2 Category of FRBSU Monoidal Categories and Crossed Modules

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December 23, 2015

Abstract

In that paper, we prove that the collection of all FRBSU monoidal categories and the collection of all crossed modules form a 2 category.

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1 Preliminaries

Throughout this paper, we are assuming that the symbol FRBSU monoidal category indicates a finite, rigid, braided monoidal category whose unit object $I$ is simple.

A category is small if the collection of all objects form a set. If $A$ is an object in a category $\mathcal{A}$, then a subobject $B$ of $A$ is an object with a monomorphism $B \to A$.

An object $A$ is simple in an abelian category $\mathcal{A}$ if for any injection $B \to A$, we get $B = 0$ or $B \cong A$.

A cover for an object $A$ in a category $\mathcal{A}$ is an object $P$ with an epimorphism $f: P \to A$. This cover is projective if $P$ is a projective object.

An object $A$ in a category is of finite length if there exists a finite sequence of monomorphisms $0 \to A_n \to A_{n-1} \to \ldots \to A_0 = A$ such that the cokernels of these monomorphisms are simple objects.

A $k$ linear abelian category is semisimple if every object is isomorphic to direct sum of simple objects.

Lemma 1.0.1. [Schur’s Lemma] If $k$ is an algebraically closed field of characteristic zero, then $\text{End}(X) = k$ whenever $X$ is a simple object in an abelian $k$ linear category $\mathcal{A}$.

Lemma 1.0.2. If $X \cong Y$ are nonzero simple objects in a $k$ linear abelian category $\mathcal{A}$ for $k$ is a perfect field, then $\text{Hom}(X, Y) = 0$.

Proof. Assume that $\mathcal{A}$ is a $k$ linear abelian category and $X$, $Y$ are nonzero simple objects. Let $f: X \to Y$ is a nonzero morphism in $\mathcal{A}$. $\text{Ker}(f) \cong 0$ since $X$ is simple and $f \neq 0$, so that morphism is a monomorphism. As a result, $X \cong Y$ that is a contradiction, hence $f = 0$. □

A $k$ linear abelian category $\mathcal{A}$ where $k$ is a perfect field is finite if for all objects $X$, $Y$ in $\mathcal{A}$, $\text{Hom}_{\mathcal{A}}(X, Y)$ is finite dimensional vector space over $k$, all objects $A \in \mathcal{A}$ has finite length, every simple object in $\mathcal{A}$ has a projective cover and that category has finitely many isomorphism classes of simple objects.

For example $\text{Vec}_f(k)$ is a finite category, because $\text{Hom}_{\text{Vec}_f(k)}(V, W)$ is isomorphic to the vector space $M_{m \times n}(k)$ of $m \times n$ matrices in which the entries are elements of the field $k$ where $\text{dim}(V) = m$ and $\text{dim}(W) = n$ for given two finite dimensional vector spaces $V$ and $W$. It is finite dimensional since $M_{m \times n}(k)$ is finite dimensional with dimension $m \times n$. The only simple object is $k$ and every object is free in that category, as a result every object is projective and $k^2 \to k$ is a surjection for example.
1.1 Monoidal Category and Braiding

We use [JoRo] as a reference for the following definitions and example.

**Definition 1.1.1.** \((\mathcal{A}, \otimes, I, a, l, r)\) is a monoidal category if for all objects \(X, Y, Z\) and \(W\) in \(\mathcal{A}\), the associativity pentagon and the unit triangle commute.

Here \(\mathcal{A}\) is a category, \(\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}\) is a functor, \(I\) is a unit object in \(\mathcal{A}\), \(a\) is the associativity constraint which is a family of natural isomorphisms

\[
a_{XYZ} : (X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z)
\]

\(l\) is a left unit constraint which is a family of natural isomorphisms

\[
l_X : I \otimes X \xrightarrow{\simeq} X
\]

and \(r\) is a right unit constraint which is a family of natural isomorphisms

\[
r_X : X \otimes I \xrightarrow{\simeq} X.
\]

**Lemma 1.1.1.** If \((\mathcal{A}, \otimes, I, a, l, r)\) is a monoidal category, then \(\mathcal{A}^{op}\) is a monoidal category.

**Proof.** We define the tensor product as \(X \otimes^{op} Y = Y \otimes X\) and associativity constraint \(a^{op}\) as a family of natural isomorphisms

\[
a_{XYZ}^{op} : (X \otimes^{op} Y) \otimes^{op} Z \xrightarrow{\simeq} X \otimes^{op} (Y \otimes^{op} Z)
\]

in \(\mathcal{A}^{op}\) for all objects \(X, Y\) and \(Z\) in \(\mathcal{A}\). This is same as the family of natural isomorphisms

\[
Z \otimes (Y \otimes X) \xrightarrow{\simeq} (Z \otimes Y) \otimes X
\]

in \(\mathcal{A}^{op}\) which can be obtained by inverting the arrows

\[
(Z \otimes Y) \otimes X \xrightarrow{\simeq} Z \otimes (Y \otimes X)
\]

in \(\mathcal{A}\) and as a result, we get \(a_{XYZ}^{op} = a_{ZYX}^{-1}\) for all objects \(X, Y\) and \(Z\) in \(\mathcal{A}\).
Here, $I^{op} = I$. $l^{op}$ is a left unit constraint which is a family of natural isomorphisms

$$l^{op}_X : I \otimes^{op} X \xrightarrow{\cong} X$$

in $\mathcal{A}^{op}$ which is same as

$$l^{op}_X : X \otimes I \xrightarrow{\cong} X$$

for all objects $X$ in $\mathcal{A}$. So, we take $l^{op}_X = r_X$ and $l^{op} = r$ in $\mathcal{A}$. Similarly, we can take $r^{op} = l$.

Also, we define a category $\mathcal{A}^{rev}$ for a given monoidal category $\mathcal{A}$ in which the objects and the arrows are the same as in $\mathcal{A}$ and the tensor product is defined as $X \otimes^{rev} Y = Y \otimes X$.

A strictly full subcategory $B$ of a monoidal category $\mathcal{A}$ is monoidal if it contains the unit object $I$ in $\mathcal{A}$ and $A \otimes B$ for all objects $A$ and $B$ in $\mathcal{A}$.

$(\mathcal{A}, \otimes)$ is an additive monoidal category if $\mathcal{A}$ is an additive category and $\otimes$ is a biadditive functor. It is abelian if $\mathcal{A}$ is an abelian category.

A monoidal category is strict if all $a$, $l$ and $r$ are identity arrows. For example, the category of all $k$ vector spaces $\text{Vec}(k)$ is not a strict monoidal category for a given field $k$. $U \otimes (V \otimes W) \neq (U \otimes V) \otimes W$ in general for all vector spaces $U$, $V$, $W$ in $\text{Vec}(k)$, even in $\text{Vec}_f(k)$, but we can obtain a family of natural isomorphisms of those products as an associativity constraint.

**Theorem 1.1.2.** [MacLane] Every monoidal category is equivalent to a strict monoidal category.

**Definition 1.1.2.** An object $A$ is invertible in a category $\mathcal{A}$ if there exists an object $B$ in $\mathcal{A}$ such that $A \otimes B \cong B \otimes A \cong I$ for $I$ is the unit object.

**Remark 1.1.1.** Invertible objects in a monoidal category $\mathcal{A}$ form a monoidal subcategory of that category. If every simple object in $\mathcal{A}$ is invertible, then we say that the category is pointed.

**Definition 1.1.3.** A braiding $c$ for a monoidal category $\mathcal{A}$ is a natural family of isomorphisms $c_{XY} : X \otimes Y \xrightarrow{\cong} Y \otimes X$ for all objects $X$, $Y$ in $\mathcal{A}$ such that two hexagon diagrams commute in $\mathcal{A}$.

**Note 1.1.1.** If $\mathcal{A}$ is a braided monoidal category, then $\mathcal{A}^{rev}$ is a braided monoidal category with the braiding $c^{rev}$ that is a family of natural isomorphisms $c^{rev}_{XY} = c_{YX}$ for all objects $X$ and $Y$ in $\mathcal{A}$. Similarly, $\mathcal{A}^{op}$ is a braided monoidal category with the braiding $c^{op}$ that is a family of natural isomorphisms $c^{op}_{XY} = c_{X^{-1}}^{-1}$ for all objects $X$ and $Y$ in $\mathcal{A}$. $\mathcal{A}^{op} \simeq \mathcal{A}^{rev}$ in that situation.
Definition 1.1.4. A monoidal category \( \mathcal{A} \) with a braiding \( c \) is called symmetric if the composition

\[
X \otimes Y \xrightarrow{c_{XY}} Y \otimes X \xrightarrow{c_{YX}} X \otimes Y
\]

is \( \text{id}_{X \otimes Y} \) for all objects \( X, Y \) in \( \mathcal{A} \).

Example 1.1.1. The category of all \( k \) vector spaces \( \text{Vec}(k) \) is a braided, symmetric monoidal category for a field \( k \).

Proof. \((c_{YX} \circ c_{XY})(x \otimes y) = (c_{YX} \circ c_{XY})(xy) = c_{YX}(yx) = xy = x \otimes y \) for all objects \( X \) and \( Y \) in \( \text{Vec}(k) \), for all elements \( x \in X, y \in Y \). As a result, the composition is the identity. \( \square \)

Example 1.1.2. Assume that \( G \) is an abelian group, \( k \) is a field, \((f, h)\) is an abelian 3 cocycle on \( G \) with coefficients in \( k \). That is, \( f : G \times G \times G \to k \) is a normalized 3 cocycle such that

\[
f(x, 0, y) = 0, \quad (5)
\]

\[
f(x, y, z) + f(w, x + y, z) + f(w, x, y + z) + f(w + x, y, z) = f(w, x, y + z) + f(w + x, y, z)
\]

and \( h : G \times G \to k \) is a function such that

\[
f(y, z, x) + h(x, y + z) + f(x, y, z) = h(x, z) + f(y, x, z) + h(x, y), \quad (7)
\]

\[
-f(z, x, y) + h(x + y, z) - f(x, y, z) = h(x, z) - f(x, z, y) + h(y, z). \quad (8)
\]

Let \( \mathcal{A} \) be a category such that the objects are families of \( k \) modules \( X = \{X_g \mid g \in G\} \) and the arrow between two families \( X, Y \) is a family \( \theta = \{X_{g_1} \xrightarrow{\theta_{g_1 g_2}} Y_{g_2}\} \) where \( \theta_{g_1 g_2} \) is a \( k \) module homomorphism for all \( g_1 \) and \( g_2 \) in \( G \),

\[
(X \otimes Y)_g = \sum_{g_1 + g_2 = g} (X_{g_1} \otimes Y_{g_2}) \quad (9)
\]

is the tensor product,

\[
a_{XYZ} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z),
\]

\[
a_{XYZ}((x \otimes y) \otimes z) = f(g_1, g_2, g_3)x \otimes (y \otimes z) \quad (10)
\]

is the associativity constraint and

\[
c_{XY} : X \otimes Y \to Y \otimes X,
\]

\[
c(x \otimes y) = h(g_1, g_2)y \otimes x \quad (13)
\]

is the braiding for \( x \in X_{g_1}, y \in Y_{g_2}, z \in Z_{g_3} \). So, this category is a braided monoidal category.
1.2 The Category of Monoidal Functors

The following materials are found in [JoRo].

**Definition 1.2.1.** For two monoidal categories \( A \) and \( B \), assume that \( F : A \to B \) is a functor, \( \gamma \) is the family of natural isomorphisms

\[
\gamma_{XY} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)
\]

for all objects \( X, Y \) in \( A \). \( \varphi : I \xrightarrow{\cong} F(I) \) is an isomorphism for the unit object \( I \). Then, \((F, \gamma, \varphi)\) is a monoidal functor if it satisfies compatible conditions.

**Note 1.2.1.** \((F, \gamma, \varphi)\) is strict if \( \gamma \) and \( \varphi \) are identities.

**Definition 1.2.2.** A monoidal functor \( F : A \to B \) between braided monoidal categories \( A \) and \( B \) is braided if the following diagram is commutative.

\[
\begin{array}{ccc}
F(X) \otimes F(Y) & \xrightarrow{\gamma} & F(X \otimes Y) \\
\downarrow{\scriptstyle{\epsilon}} & & \downarrow{\scriptstyle{F(c)}} \\
F(Y) \otimes F(X) & \xrightarrow{\gamma} & F(Y \otimes X)
\end{array}
\]

**Definition 1.2.3.** If \( F, G : A \to B \) are two monoidal functors, then a map \( \theta : F \to G \) is a natural transformation if the following two diagrams commute.

\[
\begin{array}{ccc}
F(X) \otimes F(Y) & \xrightarrow{\gamma} & F(X \otimes Y) \\
\downarrow{\scriptstyle{\theta(X) \otimes \theta(Y)}} & & \downarrow{\scriptstyle{\theta(X \otimes Y)}} \\
G(X) \otimes G(Y) & \xrightarrow{\gamma} & G(X \otimes Y) \\
F(I) & \xrightarrow{\theta(I)} & G(I)
\end{array}
\]

**Proposition 1.2.1.** The collection \( \text{Hom}(A, B) \) in which the objects are monoidal functors \( F : A \to B \) and morphisms are natural transformations between monoidal functors for given monoidal categories \( A \) and \( B \) forms a category.

**Lemma 1.2.2.** \( \text{Hom}(A, A) \) is a monoidal category in which the tensor product is the composition of functors for a given monoidal category \( A \).

We denote the category of right exact monoidal functors by \( \text{Hom}^{re}(A, B) \), the category of left exact monoidal functors by \( \text{Hom}^{le}(A, B) \) and the category of exact monoidal functors by \( \text{Hom}^{e}(A, B) \).

**Remark 1.2.1.** A monoidal functor \((F, \gamma, \varphi) : (A, \otimes_A) \to (B, \otimes_B)\) is a monoidal equivalence if \( F : A \to B \) is an equivalence of categories. In that situation, there exists a monoidal functor \((G, \gamma', \varphi') : B \to A\) and isomorphism of monoidal functors \( G \circ F \to id_A, F \circ G \to id_B \).
Proposition 1.2.3. [JoRo] If $\mathcal{A}$ is braided monoidal category, then we get a monoidal equivalence $\mathcal{A} \rightarrow \mathcal{A}^{rev}$.

Proof. We define a monoidal functor $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}^{rev}$ by sending an object $A$ in $\mathcal{A}$ to itself, $\gamma$ as a family of natural isomorphisms $\gamma_{XY} : X \otimes^{rev} Y = Y \otimes X \rightarrow X \otimes Y$ for all objects $X, Y$ in $\mathcal{A}$ and also $\varphi = id_I$. We define $\gamma_{XY} = c_{XY}^{rev}$. Then, we need to show that the following diagram commutes in $\mathcal{A}^{rev}$.

\[
\begin{array}{ccc}
(X \otimes^{rev} Y) \otimes^{rev} Z & \xrightarrow{a_{XYZ}^{rev}} & X \otimes^{rev} (Y \otimes^{rev} Z) \\
\gamma_{XY} \otimes id & & id_{Y} \otimes \gamma_{XZ} \\
\gamma_{XZ} & & \\
(X \otimes Y) \otimes^{rev} Z & \xrightarrow{id_{Y} \otimes \gamma_{XZ}} & X \otimes^{rev} (Y \otimes Z) \\
\gamma_{XZ} & & \\
(X \otimes Y) \otimes Z & \xrightarrow{a_{XYZ}} & X \otimes (Y \otimes Z)
\end{array}
\] (17)

This diagram is same as the following diagram in $\mathcal{A}$.

\[
\begin{array}{ccc}
Z \otimes (Y \otimes X) & \xrightarrow{a_{XZY}} & (Z \otimes Y) \otimes X \\
id_{Z} \otimes c_{YX} & & c_{YX} \otimes id_{X} \\
Z \otimes (X \otimes Y) & \xrightarrow{c_{ZXY}} & (Y \otimes Z) \otimes X \\
\gamma_{XZ} & & \\
(X \otimes Y) \otimes Z & \xrightarrow{a_{XYZ}} & X \otimes (Y \otimes Z)
\end{array}
\] (18)

The first and third squares commute by definition and the middle one commutes by using naturality of the braiding in the following diagram.

\[
\begin{array}{ccc}
(Z \otimes Y) \otimes X & \xrightarrow{c_{ZXY} \otimes id_{X}} & (Y \otimes Z) \otimes X \\
a_{ZXY} & & a_{YXZ}^{-1} \\
Z \otimes (Y \otimes X) & \xrightarrow{id_{Y} \otimes c_{YX}} & Y \otimes (Z \otimes X) \\
c_{Z(YX)} & & c_{YX} \otimes id_{Z} \\
(Z \otimes X \otimes Y) & \xrightarrow{c_{Z(XY)}} & (X \otimes Y) \otimes Z \\
\gamma^{-1}_{ZXY} & & a_{XYZ} \\
(Z \otimes X) \otimes Y & \xrightarrow{c_{ZXY} \otimes id_{Y}} & (X \otimes Z) \otimes Y \\
a_{ZXY}^{-1} & & a_{XYZ} \\
(Z \otimes X) \otimes (Z \otimes Y) & \xrightarrow{a_{XZY}} & X \otimes (Y \otimes Z) \\
\end{array}
\] (19)
As a result, that diagram commutes.

\[
\begin{align*}
    a_{XY} Z \circ c_{Z(Y)} \circ (id_Z \otimes c_{YX}) &= a_{XY} Z \circ a_{XYZ}^{-1} \circ (id_X \otimes c_{ZY}) \circ a_{XZ} \circ (c \otimes id_Y) \circ a_{X}^{-1} \circ (id_X \otimes c_{YX}) \\
    c_{(YZ)X} \circ (c_{ZY} \otimes id_X) \circ a_{(YX)}^{-1} &= a_{XY} Z \circ (c_{XY} \otimes id_Y) \circ a_{Y}^{-1} \circ (id_Y \otimes c_{XZ}) \circ a_{Z} \circ (c_{ZY} \otimes id_X) \circ a_{(YX)}^{-1}.
\end{align*}
\]

These two equations are same by the commutativity of Diagram 19. As a result, Diagram 17 commutes. The commutativity of other diagrams are easy to show, also it is a braided monoidal functor. The reader can show that the conditions for equivalence are satisfied.

\[ A \simeq A^{op} \] as a corollary of this proposition.

