SRB measures for hyperbolic attractor in low regularity

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Abstract
We consider a $C^1$ hyperbolic attractor, and prove the existence of a physical measure provided that the differential satisfies some summability condition which is weaker than Hölder continuity.

1 Introduction
Let $M$ be a compact Riemannian manifold, $f : M \to M$ be a $C^1$ diffeomorphism, and $\Lambda$ a closed $f$-invariant hyperbolic attractor (see section 2.1). Such map have a lot of invariant measures, however there is an interesting class called physique measures. If $f$ is $C^{1+\alpha}$, it is a classical result that $f$ has a unique physical measure (see [Yen02]). In this paper, we will be interested in the existence of a physical measure in a finer regularity. If $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus of continuity, we say that $f$ is $C^{1,\omega}$ if $f$ is $C^1$ and the modulus of continuity of $df$ is a multiple of $\omega$, i.e there is $C > 0$ such that

$$\|df_x - df_y\| \leq C \omega(d(x,y)), \forall x, y \in M.$$ 

We say that a modulus $\omega$ is Dini summable if

$$\int_0^1 \frac{\omega(t)}{t} \, dt < +\infty.$$ 

Our main results extends the work of Fan and Jiang [FJ01] from expanding maps to hyperbolic diffeomorphisms.

Theorem 1.1. If $\Lambda$ is a hyperbolic attractor of a $C^{1,\omega}$ diffeomorphism $f$, and $\omega$ is Dini summable, then $f$ has a unique physical measure.
To prove this theorem, first, we study the modulus of continuity of the unstable distribution $E^u$, and prove that

**Theorem 1.2.** Let $f : \Lambda \to \Lambda$ be a $C^1,\omega$ hyperbolic map, where $\omega$ is a Dini summable modulus of continuity, then the unstable distribution has a Dini summable modulus of continuity.

Given this theorem, we deduce that the geometric potential

$$\phi^{(u)} = -\log J^u f = -\log \det df|_{E^u}$$

has a Dini summable modulus of continuity. Finally, using Markov partitions [Bow75a], $f : \Omega(f) \to \Omega(f)$ is semiconjugated to a subshift of finite type. More precisely, there are $(\Sigma_A, \sigma)$ and a surjective Hölder map $\pi : \Sigma_A \to \Lambda$ such that $\pi \circ \sigma = f \circ \pi$, so if we take a potential with Dini summable modulus $\phi : \Lambda \to \mathbb{R}$, then $\pi \circ \sigma$ has a Dini summable modulus. To get an equilibrium state for $(\Sigma_A, \sigma, \pi \circ \phi)$ one can after several lemmas consider only the one-sided shift $(\Sigma^+_A, \sigma, \tilde{\phi})$, (where $\tilde{\phi}$ is a potential depending only on the future, and cohomologous to $\pi \circ \phi$ [PP90]) which is an expanding map, then we apply the adapted Ruelle-Perron-Frobenius theorem [FJ01a] to get an equilibrium measure for $(\sigma, \pi \circ \phi)$. We push this measure by $\pi$ to get an equilibrium measure for $(f_{\Omega(f)}, \phi)$.

**Corollary 1.3.** Let $f : \Lambda \to \Lambda$ be a $C^1,\omega$ hyperbolic map, where $\omega$ is a summable modulus of continuity, then the geometric potential $\phi^{(u)}$ has a unique equilibrium state $\mu_{\phi^{(u)}}$, in particular $\mu_{\phi^{(u)}}$ is ergodic. Furthermore, if $f$ is topologically mixing then the measure $\mu_{\phi^{(u)}}$ is Bernoulli.

The article is organized as follows. In section 2, we will recall uniform hyperbolicity and some classical tools: modulus of continuity, and equivalent formulation of the Dini summability condition. In section 3 we will recall some classical and motivational examples. In section 4 we give the proof of the regularity of the potential and proceed in the same way as in [Bow75a] to prove the existence of equilibrium measure. In section 5, we prove theorem 1.1 by adapting some volume lemmas given in [Bow75a].

## 2 Preliminaries and notations

### 2.1 Uniform hyperbolicity

Let $U$ be an open subset of $M$ and $f : U \to M$ a $C^1$ diffeomorphism. An invariant set $\Lambda \subset U$ is called hyperbolic if there are some $C > 0$ and $\lambda \in (0, 1)$
such that for all $x \in \Lambda$ we have a splitting of $T_xM = E^u_x \oplus E^s_x$ which is $f$ invariant, i.e $df_x(E^u_x) = E^u_{fx}$ and $df_x(E^s_x) = E^s_{fx}$ and such that

$$
\|df^n(v)\| \leq C\lambda^n\|v\|, \forall n \in \mathbb{N}, v \in E^s_x,
$$

(1)

$$
\|df^{-n}(v)\| \leq C\lambda^n\|v\|, \forall n \in \mathbb{N}, v \in E^u_x.
$$

(2)

In the definition, we didn’t assume any continuity on $E^s$ and $E^u$. In fact it is not hard to prove the continuity of $E^u$ and $E^s$ starting from the given definition. One can also assume that $C = 1$ by considering another equivalent Riemannian metric on $M$, and taking $\lambda' \in (\lambda, 1)$ (see Proposition 5.2.2 of [BS02]).

Some example of uniformly hyperbolic maps are: Arnold cat map, the Horseshoe, toral hyperbolic linear automorphism, the Smale solenoid.

A classical approach to deal with uniform hyperbolic maps, is to consider the space of continuous (resp. bounded) sections $\sigma : \Lambda \to T\Lambda$, which is a Banach space, and once we have the first definition, we can write this Banach space as the direct sum of two closed subspaces, corresponding to sections with value on $E^u$ or $E^s$. Once we do this, we have a natural linear action of $f$ on that Banach space which preserves the closed subspaces. This approach helps us prove a lot of result like shadowing lemma, local stability, etc.

In general it is hard to check uniform hyperbolicity using this definition (for instance we don’t know $E^u$ and $E^s$), to deal with this difficulty we study cones instead of linear subspaces.

### 2.2 Hyperbolicity via cone techniques

Let $x \in M$ and $E$ a linear subspace of $T_xM$, define the cone centered at $E$ by

$$
K^E_\alpha(x) = \{v \in T_xM : \|v_2\| \leq \alpha\|v_1\| \text{ where } v = v_1 + v_2 \text{ and } v_1 \in E, v_2 \in E^1\}.
$$

For a hyperbolic map $f$, $K^{E^u}_\alpha$ (resp $K^{E^s}_\alpha$) is called unstable (resp stable) cone field. We say that it has a small angle if $\alpha$ is small.

A cone field $K$ on $M$ is said to be invariant by $f$ if for all $x \in M$

$$
\text{d}f_x(K(x)) \subset \text{int}(K(fx)) \cup \{0\}.
$$

**Proposition 2.1.** (Proposition 5.4.3 [BS02]) Let $\Lambda$ be a compact invariant set of $f : U \to M$. Suppose that there is $\alpha > 0$ and for every $x \in \Lambda$ there are continuous subspaces $\hat{E}^s$ and $\hat{E}^u(x)$ such that $\hat{E}^s(x) \oplus \hat{E}^u(x) = T_xM$, and the cone $K^{\hat{E}^u}_\alpha(x)$ and $K^{\hat{E}^s}_\alpha(x)$ are $f$ invariant and $\|\text{d}f_x v\| < \|v\|$ for non-zero $v \in K^{\hat{E}^u}_\alpha(x)$, and $\|\text{d}f_x^{-1} v\| < \|v\|$ for non-zero $v \in K^{\hat{E}^u}_\alpha(x)$. Then $\Lambda$ is a hyperbolic set of $f$. 

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2.3 Local stable and unstable manifolds:

Define the local stable and unstable manifolds for \( x \in \Lambda \) by

\[
\begin{align*}
W_s^\epsilon(x) &= \{ y \in M \mid d(f^k x, f^k y) \leq \epsilon, \forall k \geq 0 \}, \\
W_u^\epsilon(x) &= \{ y \in M \mid d(f^{-k} x, f^{-k} y) \leq \epsilon, \forall k \geq 0 \}.
\end{align*}
\]

The definition of stable and unstable manifolds is dynamic, and the theorem of Perron-Hadamard proves that \( W_u^\epsilon \) and \( W_s^\epsilon \) are sub-manifolds for \( \epsilon \) small enough.

Let’s recall some classical definitions:

**Pseudo-orbit:** Let \( f : X \to X \) a homeomorphism of a metric space, and let \( \epsilon > 0 \). We say that a sequence \( (x_n)_{n \in \mathbb{Z}} \) is an \( \epsilon \)-pseudo-orbit if \( d(f x_n, x_{n+1}) < \epsilon \).

