Properties of $\theta$-super positive graphs

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Abstract

Let the matching polynomial of a graph $G$ be denoted by $\mu(G, x)$. A graph $G$ is said to be $\theta$-super positive if $\mu(G, \theta) \neq 0$ and $\mu(G \setminus v, \theta) = 0$ for all $v \in V(G)$. In particular, $G$ is 0-super positive if and only if $G$ has a perfect matching. While much is known about 0-super positive graphs, almost nothing is known about $\theta$-super positive graphs for $\theta \neq 0$. This motivates us to investigate the structure of $\theta$-super positive graphs in this paper. Though a 0-super positive graph may not contain any cycle, we show that a $\theta$-super positive graph with $\theta \neq 0$ must contain a cycle. We introduce two important types of $\theta$-super positive graphs, namely $\theta$-elementary and $\theta$-base graphs. One of our main results is that any $\theta$-super positive graph $G$ can be constructed by adding certain type of edges to a disjoint union of $\theta$-base graphs; moreover, these $\theta$-base graphs are uniquely determined by $G$. We also give a characterization of $\theta$-elementary graphs: a graph $G$ is $\theta$-elementary if and only if the set of all its $\theta$-barrier sets form a partition of $V(G)$. Here, $\theta$-elementary graphs and $\theta$-barrier sets can be regarded as $\theta$-analogue of elementary graphs and Tutte sets in classical matching theory.

KEYWORDS: matching polynomial, Gallai-Edmonds decomposition, elementary graph, barrier sets, extreme sets

1 Introduction

We begin by introducing matching polynomials with an interest in the multiplicities of their roots. This will lead us to a recent extension of the celebrated Gallai-Edmonds Structure Theorem by Chen and Ku [1] which will be useful later in our study of $\theta$-super positive graphs.

All the graphs in this paper are simple and finite. The vertex set and edge set of a graph $G$ will be denoted by $V(G)$ and $E(G)$, respectively.

Definition 1.1. An $r$-matching in a graph $G$ is a set of $r$ edges, no two of which have a vertex in common. The number of $r$-matchings in $G$ will be denoted by $p(G, r)$. We set $p(G, 0) = 1$ and define the matching polynomial of $G$ by

$$
\mu(G, x) = \sum_{r=0}^{[n/2]} (-1)^r p(G, r) x^{n-2r}.
$$

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We denote the multiplicity of $\theta$ as a root of $\mu(G, x)$ by $\text{mult}(\theta, G)$. Let $u \in V(G)$, the graph obtained from $G$ by deleting the vertex $u$ and all edges that contain $u$ is denoted by $G \setminus u$. Inductively if $u_1, \ldots, u_k \in V(G)$, $G \setminus u_1 \cdots u_k = (G \setminus u_1 \cdots u_{k-1}) \setminus u_k$. Note that the order in which the vertices are being deleted is not important, that is, if $i_1, \ldots, i_k$ is a permutation of $1, \ldots, k$, we have $G \setminus u_1 \cdots u_k = G \setminus u_{i_1} \cdots u_{i_k}$. Furthermore, if $X = \{u_1, \ldots, u_k\}$, we set $G \setminus X = G \setminus u_1 \cdots u_k$. If $H$ is a subgraph of $G$, by an abuse of notation, we have $G \setminus H = G \setminus V(H)$. For example, if $p = v_1 v_2 \cdots v_n$ is a path in $G$ then $G \setminus p = G \setminus v_1 v_2 \cdots v_n$. If $e$ is an edge of $G$, let $G - e$ denote the graph obtained from $G$ by deleting the edge $e$ from $G$. Inductively, if $e_1, \ldots, e_k \in E(G)$, $G - e_1 \cdots e_k = (G - e_1 \cdots e_{k-1}) - e_k$.

A graph $G$ is said to have a perfect matching if it has a $n/2$-matching ($n$ must be even). This is equivalent to $\text{mult}(0, G) = 0$, that is, 0 is not a root of $\mu(G, x)$. Recall that in the literature $\text{mult}(0, G)$ is also known as the deficiency of $G$ which is the number of vertices of $G$ missed by some maximum matching.

The following are some basic properties of $\mu(G, x)$.

**Theorem 1.2.** [2] Theorem 1.1 on p. 2]

(a) $\mu(G \cup H, x) = \mu(G, x)\mu(H, x)$ where $G$ and $H$ are disjoint graphs,

(b) $\mu(G, x) = \mu(G - e, x) - \mu(G \setminus uv, x)$ if $e = (u, v)$ is an edge of $G$,

(c) $\mu(G, x) = x\mu(G \setminus u, x) - \sum_{i \sim u} \mu(G \setminus ui, x)$ where $i \sim u$ means $i$ is adjacent to $u$,

(d) $\frac{d}{dx} \mu(G, x) = \sum_{i \in V(G)} \mu(G \setminus i, x)$ where $V(G)$ is the vertex set of $G$.

It is well known that all roots of $\mu(G, x)$ are real. Throughout, let $\theta$ be a real number. The multiplicity of a matching polynomial root satisfies the following interlacing property:

**Lemma 1.3.** [2] Corollary 1.3 on p. 97] (Interlacing) Let $G$ be a graph and $u \in V(G)$. Let $\theta$ be a real number. Then

$$\text{mult}(\theta, G) - 1 \leq \text{mult}(\theta, G \setminus u) \leq \text{mult}(\theta, G) + 1.$$ 

Lemma 1.3 suggests that given any real number $\theta$, we can classify the vertices of a graph according to an increase of 1 or a decrease of 1 or no change in the multiplicity of $\theta$ upon deletion of a vertex.

**Definition 1.4.** [3] Section 3] For any $u \in V(G)$,

(a) $u$ is $\theta$-essential if $\text{mult}(\theta, G \setminus u) = \text{mult}(\theta, G) - 1$,

(b) $u$ is $\theta$-neutral if $\text{mult}(\theta, G \setminus u) = \text{mult}(\theta, G)$,

(c) $u$ is $\theta$-positive if $\text{mult}(\theta, G \setminus u) = \text{mult}(\theta, G) + 1$.

