ON COMPLEX ZEROS OFF THE CRITICAL LINE FOR
NON-MONOMIAL POLYNOMIAL OF ZETA-FUNCTIONS

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Abstract. In this paper, we show that any polynomial of zeta or L-functions with
some conditions has infinitely many complex zeros off the critical line. This general
result has abundant applications. By using the main result, we prove that the zeta-
functions associated to symmetric matrices treated by Ibukiyama and Saito, certain
spectral zeta-functions and the Euler-Zagier multiple zeta-functions have infinitely many
complex zeros off the critical line. Moreover, we show that the Lindelöf hypothesis for
the Riemann zeta-function is equivalent to the Lindelöf hypothesis for zeta-functions
mentioned above despite of the existence of the zeros off the critical line. Next we
prove that the Barnes multiple zeta-functions associated to rational or transcendental
parameters have infinitely many zeros off the critical line. By using this fact, we show
that the Shintani multiple zeta-functions have infinitely many complex zeros under some
conditions. As corollaries, we show that the Mordell multiple zeta-functions, the Euler-
Zagier-Hurwitz type of multiple zeta-functions and the Witten multiple zeta-functions
have infinitely many complex zeros off the critical line.

1. Introduction

1.1. Universality. In 1975, Voronin [46] showed the universality theorem for the Rie-
mann zeta-function \( \zeta(s) := \sum_{n=1}^{\infty} n^{-s} \). To state it, let \( D := \{ s \in \mathbb{C} : 1/2 < \Re(s) < 1 \} \)
and \( K \subset D \) be a compact set with connected complement. Denote by \( \mu(A) \) the Lebesgue
measure of the set \( A \), and, for \( T > 0 \), write
\[ \nu_T(\{ \cdot \cdot \cdot \}) := T^{-1}\mu(\{ \tau \in [0,T] : \cdot \cdot \cdot \}) \]
where the dots stand for a condition satisfied by \( \tau \). Let \( H_0(K) \) denote the space of continuous
functions on \( K \), which are analytic in the interior, equipped with the supremum norm
\( \| \cdot \|_K \) and \( H(K) \) denote the subspace of \( H_0(K) \) consisting of non-vanishing functions.
Then the modern version of the Voronin theorem can be formulated as follows.

Theorem A. For any \( f \in H(K) \) and any \( \varepsilon > 0 \), we have
\[ \liminf_{T \to \infty} \nu_T(\{ \| \zeta(s + i\tau) - f(s) \|_K < \varepsilon \}) > 0. \]

Roughly speaking, this theorem implies that any non-vanishing analytic function can
be uniformly approximated by the Riemann zeta-function. Subsequently, many mathe-
maticians considered generalizations of universality (see for instance [38]). For example,
the strong universality property for the Hurwitz zeta-function \( \zeta(s,a) := \sum_{n=0}^{\infty} (n + a)^{-s} \)

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zeta-functions and Shintani multiple zeta-functions.

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was proved by Bagchi [5] and Gonek [10], independently (see [25, Theorem 6.1.2 and Note in Section 6]).

**Theorem B.** Let $0 < a < 1$ be transcendental. Then for any $f \in H_0(K)$ and any $\varepsilon > 0$,

$$\liminf_{T \to \infty} \nu_T \{ \| \zeta(s + i\tau, a) - f(s) \|_K < \varepsilon \} > 0.$$

As an application of strong universality for the Hurwitz zeta-function $\zeta(s, a)$, one can obtain the following theorem. The lower bound for the number of zeros of $\zeta(s, a)$ was first announced by Voronin [46] when $a$ is rational, but the proof (including transcendental $a$) was given by Bagchi [5] and Gonek [10] (see also [25, Theorems 8.4.7, 8.4.8 and 8.4.10, and Note in Section 8]). We write $f(x) \asymp x$ if there exist constants $0 < C_1 < C_2$ such that $C_1 x \leq f(x) \leq C_2 x$ for sufficiently large $x > 0$.

**Theorem C.** For any $1/2 < \sigma_1 < \sigma_2 < 1$, the Hurwitz zeta-function $\zeta(s, a)$ where $a \neq 1/2, 1$ is rational or transcendental, has $\asymp T$ nontrivial zeros in the rectangle $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$.

### 1.2. Hybrid universality

From our point of view, the most important result is so-called (strong) hybrid universality, which is a connection between the Voronin theorem and the classical Kronecker approximation theorem. Denote the distance to the nearest integer by $\| \cdot \|$. The precise definition is as follows.

**Definition.** Hybrid universality for the function $L(s)$ is the following property: Let $K \subset D$, $f \in H(K)$ and $\{\alpha_j\}_{1 \leq j \leq k}$ be real numbers linearly independent over $\mathbb{Q}$. Then for any $\varepsilon > 0$ and any real numbers $\{\theta_j\}_{1 \leq j \leq k}$, we have

$$\liminf_{T \to \infty} \nu_T \{ \| L(s + i\tau) - f(s) \|_K < \varepsilon, \| \tau\alpha_j - \theta_j \| < \varepsilon, 1 \leq j \leq k \} > 0.$$ 

Moreover, we say that the function $L(s)$ is hybridly strongly universal if a function belonging to $H_0(K)$ can be approximated.

The first result on hybrid universality was proved in weaker form by Gonek [10] and slightly improved using different method by Kaczorowski and Kulas [20]. They showed that Dirichlet $L$-functions $L(s, \chi) := \sum_{n=1}^{\infty} \chi(n)n^{-s}$ satisfy the inequality in above definition for $\alpha_n = \log p_n$, where $p_n$ denotes the $n$-th prime number. The second author [36] proved the hybrid universality in the most general form for an axiomatically defined wide class of $L$-functions having Euler product, which contains for instance Dirichlet $L$-functions. Furthermore in [37] the hybrid strong joint universality theorem was proved for $L$-functions without Euler product like Lerch zeta-functions, twists of Lerch zeta-functions and periodic Hurwitz zeta-functions.

As an application of the hybrid universality, we have the following theorem. Recall that a general Dirichlet series is an arbitrary series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad a_n \in \mathbb{C}, \quad \lambda_n \in \mathbb{R}. \tag{1}$$

The authors showed the following universality theorem for certain combinations of $L$-functions in [33, Theorem 1] by using hybrid universality.
**Theorem D.** Suppose that $L(s)$ is hybridly universal, $A(s)$ and $B(s)$ are not identically vanishing general Dirichlet series, which are absolutely convergent in the half-plane $\sigma > 1/2$, and $A(s)$ is non-vanishing in $D$. Let $F(s) := A(s)L(s) + B(s)$, $K \subset D$ and $f(s)$ be continuous on $K$ and analytic in the interior. Moreover assume that $f(s) \neq B(s)$ for all $s \in K$. Then for any $\varepsilon > 0$, it holds that

$$\liminf_{T \to \infty} \nu_T\{\|F(s + i\tau) - f(s)\|_K < \varepsilon\} > 0.$$  

Furthermore, if the function $L(s)$ is hybridly strongly universal then the assumption that $f(s) \neq B(s)$ for all $s \in K$ can be relaxed.

