Dynamics of perturbations in Gurzadyan-Xue cosmological models

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Abstract. Perturbation theory within Newtonian approximation is presented for cosmological models with varying physical constants. Analytical solutions for perturbations dynamics are obtained for each Gurzadyan-Xue model with pressureless matter and radiation. We found that perturbations grow during entire expansion within GX models, including curvature- and vacuum-dominated stage when they cease to grow in the standard cosmological model.

1. Introduction

The formula for dark energy, derived by Gurzadyan and Xue predicts the observed value for the density parameter of the dark energy without any free parameters [1]. The formula reads:

\[ \rho_{GX} = \frac{\pi \hbar c}{8 \, L_p^2 \, a^2} = \frac{\pi}{8} \frac{c^4}{G} a^2, \]  

(1)

where \( h \) is the Planck constant, the Planck length is \( L_p = (\hbar G)^{1/2} c^{-3/2} \) and \( c \) is the speed of light, \( G \) is the gravitational constant. Here \( a \) is the upper cutoff in computation of vacuum fluctuations and we take it to be the scale factor of the Universe, although other choices are possible [2]. According to Zeldovich [3], the vacuum energy (1) corresponds to the cosmological term

\[ \Lambda_{GX} = \frac{8\pi G \rho_{GX}}{c^2} = \pi^2 \left( \frac{c}{a} \right)^2, \]

(2)

In General Relativity all the quantities in (1) and (2) are constants, except for the scale factor, which is a function of cosmic time. Therefore, adopting Gurzadyan-Xue (GX) scaling (1) one has to consider possible variation of physical constants.||

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∥ Generally speaking, the vacuum energy is known to be constant only in Minkowski spacetime. However, assuming it is constant one has unique connection between the vacuum energy density and the cosmological term and can interpret one of them as a fundamental constant of physics.
Simple cosmological models, based on above ideas we proposed in [4]. In particular, models with varying cosmological term, speed of light and gravitational constant were considered. Qualitative analysis of solutions in these models [5] revealed some interesting features, in particular the presence of separatrix in the phase space of solutions. This separatrix divides the space of solutions into two classes: Friedmannian-like with initial singularity and non-Friedmannian solutions which begin with nonzero scale factor and vanishing matter density. Each solution is characterized by a single quantity, the density parameter which is defined in the same way as in the standard cosmological model:

$$\Omega_m = \frac{\mu_0}{\mu_c},$$

where $\mu$ is matter density, $H$ is Hubble parameter, $\mu_c = \frac{3H_0^2}{8\pi G} \approx 2/3$ is critical matter density, and index "0" refers to the values today. Separatrix in all models is given by the density parameter $\Omega_m \approx 2/3$ that indicated some hidden symmetries between the models, although cosmological equations look very different. Later in [11] the origin of these symmetries was discovered and attributed to existence of invariants in GX models.

Note, that the energy-momentum conservation does not lead to mass conservation of the usual matter in models with varying constants. This can be interpreted as creation of matter due to variation of constants. As a consequence, the usual power law $\mu \propto a^{-3}$ does not hold in expanding Universe. In general $\mu(a)$ is obtained by solving cosmological equations.

Analytical solutions for all GX models both for matter density and the scale factor are obtained in [6]. It turns out that the most simple solutions for the scale factor are again those of separatrix. In one model it is exponential, in the others they are polynomials.

Simple GX models were generalized to include radiation in [6], where analytical solutions were also obtained. The main difference in that case is absence of separatrix for matter density solutions in models with radiation: all solutions (with exception of model with varying cosmological constant) begin with vanishing matter density in contrast to Friedmannian solutions. There is, however, a particular density parameter in those models that divides solution again into two classes: in the one class the matter density increases in matter-dominated epoch, in the other it decreases.

Thus, GX models possess interesting and nontrivial features. The original motivation for these cosmological models, however, is the fact that the predicted dark energy density is close the value inferred from the set of current observations. Along this line, it is of crucial importance to further confront predictions of these models with observations. We have performed likelihood analysis of supernovae and radio galaxies data within GX models in [7]. Such important characteristics of models such as age and deceleration parameter were also computed. Our results show that models with density parameter smaller than the separatrix value do not pass observational constraints. This indicates the preference of high matter densities: GX models with $2/3 \leq \Omega_m \leq 1$ pass these simple but important constraints. All these results show that background dynamics of GX models is viable in front of recent observations.
In this paper we turn to analysis of perturbations dynamics within GX models. It is well known that the most stringent constraints on cosmological parameters within \( \Lambda \)CDM models were obtained from observations of large scale structure distribution of galaxies and clusters, as well as from cosmic microwave background radiation anisotropies. We will consider simple Newtonian perturbations, i.e. those with wavelengths well inside the horizon. This allows, nevertheless, to understand the main differences with respect to dynamics of perturbations in the Friedmannian models.

The paper is organized as follows. In sec. 2 GX cosmological models are reviewed. In sec. 3 we present cosmological perturbation theory with varying constants in Newtonian approximation. In sec. 4 this theory in applied to GX models. In sec. 5 discussion and conclusions follow.

2. Gurzadyan-Xue cosmological models

In this section we remind briefly the content of GX models, for detailed discussion see [6].

Adopting the scaling (1) and postulating the validity of Einstein equations when speed of light and gravitational constant vary with time we obtain four simplest cases when the speed of light, the gravitational constant or vacuum energy density vary with time in such a way that (1) is satisfied. Below we provide cosmological equations for each model. The content of the Universe is supposed to be a pressureless fluid with mass density \( \mu \) and energy density \( \mu c^2 \) as well as a radiation field with energy density \( \rho_r \) and pressure \( \rho_r/3 \).

