The Computational Complexity of the Restricted Isometry Property, the Nullspace Property, and Related Concepts in Compressed Sensing

Andreas M. Tillmann and Marc E. Pfetsch

Abstract

This paper deals with the computational complexity of conditions which guarantee that the NP-hard problem of finding the sparsest solution to an underdetermined linear system can be solved by efficient algorithms. In the literature, several such conditions have been introduced. The most well-known ones are the mutual coherence, the restricted isometry property (RIP), and the nullspace property (NSP). While evaluating the mutual coherence of a given matrix is easy, it has been suspected for some time that evaluating RIP and NSP is computationally intractable in general. We confirm these conjectures by showing that for a given matrix $A$ and positive integer $k$, computing the best constants for which the RIP or NSP hold is, in general, NP-hard. These results are based on the fact that determining the spark of a matrix is NP-hard, which is also established in this paper. Furthermore, we also give several complexity statements about problems related to the above concepts.

Index Terms

Compressed Sensing, Computational Complexity, Sparse Recovery Conditions

I. INTRODUCTION

A central problem in compressed sensing (CS), see, e.g., [1], [2], [3], is the task of finding a sparsest solution to an underdetermined linear system, i.e.,

$$\min \|x\|_0 \quad \text{s.t.} \quad Ax = b,$$

(P0)

for a given matrix $A \in \mathbb{R}^{m\times n}$ with $m \leq n$, where $\|x\|_0$ denotes the $\ell_0$-quasi-norm, i.e., the number of nonzero entries in $x$. This problem is well-known to be NP-hard, cf. [MP5] in [4]; the same is true for the denoising variant where the right hand side $b$ is assumed to be contaminated by noise and one employs the constraint $\|Ax - b\|_2 \leq \varepsilon$ instead of $Ax = b$, see [5].

Thus, in practice, one often resorts to efficient heuristics. One of the most popular approaches is known as basis pursuit (BP) or $\ell_1$-minimization, see, e.g., [6], where instead of (P0) one considers

$$\min \|x\|_1 \quad \text{s.t.} \quad Ax = b.$$  

(P1)

Here, $\|x\|_1 = \sum_{i=1}^n |x_i|$ denotes the $\ell_1$-norm. It can be shown that under certain conditions, the optimal solutions of (P0) and (P1) are unique and coincide. In this case, one says that $\ell_0-\ell_1$-equivalence holds or that the $\ell_0$-solution can be recovered by the $\ell_1$-solution. Similar results exist for the sparse approximation (denoising) version with constraint $\|Ax - b\|_2 \leq \varepsilon$.

Many such conditions employ the famous restricted isometry property (RIP) (see [7] and also [8], [9]), which is satisfied with order $k$ and a constant $\delta_k$ by a given matrix $A$ if

$$(1 - \delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k)\|x\|_2^2$$

(1)

holds for all $x$ with $1 \leq \|x\|_0 \leq k$. One is usually interested in the smallest possible constant $\delta_k$, the restricted isometry constant (RIC), such that (1) is fulfilled. For instance, a popular result states that if $\delta_{2k} < \sqrt{2} - 1$, all $x$ with at most $k$ nonzero entries can be recovered (from $A$ and $b := Ax$) via basis pursuit, see, e.g., [10]. A series of papers has been devoted to developing conditions of this flavor, often by probabilistic analyses. One of the most recent results, see [11], shows that $\ell_0-\ell_1$-equivalence for $k$-sparse solutions already holds if $\delta_k < 0.307$ is feasible (note that the RIP order is $k$, not $2k$ as above). Several probabilistic results show that certain random matrices are highly likely to satisfy the RIP with desirable values of $\delta_k$, see, for instance, [12], [9]. There also has been work on deterministic matrix constructions aiming at relatively good RIPS, see, e.g., [13], [14], [15]. The RIP also provides sparse recovery guarantees for other heuristics such as Orthogonal Matching Pursuit and variants [16], [17], as well as in the denoising case, see, e.g., [10].
In the literature, it is often mentioned that evaluating the RIP, i.e., computing the constant $\delta_k$, for some $A$ and $k$, is presumably a computationally hard problem. Most papers seem to refer to NP-hardness, but this is often not explicitly stated. This motivated the development of several (polynomial-time) approximation algorithms for $\delta_k$, e.g., the semidefinite relaxations in [18], [19]. However, while a widely accepted conjecture in the CS community, NP-hardness has, to the best of our knowledge, not been proven so far.

Recently, some first results in this direction have been obtained: In [20] and [21], hardness and non-approximability results about the RIP were derived under certain (non-standard) complexity assumptions; see also [22]. In work independent from the present paper, [23] shows that it is NP-hard to verify (1) for given $A$, $k$ and $\delta_k \in (0, 1)$.

Another popular tool for guaranteeing $\ell_0$-$\ell_1$-equivalence is the nullspace property (NSP), see, e.g., [24], [25], [26], which characterizes recoverability by $\mathcal{P}_1$ (in fact, via $\ell_p$-minimization with $0 < p \leq 1$, see [27], [28]) for sufficiently sparse solutions of $\mathcal{P}_0$. The NSP of order $k$ is satisfied with constant $\alpha_k$ if for all vectors $x$ in the nullspace of $A$ (i.e., $Ax = 0$), it holds that

$$\|x\|_{k, 1} \leq \alpha_k \|x\|_1,$$

(2)

where $\|x\|_{k, 1}$ denotes the sum of the $k$ largest absolute values of entries in $x$. The NSP guarantees exact recovery of $k$-sparse solutions to $\mathcal{P}_0$ by solving $\mathcal{P}_1$ whenever (2) holds with some constant $\alpha_k < 1/2$.

Similar to the RIP case, one is interested in the smallest constant $\alpha_k$, the nullspace constant (NSC), such that (2) is fulfilled. Indeed, if and only if $\alpha_k < 1/2$, $\mathcal{P}_1$ with $b := A\tilde{x}$, $\|\tilde{x}\|_1 \leq k$, has the unique solution $\tilde{x}$, which coincides with that of $\mathcal{P}_0$; see, e.g., [24], [25], [29]. (Thus, the NSP provides a both necessary and sufficient condition for sparse recovery, whereas the RIP is only sufficient.) Moreover, error bounds for recovery in the denoising case can be given, see, e.g., [26].

Again, the computation of $\alpha_k$ is suspected to be NP-hard, and several heuristics have been developed to compute good bounds on $\alpha_k$, e.g., the semidefinite programming approaches in [30], [31], or an LP-based relaxation in [29]. However, as far as we know, no rigorous proof of (NP-)hardness has been given.

In this paper, we show that it is NP-hard to compute the RIC and NSC of a given matrix with given $k$; see Sections III and IV, respectively. More precisely, we show that unless $P=NP$, there is no polynomial time algorithm that computes $\alpha_k$ for all given instances $(A,k)$. We also prove that certifying the RIC given $A$, $k$ and some $\delta_k \in (0, 1)$ is NP-hard.

Prior to this, in Section II, we prove NP-hardness of computing the spark of a matrix, i.e., the smallest number of linearly dependent columns. In fact, our main results concerning the complexity of determining the RIC or NSC follow from reduction of a decision problem concerning the existence of small linearly dependent column subsets. The term spark was first defined in [32], where strong results considering uniqueness of solutions to $\mathcal{P}_0$ were proven. Ever since, its value has been claimed to be NP-hard to calculate, but, to the best of our knowledge, without a proof or reference for this fact. It seems to have escaped researchers’ notice that [33] contains a proof that deciding whether the spark equals the number of rows is NP-hard, by reduction from the Subset Sum Problem (cf. [MP9] in [4]). Moreover, [34] provides a different proof for this special case, by a reduction from the (homogeneous) Maximum Feasible Subsystem problem [35]. Even earlier, in [36], the authors claim to have a proof, but give credit to the dissertation [37] for establishing NP-hardness of spark computations. However, after closer inspection, the result in [37] is in fact not about the spark, but the girth of so-called transversal matroids of bipartite graphs. Only recently, a variant of the latter proof has resurfaced in [38], where it is used to derive (non-deterministic) complexity results for constructing so-called full spark frames, i.e., matrices exhibiting the highest possible spark. Every transversal matroid can be represented by a matrix over an infinite field or finite field with sufficiently large cardinality [39], but there is no known deterministic way to construct such a matrix.

We adapt the proof idea from [37], a reduction from the $k$-Clique Problem, to vector matroids and thus establish that spark computation is NP-hard (without the restriction that the spark equals the row size). Our proof also makes use of results from [34], see Section II for more details.

Moreover, we gather several more complexity statements regarding problems related to the spark or RIP in Sections II and III, some of which are apparently new as well. In particular, we also show that solving the sparse principal component analysis problem (see, e.g., [18], [31], [40]) is strongly NP-hard, which is another widely accepted statement that appears to be lacking rigorous proof so far, and we extend this to show that the NP-hardness of RIC computation in fact holds in the strong sense; see Section III-B. Recall that strong NP-hardness implies that (unless $P=NP$) there cannot exist a fully polynomial-time approximation scheme (FPTAS), i.e., an algorithm that solves a minimization problem within a factor of $(1+\varepsilon)$ of the optimal value in polynomial time with respect to the input size and $1/\varepsilon$, see [4]; an FPTAS often exists for weakly NP-hard problems. Strong NP-hardness can also be understood as an indication that a problem’s intractability does not depend on ill-conditioning (due to the occurrence of very large numbers) of the input data.

Throughout the article, for an $m \times n$ matrix $A$ and a subset $S \subseteq \{1, \ldots, n\}$, we denote by $A_S$ the submatrix of $A$ formed by the columns indexed by $S$. Sometimes we additionally restrict the rows to some index set $R$ and write $A_{RS}$ for the resulting submatrix. Similarly, $x_S$ denotes the part of a vector $x$ containing the entries indexed by $S$. By $A^\top$ and $x^\top$, we denote the transpose of a matrix $A$ or vector $x$, respectively. For graph theoretic concepts and notation we refer to [41], for complexity theory to [42], and for matroid theory to [42].
II. Complexity issues related to the spark

In this section, we deal with complexity issues related to linearly dependent columns of a given matrix \( A \in \mathbb{Q}^{m \times n} \). Inclusion-wise minimal collections of linearly dependent columns are called circuits. More precisely, a circuit is a set \( C \subseteq \{1, \ldots, n\} \) of column indices such that \( A_C \mathbf{x} = 0 \) has a nonzero solution, but every proper subset of \( C \) does not have this property, i.e., \( \text{rank}(A_C) = |C| - 1 = \text{rank}(A_{C \setminus \{j\}}) \) for every \( j \in C \). For notational simplicity, we will sometimes identify circuits \( C \) with the associated solutions \( \mathbf{x} \in \mathbb{R}^n \) of \( A \mathbf{x} = 0 \) having support \( C \). The spark of \( A \) is the size of its smallest circuit.

