PACKING SETS

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ABSTRACT. For a given subset \( A \subseteq G \) of a finite abelian group \((G, \circ)\), we study the problem of finding a large packing set \( B \) for \( A \), that is, a set \( B \subseteq G \) such that \( |A \circ B| = |A||B| \). Ruzsa’s covering lemma and the trivial bound imply the existence of such a \( B \) of size \( |G|/|A|^2 \leq |A \circ A^{-1}| \leq |B| \leq |G|/|A| \). We show that these bounds are in general optimal and essentially any \( \nu(A) \) in the interval \([|G|/|A|, |G|/|A|]\) can appear for some \( |A| \).

The case that \( G \) is the multiplicative group of the finite field \( \mathbb{F}_p \) of prime order \( p \) and \( A = \{1, 2, \ldots, \lambda\} \) for some positive integer \( \lambda \) is particularly interesting in view of the construction of limited-magnitude error correcting codes. Here we construct a packing set \( B \) of size \( |B| \gg p(\lambda \log p)^{-1} \) for any \( \lambda \leq 0.9p^{1/2} \). This result is optimal up to the logarithmic factor.

1. Introduction

Given two subsets \( A \) and \( B \) of a finite abelian group \((G, \circ)\) with unit 1, the product set of \( A \) and \( B \) is defined as
\[
A \circ B := \{a \circ b : a \in A, b \in B\}.
\]
We consider the cardinality of this product set, especially those sets for which the product set is of maximal size. A simple observation is that the trivial bound
\[
|A \circ B| \leq \min\{|A||B|, |G|\}
\]
holds for any \( A, B \subseteq G \).

In this paper, we seek to answer the following question: given \( \emptyset \neq A \subseteq G \), what is the size of the largest set \( B \subseteq G \) such that \( |A \circ B| = |A||B| \)? We call any \( B \) with \( |A \circ B| = |A||B| \) an \( A \)-packing set and denote by \( \nu(A) \) the maximal size of an \( A \)-packing set:
\[
\nu(A) := \max\{|B| : B \subseteq G, |A \circ B| = |A||B|\}.
\]

Suppose that we have such a set \( B \). Since \( |A||B| = |A \circ B| \leq |G| \), it must be the case that \( |B| \leq |G|/|A| \) and thus
\[
\nu(A) \leq \left\lfloor \frac{|G|}{|A|} \right\rfloor.
\]
For some interesting sets \( A \), it can be easily established that \( \nu(A) \) is close to \( |G|/|A| \). For example, if \( A \subseteq G \) is a subgroup with distinct cosets \( x_1 \circ A, x_2 \circ A, \ldots, x_k \circ A \) where \( k = |G|/|A| \), we can take \( B = \{x_1, x_2, \ldots, x_k\} \) and then \( |A \circ B| = |A||B| = |G| \). Thus

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\end{itemize}
\( \nu(A) = |B| = |G|/|A| \). Conversely, if \( A = \{ x_1, x_2, \ldots, x_k \} \) with elements in different cosets of a subgroup \( B \) of order \( |G|/k \), \( B \) is an \( A \)-packing set.

The case that \( G \) is the multiplicative group \( \mathbb{F}_p^* \) of the finite field \( \mathbb{F}_p \) of \( p \) elements is particularly interesting in view of applications. More precisely, if \( p \) is prime and \( A = \{ 1, 2, \ldots, \lambda \} \) for some positive integer \( \lambda \), the authors in [8, 9, 10] used an \( A \)-packing set \( B \) to construct codes that correct single limited-magnitude errors. For more details see also [13, Section 6.2.2]. We denote \( \nu(\lambda) = \nu(\{ 1, 2, \ldots, \lambda \}) \).

Rusza’s Covering Lemma, see [18, Lemma 2.14], guarantees for any \( A \subseteq G \) the existence of \( B \subseteq G \) with

\[
\nu(A) \geq \left\lceil \frac{|G|}{|A \circ A^{-1}|} \right\rceil
\]

and since \( |A \circ A^{-1}| \leq |A|^2 \)

\[
\nu(A) \geq \left\lceil \frac{|G|}{|A|^2} \right\rceil .
\]

For the convenience of the reader we will give a very short proof of (1.1) in Section 2.

In the above result, \( A \circ A^{-1} \) denotes the set \( \{ a \circ b^{-1} : a, b \in A \} \), which we call the ratio set of \( A \). Note that the bound (1.1) is tight, up to multiplicative constants\(^{1}\) in the case when the ratio set satisfies the bound \( |A \circ A^{-1}| \ll |A| \). This generalises the result given by the simple construction above when \( A \) is a multiplicative subgroup to the broader class of sets with small ratio set.

