Dynamics of the rotor on elastic-damping supports under action of kinematic effects

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Abstract. The article describes the elements of the theory of dynamic analysis of rotor systems. The mathematical model of a gyroscopic rotor as an elementary object on elastic-damping supports. The results of simulation of the trajectories of the rotor under kinematic loading with amplitude commensurate with the clearance in bearing assemblies of fluid friction.

1. Introduction
Since the rotor is considered an object of the gyroscopic type, and is a fast rotating symmetric body, the axis of rotation of which can change its direction in space, it is natural that this circumstance should be taken into account when designing mathematical models and choosing the appropriate inertial and elastic-damping parameters [1].

The observed gyroscopic effects due to the unusual reaction of the rotor to the "long" exposure to some influence when the force tends to rotate the rotor around one axis, and it rotates around another perpendicular axis, where precession occurs. When this precessional motion occurs during the entire time of action of external force and is terminated with a termination of the external forces' action, that identifies the property of inertia-free motion. Under the action of impulse impact the axis of the rotor practically does not change the original direction, but only performs fast oscillations with a small amplitude, that is the nutation motion [2].

When developing a mathematical model of the rotor on elastic-damping supports, as a rule, one has to overcome certain difficulties, which are mainly associated with the high dimensionality of the object being studied and the problem of adequate description of external force factors. In this case, during the preliminary stage of the study, it seems justified to consider a simplified mathematical model of the rotor, which reflects the main features of the dynamics of the rotor as a gyroscopic object and allows to evaluate the intensity of its vibrations [3]. It is obvious that the rate of vibration safety of the rotor is significantly affected by penetrating vibration propagating through the chain of "driver – housing – rotor". This is because the RMS value of the vibration is generally commensurate with the size of the gap in the fluid-film bearings, and their basic frequency can be taken equal to the natural frequency of the rotor.

The presence of a penetrating vibration determines the need to consider its effect on rotor’s dynamic behavior. This can be done by reproducing the relevant kinematic effects directly in the process of integration of the mathematical model of the rotor. Note that the rotor on elastic-damping supports perceives the kinematic effects of using these supports and coupling it to the driver.
2. Model of an elementary gyroscopic object

The accepted calculation scheme of the elementary gyroscopic object in the form of a massive rapidly rotating disk in contact with the elastic-damping support shown in Figure 1.

![Diagram of the elementary gyroscopic object](image)

**Figure 1.** Diagram of the elementary gyroscopic object

Note: m – mass of the disk; \( I \) – axial momentum of inertia of the disc; \( c \) – stiffness of an elastic medium; \( b \) – viscous resistance of the medium.

It is assumed that the center of mass of the disc is at its geometric axis. The position of the disc is determined in fixed (absolute) coordinate system \( xy \). In addition, additional mobile (non-inertial) coordinate systems \( x'y' \) and \( \eta \xi \) are introduced with a common origin. The coordinate system \( x'y' \) moves forward together with the center of mass of the disk.

The \( \eta \xi \) system of coordinates is rigidly attached to the disk and rotates with it with angular velocity \( \dot{\theta} = \omega \). It is believed that the resultant restoring and dissipative forces are directed from the center of mass of the rotor to the geometric axis of the support (bearing’s sleeve). In this case, the momentum of these forces about the center of mass of the disk is zero and in the steady-state regime, when \( J\dot{\theta} = 0 \) we have \( \dot{\theta} = \text{const} \).

During the movement of the rapidly rotating disk, it is, apart from restoring and dissipative forces, influenced by the gyroscopic force. The influence of the latter will be taken into account by introducing of an additional component into the equations of motions in the form of the product of the mass of the disc \( m \) on some unknown subject and the definition of acceleration. The projection of the additional component of the \( mw \) on the axis \( xy \) is equal to: \( mw_x \) and \( mw_y \).

