Bounds on the Wireless MapReduce NDT-Computation Tradeoff

Yue Bi and Michèle Wigger

LTCE, Telecom Paris, IP Paris, 91120 Palaiseau, France
{bi, michele.wigger}@telecom-paris.fr

Abstract—We consider a full-duplex wireless Distributed Computing (DC) system under the MapReduce framework. New upper bound and lower bounds on the tradeoff between Normalized Delivery Time (NDT) and computation load are obtained. The lower bound is proved through an information-theoretic converse. The upper bound is based on a novel IA scheme tailored to the interference cancellation capabilities of the nodes and improves over existing bounds.

Index Terms—Wireless distributed computing, MapReduce, coded computing, interference alignment.

I. INTRODUCTION

Distributed Computing (DC) systems are computer networks that by parallelizing can reduce execution times of complex computing tasks such as federated learning or computer vision. MapReduce is a popular such framework and runs in three phases [1], [2]. In the first map phase, nodes calculate intermediate values (IVA) from their associated input files. In the following shuffle phase, nodes exchange these IVAs in a way that each node obtains all IVAs required to compute its assigned output function in the final reduce phase. MapReduce is primarily applied to wired systems, for which various coding schemes were proposed that can further speed up execution time compare to naive approaches [3]–[8]. DC systems are however increasingly applied also to wireless scenarios, such as vehicular networks, creating a need for good coding scheme for wireless DC systems.

A significant part of the execution time in MapReduce systems stems from the IVA delivery time during the shuffle phase [2], [6]. Two key metrics of wireless MapReduce systems thus are:

- **Computation load r**: This describes the average number of nodes to which each file is assigned. In other words, it is the ratio of the total number of assigned input files (including replications) normalized by the total number of nodes.

- **Normalized Delivery Time (NDT) Δ**: This is the wireless shuffle duration normalized by the number of reduce functions and input files and the transmission time of one IVA over a point-to-point channel in the high Signal-to-Noise Ratio (SNR) regime.

We are interested in the minimal NDT under a fixed computation load, which we define as the NDT-computation tradeoff.

Several works have considered NDT-computation tradeoffs for different wireless networks. For example, [9], [10] studied the NDT-computation tradeoff of wireless cellular networks, and proposed schemes to reduce the NDT by sending appropriate linear combinations of the IVAs and applying simple interference cancellation at the receiving nodes. The half-duplex interference network was studied in [11] where the proposed scheme converted the network into a fully connected X-channel and applied the Interference Alignment (IA) scheme in [12]. Authors of [13] and [14] considered full-duplex interference networks. In [13], a NDT reduction was achieved through one-shot beamforming and zero-forcing, and the scheme was shown to be optimal for this class of strategies. In [14], nodes were divided into groups and IA and zero-forcing was combined so that signals intended for a given node did not interfere the other nodes in the same group.

In this paper, we further improve the NDT-computation tradeoff of the wireless DC system over full-duplex wireless interference channels with a novel IA scheme. We use the same file assignment and linear combination coding of IVAs for the wired DC system as in [9], [11], [12] but the combinations of IVAs are sent with a new IA scheme.

This paper also presents an information-theoretic lower bound on the NDT-computation tradeoff. The bound is obtained by a multi-access channel (MAC) type argument that is applied in parallel to a set of well-selected sub-systems and by solving a resulting linear programme. The lower bound shows that when the storage load is at least half the number of nodes, then the one-shot beamforming and zero-forcing scheme in [13] achieves minimum NDT among the class of all schemes.

Notations: We use sans-serif font for constants, bold for vectors and matrices, and calligraphic font for most sets. The sets of complex numbers and positive integers are denoted $\mathbb{C}$ and $\mathbb{Z}^+$. For a finite set $\mathcal{A}$, let $|\mathcal{A}|$ denote its cardinality. For any $n \in \mathbb{Z}^+$, define $[n] \triangleq \{1, 2, \ldots, n\}$ and define $[\mathcal{A}]^n$ as the collection of all the subset of $\mathcal{A}$ with cardinality $n$, i.e. $[\mathcal{A}]^t \triangleq \{ T : T \subset \mathcal{A}, |T| = t \}$. In particular, $[[n]]^t$ denotes the set of all size-t subsets of $[n]$. By writing $[v_i]_{i \in S}$ or $[v_i]_{i \in S}$ we mean the matrix consisting of the columns $\{v_i\}_{i \in S}$.

II. WIRELESS MAPREDUCE FRAMEWORK

Consider a distributed computing (DC) system with a fixed number of $K$ node labelled $1, \ldots, K$; an arbitrary large number $N$ of input files $W_1, \ldots, W_N$; and $K$ output functions $\phi_1, \ldots, \phi_K$ mapping the input files to the desired computations. We assume that the output function $\phi_k$ is assigned to node $k$, for $k \in [K]$. 

A Map-Reduce System decomposes the output functions as:

$$\phi_q(W_1, \ldots, W_N) = v_q(a_{q,1}, \ldots, a_{q,N}), \quad q \in [K],$$  \hfill (1)

where $v_q$ is an appropriate reduce function and $a_{q,p}$ is an intermediate value (IVA) calculated from input file $W_p$ through an appropriate map function:

$$a_{q,p} = u_{q,p}(W_p), \quad p \in [N].$$  \hfill (2)

For simplicity, all IVAs are assumed independent and consisting of output functions. So, node $k$ observes the complex channel outputs $X_k(t)$, where the sequences of complex-valued channel coefficients $\{X_k(t)\}_{t=1}^T$ are both i.i.d. and independent of each other and of all other channel coefficients and noises.

