A linear algorithm for the identification of a relaxation kernel using two boundary measures

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Abstract
We consider a distributed system with persistent memory of a type which is encountered in the study of diffusion processes with memory and viscoelasticity for materials of Maxwell–Boltzmann type. The relaxation kernel, i.e. the kernel of the memory term, is scarcely known from first principles, and it has to be inferred from experiments taken on samples of the material. We prove that two boundary measures give a linear Volterra integral equation of the first kind for the unknown kernel. Hence, with two measures, the identification of the kernel, which in principle is a nonlinear problem, is reduced to the solution of a deconvolution problem, hence to an ill-posed but linear problem which can be solved with existing methods.

Keywords: relaxation kernel, identification, thermodynamics with memory, non-Fickian diffusion, viscoelasticity

1. Introduction
The following heat equation with memory

\[
\theta'(x, t) = \int_0^t N(t-s)\theta_{xx}(x, s) \, ds
\]  
(1.1)

(here \(\theta = \theta(x, t), t > 0, x \in (a, b)\)). Prime denotes the derivative respect to the time \(t\) is encountered in thermodynamics of materials with memory (\(\theta\) is the temperature), in non-Fickian diffusion (\(\theta\) is the concentration). In viscoelasticity, \(\theta\) represents the displacement (usually denoted \(w\)) and the equation is more commonly rewritten as

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\[
\theta^\alpha(x, t) = N(0) \Delta \theta(x, t) + \int_0^t M(t - s) \Delta \theta(x, s) \, ds, \quad M(t) = N'(t). \tag{1.2}
\]

In order to solve equation (1.1) or (1.2) we need initial and boundary conditions. We impose
\[
\theta(x, 0) = \xi(x), \quad \theta(a, t) = f(t), \quad \theta(b, t) = 0. \tag{1.3}
\]

In order to solve equation (1.2) we need also an initial velocity, but when (1.2) is an integrated version if (1.1) the initial velocity is zero.

For consistency in the presentation, \( \theta \) will be called the temperature and the terminology of diffusion processes will be used, with only few hints to the interpretation in terms of viscoelasticity.

We shall use certain properties of the systems described by (1.1), explained in the final section 4. For the moment, the important points to notice (and which are explained in section 4) are as follows:

- when \( N(0) > 0 \), signals propagate with finite speed. The speed is \( c = \sqrt{N(0)} \), see [12, formula (7.9)].
- the (density of the) flux of heat \( q(x, t) \) at the position \( x \) and time \( t \) is not \( -\theta_t(x, t) \), as in the case of the standard heat equation \( \theta_t = \theta_{xx} \), but it is given by
  \[
  q(x, t) = -\int_0^t N(t - s) \theta_s(x, s) \, ds ; \tag{1.4}
  \]
  a measure of the flux at a certain time \( t \) gives a measure of this integral (in viscoelasticity, the traction in position \( x \) and time \( t \) is \( q'(x, t) \)).

The function \( N(t) \) is often called the relaxation kernel. In this paper we assume that it is a smooth function (see below for details) and, as in most of the applications to diffusion processes and viscoelasticity, \( N(0) > 0 \).

It is a fact that the kernel \( N(t) \) is a material property. Only its qualitative properties can be deduced from general principles (see for example [7]) and the specific function \( N(t) \) has to be inferred from experiments taken on the system. The ‘experiments’ usually considered consist in the observation of the flux (or traction) on the boundary of a suitable sample of the material, subject to a known excitation. The memory kernel \( N(t) \) is identified from such observations.

**Remark 1.** The integrals in (1.1), (1.4) should extend from the time the material has been ‘created’, conceivably \(-\infty\). We assume that the measures are taken after a certain time \( t_0 \) sufficiently far from the ‘creation’ of the material and that the material was kept undisturbed for a long time, so that the effect of the previous history is negligible. This time \( t_0 \) is renamed 0.

