On the Lewis–Riesenfeld (Dodonov–Man’ko) invariant method

Julio Guerrero¹,³ and Francisco F López-Ruiz²

¹ Departamento de Matemática Aplicada, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain
² Departamento de Física Aplicada, Universidad de Cádiz, Campus de Puerto Real, 11510 Puerto Real, Cádiz, Spain

E-mail: juguerre@um.es and paco.lopezruiz@uca.es

Received 28 November 2014, revised 27 February 2015
Accepted for publication 4 March 2015
Published 29 June 2015

Abstract

We revise the Lewis–Riesenfeld invariant method for solving the quantum time-dependent harmonic oscillator in light of the quantum Arnold transformation previously introduced and its recent generalization to the quantum Arnold–Ermakov–Pinney transformation. We prove that both methods are equivalent and show the advantages of the quantum Arnold–Ermakov–Pinney transformation over the Lewis–Riesenfeld invariant method. We show that, in the quantum time-dependent and damped harmonic oscillator, the invariant proposed by Dodonov and Man’ko is more suitable, and provide some examples to illustrate it, focusing on the damped case.

Keywords: Ermakov invariant, invariant method, time-dependent Schrödinger equation, quantum Arnold transformation

1. Introduction

The Lewis–Riesenfeld invariant method [1, 2] is a technique that makes it possible to obtain a complete set of solutions to the Schrödinger equation for a time-dependent harmonic oscillator in terms of the eigenstates of a quadratic invariant. This quadratic invariant, the Lewis invariant, is built using an auxiliary variable that satisfies the Ermakov equation [3–6].

The quantum Arnold–Ermakov–Pinney transformation (QAEPT) [7] is a unitary transformation that maps solutions to a generalized Caldirola–Kanai [8–10] Schrödinger equation into solutions to another generalized Caldirola–Kanai Schrödinger equation. In particular, one of the systems can be the standard harmonic oscillator and the other a time-dependent harmonic oscillator, and in this case we will show that the Lewis–Riesenfeld invariant method is recovered.

The idea of using invariants to solve equations is rather old, going back to S Lie (1883) [11], who showed that a second-order differential equation has the maximal group of symmetries if the differential equation is up to third order in the derivative and the coefficients satisfy certain relations [12, 13].

V P Ermakov (1880) [3, 4] showed that the general solution to the non-linear equation

$$\ddot{b} + a^2(t)b = \frac{\omega_0^2}{b^3},$$

(1)

where $\omega_0$ is an arbitrary constant, can be obtained from two independent solutions $y_1, y_2$ of the corresponding linear equation

$$\ddot{y} + a^2(t)y = 0$$

(2)

by

$$b^2 = c_1y_1^2 + c_2y_2^2 + 2c_3y_1y_2,$$

(3)

with $c_1c_2 - c_3 = c_0^2$. Similar results were derived independently by W E Milne (1930) [5] and Pinney (1950) [6].

H R Lewis (1967) [1] obtained a classical and quantum quadratic invariant for a time-dependent harmonic oscillator, of the form

$$I_L = \frac{1}{2m}(bp - m\dot{x})^2 + \frac{1}{2}\omega_0^2x^2,$$

(4)

where, again, $\omega_0$ is an arbitrary constant with the dimension of frequency, in terms of an auxiliary dimensionless function $b(t)$ satisfying the Ermakov equation (1). Note that classically...
The pair of equations (1) and (5) is denoted an Ermakov system. They are uncoupled (given \( \alpha(t) \), they can be solved independently for \( x \) and \( b \)), although the system has been generalized to coupled equations and to higher dimensions [14, 15].

The reader might wonder about the comparison of \( b \) (dimensionless), satisfying (1) and providing the invariant (4), and a function \( \rho \) satisfying the usual Ermakov–Pinney equation \( \dot{\rho} + \alpha(t)^2 \rho = \frac{1}{\rho} \), which has the dimensions of the square root of time (see, e.g., [1]). The relation between \( b \) and \( \rho \) is simply \( b = \sqrt{\alpha_0} \rho \). Also, the relation between \( I_L \) and the invariant \( I \) in [1] is simply \( I_L = \omega_0 I \); i.e., \( I_L \) has the dimension of energy, whereas \( I \) has the dimension of action. The arbitrariness in the choice of \( b \) and \( I_L \) was already noted in [1, 2], and we make it explicit for later convenience. (See also [16], where the authors use the same convention.)

Lewis and Riesenfeld (1969) [2] used the eigenvectors of the quantum version of this quadratic invariant \( I_L \) written in terms of the auxiliary function \( b(t) \) satisfying the Ermakov equation to obtain solutions to the Schrödinger equation for a time-dependent harmonic oscillator. For this purpose, an extra time-dependent phase \( e^{i \int_0^\tau \frac{\alpha}{2} dt \bar{b} \bar{b}^{\dagger}} \) had to be added to the eigenfunctions to satisfy the Schrödinger equation. Lewis and Riesenfeld did not consider damping (or time-dependent mass), and they supposed that the quadratic invariant has a discrete spectrum.

V I Arnold (1978) [17], in the context of symmetries of second-order ordinary differential equations, introduced the term straightening for the linearization studied by S Lie and considered the case of linear second order differential equations (LSODEs):

\[
\ddot{x} + f \dot{x} + \omega^2 x = \Lambda, \tag{6}
\]

where \( f \), \( \omega \), and \( \Lambda \) are time-dependent functions, giving explicitly the transformation for this case:

\[
A : \mathbb{R} \times T \rightarrow \mathbb{R} \times T
\]

\[(x, t) \mapsto (\kappa, \tau) \tag{7}\]

with

\[
\tau = \frac{u_1(t)}{u_2(t)}, \quad \kappa = \frac{x - u_p(t)}{u_2(t)}, \tag{8}\]

where \( T \) and \( T \) are, in general, open intervals, \( u_1 \) and \( u_2 \) are independent solutions to the homogeneous LSODE, \( u_p \) is a particular solution to the inhomogeneous LSODE, and \( W(t) = u_1u_2 - u_1u_2 = e^{\int f dt} \) is the Wronskian of the two solutions.4

Under this transformation, the classical equation of motion (6) transforms as:

\[
\ddot{x} + f \dot{x} + \omega^2 x = \Lambda \rightarrow \frac{W}{u_2^2} \kappa = 0. \tag{9}\]

Thus, the Arnold transformation maps patches of solutions to the LSODE system to patches of free-particle trajectories.

For convenience, we will impose the canonicity conditions (see [10]):

\[
u_t(0) = u_2(0) = u_p(0) = \dot{u}_p(0) = 0, \quad \dot{u}_t(0) = u_2(0) = 1. \tag{10}\]

These conditions play a crucial role in the physical interpretation of quantities mapped from one system to the other through the Arnold transformation. More precisely, if \( \kappa(\tau) \) and \( \pi(\tau) \) are the conserved position and momentum for the free particle (verifying \( x(0) = x \) and \( p(0) = p \equiv m \pi \)), then the transformed quantities through the Arnold transformation are the conserved position \( x(\tau) \) and momentum \( p(\tau) \) in the LSODE system (verifying that \( x(0) = x \) and \( p(0) = p \equiv m \pi \)).

