Predefined-Time Antisynchronization of Two Different Chaotic Neural Networks

Lixiong Lin

School of Mechanical Engineering and Automation, Fuzhou University, Fujian 350116, China

Correspondence should be addressed to Lixiong Lin; linlixiong@126.com

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This paper is concerned with antisynchronization in predefined time for two different chaotic neural networks. Firstly, a predefined-time stability theorem based on Lyapunov function is proposed. With the help of the definition of predefined time, it is convenient to establish a direct relationship between the tuning gain of the system and the fixed stabilization time. Then, the antisynchronization is achieved between two different chaotic neural networks via active control Lyapunov function design. The designed controller presents the practical advantage that the least upper bound for the settling time can be explicitly defined during the control design. With the help of the designed controller, the antisynchronization errors converge within a predefined-time period. Numerical simulations are presented in order to show the reliability of the proposed method.

1. Introduction

Synchronization and control is one of the most important research topics in the last decades due to its wide applications in engineering [1–4], including secure communications [3] and cryptograph [4], and a variety of complex chemical and physical systems. The primary purpose of synchronization is to achieve the master and the slave systems coupled by designing an appropriate controller. The history of chaos synchronization could be traced back to the research of the interaction between coupled chaotic systems [5–7]. Since then, many different types of synchronization for chaotic systems have been proposed in the references, such as generalized synchronization [8], complete synchronization [7], lag synchronization, combination synchronization [9], and projective synchronization [10]. In practice, there also exists an interesting phenomenon in symmetrical oscillators and antisynchronization [11–15], which means that the absolute value of the state vectors between two synchronized systems is equal, but their signs are opposite. According to the related references [16, 17], antisynchronization has important applications in the communication system and laser field. Therefore, how to solve the problems of synchronization and antisynchronization in different chaotic neural networks is also an open issue to consider [12, 18].

In order to realize the synchronization of two chaotic systems, several synchronization methods have been developed such as adaptive control [19], sliding mode control [20], backstepping method control [21], active control [2], and impulsive control [22]. However, it is worth to remark that although these methods provide finite-time synchronization [23–25], the convergence time is lack of uniform boundedness. And the convergence time of finite-time synchronization depends on the initial values of two chaotic systems. In other words, if the initial values of two chaotic systems are unknown, the convergence time cannot be estimated in advance. A new stability named fixed-time stability was proposed in [26] to solve this problem. The convergence time of fixed-time stability could be globally uniformly bounded without the initial values of the system [27–30]. In order to estimate the convergence time more accurately, Xu et al. [28] proposed a simple controller without sign function and linear parts to achieve complex networks fixed-time synchronization. In [29], the inverse function of the Lyapunov function was used to realize the
synchronization in finite time and fixed time. To achieve fixed-time synchronization for a class of complex networks with impulsive effects, a controller, which was continuous and did not include the sign function, was designed by time-varying Lyapunov functions and convex combination techniques in [30]. Nevertheless, the fixed-time stability also has its disadvantages. Due to the difficulties in theoretical analysis, the convergence time cannot be accurately estimated.

For several engineering applications, such as control systems and secure communications, it is required that the system reaches the origin within a predefined time. In order to assure synchronization between two chaotic systems within a predefined time, an active controller based on predefined-time stability was proposed in [31]. Predefined-time stability is a special kind of fixed-time stability with the settling time that can be defined in advance as a parameter of the control design [32–34]. Up to present, the research on predefined-time stability is still in its infancy. Sanchez-Torres et al. [35] proposed first-order controllers based on predefined-time sliding mode for a class of dynamical systems. Muñoz-Vázquez et al. [36] proposed a controller based on second-order predefined-time stability to solve the problem of fully exact tracking of actuated mechanical systems. To the best of our knowledge, there is still no research on predefined-time antisynchronization of two different chaotic neural networks. Motivated by the above discussions, based on [31], a predefined-time stability theorem for antisynchronization of two different chaotic neural networks is proposed in this paper. The main contributions of this paper are summarized in the following two aspects:

1. A predefined-time stability theorem by Lyapunov function in [37] is proposed, the advantage of which is that it is more general and the predefined-time stability theorem in [31] is a special case of it. And its predefined-time stability criterion for the predefined-time is different from those in [31–36].