Using the above theorem the authors [33] obtained the universality theorem for zeta-functions associated to symmetric matrices treated by Ibukiyama and Saito [16] (see Theorem G). By improving the method used in [33], the authors [34] proved the universality theorem for the Euler-Zagier multiple zeta-functions (see Theorem I).

**1.3. Contents.** In the present paper, we give the lower and upper bound for the number of zeros of certain polynomials of $L$-functions in Main Theorem 1 and Main Theorem 2, respectively. Main Theorem 1 implies that any polynomial of hybrid universal function with some conditions has infinitely many complex zeros off the critical line. It should be emphasized that there are many zeta-functions whose all zeros lie on a line when they do not satisfy the assumption of Main Theorem 1 (see Remark 2.1).

By using the main theorems, we show that a lot of zeta or $L$-functions have infinitely many complex zeros off the critical line. In Theorem 3.1, we prove that the zeta-functions associated to symmetric matrices (4) treated by Ibukiyama and Saito in [16] have infinitely many zeros in $n/2 < \Re(s) < (n + 1)/2$. Next we show that certain spectral zeta-functions have infinitely many zeros in Theorem 3.2. We also prove in Theorem 3.3 that the Euler-Zagier multiple zeta-functions (9) have zeros in $D$. Furthermore, we show in Propositions 3.5, 3.6 and 3.7 that the Lindelöf hypothesis for these zeta-functions are equivalent to the Lindelöf hypothesis for the Riemann zeta-function. Hence we can say that the Lindelöf hypothesis for zeta-functions mentioned above are independent of the Riemann hypothesis for these zeta-functions.

Moreover, we prove in Theorem 3.9 that the Barnes multiple zeta-functions (11) when $\lambda_1 = \cdots = \lambda_r = 1$ and $a$ is not algebraic irrational have infinitely many zeros in the strip $r - 1/2 < \Re(s) < r$. By using this fact, we show in Theorem 3.10 that the Shintani multiple zeta-functions (13) have infinitely many zeros in $r - 1/2 < \Re(s_1) < r$ and $0 \leq \Re(s_k), 2 \leq k \leq m$ under some extra conditions. As corollaries, we show the same for the Mordell multiple zeta-functions (14), the Euler-Zagier-Hurwitz type of multiple zeta-functions (15), and the Witten multiple zeta-functions (16).

**2. Main Theorems**

**2.1. Statement of main theorems.** In this section, we state the main theorems which give the lower and upper bound for the number of zeros of certain polynomials of $L$-functions. We should mention that Kačinskaitė, Steuding, Siaučiūnas and R. Šleževičienė (see [19, Theorem 2]) showed a special case of the following theorem, namely that $P(L(s))$
has infinitely many zeros to the right of the critical line \( \Re(s) = 1/2 \) when \( P \in \mathbb{C}[X] \) of degree \( \geq 1 \) is not a monomial.

Recall \( D := \{ s \in \mathbb{C} : 1/2 < \Re(s) < 1 \} \). Let us denote by \( \mathcal{D}_s \) the ring of all general Dirichlet series defined as (1) which are absolutely convergent in the half-plane \( \Re(s) > 1/2 \). Let \( P_s \in \mathcal{D}_s[X] \) be a polynomial whose coefficients are in \( \mathcal{D}_s \).

**Main Theorem 1.** Suppose that a function \( L(s) \) is hybridly universal, \( P_s \in \mathcal{D}_s[X] \) is not a monomial but a polynomial with degree greater than zero. Then, the function \( P_s(L(s)) \) has infinitely many zeros in \( D \). More precisely, for any \( 1/2 < \sigma_1 < \sigma_2 < 1 \), there exists a constant \( C > 0 \) such that for sufficiently large \( T \), the function \( P_s(L(s)) \) has more than \( CT \) nontrivial zeros in the rectangle \( \sigma_1 < \sigma < \sigma_2 \), \( 0 < t < T \). Moreover, if the function \( L(s) \) is hybridly strongly universal, the assumption that \( P_s \) is not a monomial can be relaxed.

Note that the cases, for example, \( \zeta(s) + \zeta(2s) \), \( \zeta^2(s) - \zeta(s) \) are included but the case \( \zeta(2s)\zeta(s) \) is excluded in the above theorem by the assumption \( P_s \) is not a monomial. We obtain Theorem C by the above theorem since \( \zeta(s,a) \) where \( a \neq 1/2, 1 \) is rational or transcendental, is hybridly strongly universal.

**Remark 2.1.** It should be emphasized that any zeta-function satisfying the assumption of Main Theorem 1 has infinitely many complex zeros in \( D \). On the other hand, Taylor [41] showed that

\[
\zeta^*(s + 1/2) - \zeta^*(s - 1/2), \quad \zeta^*(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)
\]

has all its zeros on the critical line \( \Re(s) = 1/2 \). Thus there is a linear combination of the Riemann zeta-function multiplied by the Gamma function whose Riemann hypothesis holds. Namely, we can construct a zeta-function whose all zeros lie on the critical line if we relax the assumption of Main Theorem 1. Note that there are many other zeta-functions which have all their zeros on a line (see for example, Hejhal [12], Lagarias and Suzuki [26], and Ki [22]).

**Main Theorem 2.** Suppose that \( L(s) \) is a function defined as a Dirichlet series for \( \sigma > 1 \) which can be continued analytically to a meromorphic function on \( \Re(s) > 1/2 \) with a finite number of poles and all of them lie on the straight line \( \Re(s) = 1 \). Moreover assume \( L(s) \) is a function of finite order and for any fixed \( 1/2 < \Re(s) < 1 \) the square mean-value

\[
\int_0^T |L(\sigma + it)|^2\,dt \ll T \quad \text{as} \quad T \to \infty.
\]

Then, for any polynomial \( P_s \in \mathcal{D}_s \) and any \( \sigma_0 > 1/2 \) we have

\[
N_{P(L)}(\sigma_0,T) \ll T,
\]

where \( N_{P(L)}(\sigma_0,T) \) counts the number of zeros \( \rho \) with multiplicities of \( P_s(L(s)) \) with \( \Re(\rho) > \sigma_0 \) and \( 0 < \Im(\rho) < T \).

**Remark 2.2.** Let \( 0 \neq a \in \mathbb{C} \). Then following the reasoning of the proof of Main Theorems 1 and 2 for the function \( L(s) - a \), one can prove so-called \( a \)-values results of \( L \)-functions, where \( L(s) \) satisfies the assumptions of Main Theorems (see [43, Section 11]).
2.2. Proof of Main theorem 1. In order to prove Main Theorem 1, we quote the following theorem proved by the authors [34, Corollary 2.2].

