Classifying Vortex Solutions to Gauge Theories

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Abstract

We classify the spectrum, family structure and stability of Nielsen-Olesen vortices embedded in a larger gauge group when the vacuum manifold is related to a symmetric space.

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1 Introduction

The aim of this paper is to classify the structure of the solution set of embedded Nielsen-Olesen vortices for general gauge theories. To do this requires the concepts of embedded defects, first formally introduced in [1]. In many ways this work improves on and extends the initial treatment of vortex classification attempted in [2], and the companion paper to that [3] contains many illustrative examples of the formalism discussed herewithin.

This first section is devoted to a quick review of gauge theories coupled to a scalar field sector. This will serve to set the scene and establish the notation for the later sections of this paper where we shall classify vortex solutions. We emphasize in particular how the symmetry breaking relates to the group structure of gauge theory.

Fermions will be excluded from the discussion, so we shall only have to consider the effects of scalar-gauge interaction. This is appropriate because fermions seem to merely modify the form of solution, introducing fermionic zero modes around the background scalar-gauge configuration, whereas here we are interested in the background configuration itself, which is determined by the global topographical features of the gauge theory and its symmetry breaking.

1.1 Yang-Mills Theories Coupled to a Scalar Field

We are concerned with field theories whose basic dynamical variables are gauge potentials $A_\mu$ and scalar fields $\Phi$. Interaction of the scalar field $\Phi$ with the gauge potential $A_\mu$ is specified by the gauge symmetry group $G$, a compact Lie group acting upon the scalar field via the representation $D$ of $G$. We consider theories of the form described by the Lagrangian (6) below. Once the gauge group and its action upon the scalar field are specified the theory is completely determined up to the strength of scalar field self couplings and the gauge coupling constants.

The gauge potential $A_\mu$ lies in the Lie algebra of $G$, which we denote by $\mathcal{G}$. The field tensor of the gauge potential is

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \in \mathcal{G}. \quad (1)$$
The local actions of the gauge group upon the scalar field and gauge potential take the form (for \( g(x) \in G \))

\[
\Phi(x) \mapsto \Phi'(x) = D(g(x))\Phi(x),
\]

\[
A_\mu(x) \mapsto A'_\mu(x) = g(x)A_\mu(x)g(x)^{-1} - (\partial_\mu g(x))g(x)^{-1}.
\]

The field tensor transforms under a similarity transformation

\[
F_{\mu\nu}(x) \mapsto F'_{\mu\nu}(x) = g(x)F_{\mu\nu}(x)g(x)^{-1}.
\]

The covariant derivative of \( \Phi \),

\[
D_\mu \Phi = (\partial_\mu + d(A_\mu)) \Phi,
\]

transforms according to

\[
D_\mu \Phi(x) \mapsto (D_\mu \Phi)'(x) = D(g(x))D_\mu \Phi(x).
\]

Here \( d(X) \) is the derived representation of \( D \), describing how \( G \) acts on \( \Phi \) by the relation \( D(e^X) = e^{d(X)} \).

We need inner products on \( G \) and on the space \( V \) of \( \Phi \) values, both of which must be invariant under the actions of the group \( G \): on \( V \) \( \langle D(g)\Phi, D(g)\Phi \rangle = \langle \Phi, \Phi \rangle \),

while on \( G \) \( \langle \text{Ad}(g)X, \text{Ad}(g)X \rangle = \langle X, X \rangle \), where \( \text{Ad}(.) \) is the adjoint action of \( G \) on \( G \) defined by \( ge^X g^{-1} = e^{\text{Ad}(g)X} \). Suitable forms and properties of these are defined in Appendix B.

Assembling the above, the minimal gauge-invariant Lagrangian describing the gauge picture interaction of a scalar field with a gauge potential is:

\[
\mathcal{L}[\Phi, A_\mu] = -\frac{1}{4} \langle F_{\mu\nu}, F^{\mu\nu} \rangle + \frac{1}{2} \langle D_\mu \Phi, D^\mu \Phi \rangle - V[\Phi].
\]

Here \( V[\phi] \) is the scalar potential, describing self interaction of the scalar field, and is constrained to be invariant under local (gauge) transformations\(^\S\).

The gauge coupling constants manifest themselves in an interesting way. The Lie algebra \( G \) has a natural decomposition into commuting subalgebras

\[
G = G_1 \oplus \cdots \oplus G_n,
\]

\(^\S\)Note that we are using the same symbol for inner products on \( G \) and \( V \); we hope it should be clear from the context which inner product we are considering.

\(^\S\)Sensible quantisation also requires the potential to be a fourth order (or less) polynomial.
with each $G_f$ either simple or one-dimensional. Thus an $\text{Ad}(G)$ invariant inner product on $G$ has $n$ scales, related to the norm on each $G_f$. With respect to this inner product the unit norm generators are written $q_f X_f$; these $q_f$'s are interpreted as the gauge coupling constants appertaining to $G_f$, as explicitly illustrated in sec. (5.2).

Furthermore, taking the limit $q_f \to 0$ makes the symmetry $G_f$ global. For a unit norm generator $q_f X_f$, one sees that the effect of taking $q_f \to 0$ is to decouple the gauge field in Eq. (4), rendering $G_f$ global.

1.2 Symmetry Breaking

The form of the scalar potential $V$ may cause the fully symmetric theory to be unstable.

The trivial background vacuum takes field values $\Phi(x) = 0$ and $A_\mu(x) = 0$, and a perturbative quantum field theory around it has full gauge symmetry $G$. However, this may not be the global energy minimum among all background field configurations. In such a situation the unstable background will decay to the stable background vacuum, where $\Phi(x) = \Phi_0 \neq 0$. Such a background does not respect the full symmetries of the original theory, but only a compact subgroup $H$. In this case, the theory may admit solitonic scalar-gauge configurations that asymptotically tend to the stable background. In this paper we are primarily interested in the spectrum and classification of these configurations.

It is the form of the scalar potential $V[\Phi]$ that determines whether, and how, the gauge symmetry is spontaneously or dynamically broken. The stable background vacuum is the one in which $\Phi(x) = \Phi_0$, $A_\mu(x) = 0$, where $\Phi_0$ is a global minimum of the scalar potential. Then the quantum field theory with this vacuum is obtained by perturbing around $\Phi_0$, and is described by the Lagrangian

$$L_H[\Psi, A_\mu] = L_G[\Phi_0 + \Psi, A_\mu],$$

which has residual gauge symmetry

$$H = \{ h \in G : D(h) \Phi_0 = \Phi_0 \}. \quad (9)$$

The quantum field theory contains a spectrum of massive gauge bosons corresponding to those symmetries that are broken; these gauge bosons have internal directions
in \( \mathcal{M} \), where
\[
\mathcal{G} = \mathcal{H} \oplus \mathcal{M},
\]
the direct sum being defined by the inner product \( \langle \cdot, \cdot \rangle \).

