The Slicing Method: Determining Insensitivity Regions of Probability Weighting Functions

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Abstract

A popular rule of thumb, usually called “heuristic technique” in Behavioral Economics, for determining the likelihood insensitivity regions of probability weighting functions (pwf’s) is based on searching for points at which the pwf’s are twice their values at half the points. Although this technique works remarkably well for many commonly used pwf’s, it sometimes fails to provide the correct answer. In order to cover the class of pwf’s for which the heuristic technique does not work, in this paper we propose, discuss, and illustrate an extension of the technique into what we call the “slicing method,” which is capable of finding the subadditivity and insensitivity regions of any continuous pwf.

Keywords Behavioural economics · Probability weighting function · Subadditivity · Likelihood insensitivity · Probabilistic insensitivity

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1 Introduction

Extensive empirical studies, mostly conducted by psychologists and behavioural economists, have shown that people weigh objective probabilities of outcomes in a nonlinear fashion. This observation deviates from some of the axioms of expected utility theory (von Neuman & Morgenstern, 1944) and thus has inspired modern theories that aim at better explanations of human behaviour (e.g., Dhami, 2016; Wakker, 2010). A special feature of some of these theories is that objective probabilities are transformed by probability weighting functions (pwf’s), which have close connections to weighted distributions in statistics and pricing in insurance (e.g., Furman & Zitikis, 2008, 2009).

Numerous pwf’s have been proposed in the literature, and many studies have been devoted to exploring their properties (e.g., al-Nowaihi & Dhami, 2010; Fehr-Duda & Epper, 2012; Karmarkar, 1978; Stott, 2006; Wu & Gonzalez, 1996, and references therein). One of these properties is based on the likelihood insensitivity region, which reflects the tendency of people to have different perceptions of probabilities.

Many empirical studies have argued that individuals’ decision weights are less sensitive to moderate probabilities and very sensitive to extreme probabilities. In other words, pwf’s tend to be flatter for intermediate probabilities and steeper near the endpoints of the probability interval (0, 1). This intermediate range of probabilities has given rise to several notions of likelihood-insensitivity regions.

Based on this observed psychological trait, Tversky and Wakker (1995) proposed to use the class of cavex (i.e., concave-convex) pwf’s that has become a prevailing functional form in behavioural economics. In this context, the notions of lower subadditivity (henceforth simply called subadditivity) and upper subadditivity naturally arise and play important roles in decision making. In particular, using these notions, definitions of insensitivity regions have been given. The regions have been captured empirically (e.g., Abdellaoui, 2000; Bleichrodt & Pinto, 2000; Gonzalez & Wu, 1999; Kilka & Weber, 2001), although reliable analytical tools for determining them have been elusive.

As far as we know, there has been one attempt in this direction in the economics literature (Wakker, 2010) in the form of a heuristic technique that aims at finding the maximal region of subadditivity. It has been observed, and our own testing has confirmed, that the technique works well for many pwf’s that appear in the literature, but there are also examples when the technique fails to provide correct answers. Yet, as far as we are aware of, there has not been a comprehensive study to determine when the technique works. In the current paper, we shall shed light on this topic by proposing a method for determining the subadditivity regions of pwf’s.

Determining subadditivity and insensitivity regions has important decision-making implications, as they are used to explain several relevant problems in economic analysis. For instance, Baillon et al. (2020) analyzed relations between

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1 This heuristic technique of Behavioural Economics is not to be confused with heuristic optimization techniques (e.g., Chang, 2012; Doering et al., 2019; Khan et al., 2021a, 2021b) that have been prominently used when numerically solving optimization problems in Economics, Finance, and other research areas; for additional notes and references on the topic, we refer to concluding Sect. 6.
insensitivity regions and risk underprevention. Several authors used insensitivity regions in the context of probability uncertainty and applied them to study decision-making problems such as the Ellsberg paradox (Abdellaoui et al., 2011; Baillon et al., 2018), the Allais paradox (Neilson, 2003; Neilson & Stowe, 2002), tort and contract laws (Posner, 2004), terrorism (Sunstein, 2003; Phillips and Pohl, 2020).

The interest in insensitivity regions also arises when studying ambiguity aversion, called a-insensitivity (Dimmock et al., 2016). This is the tendency to treat subjective likelihood as fifty-fifty, and overweighting extreme events, as has been observed in several ambiguity aversion studies (Crockett et al., 2019; Dimmock et al., 2016). This feature is the analog of the cavex pwf’s that are usually encountered in risk analysis. It is also used to explain why source preferences between known and unknown probabilities and ambiguity aversion depend so much on the likelihood of events (Wakker, 2010), and it has been applied to study many topics in finance such as stock market participation (Dimmock et al., 2016) and asset pricing (Izhakian, 2020). The aforementioned studies and our consulting engagement to be described later in this paper have motivated our current research.

The rest of the paper is organized as follows. Mathematical preliminaries and special properties of pwf’s are presented in Sect. 2, where we also recall the heuristic technique for determining subadditivity regions. Conditions under which the technique works are discussed in the same section, alongside several examples that illustrate the technique’s strengths and limitations.

In Sect. 3 we introduce the slicing method (Theorem 3.1) for determining accurate subadditivity and thus likelihood-insensitivity regions, and we then illustrate the method by completing examples of Sect. 2.

In Sect. 4 we derive subadditivity and insensitivity regions of several classical pwf’s that are frequently used in, or have arisen from, empirical studies.

In Sect. 5 we consider an illustrative example that resembles those real-world cases that we have encountered during our consulting engagements. We shall see in the section that accurate determination of subadditivity and insensitivity regions is helpful when making decisions in real applications.

In Sect. 6 we give a short summary of our main contributions with potential directions for future research, and in Appendix A we provide a user-friendly computer code for finding subadditivity and insensitivity regions.

Throughout the paper, we use the acronyms and notation specified in Table 1, with their complete definitions, mathematical details, and references given later in the paper.

2 Preliminaries and the Heuristic Technique

Subadditive functions have been extensively studied in the mathematics literature (Beckenbach, 1964; Bruckner, 1960, 1962, 1964; Hille & Phillips, 1957). They have found numerous applications in economics, finance, and insurance (e.g., Embrechts & Wang, 2015; Föllmer & Schied, 2016; Wakker, 2010), possibly with more complex domains of definition (e.g., some classes of random variables) than the set of real numbers. For convenient referencing, next is the subadditivity definition
(e.g., Kuczma, 2009, Chapter 16) adapted to the context of our following discussion, and thus restricted to the range \([0, 1]\) of all possible probabilities.

**Definition 2.1** A pwf \( w : [0, 1] \rightarrow [0, 1] \), assumed throughout this paper to be continuous, non-decreasing, \( w(0) = 0 \) and \( w(1) = 1 \), is called subadditive on an interval \((0, b) \subseteq (0, 1)\) for some \( b \in (0, 1) \) if the bound \( w(p) + w(q) \geq w(p + q) \) holds for all \( p, q \in (0, b) \) such that \( p + q \leq b \).

**Note 2.1** Throughout the paper, we tend to define various properties using open intervals, although the properties usually hold on closed intervals. This should not cause inconvenience, or mathematical confusion, because the functions that we deal with are continuous.

The class of subadditive functions may not, however, always capture the diversity of probability distortions associated with some psychological phenomena. Hence, the class of cavex functions has arisen, whose detailed definition is next (Wakker, 2010, Appendix 7.12).

**Definition 2.2** A pwf \( w : [0, 1] \rightarrow [0, 1] \) is called cavex on the interval \((0, 1)\) if there is \( c \in (0, 1) \) such that \( w(p) \) is concave on the interval \((0, c)\) and convex on \((c, 1)\).

Of course, if \( c = 1 \), then we deal with concave and thus subadditive functions on the entire unit interval \((0, 1)\), but if \( c < 1 \), then the following question arises: do we still have subadditivity on the entire unit interval \((0, 1)\)? If not, then what is the largest \( b \in (c, 1) \) for which the pwf \( w \) is subadditive on \((0, b)\)? These questions are captured by the notion of maximal region of subadditivity, as well as by the maximal region of likelihood insensitivity (Tversky and Wakker, 1995; Wakker, 2010), which we define next.

