \sum_{n \in \mathbb{Z}^*} nC_n e^{in\pi\kappa_v \log(t_0+x)}, & \text{if } x \in (-v_1t_0,v_2t_0), \\
  2\pi i\kappa_v \sum_{n \in \mathbb{Z}^*} \left( nC_n e^{in\pi\kappa_v \log\left(\frac{1-v_2}{1+v_2}(t_0+x)\right)} \right) e^{in\pi\kappa_v \log(t_0+x)} & \text{if } x \in (v_2t_0,V_2t_0). 
\end{cases} \]

Due to (35), we can write

\[ \frac{1}{2\pi i \sqrt{2\kappa_v}} (t_0+x)\left(\tilde{\phi}_x^0 + \tilde{\phi}_x^1\right) = \sum_{n \in \mathbb{Z}^*} nC_n \sqrt{\kappa_v/2} e^{in\pi\kappa_v \log(t_0+x)}, \]  

(38)

for \(x \in (-v_1t_0,V_2t_0)\). Thanks to Lemma 4.1, we see that \(nC_n\) is the \(n\)th coefficient of the function

\[ \frac{1}{2\pi i \sqrt{2\kappa_v}} (t_0+x)\left(\tilde{\phi}_x^0 + \tilde{\phi}_x^1\right) \in L^2\left(-v_1t_0,V_2t_0, \frac{dx}{t_0+x}\right) \]  

(39)
in the basis \( \left\{ \sqrt{\kappa_v/2} e^{in\pi\kappa_v \log(t_0 + x)} \right\}_{n \in \mathbb{Z}} \). By consequence,

\[
n c_n = \frac{1}{4\pi i} \int_{-v_1 t_0}^{v_2 t_0} (t_0 + x) \left( \phi_x^0 + \phi_x^1 \right) e^{-in\pi\kappa_v \log(t_0 + x)} \frac{dx}{t_0 + x}, \quad \text{for } n \in \mathbb{Z}^*,
\]

and (10) holds as claimed.

On the other hand, the difference of (36) and (37) on the interval \((-V_1 t_0, v_2 t_0)\), yields in particular

\[
(t_0 - x) \left( \phi_x^0 - \phi_x^1 \right) = \begin{cases} 2i\pi \kappa_v \sum_{n \in \mathbb{Z}^*} n \left( c_n e^{in\pi\kappa_v \log \left( \frac{t_0 - x}{t_0} \right)} \right) e^{in\pi\kappa_v \log(t_0 - x)}, & x \in (-V_1 t_0, -v_1 t_0), \\
i\pi \kappa_v \sum_{n \in \mathbb{Z}^*} n C_n e^{in\pi\kappa_v \log(t_0 - x)), & x \in (-v_1 t_0, v_2 t_0). 
\end{cases}
\]

Noting that \( c_n e^{in\pi\kappa_v \log \left( \frac{t_0 - x}{t_0} \right)} = C_n \) we get

\[
\frac{1}{2\pi i \sqrt{2\kappa_v}} (t_0 - x) \left( \phi_x^0 - \phi_x^1 \right) = \sum_{n \in \mathbb{Z}^*} n C_n \sqrt{\kappa_v/2} e^{in\pi\kappa_v \log(t_0 - x)} ,
\]

for \( x \in (-V_1 t_0, v_2 t_0) \). Thus (11) follows since \( \left\{ \sqrt{\kappa_v/2} e^{in\pi\kappa_v \log(t_0 - x)} \right\}_{n \in \mathbb{Z}} \) is an orthonormal basis for \( L^2 \left( -V_1 t_0, v_2 t_0, \frac{dx}{t_0 - x} \right) \).

**Corollary 1.** Under the assumptions (3) and (4), the sum \( \sum_{n \in \mathbb{Z}^*} |nc_n|^2 \) is finite and is given by any of the two formula

\[
\sum_{n \in \mathbb{Z}^*} |nc_n|^2 = \frac{1}{8\pi^2 \kappa_v} \int_{-v_1 t_0}^{v_2 t_0} (t_0 + x) \left( \phi_x^0 + \phi_x^1 \right)^2 dx \quad \text{(41)}
\]

\[
= \frac{1}{8\pi^2 \kappa_v} \int_{-v_1 t_0}^{v_2 t_0} (t_0 - x) \left( \phi_x^0 - \phi_x^1 \right)^2 dx . \quad \text{(42)}
\]

**Proof.** Due to Lemma 2.2 and (38), Parseval’s equality applied to the function given in (39) yields

\[
\sum_{n \in \mathbb{Z}^*} |nc_n|^2 = \frac{1}{|2\pi i \sqrt{2\kappa_v}|^2} \int_{-v_1 t_0}^{v_2 t_0} (t_0 + x) \left( \phi_x^0 + \phi_x^1 \right)^2 \frac{dx}{t_0 + x}.
\]

Thus (41) holds as claimed. Noting that \( |C_n| = |c_n e^{in\pi\kappa_v \log \left( \frac{t_0 - x}{t_0} \right)}| = |c_n| \), the identity (42) follows from (40) in a similar manner.

3. **Energy estimates.** In this section, we prove Theorem 1.2 that gives the precise decay rate of the energy for the solution of Problem \((WP)\).
Proof of Theorem 1.2. Recalling that $S_v := 2\pi^2 \kappa_v \sum_{n \in \mathbb{Z}_*} |nc_n|^2$, then (41) and (42) means that
\[
\int_{-v_1 t}^{v_1 t} (t-x) \left( \dot{\phi}_x - \dot{\phi}_t \right)^2 dx = \int_{-v_1 t}^{v_1 t} (t+x) \left( \ddot{\phi}_x + \ddot{\phi}_t \right)^2 dx = 4S_v. \tag{43}
\]
Using (31), (32) and considering the change of variable $x = t - \frac{1-v_1}{1+v_1} (t+\xi)$, in $(-V_1 t, -v_1 t)$, we obtain
\[
\int_{-V_1 t}^{v_1 t} (t-x) \left( \dot{\phi}_x (x) - \dot{\phi}_t (x) \right)^2 dx
= - \left( \frac{1+v_1}{1-v_1} \right)^2 \int_{v_2 t}^{v_1 t} (t+\xi) \left( \frac{1-v_1}{1+v_1} (\phi_x (\xi) + \phi_t (\xi)) \right)^2 d\xi
= \int_{-v_1 t}^{v_2 t} (t+\xi) (\phi_x (\xi) + \phi_t (\xi))^2 d\xi.
\]
Taking $x = t - \frac{1+v_2}{1-v_2} (t+\xi)$ in $(v_2 t, V_2 t)$, we obtain
\[
\int_{v_2 t}^{V_2 t} (t+x) \left( \dot{\phi}_x (x) + \dot{\phi}_t (x) \right)^2 dx = \int_{-v_1 t}^{v_1 t} (t-x) (\phi_x (\xi) - \phi_t (\xi))^2 d\xi.
\]
Then, it comes that
\[
\int_{-V_1 t}^{v_1 t} (t+x) \left( \dot{\phi}_t + \dot{\phi}_x \right)^2 dx + \int_{-v_1 t}^{v_2 t} (t-x) \left( \ddot{\phi}_x - \ddot{\phi}_t \right)^2 dx
= 2 \int_{-v_1 t}^{v_2 t} (t+x) (\phi_x + \phi_t)^2 dx + 2 \int_{-v_1 t}^{v_1 t} (t-x) (\phi_t - \phi_x)^2 dx = 8S_v.
\]
Expanding $(\phi_x \pm \phi_t)^2$ and collecting similar terms, we get
\[
4t \int_{-v_1 t}^{v_2 t} \phi_x^2 + \phi_t^2 dx + 8 \int_{-v_1 t}^{v_2 t} x\phi_x\phi_t dx = 8S_v, \quad \text{for } t \geq t_0.
\]
Recalling that $E_v (t)$ is given by (12), then (13) holds as claimed.