Furthermore, if $u$ is not $\theta$-essential but it is adjacent to some $\theta$-essential vertex, we say $u$ is $\theta$-special.
It turns out that $\theta$-special vertices play an important role in the Gallai-Edmonds Decomposition of a graph (see [1]). Godsil [3, Corollary 4.3] proved that a $\theta$-special vertex must be $\theta$-positive. Note that if $\text{mult}(\theta, G) = 0$ then for any $u \in V(G)$, $u$ is either $\theta$-neutral or $\theta$-positive and no vertices in $G$ can be $\theta$-special. Now $V(G)$ can be partitioned into the following sets:

$$V(G) = D_{\theta}(G) \cup A_{\theta}(G) \cup P_{\theta}(G) \cup N_{\theta}(G),$$

where

- $D_{\theta}(G)$ is the set of all $\theta$-essential vertices in $G$,
- $A_{\theta}(G)$ is the set of all $\theta$-special vertices in $G$,
- $N_{\theta}(G)$ is the set of all $\theta$-neutral vertices in $G$,
- $P_{\theta}(G) = Q_{\theta}(G) \setminus A_{\theta}(G)$, where $Q_{\theta}(G)$ is the set of all $\theta$-positive vertices in $G$.

Note that there are no $\theta$-neutral vertices. So $N_{\theta}(G) = \emptyset$ and $V(G) = D_{\theta}(G) \cup A_{\theta}(G) \cup P_{\theta}(G)$.

**Definition 1.5.** [3, Section 3] A graph $G$ is said to be $\theta$-critical if all vertices in $G$ are $\theta$-essential and $\text{mult}(\theta, G) = 1$.

The celebrated Gallai-Edmonds Structure Theorem describes the stability of a certain canonical decomposition of $V(G)$ with respect to the zero root of $\mu(G, x)$. In [1], Chen and Ku extended the Gallai-Edmonds Structure Theorem to any root $\theta \neq 0$, which consists of the following two theorems:

**Theorem 1.6.** [1, Theorem 1.5] ($\theta$-Stability Lemma) Let $G$ be a graph with $\theta$ a root of $\mu(G, x)$. If $u \in A_{\theta}(G)$ then

(i) $D_{\theta}(G \setminus u) = D_{\theta}(G)$,
(ii) $P_{\theta}(G \setminus u) = P_{\theta}(G)$,
(iii) $N_{\theta}(G \setminus u) = N_{\theta}(G)$,
(iv) $A_{\theta}(G \setminus u) = A_{\theta}(G) \setminus \{u\}$.

**Theorem 1.7.** [1, Theorem 1.7] ($\theta$-Gallai’s Lemma) If $G$ is connected and every vertex of $G$ is $\theta$-essential then $\text{mult}(\theta, G) = 1$.

Theorem 1.6 asserts that the decomposition of $V(G)$ into $D_{\theta}(G)$, $P_{\theta}(G)$, $N_{\theta}(G)$ and $A_{\theta}(G)$ is stable upon deleting a $\theta$-special vertex of $G$. We may delete every such vertex one by one until there are no $\theta$-special vertices left. Together with Theorem 1.7 it is not hard to deduce the following whose proof is omitted.

**Corollary 1.8.**

(i) $A_{\theta}(G \setminus A_{\theta}(G)) = \emptyset$, $D_{\theta}(G \setminus A_{\theta}(G)) = D_{\theta}(G)$, $P_{\theta}(G \setminus A_{\theta}(G)) = P_{\theta}(G)$, and $N_{\theta}(G \setminus A_{\theta}(G)) = N_{\theta}(G)$.

(ii) $G \setminus A_{\theta}(G)$ has exactly $|A_{\theta}(G)| + \text{mult}(\theta, G)$ $\theta$-critical components.
(iii) If \( H \) is a component of \( G \setminus A_\theta(G) \) then either \( H \) is \( \theta \)-critical or \( \text{mult}(\theta, H) = 0 \).

(iv) The subgraph induced by \( D_\theta(G) \) consists of all the \( \theta \)-critical components in \( G \setminus A_\theta(G) \).

This paper is devoted to the study of \( \theta \)-super positive graphs. A graph is \( \theta \)-super positive if \( \theta \) is not a root of \( \mu(G, x) \) but is a root of \( \mu(G \setminus v, x) \) for every \( v \in V(G) \). It is worth noting that \( G \) is \( 0 \)-super positive if and only if \( G \) has a perfect matching. While much is known about graphs with a perfect matching, almost nothing is known about \( \theta \)-super positive graphs for \( \theta \neq 0 \). This gives us a motivation to investigate the structure of these graphs.

The outline of this paper is as follows:

In Section 2, we show how to construct \( \theta \)-super positive graphs from smaller \( \theta \)-super positive graphs (see Theorem 2.2). We prove that a tree is \( \theta \)-super positive if and only if \( \theta = 0 \) and it has a perfect matching (see Theorem 2.4). Consequently, a \( \theta \)-super positive graph must contain a cycle when \( \theta \neq 0 \). For a connected vertex transitive graph \( G \), we prove that it is \( \theta \)-super positive for any root \( \theta \) of \( \mu(G \setminus v, x) \) where \( v \in V(G) \) (see Theorem 2.8). Finally we prove that if \( G \) is \( \theta \)-super positive, then \( N_\theta(G \setminus v) = \emptyset \) for all \( v \in V(G) \) (see Theorem 2.9).

In Section 3, we introduce \( \theta \)-elementary graphs. These are \( \theta \)-super positive graphs with \( P_\theta(G \setminus v) = \emptyset \) for all \( v \in V(G) \). We prove a characterization of \( \theta \)-elementary graphs: a graph \( G \) is \( \theta \)-elementary if and only if the set of all \( \theta \)-barrier sets forms a partition of \( V(G) \) (see Theorem 3.3).

In Section 4, we apply our results in Section 3 to prove that an \( n \)-cycle \( C_n \) is 1-elementary if and only if \( n = 3k \) for some \( k \in \mathbb{N} \) (see Theorem 4.3). Furthermore, we prove that \( C_{3k} \) has exactly 3 1-barrier sets (see Corollary 4.5).

In Section 5, we introduce \( \theta \)-base graphs which can be regarded as building blocks of \( \theta \)-super positive graphs. We prove a characterization of \( \theta \)-super positive graphs, namely a \( \theta \)-super positive graph can be constructed from a disjoint union of \( \theta \)-base graphs by adding certain type of edges; moreover, these \( \theta \)-base graphs are uniquely determined by \( G \) (see Theorem 5.7 and Corollary 5.9).

## 2 \( \theta \)-super positive graphs

**Definition 2.1.** A graph \( G \) is \( \theta \)-super positive if \( \theta \) is not a root of \( \mu(G, x) \) and every vertex of \( G \) is \( \theta \)-positive.

