On the accumulation of separatrices by invariant circles

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Abstract. Let \( f \) be a smooth symplectic diffeomorphism of \( \mathbb{R}^2 \) admitting a (non-split) separatrix associated to a hyperbolic fixed point. We prove that if \( f \) is a perturbation of the time-1 map of a symplectic autonomous vector field, this separatrix is accumulated by a positive measure set of invariant circles. However, we provide examples of smooth symplectic diffeomorphisms with a Lyapunov unstable non-split separatrix that are not accumulated by invariant circles.

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1. Introduction

A theorem by Herman, ‘Herman’s last geometric theorem’, cf. [9, 12], asserts that if a smooth orientation- and area-preserving diffeomorphism \( f \) of the 2-plane \( \mathbb{R}^2 \) (or the 2-cylinder \( \mathbb{R}/\mathbb{Z} \times \mathbb{R} \)) admits a Kolmogorov–Arnold–Moser (KAM) circle \( \Sigma \) (by definition, a smooth invariant curve, isotopic in \( \mathbb{R}^2 \setminus \{0\} \) to a circle centered at the origin in the case \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) or isotopic to \( \mathbb{R}/\mathbb{Z} \times \{0\} \) in the cylinder case, on which the dynamics of \( f \) is conjugated to a Diophantine translation), then this KAM circle is accumulated by other KAM circles, the union of which has positive two-dimensional Lebesgue measure in any neighborhood of \( \Sigma \). In this paper, we investigate whether such a phenomenon holds if, instead of being a KAM circle, the invariant set \( \Sigma \) is a separatrix of a hyperbolic fixed (or periodic) point of \( f \).

More precisely, we consider the following situation (see Figure 1). Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \), \( f : (x, y) \mapsto f(x, y) \), \( f(0, 0) = (0, 0) \) be a smooth diffeomorphism which is symplectic with respect to the usual symplectic form \( \omega = dx \wedge dy \) (\( f^*\omega = \omega \)). We assume that

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$o := (0, 0)$ is a hyperbolic fixed point of $f$ (the matrix $Df(o) \in SL(2, \mathbb{R})$ has distinct real eigenvalues) and that there exists an $f$-invariant compact connected set $\Sigma \ni o$ such that $\Sigma \setminus \{o\}$ is a non-empty connected one-dimensional manifold included in both the stable and unstable manifolds $W^s_f(o)$, $W^u_f(o)$ associated to $o$:

$$\text{for all } (x, y) \in \Sigma, \quad \lim_{n \to \pm \infty} f^n(x, y) = o.$$ 

Note that because $o$ is $f$-hyperbolic, $\Sigma$ is homeomorphic to a circle and $\Sigma \setminus \{o\}$ coincides with one of the two connected components of $W^s_f(o) \setminus \{o\}$ (respectively $W^u_f(o) \setminus \{o\}$). We shall say that $\Sigma$ is a separatrix of $f$ associated to the hyperbolic fixed point $o$ or, without referring to the hyperbolic fixed point $o$, that $\Sigma$ is a separatrix of $f$.

Examples of such diffeomorphisms $f$ can be obtained in the following way. Let $X_0$ be a smooth autonomous Hamiltonian vector field of the form

$$X_0 = J \nabla H_0, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(1.1)

where $H_0 : \mathbb{R}^2 \to \mathbb{R}$, of the form

$$H_0(x, y) = \lambda xy + O^3(x, y), \quad \lambda \in \mathbb{R}^\ast,$$

(we can assume without loss of generality $\lambda > 0$) is a smooth function. The time-1 map $f_0 := \phi_{X_0}^1$ of $X_0$ is a Hamiltonian (in particular, symplectic) diffeomorphism of $\mathbb{R}^2$ admitting $o$ as a hyperbolic fixed point. We assume that it has a separatrix $\Sigma \ni o$ of the form

$$\Sigma \setminus \{o\} = \{\phi_{X_0}^t(p), \ t \in \mathbb{R}\} \text{ for some } p \in \mathbb{R}^2 \setminus \{o\} \text{ such that } \lim_{t \to \pm \infty} \phi_{X_0}^t(p) = o.$$ 

We now consider a smooth time-dependent Hamiltonian vector field $Y : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}$, $(t, (x, y)) \mapsto Y(t, x, y)$ which is 1-periodic in $t$, symplectic with respect to $(x, y)$, and tangent to $\Sigma \setminus \{o\}$:

$$\text{for all } t \in \mathbb{R}/\mathbb{Z}, \text{ for all } (x, y) \in \Sigma, \quad \det(X_0(x, y), Y(t, x, y)) = 0.$$
One can for example choose $Y(t, x, y) = J \nabla F(t, x, y)$, where $F : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}$ is a smooth time-dependent Hamiltonian that satisfies

for all $t \in \mathbb{R}/\mathbb{Z}$, for all $(x, y) \in \Sigma$, $F(t, x, y) = F(t, 0, 0)$.

Note that because $o$ is a hyperbolic fixed point of $X_0$, one has for all $t$, $Y(t, o) = 0$. For $\varepsilon \in \mathbb{R}$, define the 1-periodic in $t$ symplectic vector field $\mathbb{R}^2 \to \mathbb{R}^2$ as

$$X_\varepsilon^t(x, y) := X_\varepsilon(t, x, y) = X_0(x, y) + \varepsilon Y(t, x, y).$$

(1.2)

For $\varepsilon$ small enough, the time-0-to-1 map,

$$f_\varepsilon = \phi_{X_\varepsilon}^{1,0},$$

(1.3)

of the symplectic vector field $X_\varepsilon$ is a symplectic diffeomorphism of $\mathbb{R}^2$ admitting $o$ as a hyperbolic fixed point and still $\Sigma$ as a separatrix. (If $X(t, z)$ is a time dependent vector field, the time-$s$-to-$t$ map of $X$ is defined by $\phi_X^{t,s}(z(s)) = z(t)$ for any $z(\cdot)$ solution of $\dot{z}(t) = X(z(t))$. When $X$ is time independent, the notation $\phi_X^t$ stands for $\phi_X^{t,0}$.) Note that $f_\varepsilon$ is a Hamiltonian diffeomorphism (for more details on Hamiltonian diffeomorphisms, see [16]).

Here is the analogue of the aforementioned last geometric theorem of Herman.

**Theorem A.** For any $r \in \mathbb{N}^*$, there exists $\varepsilon_r > 0$ such that, for any $\varepsilon \in ]-\varepsilon_r, \varepsilon_r[$, there exists a set of $f_\varepsilon$-invariant $C^r$ KAM circles accumulating the separatrix $\Sigma$ and which covers a set of positive Lebesgue measure of $\mathbb{R}^2$ in any neighborhood of $\Sigma$.

Let us clarify some points made in the preceding statement.

By a $C^r$ circle, $r \geq 0$, we mean a $C^r$ non-self-intersecting closed curve (or equivalently, if $r \geq 1$, a non-empty connected one-dimensional $C^r$ submanifold of $\mathbb{R}^2$) which is isotopic in $\mathbb{R}^2 \setminus \{o\}$ to the separatrix $\Sigma$. Such a set $\Gamma$ is invariant by $f_\varepsilon$ if $f_\varepsilon(\Gamma) = \Gamma$.

We say that a set $G$ of $f_\varepsilon$-invariant circles accumulates the set $\Sigma$ if for any $\xi > 0$, the set of $\Gamma \in G$ such that $\text{dist}(\Gamma, \Sigma) < \xi$ is not empty, where $\text{dist}$ denotes the Hausdorff distance,

$$\text{dist}(A, B) = \max \left( \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right)$$

(here $d(x, C) = \inf_{c \in C} \|x - c\|_{\mathbb{R}^2}$).

The $f_\varepsilon$-invariant circles obtained in Theorem A are KAM circles: the restrictions of $f_\varepsilon$ on each of these curves are $C^r$ circle diffeomorphisms that are conjugated to Diophantine translations. A real number $\alpha$ is Diophantine if there exist positive constants $\kappa$, $\tau$ such that, for any $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$, $|\alpha - (p/q)| \geq \kappa/q^\tau$. The constants $\tau$ and $\kappa$ are respectively called the exponent and the constant of the Diophantine condition. The set of Diophantine numbers with fixed exponent $\tau > 2$ has full Lebesgue measure if the constant is not specified and positive measure if the constant is also fixed (and small). In our case, the exponent of the Diophantine condition can be chosen to be independent of $\varepsilon$ (it depends only on $\lambda$).
Remark 1.1. However, and this is a difference with the situation of Herman’s last geometric theorem, the constants of these Diophantine numbers are arbitrarily small. Moreover, as these circles accumulate the separatrix, their $C^2$-norm must explode.

Remark 1.2. The phase space $\mathbb{R}^2$ can be replaced by the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ in the statement of the main theorem.

The smallness condition in Theorem A is indeed necessary as shown by the following theorem.

Let $\Delta_\Sigma$ be the bounded connected component of $\mathbb{R}^2 \setminus \Sigma$.

**Theorem B.** There exists a smooth symplectic diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ admitting a separatrix $\Sigma$ which is included in an open set $W$ of $\Sigma \cup \Delta_\Sigma$ that contains no $f$-invariant circle in $W \setminus \Sigma$.

The situation described in Theorems A and B is not generic. Indeed, as Poincaré discovered, in general, the stable and unstable manifolds of a hyperbolic fixed or periodic point of a symplectic map intersect transversally (one usually refers to this phenomenon as the splitting of separatrices), a fact that forces the dynamics of $f$ to be ‘quite intricate’. This was Poincaré’s key argument in his proof of the fact that the Three-body problem in Celestial Mechanics does not admit a complete set of independent commuting first integrals. Later, Smale [18] showed that this splitting of separatrices has an even more striking consequence on the dynamics of $f$, namely the existence of a horseshoe, that is, a uniformly hyperbolic $f$-invariant compact set (locally maximal) with positive topological entropy and on which the dynamics of $f$ is ‘chaotic’ (isomorphic to a two-sided shift). By a result of the first author [13], in this situation, positive topological entropy is indeed equivalent to the existence of a horseshoe. A consequence of the splitting of a separatrix is thus the existence of a Birkhoff instability zone (open region without invariant circles) in the vicinity of this split separatrix (see [11] for a detailed exposition on the topic). In some sense, Theorem A shows that in the perturbative situation of equations (1.2)–(1.3) ($\varepsilon$ small enough), the splitting of separatrices is essentially the only mechanism responsible for the creation of instability zones. However, in a ‘non-perturbative’ situation, Theorem B points in the opposite direction. Figures 4 and 7 illustrate the role that plays the smallness assumption in Theorem A (or its absence in Theorem B).

1.1. On the proofs of Theorems A and B. As suggests Remark 1.1, the invariant circles of Theorem A cannot be obtained directly via a classical KAM approach. However, the existence of the (non-split) separatrix $\Sigma$ allows to associate to each diffeomorphism $f_\varepsilon$ a regular diffeomorphism $\hat{f}_\varepsilon$, defined on a standard open annulus and preserving a finite probability measure, to which one can apply Moser’s or Rüssmann’s invariant (or translated) curve theorem [15, 17] (see §6). The thus obtained invariant curves for $\hat{f}_\varepsilon$ yield invariant curves for $f_\varepsilon$. The construction of the diffeomorphism $\hat{f}_\varepsilon$ is done as follows. We first make preliminary reductions involving some Birkhoff and symplectic Sternberg-like normal forms (§2) to have a control on the dynamics in some neighborhood of the hyperbolic fixed point $o$ (§3). This allows us to define in §4 a first return map $\tilde{f}_\varepsilon$ for $f_\varepsilon$,
in a fundamental domain $F_\varepsilon$, the boundaries of which can be glued together to obtain an open abstract cylinder (or annulus). This abstract cylinder can be uniformized to become a standard annulus and the first return map $\hat{f}_\varepsilon$ then becomes a regular diffeomorphism $\hat{f}_\varepsilon$ of a standard annulus (preserving some probability measure). This is done in §5. We call normalization (see §5.3) the uniformization operation and we say that $\hat{f}_\varepsilon$ is the renormalization of $f_\varepsilon$. The term renormalization in this paper has the same acceptation as in the theories of circle diffeomorphisms, holomorphic germs, or quasi-periodic cocycles; cf. [6, 14, 23, 24]. The dynamics of $\hat{f}_\varepsilon$ is closely related to that of $f_\varepsilon$ in the sense that the existence of invariant curves for $\hat{f}_\varepsilon$ translates into a similar statement for $f_\varepsilon$ (see §7). The renormalized diffeomorphism $\hat{f}_\varepsilon$ has a large twist (this is reminiscent of the hyperbolicity of $f_\varepsilon$ at $o$) and we are thus led to rescale it to obtain the aforementioned diffeomorphism $\hat{f}_\varepsilon$ which is now a small $C^r$-perturbation of an integrable twist map (this is where the smallness assumption of Theorem A appears) with a controlled twist (see §6). The proof of Theorem A is completed in §8.

To prove Theorem B (cf. §9), we construct a symplectic diffeomorphism $f$ (named $f_{\text{pert}}$ in that section) so that the associated renormalized diffeomorphism $\hat{f}$ has an orbit accumulating the boundary of the aforementioned annulus: this prevents the existence of $\hat{f}$-invariant curves close to this boundary and therefore of $f$-invariant curves close to the separatrix $\Sigma$.

We note that the authors of [21] introduce the ‘separatrix map’ constructed by a gluing construction to investigate the size of the instability zones. Our approach here, which is focused on a renormalization point of view, is different. The technique we use to prove Theorem A might be useful to study the dynamics of symplectic twist maps with zero topological entropy. That is, to which extent are they integrable? Angenent, [1], proves they are $C^0$-integrable in the sense that, for any rotation number, one can find a $C^0$-invariant curve with this rotation number. Can one prove $C^k$-integrability? The word ‘integrable’ is meant in a broad sense. Additionally, the construction of Theorem B might give a hint to provide examples of smooth twist maps admitting isolated invariant circles with irrational rotation number (if they exist, these curves bound two instability zones). A modification of the example of Theorem B yields examples of such isolated invariant curves with rational rotation numbers. For the existence of curves with irrational rotation number in low regularity and related results, see [2–5].

2. Normal forms

The main result of this section is the following Sternberg-like symplectic normal form theorem (Proposition 2.1) that will allow us in §3 to control the long-time dynamics of $f_\varepsilon$ in a neighborhood of the hyperbolic point $o$. This will be useful when we shall define first return maps for $f_\varepsilon$ in a convenient fundamental domain, see §4.

Let $f_\varepsilon$ be defined by (1.2) and (1.3).

**Proposition 2.1.** For any $k \in \mathbb{N}^*$ large enough, there exists $\varepsilon_k > 0$ for which the following holds. There exist a smooth family $(q_{\varepsilon,k})_{\varepsilon \in I}$ ($I \ni 0$ some open interval of $\mathbb{R}$) of polynomials $q_{\varepsilon,k}(s) = \lambda s + O(s^2) \in \mathbb{R}[s]$ and a continuous family $(\Theta_{\varepsilon,k})_{\varepsilon \in I}$ of symplectic $C^k$-diffeomorphism of $\mathbb{R}^2$ such that $\Theta_{\varepsilon,k}(o) = o$, $D\Theta_{\varepsilon,k}(o) = \text{id}$, and on a neighborhood
$V_k$ of $o$ one has, provided $\varepsilon \in ]-\varepsilon_k, \varepsilon_k[$:

$$f_{\varepsilon, k} = \Theta_{\varepsilon, k} \circ f_\varepsilon \circ \Theta_{\varepsilon, k}^{-1}$$

(2.4)

and

$$\Theta_{0, k} \ast X_0 = J \nabla Q_{0, k}.$$  

(2.6)

Note that $o$ is still a hyperbolic fixed point of $f_{\varepsilon, k}$ and that

$$\Sigma_{\varepsilon, k} := \Theta_{\varepsilon, k}(\Sigma)$$

is still a separatrix for $f_{\varepsilon, k}$.

2.1. Reduction of Theorem A to Theorem 2.1. After applying Proposition 2.1, we are thus left with a family $(f_{\varepsilon, k})$ of $C^k$-symplectic diffeomorphisms, each $f_{\varepsilon, k}$ being conjugated to $f_\varepsilon$ and admitting a separatrix $\Sigma_{\varepsilon, k}$. Because the conclusions of Theorem A are clearly invariant by conjugation, to prove Theorem A, we just need to prove that if $k \geq r$ and $\varepsilon$ is small enough, each separatrix $\Sigma_{\varepsilon, k}$ is accumulated by a set of positive measure of KAM circles for $f_{\varepsilon, k}$. This is the content of Theorem 2.1 below that we shall apply to the family of $C^k$-diffeomorphisms $f_{\varepsilon, k}$ defined by (1.2), (1.3), and (2.4), but that holds for any family (that we still denote in what follows by $(f_\varepsilon)_{\varepsilon \in I}$ to alleviate the notations) of symplectic $C^k$-diffeomorphisms satisfying the following hypothesis.

Let $(f_\varepsilon)_{\varepsilon \in I}, (I \ni 0$ open interval of $\mathbb{R})$ be a family of $C^k$-symplectic diffeomorphisms of $\mathbb{R}^2$ that satisfies:

(H1) each $f_\varepsilon$ has a (non-split) separatrix $\Sigma_\varepsilon$ associated to the hyperbolic point $o$;
(H2) the map $I \ni \varepsilon \rightarrow f_\varepsilon - \text{id} \in C^k(\mathbb{R}^2, \mathbb{R}^2)$ is continuous (the norm on $C^k$ is the usual $C^k$-norm);
(H3) on some neighborhood $V$ of $o$, each $f_\varepsilon$ coincides with the time-1 map of a symplectic vector field $J \nabla Q_\varepsilon(x, y)$ where $Q_\varepsilon(x, y) = q_\varepsilon(xy), q_\varepsilon \in C^{k+1}(\mathbb{R}^2)$

$$q_\varepsilon(t) = \lambda t + O^2(t), \quad \lambda > 0;$$

(H4) on $\mathbb{R}^2$, $f_0 = \phi_{X_0}^1$, where $X_0 = J \nabla H_0$ is a Hamiltonian vector field that coincides with $J \nabla Q_0$ on $V$.

Remarks 2.1. On $V$, the orbits of $f_\varepsilon|_V = \phi_{J \nabla Q_\varepsilon}^1$ are pieces of hyperbolae $\{xy = \text{constant}\}$ (condition (H3)).

When $\varepsilon = 0$, for any $z \in \{xy = c\} \cap V$, $N \in \mathbb{Z}$ such that $f_0^N(z) \in V$, one has $f_0^N(z) \in \{xy = c\} \cap V$ (condition (H4)).

