Field-theoretic methods for systems of particles with exotic exclusion statistics

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Abstract

We calculate the partition function of a gas of particles obeying Haldane exclusion statistics, using a definition of a Hilbert space having a ‘fractional dimension’ and constructing appropriate coherent states. The fractional dimension is expressed though the form of the identity operator in the Hilbert space.

We find that there many possible generalisations of the Pauli exclusion principle, with particular choices of the scalar product leading to consistency either with Haldane’s original definition of the effective dimensionality of the Hilbert space or with the combinatorial procedure invoked by Haldane and Wu. We explicitly demonstrate that at low particle densities these definitions are equivalent.

1 Introduction

Haldane introduced in [1] a generalized exclusion principle defining a quantity $d(N)$, the Haldane dimension, which is the dimension of the one-particle Hilbert space associated with the $N$-th particle, keeping the coordinates of the other $N-1$ particles fixed. The statistical parameter, $g$, of a particle (‘$g$-on’) is defined by (where we add $m$ particles)

$$g = -\frac{d(N+m) - d(N)}{m}$$

and the conditions of homogeneity on $N$ and $m$ are imposed. The system is assumed to be confined to a finite region where the number $K$ of independent
single-particle states is finite and fixed. Here the usual Bose and Fermi ideal
gases have \( g = 0 \) for Bose case (i.e. \( d(N) \) does not depend on \( N \)) and \( g = 1 \)
for Fermi case – that is the dimension is reduced by one for each added
fermion, which is the usual Pauli principle.

Haldane also introduced a combinatorial expression (which we will term
the Haldane-Wu state-counting procedure \cite{1, 2}) for the number of ways, \( W \),
to place \( N \) \( g \)-ons into \( K \) single–particle states. Then

\[
W = \frac{[d(N) + N - 1]!}{[d(N) - 1]!N!} \quad d(N) = K - g(N - 1),
\]

which was subsequently used by many authors \cite{3, 4, 5, 6, 7, 8} to describe
thermodynamical properties of \( g \)-ons. In particular Bernard and Wu \cite{4} and
Murthy and Shankar \cite{9} showed that the behavior of the excitations in the
Calogero–Sutherland model is consistent with Eqn.(2) \cite{2} for \( g \)-ons, with
fractional \( g \), in general.

In Ref \cite{10} the microscopic origin of the Haldane-Wu state-counting pro-
cedure was examined. The notion of statistics was considered in a proba-
bilistic spirit. The author assumed that a single level may be occupied by
any number of particles, and each occupancy is associated with an a priori
probability. These probabilities are determined by enforcing consistency with
the Haldane-Wu state-counting procedure and not with Haldane’s definition
of exclusion statistics. There was no construction of a Hilbert space and
the a priori probabilities may be negative. This approach has been further
elaborated in a number of papers \cite{11, 12, 13}.

Another probabilistic approach has been developed in \cite{14, 15, 16}. It
was pointed out that there is a distinction between Haldane’s dimension and
Haldane-Wu state counting procedures. A ‘fractional’ Hilbert space (associ-
ated with the non-integer nature of \( d(N) \)) and the corresponding creation-
annihilation operators were constructed and a set of probabilities which give
Haldane’s dimension was obtained.

The paper is organized as follows. In the next section we introduce the
notion of a fractional Hilbert space and creation-annihilation operators as-
associated with it. In section 3 we obtain a generalised resolution of unity
in terms of coherent states. In section 4 the definition of Haldane’s dimen-
sion is considered in detail. We calculate the partition function and the
state-counting expression. In section 5 we consider the Haldane-Wu state-
counting procedure and make a comparison with the definition of Haldane’s
dimension.
2 Hilbert space and creation-annihilation operators

In this section we recall the main ideas introduced in Ref [15].

The definition of a fractional dimension Hilbert space is connected with state-counting, which we need to calculate the entropy and other thermodynamical quantities of \( g \)-particles.

The main idea is to consider the process of inserting the \( N \)-th particle into the system as a probabilistic process (in spirit of Gibbs), i.e. we assume that the probability of such insertion plays the role of Haldane’s measure of the probability to add the \( N \)-th particle to the system. Let us illustrate the idea for the case of a single degree of freedom, \( g = 1/p \), and provide an interpretation of \( d(N) \) for that case.