By Lemma 1.3 this is equivalent to \( \text{mult}(\theta, G) = 0 \) and \( \text{mult}(\theta, G \setminus v) = 1 \) for all \( v \in V(G) \). There are a lot of \( \theta \)-super positive graphs. For instance the three cycle, \( C_3 \) and the six cycle, \( C_6 \) are 1-super positive. In the next theorem, we will show how to construct \( \theta \)-super positive graphs from smaller \( \theta \)-super positive graphs.

**Theorem 2.2.** Let \( G_1 \) and \( G_2 \) be two \( \theta \)-super positive graphs and \( v_i \in V(G_i) \) for \( i = 1, 2 \). Let \( G \) be the graph obtained by adding the edge \( (v_1, v_2) \) to the union of \( G_1 \) and \( G_2 \). Then \( G \) is \( \theta \)-super positive.

**Proof.** Let \( e = (v_1, v_2) \). First we prove that \( \mu(G, \theta) \neq 0 \). By part (b) of Theorem 1.2 we have \( \mu(G, x) = \mu(G - e, x) - \mu(G \setminus v_1v_2, x) \). It then follows from part (a) of Theorem 1.2 that \( \mu(G, x) = \mu(G_1, x)\mu(G_2, x) - \mu(G_1 \setminus v_1, x)\mu(G_2 \setminus v_2, x) \). Since \( G_1 \) and \( G_2 \) are \( \theta \)-super positive, \( \mu(G, \theta) = \mu(G_1, \theta)\mu(G_2, \theta) \neq 0 \).
It is left to prove that $\mu(G \setminus v, \theta) = 0$ for all $v \in V(G)$. Let $v \in V(G_1)$. Suppose $v = v_1$. Then by part (a) of Theorem 1.2 $\mu(G \setminus v, x) = \mu(G_1 \setminus v_1, x)\mu(G_2, x)$, and thus $\mu(G \setminus v, \theta) = 0$. Suppose $v \neq v_1$. By part (b) of Theorem 1.2 $\mu(G \setminus v, x) = \mu((G \setminus v) - e, x) - \mu((G \setminus v) \setminus v_1v_2, x)$. Note that $(G \setminus v) - e = (G_1 \setminus v) \cup G_2$ and $(G \setminus v) \setminus v_1v_2 = (G_1 \setminus vv_1) \cup (G_2 \setminus v_2)$. Hence $\mu(G \setminus v, \theta) = \mu(G_1 \setminus v, \theta)\mu(G_2, \theta) - \mu(G_1 \setminus vv_1, \theta)\mu(G_2 \setminus v_2, \theta) = 0$ (part (a) of Theorem 1.2).

The case $v \in V(G_2)$ is proved similarly.

The graph $G$ in Figure 1 is constructed by using Theorem 2.2 with $G_1 = C_6$ and $G_2 = C_3$. Therefore it is 1-super positive graph.

It is clear that a 0-super positive may or may not contain any cycle. However, we will show later that if $G$ is $\theta$-super positive and $\theta \neq 0$, then it must contain a cycle (see Corollary 2.5). Note that any tree $T$ with at least three vertices can be represented in the following form (see Figure 2), where $u$ is a vertex with $n + 1$ neighbors $v_1, \ldots, v_{n+1}$ such that all of them except possibly $v_1$ have degree 1 and $T_1$ is a subtree of $T$ that contains $v_1$. Such a representation of $T$ is denoted by $(T_1, u; v_1, \ldots, v_{n+1})$.

**Lemma 2.3.** Let $T$ be a tree with at least three vertices. Suppose $T$ has a representation $(T_1, u; v_1, \ldots, v_{n+1})$. Then $\theta$ is a root of $\mu(T, x)$ if and only if

$$(n - \theta^2)\theta^{n-1} \mu(T_1, \theta) + \theta^n \mu(T_1 \setminus v_1, \theta) = 0.$$

**Proof.** By part (c) of Theorem 1.2 $\mu(T, \theta) = \theta \mu(T \setminus u, \theta) - \sum_{i=1}^{n+1} \mu(T \setminus uv_i, \theta)$ (see Figure 2), which implies (using part (a) of Theorem 1.2),

$$\mu(T, \theta) = (\theta^2 - n)\theta^{n-1} \mu(T_1, \theta) - \theta^n \mu(T_1 \setminus v, \theta).$$

Hence the lemma holds
Theorem 2.4. Let $T$ be a tree. Then $T$ is $\theta$-super positive if and only if $\theta = 0$ and it has a perfect matching.

Proof. Suppose $T$ is $\theta$-super positive and $\theta \neq 0$. Then $T$ must have at least three vertices. By Lemma 2.3,

$$(n - \theta^2)\theta^{n-1}\mu(T_1, \theta) + \theta^n\mu(T_1 \setminus v_1, \theta) \neq 0.$$ 

By part (a) of Theorem 1.2, $0 = \mu(T \setminus u, \theta) = \theta^n\mu(T_1, \theta)$ (see Figure 2). Therefore $\mu(T_1, \theta) = 0$ and $\mu(T_1 \setminus v_1, \theta) \neq 0$. Now $\mu(T \setminus v_{n+1}, \theta) = 0$. By part (c) of Theorem 1.2, $\mu(T \setminus v_{n+1}, \theta) = \theta\mu(T \setminus uv_{n+1}, \theta) - \sum_{i=1}^{n} \mu(T \setminus uv_{i+1}, \theta) = \theta^n\mu(T_1, \theta) - (n-1)\theta^{n-2}\mu(T_1, \theta) - \theta^{n-1}\mu(T_1 \setminus v_1, \theta)$. This implies that $\mu(T_1 \setminus v_1, \theta) = 0$, a contradiction. Hence $\theta = 0$. Since $0$ is not a root of $\mu(T, x)$, $T$ must have a perfect matching.

The converse is obvious. 

A consequence of Theorem 2.3 is the following corollary.

Corollary 2.5. If $G$ is $\theta$-super positive for some $\theta \neq 0$, then $G$ must contain a cycle.

We shall need the following lemmas.

Lemma 2.6. [4] Theorem 6.3 (Heilmann-Lieb Identity) Let $u, v \in V(G)$. Then

$$\mu(G \setminus u, x)\mu(G \setminus v, x) - \mu(G, x)\mu(G \setminus uv) = \sum_{p \in \mathcal{P}(u,v)} \mu(G \setminus p, x)^2,$$

where $\mathcal{P}(u, v)$ is the set of all the paths from $u$ to $v$ in $G$.

