The Wave Function of the Universe by the New Euclidean Path-integral Approach in Quantum Cosmology

Atushi Ishikawa\textsuperscript{1}, and Haruhiko Ueda\textsuperscript{2,3}

\textsuperscript{1} Department of Physics
Osaka University, Toyonaka, Osaka 560, Japan

\textsuperscript{2} Uji Research Center, Yukawa Institute for Theoretical Physics
Kyoto University, Uji 611, Japan

\textsuperscript{3} Department of Physics
Hiroshima University, Higashihiroshima 724, Japan

ABSTRACT

The wave function of the universe is evaluated by using the Euclidean path integral approach. As is well known, the real Euclidean path integral diverges because the Einstein-Hilbert action is not positive definite. In order to obtain a finite wave function, we propose a new regularization method and calculate the wave function of the Friedmann- Robertson-Walker type minisuperspace model. We then consider a homogeneous but anisotropic type minisuperspace model, which is known as the Bianch type I model. The physical meaning of the wave function by this new regularization method is also examined.
Quantum cosmology is one of the most fascinated subjects in modern physics.\[^{[1,2]}\]

In quantum theory, the wave function characterizes each theory. Therefore the main purpose in quantum cosmology is to evaluate the wave function of the universe. As is well known, the wave function of the universe is obtained as the solution of the Wheeler-DeWitt second-order functional differential equation. This equation, however, is very difficult to solve even in minisuperspace models, although in minisuperspace models the Wheeler-DeWitt equation is reduced to an ordinary differential equation. In addition, we have to know the initial condition of the universe which is hard to understand completely. Another approach, which gives the wave function by the Euclidean path integral, has been proposed by Hartle and Hawking\[^{[3,4]}\]. They claimed that the path integral should be performed over regular and closed four-dimensional manifolds. This idea is called the boundary condition of no-boundary. The four-dimensional manifold must be Euclidean to avoid the cone singularity at an initial point. Their proposal solves the initial condition problem but the real Euclidean path integral does not converge because the Einstein-Hilbert action is not positive definite. The question is therefore whether we can construct the Euclidean path integral approach which is free from a divergence problem.

A first way to avoid this difficulty was proposed by Gibbons, Hawking and Perry\[^{[5]}\]. They suggested that the path integral could converge if we performed the conformal rotation. This proposal, however, has several objections\[^{[6]}\]. Hartle proposed another path integral approach. He suggested that the path integral should be taken along the steepest-descent path in the space of complex four-metrics. Following the Hartle’s alternative proposal, we can make the Euclidean path integral convergent by changing an integration contour from a real axis to a curve on a complex plane. Halliwell and Louko calculated the wave function of the universe following this method\[^{[8,9,10]}\]. The integral along this complex curve is so difficult to perform that they evaluated the wave function in a saddle point approximation\[^{[8]}\]. Moreover the physical interpretation of this wave function is hard because the path integral is defined on the complex four-manifold\[^{[11]}\].
The contour changing method seems not to complete and it is worthy investigating a possibility of another regularization method.

In this paper, we propose a new regularization method and evaluate the wave function in minisuperspace models. We first propose a new regularization method and calculate the wave function of the Friedmann-Robertson-Walker type minisuperspace model. We then consider a more general case, i.e. a homogeneous but anisotropic type minisuperspace model, which is known as the Bianchi type I model. We give a physical interpretation of the wave function by this new regularization method.

We first calculate the wave function of the closed Friedmann-Robertson-Walker type minisuperspace model\(^{7,8}\). The metric of the Friedmann-Robertson-Walker type is

\[
ds^2 = \sigma^2 \left\{ \frac{N^2(\tau)}{q(\tau)} d\tau^2 + q(\tau)d\Omega_3^2 \right\},
\]

where \(\sigma^2 = 2G/3\pi\) and \(d\Omega_3^2\) is the metric on the unit three-sphere. The Einstein-Hilbert action with the rescaled cosmological constant \(\lambda\) then becomes

\[
I[q(\tau)] = \frac{1}{2} \int_{\tau'}^{\tau''} d\tau N \left[ -\frac{q'^2}{4N^2} + \lambda q - 1 \right].
\]

We can rewrite this action to the Hamiltonian form as follows.

\[
I[p(T), q(T)] = \int_0^T d\tau \left\{ p\dot{q} - \frac{1}{2}(-4p^2 - \lambda q + 1) \right\},
\]

where \(T = N \cdot (\tau'' - \tau')\) and \(p = -\dot{q}/4N\). The Wheeler-DeWitt equation can be easily obtained by replacing \(p\) in the Hamiltonian with \(-d/dq\), therefore the Wheeler-DeWitt equation in the Friedmann-Robertson-Walker type becomes

\[
H\Psi = \frac{1}{2} \left[ -4 \frac{d^2}{dq^2} - \lambda q + 1 \right] \Psi = 0.
\]

