THE HOMOLOGY OF \( \text{tmf} \)

AKHIL MATHEW

Abstract. We compute the mod 2 homology of the spectrum \( \text{tmf} \) of topological modular forms by proving a 2-local equivalence \( \text{tmf} \wedge DA(1) \simeq BP(2) \), where \( DA(1) \) is an eight cell complex whose cohomology “doubles” the subalgebra \( \mathcal{A}(1) \) of the Steenrod algebra generated by \( Sq^1 \) and \( Sq^2 \). To do so, we give, with use of the language of stacks, a modular description of the elliptic homology of \( DA(1) \) via level three structures. We briefly discuss analogs at odd primes and recover the stack-theoretic description of the Adams-Novikov spectral sequence for \( \text{tmf} \).

Contents

1. Introduction 1
2. The language of stacks 2
3. Calculating \( H_*(kO; \mathbb{Z}/2) \) 8
4. A vector bundle on \( M_{\text{cub}} \) 25
5. Calculation of \( \text{Tmf}_*(DA(1)) \) 30
6. \( \text{tmf} \wedge DA(1) \) and calculation of the homology 38
7. \( \text{tmf} \) at odd primes 44
8. The stack for \( \text{tmf} \) 47
9. References 49

1. Introduction

Let \( \text{tmf} \) be the spectrum of (connective) topological modular forms. The spectrum \( \text{tmf} \) is constructed from a derived version of the moduli stack \( M_{\text{Ell}} \) of elliptic curves, and its homotopy groups approximate both the stable homotopy groups of spheres and the ring of integral modular forms.

The goal of this paper is to compute the mod 2 cohomology of \( \text{tmf} \). Namely, we show that there is an isomorphism (due to Hopkins and Mahowald in [HM98])

\[
H^*(\text{tmf}; \mathbb{Z}/2) \simeq \mathcal{A}/(\mathcal{A}(2)) \overset{\text{def}}{=} \mathcal{A} \otimes_{\mathcal{A}(2)} \mathbb{Z}/2,
\]

where \( \mathcal{A} \) is the (mod 2) Steenrod algebra and \( \mathcal{A}(2) \subset \mathcal{A} \) is the subalgebra generated by \( Sq^1, Sq^2, Sq^4 \). The computation is carried out by exhibiting a 2-local eight cell complex \( DA(1) \) and demonstrating an equivalence (also due to [HM98]),

\[
\text{tmf} \wedge DA(1) \simeq BP(2),
\]

which is a \( \text{tmf} \)-analog of the equivalence in \( K \)-theory (due to Wood) \( kO \wedge \Sigma^{-2}CP^2 \simeq ku \). This equivalence implies the result on \( H^*(\text{tmf}; \mathbb{Z}/2) \) using Hopf algebra manipulations, and

Date: June 3, 2013.
it enables the computation of the stack associated to tmf (that is, a description of the Adams-Novikov spectral sequence).

The proof of (1) proceeds by calculating $\text{Tmf}_*(DA(1))$ (where Tmf is the non-connective version of tmf). To do so, we give an algebro-geometric description of the module $E_0(DA(1))$, for $E$ an elliptic homology theory, in terms of $M_{\text{cub}}(2)$; in other words, we identify a certain vector bundle on the moduli stack of elliptic curves as arising from an eight-fold cover of $(M_{\text{cub}}(2))$. This identification relies on the observation that the association $E \mapsto E_0(DA(1))$, which defines a vector bundle on $(M_{\text{cub}}(2))$, actually canonically extends to the larger stack $(M_{\text{cub}})$ of cubic curves, and $M_{\text{cub}}$ contains a special point (corresponding to the cuspidal cubic over $\mathbb{Z}/2$) from which it is easy to extract information. The description of $\text{tmf}_*(DA(1))$ from that of $\text{Tmf}_*(DA(1))$ follows from the “gap theorem” in $\pi_*\text{Tmf}$.

This paper is organized as follows. Section 2 reviews the language of stacks and, in particular, the role of the moduli stack $M_{FG}$ of formal groups; it also states the results we need about tmf. Section 3 describes how Wood’s theorem can be used to compute $H^*(ko; \mathbb{Z}/2)$ and introduces some of the relevant ideas. Section 4 is purely algebraic and describes a vector bundle on the stack $M_{\text{cub}}$. Section 5 shows that this vector bundle arises from an eight cell complex. Section 6 contains the proofs of the main results at the prime 2. In Sections 7 and 8, we describe the modifications at odd primes and obtain the Adams-Novikov spectral sequence for tmf.

We emphasize that the results of this paper are not new, but proofs of some of them have not formally appeared in the literature; as such, we have tried to write this paper in an expository manner.

Acknowledgments. I would like to thank heartily Mike Hopkins for introducing me to the subject and for suggesting this project. I would also like to thank Mark Behrens, Dustin Clausen, Tyler Lawson, Jacob Lurie, and Vesna Stojanoska for many helpful answers to my questions, and Lennart Meier and Niko Naumann for several comments on a draft of this paper. This work was supported by the Herchel Smith summer fellowship at Harvard.

2. The language of stacks

Let $X$ be a spectrum. The homotopy groups $\pi_*X$ may be complicated, but often their calculation can be attacked by choosing an appropriate resolution of $X$ by simpler spectra. This formalism can be expressed efficiently using the language of stacks. The language has been described in the course notes [Hop99] and in the talk [Hop07]; we also found the notes [Lur10] helpful. Other references on the stacky viewpoint, especially on the stack of formal groups, are [Nau07] and [Goe]. We will briefly summarize what we need below.

2.1. Stacks and spectra. Suppose, for example, that $R$ is a fixed $A_{\infty}$-ring spectrum. The structure on $R$ enables one to build the cobar construction; this is a cosimplicial spectrum $CB^*(R)$ with $CB^*(R) = R^{\wedge(s+1)}$ and with the coface and codegeneracy maps arising in a standard manner from the unit $S^0 \rightarrow R$ and the multiplication $R \wedge R \rightarrow R$. Under good conditions, for a spectrum $X$, the cosimplicial diagram

$$R^*(X) \overset{\text{def}}{=} X \wedge CB^*(R) = \{ X \wedge R \overset{\nu}{\rightarrow} X \wedge R \wedge R \overset{\nu}{\rightarrow} \ldots \}$$

will be a resolution of $X$ in the sense that the natural map

$$X \rightarrow \text{Tot}(R^*(X))$$

is an equivalence: in other words, $X$ can be recovered as a homotopy limit from the smash products $R^{\wedge s+1} \wedge X$, $s \geq 0$. 

Remark 2.1. This phenomenon is analogous to the theory of faithfully flat descent. Given a faithfully flat morphism $A \to B$ of commutative rings, an $A$-module $M$ can be recovered from the $B$-module $N = M \otimes_A B$ together with the “descent data” on $N$ via an equalization construction $M \otimes_A B \overset{\eta}{\to} M \otimes_A B \otimes_A B$. Here, one is trying to recover a spectrum $X$ (i.e., a module over the sphere spectrum $S^0$) from its base-change $R \wedge X$ along $S^0 \to R$ and the “descent data” given by the maps in $R^\bullet(X)$ (an approximation to which is given by the cooperations in $R$-homology). The Barr-Beck theorem (or its ∞-categorical version) provides a general framework for this phenomenon. See Chapter 6 of [Lur].

The benefit of this approach is that the spectra $R^{\wedge(s+1)} \wedge X$ are often simpler than $X$ itself: for instance, they are $R$-module spectra, and their homotopy groups are the $R^{\wedge(s+1)}$-homology groups of $X$. If (3) is an equivalence, then one has a homotopy spectral sequence

$$E_2^{s,t} = \pi^s \pi_t R^\bullet(X) \implies \pi_{t-s}X.$$ 

We recall two important examples:

Example 2.2. If $R$ is taken to be $HZ/2$ and $X$ is connective with finitely generated homology in each dimension, then the natural map $X \to \text{Tot}(HZ/2)^\bullet(X)$ exhibits $(HZ/2)^\bullet(X)$ as the 2-adic completion of $X$. (As a homotopy limit of $R$-module spectra, it is always true that $R^\bullet(X)$ is $R$-local, so the best we can hope for is that $\text{Tot}R^\bullet(X)$ is the $R$-localization of $X$.) The associated spectral sequence is the classical (mod 2) Adams spectral sequence $\text{Ext}_{A_s}^s(Z/2, H_4(X; Z/2)) \implies \pi_{t-s}(X) \otimes Z_2$ where $A_s$ is the dual Steenrod algebra and the $\text{Ext}$ is computed in the category of comodules over $A_s$.

Example 2.3. If $R = MU$ is complex bordism, and $X$ is connective with finitely generated homology in each dimension, then (3) is an equivalence, and the associated spectral sequence is the Adams-Novikov spectral sequence.

Suppose now that $R$ is homotopy commutative, and for each $s$, $\pi_s R^{\wedge(s+1)}$ is either connective in even degrees or satisfies $2 = 0$. Then we get a cosimplicial commutative ring $\pi_s R^{\wedge(s+1)}$, over which $\pi_s (R^{\wedge(s+1)} \wedge X)$ is a cosimplicial module. The language of stacks often enables us to interpret the $E_2$-page of this spectral sequence in terms of algebraic geometry. If $R_*$ is flat over $R_+$, the calculation of $\pi_* (R^{\wedge(s+1)} \wedge X)$ can be reduced to that of $R_+ (X)$ and the “cooperations.” In this case, the diagram

$$\pi_* R \overset{\eta}{\to} \pi_* (R \wedge R) \overset{\eta}{\to} \cdots$$

is determined by its 2-truncation, and it is a commutative Hopf algebroid (that is, a cogroupoid object in the category of commutative rings). The dual simplicial scheme

$$\cdots \overset{\eta}{\to} \text{Spec} \pi_* (R \wedge R) \overset{\eta}{\to} \text{Spec} \pi_* R$$

is the simplicial object presenting a stack $\mathfrak{X}$, which can be described as the geometric realization of the above simplicial scheme (in the 2-category of stacks).

Definition 2.4. In this paper, a stack is a sheaf of groupoids on the category of affine schemes, in the fpqc topology. To avoid set-theoretic technicalities, we restrict our stacks to take values on the category of rings of some cardinality $\leq \kappa$, where $\kappa$ is an uncountable cardinal. Most of the stacks we encounter, while infinite-dimensional, will be nonetheless reasonably well-behaved in that they will have affine diagonals.

Furthermore, the simplicial $\pi_* R^\bullet$-module $\pi_* R^\bullet(X)$ defines a quasi-coherent sheaf on $\mathfrak{X}$: that is, it defines a $\pi_* R$-module, a $\pi_* (R \wedge R)$-module, and a host of compatibility isomorphisms. Stated another way, it defines a comodule over the Hopf algebroid $(R_+, R_* R)$, and
this comodule structure is precisely given by the cooperations in \( R \)-homology. Note that the descent data alone here does not account for the grading involved.

We have a functor
\[
\mathcal{F} : \text{Ho}(\text{Sp}) \to \text{Mod}(\mathcal{X}),
\]
from the homotopy category \( \text{Ho}(\text{Sp}) \) of spectra to the category \( \text{Mod}(\mathcal{X}) \) of quasi-coherent sheaves on \( \mathcal{X} \), which is equivalent to the category of (ungraded) \((R_s, R_R)\)-comodules. We note that \( \mathcal{F} \) is not a symmetric monoidal functor in general, but if \( X, Y \) are spectra with the property that one of \( \mathcal{F}(X), \mathcal{F}(Y) \) is flat as a sheaf (i.e., \( R_sX \) or \( R_sY \) is a flat \( R_s \)-module), then \( \mathcal{F}(X) \otimes \mathcal{F}(Y) \simeq \mathcal{F}(X \wedge Y) \) by the Künneth theorem.

The chain complex
\[
\pi_s(R \wedge X) \to \pi_s(R \wedge R \wedge X) \to \ldots,
\]
whose cohomology is the \( E_2 \) page of \((4)\), can be identified with the the cobar complex
\[
R_s(X) \to R_s(R) \otimes_{R_s} R_sX \to \ldots,
\]
which computes the cohomology of the sheaf \( R_sX \). In particular, the spectral sequence \((4)\) can be written as
\[
H^*(\mathcal{X}, \mathcal{F}(X)) \implies \pi_{t-s}X.
\]

In practice, it is convenient to make one further modification. We observe that the rings \( R_s, R_R \) are graded rings, and the sheaves \( \mathcal{F}(X), X \in \text{Sp} \) come from graded comodules over the Hopf algebroid \((R_s, R_R)\). Let us now suppose that \( R_s, R_R \) are \emph{evenly graded}. The grading determines (and is equivalent to) a \( \mathbb{G}_m \)-action on the Hopf algebroid \((R_s, R_R)\), or on the stack \( \mathfrak{Y} \), such that an element in degree \( 2k \) is acted on by \( \mathbb{G}_m \) with eigenvalue given by the character \( \chi_k : \mathbb{G}_m \to \mathbb{G}_m, \chi_k(u) = u^k \). While \( \mathcal{F}(X) \) may have terms in odd grading, we can regard \( \mathcal{F}(X) \) as the sum of comodules \( \mathcal{F}_{\text{even}}(X) \oplus \mathcal{F}_{\text{odd}}(X) \), where each of the two summands inherits a \( \mathbb{G}_m \)-action in a similar manner. That is, \( \mathcal{F}_{\text{even}}(X) \) is given a \( \mathbb{G}_m \)-action in the same manner, and \( \mathbb{G}_m \) acts on \( \mathcal{F}_{2k+1}(X) \) by the character \( \chi_{k+1} \).

If we form the stack \( \mathfrak{Y} = \mathcal{X}/\mathbb{G}_m \), then \( \mathfrak{Y} \) comes with a tautological line bundle \( \omega \), from which the \( \mathbb{G}_m \)-torsor \( \mathcal{X} \to \mathfrak{Y} \) is constructed: \( \omega \) corresponds to the \((R_s, R_R)\)-comodule \( R_{s+2} \). Moreover, \( \mathcal{F}_{\text{even}} \) and \( \mathcal{F}_{\text{odd}} \) descend to functors into \( \text{Mod}(\mathfrak{Y}) \), which is equivalent to the category of \emph{evenly graded} comodules over \((R_s, R_R)\). The spectral sequence can be written
\[
E_2^{s,t} \implies \pi_{t-s}X,
\]
\[
E_2^{s,t} = \begin{cases} H^s(\mathfrak{Y}, \mathcal{F}_{\text{even}}(X) \otimes \omega^{s+t}) & \text{if } t = 2t' \\ H^s(\mathfrak{Y}, \mathcal{F}_{\text{odd}}(X) \otimes \omega^{s+t}) & \text{if } t = 2t' + 1 \end{cases}.
\]

2.2. \( M_{FG} \) and \( MU \). Remarkably, there exist choices of \( R \) for which the associated stack has a clear algebro-geometric interpretation.

\textbf{Definition 2.5.} \( M_{FG} \) is the moduli stack of formal groups. In other words, \( M_{FG} \) is the 2-functor
\[
M_{FG} : \text{Ring} \to \text{Gpd}
\]
assigning to any (commutative) ring \( A \) the groupoid of formal group schemes \( X \to \text{Spec} A \) which are Zariski locally (on \( A \)) isomorphic to \( \text{Spf} A[[x]] \) as formal schemes.

When \( R \) is \emph{complex bordism} \( MU \) (as we will henceforth assume), Quillen’s theorem states that there is an equivalence
\[
\mathfrak{Y} \overset{\text{def}}{=} \text{hocolim} \left( \text{Spec} (\pi_* MU^{\wedge(s+1)}) \right) / \mathbb{G}_m \simeq M_{FG}
\]
between the stack that one builds from the evenly graded Hopf algebroid \((MU_s, MU_s, MU)\) and the moduli stack of formal groups. The equivalence comes from the theory of complex...
orientations [Ada95]. In fact, Quillen showed that $MU_*$ is the Lazard ring that classifies formal group laws, while $MU_*MU$ classifies pairs of formal group laws with a strict isomorphism (i.e., an isomorphism which is the identity on the Lie algebra) between them. The operation of taking the $\mathbb{G}_m$-quotient accounts for non-strict isomorphisms of formal groups.

We can often describe the quasi-coherent sheaves $\mathcal{F}_{\text{even}}(X), \mathcal{F}_{\text{odd}}(X)$ on $M_{FG}$ in terms of the geometry of formal groups. We will have many more examples in the future, but we start with a simple one.

**Example 2.6.** The line bundle $\omega$ described above on $M_{FG}$ (that is, associated to the $(MU_*, MU_*MU)$-comodule $MU_{s+2}$, which arises topologically from $S^{-2}$) assigns to a formal group $X$ over $R$ the cotangent space $\mathcal{O}_X(-e)/\mathcal{O}_X(-2e)$ of functions on $X$ that vanish at zero, modulo functions that vanish to order two (that is, the dual to the $MU$ diagram of commutative rings if $X$ is a diagram of ring spectra itself, and the associated diagram of homotopy groups is a zero, modulo functions that vanish to order two (that is, the dual to the Lie algebra).

2.3. **Stacks associated to ring spectra.** We will also need a means of extracting stacks from ring spectra which may not be as well-behaved as $MU$. Suppose $X$ is a homotopy commutative ring spectrum. Then the above cosimplicial diagram

$$X \land MU \xrightarrow{e} X \land MU \land MU \xrightarrow{e} \ldots$$

is a diagram of ring spectra itself, and the associated diagram of homotopy groups is a diagram of commutative rings if $MU_*X$ is evenly graded (a condition which we will see is often satisfied); observe that this implies that $\pi_*(MU^{\land(s+1)} \land X)$ is evenly graded for each $s \geq 0$ because $MU \land MU$ is a wedge of even suspensions of $MU$. Consequently, the associated sheaf $\mathcal{F}(X) = \mathcal{F}_{\text{even}}(X)$ on the stack $M_{FG}$ is a sheaf of commutative rings, and can be used to present another stack, affine over $M_{FG}$.

**Definition 2.7.** We write $\text{Stack}(X)$ for the stack built in the above manner.

In other words, $\text{Stack}(X)$ is the geometric realization (in the category of stacks) of the simplicial object $(\text{Spec}_*(MU^{\land(s+1)} \land X))/\mathbb{G}_m$. Observe that a reformulation of the Künneth theorem in this context is that for commutative ring spectra $X, Y$, we have

$$\text{Stack}(X \land Y) \simeq \text{Stack}(X) \times_{M_{FG}} \text{Stack}(Y),$$

if $MU_*(X)$ is flat over $\pi_*MU$.

There are several important examples of this construction:

**Example 2.8.** Consider the spectrum $X(n)$, which is the Thom spectrum of the composite $\Omega SU(n) \to \Omega SU \simeq BU$.

The stack associated (in the above way) to $X(n)$ is known to be the stack of formal groups together with a coordinate modulo degree $n$.

**Example 2.9.** Consider $X = MU$. Then the stack one gets is the $\mathbb{G}_m$-quotient of the geometric realization of

$$\ldots \rightarrow \text{Spec}_*(MU \land MU \land MU) \rightarrow \text{Spec}_*(MU \land MU).$$

Recall that $\text{Spec}_*(MU \land MU)$ classifies a pair of formal group laws with a strict isomorphism between them. The scheme $\text{Spec}_*(MU \land MU \land MU)$ corresponds to a triple of formal group laws with strict isomorphisms between them. It follows that the associated stack is equivalent to the $\mathbb{G}_m$-quotient of $\text{Spec}L$, where $L$ is the Lazard ring.

More generally, whenever $R$ is a complex-orientable ring spectrum with $\pi_*R$ evenly graded, the stack $\text{Stack}(R)$ is the $\mathbb{G}_m$-quotient of $\text{Spec}R_*$. It follows in particular that for such $R$, the sheaf $\mathcal{F}(R)$ is the push-forward of the structure sheaf under the map $(\text{Spec}R_*)/\mathbb{G}_m \to M_{FG}$. 

THE HOMOLOGY OF tmf
2.4. Even periodic ring spectra. Let $X$ be a spectrum. As we saw, $X$ defines quasi-coherent sheaves $\mathcal{F}_{\text{odd}}(X), \mathcal{F}_{\text{even}}(X)$ on the moduli stack $M_{FG}$ of formal groups. These come from $MU_*(X)$ (alternatively, $MP_0X$ and $MP_1X$ where $MP = \bigvee_{n \in \mathbb{Z}} \Sigma^n MU$ is the periodic complex bordism spectrum) together with the comodule structure over the Hopf algebroid $(MU_*, MU_*MU)$ and the grading. On the flat site of $M_{FG}$, there is a classical topological interpretation of these sheaves.

Definition 2.10 ([AHS01]). A homotopy commutative ring spectra $E$ is even periodic if $\pi_i E = 0$ for $i$ odd, and if $\pi_2 E$ is an invertible module over $\pi_0 E$ with the property that the multiplication map

$$\pi_2 E \otimes \pi_{-2} E \to \pi_0 E,$$

is an isomorphism. If $\pi_2 E \cong \pi_0 E$ as $\pi_0 E$-modules, so that $\pi_* E \cong (\pi_0 E)[t, t^{-1}]$ for $t \in \pi_2 E$ a generator, we will say that $E$ is strongly even periodic.

Given an even periodic ring spectrum $E$, the formal scheme $\text{Spf} E \otimes (\mathbb{CP}^\infty)$ is a formal group over $E_0$ (see [AHS01]), and is classified by a morphism $q: \text{Spec} E_0 \to M_{FG}$. If $q: \text{Spec} E_0 \to M_{FG}$ is flat, then $E$ is called a Landweber-exact theory and one has functorial isomorphisms:

$$q^*(\mathcal{F}_{\text{even}}(X)) \cong E_0(X), \quad q^*(\mathcal{F}_{\text{odd}}(X)) \cong E_1(X).$$

More generally,

$$q^*(\omega^j \otimes \mathcal{F}_{\text{even}}(X)) \cong E_{2j}(X), \quad q^*(\omega^j \otimes \mathcal{F}_{\text{odd}}(X)) \cong E_{2j+1}(X).$$

In particular, for a Landweber-exact theory, $E_*(X)$ can be recovered from $MU_*(X)$ together with the cooperations $MU_*(X) \to MU_* MU \otimes_{MU_* MU} X$. The stack associated to the ring spectrum $E$ is precisely $\text{Spec} E_0$.

Conversely, given a ring $R$ and a flat morphism $q: \text{Spec} R \to M_{FG}$, the functor

$$X \mapsto \bigoplus_j q^*(\omega^j \otimes (\mathcal{F}_{\text{even}}(X) \oplus \mathcal{F}_{\text{odd}}(X))),$$

defines a multiplicative homology theory, representable by a Landweber-exact ring spectrum. The result is a presheaf of multiplicative homology theories on the flat site of $M_{FG}$.

One reason this point of view is so useful is the following concrete criterion for flatness over $M_{FG}$, known as the Landweber exact functor theorem:

Theorem 2.11 (Landweber [Lan76]). Let $R$ be a commutative ring, and let $F(x, y) \in R[[x, y]]$ be a formal group law over $R$. Then the formal group that $F$ defines classifies a map $\text{Spec} R \to M_{FG}$, which is flat if and only if the following condition holds, for every prime number $p$.

For a prime number $p$, define a sequence $v_0, v_1, \ldots$, such that $v_i$ is the coefficient of $x^{p^i}$ in the expansion of the series $[p]_F$ (so $v_0 = p$). Then the condition is that

$$p, v_1, \ldots, v_n, \ldots \in R,$$

should be a regular sequence.

Landweber’s original formulation of the theorem did not use the language of stacks. A proof of the theorem using this language is in Lecture 16 of [Lur10]. We note that Landweber’s theorem can be used to produce spectra which are not even periodic as well. Fix an evenly graded ring $R$, and a formal group law $F(x, y) \in R[[x, y]]$ satisfying the grading convention that if $x, y$ have degree $-2$, then $F(x, y)$ has degree $-2$ as well. Then the
formal group law $F$ is classified by a map $MU_* \to R_*$, and if this map satisfies Landweber’s condition, then one can define a functor

$$E_*(X) \overset{\text{def}}{=} MU_*(X) \otimes_{MU_*} R_*,$$

which (as a consequence of Landweber’s theorem) is in fact a homology theory. Note that this homology theory is not even periodic: in a sense, it arises via the previous construction from a map $\text{Spec}R_*/\mathbb{G}_m \to MF_G$.

Finally, we note that this last construction does not require $R_*$ to be a ring, but only an $MU_*$-module.

2.5. Topological modular forms. Although $MF_G$ is a very large stack, there are smaller stacks that can be used to approximate it.

**Definition 2.12.** Let $M_{\text{ell}}$ be the moduli stack of possibly nodal elliptic curves. In other words, $M_{\text{ell}}$ assigns to each commutative ring $R$ the groupoid of all pairs $(p: C \to \text{Spec}R, e: \text{Spec}R \to C)$, where $p, e$ are such that:

1. $p$ is a proper, flat morphism of finite presentation.
2. The fibers of $p$ have arithmetic genus one, and they are either smooth curves or curves with a single nodal singularity.
3. $e$ is a section of $p$ whose image is contained in the smooth locus of $p$.

There is a flat map $M_{\text{ell}} \to MF_G$ which sends each elliptic curve (or, more generally, nodal cubic curve) to its formal completion at the identity, which acquires the structure of a formal group. Consequently, any spectrum $X$ defines by pull-back a quasi-coherent sheaf on $M_{\text{ell}}$ as well; we will denote this too by $\mathcal{F}(X)$.

