A comprehensive theory is developed that describes the coexistence of p-wave, spin-triplet superconductivity and itinerant ferromagnetism. It is shown how to use field-theoretic techniques to derive both conventional strong-coupling theory, and analogous gap equations for superconductivity induced by magnetic fluctuations. It is then shown and discussed in detail that the magnetic fluctuations are generically stronger on the ferromagnetic side of the magnetic phase boundary, which substantially enhances the superconducting critical temperature in the ferromagnetic phase over that in the paramagnetic one. The resulting phase diagram is compared with the experimental observations in UGe$_2$ and ZrZn$_2$.

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I. INTRODUCTION

It is well known that, in principle, the exchange of magnetic fluctuations between electrons can induce superconductivity in both the paramagnetic and ferromagnetic phases of metals. In general, for superconductivity to occur one needs both some sort of attraction between quasi-particles, which can be provided by magnetic fluctuations, and low temperatures. Since magnetic fluctuations become large near continuous magnetic phase transitions, ideal candidates for this phenomenon would seem to be itinerant ferromagnets with a low Curie temperature. In contrast to the much more common phonon-exchange case, which usually leads to spin-singlet, s-wave superconductivity, the magnetically mediated pairing is believed to usually be strongest in the spin-triplet, p-wave channel. This type of superconductivity is very sensitive to nonmagnetic disorder, so that very clean samples are also required. The combined requirements of low temperatures, high purity, and vicinity to a continuous ferromagnetic transition severely restricts the number of materials where ferro-magnetically induced superconductivity might be observed. Furthermore, recent theoretical work has shown that at low temperature the ferromagnetic transition in an itinerant electron system should be generically of first order, and there is some experimental evidence for this as well. If this is the case, then there are no divergent magnetic fluctuations even at the transition point itself.

Very recently, however, coexistence of ferromagnetism and superconductivity has indeed been observed, in UGe$_2$. The f-electron structure of the uranium notwithstanding, this material is more similar to d-band metals than to heavy fermion systems, and ferromagnetism and superconductivity are believed to be caused by itinerant electrons in the same band. The persistence of ferromagnetic order within the superconducting phase has been ascertained by neutron scattering. Since superconductivity in the presence of ferromagnetism is likely to be of spin-triplet type, magnetic-fluctuation induced pairing is a possible mechanism. However, two aspects of the experiments are not readily reconcilable with previous theories of superconductivity in ferromagnets. First, the magnetic transition is observed to be of first order, so that there are no divergent fluctuations as the transition is approached. Second, the superconducting state is found only on the ferromagnetic side of the phase boundary. A schematic phase diagram for this situation is shown in Fig. 1(a). The theoretical prediction, on the other hand, has been a phase diagram like the one shown in Fig. 1(b), with superconductivity existing, more or less symmetrically, on both sides of the magnetic phase boundary.

Since the initial experimental observation of superconductivity in UGe$_2$ there has been considerable new experimental and theoretical work. Experimentally, the qualitatively same phenomenon has been observed in ZrZn$_2$, and in URhGe. In ZrZn$_2$, the paramagnetic-to-ferromagnetic phase transition appears to be either continuous or only weakly first order; in URhGe the nature of the magnetic transition at very low temperatures is not known. In both of these materials, superconductivity is observed far inside the ferromagnetic phase. This is in contrast to UGe$_2$, where it is observed only close to the magnetic phase boundary, see Fig. 1(a). Theoretically, Shick and Pickett have included correlation effects in a local density approximation approach and have concluded that magnetically mediated triplet superconductivity can occur in UGe$_2$. In contrast, Shimahara and Kohmoto have argued that triplet superconductivity can be induced by screened phonon interactions in ferromagnetic compounds such as UGe$_2$. Santi et al.
have performed ab initio calculations for ZrZn$_2$, concluding that magnetically mediated triplet superconductivity can occur, with a critical superconductivity temperature as high as 1 K. Watanabe and Miyake$^{13}$ have argued that coupled charge density waves and spin density waves can be used to understand the superconductivity in UGe$_2$, and a Larkin-Ovchinnikov-Fulde-Ferrell state is also a possibility.$^{14}$ Machida and Ohmi,$^{15}$ and more recently Samokhin and Walker,$^{16}$ have investigated possible order parameters for magnetically induced superconductivity, and a $d$-electron exchange mechanism has been proposed.$^{17}$ Finally, the experiments have prompted investigations of the detailed role of the normal self energy in the Eliashberg equations for magnetically induced superconductivity.$^{18-20}$

In a previous short paper,$^{21}$ the current authors and collaborators have shown that the critical temperature for spin-triplet, $p$-wave superconductivity mediated by spin fluctuations is generically much higher in a Heisenberg ferromagnetic phase than in a paramagnetic one, due to the coupling of the magnons to the longitudinal magnetic susceptibility, and that this qualitatively explains the observed phase diagram. In the current paper we explain and expand on this work. There are two important aspects. The first, formal one, is to show how to use field-theoretic techniques, rather than many-body perturbation theory, to derive Eliashberg-like equations for magnetic-fluctuation induced superconductivity. For completeness, and to motivate the more complicated case of magnetic-fluctuation induced superconductivity, we also show how these ideas can be used to derive conventional Eliashberg theory. The second one is a detailed explanation of our previous observation that magnetic fluctuations are generically stronger on the ferromagnetic side of the phase boundary, which in turn is shown to lead to a much higher critical temperature for superconductivity in the ferromagnetic phase. This result is consistent with the observed phase diagrams in UGe$_2$ and ZrZn$_2$, schematically shown in Fig. 1(b). The formalism developed here will also lay the ground for further investigations of the superconductor-ferromagnet phase diagram. In particular, we will argue that as a function of temperature and magnetization, one expects phase transitions to other, more exotic, superconducting states inside the superconducting phase.

The plan of this paper is as follows. In Section II we derive an effective action or field theory for superconductivity due to magnetic fluctuations. The field theory is made explicit by keeping magnetic fluctuations to one-loop order. The resulting superconducting field theory is then solved in a saddle-point approximation in Section III. The resulting gap equations are consistent with earlier results,$^6$ except that the interaction is an exact longitudinal magnetic susceptibility rather than an approximate one. The field theoretic derivation also has structural advantages that we plan to exploit in the future, as is discussed in Section V. In the last part of Section III, explicit expressions for the magnetic susceptibility in either phase are given. Emphasis is on the difference between magnetic fluctuations in the two phases. In Section IV the linearized gap equations derived in Section III are solved using a McMillan-type approximation. Phase diagrams are derived showing the paramagnetic, ferromagnetic and superconducting phases. It is concluded that the critical temperature for superconductivity is much higher in the ferromagnetic phase than in the paramagnetic phase. The paper is concluded in Section V with a discussion of our results, of possible future applications of the formalism developed here, and of some open questions. The derivation of conventional Eliashberg theory by field-theoretic methods is relegated to an appendix.

II. COUPLED FIELD THEORY FOR NORMAL STATE, SUPERCONDUCTING, AND MAGNETIC FLUCTUATIONS

A. The model

Our starting point is an interacting electron gas model. In terms of Grassmann variables, the partition function is

$$Z = \int D[\bar{\psi}, \psi] e^{S[\bar{\psi}, \psi]} .$$  \hspace{1cm} (2.1)

Here the functional integration is with respect to Grassmann-valued fields $\psi$ and $\bar{\psi}$, and the action $S$ reads

$$S = -\int dx \sum_\sigma \bar{\psi}_\sigma(x) \partial_\tau \psi_\sigma(x) + S_0 + S_{\text{int}} .$$  \hspace{1cm} (2.2a)

We use a $(d+1)$-vector notation, with $x \equiv (x, \tau)$, and $\int dx \equiv \int_0^\beta d\tau$. $x$ denotes position, $\tau$ imaginary.
time, $V \to \infty$ the system volume, and $\beta = 1/T$ the inverse temperature. $\sigma$ is a spin label, and we use units such that $\hbar = k_B = 1$. $S_0$ describes free electrons with chemical potential $\mu$,

$$ S_0 = \int dx \sum_{\sigma} \bar{\psi}_\sigma(x) \left( \frac{\nabla^2}{2m_e} + \mu \right) \psi_\sigma(x) \ , \tag{2.2b} $$

with $m_e$ the electron mass. $S_{\text{int}}$ denotes the two-body electron-electron interaction. The underlying Coulomb interaction becomes screened, and after renormalization from a microscopic to a mesoscopic scale, $S_{\text{int}}$ separates into spin-singlet and spin-triplet particle-hole and particle-particle interactions, respectively.\textsuperscript{22} The interaction in the particle-particle channel is what causes superconductivity, and we do not include it in our bare action. Rather, one of the main points of this paper is to see how this interaction is generated by magnetic fluctuations. The interactions in the particle-hole channel are

$$ S_{\text{int}} = S_b^{\text{ph}} + S_t^{\text{ph}} \ , \tag{2.2c} $$

with,

$$ S_b^{\text{ph}} = \frac{\Gamma_s}{2} \int dx \, n(x) n(x) \ , \tag{2.2d} $$

$$ S_t^{\text{ph}} = \frac{\Gamma_t}{2} \int dx \, \mathbf{n}_s(x) \cdot \mathbf{n}_s(x) \ . \tag{2.2e} $$

Here $\Gamma_{s,t}$ are the particle-hole spin-singlet and spin-triplet interaction amplitudes, respectively.

$$ n(x) = \sum_{\sigma} \bar{\psi}_\sigma(x) \psi_\sigma(x) \ , \tag{2.3a} $$

is the electron number density field, and

$$ n^i_s(x) = \sum_{a,b} \bar{\psi}_a(x) \sigma^{ab}_i \psi_b(x) \ , \ (i = 1, 2, 3) \ , \tag{2.3b} $$

are the components of the electron spin density field $\mathbf{n}_s$. $\sigma_{1,2,3}$ denote the Pauli matrices.

