EIGENVALUES AND EIGENFUNCTIONS FOR A ONE-DIMENSIONAL GEL’FAND PROBLEM

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Abstract. It is known that every positive solution of a one-dimensional Gel’fand problem can be written explicitly. In this paper we obtain exact expressions of all the eigenvalues and eigenfunctions of the linearized eigenvalue problem at each solution. We also study asymptotic behaviors of eigenvalues and eigenfunctions as the $L^\infty$-norm of the solution goes to the infinity.

1. Introduction and main results

We consider the one-dimensional Gel’fand problem

\begin{equation}
\begin{cases}
  u'' + \lambda e^u = 0, & -1 < x < 1, \\
  u(-1) = u(1) = 0,
\end{cases}
\end{equation}

where $\lambda > 0$ is a parameter. A history of (1.1) including multi-dimensional cases can be found in \cite[Section 1]{2}. The solution set of (1.1) is a curve $C := \{(\lambda, u)\}$ which can be parametrized by $\|u\|_{L^\infty([-1,1])}$. Each solution on $C$ has the exact expression

\begin{equation}
(\lambda, u) = \left(2e^{-\alpha} \left(\text{arcosh} \left(e^{\alpha/2}\right)\right)^2, \alpha - 2 \log \cosh \left(\sqrt{\frac{\lambda}{2}} e^{\alpha/2} x\right)\right),
\end{equation}

where $\alpha = \|u\|_{L^\infty([-1,1])}$ and $w = \text{arcosh}(z)$ denotes the inverse function of $z = \cosh(w) := (e^w + e^{-w})/2$.

Let $\tau(\alpha) := \text{arcosh} \left(e^{\alpha/2}\right)$. Then $\tau$ is a homeomorphism from $[0, \infty)$ to $[0, \infty)$, $\tau(0) = 0$, and $\tau(\infty) = \infty$. It is convenient to use $\tau$ as an independent variable instead of $\alpha$. The above solution $(\lambda, u)$ can be written as follows:

\begin{equation}
(\lambda(\tau), u(x, \tau)) = \left(\frac{2\tau^2}{\cosh^2 \tau}, 2 \log \frac{\cosh \tau}{\cosh(\tau x)}\right).
\end{equation}

Then, $C = \{(\lambda(\tau), u(x, \tau)) | \tau > 0\}$. It is well known that the solution curve $C$ starts from $(\lambda, u) = (0, 0)$, bends back once, and blows up at $\lambda = 0$. Since

\begin{equation}
\lambda'(\tau) = \frac{4\tau}{\cosh^2 \tau} (1 - \tau \tanh \tau),
\end{equation}

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we see the following: \( \lambda'(\tau) > 0 \) for \( 0 < \tau < \tau_1 \), \( \lambda'(\tau) = 0 \), \( \lambda'(\tau) < 0 \) for \( \tau > \tau_1 \), and \( \lim_{\tau \to \infty} \lambda(\tau) = 0 \). Here \( \tau_1 \) is the (unique) solution of the equation

\[
\tau \tanh \tau = 1 \quad \text{and} \quad \tau > 0.
\]

Eigenvalues and eigenfunctions are important in both qualitative and quantitative studies of a steady state of the associated parabolic problem. In this paper we study the linearized eigenvalue problem at the solution \((\lambda(\tau), u(x, \tau))\)

\[
\begin{cases}
\varphi'' + \lambda e^\mu \varphi = -\mu \varphi, & -1 < x < 1, \\
\varphi(-1) = \varphi(1) = 0.
\end{cases}
\]

The first main result is about the exact expression of the eigenvalue.

**Theorem 1.1.** Let \( \{\mu_j\}_{j=1}^\infty \), \( \mu_1 < \mu_2 < \cdots \), be the eigenvalues of (1.6). Let \( \tau_1 \) be the solution of (1.5). Then the following hold:

(i) If \( 0 < \tau < \tau_1 \), then \( \mu_j > 0 \) for \( j \geq 1 \) and \( \mu_j, j \geq 1 \), is the (unique) solution of the problem

\[
\tan \left( \sqrt{\mu_j} - \frac{\pi}{2} (j - 1) \right) = \frac{\sqrt{\mu_j}}{\tau \tanh \tau} \quad \text{and} \quad \frac{\pi}{2} (j - 1) < \sqrt{\mu_j} < \frac{\pi}{2} j.
\]

(ii) If \( \tau = \tau_1 \), then \( \mu_1 = 0 \), \( \mu_j > 0 \) for \( j \geq 2 \), and \( \mu_j, j \geq 2 \), satisfies (1.7).

(iii) If \( \tau > \tau_1 \), then \( \mu_1 < 0 \), \( \mu_1 \) is the (unique) negative solution of

\[
\tanh \sqrt{-\mu_1} = \frac{\sqrt{-\mu_1}}{\tau \tanh \tau},
\]

\( \mu_j > 0 \) for \( j \geq 2 \), and \( \mu_j, j \geq 2 \), satisfies (1.7).