### 1.3 Rigid Monoidal Categories

An object \( Y \) in a monoidal category \( A \) is a right dual for a given object \( X \) in \( A \) if there are morphisms \( ev_{rX} : Y \otimes X \to I \) and \( coev_{rX} : I \to X \otimes Y \) such that the following compositions are the identities.

\[
\begin{align*}
    Y &= Y \otimes I \xrightarrow{id_Y \otimes coev_{rX}} Y \otimes X \otimes Y \xrightarrow{ev_{rX} \otimes id_Y} I \otimes Y = Y \tag{20} \\
    X &= I \otimes X \xrightarrow{coev_{rX} \otimes id_X} X \otimes Y \otimes X \xrightarrow{id_X \otimes ev_{rX}} X \otimes I = X \tag{21}
\end{align*}
\]

Similarly, an object \( Z \) is a left dual object for the object \( X \) in that category if there are morphisms \( ev_{lX} : X \otimes Z \to I \) and \( coev_{lX} : I \to Z \otimes X \) such that the following compositions are the identities.

\[
\begin{align*}
    Z &= Z \otimes I \xrightarrow{coev_{lX} \otimes id_Z} Z \otimes X \otimes Z \xrightarrow{id_Z \otimes ev_{lX}} Z \otimes I = Z \tag{22} \\
    X &= X \otimes I \xrightarrow{id_X \otimes coev_{lX}} X \otimes Z \otimes X \xrightarrow{ev_{lX} \otimes id_X} I \otimes X = X \tag{23}
\end{align*}
\]

We denote the left dual with \( +X \) and the right dual with \( X^+ \).

**Lemma 1.3.1.** A left dual \( +X \) and a right dual \( X^+ \) in a monoidal category \( A \) is unique up to a unique isomorphism.

**Proof.** See [Baki] for the proof. \( \square \)

**Definition 1.3.1.** A monoidal category is rigid if every object \( X \) in that category has both a right and a left dual object.
Example 1.3.1. The category of finite dimensional vector spaces $Vec_k$ over a field $k$ is rigid. If that category is consisting of all $k$ vector spaces without finiteness assumption, then it is not rigid.

Proof. For a given finite dimensional $k$ vector space $V$, the right and left dual object for $V$ is the dual space $Hom_k(V, k)$ with the evaluation map

$$ev_{rV} : Hom_k(V, k) \otimes V \to k, \ (f, v) \mapsto f(v)$$

and the coevaluation map $coev_{rV} : k \to V \otimes Hom_k(V, k)$ which is an embedding. We may see that the following compositions are the identities.

$$V = k \otimes V \xrightarrow{coev_{rV} \otimes id_V} V \otimes Hom_k(V, k) \otimes V \xrightarrow{id_V \otimes ev_{rV}} V \otimes k = V$$

Similarly, we may show that the compositions 22 and 23 are the identities which shows that $Hom_k(V, k)$ is a left dual for the object $V$.

Second part follows since infinite dimensional spaces don’t have any coevaluation map.

Remark 1.3.1. [Baki] Tensor product functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is exact in each variable in an abelian, rigid monoidal category $\mathcal{A}$.

Lemma 1.3.2. If a monoidal category $\mathcal{A}$ is rigid, then $\mathcal{A}^{op}$ is rigid, too.

Proof. If $X$ is an object in a rigid monoidal category $\mathcal{A}$, then $X$ has both a left and a right dual objects $^+X$ and $X^+$ which are unique up to a unique isomorphism by Lemma 1.3.1 such that the following compositions are the identities by definition where $ev_{rX} : X_+ \otimes X \to I$, $coev_{rX} : I \to X \otimes X^+$, $ev_{lX} : X \otimes +X \to I$ and $coev_{lX} : I \to +X \otimes X$ are morphisms in $\mathcal{A}$.

$$X^+ = X^+ \otimes I \xrightarrow{id_{X^+} \otimes coev_{rX}} X^+ \otimes X \otimes X^+ \xrightarrow{ev_{rX} \otimes id_{X^+}} I \otimes X^+ = X^+$$

$$X = I \otimes X \xrightarrow{coev_{rX} \otimes id_X} X \otimes X^+ \otimes X \xrightarrow{id_X \otimes ev_{rX}} X \otimes I = X$$

$$^+X = I \otimes ^+X \xrightarrow{coev_{lX} \otimes id_{^+X}} +X \otimes X \otimes +X \xrightarrow{id_{^+X} \otimes ev_{lX}} +X \otimes I = +X$$

$$X = X \otimes I \xrightarrow{id_X \otimes coev_{lX}} X \otimes +X \otimes X \xrightarrow{ev_{lX} \otimes id_X} I \otimes X = X$$
We know that $ev_r X \otimes id_{X^+} \circ id_{X^+} \otimes coev_r X = id_{X^+}$ in $A$ by the first composition, so $(ev_r X \otimes id_{X^+} \circ id_{X^+} \otimes coev_r X)^{op} = (id_{X^+})^{op} = id_{X^+}$. This implies that

$$(id_{X^+} \otimes coev_r X)^{op} \circ (ev_r X \otimes id_{X^+})^{op} = coev_r {op} \otimes X \circ X^+ \otimes op \ ev_r {op} = id_{X^+}.$$  

As a result, the following composition is the identity of $X^+$ in $A^{op}$

$$X^+ = I \otimes X^+ \xrightarrow{id_{X^+} \otimes op \ ev_r {op}} X^+ \otimes X \otimes X^+ \xrightarrow{coev_r X \otimes op \ id_{X^+}} X^+ \otimes I = X^+$$

which is same as the following one

$$X^+ = X^+ \otimes op \ I \xrightarrow{id_{X^+} \otimes op \ ev_r {op}} X^+ \otimes op \ X \otimes op \ X^+ \xrightarrow{coev_r X \otimes op \ id_{X^+}} I \otimes op \ X^+ = X^+.$$

Also, we get an identity of $X$ by the composition

$$X = X \otimes I \xrightarrow{(id_X \otimes ev_r X)^{op}} X \otimes X^+ \otimes X \xrightarrow{(coev_r X \otimes id_X)^{op}} I \otimes X = X$$

which is same as the following one

$$X = I \otimes op \ X \xrightarrow{ev_r X \otimes op \ id_X} X \otimes op \ X^+ \otimes op \ X \xrightarrow{id_X \otimes op \ coev_r X} X \otimes op \ I = X.$$  

These two identities show that $X^+$ is right dual for $X$ in $A^{op}$. By using same technique, we may show that $+X$ is left dual for $X$ in $A^{op}$. Those are unique objects in $A$ by Lemma 1.3.1, so they are also unique in $A^{op}$ as objects. This shows that $A^{op}$ is a rigid category.

**Lemma 1.3.3.** If $I$ is a unit object in a rigid monoidal category $A$, then $I^+ = I$.

**Proof.** The composition $I \longrightarrow I^+ \longrightarrow I$ is the identity by [21] Also, the other condition is satisfied. It is easy to see that $I$ satisfies the required conditions, too. Hence, $I = I^+$ by uniqueness of a right dual. □

**Lemma 1.3.4.** If a monoidal category $A$ is rigid, then for all objects $X$ and $Y$ in $A$, $(X \otimes_A Y)^+ = Y^+ \otimes_A X^+$.

**Example 1.3.2.** Assume that $A$ and $B$ are two monoidal categories and $(F, \gamma, \varphi)$ is a monoidal functor between those categories. If $X$ is an object in $A$ with a right dual $X^+$, then $F(X^+)$ is a right dual of $F(X)$.
Proof. We define the evaluation map as $ev_{rF(X)} = F(ev_{rX}) \circ \gamma$ that is shown with the following diagram
\[
ev_{rF(X)} : F(X^+) \otimes F(X) \to F(X^+ \otimes X) \to F(I)
\]
and the coevaluation map as $coev_{rF(X)} = \gamma^{-1} \circ F(coev_{rX})$ that is shown with the following diagram
\[
coev_{rF(X)} : F(I) \to F(X \otimes X^+) \to F(X) \otimes F(X^+)
\]
by using $\gamma$. It is obvious that the following compositions are the identities since $F$ is a monoidal functor and $X^+$ is a right dual for $X$.

$F(X^+) = F(X^+) \otimes F(I) \xrightarrow{\gamma^{-1}} F(X) \otimes F(X^+) \otimes F(X^+) \xrightarrow{F(I) \otimes F} F(I \otimes F(X^+)) = F(X^+)$

$F(X) = F(I) \otimes F(X) \xrightarrow{\gamma^{-1}} F(X) \otimes F(X^+) \otimes F(X) \xrightarrow{F} F(X) \otimes F(I) = F(X)$

As a result, $F(X^+)$ is a right dual for $F(X)$. \qed

Definition 1.3.2. A monoidal subcategory of a monoidal category $\mathcal{A}$ is a monoidal category under the induced monoidal structure of $\mathcal{A}$ and it is a rigid monoidal subcategory of a rigid monoidal category $\mathcal{A}$ if it contains $X^+$ and $+X$ whenever it contains an object $X$.

Proposition 1.3.5. If $\mathcal{A}$ is a rigid monoidal category, then an object $X$ in $\mathcal{A}$ is invertible if and only if $ev_{rX} : X^+ \otimes X \to I$ and $coev_{rX} : I \to X \otimes X^+$ are isomorphisms. In that situation, $X^+ \cong X^+$. If $Y$ is another invertible object, then $X \otimes Y$ is invertible.

Proof. If the above maps are isomorphisms, then we get $X^+ \otimes X \cong I \cong X \otimes X^+$, so $X^+$ is the required object in the definition of an invertible object. Thus, $X$ is invertible. Similarly, we see that $X^+$ is invertible.

Conversely, if $X$ is invertible, then there exists an object $Z$ such that $X \otimes Z \cong Z \otimes X \cong I$, so we can use $Z$ as a right dual, hence $Z \cong X^+$ by uniqueness of a right dual. Then the above maps are isomorphisms. With the same idea, we may consider $Z \cong +X$ by the isomorphism and we reverse the arrows if required and see $Z$ is a left dual. As a result, $X^+ \cong +X$.

Now, assume that $X$ and $Y$ are two invertible objects in the category. Then, $X^+ \otimes X \cong I \cong X \otimes X^+$ and $Y^+ \otimes Y \cong I \cong Y \otimes Y^+$. So, we get $Y^+ \otimes X^+ \otimes X \otimes Y \cong I \cong X \otimes Y \otimes Y^+ \otimes X^+$. This shows that $Y^+ \otimes X^+$ is the inverse of $X \otimes Y$. \qed

Proposition 1.3.6. Assume that $(\mathcal{F}, \gamma, \varphi)$ and $(\mathcal{G}, \gamma', \varphi')$ are two monoidal functors from $\mathcal{A} \to \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are rigid monoidal categories, then every morphism of monoidal functors from $\mathcal{F}$ to $\mathcal{G}$ is an isomorphism.
1.4 Drinfeld Center of A Monoidal Category

Assume that $\mathcal{A}$ is a monoidal category. We denote the Drinfeld center of $\mathcal{A}$ by $Z(\mathcal{A})$.

Objects in $Z(\mathcal{A})$ are $(Z, \gamma_Z)$ where $Z$ is an object in $\mathcal{A}$ and $\gamma_Z$ is a family of natural isomorphisms $\gamma_{ZX} : Z \otimes X \sim X \otimes Z$ for all objects $X$ in $\mathcal{A}$ such that the following diagrams commute.

$$
\begin{array}{cccc}
Z \otimes (X \otimes Y) & \xrightarrow{\gamma_{Z(X \otimes Y)}} & (X \otimes Y) \otimes Z \\
\downarrow a & & \downarrow a \\
(Z \otimes X) \otimes Y & & X \otimes (Y \otimes Z) \\
\gamma_{ZX} \otimes id_Y & & id_X \otimes \gamma_{ZY} \\
\downarrow & & \downarrow \\
(X \otimes Z) \otimes Y & & X \otimes (Z \otimes Y)
\end{array}
$$

(24)

$$
\begin{array}{cccc}
Z \otimes I & \xrightarrow{\gamma_{ZI}} & I \otimes Z \\
\downarrow & & \downarrow r & \downarrow l \\
Z & & &
\end{array}
$$

(25)

A morphism $f : (Z, \gamma_Z) \to (W, \gamma_W)$ in $Z(\mathcal{A})$ is an arrow $f : Z \to W$ such that the following diagram commutes.

$$
\begin{array}{cccc}
Z \otimes X & \xrightarrow{\gamma_{ZX}} & X \otimes Z \\
\downarrow f \otimes id_X & & \downarrow id_X \otimes f \\
W \otimes X & \xrightarrow{\gamma_{WX}} & X \otimes W
\end{array}
$$

(26)

Lemma 1.4.1. $Z(\mathcal{A})$ is a braided monoidal category.