Dodonov and Man’ko (1979) [18, 19] (and Malkin–Man’ko–Trifonov [20] [1969] without considering damping) computed the coherent states for the generalized Caldirola–Kanai model (the quantum version of a general LSODE system) with Hamiltonian

\[
\hat{H}_{GCK} = \frac{\hat{b}^2}{2m} + \left( \frac{1}{2} m \omega^2 \hat{x}^2 - m \lambda \hat{x} \right) e^{\int f dt}, \tag{11}\]

using first-order invariants as annihilation and creation operators. The number operator associated with these annihilation and creation operators is a quadratic invariant that will be denoted the Dodonov–Man’ko invariant \( I_{DM} \). Later other authors used first-order invariants to solve time-dependent problems [21].

Hartley and Ray (1981) [22] and Lewis and Leach (1982) [23] generalized the construction of the Lewis invariant to some non-linear systems.

Pedrosa (1987) [24] constructed the Lewis invariant for the Ermakov equation with a damping term using canonical transformations.

V Aldaya et al (2011) [10] extended to the quantum case the Arnold transformation and denoted it the quantum Arnold transformation (QAT):

\[
\hat{A} : \quad \hat{H}_t \rightarrow \hat{H}_t^Q \nonumber
\]

\[
\phi(x, t) \mapsto \phi(\kappa, \tau) = \hat{A}(\phi(x, t))
\]

\[
\quad = \hat{A} \left( \sqrt{u_2(t)} \hat{x} e^{\int f dt} \right) \left( \frac{1}{2} m \frac{1}{W(t)} \frac{u_2(t)}{u_1(t)} \right) \phi(x, t). \tag{12}\]

Here \( \hat{A}^* \) is defined as \( \hat{A}^*(f(x, t)) = f(\hat{A}^{-1}(\kappa, \tau)) \), \( \hat{H}_t \) is the Hilbert space of solutions to the generalized Caldirola–Kanai Schrödinger equation at time \( t \), and \( \hat{H}_t^Q \) is the Hilbert space of solutions to the Schrödinger equation for the free Galilean particle at time \( \tau \), where \( t \) and \( \tau \) are related by the Arnold transformation. Note that the QAT transforms solutions to the time-dependent Schrödinger equation of the generalized Caldirola–Kanai system into free-particle wave functions and that this is achieved by applying the Arnold transformation together with multiplying the wave function by a suitable phase and a rescaling factor. These factors also render the QAT unitary [7, 10].

\[\text{x satisfies the equation of motion}\]

\[
\ddot{x} + \alpha^2(t)x = 0. \tag{5}\]
Some applications of the QAT were given in [25], where states from the harmonic oscillator were mapped into the free particle, giving rise to Hermite–Gauss and Laguerre–Gauss wave packets; in [26, 27], where processes of release and recapture of a particle by a harmonic trap were studied using the QAT; and in [28], where the QAT, which is a local diffeomorphism in time, is extended beyond the ‘focal’ points, correctly reproducing the change in phase of the wave function (the Maslov correction; see, for instance, [29]).

Castaños, Schuch, and Rosas–Ortiz (2013) [30] constructed coherent states for different models (time-dependent and non-linear Hamiltonians) through complex Riccati equations and found the corresponding Lewis invariants.

Since its introduction, the Lewis invariant and its associated Ermakov equation have entered an inflationary scenario, with applications in many areas. Some of the most remarkable applications are those concerning Bose–Einstein condensates (BECs) [31, 32], where a transformation similar to that of Arnold (and known as the scaling transformation in this context), taking the time-dependent harmonic trap in the Gross–Pitaevskii equation into a stationary one, is applied. Although these applications did not use the Lewis invariant, the scaling parameter satisfies the Ermakov equation (1). In this context, equation (1) also appears in [33].

Recently the Lewis–Riesenfeld invariant method was used to inverse-engineer shortcuts to adiabaticity [16], to speed up cooling processes and transport in electromagnetic traps and BECs, and to manipulate states in waveguides [34], where the relation with generalized Caldirola–Kanai systems has been established [35]. The main idea here is to design a Lewis invariant satisfying the property of commuting with the Hamiltonian at initial and final times, and this can be achieved by building up a function \( b \) satisfying certain boundary conditions and then determining, through the Ermakov equation, the time-dependent frequency that should be applied to take the system from the initial state to the desired final state without affecting the population of the levels.

Another recent application of the Lewis–Riesenfeld invariant method is in mesoscopic RLC electric circuits [36], where the quantum evolution (even in the case of time-dependent \( R(t) \), \( L(t) \), and \( C(t) \) and a source term) is described.

The content of the paper is as follows. In section 2 we revise the Lewis–Riesenfeld method and explain it in terms of the QAT and the QAEPT, showing that the use of the Dodonov–Man’ko invariant is more appropriate for damped systems. In section 3 the examples of the Caldirola–Kanai and Hermite oscillators are studied in detail.

2. The Lewis–Riesenfeld method in light of the quantum Arnold transformation

In their original paper Lewis and Riesenfeld [2] provided a method to obtain a family of exact wave functions for the time-dependent harmonic oscillator spanning the entire Hilbert space. In a first step the method looks for an invariant Hermitian operator \( \hat{I}_L \), a task which can proceed along the lines of [1]. Imposing the invariance condition

\[
\frac{d\hat{I}_L}{dt} \equiv \frac{d\hat{I}_L}{\partial t} + i\hbar \left[ \hat{H}(t), \hat{I}_L \right] = 0,
\]

where \( \hat{H}(t) \) is the Hamiltonian for the time-dependent harmonic oscillator and assuming the most general quadratic invariant, the authors arrived at the quantum version of (4), where the auxiliary function \( b(t) \) satisfies the Ermakov equation (1). In this equation \( \alpha_0 \) is just an arbitrary constant. The possibility exists of giving a generic form for the invariant (i.e., quadratic or linear, Hermitian or complex combinations of the basic operators \( \hat{p} \) and \( \hat{x} \), etc.) and then solving for the coefficients to fulfill equation (13).

The second step in the method is realizing that finding eigenfunctions \( \psi_f(x, t) \) of \( \hat{I}_L \), \( \hat{I}_L \psi_f(x, t) = \lambda \psi_f(x, t) \), amounts to finding solutions to the Schrödinger equation except for a time-dependent phase, which must be computed.

That is, the solutions \( \psi_f(x, t) \) to the Schrödinger equation

\[
i\hbar \frac{d\psi_f}{dt} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_f}{\partial x^2} + \frac{1}{2m} \omega_0^2(t) x^2 \psi_f
\]

\[
\equiv \hat{H}(t)\psi_f
\]

may be of the form

\[
\psi_f(x, t) = e^{\alpha_x(t)} \phi_f(x, t),
\]

where \( \alpha_x(t) \) satisfies

\[
\hbar \frac{d\alpha_x}{dt} \phi_f(x, t) = \left( i\hbar \frac{\partial}{\partial t} - \hat{H}(t) \right) \phi_f(x, t).
\]

That is just a nice consequence of the fact that \( \hat{I}_L \) applied to a solution to equation (14) is again a solution:

\[
i\hbar \frac{d\hat{I}_L \psi_f}{dt} = \hat{H}(t) \left( \hat{I}_L \psi_f \right).
\]

The phase can be solved in terms of \( b \) to give

\[
\alpha_x(t) = -\frac{\lambda}{\hbar} \int \frac{dt}{b(t)^2}.
\]

Two observations can be made. First, the way in which the eigenfunctions \( \phi_f(x, t) \) of the invariant are found is left to the ability of the user of the method. In this respect, some authors have developed a unitary transformation from the Hamiltonian of the simple harmonic oscillator into the invariant [37] (resembling very much the QAEPT; see the following). For this purpose, transforming the time-independent Schrödinger equation for the simple harmonic oscillator, including the wave functions, would do the trick. Second, although the Lewis–Riesenfeld method can provide all (quadratic) invariants for the time-dependent harmonic oscillator (by taking all possible solutions to the Ermakov equation), it does not provide insight into their physical interpretation (such as their

5 Although we are working in the Schrödinger picture of quantum mechanics, i.e., wave functions depend explicitly on time and common operators like position \( \hat{x} \) and momentum \( \hat{p} \) do not depend on time, other quantum operators may depend explicitly on time. This is precisely the case of the Hamiltonian for non-conservative systems and in general for invariant operators.
spectra). In our case, we provide a method that allows a neat physical interpretation of the invariants since each one preserves the same character as in the harmonic oscillator.