**Theorem E.** Suppose that a function \( L(s) \) is hybridly universal and \( P_s \in D_s[X] \) is a polynomial with degree greater than zero. Then, for any \( \varepsilon > 0 \) and any function \( f(s) \in H(K) \), it holds that

\[
\liminf_{T \to \infty} \nu_T \left\{ \| P_{s+i\tau}(L(s+i\tau)) - P_s(f(s)) \|_K < \varepsilon \right\} > 0.
\]

Moreover if the function \( L(s) \) is hybridly strongly universal, then \( P_s(f(s)) \) for any function \( f \in H_0(K) \) can be approximated.

**Proof.** For the reader’s convenience, we give a sketch of the proof. H. Bohr in [8] showed that for any Dirichlet series \( \sum_{n=1}^{\infty} a_n n^{-s} \) and every \( \varepsilon > 0 \) we have for an arbitrary \( s \) lying in the half-plane of absolute convergence that

\[
\lim_{T \to \infty} \nu_T \left\{ \sum_{n=1}^{\infty} \frac{|a_n|}{n^{s+i\tau}} - \sum_{n=1}^{\infty} \frac{|a_n|}{n^s} < \varepsilon \right\} > 0.
\]

In fact, Bohr applied the Kronecker theorem only for the sequence \( \{\log p\} \), where \( p \) runs over all primes. To obtain an analogous result for general Dirichlet series defined as (1) it suffices to replace the sequence \( \{\log p\} \subset \{\log n\} \) by the sequence \( \{\lambda_{n_k}\} \subset \{\lambda_n\} \), where \( \{\lambda_{n_k}\} \) is the basis of the vector space over \( \mathbb{Q} \) generated by all \( \lambda_n \). Then since hybrid universality combines diophantine approximations and the universality property, we obtain the theorem. \( \square \)

**Proof of Main Theorem 1.** Firstly, suppose that the function \( L(s) \) is hybridly universal. Let \( z \in \mathbb{C} \) and fix a complex number \( s_0 \in D \). Then \( P_{s_0}(z) \) is a polynomial of variable \( z \). Let \( z_0 \in \mathbb{C} \) be a root of the equation \( P_{s_0}(z) = 0 \). By the assumption \( P_s \) is not a monomial, we can take \( z_0 \neq 0 \). Then we consider the function \( f(s) := z_0 \exp(s - s_0) \) in Theorem E. Because of the definition of \( z_0 \), one has \( P_{s_0}(f(s_0)) = 0 \).

Let \( \mathcal{K} \) be the closed disk whose center is \( s_0 \) and the radius is \( r \). Moreover assume that \( P_s(f(s)) \) does not have zeros on the boundary of \( \mathcal{K} \). Then by (2), we have

\[
\max_{|s-s_0|=r} \left| P_{s+i\tau}(L(s+i\tau)) - P_s(f(s)) \right| < \min_{|s-s_0|=r} \left| P_s(f(s)) \right|.
\]

An application of Rouché’s theorem shows that whenever the inequality (3) holds, the function \( P_{s+i\tau}(L(s+i\tau)) \) has a zero in the interior of \( \mathcal{K} \) (see also [38, Section 8.1]). According to (2), the measure of such \( \tau \in [0, T] \) is greater than \( CT \). This proves the theorem (see also [25, Proof of Theorem 8.4.7]).

If \( P_s \) is a monomial and the function \( L(s) \) is hybridly strongly universal, it is sufficient to consider the function \( s - s_0 \) instead of \( z_0 \exp(s - s_0) \). \( \square \)

2.3. Proof of Main theorem 2. We use the following two lemmas to show Main Theorem 2. The first one is called Littlewood’s lemma (see for example [38, lemma 7.2]).

**Lemma 2.3** (Littlewood). Let \( g(s) \) be regular in and upon the boundary of the rectangle \( \mathcal{R} \) with vertices \( a, a+iT, b+iT, b \), and not zero on \( \Re(s) = b \). Denote by \( \nu(\sigma,T) \) the
number of zeros \( \rho \) inside the rectangle with \( \Re(\rho) > \sigma \) including those with \( \Im(\rho) = T \) but not \( \Im(\rho) = 0 \). Then

\[
\int_{\mathcal{R}} \log g(s) ds = -2\pi i \int_{a}^{b} \nu(\sigma, T) d\sigma.
\]

Lemma 2.4. Assume that \( f(s) \) is defined as \((1)\) for \( \Re(s) > \sigma_0 \). Then there exists \( \theta \in \mathbb{R} \) such that \( e^{\theta_0} f(s) \to c \neq 0 \) as \( \Re(s) \to \infty \). Particularly, there exists \( \sigma_f \geq \sigma_0 \) such that \( f(s) \neq 0 \) for \( \Re(s) > \sigma_f \).

Proof. Without loss of generality, we can assume that \( a_n \neq 0 \) and \( \lambda_n \neq \lambda_m \) for \( n \neq m \). Put \( \lambda_{n_0} := \min\{\lambda_m : m = 0, 1, 2, \ldots\} \). Obviously \( \lambda_{n_0} > -\infty \). Then, if \( \lambda_{n_0} > 0 \), we easily see that \( e^{\lambda_{n_0} s} f(s) \to a_{n_0} \neq 0 \) as \( \Re(s) \to \infty \). Otherwise, we have \( e^{-\lambda_{n_0} s} f(s) \to a_{n_0} \neq 0 \) as \( \Re(s) \to \infty \). \( \square \)

Proof of Main Theorem 2. Without loss of generality, by Lemma 2.4, we can assume that there is \( \sigma_2 > 1 \) such that \( F(s) := P_s(L(s)) \neq 0 \) for \( \Re(s) > \sigma_2 \).

Let \( 1/2 < \sigma_1 < \sigma_0 \), and \( T_0 > 0 \) be such that \( F(\sigma + it) \) is regular for \( |t| > T_0 \) and \( \sigma > 1/2 \). Then using Littlewood’s lemma and comparing the imaginary parts gives

\[
2\pi \int_{\sigma_1}^{\sigma_2} N_F(\sigma, T)d\sigma = \int_{T_0}^{T} \log |F(\sigma + it)| dt - \int_{T_0}^{T} \log |F(\sigma + it)| dt \\
\quad - \int_{\sigma_1}^{\sigma_2} \arg F(\sigma + i(T_0 + T)) d\sigma + \int_{\sigma_1}^{\sigma_2} \arg F(\sigma + iT_0) d\sigma
\]

\[
= I_1 + I_2 + I_3 + I_4, \quad \text{say}.
\]

One can easily show that \( I_2 + I_4 \ll T \) (see for example [38, the proof of Theorem 7.1]). Let \( m \) be the degree of \( P_s(X) \). By Jensen’s inequality and convexity of logarithm we get

\[
I_1 = \frac{m}{2} \int_{T_0}^{T} \log |F(\sigma + it)|^{2/m} dt \ll T \log \frac{1}{T - T_0} \int_{T_0}^{T} |F(\sigma + it)|^{2/m} dt
\]

\[
\ll T \log \frac{1}{T} \sum_{n=0}^{m} \int_{0}^{T} |L(\sigma + it)|^{2n/m} dt \ll T \log \frac{1}{T} \int_{0}^{T} |L(\sigma + it)|^{2} dt \ll T.
\]

