Spin-1/2 XXZ Heisenberg chain in a longitudinal magnetic field

Mykhailo V. Rakov and Michael Weyrauch

Physikalisch-Technische Bundesanstalt, Bundesallee 100, D-38116 Braunschweig, Germany

Technische Universität Braunschweig, Mendelssohnstraße 3, D-38106 Braunschweig, Germany;
Kyiv National Taras Shevchenko University, 64/13 Volodymyrska st., 01601 Kyiv, Ukraine

We study the XXZ Heisenberg model in a longitudinal magnetic field using a tensor renormalization method. Built into the tensor representation of the XXZ model is the U(1) symmetry, which is systematically maintained at each renormalization step. This enables rather large tensor representations. We extract ground state properties as well as the low-lying spectrum from the fixed point tensors. With rather moderate numerical effort we achieve a very good accuracy as demonstrated by comparison with Bethe Ansatz calculations. The phase structure of the model can be accurately reproduced just from the largest fixed point tensor elements.

I. INTRODUCTION

Spin models are of interest for two reasons: firstly, they model in a simple way the magnetic properties of various crystals and therefore provide a suitable basis for physical understanding. Secondly, they are often amenable to rigorous mathematical analysis. A well-known example is the spin-1/2 XXZ Heisenberg chain. The model is integrable and therefore most of its properties may be obtained exactly. An elementary survey of the physics of this model may be found in Ref. [1].

The mathematical tools required for the analytic solution of a spin model are often rather advanced and specific for a particular model. However, not all spin models are integrable, and numerical tools are required to gain quantitative insight into the physics of such models. A famous example for a widely applicable numerical method for the study of spin models is the density matrix renormalization group (DMRG) which was first applied to the spin-1 Heisenberg model with enormous success [2, 3].

Recently, tensor network methods emerged as promising numerical tools to describe classical and quantum many-body systems. This is based on the fact that the partition function of a classical statistical system or the path integral representation of a quantum system may be approximated by the tensor trace of a tensor network [4, 5]. In principle, numerical evaluation of physical quantities just requires the evaluation of a tensor trace. Obviously, such a calculation is exponentially hard in two and more dimensions, and it is precisely this issue which is addressed by tensor renormalization methods, which convert this exponentially hard problem into a problem which can be solved in polynomial time with polynomial memory resources. By now there are many different methods and algorithms which essentially implement the classical real-space coarse-graining idea of Kadanoff [6], i.e. a coarse graining iteration until a fixed point tensor is found. From the fixed-point tensor, which is (approximately) invariant under coarse graining, one determines the physical properties of the particular model. A deeper look into such methods reveals that they are, in fact, closely related to DMRG.

A first practical implementation was described by Levin and Nave [4] and named ‘Tensor Renormalization Group’ (TRG). A deeper understanding of the method and its application were established in Ref. [5]. A more efficient renormalization method based on the higher order SVD was proposed in Ref. [7] and was named HOTRG. Later Evenbly and Vidal [8] introduced a coarse graining method (TNR) which proved to be closely related to the MERA (multi-scale entanglement renormalization ansatz) [9] and, therefore, enables particularly precise calculations close to critical points. TNR shares this advantage with the ‘loop’ algorithm introduced in Ref. [10].

In the present study we apply the HOTRG method proposed in Ref. [1] to the spin-1/2 XXZ model. This model shows a rather rich phase structure, and is therefore ideally suited for a test of tensor renormalization methods. We not only study the ground state properties but also the low-lying spectrum as obtained from the fixed-point tensors. This enables significantly more detailed investigations and tests than were previously available. Most studies up to now concentrated either on the classical or quantum Ising model [4, 7, 10, 11] or the phase boundaries only [5]. The low-lying spectrum is not usually considered, since precision calculations require large tensor sizes. In order to handle large tensor sizes we introduce U(1) symmetric tensors. The U(1) tensors introduced here differ from previous studies [12, 13] and cannot be represented as arrays as briefly explained in the Appendix. This requires a specific implementation of their algebra.

In section II we briefly review and explain the HOTRG tensor renormalization method as far as necessary for our application. A discussion of the results we obtain for the XXZ Heisenberg model is presented in section III. A brief description of the U1-Tensors we implement is given in Appendix B.

