ON THE PERIODIC APPROXIMATION OF LYAPUNOV EXPONENTS FOR SEMI-INVERTIBLE COCYCLES

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Abstract. We prove that, for semi-invertible linear cocycles, Lyapunov exponents of ergodic measures may be approximated by Lyapunov exponents on periodic points.

1. Introduction. A very general and also vague idea that appears in the study of dynamical systems is that “if a system exhibits enough hyperbolicity then most of its dynamical interesting information is concentrated in its periodic orbits”. There are many examples supporting this idea. For instance, it is known that cohomology classes of Hölder cocycles over hyperbolic systems are characterized by its information on periodic points (see for instance [14, 15, 10, 7, 1, 17, 2, 13] and references therein), equilibrium states associated to different potentials coincide whenever those potentials have the same information on periodic points [4] and so on.

In the present work we also present an example supporting the previous “belief” in the context of Lyapunov exponents of semi-invertible linear cocycles. More precisely, we show that (see Section 2 for precise definitions and statements)

Theorem 1.1. Lyapunov exponents of ergodic measures may be approximated by Lyapunov exponents on periodic points.

In other words, all the information carried by the Lyapunov exponents of \((f, A)\) is indeed concentrated on periodic points.

The objects involved in our example are very classical in the fields of Dynamical Systems and Ergodic Theory and can be defined as follows: given an invertible ergodic measure preserving dynamical system \(f : M \to M\) defined on a measure space \((M, A, \mu)\) and a measurable matrix-valued map \(A : M \to M(d, \mathbb{R})\), the pair \((f, A)\) is called a semi-invertible linear cocycle (or just linear cocycle for short). Sometimes one calls linear cocycle (over \(f\) generated by \(A\)), instead, the sequence \(\{A^n\}_{n \in \mathbb{N}}\) defined by

\[
A^n(x) = \begin{cases} 
A(f^{n-1}(x)) \ldots A(f(x))A(x) & \text{if } n > 0 \\
Id & \text{if } n = 0
\end{cases}
\]
for all \( x \in M \). The word ‘semi-invertible’ refers to the fact that the action of the underlying dynamical system \( f \) is invertible while the action on the fibers given by \( A \) may fail to be invertible. We refer to the Introduction of [8] for some interesting applications of semi-invertible cocycles.

Under certain integrability conditions, it was proved in [9] that there exists a full \( \mu \)-measure set \( R^\mu \subset M \), whose points are called \( \mu \)-regular points, such that for every \( x \in R^\mu \) there exist numbers \( \lambda_1 > \ldots > \lambda_l \geq -\infty \), called Lyapunov exponents, and a direct sum decomposition \( \mathbb{R}^d = E^{1,A}_x \oplus \ldots \oplus E^{l,A}_x \) into vector subspaces which are called Oseledets subspaces and depend measurable on \( x \) such that, for every \( 1 \leq i \leq l \),

- \( \dim(E_i^{i,A}_x) \) is constant,
- \( A(x)E_i^{i,A}_x \subseteq E_i^{i,A}_{f(x)} \) with equality when \( \lambda_i > -\infty \)

and

- \( \lambda_i = \lim_{n \to +\infty} \frac{1}{n} \log \| A^n(x) \| \) for every non-zero \( v \in E_i^{i,A}_x \).

This result extends a famous theorem due to Oseledets [10] known as the multiplicative ergodic theorem which was originally stated in both, invertible (both \( f \) and the matrices are assumed to be invertible) and non-invertible (neither \( f \) nor the matrices are assumed to be invertible) settings (see also [18]). While in the invertible case the conclusion is similar to the conclusion above (except that all Lyapunov exponents are finite), in the non-invertible case, instead of a direct sum decomposition into invariant vector subspaces, one only get an invariant filtration (a sequence of nested subspaces) of \( \mathbb{R}^d \).

In the invertible setting, Theorem 1.1 was already gotten by Kalinin in [10] extending a theorem of Wang and Sun [19] on the approximation of Lyapunov exponents of hyperbolic invariant measures for diffeomorphisms. In fact, the proof of our main result is based on ideas from those works. The lack of invertibility of the matrices, however, brings in some additional difficulties. To deal with it, we introduce the notion of Lyapunov norm for semi-invertible cocycles and present some useful properties about these objects.

It is also worth noticing that a similar approximation result was gotten by Dai [6] in the case when just the matrices are assumed to be invertible. More recently, Kalinin and Sadovskaya [11] addressed a similar problem in the invertible setting but when the cocycle takes values in the set of invertible operators of a Banach space. In such setting, Theorem 1.1 can not be fully recovered.

2. Statements. Let \((M,d)\) be a compact metric space, \( \mu \) a measure defined on the Borel sets of \((M,d)\) and \( f : M \to M \) a measure preserving homeomorphism. Assume also that \( \mu \) is ergodic.

We say that \( f \) satisfies the Anosov Closing property if there exist \( C_1, \varepsilon_0, \theta > 0 \) such that if \( z \in M \) satisfies \( d(f^n(z), z) < \varepsilon_0 \) then there exists a periodic point \( p \in M \) such that \( f^n(p) = p \) and

\[
d(f^j(z), f^j(p)) \leq C_1 e^{-\theta \min\{j,n-j\}} d(f^n(z), z)
\]

for every \( j = 0, 1, \ldots, n \). Notice that shifts of finite type, basic pieces of Axiom A diffeomorphisms and more generally, hyperbolic homeomorphisms are particular examples of maps satisfying the Anosov Closing property. See for instance, [12] p.269, Corollary 6.4.17.
Given a continuous map \( A : M \to M(d, \mathbb{R}) \) such that \( \int \log^+ \| A(x) \| d\mu(x) < \infty \), let us denote by

\[
\lambda_1(A, \mu) > \lambda_2(A, \mu) > \cdots > \lambda_t(A, \mu) \geq -\infty
\]

the Lyapunov exponents of the cocycle \((f, A)\) with respect to the measure \( \mu \) and by

\[
\gamma_1(A, \mu) \geq \gamma_2(A, \mu) \geq \cdots \geq \gamma_d(A, \mu)
\]

the Lyapunov exponents of \((f, A)\) with respect to \( \mu \) counted with multiplicities.

Given a periodic point \( p \), we denote its Lyapunov exponents and Lyapunov exponents counted with multiplicities by \( \{\lambda_i(A, p)\}_{i=1}^d \) and \( \{\gamma_i(A, p)\}_{i=1}^d \), respectively. When there is no risk of ambiguity, we suppress the index \( A \) or even both \( A \) and \( \mu \) from the previous objects.

In what follows we are also going to assume that \( A : M \to M(d, \mathbb{R}) \) is an \( \alpha \)-Hölder continuous map. This means that there exists a constant \( C_2 > 0 \) such that

\[
\| A(x) - A(y) \| \leq C_2 d(x, y)\alpha
\]

for all \( x, y \in M \) where \( \| A \| \) denotes the operator norm of a matrix \( A \), that is, \( \| A \| = \sup \{ \| Av \| / \| v \| : \| v \| \neq 0 \} \).