In fact, the weaker bound (1.2) is also optimal up to multiplicative constants in general, as the construction described in Section 2 shows. This construction can be modified to see that essentially any integer value \( \nu(A) \) in the interval \( |G|/|A|^2, |G|/|A| \) can be attained.

Section 3 deals with the special case when \( G = \mathbb{F}_p^* \) with a prime \( p \) and \( A = \{ 1, 2, \ldots, \lambda \} \). In this case we use the standard notation \( AB \) for the product set, rather than \( A \circ B \) as above. Since \( |AA^{-1}| \gg \min\{\lambda^2, p\} \), (1.1) is only of limited power in this case. However, we give a simple construction which proves that\(^2\)

\[
\nu(\lambda) \gg \frac{p}{\lambda \log p}
\]

under the condition that \( \lambda \leq 0.9p^{1/2} \).

Section 4 contains a result on the group of symmetries \( \text{Sym}(B) = \{ x \in G : x \circ B = B \} \) of any \( A \)-packing set \( B \) of maximal size.

Finally in Section 5, we briefly discuss the related problem of finding a small \( A \)-covering set \( B \), that is, a set \( B \subseteq G \) such that \( A \circ B = G \).

\(^{1}\)Here and throughout the paper, the notation \( X \ll Y \) and \( Y \gg X \) indicates that there exists an absolute constant \( c > 0 \) such that \( X \leq cY \). If both \( X \ll Y \) and \( Y \ll X \), we write \( X \approx Y \).

\(^{2}\)We denote by \( \log \) the natural logarithm.
2. Proof of (1.1) and proof of the optimality of (1.2)

Proof of (1.1). Let $B \subseteq G$ be any set with $\nu(A) = |B|$. Then, by the maximality of $B$, for each $x \in G$ we have $(A \circ x) \cap (A \circ B) \neq \emptyset$, that is, $G \subseteq A^{-1} \circ A \circ B$ and hence $|G| \leq |A^{-1} \circ A \circ B| \leq |A \circ A^{-1}||B|$. Thus $|B| \geq |G|/|A \circ A^{-1}|$. \hfill \Box

The following construction shows that (1.2) is (up to a multiplicative constant) optimal.

Let $H = \{g, g^2, \ldots, g^k\} \subseteq G$ be any cyclic subgroup of $G$ with $|H| = k \geq 2$. Let $d = \lceil \sqrt{k} \rceil \geq 2$ and define

$$A_1 = \{g, g^2, \ldots, g^d\}, \quad A_2 = \{g^d, g^{2d}, \ldots, g^{(d-1)d}, g^{d^2}\}.$$ 

Define $A = A_1 \cup A_2$. Note that $|A| < 2d$ and that $A \circ A^{-1} = H$.

Now suppose that $|A \circ B| = |A||B|$ for some $B \subseteq G$. This is true if and only if there are no non-trivial solutions to the equation

$$a_1 \circ b_1 = a_2 \circ b_2, \quad (a_1, a_2, b_1, b_2) \in A \times A \times B \times B,$$

which happens if and only if

$$(A \circ A^{-1}) \cap (B \circ B^{-1}) = \{1\}.$$ 

We want to show that $B$ cannot be too large. Since $A \circ A^{-1} = H$, it must be the case that $(B \circ B^{-1}) \cap H = \{1\}$. But then $B$ cannot contain more than one element from each coset of $H$. Indeed, if $b_1, b_2 \in B$ with $b_1 = x \circ h_1$ and $b_2 = x \circ h_2$ and with $h_1, h_2 \in H$ distinct, it follows that

$$b_1 \circ b_2^{-1} = h_1 \circ h_2^{-1} \in H \setminus \{1\} = A \circ A^{-1} \setminus \{1\}.$$ 

Therefore

$$|B| \leq \frac{|G|}{k} < \frac{|G|}{(d-1)^2} \leq 16|G|/|A|^2.$$ 

This shows that $\nu(A) \ll |G|/|A|^2$. Furthermore, one can modify this construction by adding more elements from $H$ to the set $A$ in order to obtain, for any $0 \leq \alpha \leq 1$, a set $A'$ with $|A' \circ A^{-1}| \approx |A'|^{1+\alpha}$ and with $\nu(A') \ll |G|/|A' \circ A^{-1}|$. This gives a broader class of sets for which the bound (1.1) is tight up to multiplicative constants.

