Practical Algorithms for Linear Boolean-width

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Abstract

In this paper, we give a number of new exact algorithms and heuristics to compute linear boolean decompositions, and experimentally evaluate these algorithms. The experimental evaluation shows that significant improvements can be made with respect to running time without increasing the width of the generated decompositions. We also evaluated dynamic programming algorithms on linear boolean decompositions for several vertex subset problems. This evaluation shows that such algorithms are often much faster (up to several orders of magnitude) compared to theoretical worst case bounds.

1998 ACM Subject Classification G.2.2 [Discrete Mathematics]: Graph Theory — Graph algorithms; F.2.2 [Analysis of Algorithms and Problem Complexity]: Nonnumerical Algorithms and Problems — Computations on discrete structures

Keywords and phrases graph decomposition, boolean-width, heuristics, exact algorithms, vertex subset problems

Digital Object Identifier 10.4230/LIPIcs.xxx.yyy.p

1 Introduction

Boolean-width is a recently introduced graph parameter \[2\]. Similarly to treewidth and other parameters, it measures some structural complexity of a graph. Many NP-hard problems on graphs become easy if some graph parameter is small. We need a derived structure which captures the necessary information of a graph in order to exploit such a small parameter. In the case of boolean-width, this is a binary partition tree, referred to as the decomposition tree. However, computing an optimal decomposition tree is usually a hard problem in itself. A common approach to bypass this problem is to use heuristics to compute decompositions with a low boolean-width.

Algorithms for computing boolean decompositions have been studied before in \[17, 10, 12, 7\], but in this paper we study the specific case of linear boolean decompositions, which are considered in \[1, 10, 12\]. Linear decompositions are easier to compute and the theoretical running time of algorithms for solving practical problems is lower on linear decompositions than on tree shaped ones. For instance, vertex subset problems can be solved in \(O^*(necc^2)\) due to a dynamic programming algorithm by Bui-Xuan et al. \[3\], but this can be improved to \(O^*(necc^2)\) for linear decompositions. Here, \(necc\) is the number of d-neighborhood equivalence classes, i.e., the maximum size of the dynamic programming table.
2 Practical Algorithms for Linear Boolean-width

We first give an exact algorithm for computing optimal linear boolean decompositions, improving upon existing algorithms, and subsequently investigate several new heuristics through experiments, improving upon the work by Sharmin [12, Chapter 8]. We then study the practical relevance of these algorithms in a set of experiments by solving an instance of a vertex subset problem, investigating the number of equivalence classes compared to the theoretical worst case bounds.

2 Preliminaries

A graph $G = (V, E)$ of size $n$ is a pair consisting of a set of $n$ vertices $V$ and a set of edges $E$. The neighborhood of a vertex $v \in V$ is denoted by $N(v)$. For a subset $A \subseteq V$ we denote the neighborhood by $N(A) = \bigcup_{v \in A} N(v)$. In this paper we only consider simple, undirected graphs and assume we are given a total ordering on the vertices of a graph $G$. For a subset $A \subseteq V$ we denote the complement by $\overline{A} = V \setminus A$. A partition $(A, \overline{A})$ of $V$ is called a cut of the graph. Each cut $(A, \overline{A})$ of $G$ induces a bipartite subgraph $G[A, \overline{A}]$.

The neighborhood across a cut $(A, \overline{A})$ for a subset $X \subseteq A$ is defined as $N(X) \cap \overline{A}$.

Definition 1 (Unions of neighborhoods). Let $G = (V, E)$ be a graph and $A \subseteq V$. We define the set of unions of neighborhoods across a cut $(A, \overline{A})$ as

$$\mathcal{UN}(A) = \{ N(X) \cap \overline{A} \mid X \subseteq A \}. $$

The number of unions of neighborhoods is symmetric for a cut $(A, \overline{A})$, i.e., $|\mathcal{UN}(A)| = |\mathcal{UN}(\overline{A})|$ [3 Theorem 1.2.3]. Furthermore, for any cut $(A, \overline{A})$ of a graph $G$ it holds that $|\mathcal{UN}(A)| = \#\text{MIS}(G[A, \overline{A}])$, where $\#\text{MIS}(G)$ is the number of maximal independent sets in $G$ [17 Theorem 3.5.5].

Definition 2 (Decomposition tree). A decomposition tree of a graph $G = (V, E)$ is a pair $(T, \delta)$, where $T$ is a full binary tree and $\delta$ is a bijection between the leaves of $T$ and vertices of $V$. If $a$ is a node and $L$ are its leaves, we write $\delta(a) = \bigcup_{l \in L} \delta(l)$. So, for the root node $r$ of $T$ it holds that $\delta(r) = V$. Furthermore, if nodes $a$ and $b$ are children of a node $w$, then $(\delta(a), \delta(b))$ is a partition of $\delta(w)$.

In this paper we consider a special type of decompositions, namely linear decompositions.

Definition 3 (Linear decomposition). A linear decomposition, or caterpillar decomposition, is a decomposition tree $(T, \delta)$ where $T$ is a full binary tree and for which each internal node of $T$ has at least one leaf as a child. We can define such a linear decomposition through a linear ordering $\pi = \pi_1, \ldots, \pi_n$ of the vertices of $G$ by letting $\delta$ map the $i$-th leaf of $T$ to $\pi_i$.

Definition 4 (Boolean-width). Let $G = (V, E)$ be a graph and $A \subseteq V$. The boolean dimension of $A$ is a function $\text{bool-dim} : 2^V \rightarrow \mathbb{R}$.

$$\text{bool-dim}(A) = \log_2 |\mathcal{UN}(A)|. $$

Let $(T, \delta)$ be a decomposition of a graph $G$. We define the boolean-width of $(T, \delta)$ as the maximum boolean dimension over all cuts induced by nodes of $(T, \delta)$.

$$\text{boolw}(T, \delta) = \max_{w \in T} \text{bool-dim}(\delta(w)) $$

The boolean-width of $G$ is defined as the minimum boolean-width over all possible full decompositions of $G$, while the linear boolean-width of a graph $G = (V(G), E(G))$ of size $n$ is defined as the the minimum boolean-width over all linear decompositions of $G$.

$$\text{boolw}(G) = \min_{(T, \delta) \text{ of } G} \text{boolw}(T, \delta) $$
\[ \text{lboolw}(G) = \min_{(T, \delta)} \text{boolw}(T, \delta) \]

It is known that for any graph \( G \) it holds that \( \text{boolw}(G) \leq \text{treewidth}(G) + 1 \) [17, Theorem 4.2.8]. The linear variant of treewidth is called \( \text{pathwidth} \) [11], or \( \text{pw} \) for short.

\begin{itemize}
  \item \textbf{Theorem 5 (Appendix A.1).} For any graph \( G \) it holds that \( \text{lboolw}(G) \leq \text{pw}(G) + 1 \).
\end{itemize}

The algorithms in this paper make extensive use of sets and set operations, which can be implemented efficiently by using bitsets. By using a mapping from vertices to bitsets that represent the neighborhood of a vertex we can store the adjacency matrix of a graph efficiently. We assume that bitset operations take \( \mathcal{O}(n) \) time and need \( \mathcal{O}(n) \) space, even though in practice this may come closer to \( \mathcal{O}(1) \). If one assumes that these requirements are constant, several time and space bounds in this paper improve by a factor \( n \).

In this paper we assume that the graph \( G \) is connected, since if the graph consists of multiple connected components we can simply compute a linear decomposition for each connected component, after which we glue them together, in any arbitrary order.

## 3 Exact Algorithms

We can characterize the problem of finding an optimal linear decomposition by the following recurrence relation, in which \( P \) is a function mapping a subset of vertices \( A \) to the linear boolean-width of the induced subgraph \( G[A, \overline{A}] \).

\[
\begin{align*}
P(\{v\}) &= |\mathcal{U}N(\{v\})| = \begin{cases} 
1 & \text{if } N(v) = \emptyset \\
2 & \text{if } N(v) \neq \emptyset
\end{cases} \\
P(A) &= \min_{v \in A} \{\max\{|\mathcal{U}N(A)|, P(A \setminus \{v\})\}\}
\end{align*}
\]

The boolean-width of the graph \( G \) is now given by \( \log_2(P(V)) \). Adaptation of existing techniques lead to the following algorithms for linear boolean-width, upon we hereafter improve:

\begin{itemize}
  \item With dynamic programming a running time of \( \mathcal{O}(2.7284^n) \) is achieved. (See Theorem 19, Appendix A.2)
  \item With adaptation of the exact algorithm for boolean-width by Vatshelle [17], a running time of \( \mathcal{O}(n^3 \cdot 2^{n + \text{lboolw}(G)}) \) is achieved. (See Theorem 20, Appendix A.2)
\end{itemize}

### 3.1 Improving the running time

We present a faster and easier way to precompute for all cuts \( A \subseteq V \) the value \( |\mathcal{U}N(A)| \), which results in a new algorithm displayed in Algorithm 2. In the following it is important that the \( \mathcal{U}N \) sets are implemented as hashmaps, which will only save distinct neighborhoods.

\begin{algorithm}[h]
\caption{Compute \( \mathcal{U}N(X \cup \{v\}) \) given \( \mathcal{U}N(X) \).}
1: \textbf{procedure} \text{INCREMENT-UN}(G, X, \mathcal{U}N_X, v)
2: \hspace{1cm} \mathcal{U} \leftarrow \emptyset
3: \hspace{1cm} \textbf{for} \ S \in \mathcal{U}N_X \ \textbf{do}
4: \hspace{1.5cm} \mathcal{U} \leftarrow \mathcal{U} \cup \{S \setminus \{v\}\}
5: \hspace{1.5cm} \mathcal{U} \leftarrow \mathcal{U} \cup \{(S \setminus \{v\}) \cup (N(v) \cap (X \setminus \{v\}))\}
6: \hspace{1cm} \textbf{return} \mathcal{U}
\end{algorithm}
Lemma 6 (Appendix A.3). The procedure Increment-UN is correct and runs in $O(n \cdot |UN_X|)$ time using $O(n \cdot |UN_X|)$ space.

Algorithm 2 Return $lboolw(G)$, if it is smaller than $\log K$, otherwise return $\infty$.

```
1: procedure Incremental-UN-exact(G, K)
2: \( T_{UN}(\emptyset) \leftarrow 0 \)
3: Compute-count-UN(G, K, T_{UN}, \emptyset, \{\emptyset\})
4: \( P(X) \leftarrow \infty \), for all \( X \subseteq V \)
5: \( P(\emptyset) \leftarrow 0 \)
6: for \( i \leftarrow 0, \ldots, |V| - 1 \) do
7: for \( X \subseteq V \) of size \( i \) do
8: for \( v \in V \setminus X \) do
9: \( Y \leftarrow X \cup \{v\} \)
10: if \( P(X) \leq K \) then
11: \( P(Y) \leftarrow \min(P(Y), \max(T_{UN}(Y), P(X))) \)
12: return \( \log_2(P(V)) \)
13: procedure Compute-count-UN(G, K, T_{UN}, X, \mathcal{UN}_X)
14: for \( v \in V \setminus X \) do
15: \( Y \leftarrow X \cup \{v\} \)
16: if \( T_{UN}(Y) \) is not defined then
17: \( \mathcal{UN}_Y \leftarrow \text{Increment-UN}(G, X, \mathcal{UN}_X, v) \)
18: \( T_{UN}(Y) \leftarrow |\mathcal{UN}_Y| \)
19: if \( T_{UN}(Y) \leq K \) then
20: Compute-count-UN(G, K, T_{UN}, Y, \mathcal{UN}_Y)
```

Theorem 7 (Appendix A.4). Given a graph \( G \), Algorithm 2 can be used to compute $lboolw(G)$ in $O(n \cdot 2^n + lboolw(G))$ time using $O(n \cdot 2^n)$ space.

This new algorithm improves upon the time in Theorem 20 by a factor $n^2$, while the space requirements stay the same. Since the tightest known upperbound for linear boolean-width is $\frac{2}{7} - \frac{1}{133} + O(1)$ [10], this algorithm can be slower than dynamic programming, since $O(2^n + \frac{2}{7} - \frac{1}{133} + O(1)) = O(2.8148^n + O(1)) \geq O(2.7284^n)$, but this is very unlikely to happen in practice.