**Remark 2.13.** Flatness of the map $M_{\text{ell}} \to MF_G$ can be checked using the Landweber exact functor theorem. Namely, $M_{\text{ell}}$ is a torsion-free stack, and (after one sets $p = 0$), the element $v_1$ (the Hasse invariant) is regular and cuts out the divisor of supersingular curves in $M_{\text{ell}} \times \text{Spec}F_p$. On that locus, $v_2$ is invertible, since the formal group of an elliptic curve (or nodal cubic) has height $\leq 2$.

As before, the quasi-coherent sheaf $\mathcal{F}(X)$ on $M_{\text{ell}}$ has a topological interpretation. Given a flat map $q: \text{Spec}R \to M_{\text{ell}}$ classifying an elliptic curve $C \to \text{Spec}R$, one can construct an elliptic spectrum $E$: in other words, $E$ is even periodic with $E_0 = R$, and there is an isomorphism of formal groups

$$\text{Spf} E^0(\mathbb{CP}^\infty) \simeq \hat{C},$$

between the formal group of $E$ and the formal completion of $C$. In this case, one has $E_0(X) \oplus E_1(X) \simeq q^*(\mathcal{F}(X))$. Moreover, there is a functorial isomorphism

$$E_{2j}(*) \simeq \omega^j, \quad E_{2j+1}(*) = 0,$$

where $\omega$ is the line bundle on $M_{\text{ell}}$ obtained by pulling back $\omega$ on $MF_G$.

The work of Goerss, Hopkins, Miller, and Lurie shows that, when we restrict to étale affines over $M_{\text{ell}}$, $E$ actually can be taken to be an $E_\infty$-ring spectrum, and that the above construction is (or rather, can be made to be) functorial: it defines a sheaf $\mathcal{O}^{\text{top}}$ of $E_\infty$-ring spectra on the étale site of $M_{\text{ell}}$.

**Theorem 2.14** (Goerss, Hopkins, Miller, Lurie). (1) (Existence) There is a sheaf $\mathcal{O}^{\text{top}}$ of $E_\infty$-ring spectra on the étale site of $M_{\text{ell}}$, such that for $\text{Spec}R \to M_{\text{ell}}$ an affine étale open classifying an elliptic curve $C \to \text{Spec}R$, the $E_\infty$-ring $\mathcal{O}^{\text{top}}(\text{Spec}R)$ is an elliptic spectrum corresponding to $C$. 

(2) (Gap theorem) Moreover, if \( \text{tmf} = \Gamma(M_{\text{cl}}, O^{\text{top}}) \), then \( \pi_j(\text{tmf}) = 0 \) for \(-21 < j < 0\).

They define
\[
\text{tmf} = \Gamma(M_{\text{cl}}, O^{\text{top}}), \quad \tau_0 = \tau_{\geq 0}(\text{tmf}),
\]
where \( \tau_0 \) denotes the connective cover. In other words, \( \text{tmf} \) is the spectrum of global sections of \( O^{\text{top}} \) (constructed as a homotopy limit of the associated elliptic spectra as \( \text{Spec} R \to M_{\text{cl}} \) ranges over étale morphisms). Since \( \text{tmf} \) is constructed as a homotopy limit, we have a descent spectral sequence
\[
H^i(M_{\text{cl}}, \pi_j O^{\text{top}}) \implies \pi_{j-i}(\text{tmf}).
\]
Equivalently, the spectral sequence runs
\[
H^i(M_{\text{cl}}, \omega^j) \implies \pi_{2j-i}(\text{tmf}),
\]
in view of the identification of the homotopy groups \( \pi_j O^{\text{top}} \) on affine étale opens. More generally, if \( X \) is a finite spectrum, then smashing with \( X \) commutes with homotopy limits, and we get a spectral sequence
\[
H^i(M_{\text{cl}}, \pi_j (O^{\text{top}} \wedge X)) \implies \pi_{j-i}(\text{tmf} \wedge X).
\]
Observe that \( \pi_{j+2k}(O^{\text{top}} \wedge X) \simeq \pi_j(O^{\text{top}} \wedge X) \otimes \omega^k \), so a knowledge of \( F(X) \) (that is, of \( \pi_0(O^{\text{top}} \wedge X) \) and \( \pi_1(O^{\text{top}} \wedge X) \)) is the information necessary to run the spectral sequence. The following case, which is of interest to us, offers a simplification of the spectral sequence:

**Definition 2.15** [*AHS01*]. A connective spectrum \( X \) is even if \( H_*(X; \mathbb{Z}) \) is free and concentrated in even dimensions. We can make a similar definition for a \( p \)-local spectrum, for a prime \( p \).

In the even case, the sheaf \( F_{\text{even}}(X) \) on \( M_{\text{FG}} \) can be interpreted in the following way: for a flat morphism \( q: \text{Spec} R \to M_{\text{FG}} \), the \( R \)-module \( q^* F_{\text{even}}(X) \) is identified with \( E_0(X) \) for \( E \) the Landweber-exact, even-periodic ring spectrum associated with \( q \). It follows from the (degenerate) Atiyah-Hirzebruch spectral sequence that \( E_1(X) = 0 \), and in particular \( F_{\text{odd}}(X) = 0, F(X) = F_{\text{even}}(X) \). We find that \( F(X) \) can be simply viewed as a sheaf on \( M_{\text{FG}} \) or \( M_{\text{cl}} \). In this case, the Adams-Novikov spectral sequence can be written as
\[
H^i(M_{\text{cl}}, F(X) \otimes \omega^j) \implies \pi_{2j-i}(X),
\]
and the descent spectral sequence can be written as
\[
H^i(M_{\text{cl}}, \omega^j \otimes F(X)) \implies \pi_{2j-i}(\text{tmf} \wedge X).
\]

The strategy of this paper is to identify the sheaf \( F(X) \) on \( M_{\text{cl}} \) in algebro-geometric terms, for a suitable even 8-cell complex \( X \), in such a way that the spectral sequence is forced to degenerate. This will compute \( \pi_*(\text{tmf} \wedge X) \), and ultimately give a description of \( \text{tmf} \wedge X \).

3. Calculating \( H_*(ko; \mathbb{Z}/2) \)

In this section, we perform a toy analog of the main calculation of this paper. Let \( ko = \tau_{\geq 0}(KO) \) be the connective cover of \( KO \)-theory. Then we have a map
\[
ko \to \tau_{<0}(ko) = H\mathbb{Z} \to H\mathbb{Z}/2,
\]
inducing a morphism \( H_*(ko; \mathbb{Z}/2) \to H_*(H\mathbb{Z}/2; \mathbb{Z}/2) \), where \( H_*(H\mathbb{Z}/2; \mathbb{Z}/2) \) is the dual Steenrod algebra \( A_* \).
Let us recall the structure of the (mod 2) dual Steenrod algebra. $A_*$ is a commutative, non-cocommutative Hopf algebra such that $\Spec A_*$ is the group scheme of strict automorphisms of the formal additive group $\widehat{G}_a$ over $\mathbb{Z}/2$. As a graded algebra,

$$A_* = \mathbb{Z}/2[\xi_1, \xi_2, \ldots], \quad |\xi_i| = 2^i - 1,$$

where the coalgebra structure map $\Delta: A_* \to A_* \otimes A_*$ is given by

$$\Delta(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^2 \otimes \xi_i. \quad (8)$$

A morphism $f: A_* \to R$ (that is, an $R$-valued point of $\Spec A_*$) corresponds to the inverse of the power series

$$t + f(\xi_1)t^2 + f(\xi_2)t^4 + \ldots,$$

which is an automorphism of $\widehat{G}_a$ over $R$.

**Theorem 3.1 (Stong).** The map $ko \to H\mathbb{Z}/2$ induces an injection in mod 2 homology, and

$$H_*(ko; \mathbb{Z}/2) = \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \xi_4, \ldots] \subset A_* = H_*(H\mathbb{Z}/2; \mathbb{Z}/2),$$

where $\xi$ for $x \in A_*$ denotes Hopf conjugation.

For instance, this result can be used (together with the Adams spectral sequence) to work out the homotopy groups $\pi_*$ of $\text{Spec} A_*$ Moreover, since $H_*(ko; \mathbb{Z}/2)$ is considerably smaller than $A_*$, we can think of $ko$ as a sort of first-order approximation to the sphere $S^0$.

The purpose of this section is to describe another way of performing this calculation, based on the calculation of $H_*(ko; \mathbb{Z}/2)$. The key step is the use of a theorem of Wood. Namely, one has:

**Theorem 3.2 (Wood).** $KO \wedge \Sigma^{-2}CP^2 \simeq KU$.

In this section, we will describe how one can prove Theorem 3.2 in a stack-theoretic manner, and how it enables the computation of $H_*(ko; \mathbb{Z}/2)$. We will later imitate the above strategy for tmf.

### 3.1. The stack for $KO$-theory.

Complex $K$-theory $KU$ is an even-periodic cohomology theory, and one has an isomorphism

$$\text{Spf} KU^0(\mathbb{CP}^\infty) \simeq \widehat{G}_m,$$

between the formal group of $KU$-theory and the formal multiplicative group (over $\mathbb{Z}$). The formal multiplicative group is classified by a map $\Spec \mathbb{Z} \to M_{FG}$, which is flat by Landweber’s criterion (Theorem 2.11): $\pi$ is regular and $v_1$ is invertible (the formal group is of height one). Consequently, we get isomorphisms, for any spectrum $X$ (which were originally proved differently by Conner and Floyd),

$$KU_0(X) \simeq MP_0(X) \otimes_{MP_0} \mathbb{Z}, \quad KU_1(X) \simeq MP_1(X) \otimes_{MP_0} \mathbb{Z}.$$

Observe now that we have an isomorphism of group schemes $\text{Aut}(\mathbb{G}_m) \simeq \mathbb{Z}/2$: the only automorphisms of $\mathbb{G}_m$ (over any ring) are given by the identity and inversion. Consequently, we have a $\mathbb{Z}/2$-action on the formal group $\widehat{G}_m$, which gives an action of $\mathbb{Z}/2$ on $KU_0(X)$ for any spectrum $X$. This action is precisely dual to complex conjugation on vector bundles, or the Adams operation $\Psi^{-1}$, and it lifts to an action of $\mathbb{Z}/2$ on the spectrum $KU$ itself.

Moreover, we have $KO = (KU)^{h\mathbb{Z}/2}$. For our purposes, we can take this as a definition. For a finite spectrum $X$, we then have

$$KO \wedge X \simeq (KU \wedge X)^{h\mathbb{Z}/2},$$
and we get thus a homotopy fixed point spectral sequence

$$H^i(\mathbb{Z}/2, KU_j(X)) \Rightarrow KO_{j-i}(X).$$

We can state this in language that highlights the parallel with Tmf more precisely. As above, there is a $\mathbb{Z}/2$-equivariant flat morphism of stacks $\text{Spec} \mathbb{Z} \to M_{FG}$ (where $\mathbb{Z}/2$ acts trivially on both $M_{FG}$ and $\text{Spec} \mathbb{Z}$) classifying the formal group $\widehat{\mathbb{G}}_m$. Note that the $\mathbb{Z}/2$-equivariance here is not a condition: it is extra data, since the collection of maps $\{\text{Spec} \mathbb{Z} \to M_{FG}\}$ is a groupoid and not a set; in this case, the extra data encodes the involution of $\widehat{\mathbb{G}}_m$. For instance, one could also make the map $\text{Spec} \mathbb{Z} \to M_{FG}$ into a $\mathbb{Z}/2$-equivariant map by taking the trivial involution of $\widehat{\mathbb{G}}_m$.

Passing to the quotient defines a (flat) morphism of stacks

$$B \mathbb{Z}/2 \to M_{FG},$$

which we can think of as a loose approximation to the stack $M_{FG}$; in the same way that $KO$-theory is a (very coarse) approximation to the sphere spectrum. The map $B \mathbb{Z}/2 \to M_{FG}$ sends a $\mathbb{Z}/2$-torsor to the formal completion of the associated non-split torus: since $\text{Aut}(\mathbb{G}_m) \simeq \mathbb{Z}/2$, $\mathbb{Z}/2$-torsors correspond to one-dimensional tori.

In particular, any spectrum $X$ defines sheaves $\mathcal{F}_\text{even}(X), \mathcal{F}_\text{odd}(X)$ on the stack $B \mathbb{Z}/2$. When $X$ is finite, there is a descent spectral sequence from the cohomology of these sheaves on $B \mathbb{Z}/2$ to the $KO$-theory of $X$. In simpler terms, any finite spectrum $X$ defines an abelian group $KU_0(X) \oplus KU_1(X)$ with a $\mathbb{Z}/2$-action. When $X$ is even (so $KU_\text{odd}(X) = 0$), there is a descent spectral sequence from the $\mathbb{Z}/2$-group cohomology of $\{KU_{2j}(X) \simeq KU_0(X) \otimes \omega^j\}_{j \in \mathbb{Z}}$ to $KO_4(X)$.

In fact, $B \mathbb{Z}/2$ is a derived stack in a similar manner as $M_{\text{tr}}$. That is, there is a sheaf $\mathcal{O}_\text{top}$ of $E_\infty$-rings on the category $\text{Aff}_{/B \mathbb{Z}/2}$ of affine schemes equipped with an étale map to $B \mathbb{Z}/2$.

It has the property that for any étale map $\text{Spec} R \to B \mathbb{Z}/2$ classifying a one-dimensional torus $T \to \text{Spec} R$, the $E_\infty$-ring $\mathcal{O}_\text{top}(\text{Spec} R)$ is even periodic with functorial isomorphisms

$$\mathcal{O}_\text{top}(\text{Spec} R)_0 \simeq R, \quad \text{Spf}\mathcal{O}_\text{top}(\text{Spec} R)^0(\mathbb{CP}^\infty) \simeq \hat{T}.$$  

The global sections of $\mathcal{O}_\text{top}$ are given by $KO$, and the above spectral sequence is the descent spectral sequence. This is described in [LN12a], Theorem A.2, as a consequence of the $\mathbb{Z}/2$-action on the $E_\infty$-ring corresponding to $KO$-theory, and a derived version of the “topological invariance of the étale site.”

**Remark 3.3.** After $p$-adic completion, the $E_\infty$-structure on $KU$-theory together with the $\mathbb{Z}/2$-action (by $E_\infty$-maps) is a special case of the Hopkins-Miller theorem that constructs functorial $E_\infty$-structures on the more general Morava $E$-theory spectra $E_n$. The required structure on integral $KU$-theory then follows from an arithmetic square.

### 3.2. The Wood equivalence.

Our goal is to use the stack-theoretic viewpoint to explain the Wood equivalence $KO \wedge \Sigma^{-2} \mathbb{CP}^2 \simeq KU$.

Recall that $\Sigma^{-2} \mathbb{CP}^2$ is the cone on the Hopf map $\eta: S^1 \to S^0$, and we can draw the 2-cell complex as in Figure 11 where the vertical line represents cohomology classes connected by a $S^1$.

In particular, $\Sigma^{-2} \mathbb{CP}^2$ is a two-cell complex with the attaching map given by $\eta$. Consequently, $\Sigma^{-2} \mathbb{CP}^2$ defines a rank two vector bundle $V = \mathcal{F}(\Sigma^{-2} \mathbb{CP}^2)$ on $B \mathbb{Z}/2$, and there is a spectral sequence $H^i(B \mathbb{Z}/2, V \otimes \omega^j) \Rightarrow \pi_{2j-i}(KO \wedge \Sigma^{-2} \mathbb{CP}^2)$. The claim is that we can identify $V$ very concretely.
Proposition 3.4. The vector bundle \( \mathcal{V} \) on \( \mathbb{Z}/2 \) associated to \( \Sigma^{-2}\mathbb{CP}^2 \) is the vector bundle obtained from the two-fold cover
\[
p: \text{Spec}\mathbb{Z} \to \mathbb{Z}/2,
\]
given as \( p_*(\mathcal{O}) \). More concretely, the \( \mathbb{Z}/2 \)-representation that \( \tilde{KU}_0(\mathbb{CP}^2) \) defines is the permutation representation.

In other words, if one constructs the form \( KU_{(T,R)} = O^\text{top}(\text{Spec}R) \) of \( KU \)-theory corresponding to a one-dimensional algebraic torus \( T \) over a ring \( R \) (such that the map \( \text{Spec}R \to \mathbb{Z}/2 \) is étale), then we get a functorial identification of \( (KU_{(T,R)})_0(\Sigma^{-2}\mathbb{CP}^2) \) with the \( R \)-algebra that classifies isomorphisms of \( T \) with \( G_m \).

Proof. It suffices to prove this for the group \( 
\tilde{KU}^0(\mathbb{CP}^2) \cong \text{Hom}(\tilde{KU}_0(\mathbb{CP}^2), \mathbb{Z}),
\) which can be identified with functions on the formal group of \( KU \) (that is, the formal completion \( \hat{G}_m \) of \( G_m = \text{Spec}\mathbb{Z}[x,x^{-1}] \) at \( x = 1 \) vanishing at 1, modulo functions vanishing at 1 to order \( \geq 3 \). The identification holds in fact for \( KU \) replaced by any even periodic cohomology theory. This is now a purely algebraic calculation (although it could of course have been done topologically).

In particular, we find that
\[
\tilde{KU}^0(\mathbb{CP}^2) \cong \left( \mathbb{Z}[x,x^{-1}]/(x-1) \right)/\left( \mathbb{Z}[x,x^{-1}]/(x-1)^3 \right),
\]
has a basis given by \( \{ \alpha = x - 1, \beta = (x - 1)^2 \} \). The \( \mathbb{Z}/2 \)-action on \( G_m \) corresponds to the action \( x \mapsto x^{-1} \) on \( \mathbb{Z}[x,x^{-1}] \), and this means that we find for the action of the involution on \( \tilde{KU}^0(\mathbb{CP}^2) \):
\[
\alpha = (x - 1) \mapsto \frac{1}{x} - 1 = -x - 1 + (x - 1)^2 = -\alpha + \beta
\]
\[
\beta = (x - 1)^2 \mapsto (x^{-1} - 1)^2 = (\alpha + \beta)^2 = \beta.
\]

In particular, the basis \( \{ \alpha, -\alpha + \beta \} \) for \( \tilde{KU}^0(\mathbb{CP}^2) \) shows that \( \tilde{KU}^0(\mathbb{CP}^2) \) is the permutation representation of \( \mathbb{Z}/2 \). \( \square \)

In view of this, we can write
\[
H^i(\mathbb{Z}/2, \mathcal{F}(\mathbb{CP}^2) \otimes \omega^j) = H^i(\text{Spec}\mathbb{Z}, \mathcal{O}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases},
\]
so that the spectral sequence collapses, and gives precisely the homotopy groups of \( KU \)-theory.

Proof of Wood’s theorem, Theorem 3.3. We observe that \( KU \wedge \Sigma^{-2}\mathbb{CP}^2 \) has an action of \( \mathbb{Z}/2 \) from the first factor. As a spectrum with a \( \mathbb{Z}/2 \)-action, it is equivalent to \( KU \vee KU \) with the action swapping the two factors (and ignoring \( KU \)): this is the content of the above result. In more detail, choose \( \alpha \in \pi_0(KU \wedge \Sigma^{-2}\mathbb{CP}^2) \) such that \( \pi_0(KU \wedge \Sigma^{-2}\mathbb{CP}^2) \) is free on \( \alpha \) and its conjugate; then the induced \( \mathbb{Z}/2 \)-equivariant map \( S^0 \oplus S^0 \to KU \wedge \Sigma^{-2}\mathbb{CP}^2 \) is
an isomorphism on \( \pi_0 \), so the induced \( \mathbb{Z}/2 \)-equivariant map \( KU \vee KU \to KU \wedge \Sigma^{-2}\mathbb{CP}^2 \) is an equivalence.

Consequently,
\[
KO \wedge \Sigma^{-2}\mathbb{CP}^2 \simeq (KU \wedge \Sigma^{-2}\mathbb{CP}^2)_{h\mathbb{Z}/2} \simeq (KU \vee KU)_{h\mathbb{Z}/2} \simeq KU.
\]

Observe that these manipulations can be done in the category of \( KO \)-modules.

\[\Box\]

**Corollary 3.5.** There is an equivalence of \( ko \)-modules \( k \wedge \Sigma^{-2}\mathbb{CP}^2 \simeq k \).

**Proof.** In fact, Theorem 3.2 gives an equivalence of \( KO \)-modules \( KO \wedge \Sigma^{-2}\mathbb{CP}^2 \simeq KU \), which is in particular an equivalence of \( ko \)-modules. We now take connective covers in this equivalence, and use the fact that
\[
\tau_{\geq 0}(KO \wedge \Sigma^{-2}\mathbb{CP}^2) \simeq (\tau_{\geq 0}KO) \wedge \Sigma^{-2}\mathbb{CP}^2,
\]
because \( \pi_i KO = 0 \) for \( -4 < i < 0 \), so that \( \tau_{<0}KO \wedge \Sigma^{-2}\mathbb{CP}^2 \) has homotopy groups only in negative degrees.

\[\Box\]

### 3.3. Finiteness considerations.

We recall that the equivalence \( KO \simeq (KU)_{h\mathbb{Z}/2} \) implies that for any finite spectrum \( X \), \( KO \wedge X \simeq (KU \wedge X)_{h\mathbb{Z}/2} \), since the collection of spectra \( X \) satisfying the equivalence contains the sphere spectrum and is closed under suspensions and homotopy limits. However, Wood’s theorem enables us to say the same for any spectrum. This observation will be useful in computing \( KO_*(MU) \), and it will generalize to \( \text{Tmf} \).

**Corollary 3.6.** If \( X \) is any spectrum, then \( KO \wedge X \simeq (KU \wedge X)_{h\mathbb{Z}/2} \).

**Proof.** We learned this argument from Hopkins. Consider the category \( \mathcal{C} \) of all finite spectra \( T \) such that the natural map
\[
KO \wedge X \wedge T \to (KU \wedge X \wedge T)_{h\mathbb{Z}/2}
\]
is an equivalence. This is a thick subcategory of the homotopy category of finite spectra: that is, it is closed under retracts, and if \( \mathcal{C} \) contains two of the terms of a cofiber sequence of finite spectra
\[
T' \to T \to T''
\]
then it contains the third too. The thick subcategory theorem of [HS98] now implies that to show that \( \mathcal{C} \) is the category of all finite spectra (in particular, that it contains the sphere), it suffices to show that \( \mathcal{C} \) contains a finite spectrum with nontrivial rational homology. In fact, \( \Sigma^{-2}\mathbb{CP}^2 \) will suffice for this, as \( KU \wedge \Sigma^{-2}\mathbb{CP}^2 \wedge X \) is equivalent as a “naive” \( \mathbb{Z}/2 \)-spectrum to \( (KU \vee KU) \wedge X \) with permutation action on \( KU \vee KU \), and consequently \( (KU \wedge \Sigma^{-2}\mathbb{CP}^2 \wedge X)_{h\mathbb{Z}/2} \simeq KU \wedge X \simeq KO \wedge \Sigma^{-2}\mathbb{CP}^2 \wedge X \). Therefore \( S^0 \in \mathcal{C} \), proving the corollary.

\[\Box\]

**Remark 3.7.** We do not need the full strength of the thick subcategory theorem to run the above argument, but only a small piece of the idea ultimately used in its proof. Namely, let \( \mathcal{C} \) be any thick subcategory of the homotopy category of finite spectra containing \( \Sigma^{-3}\mathbb{CP}^2 \). Since \( \Sigma^{-2}\mathbb{CP}^2 \) is the cofiber of \( \eta : S^3 \to S^0 \), we can use the octahedral axiom applied to the composite \( S^2 \stackrel{\eta^2}{\to} S^1 \stackrel{\eta}{\to} S^0 \) to get a cofiber sequence
\[
\Sigma^{-1}\mathbb{CP}^2 \to \text{cof}(S^2, \eta^2, S^0) \to \Sigma^{-2}\mathbb{CP}^2,
\]
which implies that the cofiber of \( \eta^2 \) belongs to \( \mathcal{C} \). Repeating the same argument, we find that the cofiber of \( \eta^3 \), and then the cofiber of \( \eta^4 \), all belong to \( \mathcal{C} \). However, since \( \eta^4 = 0 \), we conclude that a wedge of spheres is in \( \mathcal{C} \). Taking retracts shows that \( S^0 \in \mathcal{C} \), and \( \mathcal{C} \) is thus the category of finite spectra. This observation goes back (at least) to [Rav84].
Figure 2. The homotopy fixed point spectral sequence for $KO \simeq K^{h\mathbb{Z}/2}$.

Arrows in red indicate differentials, while arrows in black indicate recurring patterns. Dots indicate copies of $\mathbb{Z}/2$, while squares indicate copies of $\mathbb{Z}$.

Remark 3.8. The same argument shows that the Tate spectrum of $\mathbb{Z}/2$ acting on $KU$-theory is trivial, since it is trivial after smashing with $\Sigma^{-2}\mathbb{C}P^2$.

The previous observations express the fact that the infinite homotopy colimit that computes $K^{h\mathbb{Z}/2}$ is in some sense a finite homotopy colimit; this is expressed, for instance, in the behavior of the homotopy spectral sequence $H^i(\mathbb{Z}/2, \omega^{\otimes j}) \Rightarrow \pi_{2j-i}KO$, for $\omega$ the sign representation of $\mathbb{Z}/2$. The spectral sequence is drawn in Figure 2. At $E_2$, the spectral sequence is quite complicated, with elements of arbitrarily high filtration; however, at $E_4$ and beyond, the spectral sequence is degenerate with a horizontal vanishing line.

3.4. Calculation of $H_\ast(ku; \mathbb{Z}/2)$. Before calculating $H_\ast(ko; \mathbb{Z}/2)$, it will be necessary to calculate $H_\ast(ku; \mathbb{Z}/2)$. The calculation is performed in [Ada95] using an explicit description of the spaces $\Omega^{\infty-n}ku$. An alternative way of handling the calculation, which will generalize below to tmf, is to use the Landweber-exactness of $KU$-theory.