Based on its outputs $Y_j(t) = (Y_j(1), \ldots, Y_j(T))^T$ and the IVAs $\{a_{q,p}: p \in M_k, q \in [K]\}$ it computed during the Map phase, Node $k$ decodes the missing IVAs $\{a_{k,p}: p \not\in M_k\}$ required to compute its assigned output functions $\phi_k$ as:

$$\hat{a}_{k,p} = g_{k,p}(\{a_{i,i}, \ldots, a_{k,i}\}_{i \in M_k}, Y_k), \quad p \not\in M_k.$$  \hfill (6)

Reduce phase: Each node applies the reduce functions to the appropriate IVAs calculated during the Map phase or decoded in the Shuffle phase.

The performance of the distributed computing system is measured in terms of its computation load

$$r \triangleq \sum_{k \in [K]} \frac{|M_k|}{N},$$  \hfill (7)

and the normalized delivery time (NDT)

$$\Delta \triangleq \lim_{P \to \infty} \lim_{A \to \infty} \frac{T}{A \cdot K \cdot N} \cdot \log P.$$  \hfill (8)

We focus on the fundamental NDT-computation tradeoff $\Delta^*(r)$, which is defined as the infimum over all values of $\Delta$ satisfying (8) for some choice of file assignments $\{M_k\}$, transmission time $T$, and encoding and decoding functions $\{f_k(T)\}$ and $\{g_{k,p}\}$ in (3) and (6), all depending on $A$ so that the probability IVA decoding error

$$\Pr\left[ \bigcup_{k \in [K]} \bigcup_{p \not\in M_k} \hat{a}_{k,p} \neq a_{k,p} \right] \to 0 \quad \text{as} \quad A \to \infty.$$  \hfill (9)

III. MAIN RESULTS

The main results of this paper are new upper and lower bounds on the NDT-computation tradeoff of the wireless DC system described in Section II. We focus on the case that $r < K$, because for $r = K$ the nodes are able to calculate the output function independently and trivially $\Delta^*(r) = 0$.

Theorem 1. For fixed $K$, define for each $r \in [K]$:

$$\Delta_{UB}(r) \triangleq \begin{cases} \left(1 - \frac{r}{K}\right) \cdot \frac{1}{K}, & \text{if } r \geq K/2, \\ \left(1 - \frac{r}{K}\right) \cdot \frac{1}{r(K - 1)^2 + r(K - 2)}, & \text{if } r < K/2, \end{cases}$$

Also, let

$$\Delta_{LB}(r) \triangleq \begin{cases} \frac{1}{K} \cdot \left(\frac{2}{K} - 3\right), & \text{if } r = 1, \\ \frac{1}{K} \cdot \left(1 - \frac{r}{K}\right) + \max_{t \in [\lceil K/2 \rceil]} \tilde{C}_t(r), & \text{if } r \in (1, 2), \\ \frac{1}{K} \cdot \left(1 - \frac{r}{K}\right) + \tilde{C}_t(r), & \text{if } r \in [2, K], \end{cases}$$

where for any $t \in [\lceil K/2 \rceil]$, the function $\tilde{C}_t(\cdot)$ denotes the lower-convex envelope of the pairs \{(i, C_t(i))\} for

$$C_t(i) = \begin{cases} \left(\frac{K - 1}{K}\right)^t (K - t - i), & \text{if } i \in [t], \\ 0, & \text{if } i \in [K] \setminus [t]. \end{cases}$$

The NDT-computation tradeoff $\Delta^*(r)$ is upper- and lower-bounded as:

$$\Delta_{UB}(r) \leq \Delta^*(r) \leq \Delta_{LB}(r).$$

where $\Delta_{LB}(r)$ denotes the lower-convex envelope of the pairs \{(r, \Delta_{LB}(r))\} defined in (10).

Proof: See Sections IV and V.

For all $r \geq \lceil K/2 \rceil$ the lower bound (10) and the upper bound (11) match (see the following corollary). In this regime, the optimal NDT-computation tradeoff can be achieved with a linear zero-forcing and one-shot beamforming scheme. As the converse result in [13] and the upper bound on the NDT tradeoff in Theorem 1 show, this is not the case for smaller values of $r$.

Corollary 1. For all $r \geq \lceil K/2 \rceil$:

$$\Delta^*(r) = \left(1 - \frac{r}{K}\right) \cdot \frac{1}{K}.$$  \hfill (14)
and it can be achieved with linear zero-forcing, one-shot beamforming, and side-information cancellation.

Proof: For \( r \geq \lceil K/2 \rceil \) the upper bound \( \Delta_{LB}(r) \) is equal to the lower bound \( \Delta_{LB}(r) \) because \( C_{[K/2]}(i) = 0 \) for all \( i \geq \lceil K/2 \rceil \).

