ON STABLE INVERTIBILITY AND GLOBAL NEWTON CONVERGENCE FOR CONVEX MONOTONIC FUNCTIONS

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Abstract. We derive a simple criterion that ensures uniqueness, Lipschitz stability and global convergence of Newton’s method for finite dimensional inverse problems with a continuously differentiable, componentwise convex and monotonic forward function. Our criterion merely requires to evaluate the directional derivative of the forward function at finitely many evaluation points and finitely many directions.

Using a relation to monotonicity and localized potentials techniques for inverse coefficient problems in elliptic PDEs, we will then show that a discretized inverse Robin transmission problem always fulfills our criterion if enough measurements are being used. Thus our result enables us to determine those boundary measurements from which an unknown coefficient can be uniquely and stably reconstructed with a given desired resolution by a globally convergent Newton iteration.

1. Disclaimer. This is a preliminary draft version. It is missing references, an introduction and numerical examples. It also requires significant proofreading and polishing.

2. Uniqueness, stability and global convergence of the Newton method. We consider a continuously differentiable, componentwise convex and monotonic function

\[ F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m \]

where \( n, m \in \mathbb{N}, m \geq n \geq 2, \) and \( U \) is a convex open set. In this section, we will derive a simple criterion that implies injectivity of \( F \) on a multidimensional interval, allows us to estimate the Lipschitz stability constant of the left inverse \( F^{-1} \) and, for \( n = m \), ensures global convergence of Newton’s method.

Remark 2.1. Throughout this work, “≤” is always understood componentwise for finite-dimensional vectors and matrices, and \( x \not\leq y \) denotes the converse, i.e., that \( x - y \) has at least one positive entry.

Monotonicity and convexity are understood with respect to this componentwise partial order, i.e., \( F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m \) is monotonic if

\[ x \leq y \quad \text{implies} \quad F(x) \leq F(y) \quad \text{for all } x, y \in U. \]

and \( F \) is convex if

\[ F((1 - t)x + ty) \leq (1 - t)F(x) + tF(y) \quad \text{for all } x, y \in U, \ t \in [0, 1]. \]

For continuously differentiable functions, it is easily shown that monotonicity is equivalent to

\[ F'(x)y \geq 0 \quad \text{for all } x \in U, \ 0 \leq y \in \mathbb{R}^n, \]

(2.1)

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and thus equivalent to \( F'(x) \geq 0 \). It is also well known (cf., e.g., [4, Thm. 13.3.2]) that convexity is equivalent to

\[
F(y) - F(x) \geq F'(x)(y - x) \quad \text{for all } x, y \in U.
\]

(2.2)

All the proofs in this section use the monotonicity and convexity assumption in the form (2.1) and (2.2).

Throughout this work, we denote by \( e_j \in \mathbb{R}^n \) the \( j \)-th unit vector in \( \mathbb{R}^n \), \( \mathbb{1} := (1, 1, \ldots, 1)^T \in \mathbb{R}^n \), and \( e_j' := 1 - e_j \). \( I_n \in \mathbb{R}^{n \times n} \) denotes the \( n \)-dimensional identity matrix, and \( \mathbb{1} \mathbb{1}^T \in \mathbb{R}^{n \times n} \) is the matrix containing 1 in all of its entries.

### 2.1. A simple criterion for uniqueness and Lipschitz stability

Before we state our result in its final form in subsection 2.3, we derive two weaker results that motivate our arguments and may be of independent interest. We first show a simple criterion that yields injectivity of \( F \) and allows us to estimate the Lipschitz stability constant of its left inverse \( F^{-1} \).

**Theorem 2.2.** Let \( F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m \), \( m \geq n \geq 2 \), be a continuously differentiable, componentwise convex and monotonic function on an open set \( U \) containing \([-1, 3]^n\).

If, additionally,

\[
F'(-e_j + 3e_j') \left( e_j - 3e_j' \right) \not\leq 0 \quad \text{for all } j \in \{1, \ldots, n\},
\]

then

(a) \( F \) is injective on \([0, 1]^n\).
(b) \( F'(x) \) is injective for all \( x \in [0, 1]^n \).
(c) With

\[
L := 2 \left( \min_{j=1,\ldots,n} \max_{k=1,\ldots,n} e_k^T F'(-e_j + 3e_j') \left( e_j - 3e_j' \right) \right)^{-1} > 0,
\]

(2.4)

we have that for all \( x, y \in [0, 1]^n \)

\[
\|x - y\|_\infty \leq L\|F(x) - F(y)\|_\infty \quad \text{and} \quad \|F'(x)^{-1}\|_\infty \leq L,
\]

where \( F'(x)^{-1} \in \mathbb{R}^{m \times m} \) denotes the left inverse of \( F'(x) \in \mathbb{R}^{m \times n} \).

**Proof.** We first note that (2.2) also implies that

\[
F'(y)(y - x) \geq F(y) - F(x) \geq F'(x)(y - x) \quad \text{for all } x, y \in U.
\]

Let \( j \in \{1, \ldots, n\} \). Writing \( \tilde{x}^{(j)} := -e_j + 3e_j' \), we have that

\[
e_j - 3e_j' \leq x - \tilde{x}^{(j)} \leq 2e_j - 2e_j' \quad \text{for all } x \in [0, 1]^n.
\]

Thus we deduce from (2.1) and (2.2)

\[
F'(x) \left( e_j - e_j' \right) = \frac{1}{2} F'(x) \left( 2e_j - 2e_j' \right) \geq \frac{1}{2} F'(x) \left( x - \tilde{x}^{(j)} \right) \geq \frac{1}{2} \left( F(x) - F(\tilde{x}^{(j)}) \right)
\]

\[\geq \frac{1}{2} F'(\tilde{x}^{(j)}) \left( x - \tilde{x}^{(j)} \right) \geq \frac{1}{2} F'(\tilde{x}^{(j)}) \left( e_j - 3e_j' \right).\]
With the definition of $L$ in (2.4) this shows that
\[
\max_{k=1,\ldots,n} e_k^T F'(x) (e_j - e'_j) \geq L^{-1} \quad \text{for all } x \in [0,1]^n, \ j \in \{1,\ldots,n\}. \tag{2.5}
\]

To prove injectivity of $F$ on $[0,1]^n$ and the Lipschitz bound on its inverse, let $x, y \in [0,1]^n$ with $x \neq y$. Then at least one of the entries $\|y-x\|_\infty$ must be either 1 or $-1$.

(i) In the case that $\|y-x\|_\infty = 1$ with $j \in \{1,\ldots,n\}$, we have that
\[
\frac{y-x}{\|y-x\|_\infty} \geq e_j - e'_j,
\]
so that we obtain using (2.2) and (2.1)
\[
\frac{F(y) - F(x)}{\|y-x\|_\infty} \geq F'(x) \frac{y-x}{\|y-x\|_\infty} \geq F'(x) (e_j - e'_j).
\]
Using (2.5) this shows that
\[
\frac{\|F(y) - F(x)\|_\infty}{\|y-x\|_\infty} \geq \max_{k=1,\ldots,n} e_k^T F(y) - F(x) = L^{-1}. \tag{2.6}
\]

(ii) In the case that $\|y-x\|_\infty = -1$ with $j \in \{1,\ldots,n\}$, we use (i) with interchanged roles of $x$ and $y$ and also obtain
\[
\frac{\|F(y) - F(x)\|_\infty}{\|y-x\|_\infty} \geq L^{-1}.
\]

With the same arguments, we obtain that for all $x \in [0,1]^n$ and $0 \neq y \in \mathbb{R}^n$, at least one of the entries of $\frac{y-x}{\|y\|_\infty}$ must be either 1 or $-1$, so that there exists $j \in \{1,\ldots,n\}$ with either
\[
F'(x) \frac{y}{\|y\|_\infty} \geq F'(x)(e_j - e'_j), \quad \text{or} \quad F'(x) \frac{-y}{\|y\|_\infty} \geq F'(x)(e_j - e'_j).
\]
In both cases it follows from (2.5) that
\[
\frac{\|F'(x)y\|}{\|y\|_\infty} \geq \max_{k=1,\ldots,n} e_k^T F'(x) (e_j - e'_j) \geq L^{-1}.
\]
This proves injectivity of $F'(x)$ and the Lipschitz bound on its left inverse. \qed

2.2. A simple criterion for global convergence of the Newton iteration.

We will now show that we can also ensure that a convex monotonic function $F$ has a unique zero, and that the Newton method globally converges against this zero.

**Theorem 2.3.** Let $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, be a continuously differentiable, componentwise convex and monotonic function on an open set $U$.

If $[-2,n(n+3)]^n \subset U$, and
\[
F'(z^{(j)})d^{(j)} \not\leq 0 \quad \text{for all } j \in \{1,\ldots,n\}, \tag{2.6}
\]
with
\[
z^{(j)} := -2e_j + n(n+3)e'_j, \quad \text{and} \quad d^{(j)} := e_j - (n^2 + 3n + 1)e'_j,
\]
then the following holds:
(a) $F$ is injective on $[-1, n]^n$, $F'(x)$ is injective for all $x \in [-1, n]^n$, and for all $x, y \in [-1, n]^n$

$$\|x - y\|_\infty \leq L\|F(x) - F(y)\|_\infty,$$  \hspace{1em} and \hspace{1em} $$\|F'(x)^{-1}\|_\infty \leq L,$$  \hspace{1em} (2.7)

where

$$L := (n + 2) \left( \min_{j=1, \ldots, n} \max_{k=1, \ldots, n} e_k^T F'(x(j)) d(j) \right)^{-1} > 0.$$

(2.8)

(b) If, additionally, $F(0) \leq 0 \leq F(1)$, then there exists a unique $\hat{x} \in \left(-\frac{1}{n-1}, 1 + \frac{1}{n-1}\right)^n \subset (-1, 2)^n$ with $F(\hat{x}) = 0$.