Because the theory has gauge symmetry \( \mathcal{G} \) (about the trivial vacuum), the global minimum of the scalar potential generally has degenerate values, forming a compact manifold. This is the vacuum manifold,
\[
M = D(G)\Phi_0 \cong G/H;
\]
it is the shape of this manifold that determines the non-perturbative, solitonic spectrum of the scalar-gauge theory. The coset space \( G/H \) is homogenous, and at the identity element has tangent space \( \mathcal{M} \). The group \( H \) acts as rotations on points of \( G/H \) around the identity element. This picture is echoed in the space \( D(G)\Phi_0 \subset \mathcal{V} \), where the tangent space at \( \Phi_0 \) is \( d(\mathcal{M})\Phi_0 \) and \( D(H) \) acts as rotations of points of \( D(G)\Phi_0 \) around \( \Phi_0 \). One thus interprets \( \mathcal{M} \) as vectors that move \( \Phi_0 \), and \( H \) as actions that rotate around \( \Phi_0 \).

The set of all symmetries of the vacuum manifold form the isometry group
\[
I = \{ a \in \text{aut}(\mathcal{V}) : aM = M \},
\]
which is generally larger than \( \mathcal{G} \). This has a corresponding induced representation \( \tilde{D} \) of \( I \), reducing to \( D \) of \( G \), and a subgroup
\[
J = \{ j \in I : \tilde{D}(j)\Phi_0 = \Phi_0 \},
\]
containing \( H \). The vacuum manifold may then be re-expressed
\[
M \cong \frac{G}{H} \cong \frac{I}{J}.
\]

In some cases these additional elements of \( I \) represent hidden global symmetries of the theory; we discuss the \( SU(2) \to 1 \) example in Sec. (5.1). In other cases, however, the additional elements are not symmetries of the theory. An example is the electroweak model, discussed in Sec. (5.2), where \( I = SU(2) \times SU(2) \) is not a symmetry of the model unless \( \sin \theta_w = 0 \). However \( M \) is isomorphic to the symmetric space \( I/J \cong S^3 \).

In what follows we shall assume that all hidden symmetries have been included in the symmetry group \( G \).
2 Vortices

This is the pivotal section of the paper: here we describe how vortices are classified; whereas the rest of the paper will be used to explain and establish these results.

The tactic is to consider the archetypal Abelian-Higgs vortex, and then to embed it into a larger theory — one may then examine the associated gauge freedom by moving it continuously around within the larger theory. This will naturally partition the set of all vortex solutions into families of gauge equivalent vortices. We will find that this process is equivalent to a partitioning of the tangent space to the vacuum manifold, \( M \), in a very natural way; this equivalence being provided by a natural association between vortices and vectors in \( M \). Thus by explicitly partitioning \( M \) into its constituent parts a classification of vortices is achieved.

Our classification of vortices is on a level separate to the question of stability. Classification relies on the group actions on the tangent space, whereas stability relies on the topology of the vacuum manifold.

2.1 The Abelian-Higgs Vortex

This is the archetypal model of a vortex, the Nielsen-Olesen vortex, within the simplest theory that contains such vortices.

For simplicity, we consider only straight, static vortices, and impose cylindrical symmetry.

The Abelian-Higgs model consists of a one-dimensional complex scalar field coupled to a \( U(1) \) gauge potential \( A_\mu \), with a symmetry breaking Landau potential. (Note that, to conform with our general notation, we take the gauge potential \( A_\mu \) to be pure imaginary.) The Lagrangian is of the form

\[
\mathcal{L}[\Phi, A_\mu] = -\frac{1}{4} F_{\mu\nu}^* F^{\mu\nu} + \frac{1}{2} D_\mu \Phi^* D^\mu \Phi - V[\Phi],
\]

with

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (16a)
\]

\[
D_\mu \Phi = (\partial_\mu + q A_\mu) \Phi, \quad (16b)
\]

\[
V[\Phi] = \frac{1}{2} \lambda (\Phi^* \Phi - \eta^2)^2. \quad (16c)
\]

*Semi-local, or dynamical, stability is slightly more subtle; we discuss this later.
Here $\lambda > 0$ and $\eta$ are parameters of the potential, and $q$ the gauge coupling constant. For $\eta^2 > 0$ the trivial vacuum is unstable and decays to a vacuum of the form $\Phi_0 = \eta e^{i\chi}$. Perturbations around this comprise a field theory consisting of a massive gauge field interacting with a massive scalar field; this theory having no manifest gauge symmetry. All this information is encoded within the symmetry breaking

$$U(1) \rightarrow 1. \quad (17)$$

The gauge symmetry breaking leads to the existence of vortex solutions. In the temporal gauge $A_0 = 0$, they have the form of the Nielsen-Olesen Ansatz:

$$\Phi(r, \theta) = \eta f_{NO}(r) e^{in\theta}, \quad (18a)$$

$$A(r, \theta) = \frac{g_{NO}(r)}{r} \left( \frac{in}{q} \right) \hat{\theta}, \quad (18b)$$

with boundary conditions

$$f_{NO}(0) = g_{NO}(0) = 0, \quad \text{and} \quad f_{NO}(\infty) = g_{NO}(\infty) = 1, \quad (19)$$

Asymptotically the configurations wind around the possible degenerate vacua, and then over the intervening space the fields continuously interpolate. The $n = \pm 1$ solutions are stable because there is no way they may be continuously deformed to the vacuum $\Phi(x) = \Phi_0$ whilst keeping the asymptotic field within the vacuum manifold. The asymptotic field configuration, which forms a circle in the space of scalar field values, winds around the spatial circle at infinity — since this map has topological degree it may not be continuously deformed to other inequivalent situations.

### 2.2 Vortices in General Gauge Theories

We now embed the archetypal $U(1) \rightarrow 1$ vortex (18a, 18b) within a larger theory:

$$G \rightarrow H \cup \cup \quad H \cap U(1) = 1 \quad (20)$$

$$U(1) \rightarrow 1.$$  

The general vortex Ansatz is

$$\Phi(r, \theta) = f_{NO}(r) D(e^{X\theta}) \Phi_0, \quad (21a)$$

$$A(r, \theta) = \frac{g_{NO}(r)}{r} X \hat{\theta}. \quad (21b)$$
where \( X \in \mathcal{M} \) is the generator of the embedded \( U(1) \). In this paper we shall classify vortex solutions of this form.

It should be remarked that the Ansatz (21a, 21b) is not the most general we could assume. The lowest-energy configuration with these boundary conditions may in some cases be distorted from the pure embedded-vortex solution, though the distortions are irrelevant for the classification.

Asymptotically the scalar field winds around a geodesic \( D(e^{X\theta})\Phi_0 \) on the vacuum manifold \( D(G)\Phi_0 \), and nearer the core the solution continuously interpolates over a two dimensional surface within \( \mathcal{V} \) containing this geodesic. There could be components of the scalar field \( \Phi \) out of this surface that vanish at infinity but are non-zero in the core [4]. However, such components do not affect the classification of the vortex solutions, so we may ignore them here.

The asymptotic gauge field in the minimum-energy configuration is in the direction of \( X \in \mathcal{M} \), because if we were to add any component in \( \mathcal{H} \), the effect would be to leave the scalar field unchanged but to add an extra term to the gauge-field magnetic energy. Though one should note that when gauge field takes components within more than one simple part of \( G \) the various components may have different radial dependence, and so near the core there may be components in \( \mathcal{H} \). Again, however, these components in \( \mathcal{H} \) do not affect the classification.