In Fig.1, we compare the bounds in Theorem I to the upper bound given in [13] for \( K = 12 \). The upper bound in [13] is given as following:

\[
\Delta^*(r) \leq \text{lowc} \left( \left\{ \left( r, \frac{1 - r}{\min(K, 2r)} \right) : r \in [K] \right\} \right) \tag{15}
\]

The bound is tight under the assumption of zero-forcing, one-shot beamforming, and side-information cancellation. The upper bound in [14] has the following form:

\[
\Delta^*(r) \leq \text{lowc} \left( (K, 0) \cup \left\{ \left( r, \frac{1 - r/K}{\text{Sum-DoF}_{LB}} \right) : 1 \leq r < K, r|K \right\} \right), \tag{16}
\]

where

\[
\text{Sum-DoF}_{LB} \triangleq \begin{cases} 
2r/K - r^2/2K & \text{if } K/r \in \{2, 3\}, \\
0 & \text{if } K/r \geq 4,
\end{cases} \tag{17}
\]

For \( r \geq K/2 \), our upper bound in Theorem I is strictly better than the bounds in [13] and [14], but still higher than the lower bound. For \( r < K/2 \), our bounds in Theorem I match.

After learning the files, Node \( k \) computes all associated IVAs. Notice that the proposed assignment satisfies the constraint on the computation load, because

\[
|M_k| = \left( \frac{K - 1}{r - 1} \right) \frac{N}{r} = \frac{r}{K}N. \tag{19}
\]

2) Wireless Shuffle Phase: The shuffle phase is in rounds. In each round, every node sends information to each other node, except for a single node \( k_1 \in [K] \) that communicates to all other nodes except for node \( k_2 \in [K] \setminus \{k_1\} \). In this sense, each round can be associated with a unique ordered pair \((k_1, k_2) \in [K]^2 \). We shall consider a completely symmetric scheme with \( K \cdot (K - 1) \) rounds, each round of length \( T_{\text{round}} \triangleq T/(K(K - 1)) \) and associated with a different ordered pair \((k_1, k_2) \in [K]^2 \).

To prepare the shuffle phase, each IVA is split into \( r \) segments, and each segment is uniquely assigned to one of the \( r \) nodes that computed this IVA. Each node further splits its assigned IVA segments into \( K \cdot (K - 1) - 1 \) subsegments, each of which it will be sending in a round where it communicates with the node to which the IVA is intended.

We describe communication for the round associated with the ordered pair \((K, 1)\), i.e., the round where Node \( K \) does not communicate to Node \( 1 \). The other rounds are analogous.

For any distinct pair of nodes \((j, k) \in [K]^2 \setminus \{(1, K)\} \) and any set of nodes \( T = [K] \setminus \{j\} \) with \( k \in T \), let \( a_{k,T}^d \) denote the vector of subsegments that Node \( k \) sends in this round to Node \( j \) and that can be computed by all nodes in \( T \) (because all nodes in \( T \) know the corresponding IVAs).

We fix a large parameter \( \eta \in \mathbb{Z}^+ \) (which we shall let tend to \( \infty \)) and define

\[
\Gamma \triangleq K \cdot (K - r - 1) \tag{20}
\]

\[
\mu \triangleq (K - 2) \cdot \left( \frac{K - 2}{r - 1} \right) \cdot \eta^\Gamma + \left( \frac{K - 1}{r} \right) \cdot (\eta + 1)^\Gamma. \tag{21}
\]

Split the block length \( T_{\text{round}} \) into \( T' \triangleq T_{\text{round}}/\mu \) subblocks of length \( \mu \). We construct a Gaussian codebook of power \( \mu/((K^2 - r^2)/(r - 1)) \) and length \( T' \eta^\Gamma \) to encode each segment vector \( a_{k,T}^d \) into a codeword. We describe transmission in the first subblock, the following subblocks are similar. Denote by \( b_{k,T}^j \) the first \( \eta^\Gamma \) symbols of the codeword associated to IVA segment \( a_{k,T}^d \).

As we explain next, each Node \( k \in [K] \) multiplies the vector \( \{b_{k,T}^j\} \) with appropriate precoding matrices ensuring that the receiving nodes can perfectly separate their intended vectors from non-intended interference and the interference-alignment (IA) principle concentrates interference at each node in a relatively small subspace. Before describing our choice of IA precoding-matrices, notice that codeword \( b_{k,T}^j \) is transmitted by Node \( k \) but also known to all other nodes in \( T \). Since we are using a linear precoding scheme, Node \( j \in [K] \) can thus mitigate the influence of the codewords

\[
\{b_{k,T}^j\} \forall T : k,j \in T, i \notin T. \tag{22}
\]

As a consequence, for each set \( \mathcal{R} \), without causing non-desired interference to nodes in \( \mathcal{R} \), we can use the same...
precoding matrix \( \mathbf{U}_\mathcal{R} \) (whose choice we describe later) for all the codewords:

\[
\{ \mathbf{b}_{k,\mathcal{R}\cup\{k\}\setminus\{j\}} \}_{k \in [K]\setminus\mathcal{R}}, \quad \text{for all } j \in \mathcal{R}.
\] (23)

This idea was already used in the related works [11], [12]. In contrast to these previous works, here we do not introduce the precoding matrices \( \mathbf{U}_\mathcal{R} \) for sets \( \mathcal{R} \) containing 1. Instead, for any \( \mathcal{R} \) not containing 1 and any \( k \in \mathcal{R} \), we use the matrix \( \mathbf{U}_\mathcal{R} \) also to precode the set of codewords \( \{ \mathbf{b}_{k,\mathcal{R}\cup\{k\}\setminus\{j\}} \} \). Nodes in \( \mathcal{R} \) can subtract these interferences from their receive signals because they know the codewords. This trick allows us to reduce the dimension of the interference space and thus improve performance.

