Brittle failure of incompressible material in plane strain

AI Chanyshev\textsuperscript{a,b}, OE Belousova\textsuperscript{**} and OA Lukyashkina\textsuperscript{***}
\textsuperscript{a}Chulaloon Institute of Mining, Siberian Branch, Russian Academy of Sciences, Novosibirsk, Russia
\textsuperscript{b}Novosibirsk State University of Economics and Management, Novosibirsk, Russia

E-mail: *a.i.chanyshev@gmail.com; **o.e.belousova@mail.ru; ***lykola@yandex.ru

Abstract. The stress–strain behavior of rock mass under plane strain is studied in the cases of incompressibility and perfect brittle failure. The characteristics of of differential equations of equilibrium and the relations at them are obtained using the condition of coaxiality of the stress and strain tensors. The boundary problem is formulated for determining the stress–strain behavior in the failure zone of rocks. By way of illustration, the equations of the perfect brittle post-limit deformation of rock mass in the form of a rectangular plate (pillar) under uniform compression are analyzed. It is shown that by monitoring displacement of the side boundary of the plate, it is possible to predict the plate failure.

1. Introduction
The stress–strain curve describes deformation resistance of a material [1–4]. An increase in the stress means the growing resistance; a decrease in the stress implies that the strength of the material drops, i.e. the material fails. The post-limit deformation curve may be smooth. The curve in the form of a vertical straight line means perfect brittle failure [5–11]. Plane strain of a perfect brittle, incompressible material was studied in [8]. The study used the incompressibility condition and two compatibility conditions of strains with derivatives of components of the rotation vector in the line of the axis $Z$. Two characteristics of the system of differential equations for displacements and two relations at them, connecting the rotation vector component and the angle between the major axes of the strain tensor were obtained. However, the boundary problem was not formulated in [8], no equations were proposed for the determination of stresses. The present paper formulates the equations for stress estimation and the boundary problem to find both stresses and strains.

2. Determination of stresses and strain under plane strain deformation
Problem formulation. Let in medium in the Cartesian coordinates the equations below be fulfilled:

\[ \varepsilon_z = 0, \quad \varepsilon_x + \varepsilon_y = 0, \quad (1) \]

\[ \frac{\varepsilon_x - \varepsilon_y}{2} = \Gamma \cos 2\Omega, \quad \varepsilon_{xy} = \Gamma \sin 2\Omega, \quad \Gamma = \Gamma_p, \quad (2) \]

where $\Gamma_p$ is the maximum tangent deformation:
\[ \Gamma = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} \]

in brittle failure diagram \( T = T(\Gamma) \), where \( T \) is the maximum shear stress (Figure 1).

\[ \theta = \frac{2 \varepsilon_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{2 \tau_{xy}}{\sigma_x - \sigma_y}, \quad \Omega = \frac{2 \tau_{xy}}{\sigma_x - \sigma_y}, \quad \Omega = \frac{2 \tau_{xy}}{\sigma_x - \sigma_y}, \]

where \( \theta \) is the angle between the first major direction of the stress tensor \( T_\sigma \) and axis \( x \); \( \Omega \) is the angle between the first major direction of the strain tensor \( T_\varepsilon \) and axis \( x \). Equality (3) means that the major axes of the tensors \( T_\sigma \) and \( T_\varepsilon \) coincide during deformation of the medium.

In addition to (1)–(3), the compatibility of strains is conditioned to be true:

\[ \begin{cases} \frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \varepsilon_{xy}}{\partial y} - \frac{\partial \omega_z}{\partial y} = 0, \\ \frac{\partial \varepsilon_x}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial x} + \frac{\partial \omega_z}{\partial x} = 0, \end{cases} \]

where \( \omega_z \) is the rotation vector component:

\[ \omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right). \]

\( u, v, \omega \) are the displacement vector components. Furthermore, it is assumed that the equilibrium equations are fulfilled:

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \]

In [8] relations (1), (2) were inserted in (4), which produced:
System (7) is hyperbolic and has the characteristics:
\[
\frac{dy}{dx} = \tan \Omega, \quad \frac{dy}{dx} = -\cot \Omega.
\]  

They have the same direction as the major axes of the strain tensor \( T_\varepsilon \). The relations at the characteristics are given by [5]:
\[
\omega_z - \Omega = \xi \quad \omega_z + \Omega = \eta,
\]  
where \( \xi, \eta \) are the constants.