Below we will often use a Fourier representation of the Grassmann field,

$$ \psi_{\sigma,n}(\mathbf{k}) = \sqrt{\frac{\mathcal{V}}{T}} \int dx \, e^{i(\omega_n \tau - \mathbf{k} \cdot \mathbf{x})} \, \psi_\sigma(x) \ , \tag{2.4a} $$

$$ \bar{\psi}_{\sigma,n}(\mathbf{k}) = \sqrt{\frac{\mathcal{V}}{T}} \int dx \, e^{-i(\omega_n \tau - \mathbf{k} \cdot \mathbf{x})} \, \bar{\psi}_\sigma(x) \ . \tag{2.4b} $$

Finally, to introduce the relevant order parameters, it will be convenient to introduce generalized Nambu spinors,

$$ \Psi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_\uparrow(x) \\ \bar{\psi}_\downarrow(x) \\ \bar{\psi}_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix} \ , \tag{2.5a} $$

$$ \bar{\Psi}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\psi}_\uparrow(x) \\ \psi_\downarrow(x) \\ \bar{\psi}_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix} \ . \tag{2.5b} $$

and corresponding composite variables as the tensor product of the spinors,

$$ \Phi_{ij}(x, y) = \bar{\Psi}_{i}(x) \Psi_{j}(y) \ , \tag{2.5c} $$

It is also useful to define a four-vector of $4 \times 4$ matrices

$$ (\gamma_0, \gamma_i) = (\sigma_3 \otimes \sigma_0, \sigma_3 \otimes \sigma_1, \sigma_0 \otimes \sigma_2, \sigma_3 \otimes \sigma_3) \ , \tag{2.6} $$

with $\sigma_0$ the $2 \times 2$ unit matrix.\textsuperscript{23} In terms of these spinors and the $\gamma_i$, the noninteracting part of the action reads

$$ S - S_{\text{int}} = \int dx \, \bar{\Psi}(x) \left[ -\partial_x + \gamma_0 \left( \nabla^2 / 2m_e + \mu \right) \right] \Psi(x) \ , \tag{2.7a} $$

and the number density and spin density fields that make up the interacting part can be written

$$ n(x) = \bar{\Psi}(x) \gamma_0 \Psi(x) \ , \tag{2.7b} $$

$$ n^i_s(x) = \bar{\Psi}(x) \gamma_i \Psi(x) \ , \ (i = 1, 2, 3) \ , \tag{2.7c} $$

with the product between spinors defined as the matrix product.

B. Superconducting and magnetic order parameter field theory

1. Intuitive argument for magnetic-fluctuation induced superconductivity

To motivate our technical considerations, we recapitulate a simple physical argument given by De Gennes.\textsuperscript{24} Let us suppose that we have a local magnetization fluctuation, $\delta \mathbf{M}(\mathbf{x})$, that couples to the electronic spin density. Since the coupling of $\delta \mathbf{M}$ to $\mathbf{n}_s$ is analogous to that of a magnetic field, the change in energy of the electron system is given by a Zeeman term,

$$ \delta H = -\Gamma \int dx \, \mathbf{n}_s(x) \cdot \delta \mathbf{M}(\mathbf{x}) \ , \tag{2.8a} $$

with $\Gamma$ a coupling strength, and $\mathbf{n}_s(x)$ the electronic spin density. To first order in $\Gamma$, the magnetization induced at point $\mathbf{x}$ by an electronic spin density at point $\mathbf{y}$ is given by

$$ \delta M_i(x) = \Gamma \int dy \sum_j \chi_{ij}(\mathbf{x} - \mathbf{y}) \, n^j_s(y) \ , \tag{2.8b} $$

where $\chi_{ij}$ is the electronic magnetic susceptibility tensor. This implies

$$ \delta H = -\Gamma^2 \int dx \, \int dy \sum_{ij} n^i_s(x) \chi_{ij}(x - y) \, n^j_s(y) \ . \tag{2.9} $$
Equation (2.9) shows that there is an effective electronic spin-spin interaction potential tensor given by

\[ V_{ij}(x - y) = -\Gamma^2 \chi_{ij}(x - y) \]  

(2.10)

This result has two important features that motivate our formal considerations below. First, we anticipate that the pairing potential responsible for superconductivity will be proportional to the magnetic susceptibility. Second, the interaction will be attractive if the spins form a relative spin-triplet, and repulsive if they form a spin-singlet.  

2. Order parameter field theory

The above considerations motivate transforming the fermionic field theory to an effective one in terms of an order parameter field for spin-triplet superconductivity, and a magnetization field. Since we are considering an electronic mechanism for superconductivity, a (relatively) high \( T_c \) can be achieved only if electron-electron interaction effects are in some sense strong. This implies that for a consistent theory we need to also worry about strong-coupling effects as described by normal-state self energies, i.e., an Eliashberg-type theory is needed. We will accomplish this by introducing an additional field related to normal-state effects.

Because we will eventually use a saddle-point, or free energy variational, technique we will have to choose a superconducting order parameter as an input. In the ferromagnetic phase it seems likely that the triplet superconducting order parameter will be proportional to the magnetic susceptibility. In what follows we assume that this is the case.

In Appendix A we show how field-theoretic techniques can be used to derive standard strong coupling theory for conventional superconductors. Here we essentially repeat the formalism of Appendix A, except that the coupling constant \( \gamma \) is proportional to the magnetic susceptibility.

\[ \langle \psi_\uparrow(x), \psi_\downarrow(y) \rangle = 2 \gamma(x, y) \]  

(2.11a)

Here we have constrained the field \( \Phi \), Eq. (2.5c), to a bosonic field \( \mathcal{G} \) by means of the Lagrange multiplier \( \Lambda \), and the superscript \( T \) denotes a transposition operation. The interaction will be proportional to the magnetic susceptibility. In terms of the matrix elements of \( \mathcal{G} \), which are isomorphic to those of \( \Phi \), this order parameter and its hermitian adjoint are

\[ 2\langle \mathcal{G}_{13}(x, y) \rangle \equiv F^+(x - y) \neq 0 \]  

(2.11b)

We also define anomalous self energies, or gap functions, as

\[ \langle \Lambda_{13}(x, y) \rangle \equiv \Delta^+(x - y) \neq 0 \]  

(2.11c)

We are leading order in order to derive a theory for magnetic-fluctuation induced spin-triplet superconductivity. Since the magnetic transition is already understood, we will model its first order nature in a simple way. Generally, our guiding philosophy will be to keep fluctuations to leading order if they lead to a qualitatively new effect that is relevant for the superconducting transition temperature, but to neglect them otherwise.

The first formal step is to introduce the magnetization field, \( M \), by performing a Hubbard-Stratonovich transformation on the triplet interaction term, Eq. (2.2c). The other order parameter fields are introduced via a delta-function constraint as in Appendix A, and we integrate out the fermions. This allows us to write the partition function, in analogy to Eq. (A4), as

\[ Z = \int D[M, \mathcal{G}, \Lambda] e^{-\mathcal{A}[M, \mathcal{G}, \Lambda]} \]  

(2.11a)

with the action \( \mathcal{A} \) given by

\[ \mathcal{A}[M, \mathcal{G}, \Lambda] = \frac{\Gamma_s}{2} \int dx (\text{tr}[\mathcal{G}(x, x)])^2 + \text{Tr}(\Lambda \mathcal{G}) \]  

(2.11b)

\[ + \frac{1}{2} \text{Tr} \ln(G_0^{-1} + \sqrt{2\Gamma_s \gamma} \cdot M - \Lambda^T) \]  

(2.11c)

\[ - \int dx \, M(x) \cdot M(x) \]  

(2.11d)

In terms of the matrix elements of \( \mathcal{G} \), which are isomorphic to those of \( \Phi \), this order parameter and its hermitian adjoint are

\[ 2\langle \mathcal{G}_{13}(x, y) \rangle \equiv F^+(x - y) \neq 0 \]  

(2.12a)

We also define anomalous self energies, or gap functions, as

\[ \langle \Lambda_{13}(x, y) \rangle \equiv \Delta^+(x - y) \neq 0 \]  

(2.12b)

The normal state order parameters and self energies are defined analogously to those in Appendix A, except that isotropy in spin space is lost in the ferromagnetic state, and

\[ 2\langle \mathcal{G}_{11}(x, y) \rangle = -2\langle \mathcal{G}_{33}(y, x) \rangle \equiv G_t(x - y) \]  

(2.12c)

\[ 2\langle \mathcal{G}_{22}(x, y) \rangle = -2\langle \mathcal{G}_{44}(y, x) \rangle \equiv G_s(x - y) \]  

(2.12d)
\begin{equation}
\langle \Lambda_{11}(x, y) \rangle = -\langle \Lambda_{33}(x, y) \rangle = \Sigma_{1}(x - y) , \\
\langle \Lambda_{22}(x, y) \rangle = -\langle \Lambda_{44}(x, y) \rangle = \Sigma_{2}(x - y) .
\end{equation}

Note that we are assuming that the equilibrium state is spatially homogeneous. For our present discussion, which will focus on the superconducting phase transition, this assumption is legitimate. However, within the superconducting phase it cannot be correct in general, as we discuss in Section V.