II. TENSOR NETWORK RENORMALIZATION

We start by expressing the partition function of a 1D quantum system as a tensor trace

$$Z = \text{Tr} e^{-\beta H} = \text{cTr} T^\otimes K$$

(1)
following Levin and Nave [1] and Gu and Wen [3]. Here, $H$ is the Hamiltonian of the many-body system under consideration and $\beta$ the inverse temperature. The four-index tensors $T_{ijkl}$ are layed out on a two-dimensional rectangular grid with one time and one space dimension, and the tensor trace includes summation over all connected indices of the $N$ tensors $T^{\otimes K}$ as illustrated by Fig. (1a). The (imaginary) time dimension is discretized into $\tau = \beta/M$ time intervals, and the system size in space direction is $N$ spins, such that $K = NM$ corresponds to the total number of tensors. We try to achieve as large a grid as possible in order to approximate a system in the thermodynamic limit.

In principle, there are many different ways to express the partition sum for a given Hamiltonian as a tensor network. A simple method for Hamiltonians with only the partition sum for a given Hamiltonian as a tensor is described in Refs. [5, 14]. We emphasize that the tensors $T$ all have identical structure at each space-time point, i.e. we are dealing with a homogeneous tensor network. As a consequence, only at most two of them must be stored in computer memory.

Explicit contraction (summation) of the tensor network in order to calculate the partition sum is exponentially hard. The network can only be contracted using suitable approximation schemes, often termed tensor network renormalization (TNR). Here, we will use an iterative coarse-graining scheme, which reduces the size of the tensor network by a factor of two at each iteration step. Such methods are related to Kadanoff’s block spin idea [8], and can be implemented rather efficiently. There are various ways to implement such a scheme, and here we use the higher order SVD (HOSVD) introduced in Ref. [9], which provides a very simple way to coarse grain a tensor network. The essentials of this method are summarized in Fig. (1) Two tensors $T$ are contracted as indicated in Fig. (1b), then the two left and the two right indices are contracted with unitary three-leg projectors determined such that the size of the resulting tensor $T''$ (Fig. (1c)) does not increase with respect to the original tensor $T$. Technically speaking, one approximates the tensor $T$ by the lower rank tensor $T''$. The coarse-grained homogeneous tensor network shown in Fig. (1d) is made up of these renormalized tensors $T''$. The higher order (tensor) SVD [2] used for this purpose is a generalization of the standard matrix SVD, which is a well-known tool in order to approximate a matrix by a lower rank matrix by eliminating small singular values.

After several coarse-graining steps, alternating between the space and time directions, the tensors do change only very little from step to step and we find an approximate fixed point tensor. From this fixed point tensor we obtain the ground state energy as well as the spectrum rather precisely. Other properties of the system, e.g. the magnetization, can be calculated as well. Of course, the actual precision we may achieve in our calculations depends on the size of the dimensions of the tensors we can handle numerically. This depends on the amount of computer memory available and on the internal structure of the tensors.

Unfortunately, the fixed point tensors we determine are not perfectly stable. If our procedure reaches an approximate fixed point after about 20 iterations alternating between space and time direction, the fixed point destabilizes and the calculation departs from the physical fixed point for numerical reasons. This corresponds to system sizes of about 30.000 spins in most iterations and the fixed point approximates the thermodynamic limit sufficiently for our purposes. Gu and Wen [5] pointed out, that the fixed point tensors are contaminated by residual short range entanglement, which they tried to remove by various entanglement filtering procedures. Here we do not filter out short range entanglement, nevertheless we find characteristically different fixed point tensors for each quantum phase.

In our particular implementation we take advantage of the fact that all tensors are $U(1)$ symmetric due to the fact that the XXZ model is $U(1)$ symmetric. The structure of symmetric tensors was elucidated by Singh and Vidal [12, 16], who showed that the tensors decompose into a structural and a degeneracy part on the basis of a generalized Wigner-Eckart theorem. The structural part is given by the symmetry and only the degeneracy part contains the parameters of the tensors. In fact, only the degeneracy part of symmetric tensors must be stored in memory. In this way we can handle tensor sizes of 130-140 in each dimension, about a factor 4 larger than in previous calculations [5]. Some details of our $U(1)$ symmetric tensor network implementation are given in Appendix [12].