The problem solution needs knowing boundary values of the functions \( \omega_z = \omega_z(x, y), \Omega = \Omega(x, y) \). To this effect, it is assumed that the boundary \( L \) the displacements are preset:
\[
u = u(x, y), \quad v = v(x, y),
\]  
where the coordinates \( x, y \in L \). Calculation of the total differentials of \( u \) and \( v \) gives:
\[
du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \quad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy
\]  
or
\[
du = \varepsilon_x dx + \left( \varepsilon_{xy} - \omega_z \right) dy, \quad dv = \left( \varepsilon_{xy} + \omega_z \right) dx + \varepsilon_y dy.
\]  

Placement of (11) in (1), (2) yields:
\[
du = \Gamma_p \cos 2\Omega dx + \left( \Gamma_p \sin 2\Omega - \omega_z \right) dy,
\]  
\[
dv = \left( \Gamma_p \sin 2\Omega + \omega_z \right) dx - \Gamma_p \cos 2\Omega dy.
\]  

By dividing (12) by \( ds = \sqrt{(dx)^2 + (dy)^2} \), we obtain:
\[
\begin{align*}
\frac{u_s'}{u_s} = & -\Gamma_p \cos 2\Omega \sin \varphi + \Gamma_p \sin 2\Omega \cos \varphi - \omega_z \cos \varphi, \\
\frac{v_s'}{v_s} = & -\Gamma_p \sin 2\Omega \sin \varphi - \Gamma_p \cos 2\Omega \cos \varphi - \omega_z \sin \varphi,
\end{align*}
\]  
where \( u_s', v_s' \) are the derivatives of the displacements \( u, v \) along the tangent to \( L \), and \( \varphi \) is the angle between the normal to \( L \) and the axis \( x \). From (13) we find:
\[
\cos 2(\Omega - \varphi) = \frac{\left( u_s' \sin \varphi - v_s' \cos \varphi \right)}{\Gamma_p}, \quad \omega_z = -\frac{u_s' \cos(2\Omega - \varphi) + v_s' \sin(2\Omega - \varphi)}{\cos 2(\Omega - \varphi)}.
\]  

Formulas (14) allow determining the boundary values of the angle \( \Omega \) and the rotation vector component \( \omega_z \). Now, how to determine stress state in the failure domain?
For finding stresses, we have (3), i.e. \( \theta = \Omega \), and the common relations:
\[
\sigma_x = \sigma + T \cos 2\theta, \quad \sigma_y = \sigma - T \cos 2\theta, \quad \tau_{xy} = T \sin 2\theta,
\]
where the angle \( \theta = \theta(x, y) \) from the previous analysis is assumed as the known function of the coordinates \( x, y \). Insert (15) in (6) at the known \( \theta = \theta(x, y) \) and obtain the system of equations:
\[
\begin{aligned}
\frac{\partial \sigma}{\partial x} + \frac{\partial \tau}{\partial y} \cos 2\theta + \frac{\partial \tau}{\partial y} \sin 2\theta &= 2T \left[ \sin 2\theta \frac{\partial \theta}{\partial x} - \cos 2\theta \frac{\partial \theta}{\partial y} \right], \\
\frac{\partial \sigma}{\partial y} + \frac{\partial \tau}{\partial x} \sin 2\theta - \frac{\partial \tau}{\partial x} \cos 2\theta &= -2T \left[ \cos 2\theta \frac{\partial \theta}{\partial x} + \sin 2\theta \frac{\partial \theta}{\partial y} \right].
\end{aligned}
\]
(16)

Here, the right-hand sides are considered the known functions accurate to \( T \). From the comparison of (16) and (7), the right-hand sides of (16) respectively equal the expressions below:
\[
\frac{\partial \omega_z}{\partial y} = \frac{T}{\Gamma_p}, \quad -\frac{\partial \omega_z}{\partial x} = \frac{T}{\Gamma_p}.
\]
(17)