Example 1: Consider the matrix

\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}.
\]

Clearly, the first two columns yield the minimum-size circuit (in fact, the only one), i.e., \( \text{spark}(A) = 2 \). In particular, note that generally, \( \text{spark}(A) \leq k \) does not guarantee that there also exists a \textit{vector with } \( k \text{ nonzeros} \) in the nullspace of \( A \); e.g., take \( k = 3 \) for the above \( A \). On the other hand, it is immediately clear that a nullspace vector with support size \( k \) does not yield \( \text{spark}(A) = k \), but only \( \text{spark}(A) \leq k \). This distinction between circuits and nullspace vectors in general will be crucial in the proofs below.

The main result of this section is the following.

Theorem 1: Given a matrix \( A \in \mathbb{Q}^{m \times n} \) and a positive integer \( k \), the problem to decide whether there exist a circuit of \( A \) of size at most \( k \) is \( \text{NP-complete} \).

For our proof, we employ several auxiliary results:

Lemma 1: The vertex-edge incidence matrix of an undirected simple graph with \( N \) vertices, \( B \) bipartite components, and \( Q \) isolated vertices has \( \text{rank} \ N - B - Q \).

This result seems to be rediscovered every once in a while. The earliest proof we are aware of is due to van Nuffelen [43] and works through various case distinctions considering linear dependencies of the rows and consequences of the existence of isolated or bipartite components.

Lemma 2: Let \( G = (V, E) \) be a simple undirected graph with vertex set \( V \) and edge set \( E \). Let \( A \) be its vertex-edge incidence matrix, and let \( k > 4 \) be some integer. Suppose \( G \) only has connected components with at least four vertices each, \( |E| = \binom{k}{2} \), and \( \text{rank}(A) = k \). Then the graph \( G \) has exactly \( |V| = k \) vertices.

Proof: Let \( G = (V, E) \) and \( k > 4 \) be the graph and integer given in the statement of the lemma. Assume that \( G \) has no component with less than four vertices, has \( |E| = \binom{k}{2} \) edges, and that its incidence matrix \( A \) has \( \text{rank}(A) = k \).

Since consequently, \( G \) has no isolated vertices, Lemma 1 tells us that the number of vertices is

\[
N = \text{rank}(A) + B = k + B,
\]

where \( B \) is the number of bipartite components in \( G \). Assume that \( B > 0 \), since otherwise the lemma is trivially true.

We claim that the number of edges in \( G \) can be at most

\[
|E| \leq \binom{N}{2} - \frac{4(N-4)}{2} B - 2B.
\]

(3)

To see this, recall that \( G \) can have at most \( \binom{N}{2} \) edges. Each connected component has at least four vertices. Since there are no edges between such a component and vertices outside, the total number of possible edges is reduced by at least \( 4(N-4)/2 \) per component (the factor \( 1/2 \) ensures that we do not count any edges twice). Since \( G \) has at least \( B \) connected components, the possible number of edges is hence decreased at least by the second term in (3). Moreover, since each bipartite component has at least four vertices, at least two of the potential edges cannot be present inside each such component, which yields the last term in (3). Note that the bound (3) is sharp if \( G \) consists only of bipartite components with four vertices each.

Expanding (3) using \( N = k + B \), we obtain

\[
|E| \leq \binom{N}{2} - \frac{4(N-4)}{2} B - 2B = \frac{(k+B)(k+B-1)}{2} \leq \frac{4(k+B-4)}{2} B - 2B
\]

\[= \frac{k^2}{2} - \frac{k}{2}k - B - \frac{3}{2} B^2 + \frac{11}{2} B = \binom{k}{2} - \left( kB + \frac{3}{2} B^2 - \frac{11}{2} B \right), \]

and observe that

\[kB + \frac{3}{2} B^2 - \frac{11}{2} B \geq 5B + \frac{3}{2} B^2 - \frac{11}{2} B = \frac{3}{2} B^2 - \frac{1}{2} B > 0 \]

if \( B > 0 \). Thus, there are strictly less than \( \binom{k}{2} \) edges, contradicting the requirement \( |E| = \binom{k}{2} \). Hence, \( B = 0 \).
Lemma 3: Let $H = (h_{ij}) \in \mathbb{Z}^{m \times n}$ be a full-rank integer matrix with $m \leq n$ and let $\alpha := \max |h_{ij}|$. Let $q \in \{m, \ldots, n\}$ and define

$$H(x) := \begin{pmatrix} 1 & x & (x)^2 & \ldots & (x)^{n-1} \\ 1 & x + 1 & (x + 1)^2 & \ldots & (x + 1)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x + q - m - 1 & (x + q - m - 1)^2 & \ldots & (x + q - m - 1)^{n-1} \end{pmatrix} \in \mathbb{Z}^{q \times n}.$$ 

For any column subset $S$ with $|S| = q$, if $\text{rank}(H_S) = m$ and $|x| \geq \alpha^m q^m + 1$, then $H(x)_S \in \mathbb{Z}^{q \times q}$ has full rank $q$.

Proof: This result is a combination of Lemma 1 and (ideas from the proof of) Proposition 4 from [34]. For clarity, we give the details here. Consider some $S \subseteq \{1, \ldots, n\}$ with $|S| = q$ (w.l.o.g., $q > m \geq 1$; otherwise there is nothing to show). Assume that $\text{rank}(H_S) = m$ and note that the last $q - m$ rows of $H(x)_S$ form a submatrix of a generalized Vandermonde matrix with distinct nodes (see, e.g., [44]), which is easily seen to have full rank as well; see also [34]. Consider the polynomial $p(x) := \det(H(x)_S)$. From [34, Lemma 1], we know that there exists some $x$ for which the subspace spanned by the last $q - m$ rows of $H(x)_S$ and the row space of $H_S$ are transversal, i.e., they only intersect trivially. In particular, this shows that $p$ cannot be identical to the zero polynomial (both transversal parts have full rank). Let $d$ be the degree of $p(x)$ (which depends on the choice of $S$); thus, $p(x) = \beta_0 + \beta_1 x + \ldots + \beta_d x^d$ with $\beta_d \neq 0$. Expanding the determinant $\det(H(x)_S)$ using Leibniz’s formula, one can derive that $|\beta_i| \leq \alpha^m q^m$ for all $i$, by noting that the expansion consists of $q! < q^d$ summands which each are the product of precisely one entry per matrix row and column—in absolute value terms, we can extract a factor of $\alpha^m$ from this sum (from the $m$ rows corresponding to $H_S$), and upper-bound all absolute values of coefficients of $x$ by the highest possible value (occurring when the last $q - m$ columns of $H(x)$ are contained in $S$), which can be no larger than $(q - m)(n - 1)(q - m) < q^m - q$. Moreover, it is easy to see that all $\beta_i \in \mathbb{Z}$ for all $i$. Applying Cauchy’s bound [45] to the monic polynomial obtained from dividing $p(x)$ by $\beta_d$ yields

$$|x| < 1 + \max_{0 \leq \ell \leq d - 1} \left| \frac{\beta_{\ell}}{\beta_d} \right| \leq 1 + \max_{0 \leq \ell \leq d - 1} |\beta_{\ell}| \leq 1 + \alpha^m q^m \quad \text{for all} \quad r \text{ with } p(r) = 0.$$ 

Thus, any $x$ with $|x| \geq \alpha^m q^m + 1$ is not a root of $p(x)$; equivalently, $\text{rank}(H(x)_S) = q$ for such $x$.

Proof of Theorem 1: The problem is clearly in NP: Given a subset $C$ of column indices of $A$, it can be verified in polynomial time that $|C| \leq k$ and that $\text{rank}(A_C) = |C| - 1 = \text{rank}(A_{C \setminus \{j\}})$ for every $j \in C$ (by Gaussian elimination, see, e.g., [46]).

To show hardness, we reduce the NP-complete $k$-Clique Problem (cf. [GT19] in [4], or [47]): Given a simple undirected graph $G$, decide whether $G$ has a clique, i.e., a vertex-induced complete subgraph, of size $k$. We may assume without loss of generality that $k > 4$.

For the given graph $G$ with $n$ vertices and $m$ edges construct a matrix $A = (a_{ie})$ of size $(n + \binom{k}{2} - k - 1) \times m$ as follows: Index the first $n$ rows of $A$ by the vertices of $G$ and its columns by the edges of $G$ (we will also identify the vertices and edges with their indices). Let the first $n$ rows of $A$ contain the vertex-edge incidence matrix of $G$ (i.e., set $a_{ie} = 1$ if $i \in e$, and 0 otherwise). For the non-vertex rows $n + i$, $i \in \{1, \ldots, \binom{k}{2} - k - 1\}$, set $a_{n+i} = (U + i - 1)e^{-1}$ with $U := k2^{k-2} + 1$; note that this corresponds to the bottom part of $A$ consisting of a Vandermonde matrix (each row consists of increasing powers of the distinct numbers $U, \ldots, U + \binom{k}{2} - k - 2$). Clearly, this matrix $A$ can be constructed in polynomial time, and its encoding length is polynomially related to that of the input (in particular, that of its largest entry is $O(k^2 m^2 \log_2(k))$).

We first show that $G$ has a $k$- clique if and only if $A$ has a circuit of size $\binom{k}{2}$. Suppose that $G$ has a $k$-clique, $k > 4$, say on the vertices in the set $R$ (so that $|R| = k$), and with its $\binom{k}{2}$ edges in the set $S$. Since $A_C$ has all-zero rows for each vertex outside of $R$, $|R| + $ (number of non-vertex rows) = $\binom{k}{2} - 1 = |C| - 1 \geq \text{rank}(A_C)$. Clearly, a clique is never bipartite (it always contains odd cycles, for $k \geq 3$). Hence, by Lemma 1, the rows of $A_C$ indexed by $R$ are linearly independent. Now observe that removing any edge from a $k$-clique does not affect the rank of the associated incidence matrix, since the subgraph remains connected and non-bipartite with less vertices than edges (for $k \geq 4$). Thus, by Lemma 1, the rank of the nonzero vertex row part of $A_C$ remains $k$ if any column from $C$ is removed. Therefore, since $a_{ie} \leq 1$ for all $i \leq n$ and all $e \leq m$, Lemma 3 applies to $A_{C \setminus \{e\}}$ for every $e \in C$ (with $x = U$, $H(x) = A$, $S = C \setminus \{e\}$, and $q = \binom{k}{2} - 1$ and yields $\text{rank}(A_{C \setminus \{e\}}) = q = |C| - 1$, whence also $\text{rank}(A_C) = |C| - 1$. Thus, $C$ is a circuit.