2. Different from [14], this paper investigates the predefined-time antisynchronization schemes for two different chaotic neural networks, which assure the antisynchronization execution before a time that is predefined as a parameter during the control design.

This paper is organized as follows. Useful results in relation to the predefined-time stability and other preliminaries are presented in Section 2. In Section 3, a predefined-time stability theorem based on Lyapunov function is proposed, which is applied to achieve two different chaotic neural networks antisynchronized. Section 4 gives some simulation results, and finally, some conclusions are drawn in Section 5.

2. Preliminaries

Some basis on fixed-time stability and predefined-time stability are addressed below. Consider the system

\[ \dot{x} = f(x; r), \]

where \( x \in \mathbb{R}^n \) is the system state, \( r \in \mathbb{R}^l \) with \( l \geq 0 \) represents the parameters of system (1). \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is nonlinear function. For this system, the initial condition is \( x_0 = x(0) \).

Definition 1 (see [26]). The origin is said to be a globally fixed-time stable equilibrium point for system (1) if it is globally finite-time stable and the settling time function is bounded and independent of the initial conditions, i.e., \( \exists T_{\max} > 0: \forall x_0 \in \mathbb{R}^n, T(x_0) \leq T_{\max} \).

Definition 2 (see [38]). For a predefined constant \( T_c > 0 \), the origin of system (1) is said to be predefined-time stable if it is fixed-time stable and the settling time function \( T: \mathbb{R}^n \rightarrow \mathbb{R} \) is such that

\[ T(x_0) \leq T_c, \quad \forall x_0 \in \mathbb{R}^n. \]

If this is the case, \( T_c \) is called a predefined time.

Remark 1. Let the origin of system (1) be fixed-time stable. Notice that it is difficult to find an explicit relationship between the system parameters and the fixed-time \( T_{\max} \). In some cases, it is difficult to reduce time \( T_{\max} \) even by tuning system parameters. The predefined-time \( T_c \), which is defined during the control design, can solve the above problems, so system (1) can achieve stability before the predefined-time \( T_c \).

Lemma 1 (inequalities [39]). If \( z_1, z_2, \ldots, z_n \geq 0, 0 < \theta \leq 1, \gamma > 1 \), then it gets that

\[ \sum_{i=1}^{n} z_i^\theta \geq \left( \sum_{i=1}^{n} z_i \right)^\theta, \]

\[ \sum_{i=1}^{n} z_i^{-\gamma} \geq n^{-\gamma} \left( \sum_{i=1}^{n} z_i \right)^{-\gamma}. \]

Lemma 2 (see [26]). Supposing \( V(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\} \) is a continuous strictly monotonically decreased function and satisfies the following:

1. \( V(x) = 0 \implies x \in M, \) where \( M \in \mathbb{R}^n \) is a nonempty set and said to be globally fixed time attractive for system (1).

2. For any \( V(x) > 0 \), there exist \( a, \eta > 0, \) \( p > 1, \) and \( 0 < q < 1 \) such that

\[ \dot{V} \leq -(aV^p + \eta V^q). \]

Then, system (1) can realize globally fixed-time stability and

\[ T = \frac{1}{\alpha \left(1 - p \right)} + \frac{1}{\eta \left(1 - q \right)}. \]
Complexity

$V(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup 0$ is a continuous strictly monotonically decreased function and satisfies

$$
\dot{V} \leq -\frac{\pi}{g T_c} \left( V^{1-(g/2)} + V^{1+(g/2)} \right)
$$

(6)

for $g \in (0, 1)$. Then, system (1) can realize globally predefined-time stability with the time $T_c$.