The Newton iteration sequence \hspace{1em} $x^{(k+1)} := x^{(k)} - F'(x^{(k)})^{-1} F(x^{(k)})$ \hspace{1em} with initial value $x^{(0)} := 1 \hspace{1em} (2.9)$

is well defined (i.e., $F'(x^{(k)})$ is invertible in each step) and converges against $\hat{x}$. For all $k \in \mathbb{N}$

$x^{(k)} \in (-1, n)^n$ \hspace{1em} and \hspace{1em} $0 \leq M \hat{x} \leq M x^{(k)} \leq M x^{(0)} = (n + 1)1$,

where $M := 1^T n I_n \in \mathbb{R}^{n \times n}$. The rate of convergence of $x_k \to \hat{x}$ is superlinear. If $F'$ is locally Lipschitz in

$\left(-\frac{1}{n-1}, 1 + \frac{1}{n-1}\right)^n$ then the rate of convergence is quadratic.

To prove Theorem 2.3 we will first show the following lemma.

**Lemma 2.4.** Under the assumptions and with the notations of Theorem 2.3, the following holds:

(a) For all $x \in [-1, n]^n$,

$$\max_{k=1, \ldots, n} e_k^T F'(x)(e_j - ne_j) \geq L^{-1}.$$

(b) $F$ is injective on $[-1, n]^n$, $F'(x)$ is injective for all $x \in [-1, n]^n$, and for all $x, y \in [-1, n]^n$

$$\|x - y\|_\infty \leq L\|F(x) - F(y)\|_\infty,$$  \hspace{1em} and \hspace{1em} $$\|F'(x)^{-1}\|_\infty \leq L.$$

(c) For all $x \in [-1, n]^n$, and $0 \neq y \in \mathbb{R}^n$

$$F'(x)y \geq 0 \hspace{1em} \text{implies} \hspace{1em} \max_{j=1, \ldots, n} y_j = \|y\|_\infty \hspace{1em} \text{and} \hspace{1em} \min_{j=1, \ldots, n} y_j > -\frac{1}{n}\|y\|_\infty.$$

(d) With $M := 1^T n + I_n \in \mathbb{R}^{n \times n}$, for all $x \in [-1, n]^n$, and $y \in \mathbb{R}^n$

$$F'(x)y \geq 0 \hspace{1em} \text{implies} \hspace{1em} My \geq 0.$$

$M$ is invertible and $M^{-1} = I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T$.

c) For all $x \in [-1, n]^n$

$$MF'(x)^{-1} \geq 0.$$
Proof. The proof is similar to that of Theorem 2.2.

(a) Let \( j \in \{1, \ldots, n\} \). Using \( z^{(j)} = -2e_j + n(n+3)e'_j \), we have that for all \( x \in [-1, n]^n \)
\[
d^{(j)} = e_j - (n^2 + 3n + 1)e'_j \leq x - z^{(j)} \leq (n+2)e_j - n(n+2)e'_j
\]
and thus
\[
F'(x) (e_j - ne'_j) = \frac{1}{n+2} F'(x) ((n+2)e_j - n(n+2)e'_j) \\
\geq \frac{1}{n+2} F'(x) (x - z^{(j)}) \geq \frac{1}{n+2} (F(x) - F(z^{(j)})) \\
\geq \frac{1}{n+2} F'(z^{(j)}) (x - z^{(j)}) \geq \frac{1}{n+2} F'(z^{(j)}) d^{(j)},
\]
which proves (a).

(b) Since (a) implies a fortiori that
\[
\max_{k=1, \ldots, n} e_k^T F'(x)(e_j - e'_j) \geq L^{-1},
\]
the assertion (b) follows by the same arguments as in the proof of Theorem 2.2.

(c) Let \( x \in [-1, n]^n \), and \( 0 \neq y \in \mathbb{R}^n \). If there exists an index \( j \in \{1, \ldots, n\} \) with
\[
y_j \leq -\frac{1}{n} \|y\|_\infty, \text{ then } y \leq -\frac{1}{n} \|y\|_\infty e_j + \|y\|_\infty e'_j, \text{ so that } -\frac{n}{\|y\|_\infty} y \geq e_j - ne'_j \text{ and thus } -\frac{n}{\|y\|_\infty} F'(x)y \leq 0.
\]
By contraposition, this shows that
\[
F'(x)y \geq 0 \quad \text{implies} \quad \min_{j=1, \ldots, n} y_j > -\frac{1}{n} \|y\|_\infty,
\]
which also shows that \( \|y\|_\infty = \max_{j=1, \ldots, n} y_j \).

(d) Using (c) it follows that \( F'(x)y \geq 0 \) implies that for all \( k \in \mathbb{N} \),
\[
\sum_{j=1}^n y_j + y_k \geq \max_{j=1, \ldots, n} y_j + n \min_{j=1, \ldots, n} y_j \geq 0.
\]
so that \( F'(x)y \geq 0 \) implies \( My = (11^T + I_n) \geq 0 \). Also, it is easily checked that
\[
(I_n - \frac{1}{n+1} 11^T) (11^T + I_n) = 11^T + I_n - \frac{1}{n+1} 11^T 11^T - \frac{1}{n+1} 11^T
\]
\[
= \frac{n}{n+1} 11^T + I_n - \frac{1}{n+1} 11^T = I_n
\]
\[
\]
so that \( F'(x)y \geq 0 \) implies \( MF'(x)^{-1}d = d \geq 0 \) implies \( M F'(x)^{-1}d \geq 0 \),
which proves (e).

\( \square \)

Proof of Theorem 2.3. The assertion (a) has already been proven in lemma 2.3 (b). To motivate the proof of (b), let us first note that, by lemma 2.3 (c), \( \hat{F}(x) := F(M^{-1}x) \) is a convex function with Collatz monotone derivative \( \hat{F}'(x) \), i.e. \( \hat{F}'(x)^{-1} = MF'(x)^{-1} \geq 0 \). If the Newton iterates do not leave the region where convexity and Collatz monotony holds, then classical results on monotone Newton methods (cf., e.g., 7, Thm. 13.3.4)
yield global Newton convergence for $\tilde{F}$, and thus for $F$ since the Newton method is invariant under linear transformation. The following proof combines the classical arguments in [7, Thm. 13.3.4] with a homotopy argument to bound the Newton iterates.

We first prove that for all $x^{(k)} \in (-1, n)^n$ with $F(x^{(k)}) \geq 0$ and $M x^{(k)} \leq M I$, the next Newton iterate $x^{(k+1)}$ is well-defined and fulfills

$$x^{(k+1)} \in (-1, n)^n, \quad F(x^{(k+1)}) \geq 0, \quad 0 \leq M x^{(k+1)} \leq M x^{(k)} \leq M I.$$  \hfill (2.10)

To show this let $x^{(k)} \in (-1, n)^n$ fulfill $F(x^{(k)}) \geq 0$ and $M x^{(k)} \leq M I$. Then $F'(x^{(k)}) \in \mathbb{R}^{n \times n}$ is invertible, so that we can define the intermediate Newton steps

$$x^{(k+t)} := x^{(k)} - t F'(x^{(k)})^{-1} F(x^{(k)})$$ for all $t \in [0, 1]$.

Then, by convexity, we have for all $t \in [0, 1]$

$$F(x^{(k+t)}) = F(x^{(k)} - t F'(x^{(k)})^{-1} F(x^{(k)})) \geq F(x^{(k)}) - t F(x^{(k)}) \geq 0.$$  

Moreover, it follows from $MF'(x)^{-1} \geq 0$, cf. lemma 2.4(c), that

$$M x^{(k)} - M x^{(k+t)} = t M F'(x^{(k)})^{-1} F(x^{(k)}) \geq 0.$$  

Using also that $F(0) \leq 0$ and the convexity assumption, we have that

$$0 \leq -t M F'(x^{(k)})^{-1} F(0)$$

$$= 0 + M x^{(k+t)} - \left( M x^{(k)} - t M F'(x^{(k)})^{-1} F(x^{(k)}) \right) - t M F'(x^{(k)})^{-1} F(0)$$

$$\leq M x^{(k+t)} - M x^{(k)} + t M F'(x^{(k)})^{-1} F(x^{(k)}) \left( x^{(k)} - 0 \right)$$

$$= M x^{(k+t)} - (1-t)M x^{(k)}.$$  

Hence, for all $t \in [0, 1]$

$$0 \leq (1-t)M x^{(k)} \leq M x^{(k+t)} \leq M x^{(k)} \leq M I = (n+1)I.$$  

It remains to prove that $x^{(k+1)} \in (-1, n)^n$. We argue by contradiction and assume that this is not the case. Then, by continuity, there exists $t \in (0, 1]$ with $x^{(k+t)} \in [-1, n]^n \setminus (-1, n)^n$. Hence, by convexity,

$$F'(x^{(k+t)}) (x^{(k+t)} - 0) \geq F(x^{(k+t)}) - F(0) \geq 0$$  

and using lemma 2.4(c) this would imply

$$\min_{j=1, \ldots, n} x_j^{(k+t)} > \frac{1}{n} \max_{j=1, \ldots, n} x_j^{(k+t)} \geq -1.$$  \hfill (2.11)

Let $l \in \{1, \ldots, n\}$ be an index so that $x_j^{(k+t)}$ attains its maximum for $j = l$. Then the $l$-th component of $M x^{(k+t)} \leq M I = (n+1)I$ gives the inequality

$$2 x_l^{(k+t)} + \sum_{j=1 \atop j \neq l}^n x_j^{(k+t)} \leq n + 1.$$
and (2.11) yields that \( x^{(k+1)} < n \). Hence, \( x^{(k+1)} \in (-1, n)^n \) which contradicts the assumption, and thus shows that \( x^{(k+1)} \in (-1, n)^n \). This finishes the proof of (2.10).