Proof. $(Z, \gamma_Z) \otimes (W, \gamma_W) = (Z \otimes W, (\gamma_Z \otimes id_W) \circ (id_Z \otimes \gamma_W))$ is a tensor product for all objects $Z$ and $W$ in $\mathcal{A}$ and $c_{(Z, \gamma_Z)(W, \gamma_W)} : (Z, \gamma_Z) \otimes (W, \gamma_W) \to (W, \gamma_W) \otimes (Z, \gamma_Z)$ is a braiding which is same as

$$
c : (Z \otimes W, (\gamma_Z \otimes id_W) \circ (id_Z \otimes \gamma_W)) \to (W \otimes Z, (\gamma_W \otimes id_Z) \circ (id_W \otimes \gamma_Z)).$$
1.5 Module Categories

**Definition 1.5.1.** ([Os]) A category $\mathcal{M}$ is a left module category on a finite monoidal category $\mathcal{A}$ if there exists an exact bifunctor

$$\otimes_{\mathcal{L}M} : \mathcal{A} \times \mathcal{M} \to \mathcal{M}, \ (X, M) \mapsto X \otimes_{\mathcal{L}M} M$$

for all objects $X$ in $\mathcal{A}$, $M$ in $\mathcal{M}$, associativity constraint $a_{\mathcal{L}M}$ consisting of associativity isomorphisms $a_{XY,M} : (X \otimes_{\mathcal{A}} Y) \otimes_{\mathcal{L}M} M \to X \otimes_{\mathcal{L}M} (Y \otimes_{\mathcal{L}M} M)$ and left unit constraint $l_{\mathcal{L}M}$ which is a family of left unit isomorphisms $l_M : I \otimes_{\mathcal{L}M} M \to M$ such that for any $X, Y, Z$ in $\mathcal{A}$, $M$ in $\mathcal{M}$, the following diagrams commute.

\[
\begin{array}{ccc}
((X \otimes_{\mathcal{A}} Y) \otimes_{\mathcal{A}} Z) \otimes_{\mathcal{L}M} M & \xrightarrow{a_{XYZ,M}} & (X \otimes_{\mathcal{A}} (Y \otimes_{\mathcal{A}} Z)) \otimes_{\mathcal{L}M} M \\
(X \otimes_{\mathcal{A}} Y) \otimes_{\mathcal{L}M} (Z \otimes_{\mathcal{L}M} M) & \xrightarrow{a_{X,Y,Z,M}} & X \otimes_{\mathcal{L}M} ((Y \otimes_{\mathcal{A}} Z) \otimes_{\mathcal{L}M} M) \\
(X \otimes_{\mathcal{L}M} (Y \otimes_{\mathcal{L}M} (Z \otimes_{\mathcal{L}M} M))) & \xrightarrow{id_X \otimes_{\mathcal{L}M} a_{Y,Z,M}} & X \otimes_{\mathcal{L}M} (Y \otimes_{\mathcal{L}M} (Z \otimes_{\mathcal{L}M} M))
\end{array}
\]

\[
\begin{array}{ccc}
(X \otimes_{\mathcal{A}} I) \otimes_{\mathcal{L}M} M & \xrightarrow{a_{XIM}} & X \otimes_{\mathcal{L}M} (I \otimes_{\mathcal{L}M} M) \\
r_{X} \otimes_{\mathcal{L}M} id_M & \xrightarrow{id_X \otimes_{\mathcal{L}M} l_M} & X \otimes_{\mathcal{L}M} M
\end{array}
\]

**Definition 1.5.2.** A category $\mathcal{M}$ is right module category on a finite monoidal category $\mathcal{A}$ if there exists an exact bifunctor

$$\otimes_{\mathcal{R}M} : \mathcal{M} \times \mathcal{A} \to \mathcal{M}, \ (M, X) \mapsto M \otimes_{\mathcal{R}M} X$$

for all objects $X \in \mathcal{A}$, $M \in \mathcal{M}$, associativity constraint $a_{\mathcal{R}M}$ consisting of associativity isomorphisms $a_{MXY} : M \otimes_{\mathcal{R}M} (X \otimes_{\mathcal{A}} Y) \to (M \otimes_{\mathcal{R}M} X) \otimes_{\mathcal{R}M} Y$ and right unit constraint $r_{\mathcal{L}M}$ which is a family of right unit isomorphisms $r_M : M \otimes_{\mathcal{R}M} I \to M$ such
that for any $X, Y, Z$ in $A$, $M$ in $\mathcal{M}$, the following diagrams commute.

\[
\begin{array}{c}
\begin{array}{c}
\sum_{\mathcal{M} \otimes r_{\mathcal{M}}} (\sum_{(X \otimes_A Y) \otimes_A Z} M \otimes_{r_{\mathcal{M}}} (X \otimes_A (Y \otimes_A Z))) \\
\xrightarrow{id_{\mathcal{M}} \otimes_{r_{\mathcal{M}}} a_{XYZ}} \sum_{\mathcal{M} \otimes r_{\mathcal{M}}} (M \otimes_{r_{\mathcal{M}}} \sum_{(X \otimes_A Y) \otimes_A Z} X \otimes_Y (Y \otimes_A Z))
\end{array}
\end{array}
\]

(30)

\[
\begin{array}{c}
\begin{array}{c}
\sum_{\mathcal{M} \otimes r_{\mathcal{M}}} (\sum_{(X \otimes_A Y) \otimes_A Z} M \otimes_{r_{\mathcal{M}}} (X \otimes_A (Y \otimes_A Z))) \\
\xrightarrow{a_{(MX)YZ}} \sum_{\mathcal{M} \otimes r_{\mathcal{M}}} (M \otimes_{r_{\mathcal{M}}} \sum_{(X \otimes_A Y) \otimes_A Z} X \otimes_Y (Y \otimes_A Z))
\end{array}
\end{array}
\]

(31)

**Proposition 1.5.1.** For a left $A$ module category $\mathcal{M}$ where $A$ is a finite, rigid monoidal category, $\mathcal{M}^{op}$ is a right $A$ module category obtained from $\mathcal{M}$ by reversing the arrows.

**Proof.** We define the action as

\[
\sum_{\mathcal{M}^{op} \otimes r_{\mathcal{M}^{op}}} : \mathcal{M}^{op} \times A \to \mathcal{M}^{op}, \ (M, X) \mapsto M \otimes_{r_{\mathcal{M}^{op}}} X
\]

for all objects $M \in \mathcal{M}^{op}$ and for all objects $X \in A$ where $M \otimes_{r_{\mathcal{M}^{op}}} X = X^+ \otimes_{l_{\mathcal{M}}} M$. $\sum_{\mathcal{M}^{op} \otimes r_{\mathcal{M}^{op}}}$ is an exact bifunctor since $\sum_{l_{\mathcal{M}}}$ is an exact bifunctor.

Can we find an associativity constraint $a_{r_{\mathcal{M}^{op}}}$ consisting of associativity isomorphisms $a_{MYX} : M \otimes_{r_{\mathcal{M}^{op}}} (X \otimes_A Y) \to (M \otimes_{r_{\mathcal{M}^{op}}} X) \otimes_{r_{\mathcal{M}^{op}}} Y$ for all objects $X, Y$ in $A$, $M$ in $\mathcal{M}$ and a right unit constraint $r_{\mathcal{M}^{op}}$ which is a family of right unit isomorphisms $r_M : M \otimes_{r_{\mathcal{M}^{op}}} I \to M$ such that for any $X, Y, Z$ in $A$, $M$ in $\mathcal{M}$, the Diagram 30 and Diagram 31 commute?

\[
M \otimes_{r_{\mathcal{M}^{op}}} (X \otimes_A Y) = (X \otimes_A Y)^+ \otimes_{l_{\mathcal{M}}} M = (Y^+ \otimes_A X^+) \otimes_{l_{\mathcal{M}}} M \text{ by Lemma 1.3.4}
\]

\[
(M \otimes_{r_{\mathcal{M}^{op}}} X) \otimes_{r_{\mathcal{M}^{op}}} Y = Y^+ \otimes_{l_{\mathcal{M}}} (X^+ \otimes_{l_{\mathcal{M}}} M).
\]

We know that there is an isomorphism

\[
a_{Y^+, X^+} : (Y^+ \otimes_A X^+) \otimes_{l_{\mathcal{M}}} M \to Y^+ \otimes_{l_{\mathcal{M}}} (X^+ \otimes_{l_{\mathcal{M}}} M)
\]
since $\mathcal{M}$ is a left $\mathcal{A}$ module category, so, we can take $a_{MXY} = a_{Y+X+M}$.

$M \otimes_{r,\mathcal{M}^{op}} I = I^+ \otimes_{l,M} M$ and there is an isomorphism $l_M : I^+ \otimes_{l,M} M \to M$, so we get the family of right unit constraints $r_M = l_M$ since $I^+ = I$. The reader can show that the Diagram 30 and Diagram 31 commute.

Similarly, for a right $\mathcal{A}$ module category $\mathcal{M}$, $\mathcal{M}^{op}$ is the category obtained from $\mathcal{M}$ by reversing the arrows which is a left $\mathcal{A}$ module category with $X \times M = M \otimes X$ for all objects $M \in \mathcal{M}$, $X \in \mathcal{A}$.

**Lemma 1.5.2.** Assume that $\mathcal{A}$ is a fusion category and $\mathcal{M}$ is a module category over $\mathcal{A}$. $(\mathcal{M}^{op})^{op} \simeq \mathcal{M}$ canonically as an $\mathcal{A}$ module category.

**Lemma 1.5.3.** Assume that $\mathcal{A}$ is a finite monoidal category and $\mathcal{A}$ is an algebra in $\mathcal{A}$. Then, the category of right $\mathcal{A}$ modules $\mathcal{A}\mathcal{A}$ is a left $\mathcal{A}$ module category and the category of left $\mathcal{A}$ modules $\mathcal{A}\mathcal{A}$ is a right $\mathcal{A}$ module category.

**Note 1.5.1.** A tensor category $\mathcal{A}$ is a $Z(\mathcal{A})$ module category with the action

$$Z(\mathcal{A}) \times \mathcal{A} \to \mathcal{A}, \quad ((Z, \gamma), X) \mapsto Z \otimes X.$$ 

### 1.6 Indecomposable, Exact and Semisimple Module Category

**Proposition 1.6.1.** Assume that $\mathcal{M}$ and $\mathcal{N}$ are module categories over a finite monoidal category $\mathcal{A}$. Then, the direct sum $\mathcal{P} = \mathcal{M} \oplus \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ is a module category over $\mathcal{A}$ with $\otimes_{\mathcal{P}} = \otimes_{\mathcal{M}} \oplus \otimes_{\mathcal{N}}$, $a_{\mathcal{P}} = a_{\mathcal{M}} \oplus a_{\mathcal{N}}$, $l_{\mathcal{P}} = l_{\mathcal{M}} \oplus l_{\mathcal{N}}$.

**Definition 1.6.1.** A module category $\mathcal{P}$ over a finite monoidal category $\mathcal{A}$ is indecomposable if $\mathcal{M} = 0$ or $\mathcal{N} = 0$ whenever $\mathcal{P} \simeq \mathcal{M} \oplus \mathcal{N}$.

**Definition 1.6.2.** A module category $\mathcal{M}$ over a monoidal category $\mathcal{A}$ is exact if for all projective objects $P$ in $\mathcal{A}$ and all objects $M$ in $\mathcal{M}$, $P \otimes M$ is a projective object in $\mathcal{M}$.

**Lemma 1.6.2.** Every finite monoidal category $\mathcal{A}$ is an exact module category over itself.

**Example 1.6.1.** Every object in an exact module category $\mathcal{M}$ over $Vec_f(k)$ is projective.

**Proof.** $k$ is free, hence a projective object in $Vec_f(k)$, for all objects $M$ in $\mathcal{M}$, we have $M = k \otimes M$ that is projective. As a result, every object is projective in $\mathcal{M}$. $\square$

**Lemma 1.6.3.** If $\mathcal{M}$ is a module category over a rigid monoidal category $\mathcal{A}$, then for all objects $A \in \mathcal{A}$ and projective objects $P \in \mathcal{M}$, $A \otimes P$ is a projective object in $\mathcal{M}$. 

15
Proof. Assume that \( \mathcal{M} \) is a module category over \( \mathcal{A} \) given as above. Let \( A \) be an object in \( \mathcal{A} \) and \( P \) be projective in \( \mathcal{M} \). For all epimorphisms \( f : X \to Y \) and morphisms \( g : A \otimes P \to Y \) in \( \mathcal{M} \), can we find a morphism \( k : A \otimes P \to X \) such that \( g = k \circ f \).

\[
\begin{align*}
\text{Hom}_{\mathcal{M}}(X, Y) & \cong \text{Hom}_{\mathcal{M}}(\mathcal{A} \otimes X, \mathcal{A} \otimes Y), \\
\text{Hom}_{\mathcal{M}}(\mathcal{A} \otimes P, Y) & \cong \text{Hom}_{\mathcal{M}}(\mathcal{A} \otimes P, \mathcal{A} \otimes Y), \\
\text{Hom}_{\mathcal{M}}(\mathcal{A} \otimes P, X) & \cong \text{Hom}_{\mathcal{M}}(\mathcal{A} \otimes P, \mathcal{A} \otimes X).
\end{align*}
\]

We find an epimorphism \( f' : \mathcal{A} \otimes X \to \mathcal{A} \otimes Y \) corresponding to \( f \) and a morphism \( g' : P \to \mathcal{A} \otimes X \) such that \( g' = f' \circ k' \). After that, we get a morphism \( k : \mathcal{A} \otimes P \to X \) corresponding to \( k' \) such that \( f \circ k = g \). As a result, \( \mathcal{A} \otimes P \) is a projective object in \( \mathcal{M} \).

Lemma 1.6.4. If \( \mathcal{A} \) is a finite semisimple monoidal category, then the unit object \( I \) is projective in that category. Plus, all objects are projective in such a category.

Proof. We want to show that \( I \) is projective in \( \mathcal{A} \). Assume that we are given an epimorphism \( f : \mathcal{A} \to \mathcal{B} \) and a map \( g : I \to \mathcal{B} \). Can we find a map \( h : I \to \mathcal{A} \) such that \( f \circ h = g \)?

\[
\begin{align*}
A & = \oplus_i A_i, I = \oplus_j I_j \text{ and } B = \oplus_k B_k \text{ for simple objects } A_i, I_j \text{ and } B_k \text{ for all } i, j, \text{ and } k. \\
& \text{We can decompose } f \text{ and } g \text{ as } f = \oplus_{ik} f_{ik} \text{ and } g = \oplus_{jk} g_{jk} \text{ where } f_{ik} : A_i \to B_k, \text{ } g_{jk} : I_j \to B_k \text{ are morphisms in } \mathcal{A}. \\
& \text{By Proposition 1.0.2 we get } f_{ik} = g_{jk} = 0, \text{ so any morphism } I_j \to A_i \text{ works, then we take } h \text{ as the direct sum of those morphisms and the result follows afterwards.}
\end{align*}
\]

\[ A = I \otimes A \text{ for all objects } A \in \mathcal{A} \text{ by left unit associativity. } \mathcal{A} \text{ is an exact left module category over itself by Lemma 1.6.2 } I \text{ is projective, thus } I \otimes A \text{ is projective by exactness. Hence, every object is projective in } \mathcal{A}. \]

Corollary 1.6.1. If a module category \( \mathcal{M} \) over a finite monoidal category \( \mathcal{A} \) is semisimple, then it is exact.

Proof. Assume that \( \mathcal{M} \) is a semisimple module category over a finite monoidal category \( \mathcal{A} \). Any object in a semisimple category is projective, so \( \mathcal{M} \) is exact.

Corollary 1.6.2. [ETOS] A module category \( \mathcal{M} \) over a fusion category \( \mathcal{A} \) is exact if and only if it is semisimple.

Lemma 1.6.5. Assume that \( \mathcal{A} \) is a finite, rigid monoidal category with simple unit object \( I \). Then, any exact module category \( \mathcal{M} \) over \( \mathcal{A} \) has enough projectives.

Proof. Assume that \( \mathcal{A} \) is given as above and \( \mathcal{M} \) is an exact left module category over \( \mathcal{A} \). Then, we find a projective object \( P \) in \( \mathcal{A} \) with an epimorphism \( P \to I \) since \( \mathcal{A} \) is a finite category. Hence, for all objects \( M \) in \( \mathcal{M} \), we get an epimorphism \( P \otimes M \to I \otimes M \cong M \) in \( \mathcal{M} \). \( \mathcal{M} \) is exact, so \( P \otimes M \) is projective by exactness. As a result, we see that \( \mathcal{M} \) has enough projectives and there exists a projective cover for every simple object in \( \mathcal{M} \).
1.7 The Category of Module Functors

**Definition 1.7.1.** Assume that \( \mathcal{M} \) and \( \mathcal{N} \) are two left module categories over a finite monoidal category \( \mathcal{A} \). A module functor between them is a pair \((\mathcal{F}, f)\) where \( \mathcal{F} : \mathcal{M} \to \mathcal{N} \) is a functor and \( f \) is a family of natural isomorphisms

\[
f_{XM} : \mathcal{F}(X \otimes M) \to X \otimes \mathcal{F}(M)
\]

for all objects \( X \) in \( \mathcal{A} \) and \( M \) in \( \mathcal{M} \) such that for any \( X, Y \) in \( \mathcal{A} \), \( M \) in \( \mathcal{M} \), the following diagrams are commutative.

\[
\begin{align*}
\mathcal{F}((X \otimes Y) \otimes M) & \xrightarrow{\mathcal{F}(a_{XY,M})} \mathcal{F}(X \otimes (Y \otimes M)) \xrightarrow{f_{X(Y \otimes M)}} X \otimes \mathcal{F}(Y \otimes M) \\
(X \otimes Y) \otimes \mathcal{F}(M) & \xrightarrow{a_{XY,\mathcal{F}(M)}} X \otimes (Y \otimes \mathcal{F}(M))
\end{align*}
\]

\[
\begin{align*}
\mathcal{F}(I \otimes M) & \xrightarrow{f_{IM}} I \otimes \mathcal{F}(M) \\
\mathcal{F}(M) & \xrightarrow{\mathcal{F}(f_{M})} X \otimes \mathcal{F}(M)
\end{align*}
\]

The collection of all module functors \((\mathcal{F}, f) : \mathcal{M} \to \mathcal{N}\) between two module categories \( \mathcal{M} \) and \( \mathcal{N} \) over a finite monoidal category \( \mathcal{A} \) is denoted by \( \text{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \).