In that sense the QAT and its generalization, the QAEPT, turn out to be very useful.

2.1. The quantum Arnold–Ermakov–Pinney transformation

The QAEPT is obtained when two different LSODE systems are related by QATs with the free-particle system as an intermediary, that is, when a QAT and an inverse QAT are composed. That was shown in [7]. Like the QAT, the QAEPT relies on the symmetry structure of the systems of the generalized Caldirola–Kanai type (see [10]).

In the QAEPT, it is the full set of invariant operators and the corresponding eigenstates, including time dependence (with no need for searching for a phase because it is given by the transformation), which are mapped from the simple harmonic oscillator system to a generalized Caldirola–Kanai system. The interpretation of the eigenstates may be the same on both sides of the mapping.

Let $A_1$ and $A_2$ denote Arnold transformations relating LSODE system 1 and LSODE system 2 to the free particle, respectively; then $E = A_1^{-1} A_2$ relates LSODE system 2 to LSODE system 1. $E$ can be written as

$$
E : \mathbb{R} \times T_2 \to \mathbb{R} \times T_1
$$

$$(x_2, t_2) \mapsto (x_1, t_1) = E(x_2, t_2).$$

The explicit form of the transformation can be easily computed by composing the two Arnold transformations, resulting in

$$
x_1 = b(t_2) x_2,
$$

$$
W_1(t_1) d_1 = \frac{W_2(t_2)}{b(t_2)^2} d_2,
$$

where $b(t_2) = \frac{u^{(2)}(t_2)}{u^{(1)}(t_2)}$ satisfies the non-linear SODE:

$$
b + f b + \omega_1^2 b = \frac{W_2}{W_1} \frac{1}{b^3} \left[\omega_1^2 + \frac{f}{u^{(1)}} u^{(1)} \left(1 - b^2 \frac{W_1}{W_2}\right)\right],$$

and where $u^{(i)}$ refers to the $i$th particular solution for the system $j$; $W_j$, $f_j$, and $\omega_j$ stand for the Wronskian and LSODE coefficients for the system $j$; and the dot means derivation with respect to the corresponding time variable. If all $u^{(i)}$ satisfy the corresponding canonicity conditions (10), then $b(t_2)$ satisfies the corresponding canonicity conditions

$$
b(0) = 1, \quad \dot{b}(0) = 0.$$

Equation (21) constitutes a generalization of the Ermakov equation. That equation, together with the LSODE of system 2, is a generalized Ermakov pair [14, 15]. Also, any (quadratic) conserved quantity which is shared by the two LSODE systems constitutes a generalized Lewis invariant. Equation (21) actually defines a generalized Arnold transformation, to be named the (classical) Arnold–Ermakov–Pinney transformation, which transforms solutions to LSODE 1 into solutions to LSODE 2.

The quantum version of the Arnold–Ermakov–Pinney transformation, $\hat{E}$, can be obtained by computing the composition of a QAT and an inverse QAT to give

$$
\hat{E} : H_{O2}^{(2)} \longrightarrow H_{O1}^{(1)}
$$

$$
\phi(x_2, t_2) \longmapsto \phi(x_1, t_1) = \hat{E}(\phi(x_2, t_2)) = E^\dagger \left( \beta(t_2) e^{-i \frac{1}{\hbar} \int \frac{1}{W_1} W_2 b(t_2)^{-2} \dot{b}(t) \phi(x_2, t) dt} \right).
$$

(23)

The QAEPT maps solutions to a generalized Caldirola–Kanai Schrödinger equation to solutions to a different, auxiliary, generalized Caldirola–Kanai Schrödinger equation, and by construction it is also a unitary transformation. The auxiliary system might be, in particular, one corresponding to a harmonic oscillator with frequency $\omega_1(t) = \omega_0$ and $\dot{f}_1 = 0$. In this case, equations (20) and (21) reduce to

$$
x_1 = \frac{x_2}{b(t_2)},
$$

$$
t_1 = \int_0^{t_2} \frac{W_2(t)}{b(t)^2} dt, \tag{24}
$$

and

$$
b + \dot{f}_2 b + \omega_2^2 b = \frac{W_2^2 \omega_0^2}{b^3}. \tag{25}
$$

2.2. Ermakov system and interpretation of the Lewis invariant

Consider the particular case where LSODE system 1 is a harmonic oscillator ($\omega_1(t_1) \equiv \omega_0$ and $\dot{f}_1 = 0$), which can be described by the Hamiltonian

$$
H_{HO} = \frac{p^2}{2m} + \omega_0^2 x^2. \tag{26}
$$

and LSODE system 2 is a time-dependent harmonic oscillator with frequency $\omega_2(t_2) \equiv \omega(t)$ and $\dot{f}_2 = 0$, with the Hamiltonian $H(t)$ given by equation (14). Then expressions (21) and (25) simplify to equation (1). Obviously, for $\omega_0 = 0$ the Arnold–Ermakov–Pinney transformation reduces to the ordinary Arnold transformation, i.e., $E = A$.

Now note that the LSODE 1 Hamiltonian, $H_{HO}$, is conserved and that it is so on both sides of the transformation $E$, given by (see equation (20))

$$
x_1 = \frac{x}{b}, \quad t_1 = \int \frac{1}{b^2} dt = \frac{1}{\omega_0} \arctan \omega_0 \tau, \tag{27}
$$

where $\tau$ denotes the (common) time in the free particle given by the Arnold transformations $A_1$ and $A_2$. Also, $b(t) = u^{(2)}(t) \sqrt{1 + \omega_0^2 \tau^2} = \sqrt{(u^{(2)})^2 + \omega_0^2 u^{(1)}^2}$ satisfies the Ermakov equation (1) together with the canonicity conditions (22).
It should be stressed that $b(t)$ never vanishes; otherwise, the Wronskian $W_2(t)$ of the two independent solutions would also vanish. And since in the quantum case the time $t_1$ appears in the form $e^{-i\omega_0t_1}$, this expression is well defined for all times (even in the case where $\tau$ has singularities). This means that the QAEPT transformation is well defined for all times (i.e., $E: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$). This is an important advantage with respect to the QAT, which was defined only locally in time.

Computing the momentum $p_1 = m\dot{x}_1 = m\frac{dx_1}{dt} = m\frac{d\dot{x}_1}{dt} + \dot{x}_1 = \dot{m}\dot{x}_1 = \dot{m}(xb - bx)$, we can write $H_{HO}$ in variables corresponding to system 2:

$$H_{HO} = \frac{1}{2m}(pb - mbx)^2 + \frac{1}{2}m\omega_0^2\left(\frac{x}{b}\right)^2 \equiv \tilde{I}_L.$$ (28)

This is easily recognized as the usual Lewis invariant $I_L$. Thus, we have found a way to characterize it through the Arnold–Ermakov–Pinney transformation: $I_L$ corresponds to the conserved quantity $H_{HO}$ imported from the simple harmonic oscillator, which is used as an auxiliary system. Because the auxiliary system is arbitrary, $I_L$ is conserved for any $\omega_0$, provided equation (1) is satisfied. Note that to establish the identification $H_{HO} \equiv I_L$, it is essential to impose the canonicity conditions (22).