Let $BP$ be the Brown-Peterson spectrum at a prime $p$. Recall that

$$\pi_\ast BP \simeq \mathbb{Z}_{(2)}[v_1, v_2, \ldots], \quad |v_i| = 2(p^i - 1),$$

which is the ring classifying the universal $p$-typical formal group law over a $\mathbb{Z}_{(p)}$-algebra. There are many possible choices of the generators $v_i$; in this subsection, we make the following choice: we let $v_1$ be the coefficient of $x^p$ in the formal power series $[p](x) \in \pi_\ast(BP)[[x]]$.

(By this convention, $v_0 = p$.)

Now take $p = 2$. Then observe that, when localized at 2, one has the Conner-Floyd isomorphism:

$$KU_\ast(X) \simeq BP_\ast(X) \otimes_{BP} (BP_\ast/(v_2, v_3, \ldots))[v_1^{-1}] = BP_\ast(X) \otimes_{BP} \mathbb{Z}_{(2)}[v_1, v_1^{-1}].$$

This isomorphism is precisely the Landweber-exactness of $KU$-theory.

Let $BP \langle 1 \rangle = BP/(v_2, v_3, \ldots)$; see Proposition 3.17 below. Then we get a map

$$BP \langle 1 \rangle \to BP \langle 1 \rangle [v_1^{-1}] \simeq KU,$$

whose map on homotopy groups is $\mathbb{Z}_{(2)}[v_1] \to \mathbb{Z}_{(2)}[v_1^{\pm 1}]$ with $|v_1| = 2$. In fact, it is known that Landweber-exact spectra associated to a given formal group over a ring are unique up to homotopy; see Lecture 18 of [Lur10] for a proof. The map lifts uniquely to the connective cover, giving a map

$$BP \langle 1 \rangle \to ku,$$

and a look at homotopy groups shows that this is an equivalence.

It now remains to calculate $H_\ast(BP \langle 1 \rangle; \mathbb{Z}/2)$. In the next section, we will review the calculation of $H_\ast(BP \langle n \rangle; \mathbb{Z}/2)$ for any $n$. 
3.5. The homology of $\mathcal{B}P\langle n \rangle$. In this section, we work through the computation of $H_\ast(\mathcal{B}P\langle n \rangle; \mathbb{Z}/2)$, which we need to calculate $H_\ast(ku; \mathbb{Z}/2)$ (and later for the calculation of $H_\ast(\text{tuf}; \mathbb{Z}/2)$). This calculation is well-known (for instance, the result is stated in [Rav86]), but we include it to be self-contained. A different approach to this calculation is presented in [LN12a].

We will use the following (slightly abusive) definition of $\mathcal{B}P\langle n \rangle$. Recall that $MU$ is an $E_\infty$-ring spectrum and that $\mathcal{B}P$ is a module over it.

**Definition 3.9.** A $MU$-module spectrum $M$ is said to be a form of $\mathcal{B}P\langle n \rangle$ (at the prime $p$) if there exist elements $v_{n+1}, v_{n+2}, \ldots \in \pi_\ast MU$ such that one has an equivalence of $MU$-modules

$$BP/\langle v_{n+1}, v_{n+2}, \ldots \rangle \simeq M,$$

and such that the $v_k \in \pi_\ast MU, k \geq n + 1$ project to generators of the indecomposables in $\pi_{2(p^k-1)}BP$ (not necessarily the ones described earlier). In particular, the homotopy groups of $M$ are given by $BP_\ast/\langle v_{n+1}, v_{n+2}, \ldots \rangle \simeq \mathbb{Z}_p[v_1, \ldots, v_n]$.

In general, it does not seem easy to tell whether an $MU$-module spectrum is a form of $\mathcal{B}P\langle n \rangle$ simply from looking at its homotopy groups. However, the following examples are useful:

**Example 3.10.** Suppose $M$ is an $MU$-module spectrum with $M_\ast \simeq BP_\ast/\langle v_{n+1}, v_{n+2}, \ldots \rangle \simeq \mathbb{Z}_p[v_1, v_2, \ldots, v_n]$ Suppose $M$ admits the structure of a ring spectrum inducing the natural ring structure on $\pi_\ast$. Then $M$ is a form of $\mathcal{B}P\langle n \rangle$.

In fact, we choose indecomposables $v_{n+1}, v_{n+2}, \ldots$ generating the kernel of $BP_\ast \to M_\ast$ under the specified map $\phi: BP \to M$ of $MU$-modules. Then consider the cofiber sequence in $MU$-modules

$$\Sigma^{[v_{n+1}]} BP/\langle v_{n+1}, v_{n+2}, \ldots \rangle \to BP/\langle v_{n+1}, v_{n+2}, \ldots \rangle \to \Sigma^{[v_{n+1}]+1} BP.$$

This produces a long exact sequence beginning

$$[\Sigma^{[v_{n+1}]}+1 BP, M]_k \to [BP/\langle v_{n+1}, v_{n+2}, \ldots \rangle, M]_k \to [BP, M]_k \to [\Sigma^{[v_{n+1}]} BP, M]_k,$$

where the maps are in the category of $MU$-modules and where the subscript $k$ denotes maps of degree $k$. Suppose further that $k$ is even. The last map is the zero map and $[\Sigma^{[v_{n+1}]}+1 BP, M]_k = 0$ as $\pi_\ast M$ is concentrated in even degrees. We find that, for even $k$,

$$[BP/\langle v_{n+1}, v_{n+2}, \ldots \rangle, M]_k \simeq [BP, M]_k,$$

and in particular, taking $k = 0$, we can produce a map $BP/\langle v_{n+1}, v_{n+2}, \ldots \rangle \to M$ inducing the natural map on $\pi_\ast$. We can similarly produce maps $BP/\langle v_{n+m} \rangle \to M$ for $m \geq 1$, and smashing these all together via the ring structure on $M$ (and taking a colimit) produces the desired equivalence

$$BP/\langle v_{n+1}, v_{n+2}, \ldots \rangle \simeq M.$$

**Example 3.11.** Suppose $M$ is an $MU$-module spectrum with $M_\ast \simeq BP_\ast/\langle v_3, v_4, \ldots \rangle \simeq \mathbb{Z}_p[v_1, v_2]$. Then the underlying spectrum of $M$ is equivalent to a form of $\mathcal{B}P\langle 2 \rangle$.

To see this, choose elements $v_1, v_2, v_3, v_4, \ldots \in \pi_\ast MU$ lifting the elements in $\pi_\ast BP$. Note first that $M[v_1^{-1}]$ and $M[v_2^{-1}]$ are Landweber-exact homology theories. For example, we have an isomorphism for any spectrum $X$,

$$M[v_1^{-1}]_\ast(X) \simeq BP_\ast(X) \otimes_{BP_\ast} \mathbb{Z}_p[v_1^{-1}, v_2].$$

In fact, the map comes from the $MU$-module structure on $M$, and since it is an equivalence for $X = \ast$ and both sides are homology theories (by Landweber’s criterion, Theorem 2.11), it is an equivalence for every $X$. It follows that we can promote $M[v_1^{-1}]$ to a ring spectrum. Similarly, $M[v_2^{-1}]$ is Landweber-exact, as is $M[(v_1v_2)^{-1}]$, and these two spectra become
ring spectra (in a manner inducing the natural ring structure on \( \pi_* \)). Moreover, these must be equivalent to the Landweber-exact spectra \( BP/(v_3, v_4, \ldots)[v_1^{-1}] \), \( BP/(v_3, v_4, \ldots)[v_2^{-1}] \), and \( BP/(v_3, v_4, \ldots)[(v_1v_2)^{-1}] \).

Now we consider a commutative diagram:

\[
\begin{array}{ccc}
M & \longrightarrow & M[v_1^{-1}] \\
\downarrow & & \downarrow \\
M[v_2^{-1}] & \longrightarrow & M[(v_1v_2)^{-1}]
\end{array}
\]

where the maps are the natural localization homomorphisms. We will now use the uniqueness of Landweber-exact spectra to identify this diagram with an analogous diagram for \( BP(2) \).

The homotopy category of (evenly graded) Landweber-exact ring spectra is equivalent to the category of homology theories that they define: for instance, there are no phantom maps between Landweber-exact ring spectra. In fact, we have the more precise result below.

A weak \( MU \)-module will mean a spectrum \( E \) together with a map (up to homotopy) \( MU \wedge E \to E \), such that the module axioms are satisfied up to homotopy (as opposed to coherent homotopy). There is a natural category of weak \( MU \)-modules and weak homomorphisms, where maps are homotopy classes of maps of spectra that respect the structure maps in the homotopy category. Given a weak \( MU \)-module \( E \), the homology theory \( E_* \) takes values in the homotopy category of \( MU_* \)-modules. We will say that \( E \) is Landweber-exact if the \( MU_* \)-module \( E_* \) satisfies the Landweber-exactness condition of Theorem 2.11.

**Proposition 3.12.** The following two categories are equivalent under \( \pi_* \):

(1) The category of evenly graded, Landweber-exact (unstructured) weak \( MU \)-module spectra.

(2) The category of \( MU_* \)-modules that satisfy the Landweber-exactness axiom.

This result is proved in Section 2.1 of [HS99].

In particular, in the category of weak \( MU \)-modules, we can identify the localizations of \( M \) with the appropriate localizations of \( BP(2) = BP/(v_3, v_4, \ldots) \) and form a square of spectra which commutes up to homotopy,

\[
\begin{array}{ccc}
M & \longrightarrow & BP(2)[v_1^{-1}] \\
\downarrow & & \downarrow \\
BP(2)[v_2^{-1}] & \longrightarrow & BP(2)[(v_1v_2)^{-1}]
\end{array}
\]

Choosing a homotopy, and taking connective covers, we get a map

\[
M \to \tau_{\geq 0} \left( BP(2)[v_1^{-1}] \times_{BP(2)[(v_1v_2)^{-1}]} BP(2)[v_2^{-1}] \right) \simeq BP(2),
\]

which we see is an equivalence by considering homotopy groups. Once again, this is an equivalence of spectra and not of \( MU \)-modules.

**Example 3.13.** Let \( p \geq 3 \). In a similar manner, it follows that if \( M \) is a \( MU \)-module spectrum with \( M_* \simeq (BP(2) \vee \Sigma^3 BP(2)) \), as \( MU_* \)-modules, then the underlying spectrum of \( M \) is equivalent to that of \( BP(2) \vee \Sigma^3 BP(2) \).

We will abuse notation and write \( BP(n) \) for any form of \( BP(n) \), and the \( v_i \) for any collection of indecomposable generators in \( \pi_{2(p^i-1)}(BP) \).
Let $p = 2$, throughout. Then we recall that

$$H_*(BP; \mathbb{Z}/2) \cong \mathbb{Z}/2[t_1, t_2, \ldots], \quad |t_i| = 2(2^i - 1).$$

In fact, we can express $\pi_*(HZ/2 \wedge BP)$ as the ring classifying a 2-typical formal group law together with an isomorphism with the additive one, together with the condition that $2 = 0$. We can see that the Hurewicz homomorphism $\pi_*(BP) \cong \mathbb{Z}[v_1, v_2, \ldots] \to H_*(BP; \mathbb{Z}/2)$ annihilates each $v_i$, because the formal group of $HZ/2 \wedge BP$ is isomorphic to the additive one.

There is a unique nonzero map $BP \to HZ/2$, and the map $H_*(BP; \mathbb{Z}/2) \to H_*(HZ/2; \mathbb{Z}/2)$ is an imbedding, realizing $H_*(BP; \mathbb{Z}/2)$ as the subcomodule $\mathbb{Z}/2[\xi_i] = \mathbb{Z}/2[\xi_2^2]$ of the dual Steenrod algebra $A_* = \mathbb{Z}/2[\xi_i]$. (This calculation is a fundamental piece of the computation of $\pi_*MU$ via the Adams spectral sequence: one first computes $H_*(MU; \mathbb{Z}/2)$.) We are going to see that $H_*(BP; \mathbb{Z}/2)$ is a larger subcomodule of $A_*$. First, fix an indecomposable generator $v_i \in \pi_{2(2^i - 1)}(BP)$ (where $v_0 = 2$) and consider the $BP$-module spectrum $BP/v_i$. There is a cofiber sequence

$$\Sigma^{|v_i|}BP \to BP \to BP/v_i \to \Sigma^{|v_i|+1}BP,$$

in the category of $MU$-module spectra. This induces a cofiber sequence of $HZ/2$-module spectra

$$HZ/2 \wedge \Sigma^{|v_i|}BP \to HZ/2 \wedge BP \to HZ/2 \wedge BP/v_i \to HZ/2 \wedge \Sigma^{|v_i|+1}BP.$$

However, the first map is nullhomotopic, because $v_i$ has image zero in homology. We thus get a short exact sequence of graded vector spaces:

$$0 \to H_*(BP; \mathbb{Z}/2) \to H_*(BP/v_i; \mathbb{Z}/2) \to H_*(BP/v_i; \mathbb{Z}/2) \to 0.$$

This is split exact in the category of vector spaces, but it need not be in the category of $A_*$-comodules.

**Lemma 3.14.** The above sequence (9) is not split exact in the category of $A_*$-comodules.

**Proof.** In fact, suppose (9) were split exact in the category of comodules. For simplicity, let $C = \mathbb{Z}/2[\xi_2]$ denote the subcomodule (in fact, subcoalgebra) of $A_*$ which is the homology of $BP$. Then

$$H_*(BP/v_i; \mathbb{Z}/2) = C \oplus C[2i+1 - 1],$$

where the brackets denote a shift. The Adams spectral sequence for $BP/v_i$ then has $E_2$-page equal to

$$\text{Ext}_{A_*}^{s,t}(\mathbb{Z}/2, C) \oplus \text{Ext}_{A_*}^{s,t-(2i+1-1)}(\mathbb{Z}/2, C),$$

and this is a module over the $E_2$-page of the (degenerate) ASS for $BP$ because $BP/v_i$ is a $MU$-module. As a module over the $E_2$-page of the ASS for $BP$, it is free on two generators.

The $E_2$-page of the ASS for $BP$ has a polynomial generator in position $(1, 2i+1 - 1)$ corresponding to $v_i$. This must remain a nonzero permanent cycle in the ASS for $BP/v_i$ if we have a splitting as above, and it cannot be a boundary since it is in filtration one. This means that $v_i$ actually represents a nontrivial element in homotopy for $BP/v_i$, which is absurd. It follows that there can be no such splitting.

Next, let us return to the reduction morphism $BP \to HZ/2$. It canonically factors $BP \to BP/v_i \to HZ/2$.

**Lemma 3.15.** The map $BP/v_i \to HZ/2$ induces an injection on homology.
Proof. In fact, suppose otherwise. Let us use the notation $C = H_*(BP; \mathbb{Z}/2)$ of the previous proof. Recall that $H_*(BP/v; \mathbb{Z}/2)$ sits inside an exact sequence of $C$-modules $0 \to C \to H_*(BP/v; \mathbb{Z}/2) \to \mathbb{C}^{[2i+1]} \to 0$. It is thus a free module of rank two over $H_*(BP; \mathbb{Z}/2) = C$, and the first summand injects into $A_*$. Since one summand (in $C$-modules) is in even degrees and the other summand is in odd degrees, we find that if the map

$$H_*(BP/v; \mathbb{Z}/2) \to A_*$$

is not an injection, then the (unique) element in degree $2i+1 - 1$ must be sent to zero. It follows that the kernel of $H_*(BP/v; \mathbb{Z}/2) \to A_*$ is an $A_*$-submodule of $H_*(BP/v; \mathbb{Z}/2)$ which gives a splitting of $H_*(BP/v; \mathbb{Z}/2)$ in comodules, which we saw above cannot occur. □

It remains to determine the image of $BP/v \to \mathcal{H}/2$ in homology. We know that, as a $C$-module, the image is generated by 1 and by an element in degree $2i+1 - 1$. The next result identifies this element.

**Proposition 3.16.** The image of $BP/v \to \mathcal{H}/2$ in homology is given by the subring (of $A_*$) $\mathbb{Z}/2[\xi_1^2, \xi_2^2, \ldots, \xi_n^2, \xi_{n+1}, \xi_{n+2}, \ldots]$. 

Proof. In fact, we know that there is a nonzero element $x \in A_*$ which is in degree $2i+1 - 1$ and which is in the image of $H_*(BP/v; \mathbb{Z}/2)$. We observe that this element is primitive in $A_*/C$, as it is an element of minimal degree in the submodule $H_*(BP/v; \mathbb{Z}/2)/C$.

But the claim is that there is precisely one primitive element of $A_*/C$ in degree $2i+1 - 1$. In fact, primitive elements are given by $\text{Prim}(\cdot) = \text{Hom}_{A_*}(\mathbb{Z}/2, \cdot)$, so we have to compute $\text{Hom}_{A_*}(\mathbb{Z}/2, A_*/C)$. We can do this using the short exact sequence of comodules

$$0 \to C \to A_* \to A_*/C \to 0,$$

and applying $\text{Hom}_{A_*}(\mathbb{Z}/2, \cdot)$ to this gives an exact sequence

$$0 \to \text{Prim}(C) \to \text{Prim}(A_*) \to \text{Prim}(A_*/C) \to \text{Ext}^1_{A_*}(\mathcal{H}/2, C) \to 0.$$

Now $\text{Prim}(C) = \text{Prim}(A_*) = \mathbb{Z}/2$ in degree zero; note that Prim refers to comodule primitives. So we have an isomorphism, by the change-of-rings theorem (see appendix 1 of [Rav86]),

$$\text{Prim}(A_*/C) \cong \text{Ext}^1_{A_*}(\mathcal{H}/2, C) \cong \text{Ext}^1_{A_*/C}(\mathbb{Z}/2, \mathcal{H}/2).$$

However, $A_*/C$ is an exterior algebra on generators of degree $2j - 1$, $j \in \mathbb{N}$. So the relevant $\text{Ext}^1$ group has one generator in bidegree $(1, 2j - 1)$ for each $j$. It follows that in degree $2i+1 - 1$, there is only one nonzero primitive element of $A_*/C$. One checks easily that $\xi_{n+1}$ is primitive from (S). We then conclude that $H_*(BP/v; \mathbb{Z}/2)$ is free over $C$ on 1 and $\xi_{n+1}$. □

It is now straightforward to deduce the following.

**Proposition 3.17.** The map $BP \langle n-1 \rangle \to \mathcal{H}/2$ induces an injection on homology, with image

$$\mathbb{Z}/2[\xi_1^2, \xi_2^2, \ldots, \xi_n^2, \xi_{n+1}, \xi_{n+2}, \ldots].$$

Proof. It suffices to compute the homology of $BP/(v_n, v_{n+1}, \ldots, v_{n+k})$, as we can obtain $BP \langle n-1 \rangle$ as a colimit of such $BP$-modules. In fact, we claim that the map

$$BP/(v_n, v_{n+1}, \ldots, v_{n+k}) \to \mathcal{H}/2$$

induces an injection in homology with image equal to the subring of $A_*$ given by $\mathbb{Z}/2[\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+k+1}, \xi_{n+k+2}, \ldots]$. We have proved this when $k = 0$. In general, we find by the same logic that the homology $H_*(BP/(v_n, v_{n+1}, \ldots, v_{n+k}); \mathbb{Z}/2)$ is a
free module of rank $2^{k+1}$ over $C$. Moreover, we can describe $BP/(v_n, v_{n+1}, \ldots, v_{n+k})$ as the smash product

$$BP/v_n \wedge_{MU} BP/v_{n+1} \wedge_{MU} \cdots \wedge_{MU} BP/v_{n+k},$$

which produces a map $BP/v_n \wedge \cdots \wedge BP/v_{n+k} \to BP/(v_n, \ldots, v_{n+k})$. Taking images in homology now shows that the image of $BP/(v_n, v_{n+1}, \ldots, v_{n+k}) \to HZ/2$ in homology contains the desired ring. Since this ring is free of rank $2^{k+1}$ over $H_*(BP; \mathbb{Z}/2)$, this completes the proof. □

**Remark 3.18.** We will see in the next section that comodule subalgebras of $A_*$ are fairly rigid objects. If we knew that $H_*(BP(\langle n \rangle ; \mathbb{Z}/2) \subset A_*$ and that it was a comodule algebra, then we could argue directly using the arguments of the next section: for instance, the proof of Proposition 3.16 could be arranged analogously to Proposition 3.19.

**3.6. Calculation of $H_*(ko; \mathbb{Z}/2)$**. Let us now describe how Wood’s theorem implies Theorem 3.1. Namely, we know from Proposition 3.17 and the 2-local equivalence $ku \simeq BP \langle 1 \rangle$ that the map $H_*(ku; \mathbb{Z}/2) \to H_*(HZ/2; \mathbb{Z}/2) = A_*$ is an injection, with image

$$\mathbb{Z}/2[\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \ldots] \subset A_*.$$

Recall that $\Sigma^{-2}\mathbb{CP}^2$ is the cofiber of the Hopf map $\eta: S^1 \to S^0$, and in particular receives a map from $S^0$. We have a commutative diagram

$$
\begin{array}{ccc}
ko & \xrightarrow{f} & ku \\
\downarrow & & \downarrow \\
HZ/2 & \xrightarrow{\simeq} & ko \wedge \Sigma^{-2}\mathbb{CP}^2
\end{array}
$$

such that $f$ (which is the natural map) is an injection on mod 2 homology by Wood’s theorem. It follows that the map $H_*(ko; \mathbb{Z}/2) \to A_*$ is an injection, and imbeds $H_*(ko; \mathbb{Z}/2)$ as a subcomodule of $A_*$. Since $ko \to HZ/2$ is also a morphism of ring spectra, $H_*(ko; \mathbb{Z}/2)$ is actually a comodule subalgebra of $A_*$, contained in $H_*(ko; \mathbb{Z}/2)$. Moreover, we know that as an $H_*(ko; \mathbb{Z}/2)$-module, we have

$$H_*(ko; \mathbb{Z}/2) \simeq H_*(ko; \mathbb{Z}/2) \oplus H_{*-2}(ko; \mathbb{Z}/2)$$

in virtue of Wood’s theorem again. This in particular gives the graded dimension of $H_*(ko; \mathbb{Z}/2)$.

The calculation of $H_*(ko; \mathbb{Z}/2)$ given in Theorem 3.1 will now follow from the next purely algebraic proposition.

**Proposition 3.19.** A comodule subalgebra $C \subset H_*(ko; \mathbb{Z}/2)$ of the above graded dimension is necessarily $\mathbb{Z}/2[\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \ldots]$.\]

We need a few lemmas. Let $k$ be a field. As usual, a nonnegatively graded $k$-algebra $R$ is called **connective** if $k \to R_0$ is an isomorphism, and a graded $R$-module $M$ is called **connective** if it is zero in negative degrees. Given $R$, we let $\overline{R} = \bigoplus_{i>0} R_i$. We recall the following:

**Lemma 3.20** (Nakayama’s lemma). Let $R$ be a graded, connected $k$-algebra with augmentation ideal $\overline{R}$.

1. Let $M \to N$ be a morphism of connective $R$-modules such that $M/\overline{R}M \to N/\overline{R}N$ is surjective. Then $M \to N$ is surjective.
2. If $M$ is a connective $R$-module which is flat, then it is free (in particular, if in addition $M \neq 0$, then $M$ is faithfully flat).
(3) If $M \to N$ is a morphism of free $R$-modules such that $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is an injection, then $M \to N$ is a split injection.

An inclusion of commutative Hopf algebras over a field is always faithfully flat (see [Wat79]). We need a slight modification of this fact, even though we already know the statement of the lemma for $H_*(ko;\mathbb{Z}/2) \subset H_*(ku;\mathbb{Z}/2)$.

**Lemma 3.21.** Let $B$ be a commutative graded, connected Hopf algebra over a field $k$, which is a domain, and let $A \subset A' \subset B$ be comodule subalgebras. Then $A'$ is graded free over $A$.

By “commutative graded,” we do not mean “graded-commutative.” The lemma is false without the graded and connected hypotheses: for instance, consider the ring of functions on the multiplicative group, $k[t, t^{-1}]$ (a Hopf algebra with $\Delta(t) = t \otimes t$), and the comodule subalgebra $k[t]$. Geometrically, this corresponds to the $\mathbb{G}_m$-variety $\mathbb{A}^1$: the natural inclusion map $\mathbb{G}_m \to \mathbb{A}^1$ (of $\mathbb{G}_m$-varieties) is not faithfully flat. We remark that the existence of such structure theorems for comodule algebras in the graded, connected case goes back to [MM65].

**Proof.** We will show that $B$ is faithfully flat over $A$. This also implies that $B$ is faithfully flat over $A'$, and combining these observations shows that $A'$ is faithfully flat over $A$. Lemma 3.20 will then imply that $A'$ is a free $A$-module. In other words, we may assume $A' = B$.

Observe first that we can write the inclusion $A \subset B$ as a filtered colimit of inclusions

$$A_j \subset B_j, \quad j \in J,$$

where $B_j \subset B$ is a finitely generated commutative graded, connected Hopf subalgebra, and $A_j \subset B_j$ are finitely generated subalgebras which are also comodules for $B_j$. We will prove that each of the inclusions $A_j \subset B_j$ is faithfully flat, which will suffice; in fact we only need to check flatness by Lemma 3.20 again. Thus, we may assume $A$ and $B$ are finitely generated. Then $\text{Spec}B$ is an affine group scheme $G$, of finite type over $k$, and $X = \text{Spec}A$ is a scheme acted on by $G$. There is a map of $G$-schemes

$$p: G \to X$$

which is dominant (since $A \subset B$). Since $X$ is an integral scheme, generic flatness implies that there exists a nonempty open subset $U \subset X$ such that $p^{-1}(U) \to U \to X$ is flat. Thus for any $g \in G(k)$, we have that

$$gp^{-1}(U) \to X$$

is flat, and since $G$ is the union of the $gp^{-1}(U), g \in G$, we find that $G \to X$ is flat.