## 3. Main Results

### 3.1. Predefined-Time Stability

**Theorem 1.** Supposing $V(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup 0$ is a continuous strictly monotonically decreased function and satisfies the following:

1. $V(x) = 0 \implies x \in M$, where $M \in \mathbb{R}^n$ is a nonempty set and said to be globally fixed time attractive for system (1).
2. $T_c \in \{r_1, \ldots, r_k\} > 0$ is a user-defined parameter.
3. For any $V(x) > 0$, there exist $\alpha, \eta, k, p, q > 0$, $pk > 1$, and $0 < qk < 1$ such that

$$
\dot{V} \leq -\frac{C_v}{T_c} (aV^p + \eta V^q)^k
$$

(7)

where

$$
C_v = \begin{cases}
\frac{1}{1-qk} & \text{if } 1 > \gamma, \\
\frac{2^{(\gamma-1)k}}{a^{1/\gamma} \alpha^{1/\gamma} (1-qk) \eta^{1-\gamma}} & \text{if } 0 < \gamma \leq 1,
\end{cases}
$$

(8)

$$
\gamma = \frac{p-q}{1-qk}
$$

Then, system (1) is globally predefined-time stable, with predefined-time $T_c$.

**Proof.** As we know, for any $V(x) > 0$, it gets that

$$
\dot{V} \leq -\frac{C_v}{T_c} (aV^p + \eta V^q)^k
$$

(9)

Since $V^{qk} > 0$, it follows that

$$
\frac{1}{V^{qk}} \frac{dV}{dt} \leq -\frac{C_v}{T_c} (aV^p + \eta V^q)^k.
$$

(10)

Let $z = V^{1-qk}$, and equation (10) can be written as

$$
\frac{dz}{dt} \leq -\frac{C_v}{T_c} (1-qk) (aV^p + \eta V^q)^k = -\frac{C_v}{T_c} (1-qk)(az^\gamma + \eta)^k.
$$

(11)

The settling time function is

$$
T(x_0) = \int_0^{T_c(x_0)} dt.
$$

(12)

Furthermore, since the function $V(x)$ is continuous and strictly monotonically decreased, one has that

$$
T(x_0) = \int_0^{T_c(x_0)} dt \leq -\int_T^{T_c(x_0)} \frac{T_c}{V_{zQ} \alpha} \frac{1}{1-qk} \frac{1}{(az^\gamma + \eta)^k} dz.
$$

(13)

The following proof is divided in two cases.

**Case 1.** If $y > 1$, by Lemma 1, it gets that

$$
a z^\gamma + \eta \geq 2^{1-\gamma} (a^{1/\gamma} z + \eta^{1/\gamma})^\gamma.
$$

(14)

Thus,

$$
T(x_0) \leq \int_0^{T_c(x_0)} \frac{T_c}{V_{zQ} \alpha} \frac{1}{1-qk} \frac{1}{2^{1-\gamma/k}} (a^{1/\gamma} z + \eta^{1/\gamma})^k dz.
$$

(15)

Then,

$$
T(x_0) \leq \int_0^{T_c(x_0)} \frac{T_c}{V_{zQ} \alpha} \frac{1}{1-qk} \frac{2^{(\gamma-1)k}}{a^{1/\gamma} \alpha^{1/\gamma} (1-qk) \eta^{1-\gamma}} \left[ 2^{(\gamma-1)k} (a^{1/\gamma} V(x_0) + \eta^{1/\gamma})^{1-\gamma} - \eta^{(1-\gamma)\gamma} \right].
$$

(16)

By virtue of

$$
yk = \frac{p-q}{1-qk} \cdot k = 1 + \frac{pk-1}{1-qk} > 1.
$$

(17)

Thus,

$$
T(x_0) \leq \int_0^{T_c(x_0)} \frac{T_c}{V_{zQ} \alpha} \frac{1}{1-qk} \frac{2^{(\gamma-1)k}}{a^{1/\gamma} \alpha^{1/\gamma} \eta^{1-\gamma}} \left[ \eta^{(1-\gamma)\gamma} - (a^{1/\gamma} V(x_0) + \eta^{1/\gamma})^{1-\gamma} \right].
$$

(18)

If $V(x_0) = 0$, one can obtain that
Corollary 1. Supposing $V(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{0\}$ is a continuous strictly monotonically decreased function and satisfies the following:

(1) $V(x) = 0 \implies x \in M$, where $M \in \mathbb{R}^n$ is a nonempty set and said to be globally fixed time attractive for system (1).