**Definition 2.3** Given a pwf \( w \), its maximal region of subadditivity (MS) is \((0, w_{rb})\), where \( w_{rb} \) is the largest \( b \in (0, 1) \) such that \( w(p) + w(q) \geq w(p + q) \) holds for all \( p, q \in (0, b) \) such that \( p + q \leq b \). The region of likelihood insensitivity (LI) is \((b_{lb}, w_{rb})\), where

\[
 b_{rb} = 1 - w_{rb}^*
\]

with \( w_{rb}^* \) denoting the largest \( b \in (0, 1) \) such that \( w^*(p) + w^*(q) \geq w^*(p + q) \) for all \( p, q \in (0, b) \) satisfying \( p + q \leq b \), where \( w^* \) is the dual pwf defined by

\[
 w^*(p) = 1 - w(1 - p)
\]
\[ w^*(p) = 1 - w(1 - p). \] (2.1)

Hence, the interval \((0, w^{*\text{rb}})\) is the MS region of the dual pwf \(w^*\).

Subadditivity reflects the fact that pwf’s for lower probabilities have more impact on decision making than in the case of moderate probabilities. For example, this implies that individuals prefer to reduce risk from probability \(p > 0\) to 0 than from probability \(p + q\) to \(q > 0\). In other words, the probability of the whole is judged less than the probability of the sum of any parts of it (Baron, 2006).

Another way to define likelihood insensitivity has recently been proposed by Baillon et al. (2020), which is as follows.

**Definition 2.4** For a continuously differentiable pwf \(w\), its region of probabilistic insensitivity (PI) is \((p_1, p_2)\), where \(p_1\) is the smallest and \(p_2\) is the largest numbers such that \(w'(p_1) = w'(p_2) = 1\) and \(w'(p) < 1\) for all \(p \in (p_1, p_2)\).

**Note 2.2** Since later in this paper we shall calculate and compare the regions of likelihood insensitivity proposed by Tversky and Wakker (1995) as well as by Baillon et al. (2020), from now on we shall follow the suggestion of Wakker (2021) and use the term “probabilistic insensitivity” for the region proposed by Baillon et al. (2020).

Note that Definition 2.4 requires \(w\) to be differentiable, and we shall therefore later calculate PI regions only for such pwf’s. The intuition behind the definition is that for differentiable pwf’s, we have \(\int_0^1 w'(p)dp = 1\), which means that the average rate of change of \(w\) on \((0, 1)\) is equal to 1. Hence, the PI region is where the rate of change is less than the aforementioned average. We shall see from our following examples that the types of regions that arise from Definitions 2.3 and 2.4 are, in general, different.

### 2.1 The Heuristic Technique

Determining the lower and upper limits of the PI region is not particularly challenging, at least numerically, but the same cannot be said about the LI region. For the latter task, we only need to understand how to obtain the subadditivity upper bound \(w_{rb}\) of the pwf \(w\), since \(b_{rb}\) can be obtained via the dual pwf \(w^*\), as explained in Definition 2.3. But deriving \(w_{rb}\) even numerically, let alone in closed form, is challenging. The most well-known test in the case of cavedx functions can be traced back to Bruckner (1962), who established a general result that we next reformulate for pwf’s.

**Theorem 2.1** Given \(b \in (0, 1]\), a necessary and sufficient condition for any continuous cavedx pwf \(w : [0, 1] \rightarrow [0, 1]\) to be subadditive on \((0, b)\) is the bound

\[
\inf_{p \in (0, b)} \{w(p) + w(b - p)\} \geq w(b).
\]
Note 2.3 The original result of Bruckner (1962) concerns with super-additivity of convex-concave functions (recall that cavex means concave-convex), and having thus reformulated the result as in Theorem 2.1 makes our subsequent referencing more straightforward.

Although Theorem 2.1 makes the search of the worst-rank bound \( \text{wrb} \) easier, the task is still fairly challenging for many classical pwf’s. As a reprieve, it has been observed that the theorem implies the bound \( 2w(b/2) \geq w(b) \). This observation has inspired researchers to propose a rule of thumb, often called “heuristic technique” in Behavioural Economics, for checking subadditivity of cavex pwf’s.

**Heuristic technique** (Wakker, 2010, p. 227) A cavex pwf \( w \) is subadditive on the interval \((0, b)\) if \( b > 0 \) is the smallest number that solves the equation
\[
h(b) = 0,
\]
where \( h(b) \) is the “heuristic” function defined by
\[
h(b) = 2w(b/2) - w(b).
\]

Note 2.4 In our following considerations, we shall frequently contrast the function \( h(b) \) with the “slicing” function \( s(b) \), which we shall formally define in Sect. 3.

Instead of searching for the smallest root (if there is any) of the heuristic function \( h(b) \), a more encompassing way to define the heuristic upper bound of subadditivity, that is, the heuristic worst-rank bound \( w^H_{rb} \), is as follows.

**Definition 2.5** With the heuristic worst-rank bound defined by
\[
w^H_{rb} = \inf\{b \in (0, 1] : h(b) < 0\},
\]
we call \((0, w^H_{rb})\) the heuristic maximal-region of subadditivity (HMS).

By definition, the bound \( w^H_{rb} \) is an infimum, that is, \( w^H_{rb} \) is the greatest lower bound of the set of those \( b \in (0, 1] \) for which the property \( h(b) < 0 \) holds. Hence, we need to keep in mind that if the infimum is taken over an empty set, the infimum is, by definition, the largest possible value, which is 1 in our case. Hence, if it happens that \( h(b) \geq 0 \) for all \( b \in (0, 1) \), as will be the case in, e.g., Example 2.1 below, then the infimum on the right-hand side of Eq. (2.2) is taken over the empty set, because there is not any \( b \in (0, 1] \) such that \( h(b) < 0 \). Therefore, we set \( w^H_{rb} = 1 \), and so the unit interval \((0, 1)\) is the HMS region of \( w \).

The quantity \( w^H_{rb} \) is easy to compute numerically. Interestingly, for most of the pwf’s, \( w^H_{rb} \) happens to be the true subadditivity upper bound \( w_{rb} \). This explains the popularity of this heuristic technique in the behavioural economics literature (Wakker, 2010, p. 227). However, as we shall soon see, there are pwf’s for which this technique fails to identify the true upper bound \( w_{rb} \) of subadditivity. Therefore, in general, a different technique is needed for the purpose, and we shall introduce it in Sect. 3.
Analogously to Definition 2.5 but using the dual pwf $w^*$ given by Eq. (2.1), we next define the heuristic lower bound of likelihood insensitivity and, in turn, the heuristic region of likelihood insensitivity.

**Definition 2.6** The heuristic lower bound $b_H^{rb}$ of likelihood insensitivity is given by the equation

$$b_H^{rb} = 1 - w_H^{rb},$$

where $w_H^{rb}$ is the heuristic upper bound of subadditivity of the dual pwf $w^*$, that is,

$$w_H^{rb} = \inf\{ b \in (0, 1] : g(b) < 0 \},$$

where

$$g(b) = 2w^*(b/2) - w^*(b).$$

We call $(b_H^{rb}, w_H^{rb})$ the heuristic region of likelihood insensitivity (HLI).

### 2.2 Cautionary Examples

We have already noted that the heuristic technique should be used with caution as it does not always completely describe the region of subadditivity, and we shall next provide several concrete examples illustrating the point. Namely:

- **Example 2.1** illustrates the case when the heuristic function $h(b)$ does not dip below 0 anywhere on the interval $(0, 1)$, and thus the modified version of the heuristic technique (see Definition 2.5) needs to be applied to identify the MS region.
- **Example 2.2** illustrates the case when $h(b)$ does not dip below 0 on the interval $(0, 1)$ and fails to correctly identify the MS region.
- **Example 2.3** deals with a pwf for which $h(b)$ has a single root on the interval $(0, 1)$ and fails to correctly identifies the MS region.
- **Example 2.4** deals with a pwf for which $h(b)$ has many roots and correctly identifies the MS region.

Some of these examples are based on pwf’s borrowed from the literature while others have been specially created for the purpose.