In order to evaluate \( I_3 \) let us observe that \( |\arg F(\sigma + i(T_0 + T))| \leq (N + 1)\pi \), where \( N \) denotes the number of zeros of \( \Re(F(\sigma + i(T_0 + T))) \) for \( \sigma \in [\sigma_1, \sigma_2] \). To estimate \( N \), put

\[
g(z) = \frac{1}{2} \left( F(z + i(T_0 + T)) + \overline{F(z + i(T_0 + T))} \right),
\]

\( R = \sigma_2 - \sigma_1 \) and \( T \) larger than \( 2R \). Then it is easy to see that \( g(z) \) is analytic for \( |z - \sigma_2| < T \) and \( N \leq n(R) \), where \( n(r) \) denotes the number of zeros of \( g(z) \) in \( |z - \sigma_2| \leq r \). Then using Jensen’s formula (see [43, § 3.61]), the fact that \( L(s) \) is a function of finite order and the following inequality

\[
\int_{0}^{2R} \frac{n(r)}{r} dr \geq n(R) \log 2
\]

gives the estimate \( I_3 \ll \log T \).
Collecting all estimates, we obtain
\[ N_F(\sigma_0, T) \leq \frac{1}{\sigma_0 - \sigma_1} \int_{\sigma_1}^{\sigma_0} N_F(\sigma, T) d\sigma \ll \int_{\sigma_1}^{\sigma_2} N_F(\sigma, T) \ll T, \]
which completes the proof. \qed

3. Applications

3.1. Zeros of the zeta-functions associated to symmetric matrices. Zeta-functions associated to the prehomogeneous vector space \( V_n \) of \( n \times n \) rational symmetric matrices are interesting for several reasons. For example, their special values at non-positive integers describe the contribution of central unipotent elements to the dimensions of the Siegel modular form. For the case \( n = 2 \), these zeta-functions have been investigated by Siegel, Morita, Shintani, Sato and Arakawa. For the case \( n \geq 3 \), explicit forms of the zeta-functions associated to symmetric matrices have been proved by Ibukiyama and Saito [16].

We use the same definitions and notation as in [16]. For each field \( F \) in \( \mathbb{C} \), we put \( V_n(F) = V_n \otimes_{\mathbb{Q}} F \). Then the pair \((GL_n(\mathbb{C}), V_n(\mathbb{C}))\) is a prehomogeneous vector space through the action: \( V_n(\mathbb{C}) \to gV_n(\mathbb{C}) \) for \( g \in GL_n(\mathbb{C}) \). Denote by \( V_n^j \) the subset of \( V_n(\mathbb{R}) \) consisting of matrices with \( j \) positive and \( n-j \) negative eigenvalues. We fix a lattice \( L \subset V_n(\mathbb{R}) \) which is invariant under \( SL_n(\mathbb{Z}) \) and put \( L^{(j)} := L \cap V_n^j \) and denote by \( L^{(j)}/\sim \) the set of \( SL_n(\mathbb{Z}) \) equivalence classes in \( L^{(j)} \). Then, except for the case \( n = 2 \) and \( j = 1 \), the zeta-functions \( \zeta_j(s, L) \) of signature \((j, n-j)\) attached to \( L \) are given by the following series

\[ \zeta_j(s, L) := \frac{2 \prod_{k=1}^{n} \Gamma(k/2)}{\pi^{n(n+1)/4}} \sum_{x \in L^{(j)}/\sim} \frac{\mu(x)}{|\det x|^s} \]

where \( \mu(x) \) is defined as follows: let \( G_+ := \{ g \in GL_n(\mathbb{R}) : \det g > 0 \} \), and \( dg := (\det g)^{-n} \prod_{1 \leq j, k \leq n} dg_{jk} \) be the measure on \( G_+ \). As a measure on \( V_n^j \), we take \( dy := \prod_{1 \leq j, k \leq n} dy_{jk} \). For \( x \in L^{(j)} \), let \( U \) be a relatively compact open set in \( V_n^j \) and let \( Y \) be the domain in \( G_+ \) which is mapped to \( U \) by \( x \mapsto gx^t \). Let \( \Gamma_x \) be the stabilizer of \( x \) in \( SL_n(\mathbb{Z}) \), and \( Y_0 \) a fundamental domain with respect to the right action of \( \Gamma_x \). The ratio \( \mu(x) := \int_{Y_0} dg / \int_U |y|^{-(n+1)/2} dy \) is finite and independent of the choice of \( U \) unless \( n = 2 \) and \( j = 1 \). For a ring \( R \), we denote by \( S_n(R) \) the set of all symmetric matrices with coefficients in \( R \) and by \( S_n(R)_e \) its subset consisting of even elements, that is, the elements whose diagonal elements are contained in \( 2R \). By [16, Lemma 1.1], it is enough to consider the cases \( L_n := S_n(\mathbb{Z}) \) or \( L_n^* := S_n(\mathbb{Z})_e/2 \) when \( n \geq 3 \). Note that \( \zeta_j(s, L) \), where \( L = L_n \) or \( L_n^* \) depends only on the determinant \( \eta := (-1)^{n-j} \) and the Hasse invariant \( \theta := (-1)^{(n-j)(n-j+1)/2} \) of \( x \in V_n^j \) (see [16, Section 2]). Hence we denote

\[ \zeta_{\eta, \theta}(s, L) := \zeta_j(s, L) \]
that the spectral zeta function henceforward we consider only such nonnegative increasing sequences. Then one can show

$$\sigma \in \text{the rectangle}$$

Theorem 3.1.

have functional equations (see [16, Introduction]).

Main Theorems. It should be noted that zeta-functions associated to symmetric matrices of constant curvature in terms of special values of the Selberg zeta-function.

considerable attention by many researchers including D’hoker and Phong, Sarnak, and

of evaluating the determinants of the Laplacians on Riemann manifolds has received

(5) $$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots; \quad \lambda_k \to \infty, \quad k \to \infty;$$

henceforward we consider only such nonnegative increasing sequences. Then one can show

that the spectral zeta function

$$Z(s) := \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s};$$
which is known to converge absolutely in a half-plane $\Re(s) > \sigma_0$ for some $\sigma_0 \in \mathbb{R}$. Osgood, Phillips and Sarnak [35] defined the determinant of the Laplacian $\Delta$ on the compact manifold $M$ by $\det'\Delta := \prod_{\lambda_k \neq 0} \lambda_k$, where $\{\lambda_k\}$ is the sequence of eigenvalues of the the Laplacian $\Delta$ on $M$. The sequence $\{\lambda_k\}$ is known to satisfy the condition (5), but the product $\prod_{\lambda_k \neq 0} \lambda_k$ is always divergent. In order to make sense for the product, we must use some sort of regularization. It is easily seen that, formally, $e^{-Z'(0)}$ is the product of nonzero eigenvalues of $\Delta$. This product does not converge, but $Z(s)$ can continued analytically to a neighborhood of $s = 0$. Thus we can give a meaningful definition

$$\det'\Delta := \exp(-Z'(0)),$$

which is called the Functional Determinant of the Laplacian $\Delta$ on $M$.