Conversely, suppose that $A$ has a circuit $C$ of size $|C| = \binom{k}{2}$ with $k > 4$. Then, by definition of a circuit, $\text{rank}(A_C) = |C| - 1$, so $A_C$ has at least $|C| - 1$ nonzero rows. Since these include the $|C| - k - 1$ non-vertex rows, the set $R$ of nonzero vertex rows of $A_C$ has size $|R| \geq (|C| - 1) - (|C| - k - 1) = k$. Let $A_{RC}$ and $A_{NC}$ denote the vertex and non-vertex row submatrices of $A_C$, respectively. Since $\text{rank}(A_C) \leq \text{rank}(A_{NC}) + \text{rank}(A_{RC}) = |C| - k - 1 + \text{rank}(A_{RC})$ and $|R| \geq k$, clearly $\text{rank}(A_{RC}) \geq k$. Suppose that $\text{rank}(A_{RC}) > k$; then there would exist a subset $R' \subseteq R$ with $|R'| = k + 1$ and $\text{rank}(A_{RC}') = k + 1$. But by Lemma 3, the square matrix $(A_{RC}^T A_{NC})^{-1}$ would then have full rank $|C|$, whence $\text{rank}(A_C) = |C| > |C| - 1$, contradicting the fact that $C$ is a circuit. Thus, the upper part $A_{RC}$ of $A_C$ must in fact have rank exactly $k$.

Observe that the subgraph $(R, C)$ of $G$ with vertex set $R$ and edge set $C$ cannot contain components with less than 4 vertices: such a subgraph $(R', C')$ could have at most as many edges as vertices, so that the associated incidence matrix $A_{C'}$ has full
column rank. Removing a column corresponding to an edge \( e \in C' \) would reduce the rank, i.e., \( \text{rank}(A_{C'\setminus \{e\}}) < \text{rank}(A_{C'}) \). Moreover, note that \( A_{RC} \) has block diagonal form where the blocks are the incidence matrices of the separate graph components within \( (R,C) \), so that the rank is the sum of the ranks of the blocks (one of which is \( A_{RC'} \)). In particular, the non-vertex row part of \( A_C \) maintains full (row) rank when any column is removed from \( C' \), so that deleting \( e \in C' \) would yield \( \text{rank}(A_{C'\setminus \{e\}}) = \text{rank}(A_C) - 1 \), contradicting the fact that \( C \) is a circuit. Thus, Lemma 2 applies to the graph \( (R,C) \) and yields that \( |R| = k \). This implies that the vertices in \( R \) form a \( k \)-clique, because \( R \) can induce at most \( \binom{k}{2} \) edges and the \( \binom{k}{2} \) edges in \( C \) are among them.

We now show that each circuit of \( A \) has size at least \( \binom{k}{2} \). This proves the claim, since by the arguments above, it shows that there exists a circuit of size at most \( (\binom{k}{2}) \) if and only if \( G \) has a \( k \)-clique, i.e., for the given construction a solution to the spark problem yields a solution to the clique problem as well.

Suppose that \( A \) has a circuit \( C \) with \( c := |C| < \binom{k}{2} \). Let \( d := \binom{k}{2} - c > 0 \). Clearly, not all vertex rows restricted to any column subset can be zero, and any submatrix of the non-vertex part of \( A \) with fewer than \( \binom{k}{2} - k \) columns is of (full) column rank. Therefore, \( c > \binom{k}{2} - k \) necessarily. Since \( C \) is a circuit, \( A_C \) has \( c - 1 \) nonzero rows (similar to the arguments above, it can be seen that the lower bound \( c - 1 \) holds with equality). Because the \( \binom{k}{2} - k \) non-vertex rows are among these, and by Lemmas 2 and 3, \( A_C \) has \( (c - 1) - (\binom{k}{2} - k - 1) = k - d > 0 \) nonzero vertex rows. Denote the set of such rows by \( R \), and let \( r \) be the number of edges in the subgraph \( G \) induced by the vertices in \( R \). Since \( |R| = k - d \) vertices can induce at most \( \binom{k-d}{2} \) edges, \( r \leq \binom{k-d}{2} \). But surely all the edges in \( C \) are among those induced by \( R \), so that \( r \geq c = \binom{k}{2} - d \). Putting these inequalities together yields \( \binom{k}{2} - d \leq r \leq \binom{k-d}{2} \). However, since we assumed \( k > 4 \), it holds that \( \binom{k}{2} - d > \binom{k-d}{2} \), yielding a contradiction. Consequently, every circuit \( C \) of \( A \) must satisfy \( |C| \geq \binom{k}{2} \).

The smallest size (cardinality) of a circuit can be expressed as

\[
\text{spark}(A) := \min \{ \|x\|_0 : Ax = 0, \ x \neq 0 \}.
\]

Clearly, there exists a circuit of size at most \( k \) if and only if the spark is at most \( k \). Hence, Theorem 1 immediately yields the following.

**Corollary 1:** Computing \( \text{spark}(A) \) is \( \text{NP} \)-hard.

**Remark 1:** The idea of reducing from the clique problem to prove Theorem 1 is due to Larry Stockmeyer and appears in Theorem 3.3.6 of [37] (see also [36], [38]). However, [37] uses generic matrices that, in fact, represent transversal matroids (of bipartite graphs) and therefore have certain properties needed in the proof. The entries of these generic matrices are not specified, and to date there is no known deterministic way to do so such that the matrix represents a transversal matroid. We replaced the corresponding machinery by our explicit matrix construction and the arguments using Lemmas 1, 2 and 3 to become independent of transversal matroid representations and work directly on vector matroids. Note that the proof of Theorem 1 also shows \( \text{NP} \)-completeness of the problem to decide whether \( A \) has a circuit \( C \) with equal to \( k \); see also Example 1.

**Remark 2:** The results above are related to, but different from, the following.

1) Theorem 1 in [33] shows that for an \( m \times n \) matrix \( A \), it is \( \text{NP} \)-complete to decide whether \( A \) has an \( m \times m \) submatrix with zero determinant. This implies \( \text{NP} \)-completeness of deciding whether \( \text{spark}(A) \leq k \) for the special case \( k = m \). This restriction of \( k \) to the row number \( m \) of \( A \) could in principle be removed by appending all-zero rows, but one would then no longer be in the interesting case where the matrix has full (row) rank. Our proof admits spark values other than the row number for full-rank matrices; however, the row and column numbers in the reduction depend on the instance.

2) In contrast to the results above, for graphic matroids, the girth can be computed in polynomial time [48], [37].

3) The paper [49] proves \( \text{NP} \)-hardness of computing the girth of the binary matroid, i.e., a vector matroid over \( \mathbb{F}_2 \). This, however, does not imply \( \text{NP} \)-hardness over the field of rational or real numbers, and the proof cannot be extended accordingly. Similarly, in [50] it was shown that, over \( \mathbb{F}_2 \), it is \( \text{NP} \)-complete to decide whether there exists a vector with \( k \) nonzero entries in the nullspace of a matrix. However, while the proof for this result can straightforwardly be extended to the rational case, it does not imply hardness of computing the spark either. Since in [50], there is no lower bound on the spark (such as we provide in the last paragraph of the proof of Theorem 1), a situation as in Example 1 is not explicitly avoided there. (Note also that it was already remarked in [50] that the problem to decide whether an \( (\mathbb{F}_2) \)-nullspace vector with at most \( k \) nonzeros exists is not covered by their proof.)

We also have the following result, which shows another relation between minimum cardinality circuits and the task of finding sparsest solutions to underdetermined linear systems.

**Corollary 2:** Given a matrix \( B \), a specific column \( b \) of \( B \), and a positive integer \( k \), the problem of deciding whether there exists a circuit of \( B \) of size \( k \) which contains \( b \) is \( \text{NP} \)-complete in the strong sense. Consequently, it is strongly \( \text{NP} \)-hard to determine the minimum cardinality of circuits that contain a specific column \( b \) of \( B \).

**Proof:** Denote by \( A \) the matrix \( B \) without the column \( b \). Then it is easy to see that \( B \) has a circuit of size \( k \) which contains \( b \) if and only if \( Ax = b \) has a solution with \( k - 1 \) nonzero entries. Deciding the latter is well-known to be \( \text{NP} \)-complete in the strong sense (it amounts to the decision version of \((P_0))\), see [MP5] in [4].
Remark 3: As mentioned in [32], one can reduce spark computations to \((P_0)\) as follows: For each column of \(A \in \mathbb{Q}^{m \times n}\) in turn, add a new row with a 1 in this column and 0 elsewhere. The right-hand side \(b\) is the \((m + 1)\)-th unit vector. Now solve each such \((P_0)\) problem, and take the solution with smallest support. (Note that this is a (Turing-)reduction, using Theorem 1 to show \(\text{NP}\)-hardness of \((P_0)\).) Interestingly, we do not know an easy reduction for the reverse direction.

Let us now briefly consider full spark frames. An \(m \times n\) matrix \(A\) with full rank \(m\) \((m \leq n)\) is said to be full spark if \(\text{spark}(A) = m + 1\), i.e., every submatrix consisting of at most \(m\) columns of \(A\) has full rank. In [38], it was shown that testing a matrix for this property is hard for \(\text{NP}\) under randomized reductions. In fact, the following stronger result holds:

Corollary 3: Given a rational matrix \(A\), deciding whether \(A\) is a full spark frame is \(\text{coNP}\)-complete.

Proof: We can assume w.l.o.g. that \(A \in \mathbb{Q}^{m \times n}\) with rank \(m \leq n\). Thus, \(A\) is full spark if \(\text{spark}(A) = m + 1\). Clearly, \(\text{spark}(A) = m + 1\) holds if and only if the question whether \(A\) has a singular \(m \times m\) submatrix has a negative answer. Since the latter decision problem is \(\text{NP}\)-complete by [33] (or [34, Proposition 4] and the results in [35]), deciding whether \(A\) is a full spark frame is \(\text{NP}\)-hard. Moreover, this problem is contained in \(\text{coNP}\), since the “no” answer can be certified in polynomial time by specifying a singular (square) submatrix and because singularity can be verified in polynomial time.

Note that for the above proof, we cannot employ Theorem 1, because this would require considering the matrix construction used in the reduction from the clique problem (see the proof of the theorem) with \(k = n - 1\), and the clique problem can be solved in polynomial time for any \(k = n - \ell\) with constant \(\ell\).

Remark 4: Above, and in the related complexity results to follow, we show that the decision problem under consideration has a negative answer if and only if a known \(\text{NP}\)-complete problem has a positive answer. Since, by definition, the complementary problem of an \(\text{NP}\)-complete problem is \(\text{coNP}\)-complete, the respective hardness results follow—see also [4]. Both \(\text{NP}\)- and \(\text{coNP}\)-completeness imply that no polynomial time algorithm exists unless \(\text{P} = \text{NP}\) (or equivalently \(\text{P} = \text{coNP}\)). Since a problem is \(\text{NP}\)-hard if and only if it is \(\text{coNP}\)-hard (every problem in \(\text{coNP}\) can be \(\text{Turing}\)-reduced to it; cf [41, Ch. 15.7]), we use the term \(\text{NP}\)-hard throughout the article.

In the next sections we shall see how we can deduce \(\text{NP}\)-hardness of computing restricted isometry or nullspace constants from the above results. (In particular, the extension to \(k < m\) provided by Theorem 1, cf. Remark 2, will allow for making statements about \(\text{RIP}\) or \(\text{NSP}\) orders other than the row number.)