4 Heuristics

4.1 Generic form of the heuristics

The goal when using a heuristic is to find a linear ordering of the vertices in a graph in such a way that the decomposition that corresponds to this ordering will be of low boolean-width. A basic strategy to accomplish this is to start the ordering with some vertex and then by some selection criteria append a new vertex to the ordering that has not been appended yet. This strategy is used in heuristics introduced by Sharmin [12, Chapter 8], and a similar approach is shown in Algorithm 3.
Algorithm 3 Greedily generate an ordering based on the score function and the given starting vertex.

```plaintext
1: procedure GENERATEVERTEXORDERING(G, ScoreFunction, init)
2:     Decomposition ← (init)
3:     Left ← {init}
4:     Right ← V \ {init}
5:     while Right ≠ ∅ do
6:         Candidates ← set returned by candidate set strategy
7:         if there exists v ∈ Candidates belonging to a trivial case then
8:             chosen ← v
9:         else
10:             chosen ← argmin_v ∈ Candidates (ScoreFunction(G, Left, Right, v))
11:         Decomposition ← Decomposition · {chosen}
12:     Left ← Left ∪ {chosen}
13:     Right ← Right \ {chosen}
14: return Decomposition
```

At any point in the algorithm we denote the set of all vertices contained in the ordering by Left, and the remaining vertices by Right. While Right is not empty, we choose a vertex from a candidate set Candidates ⊆ Right, based on a set of trivial cases, and, if no trivial case applies, by making a local greedy choice using a score function that indicates the quality of the current state Left, Right.

4.1.1 Selecting the initial vertex

Selecting a good initial vertex can be of great influence on the quality of the decomposition. Sharmin proposes to use a double breadth first search (BFS) in order to select the initial vertex. This is done by initiating a BFS, starting at an arbitrary vertex, after which a vertex of the last level of the BFS is selected. This process is then repeated by using the found vertex as a starting point for the second BFS. However, the fact that an arbitrary vertex is used for the first BFS already influences the boolean-width of the computed decomposition. During our experiments we noticed that performing a single BFS sometimes gave better results. But since we will see in Chapter 5 that applications are a lot more expensive in terms of running time, it is wise to use all possible starting vertices when trying to find a good decomposition.

4.1.2 Pruning

Starting from multiple initial vertices allows us to do some pruning. If we notice during the algorithm that the score of the decomposition that is being constructed exceeds the score of the best decomposition found so far, we can stop immediately and move to the next initial vertex. For this reason, it is wise to start with the most promising initial vertices (e.g. obtained by the double BFS method), and after that try all other initial vertices.

4.1.3 Candidates

The most straightforward choice for the set Candidates is to take Right entirely. However, we may do unnecessary work here, since vertices that are more than 2 steps away from any
A vertex is chosen to be the next vertex in the ordering if it can be guaranteed that it is an optimal choice by means of a trivial case. Lemma 8 generalizes results by Sharmin [12], since the two trivial cases given by her are subcases of our lemma, namely \( X = \emptyset \) and \( X = \{u\} \) for all \( u \in \text{Left} \). Note that we can add a wide range of trivial cases by varying \( X \), such as \( X = \text{Left} \) and \( \forall u, w \in \text{Left} : X = \{u, w\} \), but this will increase the complexity of the algorithm.

Lemma 8 (Appendix A.5). Let \( X \subseteq \text{Left} \). If \( \exists v \in \text{Right} \) such that \( N(v) \cap \text{Right} = N(X) \cap \text{Right} \), then choosing \( v \) will not change the boolean-width of the resulting decomposition.

4.1.5 Relative Neighborhood Heuristic

For a cut \((\text{Left}, \text{Right})\) and a vertex \( v \) define

\[
\text{Internal}(v) = (N(v) \cap N(\text{Left})) \cap \text{Right} \\
\text{External}(v) = (N(v) \setminus N(\text{Left})) \cap \text{Right}
\]

In the original formulation by Sharmin [12], \( \frac{\text{External}(v)}{\text{Internal}(v) + \text{External}(v)} \) is used as a score function. However, if we use \( \frac{\text{External}(v)}{N(v) - \text{Internal}(v) \cap \text{Right}} \) we get the same ordering by Lemma 9, without having an edge case for dividing by zero. Furthermore, in contrast to Sharmin’s proposal of checking for each vertex \( w \in N(v) \) if \( w \in N(\text{Left}) \cap \text{Right} \) or not, we can compute these sets directly by performing set operations. We will refer to this heuristic by \( \text{RelativeNeighborhood} \).

Lemma 9 (Appendix A.6). The mapping \( \frac{a}{b} \mapsto \frac{a}{a + b} \) is order preserving.

Two variations on this heuristic can be obtained through the score functions \( \frac{\text{External}(v)}{\text{Internal}(v)} \) and \( 1 - \frac{\text{Internal}(v)}{N(v)} \), which work slightly better for sparse random graphs and extremely well for dense random graphs respectively. We will refer to these two variations by \( \text{RelativeNeighborhood}^2 \) and \( \text{RelativeNeighborhood}^3 \).

One can easily see that the running time of these three algorithms is \( O(n^3) \) and the required space amounts to \( O(n) \). Notice however that this algorithm only gives us a decomposition. If we need to know the corresponding boolean-width we need to compute it afterwards, for instance by iteratively applying \( \text{INCREMENT-UN} \) on the vertices in the decomposition, and taking the maximum value. This would require an additional \( O(n^2 \cdot 2^k) \) time and \( O(n \cdot 2^k) \) space, where \( k \) is the boolean-width of the decomposition.

4.1.6 Least Cut Value Heuristic

The \( \text{LeastCutValue} \) heuristic by Sharmin [12] greedily selects the next vertex \( v \in \text{Right} \) that will have the smallest boolean dimension across the cut \((\text{Left} \cup \{v\}, \text{Right} \setminus \{v\})\). This vertex is obtained by constructing the bipartite graph \( BG = G[\text{Left} \cup \{v\}, \text{Right} \setminus \{v\}] \) for each \( v \in \text{Right} \), and counting the number of maximal independent sets of \( BG \) using the \( \text{CCM}_{1S} \) algorithm on \( BG \), with the time of \( \text{CCM}_{1S} \) being exponential in \( n \).
4.1.7 Incremental Unions of Neighborhoods Heuristic

Generating a bipartite graph and then calculating the number of maximal independent sets is a computationally expensive approach. A different way to compute the boolean dimension of each cut is by reusing the neighborhoods from the previous cut, similarly to Incremental-UN-exact. We present a new algorithm, called the Incremental-UN-heuristic, in Algorithm 4. A useful property of this algorithm is that the running time is output sensitive. It follows that if a decomposition is not found within reasonable time, then the decomposition that would have been generated is not useful for practical algorithms.

Algorithm 4 Greedy heuristic that incrementally keeps track of the Unions of Neighborhoods.
\[\text{procedure Incremental-UN-Heuristic}(G, init)\]
1: \(\text{Decomposition} \leftarrow (\text{init})\)
2: \(\text{Left, Right} \leftarrow \{\text{init}\}, V \setminus \{\text{init}\}\)
3: \(\text{UN}_{\text{Left}} \leftarrow \emptyset, N(\text{init}) \cap \text{Right}\)
4: \text{while} \ Right \neq \emptyset \text{ do}
5: \text{Candidates} \leftarrow \text{set returned by candidate set strategy}
6: \text{if} \ there \ exists \ v \in \text{Candidates} \ \text{belonging to a trivial case} \ \text{then}
7: \quad \text{chosen} \leftarrow v
8: \quad \text{UN}_{\text{chosen}} \leftarrow \text{Increment-UN}(G, \text{Left}, \text{UN}_{\text{Left}}, v)
9: \text{else}
10: \quad \text{for all} \ v \in \text{Candidates} \ \text{do}
11: \quad \text{UN}_{v} \leftarrow \text{Increment-UN}(G, \text{Left}, \text{UN}_{\text{Left}}, v)
12: \quad \text{if} \ \text{chosen} \ \text{is undefined} \ \text{or} \ |\text{UN}_{v}| < |\text{UN}_{\text{chosen}}| \ \text{then}
13: \quad \quad \text{chosen} \leftarrow v
14: \quad \quad \text{UN}_{\text{chosen}} \leftarrow \text{UN}_{v}
15: \text{Decomposition} \leftarrow \text{Decomposition} \cdot \text{chosen}
16: \text{Left} \leftarrow \text{Left} \cup \{\text{chosen}\}
17: \text{Right} \leftarrow \text{Right} \setminus \{\text{chosen}\}
18: \text{UN}_{\text{Left}} \leftarrow \text{UN}_{\text{chosen}}
19: \text{return} \text{Decomposition}

\[\text{Theorem 10 (Appendix A.7).} \ \text{The Incremental-UN-heuristic procedure runs in} \ O(n^3 \cdot 2^k) \text{ time using} \ O(n \cdot 2^k) \text{ space, where} \ k \text{ is the boolean-width of the resulting linear decomposition.}\]

4.1.8 Unsuccessful ideas

- First Improvement — Preliminary experiments pointed out that it not only gave worse results in terms of boolean-width, but it also increased the time needed to compute a decomposition, which can be explained by the output sensitivity of the Incremental-UN-heuristic. In other words, even though the best improvement strategy takes more time to determine the next vertex for a single iteration, it is worthwhile to put effort in finding a good cut, as it also decreases the time for future cuts.

- Lookaheads — This technique does not only look at the change of UN resulting from choosing a candidate v, but also recursively considers the changes of the algorithm after v has been chosen, up to a fixed depth. With each level of depth added, the time complexity increases with a factor n, but experiments turned out that the benefits were only marginal.
Minimal Neighborhood Cover — This heuristic tries to minimize the number of neighborhoods in Left that are needed to cover the neighborhood of the vertex to be chosen.

Max Cardinality Search — This heuristics selects vertices in such an order that at each step the vertex with most neighbors in Left is chosen. In practice this heuristic performed similar to other already known polynomial heuristics.

5 Vertex subset problems

Boolean decompositions can be used to efficiently solve a class of vertex subset problems called $(\sigma, \rho)$ vertex subset problems, which were introduced by Telle [13]. This class of problems consists of finding a $(\sigma, \rho)$-set of maximum or minimum cardinality and contains well known problems such as the maximum independent set, the minimum dominating set of equivalence classes of a problem specific equivalence relation, which can be bounded in terms of boolean-width. In Section 6 we investigate how close the value of $\text{nec}_d(T, \delta)$ comes to any of the theoretical bounds.

5.1 Definitions

- **Definition 11** $(\sigma, \rho)$-set. Let $G = (V, E)$ be a graph. Let $\sigma$ and $\rho$ be finite or co-finite subsets of $\mathbb{N}$. A subset $X \subseteq V$ is called a $(\sigma, \rho)$-set if the following holds

  \[
  \forall v \in V : |N(v) \cap X| \in \{\sigma \text{ if } v \in X, \rho \text{ if } v \in V \setminus X.\}
  \]

  In order to confirm if a set $X$ is a $(\sigma, \rho)$-set we have to count the number of neighbors each vertex $v \in V$ has in $X$, where it suffices to count up until a certain number of neighbors. As an example, when we want to confirm if a set $X$ is an independent set, it is equivalent to checking if $X$ is a $([0], \mathbb{N})$-set, it is irrelevant if a vertex $v$ has more than one neighbor in $X$. We capture this property in the function $d : 2^\mathbb{N} \rightarrow \mathbb{N}$, which is defined as follows:

- **Definition 12** (d-function). Let $d(\emptyset) = 0$. For every finite or co-finite set $\mu \subseteq \mathbb{N}$, let $d(\mu) = 1 + \min_{x \in \mathbb{N}}(\max_{x \in \mu} x, \max_{x \in \mathbb{N} \setminus \mu} x)$. Let $d(\sigma, \rho) = \max(d(\sigma), d(\rho))$.

