Interaction with an obstacle in the 2D focusing nonlinear Schrödinger equation

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Abstract

We present a numerical study of solutions to the 2D cubic and quintic focusing nonlinear Schrödinger equation in the exterior of a smooth, compact and strictly convex obstacle (a disk) with Dirichlet boundary condition. We first investigate the effect of the obstacle on the behavior of solutions traveling towards the obstacle at different angles and with different velocities directions. We introduce a new concept of weak and strong interactions of the solutions with the obstacle. Next, we study the existence of blow-up solutions depending on the type of the interaction and show how the presence of the obstacle changes the overall behavior of solutions (e.g., from blow-up to global existence), especially in the strong interaction case, as well as how it affects the shape of solutions compared to their initial data (e.g., splitting into transmitted and reflected parts). We also investigate the influence of the size of the obstacle on the eventual existence of blow-up solutions in the strong interaction case in terms of the transmitted and the reflected parts of the mass. Moreover, we show that the sharp threshold for global existence vs. finite time blow-up solutions in the mass critical case in the presence of the obstacle is the same as the one given by Weinstein for NLS in the whole Euclidean space \( \mathbb{R}^d \). Finally, we construct new wall-type initial data that blows up in finite time after a strong interaction with an obstacle and having a very distinct dynamics compared with all other blow-up scenarios and dynamics for the NLS in the whole Euclidean space \( \mathbb{R}^d \).

Keywords Focusing NLS equation · Convex obstacle · Exterior domain · Soliton-obstacle interaction · Scattering · Blow-up

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1 Introduction

We consider the 2d focusing nonlinear Schrödinger equation (NLS) outside of a smooth, compact, and strictly convex obstacle with Dirichlet boundary conditions, denoted by NLS$_\Omega$:

\[
\begin{align*}
&i\partial_t u + \Delta_\Omega u + |u|^{p-1} u = 0, \quad (t, x) \in \mathbb{R} \times \Omega, \\
&u(t_0, x) = u_0(x), \quad \forall x \in \Omega, \\
&u(t, x) = 0, \quad (t, x) \in \mathbb{R} \times \partial \Omega,
\end{align*}
\]

where $t_0 \in \mathbb{R}$ is the initial time, $\Omega$ is an exterior domain in $\mathbb{R}^2$, and $\Delta_\Omega$ is the Dirichlet Laplace operator defined by $\Delta_\Omega := \partial_x^2 + \partial_y^2$, $(x, y) \in \mathbb{R}^2$.

Here, $u$ is a complex-valued function, $u : \mathbb{R} \times \Omega \rightarrow \mathbb{C}$, $(t, x) \mapsto u(t, x)$.

We consider $u_0 \in H^1_0(\Omega)$, where the Sobolev space $H^1_0(\Omega)$ is the set of functions in $H^1(\Omega)$ that satisfy Dirichlet boundary conditions, i.e., $u = 0$ on $\partial \Omega$.

The NLS$_\Omega$ equation is locally well-posed in $H^1_0(\Omega)$ in dimension $d = 2$, see [8, 9] for a non-trapping obstacle and [21, 22] for a strictly convex obstacle. The solution $u$ can be extended to a maximal time interval $I = (-T_-, T_+)$ of existence and the following alternative holds:

either $T_+ = \infty$ (respectively, $T_- = \infty$), or $T_+ < \infty$ (respectively, $T_- < \infty$)

with

\[
\lim_{t \to T_+} \|u(t, \cdot)\|_{H^1_0(\Omega)} = \infty \quad \text{respectively, } \lim_{t \to T_-} \|u(t, \cdot)\|_{H^1_0(\Omega)} = \infty.
\]

During their lifespans, solutions to the nonlinear Schrödinger equation outside an obstacle conserve both mass and energy:

\[
M_\Omega[u(t)] := \int_\Omega |u(t, x)|^2 \, dx = M_\Omega[u_0],
\]

\[
E_\Omega[u(t)] := \int_\Omega |\nabla u(t, x)|^2 \, dx - \frac{1}{p+1} \int_\Omega |u(t, x)|^{p+1} \, dx = E_\Omega[u_0].
\]

Unlike the nonlinear Schrödinger equation NLS$_{\mathbb{R}^d}$ posed on the whole Euclidean space $\mathbb{R}^d$, the NLS$_\Omega$ equation does not preserve the momentum $P_\Omega[u] = \text{Im} \int_\Omega \bar{u}(t, x) \nabla u(t, x) \, dx$, since the derivative of the momentum $P_\Omega$ with respect to the time variable is equal to a non-zero boundary term.

Furthermore, the NLS$_{\mathbb{R}^2}$ equation, posed on the whole Euclidean space $\mathbb{R}^2$, is invariant under the scaling transformation, that is, if $u(t, x)$ is a solution to the NLS$_{\mathbb{R}^2}$
equation, then $\lambda \frac{2}{p-1} u(\lambda x, \lambda^2 t)$ is also a solution for $\lambda > 0$. This scaling identifies the critical Sobolev space $\dot{H}^{\infty}_x$, where the critical regularity $s_c$ is given by

$$s_c := \frac{p - 3}{p - 1}.$$  

The equation, when $s_c = 0$, is referred to as the mass-critical (or the $L^2$-critical), and when $0 < s_c < 1$, is called the mass-supercritical (or $L^2$-supercritical) and energy-subcritical (or $H^1$-subcritical). Throughout this paper, we consider the 2d cubic ($p = 3$) and quintic ($p = 5$) NLS/\Omega equations. Since the presence of the obstacle does not change the intrinsic dimensionality of the problem, we may regard the cubic NLS/\Omega equation as being the mass-critical equation and the quintic one as the mass-supercritical and energy-subcritical (or intercritical) equation.

The focusing NLS equation, posed on the whole space, admits soliton solutions that are periodic in time, that is, $u(t, x) = e^{it\omega} Q_\omega(x)$, where $\omega > 0$ and $Q_\omega$ is an $H^1$ smooth solution of the nonlinear elliptic equation,

$$-\Delta Q_\omega + \omega Q_\omega = |Q_\omega|^{p-1} Q_\omega. \quad (1.4)$$

In this paper, we denote by $Q_\omega$ the ground state solution, that is, the unique, positive, vanishing at infinity $H^1$ solution of (1.4). The ground state solution turns out to be radial, smooth, and exponentially decaying function (for $s_c < 1$); see [4, 10, 18, 29].

Moreover, $Q_\omega$ characterized as the unique minimizer for the Gagliardo-Nirenberg inequality up to scaling, space translation, and phase shift; see [29]. For simplicity, we denote by $Q$ the ground state solution of (1.4), when $\omega = 1$.

The NLS equation, posed on the whole Euclidean space $\mathbb{R}^d$, also enjoys Galilean invariance: if $u(t, x)$ is a solution, then so is $u(t, x - vt) e^{i \left( \frac{1}{2} x \cdot v - \frac{1}{4} |v|^2 t + t \omega \right)}$, $v \in \mathbb{R}^d$.

Applying the Galilean transform to the solution $e^{it\omega} Q_\omega(x)$ of the NLS on $\mathbb{R}^d$, we obtain a soliton solution, moving on the line $x = tv$ with a velocity $v \in \mathbb{R}^d$:

$$u(t, x) = e^{i \left( \frac{1}{2} x \cdot v - \frac{1}{4} |v|^2 t + t \omega \right)} Q_\omega(x - tv). \quad (1.5)$$

The soliton solution is a global solution of the focusing NLS equation, but it is not a soliton solution for the NLS/\Omega equation: this soliton solution does not satisfy the Dirichlet boundary conditions.

In [31], the first author constructed a solitary wave solution for the 3d focusing $L^2$-supercritical NLS/\Omega equation for large $t$, which behaves asymptotically as a soliton on the Euclidean space $\mathbb{R}^3$, traveling with a velocity $v$, and moving away from the obstacle. Indeed, let $T_0 > 0$, $c_\omega > 0$ and let $\Psi$ be a $C^\infty$ function such that $\Psi = 0$ near the obstacle and $\Psi = 1$ for $|x| \gg 1$, then

$$\|u(t, x) - e^{i \left( \frac{1}{2} (x \cdot v) - \frac{1}{4} |v|^2 t + t \omega \right)} Q_\omega(x - tv) \Psi(x)\|_{H^1_0(\omega)} \leq e^{-c_\omega |v| t} \forall (t, x) \in [T_0, +\infty) \times \Omega,$$
is a solution of the NLS$\Omega$ equation. This solution is global in time, however, it does not scatter. For an arbitrary small velocity, this solution proves the optimality of the following threshold for the global existence and scattering given in [27] for the cubic NLS$\Omega$ equation, in dimension $d = 3$: let $u_0 \in H^1_0(\Omega)$ satisfy

$$E_\Omega[u_0]M_\Omega[u_0] < E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q], \quad (1.6)$$

$$\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}, \quad (1.7)$$

Then, the solution $u(t)$ scatters in $H^1_0(\Omega)$ in both time directions. This threshold was first proved for the 3d cubic NLS equation on the whole space $\mathbb{R}^3$ by the second author with Holmer in [20] (in the radial setting) and with Duyckaerts and Holmer in [14] (nonradial case); further generalizations can be found in [17, 19].

In [15], the first two authors with Duyckaerts studied the dynamics of the focusing 3d cubic NLS$\Omega$ equation in the exterior of a strictly convex obstacle at the mass-energy threshold, namely, when

$$E_\Omega[u_0]M_\Omega[u_0] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q],$$

with the initial mass-gradient bound on $u_0 \in H^1_0(\Omega)$,

$$\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)},$$

where $Q$ is the ground state solution of (1.4), with $\omega = 1$. The same problem was studied in the whole Euclidean space by Duyckaerts and the second author in [16] for the focusing cubic NLS equation on $\mathbb{R}^3$. The dynamics of the NLS equation on the whole Euclidean space is more involved. Indeed, the authors proved that if the initial datum $u_0 \in H^1(\mathbb{R}^3)$ satisfies the same mass-gradient condition as above, then the solution $u(t)$ scatters or $u(t)$ is a “special solution” $Q^+$, up to symmetries, that scatters in negative time and converges to the soliton $e^{it}Q$ (up to symmetries) in positive time. We showed in [15] that this special solution does not have an analog for the problem in the exterior of an obstacle and prove that such solutions are globally defined and scatter in the positive time direction. The existence of blow-up solutions at the mass-energy threshold for the NLS equation on the whole space was also proved in [16] and the behavior of solutions is related to another special solution $Q^-$. It was proved that if $E_{\mathbb{R}^3}[u_0]M_{\mathbb{R}^3}[u_0] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]$ and $\|u_0\|_{L^2(\mathbb{R}^3)} \|\nabla u_0\|_{L^2(\mathbb{R}^3)} > \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, then the solution $u(t)$ blows up in finite time or $u(t)$ is a special solution $Q^-$, up to symmetries. The existence of blow-up solutions at the mass-energy threshold $E_\Omega[u_0]M_\Omega[u_0] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]$ and $\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} > \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, for the NLS$_\Omega$ equation is currently an open question.

All results obtained for the NLS$_\Omega$ equation are for the globally existing and scattering solutions; however, the existence of blow-up solutions has been an open question for some time. The classical proof by the convexity argument on the Euclidean space
\( \mathbb{R}^d \) fails in the exterior of an obstacle due to the appearance of the boundary terms with an unfavorable sign in the second derivative of the variance \( V(u(t)) \), that is, if

\[
V(u(t)) := \int_{\mathbb{R}^d} |x|^2 |u(t, x)|^2 \, dx,
\]

then

\[
\frac{1}{16} \frac{d^2}{dt^2} V(u(t)) = E[u] - \frac{1}{2} \left( \frac{d}{2} - \frac{d + 2}{p + 1} \right) \int_{\Omega} |u|^{p+1} \, dx - \frac{1}{4} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \tilde{n}) \, d\sigma(x),
\]

where \( \tilde{n} \) is the unit outward normal vector. One can see that in the last term,

\[ x \cdot \tilde{n} \leq 0, \text{ for all } x \in \partial\Omega. \]

Recently, the first author in [32] (see also [30]) proved the existence of blow-up solution to the NLS equation in the exterior of a smooth, compact, convex obstacle. This was the first step in the study of the existence of blow-up solutions to the NLS equation. A new modified variance \( V(u(t)) \), which is bounded from below and is strictly concave for the solutions considered, was introduced

\[
V(u(t)) := \int_{\Omega} \left( d(x, \Omega^c) + 10 \right)^2 |u(t, x)|^2 \, dx,
\]

where \( d(x, \Omega^c) = |x| - R \) is the distance to the obstacle and \( R \) is the radius of the obstacle (a ball in \( \mathbb{R}^d \)). In [32] (see also [30]), it was shown that solutions with finite variance and negative energy blow up in finite time (for a ball and also any smooth, compact, and convex obstacle). Furthermore, it was proved that finite variance solutions to the NLS equation for \( p \geq 1 + \frac{4}{d} \), which satisfy (1.6), \( \|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} > \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)} \) and a certain symmetry condition, will blow up in finite time.

From the above review, one notices that an obstacle does influence the behavior of solutions, and while some properties and criteria remain robust and almost unchanged (except for the restriction of the whole space to the exterior domain problem \( \Omega \)), other properties either get significantly modified or, even more, become unclear in the obstacle setting. Further analytical investigations of the interaction between solutions and an obstacle are needed, though the presence of the obstacle breaks down quite a few invariance properties of the equation, creating additional difficulties for theoretical study.