Let us assume that the bearing’s sleeve is stationary. Then, the result of the application of the theorem of motion of center of mass is the following interrelated differential equations:

\[
\begin{align*}
mx' + mwx + bx + cx &= 0 \\
m\dot{y} + mw_y + b\dot{y} + cy &= 0
\end{align*}
\]  

(1)

To obtain a closed system of differential equations we use the kinematic relations that connect the projection of the velocity and acceleration of the center of mass of the disk on the axis of the stationary \( xy \) and rotating \( \eta \xi \) coordinate systems:

\[
\begin{align*}
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= R(\theta) \begin{pmatrix} \dot{\eta} \\ \dot{\zeta} \end{pmatrix}, \\
\begin{pmatrix} \dot{x} + w_x \\ \dot{y} + w_y \end{pmatrix} &= R(\theta) \begin{pmatrix} \dot{\eta} \\ \dot{\zeta} \end{pmatrix}
\end{align*}
\]  

(2)

Here the rotation matrix is used:
\[
R(\theta)=\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]  
(3)

which has the following properties:

\[
R(\theta)R^{-1}(\theta)=\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}, \quad R(\theta)\dot{R}^{-1}(\theta) = \begin{pmatrix}0 & +\dot{\theta} \\ -\dot{\theta} & 0\end{pmatrix}
\]  
(4)

Performing the obvious sequence of matrix transformations:

\[
\begin{pmatrix}\dot{\eta} \\ \dot{\zeta}\end{pmatrix} = R^{-1}(\theta)\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} = \dot{R}^{-1}(\theta)\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} + R^{-1}(\theta)\begin{pmatrix}0 \\ +\dot{\theta}\end{pmatrix}
\]

\[
\begin{pmatrix}\ddot{x} + w_x \\ \ddot{y} + w_y\end{pmatrix} = R(\theta)\begin{pmatrix}0 & +\dot{\theta} \\ -\dot{\theta} & 0\end{pmatrix}\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} + \begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix}
\]  
(5)

From the last matrix relationship (5) implies that acceleration:

\[
w_x = \dot{\theta} \dot{y}, \quad w_y = -\dot{\theta} \dot{x}
\]  
(6)

With (6) the system of differential equations of motion of the disc (1) can be written in the form:

\[
m\ddot{x} + m\dot{\theta} \dot{y} + b\dot{x} + cx = 0
\]

\[
m\ddot{y} - m\dot{\theta} \dot{x} + b\dot{y} + cy = 0
\]  
(7)

As it can be seen, the total force component of the gyroscopic force is zero.

Note that in the case of the kinematic effects, when the bearing’s sleeve is not stationary, the system of differential equations of motion of the disk is transformed to the following:

\[
m\ddot{x} + m\dot{\theta} \dot{y} + b(\ddot{x} - \dot{g}_x) + c(x - g_x) = 0
\]

\[
m\ddot{y} - m\dot{\theta} \dot{x} + b(\ddot{y} - \dot{g}_y) + c(y - g_y) = 0
\]  
(8)

Here \( g_x, g_y \) and \( \dot{g}_x, \dot{g}_y \) – projections of the displacement and velocity of the bearing’s sleeve on the \( xy \) axis.

3. **Analysis of the equations of motion**

Considering the system of differential equations (7). Using substitutions \( c = mk^2 \), \( b = mn \) and \( \dot{\theta} = \omega \), we get:

\[
\ddot{x} + \omega \dot{y} + n\dot{x} + k^2 x = 0
\]

\[
\ddot{y} + \omega \ddot{x} + n\dot{y} + k^2 y = 0
\]  
(9)

Let us seek the solution to (9) as functions:

\[
x = Ae^{\xi t}, \quad y = Be^{\eta t}
\]  
(10)

Let us find the pairs \( \lambda, \nu \) of the system matrix.
\[
M = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-k^2 - n & 0 & -\omega \\
0 & 0 & 0 & 1 \\
0 & \omega & -k^2 & -n
\end{pmatrix}
\] (11)