**Shadowing lemma:** We say that a homeomorphism \( f : X \to X \) have the shadowing property if for all \( \epsilon > 0 \) there is \( \delta > 0 \) such that for all \( \epsilon \)-pseudo-orbit \( (x_n)_{n \in \mathbb{Z}} \) there is \( y \in X \) such that for all \( n \in \mathbb{Z} \) we have \( d(f^n y, x_n) \leq \delta \).

It is known that Anosov diffeomorphisms and Axiom A diffeomorphisms have the shadowing property, in fact they have a stronger property called specification.

**Expansiveness:** A homeomorphism \( f : X \to X \) is expansive if there is \( \epsilon_0 > 0 \) such that for all \( x, y \in X \) there is \( n \in \mathbb{Z} \) such that \( d(f^n x, f^n y) \geq \epsilon_0 \). For instance an isometry is not expansive, and hyperbolic maps are expansive.

**Wandering set:** \( x \) is non-wandering, if for all neighborhood \( U \) of \( x \) there is \( n > 0 \) such that \( U \cap f^n U \neq \emptyset \). we denote the wandering set of a diffeomorphism \( \Omega(f) \).

**Axiom A diffeomorphism:** \( f \) is an Axiom A diffeomorphism if \( \Omega(f) \) is hyperbolic, and periodic orbits are dense in \( \Omega(f) \).

**Remark 2.2.** Local stable and unstable manifolds are used to construct Markov partitions, then shadowing, expansiveness and density of periodic orbits are used to prove that an Axiom A diffeomorphism is semiconjugated to a subshift of finite type via a Hölder map (see [Bow75a]).

**Spectral Decomposition:** (Proposition 3.5 [Bow75a]) Let \( f \) be an Axiom A diffeomorphism, then one can write \( \Omega(f) = \Omega_1 \cup \cdots \cup \Omega_s \) where the \( \Omega_i \) are pairwise disjoint closed sets (called basic sets) such that

1. \( f(\Omega_i) = \Omega_i \) and \( f|_{\Omega_i} \) is topologically transitive;
2. $\Omega_i = X_{1,i} \cup \cdots \cup X_{n,i}$ with $X_{j,i}$'s pairwise disjoint closed sets, $f(X_{j,i}) = X_{j+1,i}$ ($X_{n_j+1,i} = X_{1,i}$) and $f^n|_{X_{j,i}}$ is topologically mixing.

**Remark 2.3.** Using the spectral decomposition theorem, one can assume without loss of generality that an Axiom A diffeomorphism is transitive.

**Physical measure:** Let $f : M \to M$ be a continuous map, $X$ a closed $f$-invariant subset of $M$ and $\mu$ a $f$-invariant probability measure with support in $X$. Define the basin of the measure $\mu$ by:

$$B_\mu = \left\{ x \in M \left| \forall g \in C^0(M, \mathbb{R}), \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g \circ f^k(x) = \int g d\mu \right. \right\}.$$

We say that $\mu$ is physical if $B_\mu$ has positive measure with respect to the Lebesgue measure of $M$.

**Attractor:** A basic set $\Omega_s$ is called an attractor if it has a neighborhood $U$ such that $f(U) \subset U$.

### 2.4 Grassmanian bundle

Let $G^q(M) = \bigsqcup_{x \in M} G(q, T_x M)$ be the fiber bundle over $M$ with fiber $G(q, T_x M)$ (the set of subspaces of $T_x M$ of dimension $q$). A continuous distribution $E$ is a continuous section of $G^q(M)$, the latter has a Riemannian metric, so one can talk about the modulus of continuity of a distribution or the distance $d_{\text{grass}}$ between distributions that comes from the Riemannian structure.

Consider a continuous distribution $E$. Let $x_0 \in M$, and consider a chart $\psi : U \to \mathbb{R}^m$, where $U$ is a small neighborhood of $x_0$ such that for all $x \in U$, $E(x)$ is sufficiently close to $E(x_0)$ and $d_{\psi(x_0)}E(x_0) = \mathbb{R}^q \times \{0\}$, where $q = \dim E$. Define the distance $d = d_{x_0,\psi,E(x_0)}$ on a small neighborhood $\tilde{U}$ of $(x_0, E_0)$ in $G^q(M)$ by:

$$d(F_1^x, F_2^y) := d(x, y) + \| L_{F_1^x} - L_{F_2^y} \|,$$

where $L_{F_1^x} : \mathbb{R}^q \to \mathbb{R}^{m-q}$ (resp $L_{F_2^y}$) is the linear map whose graph is $d\psi_x(F_1^x)$ (resp $d\psi_y(F_2^y)$). This distance induces locally the usual topology of $G^q(M)$.

### 2.5 Pressure and equilibrium measure

In this part, let $(X, d)$ be a compact metric space, $f : X \to X$ a continuous function, and $\phi : X \to \mathbb{R}$ a potential. Define the dynamical distance $d_\phi$ for
\[ n \in \mathbb{N} \text{ and } x, y \in X \text{ by:} \]
\[ d_n(x, y) = \sup_{0 \leq k < n} d(f^k x, f^k y). \]

We denote by \( B_n(x, \epsilon) \) the ball of center \( x \) and radius \( \epsilon \) with respect to the distance \( d_n \).

A set \( A \) is called \((n, \epsilon)\)-spanning (resp. \((n, \epsilon)\)-separated) if \( X = \bigcup_{x \in A} B_n(x, \epsilon) \) (resp. for any \( x, y \in A \) we have \( B_n(x, \epsilon) \cap B_n(y, \epsilon) = \emptyset \)).

Consider the two following numbers that depend on \( f, \phi, \epsilon \) and \( n \geq 1 \)

\[ Q_n(f, \phi, \epsilon) = \inf \left\{ \sum_{x \in A} e^{(S_n f)(x)} : A \text{ is a } (n, \epsilon) - \text{spanning set for } X \right\}, \]
\[ P_n(f, \phi, \epsilon) = \sup \left\{ \sum_{x \in A} e^{(S_n f)(x)} : A \text{ is a } (n, \epsilon) - \text{separated set for } X \right\}, \]

where \( (S_n f)(x) = \sum_{k=0}^{n-1} \phi \circ f^k(x) \). If \( \phi \) is continuous, then

\[ \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \left( Q_n(f, \phi, \epsilon) \right) \]

exists, is finite and is equal to

\[ \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \left( P_n(f, \phi, \epsilon) \right). \]

The limit is called the topological pressure with respect to the potential \( \phi \), we denote it by \( P(\phi) \).

Denote by \( \mathcal{M}_f(X) \) the space of \( f \)-invariant probability measures on \( X \). Let \( \mu \in \mathcal{M}_f(X) \), then define the pressure with respect to this measure by:

\[ P_\mu(\phi) = h_\mu(f) + \int_X \phi \, d\mu, \]

where \( h_\mu(f) \) is the entropy of \( f \) with respect to the measure \( \mu \). The variational principle (see theorem 9.10 in [?]) gives the following formula:

\[ P(\phi) = \sup \left\{ h_\mu(f) + \int \phi \, d\mu : \mu \in \mathcal{M}_f(X) \right\}. \]

If \( \mu \in \mathcal{M}_f(X) \) is such that \( P_\mu(\phi) = P(\phi) \), then \( \mu \) is called an equilibrium measure for the potential \( \phi \).
2.6 Modulus of continuity

In this section, we give a formal definition of the summability condition introduced in the abstract. We start by recalling the classical definition of a modulus of continuity.

**Definition 2.4.** A modulus of continuity is a continuous, increasing and concave map $\omega: \mathbb{R}^+ \to \mathbb{R}^+$, such that $\omega(0) = 0$.

We say that the modulus $\omega$ is Dini summable if

$$\int_0^1 \frac{\omega(t)}{t} dt < +\infty.$$  

For instance, for any $\alpha \in (0, 1)$, the map $\omega(t) = t^\alpha$ is a modulus of continuity which is Dini summable. We assumed the concavity condition in the definition of a modulus of continuity for technical issues (see next proposition and lemma (4.4)), but note that any uniformly continuous map admits a concave modulus of continuity (See the end of this section).

The following proposition gives an equivalent condition on a modulus $\omega$ to be Dini summable.