Remark 2.2. The intersection $\Sigma_\varepsilon \cap V$ is the union

$$\Sigma_\varepsilon \cap V = (W^s_{f_\varepsilon}(o) \cap V) \cup (W^u_{f_\varepsilon}(o) \cap V)$$

and

$$W^s_{f_\varepsilon}(o) \cap V = (\mathbb{R} \times \{0\}) \cap V, \quad W^u_{f_\varepsilon}(o) \cap V = (\{0\} \times \mathbb{R}) \cap V.$$

One then has the following.
THEOREM 2.1. There exists $k_0 \in \mathbb{N}$ for which the following holds. Let $k \geq k_0 + 2$ and let $(f_\varepsilon)_{\varepsilon \in I}$ be a family of $C^k$-symplectic diffeomorphisms of $\mathbb{R}^2$ satisfying the previous conditions (H1)–(H4). Then, there exists $\varepsilon_1 > 0$ such that, for any $\varepsilon \in ]-\varepsilon_1, \varepsilon_1[$, the diffeomorphism $f_\varepsilon$ admits a set of positive Lebesgue measure of invariant $C^{k-k_0-2}$-circles in any neighborhood of the separatrix $\Sigma_{\varepsilon}$.

Moreover, if $k - k_0 - 2 \geq k_1$ ($k_1$ depending only on $\lambda$), these circles are KAM circles.

We shall give the proof of Theorem 2.1 in §8.

The proof of Proposition 2.1 occupies the rest of this section. It will be based on a first reduction obtained by performing some steps of Birkhoff normal forms (Proposition 2.3) and then the application of various Sternberg-like normal forms (Corollary 2.4 and Proposition 2.5).

2.2. Birkhoff normal form for the time-periodic vector field $X^t_\varepsilon$. A preliminary step in Sternberg’s classical linearization theorem is to first conjugate the considered system (diffeomorphism or vector field) defined in the neighborhood of the hyperbolic fixed point $o$ to a system which is tangent to an integrable model to some high enough order. This is what we do in this subsection and in a symplectic framework (see Proposition 2.3) by using Birkhoff normal form techniques.

2.2.1. Periodically forced vector fields. Let $X : \mathbb{R} \times \mathbb{R}^2 \ni (t, x) \mapsto X(t, z) \in \mathbb{R}^2$ be a smooth time-dependent symplectic vector field: for each $t$, the 1-form $i_{X_t} \omega$ is closed (and hence locally exact). For $t, s \in \mathbb{R}$, we denote by $\phi_{t,s}^X$ the flow of $X$ between times $s$ and $t$ when it is defined (see page 3 for the definition of $\phi_{t,s}^X$). If $t \mapsto g^t(\cdot)$ is a one-parameter family of symplectic diffeomorphisms, one has

$$g^t \circ \phi_{t,s}^X \circ (g^s)^{-1} = \phi_{t,s}^{\tilde{X}},$$

where $\tilde{X} : (t, z) \mapsto \tilde{X}(t, z)$ is the smooth time-dependent symplectic vector field

$$\tilde{X} = \partial_t g^t \circ (g^t)^{-1} + (g^t)_* X^t.$$  \hfill (2.8)

Conversely, if (2.8) is satisfied, then so is (2.7). Note that if $g^t$ depends 1-periodically on $t$, then (2.7) yields the more classical conjugation equation

$$g \circ \phi_{1,0}^{X} \circ g^{-1} = \phi_{1,0}^{\tilde{X}},$$

where $g = g^0 = g^1$ ($g^t$ is 1-periodic in $t$).

Assume now that $X^t$ depends 1-periodically in $t$ and, in a smooth way, on a small parameter $\varepsilon \in \mathbb{R}$; we furthermore assume that it is of the form

$$X^t_\varepsilon(z) = J \nabla H^t_\varepsilon(z),$$ \hfill (2.9)

where $(z = (z_1, z_2) \in \mathbb{R}^2)$

$$H^t_\varepsilon(z) = \lambda_\varepsilon(t) z_1 z_2 + O^3(z), \quad \int_T \lambda_\varepsilon(t) \, dt > 0, \quad \lambda_0(t) = \lambda \in \mathbb{R}_+^*,$$ \hfill (2.10)
$H_\varepsilon : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}, \quad H_\varepsilon : (t, z) \mapsto H_\varepsilon(t, z) := H_\varepsilon^j(z)$ being a smooth function. Assume also that, for some $j \in \mathbb{N}^*$,

$$g_\varepsilon^j(z) = \phi_{j}^{-1}\nabla G_\varepsilon^j(z) = \text{id} + O^j(z), \quad G_\varepsilon^j(z) = O^{j+1}(z),$$

where $G : I \times \mathbb{R}/\mathbb{Z} \times (\mathbb{R}^2, o) \ni (\varepsilon, t, z) \mapsto G_\varepsilon(t, z) := G_\varepsilon^j(z) \in \mathbb{R}$ is a smooth function. Then, one has

$$\partial_t g_\varepsilon^j \circ (g_\varepsilon^j)^{-1} = J\nabla \partial_t G_\varepsilon^j + O^j(z),$$

(here $\{A, B\}$ denotes the Poisson bracket $\{A, B\} = \langle \nabla A, J\nabla B \rangle$) so that $\tilde{X}_\varepsilon^j$ defined by (2.8) is of the form

$$\tilde{X}_\varepsilon^j = J\nabla \tilde{H}_\varepsilon^j,$$

with

$$\tilde{H}_\varepsilon^j = H_\varepsilon^j + \partial_t G_\varepsilon^j + \{G_\varepsilon^j, H_\varepsilon^j\} + O^{j+2}(z)$$

(2.12)

$$= H_\varepsilon^j + \partial_t G_\varepsilon^j + \{G_\varepsilon^j, H_\varepsilon^j\} + O^{j+2}(z),$$

(2.13)

where we have denoted $H_\varepsilon^j(z_1, z_2) = \lambda_\varepsilon(t) z_1 z_2$.

If in the preceding equation, one chooses $G_\varepsilon^j = G_\varepsilon^j, 2$ with $G_\varepsilon^j, 2(z) = a_\varepsilon, 0(t) z_1 z_2$, where $a_\varepsilon, 0$ is the 1-periodic function defined by

$$a_\varepsilon, 0(t) = -\int_0^t (\lambda_\varepsilon(s) - \int_T \lambda_\varepsilon(u) \, du) \, ds,$$

one has

$$H_\varepsilon^j(z) = \tilde{\lambda}_\varepsilon z_1 z_2 + O^3(z),$$

where $\tilde{\lambda}_\varepsilon = \int_T \lambda_\varepsilon(t) \, dt$. In other words, performing a change of coordinates (2.8) on $X_\varepsilon^j$ with $g_\varepsilon^j = g_\varepsilon^j, 2 = \phi_{j}^{-1}\nabla G_\varepsilon^j, 2$, we can assume that in (2.10), $\lambda_\varepsilon(t)$ does not depend on $t$:

$$H_\varepsilon^j(z) = \lambda_\varepsilon z_1 z_2 + O^3(z), \quad \lambda_\varepsilon \in \mathbb{R}_+^*$$

(2.14)

(we write $\lambda_\varepsilon$ in place of $\tilde{\lambda}_\varepsilon$).

2.2.2. Birkhoff normal form. Having put $H_\varepsilon^j$ under the form (2.14), we now eliminate by successive conjugations (2.8) non-diagonal higher-order terms in $z$ from $H_\varepsilon^j$ (note that they depend on $t$).

The following lemma describes this elimination procedure.

**Lemma 2.2.** Let $j \in \mathbb{N}, \ j \geq 2$. Assume that, for some polynomials $q_\varepsilon(s) = \lambda s + O(s^2) \in \mathbb{R}[s]$ of degree $\leq [j/2]$ depending smoothly on $\varepsilon$,

$$H_\varepsilon^j(z) = q_\varepsilon(z_1 z_2) + O^{j+1}(z).$$

Then, there exist a smooth family $(\tilde{q}_\varepsilon)_\varepsilon$ of polynomials $\tilde{q}_\varepsilon(s) = \lambda s + O(s^2) \in \mathbb{R}[s]$ of degree $\leq [(j + 1)/2]$ and a smooth family of smooth maps $G_\varepsilon : \mathbb{R}/\mathbb{Z} \times (\mathbb{R}^2, o) \ni (t, z)$ →
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\begin{equation}
G_\varepsilon(t, z) = G^t_\varepsilon(z) \in \mathbb{R}^2 \text{ such that on a neighborhood of } o,
\begin{align*}
G^t_\varepsilon(z) &= O^{j+1}(z), \\
H^t_\varepsilon(z) &= \partial_t G^t_\varepsilon(z) + \{G^t_\varepsilon, H^t_\varepsilon\}(z) = \hat{q}_\varepsilon(z_1 z_2) + O^{j+2}(z).
\end{align*}
\tag{2.15}
\end{equation}

Moreover, if for } \varepsilon = 0, H^t_0 \text{ does not depend on } t, \text{ one can choose } G^t_0 \text{ to be independent of } t.

\textbf{Proof.} See Appendix A.

Let now } X^t_\varepsilon \text{ be the family of vector fields of (1.2).

\textbf{Proposition 2.3.} For any } N \geq 1 \text{ there exist an open neighborhood } V_N \text{ of } o, \text{ a smooth two-parameters family } (b^t_\varepsilon)_{\varepsilon \in I, t} \in \mathbb{R}/\mathbb{Z} \text{ (I some open interval containing 0) of smooth symplectic diffeomorphisms } b^t_\varepsilon : (\mathbb{R}^2, o) \circlearrowright \text{ satisfying } b^t_\varepsilon(o) = o, \text{ Db}^t_\varepsilon(o) = \text{id} \text{ and a smooth family of polynomials } q_{\varepsilon, N}(s) = \lambda s + O(s^2) \text{ of degree } \leq [(N + 1)/2], \text{ such that, for any } \varepsilon \in I, t \in \mathbb{R}/\mathbb{Z}, (x, y) \in V_N \text{ one has}

\begin{align*}
X^t_\varepsilon(x, y) &= \partial_t b^t_\varepsilon \circ (b^t_\varepsilon)^{-1} + (b^t_\varepsilon)_* X^t_\varepsilon \\
&= J \nabla Q_{\varepsilon, N} + O^{N+1}(x, y) \text{ with } Q_{\varepsilon, N}(x, y) = q_{\varepsilon, N}(xy),
\end{align*}

and for } \varepsilon = 0, b^t_0 \text{ is independent of } t.

\textbf{Proof.} Applying the preceding Lemma 2.2 and relations (2.8)–(2.12) inductively (starting from (2.14)), we thus construct polynomials } q_{\varepsilon, j} \text{ of degree } \leq [j/2] \text{ (} j \geq 2 \text{) and functions } G^t_{\varepsilon, j} = O^{j+1}(z) \text{ such that if one defines}

\begin{align*}
b^t_\varepsilon &= g^t_{\varepsilon, N} \circ \cdots \circ g^t_{\varepsilon, 2} = \text{id} + O^2(z), \\
g^t_{\varepsilon, j} &= \phi^t_{J \nabla G^t_{\varepsilon, j}} = \text{id} + O^j(z),
\end{align*}

one has

\begin{align*}
\tilde{X}^t_\varepsilon &= \partial_t b^t_\varepsilon \circ (b^t_\varepsilon)^{-1} + (b^t_\varepsilon)_* X^t_\varepsilon \\
&= J \nabla Q_{\varepsilon, N} + O^{N+1}(z) \text{ with } Q_{\varepsilon, N}(z) = q_{\varepsilon, N}(z_1 z_2),
\end{align*}

all depending on } \varepsilon \text{ being smooth. Moreover, if } X^t_0 \text{ is independent of } t, \text{ the diffeomorphism } b^t_0 \text{ is independent of } t.

\textbf{Remark 2.3.} Note that because } b^t_0 = b_0 \text{ is independent of } t, \text{ the vector field }

\begin{equation}
X^{(1)}_0 = (b_0)_* X_0
\end{equation}

is autonomous.

2.3. \textbf{Symplectic Sternberg theorem for the autonomous vector field } X^{(1)}_0. \textbf{ We shall need a symplectic version of the famous theorem by S. Sternberg (on smooth linearization of hyperbolic germs of smooth vector fields, see [19]), as proved in [7] or [8] (see also [20]). We follow here the exposition of [7].}
Let $Z_i, i = 1, 2$, be two symplectic smooth autonomous vector fields such that, for some $\lambda \in \mathbb{R}^*$ and $N \in \mathbb{N}$, one has
\[
\begin{align*}
    Z_i(x, y) &= -\lambda x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} + O^2(x, y) \quad (i = 1, 2), \\
    Z_1(x, y) - Z_2(x, y) &= O^{N+1}(x, y).
\end{align*}
\tag{2.16}
\]

**Theorem 2.2.** [7, Theorem 1.2] There exist positive constants $A, B$ for which the following holds. Let $m \in \mathbb{N}^*$ large enough and $N = [(m + B)/A] + 1 \geq 1$. If (2.16) is satisfied, then there exists a $C^m$ symplectic change of coordinates $S_0 : (\mathbb{R}^2, 0) \to \mathbb{R}^2$ such that on a neighborhood of $o$,
\[
\begin{align*}
    (S_0)_* Z_1 &= Z_2, \\
    S_0(o) &= 0, \\
    D S_0(o) &= \text{id}.
\end{align*}
\tag{2.17}
\]

We apply the preceding theorem to the case $Z_1 = X_0^{(1)}$ and $Z_2 = J \nabla Q_{0,N}$ ($X_0^{(1)}, Q_{0,N}$ given by Proposition 2.3 when $\varepsilon = 0$). In view of Proposition 2.3, the condition (2.16) is satisfied and we hence get a symplectic diffeomorphism $S_0$ satisfying $S_0(o) = 0$, $D S_0(o) = \text{id}$, and such that on a neighborhood of $o$,
\[
(S_0)_* X_0^{(1)} = J \nabla Q_{0,N}.
\]

For each value of $t \in \mathbb{R}/\mathbb{Z}$ and $\varepsilon \in I$, the diffeomorphism $(S_0 \circ b_t^\varepsilon)$ fixes the origin and its derivative at the origin is the identity. It can thus be extended as a symplectic $C^m$-diffeomorphism $R_\varepsilon^t$ of $\mathbb{R}^2$ (cf. Lemma B.1). Note that the dependence of $R_\varepsilon^t$ with respect to $t$ is smooth and 1-periodic ($t \in \mathbb{R}/\mathbb{Z}$). We now define on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}^2$ the time-periodic vector field $X_\varepsilon^{(2)} : (t, (x, y)) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}^2$ by
\[
X_\varepsilon^{(2),t} = (\partial_t R_\varepsilon^t) \circ (R_\varepsilon^t)^{-1} + (R_\varepsilon^t)_* X_\varepsilon^t,
\tag{2.18}
\]
and we observe that on a neighborhood of $o$,
\[
(R_0^0)_* X_0 = X_0^{(2)} = J \nabla Q_{0,N}.
\]

Because the conjugacy relation (2.18) is equivalent to (see §2.2.1)
\[
\text{for all } t, s, \quad R_\varepsilon^t \circ \phi_\varepsilon^{t,s} \circ (R_\varepsilon^s)^{-1} = \phi_\varepsilon^{t,s},
\]
we get by taking $t = 1, s = 0$, and setting $R_\varepsilon := R_\varepsilon^1 = R_\varepsilon^0$, the following corollary.

**Corollary 2.4.** If $m \in \mathbb{N}^*$ is large enough and $N = [(m + B)/A] + 1$, there exists a smooth family $(R_\varepsilon)$ of $C^m$ symplectic diffeomorphisms of $\mathbb{R}^2$ such that $R_\varepsilon(o) = o$, $D R_\varepsilon(o) = \text{id}$, and on a neighborhood of $o$,
\[
f_\varepsilon^{(1)} : = R_\varepsilon \circ f_\varepsilon \circ (R_\varepsilon)^{-1} = \phi_\varepsilon^{1}(J \nabla Q_{\varepsilon,N}) + O^{N+1}(x, y).
\tag{2.19}
\]

Moreover,
\[
(R_0)_* X_0 = J \nabla Q_{0,N}.
\tag{2.21}
\]
Note that the last equation shows that
\[ f_0^{(1)} = \phi^1 J \nabla Q_{0,N}. \]  

2.4. Symplectic Sternberg normal form for the diffeomorphism \( f_ε^{(1)} \). Theorem 2.2 has a version for smooth germs of symplectic diffeomorphisms which are hyperbolic at the origin. This is theorem 1.1 of [7]. In our paper, we shall need a parametric version of that result, which is not explicitly stated in [7] but that can be checked after close examination of the proof.