The purpose of this paper is to investigate this question numerically and to gain further insights of the obstacle influence. We are specifically interested in how solitary wave-type data (e.g., as in (1.5)) interacts with an obstacle, depending on the distance and the angle to the obstacle as well as the size of the obstacle.

In our simulations, we distinguish two types of interaction between a solitary wave solution moving on the line \( \bar{x} = t \bar{v} \) (\( \bar{v} \) is the velocity vector) and the obstacle:

**Strong interaction** We call the interaction strong, when a soliton-type solution is moving, towards the obstacle, in the same direction as the outward normal \( \tilde{n} \) vector of
the obstacle, i.e., the velocity vector is collinear to the normal vector, \( \mathbf{v} = \alpha \mathbf{n} \), \( \alpha > 0 \),
(e.g., see Fig. 20). In this case, after the collision or the shock, the solitary wave solution does not preserve the shape of the initial or the original soliton, but the solution splits into several solitons or bumps, with a substantial amount of backward reflected waves.

**Weak interaction** We call the interaction *weak*, when the velocity vector of the moving soliton solution, towards the obstacle, is not pointed in the same direction as the outward normal vector, i.e., the solution hits the obstacle at an angle \( 0 < \theta \leq \frac{\pi}{2} \) between the velocity vector and the outward normal vector; see Fig. 4. In this case, after the interaction, the solitary wave solution is transmitted almost with the same shape and with backward reflected waves of insignificant size.

The interaction between a solitary wave-type solution and an obstacle does not depend only on the direction of the velocity vector and the angle of the collision; it also depends on the initial distance between the solitary wave solution and the obstacle. For that, we also study the dependence on the distance. Throughout this paper, we denote by

\[
    d^* := \min_{x \in \text{supp}(u_0)} \text{dist}(x, \Omega^c),
\]

the distance between the obstacle \( \Omega^c \) and the essential support of the initial data \( u_0 \) such that \( u_0 \) is well-defined, i.e., \( u_0 \) is smooth and satisfies Dirichlet boundary conditions. Note that if we consider the initial condition with the distance \( d \gg d^* \), then the presence of the obstacle does not effectively influence the behavior of the solution (provided that \( u_0 \) has essentially a compact support, for instance, the Gaussian \( u_0 = Ae^{-x^2} \) will suffice for computational purposes). If we consider \( u_0 \) with a large mass such that \( d \gg d^* \), then the solution will blow-up in finite time before it could reach the obstacle for all velocity directions; see Fig. 4 for such scenarios. In such a case, there is no interaction between the obstacle and the solution. Moreover, the computation of the boundary value terms in (1.9) vanishes to 0 when \( d \gg d^* \), and one can see that the expression of the second derivative of the variance \( V(t) \) in (1.9) is close to the corresponding value of the variance defined on the whole space \( \mathbb{R}^2 \).

In this case, numerically, the soliton-type solution behaves as a solution posed on a computational domain without an obstacle.

For the purpose of this work, and in order to study the influence and the interaction of a generic solution (a solitary wave-type solution) with an obstacle, we always consider the distance \( d \) to be the minimal distance \( d^* \) such that even a slight modification of the velocity direction or the translation parameters would produce at least a weak interaction.

In this paper, we present our numerical results about the behavior of solutions influenced by an obstacle in the NLS\( _{\Omega} \) equation outside of a ball or a disk or radius \( r_* \), in dimension \( d = 2 \). Our goal is to understand the interaction between a solitary wave (for example, traveling with a velocity \( v \)) and the obstacle, as well as the influence of the obstacle on the nonlinear dynamics of the NLS\( _{\Omega} \) equation. We also study the existence of blow-up solutions to the NLS\( _{\Omega} \) equation, in dimension \( d = 2 \), in particular, we investigate the influence of the obstacle on the behavior of finite time blow-up solutions and its dependence on different types of interaction, which is affected by the direction of the velocity \( v \) and the angle at the collision. According to our
numerical simulations, the solitary wave amplitudes decrease at the collision or at any interaction (even small) between the soliton and the obstacle. This could be explained by the appearance of reflection or reflection waves, due to the Dirichlet boundary conditions at the obstacle. After the collision, our numerical results show that if there is a weak or small interaction, then the solitary wave is transmitted almost completely with little or insignificant backward reflection. If there is a strong interaction, then the solution does not typically preserve the shape of the original solitary wave. First, a single bump will split into two bumps with some substantial backward reflection. After that, the two bumps will start to merge together with a creation of a third bump in the middle, and then all of that will continue as a sum of several solitary waves. Typically, the first two bumps will have a dispersive behavior due to a creation of the third middle bump, which will be the main part of the (after-interaction) solitary wave.

We also observe that the leading reflected wave has a dispersive behavior, which radiates away. The reflection phenomenon, the loss of the amplitude, and the change in the shape of the solitary wave make it very challenging to show (even numerically) the existence of blow-up solutions. Nevertheless, we confirm numerically the existence of blow-up solutions after the collision for the 2d focusing NLS\(_\Omega_1\) equation in several cases of (i) the weak interaction, depending on the velocity direction (see Sect. 5), and (ii) the strong interaction, depending on the radius \(r_\star\) of the obstacle. Moreover, we investigate the influence of the size of the obstacle on the behavior of the solutions in terms of the transmitted and reflected mass: (i) if the radius of the obstacle is very small (e.g., \(r_\star \approx 0.1\)), so that the interaction region of the solution with the obstacle is insignificant or negligible, then the solution is mostly transmitted with a small reflected part and it blows up in finite time after the interaction; (ii) if the radius of the obstacle is large enough, so that the interaction region is relevant and it is larger than the contour of the solution (e.g., \(r_\star \approx 2\)), then there is almost no transmission of the solution and it blows up in finite time at the boundary of the obstacle; see Sect. 7. Furthermore, we design a new type of initial data (with a single maximum peak bump) in context of NLS, which we refer to as the Wall-type, where after a strong interaction with the obstacle, the corresponding solution blows up in finite time, possibly not even at a single point, but in various locations (for example, we observe simultaneous blow up at two different locations; this happens even for a larger obstacle size). The wall-type data have been used on a rectangular grid in a context, for example, of the 2d ZK (see [28]); however, crafting of the data to produce the blow-up after the strong interaction with an obstacle is very delicate and new in a context of NLS. To be more precise, we use a Galilean transform on a super-Gaussian in radial and angular variables. These specific solutions to the NLS\(_\Omega_1\) equation have a very distinct dynamics compared with all other blow-up scenarios and dynamics we observed, since as mentioned they can produce a blow up not at a single location (see Sect. 8); also, there is interesting dynamics for certain parameters when the blow-up happens at a single point after the strong interaction (see Fig. 35).

In addition, we study the sharp threshold for global existence vs. finite time blow-up solutions (in the 2d focusing mass-critical NLS\(_\Omega_1\) equation); see Sect. 3. This threshold was first obtained by Weinstein in [47] for the focusing mass-critical NLS equation.
in the whole Euclidean space $\mathbb{R}^d$ (for example, $2d$ cubic NLS). He showed a sharp threshold for the global existence using the Gagliardo-Nirenberg inequality combined with the energy conservation,

$$\|\nabla u\|_{L^2}^2 \leq \left(1 - \frac{\|u\|_{L^2}^2}{\|Q\|_{L^2}^2}\right)^{-1} E[u],$$

which implies that (i) if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then an $H^1$ solution exists globally in time and (ii) if $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$, then the solution may blow up in finite time. Recently, Dodson proved in [13] that initial data $u_0 \in L^2(\mathbb{R}^d)$ with $\|u_0\|_{L^2} < \|Q\|_{L^2}$ generates a corresponding solution that is global and scatters in $L^2(\mathbb{R}^d)$. To confirm this threshold in our setting of the NLS with an obstacle, we consider initial data $u_0$ as a small perturbation of a shifted soliton, $u_0 = A Q(x - x_0)$, with either $A < 1$ (e.g., $A = 0.9$) for global existence or $A > 1$ (e.g., $A = 1.1$) for a finite time of existence.

In physics, the study of the reflected, diffracted, or scattered (mechanical, electromagnetic, or gravitational) waves after encountering an object, an obstacle, or a body is related to the study of boundary value problems. These problems are usually described mathematically as an exterior domain or obstacle problem for the wave-type equations with Dirichlet or Neuman boundary conditions. The study of the wave-type equations in the exterior of an obstacle started in the late 1950s and early 1960s, and until now, the understanding of the dynamics of the evolution equations on exterior domains has been a widely open area for investigations. Let us mention some relevant works on the wave-type equation in an exterior domain. H. W. Calvin and Morawetz have studied the local-energy decay of the solutions to the linear wave equation in an exterior of a sphere and star-shape obstacles, with Dirichlet and Neuman boundary conditions; see [48] and [35, 36]. For later works, see [33, 34]. Different results were obtained for almost-star shape, non-trapping and moving obstacles; see [24, 37, 38] and [11]. In that period of time, the authors considered a classical solutions with $C^2$ initial data. In 2004, the Cauchy theory in $H^1_T(\Omega)$ for the NLS$_\Omega$ equation was initiated by Burq, Gérard and Tzvetkov in [9], for a non-trapping obstacle. After that, the well-posedness problem for the NLS$_\Omega$ equation was investigated by others; see, for example, [3, 8, 22, 23, 42]. In [31], the first author proved the local well-posedness for the 3d NLS$_\Omega$ equation in the critical Sobolev space using the fractional chain rule in the exterior of a compact convex obstacle given in [26].

This paper is organized as follows: in Sect. 2, we present the numerical method that we design for this study. In Sect. 3, we study the sharp threshold for global existence vs. blow-up solutions for the critical NLS$_\Omega$ equation in terms of the ground state perturbations. In Sect. 4, we fix the radius $r_\star$ of the obstacle (e.g., $r_\star = 0.5$) and study the dependence of the interaction on the initial distance between the solution and the obstacle, depending on the velocity direction. In Sects. 5 and 6, we study the weak and strong interactions between the traveling solutions and the obstacle. In Sect. 7, we investigate the influence of the size of the obstacle $r_\star$ on the behavior of solutions, especially in the strong interaction case. Finally, in Sect. 8, we study the existence of blow-up solutions, with the new Wall-type initial data, for a variety of large size obstacles and different initial amplitude of the data. We summarize our findings in...
conclusions’ Sect. 9. In all our simulations, we consider both the cubic (mass-critical) and quintic (mass-supercritical) NLSΩ equations.

2 Numerical approach

2.1 The scheme and initial data

Various numerical methods are used in order to approximate the nonlinear Schrödinger equation ranging from the explicit and implicit schemes in time to the finite difference or Fourier pseudo-spectral methods in space. There are different methods for the time discretization, for example, the Crank-Nicolson scheme [12], Runge-Kutte type [1, 2, 25], symplectic and splitting type, [44, 45] and [7, 46], and relaxation methods [5] and [6].

We use the well-known Crank-Nicolson scheme for the time discretization of the NLSΩ equation. The scheme is based on a time centering approximation $u^{n+1/2} \approx u^n + u^2$. The Crank-Nicolson-type scheme is the 2nd order implicit method. On the plus side, this scheme preserves both the discrete mass and the discrete energy exactly during the time evolution. On the negative side, the schemes have to deal with solving the resulting nonlinear algebraic system; consequently, the Newton’s iterative method (2.5) is used for solving the nonlinear system at each time step.

We consider exponentially decaying data only, and we take a large enough computational domain to approximate the convex domain Ω containing an obstacle (of radius $r^*$). To be specific, we consider the polar coordinates $(r, \theta)$ with $0 < r^* < r < R$ and $0 \leq \theta \leq 2\pi$ and use the following domain in our simulations:

$$\Omega = \{(r, \theta) \in \mathbb{R}^2 : r^* \leq r \leq R, \ 0 \leq \theta \leq 2\pi\}.$$

We note that the obstacle size is fixed as $r^* = 0.5$, unless stated otherwise (e.g., in Sects. 7 and 8).

To approximate our model, we impose the Dirichlet boundary condition on the variable $r$, i.e., $u(r^*, \theta, t) = u(R, \theta, t) = 0$, and the periodic boundary conditions on the variable $\theta$, i.e., $u(r, 0, t) = u(r, 2\pi, t)$.

Initial data: We consider the following initial data:

$$u_0(r, \theta) = A_0 \ F (r \cos \theta - x_c, r \sin \theta - y_c) \ e^{i \frac{1}{2}(v_x \cdot r \cos \theta + v_y \cdot r \sin \theta)}, \quad (2.1)$$

where $A_0$ is the initial amplitude, $(x_c, y_c)$ is the translation, $v = (v_x, v_y)$ is the velocity vector, and $F$ is the profile of the solutions, which is typically taken as the Gaussian; for example, $F = e^{-r^2/2}$. To be precise, we consider

$$u_0(r, \theta) = A_0 \ e^{-\frac{1}{2}[(r \cos \theta - x_c)^2 + (r \sin \theta - y_c)^2]} \ e^{i \frac{1}{2}(v_x \cdot r \cos \theta + v_y \cdot r \sin \theta)}, \quad (2.2)$$
Throughout the paper, in most cases, we use the same Gaussian initial data \((2.2)\), (unless indicated otherwise as in Sects. 3 and 8). The amplitude, translation, and velocity parameters vary according to the specific examples considered.

