The residue formula and the Tolman-Weitsman theorem

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Abstract

We give a simple direct proof (for the case of Hamiltonian circle actions with isolated fixed points) that Tolman and Weitsman’s description of the kernel of the Kirwan map in [9] (in other words the sum of those equivariant cohomology classes vanishing on one side of a collection of hyperplanes) is equivalent to the characterization of this kernel given by the residue theorem [6].

1 Introduction

Let $M$ be a symplectic manifold of dimension $2n$ equipped with the Hamiltonian action of a torus $T$. The $T$-equivariant cohomology of $M$ is $H^*_T(M)$. This is a module over $H^*_T(pt)$. The moment map for the torus action is denoted by $\mu : M \to \mathfrak{t}^*$. We denote by $M_c$ the reduced space at a value $c$ of the moment map:

$$M_c \overset{\text{def}}{=} \mu^{-1}(c)/T.$$ 

The Kirwan map is the map

$$\kappa : H^*_T(M) \to H^*(M_c)$$

induced by the restriction map

$$H^*_T(M) \to H^*_T(\mu^{-1}(c)).$$

Provided $c$ is a regular value of $\mu$ (which we shall assume) we have that $H^*_T(\mu^{-1}(c)) \cong H^*(M_c)$.

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1In this paper all cohomology groups are with complex coefficients.
Kirwan proved in [8] that the Kirwan map is surjective. Thus to find the cohomology ring of $M_c$ it suffices to find the kernel $K$ of $\kappa$:

$$H^*(M_c) \cong H^*_T(M)/K.$$  

We denote by $\mathcal{F}$ the set of components of the fixed point set of $T$. We define a distinguished subset $\mathcal{F}_+$ of $\mathcal{F}$: in the case when $T = S^1$ it is simply

$$\mathcal{F}_+ = \{ F \in \mathcal{F} | \mu(F) > c \}.$$

In the case $T = S^1$ we also introduce

$$\mathcal{F}_- = \{ F \in \mathcal{F} | \mu(F) < c \}.$$

Because $c$ is a regular value of $\mu$, there are no $F$ with $\mu(F) = 0$.

In [6] the following is proved:

**Theorem 1.1** Let $\eta \in H^*_T(M)$. Then

$$\kappa(\eta)[M_c] = \sum_{F \in \mathcal{F}_+} \text{Res} \left( \frac{\eta}{e_F} \right) [F].$$

Here $e_F$ is the equivariant Euler class of the normal bundle to $F$, and $\text{Res}$ is an iterated residue. In the case when $T$ has rank one it is simply $\text{Res}_{X=0}$ where the variable $X$ is the generator of $H^*_T(pt)$.

Since $\kappa$ is a ring homomorphism, we have from Theorem 1.1

**Corollary 1.2**

$$\kappa(\eta)\kappa(\zeta)[M_c] = \sum_{F \in \mathcal{F}_+} \text{Res} \left( \frac{\eta\zeta}{e_F} \right) [F]. \quad (1.1)$$

Assuming $c$ is a regular value of $\mu$, $M_c$ has at worst orbifold singularities and so satisfies Poincaré duality. Thus according to Theorem 1.1, an element $\eta \in H^*_T(M)$ is in $K$ if and only if

$$\sum_{F \in \mathcal{F}_+} \text{Res} \left( \frac{\eta\zeta}{e_F} \right) [F] = 0$$

for all $\zeta \in H^*_T(M)$.

In [9] Tolman and Weitsman give an alternative characterization of the kernel $K$:

**Theorem 1.3** The kernel $K$ of the Kirwan map is the sum

$$K = \sum_{\xi \in t} K_\xi$$

where $K_\xi$ is the set of cohomology classes vanishing on all components $F$ of the fixed point set for which $(\mu(F) - c, \xi) > 0$.