It now follows from (2.10) that for \( x^{(0)} = 1 \), the Newton algorithm produces a well-defined sequence \( x^{(k)} \in (-1, n)^n \) for which \( Mx^{(k)} \) is monotonically non-increasing and bounded. Hence, \((Mx^{(k)})_{k \in \mathbb{N}}\) and thus also \((x^{(k)})_{k \in \mathbb{N}}\) converge. We define

\[
\hat{x} := \lim_{k \to \infty} x^{(k)} \in [-1, n]^n.
\]

Since \( F \) is continuously differentiable and \( F'(\hat{x}) \) is invertible, it follows from the Newton iteration formula (2.9) that \( F(\hat{x}) = 0 \). Also, the monotone convergence of \((Mx^{(k)})_{k \in \mathbb{N}}\) shows that

\[
0 \leq M\hat{x} \leq Mx^{(k)} \leq M1 \quad \text{for all } k \in \mathbb{N}.
\]

To show that \( \hat{x} \in (\frac{1}{n-1}, \frac{n}{n-1})^n \subset (-1, 2)^n \), we use the convexity to obtain

\[
F'(\hat{x})(\hat{x} - 0) \geq F(\hat{x}) - F(0) \geq 0, \\
F'(1)(1 - \hat{x}) \geq F(1) - F(\hat{x}) \geq 0,
\]

which then implies by lemma 2.4(c) that

\[
\min_{j=1,\ldots,n} \hat{x}_j > -\frac{1}{n} \max_{j=1,\ldots,n} \hat{x}_j, \\
\min_{j=1,\ldots,n} (1 - \hat{x}_j) > -\frac{1}{n} \max_{j=1,\ldots,n} (1 - \hat{x}_j).
\]

From this we obtain that

\[
\min_{j=1,\ldots,n} \hat{x}_j > -\frac{1}{n} \max_{j=1,\ldots,n} \hat{x}_j = \frac{1}{n} \min_{j=1,\ldots,n} (1 - \hat{x}_j) - \frac{1}{n} \\
> -\frac{1}{n^2} \max_{j=1,\ldots,n} (1 - \hat{x}_j) - \frac{1}{n} = \frac{1}{n^2} \min_{j=1,\ldots,n} \hat{x}_j - \frac{1}{n^2} - \frac{1}{n},
\]

which yields \( \min_{j=1,\ldots,n} \hat{x}_j > -\frac{1}{n-1} \geq -1 \). Using (2.12) again, we then obtain

\[
-\frac{1}{n} \max_{j=1,\ldots,n} \hat{x}_j > \frac{1}{n^2} \min_{j=1,\ldots,n} \hat{x}_j - \frac{1}{n^2} - \frac{1}{n} > -\frac{1}{n-1},
\]

which shows \( \max_{j=1,\ldots,n} < \frac{n}{n-1} \leq 2 \).

Finally, since this is the standard Newton iteration, the convergence speed is super-linear and the speed is quadratic if \( F' \) is Lipschitz continuous in a neighbourhood of \( \hat{x} \).

### 2.3. A result with tighter domain assumptions.

Our results in subsections 2.1 and 2.2 require the considered function to be defined (and convex and monotonic) on a much larger set than \((0, 1)^n\). For some applications (such as the inverse coefficient problem in section 3), the following more technical variant of Theorem 2.3 may be useful, that allows us treat the case where the domain of definition is an arbitrarily small neighbourhood of \([0, 1]^n\).

**Theorem 2.5.** Let \( \epsilon > 0 \) and \( c \geq 2 + \frac{2}{\epsilon} \). Let \( F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n, n \geq 2, \) be a continuously differentiable, componentwise convex and monotonic function on an open set \( U \).
If $[-\frac{1+\epsilon}{cn}, 1+2\epsilon]^n \subset U$, and
\[ F'(z^{(j,k)})d^{(j)} \leq 0 \quad \text{for all } j \in \{1, \ldots, n\}, k = \{1, \ldots, K\}, \tag{2.13} \]
then the following holds:
(a) $F$ is injective on $[-\frac{1+\epsilon}{cn}, 1+\epsilon]^n$, $F'(x)$ is injective for all $x \in [-\frac{1+\epsilon}{cn}, 1+\epsilon]^n$, and for all $x, y \in [-\frac{1+\epsilon}{cn}, 1+\epsilon]^n$
\[ \|x - y\|_\infty \leq L\|F(x) - F(y)\|_\infty, \quad \text{and} \quad \|F'(x)^{-1}\|_\infty \leq L, \]
where
\[ L := \left( \min_{j=1,\ldots,n} \max_{k=1,\ldots,n} e_k^T F'(z^{(j)})d^{(j)} \right)^{-1} > 0. \tag{2.16} \]
(b) If, additionally, $F(0) \leq 0 \leq F(1)$, then there exists a unique
\[ \hat{x} \in [-\frac{1+\epsilon}{cn}, 1+\epsilon]^n \quad \text{with} \quad F(\hat{x}) = 0. \]
The Newton iteration sequence
\[ x^{(k+1)} := x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}) \quad \text{with initial value } x^{(0)} := 1 \tag{2.17} \]
is well defined (i.e., $F'(x^{(k)})$ is invertible in each step) and converges against $\hat{x}$. For all $k \in \mathbb{N}$
\[ x^{(k)} \in (-\frac{1+\epsilon}{cn}, 1+\epsilon)^n, \quad \text{and} \quad 0 \leq Mx^{(k+1)} \leq Mx^{(k)} \leq Mx^{(0)} = (1+cn)1, \]
where $M := \Pi^T + (1 + (c-1)n)I_n \in \mathbb{R}^{n \times n}$.
The rate of convergence of $x_k \to \hat{x}$ is superlinear. If $F'$ is locally Lipschitz in $U$ then the rate of convergence is quadratic.

To prove Theorem 2.5 we first prove a variant of lemma 2.4 with tighter domain assumptions.

**Lemma 2.6.** Under the assumptions and with the notations of Theorem 2.5, the following holds:
(a) For all $x \in [-\frac{1+\epsilon}{cn}, 1+\epsilon]^n$,
\[ \max_{k=1,\ldots,n} e_k^T F'(x)(e_j - cne'_j) \geq L^{-1}. \]
(b) $F$ is injective on $[-\frac{1+\epsilon}{cn}, 1+\epsilon]^n$, $F'(x)$ is injective for all $x \in [-\frac{1+\epsilon}{cn}, 1+\epsilon]^n$, and for all $x, y \in [-\frac{1+\epsilon}{cn}, 1+\epsilon]^n$
\[ \|x - y\|_\infty \leq L\|F(x) - F(y)\|_\infty, \quad \text{and} \quad \|F'(x)^{-1}\|_\infty \leq L. \]
(c) For all \( x \in \left[-\frac{1+\epsilon}{cn}, 1+\epsilon\right]^n \), and \( 0 \neq y \in \mathbb{R}^n \)

\[
F'(x) y \geq 0 \quad \text{implies} \quad \max_{j=1,\ldots,n} y_j = \|y\|_\infty \quad \text{and} \quad \min_{j=1,\ldots,n} y_j > -\frac{1}{cn} \|y\|_\infty.
\]

(d) With \( M := \mathbb{I}^T + (1 + (c-1)n)I_n \in \mathbb{R}^{n \times n} \), for all \( x \in \left[-\frac{1+\epsilon}{cn}, 1+\epsilon\right]^n \), and \( y \in \mathbb{R}^n \)

\[
F'(x) y \geq 0 \quad \text{implies} \quad My \geq 0.
\]

\( M \) is invertible and \( M^{-1} = \frac{1}{1+(c-1)n} \left( I_n - \frac{1}{1+cn} \mathbb{I}^T \right) \).

(e) For all \( x \in \left[-\frac{1+\epsilon}{cn}, 1+\epsilon\right]^n \)

\[
MF'(x)^{-1} \geq 0.
\]

**Proof.** To prove (a) let \( j \in \{1,\ldots,n\} \) and \( x \in \left[-\frac{1+\epsilon}{cn}, 1+\epsilon\right]^n \). Then, by the definition of \( K \), there exists \( k \in \{1,\ldots, K\} \), so that

\[
-\frac{1+\epsilon}{cn} + (k-1) \frac{\epsilon}{2cn} \leq x_j \leq -\frac{1+\epsilon}{cn} + k \frac{\epsilon}{2cn}.
\]

Then it follows from the definition of \( z^{(j,k)} \) and \( d^{(j)} \) in (2.14) and (2.15) that

\[
x - z^{(j,k)} \geq \frac{\epsilon}{2cn} e_j - \left( \frac{1+\epsilon}{cn} + 1 + 2\epsilon \right) e'_j = \frac{\epsilon}{cn} d^{(j)},
\]

\[
x - z^{(j,k)} \leq \frac{\epsilon}{cn} e_j - \epsilon e'_j.
\]

We thus obtain

\[
F'(x) \left( e_j - cne'_j \right)
\]

\[
= \frac{cn}{\epsilon} F'(x) \left( \frac{\epsilon}{cn} e_j - \epsilon e'_j \right) \geq \frac{cn}{\epsilon} F'(x) \left( x - z^{(j,k)} \right) \geq \frac{cn}{\epsilon} F(x - F(z^{(j,k)}))
\]

\[
\geq \frac{cn}{\epsilon} F'(z^{(j,k)}) \left( x - z^{(j,k)} \right) \geq F'(z^{(j,k)})d^{(j)},
\]

which proves (a).

(b) and (c) are analogous to the proof of lemma 2.1.