We next describe our algorithm. We first consider the time discretization. Let \(T_{\text{max}}\) be the existence time of the solution and \(T_{\Delta t}\) be the computational time \((T_{\Delta t} < T_{\text{max}})\). We use \(N\) points for the time discretization, thus defining a time step \(\Delta t = \frac{T_{\text{max}}}{N}\). We discretize the NLS\(_\Omega\) equation at times \(t_n = n\Delta t\), \(n = 0, \ldots, N\), by considering the semi-discretization in time \(u^n \approx u(x, t_n)\) with \(u^0 := u_0\). This yields the following time evolution:

\[
\frac{i u^{n+1} - u^n}{\Delta t} + \frac{1}{2} \Delta u^{n+1} + \frac{1}{2} \Delta u^n = -F(u^{n+1}, u^n),
\]

where \(F\) is the nonlinear term \(|u|^{p-1}u\) approximated by

\[
F(u^{n+1}, u^n) := \frac{2}{p + 1} \frac{|u^{n+1}|^{p+1} - |u^n|^{p+1} + u^{n+1} + u^n}{2}.
\]

Note that, \(u^n\) is the known variable and for \(n = 0\), \(u^0 = u_0\) is the given initial condition. We compute the evolution \(u^n \rightarrow u^{n+1}\) by solving the above system \((2.3)\).

For that, we use the Newton iteration to solve the nonlinear implicit system \((2.3)\). We denote \(u^{n+1}\) at the iteration \(l\) by \(u^{n+1,l}\) and assume \(u^{n+1} = u^{n+1,\infty}\), which gives

\[
\begin{cases}
  u^{n+1,l+1} = u^{n+1,l} - J^{-1} \cdot G(u^{n+1,l}), \\
  u^{n+1,0} = 1.001 \cdot u^n,
\end{cases}
\]

where \(G(u^{n+1}) = u^{n+1} - u^n + \frac{\Delta t}{2i} \Delta u^{n+1} + \frac{\Delta t}{2i} \Delta u^n + F(u^{n+1}, u^n)\) and \(J\) is the Jacobian of \(G\).

The stopping criterion for \((2.5)\) is \(\|u^{n+1,l+1} - u^{n+1,l}\|_{L^\infty} < \text{Tol}\) for some small constant \(\text{Tol}\). In our simulation, we take \(\text{Tol} < 10^{-13}\), which is close to the machine precision. In order to reach the blow-up time (or the closest time), we slightly decrease the tolerance according to the examples treated.

We employ the polar transformation in space \(x = r \cos \theta\) and \(y = r \sin \theta\) to convert the problem into the polar coordinate setting. Thus, we write the Laplacian in polar coordinates as

\[
\Delta u(r, \theta) = \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} u(r, \theta) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} u(r, \theta), \quad (r, \theta) \in \Omega.
\]

We then rewrite the NLS\(_\Omega\) equations for \(t \in (0, T), \ (r, \theta) \in \Omega\), as

\[
i \frac{\partial}{\partial t} u(t, r, \theta) + \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} u(t, r, \theta) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} u(t, r, \theta) = -|u(t, r, \theta)|^{p-1} u(t, r, \theta)
\]
with the periodic boundary condition on $\theta$

$$u(t, r, 0) = u(t, r, 2\pi), \quad \forall t \in [0, T], \; \forall r \in [r_*, R],$$

(2.7)

and the Dirichlet boundary condition on $r$

$$u(t, r_*, \theta) = u(t, R, \theta) = 0, \quad \forall t \in [0, T], \; \forall \theta \in [0, 2\pi].$$

(2.8)

We use $N_r$ and $N_\theta$ to denote the number of points for the space discretization, setting

$$\Delta r = \frac{R - r_*}{N_r} \quad \text{and} \quad \Delta \theta = \frac{2\pi}{N_\theta}.$$  

We denote the full discretization by $u_{k,j}^n \approx u(r_k, \theta_j, t_n), r_k = r_* + k \Delta r, \; \theta_j = j \Delta \theta,$ for $n = 0, \ldots, N, \; k = 0, \ldots, N_r$ and $j = 0, \ldots, N_\theta$.

We use the second order finite difference scheme in space, to approximate the NLS$_\Omega$ equation:

$$i \frac{u_{k,j}^{n+1} - u_{k,j}^{n}}{\Delta t} + \frac{1}{2} \left[ D_r + D_\theta \right] u_{k,j}^{n+1} + \frac{1}{2} \left[ D_r + D_\theta \right] u_{k,j}^{n} = -F(u_{k,j}^{n+1}, u_{k,j}^{n}),$$

(2.9)

where $F(u_{k,j}^{n+1}, u_{k,j}^{n})$ is defined in (2.4), and $D_r$ and $D_\theta$ are the second order finite difference operators:

$$D_r u_{k,j}^{n} = \frac{1}{\Delta r} \frac{1}{r_k} \left( \frac{u_{k+1,j}^{n} - u_{k,j}^{n}}{\Delta r} - r_{k-\frac{1}{2}} \frac{u_{k,j}^{n} - u_{k-1,j}^{n}}{\Delta r} \right),$$

(2.10)

$$D_\theta u_{k,j}^{n} = \frac{1}{\Delta \theta} \frac{1}{r_k} \left( \frac{u_{k,j+1}^{n} - 2u_{k,j}^{n} + u_{k,j-1}^{n}}{(\Delta \theta)^2} \right),$$

(2.11)

$$F(u_{k,j}^{n+1}, u_{k,j}^{n}) = \frac{2}{p+1} \frac{|u_{k,j}^{n+1}|^{p+1} - |u_{k,j}^{n}|^{p+1}}{|u_{k,j}^{n+1}|^2 - |u_{k,j}^{n}|^2} \frac{u_{k,j}^{n+1} + u_{k,j}^{n}}{2},$$

(2.12)

where $r_{k+\frac{1}{2}} = \frac{2k+1}{2} \Delta r + r_*$, and we set $u_{0,j}^{n} = 0$ (from the Dirichlet boundary condition (2.8)), $u_{k,0}^{n} = u_{k,0}^{n}$ and $u_{k,1}^{n} = u_{k,N_\theta+1}^{n}$ (from the periodic boundary condition (2.7)). The scheme (2.9) is the second order approximation to the original equation (1.1) both in space and time, $O(\Delta t^2 + \Delta r^2 + \Delta \theta^2)$. Indeed this is the case as we show below this convergence rate in our numerical tests.

To solve the above system (2.9) with (2.10) and (2.11), we consider the initial condition such that $u_0$ (at least, approximately) satisfies Dirichlet boundary conditions. In almost all examples that we present (except Sect. 3), we consider a shifted Gaussian as an initial condition, and we define the translation parameters $(x_c, y_c)$ such that $u_0$ is smooth and vanishes to 0 near both the obstacle and the boundary of the computational domain (while it may not always be the computational accuracy zero, it vanishes to a sufficient order not to affect the results).
2.2 Mass and energy conservation

The Crank-Nicolson scheme (2.3) conserves the following discretized quantities: the discretized $L^2$-norm, or often referred to as the discrete mass $\mathcal{M}$, and the discretized energy, called the discrete energy $\mathcal{E}$, which are the discrete analogs of the mass and energy conservation in (1.2) and (1.3).

If we consider the rectangular coordinates $(x, y)$, and the discretization of the Laplacian term $\Delta u$ by the standard five points stencil finite difference approximation (e.g., see [12]), i.e.,

\[
\Delta u_{k,j} \approx \frac{u_{k-1,j} + u_{k+1,j} + u_{k,j-1} + u_{k,j+1} - 4u_{k,j}}{\Delta x \Delta y},
\]

by assuming $\Delta x = x_{k+1} - x_k = \Delta y$, then the conservation of the discrete mass for the scheme (2.3) with the Laplacian $\Delta u$ approximated by (2.13) on the rectangular domain is given by

\[
\mathcal{M}[u^n] = \sum_{k=0}^{N_x} \sum_{j=0}^{N_y} |u^n_{k,j}|^2 \Delta x \Delta y = \mathcal{M}[u^0], \quad \text{for } n \geq 0,
\]

which can be proved by multiplying the equation (2.3) by $(\bar{u}_{k,j}^{n+1} + \bar{u}_{k,j}^n)\Delta x \Delta y$ and summing from $k = 0$ to $N_x$, and $j = 0$ to $N_y$ for each $k, j$.

Similarly, the conservation of the discrete energy in rectangular coordinates $(x, y)$ is obtained by multiplying (2.3) with $(\bar{u}_{k,j}^{n+1} - \bar{u}_{k,j}^n)\Delta x \Delta y$, summing from $k = 0$ to $N_x$ and $j = 0$ to $N_y$ for each $k, j$ and taking the real part:

\[
\mathcal{E}[u^n] = \frac{1}{2} \sum_{k=0}^{N_x-1} \sum_{j=0}^{N_y-1} \left( \frac{|u^n_{k+1,j} - u^n_{k,j}|}{\Delta x} \right)^2 + \left( \frac{|u^n_{k,j+1} - u^n_{k,j}|}{\Delta y} \right)^2 - \frac{2}{p+1} |u^n_{k,j}|^{p+1} \Delta x \Delta y = \mathcal{E}[u^0], \quad \text{for } n \geq 0.
\]

For brevity, we omit the above standard proofs.

In polar coordinates $(r, \theta)$, the scheme (2.9) also conserves the discrete mass and energy exactly, similarly to the above. More specifically, we define the discrete mass at $t = t_n$ by

\[
\mathcal{M}[u^n] = \sum_{k=0}^{N_x} \sum_{j=0}^{N_y} |u^n_{k,j}|^2 r_k \Delta r \Delta \theta, \quad \text{for } n \geq 0.
\]
One can see that the definition (2.16) is an analog to the mass in (1.2). In the same spirit, we define the discrete energy as

$$E[u^n] = \frac{1}{2} \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} \left( \frac{|u^n_{k+1,j} - u^n_{k,j}|^2}{\Delta r} r_{k+\frac{1}{2}} \Delta \theta + \frac{1}{2} \frac{1}{r_k} \left| \frac{u^n_{k,j+1} - u^n_{k,j}}{\Delta \theta} \right|^2 \Delta r \Delta \theta \right)$$

$$- \frac{1}{p + 1} \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} |u^n_{k,j}|^{p+1} r_k \Delta r \Delta \theta, \quad \text{for } n \geq 0,$$

(2.17)

which is an analog of the energy conservation in (1.3).

We have the following theorem:

**Theorem 1** The numerical scheme (2.9) conserves the discrete mass (2.16) and the discrete energy (2.17) for all $n \in \mathbb{N}$, that is,

$$\mathcal{M}[u^n] = \mathcal{M}[u_0] \quad \text{and} \quad \mathcal{E}[u^n] = \mathcal{E}[u_0].$$

**Proof** The proof of the mass conservation is similar to the proof in the case of the rectangular coordinates $(x, y)$, it suffices to multiply (2.9) with $(\tilde{u}^n_{k,j} + \hat{u}^n_{k,j}) r_k \Delta r \Delta \theta$, sum up over $k$ and $j$ from 0 to $N_r$, 0 to $N_\theta$, respectively, and then take the imaginary part.

For the energy-conservation, the proof is slightly different than the one in the rectangular coordinates $(x, y)$, due to the space discretization of the Laplacian in (2.10). For that, we write the scheme (2.9) for $u^n_{k,j}$, using (2.10), (2.11) and (2.12), to obtain

$$\frac{u_{k,j}^{n+1} - u_{k,j}^n}{\Delta t} - \frac{1}{2} \frac{1}{r_k} \left( \frac{u_{k+1,j}^{n+1} - u_{k,j}^{n+1}}{\Delta r} r_{k+\frac{1}{2}} - r_{k-\frac{1}{2}} \frac{u_{k,j}^{n+1} - u_{k-1,j}^{n+1}}{\Delta r} \right)$$

$$+ \frac{1}{2} \frac{1}{r_k} \left( \frac{u_{k,j+1}^{n+1} - u_{k,j}^n}{\Delta r} r_{k+\frac{1}{2}} - r_{k-\frac{1}{2}} \frac{u_{k,j}^n - u_{k-1,j}^n}{\Delta r} \right)$$

$$+ \frac{1}{2} \frac{1}{(r_k)^2} \frac{u_{k,j+1}^{n+1} - 2u_{k,j}^{n+1} + u_{k,j-1}^{n+1}}{(\Delta \theta)^2} + \frac{1}{2} \frac{1}{(r_k)^2} \frac{u_{k,j}^{n+1} - 2u_{k,j}^n + u_{k,j}^{n+1}}{(\Delta \theta)^2}$$

$$= - \frac{1}{p + 1} \left( |u_{k,j}^{n+1}|^{p+1} - |u_{k,j}^n|^{p+1} \right) \left( u_{k,j}^{n+1} + u_{k,j}^n \right)$$

(2.18)
Multiplying (2.18) with \((\tilde{\mu}_{k,j}^{n+1} - \tilde{\mu}_{k,j}^n) r_k \Delta r \Delta \theta\), taking the real part and summing up over \(k\) and \(j\), yields

\[
\text{Re} \left[ \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} (I_{1})_{k,j} \times (\tilde{\mu}_{k,j}^{n+1} - \tilde{\mu}_{k,j}^n) r_k \Delta r \Delta \theta \right] = 0. \tag{2.19}
\]

\[
\text{Re} \left[ \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} ((I_{2,1})_{k,j} + (I_{2,2})_{k,j}) \times (\tilde{\mu}_{k,j}^{n+1} - \tilde{\mu}_{k,j}^n) r_k \Delta r \Delta \theta \right] \]

\[
= \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} \frac{1}{2} r_k \Delta r \theta \left| \frac{u_{k+1,j}^n - u_{k-1,j}^n}{\Delta r} \right|^2 \Delta r \Delta \theta
\]

\[
+ \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} \frac{1}{2} r_k \Delta r \theta \left| \frac{u_{k+1,j}^n - u_{k,j}^n}{\Delta r} \right|^2 \Delta r \Delta \theta. \tag{2.20}
\]

