Contractibility of vector-valued Köthe echelon algebras

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Abstract
We give a characterization of those vector-valued Köthe echelon algebras which are contractible.

Keywords Fréchet space/algebra · Köthe echelon space/algebra · Bimodule · Contractible algebra

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1 Introduction

Let \( S \) be some sequence (function) space, i.e. a topological vector space of sequences (functions) in which convergence in the original topology implies the coordinate-wise (point-wise) convergence and let \( X \) be some topological vector space such that the space \( S(X) \) of \( X \)-valued sequences has a canonical definition. This is the case if e.g. \( S \) and \( X \) are Banach or Fréchet spaces. Assume that \( X \) satisfies some property \( P \). It is a natural question to ask whether or not \( S(X) \) has the same property too. The specific property \( P \) that we are concerned with will be amenability/contractibility of an algebra. Such problems have already been considered in [4, 14] where the authors study the relation between amenability properties of some Banach algebra \( A \) and the algebra \( C(\Omega, A) \) of \( A \)-valued continuous functions on some compact Hausdorff space \( \Omega \).

In this paper, we will study the relation between contractibility of some Fréchet algebra \( A \) and a Köthe echelon algebra \( \lambda_p(A) \) on the one hand and the \( A \)-valued Köthe echelon algebra \( \lambda_p(A, A) \) on the other (see the next section for definitions). Recall that in the Banach algebra category contractibility seems to be a much too strong property. Indeed, if a contractible Banach algebra possesses the approximation...
property then it has to be finite dimensional—see [13, Theorem 4.1.2]. In the Fréchet algebra category the situation is different and there exist infinite dimensional contractible Fréchet algebras (even with a Schauder basis)—see [12, Theorem 14].

The next section collects all the necessary definitions and notation together with some well-known results while the last section contains the main result. For issues that are not explained here we refer the reader to [7, 8] (functional analysis), [3] (topological algebra theory), [5] (homological methods).

2 Notation and preliminaries

We start by recalling some facts and definitions that will be used in the paper. It will be enough to restrict ourselves to the category of Fréchet spaces. Let $\mathcal{A}$ be a Fréchet algebra and let $m$ be the multiplication in $\mathcal{A}$. The product map is denoted by $\pi$ and it is the unique linearization of $m$, i.e. $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}$ and it is defined on elementary tensors as

$$\pi(a \otimes b) := ab.$$ 

Let now $P$ be a Fréchet $\mathcal{A}$-bimodule. We say it is projective (see [3, Definition 2.8.35]) if for all Fréchet $\mathcal{A}$-bimodules $Y, Z$, every admissible epimorphism $T \in \mathcal{A}L_A(Y, Z)$ and every $S \in \mathcal{A}L_A(P, Z)$ there exists $R \in \mathcal{A}L_A(P, Y)$ such that $T \circ R = S$, i.e. the diagram

$$\begin{array}{ccc}
P & \xrightarrow{R} & S \\
\downarrow & & \downarrow \\
Y & \xrightarrow{T} & Z
\end{array}$$

is commutative. A Fréchet algebra $\mathcal{A}$ is biprojective if it is a projective $\mathcal{A}$-bimodule. We will also need an equivalent formulation.

**Proposition 2.1** [3, Proposition 2.8.41] A Fréchet algebra $\mathcal{A}$ is biprojective if and only if the product map is a retract, i.e. there exists a bimodule map $\sigma : \mathcal{A} \to \mathcal{A} \hat{\otimes} \mathcal{A}$ which is a right inverse to $\pi$.

The following result is crucial for our definition.