Finally, using the algebraic inequality $\pm ab \leq (a^2 + b^2)/2$ and the fact that $|x| \leq v_{\text{max}} t$, we obtain
\[
\pm \int_{-v_1 t}^{v_2 t} x\phi_x\phi_t dx \leq v_{\text{max}} t E_v (t), \quad \text{for } t \geq t_0.
\]
Due to (13), it comes that
\[
S_v \leq (1 + v_{\text{max}}) t E_v (t) \quad \text{and} \quad (1 - v_{\text{max}}) t E_v (t) \leq S_v, \quad \text{for } t \geq t_0. \tag{44}
\]
This implies (14) and the theorem follows. \qed

The next corollary compares $E_v (t)$ to the initial energy $E_v (t_0)$.
Corollary 2. Under the assumptions (3) and (4), the energy of the solution of Problem (WP) satisfies
\[
\left( \frac{1 - v_{\text{max}}}{1 + v_{\text{max}}} \right) \frac{t_0 E_v(t_0)}{t} \leq E_v(t) \leq \left( \frac{1 + v_{\text{max}}}{1 - v_{\text{max}}} \right) \frac{t_0 E_v(t_0)}{t}, \quad \text{for } t \geq t_0. \tag{45}
\]

Proof. Since (44) holds also for \( t = t_0 \), then (45) follows by combining the two inequalities
\[
(1 - v_{\text{max}}) t E_v(t) \leq (1 + v_{\text{max}}) t_0 E_v(t_0),
\]
\[
(1 + v_{\text{max}}) t E_v(t) \geq (1 - v_{\text{max}}) t_0 E_v(t_0),
\]
for \( t \geq t_0 \). \hfill \qed

4. Observability and controllability at one endpoint. In this section, we show the observability of (WP) at the endpoint \( x = v_2 t \), then by applying HUM we deduce the exact boundary controllability for (CWP).

4.1. Observability at one endpoint. First, we show the following lemma.

Lemma 4.1. Let \( M \) be a positive integer and \( \phi \) the solution of (WP). Under the assumptions (3) and (4), we have
\[
\int_{t_0}^{(\alpha_v \beta_v)^M t_0} t \phi_x^2(v_2 t, t) dt = \frac{4M}{(1 - v_2^2)^2} S_v, \tag{46}
\]
and it holds that
\[
\frac{4M (1 - v_{\text{max}}) t_0}{(1 - v_2^2)^2} E_v(t_0) \leq \int_{t_0}^{(\alpha_v \beta_v)^M t_0} t \phi_x^2(v_2 t, t) dt \leq \frac{4M (1 + v_{\text{max}}) t_0}{(1 - v_2^2)^2} E_v(t_0). \tag{47}
\]

Proof. Thanks to (33), we can evaluate \( \phi_x \) at the endpoint \( x = v_2 t \). We obtain
\[
\phi_x(v_2 t, t) = \pi \kappa_v \sum_{n \in \mathbb{Z}^*} \text{inc}_n \left( \frac{e^{in \pi \kappa_v \log(1 + v_2) t} + e^{in \pi \kappa_v \log(1 + v_2) t}}{(1 + v_2) t} \right)
\]
\[
= \pi \kappa_v \sum_{n \in \mathbb{Z}^*} \text{inc}_n \left( \frac{1}{(1 + v_2) t} + \frac{1}{(1 - v_2) t} \right) e^{in \pi \kappa_v \log(1 + v_2) t},
\]
due to (35). Thus
\[
t \phi_x(v_2 t, t) = \frac{2 \pi \kappa_v}{(1 - v_2^2)^2} \sum_{n \in \mathbb{Z}^*} \text{inc}_n e^{in \pi \kappa_v \log(1 + v_2)} e^{in \pi \kappa_v \log t}. \tag{48}
\]

Lemma 2.1 and Parseval’s equality applied to the function
\[
t \phi_x(v_2 t, t) \in L^2 \left( (\alpha_v \beta_v)^m t_0, (\alpha_v \beta_v)^{m+1} t_0, \frac{dt}{t} \right), \quad \text{for } m = 0, \ldots, M - 1,
\]
yields, after summing up the integrals for all the subintervals of \((t_0, (\alpha_v\beta_v)^M t_0)\),
\[
\int_{t_0}^{(\alpha_v\beta_v)^M t_0} t \phi_x^2(v(t), t) dt = M \frac{4\pi^2 \kappa_v^2}{(1 - v_2^2)} \left( \frac{2}{\kappa_v} \right) \sum_{n \in \mathbb{Z}^*} |nc_n e^{in\pi \kappa_v \log(1 + v_2)}|^2
\]
\[= 8M \pi^2 \kappa_v \sum_{n \in \mathbb{Z}^*} |nc_n|^2.
\]
This shows the identity (46). The estimate (47) follows by using (44) for \(t = t_0\).