III. \(\text{NP}\)-Hardness of Computing the Restricted Isometry Constant

Recall that for a positive integer \(k\), \(A\) satisfies the RIP of order \(k\) with constant \(\delta_k\) if (1) holds. Given \(A\) and \(k\), the smallest such constant is the RIC \(\delta_k\). Note that (1) holds for \(\delta_1 = 0\) if and only if \(A\) is orthogonal, and that \(\delta_2 < 0\) is impossible.

The suspicions about computational intractability of the RIP are based on the observation that a brute-force method would have to inspect all submatrices induced by column subsets of sizes up to \(k\). Of course, by itself, this does not generally rule out the possible existence of an efficient algorithm. However, we show below that (given \(A\) and \(k\)) deciding whether there exists a constant \(\delta_k < 1\) such that (1) holds is \(\text{coNP}\)-complete; consequently, computing the RIC is \(\text{NP}\)-hard. Moreover, we show that RIP certification (for a given \(\delta_k \in (0,1)\)) is \(\text{NP}\)-hard. Thus, unless \(\text{P} = \text{NP}\), there exists no polynomial time algorithm for any of these problems (cf. Remark 4).

We will need the following technical result.

Lemma 4: Let \(A = (a_{ij}) \in \mathbb{Q}^{m \times n}\) be a matrix. Define \(\alpha := \max\{|a_{ij}| : i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}\}\) and set \(C := 2^{\lceil \log_2(\alpha \sqrt{mn}) \rceil}\). Then \(A := \frac{1}{\sqrt{C}}A\) has encoding length polynomial in that of \(A\), and satisfies
\[
\|\tilde{A}\|_2^2 \leq (1 + \delta)\|x\|_2^2 \quad \text{for all } x \in \mathbb{R}^n \text{ and } \delta \geq 0.
\]

Proof: First, observe that the largest singular value of \(A\), \(\sigma_{\max}(A)\), satisfies
\[
\sigma_{\max}(A) = \|A\|_2 \leq \alpha \sqrt{mn} \leq 2^{\lceil \log_2(\alpha \sqrt{mn}) \rceil} = C.
\]
It follows that
\[
\|\tilde{A}\|_2^2 \leq \|\frac{A}{\sqrt{C}}\|_2^2 \|x\|_2^2 \leq \frac{1}{\sigma_{\max}(A)^2} \|A\|_2^2 \|x\|_2^2 = \|x\|_2^2 \leq (1 + \delta)\|x\|_2^2
\]
for any \(\delta \geq 0\). Moreover, the encoding length of \(C\) and of \(\tilde{A}\) is clearly polynomially bounded by that of \(A\), \(m\) and \(n\).

By the singular value interlacing theorem (see, e.g., [51]), \(\sigma_{\max}(A)\) is an upper bound for the largest singular value of every submatrix of \(A\). Thus, the above lemma essentially shows that by scaling the matrix \(A\), one can focus on the lower part of the RIP (1) (a similar argument has been derived independently in [23]). This leads to the following complexity result.

Theorem 2: Given a matrix \(A \in \mathbb{Q}^{m \times n}\) and a positive integer \(k\), the problem to decide whether there exists some rational constant \(\delta_k < 1\) such that \(A\) satisfies the RIP of order \(k\) with constant \(\delta_k\) is \(\text{coNP}\)-complete.

Proof: We first show that the problem is in \(\text{coNP}\). To certify the “no” answer, it suffices to consider a vector \(\tilde{x}\) with \(1 \leq \|\tilde{x}\|_0 \leq k\) that tightly satisfies (1) for \(\delta_1 = 1\). This implies \(A\tilde{x} = 0\). Clearly, since \(\tilde{x}\) is contained in the nullspace of \(A\), we can assume that \(\tilde{x}\) is rational with encoding length polynomially bounded by that of \(A\). Then, we can verify \(1 \leq \|\tilde{x}\|_0 \leq k\) and \(A\tilde{x} = 0\) in polynomial time, which shows that the “no” answer can be certified in polynomial time.

To show hardness, we reduce the problem to decide whether there exists a circuit of size at most \(k\), which is \(\text{NP}\)-complete by Theorem 1. Consider the matrix \(\tilde{A}\) as defined in Lemma 4; note that the circuits of \(A\) and \(\tilde{A}\) coincide, since nonzero
Consequently, we have that linearly independent columns. Consider an arbitrary such \( \lambda \) of order \( A \) has a circuit of size at most \( A \), which influences the least common denominator of the matrix entries, which influences scaling does not affect linear dependencies among columns. We claim that there exists a circuit \( \hat{x} \) with \( 1 \leq \| \hat{x} \|_0 \leq k \) if and only if \( A \) violates (1) for all \( \delta_k < 1 \). Since deciding the former question is NP-complete, this completes the proof.

Clearly, if such an \( \hat{x} \neq 0 \) exists, then
\[
(1 - \delta_k)\| \hat{x} \|_2^2 \leq \| \hat{A} \hat{x} \|_2^2 = 0
\]
implies that we must have \( \delta_k \geq 1 \).

For the converse, assume that there does not exist \( \delta_k < 1 \) for which (1) holds. By Lemma 4, the upper part of (1) is always satisfied. Consequently, there must exist a vector \( \hat{x} \) with \( 1 \leq \| \hat{x} \|_0 \leq k \) such that the lower part is tight, i.e.,
\[
0 \geq (1 - \delta_k)\| \hat{x} \|_2^2 = \| \hat{A} \hat{x} \|_2^2 \geq 0.
\]
This implies that \( \hat{A} \hat{x} = 0 \). Thus, there also exists a circuit \( \hat{x} \) with \( 1 \leq \| \hat{x} \|_0 \leq k \), which shows the claim.  

Remark 5: For a given matrix \( A \in \mathbb{C}^{m \times n} \) and positive integer \( k \), it is NP-hard to compute the RIP \( \delta_k \).

A. RIP certification

In this section, we show that the RIP certification problem, i.e., deciding whether a matrix \( A \) satisfies the RIP with given order \( k \) and given constant \( \delta_k \in (0, 1) \), is (co)NP-hard. The main arguments used in the proofs of the following Lemma and Theorem have been independently derived by us and the authors of [23].

The following observation is essential.

**Lemma 5:** Given a matrix \( A \in \mathbb{C}^{m \times n} \) and a positive integer \( k \leq n \), if \( \text{spark}(A) > k \), there exists a constant \( \varepsilon > 0 \) with encoding length polynomially bounded by that of \( A \) such that \( \| Ax \|_2^2 \leq \varepsilon \| x \|_2^2 \) for all \( x \) with \( \| x \|_0 \leq k \).

**Proof:** Assume without loss of generality that \( A \) has only integer entries (this can always be achieved by scaling with the least common denominator of the matrix entries, which influences \( \varepsilon \) by a polynomial factor only).

Define \( \alpha \) as in Lemma 4. Note that \( \text{spark}(A) > k \) implies that every submatrix \( A_S \) with \( S \subseteq \{1, \ldots, n\}, |S| \leq k \), has linearly independent columns. Consider an arbitrary such \( S \). Then, \( A_S^\top A_S \) is positive definite, so its smallest eigenvalue fulfills \( \lambda_{\min}(A_S^\top A_S) > 0 \), and also \( \text{det}(A_S^\top A_S) > 0 \). Moreover, since the absolute values of entries of \( A \) are integers in \( \{0, 1, \ldots, \alpha\} \), the entries of \( A_S^\top A_S \) are also integral and lie in \( \{0, 1, \ldots, m \alpha^2\} \). Therefore, it must in fact hold that \( \text{det}(A_S^\top A_S) \geq 1 \) and \( \lambda_{\max}(A_S^\top A_S) \geq 1 \). It follows that
\[
1 \leq \text{det}(A_S^\top A_S) = \prod_{i=1}^{n} \lambda_i(A_S^\top A_S) \leq \lambda_{\min}(A_S^\top A_S) \cdot \lambda_{\max}(A_S^\top A_S)^{k-1}
\]
\[
\leq \lambda_{\min}(A_S^\top A_S)^{\frac{|S| \cdot \max_{1 \leq i, j \leq |S|} |(A_S^\top A_S)_{ij}|}{k}} \leq \lambda_{\min}(A_S^\top A_S)(k m \alpha^2)^{k-1}.
\]
Consequently, we have that
\[
\lambda_{\min}(A_S^\top A_S) \geq \frac{1}{(k m \alpha^2)^{k-1}} =: \varepsilon > 0.
\]

Since \( S \) was arbitrary, \( \| Ax \|_2^2 \geq \lambda_{\min}(A_S^\top A_S)\| x \|_2^2 \geq \varepsilon \| x \|_2^2 \) for all \( x \) with support \( S \subseteq \{1, \ldots, n\}, |S| \leq k \). Moreover, the encoding length of \( \alpha \), and therefore that of \( \varepsilon \), is clearly polynomially bounded by the encoding length of \( A \), which completes the proof.  

**Theorem 3:** Given a matrix \( A \in \mathbb{C}^{m \times n} \), a positive integer \( k \), and some constant \( \delta_k \in (0, 1) \), it is NP-hard to decide whether \( A \) satisfies the RIP of order \( k \) with constant \( \delta_k \).

**Proof:** Consider a matrix \( A \) as in Lemma 4, so we can again focus on the lower inequality of the RIP. Clearly, if \( A \) has a circuit of size at most \( k \), \( A \) cannot satisfy the RIP of order \( k \) with any given \( \delta_k \in (0, 1) \), since in this case, \( \| \hat{A} \hat{x} \|_2^2 = 0 \) or \( (1 - \delta_k)\| \hat{x} \|_2^2 \leq \| \hat{A} \hat{x} \|_2^2 = 0 \) for some \( \hat{x} \) with \( 1 \leq \| \hat{x} \|_0 \leq k \). Moreover, Lemma 5 shows that if \( A \) has no circuit of size at most \( k \), \( A \) satisfies the RIP of order \( k \) with constant \( 1 - \varepsilon \in (0, 1) \), where \( \varepsilon \) has size polynomially bounded by \( k \) and that of \( A \), cf. (6). By Theorem 2, it is coNP-complete to decide whether there exists a constant \( \delta_k < 1 \) such that \( A \) satisfies the RIP of a given order \( k \) with this constant. But as seen above, such a constant exists if and only if \( A \) satisfies the RIP of \( k \) with the constant \( 1 - \varepsilon \), too. Thus, deciding whether the RIP holds for a given matrix, order, and constant, is (co)NP-hard.