### III. XXZ MODEL: GROUND STATE ENERGY, GAPS AND TENSOR PROPERTIES

The anisotropic spin-1/2 XXZ model is given by the Hamiltonian

$$\sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z - h S_i^z, \tag{2}$$

Here, the $S_i^\lambda$ are spin-1/2 matrix representations of SU(2). The model depends on two parameters: $\Delta$ (anisotropy) and $h$ (magnetic field); we will investigate its properties as a function of these two parameters. Well known special cases are the Heisenberg model ($\Delta = 1, h = 0$), and the XX model ($\Delta = 0, h = 0$).

The XXZ model is $U(1)$ symmetric and its states may be labelled by $U(1)$ quantum numbers $S_z$. Furthermore, at $h = 0$ the model is $Z_2$ spin reflection symmetric, and at the Heisenberg point ($\Delta, h = (1, 0)$) there is an SU(2) symmetry. These symmetries are reflected in the spectra as well as in the phase structure one obtains. However, it is only the $U(1)$ symmetry which we build explicitly into our numerical algorithm.

The spin-1/2 XXZ model may be solved using the Bethe Ansatz [17, 18]. In the special case of the XX
FIG. 1. The essentials of the HOTRG coarse graining method: (a) a symmetric tensor network as a directed graph. (b) two tensors $T$ symbolized by the small black squares are contracted into one. (c) unitary projection and approximation using HOTRG. (d) the renormalized tensor. (e) the coarse grained tensor network. (Arrows are omitted in (b), (c), (d) for simplicity.)

FIG. 2. Phase diagram of the XXZ model in the $(\Delta, h)$-plane. The orange lines separate the ferromagnetic phase (FM) from the XY phase, and the blue lines separate the XY phase from the antiferromagnetic phase (AFM). The dashed grey lines indicate cross sections in the parameter plane, for which numerical results will be presented.

model $(\Delta = 0)$ an analytical solution is found by a transformation to a non-interacting Fermion model \[19\]. Results of such calculations serve as benchmarks for our numerical investigations. For reference we collect a number of Bethe Ansatz results in Appendix \[A\]. The phase diagram \[20\] of the model is shown in Fig. 2. We distinguish a ferromagnetic phase (FM), a critically disordered spin liquid phase (XY), and an antiferromagnetic phase (AFM). These phases are separated by the critical lines $h_s$ and $h_c$ which start at the critical points $\Delta = \pm 1$.

We now present numerical results obtained with our $U(1)$ symmetric HOTRG code and compare to corresponding Bethe ansatz results. In particular, we present results for the spectrum and magnetization along the light-grey dashed lines depicted in the phase diagram (Fig. 2). Strictly speaking, the numerical results correspond to finite size systems of about 500 spins, which are compared to Bethe ansatz results in the thermodynamic limit. We do not present finite size scaling extrapolations of our numerical results, since such extrapolations are within the resolution of the presented figures. Typically, the calculations are made with nominal tensor sizes of $m = 130$ in each tensor dimension, not counting savings due to $U(1)$ symmetry, and about 30,000 imaginary time steps with step size $\tau = 0.002$. There are more imaginary time steps than spins, because we initially perform 6 HOTRG renormalizations in the imaginary time direction only.

The ground state energy at $h = 0$ is shown in Fig. 3(a) as a function of the anisotropy $\Delta$. The difference of the numerical results to Bethe ansatz calculations is illustrated in the inset. This difference includes finite size, finite (imaginary) time as well as truncation contributions. Such effects often contribute with opposite sign and (partly) cancel each other. A detailed analysis of such effects is beyond the present paper. Not surprisingly, our results indicate that numerical calculations in the critical XY phase are the most difficult. But still, the relative difference $\Delta E/E$ is of the order of $10^{-4}$ over the whole parameter range.

From the ground state energy the nearest neighbor spin-spin correlator $\langle S_i^z S_{i+1}^z \rangle$ (Fig. 3(b)) can be evaluated as a function of $\Delta$. The numerical result reproduces the jump at the critical point $\Delta = -1$ and the rather smooth $\Delta$ dependence at $\Delta = 1$. Above $\Delta = 1$, in the AFM phase, a small ‘bump’ in the magnetization can be identified, which our numerical analysis is able to reproduce precisely.

