Multiscalar production amplitudes beyond threshold

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Abstract

We present exact tree-order amplitudes for $H^* \rightarrow n H$, for final states containing one or two particles with non-zero three-momentum, for various interaction potentials. We show that there are potentials leading to tree amplitudes that satisfy unitarity, not only at threshold but also in the above kinematical configurations and probably beyond. As a by-product, we also calculate $2 \rightarrow n$ tree amplitudes at threshold and show that for the unbroken $\phi^4$ theory they vanish for $n > 4$, for the Standard Model Higgs they vanish for $n \geq 3$ and for a model potential, respecting tree-order unitarity, for $n$ even and $n > 4$. Finally, we calculate the imaginary part of the one-loop $1 \rightarrow n$ amplitude in both symmetric and spontaneously broken $\phi^4$ theory.

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1 Introduction

The problem of calculating amplitudes for high-multiplicity production of weakly interacting particles has recently received considerable attention. The tree-level contributions for $1 \to n$ processes at the threshold point (all produced particles at rest) have been calculated exactly for $\phi^4$ scalar theory in both the symmetric case and broken symmetry (Standard Model Higgs) case [1, 2]. In addition the one-loop contribution at threshold has also been calculated exactly in both cases [3, 4], using the method proposed in ref. [5].

These amplitudes, growing as $n!$ with the multiplicity $n$, lead to unitarity-violating cross sections. In a recent paper [6], we discussed a potential, which allowed the tree-order amplitudes to satisfy the unitarity bound, at least at the threshold point.

In this paper, we extend the calculation of tree amplitudes beyond the threshold point. We present exact results for the cases when one or two of the produced particles have non-zero momentum and we show that the tree amplitudes calculated using our unitarity-respecting potential (UR potential) continue to satisfy the unitarity bound. This strongly suggests that the UR potential leads to unitarity-respecting cross sections.

As a by-product of our calculation, inspired by the work of ref. [3], we also obtain the exact amplitude for two on-shell scalars to go to $n$ scalars at rest for various potentials, with the remarkable result that in symmetric $\phi^4$ theory all amplitudes are zero for $n > 4$; for the symmetry-broken case this happens for $n \geq 3$ and for the UR potential we get a nullification when $n$ is even and $n > 4$. We also show that this property holds for $\phi^m$ only when $m \leq 4$. Finally, using the result for the amplitude $1 \to n$ when two of the final momenta are non-zero, the above property and the well-known Cutkosky rules, we calculate the imaginary part of the one-loop amplitude at threshold.

The paper is organized as follows. In section 2, we discuss the general $\phi^m$ theory, the special case of unborken $\phi^4$ theory, and its spontaneously broken version, for the case where one of the final-state three-momenta is non-zero. In section 3, we calculate tree amplitudes for the UR potential, in the cases where one or two of the final-state momenta are non-zero, and we show that they respect the unitarity bound. We also derive some results for the $\phi^4$ theory. In section 4, we calculate the imaginary part of the one-loop amplitude for the $1 \to n$ process, in both symmetric and broken-symmetry $\phi^4$ theories. Finally, section 5 contains our conclusions.

2 Amplitudes for final states with one non-zero momentum

We consider the process $H^* \to nH$ with the following configuration of the final-state momenta:

\[ p_1^\mu = (E, \vec{p}), \quad p_i^\mu = (1, \vec{0}) \quad \text{for } n = 2, 3, \ldots, n, \]

(1)
where we put the mass of the Higgs equal to unity, so that $E^2 - \vec{p}^2 = 1$. Let us first consider a general potential

$$V(\phi) = \sum_{m \geq 3} \frac{\lambda_m}{(m)!} \phi^m .$$

(2)

The tree amplitude is given by the following recursion relation (see fig.1):

$$a(n, p_1) = a_1(n)$$

$$= -i \sum_{p=2}^{n} \frac{\lambda_{p+1}}{(p-1)!} \sum_{n_1, ..., n_p \geq 1} \frac{ia_1(n_1)}{P(n_1)} \frac{ia(n_2)}{(n_2 - 1)!} \cdots \frac{ia(n_p)}{(n_p - 1)! n_1! n_2! \cdots n_p!} n!$$