**Remark 6:** Clearly, Theorem 3 leads to another easy proof for Corollary 4 (and Corollary 5 below); computing the (lower asymmetric) RIP would also decide the RIP certification problem. On the other hand, our proof of Theorem 3 essentially reduces the RIP certification problem to the setting of Theorem 2, which therefore can be seen as the core RIP hardness result (by establishing the direct link to spark computations); see also Remark 9 below.
B. Asymmetric restricted isometry constants

It has been remarked in [9] that the symmetric nature of the RIP can be overly restrictive. In particular, the influence of the upper inequality in (5) is often stronger, although the lower inequality is more important in the context of sparse recovery. For instance, the often stated condition \( \delta_{2k}^L < 1 \) for uniqueness of \( k \)-sparse solutions (see, e.g., [10]) should in fact read \( \delta_{2k}^U < 1 \), where

\[
\delta_{2k}^L := \min_{\delta \geq 0} \delta \quad \text{s.t.} \quad (1 - \delta) \| x \|_2^2 \leq \| A x \|_2^2 \quad \forall x : \| x \|_0 \leq k
\]

is the lower asymmetric restricted isometry constant [9] (see also [52]). Correspondingly, the upper asymmetric RIC is

\[
\delta_{2k}^U := \min_{\delta \geq 0} \delta \quad \text{s.t.} \quad (1 + \delta) \| x \|_2^2 \geq \| A x \|_2^2 \quad \forall x : \| x \|_0 \leq k.
\]

The central argument in the proof of Theorem 2 in fact shows the following:

**Corollary 5:** Given a matrix \( A \in \mathbb{Q}^{m \times n} \) and a positive integer \( k \), it is NP-hard to compute \( \delta_{2k}^U \).

Moreover, the next result settles the computational complexity of computing the upper asymmetric RIC \( \delta_{2k}^U \).

**Theorem 4:** Given a matrix \( A \in \mathbb{Q}^{m \times n} \), a positive integer \( k \leq m \) and a parameter \( \delta > 0 \), it is NP-hard in the strong sense to decide whether \( \delta_{2k}^U < \delta \), even in the square case \( m = n \). Consequently, it is strongly NP-hard to compute \( \delta_{2k}^U \).

To prove this theorem, we need some auxiliary results.

**Lemma 6:** Let \( G = (V, E) \) be a simple undirected graph with \( n = |V| \) and let \( A_G \) be its \( n \times n \) adjacency matrix, i.e., \((A_G)_{ij} = 1\) if \( \{i,j\} \in E \) and 0 otherwise. Denote by \( K_n \) the complete graph with \( n \) vertices.

1. If \( G = K_n \), i.e., \( G \) is a clique, then \( A_G \) has eigenvalues \(-1 \) and \( n - 1 \) with respective multiplicities \( n - 1 \) and \( 1 \).
2. Removing an edge from \( G \) does not increase \( \lambda_{\max}(A_G) \). In fact, if \( G \) is connected, this strictly decreases \( \lambda_{\max}(A_G) \).
3. If \( G = K_n \setminus e \), i.e., a clique with one edge removed, then the largest eigenvalue of \( A_G \) is \((n - 3 + n^2 - 2n - 3)/2\).

**Proof:** The first two statements can be found in, or deduced easily from, [53, Ch. 1.4.1 and Prop. 3.1.1], respectively. The third result is a special case of [54, Theorem 1].

**Remark 7:** Lemma 6 shows that, in a graph \( G = (V, E) \) with \( |V| = n \), the largest eigenvalue of the adjacency matrix of any induced subgraph with \( k \in \{2, \ldots, n\} \) vertices is either \( k - 1 \) (if and only if the subgraph is a \( k \)-clique) or at most \((k - 3 + \sqrt{k^2 + 2k - 7})/2\).

**Proposition 1:** Given a matrix \( H \in \mathbb{Q}^{n \times n} \), a positive integer \( k \leq n \) and a parameter \( \lambda > 0 \), it is coNP-complete in the strong sense to decide whether \( \lambda_{\max}^{(k)} < \lambda \), where

\[
\lambda_{\max}^{(k)} := \max\{ x^T H x : \| x \|_2^2 = 1, \| x \|_0 \leq k \} = \max\{ \lambda_{\max}(H_{SS}) : S \subseteq \{1, \ldots, n\}, |S| \leq k \}.
\]

Consequently, solving the sparse principal component analysis (Sparse PCA) problem is strongly NP-hard.

**Proof:** We reduce from the \( k \)-Clique Problem. Let \( G = (V, E) \) be a simple undirected graph with \( n \) vertices (w.l.o.g., \( n \geq 2 \)). From \( G \), construct its \( n \times n \) adjacency matrix \( H := A_G \). By the previous Lemma, \( G \) contains a \( k \)-clique if and only if \( \lambda_{\max}^{(k)}(H) = k - 1 =: \lambda \). (Note that \( \lambda_{\max}^{(k)} \leq k - 1 \) always holds by construction.) Hence, the Sparse PCA decision problem has a negative answer for the instance \((H, k, \lambda)\) if and only if the \( k \)-Clique Problem has a positive answer. Since the latter is NP-complete in the strong sense, and because all numbers appearing in the constructed Sparse PCA instance, and their encoding lengths, are polynomially bounded by \( n \), the former is strongly (co)NP-hard.

Moreover, consider a “no” instance of the Sparse PCA decision problem. Then, as we just saw, there is a \( k \)-clique \( S \) in \( G \), and it is easily verified that \( \hat{x} \) with \( \hat{x}_i = 1/\sqrt{k} \) for \( i \in S \), and zeros everywhere else, achieves \( \hat{x}^T H \hat{x} = \lambda_{\max}^{(k)}(H) = \lambda \). Scaling (9) by \( k \), we see that equivalently,

\[
k \cdot \lambda_{\max}^{(k)} = \max\{ x^T H x : \| x \|_2^2 = k, \| x \|_0 \leq k \} = k\lambda = k^2 - k.
\]

Thus, a rational certificate for the “no” answer is given by the vector \( \hat{x} \) with \( \hat{x}_i = 1 \) for \( i \in S \), and zeros everywhere else. Since we can clearly check all constraints on \( \hat{x} \) (from the scaled problem) and that \( \hat{x}^T H \hat{x} = k\lambda = k^2 - k \) in polynomial time, the Sparse PCA decision problem is contained in coNP.

**Remark 8:** The Sparse PCA problem (see, e.g., [40], [31], [18]) is often mentioned to be (NP-)hard, but we could not locate a rigorous proof of this fact. In [55, Section 6], the authors sketch a reduction from the \( k \)-Clique Problem but do not give the details; the central spectral argument mentioned there, however, is exactly what we exploit in the above proof.

We will extend the proof of Proposition 1 to show Theorem 4 by suitably approximating the Cholesky decomposition of a matrix very similar to the adjacency matrix; the following technical result will be useful for this extension.

**Lemma 7:** Let \( A_G \) be the adjacency matrix of a simple undirected graph \( G = (V, E) \) with \( n \) vertices, and let \( H := A_G + n^2 I \), where \( I \) is the \( n \times n \) identity matrix. Then \( H \) has a unique Cholesky factorization \( H = LDL^T \) with diagonal matrix \( D = \text{Diag}(d_1, \ldots, d_n) \in \mathbb{Q}^{n \times n} \) and unit lower triangular matrix \( L = (l_{ij}) \in \mathbb{Q}^{n \times n} \), whose respective entries fulfill \( d_i = [(n^4 - 2n^2 + 2n)/n^2, i^2] \) and \( l_{ij} \in [(2 - n^2 - 2n)/(n^4 - 2n^2 + 2n), (2 - 2n)/(n^4 - 2n^2 + 2n)] \) for all \( i, j \in \{1, \ldots, n\}, i \neq j \).

**Proof:** With \( \deg(v) \) denoting the degree of a vertex \( v \) of \( G \), and \( \lambda_i(A_G), i = 1, \ldots, n \), the eigenvalues of \( A_G \),

\[
\| A_G \|_2 = \max_{1 \leq i \leq n} |\lambda_i(A_G)| \leq \sqrt{n} \| A_G \|_\infty < n \cdot \max_{v \in V} \deg(v) < n^2.
\]
Thus, it is easy to see that $H = A_G + n^2I$ is (symmetric) positive definite; in particular, the eigenvalues of $H$ obey $\lambda_i(H) = \lambda_i(A_G) + n^2$ for all $i$. Then, $H$ has a unique Cholesky factorization $H = LDL^\top$, and $D \in \mathbb{Q}^{n \times n}$ (unit lower triangular) and $L \in \mathbb{Q}^{n \times n}$ (diagonal) can be obtained by Gaussian elimination; see, e.g., [56, Section 4.9.2] or [57, Section 4.2.3].

Let $H^{(0)} = H$ and let $H^{(k)} = (h_{ij}^{(k)})$ be the matrix obtained from $H$ after $k$ iterations of (symmetric) Gaussian elimination. There are $n - 1$ such iterations, and each matrix $H^{(k)}$ has block structure with $\text{Diag}(d_1, \ldots, d_k)$ in the upper left part and a symmetric matrix in the lower right block. We show by induction that for all $k = 1, \ldots, n$, and all $i, j \in \{0, \ldots, n - k\}$,

$$h_{k+i,k+j}^{(k-1)} \in \left[ n^2 - \frac{2(k-1)}{n^2}, n^2 \right] \quad \text{and} \quad (i \neq j) \quad h_{k+i,k+j}^{(k-1)} \in \left[ - \frac{2(k-1)}{n^2}, 1 + \frac{2(k-1)}{n^2} \right].$$

(10)

Clearly, by construction of $H$, $h^{(0)}_{ii} = n^2$ and $h^{(0)}_{ij} \in \{0, 1\}$ for all $i, j \in \{1, \ldots, n\}, i \neq j$, so (10) holds true for $k = 1$. Suppose (10) holds for some $k \in \{1, \ldots, n - 1\}$, i.e., throughout the first $k - 1$ iterations of the Gaussian elimination process. Performing the $k$-th iteration, we obtain $h_{ik}^{(k)} = h_{kk}^{(k-1)} = 0$ for all $i > k$, $h_{kk}^{(k)} = h_{kk}^{(k-1)}$, and

$$h_{k+i,k+j}^{(k)} = h_{k+i,k+j}^{(k-1)} - \frac{h_{k+i,k}^{(k-1)}}{h_{kk}^{(k-1)}} \quad \text{for all} \quad i, j \in \{1, \ldots, n - k\}.$$ \hfill (11)

Thus, in particular, by symmetry of $H^{(k-1)}$, for any $i \in \{1, \ldots, n - k\}$,

$$h_{k+i,k+i}^{(k)} = h_{k+i,k+i}^{(k-1)} = h_{k+i,k+i}^{(k-1)} = h_{k+i,k+i}^{(k-1)} - \frac{(h_{k+i,k+i}^{(k-1)})^2}{h_{kk}^{(k-1)}}.$$ \hfill (12)

Applying the induction hypothesis (10) to (12) yields the first interval inclusion (note that $h_{kk}^{(k-1)} > 0$, so $h_{k+i,k+i}^{(k)} \leq h_{k+i,k+i}^{(k-1)}$):

$$n^2 \geq h_{k+i,k+i}^{(k)} \geq n^2 - \frac{2(k-1)}{n^2} - \frac{(1 + \frac{2(k-1)}{n^2})^2}{n^2} = n^2 - \frac{2(k-1)}{n^2} - \frac{n^2 + 4(k-1) + \frac{4(k-1)^2}{n^2}}{n^4 - 2k + 2} \geq n^2 - \frac{2k}{n^2} \quad \leq \frac{2}{\sqrt{n^2}}.$$ \hfill (13)

Similarly, for the off-diagonal entries $h_{k+i,k+j}^{(k)}$ with $i, j \in \{1, \ldots, n - k\}, i \neq j$, from (11) and (10) we obtain

$$h_{k+i,k+j}^{(k)} \leq 1 + \frac{2(k-1)}{n^2} - \frac{- \frac{2(k-1)}{n^2} \left(1 + \frac{2(k-1)}{n^2}\right)}{n^2 - \frac{2(k-1)}{n^2}} = 1 + \frac{2(k-1)}{n^2} + \frac{2(k-1) + \frac{4(k-1)^2}{n^2}}{n^4 - 2k + 2} \leq 1 + \frac{2k}{n^2} \quad \leq \frac{2}{\sqrt{n^2}}.$$ \hfill (14)

and (compare with (13))

$$h_{k+i,k+j}^{(k)} \geq - \frac{2(k-1)}{n^2} - \frac{(1 + \frac{2(k-1)}{n^2})^2}{n^2 - \frac{2(k-1)}{n^2}} \geq \frac{2 - 2k}{n^2} - \frac{2}{n^2} = - \frac{2k}{n^2},$$

which shows the second interval inclusion and concludes the induction.