**Proposition 2.5.** The following conditions are equivalent:

1. $\omega$ is Dini summable.
2. $\forall c \in (0, 1)$ and $t \geq 0$, $\sum_{k=0}^{+\infty} \omega(c^k t) < +\infty$
3. $\exists c \in (0, 1)$ and $t > 0$, $\sum_{k=0}^{+\infty} \omega(c^k t) < +\infty$

**Proof.** Since $\omega$ is concave, the map $t \mapsto \frac{\omega(t)}{t}$ is decreasing, hence we have the following inequalities for all $n$ and small $t$:

$$\sum_{k=0}^{n-1} (c^k - c^{k+1}) \frac{\omega(c^k t)}{c^k t} \leq \int_{c^k t}^{c^{k+1} t} \frac{\omega(x)}{x} dx \leq \sum_{k=0}^{n-1} (c^k - c^{k+1}) \frac{\omega(c^{k+1} t)}{c^{k+1} t}. \quad (3)$$

We deduce the proposition from these inequalities. $\square$

Let $\omega$ be a Dini summable modulus, and for $c \in (0, 1)$ define

$$\hat{\omega}_c(t) = \sum_{k=0}^{+\infty} \omega(c^k t), \forall t \geq 0. \quad (4)$$

It follows immediately that $\hat{\omega}_c$ is a modulus of continuity.
**Definition 2.6.** If $\omega$ is a Dini summable modulus, we denote $\tilde{\omega}$ the modulus defined by

$$\tilde{\omega}(t) = \int_0^t \omega(s) \frac{ds}{s}.$$ 

Using the previous proposition, the modulus $\tilde{\omega}$ and $\omega_c$ are equivalent for any $c \in (0, 1)$ i.e there is $C > 1$ such that:

$$C^{-1} \omega_c \leq \tilde{\omega} \leq C \omega_c.$$ 

**Remark 2.7.** The Dini summability condition might seem artificial, but it becomes more natural once we see in [FJ01a] how it is used to prove Ruelle theorem for transfer operator.

**Examples:**

- For $\alpha \in (0, 1]$, $\omega(t) = t^\alpha$ is Dini summable, because $\tilde{\omega}(t) = \frac{1}{\alpha} \omega(t)$.
- The modulus $\omega_{\beta \log}(t) = \frac{1}{(\log(\frac{1}{t}))^\beta}$ is Dini summable if and only if $\beta > 1$.

In this example $\omega$ is defined only for small $t$, then we extend it by an affine map.

**Definition 2.8.** Let $X, Y$ be two metric spaces, and $\omega$ a modulus of continuity, we say that a map $f : X \to Y$ is $C^{0, \omega}$ if there is a $C > 0$ such that:

$$d(f(x), f(y)) \leq C \omega(d(x, y)), \quad \forall x, y \in X.$$ 

(5)

- If $\omega(t) = t$, then $C^{0, \omega}$ is the set of Lipschitz maps.
- If $\omega(t) = t^\alpha$, where $0 < \alpha < 1$, then $C^{0, \omega}$ is the set of Hölder maps with exponent $\alpha$.

Given a continuous map $g : M \to M$ of a compact manifold, a natural way to define the modulus of continuity of $g$ would be to take:

$$\tilde{\omega}_g(t) = \sup_{x, y \in M, d(x, y) \leq t} d(gx, gy),$$ 

(6)

but $\tilde{\omega}_g$ is not concave. To get concavity, we take:

$$\omega_g = \inf \{ h \mid h \text{ continuous, concave and increasing and } h \geq \tilde{\omega}_g \}.$$ 

(7)

It is clear that $\omega_g$ is a modulus of continuity, and it satisfies the inequality (5) with constant $C = 1$. 

8
3 Motivating example: A Horseshoe with Positive Measure

In this section we will see the importance of the summability condition by comparing proposition 5.7 and the example given in [Bow75b]. Let $I = [a, b]$ be a closed interval, and $(\alpha_n)$ a sequence of positive numbers such that $\sum_{n=0}^{\infty} \alpha_n < l(I)$. Let $a = a_1a_2 \cdots a_n$ denote a sequence of 0’s and 1’s of length $n = n(a)$. Define $I_0^a = I$, $I_0^{a_n} = \left[ a_n, a_n \cdot \frac{a_n b}{a_n a} \right]$ and $I^{a_n}_k \subset I^a_k$ recursively as follows. Let $I^a_0$ and $I^a_1$ be the left and right intervals remaining when the interior of $I^a_1$ is removed from $I^a_2$. Let $I^a_k(k = 0, 1)$ be the closed interval of length $\frac{\alpha_n(a_n)}{2n(a_n)}$ and having the same center as $I^a_k$.

The set $K = \bigcap_{m=0}^{\infty} \bigcup_{n(a)=m} I^a_k$ is the standard Cantor set, and by construction the Lebesgue measure of $K$ is $l(I) - \sum_{n=0}^{\infty} \alpha_n$. Let $J$ be another interval, and $(\beta_n)$ be a sequence of positive numbers. We construct $J^a, J^a_k$ and $K_J$ as above. Let us take $\beta_n = \frac{1}{(n+1)!}$, $\alpha_n = \frac{\beta_n}{\alpha_n}$, $\delta_n = 2\frac{\beta_n}{\beta_{n+1}} - 2$, $I = \left[ \frac{\beta_1}{2}, 1 \right]$ and $J = [-1, 1]$. For each $a$ let $g : I^a_k \to J^a_k$ be a $C^1$ diffeomorphism so that

i. $g'(x) = 2$ for $x$ and endpoint of $I^a_k$,

ii. $2 - \delta_n \leq g'(x) \leq \frac{\beta_n}{\alpha_n} + \delta_n$ for $x \in I^a_k$.

Then $g$ extends from $\bigcup_a I^a_k$ to a homeomorphism $g : I \to J$; $g$ is in fact a $C^1$ diffeomorphism with derivative 2 at each point of $K_I$. One defines a diffeomorphism $f$ of the square $S = J \times J$ into $\mathbb{R}^2$ by

i. $f(x, y) = (g(x), g^{-1}(y))$ for $(x, y) \in I \times J$,

ii. $f(x, y) = (g(-x), -g^{-1}(y))$ for $(x, y) \in (-I) \times J$ and $f(T) \cap (J \times J) = \emptyset$ where $T = (-\frac{\delta_n}{2}, \frac{\beta_k}{2}) \times J$.

The mapping $f$ can be extended to the sphere. Then $\Lambda = \bigcap_{n=-\infty}^{+\infty} f(S) = K_I \times K_J$ has Lebesgue measure $(2 - \sum_{n=0}^{\infty} \beta_n)^2 > 0$. The modulus of continuity of $g'$ does not satisfy Dini condition, in particular the modulus of continuity of $df$ does not satisfy Dini condition. This hyperbolic horseshoe has a positive measure but it is not an attractor. Proposition 5.7 gives a sufficient
condition on the regularity of the hyperbolic map so that we have an equivalence between \( \Lambda \) being an attractor and \( W^s(\Lambda) \) having a positive Lebesgue measure.

**Modulus of continuity of \( g' \):** Let \( \omega \) be the modulus of continuity of \( g' \), then we have for all \( x \in I \), and \( x_0 \in K_I \)

\[
|g'(x) - g'(x_0)| \leq \omega(d(x, x_0)).
\]

(8)

Consider some interval \( I_a^* \), where \( \text{length}(a) = n \), since \( g \) send \( I_a^* \) of length \( \frac{a}{2n} \) to some interval of length \( \frac{\beta_n}{2n} \), there is \( x \in I_a^* \) such that \( g(x) \geq 2 \frac{\beta_n}{\beta_{n+1}} \), and if we take \( x_0 \) in the boundary of \( I_a^* \) then \( g(x_0) = 2 \), we deduce that:

\[
\delta_n = 2 \frac{\beta_n}{\beta_{n+1}} - 2 \leq |g(x) - g(x_0) | \leq \omega\left(\frac{1}{2n}\right),
\]

which implies that:

\[
\delta_n \leq \omega\left(\frac{1}{2n}\right), \forall n \in \mathbb{N},
\]

(9)

but it is clear that \( \sum_{k \geq 1} \delta_n = +\infty \), which implies that the modulus does not satisfy Dini condition.

**Remark 3.1** (Perturbation of hyperbolic toral automorphisms). We can construct an example of a diffeomorphism \( f \) close to the identity, with derivative Dini summable modulus, but which is not Hölder continuous either by perturbing a toral hyperbolic linear automorphism \( A \), or by considering a diffeomorphism \( g \) close to the identity with derivative having a Dini summable modulus, then considering the map \( A^n \circ g \circ A^n \) which is hyperbolic for large \( n \) (using the cone field criterion one can prove the hyperbolicity of such a map).

4 Proof of theorem 1.2

The strategy to prove theorem 1.2 is to study the regularity of the unstable distribution restricted to a stable leaf, then determining the regularity of unstable leaves, then deducing by lemma 4.6 the desired regularity of the unstable distribution.

Let \( A : \mathbb{R}^n \to \mathbb{R}^n \) be a hyperbolic linear map, and consider the set \( \xi_s \) defined by

\[
\xi_s = \{ E \in G^n \mathbb{R}^n \mid E \oplus E^s = \mathbb{R}^n \},
\]

where \( E^s \) is the stable subspace of \( A \). Consider the distance \( d_{E^s \oplus E^s} \) on \( \xi_s \) given by

\[
d_{E^s \oplus E^s}(E, F) = \| L_E - L_F \|, \quad E, F \in \xi_s.
\]
where \( L_E : E^u \to E^s \) (resp \( L_F \)) is the unique linear map whose graph is \( E \) (resp \( F \)), and \( \| \cdot \| \) is any operator norm on the space of linear maps from \( E^u \) to \( E^s \) (See Chapter 4.3 of [Man12]).