2.1. Main results. The main result of this work is the following one

**Theorem 2.1.** Let \( f : M \to M \) be a homeomorphism satisfying the Anosov Closing property, \( \mu \) an ergodic \( f \)-invariant probability measure and \( A : M \to M(d, \mathbb{R}) \) an \( \alpha \)-Hölder continuous map. Then, there exists a sequence of periodic points \((p_k)_{k \in \mathbb{N}}\) such that

\[
\gamma_i(A, p_k) \xrightarrow{k \to +\infty} \gamma_i(A, \mu)
\]

for every \( i = 1, \ldots, d \).

As a simple consequence of our main result we get the following corollary. In order to state it, let us assume that \( f \) and \( A \) satisfy the hypotheses of Theorem 2.1.

**Corollary 1.** If all Lyapunov exponents of \((f, A)\) on periodic points are uniformly bounded by below then \( A(x) \in GL(d, \mathbb{R}) \) for almost every \( x \in M \) with respect to every \( f \)-invariant probability measure \( \mu \).

**Proof.** If there exist a measure \( \mu \), which by the the Ergodic Decomposition Theorem may be assumed to be ergodic, and a set \( B \subset M \) with positive \( \mu \)-measure such that \( A(x) \notin GL(d, \mathbb{R}) \) for every \( x \in B \) then \( \gamma_d(A, \mu) = -\infty \) which in light of Theorem 2.1 contradicts our assumption.

We observe that satisfying \( A(x) \in GL(d, \mathbb{R}) \) for almost every \( x \in M \) with respect to every \( f \)-invariant probability measure as in the previous corollary does not imply, in general, that \( A(x) \in GL(d, \mathbb{R}) \) for every \( x \in M \). Indeed,

**Example 2.2.** Let \( M = \{0, 1\}^\mathbb{Z} \) be the space of bilateral sequences in zeros and ones and \( f : M \to M \) be the left shift \( f((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}} \). Given \( \theta \in (0, 1) \) we endow \( M \) with the distance

\[
d(x, y) = \theta^{N(x, y)} \text{, where } N(x, y) = \max\{N \geq 0; x_n = y_n \text{ for all } |n| < N\}
\]

for \( x = (x_i)_{i \in \mathbb{Z}} \) and \( y = (y_i)_{i \in \mathbb{Z}} \). A very well known fact is that \((M, d)\) is a compact metric space and \( f \) is a homeomorphism satisfying the Anosov Closing property.
Let $q = (q_i)_{i \in \mathbb{Z}} \in M$ be such that $q_i = 1$ for every $i \neq -1$ and $q_{-1} = 0$ and fix $a > 1$ so that $a\theta > 1$. Consider $A : M \to \mathbb{R}$ given by

$$A(x) = \begin{cases} \frac{a \theta}{a} d(x, q) & \text{if } d(x, q) \leq \theta^3 \\ a & \text{if } d(x, q) > \theta^3. \end{cases}$$

Then, $A$ is a Hölder map and generates a semi-invertible cocycle over $f$. Moreover, $A(q) = 0$ and $(f, A)$ has all Lyapunov exponents on periodic points uniformly bounded by below by $\log(a\theta) > 0$. In fact, if a periodic point $p \in M$ is such that $d(f^j(p), q) > \theta^3$ for every $j \in \mathbb{N}$ then obviously $\lambda_1(p) = \log a > 0$. Now, suppose there exists $j \in \mathbb{N}$ so that $d(f^j(p), q) \leq \theta^3$. We may assume without loss of generality that $j = 0$. More precisely, suppose $p = (p_i)_{i \in \mathbb{Z}}$ satisfies $p_i = q_i$ for every $|i| \leq n$ and $p_i \neq q_i$ for some $i$ so that $|i| = n + 1$ with $n \geq 2$. Then, $A(p) = \frac{a \theta}{a} \theta^{n+1}$ and $A(f^j(p)) = \frac{a}{a}$, for, at least, $j = 1, 2, \ldots, n + 1$. In particular, $A^{n+2}(p) = a^{n+2} \theta^{n-2}$ and $\frac{1}{n+2} \log \|A^{n+2}(p)\| \geq \log(a\theta) > 0$. In other words, whenever a periodic point $p$ is $\theta^{n+1}$ close to $q$, its next $n + 1$ iterations are going to be out of the ball of radius $\theta^3$ and centered at $q$. Repeating this argument we can construct a sequence $(n_k)_{k \in \mathbb{N}}$ going to $+\infty$ such that

$$\frac{1}{n_k} \log \|A^{n_k}(p)\| \geq \log(a\theta) > 0$$

for every $k \in \mathbb{N}$. In particular, $\lambda_1(p) \geq \log(a\theta) > 0$ as claimed. Consequently, by Corollary \ref{corollary-invariance} $A(x) \in GL(1, \mathbb{R})$ for almost every $x \in M$ with respect to every $f$-invariant probability measure and $A(q) \notin GL(1, \mathbb{R})$. Observe that, besides the choice of $A$, the main feature underlying our construction is that $q \in W^s(p) \cap W^u(p)$ where $p$ is the fixed point $p = (p_i)_{i \in \mathbb{Z}}$ such that $p_i = 1$ for every $i \in \mathbb{Z}$. Another simple remark is that such example can be constructed in any dimension. We take $A : M \to M(d, \mathbb{R})$ to be such that $A(x)$ is a diagonal matrix for every $x \in M$ where one of its entries is the map constructed above while the others are all constant.

The previous example also reveals very different behaviors of invertible and semi-invertible cocycles. Indeed, it was proved by Cao in \cite{Cao2008} that if an invertible cocycle $A$ defined over a homeomorphism $f$ has only positive Lyapunov exponents with respect to every $f$-invariant probability measure then it is uniformly expanding. The previous example shows us that this is no longer true in the semi-invertible context.

3. Lyapunov norm. In order to estimate the growth of the cocycle $A$ along an orbit we introduce the notion of Lyapunov norm for semi-invertible cocycles. This is based on a similar notion for invertible cocycles (see for instance \cite{HasselblattKatok}).

Let $x \in \mathcal{R}^\mu$ be a regular point and $\mathbb{R}^d = E_{x}^{1,A} \oplus \ldots \oplus E_{x}^{l,A}$ be the Oseledets decomposition at $x$. Given $i \in \{1, \ldots, l-1\}$ and $n \in \mathbb{N}$, let us consider the map

$$A^n(f^{-n}(x))|_{E_{f^{-n}(x)}^{i,A}} : E_{f^{-n}(x)}^{i,A} \to E_{x}^{i,A}$$

which is invertible and let us denote its inverse by $(A^n(f^{-n}(x)))^{-1}$. Now, for every $n \in \mathbb{Z}$ and $x \in \mathcal{R}^\mu$ let us consider the linear map $A^n_i(x) : E_{x}^{i,A} \to E_{f^n(x)}^{i,A}$ given by

$$A^n_i(x)u = \begin{cases} A^n(x)|_{E_{x}^{i,A}}u & \text{if } n \geq 0 \\ (A^{-n}(f^n(x)))^{-1}_i u & \text{if } n < 0. \end{cases}$$

Observe that, for every $m, n \in \mathbb{Z}$,

$$A^{m+n}_i(x) = A^m_i(f^m(x))A^n_i(x).$$
Indeed, for \( m, n \geq 0 \) it follows readily from the definition and (1). Suppose now \( m, n > 0 \) and let us prove that \( A_i^{-m-n}(x) = A_i^{-n}(f^{-m}(x))A_i^{-m}(x) \). We start observing that, since

\[
(A^{m+n}(f^{-m-n}(x)))|_{E_i^{i,A}} = (A^m(f^{-m}(x)))A^n(f^{-m-n}(x)))|_{E_i^{i,A}}
\]

it follows by the invariance of the Oseledets spaces that

\[
(A^{m+n}(f^{-m-n}(x)))|_{E_i^{i,A}} = A^m(f^{-m}(x))|_{E_i^{i,A}} A^n(f^{-m-n}(x))|_{E_i^{i,A}}.
\]

Thus, taking the inverses on both sides the result follows. The case when \( m \) and \( n \) have different signs may be deduced by combining the previous two.