For the proof of (d) note that using (c) \( F'(x) y \geq 0 \) implies that for all \( k \in \mathbb{N} \),

\[
\sum_{j=1}^{n} y_j + (1 + (c-1)n)y_k \geq \max_{j=1,\ldots,n} y_j + (n-1) \min_{j=1,\ldots,n} y_j + (1 + (c-1)n)y_k
\]

\[
= \max_{j=1,\ldots,n} y_j + cn \min_{j=1,\ldots,n} y_j \geq 0.
\]

so that \( F'(x) y \geq 0 \) implies \( My = (\mathbb{I}^T + (1 + (c-1)n)I_n) y \geq 0 \). Also, it is easily checked that

\[
\frac{1}{1 + (c-1)n} \left( I_n - \frac{1}{1+cn} \mathbb{I}^T \right) \left( \mathbb{I}^T + (1 + (c-1)n)I_n \right) = I_n
\]

(e) follows from (d) as in the proof of lemma 2.4. \( \square \)
Proof of Theorem 2.5. We proceed as in the proof of Theorem 2.3. Assertion (a) has already been proven in lemma 2.0(b). To show assertion (b), we first prove that for all \( x^{(k)} \in (-\frac{1+c}{cn}, 1+\epsilon)^n \) with \( F(x^{(k)}) \geq 0 \) and \( Mx^{(k)} \leq M1 \), the next Newton iterate \( x^{(k+1)} \) is well-defined and fulfills

\[
x^{(k+1)} \in (-\frac{1+c}{cn}, 1+\epsilon)^n, \quad F(x^{(k+1)}) \geq 0, \quad 0 \leq Mx^{(k+1)} \leq Mx^{(k)} \leq M1. \tag{2.18}
\]

To show this let \( x^{(k)} \in [-\frac{1+c}{cn}, 1+\epsilon]^n \) fulfill \( F(x^{(k)}) \geq 0 \) and \( Mx^{(k)} \leq M1 \). Then \( F'(x^{(k)}) \in \mathbb{R}^{n \times n} \) is invertible, so that we can define the intermediate Newton steps

\[
x^{(k+t)} := x^{(k)} - tF'(x^{(k)})^{-1}F(x^{(k)}) \quad \text{for all } t \in [0,1].
\]

With the same arguments as in Theorem 2.3 we obtain that for all \( t \in [0,1] \)

\[
(1-t)Mx^{(k)} \leq Mx^{(k+t)} \leq Mx^{(k)} \leq M1 = (1+cn)1.
\]

It remains to prove that \( x^{(k+1)} \in (-\frac{1+c}{cn}, 1+\epsilon)^n \). We argue by contradiction and assume that this is not the case. Then, by continuity, there exists \( t \in (0,1] \) with \( x^{(k+t)} \in [-\frac{1+c}{cn}, 1+\epsilon]^n \setminus (-\frac{1+c}{cn}, 1+\epsilon)^n \). Hence, by convexity,

\[
F'(x^{(k+t)})(x^{(k+t)} - 0) \geq F(x^{(k+t)}) - F(0) \geq 0
\]

and using lemma 2.0(c) this would imply

\[
\min_{j=1,\ldots,n} x^{(k+t)}_j > \frac{1}{cn} \max_{j=1,\ldots,n} x^{(k+t)}_j \geq -\frac{1+\epsilon}{cn}. \tag{2.19}
\]

Let \( l \in \{1,\ldots,n\} \) be an index so that \( x^{(k+t)}_j \) attains its maximum for \( j = l \). Then the \( l \)-th component of \( Mx^{(k+t)} \leq M1 = (1+cn)1 \) together with (2.19) gives the inequality

\[
(2+(c-1)n)x^{(k+t)}_l - (n-1)\frac{1+\epsilon}{cn} \leq (2+(c-1)n)x^{(k+t)}_l + \sum_{j=1}^{n} x^{(k+t)}_j \leq 1 + cn,
\]

which yields

\[
x^{(k+t)}_l \leq \frac{1 + cn + (n-1)\frac{1+\epsilon}{cn}}{2+(c-1)n} = 1 + \frac{(n-1)(1+\epsilon + cn)}{(2+(c-1)n)cn}.
\]

An elementary computation shows that

\[
\frac{1+\epsilon + cn}{(2+(c-1)n)cn} < \frac{(1+\epsilon)cn + cn}{(2+(c-1)n)cn} = \frac{2+\epsilon}{2+(c-1)n} < \frac{2+\epsilon}{(c-1)n} \leq \frac{a}{n} \tag{2.20}
\]

where we used \( cn > 1 \) for the first inequality, and the assumption \( c \geq 2 + \frac{2}{n} = \frac{2(n+1)}{n} \) for the last inequality. Hence, \( x^{(k+t)}_l < 1 + \frac{cn-1}{n} < 1+\epsilon \), so that \( x^{(k+t)} \in (-\frac{1+c}{cn}, 1+\epsilon)^n \). This contradicts the assumption, and thus shows that \( x^{(k+1)} \in (-\frac{1+c}{cn}, 1+\epsilon)^n \). This finishes the proof of 2.15.
As in the Theorem 2.3, it follows from (2.10) that for \( x^{(0)} = \mathbb{1} \), the Newton algorithm produces a well-defined sequence \( x^{(k)} \in (\frac{1 + \varepsilon}{cn}, 1 + \varepsilon)^n \) that converges against \( \hat{x} \in [-\frac{1 + \varepsilon}{cn}, 1 + \varepsilon]^n \) with \( F(\hat{x}) = 0 \), and that
\[
0 \leq M \hat{x} \leq M x^{(k+1)} \leq M x^{(k)} \leq M \mathbb{1} \quad \text{for all } k \in \mathbb{N}.
\]
Again, since this is the standard Newton iteration, the convergence speed is superlinear and the speed is quadratic if \( F' \) is Lipschitz continuous in a neighbourhood of \( \hat{x} \).

2.4. Academic examples. We give two simple academic examples to illustrate our results. The first example shows that convex monotonic functions do not have to be injective.

**Example 2.7.** Consider
\[
F : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} e^{x_1} + e^{x_2} \\ x_1 + x_2 \end{pmatrix}.
\]
Then
\[
F'(x) = \begin{pmatrix} e^{x_1} & e^{x_2} \\ 1 & 1 \end{pmatrix}.
\]
Clearly \( F \) is convex and monotonic since all components of \( F \) are convex, and \( F'(x) \) has only positive entries, cf. Remark 2.1. However, \( F \) is not injective on any interval \([a,b]^2 \subseteq \mathbb{R}^2 \) with \( a, b \in \mathbb{R} \), \( a < b \), since \( F(x_1, x_2) = F(x_2, x_1) \).

The second example is on a situation where Theorem 2.3 applies.

**Example 2.8.** Given \( \hat{x} := (\hat{x}_1, \hat{x}_2)^T \in (0, 1)^2 \subseteq \mathbb{R}^2 \), we consider
\[
F : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} e^{x_1+\hat{x}_1} + x_2 \\ x_1 + e^{x_2+\hat{x}_2} \end{pmatrix} - \begin{pmatrix} e^{\hat{x}_1+\hat{x}_2} + \hat{x}_2 \\ \hat{x}_1 + e^{\hat{x}_2+\hat{x}_2} \end{pmatrix},
\]
so that, by construction, \( F(\hat{x}) = 0 \). Since the components of \( F \) are convex and
\[
F'(x) = \begin{pmatrix} e^{x_1+\hat{x}_1} & 1 \\ 1 & e^{x_2+\hat{x}_2} \end{pmatrix},
\]
the monotonicity and coercivity assumptions are fulfilled.

We have that
\[
F' \left( \begin{pmatrix} -2 \\ n(n+3) \end{pmatrix} \right) = \begin{pmatrix} 1 \\ (-n^2 + 3n + 1) \end{pmatrix} = \begin{pmatrix} e^3 - 8 \\ 1 - 8e^{15} \end{pmatrix},
\]
\[
F' \left( \begin{pmatrix} n(n+3) \\ -2 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ (-n^2 + 3n + 1) \end{pmatrix} = \begin{pmatrix} 1 - 8e^{15} \\ e^3 - 8 \end{pmatrix},
\]
so that the assumptions of Theorem 2.3 are fulfilled with (using \( e^3 \geq 20 \))
\[
L = (n + 2)(e^3 - 8)^{-1} \leq \frac{1}{3}.
\]

Theorem 2.3 thus shows that \( F \) is injective on \([1, 2]^2 \) and that for all \( x, y \in [-1, 2]^2 \)
\[
\|x - y\|_\infty \leq \frac{1}{3} \|F(x) - F(y)\|_\infty \quad \text{and} \quad \|F^{-1}(x)\|_\infty \leq \frac{1}{3}
\]
(and probably also that \( L \|F(x) - F(y)\|_\infty \geq \|x - y\|_\infty \) holds with \( L = e^2 - 7 > 0.3 \)).

Moreover, the theorem guarantees that the Newton method will converge when applied with starting value \((1, 1)\).
3. Application to an inverse boundary value problem. We will now show how to use our result to obtain global convergence for a discretized inverse problem of determining a coefficient in an elliptic partial differential equation from boundary data. We consider the inverse Robin transmission problem from [6], that is motivated by corrosion detection.

3.1. The setting. We first introduce the idealized infinite-dimensional forward problem following [6] and then describe its discretization.

3.1.1. The forward problem. Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$), be a bounded domain with Lipschitz boundary $\partial \Omega$ and let $\Omega_1$ be an open subset of $\Omega$, with $\Omega_1 \subset \Omega$. Furthermore let $\Gamma := \partial \Omega_1$ be Lipschitz, and $\Omega_2 := \Omega \setminus \overline{\Omega_1}$ be connected. Thus, we have $\partial \Omega_2 = \Gamma \cup \partial \Omega$.

