SOME CHARACTERIZATIONS OF ROBUST SOLUTION SETS
FOR UNCERTAIN CONVEX OPTIMIZATION PROBLEMS WITH
LOCALLY LIPSCHITZ INEQUALITY CONSTRAINTS

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(Communicated by Renata Sotirov)

Abstract. In this paper, we consider an uncertain convex optimization problem with a robust convex feasible set described by locally Lipschitz constraints. Using robust optimization approach, we give some new characterizations of robust solution sets of the problem. Such characterizations are expressed in terms of convex subdifferentials, Clarke subdifferentials, and Lagrange multipliers. In order to characterize the solution set, we first introduce the so-called pseudo Lagrangian function and establish constant pseudo Lagrangian-type property for the robust solution set. We then used to derive Lagrange multiplier-based characterizations of robust solution set. By means of linear scalarization, the results are applied to derive characterizations of weakly and properly robust efficient solution sets of convex multi-objective optimization problems with data uncertainty. Some examples are given to illustrate the significance of the results.

1. Introduction. The study of characterizations of solution sets has become an important research direction for many mathematical programming problems. Based on understanding characterizations of solution sets, solution methods for solving mathematical programs that have multiple solutions can be developed. The notion of characterizations of solution sets was first introduced and studied by Mangasarian

2010 Mathematics Subject Classification. Primary: 90C25, 90C46; Secondary: 90C29.
Key words and phrases. Robust optimal solutions, subdifferential, uncertain convex optimization, multi-objective optimization.

This research was supported by the Thailand Research Fund through the Royal Golden Jubilee Ph.D. Program (Grant No. PHD/0026/2555), the Thailand Research Fund, Grant No. RSA6080077 and Naresuan University, and the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (Grant No. 2017R1E1A1A03069931).

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for a convex extrema problem with differentiable function [29]. Some useful examples clarifying such characterizations of solution sets can be found in [7] for characterizing the problems that have weak sharp minimum. This being a reason why several characterizations of solution sets for some classes of constrained optimization problems have appeared in the literature (see [6, 8, 13, 14, 19, 23, 32, 33, 36, 38, 39] and other references therein).

However, dealing with real-world optimization problems, the input data associated with the objective function and the constraints of programs are uncertain due to prediction error or measurement errors (see [1, 2, 3, 4]). Moreover, in many situations often we need to make decisions now before we can know the true values or have better estimations of the parameters. Robust optimization is one of the basic methodologies to protect the optimal solution that it is no longer feasible after realization of actual values of parameters. This means that any feasible points must satisfy all constraints including each set of constraints corresponding to a possible realization of the uncertain parameters from the uncertainty sets. Precisely stated, let us first consider the following optimization problem:

$$\min_{x \in \mathbb{R}^n} \{ f(x) : g_i(x) \leq 0, \ i = 1, \ldots, m \}, \quad (P)$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, \ldots, m$, are functions. The problem (P) in the face of data uncertainty both in the objective and constraints can be written by the following optimization problems:

$$\min_{x \in \mathbb{R}^n} \{ f(x, u) : g_i(x, v_i) \leq 0, \ i = 1, \ldots, m \}, \quad (UP)$$

where $f : \mathbb{R}^n \times \mathbb{R}^{q_0} \to \mathbb{R}$, and $g_i : \mathbb{R}^n \times \mathbb{R}^{q_i} \to \mathbb{R}, \ i = 1, \ldots, m$, are functions, $u$ and $v_i$ are uncertain parameters and they belong to the specified nonempty convex and compact uncertainty sets $\mathcal{U} \subseteq \mathbb{R}^{q_0}$ and $\mathcal{V}_i \subseteq \mathbb{R}^{q_i}$, respectively. The robust (worst case) counterpart of (UP), by construction in [3], is obtained by solving the single problem:

$$\min_{x \in \mathbb{R}^n} \{ \max_{u \in \mathcal{U}} f(x, u) : g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m \}, \quad (RP)$$

where the objective and constraints are enforced for every possible value of the parameters within their prescribed uncertainty sets $\mathcal{U}$ and $\mathcal{V}_i$. The set of feasible solutions of problem (RP),

$$F := \{ x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m \},$$

refer to robust feasible set of the problem (UP). The optimal solution to the problem (RP) is known as a robust optimal solution of (UP). A successful treatment of the robust optimization approaches for treating convex optimization programs with data uncertainty to derive characterizations of robust optimal solution sets was given in [15, 27, 34, 35]. For issues related to optimality conditions and duality properties, see [5, 11, 16, 17, 18, 25, 26] and other references therein.

This paper is an attempt to investigate optimality conditions and to derive characterizations of robust solution sets of (UP). Unlike various related works in the literature mentioned above, in the present paper, appearing constraint functions are not convex necessarily while the robust feasible set $F$ is convex. In this way, we refer to convex problems without convex representation in the sense that the constraint functions to represent the convex feasible set are non necessarily convex. Optimality conditions and characterizations of convexity of feasible set for such problems
in the absence of data uncertainty can be found in [24] for differentiable case, and in [10, 30, 21] for non-differentiable case.

To the best of our knowledge, completely characterizations of robust solutions for uncertain scalar and multi-objective optimization problems over a robust convex feasible set described by non necessarily convex functions within the framework of robust optimization approach are not available in the literature. So, in this paper we examine a robust optimization framework for studying characterizations of the robust optimal solution set for uncertain convex optimization problems with a robust convex feasible set described by locally Lipschitz constraints. First, complete optimality conditions for uncertain convex optimization problems are given. In order to characterize the robust optimal solution set of a given problem, we introduce the so-called pseudo-Lagrange function and then, we show that pseudo-Lagrange function is constant on the robust optimal solution set. Afterwards, we then use this property to derive various characterizations of the robust optimal solution set that these are expressed in terms of convex subdifferentials, Clarke subdifferentials and Lagrange multipliers. Finally, the results are then applied to derive characterizations of weakly robust efficient solution set and properly robust efficient solution set of uncertain convex multi-objective optimization problems without convexity assumption on constraint functions.

The remainder of the present paper is organized as follows. In Sect. 2, we gives some notations, definitions and preliminary results. In Sect. 3, we establish a multiplier characterization for the robust optimal solution of uncertain convex optimization problem. Sect. 4 provides characterizations of robust solution set of uncertain convex optimization without convexity assumption on constraint functions. In Sect. 5, we give a sufficient condition that a robust efficient solution of uncertain multi-objective convex optimization problems can be a properly robust efficient solution. Moreover, characterizations of weakly robust efficient solution set and properly robust efficient solution set of such problem are given.

2. Preliminaries. We begin this section by fixing certain notations, definitions and preliminary results that will be used throughout the paper. We denote by $\mathbb{R}^n$ the Euclidean space with dimension $n$ whose norm is denoted by $\| \cdot \|$ and $(x, y)$ denotes the usual inner product between two vectors $x, y$ in $\mathbb{R}^n$, that is, $(x, y) = x^T y$.

Let $\mathbb{R}_+^n := \{x := (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \ i = 1, \ldots, n\}$ be non-negative orthant of $\mathbb{R}^n$. Note also that the interior non-negative orthant of $\mathbb{R}^n$ is denoted by $\text{int}\mathbb{R}_+^n$ and is defined by $\text{int}\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i > 0, \ i = 1, \ldots, n\}$. Given a set $A \subseteq \mathbb{R}^n$, we recall that a set $A$ is convex whenever $\lambda x + (1 - \lambda)y \in A$ for all $\lambda \in [0, 1]$, $x, y \in A$. A set $A$ is said to be a cone if $\lambda A \subseteq A$ for all $\lambda \geq 0$. We denote the convex hull and the conical hull generated by $A$, by conv$A$ and cone$A$, respectively. The normal cone at $x$ to a closed convex set $A$, denoted by $N(A, x)$, is defined by

$$N(A, x) := \{\xi \in \mathbb{R}^n : \langle \xi, y - x \rangle \leq 0, \ \forall y \in A\}.$$ 

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if for all $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}^n$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

It is a well known fact that a convex function need not be differentiable everywhere. However if $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function then the one-sided or rather right-sided directional derivative always exists and is finite. The right-sided directional derivative of $f$ at $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is denoted by $f'(x; d)$, is defined as
Clarke generalized subdifferential

The derivative Let \( q \) be a convex function on \( \mathbb{R} \) and for each fixed \( x \in \mathbb{R} \), \( f(x, \cdot) \) is a concave function on \( \mathbb{R}^n \). Then, \[
\partial f(x) := \{ \xi \in \mathbb{R}^n : f(y) \geq f(x) + \langle \xi, y-x \rangle, \text{ for all } y \in \mathbb{R}^n \}.
\]

It is important to note that for every fixed \( x \) the function \( f'(x, \cdot) \) is a positively homogeneous convex function. The subdifferential of convex function \( f \) at \( x \) is defined as

\[
\partial f(x) := \{ \xi \in \mathbb{R}^n : f(y) \geq f(x) + \langle \xi, y-x \rangle, \text{ for all } y \in \mathbb{R}^n \}.
\]

We now recall the following useful result, which is a subdifferential max-function rule of convex functions over a compact set, that will be used later in the paper.

**Lemma 2.1.** [15, Lemma 2.1] Let \( \mathcal{U} \subseteq \mathbb{R}^p \) be a convex compact set, and let \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) be a function such that for each fixed \( u \in \mathcal{U} \), \( f(\cdot, u) \) is a convex function on \( \mathbb{R}^n \) and for each fixed \( x \in \mathbb{R}^n \), \( f(x, \cdot) \) is a concave function on \( \mathbb{R}^m \). Then,

\[
\partial \left( \max_{u \in \mathcal{U}} f(\cdot, u) \right)(\bar{x}) = \bigcup_{u \in \mathcal{U}(\bar{x})} \partial f(\cdot, u)(\bar{x}),
\]

where \( \mathcal{U}(\bar{x}) := \{ \bar{u} \in \mathcal{U} : f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u) \}. \]

**Definition 2.2.** A function \( h : \mathbb{R}^n \to \mathbb{R} \) is said to be locally Lipshitz at \( x \in \mathbb{R}^n \), if there exists a positive scalar \( L \) and a neighborhood \( N \) of \( x \) such that, for all \( y, z \in N \), one has

\[
|h(y) - h(z)| \leq L \|y-z\|.
\]

**Definition 2.3.** [9] Let \( h : \mathbb{R}^n \to \mathbb{R} \) be locally Lipschitz at a given point \( x \in \mathbb{R}^n \). The Clarke generalized directional derivative of \( h \) at \( x \) in the direction \( d \in \mathbb{R}^n \), denoted \( h^o(x; d) \), is defined as

\[
h^o(x; d) := \limsup_{y \to x, t \to 0^+} \frac{h(y + td) - h(y)}{t},
\]

**Definition 2.4.** [9] Let \( h : \mathbb{R}^n \to \mathbb{R} \) be locally Lipschitz at a given point \( x \in \mathbb{R}^n \). The Clarke generalized subdifferential of \( h \) at \( x \), denoted by \( \partial h^o(x) \), is defined as

\[
\partial h^o(x) := \{ \xi \in \mathbb{R}^n : h^o(x; d) \geq \langle \xi, d \rangle \text{ for all } d \in \mathbb{R}^n \}.
\]

From the definition of the Clarke generalized subdifferential, it follows that

\[
h^o(x; d) = \max_{\xi \in \partial h^o(x)} \langle \xi, d \rangle, \text{ \forall d } \in \mathbb{R}^n.
\]

**Definition 2.5.** Let \( h : \mathbb{R}^n \to \mathbb{R} \) be locally Lipschitz at a given point \( x \in \mathbb{R}^n \). The function \( h \) is said to be regular at \( x \in \mathbb{R}^n \) if, for each \( d \in \mathbb{R}^n \), the directional derivative \( h'(x; d) \) exists and coincides with \( h^o(x; d) \).

