A class large solution of the 2D MHD equations with velocity and magnetic damping

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Abstract
In this paper, we construct a class global large solution to the two-dimensional MHD equations with damp terms in the nonhomogeneous Sobolev framework.

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1. Introduction

This paper focuses on the following 2D incompressible magnetohydrodynamics (MHD) equations

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \mu (-\Delta)^{\alpha} u + \nabla p &= b \cdot \nabla b, \quad x \in \mathbb{R}^2, \quad t > 0, \\
\partial_t b + u \cdot \nabla b + \nu (-\Delta)^{\beta} b &= b \cdot \nabla u, \quad x \in \mathbb{R}^2, \quad t > 0, \\
\text{div} u &= \text{div} b = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0, \\
(u, b)|_{t=0} &= (u_0, b_0), \quad x \in \mathbb{R}^2, 
\end{aligned}
\]  

(1.1)

where \( u = (u_1(t, x), u_2(t, x)) \in \mathbb{R}^2 \) and \( b = (b_1(t, x), b_2(t, x)) \in \mathbb{R}^2 \) denote the divergence free velocity field and magnetic field, respectively, \( p \in \mathbb{R} \) is the scalar pressure. \( \mu \) is the viscosity and \( \nu \) is the magnetic diffusivity. The fractional power operator \( (-\Delta)^{\gamma} \) with \( 0 < \gamma < 1 \) is defined by Fourier multiplier with symbol \( |\xi|^{2\gamma} \) (see e.g. \([7,12]\))

\[ (-\Delta)^{\gamma} u(x) = \mathcal{F}^{-1}|\xi|^{2\gamma}\mathcal{F}u(\xi). \]

We make the convention that by \( \gamma = 0 \) we mean that \( (-\Delta)^{\gamma} u \) is a damp term \( u \). The magnetohydrodynamic (MHD) equations which can be view as a coupling of incompressible Navier–Stokes and Maxwell’s equations govern the motion of electrically conducting fluids such as plasmas, liquid metals and electrolytes, and play a fundamental role in geophysics, astrophysics, cosmology and engineering (see e.g. \([10, 4, 9]\)). Due to the profound physical background and important mathematical significance, the MHD equations attracted quite a lot of attention from many physicists and mathematicians in the past few years. Let us review some progress has been made about the MHD equations (1.1) which are more relatively with our problem. It is well known that the 2D MHD equations (1.1) with \( -\Delta u \) and \( -\Delta b \) (namely, \( \alpha = \beta = 1 \)) have the global smooth solution(\([6]\)). In the completely inviscid case \( (\mu = \nu = 0) \), the question of whether smooth solution of the MHD equations (1.1) with large initial data develops singularity in finite time remains completely open. Besides these

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the two extreme cases, many intermediate cases, for example, the 2D MHD equations with partial dissipation, has been studied by various authors. The issue of the global regularity for the MHD equations (1.1) with \( \mu > 0, \nu > 0, \alpha > 0, \beta = 1 \) has been solved by Fan et al.\[5\]. Recently, Yuan and Zhao \[13\] considered the MHD equations (1.1) with the dissipative operators weaker than any power of the fractional Laplacian and obtained the global regularity of the corresponding system. On the other hand, Cao et al.\[3\], Jiu and Zhao \[8\] established the global regularity of smooth solutions to the MHD equations (1.1) with \( \mu = 0, \nu > 0, \beta > 1 \) by different approach. Subsequently, Agelas \[1\] improved this work with the diffusion \((-\Delta)^\beta b(\beta > 1)\) replaced by \((-\Delta)\log(\varepsilon - \Delta)b(\kappa > 1)\).

As mentioned above, the global regularity for the completely inviscid MHD equations (1.1) with large initial data is still a challenging open problem. When \( \alpha = \beta = 0 \), Wu et al \[11\] obtained that the d-dimensional MHD equations (1.1) always possesses a unique global solution provided that the initial datum is sufficiently small in the nonhomogeneous functional setting \( H^s \) with \( s > 1 + \frac{d}{2} \). Our main goal is to prove the global existence of solutions to (1.1) with \( \alpha = \beta = 0 \) for a class of large initial data.