We assume that $\Omega$ describes an imaging domain with conductivity $\sigma = \sigma_1 \chi_{\Omega_1} + \sigma_2 \chi_{\Omega_2}$, where $\sigma_1, \sigma_2 > 0$ are known constants, and an unknown Robin transmission parameter $\gamma \in L^\infty(\Gamma)$, where $L^\infty$ denotes the subset of $L^\infty$-functions with positive essential infima. For given Neumann boundary data $g \in L^2(\partial \Omega)$, we then consider the problem to find $u \in H^1(\Omega)$ solving

$$-\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega_1 \cup \Omega_2, \quad (3.1)$$

$$\sigma_2 \partial_\nu u|_{\partial \Omega} = g \quad \text{on } \partial \Omega, \quad (3.2)$$

$$[u]|_{\Gamma} = 0 \quad \text{on } \Gamma, \quad (3.3)$$

$$[\sigma \partial_\nu u]|_{\Gamma} = \gamma u \quad \text{on } \Gamma, \quad (3.4)$$

where $\nu$ is the unit normal vector to the interface $\Gamma$ or $\partial \Omega$ pointing outward of $\Omega_1$ or $\Omega$.

$$[u] := u^+|_\Gamma - u^-|_\Gamma, \quad \text{and} \quad [\sigma \partial_\nu u] := \sigma_2 \partial_\nu u^+|_\Gamma - \sigma_1 \partial_\nu u^-|_\Gamma,$$

denote the jump of the Dirichlet, resp., Neumann trace values on $\Gamma$, with the superscript ”+” denoting that the trace is taken from $\Omega_2$ and ”−” denoting the trace taken from $\Omega_1$.

It is easily seen that this problem is equivalent to the variational formulation of finding $u \in H^1(\Omega)$ such that

$$\int_{\Omega_1} \sigma \nabla u \cdot \nabla w \, dx + \int_{\Gamma} \gamma uw \, ds = \int_{\partial \Omega} gw \, ds \quad \text{for all } w \in H^1(\Omega), \quad (3.5)$$

and that $(3.5)$ is uniquely solvable by the Lax-Milgram-Theorem. Hence, we can define the Neumann-to-Dirichlet map

$$\Lambda(\gamma) : L^2(\partial \Omega) \rightarrow L^2(\partial \Omega), \quad g \mapsto u|_{\partial \Omega}, \quad \text{where } u \in H^1(\Omega) \text{ solves } (3.5).$$

It is easy to show that $\Lambda(\gamma)$ is a compact self-adjoint linear operator. We summarize some more properties of $\Lambda$ in the following lemma.

**Lemma 3.1.** The non-linear mapping

$$\Lambda : L^\infty(\Gamma) \rightarrow C(\partial \Omega), \quad \gamma \mapsto \Lambda(\gamma)$$

has the following properties:
(a) \( \Lambda \) is Fréchet differentiable. Its derivative \( \Lambda' : L^\infty_+ (\Gamma) \to L(L^\infty(\Gamma), L(L^2(\partial\Omega))) \) is given by the bilinear form
\[
\int_{\partial\Omega} g (\Lambda'(\gamma) \delta) \, h \, ds = - \int_{\Gamma} \delta u_{\gamma}^{(g)} u_{\gamma}^{(h)} \, ds,
\]
for all \( \gamma \in L^\infty(\Gamma) \), \( \delta \in L^\infty(\Gamma) \), and \( g, h \in L^2(\partial\Omega) \), where \( u_{\gamma}^{(g)} \in H^1(\Omega) \) solves (3.5). \( \Lambda' \) is Lipschitz continuous and \( \Lambda'(\gamma) \delta \in L(L^2(\partial\Omega)) \) is compact and self-adjoint.

(b) For all \( g \in L^2(\partial\Omega) \) and all \( \gamma_1, \gamma_2 \in L^\infty_+ (\Omega) \),
\[
\int_{\partial\Omega} g (\Lambda(\gamma_2) - \Lambda(\gamma_1)) \, g \, ds \geq \int_{\partial\Omega} g (\Lambda'(\gamma_1)(\gamma_2 - \gamma_1)) \, g \, ds.
\]

(c) For all \( \gamma \in L^\infty_+ (\Omega) \), \( \delta \in L^\infty(\Gamma) \), and \( g \in L^2(\partial\Omega) \),
\[
\delta(x) \geq 0 \text{ for } x \in \Omega \text{ a.e. } \implies \int_{\partial\Omega} g (\Lambda'(\gamma) \delta) \, g \, ds \leq 0.
\]

Proof. Obviously, for all \( \gamma \in L^\infty_+ (\Gamma) \) and \( \delta \in L^\infty(\Gamma) \), (3.5) defines a compact self-adjoint linear operator \( \Lambda'(\gamma) \delta \in L(L^2(\partial\Omega)) \). Moreover, it follows from the monotonicity estimate in [6] Lemma 4.1 that for all \( \delta \in L^\infty(\Gamma) \) (that are sufficiently small so that \( \gamma + \delta \in L^\infty_+ (\Gamma) \))
\[
\int_{\Gamma} \delta |u_{\gamma}^{(g)}|^2 \, ds \geq \int_{\partial\Omega} g (\Lambda(\gamma) - \Lambda(\gamma + \delta)) \, ds \geq \int_{\Gamma} \left( \gamma - \frac{\gamma^2}{\gamma + \delta} \right) |u_{\gamma}^{(g)}|^2 \, ds
\]
and thus
\[
\| \Lambda(\gamma + \delta) - \Lambda(\gamma) - \Lambda'(\gamma) \delta \|_{L(L^2(\partial\Omega))} \leq \int_{\Gamma} \left( \frac{\delta^2}{\gamma + \delta} \right) |u_{\gamma}^{(g)}|^2 \, ds = O(\|\delta\|^2).
\]
This shows that \( \Lambda \) is Fréchet differentiable for all \( \gamma \in L^\infty_+ (\Gamma) \), and that \( \Lambda'(\gamma) \) is its derivative. Since it is easily shown that \( u_{\gamma}^{(g)} \in H^1(\Omega) \) depends Lipschitz continuously on \( \gamma \in L^\infty_+ (\Gamma) \), it also follows that \( \Lambda' \) is locally Lipschitz continuous. This proves (a).

(b) is shown in [6] Lemma 4.1, and (c) follows from (3.6). \( \square \)

Note that lemma 4.1 shows that \( \Lambda \) is a convex non-increasing function with respect to the pointwise partial order on \( L^\infty_+ (\Omega) \), and the Loewner partial order in the space of compact self-adjoint operators on \( L^2(\partial\Omega) \). Using this property, we are able to formulate the discretized inverse problem with a componentwise convex and monotonic forward function in the following subsection.

3.1.2. Discretization of the inverse problem. The idealized infinite-dimensional inverse Robin transmission problem is to determine the true Robin coefficient function \( \hat{\gamma} \in L^\infty_+ (\partial\Omega) \) from measuring the full Neumann-to-Dirichlet operator
\[
\hat{\Lambda} := \Lambda(\hat{\gamma}) \in L(L^2(\partial\Omega)).
\]
In practical applications, it is natural to discretize this problem by considering only piecewise constant Robin coefficient functions \( \gamma = \sum_{j=1}^{n} \gamma_j \), with \( \gamma_j > 0 \) and \( \chi_j := \chi_{\Gamma_j} \) being the characteristic functions of a given partition \( \Gamma = \bigcup_{j=1}^{n} \Omega_j \) into \( n \in \mathbb{N} \), \( n \geq 2 \), disjoint measurable subsets \( \Gamma_j \subset \Gamma \) with positive measure. Moreover, it seems natural to assume that one knows bounds \( a, b \in \mathbb{R} \) with \( 0 < a \leq \gamma_j \leq b \) for all \( j = 1, \ldots, n \). Hence, a semi-discretized inverse problem is to reconstruct the coefficients \( (\hat{\gamma}_1, \ldots, \hat{\gamma}_n)^T \in [a,b]^n \subseteq \mathbb{R}^n \) of a piecewise-constant Robin transmission function \( \hat{\gamma} = \sum_{j=1}^{n} \hat{\gamma}_j \chi_j \) from knowledge of the infinite-dimensional measurement \( \hat{\Lambda} = \Lambda(\hat{\gamma}) \). The results in [6, Thm. 2.1 and 2.2] show that this semi-discretized inverse problem is uniquely solvable and that Lipschitz stability holds. [6, Thm. 5.2] shows how to explicitly calculate the Lipschitz constant for a given setting using arguments similar to (and inspiring) section 2 in this work.

In practical applications, it will only be possible to measure finitely many components of \( \Lambda = \Lambda(\hat{\gamma}) \), i.e., one can measure

\[
\hat{\Lambda}_{gh} = \int_{\partial \Omega} g \Lambda(\hat{\gamma}) h \, ds
\]

for a finite number of Neumann boundary data \( g, h \in L^2(\partial \Omega) \). Hence, the fully discretized inverse Robin transmission problem is to identify the finitely many unknown parameters \( \hat{\gamma}_1, \ldots, \hat{\gamma}_n \in [a,b] \) from finitely many measurements of the form (3.7).

The following practical important questions remain to be answered: Given bounds \([a,b]\) and a partition of \( \Gamma \) (i.e., a desired resolution), how many (and which) measurements are sufficient to uniquely determine \( \gamma \)? How good is the stability of the resulting inverse problem with finitely many measurements? And how can one construct a globally convergent algorithm to practically determine \( \gamma \)? The following subsections show how these questions can be answered using the theory developed in section 2.

### 3.2. Uniqueness, stability and global Newton convergence.

We summarize the assumptions on the setting: Let \( \Omega \subset \mathbb{R}^d \) \( (d \geq 2) \), be a bounded domain with Lipschitz boundary \( \partial \Omega \) and let \( \Omega_1 \) be an open subset of \( \Omega \), with \( \overline{\Omega_1} \subset \Omega \). Let \( \Gamma := \partial \Omega_1 \) be Lipschitz, and \( \Omega_2 := \Omega \setminus \overline{\Omega_1} \) be connected. We assume that the true unknown Robin coefficient \( \hat{\gamma} \in L^\infty(\Gamma) \) is bounded by \( b \geq \hat{\gamma}(x) \geq a \) with known bounds \( b > a > 0 \), and that \( \hat{\gamma} = \sum_{j=1}^{n} \hat{\gamma}_j \chi_j , \chi_j := \chi_{\Gamma_j} \), is piecewise constant on a known partition \( \Gamma = \bigcup_{j=1}^{n} \Gamma_j \) into \( n \in \mathbb{N} \), \( n \geq 2 \), disjoint measurable subsets \( \Gamma_j \subset \Gamma \) with positive measure.