For a given compact subset \( V \) of \( \mathbb{R}^q \) and a given function \( g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R} \), the following conditions will be considered in this paper.

1. For every \( x \in \mathbb{R}^n \) the function \( V \ni v \mapsto g(x, v) \) is upper semicontinuous;
2. \( g \) is locally Lipschitz in \( x \), uniformly for \( v \in V \), that is, for each \( x \in \mathbb{R}^n \), there exist an open neighborhood \( U \) of \( x \) and a constant \( L > 0 \) such that for all \( y \) and \( z \) in \( U \), and \( v \in V \), one has

\[
|g(y, v) - g(z, v)| \leq L \|y-z\|.
\]
(C3) for each \((x, v) \in \mathbb{R}^n \times V\), the function \(g(\cdot, v)\) is regular at \(x\), that is,
\[ g^*(x, v; \cdot) = g'_d(x, v; \cdot) \]
(the derivatives being with respect to \(x\));

(C4) set-valued map \(\mathbb{R}^n \times V \ni (x, v) \mapsto \partial^o g(\cdot, v)(x)\) is upper semicontinuous where \(\partial^o g(\cdot, v)(x)\) denotes the Clarke subdifferential of \(g\) with respect to \(x\).

**Remark 1.** In a suitable setting, if the function \(g\) is convex in \(x\) and continuous in \(v\), the conditions (C2), (C3), and (C4) are then automatically satisfied. These conditions also hold whenever the derivative \(\nabla v\) is continuous in \((x, v)\).

**Remark 2.** [25] Under the conditions (C1) and (C2) the function \(\psi : \mathbb{R}^n \to \mathbb{R}\),
\[ \psi(x) := \max\{g(x, v) : v \in V\}, \]
is defined and finite. Further, \(\psi\) is locally Lipschitz on \(\mathbb{R}^n\), and hence for each \(x \in \mathbb{R}^n\) the set \(V(x)\) defined as
\[ V(x) := \{v \in V : g(x, v) = \psi(x)\}, \]
is a nonempty closed subset of \(\mathbb{R}^q\).

We conclude this section by the following lemmas which will be useful in our later analysis.

**Lemma 2.6.** [9] Let the function \(\psi\) be defined in Remark 2. Suppose that the conditions (C1) - (C4) are fulfilled. Then the usual one-sided directional derivative \(\psi'(x; d)\) exists, and satisfies the following: for each \(x, d \in \mathbb{R}^n\),
\[ \psi'(x; d) = \psi^0(x; d) = \max\{g^*\cdot(x, v; d) : v \in V(x)\} = \max\{\langle \xi, d \rangle : \xi \in \partial^o g(\cdot, v)(x), v \in V(x)\}. \]

**Lemma 2.7.** [26] For a given compact convex subset \(V\) of \(\mathbb{R}^q\) and a given function \(g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}\), suppose that the basic conditions (C1) - (C4) are fulfilled. Further, suppose that \(g(\cdot, \cdot)\) is concave on \(V\), for each \(x \in \mathbb{R}^n\). Then
\[ \partial^o \psi(x) = \{\xi \in \mathbb{R}^n : \exists v \in V(x) \text{ such that } \xi \in \partial^o g(\cdot, v)(x)\}. \]

3. Multiplier characterization for the robust solution. In this section, we give a multiplier characterization for the robust optimal solution of (UP), which will play an important role in deriving characterizations of the robust optimal solution sets in the next section. Let us recall the following robust (worst case) counterpart optimization problem of (UP):
\[
\min_{x \in \mathbb{R}^n} \left\{ \max_{u \in U} f(x, u) : g_i(x, v_i) \leq 0, \forall v_i \in V_i, \ i = 1, \ldots, m \right\}, \tag{RP}
\]
where \(f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}\), and \(g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}\), \(i = 1, \ldots, m\), are given functions and for each \(i = 1, 2, \ldots, m\), \((u, v_i) \in U \times V_i \subseteq \mathbb{R}^m \times \mathbb{R}^q\), where \(U\) and \(V_i\) are the specified nonempty convex and compact uncertainty sets. The robust feasible set of (UP) is defined by
\[ F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in V_i, \ i = 1, \ldots, m\}. \]

**Assumption 3.1.** Throughout this paper, we always assume that \(F \neq \emptyset\), \(f : \mathbb{R}^n \times U \to \mathbb{R}\) is a convex-concave in the sense that \(f(\cdot, u)\) is a convex function for any \(u \in U\), and \(f(\cdot, \cdot)\) is a concave function for any \(x \in \mathbb{R}^n\) while \(g_i(x, \cdot), \ i = 1, \ldots, m\), are concave functions for any \(x \in \mathbb{R}^n\). Further, let the functions \(g_i, \ i = 1, \ldots, m\), be satisfied the conditions (C1) and (C2).
Definition 3.1. We say that \( \bar{x} \in F \) is a robust optimal solution of (UP) if and only if \( \bar{x} \) is an optimal solution of (RP).

By using Proposition 2.2 in [10], we can derive the following characterization of convexity for robust feasible set of (UP) in terms of the Clarke directional derivative. Before doing so let us denote, for each \( x \in F \),

\[
I(x) := \left\{ i \in \{1, \ldots, m\} : \max_{v_i \in V_i} g_i(x, v_i) = 0 \right\},
\]

and for all \( i = 1, \ldots, m \),

\[
V_i(x) := \left\{ \bar{v}_i \in V_i : g_i(x, \bar{v}_i) = \max_{v_i \in V_i} g_i(x, v_i) \right\}.
\]

Proposition 1. Let the system \( g_i(x, v_i) \leq 0, \forall v_i \in V_i, \ i = 1, \ldots, m \), be satisfied the robust Slater constraint qualification, that is, there exists \( x_0 \in \mathbb{R}^n \) such that

\[ g_i(x_0, v_i) < 0, \text{ for any } v_i \in V_i, \ i = 1, \ldots, m. \]

For each \( x \in F \) and \( i \in I(x) \), let the function \( g_i \) be satisfied the conditions \((C3),(C4)\), and \( 0 \notin \partial^o g_i(\cdot, v_i) (x) \) whenever \( v_i \in V_i(x) \). Then \( F \) is convex if and only if

\[
F = \{ y \in \mathbb{R}^n : g_{ix}^o(x, v_i; y - x) \leq 0, \forall x \in F, \forall i \in I(x), \forall v_i \in V_i(x) \}.
\]

Proof. For each \( i = 1, \ldots, m \), define a function \( \psi_i : \mathbb{R}^n \to \mathbb{R} \) by

\[
\psi_i(x) := \max_{v_i \in V_i} g_i(x, v_i) \quad \text{for all } x \in \mathbb{R}^n.
\]

Applying the conditions \((C1),(C2)\), we have, for each \( i = 1, \ldots, m \), \( \psi_i \) is locally Lipschitz on \( \mathbb{R}^n \). To achieve the result, we will use Proposition 2.2 in [10] and then we need to justify that for any \( x \in F, \psi_i, i \in I(x) \), are regular in the sense of Clarke and \( 0 \notin \partial^o \psi_i(x) \), and the system \( \psi_i(x) \leq 0, \ i = 1, \ldots, m \), satisfies the Slater condition. The first and the second requirements will follow from Lemma 2.6 and Lemma 2.7 that for any \( x \in F \),

\[
\psi_i^\prime(x; d) = \psi_i^\prime(x; d) = \max\{g_{ix}^o(x, v_i; d) : v_i \in V_i(x)\}, \forall i \in I(x),
\]

and for each \( i \in I(x) \)

\[
0 \in \bigcap_{v_i \in V_i} \mathbb{R}^n \setminus \left( \partial^o g_i(\cdot, v_i)(x) \right) = \mathbb{R}^n \setminus \bigcup_{v_i \in V_i, g_i(\cdot, v_i) = \psi_i(x)} \partial^o g_i(\cdot, v_i)(x) = \mathbb{R}^n \setminus \partial^o \psi_i(x).
\]

Finally, the robust Slater constraint qualification leads us to the following strict inequality

\[
\psi_i(x_0) = \max\{g_i(x_0, v_i) : v_i \in V_i \} < 0, \forall i = 1, \ldots, m,
\]

which means that the system \( x \in \mathbb{R}^n \), \( \psi_i(x) \leq 0 (i = 1, \ldots, m) \) satisfies the Slater’s condition\(^1\). Now applying [10, Proposition 2.2] and taking (1) into consideration, we obtain the desired results.

Remark 3. It should be noted that in Proposition 1 without robust Slater constraint qualification and \( 0 \notin \partial^o g_i(\cdot, v_i)(x) \) whenever \( x \in F, i \in I(x), \) and \( v_i \in V_i(x) \), we easily obtain that if \( F \) is convex then

\[
F \subseteq \{ y \in \mathbb{R}^n : g_{ix}^o(x, v_i; y - x) \leq 0, \forall x \in F, \forall i \in I(x), \forall v_i \in V_i(x) \}.
\]

\(^1\) the system \( x \in \mathbb{R}^n \), \( g_i(x) \leq 0 (i = 1, \ldots, m) \) satisfies the Slater’s condition if there exists \( x_0 \in \mathbb{R}^n \) such that \( g_i(x_0) < 0 \) for all \( i = 1, \ldots, m \).
Furthermore, for every $x \in F$ one has

$$\partial^o g_i(\cdot, v_i)(x) \subseteq N(F, x)$$

whenever $i \in I(x)$ and $v_i \in \mathcal{V}_i(x)$.

In order to establish a multiplier characterization for the robust optimal solution of (UP), we first recall a robust basic constraint qualification which was introduced in [5], where the constraint data uncertainty $g_i(\cdot, v_i), i = 1, \ldots, m,$ are assumed to be convex for each $v_i \in \mathcal{V}_i$.