**Theorem 2.2** [3, Theorem 2.8.48] Let $\mathcal{A}$ be a non-zero Fréchet algebra. Then the following conditions are equivalent:

(i) every continuous derivation from $\mathcal{A}$ into any Fréchet $\mathcal{A}$-bimodule is inner,
(ii) $\mathcal{A}$ is unital and biprojective,
(iii) $\mathcal{A}$ is unital and has a projective diagonal in $\mathcal{A} \hat{\otimes} \mathcal{A}$, i.e. an element $d \in \mathcal{A} \hat{\otimes} \mathcal{A}$ such that

\[ \pi(a \otimes b) = ab. \]
\[ \pi(d) = 1, \quad a \cdot d = d \cdot a(a \in A). \]

A Fréchet algebra is \textit{contractible}—see [6, Definition VII.1.59]—if it satisfies any of the conditions in Theorem 2.2.

Throughout the article we denote by \( \mathbb{N} := \{1, 2, 3, \ldots\} \) the set of natural numbers and by \( \mathbb{K} \) the field of scalars which is either \( \mathbb{R} \) or \( \mathbb{C} \). The sequence of standard unit vectors is denoted by \((e_j)_{j \in \mathbb{N}}\). Let \((\xi_j)_{j \in \mathbb{N}}\) be a sequence of non-negative numbers. If there is some \( n \in \mathbb{N} \) such that \( \xi_j > 0 \) for all \( j \geq n \) then we say that \( \xi_j > 0 \) for large \( j \in \mathbb{N} \).

A matrix \( A := (a_n(j))_{n,j \in \mathbb{N}} \) of non-negative numbers is a Köthe matrix if

(i) \( \forall j \in \mathbb{N} \exists n \in \mathbb{N} : a_n(j) > 0, \)

(ii) \( \forall j, n \in \mathbb{N} : a_n(j) \leq a_{n+1}(j). \)

A Köthe matrix \( A := (a_n(j))_{n,j \in \mathbb{N}} \) is called \textit{bounded} if \( a_n \in \ell_\infty \) for all \( n \in \mathbb{N} \), i.e. if all the weights \( a_n \) are bounded.

For a Köthe matrix \( A \) the Köthe echelon space of order \( p \), \( \lambda_p(A) \) is the space of scalar-valued sequences defined as

\[ \lambda_p(A) := \{ \xi \in \mathbb{K}^\mathbb{N} : \|\xi\|_{n,p} := \| (\xi_j, a_n(j))_{j \in \mathbb{N}} \|_{\ell_p} < \infty \text{ for all } n \in \mathbb{N} \} \]

if \( 1 \leq p \leq \infty \) and

\[ \lambda_0(A) := \{ \xi \in \mathbb{K}^\mathbb{N} : \lim_{j \to \infty} \xi_j a_n(j) = 0 \text{ for all } n \in \mathbb{N} \}. \]

With the topology defined by the sequence of seminorms \( (\| \cdot \|_{n,p})_{n \in \mathbb{N}} \) they become Fréchet spaces. Clearly, \( \lambda_0(A) \) is considered with the topology inherited from \( \lambda_\infty(A) \). Equivalently—at least in the case when all the entries of the Köthe matrix are positive—one may say that \( \lambda_p(A) \) is the intersection of Banach spaces \( \ell_p(a_n) \) (resp. \( c_0(a_n) \)) endowed with the weakest locally convex topology under which the natural injections \( \lambda_p(A) \hookrightarrow \ell_p(a_n) \) are continuous for all \( n \in \mathbb{N} \). A thorough investigation of Köthe echelon spaces can be found in [1] or [8, Chapter 27]. Observe that \((e_j)_{j \in \mathbb{N}}\) is a Schauder basis in \( \lambda_p(A) \) if \( p \) is finite (this includes also \( p = 0 \)).

In this paper we will be focused on those Köthe echelon spaces which are Fréchet algebras with respect to the coordinate-wise multiplication—see [2] for an expository account. A Köthe echelon space is a Fréchet algebra if and only if

\[ \forall n \in \mathbb{N} \exists k \in \mathbb{N} : a_n / a_k^2 \in \ell_\infty. \]  

Without loss of generality we may assume that \( k \geq n \), if necessary. Amenability/contractibility of these algebras has been characterized by Pirkovskii [9] and the author [10–12]. Below we recall those results.