(2) $T_c \in \{r_1, \ldots, r_b\} > 0$ is a user-defined parameter.

(3) For any $V(x) > 0$, there exist $\alpha, \eta > 0$, $p > 1$, and $0 < q < 1$ such that

$$\dot{V} \leq -\frac{C_v}{T_c} (\alpha V^p + \eta V^q),$$

(28)
where
\[
C_v = \frac{1}{1-q} \frac{2^{r-1}}{a^r(y-1)\eta^{(1-\gamma)y}},
\]
(29)
\[
y' = 1 + \frac{p-1}{1-q}.
\]

Then, system (1) is globally predefined-time stable, with predefined-time \(T_c\).

**Proof.** Proof of Corollary 1 in Appendix A.

Compared with Lemma 2, predefined-time stability can be achieved in Corollary 1.

**Remark 2.** In Corollary 1, \(\alpha, \eta, p, \) and \(q\) are required to satisfy \(\alpha, \eta > 0, p > 1, \) and \(0 < q < 1\). However, in Lemma 3, \(\alpha, \eta, p, \) and \(q\) should satisfy \(\alpha = \eta = (\pi/gC_v), g = p - q, \) and \(p + q = 2\), which will constrain the application of Lemma 3 in some scenarios.

**Remark 3.** Different from the proof of Lemma 3, the proof of Theorem 1 uses Lemma 1 to solve the inequality problem. In order to reflect the convergence time of system (1) more intuitively, the convergence of Lyapunov function is regarded as an implicit inverse function of time. Furthermore, using these techniques, it easily derives the results of Lemma 2, Lemma 3, and Corollary 1.

### 3.2. Controller Design for Predefined-Time Antisynchronization

In this subsection, the predefined-time antisynchronization of two different delayed chaotic neural networks is realized by using active control strategy. Consider the following delayed chaotic neural network

\[
\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(t - \tau)),
\]
(30)
i = 1, 2, \ldots, n, where \(x_i(t)\) represents the state of the \(i\)-th neuron; \(a_i > 0\) denotes the self-inhibition; \(g_j(\cdot)\) is the activation function; \(b_{ij} \in R\) stands for the connection weight; \(c_{ij} \in R\) stands the delayed connection weight; \(t(t)\) represents the discrete delay and satisfies \(0 \leq \tau(t) \leq \tau\).

Let (30) be the master delayed chaotic neural network, and the slave delayed chaotic neural network is given by

\[
\dot{y}_i(t) = -d_i y_i(t) + \sum_{j=1}^{n} h_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} m_{ij} f_j(y_j(t - \tau)) + u_i(t),
\]
(31)
i = 1, 2, \ldots, n, where \(y_i(t)\) represents the state of the \(i\)-th neuron; \(d_i > 0\) denotes the self-inhibition; \(f_j(\cdot)\) denotes the neuron activation function; \(h_{ij} \in R\) stands for the connection weight; \(m_{ij} \in R\) stands the delayed connection weight; and \(u_i(t)\) is a controller to be designed late.

**Assumption 1** (see [40]). The neuron activation functions in system (30) and (31) satisfy

\[
|g_j(\cdot)| \leq N_j, \quad |f_j(\cdot)| \leq M_j,
\]
(32)
where \(N_j > 0\) and \(M_j > 0, j = 1, 2, \ldots, n\).

