A Normal Form for Two-Input Flat Nonlinear Discrete-Time Systems

Johannes Diwold, Bernd Kolar and Markus Schöberl

Abstract—We show that every flat nonlinear discrete-time system with two inputs can be transformed into a structurally flat normal form by state- and input transformations. This normal form has a triangular structure and allows to read off the flat output, as well as a systematic construction of the parameterization of all system variables by the flat output and its forward-shifts. For flat continuous-time systems no comparable normal form exist.

I. INTRODUCTION

In the 1990s, the concept of flatness has been introduced by Flies, Lévine, Martin and Rouchon for nonlinear continuous-time systems (see e.g. [1], [2] and [3]). Flat continuous-time systems have the characteristic feature that all system variables can be parametrized by a flat output and its time derivatives. They form an extension of the class of static feedback linearizable systems and can be linearized by endogenous dynamic feedback. Their popularity stems from the fact that a lot of physical systems possess the property of flatness and that the knowledge of a flat output allows an elegant solution to motion planning problems and a systematic design of tracking controllers.

For nonlinear discrete-time systems, flatness can be defined analogously to the continuous-time case. The main difference is that time derivatives have to be replaced by forward-shifts. Like in the continuous-time case, flat systems form an extension of static feedback linearizable systems. The problem of static feedback linearization for discrete-time systems is already solved, see [4], [5] and [6]. An important difference to the continuous-time case is the existence of discrete-time systems that can be linearized by exogenous dynamic feedback only. This fact has been pointed out first in [7]. The corresponding linearizing output contains not only forward-shifts but also backward-shifts of system variables. This gives rise to the question whether the definition of discrete-time flatness should be extended to both forward- and backward-shifts, as proposed in [8]. However, within this contribution, we follow [9], [10] and define flatness in such a manner that it corresponds to the endogenous dynamic feedback linearization problem. Therefore, we only consider forward-shifts in the flat output. In order to avoid any confusion, we also use the term endogenous flatness.

In general, the analysis of flat systems can be divided into two separate tasks. First, we are interested in checking whether a system is flat or not in order to clarify if flatness based control strategies can be applied in principle. In [4] and [11], an efficient test for static feedback linearizable systems, which is based on the computation of certain distributions, can be found. As we have shown in [12], this test can be generalized to systems that possess the property of endogenous flatness. The test is based on the results of [13]. It should be noted that for continuous-time flat systems no comparable test is available so far. Second, in order to use the flatness property for control strategies, the knowledge of a flat output as well as the corresponding parameterization of all system variables is necessary. For this purpose, the use of structurally flat normal forms (see e.g. [14], [15] and [16]) has turned out to be helpful. Structurally flat normal forms allow to read off the flat output, as well as a systematic construction of the parameterization of all system variables. The most famous example for such a normal form is the Brunovsky normal form. However, a transformation to Brunovsky normal form is possible if and only if the system is static feedback linearizable. In [12], we have shown that every system that possesses the property of endogenous flatness can be transformed into a structurally flat implicit normal form. The main feature of this normal form is that the equations depend on the system variables in a triangular manner. The reason for the implicit character of this normal form is that the required coordinate transformations are possibly more general than the usual state- and input transformations. In the present contribution, we show that systems with two inputs are an exception and can be transformed into a structurally flat normal form by using state- and input transformations only. Thus, the state representation is preserved. For this reason, we also use the term explicit triangular normal form. We want to emphasize that for flat continuous-time systems no comparable normal form exist.

The paper is organized as follows: In Section II we recapitulate the concept of endogenous flatness and the corresponding test according to [12]. In Section III we discuss certain coordinate transformations, which will be useful later on. In Section IV we introduce a structurally flat explicit triangular form. Then, we prove that every two-input discrete-time system that possesses the property of endogenous flatness can be transformed into such a representation by successive state- and input transformations. Finally, in Section V we illustrate our results by an example.

II. FLATNESS OF DISCRETE-TIME SYSTEMS

Throughout this contribution we consider discrete-time nonlinear systems in explicit state representation of the form
with $\dim(x) = n$, $\dim(u) = m$ and smooth functions $f^i(x, u)$. Geometrically, the system (1) can be interpreted as a map $f$ from a manifold $\mathcal{X} \times \mathcal{U}$ to a manifold $\mathcal{X}^+$ with coordinates $(x, u)$ to coordinates $x^+$. Furthermore, we assume that the system meets $\rank(\partial_{(x,u)} f) = n$, which is a necessary condition for accessibility and consequently also for flatness. Apart from this, we assume that the system possesses no redundant inputs, i.e. $\rank(\partial_u f) = m$, and define endogenous flatness according to [12].

**Definition 1:** A system (1) possesses the property of endogenous flatness around an equilibrium $(x_0, u_0)$, if there exists an $m$-tuple of functions

$$y^j = \varphi^j(x, u, u[1], u[2], \ldots, u[q]), \quad j = 1, \ldots, m,$$

(2)

where $u[i]$ denotes the $i$-th forward-shift of $u$, such that the $n+m$ coordinate functions $x$ and $u$ can be expressed locally by $y$ and forward-shifts of $y$ up to some finite order, i.e.

$$x^i = F^i_1(y, y[1], y[2], \ldots, y[r_{i-1}]), \quad i = 1, \ldots, n$$

$$u^j = F^j_u(y, y[1], y[2], \ldots, y[r_j]), \quad j = 1, \ldots, m.$$  

The $m$-tuple (3) is called a flat output.

The test for flatness, as stated in [12], is based on the construction of sequences of nested distributions on $\mathcal{X} \times \mathcal{U}$ and $\mathcal{X}^+$. The construction makes use of the system equations (1) and the map

$$\pi : \mathcal{X} \times \mathcal{U} \to \mathcal{X}^+$$

defined by

$$x^{i+} = x^i, \quad i = 1, \ldots, n.$$  

**Algorithm 1:**

**Step $k = 0$:** Define the distribution

$$\Delta_0 = 0$$

on $\mathcal{X}^+$ and

$$E_0 = \pi_*^{-1}(\Delta_0) = \text{span}\{\partial_u\}$$

on $\mathcal{X} \times \mathcal{U}$. Then compute the largest subdistribution

$$D_0 \subset E_0$$

(3)

which is projectable with respect to the map $f$ of (1). The distribution $D_0$ is involutive and its pushforward

$$\Delta_1 = f_*(D_0)$$

is a well-defined involutive distribution on $\mathcal{X}^+$.