Firstly, we have the vacuum state to which we add the first particle. We assume that the nature of the statistics reveals itself at the level of two particles, so \( d(1) = 1 \). Now let us assume that the process of insertion of the second particle is a probabilistic one with the probability \((1 - g)\) of success. We interpret this as fractional dimension, \( d(2) \), of the subspace (corresponding to double occupation) and \( d(2) = 1 - g \). The conditional probability to add a third particle (with two assumed present) is \( 1 - 2g \). Hence the probability of success in adding three particles is \( 1 \times (1 - g)(1 - 2g) \). This leads us to the probability of adding \( n \) particles is:

\[
\alpha_n = [1 - g][1 - 2g] \cdots [1 - (n - 1)g]
\]

We see that the probability to find \( N > p \) particles in the system is equal to zero.

Drawing parallels with dimensional regularization we can formulate a geometrical definition of the fractional dimension. In that case the trace of the identity matrix is identified with the value of (non-integer) dimension, \( d(N) \). In the calculation of thermodynamical quantities such as the partition function or the mean value of an arbitrary operator \( \hat{O} \) we must compute the following traces:

\[
Z = \text{Tr} \left[ \text{Id} \cdot e^{-\beta H} \right], \quad \langle \hat{O} \rangle = \frac{1}{Z} \text{Tr} \left[ \text{Id} \cdot e^{-\beta H} \hat{O} \right]
\]

where the Hamiltonian \( H \), e.g. for an ideal gas, is

\[
H = \sum_{i=1}^{K} \epsilon_i n_i
\]
and the “unit operator”, \( \text{Id} \), which completely defines the exclusion statistics of the particles is defined by

\[
\text{Id} = \sum_{n_1, \ldots, n_K = 0}^{\infty} \alpha_{n_1, \ldots, n_K} |n_1, \ldots, n_K\rangle \langle n_K, \ldots, n_1|
\]

where \( \alpha_{n_1, \ldots, n_K} \) is the probability to find the state \( |n_1, \ldots, n_K\rangle \). Then the full dimension of the \( N \)-particle subspace is given by the formula

\[
W(N) = \text{Tr} \left( \text{Id} | \sum_{i=1}^{K} n_i = N \rangle = \sum_{n_1 + \ldots + n_K = N} \alpha_{n_1, \ldots, n_K} \right)
\]

An analog of Haldane’s dimension, \( d(N) \), for the \( N \)-particle subspace with an arbitrary fixed \((N-1)\)-particle substate is described by the relation

\[
d(N) = \sum_{\ell=1}^{N} \frac{\alpha_{n_1, \ldots, n_{\ell+1}, \ldots, n_K}}{\alpha_{n_1, \ldots, n_K} \mid \sum_{n_i = N-1}}
\]

The above procedure is completely general, a concrete choice of the probabilities \( \alpha_{n_1, \ldots, n_K} \) is not required.

On the basis of the following two assumptions:

1. the definition of the \( N \)-th particle dimension \( d(N) \) actually yields Haldane’s conjecture \( d(N) = K - g(N - 1) \);

2. the Hilbert space of the system with \( K \) degrees of freedom is factorized into a product of Hilbert spaces corresponding to each degree of freedom. This means

\[
\text{Id} = \text{Id}_1 \otimes \text{Id}_2 \otimes \ldots \otimes \text{Id}_K
\]

it was shown, in Ref \[13\], that there is a single self-consistent way to define \( \alpha_{n_1, \ldots, n_K} \):

\[
\alpha_{n_1, \ldots, n_K} = \prod_{i=1}^{K} [1 - g][1 - 2g]\ldots[1 - (n_i - 1)g]
\]

where \( g = 1/p \), \( p \) integer. In this case the statistical parameter \( g \) can take values between 0 and 1.