**Proof of Proposition 3.19.** Suppose $\mathcal{C}$ is any comodule subalgebra of $H_*(ku;\mathbb{Z}/2)$ with the graded dimension of $\mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \ldots]$ (which we observe is in fact a comodule algebra). We have just seen that $H_*(ku;\mathbb{Z}/2)$ is a free graded $\mathcal{C}$-module. Counting dimensions shows that it has rank two.

Suppose now $\mathcal{C}, \mathcal{C}' \subset H_*(ku;\mathbb{Z}/2)$ are two comodule algebras with the same graded dimension as the desired object $\mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \ldots]$. Then the algebra $\mathcal{C}'' \subset H_*(ku;\mathbb{Z}/2)$ generated by $\mathcal{C}, \mathcal{C}'$ is still a comodule algebra contained in $A_*$. Observe that $\mathcal{C}'' \neq H_*(ku;\mathbb{Z}/2)$ (there is no element in degree two). However, $H_*(ku;\mathbb{Z}/2)$ is a free module over $\mathcal{C}''$, and since $\mathcal{C}'' \neq H_*(ku;\mathbb{Z}/2)$, the rank must be at least two. This contradicts the fact that the graded dimension of $\mathcal{C}''$ is too large unless $\mathcal{C} = \mathcal{C}' = \mathcal{C}''$: in particular $\mathcal{C} = \mathcal{C}' = \mathcal{C}'' = \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \ldots]$.

\[\square\]
As a consequence, if $A(1) \subset A$ is the eight-dimensional subalgebra generated by $\text{Sq}^1$ and $\text{Sq}^2$, then

$$H^*(ko; \mathbb{Z}/2) \simeq A \otimes_{A(1)} \mathbb{Z}/2,$$

and (by the change-of-rings theorem) the Adams spectral sequence for $ko$ runs

$$\text{Ext}_{A(1)}^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow \pi_{t-s}(ko) \otimes \mathbb{Z}_2.$$ 

### 3.7. The stack for $ko$

Using these ideas, we can re-derive an explicit description of the stack for $ko$, originally due to Hopkins (sketched in [Hop07]). The strategy is to write down an explicit graded Hopf algebroid presentation of the stack $B\mathbb{Z}/2$ (which corresponds to $KO$-theory), and then to take a “connective” version of this stack to match $ko$. We will take the same point of view in studying the stack associated to $\text{tmf}$ later.

**Definition 3.22.** We define the *moduli stack of quadratic equations* $\mathcal{M}_{qd}$ as follows.

$\mathcal{M}_{qd}$ is the stack associated to the prestack $\text{Ring} \to \text{Gpd}$ assigning to a ring $R$ the groupoid whose objects are quadratic equations $x^2 + bx + c = 0$ (with $b, c \in R$) and where an isomorphism between two quadratic equations $x^2 + bx + c = 0$ and $x^2 + b'x + c' = 0$ is an affine-linear transformation $x \mapsto ux + t$ (with $u \in R^*, t \in R$) that takes the first equation into the second. In other words, $\mathcal{M}_{qd}$ is the stack associated to the graded Hopf algebroid

$$\mathbb{Z}[b, c] \xrightarrow{\eta_R} \mathbb{Z}[b, c, t], \quad |b| = 2, |c| = 4, |t| = 2,$$

where the right unit is given by

$$\eta_R(b) = b + 2t, \quad \eta_R(c) = c + t^2 + bt,$$

and where the diagonal map is given via

$$\Delta(t) = t \otimes 1 + 1 \otimes t.$$

The stack $\mathcal{M}_{qd}$ contains an open substack $\mathcal{M}_{qd}^{ns}$ of nonsingular quadratic equations: these are quadratic equations such that the subscheme of $\mathbb{A}^1_R$ cut out by them is smooth. (More explicitly, a quadratic equation $x^2 + bx + c$ over $R$ is nonsingular if and only if for every base-change $R \to k$, for $k$ an algebraically closed field, the quadratic has distinct roots in $k$.) There is an equivalence

$$\mathcal{M}_{qd}^{ns} \simeq B\mathbb{Z}/2,$$

because any two-fold étale cover of a ring $R$ comes from adjoining an element $\alpha$ subject to a nonsingular quadratic equation.

More generally, we have the following modular interpretation of $\mathcal{M}_{qd}$:

**Proposition 3.23.** Given a commutative ring $R$, the $R$-valued points $\mathcal{M}_{qd}$ are the two-fold flat covers of $\text{Spec} R$.

**Proof.** In fact, we will check this at the level of prestacks. The prestack assigning to $R$ the groupoid quadratic equations over $R$ and affine-linear isomorphisms between them is equivalent to the prestack assigning to $R$ the groupoid of $R$-algebras which are free of rank two. Stackifying forms $\mathcal{M}_{qd}$ on the one hand, and the moduli stack of flat covers of rank two on the other. \qed

Since $\mathcal{M}_{qd}^{ns}$ is equivalent to $B\mathbb{Z}/2$, we should expect a two-fold cover of $\mathcal{M}_{qd}$ by $\text{Spec} \mathbb{Z}$. We can explicitly exhibit this as follows. Observe that there is a map

$$\text{Spec} \mathbb{Z} \simeq (\text{Spec} \mathbb{Z}[s, s^{-1}])/\mathbb{G}_m \to \mathcal{M}_{qd}^{ns}, \quad |s| = 2,$$

classifying the quadratic equation $x^2 + sx = 0$ over $\mathbb{Z}[s, s^{-1}]$, or the quadratic equation $x^2 + x = 0$ over $\text{Spec} \mathbb{Z}$.
Moreover, translation by \( s \) defines an isomorphism of this quadratic equation with the quadratic equation \( x^2 - sx = 0 \); that is, we get a factorization
\[
\text{(Spec}\mathbb{Z})/(\mathbb{Z}/2) \simeq \left(\text{Spec}\mathbb{Z}[s, s^{-1}]\right)/\mathbb{G}_m \to \mathcal{M}_{\text{qd}}^{\text{ns}}
\]
where the \( \mathbb{Z}/2 \)-action on \( \mathbb{Z}[s, s^{-1}] \) sends \( s \mapsto -s \). The map (10) is an equivalence, because any nonsingular quadratic equation is étale locally isomorphic to one of the form \( x^2 + sx = 0 \) for \( s \) invertible, and the automorphisms are accounted for. We get the “tautological” two-fold étale cover of \( \mathcal{M}_{\text{qd}}^{\text{ns}} \).

More generally, we have a morphism:
\[
(\text{Spec}\mathbb{Z}[s])/\mathbb{G}_m \to \mathcal{M}_{\text{qd}}
\]
classifying the quadratic equation \( x^2 + sx = 0 \).

**Proposition 3.24.** The map \( (\text{Spec}\mathbb{Z}[s])/\mathbb{G}_m \to \mathcal{M}_{\text{qd}} \) is the tautological two-fold flat cover.

In fact, we will see below that it corresponds to the morphism \( ko \to ku \).

**Proof.** We need to show that for any map \( \text{Spec}R \to \mathcal{M}_{\text{qd}} \) classifying a quadratic equation \( x^2 + bx + c = 0 \) for \( b, c \in R \), the fiber product \( \text{Spec}R \times_{\mathcal{M}_{\text{qd}}} (\text{Spec}\mathbb{Z}[s])/\mathbb{G}_m \) is the associated finite flat cover of rank two. The above fiber product parametrizes isomorphisms given by translations between the quadratic equation \( x^2 + bx + c = 0 \) and a quadratic equation of the form \( x^2 + sx = 0 \), and any such is given by \( x \mapsto x + w \) where \( w^2 + bw + c = 0 \). In other words, the fiber product is precisely \( \text{Spec}R[w]/(w^2 + bw + c) \); this is the “tautological” flat cover.

We also note that the stack \( \mathcal{M}_{\text{qd}} \) simplifies considerably when 2 is inverted. Namely, when 2 is inverted, one can complete the square, and write every quadratic equation uniquely (up to a square) in the form \( x^2 + a \). It follows that
\[
\mathcal{M}_{\text{qd}} \times_{\text{Spec}\mathbb{Z}} \text{Spec}\mathbb{Z}[1/2] \simeq (\text{Spec}\mathbb{Z}[1/2, a])/\mathbb{G}_m, \quad |a| = 4.
\]
We will see below that this is an algebraic analog of the fact that \( ko[1/2] \) is complex-orientable: in fact, it is a summand of \( ku \).

**Proposition 3.25** (Hopkins [Hop07]). We have equivalences \( \text{Stack}(KO) \simeq B\mathbb{Z}/2 \simeq \mathcal{M}_{\text{qd}}^{\text{ns}} \) and \( \text{Stack}(ko) \simeq \mathcal{M}_{\text{qd}} \).

**Proof.** Note first that the stacks for \( ko \) and \( KO \) exist: that is, \( MU_* (KO) \) and \( MU_* (ko) \) are evenly graded. Since \( KO \) is obtained by inverting an eight-dimensional class in \( ko \), it suffices to check this for \( ko \). Since \( ko \land \Sigma^{-2}\mathbb{CP}^2 \simeq ku \) is a complex-orientable theory, we find that
\[
MU_* (ko \land \Sigma^{-2}\mathbb{CP}^2) \simeq MU_* (ku) \simeq ku_* [b_1, b_2, \ldots], \quad |b_i| = 2i;
\]
in fact for any complex-oriented theory \( E \), we have \( E_* (MU) \simeq E_* [b_1, b_2, \ldots] \). This is evenly graded. However, since \( MU \land ko \) is complex-orientable (thanks to the first factor), we find that
\[
\pi_* (MU \land ko \land \Sigma^{-2}\mathbb{CP}^2) \simeq MU_* (ko) \oplus MU_* (ko),
\]
and if the left side is evenly graded, \( MU_* (ko) \) must be as well.

We begin by computing \( \text{Stack}(KO) \). By Corollary 3.6, we know that \( MU \land KO = (MU \land KU)^{h\mathbb{Z}/2} \), so we need an algebro-geometric description of the sheaf \( \mathcal{F}(MU) = \mathcal{F}_{\text{even}}(MU) \) on \( B\mathbb{Z}/2 \) in order to run the descent spectral sequence for \( KO_* (MU) \). The sheaf \( \mathcal{F}(MU) \) associated to \( MU \) on \( M_{FG} \) comes from the pushing forward the structure sheaf under the map \( \text{Spec}L/\mathbb{G}_m \to M_{FG} \), where \( L = MU_* \) is the Lazard ring. See Example 2.9.
Consequently, the sheaf associated to $MU$ on $\mathbb{BZ}/2$ comes from vertically pushing forward the structure sheaf in the diagram:

$$\begin{align*}
\text{Spec}L/\mathbb{G}_m \times_{\mathcal{M}_{FG}} \mathbb{BZ}/2 & \longrightarrow \text{Spec}L/\mathbb{G}_m. \\
\mathbb{BZ}/2 & \longrightarrow \mathcal{M}_{FG}.
\end{align*}$$

**Lemma 3.26.** The morphism $\mathbb{BZ}/2 \to \mathcal{M}_{FG}$ is affine.

**Proof.** We note that the lemma is plausible for the following reason. Let $L$ be the Lazard ring. There is a faithfully flat, affine cover $\text{Spec}L \to \mathcal{M}_{FG}$. We form the pull-back $\mathbb{BZ}/2 \times_{\mathcal{M}_{FG}} \text{Spec}L$; the lemma is equivalent to the statement that this is an affine scheme. We first show that the stack $\mathbb{BZ}/2 \times_{\mathcal{M}_{FG}} \text{Spec}L$ is set-valued rather than groupoid-valued. Given a ring $R$, the $R$-valued points of the stack $\mathbb{BZ}/2 \times_{\mathcal{M}_{FG}} \text{Spec}L$ are given by one-dimensional algebraic tori over $R$, together with a coordinate on their formal group (since $\text{Spec}L$ classifies a formal group with a coordinate). The groupoid of algebraic tori together with a coordinate is discrete: in fact, any automorphism $f : T \to T$ of a one-dimensional algebraic torus $T$ whose formal completion $\hat{f} : \hat{T} \to \hat{T}$ is the identity to order three is the identity.

It is possible to show that any stack, affine over $\mathbb{BZ}/2$ and taking values in sets (i.e., discrete groupoids), is itself affine if the stack is of finite type: this reduces to the assertion that if $X = \text{Spec}A$ is an affine scheme with a $\mathbb{Z}/2$-action that is free on the “functor of points,” then the induced map $X \to \text{Spec}A^{\mathbb{Z}/2}$ is an $\mathbb{Z}/2$-torsor (so that $\text{Spec}A^{\mathbb{Z}/2}$ is the stacky quotient of $\text{Spec}A$ by $\mathbb{Z}/2$). This fact is known when $X$ is noetherian. We do not know how to apply this to the very non-noetherian setting we are in here, though.

However, it is possible to make the following argument. Rather than using $L$, let $\mathcal{M}_{FG}(n)$ be the moduli stack of formal groups together with a coordinate modulo degree $n$; then $\mathcal{M}_{FG}(n)$ is an affine flat cover of $\mathcal{M}_{FG}$ and is of finite type over $\mathcal{M}_{FG}$. It thus suffices to show that $\mathbb{BZ}/2 \times_{\mathcal{M}_{FG}} \mathcal{M}_{FG}(n)$, which is a stack affine over $\mathbb{BZ}/2$ and of finite type, is in fact an affine scheme. Note that if $n \geq 3$, then the stack $\mathbb{BZ}/2 \times_{\mathcal{M}_{FG}} \mathcal{M}_{FG}(n)$ takes values in sets instead of groupoids. We now appeal to the following sublemma to complete the proof.

**Sublemma 3.27.** Let $\mathfrak{X} \to \mathbb{BZ}/2$ be an affine morphism of finite type. Suppose $\mathfrak{X}$ takes values in sets. Then $\mathfrak{X}$ is an affine scheme.

We learned this argument from [KMS5].

**Proof.** By definition, there is a $\mathbb{Z}/2$-torsor $\mathfrak{X}' \to \mathfrak{X}$, classified by the map $\mathfrak{X} \to \mathbb{BZ}/2$. In other words, $\mathfrak{X}' = \mathfrak{X} \times_{\mathbb{BZ}/2} \text{Spec} \mathbb{Z}$ is an affine scheme, equal to $\text{Spec} R$ for some noetherian ring $R$, and $\mathfrak{X} \simeq \mathfrak{X}'/\mathbb{Z}/2)$. However, the action of $\mathbb{Z}/2$ on the functor represented by $\text{Spec} R$ must be free, since there are no nontrivial automorphisms in the values of the quotient stack $\mathfrak{X}'/\mathbb{Z}/2$. It follows that $R^{\mathbb{Z}/2}$ is étale (by Corollary 2.4 of Exposé V in [SGA03]) and a $\mathbb{Z}/2$-torsor; therefore $\mathfrak{X} \simeq \text{Spec} R^{\mathbb{Z}/2}$. □

We now apply the descent spectral sequence to compute $MU \wedge KO$. Recall that the sheaf $\mathcal{F}(MU)$ on $\mathcal{M}_{FG}$ is the push-forward of the structure sheaf under $(\text{Spec}L)/\mathbb{G}_m \to \mathcal{M}_{FG}$; see Example 2.24. In particular, the pull-back of $\mathcal{F}(MU)$ to $\mathbb{BZ}/2$ is the push-forward of the structure sheaf under

$$(\text{Spec}L)/\mathbb{G}_m \times_{\mathcal{M}_{FG}} \mathbb{BZ}/2 \to \mathbb{BZ}/2.$$
Since the morphism $B\mathbb{Z}/2 \to M_{FG}$ is affine, the source of the above morphism is an affine scheme, and we find that the descent spectral sequence for $MU \wedge KO$ degenerates. Therefore:

$$(MU \wedge KO)_s = \bigoplus_{j \in \mathbb{Z}} \Gamma(\text{Spec} L/G \times_{M_{FG}} B\mathbb{Z}/2, \omega^j) = \Gamma(\text{Spec} L \times_{M_{FG}} B\mathbb{Z}/2, \mathcal{O}).$$

We can run a similar argument with $MU$ replaced by the smash powers $MU \wedge s$, $s > 0$, which are wedges of suspensions of $MU$.

We conclude that $\text{Stack}(KO)$ is the homotopy colimit (in stacks) of

$$\ldots \to \text{Spec} L/G \times_{M_{FG}} \text{Spec} L \times_{M_{FG}} B\mathbb{Z}/2 \to \text{Spec} L \times_{M_{FG}} B\mathbb{Z}/2,$$

which is precisely a presentation of the stack $B\mathbb{Z}/2$. We also note that the map $\text{Stack}(KU) \to \text{Stack}(KO)$ is identified with the tautological étale cover of $B\mathbb{Z}/2$.

We now describe $\text{Stack}(ko)$. There is a diagram of $E_\infty$-ring spectra

$$ko \longrightarrow KO,$$

$$ku \longrightarrow KU$$

which leads to a diagram of stacks

$$\text{Stack}(KU) \longrightarrow \text{Stack}(ku).$$

$$\text{Stack}(KO) \longrightarrow \text{Stack}(ko)$$

Observe that $\text{Stack}(KU) \simeq (\text{Spec} \mathbb{Z}[t^{\pm 1}])/G_m$ and $\text{Stack}(ku) \simeq (\text{Spec} \mathbb{Z}[t])/G_m$ as $KU$ and $ku$ are complex-orientable. Moreover, the diagram is cartesian, because $KU \simeq ku \otimes_{ko} KO$, and the vertical maps are two-fold flat covers by Wood’s theorem.

We can thus rewrite the diagram as:

$$(\text{Spec} \mathbb{Z}[t^{\pm 1}])/G_m \longrightarrow (\text{Spec} \mathbb{Z}[t])/G_m.$$

$$\text{Stack}(KO) \longrightarrow \text{Stack}(ko)$$

Observe that $(\text{Spec} \mathbb{Z}[t^{\pm 1}])/G_m \to \text{Stack}(KO)$ is the tautological two-fold cover of $B\mathbb{Z}/2$. Since $(\text{Spec} \mathbb{Z}[t])/G_m \to \text{Stack}(ko)$ is a two-fold cover, we can extend this diagram out to

$$(\text{Spec} \mathbb{Z}[t^{\pm 1}])/G_m \longrightarrow (\text{Spec} \mathbb{Z}[t])/G_m \simeq (\text{Spec} \mathbb{Z}[t])/G_m.$$

$$\text{Stack}(KO) \longrightarrow \text{Stack}(ko) \longrightarrow M_{qd}$$

We can see that the top map $(\text{Spec} \mathbb{Z}[t])/G_m \to (\text{Spec} \mathbb{Z}[t])/G_m$ is an equivalence (in fact, the identity!) because we identified the map $(\text{Spec} \mathbb{Z}[t^{\pm 1}])/G_m \to (\text{Spec} \mathbb{Z}[t])/G_m$ with the inclusion. We note that all maps in question of quotient stacks of the form $(\text{Spec} R)/G_m$ that arise here can be simply thought of as maps of graded rings: equivalently, all maps $(\text{Spec} R)/G_m \to (\text{Spec} R’)/G_m$ preserve the tautological bundles over each.

Since the diagram is cartesian, and since equivalences descend under faithfully flat base change, we conclude that $\text{Stack}(ko) \to M_{qd}$ is an equivalence. \qed
In particular, for any connective even spectrum $X$, we find that $X$ defines a quasi-coherent sheaf $F(X)$ on $\mathcal{M}_{qd}$, and there is a spectral sequence

$$H^i(\mathcal{M}_{qd}, F(X) \otimes \omega^j) \Rightarrow \pi_{2j-i}(ko \wedge X).$$

Since $\mathcal{M}_{qd}$ is a stack presented by a fairly small Hopf algebroid, calculations of the relevant cohomology groups can often be done explicitly (e.g. using the cobar complex). This is effectively the Adams-Novikov spectral sequence applied to $ko \wedge X$. In particular, we can use this strategy to compute $\pi_*ko$, although we note that the gap in $\pi_*KO$ was necessary to run the argument.

In the rest of the paper, we will carry out the analogous computation for tmf.

### 3.8. A modular interpretation of $\mathcal{M}_{qd} \rightarrow M_{FG}$.

Although we have completed the calculation of $\text{Stack}(ko)$, it seems worthwhile to describe the morphism of stacks $\text{Stack}(ko) \simeq \mathcal{M}_{qd} \rightarrow M_{FG}$, and we digress in this subsection to do so. For $\text{Stack}(KO) \simeq B\mathbb{Z}/2$, we observed that a $\mathbb{Z}/2$-torsor over a scheme $X$ could produce a one-dimensional torus over $X$, since $\text{Aut}(\mathbb{G}_m) \simeq \mathbb{Z}/2$, and the formal completion of that was a formal group law. Here we give a more explicit version of that construction, which generalizes to two-fold flat covers.

Let $f \colon Y \rightarrow X$ be a two-fold flat cover. One has a morphism

$$\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y,$$

which is a locally split injection of vector bundles on $X$, whose cokernel is a line bundle $\mathcal{L}$ on $X$. If 2 is invertible on $X$, then the trace gives a splitting of the above morphism. (See Exercise 2.7 of Chapter 4 of [Har77] for a discussion of this construction when $X$ is a curve in characteristic $\neq 2$, in the étale case.) If we pull back to $Y$, we have an exact sequence

$$0 \rightarrow \mathcal{O}_Y \rightarrow f^*f_*\mathcal{O}_Y \rightarrow f^*\mathcal{L} \rightarrow 0,$$

and the last surjection induces a map

$$Y \rightarrow \mathbb{P}(f_*\mathcal{O}_Y),$$

where $\mathbb{P}(f_*\mathcal{O}_Y)$ is the projectivization of the two-dimensional vector bundle $f_*\mathcal{O}_Y$ on $X$. In local coordinates, if $X = \text{Spec} R, Y = \text{Spec} R[y]/(y^2 + ay + b)$, then one has $\mathbb{P}(f_*\mathcal{O}_Y) \simeq \mathbb{P}^1_X$ and the map $Y \rightarrow \mathbb{P}^1_X$ is the natural map

$$\text{Spec} R[y]/(y^2 + ay + b) \hookrightarrow \mathbb{A}^1_X \hookrightarrow \mathbb{P}^1_X.$$

It follows that for any two-fold cover $Y \rightarrow X$, the natural map $Y \rightarrow \mathbb{P}(f_*\mathcal{O}_Y)$ is an imbedding, and $\mathbb{P}(f_*\mathcal{O}_Y)$ is a $\mathbb{P}^1$-bundle over $X$.

**Definition 3.28.** Given a two-fold flat cover $f \colon Y \rightarrow X$, we define the group scheme $G_{Y/X} \overset{\text{def}}{=} \text{Aut}(Y, \mathbb{P}(f_*\mathcal{O}_Y))$ to be the automorphism group scheme of the pair $(Y, \mathbb{P}(f_*\mathcal{O}_Y))$.

If $X = \text{Spec} k$ for an algebraically closed field, then the natural map above imbeds $Y$ as a divisor of degree two inside $\mathbb{P}^1_k$. If the cover is étale, then $Y$ consists of two distinct points which one may assume to be 0, $\infty \in \mathbb{P}^1_k$. Moreover, the automorphism group scheme is given by the automorphisms of $\mathbb{P}^1_k$ that preserves 0 and $\infty$: that is, $\mathbb{G}_m$. The automorphism that flips 0 and $\infty$ induces the inversion automorphism of $\mathbb{G}_m$. It follows that the induced morphism

$$B\mathbb{Z}/2 \rightarrow M_{FG},$$

sending a two-fold étale cover $Y \rightarrow X$ to the formal completion of $G_{Y/X}$, coincides with the map $B\mathbb{Z}/2 \rightarrow M_{FG}$ described before.
If the cover is not étale, then $Y$ may assumed to be the subscheme $2(\infty)$ in $\mathbb{P}_k^1$. In this case, the automorphism group scheme is given by $G_a$. In any event, though, one does get a group scheme, and an induced map

$$\mathcal{M}_{gd} \to M_{FG},$$

which coincides with the map $\text{Stack}(ko) \to \text{Stack}(S^0) \simeq M_{FG}$. To see this, observe that both maps are affine and all stacks are reduced, and we have already checked that the two maps agree on the dense open substack $B\mathbb{Z}/2 \subset \mathcal{M}_{gd}$.

4. A vector bundle on $M_{cub}$

Let $X$ be a finite even spectrum. As in section 2, there is associated to $X$ a coherent sheaf $F(X)$ on the stack $\mathcal{M}_{\text{aff}}$, and a spectral sequence from the cohomology of $F(X)$ converging to $\pi_*(\text{Tmf} \wedge X)$. It is therefore of interest to describe such sheaves in algebrao-geometric terms. Equivalently, for an elliptic spectrum $E$ associated to an elliptic curve $C \to \text{Spec}R$, it is desirable to find a description of $E_0(X)$ in terms of the algebraic geometry of $C$ (in particular, in a way that is natural in $E$).