**Step $k \geq 1$:** Compute

$$E_k = \pi_*^{-1}(\Delta_k)$$

(4)

and the largest subdistribution

$$D_k \subset E_k$$

which is projectable with respect to the map $f$ of (1). The distribution $D_k$ is involutive and its pushforward

$$\Delta_{k+1} = f_*(D_k)$$

is a well-defined involutive distribution on $\mathcal{X}^+$.

**Stop** if for some $k = \bar{k}$,

$$\dim(\Delta_{\bar{k}+1}) = \dim(\Delta_{\bar{k}}).$$

The procedure according to Algorithm 1 yields a unique nested sequence of projectable and involutive distributions

$$D_0 \subset D_1 \subset \ldots \subset D_{\bar{k}-1}$$

(5)

on $\mathcal{X} \times \mathcal{U}$ and a unique nested sequence of involutive distributions

$$\Delta_1 \subset \Delta_2 \subset \ldots \subset \Delta_{\bar{k}}$$

(6)

on $\mathcal{X}^+$ so that

$$f_*(D_k) = \Delta_{k+1}, \quad k = 0, \ldots, \bar{k} - 1.$$  

(7)

Whether a system is flat or not can now be checked by the use of the following theorem.

**Theorem 1:** A system (1) with $\rank(\partial_u f) = m$ is flat if and only if $\dim(\Delta_{\bar{k}}) = n$.

For the proof we refer to [12]. The test for flatness contains the test for static feedback linearizability (see [11]) as a special case. The only difference is that the distributions $\Delta_k$ according to Algorithm 1 are defined as the largest projectable subdistributions $D_k \subset E_k$, while in the static feedback linearizable case these distributions coincide, i.e. $D_k = E_k$.

**Theorem 2:** A system (1) with $\rank(\partial_u f) = m$ is static feedback linearizable if and only if $D_k = E_k$, $k \geq 0$ and $\dim(\Delta_{\bar{k}}) = n$.

Since all distributions $E_k$ must be completely projectable, the test for static feedback linearizability is more restrictive. For a system that meets Theorem 1 a single step where $D_k \neq E_k$ can be interpreted as a defect in the test of static feedback linearizability, as we will demonstrate by the following example.

**Example 1:** For the system

$$x^{5+} = x^4 + x^1 + x^5$$

$$x^{4+} = x^3(x^4 + 1) + x^3$$

$$x^{3+} = x^1 + x^2$$

$$x^{2+} = u^1$$

$$x^{1+} = u^2$$

(8)
we obtain the sequence of distributions
\[ D_0 = \text{span}\{\partial_{u^1}, \partial_{u^2}\} = E_0 \]
\[ D_1 = \text{span}\{\partial_{u^1}, \partial_{x^2}, \partial_{x^3}\} \subset E_1 = \text{span}\{\partial_{u^1}, \partial_{u^2}, \partial_{x^1}, \partial_{x^2}\} \]
\[ D_2 = \text{span}\{\partial_{u^1}, \partial_{u^2}, \partial_{x^2}, \partial_{x^3}\} = E_2 \]
on \(X \times \mathcal{U}\) and
\[ \Delta_1 = \text{span}\{\partial_{x^1,+}, \partial_{x^2,+}\} \]
\[ \Delta_2 = \text{span}\{\partial_{x^1,+}, \partial_{x^2,+}, \partial_{x^3,+}\} \]
\[ \Delta_3 = \text{span}\{\partial_{x^1,+}, \partial_{x^2,+}, \partial_{x^3,+}, \partial_{x^4,+}\} \]
on \(X^+\). Despite the fact that \(E_1\) is not completely projectable, i.e. \(D_1 \neq E_1\), the distribution \(\Delta_3\) meets \(\dim(\Delta_3) = n\) and the system possesses the weaker property of flatness instead of static feedback linearizability. A flat output is given by \(y = (x^4, x^5)\).

Steps with \(D_k \neq E_k\) may occur several times through the algorithm. However, like in Example 1, for the last distribution the relation \(D_{k-1} = E_{k-1}\) holds. Since we will use this relation in Section IV, we establish the following lemma.

**Lemma 1:** For a flat system \(\mathbf{I}\) the distribution \(E_{k-1}\) is completely projectable, i.e. \(D_{k-1} = E_{k-1}\).

**Proof:** The pushforward of the last distribution \(D_{k-1}\) meets
\[ f_*(D_{k-1}) = \Delta_k \]
with
\[ \Delta_k = \text{span}\{\partial_{x^1,+}, \ldots, \partial_{x^n,+}\} . \]
Because of \(D_{k-1} \subset E_{k-1}\) and \(\dim(\Delta_k) = \dim(X^+) = n\), we also get
\[ f_*(E_{k-1}) = \text{span}\{\partial_{x^1,+}, \ldots, \partial_{x^n,+}\} . \]
Thus, \(E_{k-1}\) is projectable and according to the definition of \(D_{k-1}\) we have \(D_{k-1} = E_{k-1}\). \(\blacksquare\)

**III. Coordinate Transformations**

For static feedback linearizable systems, the sequences of distributions \(\mathbf{5}\) and \(\mathbf{6}\) can be straightened out simultaneously by suitable state transformations. As shown in \(\mathbf{11}\), in such coordinates the system \(\mathbf{1}\) exhibits a triangular form. The primary objective of the present paper is to prove that every flat system with two inputs allows a similar explicit triangular representation. The key tool is again to straighten out the sequences of distributions \(\mathbf{5}\) and \(\mathbf{6}\) simultaneously (in Example \(\mathbf{1}\) we already have such special coordinates). However, now we will need in general both state- and input transformations. For systems with more than two inputs, there is no guarantee that an explicit triangular representation exists at all.