Using the explicit form of the inverse $E^{-1}$ of equation (23) in this case, it is straightforward to arrive at solutions $\phi(x, t)$ to the Schrödinger equation for the time-dependent harmonic oscillator in terms of solutions to the Schrödinger equation for the simple harmonic oscillator $\varphi(x_1, t_1)$:

$$\phi(x, t) = \frac{1}{\sqrt{b}}e^{\frac{i}{2}b^2\frac{x^2}{b^2}}\varphi\left(\frac{x}{b}, \int \frac{1}{b^2}dt\right),$$ (29)

where $b$ is any solution to equation (1) satisfying the canonicity conditions equation (22). Note that if $\varphi(x_1, t_1)$ is chosen to be, for instance, an eigenfunction of the quantum operator corresponding to equation (26), $\tilde{H}_{HO}$, then the transformed wave function $\phi(x, t)$ is an eigenfunction of the quantum operator $\tilde{I}_L$ corresponding to the invariant (28). (The explicit form of such operators is easily obtained from their classical counterparts by the canonical quantization prescription.) This shows that $\tilde{I}_L$ has a discrete spectrum.

In the wave functions equation (29), two phases can be distinguished: the one corresponding to the transformation itself, explicit in equation (29), and the phase mapped from $e^{-i(\sigma + \frac{i}{2}\omega_0t_1)}$ (which is the only time dependence for stationary states in the harmonic oscillator) to $e^{-i(\sigma + \frac{i}{2}\omega_0t_1)}\int \frac{1}{t_1}dt$. The latter accounts for the phase of the Lewis–Riesenfeld method. The former accounts for the phase (and the factor) which appears, for instance, in [37].

Regarding the canonicity conditions (22), they play an important role in short-cuts to adiabaticity processes for time-dependent harmonic oscillators (see [16]), since they imply that $\tilde{I}_L$ commutes with the Hamiltonian at the initial time $t = 0$. If we further impose $\dot{b} = 0$, then $\omega_0 = \omega_2(0)$ holds, and the Hamiltonian at $t = 0$ will coincide with the invariant $\tilde{I}_L$ at $t = 0$. In the following, we will assume that the Lewis invariant $\tilde{I}_L$ verifies these conditions.

The same process can be repeated for any other operator representing an invariant in the simple harmonic oscillator (LSODE 1), showing the usefulness of the QAEPT to perform quick computations.

In conclusion, it is the full set of invariant operators and the corresponding eigenstates (without searching a phase) that are mapped from the simple system to the generalized Caldirola–Kanai system through the QAEPT. Also, a word of caution is in order: the Hamiltonian of one system is not mapped to the Hamiltonian of the other system, which may not be invariant itself.

2.3. The Lewis–Riesenfeld (Dodonov–Man’ko) invariant method for the generalized Caldirola–Kanai oscillator through the QAEPT

In a generalized Caldirola–Kanai system, the easiest way to find eigenstates of an invariant operator and its eigenfunctions as solutions to the generalized Caldirola–Kanai Schrödinger equation is to focus on an auxiliary system (the harmonic oscillator in the preceding subsection) with its Hamiltonian being the invariant operator and perform the QATs or QAEPT necessary to map the Schrödinger equation for such an auxiliary system into the generalized Caldirola–Kanai Schrödinger equation. In this process, Hamiltonian is not mapped to Hamiltonian, but conserved operators are mapped to conserved operators. This procedure takes advantage of the fact that the eigenstates of the harmonic oscillator Hamiltonian have a simple time dependence, as was just noted in the preceding subsection.

Let us now describe a different way of constructing an invariant. The idea is to consider any linear combination of quadratic invariants in such a way that its eigenfunctions solve the generalized Caldirola–Kanai Schrödinger equation. The most general invariant can be written in the form [10]

$$\hat{I} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega_0^2\hat{x}^2 + \frac{\tilde{\rho}}{2}\hat{x}\hat{p} + \hat{p}\hat{x}$$

(30)

where $\tilde{\rho}$ and $\tilde{\rho}$ are arbitrary real numbers and $\hat{x}$, $\hat{p}$ are conserved position and momentum operators satisfying the condition that at $t = 0$ they coincide with the usual $\hat{x}$, $\hat{p}$, namely [10],

$$\hat{p} = -i\hbar\frac{d}{dx} - m\frac{\dot{u}_2}{W},$$

$$\hat{x} = \frac{\dot{u}_1}{W} + i\hbar\frac{d}{dx}.$$ (31)

The new invariant $\hat{I}$ now plays the role of $\hat{H}_{HO}$ in this more general setting. The eigenfunctions of this operator, solutions to the generalized Caldirola–Kanai Schrödinger

Phys. Scr. 90 (2015) 074046 J Guerrero and F F López-Ruiz

J. Guerrero and F F López-Ruiz

(2015) 074046 J Guerrero and F F López-Ruiz
are the parabolic

\( b(t) = \frac{u_2^2 + \Omega^2 u_1^2}{u_2 - \dot{u}_2/2} = \sqrt{(u_2 - \dot{u}_2/2)^2 + \Omega^2 u_1^2} \) (33)

plays the role of \( b(t) \), i.e., it satisfies the generalized Ermakov equation (25) with \( \omega_0 = \Omega \), but with different initial conditions, namely, \( \dot{b}(0) = 1 \) and \( \dot{b}(0) = -\frac{1}{2} \).

As in the case without damping, neither \( b(t) \) nor \( \dot{b}(t) \) vanishes; otherwise the Wronskian \( W(t) \) of the two solutions \( u_1 \) and \( u_2 \) would also vanish.

Thus, the invariant \( \tilde{I} \) can be seen as \( \tilde{H}_{HO} \) mapped from the harmonic oscillator with a frequency \( \tilde{\Omega} \) through a QAEPT (equation (24)) characterized by \( \tilde{b}(t) \) satisfying equation (25) with \( \omega_0 = \Omega \).

The associated spectrum of \( \tilde{I} \) is

\[
\lambda_\nu = \hbar \tilde{\Omega} \left( \nu + \frac{1}{2} \right). \tag{34}
\]

The integer, real, or complex character of \( \nu \) depends on the value of \( \tilde{\Omega} \) (and this in turn depends on the particular values of \( \dot{\omega} \) and \( \dot{\gamma} \)). See [18] for a discussion that deals with the case of the damped harmonic oscillator.

The choice \( \dot{\omega} = \omega_2(0) \) and \( \dot{\gamma} = 0 \) leads to a generalized Lewis invariant \( \tilde{I}_L \), where the arbitrary frequency has been chosen as \( \omega_0 = \omega_2(0) \), providing an invariant that commutes with the generalized Caldirola–Kanai Hamiltonian (11), and in fact coincides with it, at \( t = 0 \). This invariant may be useful in shortcuts to adiabaticity processes for damped systems or with time-dependent mass (as in waveguides [35]), where an invariant commuting with the Hamiltonian is needed.