**Lemma 1.7.1.** If \((\mathcal{F}, f) : \mathcal{M} \to \mathcal{N}\) and \((\mathcal{G}, g) : \mathcal{N} \to \mathcal{K}\) are two module functors, then \((\mathcal{G} \circ \mathcal{F}, e) : \mathcal{M} \to \mathcal{K}\) is a module functor where \( e = g \circ \mathcal{G}(f) \).

A morphism between \((\mathcal{F}, f)\) and \((\mathcal{G}, g)\) is a natural transformation \( h : \mathcal{F} \to \mathcal{G} \) such that for any \( X \) in \( \mathcal{A} \), \( M \) in \( \mathcal{M} \), the following diagram commutes.

\[
\begin{align*}
\mathcal{F}(X \otimes M) & \xrightarrow{h_{X \otimes M}} \mathcal{G}(X \otimes M) \\
X \otimes \mathcal{F}(M) & \xrightarrow{id_X \otimes h(M)} X \otimes \mathcal{G}(M)
\end{align*}
\]

**Lemma 1.7.2.** \( \text{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \) is a category of module functors \((\mathcal{F}, f) : \mathcal{M} \to \mathcal{N}\) for all module categories \( \mathcal{M} \) and \( \mathcal{N} \) over a given finite monoidal category \( \mathcal{A} \).

Two module categories \( \mathcal{M} \) and \( \mathcal{N} \) over a finite monoidal category \( \mathcal{A} \) are equivalent if there exist module functors \((\mathcal{F}, f) : \mathcal{M} \to \mathcal{N}, (\mathcal{G}, g) : \mathcal{N} \to \mathcal{M}\) and natural isomorphisms

\[
h : id_{\mathcal{N}} \to (\mathcal{F} \circ \mathcal{G}), \quad k : id_{\mathcal{M}} \to (\mathcal{G} \circ \mathcal{F}).
\]

We denote the full subcategory of \( \text{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \) consisting of right exact \( \mathcal{A} \) module functors by \( \text{Hom}_{\mathcal{A}}^{re}(\mathcal{M}, \mathcal{N}) \). Similarly, we use \( \text{Hom}_{\mathcal{A}}^{le}(\mathcal{M}, \mathcal{N}) \) to denote the full subcategory of left exact \( \mathcal{A} \) module functors and \( \text{Hom}_{\mathcal{A}}^{ex}(\mathcal{M}, \mathcal{N}) \) to denote the full subcategory of exact \( \mathcal{A} \) module functors.
Theorem 1.7.3. [EtOs] Every additive module functor $F : \mathcal{M} \to \mathcal{N}$ between two module categories $\mathcal{M}$ and $\mathcal{N}$ over an FRBSU monoidal category $\mathcal{A}$ is exact if $\mathcal{M}$ is exact.

1.8 Bimodule Category and Some Properties

[DaNi] Assume that $\mathcal{A}$ and $\mathcal{B}$ are two finite monoidal categories. A category $\mathcal{M}$ is an $(\mathcal{A}−\mathcal{B})$ bimodule category if it is a left $\mathcal{A}$ module category and right $\mathcal{B}$ module category such that there exists a middle associativity constraint $\alpha$ consisting of a collection of isomorphisms

$$a_{XMY} : X \otimes (M \otimes Y) \to (X \otimes M) \otimes Y$$

natural in $X \in \mathcal{A}, Y \in \mathcal{B}, M \in \mathcal{M}$ which satisfies the commutativity of two pentagons.

Lemma 1.8.1. If $\mathcal{M}$ is an $(\mathcal{A}−\mathcal{B})$ bimodule category, then $\mathcal{M}^{op}$ is a $(\mathcal{B}−\mathcal{A})$ bimodule category.

Proof. Assume that $\mathcal{M}$ is an $(\mathcal{A}−\mathcal{B})$ bimodule category. In that situation $\mathcal{M}$ is a left $\mathcal{A}$ module category and a right $\mathcal{B}$ module category, so $\mathcal{M}^{op}$ is a left $\mathcal{B}$ module category and right $\mathcal{A}$ module category by Proposition 1.5.1.

We have an associativity constraint $\alpha$ consisting of a family of isomorphisms

$$a_{XMY} : X \otimes (M \otimes Y) \to (X \otimes M) \otimes Y$$

natural in $X \in \mathcal{A}, Y \in \mathcal{B}, M \in \mathcal{M}$ as in 37 which satisfies the commutativity of the required diagrams for all $X, Y \in \mathcal{A}, Z, W \in \mathcal{B}$ and $M \in \mathcal{M}$ to be an $(\mathcal{A}−\mathcal{B})$ bimodule. We need to define $\alpha^{op}$ consisting of associativity constraints

$$a_{X^{op}MY}^{op} : X^{op} \otimes^{op} (M^{op} \otimes^{op} Y) \to (X^{op} \otimes^{op} M^{op}) \otimes^{op} Y.$$  

38 is obtained by reversing the morphism $Y \otimes (M \otimes X) \to (Y \otimes M) \otimes X$ in $\mathcal{M}$ which is same as $a_{YMX}$. We can prove the compatibility conditions without no difficulty.

Lemma 1.8.2. Every finite, rigid monoidal category $\mathcal{A}$ is a bimodule category over itself.

Proof. Assume that $\mathcal{A}$ is a finite, rigid monoidal category. We can take $\mathcal{M} = \mathcal{A}$. We have a bifunctor $\mathcal{F} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ taking $(X, Y)$ to $X \otimes Y$. $\mathcal{F}$ is exact in each variable by Remark 1.3.1.

We can use the associativity constraint $\alpha$ and left unit constraint $l$ in the definition of a monoidal category. We can see that these satisfy the commutativity of the required diagrams to be a left $\mathcal{A}$ module category. Similarly, it is a right $\mathcal{A}$ module category with the same associativity constraint and right unit constraint $r$ by the definition of a monoidal category. Also, we use same associativity constraint and a middle associativity constraint. These satisfy the compatibility conditions.
Lemma 1.8.3. If $\mathcal{A}$ and $\mathcal{B}$ are finite monoidal categories, then every exact $(\mathcal{A} - \mathcal{B})$ bimodule category $\mathcal{M}$ is finite.

Lemma 1.8.4. [DaNi] If $\mathcal{A}$ is a braided monoidal category, then any left $\mathcal{A}$ module category $\mathcal{M}$ is an $(\mathcal{A} - \mathcal{A})$ bimodule category.

Remark 1.8.1. $\mathcal{A}\mathcal{A} = \mathcal{A}\mathcal{I}$ and $\mathcal{A}\mathcal{B} = \mathcal{I}\mathcal{A}\mathcal{B}$.

Proof. Assume that $M$ is an object in $\mathcal{A}\mathcal{A}$, so it is a left $\mathcal{A}$ module. $M \otimes I \cong M$, so it is a right $I$ module at the same time. As a result, it is an object in $\mathcal{A}\mathcal{A}I$. Same for $I\mathcal{A}\mathcal{B}$.

Lemma 1.8.5. The category $\mathcal{A}\mathcal{A}\mathcal{B}$ consisting of $(\mathcal{A} - \mathcal{B})$ bimodules is an $(\mathcal{A} - \mathcal{A})$ bimodule category.

Proof. Every $(\mathcal{A} - \mathcal{B})$ bimodule $M$ is a left $\mathcal{A}$ module and a right $\mathcal{B}$ module in $\mathcal{A}$ that satisfies the compatible conditions, so $M$ is an object in $\mathcal{A}\mathcal{A}$ and an object in $\mathcal{A}\mathcal{B}$. This means that $\mathcal{A}\mathcal{A}\mathcal{B}$ is a subcategory of $\mathcal{A}\mathcal{A}$ and a subcategory of $\mathcal{A}\mathcal{B}$. $\mathcal{A}\mathcal{A}$ is a left $\mathcal{A}$ module category and $\mathcal{A}\mathcal{B}$ is left $\mathcal{A}$ module category. As a result, $\mathcal{A}\mathcal{A}\mathcal{B}$ is both a left $\mathcal{A}$ and right $\mathcal{A}$ module category. We need to define an associativity constraint $\alpha$ consisting of associativity isomorphisms as in [37] for all objects $X, Y$ in $\mathcal{A}$, $M$ in $\mathcal{A}\mathcal{A}\mathcal{B}$ that satisfies the required conditions.

We have two actions $\mathcal{A} \times \mathcal{A}\mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{A}\mathcal{B}$ taking $(X, M)$ to $X \otimes_{l(\mathcal{A}\mathcal{B})} M$ and $\mathcal{A}\mathcal{A}\mathcal{B} \times \mathcal{A} \to \mathcal{A}\mathcal{A}\mathcal{B}$ taking $(M, Y)$ to $M \otimes_{r(\mathcal{A}\mathcal{A})} Y$.

We know that $X \otimes_{l(\mathcal{A}\mathcal{B})} M$ is a right $\mathcal{B}$ module and we may show that it is a left $\mathcal{A}$ module which means that it is an $(\mathcal{A} - \mathcal{B})$ bimodule. Similarly, $M \otimes_{r(\mathcal{A}\mathcal{A})} Y$ is an $(\mathcal{A} - \mathcal{B})$ bimodule.

$X \otimes_{l(\mathcal{A}\mathcal{B})} (M \otimes_{r(\mathcal{A}\mathcal{A})} Y) \to (X \otimes_{l(\mathcal{A}\mathcal{B})} M) \otimes_{r(\mathcal{A}\mathcal{A})} Y$ is an isomorphism since $M$ is an object in $\mathcal{A}$ and the above actions are exactly same as the tensor product in $\mathcal{A}$, we can use the associativity constraint in $\mathcal{A}$ as a middle associativity constraint. It is obvious, this gives the commutativity of the diagrams in the definition.

The following proposition and its proof is found in [DaNi] and we want to repeat the proof here.

Proposition 1.8.6. The functor $\mathfrak{F}: \mathcal{A}\mathcal{B} \to \mathcal{H}_{\mathcal{A}}(\mathcal{A}\mathcal{A}, \mathcal{A}\mathcal{B})$ taking $M$ to $- \otimes_{\mathcal{A}} M$ is an equivalence of categories for all algebras $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{A}$ for $\mathcal{A}$ is a finite monoidal category.

Proof. We must show that

$$f: \mathcal{H}_{\mathcal{A}}(\mathcal{A}\mathcal{A}\mathcal{B}, \mathcal{N}) \to \mathcal{H}_{\mathcal{A}}(\mathcal{A}\mathcal{A}\mathcal{B}, \mathcal{M})$$
is an isomorphism and $\mathcal{F}$ is essentially surjective for all $(A - B)$ bimodules $M$ and $N$.

We send each morphism $M \xrightarrow{a} N$ in the category of $(A - B)$ bimodules to a natural transformation $- \otimes_A M \xrightarrow{f(a)} - \otimes_A N$ in the category of module functors $\text{Hom}_\mathcal{A}^r(\mathcal{A}A, \mathcal{A}B)$. Here, $- \otimes_A M, - \otimes_A N : \mathcal{A}A \to \mathcal{A}B$ are $\mathcal{A}$ module functors for all $(A - B)$ bimodules $M$ and $N$ in $\mathcal{A}$.

To show that they are module functors, we show that $\text{Hom}(f, g) = f \circ g$ for all morphisms $f, g : M \to N$. For all morphism $f : M \to N$, we get $f(a)(K) = f(b)(K) : K \otimes_A M \to K \otimes_A N$.

For all elements $k$ in $K$ and $m$ in $M$, we have $k \otimes a(m) = k \otimes b(m)$, so $k \otimes (a(m) - b(m)) = 0$ and $a(m) = b(m)$. This says that $a = b$. Surjectivity is clear. As a result $f$ is an isomorphism.

For all right exact $\mathcal{A}$ module functors $\mathcal{G} : \mathcal{A}A \to \mathcal{A}B$, can we find an $(A - B)$ bimodule $M$ such that $\mathfrak{F}(M) \cong \mathcal{G}$? We take $M = \mathcal{G}(A)$. It is $(A - B)$ bimodule.

Now we want to show the commutativity of the required diagram for the natural isomorphism. For all morphism $\alpha : T \to S$ in $\mathcal{A}A$, we get the following commutative diagram.

\[
\begin{array}{ccc}
T \otimes_A \mathcal{G}(A) & \xrightarrow{\alpha \otimes \mathcal{G}(A)} & \mathcal{G}(S) \\
\mathfrak{F} & \xrightarrow{\mathfrak{F}} & \mathfrak{F}
\end{array}
\]

$T \otimes_A \mathcal{G}(A) \cong T \otimes_A A \cong T \cong \mathcal{G}(T)$ since $\mathcal{G}(A) \cong A$ and $\mathcal{G}(T) \cong T$. Similarly, $\theta$ is an isomorphism, so the diagram commutes.

As a result, we see that $\text{Hom}_\mathcal{A}^r(\mathcal{A}A, \mathcal{A}B)$ is a finite category since the category
$AAB$ is finite.

Similarly, the functor $AA \to \text{Hom}^r_A(AA, AB)$ taking $M$ to $\text{Hom}_{AA}(-, M) : AA \to AB$ is an equivalence of categories for all given algebras $A$ and $B$ in $A$. So, $\text{Hom}^r_A(AA, AB)$ is a finite category.

**Lemma 1.8.7.** $\text{Hom}^r_A(M, N)$ is finite if $M$ and $N$ are exact module categories over $A$ and satisfies the required conditions in Proposition 2.0.13.

**Proof.** $M \simeq AA$ and $N \simeq AB$ for some algebras $A$ and $B$ in $A$. So, the category $\text{Hom}^r_A(M, N)$ is equivalent to the category $AAB$ and $AAB$ is finite.

**Lemma 1.8.8.** $\text{Hom}^r_A(M, M)$ is a monoidal category of endofunctors of $M$ for $M$ is a left module category over a finite monoidal category $A$.

**Proof.** For given two module categories $F, G : M \to M$, we define their tensor product as the composition $G \circ F : M \to M$ and the unit functor as the identity of $M$ which is $I = id_M : M \to M$. We get an associativity constraint $a$ which is a family of associative isomorphisms $a_{FGH} : (F \circ G) \circ H \to F \circ (G \circ H)$, left and right unit constraints satisfying the required compatibility conditions.

**Proposition 1.8.9.** The monoidal category $\text{Hom}^r_A(AA, AA)$ is strict and rigid. If $A = \text{Vec}(k)$, then it is not a rigid category.

**Proof.** The right and left duals are the right and left adjoint functors.

**Theorem 1.8.10.** If $A$ is a multifusion category, $M$ and $N$ are module categories over $A$, then the category $\text{Hom}^r_A(M, N)$ is semisimple module category over $\text{Hom}^r_A(M, M)$ with action given by composition of functors. It is exact if $M$ and $N$ are exact module categories over $A$.

**Proof.** We define the action as $\text{Hom}^r_A(M, M) \times \text{Hom}^r_A(M, N) \to \text{Hom}^r_A(M, N)$ with $(F, G) \mapsto G \circ F$.

### 1.9 The Center of A Bimodule Category

The center $Z_A(M)$ of an $A$ bimodule category $M$ is defined in [Gr] as below. Here, $A$ is a finite rigid, monoidal category whose unit object is simple.

The objects are $(M, \gamma_M)$ where $M$ is an object in $M$ and $\gamma_M$ is a family of natural isomorphisms $\gamma_{MX} : X \otimes M \to M \otimes X$ which satisfy the commutativity of the following...
A morphism between $(M, \gamma_M)$ and $(N, \gamma_N)$ in $Z_A(M)$ is a morphism $f : M \to N$ in $M$ satisfying the condition $\gamma_N(X)(id_X \otimes f) = (f \otimes id_X)\gamma_M(X)$.

### 1.10 Definition of A Bicategory

The following definitions are found in [Le1] and [Le2] in detail.