**Proof of Theorem 1.3.** 

1. The right-hand side of (47) yields
\[
t_0 \int_{t_0}^{(\alpha_v\beta_v)^M t_0} \phi_x^2(v(t), t) dt \leq \frac{4M (1 + v_{\text{max}}) t_0}{(1 - v_2^2)} E_v(t_0).
\]

Noting that \(\alpha_v\beta_v > 1\), then for every \(T \geq 0\), we can take the integer \(M\) large enough to satisfy \((\alpha_v\beta_v)^M t_0 \geq t_0 + T\). Since the integrated function is nonnegative, then the direct inequality (17) holds for
\[K_v(T) := 4M (1 + v_{\text{max}}) / (1 - v_2^2).
\]

2. The left-hand side of (47), for \(M = 1\), yields
\[
4 \frac{(1 - v_{\text{max}}) t_0}{(1 - v_2^2)} E_v(t_0) \leq \alpha_v \beta_v t_0 \int_{t_0}^{(\alpha_v\beta_v) t_0} \phi_x^2(v(t), t) dt
\]
and thus inequality (18) holds for \(T = \alpha_v \beta_v t_0 - t_0\) and therefore for every \(T \geq \alpha_v \beta_v t_0 - t_0\) as well. Recalling that \(L \equiv (v_1 + v_2) t_0\), we check easily that \(\alpha_v \beta_v t_0 - t_0 = L_0 / (1 - v_1) (1 - v_2) = T_v\).

It remains to show that the observability does not hold for every \(T < T_v\). Let \(\varepsilon > 0\), \(\varepsilon \in (0, 1)\), and consider a real valued function \(g \in L^2(t_0, \alpha_v \beta_v t_0, dt/t)\), not identically zero and satisfying
\[
\text{supp}(g) \subset ((1 - \varepsilon) \alpha_v \beta_v t_0, \alpha_v \beta_v t_0), \quad \text{for } t \in (t_0, \alpha_v \beta_v t_0)
\]
\[
\int_{t_0}^{\alpha_v \beta_v t_0} g(t) \frac{dt}{t} = 0.
\]

Since \(\left\{ \sqrt{\kappa_v/2} e^{i n \pi \kappa_v \log t} \right\}_{n \in \mathbb{Z}}\) is a complete basis for \(L^2(t_0, \alpha_v \beta_v t_0, dt/t)\), then \(g\) can be written as
\[
g(t) = \sum_{n \in \mathbb{Z}} g_n \sqrt{\kappa_v/2} e^{i n \pi \kappa_v \log t}, \quad t \in (t_0, \alpha_v \beta_v t_0).
\]

Note that (50) ensures that \(g_0 = 0\). Let us define
\[
c_0 = 0, \quad \text{and } c_n = \frac{g_n}{i n} e^{-i n \pi \kappa_v \log(1 + v_2)}, \quad n \in \mathbb{Z}^*.
\]

Then, the series
\[
\sum_{n \in \mathbb{Z}} inc_n \sqrt{\kappa_v/2} e^{i n \pi \kappa_v \log(t_0 + x)}
\]
converges and defines a non-identically zero function
\[
g \in L^2(-v_1 t_0, V_2 t_0, dx / (t_0 + x)) .
\]
Having (38) in mind, we need to choose two functions \( \tilde{\phi}^0 \in L^2(-v_1 t_0, V_2 t_0) \) and \( \tilde{\phi}^1 \in L^2(-v_1 t_0, V_2 t_0) \), satisfying respectively the even-like and odd-like symmetries considered in (31) and (32), and such that

\[
\frac{1}{2\pi \sqrt{2\kappa_v}} (t_0 + x) \left( \tilde{\phi}^0 + \tilde{\phi}^1 \right) = g(x), \quad \text{for } x \in (-v_1 t_0, V_2 t_0),
\]

i.e.,

\[
\tilde{\phi}^0_x + \tilde{\phi}^1 = \mathbf{G}(x) := 2\pi \sqrt{2\kappa_v} \frac{g(x)}{t_0 + x}, \quad \text{for } x \in (-v_1 t_0, V_2 t_0).
\]

This is realised in the following way. Consider the function \( \tilde{\mathbf{G}} \) defined as

\[
\tilde{\mathbf{G}}(x) = \begin{cases} 
\frac{1 + v_2}{1 - v_2} \mathbf{G} \left( t_0 + \frac{1 + v_2}{1 - v_2} (t_0 - x) \right), & x \in (-v_1 t_0, v_2 t_0), \\
\frac{1 - v_2}{1 + v_2} \mathbf{G} \left( t_0 - \frac{1 - v_2}{1 + v_2} (t_0 + x) \right), & x \in (v_2 t_0, V_2 t_0),
\end{cases}
\]

then it suffices to take

\[
\tilde{\phi}^0_x (x) = \frac{\mathbf{G}(x) + \tilde{\mathbf{G}}(x)}{2} \quad \text{and} \quad \tilde{\phi}^1_x (x) = \frac{\mathbf{G}(x) - \tilde{\mathbf{G}}(x)}{2}, \quad \text{for } x \in (-v_1 t_0, V_2 t_0).
\]

One can check that \( \tilde{\phi}^0 \) and \( \tilde{\phi}^1 \) are respectively an even-like and an odd-like function in the interval \((-v_1 t_0, V_2 t_0)\) with respect to \(x = v_2 t_0\). In particular

\[
\int_{-v_1 t_0}^{v_2 t_0} \tilde{\phi}^1 (x) \, dx = 0
\]

and by (54) it follows that

\[
\tilde{\phi}^0(v_2 t_0) - \tilde{\phi}^0(-v_1 t_0) = \int_{-v_1 t_0}^{v_2 t_0} \tilde{\phi}^0_x (x) \, dx = \frac{1}{2} \int_{-v_1 t_0}^{v_2 t_0} \tilde{\phi}^0_x (x) \, dx = \pi \sqrt{2\kappa_v} \int_{-v_1 t_0}^{v_2 t_0} \mathbf{G}(x) \frac{1}{t_0 + x} \, dx,
\]

i.e. \( \tilde{\phi}^0(v_2 t_0) - \tilde{\phi}^0(-v_1 t_0) = 0 \) thus \( c_0 = 0 \). Thus, we can always assume that \( \tilde{\phi}^0 \) belongs to \( H_0^1((-v_1 t_0, V_2 t_0)) \).