In Fig. 3(c) we plot the low energy spectrum as a function of $\Delta$ with respect to the ground state energy as determined from the fixed point tensors: for all $\Delta$ one finds a degenerate ground state, which is a consequence of the spin reflection symmetry of the XXZ Hamiltonian for $h = 0$. Numerically this degeneracy is not easy to obtain close to $\Delta = 1$ in the AFM phase, and one needs large tensor sizes here. Of course, degeneracies are never exact due to remaining finite size effects, however, those are smaller than the resolution of the plot. Our numerical procedure assigns spin quantum numbers $S_z$ to the ground state: $S_z = \pm N/2$ in the ferromagnetic regime,
several different quantum numbers in the XY phase, and $S_z = 0$ in the AFM phase as expected from analytical considerations.

The lowest gap in the ferromagnetic regime is given by $\delta = -\Delta - 1$ with quantum number $S_z = \pm (N/2 - 1)$. This result may be obtained by spin wave theory [21] or Bethe ansatz [20]. Numerically we reproduce this finding quantitatively and find this state to be highly degenerate. Moreover we clearly identify a second gap with quantum number $S_z = \pm (N/2 - 2)$. Both gaps close at the critical point $\Delta = -1$ and remain closed in the whole XY phase. The gap opens again at $\Delta = 1$, however, we see a completely different $\Delta$ dependence here as compared to the ferromagnetic phase due to a Kosterlitz-Thouless phase transition into the AFM phase. In fact, close to the critical point the gap shows a nonanalytic dependence on $\Delta$,

$$\delta = 4\pi \exp\left(-\frac{\pi^2}{2\sqrt{2}} \frac{1}{\sqrt{\Delta - 1}}\right),$$

as obtained from Eq. (3).

Numerically one finds this gap to be highly degenerate with a triplet of quantum numbers $S_z = -1, 0, 1$ for all $\Delta > 1$. This may be qualitatively understood in the limit $\Delta \to \infty$ as excitations from the Neel state [1]. Moreover, one finds a second highly degenerate state with quantum numbers $S_z = \pm 2$. All these states merge at $\Delta = 1$ into the critical disordered ground state of the XY phase. This analysis illustrates that the rather intricate low-lying spectrum of the XXZ model can be obtained rather precisely from the fixed point tensors.

It was proposed by Gu and Wen [3] that the structure of fixed point tensor as a function of the control parameter may directly reveal the phase structure of a given Hamiltonian. Here, we follow up on this proposal by plotting the largest elements of the fixed point tensors as shown in Fig. (d), where the largest tensor element is always normalized to 1. One finds that the fixed-point tensors $T$ always have at least two non-zero elements which are numerically close or equal to 1 as seen in Fig. (d). This corresponds to the fact that the ground state is always at least doubly degenerate in the thermodynamic limit.

Since our results are calculated using a $U(1)$ symmetric tensor representation, each element of the fixed point tensor can be labeled by four $U(1)$ quantum numbers. The largest elements are found with completely different quantum numbers in different quantum phases. In fact, in the FM phase one finds just two nonzero elements with quantum numbers $(\pm N/2, 0, \pm N/2, 0)$. All other tensor entries are small. In the XY phase the tensor consists of many elements close to 1 with different quantum numbers. This reflects the fact that the phase is critical. The three largest elements have quantum numbers $(0, 0, 0, 0)$ and $(\pm 1, 0, \pm 1, 0)$, respectively. Finally, in the AFM phase the two largest elements have quantum numbers $(0, 0, 0, 0)$ at sufficiently large system sizes cor-
responding to two states with $S_z = 0$. However, in the AFM one finds tensor elements smaller than one but essentially non-zero with quantum numbers $(0, 0, 0, 0)$ and $(±1, 0, ±1, 0)$. This illustrates that the tensor structure is fundamentally different in the different phases, and we conclude from the results shown in Fig. 4(d) that the phase structure can indeed be immediately read off from the tensor structure. The critical points can be identified precisely.

We now start the analysis of results for nonzero magnetic field $h$ at a few fixed values of $\Delta$. Due to the symmetry in parameter space $E(−h) = E(h)$ and $S_z(−h) = −S_z(h)$ we only need to consider $h > 0$. At $\Delta = 0$ we obtain the results shown in Fig. 4. For small magnetic fields the system is in the XY phase, and it shows an Ising-like phase transition to the FM phase at $h_s = 1$. This Ising-like phase transition is easily recognized from the magnetization plotted in Fig. 4(b). It has a cusp at $h = 1$ where the ground state becomes fully polarized with $S_z = N/2$. The numerical results agree precisely with Bethe Ansatz predictions. The low lying spectrum is plotted in Fig. 4(c). Obviously the system is gapless in the XY phase, and the gap opens at the critical point $h_s = 1$. In the ferromagnetic phase the lowest gap is given by $\delta = h − 1$ with quantum number $S_z = N/2 − 1$. Our calculation also clearly identifies the second gap with quantum number $S_z = N/2 − 2$.