It is known that the portion of the curve \( \{(\lambda(\tau), u(x, \tau)) : 0 < \tau < \tau_1\} \subset \mathcal{C} \) consists only of minimal solutions. Hence, each solution is stable. Indeed, Theorem 1.1 (i) says that \( \mu_j > 0 \) for \( j \geq 1 \).

The second main result is about the exact expression of the eigenfunction.

**Theorem 1.2.** Let \( \{\mu_j\}_{j=1}^\infty \) be the eigenvalues given in Theorem 1.1, and let \( \varphi_j(x) \) be the eigenfunction corresponding to \( \mu_j \). Let \( \tau_1 \) be the solution of (1.5). Then the following hold:

(i) If \( 0 < \tau < \tau_1 \), then, for \( j \geq 1 \),

\[
\varphi_j(x) = \sqrt{\frac{\mu_j}{\tau^2} + \tanh^2(\tau x)} \sin \left( \sqrt{\mu_j} x + \arctan \left( \frac{\tau \tanh(\tau x)}{\sqrt{\mu_j}} \right) \right) + \frac{\pi}{2} j).
\]

(ii) If \( \tau = \tau_1 \), then

\[
\varphi_1(x) = \tanh \tau_1 - x \tanh(\tau_1 x),
\]

and \( \varphi_j(x), j \geq 2 \), is given by (1.9).

(iii) If \( \tau > \tau_1 \), then

\[
\varphi_1(x) = \sqrt{-\mu_1} \cosh(\sqrt{-\mu_1} x) - \tau \sinh(\sqrt{-\mu_1} x) \tanh(\tau x),
\]

and \( \varphi_j(x), j \geq 2 \), is given by (1.9).

Using Theorems 1.1 and 1.2 we obtain the asymptotic behavior.
Corollary 1.3. Let \( \{\mu_j\}_{j=1}^{\infty} \) be the eigenvalues given in Theorem 1.1. Then the following hold:

(i) For \( j \geq 1 \),

\[
(1.12) \quad \lim_{\tau \downarrow 0} \sqrt{\mu_j} = \frac{\pi}{2} j.
\]

(ii)

\[
(1.13) \quad \lim_{\tau \to \infty} \mu_1 = -\infty, \quad \text{and, for } j \geq 2, \quad \lim_{\tau \to \infty} \sqrt{\mu_j} = \frac{\pi}{2} (j - 1).
\]

Corollary 1.4. Let \( \varphi_j(x) \) be the eigenfunction given in Theorem 1.2.

(i) As \( \tau \to \infty \),

\[
\frac{1}{\tau} \varphi_1 \left( \frac{y}{\tau} \right) \to \frac{1}{\cosh y} \quad \text{in } C_{\text{loc}}(\mathbb{R}).
\]

(ii) For \( j \geq 2 \), as \( \tau \to \infty \),

\[
\varphi_j(x) \to \bar{\varphi}_j(x) \quad \text{in } C_{\text{loc}}([-1, 0) \cup (0, 1]),
\]

where

\[
\bar{\varphi}_j(x) := \begin{cases} 
\sin \left( \frac{\pi}{2} (j - 1) x + \frac{\pi}{2} (j + 1) \right) & \text{if } 0 < x \leq 1, \\
\sin \left( \frac{\pi}{2} (j - 1) x + \frac{\pi}{2} (j - 1) \right) & \text{if } -1 \leq x < 0.
\end{cases}
\]

It follows from Corollary 1.4 (i) that the first eigenfunction \( \varphi_1(x) \) converges to \( \pi \delta_0(x) \) in a weak sense, where \( \delta_0(x) \) is the delta measure.

Remark 1.5. Let \( \Omega \) be a two-dimensional bounded domain with smooth boundary. Let \( \{(\lambda_i, u_i)\}_{i=1}^{\infty} \) be a sequence of solutions of the problem

\[
\begin{align*}
\Delta u + \lambda e^u &= 0, & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

Nagasaki-Suzuki \[3\] showed that as \( \lambda_i \downarrow 0 \), the sequence \( \{\int_{\Omega} \lambda_i e^{u_i} dx\}_{i=1}^{\infty} \) accumulates \( 0, 8\pi m \) \((m \in \{1, 2, 3, \ldots\})\), or \( \infty \). In one-dimensional case by direct calculation we have

\[
\int_{-1}^{1} \lambda(\tau)e^{u(\tau)} dx = 2\tau \cos(2 \arctan e^{-\tau} - \cos(2 \arctan e^\tau)) \to \infty \quad \text{as } \tau \to \infty, \quad \text{and}
\]

\[
\int_{-1}^{1} \lambda(\tau)e^{u(\tau)} dx \to 0 \quad \text{as } \tau \to 0.
\]

Therefore, \( \left\{ \int_{-1}^{1} \lambda(\tau_i)e^{u(\tau_i)} dx \right\}_{i=1}^{\infty} \) does not accumulate a finite positive number as \( \lambda(\tau_i) \downarrow 0 \).