Lemma 2.7. [3] Lemma 3.1] Suppose $\text{mult}(\theta, G) > 0$. Then $G$ contains at least one $\theta$-essential vertex.

Theorem 2.8. Let $G$ be connected, vertex transitive and $z \in V(G)$. If $\theta$ is a root of $\mu(G \setminus z, x)$ then $G$ is $\theta$-super positive.

Proof. Since $G \setminus z$ is isomorphic to $G \setminus y$ for all $y \in V(G)$, $\mu(G \setminus z, x) = \mu(G \setminus y, x)$ for all $y \in V(G)$. So $\text{mult}(\theta, G \setminus z) = \text{mult}(\theta, G \setminus y)$. This implies that $\theta$ is a root of $\mu(G \setminus y, x)$ for all $y$.

Now it remains to show that $\mu(G, \theta) \neq 0$. Suppose the contrary. Then by Lemma 2.7, $G$ has at least one $\theta$-essential vertex. Since $G$ is vertex transitive, all vertices in $G$ are $\theta$-essential. By Theorem 1.7, $\text{mult}(\theta, G) = 1$. But then $\text{mult}(\theta, G \setminus z) = 0$, a contradiction. Hence $\mu(G, \theta) \neq 0$ and $G$ is $\theta$-super positive.

However, a $\theta$-super positive graph is not necessarily vertex transitive (see Figure 1). Furthermore a $\theta$-super positive graph is not necessary connected, for the union of two $C_3$ is 1-super positive.

Theorem 2.9. Let $G$ be $\theta$-super positive. Then $N_\theta(G \setminus v) = \emptyset$ for all $v \in V(G)$.

Proof. Suppose $N_\theta(G \setminus v) \neq \emptyset$ for some $v \in V(G)$. Let $u \in N_\theta(G \setminus v)$. By Lemma 2.6,

$$\mu(G \setminus u, x)\mu(G \setminus v, x) - \mu(G, x)\mu(G \setminus uv) = \sum_{p \in \mathcal{P}(u,v)} \mu(G \setminus p, x)^2.$$


Note that the multiplicity of \( \theta \) as a root of \( \mu(G \setminus u, x)\mu(G \setminus v, x) \) is 2, while the multiplicity of \( \theta \) as a root of \( \mu(G, x)\mu(G \setminus vu, x) \) is 1 since \( u \) is \( \theta \)-neutral in \( G \setminus v \). Therefore the multiplicity of \( \theta \) as a root of the polynomial on the left-hand side of the equation is at least 1. But the multiplicity of \( \theta \) as a root of the polynomial on the right-hand side of the equation is even and so, in comparison with the left-hand side, it must be at least 2. This forces the multiplicity of \( \theta \) as a root of \( \mu(G, x)\mu(G \setminus vu, x) \) to be at least 2, a contradiction. Hence \( N_\theta(G \setminus v) = \emptyset \) for all \( v \in V(G) \).

Now we know that for a \( \theta \)-super positive graph \( G \), \( N_\theta(G \setminus v) = \emptyset \) for all \( v \in V(G) \). So it is quite natural to ask whether \( P_\theta(G \setminus v) = \emptyset \) for all \( v \in V(G) \). Well, this is not true in general (see Figure 1). This motivates us to study the \( \theta \)-super positive graph \( G \), for which \( P_\theta(G \setminus v) = \emptyset \) for all \( v \in V(G) \). We proceed to do this in the next section.

### 3 \( \theta \)-elementary graphs

**Definition 3.1.** A graph \( G \) is said to be \( \theta \)-elementary if it is \( \theta \)-super positive and \( P_\theta(G \setminus v) = \emptyset \) for all \( v \in V(G) \).

The graph \( G \) in Figure 3 is 1-elementary. Not every \( \theta \)-positive graph is \( \theta \)-elementary. For instance, the graph in Figure 1 is not 1-elementary.

![Figure 3](image)

#### Theorem 3.2. A graph \( G \) is \( \theta \)-elementary if and only if \( \text{mult}(\theta, G) = 0 \) and \( P_\theta(G \setminus v) \cup N_\theta(G \setminus v) = \emptyset \) for all \( v \in V(G) \).

*Proof.* Suppose \( \text{mult}(\theta, G) = 0 \) and \( P_\theta(G \setminus v) \cup N_\theta(G \setminus v) = \emptyset \) for all \( v \in V(G) \). Then for each \( v \in V(G) \), \( \text{mult}(\theta, G \setminus v) = 1 \), for otherwise \( G \setminus v \) would only consist of \( \theta \)-neutral and \( \theta \)-positive vertices whence \( P_\theta(G \setminus v) \cup N_\theta(G \setminus v) \neq \emptyset \). Therefore \( G \) is \( \theta \)-super positive and it is \( \theta \)-elementary.

The other implication follows from Theorem 2.9.

It turns out that the notion of a \( \theta \)-elementary graph coincide with the classical notion of an elementary graph. Properties of elementary graphs can be found in Section 5.1 on p. 145 of [7].

The number of \( \theta \)-critical components in \( G \) is denoted by \( c_\theta(G) \).

**Definition 3.3.** A \( \theta \)-barrier set is defined to be a set \( X \subseteq V(G) \) for which \( \text{mult}(\theta, G) = c_\theta(G \setminus X) - |X| \).

A \( \theta \)-extreme set is defined to be a set \( X \subseteq V(G) \) for which \( \text{mult}(\theta, G \setminus X) = \text{mult}(\theta, G) + |X| \).
\(\theta\)-barrier sets and \(\theta\)-extreme sets can be regarded as \(\theta\)-analogue of Tutte sets and extreme sets in classical matching theory. Properties of \(\theta\)-barrier sets and \(\theta\)-extreme sets have been studied by Ku and Wong [5]. In particular, the following results are needed.

**Lemma 3.4.** [5] Lemma 2.5] A subset of a \(\theta\)-extreme set is a \(\theta\)-extreme set.

**Lemma 3.5.** [5] Lemma 2.6] If \(X\) is a \(\theta\)-barrier set and \(Y \subseteq X\) then \(X \setminus Y\) is a \(\theta\)-barrier set in \(G \setminus Y\).