**Theorem 2.** The slave delayed chaotic neural network (31) can achieve predefined-time antisynchronization with the master delayed chaotic neural network (30) via the control law as below:

If \(k > 1\),

\[
u_i(t) = -k_i e_i - k_2 \text{sign}(e_i) - \frac{2^{k-1} C_v}{2T_c} \left( a_i \text{sign}(e_i) |e_i|^q + \eta_i \text{sign}(e_i) |e_i|^p \right) - (d_i - a_i)x_i.
\]
(33)

If \(0 < k \leq 1\),

\[
u_i(t) = -k_i e_i - k_2 \text{sign}(e_i) - \frac{C_v}{2T_c} \left( a_i \text{sign}(e_i) |e_i|^q + \eta_i \text{sign}(e_i) |e_i|^p \right) - (d_i - a_i)x_i.
\]
(34)

**Proof.** The antisynchronization error system between (30) and (31) can be defined as

\[e_i(t) = y_i(t) + x_i(t).
\]
(35)

According to the error system (35) of master-slave delayed chaotic neural networks (30) and (31), it gets that

\[
\dot{e}_i(t) = -d_i e_i + \sum_{j=1}^{n} h_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} m_{ij} f_j(y_j(t - \tau)) + u_i(t) + \left( a_i e_i + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(t - \tau)) \right)
\]
\[
= -d_i e_i + \sum_{j=1}^{n} h_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} m_{ij} f_j(y_j(t - \tau)) + u_i(t) + (d_i - a_i)x_i + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(t - \tau)).
\]
(36)

To simplify the formula, the variable \(G\) is defined as follows:

\[
G = \sum_{j=1}^{n} h_{ij} f_j(y_j) + \sum_{j=1}^{n} m_{ij} f_j(y_j(t - \tau)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(t - \tau)).
\]
(37)
If Assumption 1 holds, it can be derived that
\[
G \leq |G| \leq \sum_{j=1}^{n} h_{ij}M + \sum_{j=1}^{n} m_{ij}M + \sum_{j=1}^{n} b_{ij}N + \sum_{j=1}^{n} c_{ij}N.
\]
(38)

Thus,
\[
\dot{e}_i(t) = -d_i e_i + (d_i - a_i)x_i + G + u_i.
\]
(39)

Consider the candidate Lyapunov function
\[
V(e(t)) = \sum_{i=1}^{n} \text{sign}(e_i)\dot{e}_i.
\]

Since \( k_{1i} \geq d_i \) and \( k_{2i} \leq -\sum_{j=1}^{n} (h_{ij} + m_{ij})M - \sum_{j=1}^{n} (b_{ij} + c_{ij})N \), it follows that
\[
\dot{V}(e(t)) \leq -\frac{2^{k-1}C_v}{T_c} \left( \sum_{i=1}^{n} \alpha_i |e_i|^{p_k} + \sum_{i=1}^{n} \eta_i |e_i|^{p_k} \right).
\]
(43)

Let \( \alpha = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \), \( \eta = \min\{\eta_1, \eta_2, \ldots, \eta_n\} \), and by Lemma 1, then
\[
\dot{V}(e(t)) = \sum_{i=1}^{n} \text{sign}(e_i)\left[-d_i e_i + (d_i - a_i)x_i + G + (-k_{1i} e_i - k_{2i}\text{sign}(e_i))\right]

- \frac{C_v}{2T_c}\left( \alpha_i |e_i|^{p_k} + \eta_i |e_i|^{p_k} \right) - (d_i - a_i)x_i \right) \]
(45)

This proof is completed. \( \square \)

Remark 4. Theorem 2 states that two different chaotic neural networks can achieve antisynchronization within a predefined-time period. Compared to other antisynchronization of
two different chaotic neural networks, such as asymptotic antisy-
chronization, finite-time antisynchronization, or fixed-time antisynchronization, the new predefined-time antisynchronization Theorem 2 can provide a tuning stable time $T_c$, which could be tuned according to the actual needs of the antisynchronization of neural networks.

Li et al. [22] realized the exponential synchronization of
dynamical networks by hybrid pinning impulsive controller, and the convergence time was uncertain. In [30], fixed-time synchronization of complex networks with impulsive effects was realized by nonchattering control, but the convergence time could not be accurately estimated. Theorem 2 can achieve the predefined-time antisynchronization of two different chaotic networks with the convergence time explicitly defined during the control design. Theorem 2 also has many applications, such as communication system or image field.