A collection $\mathcal{X}$ consisting of the objects $A, B, ...$ is a bicategory if the following conditions are satisfied.

1. $\mathcal{X}(A, B)$ is a category whose objects are 1 arrows $f : A \to B$, $g : A \to B$, ... and morphisms are 2 arrows $\gamma : f \Rightarrow g$, $\theta : f \Rightarrow g$, ... as shown in the following diagram.

   $\begin{tikzcd}
   A \arrow[swap]{r}{g} \arrow[anchor=base east]{rr}{\gamma} & B
   \end{tikzcd}$

2. $\mathcal{F}_{ABC} : \mathcal{X}(B, C) \times \mathcal{X}(A, B) \to \mathcal{X}(A, C)$ is a functor taking the pairs $(g, f)$ to $g \circ f = gf$ and $(\theta, \gamma)$ to $\theta \ast \gamma$. $\theta \ast \gamma$ is shown as in the following diagram.

   $\begin{tikzcd}
   A \arrow{r}{f} \arrow{r}{\gamma} & B \arrow{r}{\theta} & C = A \arrow{r}{\theta \ast \gamma} & C
   \end{tikzcd}$

3. $\mathcal{F}_A : 1 \to \mathcal{X}(A, A)$ is a functor sending the object $\ast$ in 1 to the arrow $id_A$ where 1 is a category with one object.
4. \( a_{ABCD} : F_{ABD} \circ (F_{BCD} \times 1) \rightarrow F_{ACD} \circ (1 \times F_{ABC}) \)

\[
\begin{array}{ccc}
\mathcal{X}(C, D) \times \mathcal{X}(B, C) \times \mathcal{X}(A, B) & \xrightarrow{F_{BCD} \times 1} & \mathcal{X}(B, D) \times \mathcal{X}(A, B) \\
\downarrow 1 \times F_{ABC} & & \downarrow F_{ABD} \\
\mathcal{X}(C, D) \times \mathcal{X}(A, C) & \xrightarrow{a_{ABCD}} & \mathcal{X}(A, D)
\end{array}
\]

(40)

is a natural isomorphism. \( a_{ABCD}(f, g, h) : (fg)h \sim f(gh) \) are 2 arrows for all 1 arrows \( f : C \rightarrow D, g : B \rightarrow C \) and \( h : A \rightarrow B \) such that the following pentagon commutes for all 1 arrows \( f, g, h, k \).

\[
\begin{array}{ccc}
(fg)(hk) & \xrightarrow{a} & (fgh)k \\
\downarrow a & & \downarrow a \\
(fg)(hk) & \xrightarrow{id_f \ast a} & f((gh)k) \\
\downarrow a & & \downarrow f(g(hk))
\end{array}
\]

(41)

5. \( r_{AB} : F_{AAB} \circ (1 \times F_A) \rightarrow \mathcal{G} \) and \( l_{AB} : F_{ABB} \circ (F_B \times 1) \rightarrow \mathcal{H} \)

\[
\begin{array}{ccc}
\mathcal{X}(A, B) \times \mathcal{X}(A, A) & \xrightarrow{F_{AAB}} & \mathcal{X}(B, B) \times \mathcal{X}(A, B) \\
\downarrow 1 \times F_A & & \downarrow F_{ABB} \\
\mathcal{X}(A, B) & \xrightarrow{\sim} & \mathcal{X}(A, B) \times 1 \\
\downarrow r_{AB} & & \downarrow l_{AB} \\
\mathcal{X}(A, B) & \xrightarrow{\sim} & 1 \times \mathcal{X}(A, B)
\end{array}
\]

(42)

are natural isomorphisms. \( r_{AB}(f, \ast) : f \circ id_A \sim f \) and \( l_{AB}(\ast, f) : id_B \circ f \sim f \) are 2 arrows for all 1 arrows \( f : A \rightarrow B \) such that the following triangle commutes.

\[
\begin{array}{ccc}
(fid_A)g & \xrightarrow{a} & f(id_Bg) \\
\downarrow r_*id_g & & \downarrow id_f \ast l \\
f \circ g & \xrightarrow{id_f \ast l}
\end{array}
\]

(43)

Remark 1.10.1. If all natural isomorphisms \( a, r, l \) are identities such that \( (fg)h = f(gh) \), \( 1f = f = f1 \) and same conditions are true for the composition of 2 arrows, then \( \mathcal{X} \) is called a 2-category.
2 Internal Hom of Two Objects in A Module Category

In this section, we are assuming that $\mathcal{M}$ is an exact module category over a finite, rigid monoidal category $\mathcal{A}$ whose unit object $I$ is simple and we are given objects $M, N$ in $\mathcal{M}$.

**Lemma 2.0.1.** The functor $\text{Hom}_\mathcal{M}(- \otimes M, N) : \mathcal{A} \to \text{Set}$ is left exact.

**Proof.** Assume that we have an exact sequence $0 \to A \to B \to C \to 0$ for all objects $A, B$ and $C$ in $\mathcal{A}$. Then, the sequence $A \otimes M \to B \otimes M \to C \otimes M \to 0$ is exact since $- \otimes M$ is an exact functor by 27. So, the sequence

$$0 \to \text{Hom}_\mathcal{M}(C \otimes M, N) \to \text{Hom}_\mathcal{M}(B \otimes M, N) \to \text{Hom}_\mathcal{M}(A \otimes M, N)$$

is exact since $\text{Hom}_\mathcal{M}(-, N)$ is left exact contrafunctor by Example ???. This proves the left exactness. 

**Definition 2.0.1.** Internal hom of $M$ and $N$ is an object $\text{Hom}_\mathcal{M}(M, N)$ in $\mathcal{A}$ which represents the functor $\text{Hom}_\mathcal{M}(- \otimes M, N) : \mathcal{A} \to \text{Set}$ whenever it is representable.

This means that there exists a natural isomorphism between the functors $\text{Hom}_\mathcal{M}(- \otimes M, N)$ and $\text{Hom}_\mathcal{A}(-, \text{Hom}_\mathcal{M}(M, N))$.

**Lemma 2.0.2.** The functor $\text{Hom}_\mathcal{M}(- \otimes M, N) : \mathcal{A} \to \text{Set}$ is exact if the internal hom $\text{Hom}_\mathcal{M}(M, N)$ exists and projective in $\mathcal{A}$.

**Proof.** $\text{Hom}_\mathcal{M}(X \otimes M, N) \cong \text{Hom}_\mathcal{A}(X, \text{Hom}_\mathcal{M}(M, N))$ for all objects $X$ in $\mathcal{A}$ by existence of representing object. We know that this functor is left exact contrafunctor by Lemma 2.0.1. We want to show that it is right exact. For this, we need to show that the contrafunctor $\text{Hom}_\mathcal{A}(-, \text{Hom}_\mathcal{M}(M, N)) : \mathcal{A} \to \text{Set}$ is right exact which means that the covariant functor $\text{Hom}_{\mathcal{A}^{\text{op}}}(\text{Hom}_\mathcal{M}(M, N), -) : \mathcal{A}^{\text{op}} \to \text{Set}$ is right exact.

Assume that $0 \longrightarrow X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z \longrightarrow 0$ is an exact sequence in $\mathcal{A}^{\text{op}}$. We want to show that the sequence

$$0 \longrightarrow \text{Hom}_{\mathcal{A}^{\text{op}}}(\text{Hom}_\mathcal{M}(M, N), X) \overset{F}{\longrightarrow} \text{Hom}_{\mathcal{A}^{\text{op}}}(\text{Hom}_\mathcal{M}(M, N), Y) \overset{G}{\longrightarrow} \text{Hom}_{\mathcal{A}^{\text{op}}}(\text{Hom}_\mathcal{M}(M, N), Z) \longrightarrow 0$$

is exact in $\text{Set}$. We just need to show that $G$ is an epimorphism since that sequence is left exact. It is obvious since the internal hom is projective. 

This functor is always exact if $\mathcal{A}$ is semisimple by Lemma 1.6.4.
Lemma 2.0.3. \( \text{Hom}_\mathcal{M}(X \otimes M, N) \cong \text{Hom}_\mathcal{A}(X, \text{Hom}_\mathcal{M}(M, N)) \) canonically for all objects \( X \) in \( \mathcal{A} \) by definition since the functor \( \text{Hom}_\mathcal{M}(\cdot \otimes M, N) \) is contravariant.

Lemma 2.0.4. \( \text{Hom}_\mathcal{M}(M, X \otimes N) \cong \text{Hom}_\mathcal{A}(I, X \otimes \text{Hom}_\mathcal{M}(M, N)) \) canonically for all objects \( X \) in \( \mathcal{A} \).

Proof. For all morphisms \( M \to X \otimes N \) in \( \mathcal{M} \), we find a morphism
\[
X^+ \otimes M \to X^+ \otimes X \otimes N \to I \otimes N = N
\] (44)
by using rigidity of \( \mathcal{A} \). Here, we use the evaluation map \( ev_{X^+} : X^+ \otimes X \to I \). So,
\[
\text{Hom}_\mathcal{M}(M, X \otimes N) \cong \text{Hom}_\mathcal{M}(X^+ \otimes M, N) \cong \text{Hom}_\mathcal{A}(X^+, \text{Hom}_\mathcal{M}(M, N)).
\] (45)
For all morphisms \( X^+ \to \text{Hom}_\mathcal{M}(M, N) \), we get a morphism
\[
I \to X \otimes X^+ \to X \otimes \text{Hom}_\mathcal{M}(M, N)
\] (46)
by using the coevaluation map. As a result, we get an isomorphism
\[
\text{Hom}_\mathcal{M}(X^+, \text{Hom}_\mathcal{M}(M, N)) \cong \text{Hom}_\mathcal{A}(I, X \otimes \text{Hom}_\mathcal{M}(M, N)).
\] (47)

Lemma 2.0.5. \( \text{Hom}_\mathcal{M}(X \otimes M, N) \cong \text{Hom}_\mathcal{M}(M, N) \otimes X^+ \) canonically.

Proof. \[\text{Os}\] We have
\[
\text{Hom}_\mathcal{A}(K, \text{Hom}_\mathcal{M}(M, N) \otimes X^+) \cong \text{Hom}_\mathcal{A}(K \otimes X, \text{Hom}_\mathcal{M}(M, N)) \cong \text{Hom}_\mathcal{M}((K \otimes X) \otimes M, N) \cong \text{Hom}_\mathcal{M}(K \otimes (X \otimes M), N) \cong \text{Hom}_\mathcal{A}(K, \text{Hom}_\mathcal{M}(X \otimes M, N))
\]
for all \( K \) in \( \mathcal{M} \), so \( \text{Hom}_\mathcal{M}(M, N) \otimes X^+ \cong \text{Hom}_\mathcal{M}(X \otimes M, N) \) canonically. \[\Box\]

Lemma 2.0.6. \( \text{Hom}_\mathcal{M}(M, X \otimes N) \cong X \otimes \text{Hom}(M, N) \) canonically.

Proof. See \[\text{Os}\]. \[\Box\]

Lemma 2.0.7. \[\text{EHOS}\] If we assume that \( \mathcal{A} \) is a braided monoidal category, then \( \text{Hom}_\mathcal{M}(M, M) \) is an algebra for given object \( M \) in \( \mathcal{M} \) if it exists as a representing object of the functor \( \text{Hom}_\mathcal{M}(\cdot \otimes M, M) \).

Proof. We need to define a multiplication morphism
\[
m : \text{Hom}_\mathcal{M}(M, M) \otimes \text{Hom}_\mathcal{M}(M, M) \to \text{Hom}_\mathcal{M}(M, M)
\]
and a unit morphism \( u : I \to \text{Hom}_\mathcal{M}(M, M) \) satisfying the required compatibility conditions.
Os finds a multiplication morphism $m$ as below.

$id_{\text{Hom}_A(M, M)}$ is in $\text{Hom}_A(\text{Hom}_A(M, M), \text{Hom}_A(M, M))$ since $\text{Hom}_A(M, M)$ is an object in $\mathcal{A}$.

$\text{Hom}_A(\text{Hom}_A(M, M), \text{Hom}_A(M, M)) \cong \text{Hom}_A(\text{Hom}_A(M, M) \otimes M, M)$ by definition. So, we get a unique morphism $f : \text{Hom}_A(M, M) \otimes M \to M$ corresponding to $id_{\text{Hom}_A(M, M)}$. Using this morphism, we get a composition

$$
\text{Hom}_A(M, M) \otimes (\text{Hom}_A(M, M) \otimes M) \xrightarrow{id \otimes f} \text{Hom}_A(M, M) \otimes M \xrightarrow{f} M
$$

This is same as the morphism $(\text{Hom}_A(M, M) \otimes \text{Hom}_A(M, M)) \otimes M \to M$ since $\text{Hom}_A(M, M) \otimes (\text{Hom}_A(M, M) \otimes M) \cong (\text{Hom}_A(M, M) \otimes \text{Hom}_A(M, M)) \otimes M$.

This defines a multiplication morphism

$$\text{Hom}_A(M, M) \otimes \text{Hom}_A(M, M) \to \text{Hom}_A(M, M)$$

as shown in Os.

This multiplication is associative since $\text{Hom}_A(M, M)$ is an object in $\mathcal{A}$ and we have the associativity constraint.

For the compatibility conditions, we need to show that the following diagrams commute.

$$
\begin{array}{ccc}
\text{Hom}_A(M, M) \otimes \text{Hom}_A(M, M) & \xrightarrow{m \otimes id} & \text{Hom}_A(M, M) \otimes \text{Hom}_A(M, M) \\
& \xrightarrow{id \otimes m} & \text{Hom}_A(M, M) \\
\end{array}
$$

$$
\begin{array}{ccc}
\text{Hom}_A(M, M) \otimes \text{Hom}_A(M, M) & \xrightarrow{m} & \text{Hom}_A(M, M) \\
\end{array}
$$

$$
\begin{array}{ccc}
\text{Hom}_A(M, M) \otimes \text{Hom}_A(M, M) & \xrightarrow{u \otimes id} & \text{Hom}_A(M, M) \\
& \xrightarrow{l} & \text{Hom}_A(M, M) \\
\end{array}
$$

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We have the isomorphisms

\[ \text{Hom}_\mathcal{M}(M, M) \otimes \text{Hom}_\mathcal{M}(M, M) \cong \text{Hom}_\mathcal{M}(M, M) \cong \text{Hom}_\mathcal{M}(\text{Hom}_\mathcal{M}(M, M) \otimes \text{Hom}_\mathcal{M}(M, M) \otimes M, M), \]

\[ \text{Hom}_\mathcal{M}(\text{Hom}_\mathcal{M}(M, M), \text{Hom}_\mathcal{M}(M, M)) \cong \text{Hom}_\mathcal{M}(\text{Hom}(M, M) \otimes M, M). \]

\( m \) is a morphism in the first hom set and \( id \) is a morphism in the second one.

Also, we have a composition of hom sets

\[ \text{Hom}_\mathcal{A}(A \otimes A \otimes A, A \otimes A) \times \text{Hom}_\mathcal{A}(A \otimes A, A) \to \text{Hom}_\mathcal{A}(A \otimes A \otimes A, A) \]

for \( A = \text{Hom}_\mathcal{M}(M, M) \) taking \((m \otimes id, m)\) to \( m \circ (m \otimes id)\) and \((id \otimes m, m)\) to \( m \circ (id \otimes m)\). We may prove that \( m \circ (m \otimes id) = m \circ (id \otimes m)\) for the commutativity of the first diagram.

Now, we want to find a unit morphism \( u : I \to \text{Hom}_\mathcal{M}(M, M) \) satisfying the commutativity of the required diagrams.

\[ \text{Hom}_\mathcal{M}(M, X \otimes N) \cong \text{Hom}_\mathcal{A}(I, X \otimes \text{Hom}_\mathcal{M}(M, N)) \] for all objects \( X \) in \( \mathcal{A} \). Taking \( X = I \) and \( M = N \), we get an isomorphism \( \text{Hom}_\mathcal{M}(M, M) \cong \text{Hom}_\mathcal{A}(I, \text{Hom}_\mathcal{M}(M, M)) \). So, for the identity morphism \( id_M : M \to M \), we get a unique morphism \( u : I \to \text{Hom}_\mathcal{M}(M, M) \).