We also assume that evolution of radiation energy density is described by the following equation

\[
\rho_r = \rho_{r0} \left( \frac{a}{a_0} \right)^{-4},
\]

which means there is no energy flows into radiation field. This assumption allows to avoid problems with anisotropies of cosmic microwave background and large scale structure data for models with varying constants [8], [9].

All GX model with radiation are characterized by two density parameters: \( \Omega_m \) and \( \Omega_r = \rho_{r0}/(\mu c^2) \).

2.1. Model 1, with radiation

We suppose that neither the speed of light, nor the gravitational constant change with time, but the vacuum energy density does, so \( \rho_{GX} \propto a^{-2} \). This is the unique case when the radiation does not couple to matter [6].

This model is defined by the following cosmological equations [6]

\[
H^2 + \frac{k'c^2}{a^2} = \frac{8\pi G}{3} \left( \mu + \frac{\rho_{r0}}{c^2} \left( \frac{a}{a_0} \right)^{-4} \right),
\]  

(3)
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\[
\frac{\rho_{r} + \mu c^2}{\rho_c} = \frac{\pi}{4G} \left( \frac{c}{a} \right)^2 H,
\]

where \( \rho_{r0} \) is radiation energy density today, and effective curvature is \( k' = k - \frac{a^2}{3} \), \( k = -1, 0, 1 \). The first equation looks identical to Friedmann equation without cosmological constant. The second equation corresponds to continuity equation with the source term due to vacuum energy variation. This term is positive during expansion and may be interpreted as creation of matter.

Solutions of these equations are represented at fig. 2. Matter density and energy density of pressureless fluid are related through constant speed of light.

This model contains radiation-dominated (RD) stage which begins from Friedmannian singularity if \( \Omega_m < \Omega_{sep} \), where density parameter for separatrix is \( \Omega_{sep} = \frac{\Omega_r - 1}{\Omega_r} \), or with finite value of scale factor and vanishing energy density if \( \Omega_m > \Omega_{sep} \). Then RD stage is followed by matter-dominated (MD) stage. On last (kD) stage curvature dominates.

2.2. Model 2, with radiation

In this model the speed of light changes with time in such a way that vacuum energy density (1) together with (2) remain constants, so that the latter may be interpreted as cosmological term. The speed of light increases during cosmic expansion: \( c \propto a^2 \). Our cosmological equations are

\( \mu \rangle More precisely, terms of the order \( a^{-2} \) dominate in the first cosmological equation.
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Figure 2. Scale factor as a function of time for model 1 with radiation. Here time is measured in inverse Hubble parameter today, for which we take $H_0 = 70 \text{ km/(s Mpc)}$, so $H^{-1}_0 = 14 \text{ GYr}$ for this and all subsequent figures.

\[
H^2 - \frac{\Lambda'}{3} = \frac{8\pi G}{3} \left( \mu + \frac{\pi^2 \rho r_0 a_0^4}{\Lambda a^6} \right),
\]

(5)

\[
\dot{\mu} + H \left( \mu - \frac{\Lambda}{4\pi G} \right) = \frac{4\pi^2 \rho r_0 a_0^4}{\Lambda a^6} H,
\]

(6)

where $\Lambda' = \Lambda \left( 1 - 3k/\pi^2 \right)$ is effective cosmological constant.

Solutions of these equations are shown at fig. 3, 4 and 5.

The difference with model 1 is that now energy density and matter density behave differently (see fig. 3 and 4) since in the simple relation $\rho = \mu c^2$ the speed of light is no more a constant but a function of the scale factor, so now $\rho \propto \mu a^2$. Separatrix solution is very special since the total energy density for this solution begins in infinity, decreases, then vanishes at some value of the scale factor, and increases without bound again for large $a$. For smaller density parameter, than the separatrix one, solutions admit negative energy density for finite interval of scale factors, and for larger density parameter the energy density is always positive definite. In contrast (as can be seen at fig. 4), matter density is always negative for small scale factors (RD stage). For separatrix solution the matter density stays constant during "vacuum-dominated" (VD) (dominance of the vacuum energy density in cosmological equations) epoch, which follows after MD stage; there matter density decreases for $\Omega_m > \Omega_{sep}$ and increases otherwise.

2.3. Model 3, with radiation

Here gravitational constant decreases with cosmic expansion $G \propto a^{-2}$ and it is assumed that vacuum matter density $\mu_{GX}$ is a new constant. Cosmological equations read in this
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\[ \frac{\rho_c + \mu c^2}{\rho_c} \]

\[ \frac{\mu}{\mu_0} \]

**Figure 3.** Total energy density as a function of scale factor for model 2 with radiation.

**Figure 4.** Matter density of pressureless component in units of critical density as a function of scale factor for model 2 with radiation. For solution with \( \Omega_m > \Omega_{sep} \) the matter density has a maximum which is not shown here.

\[ H^2 + \frac{k c^2}{a^2} = \frac{\pi^2}{3} \left( \frac{c}{a} \right)^2 \left[ 1 + \frac{1}{\mu_{GX}} \left( \mu + \frac{\rho_r a}{c^2} \left( \frac{a}{a_0} \right)^{-4} \right) \right], \]  \( \text{Eqn. (7)} \)

\[ \dot{\mu} + H(\mu - 2\mu_{GX}) = 2H \frac{\rho_r a}{c^2} \left( \frac{a}{a_0} \right)^{-4}. \]  \( \text{Eqn. (8)} \)

At fig. 6, 7 and 8 we illustrate solutions of these equations.
One can see that during the VD stage both total energy density and matter density of pressureless fluid stay constants for separatrix solution; they both increase for \( \Omega_m > \Omega_{sep} \) and decrease otherwise.