As in (48), the solution of Problem \((WP)\) corresponding to these initial conditions \( \tilde{\phi}^0_x \) and \( \tilde{\phi}^1 \) satisfies

\[
\phi_x(v_2 t, t) = \frac{2\pi \sqrt{2\kappa_v}}{(1 - v_2^2)t} \sum_{n \in \mathbb{Z}} \text{inc}_n e^{in\pi \kappa_v \log(1 + v_2)} \sqrt{\kappa_v/2} e^{in\pi \kappa_v \log t} = \frac{2\pi \sqrt{2\kappa_v}}{(1 - v_2^2)t} \frac{g(t)}{t},
\]

since the function \( g \) is given by (51) and the coefficients \( c_n \) where chosen to satisfy

\[\text{inc}_n e^{in\pi \kappa_v \log(1 + v_2)} = g_n.\]

Taking the squares and integrating on \((t_0, (1 - \varepsilon) t_0)\), we get

\[
\int_{t_0}^{(1 - \varepsilon) t_0} \phi_x^2(v_2 t, t) \, dt = \frac{8\pi^2 \kappa_v}{(1 - v_2^2)^2} \int_{t_0}^{(1 - \varepsilon) t_0} g^2(t) \, dt t^2 = 0,
\]

since \( \text{supp}(g) \cap (t_0, (1 - \varepsilon) t_0) = \emptyset \) for all \( \varepsilon > 0 \). This means that the observability inequality (18) does not hold for every \( T < T_v \). \( \Box \)

**Remark 6.** The time of observability \( T_v \) can be predicted by a simple argument, see Figure 2. An initial disturbance concentrated near \( x = v_2 t_0 \) may propagate to the left, as \( t \) increases, and bounce back on the left boundary, then travel to reach the right boundary, when \( t \) is close to \( (\alpha_v \beta_v - 1) t_0 \), see Figure 2 (left). Thus, the needed time to complete this journey is close to \( (\alpha_v \beta_v - 1) t_0 \), which is the sharp time of
observability $T_v$. Figure 2 (right) shows that we need the same time $T_v$ for an initial disturbance concentrated near $x = -v_1 t_0$.}

4.2. **Controllability at one endpoint.** First, arguing as in [8, page 54], we can easily check that the controllability problem ($CWP$) can be reduced to a null-controllability one, i.e., we can always assume that

$$u(t_0 + T) = u(t_0 + T) = 0 \text{ on } I_{t_0 + T}. \quad (55)$$

Then, the null-controllability of ($CWP$) is derived by mean of HUM.

**Theorem 4.2.** Under the assumptions (3) and (19), Problem ($CWP$) is exactly controllable at the endpoint $x = v_2 t$ for $T \geq T_v$. Moreover, we can choose a control $f$ satisfying

$$\int_{t_0}^{t_0 + T} f^2(t) \, dt \leq K_v(T) E_v(t_0), \quad (56)$$

where $K_v(T)$ is a constant depending on $v_1, v_2$ and $T$.

Conversely, if $T < T_v$, Problem ($CWP$) is not controllable at $x = v_2 t$.

**Proof.** Let $\phi$ be the solution of problem ($WP$). The idea of HUM is to seek a control $f$ in the special form

$$f = \phi_x(v_2 t, t) \in L^2(t_0, t_0 + T),$$

for a suitable choice of $\phi^0$ and $\phi^1$. First, we consider the backward problem

$$\begin{cases} \psi_{tt} - \psi_{xx} = 0, & \text{in } Q_{t_0 + T}, \\ \psi(-v_1 t, t) = 0, & \psi(v_2 t, t) = \phi_x(v_2 t, t), \quad \text{for } t \in (t_0, T), \\ \psi(x, t_0 + T) = 0, & \psi_1(x, t_0 + T) = 0, \quad \text{for } x \in I_{t_0 + T}. \end{cases} \quad (57)$$

We obtain a linear map, that relates $(\phi^0, \phi^1)$ to the initial data $(\psi_t(t_0), -\psi(t_0))$,

$$\Lambda_1 : H^1_0(I_{t_0}) \times L^2(I_{t_0}) \rightarrow H^{-1}(I_{t_0}) \times L^2(I_{t_0})$$

$$(\phi^0, \phi^1) \mapsto (\psi_t(t_0), -\psi(t_0)).$$

The space $H^1_0(I_{t_0}) \times L^2(I_{t_0})$ is equipped with the energy norm. To show that $(\phi^0, \phi^1)$ can be chosen such that $(\psi_t(t_0), -\psi(t_0)) = (u^1, -u^0)$, we argue as in [8].
Indeed, since the solution of (57) is taken in the transposition sense, it comes that
\[
0 = \int_{t_0}^{t_0+T} \langle (\psi_{tt} - \psi_{xx}) , \phi \rangle_{H^1_0(\Omega)} \, dt
\]
\[
= - \langle \psi_t(t_0), \phi^0 \rangle_{H^1_0(\Omega_0)} + \int_{-v_1t_0}^{v_2t_0} \psi(t_0) \phi^1 \, dx
\]
\[
+ v_2 \int_{t_0}^{t_0+T} \psi(v_2t, t) \phi_t(v_2t, t) \, dt + \int_{t_0}^{t_0+T} \psi(v_2t, t) \phi_x(v_2t, t) \, dt
\]
(58)
where \( \langle , \rangle_X \) denotes the duality product between a Banach space \( X \) and its dual. Observing that the boundary condition \( \phi(v_2t, t) = 0 \) implies that \( v_2 \phi_x(v_2t, t) + \phi_t(v_2t, t) = 0 \), i.e.,
\[
\phi_t(v_2t, t) = -v_2 \phi_x(v_2t, t).
\]
Then, we can rewrite (58) as
\[
\langle \psi_t(t_0), \phi^0 \rangle_{H^1_0(\Omega_0)} - \int_{-v_1t_0}^{v_2t_0} \psi(t_0) \phi^1 \, dx
\]
\[
= -v_2^2 \int_{t_0}^{t_0+T} \phi_x^2(v_2t, t) \, dt + \int_{t_0}^{t_0+T} \phi_x^2(v_2t, t) \, dt,
\]
hence
\[
\langle A_1(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle_{H^1_0(\Omega_0) \times L^2(\Omega_0)} = (1 - v_2^2) \int_{t_0}^{t_0+T} \phi_x^2(v_2t, t) \, dt.
\]
Thanks to Theorem 1.3, we deduce that
\[
\frac{4}{\alpha \sqrt{\beta}} \frac{(1 - v_{\text{max}})}{(1 - v_2^2)} E_v(t_0) \leq \langle A_1(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle \leq (1 - v_2^2) K_v(T) E_v(t_0),
\]
for \( T \geq T_v \). This means that \( A_1 \) is an isomorphism for \( T \geq T_v \) and therefore \( (\phi^0, \phi^1) \) can be determined such that the control \( f = \phi_x(v_2t, t) \) drive the solution of \( (W_P) \) from the initial data \( u^0, u^1 \) to the rest, i.e.
\[
u(t_0 + T) = u_t(t_0 + T) = 0.
\]
- If \( T < T_v \), then \( (W_P) \) is not observable by Theorem 1.3. This means that we can find non-zero initial data \( (\phi^0, \phi^1) \in H^1_0(\Omega_0) \times L^2(\Omega_0) \) such that
\[
\phi_x(v_2t, t) = 0, \quad \forall t \in (t_0, t_0 + T).
\]
(59)
Choose \( (u^1, u^0) \in H^{-1}(\Omega_0) \times L^2(\Omega_0) \) such that
\[
\langle u^1, \phi^0 \rangle_{H^1_0(\Omega_0)} - \int_{-v_1t_0}^{v_2t_0} u^0 \phi^1 \, dx \neq 0.
\]
Then, the solution of \( (W_P) \), in the transposition sense, satisfies
\[
\langle u_t(t_0 + T), \phi(t_0 + T) \rangle_{H^1_0(\Omega_0 + T)} - \int_{-v_1(t_0 + T)}^{v_2(t_0 + T)} u(t_0 + T) \phi_t(t_0 + T) \, dx
\]
\[
- \langle u^1, \phi^0 \rangle - \int_{-v_1t_0}^{v_2t_0} u^0 \phi^1 \, dx + (1 - v_2^2) \int_{t_0}^{t_0+T} f(t) \phi_x(v_2t, t) \, dt = 0.
\]
Whatever the choice of the control function $f \in L^2(t_0, t_0 + T)$, the last integral term always vanishes due to (59). Hence