**Example 2.1** Let the pwf be

$$w(p) = \frac{\sqrt{p}}{\left(\sqrt{p} + \sqrt{1-p}\right)^{1/5}}, \quad 0 \leq p \leq 1,$$

which is a member of a large class of pwf’s considered by Goldstein and Einhorn (1987); we shall look at this class in more detail in Sect. 4. We easily check the bound $2w(b/2) > w(b)$ for all $b \in (0, 1]$, and thus the heuristic function $h(b)$ does not have any root in the interval $(0, 1)$ (see Fig. 1). This implies $w_H^{rb} = 1$, and we
thus conclude that the HMS is the entire unit interval (0, 1). To calculate the heuristic lower bound $b^H_{rb}$, we first numerically obtain the heuristic upper bound $w^H_{rb} = 0.938$ of subadditivity of the dual pwf $w^*$ (recall Definition 2.6) and then obtain $b^H_{rb} = 1 - w^H_{rb} = 0.062$. Hence, the HLI region is $(b^H_{rb}, w^H_{rb}) = (0.062, 1)$.

For comparison, the PI region (see next Note 2.5 for an explanation) is $(p_1, p_2) = (0.159, 0.937)$.

This concludes Example 2.1.

**Note 2.5** We have used numerical calculations to derive the endpoints $p_1$ and $p_2$ of the PI region in Example 2.1, as well as of those in other examples throughout this paper. Namely, we have employed Wolfram Research (2021) to calculate the first derivative $w'(p)$ and then numerically obtained those points $p$ for which $w'(p) = 1$.

**Example 2.2** Let (see Fig. 2)

$$w(p) = \begin{cases} 
8p/7 & \text{when } 0 \leq p \leq 0.7, \\
0.25p + 0.625 & \text{when } 0.7 < p \leq 0.9, \\
1.5p - 0.5 & \text{when } 0.9 < p \leq 1.
\end{cases}$$

The heuristic function $h(b)$ never dips below zero and so $w^H_{rb} = 1$, implying that the HMS region is $(0, w^H_{rb}) = (0, 1)$.

However, this is not the MS region. The correct MS region is $(0, 0.9)$, which we shall obtain using the slicing method in Sect. 3 (see Table 2 there). Finally, a simple numerical computation yields $b^H_{rb} = 0.56$ and thus the HLI region is $(b^H_{rb}, w^H_{rb}) = (0.56, 1)$.
This concludes Example 2.2.

**Example 2.3** Let (see Fig. 3)

\[
\begin{align*}
  w(p) &= \begin{cases} 
    0.2 - (p - 0.4)^2/0.8 & \text{when } 0 \leq p \leq 0.4, \\
    0.2 + (p - 0.4)^2/0.45 & \text{when } 0.4 < p \leq 1.
  \end{cases} \\
\end{align*}
\]  

(2.5)

Note that the equation \(2w(b/2) = w(b)\) is solved at \(w_{rb}^H = 0.695\). Similarly, using the dual pwf \(w^*\), we obtain \(b_{rb}^H = 0\). Hence, the HMS and HLI regions coincide and are

\[
(0, w_{rb}^H) = (b_{rb}^H, w_{rb}^H) = (0, 0.695).
\]

However, when \(p = 0.2\) and \(b = 0.69\), we have the bound

\[
w(0.2) + w(0.49) = 0.368 < w(0.69) = 0.3868,
\]

which shows that the pwf \(w\) is not subadditive on the entire interval \((0, w_{rb}^H)\), thus
invalidating the heuristic technique. We shall correctly determine MS and LI regions in Sect. 3. This concludes Example 2.3.

**Example 2.4** Let

\[
  w(p) = \begin{cases} 
  \sqrt{0.2p} & \text{when } p \leq 0.2, \\
  p & \text{when } 0.2 \leq p. 
  \end{cases} 
\]  

\hspace{1cm} (2.6)

The pwf is not concave, although it is piece-wise concave. Hence, we cannot conclude that the MS region is the entire unit interval \((0, 1)\). We appeal to the heuristic technique and find that the equation \(2w(p/2) = w(p)\) holds for all \(p \geq 0.4\). The original heuristic technique (Wakker, 2010, p. 227) would imply that the heuristic worst-rank bound is 0.4, but this is not the case. Indeed, since the heuristic function \(h(b)\) never dips below 0 (see Fig. 4), Definition 2.5, which is a modification of the original heuristic technique, implies \(w_{rb}^H = 1\). Furthermore, from Theorem 2.1 and the easy-to-verify bound \(w(p) + w(1 - p) \geq w(1)\) for all \(p \in (0, 1)\), we conclude that \(w_{rb} = 1\) and thus

\[
  (0, w_{rb}^H) = (0, w_{rb}) = (0, 1).
\]

(Pwf (2.6) is concave-concave and also concave-convex, that is, cavex, and thus Theorem 2.1 applies.) Finally, by a simple numerical computation, we get \(b_{rb}^H = 0\), and so the HLI region is

\[
  (b_{rb}^H, w_{rb}^H) = (0, 1).
\]

This concludes Example 2.4.
2.3 The Heuristic Technique Revisited

We have seen that for some pwf’s the heuristic technique does not provide the correct MS region. Hence, a natural question arises: when does the technique work? For this, we recall the Boas test (e.g., Beckenbach, 1964; Johnson, 1972), which says that in addition to the equation \( w(b) = 2w(b/2) \), two other conditions need to be satisfied.

**Theorem 2.2 (Boas’ test)** Let \( w : [0, 1] \to [0, 1] \) be a continuously differentiable and convex pwf. Then \( (0, b) \subseteq (0, 1) \) is its MS region whenever the following three conditions are satisfied:

(a) \( w(b) = 2w(b/2) \);
(b) \( w'(0) > w'(b) \);
(c) the equation \( w'(p) = w'(b - p) \) is solved in \( p \) no more than once on the interval \( (0, b/2) \).

The test is appealing, although its conditions involve differentiability, exclude some important pwf’s, and can often be verified only numerically. Nevertheless, in some instances (see, e.g., Sect. 4) we shall employ the test to analyze several pwf’s. The slicing method that we shall introduce in the next section successfully tackles all continuous pwf’s.

3 The Slicing Method

Solving the equation \( h(b) = 0 \) is convenient because the task relies only on one variable \( b \), but coming back to the very basic Definition 2.3 of subadditivity—which relies on two variables \( p \) and \( q \)—is necessary because we already know that the heuristic technique occasionally fails to determine the correct MS region. This leads...
us to our “slicing” method that overcomes the drawbacks of the heuristic technique. We shall present the method as next Theorem 3.1, where the function

$$W_b(p) = w(p) + w(b - p) - w(b)$$

is used. Note that the pwf $w$ is subadditive on the interval $(0, b)$ whenever $W_b(p) \geq 0$ for all $p \in (0, b)$.

**Theorem 3.1 (slicing method)** For any continuous pwf $w : [0, 1] \rightarrow [0, 1]$, the worst-rank bound $w_{rb}$ of the pwf $w$ can be expressed as

$$w_{rb} = \inf\{b \in (0, 1] : s(b) < 0\}, \quad (3.1)$$

where, by definition,

$$s(b) = \inf_{p \in (0, b)} W_b(p) \quad (3.2)$$

is the “slicing” function, which can equivalently be rewritten as

$$s(b) = \inf_{p \in (0, b/2)} W_b(p). \quad (3.3)$$

**Proof** Assume for the sake of contradiction that the pwf $w$ is not subadditive on the interval $(0, w_{rb})$. Then we can find $x < y < w_{rb}$ such that $x + y \leq w_{rb}$ and $w(x) + w(y) - w(x + y) < 0$. Hence, we have $W_{x+y}(x) < 0$. By continuity, there is $b < x + y \leq w_{rb}$ such that $W_b(x) < 0$. Consequently, $s(b) < 0$ and thus $w_{rb}$ cannot be the infimum with this property. Hence, $w$ has to be subadditive on the interval $(0, w_{rb})$.

On the other hand, if we take $q > w_{rb}$, then by the definition of infimum, there exists $b$ satisfying $q > b > w_{rb}$ and such that $s(b) < 0$. But when $s(b) < 0$, we have $W_b(p) < 0$ for some $p \in (0, b)$ and therefore the pwf $w$ cannot be subadditive on the interval $(0, b)$, which in turn implies that it cannot be subadditive on the interval $(0, q) \supset (0, b)$.

In summary, the worst-rank bound $w_{rb}$ can be written as in Eq. (3.1) using $s(b)$ defined by Eq. (3.2).

