Using Green functions to solve potentials in electrostatics

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Abstract

In this paper, we summarize the technique of using Green functions to solve electrostatic problems. We start by deriving the electric potential in terms of a Green function and a charge distribution. We then provide a variety of example problems in spherical, Cartesian, and cylindrical coordinates. For a given coordinate system, we derive the corresponding Green function for some geometries, and then place an arbitrary charge distribution in the region; we then calculate the corresponding electric potential.

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0.1 Introduction

In §1 we derive the electric potential in terms of Green functions, in §2, §3, and §4 we derive Green functions and solve for potentials in spherical, Cartesian, and cylindrical coordinates respectively. §5 contains closing remarks and §6 contains a set of miscellaneous geometries and potentials.

We use a variety of texts in this paper to help with derivations and example problems. These are listed on the references page and cited in-line throughout the paper.

The notation we will be using for this paper is as follows. The position vector, \( \mathbf{r} \) is an arbitrary point in space where the potential is being solved. The vector \( \mathbf{r}' \) is a point on the geometry of the object that is being considered. \( \Phi \) will denote the electric potential function of some space. Finally, \( G(\mathbf{r},\mathbf{r}') \) denotes the Green function of two vectors \( \mathbf{r} \) and \( \mathbf{r}' \).

1 Potential in Terms of a Green Function

The idea behind a Green function for electrostatics, is that any electrostatic problem, if a) a Green function for the geometry exists, and b) The potential on the boundary of this geometry (Dirichlet condition), or the normal of the electric field on this geometry (Neumann condition) is specified, then the electrostatic...
problem has essentially been solved. From this electric potential, almost all other necessary quantities (like electric field) can be solved for (using \( E = -\nabla \Phi \)). This derivation will be on finding the solution to the Poisson equation which is:

\[
\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}
\]

(1)

where \( \Phi \) is the electrostatic potential. Using Green functions, the solution to this potential for any set boundary and charge density can be found. Starting from the divergence theorem:

\[
\int_V d^3r (\nabla \cdot A) = \int_S dS \cdot A
\]

We can choose \( A = \phi \nabla \psi \) to obtain:

\[
\int_V d^3r (\nabla \cdot (\phi \nabla \psi)) = \int_S dS \cdot \phi \nabla \psi
\]

By interchanging \( \psi \) and \( \phi \) and taking the difference, we reach:

\[
\int_V d^3r (\phi \nabla^2 \psi - \psi \nabla^2 \phi) = \int_S dS \cdot (\phi \nabla \psi - \psi \nabla \phi)
\]

(2)

Allowing \( \Phi \) to be \( \phi \) the function that satisfies eq. (1), and \( G(\mathbf{r}, \mathbf{r}') \) to be \( \psi \), where:

\[
\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi \delta(\mathbf{r} - \mathbf{r}')
\]

(3)

The solution to eq. (3) is \( G(\mathbf{r}, \mathbf{r}') = (\mathbf{r} - \mathbf{r}')^{-1} + F(\mathbf{r}, \mathbf{r}') \) where \( \nabla^2 F(\mathbf{r}, \mathbf{r}') = 0 \). The necessity of the term \( F(\mathbf{r}, \mathbf{r}') \) will become apparent later. Using eq. (1) and eq. (3), the equation becomes:

\[
\int_V d^3r \left[ \Phi \left[ -4\pi \delta(\mathbf{r} - \mathbf{r}') \right] - G \left( -\frac{\rho(\mathbf{r}')}{\varepsilon_0} \right) \right] = \int_S dS \cdot (\Phi \nabla G - G \nabla \Phi)
\]

Solving for \( \Phi \), we obtain the general form of:

\[
\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V G(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}') d^3\mathbf{r}' - \frac{1}{4\pi} \int_S dS \cdot (\Phi \nabla G - G \nabla \Phi)
\]

(4)

The necessity of the \( F(\mathbf{r}, \mathbf{r}') \) becomes apparent in eq. (4). \( F \) allows for an extra degree of freedom with which to cancel one of the terms in \( (\Phi \nabla G - G \nabla \Phi) \). Only one term must be chosen in order to avoid redundancy. The right hand side of eq. (4) will change based on the boundary conditions given, i.e. whether \( \Phi \) or \( \nabla \Phi \cdot \hat{n} \) is given. This will determine whether the problem will be subject to Dirichlet conditions or Neumann conditions, respectively. In this paper, we will only be dealing with Dirichlet conditions, this allows us to work the electrostatic problem under the assumption we are given the potential \( \Phi \) on the surface of the object. Therefore, we can choose \( \nabla^2 G_D(\mathbf{r}, \mathbf{r}') = 0 \) on the surface, \( S \). This simplifies eq. (4) to the following:

\[
\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V G_D(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}') d^3\mathbf{r}' - \frac{1}{4\pi} \int_S (\Phi(\mathbf{r}') \nabla G_D(\mathbf{r}, \mathbf{r}') \cdot \hat{n} dS)
\]

(5)

This is the main equation we will be using to solve the electrostatics problems. Although we will not be solving any Neumann condition problems, the corresponding Potential equation is given here:

\[
\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V G_N(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}') d^3\mathbf{r}' + \frac{1}{4\pi} \int_S G_N(\mathbf{r}, \mathbf{r}') \nabla \Phi(\mathbf{r}') \cdot \hat{n} dS + \langle \Phi \rangle_S
\]

(6)

This can be derived from eq. (3) and \( \int_S \partial G_N / \partial n' dS = 4\pi \). Whereas in Dirichlet conditions, one of the terms in \( (\Phi \nabla G - G \nabla \Phi) \) evaluates to 0, in Neumann conditions, one of the terms evaluates to the average potential across the surface of the geometry. This derivation is left as an exercise to the reader.
2 Spherical Green Functions