$$
\langle u_t (t_0 + T), \phi (t_0 + T) \rangle_{H^1_t (I_{t_0 + T})} - \int_{-v_1(t_0 + T)}^{v_2(t_0 + T)} u (t_0 + T) \phi_t (t_0 + T) \, dx
= \langle u^1, \phi^0 \rangle_{H^1_t (I_0)} - \int_{-v_1 t_0}^{v_2 t_0} u^0 \phi^1 \, dx \neq 0
$$

and therefore we cannot have $u_t (t_0 + T) = u(t_0 + T) = 0$ on $I_{t_0 + T}$. This completes the proof. \hfill \Box

**Remark 7.** The proof of Theorem 4.2 shows that the controllability of $(CWP)$, at the endpoint $x = v_2 t$, is equivalent to the observability of $(CWP)$ at the same endpoint.

**Remark 8.** The equation $\phi_{tt} - \phi_{xx} = 0$ is invariant under the map $x \mapsto -x$, from $(-v_1 t, v_2 t)$ to $(-v_2 t, v_1 t)$, then it suffices to change the roles of $v_1$ and $v_2$ in the statement of Theorems 1.3 and 4.2 to obtain the observability and controllability results at to the other endpoint $x = -v_1 t$. Clearly, the value of $T_c$, given by (16), remain the same for the two cases.

5. **Observability and controllability at both endpoints.** If we can observe simultaneously the two endpoints of the interval, one expects a shorter time of observability. The proof is more challenging in this case.

5.1. **Observability at both endpoints.** Let us start by showing the following lemma.

**Lemma 5.1.** Under the assumptions (3) and (4), the solution of $(WP)$ satisfies

$$
(1 - v_1^2)^2 \int_{t_0}^{\beta_{v_1} t_0} t \phi_x^2 (-v_1 t, t) \, dt + (1 - v_2^2)^2 \int_{t_0}^{\alpha_{v_2} t_0} t \phi_x^2 (v_2 t, t) \, dt = 4M S_v
$$

and it holds that

$$
4M (1 - v_{\max}) t_0 E_v (t_0)
\leq (1 - v_1^2)^2 \int_{t_0}^{\beta_{v_1} t_0} t \phi_x^2 (-v_1 t, t) \, dt + (1 - v_2^2)^2 \int_{t_0}^{\alpha_{v_2} t_0} t \phi_x^2 (v_2 t, t) \, dt
\leq 4M (1 + v_{\max}) t_0 E_v (t_0).
$$

**Proof.** First, we establish (60) for smooth initial data. Assume that $\phi_x^0$ and $\phi_x^1$ are continuous functions. This ensures in particular that their generalized Fourier series are absolutely converging, see [3, 16]. More precisely, the coefficients $c_n$, given by (10), satisfy

$$
\sum_{n \in \mathbb{Z}^*} n |c_n| < +\infty.
$$

On one hand, taking $x = -v_1 t$ in (33), we get

$$
\phi_x (-v_1 t, t) = \pi \kappa_n \sum_{n \in \mathbb{Z}^*} in c_n \left( \frac{e^{i \pi \kappa_n \log (1 - v_1) t}}{(1 - v_1) t} + \frac{e^{i \pi \kappa_n \log (1 + v_1) t}}{(1 + v_1) t} \right)
= \pi \kappa_n \sum_{n \in \mathbb{Z}^*} in c_n \left( \frac{1}{1 - v_1} + \frac{e^{i \pi \kappa_n \log \alpha_n \beta_n}}{1 + v_1} \right) \frac{e^{i \pi \kappa_n \log (1 - v_1) t}}{t},
$$
hence
\[ \phi_x (-v_1 t, t) = \frac{2\pi \kappa_v}{1 - v_1^2} \sum_{m \in \mathbb{Z}} \text{i} m c_m e^{im\pi \kappa_v \log(1 - v_1) t}. \] (63)

Let \( m \in \mathbb{Z}^* \). Multiplying (63) by \((1 - v_1^2) \text{i} m c_m e^{im\pi \kappa_v \log(1 - v_1) M t}\) and integrating term-by-term, we obtain
\[ (1 - v_1^2) \int_{t_0}^{\alpha^M t_0} \phi_x (-v_1 t, t) \text{i} m c_m e^{im\pi \kappa_v \log(1 - v_1) M t} dt = 2\pi \kappa_v m \bar{c}_m \int_{t_0}^{\alpha^M t_0} \left( \sum_{n \in \mathbb{Z}^*} n c_n e^{i(n-m)\pi \kappa_v \log(1 - v_1) M t} \right) dt. \]

Since \(|n c_n e^{i(n-m)\pi \kappa_v \log(1 - v_1) M t}| = |n c_n|\), then due to (62) the series in the right hand side is absolutely converging. Applying Lebesgue’s dominated convergence theorem and integrating term-by-term, we obtain
\[ (1 - v_1^2) \int_{t_0}^{\alpha^M t_0} \phi_x (-v_1 t, t) \text{i} m c_m e^{im\pi \kappa_v \log(1 - v_1) M t} dt = 2\pi \kappa_v \sum_{n \in \mathbb{Z}^*} n m c_n \bar{c}_m e^{i(n-m)\pi \kappa_v \log(1 - v_1) M t} \int_{t_0}^{\alpha^M t_0} e^{i(n-m)\pi \kappa_v \log(t)} \frac{dt}{t} = \sum_{n \in \mathbb{Z}^*} A_{nm} \] (64)

where
\[ A_{nm} = 2M \pi \kappa_v |m c_m|^2 \log \beta_v, \]
\[ A_{nm} = \frac{2n m c_n \bar{c}_m}{i(n-m)} \left( e^{i(n-m)M \pi \kappa_v \log \beta_v} - 1 \right) e^{i(n-m)M \pi \kappa_v \log(1 - v_1) t_0} \text{ if } n \neq m. \]