The phase structure may again be determined just by plotting the largest tensor elements as done in Fig. 4(d). The largest tensor elements fluctuate as a function of the magnetic field, but still the critical point at $h_s = 1$ can be extracted easily. While in the XY phase one finds several tensor elements with absolute value close to one, in the FM phase there is only one such element. Furthermore, the quantum numbers of the leading elements differ significantly between the two phases ($S_z = N/2$ in the FM phase and in the range from 0 to $N/2$ in the XY phase).

Let us continue our analysis at $\Delta = 1$ with results shown in Fig. 5. At this $\Delta$ the Ising-like phase transition to the FM phase occurs at $h_s = 2$, but in general the structure seen is very similar to the results obtained at $\Delta = 0$. We just note that the numerical procedure is capable to capture this structure surprisingly well.

A significantly different structure for the results is observed at $\Delta = 2.5$: All three phases of the XXZ model are clearly identified by the tensor method. This is easily seen from the magnetization shown in Fig. 5(b) as well as the lowest lying spectrum (Fig. 5(c)). Ground state energy and magnetization agree quantitatively with Bethe ansatz results numerically obtained from integral equation given Ref. [21]. The magnetization is zero to a high precision in the AFM phase, however the vertical slope at the critical point $h_c ≃ 0.787$ is hard to obtain numerically. The magnetization increases until the system is fully magnetized in the FM phase and again shows a vertical slope at the critical point.

In Fig. 5(c) we also indicate the ‘triplet’ excited state at $h = 0$, which was shown already in Fig. 5. This state is split by the magnetic field as depicted in the figure. The $S_z = 1$ state is the lowest throughout the AFM phase and...
joins the lowest state in the XY phase at the critical point \( h_c \approx 0.787 \). The \( S_z = 0 \) and \( S_z = -1 \) components of the triplet state can be easily tracked within the AFM phase, however it must be noted that there are states with larger \( S_z \) below these states which are not shown. Moreover, as these states are lying high up in the spectrum they cannot be tracked through the XY phase in our calculation.

The spectrum remains degenerate as a function of the magnetic field in the AFM and XY phase. In the ferromagnetic phase we obtain the lowest gap \( \delta = h - h_s \), where \( h_s \) is given by Eq. (A1), while in the antiferromagnetic phase one finds a doubly degenerate ground state and a lowest gap \( \delta = h_c - h \), where \( h_c \) is given by Eq. (A2).

The behavior of largest tensor elements is shown in Fig. (d). The figure indicates two leading elements in the AFM phase, one leading element in the FM phase and several in the XY phase. The quantum numbers assigned to the leading elements in the XY and FM phase are the same as for \( \Delta = 0 \) and \( \Delta = 1 \). In the AFM phase the two leading elements have quantum numbers \((0,0,0,0)\) while the next two have quantum numbers \((\pm 1,0,\pm 1,0)\). From these observations the phase boundaries are easily determined.

**IV. CONCLUSIONS**

The purpose of this paper is to provide a stringent quantitative test for the HOTRG tensor renormalization method on the basis of the spin-1/2 Heisenberg XXZ model. To this end we implement this method for U(1) symmetric tensors on the basis of a new U(1) tensor representation and its algebra. Several previous papers concentrated on the Ising model \(\sqrt{2} \) or phase boundaries of spin 1 models \(\sqrt{2} \). The XXZ model has a rich phase structure including a extended critical phase. We find that the numerics quantitatively reproduces Bethe ansatz results for ground state properties as well as the low lying spectrum in a large parameter space. This encourages applications of the HOTRG method to systems with spins larger than 1/2 where no analytical results are available. Such studies have been done previously by exact diagonalization with finite size extrapolations \(\sqrt{2} \).

**ACKNOWLEDGMENTS**

M.V.R. acknowledges funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – EXC-2123/1.