For the damped harmonic oscillator with constants \( \omega \) and \( \gamma \) (see section 3.1), a different choice \( \dot{\omega} = \omega \) and \( \dot{\gamma} = \gamma \) exists, leading to the only quadratic invariant \( \tilde{I} \equiv \tilde{I}_{DM} \), the Dodonov–Man’ko invariant, whose unique, explicit time dependence is through the Wronskian \( W(t) \) (like the Caldirola–Kanai Hamiltonian itself). Furthermore, the coherent states associated with this invariant (through a factorization of the form \( \tilde{I}_{DM} = \frac{1}{2}(\hat{\lambda}^2 + \hat{\lambda}^2) \), \( \hat{\lambda} \), \( \hat{\lambda} \) being conserved creation–annihilation operators) are the only ones with minimal time-independent uncertainty relations (see Dodonov and Man’ko [18]).

In conclusion, when studying generalized Caldirola–Kanai systems with the Lewis–Riesenfeld invariant method or with the more general method given by the QAEPT, a choice for the invariant operator other than \( \tilde{I}_L \) should be made; in particular \( \tilde{I} \) (with suitable coefficients \( \dot{\omega} \) and \( \dot{\gamma} \)) may be more appropriate (as in the case of \( \tilde{I}_{DM} \) for the damped harmonic oscillator). We will denote this invariant a generalized Dodonov–Man’ko invariant, \( \tilde{I}_{GDM} \).

2.4. Engineering a suitable QAEPT to build a generalized Dodonov–Man’ko invariant

Once the general setting has been established, let us apply the method in a suitable way to obtain a proper invariant \( \tilde{I}_{GDM} \). The preceding discussion of the choice of an appropriate invariant, together with the corresponding analysis of the Caldirola–Kanai oscillator (with constant damping and frequency, see the following), suggests that it would be helpful to construct a QAEPT from a generalized Caldirola–Kanai system 2 to a yet undetermined generalized Caldirola–Kanai system 1, but satisfying certain requirements implemented in the choice of \( b(t_2) \). In other words, we look for an auxiliary system to help solve a generalized Caldirola–Kanai oscillator in such a way that the QAEPT is as simple as possible.

In particular, it is easy to verify that if we choose \( b = W_{1/2}/2 \), the generalized Caldirola–Kanai system 1 is a time-dependent harmonic oscillator, i.e., it has no damping term.

In fact, with \( b = W_{1/2}/2 \) the corresponding QAEPT is given by

\[
\begin{align*}
\lambda_1 &= \frac{W_1}{W_2^{1/2}}, \\
\phi(x_2, t) &= W_2^{1/4} e^{i x_2 W_1^{1/2} t} \times \phi \left( \frac{x_2}{W_2^{1/2}}, t \right). \tag{36}
\end{align*}
\]

The wave function \( \phi \) satisfies the Schrödinger equation for a time-dependent harmonic oscillator with frequency

\[
\begin{align*}
\omega_1(t)^2 &= \omega_2(t)^2 + \frac{2W_1 W_2 - 3W_2^2}{4W_2^2} \\
&= \omega_2(t)^2 - \frac{1}{4} \left( \frac{x_2}{W_2^{1/2}} \right)^2 - \frac{1}{2} \dot{\gamma}^2, \tag{37}
\end{align*}
\]

and the Wronskian for system 1 turns out to be \( W_1 = 1 \); i.e., the auxiliary system 1 is not damped. However, \( \omega_1(t) \) is not arbitrary but specifically designed to simplify the mapping. Therefore, it is straightforward to map results (such as
invariants) and computations from the known, auxiliary, system 1 to the generalized Caldirola–Kanai system 2.

In this case the canonicity conditions (22) are not satisfied in general since \( b(0) = 1 \) but \( \dot{b}(0) = \frac{1}{2} \delta(t) \) (see the examples in section 3).

Note that the transformation (35), together with its corresponding extension to velocity, is nothing other than a generalized version [24] of the time-dependent canonical transformation that removes the damping in the damped harmonic oscillator [46] (see also [47, 48]).

Constructing the Lewis invariant \( \hat{I}_L \) for this time-dependent harmonic oscillator and mapping it back to our generalized Caldirola–Kanai system 2 through the previous QAEPT leads to a generalized Dodonov–Man’ko invariant \( \hat{I}_{GDM} \) for system 2. More precisely, the invariant can be written as

\[
\hat{I}_{GDM} = \frac{\dot{\hat{p}}^2}{2m} + \frac{1}{2} m \left( \omega^2 (0)^2 - \frac{1}{2} \dot{j}_L^2 (0) \right) \hat{X}^2 + \frac{\dot{j}_L (0)}{2} \hat{X} \dot{\hat{p}} + \hat{\dot{p}} \hat{X}.
\]  

(38)

We will provide examples of this construction in the next section.

3. Examples

Let us discuss some examples of damped systems, where the previous ideas can be applied to construct invariants and simplify analytical computations.

3.1. Caldirola–Kanai oscillator

The simplest example that can be studied is the quantum damped harmonic oscillator, also known as the Caldirola–Kanai oscillator [8, 9]. From the point of view of invariant operators, it was first studied by Dodonov and Man’ko [18], who constructed first-order invariants in the form of conserved creation and annihilation operators and derived a basis of number states and a family of coherent states satisfying minimal time-independent uncertainty relations. The classical equation of motion is given by (we will not consider the external force term)

\[ \ddot{x} + \gamma \dot{x} + \omega^2 x = 0, \]  

(39)

where \( \omega \) and \( \gamma \) are constants. Even though this is a linear equation with constant coefficients, the system is not conservative since the Hamiltonian describing this equation is time-dependent:

\[ H_{CK} = e^{-\gamma t} \frac{\dot{p}^2}{2m} + \frac{1}{2} m \omega^2 e^{\gamma t} x^2. \]  

(40)

There is an old controversy around the quantum version of the Caldirola–Kanai oscillator concerning the dissipative character of this system [40]. The main drawback is that the evolution is unitary for all times and there is no loss of coherence, something that it is considered inherent in a quantum dissipative system. Some proposals have been made to address this paradoxical situation [41] (see also [42]).

An alternative physical interpretation of the damping term in equation (39) is that of a time-dependent mass; that is, the mass is actually of the form

\[ m(t) = m e^{\gamma t} \Rightarrow H_{CK} = \frac{\dot{p}^2}{2m(t)} + \frac{1}{2} m(t) \omega^2 x^2. \]  

(41)

Thus the Caldirola–Kanai oscillator describes an oscillator whose mass is growing exponentially.

The solutions to the classical equations (39) satisfying the canonical conditions \( u_1(0) = u_2(0) = 0, u_3(0) = u_1(0) = 1 \) (see [10]) are

\[ u_1(t) = e^{\frac{\gamma}{2} t} \sin \Omega t, \]

\[ u_2(t) = e^{-\frac{\gamma}{2} t} \left( \cos \Omega t + \frac{\gamma}{2\Omega} \sin \Omega t \right), \]  

(42)

where \( \Omega = \sqrt{\omega^2 - \frac{\gamma^2}{4}} \) and the Wronskian is given by \( W(t) \equiv u_1(t)u_2(t) - u_1(t)u_2(t) = e^{-\gamma t} = m/m(t) \). The solutions are oscillatory if \( \omega > \frac{\gamma}{2} \) (underdamping) and have a single zero for \( \omega < \frac{\gamma}{2} \) (overdamping) and \( \omega = \frac{\gamma}{2} \) (critical damping). In this last case the solutions are \( u_1(t) = t e^{-\gamma t} \) and \( u_2(t) = e^{\gamma t} \left( 1 + \frac{\gamma}{2} t \right) \).