(3)

where $a(n)$ is the amplitude at threshold (all outgoing momenta zero), and $P_1(n)$ is the inverse propagator, given by

$$P_1(n) = (p_1 + (n - 1)p_2)^2 - 1 = (n - 1)(n + \omega) ;$$

(4)

here, we have introduced $\omega = 2E - 1$. The ansätze

$$a(n) = -in!(n^2 - 1)b(n) , \quad a_1(n) = -i(n-1)!P_1(n)b_1(n) ,$$

(5)

and the introduction of the generating functions

$$f(x) = \sum_{n \geq 1} b(n)x^n , \quad f_1(x) = \sum_{n \geq 1} b_1(n)x^n$$

(6)

transform Eq.(3) into the following differential equation for $f_1(x)$:

$$x^2 f_1''(x) + \omega x f_1'(x) - [\omega + V''(f(x))] f_1(x) = 0 .$$

(7)

Note that this equation for the $f_1$ is linear; the additional, nonlinear equation for the $f(x)$ has been discussed in [2, 4]. The boundary conditions for $f_1$ are, obviously, $f_1(0) = f(0) = 0, f_1'(0) = f'(0) = 1$. We shall now consider the solution of Eq.(7) for a number of different potentials.

2.1 Monomial $\phi^m$ interactions

When $V(f) = \lambda f^m/m!$, we know, from [2], the generating function when all outgoing particles are at rest:

$$f(x) = x \left( 1 - \frac{\lambda x^q}{2m!} \right)^{-2/q} ,$$

(8)

where $q = m - 2$ . Thus, in this case Eq.(7) becomes

$$x^2 f_1''(x) + \omega x f_1'(x) - \left[ \omega + 2m! \frac{y}{q! (1-y)^2} \right] f_1(x) = 0 .$$

(9)
with \( y = \lambda x^q/2m! \). We now introduce \( G(y) \) and \( r > 0 \) as follows:

\[
f_1(x) = x(1 - y)^r G(y) \quad , \quad r(r - 1) = \frac{2m!}{q^2 q!} ,
\]

(10)
to find

\[
y(1 - y)G''(y) + \left\{ \frac{q + 1 + \omega}{q} - \left[ 2r + \frac{1 + \omega}{q} + 1 \right] y \right\} G'(y) - r \left( r + \frac{1 + \omega}{q} \right) G(y) = 0 ,
\]

(11)
with boundary condition \( G(0) = 1 \). This is the hypergeometric equation \cite{1}. The resulting solution for \( f_1 \) can be written as

\[
f_1(x) = x \left( 1 - \frac{\lambda x^q}{2m!} \right)^{1-r} F \left( 1 - r, \frac{q(1-r) + \omega + 1}{q}; \frac{q + \omega + 1}{q}; \frac{\lambda x^q}{2m!} \right) ,
\]

\[
F(a, b; c; t) \equiv \sum_{n \geq 0} \frac{(a)_n (b)_n x^n}{(c)_n n!} ,
\]

(12)
where \((a)_n = (a + n - 1)!/(a - 1)!\) is the Pochhammer symbol. If it happens that \( r - 1 \) is an integer, the summation will end at \( n = r - 1 \) and the hypergeometric function \( F \) is a polynomial of degree \( r - 1 \). The denominator \(((\omega + 1)/q + 1)_n\) vanishes whenever \( \omega = -qk - 1 \) for \( k = 1, 2, \ldots, r - 1 \).

Having determined \( f_1(x) \) we can, in principle, get the \( b_1(n) \) by expanding in powers of \( x \), and this gives us the amplitudes \( a_1(n) \). This \( 1 \to n \) amplitude is equal, by crossing, to the \( 2 \to n - 1 \) amplitude, if we take for \( p_k \) an unphysical value with negative energy. Since \( n \) must, in a \( \phi^m \) theory, be of the form \( qk + 1 \), this corresponds to taking \( \omega = -qk - 1 \). Then, \( a_1(qk + 1) \) will contain the factor \( P_1(qk + 1) \) and will hence be zero - unless this zero is cancelled by a corresponding simple pole in the function \( f_1(x) \). It follows that the only \( a_1(qk + 1) \) that are non-zero (with the above choice for \( \omega \)) are those with \( k \leq r - 1 \), provided, of course, that \( r \) is indeed an integer. Since Eq.\,(10) implies \( r = 2(m - 1)/(m - 2) \), we see that this can only be the case if \( m = 3 \) \((r = 4)\) or \( m = 4 \) \((r = 3)\). We conclude that in a pure \( \phi^3 \) theory all threshold amplitudes \( 2 \to n \) with \( n > 3 \), and in a pure \( \phi^4 \) theory all such amplitudes with \( n > 4 \), will vanish. In theories with higher purely monomial interaction, no such ‘nullification’ takes place.