We consider the sequence of eigenvalues on the standard Laplacian $\Delta_{S^n}$ on the $n$-dimensional sphere $S^n$ (see for example [42, Chapter 8.3]). The Laplacian $\Delta_{S^n}$ has eigenvalues $\mu_k := k(n+1)$ with multiplicity

$$\lambda_k := \mu_k + \left(\frac{n-1}{2}\right)^2 = \left(k + \frac{n-1}{2}\right)^2, \quad k \in \mathbb{N}.$$  

From now on we consider the shifted sequence $\{\lambda_k\}$ of $\{\mu_k\}$ by $(n-1)^2/4$ as a fundamental sequence. Then the sequence $\{\lambda_k\}$ is written in the following tractable form:

$$\lambda_k := \mu_k + \left(\frac{n-1}{2}\right)^2 = \left(k + \frac{n-1}{2}\right)^2$$

with the same multiplicity as in (6). Hence their corresponding spectral zeta-functions $Z_{S^n}(s)$, $n = 1, 2, 3$ are as follows (see [40, Chapter 5]):

$$Z_{S^1}(s) = 2\zeta(2s), \quad Z_{S^2}(s) = (2^{2s} - 2)\zeta(2s - 1) - 4^s, \quad Z_{S^3}(s) = \zeta(2s - 2) - 1,$$

$$Z_{S^4}(s) = \frac{1}{3}(2^{2s-3} - 1)\zeta(2s - 3) - \frac{1}{3}\left(2^{2s-3} - \frac{1}{4}\right)\zeta(2s - 1) - \frac{1}{3}\left(\frac{2}{3}\right)^{2s-3} + \frac{1}{8}\left(\frac{2}{3}\right)^{2s}.$$

By the definition of the sequence $\{\lambda_k\}$ and their multiplicity, we have

$$Z_{S^n}(s) = A_{S^n}(s)\zeta(2s - n + 1) + B_{S^n}(s),$$

where $A_n(s)$ is a Dirichlet polynomial and $B_n(s)$ is a general Dirichlet series which is absolutely convergent in the half-plane $\sigma > n/2 - 1/4$ (see also the proof of Proposition 3.6). Therefore we obtain the following theorem by Main Theorems.

**Theorem 3.2.** Let $n \geq 2$. Then for any $n/2 - 1/4 < \sigma_1 < \sigma_2 < n/2$ the spectral zeta-function $Z_{S^n}(s)$ has $\asymp T$ nontrivial zeros in the rectangle $\sigma_1 < \sigma < \sigma_2, 0 < t < T$.

### 3.3. Zeros of the Euler-Zagier multiple zeta-functions

The Euler-Zagier multiple zeta-functions $\zeta_r(s_1, \ldots, s_r)$ are defined by

$$\zeta_r(s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}, \quad \Re(s_1) > 1, \quad \Re(s_j) \geq 1, \quad 2 \leq j \leq r.$$  

When $r = 2$, Atkinson [4] obtained an analytic continuation for $\zeta_2(s_1, s_2)$ in order to study the mean square $\int_0^T |\zeta(1/2 + it)|^2 dt$. He obtained an explicit formula for $\int_0^T |\zeta(1/2 + it)|^2 dt$.
by using the harmonic product formula
\[ \zeta(s_1)\zeta(s_2) = \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) + \zeta(s_1 + s_2). \]

More than fifty years later, Zhao [49, Theorem 5], and Akiyama, Egami and Tanigawa [1, Theorem 1], independently, published the following analytic continuation for the Euler-Zagier multiple zeta-functions (see also a survey of Matsumoto [29]).

**Theorem H** ([49, Theorem 5]). The Euler-Zagier multiple zeta-function \( \zeta_r(s_1, \ldots, s_r) \) can be analytically continued to a meromorphic function on all of \( \mathbb{C}^r \) with possible poles at \( s_1 = 1 \) and \( s_1 + \cdots + s_k = k - n \) for \( 1 \leq k \leq r \) and non-negative integer \( n \). Moreover all the poles are simple.

The values of the Euler-Zagier zeta-functions (after the continuation) at negative integer points are discussed by Akiyama and Tanigawa [3]. Moreover, Matsumoto showed a functional equation for the Euler-Zagier double zeta-functions in [28].

In the 1990s it was recognized that the values of the Euler-Zagier multiple zeta-function at integer points (multiple zeta-values) are quite important in the theory of quantum groups, knot theory, and so on. A lot of research on multiple zeta-values has then been done. For example, explicit expressions such as
\[ \zeta_r(2, \ldots, 2) = \frac{\pi^{2r}}{(2r+1)!}, \quad \zeta_{2r}(3, 1, \ldots, 3, 1) = \frac{2r^{4r}}{(4r+2)!} \]

have been shown. In recent years, many relations among multiple zeta-values were discovered by a lot of mathematicians, for instance, Ihara, Hoffman, Kaneko, Kawashima, Ohno and Zagier (see Hoffman’s web page).

Using the proof of Hoffman’s relation [15, Theorem 2.1], one can show the following relation (see also [34, Lemma 3.4]). Let \( \Pi \) denote a partition of the set \( \{1, 2, \ldots, r\} \). Moreover, for \( \Pi = \{\varpi_1, \ldots, \varpi_l\} \), put
\[ c(\Pi) = (-1)^{r-1} \prod_{j=1}^{l} (|\varpi_j| - 1)! \quad \text{and} \quad \zeta(s_1, \ldots, s_r; \Pi) = \prod_{j=1}^{l} \zeta\left( \sum_{k \in \varpi_j} s_k \right). \]

Then, one can prove the following. For any \( s_1, \ldots, s_r \) except for singularity, we have
\[ \sum_{\sigma \in \Sigma_r} \zeta_r(s_{\sigma(1)}, \ldots, s_{\sigma(r)}) = \sum_{\text{partitions } \Pi \text{ of } \{1, \ldots, r\}} c(\Pi)\zeta(s_1, \ldots, s_r; \Pi), \]

where \( \Sigma_r \) denotes the symmetric group of degree \( r \). Hence, there exists a polynomial \( Z^\#_r(s) \) whose coefficients are Dirichlet series absolutely convergent in the half-plane \( \Re(s) > 1/2 \), such that \( \zeta_r(s, s, \ldots, s) = Z^\#_r(\zeta(s)) \). For example, we have
\[ \zeta_2(s, s) = \frac{1}{2} \zeta(s)^2 - \frac{1}{2} \zeta(2s), \quad \zeta_3(s, s, s) = \frac{1}{6} \zeta(s)^3 - \frac{1}{2} \zeta(s)\zeta(2s) + \frac{1}{3} \zeta(3s). \]

By the relation (10), the authors showed the following theorem.