We shall next prove Eq. (3.3), that is, we shall show that the infimum in definition (3.2) can be taken over only $p \in (0, b/2)$. For any $b \in (0, 1)$, the function $W_b(p)$ is symmetric around $p = b/2$. Indeed, for every $\epsilon \in (0, b/2]$, we have

$$W_b(b/2 + \epsilon) = w(b/2 + \epsilon) + w(b/2 - \epsilon) - w(b)$$

$$= w(b/2 - \epsilon) + w(b/2 + \epsilon) - w(b)$$

$$= W_b(b/2 - \epsilon).$$
Consequently,
\[
\inf_{p \in (0, b)} W_b(p) = \min \left\{ \inf_{p \in (0, b/2)} W_b(p), \inf_{p \in (b/2, b)} W_b(p), W_b(b/2) \right\}
\]
\[
= \min \left\{ \inf_{p \in (0, b/2)} W_b(p), W_b(b/2) \right\}
\]
\[
= \inf_{p \in (0, b/2)} W_b(p),
\]
where the right-most equation holds due to the continuity of the pwf \( w \) and thus of the function \( p \mapsto W_b(p) \). This explains the reason why the slicing function \( s(b) \) in the formulation of Theorem 3.1 can be defined in the minimalist fashion as the infimum over \( p \in (0, b/2) \). This completes the proof of Theorem 3.1. \( \square \)

A few clarifying notes concerning Theorem 3.1 follow.

**Note 3.1** The upper bound \( w_{rb} \) is an infimum, and thus we need to keep in mind that when it is taken over an empty set, the infimum is, by definition, the right-most point of the domain of all possible values. For example, the pwf \( w(p) = p \), which is linear and thus subadditive on the entire unit interval \((0, 1)\), gives rise to \( W_b(p) = 0 \) for all \( p \) and we thus have \( s(b) = 0 \) for all \( b \). This makes the set over which the infimum is taken empty, and we therefore conclude that \( w_{rb} = 1 \).

**Note 3.2** Since \( W_b(b/2) = h(b) \), if the function \( W_b(p) \) of the variable \( p \in (0, b/2) \) achieves its minimum at the point \( p = b/2 \), then \( s(b) = h(b) \). Because of this reason, we shall see in some of the following figures that if \( w_{rb} = w_{rf}^H \), then the equation \( s(b) = h(b) \) holds for all \( b \geq w_{rb} \).

In Table 2 and Fig. 5, we present the insensitivity regions of the four examples of Sect. 2.2, as well as the heuristic and slicing functions used to determine the regions. Numerical computations and graphing have been indispensable for the task, and in Appendix A we give a ready-to-use computer code in R (R Core Team, 2018) for drawing the heuristic and slicing functions, as well as for calculating the endpoints of heuristic (HLI) and likelihood insensitivity (LI) regions. We note in this regard that could not calculate the PI regions for pwf’s (2.4)–(2.6) because the pwf’s are only piecewise differentiable.

Although we can now easily determine subadditivity and insensitivity regions numerically, it is also instructive to occasionally arrive at the regions theoretically. We conclude this section with one such example.

**Example 3.1** (Continuation of Example 2.3) We start with the note that pwf (2.5) is concave on the interval \([0, 0.4]\) and thus subadditive. We therefore search for the largest \( b \geq 0.4 \) such that the pwf is subadditive on the entire interval \([0, b]\). In other words, we are searching for the smallest \( b \) in the interval \([0.4, 1]\) such that \( s(b) < 0 \). We split the interval \([0.4, 1]\) into two parts: \([0.4, 0.8]\) and \((0.8, 1]\). Since we shall find \( b^* \in [0.4, 0.8] \) such that \( s(b^*) = 0 \) and \( s(b) < 0 \) for all \( b > b^* \), we do not need to
consider the interval \((0.8, 1]\). Hence, our focus is on the \(b\) values in the interval \([0.4, 0.8]\) where the bound \(b/C_0 = 4/C_2\) is always satisfied. By considering \(p\)’s below and above \(b/C_0 = 4/C_2\), we arrive at the representation

\[
W_b(p) = \frac{1}{36} \times \begin{cases} 
5p(20 + 7p - 32b) & \text{when } 0 \leq p \leq b - 0.4, \\
-125b^2 + 90bp + 100b - 90p^2 - 20 & \text{when } b - 0.4 < p \leq b/2. 
\end{cases}
\]

Next we split our analysis into two cases:

(a) \(0 \leq p \leq b - 0.4\);  
(b) \(b - 0.4 < p \leq b/2\).

If condition (a) holds and \(b \leq 5/8\), then \(W_b(p)\) attains its minimum at \(p = 0\), with the value \(W_b(0) = 0\). If, however, \(b \geq 5/8\), then \(W_b(p)\) attains its minimum at...
\[ p = 2(8b - 5)/7 \], where it gives the value \[ W_b(b) = -5(5 - 8b)^2/63 \], which is negative for all \( b > 5/8 \).

If condition (b) holds, then \( W_b(p) \) is increasing in \( p \) and thus the minimum is attained at \( p = b - 0.4 \), at which it gives the value \( W_b(b - 0.4) = -125(b - 0.688)(b - 0.4)/36 \). The latter function of \( b \) has two roots, \( b = 0.4 \) and \( b = 0.688 \), thus implying that \( W_b(b) > 0 \) for all \( b \) in between the two roots.

In summary, we have \( \omega_{pb} = 5/8 = 0.625 \). This concludes Example 3.1.

4 Insensitivity Regions of Classical pwf’s

In this section, we provide a theoretical derivation and analysis of the insensitivity regions of several pwf’s that have commonly appeared in empirical and theoretical studies of Behavioural Economics. We note that many previous studies have also explored these pwf’s, but the following analysis—to the best of our knowledge—is more detailed and leads to full characterization of the insensitivity regions. As we shall see in next Sect. 5, knowing insensitivity regions as accurately as possible facilitates practical decision making.

4.1 Pwf Class 1

Consider the pwf (see Goldstein & Einhorn, 1987, for details)

\[
W(p) = \frac{p}{(p^\gamma + (1-p)^\gamma)\alpha}, \quad 0 \leq p \leq 1, \tag{4.1}
\]

where \( \alpha > 0 \) and \( \gamma \in (0, 1) \) are parameters. When \( \alpha = 1 \), the pwf was analyzed by Karmarkar (1978).

**Proposition 4.1** When \( \alpha \in (0, 1] \), pwf (4.1) is subadditive on the entire unit interval \( (0, 1) \), but when \( \alpha > 1 \), the pwf is not subadditive on the entire interval \( (0, 1) \).

**Proof** Consider first the case \( \alpha \in (0, 1] \). We shall use Theorem 2.1 and verify the bound

\[
\min_{p \in (0, b)} \{W(p) + W(b-p)\} \geq W(b)
\]

when \( b = 1 \). To this end, we rewrite \( W(p) + W(1-p) \) as \( f^{1-\gamma}(p) \), where \( f(p) = p^\gamma + (1-p)^\gamma \). Since \( \alpha \in (0, 1] \), we have \( f^{1-\gamma}(p) \geq 1 \) because \( f(p) \geq 1 \). This implies \( W(p) + W(1-p) \geq 1 = W(1) \) and verifies the condition of Theorem 2.1.

When \( \alpha > 1 \), the minimum of the function \( f^{1-\gamma}(p) \) over the interval \( (0, 1) \) is strictly less than 1, and thus the necessary and sufficient condition of Theorem 2.1 is violated. This completes the entire proof of Proposition 4.1.

In following Sect. 4.1.1, we shall illustrate the insensitivity regions of pwf (4.1) when \( \gamma = 0.93 \) and \( \alpha = 0.89 \), which are the values reported by Stott (2006).
Furthermore, note that in the special case $a = 1/\gamma$, pwf (4.1) becomes the one-parameter pwf proposed by Tversky and Kahneman (1992). As pointed out by Rieger and Wang (2006) and Ingersoll (2008), this pwf is not monotone for all $\gamma \in (0,1)$, although monotone for $\gamma > 0.279$. Based on empirical data, Camerer and Ho (1994) reported point estimates of $\gamma$ ranging from 0.28 to 1.87, with the average value of about 0.56. Therefore, to illustrate the insensitivity regions of this pwf, in Sect. 4.1.2 we shall use $\gamma = 0.56$ and thus $a = 1/\gamma = 1.785714$.