**Example 4.1.** Given an elliptic spectrum $E$, $E^0(\mathbb{CP}^\infty)$ is the ring of functions on the formal group of $E$, which by definition is $\text{Spf}E^0(\mathbb{CP}^\infty)$. Consequently, $E_0(\mathbb{CP}^\infty)$ is the (pre)dual space to this: that is, the space of distributions on the associated formal group. More generally, $E_0(\mathbb{CP}^n)$ is the space of distributions on order at most $n$ (i.e., which vanish on functions which vanish to order $\geq n + 1$ at zero). In fact, this analysis is valid for any even-periodic ring spectrum.

Such descriptions can be given for many other spaces and spectra: for instance, $BU, MU, \mathbb{HP}^\infty$, and the Thom spectra $MU(2n)$ for $n \leq 3$; see [AHS01] and [Rez07]. These analyses are based on the description of $\mathbb{CP}^\infty$ and usually require producing suitable maps of spaces.

4.1. The stack $M_{cub}$. For our purposes, it will be convenient to work over the larger stack of all cubic curves (rather than simply the stable ones). The results on cubic curves that we need can be found in [Del75].

**Definition 4.2.** Given a scheme $S$, a cubic curve over $S$ is a map $p: E \to S$ which is flat and proper of finite presentation, together with a section $e: S \to E$ whose image is contained in the smooth locus of $p$. The geometric fibers of $p$ are required to be reduced, irreducible curves of arithmetic genus one.

We denote by $M_{cub}$ the stack which assigns to each commutative ring $R$ the groupoid of cubic curves over $\text{Spec}R$. (It is a consequence of the local structure of cubic curves that they actually do satisfy descent.)

**Definition 4.3.** There is a line bundle $\omega$ on the stack $M_{cub}$, which assigns to a cubic curve $p: C \to \text{Spec}R, e: \text{Spec}R \to C$ the $R$-module of sections of $\mathcal{O}_C(-e)/\mathcal{O}_C(-2e)$: that is, the cotangent space along the “zero section” $e$, or the dual to the Lie algebra.

If $p: E \to S$ is a cubic curve, there are three possibilities for each geometric fiber: it can be an elliptic curve, a nodal cubic in $\mathbb{P}^2$, or a cuspidal cubic in $\mathbb{P}^2$ (isomorphic to the projective closure of the curve defined by $y^2 = x^3$). In the first two cases, there are no infinitesimal automorphisms. However, the multiplicative group $G_m$ acts on the cuspidal curve. In particular, $M_{cub}$ is only an Artin stack, which contains the Deligne-Mumford stack $\mathcal{M}_{\text{aff}}$ as an open substack. The complement of $\mathcal{M}_{\text{aff}}$ in $M_{cub}$ is given by the vanishing locus of the modular forms $c_4 \in H^0(M_{cub}, \omega^4)$ and $\Delta \in H^0(M_{cub}, \omega^{12})$. 
Zariski locally on $S$, a cubic curve can be described as a subscheme of $\mathbb{P}^2_S$ cut out by a cubic equation
\begin{equation}
y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.
\end{equation}
In order to show this, one has to choose coordinates $x, y$ (which are chosen as sections of appropriate line bundles). Consequently, one can write down an explicit presentation of this stack via a Hopf algebroid. We sketch this below.

Let $E \to \text{Spec}R$ be a cubic curve, given by a cubic equation (11). Given one choice of $x, y$, the collection of other choices of coordinates is parametrized by
\begin{align}
x &= u^2x' + r \\
y &= u^3y' + su^2x' + t
\end{align}
where $u \in R^*$ and $r, s, t \in R$. These are the isomorphisms between Weierstrass curves.

In particular, we can represent $M_{\text{cub}}$ as a Hopf algebroid (the \textit{Weierstrass Hopf algebroid}) over the ring $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. The left and right units come from the transformation laws of a cubic equation, and the comultiplication comes from composition of isomorphisms.

Suppose given a cubic curve (11), and suppose one makes a change of coordinates as in (12). Then the new Weierstrass cubic, in coordinates $x', y'$, has the form
\begin{equation}
y'^2 + a_1'x'y' + a_3'y' = x'^3 + a_2'x'^2 + a_4'x' + a_6',
\end{equation}
where:
\begin{align}
u a_1' &= a_1 + 2s \\
u^2 a_2' &= a_2 - sa_1 + 3r - s^2 \\
u^3 a_3' &= a_3 + ra_1 + 2t \\
u^4 a_4' &= a_4 - sa_3 + 2a_2r - (t + rs)a_1 + 3r^2 - 2st \\
u^6 a_6' &= a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1
\end{align}
The stack $M_{\text{cub}}$ is presented either by the Hopf algebroid $(\mathbb{Z}[a_1, \ldots, a_6], \mathbb{Z}[a_1, \ldots, a_6][u^{\pm 1}, r, s, t])$, or by the \textit{graded} Hopf algebroid
\begin{align}(\mathbb{Z}[a_1, \ldots, a_6], \mathbb{Z}[a_1, \ldots, a_6][r, s, t]), \quad |a_i| = 2i \text{ and } |r| = 4, |s| = 2, |t| = 6.
\end{align}
The grading is doubled as in topology.

Given a cubic curve $E \to \text{Spec}R$, there is induced a structure of commutative group scheme on the smooth locus $E^0 \to \text{Spec}R$ (see Proposition 2.7 of \cite{DR73}). In fact, $E^0$ can be described as the \textit{relative Picard scheme} $	ext{Pic}^0_{E/\text{Spec}R}$; see Theorem 2.6 of \cite{DR73}. This is also described in Proposition 2.5 of III in \cite{Sil09}. In particular, we can take the formal completion at the zero section and get a \textit{formal group} over $R$. This gives a morphism of stacks
\[M_{\text{cub}} \to M_{\text{FG}}.\]

\textbf{Example 4.4.} The geometric points of $M_{\text{cub}}$ fall into four types in characteristic $p > 0$.

There are the \textit{ordinary} elliptic curves, which map to height one formal groups, as do the \textit{nodal} elliptic curves. There are \textit{supersingular} elliptic curves, which map to height two formal groups. Finally, there is the \textit{cuspidal cubic}, which maps to the additive formal group (of infinite height).

Consequently, given a spectrum $X$, defining sheaves $\mathcal{F}_{\text{even}}(X), \mathcal{F}_{\text{odd}}(X)$ on $M_{\text{FG}}$, we can pull back to define a sheaves (which we will still denote by the same notation) on $M_{\text{cub}}$. If
X is an even spectrum, then \(F(X) = F_{\text{even}}(X)\) will be a vector bundle on \(M_{\text{cub}}\) too. Note, however, that the map \(M_{\text{cub}} \to M_{\text{FG}}\) is no longer flat, unlike the map \(M_{\text{ell}} \to M_{\text{FG}}\).

We note that the line bundle \(\omega\) on \(M_{\text{FG}}\) pulls back to the line bundle \(\omega\) on \(M_{\text{cub}}\); given an elliptic curve \(p: C \to \text{Spec} \mathbb{Z}, e: \text{Spec} \mathbb{Z} \to C,\) the \(R\)-module \(O_C(-e)/O_C(-2e)\) is also the Lie algebra of the formal group. In particular, it can also be described as associated to the graded comodule \(\mathbb{Z}[a_1, \ldots, a_6]\) over the Weierstrass Hopf algebroid with the grading shifted by 2.

4.2. An eight-fold cover of \(M_{\text{cub}}\). Henceforth, in this section, we work localized at 2 throughout: in particular, stacks such as \(M_{\text{cub}}\) will really mean \(M_{\text{cub}} \times_{\text{Spec} \mathbb{Z}} \text{Spec} \mathbb{Z}(2)\). We will exhibit an eight-dimensional vector bundle on \(M_{\text{cub}}\) (which we will later see corresponds to a finite spectrum) by producing a finite flat cover \(p: T \to M_{\text{cub}},\) for \(T\) a simpler stack.

The associated vector bundle will be \(p_* \mathcal{O}_T\).

Namely, we take for \(T\) the stack-theoretic quotient

\[ T = \text{Spec} \mathbb{Z}(2)[\alpha_1, \alpha_3]/\mathbb{G}_m, \]

where the \(\mathbb{G}_m\)-action corresponds to the grading of the ring \(\mathbb{Z}(2)[\alpha_1, \alpha_3]\) with \(|\alpha_1| = 2, |\alpha_3| = 6\), where the grading is doubled in accordance with topology. In other words, we can think of \(T\) as the stack associated to the prestack \(\text{Ring} \to \text{Gpd}\) sending a ring \(R\) to the groupoid of pairs of elements \((\alpha_1, \alpha_3) \in R,\) with

\[ \text{Hom}((\alpha_1, \alpha_3), (\alpha'_1, \alpha'_3)) = \{ u \in R^* : u\alpha_1 = \alpha'_1, u^3\alpha_3 = \alpha'_3 \}. \]

Equivalently, \(T\) is the functor \(\text{Ring} \to \text{Gpd}\) which sends a ring \(R\) to the groupoid of triples \((\mathcal{L}, s, t)\) where \(\mathcal{L}\) is a line bundle on \(\text{Spec} R\) and \(s \in \mathcal{L}, t \in \mathcal{L}^{\otimes 3}\) are global sections.

To produce the cover \(p: T \to M_{\text{cub}},\) observe first that there is a map \(\text{Spec} \mathbb{Z}(2)[\alpha_1, \alpha_3] \to M_{\text{cub}}\) classifying the cubic curve

\[ y^2 + \alpha_1xy + \alpha_3y = x^3. \]

Observe that \(\mathbb{G}_m\) also acts on the cubic curve in a corresponding fashion as in the action on \(\mathbb{Z}(2)[\alpha_1, \alpha_3].\) Namely, given an invertible element \(u,\) we have the transformation

\[ x \mapsto u^2 x, \quad y \mapsto u^3 y \]

from the curve \(y^2 + \alpha_1xy + \alpha_3y = x^3\) into the curve \(y^2 + u\alpha_1xy + u^3\alpha_3y = x^3.\) These isomorphisms, together with the description of \(T'\) above, allow us to produce the morphism of stacks \(T \to M_{\text{cub}},\) as desired. Alternatively, we can construct a cubic curve out of the data of a line bundle \(\mathcal{L}\) and sections of \(\mathcal{L}\) and \(\mathcal{L}^{\otimes 3}.

Proposition 4.5. The map \(p: T \to M_{\text{cub}}\) is a finite, flat cover of rank eight.

Proposition 4.5 will be proved in the next section. We remark that over the locus \(M_{\text{cub}}[\Delta^{-1}]\) of (nonsingular) elliptic curves, the restriction of the cover \(T \to M_{\text{cub}}\) is the forgetful functor from the moduli stack of elliptic curves with a \(\Gamma_1(3)\)-structure (i.e., a choice of nonzero 3-torsion point, which here is \((0,0)) to \(M_{\text{cub}}[\Delta^{-1}]\); see [MR09]. We will not use this fact.

4.3. The rank eight cover. The goal of this subsection is to establish Proposition 4.5. To check that a morphism \(\mathfrak{X} \to \mathfrak{Y}\) of stacks is finite and flat, we just need to show that for every map \(\text{Spec} R \to \mathfrak{Y},\) the pull-back \(\text{Spec} R \times_{\mathfrak{Y}} \mathfrak{X}\) is a finite flat cover of \(\text{Spec} R.\) In our case, we need to show that for every morphism \(\text{Spec} R \to M_{\text{cub}},\) when one forms the
In other words, the relations defining $I$ over $\mathbb{Z}(2)[\alpha_1, \alpha_3]/\mathbb{G}_m$ can now prove finiteness by quotienting by the augmentation idea: that is, by working under the above three relations

$$\begin{align*}
\text{(19)} & \quad 0 = a_2 - sa_1 + 3r - s^2 \\
\text{(20)} & \quad 0 = a_4 - sa_3 + 2a_2r - (t + rs)a_1 + 3r^2 - 2st \\
\text{(21)} & \quad 0 = a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1
\end{align*}$$

In other words, it is $R[u^{\pm 1}, r, s, t]$ modulo the above relations. Taking the pull-back $P = \text{Spec}R \times_{\mathbb{M}_{\text{cub}}} T$ corresponds to taking the $\mathbb{G}_m$-quotient under a $\mathbb{G}_m$-action with the invertible $u$ in degree two: in other words, we just have to ignore $u$.

We find that $P$ is a closed subscheme of affine space $\mathbb{A}^3_R$ cut out by the three equations above; by the first relation, $r$ is determined in terms of $s$, and $P$ is even a closed subscheme of $\mathbb{A}^2_R$ cut out by two equations. Our goal is to show that it is finite flat over $\text{Spec}R$, of rank eight. Let us first show finiteness.

**Lemma 4.6.** Let $R$ be a ring, and let $a_1, a_2, a_3, a_4, a_6 \in R$. Then the quotient of $R[r, s, t]$ under the above three relations (19), (20), (21) is a finite $R$-module.

**Proof.** We can do this in the “universal” case where $R = \mathbb{Z}(2)[a_1, \ldots, a_6]$. In this case, observe that $R$ is naturally a graded ring (where we set $|a_i| = 2i$), and the ring $R[r, s, t]/I$ is a graded ring too, if we set

$$|r| = 4, \quad |s| = 2, \quad |t| = 6.$$  

In other words, the relations defining $I$ are homogeneous. In view of Nakayama’s lemma, we can now prove finiteness by quotienting by the augmentation ideal: that is, by working over $\mathbb{Z}(2)$, and by setting all the $a_i = 0$ (so that we have the cuspidal cubic $y^2 = x^3$, and the grading of $R[r, s, t]/I$ comes from this grading). Then the relevant ring is the quotient of $\mathbb{Z}(2)[r, s, t]$ by the relations

$$\begin{align*}
\text{(22)} & \quad 3r - s^2 = 0, \quad 3r^2 - 2st = 0, \quad r^3 - t^2 = 0.
\end{align*}$$

Alternatively, this is the quotient of $\mathbb{Z}(2)[s, t]$ under the relations

$$3(s^2/3)^2 = 2st, \quad (s^2/3)^3 = t^2$$
Since this is a graded $\mathbb{Z}_{(2)}$-module each of whose graded pieces is finitely generated, we may as well prove finiteness after tensoring with $\mathbb{Z}/2$, by the ungraded version of Nakayama’s lemma. We then get the $\mathbb{Z}/2$-algebra $\mathbb{Z}/2[s, t]/(s^4, t^2)$, which is evidently finite over $\mathbb{Z}/2$. □

If $R$ is any ring and we form the quotient $R[r, s, t]/I$ as above, the above proof also shows that there is a set of eight generators of the $R$-module $R[r, s, t]/I$, given by

$$\{1, s, s^2, s^3, t, st, s^2t, s^3t\}.$$ 

Namely, we just need to prove this in the universal case $\mathbb{Z}_{(2)}[a_1, a_2, a_3, a_4, a_6]$, which reduces by the Nakayama-type argument above to the case of the cuspidal cubic over $\mathbb{Z}/2$. We have seen that the above elements generate in that case.

**Remark 4.7.** We can conceptually understand the previous argument in the following terms. We had a ring $A$ with an action of the multiplicative monoid $\mathbb{G}_m^2$ (that is, the commutative monoid-scheme which assigns to a ring $R$ its monoid of elements under multiplication): this corresponds to a nonnegative grading of $A$. Consider a $\mathbb{G}_m^2$-equivariant quasi-coherent sheaf $\mathcal{G}$ on $\text{Spec}A$, or in other words a nonnegatively graded $A$-module. These form a full subcategory of the category of quasi-coherent sheaves on the (stacky) quotient $\text{Spec}A/\mathbb{G}_m^2$, which is the category of $\mathbb{Z}$-graded $A$-modules.

In this case, $\mathcal{G}$ is coherent if and only if the pull-back of $\mathcal{G}$ under the map

$$\text{Spec}A/\mathbb{G}_m^2 \to \text{Spec}A$$

is coherent. This restatement of Nakayama’s lemma suggests that we can check many assertions about $M_{\text{cub}}$ by working over the cuspidal cubic, an observation which we will find useful in the future.

Now that we know that the map $T \to M_{\text{cub}}$ is finite, flatness is automatic. In fact, we saw in the proof that for any map $\text{Spec}R \to M_{\text{cub}}$, the pull-back $T \times_{M_{\text{cub}}} \text{Spec}R$ was of the form (at least Zariski locally on $R$) $R[r, s, t]/I$, where $I$ was an ideal generated by three elements; geometrically, this is a complete intersection in $\mathbb{A}^4_R$. This implies flatness by the following well-known lemma.

**Lemma 4.8.** Let $X$ be a scheme, and let $Z \subset \mathbb{A}^n_X$ be a closed subscheme locally cut out by $n$ equations. Suppose the fibers $Z_x, x \in X$ are zero-dimensional. Then $Z \to X$ is flat.

**Proof.** In fact, we may suppose $X = \text{Spec}R$, for $(R, \mathfrak{m})$ a local noetherian ring. We need to show that the local rings $\mathcal{O}_{Z, z}$ are flat $R$-modules for each $z \in Z$ lying over the maximal ideal of $R$. The ring $\mathcal{O}_{Z, z}$ is obtained from a localization of $R[t_1, \ldots, t_n]$ at a maximal ideal $\mathfrak{m} = \mathfrak{n} R[t_1, \ldots, t_n]$ is flat over $R$, and $S/(f_1, \ldots, f_n)$ is the localization of a finite $R$-module.

In particular, let $k$ be the residue field of $R$. Then $S \otimes_R k$ is an $n$-dimensional regular local ring and $(S \otimes_R k)/(f_1, \ldots, f_n)$ is artinian, so $f_1, \ldots, f_n$ form a regular sequence on $S \otimes_R k$. It follows now from Proposition 15.1.6 in [Groth] that $f_1, \ldots, f_n$ are regular on $S$ itself, and that $\mathcal{O}_{Z, z} = S/(f_1, \ldots, f_n)$ is a flat $R$-module. □

Finally, let us show that the map $p: \text{Spec}\mathbb{Z}_{(2)}[\alpha_1, \alpha_3]/\mathbb{G}_m \to M_{\text{cub}}$ is a cover, i.e., that it is surjective, a condition which can be checked on closed points (i.e., on the associated topological space). Surjectivity follows because $p$ is finite flat, so that the image is both open and closed; however, $M_{\text{cub}}$ admits a cover by $\text{Spec}\mathbb{Z}_{(2)}[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6]$ and is thus connected.

One can also give a more explicit argument for surjectivity. For instance, we can check surjectivity at the level of points in $\text{Spec}\mathbb{Z}/2$. Points from $\text{Spec}\mathbb{Z}/2$ modulo isomorphism fall into three categories:
(1) Elliptic curves, classified by the $j$-invariant. For this, one can explicitly compute the $j$-invariant of $y^2 + a_1 x + a_3 x = x^3 \pmod{2}$, and show that every $j$-invariant is obtained.

(2) Nodal curves. Here one has to show that one can choose $a_1, a_3$ so as to make $c_4 \neq 0$ but $\Delta = 0$ (this is a calculation).

(3) Cuspidal curves (which we have seen are in the image as above). Similarly, one needs to make an argument for $\mathbb{Q}$-points.

The upshot of all this is that the map $p: T = \text{Spec} \mathbb{Z}(2)[\alpha_1, \alpha_3]/G \to M_{\text{cub}}$ is an eight-fold flat cover, and consequently the push-forward of the structure sheaf gives us an eight-dimensional vector bundle on $M_{\text{cub}}$. Our goal is to show that this is the vector bundle which comes from an $8$-cell complex.

5. Calculation of $\text{Tmf}_*(DA(1))$

In this section, we recall the $2$-local complex $DA(1)$ and show that the vector bundle it induces on $M_{\text{ell}}$ is the one constructed algebraically in the previous section. We will do this by producing (by topology) a map $\mathcal{F}(DA(1)) \to \mathcal{F}(MU)$ and (by algebra) a map from $\mathcal{F}(MU)$ to the vector bundle of the previous section, over $M_{\text{cub}}$ or rather its localization at $2$. We will check that the composite is an isomorphism over the cuspidal cubic over $\mathbb{Z}/2$.

This analysis leads to the computation of $\text{Tmf}_*(DA(1))$ by the descent spectral sequence.

Let us start by motivating the choice of complex. Consider the flat cover $p: T \to M_{\text{cub}}$ of the previous section. If we are to realize the vector bundle $p_*(\mathcal{O})$ on $M_{\text{cub}}$ by an even cell complex (that is, find an even complex $X$ such that the pull-back of $F(X)$ on $M_{FG}$ to $M_{\text{cub}}$ is isomorphic to $p_*(\mathcal{O})$), then in particular

$$H_* (X; \mathbb{Z}/2) \simeq MP_0(X) \otimes_{MP_0} \mathbb{Z}/2 \simeq \bigoplus_{j \in \mathbb{Z}} \Gamma(\text{Spec} \mathbb{Z}/2 \times_{M_{\text{cub}}} T, \omega^j),$$

as we will see below. We saw in the previous section that the graded ring corresponding to $\text{Spec} \mathbb{Z}/2 \times_{M_{\text{cub}}} T$ is $\mathbb{Z}/2[s, t]/(s^4, t^2)$ with $|s| = 2, |t| = 6$. It follows that we want to look for a complex with such homology.

5.1. The complex $DA(1)$. In this section, we describe a $2$-local complex with the homology specified above.

We start by describing the “question mark” complex, a variant of which is constructed in Lemma 7.2 of [Hop89]. First, let $\nu: S^3 \to S^0$ and $\eta: S^1 \to S^0$ be the usual Hopf maps. We have $\eta \nu = 0 \in \pi_4(S^0) = 0$. We draw the cofiber sequence for $\eta$, which runs $S^1 \xrightarrow{\eta} S^0 \to \Sigma^{-2}\mathbb{CP}^2 \to S^2 \to \ldots$, and consider the diagram:

$$S^1 \xrightarrow{\eta} S^0 \xrightarrow{\Sigma^{-2}\mathbb{CP}^2} S^2 \xrightarrow{\Sigma \eta} S^1$$

The map $t$ drawn as a dotted arrow exists because $\eta \nu = 0$; it is even unique as $\pi_5(S^0) = 0$.

**Definition 5.1.** The *question mark complex* $Q$ is defined to be the cofiber of $t$. In particular, it is a three cell complex: informally, the cells are attached via the Hopf maps $\eta$ and $\nu$. 
Figure 3. The cohomology of the question mark complex

Consequently, $Q$ comes with a map $Q \to \Sigma^{-2} \mathbb{H}^2$, leading to a commutative square:

$$
\begin{array}{ccc}
\Sigma^{-2} \mathbb{C}P^2 & \to & S^2 \\
\downarrow & & \downarrow \\
Q & \to & \Sigma^{-2} \mathbb{H}^2
\end{array}
$$

A look at the diagram shows that $H^*(Q; \mathbb{Z}/2)$ is three-dimensional, with a basis given by the element $x_0$ in degree zero, and $Sq^2 x_0$ and $Sq^4 Sq^2 x_0$. See Figure 5 for a picture of $H^*(Q; \mathbb{Z}/2)$; here the dots represent basis elements, a vertical arrow connecting dots indicates a $Sq^2$, and a curved arrow connecting dots indicates a $Sq^4$.

**Definition 5.2.** Let $DQ$ be the Spanier-Whitehead dual to $Q$. We define the (2-local) complex $DA(1)$ to be the six-fold suspension of the cofiber of the coevaluation map $S^0 \to Q \wedge DQ$.

Recall that Spanier-Whitehead duality $D = \text{Fun}_\text{Sp}(\cdot, S^0)$ can be identified with duality in the symmetric monoidal category $\text{Ho}(\text{Sp}^\omega)$ (the homotopy category of finite spectra). As a result, one has evaluation and coevaluation maps. Moreover, observe that the composite of the coevaluation and evaluation maps

$$
S^0 \to Q \wedge DQ \to S^0,
$$

is multiplication by the Euler characteristic (the "dimension") $\chi(Q) = 3$. Since we are working 2-locally, we find that there is a splitting

$$
Q \wedge DQ \simeq S^0 \vee DA(1),
$$

so that $DA(1)$ is actually an *even* 2-local spectrum.

The cohomology $H^*(DQ; \mathbb{Z}/2)$ is, up to a dimension shift, an upside down question mark: there is $y_6 \in H^{-6}(DQ, \mathbb{Z}/2)$ such that $H^*(DQ; \mathbb{Z}/2)$ has as basis $\{y_6, Sq^6 y_6, Sq^2 Sq^4 y_6\}$. We can write down a basis of the nine-dimensional space $H^*(Q \wedge DQ) \simeq H^*(Q; \mathbb{Z}/2) \otimes H^*(DQ; \mathbb{Z}/2)$, and isolate the one-dimensional summand that comes from $S^0$. A little computation shows that the cohomology $H^*(DA(1); \mathbb{Z}/2)$ is a free module over the eight-dimensional algebra $DA(1) \subset A/ASq^1 A$ generated by $Sq^2, Sq^4$; the cohomology is drawn in Figure 3. Its homology, as a comodule over $A_\ast$, can be described as $\mathbb{Z}/2[\xi^2]/(\xi^2) \otimes \mathbb{Z}/2[\xi^2]/(\xi^2)$.

**Remark 5.3.** The complex is so named because its cohomology is free over the subalgebra of the Steenrod algebra generated by $Sq^2$ and $Sq^4$, which “doubles” the subalgebra $A(1) \subset A$ generated by $Sq^1, Sq^2$. 
Since $DA(1)$ is an even spectrum, the Atiyah-Hirzebruch spectral sequence for $MU^*(DA(1))$ degenerates, and we can find a morphism

$$DA(1) \to MU$$

which induces an isomorphism on $\pi_0$, or on homology $H_0(\cdot; \mathbb{Z}/2)$. (By abuse of notation, we write $MU$ for $MU_{(2)}$ here.) This map is not unique, but we have specified its image in mod two homology. For instance, if one writes $H_*(BP; \mathbb{Z}/2) = \mathbb{Z}/2[\xi^1] \subset A_*$, so that $H_*(BP; \mathbb{Z}/2)$ is a summand of $H_*(MU; \mathbb{Z}/2)$, then the image of $H_*(DA(1); \mathbb{Z}/2)$ in $H_*(BP; \mathbb{Z}/2)$ is the tensor product of the exterior algebras $E(\xi^1) \otimes E(\xi^1) \otimes E(\xi^2)$.