The wave function of the universe can be obtained if one solves this equation\(^{12}\). As
is stated before, we can estimate the wave function by means of the path integral

$$\Psi = \int dT \int DpDq e^{-I[p,q]} \equiv \int dT \psi(T), \quad (5)$$

instead of solving the Wheeler-DeWitt equation$^7$. The Wheeler-DeWitt equation will be a guide for the path integral to define its contour. As is easily seen, the integration with respect to $T$ is not easy to evaluate. We first consider the $p$ and $q$ integration$^{13}$. To perform the $p$ and $q$ integration, we divide the interval $T$ into $L + 1$ pieces. Then $\psi(T)$ is

$$\psi(T) = \lim_{L \to \infty} \int \prod_{i=1}^L dq_i \prod_{i=0}^L dp_i \frac{1}{2\pi} \exp\left\{ -\Delta T \sum_{i=0}^L \left( \frac{p_i q_i+1 - q_i}{\Delta T} + 2p_i^2 + \frac{\lambda}{2} q_i - \frac{1}{2} \right) \right\}, \quad (6)$$

where $\Delta T = T/(L + 1)$, $T_i = i \cdot \Delta T$, $q_i = q(T_i)$, $p_i = p(T_i)$. The $p$ integration is carried out immediately because of the Gaussian form with respect to $p$. In order to perform the $q$ integration, we use a following formula which is obtained by the mathematical induction

$$\int dx_1 dx_2 \cdots dx_n e^{-\alpha \{ (x_1 - a)^2 + \beta x_2 + (x_2 - a)^2 + \beta x_3 + \cdots + (b - x_n)^2 + \beta x_n \}}$$

$$= \frac{1}{(n + 1)^{1/2} \left( \frac{\pi}{\alpha} \right)^{n/2}} \times \exp\left\{ \frac{-\alpha}{n + 1} (b - a)^2 - \alpha \beta \frac{1}{2} \left( \frac{n + 2}{a} \right)^2 + \frac{n b}{2} - \frac{n}{48} (n + 1)(n + 2) \beta \right\}. \quad (7)$$

We can evaluate the $p, q$ integration completely and the result is

$$\psi(T) = \left\{ \frac{1}{8\pi T} \right\}^{1/2} e^{-I(T)}, \quad (8)$$

where

$$I(T) = \frac{\lambda^2}{24} T^3 + \left\{ \frac{\lambda (q'' + q')}{4} - \frac{1}{2} \right\} T - \frac{(q'' - q')^2}{8T}. \quad (9)$$

In Ref.$[8]$, Halliwell and Louko divided the variables $p(T)$ and $q(T)$ into the classical
part and the quantum one. On the other hand, we can calculate the $p, q$ integration directly by using above induction (7).

We now evaluate the $T$ integration. $T$ is the rescaled lapse function and the range of the $T$ integration must preserve the time reparametrization invariance. The $T$ integration should be performed from $-\infty$ to $+\infty$ naively. This integration, however, diverges if we take the contour along the real axis. From this reason, a regularization is necessary to remove this divergence. In Ref.[8, 9, 10], the integration is regularized by means of changing a contour of the $T$ integration and, by using a saddle point approximation, they evaluated the wave function. While the contour changing method is a very natural regularization, the full integration is very difficult. It seems to be hard to study the wave function beyond a saddle point approximation. We adopt another strategy, i.e. instead of changing the contour directly, we extend the integrated function $\psi(T)$ in (8) such that the $T$ integration will converge.

We introduce a complex parameter $\alpha$ and define the wave function as an analytic function with respect to $\alpha$ as follows (up to a normalization factor).

$$
\Psi(\alpha) = \int_{-\infty}^{\infty} \frac{dT}{(\alpha T)^{1/2}} e^{-I(\alpha T)},
$$

where the rescaled lapse function $T$ remains real and the complex parameter $\alpha$ must be determined in such a way that the $T$ integration will converge. The strategy is as follows. We temporally fix $\alpha$ as a constant complex number during the integration with respect to $T$. After the integration, we take the analytic continuation $\alpha \rightarrow 1$ and get the wave function.