### 4.1.1 An Illustration of the Case $a \in (0,1]$ 

When $a \in (0,1]$, the equation $2w(b/2) = w(b)$ does not have a solution, and thus we set $w_{rb}^H = 1$. Proposition 4.1 tells us that $w_{rb} = 1$. Hence, in summary, the HMS and MS regions coincide and are

$$(0,w_{rb}^H) = (0,w_{rb}) = (0,1).$$

Figure 6 depicts pwf (4.1) as well as its heuristic and slicing functions when $\gamma = 0.93$ and $a = 0.89$. Applying the slicing method on the dual pwf $w^*$, we obtain $w_{rb}^H = 0.987$ and thus $b_{rb} = 0.013$. We also obtain $b_{rb}^H = 0.013$ using numerical methods. Consequently, the HLI and LI regions are

$$(b_{rb}^H,w_{rb}^H) = (b_{rb},w_{rb}) = (0.013, 1).$$

For comparison, the PI region (see Note 2.5) is

$$(p_1,p_2) = (0.179, 0.844).$$

![Figure 6](image-url)
4.1.2 An Illustration of the Case $a > 1$

When $a > 1$, the conditions of Boas’ test (Theorem 2.2) are satisfied and thus the heuristic and slicing methods give the same values of $w^H_{rb}$ and $w_{rb}$, as illustrated in Fig. 7 when $a = 1/\gamma = 1.785714$ under the parameter value $\gamma = 0.56$ borrowed from Camerer and Ho (1994). Specifically, we obtain $w^H_{rb} = w_{rb} = 0.959$. Furthermore, applying the slicing method on the dual pwf $w^*$, we obtain $w^*_{rb} = 1$ and thus $b_{rb} = 0$. We also obtain $b^H_{rb} = 0$ using numerical methods. Consequently, the HMS, MS, HLI and LI regions coincide:

$$(0, w^H_{rb}) = (0, w_{rb}) = (b^H_{rb}, w^H_{rb}) = (b_{rb}, w_{rb}) = (0, 0.959).$$

To compare, the PI region (see Note 2.5) is

$$(p_1, p_2) = (0.073, 0.852).$$

4.2 Pwf Class 2

Consider the pwf (see Goldstein & Einhorn, 1987, for details)

$$w(p) = \frac{zp^\gamma}{zp^\gamma + (1-p)^\gamma}, \quad 0 \leq p \leq 1,$$

(4.2)

with parameters $\alpha > 0$ and $\gamma \in (0, 1)$. Note that when $\alpha = 1$, pwf’s (4.2) and (4.1) are identical.

**Proposition 4.2** When $\alpha \geq 1$, pwf (4.2) is subadditive on the entire unit interval $(0, 1)$, but when $\alpha \in (0, 1)$, the pwf is not subadditive on the entire unit interval $(0, 1)$.
Proof Since \( w(1) = 1 \), we employ Theorem 2.1 with \( b = 1 \). To verify its necessary and sufficient condition, we start with the equation

\[
\frac{w(p) + w(1 - p)}{w(1) + w(1 - 1)} = 1 + \frac{(x^2 - 1)p(x^2 - 1) + \alpha(1 + (1 - x)^2)}{(x^2 + 1)p(x^2 + 1) + \alpha(1 + (1 - x)^2)}.
\]

Clearly now, \( w(p) + w(1 - p) \geq 1 \) whenever \( x \geq 1 \), but when \( x < 1 \), the minimum of \( w(p) + w(1 - p) \) over \( p \in (0, 1) \) is (strictly) smaller than 1. This concludes the proof of Proposition 4.2.

The conditions of Boas’s test (Theorem 2.2) hold for pwf (4.2) and thus the heuristic and slicing methods give the same values of \( w_H \) and \( w_{rb} \).

In following Sections 4.2.1 and 4.2.2, we shall numerically and graphically illustrate the cases \( x \in (0, 1] \) and \( x > 1 \) using specific parameter values borrowed from Wu and Gonzalez (1996) and Stott (2006), respectively. The corresponding functions will exhibit different patterns, and thus the resulting insensitivity intervals will also be different.

4.2.1 An Illustration of the Case \( x \in (0,1] \)

To illustrate, in Fig. 8 we use \( \gamma = 0.68 \) and \( \alpha = 0.84 \), as reported by Wu and Gonzalez (1996). Under these parameter values, we have \( w_{rb}^* = 0.965 \). Furthermore, we obtain \( w_{rb}^* = 1 \) and thus \( b_{rb} = 0 \). Consequently, the HMS, MS, and LI regions are the same:

\[
(0, w_{rb}^*) = (0, w_{rb}) = (b_{rb}, w_{rb}) = (0, 0.965).
\]

The PI region (see Note 2.5) is

\[
(p_1, p_2) = (0.102, 0.834).
\]
4.2.2 An Illustration of the Case $x > 1$

Figure 9 has been produced using $\gamma = 0.96$ and $x = 1.4$ (Stott, 2006). Since the heuristic and slicing functions do not dip below zero on the unit interval $(0, 1)$, we therefore conclude that $w_{rb}^H = w_{rb} = 1$, and thus the HMS and MS regions are $(0, w_{rb}^H) = (0, w_{rb}) = (0, 1)$.

However, $w_{rb}^a = 0.231$ and thus $b_{rb} = 0.769$. Furthermore, we obtain $b_{rb}^H = 0.769$ using numerical methods. Consequently, the HLI and LI regions are $(b_{rb}^H, w_{rb}^H) = (b_{rb}, w_{rb}) = (0.769, 1)$.

For comparison, the PI region (see Note 2.5) is $(p_1, p_2) = (0.399, 1)$.

4.3 Pwf Class 3

Based on a system of axioms, Prelec (1998) derived the pwf

$$w(p) = \exp\{-\beta(-\log p)^2\}, \quad 0 \leq p \leq 1,$$

with parameters $x > 0$ and $\beta > 0$. When $x \in (0, 1)$, the pwf is caved, and when $x > 1$, it is S-shaped. When $x = 1$, we have $w(p) = p^\beta$ and thus concavity (or convexity) of this pwf is determined by the value of $\beta$. In general, $x$ controls the concavity-convexity of the pwf and is used as an index of optimism, whereas $\beta$ indicates the locus of the inflexion point relative the 45 degree line and is used as an
index of sensitivity. For more details, we refer to al-Nowaihi and Dhami (2010). Since our focus is on cavex pwf’s, we shall next consider only the case $\alpha \in (0, 1)$.

**Proposition 4.3** When $\alpha \in (0, 1)$, the following statements hold for pwf (4.3):

(a) if $\beta \leq (\log 2)^1 - \alpha$, then $w_{rb}^H = 1$;
(b) if $\beta > (\log 2)^1 - \alpha$, then $w_{rb}^H < 1$.

**Proof** To prove part (a), note that the heuristic function $h(b)$ is non-negative if and only if

$$\log 2 - \beta(-\log p + \log 2)^x + \beta(-\log p)^x \geq 0. \quad (4.4)$$

The bound $(-\log p + \log 2)^x \leq (-\log p)^x + (\log 2)^x$ holds for all $p \in (0, 1]$, and so

$$\log 2 - \beta(-\log p + \log 2)^x + \beta(-\log p)^x,$$

$$\geq \log 2 - \beta(-\log p)^x - \beta(\log 2)^x + \beta(-\log p)^x,$$

$$= \log 2 - \beta(\log 2)^x.$$

The right-hand side is non-negative due to $\beta \leq (\log 2)^1 - \alpha$, which establishes statement (4.4) and completes the proof of part (a).

To prove part (b), we first note that since $\beta > (\log 2)^1 - \alpha$, we have $h(0) > 0$ and also $h(1) < 0$. Since $h(b)$ is continuous on $[0, 1]$, classical Bolzano’s theorem says that there must be at least one $p \in (0, 1)$ such that $h(b) = 0$. This completes the entire proof of Proposition 4.3.