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2For more details see Section 8 of [6]; see also [7] and [5].
In the special case $T = S^1$ the Tolman-Weitsman theorem reads as follows:

**Corollary 1.4** Let $T = S^1$. Then the kernel $K$ of the Kirwan map is the sum

$$K = K_+ \oplus K_-$$

where

$$K_\pm = \{ \eta \in H^*_T(M) \mid \eta[F] = 0 \text{ for all } F \in \mathcal{F} \text{ for which } \pm \mu(F) > c \}. $$

Tolman and Weitsman prove their theorem by establishing directly that $\sum \xi K_\xi$ is the kernel of the Kirwan map $\kappa$, and that all elements of this kernel take this form.

In this paper we give a direct proof in the case $T = S^1$ that the classes satisfying the condition that the residue on the right hand side of (1.1) is zero are contained in the Tolman-Weitsman kernel $\oplus \xi K_\xi$. The reverse inclusion is clear from the definition of the residue: if a class $\eta \in H^*_T(M)$ satisfies $\eta|_F = 0$ for $F \in \mathcal{F}_+$, then clearly

$$\sum_{F \in \mathcal{F}_+} \text{Res}_{X=0} \left( \frac{\eta \xi}{\varepsilon_F} \right) [F] = 0 \tag{1.2}$$

for all $\xi \in H^*_T(M)$. Similarly if $\eta$ satisfies $\eta|_F = 0$ for all $F \in \mathcal{F}_-$ then the residue in (1.2) is equal to zero. Thus we have shown that the description of the kernel of the Kirwan map given by the residue theorem is equivalent to the description given by Tolman and Weitsman. We expect that our methods could be enlarged to treat the case of general torus actions.

## 2 Results from Morse theory

We begin by stating a number of fundamental results related to equivariant Morse theory and the Thom-Gysin map. These results are found in the work of Atiyah and Bott [1] and Kirwan [3]: the formulation we present appears in the work of Goldin [4]. We introduce a partial ordering $<$ on the components of the critical set of a Morse function $f$: denote these components by $F_i$. The ordering is given by the value of the Morse function $f$: in other words $i > j$ if and only if $f(F_i) > f(F_j)$.

In our situation the most important Morse function $f$ is $-\mu_\xi$, where $\mu$ is the moment map and $\mu_\xi = (\mu, \xi)$ is the component of the moment map in the direction $\xi$ for nonzero values $\xi \in t$. This is an equivariantly perfect Morse function (as proved by Atiyah-Bott and Kirwan [1, 3]).

We denote by $\text{Ind}(F)$ the Morse index, in other words the number of negative eigenvalues of the Hessian of $f$ on the critical set $F$.

In this paper we assume for simplicity that all components $F$ of the critical set of $f$ are isolated (so $f$ is a Morse function rather than a Morse-Bott function).

**Theorem 2.1** Suppose $\eta \in H^*_T(M)$ restricts to 0 on all $F_i$ for which $i < j$. Then $\eta|_{F_j}$ is some multiple of $e\left(\nu_i^* F_j\right)$, the equivariant Euler class of the negative normal bundle of $F_j$. 

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Remark 2.2 An important consequence of Theorem 2.1 is that if \( \eta \) restricts to 0 on all \( F_i \) with \( i < j \) then either \( \eta|_{F_j} = 0 \) or the degree of \( \eta \) is greater than or equal to the degree of \( e(\nu^-(F_j)) \).

Definition 2.3 We say that an equivariant cohomology class \( \zeta \in H^*_S(M) \) is supported on a subset \( S \) of the critical set of \( f \) if \( \zeta|_G = 0 \) for all \( G \) in the critical set for which \( G \notin S \).

Definition 2.4 Let \( F \) be a component of the critical set of \( f \). The equivariant stable manifold of \( F \) under \( f \) is the set of points \( x \in M \) for which there is a trajectory which converges to \( F \) under the flow of \(-\text{grad}(f)\).

Definition 2.5 Let \( F \) be as in Definition 2.4. The equivariant extended stable manifold of \( F \) under \( f \) is the set of points \( x \in M \) which have a sequence of trajectories of the flow of \(-\text{grad}(f)\) (passing through a number of components \( F_1, \ldots, F_l \) of the critical set of \( f \)) starting at \( x \) and converging finally to \( F \).