Let us restrict our discussion to the underdamping case. From the preceding solutions the Arnold transformation (7) is given by

\[ \tau = \frac{u_1(t)}{u_2(t)} = \frac{\sin \Omega t}{\cos \Omega t + \frac{\gamma}{2\Omega} \sin \Omega t}, \]

\[ \kappa = \frac{x}{u_2(t)} = e^{-\frac{\gamma}{2} t} \left( \cos \Omega t + \frac{\gamma}{2\Omega} \sin \Omega t \right). \]  

(43)

The Arnold–Ermakov–Pinney transformation mapping the Caldirola–Kanai oscillator to the standard harmonic oscillator (and its corresponding quantum version) is also easily derived, resulting in

\[ r' = \int \frac{W(t)}{b(t)^2} dt = \frac{1}{\omega_0} \arctan \omega_0 \tau, \]

\[ x' = \frac{x}{b(t)}, \]  

(44)

where \( \omega_0 \) is the (arbitrary) frequency of the auxiliary harmonic oscillator and

\[ b(t) = u_2(t) \sqrt{1 + \omega_0^2 \tau^2} = \sqrt{u_2(t)^2 + \omega_0^2 u_1(t)^2} = e^{\frac{\gamma}{2} t} \sqrt{\left( \cos \Omega t + \frac{\gamma}{2\Omega} \sin \Omega t \right)^2 + \omega_0^2 \frac{\sin^2 \Omega t}{\Omega^2}}. \]  

(45)
Note that $b(t)$ never vanishes since this would imply $u_1(t) = u_2(t) = 0$ at some time instant, and thus the Wronskian would also be zero at that time instant, contradicting the fact that $u_1$ and $u_2$ are independent solutions to equation (39). Note also that $b(t)$ satisfies the canonicity conditions (22).

Let us construct an invariant for the Caldirola–Kanai oscillator. The first possibility is to construct a generalized Lewis invariant $\tilde{I}_L$ with $\tilde{\omega} = \omega$ and $\tilde{\gamma} = 0$:

$$\tilde{I}_L = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{X}^2,$$

leading to the invariant constructed by Pedrosa [24].

According to the discussion in section 2, for a damped system it is more appropriate to use the Dodonov–Man’ko invariant, which is built using the solution $u_1(t)$ and the (noncanonical) solution $\tilde{u}_2(t)$, with $\tilde{\omega} = \omega$ and $\tilde{\gamma} = \gamma$:

$$u_1(t) = e^{-\frac{\tilde{\gamma}}{2} t} \sin \frac{\Omega t}{\tilde{\Omega}},$$

$$\tilde{u}_2(t) = u_2(t) - \frac{\tilde{\gamma}}{2} u_1(t) = e^{-\frac{\tilde{\gamma}}{2} t} \cos \frac{\Omega t}{\tilde{\Omega}}. \quad (47)$$

Therefore $\tilde{b}(t) = \sqrt{\tilde{u}_2(t)^2 + \Omega^2 u_1(t)^2} = e^{-\frac{\tilde{\gamma}}{2} t}$, which also never vanishes. The Dodonov–Man’ko invariant (30) for these solutions is given by

$$\tilde{I}_{DM} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega \tilde{\omega} \hat{X}^2 + \frac{\tilde{\gamma}}{2} \hat{X} \hat{P} + \hat{P} \hat{X}$$

$$= \tilde{H}_{CK} + \frac{\tilde{\gamma}}{2} \hat{X} \hat{P} + \hat{P} \hat{X}. \quad (48)$$

It can be shown that, for the specific choice $\tilde{\omega} = \omega$ and $\tilde{\gamma} = \gamma$ made earlier, the invariant $\tilde{I}_{DM}$ coincides with that provided by Nassar [44, 45]. It also coincides with the invariant discussed by Cerveró and Villarroel [38].

The eigenstates of this invariant, solutions to the Caldirola–Kanai Schrödinger equation (note that the time-dependent phase appearing in the Lewis–Riesenfeld invariant method is included in the QAEPT), are derived from equation (32):

$$\phi_n(x, t) = \frac{1}{\sqrt{\sqrt{2 \pi} 2^n n!}} e^{-\left(i\omega t + \frac{1}{2} (2n + 1) \tilde{\gamma} t^2\right)}$$

$$\times e^{-\frac{m}{2\hbar} \frac{\tilde{\Omega}}{\tilde{\omega}} \left(\frac{2n + 1}{2}\right)^{\tilde{\gamma}} \hat{X}^2} H_n \left(\frac{m \Omega}{\hbar} e^{\frac{\tilde{\gamma}}{2} t} \hat{X}\right). \quad (49)$$

with eigenvalues $\hbar \Omega (n + \frac{1}{2})$, with $n = 0, 1, \ldots$ (assuming $\Omega > 0$). Thus, the result of [18] is recovered.

Note that in deriving (49) it was crucial to choose the solutions $u_1(t)$ and $\tilde{u}_2(t)$, since in this case $\tilde{b}(t) = e^{-\frac{\tilde{\gamma}}{2} t} = W^{1/2}$ and thus the change of variables in time in (44) is trivial, $t' = t$ (as explained in section 2.4). Thus, for the case of the damped harmonic oscillator the generalized Dodonov–Man’ko invariant $I_{GDM}$ coincides with $I_{DM}$. This is a special feature of the damped harmonic oscillator since $\tilde{f}(0) = 0$ in this case.

For an arbitrary wave function $\phi_{HO}$ of the harmonic oscillator, the corresponding wave function $\phi_{CK}$ in the Caldirola–Kanai system is given by

$$\phi_{CK}(x, t) = \frac{1}{\sqrt{\tilde{b}}} e^{i \frac{m}{\hbar} \frac{\tilde{\Omega}}{\tilde{\omega}} \frac{\tilde{b}^{1/2}}{2} \phi_{HO}\left(\frac{x}{\tilde{b}}, t\right)}$$

$$= e^{\frac{\tilde{\gamma}}{2} t} e^{-\frac{m}{2\hbar} \frac{\tilde{\Omega}}{\tilde{\omega}} \tilde{b}^{1/2} x^2} \phi_{HO}\left(\frac{e^{\frac{\tilde{\gamma}}{2} t} x}{\tilde{b}}, t\right). \quad (50)$$

This makes apparent the general strategy: make computations in a (simpler) system (e.g., compute $\phi_{HO}$) and map them to the system of interest (get $\phi_{CK}$).

It should be stressed that in deriving equation (50) it was not necessary to solve an eigenvalue equation for an invariant in the generalized Caldirola–Kanai variables; equation (50) is the result of a unitary transformation between two Hilbert spaces. This is a great improvement relative to the Lewis–Riesenfeld or the Dodonov–Man’ko invariant method, where first an invariant is found and then its eigenvectors are computed to provide a basis of the Hilbert space. With the QAEPT it is possible to map any harmonic oscillator wave function $\phi_{HO}$ to its corresponding function $\phi_{CK}$ in the Caldirola–Kanai Hilbert space. This applies in particular to number states, coherent states, squeezed states, and even density matrices (see [25]).