**Definition 3.2.** Let $x \in F$ be a robust feasible solution of (UP). The **robust basic constraint qualification** is satisfied at $x$ if

$$N(F, x) = \bigcup_{\lambda_i \geq 0, v_i \in \mathcal{V}_i} \sum_{i=1}^m \lambda_i \partial^o g_i(\cdot, v_i)(x).$$

Now the following theorem declares a result that the robust basic constraint qualification defined in Definition 3.2 is a necessary and sufficient constraint qualification of a robust optimal solution for the given problem, that is, the robust basic constraint qualification holds if and only if the Lagrange multiplier conditions are satisfied for a robust optimal solution.

**Theorem 3.3 (Characterizing the robust basic constraint qualification).** Suppose that for each $x \in F$ and $i \in I(x)$, the function $g_i$ satisfies the conditions (C3) and (C4). Then, the following statements are equivalent:

(i) the robust basic constraint qualification holds at $\bar{x} \in F$;
(ii) for each real-valued convex-concave function $f$ on $\mathbb{R}^n \times \mathcal{U}$, the following statements are equivalent:

(a) $\max_{u \in \mathcal{U}} f(x, u) \geq \max_{u \in \mathcal{U}} f(\bar{x}, u)$ for all $x \in F$,
(b) there exist $\bar{u} \in \mathcal{U}$, $\bar{\lambda}_i \geq 0$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ such that

$$0 \in \partial f(\cdot, \bar{u})(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}), \quad \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0, \quad \forall i = 1, \ldots, m,$$

(2)

and

$$f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u).$$

(3)

**Proof.** [(i) $\Rightarrow$ (ii)] Suppose that (i) holds. Let $f$ be a real-valued convex-concave function on $\mathbb{R}^n \times \mathcal{U}$. Firstly, we assume that (a) holds. Then, $\bar{x}$ is a solution of the following constrained convex optimization problem:

Minimize $\max_{u \in \mathcal{U}} f(x, u)$ subject to $x \in F$,

which can be equivalently expressed as,

$$0 \in \partial (\max_{u \in \mathcal{U}} f(\cdot, u))(\bar{x}) + N(F, \bar{x}).$$

By (i), there are $\bar{\lambda}_i \geq 0$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ such that

$$0 \in \partial (\max_{u \in \mathcal{U}} f(\cdot, u))(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x})$$

and

$$\bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0, \quad \forall i = 1, \ldots, m.$$ 

Then, it follows from Lemma 2.1 that there exists $\bar{u} \in \mathcal{U}$ such that (2) and (3) hold.
To prove sufficiency, assume that there exist $\bar{u} \in \mathcal{U}$, $\bar{\lambda}_i \geq 0$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ such that (2) and (3) hold. According to (2), we can find $\xi \in \partial f(\cdot, \bar{u})(\bar{x})$ and $\eta_i \in \partial^\circ g_i(\cdot, \bar{v}_i)(\bar{x})$, $i = 1, \ldots, m$, such that
\[
\xi + \sum_{i=1}^m \bar{\lambda}_i \eta_i = 0. \tag{4}
\]
It stems from $\xi \in \partial f(\cdot, \bar{u})(\bar{x})$ and $\eta_i \in \partial^\circ g_i(\cdot, \bar{v}_i)(\bar{x})$, $i = 1, \ldots, m$, we get
\[
f(x, \bar{u}) - f(\bar{x}, \bar{u}) \geq \langle \xi, x - \bar{x} \rangle \tag{5}
\]
and
\[
g_i^u(\bar{x}, \bar{v}_i; x - \bar{x}) \geq \langle \eta_i, x - \bar{x} \rangle \quad \forall i = 1, \ldots, m, \tag{6}
\]
for any $x \in \mathbb{R}^n$. Multiplying each of inequalities in (6) by $\bar{\lambda}_i$ and summing up the obtained inequalities with (5), we obtain that, for all $x \in \mathbb{R}^n$,
\[
f(x, \bar{u}) - f(\bar{x}, \bar{u}) + \sum_{i=1}^m \bar{\lambda}_i g_i^u(\bar{x}, \bar{v}_i; x - \bar{x}) \geq \langle \xi + \sum_{i=1}^m \bar{\lambda}_i \eta_i, x - \bar{x} \rangle.
\]
Taking (4) into account together with the condition $\bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0$, $i = 1, \ldots, m$, we deduce
\[
f(x, \bar{u}) - f(\bar{x}, \bar{u}) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i g_i^u(\bar{x}, \bar{v}_i; x - \bar{x}) \geq 0, \quad \forall x \in \mathbb{R}^n.
\]
Note that for each $i \in I(\bar{x})$ with $g_i(\bar{x}, \bar{v}_i) \neq 0$, $\bar{\lambda}_i = 0$. So, we consider in the case of $g_i(\bar{x}, \bar{v}_i) = 0$ for $i \in I(\bar{x})$, and hence $\bar{v}_i \in \mathcal{V}_i(\bar{x})$. By Remark 3, the last inequality becomes
\[
f(x, \bar{u}) - f(\bar{x}, \bar{u}) \geq 0 \quad \text{for all } x \in F.
\]
Thus, together with $\max_{u \in \mathcal{U}} f(x, u) \geq f(x, \bar{u})$ for all $x \in \mathbb{R}^n$ and (3), we obtain
\[
\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\bar{x}, u) \geq 0, \quad \forall x \in F.
\]
It means that $\bar{x}$ is a robust optimal solution of problem (UP).

[(ii) ⇒ (i)] The proof is similar to the one in [35, Theorem 3.1], and so is omitted. \hfill \Box

In the uncertainty free case, we can easily obtain the following result, which was obtained by Yamamoto and Kuroiwa in [37].

**Corollary 1.** [37, Theorem 3.2] Let $\bar{x} \in F' := \{x \in \mathbb{R}^n : g_i(x) \leq 0, \forall i = 1, \ldots, m\}$ be a feasible solution, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, be locally Lipschitz on $\mathbb{R}^n$. Assume further that for any $x \in F'$ and any $i = 1, \ldots, m$ such that $g_i(x) = 0$, the function $g_i$ is regular, and $F'$ is convex. Then the following statement are equivalent:

(i) $N(F', \bar{x}) = \bigcup_{\lambda_{g_i}(\bar{z}) = 0, i = 1, \ldots, m} \sum_{i=1}^m \lambda_i \partial^\circ g_i(\bar{x})$;

(ii) for each real-valued convex function $f$ on $\mathbb{R}^n$, the following statements are equivalent:

(a) $f(x) \geq f(\bar{x})$ for all $x \in F'$;

(b) there exist $\bar{\lambda}_i \geq 0, i = 1, \ldots, m$ such that

\[
0 \in \partial f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial^\circ g_i(\bar{x}) \quad \text{and} \quad \bar{\lambda}_i g_i(\bar{x}) = 0, \forall i = 1, \ldots, m.
\]
Remark 4. Both the robust Slater constraint qualification condition and robust non-degeneracy at $\bar{x}$, i.e.,

$$0 \notin \partial^o g_i(\cdot, v_i)(\bar{x})$$

whenever $i = 1, \ldots, m$ and $v_i \in \mathcal{V}_i$ such that $g_i(\bar{x}, v_i) = 0$, is a sufficient condition for the robust basic constraint qualification holds at $\bar{x}$. Indeed, according to Remark 3, we only have to show that

$$N(F, \bar{x}) \subseteq \bigcup_{\lambda_i \geq 0, v_i \in \mathcal{V}_i} \sum_{i=1}^m \lambda_i \partial^o g_i(\cdot, v_i)(\bar{x}).$$

Let $\eta \in N(F, \bar{x})$ be arbitrarily. Since the robust Slater constraint qualification condition and robust non-degeneracy are satisfied at $\bar{x}$, by Theorem 2.4 in [10] with $f := \langle -\eta, \cdot \rangle$, and $g_i := \psi_i$, $i = 1, \ldots, m$, there exist $\lambda_i \geq 0$, $i = 1, \ldots, m$, such that

$$0 \in -\eta + \sum_{i=1}^m \lambda_i \partial^o \psi_i(\bar{x}) \text{ and } \lambda_i \psi_i(\bar{x}) = 0, \forall i = 1, \ldots, m.$$ 

For $i \notin I(\bar{x})$, we get $\lambda_i = 0$. In the case of $i \in I(\bar{x})$, Lemma 2.7 together with $\lambda_i \psi_i(\bar{x}) = 0$, $\forall i = 1, \ldots, m$, implies there exist $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$, such that

$$\eta \in \sum_{i=1}^m \lambda_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}) \text{ and } \lambda_i g_i(\bar{x}, \bar{v}_i) = 0.$$ 

This shows that

$$\eta \in \bigcup_{\lambda_i \geq 0, v_i \in \mathcal{V}_i} \sum_{i=1}^m \lambda_i \partial^o g_i(\cdot, v_i)(\bar{x}),$$

the result as require.

The following example is given to illustrate the condition (i) of Theorem 3.3 is essential.

Example 3.2. Let $x := (x_1, x_2) \in \mathbb{R}^2$, $v_1 := (v_{1,1}, v_{1,2})$, $v_2 := (v_{2,1}, v_{2,2})$, $v_3 := (v_{3,1}, v_{3,2})$, $\mathcal{V}_1 := \{(v_1, v_2) \in \mathbb{R}^2 : v_1^2 + v_2^2 \leq 1\}$, $\mathcal{V}_2 := [0, 1] \times [1, 2]$, $\mathcal{V}_3 := [2, 3] \times [0, 1]$, $g_1(x, v_1) := v_{1,1} x_1 + v_{1,2} x_2 - x_1^2 - 2$, $g_2(x, v_2) := -v_{2,1} x_1^3 + v_{2,2} \max\{-x_2, -x_3^2\}$, $g_3(x, v_3) := v_{3,1} x_1 + v_{3,2} x_2$, $F := \{x \in \mathbb{R}^2 : g_1(x, v_1) \leq 0, g_2(x, v_2) \leq 0, g_3(x, v_3) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, 2, 3\}$ and $\bar{x} := (0, 0)$. Then $F = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} - x_1^2 - 2 \leq 0, 2x_1 + x_2 \leq 0, -x_1 - x_2 \leq 0\}$, $I(\bar{x}) = \{2, 3\}$, $\partial^o g_2(\cdot, v_2)(\bar{x}) = \{0\} \times [-v_{2,2}, 0]$ and $\partial^o g_3(\cdot, v_3)(\bar{x}) = \{(v_{3,1}, v_{3,2})\}$. It can be observed that

$$N(F, \bar{x}) = \text{cone } \{(-1, -1), (2, 1)\}$$

and

$$\bigcup_{\lambda_i \geq 0, v_i \in \mathcal{V}_i} \sum_{i=1}^3 \lambda_i \partial^o g_i(\cdot, v_i)(\bar{x}) = \text{cone } \{(0, -1), (2, 1)\}.$$ 