We assume from now on that the damping coefficients \( \mu = \nu = 1 \), just for simplicity. Our main result is stated as follows.

**Theorem 1.1.** Let \( \alpha = \beta = 0 \) and \( s > 2 \). Assume that the initial data fulfills \( \text{div} \ u_0 = \text{div} \ b_0 = 0 \) and

\[
\begin{align*}
    u_0 &= U_0 + v_0 \quad \text{and} \quad b_0 = B_0 + c_0,
\end{align*}
\]

where

\[
\begin{align*}
    U_0 &= \left( \frac{\partial_2 a_0}{-\partial_1 a_0} \right) \quad \text{and} \quad B_0 = \left( \frac{\partial_2 m_0}{-\partial_1 m_0} \right)
\end{align*}
\]

with

\[
\text{supp } \hat{a}_0(\xi), \text{supp } \hat{m}_0(\xi) \subseteq C := \left\{ \xi \mid |\xi_1 - \xi_2| \leq \varepsilon \right\}.
\]

(1.2)

There exists a sufficiently small positive constant \( \delta \), and a universal constant \( C \) such that if

\[
\left( \|v_0\|_{H^s}^2 + \|c_0\|_{H^s}^2 + \varepsilon^2 (\|a_0\|_{H^s}^2 + \|m_0\|_{H^s}^2) \right) \exp \left( C (\|a_0\|_{H^{s+2}} + \|m_0\|_{H^{s+2}}) \right) \leq \delta,
\]

(1.3)

then the system (1.1) has a unique global solution.

**Remark 1.1.** Let \( v_0 = c_0 = 0 \) and \( a_0 = m_0 = \varepsilon^{-\frac{1}{2}} \log \log \frac{1}{\varepsilon} \chi \), where the smooth function \( \chi \) satisfying

\[
\text{supp } \hat{\chi} \in \tilde{C}, \quad \hat{\chi}(\xi) \in [0, 1] \quad \text{and} \quad \hat{\chi}(\xi) = 1 \quad \text{for} \quad \xi \in \frac{1}{2} \tilde{C},
\]

where

\[
\tilde{C} := \left\{ \xi \mid |\xi_1 - \xi_2| \leq \varepsilon, \quad 1 \leq \xi_1^2 + \xi_2^2 \leq 2 \right\}.
\]

Then, direct calculations show that the left side of (1.3) becomes

\[
C \varepsilon^2 \left( \log \log \frac{1}{\varepsilon} \right)^4 \exp \left( C \log \log \frac{1}{\varepsilon} \right).
\]

Therefore, choosing \( \varepsilon \) small enough, we deduce that the system (1.1) has a global solution.

Moreover, we also have

\[
\|u_0\|_{L^2} \gtrsim \log \log \frac{1}{\varepsilon} \quad \text{and} \quad \|b_0\|_{L^2} \gtrsim \log \log \frac{1}{\varepsilon}.
\]
Remark 1.2. Considered the system (1.1) with \(0 < \alpha, \beta < 1\), if the support condition (1.2) of the Theorem 1.1 were replaced by
\[
\text{supp } \hat{a}_0(\xi), \text{supp } \hat{m}_0(\xi) \subset C := \left\{ \xi \mid |\xi_1 - \xi_2| \leq \varepsilon, \ 1 \leq \xi_1^2 + \xi_2^2 \right\},
\]
the Theorem 1.1 holds true.

Notations: For the sake of simplicity, \(a \lesssim b\) means that there is a uniform positive constant \(C\) such that \(a \leq Cb\). \([A, B]\) stands for the commutator operator \(AB - BA\), where \(A\) and \(B\) are any pair of operators on some Banach space. In the paper, we will use the Besov space \(B^s_{p,q}\), for more details, we refer the readers to see the Chapter 2 in [2]. It is worth mentioning that the Besov space \(B^s_{2,2}\) coincides with the nonhomogeneous Sobolev spaces \(H^s\) for \(s > 0\), namely, \(B^s_{2,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)\), where
\[
H^s(\mathbb{R}^d) := \left\{ f \in S'(\mathbb{R}^d) : ||f||_{H^s(\mathbb{R}^d)} < \infty \right\}
\]
with the norm
\[
||f||_{H^s(\mathbb{R}^d)} := \left( \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]