When determined, the characteristics of (16) coincide with (8), where \( \theta = \Omega \). In this case, the relations at the characteristics take the form of:
\[
\begin{aligned}
\frac{dy}{dx} &= \tan \theta, \quad d\sigma + dT = \frac{T}{\Gamma_p} \left( \frac{\partial \omega_z}{\partial x} dx - \frac{\partial \omega_z}{\partial y} dy \right), \\
\frac{dy}{dx} &= -\cot \theta, \quad d\sigma - dT = \frac{T}{\Gamma_p} \left( \frac{\partial \omega_z}{\partial y} dx - \frac{\partial \omega_z}{\partial x} dy \right).
\end{aligned}
\]
(18)

The right-hand sides of (18) are the vector product:
\[
\text{grad} \omega_z = \left( \frac{\partial \omega_z}{\partial x}, \frac{\partial \omega_z}{\partial y}, 0 \right) u \overrightarrow{ds} = (dx, dy, 0).
\]

This means that when integrating the relations at the characteristics, the integrals will depend on the integrating path as (18) via the vector product are connected with the areas of the parallelograms plotted at the vectors \( \text{grad} \omega_z, \overrightarrow{ds} \).

Again, how to find the boundary values of the functions \( \sigma = \sigma(x, y), \ T = T(x, y) \) included in (18)? We have the Cauchy vector of stresses at the boundary \( L \). The coordinates of the vector along the axes \( x, y \), respectively, are given by:
\[
\begin{aligned}
p_n^x &= (\sigma + T \cos 2\theta) \cos \varphi + T \sin 2\theta \sin \varphi, \\
p_n^y &= T \sin 2\theta \cos \varphi + (\sigma - T \cos 2\theta) \sin \varphi,
\end{aligned}
\]
(19)

where \( \varphi \) is the angle between the normal to \( L \) and the axis \( x \). From (19) we obtain the wanted boundary values:
\[
\begin{aligned}
T &= \frac{p_n^y \cos \varphi - p_n^x \sin \varphi}{\sin 2(\theta - \varphi)}, \quad \sigma = \frac{p_n^y \sin(2\theta - \varphi) - p_n^x \cos(2\theta - \varphi)}{\sin 2(\theta - \varphi)}.
\end{aligned}
\]
(20)
Thus, the stress state in the failure domain is found using (20) and (18).

3. Determination of stresses and strains in a plate in post-limit deformation under compression

This problem relates stability of pillars in mineral mining and is a subject of many studies [12–20].

By way of illustration, we present the equations and the simplest solution for the problem on post-limit deformation of a plate in the plain strain under uniform compression. Let during loading:

\[ \varepsilon_{xy} = \tau_{xy} = 0. \]  

(21)

This mean that the angles \( \Omega = \theta = 0 \), i.e. major directions of the tensors \( T_\sigma \) and \( T_\varepsilon \), coincide with the direction of the axis \( x \). Then, in the attained stress state:

\[ \varepsilon_x = \Gamma_p, \varepsilon_y = -\Gamma_p. \]

where \( \Gamma_p \) is the limit strength (Figure 1).

Find the displacements \( u, v \) given that \( \varepsilon_{xy} = \omega_z = 0 \):

\[ u = \Gamma_p x, \ v = -\Gamma_p y. \]

The stress is in accord with the equations of equilibrium:

\[ \frac{\partial \sigma_x}{\partial x} = 0, \frac{\partial \sigma_y}{\partial y} = 0. \]

Then:

\[ \sigma_y = f(x), \sigma_x = \varphi(y), \]

where \( f, \varphi \) are the preset functions of the coordinates.

This situation is depicted in Figure 2.

It is evident from the formulas and Figure 2 that to detect the moment of failure, it is necessary to monitor the change in the lateral displacement \( u \). As soon as it reaches the critical value \( u = \Gamma_p L \), failure of the plate immediately takes place.

\[ \text{Figure 2.} \text{ Plate with height } H \text{ and width } L \text{ is subjected to vertically oriented compression } \sigma_y = f(x). \text{ The characteristics of the system of differential equations are the lines } x = \text{const}, \ y = \text{const}. \]
4. Comments
It is unclear from the previous analysis when $\Gamma$ reaches the value $\Gamma_p$. In order to solve this problem, the previous solutions should be considered, for instance, in elasticity (Figure 1).
We have for elasticity:

$$du = \varepsilon_x dx + (\varepsilon_{xy} - \varepsilon_z) dy, \quad dv = (\varepsilon_{xy} + \varepsilon_z) dx + \varepsilon_y dy.$$  