If we weaken the second condition and allow matrix \( \text{Id} \) to be a direct product of \( \text{Id} \)'s which correspond to some elementary ‘exclusion cell’ (block)
with dimension $q > 1$ but keeping the first condition, we obtain the following set of probabilities:

$$
\alpha(\{n_{ij}\}_{i,j=1}^{K,q}) = \prod_{i=1}^{K} \left[ 1 - \frac{1}{p} \left( 1 - \frac{2}{p} \right) \left( 1 - \frac{q - 1}{p} \sum_{j=1}^{q} n_{ij} - 1 \right) \right] \quad (9)
$$

with statistical parameter $g = q/p$, $p$ integer and $K' = qK$ the full number of single particle states [14]. Note that the statistical parameter can take values greater than 1.

Next we can weaken in addition the first condition and allow $d(N)$ to be a non-linear function of $N = \sum_{i=1}^{K} n_i$. Then we find a large variety of probabilities, among them there is a set of probabilities corresponding to the Haldane-Wu state-counting procedure.

To allow interactions between exclusions, ‘hopping’ or interaction with some random potential we should develop a second-quantized formalism. Representation for these operators can be found from the following conditions:

$$
a_i^\dagger |n_1 \ldots n_i \ldots n_K\rangle = \beta_{n_1 \ldots n_K} |n_1 \ldots n_i + 1 \ldots n_K\rangle \quad (10)
$$

$$
(a_i^\dagger)^\dagger = a_i = \text{Id}^{-1} (a_i^\dagger)^* \text{Id} \quad (11)
$$

$$
N_i |n_1 \ldots n_K\rangle = n_i |n_1 \ldots n_K\rangle \quad (12)
$$

The most interesting case is a system consisting from one exclusion cell, that is $g = K/p$, $K$ the full number of single particle states. In this case coefficients $\beta$ depend only on $n_i$ and $n = \sum_k n_k$:

$$
\beta_{n_1 \ldots n_K} = \beta_{n_i,n} = \sqrt{\frac{n_i + 1}{\alpha_n}} \frac{\alpha_n}{\alpha_{n+1}} \quad (13)
$$

A remarkable result is that for the hopping term $(a_i^\dagger a_j)$ the dependence on $\alpha$ disappears and it can be represented as

$$
a_i^\dagger a_j = b_i^\dagger b_j P(p) \quad (14)
$$

where $b_i^\dagger, b$ are the bosonic operators and $P(p)$ the projector on a subspace with the number of particles less than or equal to $p$. 
3 Coherent states

To illustrate the idea consider the simplest case $K = 1, g = 1/p$ and exchange statistics between $g$-particles to be bosonic one. Let us confine ourself for a moment by considering Hamiltonians depending on the number of particles only. Then states $|n\rangle$ can be chosen to be bosonic ones:

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \quad n = a^\dagger a$$

$a^\dagger, a_i$ are bosonic operators.

There is a well-known expressions for a trace and a expansion of the unit in terms of the bosonic coherent states:

$$\text{Tr}[\hat{O}] = \frac{1}{\pi} \int d\bar{z}dz \ e^{-\bar{z}z} \langle \bar{z}|\hat{O}|z\rangle$$

$$I = \frac{1}{\pi} \int d\bar{z}dz \ e^{-\bar{z}z} |z\rangle\langle \bar{z}| = \sum_n |n\rangle\langle n|$$

$$|z\rangle = e^{a^\dagger z}|0\rangle, \quad \langle \bar{z}| = \langle 0|e^{a\bar{z}}$$

If we express the matrix Id in terms of the bosonic coherent states then we can directly apply the bosonic technique to the system of exclusions. Having a look on the usual resolution of the unit we can conclude that if we find a function $F$ such that

$$\frac{1}{\pi} \int d\bar{z}dz \ F(\bar{z}z)|z\rangle\langle \bar{z}| = \sum_n \alpha_n |n\rangle\langle n| = \text{Id}$$

we solve the problem. Rewriting the probabilities in the form

$$\alpha_n = \frac{p!}{p^n(p-n)!}$$

it can be shown that the following relation for the matrix Id holds

$$\text{Id} = \int_C dt \ F_p(t) \ \int d\bar{z}dz \ e^{-\bar{z}z} |zt^{1/2}\rangle\langle \bar{z}t^{1/2}|$$