Using that \(\tilde{\mu}_{k,0}^n = \tilde{\mu}_{k,0},\ u_{k,0}^n = u_{k,1}^n,\ \tilde{\mu}_{k,0}^n = u_{k,0}^n\) and \(u_{0,j}^n = u_{N_r,j}^n = 0\), for all \(n \in \mathbb{N}\), we get

\[
\text{Re} \left[ \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} (I_{3,1})_{k,j} \times (\tilde{\mu}_{k,j}^{n+1} - \tilde{\mu}_{k,j}^n) r_k \Delta r \Delta \theta \right] = \frac{1}{2} \frac{1}{r_k} \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} \left| \frac{u_{k,j}^{n+1} - u_{k,j-1}^{n+1}}{\Delta \theta} \right|^2 \Delta r \Delta \theta
\]

\[
- \frac{1}{2} \frac{1}{r_k} \frac{1}{(\Delta \theta)^2} \text{Re} \left[ \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} u_{k,j}^{n+1} u_{k,j}^n + 2u_{k,j}^{n+1} u_{k,j}^n - u_{k,j}^{n+1} u_{k,j}^n \right] \Delta r \Delta \theta. \tag{2.21}
\]

Similarly, we deduce

\[
\text{Re} \left[ \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} (I_{3,2})_{k,j} \times (\tilde{\mu}_{k,j}^{n+1} - \tilde{\mu}_{k,j}^n) r_k \Delta r \Delta \theta \right] = \frac{1}{2} \frac{1}{r_k} \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} \left| \frac{u_{k,j}^{n+1} - u_{k,j-1}^{n+1}}{\Delta \theta} \right|^2 \Delta r \Delta \theta
\]

\[
- \frac{1}{2} \frac{1}{r_k} \frac{1}{(\Delta \theta)^2} \text{Re} \left[ \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} - \tilde{\mu}_{k,j}^{n+1} u_{k,j}^n + 2\tilde{\mu}_{k,j}^{n+1} u_{k,j}^n + \tilde{\mu}_{k,j}^{n+1} u_{k,j}^n \right] \Delta r \Delta \theta. \tag{2.22}
\]

By (2.22) and (2.21), we obtain

\[
\text{Re} \left[ \sum_{k=0}^{N_r} \sum_{j=0}^{N_\theta} ((I_{3,1})_{k,j} + (I_{3,2})_{k,j}) \times (\tilde{\mu}_{k,j}^{n+1} - \tilde{\mu}_{k,j}^n) r_k \Delta r \Delta \theta \right] \tag{2.23}
\]
Interaction with an obstacle...

For that, we define the relative mass and energy errors as follows:

\[ \text{Re} \left[ \sum_{k=0}^{N_r} \sum_{j=0}^{N_0} \left( I_4 \right)_{k,j} \times (\tilde{u}_{k,j}^{n+1} - \tilde{u}_{k,j}^n) r_k \Delta r \Delta \theta \right] \]

\[ = -\frac{1}{p+1} \sum_{k=0}^{N_r} \sum_{j=0}^{N_0} \left( |u_{k,j}^{n+1}|^{p+1} - |u_{k,j}^n|^{p+1} \right) r_k \Delta r \Delta \theta. \tag{2.24} \]

Summing up (2.19), (2.20), (2.23) and (2.24), we finally arrive at

\[ \mathcal{E}[u^n] = \frac{1}{2} \sum_{k=0}^{N_r} \sum_{j=0}^{N_0} r_{k+\frac{1}{2}} \frac{|u_{k+1,j}^n - u_{k,j}^n|}{\Delta r} \left( \Delta r \Delta \theta + \frac{1}{2} r_k \frac{|u_{k,j+1}^n - u_{k,j}^n|}{\Delta \theta} \right)^2 \Delta r \Delta \theta \]

\[ -\frac{1}{p+1} \sum_{k=0}^{N_r} \sum_{j=0}^{N_0} |u_{k,j}^n|^{p+1} r_k \Delta r \Delta \theta \]

\[ = \frac{1}{2} \sum_{k=0}^{N_r} \sum_{j=0}^{N_0} r_{k+\frac{1}{2}} \frac{|u_{k+1,j}^{n+1} - u_{k,j}^n|}{\Delta r} \left( \Delta r \Delta \theta + \frac{1}{2} r_k \frac{|u_{k,j+1}^{n+1} - u_{k,j}^n|}{\Delta \theta} \right)^2 \Delta r \Delta \theta \]

\[ -\frac{1}{p+1} \sum_{k=0}^{N_r} \sum_{j=0}^{N_0} |u_{k,j}^{n+1}|^{p+1} r_k \Delta r \Delta \theta = \mathcal{E}[u^{n+1}]. \]

### 2.3 Numerical conservation and convergence test

We verify the mass and energy conservation and the convergence rate of the scheme via numerical tests. We first show the conservation of the discrete mass and energy. For that, we define the relative mass and energy errors as follows:

\[ \mathcal{E}M = \frac{\max_{0 \leq m \leq n} (\mathcal{M}[u^m]) - \min_{0 \leq m \leq n} (\mathcal{M}[u^m])}{\mathcal{M}[u_0]} \tag{2.25} \]

and

\[ \mathcal{E}E = \frac{\max_{0 \leq m \leq n} (\mathcal{E}[u^m]) - \min_{0 \leq m \leq n} (\mathcal{E}[u^m])}{\mathcal{E}[u_0]} \tag{2.26} \]

By tracking the quantities \( \mathcal{E}M \) and \( \mathcal{E}E \) for the numerical solution from the scheme (2.9), with initial condition (2.2), \( \Delta r = 200 \times 4 \) (\( \Delta r = 0.0306 \)), \( \Delta \theta = 180 \times 4 \) (\( \Delta \theta = 0.0087 \)), and \( \Delta t = 10^{-2} \) until time \( T = 20 \), we noticed that \( \mathcal{E}M \) and \( \mathcal{E}E \) remains on the order of \( 10^{-14} \). For \( p = 5 \) the relative errors of the discrete mass and energy are also on the same order.

We next show the convergence rate of our scheme (2.9) in Fig. 1.
We take \( u_0 = 2 \exp(-(x - 5)^2 - y^2) \) on \( r \in [1, 21] \) (recall \( r^2 = x^2 + y^2 \)), and use the scheme (2.9) to simulate the solution till the time \( T = 1 \) with different time steps \( \Delta t \) and spatial steps \( \Delta r \) and \( \Delta \theta \). Then, we track the \( L^2 \) error \( \|u_h - u_{ref}\|_{L^2} \) (here, \( u_h \) is our numerical solution from the given parameters) at time \( T = 1 \), where the reference solution \( u_{ref} \) can be viewed as the “exact” solution, obtained from taking \( \Delta t = 1.25e - 2 \), \( \Delta r = 0.005 \) and \( \Delta \theta = 2\pi/720 \).

The left subplot in Fig. 1 is obtained from taking different time steps \( \Delta t = 0.2, 0.1, 0.05 \) with fixed \( \Delta r = 0.005 \) and \( \Delta \theta = 2\pi/720 \). One can see that the slope is approximately \(-2\), which implies the second order accuracy of the scheme (2.9) in time as expected.

The right subplot in Fig. 1 is obtained from taking different spatial steps \( (\Delta r, \Delta \theta) = (0.4, 2\pi/45), (0.2, 2\pi/90), \) and \((0.1, 2\pi/180)\) with fixed time step \( \Delta t = 1.25e - 2 \). Again, the slope is still approximately \(-2\). This implies that the scheme (2.9) is of second order accuracy in space. We point out that we shrink \( \Delta r \) and \( \Delta \theta \) by two simultaneously, since \( \Delta r \) and \( \Delta \theta \) are not on the same scale (e.g., \((0.4, 2\pi/45) \approx (0.4, 0.1396)\)). Therefore, to properly evaluate the convergence rate, both need to be changed at the same time.

### 3 Perturbations of the soliton in the NLS with obstacle

We start investigating the 2d NLS on the domain with an obstacle, \( \text{NLS}_\Omega \), with a multiple of the ground state that is shifted by \((x_c, y_c)\) as in (2.1). We refer to this evolution as a “perturbed soliton,” since the initial condition has the form

\[
u_0(r, \theta) = \lambda \, Q(r \cos \theta - x_c, r \sin \theta - y_c), \quad \lambda \in \mathbb{R},
\]

where \( Q \) is the ground state solution to (1.4) on the whole space. This ground state solution is obtained numerically via the Petviashvili’s iteration, for example, see [39–41] or the work of the last two authors in [43] or [49, 50].
In this simulation, we consider various shifts by \((x_c, y_c)\) of the perturbed soliton, so that the zero on the boundary (at the obstacle and outside boundaries) would have reasonable vanishing. Typically, we shift the ground state from the origin by the distance \(|(x_c, y_c)|\) from about 5 to 15. The value of the shifted ground state at the obstacle boundary is typically \(10^{-6}\) or smaller, and when the distance is about 5 away from the boundary (e.g., \(x_c = 4.5, y_c = 0\)), then the accuracy at the boundary where we cut off the ground state goes down to about \(10^{-3}\); this is the closest distance to the obstacle, placed at the origin, that we shift the ground state to, while still producing reasonable results in our simulations. Then, at the very first computational step, we make the boundary values to be zero.

Perturbations of the soliton solution to the \(\text{NLS}_\Omega\) equation with a large mass initial condition (greater than the mass of a soliton), for example, \(\lambda = 1.1\), lead to blow-up solutions. For example,

\[
 u_0(r, \theta) = 1.1 \, Q(r \cos \theta + 4.5, r \sin \theta), \quad u_0(r_0, \theta) = u_0(R, \theta) = 0,
\]

blows up at time \(t = 0.954\) with the diverging \(L^\infty\) norm as shown in Fig. 2. We use the Newton iteration to solve the implicit scheme (2.3) and to reach the desired accuracy, with \(N_r = 200 \times 4\) (\(\Delta r = 0.0306\)), \(N_\theta = 180 \times 4\) (\(\Delta \theta = 0.0087\)) and \(\Delta t = 10^{-3}\).

It is quite challenging to approach the blow-up time while maintaining the convergence of the Newton iteration (2.5). To address this issue, we run the scheme with a more refined mesh in order to maintain the convergence of (2.5). This is not a simple task to perform in the 2d non-radial case and can become computationally prohibitive. However, for our results of identifying the blow-up and the type of interaction, it suffices to investigate the \(L^\infty\)-norm (or the height) and label the solution as “the blow-up” if, for example, the amplitude becomes several times higher than the initial one or starts forming a vertical line (e.g., see the right graph in Fig. 2).

Examining the initial condition of the perturbed soliton with the mass smaller than that of the ground state (i.e., \(\|u_0\|_{L^2} < \|Q\|_{L^2}\)), for example,

\[
 u_0(r, \theta) = 0.9 \, Q(r \cos \theta + 4.5, r \sin \theta), \quad u_0(r_0, \theta) = u_0(R, \theta) = 0,
\]

Fig. 2 Ground state solution to the 2d cubic \(\text{NLS}_\Omega\) equation with \(u_0(x, y) = 1.1 \, Q(x+4.5, y)\) at \(t = 0.954\) (left) and its \(L^\infty\) norm depending on time (right)
we observe that the solution disperses, see Fig. 3: a snapshot at time \( t = 1.5 \) on the left and the \( L^\infty \) time dependence on the right.

To conclude that this solution disperses in a long run, as expected in the \( L^2 \)-critical case for the perturbations with smaller mass than that of the soliton (note that \( \lambda = 0.9 < 1 \)), we run the simulations till \( t = 12 \) and track the \( L^\infty \) norm, shown on the right of Fig. 3, with \( N_r = 200 \times 4 (\Delta r = 0.0306) \), \( N_\theta = 180 \times 4 (\Delta \theta = 0.0087) \) and \( \Delta = 10^{-2} \).

Other perturbations of \( Q \) with different amplitudes and various (acceptable) shifts produce similar behavior. (We take shifts no closer than 4.5 due to the Dirichlet boundary condition.) Therefore, we confirm that the threshold for blow-up vs. scattering for the perturbed soliton is indeed given by the ground state.

**Remark** In the rest of the paper, we consider a shifted Gaussian initial data \( u_0 \) as in (2.2) (in Sect. 8, it is a variation of a super-Gaussian on an annulus support). One of the reasons to use Gaussian is that it has a faster decay than the ground state (though both decay exponentially), which ensures that the simulations close to the obstacle satisfy Dirichlet boundary condition (even a slightly faster exponential decay makes computations easier). Another reason is that in order to study various interactions with an obstacle, we consider initial data \( u_0 \) with the minimal possible distance \( d^* \) to the obstacle (as defined in (1.11)) so that \( u_0 \) is smooth and still satisfies Dirichlet boundary condition.

### 4 Dependence on the distance

From now on, we study both the 2\( d \) cubic and quintic NLS\( _\Omega \) equations (\( p = 3, 5 \)) with the radius of the obstacle \( r_\star = 0.5 \). Our goal in this section is to consider solutions with data \( u_0 \) such that the distance \( d \) between the obstacle and the initial condition is larger than the minimal distance \( d^* \).