\textbf{Theorem 2.3} [2, Theorems 8.2 and 8.4] Let \( 1 \leq p \leq \infty \) or \( p = 0 \) and let \( \lambda_p(A) \) be a Köthe echelon algebra.
(1) If $1 \leq p < \infty$ then the following conditions are equivalent:
   (i) $\lambda_p(A)$ is amenable,
   (ii) $\lambda_p(A)$ is contractible,
   (iii) $\lambda_p(A)$ is unital,
   (iv) $\lambda_p(A)$ is nuclear and $A$ is bounded.

(2) If $p = 0, \infty$ then the following conditions are equivalent:
   (i) $\lambda_0(A)$ is contractible,
   (ii) $\lambda_\infty(A)$ is contractible,
   (iii) $\lambda_0(A)$ is unital.

(3) If $p = 0, \infty$ then the following conditions are equivalent:
   (i) $\lambda_0(A)$ is amenable,
   (ii) $\lambda_\infty(A)$ is amenable,
   (iii) $\lambda_\infty(A)$ is unital,
   (iv) $A$ is bounded.

Let now $\lambda_p(A)$ be a Köthe echelon space and $X$ some Fréchet space. The vector-valued Köthe echelon space $\lambda_p(A, X)$ is defined as

$$\lambda_p(A, X) := \{ x = (x_j)_{j \in \mathbb{N}} \subseteq X : (\|x_j\|_n a_n(j))_{j \in \mathbb{N}} \in \ell_p \quad \forall n \in \mathbb{N} \}$$

if $1 \leq p \leq \infty$ and

$$\lambda_0(A, X) := \{ x = (x_j)_{j \in \mathbb{N}} \subseteq X : (\|x_j\|_n a_n(j))_{j \in \mathbb{N}} \in c_0 \quad \forall n \in \mathbb{N} \}.$$

It comes equipped with the Fréchet space topology given by the sequence of semi-norms

$$\|x\|_{n,p} := \|(\|x_j\|_n a_n(j))_{j \in \mathbb{N}}\|_{\ell_p} \quad (x \in \lambda_p(A, X)).$$

Clearly, $\lambda_0(A, X)$ is a Fréchet subspace of $\lambda_\infty(A, X)$. If $X$ and $\lambda_p(A)$ are Fréchet algebras then $\lambda_p(A, X)$ is a Fréchet algebra with coordinate-wise multiplication. The product map is clearly defined as $\pi_{\lambda_p(A, X)} := \pi_{\lambda_p(A)} \otimes \pi_X$. For notational convenience we let $x \otimes e_j$ stand for the sequence in $\lambda_p(A, X)$ which has $x$ on the $j$-th coordinate and zeros elsewhere. Observe that if $p$ is finite (this includes $p = 0$) then

$$x = \sum_{j=1}^\infty x_j \otimes e_j \quad (x \in \lambda_p(A, X)).$$

3 Main result

We are now in the position to state the main result of the paper.
**Theorem 3.1** Let $1 \leq p < \infty$ or $p = 0$, let $\lambda_p(A)$ be a Köthe echelon algebra and let $A$ be a Fréchet algebra. The vector-valued Köthe echelon algebra $\lambda_p(A, A)$ is contractible if and only if $\lambda_p(A)$ and $A$ are both contractible.

The proof splits into two separate cases depending on whether we have $1 \leq p < \infty$ or $p = 0$. Therefore the main theorem will be a simple consequence of the following two results.

**Proposition 3.2** Let $1 \leq p < \infty$, let $\lambda_p(A)$ be Köthe echelon algebra and let $A$ be a Fréchet algebra. Then $\lambda_p(A, A)$ is contractible if and only if $\lambda_p(A)$ and $A$ are both contractible.