The following lemma states that this distance is contracted under the action of \( A \).

**Lemma 4.1.** We have \( d_{E^u \oplus E^s}(AE, AF) \leq \lambda^2 d_{E^u \oplus E^s}(E, F), \forall E, F \in \xi \), where \( \lambda \in (0, 1) \) is a hyperbolicity constant of \( A \).

**Proof.** We have by definition \( L_{AE} = A \circ L_E \circ A^{-1}_{|E^u} \), so

\[
d_{E^u \oplus E^s}(AE, AF) = \| L_{AE} - L_A F \| = \| A(L_E - L_F)A^{-1}_{|E^u} \| \\
\leq \| A_{|E^u} \| \| A^{-1}_{|E^u} \| \| L_E - L_F \| \\
\leq \lambda^2 d_{E^u \oplus E^s}(E, F).
\]

The following two lemmas show that the defined distance depends in a bi-Lipschitz way on the reference spaces \( E^u \) and \( E^s \).

**Lemma 4.2.** If \( B \) is a linear map such that \( \| A - B \| \leq \epsilon \), for some small \( \epsilon > 0 \), then we have \( d_{E^u \oplus E^s}(BE^u, E^u) \leq \epsilon \).

**Proof.** Since we take a small \( \epsilon \), \( BE^u \) is transverse to \( E^s \). Let us show that \( L_{BE^u} = A L_{A^{-1}BE^u} A^{-1} \). Take \( x \in E^u \), then by definition of \( L_{BE^u} \) we have \( x + L_{BE^u} x \in BE^u \), so \( A^{-1} x + A^{-1} L_{BE^u} x \in A^{-1} BE^u \). We have also \( A^{-1} x + L_{A^{-1}BE^u} A^{-1} x \in A^{-1} BE^u \), so we deduce that

\[
A^{-1} L_{BE^u} = L_{A^{-1}BE^u} A^{-1}_{|E^u}.
\]

We have also \( L_{E^u} = 0 \), hence

\[
d_{E^u \oplus E^s}(BE^u, E^u) = \| A_{|E^u} L_{A^{-1}BE^u} A^{-1}_{|E^u} - 0 \| \leq \lambda^2 \| L_{A^{-1}BE^u} \|,
\]

and since \( A^{-1} B \) is close to the identity we deduce the lemma.

**Lemma 4.3.** Let \( \epsilon > 0 \) small, and take \( E_0 \) (resp \( F_0 \)) a subspace of dim \( q \) (resp \( m - q \)) such that \( d(E_0, E^u) \leq \epsilon \) (resp \( d(F_0, E^s) \leq \epsilon \)). Then, there is \( \delta = \delta(\epsilon) \), such that \( \delta(\epsilon) \to 1 \) and for all \( E, E' \) transversal to \( E^s \) and \( F_0 \) and close enough to \( E^u \) we have

\[
\frac{1}{\delta} d_{E^u \oplus E^s}(E, E') \leq d_{E_0 \oplus F_0}(E, E') \leq \delta d_{E^u \oplus E^s}(E, E').
\]
Proof. Let $M_1$ (resp $M_2$) the linear map from $E^u$ to $E^s$ (resp $E^s$ to $E^u$) whose graph is $E_0$ (resp $F_0$). Since $E_0$ (resp $F_0$) is close to $E^u$ (resp $E^s$) we deduce using the previous lemma that $\|M_i\| < \epsilon$, for $i = 1, 2$. Let $B_\epsilon$ the linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined on $E^u$ as $Id_{E^u} + M_1$ and on $E^s$ as $Id_{E^s} + M_2$.

Let $E$ and $E'$ be two subspaces transverse to $E^s$ and $F_0$, and let $L_E$ (resp $L'_E$) the linear map from $E^u$ to $E^s$ (resp $E_0$ to $F_0$) whose graph is $E$.

First, assume that $F_0 = E^s$. Let $x \in E^u$, then we have by definition of $L_E$ :

$$B_\epsilon x + (L_E x - M_1 x) = x + M_1 x + (L_E x - M_1 x) \in E,$$

and since $L^0_E(B_\epsilon x)$ is the unique vector in $F_0$ such that $B_\epsilon x + L^0_E(B_\epsilon x) \in E$, we deduce that

$$L^0_E B_\epsilon x = L_E x - M_1 x,$$

using the last equality we deduce that

$$\|L^0_E - L^0_{E'}\| = \|(L^0_E - L^0_{E'}) B_\epsilon B_\epsilon^{-1}\| \leq \|B_\epsilon^{-1}\| \|L^0_E B_\epsilon - L_{E'} B_\epsilon\| \\ \leq \|B_\epsilon^{-1}\| \|L_E - L_{E'}\|.$$  

Since $B_\epsilon$ is a perturbation of $Id$, the norm $\|B_\epsilon^{-1}\|$ is close to 1, and goes to 1 when $\epsilon$ goes to 0, which proves the lemma in the special case where $F_0 = E^s$. Now assume that $E_0 = E^u$, and define $M_\epsilon$ from $E^u$ to $E^u$

$$M_\epsilon x = x + L_E x - B_\epsilon L_E x = x - M_2 L_E x \in E^u,$$

notice that $M_\epsilon$ does not depend on $E$. Since $E_0 = E^u$, we have $L^0_E M_\epsilon x = B_\epsilon L_E x$, for all $x \in E^u$. The map $M_\epsilon$ is a small perturbation of $Id_{E^u}$, so it is invertible and we have

$$L^0_E = B_\epsilon L_E M_\epsilon^{-1},$$

we deduce that

$$\|L^0_E - L^0_{E'}\| = \|B_\epsilon(L_E - L^0_E) M_\epsilon^{-1}\| \leq (1 + \epsilon)(1 + \epsilon)\|L_E - L_{E'}\|,$$

which proves the lemma in the special case where $E_0 = E^u$. Combining the two cases we get the proof in general.

We will need a few lemmas to get the regularity of the unstable distribution when restricted to a small piece of stable manifold. The strategy is to use the dynamic of $f$ to give a upper bound for the modulus of continuity of $E^u$ when restricted to a local stable leaf.
Lemma 4.4. Let $f : \Lambda \to \Lambda$ be a $C^{1,\omega}$ hyperbolic map and $\lambda$ the constant given by equation (1) with respect to an adapted norm. Define for $x_0 \in \Lambda$, $n \in \mathbb{N}$ and $c \in (0, \lambda)$, the map $\Omega_n^c : W^s(x_0) \times W^s(x_0) \to \mathbb{R}^+$

$$\Omega_n^c(x, y) = \sum_{k=0}^{n-1} c^{n-k} \omega(d(f^k x, f^k y)), \quad (12)$$

then we have for all $n \in \mathbb{N}$:

$$\Omega_n^c(x, y) \leq \frac{c}{\lambda - c} \cdot \omega(\lambda^n d(x, y)). \quad (13)$$

Proof. Let $x, y \in W^s(x_0)$. By the concavity of $\omega$:

$$\frac{\omega(\lambda^nt)}{\lambda^nt} \geq \frac{\omega(\lambda^kt)}{\lambda^kt}, \quad \forall n, k, t \in \mathbb{R}^+ \text{ and } k \leq n.$$ 

Now using the fact that $d(f^k x, f^k y) \leq \lambda^k d(x, y)$ for all $k \in \mathbb{N}$, we get:

$$\Omega_n^c(x, y) \leq \sum_{k=0}^{n-1} c^{n-k} \omega(\lambda^k d(x, y))$$

$$\leq \sum_{k=0}^{n-1} \left( \frac{c}{\lambda} \right)^{n-k} \omega(\lambda^n t) \leq \frac{c}{\lambda - c} \omega(\lambda^n t).$$

\hfill \Box

Lemma 4.5. Let $f : \Lambda \to \Lambda$ be a $C^{1,\omega}$ hyperbolic map, then local unstable manifolds are $C^{1,\omega}$.

Proof. Consider $E$ (resp $F$) a smooth distribution close to $E^u$ (resp $E^s$) over $\Lambda$, then consider the distribution $E^n := f^n_* E$. By definition $E^n$ converges exponentially to $E^u$ with respect to the distance $d = d_{E \oplus F}$. So in order to prove the lemma, it is sufficient to prove that there is $C > 0$ such that for all $n \geq 0$ the distribution $E^n$ has $C \omega$ as a modulus of continuity when restricted to an unstable leaf.