We take (2.2) with \( x_c \) and \( y_c \) such that \( d >> d^* \). As before, \( v = (v_x, v_y) \) is the velocity vector, which governs the moving direction of the initial bump. Top left of Fig. 4 shows different directions of propagation for this solitary wave-type data depending on the velocity \( \vec{v} \).

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**Fig. 3** Ground state solution to the 2\( d \) cubic NLS\( _\Omega \) with \( u_0(x, y) = 0.9 \, Q(x + 4.5, y) \) at \( t = 1.5 \) (left) and the \( L^\infty \) norm for \( 0 < t < 12 \) (right)
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**Fig. 4** Top left: directions of the velocity $\vec{v}$ of the single bump initial data relative to the obstacle. If the initial bump is relatively far from the obstacle, $d \gg d^\ast$, then (for $u_0$ with large enough mass) the blow-up occurs in any direction of the initial velocity $\vec{v}$ shown on the plot (an example is shown in Fig. 5). Top right: directions of the velocity $\vec{v}$ of the initial two single bumps that move along either the line $y = 5$ or $y = 2$ with $d \gg d^\ast$, discussed in §4.2 (an example is shown in Fig. 6). Bottom left: directions of the velocity for the examples in §5.1, Fig. 7 (no significant interaction). Bottom right: directions of the velocity for the examples in §5.1, Figs. 8 to 11 (weak interaction).

### 4.1 The $L^2$-critical case

For the $2d$ cubic NLS$_\Omega$ equation we take the initial data (2.2) with large enough mass and $d \gg d^\ast$. Then, the corresponding solution to our numerical scheme (2.3) blows up in finite time before reaching or interacting with an obstacle in any direction of the velocity vector $v$, see Fig. 5.

Later, we study the case when $d \equiv d^\ast$ and the solution concentrates in its (blow-up) core after the obstacle, for the same initial data but with a different velocity direction. We also investigate the influence of the obstacle when there is an interaction between the traveling wave and the obstacle. In Sect. 5.1, we consider the weak interaction for the $2d$ cubic NLS$_\Omega$ equation ($L^2$-critical case), and in Sect. 6.1, we study the strong interaction.

**Fig. 5** Snapshots of the initial data $u_0$ as in (2.2) with $A_0 = 2.25$, $(x_c, y_c) = (-8, -8)$, $(v_x, v_y) = (10, 10)$ and the solution $u(t)$ to the $2d$ cubic NLS$_\Omega$ (critical case) at $t = 0$ and $t = 0.583$ (left and middle); the $L^\infty$-norm depending on time (right).
interaction. We observe that in those cases, the solution exhibits a different behavior on a longer time interval.

### 4.2 The $L^2$-supercritical case

Next, we consider the 2d quintic NLS$_\Omega$ equation and take the initial condition (2.2) with a large mass and $d >> d^*$. In the following scenario, we fix parameters $A_0$ and $v = (v_x, 0)$ and vary the translation parameter $y_c$ in the translation $(x_c, y_c)$ as shown in Fig. 4 top right.

Snapshots of the corresponding solution to the 2d quintic NLS$_\Omega$ equation are plotted in Fig. 6. As in the previous example, the solution blows up in finite time before the obstacle (for large $x_c$).

We later investigate the case when the solution blows up in finite time, after the obstacle and when $d = d^*$, with the same initial data $u_0$ as in (2.2) for a fixed amplitude $A_0 = 1.25$ and velocity direction $v = (15, 0)$, and $x_c = -4.5$, but for different space translation $y_c$, see Table 2. This will lead to the weak or strong interaction for the 2d quintic NLS$_\Omega$ equation ($L^2$-supercritical case), see Sects. 5.2 and 6.2.

### 5 Weak interaction with an obstacle

#### 5.1 The $L^2$-critical case

We return to the cubic NLS$_\Omega$ setting (critical case) and consider the initial data (2.2) with $A_0 = 2.25$, $x_c = -4.5$, $y_c = -4.5$, while varying the direction of the velocity vector, two scenarios shown in Fig. 4 bottom left.

Taking $v_1 = (v_x, 0) = (15, 0)$, we observe that the solution blows up at time $t = 0.568$. It basically does not interact with the obstacle; its behavior is the same as it would be of a solitary wave on the whole space; see Fig. 7. When taking $v_2 = (0, v_y) = (0, 15)$, that is, perpendicular direction to $v_1$ (as shown in Fig. 4 bottom left), we obtain the same behavior with the solution blowing up at the same time.

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**Fig. 6** A blow-up solution to the 2d quintic NLS$_\Omega$ (supercritical case) with the initial condition (2.3) and $A_0 = 1.25$, $(v_x, v_y) = (15, 0)$, $(x_c, y_c) = (-15, 5)$. The initial profile (left), a snapshot of the solution at $t = 0.63$ (middle), and the time dependence of the $L^\infty$-norm (right)
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In our third example, we take the initial condition $u_0$ from (2.2) with $A_0 = 2.25$, $x_c = -4.5$, $y_c = -4.5$ and $v_1 = (15, 0)$ (left); the time evolution at $t = 0.524$ (middle); the time dependence of the $L^\infty$-norm (right). The same behavior occurs if the velocity is $v_2 = (0, 15)$ as shown in the bottom left of Fig. 4.

The direction $v_1 = (15, 9)$ gives a small interaction, or collision, with the obstacle. After the collision, we observe that the solution has almost the same behavior (as in Fig. 7), i.e., it blows up in finite time but with a slightly dispersive reflection part, preserving the shape of the soliton, similar to the previous case. The solution blows up at time $t = 0.57$ after the interaction with the obstacle, see Fig. 8. Moreover, we see that at the collision time the $L^\infty$-norm has a slight perturbation (or a small oscillation); however, shortly afterwards, it continues to increase: such perturbation is not sufficient to prevent the overall growing of the $L^\infty$-norm and the occurrence of the blow-up.

Besides Fig. 8 with the $L^\infty$ norm, we also show snapshots of the solution at different times for a symmetric velocity direction $v_2 = (9, 15)$ in Fig. 9.

In our fourth example, here, we consider velocity vectors even closer to the diagonal (head-on) direction as on the bottom right of Fig. 4; thus, we take the initial condition $u_0$ from (2.2) with $A_0 = 2.25$, $x_c = -4.5$, $y_c = -4.5$ and the velocity $\vec{v}$ that has a different direction but has the same magnitude $|\vec{v}|$, as in the previous two examples: we choose $v_1 = (v_x, v_y)$ and $v_2 = (v_y, v_x)$ as shown on the bottom right of Fig. 4.

The right subplot shows the $L^\infty$-norm depending on time, which appears to grow quite fast in the beginning of the simulation, but after the collision, it starts to decrease monotonically. This solution disperses, or, in other words, it becomes a scattering solution. Thus, the obstacle arrests the blow-up. This is a different behavior compared

\[Fig. 7 \text{ The 2d cubic NLS}_2 \text{ with } u_0 \text{ from (2.2), } A_0 = 2.25, x_c = -4.5, y_c = -4.5 \text{ and } v_1 = (15, 0) \text{ (left); the time evolution at } t = 0.524 \text{ (middle); the time dependence of the } L^\infty\text{-norm (right). The same behavior occurs if the velocity is } v_2 = (0, 15) \text{ as shown in the bottom left of Fig. 4.} \]

\[Fig. 8 \text{ Solution to the 2d cubic NLS}_2 \text{ equation with the initial condition } u_0 \text{ as in (2.2), } A_0 = 2.25, x_c = -4.5, y_c = -4.5, \text{ and } v_1 = (15, 9) \text{ at time } t = 0.554, \text{ moving on the line } y = \frac{3}{5} x \text{ (left); enlargement of } L^\infty\text{-norm around time } 0.3 \text{ (middle); the time dependence of } L^\infty\text{-norm (right).} \]
to the previous examples, where the solutions were transmitted almost with the same shape after the interaction and the soliton core was preserved. Unlike the previous examples, the collision of the solution with the obstacle here creates reflected waves, which then disperse the solution. The reflection causes the loss of the mass in the main part of the solution, arresting the blow-up in finite time unlike the examples above, where the reflection does not affect the blow-up of the solution and only delays the blow-up time. In this case, the interaction between the soliton and the obstacle has a substantial influence on the behavior of the solution, which is a completely new dynamics compared to the dynamics on the whole space. For better understanding of this dynamics, we provide snapshots of the behavior of the solution for different time steps for $v_2 = (15, 12)$; see Fig. 11. This is an example where the solution has a behavior close to the strong interaction case.

We record the results of our simulations for the $L^2$-critical case ($2d$ cubic NLS$_\Omega$) in Table 1, considering different velocity directions (but the same magnitude) and observe the final behavior and the type of the interaction (the initial condition as in (2.2) with parameters $A_0 = 2.25$, $x_c = -4.5$, $y_c = -4.5$).
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Fig. 11 Snapshots of the solution $u(t)$ to the 2d cubic NLS$_\Omega$ equation, with the initial condition $u_0$ from (2.2), $A_0 = 2.25$, $(x_c, y_c) = (-4.5, -4.5)$ and $v_2 = (15, 12)$ moving on the line $y = \frac{5}{2}x$.

5.2 The $L^2$-supercritical case

We now consider the 2d quintic NLS$_\Omega$ equation ($p = 5$). Again, we try to carefully examine the interaction between the obstacle and the solution. In the following simulations, we fix $A_0$ and the velocity $v$, but vary the translation parameters.

We start with an example, where there is no interaction in order to compare the behavior of the solution for different scenarios later, especially when there will be a strong interaction. For that, we consider the initial data $u_0$ from (2.2) with

$$A_0 = 1.25, \quad x_c = -4.5, \quad y_c = 5, \quad \text{and} \quad v = (15, 0), \quad (5.1)$$

which can be seen on the left of Fig. 12. The middle subplot shows that the corresponding solution to the 2d quintic NLS$_\Omega$ equation blows up in finite time at $t = 0.65$ with the diverging $L^\infty$-norm. Snapshots of the solution in time (top view onto the $xy$-plane) are plotted in Fig. 13. We observe that the solution blows up in finite time and there is no interaction between the solution and the obstacle.

Next, we take the same initial data $u_0$ as in the previous example (i.e., $A_0 = 1.25$, $v = (15, 0)$, $x_c = -4.5$) except for the $y_c$ value we choose $y_c = 2$ as shown in Fig. 4. In this case, we expect that the traveling wave solution has some weak interaction with the obstacle; see Fig. 14.

We observe that with this weak interaction, the solution still blows up in finite time at $t = 0.66$, but the blow-up time is delayed compared to the case, where there was no interaction between the solution and the obstacle, compare Figs. 6 and 14. Moreover, we observe a slight perturbation of the growth in the $L^\infty$-norm: at the collision, the amplitude of the solution starts decreasing, but after the weak interaction, the solution is back to the concentration leading to the blow-up. This can be explained by the
Table 1  Different velocity directions $\vec{v} = (v_x, v_y)$ and the corresponding behavior of the solution $u(t)$ to the 2d cubic (NLS$_{\Omega}$) with the indicated discrete mass and energy (the value of energy differs due to the phase)

| $\vec{v}$ | Discrete mass | Discrete energy | Behavior of the solution | Type of interaction |
|-----------|---------------|----------------|--------------------------|--------------------|
| $(15, 0)$ | 15.9043       | 442.9353       | Blow up at $t \approx 0.52$ | No interaction     |
| $(0, 15)$ | 15.9043       | 442.9353       | Blow up at $t \approx 0.52$ | No interaction     |
| $(15, 8)$ | 15.9043       | 570.4814       | Blow up at $t \approx 0.56$ | Weak interaction   |
| $(8, 15)$ | 15.9043       | 570.4814       | Blow up at $t \approx 0.56$ | Weak interaction   |
| $[9, 15)$ | 15.9043       | 604.0182       | Blow up $t \approx 0.57$   | Weak interaction   |
| $(15, 9)$ | 15.9043       | 604.0182       | Blow up $t \approx 0.57$   | Weak interaction   |
| $(10, 15)$| 15.9043       | 641.4737       | Blow up $t \approx 0.63$   | Weak interaction   |
| $(15, 10)$| 15.9043       | 641.4737       | Blow up $t \approx 0.63$   | Weak interaction   |
| $(15, 12)$| 15.9043       | 728.1404       | Scattering                 | Weak interaction   |
| $(12, 15)$| 15.9043       | 728.1404       | Scattering                 | Weak interaction   |
| $(15, 15)$| 15.9043       | 887.5015       | Scattering                 | Strong interaction |

Fig. 12  Solution to the 2d quintic NLS$_{\Omega}$ equation with $u_0$ from (2.2) and (5.1) (left) close to blow-up time (middle); time dependence of the $L^\infty$-norm (right)

Fig. 13  Snapshots of the evolution of $u_0$ from (2.2) with (5.1) at $t = 0$, $t = 0.38$ and $t = 0.65$. 

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Fig. 14 Solution $u(t)$ to the 2d quintic $\text{NLS}_\Omega$ equation with initial condition $u_0$ from (2.2), where $A_0 = 1.25$, $v = (15, 0)$ and $(x_c, y_c) = (-4.5, 2)$. A snapshot of $u(t)$ at $t = 0.66$ (left), the time dependence of the $L^\infty$-norm (right).

appearance of small reflected waves after the collision, which scatter at the end of the simulation. They can be seen in the snapshots of the solution in Fig. 15 with the view onto the $xy$-plane and zooming near the obstacle.