2.4. Model 4, with radiation

In this model it is assumed that the speed of light changes with time, but the vacuum energy density \( \rho_{GX} \) stays constant. This leads to \( c \propto a^{1/2} \) dependence.

Cosmological equations are
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\[
\frac{\dot{\mu}}{\mu_c} = \frac{\mu + \frac{1}{a} \sqrt{\frac{\pi}{8G\rho_{GX}}} \rho_{r0} \left( \frac{a}{a_0} \right)^{-4}}{2H} + \frac{\beta}{a},
\]

(9)

\[
\dot{\mu} + 2H\mu = \frac{H}{a} \sqrt{\frac{\pi \rho_{GX}}{2G}} \left( 1 + \frac{\rho_{r0}}{\rho_{GX}} \left( \frac{a}{a_0} \right)^{-4} \right),
\]

(10)

where \( \beta = \frac{2\sqrt{2\pi}}{3} \left( G\rho_{GX} \right)^{1/2} \left( 1 - \frac{3k}{5} \right) \).

Solutions in this case are represented at fig. 9, 10 and 11. The role of separatrix in this model is similar to the model 3 with respect to the
energy density, but the matter density of pressureless matter behaves differently since now $\rho \propto a\mu$.

3. Cosmological perturbations with varying constants in Newtonian approximation

Since cosmological equations and their solutions for GX models are known, it is of interest to consider evolution of perturbations in these models. This is relevant for understanding the dynamics of large scale structure, its analogy and difference with the
standard cosmological model. Due to some peculiarities of GX models this dynamics turns out to be quite different.

We remind that Newtonian equations for perturbations may be used to describe evolution of perturbations in nonrelativistic component and for modes well inside horizon.

In general, when both speed of light and gravitational constant change with time we have (see Appendix):

continuity equation

$$\frac{\partial \mu}{\partial t} + \partial_\alpha (\mu v^\alpha) = -\mu \left( \frac{\dot{G}}{G} - 2 \frac{\dot{c}}{c} \right),$$

(11)

Euler equation

$$\frac{\partial v_\alpha}{\partial t} + v_\beta \frac{\partial v_\alpha}{\partial x^\beta} + \frac{\partial p}{\mu \partial x^\alpha} + \frac{\partial \varphi}{\partial x^\alpha} = -v_\alpha \frac{\dot{c}}{c},$$

(12)

and Poisson equation

$$\nabla^2 \varphi - 4\pi G \mu = 0,$$

(13)

where $\mu$ and $p$ are density and pressure of the perfect fluid, which is moving with velocity $v_\alpha$ in a gravitational potential $\varphi$. Both the continuity and Euler equations contain additional terms: $\mu \left( \frac{\dot{G}}{G} - 2 \frac{\dot{c}}{c} \right)$ in continuity equation and $-v_\alpha \frac{\dot{c}}{c}$ term in Euler equation. This means, that mass in comoving frame is not conserved in models with varying constants, so there is continuous creation of matter. Also velocity flow changes not only due to pressure and gravitational field gradients and universal expansion, but also due to variation of the speed of light.

Density perturbations in expanding Universe obey second order differential equation. In Newtonian approximation, neglecting the sound velocity this equation
reads (see Appendix)

\[ \delta''(a) + \delta'(a) \left( \frac{3}{a} + \frac{1}{H a c} + \frac{H'}{H} \right) - \frac{4\pi G \bar{\mu}}{H^2 a^2} \delta = 0. \]

where \( \delta = (\mu - \bar{\mu})/\bar{\mu} \) is dimensionless density contrast, \( \bar{\mu} \) is unperturbed density. Here again, in contrast with the corresponding equation for Friedmannian models, additional term appears due to variation of the speed of light. These equations are well defined to describe perturbations in pressureless component which represents e.g. cold dark matter, or baryons after recombination in large wavelength limit.

4. Linear Newtonian perturbations in GX models

4.1. Friedmannian models

Solutions for density perturbations in Friedmann models are well known. During RD stage density contrast of dark matter grows with the scale factor logarithmically*,

\[ \delta \propto \ln a. \]

During MD epoch perturbations both in baryonic and dark matter components grow linearly with the scale factor,

\[ \delta \propto a. \]

Finally, in VD, or curvature-dominated (kD) epochs perturbations cease to grow.

Now we turn to perturbations dynamics specifically within GX models.

4.2. Model 1 with radiation

The solution for matter density is

\[ \bar{\mu} = \frac{c^2 \pi}{4 a^3 G} \left( 1 - \frac{a_0}{a} \right) + \left( \frac{a}{a_0} \right)^{-3} \mu_0 \]

and one recovers Friedmannian behavior \( \bar{\mu} \propto a^{-3} \) in the limit of small scale factors. However, it is during RD stage where dynamics is determined by radiation. According to this solution at late expansion dominates the curvature.

In this model \( c = \text{const}, G = \text{const} \), so the perturbation equation is

\[ \delta''(a) + \delta'(a) \left( \frac{3}{a} + \frac{H'}{H} \right) - \frac{4\pi G \bar{\mu}}{H^2 a^2} \delta = 0. \]  

(14)

* In this paper we consider only perturbations with wavelength larger than the corresponding Jeans wavelength.

* We are interested in growing modes only.
4.2.1. Radiation-dominated epoch. Within Newtonian approximation one can still follow dynamics of perturbations in the pressureless component during RD stage \[10\]. This results in the following solution

\[ \delta \propto \ln a, \]

so perturbations evolve exactly as they do in Friedmannian models. It is not surprising, since during this stage we have usual dependence of the matter density on the scale factor, and \[11\] is also the same as in Friedmannian models.