Before we state our main results in Section IV, we discuss certain state- and input transformations which will be useful for straightening out the sequences of distributions. In order to preserve an explicit system representation like \(\mathbf{1}\), within this contribution we restrict ourselves to state- and input transformations
\[ \hat{x}^i = \Phi^k_x(x), \quad i = 1, \ldots, n \]
\[ \hat{u}^j = \Phi^k_u(x, u), \quad j = 1, \ldots, m \]
where both \(x\) and \(x^+\) are transformed equally. The transformed system is given by
\[ \hat{x}^{i+} = \Phi^k_{x^+}(x^+) \circ f(x, u) \circ \Phi^{-1}(\hat{x}, \hat{u}), \quad i = 1, \ldots, n \]
where \(\Phi^{-1}(\hat{x}, \hat{u})\) denotes the inverse of \(\mathbf{9}\). Like in the static feedback linearizable case, the first step in achieving a triangular representation is to straighten out the sequence \(\mathbf{6}\). Since \(\mathbf{6}\) is a nested sequence of involutive distributions on \(X^+\), by an extension of the Frobenius theorem there exists a state transformation
\[ (\hat{x}_1, \ldots, \hat{x}_{\bar{k}}) = \Phi_x(x), \]
with \(\dim(\hat{x}_{\bar{k}}) = \dim(\Delta_k) - \dim(\Delta_{k-1})\), which straightens out the distributions
\[ \Delta_1 = \text{span}\{\partial_{\hat{x}_1^+}\} \]
\[ \Delta_2 = \text{span}\{\partial_{\hat{x}_1^+, \partial_{\hat{x}_2^+}}\} \]
\[ \vdots \]
\[ \Delta_k = \text{span}\{\partial_{\hat{x}_1^+, \partial_{\hat{x}_2^+, \ldots, \partial_{\hat{x}_k^+}}\} \]
simultaneously. The system in new coordinates reads at
\[ \hat{x}_k^+ = f_k(\hat{x}, u), \quad k = 1, \ldots, \bar{k} \]
and meets
\[ f_*(D_{k-1}) = \text{span}\{\partial_{\hat{x}_1^+, \partial_{\hat{x}_2^+, \ldots, \partial_{\hat{x}_k^+}}\} \]
for \(k = 1, \ldots, \bar{k}\). Additionally, from the definition of \(E_k\) according to \(\mathbf{4}\) and \(\mathbf{12}\), it follows automatically that \(E_k\) is also straightened out and reads as
\[ E_k = \pi_*^{-1}(\Delta_k) = \text{span}\{\partial_{\hat{x}_1}, \ldots, \partial_{\hat{x}_k}, \partial_{\hat{u}}\} \]
for \(k = 0, \ldots, \bar{k} - 1\). The D-sequence, in contrast, is only straightened out automatically if the system possesses the stronger property of static feedback linearizability, since then \(D_k = E_k\). Thus, for finding coordinates that straighten out both sequences of distributions \(\mathbf{5}\) and \(\mathbf{6}\), the transformation \(\mathbf{11}\) alone is not sufficient. Therefore, we need an additional transformation that straightens out the D-sequence while the \(\Delta\)-sequence remains straightened out. In the following, we introduce transformations that meet the latter condition.

**Lemma 2:** State- and input transformations of the form
\[ \hat{x}_k = \Phi_{x,k}(\hat{x}_{\bar{k}}, \ldots, \hat{x}_k), \quad k = 1, \ldots, \bar{k} \]
\[ \hat{u} = \Phi_u(\hat{x}, u) \]
\[\footnote{By the term “explicit” we refer to a state representation \(\mathbf{1}\), in order to distinguish it from the implicit triangular representation discussed in \(\mathbf{12}\).} \]
\[\footnote{Note that subsequently we will use the bar notation for system representations where the \(\Delta\)-sequence is already straightened out.}\]
preserve the structure of the sequence of distributions, i.e.
\[
\Delta_1 = \text{span}\{\partial_{x_1^1}\}
\]
\[
\Delta_2 = \text{span}\{\partial_{x_1^1}, \partial_{x_2^2}\}
\]
\[
\vdots
\]
\[
\Delta_k = \text{span}\{\partial_{x_1^1}, \partial_{x_2^2}, \ldots, \partial_{x_k^k}\}.
\]

The proof follows from the triangular structure of the state transformation of (16). The input transformation does not affect the \(\Delta\)-sequence.

For systems with two inputs the distributions of the \(D\)-sequence have a very special structure. Since we will deal with two-input systems in Section IV-B we state the following important lemma.

**Lemma 3:** Consider an \(n\)-dimensional manifold \(Z\) with coordinates \(\zeta = (\zeta^1, \ldots, \zeta^n)\) and an involutive distribution
\[
D = \text{span}\{\partial_{\zeta^1}, \ldots, \partial_{\zeta^{i-1}}, \partial_{\zeta^i} + \alpha(\zeta)\partial_{\zeta^j}\},
\]
for some \(i, j \geq k\). There exists a transformation
\[
\hat{\zeta}^j = \Phi^j(\zeta^k, \ldots, \zeta^n)
\]
of the coordinate \(\hat{\zeta}^j\) such that
\[
D = \text{span}\{\partial_{\hat{\zeta}^1}, \ldots, \partial_{\hat{\zeta}^{i-1}}, \partial_{\hat{\zeta}^i}\}.
\]
The proof can be found in the appendix. For two-input systems we will encounter distributions \(D_k\) of the form (17) on the manifold \(\mathcal{X} \times \mathcal{U}\), and straighten them out by state- or input transformations of the type (18). These transformations will also exhibit the structure-preserving form (16) with respect to the \(\Delta\)-sequence.