### 2.2.1 Vortex Generators

We shall always consider vortices in the temporal gauge, and assume cylindrical symmetry, i.e. their field values are cylindrically symmetric, and stationary in time. Non-trivial gauge transformations that respect this gauge are the global (or rigid) gauge transformations where \( g(x) \) in Eq. (2a,2b) is independent of \( x \).

By Eq. (11) we can use the gauge freedom of \( G \) to fix \( \Phi_0 \). Then vortex solutions are defined by just one variable, the vortex generator \( X \in \mathcal{M} \). To classify vortex solutions, we must answer the questions: which generators \( X \in \mathcal{M} \) define vortex solutions? And which of those are gauge equivalent?

The remaining gauge freedom corresponds to global transformations by elements \( h \in H \), the residual symmetry group. Two vortices described by Ansätze of the form (21a, 21b) are gauge equivalent if and only if they are related by such a transforma-
tion. Since $\Phi_0$ is unaltered, the only effect is to transform the generator $X \in \mathcal{M}$:

$$X \mapsto \text{Ad}(h)X, \quad \text{with } h \in H,$$

where $\text{Ad}$ is the adjoint representation of $G$ acting on $\mathcal{G}$.

Geometrically, the scalar boundary conditions of a vortex describe a geodesic $D(e^{X\theta})\Phi_0$ on the vacuum manifold, with the gauge potential being defined from the tangent vector to the corresponding curve $e^{X\theta}$ in $G$. Global gauge transformations defined by elements of $H$ keep the point $\Phi_0$ fixed, and rotate the geodesic around this point. Thus vortex generators that may be rotated into one another by $\text{Ad}(H)$ define gauge equivalent vortices.

There are two constraints that the vortex Ansatz must obey [5]:

(i) Closure. The asymptotic scalar field forms a closed geodesic with $D(e^{2\pi X})\Phi_0 = \Phi_0$. Equivalently, $e^{2\pi X} \in H$.

(ii) The Ansatz is a solution to the equations of motion. An argument of [1] identifies this to be when: fields in the vortex do not induce currents perpendicular (in Lie algebra space) to it. This can be shown to be equivalent to (see appendix A):

if $X$ is a vortex generator, then for all $X^\perp$ such that $\langle X, X^\perp \rangle = 0$ one has $\langle d(X)\Phi_0, d(X^\perp)\Phi_0 \rangle = 0$.

In other words the map $X \mapsto d(X)\Phi_0$ is conformal over the classes of generators that define vortex solutions.

### 2.2.2 Family Structure of Vortex Solutions

Consider the action of $\text{Ad}(H)$ on $\mathcal{M}$. In general, $\mathcal{M}$ may be reducible; let us write

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$$

where $\mathcal{M}_i$ is (real) irreducible under the action of $\text{Ad}(H)$. In Appendix B we show that the (real) inner products on $\mathcal{V}$ and $\mathcal{M}$ are related upon this decomposition by:

$$\langle d(X_i)\Phi_0, d(Y_j)\Phi_0 \rangle = \lambda_i \lambda_j \langle X_i, Y_j \rangle, \quad X_i \in \mathcal{M}_i, Y_j \in \mathcal{M}_j,$$

where $\lambda_i = \frac{\|d(X_i)\Phi_0\|}{\|X_i\|}$.

and $\lambda_i$ is constant upon its particular $\mathcal{M}_i$. 

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Comparing Eq. (24) with condition (ii) above we see that generally, vortex generators lie only in the individual \(M_i\)’s, but for some pairs of values of gauge coupling constants, defined by the condition,

\[\lambda_i(q_1, \cdots, q_n) = \frac{\|d(X_i)\Phi_0\|}{\|X_i\|} = \frac{\|d(X_j)\Phi_0\|}{\|X_j\|} = \lambda_j(q_1, \cdots, q_n), \tag{25}\]

the solution space increases, encompassing combinations of vortices with generators in \(M_i\) and vortices with generators in \(M_j\). Then the vortex generators are those generators that close in \(M_i \oplus M_j\); such vortices we dub combination vortices.

Leaving aside the special case of combination vortices, we may concentrate on vortices described by generators \(X\) in one specific irreducible subspace \(M_i\). First, we show that every \(X\) is equivalent to one lying in a subspace that plays the same role as a Cartan subalgebra in a Lie algebra.

For each \(M_i\) one may define a rank analogous to the rank of a Lie group: the rank \(r\) of \(M_i\) is the maximum number of linearly independent commuting generators in \(M_i\). Let \(T_1, \ldots, T_r\) be such a set, and consider the algebra, \(\mathcal{T}\) say, that they generate. It is an Abelian subalgebra of \(G\), and \(T = \exp \mathcal{T}\) is an Abelian subgroup of \(G\), a torus.

Then, in general we conjecture that vortex generators are given by the following crucial result:

\[\text{for } M_i \text{ of rank } r, \text{ we can find a set of linearly independent commuting generators } \{T_1, \ldots, T_r\} \text{ such that the } X \in M_i \text{ satisfying the closure condition may be written }\]

\[X = \text{Ad}(h) \sum_{k=1}^{r} l_k T_k, \tag{26}\]

\[\text{with } h \in H, \text{ and } l_k \in \mathbb{Z}.\]

In sec. (3) we prove the above result for two categories of vacuum manifolds: symmetric spaces, and symmetric spaces that are modified to admit semi-local vortices. Practically, these two situations cover most cases of physical relevance, although we suspect that the result is true generally.
2.2.3 Gauge Equivalent Embedded Vortices

Each of the vortices labelled by the set of coefficients \( \{l_j\} \) has an orbit of gauge equivalent solutions associated with it,

\[
\text{Ad}(H) \sum_{k=1}^{r} l_k T_k \cong H \, H_{\{l_j\}},
\]

(27a)

where

\[
H_{\{l_j\}} = \{ h \in H : \text{Ad}(h) \sum_{k=1}^{r} l_k T_k = \sum_{k=1}^{r} l_k T_k \}.
\]

(27b)

Generally these orbits differ in structure depending upon which element of the lattice of coefficients is considered.

In addition, these orbits may contain vortex generators with different sets of coefficients \( \{l_j\} \) that are gauge equivalent. In fact, \( H \) acts as a discrete transformation group on the lattice \( L \). We can construct this group as follows.

First, define the subgroup of \( H \) that maps \( T \) into itself:

\[
N(T) = \{ h \in H : \text{Ad}(h) T = T \}.
\]

(28)

Then, introduce the subgroup which leaves each point of the lattice unaltered:

\[
B(T) = \{ h \in H : \text{Ad}(h) T_k = T_k, k = 1, \ldots, r \}.
\]

(29)

The effective transformation group on the lattice \( L \) is the quotient group \( K = N(T)/B(T) \). Thus finally the set of gauge-inequivalent vortex generators in \( M_i \) is the quotient \( L/K \), comprising equivalence classes of lattice points under the induced action of \( K \).

We conjecture that the effect of \( K \) upon the lattice \( L \) allows us to enumerate gauge inequivalent vortex generators by \( l_1 \geq \cdots \geq l_r \) for those \( M_i \)'s of dimension larger than one. This conjecture is largely motivated by example (5.3), and is beyond the scope of the present work to prove.