Our goal is to determine \( \hat{\gamma} \) (i.e., \( \hat{\gamma}_1, \ldots, \hat{\gamma}_n \in \mathbb{R} \)) from finitely many measurements of the form (3.7).

#### 3.2.1. Formulation as a zero finding problem.

The following lemma shows that we can formulate this problem as a zero finding problem for a componentwise coercive and monotonic nonlinear function.

**Lemma 3.2.** Let \( g_1, \ldots, g_n \in L^2(\partial \Omega) \). Define

\[
F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n , \\
\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \mapsto \begin{pmatrix} F_1(\xi) \\ \vdots \\ F_n(\xi) \end{pmatrix} := \frac{1}{b-a} \left( \int_{\partial \Omega} g_1 \Lambda(r(\xi)) \, g_1 \, ds - \int_{\partial \Omega} g_1 \hat{\Lambda} g_1 \, ds \right)_{l=1}^n ,
\]

where
Global Newton convergence for finite dimensional convex monotonic functions

where $U := (-\infty, \frac{b}{b-a})^n \subset \mathbb{R}^n$, and

$$r : \mathbb{R}^n \to L^\infty(\Gamma), \quad r(\xi) := \sum_{j=1}^n (b - (b - a)\xi_j) \chi_j.$$ 

Then $F$ is continuously differentiable with locally Lipschitz continuous derivative, $F$ is componentwise convex and monotonic, and for all $\xi = (\xi_1, \ldots, \xi_n) \in U$

$$\frac{\partial F_l(\xi)}{\partial \xi_j} = \int_{\Gamma_j} |u_{r(\xi)}^{|\cdot|}|^2 \, ds.$$ 

Furthermore, for $\hat{\xi} := (\frac{b-\gamma_1}{b-a}, \ldots, \frac{b-\gamma_n}{b-a}) \in [0, 1]^n$,

$$F(0) \leq 0 = F(\hat{\xi}) \leq F(\mathbb{1}), \quad \text{and} \quad \hat{\gamma} = r(\hat{\xi}).$$

Proof. Obviously, $r(U) \subset L^\infty_\infty(\Gamma)$, and $r$ is continuously differentiable with

$$\frac{\partial r(\xi)}{\partial \xi_j} = -(b - a)\chi_j.$$ 

Using lemma 3.1 it follows that $F$ is continuously differentiable with

$$\frac{\partial F_l(\xi)}{\partial \xi_j} = \frac{1}{b - a} \int_{\partial \Omega} g_l \Lambda' (r(\xi)) \frac{\partial r(\xi)}{\partial \xi_j} g_l \, ds = \int_{\Gamma_j} |u_{r(\xi)}^{|\cdot|}|^2 \, ds \geq 0,$$ 

and that $F'$ is locally Lipschitz continuous. This also shows that $F$ is componentwise monotonic, cf. remark 2.1.

Moreover, again using lemma 3.1 it also follows that for all $\xi, \eta \in U$, and $l \in \{1, \ldots, n\}$

$$e_l^T (F(\eta) - F(\xi)) = F_l(\eta) - F_l(\xi) = \frac{1}{b - a} \int_{\partial \Omega} g_l \left( \Lambda (r(\eta)) - \Lambda (r(\xi)) \right) g_l \, ds$$

$$\geq \frac{1}{b - a} \int_{\partial \Omega} g_l \Lambda' (r(\eta)) (r(\eta) - r(\xi)) g_l \, ds$$

$$= \sum_{j=1}^n (\eta_j - \xi_j) \int_{\Gamma_j} |u_{r(\xi)}^{|\cdot|}|^2 \, ds = \sum_{j=1}^n \frac{\partial F_l(\xi)}{\partial \xi_j} (\eta_j - \xi_j)$$

$$= e_l^T F'(\xi)(\eta - \xi),$$ 

which shows that $F$ is also componentwise convex.

Obviously, $\hat{\xi} := (\frac{b-\gamma_1}{b-a}, \ldots, \frac{b-\gamma_n}{b-a})$ fulfills $\hat{\gamma} = r(\hat{\xi})$, $F(\hat{\xi}) = 0$, and $0 \leq \hat{\xi} \leq \mathbb{1}$. By monotonicity, this also implies

$$F(0) \leq 0 = F(\hat{\xi}) \leq F(\mathbb{1}), \quad \text{and} \quad \hat{\gamma} = r(\hat{\xi}),$$

so that lemma 3.2 is proven. \(\square\)
3.2.2. Choosing the measurements (for specific bounds). Lemma 3.2 showed that for any choice of \( n \) Neumann boundary data \( g_1, \ldots, g_n \in L^2(\partial \Omega) \), the discretized inverse coefficient problem leads to a zero finding problem for a componentwise monotonic and convex function \( F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \). In this subsection we will use a relation to the concept of localized potentials \( \mathcal{L} \), to show that there exists a choice of Neumann boundary data, for which \( F \) fulfills the additional assumptions from section 2, which yields that the discretized inverse coefficient problem is uniquely solvable, Lipschitz stability holds, and that Newton’s method globally converges. More precisely, we will show that every large enough finite dimensional subspace of \( L^2(\partial \Omega) \) contains such Neumann data and derive a constructive method to identify this Neumann data.

To demonstrate the key idea, we will first consider the specific (and rather restrictive) case where the bounds \( a, b \) fulfill
\[
n(n + 3) < \frac{b}{b - a}
\]
since this case can be treated by simply combining Theorem 2.3 with a known localized potentials result from \( \mathcal{L} \). The general case with arbitrary bounds \( b > a > 0 \) requires an extended result on simultaneously localized potential, and will be treated in the next subsection.

**Theorem 3.3.** Let \( n(n + 3) < \frac{b}{b - a} \), and let \( (f_m)_{m \in \mathbb{N}} \subseteq L^2(\partial \Omega) \) be a sequence of vectors with dense linear span in \( L^2(\partial \Omega) \).

For \( j \in \{1, \ldots, n\} \), and \( m \in \mathbb{N} \), let \( M_{\kappa}^{(j)} \in \mathbb{N}^{m \times m} \) be the symmetric matrix with entries
\[
e_l^T M_{\kappa}^{(j)} e_k := \int_{\Gamma_j} u_{\kappa_{\lambda}^{(j)}}^l u_{\kappa_{\lambda}^{(j)}}^k \, ds - (n^2 + 3n + 1) \int_{\Gamma \setminus \Gamma_j} u_{\kappa_{\lambda}^{(j)}}^l u_{\kappa_{\lambda}^{(j)}}^k \, ds,
\]
where \( l, k \in \{1, \ldots, m\} \), and
\[
\kappa_{\lambda}^{(j)} := r(2e_j + n(n + 3)e_j) = \begin{cases} 3b - 2a & \text{on } \Gamma_j, \\ b - n(n + 3)(b - a) & \text{on } \Gamma \setminus \Gamma_j. \end{cases}
\]

Then the following holds:

(a) For each \( j \in \{1, \ldots, n\} \), and sufficiently large dimension \( m \in \mathbb{N} \), \( M_{\kappa}^{(j)} \) has a positive eigenvalue \( \lambda^{(j)} > 0 \) with a corresponding normalized eigenvector \( u^{(j)} = (v_{\lambda_1}^{(j)}, \ldots, v_{\lambda_m}^{(j)}) \in \mathbb{R}^m \). The Neumann boundary data
\[
g^{(j)} := \sum_{k=1}^m v_{\lambda}^{(j)} f_k \in \text{span}\{f_1, \ldots, f_m\} \subset L^2(\partial \Omega)
\]
fulfill
\[
\int_{\Gamma_j} |u_{\kappa_{\lambda}^{(j)}}^{(j)}|^2 \, ds - (n^2 + 3n + 1) \int_{\Gamma \setminus \Gamma_j} |u_{\kappa_{\lambda}^{(j)}}^{(j)}|^2 \, ds = \lambda^{(j)} > 0. \quad (3.8)
\]

(b) With \( g^{(j)} \in L^2(\partial \Omega) \) fulfilling (3.8) in (a) for \( j = 1, \ldots, n \), define \( F \) as in lemma 3.2. Then \( F \) is injective on \([-1, n^n] \), and, for all \( x \in [-1, n^n] \), \( F'(x) \in \mathbb{R}^{n \times n} \) is invertible. Their (left) inverses are Lipschitz continuous with Lipschitz constant
\[
L := (n + 2) \left( \min_{j=1, \ldots, n} \lambda^{(j)} \right)^{-1}
\]
in the sense of \( \text{[24]} \). The Newton iteration sequence
\[
\xi^{(k+1)} := \xi^{(k)} - F'(\xi^{(k)})^{-1} F(\xi^{(k)}) \quad \text{with initial value } \xi^{(0)} := 1
\]
converges (with quadratic speed) to \( \hat{\xi} \in [0, 1]^m \) with \( \dot{\gamma} = r(\hat{\xi}) \).