4.2.2. Matter-dominated epoch. Here we take into account only \( a^{-3} \) terms in \[4.2\] and also put \( \rho_{r0} \to 0, \dot{k} = 0 \) when calculate \( H(a) \). The solution is

\[ \delta \propto a, \tag{15} \]

and it again coincides with the corresponding solution in Friedmannian models. The reason is the same as for RD stage.

4.2.3. Curvature dominated epoch. Here we account only for \( a^{-2} \) terms in \[4.2\] and also put \( \rho_{r0} \to 0 \) when calculate the Hubble rate.

Thus we get the solution

\[ \delta \propto a^\frac{1}{2}\left(\sqrt{\frac{5}{2}} - \frac{4k}{\pi^2 - k} - 1\right). \tag{16} \]

Interesting, that for zero spatial curvature this is "golden section" solution

\[ \delta \propto a^\frac{1}{2}\left(\sqrt{\frac{5}{2}} - 1\right) \tag{17} \]

Perturbations grow at kD stage due to existence of energy transfer from dark energy into pressureless matter which manifests itself in the presence of the source in continuity equation of this model.

As we see, in contrast with the Friedmann models, perturbations grow during entire cosmological expansion.

4.3. Model 2, with radiation

Matter density depends on the scale factor as follows

\[ \bar{\mu} = \frac{\Lambda}{4\pi G} \left(1 - \frac{a_0}{a}\right) + \frac{a_0}{a} \mu_0 + \frac{4\pi^2 \rho_{r0}}{5a_0 \Lambda} \left[1 - \left(\frac{a_0}{a}\right)^5\right] \tag{18} \]

As in this model \( c \propto a \), the perturbation equation becomes

\[ \delta''(a) + \delta'(a) \left(\frac{4}{a} + \frac{H'}{H}\right) - \frac{4\pi G \bar{\mu}}{H^2 a^2} \delta = 0. \tag{19} \]

In all GX models with radiation, except for model 1, matter density is negative in the beginning of cosmological equations (see figures above). This leads to anomalous behavior of perturbations during RD stage and in what follows we do not consider dynamics of perturbations for RD epoch.
4.3.1. Matter-dominated epoch. In contrast to model 1, matter-dominated epoch corresponds to the prevalence of $a^{-1}$ terms in (18). We also put $\dot{\Lambda} = 0, \rho_{r0} \to 0$ in the background equations when calculate the Hubble rate.

The solution is
\[
\delta \propto \sqrt{a}. \tag{20}
\]

Thus perturbations growth is somewhat slower comparing to Friedmannian case in MD epoch. This is a consequence of matter creation in all space due to variation of the speed of light with cosmic time.

4.3.2. Vacuum-dominated epoch. Here we take into account only $a^0$ terms in (18) and also put $\rho_{r0} \to 0$ when calculate the Hubble rate.

In this way we obtain the solution
\[
\delta \propto a^{\frac{1}{2}} \left( \sqrt{\frac{13\pi^2 - 9k\pi^2}{a^2}} - 3 \right). \tag{21}
\]

Matter is also created during VD stage but this leads to an opposite effect, comparing to the case discussed above, namely it stimulates the growth of perturbations, like at curvature-dominated stage in the model 1.

For zero spatial curvature $k = 0$ it follows
\[
\delta(a) = c_1 a^{\frac{\sqrt{13\pi^2 - 9}}{2}}. \tag{22}
\]

Again, perturbations grow at all stages of expansion.

4.4. Model 3, with radiation

Cosmological equations lead to the following solution for matter density
\[
\bar{\mu} = 2\mu_{GX} + \frac{a_0}{a} (\mu_0 - 2\mu_{GX}) + \frac{2\rho_{r0}a_0}{3c^2} a \left( 1 - \left( \frac{a_0}{a} \right)^3 \right). \tag{23}
\]

In this model $c = const, G \propto a^{-2}$, so the perturbation equation is the same as in model 1.

Solutions for perturbations within this model during MD and VD epochs are exactly the same as in model 1 during MD and kD epochs respectively.

4.5. Model 4, with radiation

Matter density depends on the scale factor as follows
\[
\bar{\mu} = \left( \frac{a_0}{a} \right)^2 \mu_0 + \frac{1}{a} \left( 1 - \frac{a_0}{a} \right) \sqrt{\frac{\pi \rho_{r0} a_0}{2G}} + \frac{\rho_{r0}a_0}{3c^2} a \left( 1 - \left( \frac{a_0}{a} \right)^3 \right). \tag{24}
\]

In this model $c \propto a^{\frac{1}{2}}, G = const$, so the perturbation equation is modified
\[
\delta''(a) + \delta'(a) \left( \frac{7}{2a} + \frac{H'}{H} \right) - \frac{4\pi G \bar{\mu}}{H^2 a^2} \delta = 0. \tag{25}
\]
4.5.1. Matter-dominated epoch. Considering only $a^{-2}$ terms in (24) and assuming in addition $\beta = 0, \rho_0 \rightarrow 0$ in the cosmological equations we find the solution
\[
\delta \propto a^{\sqrt{\frac{2\pi^2}{4} - \frac{3}{4}}}. \tag{26}
\]
This is slower growth than in the Friedmannian model and model 1, but faster than in the model 2.