2.1 Green Function in Spherical Coordinates

The sphere can be understood as a geometry centered at the origin with radius $a$ and surface potential $\Phi_S$ with some arbitrary charge distribution in space $\rho$. Using a test charge and method of images, it can be found that the $F(r,r')$ which satisfies the condition $\nabla^2 F(r,r') = 0$ applies in spherical coordinates in the following way. The Green function is thus:

$$G(r,r') = \frac{1}{|r-r'|} - \frac{a}{r'|r-(a/r')^2 r|}$$

(7)

This is a good opportunity to look at the term $|r-r'|^{-1/2}$. There are a number of ways to expand this, one of which is $(r^2 + r'^2 - 2rr'\cos \gamma)^{-1/2}$ where $\gamma$ is the angle between $r$ and $r'$. From here, $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')$. Without loss of generality, we can align our point of interest along the $z$-axis such that $\theta' = \phi' = 0$. While this is a valid way of expanding the terms of the Green function, it is often much easier if the Green function is expanded in spherical harmonics via the addition formula.

$$\frac{1}{|r-r'|} = \frac{1}{r} \left( 1 + \frac{r'^2}{r^2} - \frac{2r'}{r} \cos \gamma \right)^{-\frac{1}{2}} = \frac{1}{r} \sum_{l=0}^{\infty} \left( \frac{r'}{r} \right)^l P_l(\cos \gamma)$$

In the case where $r' < r$. In the opposite case, it is sufficient to interchange the two. It is now (and for the rest of the problem) convenient to change notation such that $r_\text{<} = \min(r,r')$ and $r_\text{>} = \max(r,r')$. The addition theorem for spherical harmonics then gives the following:

$$P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^m(\theta,\phi) Y_l^m(\theta',\phi')^*$$

Thus, along with our previous notation, we have the following:

$$\frac{1}{|r-r'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{1}{r_\text{<}^l} \frac{r_\text{>}^l}{r_\text{>}^{l+1}} Y_l^m(\theta,\phi) Y_l^m(\theta',\phi')^*$$

(8)

As such, inserting these equations into eq. (7) gives two Green functions. One of which applies for the interior of the sphere and the other which applies for the exterior of the sphere. The external Green function (note that this one function represents two different scenarios, $r < r'$ and $r > r'$) is therefore:

$$G_{\text{outside}}(r,r') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left( \frac{r_\text{<}^l}{r_\text{>}^{l+1}} - \frac{r_\text{>}^l}{r_\text{>}^{l+1}} \right) Y_l^m(\theta,\phi) Y_l^m(\theta',\phi')^*$$

(9)

The corresponding internal Green function is:

$$G_{\text{inside}}(r,r') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left( \frac{r_\text{<}^l}{r_\text{<}^{l+1}} - \frac{r_\text{>}^l}{r_\text{>}^{l+1}} \right) Y_l^m(\theta,\phi) Y_l^m(\theta',\phi')^*$$

(10)

2.2 A Line of Charge

Find the potential due to a charge $Q$ uniformly distributed over a line of length $2b$ running from the north to south pole of a grounded hollow sphere of radius $b$ inside the sphere. See fig. (11), §3.10

We get the Green function from eq. (10). If we put the line on the $z$ axis, we have symmetry with respect to $\phi$, so $m = 0$. We have

$$G(r,r') = 4\pi \sum_{l=0}^{\infty} \frac{P_l(\cos \theta) P_l(\cos \theta')^*}{2l+1} \frac{r_\text{<}^l}{r_\text{>}^{l+1}} \left( \frac{1}{r_\text{<}^l} - \frac{r_\text{>}^l}{b^{2l+1}} \right)$$

(11)
We use some Dirac delta functions to describe the charge density \( \rho(r') \) within the sphere. We want \( \rho(r') \) to blow up at \( \theta' = 0 \) and \( \theta' = \pi \). We can use \( \cos \theta' \) and write

\[
\rho(r') = A[\delta(\cos \theta' + 1) + \delta(\cos \theta' - 1)],
\]

where \( A \) is some factor (not necessarily a constant) which is determined by the fact that total charge is \( Q \) and is uniformly distributed over the line. We compute \( A \) by integrating over the volume enclosed by the sphere of radius \( r' < b \). We already know that the result of this operation should be \( r'Q/b \).

\[
\frac{r'Q}{b} = \int_0^{2\pi} \int_0^\pi \int_0^{r'} A[\delta(\cos \theta' + 1) + \delta(\cos \theta' - 1)]r^2 \sin \theta' \, dr' \, d\theta' \, d\phi'
\]

\[
= 2\pi \int_0^1 \int_0^{r'} A[\delta(u + 1) + \delta(u - 1)]r^2 \, dr' \, du
\]

\[
= 4\pi \int_0^{r'} Ar^2 \, dr'.
\]

For this to work, we can say \( A = B/r^2 \) where \( B \) is a constant. This will work because we want that integral to evaluate to something times \( r' \) and so the integral should be constant.

\[
4\pi Br' = \frac{r'Q}{b}
\]

\[
B = \frac{Q}{4\pi b}
\]

So

\[
\rho(r') = \frac{Q}{4\pi br^2} [\delta(\cos \theta' + 1) + \delta(\cos \theta' - 1)].
\]

When we substitute this and eq. (11) into eq. (5) we get

\[
\Phi(r) = \frac{Q}{8\pi\epsilon_0 b} \sum_{l=0}^{\infty} [P_l(1) + P_l(-1)]P_l(\cos \theta) \int_0^b r' \left[ \left( \frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \right] dr'.
\]