**Proof** *Necessity.* The mapping $\theta : \lambda_p(A, A) \to A$. $\theta((x_j)_j) := x_1$, is a surjective Fréchet algebra homomorphism therefore contractibility of $A$ follows from [3, Proposition 2.8.64]. Let now $(u_j)_{j \in \mathbb{N}}$ be the unit in $\lambda_p(A, A)$. Then for any $k \in \mathbb{N}$ we have

$$(x \otimes e_k)(u_j)_j = x u_k \otimes e_k = x \otimes e_k = u_k x \otimes e_k = (u_j)_j(x \otimes e_k) \quad (x \in \lambda_p(A, A)).$$

Consequently, $u := u_j = u_i$ for all $i, j \in \mathbb{N}$ and it is the unit in $A$. This implies that $(u_j)_j = \sum_{j=1}^{\infty} u \otimes e_j$ and

$$\|(u_j)_j\|_{n,p} = \|u\|_n \|1\|_{n,p} < \infty \quad (n \in \mathbb{N})$$

where we have denoted $1 := \sum_{j=1}^{\infty} e_j$. Since $\|u\|_n > 0$ for large $n$, we obtain

$$\|1\|_{n,p} < \infty \quad (\text{large } n).$$

Hence $\lambda_p(A)$ is unital thus contractible by Theorem 2.3.

*Sufficiency.* From Theorem 2.3 if follows that $\lambda_p(A)$ is nuclear therefore [8, Proposition 28.16] implies that $\lambda_p(A) = \lambda_1(A)$ and, consequently, $\lambda_p(A, A) = \lambda_1(A, A)$. From [7, Corollary 15.7.2] it now follows that $\lambda_1(A, A)$ and $\lambda_1(A) \hat{\otimes} A$ are isomorphic as Fréchet algebras. Therefore

$$\lambda_1(A) \hat{\otimes} \lambda_1(A) \hat{\otimes} A \hat{\otimes} A = \lambda_1(A, A) \hat{\otimes} \lambda_1(A, A)$$

as well. Let now

$$\sigma_1 : \lambda_1(A) \to \lambda_1(A) \hat{\otimes} \lambda_1(A), \quad \sigma : A \to A \hat{\otimes} A$$

be the right inverse bimodule maps to $\pi_1$ and $\pi$, respectively. The assignment

$$\xi \otimes x \mapsto \sigma_1(\xi) \otimes \sigma(x)$$

gives rise to a continuous map

\[ \text{Birkhäuser} \]
\[ \sigma_1 \otimes \sigma : \lambda_1(A) \hat{\otimes} \mathcal{A} \to \lambda_1(A) \hat{\otimes} \lambda_1(A) \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A}. \]

This is clearly a bimodule map since for elementary tensors we have
\[
(\eta_1 \otimes y_1)(\sigma_1 \otimes \sigma(\xi \otimes x))(\eta_2 \otimes y_2) = \eta_1 \sigma_1(\xi) \eta_2 \otimes y_1 \sigma(x) y_2
\]

Moreover,
\[
\sigma_1(\eta_1 \xi \eta_2 \otimes y_1 \sigma y_2)
\]

Thus \( \sigma_1 \otimes \sigma \) is a right inverse bimodule map and \( \lambda_1(\mathcal{A}, \mathcal{A}) \) is biprojective.

By assumption \( \lambda_1(\mathcal{A}) \) and \( \mathcal{A} \) are unital with units, say \( \mathbf{1} \) and \( u \), respectively. Then
\[
\sum_{j=1}^{\infty} u \otimes e_j = u \otimes 1
\]
is the unit in \( \lambda_p(\mathcal{A}, \mathcal{A}) \). Consequently, \( \lambda_p(\mathcal{A}, \mathcal{A}) \) is contractible. \( \square \)

**Proposition 3.3** Let \( \lambda_0(\mathcal{A}) \) be a Köthe echelon algebra and let \( \mathcal{A} \) be a Fréchet algebra. The following conditions are equivalent:

(i) \( \lambda_0(\mathcal{A}, \mathcal{A}) \) is contractible,

(ii) \( \lambda_0(\mathcal{A}) \) is contractible and \( \mathcal{A} \) is contractible.