In order to define the Lyapunov norm associated to the cocycle \( A \) at a regular point \( x \in \mathcal{R}^n \), we start by defining the Lyapunov inner product: given \( \delta > 0 \) and two vectors \( u = u_1 + \ldots + u_l \) and \( v = v_1 + \ldots + v_l \) in \( \mathbb{R}^d \) where \( u_i, v_i \in E_x^{i,A} \) for every \( 1 \leq i \leq l \), the \( \delta \)-Lyapunov inner product of \( u \) and \( v \) is defined by

\[
\langle u, v \rangle_{x,\delta} = \sum_{i=1}^l \langle u_i, v_i \rangle_{x,\delta,i}
\]

where

\[
\langle u_i, v_i \rangle_{x,\delta,i} = \sum_{n \in \mathbb{Z}} \langle A_i^n(x)u_i, A_i^n(x)v_i \rangle e^{-2\lambda_i n-2\delta|u|}
\]

for every \( i \) for which \( \lambda_i \) is finite and

\[
\langle u_i, v_i \rangle_{x,\delta,i} = \sum_{n=0}^{+\infty} \langle A_i^n(x)u_i, A_i^n(x)v_i \rangle e^{\frac{\delta}{n}}
\]

in the case when \( \lambda_i = -\infty \). Observe that both series (3) and (4) converge for any \( x \in \mathcal{R}^n \). Indeed, convergence of the second one is easily to verify while the convergence of the first one follows from the next lemma whose proof is also going to be used in the sequel.

**Lemma 3.1.** For every \( u \in E_x^{i,A} \setminus \{0\} \),

\[
\lim_{n \to +\infty} \frac{1}{n} \log \| A_i^n(x)u \| = \lambda_i.
\]

**Proof.** The fact that \( \lim_{n \to +\infty} \frac{1}{n} \log \| A_i^n(x)u \| = \lambda_i \) follows from the definition and Oseledets’ Theorem. Let us prove that \( \lim_{n \to +\infty} \frac{1}{n} \log \| A_i^n(x)u \| = \lambda_i \). Given \( \varepsilon > 0 \), it follows from Theorem 2 of [8] that there exists a measurable map \( C : M \to (0, \infty) \) such that

\[
\frac{1}{C(f^{-n}(x))} e^{(\lambda_i-\varepsilon)n} \| v \| \leq \| A_i^n(f^{-n}(x))v \| \leq C(f^{-n}(x)) e^{(\lambda_i+\varepsilon)n} \| v \|
\]

for every \( n \in \mathbb{N} \) and \( v \in E_f^{i,A} \setminus \{0\} \) and

\[
C(f^n(x)) \leq C(x) e^{\varepsilon|n|}
\]

for any \( n \in \mathbb{Z} \). Combining this two inequalities we get that

\[
\frac{1}{C(x)} e^{(\lambda_i-2\varepsilon)n} \| v \| \leq \| A_i^n(f^{-n}(x))v \| \leq C(x) e^{(\lambda_i+2\varepsilon)n} \| v \|
\]

for every \( n \in \mathbb{N} \) and \( v \in E_f^{i,A} \setminus \{0\} \).
Let $u \in E^{i,A}_x \setminus \{0\}$ and $n \in \mathbb{N}$. Then, applying the previous inequality to $v = A_i^{-n}(x)u$ we get that
\[
\frac{1}{C(x)} e^{(\lambda_i - 2\varepsilon)n} \| A_i^{-n}(x)u \| \leq \| u \| \leq C(x) e^{(\lambda_i + 2\varepsilon)n} \| A_i^{-n}(x)u \|.
\]
Taking the logarithm at each term, dividing by $-n$ and making $n \to +\infty$ it follows that
\[\lambda_i - 2\varepsilon \leq \lim_{n \to +\infty} \frac{1}{-n} \log \| A_i^{-n}(x)u \| \leq \lambda_i + 2\varepsilon.\]
Now, since $\varepsilon > 0$ is arbitrary the lemma follows.

In particular, it follows that $\langle \cdot, \cdot \rangle_{x,\delta}$ is actually an inner product in $\mathbb{R}^d$. We then define the $\delta$-Lyapunov norm $\| \cdot \|_{x,\delta}$ associated to the cocycle $A$ at $x \in \mathcal{R}^n$ as the norm generated by $\langle \cdot, \cdot \rangle_{x,\delta}$. When there is no risk of ambiguity, we just write $\| \cdot \|_x$ and $\| \cdot \|_{x,i}$ instead of $\| \cdot \|_{x,\delta}$ and $\| \cdot \|_{x,\delta,i}$ and call it just Lyapunov norm.

Given a linear map $B \in M(d, \mathbb{R})$, its Lyapunov norm is defined for any regular points $x, y \in \mathcal{R}^n$ by
\[
\| B \|_{y \leftarrow x} = \sup \{ \| Bu \|/\| u \|_x; \ u \in \mathbb{R}^d \setminus \{0\} \}.
\]

The next proposition gives us some useful properties of the Lyapunov norm that we are going to use in the sequel. In the case when the cocycle is invertible, similar properties of the Lyapunov norm are very well known. We now prove them in the semi-invertible setting.

**Proposition 1.** Let $x \in \mathcal{R}^n$.

i) For every $1 \leq i < l$ and $u \in E^{i,A}_x$ we have
\[
e^{(\lambda_i - \delta)n} \| u \|_{x,i} \leq \| A_i^n(x)u \|_{f^n(x),i} \leq e^{(\lambda_i + \delta)n} \| u \|_{x,i}
\]
for every $n \in \mathbb{N}$;

ii) If $\lambda_i = -\infty$ then for every $u \in E^{l,A}_x$ and $n \in \mathbb{N}$ we have
\[
\| A_i^n(x)u \|_{f^n(x),i} \leq e^{-\frac{n}{2\delta}} \| u \|_{x,i};
\]

iii) For $\delta > 0$ such that $-\frac{1}{\delta} < \lambda_1$ we have
\[
\| A_i^n(x) \|_{f^n(x)\leftarrow x} \leq e^{(\lambda_1 + \delta)n}
\]
for every $n \in \mathbb{N}$;