We split the integral into the two ranges \( 0 \leq r' < b \), and \( r \leq r' < b \). The \( < \) or \( > \) in the subscript tells us which integral each \( r \) belongs to. This yields, for the integral only,

\[
\left( \frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \int_0^r r' \, dr' + \int_r^b \frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \, dr' = \frac{2l + 1}{l(l+1)} \left[ 1 - \left( \frac{r}{b} \right)^l \right].
\]

When \( l = 0 \), the above expression involves dividing by \( 0 \). So to find the \( l = 0 \) term, we use L’Hospital’s rule to take the limit as \( l \to 0 \) and get \( \ln(b/r) \). \( P_l(-1) = (-1)^l P_l(1) \), so only \( l \in \mathbb{E} \) terms will survive in the expression for \( \Phi(r) \). Putting this all together we get

\[
\Phi(r) = \frac{Q}{4\pi\epsilon_0 b} \left\{ \ln \left( \frac{b}{r} \right) + \sum_{l \in \mathbb{E}} \frac{l + 1}{l(l+1)} \left[ 1 - \left( \frac{r}{b} \right)^l \right] P_l(\cos \theta) \right\}.
\] (12)

Because of the \( \ln \left( b/r \right) \) term, the potential has the expected behavior at \( r \ll b \); it diverges to positive infinity.
2.3 A Ring of Charge

Find the potential due to a charge $Q$ uniformly distributed over a ring of radius $a$. (II, §3.10)

To evaluate this equation similar to the previous problem we will stick the ring inside a grounded conducting sphere of radius $b$ and will just make take the limit as $b \to \infty$. See fig. 1b.

The volumetric density can be expressed using Dirac delta functions as the following expression:

$$
\rho(r') = \frac{Q}{2\pi a^2} \delta(r' - a) \delta(\cos \theta')
$$

(13)

This will produce a ring of charge on the $x-y$ plane of radius $a$ which is exactly what we want.

Due to azimuthal symmetry, we know that for the general Laplace equation in spherical coordinates only terms with $m = 0$ will remain. This will give us the following potential equations:

$$
\Phi_{r>a}(r) = \frac{1}{4\pi \varepsilon_0} \int_V G(r, r') \rho(r') d^3 r = \frac{Q}{4\pi \varepsilon_0} \sum_{l=0}^{\infty} P_l(0) a^l \left( \frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) P_l(\cos \theta)
$$

$$
\Phi_{r<a}(r) = \frac{1}{4\pi \varepsilon_0} \int_V G(r, r') \rho(r') d^3 r = \frac{Q}{4\pi \varepsilon_0} \sum_{l=0}^{\infty} P_l(0) r^l \left( \frac{1}{a^{l+1}} - \frac{a^l}{b^{2l+1}} \right) P_l(\cos \theta)
$$

$P_{2n}(o)$ can be written as a fraction with factorials. Let us set $l = 2n$. Also, we can see that since $b \to \infty$ cancels out one of the fractions in the series, we get the following equations.

$$
\Phi_{r>a}(r) = \frac{Q}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n - 1)!!}{2^n n!} a^{2n} \left( \frac{1}{r^{2n+1}} \right) P_{2n}(\cos \theta)
$$

$$
\Phi_{r<\theta}(r) = \frac{Q}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n - 1)!!}{2^n n!} a^{2n} \left( \frac{1}{a^{2n+1}} \right) P_{2n}(\cos \theta)
$$

(14)

This matches with the general solution of a ring in free space that can be derived in other methods. This shows the ability to use the Green function even for simpler problems.

2.4 Surface Potential Contributions

Note that in the previous two problems, it has been assumed that the surface of the sphere has been grounded. However, if the surface of the sphere has some potential $V(\theta', \phi')$, it is simply sufficient to subtract the surface contribution from the volume term (second term of eq. [5]).
2.4.1 Interior Surface Contribution

Let us first look for general values of a sphere with surface potential $V(\theta', \phi')$. The Green function for the inside of the sphere, eq. (10) becomes the following,

$$G_{\text{inside}}(r, r') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left( \frac{r'^l}{r^{l+1}} - \frac{b^{l+1}}{r^{l+1}} \right) Y_l^m(\theta, \phi) Y_l^m(\theta', \phi')$$

with $0 < r < r'$. Taking the derivative with respect to $r'$ and evaluating at $r' = b$ where $b$ is the radius of the sphere give the following:

$$\frac{\partial G}{\partial r'}|_{r'=b} = -\frac{4\pi}{b^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \frac{r}{b} \right)^l Y_l^m(\theta, \phi) Y_l^m(\theta', \phi')$$

We can insert this into the second hand term of eq. (5) to get the contribution from the potential on the surface of the sphere.

$$-\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \int V(\theta', \phi') Y_l^m(\theta', \phi')^* d\Omega' \right] \left( \frac{r}{b} \right)^l Y_l^m(\theta, \phi)$$

Note that $d\Omega' = \sin(\theta') d\theta' d\phi'$. Therefore, eq. (5) becomes:

$$\Phi(r) = \frac{1}{4\pi\epsilon_0} \int_V G_D(r, r') \rho(r') d^3r' + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int V(\theta', \phi') Y_l^m(\theta', \phi')^* d\Omega' \left( \frac{r}{b} \right)^l Y_l^m(\theta, \phi)$$  \hspace{1cm} (15)

2.4.2 Exterior Surface Contribution

We can accomplish the same derivation for outside the sphere.

$$G_{\text{outside}}(r, r') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left( \frac{r'^l}{r^{l+1}} - \frac{b^{l+1}}{r^{l+1}} \right) Y_l^m(\theta, \phi) Y_l^m(\theta', \phi')$$

with $r' < r < \infty$. Once again,

$$\frac{\partial G}{\partial r'}|_{r'=b} = \frac{4\pi}{br} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \frac{b}{r} \right)^l Y_l^m(\theta, \phi) Y_l^m(\theta', \phi')$$

which transforms the right hand side to get the final potential:

$$\Phi(r) = \frac{1}{4\pi\epsilon_0} \int_V G_D(r, r') \rho(r') d^3r' - \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int V(\theta', \phi') Y_l^m(\theta', \phi')^* d\Omega' \left( \frac{b}{r} \right)^{l+1} Y_l^m(\theta, \phi)$$  \hspace{1cm} (16)

Note that for any charge distribution $\rho(r')$, if there is a nonzero potential on the surface of the sphere, all that must be done is to simply subtract the surface voltage component from the 1st term of eq. (5).