**Proof** (i)\( \Rightarrow \) (ii): The mapping
\[
\theta : \lambda_0(\mathcal{A}, \mathcal{A}) \to \mathcal{A}, \quad \theta((x_j)_j) : = x_1,
\]
is a surjective Fréchet algebra homomorphism therefore contractibility of \( \mathcal{A} \) follows from [3, Proposition 2.8.64]. Observe now that \( u \otimes 1 = \sum_{j=1}^{\infty} u \otimes e_j \) (\( u \)—the unit in \( \mathcal{A} \)) is the unit in \( \lambda_0(\mathcal{A}, \mathcal{A}) \). Therefore for every \( n \in \mathbb{N} \) we have
\[
\|u\|_n a_n(j) \xrightarrow{j \to \infty} 0.
\]
Since \( \|u\|_n > 0 \) for large \( n \) this implies that \( 1 \in \lambda_0(\mathcal{A}) \). From Theorem 2.3 it now follows that \( \lambda_0(\mathcal{A}) \) is contractible.

(ii)\( \Rightarrow \) (i): Let \( u \) be the unit in \( \mathcal{A} \). Then \( u \otimes 1 = \sum_{j=1}^{\infty} u \otimes e_j \) is the unit in \( \lambda_0(\mathcal{A}, \mathcal{A}) \) and \( \|u \otimes 1\|_{n,\infty} = \|u\|_n \|1\|_{n,\infty} \). It remains to show that there is a projective diagonal in \( \lambda_0(\mathcal{A}, \mathcal{A}) \hat{\otimes} \lambda_0(\mathcal{A}, \mathcal{A}) \). To this end, let the mapping \( i : \mathcal{A} \hat{\otimes} \mathcal{A} \to \lambda_0(\mathcal{A}, \mathcal{A}) \hat{\otimes} \lambda_0(\mathcal{A}, \mathcal{A}) \) be defined as...
We need to show that \( \iota \) is well-defined and continuous. Indeed, let \( a, b \in \mathcal{A} \) and \( k \in \mathbb{N} \). By the Rademacher averaging we get

\[
\| \iota(a \otimes b) \|_{k,\infty} = \left\| \sum_{j=1}^{\infty} (a \otimes e_j) \otimes (b \otimes e_j) \right\|_{k,\infty}
= \left\| \int_{0}^{1} \left( \sum_{j=1}^{\infty} r_j(t)a \otimes e_j \right) \otimes \left( \sum_{j=1}^{\infty} r_j(t)b \otimes e_j \right) dt \right\|_{k,\infty}
\leq \sup_{r \in [0,1]} \left\| \left( \sum_{j=1}^{\infty} r_j(t)a \otimes e_j \right) \right\|_{k,\infty} \left\| \left( \sum_{j=1}^{\infty} r_j(t)b \otimes e_j \right) \right\|_{k,\infty}
\leq \left\| \left( \sum_{j=1}^{\infty} a \otimes e_j \right) \right\|_{k,\infty} \left\| \left( \sum_{j=1}^{\infty} b \otimes e_j \right) \right\|_{k,\infty}
= \|1\|_{k,\infty}^{2} \|a\|_{k} \|b\|_{k}.
\]

Thus for any \( \varepsilon > 0 \) and any element \( v = \sum_{n=1}^{\infty} a_n \otimes b_n \) satisfying

\[
\sum_{n=1}^{\infty} \|a_n\|_{k} \|b_n\|_{k} < \|v\|_{\mathcal{A} \hat{\otimes} \mathcal{A}} + \varepsilon
\]
we obtain

\[
\|\iota(v)\|_{k} \leq \|1\|_{k,\infty}^{2} \sum_{n=1}^{\infty} \|a_n\|_{k} \|b_n\|_{k} < \|1\|_{k,\infty}^{2} (\|v\|_{\mathcal{A} \hat{\otimes} \mathcal{A}} + \varepsilon).
\]