**IV. EXPLICIT TRIANGULAR FORM**

In [11] it is shown how a static feedback linearizable system can be transformed into Brunovsky normal form. In the first step, a state transformation is performed that straightens out the sequences of distributions simultaneously. This yields an explicit triangular system representation which can be interpreted as a composition of smaller subsystems. With respect to the inputs of these subsystems, there may occur redundancies. Following [11], further state- and input transformations are successively performed in order to obtain the Brunovsky normal form. The above mentioned redundancies appear if the chains of the Brunovsky normal form have different lengths.

For flat systems that are not static feedback linearizable, a transformation to Brunovsky normal form is not possible. Thus, we introduce a more general structurally flat explicit triangular form that can be obtained by straightening out the \(D\)- and \(\Delta\)-sequences by suitable state- and input transformations. For two-input systems, we prove that such a transformation which straightens out both sequences of distributions simultaneously always exists. Subsequently, similar to the static feedback linearizable case, redundant inputs of the subsystems can be eliminated by further structure-preserving state- and input transformations. For the resulting system representation we use the term explicit triangular normal form. It allows to read off a flat output and the corresponding parameterizing map, according to Definition 1 in a systematic way.

**A. Explicit Triangular Form for Multi-Input Systems**

In the following we present an explicit triangular representation for flat systems. Note, we do not refer to it as a normal form, since for systems with an arbitrary number of inputs the existence of such coordinates is not guaranteed in general.

**Theorem 3:** Assume there exists a state- and input transformation
\[
\hat{x} = \Phi_x(x)
\]
\[
\hat{u} = \Phi_u(x, u)
\]
that straightens out the sequences (5) and (6) simultaneously, i.e.
\[
\Delta_j = \text{span}\{\partial_{x_1^j}, \ldots, \partial_{x_k^j}\}, \quad j = 1, \ldots, k
\]
with \(\dim(\hat{x}_j) = \dim(\Delta_j) - \dim(\Delta_{j-1})\) and
\[
D_k = \text{span}\{\partial_{z_0}, \ldots, \partial_{z_k}\}, \quad k = 0, \ldots, \tilde{k} - 1.
\]

Here \(z_k\) denotes a selection of components of \((\hat{u}, \hat{x}_1, \ldots, \hat{x}_k)\) with \(\dim(z_0) = \dim(D_0)\) and \(\dim(z_k) = \dim(D_{k-1}) - \dim(D_{k-1})\) for \(k = 1, \ldots, \tilde{k} - 1\). In such coordinates, the system (11) has the triangular form
\[
\hat{x}_k^+ = f_k(\hat{x}_k, z_{k-1})
\]
\[
\hat{x}_{k-1}^+ = f_{k-1}(\hat{x}_{k-1}, z_{k-1}, z_{k-2})
\]
\[
\vdots
\]
\[
\hat{x}_1^+ = f_1(\hat{x}_1, z_{k-1}, \ldots, z_0)
\]
with
\[
\hat{x}_k \subset (z_k, \ldots, z_{k-1}), \quad k = 1, \ldots, \tilde{k} - 1
\]
and meets
\[
\text{rank}(\partial_{z_j-1, f_j}) = \dim(\hat{x}_j), \quad j = 1, \ldots, \tilde{k}.
\]
The proof can be found in the appendix. Hereinafter, we state the intention of defining \(z_k\). Since the \(\Delta\)-sequence is straightened out, likewise is \(E_k = \text{span}\{\partial_{\hat{\zeta}_3}, \ldots, \partial_{\hat{\zeta}_1}, \partial_{\hat{\zeta}_0}\}\). Furthermore, by assumption the \(D\)-sequence is also straightened out, and due to \(D_k \subset E_k\) it follows that \(D_k\) might not contain all components of \(\partial_{\hat{\zeta}_3}, \ldots, \partial_{\hat{\zeta}_1}, \partial_{\hat{\zeta}_0}\). Therefore, we introduce the variable \(z_k\), which acts as placeholder and describes states and/or inputs of \((\hat{u}, \hat{x}_1, \ldots, \hat{x}_k)\) so that \(D_k\) reads as (23). It is important to mention that the variables \(z_0, \ldots, z_{k-1}\) contain all inputs and states except \(\hat{x}_k\) (see the proof in the appendix). Note, the system (23) is still in an explicit state representation (11), as we can always replace \(z_k\) by the corresponding states and inputs. We clarify the definition of \(z_k\) using the system of Example 1.

**Example 2:** Consider the system (8) of Example 1. In this example, both the \(D\)- and the \(\Delta\)-sequence are already straightened out. Thus, the coordinate transformation of Theorem 3 is just a renaming
\[
\hat{x}_1 = x^5, \quad \hat{x}_2 = x^3, \quad \hat{x}_1 = x^2, \quad \hat{u}_1 = u^1
\]
\[
\hat{x}_2 = x^4, \quad \hat{x}_2 = x^3, \quad \hat{u}_2 = u^2.
\]
According to Theorem 3 we define
\[ z_0 = (\hat{u}^1, \hat{u}^2), \quad z_1 = (\hat{x}_1^1), \quad z_2 = (\hat{x}_1^2, \hat{x}_2^2) \] (26)
such that the \(D\)-sequence of distributions reads as
\[
D_0 = \text{span}\{\partial_{z_0}\} \\
D_1 = \text{span}\{\partial_{z_0}, \partial_{z_1}\} \\
D_2 = \text{span}\{\partial_{z_0}, \partial_{z_1}, \partial_{z_2}\}
\]
and the system follows as
\[
x_3^{1,+} = x_3^2 + z_2 + x_3^1 \\
x_3^{2,+} = z_2^2(x_3^2 + 1) + z_2^1 \\
x_2^{1,+} = z_2^0 + z_1^1 \\
x_1^{1,+} = z_0^1 \\
x_1^{2,+} = z_0^2
\]
The system has the structure of (23), and a flat output is given by \(y = (x_3^1, x_2^2)\).

It is important to emphasize that for systems with \(m > 2\) inputs the existence of a state- and input transformation (20) that straightens out both sequences (5) and (6) simultaneously is not guaranteed. Thus, an explicit triangular form (23) does not necessarily exist. However, at least a transformation into an implicit triangular form as discussed in [12] is always possible.