2.4.3 Concentric Spheres Surface Contribution

A far more general Green function exists. For concentric spheres, with inner radius $a$ and outer radius $b$, there is the Green function as follows:

$$G(r, r') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( Y_l^m(\theta, \phi) Y_l^m(\theta', \phi')^* \right) \left( \frac{r'^l}{r^{l+1}} - \frac{a^{l+1}}{r^{l+1}} \right) \left( \frac{1}{r_{r'>}} - \frac{r_{r'<}^{l+1}}{b^{l+1}} \right)$$  \hspace{1cm} (17)
Note that if we allow $a \to 0$ or $b \to \infty$, we recover the Green functions above for those respective cases. We can evaluate the surface potential contributions in the following way. Let us say the sphere with radius $a$ has surface potential $V_a(\theta', \phi')$ and the sphere with radius $b$ has potential $V_b(\theta', \phi')$. It is then sufficient to solve the problem by separating it into two parts. Solving the problem as if the potential of $b$ is $0$ (with the previously stated green function), and then adding the surface contribution from the interior example (the second term of eq. (15)). Note we are excluding the volume component.

Let us take a look at solving the problem where there is a zero potential $V_a(\theta', \phi')$ and $V_b(\theta', \phi') = 0$. Eq. (17) becomes

$$G(r, r') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\theta, \phi) Y_l^m(\theta', \phi')^* \left( r - \frac{a^{l+1}}{r^{l+1}} - \frac{b^{l+1}}{b^{l+1}} \right)$$

But

$$\frac{\partial G}{\partial \theta} = \frac{\partial G}{\partial \phi}\big|_{r'=a} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \frac{a^{l-1} r^{-1} - b^{l+1}}{a^{l+1} - b^{l+1}} \right) Y_l^m(\theta, \phi) Y_l^m(\theta', \phi')^*$$

Thus

$$\Phi(r)^d_{\text{SC}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \int V_a(\theta', \phi') Y_l^m(\theta', \phi')^* d\Omega \right) \left( \frac{a^{l-1} r^{-1} - b^{l+1}}{a^{l+1} - b^{l+1}} \right) Y_l^m(\theta, \phi)$$

(18)

### 2.4.4 Surface Potential Contribution Example

Let us take a sphere of radius $b$ with $V(\theta', \phi') = V_o(\cos \phi' \sin \theta' + \cos \theta')$. Due to our choice to use spherical harmonics as well as the orthogonality of such, solving for the surface potential contribution is easy. Looking at the integral first, as they are shared between the two regions (internal and external):

$$\int_0^{2\pi} \int_0^\pi V(\theta', \phi') Y_l^m(\theta', \phi')^* \sin(\theta') d\theta' d\phi' = V_o \int_0^{2\pi} \int_0^\pi (\cos \phi' \sin \theta' + \cos \theta') Y_l^m(\theta', \phi')^* \sin(\theta') d\theta' d\phi'$$

However,

$$\cos \phi' \sin \theta' = \sqrt{\frac{4\pi}{3}} Y_1^1 \quad \text{and} \quad \cos \theta' = \sqrt{\frac{4\pi}{3}} Y_1^0$$

Which transforms the integral into the following:

$$V_o \sqrt{\frac{4\pi}{3}} \int_0^{2\pi} \int_0^\pi (Y_1^1 + Y_1^0) Y_l^m(\theta', \phi')^* \sin(\theta') d\theta' d\phi' = V_o \sqrt{\frac{4\pi}{3}} (\delta_{l1} \delta_{m1} + \delta_{l1} \delta_{m0})$$

From here, we must take it by case. However, this is still easy due to the properties of Dirac delta functions. For the outside of the sphere, we get the following:

$$\int V(\theta', \phi') Y_l^m(\theta', \phi')^* d\Omega = \sqrt{\frac{4\pi}{3}} \left( \cos \phi' \sin \theta' + \cos \theta' \right)$$
Therefore, the magnitude of the contribution from the voltage on the surface of the sphere is as follows:

\[
\Phi_{\text{outside}}(r)_{\text{SC}} = V_o \left( \frac{b}{r} \right)^2 \left[ \cos \phi' \sin \theta' + \cos \theta' \right]
\]

\[
\Phi_{\text{inside}}(r)_{\text{SC}} = V_o \frac{r}{b} \left[ \cos \phi' \sin \theta' + \cos \theta' \right]
\]

(19)

3 Cartesian Green Functions

3.1 Green Functions in Cartesian Coordinates

Consider a long grounded rectangular pipe with axis lying on the z-axis, with sides \(x = 0\), \(x = a\), \(y = 0\), and \(y = b\). Now, consider a point charge \(q\) within this region located at \((x', y')\).