Both $DA(1)$ and $MU$ define vector bundles on $M_{FG}$, and thus on $M_{\text{odd}}$ and even $M_{\text{cub}}$; these are denoted $F(DA(1))$ and $F(MU)$. The map $DA(1) \to MU$ defines a morphism of vector bundles

$$F(DA(1)) \to F(MU),$$

which is locally a split injection, again by the degeneration of the AHSS. Our goal will be to produce a map $F(MU) \to V$, for $V$ the eight-dimensional vector bundle constructed in the previous section, such that the composite is an isomorphism. We will construct the map, and check that it is an isomorphism on the cuspidal curve over $\mathbb{Z}/2$.

5.2. Construction of a map. Let us start by describing the (infinite-dimensional) vector bundle $F(MU)$ on the stack $M_{FG}$. This is the vector bundle which assigns to a formal group over a ring $R$ the ring parametrizing coordinates on this formal group, modulo action of $\mathbb{G}_m$.

Namely, let $M_{FG}^{\text{coord}}$ be the moduli stack of formal groups together with a coordinate (i.e., an isomorphism of formal schemes with the formal affine line $\hat{A}^1$), so that $M_{FG}^{\text{coord}}$ is simply the spectrum of the Lazard ring. It parametrizes formal group laws (and no isomorphisms).

There is a $\mathbb{G}_m$-action on $M_{FG}^{\text{coord}}$, corresponding to twisting a coordinate: this induces the usual grading of the Lazard ring. There is a morphism of stacks

$$q: M_{FG}^{\text{coord}}/\mathbb{G}_m \to M_{FG}.$$  

The associated (infinite-dimensional) vector bundle $F(MU)$ on $M_{FG}$ comes from $q_*\mathcal{O}$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{The cohomology of $DA(1)$}
\end{figure}
Consider next the pull-back diagram

\[ M^{\text{coord}}_{\text{cub}} / \mathbb{G}_m \longrightarrow M^{\text{coord}}_{\text{FG}} / \mathbb{G}_m. \]

Here \( M^{\text{coord}}_{\text{cub}} / \mathbb{G}_m \) is the stack parametrizing cubic curves together with a coordinate on the formal group, modulo \( \mathbb{G}_m \)-action. If we do not quotient by the \( \mathbb{G}_m \)-action, this is the almost-affine stack \( \text{Spec} \mathbb{Z}(2)[a_1, \ldots, a_6, e_4, e_5, \ldots] \): the choice of a coordinate modulo degree five on the formal group of a cubic curve is equivalent to the choice of a Weierstrass equation, and the remaining \( e_i \) allow one to modify the coordinate in higher degrees. See [Rez07].

We next construct a map \( g : \text{Spec} \mathbb{Z}(2)[\alpha_1, \alpha_3] / \mathbb{G}_m \rightarrow M^{\text{coord}}_{\text{cub}} / \mathbb{G}_m \) giving a commutative diagram:

\[ \text{Spec} \mathbb{Z}(2)[\alpha_1, \alpha_3] / \mathbb{G}_m \xrightarrow{g} M^{\text{coord}}_{\text{cub}} / \mathbb{G}_m \longrightarrow M^{\text{coord}}_{\text{FG}} / \mathbb{G}_m. \]

Namely, we define a map \( g : \text{Spec} \mathbb{Z}(2)[\alpha_1, \alpha_3] / \mathbb{G}_m \rightarrow M^{\text{coord}}_{\text{cub}} / \mathbb{G}_m \) (which is a closed immersion) by sending the cubic curve \( y^2 + \alpha_1 xy + \alpha_3 y = x^3 \) to the same cubic curve with the canonical coordinate \(-x/y\). In other words, \( a_1 \mapsto \alpha_1, a_3 \mapsto \alpha_3 \), and all the other polynomial generators are mapped to zero.

We let \( V \) be the vector bundle of the previous section: that is, \( V = p_*(O) \). Then \( V \) is actually a bundle of finite, flat \emph{commutative algebras} on \( M_{\text{cub}} \). The diagram naturally furnishes a map

\[ q'_*(O) \rightarrow q'_*(g_*(O)) \cong p_*(O) = V, \]

which is a surjection of sheaves of algebras.

Since all the morphisms of stacks shown here are affine morphisms, we can appeal to base-change for push-forwards of quasi-coherent sheaves. Therefore, we know that \( q'_*(O) \cong \mathcal{F}(MU) \), and the map \( \mathcal{F}(DA(1)) \rightarrow \mathcal{F}(MU) \) combined with the surjection \( q'_*(O) \rightarrow V \) induces a map \( \mathcal{F}(DA(1)) \rightarrow V \). This is a morphism of eight-dimensional vector bundles on \( M_{\text{cub}} \).

5.3. \textbf{Identification of} \( \mathcal{F}(DA(1)) \). Our goal is to prove that the map \( \mathcal{F}(DA(1)) \rightarrow V \) constructed in the previous section is an isomorphism, or equivalently, that it is a surjection. The following lemma will be useful.

\textbf{Lemma 5.4.} Let \( \mathcal{E} \rightarrow \mathcal{F} \) be a morphism of coherent sheaves on the stack \( M_{\text{cub}} \). Let \( x : \text{Spec} \mathbb{Z}/2 \rightarrow M_{\text{cub}} \) classify the cuspidal cubic curve. If \( x^* \mathcal{E} \rightarrow x^* \mathcal{F} \) is a surjection of \( \mathbb{Z}/2 \)-vector spaces, then \( \mathcal{E} \rightarrow \mathcal{F} \) is a surjection of coherent sheaves.

This lemma is a consequence of Nakayama’s lemma (both in the graded and ungraded forms) applied to the ring \( \mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \), though we will state the argument in stack-theoretic terms to emphasize the role of the cuspidal curve.

\textit{Proof.} In fact, we need to show that if \( \mathcal{C} \) is the cokernel of \( \mathcal{E} \rightarrow \mathcal{F} \), then \( \mathcal{C} = 0 \). We are given that \( x^*(\mathcal{C}) = 0 \), so that the closed substack \( \text{Supp}(\mathcal{C}) \) (that is, the maximal reduced closed substack on which every fiber of \( \mathcal{C} \) is nonzero) does not contain the point corresponding to \( x \). The next lemma shows that this implies the closed substack must be empty. \( \square \)
Lemma 5.5. Let $\mathfrak{Z} \subset M_{\text{cub}}$ be a closed substack. Suppose that the geometric point corresponding to the cuspidal curve over $\mathbb{Z}/2$ does not factor through $\mathfrak{Z}$. Then $\mathfrak{Z}$ is empty.

Proof. This lemma is based on the observation that the geometric point associated to the cuspidal curve is in the closure of any point in $M_{\text{cub}}$. In fact, consider a cubic curve $E$ classified by $\text{Spec} k \to M_{\text{cub}}$, which we will assume comes from a field $k$ of characteristic two. If $E$ is cut out by the equation

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

then we can draw a family of cubic curves over $A^1_k = \text{Spec} [u]$, $y^2 + u a_1 xy + u^3 a_3 y = x^3 + u^2 a_2 x^2 + u^4 a_4 x + u^6 a_6,$

with the property that for $u \in A^1_k \setminus \{0\}$, these fibers are isomorphic to the original one. For $u = 0$, though, one gets the cuspidal curve. The associated map $A^1_k \to M_{\text{cub}}$ shows thus that the image of zero is in the closure of the image of $A^1_k \setminus \{0\}$, which proves the lemma in characteristic two.

The analogous proof shows that if $k$ has characteristic zero, the closure of any geometric point $\text{Spec} k \to M_{\text{cub}}$ contains the cuspidal curve $y^2 = x^3$ over $\mathbb{Q}$. The closure of this, in turn, contains the cuspidal curve over $\mathbb{Z}/2$ by a specialization argument (e.g., by using the cuspidal curve $\text{Spec} \mathbb{Z}(2) \to M_{\text{cub}}$).

We can now prove the main result of this section.

Proposition 5.6. The composite map $F(DA(1)) \to F(MU) \simeq q'_*(\mathcal{O}) \to \mathcal{V}$ is an isomorphism of vector bundles on $M_{\text{cub}}$ (in particular, on $M_{\text{cub}}^{\text{ell}}$).

Proof. Since both bundles are eight-dimensional, it suffices to show that the map is a surjection. By the discussion of the previous section, it suffices to show that if $x : \text{Spec} \mathbb{Z}/2 \to M_{\text{cub}}$ classifies the cuspidal cubic, then

$$x^* F(DA(1)) \to x^* \mathcal{V}$$

is a surjection of vector spaces.

Now, $DA(1)$ is an even 2-local spectrum: in particular, it has free $\mathbb{Z}(2)$-homology and the AHSS degenerates for $MU_*(DA(1))$. This means in particular that, if we write $HP$ for even periodic mod 2 homology,

$$x^* (F(DA(1))) \simeq MP_0(DA(1)) \otimes_{MP_0 \mathbb{Z}/2} \mathcal{V} \simeq HP_0(DA(1)) \simeq H_*(DA(1); \mathbb{Z}/2);$$

that is, we can get a description of $F(DA(1))$ even over the cuspidal locus in terms of topology. The same is true for $x^* (F(MU)) \simeq HP_0(MU)$.

Let us now describe the two maps

$$HP_0(DA(1)) \to HP_0(MU), \quad HP_0(MU) \to x^* \mathcal{V}$$

algebro-geometrically. The cuspidal curve $y^2 = x^3$ admits a $\mathbb{G}_m$-action, and consequently all the vector spaces in question acquire a canonical $\mathbb{G}_m$-action (i.e., grading); these extra gradings will be indicated by superscripts. This is the ordinary grading in homology, and the grading on $x^* \mathcal{V}$ that we encountered earlier (see below).

In view of this grading, we can describe the image of the map $HP_0(DA(1)) \to HP_0(MU)$ as follows. There is an indecomposable element $S$ in $HP_0^0(MU)$ and an indecomposable element $T$ in $HP_0^0(MU)$. These have the property that

$$\{1, S, S^2, S^3, T, ST, S^2T, S^3T \}$$

forms a basis of the image of $HP_0(DA(1)) \to HP_0(MU)$. 
Consider now the map $HP_0(MU) \to x^*(\mathcal{V})$; observe that this is a morphism of algebras. We can describe the image of the associated graded here. Observe that $x^*(\mathcal{V})$ and its grading was described earlier; one has an element $s$ in degree 2, an element $t$ in degree 6, and one has a basis for the algebra

$1, s, s^2, s^3, t, s^2t, s^3t.$

In fact, as we saw earlier, the algebra is $\mathbb{Z}/2[s, t]/(s^4, t^2)$. Since $HP_0(MU) \to x^*(\mathcal{V})$ is a surjection of graded algebras, this means that the indecomposable element $S$ in degree 2 of $HP_0(MU)$ must map to $s$ in $x^*(\mathcal{V})$, and the indecomposable element $T$ in degree 6 must map to $t$ mod decomposables. Combining these two observations, it now follows that the composite map is an isomorphism, as desired.

\begin{remark}
This remark may motivate the use of $DA(1)$. Given any even spectrum $X$, one can associate to $X$ a quasi-coherent sheaf $F(X)$ on $M_{cub}$. If $F : \text{Spec}\mathbb{Z}/2 \to M_{cub}$ classifies the cuspidal cubic $C$, we can recover the homology groups $H_*(X; \mathbb{Z}/2)$ from the sheaf $F(X)$ via $x^*F(X) \cong H_*(X; \mathbb{Z}/2)$.

More is true, though: the stack presented by the truncated simplicial diagram

$$\text{Spec}\mathbb{Z}/2 \times M_{cub} \to \text{Spec}\mathbb{Z}/2 \times M_{cub} \text{Spec}\mathbb{Z}/2 \to \text{Spec}\mathbb{Z}/2 \times M_{cub} \text{Spec}\mathbb{Z}/2 \to \text{Spec}\mathbb{Z}/2$$

is $B\text{Aut}(C)$, and a description of $F(X)$ gives an action of $\text{Aut}(C)$ on $x^*(F(X))$. On the other hand, the stack presented by the analogous simplicial diagram

$$\text{Spec}\mathbb{Z}/2 \times M_{FG} \to \text{Spec}\mathbb{Z}/2 \times M_{FG} \text{Spec}\mathbb{Z}/2 \to \text{Spec}\mathbb{Z}/2 \times M_{FG} \text{Spec}\mathbb{Z}/2 \to \text{Spec}\mathbb{Z}/2,$$

the classifying stack for the automorphism group $\text{Aut}(\mathcal{E}_a)$ of the additive formal group over $\mathbb{Z}/2$. The action of $\text{Aut}(\mathcal{E}_a)$ on the homology groups of a space recovers precisely the action of the Steenrod operations on it. It follows that the action of $\text{Aut}(C)$ on $x^*(F(X))$ is precisely a first-order approximation to the action of the full Steenrod algebra on $H_*(X; \mathbb{Z}/2)$ (that is, the restriction of the $\text{Aut}(\mathcal{E}_a)$-action to the $\text{Aut}(C)$-action), and some of this information is already contained in the sheaf $F(X)$. Given the eight-fold cover $p : T \to M_{cub}$, we could already read off the Steenrod operations in the (co)homology of any spectrum realizing the associated vector bundle.

\end{remark}

5.4. Calculation of $\text{Tmf}_s(DA(1))$. We now have done the work necessary to compute $\text{Tmf}_s(DA(1))$. As before, let $T = \text{Spec}\mathbb{Z}/2[\alpha_1, \alpha_3]/\mathbb{G}_m$. Note that it is not $T$ and $M_{cub}$ which are relevant to this computation, but rather $T \times M_{cub} M_{cub}$ and $M_{cub}$. We will use the descent spectral sequence \cite{[13]} and the determination of $F(DA(1))$ of the previous subsections.

\begin{lemma}
$M_{cub} \times M_{cub} T \simeq \text{P}(1, 3)$ is the weighted projective stack: that is, the complement of the intersection $V(\alpha_1) \cap V(\alpha_3)$ in $\text{Spec}\mathbb{Z}/2[\alpha_1, \alpha_3]/\mathbb{G}_m$.
\end{lemma}

\begin{proof}
We recall (cf. \cite{[13], [10]}) that the substack $M_{cub} \subset M_{cub}$ is the complement of the closed substack of cuspidal curves cut out by the vanishing of the modular forms $c_4, \Delta$. It follows that $M_{cub} \times M_{cub}$ is the substack of $T \simeq \text{Spec}\mathbb{Z}/2[\alpha_1, \alpha_3]/\mathbb{G}_m$ complementary to that cut out by the vanishing of $c_4, \Delta$. In other words, we need to show that $c_4, \Delta$ generate an ideal in $\mathbb{Z}/2[\alpha_1, \alpha_3]$ which contains all elements of sufficiently large degree.

We can again show this after reducing mod 2. Here we use the expressions mod 2 for $c_4, \Delta$ of the cubic curve $y^2 + \alpha_1 xy + \alpha_3 y = x^3$: they are given by

$$c_4 \equiv \alpha_1^4, \quad \Delta \equiv (\alpha_1 \alpha_3)^3 + \alpha_3^4.$$

These together imply that $c_4, \Delta$ cut out the empty subscheme of $\mathbb{P}_{\mathbb{Z}/2}^1$ and consequently generate a power of the irrelevant ideal.
\end{proof}
We saw in Proposition 5.6 that if \( p: \mathbb{P}(1, 3) \to M_{\text{ell}} \) was the eight-fold cover as above, then \( \mathcal{F}(DA(1)) \simeq p_*(O) \), so that by the projection formula,
\[
\mathcal{F}(DA(1)) \otimes \omega^j \simeq p_*(\omega^j).
\]
Here \( \omega \) refers to the vector bundle on \( \mathbb{P}(1, 3) \) given by the Serre twist \( O(1) \) (i.e., the line bundle associated to the graded module \( Z_{(2)}[\alpha_1, \alpha_3] \) shifted by two) as well as the usual (Lie algebra) bundle on \( M_{\text{ell}} \) or \( M_{\text{ell}}^\omega \).

Consequently, we have the \( E_2 \)-term of the descent spectral sequence:
\[
H^1(M_{\text{ell}}, \mathcal{F}(DA(1)) \otimes \omega^j) = H^1(\mathbb{P}(1, 3), \omega^j).
\]
The cohomology of a weighted projective stack is the same as the classical cohomology of projective space, but the grading is modified. Namely, one has to compute the cohomology only occurs in dimension zero and one, and in dimension zero it is the same as that of \( \mathbb{A}^2 \). The collection \( \{ \alpha_1 \pm 1, \alpha_3 \} \) and keep track of the grading. The cohomology of \( (\mathbb{A}^2 \setminus \{(0, 0)\}) \) is given by the Serre twist \( V(\alpha_1, \alpha_3) \) and in dimension zero it is the same as that of \( \mathbb{A}^2 \).

More precisely, \( H^*(\mathbb{A}^2 \setminus \{(0, 0)\}, O) \) is the cohomology of the two-term (Cech) complex:
\[
Z_{(2)}[\alpha_1^{\pm 1}, \alpha_3] \oplus Z_{(2)}[\alpha_1, \alpha_3^{\pm 1}] \to Z_{(2)}[\alpha_1^{\pm 1}, \alpha_3^{\pm 1}],
\]
and the cohomology of \( \mathbb{P}(1, 3) \) is the same, with the grading taken into account. In particular, the spectral sequence \( (23) \) is concentrated in the bottom two rows, and each row is easy to describe. We have:
\[
H^2(\mathbb{A}^2 \setminus \{(0, 0)\}, O) = Z_{(2)}[\alpha_1, \alpha_3], \quad H^1(\mathbb{A}^2 \setminus \{(0, 0)\}, O) = Z_{(2)} \{ \alpha_1^{-1} \alpha_3^{-1}, \alpha_1^{-2} \alpha_3^{-1}, \alpha_1^{-1} \alpha_3^{-2}, \ldots \}.
\]
We get that:

**Proposition 5.9.** The descent spectral sequence for \( \pi_*(\text{Tmf} \wedge DA(1)) \) collapses. The terms in nonnegative degrees are given (additively) by \( Z_{(2)}[\alpha_1, \alpha_3] \): that is, \( \pi_* \tau_{\geq 0}(\text{Tmf} \wedge DA(1)) \simeq Z_{(2)}[\alpha_1, \alpha_3] \). Here \( |\alpha_1| = 2, |\alpha_3| = 6 \).

By the gap theorem in Theorem 2.14, we have:
\[
\tau_{\geq 0}(\text{Tmf} \wedge DA(1)) \simeq \text{tmf} \wedge DA(1).
\]
In particular, we also get:

**Corollary 5.10.** We have an additive isomorphism: \( \text{tmf}_*(DA(1)) \simeq Z_{(2)}[\alpha_1, \alpha_3] \).

A consequence of this analysis is that the homotopy limit associated to the sheaf \( O^{\text{top}} \) on \( M_{\text{ell}} \) commutes with homotopy colimits.

**Corollary 5.11.** Let \( X \) be a spectrum. Then \( \Gamma(M_{\text{ell}}, O^{\text{top}} \wedge X) \simeq \text{Tmf} \wedge X \).

Recall that every functor \( \mathbf{Sp} \to \mathbf{Sp} \) (where \( \mathbf{Sp} \) is the \( \infty \)-category of spectra) commuting with (homotopy) colimits is given by smashing with a spectrum. More precisely, we have an equivalence of \( \infty \)-categories
\[
\text{Fun}_L(\mathbf{Sp}, \mathbf{Sp}) \simeq \mathbf{Sp}
\]
between the \( \infty \)-category of colimit-preserving functors \( \mathbf{Sp} \to \mathbf{Sp} \) and \( \mathbf{Sp} \) itself, given by evaluation on \( S^0 \). For our purposes, this means that we need only show that \( X \mapsto \Gamma(M_{\text{ell}}, O^{\text{top}} \wedge X) \) commutes with homotopy colimits: in fact, we only need to know that it commutes with infinite wedges. See [Lur].

**Proof.** As in Corollary 3.6, we consider the collection \( \mathcal{C} \) of finite spectra \( T \) such that the functor
\[
F_T: X \mapsto \Gamma(M_{\text{ell}}, O^{\text{top}} \wedge X \wedge T)
\]
commutes with homotopy colimits (equivalently, arbitrary wedges, since \( F \) is exact) in \( X \).

The collection \( \mathcal{C} \) forms a thick subcategory of the homotopy category of finite (2-local)
spectra, and to show that $S^0 \in \mathcal{C}$ it suffices to show that $\mathcal{C}$ contains a spectrum with nontrivial rational homology, by the thick subcategory theorem.

In fact, we will show that $DA(1) \in \mathcal{C}$, which will prove the result by the previous paragraph. To see this, we consider the descent spectral sequence for $\Gamma(M_{\text{et}}, \mathcal{O}^{\text{top}} \times X \times DA(1))$, which runs

$$H^i(M_{\text{et}}, \pi_j(\mathcal{O}^{\text{top}} \times X \times DA(1))) \implies \pi_{j-i}(\Gamma(M_{\text{et}}, \mathcal{O}^{\text{top}} \times X \times DA(1))),$$

where cohomology is really of the sheaf associated to $\pi_j(\mathcal{O}^{\text{top}} \times X \times DA(1)))$. Given an affine étale map $\text{Spec} R \to M_{\text{et}}$, with associated elliptic spectrum $E$, we find:

$$E_\ast (X \times DA(1)) \simeq E_\ast (X) \otimes_{E_0} E_\ast (DA(1)) \simeq E_\ast (X) \otimes_{E_0} E_0 (DA(1)),$$

by the Künneth theorem. In particular, for each $j$, we have an isomorphism of quasi-coherent sheaves on $M_{\text{et}}$,

$$E_j(X \times DA(1)) \simeq p_\ast (\mathcal{O}) \otimes \pi_j(\mathcal{O}^{\text{top}} \times X) \simeq p_\ast p^\ast (\pi_j(\mathcal{O}^{\text{top}} \times X)).$$

It follows that the spectral sequence for $\Gamma(M_{\text{et}}, \mathcal{O}^{\text{top}} \times X \times DA(1))$ is

$$H^i (\mathbb{P}(1,3), p^\ast \pi_j(\mathcal{O}^{\text{top}} \times X)) \implies \pi_{j-i}(F_{DA(1)}(X)).$$

In particular, the descent spectral sequence for $\Gamma(M_{\text{et}}, \mathcal{O}^{\text{top}} \times X \times DA(1))$ degenerates as the $E_2$ page is concentrated in the bottom two rows: $\mathbb{P}(1,3)$ has cohomological dimension one. Moreover, the terms $H^i (\mathbb{P}(1,3), p^\ast \pi_j(\mathcal{O}^{\text{top}} \times X))$ of the $E_2$ page send arbitrary wedges in $X$ to direct sums because $\mathbb{P}(1,3)$ is noetherian. Since the $E_\infty$ page is concentrated in the bottom two rows, there are no infinite filtrations involved, and $\pi_\ast (F_{DA(1)}(X))$ sends arbitrary wedges to direct sums.

Consequently, the functor $X \mapsto F_{DA(1)}(X) = \Gamma(M_{\text{et}}, \mathcal{O}^{\text{top}} \times X \times DA(1))$ commutes with all homotopy colimits and is therefore equivalent to $\Gamma(M_{\text{et}}, \mathcal{O}^{\text{top}} \times DA(1)) \times X \simeq \text{Tmf} \times DA(1) \times X$. It follows that we can replace $DA(1)$ by $S^0$, by the thick subcategory theorem.

Equivalently, we can consider the exact functor $\mathbf{Sp} \to \mathbf{Sp}$ given by $X \mapsto \Gamma(M_{\text{et}}, \mathcal{O}^{\text{top}} \times X)$. Since it commutes with finite limits, it is a pro-spectrum: modulo set-theoretic issues, the ∞-category of pro-spectra is equivalent to the ∞-category of exact functors $\mathbf{Sp} \to \mathbf{Sp}$. The above result is that the pro-spectrum is constant.

In particular, there is a descent spectral sequence for $\text{Tmf}_\ast (X)$ for any spectrum $X$. We will use this fact below.

**Remark 5.12.** An analogous argument gives a quick proof that $\pi_\ast \text{Tmf}$ is finitely generated in each dimension; this is not obvious from the fairly complicated $E_2$ page of the descent spectral sequence. (This fails for periodic TMF, though.)

**Remark 5.13.** We also get a quick proof that there can be no analog of the thick subcategory theorem for the perfect derived category $D_{\text{perf}}(M_{FG})$ of finite length complexes of graded comodules over the Hopf algebroid $(MU_\ast, MU_\ast MU)$ whose constituents are finitely generated projective $MU_\ast$-modules. Recall, however, that there is a version of the thick subcategory theorem for the abelian category $\text{QCoh}^{\omega}(M_{FG})$ of coherent sheaves on $M_{FG}$ (i.e., finitely presented graded comodules over $(MU_\ast, MU_\ast MU)$).