As already stressed, the crucial interactions leading to the superconducting state and the most important normal-state self-energy effects are due to magnetic fluctuations. To take these fluctuations into account, we insert Eq. (2.13a) into Eq. (2.12a), expand to order \((\delta M)^{2}\), and then integrate out the magnetic fluctuations. The partition function and action can then be written

\begin{equation}
Z = \int D[\mathcal{G}, \Lambda] e^{A[\mathcal{G}, \Lambda]} ,
\end{equation}

with

\begin{equation}
A[\mathcal{G}, \Lambda] = \frac{\Gamma_s}{2} \int dx \left( \text{tr} \left[ \gamma_{00} \mathcal{G}(x, x) \right] \right)^{2} + \text{Tr} \left( \Lambda \mathcal{G} \right) + \frac{1}{2} \text{Tr} \ln \tilde{\mathcal{G}}^{-1} \Lambda - \frac{1}{2} \text{Tr} \ln \chi_{\Lambda}^{-1} .
\end{equation}

Here we have defined a 4 \times 4 matrix Green operator,

\begin{equation}
\tilde{\mathcal{G}}^{-1}_{\Lambda} = \tilde{\mathcal{G}}^{-1}_{0} + \Gamma_{s} m \gamma_{3} - \Lambda^{T} ,
\end{equation}

and a normalized inverse magnetic susceptibility tensor,

\begin{equation}
(\chi_{\Lambda}^{-1})_{ij}(x - y) = \delta_{ij} \delta(x - y) \\
+ \frac{\Gamma_{s}}{2} \text{tr} \left[ \tilde{\mathcal{G}}_{\Lambda}(y, x) \gamma_{i} \tilde{\mathcal{G}}_{\Lambda}(x, y) \gamma_{j} \right] , (i, j = 1, 2, 3) ,
\end{equation}

Notice that \(\tilde{\mathcal{G}}\) and \(\chi\) both depend on \(\Lambda\), as we explicitly indicate with our notation.

The action as written in Eqs. (2.15) does not yet allow for a saddle-point solution that describes a superconducting state, i.e. one with a broken \(U(1)\) gauge symmetry that corresponds to particle number conservation. This can be seen by taking the variation with respect to, say, \(\tilde{\mathcal{G}}_{13}\), which yields \(\Lambda_{31} = 0\), which characterizes a non-superconducting state. Indeed, such a solution is more than just a saddle-point solution, as can be seen by formally integrating out \(\tilde{\mathcal{G}}_{13}\). However, while always a solution of the field theory, such a state is not necessarily stable. To allow for superconducting states with a spontaneously broken gauge symmetry, we write \(\Lambda\) as its average value plus a fluctuation,

\begin{equation}
\Lambda = \langle \Lambda \rangle + \delta \Lambda .
\end{equation}

We then formally integrate over the \(\delta \Lambda\) fluctuations to obtain

\begin{equation}
Z = \int D[\mathcal{G}] e^{A[\mathcal{G}, \langle \Lambda \rangle]} ,
\end{equation}

with

\begin{equation}
A[\mathcal{G}, \langle \Lambda \rangle] = \frac{\Gamma_s}{2} \int dx \left[ \text{tr} \left[ \gamma_{00} \mathcal{G}(x, x) \right] \right]^{2} + \text{Tr} \left( \langle \Lambda \rangle \mathcal{G} \right) + \frac{1}{2} \text{Tr} \ln \tilde{\mathcal{G}}^{-1}_{\langle \Lambda \rangle} \frac{1}{2} \text{Tr} \ln \chi_{\langle \Lambda \rangle}^{-1} + \frac{1}{2} \text{Tr} \ln \chi_{\langle \Lambda \rangle}^{-1} \frac{1}{2} \text{Tr} \ln \chi_{\langle \Lambda \rangle}^{-1} \frac{1}{2} .
\end{equation}

Notice that \(\langle \Lambda_{13} \rangle = \Delta^{+}\), which couples to \(\mathcal{G}_{11}\) in the \(\text{Tr} \left( \langle \Lambda \rangle \mathcal{G} \right)\) term, acts like an external field that breaks gauge invariance. The electron system with such an external field defines an ensemble that allows for a nonzero superconducting order parameter, with \(\langle \Lambda \rangle\) (and \(\langle \mathcal{G} \rangle\)) to be determined self-consistently.\(^{28}\)

The Green function \(\tilde{\mathcal{G}}_{\langle \Lambda \rangle}(x - y) \equiv \tilde{\mathcal{G}}(x - y)\) will play a central role in our description of the superconducting phase transition, so we explicitly give it here. It is convenient to express the matrix elements of \(\tilde{\mathcal{G}}\) in terms of a normal-state Green function \(\mathcal{G}\). In frequency-wavenumber space, with a four-vector \(k = (\omega, n, k)\), we define

\begin{equation}
g_{\sigma}^{-1}(k) = i \omega_{n} - \xi_{k, \sigma} - \Sigma_{\sigma}(k) ,
\end{equation}

with

\begin{equation}
\xi_{k, \sigma} = \frac{k^{2}}{2m_{e}} - \mu - \sigma \delta ,
\end{equation}

where \(\sigma = (\uparrow, \downarrow) \equiv (+, -)\) is the spin projection index, and \(\delta = \Gamma_{t} m\) reflects the Stoner splitting of the Fermi surface in the ferromagnetic phase. We also define a function

\begin{equation}
D(k) = g_{\uparrow}^{-1}(k) g_{\downarrow}^{-1}(k) + \Delta(k) \Delta^{+}(k) .
\end{equation}

Finally we note the symmetry relations,

\begin{equation}
\Delta(k) = -\Delta(-k) ,
\end{equation}

\begin{equation}
\Delta^{+}(k) = -\Delta(-k)^{*} ,
\end{equation}

with \(\delta\) denoting a complex conjugate. The matrix inversion problem for \(\tilde{\mathcal{G}}\) decouples into one \(2 \times 2\) problem and two \(1 \times 1\) problems, and we obtain

\begin{align}
\tilde{\mathcal{G}}_{11}(k) &= g_{\downarrow}^{-1}(k) / D(k) , \\
\tilde{\mathcal{G}}_{13}(k) &= \Delta(k) / D(k) , \\
\tilde{\mathcal{G}}_{22}(k) &= g_{\downarrow}(k) ,
\end{align}

Symmetry gives
with the subscripts $L(2.15)$, (2.13), and (2.12b) we obtain an external magnetic field. For later reference we give the inverse susceptibility has both diagonal and off-diagonal transverse components, respectively, and the transverse magnetic susceptibility of a ‘reference ensemble’ of electrons that are subject to an external magnetic field equal to $\delta$, and whose self energy operator is given by $\Sigma_\sigma$. General symmetry arguments show that $\chi^{-1}$ has the form

$$\chi_{ij}^{-1}(k) = \delta_{ij} \left[ \delta_{i3} \chi_{L}^{-1}(k) + (1 - \delta_{i3}) \chi_{T,+}^{-1}(k) \right] + (\delta_{i2} - \delta_{21}) \chi_{T,-}^{-1}(k) \ , \quad (2.23a)$$

with the subscripts $L$ and $T$ denoting longitudinal and transverse components, respectively, and the transverse inverse susceptibility has both diagonal and off-diagonal pieces. Notice that the latter are nonzero only in a phase with a nonvanishing magnetization (or in the presence of an external magnetic field). For later reference we give the explicit expressions that result from our Gaussian approximation for the magnetic fluctuations. With Eqs. (2.15), (2.13), and (2.12b) we obtain

$$\chi_{L}^{-1}(k) = 1 + \Gamma_1 \int_q \left[ \tilde{G}_{11}(q) \tilde{G}_{11}(q - k) + \tilde{G}_{22}(q) \tilde{G}_{22}(q - k) - \tilde{G}_{31}(q) \tilde{G}_{13}(q - k) \right] \ , \quad (2.23b)$$

$$\chi_{T,+}^{-1}(k) = 1 + \Gamma_1 \int_q \tilde{G}_{11}(q) \left[ \tilde{G}_{22}(q + k) + \tilde{G}_{22}(q - k) \right] \ , \quad (2.23c)$$

$$\chi_{T,-}^{-1}(k) = i \Gamma_1 \int_q \tilde{G}_{11}(q) \left[ \tilde{G}_{22}(q + k) - \tilde{G}_{22}(q - k) \right] \ . \quad (2.23d)$$

where $\int_q = T \sum_n \int d\mathbf{q}/(2\pi)^d$. More explicit expressions for these quantities will be discussed in Sec. III D below.

### III. SADDLE-POINT THEORY FOR SUPERCONDUCTIVITY DUE TO MAGNETIC FLUCTUATIONS

#### A. Saddle-point solution

The next step is to solve the field theory defined by the Eqs. (2.17) in a saddle-point approximation. Notice that there is only one fluctuating field left, namely $G$. For fixed $\langle \Lambda \rangle$ we therefore have only one saddle-point condition,

$$\delta A/\delta G|_{sp} = 0 \ . \quad (3.1)$$

and we will be looking for a translationally invariant saddle-point solution, $G_{sp}(x, y) = G_{sp}(x - y) \approx \langle G \rangle(x - y)$ (cf., however, the remarks in Sec. V below). Equation (3.1) yields an expression for the normal self energies, Eq. (2.13e),

$$\Sigma_1(x - y) = -\langle \delta \Lambda_{11} \rangle \Lambda(x - y)$$

$$-\Gamma_1 \delta(x - y) \sum_{\sigma} G_{\sigma}(x - y) \ , \quad (3.2a)$$

$$\Sigma_4(x - y) = -\langle \delta \Lambda_{22} \rangle \Lambda(x - y)$$

$$-\Gamma_1 \delta(x - y) \sum_{\sigma} G_{\sigma}(x - y) \ , \quad (3.2b)$$

with $G_\sigma$ the normal Green functions, Eq. (2.13d). For the anomalous self energies, Eq. (2.13c), on obtains

$$\Delta^+(x - y) = -\langle \delta \Lambda_{13} \rangle \Lambda(x - y) \ , \quad (3.2c)$$

$$\Delta(x - y) = -\langle \delta \Lambda_{31} \rangle \Lambda(x - y) \ , \quad (3.2d)$$