The statement of the Lemma now follows from observing that $d_k = h_{kk}^{(k-1)}$ for all $k = 1, \ldots, n$, and because the entries in the lower triangular part of $L$, i.e., $\ell_{ij}$ with $i > j, j = 1, \ldots, n - 1$, contain precisely the negated elimination coefficients (from the $j$-th iteration, respectively), whence

$$\frac{2 - n^2 - 2n}{n^4 - 2n + 2} = - \frac{(1 + \frac{2(n-1)}{n^2})^2}{n^4 - \frac{2(n-1)}{n^2}} \leq \ell_{ij} \leq - \frac{h_{ij}^{(j-1)}}{h_{ij}^{(j-1)}} = - \frac{h_{ij}^{(j-1)}}{d_j} \leq - \frac{- \frac{2(n-1)}{n^2}}{n^4 - \frac{2(n-1)}{n^2}} = \frac{2n^2 - 2n}{n^4 - 2n + 2}.$$

(By construction, $\ell_{ii} = 1$ and $\ell_{ij} = 0$ for all $i = 1, \ldots, n, j > i$.) This completes the proof.

Proof of Theorem 4: Given an instance $(G, k)$ for the $k$-Clique Problem, we construct $H = A_G + n^2I$ from the graph’s adjacency matrix $A_G$ (w.l.o.g., $n \geq 2$). From the proof of Proposition 1, recall that $G$ has no $k$-clique if and only if $\lambda^{(k)}(A_G) < k - 1$, or equivalently $\lambda^{(k)}(H) < n^2 + k - 1$ (cf. the beginning of the proof of Lemma 7). Let $D$ and $L$ be the Cholesky factors of $H$, i.e., $H = LDL^\top$. Letting $D^{1/2} := \text{Diag}(\sqrt{d_1}, \ldots, \sqrt{d_n})$, observe that the upper asymmetric RIC for the matrix $A' := D^{1/2}L^\top$ can be written as

$$\delta^{(k)}(A') = \max \{ x^\top LD^{1/2}D^{1/2}L^\top x : \|x\|^2 = 1, \|x\|_0 \leq k \} - 1 = \lambda^{(k)}(H) - 1.$$\hfill (15)

Consequently, $G$ has a $k$-clique $S$ if and only if $H$ has a $k \times k$ submatrix $H_{SS}$ with largest eigenvalue $n^2 + k - 1$, i.e., $\lambda^{(k)}(A') = n^2 + k - 2$. Moreover, by Lemma 6, $\lambda^{(k)}(H_{TT}) \leq n^2 + (k-3 + \sqrt{k^2 + 4k - 7})/2 < n^2 + k - 1$ for any incomplete induced subgraph of $G$ with vertex set $T, |T| = k$. However, while $L$ and $D$ are rational, $D^{1/2}$ can contain irrational entries, so we cannot directly use $A'$ as the input matrix for the upper asymmetric RIC decision problem. The remainder of this proof shows that we can replace $D^{1/2}$ by a rational approximation to within an accuracy that still allows us to distinguish between eigenvalues associated to $k$-cliques and those belonging to incomplete induced subgraphs.
Let us consider the rational approximation obtained by truncating after the \( p \)-th decimal number (we will specify \( p \) later), i.e., let \( \tilde{D}^{1/2} = \text{Diag}(r_1, \ldots, r_n) \) with \( r_i := [10^p \cdot \sqrt{d_i}] / 10^p \). Thus, \( \sqrt{d_i} - r_i \leq 10^{-p} \) for all \( i \) by construction, and in particular, since \( d_1 = n^2 \) (see Lemma 7), \( r_1 = n \). Consequently,

\[
\|D^{1/2} - \tilde{D}^{1/2}\|_2 = \max_{1 \leq i \leq n} \{ |\sqrt{d_i} - r_i| \} = \max_{2 \leq i \leq n} \{ |\sqrt{d_i} - r_i| \} \leq 10^{-p}.
\]

Denoting \( \tilde{D} := D^{1/2} \tilde{D}^{1/2} \) and using \( d_i \leq n^2 \) (by Lemma 7), we obtain

\[
\|D - \tilde{D}\|_2 = \max_{2 \leq i \leq n} \{ d_i - r_i^2 \} = \max_{2 \leq i \leq n} \{ (\sqrt{d_i} + r_i)(\sqrt{d_i} - r_i) \} \leq 10^{-p} \cdot 2 \cdot \max \{ \sqrt{d_i}, r_i \} \leq 2 \cdot 10^{-p} \cdot n.
\]

Let \( \tilde{H} := LDL^\top \) and note that, for all \( i, j = 1, \ldots, n \), we have \( h_{ij} = \sum_{q=1}^n d_q \ell_{iq} \ell_{jq} \) and \( \tilde{h}_{ij} = \sum_{q=1}^n r_q \ell_{iq} \ell_{jq} \). Since \( |\ell_{ij}| \leq 1 \) for all \( i, j \) (by Lemma 7), and by symmetry of \( H \) and \( \tilde{H} \), it follows that

\[
|\tilde{h}_{ij}| = |(H - \tilde{H})_{ij}| = \left| \sum_{q=1}^n (d_q - r_q^2) \ell_{iq} \ell_{jq} \right| \leq \sum_{q=1}^n |d_q - r_q^2| |\ell_{iq}| |\ell_{jq}| \leq \sum_{q=1}^n (d_q - r_q^2) \leq 2 \cdot 10^{-p} \cdot n^2.
\]

Thus, we have \( \tilde{H} = H + \tilde{E} \), where \( \|\tilde{E}_{ij}\| \leq 2 \cdot 10^{-p} \cdot n^2 \) and \( \tilde{E} \) is also symmetric. Note that, for any \( S \subseteq \{1, \ldots, n\} \),

\[
\lambda_{\max}(\tilde{E}_{SS}) \leq \lambda_{\max}(\tilde{E}) = \|\tilde{E}\|_2 \leq \sqrt{\|\tilde{E}\|_1 \cdot \|\tilde{E}\|_\infty} \leq \sqrt{(n \cdot 2 \cdot 10^{-p} \cdot n^2)(n \cdot 2 \cdot 10^{-p} \cdot n^2)} = 2 \cdot 10^{-p} \cdot n^3.
\]

Therefore (cf., e.g., [57, Corollary 8.1.6]), we have for all \( S \subseteq \{1, \ldots, n\} \) that

\[
|\lambda_i(H_{SS}) - \lambda_i(\tilde{H}_{SS})| \leq \|\tilde{E}_{SS}\|_2 = \lambda_{\max}(\tilde{E}_{SS}) \leq 2 \cdot 10^{-p} \cdot n^3
\]

for all \( i = 1, \ldots, |S| \). Consequently, if \( S \) is a \( k \)-clique, we have

\[
\lambda_{\max}(\tilde{H}_{SS}) \geq n^2 + k - 1 - 2 \cdot 10^{-p} \cdot n^3,
\]

whereas for any \( T \subseteq \{1, \ldots, n\} \) with \( |T| = k \) which induces no clique, it holds that

\[
\lambda_{\max}(\tilde{H}_{TT}) \leq n^2 + (k - 3 + \sqrt{k^2 + 2k - 7})/2 + 2 \cdot 10^{-p} \cdot n^3.
\]

Now fix \( p := 1 + \lceil 4 \log_{10}(n) \rceil \), and let an instance for the upper asymmetric RIC decision problem be given by \( A := \tilde{D}^{1/2}L^\top \) (where \( \tilde{D}^{1/2} \) is computed from the Cholesky factors \( L \) and \( D \) of \( H = A_G + n^2I \) using this precision parameter \( p \)), \( \delta := n^2 + k - 2 - 2 \cdot 10^{-p} \cdot n^3 \), and \( k \).

If \( G \) has a \( k \)-clique, then by (14), \( \tilde{d}_k(A) \geq \delta \), and if not, \( \tilde{d}_k(A) \leq n^2 + (k - 3 + \sqrt{k^2 + 2k - 7})/2 + 2 \cdot 10^{-p} \cdot n^3 - 1 \) by (15). In fact, our choice of \( p \) implies

\[
p > \log_{10}(8) + 4 \log_{10}(n) \quad \Rightarrow \quad 10^p > 8n^4 > \frac{8n^3}{k + 1 - \sqrt{k^2 + 2k - 7}},
\]

from which we can derive that

\[
n^2 + k - 1 - 2 \cdot 10^{-p} \cdot n^3 > n^2 + \frac{k - 3 + \sqrt{k^2 + 2k - 7}}{2} + 2 \cdot 10^{-p} \cdot n^3,
\]

which shows that \( \tilde{d}_k(A) < \delta \) if no \( k \)-clique is contained in \( G \). Therefore, \( G \) has a \( k \)-clique if and only if \( \tilde{d}_k(A) \geq \delta \), i.e., the upper asymmetric RIC decision problem under consideration has a negative answer.