Hence, we have the condition (i) of Theorem 3.3 does not hold. Thus for some convex-concave function $f : \mathbb{R}^2 \times \mathcal{U} \to \mathbb{R}$, it is impossible to characterize a sufficient
condition for robust optimal solution for the following uncertain problem by using Theorem 3.3,

\[
\begin{align*}
\min_{x \in \mathbb{R}^2} f(x, u) & \quad \text{subject to } x \in \mathbb{R}^2, \ g_i(x, v_i) \leq 0, \ i = 1, 2, 3.
\end{align*}
\]

Actually, let \( u := (u_1, u_2) \) be an uncertain parameter belong to uncertainty set \( \mathcal{U} := \{(u_1, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 1 \} \), and \( f(x, u) := e^{x_1} - u_1 x_1 - u_2 x_2 \). Selecting \( \bar{u} := (1, 0), \bar{v}_1 := (1, 0), \bar{v}_2 := (1, 1), \bar{v}_3 := (2, 0), \bar{\lambda}_1 := 0, \bar{\lambda}_2 := 1 \) and \( \bar{\lambda}_3 := 1 \) we obtain \( \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0 \) for all \( i = 1, 2, 3 \),

\[
f(\bar{x}, \bar{u}) = 1 = \max_{u \in \mathcal{U}} f(\bar{x}, u)
\]

and

\[
(0, 0) \in \{(-2, 0) + \{0\} \times [-1, 0] + \{(2, 0)\} = \partial f(\cdot, \bar{u})(\bar{x}) + \sum_{i=1}^{3} \bar{\lambda}_i \partial^2 g_i(\cdot, \bar{v}_i)(\bar{x}).
\]

However, by taking \( x := (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \in F \), \( \max_{u \in \mathcal{U}} f(\bar{x}, u) = e^{1/2} - 1 < 1 = f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u) \) which shows \( \bar{x} \) is not a minimizer of \( \max_{u \in \mathcal{U}} f(\cdot, u) \) on \( F \).

**Remark 5.** According to Remark 4, Example 3.2 demonstrates that only robust Slater constraint qualification condition is not sufficient to ensure the robust basic constraint qualification holds at consideration point. The reason is that the robust non-degeneracy condition at such a point is destroyed.

4. **Characterizations of the robust solution sets.** In this section, we will establish some characterizations of robust optimal solution set in terms of a given robust solution point of the given problem.

We begin by recalling the following constrained convex optimization problem in the face of data uncertainty (UP):

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} \{ f(x, u) : g_i(x, v_i) & \leq 0, \ i = 1, \ldots, m \},
\end{align*}
\]

(UP)

where \( f : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R} \) is a convex-concave function, the functions \( g_i, i \in I \), satisfy the condition (C1) and (C2), \( g_i(x, \cdot) : \mathcal{V}_i \rightarrow \mathbb{R}, i \in I \), are concave functions for any \( x \in \mathbb{R}^n \), and the robust feasible set \( F \) is convex. Assume that the robust solution set of the problem (UP), denoted by

\[
S := \{ a \in F : \max_{u \in \mathcal{U}} f(a, u) \leq \max_{u \in \mathcal{U}} f(x, u), \forall x \in F \},
\]

is nonempty. In what follows, for any given \( y \in \mathbb{R}^n \), \( \lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \), \( u \in \mathcal{U} \), \( v_i \in \mathcal{V}_i, i \in I \) and \( v := (v_1, \ldots, v_m) \), we introduce the so-called pseudo Lagrangian-type function \( L^P(\cdot, y, \lambda, u, v) \) by, for all \( x \in \mathbb{R}^n \),

\[
L^P(x, y, \lambda, u, v) := f(x, u) + \sum_{i \in I(y)} \lambda_i g^P_{i \lambda}(y, v_i; x - y).
\]

Now, we show that the pseudo Lagrangian-type function associated with a Lagrange multiplier vector and uncertainty parameters according to a solution is constant on \( S \).

**Proposition 2.** Assume all conditions of Theorem 3.3 hold. Let \( a \in S \) be a robust optimal solution fulfilling the robust basic constraint qualification. Then there exist a Lagrange multiplier vector \( \lambda^a := (\lambda^a_1, \ldots, \lambda^a_m) \in \mathbb{R}^m_+ \), and uncertainty parameters \( u^a \in \mathcal{U}, v_i^a \in \mathcal{V}_i, i \in I \), such that for any \( x \in S \), \( \lambda^a_i g^P_{i \lambda}(a, v_i^a; x - a) = 0, \forall i \in I(a), f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u), \) and \( L^P(\cdot, a, \lambda^a, u^a, v^a) \) is constant on \( S \).
Remark 6. It is worth noting that if \( g_i(\cdot, v_i), i \in I, \) are convex functions for any \( v_i \in \mathcal{V}_i \), then, for each \( i \in I, \) Proposition 2 gives

\[
\lambda_i^a g_i(x, v_i^a) - \lambda_i^a g_i(x, v_i^a) \geq \lambda_i^a g_i(x, v_i^a; x - a) = \lambda_i^a g_i(x, v_i^a; x - a) = 0 \quad \text{for any } x \in S.
\]

This together with \( x \in F \) and \( \lambda_i^a g_i(x, v_i^a) = 0, i \in I, \) arrives \( \lambda_i^a g_i(x, v_i^a) = 0, i \in I. \) Furthermore, it yields

\[
L^P(x, a, \lambda^a, u^a, v^a) = f(x, u^a) + \sum_{i \in I(a)} \lambda_i^a g_i^a(x, v_i^a; x - a)
\]

\[
= f(x, u^a)
\]

\[
= f(x, u^a) + \sum_{i=1}^m \lambda_i^a g_i(x, v_i^a), \forall x \in S.
\]

This shows that pseudo Lagrangian-type function collapses to the well-known Lagrangian-type function on the robust solution set \( S. \)
In the sequel, we are now in a position to establish the characterizations of the robust solution set for problem (UP) in terms of convex subdifferentials, Clarke subdifferentials and Lagrange multipliers. But before doing so it will thus be convenient to denote the following:

\[ \bar{I}(a) := \{ i \in I(a) : \lambda^a_i > 0 \}, \]

\[ C(x) := \{ \xi \in \partial f(\cdot, u^a)(x) : \langle \xi, x - a \rangle \geq 0 \} \text{ for any given } x \in F. \]

**Theorem 4.1 (Characterizing the robust solution set).** Assume all conditions of Theorem 3.3 hold. Let \( a \in S \) be a robust optimal solution fulfilling the robust basic constraint qualification. Then there exist a Lagrange multiplier vector \( \lambda \in \mathbb{R}^{n^a}_+ \), and uncertainty parameters \( u^a \in \mathcal{U}, v^a_i \in \mathcal{V}_i, i \in I, \) such that the robust solution set for the problem (UP) is characterized by

\[
S = S_1 = S_2 = S_3 = S_4 = S_5 = S_6 = S_7,
\]

where

\[
S_1 := \{ x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^a g_i(\cdot, v^a_i)(a), \forall i \in \bar{I}(a); \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \cap \partial f(\cdot, u^a)(a); f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u) \},
\]

\[
S_2 := \{ x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^a g_i(\cdot, v^a_i)(a), \forall i \in \bar{I}(a); \langle \zeta, a - x \rangle \geq 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \cap \partial f(\cdot, u^a)(a); f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u) \},
\]

\[
S_3 := \{ x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^a g_i(\cdot, v^a_i)(a), \forall i \in \bar{I}(a); \langle \xi, x - a \rangle = \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \text{ and } \xi \in C(x); f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u) \},
\]

\[
S_4 := \{ x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^a g_i(\cdot, v^a_i)(a), \forall i \in \bar{I}(a); \langle \xi, x - a \rangle = \langle \zeta, a - x \rangle \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \text{ and } \xi \in C(x); f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u) \},
\]

\[
S_5 := \{ x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^a g_i(\cdot, v^a_i)(a), \forall i \in \bar{I}(a); \langle \xi, x - a \rangle \leq \langle \zeta, a - x \rangle \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \text{ and } \xi \in C(x); f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u) \},
\]

\[
S_6 := \{ x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^a g_i(\cdot, v^a_i)(a), \forall i \in \bar{I}(a); \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x); f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u) \},
\]

\[
S_7 := \{ x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^a g_i(\cdot, v^a_i)(a), \forall i \in \bar{I}(a); \langle \zeta, a - x \rangle \geq 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x); f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u) \}.
\]

**Proof.** Evidently, the following containments hold:

\[ S_1 \subseteq S_2 \subseteq S_7, \]
Hence, we only have to show that $S \subseteq S_1$ and $S_7 \subseteq S$. In order to establish $S \subseteq S_1$, let $x \in S$ be arbitrarily given. It follows from (2), we therefore obtain vectors $\zeta \in \partial f(\cdot, u^a)(a)$ and $\xi_i \in \partial^o g_i(\cdot, v^a_i)(a)$, $i \in I(a)$, such that
\[
\zeta + \sum_{i \in I(a)} \lambda^o_i \xi_i = 0
\]  
(since $\lambda^o_i = 0$ for $i \notin I(a)$). According to $\zeta \in \partial f(\cdot, u^a)(a)$, $\xi_i \in \partial^o g_i(\cdot, v^a_i)(a)$, $i \in I(a)$, and $x, a \in S$, one has
\[
f(x, u^a) - f(a, u^a) \geq \langle \zeta, x - a \rangle
\]  
and
\[
g^a_i(a, v^a_i; x - a) \geq \langle \xi_i, x - a \rangle, \forall i \in I(a).
\]  
Once we have shown, in Proposition 2, that $\lambda^o_i g^a_i(a, v^a_i; x - a) = 0$, $\forall i \in I(a)$, after multiplying both sides of (13) by $\lambda^o_i$, $i \in I(a)$ we get
\[
0 \geq \langle \lambda^o_i \xi_i, x - a \rangle, \forall i \in I(a).
\]  
Summing up these inequalities and using (11) we obtain that
\[
0 \geq \left( \sum_{i \in I(a)} \lambda^o_i \xi_i, x - a \right) = \langle -\zeta, x - a \rangle.
\]  
Again, it follows from Proposition 2 that
\[
f(x, u^a) = \max_{u \in U} f(x, u),
\]  
and for each $i \in I(a)$, $\max_{\eta_i \in \partial^o g_i(\cdot, v^a_i)(a)} \langle \eta_i, x - a \rangle = g^a_i(a, v^a_i; x - a) = 0$, the latter which in turn leads to there exists $\eta_i \in \partial^o g_i(\cdot, v^a_i)(a)$ such that
\[
\langle \eta_i, x - a \rangle = 0.
\]  
On the one hand, taking (3) and (15) into account (12) we obtain
\[
\langle \zeta, x - a \rangle \leq f(x, u^a) - f(a, u^a) = \max_{u \in U} f(x, u) - \max_{u \in U} f(a, u) = 0.
\]  
This together with (14) arrives at
\[
\langle \zeta, x - a \rangle = 0.
\]  
Now, we only need to prove that $\zeta \in \partial f(\cdot, u^a)(x)$. In fact, for any $y \in \mathbb{R}^n$,
\[
f(y, u^a) - f(x, u^a) = f(y, u^a) - f(a, u^a)
\]
\[
\geq \langle \zeta, y - a \rangle
\]
\[
= \langle \zeta, y - x \rangle + \langle \zeta, x - a \rangle = \langle \zeta, y - x \rangle,
\]  
which means $\zeta \in \partial f(\cdot, u^a)(x)$ and so, $x \in S_1$. This proves $S \subseteq S_1$.