**B. Explicit Triangular Form for Two-Input Systems**

In the following we restrict ourselves to flat systems with two inputs and state our main result.

**Theorem 4:** A two-input flat system \((1)\) is locally transformable into an explicit triangular representation (23).

According to Theorem 3 we must show that there exists a state- and input transformation (20) which straightens out both sequences (5) and (6) simultaneously. We start with the system representation (13), where the \(D\)-sequence has already been straightened out by a suitable state transformation. Next, we want to straighten out the \(D\)-sequence step by step, starting with \(D_0\). For this purpose, we can exploit the fact that for systems with two inputs the dimension of these distributions grows in every step by either one or two. In the first case, the distribution \(D_k\) is of the form (17) and can be straightened out by a transformation (18), whereas in the latter case the distribution \(D_k\) is already straightened out and no transformation is required.

The following algorithm straightens out the \(D\)-sequence step by step with state- and input transformations that preserve the structure of the \(D\)-sequence according to Lemma 2. In every step \(k\), after performing the transformation the corresponding states or inputs are renamed by \(z_k\) and \(z_{k,c}\). As mentioned before, \(z_k\) is just a selection of states and inputs so that
\[ D_k = \text{span}\{\partial_{z_0}, \ldots, \partial_{z_k}\}, \]
whereas \(z_{k,c}\) denotes the complementary state or input so that
\[ E_k = \text{span}\{\partial_{z_0}, \ldots, \partial_{z_k}, \partial_{z_{k,c}}\}. \]
To keep the successive transformations readable, after each step \(k\) we return from the hat notation for the transformed variables again to the bar notation. However, for the final system representation after the last step we use the hat notation.

**Algorithm 2:**

- **Step 1:** We distinguish between the two cases:
  a) If the entire input distribution is projectable, i.e. \(D_0 = E_0\), then there is no need for an input transformation because \(E_0\) is already straightened out. We define \(z_0 = (u^1, u^2)\) and \(z_{0,c}\) is empty.
  b) If \(D_0 \neq E_0\), then
    \[
    D_0 = \text{span}\{\partial_{z_0}\} \\
    E_0 = \text{span}\{\partial_{z_0}, \partial_{z_{0,c}}\}.
    \]

- **Step 2:** \(k = 1, \ldots, k - 1\), we repeat the procedure with the distribution
  \[ D_k \subset E_k = \text{span}\{\partial_{z_0}, \ldots, \partial_{z_{k-1}}, \partial_{z_{k-1,c}}, \partial_{\bar{z}_k}\}. \]
  It can be shown that the dimensions of \(z_{k-1,c}\) and \(\bar{x}_k\) meet
  \[
  \dim(z_{k-1,c}) \leq 1 \\
  \dim(\bar{x}_k) \geq 1
  \]
  \[
  \dim(z_{k-1,c}) + \dim(\bar{x}_k) \leq 2.
  \]
  Thus, in every step we must distinguish three cases:
  a) If the entire distribution \(E_k\) is projectable, i.e. \(D_k = E_k\), then there is no need for a transformation because \(E_k\) is already straightened out. We define \(z_k = (\bar{x}_k, z_{k-1,c})\) and \(z_{k,c}\) is empty.
  b) If \(D_k \neq E_k\) and \(\dim(\bar{x}_k) = 2\), then \(z_{k-1,c}\) is empty and
  \[
  D_k = \text{span}\{\partial_{z_0}, \ldots, \partial_{z_{k-1}}, \alpha(z, \bar{x}, \partial_{\bar{z}_k}^1 + \partial_{\bar{z}_k}^2)\},
  \]
  up to a renumbering of the components of \(\bar{x}_k\). According to Lemma 3 there exists a state transformation
  \[
  \hat{x}_k^1 = \Phi_{\bar{z}_k}^1(\bar{x}_k, \ldots, \bar{x}_k)
  \]
  such that \(D_k = \text{span}\{\partial_{z_0}, \ldots, \partial_{z_{k-1}}, \partial_{\hat{x}_k}^1\}\). We define \(z_k = \hat{x}_k^2\) and \(z_{k,c} = \hat{x}_k^1\).
  c) If \(D_k \neq E_k\) and \(\dim(\bar{x}_k) = 1\), then necessarily also \(\dim(z_{k-1,c}) = 1\). Otherwise, we would have \(D_k = E_k\) and case (a) would apply. Thus,
  \[
  D_k = \text{span}\{\partial_{z_0}, \ldots, \partial_{z_{k-1}}, \alpha(z, z_{k-1,c}, \bar{x}, \partial_{z_{k-1,c}} + \partial_{\bar{z}_k})\}. 
  \]
  According to Lemma 3 there exists a transformation
  \[
  \hat{z}_{k-1,c} = \Phi_{z_{k-1,c}}(\bar{x}_k, \ldots, \bar{x}_k, z_{k-1,c})
  \]
  (28)
such that $D_k = \text{span}\{\partial_{z_0}, \ldots, \partial_{z_{k-1}}, \partial_{x_k}\}$. Since $z_{k-1,c}$ could represent both an input or state variable of the system, the transformation is either an input- or a state transformation. We define $z_k = \hat{x}_k$ and $z_{k,c} = \hat{z}_{k-1,c}$.

Finally, the distributions are given by

$$D_k = \text{span}\{\partial_{z_0}, \ldots, \partial_{z_k}\}$$
$$E_k = \text{span}\{\partial_{z_0}, \ldots, \partial_{z_k}, \partial_{z_{k,c}}\}.$$ 

Within the algorithm, only state transformations (27) and input- or state transformations (28) are performed. Since both are of the structure-preserving form (16), the resulting entire transformation law is also of the form (16) and the $\Delta$-sequence remains straightened out. Thus, after the last step of the algorithm, both sequences of distributions are straightened out according to (21) and (22) in Theorem 3.