Let us work $p$-locally for a fixed prime $p$; then the stack $M_{FG}$ has a stratification by height (see [Gois]). There is a sequence of closed substacks

$$M_{FG} \supset M_{FG}^{\geq 1} \supset M_{FG}^{\geq 2} \supset \ldots$$

where $M_{FG}^{\geq n}$ is the substack of formal groups of height at least $n$ ($M_{FG}^{\geq 1}$ is the substack where $p = 0$); the locally closed substacks $M_{FG}^{\geq n} \setminus M_{FG}^{\geq n+1}$ are pro-étale quotients of $\text{Spec} \mathbb{F}_p$, and in
particular every reduced closed substack of $M_{FG}$ is equal to some $M_{FG}^{>n}$ (including the limit cases $n = 0, \infty$). We say that a subcategory $\mathcal{C} \subset \text{Qcoh}^\omega(M_{FG})$ is thick if whenever one has an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in $\text{Qcoh}^\omega(M_{FG})$, then $M', M'' \in \mathcal{C}$ if and only if $M \in \mathcal{C}$. It is known (see [Rav92]) that given any thick subcategory $\mathcal{C}$, there exists $n$ such that $\mathcal{C}$ consists of those sheaves whose support is contained in $M_{FG}^{>n}$. There is, consequently, a correspondence between thick subcategories and (finitely presented) reduced closed substacks.

An analogous result for $D_{\text{perf}}(M_{FG})$ (using the definition of “thick subcategory” appropriate to the triangulated context) fails. (We remark that there is a thick subcategory theorem for perfect complexes over an affine scheme giving a correspondence between thick subcategories and reduced closed subschemes; see [Hop87].) We can in fact state:

**Proposition 5.14.** There is a proper thick subcategory $\mathcal{C} \subseteq D_{\text{perf}}(M_{FG})$ which contains an object whose cohomology is torsion-free. In particular, $D_{\text{perf}}(M_{FG})$ has a thick subcategory which does not consist of complexes cohomologically supported on some closed substack.

**Proof.** In fact, we consider the flat morphism $f: M_{\text{ell}} \to M_{FG}$ and consider the collection $\mathcal{C}$ of complexes $\mathcal{C} \in D_{\text{perf}}(M_{FG})$ such that for $N \gg 0$, $H^N(M_{\text{ell}} f^* C \otimes \omega^j) = 0$ for all $j$ (where $H$ is hypercohomology). This is a thick subcategory, and the object $\mathcal{O}_{M_{FG}}$ does not belong to it, as the cohomology of $M_{\text{ell}}$ occurs in arbitrarily large dimensions. (In fact, even the cohomology of $M_{\text{ell}}(\Delta^{-1})$ occurs in arbitrarily large dimensions; see [Rez07].) However, the finite even spectrum $DA(1)$ determines a vector bundle on $M_{FG}$, hence an object of $D_{\text{perf}}(M_{FG})$, whose pull-back to $M_{\text{ell}}$ has bounded cohomology. This gives an example of a torsion-free object in $\mathcal{C}$.

Note that we could have equally well run this argument with the flat morphism $BZ/2 \to M_{FG}$, and the vector bundle on $M_{FG}$ corresponding to $\Sigma^{-2}\mathbb{CP}^2$; this has the advantage that the cohomology of $BZ/2$ is easier to work out than that of $M_{\text{ell}}$. The point is essentially that the nilpotence of elements in the stable stems (e.g., $\eta$) is not verified in the $E_2$-term of the ANSS, but only after the differentials are allowed to strike. \qed

### 6. tmf $\land DA(1)$ and Calculation of the Homology

This section contains the main results at the prime 2. We will identify the spectrum $\text{tmf} \land DA(1)$ (whose homotopy groups were computed in the previous section) as a form of $BP(2)$. In order to do this, we first use the map $DA(1) \to MU$ (constructed by obstruction theory) to map $\text{tmf} \land DA(1)$ into the $E_\infty$-ring $\text{Tmf} \land MU$. We then quotient this by a regular sequence in $\text{Tmf}_* (MU)$, and finally take the connective cover to obtain an $MU$-module that we check to be $BP(2)$.

A consequence is the computation of $H_*(\text{tmf}; \mathbb{Z}/2)$ from that of $BP(2)$, which was the initial goal of this paper.

#### 6.1. Calculation of $\text{Tmf}_* (MU)$

Our next goal is to study the spectrum $\text{tmf} \land DA(1)$, and show that this spectrum is equivalent to a form of $BP(2)$. In the previous section, we computed the homotopy groups; in this section we shall produce a map. In fact, we will give an alternative description of $\text{tmf} \land DA(1)$ as a tmf-module.

In order to show that $\text{tmf} \land DA(1)$ (which, so far, has not been given an $MU$-module structure) is a form of $BP(2)$, we will first map it to an $MU$-module spectrum, which we will then modify to obtain a form of $BP(2)$. Namely, choose any map $DA(1) \to MU$ inducing an isomorphism on $\pi_0$, as before. This induces a map of $\text{Tmf}$-modules

$$\text{Tmf} \land DA(1) \to \text{Tmf} \land MU \simeq \Gamma(M_{\text{ell}}, \mathcal{O}_{\text{top}} \land MU),$$
where the last equivalence follows from Corollary 5.11 and implies that there is a descent spectral sequence computing \( \pi_*(\text{Tmf} \wedge \text{MU}) \). Observe that \( \text{Tmf} \wedge \text{MU} \) is an \( \text{MU} \)-module. In fact, it is much better, as it is an \( E_{\infty} \)-algebra over \( \text{MU} \).

We can compute the homotopy groups of \( \text{Tmf} \wedge \text{MU} \) easily.

**Proposition 6.1.** Let \( R = \mathbb{Z}_{(2)}[a_1, \ldots, a_6, e_n]_{n \geq 4} \). Then \( \text{Tmf}_*(\text{MU}) \) is a module over the ring \( R \). As an \( R \)-module, it is isomorphic to \( R \oplus C \), where \( C \) is the cokernel of \( R[c_4^{-1}] \oplus R[\Delta^{-1}] \to R[(c_4 \Delta)^{-1}] \).

**Proof.** Namely, recall that the sheaf \( \mathcal{F}(\text{MU}) \) on \( M_{\text{ell}} \) is obtained by pushing forward the structure sheaf under the map

\[
\phi: (\text{Spec} R \setminus V(c_4, \Delta))/\mathbb{G}_m \simeq M_{\text{ell}}^\text{coord}/\mathbb{G}_m \to M_{\text{ell}},
\]

where the first stack classifies a possibly nodal elliptic curve together with a coordinate, modulo \( \mathbb{G}_m \)-action. This is a consequence of Example 2.9 together with base-change for affine morphisms and open immersions. Here \( \phi \) is an affine morphism; in fact, it is a base-change of the map \( (\text{Spec} R)/\mathbb{G}_m \to \text{M}_{\text{cub}} \) studied earlier.

Let \( B \) be the scheme \( \text{Spec} R \setminus V(c_4, \Delta) \). Consequently, we get for the \( E_2 \) page of the descent spectral sequence for \( \text{Tmf}_*(\text{MU}) \),

\[
E_2^{i,j} = H^i(M_{\text{ell}}, \mathcal{F}(\text{MU}) \otimes \omega^j) \simeq H^i(M_{\text{ell}}, \phi_*(\mathcal{O}) \otimes \omega^j)
\]

\[
\simeq H^i(M_{\text{ell}}, \phi_*(\omega^j)) \quad \text{by the projection formula}
\]

\[
\simeq H^i(B/\mathbb{G}_m, \omega^j) \quad \text{since } \phi \text{ is affine}
\]

\[
\simeq H^i(B, \mathcal{O})_j,
\]

where the subscript \( j \) denotes taking the \( j \)th piece; recall that the \( \mathbb{G}_m \)-action simply serves to record the grading when computing such cohomology. The scheme \( B \) is not quite affine, but it covered by two affine opens (the localizations at \( \Delta \) and \( c_4 \)). In particular, it has cohomological dimension one, and for dimensional reasons the descent spectral sequence for \( \text{Tmf} \wedge \text{MU} \) degenerates.

Since \( B \) is covered by the two affine open subschemes (corresponding to the nonvanishing locus of \( c_4 \) and that of \( \Delta \)), we can describe its cohomology as that of the Cech complex

\[
R[c_4^{-1}] \oplus R[\Delta^{-1}] \to R[(c_4 \Delta)^{-1}].
\]

First, \( H^0(B, \mathcal{O}) = R \), since \( B \) is obtained from the normal affine scheme \( \text{Spec} R \) by removing the intersection of two divisors intersecting transversely: more precisely, the sequence \( c_4, \Delta \) is regular in \( R \) as we will see below in Lemma 6.3. The \( H^1 \) of this complex is \( C \). Since the descent spectral sequence for \( \text{Tmf} \wedge \text{MU} \) degenerates, we get the result once we choose a lift \( R \to \text{Tmf}_*(\text{MU}) \) of the edge homomorphism \( \text{Tmf}_*(\text{MU}) \to R \).

We observe that \( C \) can be identified with the ideal of \( \text{Tmf}_*(\text{MU}) \) given by elements in positive filtration degree (in the filtration from the descent spectral sequence), and consequently \( C^2 = 0 \).

6.2. **Description of** \( \text{tmf} \wedge DA(1) \). We keep the same notation of the previous subsection.

In section 5, we gave a map \( DA(1) \to \text{MU} \) and saw that it had the following property in \( \text{Tmf} \)-homology. There is a morphism from the descent spectral sequence of \( DA(1) \) to that of \( \text{MU} \), and the map on the zero-line imbeds \( \pi_{* + 2}(\text{Tmf} \wedge DA(1)) \) (global sections of \( \mathcal{F}(DA(1)) \)) as a subgroup of \( R \) which maps isomorphically to \( R/(a_2, a_4, a_6, \{e_n\}) \). This in turn is the ring of global sections of the vector bundle \( V \overset{\text{def}}{=} p_*\mathcal{O}_T \) for \( p: T \to M_{\text{cub}} \) the eight-fold cover of section 4 (or rather its restriction to \( M_{\text{ell}} \)).
More precisely, we have filtrations
\[ 0 \subset F_1 \text{Tmf}_*(DA(1)) \subset \text{Tmf}_*(DA(1)), \quad 0 \subset F_1 \text{Tmf}_*(MU) \subset \text{Tmf}_*(MU), \]
such that \( \text{Tmf}_*(MU)/F_1 \text{Tmf}_*(MU) \) is multiplicatively isomorphic to \( R \). Moreover, the composite map
\[ (24) \]
\[ \text{Tmf}_*(DA(1))/F_1 \text{Tmf}_*(DA(1)) \to \text{Tmf}_*(MU)/F_1 \text{Tmf}_*(MU) \simeq R \to R/(a_2, a_4, a_6, \{e_n\}) \]
is an isomorphism. In fact, this was precisely the analysis done earlier in Proposition 5.6 together with the degeneracy of both descent spectral sequences. The next step is to modify the spectrum \( \text{Tmf} \wedge MU \) so as to produce a form of \( BP(2) \), which will require dealing with \( R \) as well as \( C \).

**Remark 6.2.** The filtration of \( \text{Tmf}_*(DA(1)) \) is such that \( F_1 \text{Tmf}_*(DA(1)) \) is concentrated in negative degrees. This is no longer true for \( F_1 \text{Tmf}_*(MU) \), so our strategy will be to quotient first and take connective covers later.

Observe that \( \text{Tmf} \wedge MU \) is an \( E_\infty \)-ring spectrum, so that we can make various quotients in the category of \( \text{Tmf} \wedge MU \)-modules. Consider in particular the quotient \( (\text{Tmf} \wedge MU)/(a_2, a_4, a_6, \{e_n\}) \). The next lemma will enable us to analyze it.

**Lemma 6.3.** The sequence \( a_2, a_4, a_6, \{e_n\}_{n \geq 4} \) is regular on \( \pi_* \text{Tmf} \wedge MU \). Moreover, the homotopy groups of the MU-module
\[ \tau_{\geq 0}((\text{Tmf} \wedge MU)/(a_2, a_4, a_6, \{e_n\})) \]
are given by \( \mathbb{Z}_2[a_1, a_3] \); in particular, it is a form of \( BP(2) \).

Let \( I \) denote the ideal \( (a_2, a_4, a_6, \{e_n\}) \subset R \); we will (by abuse of notation) write \( (\text{Tmf} \wedge MU)/I \) for the quotient \( (\text{Tmf} \wedge MU)/(a_2, a_4, a_6, \{e_n\}) \).

**Proof.** Since \( (\text{Tmf} \wedge MU)_* \simeq R \oplus C \), and since the sequence \( a_2, a_4, a_6, \{e_n\} \) is regular on \( R \), we just need to show that they form a regular sequence on \( C \). The \( \{e_n\} \) are irrelevant, so we show that the sequence \( a_2, a_4, a_6 \) is regular on the cokernel of
\[ \mathbb{Z}_2[a_1, a_2, a_3, a_4, a_6\{c_4^{-1}\}] \oplus \mathbb{Z}_2[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}] 
\to \mathbb{Z}_2[a_1, a_2, a_3, a_4, a_6][c_4 \Delta^{-1}]. \]
This two-term complex is a piece of a colimit of Koszul complexes for the sequence \( c_4^m, \Delta^m \), each of whose cokernels is isomorphic to a shift of \( \mathbb{Z}_2[a_1, a_2, a_3, a_4, a_6]/(c_4^m, \Delta^m) \). It suffices to show that for any \( m \), the sequence
\[ (25) \]
\[ c_4^m, \Delta^m, a_2, a_4, a_6 \]
is regular on \( \mathbb{Z}_2[a_1, a_2, a_3, a_4, a_6] \) (actually, it suffices to check this for \( m = 1 \)). As \( \mathbb{Z}_2[a_1, a_2, a_3, a_4, a_6] \) is a graded ring and as the elements in question are homogeneous, it suffices for regularity of the sequence (25) that
\[ \dim \mathbb{Z}_2[a_1, a_2, a_3, a_4, a_6]/(c_4^n, \Delta^n, a_2, a_4, a_6) = 1; \]
see Lemma 5.4 below. This in turn is a consequence of Lemma 5.8.

Next, we need to work out \( \pi_* (\text{Tmf} \wedge MU)/I \), which by regularity is \( \pi_* (\text{Tmf} \wedge MU/I) \). Since the sequence \( a_2, a_4, a_6, \{e_n\} \) is regular on the cohomology groups of the complex \( R[c_4^{-1}] \oplus R[\Delta^{-1}] \to R[(c_4 \Delta)^{-1}] \), we find that \( (\text{Tmf} \wedge MU)_*/I \) can itself be described via a two-term complex, that is:
\[ (\text{Tmf} \wedge MU)_*/I = H^*(R[c_4^{-1}] \oplus R[\Delta^{-1}] \to R[(c_4 \Delta)^{-1}]), \]
where $R' = R/(a_2, a_4, a_6, \{e_n\}) \cong \mathbb{Z}_2[\alpha_1, \alpha_3]$. We now recognize this complex as the Cech complex that computes the cohomology of $\mathcal{P}(1, 3)$. It follows that

$$\pi_*(\text{Tmf} \wedge MU/I) \cong \mathbb{Z}_2[\alpha_1, \alpha_3] \oplus C',$$

where $C' \overset{\text{def}}{=} \text{coker } (R'[\epsilon^{-1}] \oplus R'[\Delta^{-1}] \rightarrow R'[\epsilon \Delta^{-1}])$ is concentrated in negative degrees.

Finally, we have to show that the $MU$-module spectrum $M = \tau_{\geq 0}((\text{Tmf} \wedge MU)/I)$ is a form of $BP(2)$. For this it suffices to describe the Hurewicz map $MU_* \rightarrow M_*$, in view of Example 3.11. Given an elliptic spectrum $E$ associated to an elliptic curve $C \rightarrow \text{Spec} R$, the map $MU_* \rightarrow E_*(MU)$ describes algebro-geometrically the map $\text{Spec} E_*(MU) \rightarrow \text{Spec} MU$ from the ring classifying coordinates on $\hat{C}$ to the ring classifying formal group laws. In particular, the map

$$MU_* \rightarrow \text{Tmf}_*(MU) \rightarrow \text{Tmf}_*(MU)/F_1 \text{Tmf}_*(MU)$$

classifies the formal group law constructed by choosing the coordinate $y/x = \sum_{n \geq 4} c_n (-y/x)^{n+1}$ on the formal group of $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$. The composite $MU_* \rightarrow (\text{Tmf}_*(MU)/F_1 \text{Tmf}_*(MU))/I$ (which we note is a morphism of rings) classifies the formal group law associated to the coordinate $y/x$ on $y^2 + a_1 xy + a_3 y = x^3$.

Here we use the formulas given in Example 4.5 of Chapter 4 of [Sil09]. The expansion of the power series $[2](z)$ for the formal group law associated to a Weierstrass equation is

$$[2](z) = 2z - a_1 z^2 - 2a_2 z^3 + (a_1 a_2 - 7a_3) z^4 + \ldots,$$

where in our case $a_2 = 0$ and 7 is invertible. In particular, if we take $v_1, v_2$ to be indecomposable elements of $MU$ in degrees 2 and 6 (e.g., the coefficients of $z^2, z^4$ in $[2](z)$), and take our equation to be $y^2 + a_1 xy + a_3 y = x^3$, then the standard $v_1$ maps to a unit times $a_1$, and $v_2$ maps to a unit times $a_3$. It follows that there exist indecomposable generators $v_3, v_4, \ldots$, which generate the kernel of the surjective map $BP_* \rightarrow MU_* \rightarrow \mathbb{Z}_2[\alpha_1, \alpha_3]$. This proves the result.

We need an elementary algebraic lemma.

**Lemma 6.4.** Let $R$ be a commutative, graded, connected noetherian ring, and let $x_1, \ldots, x_n \in R$ be homogeneous elements. Suppose that $R$ is smooth and that $\dim R/(x_1, \ldots, x_n) = \dim R - n$. Then $x_1, \ldots, x_n$ is a regular sequence on $R$.

**Proof.** Let $R_+$ be the augmentation ideal. We note that the $R_+$-adic completion functor from finitely generated graded $R$-modules to finitely generated $\hat{R}$ modules (where $\hat{R} = \prod R_i$) is exact and reflects isomorphisms. It follows that we may as well prove regularity in the complete local ring $\hat{R}$. Now the result is classical. \hfill $\square$

**Corollary 6.5.** The composite map of $\text{tmf}$-modules

$$\text{tmf} \wedge DA(1) \rightarrow \text{Tmf} \wedge MU \rightarrow (\text{Tmf} \wedge MU)/I$$

exhibits an equivalence $\text{tmf} \wedge DA(1) \cong \tau_{\geq 0}(\text{Tmf} \wedge MU)/I$. Moreover, $\text{tmf} \wedge DA(1) \rightarrow (\text{Tmf} \wedge MU)/I$ is an equivalence.

**Proof.** In fact, we have seen that the above composite (modulo taking the connective cover) realizes $[24]$ precisely.

Moreover, we also saw in the proof of Lemma 6.3 that the negative homotopy groups of $(\text{Tmf} \wedge MU)/I$ were in correspondence with those of $\text{Tmf} \wedge DA(1)$. More precisely, the morphism $DA(1) \rightarrow MU$ defines a map $H^1(M_\text{cp}, F(DA(1)) \otimes \omega) \rightarrow H^1(M_\text{cp}, F(MU) \otimes \omega)$ with the property that the composite $H^1(M_\text{cp}, F(DA(1)) \otimes \omega) \rightarrow H^1(M_\text{cp}, V \otimes \omega)$ is an
isomorphism, for all $j$. Observe, however, that $H^1(M_{\text{et}}, F(MU) \otimes \omega^j) \to H^1(M_{\text{et}}, V \otimes \omega^j)$ is equivalent to the map $F_1(Tmf_{+}(MU)) \to F_1(Tmf_{+}(MU))/I$. In particular, $Tmf \wedge DA(1) \to Tmf \wedge MU \to (Tmf \wedge MU)/I$ is an equivalence.

Now, take connective covers and use the gap theorem to conclude $tmf \wedge DA(1) \simeq \tau_{\geq 0}(Tmf \wedge DA(1)) \simeq \tau_{\geq 0}(Tmf \wedge MU)/I$.

Alternatively, we could argue as follows: if we know that $\tau_{\geq 0}(Tmf \wedge DA(1)) \to \tau_{\geq 0}(Tmf \wedge MU)/I$ is an equivalence, then the cofiber of the map $Tmf \wedge DA(1) \to (Tmf \wedge MU)/I$ is coconnective. One can check directly that the homotopy groups of the cofiber are torsion, and consequently the cofiber smashes to zero with Morava $E$-theory $E(2)$. However, the cofiber is $E(2)$-local, because $Tmf$ is and $E(2)$-local spectra are closed under homotopy colimits by the Hopkins-Ravenel smash product theorem (see [Rav92]). It follows that the cofiber must be zero.

Combining Lemma 6.3 and Corollary 6.5, we find:

**Theorem 6.6** (Hopkins-Mahowald [HM98]). $tmf \wedge DA(1) = \tau_{\geq 0}(Tmf \wedge DA(1))$ is a form of $BP(2)$.

**Remark 6.7.** Strictly speaking, we did not need Corollary 5.11 (which relied on the thick subcategory theorem) to run the preceding proof; we could have worked just as well with $\Gamma(M_{\text{et}}, O^{top} \wedge MU)$.

We suspect that $Tmf \wedge DA(1)$ can be realized as the spectrum of topological modular forms of level three. That is, the restriction of the eight-fold flat cover $p: T \to M_{\text{et}}$ over the locus $M_{\text{et}}$ can be realized topologically via a derived stack, whose homotopy limit could be denoted $Tmf_{1}(3)$. The theory of topological modular forms with level structure is discussed in [Sto12] away from the prime 2. One assumes that a similar picture holds at the prime 2. (There is no issue over the moduli stack of nonsingular elliptic curves, where the cover is étale.)

For the purposes of the next section, we need a consequence of the above observations:

**Corollary 6.8.** $Tmf \wedge DA(1)$ (and thus $\tau_{\geq 0}(Tmf \wedge DA(1)) = tmf \wedge DA(1)$ too) admits the structure of a module over $MU$.

In particular, we need the fact that for any $MU$-module $M$, the Atiyah-Hirzebruch spectral sequence for $\pi_*(MU \wedge M) = M_*(MU)$ degenerates. That is,

$(26) \quad M_*(MU) \simeq M_1[b_1, b_2, \ldots].$

We refer the reader to [Axi95].

### 6.3. The homology of $tmf$.

In the previous subsection, we established the 2-local equivalence $tmf \wedge DA(1) \simeq BP(2)$. Using Hopf algebra manipulations and the homology of $BP(2)$, we can now calculate the homology of $tmf$. (Note that $H_*(Tmf; Z/2) = 0$ since $Tmf$ is $E_2$-local.)

**Theorem 6.9** (Hopkins-Mahowald [HM98]). The map $tmf \to HZ/2$ induces an injection on mod 2 homology, and we have an identification

$H_*(tmf; Z/2) = Z/2[\xi_1^8, \xi_2^4, \xi_3^2, \xi_4, \xi_5, \ldots] \subset A_*$.

**Proof.** We will prove this analogously to Theorem 3.1. In fact, we know that

$H_*(BP(2); Z/2) \simeq Z/2[\xi_1^2, \xi_2^2, \xi_3^2, \xi_4, \xi_5^2, \ldots],$
by Proposition 3.17: the map comes from the truncation $BP \langle 2 \rangle \to HZ_{(2)} \to HZ/2$ which imbeds $H_*(BP \langle 2 \rangle; \mathbb{Z}/2)$ as a submodule of $A_*$. We also know that $H_*(\text{tmf}; \mathbb{Z}/2)$ is a comodule algebra, and the factorization $tmf \to tmf \wedge DA(1) \simeq BP \langle 2 \rangle \to HZ/2$

shows that it is a comodule algebra of $H_*(BP \langle 2 \rangle; \mathbb{Z}/2) \subset A_*$. Moreover, the Künneth formula shows that the graded dimension of $H_*(BP \langle 2 \rangle; \mathbb{Z}/2)$ is that of $Z/2[\xi^1_1, \xi^2_2, \xi^3_3, \xi^4_4, \ldots]$. We will show that if $C$ is any submodule algebra of $H_*(BP \langle 2 \rangle; \mathbb{Z}/2)$ with the same graded dimension as $Z/2[\xi^1_1, \xi^2_2, \xi^3_3, \xi^4_4, \ldots]$, then $C$ is in fact $Z/2[\xi^1_1, \xi^2_2, \xi^3_3, \xi^4_4, \ldots]$ (which is easily checked to be a valid comodule algebra). Here the proof is analogous to the argument used when computing $H_*(ko; \mathbb{Z}/2)$: we have to show that any two $C, C'$ satisfying that condition are equal.

In fact, if $C, C'$ satisfy the condition, then Lemma 3.21 shows that $H_*(BP \langle 2 \rangle; \mathbb{Z}/2)$ is free (necessarily of rank eight) over each of $C, C'$. Consider the submodule algebra $C'' \subset H_*(BP \langle 2 \rangle; \mathbb{Z}/2)$ generated by $C, C'$. It also has the property that $H_*(BP \langle 2 \rangle; \mathbb{Z}/2)$ is free over $C''$, by Lemma 3.21 again. By counting the rank, we will arrive at a contradiction. We need first:

**Lemma 6.10.** The only elements of $C''$ in degrees $< 12$ are $1$ and $\xi^8_1 = \xi^8_1$.

**Proof.** In fact, we know that $\dim C_8 = \dim C'_8 = 1$, and since this is the smallest dimension of a nonzero element in each of $C, C'$, the generating element must be primitive. The primitive elements in the dual Steenrod algebra $A_*$ are $1$ and $\{\xi^2_n\}_{n > 0}$, though, so $C_8 = C'_8$ is generated by $\xi^8_1$. There are no other elements in degrees $< 12$ in $C$ or $C'$.