**Proposition 2.5.** There exist constants \( A_1, B_1 \) depending on \( \lambda \in \mathbb{R}^* \) such that the following holds. Let \( m \in \mathbb{N}^* \) large enough and \( N = \lfloor m/2 \rfloor - 3 \). If \( (g_{1,ε})_{ε \in I} \) and \( (g_{2,ε})_{ε \in I} \) \((I \ni 0 \text{ some open interval of } \mathbb{R})\) are two continuous \((\text{with respect to } ε \in I)\) families of \( C^m \) symplectic diffeomorphisms \((\mathbb{R}^2, o)\) such that
\[ \begin{align*}
  &\quad \text{for all } ε, \quad g_{1,ε}(o) = g_{2,ε}(o) = o, \\
  &\quad Dg_{1,0}(o) = \text{diag}(\lambda, \lambda^{-1}) \quad (\text{is hyperbolic}), \\
  &\quad g_{1,ε}(x, y) = g_{2,ε}(x, y) + O^{N+1}(x, y), \quad (2.23)
\end{align*} \]

then, there exists a continuous family \((S_{ε}^{(1)})_{ε}\) \((\text{with respect to } ε \in I \text{ small enough})\) of \( C^k \) symplectic diffeomorphisms such that \( S_{ε}^{(1)}(o) = o, \ D(S_{ε}^{(1)})(o) = \text{id} \) with \( k = \lfloor NA_1 - B_1 \rfloor - 1 \), and
\[ \begin{align*}
  &\quad \left\{ \begin{array}{l}
    S_{ε}^{(1)} \circ g_{1,ε} \circ (S_{ε}^{(1)})^{-1} = g_{2,ε}, \\
    S_{0}^{(1)} = \text{id}.
  \end{array} \right. \quad (2.24)
\end{align*} \]

2.5. Proof of Proposition 2.1. It will be a consequence of Corollary 2.4 and Proposition 2.5.

We first choose \( N \) so that \( k = \lfloor NA_1 - B_1 \rfloor - 1 \) and we define \( m \) by \( N = \lfloor (m + B)/A \rfloor + 1 \). If \( k \) is large enough, \( m \) will satisfy the assumption of Corollary 2.4. We then apply Proposition 2.5 to \( g_{1,ε} = f_{ε}^{(1)}, g_{2,ε} = \phi^1 J \nabla Q_{ε,N}, \) which satisfies (2.23) \((\text{note that } (2.20) \text{ is satisfied})\). This provides us with a continuous family \((S_{ε}^{(1)})_{ε}\) of \( C^k \) symplectic diffeomorphisms defined in a fixed neighborhood of \( o \) such that \( S_{ε}^{(1)}(o) = o, \ D(S_{ε}^{(1)})(o) = \text{id} \), and on a neighborhood of \( o \),
\[ \begin{align*}
  &\quad \left\{ \begin{array}{l}
    S_{ε}^{(1)} \circ f_{ε}^{(1)} \circ (S_{ε}^{(1)})^{-1} = \phi^1 J \nabla Q_{ε,N}, \\
    S_{0}^{(1)} = \text{id}.
  \end{array} \right. \quad (2.24)
\end{align*} \]

We can extend these \( S_{ε}^{(1)} \) as symplectic \( C^k \) diffeomorphisms \( S_{ε}^{(2)} \) of \( \mathbb{R}^2 \) which depend continuously on \( ε \) \((\text{cf. Lemma B.1})\). We then define
\[ \Theta_{ε,k} = S_{ε}^{(2)} \circ R_{ε} \]
and we observe that on a neighborhood of \( o \),
\[ \left\{ \begin{array}{l}
    \Theta_{ε,k} \circ f_{ε} \circ \Theta_{ε,k}^{-1} = \phi^1 J \nabla Q_{ε,N}, \\
    (\Theta_{0,k})_* X_o = J \nabla Q_{0,N};
  \end{array} \right. \]
3. Dynamics in a neighborhood of the origin

The purpose of this section is to estimate the time spent by the orbits of the flow \( \Phi^f_{Q,\epsilon} \) in the neighborhood \( V \) of the hyperbolic point \( o \).

To do that, we perform one more change of coordinates. Let us define the following diffeomorphisms \( \Xi_1, \Xi_2 \)

\[
\begin{aligned}
\Xi_1(x, y) &= (\ln x, xy), \\
\Xi_2(x, y) &= (-\ln y, xy).
\end{aligned}
\]

(3.25)

Because \( d(\ln x) \wedge d(xy) = d(-\ln y) \wedge d(xy) = dx \wedge dy \), we see that \( \Xi_i, i = 1, 2 \), are symplectic.

Let \( I_1, I_2 \subset \mathbb{R}_+^* \) be some open intervals such that \( I_1 \times \{0\} \) and \( \{0\} \times I_2 \) are both contained in \( V \).

**Lemma 3.1.** Let \((x_\ast, y_\ast) \in (I_1 \times \mathbb{R}) \cap V \) and \( \tilde{I}_f(x_\ast, y_\ast) = \inf\{t > 0 : \phi^f_{Q,\epsilon}(x_\ast, y_\ast) \in (\mathbb{R} \times I_2) \cap V\} \). Then the following hold.

1. There exists \( c(I_1, I_2) \geq 1 \) such that if \( 0 < x_\ast y_\ast \leq \bar{I}_f(x_\ast, y_\ast) \), one has

\[
c(I_1, I_2)^{-1} \frac{|\ln(x_\ast y_\ast)|}{\lambda} \leq \tilde{I}_f(x_\ast, y_\ast) \leq c(I_1, I_2) \frac{|\ln(x_\ast y_\ast)|}{\lambda}.
\]

(3.26)

2. For any \((x, y)\) in a neighborhood of \((x_\ast, y_\ast)\) and any \( t \) in a neighborhood of \( \tilde{I}_f(x_\ast, y_\ast) \),

\[
\Xi_2 \circ \phi^f_{Q,\epsilon} \circ \Xi_1^{-1} : (u, v) \mapsto (u + \tau^f_e(v), v),
\]

(3.27)

with

\[
\tau^f_e(v) = tq^f_e(v) - \ln v.
\]

(3.28)

**Proof.** (1) We evaluate \( \tilde{I}_f(x_\ast, y_\ast) \). Because

\[
\phi^f_{Q,\epsilon}(x_\ast, y_\ast) = (e^{-q^f_e(x_\ast y_\ast)x_\ast}, e^{q^f_e(x_\ast y_\ast)}y_\ast),
\]

we have \( e^{q^f_e(x_\ast y_\ast)}y_\ast \in I_2 \) if and only if

\[
t \in \left[ \frac{\ln((x_\ast y_\ast)^{-1} \times x_\ast \min I_2)}{q^f_e(x_\ast y_\ast)}, \frac{\ln((x_\ast y_\ast)^{-1} \times x_\ast \max I_2)}{q^f_e(x_\ast y_\ast)} \right].
\]

Hence for \( x_\ast y_\ast \) small enough,

\[
\left| \tilde{I}_f(x_\ast, y_\ast) - \frac{|\ln(x_\ast y_\ast)|}{q^f_e(x_\ast y_\ast)} \right| \lesssim \max(|\ln(x_\ast \min I_2)|, |\ln(x_\ast \max I_2)|) \frac{|\ln(x_\ast y_\ast)|}{q^f_e(x_\ast y_\ast)}.
\]
Because for $0 < x_\ast y_\ast \lesssim 1$ one has $q'_\varepsilon(x_\ast y_\ast) \asymp \lambda$, there exists $c(I_1, I_2)$ such that if $x_\ast y_\ast$ small enough (how small depends on $I_1$, $I_2$, $\lambda$), the inequality (3.26) is satisfied.

(2) We write

$$\Xi_2 \circ \phi_\varepsilon^\prime Q_\varepsilon \circ \Xi_1^{-1} = \Xi_2 \circ \Xi_1^{-1} \circ \Xi_1 \circ \phi_\varepsilon^\prime Q_\varepsilon \circ \Xi_1^{-1} = \Xi_2 \circ \Xi_1^{-1} \circ \phi_\varepsilon^\prime \hat{Q}_\varepsilon,$$

with $\hat{Q}_\varepsilon(u, v) = (Q_\varepsilon \circ \Xi_1^{-1})(u, v) = q_\varepsilon(v)$. Because $\phi_\varepsilon^\prime \hat{Q}_\varepsilon(u, v) = (u - tq'_\varepsilon(v), v)$ and $\Xi_2 \circ \Xi_1^{-1}(u, v) = (u - \ln v, v)$, we get (3.28).

4. Fundamental domains and first return maps

We construct in this section adapted fundamental domains $\mathcal{F}_{\varepsilon, x_\ast y_\ast}$ for the maps $(f_\varepsilon)_\varepsilon$ satisfying conditions (H1)–(H4) of Theorem 2.1 and define their first return maps $\hat{f}_\varepsilon$ in $\mathcal{F}_{\varepsilon, x_\ast y_\ast}$.

4.1. Fundamental domains. Let $V$ be the domain of Theorem 2.1. One can choose $x_\ast > 0$ such that, for any $\varepsilon$ small enough,

$$(x_\ast, 0) \in V \quad \text{and} \quad f_\varepsilon^{-1}(x_\ast, 0) \notin V.$$ 

For $y_\ast > 0$ small enough, we define the vertical segment

$$L_{x_\ast y_\ast} := \{(x_\ast, ty_\ast), \ 0 < t < 1\}$$

and the domain

$$\mathcal{F}_{\varepsilon, x_\ast y_\ast}$$

as the interior of the contour defined by (see Figure 2)

(a) the segment $[f_\varepsilon(x_\ast, 0), (x_\ast, 0)];$
(b) the transversal $L_{x_\ast y_\ast};$
(c) the piece of hyperbola joining $(x_\ast, y_\ast)$ to $f_\varepsilon(x_\ast, y_\ast)$ (cf. Remark 2.1);
(d) the curve $f_\varepsilon(L_{x_\ast y_\ast}).$

We shall often drop the index $x_\ast$ in the notations of $L_{x_\ast y_\ast}$, $\mathcal{F}_{\varepsilon, x_\ast y_\ast}$ and simply set

$$L_{y_\ast} := L_{x_\ast y_\ast} \quad \text{and} \quad \mathcal{F}_{\varepsilon y_\ast} = \mathcal{F}_{\varepsilon, x_\ast y_\ast}.$$ 

If $y_\ast$ is small enough, one has $\mathcal{F}_{\varepsilon y_\ast}, \overline{L_{y_\ast}} \subset V$. We set

$$\tilde{\mathcal{F}}_{\varepsilon, y_\ast} = \mathcal{F}_{\varepsilon y_\ast} \cup L_{y_\ast}.$$ 

4.2. First return maps. Our aim in this subsection is to define the first return map of $f_\varepsilon$ in $\tilde{\mathcal{F}}_{\varepsilon, y_\ast}$.

Because $\Sigma_\varepsilon$ is a separatrix for $f_\varepsilon$, we can define (see Remark 2.2)

$$N(\varepsilon) = \min_{\text{defin.}} \{n \in \mathbb{N}^*, f_\varepsilon^{-n}(\{f_\varepsilon(x_\ast), x_\ast\}) \subset V\}.$$ 

We note that if $\varepsilon$ is small enough, $N(\varepsilon)$ is independent of $\varepsilon$, so we shall denote it by $N$. Moreover, if $\varepsilon$ and $y_\ast$ are small enough,

$$N = \min_{\text{defin.}} \{n \in \mathbb{N}^*, f_\varepsilon^{-n}(\tilde{\mathcal{F}}_{\varepsilon, y_\ast}) \subset V\}. \quad (4.29)$$
LEMMA 4.1. There exists a constant $0 < c_* < 1$ such that, for $(x, y) \in \mathcal{F}_{\varepsilon, y*}$,

$$\tilde{n}_\varepsilon(x, y) : = \min\{j \in \mathbb{N}^*, f_{\varepsilon}^j(x, y) \in f_{\varepsilon}^{-N}(\mathcal{F}_{\varepsilon, y*})\} < \infty.$$  

(4.30)

One has

$$\tilde{n}_\varepsilon(x, y) \asymp \ln(xy)/\lambda.$$  

(4.31)

Proof. Note that the domain $f_{\varepsilon}^{-N}(\mathcal{F}_{\varepsilon, y*}) \subset \tilde{V}$ is the interior of the contour defined by:

(a) the segment $[f_{\varepsilon}^{-(N-1)}(x_*, 0), f_{\varepsilon}^{-N}(x_*, 0)] \subset W_f^u(o) \cap V \subset \{0\} \times \mathbb{R};$

(b) the curve $f_{\varepsilon}^{-N}(L_{y*});$

(c) a curve joining $f_{\varepsilon}^{-N}(x_*, y_*)$ to $f_{\varepsilon}^{-(N-1)}(x_*, y_*);$  

(d) the curve $f_{\varepsilon}^{-(N-1)}(L_{y*}),$

and

$$f_{\varepsilon}^{-N}(\mathcal{F}_{\varepsilon, y*}) = f_{\varepsilon}^{-N}(\mathcal{F}_{\varepsilon, y*}) \cup f_{\varepsilon}^{-N}(L_{y*}).$$

Note that the lines $f_{\varepsilon}^{-N}(L_{y*}), f_{\varepsilon}^{-(N-1)}(L_{y*})$ are transversal to the segment $[f_{\varepsilon}^{-(N-1)}(x_*, 0), f_{\varepsilon}^{-N}(x_*, 0)].$

Now let $(x, y) \in \mathcal{F}_{\varepsilon, y*}$. We denote by $\mathcal{H}_{x,y}$ the hyperbola

$$\mathcal{H}_{x,y} := \{(x', y'), x'y' = xy\}. $$
and if \( z, z' \in \mathcal{H}_{x,y} \), by \( \mathcal{H}_{x,y}(z, z') \), the arc of hyperbola of \( \mathcal{H}_{x,y} \) between \( z \) and \( z' \) which is open in \( z \) and closed in \( z' \). If \( y > 0 \) is small enough, \( \mathcal{H}_{x,y} \) intersects \( f^{-N}_e(\tilde{\mathcal{F}}_{e,y_*}) \subset V \) in an arc of hyperbola of the form \( \mathcal{H}_{x,y}(p, f^{-1}_e(p)) \) with \( p \in f^{-N}_{e-N+1}(L_{y_*}) \) and \( f^{-1}_e(p) \in f^{-N}_e(L_{y_*}) \). The sets \( f^{-j}_e(\mathcal{H}_{x,y}(p, f^{-1}_e(p)), j \geq 0 \) form a partition of the semi-arc of parabola \( \bigcup_{n \geq 0} \mathcal{H}_{x,y}(p, f^{-n}_e(p)) \) which contains \( (x, y) \). In particular, there exists \( j \geq 0 \) (in fact \( j \geq 1 \)) such that \( (x, y) \in f^{-j}_e(\mathcal{H}_{x,y}(p, f^{-1}_e(p))) \) or equivalently

\[
\hat{f}^j_e(x, y) \in \mathcal{H}_{x,y}(p, f^{-1}_e(p)) \subset f^{-N}_e(\tilde{\mathcal{F}}_{e,y_*}).
\]

This proves (4.30).

To prove (4.31), we note that there exists an interval \( I_2 \) not containing 0 and depending only on \( x_*, y_* \) such that \( f^{-N}_e(\tilde{\mathcal{F}}_{e,y_*}) \subset \mathbb{R} \times I_2 \). We then use Lemma 3.1 and the fact that

\[
|\tilde{I}_2(x, y) - \tilde{n}_e(x, y)| \leq 1.
\]

We now define

\[
n_e = N + \tilde{n}_e. \tag{4.32}
\]

By (4.31), one has

\[
n_e(x, y) \asymp \ln(xy)/\lambda. \tag{4.33}
\]

The map \( \hat{f}_e : \tilde{\mathcal{F}}_{e,c_*y_*} \to \tilde{\mathcal{F}}_{e,y_*} \), defined by

\[
\hat{f}_e = f^{n_e}_e,
\]

is the first return map of \( f_e \) in \( \tilde{\mathcal{F}}_{e,y_*} \) (for points starting in \( \tilde{\mathcal{F}}_{e,c_*y_*} \)). Note that \( \hat{f}_e \) is not \( C^k \) on the whole domain \( \tilde{\mathcal{F}}_{e,c_*y_*} \).

4.3. Estimates on first return maps. We denote for \( a \in \mathbb{R} \)

\[
T_a : (u, v) \mapsto (u + a, v)
\]

and we recall the definition (3.25) of the symplectic diffeomorphisms \( \Xi_1, \Xi_2 \).

We observe that there exist open sets \( W_1 \subset \mathbb{R}_+^* \cap \mathbb{R}, W_2 \subset \mathbb{R} \times \mathbb{R}_+^* \) such that, for any \( \varepsilon \) and \( y_* > 0 \) small enough,

\[
\tilde{\mathcal{F}}_{e,y_*} \subset W_1 \subset V, \quad f^{-N}_0(\tilde{\mathcal{F}}_{e,y_*}) \subset W_2 \subset V.
\]

**Lemma 4.2.** There exists a \( C^k \) function \( \sigma_{0,N} \in C^k(\mathbb{R}_+^*, \mathbb{R}) \) such that on \( \Xi_2(W_2) \), one has

\[
\Xi_1 \circ f^N_0 \circ \Xi_2^{-1} = T_{\sigma_{0,N}}. \tag{4.35}
\]

**Proof.** From condition (H4), one can write on \( \mathbb{R}^2 \)

\[
f_0 = \phi^1_{J \nabla H_0}
\]

and hence

\[
f^N_0 = \phi^N_{J \nabla H_0} \quad \text{where} \quad H_0|_V = Q_0.
\]
If \((u, v) \in \Sigma_2(W_2)\) and \((\tilde{u}, \tilde{v}) = \Xi_1(f_0^N(\Sigma_2^{-1}(u, v)))\), one then has
\[
Q_0(\Sigma_2^{-1}(\tilde{u}, \tilde{v})) = Q_0(f_0^N(\Sigma_2^{-1}(u, v))) = Q_0(\Sigma_2^{-1}(u, v))
\]
and hence
\[
q_0(\tilde{v}) = q_0(v)
\]
and thus \(\tilde{v} = v\). Because the map \((u, v) \mapsto (\tilde{u}, \tilde{v})\) is symplectic, this forces \(\tilde{u} = u + \tilde{\sigma}_{0,N}(v)\) for some \(C^k\) function \(\tilde{\sigma}_{0,N}\); this function can be extended as a \(C^k\) function \(\sigma_{0,N} : \mathbb{R} \to \mathbb{R}\).

Recall the definition (4.34) of \(\hat{f}_\varepsilon\).