Remark 1: In the last step $k = \bar{k} - 1$, according to Lemma 1 the distribution $E_{k-1}$ is completely projectable and case (a) applies. Consequently, the last distribution meets $D_{k-1} = E_{k-1}$ and $z_{k-1,c}$ is empty. Thus, it is ensured that the variables $z_0, \ldots, z_{k-1}$ indeed contain all inputs and states except $\hat{x}_k$.

C. Explicit Triangular Normal Form for Two-Input Systems

The explicit triangular form (23) consists of the $k$ subsystems

$$\hat{x}_k^+ = f_k(\hat{x}_k, z_{k-1})$$
$$\hat{x}_{k-1}^+ = f_{k-1}(\hat{x}_k, z_{k-1}, z_{k-2})$$
$$\vdots$$
$$\hat{x}_1^+ = f_1(\hat{x}_k, z_{k-1}, \ldots, z_1)$$

with $k = 1, \ldots, \bar{k}$. The parameterization of the system variables of the system (23) by the flat output can be obtained by determining step by step the parameterization of the system variables of the subsystems (29), starting with the topmost subsystem

$$\hat{x}_1^+ = f_1(\hat{x}_1, z_{k-1}).$$

If $\dim(\hat{x}_1) = \dim(z_{k-1})$ for all $k = 1, \ldots, \bar{k}$, then due to the rank conditions (23) this is particularly simple, and $y = \hat{x}_1$ with $\dim(\hat{x}_1) = m$ is a flat output (see Example 2). By applying the implicit function theorem to the topmost subsystem (30) we immediately get the parameterization of the variables $z_{k-1}$. Next, since $z_{k-1}$ contains the state variables $\hat{x}_{k-1}$ (see (24), by applying the implicit function theorem to the equations

$$\hat{x}_{k-1}^+ = f_{k-1}(\hat{x}_k, z_{k-1}, z_{k-2})$$
we get the parameterization of the variables $z_{k-2}$. Continuing this procedure finally yields the parameterization of all system variables by the flat output $y = \hat{x}_k$. However, if $\dim(\hat{x}_k) < m$, then for at least one $k \in \{2, \ldots, \bar{k}\}$ we have $\dim(\hat{x}_k) < \dim(z_{k-1})$, which means that the equations

$$\hat{x}_k^+ = f_k(\hat{x}_k, z_{k-1}, \ldots, z_{k-1})$$
cannot be solved for all components of $z_{k-1}$. In this case, the subsystem (29) has redundant inputs, and in addition to $\hat{x}_k$ the flat output has further components. The redundant inputs can be eliminated from the subsystem by suitable coordinate transformations of the structure-preserving form (16). The flat output of the complete system (23) consists of $\hat{x}_k$ and the eliminated redundant inputs of all subsystems.

For systems with two inputs there can occur exactly two cases. If $\dim(\hat{x}_k) = 2$, then $y = \hat{x}_1$ is a flat output and none of the subsystems has redundant inputs. Otherwise, if $\dim(\hat{x}_k) = 1$, then there is exactly one $k \in \{2, \ldots, \bar{k}\}$ with $\dim(z_{k-1}) = 1 < \dim(z_{k-1}) = 2$. Thus, the corresponding subsystem (29) has one redundant input. By the use of the regular transformation

$$z_{k-1}^2 = f_k(\hat{x}_k, z_{k-1-1}, \ldots, z_{k-1}),$$

which is either an input- or state transformation but still of type (16), the subsystem reads as

$$\hat{x}_1^+ = f_k(\hat{x}_k, z_{k-1})$$
$$\hat{x}_{k-1}^+ = f_{k-1}(\hat{x}_k, z_{k-1-1}, z_{k-2})$$
$$\vdots$$
$$\hat{x}_1^+ = z_{k-1}^2$$

and is independent of $z_{k-1}$. The flat output of the complete system is given by $y = (\hat{x}_k, z_{k-1})$. After the elimination of the possibly occurring redundant input of the subsystem (29) by the coordinate transformation (31), refer to the resulting system representation as explicit triangular normal form for two-input systems.

V. Example

In this section, we demonstrate our results with an example already discussed in [12]. The system reads as

$$x_1^+ = \frac{x_1^2 + x_1^3 + 3x_1}{u_1 + 2u_1 + 1}$$
$$x_2^+ = x_1(x_1 + 1)(u_1 + 2u_2 - 3) + x_2 - 3u_2^2$$
$$x_3^+ = u_1 + 2u_2$$
$$x_4^+ = x_1(x_1 + 1) + u_2,$$

and the sequences of distributions (5) and (6) are given by

$$D_0 = \text{span}\{-\partial u_1 + \partial u_2\} \subset E_0 = \text{span}\{\partial u_1, \partial u_2\}$$
$$D_1 = \text{span}\{\partial u_1, \partial u_2, -3\partial x_2 + \partial x_3\} = E_1$$
$$D_2 = \text{span}\{\partial u_1, \partial u_2, -3\partial x_2 + \partial x_3, \frac{\partial x_1}{\partial x_1} \partial x_1 - 2\partial x_3 - \partial x_4\} = E_2$$

for $X \times U$ and

$$\Delta_1 = \text{span}\{-3\partial x_2^+, + \partial x_4^+\}$$
$$\Delta_2 = \text{span}\{-3\partial x_2^+ + \partial x_4^+, \frac{\partial x_1}{\partial x_1} \partial x_1^+ - \partial x_3^+, \frac{\partial x_1}{\partial x_1} \partial x_1^+ - 2\partial x_3^+ - \partial x_4^+\}$$
$$\Delta_3 = \text{span}\{\partial x_1^+, \partial x_2^+, \partial x_3^+, \partial x_4^+\}$$

on $X^+$. Following the procedure of Section IV first we straighten out the $\Delta$-sequence by a state transformation of
Similarly, the last distribution just define $z$ projectable, case (a) applies. We just define the system has the structure of (23) and reads as
\[
\begin{align*}
\bar{x}_3 &= x^1(x^3 + 1) \quad \bar{x}_2 = x^2 + 3x^4 \quad \bar{x}_1 = x^4, \\
\bar{x}_3^1 &= x^1 \quad \bar{x}_2^1 = x^2, \\
\bar{x}_2^2 &= x^2, \\
\bar{x}_1^1 &= x^4.
\end{align*}
\]