Using Dirichlet boundary conditions, find the Green function to solve Poisson’s Equation in the region. ([21], §C.4)

![Figure 2](image.png)

Since we are assuming translational symmetry in the z-direction, this problem reduces to two dimensions. The potential satisfies Poisson’s Equation

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -4\pi \sigma(x, y)
\]

where the right hand side describes the sources for the potential (in our case, a point charge). The Green function should therefore satisfy the differential equation

\[
\frac{\partial^2 G(r, r')}{\partial x^2} + \frac{\partial^2 G(r, r')}{\partial y^2} = -4\pi \delta(r - r')
\]

The delta function as divides the rectangular region into two pieces. Without loss of generality, we can choose to divide in the y-direction. Let the boundary between the two regions be \(y = y'\). Define "Region I" to be the region within the pipe below the boundary and "Region II" to be the region within the pipe above the boundary. See fig. 2.

Let \(G = X(x)Y(y)\). Then a simple separation of variables yields the general form of Green function for this problem.

\[
G_I(r, r') = \sum_{n=1}^{\infty} C_n(x', y') \sin \left( \frac{n\pi x}{a} \right) \sinh \left( \frac{n\pi y}{a} \right)
\]

\[
G_{II}(r, r') = \sum_{n=1}^{\infty} D_n(x', y') \sin \left( \frac{n\pi x}{a} \right) \sinh \left( \frac{n\pi (b - y)}{a} \right)
\]
where \( G_I \) corresponds to the region \( 0 \leq y < y' \) and \( G_{II} \) corresponds to the region \( y' < y \leq b \). Continuity at \( y = y' \) gives us the condition

\[
\sum_{n=1}^{\infty} C_n(x', y') \sin \left( \frac{n\pi x}{a} \right) \sinh \left( \frac{n\pi y}{a} \right) = \sum_{n=1}^{\infty} D_n(x', y') \sin \left( \frac{n\pi x}{a} \right) \sinh \left( \frac{n\pi (b - y')}{a} \right)
\]

By orthogonality of the eigenfunctions

\[
C_n(x', y') \sinh \left( \frac{n\pi y}{a} \right) = D_n(x', y') \sinh \left( \frac{n\pi (b - y')}{a} \right)
\]

This allows us to define a new "constant" \( E_n(x', y') \)

\[
E_n = \frac{C_n(x', y')}{\sinh(n\pi (b - y')/a)} = \frac{D_n(x', y')}{\sinh(n\pi y/a)}
\]

Thus, the Green function for both regions becomes

\[
G_I = \sum_{n=1}^{\infty} E_n(x', y') \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \sinh \left( \frac{n\pi (b - y')}{a} \right)
\]

\[
G_{II} = \sum_{n=1}^{\infty} E_n(x', y') \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{n\pi (b - y)}{a} \right) \sinh \left( \frac{n\pi y'}{a} \right)
\]

There is one more boundary conditions we have to implement to find \( E_n \). This can be taken from the discontinuity in the normal derivatives of the Green function at the boundary. The implementation of this boundary condition to find the coefficient is lengthy, so we will omit it and give the value of \( E_n \) instead. This yields a value of \( E_n \) such that the Green functions are

\[
G_I = \sum_{n=1}^{\infty} \frac{8}{n \sinh(n\pi b/a)} \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \sinh \left( \frac{n\pi (b - y')}{a} \right)
\]

\[
G_{II} = \sum_{n=1}^{\infty} \frac{8}{n \sinh(n\pi b/a)} \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{n\pi (b - y)}{a} \right) \sinh \left( \frac{n\pi y'}{a} \right)
\]

Here, we can note the symmetry of the function: \( G(r, r') = G(r', r) \). Now let \( y_\text{min} = \min\{y, y'\} \) and \( y_\text{max} = \max\{y, y'\} \). Therefore, the final green function can be written as

\[
G(r, r') = \sum_{n=1}^{\infty} \frac{8}{n \sinh(n\pi b/a)} \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \sinh \left( \frac{n\pi y_\text{max}}{a} \right) \sinh \left( \frac{n\pi (b - y_\text{min})}{a} \right)
\]

(20)

### 3.1.1 Cartesian Example

Now let us calculate the potential within the rectangular pipe using the derived Green function. We can use eq. 6 and neglect the “surface contribution” since the normal derivative of the Green function will be zero. The potential is therefore

\[
\Phi(r) = \frac{1}{4\pi\varepsilon_0} \iiint_V \sum_{n=1}^{\infty} \frac{8}{n \sinh(n\pi b/a)} \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \sinh \left( \frac{n\pi y_\text{min}}{a} \right) \sinh \left( \frac{n\pi (b - y_\text{max})}{a} \right) \rho(r') d^3r'
\]

where \( \rho(r') = \delta(x - x')\delta(y - y')\delta(z) \) and \( d^3r' = dx'dy'dz \). When the integrals are evaluated from \(-\infty\) to \(\infty\), the result becomes

\[
\Phi(r) = \frac{1}{4\pi\varepsilon_0} \sum_{n=1}^{\infty} \frac{8}{n \sinh(n\pi b/a)} \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \sinh \left( \frac{n\pi y_\text{max}}{a} \right) \sinh \left( \frac{n\pi (b - y_\text{min})}{a} \right)
\]
Therefore our final potential inside the rectangular pipe is

$$\Phi(r) = \frac{1}{4\pi \varepsilon_0} \sum_{n=1}^{\infty} \frac{8}{n \sinh(n \pi b / a)} \sin^2 \left( \frac{n \pi x'}{a} \right) \sinh \left( \frac{n \pi y'}{a} \right) \sinh \left( \frac{n \pi (b - y')}{a} \right)$$  \hspace{1cm} (21)

This example provides an excellent demonstration of the power of the Green function. Once the function was found for this particular problem, solving for the potential was a matter of only three lines. Solving this problem using traditional methods would require a much greater amount of calculation.

