Relation between quark-antiquark potential and quark-antiquark free energy in hadronic matter

Zhen-Yu Shen and Xiao-Ming Xu
Department of Physics, Shanghai University, Baoshan, Shanghai 200444, China

Abstract

We study the relation between the quark-antiquark potential and the quark-antiquark free energy in hadronic matter. While a temperature is over the critical temperature, the potential of a heavy quark and a heavy antiquark almost equals the free energy, otherwise the quark-antiquark potential is substantially larger than the quark-antiquark free energy. While a temperature is below the critical temperature, the quark-antiquark free energy can be taken as the quark-antiquark potential.

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1. Introduction

Quark-antiquark free energy $F$ is defined as quark-antiquark internal energy $U$ minus the product of temperature $T$ and quark-antiquark entropy $S$. The internal energy of a quark and an antiquark at rest is the quark-antiquark potential. When the temperature is above the critical temperature $T_c$, the quark-antiquark free energy can not be identified as the quark-antiquark potential [1]. At $T < T_c$ (in hadronic matter), whether the free energy can be taken as the potential is not understood even though the potential determined in quenched lattice QCD via a correlator of a very heavy quark-antiquark pair at very large times may agree with the free energy in the Coulomb gauge [2, 3]. In order to study mesons and meson-meson reactions in hadronic matter with a temperature-dependent quark potential [4, 5], the quark-antiquark free energy obtained in lattice gauge theory calculations is taken as the quark-antiquark potential. Therefore, we must study the relation between the quark-antiquark potential and the quark-antiquark free energy.

In thermodynamics entropy indicates the disorder of random motion of particles in a system that consists of a large number of particles. It equals the negative of the derivative of the system free energy with respect to temperature while the system volume $V$ is fixed. While $V$ is fixed, the distance $r$ between any quark and any antiquark always changes. Then, the quark-antiquark entropy cannot be taken as the negative of the derivative of the quark-antiquark free energy $F(T, r)$ with respect to temperature while $r$ is fixed. How to get the quark-antiquark entropy from the quark-antiquark free energy is a problem in lattice QCD. A formula for calculating the entropy from the QCD partition function with additional heavy quarks is given in Ref. [6]. But the formula is complicated and, in particular, not suitable for lattice simulations.

We must find another way to calculate the quark-antiquark entropy. Entropy is the thermodynamic quantity that characterizes the disorder of thermal motion of particles in a system. It is obtained from observables in thermodynamics. In the present work we evaluate the quark-antiquark entropy from energy densities and pressure of particle systems.
2. Free energy and potential

The quark-antiquark internal energy is related to the quark-antiquark free energy by

\[ U(T, r) = F(T, r) + TS. \] (1)

The free energy is given by the Polyakov loop correlation function in lattice QCD. If \( TS \) is small, the quark-antiquark free energy can be considered as the quark-antiquark potential. If the free energy is of a heavy quark and a heavy antiquark, we calculate the entropy of this heavy quark-antiquark pair. If the free energy is of a light quark and a light antiquark, we calculate the entropy of this light quark-antiquark pair. In the following we show that \( TS \) is generally comparable to the free energy in a quark-gluon plasma, but is small in hadronic matter.

A. In the case of massless particles

In a quark-gluon plasma with three massless flavors, the quark energy density is \[ \epsilon_q = g_Q \frac{7\pi^2}{240} T^4, \] (2)

with the color-spin-flavor degeneracy factor \( g_Q = 18 \), and the antiquark energy density is \[ \epsilon_{\bar{q}} = g_{\bar{Q}} \frac{7\pi^2}{240} T^4, \] (3)

with \( g_{\bar{Q}} = 18 \). The pressure due to quarks is \( P_q = \frac{1}{3} \epsilon_q \), and the pressure due to antiquarks is \( P_{\bar{q}} = \frac{1}{3} \epsilon_{\bar{q}} \). The quark entropy density is \[ s_q = \frac{\epsilon_q + P_q}{T} = \frac{4}{3} \frac{\epsilon_q}{T}, \] (4)

and the antiquark entropy density is \[ s_{\bar{q}} = \frac{\epsilon_{\bar{q}} + P_{\bar{q}}}{T} = \frac{4}{3} \frac{\epsilon_{\bar{q}}}{T}. \] (5)

The quark number density is \[ n_q = g_Q \frac{3\zeta(3)}{4\pi^2} T^3, \] (6)
with $\zeta(3) = 1.20205$, and the antiquark number density is

$$n_{\bar{q}} = g_Q \frac{3\zeta(3)}{4\pi^2} T^3.$$  

(7)

There are one quark and one antiquark in the volume $V_c = 1/n_q = 1/n_{\bar{q}}$. The entropy of the pair of quark and antiquark is

$$S_{q\bar{q}} = s_q V_c + s_{\bar{q}} V_c = \frac{14\pi^4}{135\zeta(3)} \approx 8.408,$$