- **Definition 13** (d-neighborhood). Let $G = (V, E)$ be a graph. Let $A \subseteq V$ and $X \subseteq A$. The d-neighborhood of $X$ with respect to $A$, denoted by $N^d_A(X)$, is a multiset of vertices from $A$, where a vertex $v \in A$ occurs $\min(d, |N(v) \cap X|)$ times in $N^d_A(X)$. A d-neighborhood can be represented as a vector of length $|A|$ over $\{0, 1, \ldots, d\}$.

- **Definition 14** (d-neighborhood equivalence). Let $G = (V, E)$ be a graph and $A \subseteq V$. Two subsets $X, Y \subseteq A$ are said to be d-neighborhood equivalent with respect to $A$, denoted by $X \equiv_A Y$, if it holds that $\forall v \in A : \min(d, |X \cap N(v)|) = \min(d, |Y \cap N(v)|)$. The number of equivalence classes of a cut $(A, \overline{A})$ is denoted by $\text{nec}(\equiv_A)$. The number of equivalence classes $\text{nec}_d(T, \delta)$ of a decomposition $(T, \delta)$ is defined as $\max(\text{nec}(\equiv_A), \text{nec}(\equiv_{\overline{A}}))$ over all cuts $(A, \overline{A})$ of $(T, \delta)$.

Note that $N^1_A(X) = N(X) \cap A$. It can then be observed that $|\mathcal{U}N(A)| = \text{nec}(\equiv_A)$ [17, Theorem 3.5.5] Also note that $X \equiv_A Y$ if and only if $N^d_A(X) = N^d_A(Y)$. 
5.2 Bounds on the number of equivalence classes

We present a brief overview of the most relevant bounds that are currently known, for which we make use of a twin class partition of a graph.

Definition 15 (Twin class partition). Let $G = (V, E)$ be a graph of size $n$ and let $A \subseteq V$. The twin class partition of $A$ is a partition of $A$ such that for all $x, y \in A$, $x$ and $y$ are in the same partition class if and only if $N(x) \cap \overline{A} = N(y) \cap \overline{A}$. The number of partition classes of $A$ is denoted by $ntc(A)$ and it holds that $ntc(A) \leq \min(n, 2^{\text{bool-dim}(A)})$.[2]

For all bounds listed below, let $G = (V, E)$ be a graph of size $n$ and let $d$ be a non-negative integer. Let $(A, \overline{A})$ be a cut induced by any node of a decomposition $(T, \delta)$ of $G$, and let $k = \text{bool-dim}(A) = \text{nec}(\equiv^d_A)$.

Lemma 16. [3, Lemma 5] $\text{nec}(\equiv^d_A) \leq 2^d k^2$.

Lemma 17. [17, Lemma 5.2.2] $\text{nec}(\equiv^d_A) \leq (d + 1) \min(ntc(A), ntc(\overline{A}))$.

Lemma 18 (Appendix A.8). $\text{nec}(\equiv^d_A) \leq ntc(A)^d k$.

By Lemma 16 we conclude that we can solve $(\sigma, \rho)$ problems in $O^*(8^d k^2)$. This shows that applications are more computationally expensive than using heuristics to find a decomposition.

6 Experiments

The experiments in this section are performed on a 64-bit Windows 7 computer, with a 3.40 GHz Intel Core i5-4670 CPU and 8GB of RAM. We implemented the algorithms using the C# programming language and compiled our programs using the csc compiler that comes with Visual Studio 12.0.

6.1 Comparing Heuristics on random graphs

We will look at the performance of heuristics on randomly generated graphs, for which we used the Erdös-Rényi-model [15] to generate a fixed set of random graphs with varying edge probabilities. By using the same set of graphs for each heuristic, we rule out the possibility that one heuristic can get a slightly easier set of graphs than another. In these experiments we start a heuristic once for each possible initial vertex, so $n$ times in total. For the RELATIVENEIGHBORHOOD heuristic we select the best decomposition based upon the sum of the score function for all cuts, since computing all actual linear boolean-width values would take $O(n^3 \cdot 2^k)$ time, thereby removing the purpose of this polynomial time heuristic. For the set Candidates we take $N^2(Left) \cap Right$, which avoids that we exclude certain optimal solutions, as opposed to Sharmin [12], who restricted this set to $N(Left) \cap Right$. However, this does not affect the results significantly.

We let the edge probability vary between 0.05 and 0.95 with steps of size 0.05. For each edge probability value, we generated 20 random graphs. The result per edge probability is taken to be the average boolean-width over these 20 graphs, which are shown in Figure 1. It can be observed that the INCREMENTAL-UN-HEURISTIC procedure performs near optimal. Furthermore we see that the RELATIVENEIGHBORHOOD variants perform somewhere in between the optimal value and the value of random decompositions.

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1 Source code of our implementations can be found on [https://github.com/Chiel92/boolean-width](https://github.com/Chiel92/boolean-width) and [https://github.com/FrankvH/BooleanWidth](https://github.com/FrankvH/BooleanWidth)
6.2 Comparing heuristics on real-world graphs

In order to get an idea of how the INCREMENTAL-UN-HEURISTIC compares to existing heuristics we compare them by both the boolean-width of the generated decomposition and the time needed for computation. We cannot compare the heuristics to the optimal solution, because computing an exact decomposition is not feasible on these graphs. The graphs that were used come from Treewidthlib [14], a collection of graphs that are used to benchmark algorithms using treewidth and related graph problems.

We ran the three different heuristics mentioned in Section 4 with Candidates = Right and with an additional two variations on the INCREMENTAL-UN-HEURISTIC (IUN) by varying the set of start vertices. The first variation, named 2-IUN, has two start vertices which are obtained through a single and double BFS respectively. The n-IUN heuristic uses all possible start vertices. For all other heuristics we obtained the start vertex through performing a double BFS. In Table 1 and 2 we present the results of our experiments.

Table 1: Linear boolean-width of the decompositions returned by different heuristics.

| Graph       | $|V|$ | Edge Density | Relative | LeastCut | IUN | 2-IUN | n-IUN |
|-------------|-----|--------------|----------|----------|-----|-------|-------|
| barley      | 48  | 0.11         | 5.70     | 5.91     | 5.91| 4.70  | 4.58  |
| pigs-pp     | 48  | 0.12         | 10.35    | 7.13     | 7.13| 7.13  | 6.64  |
| david       | 87  | 0.11         | 9.38     | 6.27     | 6.27| 6.27  | 5.86  |
| celar04-pp  | 114 | 0.08         | 11.67    | 7.27     | 7.27| 7.27  | 7.27  |
| 1bkb-pp     | 127 | 0.18         | 16.81    | 9.98     | 9.98| 9.53  | 9.53  |
| miles1500   | 128 | 0.64         | 8.17     | 5.58     | 5.58| 5.58  | 5.29  |
| celar10-pp  | 133 | 0.07         | 10.32    | 11.95    | 11.95| 7.64  | 6.91  |
| mumin2-pp   | 167 | 0.03         | 15.17    | 9.61     | 9.61| 9.61  | 7.61  |
| mulsol.i.5  | 186 | 0.23         | 7.55     | 5.29     | 5.29| 5.29  | 3.58  |
| zeroin.i.2  | 211 | 0.16         | 7.92     | 4.46     | 4.46| 4.46  | 3.81  |
| boblo       | 221 | 0.01         | 19.00    | 4.32     | 4.32| 4.32  | 4.00  |
| fpsol2.i-pp | 233 | 0.40         | 5.58     | 6.07     | 6.07| 5.78  | 4.81  |
| munini4-wpp | 271 | 0.02         | 13.04    | 9.27     | 9.27| 9.27  | 7.61  |
It is expected that the IUN heuristic and LeastCutValue heuristic give the same linear boolean-width, since both these heuristics greedily select the vertex that minimizes the boolean dimension. The RelativeNeighborhood heuristic performs worse than all other heuristics in nearly all cases. While the difference might not seem very large, note that algorithms parameterized by boolean-width are exponential in the width of a decomposition. The 2-IUN heuristic outperforms IUN in three cases while n-IUN gives a better decomposition in 11 out of 13 cases, which shows that a good initial vertex is of great influence on the width of the decomposition.

Looking at the times displayed in Table 2 for computing each decomposition we see that the RelativeNeighborhood heuristic is significantly faster. This is to be expected because of the $O(n^3)$ time, compared to the exponential time for all other heuristics. The interesting comparison that we can make is the difference between the IUN heuristic and LeastCutValue heuristic. While both of these heuristics give the same decomposition, IUN is significantly faster. Additionally, even 2-IUN and n-IUN are often faster than the LeastCutValue heuristic.

In Table 3 (Appendix B.2) we show linear boolean-width upperbounds that are obtained through using the IUN heuristic with all starting vertices and candidates = Right. We compare this with the best known tree-width and boolean-width values. Examining these results, it seems that linear boolean-width seem to be more useful in practice than boolean-width heuristics. However, one should note that on certain graph classes, for instance graphs which look like trees, boolean-width is a lot lower than linear boolean-width.

### 6.3 Vertex subset experiments

We have used the linear decompositions given by the n-IUN heuristic to compute the size of the maximum induced matching (MIM) in a selection of graphs, of which the results are presented in Table 4. The maximum induced matching problem is defined as finding the largest $\{1\}, N$ set, with $d(\{1\}, N) = 2$. The choice for the MIM problem is arbitrary, any vertex subset problem with $d = 2$ will have the same number of equivalence classes and therefore they all require the same time when computing a solution. We present the computed value of $\text{nec}_d(T, \delta)$, together with theoretical upperbounds. For $d = 2$ a tight
upperbound in terms of boolean-width is not known. Note that we take the logarithm of each value, since we find this value easier to interpret and compare to other graph parameters. We let $UB_1 = 2^d \cdot \text{boolw}$, $UB_2 = (d + 1) \min_{w} \text{ntc}$ and $UB_3 = ntc_d \cdot \text{boolw}$, with $ntc = \max_{w \in T} ntc(\delta(w))$ and $\min_{w \in T} \text{ntc}(\delta(w))$.

The column $MIM$ displays the size of the MIM in the graph, while the time column indicates the time needed to compute this set. Missing values for $nec$ and MIM are caused by a lack of internal memory, because of the $O^*(nec_d(T, \delta)^2)$ space requirement. One can immediately see that there is a large gap between the upperbound for $nec_2$ in terms of boolean-width and $nec_2$ itself. Another interesting observation we can make by looking at the graphs zeroin.i.2 and boblo, is that a lower boolean-width does not imply a lower $nec_2$. We even encountered this for decompositions of the same graph: for the graph barley we observed $\text{boolw}(T, \delta) = 4.58$ and $\text{boolw}(T', \delta') = 4.81$, while $\log_2(nec_2(T, \delta)) = 7.00$ and $\log_2(nec_2(T', \delta')) = 6.75$. This suggests that this upperbound does not justify minimizing $nec_2$ through boolean-width in practice.

| Table 3 | Results of using the algorithm by Bui-Xuan et al. [3] for solving $(\sigma, \rho)$ problems on graphs, using decompositions obtained through the n-IUN heuristic. |

| Graph       | boolw | $\log_2(nec)$ | $\log_2(UB_1)$ | $\log_2(UB_2)$ | $\log_2(UB_3)$ | MIM | Time (s) |
|-------------|-------|---------------|-----------------|-----------------|-----------------|-----|----------|
| barley      | 4.58  | 7.00          | 42.04           | 12.68           | 27.51           | 22  | 3        |
| pigs-pp     | 6.64  | 10.31         | 88.28           | 19.02           | 49.17           | 22  | 1147     |
| david       | 5.86  | 9.37          | 68.63           | 22.19           | 44.61           | 34  | 919      |
| celar04-pp  | 7.27  | 11.15         | 105.61          | 28.53           | 65.74           | -   | -        |
| 1bkb-pp     | 9.53  | -             | 181.47          | 52.30           | 98.49           | -   | -        |
| miles1500   | 5.29  | 9.30          | 55.87           | 34.87           | 49.69           | 8   | 4038     |
| celar10-pp  | 6.91  | 10.34         | 95.41           | 25.36           | 59.70           | 50  | 10179    |
| mumin2-pp   | 7.61  | 11.82         | 115.97          | 19.02           | 54.60           | -   | -        |
| mulsol.1.5  | 3.58  | 6.11          | 25.70           | 14.26           | 24.80           | 46  | 22       |
| zeron.1.2   | 3.81  | 6.58          | 28.99           | 20.60           | 28.18           | 30  | 59       |
| boblo       | 4.00  | 6.17          | 32.00           | 9.51            | 20.68           | 148 | 41       |
| fpsol2.i-pp | 4.81  | 8.07          | 46.22           | 22.19           | 36.61           | 46  | 934      |
| mumin4-wpp  | 7.61  | 12.13         | 115.97          | 19.02           | 57.98           | -   | -        |

7 Conclusion

We have presented a new heuristic and a new exact algorithm for finding linear boolean decompositions. The heuristic has a running time that is several orders of magnitude lower than the previous best heuristic and finds a decomposition in output sensitive time. This means that if a decomposition is not found within reasonable time, then the decomposition that would have been generated is not useful for practical algorithms. Running the new heuristic once for every possible starting vertex results in significantly better decompositions compared to existing heuristics.