Here we have defined a ‘$\Lambda$-ensemble’ that is characterized by a statistical weight given by the integrand in the last term in Eq. (2.17b), with $G_{sp} \approx \langle G \rangle$,

$$\langle \ldots \rangle_\Lambda = \int D[\delta \Lambda] \ldots e^{B[\delta \Lambda]} / \int D[\delta \Lambda] e^{B[\delta \Lambda]} \ , \quad (3.3a)$$

where

$$B[\delta \Lambda] = \text{Tr} (G_{sp} \delta \Lambda) + \frac{1}{2} \text{Tr} \ln \left[ \frac{G_{sp} - 1}{\langle G \rangle} \right] \ . \quad (3.3b)$$

In order to make the Eqs. (3.2) more explicit, we now calculate $\langle \delta \Lambda \rangle_\Lambda$ in Gaussian approximation. Expanding the ‘action’ $B[\delta \Lambda]$ to second order in $\delta \Lambda$, and performing the Gaussian integral, we obtain

$$\langle \delta \Lambda \rangle_\Lambda = 2 \tilde{G}^{-1} \left( G_{sp} - \frac{1}{2} \tilde{G} \right) \tilde{G}^{-1} \ . \quad (3.4)$$

Here we have neglected the contribution of the last term in $B[\delta \Lambda]$, since it represents corrections to mean-field superconductivity due to magnetic fluctuations. In agreement with our general philosophy of keeping only leading fluctuation effects, we drop these terms. Inserted in Eqs. (3.2), Eq. (3.4) provides a relation between the self energies and the saddle-point Green function. The free energy in saddle-point approximation, $F_{sp} = -T \ln Z_{sp}$ now reads

$$F_{sp} = \frac{1}{2} V T \Gamma_m^2 - 2 TT_s \int dx \left( \sum_\sigma G_{\sigma}(x, x) \right)^2$$

$$-TT_s \langle \Lambda \rangle G_{sp} - \frac{T}{2} \text{Tr} \ln \tilde{G}^{-1} + \frac{T}{2} \text{Tr} \ln \chi^{-1}$$

$$-T \ln \int D[\delta \Lambda] e^{B[\delta \Lambda]} \ , \quad (3.5)$$

and it depends on only one independent parameter, e.g., $\langle \Lambda \rangle$.  


B. Minimization of the free energy, and strong-coupling equations

As the last step in our formal development we now require that the physical value of $\langle \Lambda \rangle$ minimizes the saddle-point free energy. The condition

$$\frac{\delta F_{sp}}{\delta \langle \Lambda \rangle} = 0 \quad ,$$

leads to

$$\langle \delta \Lambda \rangle = \tilde{G}^{-1} \left( \chi \left| \frac{\delta}{\delta \langle \Lambda \rangle} \chi^{-1} \right| \right) \tilde{G}^{-1} \quad .$$

(3.7a)

Here we have used Eq. (3.4), and a symbolic scalar product notation

$$\left( \chi \left| \frac{\delta}{\delta \langle \Lambda \rangle} \chi^{-1} \right| \right) = \int dx \, dy \sum_{ij} \chi_{ij}(x-y) \frac{\delta}{\delta \langle \Lambda \rangle} \chi_{ji}^{-1}(y-x) \quad .$$

(3.7b)

Notice that $\langle \delta \Lambda \rangle$, $\tilde{G}^{-1}$, and $\langle \Lambda \rangle$ are still $4 \times 4$ matrices in spinor space (as well as functions of space-time), while $\chi$ is a scalar in spinor space. Again in agreement with our general philosophy, we have neglected the contribution of the last term in Eq. (3.5), which would lead to superconducting order-parameter fluctuation corrections to mean-field superconductivity. Equations (3.2) and Eq. (3.7a) now form a closed set of equations for the various self energies. Due to their matrix nature, they are complicated, and for our purposes we restrict ourselves to the linearized gap equations. That is, we evaluate Eqs. (3.7a) and (3.2) only to linear order in the gap parameter $\Delta$. This procedure is sufficient for deriving the superconducting phase boundary in both the paramagnetic and the ferromagnetic phases. We find

$$\Delta(k) = 2\Gamma_i \int_q \chi_L(k-q) \Delta(q) \left| g_i(q) \right|^2 \quad ,$$

(3.8a)

$$\Sigma_i(k) = -\Gamma_s n + 2\Gamma_i \int_q \left[ \chi_L(k-q) g_i(q) + 2\chi_{T,+}(k-q) g_i(q) \right] \quad ,$$

(3.8b)

$$\Sigma_i(k) = -\Gamma_s n + 2\Gamma_i \int_q \left[ \chi_L(k-q) g_i(q) + 2\chi_{T,+}(k-q) g_i(q) \right] \quad ,$$

(3.8c)

with $g_i$ and $g_i$ given by Eqs. (2.18). A comparison with Eqs. (A10) shows that Eqs. (3.8) are an obvious generalization of the conventional Eliashberg equations to the case of superconductivity mediated by spin fluctuations, with the magnetic susceptibility essentially replacing the phonon propagator, and the inequivalency of the two spin projections taken into account.

The susceptibilities that appear in the Eqs. (3.8) are given explicitly by Eqs. (2.23). It is obvious from our derivation, however, that these explicit expressions are just a particular approximation for the exact susceptibilities, and that the latter will appear in the theory if one takes loop corrections into account. We will therefore consider the $\chi$ as physical susceptibilities that we model in the next subsections, using general physical arguments. We also note that in the paramagnetic phase, $\chi_L = \chi_T$. However, in the ferromagnetic phase the two susceptibilities are fundamentally different, as we will discuss below.

Finally, we have yet to discuss the magnetization as a function of the control parameters that drive the magnetic transition. This is obtained by minimizing the free energy with respect to $m$, in order to find the physical values of the magnetization,

$$\frac{\delta F_{sp}}{\delta m} = 0 \quad .$$

(3.9)

A detailed theory of the itinerant ferromagnetic phase transition at low temperatures has been given elsewhere. In this paper, where we are interested in the superconducting transition, we restrict ourselves to treating the magnetization in mean-field theory. That is, we neglect the fluctuation effects that are represented by the last two terms on the right-hand side of Eq. (3.5). We then obtain the magnetic mean-field equation of state in the form

$$m = \frac{T}{2V} \mathrm{Tr} \left( \gamma_3 \tilde{G} \right) \quad .$$

(3.10)

We will discuss this expression explicitly in Sec. III D 2 below.

C. Gap equation for p-wave superconductivity

So far the symmetry of the superconducting order parameter has not been specified. To do so, we expand $\Delta(k)$ in Legendre polynomials,

$$\Delta(k) = \Delta(\iota \omega_n, |k|, \cos \theta) = \sum_{l=0}^{\infty} \Delta^{(l)}(\iota \omega_n, |k|) P_l(\cos \theta) \quad ,$$

(3.11)

where $\theta$ is the azimuthal angle on the spherical Fermi surface. Since $\Delta(k) = -\Delta(-k)$, the $\Delta^{(l)}$ are odd (even) functions of the frequency for even (odd) values of $l$. If we insert this expansion in Eq. (3.8a), we see that in the static limit only odd values of $l$ contribute. Of the contributions with odd angular momenta, $l = 1$ will be the strongest for phase space reasons. We thus specialize to the p-wave case, and use the notation $\Delta^{(1)} = \Delta$. We further split off the frequency dependence of the normal self energy and write, as in Eliashberg theory,

$$G_{n,\uparrow}^{-1}(q) = \iota \omega_n Z_{\uparrow}(q) - \hat{\xi}_{q,\uparrow} \quad ,$$

(3.12a)

with
\[ Z_1(q) = 1 - \left| \Sigma_1(i\omega_n, q) - \Sigma_1(0, q) \right| / i\omega_n \quad , \] (3.12b)

and
\[ \tilde{\xi}_{q, \uparrow} = \xi_{q, \uparrow} + \Sigma_1(0, q) \quad . \] (3.12c)

The normal state Green functions in Eq. (3.8a) then pin the momentum to the up-spin Fermi surface, and we obtain for \( \Delta^{(1)}(i\omega_n, k_F^\uparrow) \equiv \Delta(i\omega_n) \) the equation
\[
\Delta(i\omega_n) = \pi T \sum_{i\omega_m} d_L^1(i\omega_n - i\omega_m) \Delta(i\omega_m) / |\tilde{\omega}_m| \quad ,
\] (3.13a)

where \( i\tilde{\omega}_m = i\omega_m Z_1(i\omega_m) \). In the same way we obtain an equation for \( Z_1 \),
\[
\omega_n \left[ 1 - Z_1(i\omega_n) \right] = -\pi T \sum_{i\omega_m} \text{sgn} (\omega_m) \left[ d_L^0(i\omega_n - i\omega_m) + 2d_T^0(i\omega_n - i\omega_m) \right] \quad .
\] (3.13b)

The kernels in these integral equations are given by
\[
d_L^1(i\Omega_n) = \Gamma_L N_F^\uparrow \int_0^2 dx x (1 - x^2/2) \chi_L(xk_F^\uparrow, i\Omega_n) \quad ,
\] (3.14a)
\[
d_L^0(i\Omega_n) = \Gamma_L N_F^\uparrow \int_0^2 dx x \chi_L(xk_F^\uparrow, i\Omega_n) \quad ,
\] (3.14b)
\[
d_T^0(i\Omega_n) = \Gamma_T N_F^\uparrow \int_{x=0}^{1+} dx x \chi_T(xk_F^\uparrow, i\Omega_n) \quad .
\] (3.14c)

Here \( N_F^\uparrow \) and \( N_F^\uparrow \) are the density of states per spin at the Fermi level in the paramagnetic phase, and the density of states at the Fermi level for up-spin electrons in the ferromagnetic phase, respectively, and \( x_{\pm} = 1 \pm k_F^\uparrow / k_F^\uparrow \).