Remark 1.6. By (1.3) we see that \( \sqrt{\lambda(\tau)}e^{u(\tau)} = 2\tau / \cosh(\tau x) \). Since

\[
\int_{-1}^{1} \sqrt{\lambda(\tau)}e^{u(\tau)} dx = 2\sqrt{2} \arctan e^{\tau} - \arctan e^{-\tau} \to \sqrt{2} \pi \quad \text{as } \tau \to \infty,
\]

we can easily see that \( \sqrt{\lambda(\tau)}e^{u(\tau)} \to \sqrt{2} \pi \delta_0(x) \) as \( \tau \to \infty \), i.e.,

\[
\int_{-1}^{1} \sqrt{\lambda(\tau)}e^{u(\tau)}\psi(x) dx \to \sqrt{2} \pi \psi(0) \quad \text{for all } \psi \in C([-1, 1]) \quad \text{as } \tau \to \infty.
\]
Let us mention technical details. Wakasa-Yotsutani [4] constructed a theory, which is explained in Section 2 of the present paper, in order to obtain exact eigenvalues and eigenfunctions for a one-dimensional elliptic problem with general nonlinearity. They derived a key ODE (2.6) below, and show that if the ODE has an exact solution, then all the eigenvalues and eigenfunctions can be written explicitly. Then they applied the theory to a Neumann problem in the case \( f(u) = \sin u \) (resp. \( f(u) = u - u^3 \)) and obtained all exact eigenvalues in [4] (resp. [6]) and all exact eigenfunctions in [5] (resp. [7]). In this paper we obtain an exact solution (2.12) of the key ODE (2.6) and apply the theory to a Dirichlet problem in the case \( f(u) = e^u \). In this paper we do not use the Jacobi elliptic functions, while they appear in the case \( f(u) = u - u^3 \) or \( f(u) = \sin u \).

This paper consists of five sections. In Section 2 we recall basic properties about eigenvalues and eigenfunctions. We also recall the theory of [4] about exact eigenvalues and eigenfunctions. In Section 3 we obtain an exact expression of the eigenfunction when the associated eigenvalue is positive. We see that all the eigenvalues except the first one are positive. Thus, only the first eigenvalue may be non-positive. In Section 4 we study the case where the first eigenvalue is zero. In Section 5 we study the case where the first eigenvalue is negative. Moreover, we prove Theorems 1.1 and 1.2 and Corollaries 1.3 and 1.4 using results obtained in Sections 3, 4, and 5.

2. General nonlinearity

Let \( f \) be an arbitrary \( C^2 \)-function. We consider the one-dimensional elliptic Dirichlet problem with general nonlinearity

\[
\begin{aligned}
  &u'' + \lambda f(u) = 0, \quad -1 < x < 1, \\
  &u(-1) = u(1) = 0.
\end{aligned}
\]  

Let \((\lambda, u(x))\) be a solution of (2.1). The linearized eigenvalue problem at \((\lambda, u(x))\) becomes

\[
\begin{aligned}
  &\varphi'' + \lambda f'(u) \varphi = -\mu \varphi, \quad -1 < x < 1, \\
  &\varphi(-1) = \varphi(1) = 0.
\end{aligned}
\]

First, we recall basic properties about the eigenvalue and eigenfunction for the one-dimensional problem (2.2).

Proposition 2.1. Let \( \{\mu_j\}_{j=1}^\infty \) be the eigenvalues of (2.2), and let \( \varphi_j(x) \) be the eigenfunction corresponding to \( \mu_j \). Then the following hold:

(i) Each eigenvalue \( \mu_j \) is real and simple, and \( \{\mu_j\}_{j=1}^\infty \) satisfies \( \mu_1 < \mu_2 < \cdots \).

(ii) For \( j \geq 1 \), the zero number \#\{\( x \in (-1, 1) | \varphi_j(x) = 0 \)\} is \( j - 1 \).

(iii) Assume that the solution \( u(x) \) of (2.1) is even. If \( j(\geq 1) \) is odd, then \( \varphi_j(x) \) is an even function. If \( j(\geq 2) \) is even, then \( \varphi_j(x) \) is an odd function.

We omit the proof of Proposition 2.1. See [1, Theorem 2.1 in p.212] for (i) and (ii). The assertion (iii) follows from an easy symmetric argument.

We consider the positive solution of (2.1). Then every solution is an even function, and hence the conclusions of Proposition 2.1 (iii) hold.