Equation (1.1) or (1.2) are studied also in regions \( \Omega \subseteq \mathbb{R}^d \) (in practice, \( d = 2 \) or \( d = 3 \)) but most of the papers in Engineering Journals on kernel identification assume \( \Omega = (a, b) \). The reason is that the shape of the sample of the material usually does not influence the memory kernel, otherwise \( N \) would depend also on the position \( x \) (see [3]). So, we can try to determine \( N(t) \) from samples which have simple geometries and most of the references in Engineering Journals (commented in section 1.2) use samples which are in the shape of a slab, i.e. \( x \in (a, b) \), as we do in this paper. In order to present simpler formulas, we normalize \( (a, b) = (0, \pi) \).
The case $x \in \Omega \subseteq \mathbb{R}^d$, $d > 1$, may have an interest in nondestructive testing. We reserve this problem for a future study.

The identification of $N(t)$ is a nonlinear problem since if $N(t)$ is not yet known, the function $\theta(x, t)$ cannot be computed and the problem consists in the identification of the pair $(N(t), \theta(x, t))$. This pair enters in a nonlinear way in the equation. It is a fact that one boundary measure obtained using a suitable and known initial datum is sufficient for the identification of the relaxation kernel, but at the expenses of solving a complex nonlinear problem in a Hilbert space (see the references in section 1.2). In this paper we shall see that using two boundary measures we can construct a linear Volterra integral equation of the first kind solely for $N(t)$. So, the identification of $N(t)$ is reduced to a deconvolution problem. This is a first difference with existing literature. A second difference is described in section 1.2.

1.1. The assumption on $N(t)$ and informal description of the procedure

The first of our assumptions is that the kernel $N(t)$ is of class $C^3$ (so that the memory kernel $M(t) = N'(t)$ of the viscoelastic system is of class $C^2$ as often assumed in this kind of problems, see [1]).

As noted above, signals propagate in the body with finite velocity $c = \sqrt{N(0)}$. So the value of $c$ is easily computed: just measure the velocity of propagation of the signals in the sample of the material. We assume that the first step of the identification of the value $c = \sqrt{N(0)}$ has already been done. Then, solely to present simple formulas, we change the time scale via $t \mapsto t/c$. This change in the time scale replaces $N(t)$ with $(1/c^2)N(t/c)$ which is renamed $N(t)$ (and so we have $N(0) = 1$). Moreover we recall that also the interval $(a, b)$ was normalized to $(0, \pi)$.

Now we describe informally the identification procedure. We recall the goal: we fix any time $T$ and we want to identify $N(t)$ for $t \in [0, T]$ (we already reduced to problem to have $N(0) = 1$).

We perform the following two measures, both on intervals of duration $T$ (it is not restrictive to rename such intervals as $[0, T]$):

1. We impose null initial condition and $\theta(\pi, t) = 0$. We excite the system from the left end: we impose $\theta(0, t) = f(t)$ and we measure $Y^f(t) = \varphi(\pi, t)$, i.e. the flux at the opposite end (we use the upper case $Y$ to distinguish the flux due to a boundary source from that due to a nonzero initial condition, which will be denoted $\varphi$).

We don’t need to use any special function $f(t)$ but in practice the function $f(t)$ will be smooth. We take $f(0) = 0$ (in accordance with $\theta(x, 0) = 0$ for $x \in (0, \pi)$) and $f(t)$ approaches $\lim_{t \to +\infty} f(t)$ (which in practice is finite) quite fast.

We shall see that using this measure we can write a linear Volterra integral equation of the first kind whose solution is the function

$$K(t) = \frac{1}{2} \int_0^t N(s) \, ds$$

where $K(t)$ has the special expression given in (3.4). So, in order to identify $N(t)$ we need a second measure which gives $K(t)$.

2. The second measure is as follows: the temperature $\theta(x, t)$ at the ends $x = 0$ and $x = \pi$ is kept equal zero, $\theta(0, t) = 0$, $\theta(\pi, t) = 0$. We impose a special and practically realizable initial condition $\theta(x, 0) = \xi(x)$, i.e. a ramp, and we read $y^\xi(t) = \varphi(\pi, t)$ (note the lower case $\varphi$). The function $K(t)$ is given by this measure and so $N(t)$ can be identified.
We note that the second step might be done by imposing \( \theta(x, 0) = 0 \) and \( \theta'(x, 0) = \eta(x) \) for \( x \in (0, \pi) \), with a special initial velocity \( \eta(x) \) (integrating (1.1)) we see that in this case equation (1.2) must have an additive term \( \eta(x) \) on the right hand side. We study \( \eta = 0 \) and \( \xi = 0 \) but in practical applications to viscoelasticity \( \xi = 0 \) and \( \eta \neq 0 \) might be easier to impose.