Proof.
(a) Let \( j \in \{1, \ldots, n\} \). From the localized potentials result in \([9]\) Lemma 4.3], it follows that there exists \( g \in L^2(\partial \Omega) \) with
\[
\int_{\Gamma_j} |u_{\kappa(j)}|^2 \, ds - (n^2 + 3n + 1) \int_{\Gamma \setminus \Gamma_j} |u_{\kappa(j)}|^2 \, ds > 1.
\]
By density and continuity, for sufficiently large \( m \in \mathbb{N} \), there exists a function \( f \in \text{span}\{ f_1, \ldots, f_m \} \) with
\[
\int_{\Gamma_j} |u_{\kappa(j)}|^2 \, ds - (n^2 + 3n + 1) \int_{\Gamma \setminus \Gamma_j} |u_{\kappa(j)}|^2 \, ds > \frac{1}{2}.
\]
Writing \( f = \sum_{i=1}^m v_i f_i \) with \( v_i \in \mathbb{R} \), and \( v := (v_i)_{i=1}^m \in \mathbb{R}^m \), we thus have
\[
v^T M_m^j v = \int_{\Gamma_j} |u_{\kappa(j)}|^2 \, ds - (n^2 + 3n + 1) \int_{\Gamma \setminus \Gamma_j} |u_{\kappa(j)}|^2 \, ds > \frac{1}{2},
\]
which shows that the symmetric matrix \( M_m^j \in \mathbb{R}^{m \times m} \) must have a positive eigenvalue \( \lambda^j > 0 \). A corresponding normalized eigenvector \( v^j = (v_1^j, \ldots, v_m^j) \in \mathbb{R}^m \) then fulfills
\[
\int_{\Gamma_j} |u_{\kappa(j)}|^2 \, ds - (n^2 + 3n + 1) \int_{\Gamma \setminus \Gamma_j} |u_{\kappa(j)}|^2 \, ds = (v_j)^T M_m^j v_j = \lambda^j > 0,
\]
with \( g^j := \sum_{k=1}^m v_k^j f_k \in L^2(\partial \Omega) \), so that (a) is proven.
(b) With \( F \) defined as in lemma \([24]\) we have for all \( j \in \{1, \ldots, n\} \)
\[
e_j^T F'(-2e_j + n(n+3)e_j') \left( e_j - (n^2 + 3n + 1)e_j' \right)
= \int_{\Gamma_j} |u_{\kappa(j)}|^2 \, ds - (n^2 + 3n + 1) \int_{\Gamma \setminus \Gamma_j} |u_{\kappa(j)}|^2 \, ds = \lambda^j > 0,
\]
so that (b) follows from Theorem \([24]\).

3.2.3. Choosing the measurements (for general bounds). We now show how to treat the case of general bounds \( b > a > 0 \).

**Theorem 3.4.** Given \( b > a > 0 \), we set \( \epsilon := \frac{a}{(b-a)} \), \( q := 2 + \frac{2}{n} \), and let \( K \in \mathbb{N} \) be the smallest natural number with \( K \geq \frac{2a}{\epsilon} \left( 1 + \epsilon + \frac{1}{cn} \right) \).
Let \( (f_m)_{m \in \mathbb{N}} \subseteq L^2(\partial \Omega) \) be a sequence of vectors with dense linear span in \( L^2(\partial \Omega) \).
For \( j \in \{1, \ldots, n\} \), and \( m \in \mathbb{N} \), let \( M_m^j \in \mathbb{N}^{m \times m} \) be the symmetric matrix with entries
\[
e_j^T M_m^j e_l := \left( 2 + \frac{4b}{a} - \frac{6n-1}{2n} \right)^{-1} \sum_{\Gamma_j} |u_{\kappa(j,1)}|^2 |u_{\kappa(j,l)}|^2 \, ds
- \frac{1+\epsilon+cn+2cn}{\epsilon} \sum_{k=1}^K \int_{\Gamma \setminus \Gamma_j} |u_{\kappa(j,k)}|^2 |u_{\kappa(j,l)}|^2 \, ds,
\]
where \( i, l \in \{1, \ldots, m\} \), and
\[
\kappa^{(j,k)} := \begin{cases} \frac{b - (b - a) \left(-\frac{1}{cn} + (k - 2)\frac{1}{2cn}\right)}{2} & \text{on } \Gamma_j, \\ \frac{b - (b - a) \left(-\frac{1}{cn} + (k - 2)\frac{1}{2cn}\right)}{2} & \text{on } \Gamma \setminus \Gamma_j. \end{cases}
\]

Then the following holds:

(a) For each \( j \in \{1, \ldots, n\} \), and sufficiently large dimension \( m \in \mathbb{N} \), \( M^{(j)}_m \) has a positive eigenvalue \( \lambda^{(j)} > 0 \) with a corresponding normalized eigenvector \( \nu^{(j)} = (v^{(j)}_1, \ldots, v^{(j)}_m) \in \mathbb{R}^m \). For all \( k \in \{1, \ldots, K\} \), the Neumann boundary data
\[
g^{(j)} := \sum_{k=1}^m v^{(j)}_k f_k \in \text{span}\{f_1, \ldots, f_m\} \subset L^2(\partial \Omega)
\]
fulfill
\[
\lambda_j = \left(2 + \frac{4b}{a} - \frac{6n - 1}{2n}\right)^{-1} \int_{\Gamma_j} \left|u^{(j)}_{\kappa^{(j,k)}}\right|^2 ds - \frac{1 + \epsilon + cn + 2\epsilon cn}{\epsilon} \int_{\Gamma \setminus \Gamma_j} \left|u^{(j)}_{\kappa^{(j,k)}}\right|^2 ds > 0.
\]

(b) With \( g^{(j)} \in L^2(\partial \Omega) \) fulfilling (3.9) in (a) for \( j = 1, \ldots, n \), define \( F \) as in lemma 3.2. Then \( F \) is injective on \([- \frac{1 + \epsilon}{\epsilon}, 1 + \epsilon] \) and, for all \( x \in [- \frac{1 + \epsilon}{\epsilon}, 1 + \epsilon] \), \( F'(x) \in \mathbb{R}^{n \times n} \) is invertible. Their (left) inverses are Lipschitz continuous with Lipschitz constant
\[
L := \left(\min_{j=1,\ldots,n} \lambda^{(j)}\right)^{-1} > 0.
\]
in the sense of (2.7). The Newton iteration sequence
\[
\xi^{(k+1)} := \xi^{(k)} - F'(\xi^{(k)})^{-1} F(\xi^{(k)}) \quad \text{with initial value } \xi^{(0)} := \mathbb{1}
\]
converges (with quadratic speed) to \( \hat{\xi} \in [0,1]^n \) with \( \hat{\gamma} = r(\hat{\xi}) \).

Before we prove Theorem 3.4 let us summarize its consequences for the inverse Robin transmission coefficient problem and remark on its implementation.

**Remark 3.5.** Theorem 3.4 shows that \( n \) unknown parameters \( \gamma_1, \ldots, \gamma_n \in [a,b] \) of a piecewise-constant Robin coefficient \( \gamma = \sum_{j=1}^n \gamma_j \chi_j \), with a-priori known partition and bounds \( b > a > 0 \), are uniquely determined by \( n \) measurements of the associated Neumann-to-Dirichlet operator
\[
\int_{\partial \Omega} g_j \Lambda(\gamma) g_j\, ds, \quad j = 1, \ldots, n,
\]
when \( g_j \) are chosen according to Theorem 3.4. Any sufficiently large subspace of \( L^2(\partial \Omega) \) contains such measurements \( g_j \).

The discretized non-linear inverse problem of determining
\[
(\gamma_j)_{j=1}^n \in \mathbb{R}^n \text{ from the measurements } \left(\int_{\partial \Omega} g_j \Lambda(\gamma) g_j\, ds\right)_{j=1}^n \in \mathbb{R}^n
\]
has a unique solution in \([a, b]^n\), the solution depends Lipschitz continuously on the measurements (with Lipschitz constant given in Theorem 3.3), and the solution can be found by applying the Newton method with initial value \(\gamma_1 = \ldots = \gamma_n = a\). (Note that the Newton iteration is invariant under linear transformations, so that one may omit the rescaling function \(r(\cdot)\) in the implementation.)

We now turn to the proof of Theorem 3.4. As in the previous subsection, we need to make sure that there exists Neumann data \(g^{(j)} \in L^2(\partial \Omega)\) so that the corresponding solutions \(u_{\kappa^{(j,k)}}\) are much larger on \(\Gamma_j\) than on \(\Gamma \setminus \Gamma_j\), but, additionally, this property now has to hold for several Robin coefficients \(\kappa^{(j,k)}\), \(k = 1, \ldots, K\), simultaneously. To show this, we prove two lemmas. The first one shows that solutions on a boundary part \(\Gamma_0 \subseteq \Gamma\) for different Robin coefficients are bounded by each other, when the Robin coefficients only differ on \(\Gamma_0\).

**Lemma 3.6.** Let \(\gamma^{(1)}, \gamma^{(2)} \in L^\infty_+ (\Gamma)\) with \(\gamma^{(1)} = \gamma^{(2)}\) on \(\Gamma \setminus \Gamma_0\), where \(\Gamma_0\) is a measurable subset of \(\Gamma\). Then, for all \(g \in L^2(\partial \Omega)\) the corresponding solutions \(u_1, u_2 \in H^1(\Omega)\) of (3.1), (3.4) with \(\gamma = \gamma^{(1)}\), and \(\gamma = \gamma^{(2)}\), respectively, fulfill

\[
\left(1 + \frac{\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Gamma)}}{\inf(\gamma^{(1)})}\right)^{-1} \|u_1\|_{L^2(\Gamma_0)} \leq \|u_2\|_{L^2(\Gamma_0)} \leq \left(1 + \frac{\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Gamma)}}{\inf(\gamma^{(2)})}\right) \|u_1\|_{L^2(\Gamma_0)}.
\]

**Proof.** We proceed analogously to [3] Lemma 3.6. It follows from the variational formulation (3.5) that

\[
\inf(\gamma^{(2)}) \||u_2 - u_1||_{L^2(\Gamma)}^2 \leq \int_\Omega |\nabla (u_2 - u_1)|^2 dx + \int_\Gamma \gamma^{(2)} |u_2 - u_1|^2 ds
\]

\[= \int_\Omega gw ds - \int_\Omega \sigma \nabla u_1 \cdot \nabla (u_2 - u_1) dx - \int_\Gamma \gamma^{(2)} u_1 (u_2 - u_1) ds
\]

\[= \int_\Gamma (\gamma^{(1)} - \gamma^{(2)}) u_1 (u_2 - u_1) ds \leq \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Gamma)} \|u_1\|_{L^2(\Gamma_0)} \|u_2 - u_1\|_{L^2(\Gamma)}.
\]

Hence, we obtain

\[\|u_2\|_{L^2(\Gamma_0)} - \|u_1\|_{L^2(\Gamma_0)} \leq \|u_2 - u_1\|_{L^2(\Gamma)} \leq \frac{\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Gamma)}}{\inf(\gamma^{(2)})} \|u_1\|_{L^2(\Gamma_0)},
\]

and thus

\[\|u_2\|_{L^2(\Gamma_0)} \leq \left(1 + \frac{\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Gamma)}}{\inf(\gamma^{(2)})}\right) \|u_1\|_{L^2(\Gamma_0)}.
\]

The other inequality follows from interchanging \(\gamma^{(1)}\) and \(\gamma^{(2)}\).