**Lemma 3.6.** [5] Lemma 2.7] Every \(\theta\)-extreme set of \(G\) lies in a \(\theta\)-barrier set.

**Lemma 3.7.** [5] Lemma 2.8] Let \(X\) be a \(\theta\)-barrier set. Then \(X\) is a \(\theta\)-extreme set.

**Lemma 3.8.** [5] Lemma 3.1] If \(X\) is a \(\theta\)-barrier set then \(X \subseteq A_\theta(G) \cup P_\theta(G)\).

**Lemma 3.9.** [5] Theorem 3.5] Let \(X\) be a \(\theta\)-barrier set in \(G\). Then \(A_\theta(G) \subseteq X\).

**Lemma 3.10.** Let \(G\) be a graph. If \(X\) is a \(\theta\)-barrier set in \(G\), \(x \in X\) and \(P_\theta(G \setminus x) = \emptyset\), then \(A_\theta(G \setminus x) = X \setminus x\).

**Proof.** By Lemma 3.5, \(X \setminus x\) is a \(\theta\)-barrier set in \(G \setminus x\). By Lemma 3.8, \(X \setminus x \subseteq A_\theta(G \setminus x) \cup P_\theta(G \setminus x)\). Therefore, \(X \setminus x \subseteq A_\theta(G \setminus x)\). It then follows from Lemma 3.9 that \(A_\theta(G \setminus x) = X \setminus x\).

**Definition 3.11.** We define \(\mathcal{P}(\theta, G)\) to be the set of all the \(\theta\)-barrier sets in \(G\).

Note that in Figure 3, \(\mathcal{P}(1, G) = \{\{u_1\}, \{u_2\}, \{u_3, u_4\}, \{u_5\}, \{u_6\}\}\). Now Lemma 3.12 follows from part (c) of Theorem 1.2.

**Lemma 3.12.** Suppose \(G\) is \(\theta\)-super positive. Then for each \(v \in V(G)\) there is a \(u \in V(G)\) with \((u, v) \in E(G)\) and \(\text{mult}(\theta, G \setminus uv) = 0\).

**Theorem 3.13.** A graph \(G\) is \(\theta\)-elementary if and only if \(\mathcal{P}(\theta, G)\) is a partition of \(V(G)\).

**Proof.** Let \(\mathcal{P}(\theta, G) = \{S_1, \ldots, S_k\}\).

(\(\Rightarrow\)) Suppose \(G\) is \(\theta\)-elementary. Then for each \(v \in V(G)\), \(\{v\}\) is a \(\theta\)-extreme set. By Lemma 3.6, it is contained in some \(\theta\)-barrier set. Therefore \(V(G) = S_1 \cup \cdots \cup S_k\). It remains to prove that \(S_i \cap S_j = \emptyset\) for \(i \neq j\). Suppose the contrary. Let \(x \in S_i \cap S_j\). By Lemma 3.10, \(S_i \setminus \{x\} = A_\theta(G \setminus x) = S_j \setminus \{x\}\) and so \(S_i = S_j\), a contradiction. Hence \(S_i \cap S_j = \emptyset\) for \(i \neq j\) and \(\mathcal{P}(\theta, G)\) is a partition of \(V(G)\).

(\(\Leftarrow\)) Suppose \(\mathcal{P}(\theta, G)\) is a partition of \(V(G)\). Let \(v \in V(G)\). Then \(v \in S_i\) for some \(\theta\)-barrier set \(S_i\). By Lemma 3.8, \(v \in A_\theta(G) \cup P_\theta(G)\). Therefore \(V(G) \subseteq A_\theta(G) \cup P_\theta(G)\). This implies that \(\text{mult}(\theta, G) = 0\), for otherwise \(D_\theta(G) \neq \emptyset\) by Lemma 2.7. Hence \(A_\theta(G) = \emptyset\) and \(V(G) = P_\theta(G)\), i.e., \(G\) is \(\theta\)-super positive. It remains to show that \(P_\theta(G \setminus v) = \emptyset\) for all \(v \in V(G)\). Suppose the contrary. Then \(P_\theta(G \setminus v) \neq \emptyset\) for some \(v_0 \in V(G)\). We may assume \(v_0 \in S_1\). By Corollary 1.8, \((G \setminus v_0) \setminus A_\theta(G \setminus v_0)\) has a component \(H\) for which \(\text{mult}(\theta, H) = 0\). By Theorem 2.9, \(N_\theta(G \setminus v_0) = \emptyset\). So we conclude that \(H\) is \(\theta\)-super positive. Let \(w \in H\). By Lemma 3.12, there is a \(z \in V(H)\) with \((w, z) \in E(H)\) and \(\text{mult}(\theta, H \setminus wz) = 0\). By part (a) of Theorem 1.2 and, (ii) and (iii) of Corollary 1.8, \(\text{mult}(\theta, (G \setminus v_0) \setminus A_\theta(G \setminus v_0)) \setminus wz) = 1 + |A_\theta(G \setminus v_0)|\).

On the other hand, by Lemma 3.5, \(S_1 \setminus \{v_0\}\) is a \(\theta\)-barrier set in \(G \setminus v_0\). So by Lemma 3.9, \(A_\theta(G \setminus v_0) \subseteq S_1 \setminus \{v_0\}\). By Lemma 3.5 again, \(S_1 \setminus (\{v_0\} \cup A_\theta(G \setminus v_0))\) is a \(\theta\)-barrier set in \((G \setminus v_0) \setminus A_\theta(G \setminus v_0)\). Note
that \( w \) is \( \theta \)-positive in \( G \setminus v_0 \) (by Corollary 1.8). Therefore \( \{ w, v_0 \} \) is an \( \theta \)-extreme set. By Lemma 3.6 \( \{ w, v_0 \} \) is contained in some \( \theta \)-barrier set. Since \( \mathfrak{P}(\theta, G) \) is a partition of \( V(G) \) and \( v_0 \in S_1 \), we must have \( \{ w, v_0 \} \subseteq S_1 \). Note also \( z \) is \( \theta \)-positive in \( G \setminus v_0 \) (recall that \( H \) is \( \theta \)-super positive). Using a similar argument, we can show that \( \{ z, v_0 \} \subseteq S_1 \). By Lemma 3.4 and Lemma 3.7 we conclude that \( \{ w, z \} \subseteq S_1 \setminus (\{ v_0 \} \cup A_\theta(G \setminus v_0)) \) is a \( \theta \)-extreme set in \( (G \setminus v_0) \setminus A_\theta(G \setminus v_0) \). This implies that \( \mu(\theta, ((G \setminus v_0) \setminus A_\theta(G \setminus v_0)) \setminus wz) = 3 + |A_\theta(G \setminus v_0)| \), contradicting the last sentence of the preceding paragraph. Hence \( P_\theta(G \setminus v) = \emptyset \) for all \( v \in V(G) \) and \( G \) is \( \theta \)-elementary.