4 Cylindrical Green Functions

4.1 Green Function in Polar Coordinates

The Dirichlet Green function for the unbounded space between the planes at \( z = 0 \) and \( z = L \) allows discussion of a point charge or a distribution of charge between parallel conducting planes held at zero potential. Using cylindrical coordinates, find the Green function for this problem. ([11], [3], problem 3.17)

First, we should note our boundary conditions:

$$\Phi(0) = \Phi(2\pi) = \ldots \text{ and } G(z = 0) = G(z = L) = 0$$

The \( \Phi \) boundary conditions imply the Green function could be expanded as a Fourier series in \( e^{im\phi} \). Similarly, the \( z \) boundary condition seems to imply the use of a Fourier sine series \( \sin(n\pi z / L) \) in the \( z \) coordinate. A complete Fourier expansion in \( \phi \) and \( z \) would give

$$G(r, r') = \sum_{m=-\infty}^{\infty} g(\rho, \rho') e^{im\phi} e^{im'\phi'} \sin \left( \frac{n\pi z}{L} \right) \sin \left( \frac{n'\pi z}{L} \right)$$

The Green function equation, \( \nabla^2 G(r, r') = -4\pi \delta(r - r') \), once converted to cylindrical coordinates, becomes

$$\left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right) G(\rho, \phi, z; \rho', \phi', z') = -\frac{4\pi}{\rho} \delta(\rho - \rho') \delta(\phi - \phi') \delta(z - z')$$  \hspace{1cm} (22)

From here, it can be seen that \( m \) and \( m' \) as well as \( n \) and \( n' \) do not need to be chosen to be independent. Using the completeness relations, \( \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} = 2\pi \delta(\phi - \phi') \) and \( \sum_{n=1}^{\infty} \sin(n\pi z / L) \sin(n'\pi z / L) = \frac{L}{2} \delta(z - z') \). So our Green function is of the form

$$G(r, r') = \sum_{m=-\infty}^{\infty} g(\rho, \rho') e^{im\phi} e^{im'\phi'} \sin \left( \frac{n\pi z}{L} \right) \sin \left( \frac{n'\pi z}{L} \right)$$  \hspace{1cm} (23)

Substituting eq. 23 into eq. 22 yields

$$\left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} - \left( \frac{n\pi}{L} \right)^2 \right) g(\rho, \rho') = -\frac{4}{L\rho} \delta(\rho, \rho')$$

We can now make the substitution \( r = n\pi \rho / L \) to get a modified Bessel equation

$$\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left( 1 + \frac{m^2}{r^2} \right) \right) g(r, r') = -\frac{4}{Lr} \delta(r, r')$$

We should note that the modified Bessel function \( I_m(x) \) blows up as \( x \to \infty \) and the function \( K_m(x) \) blows up as \( x \to 0 \), which leaves us with

$$g(r, r') = \begin{cases} A I_m(r) & r < r' \\ B K_m(r) & r > r' \end{cases}$$
where coefficients A and B can be determined by the boundary conditions at \( r = r' \):
\[ g_\leq = g_\geq \text{ and } \frac{\text{d}}{\text{d}r} g_\leq = \frac{\text{d}}{\text{d}r} g_\geq + \frac{1}{r}. \]
The coefficients can be solved for and yield
\[ A = \frac{4}{L r'} I_m(r') \frac{K_m(r')}{I_m(r') K_m(r')} - I_m(r') K_m(r') \]
\[ B = \frac{4}{L r'} I_m(r') K_m(r') \]

It can be seen that the denominator of both coefficients (modified Bessel functions) satisfy the Wronskian formula, \( I'_v(r')K_v(r') - I_v(r')K'_v(r') = \frac{1}{L} \). This finally gives an equation for radial component of the Green function of this system.
\[ g(r, r') = \frac{4}{L} \begin{cases} I_m(r)K_m(r') & r < r' \\ I_m(r')K_m(r) & r > r' \end{cases} \]

Let \( \rho_\leq = \text{min}\{r, r'\} \) and \( \rho_\geq = \text{max}\{r, r'\} \). We can now convert \( r \) back to \( \rho \) and substitute this equation into the general form of Equation to get our final Green function.
\[ G(\rho, r') = \frac{4}{L} e^{im(\phi - \phi')} \sin \left( \frac{n\pi \rho}{L} \right) \sin \left( \frac{n\pi r'}{L} \right) I_m \left( \frac{n\pi \rho_\leq}{L} \right) K_m \left( \frac{n\pi \rho_\geq}{L} \right) \]

### 4.1.1 Polar Example

Let us consider a flat disk with radius \( R \) and its axis pointing in the \( z \) direction with charge density given by \( \sigma(s, \phi) = cs \sin \phi \) where \( c \) is a constant and \( s \) is the cylindrical radius. It is placed at \( z = z' \) where \( o < z' < L \). See fig. 5.

![Figure 3](image)

For Dirichlet boundary conditions, we can use eq. (5).
\[ \Phi(r) = \frac{1}{4\pi \varepsilon_0} \int_V \rho(r') G(r, r') d^3 r' - \frac{1}{4\pi} \int_S \Phi(r') \frac{\partial G}{\partial n} d\alpha' \]