2. Reformulation of the System

Let \((a, m)\) be the solutions of the following system
\[
\begin{align*}
\partial_t a + a &= 0, \\
\partial_t m + m &= 0, \\
(a, m)|_{t=0} &= (a_0, m_0).
\end{align*}
\]
(2.1)

Setting
\[
U = \begin{pmatrix} \partial_2 a \\ -\partial_1 a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \partial_2 m \\ -\partial_1 m \end{pmatrix},
\]
we can deduce from (2.1) that
\[
\begin{align*}
\partial_t U + U &= 0, \\
\partial_t B + B &= 0, \\
\text{div} U &= \text{div} B = 0, \\
(U, B)|_{t=0} &= (U_0, B_0).
\end{align*}
\]
(2.2)

Denoting \(v = u - U\) and \(c = b - B\), the system (1.1) can be written as follows
\[
\begin{align*}
\partial_t v + u \cdot \nabla v + v \cdot \nabla U + u + \nabla p &= b \cdot \nabla c + c \cdot \nabla B + f, \\
\partial_t c + u \cdot \nabla c + v \cdot \nabla B + c &= b \cdot \nabla v + c \cdot \nabla U + g, \\
\text{div} v &= \text{div} c = 0, \\
(v, c)|_{t=0} &= (v_0, c_0).
\end{align*}
\]
(2.3)

where
\[
f = -U \cdot \nabla U + B \cdot \nabla B \quad \text{and} \quad g = -U \cdot \nabla B + B \cdot \nabla U.
\]
3. The Proof of Theorem 1.1

Before proceeding on, we present some estimates which will be used in the proof of Theorem 1.1.

Lemma 3.1. For $s > 2$, under the assumptions of Theorem 1.1, the following estimates hold

$$||f||_{H^s} + ||g||_{H^s} \leq Ce^{-t}\varepsilon(||a_0||_{H^{s+2}}^2 + ||m_0||_{H^{s+2}}^2)$$ (3.1)

and

$$||\nabla U||_{H^s} + ||\nabla B||_{H^s} \leq Ce^{-t}(||a_0||_{H^{s+2}} + ||m_0||_{H^{s+2}}).$$ (3.2)

Proof of Lemma 3.1 Notice that

$$f^1 = -U \cdot \nabla U^1 + B \cdot \nabla B^1 = \partial_1 a \partial_2 \partial_2 a - \partial_2 a \partial_1 \partial_1 a - (\partial_1 m \partial_2 \partial_3 m - \partial_2 m \partial_1 \partial_2 m)$$

$$= (\partial_1 - \partial_2) a \partial_2 \partial_2 a + \partial_2 a \partial_2 (\partial_2 - \partial_1) a + (\partial_2 - \partial_1) m \partial_2 \partial_2 m + \partial_2 m \partial_2 (\partial_2 - \partial_1) m,$$

due to the fact that $H^s$ with $s > 2$ is a Banach algebra, then we have

$$||f^1||_{H^s} \lesssim ||(\partial_1 - \partial_2) a||_{H^s} ||a||_{H^{s+2}} + ||a||_{H^{s+1}} ||(\partial_2 - \partial_1) a||_{H^{s+1}}$$

$$+ ||(\partial_1 - \partial_2) m||_{H^s} ||m||_{H^{s+2}} + ||m||_{H^{s+1}} ||(\partial_2 - \partial_1) m||_{H^{s+1}}.$$ (3.3)

Direct calculations show that for $\tau \geq 0$

$$||a||_{H^s} + ||m||_{H^s} \leq e^{-\tau}(||a_0||_{H^s} + ||m_0||_{H^s})$$ (3.4)

and

$$||(\partial_1 - \partial_2) a||_{H^s} + ||(\partial_1 - \partial_2) m||_{H^s} \leq e^{-\tau}(||a_0||_{H^s} + ||(\partial_1 - \partial_2) m||_{H^s})$$

$$\leq e^{-\tau}\varepsilon(||a_0||_{H^s} + ||m_0||_{H^s}).$$ (3.5)

where we have used the conditions supp $\hat{a}_0(\xi) \subseteq \mathcal{C}$ and supp $\hat{m}_0(\xi) \subseteq \mathcal{C}$.