$$F_p(t) = \frac{1}{2\pi t} p! e^{pt}t^{-p-1}p^{-p}$$
where the contour $C$ runs around the origin in the complex plane in the counter clockwise direction. Noting that
\[
\int_C dt \, F_p(t) t^k = \frac{p!}{p^k(p-k)!}
\] (21)
we see that the partition function takes the form:
\[
Z_{1/p} = \int_C dt \, F_p(t)(1 - te^{-\beta\epsilon})^{-1} = \sum_{k=0}^{p} \frac{p!}{p^k(p-k)!} e^{-k\beta\epsilon}
\] (22)
For fermions ($p = 1$):
\[
Z_f = Z_1 = 1 + e^{-\beta\epsilon}
\]
To investigate the bosonic limit the following representation for the partition function is useful:
\[
Z_{1/p} = \int_0^\infty dt \, e^{-t} \left[ 1 + \frac{te^{-\beta\epsilon}}{p} \right]^p
\] (23)
when $p \to \infty$ (bosonic limit) we have
\[
Z_{1/p} \xrightarrow{p \to \infty} Z_\infty = \int_0^\infty dt \, e^{-t+te^{-\beta\epsilon}} = \frac{1}{1 - e^{-\beta\epsilon}} = Z_b
\]

4 Haldane’s dimension procedure

In this section we consider in detail the set of probabilities (9) with $K = 1$, i.e. one exclusion cell ($q$ is the number of states):
\[
\alpha(\{n_j\}_{j=1}^q) = \prod_{j=1}^{N-1} \left[ 1 - \frac{j}{p} \right] = \frac{1}{p^N(p-N)!} \cdot N = \sum_{j=1}^{q} n_j
\] (24)
From (24) and (7) we have obviously
\[
d(N) = q \left[ 1 - \frac{N-1}{p} \right] = q - g(N-1) , \quad g = \frac{q}{p}
\]
The matrix $\text{Id}$ in this case has the form
\[
\text{Id} = \int_C dt \, F_p(t) \frac{1}{\pi^q} \int \prod_{j=1}^{q} [d\bar{z}_j dz_j \, e^{-\bar{z}_j z_j}] \, |\{z_j \sqrt{t}\}_{j=1}^q \rangle \langle \{\bar{z} \sqrt{t}\}_{j=1}^q |
\] (25)
with the same function $F_p(t)$ defined in (20) and the usual bosonic coherent states.

To express trace of an operator in terms of bosonic coherent states we use the following relation

$$\text{Tr}[\text{Id} \cdot \hat{O}] = \frac{1}{\pi^q} \int \prod_{j=1}^{q} \tilde{w}_j dw_j \ e^{-\sum_j \bar{w}_j w_j} \langle \{ \bar{w}_j \} | \text{Id} \cdot \hat{O} | \{ w \} \rangle$$  \hspace{1cm} (26)

The last relation allows us to calculate a partition function of $g$-ons with Hamiltonian

$$\hat{H} = \epsilon \hat{N} , \quad \hat{N} = \sum_{i=1}^{n} n_i = \sum_{i=1}^{q} a_i^\dagger a_i$$  \hspace{1cm} (27)

with $a_i^\dagger, a_i$ being bosonic creation-annihilation operators. From (26,25) we have

$$Z_{q/p} = \int_C dt \ F_p(t) \frac{1}{\pi^{2q}} \int D\bar{w} Dw D\bar{z} Dz \ e^{-\bar{w}w - \bar{z}z} \langle \bar{w} | z \sqrt{t} \rangle \langle z \sqrt{t} | e^{-\beta (\hat{H} - \mu \hat{N})} | w \rangle$$

where

$$D\bar{w} Dw \equiv \prod_{j=1}^{q} d\bar{w}_j dw_j$$

and summations in the exponential are implied.