Definition 2.6 Let \( F \) be as in Definition 2.4. The unstable manifold and equivariant extended unstable manifold of \( F \) are specified as in Definitions 2.4 and 2.5 but using the gradient flow of \( \text{grad}(f) \) rather than that of \(-\text{grad}(f)\).

Theorem 2.7 Suppose \( F \) is a connected component of the critical set of a Morse function \( f \). Then there is a class \( \alpha^-(F) \) with the following properties:

1. \( \alpha^-(F)|_G = 0 \) if \( G \) is a component of the critical set of \( f \) which is not in the equivariant extended stable manifold of \( F \)
2. \( \alpha^-(F)|_F = e(\nu_F^- F) \) where \( e(\nu_F^- F) \) is the equivariant Euler class of the negative normal bundle (defined by \( f \)) of \( F \). Thus the degree of \( \alpha^-(F) \) is \( \text{Ind}(F) \).

In the same way there is a class \( \alpha^+(F) \) such that

1. \( \alpha^+(F)|_G = 0 \) if \( G \) is a component of the critical set of \( f \) which is not in the equivariant extended unstable manifold of \( F \)
2. \( \alpha^+(F)|_F = e(\nu_F^+ F) \) where \( e(\nu_F^+ F) \) is the equivariant Euler class of the positive normal bundle (defined by \( f \)) of \( F \)

The degree of \( \alpha^+(F) \) is \( 2n - \text{Ind}(F) \).
3 The $S^1$ case

We now assume that the torus $T$ is the circle group $S^1$, and also that the fixed points of the action of $T$ are isolated. Suppose we are forming the reduced space at a regular value $c$ of the moment map. The purpose of this paper is to give a direct proof of the following result.

**Theorem 3.1** Let $\eta \in H^*_T(M)$ satisfy the hypothesis that

$$\sum_{F \in \mathcal{F}_+} \text{Res}_{X=0} \left( \eta \frac{\zeta}{e_F} \right) [F] = 0$$

for all $\zeta \in H^*_T(M)$. Then $\eta \in K_+ \oplus K_-$. 

**Proof of Theorem 3.1:** Let $\eta \in H^d_{S^1}(M)$, in other words $d$ is the degree of $\eta$.

**Lemma 3.2** We can write

$$\eta = \eta_+ + \eta_- \quad (3.3)$$

where

$$\eta_- = \sum_{F \in \mathcal{F}_-, \text{Ind}(F) \leq d} c_F X^{n_F} \alpha^-(F) \quad (3.4)$$

while

$$\eta_+ = \sum_{F \in \mathcal{F}_+, \text{Ind}(F) \leq d} c_F X^{n_F} \alpha^-(F) \quad (3.5)$$

and

$$2n_F = d - \text{Ind}(F). \quad (3.6)$$

Here, the $c_F \in \mathbb{C}$ are constants.

**Proof:** This is straightforward. We simply have to identify the coefficients of the restrictions to components $F$ of the fixed point set. We find that we can adjust the coefficients in such a way that both sides have the same restriction to all components of the fixed point set, which by Kirwan’s injectivity theorem [8] (as well as Remark 2.2) guarantees that the two are equal in equivariant cohomology.

We define the constants $c_F$ by induction using $\alpha^-(F)|_F = e(\nu_F F)$. Adding $c_F X^{n_F} \alpha^-(F)$ will alter only the values of the restrictions to those elements $G \in \mathcal{F}$ for which $\mu(G) \geq \mu(F)$. □

**Lemma 3.3** We have that

$$\eta_+ \in K_-.$$ 

**Proof:** For $F \in \mathcal{F}_+$, $\alpha^-(F)$ is supported on $\mathcal{F}_+$ (since if $G$ is a component of $\mathcal{F}$ for which $\alpha(F)|_G \neq 0$, $G$ is in the equivariant extended stable manifold of $F$ so $\mu(G) \geq \mu(F))$. □
**Lemma 3.4** We may add terms 

$$b_F X^{n_F} \alpha^-(F)$$

for $F \in \mathcal{F}_+$, in such a way that

$$\eta_- - \sum_{F \in \mathcal{F}_+} b_F X^{n_F} \alpha^-(F)$$

restricts to zero on $G$ for all $G \in \mathcal{F}_+$ for which $\text{Ind}(G) \leq d$. Here, $n_F$ is as in (3.4).