As earlier, by dividing the above expression by $ds = \sqrt{(dx)^2 + (dy)^2}$, we obtain:

$$u'_s = -\varepsilon_x \cos \phi + (\varepsilon_{xy} - \varepsilon_z) \sin \phi, \quad v'_s = (\varepsilon_{xy} + \varepsilon_z) \cos \phi + \varepsilon_y \sin \phi$$  

(22)

These equations are added with:

$$\sigma_x \cos \phi + \tau_{xy} \sin \phi, \quad \sigma_y \cos \phi + \sigma_{xy} \sin \phi$$  

(23)

$$(\sigma_x, \sigma_y, \tau_{xy})$$  

(24)

($p_n^x, p_n^y$ are the preset values at the boundary $L$). As a result, we have seven equations (22)–(24) to determine seven unknowns $\varepsilon_x, \varepsilon_y, \varepsilon_z, \omega_x, \sigma_x, \sigma_y, \tau_{xy}$. The system has a unique solution from which the value $\Gamma$ is found.

5. Conclusion
The stress–strain behavior is determined in the domain of post-limit deformation in perfect brittle failure of incompressible medium. The solution to the problem on uniform compression of a plate under conditions of perfect brittle failure is analyzed. It is show that pillar control in rock mass needs monitoring the change in the lateral displacement of the pillar.

Acknowledgements
The study was supported by the Russian Foundation for Basic Research, Project No. 18-05-00757 A.

References
[1] Kurlenya MV, Afinogenov YuA, Zhigalkin VM, Oparin VN, Usoltseva OM and Chanyshchev AI 1998 Block phenomenological mechanical model of an element of a deformable medium. Definitions, basic properties. Part II: Dynamic effects Journal of Mining Science Vol 34 No 5 pp 402–413
[2] Chanyshchev AI and Belousova OE 2009 An interpretation for zonal rock disintegration around underground excavations Fiz. Mezomekh. Vol 12 No 1 pp 89–99
[3] Fedosiev VI 1999 Strength of Materials; University Textbook Moscow: MGTU Baumana Vol 2 (in Russian)
[4] Starovoitov EI 2008 Strength of Materials Moscow: Fizmatlit (in Russian)
[5] Nikiforovsky VS and Shemyakin EI 1979 Dynamic Failure of Solids Novosibirsk: Nauka (in Russian)
[6] Shemyakin EI 1996 Problem on brittle hinge Mekh. Tverd. Tela No 2 pp 138–144
[7] Leonov MYa and Panasyuk VV 1959 Propagation of the finest cracks in a solid Prikl. Mekh., No 5 pp 49–61
[8] Kachanov LM Foundations of Failure Mechanics Moscow: Nauka (in Russian)
[9] Shemyakin EI 1997 Brittle failure of solids (plain strain) Mekh. Tverd. Tela No 2 pp 145–151
[10] Cherepanov GP 1974 *Brittle Failure Mechanics* Moscow: Nauka (in Russian)
[11] Parton VZ and Morozov EM 1985 *Mechanics of Elastoplastic Failure* Moscow: Nauka (in Russian)
[12] Borisov AA 1980 *Mechanics of Rocks and Rock Masses* Moscow: Nedra (in Russian)
[13] Proskuryakov NM 1991 *Rock Mass Control* Moscow: Nedra (in Russian)
[14] Sherman DI 1952 Stress state of rib pillars. Elastic ponderous medium weakened by two elliptical openings *Izv. AN SSSR. Otdel Tekh. Nauk* No 6 pp 840–857
[15] Slesarev VD 1948 *Determination of Optimal Sizes of Various Purpose Pillars* Moscow: Ugletekhizdat (in Russian)
[16] Ruppeneit KV 1954 *Some Questions in Rock Mechanics* Moscow: Ugletekhizdat (in Russian)
[17] Nesterov MP 1968 Engineering methods of chain pillar design *Gornyi Zhurnal* No 9
[18] Mikhlin SG 1942 Stress in rocks overlying a coal seam *Izv. AN SSSR. Otdel Tekh. Nauk* No 7 pp 133–28
[19] Dinnik AN 1925 Rock pressure and design of support system for spheric mine *Inzh. Rabotn*. No 7 pp 15–18
[20] Vorobiev AN and Prutkov SN 2006 Desing method for room spans as well as safety and rib pillars based on the theory of arch systems *GIAB* No 10 pp 201–205