This completes the classification of vortex generators in each of the subspaces \( M_i \). Of course, if the coupling constants allow then one may also have combination vortex generators, as described above.
3 Symmetric Spaces and the Conjugacy of Maximal Tori

The goal of this section is to discuss the mathematics of, and prove, the following crucial result upon vacuum manifolds that are either compact symmetric spaces or compact symmetric spaces that are modified to admit semi-local vortices:

**for \( \mathcal{M}_i \) of rank \( r \), we can find a set of linearly independent commuting generators \( \{T_1, \ldots, T_r\} \) such that the \( X \in \mathcal{M}_i \) satisfying the closure condition may be written**

\[
X = \text{Ad}(h) \sum_{k=1}^{r} l_k T_k, \tag{30}
\]

**with \( h \in H \), and \( l_k \in \mathbb{Z} \).**

Before establishing the above result for our cases of interest, we shall discuss some basic properties of symmetric spaces that will be useful as a background to the work in this paper [6].

There are several ways of characterising when the vacuum manifold \( G/H \) is a symmetric space and we find the most convenient in the context of this work to be:

\( G/H \) is a symmetric space if the decomposition \( G = H \oplus M \) obeys

\[
[M, M] \subseteq H. \tag{31}
\]

This has the consequence (Cartan, [7]) that \( M \) may be written as \( M \cong \exp(M) \), and then \( G \) has the polar decomposition into cosets \( G = \exp(M)H \). Geodesics in \( \exp(M) \) are easily found, and those passing through \( 1 \) may be written as \( \gamma_X(t) = \exp(Xt) \) with \( X \in M \).

Other important features of symmetric spaces may be found from the above structure, such as a well defined composition of elements in \( M \) relating to a symmetry operation upon geodesics. However these features are not central to this paper and we shall not discuss them further.

Cartan has proved that the compact symmetric spaces can be written as \( M = M_0 \times M_+ \); where \( M_0 \) is the toroidal part of \( M \), written \( M_0 = U(1)^k \), and \( M_+ \cong G/H \), with \( G \) semi simple, is the rest. We shall establish the main result of this section firstly upon \( M_0 \), and then \( M_+ \), before discussing the case when \( M \) has been modified from a symmetric space so as to admit semi-local vortices.
3.1 Toroidal $M_0$

For the toroidal part of the vacuum $M_0$ we may write $M_0 \cong G/H$ with $G = U(1)^k$ and $H = 1$. Trivially then $\mathcal{M}_0 = \mathcal{G} = u(1)^k$ and then $[\mathcal{M}_0, \mathcal{M}_0] = 0 \subseteq \mathcal{H}$. Eq. (26) is trivial in such a case, since each $\mathcal{M}_i$ is one-dimensional.

3.2 Non-toroidal $M_+$

For the non-toroidal part of a symmetric vacuum $M_+$, Cartan has shown that $M_+ \cong G/H$ where $G$ is a semi-simple Lie group and $H$ is embedded in $G$ according to Eq. (31) above. We now establish the main result of this section for such cases.

Firstly we establish the following, a variant of Hunt’s lemma [8]:

for $X, Y \in \mathcal{M}_i$ there exists $h_0 \in H$ such that $[\text{Ad}(h_0)X, Y] = 0$.

This result is proved by considering a function $f : H \to \mathbb{R}$ defined by

$$f(h) = \langle \text{Ad}(h)X, Y \rangle. \quad (32)$$

Since this is a bounded real function on a compact domain, it attains its maximum, say at a point $h_0$. Then for any $K \in \mathcal{H}$,

$$0 = \frac{d}{dt} f(e^{Kt}h_0) \bigg|_{t=0} = \langle \text{ad}(K)\text{Ad}(h_0)X, Y \rangle = \langle K, [\text{Ad}(h_0)X, Y] \rangle. \quad (33)$$

Finally, since $G \to H$ defines a symmetric space, $[\text{Ad}(h_0)X, Y] \in \mathcal{H}$ and therefore vanishes, proving the lemma.

From this result, in three steps we may establish our main result when the vacuum is a symmetric space.

Firstly, there exists $Y \in \mathcal{M}_i$ such that the centraliser $\mathcal{C}$ of $Y$ in $\mathcal{M}_i$

$$\mathcal{C}(Y) = \{ U \in \mathcal{M}_i : [Y, U] = 0 \}, \quad (34)$$

is precisely $\mathcal{T}$. This follows by considering elements $Y$ that define paths $\{ \exp(\alpha Y) : \alpha \in \mathbb{R} \}$ dense in $T = \exp(\mathcal{T})$. Then for any $Z \in \mathcal{M}_i$ such that $[Z, Y] = 0$, also $[Z, \mathcal{T}] = 0$ and thus, by maximality, $Z \in \mathcal{T}$.

Next, we use the result established above. Thus for any $X \in \mathcal{M}_i$ there exists $h_0 \in H$ such that $\text{Ad}(h_0)X \in \mathcal{C}(Y)$. Hence, we have shown that there exists $h_0 \in H$ with $\text{Ad}(h_0)X \in \mathcal{T}$. 

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Finally, we note that within $\mathcal{T}$, the generators $X$ that satisfy the condition of closure, $e^{2\pi X} \in H$, obviously form a lattice, $\mathcal{L}$ say, and we can choose the generators $T_j$ so that this comprises precisely the linear combinations with integer coefficients, $\sum_j l_j T_j, l_j \in \mathbb{Z}$.

3.3 $M$ Admits Semi-local Vortices

Here we deal with the situation when $M \cong G/H$ admits semi-local vortices, and has been modified from a symmetric space. Specifically we are going to consider vacuum manifolds of the form

$$M \cong \frac{K \times A/Z}{H},$$

(35)

where $K$ is semi-simple, $A$ is Abelian and connected, $Z$ is a finite subgroup, and $H$ is isomorphic to a subgroup $L$ of $K$, such that $K/L$ is a compact symmetric space.

Perhaps the easiest way to see the structure of Eq. (35) is to represent it by a two step symmetry breaking:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{A} \rightarrow \mathcal{L} \oplus \mathcal{A} = \mathcal{H}' \oplus \mathcal{C}(L) \oplus \mathcal{A}$$

$$\rightarrow \mathcal{H} = \mathcal{H}' \oplus \mathcal{B},$$

(36)

where here $H'$ is the semi-simple part of $L$, $\mathcal{C}(L)$ is the connected centre of $\mathcal{L}$, and $\mathcal{B} \cong \mathcal{C}(L)$.

Decomposing $\mathcal{G} = \mathcal{H} \oplus \mathcal{M}$, we see that $\mathcal{M}$ is made up of two parts, and may be written as $\mathcal{M} = \mathcal{P} \oplus \mathcal{B}^\perp$, where

$$\mathcal{K} = \mathcal{L} \oplus \mathcal{P},$$

(37a)

$$\mathcal{C}(L) \oplus \mathcal{A} = \mathcal{B} \oplus \mathcal{B}^\perp.$$  

(37b)

From this decomposition the main result (26) of this section can be established: it is trivially true upon $\mathcal{B}^\perp$ and, because $K/L$ is a symmetric space, it is true also on $\mathcal{P}$. 