**Proof:** We choose the coefficients $b_F$ by induction on the value of $\mu(F)$, first choosing $b_F$ for the lowest value of $\mu(F)$ for $F \in \mathcal{F}_+$. This will alter only the value of the restrictions to those $G$ for which $\mu(G) \geq \mu(F)$. This process can be used to make the values of $\eta_-(G) = 0$ for all $G$ for which $\mu(G) > c$ and $\text{Ind}(G) \leq d$. $\square$

As described in the proof of Lemma 3.3, for $G \in \mathcal{F}_+$, $\alpha^-(G)$ is supported on $\mathcal{F}_+$. So we have the following corollary:

**Corollary 3.5** Suppose $G \in \mathcal{F}_+$. Then we see that $\alpha^-(G) \in K_-$ and thus

$$\text{Res}_{X=0} \sum_{G \in \mathcal{F}_-} \frac{\alpha_-(G)|_G \zeta|_G}{e_G} = 0$$

(3.7)

for any class $\zeta \in H^*_T(M)$. Therefore also

$$\text{Res}_{X=0} \sum_{G \in \mathcal{F}_+} \frac{\alpha_-(G)|_G \zeta|_G}{e_G} = 0,$$

(3.8)

since the sum of residues for $F \in \mathcal{F}_-$ is equal to minus the sum of residues for $F \in \mathcal{F}_+$ for any element of $H^*_T(M)$. Because

$$\sum_{G \in \mathcal{F}_+ \cup \mathcal{F}_-} \frac{\alpha_-(G)|_G \zeta|_G}{e_G}$$

is a smooth function of $X$ (by the localization theorem in equivariant cohomology [\cite{3}], [\cite{4}]), it does not have any residue at $X = 0$, so (3.7) implies (3.8).

**Lemma 3.6** Write

$$\eta_-(F) = a_F X^{d/2}$$

for some $a_F \in \mathbb{C}$. If $\eta_-(F) = 0$ for all $F \in \mathcal{F}_+$ for which $\text{Ind}(F) \leq d$, then also $\eta_-(F) = 0$ for all $F \in \mathcal{F}_+$. In other words $\eta_- \in K_+$.

**Remark 3.7** The hypothesis of Lemma 3.6 can be achieved because of Lemma 3.4.
Proof of Lemma 3.6:

By Corollary 3.5 we may assume

\[ \text{Res}_{X=0} \sum_{G \in F_+} \frac{\eta \mid_G \zeta \mid_G}{e_G} = 0 \]

for all classes \( \zeta \). We put a partial order on the elements \( F \in F_+ \) according to the values of \( \mu(F) \), as described above. We choose

\[ \zeta = X^{m_F} \alpha_+(F) \]  

for

\[ 2n - \text{Ind}(F) + d \leq 2n - 2 \]  

and an appropriate nonnegative integer \( m_F \) (in other words

\[ \text{Ind}(F) \geq d + 2 : \]  

this hypothesis is needed in order to obtain terms with nonzero residue in the sum). We choose \( m_F \) so that \( \deg(\zeta \eta) = 2n - 2 \), in other words

\[ 2m_F + 2 + d = \text{Ind}(F). \]  

We allow \( \zeta \) to be determined by (3.9) and (3.12) for all \( F \) for which \( \text{Ind}(F) \geq d + 2 \). Notice that because the fixed point set consists of isolated points, Kirwan’s injectivity theorem implies that all \( \eta \in H^*_T(M) \) have even degree. Hence there are no components \( F \) for which

\[ \eta \mid_F \neq 0 \text{ and } \text{Ind}(F) = d + 1 \]  

(using (3.12)).