Next, we study exact expressions of the eigenvalue and eigenfunction. We look for the eigenfunction of the form \( \varphi(x) = \sqrt{\phi(x)} \). Substituting \( \sqrt{\phi(x)} \) into (2.2), we see that \( \phi(x) \) satisfies

\[
2\phi\phi'' - \phi'^2 + 4(\lambda f'(u) + \mu)\phi^2 = 0.
\]
Let \( \Phi(x) := 2\phi'' \phi - \phi'^2 + 4(\lambda f'(u) + \mu)\phi^2 \). Then,
\[
\Phi'(x) = 2\phi \{\phi'' + 4(\lambda f'(u) + \mu)\phi' + 2f''(u)u'\phi\}.
\]

We consider the equation
\[
(2.4) \quad \phi'' + 4(\lambda f'(u) + \mu)\phi' + 2f''(u)u'\phi = 0.
\]

If \( \phi(x) \) is a solution of (2.4), then \( \Phi'(x) = 0 \), and hence \( \Phi(x) \) is constant. In particular, \( \Phi(x) = \Phi(0) \), i.e.,
\[
(2.5) \quad 2\phi''(x)\phi(x) - \phi'(x)^2 + 4(\lambda f'(u(x)) + \mu)\phi(x)^2 \\
= 2\phi''(0)\phi(0) - \phi'(0)^2 + 4(\lambda f'(u(0)) + \mu)\phi(0)^2.
\]

When \( \Phi(0) = 0 \), then \( \sqrt{\phi(x)} \) is a candidate of the eigenfunction, since (2.3) holds. However, we show that a candidate of the eigenfunction can be constructed even if \( \Phi(0) \neq 0 \).

We study (2.4). We look for a solution of the form \( \phi(x) = h(u(x)) \). In other words we construct the eigenfunction, utilizing the shape of a general solution \( u(x) \). Substituting \( h(u(x)) \) into (2.4), we see that \( h(u) \) satisfies the following key equation:
\[
(2.6) \quad 2(F(\alpha) - F(u))h''' - 3f(u)h'' + \left(3f'(u) + 4\frac{\mu}{\lambda}\right)h' + 2f''(u)h = 0,
\]
where we use \( \alpha := u(0) \), \( u'(0) = 0 \), \( u'(x)^2 = 2\lambda(F(\alpha) - F(u(x))) \), and \( u'' = -\lambda f(u) \). It is worth noting that the independent variable of (2.6) becomes \( u \) and that the explicit \( x \)-dependence disappears. Thus, (2.6) does not depend on each solution \( u \) of (2.1). One of important results in [4] is finding the equation (2.6). Substituting \( h(u(x)) \) into (2.5), we have
\[
(2.7) \quad (F(\alpha) - F(u))(2h''h - h^2) - f(u)hh' + 2\left(f(u) + \frac{\mu}{\lambda}\right)h^2 = \rho,
\]
where
\[
(2.8) \quad \rho := -f(\alpha)h(\alpha)h'(\alpha) + 2\left(f'(\alpha) + \frac{\mu}{\lambda}\right)h(\alpha)^2.
\]

Hereafter, we construct an eigenfunction. We assume that a solution \( h(u) \) of (2.6) is obtained. Then we look for the eigenfunction of the following form:
\[
(2.9) \quad \varphi(x) = \sqrt{h(u(x))}W(\theta(x)).
\]

The functions \( W(\theta) \) and \( \theta(x) \) are defined later. By (2.7) we have
\[
(2.10) \quad \frac{d^2}{dx^2}\sqrt{h(u(x))} + \lambda f'(u)\sqrt{h(u(x))} + \mu \sqrt{h(u(x))} \\
= \frac{1}{4h\sqrt{h}} \left\{2\lambda(F(\alpha) - F(u))(2hh'' - h^2) - 2\lambda f(u)hh' + 4(\lambda f'(u) + \mu)h^2\right\} = \frac{2\lambda \rho}{4h\sqrt{h}}.
\]
Substituting (2.9) into (2.2), by (2.10) we have
\[
0 = \frac{d^2}{dx^2} \sqrt{h} W + 2 \frac{d}{dx} \sqrt{h} W' \theta' + \sqrt{h} (W'' \theta^2 + W' \theta'') + \lambda f'(u) \sqrt{h} W + \mu \sqrt{h} W
\]
\[
= \sqrt{h} \theta'^2 \left( W'' + \frac{d}{dx} \sqrt{h} \theta' W' + \frac{2}{\sqrt{h} \theta^2} W' \theta'' + \lambda f'(u) \sqrt{h} + \mu \sqrt{h} \right) W
\]

(2.11)
\[
= \sqrt{h} \theta'^2 \left( W'' + \frac{1}{h \theta'^2} d(h \theta') W' + \frac{\lambda \rho}{2h^2 \theta'^2} W' \right).
\]

If \( h(u(x))\theta'(x) \) is constant \( C \), then
\[
\theta(x) = \theta_0 + C \int_0^x \frac{dy}{h(u(y))} \quad \text{for some } \theta_0.
\]
Moreover, it follows from (2.11) that \( W \) satisfies a second order linear ODE with constant coefficients, and hence the solution \( W \) has an exact expression. If (2.6) has an exact solution \( h(u) \), then \( \theta \) can be written explicitly. Because of (2.9), the eigenfunction \( \varphi(x) \) has an exact expression. What we have to do is to find an exact solution \( h(u) \) of (2.6).