(8)

which is independent of temperature. On the average the spatial separation of the quark and the antiquark is

$$\sqrt[3]{V_c} = \frac{1}{T} \sqrt[3]{\frac{4\pi^2}{3g_Q\zeta(3)}}.$$

(9)

At $T = 1.01T_c$ with the critical temperature $T_c = 0.175$ GeV [8,9], the spatial separation is 0.946 fm where the free energy given in the lattice calculations is about 0.123 GeV [8]. The free energy is only for a pair of quark and antiquark in a flavor and in the color singlet. The entropy that corresponds to the free energy is $\frac{4}{3} S_{q\bar{q}}$, and $\frac{4}{3} T S_{q\bar{q}}$ takes the value 0.165 GeV which is larger than the free energy 0.123 GeV. In the evolution of the quark-gluon plasma entropy is conserved. While temperature increases from $T_c$, free energy decreases [8], and the ratio of $\frac{4}{3} T S_{q\bar{q}}$ to the free energy increases. Therefore, the quark-antiquark free energy can not be taken as the quark-antiquark potential in the quark-gluon plasma in the case of massless quarks and antiquarks.

The energy density of a system of massless pions is [7]

$$\epsilon_\pi = g_\pi \frac{\pi^2}{30} T^4,$$

(10)

with $g_\pi = 3$ for $\pi^+, \pi^0$, and $\pi^-$. The pressure due to massless pions is $P_\pi = \frac{1}{3} \epsilon_\pi$. The pion entropy density is

$$s_\pi = \frac{\epsilon_\pi + P_\pi}{T} = \frac{4\epsilon_\pi}{3T}.$$

(11)

At $T_c$ the quark and the antiquark in the volume $V_c$ transit into a pion. The pion entropy in this volume is given by

$$S_\pi = s_\pi V_c = \frac{8\pi^4 g_\pi}{135g_Q\zeta(3)} \approx 0.8,$$

(12)
which is independent of temperature. The entropy that corresponds to the quark-antiquark free energy is \( \frac{1}{3} S_\pi \) for one isospin component. The free energy slightly below \( T_c \) is about 0.29 GeV \([8]\). The ratio of \( \frac{1}{3} T S_\pi \) to the free energy is about 0.16. While hadronic matter expands, entropy is conserved, temperature decreases, free energy increases \([8]\), and the ratio of \( \frac{1}{3} T S_\pi \) to the free energy decreases. Hence, we conclude that \( U(T, r) \approx F(T, r) \) in hadronic matter.

B. In the case of particles with masses

The grand partition function for a quark system is

\[
\Xi = \prod_l (1 + e^{\beta(\mu - \epsilon_l)})^{\omega_l},
\]

where \( \beta = 1/T \), \( \mu \) is the quark chemical potential, and \( \omega_l \) is the number of states corresponding to the quark energy \( \epsilon_l \). In the relativistic case a quark has the energy \( \epsilon_l = \sqrt{\vec{p}^2 + m^2} \). In a volume \( V \) of quark-gluon plasma created at RHIC and LHC, the quark chemical potential is nearly zero, and we have

\[
\ln \Xi = \frac{4\pi g_Q V}{(2\pi)^3} \int_0^\infty d|\vec{p}| \frac{|\vec{p}|^2}{e^{\beta \sqrt{\vec{p}^2 + m^2}} - 1} \ln(1 + e^{-\beta \sqrt{\vec{p}^2 + m^2}}).
\]

The total energy of the quark system is

\[
E_q = -\frac{1}{\beta} \frac{\partial}{\partial V} \ln \Xi = \frac{4\pi g_Q V}{(2\pi)^3 \beta^4} \int_{m\beta}^\infty dz z^2 \sqrt{z^2 - m^2 \beta^2} \ln(1 + e^{-z})
\]

\[
+ \frac{4\pi g_Q m^2 V}{(2\pi)^3 \beta^2} \int_{m\beta}^\infty dz \frac{z \ln(1 + e^{-z})}{\sqrt{z^2 - m^2 \beta^2}}.
\]

and the pressure of the quark system is

\[
P_q = \frac{1}{\beta} \frac{\partial}{\partial V} \ln \Xi = \frac{4\pi g_Q V}{(2\pi)^3 \beta^4} \int_{m\beta}^\infty dz z^2 \sqrt{z^2 - m^2 \beta^2} \ln(1 + e^{-z})
\]

\[
- \frac{1}{3} \frac{E_q}{V} - \frac{4\pi g_Q m^2 V}{3(2\pi)^3 \beta^2} \int_{m\beta}^\infty dz \frac{z \ln(1 + e^{-z})}{\sqrt{z^2 - m^2 \beta^2}}.
\]

The quark entropy density is

\[
s_q = \frac{1}{T} \left( \frac{E_q}{V} + P_q \right).
\]
The total energy, pressure and entropy density of the antiquark system are written similarly. For a quark and an antiquark in the volume $V_c$, their entropy is

$$S_{q\bar{q}} = s_q V_c + s_{\bar{q}} V_c.$$  \hfill (18)