For all \( G \in F_+ \) we define \( b_{G}(F) \in \mathbb{C} \) by

\[ \alpha_+(F) \mid_G = b_G(F) X^{n - (\text{Ind}(F)/2)}. \]  

Note that \( b_G(G) \neq 0 \) since \( \alpha_+(G) \mid_G = \epsilon^+_G(G) \neq 0 \). We also write

\[ e_F \mid_F = \epsilon_F X^n \]  

for some nonzero \( \epsilon_F \in \mathbb{C} \). We define \( a_G \in \mathbb{C} \) by

\[ \eta \mid_G = a_G X^{d/2}. \]

The equation

\[ \text{Res}_{X=0} \sum_{G \in F_+} \frac{\eta \mid_G \zeta \mid_G}{e_G} = 0 \]

then becomes (combining (3.9), (3.13), (3.14) and (3.15))

\[ \sum_{G \in F_+} \frac{a_G b_G(F)}{e_G} = 0. \]  

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For the value of $F$ for which $\mu(F)$ is maximal, the equation (3.16) is simply
\[
\sum_{G \in \mathcal{F}_+} \frac{a_G}{e_G} = 0,
\]
(3.17)
since in this case the class $\alpha_+(F)$ has degree zero and is represented in the Cartan model by a constant in $C$, so the $b_G(F)$ all take the same value.

Notice that $b_G(F) = 0$ if $G$ is not in the equivariant extended unstable manifold of $\mu$ for $F$ (for example, if $\mu(G) > \mu(F)$ then $b_G(F) = 0$). Thus $b_G(F) \neq 0$ only for $G$ in the equivariant extended unstable manifold of $F$. By Lemma 3.4, we can also assume that $a_G = 0$ whenever $\text{Ind}(G) \leq d$. Also if $\text{Ind}(G) = d + 1$ then $a_G = 0$ since $\text{Ind}(G)$ must be odd so $\eta|_G = 0$ (by 3.12). Thus the residue formula implies that (3.16) holds for all $F$ for which $\text{Ind}(F) \geq d + 2$. By (3.10) and (3.11), we see that (3.16) becomes the equation
\[
\sum_{G \in \mathcal{F}_+, \text{Ind}(G) \geq d + 2} \frac{a_G}{e_G} b_G(F) = 0
\]
(3.18)
which is valid whenever $\text{Ind}(F) \geq d + 2$.

To deduce from this that $a_G = 0$ for all $G \in \mathcal{F}_+$, we think of (3.16) as a matrix equation involving an $N \times N$ matrix $B$ (where $N$ is the number of elements $F \in \mathcal{F}_+$ with $\text{Ind}(F) \geq d + 2$): the matrix is
\[
B_{FG} = b_G(F)
\]
for all $F$ and $G$ with $\text{Ind}(F) \geq d + 2$ and $\text{Ind}(G) \geq d + 2$. (Here the $F$ index the rows and $G$ index the columns of the matrix.) We wish to solve (3.16) for the vector $(a_1, \ldots, a_N)$. We note that $b_G(F) = 0$ if $\mu(G) > \mu(F)$ or $\mu(G) = \mu(F)$ and $G$ is not in the equivariant extended unstable manifold of $F$. Thus the matrix $B$ is upper triangular; furthermore its entries on the diagonal are nonzero. Thus the matrix is invertible, so the only solution to the equations (3.16) is $a_G = 0$ for all $G$. This completes the proof of Lemma 3.6. \hfill $\square$

Proof of Theorem 3.1: We now have completed the proof of the Theorem, since we have shown that $\eta$ can be written
\[
\eta = \eta_+ + \eta_-
\]
where it is possible to find $\eta_+ \in K_+$ and $\eta_- \in K_-$. \hfill $\square$

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