Using the following relation

$$\exp[ca^\dagger a] = N \left[ \exp\left( (e^c - 1) a^\dagger a \right) \right]$$  \hspace{1cm} (28)

(here $N$ stands for normal form of an operator expression) and the following properties of the coherent states:

$$\langle \bar{w} | NQ(a^\dagger, a) | z \rangle = Q(\bar{w}, z) \langle \bar{w} | z \rangle , \quad \langle \bar{w} | z \rangle = \exp(\bar{w}z)$$  \hspace{1cm} (29)

we obtain

$$Z_{q/p}(\epsilon) = \int_C dt \ F_p(t) \left( 1 - te^{-\beta(\epsilon - \mu)} \right)^{-q}$$  \hspace{1cm} (30)

Using (21), after some algebra, expression (30) can be transformed to

$$Z_{q/p}(\epsilon) = \frac{1}{(q-1)!} \int_0^{\infty} dt \ t^{q-1} e^{-t} \left[ 1 + \frac{te^{-\beta(\epsilon - \mu)}}{p} \right]^p$$  \hspace{1cm} (31)
Taking $p \to \infty$ with $q$ fixed corresponds to the bosonic case. From (31) we readily obtain
\[ Z_{q/p}(\epsilon) \bigg|_{p \to \infty} = \left[ 1 - e^{-\beta(\epsilon - \mu)} \right]^{-q} \]
which is obviously the bosonic partition function.

Calculating (31) in the thermodynamical limit ($q, p \to \infty$ with $g = q/p$ fixed) we have
\[ Z_{q/p}(\epsilon) = \left[ h^{-1+g} (h + gz)^{1/g} e^{-1/h+1} \right]^q \]
where
\[ h = h(g) = \frac{1}{2} \left[ 1 - (1 + g)z + \sqrt{[1 - (1 + g)z]^2 + 4gz} \right] \]
and
\[ z \equiv e^{-\beta(\epsilon - \mu)} \]

The distribution function is defined by the relation:
\[ n = \langle \hat{N} \rangle = \frac{\partial \mu}{q \beta Z} \]
From (35) and (32) we have
\[ n = \frac{z}{h + gz} \]
Putting in (32) and (33) $g = 1$ we obtain
\[ Z_1(\epsilon) = \left[ h^{-1} (h(1) + z) e^{-1/h(1)+1} \right]^q \]
where
\[ h(1) = \frac{1}{2} \left[ 1 - 2z + \sqrt{1 + 4z^2} \right] \]
If we further consider the case of low densities ($z \ll 1$) we have from (37):
\[ Z_1(\epsilon) \approx [1 + z]^q \]
which obviously coincides with usual fermionic partition function.

Let us turn now our attention to the state-counting corresponding to the Haldane’s dimension formula. In the case of one exclusion cell we have from eq. (3):
\[ W(N) = \alpha(N) \sum_{n_1, \ldots, n_q = 0}^{\infty} \delta_{n_1 + \ldots + n_q, N} = \alpha(N) \cdot W_h(N) \]
where
\[ W_b(N) = \frac{(q + N - 1)!}{N!(q - 1)!} \] (40)
is the bosonic statistical weight. For the set of probabilities (24) we obtain
the following expression for the state-counting
\[ W(N) = \frac{1}{p^N} \frac{p!}{(p - N)!} \cdot \frac{(q + N - 1)!}{N!(q - 1)!} \] (41)
which is obviously different from Haldane-Wu state-counting [1, 2]:
\[ W_H(N) = \frac{[q + (1 - g)(N - 1)]!}{N![q - g(N - 1) - 1]!} \] (42)

5 Haldane-Wu state-counting procedure

Although the procedure of the last section does not lead to the combinatorial
expression derived by Haldane and Wu, we may modify the probabilities, \( \alpha \), so that this is obtained.