Consider the case $\beta = 1$. Since we always assume $\alpha \in (0, 1)$, the condition $\beta > (\log 2)^1 - \alpha$ holds and thus $w_{rb}^H < 1$. As to $w_{rb}$, we check that the conditions of Boas’ test (Theorem 2.2) are satisfied and thus $w_{rb} = w_{rb}^H$. Hence, to determine $w_{rb}$, we can apply the slicing method or the heuristic technique. To illustrate, let $\alpha = \ldots$

![Fig. 10 Pwf (4.3) (left panel, solid), its heuristic (right panel, dashed) and slicing (right panel, solid) functions when $\beta = 1$ and $\alpha = 0.533$](image-url)
0.533 as reported by Bleichrodt and Pinto (2000). Figure 10 depicts the pwf as well as its heuristic and slicing functions. The roots of these two functions are $w_{rb}^H = w_{rb} = 0.973$. An application of the slicing method on the dual pwf $w^*$ gives $w_{rb}^* = 1$ and thus $b_{rb} = 0$. Furthermore, we obtain $b_{rb}^H = 0$ using numerical methods. Consequently, the HMS, MS, HLI and LI regions are the same:

$$(0, w_{rb}^H) = (0, w_{rb}) = (b_{rb}^H, w_{rb}^H) = (b_{rb}, w_{rb}) = (0, 0.973).$$

For comparison, the PI region (see Note 2.5) is

$$(p_1, p_2) = (0.056, 0.849).$$

### 4.4 Pwf Class 4

Consider the polynomial pwf (see Rieger and Wang, 2006, for details)

$$w(p) = \frac{3(1 - \beta)}{x^2 - x + 1}(p^3 - (x + 1)p^2 + xp) + p, \quad 0 \leq p \leq 1, \quad (4.5)$$

with parameters $x \in (0, 1)$ and $\beta \in (0, 1)$. For other similar forms, we refer to, e.g., Walther (2003) and Pfiffelmann (2011). Figure 10 depicts the pwf when $\beta = 0.2$ and $x = 0.3$, the parameter values we have borrowed from an illustrative example of Rieger and Wang (2006). We obtain $w_{rb}^H = w_{rb} = 0.867$. Note that, barring some rounding errors, 0.867 is equal to $(2x + 2)/3$, an observation whose meaning will become clear from the next proposition. Applying the slicing method on the dual pwf $w^*$, we obtain $w_{rb}^* = 1$ and thus $b_{rb} = 0$. Furthermore, we obtain $b_{rb}^H = 0$ using numerical methods. Consequently, the HMS, MS, HLI and LI regions are the same:

$$(0, w_{rb}^H) = (0, w_{rb}) = (b_{rb}^H, w_{rb}^H) = (b_{rb}, w_{rb}) = (0, 0.867).$$

The PI region (see Note 2.5) is

$$(p_1, p_2) = (0.137, 0.730).$$

The following two propositions, which are theoretical in nature, will clarify some of the above used numerical values and their relationship to the pwf parameters. Specifically, in Proposition 4.4 we shall derive a closed-form expression for the upper bound $w_{rb}$ of subadditivity of pwf (4.5), whereas in Proposition 4.5 we shall derive a closed-form expression of the corresponding lower bound $b_{rb}$ of subadditivity. To obtain the latter bound, in Proposition 4.5 we shall first derive a closed-form expression for the upper bound $w_{rb}^*$ of subadditivity of the dual pwf $w^*$.

**Proposition 4.4** When $x \geq 1/2$, pwf (4.5) is subadditive on the entire unit interval $(0, 1)$, but when $x < 1/2$, the pwf is subadditive only on the interval $(0, w_{rb})$, where
\[ \frac{w_{rb}}{3} = \frac{2x + 2}{3}. \]

**Proof** We start with the equation

\[ w(p) + w(1 - p) = 1 + \frac{3(1 - \beta)}{x^2 - x + 1} p(1 - p)(2x - 1). \] (4.6)

Since \( x \in (0, 1) \) and \( \beta \in (0, 1) \), the bound \( \min\{w(p) + w(1 - p)\} \geq w(1) = 1 \) is equivalent to \( \min\{p(1 - p)(2x - 1)\} \geq 0 \), which holds when \( x \geq 1/2 \). Theorem 2.1 implies the first part of Proposition 4.4.

When \( x < 1/2 \), we use the slicing method. For this, we first obtain the formula

\[ W_b(p) = \frac{9(1 - \beta)}{x^2 - x + 1} p(p - b) \left( b - \frac{2x + 2}{3} \right). \] (4.7)

Next, we find the infimum of the function \( W_b(p) \) with respect to \( p \in (0, b/2) \). Consider two cases: When \( b < (2x + 2)/3 \), the right-hand side of Eq. (4.7) is non-negative, and since \( W_b(p) = 0 \) at \( p = 0 \), we conclude that the infimum of the function \( W_b(p) \) with respect to \( p \in (0, b/2) \) is equal to 0. When \( b \geq (2x + 2)/3 \), to find the infimum of the function \( W_b(p) \) with respect to \( p \in (0, b/2) \) is tantamount to finding the infimum of the function \( p(p - b) \) with respect to \( p \in (0, b/2) \), which is achieved at \( p = b/2 \) and is equal to \( -(b/2)^2 \). In summary, we have

\[ s(b) = \inf_{p \in (0, b/2)} W_b(p) = \begin{cases} 0 & \text{when } b < \frac{2x + 2}{3}, \\ \frac{9(1 - \beta)}{x^2 - x + 1} \left( -\frac{b^2}{4} \right) \left( b - \frac{2x + 2}{3} \right) & \text{when } b \geq \frac{2x + 2}{3}. \end{cases} \]

Hence, the slicing function \( s(b) \) dips below 0 immediately to the right of the point \( b = (2x + 2)/3 \), thus establishing the result

\[ w_{rb} = \inf\{b \in (0, 1) : s(b) < 0\} = \frac{2x + 2}{3}. \]

This finishes the proof of Proposition 4.4. □

**Note 4.1** The heuristic function is

\[ h(b) = \frac{9(1 - \beta)}{x^2 - x + 1} \left( -\frac{b^2}{4} \right) \left( b - \frac{2x + 2}{3} \right) \]

for all \( b \in [0, 1] \). Clearly, \( w_{rb}^h = w_{rb} \), and the equation \( s(b) = h(b) \) holds whenever \( b \geq w_{rb} \), but not when \( b < w_{rb} \) (recall Note 3.2).

We next derive a closed-form formula for \( w_{rb}^* \) and thus for \( b_{rb} = 1 - w_{rb}^* \). The derivation is analogous to that used in the proof of Proposition 4.4.
Proposition 4.5  When $z \leq 1/2$, the dual function $w^*$ of pwf (4.5) is subadditive on the entire unit interval $(0, 1)$, but when $z > 1/2$, the function $w^*$ is subadditive only on the interval $(0, w^*_{rb})$, where

$$w^*_{rb} = \frac{4 - 2x}{3}.$$ 

Consequently,

$$b_{rb} = \frac{2x - 1}{3}.$$ 

Proof  In view of Eq. (4.6), we have

$$w^*(p) + w^*(1-p) = 2 - w(p) - w(1-p)$$

$$= 1 - \frac{3(1-\beta)}{x^2 - x + 1} p(1-p)(2x - 1).$$

Since $x \in (0, 1)$ and $\beta \in (0, 1)$, the bound $\min\{w^*(p) + w^*(1-p)\} \geq w^*(1) = 1$ is equivalent to $\min\{p(1-p)(2x - 1)\} \leq 0$, which holds when $x \leq 1/2$. Thus, Theorem 2.1 implies the first part of Proposition 4.5.

Consider now the case $x > 1/2$. First, we derive a formula for $W_b(p)$, which is

$$W_b(p) = w^*(p) + w^*(b - p) - w^*(b)$$

$$= \frac{9(1-\beta)}{x^2 - x + 1} p(b - \frac{4 - 2x}{3}).$$

Following virtually the same arguments as in the second part of the proof of Proposition 4.4, we obtain

$$s(b) = \inf_{p \in (0, b/2)} W_b(p) = \begin{cases} 0 & \text{when } b < \frac{4-2x}{3}, \\ \frac{9(1-\beta)}{(x^2 - x + 1)} \left(\frac{-b^2}{4}\right) \left(\frac{b - \frac{4 - 2x}{3}}{3}\right) & \text{when } b \geq \frac{4-2x}{3}. \end{cases}$$

Consequently,

$$w^*_{rb} = \inf\{b \in (0, 1] : s(b) < 0\} = \frac{4 - 2x}{3}$$

and thus

$$b_{rb} = 1 - w^*_{rb} = \frac{2x - 1}{3}.$$ 

This completes the proof of Proposition 4.5.  \[\square\]
5 An Illustrative Case

Many studies have shown that determining likelihood insensitivity regions facilitates decision making. To make our discussion more concrete, we next present an example that is similar—albeit simplified due to space and confidentiality considerations—to the many real world cases that we have dealt with consulting firms in the construction sector.