Next, we summarize the behavior of the solution to the 2d quintic $\text{NLS}_\Omega$ equation, depending on the initial parameters. We take different values for the space translation $y_c$ in the initial condition (2.2) and fix the following parameters:

$$A_0 = 1.25, \quad v_x = 15, \quad v_y = 0, \quad x_c = -4.5.$$ 

The results are given in Table 2.

Fig. 15 Snapshots of the time evolution of $u(t)$ with initial condition $u_0$ from (2.2), $A_0 = 1.25$, $v = (15, 0)$ and $(x_c, y_c) = (-4.5, 2)$, which eventually blows up in finite time.
Table 2  Influence of the translation parameter $y_c$ on the behavior of the solution $u(t)$ with initial data (2.2), $A_0 = 1.25$, $v = (15, 0)$

| $(x_c, y_c)$ | Discrete mass | Discrete energy | Behavior of the solution | Type of interaction |
|-------------|--------------|----------------|--------------------------|---------------------|
| $(-4.5, 5)$ | 4.9087       | 138.9766       | Blow up at $t \approx 0.65$ | No interaction      |
| $(-4.5, 4)$ | 4.9087       | 139.2859       | Blow up at $t \approx 0.68$ | No interaction      |
| $(-4.5, 3)$ | 4.9087       | 139.4553       | Blow up at $t \approx 0.65$ | Weak interaction    |
| $(-4.5, 2)$ | 4.9087       | 139.5022       | Blow up at $t \approx 0.66$ | Weak interaction    |
| $(-4.5, 1.5)$ | 4.9087   | 139.4946       | Blow up at $t \approx 0.51$ | Weak interaction    |
| $(-4.5, 1)$ | 4.9087       | 139.4784       | Blow up at $t \approx 0.41$ | Weak interaction    |
| $(-4.5, 0.5)$ | 4.9087  | 139.4636       | Scattering               | Weak interaction    |
| $(-4.5, 0)$  | 4.9087       | 139.4578       | Scattering               | Strong interaction  |
| $(-4.5, -0.5)$ | 4.9087  | 139.4636       | Scattering               | Weak interaction    |
| $(-4.5, -1)$  | 4.9087       | 139.4946       | Blow up at $t \approx 0.4$ | Weak interaction    |
| $(-4.5, -1.5)$ | 4.9087  | 139.4946       | Blow up at $t \approx 0.5$ | Weak interaction    |
| $(-4.5, -2)$  | 4.9087       | 139.0924       | Blow up at $t \approx 0.63$ | Weak interaction    |

Note that a tiny difference in the values of the discrete energy for different $y_c$ results from the varying density of the mesh grid: this is due to the fact that the polar coordinates $(r, \theta)$ form a uniform mesh, however, $(x, y) = (r \cos \theta, r \sin \theta)$ will not be uniform. Nevertheless, the discrete energy is conserved in each case from the start, i.e., $\mathcal{E}[u^0] = \mathcal{E}[u^1]$ in each simulation.

6 Strong interaction with an obstacle

6.1 The $L^2$-critical case

We now consider a direct interaction of the solution with an obstacle, which we term as a strong interaction, starting with the $L^2$-critical case. The depiction of the velocity direction and the initial location is in Fig. 16.

We consider the cubic $\text{NLS}_\Omega$ equation with the same initial data (2.2), with the same amplitude and space translation ($A_0 = 2.25$, $(x_c, y_c) = (-4.5, -4.5)$), as in Subsection 5.1, but in this case, we take the velocity directly pointed at the obstacle.

Fig. 16  The directions of the velocity of the solution $u(t)$ on the line $y = x$ and the same direction of the outward normal vector $\vec{n}$
\( v = (15, 15) \), meaning that the solution \( u(t) \) is moving along the line \( y = x \), i.e., in the same direction as the outward normal vector \( \vec{n} \) as shown in Fig. 16.

In this scenario, we observe that the solution has a scattering behavior and does not conserve the same profile or shape of the initial form of the solitary wave (see Fig. 17), thus exhibits the strong interaction. Snapshots of the time evolution of this solution \( u(t) \) to show the strong interaction are given in Fig. 18: the solution hugs the obstacle while transferring the mass forward, then it forms the two main bumps in front of the obstacle; later, they connect together, which creates a third (middle) bump and shifts more and more mass into this central lump while propagating it forward along the main velocity line (\( y = x \) in this case). The circle of dispersive reflective waves (including backward reflective waves) forms and expands, radiating out all of the reflective waves.

The obstacle transforms the blow-up behavior into what seems to be scattering, that we investigate further. Before doing that, we emphasize that the strong interaction has a substantial influence on the dynamics of the solution. We point out that in the weak interaction case in a similar example in Sect. 5.1 (\( L^2 \)-critical case), the solution blows up in finite time. In the considered case, the \( L^\infty \)-norm starts increasing, manifesting a blow-up behavior (see Fig. 17); however, after the collision, the amplitude of the solution starts decreasing. After that, we observe that the \( L^\infty \)-norm appears to stabilize, as shown in the right graph in Fig. 17, where \( \|u(t)\|_\infty \approx 0.8 \) after the interaction, indicating that the solution might not be scattering (for example, it could approach a rescaled soliton). We check this case simulating it for a longer time and observe that the \( L^\infty \)-norm continues to decrease as shown in the left subplot of Fig. 19. In the right graph, we show the solution amplitude change in time on a log scale for \( t \in [2, 3.32] \), which shows that it is decreasing as \( \frac{1}{\sqrt{t}} \). Thus, it is plausible to conclude that the solution scatters (however, this would have to be proved analytically, as it is possible that it might approach an asymptote at a later time).

### 6.2 The \( L^2 \)-supercritical case

In the 2d quintic NLS\(_\Omega \) equation (\( p = 5 \)), we also investigate the strong interaction between the obstacle and the solution, where the solution is moving in the same direction as the outward normal vector of the obstacle, see Fig. 20.

---

**Fig. 17** The initial condition \( u_0 \) at \( t = 0 \) from (2.2), \( A_0 = 2.25 \), \( (x_c, y_c) = (-4.5, -4.5) \) and \( v = (15, 15) \) (left); the corresponding solution \( u(t) \) to the 2d cubic NLS\(_\Omega \) equation at \( t = 1.2 \) (middle); time dependence of \( L^\infty \) norm (right)
We consider the same initial data (2.2), with the same phase as for the quintic NLS$_{\Omega}$ equation described in Sect. 5.2 but now with $y_c = 0$ (i.e., $A_0 = 1.25$, $x_c = -4.5$ and $v = (v_x, 0)$ are fixed parameters). In the present situation, the solution is moving on the line $y = 0$, i.e., in the same direction as the outward normal vector of the obstacle. The solitary wave hits the obstacle straight on, causing a strong interaction between the wave and the obstacle; see Fig. 21.

**Fig. 18** Snapshots of the solution $u(t)$ to the 2d cubic NLS$_{\Omega}$ equation for different time steps of the strong-interaction between the solution and the obstacle with the initial data as in Fig. 17.

**Fig. 19** The time dependence of $L^\infty$ norm (left); the slope of the $L^\infty$ norm from $t = 2$ to $t = 3.32$ on a log scale (right).
In this case, the solution scatters and does not preserve the shape of the original solitary wave. After the collision, the solitary wave solution forms two (then later possibly three, and eventually just one) bumps with a circular reflecting waves, part of which reflects backward; see Fig. 22. We observe also that the leading reflected wave has a dispersive behavior. Moreover, one can see that the presence of the obstacle completely prevents blow-up. Before the interaction, the $L^\infty$-norm of the solution starts increasing, indicating a possible blow-up behavior; however, after the interaction with the obstacle, the amplitude of the solution decreases towards 0, which confirms the dispersion of the solution in a long term, thus scattering.

7 Dependence on the obstacle size

In this section, we describe the behavior of the solution to the cubic and quintic NLS$_\Omega$ equations in the strong interaction case, as in Figs. 16 and 20, depending on the size of the obstacle (a disk of radius $r_\ast$). We study the strong interaction of the solution with the obstacle in terms of the transmitted and the reflected parts of the mass: we call the transmitted mass $M_T[u^n]$, the discrete $L^2$-norm of the solution $u^n$ at time $t = t^n$ on the right-half plane

$$\Omega_+ := \{(r, \theta) \in [r_\ast, R] \times [0, 2\pi] : 0 \leq \theta \leq \frac{\pi}{2} \text{ and } \frac{3\pi}{2} \leq \theta \leq 2\pi\},$$

Fig. 21 Solution $u(t)$ to the 2d quintic NLS$_\Omega$ equation with the initial condition $u_0$ from (2.2), $A_0 = 1.25$, $(x_c, y_c) = (-4.5, 0)$ and $v = (15, 0)$ (left), the solution $u(t)$ moving on the line $y = 0$ at time $t = 1.36$ (middle); time dependence of the $L^\infty$-norm (right)
Fig. 22  Snapshots of the behavior of the solution $u(t)$ to the $2d$ quintic NLS $Ω$ equation for different times in the strong interaction between the solution and the obstacle, with $(x, y)$ view.

thus, we obtain the mass on the right-half plane $\{(x, y) \in Ω : x \geq 0\}$,

$$\mathcal{M}_T[u^n] = \sum_{k=0}^{N_r} \sum_{0 \leq j \leq \frac{\pi}{2}} |u^n_{k,j}|^2 r_k \Delta r \Delta \theta, \quad \text{for } n \geq 0.$$ 

We call the reflected mass $\mathcal{M}_R[u^n]$, the discrete $L^2$-norm of the solution $u^n$ at time $t = t^n$ on the left-half plane

$$\Omega_- := \{(r, \theta) \in [r_*, R] \times [0, 2\pi] : \frac{\pi}{2} < \theta < \frac{3\pi}{2}\},$$

so that, we obtain the mass on the left-half plane $\{(x, y) \in Ω : x \leq 0\}$,

$$\mathcal{M}_R[u^n] = \sum_{k=0}^{N_r} \sum_{\frac{\pi}{2} < j \leq \frac{3\pi}{2}} |u^n_{k,j}|^2 r_k \Delta r \Delta \theta, \quad \text{for } n \geq 0.$$
We consider different values for the radius \( r_\ast \) of the obstacle and investigate solutions with initial data (2.2) having the amplitude \( A_0 \) and the velocity \( \mathbf{v} = (v_x, v_y) \). The spatial translation \((x_c, y_c)\) in the initial data depends on the radius \( r_\ast \) of the obstacle so that the initial conditions would satisfy the Dirichlet boundary condition. In §7.1, we first study the cubic, \( L^2 \)-critical, NLS\(_{\Omega} \) equation and in §7.2, we investigate the super-critical NLS\(_{\Omega} \) equation.

### 7.1 The \( L^2 \)-critical case

We consider the cubic NLS\(_{\Omega} \) equation with the initial condition as in (2.2) with the following parameters:

\[
A_0 = 2.5, \quad v_x = 15, \quad v_y = 15, \quad x_c = -4 - r_\ast, \quad y_c = -4 - r_\ast.
\]  

(7.1)

The solution \( u \) moves on the line \( y = x \), i.e., in the same direction as the outward normal vector of \( \Omega \). The dependence of the solution behavior on the obstacle size in the initial data depends on the radius \( r_\ast \) and the corresponding behavior of the solution \( u \) at the obstacle size in the strong interaction case is as follows (the summary is given in Table 3):

- For \( r_\ast = 0.1 \), the solution splits into two bumps with a small backward reflection, which has a dispersive behavior. As there is a sufficient mass transmitted, the two bumps get back together and behave as a single solitary wave, which blows up in finite time; see Fig. 23.

| \( r_\ast \)   | Discrete total mass | Behavior of the solution | Discrete reflected mass | Discrete transmitted mass |
|-----------|---------------------|--------------------------|------------------------|--------------------------|
| 0.1       | 19.6350             | Blow up at \( t \approx 0.888 \) | 3.7746 at \( t \approx 0.888 \) | 15.8603 at \( t \approx 0.888 \) |
| 0.2       | 19.6350             | Scattering               | 5.2641 at \( t \approx 1.2 \) | 14.3709 at \( t \approx 1.2 \) |
| 0.3       | 19.6350             | Scattering               | 7.1144 at \( t \approx 1.2 \) | 12.5206 at \( t \approx 1.2 \) |
| 0.4       | 19.6350             | Scattering               | 8.8085 at \( t \approx 1.2 \) | 10.8264 at \( t \approx 1.2 \) |
| 0.5       | 19.6350             | Scattering               | 10.1753 at \( t \approx 1.2 \) | 9.4596 at \( t \approx 1.2 \) |
| 0.6       | 19.6350             | Scattering               | 11.4460 at \( t \approx 1.2 \) | 8.1890 at \( t \approx 1.2 \) |
| 0.7       | 19.6350             | Scattering               | 12.5008 at \( t \approx 1.2 \) | 7.1341 at \( t \approx 1.2 \) |
| 0.8       | 19.6350             | Scattering               | 13.4017 at \( t \approx 1.2 \) | 6.2332 at \( t \approx 1.2 \) |
| 0.9       | 19.6350             | Scattering               | 14.1658 at \( t \approx 1.2 \) | 5.4691 at \( t \approx 1.2 \) |
| 1.0       | 19.6350             | Scattering               | 14.8208 at \( t \approx 1.2 \) | 4.8141 at \( t \approx 1.2 \) |
| 2.0       | 19.6350             | Scattering               | 18.0537 at \( t \approx 1.2 \) | 1.5813 at \( t \approx 1.2 \) |
| 3.0       | 19.6350             | Scattering               | 19.0024 at \( t \approx 1.2 \) | 0.6325 at \( t \approx 1.2 \) |
| 4.0       | 19.6350             | Blow up at \( t \approx 0.36 \) | 19.6349 at \( t \approx 0.36 \) | 5.6838e - 05 at \( t \approx 0.36 \) |
| 5.0       | 19.6350             | Blow up at \( t \approx 0.36 \) | 19.6349 at \( t \approx 0.36 \) | 5.6838e - 05 at \( t \approx 0.36 \) |
For $0.2 \leq r_* \leq 0.4$, we observe that the solution splits into several bumps with a small reflected part that leads to some loss of the transmitted mass and results in an overall scattering behavior of the solution (the transmitted part reconstructs back into one bump).