We have seen that if $l\text{boolw}(T, \delta) < l\text{boolw}(T', \delta')$, then there is no guarantee that $nec(T, \delta) < nec(T', \delta')$. While in general it holds that minimizing boolean-width results in a low value of number of equivalence classes, we think that it can be worthwhile to focus on minimizing the $nec_d$ instead of the boolean-width when solving vertex subset problems. However, the number of equivalence classes is not symmetric, i.e., for a cut $(A, \overline{A})$
nec_d(A) ≠ nec_d(\overline{A})$, which makes it harder to develop fast heuristics that focus on minimizing $nec_d$ since we need to keep track of both the equivalence classes of $A$ and $\overline{A}$.

Further research can be done in order to obtain even better heuristics and better upperbounds on both the linear boolean-width and boolean-width on graphs. For instance, combining properties of the **Incremental-UN-heuristic** and the **RelativeNeighborhood** heuristic might lead to better decompositions, as they make use of complementary features of a graph. Another approach for obtaining good decompositions could be a branch and bound algorithm that makes us of trivial cases that are used in the heuristics. To decrease the time needed by the heuristics one can investigate reduction rules for linear boolean-width. While most reduction rules introduced by Sharmin [12] for boolean-width do not hold for linear boolean-width, they can still be used on a graph after which we can use our heuristic on the reduced graph. Although the resulting decomposition after reinserting the reduced vertices will not be linear, the asymptotic running time for applications does not increase [15]. Another topic of research is to compare the performance of vertex subset algorithms parameterized by boolean-width to algorithms parameterized by treewidth [16].

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A.1 Proof of Theorem 5

Claim. For any graph $G$ it holds that $lboolw(G) \leq pw(G) + 1$.

Proof. We give a method of construction that gives us a linear boolean decomposition of a graph $G$ from a path decomposition of $G$. Recall that a linear boolean decomposition can be defined through a linear ordering $\pi = \pi_1, \ldots, \pi_n$ of $V$. The idea is that given a path decomposition $X_1, \ldots, X_n$ we select vertices one by one from a subset $X_i$ and append them to the linear ordering $\pi$, after which we move on to $X_{i+1}$. For shorthand notation we denote $X_i = \bigcup_{j=1}^{i} X_j$.

Let $S_i = \{u \mid u \in X_i : N(u) \cap X_i \neq \emptyset\}$. For each $u \in S_i$ it holds that $\exists j > i \exists w \in X_j$ for which $\{u, w\} \in E$. By definition of a path decomposition we know that there is a subset $X_j$ with $u, w \in X_j$, and since all subsets containing a certain vertex are subsequent in the path decomposition, it follows that $u \in X_i$ and $u \in X_{i+1}$, implying that $S_i \subseteq X_i$ and $S_i \subseteq X_{i+1}$.

By definition, the unions of neighborhoods of $X_i$ can only consist of neighborhoods of subsets of $S_i$, thus it follows that $|\mathcal{UN}(X_i)| = 2^{bool\text{-}dim(X_i)} \leq 2^{|S_i|} \leq 2^{pw(X_i)} \leq 2^{pw(G)+1}$. What remains to be shown is that while appending vertices one by one from a subset $X_{i+1}$, the number of unions of neighborhoods will not exceed $2^{X_{i+1}}$ at any point. For each vertex $v \in X_{i+1}$ there are two possibilities. If $v \in S_i$, then appending $v$ to the linear ordering will not increase the boolean dimension, since $v$’s neighborhood was already an element of the unions of neighborhoods constructed so far. If $v \notin S_i$, then it is possible that $v$ will contribute a new neighborhood to the unions of neighborhoods, which will cause factor 2 increase in the worst case. There are at most $|X_{i+1} \setminus S_i|$ such vertices, and because $S_i \subseteq X_{i+1}$, it follows that $|X_{i+1} \setminus S_i| = |X_{i+1}| - |S_i|$. We conclude that at any point during construction it holds that

$$\mathcal{UN}(X_{i+1}) = 2^{bool\text{-}dim(X_{i+1})} \leq 2^{|S_i|} \cdot 2^{|X_{i+1}| - |S_i|} = 2^{|X_{i+1}|} \leq 2^{pw(G)+1}$$

A.2 Adaptation of existing exact algorithms

Straightforward dynamic programming leads to the following result.

Theorem 19. A linear boolean decomposition of minimum boolean-width can be computed in $O(2.7284^n)$ time using $O(n \cdot 2^n)$ space.

Proof. As a preprocessing step we compute for all cuts $A \subseteq V$ the values $|\mathcal{UN}(A)|$ by computing $\#MIS(G[A, \overline{A}])$. Computing $\#MIS$ for any graph can be done in $O(1.3642^n)$ time [3]. Doing this for all $A$ takes $O(2.7284^n)$ time.

We solve recurrence relation [1] in a bottom-up fashion. For each iteration, the minimum of $|A|$ numbers has to be taken. Suppose $|A| = k$, then this takes $O(k)$ time for each iteration. When solving the recurrence relation, $|A|$ goes from 1 to $n$. Since there are $\binom{n}{k}$ subsets of size $k$, it takes $\sum_{k=1}^{n} \binom{n}{k} k = O(n \cdot 2^{n-1}) = O(n \cdot 2^n)$ time to compute all values for $lboolw$.

Because the preprocessing step of computing bool-dim is the bottleneck, the total time is $O(2.7284^n)$. The space requirements amount to $O(n \cdot 2^n)$, since bool-dim and lboolw contain at most $2^n$ entries of integers of at most $n$ bits.

The currently fastest known exact algorithm for boolean-width runs in $O^*(2^{n+K})$ [17], where $K$ is a known upperbound for the boolean-width of the current graph. By performing a
binary search on $K$, we can achieve an output sensitive asymptotic running time. Theorem 20 is a direct adaptation to linear boolean-width.

**Theorem 20.** A linear boolean decomposition of minimum boolean-width for a graph $G$ can be computed in $O(n^3 \cdot 2^{n+\text{boolw}(G)})$ time using $O(n \cdot 2^n)$ space.

**Proof.** As a preprocessing step we compute for all cuts $A \subseteq V$ the values $|UN(A)|$, using a polynomial time delay algorithm, which lists maximal independent sets in $G[A, \overline{A}]$ with at most $O(n^3)$ time in between two results \[4\]. We can use the upperbound $K$ as a limit for this algorithm, such that computing $\max(|UN(A)|, K)$ takes at most $O(n^3 \cdot K)$ time.

Now consider relation \[4\]. This can be solved in $O(n \cdot 2^n)$ time by the same reasoning as in Theorem 19. This results in a total running time of $O(n^3 \cdot 2^{n+\text{boolw}(G)})$ by binary search on $K$. The space requirements amount to $O(n \cdot 2^n)$, since the tables bool-dim and boolw contain at most $2^n$ entries of integers of at most $n$ bits. ▮

### A.3 Proof of Lemma 6

**Claim.** The procedure Increment-UN is correct and runs in $O(n \cdot |UN_X|)$ time using $O(n \cdot |UN_X|)$ space.

**Proof.** For proof by induction, assume that all unions of neighborhoods for the cut $(X, \overline{X})$ saved inside the set $UN_X$ are computed correctly. For each neighborhood in $UN_X$ we only perform two actions to obtain new neighborhoods. The first action is removing $v$, since $v$ cannot be in any neighborhood of $X \cup \{v\}$. The second operation is adding $N(v)$ to an existing neighborhood, which also results in a valid new neighborhood across the cut. It is clear that if a neighborhood is added to $U$, then it is a valid neighborhood across the cut $(X \cup \{v\}, \overline{X} \setminus \{v\})$. We now show that all valid neighborhoods of the cut $(X \cup \{v\}, \overline{X} \setminus \{v\})$ are contained in $U$. Assume for contradiction that $S$ is a valid neighborhood not contained in $U$. By definition, there is a set $R$ for which $N(R) \cap (X \setminus \{v\}) = S$. If $v \notin R$, then $N(R) \cap \overline{X} \in UN_X$, meaning that we add $N(R) \cap (X \setminus \{v\}) \in U$, contradicting our assumption. If $v \in R$, then $N(R \setminus \{v\}) \cap \overline{X} \in UN_X$. During the algorithm we construct $(N(R \setminus \{v\}) \cup N(v)) \cap (X \setminus \{v\})$, which is equal to $N(R) \cap (X \setminus \{v\})$. This means that $N(R) \cap (X \setminus \{v\})$ is added to $U$, also contradicting our assumption. It follows that a neighborhood is contained in the set $U$ if and only if it is a valid neighborhood across the cut $(X \cup \{v\}, \overline{X} \setminus \{v\})$.

The time is determined by the number of sets $S$ saved in $UN_X$. The number of unions of neighborhoods that we iterate over does not exceed $|UN_X|$. The set operations that are performed for each $S$ take at most $O(n)$ time. This results in the total time for this algorithm to be $O(n \cdot |UN_X|)$. The space requirements amount to $O(n \cdot |UN_X|)$, for storing $U$ which contains at most $O(|UN_X|)$ sets of size at most $O(n)$. ▮

### A.4 Proof of Theorem 7

**Claim.** Given a graph $G$, Algorithm 2 can be used to compute $\text{boolw}(G)$ in $O(n \cdot 2^{n+\text{boolw}(G)})$ time using $O(n \cdot 2^n)$ space.

**Proof.** Iteratively double $K$ in Algorithm 2 starting with $K = 1$, until it returns a number that is not $\infty$. By Lemma 21 (this will take $O(\sum_{K=1}^{\log K} n \cdot 2^{n+\text{log} K}) = O(n \cdot 2^{n+\text{boolw}(G)+1}) = O(n \cdot 2^{n+\text{boolw}(G)})$) and take $O(n \cdot 2^n)$ space. ▮

**Lemma 21.** Given a graph $G = (V, E)$ of size $n$ and an integer $K$, Algorithm 2 computes the linear boolean width, if it is at most $\log K$, in $O(n \cdot K \cdot 2^n)$ time using $O(n \cdot 2^n)$ space.
Proof. Consider the first part of procedure Incremental-UN-exact, where the call to the procedure Compute-count-UN is made. It may not be immediately clear that $T_{UN}$ is always computed when necessary, since there may be $X$ such that $T_{UN}(X)$ is not computed, while $T_{UN}(X) \leq K$. Suppose that $X \subseteq V$ of size $i$ occurs in an optimal decomposition and $T_{UN}(X)$ has not been computed. Since we are dealing with linear decompositions, there exists an ordering $v_1, \ldots, v_i$ of $X$ such that for all $1 \leq j \leq i$, the set $X_j = \bigcup_{0 \leq j' \leq j} v_{j'}$ also occurs in the optimal decomposition. Obviously this implies that $T_{UN}(X_j) \leq K$ for all $j$. But this means that for all these $X_j$, the if-statement on line 23 evaluates to true. But that means that $T_{UN}(X)$ must be computed, contradiction. Thus we conclude that $T_{UN}$ is computed correctly throughout the algorithm. The second part of procedure Incremental-UN-exact simply solves the recurrence in a bottom-up dynamic programming fashion. Finally, the procedure Incremental-UN is correct by Lemma 6.