5.1 Description

Suppose that a property developer owns a piece of land in the resort town of Punta del Este, Uruguay, and wishes to decide whether to construct twin 12 storey or a single 23 storey high-rise buildings on the land. (In Punta del Este, the local Government has set up a limit of 23 storey height.) There are pros and cons of building taller buildings. First, there is evidence of a positive premium on the price per square meter when units are higher, because of factors such as better view, less noise pollution, feel of prestige or reputation (e.g., Chau et al., 2007; Picken & Ilozor, 2003; Wong et al., 2011). Second, taller buildings could benefit from economies of scale, mainly from the cost of land, although empirical evidence suggests that construction marginal costs are increasing with height (Barr, 2016). However, taller buildings could be riskier than shorter ones, primarily because the average price per square meter is higher for taller high-rise buildings than for shorter ones, and thus their demand can be lower. Third, twin high-rise buildings on the same land could imply less green area and less open-air amenities than in the case of one taller high-rise building with a smaller footprint area, which could make it less attractive to potential buyers.

In view of these considerations, new projects of taller buildings have been developed in Punta del Este during the last few years. Among them are Cipriani (24 storeys), Fendi Château (27 storeys), Trump (25 storeys), and Venetian (27 storeys) Towers, which are some of the examples of high-rise buildings above the height restriction (Punta del Este Internacional, 2019). Indeed, in some special cases, the local Government can wave the height requirement and allow new buildings higher than 23 storeys.

In Punta del Este, the main demand for tall high-rise residential buildings comes from foreign buyers, primarily Argentineans and Brazilians. However, during the year 2020, due to the COVID-19 pandemic and economic difficulties in Argentina, the real estate market in Punta del Este considerably slowed down. Consequently, several of the aforementioned projects faced significant financial problems (Drucker & Andreoni, 2019; Foster, 2018; Parks, 2021), and most of the new building projects concentrated on mid high-rise residential buildings oriented toward the domestic demand.

Hence, suppose that the developer wishes to decide between tall high-rise buildings, expecting an increase in foreign demand in the future, or lower high-rise buildings oriented toward the domestic demand. Consider two options as an illustration:
A: Twin 12 storey towers;  
B: Single 23 storey tower.

For each option, we assume binary (i.e., two-outcome) prospects similar to the many illustrative examples of Behavioural Economics (e.g., Wakker, 2010, Sect. 7.11). Relevant data, which are aligned with recent market prices and profit margins in Punta del Este (INE, 2019), are in Table 3.

The sales, costs, and profits are in millions of USD, because in Punta del Este most of the real estate transactions are in US Dollars.

We see from Table 3 that the (unknown) profit probabilities \( p_A \) and \( p_B \) are transformed by a pwf \( w \) that reflects developer’s psychological traits. Generally speaking, this pwf is unknown but its form can be reasonably guessed and the underlying parameters estimated, as has been done in numerous scholarly studies (e.g., Stott, 2006, and references therein).
5.2 Decision Making

We assume that the developer aims to maximize the expected profit, which we calculate as in the dual utility theory (DU) of Yaari (1987). Hence, in order to decide which of the two options A or B is preferable, we compare the expected profits

$$DU(A) = 6w(p_A) + 2(1 - w(p_A)) = 2 + 4w(p_A)$$  \hspace{1cm} (5.1)$$

and

$$DU(B) = 4w(p_B) + 0(1 - w(p_B)) = 4w(p_B).$$  \hspace{1cm} (5.2)$$

Assume for the sake of argument that the developer transforms the outcome probabilities according to $pwf$ (4.3) with $\beta = 1$ and $\alpha = 0.533$, as discussed in Sect. 4.3. Using Eqs. 5.1) and (5.2), we then compare the expected profits $DU(A)$ and $DU(B)$ using the contours depicted in Fig. 12. In the figure, the bluer the region, the more positive the difference $DU(A) - DU(B)$ is, and thus the more preferable the option A is. The redder the region, the reverse ranking is truer: option B is more preferable.

We now reflect on these findings in a more general setting, given that the actual probabilities $p_A$ and $p_B$ are unknown in practice. Yet, it is quite reasonable to assume that given the subject matter knowledge of experienced developers, these probabilities belong to a certain known region of admissible probabilities, that is, let

$$p_A, p_B \in (p_{\text{min}}, p_{\text{max}})$$  \hspace{1cm} (5.3)$$

for some known $p_{\text{min}}$ and $p_{\text{max}}$. 

Fig. 12 The contour graph of $DU(A) - DU(B)$ with respect to $p_A$ and $p_B$
In addition to condition (5.3), it is also realistic to assume—given of course a reasonably informed developer—that the probabilities \( p_A \) and \( p_B \) of favourable outcomes in both options \( A \) and \( B \) are larger than the probabilities \( 1 - p_A \) and \( 1 - p_B \) of the less favourable, yet possible, outcomes. Hence, we let

\[
p_A, p_B > 1/2. \tag{5.4}
\]

As a consequence, we can now also assume without loss of generality that \( p_{\min} > 1/2 \).

**Definition 5.1** We call the decision-maker *adequately informed* (relative to options \( A \) and \( B \)) when both conditions (5.3) and (5.4) are satisfied.

We shall see from the following proposition that decision making depends on how the endpoints \( p_{\min} \) and \( p_{\max} \) of the knowledge-based interval (5.3) interact with the subadditivity regions of the \( w \)-function and its dual function \( w^* \), whose accurate determination has been the main topic of the present paper.

**Proposition 5.1** For an adequately-informed decision maker, scenario \( A \) is preferable to \( B \) whenever the following two conditions hold:

1. the \( w \)-function is subadditive on the interval \((0, p_{\max})\);
2. the dual \( w^* \)-function is subadditive on the interval \((0, 1 - p_{\min})\).

Before we prove the proposition, we want to clarify its role in decision making. First, just like probabilities \( p_A \) and \( p_B \), the \( w \)-function is unknown in practice, but its form can be guessed and parameters estimated from empirical studies (e.g., Stott, 2006, and references therein). These arguments, however, cannot provide a clear-cut region of subadditivity \((b_{rb}, w_{rb})\), as it depends on the decision maker. Consequently, it is rather challenging to say whether or not the inclusion \((p_{\min}, p_{\max}) \subseteq (b_{rb}, w_{rb})\) holds, and thus whether or not \( p_A, p_B \in (b_{rb}, w_{rb}) \). Nevertheless, it is clear that having more information about the probabilities \( p_A \) and \( p_B \) allows the researcher to specify a narrower interval \((p_{\min}, p_{\max})\) and thus reduce uncertainty about the validity of the inclusion \((p_{\min}, p_{\max}) \subseteq (b_{rb}, w_{rb})\); if it holds, scenario \( A \) is preferable to \( B \), but if not, then we cannot confidently rank \( A \) and \( B \). Despite the latter note, making practical (as opposed to mathematically rigorous) decisions may not be so difficult: the probability region \( p_A, p_B > 1/2 \) in Fig. 12 is mostly blue, and the subadditivity region \((b_{rb}, w_{rb}) = (0, 0.973)\) (see Sect. 4.3) covers all practically plausible probabilities.