**Lemma 4.3.** There exists a continuous family \((\hat{\eta}_\varepsilon)\) of \(C^k\) symplectic diffeomorphisms defined on \(\mathbb{R}^2\) and a neighborhood \(W\) of \(f_{-1}(\hat{F}_{\varepsilon,y_*} \cup \tilde{T}_{\varepsilon y_*} \cup f_{\varepsilon}(\hat{F}_{\varepsilon,y_*})\) such that
\[
\lim_{\varepsilon \to 0} \|\hat{\eta}_\varepsilon - \text{id}\|_k = 0,
\]
and on a neighborhood of \(\hat{F}_{\varepsilon,c,y_*}\), one has
\[
\Xi_1 \circ \hat{f}_\varepsilon \circ \Xi_1^{-1} = \hat{\eta}_\varepsilon \circ T_{\hat{\varepsilon}} ,
\]
where
\[
\hat{\varepsilon}(v) = \sigma_{0,N}(v) + \hat{n}_\varepsilon(u, v)q_\varepsilon'(v) - \ln v \quad \text{with} \quad \hat{n}_\varepsilon = \tilde{n}_\varepsilon \circ \Xi_1^{-1} .
\]

**Proof.** We write (we use (4.34), (4.32), (H3)):
\[
\hat{f}_\varepsilon = f_\varepsilon^{N+\hat{n}_\varepsilon}
\]
\[
= f_\varepsilon^N \circ \phi_{f_\varepsilon^N Q_\varepsilon}
\]
\[
= \eta_\varepsilon \circ f_0^N \circ \phi_{f_\varepsilon^N Q_\varepsilon},
\]
with
\[
\eta_\varepsilon = f_\varepsilon^N \circ f_0^{-N} .
\]
As a consequence, if we set
\[
\hat{\eta}_\varepsilon = \Xi_1 \circ \eta_\varepsilon \circ \Xi_1^{-1} \quad \text{and} \quad \hat{n}_\varepsilon = \tilde{n}_\varepsilon \circ \Xi_1^{-1},
\]
we have, using (4.35),
\[
\Xi_1 \circ \hat{f}_\varepsilon \circ \Xi_1^{-1} = \hat{\eta}_\varepsilon \circ (\Xi_1 \circ f_0^N \circ \Xi_2^{-1}) \circ (\Xi_2 \circ \phi_{f_\varepsilon^N Q_\varepsilon} \circ \Xi_2^{-1}) \circ (\Xi_2 \circ \Xi_1^{-1})
\]
\[
= \hat{\eta}_\varepsilon \circ T_{\sigma_{0,N}} \circ \phi_{f_\varepsilon^N Q_\varepsilon} \circ T_{-\ln \varepsilon}
\]
\[
= \hat{\eta}_\varepsilon \circ T_{\sigma_{0,N}} \circ \hat{\varepsilon}_\varepsilon \circ T_{-\ln \varepsilon}
\]
\[
= \hat{\eta}_\varepsilon \circ T_{\sigma_{0,N} + \hat{\varepsilon}_\varepsilon - \ln \varepsilon},
\]
which is (4.37) together with (4.38).
5. Renormalization

We define in this section a renormalization $\hat{f}_\varepsilon$ of the map $f_\varepsilon$. The first return map $\hat{f}_\varepsilon$ of $f_\varepsilon$ in the fundamental domain $F_{\varepsilon, y_\varepsilon}$ that we have constructed in §4 is not differentiable at every point (see (4.37), (4.38), and the fact that the integer valued function $n_\varepsilon$ has, in general, discontinuity points). However, if one glues the ‘vertical’ boundaries of $F_{\varepsilon, y_\varepsilon}$ by $f_\varepsilon$, we obtain an abstract open annulus $F_{\varepsilon, y_\varepsilon}/f_\varepsilon$ (see §§5.1 and 5.2) and the map $\hat{f}_\varepsilon$ is now $C^k$ on it. We can uniformize this abstract annulus so that it becomes the standard (with the usual topology) open annulus $\mathbb{R}/\mathbb{Z} \times ]0, c[$ (some $c > 0$), see §5.3, and the map $\hat{f}_\varepsilon$ in these new coordinates turns into a $C^k$ diffeomorphism $\hat{f}_\varepsilon$ defined on (part of) this standard annulus. This is the (one should say ‘a’ instead of ‘the’ since the uniformizing/normalizing procedure is not unique) renormalized diffeomorphism associated to $f_\varepsilon$. Uniformizing the annulus is equivalent to conjugating $f_\varepsilon$ to $(x, y) \mapsto (x + 1, y)$ on a domain containing $F_{\varepsilon, y_\varepsilon}$. This procedure is, in a different context, the one described in [24]. We shall often call the uniformization operation normalizing in reference to the corresponding renormalization procedure defined for quasi-periodic cocycles, cf. [6, 14].

5.1. Gluing. Let $F$ be an open set of $\mathbb{R}^2$, $L$ a one-dimensional submanifold of $\mathbb{R}^2$ and $f$ an orientation preserving smooth diffeomorphism from a neighborhood of $F \cup L$ to a neighborhood of $f(F \cup L)$. We assume that:

(1) $f(F \cup L) \cap (F \cup L) = \emptyset$;

(2) $F \cup L$ is a two-dimensional submanifold of $\mathbb{R}^2$ with boundary and this boundary is $\partial(F \cup L) = L$; in particular, for any point $p \in L$, there exists an open set $U_p$, $p \in U_p \subset \mathbb{R}^2$, and a smooth diffeomorphism $\varphi_p : U_p \to \varphi_p(U_p) \subset \mathbb{R}^2$ such that $\varphi_p(U_p \cap L) = \varphi_p(U_p) \cap (\mathbb{R} \times \{0\})$ and $\varphi_p(U_p \cap F) = \varphi_p(U_p) \cap (\mathbb{R} \times \mathbb{R}^*_+)$;

(3) for any $p \in F \cup L$ and $U_p$, one has $U_p \cap f(F \cup L) = \emptyset$;

(4) for any $p \in L$ one has $f^{-1}(f(U_p) \cap F) = \varphi_p^{-1}(\varphi_p(U_p) \cap (\mathbb{R} \times \mathbb{R}^*_+))$, for any of the previous chart $(U_p, \varphi_p)$ at $p$.

We define the topological space $(F \cup L, \mathcal{T})$ as being the set $F \cup L$ endowed with the following topology $\mathcal{T}$: a subset $S$ of $F \cup L$ is an element of $\mathcal{T}$ (that is an open set) if for every $p \in S$, there exists an open set $V \subset \mathbb{R}^2$ (contained in a neighborhood of $F \cup L$ where $f$ is defined) such that $V \cap f(F \cup L) = \emptyset$ and $p \in (V \cup f(V)) \cap (F \cup L) \subset S$.

We can then define the following differentiable structure on $(F \cup L, \mathcal{T})$ as follows:

(a) if $p \in F$, we define the local chart $C_p := (W_p, \text{id})$, where $W_p$ is an open set of $\mathbb{R}^2$ such that $p \in W_p \subset F$; and (b) if $p \in L$, we define the local chart $C_p := (W_p, \psi_p)$ where $W_p$ is the open set of $F \cup L$ (see condition (3)), $W_p = (F \cup L) \cap (U_p \cup f(U_p))$ (here $(U_p, \varphi_p)$ is the local chart for $p \in L$ as defined in (2)), and where $\psi_p$ is defined by

\[
\lim_{\varepsilon \to 0} \|\hat{n}_\varepsilon - \text{id}\|_{C^k} = 0, \\
\hat{n}_\varepsilon(W \cap (\mathbb{R} \times \{0\})) \subset \mathbb{R} \times \{0\}.
\]
FIGURE 3. Gluing: $(\mathcal{F}_{\varepsilon,y_*} \cup L_{y_*})/f_{\varepsilon}$.

(we use condition (4))

\[
\begin{cases}
\psi_p = \varphi_p & \text{on } U_p \cap (\mathcal{F} \cup L) = \varphi_p^{-1}((\varphi_p(U_p) \cap (\mathbb{R} \times \mathbb{R}_+)), \\
\psi_p = \varphi_p \circ f^{-1} & \text{on } f(U_p) \cap \mathcal{F} = f \circ \varphi_p^{-1}((\varphi_p(U_p) \cap (\mathbb{R} \times \mathbb{R}_+)').
\end{cases}
\]

We denote by $\mathcal{A}$ the collection of all these local charts $C_p$ and we set $(\mathcal{F} \cup L)/f = (\mathcal{F} \cup L, \mathcal{T}, \mathcal{A})$.

Remark 5.1. If we assume in addition that $f$ preserves the standard symplectic form $dx \wedge dy$ on $\mathbb{R}^2$, we can endow $(\mathcal{F} \cup L)/f$ with a symplectic form $\omega$.

Remark 5.2. If $g : \mathcal{F} \to g(\mathcal{F})$ is a smooth diffeomorphism defined in a neighborhood of $\mathcal{F}$, it induces a smooth diffeomorphism (that we still denote $g$) $g : (\mathcal{F} \cup L)/f \to (g(\mathcal{F}) \cup g(L))/(g \circ f \circ g^{-1})$.

Remark 5.3. If $\mathcal{F} = [0, 1[ \times ]0, 1[ \cup L = ]0, 1[ \times \mathbb{R}$ and $f = T_1 : (x, y) \mapsto (x + 1, y)$, one sees that $(\mathcal{F} \cup L)/T_1$ is (diffeomorphic to) the standard open annulus $(\mathbb{R}/\mathbb{Z} \times ]0, 1[,$ canonical) endowed with its canonical differentiable structure.

5.2. The space $(\mathcal{F}_{\varepsilon,y_*} \cup L_{y_*})/f_{\varepsilon}$. If $\varepsilon$ and $y_*$ are small enough, item (1) is satisfied and we can find charts $(p, U_p)$ such that items (2)–(4) are satisfied. See Figure 3. We can then define the manifold $(\mathcal{F}_{\varepsilon,y_*} \cup L_{y_*})/f_{\varepsilon}$. We shall see that it is an annulus without boundary, cf. Lemma 5.3.

Note that if $0 < c_* < 1$, the smaller set $\tilde{\mathcal{F}}_{\varepsilon,c_*y_*} = \mathcal{F}_{\varepsilon,c_*y_*} \cup L_{c_*y_*}$ is an open subset of $(\mathcal{F}_{\varepsilon,y_*} \cup L_{y_*})/f_{\varepsilon}$ (which means that it belongs to $\mathcal{F}$) and it can be endowed with the topology and differentiable structure induced by the inclusion. We denote $(\mathcal{F}_{\varepsilon,c_*y_*} \cup L_{c_*y_*})/f_{\varepsilon}$ the thus obtained submanifold of $(\mathcal{F}_{\varepsilon,y_*} \cup L_{y_*})/f_{\varepsilon}$. The following lemma is then tautological.

LEMMA 5.1. The map $\hat{f}_{\varepsilon}$ induces a $C^k$ map $\tilde{\mathcal{F}}_{\varepsilon,c_*y_*}/f_{\varepsilon} \to \tilde{\mathcal{F}}_{\varepsilon,y_*}/f_{\varepsilon}$.

We shall need in §6 the following lemma.
LEMMA 5.2. There exists a probability measure with positive density \( \pi_{\varepsilon, y_0} \) on \( \tilde{F}_{\varepsilon, y_0} / f_\varepsilon \) which is \( \hat{f}_\varepsilon \) invariant; for any measurable set \( A \in \tilde{F}_{\varepsilon, y_0} / f_\varepsilon \) such that \( \hat{f}_\varepsilon^{-1}(A) \in \tilde{F}_{\varepsilon, y_0} / f_\varepsilon \), one has \( \pi_{\varepsilon, y_0}(A) = \pi_{\varepsilon, y_0}(\hat{f}_\varepsilon^{-1}(A)) \).

**Proof.** We shall in fact construct this measure \( \pi_{\varepsilon, y_0} \) on the bigger set \( \tilde{F}_{\varepsilon, y_0} / f_\varepsilon \),

\[
\tilde{F}_{\varepsilon, y_0} = \tilde{F}_{\varepsilon, y_0} \cup \sigma(\tilde{F}_{\varepsilon, y_0}),
\]

where \( \sigma : \mathbb{R}^2 \to \mathbb{R}^2 \) is the reflection \( (x, y) \mapsto (x, -y) \) (it commutes with \( f_\varepsilon \) in \( V \), see condition (H3)). From Remark 5.1, there exists a symplectic form \( \omega_\varepsilon \) on \( \tilde{F}_{\varepsilon, y_0} / f_\varepsilon \). Note that the first return map \( \hat{f}_\varepsilon \) is not defined on the whole set \( \tilde{F}_{\varepsilon, y_0} / f_\varepsilon \) but nevertheless

\[
(\hat{f}_\varepsilon)^* \omega_\varepsilon = \omega_\varepsilon
\]

whenever this formula makes sense. The probability measure \( \pi_{\varepsilon, y_0} \) defined by

\[
\pi_{\varepsilon, y_0}(A) = \int_A |\omega_\varepsilon| \int_{\tilde{F}_{\varepsilon, y_0}} |\omega_\varepsilon| \nonumber
\]

is \( \hat{f}_\varepsilon \) invariant. \( \square \)

5.3. **Normalization of \( f_\varepsilon \).** We now uniformize the abstract annulus \( \tilde{F}_{\varepsilon, y_0} / f_\varepsilon \). To do that, it is enough to normalize \( f_\varepsilon \) in the sense of item 2 of the following lemma.

**LEMMA 5.3. (Normalization Lemma)** There exists a continuous family \( (h_\varepsilon)_\varepsilon \) of (not necessarily symplectic) \( C^k \)-diffeomorphisms defined on a neighborhood of \( \tilde{F}_{\varepsilon, y_0} \) such that for some \( c > 0 \):

1. \( h_\varepsilon \) sends \( \tilde{F}_{\varepsilon, y_0} / f_\varepsilon \) to the standard open annulus \( ([\mathbb{R} / \mathbb{Z}] \times ]0, c[ \), canonical);  
2. \( h_\varepsilon \circ f_\varepsilon \circ h_\varepsilon^{-1} = T_1 : (x, y) \mapsto (x + 1, y) \);  
3. \( h_\varepsilon(x_0, f_\varepsilon(x_0)) \times \{0\} = [0, 1[ \times \{0\} \).

**Proof.** Using condition (H3) and the change of coordinates (3.25) of §3, we see that on a neighborhood of \( \tilde{F}_{\varepsilon, y_0} \), one has (we use the notation \( (x, y) \) for \( (u, v) \))

\[
\Xi_1 \circ f_\varepsilon \circ \Xi_1^{-1} = T_{q_\varepsilon'} : (x, y) \mapsto (x + q_\varepsilon'(y), y) .
\]

If \( g_\varepsilon \) is the (not necessarily symplectic) smooth diffeomorphism

\[
g_\varepsilon : (x, y) \mapsto \left( \frac{x}{q_\varepsilon'(y)}, y \right) , \tag{5.41}
\]

one has

\[
g_\varepsilon \circ \Xi_1 \circ f_\varepsilon \circ \Xi_1^{-1} \circ g_\varepsilon^{-1} = T_1 . \tag{5.42}
\]

The set \( (g_\varepsilon \circ \Xi_1)(\tilde{F}_{\varepsilon, y_0}) \) is of the form

\[
(g_\varepsilon \circ \Xi_1)(\tilde{F}_{\varepsilon, y_0}) = \{(x, y), \ y \in [0, c], \ \gamma_\varepsilon(y) \leq x \leq \gamma_\varepsilon(y) + 1\},
\]

where \( c > 0, \ \gamma_\varepsilon : [0, c] \to \mathbb{R}_+ \) is \( C^k \), \( \gamma_\varepsilon(0) = 0 \), and the map \( \mathbb{R} \ni \varepsilon \mapsto \gamma_\varepsilon \in C^k([0, c], \mathbb{R}) \) is continuous. This indeed follows from the definition of \( \tilde{F}_{\varepsilon, y_0} \) in §4.1.
the definition of $\Xi_1$ (3.25), and (5.41), (5.42). As a consequence, if we denote
\[ j_\varepsilon : (x, y) \mapsto (x - \gamma_\varepsilon(y), y), \tag{5.43} \]
we have
\[
\begin{cases}
  j_\varepsilon \circ T_1 = T_1 \circ j_\varepsilon, \\
  j_\varepsilon((g_\varepsilon \circ \Sigma_1)(T_1)) = \{0\} \times [0, c[,
\end{cases}
\]
By Remarks 5.2 and 5.3, the map
\[ h_\varepsilon = j_\varepsilon \circ g_\varepsilon \circ \Xi_1 \tag{5.45} \]
is a diffeomorphism that sends $\tilde{F}_{\varepsilon,Y_\varepsilon}/f_\varepsilon$ to the standard annulus $([0, 1[ \times ]0, c[)/T_1 \simeq (\mathbb{R}/\mathbb{Z}) \times ]0, c[)$ and such that
\[ h_\varepsilon \circ f_\varepsilon \circ h_\varepsilon^{-1} = T_1. \]

To conclude the proof, we notice (2) is an immediate consequence of the definition (5.45) of $h_\varepsilon$. \hfill \square

Remark 5.4. Note that if $T_\alpha(x, y) = (x + \alpha(y), y)$, one has
\[ (h_\varepsilon \circ \Xi_1^{-1}) \circ T_\alpha \circ (h_\varepsilon \circ \Xi_1^{-1})^{-1} = T_{\tilde{\alpha}}, \quad \tilde{\alpha}(y) = \alpha(y)/q_\varepsilon'(y). \]

5.4. The renormalization $\tilde{f}_\varepsilon$ of $f_\varepsilon$. There exists $\delta \in ]0, c[$ such that the map
\[ \tilde{f}_\varepsilon = h_\varepsilon \circ \tilde{f}_\varepsilon \circ h_\varepsilon^{-1} : \mathbb{R}/\mathbb{Z} \times ]0, \delta[ \rightarrow \mathbb{R}/\mathbb{Z} \times ]0, c[ \tag{5.46} \]
is well defined and is a $C^k$ diffeomorphism onto its image.

Proposition 5.4. One has
\[ \tilde{f}_\varepsilon = \tilde{\eta}_\varepsilon \circ T_{l_\varepsilon}, \tag{5.47} \]
where $\tilde{\eta}_\varepsilon$ is a $C^k$ diffeomorphism defined on $\mathbb{R}/\mathbb{Z} \times ]0, \delta[$ and $l_\varepsilon \in C^k(]0, c[ \setminus \mathbb{R}/\mathbb{Z})$; they satisfy
\[ l_\varepsilon(y) = \frac{\sigma_0(y)}{q_\varepsilon'(y)} - \frac{\ln y}{q_\varepsilon'(y)} \mod \mathbb{Z}, \tag{5.48} \]
\[ \lim_{\varepsilon \to 0} \|\tilde{\eta}_\varepsilon - \text{id}\|_{C^k} = 0, \tag{5.49} \]
\[ \tilde{\eta}_\varepsilon : (x, y) \mapsto (x + a_\varepsilon(x, y), y + b_\varepsilon(x, y)), \tag{5.50} \]
where $a_\varepsilon \in C^k, b_\varepsilon \in C^{k-1}$ are functions defined on $\mathbb{R}/\mathbb{Z} \times (0, \delta)$. Moreover, the map $\tilde{f}_\varepsilon$ preserves a probability measure $\tilde{\pi}_{\varepsilon,Y_\varepsilon}$ with positive density defined on $\mathbb{R}/\mathbb{Z} \times ]0, c[$.