The $\Delta$-sequence reads as
\[
\begin{align*}
\Delta_1 &= \text{span}\{\partial_{\bar{x}_1^1}\} \\
\Delta_2 &= \text{span}\{\partial_{\bar{x}_1^1}, \partial_{\bar{x}_2}, \partial_{\bar{x}_3^2}\} \\
\Delta_3 &= \text{span}\{\partial_{\bar{x}_1^1}, \partial_{\bar{x}_2}, \partial_{\bar{x}_3^2}, \partial_{\bar{x}_3^1}\},
\end{align*}
\]

and the system in new coordinates is given by
\[
\begin{align*}
\bar{x}_3^{1+} &= \bar{x}_2^1 + \bar{x}_2^2 \\
\bar{x}_2^{1+} &= \bar{x}_3^1 + \bar{x}_3^2(u^1 + 2u^2) \\
\bar{x}_2^2 &= u^1 + 2u^2 \\
\bar{x}_1^{1+} &= \bar{x}_3^1 + u^1.
\end{align*}
\]

The $D$-sequence in new coordinates reads as
\[
\begin{align*}
D_0 &= \text{span}\{-2\partial_{u^1} + \partial_{u^2}\} \subset E_0 \\
D_1 &= \text{span}\{\partial_{u^1}, \partial_{u^2}, \partial_{\bar{x}_1^1}\} = E_1 \\
D_2 &= \text{span}\{\partial_{u^1}, \partial_{u^2}, \partial_{\bar{x}_1^1}, \partial_{\bar{x}_2} \partial_{\bar{x}_3^2}\} = E_2.
\end{align*}
\]

Next, we use Algorithm 2 in order to straighten out the $D$-sequence and transform the system into the explicit triangular representation (33). Due to the fact that $E_0$ is not completely projectable, the case (b) applies and we need to perform an input transformation
\[
\hat{u}^1 = u^1 + 2u^2
\]

which yields $D_0 = \text{span}\{\partial_{u^2}\}$. We define $z_0 = u^2, \ z_{0,c} = \hat{u}^1$ and the distribution reads as
\[
D_0 = \text{span}\{\partial_{z_0}\}.
\]

In the second step, due to the fact that $E_1$ is completely projectable, case (a) applies. We just define $z_1 = (\bar{x}_1^1, z_{0,c}) = (\bar{x}_1^1, \hat{u}^1)$, and the distribution $D_1$ reads as
\[
D_1 = \text{span}\{\partial_{z_0}, \partial_{z_1}\}.
\]

Similarly, the last distribution $E_2$ is also completely projectable (cf. Lemma 1) and thus we have again case (a). We just define $z_2 = (\bar{x}_2^2, \bar{x}_2^3)$, and the distribution $D_2$ reads as
\[
D_2 = \text{span}\{\partial_{z_0}, \partial_{z_1}, \partial_{z_2}\}.
\]

Consequently, with
\[
z_0 = (u^2), \quad z_1 = (\bar{x}_1^1, \hat{u}^1), \quad z_2 = (\bar{x}_2^2, \bar{x}_2^3)
\]
the system has the structure of (23) and reads as
\[
\begin{align*}
\hat{x}_3^{1+} &= z_2^1 + z_2^2 \\
\hat{x}_2^{1+} &= \hat{x}_3^{1+}z_1^2 + z_1^1 \\
\hat{x}_2^2 &= z_1^2 \\
\hat{x}_1^{1+} &= \hat{x}_3^1 + z_1^1.
\end{align*}
\] (33)

As mentioned before, the subsystems of (33) may still have redundant inputs. Indeed, because of $\dim(\hat{x}_3^1) < \dim(z_2)$, the inputs $z_2^1$ and $z_2^2$ of the topmost subsystem
\[
\hat{x}_3^{1+} = z_2^1 + z_2^2
\]
are redundant. This redundancy can be eliminated by the final transformation
\[
z_2^2 = z_2^1 + z_2^2. \quad (34)
\]

Since $z_2^2$ represents a state variable, the equation (34) defines a state transformation and can be rewritten as
\[
\hat{x}_2^2 = \hat{x}_2^1 + \hat{x}_2^2.
\]

Collecting all transformations we performed so far, we obtain the complete transformation
\[
\begin{align*}
\hat{x}_3^1 &= x^1(x^3 + 1) \quad \hat{x}_1^1 = x^4 \\
\hat{x}_2^1 &= x^2 + 3x^4 \quad \hat{u}^1 = u^1 + 2u^2 \\
\hat{x}_2^2 &= x^3 + x^2 + 3x^4 \quad u^2 = u^2,
\end{align*}
\] (35)

which transforms the system (32) into the explicit triangular normal form
\[
\begin{align*}
\hat{x}_3^{1+} &= \hat{x}_2^2 \\
\hat{x}_2^{1+} &= \hat{u}^1 \hat{x}_3^1 + \hat{x}_1 \\
\hat{x}_2^2 &= \hat{u}^1 \hat{x}_3^2 + \hat{u}^1 + \hat{x}_1 \\
\hat{x}_1^{1+} &= \hat{x}_3^1 + u^2.
\end{align*}
\]

with the flat output $y = (\hat{x}_3^1, \hat{x}_2^1)$. The parameterizing map can now be constructed in a systematic way. From the first equation we immediately get the parameterization of $\hat{x}_2^2$. Inserting this parameterization into the second and the third equation yields the parameterization of $\hat{x}_3^1$ and $\hat{u}^1$. Finally, inserting the parameterization of $\hat{x}_3^1$ into the last equation yields the parameterization of $u^2$. With the inverse of (35), the parameterization of the original system variables $x$ and $u$ follows.