In the special case of a \( \phi^4 \) theory, we find the following explicit results:

\[
f_1(x) = \left( 1 - \frac{\lambda_4 x^2}{48} \right)^{-2} \left\{ x + \frac{2(3 - \omega)}{3 + \omega} \left( \frac{\lambda_4}{48} \right) x^3 + \frac{(3 - \omega)(1 - \omega)}{(3 + \omega)(5 + \omega)} \left( \frac{\lambda_4}{48} \right)^2 x^5 \right\} ,
\]

(13)
and

\[
a_1(3) = -i\lambda_4 ,
\]

\[
a_1(2k + 1) = -i(2k)!(2k)(2k + 1 + \omega) \left( \frac{\lambda_4}{48} \right)^k \times \left\{ k + 1 + \frac{2(3 - \omega)}{3 + \omega} k + \frac{(3 - \omega)(1 - \omega)}{3 + \omega}(k - 1) \right\} ,
\]

(14)
where $k \geq 2$ in the last line. It is easily checked that $a_1(n)$ reduces to $a(n)$ when $\omega \to 1$, as it should. Moreover, by letting $\omega$ approach $-2k-1$ we immediately find the following $2 \to n$ amplitudes:

$$
\begin{align*}
a(2 \to 2) &= -i\lambda_4, \\
a(2 \to 4) &= i\lambda_4^2, \\
a(2 \to n) &= 0 \text{ for all } n > 4. 
\end{align*}
$$

These are the threshold zeros noted in ref. [3]. Note that we have obtained, in addition to those results, the explicit form of the amplitudes.

### 2.2 The Minimal Standard Model

In the Minimal Standard Model, with spontaneous symmetry breaking, we have the potential

$$
V(\phi) = \sqrt{\frac{\lambda_4}{12}}\phi^3 + \frac{\lambda_4}{24}\phi^4, 
$$

and, from [2]:

$$
f(x) = x \left(1 - x \sqrt{\frac{\lambda_4}{12}}\right)^{-1},
$$

so that Eq.(7) becomes

$$
x^2 f''_1(x) + \omega x f'_1(x) - \left[\omega + \frac{6y}{1-y} + \frac{6y^2}{(1-y)^2}\right] f_1(x) = 0,
$$

with $y = x \sqrt{\lambda_4/12}$. We now put $f_1(x) = x(1 - y)^3 G(y)$, to find the equation

$$
y(1 - y)G''(y) + (2 + \omega - [(4 + \omega) + 4y]) G'(y) - 3(4 + \omega)G(y) = 0,
$$

which is again of the hypergeometric type, this time with $b = 3$. We find, for the generating function:

$$
f_1(x) = x \left(1 - x \sqrt{\frac{\lambda_4}{12}}\right)^{-2} \left\{1 + \sqrt{\frac{\lambda_4}{3}} \cdot \frac{1 - \omega}{2 + \omega} - \frac{\lambda_4}{12} \cdot \frac{\omega(1 - \omega)}{(2 + \omega)(3 + \omega)} x^2\right\},
$$

and for the tree amplitude:

$$
a_1(n) = -i(n-1)!(n-1)(n+\omega) \left(\frac{\lambda_4}{12}\right)^{(n-1)/2} \times \left\{n + 2(n-1) \frac{1 - \omega}{2 + \omega} - \frac{\omega(1 - \omega)}{(2 + \omega)(3 + \omega)}\right\}. 
$$

As a check of this result, observe that indeed

$$
\lim_{\omega \to 1} a_1(n) = -in!(n^2 - 1) \left(\frac{\lambda_4}{12}\right)^{(n-1)/2} = a(n),
$$
as it should. Letting again $\omega \to -n$, we get for the threshold amplitudes

$$
a(2 \to 1) = -i \sqrt{3\lambda_4} , \\
a(2 \to 2) = 4i\lambda_4 , \\
a(2 \to n) = 0 \text{ for all } n \geq 3 .
$$

These are the threshold zeros of ref. [4].