Let us evaluate the volume contribution first.
\[ \Phi_V(r) = \frac{1}{4\pi \varepsilon_0} \int_V \int_{r'} (c \rho' \sin(\phi')) \frac{4}{L} e^{im(\phi - \phi')} \sin \left( \frac{n\pi \rho'}{L} \right) \sin \left( \frac{n\pi r'}{L} \right) I_m \left( \frac{n\pi \rho_\leq}{L} \right) K_m \left( \frac{n\pi \rho_\geq}{L} \right) \delta(z' - z) \rho' d\rho' d\phi' dz' \]
\[ = \frac{c}{L\pi \varepsilon_0} \int_o^R \int_{2\pi}^{2\pi} \rho' \sin\left( \frac{n\pi \rho'}{L} \right) \sin \left( \frac{n\pi r'}{L} \right) \delta(z' - z) d\rho' d\phi' \]
\[ = \frac{c}{L\pi \varepsilon_0} \sin^2 \left( \frac{n\pi z}{L} \right) \int_o^R \rho' \sin\left( \frac{n\pi \rho'}{L} \right) \sin \left( \frac{n\pi r'}{L} \right) d\rho' \]
where \( \eta = \eta(\phi, \phi') \equiv \sin(\phi')e^{im(\phi-\phi')} \) and \( \zeta = \xi(\rho, \rho') \equiv \rho'^2 I_m\left(\frac{n\pi\rho}{L}\right)K_m\left(\frac{n\pi\rho}{L}\right) \). Next, we can evaluate the surface contribution.

\[
\Phi_S(r) = -\frac{1}{4\pi} \int_0^R \int_0^{2\pi} \rho' \sin(\phi') \frac{d}{dz}\left(\frac{4}{L} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi \rho}{L}\right) K_m\left(\frac{n\pi \rho}{L}\right) \right) d\rho' d\phi'
\]

\[
= -\frac{c}{nL} \int_0^R \rho' \sin(\phi') I_m\left(\frac{n\pi \rho}{L}\right) K_m\left(\frac{n\pi \rho}{L}\right) e^{im(\phi-\phi')} \sin\left(\frac{n\pi z'}{L}\right) L \cos\left(\frac{n\pi z'}{L}\right) \rho' d\rho' d\phi'
\]

Therefore, our final potential function is

\[
\Phi(r) = \frac{c}{L\epsilon_0} \sin^2\left(\frac{n\pi z}{L}\right) \int_0^R \xi d\rho' \int_0^{2\pi} \eta d\phi' \left[ \frac{1}{\xi L\epsilon_0} \sin\left(\frac{n\pi z'}{L}\right) - \frac{1}{n\pi} \cos\left(\frac{n\pi z'}{L}\right) \right] \int_0^R \xi d\rho' \int_0^{2\pi} \eta d\phi'
\]

where \( \rho_- = \min(r, r') \) and \( \rho_+ = \max(r, r') \).

### 4.2 Cylindrical Example

A point charge \( q \) is located at \((\rho', \phi', z')\) inside a grounded cylindrical box defined by the surfaces \( z = 0, z = L, \rho = a \). Find the electric potential. ([11], problem 3.23)

We must derive a valid Green’s Function that will satisfy the equation.

\[
\nabla^2 G(r, r') = -4\pi \frac{\delta(\rho - \rho')}{\rho'} \delta(\phi - \phi') \delta(z - z')
\]

Using the completeness relations for \( \delta(\phi - \phi') \) and \( \delta(z - z') \) we can change this to be:

\[
\nabla^2 G(r, r') = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} -\frac{4}{L} e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right) \delta(\rho - \rho') e^{im\phi} \sin\left(\frac{n\pi z}{L}\right) \quad (25)
\]

Through expanding the Green’s Function in the \( e^{im\phi} \sin\left(\frac{n\pi z}{L}\right) \) term we get the following equation:

\[
\nabla^2 G(r, r') = \nabla^2 \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} e^{im\phi} \sin\left(\frac{n\pi z}{L}\right) A_{mn}(\rho|\rho', z', \phi')
\]

By applying the Laplacian in cylindrical coordinates to the function we get

\[
\nabla^2 G(r, r') = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} - \frac{n^2}{\rho^2} - \frac{m^2}{\rho^2} \right) A_{mn}(\rho|\rho', z', \phi') e^{im\phi} \sin\left(\frac{n\pi z}{L}\right) \quad (26)
\]

By setting eq. (25) and eq. (26) equal to each other and dividing by \( -\frac{4}{L} e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right) \) we get:

\[
\left( \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} - \frac{n^2}{\rho^2} - \frac{m^2}{\rho^2} \right) g_{mn}(\rho, \rho') = \frac{\delta(\rho - \rho')}{\rho'}
\]

(27)

Where \( g_{mn}(\rho, \rho') \) is the reduced Green’s function:

\[
g_{mn}(\rho, \rho') = \frac{LA_{mn}(\rho|\rho', z', \phi')}{4e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right)}
\]
By solving the differential equation given in eq. \[27\] and the following boundary conditions ($\rho = 0$ should be regular, $\rho = a$ should be vanishing, and the equation should be continuous at $\rho = \rho'$), we get the following equation:

$$g_{mn}(\rho, \rho') = C_n I_m\left(\frac{n\pi}{L}\rho\right)\left[I_m\left(\frac{n\pi}{L}\rho\right) - I_m\left(\frac{n\pi}{L}\rho\right)K_m\left(\frac{n\pi}{L}\rho\right)\right]$$

Where $\rho_c = \min(\rho, \rho')$ and $\rho_s = \max(\rho, \rho')$.