(1) By using predefined-time antisynchronization in communication system, one may transmit digital signals continuously by setting the synchronization and antisynchronization time in advance, which will strengthen the security and secrecy.

(2) In image secure communication, due to the complex behavior of neural networks, it is possible to add a predefined-time antisynchronization of neural networks in the process of image encryption and decryption to build a strong encryption and decryption model.

4. Numerical Simulations

In this section, some simulation results are used to verify the correctness of our conclusions.

4.1. Predefined-Time Antisynchronization of Two Different Chaotic Neural Networks. Consider the following delayed chaotic neural network:

$$x_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} g(x_j(t)) + \sum_{j=1}^{n} c_{ij} f(y_j(t - \tau(t))).$$

(46)

Let (46) be the master delayed chaotic neural network, the slave delayed chaotic neural network is designed as

$$y_j(t) = -d_j y_j(t) + \sum_{k=1}^{n} h_{jk} f(y_j(t))$$

$$+ \sum_{k=1}^{n} m_{jk} f(y_j(t - \tau(t))) + u_i(t),$$

(47)

where $i, j = 1, 2, \ldots, n$, denotes the number of neurons. In this example, we set $n = 2$, and some parameters are set as follows: $a_1 = a_2 = 1.0$, $b_{11} = 2$, $b_{12} = -0.1$, $b_{21} = -5$, $b_{22} = 3$, $c_{11} = -1.5$, $c_{12} = -0.1$, $c_{21} = -0.2$, $c_{22} = -2.5$, $d_1 = d_2 = 1.0$, $h_{11} = 0.02$, $h_{12} = 0.01$, $h_{21} = 0.02$, $h_{22} = 0.1$, $m_{11} = -0.01$, $m_{12} = 0.01$, $m_{21} = 0.01$, $m_{22} = 0.05$, $\tau(t) = 1$, $p = (2/3)$, $q = 2$, and

$$g(x(t)) = \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix},$$

$$f(y(t)) = \begin{bmatrix} \tanh(0.2 \ast y_1(t)) \\ \tanh(0.4 \ast y_2(t)) \end{bmatrix}.$$  

(48)

The initial values of the master delayed chaotic neural network (46) are $x(t) = [2, -6]^T, t \in [-1, 0]$. The initial values of the slave delayed chaotic neural network (47) are $y(t) = [6, -2]^T, t \in [-1, 0]$. The antisynchronization of master-slave delayed chaotic neural networks begins at $t_0 = 0$, and the goal is to achieve antisynchronization within the predefined-time $T_c$. Simulation results in Figures 1 and 2 verify the correctness and advantages of the proposed methods. Note that, for the case of

$$u_i(t) = -k_i e_i - \omega_i \text{sign}(e_i),$$

(49)

simulation results are shown in Figure 3 with the initial values $y(t) = [6, -2]^T, t \in [-1, 0]$, and $x(t) = [2, -6]^T$ and in Figure 4 with the initial values $y(t) = [-6, -4]^T, t \in [-1, 0]$, and $x(t) = [4, 2]^T$. The antisynchronization is finite-time stable in theory, but the drawback of which is impossible to estimate the settling time in advance. For the case of

$$u_i(t) = -k_i e_i - \omega_i \text{sign}(e_i) - \alpha_i \text{sign}(e_i) |e_i|^{\eta} - \eta_i \text{sign}(e_i) |e_i|^{\eta},$$

(50)

simulation results are shown in Figure 5. Theoretically, the antisynchronization is fixed-time stable and the settling time can be calculated as follows:

$$T_{set} = \frac{1}{1 - q} 2^{(p-1)} \left(1 - \frac{1}{\gamma} \right)^{1-1/\gamma} = 1.9,$$

(51)

with the parameters $q = 0.8$, $p = 1.5$, $\alpha = 6$, and $\eta = 6$. As can be seen from Figure 5, the master-slave delayed chaotic neural networks can realize fixed-time antisynchronization under controller (50) and the convergence time $T \leq T_{set}$.