### 2.3 Unitarity-respecting model potential

In a previous publication [6], we studied a model potential that allows the tree-level amplitudes to satisfy unitarity at the threshold point. We now show that unitarity is also respected at phase-space points of the type of Eq.(1). The UR model has a potential $V(\phi)$, and a generating function $f(x)$, given by

$$
V(\phi) = \frac{1}{2} (1 + \phi)^2 (\log(1 + \phi))^2 , \\
f(x) = e^x - 1 ,
$$

and, consequently,

$$
V''(f) = x^2 + 3x .
$$

Inserting this into Eq.(7), and writing

$$
f_1(x) = xe^{-x}G(y) , \quad y = 2x ,
$$

we get the following equation for $G$:

$$
yG''(y) + (\omega + 2 - y)G'(y) + \frac{\omega + 5}{2}G(y) = 0 ,
$$

whose solution is the confluent hypergeometric function [7]. The final solution with boundary conditions $f_1(0) = 0$ and $f_1'(0) = 1$ is

$$
f_1(x) = xe^{-x}M \left( \frac{\omega + 5}{2}; \omega + 2; 2x \right) ,
$$

where $M$ is a Kummer function [7]:

$$
M(a, b; z) = \sum_{n \geq 0} \frac{(a)_n z^n}{(b)_n n!} .
$$

Since this series has an infinite radius of convergence, so does $f_1(x)$, and hence the coefficients $b_1(n)$ decrease sufficiently fast with $n$ to satisfy unitarity. In fact, we can read off the explicit form for $b_1(n)$ (and hence that for $a_1(n)$ immediately from Eq.(28): after some trivial algebra, we obtain

$$
a_1(n) = -i(n - 1)(n + \omega) \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} \prod_{j=2}^{k+1} \frac{\omega + 2j + 1}{\omega + j} .
$$
Note that we define the empty product for $k = 0$ to be unity. The check that $a_1(n) \to n^2 - 1$ as $\omega \to 1$ follows immediately, and we also find, by letting $\omega \to -p - 1$ with integer $p$, that $a_1(p + 1)$, and therefore the threshold amplitude for the $2 \to p$ process, will vanish whenever $p$ is even and larger than 4. So also in this case, there is nullification. Note, however, that the technical reason for this nullification is slightly different from that in the polynomial scalar theories: there, the fact that the hypergeometric function turns out to be a finite polynomial limits the number of different poles, whereas in this case some of the poles cancel against corresponding numerators, without the Kummer function having to reduce to a finite polynomial (although, in fact, it does).

Before finishing this section, we want to remark that we have also reproduced the threshold nullification described in ref. [8], for the annihilation of a fermion-antifermion pair, and of a vector boson pair, into scalars. In addition, we have found explicit forms for the amplitudes. Since these processes, however, involve more than just scalar particles, we defer their study to a forthcoming publication [9].

3 Final states with two non-zero momenta

3.1 Back-to-back momenta

We now turn to the case where two of the final-state momenta have nonvanishing space-like components. As a first step, we take the momentum configuration

$$\begin{align*}
p_1^\mu &= (E, \vec{p}) \quad \text{and} \quad p_2^\mu = (E, -\vec{p}) \quad \text{and} \quad p_i^\mu = (1, \vec{0}) \quad \text{for} \quad i = 3, 4, \ldots, n.
\end{align*}$$