Hence

\[
\|\iota(v)\|_{k} \leq \|1\|_{k,\infty}^{2} \|v\|_{\mathcal{A} \hat{\otimes} \mathcal{A}} \quad (v \in \mathcal{A} \hat{\otimes} \mathcal{A}).
\]

Consequently, \( \iota \) is well-defined and continuous. Let now

\[
d := \sum_{n=1}^{\infty} x_n \otimes y_n \in \mathcal{A} \hat{\otimes} \mathcal{A}
\]

be a projective diagonal. We will show that \( \iota(d) \) is a projective diagonal in \( \lambda_{0}(\mathcal{A} \hat{\otimes} \mathcal{A}) \). To this end, let \( (a_j)_{j \in \mathbb{N}} \in \lambda_{0}(\mathcal{A}) \) be given. Then

\[
(a_j)_{j} \cdot \iota(d) = \sum_{n,j=1}^{\infty} (a_j x_n \otimes e_j) \otimes (y_n \otimes e_j)
\]

and

\[
\iota(d) \cdot (a_j)_{j} = \sum_{n,j=1}^{\infty} (x_n \otimes e_j) \otimes (y_n a_j \otimes e_j).
\]
We claim that these are equal. Indeed, let $B \in (\lambda_0(A, A) \otimes \lambda_0(A, A)) = B(\lambda_0(A, A) \times \lambda_0(A, A))$ be a given continuous bilinear form. We define

$$B_j : A \times A \to \mathbb{K}, \quad B_j(a, b) := B(a \otimes e_j, b \otimes e_j) \quad (j \in \mathbb{N}).$$

Clearly, each $B_j, j \in \mathbb{N}$ is continuous and bilinear. Since $d \in A \otimes A$ is a projective diagonal, for any $j \in \mathbb{N}$ we obtain

Consequently,

$$B \left( (a_j) \cdot \iota(d) \right) = B \left( \iota(d) \cdot (a_j) \right)$$

for every continuous bilinear form $B$. By Hahn–Banach we finally get

$$(a_j) \cdot \iota(d) = \iota(d) \cdot (a_j) \quad ((a_j) \in \lambda_0(A, A)).$$

Moreover,

$$\pi \circ \iota(d) = \sum_{j,n=1}^{\infty} x_n \otimes y_n \otimes e_j = \sum_{j=1}^{\infty} u \otimes e_j$$

is the unit in $\lambda_0(A, A)$. Therefore $\iota(d)$ is a projective diagonal in the space $\lambda_0(A, A) \otimes \lambda_0(A, A)$. From Theorem 2.2 it now follows that $\lambda_0(A, A) = \lambda_0(A, A)$ is contractible and the proof is thereby complete.

We end this section with a remark on the case $p = \infty$. From Theorem 2.3 it follows that if the algebra $\lambda_\infty(A)$ is contractible then in particular, $\lambda_\infty(A) = \lambda_0(A)$. Therefore we can mimic the proof of Proposition 3.3 to get that contractibility of $\lambda_\infty(A)$ and $A$ implies that of $\lambda_\infty(A, A)$. It is also clear that contractibility of $\lambda_\infty(A, A)$ implies that of $A$. Therefore it is tempting to say that contractibility of $\lambda_\infty(A)$ should follow from that of $\lambda_\infty(A, A)$. Since we do not have a proof of the last claim we state the following conjecture.

Conjecture 3.4 Let $\lambda_\infty(A)$ be a Köthe echelon algebra and let $A$ be a Fréchet algebra. The following conditions are equivalent:

\begin{enumerate}
\item $\lambda_\infty(A, A)$ is contractible,
\item $\lambda_\infty(A)$ is contractible and $A$ is contractible.
\end{enumerate}
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