We have an isomorphism

\[ \text{Hom}_\mathcal{A}(I \otimes \text{Hom}_\mathcal{M}(M, M), \text{Hom}_\mathcal{M}(M, M)) \cong \text{Hom}_\mathcal{M}(I \otimes \text{Hom}_\mathcal{M}(M, M) \otimes M, M). \]

The diagram

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{M}(M, M) \otimes \text{Hom}_\mathcal{M}(M, M) & \xrightarrow{m} & \text{Hom}_\mathcal{M}(M, M) \\
\downarrow{u \otimes id} & & \downarrow{l} \\
I \otimes \text{Hom}_\mathcal{M}(M, M) & & \\
\end{array}
\]
corresponds to the diagram

\begin{equation*}
\begin{array}{c}
\text{Hom}_\mathcal{M}(M, M) \otimes \text{Hom}_\mathcal{M}(M, M) \otimes M \\
\downarrow u \otimes id \\
I \otimes \text{Hom}_\mathcal{M}(M, M) \otimes M
\end{array}
\end{equation*}

And we get another diagram

\begin{equation*}
\begin{array}{c}
M \otimes \text{Hom}_\mathcal{M}(M, M) \otimes M \\
\downarrow id_M \otimes id \\
M \otimes \text{Hom}_\mathcal{M}(M, M) \otimes M
\end{array}
\end{equation*}

Obviously, this diagram commutes. As a result, the first one commutes and we get the result. We follow the similar way for the right associativity constraint. \hfill \Box

**Lemma 2.0.8.** [Baki] $(^+A)^+ = A$ for all objects $A$ in a rigid monoidal category $\mathcal{A}$.

**Lemma 2.0.9.** [EtOs] $\text{Hom}_{\mathcal{AA}}(M, M) = (M \otimes^+ M)^+ = (^+M)^+ \otimes M^+ = M \otimes M^+ \cong M^+ \otimes M$ for all right $\mathcal{A}$ modules $M$ in a left module category $\mathcal{AA}$ over an FRBSU monoidal category $\mathcal{A}$.

**Lemma 2.0.10.** $\text{Hom}_\mathcal{M}(M, N)$ is a right $\text{Hom}_\mathcal{M}(M, M)$ module in $\mathcal{A}$.

**Proof.** We need to define a multiplication morphism

\[ a : \text{Hom}_\mathcal{M}(M, N) \otimes \text{Hom}_\mathcal{M}(M, M) \to \text{Hom}_\mathcal{M}(M, N) \]
satisfying the commutativity of the following diagrams.

\begin{equation*}
\begin{array}{ccc}
\text{Hom}_\mathcal{M}(M, N) \otimes \text{Hom}_\mathcal{M}(M, M) & \xrightarrow{id \otimes m} & \text{Hom}_\mathcal{M}(M, N) \otimes \text{Hom}_\mathcal{M}(M, M) \\
\downarrow \alpha \otimes id & & \downarrow \alpha \otimes id \\
\text{Hom}_\mathcal{M}(M, N) \otimes \text{Hom}_\mathcal{M}(M, M) & \xrightarrow{\alpha} & \text{Hom}_\mathcal{M}(M, N) \otimes \text{Hom}_\mathcal{M}(M, M)
\end{array}
\end{equation*}
So the identity morphism $id_{\mathcal{H}om_{\mathcal{M}}(M, N)}$ corresponds to a morphism

$$k : \mathcal{H}om_{\mathcal{M}}(M, N) \otimes M \to N$$

and $a$ corresponds to a composition

$$\mathcal{H}om_{\mathcal{M}}(M, N) \otimes \mathcal{H}om_{\mathcal{M}}(M, M) \otimes M \to \mathcal{H}om_{\mathcal{M}}(M, N) \otimes M \to \mathcal{H}om_{\mathcal{M}}(M, N) \otimes M \to N$$

which is $k \circ (id_{\mathcal{H}om_{\mathcal{M}}(M, N)} \otimes f)$ and $f$ is the morphism $\mathcal{H}om_{\mathcal{M}}(M, M) \otimes M \to M$ that corresponds to $id_{\mathcal{H}om_{\mathcal{M}}(M, M)}$. It is easy to show that those diagrams commute. □

**Lemma 2.0.11.** If $A = \mathcal{H}om_{\mathcal{M}}(M, M)$ is the algebra defined as above, then $AA$ is an exact left module category over $A$.

**Proposition 2.0.12.** [EtOs] The mapping $\mathcal{H}om_{\mathcal{M}}(M, -) : \mathcal{M} \to AA$ is an exact module functor.

**Proof.** $f$ is a family of natural isomorphisms $f_{XN} : \mathcal{H}om_{\mathcal{M}}(M, X \otimes N) \to X \otimes \mathcal{H}om_{\mathcal{M}}(M, N)$ for all objects $X$ in $\mathcal{A}$, $N$ in $\mathcal{M}$ such that for all $X, Y$ in $\mathcal{A}$, $N$ in $\mathcal{M}$, the following diagrams commute.

Exactness comes from 1.7.3 and this proves the proposition. □
**Theorem 2.0.13.** Assume that $\mathcal{M}$ is an exact module category over a finite, rigid monoidal category $\mathcal{A}$ such that the unit object in $\mathcal{A}$ is simple. Let $\text{Hom}(M, M) = \mathcal{A}$ is the algebra defined as above. Assume further that there exists an object $X \in \mathcal{A}$ for all objects $N \in \mathcal{M}$ such that $\text{Hom}_\mathcal{M}(X \otimes M, N) \neq 0$. Then, the functor $\text{Hom}_\mathcal{M}(M, -) : \mathcal{M} \to \mathcal{A}$ is an equivalence of module categories.

**Proof.** We need to show that

$$\text{Hom}_\mathcal{M}(N, K) \to \text{Hom}_\mathcal{A}(\text{Hom}_\mathcal{M}(M, N), \text{Hom}_\mathcal{M}(M, K))$$

is an isomorphism for all objects $N$ and $K$ in $\mathcal{M}$ and $\mathcal{F}$ is essentially surjective for the equivalence. [Os] proves the isomorphism for all objects $N$ of the form $X \otimes M$ and for all objects $K$ in $\mathcal{M}$ first, then the author proves it for all objects $N$ and $K$ in $\mathcal{M}$ by using the exactness of the functor $\text{Hom}_\mathcal{M}(M, -)$. After that, [Os] shows that $\text{Hom}_\mathcal{M}(M, -)$ is essentially surjective.

We may apply similar way to prove the proposition. In [Os], the proposition is given whenever the category is semisimple, hence exact means semisimple in such a category. $\mathcal{M}$ is indecomposable module category there instead of the assumption for $\mathcal{M}$ in here.

**Example 2.0.1.** $\text{Vec}_f(k)$ is an FRBSU monoidal category and it is an exact left module category over itself with the tensor multiplication. Let $A = \text{Hom}_{\text{Vec}_f(k)}(M, M)$. $A$ is an algebra over $k$ for a right $A$ module $M$ in $\text{Vec}_f(k)$. For all right $A$ modules $N$ in $\text{Vec}_f(k)$, we get a surjection $N \otimes M \to N$, hence $M$ genetares $\text{Vec}_f(k)$ and as a result, $\text{Vec}_f(k) \simeq \text{Vec}_f(k) A$ by Theorem 2.0.13.

**Lemma 2.0.14.** The left module category $\mathcal{A}A$ for $A = \text{Hom}_\mathcal{M}(M, M)$ for some object $M$ in $\mathcal{M}$ is a finite category, hence $\mathcal{M}$ is a finite category.

### 3 Invertible Bimodule Categories

We use [Gr] and [EtNiOs] at most for the following information.

**Lemma 3.0.15.** Assume that $\mathcal{M}$ is an $(\mathcal{A} - \mathcal{B})$ bimodule category and $\mathcal{N}$ is a $(\mathcal{B} - \mathcal{C})$ bimodule category for given finite, rigid monoidal categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Then, $\text{Hom}^e_\mathcal{B}(\mathcal{M}^\text{op}, \mathcal{N})$ is an $(\mathcal{A} - \mathcal{C})$ bimodule category. It is abelian. If $\mathcal{A} = \mathcal{B}$, $\mathcal{M}$ and $\mathcal{N}$ are exact, then it is exact.

**Proof.** To prove that $\text{Hom}^e_\mathcal{B}(\mathcal{M}^\text{op}, \mathcal{N})$ is an $(\mathcal{A} - \mathcal{C})$ bimodule category, we define the left action of $\mathcal{A}$ on $\text{Hom}^e_\mathcal{B}(\mathcal{M}^\text{op}, \mathcal{N})$ by

$$\mathfrak{F} : \mathcal{A} \times \text{Hom}^e_\mathcal{B}(\mathcal{M}^\text{op}, \mathcal{N}) \to \text{Hom}^e_\mathcal{B}(\mathcal{M}^\text{op}, \mathcal{N}), \ (A, \mathcal{F}) \to A \otimes \mathcal{F}$$

for all objects $A$ in $\mathcal{A}$ and right exact module functors $\mathcal{F} : \mathcal{M}^\text{op} \to \mathcal{N}$. Here, we have $(A \otimes \mathcal{F})(M) = \mathcal{F}(M \otimes A)$ for all objects $M$ in $\mathcal{M}^\text{op}$. $\mathcal{M}^\text{op}$ is a right $\mathcal{A}$ module.
category, so $M \otimes A$ is an object in $\mathcal{M}^{op}$. We need to show that $\mathfrak{F}$ is a biexact bifunctor, that is $\mathfrak{F}(-, \mathcal{F})$ is exact for all right exact module functors $\mathcal{F} : \mathcal{M}^{op} \to \mathcal{N}$ which means that exact in the first variable and $\mathfrak{F}(A, -)$ is exact for all objects $A$ in $\mathcal{A}$ which means that exact in the second variable.

To prove that $\mathfrak{F}(-, \mathcal{F})$ is an exact functor, we need to show that

$$0 \to A \otimes \mathcal{F} \to B \otimes \mathcal{F} \to C \otimes \mathcal{F} \to 0$$

is an exact sequence of natural transformations of right exact module functors from $\mathcal{M}^{op}$ to $\mathcal{N}$ whenever the sequence $0 \to A \to B \to C \to 0$ is exact. That sequence of natural transformations is a sequence of morphisms

$$0 \to (A \otimes \mathcal{F})(M) \to (B \otimes \mathcal{F})(M) \to (C \otimes \mathcal{F})(M) \to 0 \quad (48)$$

for all objects $M$ in $\mathcal{M}^{op}$ which satisfies the compatibility conditions for all morphisms $M \to N$ in $\mathcal{M}^{op}$. This sequence is same as the sequence

$$0 \to \mathcal{F}(M \otimes A) \to \mathcal{F}(M \otimes B) \to \mathcal{F}(M \otimes C) \to 0 \quad (49)$$

by the above action. The sequence $0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0$ is exact since the action is an exact functor by definition of module category. As a result, (49) is an exact sequence by Lemma 1.7.3.

Assume that $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of module functors in $\text{Hom}_R^G(\mathcal{M}^{op}, \mathcal{N})$. It is clear that the sequence $0 \to A \otimes \mathcal{F} \to A \otimes \mathcal{G} \to A \otimes \mathcal{H} \to 0$ is exact. As a result, $\mathfrak{F}(A, -)$ is an exact functor.

How do we get an associativity constraint $a$ consisting of a family of associativity isomorphism $a_{ABF} : (A \otimes B) \otimes \mathcal{F} \to A \otimes (B \otimes \mathcal{F})$ for all objects $A, B$ in $\mathcal{A}$, right exact module functors $\mathcal{F} : \mathcal{M}^{op} \to \mathcal{N}$ and a unit constraint $l$ consisting of a family of unit isomorphisms $l_F : I \otimes \mathcal{F} \to \mathcal{F}$ for the unit object $I$ in $\mathcal{A}$, module functor $\mathcal{F} : \mathcal{M}^{op} \to \mathcal{N}$ making the required diagrams commute.

For all objects $A, B$ in $\mathcal{A}$, module functors $\mathcal{F} : \mathcal{M}^{op} \to \mathcal{N}$, we get

$$(A \otimes B) \otimes \mathcal{F}(M) = \mathcal{F}(M \otimes (A \otimes B)),$$

$$(A \otimes (B \otimes \mathcal{F}))(M) = (B \otimes \mathcal{F})(M \otimes A) = \mathcal{F}((M \otimes A) \otimes B).$$

$\mathcal{M}^{op}$ is a right $\mathcal{A}$ module category, so $M \otimes (A \otimes B) \cong (M \otimes A) \otimes B$. We get a short exact sequence $0 \to M \otimes (A \otimes B) \to (M \otimes A) \otimes B \to 0$. $\mathcal{F}$ is exact by Lemma 1.7.3, so the sequence $0 \to \mathcal{F}(M \otimes (A \otimes B)) \to \mathcal{F}((M \otimes A) \otimes B) \to 0$ is exact, hence we get an isomorphism $\mathcal{F}(M \otimes (A \otimes B)) \cong \mathcal{F}((M \otimes A) \otimes B)$ and use this isomorphism as an associativity constraint. Similarly, we define $l$. 

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$\mathcal{G} : \text{Hom}_B^r(\mathcal{M}^{\text{op}}, \mathcal{N}) \times \mathcal{C} \to \text{Hom}_B^r(\mathcal{M}^{\text{op}}, \mathcal{N})$ is the right action of $\mathcal{C}$ on $\text{Hom}_B^r(\mathcal{M}^{\text{op}}, \mathcal{N})$ with $(\mathcal{F}, C) \to \mathcal{F} \otimes C$ for all objects $C$ in $\mathcal{C}$ and right exact module functors $\mathcal{F} : \mathcal{M}^{\text{op}} \to \mathcal{N}$.

Here, $\mathcal{F} \otimes C : \mathcal{M}^{\text{op}} \to \mathcal{N}$ such that $(\mathcal{F} \otimes C)(M) = \mathcal{F}(M) \otimes C$ for all objects $M$ in $\mathcal{M}^{\text{op}}$. $\mathcal{N}$ is right $\mathcal{C}$ module category, so $\mathcal{F}(M) \otimes C$ is an object in $\mathcal{N}$. We can show that $\mathcal{G}$ is a biexact bifunctor in a similar way. After that we find a right associativity constraint and a right unit constraint satisfying the required conditions.

Finally, we find a middle associativity constraint satisfying two commutative diagrams, hence the lemma is proved. 

**Corollary 3.0.1.** If $\mathcal{M}$ is an $(\mathcal{A} - \mathcal{B})$ bimodule category, then

\[
\text{Hom}_B^r(\mathcal{M}^{\text{op}}, \mathcal{B}) \simeq \mathcal{M} \simeq \text{Hom}_A^r(\mathcal{A}^{\text{op}}, \mathcal{M})
\]

(50)

canonically as $(\mathcal{A} - \mathcal{B})$ bimodule categories.

**Lemma 3.0.16.** $[\text{Os}]$ $\mathcal{F}_M : \mathcal{A} \to \text{Hom}_A^r(\mathcal{M}, \mathcal{M})$ is a monoidal functor that takes any object $A$ in $\mathcal{A}$ to $A \otimes_{\mathcal{M}} -$ where $\mathcal{A}$ is a finite, braided monoidal category and $\mathcal{M}$ is a left $\mathcal{A}$ module category.

**Proof.** First, we want to show that $A \otimes -$ is a right exact $\mathcal{A}$ module functor for all objects $A$ in $\mathcal{A}$. The right exactness is clear by definition of module category.

Hom sets are vector spaces, because $\mathcal{M}$ is finite by Lemma [2.0.14]. That functor is $k$ linear since all functions $f : \text{Hom}_M(A, B) \to \text{Hom}_M(A \otimes -, B \otimes -)$ are linear maps for all objects $A$ and $B$ in $\mathcal{M}$.

For the functors $\mathcal{F}_1, \mathcal{F}_2 : A \to B$, $k_1, k_2 \in k$, we get $f(k_1 \mathcal{F}_1 + k_2 \mathcal{F}_2) = \mathcal{F}$ where $((k_1 + k_2)A) \otimes - \xrightarrow{\mathcal{F}} ((k_1 + k_2)B) \otimes -$ is a natural transformation. This natural transformation is same as

\[(k_1(A \otimes -) \to k_1(B \otimes -)) + (k_2(A \otimes -) \to k_2(B \otimes -))\]

which is same as $k_1 f(\mathcal{F}_1) + k_2 f(\mathcal{F}_2)$.