VI. CONCLUSION

We have shown that every flat nonlinear discrete-time system with two inputs can be transformed into a structurally flat explicit triangular normal form. In contrast to the implicit triangular form discussed in [12], this normal form is a state representation. The transformation is based on the sequences of distributions (5) and (6), that arise in the test for flatness introduced in [12]. If it is possible to straighten out both sequences of distributions simultaneously by state- and input transformations, then the transformed system has the triangular structure (23). For static feedback linearizable systems, even in the multi-input case with $m > 2$, this can always be achieved by a state transformation. For flat systems that are not static feedback linearizable, in contrast, there is no guarantee that both sequences can be straightened out simultaneously, even if additionally input transformations are permitted. However, for flat systems with two inputs, straightening out (5) and (6) by state- and input transformations is always possible. Thus, every flat discrete-time system
with two inputs can be transformed into an explicit triangular form.

It is important to emphasize that for flat continuous-time systems no comparable result exists. An obvious reason is that the explicit triangular form allows to read off a flat output which depends only on the state variables. In contrast to continuous-time systems, it is shown in [13] that for flat discrete-time systems such a flat output always exists.

**APPENDIX**

A. Proof of Lemma 5

Due to the involutivity, all pairwise Lie Brackets must be contained in $D$, i.e.

$$[\partial z_l, \partial z_l + \alpha(\zeta)\partial z_l] \subset D, \quad l = 1, \ldots, k - 1.$$  

Because of the special structure of the basis of $D$, this implies that all pairwise Lie brackets vanish identically.

Consequently, the coefficient $\alpha$ meets $\partial z_l = 0$ for $l = 1, \ldots, k - 1$, i.e., $\alpha$ is independent of $\zeta^1, \ldots, \zeta^{k-1}$. Next, the flow $\phi_t(\zeta_0)$ of the vector field $\partial z_l + \alpha(\zeta^k, \ldots, \zeta^n)\partial z_l$ is of the form

$$\zeta^l(t, \zeta_0) = t + \zeta^l_0,$$

i.e., it only affects the coordinates $\zeta^1$ and $\zeta^2$. According to the flow-box theorem, by setting $t = \zeta^l$, $\zeta^l_0 = 0$, $\zeta^{k} = \zeta^{l}$ and $\zeta^{k'} = \zeta^{l'}$ for $l = k, \ldots, n$ with $l \neq i, j$ on the right hand side, we obtain a coordinate transformation which transforms the above vector field into the form $\partial z_l$. In fact, only $\zeta^l$ is replaced by the transformed coordinate $\zeta^{l'}$, and all other coordinates remain unchanged. In these coordinates, the distribution $D$ reads as (19). The inverse coordinate transformation is of the form (19).

B. Proof of Theorem 3

First, we show that the variables $z_0, \ldots, z_{k-1}$ contain all inputs and states except $\hat{x}_k$. Since the distributions (12) are straightened out, according to (15) the distribution $E_{k-1}$ reads as

$$E_{k-1} = \text{span}\{\partial z_0, \partial z_1, \ldots, \partial \hat{x}_{k-1}\}.$$  

Lemma 1 guarantees that $E_{k-1}$ is completely projectable, and thus it coincides with the distribution

$$D_{k-1} = \text{span}\{\partial z_0, \ldots, \partial z_{k-1}\},$$

Thus, the variables $z_0, \ldots, z_{k-1}$ contain all inputs and states except $\hat{x}_k$. The property (24) is a consequence of $D_{k-1} \subset E_{k-1}$, i.e.

$$\text{span}\{\partial z_0, \ldots, \partial z_{k-1}\} \subset \text{span}\{\partial z_0, \partial z_1, \ldots, \partial \hat{x}_{k-1}\}$$  

and $D_{k-1} = E_{k-1}$, i.e.

$$\text{span}\{\partial z_0, \ldots, \partial z_{k-1}\} = \text{span}\{\partial z_0, \partial z_1, \ldots, \partial \hat{x}_{k-1}\}. $$

Because of (36), the variables $\hat{x}_k$ cannot be contained in $(z_0, \ldots, z_{k-1})$. However, according to (37), they must be contained in $(z_k, \ldots, z_{k-1})$.

The triangular structure of (23) is a consequence of

$$f_k(D_{k-1}) = \Delta_k = \text{span}\{\partial z_1^+, \ldots, \partial z_k^+\}, \quad k = 1, \ldots, \tilde{k}. $$  

(38)

For $k = 0$, from (38) and $D_0 = \text{span}\{\partial z_0\}$ we get $\partial z_0 f_1 = 0$ for $i = 2, \ldots, \tilde{k}$, i.e.

$$\hat{x}_k^+ = f_k(\hat{x}_k, z_{k-1}, \ldots, \hat{z}_1)$$

$$\vdots$$

$$\hat{x}_2^+ = f_2(\hat{x}_2, z_{k-1}, \ldots, \hat{z}_1)$$

$$\hat{x}_1^+ = f_1(\hat{x}_1, z_{k-1}, \ldots, \hat{z}_0).$$

Furthermore, because of $\dim(\Delta_1) = \dim(\hat{x}_1)$, the rank condition $\text{rank}(\partial z_0, f_1) = \dim(\hat{x}_1)$ follows. Next, for $k = 1$, from (38) and $D_1 = \text{span}\{\partial z_0, \partial z_1\}$ we get $\partial z_1 f_i = 0$ for $i = 3, \ldots, k$, i.e.

$$\hat{x}_k^+ = f_k(\hat{x}_k, z_{k-1}, \ldots, \hat{z}_2)$$

$$\vdots$$

$$\hat{x}_2^+ = f_2(\hat{x}_2, z_{k-1}, \ldots, \hat{z}_1)$$

$$\hat{x}_1^+ = f_1(\hat{x}_1, z_{k-1}, \ldots, \hat{z}_0).$$

Again, because of $\dim(\Delta_2) = \dim(\hat{x}_1) + \dim(\hat{x}_2)$, the rank condition $\text{rank}(\partial z_1, f_2) = \dim(\hat{x}_2)$ follows. Repeating this argumentation shows that the system has the triangular structure (23) and meets the rank conditions (25).

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