D. Explicit model for the longitudinal spin susceptibility

For an explicit evaluation of our strong-coupling theory we need the longitudinal and transverse magnetic susceptibilities as input. For the determination of the superconducting \( T_c \), or the phase diagram, it suffices to know these susceptibilities in the normal-metal state. This is equivalent to the need of the phonon spectrum as input in ordinary Eliashberg theory. As in the latter case, one would ideally want to use experimental data as input, but this information is not available for the materials of interest. We therefore resort to theoretical modeling, with the aim of developing theoretical expressions that correctly reflect the relevant qualitative physical effects. Since the latter are different in the paramagnetic and ferromagnetic phases, respectively, we will treat these two cases separately.

1. Paramagnetic phase

In the paramagnetic phase, spin rotational invariance implies \( \chi_L = \chi_T \equiv \chi \). Equation (2.23b) gives an explicit expression which, after substituting the Green functions \( G \) and using Eqs. (3.8), results in an integral equation for \( \chi_L \). This is the result of a particular approximation we made in Sec. II, that integrated out the magnetic fluctuations in a Gaussian approximation. This particular procedure results in a fairly elaborate treatment of the denominator of \( \chi \), which takes into account the self energies of the Green functions, while the numerator is approximated by a constant, neglecting its wavenumber and frequency dependence. A slightly different Gaussian approximation was derived in Ref. 30 by expanding about a Stoner saddle point within a theory focussing purely on magnetism. This approximation neglects the self energy \( \Sigma_\sigma \) in the Green function \( g_\sigma \), Eq. (2.18a), but keeps the wavenumber and frequency dependence of the numerator of \( \chi \) at the same level as that of the denominator. In the static, long-wavelength limit, and not too far from the magnetic transition, both of these zero-loop approximations reduce to the Landau expression
\[
\chi_L^{(0)}(k, i0) = \chi_T^{(0)}(k, i0) = \frac{1 - t}{t + b(k/2k_F)^2} \quad .
\] (3.15)

Here \( t = 1 - 2N_F \Gamma_L \) is the dimensionless distance from the magnetic critical point in mean-field approximation. The value of the coefficient \( b \) of course depends on the model. If one neglects the self energy in the Green function in Eq. (2.23b), or in the approximation of Ref. 30, \( b = 1/3 \), but in general there is no reason to prefer this value of \( b \) over any other value of order unity.

We have used both the model from Ref. 30, and the Landau model, Eq. (3.15), to calculate the coupling constants \( d_{L,T}^{0,1} \), Eqs. (3.14), and the resulting superconducting \( T_c \) in a McMillan approximation, see Sec. IV below. We have found that for reasonable values of \( b \) the results are qualitatively the same. In what follows, we will use the Landau approximation, Eq. (3.15), with \( b \) as a free parameter. We emphasize that this approximation correctly reflects the long-wavelength behavior of the magnetic susceptibility in the paramagnetic phase. The free parameter \( b \) just reflects deviations from the free electron model that one would expect in any real material.

2. Ferromagnetic phase, zero-loop order

In the ferromagnetic phase, any reasonable zero-loop approximation for the susceptibilities yields again a Landau expression for the longitudinal static susceptibility in the long-wavelength limit. We evaluate Eq. (2.23b), again neglecting the self energy, and find
\[
\chi_L^{(0)}(k, i0) = \frac{1 - t}{a_L|t| + b_L(k/2k_F)^2} \quad ,
\] (3.16)
where we have introduced two free parameters, $a_L$ and $b_L$. In ordinary Landau theory, $a_L = 2$. However, if one keeps the electron number density fixed, as is the case experimentally, rather than the chemical potential, then in general $a_L \neq 2$. From the point of view of Landau theory, this can be understood as follows. Since $t = 1 - 2N(\mu)\Gamma$, and the chemical potential $\mu$ depends on the magnetization, the coefficient of the magnetization squared in the Landau free energy depends itself on the magnetization, which causes the deviation of the prefactor of $|t|$ from the usual value of 2.

The actual value of $a_L$ is model dependent, and depends in particular on the equation of state, since the latter determines the relation between $|t|$ and the magnetization. If we again neglect the self energy in the Green function on the right-hand side of Eq. (3.10), the latter yields

$$6\pi^2\delta = \Gamma_1(2m_e)^{3/2} \left[ (\mu + \delta)^{3/2} - (\mu - \delta)^{3/2} \right] . \quad (3.17a)$$

This needs to be supplemented by the condition that the electron number density $n_e$ is fixed, which is

$$6\pi^2n_e = (2m_e)^{3/2} \left[ (\mu + \delta)^{3/2} + (\mu - \delta)^{3/2} \right] . \quad (3.17b)$$

This determines the chemical potential $\mu$ as a function of the magnetization. Equations (3.17) together determine the relation between $\delta$ and $|t|$.

From Equations (3.17) and Eq. (2.23b) one obtains $a_L = 1/2$. Different Gaussian approximations for $\chi$ yield different values. For instance, Ref. 30 in conjunction with Eqs. (3.17), and putting $\Gamma_s = \Gamma_1$, yields $a_L = 5/4$, and so does the RPA expression used by Fay and Appel. In order to be able to compare with the latter’s calculation of $\Gamma_s$, we adopt $a_L = 5/4$ in what follows. Other values of $O(1)$ do not lead to qualitatively different results.

In the transverse channel, a Ward identity guarantees that $\chi$ diverges in the limit of small wavenumbers and frequencies. This reflects the existence of magnetic Goldstone modes or magnons. Our mean-field equation of state, Eq. (3.10), or Eq. (3.17a), is consistent with our Gaussian treatment of the magnetic fluctuations, Eqs. (2.23), in the sense that they form a conserving approximation that correctly reflects the Goldstone modes. Inverting the susceptibility tensor given by Eqs. (2.23), dropping the self energy in the Green functions, and using Eq. (3.17a), we find for $\chi_{11}(k) \equiv \chi_{T,+}(k)$ in Gaussian approximation,\textsuperscript{31}

$$\chi^{(0)}_{T,+}(k,i\Omega_n) = \frac{-i\delta/4eF}{1 - t} \left( \frac{1}{i\Omega_n/4eF + (\delta/2eF)b_T(k/2k_F)^2} - \frac{1}{i\Omega_n/4eF - (\delta/2eF)b_T(k/2k_F)^2} \right) , \quad (3.18a)$$

and for $\chi_{12}(k) \equiv \chi_{T,-}(k)$,

$$\chi^{(0)}_{T,-}(k,i\Omega_n) = \frac{i\delta/4eF}{1 - t} \left( \frac{1}{i\Omega_n/4eF + (\delta/2eF)b_T(k/2k_F)^2} + \frac{1}{i\Omega_n/4eF - (\delta/2eF)b_T(k/2k_F)^2} \right) . \quad (3.18b)$$

Here we give the dynamical susceptibility, since we will need it later. In our approximation, $b_T = 1/3$, but in general it is some parameter of order unity, and in general it will be different from $b_L$ in the ferromagnetic phase.

3. Ferromagnetic phase, one-loop order

In contrast to the situation in the paramagnetic phase, in the ferromagnetic phase the zero-loop approximation for the longitudinal susceptibility, Eq. (3.16) is not qualitatively correct. The reason for this is as follows. In a Heisenberg ferromagnet, the transverse spin waves or magnons couple to the longitudinal spin fluctuations, and hence contribute to the longitudinal susceptibility $\chi_L$.\textsuperscript{32} At nonzero temperature, and for dimensions $d < 4$, this coupling actually leads to a magnetic susceptibility that diverges everywhere on the coexistence curve.\textsuperscript{33} More generally, and more importantly for our purposes, it causes the zero-loop or Landau expression for the longitudinal susceptibility, Eq. (3.16), to be qualitatively incorrect in the ferromagnetic phase. Going to one-loop order, on the other hand, produces a functional form of $\chi_L$ that is exact at small wavenumbers. This one-loop contribution can be calculated in a variety of ways. Suppose we expand the action, Eq. (2.12a), in powers of $M$, to quartic order. Then each of the four $M$ fields in the quartic vertex can represent either an average magnetization, or a magnetization fluctuation, see Eq. (2.13a). This leads to a one-loop contribution to the two-point $M$-vertex, or the inverse magnetic susceptibility, that has the structure shown in Fig. 2. Equivalently, an expansion in powers of $\delta M$ leads to a cubic vertex, and to a diagram with the same structure as in Fig. 2(b). Alternatively, one can use a nonlinear sigma-model to derive the same contribution, as was done in Ref. 21. Either way one obtains for the one-loop contribution to the longitudinal susceptibility\textsuperscript{31}

$$\chi^{(1)}_L(k) = \frac{2\Gamma_i}{\delta^2} \chi^{(0)}_L(k) \int_q \chi^{(0)}_{T,+}(k - q) \chi^{(0)}_{T,+}(q)$$

FIG. 2. (a) A contribution to the quartic magnetization vertex, with the dashed line representing an average magnetization, and the solid line a magnetization fluctuation. (b) A resulting contribution to the magnetic susceptibility.
Here $\chi^{(0)}_{T,\pm}$ are the zero-loop transverse susceptibility tensor elements, Eqs. (3.18), and $\chi^{(0)}_{L}$ is the zero-loop longitudinal susceptibility, Eq. (3.16). The integral in Eq. (3.19) leads to $\chi^{(0)}_{L}(k, i0) \propto |k|^{d-4}$ for the leading wavenumber dependence of $\chi^{(0)}_{L}$ in $d < 4$, which is the exact leading behavior for small wavenumbers.\(^{34}\) We note that there is no analog of the diagram shown in Fig. 2(b) in the paramagnetic phase, as there is no cubic magnetization vertex. It is therefore reasonable to use the zero-loop expression for the longitudinal susceptibility in the paramagnetic phase, and the one-loop expression given by Eq. (3.19) in the ferromagnetic one.