In result of calculationswe shall get the following:

\[
\lambda = \pm \frac{n + i\omega}{2} + \sqrt{\left(\frac{n + i\omega}{2}\right)^2 - k^2} \\
\pm \frac{n - i\omega}{2} + \sqrt{\left(\frac{n - i\omega}{2}\right)^2 - k^2} \\
\pm \frac{n + i\omega}{2} - \sqrt{\left(\frac{n + i\omega}{2}\right)^2 - k^2} \\
\pm \frac{n - i\omega}{2} - \sqrt{\left(\frac{n - i\omega}{2}\right)^2 - k^2}
\]

\[
V = \begin{pmatrix}
1 & 1 & 1 & 1 \\
-\beta_1 & \beta_1 & -\beta_2 & \beta_2 \\
-1 & 1 & -1 & 1 \\
-\beta_1 & \beta_1 & \beta_2 & -\beta_2
\end{pmatrix}
\] (13)

In particular, when \( n=0 \), the components of the vector \( \lambda \) are imaginary:

\[
\lambda_{1,2,3,4} = \pm i \sqrt{-\frac{2k^2 + \omega^2}{2} \pm \sqrt{\left(\frac{2k^2 + \omega^2}{2}\right)^2 - k^4}} \rightarrow \lambda_1,2 = \pm i\beta_1, \lambda_3,4 = \pm i\beta_2
\] (14)

Because \( \alpha_j \approx \alpha_2 \) and \( \beta_j \approx \beta_2 \), the precessionalmotion is determined respectively by the damping ratio \( \alpha_j \) and frequency \( \beta_j \), nutation movement – by damping ratio \( \alpha_2 \) and frequency \( \beta_2 \).

From the formula for determining \( \lambda \), it follows that the greater the angular rotor speed \( \omega \), the lower the frequency of the precessional motion \( \beta_j \) and, conversely, the more the frequency of nutation motion \( \beta_2 \).

Relation \( (A,B)V = \lambda_i V_i \) allows determination that \( B_1 = A_2, B_2 = -A_1, B_3 = -A_4, B_4 = A_3 \).

So, the general solution of system (9) is written in the following form:

\[
x(\alpha, t) = e^{-\alpha t} \left[ (A_1 \cos \beta_1 t + A_2 \sin \beta_1 t) + e^{-\alpha t} (A_3 \cos \beta_2 t + A_4 \sin \beta_2 t) \right] \\
y(\alpha, t) = e^{-\alpha t} \left[ -A_1 \sin \beta_1 t + A_2 \cos \beta_1 t) + e^{-\alpha t} (-A_3 \sin \beta_2 t + A_4 \cos \beta_2 t) \right]
\] (15)

From equations (15), it follows that when all initial conditions of the trajectory of precession and nutation motion are the torsion spirals (in the absence of damping it is a circle). This result displays the fact that the rotor has a pronounced error-correcting (instantaneous) properties.
Note that when $\alpha_1 \angle \alpha_2$, transient (nutation) processes are quickly damped, and decrease in the radius of the spiral path will be slow.

Let us now consider the system of differential equations (8). Using substitutions $c = mk^2$, $b = mn$ and $\dot{\theta} = \omega$, we get:

\begin{align*}
\dot{x} + \omega y + nx + k^2 x &= ng_x + k^2 g_x \\
\dot{y} - \omega x + ny + k^2 y &= ng_y + k^2 g_y
\end{align*}

Taking $g_x = y_0 \sin \omega t \to -iy_0 e^{i\omega t}$ and $g_y = y_0 \cos \omega t \to y_0 e^{i\omega t}$, we seek particular solutions of system (16) as functions:

\begin{align*}
x(t) &= x_h(t) \to K_x e^{i\omega t}, \quad y(t) = y_h(t) \to K_y e^{i\omega t}
\end{align*}

Carrying out the appropriate substitutions and transformations, we shall find the complex amplitudes $K_x$ and $K_y$.