**Theorem I** ([34, Theorem 3.5]). Let \( g(s) \) be a function such that only \( \zeta(s) \) is replaced by \( f(s) \in H(K) \) in \( \zeta(s_1, \ldots, s_r; \Pi) \) and \( Z^\#_r(f(s), s) := \sum_{\Pi_r} c(\Pi_r)g(s)/r! \). Then for every \( \varepsilon > 0 \), it holds that
\[ \liminf_{T \to \infty} \nu_T \{ ||\zeta_r(s + i\tau, \ldots, s + i\tau) - Z^\#_r(s) ||_K \} > 0. \]
It should be noted that the first author showed the universality for the Euler-Zagier-Hurwitz type of multiple zeta-functions in [32, Theorem 2.1]. Moreover he obtained relations between the zero-free region and the (joint) universality for Euler-Zagier-Hurwitz type of multiple zeta-functions (15). Afterwards, the authors showed not only a simple proof of it but also the more general cases in [34, Theorems 3.2 and 3.3] by the hybrid universality.

Zhao [49, Section 5] obtained trivial zeros of the Euler-Zagier multiple zeta-functions. By analogy with the Riemann zeta-function, he propounded the following problem: “Determine the complete set of trivial (resp. nontrivial) zeros of the multiple zeta-functions” in [49, Problem 1]. For nontrivial zeros, we obtain the following theorems and corollaries by the equation (10) and Main Theorems.

**Theorem 3.3.** For any $1/2 < \sigma_1 < \sigma_2 < 1$, the Euler-Zagier multiple zeta-function $\zeta_r(s_1, \ldots, s_r)$, $r \geq 2$ has $\asymp T$ nontrivial zeros in the rectangle $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$.

**Corollary 3.4.** The Euler-Zagier multiple zeta-function $\zeta_r(s_1, \ldots, s_r)$ has zeros in $1/2 < \Re(s_1), \ldots, \Re(s_r) < 1$.

### 3.4. Lindelöf hypothesis

The Lindelöf hypothesis for the Riemann zeta-function says that $\zeta(1/2 + it) = O(|t|^\varepsilon)$. It is widely known that the Riemann hypothesis for $\zeta(s)$ implies the Lindelöf hypothesis. Obviously, the Lindelöf hypothesis for the function

$$\zeta_N(s) := \zeta(s) - \sum_{n=1}^{N} n^{-s}$$

is equivalent to the Lindelöf hypothesis for $\zeta(s)$. Thus we can call $\Re(s) = 1/2$ the critical line for $\zeta_N(s)$ since the estimation $\zeta_N(1/2 + it) = O(|t|^\varepsilon)$ is equivalent to $\zeta(1/2 + it) = O(|t|^\varepsilon)$. However, $\zeta_N(s)$ has infinitely many zeros off the critical line for $\zeta_N(s)$ by Main Theorems. More precisely, there exist a constant $C > 0$ such that for sufficiently large $T$ the function $\zeta_N(s)$ has more than $C T$ nontrivial zeros in the rectangle $0 < t < T$, $\sigma_1 < \sigma < \sigma_2$ for any $1/2 < \sigma_1 < \sigma_2 < 1$.

Next consider the Lindelöf hypothesis for $\zeta_{n,\theta}(s, L)$ which states

$$\zeta_{n,\theta}(n/2 + it, L) = O(|t|^\varepsilon).$$

By using (4), the estimation $\zeta(1 + 2it) = O(\log t)$ (see [44, Theorem 5.16]), and the fact that $b_n(s; L)$ and $A_n(s; L)$ are absolutely convergent general Dirichlet series for any odd $n \geq 3$, we have the following proposition.

**Proposition 3.5.** The Lindelöf hypothesis for $\zeta(s)$ is true if and only if the Lindelöf hypothesis for $\zeta_{n,\theta}(s, L)$ is true for each $n \in 2\mathbb{N} + 1$.

**Proof.** It is well-know that

$$0 < \frac{\zeta(2\sigma)}{\zeta(\sigma)} = \prod_p \frac{1}{1 + p^{-\sigma}} \leq |\zeta(s)| \leq \zeta(\sigma), \quad \sigma > 1.$$ 

Hence $b_n(s; L)$ and $A_n(s; L)$ are non-vanishing and bounded when $\Re(s) = n/2$. Moreover, we have $B_n(s; L) = O(\log t)$ from its definition, $\zeta(1 + 2it) = O(\log t)$ and the estimation above. Therefore we obtain this proposition by (4). \qed
Similarly, we can define the Lindelöf hypothesis for $Z_{S^r}(s)$ which states
\[ Z_{S^r}(n/2 - 1/4 + it) = O(|t|^\varepsilon). \]
We obtain the following proposition by (8).

**Proposition 3.6.** The Lindelöf hypothesis for $\zeta(s)$ is true if and only if the Lindelöf hypothesis for $Z_{S^r}(s)$ is true for each $n \geq 2$.

**Proof.** Suppose $n \geq 2$ and $\Re(s) > n/2$. Then we have
\[
Z_{S^r}(s) = \sum_{k=1}^{\infty} \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!} \left( k + \frac{n-1}{2} \right)^{-2s} = A_{S^r}(s) \zeta(2s-n+1) + \ldots,
\]
where $A_{S^r}(s)$ is defined by
\[
A_{S^r}(s) := \frac{2}{(n-1)!} \begin{cases} 1 & n \text{ odd}, \\ 2^{2s-n+1} - 1 & n \text{ even}, \end{cases}
\]
from (6) and (7). Obviously, $A_{S^r}(s)$ does not vanish on the line $\Re(s) = n/2 - 1/4$. Therefore we obtain this proposition by (8). \qed

Finally, we can consider the Lindelöf hypothesis for the Euler-Zagier multiple zeta-function $\zeta_r(s, \ldots, s)$ which states that
\[ \zeta_r(1/2 + it, \ldots, 1/2 + it) = O(|t|^\varepsilon). \]
Related to this problem, Ishikawa and Matsumoto [18] proved an upper bound estimates for the Euler-Zagier multiple zeta-functions. Kiuchi and Tanigawa [23] considered the problem of an order of magnitude for the triple zeta-functions of Euler-Zagier type in the region $0 < \Re(s_k) \leq 1$, $1 \leq k \leq r$. It should be noted that Huxley [14] showed $\zeta(1/2 + it) = O(t^{H+\varepsilon})$, where $H := 32/205 = 0.15609\ldots$. Thus we have
\[ \zeta_r(1/2 + it, \ldots, 1/2 + it) = O(|t|^{rH+\varepsilon}) \]
by the Hoffman formula (10). As an analogue of the zeta-functions associated to symmetric matrices, we have the following result.