Fix $x_0 \in \Lambda$, and take $x, y \in W^u(x_0)$. Let $d_k = d_{E_{f^{-k}x_0} \oplus E_{f^{-k}x_0}}$ defined over distribution of dimension $q$ in a neighborhood of $f^{-k}x_0$, we have:

$$d(E^{n+1}(x), E^{n+1}(y)) = d\left( df_{f^{-1}x}(E^n(f^{-1}x)), df_{f^{-1}y}(E^n(f^{-1}y)) \right)$$

$$\leq d\left( df_{f^{-1}x}(E^n(f^{-1}x)), df_{f^{-1}x}(E^n(f^{-1}x)) \right)$$

$$+ d\left( df_{f^{-1}x}(E^n(f^{-1}y)), df_{f^{-1}y}(E^n(f^{-1}y)) \right)$$
Upper bound for $d\left(df_{f^{-1}x}(E^n(f^{-1}y)), df_{f^{-1}y}(E^n(f^{-1}y))\right)$: Since $f$ is $C^1$-smooth, we have $d\left(df_{f^{-1}x}(E^n(f^{-1}y)), df_{f^{-1}y}(E^n(f^{-1}y))\right) \leq \omega(d(f^{-1}x, f^{-1}y))$.

Upper bound for $d\left(df_{f^{-1}x}(E^n(f^{-1}x)), df_{f^{-1}x}(E^n(f^{-1}y))\right)$: using lemma 4.3 and lemma 4.1, we deduce that

$$\omega(d(f^{-1}x, f^{-1}y))$$

is the desired upper bound.

By induction, we deduce that for all $n \geq 0$ and $x, y \in W^u_{c}(x_0)$ we have:

$$d(E^n(x), E^n(y)) \leq (\delta \lambda)^2d\left(E(f^{-n}x), E(f^{-n}y)\right) + \sum_{k=1}^{n}(\delta \lambda)^k\omega(\lambda^k d(x, y))$$

$$\leq d(x, y) + \left(\sum_{k=1}^{+\infty}(\delta \lambda)^k\right)\omega(d(x, y))$$

$$\leq C\omega(d(x, y)),$$

where the constant $C = C(\epsilon, \delta)$ does not depend on $x, y$ and $n$. Finally we deduce that for all $x, y \in W^u_{c}(x_0)$ we have:

$$d(E^u_x, E^u_y) \leq C\omega(d(x, y)).$$

Lemma 4.6. Let $X$ be a metric space, and $g : M \to X$ a continuous map. Assume that there is an $\epsilon$, such that for all $x_0 \in M$

$$d(gx, gy) \leq \omega(d(x, y)), \quad \forall x, y \in W^u_{c}(x_0),$$

$$d(gx, gy) \leq \omega(d(x, y)), \quad \forall x, y \in W^s_{c}(x_0).$$

Then there is $K = K(\epsilon) > 0$ such that for all $x, y \in M$ and $d(x, y) < \epsilon$

$$d(gx, gy) \leq 2\omega(Kd(x, y)).$$

Proof. For $\epsilon > 0$ small, there is $K_1 > 0$ such that for all $x, y \in M$ with $d(x, y) < \epsilon$, we have the following inequality:

$$d(x, [x, y])^2 + d([x, y], y)^2 \leq K_1 d(x, y)^2, \quad (15)$$

where $[x, y] = W^s_{c}(x) \cap W^u_{c}(y)$ is the Bowen bracket of $x$ and $y$. Since $\omega$ is a modulus of continuity,

$$\omega(s) + \omega(t) \leq 2\omega\left(\frac{1}{\sqrt{2}} \sqrt{s^2 + t^2}\right), \forall s, t \geq 0, \quad (16)$$

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the last two inequalities implies that for all \( x, y \in M \) satisfying \( d(x, y) < \epsilon \),

\[
\begin{align*}
d(gx, gy) & \leq d(gx, g[x, y]) + d(g[x, y], gy) \\
& \leq \omega(d(x, [x, y])) + \omega(d([x, y], y)) \\
& \leq 2\omega\left(\frac{1}{\sqrt{2}}\sqrt{d(x, [x, y])^2 + d([x, y], y)^2}\right) \\
& \leq 2\omega\left(\frac{1}{\sqrt{2}}\sqrt{K_1d(x, y)}\right).
\end{align*}
\]

□

Proof of theorem \[1.3\]: Let \( x_0 \in \Lambda \subset M \), and consider the positive orbit of \( x_0 \). Take a neighborhood of \( f^k(x_0) \) and \( \psi_k : U_k \to \mathbb{R}^n \) a smooth chart. We can assume that for all \( k \in \mathbb{N}, U_k \supset B(f^kx_0, \epsilon_0) \) for some \( \epsilon_0 > 0 \), by taking \( \epsilon_0 \) less than the Lebesgue number. We can assume further that \( d_{f^kx_0}\psi_k(E^u_{f^kx_0}) = \mathbb{R}^q \times \{0\} \) and \( d_{f^kx_0}\psi_k(E^s_{f^kx_0}) = \{0\} \times \mathbb{R}^{n-q} \). Let \( g_k = \psi_{k+1} \circ f \circ \psi_k^{-1} \). We can choose \( \epsilon > 0 \) small such that for all \( k \in \mathbb{N} \), the map

\[
g_k \circ g_{k-1} \circ \cdots \circ g_0 : \psi_0(W^s_\epsilon(x_0)) \to \mathbb{R}^n,
\]

is well defined.

For \( x \in W^s_\epsilon(x_0) := \psi_0(W^s_\epsilon(x_0)) \), let \( E^u_x := d\psi_0(\psi^{-1}_0(x))E^u_{\psi^{-1}_0(x)} \). To prove the theorem it will be enough to prove that:

\[
W^s_\epsilon(x_0) \to G^u(\mathbb{R}^n) \\
x \mapsto E^u_x,
\]

has a summable modulus of continuity. By choosing \( \epsilon > 0 \) smaller, we may assume that \( E^u_x \oplus \mathbb{R}^{n-q} = \mathbb{R}^n, \forall x \in W^s_\epsilon(x_0) \).

Let \( x, y \in W^s_\epsilon(x_0), A_k = dg_k(g_{k-1}x) \) and \( B_k = dg_k(g_{k-1}y) \) \( \forall k \geq 0 \), (where \( g_{-1} := Id \)). Consider \( d = d_{\mathbb{R}^q \oplus \mathbb{R}^{n-q}} \) then we have:

\[
d(A_0E_x^u, B_0E_y^u) \leq d(A_0E_x^u, A_0E_y^u) + d(A_0E_y^u, B_0E_y^u).
\]

Using lemma \[4.3\] we can find \( \delta = \delta(W^s_\epsilon(x_0)) \) close to 1, such that

\[
d(A_0E_x^u, B_0E_y^u) \leq \delta \cdot d_{E^u_x \oplus E^u_y}(A_0E_x^u, B_0E_y^u),
\]

And using lemma \[4.1\] we can find \( \lambda = \lambda(W^s_\epsilon(x_0)) \) in \((0, 1)\) such that

\[
d_{E^u_x \oplus E^u_y}(A_0E_x^u, A_0E_y^u) \leq \lambda^2 d_{E^u_x \oplus E^u_y}(E_x^u, E_y^u),
\]
so we deduce that
\[ d(A_0 E^u_x, A_0 E^u_y) \leq (\delta \lambda)^2 \cdot d(E^u_x, E^u_y). \] (17)

Since \( f \in C^1, \omega \) we have \( \| A_0 - B_0 \| \leq \omega(d(x, y)) \), then we apply lemma 4.2 and get
\[ d(A_0 E^u_x, B_0 E^u_y) \leq \omega(d(x, y)). \]

We deduce that for all \( n \in \mathbb{N} \)
\[ d(A_n \cdots A_0 E^u_x, B_n \cdots B_0 E^u_y) \]
\[ \leq (\delta \lambda)^2 d(A_{n-1} \cdots A_0 E^u_x, B_{n-1} \cdots B_0 E^u_y) + \omega\left(d(g_{n-1} \cdots g_0 x, g_{n-1} \cdots g_0 y)\right) \]

Using induction
\[ d(A_n \cdots A_0 E^u_x, B_n \cdots B_0 E^u_y) \]
\[ \leq (\delta \lambda)^{2n} \cdot d(E^u_x, E^u_y) + \sum_{k=0}^{n-1} (\delta^2 \lambda^2)^{n-k} \omega\left(d(g_k \cdots g_0 x, g_k \cdots g_0 y)\right). \]

Since \( \delta \) is arbitrarily close to 1, we can assume that \( (\delta \lambda) \in (0, 1) \). This implies using lemma 4.4 that
\[ d(A_n \cdots A_0 E^u_x, B_n \cdots B_0 E^u_y) \leq (\delta \lambda)^{2n} \cdot d(E^u_x, E^u_y) + \frac{\delta^2 \lambda}{1 - \delta^2 \lambda} \omega(\lambda^n d(x, y)). \] (18)

Since these inequalities do not depend on charts up to multiplication by constant, we deduce that there is \( M_1, M_2, M_3 > 0 \) such that for all \( x, y \in W^s(x_0) \) and \( n \in \mathbb{N} \)
\[ d(E^u_{f^nx}, E^u_{f^ny}) \leq M_1 (\delta \lambda)^{2n} d(E^u_x, E^u_y) + M_2 \omega\left(M_3 \lambda^n d(x, y)\right). \] (19)

Let \( \omega^u \) be the modulus of continuity of \( E^u \) restricted to \( W^s(x_0) \). Using (19) we deduce that
\[ \omega^u(\lambda_0^n t) \leq M_1 (\delta \lambda)^{2n} \omega^u(t) + M_2 M_3 \omega^u(\lambda^n t), \forall n \in \mathbb{N}, t > 0, \] (20)

where \( \lambda_0 \in (0, \lambda) \) only depends on the hyperbolic map \( f \). The latter inequality implies that \( \omega^u \) is Dini summable. Using lemmas 4.5 and 4.6 we deduce that \( E^u \) has a summable modulus of continuity, hence \( \psi^{(u)} \) has a summable modulus of continuity.