Proof. By (4.37) and Remark 5.4 after Lemma 5.3,
\[
\tilde{f}_\varepsilon = (h_\varepsilon \circ \Xi_1^{-1}) \circ \tilde{\eta}_\varepsilon \circ (h_\varepsilon \circ \Xi_1^{-1})^{-1} \circ (h_\varepsilon \circ \Xi_1^{-1}) \circ T_{l_\varepsilon} \circ (h_\varepsilon \circ \Xi_1^{-1})^{-1} = \tilde{\eta}_\varepsilon \circ T_{l_\varepsilon},
\]
Applying the translated curve theorem

Because \( \tilde{\eta}_e := (h_e \circ \Xi_e^{-1}) \circ \hat{\eta}_e \circ (h_e \circ \Xi_e^{-1})^{-1} \) and \( \tilde{f}_e = f_e \), the function \( l_e : ]0, c[ \to \mathbb{R}/\mathbb{Z} \) is also \( C^k \) and

\[
l_e(y) = (1/q_e'(y))\hat{b}_e(y).
\]

By (4.38) (remember that \( \hat{\eta}_e \) takes its value in \( \mathbb{Z} \)),

\[
l_e(y) = \frac{\sigma_0,N(y)}{q_e'(y)} + \hat{\eta}_e(x, y) - \frac{\ln y}{q_e'(y)}
\]

\[
= \frac{\sigma_0,N(y)}{q_e'(y)} - \frac{\ln y}{q_e'(y)} \mod \mathbb{Z},
\]

which is (5.48).

Equation (5.49) is a consequence of the definition of \( \tilde{\eta}_e \), cf. (5.51), the first equation of (4.36), and of the fact that \( \mathbb{R} \ni \varepsilon \mapsto h_\varepsilon \in C^k \) is continuous (Lemma 5.3).

We now claim that if \( \tilde{\eta}_e(x, y) = (x + a_\varepsilon(x, y), y + \bar{b}_e(x, y)) \), one has for any \( y \),

\[
\bar{b}_e(x, 0) = 0.
\]

Indeed, because

\[
\tilde{\eta}_e := (h_e \circ \Xi_e^{-1}) \circ \hat{\eta}_e \circ (h_e \circ \Xi_e^{-1})^{-1},
\]

equality (5.52) is a consequence of the second equation of (4.36), of item (3) of Lemma 5.3, and of the fact that \( \Xi_1(\mathbb{R}_+^* \times \{0\}) = \mathbb{R}_+^* \times \{0\} \).

To prove (5.50), we thus notice that equality (5.52) gives us for \( \tilde{b}_e \) a decomposition

\[
\begin{cases}
\tilde{b}_e(x, y) = yb_e(x, y), \\
\bar{b}_e \in C^{k-1}.
\end{cases}
\]

Finally, to conclude the proof of the proposition, we observe that because the map \( \tilde{f}_e : \tilde{\Xi}_{e,\varepsilon} / f_e \to \tilde{\Xi}_{e,\varepsilon} / f_e \) preserves the probability measure \( \pi_{\varepsilon,y_\varepsilon} \), cf. Lemma 5.2, the diffeomorphism \( \tilde{f}_e : \mathbb{R}/\mathbb{Z} \times ]0, \delta[ \to \mathbb{R}/\mathbb{Z} \times ]0, c[ \) preserves the probability measure \( \tilde{\pi}_{\varepsilon,y_\varepsilon} = (h_\varepsilon)_* \pi_{\varepsilon,y_\varepsilon} \) defined on \( \mathbb{R}/\mathbb{Z} \times ]0, c[ \) (in the sense that if \( A \subset \mathbb{R}/\mathbb{Z} \times ]0, c[ \), one has \( \tilde{\pi}_{\varepsilon,y_\varepsilon}(A) = \tilde{\pi}_{\varepsilon,y_\varepsilon}(\tilde{f}_e^{-1}(A)) \)).

6. Applying the translated curve theorem

We apply in this section Rüssmann’s (or Moser’s) translated curve theorem to some rescaled version \( \tilde{f}_{e,n} \) of the renormalization \( \tilde{f}_e \) of \( f_e \) defined in §5.4.

6.1. The translated curve theorem. Let \( \psi : \mathbb{R}/\mathbb{Z} \times ]e^{-1}, 1[ \to \mathbb{R}/\mathbb{Z} \times \mathbb{R} \) be a \( C^k \) diffeomorphism defined on the annulus (or cylinder) \( \mathbb{R}/\mathbb{Z} \times ]e^{-1}, 1[ \). We say that the graph \( \text{Gr}_\gamma := \{(x, \gamma(x)) : x \in \mathbb{R}/\mathbb{Z}\} \) of a continuous map \( \gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \times ]e^{-1}, 1[ \) is translated by \( \psi \) if for some \( t \in \mathbb{R} \),

\[
\psi(\text{Gr}_\gamma) = \text{Gr}_{t + \gamma}
\]

(6.53)
and invariant if \( t = 0 \). If \( \text{Gr}_\gamma \) satisfies (6.53), there exists an orientation preserving homeomorphism of the circle \( g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) such that \( \psi(x, \gamma(x)) = \psi(g(x), t + \gamma(g(x))) \).

If \( t = 0 \) (respectively \( t \neq 0 \)), we define (respectively with a clear abuse of language) the rotation number of (\( \psi \) on) the invariant (respectively translated) graph \( \text{Gr}_\gamma \) as the rotation number of the circle diffeomorphism \( g \).

We say that \( \psi \) has the intersection property if for any continuous \( \gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \times [e^{-1}, 1] \), the curve \( \text{Gr}_\gamma := \{(x, \gamma(x)) : x \in \mathbb{R}/\mathbb{Z} \} \) intersects its image \( \psi(\text{Gr}_\gamma) \).

We state the translated curve theorem by Rüssmann [17] (which implies the invariant curve theorem by Moser [15]):

**Theorem 6.1.** (Rüssmann, [17]) There exists \( k_0 \in \mathbb{N} \) for which the following holds. Let \( k \geq k_0, C, \mu > 0 \), and \( l : \mathbb{R}/\mathbb{Z} \to \mathbb{R} \) a \( C^k \) map satisfying the twist condition,

\[
\min_y |\partial_y l(y)| > \mu > 0 \quad \text{and} \quad \|l\|_{C^{k_0}} \leq C,
\]

and define

\[
\psi_0 : (x, y) \mapsto (x + l(y), y).
\]

There exists \( \varepsilon_0 = \varepsilon_0(C, \mu) > 0 \) such that for any \( C^k \) diffeomorphism \( \psi : \mathbb{R}/\mathbb{Z} \times [e^{-\frac{3}{4}}, e^{-\frac{1}{4}}] \to \mathbb{R}/\mathbb{Z} \times \mathbb{R} \) satisfying

\[
\|\psi - \psi_0\|_{C^{k_0}} < \varepsilon_0,
\]

the diffeomorphism \( \psi \) admits a set of positive Lebesgue measure of \( C^{k-k_0} \) translated graphs contained in \( (\mathbb{R}/\mathbb{Z}) \times [e^{-\frac{3}{4}}, e^{-\frac{1}{4}}] \). Moreover, all these translated graphs have Diophantine rotation numbers (they are in a fixed Diophantine class \( DC(\kappa, \tau) \) (the exponent is \( \tau \) and the constant \( \kappa \)) that can be prescribed in advance once \( \mu \) is fixed (\( k_0 \) then depends on \( \tau \) and \( \varepsilon_0 \) on \( \kappa \) and \( \tau \))).

6.2. The rescaled diffeomorphism \( \tilde{f}_{\varepsilon,n} \). Let \( \tilde{f}_\varepsilon \) be the renormalized map defined in §5.4 and define \( u_\varepsilon, v_\varepsilon \) by

\[
\tilde{f}_\varepsilon(x, y) = (x + u_\varepsilon(x, y), y + v_\varepsilon(x, y)).
\]

Because \( \tilde{f}_\varepsilon = \tilde{\eta}_\varepsilon \circ T_{l_\varepsilon} \) (cf. (5.47)), one has using (5.50):

\[
\begin{align*}
u_\varepsilon(x, y) & = yb_\varepsilon(x + l_\varepsilon(y), y),
\end{align*}
\]

Now, let \( n \in \mathbb{N}^* \) large enough so that

\[
[e^{-(n+1)}, e^{-n}] \subset ]0, \delta[
\]

(the \( \delta \) of (5.46)) and introduce the rescaled \( C^k \) diffeomorphism \( \tilde{f}_{\varepsilon,n} \) defined on the annulus \( \mathbb{R}/\mathbb{Z} \times [e^{-1}, 1] \) by

\[
\tilde{f}_{\varepsilon,n} = \Lambda_{e^n} \circ \tilde{f}_\varepsilon \circ \Lambda_{e^{-n}}^{-1},
\]
where $\Lambda_{en} : (x, y) \mapsto (x, e^n y)$. Let us denote

$$\tilde{f}_{\varepsilon,n}(x, y) = (x + u_{\varepsilon,n}(x, y), y + v_{\varepsilon,n}(x, y)).$$

A computation shows that:

$$\begin{align*}
u_{\varepsilon,n}(x, y) &= \lambda(x + l_{\varepsilon,n}(y) + a_{\varepsilon}(x + l_{\varepsilon,n}(y), e^{-n} y), e^{-n} y), \\
v_{\varepsilon,n}(x, y) &= \lambda(y b_{\varepsilon}(x + l_{\varepsilon,n}(y), e^{-n} y), e^{-n} y),
\end{align*}$$

where

$$l_{\varepsilon,n}(y) = l_{\varepsilon}(e^{-n} y).$$

We can now state the following important proposition the proof of which occupies the next subsection.

**Proposition 6.1.** Assume that $k \geq k_0 + 2$ ($k$ is the regularity in conditions (H1)–(H4) and $k_0$ is the one of Theorem 6.1). There exists $\varepsilon_1 > 0$ such that the following holds. If $|\varepsilon| \leq \varepsilon_1$ and $n \gg 1$, $\tilde{f}_{\varepsilon,n}$ admits a set of positive Lebesgue measure of invariant $C^{k - k_0 - 2}$-graphs in $\mathbb{R}/\mathbb{Z} \times [e^{-n+1}, e^{-n}]$.

6.3. **Proof of Proposition 6.1.**

6.3.1. **Twist condition for $l_{\varepsilon,n}$.**

**Lemma 6.2.** There exist $C, \mu > 0$ such that, for any $\varepsilon$ small enough and any $n \gg 1$, the map $l_{\varepsilon,n}$ satisfies the twist condition (6.54) provided $k \geq k_0 + 1$.

**Proof.** Using (5.48), (6.59), we have

$$l_{\varepsilon,n}(y) = l_{\varepsilon}(e^{-n} y) = \frac{\sigma_{0,N}(e^{-n} y)}{q_{\varepsilon}'(e^{-n} y)} + \frac{n}{q_{\varepsilon}'(e^{-n} y)} - \frac{\ln y}{q_{\varepsilon}'(e^{-n} y)} \mod \mathbb{Z} = \frac{\sigma_{0,N}(0) + n}{\lambda} \frac{\ln y}{\lambda} + \theta_{\varepsilon,n}(y) \mod \mathbb{Z},$$
where
\[ \| \theta_{\varepsilon,n} \|_{C^{k-1}([e^{-1},1])} = O(e^{-n}); \]
this last inequality is a consequence of the fact that \( q_{\varepsilon}(s) = \lambda s + O(s^2) \) is continuous with respect to \( \varepsilon \) (cf. condition H2) and of the fact that \( \sigma_{0,N} \) is \( C^k \) (cf. Lemma 4.2). In particular, for some \( C_k > 0 \) (depending on \( \lambda \)),
\[ \| l_{\varepsilon,n} \|_{C^{k-1}} \leq C_k \]
and because \( \partial_y l_{\varepsilon,n}(y) = -1/(\lambda y) + \partial_y \theta_{\varepsilon,n}(y) \) and \( y \in ]e^{-1}, 1[ \),
\[ |\partial_y l_{\varepsilon,n}(y)| \geq 1/(2\lambda). \]
Hence (6.54) holds uniformly in \( \varepsilon, n \) with \( C = C_{k_0+1} \) and \( \mu = 1/(2\lambda) \) as soon as \( n \) is large enough.

6.3.2. \( \hat{f}_{\varepsilon,n} \) is close to a twist. We observe that from (5.49), (5.50), (6.58), and Lemma 6.2, one has uniformly in \( n \),
\[ \lim_{\varepsilon \to 0} \max(\| u_{\varepsilon,n} - l_{\varepsilon,n} \|_{C^{k-2}}, \| v_{\varepsilon,n} \|_{C^{k-2}}) = 0. \]
(6.60)
In particular, if \( n \) is large enough, inequality (6.55) is satisfied if \( k \geq k_0 + 2 \) with \( \psi = \hat{f}_{\varepsilon,n} \) and \( \psi_0 : (x, y) \mapsto (x + l_{\varepsilon,n}(y), y) \).
We see from §§6.3.1 and 6.3.2 that, if \( |\varepsilon| \leq \varepsilon_1 = \varepsilon_0(C_{k_0+1}, 1/(2\lambda)) \)
and \( n \gg 1 \), the assumptions of Theorem 6.1 are then satisfied by \( \hat{f}_{\varepsilon,n} \) with \( k - 2 \) in place of \( k \). Under these conditions, there thus exists a set \( \hat{\mathcal{G}}_{\varepsilon,n} \) of \( C^{k-k_0-2} \) \( \hat{f}_{\varepsilon,n} \)-translated graphs, the union of which covers a set of positive Lebesgue measure in \( (\mathbb{R}/\mathbb{T}) \times ]e^{-3/4}, e^{-1/4}[. \)
We just have to check that these translated graphs are indeed invariant.

6.3.3. \( \hat{f}_{\varepsilon,n} \)-translated graphs are invariant. Let \( \tilde{\gamma} \subset (\mathbb{R}/\mathbb{T}) \times ]e^{-3/4}, e^{-1/4}[ \) be a \( \hat{f}_{\varepsilon,n} \)-translated graph: \( \hat{f}_{\varepsilon,n}(\tilde{\gamma}) = \tilde{\gamma} + (0, t) \) for some \( t \in \mathbb{R} \). We shall prove that \( t = 0 \).
We can without loss of generality assume that \( t \geq 0 \) (the case \( t \leq 0 \) is treated in a similar way).
Formula (6.60) shows that if \( n \gg 1 \), one has \( \hat{f}_{\varepsilon,n}(\tilde{\gamma}) \subset (\mathbb{R}/\mathbb{T}) \times ]e^{-1}, 1[. \) From the conjugation relation (6.57), we see that (cf. (6.56))
\[ \tilde{\gamma} := \Lambda^{-1}_{\varepsilon,n}(\tilde{\gamma}) \subset (\mathbb{R}/\mathbb{Z}) \times ]e^{-n-3/4}, e^{-n-1/4}[ \subset (\mathbb{R}/\mathbb{Z}) \times ]0, \delta[ \]
is a \( \hat{f}_{\varepsilon} \)-translated graph such that
\[ \hat{f}_{\varepsilon}(\tilde{\gamma}) = \tilde{\gamma} + (0, e^{-n}t) \subset (\mathbb{R}/\mathbb{Z}) \times ]e^{-(n+1)}, e^{-n}[ \subset (\mathbb{R}/\mathbb{Z}) \times ]0, \delta[. \]
Let \( A \) be the open domain of \( (\mathbb{R}/\mathbb{Z}) \times ]0, c[ \) between \( (\mathbb{R}/\mathbb{Z}) \times ]0, \tilde{\gamma} \). Because \( t \geq 0 \), one has \( A \subset \hat{f}_{\varepsilon}(A) \subset (\mathbb{R}/\mathbb{Z}) \times ]0, c[ \).
Assume by contradiction that \( t > 0 \); then the set \( \hat{f}_{\varepsilon}(A) \setminus A \) contains a non-empty open set. We have seen (cf. Proposition 5.4) that \( \hat{f}_{\varepsilon} \) preserves a probability measure \( \pi_{\varepsilon,y} \).
with positive density defined on $(\mathbb{R}/\mathbb{Z}) \times]0, c[$. so $\tilde{\pi}_{\epsilon,y}((f_\epsilon(A) \setminus A) > 0$. However, this contradicts the invariance of $\tilde{\pi}_{\epsilon,y}$ by $f_\epsilon$.

The proof of Proposition 6.1 is complete. \qed

6.4. Invariant curves for $f_\epsilon$. We can now state the following.

**Theorem 6.2.** Let $k \geq k_0 + 2$ and $|\epsilon| \leq \epsilon_1$. There exists $\nu \in ]0, \delta[\ such that, for any $\nu \in ]0, \nu[\$, there exists a set $\mathcal{G}_{\epsilon,v}$ of $C^{k-k_0-2}$, $f_\epsilon$-invariant graphs contained in $(\mathbb{R}/\mathbb{Z}) \times ]\nu^{-1}v, v[\$ such that

$$\text{Leb}_{\mathbb{R}^2} \left( \bigcup_{\nu \in G_{\epsilon,v}} \nu \right) > 0.$$  

**Proof.** We choose $n$ so that

$$|e^{-(n+1)} - e^n| = 0, \nu[\]$$

and we observe that when $\nu \to 0$, one has $n \to \infty$. Define

$$\hat{f}_{\epsilon,n} = \Lambda_{e^n} \circ \hat{f}_\epsilon \circ \Lambda_{e^n}^{-1}.$$

By Proposition 6.1, there exists $\nu_1 > 0$ such that if $\nu \in ]0, \nu_1[$ satisfying (6.61) is then large enough), the diffeomorphism $\hat{f}_{\epsilon,n}$ admits $C^{k-k_0-2}$-invariant curves in $\mathbb{T} \times ]\nu^{-1}v, v[$ covering a set of positive Lebesgue measure; hence, $\hat{f}_{\epsilon,n}$ has $C^{k-k_0-2}$-invariant curves in $\mathbb{T} \times ]\nu^{-1}v, v[$ covering a set of positive Lebesgue measure. \qed

We shall denote

$$\tilde{G}_\epsilon = \bigcup_{\nu \in ]0, \nu_1[} \tilde{G}_{\epsilon,\nu}.$$

**Remark 6.1.** For all $\tilde{\gamma} \in \tilde{G}_{\epsilon,\nu}$, the rotation number of the circle diffeomorphism $f_\epsilon |\tilde{\gamma}$ is Diophantine in a fixed Diophantine class $DC(\kappa, \tau)$ (see the comment at the end of the statement of Theorem 6.1).

7. Invariant curves for $f_\epsilon$

We define

$$r = k - k_0 - 2$$

and assume that $|\epsilon| \leq \epsilon_1$.

Let $\tilde{\gamma} \subset (\mathbb{R}/\mathbb{Z}) \times]0, \delta[\$, $\tilde{\gamma} \in \tilde{G}_\epsilon$ be a $C^r$ invariant graph for $f_\epsilon : (\mathbb{R}/\mathbb{Z}) \times]0, \delta[\to (\mathbb{R}/\mathbb{Z}) \times]0, c[\$. Note that there exists $\delta_1 > 0$ such that $\tilde{\gamma} \subset (\mathbb{R}/\mathbb{Z}) \times]\delta_1, \delta[\$.