3. The case when $G = \mathbb{F}_p^*$ and $A = \{1, 2, \ldots, \lambda\}$

In this Section, we consider the case of the multiplicative group $\mathbb{F}_p^*$ of a finite prime field and fix $A$ to be the interval $A = \{1, 2, \ldots, \lambda\} \subseteq \mathbb{F}_p^*$. Recalling the notation from the introduction, we seek lower bounds for $\nu(\lambda)$. Inequality (1.1) does not immediately give a strong result because of the following proposition.

Proposition 3.1. For $A = \{1, 2, \ldots, \lambda\} \subseteq \mathbb{F}_p^*$ we have $|AA^{-1}| \gg \min\{\lambda^2, p\}$. 
Proof. For the set $A_Z = \{1, 2, \ldots, \lambda\}$ of integers we have

$$A_Z A_Z^{-1} = \left\{ ab^{-1} : a, b \in A_Z, \gcd(a, b) = 1 \right\}$$

and thus

$$|A_Z A_Z^{-1}| = \varphi(1) + 2(\varphi(2) + \varphi(3) + \ldots + \varphi(\lambda)) = \frac{6}{\pi^2} \lambda^2 + O(\lambda \log \lambda)$$

by [7, Theorem 330], where $\varphi$ is Euler’s totient function. If $\lambda < p^{1/2}$ and $1 \leq a_1, b_1, a_2, b_2 \leq \lambda$, then the congruence $a_1 b_1^{-1} \equiv a_2 b_2^{-1} \mod p$ is equivalent to the integer equation $a_1 / b_1 = a_2 / b_2$. Hence, the number of different elements of $AA^{-1}$ is the same as of $A_Z / A_Z$. If $\lambda \geq p^{1/2}$, $A$ contains the subset $A' = \{0, 1, \ldots, \lfloor p^{1/2} \rfloor\}$ and thus $|AA^{-1}| \gg p^{1/2}$. $\square$

Remark. For $\lambda \geq p^{1/2} \log^{1+\varepsilon} p$ we have $|AA^{-1}| = (1 + o(1))p$, see [5]. This result was later extended to all $\lambda$ with $p^{1/2} = o(\lambda)$, see [6, Theorem 1.7]. Also in [5], it is mentioned that $AA^{-1} = \mathbb{F}_p^*$ if and only if $\lambda \geq \frac{p + 1}{2}$.

With Proposition 3.1 in mind, (1.1) implies that $\nu(\lambda) \gg p/\lambda^2$. An explicit construction of such a set $B$ was given in [13, Section 6.2.2].

In fact, we can provide a simple construction of a set $B$ which is almost as large as possible with the property that $|AB| = |A||B|$. Identify $\mathbb{F}_p$ with the set of integers $\{1, 2, \ldots, p\}$ in the obvious way and define

$$B := \left\{ x \in \mathbb{F}_p : \lambda < x \leq \frac{p}{\lambda}, x \text{ is prime} \right\}.$$ 

This set has the property that $|AB| = |A||B|$. Indeed, suppose for a contradiction that we have a non-trivial solution to the equation

$$ab = a'b', \quad (a, a', b, b') \in A \times A \times B \times B.$$ 

Since $A$ and $B$ are both contained in sufficiently small intervals, there are no wraparound issues, and so we must have a non-trivial solution to the equation

$$(3.1) \quad ab = a'b', \quad (a, a', b, b') \in A_Z \times A_Z \times B_Z \times B_Z,$$

where

$$A_Z = \{1, 2, \ldots, \lambda\} \subseteq \mathbb{Z}, \quad B_Z = \{x \in \mathbb{Z} : \lambda < x \leq \frac{p}{\lambda}, x \text{ is prime}\}.$$ 

However, unique prime factorisation of the integers implies that the only solutions to (3.1) are trivial.

Furthermore, by the Prime Number Theorem,

$$|B| \gg \frac{p/\lambda}{\log(p/\lambda)} - \frac{\lambda}{\log \lambda}.$$ 

In particular, if $\lambda \leq 0.9 \sqrt{p}$, then we have $|B| \gg \frac{p}{\lambda \log p}$. We summarise this in the following statement:
Theorem 3.1. Let $A = \{1, 2 \ldots , \lambda \} \subset \mathbb{F}_p^*$ with $\lambda \leq 0.9 \sqrt{p}$. Then

$$\nu(A) \gg \frac{p}{\lambda \log p}.$$  

(1) Using explicit versions of the Prime Number Theorem, see [16],

$$c_1 \frac{x}{\log x} \leq \pi(x) \leq c_2 \frac{x}{\log x} \quad \text{if} \quad x \geq x_0$$

we can explicitly calculate the implied constant in Theorem 3.1.