We now analyze the running time. Consider the procedure Compute-count-UN. We observe that the procedure can only be called once for each $X \subseteq V$, because as soon as the call is made, $T_{UN}(X)$ will be defined and line 20 prevents further calls with equal $X$. At every call the for-loop has to make at most $n$ iterations, thus we obtain $O(n \cdot 2^n)$ iterations in total. If line 20 evaluates false, the body of the for-loop takes constant time. If line 20 evaluates true, the call to Incremental-UN takes $O(n \cdot 2^K)$ time (by Lemma 6), as $|UNX| \leq K$ (otherwise by line 23 the call to Compute-count-UN would not have been made). Because line 20 only returns true at most $O(2^n)$ times, the time of Compute-count-UN amounts to $O(n \cdot 2^n)$ time (by Lemma 6), as $|UNX| \leq K$ (otherwise by line 23 the call to Compute-count-UN would not have been made). Therefore we obtain a total time of $O(n \cdot 2^n)$.

For the space requirements, we observe that the tables $T_{UN}$ and $S$ are of size at most $2^n$ storing numbers of $n$ bits. Moreover, the recursion of Compute-count-UN can be at most $n$ deep, so only $n$ neighborhoods have to be stored, which are at most of size $n \cdot 2^K$. Since $O(n \cdot 2^K) \subseteq O(n \cdot 2^{n/2}) \subseteq O(n \cdot 2^n)$, the total space requirements amount to $O(n \cdot 2^n)$.

A.5 Proof of Lemma 8

**Claim.** Let $X \subseteq Left$. If $\exists v \in Right$ such that $N(v) \cap Right = N(X) \cap Right$, then choosing $v$ will not change the boolean-width of the resulting decomposition.

**Proof.** The choice for $v$ will not change the unions of neighborhoods in any way, which means that $UN(Left) = UN(Left \cup \{v\})$. Thus, for any vertex in $Right \setminus \{v\}$ it will hold that it will interact in the exact same way with $UN(Left)$ as it would with $UN(Left \cup \{v\})$, resulting in the boolean dimension of the computed ordering being the same.

A.6 Proof of Lemma 9

**Claim.** The mapping $\frac{a}{b} \mapsto \frac{a}{a+b}$ is order preserving.

**Proof.** Suppose $\frac{a}{b} \leq \frac{c}{d}$. Then $ad - bc \leq 0$. Now we see that

$$\frac{a}{a+b} - \frac{c}{c+d} = \frac{a(c+d) - c(a+b)}{(c+d)(a+b)} = \frac{ac + ad - ac - bc}{(c+d)(a+b)} = \frac{ad - bc}{(c+d)(a+b)} \leq 0$$

Thus $\frac{a}{a+b} \leq \frac{c}{c+d}$.
A.7 Proof of Theorem 10

Claim. The Incremental-UN-heuristic procedure runs in $O(n^3 \cdot 2^k)$ time using $O(n \cdot 2^k)$ space, where $k$ is the boolean-width of the resulting linear decomposition.

Proof. The time is determined by the number of sets saved in $UN_{Left}$. The worst case consisting of $Candidates = Right$ will result in at most $n$ iterations and calls to Increment-UN. This call takes $O(n \cdot 2^{\|UN_{Left}\|})$ time by Lemma 5. By definition $|UN_{Left}|$ never exceeds $2^k$, where $k$ is the boolean-width of the resulting decomposition. Because we need to make $n$ greedy choices to process the entire graph, we conclude that the total time for this algorithm is $O(n^3 \cdot 2^k)$. For the space requirements we observe that all structures in the algorithm require $O(n)$ space, except for the unions of neighborhoods. Since there are only stored two of them at any time and they require at most $O(n \cdot 2^k)$ space, the total space requirements amount to $O(n \cdot 2^k)$.

A.8 Proof of Lemma 18

Claim. $nec(\equiv^d_A) \leq ntc(A)^d \cdot k$.

Proof. We make use of a graph parameter called maximum induced matching-width [1]. Let $mim(A)$ denote the maximum matching-width of $A$. It has been shown that for a graph $G$ and for any subset $A \subseteq V$ it holds that $mim(A) \leq bool\text{-}dim(A)$ [17, Theorem 4.2.10]. From [17, Lemma 5.2.3] we know that $nec(\equiv^d_A) \leq ntc(A)^d \cdot mim(A)$, thus $nec(\equiv^d_A) \leq ntc(A)^d \cdot k$.
B Figures and Tables

B.1 Figures

- **Figure 2**: Performance of different heuristics on random generated graphs consisting of 20 vertices, with varying edge probabilities, in terms of linear boolean-width.

- **Figure 3**: Performance of different heuristics on random generated graphs consisting of 50 vertices, with varying edge probabilities. Because of feasibility limitations, the **INCREMENTAL-UN-EXACT** algorithm is only used for the in Figure 3. While the optimal values are now unknown, it is clear that **INCREMENTAL-UN-HEURISTIC** outperforms all other heuristics. Interestingly enough, **RELATIVENEIGHBORHOOD** peers with **INCREMENTAL-UN-HEURISTIC** as soon as the edge probability exceeds 0.4. Moreover, **RELATIVENEIGHBORHOOD** and **RELATIVENEIGHBORHOOD** do not perform better than a random decomposition generator after the edge probability exceeds 0.4. We also observe that the highest boolean-width values are reached when the edge probability is around 0.1–0.2, indicating that the size of the graphs has an influence on the edge-probability-boolean-width-curve. Also note that it seems that dense random graphs have lower linear boolean-width than sparse graphs. Therefore it may be profitable to use **RELATIVENEIGHBORHOOD** when dense graphs are encountered.
### B.2 Tables

**Table 4** Linear boolean-width of the decompositions returned by the heuristics described in Section 4, with Candidates = Right. For 2-IUN we use two start vertices; one is obtained through a single BFS search, while the other is obtained through a double BFS search. The n-IUN heuristic uses all n start vertices, and all other heuristics use start vertices obtained through performing a double BFS.

| Graph   | | | | | |
|---------|---|---|---|---|---|
| alarm   | 37 | 0.10 | 3.32 | 3.00 | 3.00 | 3.00 | 3.00 |
| barley  | 48 | 0.11 | 5.70 | 5.91 | 5.91 | 4.70 | 4.58 |
| pigs-pp | 48 | 0.12 | 10.35 | 7.13 | 7.13 | 7.13 | 6.64 |
| BN_100  | 58 | 0.17 | 15.84 | 11.56 | 11.56 | 10.86 | 10.86 |
| eil76   | 76 | 0.08 | 8.86 | 8.33 | 8.33 | 8.33 | 8.33 |
| david   | 87 | 0.11 | 9.38 | 6.27 | 6.27 | 6.27 | 5.86 |
| 1jhg    | 101 | 0.17 | 12.86 | 8.67 | 8.67 | 8.49 | 8.41 |
| laac    | 104 | 0.25 | 20.29 | 12.40 | 12.40 | 12.40 | 12.33 |
| celar04-pp | 114 | 0.08 | 11.67 | 7.27 | 7.27 | 7.27 | 7.27 |
| la62    | 122 | 0.21 | 18.92 | 11.68 | 11.68 | 11.28 | 11.14 |
| 1bkb-pp | 127 | 0.18 | 16.81 | 9.98 | 9.98 | 9.53 | 9.53 |
| 1dd3    | 128 | 0.17 | 16.61 | 9.98 | 9.98 | 9.90 | 9.90 |
| miles1500 | 128 | 0.64 | 8.17 | 5.58 | 5.58 | 5.58 | 5.58 |
| miles250 | 128 | 0.05 | 7.95 | 7.13 | 7.13 | 5.39 | 4.58 |
| celar10-pp | 133 | 0.07 | 10.32 | 11.95 | 11.95 | 7.64 | 6.91 |
| anna    | 138 | 0.05 | 12.65 | 8.67 | 8.67 | 8.51 | 7.94 |
| pr152   | 152 | 0.04 | 12.69 | 11.19 | 11.19 | 10.36 | 8.29 |
| mumin2-pp | 167 | 0.03 | 15.17 | 9.61 | 9.61 | 9.61 | 7.61 |
| mulsol.l.5 | 186 | 0.23 | 7.55 | 5.29 | 5.29 | 5.29 | 3.58 |
| zeroin.l.2 | 211 | 0.16 | 7.92 | 4.46 | 4.46 | 4.46 | 3.81 |
| boblo   | 221 | 0.01 | 19.00 | 4.32 | 4.32 | 4.32 | 4.00 |
| fpsol2.l-pp | 233 | 0.40 | 5.58 | 6.07 | 6.07 | 5.78 | 4.81 |
| mumin4-wpp | 271 | 0.02 | 13.04 | 9.27 | 9.27 | 9.27 | 7.61 |

**Table 5** Time in seconds of the heuristics used to find the linear boolean decompositions of which the boolean-width is displayed in Table 4.

| Graph   | | | | | | | | | |
|---------|---|---|---|---|---|---|---|---|---|
| alarm   | 37 | 0.10 | < 0.01 | 0.02 | < 0.01 | < 0.01 | < 0.01 | 0.06 |
| barley  | 48 | 0.11 | < 0.01 | 0.18 | 0.01 | 0.02 | 0.01 | 0.16 |
| pigs-pp | 48 | 0.12 | < 0.01 | 0.76 | 0.02 | 0.04 | 0.04 | 0.52 |
| BN_100  | 58 | 0.17 | < 0.01 | 25.10 | 0.41 | 1.24 | 17.17 | |
| eil76   | 76 | 0.08 | 0.02 | 5.00 | 0.13 | 0.29 | 8.35 | |
| david   | 87 | 0.11 | 0.02 | 3.15 | 0.04 | 0.06 | 1.62 | |
| 1jhg    | 101 | 0.17 | 0.03 | 24.46 | 0.21 | 0.48 | 14.75 | |
| laac    | 104 | 0.25 | 0.04 | 754.54 | 5.66 | 11.81 | 375.31 | |
| celar04-pp | 114 | 0.08 | 0.04 | 5.73 | 0.14 | 0.23 | 9.85 | |

Continued on next page
### Table 5 – Continued from previous page

| Graph     | \(|V|\) | Edge Density | Relative LeastCut | IUN   | 2-IUN | n-IUN |
|-----------|--------|--------------|-------------------|-------|-------|-------|
| 1a62      | 122    | 0.21         | 0.06              | 585.95| 3.10  | 11.57 |
| 1bkb-pp   | 127    | 0.18         | 0.06              | 198.05| 1.14  | 4.18  |
| 1dd3      | 128    | 0.17         | 0.07              | 117.21| 0.92  | 2.74  |
| miles1500 | 128    | 0.64         | 0.06              | 44.57 | 0.10  | 0.14  |
| miles250  | 128    | 0.05         | 0.02              | 0.56  | 0.05  | 0.10  |
| celar10-pp| 133    | 0.07         | 0.06              | 8.93  | 1.96  | 4.72  |
| anna      | 138    | 0.05         | 0.06              | 20.81 | 0.22  | 0.57  |
| pr152     | 152    | 0.04         | 0.10              | 50.74 | 1.76  | 5.66  |
| munin2-pp | 167    | 0.03         | 0.11              | 3.81  | 0.80  | 3.37  |
| mulsol.i.5| 186    | 0.23         | 0.09              | 37.88 | 0.13  | 0.27  |
| zeroin.i.2| 211    | 0.16         | 0.06              | 18.70 | 0.09  | 0.11  |
| boblo     | 221    | 0.01         | 0.29              | 3.39  | 0.28  | 0.56  |
| fpsol2.i-pp| 233   | 0.40         | 0.18              | 189.11| 0.36  | 0.74  |
| munin4-wpp| 271    | 0.02         | 0.61              | 57.87 | 1.98  | 6.66  |

### Table 6

Results of using the algorithm by Bui-Xuan et al. [3] for solving \((\sigma,\rho)\) problems on graphs, using decompositions obtained using the IUN heuristic using all starting vertices. The columns \(UB\) indicate theoretical upperbounds on the number of equivalence classes, with \(UB_1 = 2^{d \cdot boolw}\), \(UB_2 = (d + 1)^{\min_{w} ntc}\) and \(UB_3 = ntc \cdot boolw\), with \(ntc = \max_{w} ntc(\delta(w))\) and \(\min_{w} ntc = \max_{w} \min(\delta(w), ntc(\delta(w)))\).