Now we describe the organization of the paper. In section 1.2 we discuss few references on the identification problem and in section 2 we present preliminary properties of system (1.1). The identification algorithm is in section 3.

A similar linear algorithm can be constructed, which uses measures taken with two different, but related, initial conditions (and zero temperature at the boundary). We believe that this algorithm is not practical and it is not described in this paper.

1.2. Comments on previous references

Due to the importance of this problem, literature on kernel identification is enormous and both engineers and mathematicians addressed the problem, along different and scarcely related lines. Most of the available results can be described as follows. An approach to kernel identification, as for example in [13] and then followed by a wealth of related papers (see for example [1] for references and related results) is as follow: we associate a certain observation \( y(x, t) \) to system (1.1), which depends linearly on \( \theta(x, t) \), \( y(t) = C(\theta(·, t)) \). The idea is to compute derivatives of \( y(t) \) (of course when the data are sufficiently smooth) and to get a second equation for \( \theta(t) \). In this way we get a system of nonlinear integrodifferential equations in a Hilbert space for the pair \( (\theta(t), N(t)) \) and, using fixed point theorems, it is possible to prove the existence of solutions of this systems. If the data are not so smooth and the derivative of the output cannot be computed, it is still possible to see equation (1.1) and the output relation as a pair of nonlinear equations which, under suitable, weak, conditions, admit a unique solution \( (\theta(x, t), N(t)) \). These methods lead to the identification in particular of \( N(t) \) (according to the assumptions, either locally, for a short time, or globally). The identification requires only one measure, taken assuming known (and smooth) initial condition (or boundary function) and works equally well for systems in regions of \( \mathbb{R}^n \), \( n \geq 1 \), but the resulting algorithm requires the solution of a complex nonlinear system of equations in a Hilbert space, in the unknown \( (\theta, N) \).

Different algorithms works in the frequency domain, using Fourier or Laplace transformations, as for example in [10].

The approach that is most common in engineering literature is as follow. It is assumed that the kernel \( N(t) \) belongs to a certain class of functions, identified by a finite number of parameters. The classes that are most often considered are Dirichlet–Prony sums, i.e.

\[
N(t) = \sum_{k=0}^{K} a_k e^{-b_k t}, \quad a_k > 0, \ b_k > 0
\]  

(1.5)

as for example in [3, 18, 20] or a combination of Abel kernels (see [5, 8])

\[
N(t) = \sum_{k=0}^{K} \frac{a_k}{t^{1+\beta_k}}, \quad a_k > 0, \ 0 \leq \beta_k < 1
\]  

(1.6)

sometimes a combination of both.

The kernels (1.5) and (1.6) have a physical sense since they are encountered in ordinary differential equations of integer or fractional order. Moreover, these classes of kernels are reported to agree sufficiently well with experimental data, but not in every application, see [17].

In order to identify the parameters, a sample of the material, usually in the form of a slab, is excited using a known initial or boundary signal. The flux (or the stress) evolution in time is
measured on the boundary and the parameters \((a_k, b_k, \text{ sometime also the exponents } k_b)\) are chosen so to fit the experimental measures (usually minimizing a quadratic index). This can be done using a single measure. Multiple measures can be used to improve accuracy or to remove spurious frequencies.

Note that these algorithms require that equation (1.1) be solved for many values of the parameters, so to have candidate outputs from which the one which best fits the measure is selected.

A second difference of this paper to most of the engineering literature is that we don’t assume that the kernel belongs to any specific class of functions, except regularity and \(N(0) > 0\). In fact, no further properties of the kernel are in principle needed for the reconstruction algorithms (not even positivity for every time!) in spite of the fact that such conditions are imposed by thermodynamics (see [7]) and can be used to improve the results, see section 3.1. Note, however, that we are assuming that \(N(t)\) is smooth for \(t \geq 0\) and this rules out Abel kernels. The extension of our algorithm to Abel kernels and to systems in higher dimensional regions will be pursued in future researches.