For $0.5 \leq r_* \leq 0.7$, we observe that the solution splits into several small bumps, which later make up the transmitted part of the solution with an important reflected part. The solution has a similar behavior as in the previous case; however, the reflected mass is higher than the half of the total mass $\frac{1}{2} M[u_0] = 9.8175$, which endorses the dispersive (scattering) behavior of the solution.

For $0.8 \leq r_* < 3$, the interaction surface between the solution and the obstacle is sufficiently large; thus, the solution scatters: the main part of the solution is mostly reflected back with a small dispersive transmitted part. We observe a reverse behavior of the transmitted and the reflected part of the solution compared to case for $0.2 \leq r_* \leq 0.4$; see Fig. 24 (for an example with $r_* = 2$).

For $r_* \geq 4$, the solution behaves quite different from the previous cases: the solution blows up in finite time at the boundary of the obstacle, as the interaction region is larger than the contour of the solution and there is almost no transmission of the solution (the transmitted mass is around $10^{-5}$). The solution can not cross or get around the obstacle boundary; hence, the solution concentrates in its blow-up core at the obstacle’s boundary; see Fig. 25 (for an example with $r_* = 5$).

Next, we give examples of the solution to the cubic NLS_Ω equation with a slightly smaller initial amplitude $A_0$, depending on the size of the obstacle (the strong interaction case); see Fig. 16. We consider the initial condition (2.2), with the following
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Fig. 24 Solution to the 2d cubic $\text{NLS}_{\Omega}$ equation with the obstacle radius $r_* = 2$ and $u_0$ from (2.2) with (7.1) moving along the line $y = x$. Snapshots of the scattering solution $u(t)$ at $t = 0$ (top left), $t = 0.5$ (middle top) and $t = 1.2$ (top right). Time dependence of the $L^\infty$-norm (bottom left) and of the transmitted and the reflected mass (bottom right).

parameters:

$$A_0 = 2.25, \quad v_x = 15, \quad v_y = 15, \quad x_c = -4 - r_* \quad \text{and} \quad y_c = -4 - r_*.$$ (7.2)

We record all observations in this case in Table 4.

Fig. 25 Solution to the 2d cubic $\text{NLS}_{\Omega}$ equation with the obstacle radius $r_* = 5$ and $u_0$ from (2.2) with (7.1) moving along the line $y = x$. Snapshots of the blow-up solution $u(t)$ at $t = 0$ (top left), $t = 0.32$ (middle top) and $t = 0.36$ (top right). Time dependence of the $L^\infty$-norm (bottom left) and of the transmitted and the reflected mass (bottom right).
We observe that in the case $r_*=0.1$, in both examples (7.1) and (7.2) with $A_0 = 2.25$ and $A_0 = 2.5$, the solutions blow up in finite time, as the reflection parts of the respective solutions are small and almost all of the solution is transmitted. The time evolution proceeds as follows: First, the solution hugs around the obstacle, splitting into two bumps, and then, since the radius of the obstacle is small, the solution gets back together to form a single bump in a few time steps. One can observe that the solution has a substantial transmitted mass, which leads to a blow up in finite time. It seems as the solitary bump has a similar shape of the solution as if there would be no obstacle interaction near the blow-up time; in particular, it would be interesting to investigate the profile and other features of the blow-up solution after the interaction with the obstacle. As the radius of the obstacle increases, we see very different dynamics of the solution compared to the NLS$_{R^2}$ equation in the whole space, since this solution blows up in finite time when the obstacle is absent (or has a very small radius).

### 7.2 The $L^2$-supercritical case

In this section, we summarize the behavior of the solution to the quintic NLS$_{\Omega}$ equation depending on the radius $r_*$ of the obstacle and we study the strong interaction. As before, we take a set of values of the obstacle radius $r_*$ and investigate the behavior

| $r_*$ | Discrete total mass | Behavior of the solution | Discrete reflected mass | Discrete transmitted mass |
|-------|---------------------|--------------------------|------------------------|--------------------------|
| 0.1   | 15.9043             | Blow up at $t \approx 0.891$ | 2.448 at $t \approx 0.891$ | 13.4563 at $t \approx 0.891$ |
| 0.2   | 15.9043             | Scattering               | 3.684 at $t \approx 1.2$  | 12.2203 at $t \approx 1.2$  |
| 0.3   | 15.9043             | Scattering               | 5.0335 at $t \approx 1.2$  | 10.8708 at $t \approx 1.2$  |
| 0.4   | 15.9043             | Scattering               | 6.3137 at $t \approx 1.2$  | 9.5906 at $t \approx 1.2$  |
| 0.5   | 15.9043             | Scattering               | 7.4179 at $t \approx 1.2$  | 8.4864 at $t \approx 1.2$  |
| 0.6   | 15.9043             | Scattering               | 8.3902 at $t \approx 1.2$  | 7.5141 at $t \approx 1.2$  |
| 0.7   | 15.9043             | Scattering               | 9.2624 at $t \approx 1.2$  | 6.6419 at $t \approx 1.2$  |
| 0.8   | 15.9043             | Scattering               | 10.0278 at $t \approx 1.2$  | 5.8765 at $t \approx 1.2$  |
| 0.9   | 15.90430            | Scattering               | 10.6936 at $t \approx 1.2$  | 5.2107 at $t \approx 1.2$  |
| 1     | 15.9043             | Scattering               | 11.2791 at $t \approx 1.2$  | 4.6253 at $t \approx 1.2$  |
| 2     | 15.9043             | Scattering               | 14.3843 at $t \approx 1.2$  | 1.52 at $t \approx 1.2$  |
| 3     | 15.9043             | Scattering               | 15.3338 at $t \approx 1.2$  | 0.57049 at $t \approx 1.2$  |
| 4     | 15.9043             | Scattering               | 15.65 at $t \approx 1.2$  | 0.2543 at $t \approx 1.2$  |
| 5     | 15.9043             | Blow up at $t \approx 0.754$ | 15.8291 at $t \approx 0.36$ | 0.07523 at $t \approx 0.36$ |
| 6     | 15.9043             | Blow up at $t \approx 0.754$ | 15.8291 at $t \approx 0.36$ | 0.07523 at $t \approx 0.36$ |
of the solution moving on the line \( y = 0 \) (in the strong interaction case) as shown in Fig. 20.

We take the initial data as in (2.2) and fix the following parameters:

\[
A_0 = 1.25, \quad v_x = 15, \quad v_y = 0, \quad x_c = -4 - r_\ast \quad \text{and} \quad y_c = 0. \quad (7.3)
\]

We first discuss behavior of solutions for different sized of the obstacle; specifically, we show the snapshots of the time evolution of the above data for the obstacle radii \( r_\ast = 0.1, r_\ast = 1 \) and \( r_\ast = 3 \). Then, we provide the summary of results in Table 5.

We observe that even for a small obstacle, \( r_\ast = 0.1 \), the solution scatters with a small backward reflection; see Fig. 26. For \( r_\ast = 1 \), the solution has a similar dispersive behavior as in the previous examples; however, we note that the reflected backward part is more relevant and important, which ensures the dispersive behavior of the solutions (see Fig. 27). On the other hand, for \( r_\ast = 3 \), the solution has a different behavior: the solution blows up in finite time at the boundary of the obstacle, as the interaction region becomes larger than the solution contour, and thus, the solution can not be transmitted around the obstacle. It concentrates at its blow-up core at the obstacle’s boundary; see Fig. 28.

### 8 Blow-up: wall-type initial data

In this section, we study blow-up solutions in the strong interaction case (moving directly towards the obstacle as shown in Fig. 20) for the large obstacle size and, in particular, we observe that even for a small obstacle, \( r_\ast = 0.1 \), the solution scatters with a small backward reflection; see Fig. 26. For \( r_\ast = 1 \), the solution has a similar dispersive behavior as in the previous examples; however, we note that the reflected backward part is more relevant and important, which ensures the dispersive behavior of the solutions (see Fig. 27). On the other hand, for \( r_\ast = 3 \), the solution has a different behavior: the solution blows up in finite time at the boundary of the obstacle, as the interaction region becomes larger than the solution contour, and thus, the solution can not be transmitted around the obstacle. It concentrates at its blow-up core at the obstacle’s boundary; see Fig. 28.

### Table 5 Influence of the obstacle size \( r_\ast \) onto the behavior of the solution \( u(t) \) to the 2d quintic NLS\( \Omega \) equation with \( u_0 \) from (2.2) and (7.3) with the discrete total mass, reflected and transmitted discrete mass parts after interaction with the obstacle at time \( t \).

| \( r_\ast \) | Discrete total mass | Behavior of the solution | Discrete reflected mass | Discrete transmitted mass |
|-------------|---------------------|--------------------------|------------------------|--------------------------|
| 0.1         | 4.9087              | Scattering               | 0.67277 at \( t \approx 1.2 \) | 4.236 at \( t \approx 1.2 \) |
| 0.2         | 4.9087              | Scattering               | 1.1094 at \( t \approx 1.2 \) | 3.7994 at \( t \approx 1.2 \) |
| 0.3         | 4.9087              | Scattering               | 1.5618 at \( t \approx 1.2 \) | 3.3469 at \( t \approx 1.2 \) |
| 0.4         | 4.9087              | Scattering               | 1.9578 at \( t \approx 1.2 \) | 2.9509 at \( t \approx 1.2 \) |
| 0.5         | 4.9087              | Scattering               | 2.3141 at \( t \approx 1.2 \) | 2.5946 at \( t \approx 1.2 \) |
| 0.6         | 4.9087              | Scattering               | 2.6199 at \( t \approx 1.2 \) | 2.2888 at \( t \approx 1.2 \) |
| 0.7         | 4.9087              | Scattering               | 2.8845 at \( t \approx 1.2 \) | 2.0243 at \( t \approx 1.2 \) |
| 0.8         | 4.9087              | Scattering               | 3.1181 at \( t \approx 1.2 \) | 1.7906 at \( t \approx 1.2 \) |
| 0.9         | 4.9087              | Scattering               | 3.3263 at \( t \approx 1.2 \) | 1.5824 at \( t \approx 1.2 \) |
| 1           | 4.9087              | Scattering               | 3.5143 at \( t \approx 1.2 \) | 1.3945 at \( t \approx 1.2 \) |
| 1.5         | 4.9087              | Blow up at \( t \approx 0.377 \) | 4.7085 at \( t \approx 0.377 \) | 0.20019 at \( t \approx 0.377 \) |
| 2           | 4.9087              | Blow up at \( t \approx 0.285 \) | 4.908 at \( t \approx 0.285 \) | 7.5778e\(-4\) at \( t \approx 0.285 \) |
| 3           | 4.9087              | Blow up at \( t \approx 0.278 \) | 4.9087 at \( t \approx 0.278 \) | 6.2761e\(-8\) at \( t \approx 0.278 \) |
some cases, reuniting back into one single bump, which then blows up. We investigate the behavior of solutions to the cubic $\text{NLS}_\Omega$ equation with a special round wall-type super-Gaussian initial data. We consider the initial condition, which is defined by the product of a phase in terms of the angle $\Theta := (\Theta_j)_{1 \leq j \leq N}$ with a super-Gaussian in
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**Fig. 28** Solution to the 2d quintic NLS equation with the obstacle radius $r_*=3$ and $u_0$ from (2.2) with (7.3) moving along the line $y=0$. Snapshots of the blow-up solution $u(t)$ at $t = 0$ (top left), $t = 0.23$ (middle top) and $t = 0.278$ (top right). Time dependence of the $L^\infty$-norm (bottom left) and of the transmitted and the reflected mass (bottom right).

$$u_0(r, \Theta) := A_0 \left( e^{-(r+r_c)^4} \times e^{-\frac{1}{2}(\Theta-\pi)^4} \right) e^{i \left( \frac{1}{2}(v_x r \cos(\Theta)+v_y r \sin(\Theta)) \right)},$$

where

$$A_0 = 2.5, \quad v_x = 15, \quad v_y = 0, \quad \text{and} \quad r_c = -4 - r_*.$$  

Note that, due to the construction of this solution, the $L^2$-norm, or the mass, of $u_0$ depends on the radius of the obstacle $r_*$, i.e., the mass increases as $r_*$ becomes larger. This does not affect the conservation of the mass throughout the simulation for fixed $r_*$.

In the following simulations, we consider $r_* = 3$ (Fig. 29) and $r_* = 5$ (Fig. 31). We observe that even with the large radius of the obstacle, the solution blows up in

**Fig. 29** Solution to the 2d cubic NLS equation with the obstacle radius $r_*=3$ and $u_0$ from (8.1) with (8.2) moving along the line $y=0$. Snapshots of the blow-up solution $u(t)$ at $t = 0$ (left), $t = 0.6$ (middle) and $t = 0.79$ (right).
finite time. After the interaction, the solution splits into two bumps, with an essential backward reflection and a substantial transmitted mass. Before the two bumps could merge together, they concentrate in their own blow-up core regions, that is, each bump blows up separately at a single point location; see the right plots in Fig. 29 for $r_\star = 3$ and Fig. 31 for $r_\star = 5$; also the growth of the $L^\infty$ norm on the right plots of Figs. 30 and 32.