For all objects $B$ in $\mathcal{A}$ and for all objects $M$ in $\mathcal{M}$, we obtain

\[
f_{BM} = a_{BAM} \circ c_{AB} \circ a_{ABM}^{-1} : A \otimes (B \otimes M) \xrightarrow{\cong} B \otimes (A \otimes M)
\]

by using associativity constraint and the braiding as in the following diagram.

\[
A \otimes (B \otimes M) \xrightarrow{a_{ABM}^{-1}} (A \otimes B) \otimes M \xrightarrow{c_{AB}} (B \otimes A) \otimes M \xrightarrow{a_{BAM}} B \otimes (A \otimes M).
\]
It is easy to see that the compatibility conditions are satisfied, hence it is a module functor.

Now, we want to prove that the assignment $\mathcal{F}_M : A \rightarrow \text{Hom}^r_\mathcal{A}(\mathcal{M}, \mathcal{M})$ taking any object $A$ in $\mathcal{A}$ to a right exact module functor $\mathcal{F}(A) : \mathcal{M} \rightarrow \mathcal{M}$ defined by $\mathcal{F}(A)(M) = A \otimes M$ for all objects $M$ in $\mathcal{M}$ is a monoidal functor.

There exists a natural transformation $\gamma_{AB} : \mathcal{F}_M(A) \circ \mathcal{F}_M(B) \rightarrow \mathcal{F}_M(A \otimes B)$.

$$(\mathcal{F}_M(A) \circ \mathcal{F}_M(B))(M) = \mathcal{F}_M(A)(B \otimes M) = A \otimes (B \otimes M) \text{ and } \mathcal{F}_M(A \otimes B)(M) = (A \otimes B) \otimes M \text{ for all } M \in \mathcal{M}.$$  

We have an isomorphism $a^{-1} : A \otimes (B \otimes M) \rightarrow (A \otimes B) \otimes M$. This says that

$$a^{-1} : (\mathcal{F}_M(A) \circ \mathcal{F}_M(B))(M) \rightarrow \mathcal{F}_M(A \otimes B)(M)$$

is an isomorphism for all $M$ in $\mathcal{M}$, so $\gamma_{AB}$ is a natural isomorphism.

Also $I \rightarrow \mathcal{F}_M(I) = I \otimes M$ is an isomorphism by left unit constraint. We may show that the diagrams commute. So, we get the result. 

**Definition 3.0.2.** An $(\mathcal{A} - \mathcal{B})$ bimodule category $\mathcal{M}$ for $\mathcal{A}$ and $\mathcal{B}$ are finite monoidal categories is invertible if the monoidal functors $\mathcal{B} \rightarrow \text{Hom}^r_\mathcal{A}(\mathcal{M}, \mathcal{M})$ taking all objects $B$ in $\mathcal{B}$ to $- \otimes B$ and $\mathcal{A} \rightarrow \text{Hom}^r_\mathcal{B}(\mathcal{M}^{op}, \mathcal{M}^{op}) = \text{Hom}_\mathcal{B}(\mathcal{M}, \mathcal{M})$ taking all objects $A$ in $\mathcal{A}$ to $A \otimes -$ are equivalences as bimodule categories.

The following proposition is found in [EtNiOs] for fusion categories.

**Proposition 3.0.17.** An $(\mathcal{A} - \mathcal{B})$ bimodule category $\mathcal{M}$ for given finite monoidal categories $\mathcal{A}$ and $\mathcal{B}$ is invertible if and only if for all objects $A$ in $\mathcal{A}$ and $B$ in $\mathcal{B}$, the monoidal functor

$$\mathcal{R} : \mathcal{B}^{rev} \rightarrow \text{Hom}^r_\mathcal{A}(\mathcal{M}, \mathcal{M}), \; \mathcal{R}(B)(M) = M \otimes B$$

is an equivalence of $(\mathcal{B} - \mathcal{B})$ bimodule categories if and only if the monoidal functor

$$\mathcal{L} : \mathcal{A} \rightarrow \text{Hom}^r_\mathcal{B}(\mathcal{M}, \mathcal{M}), \; \mathcal{L}(A)(M) = A \otimes M$$

is an equivalence of $(\mathcal{A} - \mathcal{A})$ bimodule categories.

**Proof.** If $\mathcal{M}$ is invertible, then $\mathcal{B} \simeq \text{Hom}^r_\mathcal{A}(\mathcal{M}, \mathcal{M})$ as $(\mathcal{B} - \mathcal{B})$ bimodule categories. So, $\mathcal{B}^{rev} \simeq \text{Hom}^r_\mathcal{A}(\mathcal{M}, \mathcal{M})$.

Similarly, $\mathcal{A} \simeq \text{Hom}^r_\mathcal{B}(\mathcal{M}^{op}, \mathcal{M}^{op}) = \text{Hom}^r_\mathcal{B}(\mathcal{M}, \mathcal{M})$ as $(\mathcal{B} - \mathcal{B})$ bimodule categories.
Conversely, if \( R \) is an equivalence, then \( B^{rev} \simeq \text{Hom}^r_B(\mathcal{M}, \mathcal{M}) \) as \((B - B)\) bimodule categories, so \( \mathcal{B} \simeq \text{Hom}^r_A(\mathcal{M}, \mathcal{M}) \) and if \( \mathcal{L} \) is an equivalence, then

\[
\mathcal{A} \simeq \text{Hom}^r_B(\mathcal{M}, \mathcal{M}) = \text{Hom}^r_B(\mathcal{M}^{op}, \mathcal{M}^{op})
\]

as \((\mathcal{A} - \mathcal{A})\) bimodule categories. If \( R \) and \( \mathcal{L} \) are equivalences, then \( \mathcal{M} \) is invertible.

See [EtNiOs] for the rest of the proof.

**Note 3.0.1.** If \( \mathcal{A} \) is a finite braided monoidal category and \( \mathcal{M} \) is an invertible left \( \mathcal{A} \) module category such that \( \mathcal{M} \simeq \mathcal{A} \mathcal{A} \), then \( \mathcal{M} \) is an \((\mathcal{A} - \mathcal{A})\) bimodule category with right action given by \( M \otimes_{r, \mathcal{A}} A = A \otimes_{l, \mathcal{A}} M \) for all objects \( A \) in \( \mathcal{A} \) and \( M \) in \( \mathcal{M} \).

**Remark 3.0.2.** If \( \mathcal{A} \) is a finite braided monoidal category, \( \mathcal{M} \) is an invertible \((\mathcal{A} - \mathcal{A})\) bimodule category and \( A \) is an object in \( \mathcal{A} \), then we obtain two monoidal equivalences

\[
\mathcal{R} : \mathcal{A}^{rev} \to \text{Hom}^r_B(\mathcal{M}, \mathcal{M}), \mathcal{R}(A)(M) = M \otimes A \\
\mathcal{L} : \mathcal{A} \to \text{Hom}^r_B(\mathcal{M}, \mathcal{M}), \mathcal{L}(A)(M) = A \otimes M
\]

by Proposition 3.0.17 for all objects \( M \) in \( \mathcal{M} \).

**Corollary 3.0.2.** [EtNiOs] Assume that the left module category \( \mathcal{M} \) over a fusion category \( \mathcal{A} \) is invertible. Then, it is indecomposable left module category over \( \mathcal{A} \).

**Proof.** Let \( \mathcal{A} \) be a fusion category and \( \mathcal{M} \) be decomposable module category over \( \mathcal{A} \) under the given conditions. Then, \( \mathcal{M} \simeq \mathcal{P} \oplus \mathcal{Q} \) for indecomposable module categories \( \mathcal{P} \) and \( \mathcal{Q} \) over \( \mathcal{A} \) and \( A = \bigoplus A_i \) for simple objects \( A_i \) in \( \mathcal{A} \) for all objects \( A \) in \( \mathcal{A} \) since \( \mathcal{A} \) is a semisimple monoidal category. The monoidal functor \( L : \mathcal{A} \to \text{Hom}^r_B(\mathcal{M}, \mathcal{M}), L(A)(M) = A \otimes M \) between monoidal categories is an equivalence for all objects \( A \) in \( \mathcal{A} \) and for all objects \( M \) in \( \mathcal{M} \) under these conditions.

\[
A \otimes M = (\bigoplus A_i) \otimes_{l, \mathcal{M}} M \simeq (\bigoplus A_i) \otimes_{l, \mathcal{M}} (P \oplus Q) = ((\bigoplus A_i) \otimes_{l, \mathcal{P}} P) \oplus ((\bigoplus A_i) \otimes_{l, \mathcal{Q}} Q) \\
= (\bigoplus (A_i \otimes_{l, \mathcal{P}} P) \oplus (\bigoplus (A_i \otimes_{l, \mathcal{Q}} Q))) \text{ for all objects } P \text{ in } \mathcal{P} \text{ and } Q \text{ in } \mathcal{Q}.
\]

However, \( \bigoplus (A_i \otimes_{l, \mathcal{P}} P) \oplus (\bigoplus (A_i \otimes_{l, \mathcal{Q}} Q)) \) is an object in \( \mathcal{P} \oplus \mathcal{Q} \). This means that \( \bigoplus (A_i \otimes_{l, \mathcal{P}} P) \) is an object in \( \mathcal{P} \) and \( \bigoplus (A_i \otimes_{l, \mathcal{Q}} Q) \) is an object in \( \mathcal{Q} \). So, \( \mathcal{P} = \bigoplus (A_i \otimes \mathcal{P}) \) and \( \mathcal{Q} = \bigoplus (A_i \otimes \mathcal{Q}) \). This is a contradiction since \( \mathcal{P} \) and \( \mathcal{Q} \) are indecomposable module categories over \( \mathcal{A} \). As a result, \( \mathcal{M} \) is indecomposable.

## 4 2-Category of FRBSU Monoidal Categories

**Lemma 4.0.18.** The composition of two monoidal functors is again a monoidal functor.
Proof. There exists a functor $\text{Hom}(B, C) \times \text{Hom}(A, B) \to \text{Hom}(A, C)$ taking any pair $(\mathcal{F}, \mathcal{G})$ to $\mathcal{F} \circ \mathcal{G}$ for given monoidal functors $(\mathcal{F}, \beta, \varphi_1) : B \to C$ in $\text{Hom}(B, C)$ and $(\mathcal{G}, \psi, \varphi_2) : A \to B$ in $\text{Hom}(A, B)$.

$\beta$ is a family of natural isomorphisms $\beta_{XY} : \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$ for all objects $X, Y$ in $B$ and $\psi$ is a family of natural isomorphisms $\psi_{AB} : \mathcal{G}(A) \otimes \mathcal{G}(B) \to \mathcal{G}(A \otimes B)$ for all objects $A$ and $B$ in $A$.

We want to show that $\mathcal{F} \circ \mathcal{G}$ is a monoidal category. We define $\gamma$ as a family of natural isomorphisms $\gamma_{AB} = \mathcal{F}(\psi_{AB}) \circ \beta_{\mathcal{G}(A)\mathcal{G}(B)}$ for all objects $A$ and $B$ in $A$ as in the following diagram.

$I \cong \mathcal{G}(I)$, so $I \cong \mathcal{F}(I) \cong (\mathcal{F} \circ \mathcal{G})(I)$ for the unit object $I$ in $A$. It is easy to prove the commutativity of the required diagrams.

If we have natural transformations $\theta_1 : \mathcal{F}_1 \Rightarrow \mathcal{F}_2$ in $\text{Hom}(B, C)$ and $\theta_2 : \mathcal{G}_1 \Rightarrow \mathcal{G}_2$ in $\text{Hom}(A, B)$, then the composition is again a natural transformation. This gives a morphism in $\text{Hom}(A, C)$ corresponding to the pair $(\theta_1, \theta_2)$ in $\text{Hom}(B, C) \times \text{Hom}(A, B)$.

Proposition 4.0.19. The collection of all FRBSU monoidal categories forms a 2-category $\mathcal{MCT}$ in which 1 arrows are braided monoidal functors of those categories and 2 arrows are natural isomorphisms between those monoidal functors.

Proof. $\mathcal{X} = \mathcal{MCT}$. Objects are FRBSU monoidal categories.

$\mathcal{MCT}(A, B)$ is a category in which 1 arrows are braided monoidal functors $\mathcal{K} : A \to B$ and 2 arrows are natural isomorphisms $\theta : \mathcal{K} \Rightarrow \mathcal{L}$ for all monoidal functors $\mathcal{K}, \mathcal{L} : A \to B$. We show 2 arrows as in the following diagram.

$\mathcal{F}_{ABC} : \mathcal{MCT}(B, C) \times \mathcal{MCT}(A, B) \to \mathcal{MCT}(A, C)$ is a functor taking the pair $(\mathcal{L}, \mathcal{K})$ to $\mathcal{L} \circ \mathcal{K}$ and the pair $(\theta, \gamma)$ to $\theta \ast \gamma$ by Lemma 4.0.18.
\( F_A : 1 \to MCT(A, A) \) is a functor sending the object \( \star \) in 1 to 1 arrow \( id_A : A \to A \).

\[
F_{ABCD} : F_{ABD} \circ (F_{BCD} \times 1) \to F_{ACD} \circ (1 \times F_{ABC})
\]

\[
\begin{array}{c}
MCT(C, D) \times MCT(B, C) \times MCT(A, B) \\
\downarrow 1 \times F_{ABC} \\
MCT(C, D) \times MCT(A, C) \\
\downarrow F_{ACD}
\end{array}
\]

\[
\begin{array}{c}
MCT(B, D) \times MCT(A, B) \\
\downarrow F_{ABD}
\end{array}
\]

is a natural isomorphism.

\[
(F_{ABD} \circ (F_{BCD} \times 1))(\kappa, \lambda, \mu) = F_{ABD}(\kappa \circ \lambda, \mu) = (\kappa \circ \lambda) \circ \mu,
\]

\[
(F_{ACD} \circ (1 \times F_{ABC}))(\kappa, \lambda, \mu) = F_{ACD}(\kappa, \lambda \circ \mu) = \kappa \circ (\lambda \circ \mu)
\]

for all 1 arrows \( \kappa, \lambda, \mu \). \( (\kappa \circ \lambda) \circ \mu = \kappa \circ (\lambda \circ \mu) \) and \( a_{ABCD}(\kappa, \lambda, \mu) = id_{\kappa \circ \lambda \circ \mu} \).

For all morphisms \( \alpha : (K_1, L_1, M_1) \to (K_2, L_2, M_2) \) in

\[
MCT(C, D) \times MCT(B, C) \times MCT(A, B),
\]

the following diagram

\[
(K_1 \circ L_1) \circ M_1 \xrightarrow{(F_{ABD} \circ (F_{BCD} \times 1))(\alpha)} (K_2 \circ L_2) \circ M_2
\]

\[
a_{ABCD}(K_1, L_1, M_1) \downarrow \quad \downarrow a_{ABCD}(K_2, L_2, M_2)
\]

\[
K_1 \circ (L_1 \circ M_1) \xrightarrow{(F_{ACD} \circ (1 \times F_{ABC}))(\alpha)} K_2 \circ (L_2 \circ M_2)
\]

commutes. As a result \( a_{ABCD}(\kappa, \lambda, \mu) \) is a natural isomorphism for all 1 arrows \( \kappa, \lambda \) and \( \mu \). Similarly, we show \( r \) and \( l \) are natural isomorphisms. Hence, \( MCT \) is a 2-category.

\[\square\]

## 5 Crossed Modules and Morphisms Of Crossed Modules

A crossed module \( \mathcal{C} = [N \xrightarrow{h} M] \) is a pair of groups \( (M, N) \) such that \( M \) acts on \( N \) by \( M \times N \to N \) taking \( (m, n) \) to \( mn \) and \( h : N \to M \) is a group homomorphism satisfying the conditions \( h(mn) = mh(n)m^{-1} \) and \( h(n) = nn'\) for all \( n, n' \in N \) and \( m \in M \).
5.1 Strict Morphisms and Butterflies Between Crossed Modules

We use [No] and [AlNo] as references here.

Definition 5.1.1. For given crossed modules $\mathfrak{C}_1 = [N_1 \xrightarrow{h_{\mathfrak{C}_1}} M_1]$ and $\mathfrak{C}_2 = [N_2 \xrightarrow{h_{\mathfrak{C}_2}} M_2]$, a strict morphism $F = (f_1, f_2)$ between them is a pair of group homomorphisms $f_1 : M_1 \to M_2$ and $f_2 : N_1 \to N_2$ such that $h_{\mathfrak{C}_2} \circ f_2 = f_1 \circ h_{\mathfrak{C}_1}$ and $f_2(mn) = f_1(m)f_2(n)$ for all $n \in N_1$ and $m \in M_1$. We show that morphism by the following diagram.