The proposed controller (33) based on predefined-time stability

$$u_i = -k_i e_i - k_{2i} \text{sign}(e_i) - \frac{2^{p-1}C_u}{2T_c} \cdot \left(\alpha_i \text{sign}(e_i) |e_i|^{\eta} + b_i \text{sign}(e_i) |e_i|^{\eta} \right) - (d_i - a_i) x_i$$

(52)

can solve the above problems and guarantee that the master-slave delayed chaotic neural networks are antisynchronized before a predefined-time $T_c$, also verified Theorem 2. Simulation results are shown in Figures 1 and 2.

4.2. Predefined-Time Synchronization of Two Lorenz Chaotic Systems. To verify the reliability of Theorem 1, the synchronization of two Lorenz chaotic systems is considered in this subsection. The master system is described by the following equations:
\[ x = [x_1, x_2, x_3]^T \in \mathbb{R}^3 \] the system state vector and \( \sigma = 10, \rho = 28, \) and \( \beta = (8/3) \) are constant parameters. The slave system can be written as follows:

\[ \dot{x}_1 = \sigma (x_2 - x_1), \]
\[ \dot{x}_2 = x_1 (\rho - x_3) - x_2, \]
\[ \dot{x}_3 = x_1 x_2 - \beta x_3, \]

where \( x = [x_1, x_2, x_3]^T \in \mathbb{R}^3 \) is the system state vector and \( \sigma = 10, \rho = 28, \) and \( \beta = (8/3) \) are constant parameters. The slave system can be written as follows:

\[ \dot{x}_1 = \sigma (\tilde{x}_2 - \tilde{x}_1) + u_1, \]
\[ \dot{x}_2 = x_1 (\rho - \tilde{x}_3) - \tilde{x}_2 + u_2, \]
\[ \dot{x}_3 = x_1 \tilde{x}_2 - \beta \tilde{x}_3 + u_3, \]

where \( \tilde{x} = [\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]^T \in \mathbb{R}^3 \) is the slave system state vector. Then, the active controller is proposed as

\[ u = \frac{2^{k-1}C_k}{2T_c} (a^kV^{\rho k-1} + \eta^kV^{\rho k-1}) e, \]

where \( e = [e_1, e_2, e_3]^T \) and \( V = (1/2)e^Te \) the candidate Lyapunov function for the synchronization of the Lorenz chaotic systems. It can be seen from controller (55) that the design process of the predefined-time controller depends on the actual systems, and different systems will have different forms of the predefined-time controllers. In Figure 6, it is appreciable that the proposed controller (55) enforces the synchronization of two Lorenz chaotic systems before the predefined-time \( T_c \) for several different initial conditions. And the predefined-time \( T_c \) can be explicitly defined during the control design, and the simulation results are shown in Figure 6 with \( T_c = 0.06 \) and Figure 7 with \( T_c = 0.02 \). The predefined-time synchronization can be
achieved by the different settling time $T_c$. In theory, the choice of $T_c$ should only satisfy the conditions of Theorem 1, that is, $T_c = \lim_{V(x_0) \to \infty} T(x_0)$. In practice, the choice of $T_c$ should consider the actual system parameters and requirements simultaneously. It is shown in addition that Theorem 1 is also applied to other chaotic systems.

5. Conclusion

In this paper, a predefined-time stability theorem based on strict theoretical derivations has been presented. A controller has been designed to achieve the anti-synchronization of two different chaotic neural networks. The antisynchronization errors converge within a finite-time period and present the practical advantage that the least upper bound for this settling time can be explicitly defined during the control design. In order to show better performance of predefined-time controller, some comparisons have been made with the controllers by other stability theorems. Future work in this direction will apply predefined-time stability theorem to secure communications and be concerned with the following:

1. Designing output-based predefined-time controllers \cite{41, 42} for the systems with only the measured output utilized.