The next lemma is proven in Chapter 4, Lemma 4.3 of [Bow75a]. Together with the regularity of the geometric potential, it implies the existence and uniqueness of an equilibrium measure.
Lemma 4.7. Let $\sigma : \Sigma_A \rightarrow \Sigma_A$ a subshift of finite type, and $f : \Lambda \rightarrow \Lambda$ semiconjugated to $\sigma$ via $\pi$, then for any $\mu \in M_f(\Lambda)$ there is a $\nu \in M_\sigma(\Sigma_A)$ with $\pi_*\nu = \mu$.

Proof of corollary 1. Using Markov partition, the map $f$ is semiconjugated to a subshift of finite type \cite{Bow75a}. Let us consider the following commutative diagram:

\[
\begin{array}{ccc}
\Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\
\pi \downarrow & & \phi^u \circ \pi \downarrow \\
\Lambda & \xrightarrow{f} & \pi^u \\
\end{array}
\]

Since $\phi^u$ is $C^{0, \omega_0}$, where $\omega_0$ is a modulus that satisfies Dini condition (theorem 1.2) and $\pi$ is Hölder continuous, then $\phi^u \circ \pi$ is $C^{0, \omega_0}$, so Theorem 1 of \cite{FJ01a} give us a unique equilibrium state $\mu_{\phi^u \circ \pi}$ for $(\sigma, \phi^u \circ \pi)$. We push this measure by $\pi$ to get a measure $\mu_{\phi^u}$, and using the fact that $\pi$ is a bijection when restricted to a set of full $\nu$ measure, we deduce that $h_{\mu_{\phi^u}}(f) = h_{\phi^u \circ \pi}(\sigma)$, which implies that $\mu_{\phi^u}$, is an equilibrium measure. If $\mu_2$ is another equilibrium measure, we lift it using the previous lemma to an equilibrium measure of $(\sigma, \phi^u \circ \pi)$, and by uniqueness of the equilibrium measure for the shift, we get $\mu_2 = \mu_{\phi^u}$.

Since $\mu_{\phi^u}$ is the unique equilibrium state, it is ergodic. Indeed if 

$$\mu_{\phi} = t\mu_1 + (1 - t)\mu_2,$$

then we have:

\[
P(\phi^u) = h_{\mu_{\phi^u}}(f) + \int \phi^u \ d\mu \\
= th_{\mu_1}(f) + (1 - t)h_{\mu_2}(f) + t \int \phi^u \ d\mu_1 + (1 - t) \int \phi^u \ d\mu_2 \\
\leq P(\phi^u),
\]

which implies that $P_{\mu_1}(\phi^u) = P_{\mu_2}(\phi^u) = P(\phi^u)$, so $\mu_1$ and $\mu_2$ are equilibrium states, this implies that $\mu = \mu_1 = \mu_2$.

\[\square\]

5 Proof of theorem 1.1

Given Theorem 1.2 the proof of Theorem 1.1 is in the same way as in \cite{Bow75a}, but we need to adapt a few lemmas in low regularity. The fol-
Remark 5.2. If \( n \) We have also for all \( n \in \mathbb{N} \):

\[
\frac{1}{C} \leq \frac{\det df^n(x)|_{E(x)}}{\det df^n(y)|_{E(y)}} \leq C,
\]

(21)

We have also for all \( n \in \mathbb{N} \):

\[
\frac{1}{C} \leq \frac{\det df^n(x)|_{E(x)}}{J^n f^n(x)} \leq C.
\]

(22)

**Lemma 5.1** (Distortion lemma). Let \( f : \Lambda \to \Lambda \) a \( C^{1+\omega} \) hyperbolic map, where \( \omega \) is a Dini summable modulus. Fix \( \epsilon > 0 \) and an unstable invariant cone family \( (C^u_x)_{x \in \mathcal{U}} \) of sufficiently small angle, and \( \mathcal{F}_{x,n} \) a foliation of \( B_n(x, \epsilon) \) tangent to \( C^u_n \), whose leafs are \( C^{1+\omega} \) and \( E = T_{\mathcal{F}_{x,n}} \) is \( C^0,\omega \) distribution. Then there is \( C = C(\epsilon) \) such that for all \( n \in \mathbb{N}, x \in \Lambda \) and \( y \in B_n(x, \epsilon) \)

\[
\frac{1}{C} \leq \frac{\det df^n(x)|_{E(x)}}{\det df^n(y)|_{E(y)}} \leq C,
\]

(21)

Using lemma 4.1, we have

\[
\frac{1}{C} \leq \frac{\det df^n(x)|_{E(x)}}{J^n f^n(x)} \leq C.
\]

(22)

**Remark 5.2.** If \( f \) is an Anosov diffeomorphism the foliation \( \left((W^u_e(y))_{y \in W^s(x)}\right) \) satisfies the previous lemma.

**Proof.** Let \( x \in \Lambda, z \in W^s(x) \) and \( y \in P(z, \epsilon) \), where \( P(z, \epsilon) \) is a leaf of \( \mathcal{F}_{x,n} \) passing through \( z \). By the regularity of \( E \) and of \( f \), there is \( C_0 = C_0(\epsilon) \) such that for all \( n \in \mathbb{N} \)

\[
\left| \frac{\det df^n(x)|_{df^n E(x)}}{\det df^n(z)|_{df^n E(z)}} - 1 \right| \leq C_0 \left( \alpha(df^n E(x)) + \alpha(df^n E(z)) + \omega(d(f^n x, f^n z)) \right),
\]

where \( \alpha(E(*)) = d_{E^u(*)}(E^u(*), E(*)) \).

Using lemma 4.1, we have \( \alpha(df^n E(*)) \leq \lambda^{2n} \alpha(E(*)) \), so we deduce that

\[
\left| \frac{\det df^n(x)|_{df^n E(x)}}{\det df^n(z)|_{df^n E(z)}} - 1 \right| \leq C_0 \left( \lambda^{2n} \alpha(E(x)) + \lambda^{2n} \alpha(E(z)) + \omega(\lambda^n d(x, z)) \right),
\]

in particular we have

\[
\frac{\det df^n(x)|_{E(x)}}{\det df^n(z)|_{E(z)}} = \prod_{k=0}^{n-1} \frac{\det df^k(x)|_{df^k E(x)}}{\det df^k(z)|_{df^k E(z)}} \leq \prod_{k=0}^{n-1} \left( 1 + \lambda^{2k} + \omega(\lambda^k \epsilon) \right) \leq C_1 = C_1(\epsilon).
\]

Now, \( f^k(P(z, \epsilon)) \) has diameter of order \( \lambda^{n-k} \), so

\[
\frac{\det df^n(z)|_{E(z)}}{\det df^n(y)|_{E(y)}} = \prod_{k=0}^{n-1} \frac{\det df^k(z)|_{df^k E(z)}}{\det df^k(y)|_{df^k E(x)}} \leq \prod_{k=0}^{n-1} \left( 1 + \omega(\lambda^{n-k} \epsilon) \right) \leq C_2 = C_2(\epsilon).
\]
This proves the first part of the lemma. To prove the second part we use the
fact that the action of $Df_x$ on $G^u_x$ is contracting (lemma 4.1). Indeed, we
have for all $k \in \mathbb{N}$ :

$$\det \frac{d^k f(x)_{|E(x)}}{J^u f^k x} = \prod_{i=0}^{k-1} \frac{\det df(f^i x)_{|df^i E(x)}}{J^u f(f^i x)} \leq \prod_{i=0}^{+\infty} (1+\lambda^{2i} d(E^u_x,E_x)) \leq C_3 = C_3(\epsilon, E_x),$$

which finishes the proof of the second claim.

\[\square\]

**Lemma 5.3** (Bowen-Ruelle [BR75]). Let $\Lambda$ be an attractor of class $C^{1,\omega}$, where $\omega$ is a Dini summable modulus. Let $\epsilon > 0$, then there exist $C = C(\epsilon)$ such that for all $x \in \Lambda$ and $n \in \mathbb{N}$

$$C^{-1} \cdot \frac{1}{J^u f^n(x)} \leq \text{vol}^n \left( B_n(x, \epsilon) \right) \leq C \cdot \frac{1}{J^u f^n(x)}, \quad (23)$$

**Proof.** As in [BR75], consider for each $z \in \Lambda$ a local chart $\phi_z : T_z M(\epsilon) \to M$ such that $\phi_z(E^u_z) \subset W^u(z)$ and $\phi_z(E^s_z) \subset W^s(z)$ ($E^s_z$ is the open ball whose center is the origin of $E^s_z$ and radius $\epsilon$) and such that the maps $F_z = \phi_f^{-1} \circ f \circ \phi_z$ is tangent to $D_z f$ at the origin of $T_z M$.