iv) For every $\delta > 0$ such that $-\frac{1}{\delta} < \lambda_{l-1}$, there exists a measurable function $K_\delta : \mathcal{R}^n \to (0, +\infty)$ such that
\[
\| u \| \leq \| u \|_x \leq K_\delta(x)\| u \|
\]
whose growth along any regular orbit is bounded; more precisely,
\[
K_\delta(x)e^{-\delta n} \leq K_\delta(f^n(x)) \leq K_\delta(x)e^{\delta n} \quad \forall n \in \mathbb{N}.
\]
Consequently, for any linear map $B$ and any regular points $x$ and $y$
\[
K_\delta(x)^{-1}\| B \| \leq \| B \|_{y \leftarrow x} \leq K_\delta(y)\| B \|.
\]
Proof. In order to prove i) we observe that for any \( u \in E_{x}^{i,A} \),
\[
\|A(x)u\|_{f(x)}^{2,i} = \sum_{n \in \mathbb{Z}} \|A_{n}^{i}(f(x))A(x)u\|^{2} e^{-2\lambda_{n}n-2\delta|n|}
\]
\[
= \sum_{n \in \mathbb{Z}} \|A_{n}^{i+1}(x)u\|^{2} e^{-2\lambda_{n}n-2\delta|n|}
\]
\[
= \sum_{n \in \mathbb{Z}} \|A_{n}^{i+1}(x)u\|^{2} e^{-2\lambda_{n+1}n-2\delta|n+1|} e^{2\lambda_{n+1}+2\delta(|n+1|-|n|+1)}.
\]
Consequently,
\[
e^{(\lambda_i-\delta)} \|u\|_{x,i} \leq \|A(x)u\|_{f(x),i} \leq e^{(\lambda_i+\delta)} \|u\|_{x,i}
\]
which implies i). Item ii) is analogous. Indeed, we have that
\[
\|A^{n}(x)u\|_{f^{n}(x),t}^{2} = \sum_{k=0}^{\infty} \|A^{k}(f^{n}(x))A^{n}(x)u\|^{2} e^{2k}
\]
\[
= \sum_{k=0}^{\infty} \|A^{k+n}(x)u\|^{2} e^{2(k+n)} e^{-\frac{\delta}{2}n} \leq e^{-\frac{\delta}{2}n} \|u\|_{x,t}^{2},
\]
for every \( u \in E_{x}^{i,A} \).

In order to get iii) one only have to observe that, for any \( u \in \mathbb{R}^{d} \),
\[
\|A^{n}(x)u\|_{f^{n}(x),i}^{2} = \sum_{i=1}^{l} \|A_{n}^{i}(x)u_{i}\|_{f^{n}(x),i}^{2}
\]
\[
\leq \sum_{i=1}^{l} e^{2(\lambda_{i}+\delta)n} \|u_{i}\|_{x,i}^{2} + e^{-\frac{\delta}{4}n} \|u_{i}\|_{x,i}^{2}
\]
\[
\leq e^{2(\lambda_{i}+\delta)n} \sum_{i=1}^{l} \|u_{i}\|_{x,i}^{2} = e^{2(\lambda_{i}+\delta)n} \|u\|_{x,i}^{2}.
\]

The first inequality in iv) is trivial. To prove the second one, we proceed analogously to what we did in Lemma 3.1. Fix \( i \in \{1, \ldots, l-1\} \) and \( \varepsilon > 0 \) such that \( 2\varepsilon < \delta \). From Theorem 2 of [8] it follows that there exists a measurable map \( C : M \to (0, \infty) \) such that
\[
\frac{1}{C(x)} e^{(\lambda_i-\varepsilon)n} \|u\| \leq \|A_{i}^{n}(x)u\| \leq C(x) e^{(\lambda_i+\varepsilon)n} \|u\|
\]
for every \( n \in \mathbb{N} \) and \( u \in E_{x}^{i,A} \). Equation (5) from the proof of Lemma 3.1 tells us that
\[
\frac{1}{C(x)} e^{-(\lambda_i+2\varepsilon)n} \|u\| \leq \|A_{i}^{-n}(x)u\| \leq C(x) e^{-(\lambda_i-2\varepsilon)n} \|u\|
\]
for every \( n \in \mathbb{N} \). Combining this two equations we get that
\[
\|u\|_{x,i}^{2} = \sum_{n \in \mathbb{Z}} \|A_{n}^{i}(x)u\|^{2} e^{-2\lambda_{n}n-2\delta|n|}
\]
\[
\leq \sum_{n \in \mathbb{Z}} \left( C(x) e^{\lambda_{i}n+2\varepsilon|n|} \|u\| \right)^{2} e^{-2\lambda_{n}n-2\delta|n|}
\]
\[
= C(x)^{2} \sum_{n \in \mathbb{Z}} e^{(4\varepsilon-2\delta)n} \|u\|^{2}.
\]
For \( u \in E^L_x \), Theorem 2 of [8] tells us that whenever \(-\frac{1}{\lambda} < \lambda_{l-1}\),
\[
\| A^n(x)u \| \leq C(x)e^{-\frac{1}{\lambda}n} \| u \|.
\]
Thus,
\[
\| u \|_{x,l}^2 = \sum_{n \geq 0} \| A^n(x)u \|^2 e^{-2\frac{1}{\lambda}n} \leq C(x)^2 \sum_{n \geq 0} e^{-\frac{1}{\lambda}n} \| u \|^2.
\]

Thus, taking \( K = \max\{\sum_{n \in \mathbb{Z}} e^{(4\pi-2\delta)n}, \sum_{n \geq 0} e^{-\frac{1}{\lambda}n}\} \) and writing \( u \in \mathbb{R}^d \) as \( u = u_1 + \ldots + u_l \) where \( u_i \in E^L_x \) for every \( 1 \leq i \leq l \) we get that
\[
\| u \|_{x}^2 = \sum_{i=1}^{l} \| u_i \|_{x,i}^2 \leq KC^2(x) \sum_{i=1}^{l} \| u_i \|^2.
\]

It remains to obtain an upper bound for \( \| u_i \|_{x}^2 \) in terms of \( \| u \| \). This can be achieved by using the map \( K \) given by Theorem 2 of [8]. More precisely, let \( K^1 \) be the map given by [8, Theorem 2] applied for \( i = 1 \) and sufficiently small \( \epsilon > 0 \). We then have that
\[
\| u_1 \| \leq K^1(x)\| u \| \quad \text{and} \quad \| u_2 + \ldots + u_l \| \leq K^1(x)\| u \|.
\] (11)

The first inequality in (11) gives a desired bound for \( \| u_1 \| \). In order to obtain the bound for \( \| u_2 \| \), we can apply again [8, Theorem 2] but now for \( i = 2 \) (and again for \( \epsilon > 0 \) sufficiently small) to conclude that there exists \( K^2 \) such that
\[
\| u_2 \| \leq K^2(x)\| u_2 + \ldots + u_l \| \quad \text{and} \quad \| u_1 + \ldots + u_l \| \leq K^2(x)\| u_2 + \ldots + u_l \|.
\] (12)

By combining the second inequality in (11) with the first inequality in (12), we conclude that \( \| u_2 \| \leq K^1(x)K^2(x)\| u \| \). By proceeding, one can establish desired bounds for all \( \| u_j \|, \ j = 1, \ldots, l \) and construct a function \( K_{\delta} \) satisfying (9). Indeed, this follows from the fact that \( C(f^n(x)) \leq C(x)e^{\epsilon \| n \|} \) for every \( n \in \mathbb{Z} \) and similarly for the maps \( K^1, K^2, \ldots, K^l \) completing the proof.