For the moment, define the \( \mu \)-length vector of channel inputs \( \mathbf{X}_k \triangleq (X_{k1}, \ldots, X_{k\mu})^T \) for each Node \( k \). Nodes 1, \ldots, \( K \) then form the channel inputs as:

\[
\mathbf{X}_1 = \sum_{\mathcal{R} \in [K]\setminus\{1\}^\prime} \sum_{j \in \mathcal{R}} \mathbf{U}_\mathcal{R} \mathbf{b}_{j,\mathcal{R}\cup\{1\}\setminus\{j\}}.
\] (24)

\[
\mathbf{X}_k = \sum_{\mathcal{R} \in [K]\setminus\{1\}^\prime} \sum_{j \in \mathcal{R}} \mathbf{U}_\mathcal{R} \mathbf{b}_{j,\mathcal{R}\cup\{k\}\setminus\{j\}} + \sum_{\mathcal{R} \in [K]\setminus\{1\}^\prime} \sum_{j \in \mathcal{R}} \mathbf{U}_\mathcal{R} \mathbf{U}_{\mathcal{R}\cup\{1\}} \mathbf{b}_{j,\mathcal{R}\cup\{k\}\setminus\{j\}},
\] (25)

where the precoding matrices \( \{ \mathbf{U}_\mathcal{R} \}_{\mathcal{R} \in [K]\setminus\{1\}^\prime} \) are described shortly.

**Decoding:** After receiving the respective sequence of \( \mu \) channel outputs \( \mathbf{Y}_j \triangleq (Y_{j1}, \ldots, Y_{j\mu}) \), for \( j \in [K] \), each node removes the influence of the codewords corresponding to the IVAs that it can compute itself. The nodes’ “cleaned” signals can then be written as:

\[
\hat{\mathbf{Y}}_1 = \sum_{\mathcal{R} \in [K]\setminus\{1\}^\prime} \sum_{k \in [K]\setminus\mathcal{R}} \mathbf{H}_{1,k} \mathbf{U}_\mathcal{R} \mathbf{U}_{\mathcal{R}\cup\{1\}} \mathbf{b}_{j,\mathcal{R}\cup\{k\}\setminus\{1\}}
\]

\[
+ \sum_{\mathcal{R} \in [K]\setminus\{1\}^\prime} \sum_{k \in [K]\setminus\mathcal{R}} \mathbf{H}_{1,k} \mathbf{U}_\mathcal{R} \mathbf{v}_{\mathcal{R},k} + \mathbf{Z}_1 \quad \text{(27a)}
\]

\[
\hat{\mathbf{Y}}_j = \sum_{\mathcal{R} \in [K]\setminus\{1\}^\prime} \sum_{k \in [K]\setminus\mathcal{R}} \mathbf{H}_{j,k} \mathbf{U}_\mathcal{R} \mathbf{b}_{j,\mathcal{R}\cup\{k\}\setminus\{j\}}
\]

\[
+ \sum_{\mathcal{R} \in [K]\setminus\{1\}^\prime} \sum_{k \in [K]\setminus\mathcal{R}} \mathbf{H}_{j,k} \mathbf{U}_\mathcal{R} \mathbf{U}_{\mathcal{R}\cup\{k\}\setminus\{1\}} \mathbf{b}_{j,\mathcal{R}\cup\{k\}\setminus\{j\}},
\] (27b)

where for ease of notation we defined for Nodes \( k \in [K\setminus\{1\}]^\prime \):

\[
\mathbf{v}_{\mathcal{R},k} \triangleq \sum_{j \in \mathcal{R}} \mathbf{b}_{j,\mathcal{R}\cup\{k\}\setminus\{j\}}, \quad \forall \mathcal{R} \in [[K]\setminus\{k\}]^\prime \quad \text{(28)}
\]

and for the last Node \( K \), since its signal to Node 1 is absent:

\[
\mathbf{v}_{\mathcal{R},K} \triangleq \sum_{j \in [K\setminus\{1\}]} \mathbf{b}_{j,\mathcal{R}\cup\{k\}\setminus\{j\}}, \quad \forall \mathcal{R} \in [[K]\setminus\{1\}]^\prime \quad \text{(29)}
\]

Each Node \( j \) zero-forces the non-desired interference terms of its “cleaned” signal and decodes its intended IVAs \( \{a_{k,T}^j\} \).