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4 The Stability of Vortices in General Gauge Theories

The stability of the vortices is on a level separate from the family structure discussed above, though clearly all vortices in the same family have the same stability structure since all such vortices are gauge equivalent.

Classical stability is determined by the energetics of the vortex solution. If it is energetically favourable for a vortex solution to continuously relax to the vacuum then that vortex is unstable. This is the general situation for vortex solutions, and generally one needs some property of the scalar-gauge theory to ‘prop-up’ the solution, so as to make the configuration a local minimum of the energy. There are two ways this may be achieved: either through the topology of the vacuum manifold, or through the dynamics of the solution. We discuss each of these cases separately.

4.1 Topological Stability

Topological stability arises through the vacuum manifold not being simply connected, so that

$$\pi_1(G/H) \neq 0.$$  \hspace{1cm} (38)

Vortex families whose boundary conditions are the elements of the non-trivial homotopy classes of Eq. (38) are imbued with a conserved topological charge, which in appropriate cases guarantees their stability. Physically, the vacuum manifold contains loops that are incontractible, so that the vortex solutions corresponding to such loops are topologically obstructed from continuously deforming to the vacuum.

There are two distinct ways in which topological stability may occur: either through an Abelian or a non-Abelian part of the symmetry breaking. These two different situations also have contrasting properties for their family structure, and the nature of their stability. Therefore we discuss these two cases separately.

Abelian topological vortices are present for a symmetry breaking scheme such as

$$G = G' \times A/Z \to H \subseteq G',$$  \hspace{1cm} (39)
where $A$ is an Abelian subgroup, a torus

$$A = U(1)_1 \times \cdots \times U(1)_N$$

(40)

and $Z$ is a finite group. Then $\pi_1(G/H)$ takes the form

$$\pi_1(G/H) = P \times \mathbb{Z}^N,$$

(41)

where $P$ is a finite group. The Abelian topologically stable vortices have boundary conditions that belong to non-trivial $\mathbb{Z}^N$ homotopy classes.

The Lie algebra of $G$ decomposes under the adjoint action of $H$ into

$$\mathfrak{g} = \mathcal{H} \oplus \mathcal{M}' \oplus \mathcal{A}.$$  

(42)

The Abelian subalgebra $\mathcal{A}$ is generated by $N$ commuting generators $\{T_k\}$. We can always choose these so that the elements satisfying the closure condition are given by $\sum_k l_k T_k$, though only some of these may actually correspond to stable vortices. Vortices with distinct values of the quantum numbers $l_k$ are necessarily gauge inequivalent; there is no non-trivial family structure.

Note that $\mathcal{A}$ may be described as the intersection of $\mathcal{M}$ and the centre $\mathcal{C}$ of the Lie algebra $\mathfrak{g}$.

Non-Abelian topological stability can arise from elements outside the centre of the group; if for example the vacuum manifold $G/H$ has the form of the quotient of a simply connected manifold $M$ by a discrete group $\mathbb{Z}_n$, then $\pi_1(G/H) = \pi_1(M/\mathbb{Z}_n) = \mathbb{Z}_n$. This gives the vortex generators in the relevant $\mathcal{M}_i$ subspaces a mod-$n$ topological charge, imparting stability to the corresponding vortex solutions. Such stable vortices correspond to fractional quantum embedded vortices, and the solutions have non-trivial family structure because necessarily $\dim(\mathcal{M}_i) > 1$.

### 4.2 Dynamical Stability

Even when there is no topology to stabilise a vortex solution there may be non-trivial dynamics making the decay modes of the vortex energetically unfavourable: this is called dynamical stability \[9\]. Often this occurs when an Abelian symmetry couples much more strongly than a non-Abelian symmetry; then the decay modes
cost large gradient energies that cannot be compensated by inducing a non-Abelian
gauge field.

Such vortices are solutions to gauge theories that have gauge coupling constants
close to a ‘semi-local limit’, where the scalar-gauge theory admits semi-local vortices.
Preskill [10], has shown that semi-local vortices are solutions to gauge theories of
the form
\[ G_{\text{global}} \times U(1)_{\text{local}} \to H, \quad \text{with } H \cap U(1)_{\text{local}} = 1, \tag{43} \]
where the suffices ‘global’ and ‘local’ represent non-gauge and gauged symmetries,
respectively. Requiring that this does not admit topological vortices leads to the
condition \( H \not\subseteq G_{\text{global}} \). Note that Eq. (43) need only be a sub-part of a more general
symmetry breaking.

Thus we define a ‘semi-local limit’ of a gauge theory to be a limit of the gauge
coupling constants \( \{ q_f \} \) in which part of the gauge theory takes the form in Eq. (43).
Recall that the formal limit of taking a coupling constant \( q_f \to 0 \) decouples the gauge
field and makes the corresponding gauge symmetry global. Thus dynamically stable
vortices are generated by the corresponding generator of semi-local vortices, but
with the gauge coupling constants close to those in the semi-local limit.

The above may be expressed succinctly in terms of the group theory, describing
which vortex generators give solutions that may be dynamically stable. Recall that
\( C \) is the centre of the Lie algebra \( \mathcal{G} \), and denote the projection of an element \( X \in \mathcal{G} \)
into \( \mathcal{M} \) by \( \text{pr}_\mathcal{M}(X) \). We then have the following:

vortex generators \( X \in \text{pr}_\mathcal{M}(C) \) such that \( X \notin C \) define embedded vortices
that are stable in a well defined semi-local limit of the model and are thus
dynamically stable for a region of parameter space around that semi-local
limit.

We also have a useful subresult about the dimension of \( \mathcal{M}_i \)'s for such vortices:

vortices with a stable semi-local limit are always generated by generators
in one dimensional \( \mathcal{M}_i \)'s.

The proofs are given in Appendix C.
Together these give a nice interpretation of the role of semi-locality: considering \( \mathcal{M}_i \subset \text{pr}_\mathcal{M}(\mathcal{C}) \), then the embedding \( \mathcal{M}_i \subset \mathcal{G} \) is determined by the coupling constants \( \{g_i\} \). Essentially the semi-local limit is for coupling constants such that the embedding is \( \mathcal{M}_i \subset u(1)_{\text{local}} \). Then dynamical stability occurs when the embedding is close to this.

5 Examples

This section illustrates principles discussed generally in the preceding sections.

5.1 \( SU(2) \rightarrow 1 \)

This is an examples of a symmetry breaking that contains hidden symmetries.

Clearly, \( \mathcal{H} = 0 \) and \( \mathcal{M} = su(2) \). Then, since \( \mathcal{M} \) is a Lie algebra, the relation \([\mathcal{M}, \mathcal{M}] = \mathcal{M}\) holds.

However the vacuum manifold is a three-sphere embedded in \( \mathbb{C}^2 \), with an isometry group

\[
I = I(S^3) = SU(2)_l \times SU(2)_g,
\]

which is larger than the gauge group. The \( SU(2)_l \) represents the left actions upon a complex doublet, with generators represented by \( X_a = -\frac{1}{2}i\sigma_a \), and the \( SU(2)_g \) is a hidden global symmetry with generators \( Y_1 = -\frac{1}{2}\sigma_2 K \), \( Y_2 = \frac{1}{2}i\sigma_2 K \), and \( Y_3 = -\frac{1}{2}i\mathbb{I}_2 \). Here \( K \) is the complex conjugation operator and \( \sigma_a \) are the Pauli spin matrices.