4.5.2. $\beta$-dominated epoch. Taking into account only $a^{-1}$ terms in (24) and also putting $\rho_0 \rightarrow 0$ to calculate the Hubble rate we find the solution
\[
\delta \propto a^{\sqrt{\frac{2\pi^2 - k}{8\pi^2 - k}}}. \tag{27}
\]
For zero spatial curvature we obtain
\[
\delta(a) = a^{\sqrt{2-1}}. \tag{28}
\]
Again, perturbations grow during entire expansion.

5. Conclusions

Dynamics of perturbations in Gurzadyan-Xue cosmological models is studied within Newtonian approximation. With this goal relativistic hydrodynamic equations for models with varying constants are derived. Non-relativistic limit for these equations is also obtained, which provided the basis for analysis of perturbations dynamics within Newtonian approximation. Analytic solutions are presented for the model with pressureless ideal fluid and radiation. In contrast with the standard cosmological model perturbations grow during entire cosmological expansion, including curvature-dominated and vacuum-dominated epochs.

Only Poisson equation preserves its form comparing to the Newtonian hydrodynamics. Both continuity and Euler equations contain additional terms due to variation of the physical constants, namely the speed of light and the gravitational constant. Thus both at cosmological and at local level continuity equation contains source terms which may be interpreted as creation of matter.

We found no difference in dynamics of perturbations between our model 1 (varying vacuum energy density) and standard cosmological model for radiation- and matter-dominated stages. However, on the last stage perturbations grow according to the “golden-section” rule. In other models we have found polynomial dependence of the density contrast on the scale factor.

6. References

[1] V.G. Gurzadyan, S.-S. Xue in: “From Integrable Models to Gauge Theories; volume in honor of Sergei Matinyan”, ed. V.G. Gurzadyan, A.G. Sedrakian, p.177, World Scientific, 2002; Mod. Phys. Lett. A18 (2003) 561 [astro-ph/0105245], see also astro-ph/0510459.
7. Appendix

7.1. Hydrodynamic equations

It is our goal in this section to derive hydrodynamic equations for models with varying speed of light and gravitational constant in Newtonian limit. We start with the interval

\[ ds^2 = \left( c^2 + 2\varphi \right) dt^2 - dv^2, \]

where \( r_\alpha = (x, y, z) \) are cartesian coordinates, so metric coefficients are

\[ g_{00} = c^2 + 2\varphi, \quad g_{11} = g_{22} = g_{33} = -1. \]

In Newtonian limit

\[ \frac{2\varphi}{c^2} \ll 1 \]

is a small parameter, and in Taylor expansion we will keep only zeroth order terms with respect to it.

Christoffel symbol has the following nonvanishing components

\[ \Gamma^0_{0\alpha} = \frac{\dot{c} + \dot{\varphi}}{c^2 + 2\varphi}, \quad \Gamma^\alpha_{00} = \frac{\partial \varphi}{\partial x^\alpha}, \]

\[ \Gamma^0_{\alpha0} = \frac{1}{c^2 + 2\varphi} \frac{\partial \varphi}{\partial x^\alpha}. \]

Here Latin indexes take values from 0 to 3, while Greek indexes run from 1 to 3; a dot denotes differentiation with respect to time.

Ricci tensor has the following nonvanishing components

\[ R^\alpha_\beta = \delta_{\alpha\beta} \frac{1}{c^2 + 2\varphi} \left[ \frac{\partial^2 \varphi}{\partial (x^\alpha)^2} - \frac{1}{c^2 + 2\varphi} \left( \frac{\partial \varphi}{\partial x^\alpha} \right)^2 \right], \]

\[ R^0_0 = \frac{1}{c^2 + 2\varphi} \left[ \Delta \varphi - \frac{1}{c^2 + 2\varphi} (\nabla \varphi)^2 \right] \approx \frac{1}{c^2} \Delta \varphi. \]
7.1.1. Poisson equation. From Einstein’s equations

\[ R^i_k = \kappa \left( T^i_k - \frac{1}{2} g^i_k T \right), \]

where

\[ \kappa = \frac{8\pi G}{c^4}, \]

for the fluid at rest one has

\[ R^0_0 = \frac{4\pi G}{c^2} \mu, \]

so we obtain the usual Poisson equation

\[ \Delta \varphi = 4\pi G \mu. \]

Energy-momentum tensor for a perfect fluid is

\[ T^i_k = \omega u^k u_i - p g^k_i, \]

where \( \omega = \mu c^2 + p, \)

\[ u^i = \frac{\sqrt{c^2 + 2\varphi}}{\sqrt{c^2 + 2\varphi}} (1, v_\alpha), \]

\[ u^i = \frac{\sqrt{c^2 + 2\varphi}}{\sqrt{c^2 + 2\varphi}} (c^2 + 2\varphi, -v_\alpha), \]

\[ \gamma = \left( 1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}}. \]

In components

\[ T^0_0 = \omega \gamma^2 - p, \]

\[ T^0^\alpha = \omega \gamma^2 v_\alpha = -c^2 T^0_\alpha, \]

\[ T^\alpha^\beta = -\omega \left( \frac{\gamma}{c} \right)^2 v_\alpha v_\beta - p \delta_\alpha^\beta. \]

Euler and continuity equations both follow from energy-momentum conservation

\[ \frac{1}{\kappa} \left( \kappa T^i_k \right)^i_k = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^k} T^i_k + \frac{\kappa}{\kappa} T^0_0 + \frac{\partial}{\partial x^k} \left( T^i_k \right) - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} T^{kl} = 0. \]

