AN ALGORITHM AND ESTIMATES FOR THE ERDÖS-SELFRIDGE FUNCTION
ADDENDUM

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1. Introduction

This is an addendum to our paper of the same title recently published as part of the proceedings of the ANTS XIV conference, through Mathematical Sciences Publishers:

https://doi.org/10.2140/obs.2020.4.371

Due to space and time constraints, we have three minor results that did not make the official version of our paper, so we present them here.

1. Two more values of $g(k)$, for $k = 376, 377$.
2. A proof of the claim at the end of Section 6 that
   \[ \limsup_{k \to \infty} \frac{\hat{g}(k+1)}{\hat{g}(k)} = \infty. \]
3. A proof of Lemma 6.4, and hence Theorem 6.1, with an exact constant near 0.78843\ldots. The proof in the ANTS paper brackets the constant between 0.5306\ldots and 1. Thus, $\log \hat{g}(k) \sim (0.78843\ldots) \cdot (k/ \log k)$.

If you wish to cite this work, we encourage you to cite the ANTS paper linked above, and add a note to that citation with a link to this arxiv paper if you are specifically referring to the results mentioned here.

2. Two more $g(k)$ values

We have

\[
\begin{align*}
g(376) &= 7778804220120654420924631668091 \\
g(377) &= 5973303871796437264595936954237
\end{align*}
\]

$g(376)$ took one week, wall time, and $g(377)$ took about two weeks.

3. Proof of Claim from the end of Section 6

**Theorem 3.1.** We have

\[
\limsup_{k \to \infty} \frac{\hat{g}(k+1)}{\hat{g}(k)} = \infty.
\]

This proof uses some of the ideas from Section 3 in [2].
Proof. We will prove a lower bound proportional to \( \log k \) in the case when \( k + 1 \) is an odd prime. Since there are infinitely many primes, this will be sufficient to prove the theorem.

Note that \( \hat{g}(k + 1)/\hat{g}(k) = (M_{k+1}/M_k)(R_k/R_{k+1}) \).

First, we look at \( M_{k+1}/M_k \). Recall that
\[
M_k = \prod_{p \leq k} p^{\lfloor \log_p k \rfloor + 1} \quad \text{and} \quad M_{k+1} = \prod_{p \leq k+1} p^{\lfloor \log_p (k+1) \rfloor + 1}.
\]
We can write
\[
M_{k+1} = \prod_{p \leq k+1} p^{\lfloor \log_p (k+1) \rfloor + 1} \quad \text{and} \quad M_k = \prod_{p \leq k} p^{\lfloor \log_p k \rfloor + 1}.
\]
Here we use the fact that for every prime \( p \leq k \), \( \lfloor \log_p (k+1) \rfloor = \lfloor \log_p k \rfloor \) when \( k + 1 \) is prime.

Next we look at \( R_k/R_{k+1} \). Using the same notation for \( a_{ip} \) as above, and noting that the prime \( k + 1 \) will contribute \( k(k+1) \) residues, by Kummer’s theorem, we have
\[
\frac{R_k}{R_{k+1}} = \frac{\prod_{p \leq k} \prod_{i=0}^{\lfloor \log_p k \rfloor} (p - a_{ip})}{k(k+1) \cdot \prod_{p \leq k} (p - (a_{op} + 1)) \prod_{i=1}^{\lfloor \log_p (k+1) \rfloor} (p - a_{ip})} \cdot \prod_{p \leq k} \frac{p - a_{op}}{p - (a_{op} + 1)}.
\]
Again we note that \( \lfloor \log_p (k+1) \rfloor = \lfloor \log_p k \rfloor \), and observe that the representation for \( k+1 \) in base \( p \) is the same as for \( k \), with the exception of the least significant digit, \( a_{op} \), which is one larger, for all primes \( p \leq k \). This is only because \( k + 1 \) is prime; \( k + 1 \mod p \) cannot be zero unless \( p = k + 1 \).

We then bound
\[
\frac{p - a_{op}}{p - (a_{op} + 1)} \geq \frac{p}{p - 1}
\]

to obtain that
\[
\frac{R_k}{R_{k+1}} \geq \frac{1}{k(k+1)} e^\gamma \log k(1 + o(1))
\]
using Mertens’s theorem. We deduce that
\[
\frac{M_{k+1}/R_{k+1}}{M_k/R_k} \gg \frac{(k + 1)^2}{k(k+1)} \log k \geq \log k
\]
to complete the proof. \( \square \)

4. A Constant for Theorem 6.1

Here is our new proof of Theorem 6.1. All the substantive changes are in Lemma 6.4.