2. Preliminary properties of system (1.1)

We are going to use a representation of the solutions of system (1.1) when \(\Omega = (0, \pi)\), taken from [15] (see [16] for more general results, in particular for extensions to the case \(\dim \Omega > 1\)).

We consider the sequence of the functions
\[
\phi_n = \sqrt{\frac{2}{\pi}} \sin nx, \quad n \in \mathbb{N} \quad \text{so that} \quad \frac{d^2}{dx^2} \phi_n(x) = -n^2 \phi_n(x).
\]
It is known from the theory of Fourier series that \(\{ \phi_n \}\) is an orthonormal basis of \(L^2(0, \pi)\). We solve equation (1.1) with the conditions (1.3). We expand the initial condition \(\xi(x)\) and the solution \(\theta(x, t)\) of equation (1.1) in sine series:
\[
\xi(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \xi_n \sin nx, \quad \theta(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \theta_n(t) \sin nx
\]
where, for every \(n \in \mathbb{N}\), we defined
\[
\xi_n := \int_0^\pi \xi(x) \sin nx \, dx, \quad \theta_n(t) := \int_0^\pi \theta(x, t) \sin nx \, dx.
\]
The functions \(\theta_n(t)\) solve
\[
\theta_n'' = -n^2 \int_0^t N(t - s) \theta_n(s) \, ds + n \int_0^t N(t - s) f(s) \, ds, \quad \theta_n(0) = \xi_n.
\]
This is a scalar integrodifferential equation whose solution can be represented using the variation of constants formula (see for example [16, formula (5.7)])
\[
\theta_n(t) = \xi_n z_n(t) + \left[ n \int_0^t z_n(t - s) \int_s^t N(s - r) f(r) \, dr \, ds \right]
\] (2.1)
where \(z_n(t)\) solves
\[
z_n'(t) = -n^2 \int_0^t N(t - s) z_n(s) \, ds, \quad z_n(0) = 1.
\] (2.2)
The transformation \((\xi, f) \mapsto w\) is linear and continuous from \(L^2(0, \pi) \times L^2(0, T)\) to \(C([0, T]; L^2(0, \pi)) \cap C^4([0, T]; H^{-1}(0, \pi))\).
We need the following observation:

**Lemma 2.** Recall the condition $N(0) = 1$ and let $T > 0$. There exists $M = M_T$ such that, for every $t \in [0, T]$ and every $n$, we have:

\[
\left| \frac{z_n'(t)}{n} + e^{-\alpha t} \sin nt \right| \leq \frac{M}{n}, \quad \alpha = -\frac{1}{2} N'(0).
\]  

**Proof.** This inequality follows for example from the equality (5.4) in [15]. We must be careful with the notation. With $\alpha = -N'(0)/2$ let $\tilde{N}(t) = e^{2\alpha t} N(t)$ so that $\tilde{N}(0) = 1$ and $\tilde{N}'(0) = 0$. The functions denoted $z_n(t)$ and $N(t)$ in [15, formula (5.4)] are respectively $e^{2\alpha t} z_n(t)$ and $\tilde{N}(t) = e^{2\alpha t} N(t)$ (see [15, formula (5.1)]). So, we have the following equality for the function $z_n(t)$ in (2.2) (here $\beta_n = \sqrt{n^2 - \alpha^2}$):

\[
z_n(t) = e^{-\alpha t} \cos \beta_n t + \frac{\alpha}{\beta_n} e^{-\alpha t} \sin \beta_n t - \frac{n^2}{\beta_n^2} \int_0^t e^{-2\alpha s} z_n(s) \left[ \tilde{N}'(t - s) \right] e^{\alpha s} \cos \beta_n r \, dr \, ds.
\]

We compute the derivatives of both the sides, using $\tilde{N}'(0) = 0$. We get the equality

\[
z_n'(t) + \beta_n e^{-\alpha t} \sin \beta_n t = -\frac{\alpha^2}{\beta_n^2} e^{-\alpha t} \sin \beta_n t
\]

\[
- \frac{n^2}{\beta_n^2} \int_0^t e^{-2\alpha s} z_n(s) \left[ \tilde{N}''(t - s) - \tilde{N}'(0) e^{\alpha(t-s)} \cos \beta_n (t-s) \right] e^{\alpha s} \cos \beta_n r \, dr \, ds.
\]

The result follows by dividing both the sides with $n$ and noting that if $t \in [0, T]$ then there exists $M$ such that

\[
\left| \sin \beta_n t - \sin t \right| \leq \frac{M}{n}.
\]

We are going to use these properties for the identification of the kernel $N(t)$, assumed unknown.