**Appendix A: Bethe Ansatz results**

The critical lines \( h(\Delta) \) which separate the phases shown in Fig. (b) are given by

\[
\begin{align*}
    h_s &= 1 + \Delta, \\
    h_c &= \frac{\pi \sinh \lambda}{\lambda} \sum_{n=-\infty}^{\infty} \text{sech} \frac{\pi^2}{2\lambda} (2n + 1)
\end{align*}
\]

with \( \lambda = \text{arccosh} \Delta \).
For $h = 0$ the ground state energy per site as a function of $\Delta$ is obtained as [15][18]

$$E/N = \frac{\Delta}{4}$$  
for $\Delta \leq -1$,  \hspace{1cm} (A3)

$$E/N = \frac{\Delta}{4} \frac{1}{2} (1 - \Delta^2) \times$$  
for $\Delta > -1$ \hspace{1cm} (A4)

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh \pi x (\cosh (2x \arccos \Delta) - \Delta)}.$$  
At $\Delta = 1$ one obtains $E/N = 1/4 - \log 2$.

For $\Delta = 0$ one finds as a function of the magnetic field [23]

$$E/N = -\frac{1}{\pi} (\sqrt{1 - h^2} + h \arcsin h), \hspace{0.5cm} h \leq 1,$$  
(A5)

$$E/N = -\frac{h}{2}, \hspace{0.5cm} h > 1.$$  
(A6)

**Appendix B: U1Tensors**

A symmetric tensor network must be represented as a directed graph [16], and consequently one distinguishes incoming and outgoing indices for each tensor (similarly as one distinguishes covariant and contravariant indices in relativity theory). In Fig. 6 the direction of each edge is indicated by an arrow. In a symmetric tensor network contractions are always over one outgoing and one incoming index. For mathematical details we refer to Refs. [12][17] and references therein.

Generally it holds that a symmetric tensor decomposes into a structural part and a degeneracy part, where the structural part is determined by the symmetry. For U(1) symmetry this structure is particularly simple [12]. Consequently, each index $i_j$ of a U(1) symmetric tensor $T$ decomposes into a U(1) spin index $s_j$ and a degeneracy index $t_j$, $i_j = (s_j,t_j)$, where each index belongs either to the set of incoming indices $I$ or to the set of outgoing indices $O$. Then it holds that

$$(T)_{i_1,i_2,...,i_N} = \delta_{N_{in},N_{out}} P^{s_1,...,s_N}_{i_1,...,i_N},$$  \hspace{1cm} (B1)

where the Kronecker $\delta$ defines the structural tensor and implements spin conservation, $N_{in} = \sum_j s_j$ and $N_{out} = \sum_O s_j$.

For example, a U(1) tensor with one incoming index $i_1$ and one outgoing index $i_2$ which both run over the spin quantum numbers $-1,0,1$ must be a $3 \times 3$ matrix with degeneracy matrices $P_{-1,-1}, P_{0,0}, P_{1,1}$ as diagonal elements and 0 as off diagonal elements. The size of the degeneracy matrices is not determined. Similarly, a U(1) tensor with two incoming indices would have degeneracy tensors in the anti-diagonal and 0 otherwise.

Obviously, U(1) tensors are not arrays which are typically implemented by standard computer languages. There are various ways to deal with this problem: One could replace each 0 element by a corresponding 0-tensor and then use standard sparse array representations which are provided by many computer languages. However, such generic sparse array representations are not specific for U1 tensors and therefore lead to large computational overhead. Here we chose to implement a specific
data structure consistent with Eq. (B1) and implemented the corresponding algebra in order to contract, fuse, and rearrange such tensors (e.g. index splitting). This reduces significantly the required computational resources and enables to handle rather large tensors, since only the degeneracy tensors are actually stored in memory. Moreover, contractions only involve degeneracy tensors.

A similar program for SU(2) was described in Ref. [24]. The technical details of our tensor implementation will be described elsewhere. Here, we just mention that we start the calculation by representing the exponential of the Hamiltonian as a U1 tensor as described above, and then use the developed tensor algebra to implement the coarse graining procedure as outlined in Fig. 1.