\section*{Lemma 3.14}
Suppose \( G \) is \( \theta \)-elementary. Then for each \( \emptyset \neq X \subseteq S \in \mathfrak{P}(\theta, G) \), \( A_\theta(G \setminus X) = S \setminus X \) and \( P_\theta(G \setminus X) \cup N_\theta(G \setminus X) = \emptyset \).

\textbf{Proof.} Let \( x \in X \). Then \( P_\theta(G \setminus x) = \emptyset \). By Theorem 2.9 \( N_\theta(G \setminus x) = \emptyset \). Now by Lemma 3.10 \( S \setminus \{ x \} = A_\theta(G \setminus x) \) so that \( X \setminus \{ x \} \subseteq S \setminus \{ x \} = A_\theta(G \setminus x) \). By Theorem 1.6 we conclude that \( A_\theta(G \setminus X) = S \setminus X \) and \( P_\theta(G \setminus X) \cup N_\theta(G \setminus X) = \emptyset \).

\section*{Corollary 3.15}
Suppose \( G \) is \( \theta \)-elementary. Let \( S \subseteq V(G) \). Then \( S \in \mathfrak{P}(\theta, G) \) if and only if \( G \setminus S \) has exactly \( |S| \) components and each is \( \theta \)-critical.

\textbf{Proof.} Suppose \( G \setminus S \) has exactly \( |S| \) components and each is \( \theta \)-critical. Then \( c_\theta(G \setminus S) = |S| \) and \( S \) is a barrier set. Hence \( S \in \mathfrak{P}(\theta, G) \).

The other implication follows from Lemma 3.14 and Corollary 1.8.

\section*{4 1-elementary cycles}
We shall need the following lemmas.

\textbf{Lemma 4.1.} [6] Corollary 4.4| Suppose \( G \) has a Hamiltonian path \( P \) and \( \theta \) is a root of \( \mu(G, x) \). Then every vertex of \( G \) which is not \( \theta \)-essential must be \( \theta \)-special.

\textbf{Lemma 4.2.} Let \( p_0 \) be a path with \( n \geq 1 \) vertices. Then
\[
\mu(p_0, 1) = \begin{cases} 
1, & \text{if } n \equiv 0 \text{ or } 1 \pmod{6}; \\
-1, & \text{if } n \equiv 3 \text{ or } 4 \pmod{6}; \\
0, & \text{otherwise}.
\end{cases}
\]

\textbf{Proof.} Note that for \( t \geq 2 \), \( \mu(p_t, x) = x \mu(p_{t-1}, x) - \mu(p_{t-2}, x) \) (part (c) of Theorem 1.2), where we define \( \mu(p_0, x) = 1 \). Therefore \( \mu(p_t, 1) = \mu(p_{t-1}, 1) - \mu(p_{t-2}, 1) \). Now \( \mu(p_1, 1) = 1 \). So, \( \mu(p_2, 1) = 0 \), and recursively we have \( \mu(p_3, 1) = -1 \), \( \mu(p_4, 1) = -1 \) and \( \mu(p_5, 1) = 0 \). By induction the lemma holds.

\textbf{Lemma 4.3.} Let \( C_n \) be a cycle with \( n \geq 3 \) vertices. Then
\[
\mu(C_n, 1) = \begin{cases} 
1, & \text{if } n \equiv 1 \text{ or } 5 \pmod{6}; \\
-1, & \text{if } n \equiv 2 \text{ or } 4 \pmod{6}; \\
2, & \text{if } n \equiv 0 \pmod{6}; \\
-2, & \text{if } n \equiv 3 \pmod{6}.
\end{cases}
\]
Proof. By part (c) of Theorem 4.2 \( \mu(C_n, 1) = \mu(p_{n-1}, 1) - 2\mu(p_{n-2}, 1) \). The lemma follows from Lemma 4.2. \( \square \)

**Theorem 4.4.** A cycle \( C_n \) is 1-elementary if and only if \( n = 3k \) for some \( k \in \mathbb{N} \).

**Proof.** (\( \Rightarrow \)) Suppose \( C_n \) is 1-elementary. Then for any \( v \in V(C_n) \), \( C_n \setminus v = p_{n-1} \). By Lemma 4.2 \( \text{mult}(1, p_{n-1}) > 0 \) if and only if \( n - 1 \equiv 2 \) or 5 \( \mod 6 \). Thus \( n = 3k \) for some \( k \in \mathbb{N} \).

(\( \Leftarrow \)) Suppose \( n = 3k \) for some \( k \in \mathbb{N} \). By Lemma 4.3 \( \text{mult}(1, C_n) = 0 \). Note that \( 3k \equiv 3 \) or 6 \( \mod 6 \). Therefore \( 3k - 1 \equiv 2 \) or 5 \( \mod 6 \), and by Lemma 4.2 and Lemma 4.3 \( \text{mult}(1, C_n \setminus v) = \text{mult}(1, p_{n-1}) = 1 \) for all \( v \in V(C_n) \). Thus \( C_n \) is 1-super positive. By Lemma 4.1 \( P_1(C_n \setminus v) = \emptyset \) for all \( v \in V(C_n) \). Hence \( C_n \) is 1-elementary. \( \square \)

For our next result, let us denote the vertices of \( C_{3k} \) by \( 1, 2, 3, \ldots, 3k \) (see Figure 4).

![Figure 4](image_url)

**Corollary 4.5.** \( C_{3k} \) has exactly 3 1-barrier sets, that is

\[ \mathcal{P}(1, C_{3k}) = \{\{1, 4, 7, \ldots, 3k - 2\}, \{2, 5, 8, \ldots, 3k - 1\}, \{3, 6, 9, \ldots, 3k\}\}. \]

**Proof.** Note that \( C_{3k} \setminus \{1, 4, 7, \ldots, 3k - 2\} \) is a disjoint union of \( k \) number of \( K_2 \) and \( K_2 \) is 1-critical. So \( \{1, 4, 7, \ldots, 3k - 2\} \) is a 1-barrier set. Similarly \( \{2, 5, 8, \ldots, 3k - 1\} \) and \( \{3, 6, 9, \ldots, 3k\} \) are 1-barrier sets. It then follows from Theorem 4.3 and Theorem 5.13 that these are the only 1-barrier sets. \( \square \)

**5 Decomposition of \( \theta \)-super positive graphs**

**Definition 5.1.** A set \( X \subseteq V(G) \) with \( |X| > 1 \) is said to be independent in \( G \) if for all \( u, v \in X \), \( u \) and \( v \) are not adjacent to each other. A graph \( G \) is said to be \( \theta \)-base if it is \( \theta \)-super positive and for all \( S \in \mathcal{P}(\theta, G) \), \( S \) is independent.