When \( f(u) = e^u \), (2.6) has the exact solution

(2.12)
\[
h(u) = \frac{2\mu}{\lambda} + e^\alpha - e^u.
\]

3. The case \( \mu > 0 \)

From this section we consider the case \( f(u) = e^u \).

Lemma 3.1. Let \( \{\mu_j\}_{j=1}^\infty, \mu_1 < \mu_2 < \cdots \), be the eigenvalues of (1.6), and let \( \varphi_j(x) \) be the eigenfunction corresponding to \( \mu_j \).

(i) For \( j \geq 2 \), \( \mu_j > 0 \) and \( \mu_j \) is the unique solution of the problem (1.7). Moreover, \( \varphi_j(x) \), \( j \geq 2 \), can be written as (1.9).

(ii) If \( 0 < \tau < \tau_1 \), then \( \mu_1 \) is the unique solution of the problem (1.7), and \( \varphi_1(x) \) can be written as (1.9).

Proof. We consider the case \( \mu > 0 \). When \( f(u) = e^u \), one can obtain the exact solution (2.12) of (2.6). Then,

(3.1)
\[
h(u(x)) = \cosh^2(\tau) \left( \frac{\mu}{\tau^2} \frac{1}{2} + \tan(\tau x) \right).
\]

Hence, \( h(u(x)) \) > 0 for \(-1 \leq x \leq 1\), since \( \mu > 0 \).

Now we determine \( \theta \) such that

(3.2)
\[
\frac{dh(u(x))\theta'(x)}{dx} = 0 \quad \text{and} \quad \theta'(0) = \sqrt{\frac{\lambda \rho}{2h(u(0))}}.
\]

Then,
\[
\theta(x) = \sqrt{\frac{\lambda \rho}{2}} \int_0^x \frac{dy}{h(u(y))}
\]
is one solution of (3.2). Because of (2.11), \( W \) satisfies \( W'' + W = 0 \). Since \( W(\theta) = \cos(\theta + \theta_1) \), we have

(3.3)
\[
\varphi(x) = \sqrt{h(u(x))} \cos \left( \sqrt{\frac{\lambda \rho}{2}} \int_0^x \frac{dy}{h(u(y))} + \theta_1 \right).
\]
By (2.8) we have

$$\rho = \cosh^6(\tau) \frac{\mu}{\tau^2} \left( \frac{\mu}{\tau^2} + 1 \right)^2. \quad (3.4)$$

Since $\mu > 0$, we see that $\rho > 0$. By (3.4) and (1.3) we have

$$\sqrt{\lambda \rho} = \cosh^2(\tau) \sqrt{\mu} \left( \frac{\mu}{\tau^2} + 1 \right).$$

By (3.3) we see that $\varphi(x)$ is well-defined in $[-1, 1]$. Using

$$\int_0^x dy \frac{a^2 + \tanh^2(\tau y)}{a^2 + \tanh^2(\tau y)} = \frac{1}{\tau(a^3 + a)} \left( a \tau x + \arctan \left( \frac{\tanh(\tau x)}{a} \right) \right),$$

we have

$$\sqrt{\lambda \rho} \int_0^x dy h(u(y)) = \sqrt{\mu} + \arctan \left( \frac{\tau \tanh(\tau x)}{\sqrt{\mu}} \right). \quad (3.5)$$

Substituting (3.1) and (3.5) into (3.3), we have

$$\varphi(x) = \cosh(\tau) \sqrt{\frac{\mu}{\tau^2}} + \tanh^2(\tau x) \cos \left( \sqrt{\mu} \tau x + \arctan \left( \frac{\tau \tanh(\tau x)}{\sqrt{\mu}} \right) \right),$$

respectively. First, we consider $\varphi_j(x)$ in the case where $j \geq 2$ is even, i.e., $j = 2k$. Then, it follows from Proposition 2.1 (iii) that the eigenfunction is odd, and hence, $\varphi^o(x)$ is a candidate. Since the Dirichlet boundary condition is satisfied, $\varphi^o(\pm 1) = 0$, i.e.,

$$\sqrt{\mu} + \arctan \left( \frac{\tau \tanh \tau}{\sqrt{\mu}} \right) = \pi n \text{ for some } n \geq 1.$$