### 3. The identification procedure

Now we give the details of the two steps of the identification procedure described in section 1.1.

**The first measure: the flux due to a boundary excitation.** We consider system (1.1) with the following initial and boundary conditions:

\[
\theta(x, 0) = 0 \quad x \in (0, \pi), \quad \theta(0, t) = f(t), \quad \theta(\pi, t) = 0 \quad t > 0.
\]

We choose any $C^1$ input $f$ such that $f(0) = 0$ (this is consistent with the fact that the initial condition is zero). So,
\[ f(t) = \int_0^t g(s) \, ds \] where \( g(t) = f'(t) \). We assume \( g(0) = f''(0) \neq 0 \).

We observe that
\[
\frac{d}{ds} z_n(t-s) = + \frac{1}{n^2} \int_0^{t-s} N(t-s-r) z_n(r) \, dr \quad \text{i. e.}
\]
\[
\frac{d}{ds} z_n(t-s) = - \frac{d}{dr} z_n(t-s).
\]

Using (2.1) we see that (in the last step we integrate by parts)
\[
\frac{\pi}{2} \theta(x, t) = \sum_{n=1}^{+\infty} \left( \frac{1}{n} \int_0^t f(s) \int_0^{t-s} N(t-s-r) \, dr \right) \, ds
\]
\[
\quad = \sum_{n=1}^{+\infty} \left( \frac{1}{n} \int_0^t f(s) \frac{d}{ds} z_n(t-s) \, ds \right)
\]
\[
\quad = \left( \sum_{n=1}^{+\infty} \frac{1}{n} \cos(nx) \right) \left( \int_0^t g(r) \, dr - \int_0^t g(s) z_n(t-s) \, ds \right).
\]  
(3.1)

The function \((x, t) \mapsto \theta(x, t)\) belongs to \(C([0, T]; L^2(0, \pi))\) so that \((x, t) \mapsto \theta_i(x, t)\) belongs to \(C([0, T]; H^{-1}(\Omega))\),
\[
\frac{\pi}{2} \theta_i(x, t) = \left( \int_0^t g(r) \, dr \right) \sum_{n=1}^{+\infty} \frac{\cos(nx)}{n} \int_0^t g(s) z_n(t-s) \, ds
\]
\[
\quad = \pi \delta(x) \left( \int_0^t g(r) \, dr \right) - \frac{1}{2} \left( \int_0^t g(r) \, dr \right) - \sum_{n=1}^{+\infty} (\cos(nx)) \int_0^t g(s) z_n(t-s) \, ds.
\]  
(3.2)

Here \(\delta(x)\) is the Dirac’s delta (supported at \(x = 0\)). The flux at \(x = \pi\) due to \(f(t)\) is
\[
\frac{\pi}{2} Y_f(t) = \frac{\pi}{2} q(\pi, t) = \frac{1}{2} \left( \int_0^t N(t-s) \int_0^s g(r) \, dr \, ds \right)
\]
\[
\quad + \sum_{n=1}^{+\infty} (-1)^n \int_0^t g(s) \int_0^{t-s} N(t-s-r) z_n(r) \, dr \, ds
\]
\[
\quad = \frac{1}{2} \left( \int_0^t N(t-s) \int_0^s g(r) \, dr \, ds \right) - \sum_{n=1}^{+\infty} (-1)^n \frac{1}{n^2} \int_0^s g(s) z_n(t-s) \, ds
\]
\[
\quad = \frac{1}{2} \left( \int_0^t N(t-s) \int_0^s g(r) \, dr \, ds \right) - \int_0^t g(t-s) \left[ \sum_{n=1}^{+\infty} (-1)^n \frac{1}{n^2} z_n(s) \right] \, ds
\]
\[
\quad = \int_0^t g(t-s) \left( \frac{1}{2} \int_0^t N(r) \, dr - K(s) \right) \, ds
\]  
(3.3)

where
\[
K(t) = \sum_{n=1}^{+\infty} (-1)^n \frac{1}{n^2} z_n(t).  
\]  
(3.4)