### 3.2. Hermite oscillator

Similar considerations can be applied in more general systems following the same steps. Let us consider an LSODE system that has a damping rate linear in time $\gamma = \alpha t$, with $\alpha > 0$:

$$\ddot{x} + \alpha \dot{x} + \alpha^2 x = 0. \quad (51)$$

This equation is similar to the Hermite differential equation; thus this system is known as the Hermite oscillator [49]. In analogy with the oscillator with time-dependent mass, the Hermite oscillator would have a mass

$$m(t) = m e^{\alpha^2 t / 2}. \quad (52)$$

To seek wavefunctions of the quantum Hermite oscillator we need the solutions to the classical equation (51). Those satisfying the canonicity conditions (10) are

$$u_1(t) = t F_1\left(\frac{1}{2}; \frac{\alpha^2}{2\alpha}; \frac{3}{2}; -\alpha t^2 / 2\right),$$

$$u_2(t) = t F_1\left(\frac{\alpha^2}{2\alpha}; 1; \frac{1}{2}; -\alpha t^2 / 2\right) \quad (53)$$

and the Wronskian is $W(t) = e^{-\alpha t^2 / 2}$. Here $F_1(a; b; z)$ denotes the confluent hypergeometric function [39].
From these solutions the Arnold transformation (equation (7)) is given by

$$
\tau = \frac{u_1(t)}{u_2(t)} = \frac{t F(1 + \frac{\omega^2}{2\alpha}; \frac{3}{2}; -\alpha^2 t^2/2)}{\alpha F(\frac{\alpha^2}{2\alpha}; 1; -\alpha^2 t^2/2)}
$$

and

$$
x = \frac{x}{u_2(t)} = \frac{x}{\alpha F(\frac{\alpha^2}{2\alpha}; 1; -\alpha^2 t^2/2)},
$$

(54)

and with this the QAT (equation (12)) is easily derived.

The solution $u_2(t)$ has zeros if $\alpha < \omega^2$ (underdamping) and has no zeros if $\alpha > \omega^2$ (overdamping) or $\alpha = \omega^2$ (critical damping). In this last case $u_2(t) = e^\pm i\omega t$ and $u_1(t) = \frac{1}{\sqrt{2\pi}} e^{\sigma^2/2} \text{erfi}(\frac{s}{\sqrt{2}})$. Thus, for overdamping and critical damping the Arnold transformation is defined for all times, whereas for the underdamping case the transformation is local in time, mapping a patch of the Hermite oscillator to a patch of a free-particle trajectory. Note that in the underdamping case the system performs a finite number of oscillations since the zeros of the confluent hypergeometric functions are finite: $u_2(t)$ is even and has $2 \left(\frac{\omega^2 - \alpha^2}{2\alpha}\right) + 1$ zeros, whereas $u_1(t)$ is odd and has $2 \left(\frac{\omega^2 - \alpha^2}{2\alpha}\right)$ zeros.

The Arnold–Ermakov–Pinney transformation (and its corresponding quantum version, QAEPT) is also easily derived, resulting in

$$
t' = \int \frac{W(t)}{b(t)^2}\,dt
= \frac{1}{\omega_0} \arctan \omega_0 \tau
$$

(55)

and

$$
x' = \frac{x}{b(t)},
$$

(56)

where $\omega_0$ is the (arbitrary) frequency of the auxiliary harmonic oscillator and

$$
b(t) = u_2(t) \sqrt{1 + \omega_0^2 \tau^2}
= \sqrt{u_2(t)^2 + \omega_0^2 u_1(t)^2}.
$$

(57)

Note that, as before, $b(t)$ never vanishes.

With this the QAEPT is obtained and the generalized Lewis invariant $\hat{I}_L$ (i.e., the Hamiltonian for the harmonic oscillator mapped to the Hermite oscillator) can be computed, recovering the results in [49]. Let us give, instead, the construction of a generalized Dodonov–Man’ko invariant as proposed in sections 2.3 and 2.4.

Choosing $b(t) = W(t) = e^{-\omega t/2}$ and renaming $x' \equiv y$, equation (51) transforms into

$$
y' + \left(\frac{\omega^2}{2} - \frac{\alpha}{2} - \frac{1}{4} \alpha^2 t^2\right) y = 0,
$$

(58)

and the wave functions transformed as given in equation (36):

$$
\phi(x, t) = e^{\frac{1}{2} \alpha^2 t^2} e^{-\frac{1}{2} \alpha^2 m \frac{\alpha^2}{4} x^2}
\times \varphi\left(e^{\frac{1}{2} \alpha^2 t^2} x, t\right),
$$

(59)

where $\varphi(y, t)$ is a solution to the Schrödinger equation for the time-dependent harmonic oscillator with frequency $\omega^2 - \frac{\alpha^2}{2} - \frac{1}{4} \alpha^2 t^2$. Applying the Lewis–Riesenfeld invariant method to this time-dependent harmonic oscillator, we obtain a Lewis invariant that, when mapped to the original system, provides a generalized Dodonov–Man’ko invariant, given by

$$
\hat{I}_{GDM} = \frac{1}{2m} \dot{\phi}^2 + \frac{1}{4} \left(\omega^2 - \alpha^2\right) \phi^2.
$$

(60)

Note that there is no term in $\frac{\dot{\phi} + \dot{\phi}}{2}$ because in this case $\dot{f}(0) = 0$. This term would appear if we had chosen another initial time $t_0 \neq 0$.

A similar construction can be performed for the Lane–Endem oscillator:

$$
\ddot{x} + \frac{\mu}{1 + i t} \dot{x} + \alpha_0^2 x = 0,
$$

(61)

for which the construction of the Lewis invariant was given in [50]. We only provide here the expression of the generalized Dodonov–Man’ko invariant, which turns out to be:

$$
\hat{I}_{GDM} = \frac{1}{2m} \dot{\phi}^2 + \frac{1}{2} \left(\alpha^2 + \frac{1}{\mu}\right) \phi^2
+ \frac{\mu}{2} \frac{\dot{X} \dot{P} + P \dot{X}}{2}.
$$

(62)

4. Conclusion and outlook

In this paper the relation between the Lewis–Riesenfeld invariant method and the quantum Arnold–ermakov–Pinney transformation has been established. The former aims at finding an invariant for a time-dependent harmonic oscillator to build a basis of eigenstates of this invariant satisfying the Schrödinger equation (up to a time-dependent phase). The latter is a unitary transformation that relates two generalized Caldirola–Kanai systems and that makes it possible to map states and invariant operators from one system to the other. In particular, if one of the systems is a harmonic oscillator and the other is a time-dependent harmonic oscillator, the Lewis–Riesenfeld invariant method is recovered. The time-dependent phase is built into the transformation in such a way that it maps solutions to the Schrödinger equation for one system into solutions to the Schrödinger equation for the other directly.

Any invariant of the harmonic oscillator is mapped, through the QAEPT, to an invariant of the time-dependent harmonic oscillator. In particular, the Hamiltonian for the harmonic oscillator is mapped to the Lewis invariant, explaining why it has discrete eigenvalues.

In this paper we have also shown that the QAEPT is global, in contrast with the QAT, which is local in time. This
explains the robustness and wide applicability of the Lewis–Riesenfeld invariant method.

The main advantage of the QAEPT is that it can also be applied to damped systems (or with time-dependent mass) in a unified way. In the case of constant damping and frequency we have shown that a more convenient choice is the Dodonov–Man’ko invariant rather than the usual Lewis invariant. The reason is that it shares with the Hamiltonian the loss–energy property [18] (which amounts to the only explicit time dependence being through the Wronskian W(r)), and therefore its eigenstates or the associated coherent states have more desirable properties.

We thus propose in the general case to use, as an alternative to the Lewis invariant, the generalized version of the Dodonov–Man’ko invariant when dealing with time-dependent damped (generalized Caldirola–Kanai) systems. This may be useful in designing shortcuts to adiabatic processes [16] when damping or time-dependent masses are present [34, 35], as well as in constructing coherent and related states.