When $\mu$ satisfies the above equation, by the definition of $\varphi^o(x)$ we see that $\mathcal{N}\{x \in (-1, 1) | \varphi^o(x) = 0\} = 2n - 1$. It follows from Proposition 2.1 (ii) that the zero number satisfies $2n - 1 = j - 1$, i.e., $n = j/2$. We have

$$\sqrt{\mu} + \arctan \left( \frac{\tau \tanh \tau}{\sqrt{\mu}} \right) = \frac{\pi}{2} j.$$

Therefore,

$$\tan \left( \frac{\pi}{2} j - \sqrt{\mu} \right) = \frac{\tau \tanh \tau}{\sqrt{\mu}} \quad \text{and} \quad \frac{\pi}{2} (j - 1) < \sqrt{\mu} < \frac{\pi}{2} j.$$

This is equivalent to (1.7). The equation (1.7) has a unique solution $\mu_j$, which is the $j$-th eigenvalue. Consequently, $\varphi_j(x) = \varphi^o(x)/\cosh(\tau)$ with $\mu = \mu_j$.

Second, we consider $\varphi_j(x)$ in the case where $j \geq 1$ is odd, i.e., $j = 2k - 1$. By the same argument we obtain the same equation (1.7) even if $j$ is odd. When $j \in \{3, 5, 7, \ldots\}$, we see that (1.7) has a unique solution $\mu_j$, which is the $j$-th eigenvalue. When $j = 1$, we see that (1.7) has a unique solution $\mu_1$ provided that $0 < \tau < \tau_1$. 
When \( j \in \{1, 2, 3, \ldots\} \), \( \varphi^o(x) \) and \( \varphi^e(x) \) can be summarized as (1.9) up to multiple constant. The proof is complete.

In Lemma 3.1 we have obtained \( \{(\mu_j, \varphi_j(x)); \ j \geq 2\} \) for all \( \tau > 0 \) and \( (\mu_1, \varphi_1(x)) \) for \( 0 < \tau < \tau_1 \). In Sections 4 and 5 we study \( (\mu_1, \varphi_1(x)) \) for \( \tau \geq \tau_1 \).

4. THE CASE \( \mu = 0 \)

We study \( (\mu_1, \varphi_1(x)) \) in the case \( \tau = \tau_1 \).

**Lemma 4.1.** Let \( \mu_1 \) be the first eigenvalue of (1.6), and let \( \varphi_1(x) \) be the first eigenfunction. If \( \tau = \tau_1 \), then \( \mu_1 = 0 \) and \( \varphi_1(x) \) is given by (1.10).

**Proof.** Let \( (\lambda(\tau), u(\tau, x)) \) be a solution of (1.1) given by (1.3). Differentiating \( u \) with respect to \( \tau \), we have

\[
 u_\tau(x) = 2(\tanh \tau - x \tanh(\tau x)).
\]

Note that \( u_\tau(\pm 1) = 0 \). On the other hand, differentiating \( u'' + \lambda e^u = 0 \) with respect to \( \tau \), we have

\[
 u'' + \lambda e^uu_\tau = -\lambda'e^u,
\]

where \( \lambda'(\tau) \) is given by (1.4). Then, \( \tau_1 \) is the unique positive solution of \( \lambda'(\tau) = 0 \), where \( \tau_1 \) is a solution of (1.5). When \( \tau = \tau_1 \), we see that \( u_\tau(x) \) satisfies (1.6) with \( \mu = 0 \). We easily see that \( u_\tau(x) > 0 \) for \(-1 < x < 1 \). Hence, 0 is the first eigenvalue and \( u_\tau \), which is (1.10), is the first eigenfunction. The proof is complete.

5. THE CASE \( \mu < 0 \)

We study \( (\mu_1, \varphi_1(x)) \) in the case \( \tau > \tau_1 \).

**Lemma 5.1.** Let \( \mu_1 \) be the first eigenvalue of (1.6), and let \( \varphi_1(x) \) be the first eigenfunction. If \( \tau > \tau_1 \), then \( \mu_1 < 0 \), \( \mu_1 \) is a unique negative solution of (1.8), and \( \varphi_1(x) \) is given by (1.11).

**Proof.** We consider the case \( \mu < 0 \). Let \( \bar{\mu} := -\mu(> 0) \) and \( \bar{\alpha} := \sqrt{\bar{\mu}/\tau^2}(> 0) \). We see that

\[
 h(u) := \frac{2\bar{\mu}}{\lambda} - e^{\bar{\alpha}} + e^u
\]

is an exact solution of (2.6). Note that the solution (5.1) is not the same as (2.12). We see that \( h(u(x)) = \cosh^2(\tau) (\bar{\alpha}^2 - \tanh^2(\tau x)) \) and \( h(u(0)) = 2\bar{\mu}/\lambda > 0 \). We define \( \varphi(x) := \sqrt{h(u(x))}W(\theta(x)) \). Note that \( W(\theta) \) and \( \theta(x) \) are not the same functions as in Section 3. Hereafter, we work on the interval \( I := (-\text{artanh}(\bar{\alpha})/\tau, \text{artanh}(\bar{\alpha})/\tau) \), because \( h(u(x)) > 0 \) for \( x \in I \). Here, \( w = \text{artanh}(z) \) denotes the inverse function of \( z = \tanh(w) \). Note that the interval \( I \) may not include \(-1, 1\). The function \( W \) satisfies (2.11), where

\[
 \rho = -\cosh^2(\tau)\bar{\alpha}^2(-\bar{\alpha}^2 + 1)^2.
\]