The function \(g(t)\) is known and \((\pi/2) Y_f(t)\) can be measured. We shall see that \(K(t)\) can be independently measured. Hence, we can solve the deconvolution problem (3.3) and from its solution we can obtain
The determination of $N(t)$ is a second deconvolution problem (in fact, the \textit{numerical differentiation} of $N(t)$).

Before embarking in the independent determination of $K(t)$, we present few remarks.

- If $g \in C^1$, hence $f \in C^2$, then $Y(t)$ is differentiable and, upon computing the derivatives of both the sides of (3.3), we get:

$$
\frac{d}{dt} Y(t) = 2 g(0) K(t) + 2 \int_0^t g'(t-s) K(s) \, ds + \pi \left( Y'(t) \right)'
$$

We have chosen an input signal $f$ such that $f'(0) = g(0) = 0$. So, this is a Volterra integral equation of the second kind (but note the numerical computation of $Y'(t)$).

- In the special case that $g(0) = 1$ and $g'(t) = 0$ then (3.5) is an explicit formula for $N(t)$ but in this case $f(t) = t$. A steadily increasing temperature cannot be applied. In practice, $f(t)$ must approach a finite limit. For example we can choose $f(t) = t/(t+1)$ or $f(t) = 1 - e^{-t}$.

The second measure: \textbf{the flux due to the initial temperature}. The goal is the computation of the function $K(t)$ in (3.3). We consider system (1.1) with the conditions

$$
\theta(x, 0) = \xi(x) = \frac{2}{\pi} \sum_{n=1}^{+\infty} \xi_n \sin nx \in L^2(0, \pi), \quad \theta(0, t) = 0, \theta(\pi, t) = 0
$$

so that

$$
\frac{\pi}{2} \theta(x, t) = \sum_{n=1}^{+\infty} \xi_n z_n(t) \sin nx \in C \left( [0, T]; L^2(0, \pi) \right),
$$

$$
\frac{\pi}{2} \theta'(x, t) = \sum_{n=1}^{+\infty} \left( n \xi_n \right) z_n(t) \cos nx \in C \left( [0, T]; H^{-1}(0, \pi) \right),
$$

$$
\frac{\pi}{2} q(x, t) = \sum_{n=1}^{+\infty} \xi_n \left[ -n \int_0^t N(t-s) z_n(s) \, ds \right] \cos nx
$$

$$
= \sum_{n=1}^{+\infty} \xi_n z_n' \cos nx
$$

and

$$
\frac{\pi}{2} q(\pi, t) = \sum_{n=1}^{+\infty} \xi_n z_n' \cos nx
$$

(3.6)

Using the inequality (2.3), we see that $y^i(t) \in L^2(0, T)$ for every $T > 0$ and every $\xi \in L^2(0, \pi)$. If in particular we choose

$$
\xi_0(x) = \frac{1}{2} (\pi - x) = \sum_{n=1}^{+\infty} \frac{1}{n} \sin nx
$$

(3.7)

then the series in (3.7) is

$$
\sum_{n=1}^{+\infty} \frac{1}{n^2} (-1)^n z_n'(t) = K(t).
$$

(3.9)

This is the function we need in the identification process.
3.1. A mathematical simulation

We delegate to engineers on the basis of experimental data to decide about the merits of the identification algorithms we presented in this paper. Here we present a purely mathematical simulation based on theoretical computations.

We note that the algorithm proposed in this paper shows that, using two measures, kernel identification is reduced to a standard deconvolution problem: any deconvolution algorithm can be used, and no new numerical issue arises. The simulation is presented only for the sake of completeness.