For a generic potential of the form (2), the tree-level amplitude $a_2(n)$ now obeys (as depicted in fig.2) the following inhomogeneous recursion relation:

$$\begin{align*}
\sum_{p=2}^{n} \frac{\lambda_{p+1}}{(p-1)!} \sum_{n_1, \ldots, n_p \geq 1} \frac{ia(n_1) \cdots ia(n_p)}{P_2(n_1) \cdots P_2(n_p) \delta_{n_1 + \cdots + n_p, n}} \\
+ \sum_{p=2}^{n} \frac{\lambda_{p+1}}{(p-2)!} \sum_{n_1, \ldots, n_p \geq 1} \frac{ia(n_1) \cdots ia(n_p)}{P_1(n_1) \cdots P_1(n_p) \delta_{n_1 + \cdots + n_p, n}}
\end{align*}$$

where

$$P_2(n) = (p_1 + p_2 + (n - 2)p_3)^2 - 1 = (n + \rho - 1)(n + \rho + 1), \quad \rho = 2(E - 1).$$

Our ansatz for $a_2(n)$ is

$$a_2(n) = -i(n-2)!P_2(n)b_2(n);$$
with this, and the following definition of the generating function:

\[ f_2(x) = \sum_{n \geq 2} b_2(n)x^{n+\rho}, \]  

we find the analogous form of Eq.(35):

\[ x^2 f''_2(x) + x f'_2(x) - [1 + V''(f(x))] f_2(x) - x^\rho f_1(x)^2 V''(f(x)) = 0. \]  

In our model UR potential, we have

\[ V'''(f(x)) = e^{-x}(2x + 3), \]  

so, upon substituting the results of the previous section, we have the following inhomogeneous differential equation:

\[ x^2 f''_2(x) + x f'_2(x) - (1 + 3x + x^2)f_2(x) = F(x), \]  

with

\[ F(x) = x^{2+\rho}e^{-3x}(2x + 3)M(2 + E, 2E + 1, 2x^2). \]  

The boundary conditions that we have to take are

\[ \lim_{x \to 0} x^{-\rho} f_2(x) = \lim_{x \to 0} \frac{d}{dx}(x^{-\rho} f_2(x)) = 0. \]  

The corresponding solution is given by

\[ f_2(x) = -4xe^{-x} \int_0^x \{ e^{2t}U(3, 3; 2x) - e^{2x}U(3, 3; 2t) \} e^{-t} F(t), \]  

where \( U \) is the singular solution of the confluent hypergeometric equation [7]:

\[ U(3, 3; 2x) = \frac{1}{2} e^{2x}E_1(2x) - \frac{1}{4x} + \frac{1}{8x^2}, \]  

and \( E_1 \) is the exponential integral

\[ E_1(z) = \int_z^\infty dt \frac{e^{-t}}{t} = -\gamma - \log(2x) + \sum_{n \geq 1} \frac{(-2x)^n}{n!n}. \]  

As discussed in [8], if \( f_2(x) \), when expanded in powers of \( x \), has an infinite radius of convergence, the amplitude \( a_2(n) \) will not grow factorially with \( n \), and hence presumably satisfy unitarity. From Eq.(41) this is, however, not evident, since the function \( U \) has a logarithmic singularity as well as a pole at \( x = 0 \). To see that these singularities actually do cancel, consider first the contribution to \( f_2(x) \) coming from the logarithmically singular part (LS):

\[ f_2(x)_{LS} = \int_0^x \{ e^{2t} \left( -\frac{1}{2} e^{2x} \log x \right) - e^{2x} \left( -\frac{1}{2} e^{2t} \log t \right) \} e^{-t} F(t) \]

\[ = -\frac{1}{2} e^{2x} \sum_{n \geq 0} \frac{u_n}{(n + 1 + \rho)} x^{n+1+\rho}, \]  

(44)
where we wrote $e^{-t}F(t) = \sum_{n=2}^{\infty} u_n t^n$, and integrated by parts. Thus, the logarithmically singular part gives a regular contribution, and the result for $f_2(x)$ has an infinite radius of convergence. As concerns the pole terms, it is easy to see that they do not spoil the convergence of the integral since the function $F(t)$ has leading behaviour $t^{2+\rho}$. We have therefore shown that the amplitudes do not exhibit unitarity-violating growth, also at the phase-space points defined by Eq.(31).