We next consider the data (8.1) with the following parameters (changing the amplitude):

$$A_0 = 1.5, \quad v_x = 15, \quad v_y = 0, \quad \text{and} \quad r_c = -4 - r_\star.$$  \hspace{1cm} \text{(8.3)}

For the last two examples in this paper, we consider $r_\star = 2$ and $r_\star = 10$ with the data in (8.3).

First, we observe that the solution blows up in finite time even if the radius of the obstacle is large (compared to the previous example with the amplitude $A_0 = 2.5$); see Fig. 33, where $r_\star = 2$.

After the interaction, the solution splits into two bumps, with the backward reflection having a substantial amount of the transmitted mass. Then, the two bumps have sufficient time to merge together and pump the mass from both lumps (as the circle expands) into a single bump, which has enough mass to concentrates in its core to blow up in finite time; see Fig. 33 and also the $L^\infty$ norm together with the change in time in the transmitted and reflected mass in Fig. 34.

However, for a larger radius, for example, $r_\star \geq 3$, the solution has to hug around an obstacle with the bigger size, and thus, the mass gets dispersed more around; hence,

![Fig. 30](image1.png)  \hspace{1cm} ![Fig. 31](image2.png)
less of the mass is transmitted, which concludes in the overall scattering behavior. See Fig. 35 for an example of the obstacle size $r_\ast = 10$.

For the initial data (8.3) to track the dependence as the radius of the obstacle increases, we perform numerical simulations for a variety of radii and provide the summary of the results in Table 6.

9 Conclusion

In this work, we initiated a numerical study of how the behavior of solutions can change in a presence of a smooth convex obstacle. We observe that the interaction between a solitary wave and the obstacle can significantly influence the overall behavior of the solution to the NLS$_\Omega$ equation, which depends on the direction of the velocity vector $\vec{v} = (v_x, v_y)$, the size of the obstacle (radius $r_\ast$), the initial distance ($d$ vs. $d^\ast$) to the obstacle, and the translation parameters ($x_c, y_c$). The presence of the obstacle yields strong, weak, or no interaction. We observed in Sects. 5.1 and 5.2 that even a
small interaction between the obstacle and a single peak solution has some influence on the dynamics (at least, on the blow-up time). Moreover, we conclude that the strong interaction has a significant effect on the behavior of solutions depending on the size of the obstacle, for example, instead of approaching a solitary wave solution with a single bump, the shape of the solution drastically changes after the collision, splitting it into several bumps with a backward reflected wave. The appearance of the reflection waves due to the presence of the obstacle with Dirichlet boundary conditions prevents the solution from blowing up in finite time in some cases. Furthermore, this backward reflection has always a dispersive character, and this might be the reason why the solution scatters in most of the cases after a strong interaction. However, if the obstacle is very small or if the contour of the solution is significantly bigger than the radius of the obstacle, we observed the existence of blow-up solutions. In this case, the interaction surface is negligible so that the mass of the solution is almost all transmitted, which is sufficient to develop a blow-up. If the obstacle is large enough or if there is no transmission of the solution, then either the solution is completely reflected

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**Fig. 34** Time dependence of the $L^\infty$-norm (left) and of the transmitted and the reflected mass (right) for the solution in Fig. 33

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**Fig. 35** Solution to the 2d cubic NLS$_\Omega$ equation with the radius of the obstacle $r_\star = 10$ and $u_0$ from (8.1) with (8.3). Snapshots of the scattering solution $u(t)$ at $t = 0$ (top left), $t = 2.5$ (middle top) and $t = 5$ (top right). Time dependence of the $L^\infty$-norm (bottom left) and of the transmitted and the reflected mass (bottom right)
back or the solution concentrates in its blow-up core at the obstacle’s boundary, since the interaction region is relevant and it is larger than the contour of the solution. Furthermore, we construct new (Wall-type) initial condition, time evolution of which is characterized by its high mass transmission after a strong interaction, and then blowing up in finite time in a single point or in two separate locations (single points) after the strong interaction. For a weak interaction, i.e., when the solution preserves the shape as a traveling solitary wave, the solution behaves either as a solitary wave solution, constructed in [31] (which exists for all positive times), or as the one shown in [32] (see also [30]), a finite time (single point) blow-up solution (as if there would be no obstacle).

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Declarations

Conflict of interest The authors declare no competing interests.

References

1. Akrivis, G., Dougalis, V.A., Karakashian, O.: Solving the systems of equations arising in the discretization of some nonlinear PDE’s by implicit Runge-Kutta methods. RAIRO Modél. Math. Anal. Numér. 31(2), 251–287 (1997)
2. Akrivis, G.D.: Finite difference discretization of the cubic Schrödinger equation. IMA J. Numer. Anal. 13(1), 115–124 (1993)
3. Anton, R.: Global existence for defocusing cubic NLS and Gross-Pitaevskii equations in three-dimensional exterior domains. J. Math. Pures Appl. 89 (9) 4, 335–354 (2008)
4. Berestycki, H., Lions, P.-L.: Nonlinear scalar field equations. I. Existence of a ground state. Arch. Rational Mech. Anal. 82(4), 313–345 (1983)
5. Besse, C.: Schéma de relaxation pour l’équation de Schrödinger non linéaire et les systèmes de Davey et Stewartson. C.R. Acad. Sci. Paris Sér I Math. 326(12), 1427–1432 (1998)
6. Besse, C.: A relaxation scheme for the non-linear Schrödinger equation. SIAM J. Numer. Anal. 42(3), 934–952 (2004)
7. Besse, C., Bidégaray, B., Descombes, S.: Order estimates in time of splitting methods for the nonlinear Schrödinger equation. SIAM J. Numer. Anal. 40(1), 26–40 (2002)
8. Blair, M.D., Smith, H.F., Sogge, C.D.: Strichartz estimates and the nonlinear Schrödinger equation on manifolds with boundary. Math. Ann. 354(4), 1397–1430 (2012)
9. Burq, N., Gérard, P., Tzvetkov, N.: On nonlinear Schrödinger equations in exterior domains. Ann. Inst. H. Poincaré Anal. Non Linéaire 21(3), 295–318 (2004)
10. Coffman, C.V.: Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions. Arch. Ration. Mech. Anal. 46, 81–95 (1972)
11. Cooper, J., Strauss, W.A.: Energy boundedness and decay of waves reflecting off a moving obstacle. Indiana Univ. Math. J. 25(7), 671–690 (1976)
12. Delfour, M., Fortin, M., Payre, G.: Finite-difference solutions of a nonlinear Schrödinger equation. J. Comput. Phys. 44(2), 277–288 (1981)
13. Dodson, B.: Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state. Adv. Math. 285, 1589–1618 (2015)
14. Duyckaerts, T., Holmer, J., Roudenko, S.: Scattering for the non-radial 3D cubic nonlinear Schrödinger equation. Math. Res. Lett. 15(6), 1233–1250 (2008)
15. Duyckaerts, T., Landoulsi, O., and Roudenko, S.: Threshold solutions in the focusing 3D cubic NLS equation outside a strictly convex obstacle. J. Funct. Anal. 282(5), Paper No. 109326, 55 (2022)
16. Duyckaerts, T., Roudenko, S.: Threshold solutions for the focusing 3d cubic Schrödinger equation. Rev. Mat. Iberoam. 26(1), 1–56 (2010)
17. Fang, D., Xie, J., Cazenave, T.: Scattering for the focusing energy-subcritical nonlinear Schrödinger equation. Sci. China Math. 54(10), 2037–2062 (2011)
18. Guevara, C.D.: Global behavior of finite energy solutions to the $d$-dimensional focusing nonlinear Schrödinger equation. Appl. Math. Res. Express. AMRX. 2, 177–243 (2014)
19. Holmer, J., Roudenko, S.: A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation. Comm. Math. Phys. 282(2), 435–467 (2008)
20. Ivanovici, O.: Precised smoothing effect in the exterior of balls. Asymptot. Anal. 53(4), 189–208 (2007)
21. Ivanovici, O.: On the Schrödinger equation outside strictly convex obstacles. Analysis & PDE 3(3), 261–293 (2010)
22. Ivanovici, O., Planchon, F.: On the energy critical Schrödinger equation in $\mathbb{R}^3$. In Math. Analysis and Applications, Part A, vol. 7 of Adv. in Math. Suppl. Stud. Academic Press, New York pp. 369–402 (1981)
23. Ivriï, V.J.: Exponential decay of the solution of the wave equation outside an almost star-shaped region. Dokl. Akad. Nauk SSSR 189, 938–940 (1969)
24. Karakashian, O., Akrivis, G.D., Dougalis, V.A.: On optimal order error estimates for the nonlinear Schrödinger equation. SIAM J. Numer. Anal. 30(2), 377–400 (1993)
25. Killip, R., Visan, M., Zhang, X.: Riesz transforms outside a convex obstacle. Internat. Math. Res. Not. 2016(19), 5875–5921 (2015)
26. Klein, C., Roudenko, S., and Stoilov, N.: Numerical study of Zakharov-Kuznetsov equations in two dimensions. J. Nonlinear Sci. 31(2), Paper No. 26, 28 (2021)
27. Kwong, M.K.: Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. Arch. Rational Mech. Anal. 105(3), 243–266 (1989)
30. Landoulsi, O.: Dynamics of the nonlinear focusing Schrödinger equation outside of a smooth, compact and convex obstacle. PhD thesis, University Sorbonne Paris Nord, (2020)
31. Landoulsi, O.: Construction of a solitary wave solution of the nonlinear focusing Schrödinger equation outside a strictly convex obstacle in the $L^2$-supercritical case. Discrete Contin. Dyn. Syst. 41(2), 701–746 (2021)
32. Landoulsi, O.: On blow-up solutions to the nonlinear Schrödinger equation in the exterior of a convex obstacle. Dyn. Partial Differ. Equ. 19(1), 1–22 (2022)
33. Lax, P.D., Morawetz, C.S., Phillips, R.S.: The exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle. Bull. Amer. Math. Soc. 68, 593–595 (1962)
34. Lax, P.D., Morawetz, C.S., Phillips, R.S.: Exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle. Comm. Pure Appl. Math. 16, 477–486 (1963)
35. Morawetz, C.S.: The decay of solutions of the exterior initial-boundary value problem for the wave equation. Comm. Pure Appl. Math. 14, 561–568 (1961)
36. Morawetz, C.S.: The limiting amplitude principle. Comm. Pure Appl. Math. 15, 349–361 (1962)
37. Morawetz, C.S., Ralston, J.V., Strauss, W.A.: Decay of solutions of the wave equation outside non-trapping obstacles. Comm. Pure Appl. Math. 30(4), 447–508 (1977)
38. Morawetz, C.S., Ralston, J.V., and Strauss, W.A.: Correction to: “Decay of solutions of the wave equation outside nontrapping obstacles” (CPAM 30 (1977), no. 4, 447–508). Comm. Pure Appl. Math. 31(6), 795 (1978)
39. Olson, D., Shukla, S., Simpson, G., Spirm, D.: Petviashvilli’s method for the Dirichlet problem. J. Sci. Comput. 66(1), 296–320 (2016)
40. Pelinovsky, D.E., Stepanyants, Y.A.: Convergence of Petviashvili’s iteration method for numerical approximation of stationary solutions of nonlinear wave equations. SIAM J. Numer. Anal. 42(3), 1110–1127 (2004)
41. Petviashvili, V.I.: Equation of an extraordinary soliton. Fizika Plazmy 2, 469–472 (1976)
42. Planchon, F., and Vega, L.: Bilinear virial identities and applications. Ann. Sci. Éc. Norm. Supér. 42 (4) 2, 261–290 (2009)
43. Roudenko, S., Wang, Z., Yang, K.: Dynamics of solutions in the generalized Benjamin-Ono equation: a numerical study. J. Comp. Phys. 445, 110570 (2021)
44. Sanz-Serna, J.M., Calvo, M.P.: Numerical Hamiltonian problems. Applied Mathematics and Mathematical Computation, vol. 7. Chapman & Hall, London (1994)
45. Sanz-Serna, J.M., Verwer, J.G.: Conservative and nonconservative schemes for the solution of the nonlinear Schrödinger equation. IMA J. Numer. Anal. 6(1), 25–42 (1986)
46. Weideman, J.A.C., Herbst, B.M.: Split-step methods for the solution of the nonlinear Schrödinger equation. SIAM J. Numer. Anal. 23(3), 485–507 (1986)
47. Weinstein, M. I.: Nonlinear Schrödinger equations and sharp interpolation estimates. Comm. Math. Phys. 87(4), 567–576 (1982/83)
48. Wilcox, C.H.: Spherical means and radiation conditions. Arch. Rational Mech. Anal. 3, 133–148 (1959)
49. Yang, K., Roudenko, S., Zhao, Y.: Blow-up dynamics and spectral property in the $L^2$-critical nonlinear Schrödinger equation in high dimensions. Nonlinearity 31(9), 4354–4392 (2018)
50. Yang, K., Roudenko, S., Zhao, Y.: Blow-up dynamics in the mass super-critical NLS equations. Phys. D 396, 47–69 (2019)

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