**Proposition 3.7.** The Lindelöf hypothesis for $\zeta(s)$ is true if and only if the Lindelöf hypothesis for $\zeta_r(s, \ldots, s)$ is true for any $r \geq 2$.

**Proof.** Suppose the Lindelöf hypothesis for Riemann zeta-function $\zeta(s)$ is true. By the harmonic product formula, we have
\[ \zeta^2(1/2 + it) = 2\zeta_2(1/2 + it, 1/2 + it) + \zeta(1 + 2it), \quad t \neq 0. \]
Because of $\zeta(1 + 2it) = O(\log t)$, the Lindelöf hypothesis for $\zeta_2(s, s)$ is true. Obviously, the opposite is also true. Using the Hoffman formula (10), we have
\[ \zeta_3(1/2+it, 1/2+it, 1/2+it) = \zeta^3(1/2+it)/6 - \zeta(1/2+it)\zeta(1+2it)/2 + \zeta(3/2+3it), \quad t \neq 0. \]
Thus we obtain the case $r = 3$. Similarly, by the Hoffman formula (10), we have
\[
\zeta_r(1/2 + it, \ldots, 1/2 + it) = \zeta^r(1/2 + it)/r!
+ O(\zeta^{r-2}(1/2 + it) \log t) + O(\zeta^{r-3}(1/2 + it) \log t) + \cdots + O(\log^r t).
\]
This equality implies Proposition 3.7. \qed
Remark 3.8. The Lindelöf hypothesis for $\zeta_N(s)$, $\zeta_{n,\theta}(s, L)$, $n \in 2\mathbb{N} + 1$, $\zeta_{S^n}(s)$, $n \geq 2$ or $\zeta_r(s, \ldots, s)$, $r \geq 2$ is equivalent to the Lindelöf hypothesis for $\zeta(s)$. However, these functions have infinitely many zeros off the line $\Re(s) = 1/2$, $\Re(s) = n/2$, $\Re(s) = n/2 - 1/4$, or $\Re(s) = 1/2$, respectively. Hence we can say that the Lindelöf hypothesis for $\zeta_{n,\theta}(s, L)$, $n \in 2\mathbb{N} + 1$, $\zeta_{S^n}(s)$, $n \geq 2$ and $\zeta_r(s, \ldots, s)$, $r \geq 2$ are independent of the Riemann hypothesis for these zeta-functions.

3.5. Zeros of the Barnes and Shintani multiple zeta-functions. Barnes [6] considered the multiple sum of the form

$$
\zeta_r(s, a | \Lambda) := \sum_{n_1, \ldots, n_r = 0}^{\infty} (\lambda_1n_1 + \cdots + \lambda_r n_r + a)^{-s}, \quad \Re(s) > r \geq 2,
$$

where $a, \lambda_1, \ldots, \lambda_r$ are complex numbers satisfy some conditions. Nowadays this is called the Barnes $r$-tuple zeta-function. Barnes proved that the function $\zeta_r(s; a | \Lambda)$ can be continued meromorphically to the whole $s$-plane and is holomorphic except for simple poles at $s = 1, \ldots, r$. Barnes defined the multiple gamma function by $\zeta_r(s, a | \Lambda)$ and studied its properties. Afterwards many mathematicians have studied properties of the Barnes multiple zeta-functions (see for example [29, Section 1]).

In this paper, we concern only the simple case $0 < a$ and $\lambda_1 = \cdots = \lambda_r = 1$. For simplicity, in this case we write $\zeta_r(s, a)$ instead of $\zeta_r(s, a | \Lambda)$. In [40, p. 86], it was showed that

$$
\zeta_r(s, a) = \sum_{j=0}^{r-1} p_{rj}(a)\zeta(s - j, a), \quad p_{rj}(a) := \frac{1}{(r-1)!} \sum_{l=j}^{r-1} (-1)^{r+1-j}(\binom{l}{j}) s(r, l+1)a^{l-j},
$$

where $s(r, l+1)$ is the Stirling number of the first kind. For example, one has

$$
\zeta_2(s, a) = (1 - a)\zeta(s, a) + \zeta(s - 1, a),
$$
$$
\zeta_3(s, a) = (a^2 - 3a + 2)\zeta(s, a)/2 + (3 - 2a)\zeta(s - 1, a)/2 + \zeta(s - 2, a)/2.
$$

Recall that $\zeta(s, a)$ with $a = 1, 1/2$ is hybridly universal, and $\zeta(s, a)$ with rational $a \neq 1, 1/2$ or transcendental $a$ is strongly hybridly universal (see Section 1). Therefore, on zeros for $\zeta_r(s, a)$ with $r \geq 2$, we have the following theorems by the equation (12), Main Theorems and the fact mentioned above.

Theorem 3.9. Suppose that $a$ is not algebraic irrational. Then for any $r - 1/2 < \sigma_1 < \sigma_2 < r$, the function $\zeta_r(s, a)$ with $r \geq 2$ has $\asymp T$ nontrivial zeros in the rectangle $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$.

Next we consider the Shintani multiple zeta-functions defined by

$$
\zeta_r(s, a | \Lambda) := \sum_{n_1, \ldots, n_r = 0}^{\infty} \prod_{l=1}^{m} (\lambda_1(n_1 + a_1) + \cdots + \lambda_r(n_r + a_r))^{-s_l},
$$

where $a_k, \lambda_k > 0$, $1 \leq l \leq m$ and $1 \leq k \leq r$. Imai [17] and Hida [13] considered the above series or more generally with characters in the numerator. In [13, Lemma 2.4.1], it was showed that $\zeta_r(s, a | \Lambda)$ converges absolutely and uniformly on any compact subset in the region $\Re(s_l) > r/m$ for all $1 \leq l \leq m$. Moreover, in [13, Theorem 2.4.1], it was proved that $\zeta_r(s, a | \Lambda)$ can be continued to the whole space $\mathbb{C}^m$ as a meromorphic function.
Obviously, this is a generalization of the Barnes multiple zeta-function defined by (11). Shintani (see for example [39]) introduced the above series with \( s_1 = \cdots = s_m \) in order to research the Barnes multiple gamma function. Cassou-Noguès (see for example [9]) inspired by Shintani's work, considered those multiple series of the form that the numerator of (13) is multiplied by certain roots of unity with \( s_1 = \cdots = s_m \). She proved its meromorphic continuation and gave applications to \( L \)-functions and \( p \)-adic \( L \)-functions of totally real number fields (see a survey [29, Section 2]).

By Theorem 3.10, we obtain the following theorem.

**Theorem 3.10.** Suppose that \( \lambda_{11} = \cdots = \lambda_{1r} = 1 \), and \( a_1 + \cdots + a_r \) is not algebraic irrational. Then \( \zeta_r(s, a \mid \Lambda) \) has zeros in \( r - 1/2 < \Re(s_1) < r \) and \( 0 \leq \Re(s_k), 2 \leq k \leq m \).