Let us perform the above strategy and obtain the wave function of the Friedmann-Robertson-Walker minisuperspace. In this case, unfortunately, we cannot carry out the integration completely. However we can use the Wheeler-DeWitt equation and obtain the wave function, which is equivalent to calculating the integration. This method was proposed by Halliwell and Louko [8]. Apparently the wave function is
symmetric with respect to $q' = q(\tau')$ and $q'' = q(\tau'')$. In addition, the wave function satisfies the Wheeler-DeWitt equation at $q'$ and $q''$. By rescaling $T \rightarrow \lambda^{-2/3}2^{1/3}T$ and introducing a new variable $z \equiv (1-\lambda q)/(2\lambda)^{2/3}$, the wave function as a solution of the Wheeler-DeWitt equation will be expressed as Airy function’s products,

$$\Psi = a \text{Ai}(z'') \text{Ai}(z') + b \text{Bi}(z'') \text{Bi}(z') + c \left[ \text{Ai}(z'') \text{Bi}(z') + \text{Bi}(z'') \text{Bi}(z') \right].$$

(11)

In order to get coefficients $(a, b, c)$, we take $z' = z'' = z$ and expand (11) with $z$ around $z = 0$. On the other hand, (10) will be

$$\Psi(\alpha) = \frac{1}{\alpha} \int_{-\infty}^{\alpha} \frac{dT}{T^{1/2}} \exp \left\{ -\frac{1}{12} T^3 + zT \right\},$$

(12)

where $T$ is complex in this representation. We also expand this with $z$ around $z = 0$, and get coefficients $(a, b, c)$ by comparing their coefficients of $z$. We suppose that $z$ is very small value and we need not $O(z^3)$ terms.

As is stated before, the complex parameter $\alpha$ must be so determined that the integration will converge. This statement requires that $\text{Re}[(\alpha T)^3]$ is positive for $|T| \gg 1$, hence there are three $\alpha$’s areas. In each case, we get the wave function. The results are

$$\Psi = \text{Ai}(z'') \text{Ai}(z'),$$

(13)

$$\Psi = \text{Ai}(z'') \text{Bi}(z') + \text{Bi}(z'') \text{Ai}(z'),$$

(14)

$$\Psi = \text{Ai}(z'') \text{Ai}(z') - \text{Bi}(z'') \text{Bi}(z') + i \left[ \text{Ai}(z'') \text{Bi}(z') + \text{Bi}(z'') \text{Bi}(z') \right],$$

(15)

$$\Psi = 3 \text{Ai}(z'') \text{Ai}(z') + \text{Bi}(z'') \text{Bi}(z'),$$

(16)

where the overall factor is neglected. The wave function (13) has the same form of the Lorentzian one in Ref.[7]. In this case, the $\alpha$’s area is constrained on a pure
imaginary axis in our method. Therefore, up to an overall factor, the path integral form of the (13)'s is the same with the Lorentzian one. We can show that the wave function becomes the (13)'s form without using the Wheeler-DeWitt equation, by following the inverse of the Halliwell’s calculation in Ref.[7] appropriately. This fact gives us the insight that the singularities in (8) or (9) is caused for the undesirable integration. We can eliminate singuralities if we consider the products of Airy functions. This is why we take \( z' = z'' = z \). On the contrary to the (13)'s case, the path integral form of the (14)'s, (15)'s and (16)'s cases can not be calculated directly. Therefore we use above insight and we can calculate the wave function for other three cases, those are (14), (15) and (16). In our regularization method, we can follow this insight easily because we have only turned up the straight contour line by using the complex parameter \( \alpha \). Of course we must take care of a branch cut caused by the \( T^{-1/2} \) term in the complex \( T \) plane in (12). We consider both cases that the contour passes through the cut or not. From this reason, we get six results with three \( \alpha \)'s areas. On the other hand, in the contour changing method, the contour is a complicated curve. It is very difficult to perform this insight beyond a saddle point approximation and to distinguish whether the contour passes through the cut or not. Therefore the results in Ref.[8] are the part of our results.

Of course we can take another strategy to remove a divergence problem. If we extend the integrated function as

\[
\Psi(\alpha) = \int_{\beta}^{\infty} \frac{dT}{T^{1/2}} e^{-I(T)} + \int_{-\infty}^{\beta} \frac{dT}{(\alpha T)^{1/2}} e^{-I(\alpha T)},
\]

where \( \beta \) is some constant, then \( \Psi \) also converges. In this representation, we also fix \( \alpha \) as \( \text{Re}[(\alpha T)^3] \) is positive for \( |T| \gg 1 \). After the integration, we take the analytic continuation \( \alpha \to 1 \). We also calculate the wave function using this regularization and the results are almost the same as the ones by the previous strategy. However we do not express here.