Then, \( \rho < 0 \) (resp. \( \rho = 0 \)) if \( \bar{\alpha} \neq 1 \) (resp. \( \bar{\alpha} = 1 \)). Let \( \bar{\rho} := -\rho(\geq 0) \). We define \( \theta(x) \) such that

\[
 \frac{dh(u(x))\theta'(x)}{dx} = 0 \quad \text{and} \quad \theta'(0) = -\sqrt{\frac{\lambda\bar{\rho}}{2h(u(0))}}.
\]

Then \( W(\theta) \) satisfies

\[
 W'' - W = 0.
\]
We see that
\[ \theta(x) = \sqrt{\frac{\lambda \rho}{2}} \int_0^x \frac{dy}{h(u(y))} \]
is one solution of (5.3). If \( \bar{a} = 1 \), then we see by (5.2) that \( \rho = 0 \). We see by (2.10) that \( \sqrt{h(u(x))} \) is a candidate of the first eigenfunction. However, \( \sqrt{h(u(x))} > 0 \), and hence the Dirichlet boundary condition is not satisfied. The case \( \bar{a} = 1 \) does not occur.

We consider the case \( \bar{a} \neq 1 \). Using
\[ \int_0^x \frac{dy}{\bar{a}^2 - \tanh^2(\tau y)} = \frac{1}{\tau(a^2 - \bar{a})} \left( \bar{a} \tau x - \frac{1}{2} \log \frac{\bar{a} + \tanh(\tau x)}{\bar{a} - \tanh(\tau x)} \right), \]
we have
\[ \theta(x) = \begin{cases} \bar{a} \tau x - \frac{1}{2} \log \frac{\bar{a} + \tanh(\tau x)}{\bar{a} - \tanh(\tau x)} & \text{if } \bar{a} > 1, \\ -\bar{a} \tau x + \frac{1}{2} \log \frac{\bar{a} + \tanh(\tau x)}{\bar{a} - \tanh(\tau x)} & \text{if } 0 < \bar{a} < 1. \end{cases} \]

Note that \( \theta(x) \) is well-defined in \( I \). On the other hand, \( W(\theta) = \cosh(\theta) \) is a solution of (5.4) which is even. Hence,
\[ W(\theta(x)) = \frac{1}{2} \left( e^{\sqrt{\bar{\mu}x}} \sqrt{\frac{\bar{a} - \tanh(\tau x)}{\bar{a} + \tanh(\tau x)}} + e^{-\sqrt{\bar{\mu}x}} \sqrt{\frac{\bar{a} + \tanh(\tau x)}{\bar{a} - \tanh(\tau x)}} \right), \]
provided that \( 0 < \bar{a} \neq 1 \). We have
\[ \varphi(x) = \sqrt{h(u(x))} W(\theta(x)) = \cosh(\tau) \left( \bar{a} \cosh(\sqrt{\bar{\mu}x}) - \sinh(\sqrt{\bar{\mu}x}) \tanh(\tau x) \right). \]

By direct calculation we see that \( \varphi(x) \) is defined for all \( x \in (-1, 1) \) and \( \varphi'' + \lambda e^u \varphi - \bar{\mu} \varphi = 0 \) for all \( x \in (-1, 1) \), although \( \sqrt{h(u(x))} \) and \( W(\theta(x)) \) are defined only in \( I \). Thus, \( \varphi(x) \) becomes an eigenfunction provided that \( \varphi(x) \) satisfies the Dirichlet boundary condition. We study the problem \( \varphi(\pm 1) = 0 \). Then, \( \sqrt{\bar{\mu}} \cosh(\sqrt{\bar{\mu}}) - \tau \sinh(\sqrt{\bar{\mu}}) \tanh(\tau) = 0 \). We have
\[ (5.5) \quad \tanh \sqrt{\bar{\mu}} = \frac{\sqrt{\bar{\mu}}}{\tau \tanh \tau}. \]

We easily see that if \( \tau > \tau_1 \), then \( \tau \tanh \tau > 1 \), and hence (5.5) has a unique positive solution \( \bar{\mu}_1 \). Moreover, we easily see that \( \varphi(x) > 0 \) for \( -1 < x < 1 \). Let \( \mu_1 := -\bar{\mu}_1 \). When \( \tau > \tau_1 \), \( \mu_1 \) is the first eigenvalue of (1.6). Thus, \( \mu_1 \) is a unique negative solution of (1.8), and the first eigenfunction is given by (1.11). The proof is complete.