[1] H.-J. Mikeska and A.K. Kolezhuk. One-dimensional magnetism. Lect. Notes Phys., 645:1–83, 2004.
[2] Steven R. White and David A. Huse. Numerical renormalization-group study of low-lying eigenstates of the antiferromagnetic S=1 Heisenberg chain. Phys. Rev. B, 48(6):3844–3852, Aug 1993.
[3] Steven R. White. Density-matrix algorithms for quantum renormalization groups. Phys. Rev. B, 48(14):10345–10356, Oct 1993.
[4] Michael Levin and Cody P. Nave. Tensor renormalization group approach to two-dimensional classical lattice models. Phys. Rev. Lett., 99:120601, Sep 2007.
[5] Zheng-Cheng Gu and Xiao-Gang Wen. Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order. Phys. Rev. B, 80:155131, Oct 2009.
[6] L. P. Kadanoff. Scaling laws for ising models near Tc. Physics, 2:263–272, Jun 1966.
[7] Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, and T. Xiang. Coarse-graining renormalization by higher-order singular value decomposition. Phys. Rev. B, 86:045139, Jul 2012.
[8] G. Evenbly and G. Vidal. Tensor network renormalization. Phys. Rev. Lett., 115:180405, Oct 2015.
[9] G. Evenbly and G. Vidal. Tensor network renormalization yields the multiscale entanglement renormalization ansatz. Phys. Rev. Lett., 115:200401, Nov 2015.
[10] Shuo Yang, Zheng-Cheng Gu, and Xiao-Gang Wen. Loop optimization for tensor network renormalization. Phys. Rev. Lett., 118:110504, Mar 2017.
[11] Hiroshi Ueda, Kouichi Okunishi, and Tomotoshi Nishino. Doubling of entanglement spectrum in tensor renormalization group. Phys. Rev. B, 89:075116, Feb 2014.
[12] Sukhwinder Singh, Robert N. C. Pfeifer, and Guifre Vidal. Tensor network states and algorithms in the presence of a global U(1) symmetry. Phys. Rev. B, 83:115125, Mar 2011.
[13] Mykhailo V. Rakov, Michael Weyrauch, and Briiissuurs Braiіorr-Orrs. Symmetries and entanglement in the one-dimensional spin \( \frac{1}{2} \) xxz model. Phys. Rev. B, 93:054417, Feb 2016.
[14] G. Evenbly. Algorithms for tensor network renormalization. Phys. Rev. B, 95:045117, Jan 2017.
[15] L. De Lathauer, B. De. Moor, and J. Vandewalle. SIAM J. Matrix Anal. Appl., 21:1253, May 2000.
[16] Sukhwinder Singh, Robert N. C. Pfeifer, and Guifre Vidal. Tensor network decompositions in the presence of a global symmetry. Phys. Rev. A, 82:050501, Nov 2010.
[17] C. N. Yang and C. P. Yang. Ground state energy of Heisenberg-Ising lattice. Phys. Rev., 147:303–306, Dec 1965.
[18] C. N. Yang and C. P. Yang. One-dimensional chain of anisotropic spin-spin interactions. II. Properties of the ground-state energy per lattice site for an infinite system. Phys. Rev., 150:327–339, Oct 1966.
[19] E. Lieb, T. Schulz, and D. Mattis. Two soluble models of an antiferromagnetic chain. Annals of Physics, 16:407, 1961.
[20] C. N. Yang and C. P. Yang. One-dimensional chain of anisotropic spin-spin interactions. III. Applications. Phys. Rev., 151:258–264, Nov 1966.
[21] G. Albertini, V. E. Korepin, and A. Schadschneider. XXZ model as an effective hamiltonian for generalized Hubbard models with broken \( \eta \)-symmetry. J. Phys. A: Math. Gen., 28(10):L303–L309, May 1995.
[22] Wei Chen, Kazuo Hida, and B. C. Sanctuary. Ground-state phase diagram of \( S = 1 \) XXZ chains with uniaxial single-ion-type anisotropy. Phys. Rev. B, 67:104401, Mar 2003.
[23] C. N. Yang and C. P. Yang. Ground state of the anisotropic spin-\( \frac{1}{2} \) XXZ chain with uniaxial single-ion-type anisotropy. Phys. Rev. B, 89:115137, Mar 2014.
[24] G. Evenbly. Algorithms for tensor network renormalization. Phys. Rev. B, 95:045117, Jan 2017.
[25] L. De Lathauer, B. De. Moor, and J. Vandewalle. SIAM J. Matrix Anal. Appl., 21:1253, May 2000.