We again expand all terms in this equation in Taylor series with respect to small parameters

\[ \frac{v_\alpha}{c} \ll 1, \]

\[ \frac{p}{\mu c^2} \ll 1. \]
7.1.2. Continuity equation. From zeroth component of energy-momentum conservation we find

\[
\frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} T^k_0}{\partial x^k} + \frac{\kappa}{\kappa} T^0_0 + \frac{\partial}{\partial x^k} (T^k_0) - \frac{1}{2} \frac{\partial g^{kl}}{\partial t} T^{kl} = \frac{1}{\sqrt{-g}} \left( \frac{\partial \sqrt{-g} T^0_0}{\partial t} + \frac{\partial}{\partial x^\alpha} (T^0_0) + \frac{\partial}{\partial x^\alpha} (T^\alpha_0) \right) - \Gamma^0_{00} T^0_0 = \Gamma^0_{00} T^0_0 + \frac{\partial}{\partial t} (T^0_0) + \frac{\partial}{\partial x^\alpha} (T^\alpha_0) - \Gamma^0_{00} T^0_0 = \Gamma^0_{00} T^0_0 + \frac{\partial}{\partial t} (T^0_0) + \frac{\partial}{\partial x^\alpha} (T^\alpha_0) - \Gamma^0_{00} T^0_0 = \frac{1}{c^2} \frac{\partial}{\partial x^\alpha} (\omega^2 \psi_\alpha) = \frac{1}{c^2} \frac{\partial}{\partial x^\alpha} (\mu \psi_\alpha).
\]

Taking Newtonian limit of this expression, and dividing all terms by \(c^2\) we find

\[
\frac{1}{c^2} \frac{\partial}{\partial x^\alpha} (\mu \psi_\alpha) + \frac{1}{c^2} \left( \frac{\dot{G}}{G} - 4 \frac{\dot{\psi}}{c} \right) \mu c^2 + \frac{1}{c^2} \frac{\partial}{\partial t} (\mu c^2) + \frac{1}{c^2} \frac{\partial}{\partial x^\alpha} (\mu c^2 \psi_\alpha) = \frac{1}{c^2} \frac{\partial}{\partial x^\alpha} (\mu \psi_\alpha) + \left( \frac{\dot{G}}{G} - 4 \frac{\dot{\psi}}{c} \right) \mu + \frac{2 \dot{\psi}}{c} + \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x^\alpha} (\mu \psi_\alpha) \right) = \frac{\partial}{\partial t} (\mu \psi_\alpha) + \left( \frac{\dot{G}}{G} - 2 \frac{\dot{\psi}}{c} \right) \mu + \frac{1}{c^2} \frac{\partial}{\partial x^\alpha} (\mu \psi_\alpha) = 0.
\]

The last term is small comparing to the second, as can be seen from the following

\[
2 \psi \ll c^2, \quad 2 \mu \psi \ll \mu c^2, \quad 2 \frac{\partial}{\partial x^\alpha} (\mu \psi \alpha) \ll c^2 \frac{\partial}{\partial x^\alpha} (\mu \psi \alpha), \quad 2 \mu \psi \frac{\partial}{\partial x^\alpha} + 2 \psi \frac{\partial}{\partial x^\alpha} \ll c^2 \frac{\partial}{\partial x^\alpha} (\mu \psi \alpha).
\]

Therefore, continuity equation for models with varying constants in Newtonian approximation reads

\[
\frac{\partial}{\partial t} (\mu \psi_\alpha) + \frac{\partial}{\partial x^\alpha} (\mu \psi_\alpha) = \left( \frac{\dot{G}}{G} - 2 \frac{\dot{\psi}}{c} \right) \mu.
\]
7.1.3. Euler equation. For the remaining 3 components of the energy-momentum tensor we have

\[
\frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} \ T_{\alpha}^{\ k}}{\partial x^{k}} + \frac{\kappa}{\kappa} T_{\alpha}^{\ 0} + \frac{\partial}{\partial x^{\alpha}} \left( T_{\alpha}^{\ k} \right) - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^{\alpha}} T^{kl} = \\
= \frac{1}{\sqrt{-g}} \left( \frac{\partial \sqrt{-g}}{\partial t} T_{\alpha}^{\ 0} + \frac{\partial \sqrt{-g} T_{\alpha}^{\ \beta}}{\partial x^{\beta}} \right) + \frac{\kappa}{\kappa} T_{\alpha}^{\ 0} + \frac{\partial}{\partial t} \left( T_{\alpha}^{\ 0} \right) + \\
+ \frac{\partial}{\partial x^{\beta}} \left( T_{\alpha}^{\ \beta} \right) - \frac{1}{2} \frac{\partial g_{00}}{\partial x^{\alpha}} T^{00} = \Gamma_{00}^{\ 0} T_{\alpha}^{\ 0} + \Gamma_{\beta 0}^{\ 0} T_{\alpha}^{\ \beta} + \frac{\kappa}{\kappa} T_{\alpha}^{\ 0} + \frac{\partial}{\partial t} \left( T_{\alpha}^{\ 0} \right) + \\
+ \frac{\partial}{\partial x^{\beta}} \left( T_{\alpha}^{\ \beta} \right) - \Gamma_{00}^{\ 0} T_{\alpha}^{\ 0} = - \frac{\partial}{\partial x^{\beta}} \left( \frac{\dot{c}}{c^2} + \dot{\varphi} \right) \left( \frac{\gamma}{c} \right)^2 v_{\alpha} - \\
- \frac{1}{c^2 + 2 \varphi \partial x^{\beta}} \left[ \omega \left( \frac{\gamma}{c} \right)^2 v_{\alpha} v_{\beta} + p \delta_{\alpha\beta} \right] - \frac{\dot{G}}{G} \left( \frac{\dot{c}}{c} - 4 \dot{\varphi} \right) \left( \frac{\gamma}{c} \right)^2 v_{\alpha} - \\
- \frac{\partial}{\partial t} \left[ \omega \left( \frac{\gamma}{c} \right)^2 v_{\alpha} \right] - \frac{\partial}{\partial x^{\beta}} \left[ \omega \left( \frac{\gamma}{c} \right)^2 v_{\alpha} v_{\beta} + p \delta_{\alpha\beta} \right] + \frac{1}{c^2 + 2 \varphi \partial x^{\alpha}} \left( \omega \gamma^2 - p \right) \approx \\
\approx - \frac{\dot{c}}{c} \omega \left( \frac{\gamma}{c} \right)^2 v_{\alpha} - \frac{1}{c^2 + 2 \varphi \partial x^{\beta}} \left( \omega \left( \frac{\gamma}{c} \right)^2 v_{\alpha} v_{\beta} - \frac{\dot{G}}{G} \left( \frac{\dot{c}}{c} - 4 \dot{\varphi} \right) \left( \frac{\gamma}{c} \right)^2 v_{\alpha} - \\
- \frac{\partial}{\partial t} \left[ \omega \left( \frac{\gamma}{c} \right)^2 v_{\alpha} \right] - \frac{\partial}{\partial x^{\beta}} \left[ \omega \left( \frac{\gamma}{c} \right)^2 v_{\alpha} v_{\beta} + p \delta_{\alpha\beta} \right] + \frac{1}{c^2 + 2 \varphi \partial x^{\alpha}} \left( \omega \gamma^2 - 2p \right) = 0.
\]