| Graph    | boolw | \(\log_2(\text{nec})\) | \(\log_2(UB_1)\) | \(\log_2(UB_2)\) | \(\log_2(UB_3)\) | MIM | Time (s) |
|----------|-------|--------------------------|-------------------|-------------------|-------------------|-----|-----------|
| alarm    | 3.00  | 4.32                     | 18.00             | 7.92              | 13.93             | 18  | < 1       |
| barley   | 4.58  | 7.00                     | 42.04             | 12.68             | 27.51             | 22  | 3         |
| pigs-pp  | 6.64  | 10.31                    | 88.28             | 19.02             | 49.17             | 22  | 1147      |
| BN_100   | 10.86 | -                        | 235.93            | 36.45             | 105.53            | -   | -         |
| el76     | 8.33  | 12.63                    | 138.81            | 22.19             | 65.10             | -   | -         |
| david    | 5.86  | 9.37                     | 68.63             | 22.19             | 44.61             | 34  | 919       |
| l1hg     | 8.41  | 13.53                    | 141.58            | 41.21             | 81.75             | -   | -         |
| laac     | 12.33 | -                        | 304.08            | 72.91             | 141.25            | -   | -         |
| celar04-pp| 7.27 | 11.15                    | 105.61            | 28.53             | 65.74             | -   | -         |
| 1a62     | 11.14 | -                        | 248.09            | 60.23             | 121.61            | -   | -         |
| 1bkb-pp  | 9.53  | -                        | 181.47            | 52.30             | 98.49             | -   | -         |
| 1dd3     | 9.90  | -                        | 196.11            | 52.30             | 103.17            | -   | -         |
| miles1500| 5.29  | 9.30                     | 55.87             | 34.87             | 49.69             | 8   | 4038      |
| miles250 | 4.58  | 7.24                     | 42.04             | 15.85             | 31.72             | 52  | 37        |
| celar10-pp| 6.91 | 10.34                    | 95.41             | 25.36             | 59.70             | 50  | 10179     |
| anna     | 7.94  | 11.94                    | 125.98            | 33.28             | 75.48             | -   | -         |
| pr152    | 8.29  | 12.76                    | 137.45            | 22.19             | 63.13             | -   | -         |
| munin2-pp| 7.61  | 11.82                    | 115.97            | 19.02             | 54.60             | -   | -         |
| mulsol.i.5| 3.58 | 6.11                     | 25.70             | 14.26             | 24.80             | 46  | 22        |
| zeroin.i.2| 3.81 | 6.58                     | 28.99             | 20.60             | 28.18             | 30  | 59        |
| boblo    | 4.00  | 6.17                     | 32.00             | 9.51              | 20.68             | 148 | 41        |

Continued on next page
Table 7 Width of linear boolean decompositions found with the IUN heuristic using the start vertices returned by performing a double BFS, and with candidates = $N^2 (\text{Left}) \cap \text{Right}$ in order to decrease the computation time. The values of the two others heuristics are taken from [12]. Missing entries are caused by a lack of internal memory which is caused by the $O(n \cdot 2^k)$ space requirement, with $k$ being the linear boolean-width of the computed decomposition. The last column indicates the time of the IUN heuristic.

| Graph   | $|V|$ | Edge Density | LeastUncommon | Relative | IUN | Time (s) |
|---------|------|--------------|---------------|----------|-----|----------|
| link-pp | 308  | 0.02         | 34.81         | 28.68    | 17.44 | 610.09   |
| diabetes-wpp | 332  | 0.01         | 8.58          | 18.58    | 5.32  | 1.53     |
| link-wpp | 339  | 0.02         | 35.00         | 29.03    | 16.79 | 374.04   |
| celer10 | 340  | 0.02         | 20.81         | 15.00    | 10.17 | 1.83     |
| celer11 | 340  | 0.02         | 19.54         | 14.70    | 10.80 | 1.88     |
| rd400  | 400  | 0.01         | 34.73         | 21.32    | 17.01 | 1,007.03 |
| diabetes | 413  | 0.01         | 29.32         | 19.32    | -     | -        |
| fpsol2.i.3 | 425  | 0.10         | 15.87         | 8.92     | 7.67  | 2.11     |
| pigs    | 441  | 0.01         | 24.04         | 18.00    | 12.39 | 20.08    |
| celer08 | 458  | 0.02         | 24.95         | 15.00    | 10.17 | 2.12     |
| d493    | 493  | 0.01         | 20.29         | 48.10    | 16.73 | 708.57   |
| homer   | 561  | 0.01         | 36.22         | 28.49    | -     | -        |
| rat575  | 575  | 0.01         | 16.48         | 37.23    | -     | -        |
| u724    | 724  | 0.01         | 18.72         | 50.09    | -     | -        |
| initx.i.1 | 864  | 0.05         | 11.98         | 7.22     | 6.81  | 7.31     |
| munin2  | 1003 | < 0.01       | 31.25         | 12.13    | 11.91 | 61.17    |
| vm1084  | 1084 | < 0.01       | 15.21         | 48.95    | -     | -        |
| BN_24   | 1819 | < 0.01       | 4.91          | 2.32     | 2.58  | 610.72   |
| BN_25   | 1819 | < 0.01       | 4.64          | 2.32     | 2.58  | 601.41   |
| BN_23   | 2425 | < 0.01       | 8.48          | 3.17     | 2.58  | 1,808.29 |
| BN_26   | 3025 | < 0.01       | 6.98          | 2.32     | 3.58  | 4,532.83 |

Table 8 Linear boolean-width upperbounds that are obtained through using the IUN heuristic with all starting vertices and candidates = Right. The tw column gives an upperbound on the treewidth, while the bw column gives an upperbound on the boolean-width, which values are taken from [12]. Cursive graph names marked with an asterisk indicate the graphs for which, in theory, the linear boolean decomposition will give a higher bound on the running time than the boolean decomposition, i.e., graphs for which $2^{2bw} > 2^{lbw}$. From this it seems that linear boolean-width seem to be more useful in practice than boolean-width heuristics. However, one should note that on certain graph classes, for instance graphs which look like trees, boolean-width is a lot lower than linear boolean-width.

| Graph   | $|V|$ | Edge Density | tw | bw | lbw | lbw/bw |
|---------|------|--------------|----|----|-----|--------|
| celer06-pp-003 | 4    | 0.5          | 2  | 1  | 1   | 1.00   |
| Graph          | $|V|$ | Edge Density | $tw$ | $bw$ | $lbw$ | $lbw/bw$ |
|---------------|-----|-------------|------|------|-------|----------|
| diabetes-pp-001* | 6   | 0.8         | 4    | 1    | 1.58  | 1.58     |
| munin3-pp-001*  | 7   | 0.81        | 5    | 1    | 1.58  | 1.58     |
| munin3-pp-002*  | 7   | 0.81        | 5    | 1    | 1.58  | 1.58     |
| celar06-pp-000  | 8   | 0.43        | 3    | 1    | 1     | 1.00     |
| diabetes-pp-002 | 8   | 0.61        | 4    | 2.32 | 2.32  | 1.00     |
| mainuk-pp       | 9   | 0.78        | 6    | 1.58 | 1.58  | 1.00     |
| rl5934-pp-001   | 10  | 0.44        | 4    | 2.81 | 3.17  | 1.13     |
| fl3795-pp-001   | 10  | 0.44        | 4    | 2.81 | 3     | 1.07     |
| fl3795-pp-003   | 10  | 0.44        | 4    | 2.81 | 3     | 1.07     |
| fl3795-pp-002   | 10  | 0.44        | 4    | 2.81 | 3.17  | 1.13     |
| pathfinder-pp-001 | 11  | 0.58        | 5    | 2.58 | 3.32  | 1.29     |
| mycie3          | 11  | 0.36        | 5    | 3    | 3.46  | 1.15     |
| pch3038-pp-001  | 11  | 0.4         | 5    | 3    | 2.81  | 0.94     |
| fl3795-pp-004   | 11  | 0.42        | 4    | 3    | 3.46  | 1.15     |
| pathfinder-pp   | 12  | 0.65        | 6    | 2.58 | 2.81  | 1.09     |
| celar11-pp-002  | 13  | 0.59        | 7    | 2.81 | 3.17  | 1.13     |
| celar04-pp-001-000 | 15 | 0.74        | 9    | 1.58 | 2     | 1.27     |
| weeduk          | 15  | 0.47        | 7    | 1.58 | 1.58  | 1.00     |
| funguk          | 15  | 0.34        | 4    | 2    | 1.58  | 0.79     |
| pch3038-pp-002  | 15  | 0.3         | 5    | 3    | 2.81  | 0.94     |
| mildew-wpp     | 15  | 0.3         | 4    | 2.58 | 3.32  | 1.29     |
| celar04-pp-001  | 16  | 0.78        | 10   | 1.58 | 2     | 1.27     |
| celar06-pp      | 16  | 0.84        | 11   | 1.58 | 1.58  | 1.00     |
| celar10-pp-001  | 16  | 0.51        | 8    | 3    | 3.46  | 1.15     |
| celar09-pp-001  | 16  | 0.51        | 8    | 3    | 3.17  | 1.06     |
| celar08-pp-002  | 16  | 0.51        | 8    | 3    | 3.32  | 1.11     |
| celar07-pp-002  | 16  | 0.45        | 7    | 3    | 3.32  | 1.11     |
| barley-pp-001   | 16  | 0.42        | 7    | 3.32 | 3.32  | 1.00     |
| celar11-pp-004  | 16  | 0.36        | 6    | 3.17 | 3.58  | 1.13     |
| munin2-pp-005   | 16  | 0.3         | 5    | 3    | 3.58  | 1.19     |
| munin2-pp-006   | 16  | 0.3         | 5    | 3    | 3.58  | 1.19     |
| munin2-pp-003   | 16  | 0.3         | 5    | 3.17 | 3.7   | 1.17     |
| munin2-pp-004   | 16  | 0.3         | 5    | 3.17 | 3.7   | 1.17     |
| munin2-pp-007   | 17  | 0.35        | 7    | 3.46 | 3.58  | 1.03     |
| munin2-pp-011   | 17  | 0.35        | 7    | 3.46 | 3.58  | 1.03     |
| munin2-pp-010   | 17  | 0.35        | 7    | 3.46 | 3.81  | 1.10     |
| munin2-pp-008   | 17  | 0.35        | 7    | 3.46 | 3.58  | 1.03     |
| munin2-pp-009   | 18  | 0.31        | 6    | 3.46 | 3.81  | 1.10     |
| munin2-pp-012   | 18  | 0.31        | 6    | 3.46 | 3.81  | 1.10     |
| celar01-pp-002  | 19  | 0.65        | 10   | 2    | 2.32  | 1.16     |
| celar02-pp      | 19  | 0.67        | 10   | 2    | 2     | 1.00     |
| celar05-pp-001  | 19  | 0.66        | 11   | 2    | 2.32  | 1.16     |
| celar11-pp-001  | 19  | 0.65        | 10   | 2    | 2.32  | 1.16     |
| fl3795-pp-005   | 19  | 0.22        | 4    | 3.32 | 3.58  | 1.08     |
| water-pp-001    | 21  | 0.45        | 9    | 3.81 | 4.09  | 1.07     |
| anna-pp         | 22  | 0.64        | 12   | 3.46 | 3.81  | 1.10     |
Table 8 – Continued from previous page