For any \( N > 0 \), let \( \mathcal{R}^\mu_{\delta,N} \) be the set of regular points \( x \in \mathcal{R}^\mu \) for which \( K_{\delta}(x) \leq N \). Observe that \( \mu(\mathcal{R}^\mu_{\delta,N}) \to 1 \) as \( N \to +\infty \). Moreover, invoking Lusin’s theorem we may assume without loss of generality that this set is compact and that the Lyapunov norm and the Oseledets splitting are continuous when restricted to it.

As a final and simple remark about Lyapunov norms we observe that, in order to get a norm satisfying the properties given by the previous proposition, it is not necessary to use inner products. Indeed, for \( x \in \mathcal{R}^\mu, \delta > 0 \) and \( u = u_1 + \ldots + u_l \in \mathbb{R}^d \) where \( u_i \in E^L_x \) for every \( 1 \leq i \leq l \), defining
\[
\| u \|_{x,\delta} = \sum_{i=1}^{l} \| u_i \|_{x,\delta,i}
\]
where
\[
\| u_i \|_{x,\delta,i} = \sum_{n \in \mathbb{Z}} \| A^n_i(x)u_i \| e^{-\lambda_i n - \delta |n|}
\]
for every \( i \) for which \( \lambda_i \) is finite and
\[
\| u_l \|_{x,\delta,l} = \sum_{n=0}^{+\infty} \| A^n(x)u_l \| e^{\frac{1}{\lambda}n}
\]
in the case when \( \lambda_l = -\infty \) gives rise to such a norm. In particular, this can be used to define Lyapunov norms for semi-invertible cocycles taking values on Banach spaces.
4. Proof of the main result.

4.1. Approximation of the largest Lyapunov exponent. We start the proof of
Theorem 2.1 with a key proposition which tells us that the largest Lyapunov
exponent of \( A \) may be approximated by Lyapunov exponents on periodic points.
We retain all the notation introduced at the previous section.

**Proposition 2.** For every \( \delta > 0 \) small enough, there exists a periodic point \( p \in M \)
such that
\[
| \lambda_1 - \lambda_1(p) | < \delta. \tag{13}
\]
More precisely, there exists \( \delta_0 > 0 \) such that for any \( N > 0 \) and \( \delta \in (0, \delta_0) \), there
exist \( n_0 \in \mathbb{N} \) and \( \rho > 0 \) such that, for every \( n \geq n_0 \), if \( x, f^n(x) \in \mathcal{R}_{\varepsilon,N}^{\varepsilon} \) are such
that \( d(x, f^n(x)) < \rho \) and \( p \) is a periodic point associated to \( x \) by the Anosov
Closing property then
\[
| \lambda_1 - \lambda_1(p) | < \delta. \tag{13}
\]

Let \( C_1, \varepsilon_0, \theta > 0 \) be given by the Anosov Closing Property, \( \rho \in (0, \varepsilon_0) \) and
suppose \( d(x, f^n(x)) < \rho \). Thus, there exists a periodic point \( p \in M \) of period \( n \)
such that
\[
d(f^j(x), f^j(p)) \leq C_1 e^{-\min(\theta, \lambda_1)} d(f^n(x), x) \leq C_1 \rho e^{-\theta \min(\lambda_1, n-j)}
\]
for every \( j = 0, 1, \ldots, n \). We will prove that, as long as \( n \) is sufficiently large,
this periodic point satisfies the previous proposition. We split the proof into two
lemmas. In the first one we give a lower bound for \( \lambda_1(p) \) in terms of \( \lambda_1 \) while in the
second one an upper bound is given.

Consider \( 0 < \delta_0 < \frac{1}{4} \min(\theta, \lambda_1 - \lambda_2) \) if \( A \) has at least two different Lyapunov
exponents and moreover \( \lambda_2 > -\infty \) and let \( 0 < \delta_0 < \frac{1}{4} \theta \alpha \) otherwise. Assume also
that \( \delta_0 \) is small enough so that \( -\frac{1}{\delta_0} < \lambda_{l-1} \). Fix \( N > 0 \) and \( \delta \in (0, \delta_0) \) and suppose
\( x, f^n(x) \in \mathcal{R}_{\varepsilon,N}^{\varepsilon} \).

**Lemma 4.1 (Lower bound).** There exists \( n_0 \in \mathbb{N} \) such that, if \( \rho \in (0, \varepsilon_0) \) is suffi-
ciently small and \( n \geq n_0 \) then
\[
\lambda_1(p) \geq \lambda_1 - \delta. \tag{14}
\]

**Proof.** For each \( 1 \leq j \leq n \), let us consider the splitting \( \mathbb{R}^d = E_{f^j(x)}^{1,A} \oplus E_{f^j(x)}^{2,A} \) where
\( \mathbb{R}^d = E_{f^j(x)}^{1,A} \oplus \cdots \oplus E_{f^j(x)}^{k,A} \) and write \( u \in \mathbb{R}^d \) as \( u = u_E + u_F \) where \( u_E \in E_{f^j(x)}^{1,A} \)
and \( u_F \in E_{f^j(x)}^{1,A} \). Then the cone of radius \( 1 - \gamma > 0 \) around \( E_{f^j(x)}^{1,A} \) is defined as
\[
C^j = \left\{ u_E + u_F \in E_{f^j(x)}^{1,A} : \| u_F \|_{f^j(x)} \leq (1 - \gamma) \| u_E \|_{f^j(x)} \right\}.
\]
To simplify notation we write \( \| . \|_j \) for the Lyapunov norm at the point \( f^j(x) \).

We claim now that it is enough to prove that if \( \rho \) is sufficiently small then there
exists \( \gamma \in (0, 1) \) such that
\[
A(f^j(p))(C_0^j) \subset C_0^{j+1} \tag{15}
\]
and for every \( u \in C_0^j \),
\[
\left\| A(f^j(p))u \right\|_{j+1}^\gamma \geq e^{\lambda_1 - \delta} \left\| u \right\|_j. \tag{16}
\]
Indeed, since we are assuming that the Oseledets splitting and the Lyapunov norm
are continuous on \( \mathcal{R}_{\varepsilon,N}^{\varepsilon} \), it follows that if \( \rho \) is sufficiently small (and consequently
x and \( f^n(x) \) are close) then \( C_0^n \subset C_0^0 \) and thus by (15), \( A^n(p)(C_0^0) \subset C_0^0 \). Consequently, for any \( u \in C_0^0 \) and \( k \in \mathbb{N} \) we have that \( A^{kn}(p)u \in C_0^0 \). Therefore, given \( u \in C_0^0 \), invoking (16) and the fact that the Lyapunov norms at \( x \) and \( f^n(x) \) are close whenever \( \rho \) is small,

\[
\| A^n(p)u \|_n \geq \| (A^n(p)u)^n \|_n \geq e^{n(\lambda_1 - 2\delta)} \| u \|_0 \\
\geq \frac{1}{2} e^{n(\lambda_1 - 2\delta)} \| u \|_0 \geq \frac{1}{4} e^{n(\lambda_1 - 2\delta)} \| u \|_n
\]

which applied \( k \) times leads us to

\[
\| A^{kn}(p)u \|_n \geq \frac{1}{4^k} e^{kn(\lambda_1 - 2\delta)} \| u \|_n.
\]

Consequently,

\[
\lambda_1(p) \geq \lim_{k \to \infty} \frac{1}{kn} \log (\| A^{kn}(p)u \|_n) \geq \lim_{k \to \infty} \frac{1}{kn} \log \left( \frac{1}{4^k} e^{kn(\lambda_1 - 2\delta)} \| u \|_n \right) \\
= \lambda_1 - 2\delta - \frac{\log 4}{n} + \frac{1}{n} \lim_{k \to \infty} \frac{1}{k} \log (\| u \|_n) \geq \lambda_1 - 3\delta
\]

as long as \( n \) is large enough which proves our claim. So, the only thing left to do is to prove (15) and (16). Assume initially that \( \lambda_2 > -\infty \).