On the other hand, we multiply (48) by \((1 - v_2^2) \text{i} m c_n e^{im\pi \kappa_v \log(1 + v_2) M t}\), and integrate term-by-term on \((t_0, \alpha^M t_0)\). We end up with
\[ (1 - v_2^2) \int_{t_0}^{\alpha^M t_0} \phi_x (v_2 t, t) \text{i} m c_m e^{im\pi \kappa_v \log(1 + v_2) M t} dt = \sum_{n \in \mathbb{Z}^*} B_{nm} \] (65)

where
\[ B_{nm} = 2M \pi \kappa_v |m c_m|^2 \log \alpha_v, \]
\[ B_{nm} = \frac{2n m c_n \bar{c}_m}{i(n-m)} \left( e^{i(n-m)M \pi \kappa_v \log \alpha_v} - 1 \right) e^{i(n-m)M \pi \kappa_v \log(1 + v_2) t_0} \text{ if } n \neq m. \]

Summing up (64) and (65), we obtain
\[ (1 - v_1^2) \int_{t_0}^{\alpha^M t_0} \phi_x (-v_1 t, t) \text{i} m c_m e^{im\pi \kappa_v \log(1 - v_1) M t} dt + (1 - v_2^2) \int_{t_0}^{\alpha^M t_0} \phi_x (v_2 t, t) \text{i} m c_n e^{im\pi \kappa_v \log(1 + v_2) M t} dt = \sum_{n \in \mathbb{Z}^*} \left( A_{nm} + B_{nm} \right). \] (66)

Since \( \kappa_v = 2/\log \alpha_v \beta_v \), it comes that
\[ A_{nm} + B_{nm} = 2M \pi \kappa_v |m c_m|^2 (\log \alpha_v + \log \beta_v) = 4M \pi |m c_m|^2, \quad m \in \mathbb{Z}^*. \]
If \( n \neq m \), then
\[
A_{nm} + B_{nm} = \frac{2mn c_n \bar{c}_m}{i (n-m)} \left( 1 - e^{i(n-m)M \pi \kappa_v \log \alpha_v} \right)
\]
\[
\times \left( e^{i(n-m)M \pi \kappa_v \log \left( \frac{1+vt_0}{t_0} \right)} - e^{i(n-m)M \pi \kappa_v \log (1+v_2)t_0} \right)
\]
\[
= \frac{2mn c_n \bar{c}_m}{i (n-m)} \left( 1 - e^{i(n-m)M \pi \kappa_v \log \alpha_v} \right)
\]
\[
\times e^{i(n-m)M \pi \kappa_v \log (1+v_2)t_0} \left( e^{-i(n-m)M \pi \kappa_v \log \alpha_v \beta_v} - 1 \right).
\]

The last parentheses vanishes, hence
\[
A_{nm} + B_{nm} = 0 \quad \text{if } n \neq m, \quad n, m \in \mathbb{Z}^*.
\]

Thus, we can rewrite (66) as
\[
(1 - v_1^2) \int_{t_0}^{\beta_v t_0} \phi_x (-v_1 t, t) i mc_n e^{im \pi \kappa_v \log (1-v_1)t} dt
\]
\+
\[
(1 - v_2^2) \int_{t_0}^{\alpha_v t_0} \phi_x (v_2 t, t) i mc_n e^{im \pi \kappa_v \log (1+v_2)t} dt = 4M \pi |mc_n|^2, \quad m \in \mathbb{Z}^*.
\]

Summing up for \( m \in \mathbb{Z}^* \), and applying Lebesgue’s theorem to interchange summation and integration, it comes that
\[
(1 - v_1^2) \int_{t_0}^{\beta_v t_0} \phi_x (-v_1 t, t) \left( \sum_{m=-\infty}^{+\infty} \frac{imc_n e^{im \pi \kappa_v \log (1-v_1)t}}{2 \pi \kappa_v} \right) dt
\]
\+
\[
(1 - v_2^2) \int_{t_0}^{\alpha_v t_0} \phi_x (v_2 t, t) \left( \sum_{m=-\infty}^{+\infty} \frac{imc_n e^{im \pi \kappa_v \log (1+v_2)t}}{2 \pi \kappa_v} \right) dt = 4M \pi \sum_{m \in \mathbb{Z}^*} |mc_m|^2.
\]

Thanks to (48) and (63), we obtain
\[
(1 - v_1^2) \int_{t_0}^{\beta_v t_0} \phi_x^2 (-v_1 t, t) \left( \frac{1 - v_1^2}{2 \pi \kappa_v} \right) dt
\]
\+
\[
(1 - v_2^2) \int_{t_0}^{\alpha_v t_0} \phi_x^2 (v_2 t, t) \left( \frac{1 - v_2^2}{2 \pi \kappa_v} \right) dt = 4M \pi \sum_{m \in \mathbb{Z}^*} |mc_m|^2.
\]

After few rearrangements, (60) follows as claimed.

In the general case, i.e. \( \phi^0 \in H_0^1 (-v_1 t_0, v_2 t_0) \) and \( \phi^1 \in L^2 (-v_1 t_0, v_2 t_0) \), we use an argument of density. Consider two sequences
\[
\phi_j^0 \in C^1 \left( [-v_1 t_0, v_2 t_0] \right) \quad \text{and} \quad \phi_j^1 \in C \left( [-v_1 t_0, v_2 t_0] \right), \quad j \in \mathbb{N},
\]
such that
\[
(\phi_j^0) \rightarrow \phi_x^0 \quad \text{and} \quad \phi_j^1 \rightarrow \phi^1 \quad \text{in} \quad L^2 (-v_1 t, v_2 t), \quad \text{as} \quad j \rightarrow +\infty.
\]

Then, denote by \( \phi_j \) and \( E_j \) the solution and energy associated to each data \( \phi_j^0, \phi_j^1 \).

Taking into account (13) for \( t = t_0 \), and using the precedent step of the proof, we
have
\[(1 - v_1^2)^2 \int_{t_0}^{\beta_v t_0} t \phi_x^2 (v_1 t, t) dt + (1 - v_2^2)^2 \int_{t_0}^{\alpha_v t_0} t \phi_x^2 (v_2 t, t) dt = 4 t_0 E_j (t_0) + 4 \int_{-v_1 t_0}^{v_2 t_0} x (\phi_j^0) \phi_j^1 \, dx.\]

Relaying on the continuity of the solution of the wave equation with respect to the initial data, which is emphasized by (45), the last inequality holds as \(j \to +\infty\). This shows (60) for the general case.