In view of the facts (3.4) and (3.5), we obtain from (3.3) that

$$||f^1||_{H^s} \lesssim e^{-\tau}\varepsilon(2||a_0||_{H^{s+2}}^2 + ||m_0||_{H^{s+2}}^2).$$

Similarly, we also have

$$||f^2||_{H^s} \lesssim e^{-\tau}\varepsilon(||a_0||_{H^{s+2}}^2 + ||m_0||_{H^{s+2}}^2).$$

Then, we get

$$||f||_{H^s} \leq ||f^1||_{H^s} + ||f^2||_{H^s} \lesssim e^{-\tau}\varepsilon(||a_0||_{H^{s+2}}^2 + ||m_0||_{H^{s+2}}^2).$$ (3.6)
An argument similar to that used above, we get
\[ ||g||_{H^s} \lesssim e^{-t}e(||a_0||_{H^{s+2}}^2 + ||m_0||_{H^{s+2}}^2). \] (3.7)

Combining (3.6) and (3.7) yields the desired result (3.1).

(3.2) is just a consequence of (3.4). Thus, we complete the proof of Lemma 3.1. \(\square\)

**Proof of Theorem 1.1**  For notational simplicity, we set
\[ E(t) = (||v(t)||_{H^s}^2 + ||c(t)||_{H^s}^2). \]

Applying \(\Delta_j\) to (2.3) and taking the \(L^2\) inner product of the resulting equations with \(\Delta_jv\) and \(\Delta_jc\), respectively, we have
\[ \frac{1}{2} \frac{d}{dt} \left( ||\Delta_jv||_{L^2}^2 + ||\Delta_jc||_{L^2}^2 \right) + ||\Delta_jv||_{L^2}^2 + ||\Delta_jc||_{L^2}^2 =: \sum_{i=1}^5 K_i, \] (3.8)

where
\[
egin{align*}
K_1 &= -\int_{\mathbb{R}^2} [\Delta_j, u \cdot \nabla]v \cdot \Delta_jv dx - \int_{\mathbb{R}^2} [\Delta_j, u \cdot \nabla]c \cdot \Delta_jc dx, \\
K_2 &= \int_{\mathbb{R}^2} [\Delta_j, b \cdot \nabla]c \cdot \Delta_jv dx + \int_{\mathbb{R}^2} [\Delta_j, b \cdot \nabla]v \cdot \Delta_jc dx, \\
K_3 &= -\int_{\mathbb{R}^2} \Delta_j(v \cdot \nabla U) \cdot \Delta_jv dx - \int_{\mathbb{R}^2} \Delta_j(v \cdot \nabla B) \cdot \Delta_jc dx, \\
K_4 &= \int_{\mathbb{R}^2} \Delta_j(c \cdot \nabla B) \cdot \Delta_jv dx + \int_{\mathbb{R}^2} \Delta_j(c \cdot \nabla U) \cdot \Delta_jc dx, \\
K_5 &= \int_{\mathbb{R}^2} \Delta_j f \cdot \Delta_jv dx + \int_{\mathbb{R}^2} \Delta_j g \cdot \Delta_jc dx.
\end{align*}
\]

Multiplying both sides of (3.8) by \(2^{2js}\) and summing up over \(j \geq -1\) yields
\[ \frac{1}{2} \frac{d}{dt} E(t) + E(t) = \sum_{i=1}^5 \sum_{j \geq -1} 2^{2js} K_i. \] (3.9)

Next, we need to estimate the above terms involving \(K_i\) for \(i = 1, \cdots, 5\) as follows
\[
\begin{align*}
\sum_{j \geq -1} 2^{2js}|K_1| &\leq \sum_{j \geq -1} 2^{2js} ||[\Delta_j, u \cdot \nabla]v||_{L^2} ||\Delta_jv||_{L^2} + \sum_{j \geq -1} 2^{2js} ||[\Delta_j, u \cdot \nabla]c||_{L^2} ||\Delta_jc||_{L^2} \\
&\lesssim ||\nabla u||_{H^{s-1}} (||v||_{H^s}^2 + ||c||_{H^s}^2) \\
&\lesssim \left(||U||_{H^s} + ||v||_{H^s}\right) E(t), \quad (3.10)
\end{align*}
\]

where we have used the commutator estimate (see Lemma 2.6 in [11])
\[ ||[\Delta_j, u \cdot \nabla]f||_{B^2_{2,2}} \leq C ||\nabla u||_{B^{s-1}_{2,2}} ||f||_{B^2_{2,2}} \text{ with } \text{div} \, u = 0. \]