**Choice of IA Matrices \( \{ \mathbf{U}_\mathcal{R} \} \) and Analysis of Signal and Interference Spaces:** Inspired by the IA scheme in [15], we choose each \( \mu \times \eta \) precoding matrix \( \mathbf{U}_\mathcal{R} \) so that its column-span includes all power products (with powers from 1 to \( \eta \)) of the channel matrices \( \mathbf{H}_{j,k} \) that premultiply \( \mathbf{U}_\mathcal{R} \) in (27) in the non-desired interference terms. Thus, for \( \mathcal{R} \in [K]\setminus\{1\}^\prime \):

\[
\mathbf{U}_\mathcal{R} = \prod_{\mathbf{H}_j \in \mathcal{H}_\mathcal{R}} \mathbf{H}_j^{\mathbf{W}_\mathcal{R} \cdot \mathbf{H} \cdot \mathcal{E}_\mathcal{R}} \quad \forall \mathcal{R} \in [K]\setminus\{1\}^\prime \quad \text{(30)}
\]

where \( \{ \mathcal{E}_\mathcal{R} \}_{\mathcal{R} \in [K]\setminus\{1\}^\prime} \) are i.i.d. random vectors independent of all channel matrices, noises, and messages,

\[
\mathcal{H}_\mathcal{R} \triangleq \{ \mathbf{H}_{j,k} : j \in [K\setminus\mathcal{R}], k \in [K\setminus\{j\}] \} \setminus \{ \mathbf{H}_{1,k} : k \in \mathcal{R} \}, \quad \text{(31)}
\]

and \( \mathcal{E}_\mathcal{R} \triangleq \{ \alpha_{\mathcal{R},\mathbf{H}} : \mathbf{H} \in \mathcal{H}_\mathcal{R} \} \). Notice that \( |\mathcal{H}_\mathcal{R}| = \Gamma \) for any \( \mathcal{R} \in [K\setminus\{1\}]^\prime \).

Since the column-span of \( \mathbf{U}_\mathcal{R} \) contains all power products of powers 1 to \( \eta \) of the channel matrices \( \mathbf{H} \in \mathcal{H}_\mathcal{R} \), we have

\[
\text{span}(\mathbf{H} \cdot \mathbf{U}_\mathcal{R}) \subseteq \text{span}(\mathbf{W}_\mathcal{R}), \quad \mathbf{H} \in \mathcal{H}_\mathcal{R}, \quad \text{(32)}
\]

where we defined the \( \mu \times \eta \)-matrix:

\[
\mathbf{W}_\mathcal{R} = \prod_{\mathbf{H}_j \in \mathcal{H}_\mathcal{R}} \mathbf{H}_j^{\mathbf{W}_\mathcal{R} \cdot \mathbf{H} \cdot \mathcal{E}_\mathcal{R}} \quad \forall \mathcal{R} \in [K]\setminus\{1\}^\prime \quad \text{(33)}
\]

As a consequence, the signal and interference space at Rx 1 is represented by the \( \mu \times \mu \)-matrix:

\[
\mathbf{A}_1 = \left[ \sum_{\mathbf{D}_1} \mathbf{W}_\mathcal{R} \right] \quad \forall \mathcal{R} \in [K]\setminus\{1\}^\prime, \quad \text{(34)}
\]

where the signal subspace is given by a collection of \( \mu \times \mu \) matrices

\[
\mathbf{D}_1 \triangleq \left[ \sum_{k \in [K\setminus\mathcal{R}]} \mathbf{H}_{1,k} \mathbf{U}_\mathcal{R} \mathbf{U}_{\mathcal{R}\cup\{1\}} \mathbf{b}_{j,\mathcal{R}\cup\{1\}} \right]_{k \in [K\setminus\{k\}]^\prime, \quad (35)}
\]

The signal space at Rx \( j \in [K]\setminus\{1\} \) is represented by the \( \mu \times \mu \) matrix:

\[
\mathbf{A}_j \triangleq \left[ \sum_{\mathbf{D}_j} \mathbf{W}_\mathcal{R} \right] \quad \forall \mathcal{R} \in [K]\setminus\{1\}^\prime, j \notin \mathcal{R}, \quad \text{(36)}
\]
where the signal subspace is given by the collection of $\mu \times \left( r \cdot (K^{-1}) \cdot \eta^T \right)$-matrices

$$D_j \triangleq [H_j \cdot k \cdot U_R]_{j \in [K]} \cdot \prod_{k \in [K] \backslash \{j\}}^\prime$$

and

$$\bar{\mu} \triangleq r \cdot \left( \frac{K-1}{r} \right) \cdot \eta^T + \left( \frac{K-2}{r} \right) \cdot (\eta + 1)^T$$

According to Lemma 1 in [16], $\{A_j\}$ is full rank if each column has different exponent vector $\alpha$, which can be proved using the construction of the matrices $U_R$ and $W_R$ and by the following observations:

- For each $R \in [[K] \backslash \{1\}]^\prime$, matrices $U_R$ and $W_R$ are constructed using a dedicated i.i.d. vector $Z_R$ that is independent of all other random variables in the system.
- For each term $H U R$ in (35) and (37), we have $H \notin H_R$. Thus $H$ is not used in the construction of neither $U_R$ nor $W_R$ and induces a unique exponent on the columns of the signal space compared to the interference space with same $R$.