Clearly, all computations in the above reduction can be performed in polynomial time. To see that the encoding length \( \langle A \rangle \) of \( A \) is in fact polynomially bounded by \( n \), note that \( h_{ij} \in \{0, 1, n^2\} \) for all \( i, j \), whence \( \langle H \rangle \in O(\text{poly}(n)) \). Since Gaussian elimination can be implemented to lead only to a polynomial growth of the encoding lengths [46], it follows that \( \langle L \rangle, \langle D \rangle \in O(\text{poly}(\langle H \rangle)) = O(\text{poly}(n)) \). In particular, the entries of \( \tilde{D}^{1/2} \) then also have encoding length polynomially bounded by \( n \), by construction of the rational approximation. This shows that indeed \( \langle a_{ij} \rangle \in O(\text{poly}(n)) \) for all \( i, j \). Furthermore, all the numerical values \( a_{ij} \) are also polynomially bounded by \( n \) (in fact, \( |a_{ij}| \leq n \), by Lemma 7 and the construction of \( \tilde{D}^{1/2} \)). Moreover, \( 0 < \delta < n^2 + n \), and, clearly, its encoding length \( \langle \delta \rangle \) is bounded polynomially by \( n \) as well.

Thus, since the Clique Problem is well-known to be \( \text{NP} \)-complete in the strong sense, and because our polynomial reduction in fact preserves boundedness of the numbers within \( O(\text{poly}(n)) \), the upper asymmetric RIC decision problem is \((\text{co})\text{NP}\)-hard in the strong sense. This completes the proof of Theorem 4.

In fact, observe that for the matrix \( A \) constructed in the proof of Theorem 4, the upper asymmetric RIC is always larger than the lower asymmetric RIC, whence the former therefore coincides with the (symmetric) RIC \( \tilde{d}_k \) of \( A \). Thus, the following result holds true, which slightly strengthens Corollary 4.

**Corollary 6:** Given a matrix \( A \in \mathbb{Q}^{m \times n} \) and a positive integer \( k \), it is \( \text{NP} \)-hard in the strong sense to compute the RIC \( \tilde{d}_k \), even in the square case \( m = n \).
Remark 9: Theorem 2 (which yielded Corollary 4) is of interest in its own right, as it reveals, for instance, the intrinsic relation between the \(k\)-sparse solution uniqueness conditions \(2k < \text{spark}(A)\) and \(\delta_{2k} < 1\) (or \(\delta_{2k}^L < 1\), respectively, see Corollary 5); consequently, verifying uniqueness via these conditions is \(\text{NP}\)-hard because deciding whether \(\text{spark}(A) \leq k\) is.

Remark 10: We can easily extend the construction from the proof of Theorem 4 to cover the non-square case. Instead of the intractable RIP, NSP, or spark, the weaker but efficiently computable \(\text{coNP}\) of this latter problem thus remains an open question.

IV. \(\text{NP}\)-HARDNESS OF COMPUTING THE NULLSPACE CONSTANT

Recall that \(A \in \mathbb{Q}^{m \times n}\) satisfies the NSP of order \(k\) with constant \(\alpha_k\) if (2) holds for all \(x \in \mathbb{R}^n\) with \(Ax = 0\). Thus, \(\alpha_k\) is the nullspace constant (NSC), such that the NSP of order \(k\) holds with this constant. The NSC is given by

\[
\alpha_k := \min \alpha \quad \text{s.t.} \quad \|x\|_{k,1} \leq \alpha \|x\|_1 \quad \text{for all } x \text{ with } Ax = 0,
\]

or equivalently,

\[
\alpha_k := \max \|x_S\|_1 \quad \text{s.t.} \quad Ax = 0, \quad \|x\|_1 = 1, \quad S \subseteq \{1, \ldots, n\}, |S| \leq k.
\]

Sparse recovery by (P1) is ensured if and only if \(\alpha_k < 1/2\). However, the following results show that computing \(\alpha_k\) is a challenging problem.

Theorem 5: Given a matrix \(A \in \mathbb{Q}^{m \times n}\) and a positive integer \(k\), the problem to decide whether \(A\) satisfies the NSP of order \(k\) with some constant \(\alpha_k < 1\) is \(\text{coNP}\)-complete.

Proof: First of all, note that the NSP (2) is equivalent to the condition that

\[
\|x_S\|_1 \leq \alpha_k \|x\|_1
\]

holds for all \(S \subseteq \{1, \ldots, n\}, |S| \leq k, \text{ and } x \in \mathbb{R}^n\) with \(Ax = 0\). Clearly, (2) and (18) always hold for some \(\alpha_k \in [0, 1]\).

We first show that the problem is in \(\text{coNP}\). To certify the “no” answer, it suffices to consider a vector \(\hat{x}\) with \(A\hat{x} = 0\) and a set \(\emptyset \neq S \subseteq \{1, \ldots, n\}\) with \(|S| \leq k\) that tightly satisfy (18) for \(\alpha_k = 1\). This implies that \(S\) contains the support of \(\hat{x}\). Thus, \(1 \leq \|\hat{x}\|_0 \leq k\). Clearly, since \(\hat{x}\) is contained in the nullspace of \(A\), it can be assumed to be rational with encoding length polynomially bounded by that of \(A\). This shows that the “no” answer can be certified in polynomial time.

To show hardness, we claim that the matrix \(A\) has a circuit of size at most \(k\) if and only if there does not exist any \(\alpha_k < 1\) such that (2) holds. Since the former problem is \(\text{NP}\)-complete by Theorem 1, this completes the proof.

Assume \(A\) has a circuit of size at most \(k\). Then there exists a vector \(x\) in the nullspace of \(A\) with \(1 \leq \|x\|_0 \leq k\). It follows that \(\|x\|_{k,1} = \|x\|_1\). Thus, (2) implies \(\alpha_k \geq 1\). Since, trivially, \(\alpha_k \leq 1\), we conclude that \(\alpha_k = 1\).

Conversely, assume that there exists no \(\alpha_k < 1\) such that (2) holds for \(A\) and \(k\). This implies that there is a vector \(x\) with \(Ax = 0\) and \(1 \leq \|x\|_0 \leq k\) such that \(\|x\|_{k,1} = \|x\|_1\), because otherwise \(\alpha_k < 1\) would be possible. But this means that the support of \(x\) contains a circuit of \(A\) of size at most \(k\), which shows the claim.

We immediately obtain the following.

Corollary 7: Given a matrix \(A \in \mathbb{Q}^{m \times n}\) and a positive integer \(k\), it is \(\text{NP}\)-hard to compute the nullspace constant \(\alpha_k\).

V. CONCLUDING REMARKS

The results of this paper show that it is \(\text{coNP}\)-complete to answer the following questions in the case \(\gamma = 1\): “Given a matrix \(A\) and a positive integer \(k\), does the RIP or NSP hold with some constant \(\gamma < 1\)?” It is important to note that our results do not imply the hardness for every fixed constant \(\gamma < 1\). (Note that such questions become solvable in time \(O(r^{\text{poly}(k)})\) if \(k\) is fixed.) For instance, Theorem 3 asserts that it is \(\text{(co)}\text{NP}\)-hard to certify the RIP for given \(A\), \(k\) and \(\delta_k \in (0, 1)\) in general. The actual \(\delta_k\) appearing in the proof (as in the one independently derived in [23]), however, is very close to 1 and thus far from values of \(\gamma\) that could yield recovery guarantees. Similarly, the NSP guarantees \(\ell_0-\ell_1\)-equivalence for \(\alpha_k < 1/2\), while we proved \(\text{NP}\)-hardness only for deciding whether \(\alpha_k < 1\) (which implies that computing \(\alpha_k\) is \(\text{NP}\)-hard). The complexity of these related questions remains open. Nevertheless, our results provide a justification for investigating general approximation algorithms (which compute bounds on \(\delta_k\) or \(\alpha_k\)), as done in [31], [18], [29], [19], instead of searching for exact polynomial time algorithms.

Instead of the intractable RIP, NSP, or spark, the weaker but efficiently computable mutual coherence [24] is sometimes used. It can be shown that the mutual coherence yields bounds on the RIC, NSC, and the spark; see, for instance, [27], [58].
Thus, imposing certain conditions involving the mutual coherence of a matrix can yield uniqueness and recoverability (by basis pursuit or other heuristics), see, e.g., [58]. However, the sparsity levels for which the mutual coherence can guarantee recoverability are quite often too small to be of practical use. This emphasizes the importance of other concepts.

An interesting question for future research is whether it is hard to approximate the constants associated with the RIP or NSP in polynomial time to within some factor, or whether spark and NSC computations are also strongly NP-hard (as is RIC computation, see Corollary 6). A first step in this direction was taken in [21], where inapproximability of RIP parameters is shown under certain less common complexity assumptions (interestingly, also making use of Cholesky decompositions of certain matrices related to a type of adjacency matrix for random graphs), see also [20], [22]; strong inapproximability results for (P_d) appear in [59].

Moreover, (co)NP-hardness does not necessarily exclude the possibility of practically efficient algorithms. So far, to the best of our knowledge, the focus has been laid largely on relaxations or heuristics. In [20], [21], it was shown that one may sometimes do better than exhaustive search to certify the RIP, making use of the nondecreasing monotonicity of $\delta_k$ with growing $k$. A “sandwiching” procedure to compute the NSC $\alpha_k$ (exactly) was very recently proposed in [60] and empirically demonstrated to be faster than brute force. However, neither method can guarantee a running time improvement with respect to simple enumeration. Moreover, Corollary 6 shows that, in general, no pseudo-polynomial time algorithm (i.e., a method with running time polynomially bounded by the input size and the largest occurring numerical value) can exist for computing the RIC $\delta_k$, unless P=NP.

More work on exact algorithms could shed more light on the behavior of the RIP and NSP.

ACKNOWLEDGMENT

We thank the anonymous referees for their constructive comments and for spotting an error in an earlier version of this paper.