We assume that $N(t)$ is a linear combination of exponentials, as it is often assumed in applications, and for simplicity we confine ourselves to the case that $N(t)$ is the combination of three exponentials. The limitation of the simulation we present below is not so much the number of terms in $N(t)$ but the fact that the data are obtained by computations and not from concrete experimental data. In fact, the use of a sum of few exponentials is common with the simulations in the references given in section 1.2 and it is justified by the fact that the contribution of those addenda which decays too fast is blurred by the noise, and not visible in the plots.

The kernel is

$$N(t) = a_1 e^{-b_1 t} + a_2 e^{-b_2 t} + a_3 e^{-b_3 t}, \quad a_k > 0, \ b_k > 0. \quad (3.10)$$

For every $n \in \mathbb{N}$, the equation of $z_n(t)$ is

$$z_n'(t) = -n^2 \int_0^t \left[ a_1 e^{-b_1(t-s)} + a_2 e^{-b_2(t-s)} + a_3 e^{-b_3(t-s)} \right] z_n(s) \, ds.$$

It is easy to reduce the computation of $z_n(t)$ to the solution of an ordinary differential equation. In fact, $z_n(t)$ is the first component of the solution $(z, w_1, w_2, w_3)$ of the following problem:

$$
\begin{align*}
 z' &= -n^2 (a_1 w_1 + a_2 w_2 + a_3 w_3) \\
 w_1' &= -b_1 w_1 + z \\
 w_2' &= -b_2 w_2 + z \\
 w_3' &= -b_3 w_3 + z
\end{align*}
$$

So, the functions $z_n(t)$ can be easily computed and we can compute also $Y(t)$ and $K(t)$. Then we choose $f(t) = 1 - e^{-t}$ and we reconstruct the kernels $N_1(t) = \int_0^t N(s) \, ds$ and $N(t)$.

Before we present the simulation, we note that:

* the reconstruction of the relaxation kernel is in two steps: first we identify $N_1(t)$ and then numerical differentiation gives $N(t)$. The identification of $N_1(t)$ is a deconvolution problem which can be solved, essentially with the same results, using Tikonov or Lavrentev regularization. In the simulation we present below we choose to reconstruct $N_1(t)$ using Lavrentev method (see [6]). Both Tikonov and Lavrentev methods introduce fast oscillations but it is known in this problem of kernel reconstruction that $N_1(t)$ is increasing (since for physical reasons $N(t) \geq 0$) and concave since $N(t)$ must be decreasing. Hence these oscillations can be attenuated by computing the average of the reconstruction of $N_1(t)$ on nearby instants.

This has been done in the simulation below. We stress that averaging is a poor and scarcely justified method in general, but it is justified and effective in the present problem due to the a priori information on the properties of the kernel.
The reconstruction of $N(t)$ is a more delicate matter as can be seen from the fact that in principle we can reconstruct $N(t)$ in one step by solving the deconvolution problem

$$(g * H) * N = g * K + \pi Y^f$$

($H$ is the Heaviside function and $*$ denotes convolution). The difficulty stems from the fact that $g * H$ does not satisfy the sector condition in [6]. In fact, it is a 2-smoothing kernel in the terminology of [11]. In spite of this, Tikonov and Lavrentev methods (followed by regularization) could be used, but we use a different regularization: first we approximate the computed function $N(t)$ with a polynomial (in the mean square sense, using polyfit.m code in Matlab). Then we compute the derivative of this polynomial. This avoids the oscillations due to the noise and the discretization.

The special case we test is as follows: the time interval is $[0, 5]$ and we assume 10 measures every time unit. We choose

$$N(t) = (1/10)e^{-t/2} + (1/5)e^{-2t} + (1/2)e^{-3t}.$$ (3.12)

The functions $K(t)$ and $Y^f(t)$ are approximated (and corrupted by an error of 1% at every ‘measure’). Then, we ‘reconstruct’ both $N(t)$ and $N(t)$ as described above. Figure 1 (left) presents the plots of $N(t)$ and its reconstruction from Lavrentev method (the plot where the amplification of the disturbances is evident) and its regularization using the mean of three values on the left and on the right at each step. The averaged reconstruction is approximated with a polynomial of degree 7, whose derivative gives the ‘reconstruction’ of the kernel $N(t)$ (plotted on the right). The penalization parameters of Lavrentev method is 0.1 (equal to the interval of time between consecutive measures).