To obtain $S_7 \subseteq S$, we now let $x$ be arbitrary point of $S_7$. It follows that $x \in F$, and it is easy to see that
\[
\max_{u \in U} f(a, u) - \min_{u \in U} f(x, u) = f(a, u^a) - f(x, u^a) \geq \langle \zeta, a - x \rangle \geq 0.
\]  
The last inequality together with the fact that $a \in S$ gives $x \in S$, and the proof is complete. \qed
Now, we give the following example to illustrate the significance of Theorem 4.1 that at least one of the constraint functions \(g_i(\cdot, v_i)\) for some \(v_i \in \mathcal{V}_i\), is not convex while the robust feasible set is convex. Then the results in [15, 35, 34, 27] may not be relevant to this example.

**Example 4.1.** Let us denote \(\mathbf{x} := (x_1, x_2) \in \mathbb{R}^2\), \(u := (u_1, u_2)\), \(v_1 := (v_{1,1}, v_{1,2})\), \(v_2 := (v_{2,1}, v_{2,2})\), \(v_3 := (v_{3,1}, v_{3,2})\), \(\mathcal{U} := \{(u_1, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 1\}\), \(\mathcal{V}_1 := \{(v_{1,1}, v_{1,2}) \in \mathbb{R}^2 : v_{1,1}^2 + v_{1,2}^2 \leq 1\}\), \(\mathcal{V}_2 := [0, 1] \times [1, 2]\) and \(\mathcal{V}_3 := [0, 1] \times [0, 1]\). Consider the following constrained optimization problem with uncertainty data (UP):

\[
\text{Minimize } f(x, u) \quad \text{(UP)}
\]

subject to \(x \in \mathbb{R}^2\), \(g_1(x, v_1) \leq 0\), \(g_2(x, v_2) \leq 0\), \(g_3(x, v_3) \leq 0\).

where \(u \in \mathcal{U}\), \(v_i \in \mathcal{V}_i\), \(i = 1, 2, 3\),

\[
\begin{align*}
    f(x, u) &= u_1 x_1 + u_2 x_2 - x_1 - x_2, \\
    g_1(x, v_1) &= v_{1,1} x_1 + v_{1,2} x_2 - x_1^3 - 2, \\
    g_2(x, v_2) &= v_{2,1} \max(-x_1, -x_1^3) - v_{2,2} x_2, \\
    g_3(x, v_3) &= v_{3,1} x_1 - v_{3,2} x_2.
\end{align*}
\]

A robust solution of (UP) is obtained by solving its robust (worst-case) counterpart (RP)

\[
\text{Minimize } \max_{u \in \mathcal{U}} f(x, u) \quad \text{(RP)}
\]

subject to \(x \in \mathcal{F} := \left\{ \begin{array}{l}
g_1(x, v_1) \leq 0, \quad \forall v_1 \in \mathcal{V}_1, \\
g_2(x, v_2) \leq 0, \quad \forall v_2 \in \mathcal{V}_2, \\
g_3(x, v_3) \leq 0, \quad \forall v_3 \in \mathcal{V}_3,
\end{array} \right\}
\]

Then \(\mathcal{F} = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} - x_1 - x_2 \leq 0, \quad -x_1 - x_2 \leq 0, \quad x_1 \leq 0\}\). Evidently, the function \(f : \mathbb{R}^2 \times \mathcal{U} \to \mathbb{R}\) is a convex-concave function. Let us notice that

\[
\max_{u \in \mathcal{U}} f(x, u) = \sqrt{x_1^2 + x_2^2} - x_1 - x_2, \quad \text{for all } x \in \mathbb{R}^2,
\]

and

\[
\max_{u \in \mathcal{U}} f(x, u) \geq |x_2| - x_2 = 0 = \max_{u \in \mathcal{U}} f((0, 0), u), \quad \text{for all } x \in \mathcal{F}.
\]

Thus \(a := (a_1, a_2) = (0, 0) \in S\), \(I(a) = \{2, 3\}\), \(\partial^a g_2(\cdot, v_2)(a) = \{(r, -v_{2,2}) : -v_{2,1} \leq r \leq 0\}\) for each \(v_2 \in \mathcal{V}_2\) and \(\partial^a g_3(\cdot, v_3)(a) = \{(v_{3,1}, 0)\}\) for each \(v_3 \in \mathcal{V}_3\). So,

\[
N(F, a) = \text{cone}\{(-1, -1), (1, 0)\} = \bigcup_{\lambda_i \geq 0, \forall i \in I} \sum_{i=1}^{3} \lambda_i \partial^a g_i(\cdot, v_i)(a),
\]

which means that the robust basic constraint qualification holds at \(a\). Also, for each \(u \in \mathcal{U}\), the convex subdifferential of \(f(\cdot, u)\) at any point \(x\) is given by

\[
\partial f(\cdot, u)(x) = (u_1 - 1, u_2 - 1).
\]

Let us select \(\lambda^a := (\lambda^a_1, \lambda^a_2, \lambda^a_3) = (0, 0, 1)\), \(u^a := (0, 1)\), \(v^a_2 := (1, 1)\) and \(v^a_3 := (1, 0)\). Therefore, \(\bar{f}(a) = \{3\}\) and by solving the following system, for \(x \in \mathbb{R}^2\)
Using (16) and taking \(y\) the definition of \(S\) means that
\[
\begin{aligned}
\text{if } f(\cdot, u^a)(x) \cap \partial f(\cdot, u^a)(a) = \{(-1, 0)\}, \\
\langle(-1, 0), (0, x_2)\rangle = 0, \\
-x_1 = \max_{u \in U} f(x, u),
\end{aligned}
\]
the robust solution set can be described simply as
\[
S_S = S_1 = \{x \in \mathbb{R}^2 : x_1 = 0, 0 \leq x_2 \leq 2\}.
\]

With the help of Proposition 2, we see now how the robust solution set can be characterized in terms of pseudo Lagrangian-type function.

**Proposition 3.** Assume all conditions of Theorem 3.3 hold. Let \(a \in S\) be a robust optimal solution fulfilling the robust basic constraint qualification. Then there exist a Lagrange multiplier vector \(\lambda^a := (\lambda_1^a, \ldots, \lambda_m^a) \in \mathbb{R}_+^m\), and uncertainty parameters \(u^a \in U, v_i^a \in V_i, i \in I\), such that
\[
S = \{x \in F : \langle \eta_i, x-a \rangle = 0, \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in I(a); \\
0 \in \partial L^P(\cdot, a, \lambda^a, u^a, v^a)(x) \text{ and } f(x, u^a) = \max_{u \in U} f(x, u)\}.
\]

**Proof.** It will thus be convenient to denote
\[
S^* : = \{x \in F : \langle \eta_i, x-a \rangle = 0, \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in I(a); \\
0 \in \partial L^P(\cdot, a, \lambda^a, u^a, v^a)(x) \text{ and } f(x, u^a) = \max_{u \in U} f(x, u)\}.
\]

By Proposition 2, we have that for each \(x \in S\), \(\lambda_i^a g_{i,x}^a(a, v_i^a; x-a) = 0, \forall i \in I(a), \)
\(f(x, u^a) = \max_{u \in U} f(x, u)\), and \(L^P(\cdot, a, \lambda^a, u^a, v^a)\) is constant on \(S\). The latter means that
\[
\partial L^P(\cdot, a, \lambda^a, u^a, v^a)(x) = \{0\},
\]
and so, \(S \subseteq S^*\). To obtain the converse inclusion, let \(x \in S^*\) be given. Then, by the definition of \(S^*\), \(x \in F\), there exist \(\eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in I(a)\), such that
\[
\langle \eta_i, x-a \rangle = 0, \forall i \in I(a),
\]
\(f(x, u^a) = \max_{u \in U} f(x, u)\) (16)
and
\[
\begin{aligned}
f(y, u^a) + \sum_{i \in I(a)} \lambda_i^a g_{i,x}^a(a, v_i^a; y-a) &= L^P(y, a, \lambda^a, u^a, v^a) \\
&\leq L^P(x, a, \lambda^a, u^a, v^a) \\
&= f(x, u^a) + \sum_{i \in I(a)} \lambda_i^a g_{i,x}^a(a, v_i^a; x-a) \\
&= f(x, u^a) + \sum_{i \in I(a)} \lambda_i^a g_{i,x}^a(a, v_i^a; x-a) \\
&= f(x, u^a) \text{ for all } y \in \mathbb{R}^n.
\end{aligned}
\]

Using (16) and taking \(y = a\) in the last inequality, we get that
\[
\max_{u \in U} f(x, u) \geq \max_{u \in U} f(a, u) \geq f(a, u^a) \geq f(x, u^a) = \max_{u \in U} f(x, u).
\]
Hence,
\[
\max_{u \in \mathcal{U}} f(a, u) = \max_{u \in \mathcal{U}} f(x, u),
\]
which is noting else than \( x \in S \).

In the special case when \( \mathcal{U} \) and \( \mathcal{V}_i, i \in I \), are singletons, we can easily obtain the following results.