The Shintani multiple zeta-functions contain many important multiple zeta-functions. For instance, the Mordell multiple zeta-functions, the Euler-Zagier-Hurwitz type of multiple zeta-functions, and the Witten multiple zeta-functions are contained.

For example, the Mordell multiple zeta-functions are defined by

\[
\sum_{n_1, \ldots, n_r = 1}^{\infty} n_1^{-s_1} \cdots n_r^{-s_r} (n_1 + \cdots + n_r + a)^{-s_{r+1}}, \quad a \geq 0
\]

Tornheim [45] considered the values at positive integer when \( a = 0 \) and \( r = 2 \). Mordell [30] studied the case \( a = 0, r = 2 \) and \( s_1 = s_2 = s_3 \), and also studied the case \( s_1 = \cdots = s_{r+1} \). By Theorem 3.10, we can see that the Mordell multiple zeta-functions have infinitely many zeros in \( r - 1/2 < \Re(s_{r+1}) < r \) and \( 0 \leq \Re(s_j), 1 \leq j \leq r \), unless \( a \) is algebraic irrational.

Next, let us introduce the basic properties of the Euler-Zagier-Hurwitz type of multiple zeta-functions defined by

\[
\zeta_r(s_{(1, \ldots, r)}; a_{(1, \ldots, r)}) := \sum_{s_{n_1} > s_{n_2} > \cdots > s_{n_r} \geq 0} \frac{1}{(n_1 + a_1)^{s_1} (n_2 + a_2)^{s_2} \cdots (n_r + a_r)^{s_r}}
\]

\[
= \sum_{s_{n_1}, \ldots, s_{n_r} = 1}^{\infty} \frac{1}{(n_1 + \cdots + n_r + a_1)^{s_1} (n_2 + \cdots + n_r + a_2)^{s_2} \cdots (n_r + a_r)^{s_r}},
\]

where \( s_{(1, \ldots, r)} := (s_1, \ldots, s_r) \in \mathbb{C}^r \) and \( a_{(1, \ldots, r)} := (a_1, \ldots, a_r) \in (0, 1]^r \). This series absolutely converges for \( \Re(s_{j}) > 1 \) and \( \Re(s_j) \geq 1, 2 \leq j \leq r \). Obviously \( \zeta_r(s_{(1, \ldots, r)}; a_{(1, \ldots, r)}) \) is a generalization of the Euler-Zagier multiple zeta-functions (9). The meromorphic continuation of \( \zeta_r(s_{(1, \ldots, r)}; a_{(1, \ldots, r)}) \) to \( \mathbb{C}^r \) has been accomplished by Akiyama and Ishikawa [2] using the Euler-Maclaurin summation formula, Matsumoto [27] using the Mellin-Barnes formula, and by Murty and Sinha [31] using the binomial theorem and Hartog’s theorem. Moreover, Kellinger and Masri [21] not only proved a meromorphic continuation but also provided a way to locate trivial zeros of \( \zeta_r(s_{(1, \ldots, r)}; a_{(1, \ldots, r)}) \). The first author [32] showed the existence of the zeros for \( \zeta_r(s_{(1, \ldots, r)}; a_{(1, \ldots, r)}) \) in the region of absolute convergence when \( a_1, \ldots, a_r \) are algebraically independent. He also showed \( \zeta_r(s_{(1, \ldots, r)}; a_{(1, \ldots, r)}) \) has infinitely many zeros in \( 1/2 < \Re(s_1) < 1 \) when \( a_1 \) is transcendental and \( \zeta_{r-1}(s_{(2, \ldots, r)}; a_{(2, \ldots, r)}) \neq 0 \) for fixed \( \Re(s_2) > 3/2 \) and \( \Re(s_l) \geq 1, 3 \leq l \leq r \). By Theorem 3.10, we can see that the Euler-Zagier-Hurwitz type of multiple zeta-functions have infinitely many zeros in \( r - 1/2 < \Re(s_1) < r \) and \( 0 \leq \Re(s_{j}), 2 \leq j \leq r \), unless \( a_1 \) is algebraic irrational.
At the end, let us define the Witten multiple zeta-functions. Let \( g \) be a complex semisimple Lie algebra, and define
\[
\zeta(s ; g) := \sum_{\rho} \left( \dim \rho \right)^{-s},
\]
where \( \rho \) runs over all finite dimensional irreducible representations of a certain semisimple Lie algebra \( g \). Special values of this Dirichlet series were firstly studied by Witten [47] in connection with quantum gauge theory. And Zagier [48] called the above function as the Witten zeta-function associated with \( g \). Some evaluation formulas for \( \zeta(s ; g) \) at positive even integral integers were given by Zagier [48], Gunnells and Sczech [11]. In order to analyze this multiple series closely, Komori, Matsumoto and Tsumura [24] introduced the following multi-variable version of the series.

Let \( r \) be the rank of \( g \). Denote by \( \Delta = \Delta(g) \) the set of all roots of \( g \), by \( \Delta_+ = \Delta_+(g) \) the set of all positive roots of \( g \). For any \( \alpha \in \Delta \), let \( \alpha^\vee \) be the associated coroot.

Define the Witten multiple zeta-functions associated with semisimple Lie algebras by
\[
\zeta_r(s ; g) := \sum_{n_1, \ldots, n_r=1}^{\infty} \prod_{\alpha \in \Delta_+} \left( \langle \alpha^\vee, n_r \lambda_1 + \cdots + n_r \lambda_r \rangle \right)^{-s_\alpha},
\]
where \( s := (s_\alpha)_{\alpha \in \Delta_+} \in \mathbb{C}^m \) (here \( m = |\Delta_+| \) is the number of positive roots of \( g \)). When \( g \) is type of \( X_r \), where \( X = A, B, C, D, E, F, G \), Komori, Matsumoto and Tsumura [24] gave explicit formulas of (16). For example (see [24, (2.3)]), we have
\[
\zeta_r(s ; A_r) = \sum_{n_1, \ldots, n_r=1}^{\infty} \prod_{1 \leq j < k \leq r+1} (n_k + \cdots + n_{j-1})^{-s_{jk}}, \quad s := (s_{jk}) \in \mathbb{C}^{r(r+1)/2}.
\]
Hence, [24, Theorem 5.4] and Theorem 3.10 imply that the Witten multiple zeta-functions also have infinitely many zeros when \( g = A_r \) with \( r \geq 2 \) (by [24, (2.3)]), \( B_r \) with \( r \geq 2 \) (by [24, (2.9)]), \( C_r \) with \( r \geq 2 \) (by [24, (2.15)]), \( D_r \) with \( r \geq 3 \) (by \( D_2 \simeq A_1 \oplus A_1 \) and [24, (2.21)]), \( E_6, E_7, E_8, F_4, G_2 \) (by [24, diagrams in p. 383 and 384]).

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