We can view $\tilde{\gamma}$ as an invariant graph sitting in $(]0, 1[ \times]0, c[)/T_1$ (recall $T_1(x, y) = (x + 1, y)$). In particular, one can find a $C^r$, 1-periodic function

$$\tilde{z} : \mathbb{R} \to (]0, 1[ \times]0, \delta[)/T_1$$

such that, for all $t$, $(d/dt)\tilde{z}(t) \neq 0$ and

$$\tilde{\gamma} = \tilde{z}(]0, 1[), \quad \tilde{z}(0) \in ]0, c[\$, $\lim_{t \to 1^-} \tilde{z}(t) = T_1(\tilde{z}(0)) \in \{1\} \times]0, c[\$. 

Let
\[ \hat{\gamma} = h^{-1}_e(\bar{\gamma}), \]
where \( h_e \) was defined in Lemma 5.3. Because \( \tilde{f}_e = h_e \circ \hat{f}_e \circ h^{-1}_e \) (cf. (5.46)), we see that
\[ \hat{\gamma} \subset h^{-1}_e((\mathbb{R}/\mathbb{Z}) \times ]\delta_1, \delta[)) \subset \tilde{F}_e, c_y \]
is a \( C^r \) compact, connected, one-dimensional submanifold (without boundary) of \( \tilde{F}_e, c_y \), which is invariant by \( \hat{f}_e : \tilde{F}_e, c_y \to \tilde{F}_e, c_y \). Moreover, the function
\[ \hat{z} = h^{-1}_e \circ \tilde{z} : \mathbb{R} \to \tilde{F}_e, c_y \]
is a \( C^r \), 1-periodic function and
\[ \hat{\gamma} = \hat{z}([0, 1[), \quad \hat{z}(0) \in L_{y_e}, \quad \lim_{t \to 1-} \hat{z}(t) = f_e(\hat{z}(0)) \in f_e(L_{y_e}). \]

The main result of this section is the following proposition.

**Proposition 7.1.** The set
\[ \hat{\Gamma} = \bigcup_{n \in \mathbb{Z}} f^n_e(\hat{\gamma}) \subset \mathbb{R}^2 \]
is an invariant \( C^r \) curve for \( f_e \): it is a compact, connected, one-dimensional \( C^r \) submanifold of \( \mathbb{R}^2 \) which is invariant by \( f_e \).

We give the proof of this proposition in §7.2.

7.1. **Preliminary results.** We define the function \( \hat{\mathcal{Z}} : \mathbb{R} \to \mathbb{R}^2 \)
\[ \hat{\mathcal{Z}}(t) = f_{\hat{\mathcal{Z}}(t)}^{[t]}(\hat{\mathcal{Z}}([t])) \]
for all \( t \in \mathbb{R} \), \( \hat{\mathcal{Z}}(0) = \hat{\mathcal{Z}}(0) \in \tilde{F}_e, c_y \), \( k \) shows that the function \( \hat{\mathcal{Z}} \) is \( C^r \) on a neighborhood of \( t = 1 \). It is hence \( C^r \) on \( [0, 2[ \) and because for \( j \in \mathbb{Z} \), \( \hat{\mathcal{Z}}(t + j) = f_{\hat{\mathcal{Z}}(t)}^{j}(\hat{\mathcal{Z}}(t)), \) it is \( C^r \) on \( \mathbb{R} \).

**Lemma 7.2.** The function \( \hat{\mathcal{Z}} : \mathbb{R} \to \mathbb{R}^2 \) is \( C^r \).

**Proof.** Note that, for \( t \in [0, 1[ , \) \( \hat{\mathcal{Z}}(t) = \hat{\mathcal{Z}}(t) \). Also, the very definition of \( \tilde{F}_e, c_y, f_{\hat{\mathcal{Z}}(t)}^{[t]}(\hat{\mathcal{Z}}([t])) \) shows that the function \( \hat{\mathcal{Z}} \) is \( C^r \) on a neighborhood of \( t = 1 \). It is hence \( C^r \) on \( [0, 2[ \) and because for \( j \in \mathbb{Z} \), \( \hat{\mathcal{Z}}(t + j) = f_{\hat{\mathcal{Z}}(t)}^{j}(\hat{\mathcal{Z}}(t)), \) it is \( C^r \) on \( \mathbb{R} \).

Let us set
\[ \tau = \inf_{t \geq 1, \hat{\mathcal{Z}}(t) \in \tilde{F}_e, c_y \}. \]
Note that
\[ 2 \leq \tau < \infty. \] (7.62)
Indeed, the left-hand side inequality is a consequence of the fact that \( \tilde{F}_e, c_y \cap f_{\hat{\mathcal{Z}}(t)}(\tilde{F}_e, c_y) = \emptyset \).
For the right-hand side, we observe that because \( \hat{\mathcal{Z}}(0) = \hat{\mathcal{Z}}(0) \in \tilde{F}_e, c_y \), one has (see (4.34)), \( \hat{\mathcal{Z}}(n_{\hat{\mathcal{Z}}}(\hat{\mathcal{Z}}(0))) = f_{\hat{\mathcal{Z}}(0)}^{n_{\hat{\mathcal{Z}}}(\hat{\mathcal{Z}}(0))}(\hat{\mathcal{Z}}(0)) \in \tilde{F}_e, c_y \), hence \( \tau \leq n_{\hat{\mathcal{Z}}}(\hat{\mathcal{Z}}(0)) < \infty \).

**Lemma 7.3.** The map \( \hat{\mathcal{Z}} : [0, \tau[ \to \mathbb{R}^2 \) is injective.
Proof. Assume by contradiction that $\hat{Z} : [0, \tau[ \to \mathbb{R}^2$ is not injective; then, there exists $m_i \in \mathbb{N}, 0 \leq s_i < 1,$

$$0 \leq m_i + s_i < \tau, \quad i = 1, 2, \quad \hat{Z}(s_1 + m_1) = \hat{Z}(s_2 + m_2). \quad (7.63)$$

Hence, $f^n_{\hat{x}}(\hat{\gamma}) \cap f^n_{\hat{x}}(\hat{\gamma}) \neq \emptyset$ and if $m := m_2 - m_1 \geq 0,$ $f^n_{\hat{x}}(\hat{\gamma}) \cap \hat{\gamma} \neq \emptyset.$ In particular, there exists $t \in [m, m + 1]$ such that $\hat{Z}(t) \in \hat{F}_{e,ya}$ and then $t \geq \tau.$ As a consequence, $m > \tau - 1$ and because $0 \leq m < \tau$ ($m_1, m_2$ are both in the interval $[0, \tau[),$ one has $m = [\tau],$ and hence $m_2 = m = [\tau]$ and $m_1 = 0.$ We then have from (7.63), $\hat{Z}(s_2 + [\tau]) = \hat{Z}(s_1) \in \hat{F}_{e,ya}$ (because $s_1 \in [0, 1[)$ and hence by the definition of $\tau,$ $s_2 + [\tau] \geq \tau,$ which contradicts $m_2 + s_2 < \tau.$

\textbf{Lemma 7.4.} If, for some $t \geq 1,$ $\hat{Z}(t) \in \hat{F}_{e,ya},$ then $\hat{Z}(t) \in \hat{\gamma}.$

\textbf{Proof.} Indeed, writing $t = s + n,$ $s \in [0, 1[,$ $n \in \mathbb{N}^*,$ one has $\hat{Z}(t) = f^n_{\hat{x}}(\hat{z}(s)).$ The integer $n \geq 1$ is thus a $m^{th}$ return time of $\hat{z}(s)$ in $\hat{F}_{e,ya},$ $\hat{Z}(t) = f^n_{\hat{x}}(\hat{z}(s)),$ and because $\hat{\gamma}$ is invariant by $f_{\hat{x}},$ it is readily seen by induction on $m$ that $f^n_{\hat{x}}(\hat{z}(s)) \in \hat{\gamma}.$

\textbf{Lemma 7.5.} One has $\hat{Z}(\tau) = \hat{z}(0).$

\textbf{Proof.} From the definition of $\tau$ and Lemma 7.4, we have $\hat{Z}(\tau) \in \text{closure}(\hat{\gamma}) \cap \text{closure}(L_{ya} \cup f_{\hat{x}}(L_{ya}))$ and hence $\hat{Z}(\tau) \in \{\hat{z}(0), f_{\hat{x}}(\hat{z}(0))\}.$ To conclude, we observe that one cannot have $\hat{Z}(\tau) = f_{\hat{x}}(\hat{z}(0))$ because otherwise, one would have $\hat{Z}(\tau - 1) = \hat{z}(0) \in \hat{F}_{e,ya},$ which contradicts the definition of $\tau$ (from (7.62) $\tau - 1 \geq 1.$

\textbf{Lemma 7.6.} The derivative of $\hat{Z}$ at $\tau$ is transverse to $L_{ya}.$

\textbf{Proof.} (1) If there exists a sequence $t_n \in \mathbb{R},$ $\lim t_n = \tau,$ such that $Z(t_n) \in \hat{F}_{e,ya},$ then from Lemma 7.4, one has $Z(t_n) \in \hat{\gamma}$ and consequently $(d\hat{Z}/dt)(\tau)$ is tangent to $\hat{\gamma},$ thus transverse to $L_{ya}.$

(2) Otherwise, there exists an open interval $I \subset \mathbb{R}, I \ni \tau,$ such that, for all $t \in I \setminus \{\tau\},$ $\hat{Z}(t) \notin \hat{F}_{e,ya}$ and $f_{\hat{x}}(\hat{Z}(t)) \in F_{e,ya}.$ From Lemma 7.4, one then has for all $t \in I \setminus \{\tau\},$ $\hat{Z}(t + 1) = f_{\hat{x}}(\hat{Z}(t)) \in \hat{\gamma}$ (see item (2) of §5.1), and hence $Df_{\hat{x}}(f_{\hat{x}}(\hat{Z}(\tau))) \cdot (d\hat{Z}/dt)(\tau)$ is tangent to $\hat{\gamma}$ and in particular transverse to $f_{\hat{x}}(L_{ya}).$ This implies that $(d\hat{Z}/dt)(\tau)$ is transverse to $L_{ya}.$

\textbf{Lemma 7.7.} One has $\hat{Z}([\tau, \tau + 1]) = \hat{Z}([0, 1]).$

\textbf{Proof.} We define $s_\tau = \sup\{s \geq 0 : \forall t \in [\tau, \tau + s[, \hat{Z}(t) \in \hat{F}_{e,ya}\}.$ From Lemmata 7.4 and 7.6, one has: (a) $s_\tau > 0;$ (b) for any $t \in [\tau, \tau + s_\tau[, \hat{Z}(t) \in \hat{\gamma};$ and (c) $\hat{Z}(\tau + s_\tau) \in f_{\hat{x}}(L_{ya}) \cap \text{closure}(\hat{\gamma}) = f_{\hat{x}}(\hat{z}(0)) = \hat{Z}(1).$ In particular, $\hat{Z}(\tau + s_\tau - 1) = f_{\hat{x}}^{-1}(\hat{Z}(1)) = \hat{Z}(0) \in \hat{F}_{e,ya}$ and by the definition of $\tau,$ this implies $s_\tau \geq 1.$ Now we notice that one cannot have $s_\tau > 1$ because otherwise $\tau + 1 \in [\tau, \tau + s_\tau[.$ And by definition of $s_\tau,$ $\hat{Z}(\tau + 1) \notin \hat{F}_{e,ya};$ however, $\hat{Z}(\tau + 1) = f_{\hat{x}}(\hat{Z}(\tau))$ and because $\hat{Z}(\tau) = \hat{z}(0)$ (Lemma 7.5), one has $\hat{Z}(\tau + 1) = f_{\hat{x}}(\hat{z}(0)) \notin \hat{F}_{e,ya}.$ We have thus proven that $s_\tau = 1.$ This implies that $\hat{Z}([\tau, \tau + 1]) = \hat{Z}([0, 1]).$
7.2. Proof of Proposition 7.1. We first observe that
\[ \hat{\Gamma} = \bigcup_{n \in \mathbb{Z}} f^n_{\varepsilon}(\hat{\gamma}) = \hat{\mathbb{Z}}(\mathbb{R}) = \bigcup_{n \in \mathbb{Z}} f^n_{\varepsilon}(\hat{\mathbb{Z}}([0, \tau + 1])). \tag{7.64} \]

Next we note the following.

(1) One has \( \hat{\mathbb{Z}}([0, \tau + 1]) = \hat{\mathbb{Z}}([0, \tau]) \).

(2) The set \( \hat{\mathbb{Z}}([0, \tau + 1]) \) is \( f_{\varepsilon} \)-invariant.

Item (1) is a consequence of \( \hat{\mathbb{Z}}([0, \tau + 1]) = \hat{\mathbb{Z}}([0, \tau]) \cup \hat{\mathbb{Z}}([\tau, \tau + 1]) \) (Lemma 7.7). Item (2) follows from item (1) and
\[ f_{\varepsilon}(\hat{\mathbb{Z}}([0, \tau + 1])) = f_{\varepsilon}(\hat{\mathbb{Z}}([0, \tau])) = \hat{\mathbb{Z}}([1, \tau + 1]) = \hat{\mathbb{Z}}([1, \tau]) \cup \hat{\mathbb{Z}}([\tau, \tau + 1]) \]
(1 \( \leq \tau \).

Item (2) and (7.64) yield \( \hat{\Gamma} = \hat{\mathbb{Z}}([0, \tau + 1]). \)

This last identity shows that \( \hat{\Gamma} \) is a connected, compact (cf. item (1)) subset of \( \mathbb{R}^2 \) which is \( f_{\varepsilon} \)-invariant.

Let us prove that \( \hat{\Gamma} \) is a one-dimensional submanifold of \( \mathbb{R}^2 \). Because \( \hat{\mathbb{Z}}(\tau) = \hat{\mathbb{Z}}(0) \) (Lemma 7.5), one has
\[ \hat{\mathbb{Z}}([0, \tau + 1]) = \hat{\mathbb{Z}}([0, \tau + 1]) = \hat{\mathbb{Z}}([0, \tau]) \cup \hat{\mathbb{Z}}([\tau, \tau + 1]). \]
From Lemmata 7.2 and 7.3, the set \( \hat{\mathbb{Z}}([0, \tau]) \) is a one-dimensional submanifold of \( \mathbb{R}^2 \) as well as the set \( \hat{\mathbb{Z}}([\tau - 1, \tau + 1]) \) (note that \( \hat{\mathbb{Z}}([\tau - 1, \tau + 1]) = f_{\varepsilon}(\hat{\mathbb{Z}}([\tau - 1, \tau + 1])). \) The intersection of these two sets is \( \hat{\mathbb{Z}}([\tau, \tau + 1]) \) and from Lemma 7.7, it is equal to \( \hat{\mathbb{Z}}([0, 1]) \) which is a one-dimensional submanifold of \( \mathbb{R}^2 \). As a consequence, the union \( \hat{\mathbb{Z}}([0, \tau]) \cup \hat{\mathbb{Z}}([\tau - 1, \tau + 1]) \) is one-dimensional submanifold of \( \mathbb{R}^2 \).

This concludes the proof of Proposition 7.1 \( \square \)

8. Proof of Theorem 2.1 (and hence of Theorem A)

As we have mentioned in §2.5, Theorem A follows from Theorem 2.1; we describe the proof of this latter result in this section.

Let \( r = k - k_0 - 2, |\varepsilon| \leq \varepsilon_1, \) and \( \nu \leq v_1 \). Theorem 6.2 yields a set \( \tilde{G}_{\varepsilon, \nu} \) of \( C^r, \) \( \tilde{f}_\varepsilon \)-invariant graphs contained in \( (\mathbb{R}/\mathbb{Z}) \times ]e^{-1}\nu, \nu[, \) the union of which covers a set of positive Lebesgue measure.
In the previous section (cf. Proposition 7.1), for all $\nu \in ]0, \nu_1[$, we have associated to each $\tilde{f}_\varepsilon$-invariant graph $\tilde{\gamma} \in \tilde{G}_{\varepsilon, \nu}$ an $f_\varepsilon$-invariant $C^r$-curve:

$$\hat{\Gamma} = \bigcup_{n \in \mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) \quad \text{where} \quad \hat{\gamma} = h^{-1}_\varepsilon(\tilde{\gamma}).$$

(8.65)

We denote by $\hat{G}_{\varepsilon, \nu}$ the set of all such curves $\hat{\Gamma}$.

To prove Theorem 2.1, we just have to prove that, for all $\nu \in ]0, \nu_1[$,

(Positive measure) $\text{Leb}_2\left( \bigcup_{\hat{\Gamma} \in \hat{G}_{\varepsilon, \nu}} \hat{\Gamma} \right) > 0$ \hspace{1cm} (8.66)

and

(Accumulation) $\lim_{\nu \to 0} \sup_{\hat{\Gamma} \in \hat{G}_{\varepsilon, \nu}} \text{dist}(\hat{\Gamma}, \Sigma_{\varepsilon}) = 0$. \hspace{1cm} (8.67)

8.1. **Proof of (8.66) (positive measure).** This is a consequence of the inclusion (cf. (8.65))

$$h^{-1}_\varepsilon\left( \bigcup_{\tilde{\gamma} \in \tilde{G}_{\varepsilon, \nu}} \tilde{\gamma} \right) \subset \bigcup_{\hat{\Gamma} \in \hat{G}_{\varepsilon, \nu}} \hat{\Gamma}$$

and of the fact that $\text{Leb}_2(\bigcup_{\tilde{\gamma} \in \tilde{G}_{\varepsilon, \nu}} \tilde{\gamma}) > 0$ (this is the content of Theorem 6.2).