Another case, which will be of interest later on in the calculation of the one-loop correction, is that of the pure $\phi^4$ theory, with $p_1^0 = p_2^0 = 2$; the lowest non-zero $b_2(n)$ is $b_2(3)$. In this case we have $\rho = 2$, and in the $f_1(x)$ given in Eq.(20) we have, of course, to use $\omega = 3$. Inserting the results (8) and (20) into Eq.(36) we find

$$x^2 f''_2(x) + x f'_2(x) - f_2(x) = \frac{\lambda_4}{2} \frac{x^2}{(1 - \lambda_4 x^2/48)^2} f_2(x) + \lambda_4 \frac{x^5}{(1 - \lambda_4 x^2/48)^5} \quad . \quad (45)$$

The (simple) solution to this that starts with $x^5$ is

$$\phi(x) = \frac{\lambda_4}{24} \frac{x^5}{(1 - \lambda_4 x^2/48)^3} \quad . \quad (46)$$

(Note that the same solution, albeit with different normalization, also occurs in ref.[3]). The explicit form of $b_2$ follows immediately:

$$b_2(2k + 1) = k(k + 1) \left( \frac{\lambda_4}{48} \right)^k \quad . \quad (47)$$

### 3.2 General momenta

The above considerations can be generalized to evaluate the tree amplitude when two of the outgoing particles have more general momenta:

$$p_1^\mu = (E_1, \vec{p}_1) \, , \, p_2^\mu = (E_2, \vec{p}_2) \, , \, p_i^\mu = (1, 0) \quad (i = 3, \ldots, n) \quad . \quad (48)$$

Let us describe this case briefly. The inverse propagator, $P_2(n)$, is still defined as in Eq.(33), but now has the form

$$P_2(n) = (n + \rho - 1)(n + \rho + \alpha) \quad ,$$

$$\rho = E_1 + E_2 - 1 - \left( E_1^2 + E_2^2 + 2\vec{p}_1 \cdot \vec{p}_2 - 1 \right)^{1/2} \quad ,$$

$$\alpha = -1 + 2 \left( E_1^2 + E_2^2 + 2\vec{p}_1 \cdot \vec{p}_2 - 1 \right)^{1/2} \quad . \quad (49)$$

The differential equation for the generating function (35) now takes the form

$$x^2 f''_2(x) + \alpha x f'_2(x) - (\alpha + x^2 + 3x) f_2(x) = G(x) \quad , \quad (50)$$

where the inhomogeneous term is

$$G(x) = x^{2+\rho} e^{-3x(2x + 3)} M(E_1 + 2, 2E_1 + 1; 2x) M(E_{2} + 2, 2E_2 + 1; 2x) \quad , \quad (51)$$
and the boundary conditions are as before. The solution is

\[ f_2(x) = Axe^{-x} \int_0^x dt \, t^{\alpha-1} e^{-t} \{ \Phi_1(t)\Phi_2(x) - \Phi_1(x)\Phi_2(t) \} G(t) \ , \quad (52) \]

with

\[ \Phi_1(x) = M \left( \frac{5 + \alpha}{2}; 2 + \alpha; 2x \right) \ , \]
\[ \Phi_2(x) = U \left( \frac{5 + \alpha}{2}; 2 + \alpha; 2x \right) \ , \]
\[ A = -2^{a+1} \frac{\Gamma \left( \frac{5+\alpha}{2} \right)}{\Gamma(2+\alpha)} \ . \quad (53) \]

Noting that \( t^{\alpha-1}G(t) \sim t^{\alpha+1+\rho} \), and that the singular part of \( \Phi_2(x) \) goes as \( t^{-1-\alpha} \), we can easily show that the solution \( f_2(x) \) has again a series expansion with an infinite radius of convergence, with the usual implication for the high-\( n \) behaviour of \( a_2(n) \) for this phase-space point.