Given \( u \in C_0^0 \) let us consider \( v = A(f^j(x))u \). Then, it follows from (6) that \( \| v \|_{j+1} \leq e^{\lambda_1 + \delta} \| u \|_j \) and moreover that

\[
\| v^{j+1} \|_{j+1} = \| A(f^j(x))u^{j+1}_E \|_{j+1} \geq e^{\lambda_1 - \delta} \| u^{j+1}_E \|_j
\]

and

\[
\| v^{j+1} \|_{j+1} = \| A(f^j(x))u^{j+1}_F \|_{j+1} \leq e^{\lambda_2 + \delta} \| u^{j+1}_F \|_j. \tag{17}
\]

Let \( w = A(f^j(p))u \). What we want to do now is to compare the Lyapunov norms of \( w \) and its projection on \( E_{f^j+1}(x) \) and \( F_{f^j+1}(x) \) with the respective norms of \( v \). In order to do it, let us consider \( B_j = A(f^j(p)) - A(f^j(x)) \). Consequently, \( w = v + B_ju \) and thus \( w^{j+1}_E = v^{j+1}_E + (B_ju)^{j+1}_E \) and \( w^{j+1}_F = v^{j+1}_F + (B_ju)^{j+1}_F \). Moreover, for every \( 0 \leq j \leq n \),

\[
\| B_j \| = \| A(f^j(p)) - A(f^j(x)) \| \leq C_2 d(f^j(p), f^j(x))^{\alpha} \\
\leq C_1 C_2 \rho^\alpha e^{-\theta_0 \min\{j,n-j\}}.
\]

Therefore, invoking (10) and (8) it follows that

\[
\| B_ju \|_{j+1} \leq \| B_j \|_{f^j+1(x)} \| u \|_{j+1} \leq K_\delta(f^{j+1}(x))^2 \| B_j \| \| u \|.
\]

Then, using that \( K_\delta(f^{j+1}(x)) \leq N e^{\delta \min\{j+1,n-j-1\}} \) which follows from (9) and the fact that \( x \) and \( f^n(x) \) are in \( R_{\delta,N}^u \) and that \( \| u \|_j \leq 2 \| u \|_j \) since \( u \in C_0^0 \), we get

\[
\| B_ju \|_{j+1} \leq N^2 e^{2\delta \min\{j+1,n-j-1\}} C_1 C_2 \rho^\alpha e^{-\theta_0 \min\{j,n-j\}} \| u \|_j \\
\leq C_1 C_2 N^2 \rho^\alpha e^{2\delta \min\{j+1,n-j-1\}} e^{-\theta_0 \min\{j,n-j\}} \| u \|_j \\
\leq C \rho^\alpha e^{(2\delta - \theta_0) \min\{j,n-j\}} \| u \|_j.
\]
Thus, since $2\delta - \theta \alpha < 0$ we get that $\| B_j u \|_{j+1} \leq \tilde{C} \rho^\alpha \| u^j \|_j$ for some $\tilde{C} > 0$ independent of $n$ and $j$. Consequently,

$$\left\| w^{j+1}_{E} \right\|_{j+1} \geq \left\| v^{j+1}_{E} \right\|_{j+1} - \left\| (B_j u)^{j+1}_E \right\|_{j+1} \geq e^{\lambda_1 - \delta} \| u^j \|_j - \tilde{C} \rho^\alpha \| u^j \|_j \geq e^{\lambda_1 - 2\delta} \| u^j \|_j$$

whenever $\rho$ is small enough which is precisely inequality (16). To get (15) we observe initially that, analogously to the previous inequality we can get

$$\left\| w^{j+1}_{E} \right\|_{j+1} \leq e^{\lambda_1 + \delta} \| u^j \|_j + \tilde{C} \rho^\alpha \| u^j \|_j \leq \tilde{C} \| u^j \|_j.$$  

(18)

On the other hand,

$$\left\| w^{j+1}_{E} \right\|_{j+1} \geq \left\| e^{j+1}_{E} \right\|_{j+1} - \| B_j u \|_{j+1}$$

and

$$\left\| w^{j+1}_{E} \right\|_{j+1} \leq \left\| v^{j+1}_{E} \right\|_{j+1} + \| B_j u \|_{j+1}.$$  

Therefore, combining this inequalities and using again that $u \in C^j_0$, 

$$\left\| w^{j+1}_{E} \right\|_{j+1} - \left\| w^{j+1}_{E} \right\|_{j+1} \geq \left\| v^{j+1}_{E} \right\|_{j+1} - \| B_j u \|_{j+1} \geq \left( e^{\lambda_1 - \delta} - e^{\lambda_2 + \delta} - 2\tilde{C} \rho^\alpha \| u^j \|_j \right)$$

Taking $\rho$ small enough so that $e^{\lambda_1 - \delta} - e^{\lambda_2 + \delta} - 2\tilde{C} \rho^\alpha > 0$ and applying (18) to the previous inequality we get that there exists $\gamma > 0$ such that $\| w^{j+1}_{E} \|_{j+1} - \left\| w^{j+1}_{E} \right\|_{j+1} \geq \gamma \| w^{j+1}_{E} \|_{j+1}$ which implies that $w = Af^j(p)u \in C^j_{\gamma + 1}$ proving (15) and consequently the lemma whenever $\lambda_2 > -\infty$. The case when $\lambda_2 = -\infty$ is analogous. The only difference is that inequality (17) becomes

$$\left\| v^{j+1}_{E} \right\|_{j+1} = \left\| Af^j(x) u^j \right\|_{j+1} \leq e^{-\frac{\gamma}{2}} \left\| u^j \right\|_j.$$  

\[\square\]

**Lemma 4.2** (Upper bound). There exists a constant $c > 0$ such that

$$\| A^n(p) \| f^n(x) \rightarrow x \leq ce^{\rho \alpha} e^{(\lambda_1 + \delta) n}$$  

(19)

and

$$\| A^n(p) \| \leq cNe^{\rho \alpha} e^{(\lambda_1 + \delta) n}. $$  

Consequently, if $\rho > 0$ is sufficiently small and $n$ is large enough,

$$\lambda_1(p) \leq \lambda_1 + 2\delta.$$  

**Proof.** Let us consider $B_j = Af^j(p) - A(f^j(x))$. As in the proof of the previous lemma we have that, for every $0 \leq j \leq n$,

$$\| B_j \| \leq C_1 C_2 \rho^\alpha e^{-\theta \alpha \min\{j, n-j\}} = C \rho^\alpha e^{-\theta \alpha \min\{j, n-j\}}$$

and

$$\| B_j \| f^n(x) \rightarrow f^j(x) \leq K_\delta(f^j(x)) \| B_j \| \leq NC \rho^\alpha e^{(\delta - \theta \alpha) \min\{j, n-j\}}. $$  