We now have explicit model expressions, in terms of integrals, for the magnetic susceptibilities that we need as input for our theory of superconductivity. In the paramagnetic phase, the simple expression given in Eq. (3.15) suffices. In the ferromagnetic phase, we will use Eq. (3.18a) for the transverse susceptibility, and $\chi^{(0)}_{L}(k, i0) + \chi^{(1)}_{L}(k, i0)$, Eqs. (3.16) and (3.19), with $a_L = 5/4$, for the longitudinal one. This model contains two parameters of $O(1)$, viz. $b_L$ and $b_T$, and it reflects the exact functional form of the susceptibilities in the long-wavelength limit.

IV. MCMILLAN-TYPE SOLUTION OF THE GAP EQUATION

The strong coupling equations, Eqs. (3.13), have the same structure as Eliashberg theory for phonon-induced superconductivity. There are well-developed algorithms for solving these equations. It is well known that any attempt to quantitatively calculate the critical temperature requires a numerical solution of the equations, using detailed information about the kernels as input. In the case of superconductivity induced by magnetic fluctuations, the latter information is not available, and we will have to use the schematic model for the susceptibility that was developed in the last section. We will therefore calculate the superconducting $T_c$ by means of a simple expression of McMillan or Allen-Dynes type.\(^{36}\) Since we are interested in the structure of the phase diagram, and in particular in the relative values of the superconducting $T_c$ in the paramagnetic and ferromagnetic phases, respectively, these approximations will give the desired information provided we treat the two phases on equal footing. As we have explained in the last section, this is the case if we use the susceptibility model represented by Eqs. (3.15), (3.18a), and (3.19), respectively. For a rough order-of-magnitude estimate of the absolute value of $T_c$, see Sec. VA below.

A. $T_c$ formula

McMillan’s approximate solution of the linearized strong-coupling equations results in a critical temperature

$$T_c = T_0(t) e^{-(1+d_0^2+2d_1^2)/d_L^2}. \quad (4.1)$$

Here $d_0^2$ and $d_1^2$ are the zero-frequency values of the functions defined in Eqs. (3.14), and $T_0(t)$ is a temperature scale that we will define and discuss below. $d_F^2$, which involves an integral over a zero-loop susceptibility, is easily calculated analytically. For $d_1^2$, we proceed as follows. The momentum integral of the convolution in Eq. (3.19) can be done analytically. For the external legs in that expression, we replace Eq. (3.16) by a step function $\Theta(|k| - 2k_F x_c)$, with $x_c = \sqrt{5/4}$. The momentum integration in Eq. (3.14b) can then also be performed analytically. This leaves us with the frequency summation in Eq. (3.19), which needs to be done numerically. We obtain

$$d_1^2 = \frac{8\pi}{b_T^2} \left( \frac{1}{\delta} \right)^2 \frac{1}{\Delta} \frac{T_c}{\delta} \left[ \pi y_c + 2 \sum_{n=1}^N f(\Omega_m, y_c) \right], \quad (4.2a)$$

Here $\delta_+ = \sqrt{1 + \delta^2/\xi_F}$, $y_c = \text{Min}(1, x_c)$, and

$$f(\Omega_m, y) = \frac{\pi}{2} + \ln 2 - \frac{|a_m - y|}{a_m} \arctan \left( \frac{a_m}{|a_m - y|} \right) - \frac{1}{2} \ln \left( 1 + \frac{(a_m - y)^2}{a_m^2} \right) - \frac{1}{2} \ln \left( 1 + \frac{(a_m + y)^2}{a_m^2} \right) + \Theta(y - a_m) \pi \frac{y - a_m}{a_m},$$

(4.2b)

with $a_m = \sqrt{\Omega_m/b_T \delta}$. $N$ in Eq. (4.2a) is given by $N = \Omega_0/2\pi T_c$, with $\Omega_0$ a frequency cutoff that is on the order of $2b_T \delta$. Equation (3.14a) has an additional factor $1 - x^2/2$ in the integrand, which we approximately take into account, in the same spirit as our approximation for the external legs, by writing

$$d_1^2 = \frac{8\pi}{b_T^2} \left( \frac{1}{\delta} \right)^2 \frac{1}{\Delta} \frac{T_c}{\delta} \left[ \pi y_c^1 + 2 \sum_{n=1}^N f(\Omega_m, y_c^1) \right], \quad (4.2c)$$

with $y_c^1 = \text{Min}(x_c, 1/2\sqrt{2})$.

Finally, we need to specify the temperature scale $T_0(t)$. Following Refs. 6, 35, we use the prefactor of $|t|$ in Eqs. (3.15) and (3.16) as a rough measure of the magnetic excitation energy,

$$T_0(t) = T_0 \left[ \Theta(t) t + \Theta(-t) 5|t|/4 \right], \quad (4.3)$$

with $T_0$ a microscopic temperature scale that is related to the Fermi temperature, for a free electron model, or
a band width, for a band electron model. This choice of $T_0(t)$ qualitatively reflects the suppression of the superconducting $T_c$ near the ferromagnetic transition due to effective mass effects.\cite{37,6,18,20}

Equations (4.1) - (4.3) now form a closed set of transcendental equations for the superconducting $T_c$, which we have solved by numerical iteration. We discuss the results in the next subsection.

**B. Results**

We have solved the $T_c$ equations (4.1) - (4.3) for various values of the parameters, and with various implementations of our approximations. Specifically, we have also used RPA expressions for the magnetic susceptibilities\cite{35,30} instead of the simple Landau expressions given above. While this makes the calculations substantially more complicated, the results were qualitatively the same. We therefore only show results obtained by using the Landau model. We have also modified the upper frequency cutoff, and again have found only quantitative effects. Since $T_c$ depends exponentially on the coupling constants, the quantitative effect of changing the cutoff is substantial. However, the relative values of $T_c$ in the paramagnetic and ferromagnetic phases, respectively, depend only very weakly on the cutoff. We therefore use $\Omega_0 = 2b_T\delta$ throughout. Finally, in the ferromagnetic phase it is illustrative to relate the superconducting $T_c$ to the magnetization in addition to $t$, so we need again the magnetic equation of state. Within our mean-field model for the magnetization, Eqs. (3.17) it is useful to introduce a parameter $\eta$ via\cite{35}

$$t = 1 - (1 + 3\eta^2)^{1/3}/(1 + \eta^2/3) \quad . \quad (4.4a)$$

The magnetization, in units of $\mu_B n$ with $\mu_B$ the Bohr magneton and $n$ the electron number density, is related to $\eta$ by\cite{35}

$$m/\mu_B n_e = 3\eta(1 + \eta^2/3)/(1 + 3\eta^2) \quad . \quad (4.4b)$$

Finally, we take into account theoretical\cite{2} and experimental\cite{4} evidence that the quantum ferromagnetic transition in clean itinerant electron systems is of first order at least in some systems. This is because fluctuation effects first appear at one-loop order lead in general to a negative term in the Landau free energy, i.e., one has a fluctuation-induced first order transition.\cite{2,3} A very simple way to take this into account in the present context is to just impose a minimum value $t_{\min}$ on $|t|$, which corresponds to a minimum value of the magnetization in the ferromagnetic phase.

Figure 3 shows the superconducting $T_c$ as a function of $t$ with $t_{\min} = 2.66 \times 10^{-4}$, which corresponds to a discontinuity in the magnetization equal to 6% of the saturation value, $b = b_L = 0.23$, and $b_T = 0.25$. Also shown is the magnetization $m$ in the ferromagnetic phase in units of the saturation magnetization $\mu_B n_e$. $T_c$ is measured in units of the characteristic temperature $T_0$ that is given by either the Fermi temperature or the band width, depending on the model considered. The solid curves show the result in the paramagnetic and ferromagnetic phases, respectively, with the former scaled by a factor of 133.3 (right-hand scale) compared to the latter (left-hand scale). The dotted curve in the ferromagnetic phase (also scaled by a factor of 133.3, right-hand scale) represents the result that is obtained upon neglecting the mode-mode coupling effect. The zero-loop results in the paramagnetic and ferromagnetic phases are very close to one another, and also very close to the result of Fay and Appel.\cite{8} Upon inclusion of the mode-mode coupling effect at one-loop order, however, the maximum value of $T_c$ in the ferromagnetic phase is enhanced by a factor of more than 100. As we have explained above, there is no analogous contribution in the paramagnetic phase, so this comparison is physically sensible. We also reiterate that one should not take the absolute $T_c$ values very seriously. However, the relative comparison we expect to be reliable. We thus find a pronounced asymmetry between the paramagnetic and ferromagnetic phases. In the case of UGe$_2$, where the observed maximum $T_c$ in the ferromagnetic phase is about 500 mK, we thus predict that in the paramagnetic phase no superconductivity should be expected above temperatures of at most about 5 mK. This is in agreement with the experimentally observed absence of superconductivity in the paramagnetic phase. The second maximum in the Gaussian $T_c$ in the FM phase we will discuss below. Fig. 4 shows the result with the

![Fig. 3. Superconducting $T_c$ as a function of the distance from the critical point $t$, and the magnetization $m$. The solid curve in the PM phase and the dotted curve in the FM phase (right scale) show the zero-loop $T_c$ scaled by a factor of 133.3, while the solid curve in the FM phase (left scale) represents the one-loop result. See the text for parameter values and further explanation.](image-url)
sponding to a magnetization discontinuity equal to 29% of the saturation value. Figures 3 and 4 correspond to systems with a magnetic phase transition that is weakly and strongly first order, respectively.