For $x \in \Lambda$, and $n \in \mathbb{N}$ consider the set:

$$D_n(x, \epsilon) = \{ u \in T_x M : \| F^k u \|_{f^k x} \leq \epsilon, \text{ for } k = 0, \ldots, n-1 \},$$

then by definition of $D_n(x, \epsilon)$, there are $C_1, C_2 > 0$ independent of $n$ and such that

$$\phi_x(D_n(x, C_1 \epsilon)) \subset B_n(x, \epsilon) \subset \phi_x(D_n(x, C_2 \epsilon)).$$

So to estimate the volume of $B_n(x, \epsilon)$, it will be enough to estimate the volume of $D_n(x, \epsilon)$.

For $u \in T_x M$, let $(u_1, u_2)$ be the decomposition of $u$ with respect to the splitting $T_x M = E^u_x \oplus E^s_x$, then consider $v \in E^s_x$ and define the set

$$P_n(v, \epsilon) = \{ u \in T_x M = E^u_x \oplus E^s_x : u_2 = v, (F^k u)_1 \in E^u_x(f^k x) \text{ for } 0 \leq k \leq n-1 \}.$$

Let $K = K(\epsilon) > 0$ such that for all $k \geq 0$ we have $\| F^k v \| \leq (K+1)\epsilon$.

**Fact 5.4.** For $\epsilon > 0$ sufficiently small, there is $\gamma > 0$ such that for all $x \in \Lambda, n \in \mathbb{N}$ and $k \in \{0, \ldots, n-1\}$, the set $F^k(P_n(\epsilon, n))$ is a graph of a $C^1$ function $\psi_k : E^u_x(f^k x) \to E^s_x(f^k x)$ such that $\| D\psi_k \| \leq \gamma$.

Once we prove this fact, we deduce that:

$$D_n(x, \epsilon) \subset \bigcup_{v \in E^s_x(x)} P(v, n) \subset D_n(x, (K+3)\epsilon), \quad (24)$$

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so estimating the size of \( P_n(v, n) \) together with the distortion lemma (lemma 5.1) finishes the proof. Indeed, using (24) we get:

\[
\text{vol}(D_n(x, \epsilon)) \leq \text{vol}\left( \bigcup_{v \in E_n(x)} P_n(v, n) \right) \leq \text{vol}\left( D_n(x, (K + 3)\epsilon) \right),
\]

and by Fubini we get:

\[
\text{vol}\left( \bigcup_{v \in E_n(x)} P_n(v, n) \right) = \int_{E_n(x)} \text{vol}^q(P_n(v, \epsilon)) \, d\text{vol}^q(v),
\]

then the distortion lemma implies that:

\[
\frac{C^{-2}}{\text{J}_u f^n(x)} \leq \text{vol}^q(P_n(v, \epsilon)) \leq \frac{C^2}{\text{J}_u f^n(x)}.
\]

**Proof of Fact 1.** Fix \( k \in \{0, \ldots, n - 1\} \), and let \( W \) be a graph of a \( C^1 \) map \( \varphi : E_n(f^k x) \rightarrow E_n(K + 2)^n(f^k x) \) with a Lipschitz constant \( \gamma \leq 1 \).

It is clear that for \( \epsilon > 0 \) small enough, \( F_*(\text{Graph } \varphi) \) is a graph of \( C^1 \) function \( \tilde{\varphi} \).

If \( u = (u_1, u_2) \in D_{n-k}(f^k x, \epsilon) \), write \( F \) as:

\[
F(u_1, u_2) = (\tilde{F}u_1 + \alpha(u_1, u_2), \tilde{F}u_2 + \beta(u_1, u_2)),
\]

where \( \tilde{F} \) is \( DF \) at the origin of \( T_{f^n x} M \), and \( \|\alpha\|_{C^1}, \|\beta\|_{C^1} < \delta(\epsilon) \), and \( \delta(\epsilon) \to 0 \) when \( \epsilon \to 0 \).

Let \( (u_1', w_1'), (u_2', w_2') \in \text{Graph } (\tilde{\varphi}) \), then there is a point \( (u_i, w_i) \in \text{Graph } (\varphi) \) such that \( F(u_i, w_i) = (u_i', w_i') \). So we deduce the following:

\[
\|w_2' - w_1'\| = \|\tilde{F}(w_2 - w_1) + \beta(u_2, w_2) - \beta(u_1, w_1)\|
\leq \lambda\|w_2 - w_1\| + \delta(\gamma + 1)\|u_2 - u_1\|
\leq \lambda\gamma\|w_2 - w_1\| + \delta(\gamma + 1)\|u_2 - u_1\|
= (\lambda\gamma + \delta(\gamma + 1))\|u_2 - u_1\|,
\]

we have also:

\[
\|u_2' - u_1'\| = \|\tilde{F}(u_2 - u_1) + \alpha(u_2, w_2) - \alpha(u_1, w_1)\|
\geq \frac{1}{\lambda}\|u_2 - u_1\| - \delta(\gamma + 1)\|u_2 - u_1\|
= (\frac{1}{\lambda} - \delta(\gamma + 1))\|u_2 - u_1\|,
\]

so by choosing \( \epsilon \) small enough we can take any \( \gamma \leq 1 \), which finish the proof of the fact. 

\( \square \)
The following lemma is a variation of the previous lemma. It provides a lower bound of the volume of a dynamical ball centered near the hyperbolic attractor, this is crucial to find a link between being an attractor and having zero pressure with respect to the geometric potential. [BR75]

**Lemma 5.5** (Bowen-Ruelle [BR75]). For all small $\epsilon, \delta > 0$ there is $d = d(\epsilon, \delta) > 0$ such that for all $n \in \mathbb{N}, x \in \Lambda$ and $y \in B_n(x, \epsilon)$ we have:

$$\text{vol}^m (B_n(y, \delta)) \geq d \cdot \text{vol}^m (B_n(x, \epsilon)).$$

**Proof.** If $y \in \Lambda$, then the inequality of the lemma is obvious by the previous lemma. Assume that $y \notin \Lambda$. Since $W^s_\epsilon(\Lambda)$ is a neighborhood of $\Lambda$ (because $\Lambda$ is an attractor) there is $z \in \Lambda$ such that $y \in W^s_\epsilon(z)$, and since $y \in B_n(x, \epsilon)$ we have $z \in B_n(x, 2\epsilon)$. Let $A = [x, z] = W^u_\epsilon(x) \cap W^s_\epsilon(z)$. By the shadowing lemma and the expansiveness of $f$ in $\Lambda$ the point $A$ belongs to $\Lambda$.

By construction, $A \in W^u_\epsilon(x) \cap B_n(x, 3\epsilon)$, so by the previous lemma the volume of $B_n(x, \epsilon)$ and $B_n(A, \delta)$ are proportional independently of $n$, so in order to prove this lemma, it will be enough to compare the volume of $B_n(y, \delta)$ and the volume of $B_n(A, \epsilon)$.

By this remark, we may assume in that $y \in W^s_\epsilon(x)$. Using the same argument as in the previous lemma, we prove similarly that $\bigcup_{v \in W^u_\epsilon(y)} P_n(y, \delta) \subset D_n(y, \delta)$, then using distortion lemma, we get the desired inequality. \qed

The following lemma is proven in Chapter 4 Lemma 4.9 of [Bow75a].

**Lemma 5.6.** Let $\Lambda$ be a hyperbolic set of a $C^1$ diffeomorphism. If $W^u_\epsilon(x) \subset \Lambda$ for some $x$, then $\Lambda$ is an attractor. If $\Lambda$ is not an attractor, then there exists $\gamma > 0$ such that for every $x \in \Lambda$, there is $y \in W^u_\epsilon(x)$ with $d(y, \Lambda) \geq \gamma$.

The proof of the following proposition and corollary 2 is the same as Theorems 4.11 and 4.12 of [Bow75a]. For convenience, we sketch the proofs.