(21)
Our objective now is to estimate \( \| A^n(p) \|_{f^n(x)\leftarrow x} \). We start observing that

\[
\| A^n(p) \|_{f^n(x)\leftarrow x} = \| A(f^{n-1}(p)) \|_{f^{n-1}(x)\leftarrow x} \cdot \cdots \cdot \| A(p) \|_{f^n(x)\leftarrow x}
\]

\[
= \| (A(f^{n-1}(x)) + B_{n-1}) \|_{f^{n-1}(x)\leftarrow x} \cdot \cdots \cdot \| (A(x) + B_0) \|_{f^n(x)\leftarrow x}
\]

\[
\leq \| A(f^{n-1}(x)) + B_{n-1} \|_{f^{n-1}(x)\leftarrow f^{n-1}(x)} \cdot \cdots \cdot \| A(x) + B_0 \|_{f^n(x)\leftarrow x}
\]

and, for each \( 0 \leq j < n \),

\[
\| A(f^j(x)) + B_j \|_{f^{j+1}(x)\leftarrow f^j(x)} \leq \| A(f^j(x)) \|_{f^{j+1}(x)\leftarrow f^j(x)} + \| B_j \|_{f^{j+1}(x)\leftarrow f^j(x)}.
\]

Thus, since from (7)

\[
\| A(f^j(x)) \|_{f^{j+1}(x)\leftarrow f^j(x)} \leq e^{(1+\delta)}
\]

invoking (21) we get that

\[
\| A(f^j(x)) + B_j \|_{f^{j+1}(x)\leftarrow f^j(x)} \leq e^{(1+\delta)} + CNe^{\delta-\theta\alpha} \min(j,n-j)
\]

\[
= e^{(1+\delta)} (1 + CNe^{-\delta+\alpha} \min(j,n-j)).
\]

Making \( \tilde{C} = CNe^{-\delta+\alpha} \) and using the fact that \( 1+y \leq e^y \) for every \( y \geq 0 \) it follows that

\[
\| A(f^j(x)) + B_j \|_{f^{j+1}(x)\leftarrow f^j(x)} \leq e^{(1+\delta)} \exp(\tilde{C}\rho e^{\delta-\theta\alpha} \min(j,n-j)).
\]

Consequently,

\[
\| A^n(p) \|_{f^n(x)\leftarrow x} \leq \prod_{j=0}^{n-1} e^{(1+\delta)} \exp(\tilde{C}\rho e^{\delta-\theta\alpha} \min(j,n-j))
\]

\[
= e^{(1+\delta)n} e^{\sum_{j=0}^{n-1} \tilde{C}\rho e^{\delta-\theta\alpha} \min(j,n-j)}.
\]

Now, using the fact that \( \delta - \theta\alpha < 0 \) and making \( c = \exp(\tilde{C}\sum_{j=0}^{\infty} 2e^{\delta-\theta\alpha} j) \) we get the first claim of the lemma. The second one follows from the previous one observing that \( x \in \mathcal{R}_{\delta,N}^\mu \) and \( \| A^n(p) \| \leq K_\delta(x) \| A^n(p) \|_{f^n(x)\leftarrow x} \). To conclude the proof it only remains to observe that

\[
\lambda_1(p) \leq \frac{1}{n} \log(\| A^n(p) \|)
\]

which combined with the previous inequality implies

\[
\lambda_1(p) \leq \lambda_1 + \delta + \frac{1}{n} \log(cNe^{\delta}).
\]

Since \( \rho \) may be taken arbitrary small the lemma follows.

\[
\Box
\]

Proposition 2 now follows easily from these two lemmas.

4.2. Approximation of the infinite Lyapunov exponent. In this subsection we prove that even when the Lyapunov exponents are not finite we still can approximate it by Lyapunov exponents on periodic points. This will follow from the next general proposition.
**Proposition 3.** Let \( f : M \to M \) be a continuous map defined on the compact metric space \((M, d)\), \( \mu \) an ergodic \( f \)-invariant probability measure and \( B : M \to M(d, \mathbb{R}) \) a continuous map such that \( \lambda_1(B, \mu) = -\infty \). If \( \{\mu_j\}_{j \in \mathbb{N}} \) is sequence of ergodic \( f \)-invariant probability measures converging in the weak* topology to \( \mu \), then

\[
\limsup_{j \to \infty} \lambda_1(B, \mu_j) = -\infty.
\]

**Proof.** For each \( n \in \mathbb{N} \), let \( \varphi_n : M \to [-\infty, +\infty) \) be the map given by

\[
\varphi_n(x) = \frac{1}{n} \log \| B^n(x) \|.
\]

We start observing that, since \( \mu \) is ergodic, \( \varphi_1^+(x) = \max\{0, \varphi_1(x)\} \in L^1(\mu) \) and \( \{n\varphi_n\}_{n \in \mathbb{N}} \) is a subadditive sequence, it follows by Kingman’s Subadditive Theorem (see for instance [18]) that

\[
\lambda(B, \mu) = \lim_{n \to \infty} \varphi_n(x) = \inf_n \int \varphi_n(x) d\mu
\]

for \( \mu \) almost every \( x \in M \). Analogously, since for each \( j \in \mathbb{N} \) the measure \( \mu_j \) is ergodic and \( \varphi_1^+(x) \in L^1(\mu_j) \), we have

\[
\lambda(B, \mu_j) = \inf_n \int \varphi_n(x) d\mu_j.
\]

Therefore, in order to complete the proof it is enough to show that

\[
\limsup_{j \to +\infty} \inf_n \int \varphi_n(x) d\mu_j = -\infty.
\]

Given \( m \in \mathbb{N} \), let us consider \( \varphi_{n,m} : M \to (-\infty, +\infty) \) given by

\[
\varphi_{n,m}(x) = \begin{cases} 
\varphi_n(x) & \text{if } \varphi_n(x) \geq -m \\
-m & \text{if } \varphi_n(x) < -m.
\end{cases}
\]

It follows easily from the definition and from the properties of \( \varphi_n \) that, for every \( m, n \in \mathbb{N} \), \( \varphi_{n,m} : M \to (-\infty, +\infty) \) is a continuous function. Moreover,

\[
\varphi_{n,m}(x) \geq \varphi_{n,m+1}(x) \quad \text{and} \quad \varphi_{n,m}(x) \xrightarrow{m \to +\infty} \varphi_n(x)
\]

for every \( m, n \in \mathbb{N} \) and \( x \in M \).