This proves that based on the “cleaned” signal [27], each receiving node $j$ can separate the various desired signals from each other as well as from the non-desired interfering signals. Since each IVA segment $a_k^j T$ occupies $\eta^2$ dimensions out of the $\mu$ dimensions, we obtain that whenever

$$\frac{|a_k^j T|}{T_{\text{round}}} \leq \frac{\eta^2}{\mu} \log P + o(\log P),$$

for an appropriate function $o(\log P)$ that grows slower than $\log P$, the IVA subsegment can be transmitted with arbitrary small probability of error for $T' \to \infty$.

Recall that by construction, the IVA subsegments satisfy

$$|a_k^j T| = \frac{A}{r (K-1) - 1} \frac{N}{\left( \frac{K}{r} \right)}.$$ (40)

Combining (39) and (40) with equality $T_{\text{round}} K (K-1) = T$, we conclude that reliable communication is possible if

$$\frac{T \log P}{N K} \geq \frac{A}{(K-1) - 1} \frac{1}{\mu} \frac{N}{\left( \frac{K}{r} \right)} + o(\log P).$$ (41)

Since $\frac{\eta}{r} \to r \left( \frac{K-1}{r} \right) + \frac{K-2}{r}$ as $\eta \to \infty$, by letting also $P \to \infty$, we can conclude the NDT upper bound

$$\Delta^*(r) \leq \frac{(K-1) \left( r \left( \frac{K-1}{r} \right) \right)}{(K-1) - 1} \frac{1}{\left( \frac{K}{r} \right)} = \Delta_{ub}(r), \quad r \leq [K/2].$$ (43)

By standard time-sharing arguments, the lower convex-enveloppe of these points is also achievable.

V. PROOF OF THE NDT LOWER BOUND IN THEOREM 1

Consider a fixed file assignment (map phase), and for any positive power $P$ a sequence (in $T$) of wireless distributed computing systems satisfying (9) for the given file assignment. (Since for finite $N$ there are only a finite number of different file assignments irrespective of $P$ and $T$, we can fix the assignment.) The following limiting behavior must hold.

**Lemma 1.** Consider two disjoint sets $T$ and $R$ of same size

$$|T| = |R|,$$ (44)

and define $F \triangleq [K] \backslash (R \cup T)$. Let $M \subseteq [N]$ be the set of files known only to nodes $T$ but not to any other node and partition the set of all IVAs $A$ into the following disjoint subsets:

$$W_r \triangleq \{a_j, m\}_{j \in R},$$

$$W_t \triangleq \{a_j, m\}_{j \in (T \cup F)},$$

For any sequence of distributed computing systems:

$$d \triangleq \lim_{P \to \infty} \lim_{T \to \infty} \frac{T_{\text{log}} P}{A} \leq |F| + |W_r|,$$ (47)

(Notice that $W_r$ denotes the set of all IVAs intended to nodes in $R$ and $W_t$ the set of IVAs deduced from files in $M$ and intended for nodes not in $R$.)

**Proof.** Denote by $H$ the set of all channel coefficients to all nodes in the system and define $W_r \triangleq A \setminus (W_r \cup W_t)$. Since channel coefficients and IVAs are independent, we have

$$H(W_r, W_t) = H(W_t, W_r | W_c, H) = I(W_t, W_r | W_c, H) + H(W_t, W_r | W_c, Y_R, H)$$ (48)

$$= h(Y_R | W_c, H) - h(Z_R) + H(W_r | W_c, Y_R, H) + H(W_t | W_c, Y_R, H),$$ (49)

$$\leq h(Y_R | W_c, H) - h(Z_R) + T c_T + H(W_t | W_r, W_c, Y_R, H),$$ (50)

where we defined $Y_A \triangleq \{Y_j\}_{j \in A}$ for a set $A \subseteq [K]$ and $c_T$ is a vanishing sequence as $T \to \infty$. Here the inequality holds by Fano’s inequality, because $W_t$ is decoded from $Y_R$ and $W_c$, and because we impose vanishing probability of error (9).

Again by Fano’s inequality and by (9), there exists a vanishing sequence $c_T$ such that

$$H(W_t | W_r, W_c, Y_R, H) \leq I(W_t, Y_{(F \cup T)} | W_r, W_c, Y_R, H) + T c_T$$ (52)

$$= h(Y_{(F \cup T)} | W_r, W_c, Y_R, H) - h(Z_{(F \cup T)}) + T c_T,$$ (53)

$$\leq h(Y_{(F \cup T)} | Y_R, H) - h(Z_{(F \cup T)}) + T c_T,$$ (54)

where $Y_A \triangleq \{Y_j\}_{j \in A}$ and $Y_j$ denotes Node $j$’s “cleaned” signal without the inputs that do not depend on files in $M$ but only on IVAs $W_r \cup W_t$.

$$Y_j \triangleq H_{j, T} X_T + Z_j, \quad j \in T \cup F.$$
Here, $H_{A\to B}$ denotes the channel matrix from set $B$ to set $A$.