![Diagram](image)

Definition 5.1.2. The strict morphism in Diagram 51 is an equivalence of crossed modules if $\pi_2(\mathfrak{C}_1) \cong \pi_2(\mathfrak{C}_2)$ and $\pi_1(\mathfrak{C}_1) \cong \pi_1(\mathfrak{C}_2)$.

Definition 5.1.3. The commutative diagram of group homomorphisms

![Diagram](image)

is a butterfly between two crossed modules $\mathfrak{C}_1 = [N_1 \xrightarrow{h_{\mathfrak{C}_1}} M_1]$ and $\mathfrak{C}_2 = [N_2 \xrightarrow{h_{\mathfrak{C}_2}} M_2]$ if it satisfies the following axioms.

1. Both diagonal sequences are complexes,
2. The NE-SW sequence is a group extension,
3. $k(t(x)n_2) = xk(n_2)x^{-1}$ and $f(g(x)n_1) = xf(n_1)x^{-1}$ for all $x \in E$, $n_1 \in N_1$, $n_2 \in N_2$.

We denote the above butterfly with $(E, t, g, k, f)$. Also, we can denote that butterfly by $P : \mathfrak{C}_1 \to \mathfrak{C}_2$ for crossed modules $\mathfrak{C}_1$ and $\mathfrak{C}_2$.

Definition 5.1.4. A butterfly is reversible(equivalence) if both of the diagonals are extensions, that is a butterfly such that the NW-SE sequence is short exact. It is splittable if there exists a splitting homomorphism $s : M_1 \to E$ such that $g \circ s = id_{M_1}$ which is same as the condition that the NE-SW sequence is a split extension.
The inverse of the butterfly in Diagram 52 is shown as in the following diagram which is a butterfly.

\[ \begin{array}{ccc}
N_2 & \xrightarrow{k} & N_1 \\
h_{\epsilon_1} & & h_{\epsilon_1} \\
M_2 & \xleftarrow{t} & M_1 \\
\end{array} \]

Proposition 5.1.1. Every split butterfly corresponds to a unique strict morphism \((f_1, f_2)\) between two crossed modules.

Proof. Assume that \((f_1, f_2)\) is a strict morphism as in Diagram 51. We get a commutative diagram

\[ \begin{array}{ccc}
N_1 & \xrightarrow{f} & N_2 \\
h_{\epsilon_1} & & h_{\epsilon_2} \\
M_1 & \xleftarrow{t} & M_2 \\
\end{array} \]

which means that the NE-SW sequence is a split extension in that butterfly, that is \(E = N_2 \ltimes M_1\) with the product law \((n_1, m_1)(n_2, m_2) = (n_1.f_1(m_1)n_2, m_1.m_2)\) for all \(n_1, m_1 \in M_1\) and \(n_2, m_2 \in N_2\).

Here, we define \(g\) as a projection, \(k(n) = (id_{N_2}, 1)(n) = (n, 1)\) for all \(n \in N_2\), \(f(n) = (f_2(n^{-1}), h_1(n))\) for all \(n \in N_1\) and \(t(n, m) = h_2(n).f_1(m)\) for all \(n \in N_2\) and \(m \in M_1\).

Conversely, if we are given a split butterfly as in Diagram 54 we can find a canonical splitting homomorphism \(s : M_1 \rightarrow N_2 \ltimes M_1\) taking \(m\) to \((1, m)\) for all \(m \in M_1\). We define \(f_1 = t \circ s. f_2\) can be defined from the equation \(s \circ h_1 = f.(k \circ f_2)\). We can see that those group homomorphisms satisfy the required conditions.

5.2 Strict Morphisms and 2-Category of Crossed Modules

Lemma 5.2.1. [No] The collection \(XM\) consisting of crossed modules forms a category whose morphisms are strict morphisms of crossed modules as defined in 51.

Definition 5.2.1. A pointed natural transformation \(PNT : G \Rightarrow F\) between two strict morphisms \(F = (f_1, f_2)\) and \(G = (g_1, g_2)\) for the crossed modules \(C_1 = [N_1 \xrightarrow{h_{\epsilon_1}} M_1]\) and \(C_2 = [N_2 \xrightarrow{h_{\epsilon_2}} M_2]\) is a crossed homomorphism \(\gamma : M_1 \rightarrow N_2\) such that for all
whose objects are crossed modules, 1 arrows and morphisms are 2 arrows for the crossed modules \( \gamma \) which is a crossed homomorphism

**Lemma 5.2.2.** \( \text{PNT} \Rightarrow \text{PNT}' \): There exists a 2-category \( \mathcal{XM} \) whose objects are crossed modules, 1 arrows are strict morphisms between those crossed modules and 2 arrows are pointed natural transformations between those strict morphisms such that the pointed natural transformations \( \text{PNT} \Rightarrow \text{PNT}' \) are the trivial pointed natural transformations where \( G \) is a strict morphism between any crossed modules \( \mathcal{C}_1 = [N_1 \xrightarrow{h_{\epsilon_1}} M_1] \) and \( \mathcal{C}_2 = [N_2 \xrightarrow{h_{\epsilon_2}} M_2] \) in \( \mathcal{XM} \).

**Proof.** We take \( \mathcal{X} = \mathcal{XM} \). First, we need to show that \( \mathcal{XM}(\mathcal{C}_1, \mathcal{C}_2) \) is a category whose objects are 1 arrows and morphisms are 2 arrows for the crossed modules \( \mathcal{C}_1 = [N_1 \xrightarrow{h_{\epsilon_1}} M_1] \) and \( \mathcal{C}_2 = [N_2 \xrightarrow{h_{\epsilon_2}} M_2] \) in \( \mathcal{XM} \).

The identity morphism is the trivial pointed natural transformation.

We define the composition of two pointed natural transformations \( \text{PNT}_1 : G \Rightarrow F \) which is a crossed homomorphism \( \gamma_1 \) and \( \text{PNT}_2 : F \Rightarrow E \) which is a crossed homomorphism \( \gamma_2 \) between the strict morphisms as \( \gamma = \gamma_2 \cdot \gamma_1 : M_1 \rightarrow N_2 \). For all elements \( a, a' \) in \( M_1 \), we get

\[
\gamma(aa') = (f_1(a')\gamma(a))\gamma(a') = (e_1(a')\gamma_2(a))\gamma_2(a') = (e_1(a')h_{\epsilon_2}(\gamma_2((a')^{-1})))\gamma_1(a') \quad \text{since } f_1(a') = e_1(a')h_{\epsilon_2}(\gamma_2((a')^{-1}))
\]

\[
= (e_1(a')\gamma_2(a))\gamma_2(a') = (e_1(a')\gamma_2((a')^{-1}))\gamma_1(a)\gamma_2(a') \quad \text{since } h_{\epsilon_2}(\gamma_2((a')^{-1}))\gamma_1(a) = \gamma_2((a')^{-1})\gamma_1(a)\gamma_2(a') \text{ by definition of crossed module}
\]

\[
= (e_1(a')\gamma_2(a))\gamma_2(a') = (e_1(a')\gamma_2((a')^{-1}))(e_1(a')\gamma_1(a))(e_1(a')\gamma_2(a')).
\]

(55)
\[ = (e_1(a') \gamma_2(a)).\gamma_2(a').(e_1(a')\gamma_2((a')^{-1})).\gamma_2(a').\gamma_2((a')^{-1}).\gamma_2(\gamma_1(a)).\gamma_2(a') \cdot \gamma_1(a') \]

\[ = (e_1(a') \gamma_2(a)).\gamma_2((a')^{-1}a').\gamma_2((a')^{-1}).\gamma_2(\gamma_1(a)).\gamma_2(a').\gamma_2((a')^{-1}).\gamma_1(a') \]

\[ = (e_1(a') \gamma_2(a))(e_1(a')\gamma_1(a)).\gamma_2(a').\gamma_2((a')^{-1}).\gamma_1(a') \]

\[ = (e_1(a') \gamma_2(a).\gamma_1(a)).\gamma_2(a').\gamma_1(a') \]

\[ = (e_1(a') \gamma(a)).\gamma(a'). \]

\[ g_1(a) = f_1(a).h_{\xi_2}(\gamma_1(a^{-1})) \] since \( PNT_1 : G \Rightarrow F \) is a pointed natural transformation which is a crossed homomorphism \( \gamma_1 \) and \( f_1(a) = e_1(a).h_{\xi_2}(\gamma_2(a^{-1})) \) since \( PNT_2 : F \Rightarrow E \) is a pointed natural transformation which is a crossed homomorphism \( \gamma_2 \). Hence,

\[ g_1(a) = e_1(a).h_{\xi_2}(\gamma_2(a^{-1})).h_{\xi_2}(\gamma_1(a^{-1})) = e_1(a).h_{\xi_2}((\gamma_2.\gamma_1)(a^{-1})) = e_1(a).h_{\xi_2}(\gamma(a^{-1})) \]

as desired. Similarly, we may show the other part. As a result, \( \gamma \) is a crossed homomorphism and gives a pointed natural transformation \( PNT_3 : G \Rightarrow E \).

It is clear that the composition is associative.

For all pointed natural transformations \( PNT : G \Rightarrow F \), the crossed homomorphism \( \gamma \) and \( id : F \Rightarrow F \), the composition \( id \ast PNT : G \Rightarrow F \) is equal to \( PNT \). Similarly, we show the other part.

As a result, \( \mathcal{K}\mathcal{M}(\mathcal{C}_1, \mathcal{C}_2) \) is a category.

The mapping \( F_{\xi_1,\xi_2,\xi_3} : \mathcal{K}\mathcal{M}(\mathcal{C}_2, \mathcal{C}_3) \times \mathcal{K}\mathcal{M}(\mathcal{C}_1, \mathcal{C}_2) \rightarrow \mathcal{K}\mathcal{M}(\mathcal{C}_1, \mathcal{C}_3) \) is a functor. We send each pair \((G, F)\) to \( G \circ F \) where \( G = (g_1, g_2), F = (f_1, f_2) \) and \( G \circ F = (g_1 \circ f_1, g_2 \circ f_2) \) are strict morphisms between the corresponding crossed modules.

We send each pair of pointed natural transformations \( PNT_2 : E \Rightarrow K \) which is a crossed homomorphism \( \gamma_2 \) and \( PNT_1 : G \Rightarrow F \) which is a crossed homomorphism \( \gamma_1 \) to their composition \( PNT_3 : E \circ G \Rightarrow K \circ F \). Here, \( G = (g_1, g_2), F = (f_1, f_2), E = (e_1, e_2) \) and \( K = (k_1, k_2) \) are strict morphisms. We need to define a crossed homomorphism \( \gamma_3 \).

We take \( \gamma_3 = (k_2 \circ \gamma_1).\gamma_2 \circ (g_1) \) and see it satisfies the required conditions to be a crossed homomorphism. We draw the following diagrams to see the relation between
the strict morphisms.

\[
G = (g_1, g_2) \quad E = (e_1, e_2) \quad E \circ G = (e_1 \circ g_1, e_2 \circ g_2)
\]

\[
F = (f_1, f_2) \quad K = (k_1, k_2) \quad K \circ F = (k_1 \circ f_1, k_2 \circ f_2)
\]

We also draw the following diagrams to understand the group homomorphisms better.

For all \(a, a' \in M_1, b \in N_1\) and \(c \in N_2, d, d' \in M_2\), we get the following equalities by definition of \(\gamma_1\) and \(\gamma_2\).

1. \(\gamma_1(a.a') = (f_1(a'), \gamma_1(a)) \cdot \gamma_1(a')\)
2. \(g_1(a) = f_1(a) \cdot h_{\epsilon_1}(\gamma_1(a^{-1}))\)
3. \(g_2(b) = f_2(b) \cdot \gamma_1(h_{\epsilon_1}(b^{-1}))\)
4. \(\gamma_2(d.d') = (k_1(d'), \gamma_2(d)) \cdot \gamma_2(d')\)
5. \(e_1(d) = k_1(d) \cdot h_{\epsilon_1}(\gamma_2(d^{-1}))\)
6. \(e_2(c) = k_2(c) \cdot \gamma_2(h_{\epsilon_1}(c^{-1}))\)

For all \(a' \in M_1\), we have

\[
(k_1 \circ g_1)(a') = k_1(f_1(a') \cdot h_{\epsilon_1}(\gamma_1((a')^{-1}))) = (k_1 \circ f_1)(a') \cdot (k_1 \circ h_{\epsilon_1})(\gamma_1((a')^{-1})), \quad (56)
\]

\[
((k_1 \circ f_1)(a'))(k_2 \circ \gamma_1)((a')^{-1}) = k_2(f_1(a') \cdot \gamma_1((a')^{-1})) \quad (57)
\]

\[
= k_2((f_1(a') \gamma_1((a')^{-1})) \cdot \gamma_1(a') \cdot \gamma_1((a')^{-1})) \quad (58)
\]

\[
= k_2((a')^{-1} \cdot a' \cdot \gamma_1((a')^{-1})) = (k_2 \circ \gamma_1)((a')^{-1}). \quad (59)
\]

For all \(a, a' \in M_1\), we have

\[
\gamma_3(a.a') = (k_2 \circ \gamma_1)(a.a') \cdot (\gamma_2 \circ g_1)(a.a') = k_2((f_1(a') \gamma_1(a)) \cdot \gamma_1(a')) \cdot \gamma_2(g_1(a) \cdot g_1(a')) \quad \blacksquare
\]
\[
= k_2(f_1(a')\gamma_1(a)).k_2(\gamma_1(a')).((k_1 \circ \gamma_1)(a') \circ g_1)(a)).(\gamma_2 \circ g_1)(a') \text{ by } 4_1
\]
\[
= k_2(f_1(a')\gamma_1(a)).k_2(\gamma_1(a')).((k_1 \circ \gamma_1)(a') \circ g_1)(a)).(\gamma_2 \circ g_1)(a') \text{ by } 5_6
\]
\[
= k_2(f_1(a')\gamma_1(a)).k_2(\gamma_1(a')).((k_1 \circ \gamma_1)(a') \circ g_1)(a)).(\gamma_2 \circ g_1)(a') \text{ since } K
\]
\[
= k_2(f_1(a')\gamma_1(a)).k_2(\gamma_1(a')).((k_1 \circ \gamma_1)(a') \circ g_1)(a)).(\gamma_2 \circ g_1)(a') \text{ by definition of } C_3
\]
\[
= k_2(f_1(a')\gamma_1(a)).k_2(\gamma_1(a')).((k_1 \circ \gamma_1)(a') \circ g_1)(a)).(\gamma_2 \circ g_1)(a') \text{ by } 5_9
\]
\[
= k_2(f_1(a')\gamma_1(a)).k_2(\gamma_1(a')).((k_1 \circ \gamma_1)(a') \circ g_1)(a)).(\gamma_2 \circ g_1)(a') \text{ by } 5_9
\]
\[
= (k_1 \circ f_1)((k_2 \circ \gamma_1)(a') \circ g_1)(a)).(k_2 \circ \gamma_1)(a').(\gamma_2 \circ g_1)(a') \text{ as required.}
\]

The other conditions are satisfied.

For a crossed homomorphism \( \mathcal{C} = [N \xrightarrow{h_{\mathcal{C}}} M] \), the mapping \( F_{\mathcal{C}} : 1 \to X \mathcal{M}(\mathcal{C}, \mathcal{C}) \) is a functor taking the element \( \ast \) in 1 to \( id_1 = (id, id) \) and morphisms \( \ast \to \ast \) to a trivial pointed natural transformation \( PNT : (id, id) \Rightarrow (id, id) \).
\( a_{\xi,\epsilon,\xi,\epsilon}(F, G, H) : (F \circ G) \circ H \Rightarrow F \circ (G \circ H) \) is a trivial pointed natural transformation and \((F \circ G) \circ H = F \circ (G \circ H)\) for all strict morphisms \(F, G\) and \(H\). \(a_{\epsilon,\xi,\xi,\epsilon}(F, G, H)\) is the identity morphism by assumption, hence it is an isomorphism and the required diagram is commutative. As a result, \(a_{\xi,\epsilon,\xi,\epsilon}\) is a natural isomorphism.

We show the other conditions in a similar way and see \(\mathcal{X}\mathcal{M}\) is a 2-category. \(\square\)

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