Then the stability group of \( I \) upon a point of \( M \) is

\[
J = SU(2)_{l+g},
\]

the diagonal subgroup lying between \( SU(2)_l \) and \( SU(2)_g \). Hence the actual symmetry breaking is

\[
SU(2)_l \times SU(2)_g \rightarrow SU(2)_{l+g},
\]

which contains the gauge symmetry breaking \( SU(2)_l \rightarrow 1 \), and has additional global symmetry \( SU(2)_g \). Defining \( \mathcal{I} = \mathcal{J} \oplus \mathcal{N} \) one can easily verify \([\mathcal{N}, \mathcal{N}] \subseteq \mathcal{J}\).

It is interesting to note that the above model is the Weinberg-Salam model, with electroweak mixing angle \( \sin \theta_w = 0 \).
5.2 $SU(2) \times U(1) \rightarrow U(1)$

The gauge symmetry $SU(2) \times U(1)$ acts on a two-dimensional complex scalar field $\Phi$ by the fundamental representation. The generators of $SU(2)$ are $X_a = -\frac{1}{2} i \sigma_a$ and the $U(1)$ generator is $X_0 = \frac{1}{2} i 1_2$.

The vacuum manifold is

$$M \cong \frac{SU(2) \times U(1)}{U(1)} \cong SU(2) \times SU(2_{\text{diag}}),$$

(47)

thus the isometry group is $I = SU(2) \times SU(2)$. This is in general not a symmetry group of the theory unless $\sin \theta_w = 0$.

The inner product on $su(2) \oplus u(1) \subset gl(C^2)$ is obtained from Appendix B:

$$\langle X, Y \rangle = -2 s_1 \text{tr} XY - \frac{s_2}{4} \text{tr} X \text{tr} Y.$$

(48)

To obtain the usual scalar-gauge coupling [50], $s_1$ and $s_2$ are related to the hypercharge $g'$ and isospin $g$ coupling constants such that

$$\langle X, Y \rangle = -\frac{1}{g^2} \left\{ 2 \text{tr} XY + (\cot^2 \theta_w - 1) \text{tr} X \text{tr} Y \right\},$$

(49)

with $\tan \theta_w = g'/g$. The unit norm generators are then $gX_a$ and $g'X_0$, and thus the components of a gauge field $A^\mu$ couple to $\Phi$ as

$$A^\mu \Phi = (gW^\mu_a X_a + g'Y^\mu X_0) \Phi.$$

(50)

Taking the vacuum to be defined from $\Phi_0 = \nu(0 \ 1)^	op$ the decomposition $\mathcal{G} = \mathcal{H} \oplus \mathcal{M}$ takes the form

$$\mathcal{H} = \begin{pmatrix} i \alpha & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} -i \beta \cos 2 \theta_w & \gamma \\ -\gamma^* & i \beta \end{pmatrix},$$

(51)

with $\alpha, \beta$ real and $\gamma$ complex. Then under $\text{Ad}(H)$, $\mathcal{M}$ decomposes to the irreducible subspaces $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, with

$$\mathcal{M}_1 = \begin{pmatrix} -i \beta \cos 2 \theta_w & 0 \\ 0 & i \beta \end{pmatrix} \quad \text{and} \quad \mathcal{M}_2 = \begin{pmatrix} 0 & \gamma \\ -\gamma^* & 0 \end{pmatrix}.$$

(52)

Additionally $\text{pr}(u(1)_Y) = \mathcal{M}_1$.

Thus we obtain a one-parameter family of gauge equivalent unstable embedded vortices and a gauge invariant embedded vortex with a stable semi-local limit. These are identified as the $W$-strings and $Z$-string of the Weinberg-Salam model.
5.3 \( SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \)

This is an example of a symmetry breaking where the irreducible \( M_i \)'s are of a non-trivial rank.

The gauge group \( G = SU(5) \) acts on the scalar field \( \Phi \in G \) by the adjoint action. For vacuum expectation value

\[
\Phi_0 = iv \begin{pmatrix} \frac{2}{3} & 1_3 & : & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & : & -1_2 \end{pmatrix},
\]

(53)

\( G \) breaks to \( H = SU(3)_c \times SU(2)_I \times U(1)_Y \):

\[
\begin{pmatrix} SU(3)_c & : & 0 \\ \vdots & \vdots & \vdots \\ 0 & : & SU(2)_I \end{pmatrix} \times \begin{pmatrix} e^{\frac{2i\theta}{3}} & 1_3 & : & 0 \\ \vdots & \vdots & \vdots \\ 0 & : & e^{-i\theta}1_2 \end{pmatrix} \subset SU(5).
\]

(54)

Reducing \( G \) into \( G = H \oplus M \), where

\[
M = \begin{pmatrix} 0_3 & : & A \\ \vdots & \vdots & \vdots \\ -A^\dagger & : & 0_2 \end{pmatrix},
\]

(55)

one finds this to be irreducible under the adjoint action of \( H \). However it is clear that \( \text{rank}(M) = 2 \), for example with generators \( T_1, T_2 \) such that

\[
T_i = \frac{1}{2} \begin{pmatrix} 0_3 & : & A_i \\ \vdots & \vdots & \vdots \\ -A_i^\dagger & : & 0_2 \end{pmatrix}, \quad \text{with} \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

(56)

then \([T_1, T_2] = 0\) and there are no other linearly independent generators that commute with both of these.

Then using Eq. (26) the vortex generators in \( M \) are:

\[
X = \text{Ad}(h) \left( l_1 T_1 + l_2 T_2 \right),
\]

(57)

with \( l_i \in \mathbb{Z} \) and \( h \in H \). These generators form a \( \mathbb{Z}^2 \) lattice of solutions \((l_1, l_2)\), with an orbit of solutions generated by \( \text{Ad}(H) \) from each point of the lattice. However,
there is degeneracy since it is easy to find \( h_1, h_2 \in H \) such that

\[
\begin{align*}
\text{Ad}(h_1)(l_1 T_1 + l_2 T_2) &= -l_2 T_1 + l_1 T_2, \\
\text{Ad}(h_2)(l_1 T_1 + l_2 T_2) &= -l_1 T_1 + l_2 T_2.
\end{align*}
\]

(58a)  

(58b)

Together these generate the discrete group of actions \( \text{Ad}(K) \) that takes the pair \((l_1, l_2)\) to the set of eight pairs \( \{(\pm l_1, \pm l_2), (\mp l_1, \pm l_2), (\pm l_2, \pm l_1), (\mp l_2, \pm l_1)\} \). Then the set of gauge inequivalent embedded vortex generators in \( SU(5) \) are those in Eq. (57), but with \((l_1, l_2)\) restricted to the octant \( l_1 \geq l_2 \geq 0 \).