**Proof of Proposition 5.1** We have

\[
\frac{1}{2} (DU(A) - DU(B)) = 1 + 2w(p_A) - 2w(p_B). \tag{5.5}
\]

When \( p_A \geq p_B \), we have \( DU(A) > DU(B) \) because \( w \) is non-decreasing, and so \( A \) is preferred to \( B \). Next we assume \( p_A < p_B \) and write \( p_B = p_A + \Delta \), where \( \Delta = p_B - p_A > 0 \). We get
\[
\frac{1}{2} (DU(A) - DU(B)) = 1 + 2w(p_A) - 2w(p_A + \Delta) \\
= (w(p_A) + w(\Delta) - w(p_A + \Delta)) \\
+ (1 + w(p_A) - w(\Delta) - w(p_A + \Delta)).
\] (5.6)

Since \(p_A, p_B, \Delta \in [0, p_{\text{max}}]\), condition (a) implies
\[
w(p_A) + w(\Delta) - w(p_A + \Delta) \geq 0.
\]

To verify non-negativity of the remaining quantity on the right-hand side of Eq. (5.6), we convert \(w\) into \(w^*\) and thus need to show
\[
w^*(1 - \Delta) + w^*(1 - p_B) - w^*(1 - p_A) \geq 0.
\] (5.7)

In view of condition (5.4), we have \(\Delta \leq 1/2\) and thus \(w(1 - \Delta) \geq w(\Delta)\). Consequently, bound (5.7) follows if \(w^*(1) \geq w^*(1 - p_A)\), which is of course equivalent to
\[
w^*(\Delta) + w^*(1 - p_B) \geq w^*(\Delta + 1 - p_B).
\] (5.8)

Since \(\Delta, (1 - p_B), (1 - p_A) \in (0, 1 - p_{\text{min}})\), subadditivity of \(w^*\) postulated in condition (b) implies bound (5.8) and completes the entire proof of Proposition 5.1.

The next proposition tackles the same problem but under the framework of Baillon et al. (2020). Naturally, we now need to assume that the pwf \(w\) is continuously differentiable.

**Proposition 5.2** For an adequately-informed decision maker, scenario A is preferable to B whenever the bound \(w'(p) < 1\) holds on the interval \((p_{\text{min}}, p_{\text{max}})\).

The same comment as that below Proposition 5.1 can also be made now, but with the MS region \((b_{rb}, w_{rb})\) replaced by the PI region \((p_1, p_2)\). Recall in this regard that in Sect. 4.3 we obtained \((p_1, p_2) = (0.056, 0.849)\), which is slightly shorter than \((b_{rb}, w_{rb}) = (0, 0.973)\). The following proof of Proposition 5.2 illuminates the role of the Baillon et al. (2020) condition \(w'(p) < 1\) in the current context.

**Proof of Proposition 5.2** Equation (5.5) says that A is preferred to B when \(p_A \geq p_B\), because \(w\) is non-decreasing. When \(p_A < p_B\), we again start with Eq. (5.5) and note that \(DU(A) > DU(B)\) if and only if \(w(p_B) - w(p_A) < 1/2\). By the mean value theorem, there is \(p \in (p_{\text{min}}, p_{\text{max}})\) such that
\[
w(p_B) - w(p_A) = w'(p)(p_B - p_A).
\]

The right-hand side is smaller than 1/2 because \(w'(p) < 1\) and \(p_B - p_A < 1/2\), the latter being a consequence of condition (5.4). This concludes the proof of Proposition 5.2.
6 Summary and Concluding Notes

Subadditivity and likelihood-insensitivity regions are known to have important implications in decision making. So far, such regions have been determined mainly with the help of a heuristic technique (Wakker, 2010, p. 227). Although the technique works well for many pwf’s, we have shown that there are pwf’s for which the technique fails to identify correct subadditivity intervals. We have also provided conditions under which the heuristic technique correctly identifies subadditivity and thus insensitivity regions.

We have proposed a method—dubbed the slicing method—that accurately determines subadditivity and thus likelihood-insensitivity regions for all continuous pwf’s. The method is general and is not restricted to any specific class of pwf’s, although we concentrate on cavex pwf’s that are of particular interest in Behavioural Economics. The generality is important because even though cavex pwf’s are most common in the literature, differently shaped pwf’s also naturally arise (e.g., Dhami, 2016). To facilitate effortless determination of subadditivity and insensitivity regions, we have provided a ready-to-use computer code.

To illustrate the proposed method, we have analyzed the subadditivity and likelihood-insensitivity regions of a number of well-known pwf’s. We have also compared the obtained regions to those arising from the recent research of Baillon et al. (2020), where an alternative notion of likelihood-insensitivity region has been introduced.

We have also presented and analyzed an illustrative case that demonstrates the use of the derived results in practical settings.

Several interesting problems remain for future research:

1. We have developed the slicing method for continuous pwf’s, but there are situations when discontinuous pwf’s naturally arise (Wakker, 2021). To accommodate such functions, the foundational results that we relied upon, or the new ones that we have derived, need to be carefully revisited, extended, and generalized.

2. We have seen that for more complex pwf’s, closed-form formulas for the endpoints of insensitivity regions are difficult, perhaps even impossible, to derive. Given the absence of such formulas, statistical estimation procedures (e.g., confidence intervals, hypothesis tests, etc.) for the endpoints would be useful to obtain.

3. One of the most influential theories in decision theory is prospect theory (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992), which posits distortions of objective probabilities. Optimization processes using this theory are quite complex and often require algorithmic solutions (e.g., Best et al., 2014; De Giorgi et al., 2007; Gong et al., 2018). A direction for future research would be to study the relevance of insensitivity regions in financial applications, such as those in the aforementioned papers.

4. Since finding insensitivity regions often requires numerical solutions, maximization processes of utilities within insensitivity regions would rely on numerical solutions. Hence, available heuristic optimization techniques (e.g.,
Chang, 2012; Doering et al., 2019; Khan et al., 2021a, 2021b) would need to be adapted and fine-tuned to accommodate problems such as those considered in the present paper.

Appendix: Computer code

We have written a computer code in the R programming language (R Core Team, 2018) for drawing pwf’s $w$ and their dual counterparts $w^*$, their heuristic $h(b)$ and slicing $s(b)$ functions, and also for calculating the heuristic and true worst rank boundaries corresponding to $w$ and $w^*$. We next present the code in a way that allows the reader to simply copy and paste it into the R console.

A.1 Calculating $w_{rb}^H$ and $w_{rb}$

For concreteness, we have chosen pwf (2.5), which consists of two continuous functions pieced together. The code for plotting the pwf is next.

```r
rm(list = ls(all.names = TRUE))
n <- 10000
p <- seq(0.00001, 0.99999, length = n)
w <- function(p){0.2-(p<=0.4)*(p-0.4)^2/0.8+(p>0.4)*(p-0.4)^2/0.45}
plot(p, w(p), type = "l") # probability weighting function

h <- function(p){2*w(p/2)-w(p)}
plot(p, h(p), type = "l") # heuristic function
points(p, h(p)*0, type = "l") # horizontal axis
uniroot(h, c(0.00001,1))$root # heuristic worst rank boundary
```

The following piece of code, which is to be copied and pasted into the same R console, draws the slicing function and calculates the upper bound $w_{rb}$ of subadditivity. We note at the outset that since this piece of code uses a loop, it takes more time to run than any of the previous pieces.
b <- seq(0.00001, 0.99999, length = n)
min_y <- numeric(n)
for (i in 1:length(b)){
  ww <- function (p, bb=b[i]){
    return (w(p)+w(bb-p)-w(bb))
  }
  x <- seq(0.00001, b[i], length = n)
  y <- ww(x)
  min_y[i] <- min(y)
}
points(b, min_y, type = "l", lwd=3)  # slicing function
min(b[min_y < -0.00000001])  # worst rank boundary

A.2 Calculating $b_{rb}^H$ and $b_{rb}$

The above code can easily be adapted to work with the dual pwf $w^*$ and thus used to find $w_{rb}^*$ and $w_{rb}^*$. Namely, to obtain $w_{rb}^*H$ and $w_{rb}^*$, we only need to replace the line

$$w <- function(p){0.2-(p<=0.4)*(p-0.4)^2/0.8+(p>0.4)*(p-0.4)^2/0.45}$$

in the first piece of the above code by the two lines

$$v <- function(p){0.2-(p<=0.4)*(p-0.4)^2/0.8+(p>0.4)*(p-0.4)^2/0.45}$$
$$w <- function(p){1-v(1-p)}$$

We note that for this particular pwf, the code will not produce any meaningful result because the heuristic and slicing functions of the dual pwf $w^*$ do not dip below 0, and thus no roots of the two functions can be found. This automatically implies $w_{rb}^*H = w_{rb}^* = 1$ and, consequently, $b_{rb}^H = b_{rb} = 0$ via the equations $b_{rb}^H = 1 - w_{rb}^*$ and $b_{rb} = 1 - w_{rb}^*$. 

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Declarations

Conflict of interest  The authors declare no conflict of interest.

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