Theorem 4.1 (Theorem 6.1).
\[
\frac{\log \hat{g}(k)}{k/\log k} \sim 0.7884305 \ldots
\]
Applying the definitions for $M_k$ and $R_k$ above, we have

$$\hat{g}(k) = \frac{M_k}{R_k} = \frac{\prod_{p \leq k} p^{\lfloor \log_p k \rfloor + 1}}{\prod_{p \leq k} \prod_{i=0}^{\lfloor \log_p k \rfloor} (p - a_{ip})} = \prod_{p \leq k} \prod_{i=0}^{\lfloor \log_p k \rfloor} \frac{p}{p - a_{ip}}$$

$$= \prod_{p \leq \sqrt{k}} \prod_{i=0}^{\lfloor \log_p k \rfloor} \frac{p}{p - a_{ip}} \cdot \prod_{\sqrt{k} < p \leq k} \prod_{i=0}^{\lfloor \log_p k \rfloor} \frac{p}{p - a_{ip}}$$

$$= \prod_{p \leq \sqrt{k}} \prod_{i=0}^{\lfloor \log_p k \rfloor} \frac{p}{p - a_{ip}} \cdot \prod_{\sqrt{k} < p \leq k} \prod_{i=0}^{\lfloor \log_p k \rfloor} \frac{p}{p - a_{ip}} \cdot \prod_{\sqrt{k} < p \leq k} \prod_{i=0}^{\lfloor \log_p k \rfloor} \frac{p}{p - a_{1p}} p - a_{0p}.$$  

Here we observed that $\lfloor \log p \rfloor + 1 = 2$ when $p > \sqrt{k}$.

We will show that the product on the factor involving $a_{0p}$ is exponential in $k/\log k$, and is therefore significant; and the other two factors, the product on primes up to $\sqrt{k}$, and the factor with $a_{1p}$, are both only exponential in roughly $\sqrt{k}$.

We bound the first product, on $p \leq \sqrt{k}$, with the following lemma.

**Lemma 4.2** (6.2).  
\[
\prod_{p \leq \sqrt{k}} \prod_{i=0}^{\lfloor \log_p k \rfloor} \frac{p}{p - a_{ip}} \ll e^{3\sqrt{k}(1+o(1))}.  
\]

**Proof.** We note that $a_{ip} \leq p - 1$, giving

$$\prod_{p \leq \sqrt{k}} \prod_{i=0}^{\lfloor \log_p k \rfloor} \frac{p}{p - a_{ip}} \leq \prod_{p \leq \sqrt{k}} p^{\lfloor \log_p k \rfloor + 1} \leq \prod_{p \leq \sqrt{k}} p^{3\lfloor \log_p \sqrt{k} \rfloor}.$$  

From [1, Ch. 22] we have the bound $\sum_{p \leq x} [\log_p x] \log p = x(1 + o(1))$. Exponentiating and substituting $\sqrt{k}$ for $x$ gives the desired result. \qed

Next, we show that the product involving $a_{1p}$ is small.

**Lemma 4.3** (6.3).  
\[
\prod_{\sqrt{k} < p \leq k} \frac{p}{p - a_{1p}} \ll e^{O(\sqrt{k} \log \log k)}.  
\]

**Proof.** We split the product at $2\sqrt{k}$. For the lower portion, we have

$$\prod_{\sqrt{k} < p \leq 2\sqrt{k}} \frac{p}{p - a_{1p}} \leq (2\sqrt{k})^{(2\sqrt{k})} \ll e^{O(\sqrt{k})}.$$  

For the upper portion, we have

$$\prod_{2\sqrt{k} < p \leq k} \frac{p}{p - a_{1p}} \leq \prod_{2\sqrt{k} < p \leq k} \frac{p}{p - \sqrt{k}} = \prod_{2\sqrt{k} < p \leq k} \left(1 + \frac{2\sqrt{k}}{p}\right)$$

$$\leq \prod_{2\sqrt{k} < p \leq k} \left(1 + \frac{1}{p}\right)^{2\sqrt{k}+1}.$$
using the fact that \((1 + x/p) \leq (1 + 1/p)^x\) if \(x, p\) are positive integers. Mertens’s Theorem then gives the bound

\[
(e^\gamma (\log k)(1 + o(1)))^{2\sqrt{k} + 1} \ll e^{O(\sqrt{k} \log \log k)}.
\]

We now have

\[
\log \hat{g}(k) = \log \left( \prod_{\sqrt{k} < p < k} \frac{p}{p - a_0 p} \right) + O(\sqrt{k} \log \log k).
\]

The following lemma wraps up the proof of our theorem.