**Corollary 2.** For the problem \((P)\), let \( f : \mathbb{R}^n \to \mathbb{R} \) be convex function and \( F' := \{ x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I \} \) be convex. Assume that for any \( x \in F' \) and \( i \in I'(x) := \{ i \in I : g_i(x) = 0 \} \) the functions \( g_i \) are locally Lipschitz and regular in the sense of Clarke, \( a \in S' \) is an optimal solution fulfilling \( N(F', a) = \text{cone} \bigcup_{i \in I'(a)} \partial^0 g_i(a) \), and there exists a Lagrange multiplier vector \( \lambda^a := (\lambda_1^a, \ldots, \lambda_m^a) \in \mathbb{R}_+^m \) such that
\[
0 \in \partial f(a) + \sum_{i=1}^m \lambda_i^a \partial^0 g_i(a) \quad \text{and} \quad \lambda_i^a g_i(a) = 0, \quad \forall i = 1, \ldots, m.
\]

Let further \( \tilde{I}(a)' := \{ i \in I'(a) : \lambda_i^a > 0 \} \) and \( C(x)' := \{ \xi \in \partial f(a) : \langle \xi, x - a \rangle \geq 0 \} \) for any given \( x \in F' \). Then, the solution set \( S' \) of the problem \((P)\) is characterized by
\[
S' = S'_1 = S'_2 = S'_3 = S'_4 = S'_5 = S'_6 = S'_7,
\]
where
\[
S'_1 := \{ x \in F' : \langle \eta, x - a \rangle = 0 \quad \text{for some} \quad \eta_i \in \partial^0 g_i(a), \quad \forall i \in \tilde{I}(a) \};
\]
\[
\langle \zeta, a - x \rangle = 0 \quad \text{for some} \quad \zeta \in \partial f(x) \cap \partial f(a),
\]
\[
S'_2 := \{ x \in F' : \langle \eta, x - a \rangle = 0 \quad \text{for some} \quad \eta_i \in \partial^0 g_i(a), \quad \forall i \in \tilde{I}(a) \};
\]
\[
\langle \zeta, a - x \rangle \geq 0 \quad \text{for some} \quad \zeta \in \partial f(x) \cap \partial f(a),
\]
\[
S'_3 := \{ x \in F' : \langle \eta, x - a \rangle = 0 \quad \text{for some} \quad \eta_i \in \partial^0 g_i(a), \quad \forall i \in \tilde{I}(a) \};
\]
\[
\langle \xi, x - a \rangle = \langle \zeta, a - x \rangle = 0 \quad \text{for some} \quad \zeta \in \partial f(x) \quad \text{and} \quad \xi \in C(x)',
\]
\[
S'_4 := \{ x \in F' : \langle \eta, x - a \rangle = 0 \quad \text{for some} \quad \eta_i \in \partial^0 g_i(a), \quad \forall i \in \tilde{I}(a) \};
\]
\[
\langle \zeta, a - x \rangle = \langle \xi, x - a \rangle \quad \text{for some} \quad \zeta \in \partial f(x) \quad \text{and} \quad \xi \in C(x)',
\]
\[
S'_5 := \{ x \in F' : \langle \eta, x - a \rangle = 0 \quad \text{for some} \quad \eta_i \in \partial^0 g_i(a), \quad \forall i \in \tilde{I}(a) \};
\]
\[
\langle \xi, x - a \rangle \leq \langle \zeta, a - x \rangle \quad \text{for some} \quad \zeta \in \partial f(x) \quad \text{and} \quad \xi \in C(x)',
\]
\[
S'_6 := \{ x \in F' : \langle \eta, x - a \rangle = 0 \quad \text{for some} \quad \eta_i \in \partial^0 g_i(a), \quad \forall i \in \tilde{I}(a) \};
\]
\[
\langle \zeta, a - x \rangle = 0 \quad \text{for some} \quad \zeta \in \partial f(x),
\]
\[
S'_7 := \{ x \in F' : \langle \eta, x - a \rangle = 0 \quad \text{for some} \quad \eta_i \in \partial^0 g_i(a), \quad \forall i \in \tilde{I}(a) \};
\]
\[
\langle \zeta, a - x \rangle \geq 0 \quad \text{for some} \quad \zeta \in \partial f(x).
\]

**Corollary 3.** For the problem \((P)\), let \( f : \mathbb{R}^n \to \mathbb{R} \) be convex function and \( F' := \{ x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I \} \) be convex. Assume that for any \( x \in F' \) and \( i \in I'(x) \) the functions \( g_i \) are locally Lipschitz and regular in the sense of Clarke, \( a \in S' \) is an optimal solution fulfilling \( N(F', a) = \text{cone} \bigcup_{i \in I'(a)} \partial^0 g_i(a) \), and the optimality conditions \((17)\) hold with a Lagrange multiplier vector \( \lambda^a := (\lambda_1^a, \ldots, \lambda_m^a) \in \mathbb{R}_+^m \). Then,
\[
S' = \{ x \in F' : \langle \eta, x - a \rangle = 0, \exists \eta_i \in \partial^0 g_i(a), \quad \forall i \in \tilde{I}(a)' \quad \text{and} \quad 0 \in \partial L'(\cdot, a, \lambda^a)(x) \},
\]
where \( L'(x, a, \lambda^a) := f(x) + \sum_{i \in I'(a)} \lambda_i^a g_i^0(a; x - a) \).
5. Application to robust multi-objective optimization problems. In this section, as an application of the general results of the previous section, we examine the class of multiple-objective programs in the face of data uncertainty both in the objective and constraints that can be written by the following multi-objective optimization problem:

\[
\min \{ (f_1(x, u_1), \ldots, f_p(x, u_p)) : g_i(x, v_i) \leq 0, \ i = 1, \ldots, m, \}
\]

(UMP)

where \( f_k : \mathbb{R}^n \times \mathbb{R}^{q_k} \to \mathbb{R}, \ k = 1, \ldots, p, \) are convex-concave functions, \( g_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \ i = 1, \ldots, m, \) are functions satisfying the condition (C1) and (C2), \( g_i(x, \cdot) \) are concave functions for any \( x \in \mathbb{R}^n, \) and \( u_k \) and \( v_i \) are uncertain parameters and they belong to nonempty convex compact sets \( U_k \subseteq \mathbb{R}^{q_k} \) and \( V_i \subseteq \mathbb{R}^n, \) respectively.

We associate with (UMP) its robust counterpart, which is the worst case of (UMP),

\[
\min \{ \left( \max_{u_1 \in U_1} f_1(x, u_1), \ldots, \max_{u_p \in U_p} f_p(x, u_p) \right) : x \in F \},
\]

(RMP)

where \( F \) stands for the robust feasible set of (UMP), defined by

\[
F := \{ x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \ \forall v_i \in V_i, \ i = 1, \ldots, m \}.
\]

In the same way, we will give three kind robust solutions for the problems (UMP) which has been introduced in [22].

\( \bar{x} \in F \) is said to be a robust efficient solution of (UMP) if there does not exist a robust feasible solution \( x \) of (UMP) such that

\[
\max_{u_k \in U_k} f_k(x, u_k) \leq \max_{u_k \in U_k} f_k(\bar{x}, u_k) \text{ for all } k = 1, \ldots, p,
\]

and

\[
\max_{u_l \in U_l} f_l(x, u_l) < \max_{u_l \in U_l} f_l(\bar{x}, u_l) \text{ for some } l.
\]

\( \bar{x} \in F \) is called a weakly robust efficient solution of (UMP) if there does not exist a robust feasible solution \( x \) of (UMP) such that

\[
\max_{u_k \in U_k} f_k(x, u_k) < \max_{u_k \in U_k} f_k(\bar{x}, u_k) \text{ for all } k = 1, \ldots, p.
\]

\( \bar{x} \in F \) is said to be a properly robust efficient solution of (UMP) if it is a robust efficient solution of (UMP) and there is a number \( M > 0 \) such that for all \( k \in \{1, \ldots, p\} \) and \( x \in F \) satisfying \( \max_{u_k \in U_k} f_k(x, u_k) < \max_{u_k \in U_k} f_k(\bar{x}, u_k) \), there exists an index \( l \in \{1, \ldots, p\} \) such that \( \max_{u_l \in U_l} f_l(x, u_l) < \max_{u_l \in U_l} f_l(\bar{x}, u_l) \) and

\[
\max_{u_k \in U_k} f_k(\bar{x}, u_k) - \max_{u_k \in U_k} f_k(x, u_k) \leq M.
\]

According to these definitions, it is evidently that \( \bar{x} \in F \) is a robust efficient solution (resp. weakly, properly robust efficient solution) of (UMP) if and only if \( \bar{x} \in F \) is a efficient solution (resp. weakly, properly efficient solution) of (RMP). The search for an efficient solution (resp. weakly, properly efficient solution) to multi-objective optimization problem has been carried out through solving a single (scalar) or a family of single objective optimization problems, possibly depending on some appropriate parameters. We refer the reader to [28, 12, 31, 20] and other references therein for necessary and sufficient conditions for (weakly, properly) efficient solutions to a multiobjective optimization by parameterization and linear scalarization (weighted sum approach).

In this section, we present characterizations of weakly robust efficient solution set \((WRS(F))\) and properly robust efficient solution set \((PRS(F))\) of the problem.
For the problem (UMP) by using linear scalarization approach. Before presenting, in the cases of study, let us consider the following scalar convex problem of (RMP) depending on a parameter $\theta := (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p$:

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{k=1}^{p} \theta_k \max_{u_k \in U_k} f_k(x, u_k) : x \in F \right\}. \quad (RP_\theta)$$

Suppose that the solution set of problem (RP_\theta), denoted by $S_\theta$ is nonempty. It is well-known, in the literature, that weakly efficient solutions and properly efficient solutions of (RMP) can be characterized by solving some scalar parameterized convex problems (RP_\theta). More precisely,

(i) $\bar{x} \in WR(F)$ if and only if there exists $\theta \in \mathbb{R}^p_+ \setminus \{0\}$ such that $\bar{x} \in S_\theta$.

(ii) $\bar{x} \in PR(F)$ if and only if there exists $\theta \in \text{int} \mathbb{R}^p_+$ such that $\bar{x} \in S_\theta$.

Thus, by using Theorem 3.3, we can obtain immediately the following necessary and sufficient optimality conditions for weakly robust efficient solution as well as properly robust efficient solution of (UMP).

**Theorem 5.1.** For the problem (UMP), suppose all conditions of Theorem 3.3 hold and $\bar{x} \in F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in V_i, i \in I\}$ fulfilling the robust basic constraint qualification. Assume further that the set $F$ is convex. Then,

(i) $\bar{x} \in F$ is a weakly robust efficient solution of (UMP) if and only if there exist $\theta_k \geq 0, k = 1, \ldots, p$, not all zero, $\lambda_i \geq 0, i = 1, \ldots, m$, $\bar{u}_k \in U_k$, $k = 1, \ldots, p$ and $\bar{v}_i \in V_i$, $i = 1, \ldots, m$ such that

$$0 \in \sum_{k=1}^{p} \theta_k \partial f_k(\cdot, \bar{u}_k)(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial^p g_i(\cdot, \bar{v}_i)(\bar{x}),$$

$$\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \ \forall i = 1, \ldots, m, \ \text{and}$$

$$f_k(\bar{x}, \bar{u}_k) = \max_{u_k \in U_k} f_k(\bar{x}, u_k), \ k = 1, \ldots, p.$$ 

(ii) $\bar{x} \in F$ is a properly robust efficient solution of (UMP) if and only if there exist $\theta_k > 0, k = 1, \ldots, p$, $\lambda_i \geq 0, i = 1, \ldots, m$, $\bar{u}_k \in U_k$, $k = 1, \ldots, p$ and $\bar{v}_i \in V_i$, $i = 1, \ldots, m$ such that

$$0 \in \sum_{k=1}^{p} \theta_k \partial f_k(\cdot, \bar{u}_k)(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial^p g_i(\cdot, \bar{v}_i)(\bar{x}),$$

$$\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \ \forall i = 1, \ldots, m, \ \text{and}$$

$$f_k(\bar{x}, \bar{u}_k) = \max_{u_k \in U_k} f_k(\bar{x}, u_k), \ k = 1, \ldots, p.$$ 

**Proof.** (i) As $\bar{x} \in F$ is a weakly robust efficient solution of (UMP) if and only if $\bar{x} \in F$ is a weakly efficient solution of (RMP), there exist $\theta_k \geq 0, k = 1, \ldots, p$, not all zero, such that $\bar{x} \in F$ is a solution of (RP_\theta). In the other word, $\bar{x} \in F$ is a robust solution of the following uncertain (only in the constraints) convex optimization problem:

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{k=1}^{p} \theta_k \max_{u_k \in U_k} f_k(x, u_k) : g_i(x, v_i) \leq 0, \ i \in I \right\}.$$ 

Applying Theorem 3.3, we get that there exist $\bar{\lambda}_i \geq 0$, and $\bar{v}_i \in V_i$, $i \in I$ such that

$$0 \in \partial \left( \sum_{k=1}^{p} \theta_k \max_{u_k \in U_k} f_k(\cdot, u_k) \right)(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial^p g_i(\cdot, \bar{v}_i)(\bar{x}), \ \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0, \ \forall i \in I.$$ 

By employing the summation, positively homogeneous and max-function of convex subdifferential rule, the result as required.