We now discuss the effect of varying the parameters $b_L$ and $b_T$. With $b = b_L = b_T = 0.5$ and $t_{\text{min}} = 1.7 \times 10^{-3}$ we obtain the result shown in Fig. 5. $b = b_L = b_T = 1.0$ and $t_{\text{min}} = 0$ yields Fig. 6. The left and right-hand scales differ by a factor of 50 in both figures. While the (unphysical) zero-loop result in the ferromagnetic phase is very sensitive to the parameter values, we see that the enhancement of the (physical) one-loop result over the $T_c$ in the paramagnetic phase is rather robust. However, the position of the maximum of $T_c$ changes compared to Fig. 3; in Fig. 6 it occurs at the point where the magnetization reaches its saturation value, and in Fig. 5 there are two maxima of about equal height. The reason is as follows. As one approaches the magnetization saturation point from low magnetization values, the transverse coupling constant $d_{ll}^T$ vanishes, and remains zero in the saturated region. Effectively, the Heisenberg system turns into the Ising model discussed in Ref. 19. If the longitudinal coupling constant $d_L^T$ still has a substantial value at that point, then this leads to an increase in $T_c$. This is a very strong effect in the zero-loop contribution, see Figs. 5 and 6, and the effect qualitatively survives in the one-loop result. If, however, $d_L^T$ is already very small, then $d_{ll}^T$ going to zero has only a very small effect on $T_c$, as is the case in Figs. 3 and 4, although the effect is still visible in the zero-loop result. Which of these two cases is realized depends on the parameter values. This observation provides an explanation for the difference in the observed phase diagrams in UGe$_2$ and ZrZn$_2$, respectively. UGe$_2$ reaches its saturation magnetization relatively close to the magnetic phase transition, at a pressure well above ambient pressure. ZrZn$_2$, on the other hand, has not nearly reached its saturation magnetization at ambient pressure. The above discussion suggests that, as a consequence, the superconducting $T_c$ in ZrZn$_2$ is likely to monotonically increase from its onset as one goes deeper into the ferromagnetic phase by decreasing the pressure, while in UGe$_2$ one expects the superconducting $T_c$ to peak close to the magnetic phase boundary, leading effectively only to a pocket of superconductivity, as shown schematically in Fig. 1(a). This is indeed what is observed in the experiments, and the present model provides a qualitative explanation. Consistent with the experimental observation that the magnetic transition in ZrZn$_2$ is at most very weakly first order, we have used $t_{\text{min}} = 0$ in Fig. 6.

![FIG. 4. Same as Fig. 3, but with a different value of $t_{\text{min}}$, see the text.](image)

We now discuss the effect of varying the parameters $b_L$ and $b_T$. With $b = b_L = b_T = 0.5$ and $t_{\text{min}} = 1.7 \times 10^{-3}$ we obtain the result shown in Fig. 5. $b = b_L = b_T = 1.0$ and $t_{\text{min}} = 0$ yields Fig. 6. The left and right-hand scales differ by a factor of 50 in both figures. While the (unphysical) zero-loop result in the ferromagnetic phase is very sensitive to the parameter values, we see that the enhancement of the (physical) one-loop result over the $T_c$ in the paramagnetic phase is rather robust. However, the position of the maximum of $T_c$ changes compared to Fig. 3; in Fig. 6 it occurs at the point where the magnetization reaches its saturation value, and in Fig. 5 there are two maxima of about equal height. The reason is as follows. As one approaches the magnetization saturation point from low magnetization values, the transverse coupling constant $d_{ll}^T$ vanishes, and remains zero in the saturated region. Effectively, the Heisenberg system turns into the Ising model discussed in Ref. 19. If the longitudinal coupling constant $d_L^T$ still has a substantial value at that point, then this leads to an increase in $T_c$. This is a very strong effect in the zero-loop contribution, see Figs. 5 and 6, and the effect qualitatively survives in the one-loop result. If, however, $d_L^T$ is already very small, then $d_{ll}^T$ going to zero has only a very small effect on $T_c$, as is the case in Figs. 3 and 4, although the effect is still visible in the zero-loop result. Which of these two cases is realized depends on the parameter values. This observation provides an explanation for the difference in the observed phase diagrams in UGe$_2$ and ZrZn$_2$, respectively. UGe$_2$ reaches its saturation magnetization relatively close to the magnetic phase transition, at a pressure well above ambient pressure. ZrZn$_2$, on the other hand, has not nearly reached its saturation magnetization at ambient pressure. The above discussion suggests that, as a consequence, the superconducting $T_c$ in ZrZn$_2$ is likely to monotonically increase from its onset as one goes deeper into the ferromagnetic phase by decreasing the pressure, while in UGe$_2$ one expects the superconducting $T_c$ to peak close to the magnetic phase boundary, leading effectively only to a pocket of superconductivity, as shown schematically in Fig. 1(a). This is indeed what is observed in the experiments, and the present model provides a qualitative explanation. Consistent with the experimental observation that the magnetic transition in ZrZn$_2$ is at most very weakly first order, we have used $t_{\text{min}} = 0$ in Fig. 6.

![FIG. 5. Same as Fig. 3, but for different parameter values. See the text for further explanation.](image)

![FIG. 6. Same as Fig. 3, but for different parameter values. See the text for further explanation.](image)
V. DISCUSSION

We conclude with a summary of our results, and then briefly discuss several open questions.

A. Summary, and additional remarks

We have made two contributions with this paper. The first one is general in nature, and consists in a field-theoretic framework for deriving strong-coupling equations for superconductivity, regardless of the mechanism that causes superconductivity. This framework is a natural extension and generalization of Landau’s theory of phase transitions, and it has several advantages, both philosophical and practical ones, over the many-body diagrammatic formalism that is usually used to describe superconductivity. One advantage is the fact that the field theory allows for introducing several order parameters simultaneously in a very natural and transparent way. In the present case we have used this to obtain a theory of the interplay between ferromagnetism and spin-triplet, p-wave superconductivity that deals with both ordering phenomena on equal footing, and within one unified framework. No input from other theories has been necessary. We have restricted our description of the superconductivity to the mean-field level, but it is obvious how to include fluctuations within this framework.

In this context we note that for the pairing mechanism we have assumed, or for any purely electronic mechanism, there is no separation of energy scales, and hence no analog of Migdal’s theorem. As a result, Eliashberg theory cannot be expected to produce quantitative results, in constrast to the case of phonon-induced superconductivity. A rough order-of-magnitude estimate of \( T_c \) may nevertheless be of interest. Fay and Appel\(^6\) have given detailed estimates of the zero-loop \( T_c \) for ZrZn\(_2\). These ranged from 100 \( \mu \)K to 500 mK, depending on parameter values that are not well known. With a mid-range value of 5 mK, which is consistent with the observed absence of superconductivity in the paramagnetic phase, our enhancement mechanism yields a maximum \( T_c \) in the ferromagnetic phase of about 500 mK, which is the same order of magnitude as observed experimentally. Regardless of how seriously one is willing to take such estimates, it is important to note that any corrections due to the inapplicability of Migdal’s theorem are expected to equally affect both the paramagnetic and the ferromagnetic phases. They therefore will not spoil the relative comparison of the two phases that has been our main objective.

We have also restricted ourselves to discussing the linearized gap equation, but our formalism is more general, and includes the feedback of a nonvanishing superconducting order parameter on the magnetic properties. Since the magnetic properties are responsible for the superconductivity in the first place, this feedback will have to be included self-consistently in a theory of the superconducting phase. These points will be pursued in future publications. We further note that we have assumed the superconducting transition to be of second order. First order scenarios are possible, see Ref. 38.

Our second contribution is specific, and consists of a calculation of the phase diagram for p-wave, spin-triplet superconductivity that is driven by ferromagnetic spin fluctuations and coexists with ferromagnetism. The most striking qualitative feature of the result is the pronounced asymmetry of the superconducting critical temperature in the ferromagnetic and paramagnetic phases, respectively, which results from the contribution of the magnons in the ferromagnetic phase to the pairing mechanism. The results are in qualitative agreement with the experimental observations in UGe\(_2\) and ZrZn\(_2\).

B. Outlook

The most intriguing open questions concern the nature of the magnetic-fluctuation induced superconducting state, and an understanding of the phase diagram for all temperatures, magnetizations, and magnetic fields. One interesting question is whether there is more than one superconducting phase as a function of temperature, magnetization, and external magnetic fields. Since the pairing mechanism is expected to be electronic in origin, and itself sensitive to superconductivity, it is easy to imagine additional superconducting states appearing inside the superconducting phase as the temperature is lowered. In particular, the Cooper pairs that form in the superconducting state produce an additional internal magnetic field. If this field acts like an external field for the electrons, then the fermionic spin waves, i.e., the ferromagnetic Goldstone modes, will acquire a mass from this effect, leading to a smaller fluctuation contribution to the \( T_c \) mechanism. This suggest another phase inside the superconducting one, where Cooper pairs anti-aligned to the magnetization might also exist. In general, we also note that the concepts of transverse and longitudinal critical magnetic fields needs to be worked out for these superconducting states.

To understand the precise nature of the superconducting state, another fundamental problem must be addressed. In the theory presented in Section II, the only coupling of the internal magnetic field in the ferromagnetic phase to the superconducting order parameter is through the Zeeman-like term that leads to the Stoner splitting of the Fermi surface, and through the Goldstone modes (which in some sense are also due to the Zeeman term) that exist in the ferromagnetic phase. An obvious question is then whether there are orbital effects in the Cooper channel due to the internal magnetic field. If there are such effects, they will lead to an inhomogeneous superconducting state, viz., to some type of Abrikosov flux-lattice state. This magnetic field effect is of relativis-
tic origin and has been neglected in the current theory. It has not yet been discussed theoretically for pairing mechanisms that are of electronic origin and sensitive to internal magnetic field effects.