**Lemma 4.4 (6.4 - new).** There exists a constant \(c\) where \(c \approx 0.7884305\ldots\) where

\[
\log \left( \prod_{\sqrt{k} < p \leq k} \frac{p}{p - a_0 p} \right) = c \cdot \frac{k}{\log k} (1 + o(1)).
\]

**Proof.** Fix \(a_1 p = a\). Then \(k/(a + 1) < p \leq k/a\), and \(a_0 p = k \mod p = k - ap\) and \(p - a_0 p = p - (k - ap) = (a + 1)p - k\). We have

\[
\log \left( \prod_{\sqrt{k} < p \leq k} \frac{p}{p - a_0 p} \right) = \log \left( \prod_{a = 1}^{\sqrt{k}} \prod_{k/(a+1) < p \leq k/a} \frac{p}{(a+1)p - k} \right) = \sum_{a = 1}^{\sqrt{k}} \sum_{k/(a+1) < p \leq k/a} (\log p - \log((a + 1)p - k)).
\]

We split this sum into three pieces to start with:

1. The outer sum for \((\log k)^2 \leq a \leq \sqrt{k}\), and we show it is \(o(k/\log k)\).
2. The \(\log p\) term only, for \(a < (\log k)^2\), and show it is \(k + o(k/\log k)\).
3. The \(-\log((a + 1)p - k)\) term, again for \(a < (\log k)^2\), and show it is \(-k + O(k/\log k)\).

For (1), we have

\[
\sum_{a = (\log k)^2}^{\sqrt{k}} \sum_{k/(a+1) < p \leq k/a} (\log p - \log((a + 1)p - k)) \leq \sum_{a = (\log k)^2}^{\sqrt{k}} \sum_{k/(a+1) < p \leq k/a} \log p \leq \sum_{\sqrt{k} < p \leq k/(\log k)^2} \log p
\]

which is \(O(k/(\log k)^2)\) using \(\sum_{p < x} \log p = x + o(x/\log x)\). For (2), we have

\[
\sum_{a = 1}^{(\log k)^2} \sum_{k/(a+1) < p \leq k/a} \log p = \sum_{k/(\log k)^2 < p \leq k} \log p
\]

which is \(k + o(k/\log k)\).
For (3), we have

\begin{equation}
- \sum_{a=1}^{(\log k)^2} \sum_{k/(a+1) < p \leq k/a} \log((a + 1)p - k).
\end{equation}

Using the prime number theorem, this is

\[
\sum_{k/(a+1) < p \leq k/a} \log((a + 1)p - k) = o(k/\log k) + \int_{k/(a+1)}^{k/a} \frac{\log((a + 1)t - k)}{\log t} dt.
\]

Next, we substitute \( u = (a + 1)t/k \) so that \( t = ku/(a + 1) \) and \( du = ((a + 1)/k) dt \) giving \( dt = (k/(a + 1))du \).

\[
\int_{k/(a+1)}^{k/a} \frac{\log((a + 1)t - k)}{\log t} dt = \frac{k}{a + 1} \int_{1}^{A+1/a} \frac{\log(ku - k)}{\log(ku/(a + 1))} du = \frac{k}{a + 1} \int_{1}^{A+1/a} \frac{\log k + \log(u - 1)}{\log k + \log(u/(a + 1))} du.
\]

Next, for some algebra. We use the following two identities:

\[
\frac{A + C}{A + B} = 1 + \frac{C - B}{A + B}, \quad \frac{1}{A + B} = \frac{1}{A} - \frac{B}{A(A + B)}.
\]

Combining these identities gives

\[
\frac{A + C}{A + B} = 1 + \frac{C - B}{A} - \frac{B(C - B)}{A(A + B)}.
\]

Applying this, gives

\[
\frac{k}{a + 1} \int_{1}^{A+1/a} \frac{\log k + \log(u - 1)}{\log k + \log(u/(a + 1))} du + \frac{k}{a + 1} \int_{1}^{A+1/a} \frac{\log(u - 1) - \log(u/(a + 1))}{\log k} du - \frac{k}{a + 1} \int_{1}^{A+1/a} \frac{\log(u/(a + 1))(\log(u - 1) - \log(u/(a + 1)))}{(\log k)(\log k + \log(u/(a + 1)))} du.
\]

We take each of these three terms in order. We have

\[
\frac{k}{a + 1} \int_{1}^{A+1/a} du = \frac{k}{a(a + 1)}.
\]

Summing over the \( a \) values gives the \( k \) term promised above. Yes, the signs work out.
The next term gives our constant.

\[
\frac{k}{a + 1} \int_{1}^{1 + 1/a} \frac{\log(u - 1) - \log(u/(a + 1))}{\log k} du = - \frac{k}{(a + 1) \log k} \log(1 + 1/a).
\]

To see this, note that the indefinite integral of \(\log((a+1)/(u-1)/u)\) is \((u-1) \log((a+1)/(u-1)/u) - \log u\). We then obtain a constant on our \(k/\log k\) term of

\[
\sum_{a=1}^{\infty} \frac{\log(1 + 1/a)}{a + 1} = 0.7884305 \ldots
\]

With a little algebra, the third term is easily bounded by a small constant times \(k/(\log k)(\log k/a)\) which, when summed over the \(a \leq (\log k)^2\), gives \(O(k \log \log k/(\log k)^2)\) which is \(o(k/\log k)\).

\[ \square \]

**References**

[1] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, 5th edition, 1979.

[2] Richard F. Lukes, Renate Scheidler, and Hugh C. Williams. Further tabulation of the Erdös-Selfridge function. *Math. Comp.*, 66(220):1709–1717, 1997.