The next lemma shows that a Neumann boundary data can lead to a solution which is large at some boundary part \(\Gamma_0\) for one Robin coefficient, and at the same time small on \(\Gamma \setminus \Gamma_0\) for several Robin coefficients.

**Lemma 3.7.** Let \(\Gamma_0\) be a measurable subset of \(\Gamma\) with positive measure, \(K \in \mathbb{N}\), and \(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(K)} \in L^\infty_+ (\Gamma)\).
Then, for all \( C > 0 \), there exists \( g \in L^2(\partial \Omega) \), so that the corresponding solutions \( u_1, u_2, \ldots, u_K \in H^1(\Omega) \) of (3.1)–(3.4) fulfill
\[
\int_{\Gamma_0} |u_1|^2 \, ds \geq C \quad \text{and} \quad \sum_{k=1}^K \int_{\Gamma \setminus \Gamma_0} |u_k|^2 \, ds \leq \frac{1}{C}.
\]

Proof. The existence of simultaneously localized potentials for the fractional Schrödinger equation has recently been shown in [4, Theorem 3.11], and we proceed similarly in this proof. Following the original localized potentials approach in [3], we first formulate the assertion as operator range (non-)inclusions, by introducing the operators
\[
A_0 : L^2(\Gamma_0) \to L^2(\partial \Omega), \quad f \mapsto Af := v_0|_{\partial \Omega},
\]
\[
A_k : L^2(\Gamma \setminus \Gamma_0) \to L^2(\partial \Omega), \quad f \mapsto Af := v_k|_{\partial \Omega},
\]
where \( k \in \{1, \ldots, K\} \), \( v_0 \in H^1(\Omega) \) solves
\[
\int_{\Omega} \sigma \nabla v_0 \cdot \nabla w \, dx + \int_{\Gamma} \gamma(1)v_0 w \, ds = \int_{\Gamma_0} f w \, ds \quad \text{for all } w \in H^1(\Omega),
\]
and \( v_k \in H^1(\Omega) \) solves
\[
\int_{\Omega} \sigma \nabla v_k \cdot \nabla w \, dx + \int_{\Gamma} \gamma(k)v_k w \, ds = \int_{\Gamma \setminus \Gamma_0} f w \, ds \quad \text{for all } w \in H^1(\Omega).
\]

It is easily shown (see, e.g., the proof of [6, Theorem 3.1]) that the adjoints of these operators are given by
\[
A_0^* : L^2(\partial \Omega) \to L^2(\Gamma_0), \quad g \mapsto u_1|_{\Gamma_0},
\]
\[
A_k^* : L^2(\partial \Omega) \to L^2(\Gamma \setminus \Gamma_0), \quad g \mapsto u_k|_{\Gamma \setminus \Gamma_0},
\]
where \( u_1, \ldots, u_K \in H^1(\Omega) \) solve (3.1)–(3.4) with Neumann boundary data \( g \) and Robin coefficients \( \gamma(1), \ldots, \gamma(K) \), respectively.

By a simple normalization argument, the assertion is now equivalent to showing that
\[
\exists C > 0 : \|A_0^* g\|^2 \leq C \sum_{k=1}^K \|A_k^* g\|^2 = C \left\| \begin{pmatrix} A_1^* \\ \vdots \\ A_K^* \end{pmatrix} \right\|^2 \quad \text{for all } g \in L^2(\partial \Omega). \quad (3.12)
\]

Using the functional analytic relation between operator ranges and the norms or their adjoints in [3, Lemma 2.5], [2, Cor. 3.5], the property (3.12) (and thus the assertion) is proven if we can show that
\[
\mathcal{R}(A_0) \not\subseteq \mathcal{R}(A_1) + \ldots + \mathcal{R}(A_K) = \mathcal{R}(A_1) + \ldots + \mathcal{R}(A_K). \quad (3.13)
\]

We prove (3.13) by contradiction, and assume that
\[
\mathcal{R}(A_0) \subseteq \mathcal{R}(A_1) + \ldots + \mathcal{R}(A_K).
\]
Then, for every $f_0 \in L^2(\Gamma_0)$, there exist $f_1, \ldots, f_K \in L^2(\Gamma \setminus \Gamma_0)$, so that

$$A_0 f_0 = A_1 f_1 + \ldots + A_K f_K.$$ 

Let $v_0, \ldots, v_K \in H^1(\Omega)$ be the associated solutions from the definition of $A_0, \ldots, A_K$ (with $f = f_k$), and set $v := v_1 + \ldots + v_K - v_0$. Then $v|_{\partial \Omega} = 0$, and $\partial v|_{\partial \Omega} = 0$, so that by unique continuation $v = 0$ in $\Omega_2$. But this also yields that $v|_\Gamma = 0$, and from this we obtain that $v = 0$ in $\Omega_1$, so that $v = 0$ in all of $\Omega$.

Hence, using (3.10) and (3.11), we obtain for all $w \in H^1(\Omega)$,

$$0 = \int_\Omega \sigma \nabla v \cdot \nabla w \, dx + \int_\Gamma \gamma^{(1)} v w \, ds$$

$$= \sum_{k=1}^K \left( \int_\Omega \sigma \nabla v_k \cdot \nabla w \, dx + \int_\Gamma \gamma^{(k)} v_k w \, ds + \int_\Gamma (\gamma^{(1)} - \gamma^{(k)}) v_k w \, ds \right)$$

$$- \int_\Omega \sigma \nabla v_0 \cdot \nabla w \, dx - \int_\Gamma \gamma^{(1)} v_0 w \, ds$$

$$= \sum_{k=1}^K \left( \int_{\Gamma \setminus \Gamma_0} f_k w \, ds + \int_\Gamma (\gamma^{(1)} - \gamma^{(k)}) v_k w \, ds \right) - \int_{\Gamma_0} f_0 w \, ds,$$

and this shows that

$$f_0 = \sum_{k=2}^K (\gamma^{(1)} - \gamma^{(k)}) v_k|_{\Gamma_0}.$$ 

However, since this holds for all $f_0 \in L^2(\Gamma_0)$, this would imply that

$$L^2(\Gamma_0) = M_{\gamma^{(1)} - \gamma^{(k)}} \text{tr}_{\Gamma_0}(H^1(\Omega))$$

where

$$\text{tr}_{\Gamma_0} : H^1(\Omega) \to L^2(\Gamma_0), \quad \text{and} \quad M_{\gamma^{(1)} - \gamma^{(k)}} : L^2(\Gamma_0) \to L^2(\Gamma_0)$$

are the compact trace operator and the continuous multiplication operator. Hence, the closed space $L^2(\Gamma_0)$ would be the range of a compact operator, which contradicts its infinite dimensionality, cf., e.g., [8, Thm. 4.18]. Hence, (3.13) holds, and thus the assertion is proven. \[\square\]

**Proof of Theorem 3.4.**

(a) Analogously to the proof of Theorem 3.3(a), the assertion (a) follows from the simultaneously localized potentials result in lemma 3.6.

(b) To prove (b), we first note that, using the definition of $K$, it is easily checked that

$$\inf(\kappa^{(j,1)}) = a/2$$

and that

$$\|\kappa^{(j,1)} - \kappa^{(j,k)}\|_{L^\infty(\Omega)} = (k - 1) \frac{a}{8cn} \leq \frac{2cn}{\epsilon} \left( 1 + \epsilon + \frac{1 + \epsilon}{cn} \right) \frac{a}{8cn} = b - \frac{6n - 1}{8n} a.$$
Hence, it follows from lemma 3.7 that for all \( g \in L^2(\partial \Omega) \)
\[
\int_{\Gamma_j} |u^g_{\kappa(j,k)}|^2 \, ds \geq \left( 1 + \frac{\|\kappa^{(j,1)} - \kappa^{(j,k)}\|_{L^\infty(\Omega)}}{\inf(\kappa^{(j,1)})} \right)^{-1} \int_{\Gamma_j} |u^g_{\kappa(j,1)}|^2 \, ds
\]
\[
\geq \left( 1 + \frac{2b}{a} - \frac{6n-1}{4n} \right)^{-1} \int_{\Gamma_j} |u^g_{\kappa(j,1)}|^2 \, ds.
\]
To apply Theorem 2.5 we define, \( z^{(j,k)} \), \( d^{(j)} \) by (2.14) and (2.15). Then \( \kappa^{(j,k)} = r(z^{(j,k)}) \in L^\infty_+(\Gamma) \).

Hence, with \( F \) defined as in lemma 3.2 we have that for all \( j \in \{1, \ldots, n\} \) and \( k \in \{1, \ldots, K\} \)
\[
e^T_j F'(z^{(j,k)}) d^{(j)} = \frac{1}{2} \int_{\Gamma_j} |u^g_{\kappa(j,k)}|^2 \, ds - \frac{1 + \epsilon + cn + 2 \epsilon cn}{\epsilon} \int_{\Gamma_j} |u^g_{\kappa(j,1)}|^2 \, ds
\]
\[
\geq \left( 2 + \frac{4b}{a} - \frac{6n-1}{2n} \right)^{-1} \int_{\Gamma_j} |u^g_{\kappa(j,1)}|^2 \, ds
\]
\[
- \frac{1 + \epsilon + cn + 2 \epsilon cn}{\epsilon} \sum_{k=1}^K \int_{\Gamma_j \setminus \Gamma_{j,k}} |u^g_{\kappa(j,k)}|^2 \, ds = \lambda^{(j)} > 0,
\]
so that (b) follows from Theorem 2.5.

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