We will solve for $C_n$ using the following in-homogeneity at $\rho = \rho'$

$$\frac{d}{d\rho}g_{mn}(\rho, \rho') \bigg|_{\rho = \rho' + \epsilon} - \frac{d}{d\rho}g_{mn}(\rho, \rho') \bigg|_{\rho = \rho' - \epsilon} = \frac{1}{\rho'}$$

This will give us the following value for $C_n$:

$$C_n = \frac{I_m\left(\frac{n\pi}{L}\rho\right)}{I_m\left(\frac{n\pi}{L}\rho\right)}$$

Using all that we have done up until now we can finalize the Green function as the following:

$$G(r, r') = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{4}{L} e^{i m \phi} e^{-im\phi'} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi \rho}{L}\right)K_m\left(\frac{n\pi \rho}{L}\right) - K_m\left(\frac{n\pi \rho}{L}\right)I_m\left(\frac{n\pi \rho}{L}\right)$$

For a point charge $q$ at $r'$ the electric potential can be expressed as $\Phi(r, r') = kqG(r, r')$. This means the equation for the potential is as follows:

$$\Phi(r, r') = kq \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{4}{L} e^{i m (\phi - \phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi \rho}{L}\right)K_m\left(\frac{n\pi \rho}{L}\right) - K_m\left(\frac{n\pi \rho}{L}\right)I_m\left(\frac{n\pi \rho}{L}\right)$$

### 5 Conclusion and Remarks

Green functions are important equations as they allow us to solve for valid solutions in ordinary and partial differential equations. The Green function in this case is necessary to find the solution to Poisson’s equation. In this paper we motivated the derivation of the potential in terms of Green functions, derived such Green functions, and looked at situations in which the Green function would be useful in evaluating the electric potential of a system. Typically, they resembled the Laplace equations; however, some have charges inside the systems that make $\nabla^2 \Phi$ non-zero. Different charge distributions and different surface geometries require different Green functions. Because of this, we took a look at a diverse amount of systems and the derivation of several different Green functions in order to give an overview of the subject. Overall, this exploration of the Green function and its usefulness in solving Poisson’s equation has been fruitful and educational. Showing us that the real Green function is the friends we made along the way.

### 6 Miscellaneous Geometries

In this section, we solve a variety of example problems. Most of them are taken from \[11\] and \[21\]. We present the solution (potential function) to these problems using the Green function method, without derivation.

#### 6.1 Sphere with surface potential $V(\theta', \phi') = V_o \cos (\theta')$

Outside : $\Phi(r) = \frac{V_o a^2}{r^2} \cos \theta$.

Inside : $\Phi(r) = \frac{V_o}{a} r \cos \theta$. 

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6.2

Grounded sphere \( r = a \) with a ring of charge \( r = b \), linear charge density \( \lambda \) on the equatorial plane of the sphere where \( b > a \). \([21], \text{§C.7}\)

\[
\Phi(r) = \frac{\lambda b}{2\varepsilon_0} \sum_l \frac{(r^{2l+1} - a^{2l+1})}{r^{l+1} b^{l+1}} P_l(o) P_l(z)
\]

6.3

Cylinder with band of potential at \( z = -a \) to \( z = a \) and grounded everywhere else. \([21], \text{§C.7}\)

For \( |z| > a \) : \( \Phi(\rho, z) = V_o \sum u \frac{J_o(x_{on} p/a)}{J_1(x_{on})} \frac{2}{x_{on}} \sinh(x_{on}) e^{(-x_{on})} \).

For \( -a < z < a \) : \( \Phi(\rho, z) = 2V_o \sum u \frac{J_o(x_{on} p/a)}{x_{on} J_1(x_{on})} \left[ 1 - e^{(x_{on})} \right] \).

6.4

Infinitely thin, grounded, conducting plane (at \( z = 0 \)) with a circular hole of radius \( a \) cut in it, and with the electric field far from the hole being normal to the plane, constant in magnitude, and having different values on either side of the plane. \( E_z = -E_o \) for \( z > 0 \), \( E_z = -E_i \) for \( z < 0 \). \([11], \text{§3.13}\)

\[
\Phi = \begin{cases} 
E_o z + \Phi^{(1)}(z > 0) \\
E_i z + \Phi^{(1)}(z < 0)
\end{cases}
\]

with \( \Phi^{(1)}(\rho, z) = \frac{(E_o - E_i) a}{\pi} \left[ \sqrt{\frac{R - \lambda}{2}} - \frac{|z|}{a} \arctan \left( \sqrt{\frac{2}{R + \lambda}} \right) \right] \),

where \( \lambda = \frac{1}{a^2} (z^2 + \rho^2 - a^2) \), and \( R = \sqrt{\lambda^2 + 4z^2 / a^2} \).

6.5

Spherical surface of radius \( R \) with charge uniformly distributed over its surface with density \( Q/4\pi R^2 \), except for a spherical cap at the north pole defined by cap \( \cos \theta = a \). \([11], \text{problem 3.2}\)

\[
\Phi = \frac{Q}{8\pi \varepsilon_o} \sum_{l=0}^\infty \frac{1}{2l + 1} \left[ P_{l+1}(\cos \alpha) - P_{l-1}(\cos \alpha) \right] \frac{r^l_{<}}{r^l_{>} P_l(\cos \theta)}.
\]

6.6

Thin, flat, conducting, circular disc of radius \( R \) located in the \( x\)-\( y \) plane with its center at the origin, and is maintained at a fixed potential \( V \). Charge density on disc is proportional to \( (R^2 - \rho^2)^{-1/2} \). \([11], \text{problem 3.3}\)

For \( r > R \) : \( \Phi = \frac{2V}{\pi} \frac{R}{r} \sum_{l=0}^\infty \frac{(-1)^l}{2l + 1} \left( \frac{r}{R} \right)^{2l + 1} P_{2l}(\cos \theta) \).

For \( r < R \) : \( \Phi = \frac{2V}{\pi} \sum_{l=0}^\infty \frac{(-1)^l}{2l + 1} \left( \frac{r}{R} \right)^{2l + 1} P_{2l}(\cos \theta) \).
References

[1] John David Jackson. *Classical Electrodynamics* (3rd ed.). Wiley; 1999.

[2] Susan M. Lea. *Mathematics for Physicists*. Brooks/Cole; 2003.

[3] http://www-personal.umich.edu/~pran/jackson/ (An online compilation of solutions to problems from [1].)