To explicitly give an example of a vortex we give the \((1, 0)\) solution:

\[
\begin{align*}
\Phi(r, \theta) &= f_{\text{NO}}(r) \text{Ad}(e^{T_\theta}) \Phi_0, \\
A(r, \theta) &= g_{\text{NO}}(r) \frac{r}{\sqrt{r^2 + \theta^2}} \hat{X} \hat{\theta}.
\end{align*}
\]

(59a)  

(59b)

then, with a little calculation

\[
\text{Ad}(e^{T_\theta}) \Phi_0 = \Phi_0 - \frac{5 v}{3} A \sin \theta \left( \sin \frac{\theta}{2} M + \cos \frac{\theta}{2} T_1 \right),
\]

(60)

where \( A_{ij} = i(\delta_{i1}\delta_{j1} - \delta_{i4}\delta_{j4}) \). A non-trivial geodesic, that is certainly not contained within a plane through the origin.

### 6 Conclusions

We conclude by briefly summarising our classification.

We consider minimal, gauge invariant theories \( H \) of symmetry breaking \( G \to H \), where \( H \subset G \) are compact Lie groups. Then the following structure is required: writing \( G = \mathcal{H} \oplus \mathcal{M} \), one splits \( \mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n \), the irreducible subspaces of \( \mathcal{M} \) under \( \text{Ad}(H) \).

Then cylindrically symmetric, time independent vortex solutions are of the form

\[
\begin{align*}
\Phi(r, \theta) &= f_{\text{NO}}(r) D(e^{X\theta}) \Phi_0, \\
A(r, \theta) &= g_{\text{NO}}(r) \frac{r}{\sqrt{r^2 + \theta^2}} \hat{X} \hat{\theta},
\end{align*}
\]

with \( X \in \mathcal{M}_i \) specified by the following result, which has been rigorously established for \( G/H \) a symmetric space or a symmetric space modified to admit semi-local vortices:
for $\mathcal{M}_i$ of rank $r$, we can find a set of linearly independent commuting generators $\{T_1, \ldots, T_r\}$ such that the $X \in \mathcal{M}_i$ satisfying the closure condition may be written

$$X = \text{Ad}(h) \sum_{k=1}^{r} l_k T_k,$$

with $h \in H$, and $l_k \in \mathbb{Z}$.

Generally this classifies all vortices of the embedded Nielsen-Olesen type, however for certain critical values of the ratios of gauge coupling constants the solution set may increase to include combination vortices also, with generators $X$ lying between $\mathcal{M}_i$ and $\mathcal{M}_j$.

Stability of vortices may be of two types. Firstly, topological stability given by the non-trivial first homotopy classes of $G/H$. Secondly, dynamical stability, specified by the following result:

vortex generators $X \in \text{pr}_{\mathcal{M}}(\mathcal{C})$ such that $X \notin \mathcal{C}$ define embedded vortices that are stable in a well defined semi-local limit of the model and are thus dynamically stable for a region of parameter space around that semi-local limit.

Such vortex generators always lie in one-dimensional $\mathcal{M}_i$'s.

We conjecture that the above classification also holds true in general.

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Appendix A

The Lagrangian (6) gives field equations

$$D^\mu D_\mu \Phi = -\frac{\partial V}{\partial \Phi},$$  

(61a)

$$\langle D^\mu F_{\mu\nu}, X \rangle = \langle J_\nu, X \rangle = \langle d(X)\Phi, D_\nu \Phi \rangle - \langle D_\nu \Phi, d(X)\Phi \rangle.$$  

(61b)
The vortex Ansatz
\[ \Phi_{\text{vor}}(r, \theta) = f_{\text{NO}}(r)D(e^{X\theta})\Phi_0, \quad (62a) \]
\[ A_{\text{vor}}(r, \theta) = g_{\text{NO}}(r)\frac{X}{r} \hat{\theta}, \quad (62b) \]
naturally splits \( G \) globally into \( G = G_{\text{emb}} \oplus G_{\text{emb}}^\perp \) such that \( \langle A_{\text{vor}}(x), G_{\text{emb}} \rangle \neq 0 \) for all nonzero \( x \). In the scalar sector it provides a local decomposition \( \mathcal{V} = V_{\text{emb}}(\theta) \oplus V_{\text{emb}}^\perp(\theta) \) where \( V_{\text{emb}}(\theta) = D(e^{X\theta})(R\Phi_0 + Rd(X_{\text{emb}})\Phi_0) \), with \( R \) the real numberline.

Substituting this vortex Ansatz into the field equations and requiring it to be a solution yields
\[ \langle D^\mu D_\mu \Phi_{\text{vor}}(x), V_{\text{emb}}^\perp(\theta) \rangle = 0, \quad (63a) \]
\[ \langle J^\mu(x), G_{\text{emb}}^\perp \rangle = 0, \quad (63b) \]
from which one obtains,
\[ \langle \Psi, \frac{\partial V}{\partial \Phi} [\phi] \rangle = 0, \quad \Psi \in V_{\text{emb}}^\perp(\theta), \phi \in V_{\text{emb}}(\theta), \quad (64a) \]
\[ \langle d(X^\perp)\Phi, V_{\text{emb}}(\theta) \rangle = 0, \quad \Phi \in V_{\text{emb}}(\theta), X^\perp \in G_{\text{emb}}^\perp. \quad (64b) \]
The first condition constrains the scalar potential and may restrict combination vortex solutions. The second we rephrase: using the identity \( \langle d(G)\Phi, \Phi \rangle = 0 \) for \( \Phi \in \mathcal{V} \),
\[ \langle d(X^\perp)\Phi, V_{\text{emb}}(\theta) \rangle = 0 \iff \langle d(X^\perp)D(e^{X\theta})\Phi_0, D(e^{X\theta})d(X)\Phi_0 \rangle = 0 \]
\[ \iff \langle d(\text{Ad}(e^{-X\theta})X^\perp)\Phi_0, d(X)\Phi_0 \rangle = 0, \quad (65) \]
The last step by virtue of \( G_{\text{emb}} \) being a subgroup of \( G \) implies that \( \text{Ad}(e^{X\theta})G_{\text{emb}}^\perp = G_{\text{emb}}^\perp \).

One should note that the above proof is a modified version of that given in [2], to encompass the situation when \( V_{\text{emb}} \) is defined locally. In this context our definition of an embedded defect is more general than that discussed in [1], where they considered \( V_{\text{emb}} \) to be independent of \( \theta \) — which is equivalent to assuming that \( D(e^{X\theta})\Phi_0 \) is a circle centred on the origin within \( \mathcal{V} \). Generically, geodesics on non-spherical homogenous spaces are not of this form — an example of which is given in sec. (5.3).
Appendix B: Inner Product Structures on $\mathcal{V}$ and $\mathcal{G}$

The inner product on $\mathcal{V}$ is given by the (real) Euclidean inner product. Considering an element $\mathbf{v} \in \mathcal{V}$ as a column vector the form we shall use is:

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \text{Re} [\mathbf{v}_1^\dagger \mathbf{v}_2].$$ (66)

This inner product is isometric under automorphisms of $\mathcal{V}$.