4. Properties of the system (1.1)

The heat equation is derived from two fundamental physical facts, conservation of energy and the fact that temperature is a measure of the energy, associated with a suitable ‘constitutive law’. The physical facts are subsumed by the following equalities:

$$e^t(x, t) = -q(x, t), \quad \theta^t(x, t) = \gamma e^t(x, t)$$ (4.1)

where $e$ is the energy, $\gamma > 0$ is a scale factor and $q$ is the density of the flux at position $x$ and time $t$. The constitutive law relates $q$ to the temperature, i.e. to the energy. The usual Fourier law $q(x, t) = -k\theta(x, t)$ leads to the standard (memoryless) heat equation and assumes that the flux reacts immediately to the gradient of the temperature. Maxwell and later Cattaneo (see [4, 14]) proposed instead that the flux of energy reacts ‘slowly’ to variations of temperature and, once activated, persists in the future. The assumption in [4, 14] is:

$$q(x, t) = -\int_0^t ae^{-b(t-s)}\theta(x, s)\ dx, \quad a > 0, b > 0.$$ (4.2)

This constitutive law leads to equation (1.1) with an exponential kernel. In this special case equation (1.1) can be differentiated in time, and reduced to the telegraphists’ equation. Later on it was noticed that such a special kernel contains too few parameters to fit well experimental data and finally Gurtin and Pipkin in [9] proposed the constitutive equation

$$q(x, t) = -\int_0^t N(t-s)\theta(x, s)\ ds$$ (4.3)

for the (density of the) flux at time $t$ and position $x$. Equation (1.1) is obtained from this Gurtin–Pipkin law for the flux and from the relations (4.1).
Note that equation (1.2) is the (linearized) version of the equation obtained by Boltzmann in [2] as a consequence of his superposition principle. In this case, Boltzmann constitutive assumption is that the stress at time $t$ and position $x$ is $q'(x, t)$ where $q$ is given by (4.3).

The reason that stimulated the introduction of the new law (4.2) and (4.3) for the flux is to get a heat equation with the property that thermal signals propagate with finite speed. In fact, the speed of propagation is $c = \sqrt{N(0)}$, see [12, formula (7.9)]. We conclude this paper with an explanation of this property.

We consider velocity of propagation from the boundary, so we assume null initial conditions. We write equation (1.1) in the form (1.2) and we note that formally this is a Volterra integral equation in the unknown $\theta_{xt}(x, t)$. Let $R(t)$ be the resolvent kernel of $M(t)/N(0)$. Then we have (see [16, chapter 2])

$$N(0)\theta_{xt}(x, t) = \theta''(x, t) - \int_{0}^{t} R(t - s)\theta''(x, s) \, ds.$$ 

We integrate by parts and we see that

$$\theta''(x, t) = \sqrt{N(0)} \theta_{xt}(x, t) + F(x, t), \quad \left\{ \begin{array}{l} \theta(x, 0) = 0, \quad \theta'(x, 0) = 0 \quad x \in (0, \pi) \\ \theta(0, t) = f(x), \quad \theta(\pi, t) = 0 \quad t > 0 \end{array} \right.$$ 

where

$$F(x, t) = R(0)\theta'(x, t) + R'(0)\theta(x, t) - \int_{0}^{t} R''(t - s)\theta(x, s) \, ds.$$ 

Let us consider the problem on a ‘short time interval, $t < \pi/\sqrt{N(0)}$. In this case [19, formula (25) p 77] shows that

$$\theta(x, t) = f\left(\sqrt{N(0)} t - x\right) + \frac{1}{2c} \int_{0}^{t} \int_{|x-c(t-\tau)|} \int_{x-\xi(t-\tau)} F(\xi, \tau) \, d\xi \, d\tau.$$ 

The integral is a continuous function of $t$ and $x$. So, if the boundary input $f$ has a jump at some time $t_0$, this jump ‘travels’ due to the term $f\left(\sqrt{N(0)} t - x\right)$, precisely as in the wave equation, hence with velocity $c = \sqrt{N(0)}$.
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