The estimate (61) follows by using (44) for \(t = t_0\).

Now we give the proof of direct and inverse inequalities for the solution of Problem \((WP)\) at the two endpoints \(x = -v_1 t\) and \(x = v_2 t\).

**Proof of Theorem 1.4.** • The right-hand side of Inequality (61) yields

\[(1 - v_{\text{max}})^2 t_0 \left( \int_{t_0}^{\beta_v t_0} \phi_x^2 (-v_1 t, t) dt + \int_{t_0}^{\alpha_v t_0} \phi_x^2 (v_2 t, t) dt \right) \leq 4 M (1 + v_{\text{max}}) t_0 E_v (t_0),\]

and thus

\[\int_{t_0}^{\beta_v t_0} \phi_x^2 (-v_1 t, t) dt + \int_{t_0}^{\alpha_v t_0} \phi_x^2 (v_2 t, t) dt \leq \frac{4 M (1 + v_{\text{max}})}{(1 - v_{\text{max}})^2} E_v (t_0). \quad (67)\]

Since \(\min \{\alpha_v, \beta_v\} > 1\), then we can always choose the integer \(M\) such that \((\min \{\alpha_v, \beta_v\})^M t_0 \geq t_0 + T\). Thus (22) holds with

\[\hat{K}_v (T) := 4 M (1 + v_{\text{max}}) / (1 - v_{\text{max}}^2).\]

• The left-hand side of (61), for \(M = 1\), yields

\[4 (1 - v_{\text{max}}) t_0 E_v (t_0) \leq (1 - v_{\text{min}}^2)^2 \max \{\alpha_v, \beta_v\} t_0 \int_{t_0}^{\max \{\alpha_v, \beta_v\} t_0} \phi_x^2 (-v_1 t, t) + \phi_x^2 (v_2 t, t) dt\]

and thus inequality (23) holds for \(T = \max \{\alpha_v, \beta_v\} t_0 - t_0\) and therefore for every \(T \geq \max \{\alpha_v, \beta_v\} t_0 - t_0\) as well. Of course we have

\[\max \{\alpha_v, \beta_v\} t_0 - t_0 = \max \left\{ \frac{L_0}{(1 - v_1)}, \frac{L_0}{(1 - v_2)} \right\} = \frac{L_0}{(1 - v_{\text{max}})} = \hat{T}_v. \quad (68)\]

To show that (18) does not hold for \(T < \hat{T}_v\), let us assume that \(v_2 \geq v_1\). The other case can be treated similarly. We have then \(\max \{\alpha_v, \beta_v\} = \alpha_v\) and \(\hat{T}_v = (\alpha_v - 1) t_0\). Let \(\varepsilon > 0, \varepsilon \in (0, 1)\), and consider again a function \(g \in L^2 (t_0, \alpha_v \beta_v t_0, dt/t)\), non identical null, satisfying (50) and (51) with a support satisfying this time

\[\text{supp} (g) \subset C ((1 - \varepsilon) \alpha_v t_0, \alpha_v t_0). \quad (69)\]

The \(c_n\) are defined as in (52) and \(\tilde{\phi}_x, \tilde{\phi}_1\) are chosen to satisfy (53) on \((-v_1 t_0, V_2 t_0)\). The solution of \((WP)\) corresponding to these initial conditions still satisfies

\[\phi_x (v_2 t, t) = \frac{2 \pi \sqrt{2 L_v}}{(1 - v_2^2)} g (t) t. \quad (70)\]
Thanks to (63) and the fact that 

$$e^{-i\pi \kappa v \log(1-v_1)} = e^{i\pi \kappa v \log(1+v_2)\alpha \epsilon t},$$

we have

$$\phi_x(-v_1 t, t) = \frac{2\pi \sqrt{2\kappa \nu}}{(1-v_1^2)^{\frac{1}{2}}} \sum_{n \in \mathbb{Z}^*} i\kappa n e^{i\pi \kappa v \log(1+v_2)\sqrt{\kappa \nu}} e^{i\pi \kappa v \log(\alpha \epsilon t)}$$

$$= \frac{2\pi \sqrt{2\kappa \nu}}{(1-v_1^2)^{\frac{1}{2}}} \sum_{n \in \mathbb{Z}^*} g_n \sqrt{\kappa \nu} e^{i\pi \kappa v \log(\alpha \epsilon t)},$$

hence

$$\phi_x(-v_1 t, t) = \frac{2\pi \sqrt{2\kappa \nu}}{(1-v_1^2)^{\frac{1}{2}}} g(\alpha \epsilon t).$$

(71)

Taking the squares in (70) and (71), summing up and integrating on the interval 

$$(t_0, (1-\epsilon) \alpha \epsilon t_0),$$

it comes that

$$\int_{t_0}^{(1-\epsilon) \alpha \epsilon t_0} \phi_x^2(-v_1 t, t) dt + \int_{t_0}^{(1-\epsilon) \alpha \epsilon t_0} \phi_x^2(v_2 t, t) dt$$

$$= \frac{8\pi^2 \kappa \nu}{(1-v_1^2)^{\frac{1}{2}}} \int_{t_0}^{(1-\epsilon) \alpha \epsilon t_0} g^2(\alpha \epsilon t) \frac{dt}{t^2} + \frac{8\pi^2 \kappa \nu}{(1-v_1^2)^{\frac{1}{2}}} \int_{t_0}^{(1-\epsilon) \alpha \epsilon t_0} g^2(t) \frac{dt}{t^2}.$$  

(72)

Clearly, the last integral vanishes since supp($g$) $\cap$ $(t_0, (1-\epsilon) \alpha \epsilon t_0) = \emptyset$ for all $\epsilon > 0$. In addition, we have

$$\int_{t_0}^{(1-\epsilon) \alpha \epsilon t_0} g^2(\alpha \epsilon t) \frac{dt}{t^2} = \alpha \nu \int_{\alpha \epsilon t_0}^{(1-\epsilon) \alpha \epsilon t_0} g^2(s) \frac{ds}{s^2}.$$  

To evaluate this integral, we recall that the set \( \left\{ \sqrt{\kappa \nu} e^{i\pi \kappa v \log \frac{t}{2}} \right\}_{n \in \mathbb{Z}} \) is also an orthonormal basis for $L^2(\alpha \epsilon t_0, \alpha \epsilon t_0, ds/ds)$. Then, the function $g$ that is given by the series (51), can also be considered as a function in $L^2(\alpha \epsilon t_0, \alpha \epsilon t_0, ds/ds)$. In particular, due to (69) we have necessarily

$$\text{supp } (g) \subset \subset \left( (1-\epsilon) \alpha \epsilon t_0, \alpha \epsilon t_0, ds/ds \right).$$