To construct a non-degenerate $\text{Ad}(\mathcal{G})$-invariant inner product on $\mathcal{G}$, we start by splitting it into its commuting subalgebras, $\mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_n$, where each $\mathcal{G}_f$ is either simple or a one-dimensional $u(1)$ algebra. For each $\mathcal{G}_f$ there is a natural invariant scalar product $\{\cdot, \cdot\}_f$, unique up to a factor [12]. On a $u(1)$ algebra, we may set $\{X, X\}_f = 1$, where $X$ is the generator normalised so that $e^{2\pi X} = 1$. Upon a simple $\mathcal{G}_f$ it is (proportional to) the Killing form; regarding $\mathcal{G}_f$ as a matrix algebra embedded in $\text{gl}(\mathbb{C}^p)$ for some $p$, the inner product may be defined by

$$\{X, Y\}_f = -p \text{Re}[\text{tr}(X^\dagger Y)].$$ (67)

Then the most general $\text{Ad}(\mathcal{G})$-invariant scalar product on $\mathcal{G}$ is

$$\langle X, Y \rangle = \sum_{f=1}^n s_f \{X_f, Y_f\}_f,$$ (68)

for arbitrary values of the scaling factors $s_f$, where $X = X_1 + \cdots + X_n$, $Y = Y_1 + \cdots + Y_n$ and $X_f, Y_f \in \mathcal{G}_f$. The scales $s_f$ relate to the norm on each $\mathcal{G}_f$; for $X_f \in \mathcal{G}_f$ such that $\{X_f, X_f\} = 1$ the unit norm generator with respect to $\langle \cdot, \cdot \rangle$ is written $q_f X_f$, with $q_f$ a function of $\{s_1, \cdots, s_n\}$.

These two inner products over their respective spaces are related by the following result:

**Theorem** The (real) inner product on $\mathcal{V}$ is related to the inner product on $\mathcal{M}$ by

$$\langle d(X_i)\mathbf{v}, d(Y_j)\mathbf{v} \rangle = \lambda_i \lambda_j \langle X_i, Y_j \rangle, \quad X_i \in \mathcal{M}_i, Y_j \in \mathcal{M}_j,$$

where $\lambda_i = \frac{\|d(X_i)\mathbf{v}\|}{\|X_i\|}$.

Each $\lambda_i$ is constant upon its particular $\mathcal{M}_i$.

The theorem is original to this paper, although it is a fairly simple application of
linear algebra. It is a consequence of two inner products having a similar invariance, and is most easily derived from:

**Lemma** Consider $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ two (real) inner products on $\mathcal{V}$. Then $\mathcal{V}$ decomposes into $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n$ such that

$$\langle v_i, v_j \rangle_2 = \lambda_i \lambda_j \langle v_i, v_j \rangle_1, \ v_i \in M_i, \ v_j \in M_j. \quad (69)$$

Each $\lambda_i$ is constant over its particular $\mathcal{V}_i$.

**Proof** Choosing a basis for $\mathcal{V}$ orthonormal with respect to $\langle \cdot, \cdot \rangle_1$, the inner products can be written $\langle u, v \rangle_1 = u^\top v$ and $\langle u, v \rangle_2 = u^\top A v$, with $u, v \in \mathcal{V}$, with $A$ a non-degenerate symmetric matrix. Denote the eigenspaces of $A$ by $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n$, with corresponding eigenvalues $\lambda_i^2$, then for $v_i \in V_i$

$$\langle v_i, v_j \rangle_2 = v_i^\top A v_j = \lambda_i \lambda_j v_i^\top v_j. \quad (70)$$

Finally it is clear that $\lambda_i$ is constant over $\mathcal{V}_i$ because it is an eigenvalue, but also $\lambda_i = \| v_i \|_2 / \| v_i \|_1$.

Then from this lemma the theorem is easily proved:

**Proof** Firstly, let $\langle X, Y \rangle_1$ denote the usual $\text{Ad}(G)$ invariant inner product over $\mathcal{G}$, but restricted to $\mathcal{M}$; note that it is $\text{Ad}(H)$ invariant.

Then one may induce a second inner product on $\mathcal{M}$ from that on $\mathcal{V}$: this takes the form $\langle X, Y \rangle_2 = \langle d(X)v, d(Y)v \rangle$. Again this inner product is $\text{Ad}(H)$ invariant over $\mathcal{M}$.

From the lemma $\mathcal{M}$ reduces to $\tilde{\mathcal{M}}_1 \oplus \cdots \oplus \tilde{\mathcal{M}}_k$ such that for $X \in \tilde{\mathcal{M}}_i, Y \in \tilde{\mathcal{M}}_j$

$$\langle d(X)v, d(Y)v \rangle = \lambda_i \lambda_j \langle X, Y \rangle. \quad (71)$$

Here $\lambda_i = \lambda(X_i) = \| d(X_i)v \| / \| X_i \|$ is constant over each $\tilde{\mathcal{M}}_i$. However $\lambda(X)$ is also $\text{Ad}(H)$ invariant $\lambda(\text{Ad}(H)X) = \lambda(X)$, and thus also constant over each $\mathcal{M}_i$. Hence each $\tilde{\mathcal{M}}_i$ is a direct sum of $\mathcal{M}_i$'s and Eq. (71) is true for $X_i \in \mathcal{M}_i, Y_j \in \mathcal{M}_j$.

□

**Appendix C: Dynamical Stability Proofs**

For completeness we give the following proofs to results stated in section (4.2).

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Theorem  Vortex generators $X \in \text{pr}_M(C)$ such that $X \not\in C$ define embedded vortices that are stable in a well defined semi-local limit of the model and are thus dynamically stable for a region of parameter space around that semi-local limit.

Proof  Consider $X \in \text{pr}_M(C)$, such that it generates a closed geodesic. Then $X = X_c + X_h$, with $X_c \in C$, such that $\text{pr}_M(X_c) = X_c$ generating a $U(1)_c \subseteq C$ and $X_h \in H$ generating a $U(1)_h \subset H$. These define a decomposition of the symmetry breaking of the Lie algebras:

$$\mathcal{G} = \mathcal{G}' \oplus u(1)_c \to \mathcal{H} = \mathcal{H}' \oplus u(1)_h,$$

with $U(1)_c \cap H = 1$. It is now clear that the appropriate semi-local limit to obtain a symmetry breaking of the form Eq. (43) is to make coupling constants appertaining to $G'$ vanish. The corresponding vortex generator is $X$. This completes the proof.

Lemma  Vortices with a stable semi-local limit are always generated by generators in one dimensional $M_i$'s.

Proof  The proof that if $M_i \subseteq \text{pr}_M(C)$ then $\dim(M_i) = 1$ relies on the following identification: if one writes $M_{\ker}$ as the collection of one dimensional $M_i$'s then necessarily $M_{\ker} = \{X \in M : \text{Ad}(H)X = X\}$. Now we are reduced to showing $\text{pr}_M(C) \subseteq M_{\ker}$.

Consider $(X_c + X_h) \in \text{pr}_M(C)$, with $X_c \in C$ and $X_h \in H$. Then $[\mathcal{H}, X_c + X_h] \in \mathcal{M}$. But also $[\mathcal{H}, X_c + X_h] = [\mathcal{H}, X_h] \in \mathcal{H}$. Hence $[\mathcal{H}, X_c + X_h] = 0$, and equivalently $\text{Ad}(H)(X_c + X_h) = X_c + X_h$, proving the result.

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