Another interesting application of the QAEPT would be in the case of mixed states (see [25], where the QAT was discussed in this context). Let us denote by \( \hat{\rho}_{HO} \) a density matrix for the harmonic oscillator, satisfying the quantum Liouville equation

\[
\frac{\partial \hat{\rho}_{HO}}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}_{HO}, \hat{\rho}_{HO} \right].
\]

(63)

It follows that, the density matrix \( \hat{\rho}_{GCK} = \hat{E} \hat{\rho}_{HO} \hat{E}^{-1} \) is a proper density matrix (since \( \hat{E} \) is unitary: \( \text{Tr}(\hat{\rho}_{GCK}) = \text{Tr}(\hat{\rho}_{HO}) = 1 \)) satisfying the quantum Liouville equation for the generalized Caldirola–Kanai oscillator:

\[
\frac{\partial \hat{\rho}_{GCK}}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}_{GCK}, \hat{\rho}_{GCK} \right].
\]

(64)

All properties of the density matrix \( \hat{\rho}_{HO} \) are transferred to \( \hat{\rho}_{GCK} \), such as characteristic functions and quasi-probability distributions. Also, since \( \text{Tr}(\hat{\rho}_{GCK}^2) = \text{Tr}(\hat{\rho}_{HO}^2) \), the purity or mixed-state character of \( \hat{\rho}_{HO} \) is shared by \( \hat{\rho}_{GCK} \). In particular, if \( \hat{\rho}_{HO} \) describes a Gaussian state, \( \hat{\rho}_{GCK} \) also corresponds to a Gaussian state.

However, since the QAEPT does not transform Hamiltonians into one another (only Schrödinger and quantum Liouville equations do), care should be taken in the physical interpretation of the transformed density matrix. For instance, a thermal equilibrium state for the harmonic oscillator is not mapped to a thermal equilibrium state of the generalized Caldirola–Kanai oscillator (in the case where it makes physical sense, for instance, when the time scale of the time dependence of the Hamiltonian \( \hat{H}_{GCK} \) is much larger than that of relaxation to thermal equilibrium).

A deeper study of the QAEPT applied to mixed states to analyze how entanglement is transformed under \( \hat{E} \) (generalized to multipartite systems) and how it can be used to describe dissipation and decoherence, analyzing the transformation properties of master equations like the Lindblad equation under \( \hat{E} \), is the subject of a work in progress and will be presented elsewhere.

Acknowledgments

This work was partially supported by the MCYT and Junta de Andalucía under projects FIS2011-29813-C02-01 and FQM219.

References

[1] Lewis H R Jr 1967 Phys. Rev. Lett. 18 510
[2] Lewis H R Jr and Riesenfeld W B 1969 J. Math. Phys. 10 1458
[3] Ermakov V P 1880 Univ. Ev. Kieve 20 1
[4] Ermakov V P 2008 Appl. Anal. Discrete Math. 2 123
[5] Milne E W 1930 Phys. Rev. 35 86367
[6] Pinney E 1950 Proc. A.M.S I 681
[7] Guerrero J and López-Ruiz F F 2013 Phys. Scr. 87 038105
[8] Caldirola P 1941 Nuovo Cimento 18 393
[9] Kanai E 1948 Prog. Theor. Phys. 3 440
[10] Aldaya V, Cossío F, Guerrero J and López-Ruiz F F 2011 J. Phys. A 44 065302
[11] Lie S 1883 Arch. Math. VIII,IX 187
[12] Mahomed F M and Qadir A 2009 J. Nonlinear Math. Phys. 16 283
[13] Aminov A V and Aminov N 2010 N A-M Sb. Math. 201 631
[14] Maamache M 1995 Phys. Rev. A 52 936
[15] Rogers C and Hoenselaers C 1993 On higher dimensional Ermakov systems: canonical reductions, Loughborough University of Technology Technical Report ETN-93-94493
[16] Chen X, Ruschhaupt A, Schmidt S, Del Campo A, Guéry-Odelin D and Muga J G 2010 Phys. Rev. Lett. 104 063002
[17] Arnold V I 1978 Supplementary Chapters to the Theory of Ordinary Differential Equations (Moscow: Nauka)
Arnold V I 1983 English transl., Geometrical Methods in the Theory of Ordinary Differential Equations (New York: Berlin: Springer-Verlag)
[18] Dodonov V V and Man’ko V I 1979 Phys. Rev. A 20 550
[19] Dodonov V V and Man’ko V I 1989 Invariants and the Evolution of Nonstationary Quantum Systems (New York: Nova Science Publishers)
[20] Malkin I A, Man’ko V I and Trifonov D A 1969 Phys. Lett. A 30 414
[21] Pedrosa I A and Rosas A 2009 Phys. Rev. Lett. 103 010402
[22] Hartley J G and Ray J R 1982 Phys. Rev. D 25 382
[23] Lewis H R and Leach P G L 1982 J. Math. Phys. 23 165
[24] Pedrosa I A 1987 J. Math. Phys. 28 2662
[25] Guerrero J, López-Ruiz F F, Aldaya V and Cossío F 2011 J. Phys. A 44 445307
[26] López-Ruiz F F, Guerrero J and Aldaya V 2011 J. Russ. Laser Res. 32 372
[27] Guerrero J and López-Ruiz F F 2013 Nuovo Cimento C 36 127
[28] Guerrero J, Aldaya V, López-Ruiz F F and Cossío F 2012 Int. J. Geom. Meth. Mod. Phys. 9 1260011
[29] Horvathy P A 1979 Int. J. Theor. Phys. 18 245–50
[30] Castroso O, Schuch D and Rosas-Ortiz O 2013 J. Phys. A 46 075304
[31] Kagan Y, Surkov E L and Shlyapnikov G V 1996 Phys. Rev. A 54 R1753
[32] Castin Y and Dum R 1996 Phys. Rev. Lett. 77 5315
[33] García-Ripoll J J, Pérez-García V and Torres P 1999 Phys. Rev. Lett. 83 1715
[34] Stefanatos D 2014 Phys. Rev. A 90 023811
[35] Rodríguez-Lara B M, Aleahmad P, Moya-Cessa H M and
Christodoulides D N 2014 Opt. Lett. 39 2083
[36] Pedrosa I A, Melo J L and Nogueira E Jr 2014 Mod. Phys. Lett.
B 28 1450212
[37] Moya-Cessa H and Fernández Guasti M 2003 Phys. Lett. A
311 1
[38] Cerveró J M and Villarroel J 1984 J. Phys. A 17 2963
[39] Gradshteyn I S and Ryzhik I M 2007 Table of Integrals, Series
and Products (New York: Academic)
[40] Gzyl H 1983 Phys. Rev. A 27 2297
[41] Schuch D 1997 Phys. Rev. A 55 935
[42] Dekker H 1981 Phys. Rep. 80 1
[43] Dodonov V V, Malkin I A and Man’ko V I 1974 Physica 74 597
[44] Nassar A B 1986 J. Math. Phys. 27 755
[45] Nassar A B 1986 J. Math. Phys. 27 2949
[46] Gzyl H 1983 Phys. Rev. A 27 2297
[47] Schuch D 1990 Int. J. Quant. Chem., Quant. Chem. Symp.
24 767
[48] Schuch D 1999 Int. J. Quant. Chem. 72 537
[49] Um C-I, Choi J-R and Yeon K-H 2001 J. Korean Phys. Soc.
38 447
[50] Vubangsi M, Tchoffo M and Fai L C 2013 Afr. Rev. Phys.
8 341
[51] Schuch D, Guerrero J, López-Ruiz F F and Aldaya V 2015
Phys. Scr. 90 045209