(ii) The proof of (ii) is quite similar to that of (i) and so is omitted. \qed

In the following proposition, we give a sufficient condition that a robust efficient solution of (UMP) can be a properly robust efficient solution of (UMP).

**Proposition 4.** For the problem (UMP), let \( \bar{x} \in F \) be a robust feasible solution for (UMP). Assume all conditions of Theorem 3.3 hold. Assume further that the set \( F \) is convex, \( F \cap F(\bar{x}) \neq \emptyset \) and

\[
N(F \cap F(\bar{x}), \bar{x}) = \text{cone} \left\{ \left( \bigcup_{u_k \in U_k(\bar{x})} \partial f_k(\cdot, u_k)(\bar{x}) \right) \bigcup \left( \bigcup_{v_i \in V_i(\bar{x})} \partial^p g_i(\cdot, v_i)(\bar{x}) \right) \right\},
\]

where

\[
F(\bar{x}) := \{ x \in \mathbb{R}^n : \max_{u_k \in U_k} f_k(\bar{x}, u_k) \leq \max_{u_k \in U_k} f_k(\bar{x}, u_k), \forall k = 1, \ldots, p \}, \text{ and }
\]

\[
U_k(\bar{x}) := \{ \bar{u}_k \in U_k : f_k(\bar{x}, \bar{u}_k) = \max_{u_k \in U_k} f_k(\bar{x}, u_k) \}, k = 1, \ldots, p,
\]

\[
V_i(\bar{x}) := \{ \bar{v}_i \in V_i : g_i(\bar{x}, \bar{v}_i) = \max_{v_i \in V_i} g_i(\bar{x}, v_i) \}, i \in I(\bar{x}).
\]

If \( \bar{x} \) is a robust efficient solution of (UMP), then \( \bar{x} \) is a properly robust efficient solution of (UMP).

**Proof.** Let \( \bar{x} \) be a robust efficient solution of (UMP). Then \( \bar{x} \) is a minimizer of the following scalar convex problem:

\[
\min_{x \in \mathbb{R}^n} \left\{ \sum_{k=1}^p \max_{u_k \in U_k} f_k(x, u_k) : x \in F \cap F(\bar{x}) \right\},
\]

or equivalently,

\[
0 \in \sum_{k=1}^p \partial(\max_{u_k \in U_k} f_k(\cdot, u_k))(\bar{x}) + N(F \cap F(\bar{x}), \bar{x}).
\]

It follows that there exists \( \eta \in N(F \cap F(\bar{x}), \bar{x}) \) such that

\[
-\eta \in \sum_{k=1}^p \partial(\max_{u_k \in U_k} f_k(\cdot, u_k))(\bar{x}).
\]

Then, by the condition (18), there exist \( \theta_k \geq 0, \bar{u}_k \in U_k, \xi_k \in \partial f_k(\cdot, \bar{u}_k)(\bar{x}), k = 1, \ldots, p, \lambda_i \geq 0, \bar{v}_i \in V_i(\bar{x}) \) and \( \varsigma_i \in \partial^p g_i(\cdot, \bar{v}_i)(\bar{x}), i \in I(\bar{x}), \) such that

\[
\eta = \sum_{k=1}^p \theta_k \xi_k + \sum_{i \in I(\bar{x})} \lambda_i \varsigma_i
\]

and

\[
f_k(\bar{x}, \bar{u}_k) = \max_{u_k \in U_k} f_k(\bar{x}, u_k), \forall k = 1, \ldots, p.
\]

which implies that

\[
0 = -\eta + \sum_{k=1}^p \partial(\max_{u_k \in U_k} f_k(\cdot, u_k))(\bar{x}) + \sum_{k=1}^p \theta_k \partial f_k(\cdot, \bar{u}_k)(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial^p g_i(\cdot, \bar{v}_i)(\bar{x}).
\]
\[ \sum_{k=1}^{p} \partial( \max_{u_k \in U_k} f_k(\cdot, u_k))(\bar{x}) + \sum_{k=1}^{p} \theta_k \left( \bigcup_{u_k \in U_k} \partial f_k(\cdot, u_k)(\bar{x}) \right) + N(F, \bar{x}) \]

\[ = \sum_{k=1}^{p} \partial( \max_{u_k \in U_k} f_k(\cdot, u_k))(\bar{x}) + \sum_{k=1}^{p} \theta_k \partial( \max_{u_k \in U_k} f_k(\cdot, u_k))(\bar{x}) + N(F, \bar{x}) \]

\[ = \sum_{k=1}^{p} (1 + \theta_k) \partial( \max_{u_k \in U_k} f_k(\cdot, u_k))(\bar{x}) + N(F, \bar{x}). \]

Therefore,

\[ \sum_{k=1}^{p} (1 + \theta_k) \max_{u_k \in U_k} f_k(x, u_k) \geq \sum_{k=1}^{p} (1 + \theta_k) \max_{u_k \in U_k} f_k(\bar{x}, u_k) \text{ for all } x \in F, \]

which gives that \( \bar{x} \in S_{\hat{\theta}} \) with \( \hat{\theta} := (1+\theta_1, \ldots, 1+\theta_p) \in \interior \mathbb{R}_+^p \), and so \( \bar{x} \) is a properly robust efficient solution of (UMP).

Let \( \theta \in \mathbb{R}_+^p \setminus \{0\} \) (resp. \( \interior \mathbb{R}_+^p \)) and \( a^0 \in S_{\theta} \). We have seen already that if the robust basic constraint qualification holds at \( a^0 \), the set of (RPs) for \( (RPs) \) corresponding to \( a^0 \), given as

\[ M(a^0) := \left\{ (\lambda^0, u^0, v^0) \in \mathbb{R}_+^p \times \prod_{k=1}^{p} \mathbb{R}_+^n \times \prod_{i=1}^{m} \mathbb{R}_+^q : 0 \in \sum_{k=1}^{p} \theta_k \partial f_k(\cdot, u^0_k)(a^0) + \sum_{i=1}^{m} \lambda_i^0 \partial^0 g_i(\cdot, v^0_i)(a^0), \right. \]

\[ \left. \lambda_i^0 g_i(a^0, v^0_i) = 0, \forall i \in I \text{ and } f_k(a^0, u^0_k) = \max_{u_k \in U_k} f_k(a^0, u_k), \forall k = 1, \ldots, p \right\}, \]

is nonempty where \( \lambda^0 := (\lambda_1^0, \ldots, \lambda_m^0) \), \( u^0 := (u_1^0, \ldots, u_p^0) \) and \( v^0 := (v_1^0, \ldots, v_m^0). \)

Let further \( I(a^0) := \{ i \in I : \exists v^0_i \in \mathcal{V}_i \text{ such that } g_i(a^0, v^0_i) = 0 \} \) and \( \bar{I}(a^0) := \{ i \in I(a^0) : \lambda_i^0 > 0 \}. \)

By means of linear scalarization applied in Theorem 4.1, we can get characterizations of the weakly robust efficient solution set \( WR(F) \) and properly robust efficient solution set \( PR(F) \) of the problem (UMP) immediately.

**Theorem 5.2.** For the problem (UMP), assume all conditions of Theorem 3.3 hold, and the set \( F \) is convex.

(i) Suppose further that for each \( \theta \in \mathbb{R}_+^p \setminus \{0\} \), \( S_{\theta} \) is non-empty. Let \( a^0 \in S_{\theta} \) and the robust basic constraint qualification holds at \( a^0 \). Let \( (\lambda^0, u^0, v^0) \in M(a^0). \text{ Then} \)

\[ WR(F) = \bigcup_{\theta \in \mathbb{R}_+^p \setminus \{0\}} \left\{ x \in F : \langle \eta^0_i, x - a^0 \rangle = 0 \text{ for some } \eta^0_i \in \partial^0 g_i(\cdot, v^0_i)(a^0), \forall i \in \bar{I}(a^0); \right. \]

\[ \left. \langle \zeta^0, x - a^0 \rangle = 0 \text{ for some } \zeta^0 \in \sum_{k=1}^{p} \theta_k \partial f_k(\cdot, u^0_k)(x) \cap \sum_{k=1}^{p} \theta_k \partial f_k(\cdot, u^0_k)(a^0) \text{ and } \right. \]

\[ f_k(x, u^0_k) = \max_{u_k \in U_k} f_k(x, u_k), \forall k = 1, \ldots, p \right\}. \]
(ii) If for each \( \theta \in \text{int} \mathbb{R}_+^p \), \( S_\theta \) is non-empty, \( a^\theta \in S_\theta \) is fulfilled the robust basic constraint qualification, \( (\lambda^\theta, u^\theta, v^\theta) \in M(a^\theta) \), then

\[
PR(F) = \bigcup_{\theta \in \text{int} \mathbb{R}_+^p} \left\{ x \in F : \langle \eta_i^\theta, x - a^\theta \rangle = 0 \text{ for some } \eta_i^\theta \in \partial^c g_i(\cdot, \nu_i^\theta)(a^\theta), \forall i \in \tilde{I}(a^\theta) ;
\right. \\
\left. \langle \zeta^\theta, x - a^\theta \rangle = 0 \text{ for some } \zeta^\theta \in \sum_{k=1}^p \theta_k \partial f_k(\cdot, u^\theta)(x) \cap \sum_{k=1}^p \theta_k \partial f_k(\cdot, u^\theta)(a^\theta) \right) \quad \text{and} \quad f_k(x, u^\theta) = \max_{u_k \in U_k} f_k(x, u_k), \forall k = 1, \ldots, p \right\}.
\]

To close this section, we give an example illustrating Theorem 5.2 which is indicated to be conveniently applied is applicable while the aforementioned result, due to Sun et al. [34, Theorem 4.7], are not. It means that at least one of the constraint functions \( g_i(\cdot, \nu_i) \) for some \( \nu_i \in \mathcal{V}_i \), is not convex while the robust feasible set is convex.