Taking Newtonian limit of this expression we get

\[
\frac{\dot{c}}{c} \mu v_{\alpha} - \frac{\partial}{\partial x^{\beta}} \left( \frac{\dot{G}}{G} - 2 \frac{\dot{\varphi}}{c} \right) \mu v_{\alpha} - \frac{\partial}{\partial t} \left( \mu v_{\alpha} \right) - \frac{\partial}{\partial x^{\beta}} (\mu v_{\alpha} v_{\beta}) - \\
- \frac{\partial p}{\partial x^{\alpha}} - \frac{\partial}{\partial x^{\alpha}} \left( \frac{\dot{G}}{G} v_{\alpha} v_{\beta} - \frac{\partial}{\partial x^{\beta}} (\mu v_{\alpha}) - \frac{\partial}{\partial t} (\mu v_{\alpha}) - \frac{\partial}{\partial x^{\beta}} (\mu v_{\alpha} v_{\beta}) - \\
- \frac{\partial}{\partial t} \left[ \mu \left( \frac{\dot{G}}{G} - 2 \frac{\dot{\varphi}}{c} \right) + \frac{\partial}{\partial t} (\mu v_{\beta}) \right] + \frac{\mu}{c} \frac{\dot{c}}{c} + \frac{\partial}{\partial t} v_{\alpha} - \\
- \frac{\mu}{c} \frac{\partial}{\partial x^{\alpha}} v_{\beta} - \frac{\partial}{\partial x^{\alpha}} (\mu v_{\beta}) - \frac{\dot{\varphi}}{c} = 0.
\]

Finally, dividing this by \( \mu \) we find the Euler equation for models with varying constants in Newtonian approximation

\[
\frac{\partial v_{\alpha}}{\partial t} + v_{\beta} \frac{\partial v_{\alpha}}{\partial x^{\beta}} + \frac{\partial p}{\mu \partial x^{\alpha}} + \frac{\partial \varphi}{\partial x^{\alpha}} = -v_{\alpha} \frac{\dot{c}}{c}.
\]

7.2. Perturbation equations

First, introduce comoving coordinates,

\[ r_{\alpha} = a x_{\alpha}, \]

then for velocity we have

\[ v_{\alpha} = H r_{\alpha} + u_{\alpha}. \]

Perturbation equations now read

\[
\frac{\partial \mu}{\partial t} + 3 H \mu + \frac{1}{a} \partial_{\alpha} (\mu u_{\alpha}) + \mu \left( \frac{\dot{G}}{G} - 2 \frac{\dot{\varphi}}{c} \right) = 0,
\]
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\[ \frac{d^2a}{dt^2}x_\alpha + \frac{\partial u_\alpha}{\partial t} + Hu_\alpha + \frac{1}{a}u^2\partial_\beta u_\alpha + \frac{1}{a}\partial_\alpha p + \frac{1}{a}\partial_\alpha \varphi + (Hax_\alpha + u_\alpha) \frac{\dot{c}}{c} = 0, \]

(30)

\[ \partial^2 \varphi - 4\pi Ga^2 \mu = 0, \]

(31)

where \( \partial_\alpha = \frac{\partial}{\partial x_\alpha} \), \( \partial^2 = \frac{\partial^2}{\partial x^2} \). Differential operators transform from physical to comoving coordinates as \( \frac{\partial}{\partial x_\alpha} \rightarrow \frac{1}{a} \frac{\partial}{\partial x_\alpha} \), \( \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - Hx_\beta \partial_\beta \).