We claim that the eight elements $1, \xi^1_1, \xi^2_1, \xi^3_2, \xi^4_1, \xi^5_2, \xi^6_1, \xi^7_2 \in H_*(BP \langle 2 \rangle; \mathbb{Z}/2)$ are linearly independent in $H_*(BP \langle 2 \rangle; \mathbb{Z}/2) \otimes C'' Z/2$ (i.e., could be taken as a subset of generators over $C''$). This is a consequence of the fact that the only elements in $C''$ of degree less than $12$ are $1, \xi^8_1$. Moreover, in degree $12$, we observe that $\xi^1_1 \xi^2_2 \notin C''$ as it is not primitive modulo $\xi^1_1$. Consequently, the rank of $H_*(BP \langle 2 \rangle; \mathbb{Z}/2)$ as a $C''$-module must be at least eight. This means that the graded dimension of $C''$ must be equal to that of $C$, so $C = C' = C''$.

The dual assertion describes the cohomology via $H^*(\text{tmf}; \mathbb{Z}/2) \simeq A \otimes A(2) \mathbb{Z}/2$, where $A(2) \subset A$ is the subalgebra of the Steenrod algebra $A$ generated by $\text{Sq}^1, \text{Sq}^2, \text{Sq}^4$. In fact, $H^*(\text{tmf}; \mathbb{Z}/2)$ is cyclic over $A$ since $H_*(\text{tmf}; \mathbb{Z}/2) \subset A_*$. Dimensional restrictions force $\text{Sq}^1, \text{Sq}^2$, and $\text{Sq}^4$ to annihilate the generator in degree zero, and this produces a surjection $A \otimes A(2) \mathbb{Z}/2 \to H^*(\text{tmf}; \mathbb{Z}/2)$. Since $A(2) \subset A$ is a Hopf subalgebra, $A$ is free over $A(2)$ by the results of [MM65], and we find that the graded dimensions of $A \otimes A(2) \mathbb{Z}/2$ and $H^*(\text{tmf}; \mathbb{Z}/2)$ match. This proves the asserted description of the cohomology.

By the change-of-rings theorem $\text{Ext}^{*,*}_{A}(A \otimes A(2) \mathbb{Z}/2, \mathbb{Z}/2) \simeq \text{Ext}^{*,*}_{A(2)}(\mathbb{Z}/2, \mathbb{Z}/2)$, we now conclude:

**Corollary 6.11.** The (mod 2) Adams spectral sequence for $\text{tmf}$ runs $\text{Ext}^{*,*}_{A(2)}(\mathbb{Z}/2, \mathbb{Z}/2) \implies \pi_{-s}\text{tmf} \otimes \mathbb{Z}/2$. 
We remark that it is not possible to continue this process, and realize modules of the form $A \otimes A(n) \mathbb{Z}/2$ for $n \geq 3$, where $A(n) \subset A$ is generated by $\{Sq^1, \ldots, Sq^n\}$, because of the solution to the Hopf invariant one problem.

7. tmf at odd primes

In this section, we briefly indicate the modifications in the above arguments that can be used at an odd prime. Unfortunately, we do not know how to obtain the homology of tmf in the same manner, since for instance $H_*(\text{tmf}; \mathbb{Z}/p)$ does not inject into the dual Steenrod algebra when $p > 2$. Nonetheless, we provide a proof that tmf satisfies the axioms of the course notes [Rez07]. In the notes, the homology of tmf is calculated from a description Stack(tmf) that is assumed.

7.1. tmf at primes $p > 3$. When localized at a prime $p > 3$, the moduli stack $\overline{\text{M}_{	ext{ell}}}$ can be identified with the weighted projective stack $\mathbb{P}(4,6)$: that is, any (possibly nodal) elliptic curve over a $\mathbb{Z}[1/6]$-algebra $R$ can be (Zariski locally) written in the form

$$y^2 = x^3 + Ax + B,$$

where $A, B$ do not simultaneously vanish. The isomorphisms between elliptic curves are of the form $(x, y) \mapsto (u^2x, u^3y)$ for $u \in R^*$. The elements $A$ and $B$ are, up to units in $R$, the modular forms $c_4, c_6$.

In particular, the descent spectral sequence for $\text{Tmf}_p$ runs

$$H^i(\mathbb{P}(2,3), \omega^j) \Rightarrow \pi_{2j-i}\text{Tmf},$$

and degenerates, since the cohomology is concentrated in $H^0$ and $H^1$. One has therefore

$$\pi_*\text{Tmf} = \mathbb{Z}_p[4, c_6] \oplus \mathbb{Z}_p \{c_4^{-1}c_6^{-1}, c_4^{-2}c_6^{-1}, c_4^{-1}c_6^{-2}, \ldots\},$$

and Tmf is in fact complex orientable (as is any torsion-free ring spectrum).

Since the stack $\overline{\text{M}_{	ext{ell}}}$ has finite cohomological dimension, one concludes by reasoning with the (degenerate) descent spectral sequence:

Corollary 7.1. For any $p$-local spectrum $X$, one has $\text{Tmf} \wedge X \simeq \Gamma(O_{\text{top}} \wedge X)$.

Then, using the same reasoning as in Proposition 6.1, one gets:

Corollary 7.2. Let $R = \mathbb{Z}_p[a_1, \ldots, a_6, e_n]_{n \geq 4}$. Then $\text{Tmf}_* (MU)$ is a module over the ring $R$. As an $R$-module, it is isomorphic to $R \oplus C$, where $C$ is the cokernel of $R[c_4^{-1}] \oplus R[\Delta^{-1}] \to R[(c_4\Delta)^{-1}]$.

7.2. tmf at $3$. At the prime $3$, there is a three-fold cover $\overline{\rho}: R \to \text{M}_{\text{cub}}$ of the moduli stack of cubic curves. We will construct this cover, and show that it can be realized via a three-cell complex. Everything in this section is implicitly localized at $3$.

Proposition 7.3. Let $\overline{\rho}: \text{Spec}\mathbb{Z}_3[\alpha_2, \alpha_4]/\mathbb{G}_m \to \text{M}_{\text{cub}}$ classify the cubic curve $y^2 = x^3 + \alpha_2x^2 + \alpha_4x$. Then $\overline{\rho}$ is a finite flat cover of rank three.

Over the smooth locus, the stack $\text{Spec}\mathbb{Z}_3[\alpha_2, \alpha_4]/\mathbb{G}_m$ parametrizes elliptic curves together with a nonzero 2-torsion point (which in coordinates is $(0,0)$).

Proof. We will imitate the arguments of Proposition 4.3. Namely, to show finiteness, we can argue as in the proof of Proposition 4.3 and reduce to showing that the pull-back $\text{Spec}\mathbb{Z}_3[\alpha_2, \alpha_4] \times_{\text{M}_{\text{cub}}} \text{Spec}\mathbb{Z}/3$ is the spectrum of a finite $\mathbb{Z}/3$-algebra (where $\text{Spec}\mathbb{Z}/3 \to$...
Figure 5. The cohomology of the complex $F$

$M_{cub}$ classifies the cuspidal cubic. Using the change-of-variable formulas, we find that this fiber product is the spectrum of

$$\mathbb{Z}/3[r, s, t]/(2s, 2t, r^3 - t^2) = \mathbb{Z}/3[r]/(r^3),$$

which is clearly a finite $\mathbb{Z}/3$-algebra of dimension 3. We conclude that for any $R$ and for any map $\text{Spec} R \to M_{cub}$, the fiber product $\text{Spec} \mathbb{Z}/3[a_2, a_4]/G_m \times_{M_{cub}} \text{Spec} R$ is a subscheme of $A^3_R$ cut out by three equations, and that it is finite over $R$. As in the proof of Proposition 4.5, this implies flatness.

In order to realize the vector bundle $p^* (O)$ on $M_{ell}$ by a spectrum, we consider the element $\alpha_1 \in \pi_3(S^0)_{3(3)} = \mathbb{Z}/3$ (a generator of the 3-local 3-stem; $\alpha_1$ can be realized by the same quaternionic Hopf map that 2-locally realizes $\nu$). The cofiber of $\alpha_1$, which is a desuspension of $\mathbb{H}^2$, has cohomology generated by elements $x_0, x_4$ with $P^1 x_0 = x_4$. Since, furthermore, $\alpha_1^2 = 0 \in \pi_6(S^0)_{3(3)} = 0$, we conclude that there is a 3-local finite spectrum $F$ such that

$$H^*(F; \mathbb{Z}/3) \simeq \{x_0, x_4, x_8\}, \quad P^1 x_0 = x_4, \quad P^1 x_4 = x_8,$$

as in Figure 5. To construct $F$, we consider the diagram

$$
\begin{array}{ccc}
S^7 & \xrightarrow{\phi} & S^4 \\
\downarrow{\alpha_1} & & \downarrow{\Sigma a_1} \\
S^3 \xrightarrow{\alpha_1} S^0 \xrightarrow{\Sigma^{-4} \mathbb{H}^2} S^4 \xrightarrow{-\Sigma a_1} S^1
\end{array}
$$

where the horizontal line is a cofiber sequence. Since $\alpha_1^2 = 0$, we find a lifting $\phi : S^7 \to \Sigma^{-4} \mathbb{H}^2$ and let $F$ be the cofiber of $\phi$.

In the spirit of the previous sections, we prove:

**Proposition 7.4.** The vector bundle that $F$ defines on $M_{ell}$ is isomorphic to $\overline{p}_+(O)$.

**Proof.** We follow the outline of the earlier arguments. In fact, we start by producing a map $F \to MU$ (where $MU$ really means $MU(3)$) which induces an isomorphism on $\pi_0$, using the degeneration of the AHSS. In homology, the map

$$H_*(F; \mathbb{Z}/3) \to H_*(MU; \mathbb{Z}/3) \simeq \mathbb{Z}/3[x_1, x_2, \ldots], \quad |x_i| = 2i,$$

is an imbedding whose image contains an indecomposable generator in degree four, and its square. As a result, one gets a map of vector bundles on $M_{FG}$,

$$\mathcal{F}(F) \to \mathcal{F}(MU),$$
such that when one takes the fiber over the additive formal group over \( \text{Spec}\mathbb{Z}/3 \), one obtains the above map in homology.

Next, consider the cover \( q: (\text{Spec}L)/\mathbb{G}_m \to M_{FG} \) and the pullback

\[
q': \left( \text{Spec}\mathbb{Z}(3)[a_1, \ldots, a_6, \{e_n\}_{n \geq 4}] \right)/\mathbb{G}_m \cong M_{cub}^{\text{coord}} \to M_{cub}.
\]

As before, \( q'_*(\mathcal{O}) \) is the sheaf on \( M_{cub} \) that one obtains from \( MU \). One produces a map

\[
q'_*(\mathcal{O}) \to \overline{\mathcal{P}}_*(\mathcal{O}),
\]

by considering the closed imbedding

\[
(\text{Spec}\mathbb{Z}(3)[\alpha_2, \alpha_4])/\mathbb{G}_m \hookrightarrow \left( \text{Spec}\mathbb{Z}(3)[a_1, \ldots, a_6, \{e_n\}_{n \geq 4}] \right)/\mathbb{G}_m,
\]

which sends \( a_2 \mapsto \alpha_2, a_4 \mapsto \alpha_4 \), and annihilates all the other generators.

The claim is that the composite map

\[
\mathcal{F}(F) \to \mathcal{F}(MU) \to \overline{\mathcal{P}}_*(\mathcal{O}),
\]

is an isomorphism of vector bundles on \( M_{cub} \), which as before can be checked by showing that the map yields a surjection when one takes the fiber over the cuspidal cubic. To see this, we observe that when one takes the fiber over the cuspidal cubic, one gets the imbedding \( HP_0(F) \to HP_0(MU) \), whose image contains an indecomposable generator in degree 4 and its square. The map from \( HP_0(MU) \) to the fiber of \( \overline{\mathcal{P}}_*(\mathcal{O}) \) over the cuspidal cubic (which we have checked to be \( \mathbb{Z}/3[r]/(r^3) \)) is a map of algebras and induces a surjection on indecomposables. From this, the conclusion follows. \( \square \)

Using the thick subcategory theorem as before, one concludes:

**Corollary 7.5.** For any 3-local spectrum \( X \), one has \( \text{tmf} \wedge X \cong \Gamma(\mathcal{O}^{\text{top}} \wedge X) \).

Now that we know \( \text{tmf} \wedge MU \cong \Gamma(\mathcal{O}^{\text{top}} \wedge MU) \), the analog of Proposition 6.1 goes through at the prime 3. One has:

**Corollary 7.6.** Let \( R = \mathbb{Z}(3)[a_1, \ldots, a_6, e_n]_{n \geq 4} \). Then \( \text{tmf}_*(MU) \) is a module over the ring \( R \). As an \( R \)-module, it is isomorphic to \( R \oplus C \), where \( C \) is the cokernel of \( R[c_4^{-1}] \oplus R[\Delta^{-1}] \to R[\{c_4 \Delta\}]^{-1} \).

**Theorem 7.7** (Hopkins-Mahowald). \( \text{tmf} \wedge F \cong BP \langle 2 \rangle \vee \Sigma^8 BP \langle 2 \rangle \).

**Proof.** As in the proof of Theorem 6.6, we consider the map

\[
\text{tmf} \wedge F \to \text{tmf} \wedge MU = (\text{tmf} \wedge MU)/(a_1, a_3, a_5, a_6, \{e_n\}_{n \geq 4}),
\]

which we find to be an equivalence. Taking connective covers (and using the gap theorem), we conclude that \( \text{tmf} \wedge F \) admits the structure of an \( MU \)-module.

It remains to show that the homotopy groups of \( \text{tmf} \wedge F \), as a module over \( MU_* \), are those of \( BP \langle 2 \rangle \vee \Sigma^8 BP \langle 2 \rangle \). If we can prove this, then by Example 3.13, we will be done. For this, we consider the elliptic curve \( y^2 = x^3 + a_2x^2 + a_4x \) and use the formulas in Example 4.5 of Chapter 4 in \( \text{SAG} \) for the \( [3] \)-series on the elliptic curve’s formal group, which gives \( v_1 = -8a_2 \). To get \( v_2 \), we use the SAGE computer program \( \text{SAG} \) (and in particular, code written by William Stein, David Harvey, and Nick Alexander) to expand out the \( [3] \)-series to order 10 and get for the coefficient of \( t^9 \),

\[
v_2 = 2432a_4^2 + O(a_2).
\]

Let \( I \) be the kernel of the map

\[
BP_* \to MU_* \to \mathbb{Z}(3)[a_2, a_4].
\]
Then $I$ is a prime ideal, and the claim is that there exist indecomposable generators $v_3, v_4, \ldots$ such that $I = (v_3, v_4, \ldots)$ and such that $Z_{(3)}[a_2, a_4]$ is a free module of rank two over $BP_*/I$.

To see this, we show that the image of $BP_* \rightarrow Z_{(3)}[a_2, a_4]$ is contained in the subring of $R = Z_{(3)}[a_2, a_4]$ generated by the images of $v_1, v_2$. In order to do this, it suffices to show (using the connectedness of these graded rings and Nakayama’s lemma) that the images of $\{v_i\}, i \geq 3$ in $R/(3, v_1, v_2)$ all vanish. In other words, we need to show that the $v_i, i \geq 3$ vanish for the elliptic curve $y^2 = x^3 + a_4 x$ over the ring $Z/3[a_4]/(a_4^2)$. However, the $v_i$ live in grading too high to be nonzero in the ring $Z/3[a_4]/(a_4^2)$.

It follows now that the image of $BP_* \rightarrow Z_{(3)}[a_2, a_4]$ is the polynomial ring $Z_{(3)}[v_1, v_2] \subset Z_{(3)}[a_2, a_4]$, which shows that $I$ is generated by indecomposable elements in the manner as claimed above. Moreover, the inclusion $Z_{(3)}[v_1, v_2] \subset Z_{(3)}[a_2, a_4]$ is flat since $(3, v_1, v_2)$ is a regular sequence in the latter, and hence we get freeness of rank two as claimed.

\textbf{Remark 7.8.} Since all the $v_i$’s map to zero, it follows that the formal group of the elliptic curve $y^2 = x^3 + a_2 x^2$ over the ring $Z/3[a_2]/(a_2^2)$ is isomorphic to the additive one. (See for instance Lecture 13 of [Lur10].)

At $p \geq 5$, one can make a similar argument: the formal group of any cubic curve with $v_1, v_2 = 0$ is (Zariski locally) isomorphic to the additive one. Namely, it suffices to prove this for the “universal” elliptic curve $y^2 = x^3 + Ax + B$ over the (graded) ring $Z_{(p)}[A, B]$ with $|A| = 8, |B| = 12$ (where the grading is doubled as in topology). Since $(p, v_1, v_2)$ forms a regular sequence on this ring (because the sublocus where $p = v_1 = v_2 = 0$ is zero-dimensional), it follows that the inclusion $Z_{(p)}[v_1, v_2] \subset Z_{(p)}[A, B]$ exhibits $Z_{(p)}[A, B]$ as a free module over $Z_{(p)}[v_1, v_2]$ of rank $\frac{(p-1)(p^2-1)}{12}$, since $|v_1| = 2(p^2 - 1)$.

Consequently, the quotient ring $T = Z_{(p)}[A, B]/(p, v_1, v_2)$ is a $Z/p$-vector space of dimension $\frac{(p-1)(p^2-1)}{12}$. Since it is generated by elements of degree 2 and 3, it follows that

$$\max\{i : T_i \neq 0\} \leq 3 \frac{(p-1)(p^2-1)}{12},$$

which means that the $v_i, i \geq 3$ live in degree sufficiently large that they have to vanish for reasons of grading. We get that the formal group is isomorphic to the additive one.

8. The stack for $tmf$

In this section, we will describe the stack corresponding to $tmf$, or equivalently describe the structure of the Adams-Novikov spectral sequence for $tmf$. We will see that the stack associated to $tmf$ is precisely the moduli stack $M_{cub}$ of cubic curves. This produces a spectral sequence that allows computation with $tmf$. Throughout, we work \textit{integrally}.

\textbf{Corollary 8.1.} $tmf \land MU = Z[a_1, a_2, a_3, a_4, a_6, \{e_n\}]_{n \geq 4}$. The stack $\text{Stack}(tmf)$ is precisely the stack $M_{cub}$.

\textbf{Proof.} We begin by computing $tmf_*(MU)$. In fact, we know (Proposition 6.1, Corollary 7.6, and Corollary 7.2) that $\pi_*(Tmf \land MU)$ canonically surjects onto the ring in question with kernel an ideal $C \subset Tmf_*(MU)$ of square zero, and that there is a map $tmf \land MU \rightarrow Tmf \land MU$.

The claim is that the composite

$$(27) \quad tmf_*(MU) \rightarrow Tmf_*(MU) \rightarrow Tmf_*(MU)/C \simeq Tmf_*(MU)/F_1Tmf_*(MU)$$

is an isomorphism, where we use the notation of Proposition 6.1. $F_1$ refers to the filtration from the descent spectral sequence. This will compute $tmf_*(MU)$, as desired. We will prove this locally at each prime $p$. 

In order to prove this, we will use the existence of a finite even $p$-local spectrum $X$ with the following properties:

1. $\text{Tmf} \wedge X$ has torsion-free homotopy groups.
2. $\text{tmf} \wedge X = \tau_{\geq 0}(\text{Tmf} \wedge X)$.

For $p = 2$, we proved this fact with $X = DA(1)$. For $p = 3$, we proved this fact with $X = F$. For $p \geq 5$, we can take $X = S^0$, since $\text{Tmf}$ and $\text{tmf}$ are complex-orientable ring spectra with torsion-free homotopy groups.

We will now use these three facts to prove that (27) is an isomorphism at the (arbitrary) prime $p$. To start with, we note that $\text{tmf} \wedge MU$ is a complex-orientable $E_{\infty}$-ring, so that, since $X$ is an even spectrum, $\text{tmf}_*(MU \wedge X)$ is a sum of copies of $\text{tmf}_*(MU)$ (possibly shifted). The analog holds for $\text{Tmf}_*(MU \wedge X)$.

Moreover, since

$$\text{tmf} \wedge X \to \text{Tmf} \wedge X,$$

is a morphism of spectra whose map on homotopy groups is a split injection, we find that

$$\pi_*(\text{tmf} \wedge X \wedge MU) \to \pi_*(\text{Tmf} \wedge X \wedge MU)$$

is a split injection of graded abelian groups. In fact, this follows from:

**Lemma 8.2.** Let $E$ be any spectrum with torsion-free homotopy groups. Then $E_*(MU) \cong E_*[b_1, b_2, \ldots] \cong \pi_*E \otimes_\mathbb{Z} H_*(MU; \mathbb{Z})$.

**Proof.** In fact, the differentials in the Atiyah-Hirzebruch spectral sequence for $E_*(MU)$ are torsion-valued, but $E_*$ is torsion-free. This easily implies the result. We note that the isomorphism is not natural, but it is natural after passing to an associated graded. □

Since $X$ is even, we conclude that $\text{tmf}_*(MU) \to \text{Tmf}_*(MU)$ is a split injection. (Given the $X$’s we have considered, we could also appeal to (26).) It follows that both $\text{tmf}_*(MU)$ and the cokernel of $\text{tmf}_*(MU) \to \text{Tmf}_*(MU)$ are torsion-free abelian groups. To show that $\text{tmf}_*(MU) \to \text{Tmf}_*(MU)/\text{im} \pi_*(\text{tmf}/\text{Tmf}) \otimes \mathbb{Q}$ is an isomorphism, it now suffices to work rationally, since the kernel and the cokernel are torsion-free. This is much easier.

In fact, rationally, the descent spectral sequence for $\text{Tmf}$ degenerates and is concentrated in the zeroth and first row. The image of $\pi_*\text{tmf} \otimes \mathbb{Q} \to \pi_*\text{Tmf} \otimes \mathbb{Q}$ consists of the elements on the zeroth row: that is, $\pi_*\text{tmf} \otimes \mathbb{Q} \to \pi_*\text{Tmf} \otimes \mathbb{Q} \to (\pi_*\text{tmf}/\text{im} \pi_*\text{Tmf}) \otimes \mathbb{Q}$ is an isomorphism. It follows that this is true after tensoring $\text{tmf}$ with any rational spectrum.

This analysis shows that the morphism

$$\text{tmf}_*(MU) \to \text{Tmf}_*(MU) \to \text{Tmf}_*(MU)/C,$$

is an isomorphism, where $C$ is the ideal consisting of those elements of filtration 1 in the (degenerate) descent spectral sequence for $\text{Tmf} \wedge MU$. The same is therefore true when $MU$ is replaced by any wedge of suspensions of $MU$, for instance any smash power $MU^{\wedge n}$.

We find that, for any $s > 0$, $\text{tmf}_*(MU^{\wedge s})$ can be described as global sections of the stack

$$\mathcal{M}_{\text{eff}}^{\text{coord}} \times M_{\text{eff}}^{\text{coord}} \cdots \times M_{\text{eff}}^{\text{coord}}$$

(with $s$ factors): that is, the contributions of $H^1$ terms in the Tmf-homology can be ignored. The ring of global sections over this stack is the same as the ring of global sections over the larger stack

$$\mathcal{M}_{\text{cub}}^{\text{coord}} \times M_{\text{cub}} \cdots \times M_{\text{cub}},$$

which differs by a substack of codimension 2. We find that the stack

$$\text{Stack(\text{tmf})} = \text{hocolim} \left( \text{Spectmf}_*(MU \wedge MU \wedge MU) \overset{\tau}{\longrightarrow} \text{Spectmf}_*(MU \wedge MU) \overset{\tau}{\longrightarrow} \text{Spectmf}_*(MU) \right)$$

precisely writes down the presentation of $\mathcal{M}_{\text{cub}}$ (via the flat cover $M_{\text{cub}}^{\text{coord}} \to M_{\text{cub}}$). □
In particular, we can describe the spectral sequence for tmf-homology.

**Corollary 8.3.** Let $X$ be a spectrum. Then $X$ defines a quasi-coherent sheaf $\tilde{F}(X)$ on the moduli stack $M_{\text{cub}}$ (if $X$ is even, then $\tilde{F}(X)$ is the pull-back of $F(X)$ under $M_{\text{cub}} \to M_{\text{FG}}$) and a spectral sequence

$$H^i(M_{\text{cub}}, F(X) \otimes \omega^j) \Rightarrow \text{tmf}_{2j-i}(X).$$

The stack $M_{\text{cub}}$ can be presented by the Weierstrass Hopf algebroid. We remark that this spectral sequence is the Adams-Novikov spectral sequence for $\text{tmf} \wedge X$. In [Bau08], it is used to calculate $\pi_* \text{tmf}$.

**Remark 8.4.** The map $M_{\text{cub}} \to M_{\text{FG}}$ that arises from Stack($\text{tmf}$) $\to$ Stack($S^0$) can also be realized via the map that sends a cubic curve to the formal group of its nonsingular locus (with its canonical group structure). We already know this over the locus $M_{\text{ell}}$, and both maps $M_{\text{cub}} \to M_{\text{FG}}$ are affine and agree on the dense open substack $M_{\text{ell}} \subset M_{\text{cub}}$.

This method of computing the homology (and Adams-Novikov spectral sequence) of tmf is dependent on a key piece of prior computational knowledge about Tmf: namely, the gap theorem in the homotopy groups $\pi_* \text{tmf}$. It would be interesting if one could give a theoretical explanation of the gap theorem.

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HARVARD UNIVERSITY, CAMBRIDGE MA 02138
E-mail address: amathew@college.harvard.edu