8.2. **Proof of (8.67) (accumulation).** Let $\tilde{\gamma} \in \tilde{G}_{\varepsilon, \nu}$, $\tilde{\gamma} \subset (\mathbb{R}/\mathbb{Z}) \times ]0, \nu[$. From the definition (5.45) of the diffeomorphism $h_\varepsilon$, we see that, for some positive constant $C_\lambda$ depending on $\lambda$ (cf. condition (H3)),

$$\hat{\gamma} = h^{-1}_\varepsilon(\tilde{\gamma}) \subset \{(x, y) \in \tilde{F}_{\varepsilon, \nu}, \quad xy \in ]0, C_\lambda \nu[\}.$$

However,

$$\hat{\Gamma} = \bigcup_{n \in \mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) = \left( \bigcup_{n \in \mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) \cap V \right) \cup \bigcup_{n \in \mathbb{Z}, \ f_\varepsilon^n(\hat{\gamma}) \notin V} f_\varepsilon^n(\hat{\gamma}).$$

From condition (H3), one has

$$\bigcup_{n \in \mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) \cap V \subset V \cap \{(x, y), \quad xy \in ]0, C_\lambda \nu[\},$$

and hence, using Remark 2.2,

$$\text{dist}\left( \bigcup_{n \in \mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) \cap V, \Sigma_{\varepsilon} \cap V \right) = o_\varepsilon(1) \quad \text{(uniform in} \ \hat{\gamma}).$$

(8.68)

Now, recalling the definition (4.29) of the integer $N$ of §4.2, one has

$$\bigcup_{n \in \mathbb{Z}, \ f_\varepsilon^n(\hat{\gamma}) \notin V} f_\varepsilon^n(\hat{\gamma}) \subset \bigcup_{n=1}^N f_\varepsilon^{-n}(\hat{\gamma}).$$
Proof of Theorem B

is a fundamental domain of \( \gamma \)
on a fixed interval \( \text{LEMMA 8.1.} \)

which are below the initial graphs we have started from, see Figure 6. We then iterate
\[
\text{Similarly, the restriction of } f_\varepsilon \text{ to the invariant curve } \gamma \text{ yields a circle diffeomorphism } g_\gamma.
\]

Let \( \hat{\alpha} \) and \( \bar{\alpha} \) be the rotation numbers of \( g_\hat{\Gamma} \) and \( g_\gamma \).

\text{LEMMA 8.1. One has } \{1/\hat{\alpha}\} = \bar{\alpha} \text{ (here } \{\cdot\} \text{ denotes the fractional part).}

\text{Proof. We refer to the renormalization procedure defined in §§4 and 5. Let } \hat{\Gamma} \text{ be the arc}
\[
\hat{\Gamma} \ni \hat{J} , \text{ the first return map of } f_\varepsilon \text{ in } \hat{\Gamma} \text{. The map } g_\hat{\Gamma} \text{ can be identified with a circle diffeomorphism.}
\]

Similarly, the restriction of \( \hat{f}_\varepsilon \) to the invariant curve \( \hat{\gamma} \) yields a circle diffeomorphism \( g_{\hat{\Gamma}} \).

\text{Because } \bar{\alpha} \text{ can be chosen in a fixed Diophantine class } DC(\kappa, \tau) \text{ (see Remark 6.1),}
the rotation number \( \hat{\alpha} \) is Diophantine with the same exponent } \tau. \text{ By the Herman–Yoccoz}
\text{theorem on linearization of } C^r \text{-circle diffeomorphisms [10, 22], this implies that if } r \text{ is}
\text{large enough (depending on } \tau \text{ which is fixed), the diffeomorphism } g_\hat{\Gamma} \text{ is linearizable; in}
\text{other words, } \hat{\Gamma} \text{ is a KAM curve. However, one has a priori no control on the Diophantine}
\text{constant of } \hat{\alpha}.

This concludes the proof of Theorem 2.1, whence of Theorem A. \( \square \)

9. Proof of Theorem B

We construct in §§9.1 a symplectic diffeomorphism \( f_{\text{pert}} \) admitting a separatrix \( \Sigma \) (see
\text{Figure 5) and depending on a (‘large’) parameter } M. \text{ We renormalize } f_{\text{pert}} \text{ like in §§4 and}
\text{5 to get a diffeomorphism } \bar{f}_{\text{pert}} \text{ of an open annulus } \mathbb{R}/\mathbb{Z} \times [0, c]. \text{ We prove in Proposition}
\text{9.3 of §9.2 that this renormalized diffeomorphism } \bar{f}_{\text{pert}} \text{ sends some graphs projecting}
on a fixed interval } J_M \text{ (see (9.81) on graphs which project on the whole circle and}
\text{which are below the initial graphs we have started from, see Figure 6. We then iterate}
\text{this procedure in §9.3 to find an orbit of } \bar{f}_{\text{pert}} \text{ accumulating the boundary } \mathbb{R}/\mathbb{Z} \times \{0\}
of the aforementioned annulus: this prevents the existence of } \bar{f}_{\text{pert}}\text{-invariant curves close
to this boundary and therefore of $f_{pert}$-invariant curves close to the separatrix $\Sigma$. The diffeomorphism $f_{pert}$ is the searched for example of Theorem B.

9.1. Construction of the example. We start with a smooth autonomous symplectic vector field of the form $X_0 = J \nabla H_0$, where $H_0 : \mathbb{R}^2 \to \mathbb{R}$ satisfies on some neighborhood $V$ of $o = (0, 0)$

$$H_0(x, y) = xy \quad \text{on } V$$

and has the property that $\Sigma = H_0^{-1}(H_0(0, 0))$ is compact and connected. The set $\Sigma$ is a separatrix of

$$f = \phi_j^{\nabla H_0}$$

associated to the hyperbolic fixed point $o$.

Fixing $x_\ast > 0$ small enough, we can define like in §4, for $y_\ast > 0$ small enough, a fundamental domain $\tilde{F}_{y_\ast} = F_{y_\ast} \cup L_{y_\ast} \subset V$, where $F_{y_\ast}$ is defined by $(a) - (d)$ (§4.1) with $\phi_j^{\nabla H_0}$ in place of $f_\epsilon$. We can even assume that $\phi_j^{\nabla H_0}(\tilde{F}_{y_\ast}) \subset V$, $j = 1, 2$. There exists
c_*>0 such that the first return map,

\[ \hat{f} : \hat{F}_{c_*,y_*} \to F_{y_*}, \]

is well defined. We can renormalize \( f = \phi_{J_1,H_0}^1 \) like in §5 by first normalizing \( f \) (cf. Lemma 5.3):

\[ h \circ f \circ h^{-1} = T_1, \quad (9.70) \]

where

\[ h : F_{y_*} \to [0, 1[ \times ]0, c[ \]

is symplectic (see (5.45), (5.41), and the fact that we choose \( q(s) = s \)) and then setting (cf. (5.46)):

\[ \bar{f} = \text{defin.} h \circ \hat{f} \circ h^{-1} : \mathbb{R}/\mathbb{Z} \times ]0, c[. \quad (9.72) \]

By (5.48) of Proposition 5.4, we have

\[ \bar{f} = T_1, \quad l(y) = \sigma(y) - \ln y \quad (9.73) \]

for some smooth function \( \sigma \).

We can assume that \( h(\hat{F}_{y_*}) = [0, 1[ \times ]0, c[ \) and that \( T^{-j}([0, 1[ \times ]0, c[ \) \( \subset h^{-1}(V) \), \( j = 1, 2 \).

We now construct a symplectic perturbation \( f_{\text{pert}} : \mathbb{R}^2 \to \mathbb{R}^2 \) of \( f \) which admits \( \Sigma \) as a separatrix. We shall need first the following lemma.

**Lemma 9.1.** There exist \( b \in (0, 1) \) and a non-empty compact interval \( I \subset [0, 1[ \) such that, for any \( M > 0 \), there exists a smooth function \( \varphi_M : \mathbb{R} \to \mathbb{R} \) satisfying:

1. \( \varphi_M |_I \leq -bM; \)
2. \( (b^{-1}M/|I|) \geq -\varphi_M' |_I \geq (M/|I|); \)
3. the map \( s_M : \mathbb{R} \to \mathbb{R}, \) defined by

\[ s_M(t) = \int_0^t \varphi_M(u) du, \]

is an increasing smooth diffeomorphism of \( \mathbb{R} \) that coincides with the identity on \( \mathbb{R} \setminus [0, 1]. \)

**Proof.** See Appendix C. \( \square \)

Let \( \chi : \mathbb{R} \to \mathbb{R} \) be a smooth function equal to 1 on \([-c/2, c/2]\) and to 0 on \( \mathbb{R} \setminus [-3/4c, 3/4c] \), and define

\[ S_M(x, y) = (s_M(x)y)\chi(y) + xy(1 - \chi(y)). \quad (9.74) \]

The canonical (hence symplectic) mapping \( g_M \) associated to \( S_M \):

\[ g_M(x, y) = (\bar{x}, \bar{y}) \iff \begin{cases} x = \frac{\partial S_M}{\partial y} (\bar{x}, y), \\
\bar{y} = \frac{\partial S_M}{\partial \bar{x}} (\bar{x}, y), \end{cases} \quad (9.75) \]
is equal to the identity on \((\mathbb{R} \setminus [0, 1]) \times [-c, c]\) and satisfies for \((x, y) \in [0, 1[ \times ]0, c/2[\)

\[
\begin{cases}
\tilde{x} = s_M^{-1}(x), \\
\tilde{y} = s_M'(x)y.
\end{cases}
\] (9.76)

The following symplectic perturbation of \(f\):

\[
f_{\text{pert}} : \begin{align*}
&= h^{-1} \circ (g_M \circ T_1) \circ h \\
&= (h^{-1} \circ g_M \circ h) \circ f
\end{align*}
\]

(recall \(h\) satisfies (9.70)) is thus defined on \(\mathbb{R}^2\) and coincides with \(f\) outside \(f^{-1}(\tilde{F}_{\gamma_0})\). Moreover, because \(g_M(\mathbb{R} \times \{0\}) = \mathbb{R} \times \{0\}\),

\(\Sigma\) is a separatrix for \(f_{\text{pert}}\).

Now, because \(\tilde{F}_{\gamma_0}\) is a fundamental domain for \(f_{\text{pert}}\) (\(f_{\text{pert}}\) coincide with \(f\) on \(\tilde{F}_{\gamma_0}\)), for some \(c_{\text{pert}} > 0\) small enough, the first return map

\[
\hat{f}_{\text{pert}} : f_{\text{pert}}^{-1}(\tilde{F}_{\text{pert}, \gamma_0}) \to f_{\text{pert}}^{-1}(\tilde{F}_{\gamma_0}),
\]

is well defined and satisfies

\[
\hat{f}_{\text{pert}} = (\hat{f} \circ f^{-1}) \circ f_{\text{pert}}.
\]

In particular, on

\([-1, 0[ \times ]0, c/2[\),

one has (cf. (9.72), (9.73))

\[
\tilde{f}_{\text{pert}} : h \circ f_{\text{pert}} \circ h^{-1} = \tilde{f} \circ T_{-1} \circ g_M \circ T_1 = T_{-1} \circ g_M \circ T_1
\] (9.77)

and

\[
\tilde{f}_{\text{pert}} : \mathbb{R} \times ]0, \infty[ \to \mathbb{R} \times ]0, \infty[, \text{ satisies } \tilde{f}_{\text{pert}} \circ T_1 = T_1 \circ \tilde{f}_{\text{pert}};
\]

in particular, it defines a smooth map \((\mathbb{R}/\mathbb{Z}) \times ]0, \infty[ \to (\mathbb{R}/\mathbb{Z}) \times ]0, \infty[\). Note that because \(g_M\) is the identity outside \([0, 1[ \times [-c, c]\), it admits a \(T_1\)-periodization \(\tilde{g}_M : \mathbb{R} \times [-c, c] \to \mathbb{R} \times [-c, c]\) (which means that \(\tilde{g}_M\) and \(g_M\) coincide on \([0, 1[ \times [-c, c]\) and \(\tilde{g}_M\) commutes with \(T_1\)). This \(\tilde{g}_M\) is defined by the same formula (9.75) as \(g_M\), where now the new function \(\tilde{s}_M\) involved in (9.74) is the \(\mathbb{Z}\)-periodization of \(s_M\). To simplify the notation, we shall continue to denote \(\tilde{g}_M\) and \(\tilde{s}_M\) by \(g_M\) and \(s_M\).

Let

\[
t := t(x) := s_M^{-1}(x + 1).
\] (9.79)

**Lemma 9.2.** For \((x, y) \in [-1, 0[ \times ]0, c/2[\), the point \((\tilde{x}, \tilde{y}) := \tilde{f}_{\text{pert}}(x, y)\) satisfies with the notation (9.79)

\[
\begin{cases}
\tilde{x} = t - 1 + \sigma(s_M'(t) \times y) - \ln(s_M'(t)) - \ln y, \\
\ln \tilde{y} = \ln(s_M'(t)) + \ln y.
\end{cases}
\] (9.80)
Proof. Let \((x, y) \in [-1, 0[ \times ]0, c/2[\); with the notation \((x_1, y_1) = (g_M \circ T_1)(x, y) = g_M(x + 1, y)\), one has from (9.78), \((\bar{x}, \bar{y}) = T_{l-1}(x_1, y_1)\) and from (9.76), (9.73),
\[
\begin{align*}
&\begin{cases}
x_1 = s_M^{-1}(x + 1), \\
y_1 = s_M'(s_M^{-1}(x + 1)) \times y,
\end{cases} \\
&\begin{cases}
\bar{x} = x_1 - 1 + \sigma(y_1) - \ln y_1, \\
\bar{y} = y_1,
\end{cases}
\end{align*}
\]
and hence (9.80). □

9.2. Image of a piece of graph by \(\tilde{f}_{\text{pert}}\). We take \(M > 0\) (from Lemma 9.1) large enough and we define
\[
J_M = s_M(I) - 1 \subset [-1, 0[,
\]
where \(I\) is the interval introduced in Lemma 9.1.

If \(y : J \to \mathbb{R}^*_+, x \to y(x)\) is a differentiable function, we denote by \(\gamma_{J,y}\) its graph:
\[
\gamma_{J,y} = \{(x, y(x)), x \in J \} \subset [-1, 0[ \times ]0, c/2[.
\]

PROPOSITION 9.3. There exists a constant \(y_{\text{pert}} > 0\) for which the following holds. Assume that \(y : J_M \to [0, y_{\text{pert}}], x \mapsto y(x)\) is a differentiable function such that
\[
\text{for all } x \in J_M, \quad \left| \frac{d \ln y}{dx} + 1 \right| \leq 1/2.
\]
Then, \(\tilde{f}_{\text{pert}}(\gamma_{J_M,y}) + (\mathbb{Z}, 0)\) contains the graph \(\gamma_{[-1,0[,\bar{y}}\) of a differentiable function \(\bar{y} : [-1, 0[ \to \mathbb{R}^*_+\) (see Figure 6)
\[
\gamma_{[-1,0[,\bar{y}} = \{(\bar{x}, \bar{y}(\bar{x})), \ \bar{x} \in [-1, 0[\},
\]
such that
\[
\text{for all } \bar{x} \in [-1, 0[, \quad \left| \frac{d \ln \bar{y}}{d \bar{x}} + 1 \right| \leq 1/2; \quad (9.82)
\]
\[
\sup_{\bar{x} \in [-1,0[} \ln \bar{y}(\bar{x}) \leq \sup_{x \in J_M} \ln y(x) - b M. \quad (9.83)
\]
Moreover, for some interval \(J_M^1 \subset J_M\), one has
\[
\gamma_{[-1,0[,\bar{y}} = \tilde{f}_{\text{pert}}(\gamma_{J_M^1,y}). \quad (9.84)
\]

We prove this proposition in §9.2.2.

9.2.1. Preliminary results. If we introduce the variable
\[
\varphi = \ln(s_M'(t)) = \ln s_M'(s_M^{-1}(x + 1)) \quad \text{(recall } t = s_M^{-1}(x + 1)),
\]
we can write (9.80) as
\[
(\bar{x}, \bar{y}) = f_{\text{pert}}(x, y(x)) \iff \begin{cases}
\bar{x} = t - 1 + \sigma(e^\varphi \times y(x)) - \varphi - \ln y(x), \\
\ln \bar{y} = \varphi + \ln y(x).
\end{cases} \quad (9.85)
\]
Note that the maps $I \ni t \mapsto \varphi = \ln s'_M(t) \in \varphi(I)$ and $J_M \ni x \mapsto \varphi = \ln s'_M \circ s_M^{-1}(x + 1) \in \varphi_M(I)$ are smooth diffeomorphisms. In particular, the maps $\varphi_M(I) \ni \varphi \mapsto \tilde{x}$ and $\varphi_M(I) \ni \varphi \mapsto \ln y, \varphi_M(I) \ni \varphi \mapsto \ln \bar{y}$ are well defined and smooth.

**Lemma 9.4.** For any $\varphi$ such that $t \in I$, one has

$$\left| \frac{dt}{d\varphi} \right| \leq |I|/M \leq 1/4.$$  

**Proof.** This follows from the identity (recall $\varphi = \ln(s'_M(t)), s'_M = e^{\varphi_M}$)

$$\frac{dt}{d\varphi} = \frac{1}{s'/M(t)}$$

and the estimates given by the second item of Lemma 9.1 ($M$ is assumed to be large enough).

**Lemma 9.5.** One has

$$\sup_{\varphi_M(I)} \left| \frac{d\tilde{x}}{d\varphi} + 1 \right| \leq 1/4,$$  

$$\sup_{\varphi_M(I)} \left| \frac{d\ln \bar{y}}{d\varphi} - 1 \right| \leq 1/4.$$  

**Proof.** Indeed, from (9.85),

$$\frac{d\tilde{x}}{d\varphi} = \frac{dt}{d\varphi} + ye^{\varphi} \sigma'(e^{\varphi} \times y) \frac{dy}{d\varphi} - 1 - \frac{d\ln y}{d\varphi}$$

$$= \frac{dt}{d\varphi} + ye^{\varphi} \sigma'(e^{\varphi} \times y) \frac{d\ln y}{d\varphi} - 1 - \frac{d\ln y}{d\varphi}$$

$$= -1 + A$$

with

$$A = \frac{dt}{d\varphi} + ye^{\varphi} \sigma'(e^{\varphi} \times y) \frac{d\ln y}{d\varphi} - \frac{d\ln y}{d\varphi}.$$  

Note that (recall $x = s_M(t) - 1, s'_M = e^{\varphi_M}$)

$$\frac{d\ln y}{d\varphi} = \frac{d\ln y}{dx} \frac{dx}{dt} \frac{dt}{d\varphi} = \frac{d\ln y}{dx} e^{\varphi} \frac{dt}{d\varphi}$$

so, by Lemma 9.4,

$$|A| \leq (|I|/M) + (ye^{-bM} \|\sigma'\|_0 + 1)e^{-bM}(|I|/M) \left| \frac{d\ln y}{dx} \right|$$

and if $M$ is large enough,

$$|A| \leq 1/4.$$  

(9.88)

In a similar way,

$$\frac{d\ln \bar{y}}{d\varphi} = 1 + \frac{d\ln y}{dx} \frac{dx}{dt} \frac{dt}{d\varphi} = 1 + \frac{d\ln y}{dx} e^{\varphi} \frac{dt}{d\varphi} = 1 + B$$
with

\[ |B| \leq 2e^{-bM} \times (1/4) \leq 1/4 \quad (M \gg 1). \] (9.89)

9.2.2. Proof of Proposition 9.3. From (9.86) of Lemma 9.5, we see that the map
\[ \varphi_M(I) \ni \varphi \mapsto \bar{x} \in \mathbb{R} \]
is a diffeomorphism onto its image \( \bar{J}_M \subset \mathbb{R} \), and hence the maps
\[ J_M \ni x \mapsto \bar{x} \in \bar{J}_M \] and \( I \ni t \mapsto \bar{x} \in \bar{J}_M \)
are diffeomorphisms. Note that from (9.86), one has
\[ |\bar{J}_M| \geq (3/4)|\varphi_M(I)| \]
and from item (2) of Lemma 9.1, one has
\[ |\bar{J}_M| \geq (3/4)(M/|I|) \times |I| > 2; \] (9.90)
there thus exists an interval \( J_{M}^{1} \subset J_M \) such that the map
\[ J_{M}^{1} \ni x \mapsto \bar{x} \in \mathbb{R}^{+}[−1, 0[ \]
for some \( n \in \mathbb{Z} \) is a differentiable homeomorphism. Replacing \( \bar{y}(\bar{x}) \) by \( \bar{y}(\bar{x} + n) \) shows (9.84).