### 4 On the one-loop correction at threshold

A by-product of the above calculation is that it enables us to obtain, without much effort, the imaginary part of the one-loop amplitude \( a(n) \). Since, as we showed in Eq.(15), for the \( \phi^4 \) potential the only non-zero \( 2 \to n \) threshold amplitudes are those for \( n = 2 \) and \( n = 4 \), and since the \( 2 \to 2 \) process has vanishing phase space, application of the Cutkosky rule (see fig.3) gives

\[ \text{Im} \ a(n)_{1\text{-loop}} = \frac{1}{2} [a_2(n - 2; p_1, p_2)]_A \left( \frac{n}{4} \right) V_2(p_1, p_2) [a_{2\to4}(p_1, p_2)]_{\text{OS}} \ . \quad (54) \]

Here, \( a_2(n - 2; p_1, p_2)_A \) is the amputated tree amplitude for the \( 1 \to n - 2 \) transition in the case studied in section 3.1, where \( p_1 \) and \( p_2 \) are back-to-back, and the other \( n - 4 \) momenta at rest. \( V_2 \) is the two-particle phase space volume. The amplitude \( a_{2\to4}(p_1, p_2)_{\text{OS}} \) describes the transition of two on-shell particles, with momenta \( p_1 \) and \( p_2 \), to four particles at rest; and the binomial factor counts the different numbers of ways in which four particles can be selected to be attached to the loop. The factor \( 1/2 \) comes from the optical theorem, which relates the imaginary part to half of the discontinuity.

Because of energy-momentum conservation in the \( 2 \to 4 \) part of the amplitude (recall that we only consider the cut part of the diagram, in which the intermediate states have to be put on shell), we have, in the rest frame of \( p_1 + p_2 \), \( p_1^0 = p_2^0 = 2 \). Thus, the phase-space factor amounts to

\[ V_2(p_1, p_2) = \frac{1}{2(2\pi)^2} \int \frac{d^3 \bar{p}_1}{2p_1^0} \frac{d^3 \bar{p}_2}{2p_2^0} \delta^3(\bar{p}_1 + \bar{p}_2) \delta(p_1^0 + p_2^0 - 4) = \frac{\sqrt{3}}{32\pi} \ , \quad (55) \]
where we have to include a factor $1/2$ for the Bose symmetry of the 2-boson final state. The $2 \to 4$ amplitude has already been given in Eq.(15). Moreover, using the result of section 3.1, we have

$$b_2(n-2) = \frac{1}{4}(n-1)(n-3) \left( \frac{\lambda_4}{48} \right)^{(n-3)/2} = k(k-1) \left( \frac{\lambda_4}{48} \right)^{k-1}, \quad n = 2k + 1. \quad (56)$$

Hence, we find for the last remaining ingredient, the amputated amplitude:

$$a_2(n-2; p_1, p_2)_A = \frac{a_2(n-2)}{P_2(n-2)} = -i(2k-3)!k(k-1) \left( \frac{\lambda_4}{48} \right)^k. \quad (57)$$

Putting everything together, and inserting the correct power of the mass $m$, we find for the imaginary part of the one-loop corrected amplitude:

$$\text{Im } a(n)_{1\text{-loop}} = (2k+1)! \left( \frac{\lambda_4}{48m^2} \right)^k k(k-1)\lambda_4 \frac{\sqrt{3}}{32\pi}. \quad (58)$$

Since for the spontaneously broken theory the only non-zero amplitudes ($2 \to 1$ and $2 \to 2$) have vanishing phase space, in that case the imaginary part is of course zero. We find ourselves in exact agreement with refs.[3, 4] (note that our $\lambda_4$ differs from the convention used there by a factor of 6).

5 Conclusions

We have shown that the calculation of the tree amplitudes in scalar theories can be extended beyond the threshold point, namely to cases where one or two of the produced particles have non-zero three-momentum. Applied to our unitarity-respecting model potential, the amplitudes satisfy unitarity also at these, slightly more general, phase space points.

We have confirmed the results of Voloshin[3] and Smith[4] that certain $2 \to n$ threshold amplitudes become zero. Our treatment is more general than theirs, in that we have derived explicit expressions for the amplitudes, also in cases where no ‘nullification’ occurs. We have shown that the nullification for purely monomial interactions is restricted to the $\phi^3$ and $\phi^4$ cases. In addition, we have shown that in our toy potential a similar nullification occurs for $n$ even and larger than 4.

Finally, we have applied our explicit results for the various amplitudes, and a simple Cutkosky rule, to rederive the imaginary part of the one-loop correction to the threshold point studied by Voloshin[3] and Smith[4]. Although with our technique we can only reproduce the imaginary, finite, part of the correction, we find that the derivation presented above is quite direct and gives a better idea of the physics behind this result.

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