Background solution for \( \mu = \bar{\mu}(t) \), \( p = \bar{p}(t) \), \( \varphi = \bar{\varphi}(t) \) and \( u_\alpha = 0 \) is given by the system

\[ \frac{\partial \bar{\mu}}{\partial t} + 3H \bar{\mu} = -\bar{\mu} \left( \frac{\dot{G}}{G} - 2\frac{\dot{c}}{c} \right), \]

\[ \frac{d^2a}{dt^2}x_\alpha + \frac{1}{a} \partial_\alpha \bar{\varphi} = -Hax_\alpha \frac{dc}{c dt}, \]

\[ \partial^2 \bar{\varphi} - 4\pi Ga^2 \bar{\mu} = 0, \]

Consider perturbations about the background solution

\[ \mu = \bar{\mu}(1 + \delta), \quad p = \bar{p}(1 + \delta p), \quad \varphi = \bar{\varphi}(1 + \delta \varphi), \quad u_\alpha = \delta u_\alpha, \]

(32)

\[ \delta \ll 1, \quad \delta p \ll 1, \quad \delta \varphi \ll 1, \quad \delta u_\alpha \ll 1. \]

(33)

From (29) and (32) we have

\[ \delta \frac{\partial}{\partial t} \left[ \bar{\mu} (1 + \delta \mu) \right] + 3H \bar{\mu} (1 + \delta \mu) + \frac{1}{a} \partial_\alpha \{ [\bar{\mu} (1 + \delta \mu)] \delta u_\alpha \} = \]

\[ = - \left( \frac{dG}{G dt} - 2 \frac{dc}{c dt} \right) \bar{\mu} (1 + \delta \mu) \]

\[ \delta \frac{\partial \bar{\mu}}{\partial t} + \frac{\partial \delta}{\partial t} + 3H \bar{\mu} \delta + \frac{\bar{\mu}}{a} \partial_\alpha (\delta u_\alpha) = -f \bar{\mu} \delta \]

\[ \delta \left[ \frac{\partial \bar{\mu}}{\partial t} + 3H \bar{\mu} + \bar{\mu} \left( \frac{\dot{C}}{G} - 2\frac{\dot{c}}{c} \right) \right] + \bar{\mu} \left[ \frac{\partial \delta}{\partial t} + \frac{1}{a} \partial_\alpha (\delta u_\alpha) \right] = 0. \]

Then

\[ \frac{\partial \delta}{\partial t} + \frac{1}{a} \partial_\alpha (\delta u_\alpha) = 0. \]

(34)

From (30) and (32) we obtain

\[ \frac{d^2a}{dt^2}x_\alpha + \frac{\partial \delta u_\alpha}{\partial t} + Hu_\alpha + \frac{1}{a\bar{\mu} (1 + \delta)} \partial_\alpha [\bar{p}(1 + \delta p)] + \]

\[ + \frac{1}{a} \partial_\alpha [\bar{\varphi}(1 + \delta \varphi)] = - \frac{dc}{c dt} (Hax_\alpha + \delta u_\alpha), \]

\[ \frac{\partial \delta u_\alpha}{\partial t} + H\delta u_\alpha + \frac{\bar{\mu}}{a\bar{\mu}} \partial_\alpha p + \frac{1}{a} \partial_\alpha (\delta \varphi) = -\delta u_\alpha \frac{dc}{c dt}. \]

Assume, as usual, the speed of sound is \( v_s^2 = \frac{\partial p}{\partial \rho} \), so \( \frac{\bar{\mu}}{\bar{\mu}} \partial_\alpha (\delta p) = v_s^2 \partial_\alpha \delta \), so

\[ \frac{\partial \delta u_\alpha}{\partial t} + H\delta u_\alpha + \frac{v_s^2}{a} \partial_\alpha \delta + \frac{1}{a} \partial_\alpha (\delta \varphi) = -\delta u_\alpha \frac{dc}{c dt}. \]

(35)
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From (31) and (32) we have

\[ \partial^2 \delta \varphi - 4\pi Ga^2 \bar{\mu} \delta = 0. \]  

(36)

Combine (34), (35), (36) to a second order differential equation

\[ \frac{\partial^2 \delta}{\partial t^2} + \frac{\partial}{\partial t} \left[ \frac{1}{a} \partial_a (\delta u_a) \right] = \frac{\partial^2 \delta}{\partial t^2} \left[ \frac{H}{a} \partial_a \delta u_a + \frac{1}{a} \partial_a \left( \frac{\partial \delta u_a}{\partial t} \right) \right] = \frac{\partial^2 \delta}{\partial t^2} - \frac{H}{a} \partial_a \delta v_a - \frac{1}{a} \partial_a \left( H \delta u_a + \frac{v^2}{a} \partial_a \delta + \frac{1}{a} \partial_a (\delta \varphi) + \delta u_a \frac{dc}{cdt} \right) = \frac{\partial^2 \delta}{\partial t^2} - \left( 2H + \frac{\dot{c}}{c} \right) \frac{1}{a} \partial_a \delta u_a - \frac{v^2}{a^2} \partial^2 \delta - 4\pi G \bar{\mu} \delta. \]

Finally, equation for density perturbations is

\[ \frac{\partial^2 \delta}{\partial t^2} + \left( 2H + \frac{\dot{c}}{c} \right) \frac{\partial \delta}{\partial t} - \frac{v^2}{a^2} \partial^2 \delta - 4\pi G \bar{\mu} \delta = 0. \]

(37)

The same equation may be rewritten where all functions depend on scale factor instead of time with conversion of the derivatives

\[ \dot{\delta} = Ha \dot{\delta} (a) \]

(38)

Then density perturbations satisfy the following equation

\[ \delta'' (a) + \dot{\delta}' (a) \left( \frac{3}{a} + \frac{H'}{H} + \frac{1}{Ha} \frac{c'(a)}{c} \right) = \frac{v^2}{H^2a^4} \partial^2 \delta - \frac{4\pi G \bar{\mu}}{H^2a^2} \delta = 0. \]

(39)

Comparing to the same equation in Newtonian cosmology [10], we find additional term \( \dot{\delta}' \) which is due to variation of constants.