Since $\beta_v > 1$, then $\alpha \epsilon t_0^2 < \alpha \epsilon t_0^2 \beta_v t_0$ and by consequence

$$\text{supp } (g) \cap (\alpha \epsilon t_0, (1-\epsilon) \alpha \epsilon t_0) = \emptyset, \text{ for all } \epsilon > 0.$$  

This means that

$$\int_{\alpha \epsilon t_0}^{(1-\epsilon) \alpha \epsilon t_0} g^2(s) \frac{ds}{s^2} = 0$$

and going back to (72), we deduce that

$$\int_{t_0}^{(1-\epsilon) \alpha \epsilon t_0} \phi_x^2(-v_1 t, t) + \phi_x^2(v_2 t, t) dt = 0, \text{ for all } \epsilon > 0.$$  

Thus, the observability inequality (23) does not hold for every $T < T_v$. \( \square \)

**Remark 9.** In the time interval $(\min \{ \alpha_v, \beta_v \} t_0, \max \{ \alpha_v, \beta_v \} t_0)$, we may only observe a fraction of the initial energy of the wave. To see this, consider a wave composed of two disturbances with small initial supports near the two endpoints $x = -v_1 t_0, x = v_2 t_0$. Then, one disturbance reaches the boundary with the smallest speed at a time close to $\min \{ \alpha_v, \beta_v \} t_0$, and the other disturbance reach the boundary with the highest speed at a time close to $\max \{ \alpha_v, \beta_v \} t_0$, see Figure 3 where the case $v_1 < v_2$ is represented.
5.2. Controllability at both endpoints. As mentioned in Subsection 4.2, it suffices to show the exact null-controllability of Problem (CWP2) using two control functions acting at the two endpoints.

**Theorem 5.2.** Under the assumptions (3) and (24), Problem (CWP2) is exactly controllable at the two endpoints \( x = -v_1 t, x = v_2 t \) for \( T \geq T_v \). Moreover, we can choose two controls \( f_1, f_2 \) satisfying

\[
\int_{t_0}^{t_0+T} f_1^2(t) \, dt, \quad \int_{t_0}^{t_0+T} f_2^2(t) \, dt \leq \tilde{K}_v(T) E_v(t_0),
\]

where \( \tilde{K}_v(T) \) is a constant depending on \( v_1, v_2 \) and \( T \).

Conversely, if \( T < T_v \), Problem (CWP2) is not controllable at both endpoints \( x = -v_1 t \) and \( x = v_2 t \).

**Proof.** We argue as in the proof of Theorem 4.2. Let \( \eta \) be the solution of the backward problem

\[
\begin{align*}
\eta_t - \eta_{xx} &= 0, \quad \text{in } Q_{t_0+T}, \\
\eta(-v_1 t, t) &= \phi_x(-v_1 t, t), \quad \eta(v_2 t, t) = \phi_x(v_2 t, t), \quad \text{for } t \in (t_0, T), \\
\eta(x, t_0 + T) &= 0, \quad \eta_t(x, t_0 + T) = 0, \quad \text{for } x \in I_{t_0+T}.
\end{align*}
\]

We obtain then a linear map

\[
\Lambda_2: H^1_0(I_{t_0}) \times L^2(I_{t_0}) \longrightarrow H^{-1}(I_{t_0}) \times L^2(I_{t_0})
\]

\[
(\phi^0, \phi^1) \longmapsto (\eta_t(t_0), -\eta(t_0)).
\]

The solution of (74), in the transposition sense, satisfies

\[
-\langle \eta_t(t_0), \phi^0 \rangle_{H^1_0(I_{t_0})} + \int_{-v_1 t_0}^{v_2 t_0} \eta(t_0) \phi^1 \, dx \\
+ v_1 \int_{t_0}^{t_0+T} \eta(-v_1 t, t) \phi_t(-v_1 t, t) \, dt - \int_{t_0}^{t_0+T} \eta(-v_1 t, t) \phi_x(-v_1 t, t) \, dt \\
+ v_2 \int_{t_0}^{t_0+T} \eta(v_2 t, t) \phi_t(v_2 t, t) \, dt + \int_{t_0}^{t_0+T} \eta(v_2 t, t) \phi_x(v_2 t, t) \, dt = 0. \tag{75}
\]
Taking into account that $\phi_t(-v_1 t, t) = v_1 \phi_x(v_2 t, t)$ and $\phi_t(v_2 t, t) = -v_2 \phi_x(v_2 t, t)$, then we can rewrite (75) as

$$0 = -\langle \eta(t_0), \phi^0 \rangle_{H^1_0(I_{t_0})} + \int_{t_0}^{v_2 t_0} \eta(t_0) \phi^1 \, dx + (1 - v_1^2) \int_{t_0}^{t_0 + T} \phi_x^2(-v_1 t, t) \, dt + (1 - v_2^2) \int_{t_0}^{t_0 + T} \phi_x^2(v_2 t, t) \, dt,$$

i.e.,

$$\langle A_2(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle = \langle \eta(t_0), \phi^0 \rangle_{H^1_0(I_{t_0})} - \int_{v_1 t_0}^{v_2 t_0} \eta(t_0) \phi^1 \, dx = (1 - v_1^2) \int_{t_0}^{t_0 + T} \phi_x^2(-v_1 t, t) \, dt + (1 - v_2^2) \int_{t_0}^{t_0 + T} \phi_x^2(v_2 t, t) \, dt.$$

Thanks to Theorem 1.4, we have

$$\frac{4(1 - v_{\max}) (1 - v_{\max}^2)}{\max\{\alpha_v, \beta_v\} (1 - v_{\min}^2)} E_v(t_0) \leq \langle A_2(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle \leq (1 - v_{\min}^2) K_v(T) E_v(t_0),$$

for $T \geq \tilde{T}_v$. This means that $A_2$ is an isomorphism for $T \geq \tilde{T}_v$ and thus $(\phi^0, \phi^1)$ can be determined such that the control $f_1 = \phi_x(-v_1 t, t)$ and $f_2 = \phi_x(v_2 t, t)$ drive the solution of $(CWP2)$ from the initial data $y^0, y^1$ to $y(t_0 + T) = y_0(t_0 + T) = 0$.

Finally, the argument used at the end of the proof of Theorem 4.2 can be easily adapted to show that the non observability of $(WP)$ at the two endpoints, for $T < \tilde{T}_v$, implies the non controllability of $(CWP2)$ for $T < \tilde{T}_v$. 

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