2. Designing active controller for other types of systems with predefined-time stability, such as fractional order neural networks, fuzzy sets, and systems.

3. Designing predefined-time active controller for discrete-time systems, such as discrete-time Boolean control networks \cite{43, 44}.

Appendix

A. Proof of Corollary 1

Proof. As we know, for any $V(x) > 0$, it gets that

\[
\dot{V} \leq -C_v V^{p/q} (a V^p + \eta) = -V^q \frac{C_v}{T_c} (a V^{p-q} + \eta). \tag{A.1}
\]

Since $V^q > 0$, it follows that

\[
\frac{1}{V^q} \frac{dV}{dt} \leq -\frac{C_v}{T_c} (a V^{p-q} + \eta). \tag{A.2}
\]

Let $z = V^{1-q}$, equation (A.2) can be written as

\[
\frac{dz}{dt} \leq -\frac{C_v}{T_c} (1-q)(a V^{p-q} + \eta) = \frac{C_v}{T_c} (1-q)(az^{1+\varepsilon} + \eta), \tag{A.3}
\]

where $\varepsilon = (p-1)/(1-q)$. Since the function $V(x)$ is continuous and strictly monotonically decreased, one has that

\[
T(x_0) = \int_0^{\infty} \frac{1}{z(x_0)^{1+\varepsilon}} dz = \int_{z(x_0)}^{\infty} \frac{1}{z^{1+\varepsilon}} dz \leq \int_0^{T(x_0)} \frac{1}{z^{1+\varepsilon}} dz.
\]

\[
T(x_0) = \int_0^{\infty} \frac{1}{z^{1+\varepsilon}} dz \leq \int_0^{\infty} \frac{1}{az^{1+\varepsilon} + \eta} dz. \tag{A.4}
\]

Since $\gamma = 1 + \varepsilon > 1$, by Lemma 1, it gets that
Thus,
\[
T(x_0) \leq \int_0^{z(x_0)^{(1/\gamma)}} \frac{T_c}{C_v} \cdot \frac{1}{1-q} \cdot \frac{1}{2^{1/\gamma} \left(a^{1/\gamma}z + \eta^{1/\gamma}\right)^{1/\gamma}} \, dz.
\] (A.6)

Then,
\[
T(x_0) \leq \frac{T_c}{C_v} \cdot \frac{1}{1-q} \cdot \frac{2^{1/\gamma}}{a^{1/\gamma}(1-\gamma)} \left(a^{1/\gamma}z + \eta^{1/\gamma}\right)^{1/\gamma}
= \frac{T_c}{C_v} \cdot \frac{1}{1-q} \cdot \frac{2^{1/\gamma}}{a^{1/\gamma}(1-\gamma)} \left(\frac{z(x_0)}{a^{1/\gamma}(1-\gamma)}\right)^{1/\gamma}.
\] (A.7)

If \( V(x_0) = 0 \), we have
\[
\lim_{V(x_0) \to 0} T(x_0) = 0.
\] (A.8)

If \( V(x_0) \to \infty \), we have
\[
T(x_0) \leq \frac{T_c}{C_v} \cdot \frac{1}{1-q} \cdot \frac{2^{1/\gamma}}{a^{1/\gamma}(1-\gamma)} \left(\frac{V(x_0)}{a^{1/\gamma}(1-\gamma)}\right)^{1/\gamma}
= \frac{T_c}{C_v} \cdot \frac{1}{1-q} \cdot \frac{2^{1/\gamma}}{a^{1/\gamma}(1-\gamma)} \left(\frac{z(x_0)/a^{1/\gamma}(1-\gamma)}{\eta^{1/\gamma}}\right)^{1/\gamma}
= T_c.
\] (A.9)

It is clear that
\[
\lim_{V(x_0) \to \infty} T(x_0) = T_c.
\] (A.10)

This proof is completed. \( \square \)

**Data Availability**

No experimental data were used to support this study. The simulation results can be reproduced with MATLAB and/or Simulink just using the mathematical model of neural networks and the proposed controllers.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

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