| Graph           | V  | Edge Density | tw  | bw  | lbw | lbw/bw |
|-----------------|----|--------------|-----|-----|-----|--------|
| water-pp        | 22 | 0.42         | 9   | 4.17| 4.32| 1.04   |
| water-wpp       | 22 | 0.42         | 9   | 4.17| 4.32| 1.04   |
| munin4-pp-001   | 23 | 0.26         | 8   | 3.58| 4   | 1.12   |
| munin4-pp-002   | 23 | 0.26         | 8   | 3.58| 4   | 1.12   |
| myciel4         | 23 | 0.28         | 10  | 5   | 5.49| 1.10   |
| BN_29           | 24 | 0.18         | 5   | 2   | 2.32| 1.16   |
| BN_28           | 24 | 0.18         | 5   | 2   | 2.32| 1.16   |
| queen5_5        | 25 | 0.53         | 18  | 5.29| 5.67| 1.07   |
| barley-pp       | 26 | 0.24         | 7   | 3.7 | 3.46| 0.94   |
| fl3795-pp-006   | 26 | 0.16         | 5   | 3.81| 4.17| 1.09   |
| david-pp        | 29 | 0.47         | 13  | 4.09| 4.32| 1.06   |
| barley-wpp      | 29 | 0.2          | 7   | 3.81| 4.38| 0.94   |
| pcb3038-pp-003  | 29 | 0.12         | 5   | 4.32| 4.75| 1.10   |
| celar02-wpp     | 30 | 0.33         | 10  | 2.81| 2.58| 0.92   |
| water           | 32 | 0.25         | 9   | 4.39| 4.75| 1.08   |
| BN_16-pp-015    | 34 | 0.28         | 11  | 3.58| 4.39| 1.23   |
| celar06-wpp     | 34 | 0.28         | 11  | 3   | 3.17| 1.06   |
| BN_16-pp-014    | 34 | 0.28         | 11  | 3.81| 4.86| 1.28   |
| 1bx7-pp         | 34 | 0.31         | 11  | 4.7 | 4.39| 0.93   |
| mildew          | 35 | 0.13         | 4   | 3   | 3.32| 1.11   |
| queen6_6        | 36 | 0.46         | 25  | 7.65| 8.08| 1.06   |
| alarm           | 37 | 0.1          | 4   | 2.58| 3   | 1.16   |
| celar03-pp-001  | 38 | 0.34         | 14  | 5.81| 6.11| 1.05   |
| munin4-pp-003*  | 38 | 0.16         | 8   | 3.58| 5.39| 1.51   |
| munin4-pp-004   | 38 | 0.16         | 8   | 4.17| 5.39| 1.29   |
| celar08-pp-001  | 39 | 0.38         | 16  | 5.09| 5.21| 1.02   |
| oesoca          | 39 | 0.09         | 3   | 2.32| 3   | 1.29   |
| 1bx7            | 41 | 0.24         | 11  | 4.91| 4.75| 0.97   |
| oesoca42        | 42 | 0.08         | 3   | 2.32| 3.17| 1.37   |
| celar07-pp-001  | 45 | 0.32         | 16  | 5.46| 5.86| 1.07   |
| celar01-pp-001  | 47 | 0.25         | 15  | 5.88| 6.36| 1.08   |
| celar05-pp-002  | 47 | 0.25         | 15  | 6.07| 5.83| 0.96   |
| myciel5         | 47 | 0.22         | 19  | 8.12| 6.49| 0.80   |
| 1ubq-pp         | 47 | 0.16         | 12  | 5.95| 8.79| 1.48   |
| pigs-pp-001     | 47 | 0.12         | 9   | 5.95| 7.07| 1.19   |
| 1brf-pp         | 48 | 0.36         | 22  | 7.01| 7.25| 1.03   |
| 1rb9            | 48 | 0.37         | 22  | 6.77| 7.17| 1.06   |
| celar11-pp-003  | 48 | 0.23         | 15  | 5.73| 4.58| 0.80   |
| mainuk*         | 48 | 0.18         | 7   | 3.58| 6.49| 1.18   |
| barley          | 48 | 0.11         | 7   | 4   | 3.7  | 0.93   |
| pigs-pp         | 48 | 0.12         | 9   | 5.7 | 6.64| 1.16   |
| 1brf            | 49 | 0.35         | 22  | 7.01| 7.3  | 1.04   |
| queen7_7        | 49 | 0.4          | 35  | 10.36| 10.97| 1.06 |
| 1kth-pp         | 51 | 0.33         | 20  | 7.06| 5.86| 0.83   |
| 1i07-pp         | 51 | 0.28         | 15  | 5.55| 7.18| 1.29   |
| eil51.tsp       | 51 | 0.11         | 9   | 5.78| 5.78| 1.00   |

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Table 8 – Continued from previous page

| Graph         | $|V|$ | Edge Density | $tw$ | $bw$ | $lbw$ | $lbw/bw$ |
|---------------|-----|--------------|------|------|-------|----------|
| 1igq-pp       | 52  | 0.37         | 23   | 6.74 | 7.45  | 1.11     |
| 1kth          | 52  | 0.32         | 20   | 7.04 | 6.87  | 0.98     |
| 1g6x          | 52  | 0.31         | 19   | 6.89 | 7.21  | 1.05     |
| 1igq          | 54  | 0.35         | 23   | 6.89 | 7.61  | 1.10     |
| zeroin.i.1-pp | 54  | 0.89         | 46   | 1.58 | 1.58  | 1.00     |
| 1e06-pp       | 55  | 0.33         | 24   | 7.69 | 8.32  | 1.08     |
| munin4-pp-006 | 55  | 0.11         | 8    | 4.32 | 5.17  | 1.20     |
| munin4-pp-005 | 55  | 0.11         | 8    | 4.39 | 5.17  | 1.18     |
| 1j75          | 56  | 0.36         | 27   | 8.51 | 8.94  | 1.05     |
| 1k61-pp       | 56  | 0.37         | 26   | 8.02 | 8.37  | 1.04     |
| 1sem-pp       | 56  | 0.37         | 26   | 8.09 | 8.5   | 1.05     |
| 1blz-pp       | 56  | 0.35         | 25   | 8.18 | 8.36  | 1.02     |
| 1bf4-pp       | 57  | 0.39         | 26   | 7.63 | 7.79  | 1.02     |
| 1cka          | 57  | 0.38         | 27   | 8.55 | 8.87  | 1.04     |
| 1sem          | 57  | 0.36         | 26   | 8.32 | 8.66  | 1.04     |
| zeroin.i.1.2-pp | 57  | 0.69        | 32   | 2.81 | 3.32  | 1.18     |
| zeroin.i.3-pp | 57  | 0.69        | 32   | 3    | 3.32  | 1.11     |
| 1bbz          | 57  | 0.34         | 25   | 8.3  | 8.36  | 1.01     |
| 1oai-pp       | 57  | 0.32         | 22   | 7.94 | 8.28  | 1.04     |
| 1jo8          | 58  | 0.37         | 27   | 8.46 | 8.73  | 1.03     |
| 1oai          | 58  | 0.32         | 22   | 7.87 | 8.15  | 1.04     |
| celar01-pp-003 | 58  | 0.19       | 15   | 6.97 | 6.89  | 0.99     |
| 1g2b-pp       | 59  | 0.37         | 28   | 8.5  | 8.99  | 1.06     |
| 1igd-pp       | 59  | 0.36         | 25   | 7.66 | 7.9   | 1.03     |
| 1kq1-pp       | 59  | 0.35         | 27   | 8.63 | 8.94  | 1.04     |
| 1pwt-pp       | 59  | 0.38         | 29   | 8.85 | 9.24  | 1.04     |
| 1o7           | 59  | 0.23         | 15   | 5.52 | 5.93  | 1.07     |
| 1k61          | 60  | 0.33         | 26   | 8.32 | 8.81  | 1.06     |
| 1kq1          | 60  | 0.34         | 27   | 8.79 | 8.89  | 1.01     |
| 1ku3-pp       | 60  | 0.33         | 23   | 7.46 | 7.53  | 1.01     |
| 1e06          | 60  | 0.29         | 24   | 8.13 | 8.42  | 1.04     |
| knights8_8-pp | 60  | 0.09        | 16   | 10.77 | 11.3  | 1.05     |
| 1gut-pp       | 61  | 0.33         | 22   | 7.19 | 7.54  | 1.05     |
| 1i2t          | 61  | 0.35         | 27   | 8.38 | 9.03  | 1.08     |
| 1igd          | 61  | 0.34         | 25   | 7.75 | 7.9   | 1.02     |
| 1pwt          | 61  | 0.36         | 29   | 8.81 | 9.27  | 1.05     |
| 1ku3          | 61  | 0.32         | 23   | 7.53 | 7.61  | 1.01     |
| 1g2b          | 62  | 0.34         | 28   | 8.72 | 9.05  | 1.04     |
| 1h3-pp        | 62  | 0.32         | 21   | 7.16 | 7.29  | 1.02     |
| celar04-pp-002 | 62  | 0.17       | 16   | 6.86 | 7.26  | 1.06     |
| 1bf4          | 63  | 0.34         | 26   | 7.9  | 8.09  | 1.02     |
| 1r69          | 63  | 0.35         | 30   | 9.12 | 9.51  | 1.04     |
| munin1-pp-001 | 63  | 0.09        | 11   | 5.58 | 6.43  | 1.15     |
| 1gcq-pp       | 64  | 0.36         | 30   | 8.95 | 9.38  | 1.05     |
| queen8_8      | 64  | 0.36         | 45   | 13.16| 14.05 | 1.07     |
| 1a8o          | 64  | 0.27         | 25   | 9.11 | 9.3   | 1.02     |

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Table 8 – Continued from previous page

| Graph        | $|V|$ | Edge Density | tw  | bw  | lbw | lbw/bw |
|--------------|-----|--------------|-----|-----|-----|--------|
| knights8_8   | 64  | 0.08         | 16  | 11.06 | 11.64 | 1.05   |
| 1fjl         | 65  | 0.29         | 26  | 7.9  | 8.49 | 1.07   |
| 1c9o         | 66  | 0.34         | 29  | 8.75 | 8.88 | 1.01   |
| 1hg7         | 66  | 0.33         | 29  | 8.81 | 9.13 | 1.04   |
| 1ezg         | 66  | 0.25         | 23  | 8.33 | 7   | 0.84   |
| 1en2-pp      | 66  | 0.21         | 17  | 7.46 | 8.54 | 1.14   |
| mumin1-pp    | 66  | 0.09         | 11  | 5.58 | 6.43 | 1.15   |
| 1c4q         | 67  | 0.34         | 31  | 9.45 | 9.71 | 1.03   |
| 1fse         | 67  | 0.33         | 27  | 8.58 | 8.75 | 1.02   |
| 1kw4         | 67  | 0.3          | 28  | 9.39 | 5.73 | 0.61   |
| 1gut         | 67  | 0.28         | 22  | 7.47 | 7.36 | 0.99   |
| 1fr3         | 67  | 0.28         | 21  | 7.29 | 7.47 | 1.02   |
| 1b67-pp      | 67  | 0.25         | 16  | 6.61 | 9.61 | 1.45   |
| 1gcq         | 68  | 0.33         | 30  | 9.36 | 9.65 | 1.03   |
| 1ail-pp      | 68  | 0.28         | 24  | 8.11 | 8.33 | 1.03   |
| 1d3b-pp      | 68  | 0.3          | 25  | 8.54 | 5.78 | 0.68   |
| 1b67         | 68  | 0.25         | 16  | 6.61 | 8.52 | 1.29   |
| 1c75         | 69  | 0.29         | 30  | 9.88 | 8.31 | 0.84   |
| 1ail         | 69  | 0.27         | 24  | 8.07 | 9.68 | 1.20   |
| 1d3b         | 69  | 0.29         | 25  | 8.44 | 8.53 | 1.01   |
| 1en2         | 69  | 0.2          | 17  | 7.24 | 7   | 0.97   |
| 1ce8         | 70  | 0.34         | 32  | 9.35 | 9.63 | 1.03   |
| 1dj7-pp      | 70  | 0.3          | 27  | 8.12 | 8.22 | 1.01   |
| 1d27-pp      | 70  | 0.3          | 27  | 8.67 | 8.82 | 1.02   |
| 1b9l         | 70  | 0.29         | 29  | 9.26 | 10   | 1.08   |
| 1ljo-pp      | 71  | 0.31         | 30  | 8.92 | 9.02 | 1.01   |
| 1dp7-pp      | 71  | 0.3          | 27  | 9.21 | 9.15 | 0.99   |
| graph03-pp-001 | 71 | 0.11       | 20  | 12.53 | 12.24 | 0.98   |
| 1mgq-pp      | 72  | 0.31         | 28  | 8.98 | 9.08 | 1.01   |
| 1d27         | 73  | 0.28         | 27  | 8.78 | 9.06 | 1.03   |
| mulsol.i1-pp | 73  | 0.83         | 50  | 2.32 | 2.58 | 1.11   |
| 1dj7         | 73  | 0.28         | 27  | 9.66 | 8.22 | 0.85   |
| 1ldd         | 74  | 0.31         | 32  | 9.6  | 9.73 | 1.01   |
| 1jio         | 74  | 0.29         | 30  | 8.88 | 9.06 | 1.02   |
| 1mgq         | 74  | 0.3          | 28  | 8.91 | 9.06 | 1.02   |
| luck         | 74  | 0.11         | 10  | 2.81 | 3.32 | 1.18   |
| 1ubq         | 74  | 0.08         | 12  | 6.61 | 7.75 | 1.17   |
| 1ig5         | 75  | 0.29         | 33  | 10.45 | 10.64 | 1.02   |
| 1dp7         | 76  | 0.27         | 27  | 9.01 | 9.3  | 1.03   |
| celular10-pp-002 | 76 | 0.15       | 16  | 7.25 | 6.58 | 0.91   |
| celular08-pp-003 | 76 | 0.15       | 16  | 7.41 | 6.58 | 0.89   |
| celular09-pp-002 | 76 | 0.15       | 16  | 7.46 | 6.58 | 0.88   |
| 1iqz         | 77  | 0.29         | 33  | 10  | 10.1 | 1.01   |
| 1qtn-pp      | 77  | 0.25         | 24  | 8.56 | 8.33 | 0.97   |
| mumin3-pp-003* | 79 | 0.09       | 7   | 4.17 | 12.73 | 3.05   |
| graph03-pp   | 79  | 0.1          | 20  | 12.99 | 5.61 | 0.43   |