By the Monotone Convergence Theorem we get

\[
\int \varphi_n d\mu = \lim_{m \to \infty} \int \varphi_{n,m} d\mu = \inf_m \int \varphi_{n,m} d\mu.
\]

On the other hand, since \( \mu_j \xrightarrow{w^*} \mu \) as \( j \) goes to infinite and \( \varphi_{n,m} \) is continuous, we have

\[
\int \varphi_{n,m} d\mu = \lim_{j \to \infty} \int \varphi_{n,m} d\mu_j \quad \forall m \in \mathbb{N}.
\]

Thus,

\[
\inf_n \int \varphi_n d\mu = \inf_n \left\{ \inf_m \left[ \lim_{j \to \infty} \int \varphi_{n,m} d\mu_j \right] \right\}.
\]

Now, for each \( j \) and \( m \) in \( \mathbb{N} \) we have \( \int \varphi_{n,m} d\mu_j \geq \inf_m \left\{ \int \varphi_{n,m} d\mu_j \right\} \) and thus

\[
\lim_{j \to \infty} \int \varphi_{n,m} d\mu_j \geq \limsup_{j \to \infty} \inf_m \left\{ \int \varphi_{n,m} d\mu_j \right\}.
\]
for every \( m \in \mathbb{N} \). Consequently,
\[
\inf_{m} \left\{ \lim_{j \to \infty} \int \varphi_{n,m} \, d\mu_j \right\} \geq \limsup_{j \to \infty} \left\{ \inf_{m} \int \varphi_{n,m} \, d\mu_j \right\}.
\] (22)

Moreover, once again by the Monotone Convergence Theorem,
\[
\limsup_{j \to \infty} \left\{ \inf_{m} \int \varphi_{n,m} \, d\mu_j \right\} = \limsup_{j \to \infty} \int \varphi_{n} \, d\mu_j.
\]

Observing then that, as in (22),
\[
\inf_{n} \left\{ \limsup_{j \to \infty} \int \varphi_{n} \, d\mu_j \right\} \geq \limsup_{n} \inf_{n} \int \varphi_{n} \, d\mu_j
\]
it follows that
\[-\infty = \lambda(B, \mu) = \inf_{n} \int \varphi_{n} \, d\mu \geq \limsup_{n} \inf_{n} \int \varphi_{n} \, d\mu_j
\]
as we want. \( \square \)

4.3. Conclusion of the proof. To complete the proof of our main result the idea is to apply Propositions 2 and 3 to the cocycle induced by \( A \) on a suitable exterior power, which by now is a very standard trick.

We may assume without loss of generality that \( \mu \) is non-atomic. Otherwise the theorem is trivial. Moreover, we assume \( \lambda_{1} = -\infty \). In the case when \( \lambda_{1} > -\infty \) we only need the first part of our argument. Recall that \( \gamma_{1} \geq \gamma_{2} \geq \ldots \geq \gamma_{d} \) are the Lyapunov exponents of \( A \) with respect to \( \mu \) counted with multiplicities and, for every \( i \in \{1, \ldots, d\} \), let \( \Lambda_{i}(\mathbb{R}^{d}) \) be the \( i \)-th exterior power of \( \mathbb{R}^{d} \) which is the space of alternate \( i \)-linear forms on the dual \( (\mathbb{R}^{d})^{*} \) and \( \Lambda' A(x) : \Lambda_{i}(\mathbb{R}^{d}) \to \Lambda_{i}(\mathbb{R}^{d}) \) the cocycle induced by \( A(x) \) on the \( i \)-th exterior power. A very well known fact about this cocycle (see for instance [18]) is that its Lyapunov exponents are
\[
\{ \gamma_{j_{1}} + \ldots + \gamma_{j_{i}} : 1 \leq j_{1} < \ldots < j_{i} \leq d \}.
\]

In particular, its largest Lyapunov exponent is given by \( \gamma_{1} + \gamma_{2} + \ldots + \gamma_{i} \).

Let \( N \) be large enough so that the intersection \( G \) of the sets \( R_{\delta, N}^{d} \) associated to all the cocycles \( \Lambda_{i} A \) for \( i = 1, \ldots, \dim(E_{x}^{1,A} \oplus \ldots \oplus E_{x}^{i-1,A}) \) have positive measure, that is, \( \mu(G) > 0 \). Let
\[
B(\mu) = \left\{ x \in M ; \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^{i}(x)} \xrightarrow{n \to \infty} \mu \text{ in the weak}^{*} \text{ topology} \right\}
\]
be the basin of \( \mu \). Since \( \mu \) is ergodic, \( B(\mu) \) has full measure. Let \( x \in B(\mu) \cap G \) be so that \( \mu(B(x, \frac{1}{k}) \cap G) > 0 \) for every \( k \in \mathbb{N} \) where \( B(x, \frac{1}{k}) \) is the ball of radius \( \frac{1}{k} \) centered at \( x \). By Poincaré’s Recurrence Theorem there exists a sequence \((n_{k})_{k \in \mathbb{N}}\) of positive integers so that \( n_{k} \to +\infty \) and \( f^{n_{k}}(x) \in B(x, \frac{1}{k}) \cap G \) for each \( k \in \mathbb{N} \).

By the Anosov Closing property it follows that, for each \( k \) sufficiently large, there exists a periodic point \( p_{k} \) of period \( n_{k} \) so that
\[
d(f^{j}(x), f^{j}(p_{k})) \leq C_{1} e^{-\theta \min\{j,n_{k}-j\}} d(f^{n_{k}}(x), x) \leq \frac{C_{1}}{k} e^{-\theta \min\{j,n_{k}-j\}} \] (23)
for every \( j = 0, 1, \ldots, n_{k} \). It follows then by Proposition 2 applied to the cocycles \( \Lambda_{i} A \) for \( i = 1, \ldots, \dim(E_{x}^{1,A} \oplus \ldots \oplus E_{x}^{i-1,A}) \) that for every \( \delta > 0 \) small, there exists \( k_{\delta} \in \mathbb{N} \) so that for any \( k \geq k_{\delta} \),
\[
| (\gamma_{1} + \gamma_{2} + \ldots + \gamma_{i}) - (\gamma_{1}(p_{k}) + \gamma_{2}(p_{k}) + \ldots + \gamma_{i}(p_{k})) | < \delta
\]
for every $i = 1, \ldots, \dim(E^1_x \oplus \cdots \oplus E^{l-1}_x, A)$ where $\{\gamma_j(p_k)\}_{j=1}^d$ are the Lyapunov exponents of $A$ at the periodic point $p_k$ counted with multiplicities. In particular,

$$\lim_{k \to +\infty} \gamma_i(p_k) = \gamma_i$$

(24)

for every $i = 1, \ldots, \dim(E^1_x \oplus \cdots \oplus E^{l-1}_x, A)$. To conclude the proof of our main result it remains to observe that (24) also holds for $i = \dim(E^1_x \oplus \cdots \oplus E^{l-1}_x, A) + 1, \ldots, d$. But this follows easily from Proposition 3 applied to $A'$ observing that $\lambda_1(A', \mu) = -\infty$ for every $i > \dim(E^1_x \oplus \cdots \oplus E^{l-1}_x, A)$ and that the sequence $(\mu_{p_k})_{k \in \mathbb{N}}$ of ergodic periodic measures given by

$$\mu_{p_k} = \frac{1}{n_k} \sum_{j=0}^{n_k-1} \delta_{f^j(p_k)}$$

converges to $\mu$ in the weak* topology which follows from the fact that $x \in B(\mu)$ and (23). Consequently,

$$\lim_{k \to +\infty} \gamma_i(p_k) = \gamma_i$$

for every $i = 1, \ldots, d$ completing the proof of Theorem 2.1.

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