**Proof of Theorems 1.1 and 1.2.** Theorems 1.1 (i) and 1.2 (i) follow from Lemma 3.1. Theorems 1.1 (ii) and 1.2 (ii) follow from Lemmas 3.1 (i) and 1.1. Theorems 1.1 (iii) and 1.2 (iii) follow from Lemmas 3.1 (i) and 3.1.

**Proof of Corollary 1.3.** By (1.3) we see that \( \lambda(\tau)e^{u(\tau)} \to 0 \) in \( C([-1, 1]) \) as \( \tau \downarrow 0 \). Hence, an eigenvalue of (1.6) converges to an eigenvalue of the problem
\[ \begin{cases} \varphi'' = -\mu \varphi, & -1 < x < 1, \\ \varphi(-1) = \varphi(1) = 0. \end{cases} \]
Thus, (1.12) holds, and the assertion (ii) holds. When \( j \geq 2 \), by (1.7) we see that
\[
\frac{\pi}{2}(j-1) \leq \sqrt{\mu_j} \leq \frac{\pi}{2}(j-1) + \arctan\left(\frac{\pi j}{2\tau \tanh \tau}\right).
\]
Since \( \arctan(\pi j/(2\tau \tanh \tau)) \to 0 \) \( (\tau \to \infty) \), (1.13) holds for \( j \geq 2 \). We consider the case \( j = 1 \). Let \( y(x) := (\tanh x)/x \), \( x > 0 \). Then we easily see that \( \lim_{x \downarrow 0} y(x) = 1 \), \( \lim_{x \to \infty} y(x) = 0 \), and \( y'(x) < 0 \) for \( x > 0 \). Thus, if \( \tau > \tau_1 \), then the solution of \( y(x) = 1/((\tau \tanh \tau)) \) is unique and it diverges as \( \tau \to \infty \). Thus, we see by (1.8) that (1.13) holds for \( j = 1 \).

**Proof of Corollary 1.4.** (i) It follows from Corollary 1.3 (ii) that \( \sqrt{-\mu_1} \to \infty \) as \( \tau \to \infty \). We see by (1.8) that
\[
\frac{\sqrt{-\mu_1}}{\tau} = \tanh(\sqrt{-\mu_1}) \tanh(\tau) \to 1 \text{ as } \tau \to \infty.
\]
As \( \tau \to \infty \),
\[
\frac{1}{\tau} \varphi_1 \left( \frac{y}{\tau} \right) = \frac{\sqrt{-\mu_1}}{\tau} \cosh \left( \frac{\sqrt{-\mu_1}}{\tau} y \right) - \sinh \left( \frac{\sqrt{-\mu_1}}{\tau} y \right) \tanh y
\]
\[
\to \cosh y - \sinh y \tanh y = \frac{1}{\cosh y} \text{ in } C_{\text{loc}}(\mathbb{R}).
\]
(ii) As \( \tau \to \infty \), \( \tanh(\tau x) \to \text{sign}(x) \) in \( C_{\text{loc}}([-1, 0) \cup (0, 1]) \), where
\[
\text{sign}(x) := \begin{cases} 
1 & x > 0, \\
0 & x = 0, \\
-1 & x < 0.
\end{cases}
\]
Thus, the conclusion follows from (1.9) and (1.13). \( \square \)

**References**

[1] E. Coddington and N. Levinson, Theory of ordinary differential equations, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955. xii+429 pp.

[2] J. Jacobsen and K. Schmitt, The Liouville-Bratu-Gelfand problem for radial operators, J. Differential Equations 184 (2002), 283–298.

[3] K. Nagasaki and T. Suzuki, Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities, Asymptotic Anal. 3 (1990), 173–188.

[4] T. Wakasa and S. Yotsutani, Representation formulas for some 1-dimensional linearized eigenvalue problems, Commun. Pure Appl. Anal. 7 (2008), 745–763.

[5] T. Wakasa and S. Yotsutani, Asymptotic profiles of eigenfunctions for some 1-dimensional linearized eigenvalue problems, Commun. Pure Appl. Anal. 9 (2010), 539–561.

[6] T. Wakasa and S. Yotsutani, Limiting classification on linearized eigenvalue problems for 1-dimensional Allen-Cahn equation I-asymptotic formulas of eigenvalues, J. Differential Equations 258 (2015), 3960–4006.

[7] T. Wakasa and S. Yotsutani, Limiting classification on linearized eigenvalue problems for 1-dimensional Allen-Cahn equation II-Asymptotic profiles of eigenfunctions, J. Differential Equations 261 (2016), 5465–5498.
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