Note that the cycle \( C_{3k} \) is \( \theta \)-base. In fact a connected \( \theta \)-base graph is \( \theta \)-elementary.

**Theorem 5.2.** A connected \( \theta \)-base graph is \( \theta \)-elementary.

**Proof.** Let \( G \) be \( \theta \)-base. Suppose it is not \( \theta \)-elementary. Then \( P_\theta(G \setminus v) \neq \emptyset \) for some \( v \in V(G) \). By Lemma 2.7 \( G \setminus v \) has at least one \( \theta \)-essential vertex.

If \( v \) is not a cut vertex of \( G \), then \( A_\theta(G \setminus v) \neq \emptyset \). By Theorem 2.9 and Corollary 1.13 \( (G \setminus v)A_\theta(G \setminus v) \) has a \( \theta \)-super positive component, say \( H \). Since \( G \setminus v \) is connected, there exists \( h \in V(H) \) that is adjacent to some element \( w \in A_\theta(G \setminus v) \). Note that \( \{h, w, v\} \) is a \( \theta \)-extreme set in \( G \). By Lemma 3.4...
\{h, w\} is a \(\theta\)-extreme set in \(G\). By Lemma 3.6, \(\{h, w\}\) is contained in some \(S \in \mathcal{P}(\theta, G)\), a contrary to the fact that \(S\) is independent.

If \(v\) is a cut vertex of \(G\), then \(G \setminus v\) contains a \(\theta\)-super positive component (for \(N_\theta(G \setminus v) = \emptyset\) by Theorem 2.4). Clearly, some vertex in this component, say \(u\), is joined to \(v\) and \(\{u, v\}\) is a \(\theta\)-extreme set in \(G\). Again, by Lemma 3.6 \(\{u, v\}\) is contained in some \(S \in \mathcal{P}(\theta, G)\), a contrary to the fact that \(S\) is independent.

Hence \(P_\theta(G \setminus v) = \emptyset\) for all \(v \in V(G)\) and \(G\) is \(\theta\)-elementary. \(\square\)

Note that the converse of Theorem 5.2 is not true. Let \(G\) be the graph in Figure 3. Note that \(\{u_3, u_4\} \in \mathcal{P}(1, G)\) but it is not independent.

**Lemma 5.3.** Let \(G\) be \(\theta\)-super positive and \(e = (u, v) \in E(G)\) such that \(\{u, v\}\) is a \(\theta\)-extreme set in \(G\). Let \(G'\) be the graph obtained by removing the edge \(e\) from \(G\). Then \(G'\) is \(\theta\)-super positive.

**Proof.** Now \(\text{mult}(\theta, G \setminus uv) = 2\). By part (b) of Theorem 1.2 \(\mu(G, x) = \mu(G', x) - \mu(G \setminus uv, x)\). This implies that \(\mu(G', \theta) = \mu(G, \theta) \neq 0\).

It is left to show that \(\mu(G' \setminus w, \theta) = 0\) for all \(w \in V(G')\). Clearly if \(w = u\) or \(v\) then \(\mu(G' \setminus w, \theta) = \mu(G \setminus w, \theta) = 0\). Suppose \(w \neq u, v\). By part (b) of Theorem 1.2 again, \(\mu(G' \setminus w, x) = \mu(G \setminus w, x) - \mu(G \setminus uvw, x)\). By Lemma 1.3 \(\text{mult}(\theta, G \setminus uvw) \geq 1\). Therefore \(\mu(G' \setminus w, \theta) = \mu(G \setminus w, \theta) = 0\). Hence \(G'\) is \(\theta\)-super positive. \(\square\)

Note that after removing an edge from \(G\) as in Lemma 5.3 \(\mathcal{P}(\theta, G') \neq \mathcal{P}(\theta, G)\) in general. In Figure 5, \(\mathcal{P}(1, G) = \{\{1, 4, 7\}, \{5, 8\}, \{6, 9\}, \{2\}, \{3\}\}\). After removing the edge \((1, 4)\) from \(G\), the resulting graph \(G' = C_9\). By Corollary 3.5 \(\mathcal{P}(1, G') = \{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\}\).

![Figure 5](image)

We shall need the following lemma.

**Lemma 5.4.** Corollary 2.5] For any root \(\theta\) of \(\mu(G, x)\) and a path \(p\) in \(G\),

\[
\text{mult}(\theta, G \setminus p) \geq \text{mult}(\theta, G) - 1.
\]

**Lemma 5.5.** Let \(G\) be \(\theta\)-super positive and \(e_1 = (u, v) \in E(G)\) with \(\{u, v\}\) is a \(\theta\)-extreme set. Let \(G' = G - e_1\) and \(e_2 = (w, z) \in E(G')\). Then \(\{w, z\}\) is a \(\theta\)-extreme set in \(G'\) if and only if it is a \(\theta\)-extreme set in \(G\).

**Proof.** **Case 1.** Suppose \(e_1\) and \(e_2\) have a vertex in common, say \(w = u\). Then \(G' \setminus wz = G \setminus wz\).

(\(\Rightarrow\)) Suppose \(\{w, z\}\) is a \(\theta\)-extreme set in \(G'\). By Lemma 5.3 \(\text{mult}(\theta, G') = 0\). Therefore \(\text{mult}(\theta, G' \setminus wz) = \text{mult}(\theta, G \setminus wz) = 2\) and \(\{w, z\}\) is a \(\theta\)-extreme set in \(G\).
\( \Leftrightarrow \) The converse is proved similarly.