Finally, our theory of the superconducting phase transition is a mean-field treatment. The validity of this type of approximation in estimating a superconducting critical temperature is not obvious. We note, however, that one would expect any fluctuation-induced degradation of the superconducting $T_c$ to be larger in the paramagnetic phase than in the ferromagnetic one, since for triplet superconductivity, the direction of the internal magnetic field caused by the magnetization will lead to an additional phase coherence of the triplet Cooper pairs in the magnetic phase. The relative importance of this effect on the phase diagram needs to be investigated.

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Appendix A: Conventional Eliashberg Theory

In this appendix we show how to derive conventional Eliashberg theory by means of field-theoretic methods. This is an extension to strong-coupling theory of previous field-theoretic derivations of BCS theory.\(^{39}\)

We start with the fermionic field theory defined in Sec. II A, but add harmonic phonons,

$$S_{ph} = -\frac{1}{2} \int dx \, dy \, \varphi(x) D^{-1}(x-y) \varphi(y) \ , \quad (A1a)$$

and an electron-phonon interaction,

$$S_{e-ph} = \int dx \, \varphi(x) \sum_{\sigma} \bar{\psi}_{\sigma}(x) \psi_{\sigma}(x) \ . \quad (A1b)$$

Here $D^{-1}$ is the phonon vertex function whose inverse, $D(x-y) = D(y-x)$, is the phonon propagator, and $g$ is the electron-phonon coupling constant. For simplicity, we consider a scalar phonon field $\varphi(x)$; the phonon polarizations can be restored in an obvious way if desired.

We now proceed to integrate out the phonons. This produces an effective electron-electron interaction, which we decompose into particle-hole and particle-particle channels, and spin-singlet and spin-triplet contributions in each channel, as usual.\(^{22}\) In the particle-particle channel, only the spin-singlet is nonzero, and in the particle-hole channel we neglect the spin-triplet contribution $S_{ph}^{S-b}$, Eq. (2.2e). The electronic part of the action then reads

$$S = -\int dx \sum_{\sigma} \bar{\psi}(x) \partial_{\tau} \psi(x) + S_0 + S_{ph}^{S-b} + S_{s-e-ph}^{S-b}$$

with

$$S_{s-e-ph}^{S-b} = g^2 \int dx \, dy \, \bar{\psi}_{\uparrow}(x) \bar{\psi}_{\uparrow}(y) D(x-y) \psi_{\uparrow}(y) \psi_{\uparrow}(x) \ , \quad (A2b)$$

and $S_0$ and $S_{ph}^{S-b}$ from Sec. II A.

Now we introduce the Nambu spinors

$$\Psi(x) = \begin{pmatrix} \psi_{\uparrow}(x) \\ \bar{\psi}_{\uparrow}(x) \end{pmatrix} , \quad \bar{\Psi}(x) = \begin{pmatrix} \bar{\psi}_{\uparrow}(x) \psi_{\uparrow}(x) \end{pmatrix} \ , \quad (A3a)$$

and the corresponding composite variables

$$\Phi_{ij}(x,y) = \bar{\Psi}_i(x) \Psi_j(y) \ . \quad (A3b)$$

The field $\Phi$ is bilinear in the fermion fields, and hence its components commute with each other as well as with all other objects. $\Phi$ is therefore isomorphic to a classical field $\mathcal{G}$, and we can transform to a description in terms of these variables by exactly rewriting the partition function, Eq. (2.1), as

$$Z = \int D[\Psi, \bar{\Psi}] e^{S[\Psi, \bar{\Psi}]} \int D[\mathcal{G}] \delta[\mathcal{G} - \Phi]$$

$$= \int D[\Psi, \bar{\Psi}] e^{S[\Psi, \bar{\Psi}]} \int D[\mathcal{G}, \Lambda] e^{Tr\{\Lambda[\mathcal{G} - \Phi]\}}$$

$$= \int D[\mathcal{G}, \Lambda] e^{A[\mathcal{G}, \Lambda]} \ . \quad (A4)$$

Here $\Lambda$ is an auxiliary field that serves as a Lagrange multiplier, and we have defined an effective action

$$A[\mathcal{G}, \Lambda] = \frac{1}{2} \text{Tr} \ln(\tilde{\mathcal{G}}^{-1} - \Lambda^T) + \text{Tr} (\Lambda \mathcal{G})$$

$$+ g^2 \int dx \, dy \, \mathcal{G}_{12}(x,y) D(x-y) \mathcal{G}_{21}(y,x)$$

$$- \frac{g^2}{4} \int dx \, dy \, \left[ \mathcal{G}_{11}(x,y) D(x-y) \mathcal{G}_{11}(y,x) + \mathcal{G}_{22}(x,y) D(x-y) \mathcal{G}_{22}(y,x) - 2 \mathcal{G}_{11}(x,y) D(x-y) \mathcal{G}_{22}(y,x) \right]$$

$$+ \frac{\Gamma_s}{2} \int dx \, \text{tr} \, \delta_3 \mathcal{G}(x,x) \ . \quad (A5a)$$

$\text{Tr}$ denotes a trace over both discrete and continuous labels, while $\text{tr}$ traces over discrete labels only. $\mathcal{G}_0^{-1}$ is an inverse free-particle Green operator,
\[ G_0^{-1} = -\partial_x + \sigma_3 \left( \frac{\nabla^2}{2m_e} + \mu \right) \]  
(A5b)

with \( \sigma_3 \) the third Pauli matrix. The matrix elements of \( G_0 \) are

\[ \hat{G}_0(x-y) = \begin{pmatrix} G_0(x-y) & 0 \\ 0 & -G_0(y-x) \end{pmatrix} \]  
(A5c)

with

\[ G_0(x-y) = \langle \bar{\psi}_\sigma(x) \psi_\sigma(y) \rangle s_0 \]

\[ = \langle x| - \partial_x + \nabla^2 / 2m_e + \mu |y\rangle \]  
(A5d)

the usual free particle Green function.

The standard strong-coupling, or Eliashberg, theory of superconductivity is just the saddle-point solution of the field theory given by Eqs. (A5). The saddle-point condition is

\[ \frac{\delta A}{\delta \hat{G}} = 0 \quad , \quad \frac{\delta A}{\delta \Lambda} = 0 \]  
(A6)

We are looking for homogeneous saddle-point solutions \( G_{sp}(x,y) = G(x-y) \) and \( \Lambda_{sp}(x,y) = \Lambda(x-y) \), and it is easy to see that the diagonal components of both \( G \) and \( \Lambda \) are related. To express this fact, and to conform with standard notation,\textsuperscript{22} we write

\[ G(x-y) = \begin{pmatrix} G(x-y) & -F^+(x-y) \\ -F^+(x-y) & -G(y-x) \end{pmatrix} \]  
(A7a)

\[ \Lambda(x-y) = \begin{pmatrix} \Sigma(x-y) & \Delta^+(x-y) \\ \Delta(x-y) & -\Sigma(y-x) \end{pmatrix} \]  
(A7b)

As we will see, \( G \) and \( F, F^+ \) are the usual normal and anomalous Green functions, and \( \Sigma \) and \( \Delta, \Delta^+ \) are the normal and anomalous self-energies, respectively.

The first equality in Eq. (A6) gives

\[ \Sigma(x-y) = \left[ g^2 D(x-y) - 2\Gamma_s \delta(x-y) \right] G(x-y) \]  
(A8a)

\[ \Delta^+(x-y) = g^2 D(x-y) F^+(x-y) \]  
(A8b)

\[ \Delta(x-y) = g^2 D(x-y) F(x-y) \]  
(A8c)

The second equality yields

\[ G(x-y) = G_0(x-y) + \int dx' dy' G_0(x-x') \Sigma(x' - y') G(y' - y) \]

\[ + \int dx' dy' G_0(x-x') \Delta(x' - y') F^+(y' - y) \]  
(A9a)

\[ F^+(x-y) = \int dx' dy' G_0(x-x') \Sigma(y' - x') F^+(y' - y) \]

\[ + \int dx' dy' G_0(x-x') \Delta^+(x'- y') G(y' - y) \]  
(A9b)

\[ F(x-y) = \int dx' dy' G_0(x-x') \Sigma(x' - y') F(y' - y) \]

\[ + \int dx' dy' G_0(x-x') \Delta(x' - y') G(y' - y) \]  
(A9c)

The Eqs. (A8) and (A9) are the standard Eliashberg equations for conventional, spin-singlet, phonon-induced superconductivity.\textsuperscript{22} \( \Gamma_s \) plays the role of the Coulomb pseudopotential.\textsuperscript{40}

The linearized gap equation, which determines the superconducting transition temperature, is obtained by expanding the Eliashberg equations to linear order in \( \Delta \). In Fourier space, with frequency-momentum four-vectors \( k = (i\omega_n, k) \), one finds for the linearized gap equation

\[ \Delta(k) = g^2 \int_q D(k-q) G(q) \Delta(q) G(-q) \]  
(A10a)

with a normal Green function

\[ G(k) = 1/ \left[ G_0^{-1}(k) - \Sigma(k) \right] \]  
(A10b)

in terms of a normal self energy

\[ \Sigma(k) = g^2 \int_q D(k-q) G(q) - \Gamma_s n_e \]  
(A10c)

Here \( \int_q \equiv T \sum_n \int dq / (2\pi)^2 \), and \( n_e = 2 \text{Tr} G \) is the electron number density.

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26 We recall that the action does not yet contain an explicit particle-particle interaction that one could decouple by means of a Hubbard-Stratonovich transformation. Rather, this interaction will be generated by integrating out the magnetic fluctuations, see also the remark after Eq. (2.2b). This is the reason why we treat the magnetic and superconducting order parameters differently.

27 Consistent with general requirements for the coexistence of superconductivity and ferromagnetism, this order parameter is a special case of non-unitary triplet pairing, see Ref. 15.

28 This feature of our theory is in exact analogy to the Bogoliubov theory of superconductivity or superfluidity, see, e.g., Ref. 24 ch. 5.1.

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