**Example 5.1.** Let \( x := (x_1, x_2) \in \mathbb{R}^2 \), \( u_1 := (u_{1,1}, u_{1,2}), u_2 := (u_{2,1}, u_{2,2}), v_1 := (v_{1,1}, v_{1,2}), v_2 := (v_{2,1}, v_{2,2}), v_3 := (v_{3,1}, v_{3,2}), v_4 := (v_{4,1}, v_{4,2}), \mathcal{U}_1 = \mathcal{U}_2 := [0, 1] \), \( \mathcal{V}_1 := \{(v_1, v_2) \in \mathbb{R}^2 : v_1^2 + v_2^2 \leq 1\} \), \( \mathcal{V}_2 := [-2, -1] \times [-2, -1] \), \( \mathcal{V}_3 := [4, 5] \times [2, 3] \) and \( \mathcal{V}_4 := [0, 1] \times [0, 1] \).

We now consider the following constrained multiobjective optimization problem with uncertainty data:

Minimize \((f_1(x, u_1), f_2(x, u_2))\)

subject to \( x \in \mathbb{R}^2 \), \( g_1(x, v_1) \leq 0 \), \( g_2(x, v_2) \leq 0 \), \( g_3(x, v_3) \leq 0 \), \( g_4(x, v_4) \leq 0 \),

where

\[
\begin{align*}
    f_1(x, u_1) & := u_1 x_1, \\
    f_2(x, u_2) & := u_2 x_2, \\
    g_1(x, v_1) & := v_{1,1} x_1 + v_{1,2} x_2 + x_1^3 - 2, \\
    g_2(x, v_2) & := v_{2,1} x_1 + v_{2,2} x_2 + 1, \\
    g_3(x, v_3) & := -v_{3,1} x_1 - v_{3,2} x_2 + 3, \\
    g_4(x, v_4) & := -v_{4,1} x_1 - v_{4,2} x_2^2,
\end{align*}
\]

and its robust counterpart

Minimize \((\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \max_{u_2 \in \mathcal{U}_2} f_2(x, u_2))\)

subject to \( x \in F := \left\{ \begin{array}{l}
    g_1(x, v_1) \leq 0, \forall v_1 \in \mathcal{V}_1, \\
    g_2(x, v_2) \leq 0, \forall v_2 \in \mathcal{V}_2, \\
    g_3(x, v_3) \leq 0, \forall v_3 \in \mathcal{V}_3, \\
    g_4(x, v_4) \leq 0, \forall v_4 \in \mathcal{V}_4, \\
    x \in \mathbb{R}^2
\end{array} \right\} .
\)

We obtain that for every \( x \in \mathbb{R}^2 \),

\[
\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1) = x_1,
\]
\[
\max_{u_1 \in \mathbb{R}} f_2(x,u_2) = x_2,
\]

\[
F = \{ x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} + x_3^2 - 2 \leq 0, \ -x_1 - x_2 + 1 \leq 0, \ -4x_1 - 2x_2 + 3 \leq 0, \ -x_1 \leq 0 \},
\]
as a straightforward calculation shows. Let us denote

\[
\Lambda := \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1 + \theta_2 = 1\},
\]
\[
\text{int}\Lambda := \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1 > 0, \ \theta_2 > 0, \ \theta_1 + \theta_2 = 1\}.
\]

We now consider the following possibilities:

(i) If \(\theta := (\theta_1, \theta_2) = (1,0)\) then \((RP_0)\) reads as follows:

\[
\begin{align*}
\text{Minimize} & \quad x_1 \\
\text{subject to} & \quad x \in F.
\end{align*}
\]

As \(\theta_1 x_1 + \theta_2 x_2 = x_1 \geq 0, \ \forall x \in F\), we can take \(a^\theta := (0,2) \in S_\theta\) and so, \(I(a^\theta) = \{1,4\}\). Let us choose \(h^\theta := (0,0,0,1)\), \(v^\theta := (1,1)\), \(v^\theta := (v_1^\theta, v_2^\theta, v_3^\theta, v_4^\theta) = ((0,1),(-1,-1),(4,2),(1,0))\), and we also have \(I(a^\theta) = \{4\}\). Let

\[
A_\theta := \{ x \in F : \langle \eta^\theta, x - a^\theta \rangle = 0 \text{ for some } \eta^\theta \in \partial^\theta g_i(\cdot, v^\theta(a^\theta)), \ \forall i \in I(a^\theta) \};
\]

\[
\exists \xi^\theta \in \Theta \partial f_1(\cdot, u^\theta_i)(x) + \Theta_2 \partial f_2(\cdot, u^\theta_i)(x)
\]
\[
\cap (\Theta_1 \partial f_1(\cdot, u^\theta_i(a^\theta)) + \Theta_2 \partial f_2(\cdot, u^\theta_i(a^\theta)));
\]

\[
\langle \xi^\theta, x - a^\theta \rangle = 0, \ \max_{u_1 \in \Theta_1} f_1(x,u_1) = f_1(x,u^\theta_i), \ \max_{u_1 \in \Theta_1} f_2(x,u_2) = f_2(x,u^\theta_i).
\]

Then \(A_\theta\) can be easily calculated \(A_\theta = \{ x \in \mathbb{R}^2 : x_1 = 0, \ \frac{3}{2} \leq x_2 \leq 2 \}\).

(ii) Similarly, if \(\theta = (0,1)\) then we can take \(a^\theta := (1,0) \in S_\theta\) and so, \(I(a^\theta) = \{1,2,4\}\). Let us choose \(h^\theta := (\frac{1}{2},1,0,0)\), \(v^\theta := (1,1), v^\theta := (v_1^\theta, v_2^\theta, v_3^\theta, v_4^\theta) = ((1,0),(-1,-1),(4,2),(1,0))\), and we also have \(I(a^\theta) = \{1,2\}\). Thus \(A_\theta = \{(1,0)\}\).

(iii) If \(\theta \in \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \frac{2}{3} < \theta_1 < 1, \ \theta_2 > 0, \ \theta_1 + \theta_2 = 1\}\) then \((RP_\theta)\) becomes

\[
\begin{align*}
\text{Minimize} & \quad \theta_1 x_1 + \theta_2 x_2 \\
\text{subject to} & \quad x \in F.
\end{align*}
\]

As \(\theta_1 x_1 + \theta_2 x_2 \geq \theta_1 x_1 - 2\theta_2 x_2 + \frac{2}{3}\theta_2 = (3\theta_1 - 2)x_1 + \frac{2}{3}\theta_2 \geq \frac{2}{3}\theta_2, \ \forall x \in F,\) then we can take \(a^\theta := (0,\frac{2}{3}) \in S_\theta\) and so, \(I(a^\theta) = \{2,4\}\). Let us choose \(h^\theta := (0,0,\frac{2}{3}, -2\theta_2)\) (note that \(\theta_1 > \frac{2}{3}\) and \(\theta_1 + \theta_2 = 1\) imply \(\theta_1 - 2\theta_2 > 0\)), \(v^\theta := (1,1), v^\theta := (v_1^\theta, v_2^\theta, v_3^\theta, v_4^\theta) = ((1,0),(-1,-1),(4,2),(1,0))\), and we also have \(I(a^\theta) = \{2,4\}\). In this case, it is easy to see that \(A_\theta = \{(0,\frac{2}{3})\}\).

(iv) Similarly, if \(\theta \in \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \frac{1}{2} < \theta_1 < \frac{2}{3}, \ \theta_2 > 0, \ \theta_1 + \theta_2 = 1\}\) then \(A_\theta = \{(0,1)\}\) and if \(\theta \in \{(\theta_1, \theta_2) \in \mathbb{R}^2 : 0 < \theta_1 < \frac{1}{2}, \ \theta_2 > 0, \ \theta_1 + \theta_2 = 1\} = \{(0,1,\theta_2) \in \mathbb{R}^2 : \frac{1}{2} < \theta_2 < 1, \ \theta_1 > 0, \ \theta_1 + \theta_2 = 1\}\) then \(A_\theta = \{(1,0)\}\).

(v) If \(\theta := (\frac{1}{2}, \frac{1}{2})\), then \(\theta_1 x_1 + \theta_2 x_2 \geq \frac{1}{2}, \ \forall x \in F\). Take \(a^\theta := (\frac{1}{2}, \frac{1}{2}) \in S_\theta\) and so, \(I(a^\theta) = \{2,3,4\}\). Let us choose \(h^\theta := (0,\frac{1}{2},0,0), v^\theta := (1,1), v^\theta := (v_1^\theta, v_2^\theta, v_3^\theta, v_4^\theta) = ((1,0),(-1,-1),(4,2),(1,0))\), and we also have \(I(a^\theta) = \{2\}\). Then elementary calculations give us \(A_\theta = \{ x \in \mathbb{R}^2 : x_1 - x_2 + 1 = 0, \ x_1 \geq 0, \ x_2 \geq 0 \} \).
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(vi) Similarly, if \( \theta = \left( \frac{2}{3}, \frac{1}{3} \right) \) then we can take \( a^\theta = (\frac{1}{3}, 1)^T \in S_\theta \), \( \lambda^\theta = (0, 0, \frac{1}{3}, 0) \) and so,

\[
A_\theta = \{ x \in \mathbb{R}^2 : -4x_1 - 2x_2 + 3 = 0, \ x_1 \geq 0, \ x_2 \geq 0 \}.
\]

Therefore, by Theorem 5.2, weakly and properly robust efficient solution sets of (UMP) look like

\[
WR(F) = \bigcup_{\theta \in \mathbb{R}^2_+ \setminus \{0\}} A_\theta = \bigcup_{\theta \in \Lambda} A_\theta
\]

\[
= \{ x \in \mathbb{R}^2 : x_1 = 0, \ 1 \leq x_2 \leq 2 \}
\]

\[
\cup \{ x \in \mathbb{R}^2 : -4x_1 - 2x_2 + 3 = 0, \ x_1 \geq 0, \ x_2 \geq 0 \}
\]

\[
\cup \{ x \in \mathbb{R}^2 : -x_1 - x_2 + 1 = 0, \ x_1 \geq 0, \ x_2 \geq 0 \}
\]

and

\[
PR(F) = \bigcup_{\theta \in \text{int} \mathbb{R}^2_+} A_\theta = \bigcup_{\theta \in \text{int} \Lambda} A_\theta
\]

\[
= \{ x \in \mathbb{R}^2 : -4x_1 - 2x_2 + 3 = 0, \ x_1 \geq 0, \ x_2 \geq 0 \}
\]

\[
\cup \{ x \in \mathbb{R}^2 : -x_1 - x_2 + 1 = 0, \ x_1 \geq 0, \ x_2 \geq 0 \}.
\]

Acknowledgments. The authors would like to thank the anonymous referees and the associate editor for their valuable suggestions and comments, which helped to improve the paper.

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Received November 2017; revised February 2018.

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