It is determined that the complex amplitudes $K_x$ and $K_y$ are equal in magnitude

\begin{equation}
|K_x| = |K_y| = K = y_0 \sqrt{\frac{\omega^2 n^2 + k^4}{4\omega^4 - 4\omega^2 k^2 + \omega^2 n^2 + k^4}}
\end{equation}

and thus have the following phases

\begin{align*}
q_x &= a \tan \frac{\omega^2 n^2 - 2\omega^2 k^2 + k^4}{2\omega^2 n} \\
q_y &= a \tan \frac{-2\omega^3 n}{\omega^2 n^2 - 2\omega^2 k^2 + k^4}
\end{align*}

Hence, particular solutions of the system (16):

\begin{align*}
x_h(t) &= K \sin(\omega t + q_x), \quad y_h(t) = K \cos(\omega t + q_y)
\end{align*}

Uniting (15) and (20) finding that

\begin{align*}
x(t) &= x_c(A,t) + x_h(t), \quad y(t) = y_c(A,t) + y_h(t)
\end{align*}

Constants $A_i$ are determined from the matrix equation:

\begin{equation}
A = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
-\alpha_1 & \beta_1 & -\alpha_2 & \beta_2 \\
-\beta_1 & -\alpha_1 & -\beta_2 & -\alpha_2
\end{pmatrix}^{-1} \begin{pmatrix}
x(0) - Kx_h(0) \\
y(0) - Ky_h(0) \\
x(0) - K\dot{x}_h(0) \\
y(0) - K\dot{y}_h(0)
\end{pmatrix}
\end{equation}

4. The results of calculations and modeling

For the calculation and modeling of systems of equations (7), (8) the following main parameters are chosen: $\omega = 820 \text{ s}^{-1}$, $m = 1 \text{ kg}$, $c = 32000 \text{ N/m}$, $b = 5 \text{ N s/m}$. Variation of these parameters was carried out with the purpose of obtaining additional information and expansion of ideas about the properties of the system.
The influence of the angular velocity of the rotor, which is a gyroscopic object, on the character of manifestation of nutation processes is shown in the Figure 2.

For transients processes to be well observed, an impulse impact was simulated by setting the initial speed of the rotor \( \dot{x}(0) \neq 0 \). The angular speed of the rotor in each numerical experiment was different: \( \omega = 500 \, \text{s}^{-1} \), \( \omega = 1000 \, \text{s}^{-1} \), \( \omega = 1500 \, \text{s}^{-1} \).

As expected, the amplitude and duration of high frequency oscillation of the rotor associated with the nutation induced processes decrease the faster, the more its angular velocity. The results of the numerical experiments confirm also that the rotor as a gyroscopic object is resistant to impact.

![Figure 2](image1.png)

**Figure 2.** The trajectory of the center of the rotor

Note: \( X^{(j)}(t) \leftrightarrow x(t), X^{(j)}(t) \leftrightarrow y(t) \).

Figure 3 shows the characteristic oscillations of the center of the rotor in the absolute and relative motion under action of kinematic effects.

![Figure 3](image2.png)

**Figure 3.** Oscillations of the center of the rotor in absolute (a) and relative (b) motion. Note:
\( X^{(j)}(t) \leftrightarrow x(t), X^{(j)}(t) \leftrightarrow y(t), X^{(0)} = X_{i,0} \leftrightarrow t, \omega = 1200 \, \text{s}^{-1}, \)
\( m = 1 \, \text{kg}, c = 16 \cdot 10^4 \, \text{N/m}, b = 0.1 \, \text{Nc/m}, y(t) = 10^{-4} \, \text{m}. \)

In the absolute motion of the rotor, excited by a vibration and nutation processes with small amplitude and high frequency close to the frequency of the kinematic effects are superimposed on the precessional motion of the rotor, the frequency of which the smaller, the more its angular velocity. Since the maximum amplitude of the complex motion of the rotor is smaller than the amplitude of the
kinematic effects, relative motion parameters of oscillograms of oscillation of the rotor practically do not differ from the parameters of the kinematic effects.

5. Conclusions
Proposed model of the rotor in the form of a massive rapidly rotating disk of elastic-damping support taking into account the accepted assumptions allows taking into account gyroscopic effects and evaluation of their influence on dynamic processes under vibrational kinematic effects.

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