To bound the first term in (55), we introduce a random variable $E$ indicating whether the matrix $H_{R\to T}$ is invertible ($E = 1$) or not ($E = 0$). If this matrix is invertible and $E = 1$, then the input vector $X_T$ can be computed from $Y_R$ up to noise terms. Based on this observation and defining the residual noise terms

$$Z_j = Z_j - H_{j,T}H_{R\to T}^{-1}Z_R,$$

if $E = 1$, (56) we obtain:

$$h(\bar{Y}_{(F\cup T)}|Y_R, H) \leq P(E = 1) \cdot h(\bar{Z}_{(F\cup T)}|Y_R, H, E = 1) + P(E = 0) \cdot h(\bar{Z}_{(F\cup T)}|Y_R, H, E = 0)$$

(57)

Since the channel coefficients follow continuous distribution, $H_{R\to T}$ is invertible almost surely, implying $P(E = 0) = 0$. By the boundedness of the entropy term $h(\bar{Z}_{(F\cup T)}|Y_R, H, E = 0)$ (since power $P$ and channel coefficients are bounded), this implies

$$h(\bar{Y}_{(F\cup T)}|Y_R, H) \leq h(\bar{Z}_{(F\cup T)}),$$

which combined with (51) and (55) yields:

$$H(W_t, W_r) \leq h(\bar{Y}_R|H) - h(Z_R) + h(\bar{Z}_{(F\cup T)}) - h(\bar{Z}_{(F\cup T)}) + T(\epsilon_T + \epsilon_T')$$

$$\leq T|\mathcal{R}| \log(P) + Tc_{T,H},$$

(59)

where $C_{T,H}$ is a function that is uniformly bounded over all realizations of channel matrices and powers $P$. Noticing

$$H(W_t, W_r) = A(|W_t| + |W_r|),$$

(60)

dividing (59) by $T \log(P)$, and letting $P \to \infty$, establishes the lemma because $|\mathcal{R}| = |T|$ and $TC_{T,H}$ is bounded. \qed

For each subset $S \subseteq [K]$, let $B_S$ denote the set of IVAs that are computed exclusively at nodes in set $S$ and intended for reduce function $j$. Define $b_S = |B_S|$, which does not depend on the index of the reduce function $j$ in $[K]|S|$. Choose two disjoint subsets $T$ and $R$ of same size $|T| = |\mathcal{R}|$. By Lemma 1 and rewriting the sets $W_t$ and $W_r$ in the lemma in terms of the sets $B_S$, we obtain:

$$\frac{|T|}{d} \geq \sum_{S \subseteq [K]} \sum_{j \in \mathcal{R}|S|} |B_S^j| + \sum_{S \subseteq [K]} \sum_{t \in [K]|\mathcal{R}|G} |B_S^G|$$

(61)

$$= \sum_{S \subseteq [K]} |\mathcal{R}| \cdot b_S + \sum_{S \subseteq [K]} (K - |\mathcal{R}| - |G|) \cdot b_G. $$

(62)

Summing up Equality (62) over all sets $T$ and $\mathcal{R}$ of constant size $t \leq K/2$, we obtain:

$$\left(\begin{array}{c} K \\ t \end{array}\right) \left(\begin{array}{c} K - t \\ t \end{array}\right) \cdot \frac{t}{d} \geq \sum_{T \subseteq [K]} \sum_{\mathcal{R} \subseteq [K]|T|} \sum_{S \subseteq [K]} |\mathcal{R}| \cdot b_S + \sum_{S \subseteq [K]} (K - |\mathcal{R}| - |G|) \cdot b_G.$$
APPENDIX A

PROOF OF MONOTONICITY AND CONVEXITY OF VALUES \( C_i^{(t)} \)

We shall prove monotonicity and convexity of the values

\[
D_i^{(t)} \overset{\text{def}}{=} \frac{K!}{t!} t \cdot C_i^{(t)}
\]

\[
= \frac{(K-i)!}{(t-i)!} (K-t-i), \quad i \in [t].
\]

Notice that

\[
D_{i-1}^{(t)} = D_{i}^{(t)} \frac{K-i+1}{t-i+1} \frac{K-t-i+1}{K-t-i}
\]

\[
D_{i+1}^{(t)} = D_{i}^{(t)} \frac{K-t-i-1}{K-i} \frac{K-t-i}{K-t-i-1}
\]

and the monotonicity

\[
D_{i-1}^{(t)} > D_{i}^{(t)}
\]

simply follows because \( K > t \) implies

\[
K-i+1 > t-i+1 \quad \text{and} \quad K-t-i+1 > K-t-i.
\]

To prove convexity, we shall prove that

\[
D_{i+1}^{(t)} + D_{i}^{(t)} \geq 2D_{i}^{(t)},
\]

or equivalently

\[
\frac{t-i}{K-i} \frac{K-t-i-1}{K-t-i} + \frac{K-i+1}{t-i+1} \frac{K-t-i+1}{K-t-i} \geq 2.
\]

Multiplying both sides with the denominators, we see that this condition is equivalent to

\[
(t-i)(K-t-i-1)(t-i+1)
\]

\[
+ (K-i+1)(K-t-i+1)(K-i)
\]

\[
\geq 2(K-i)(t-i+1)(K-t-i).
\]

and after rearranging the terms:

\[
(K-t)(K-i)(K-t-i) + (K-i)(K-t-i+1)
\]

\[
\geq (K-t)(t-i+1)(K-t-i) + (t-i)(t-i+1).
\]

This last inequality is easily verified by noting that \( K \geq t+1 \).