Beyond a homogeneous and isotropic model, we next analyze the Bianch type
I minisuperspace model\textsuperscript{[10]}. The metric of this type can be expressed as
\begin{equation}
    ds^2 = \sigma^2 \left\{ \frac{N^2(\tau)}{a^2(\tau)} \, d\tau^2 + a^2(\tau) \, dx^2 + b^2(\tau) \, dy^2 + c^2(\tau) \, dz^2 \right\}.
\end{equation}

The action of this type is
\begin{equation}
    I(a, b, c) = \frac{1}{2} \int_{\tau}^{\tau''} \, d\tau \left[ -\frac{a}{N} \dot{a} \dot{b} \dot{c} + a \dot{b} \dot{c} + a \dot{b} \dot{c} \right] + Nbc\lambda.
\end{equation}

In order to deal with the path integral more easily, we change variables of \(a, b, c\) as \(x, y, z\) where \(x \equiv (bc + a^2)/2\), \(y \equiv (bc - a^2)\), \(z^2 \equiv a^2 \dot{b} \dot{c}\). The action and the wave function are
\begin{align}
    I &= \int_{\tau}^{\tau''} \, d\tau (p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - NH), \\
    H &= -p_x^2 + p_y^2 - \frac{1}{2} p_z^2 - \frac{\lambda}{2} (x + y),
\end{align}

\begin{equation}
    \Psi = \int dN \int Dp_x Dp_y Dp_z DxDyDze^{-I} \equiv \int dN \psi(N).
\end{equation}

The integral with respect to \(x, y, z\) is obtained by means of dividing the lapse interval \(N\) into pieces. Then \(\Psi\) becomes
\begin{equation}
    \Psi = \int dN \frac{dN}{N^{3/2}} e^{-I_0},
\end{equation}

where
\begin{equation}
    I_0 = -\frac{1}{4N} \left\{ (x'' - x')^2 - (y'' - y')^2 + 2(z'' - z')^2 \right\} + \frac{\lambda N}{4} \left\{ (x' + x'') + (y' + y'') \right\}
    \equiv XN + \frac{Y}{N}.
\end{equation}

In this case, the \(N\) integration in (22) also diverges, if we take the integration range of the \(N\) from \(-\infty\) to \(+\infty\). Therefore we also use a new regularization method and perform the \(N\) integration. We define again an analytic function by introducing
a complex parameter $\alpha$. After the integration, we take the analytic continuation $\alpha \to 1$. On the contrary to a homogeneous and isotropic case, we can calculate the $N$ integration explicitly and the result becomes (up to a normalization factor)

$$\Psi = \sqrt{\frac{\pi}{Y}} e^{-2\sqrt{XY}}. \quad (24)$$

In Ref.[10], Halliwell and Louko considered the particular Bianch type I model in which $b(\tau) = c(\tau)$. Our result may be a part of the results of the contour changing method. This reason is that we have treated the integration range of the lapse function $N$ from $-\infty$ to $+\infty$ and we have only turned up the straight contour line again. By considering a branch cut, we also get a solution that is zero. However we do not express above.

We finally summarize our results. In this paper we have proposed a new regularization method by concentrating the integration of the lapse function and evaluated the wave function of the Friedmann-Robertson-Walker model along this regularization method. Our results are more general because in our method we can deal with the branch cut explicitly. We have then evaluated the wave function of the Bianch type I model. In this case our result is more limited because we treat the integration range of the lapse function from $-\infty$ to $+\infty$. While we only consider above two simple models, this regularization method will work well in the more complicated type or the matter coupled case.

There remain further works with respect to the interpretation of the wave function by this regularization method. Here we compare our method to the contour changing method. One of the most difficult points in the interpretation of the contour changing method is that its form is neither the Euclidean nor the Lorentzian path integral $^{[3,4,7,11]}$. The wave function following this method can’t, in general, represent the transition from a purely real Euclidean metric to a purely real Lorentzian one. This reason is as follows. The contour of this path integral is defined as the steepest-descent path with respect to the complex lapse function. Thus the contour is a complicated curve. In our method on the other hand, by
concentrating the lapse function we remain the real Euclidean form and evaluate the wave function in a way which is free from a divergence problem. The contour of our path integral is defined as the turned up straight line by using the complex parameter $\alpha$ in an analytic function. The contour therefore is a simple line. In our method, we can calculate the wave function of the universe without using the saddle points and the steepest-decsent path. We do not spoil above picture and can deal with the real tunneling and get the wave function of the real Euclidean path integral more easily.

The evaluation of the wave function of the universe is very difficult. The physical interpretation of the regularization method is not clear. We consider there exist some possibilities to obtain a finite wave function along the Euclidean path integral regime, and our new regularization method is nothing but one of them.

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