We now prove (9.82): for \( \bar{x} \in [-1, 0[ \),
\[ \left| \frac{d \ln \bar{y}}{d \bar{x}} + 1 \right| \leq 1/2. \] (9.91)
Indeed, let \( I_1 \subset I \) be the image of \([0, 1[ \) by \( \bar{J}_M \ni \bar{x} \mapsto t \in I \); from Lemma 9.5, for any \( \varphi \in \varphi_M(I_1) \), one has for some \( A, B \in [0, 1/4] \)
\[ \frac{d \bar{x}}{d \varphi} = -1 + A, \quad \frac{d \ln \bar{y}}{d \varphi} = 1 + B, \]
so that
\[ \left| \frac{d \ln \bar{y}}{d \bar{x}} + 1 \right| = \left| \left( \frac{d \ln \bar{y}}{d \varphi} \right) \left( \frac{d \bar{x}}{d \varphi} \right) + 1 \right| = \left| \frac{1 + B}{-1 + A} + 1 \right| \leq 1/2. \]
The preceding discussion shows that the map \( \bar{y} : [-1, 0[ \ni \bar{x} \mapsto \bar{y}(\bar{x}) \) is a well-defined differentiable function, that its graph is included in \( f_{\text{pert}}(y_{J_M,y}) + (\mathbb{Z}, 0) \), and that (9.82) holds.

There remains to prove (9.83). By the second equality of (9.85), if \( (\bar{x}, \bar{y}(\bar{x})) = f_{\text{pert}}(x, y) \), one has
\[ \ln \bar{y}(\bar{x}) \leq \ln y(x) - bM \leq \sup_{x \in J_M} \ln y - bM \]
and as a consequence, because the map \( J_M \ni J_M^{1} \ni x \mapsto \bar{x} \in [-1, 0[ \) is a bijection, (9.83) holds.

9.3. End of the proof of Theorem B. We shall prove that if \( M \) is large enough, the diffeomorphism \( f_{\text{pert}} \) constructed in §9.1 provides the searched for example of Theorem B.

Let \( M \) be large enough and \( y_0 \in ]0, y_{\text{pert}}[ \); we define the function
\[ y_0 : [-1, 0[ \to \mathbb{R}, \quad x \mapsto y_0 e^{-x}. \]
Using inductively Proposition 9.3, we construct differentiable functions

\[ y_n : [-1, 0[ \to \mathbb{R} \]

such that, for every \( n \in \mathbb{N}^* \),

\[ \text{for all } x \in J_M, \quad \left| \frac{d \ln y_n}{dx} + 1 \right| \leq 1/2, \tag{9.92} \]

\[ \gamma_{[-1,0],y_n} \subset \hat{f}_{\text{pert}}(\gamma_{J_M,y_n-1}) + (\mathbb{Z}, 0), \tag{9.93} \]

\[ \sup_{x \in [-1,0[} \ln y_n(x) \leq \sup_{x \in J_M} \ln y_n(x) - bM. \tag{9.94} \]

Inclusion (9.93) implies the existence of a decreasing sequence of non-empty compact intervals \( K_n \subset J_M \) such that

\[ \gamma_{[-3/4,-1/4],y_n} = \hat{f}_{\text{pert}}^n(\gamma_{K_n,y_0}) \mod (\mathbb{Z}, 0). \]

In particular, if \( x_\infty \subset \bigcap_{n \in \mathbb{N}^*} K_n \), one has

\[ \hat{f}_{\text{pert}}^n((x_\infty, y_0)) \in \gamma_{[-3/4,-1/4],y_n} \subset \gamma_{[-1,0],y_n} \mod (\mathbb{Z}, 0). \tag{9.95} \]

From (9.94),

\[ \sup_{x \in [-1,0[} y_n(x) \leq e^{-nbM} y_0, \]

and hence, using (9.95), we see that \( \hat{f}_{\text{pert}}^n((x_\infty, y_0)) \) accumulates \( \mathbb{R} \times \{0\} \):

\[ \hat{f}_{\text{pert}}^n((x_\infty, y_0)) \in [-1, 0[ \times ]0, e^{-nbM} y_0[ \mod (\mathbb{Z}, 0). \tag{9.96} \]

As a consequence of (9.77) and of the fact that, for some constant \( C > 0 \)

\[ \text{for all } v \in ]0, c[, \quad h^{-1}([-1, 0[ \times ]0, v[) \subset f_{\text{pert}}^{-1}(\hat{F}_{Ce^{-nbM} y_0}) \]

(this is owing to the fact that the diffeomorphism \( h \) given by (9.71) is indeed defined on a neighborhood of \( \hat{F}_{y_*} \)), one has

\[ \hat{f}_{\text{pert}}^n(h^{-1}(x_\infty, y_0)) \in f_{\text{pert}}^{-1}(\hat{F}_{Ce^{-nbM} y_0}). \]

Because \( \hat{f}_{\text{pert}} \) is the first return map of \( f_{\text{pert}} \) in \( f_{\text{pert}}^{-1}(\hat{F}_{y_*}) \), there exists a sequence \( (p_n)_{n \in \mathbb{N}} \in \mathbb{N}[\mathbb{N}, \lim_{n \to \infty} p_n = \infty \text{ such that} \]

\[ f_{\text{pert}}^{p_n}(h^{-1}(x_\infty, y_0)) \in f_{\text{pert}}^{-1}(\hat{F}_{Ce^{-nbM} y_0}). \tag{9.97} \]

However, this last fact prevents the existence of invariant circles in \( \Delta_\Sigma \) accumulating the separatrix \( \Sigma \) of \( f_{\text{pert}} \). More precisely, let \( W \) be a neighborhood of \( \Sigma \) in \( \Sigma \cup \Delta_\Sigma \) (we recall that \( \Delta_\Sigma \) is the bounded connected component of \( \mathbb{R}^2 \setminus \Sigma \), such that

\[ h^{-1}(x_\infty, y_0) \notin W. \]

We claim that \( W \setminus \Sigma \) does not contain any \( f_{\text{pert}} \)-invariant circle \( \Gamma \). Indeed, if this were not the case, the topological annulus \( \mathcal{A} \subset W \) having \( \Sigma \) and \( \Gamma \) for boundaries would be
FIGURE 7. The diffeomorphism \( \tilde{f}_{\text{pert}} \) on \( \mathbb{R}/\mathbb{Z} \times [e^{-(n+1)}, e^{-n}] \). Compare with Figures 4 and 6.

\( f_{\text{pert}} \)-invariant (by topological degree theory). However, this is impossible because one would have at the same time

\[
h^{-1}(x_\infty, y_0) \notin \mathcal{A} \quad \text{and} \quad f_{\text{pert}}^p(h^{-1}(x_\infty, y_0)) \in \mathcal{A}
\]

for some large \( p_n \) (see (9.97)).

**Remark 9.1.** If we define the renormalization \( \tilde{f}_{\text{pert}} \) of \( f_{\text{pert}} \) by considering the first return map of \( f_{\text{pert}} \) in \( F_{y*} \) instead of \( f_{\text{pert}}^{-1}(F_{y*}) \), as we have done to construct \( \tilde{f}_{\text{pert}} \), the dynamics of \( \tilde{f}_{\text{pert}} \) looks more like the one pictured in Figure 7. The comparison of this picture and that of Figure 4 illustrates the effect of the perturbative assumption in Theorem A.

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**A. Appendix. Proof of Lemma 2.2**

We write for \( j \geq 2 \)

\[
H^j_\varepsilon (z) = \lambda_\varepsilon z_1 z_2 + \sum_{2 \leq i \leq [j/2]} a_{\varepsilon,i} \times (z_1 z_2)^i + \sum_{i_1,i_2 \in \mathbb{N}} h_{\varepsilon,i_1,i_2}(i) z_1^{i_1} z_2^{i_2} + O^{j+2}(z),
\]

where \( a_{\varepsilon,i} \in \mathbb{R} \) and the \( h_{\varepsilon,i_1,i_2}(\cdot) \) are smooth 1-periodic functions. We define

\[
H_{\varepsilon,2}(z) = \lambda_\varepsilon z_1 z_2.
\]

We first observe that if \( G^j_\varepsilon \) is a solution of

\[
\begin{aligned}
G^j_\varepsilon(z) &= O^{j+1}(z), \\
H^j_\varepsilon(z) + \partial_t G^j_\varepsilon(z) + \{G^j_\varepsilon, H^j_{\varepsilon,2}\}(z) &= \tilde{q}_\varepsilon(z_1 z_2),
\end{aligned}
\]

(A.98)
for some \( \tilde{q}_\varepsilon(u) = \lambda_\varepsilon u + \sum_{2 \leq i \leq [(j+1)/2]} \tilde{a}_{\varepsilon,i} \times u^i \), \( \tilde{a}_{\varepsilon,i} \in \mathbb{R} \), then \( G^t_\varepsilon \) solves (2.15). We then have to solve (A.98) for some \( \tilde{q}_\varepsilon \) and some \( G^t_\varepsilon \) of the form

\[
\tilde{q}_\varepsilon(u) = \lambda_\varepsilon u + \sum_{2 \leq i \leq [(j+1)/2]} \tilde{a}_{\varepsilon,i} \times u^i
\]

\[
G^t_\varepsilon(z) = \sum_{i_1+i_2=j+1} g_{\varepsilon,i_1,i_2}(t) \frac{z_{i_1} \times z_{i_2}}{\Theta_1(\varepsilon)},
\]

where the \( g_{\varepsilon,i_1,i_2}(\cdot) \) are 1-periodic. This amounts to finding 1-periodic solutions to the equations

\[
\begin{align*}
\dot{h}_{\varepsilon,i_1,i_2}(t) + \partial_t g_{\varepsilon,i_1,i_2}(t) - \lambda_\varepsilon (i_1 - i_2) g_{\varepsilon,i_1,i_2}(t) &= 0 \quad \text{if} \quad i_1 \neq i_2, \\
\dot{h}_{\varepsilon,i_1,i_2}(t) + \partial_t g_{\varepsilon,i_1,i_2}(t) &= -\tilde{a}_{\varepsilon,i} \quad \text{if} \quad i_1 = i_2 = i,
\end{align*}
\]

for each couple \((i_1, i_2) \in \mathbb{N}^2\) such that \(i_1 + i_2 = j + 1\). Note that in (A.100), this last equality occurs only if \(j + 1\) is even and \(i = (j + 1)/2\). Equation (A.100) is then easily solved by setting

\[
\tilde{a}_{\varepsilon,i} = \int_{\mathbb{R}/\mathbb{Z}} h_{\varepsilon,i,i}(t) \, dt, \quad g_{\varepsilon,i,i}(t) = -\int_0^t (h_{\varepsilon,i,i}(s) - \tilde{a}_{\varepsilon,i}) \, ds.
\]

Equation (A.99) always admits unique 1-periodic solutions of the form

\[
\begin{align*}
\begin{cases}
g_{\varepsilon,i_1,i_2}(t) = e^{\lambda_\varepsilon (i_1 - i_2)} c_{\varepsilon,i_1,i_2} - \int_0^t e^{(t-s)\lambda_\varepsilon (i_1 - i_2)} h_{\varepsilon,i_1,i_2}(s) \, ds,
\end{cases}
\end{align*}
\]

where \( c_{\varepsilon,i_1,i_2} = (e^{\lambda_\varepsilon (i_1 - i_2)} - 1)^{-1} \int_0^t e^{(1-s)\lambda_\varepsilon (i_1 - i_2)} h_{\varepsilon,i_1,i_2}(s) \, ds \). In the preceding solutions, the dependence on \( \varepsilon \) is smooth and if, for \( \varepsilon = 0 \), the functions \( h_{0,i_1,i_2} \) do not depend on \( t \), we see that \( g_{0,i_1,i_2} \) is a constant.

This concludes the proof of Lemma 2.2. \( \square \)

**B. Appendix. Extension of symplectic diffeomorphisms**

**Lemma B.1.** Let \((\Theta_\varepsilon)_{\varepsilon \in [\varepsilon_0, \varepsilon_0[}\) be a smooth (or continuous) family of \( C^k \) symplectic

diffeomorphisms \( C^1 \)-close to the identity, defined on some open disk \( D(o, \delta) \) of \( \mathbb{R}^2 \), and

such that \( \Theta_\varepsilon(o) = o \). Then, there exists \((\tilde{\Theta}_\varepsilon)_{\varepsilon \in [\varepsilon_0, \varepsilon_0[}\), a smooth (or continuous)

family of \( C^k \) symplectic diffeomorphisms of \( \mathbb{R}^2 \) such that on \( D(o, \delta/2) \), one has \( \tilde{\Theta}_\varepsilon = \Theta_\varepsilon \).

**Proof.** We use the notation \( \Theta_\varepsilon(x, y) = (\tilde{x}, \tilde{y}) \). Because \( \Theta_\varepsilon \) is symplectic, the 1-form

\( \tilde{y} \, d\tilde{x} - y \, dx \) is closed and defined on a disk \( D(o, 4\delta/5) \) of center \( o \) and radius \( 4\delta/5 \) (we assume \( \Theta_\varepsilon \) \( C^1 \)-close to the identity so that we can use the implicit function theorem).

It is hence locally exact and there exists a function \( S_\varepsilon(y, \tilde{y}) \) such that \( \tilde{y} \, dx - y \, d\tilde{x} = dS_\varepsilon \).

Now the function \( F_\varepsilon(x, \tilde{y}) = -S_\varepsilon(y, \tilde{y}) + (\tilde{x} - x) \) is defined on \( D(0, 3\delta/4) \) and satisfies \( (y - \tilde{y}) \, dx + (\tilde{x} - x) \, d\tilde{y} = dF_\varepsilon \) or equivalently,

\[
\Theta_\varepsilon(x, y) = (\tilde{x}, \tilde{y}) \iff \begin{cases}
\tilde{x} = x + \partial_y F_\varepsilon(x, \tilde{y}), \\
y = \tilde{y} + \partial_x F_\varepsilon(x, \tilde{y}).
\end{cases}
\]  

(B.101)

Note that we can choose \((F_\varepsilon)\) as a \( C^k \)-family of \( C^{k+1} \)-functions such that \( F_\varepsilon(o) = 0 \), \( DF_\varepsilon(o) = 0 \).
We can then choose $\chi : \mathbb{R}^2 \to \mathbb{R}$ as a smooth function which is equal to 1 on $D(o, 2\delta/3)$ and 0 outside $D(o, 3\delta/4)$, set

$$\tilde{F}_\varepsilon = \chi \times F_\varepsilon,$$

and define $\hat{\Theta}_\varepsilon$ by (B.101) with $F_\varepsilon$ replaced by $\tilde{F}_\varepsilon$. The family of diffeomorphisms $(\hat{\Theta}_\varepsilon)_{\varepsilon}$ is a smooth (or continuous) family of exact symplectic $C^k$-diffeomorphisms.

C. Appendix. Proof of Lemma 9.1

Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth even function with support in $[-1/2, 1/2]$ such that $\chi(0) = 1$ and which is increasing on $[-1/2, 0]$. There exists $\alpha \in ]0, 1/4[ $ such that, for all $x \in ]-2\alpha, 2\alpha[$, one has $\chi(x) > 1/2$ and

$$\beta_{\text{min}} := \min_{[-2\alpha, -\alpha]} \chi' > 0, \quad \beta_{\text{max}} := \max_{[-2\alpha, -\alpha]} \chi' > 0.$$  

We define for $\rho \in ]0, 1/12[$ and $C_M > 0$,

$$\varphi_M(x) = a(\rho, C_M)\chi\left(\frac{x - 1/3}{1/12}\right) - C_M\chi\left(\frac{x - 2/3}{\rho}\right),$$

where $a(\rho, C_M) > 0$ is chosen so that

$$\int_0^1 e^{\varphi_M(u)} du = 1.$$  

Let $I = (2/3) + ]-2\alpha\rho, -\alpha\rho[$. For $x \in I$, one has

$$\varphi_M(x) \leq -C_M/2 = -C_M\alpha\beta_{\text{min}}/(2\alpha\beta_{\text{min}}),$$

$$\varphi'_M(x) \leq -(C_M/\rho)\beta_{\text{min}} = -(C_M\alpha\beta_{\text{min}})/(\alpha\rho) = -C_M\alpha\beta_{\text{min}}/|I|,$$

$$\varphi''_M(x) \geq -(C_M/\rho)\beta_{\text{max}} = -(C_M\alpha\beta_{\text{max}})/(\alpha\rho) = -(\beta_{\text{max}}/\beta_{\text{min}})C_M\alpha\beta_{\text{min}}/|I|.$$  

Fixing $\rho$ (for example $\rho = 1/12$) and taking

$$b^{-1} = \max\left(\frac{\beta_{\text{max}}}{\beta_{\text{min}}}, 2\alpha\beta_{\text{min}}\right), \quad C_M = \frac{M}{\alpha\beta_{\text{min}}},$$

provides the first two items of Lemma 9.1.

Let us check the third item is satisfied. From the definition of $s_M$, one has $s'_M(x) = e^{\varphi_M(x)} = 1$ for $x \not\in [0, 1]$. Because $s_M(0) = 0$, one has $s_M(x) = x$ for $x \leq 0$. Similarly, because

$$s_M(1) = \int_0^1 e^{\varphi_M(u)} du = 1,$$

we have $s_M(x) = x$ for $x \geq 1$.

Because in any case $s'(x) > 0$, this concludes the proof of Lemma 9.1.  

\[\square\]

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