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Table 8 – Continued from previous page

| Graph         | $|V|$ | Edge Density | $tw$ | $bw$ | $lbw$ | $lbw/bw$ |
|---------------|-----|--------------|------|------|-------|---------|
| sodoku-elim1  | 80  | 0.28         | 45   | 9.47 | 12    | 1.27    |
| *jean*       | 80  | 0.08         | 9    | 3.91 | 6.54  | 1.67    |
| celar05-pp    | 80  | 0.13         | 15   | 7.2  | 4.58  | 0.64    |
| sodoku        | 81  | 0.25         | 45   | 9    | 12.7  | 1.41    |
| celar03-pp    | 81  | 0.13         | 14   | 6.19 | 6.11  | 0.99    |
| graph03-wpp   | 84  | 0.09         | 20   | 12.74| 12.92 | 1.01    |
| 1fk5          | 85  | 0.23         | 31   | 10.76| 10.1  | 0.94    |
| 1aba          | 85  | 0.25         | 29   | 10.13| 10.81 | 1.07    |
| graph01-pp-001| 85  | 0.09        | 24   | 13.4 | 13.66 | 1.02    |
| 1ctj-pp       | 86  | 0.25         | 33   | 10.78| 11.07 | 1.03    |
| 1ctj          | 87  | 0.25         | 33   | 10.74| 11.04 | 1.03    |
| 1ptf          | 87  | 0.3          | 38   | 11.21| 10.86 | 0.97    |
| 1qt1n         | 87  | 0.21         | 24   | 9.15 | 8.97  | 0.98    |
| david         | 87  | 0.11         | 13   | 5.32 | 5.86  | 1.10    |
| graph05-pp-001| 87  | 0.1         | 24   | 12.68| 13.31 | 1.05    |
| 1awd          | 89  | 0.28         | 38   | 10.8 | 11.13 | 1.03    |
| celar03-wpp   | 89  | 0.11         | 14   | 6.17 | 6.49  | 1.05    |
| celar05-wpp   | 89  | 0.11         | 15   | 7.52 | 6.54  | 0.87    |
| graph01-pp    | 89  | 0.08         | 24   | 14.62| 13.96 | 0.95    |
| munin1-wpp    | 90  | 0.05         | 11   | 7.23 | 7.58  | 1.05    |
| l1jhg-pp      | 91  | 0.19         | 25   | 8.34 | 8.41  | 1.01    |
| graph05-pp    | 91  | 0.1          | 24   | 13.84| 13.49 | 0.97    |
| celar07-pp    | 92  | 0.12         | 16   | 6    | 6     | 1.00    |
| a280.tsp-pp   | 92  | 0.06         | 14   | 8.23 | 7.38  | 0.90    |
| kroE100.tsp-pp* | 92 | 0.06       | 10   | 6.48 | 14.84 | 2.29    |
| 1g2r-pp       | 93  | 0.26         | 37   | 11.87| 11.51 | 0.97    |
| graph01-wpp   | 93  | 0.07         | 24   | 14.69| 11.41 | 0.78    |
| 1czp          | 94  | 0.27         | 38   | 11.47| 11.6  | 1.01    |
| 1g2r          | 94  | 0.25         | 37   | 12.17| 14.19 | 1.17    |
| graph05-wpp   | 94  | 0.09         | 24   | 14.38| 13.18 | 0.92    |
| l1ce5e        | 95  | 0.26         | 36   | 11.06| 10.83 | 0.98    |
| myciel6       | 95  | 0.17         | 35   | 13.4 | 7.86  | 0.59    |
| homer-pp      | 95  | 0.17         | 31   | 14.61| 13.88 | 0.95    |
| kroA100.tsp-pp| 95  | 0.06         | 10   | 7.61 | 6.58  | 0.86    |
| celar11-pp    | 96  | 0.1          | 15   | 6.64 | 5.98  | 0.90    |
| munin3-pp     | 96  | 0.07         | 7    | 4.32 | 5.86  | 1.36    |
| celar07-wpp   | 97  | 0.01         | 16   | 6    | 7.17  | 1.20    |
| kroC100.tsp-pp* | 97 | 0.06       | 10   | 6.94 | 11.97 | 1.72    |
| 1plc          | 98  | 0.25         | 35   | 11.28| 11.1  | 0.98    |
| 1lkk-pp       | 99  | 0.24         | 34   | 11   | 10.84 | 0.99    |
| 1d4t-pp       | 99  | 0.23         | 35   | 11.88| 6.58  | 0.55    |
| celar11-wpp   | 99  | 0.1          | 15   | 7.17 | 4.91  | 0.68    |
| l1ov          | 100 | 0.24         | 41   | 12.21| 12.47 | 1.02    |
| celar02       | 100 | 0.06         | 10   | 3.32 | 4.91  | 1.48    |
| celar06*      | 100 | 0.07         | 11   | 3.81 | 14.85 | 3.90    |
| graph05       | 100 | 0.08         | 24   | 13.7 | 13.36 | 0.98    |
Table 8 – Continued from previous page

| Graph     | \(|V\) | Edge Density | tw  | bw  | lbw | lbw/bw |
|-----------|-------|--------------|-----|-----|-----|--------|
| graph01   | 100   | 0.07         | 24  | 14.61 | 14.21 | 0.97   |
| graph03   | 100   | 0.07         | 20  | 13.29 | 8.41  | 0.63   |
| 1erv      | 101   | 0.25         | 41  | 12.26 | 12.44 | 1.01   |
| 1jhg      | 101   | 0.17         | 25  | 8.87  | 11.97 | 1.35   |
| 1iib-pp   | 102   | 0.27         | 40  | 11.98 | 11.76 | 0.98   |
| 1d4t      | 102   | 0.22         | 35  | 12.87 | 10.31 | 0.80   |
| 1iib      | 103   | 0.26         | 40  | 12.62 | 11.79 | 0.93   |
| 1b0n      | 103   | 0.19         | 32  | 10.81 | 11.17 | 1.03   |
| 1lkk      | 103   | 0.22         | 34  | 11.89 | 13.56 | 1.14   |
| 1aac      | 104   | 0.25         | 41  | 12.29 | 12.33 | 1.00   |
| 1bkf-pp   | 105   | 0.23         | 36  | 11.1  | 11.4  | 1.03   |
| 1bkf      | 106   | 0.23         | 36  | 11.69 | 11.44 | 0.98   |
| 1bkr      | 107   | 0.24         | 44  | 14.4  | 13.75 | 0.95   |
| 1rro      | 107   | 0.23         | 43  | 15.36 | 3.58  | 0.23   |
| 19m3      | 109   | 0.23         | 45  | 14.27 | 13.56 | 0.95   |
| pathfinder* | 109 | 0.04         | 6   | 3.32  | 10.83 | 3.26   |
| celar04-pp | 110  | 0.09         | 16  | 7.29  | 7.27  | 1.00   |
| 1fs1      | 114   | 0.21         | 34  | 13.79 | 7.36  | 0.53   |
| celar04-wpp | 116  | 0.07         | 16  | 7.95  | 11.1  | 1.40   |
| 1gef-pp   | 117   | 0.22         | 43  | 12.93 | 13.35 | 1.03   |
| 1gef      | 119   | 0.21         | 43  | 13.6  | 13.35 | 0.98   |
| mulsol.i.5-pp | 119 | 0.36         | 31  | 3     | 3     | 1.00   |
| 1a62-pp   | 120   | 0.21         | 37  | 14.7  | 11.14 | 0.76   |
| 1a62      | 122   | 0.21         | 37  | 13.62 | 9.68  | 0.71   |
| 1dd3-pp   | 124   | 0.17         | 31  | 14.6  | 9.25  | 0.63   |
| ch130.tsp-pp | 125 | 0.05         | 12  | 8.67  | 9.53  | 1.09   |
| 1bkb-pp   | 127   | 0.18         | 30  | 15.55 | 9.9   | 0.64   |
| miles1500 | 128   | 0.64         | 77  | 4.86  | 5.29  | 1.09   |
| 1dd3      | 128   | 0.17         | 31  | 11.68 | 4.58  | 0.39   |
| miles500  | 128   | 0.14         | 22  | 9.42  | 7.04  | 0.75   |
| miles250* | 128   | 0.05         | 9   | 4.95  | 9.61  | 1.94   |
| mulsol.i.3 | 131  | 0.17         | 30  | 14.53 | 6.91  | 0.48   |
| celar10-pp | 133  | 0.07         | 16  | 9.08  | 7.7   | 0.85   |
| anna      | 138   | 0.04         | 12  | 6.67  | 7.25  | 1.09   |
| celar09-wpp | 142  | 0.06         | 16  | 8.49  | 7     | 0.82   |
| celar01-pp | 157  | 0.07         | 15  | 7.39  | 7     | 0.95   |
| celar01-wpp | 158  | 0.06         | 15  | 7.09  | 7.61  | 1.07   |
| munsin2-pp | 167  | 0.03         | 7   | 5.49  | 6.91  | 1.26   |
| mulsol.i.3 | 184  | 0.23         | 32  | 4.95  | 3.58  | 0.72   |
| mulsol.i.4 | 185  | 0.23         | 32  | 4.81  | 3.58  | 0.74   |
| mulsol.i.5 | 186  | 0.23         | 31  | 4.95  | 3.58  | 0.72   |
| mulsol.i.2 | 188  | 0.22         | 32  | 4.81  | 3.58  | 0.74   |
| celar08-wpp | 190  | 0.05         | 16  | 9.64  | 11.48 | 1.19   |
| mulsol.i.1 | 197  | 0.2          | 50  | 4     | 4.17  | 1.04   |
| zeroin.i.3 | 206  | 0.17         | 32  | 5.39  | 3.81  | 0.71   |
| zeroin.i.1 | 211  | 0.19         | 50  | 3.7   | 3.32  | 0.90   |

Continued on next page
| Graph   | $|V|$ | Edge Density | $tw$ | $bw$ | $lbw$ | $lbw/bw$ |
|---------|-----|-------------|------|------|-------|----------|
| zeroin.i.2 | 211 | 0.16        | 32   | 5.39 | 3.81  | 0.71     |
| fpsol2.i.1-pp | 233 | 0.4         | 66   | 4.91 | 4.81  | 0.98     |