**Proposition 5.7.** Let $f : \Lambda \rightarrow \Lambda$ be a transitive hyperbolic map of class $C^{1, \omega}$, where $\omega$ is Dini summable modulus, then the following are equivalent:

(i) $\Lambda$ is an attractor.

(ii) $\text{vol}^m (W^s(\Lambda)) > 0$.

(iii) $P_{f|\Lambda} (\phi^{(w)}) = 0$.  

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Proof. \((i) \Rightarrow (ii)\) This implication is in fact true if \(f\) is only \(C^1\). Indeed we have \(W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x)\), which implies that \(W^s(\Lambda)\) is a neighborhood of \(\Lambda\).

\((ii) \Rightarrow (iii)\) Define \(s(\epsilon, n)\) by:

\[
s(\epsilon, n) = \sup_{S \in S_{\epsilon, n}} \sum_{x \in S} e^{S_n \phi(u)(x)} = \frac{1}{\sum_{x \in S} J^u f(x)},
\]

where \(S_{\epsilon, n}\) is the set of \((\epsilon, n)\)–separated sets of \(\Lambda\). Using lemma \([5,3]\) we have for all \(S \in S_{\epsilon, n}\):

\[
s(\epsilon, n) \geq C^{-1} \sum_{x \in S} \text{vol}^m(B_n(x, \epsilon))
\]

\[
\geq C^{-1} \text{vol}^m\left(\bigcup_{x \in S} B_n(x, \epsilon)\right)
\]

\[
\geq C^{-1} \text{vol}^m\left(W^s_{\epsilon/2}(\Lambda)\right),
\]

which implies that:

\[
P_{f|\Lambda}(\phi^{(n)}) = \lim_{\epsilon \to 0} \lim_{n \to +\infty} \frac{1}{n} \log s(\epsilon, n) \geq 0.
\]

Similarly we have:

\[
s(\epsilon, n) \leq C \sum_{x \in S} \text{vol}^m\left(B_n(x, \epsilon)\right)
\]

\[
\leq C C_{\epsilon/2} \sum_{x \in S} \text{vol}^m\left(B_n(x, \epsilon/2)\right)
\]

\[
\leq C C_{\epsilon/2} \text{vol}^m\left(\bigcup_{x \in S} B_n(x, \epsilon/2)\right)
\]

\[
\leq C C_{\epsilon/2} \text{vol}^m\left(W^s_{\epsilon}(\Lambda)\right)
\]

where \(S\) is \((\epsilon, n)\)–separated set, and the third inequality follows from the fact that \(B_n(x, \epsilon/2)\) where \(x\) varies in \(S\) are disjoint. Hence we deduce that

\[
P_{f|\Lambda}(\phi^{(n)}) \leq 0.
\]

\((iii) \Rightarrow (i)\) Assume that \(\Lambda\) is not an attractor. We will prove that the pressure is negative. Let \(\epsilon > 0\) small, and choose \(\gamma > 0\) as in lemma \([5,9]\). Let \(N \in \mathbb{N}\) such that

\[
W^u_{\epsilon}(f^N x) \subset f^N(W^u_{\gamma/4}), \forall x \in \Lambda.
\]

Let \(S \subset \Lambda\) be \((\gamma, n)\)–separated. Using lemma \([5,8]\) there is a point \(y(x, n) \in B_n(x, \gamma/4)\) such that

\[
d(f^{n+N}y(x, n), \Lambda) > \gamma.
\]
Choose $\delta \in (0, \gamma/4)$ so that $d(f^N z, f^N y) < \gamma/2$ whenever $d(z, y) < \delta$. Then $B_n(y(x, n), \delta) \subset B_n(x, \gamma/2)$, and $f^{n+N} B_n(y(x, n), \delta) \cap B(\Lambda, \gamma/2) = \emptyset$.

So we deduce that $B_n(y(x, n), \delta) \cap B_{n+N}(\mathcal{S}, \gamma/2) = \emptyset$. Using lemma 5.5 we get

$$\text{vol}^m(B_n(\mathcal{S}, \gamma/2)) - \text{vol}^m(B_{n+N}(\mathcal{S}, \gamma/2)) \geq \sum_{x \in \mathcal{S}} \text{vol}^m(B_n(y(x, n), \delta))$$

$$\geq d(3\gamma/2, \delta) \sum_{x \in \mathcal{S}} \text{vol}^m(B_n(x, 3\gamma/2))$$

$$\geq d(3\gamma/2, \delta) \text{vol}^m(B_n(\mathcal{S}, \gamma/2)),$$

so we get for all $n > N$:

$$\text{vol}^m(B_{n+N}(\mathcal{S}, \gamma/2)) \leq (1 - d) \text{vol}^m(B_n(\mathcal{S}, \gamma/2)),$$

finally, using the upper bound of $s(n, \epsilon)$, we deduce that:

$$P_{f|\Lambda}(\phi^x) \leq \frac{1}{N} \log(1 - d) < 0.$$

The following lemma is proven in [Bow75a].

**Lemma 5.8.** Let $\phi : \Lambda \to \mathbb{R}$ be a $C^{0,\omega}$ potential, where $\omega$ is Dini summable, and $P = P_{f|\Lambda}(\phi)$ the pressure of $f$ restricted to $\Lambda$. Then for small $\epsilon > 0$ there is $b_\epsilon > 0$ such that for any $x \in \Lambda$ and $n \in \mathbb{N}$ we have:

$$\mu_\phi(B_n(x, \epsilon)) \geq b_\epsilon \exp(-Pn + S_n\phi(x)). \quad (26)$$

**Proof of theorem** Let $g : U \to \mathbb{R}$ be a continuous function. Put $\bar{g}(n, x) = \frac{1}{n} \sum_{k=0}^{n-1} g(f^k x)$ and $\bar{g} = \int g \, d\mu_\phi(\cdot)$. Fix a small $\delta > 0$, and consider the sets

$$C_n(g, \delta) = \{ x \in M \mid |\bar{g}(n, x) - \bar{g}| > \delta \}, \quad B(g, \delta) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} C_n(g, \delta).$$

We want to prove that for all $\delta > 0$ the volume of $B(g, \delta)$ is zero, which proves the physicality. Take $\epsilon > 0$, such that we have $d(gx, gy) < \delta$ whenever $d(x, y) < \epsilon$, then fix $N$ and choose $\mathcal{R}_N, \mathcal{R}_{N+1}, \cdots$ as follow: $\mathcal{R}_n$ is a maximal subset of $\Lambda \cap C_n(g, 2\delta)$, satisfying:

- $B_n(x, \epsilon) \cap B_k(y, \epsilon) = \emptyset$ for $x \in \mathcal{R}_n, y \in \mathcal{R}_k$, and $N \leq k < n$, 


• $B_n(x, \epsilon) \cap B_n(x', \epsilon) = \emptyset$ for $x, x' \in \mathcal{R}_n$ and $x \neq x'$.

Let $V_N = \bigcup_{k=N}^{\infty} \bigcup_{x \in \mathcal{R}_k} B_k(x, \epsilon)$, which is a disjoint union by definition of $(\mathcal{R}_n)_n$. We have $B_k(x, \epsilon) \subset C_k(g, \delta)$ so $V_N \subset \bigcup_{k=N}^{\infty} C_k(g, \delta)$.

Since $\mu_{\phi(u)}$ is ergodic, we have

$$0 = \mu_{\phi(u)}(B(g, \delta)) = \lim_{n \to \infty} \mu_{\phi(u)}\left( \bigcup_{n=N}^{\infty} C_n(g, \delta) \right),$$

which implies that

$$\lim_{N \to \infty} \mu_{\phi(u)}(V_N) = 0. \quad (27)$$

So using the fact that $P_{f|\Lambda}(\phi(u)) = 0$ and lemma 5.8 we get

$$\mu_{\phi(u)}(V_N) \geq b_{\epsilon} \sum_{k=N}^{\infty} \sum_{x \in \mathcal{R}_k} \exp(S_k \phi(u)(x)). \quad (28)$$

Now for $x \in \Lambda$ and $y \in W^s_\epsilon(x) \cap C_n(g, 3\delta)$ we have $x \in C_n(g, 2\delta)$, so in particular:

$$W^s_\epsilon(\Lambda) \cap \bigcup_{k=N}^{\infty} C_k(g, 3\delta) \subset \bigcup_{k=N}^{\infty} \bigcup_{x \in \mathcal{R}_k} B_k(x, 2\epsilon),$$

so using lemma 5.3 we deduce that:

$$vol^m(W^s_\epsilon(\Lambda) \cap \bigcup_{k=N}^{\infty} C_n(g, 3\delta)) \leq C_{2\epsilon} \sum_{k=N}^{\infty} \sum_{x \in \mathcal{R}_k} \exp(S_k \phi(u)(x)). \quad (29)$$

Finally, using (27), (28) and (29) we deduce that $vol(B(g, \delta) \cap W^s_\epsilon) = 0$ which ends the proof.

\[ \square \]

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