**Case 2.** Suppose \( e_1 \) and \( e_2 \) have no vertex in common. By part (b) of Theorem 1.2,

\[
\mu(G \setminus wz, x) = \mu(G' \setminus wz, x) - \mu(G \setminus wzw, x).
\]

\((\Rightarrow)\) Suppose \( \{w, z\} \) is a \( \theta \)-extreme set in \( G' \). Then \( \text{mult}(\theta, G' \setminus wz) = 2 \). Now \( \text{mult}(\theta, G \setminus uv) = 2 \) and by Lemma 5.4, \( \text{mult}(\theta, G \setminus uvwz) \geq 1 \). So we conclude that \( \text{mult}(\theta, G \setminus wz) \geq 1 \). On the other hand, \( N_0(G \setminus w) = \emptyset \) (Theorem 2.9). Therefore either \( \text{mult}(\theta, G \setminus wz) = 0 \) or \( 2 \). Hence the latter holds and \( \{w, z\} \) is a \( \theta \)-extreme set in \( G \).

\((\Leftarrow)\) Suppose \( \{w, z\} \) is a \( \theta \)-extreme set in \( G \). Then \( \text{mult}(\theta, G \setminus wz) = 2 \). As before we have \( \text{mult}(\theta, G \setminus uvwz) \geq 1 \). So we conclude that \( \text{mult}(\theta, G' \setminus wz) \geq 1 \). On the other hand, by Lemma 5.3, \( G' \) is \( \theta \)-super positive. Therefore \( N_0(G' \setminus w) = \emptyset \) (Theorem 2.9), and then either \( \text{mult}(\theta, G' \setminus wz) = 0 \) or \( 2 \). Hence the latter holds and \( \{w, z\} \) is a \( \theta \)-extreme set in \( G' \).

**Definition 5.6.** Let \( G \) be \( \theta \)-super positive. An edge \( e = (u, v) \in E(G) \) is said to be \( \theta \)-extreme in \( G \) if \( \{u, v\} \) is a \( \theta \)-extreme set.

The process described in Lemma 5.3 can be iterated. Let \( Y_0 = \{e_1, e_2, \ldots, e_k\} \subseteq E(G) \) be the set of all \( \theta \)-extreme edges. Let \( G_1 = G - e_1 \). Then \( G_1 \) is \( \theta \)-super positive (Lemma 5.3). Let \( Y_1 \) be the set of all \( \theta \)-extreme edges in \( G_1 \). Then by Lemma 5.5, \( Y_1 = Y_0 \setminus \{e_1\} \). Now let \( G_2 = G_1 - e_2 \). By applying Lemma 5.3 and Lemma 5.5, we see that \( G_2 \) is \( \theta \)-super positive and the set of all \( \theta \)-extreme edges in \( G_2 \) is \( Y_2 = Y_0 \setminus \{e_1, e_2\} \). By continuing this process, after \( k \) steps, we see that \( G_k = G - e_1 e_2 \ldots e_k \) is \( \theta \)-super positive and the set of all \( \theta \)-extreme edges in \( G_k \) is \( Y_k = \emptyset \). We claim that \( G_k \) is a disjoint union of \( \theta \)-base graphs. Suppose the contrary. Let \( H \) be a component of \( G_k \) that is not \( \theta \)-base. Since \( G_k \) is \( \theta \)-super positive, by part (a) of Theorem 1.2, we deduce that \( H \) is \( \theta \)-super positive. Therefore there is a \( S \in \mathcal{F}(\theta, H) \) for which \( S \) is not independent. Let \( e = (u, v) \in E(H) \) with \( \{u, v\} \subseteq S \). By Lemma 3.7 and Lemma 3.3, \( \{u, v\} \) is a \( \theta \)-extreme set in \( H \). This means that \( e \) is \( \theta \)-extreme in \( H \), and by part (a) of Theorem 1.2, \( e \) is \( \theta \)-extreme in \( G_k \), a contrary to the fact that \( Y_k = \emptyset \). Hence \( H \) is \( \theta \)-base and we have proved the following theorem.

**Theorem 5.7.** Let \( G \) be \( \theta \)-super positive. Then \( G \) can be decomposed into a disjoint union of \( \theta \)-base graphs by deleting its \( \theta \)-extreme edges. Furthermore, the decomposition is unique, i.e. the \( \theta \)-base graphs are uniquely determined by \( G \).

The proof of the next lemma is similar to Lemma 5.3 and is thus omitted.

**Lemma 5.8.** Let \( G \) be \( \theta \)-super positive and \( \{u, v\} \) is a \( \theta \)-extreme set with \( e = (u, v) \notin E(G) \). Let \( G' \) be the graph obtained by adding the edge \( e \) to \( G \). Then \( G' \) is \( \theta \)-super positive.

Using the process described in Lemma 5.8, we can construct \( \theta \)-super positive graph from \( \theta \)-base graphs. Together with Theorem 5.7, we see that every \( \theta \)-super positive can be constructed from \( \theta \)-base graphs.

**Corollary 5.9.** A graph is \( \theta \)-super positive if and only if it can be constructed from \( \theta \)-base graphs.

In the next theorem, we shall extend Theorem 2.2.
**Theorem 5.10.** Let $G_1$ and $G_2$ be two $\theta$-super positive graphs and $S_i \in \mathcal{P}(\theta, G_i)$ for $i = 1, 2$. Let $G$ be the graph obtained by adding the edges $e_1, e_2, \ldots, e_m$ to the union of $G_1$ and $G_2$, where each $e_j$ contains a point in $S_1$ and $S_2$. Then $G$ is $\theta$-super positive.

**Proof.** We shall prove by induction on $m$. If $m = 1$, we are done by Theorem 2.2. Suppose $m \geq 2$. Assume that it is true for $m - 1$. Let $G'$ be the graph obtained by adding the edges $e_1, e_2, \ldots, e_{m-1}$ to the union of $G_1$ and $G_2$. By induction $G'$ is $\theta$-super positive. Let $e_m = (v_1, v_2)$ where $v_i \in S_i$. Note that the number of $\theta$-critical components in $G' \setminus (S_1 \cup S_2)$ is $c_\theta(G' \setminus (S_1 \cup S_2)) = c_\theta(G_1 \setminus S_1) + c_\theta(G_2 \setminus S_2) = |S_1| + |S_2|$. So $S_1 \cup S_2$ is a $\theta$-barrier set in $G'$. By Lemma 3.7 and Lemma 3.4, \{v_1, v_2\} is a $\theta$-extreme set in $G'$. Therefore by Lemma 5.8, $G$ is $\theta$-super positive.

In Figure 6, the graph $G$ is obtained from two 1-base graphs by adding edges $e_1$ and $e_2$.

![Figure 6](image_url)

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