Comparing (39) and (42) we can easily write down a set of probabilities
which provide the Haldane-Wu state-counting:
\[ \alpha_H(N) = \frac{(q - 1)!([q - g(N - 1) + N - 1]!)}{(q + N - 1)!([q - g(N - 1) - 1]!}, \quad N = \sum_{i=1}^{q} n_i \] (43)
The operator \( \text{Id} \) in this case takes the form
\[ \text{Id}^H = \int_C dt \ F_p^H(t) \frac{1}{\pi^q} \int \prod_{j=1}^{q} [dz_j dz_j e^{-\bar{z}z_j}] \langle \{z_j \sqrt{t}\}_{j=1}^{q} \rangle \langle \{\bar{z} \sqrt{t}\}_{j=1}^{q} \rangle \] (44)
where
\[ F_p^H(t) = \frac{1}{2\pi i} \sum_{n=0}^{p} \frac{(q - 1)!([q - g(n-1) + n - 1]!}{(q + n - 1)!([q - g(n - 1) - 1]!) t^{-n-1}} \] (45)
For the partition function we have the following expression:
\[ Z_{q/p}^H(\epsilon) = \int_C dt \ F_p^H(t) \left(1 - te^{-\beta(\epsilon - \mu)}\right)^{-q} \] (46)
\[ = \sum_{N=0}^{p} \frac{[q - g(N - 1) + N - 1]!}{N![q - g(N - 1) - 1]!} \] (47)
which is identical to the one used by Wu. From (47) we can obtain the statistical distribution in the standard way \[2\]:

\[
n \equiv \frac{N}{q} = \frac{1}{w(e^{\beta(\epsilon-\mu)}) + g}
\]

where function \( w \) satisfies the following equation

\[
w(e^{\beta(\epsilon-\mu)})[1 + w(e^{\beta(\epsilon-\mu)})]^{1-g} = e^{\beta(\epsilon-\mu)}
\]

(49)

Turning our attention to an analog of Haldane’s dimension formula for Haldane-Wu state-counting procedure we have from (7) and (43):

\[
d_H(N) = \frac{\alpha_H(N)}{\alpha_H(N-1)} = q - g(N-1) + N - 1 \prod_{j=1}^{N-1} \frac{q - g(N - 2) + j - 1}{q + N - 1}
\]

Taking the thermodynamical limit leads us to the following expression

\[
d_H(N) = \frac{q - gN + N}{1 + N/q}
\]

(50)

At sufficiently small densities \( N/q \ll 1 \) we have

\[
d_H(N) = q - gN + O(N^2/q)
\]

We can conclude that at low densities Haldane’s dimension and Haldane-Wu state-counting procedures are equivalent while in general they are not.

6 Conclusion

In this paper we have demonstrated the construction of coherent states for particles obeying Haldane exclusion statistics. This construction allows considerable freedom in the definition and permits the construction of states yielding either Haldane’s dimension or Haldane-Wu state-counting. These are demonstrated in the limit of low densities.

These results will be used in a future publication to analyse “non-ideal” gas of particles obeying exclusion statistics.

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References

[1] F. D. M. Haldane, Phys. Rev. Lett. 67, 937 (1991);

[2] Y. S. Wu, Phys. Rev. Lett. 73, 922 (1994);

[3] M. V. N. Murthy and R. Shankar, Phys. Rev. Lett. 72, 3629 (1994);

[4] D. Bernard and Y. S. Wu, Preprint SPhT-94-043, UU-HEP /94-03;

[5] C. Nayak and F. Wilczek, Phys. Rev. Lett. 73, 2740 (1994);

[6] Z. N. Ha, Phys. Rev. Lett. 73, 1574 (1994); Preprint IASSNS-HEP-94/90 (1994);

[7] S. B. Isakov, Phys. Rev. Lett. 73, 2150 (1994);

[8] A. K. Rajagopal, Phys. Rev. Lett. 74, 1048 (1995);

[9] M. V. N. Murthy and R. Shankar, Phys. Rev. Lett. 73, 3331 (1994);

[10] A. P. Polychronakos, Phys. Lett. A 365, 202 (1996);

[11] S. B. Isakov, Phys. Rev. B 53, 6585 (1996);

[12] S. Chaturvedi, V. Srinivasan, Phys. Rev. Lett. 78, 4316 (1997);

[13] M. V. N. Murthy and R. Shankar, cond-mat/9903273;

[14] K. N. Ilinski, J. M. F. Gunn, Phys. Lett. A, 210, 168 (1995);

[15] K. N. Ilinski, J. M. F. Gunn and A. V. Ilinskaia, Phys. Rev. B, 53, 2615 (1996);

[16] A. V. Ilinskaia, K. N. Ilinski and J. M. F. Gunn, Nucl. Phys. B[FS], 458, 562 (1996);