APPENDIX B

PROOF OF STRUCTURE OF MINIMIZER

Start with any feasible vector \( b_1, \ldots, b_K \) and consider two indices \( i < j \) with non-zero masses, \( b_i > 0 \) and \( b_j > 0 \). Updating this vector as

\[
b_i' = b_i - \Delta, \quad \text{and} \quad b_{i+1}' = b_{i+1} + \Delta,
\]

\[
b_{j-1}' = b_{j-1} + \Delta, \quad \text{and} \quad b_j' = b_j - \Delta,
\]

for any \( \Delta \in [0, \min\{b_i, b_j\}] \), results again in a feasible solution vector, which has smaller objective function due to the convexity of the coefficients \( \{C_i^{(t)}\} \).

Applying this argument iteratively, one can conclude that there must exist an optimal solution vector where all entries are zero except for two masses \( b_k > 0 \) and \( b_{k+1} \geq 0 \). Since \( \sum_{i=1}^K b_i \leq rN \), the index \( k \) cannot exceed \( r \). By the decreasing monotonicity of the coefficients \( C_i^{(t)} \), the optimal solution must then be to choose \( b_{[r]} > 0 \) and \( b_{[r]+1} \geq 0 \) and all other masses equal to 0. Since there is a unique such choice satisfying \( \sum_{i=1}^K b_i \leq rN \) and \( \sum_{i=1}^K b_i = N \), this concludes the proof.

REFERENCES

[1] J. S. Ng, W. Y. B. Lim, N. C. Luong, Z. Xiong, A. Asheralieva, D. Niyato, C. Leung, and C. Miao, “A comprehensive survey on coded distributed computing: Fundamentals, challenges, and networking applications,” IEEE Communications Surveys & Tutorials, vol. 23, no. 3, pp. 1800–1837, Jun. 2021.

[2] J. Dean and S. Ghemawat, “MapReduce: simplified data processing on large clusters,” Communications of the ACM, vol. 51, no. 1, pp. 107–113, Jan. 2008.

[3] S. Li, M. A. Maddah-Ali, Q. Yu, and A. S. Avestimehr, “A fundamental tradeoff between computation and communication in distributed computing,” IEEE Transactions on Information Theory, vol. 64, no. 1, pp. 109–128, Jan. 2018.

[4] Q. Yan, S. Yang, and M. Wigger, “Storage-computation-communication tradeoff in distributed computing: Fundamental limits and complexity,” 2019. [Online]. Available: https://hal.telecom-paris.fr/hal-02288592/document

[5] F. Xu, S. Shao, and M. Tao, “New results on the computation-communication tradeoff for heterogeneous coded distributed computing,” IEEE Transactions on Communications, vol. 69, no. 4, pp. 2254–2270, Apr. 2021.

[6] M. Chowdhury, Zaharia, J. Ma, M. I. Jordan, and I. Stoica, “Managing data transfers in computer clusters with orchestera,” ACM SIGCOMM computer communication review, vol. 41, no. 4, pp. 98–109, 2011.

[7] S. Li, M. A. Maddah-Ali, and A. S. Avestimehr, “A unified coding framework for distributed computing with straggling servers,” in 2016 IEEE Globecom Workshops (GC Wkshps), Dec. 2016, pp. 1–6.

[8] J. Zhang and O. Simeone, “Improved latency-coding tradeoff for map-shuffle-reduce systems with stragglers,” in ICASSP 2019 - 2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), May 2019, pp. 8172–8176, ISSN: 2379-190X.

[9] S. Li, Q. Yu, M. A. Maddah-Ali, and A. S. Avestimehr, “A scalable framework for wireless distributed computing,” IEEE/ACM Transactions on Networking, vol. 25, no. 5, pp. 2643–2654, Oct. 2017.

[10] K. Yang, Y. Shi, and Z. Ding, “Data shuffling in wireless distributed computing via low-rank optimization,” IEEE Transactions on Signal Processing, vol. 67, no. 12, pp. 3087–3099, Jun. 2019.

[11] K. Yuan and Y. Wu, “Coded wireless distributed computing via interference alignment,” IEEE International Symposium on Information Theory, p. 6, 2022.

[12] J. Hachem, L. Niesen, and S. N. Diggavi, “Degrees of freedom of cache-aided wireless interference networks,” IEEE Transactions on Information Theory, vol. 64, no. 7, pp. 5359–5380, Jul. 2018.

[13] F. Li, J. Chen, and Z. Wang, “Wireless mapreduce distributed computing,” IEEE Transactions on Information Theory, vol. 65, no. 10, pp. 6101–6114, Oct. 2019.

[14] Y. Bi, P. Ciblat, M. Wigger, and Y. Wu, “DoF of a cooperative X-channel with an application to distributed computing,” in 2022 IEEE International Symposium on Information Theory (ISIT), Jun. 2022, pp. 566–571, ISSN: 2157-8117.

[15] S. Jafar and S. Shamai, “Degrees of freedom region of the MIMO X-channel,” IEEE Transactions on Information Theory, vol. 54, no. 1, pp. 151–170, Jan. 2008.

[16] V. R. Cadambe and S. A. Jafar, “Interference alignment and the degrees of freedom of wireless X networks,” IEEE Transactions on Information Theory, vol. 55, no. 9, pp. 3893–3908, Sep. 2009.