Below we show two examples of the system of quarks and antiquarks. In the first example we consider charm quarks and charm antiquarks. If they are thermalized, the above formulae can be applied with $g_Q = 6$. For the color singlet of a charm quark and a charm antiquark, $\frac{1}{3} T S_{c\bar{c}} = 1.72 \times 10^{-3}$ GeV where $T = 1.01 T_c$. This indicates that the potential of a charm quark and a charm antiquark almost equals the free energy. In the second example we consider charm quarks and massless down antiquarks. For the color singlet of a charm quark and a down antiquark, $\frac{1}{3} T S_{c\bar{d}} = 0.0834$ GeV. By comparison with the free energy 0.123 GeV, we realize that the free energy can not be taken as the potential of the charm quark and the down antiquark.

The grand partition function for a meson system is

$$\Xi = \prod_l (1 - e^{\beta (\mu - \varepsilon_l)})^{-\omega_l},$$  \hfill (19)

where $\mu$ is the meson chemical potential, and $\omega_l$ is the number of states corresponding to the meson energy $\varepsilon_l$. In the relativistic case and at $\mu = 0$,

$$\ln \Xi = -\frac{4\pi g_M V}{(2\pi)^3} \int_0^\infty d |\vec{p}| \int_\beta^\infty \frac{1}{z} \ln(1 - e^{\beta \sqrt{\vec{p}^2 + m^2}}),$$  \hfill (20)

where $g_M$ is the spin-isospin degeneracy factor. The total energy of the meson system is

$$E_m = -\frac{\partial}{\partial \beta} \ln \Xi = -\frac{4\pi g_M V}{(2\pi)^3 \beta^4} \int_{m\beta}^\infty dz z^2 \sqrt{z^2 - m^2 \beta^2} (e^{-\beta} - 1)$$

$$= -\frac{12\pi g_M V}{(2\pi)^3 \beta^4} \int_{m\beta}^\infty dz \sqrt{z^2 - m^2 \beta^2} \ln(1 - e^{-z})$$

$$-\frac{4\pi g_M m^2 V}{(2\pi)^3 \beta^2} \int_{m\beta}^\infty dz \frac{z \ln(1 - e^{-z})}{\sqrt{z^2 - m^2 \beta^2}}.$$  \hfill (21)

and the pressure of the meson system is

$$P_m = \frac{1}{\beta} \frac{\partial}{\partial V} \ln \Xi = -\frac{4\pi g_M}{(2\pi)^3 \beta^4} \int_{m\beta}^\infty dz z^2 \sqrt{z^2 - m^2 \beta^2} \ln(1 - e^{-z})$$

$$= \frac{1}{3} \frac{E_m}{V} + \frac{4\pi g_M m^2}{3(2\pi)^3 \beta^2} \int_{m\beta}^\infty dz \frac{z \ln(1 - e^{-z})}{\sqrt{z^2 - m^2 \beta^2}}.$$  \hfill (22)
The meson entropy density is
\[ s_m = \frac{1}{T} \left( \frac{E_m}{V} + P_m \right), \]  
and the meson entropy in the volume \( V_c \) is
\[ S_m = s_m V_c. \]

To get an impression on \( T S_m \), we take the unrealistic case that \( J/\psi \) is thermalized in hadronic matter. With the \( J/\psi \) mass 2.85 GeV/\( c^2 \) at \( T = 0.99T_c \) [5], we obtain \( TS_{J/\psi} = 1.98 \times 10^{-6} \) GeV at \( g_M = 3 \). Therefore, the quark-antiquark potential in a \( J/\psi \) meson equals the quark-antiquark free energy. For the system of massless pions we have already obtained the result that the free energy can be taken as the potential. While the meson mass increases from 0, \( E_m, P_m, s_m, \) and \( TS_m \) decrease. Then, the result is also true in a meson with a nonzero mass.

Finally, we note that we can not use the entropy density obtained in lattice calculations to deal with the relation between the quark-antiquark potential and the quark-antiquark free energy. This entropy density includes not only the one of quarks and antiquarks but also the one of gluons. Nobody has separated the entropy density of quarks and antiquarks from the entropy density obtained in lattice calculations.

3. Summary

The entropy of a particle system depends on the particle mass. While the system temperature is larger than the critical temperature, the product of the temperature and the entropy of a massless quark-antiquark pair is comparable to the quark-antiquark free energy, and the free energy can not be taken as the potential of a massless quark and a massless antiquark. This is also true while the quark or the antiquark is massless, but not while both the quark and the antiquark are heavy. While the system temperature is smaller than the critical temperature, the product of the temperature and a meson’s entropy is small or negligible in comparison with the quark-antiquark free energy, and the free energy can be taken as the quark-antiquark potential to a good approximation, which is in agreement with the result of the lattice calculations in Refs. [2,3].
Acknowledgements

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