Similarly, we also have
\[ \sum_{j \geq -1} 2^{2js}|K_2| \lesssim \left(||B||_{H^s} + ||c||_{H^s}\right) E(t). \] (3.11)
For the last three terms, by Hölder’s inequality, we deduce
\[ \sum_{j \geq -1} 2^{2js}|K_3| \lesssim ||\nabla U||_{H^s} ||v||_{H^s}^2 + ||\nabla B||_{H^s} ||c||_{H^s} ||v||_{H^s}, \] (3.12)
and
\[ \sum_{j \geq -1} 2^{2js}|K_4| \lesssim ||\nabla U||_{H^s} ||c||_{H^s}^2 + ||\nabla B||_{H^s} ||c||_{H^s} ||v||_{H^s}, \] (3.13)
\[ \sum_{j \geq -1} 2^{2js}|K_5| \leq \sum_{j \geq -1} 2^{2js}||\Delta_j f||_{L^2} ||\Delta_j v||_{L^2} + \sum_{j \geq -1} 2^{2js}||\Delta_j g||_{L^2} ||\Delta_j c||_{L^2} \leq C(||f||_{H^s}^2 + ||g||_{H^s}^2) + \frac{1}{2} E(t). \] (3.14)
Inserting (3.10)–(3.14) into (3.9) yields that
\[ \frac{d}{dt} E(t) + E(t) \lesssim E^2(t) + \left(||\nabla U||_{H^s} + ||\nabla B||_{H^s}\right) E(t) + ||f||_{H^s}^2 + ||g||_{H^s}^2. \] (3.15)
Utilizing the Lemma 3.1, we have from (3.15)
\[ \frac{d}{dt} E(t) + E(t) \lesssim E^2(t) + e^{-t}\left(||a_0||_{H^{s+2}} + ||m_0||_{H^{s+2}}\right) E(t) + e^{-t} \exp \left(||a_0||_{H^{s+2}}^4 + ||m_0||_{H^{s+2}}^4\right). \] (3.16)
Now, we define
\[ \Gamma := \max\{t \in [0, T^*) : \sup_{\tau \in [0, t]} E(\tau) \leq \eta\}, \]
where \( \eta \) is a small enough positive constant which will be determined later on.

Assume that \( \Gamma < T^* \). For all \( t \in [0, \Gamma] \), we obtain from (3.16) that
\[ \frac{d}{dt} E(t) \leq C e^{-t}\left(||a_0||_{H^{s+2}} + ||m_0||_{H^{s+2}}\right) E(t) + C e^{-t} \exp \left(||a_0||_{H^{s+2}}^4 + ||m_0||_{H^{s+2}}^4\right), \]
which follows from the assumption (1.3) that
\[ E(t) \leq C\left(E_0 + \varepsilon^2(||a_0||_{H^{s+2}}^4 + ||m_0||_{H^{s+2}}^4\right) \exp \left(||a_0||_{H^{s+2}} + ||m_0||_{H^{s+2}}\right) \leq C\delta. \]
Choosing \( \eta = 2C\delta \), thus we can get
\[ E(t) \leq \frac{\eta}{2} \quad \text{for} \quad t \leq \Gamma. \]

So if \( \Gamma < T^* \), due to the continuity of the solutions, we can obtain there exists \( 0 < \epsilon \ll 1 \) such that
\[ E(t) \leq \frac{\eta}{2} \quad \text{for} \quad t \leq \Gamma + \epsilon < T^*, \]
which is contradiction with the definition of \( \Gamma \).

Thus, we can conclude \( \Gamma = T^* \) and
\[ E(t) \leq C < \infty \quad \text{for all} \quad t \in (0, T^*), \]
which implies that \( T^* = +\infty \). This completes the proof of Theorem 1.1. \( \square \)
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