REFERENCES

[1] D. L. Donoho, “Compressed sensing,” IEEE Trans. Inf. Theory, vol. 52, no. 4, pp. 1289–1306, Apr. 2006.
[2] M. Elad, Sparse and redundant representations: From theory to applications in signal and image processing. Heidelberg, Germany: Springer, 2010.
[3] M. Fornasier and H. Rauhut, “Compressive sensing,” in Handbook of mathematical methods in imaging, O. Scherzer, Ed. Berlin, Heidelberg, New York, 2011, pp. 187–228.
[4] M. R. Garey and D. S. Johnson, Computers and intractability. A guide to the theory of NP-completeness. San Francisco, CA: W. H. Freeman and Company, 1979.
[5] B. K. Natarajan, “Sparse approximate solutions to linear systems,” SIAM J. Comput., vol. 24, no. 2, pp. 227–234, Apr. 1995.
[6] S. S. Chen, D. L. Donoho, and M. A. Saunders, “Atomic decomposition by basis pursuit,” SIAM J. Sci. Comput., vol. 20, no. 1, pp. 33–61, Aug. 1998.
[7] E. Candès and T. Tao, “Decoding by linear programming,” IEEE Trans. Inf. Theory, vol. 51, pp. 4203–4215, Dec. 2005.
[8] R. G. Baraniuk, M. A. Davenport, R. A. DeVore, and M. B. Wakin, “A simple proof of the restricted isometry property for random matrices,” Constr. Approx., vol. 28, no. 3, pp. 253–263, 2008.
[9] J. D. Blanchard, C. Cartis, and J. Tanner, “Compressed sensing: How sharp is the RIP?” SIAM Rev., vol. 53, no. 1, pp. 105–125, Feb. 2011.
[10] E. Candès, “The restricted isometry property and its implications for compressed sensing,” C. R. Math., vol. 346, no. 9, pp. 589–592, 2008.
[11] T. T. Cai, L. Wang, and G. Xu, “Near bounds for restricted isometry constants,” IEEE Trans. Inf. Theory, vol. 56, no. 9, pp. 4388–4394, Sept. 2010.
[12] B. Bah and J. Tanner, “Improved bounds on restricted isometry constants for Gaussian matrices,” SIAM J. Matrix Anal. Appl., vol. 31, no. 5, pp. 2882–2892, 2010.
[13] R. A. DeVore, “Deterministic constructions of compressed sensing matrices,” J. Complexity, vol. 23, pp. 918–925, Aug. 2007.
[14] M. A. Iwen, “Simple deterministically constructible RIP matrices with sublinear Fourier sampling requirements,” in Proc. 43rd Ann. Conf. Information Sciences and Systems (CISS), John Hopkins Univ., 2009, pp. 870–875.
[15] J. Bourgain, S. J. Dilworth, K. Ford, S. Konyagin, and D. Kutzarova, “Explicit constructions of RIP matrices and related problems,” Duke Math. J., 2011, to be published. [Online] Available: http://arxiv.org/abs/1008.4535
[16] M. A. Davenport and M. B. Wakin, “Analysis of orthogonal matching pursuit using the restricted isometry property,” IEEE Trans. Inf. Theory, vol. 56, no. 9, pp. 4395–4401, Sep. 2010.
[17] J. A. Tropp, “Greed is good: Algorithmic results for sparse approximation,” IEEE Trans. Inf. Theory, vol. 50, pp. 2231–2242, Oct. 2004.
[18] A. d’Aspremont, L. E. Ghaoui, M. I. Jordan, and G. R. G. Lanckriet, “A direct formulation for sparse PCA using semidefinite programming,” SIAM Rev., vol. 49, no. 3, pp. 434–447, 2007.
[19] K. Lee and Y. Bressler, “Computing performance guarantees for compressed sensing,” in Proc. IEEE Int. Conf. Acoustics, Speech and Signal Processing (ICASSP), Las Vegas, 2008, pp. 5129–5132.
[20] P. Koiran and A. Zouzias, “On the certification of the restricted isometry property,” unpublished manuscript, 2011, [Online] Available: http://arxiv.org/abs/1103.4984
[21] P. Koiran and A. Zouzias, “Hidden cliques and the certification of the restricted isometry property,” submitted for publication, 2012, [Online] Available: http://arxiv.org/abs/1211.0665
[22] Q. Berthet and P. Rigollet, “Computational Lower Bounds for Sparse PCA,” 2013, [Online] Available: http://arxiv.org/abs/1304.0828
[23] A. S. Bandeira, E. Dobriban, D. G. Mixon, and W. F. Sawin, “Certifying the restricted isometry property is hard,” IEEE Trans. Inf. Theory, vol. 59, no. 6, pp. 3448–3450, Jun. 2013.
[24] D. L. Donoho and X. Huo, “Uncertainty principles and ideal atomic decomposition,” IEEE Trans. Inf. Theory, vol. 47, no. 7, pp. 2845–2862, Nov. 2001.
[25] Y. Zhang, “Theory of compressive sensing via $\ell_1$-minimization: A non-RIP analysis and extensions,” Department of Computation and Applied Mathematics, Rice University, Houston, TX, Tech. Rep. TR08-11 (revised), 2008.
[26] A. Cohen, W. Dahmen, and R. DeVore, “Compressed sensing and best $k$-term approximation,” J. Am. Math. Soc., vol. 22, pp. 211–231, 2009.
[27] M. A. Davenport and M. B. Wakin, “Restricted isometry constants where $\delta_k$ is small for all $k$,” IEEE Trans. Inf. Theory, vol. 55, no. 2, pp. 723–727, Feb. 2009.
[28] A. Juditsky and A. Nemirovski, “On verifiable sufficient conditions for sparse signal recovery via $\ell_1$ minimization,” Math. Program., vol. 127, no. 1, pp. 57–88, 2011.
A. d'Aspremont and L. E. Ghaoui, “Optimal solutions for sparse principal component analysis,” *J. Mach. Learn. Res.*, vol. 9, pp. 1269–1294, Jul. 2008.

A. d'Aspremont and L. E. Ghaoui, “Testing the nullspace property using semidefinite programming,” *Math. Program.*, vol. 127, no. 1, pp. 123–144, 2011.

D. L. Donoho and M. Elad, “Optimally sparse representation in general (non-orthogonal) dictionaries via ℓq minimization,” *P. Natl. Acad. Sci. USA*, vol. 100, no. 5, pp. 2197–2202, Mar. 2003.

L. Khachiyan, “On the complexity of approximating extremal determinants in matrices,” *J. Complexity*, vol. 11, pp. 138–153, 1995.

A. Chistov, H. Fournier, L. Gurvits, and P. Koiran, “Vandermonde Matrices, NP-Completeness, and Transversal Subspaces,” *Found. Comput. Math.*, vol. 3, no. 4, pp. 421–427, Oct. 2003.

E. Amaldi and V. Kann, “The complexity and approximability of finding maximum feasible subsystems of linear relations,” *Theoret. Comput. Sci.*, vol. 147, no. 1–2, pp. 181–210, Aug. 1995.

T. F. Coleman and A. Pothen, “The null space problem I. Complexity,” *SIAM J. Algebra. Discr.*, vol. 7, no. 4, pp. 527–537, 1986.

S. T. McCormick, “A combinatorial approach to some sparse matrix problems,” Ph.D. dissertation, Stanford University, 1983.

B. Alexeev, J. Cahill, and D. G. Mixon, “Full spark frames,” *J. Fourier Anal. Appl.*, 2011, to be published. [Online] Available: http://arxiv.org/abs/1110.3548

M. J. Piff, and D. J. A. Welsh, “On the vector representation of matroids,” *J. London Math. Soc.*, vol. 2, no. 2, pp. 284–288, 1970.

R. Luss and M. Teboulle, “Conditional Gradient Algorithms for Rank-One Matrix Approximations with a Sparsity Constraint,” *SIAM Rev.*, vol. 55, no. 1, pp. 65–98, 2013.

B. Korte and J. Vygen, *Combinatorial optimization. Theory and algorithms*, 4th ed., ser. Algorithms and Combinatorics, vol. 21. Berlin, Germany: Springer, 2008.

J. G. Oxley, *Matroid theory*, 1st ed., ser. Oxford Graduate Texts in Mathematics, vol. 3. New York, NY, USA: Oxford University Press, 1992.

C. van Nuffelen, “On the incidence matrix of a graph,” *IEEE Trans. Circuits Syst.*, vol. CAS-23, no. 9, p. 572, Sept. 1976.

S. De Marchi, “Generalized Vandermonde determinants, Toeplitz matrices and Schur functions,” Tech. Rep. Ergebnisberichte Angewandte Mathematik, vol. 176, Universität Dortmund, 1999.

H. P. Hirst and W. T. Macey, “Bounding the roots of polynomials,” *The College Mathematics Journal*, vol. 28, no. 4, pp. 292–295, Sept. 1997.

M. Grötschel, L. Lovász, and A. Schrijver, *Geometric algorithms and combinatorial optimization*, 2nd ed., ser. Algorithms and Combinatorics, vol. 2. Heidelberg, Germany: Springer, 1993.

R. M. Karp, “Reducibility among combinatorial problems,” in *Complexity of computer computations*, R. Miller and J. W. Thatcher, Eds. New York: Plenum Press, 1972, pp. 85–103.

A. Itai and M. Rodeh, “Finding a minimum circuit in a graph,” *SIAM J. Comput.*, vol. 7, no. 4, pp. 413–423, 1978.

A. Vardy, “The intractability of computing the minimum distance of a code,” *IEEE Trans. Inf. Theory*, vol. 43, no. 6, pp. 1757–1766, Nov. 1997.

E. R. Berlekamp, R. J. McEliece, and H. C. A. van Tilborg, “On the inherent intractability of certain coding problems,” *IEEE Trans. Inf. Theory*, vol. IT-24, no. 3, pp. 384–386, May 1978.

J. F. Queiró, “On the interlacing property for singular values and eigenvalues,” *Linear Algebra Appl.*, vol. 97, pp. 23–28, 1987.

S. Foucart and M.-J. Lai, “Sparsest solutions of underdetermined linear systems via ℓq-minimization for 0 ≤ q ≤ 1,” *Appl. Comput. Harmon. A.*, vol. 26, pp. 395–407, 2009.

A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, ser. Universitext, vol. XIII. Berlin, Germany: Springer, 2012.

P. E. Gill, W. Murray, and M. H. Wright, *Numerical Linear Algebra and Optimization*, vol. 1, Redwood City, CA, USA: Addison-Wesley Publishing Company, 1991.

G. H. Golub and C. F. van Loan, *Matrix Computations*, 3rd ed. Baltimore, MD, USA: Johns Hopkins University Press, 1996.

A. M. Bruckstein, D. L. Donoho, and M. Elad, “From sparse solutions of systems of equations to sparse modeling of signals and images,” *SIAM Rev.*, vol. 51, no. 1, pp. 34–81, 2009.

E. Amaldi and V. Kann, “On the approximability of minimizing nonzero variables or unsatisfied relations in linear systems,” *Theor. Comput. Sci.*, vol. 209, pp. 237–260, 1998.

M. Cho and W. Xu, “Precisely Verifying the Null Space Conditions in Compressed Sensing: A Sandwiching Algorithm,” 2013, [Online] Available: http://arxiv.org/abs/1306.2665

**Andreas M. Tillmann** graduated in financial and business mathematics (Dipl.-Math. Occ. degree) from the TU Braunschweig, Germany, in 2009. As a doctoral candidate at the TU Braunschweig, he worked as a Teaching Assistant from 2009 to 2011, and was a Researcher on the project “SPEAR – Sparse Exact and Approximate Recovery”, funded by the Deutsche Forschungsgemeinschaft (DFG), from 03/2011 to 06/2013. In 07/2012 he has joined the Research Group Optimization at TU Darmstadt, Germany; in 07/2013 he resumed a TA position there. His research interests are mainly in compressed sensing, focussing on exact ℓq-minimization, basis pursuit, and issues related to recovery conditions.

**Marc E. Pfetsch** obtained a Diploma degree in mathematics from the University of Heidelberg, Germany, in 1997. He received a PhD degree in mathematics in 2002 and the Habilitation degree in 2008 from TU Berlin, Germany. From 2008 to 2012 he was a Full Professor for Mathematical Optimization at TU Braunschweig, Germany. Since 04/2012 he has been a Full Professor for Discrete Optimization at TU Darmstadt, Germany. His research interests are mostly in discrete optimization, in particular symmetry in integer programs, compressed sensing, and algorithms for mixed integer programs.