(2) The same approach applies to any residue class ring $\mathbb{Z}_n$ with composite $n$.

(3) We may also take the larger packing set of rough numbers $B = \{x \in \mathbb{F}_p : \lambda < x \leq \frac{p \lambda}{\lambda}, x \text{ is not divisible by a prime} \leq \lambda\}$.

We have

$$|B| \sim \frac{p}{\lambda \log \lambda} \omega(u),$$

where $\omega$ is Buchstab’s function and $u = \frac{\log(p/\lambda)}{\log \lambda}$, see [3] or [15, Paragraph IV.32]. In particular, if $p^{1/3} \leq \lambda \leq p^{1/2}$, we have $1 \leq u \leq 2$ and $\omega(u) = \frac{1}{u} = \frac{\log \lambda}{\log(p/\lambda)}$. However, for $u \to \infty$, the Buchstab function $\omega(u)$ converges to $e^{-\gamma}$, where $\gamma$ is the Euler-Mascheroni constant, see [2]. In particular, if $\lambda = e^{o(\log p)}$ is subexponential, we get $|B| \gg \frac{p}{\lambda \log \lambda}$ and so $\nu(\lambda) \gg \frac{p}{\lambda \log \lambda}$.

4. Symmetries

Let $A \subseteq G$ be a set and $B$ be an $A$-packing set. In this section we obtain a general result about symmetries of our set of translations $B$ (this is in spirit of paper [14]). Surprisingly, the set of symmetries of this extremal set $B$ does not grow after taking the ratio $B \circ B^{-1}$.

Consider an arbitrary set $T \subseteq G$. Denote by $\text{Sym}(T)$ the group of symmetries of $T$ that is

$$\text{Sym}(T) = \{x \in G : x \circ T = T\}.$$  

Notice that $1 \in \text{Sym}(T)$, $\text{Sym}(T) = \text{Sym}^{-1}(T)$ and $\text{Sym}(T) \subseteq T \circ T^{-1}$.

**Proposition 4.1.** Let $A \subseteq G$ be a set and let $B$ be an $A$-packing set of maximal size. Then

$$\text{Sym}(B) = \text{Sym}(B \circ B^{-1}).$$

Further

$$(\text{Sym}(A \circ A^{-1})) \cap (A \circ A^{-1}) = (\text{Sym}(A \circ A^{-1}) \setminus \text{Sym}(B)) \bigcup \{1\}.$$

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3We write $f(x) \sim g(x)$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

4$f(x) = o(g(x))$ means $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$.

5We denote by $A \sqcup B$ the union of two disjoint sets.
Proof. The inclusion \( \text{Sym}(B) \subseteq \text{Sym}(B \circ B^{-1}) \) is trivial. Suppose that there is an element \( x \in \text{Sym}(B \circ B^{-1}) \) but \( x \notin \text{Sym}(B) \). It follows that there is \( b \in B \) such that \( b \circ x \notin B \). As in the proof of (1.1) in Section 2 we get \( B \circ A \circ A^{-1} \supseteq G \) and see that \( b \circ x = b' \circ a_1 \circ a_2^{-1} \) for some \( a_1, a_2 \in A \) and \( b' \in B \). Hence because \( x \in \text{Sym}(B \circ B^{-1}) \), we get
\[
\tilde{b} \circ (\tilde{b}')^{-1} = b \circ x \circ (b')^{-1} = a_1 \circ a_2^{-1}
\]
for some \( \tilde{b}, \tilde{b}' \in B \). But \( |A \circ B| = |A||B| \) and thus \( a_1 = a_2, \tilde{b} = b' \). It gives us \( b \circ x = b' \in B \) and this is a contradiction.

Taking \( x \in \text{Sym}(A \circ A^{-1}) \setminus \text{Sym}(B) \) and repeating the previous arguments, we obtain
\[
b \circ (b')^{-1} = a_1 \circ (x \circ a_2)^{-1} = \tilde{a}_1 \circ (\tilde{a}_2)^{-1}
\]
and hence \( b = b', \tilde{a}_1 = \tilde{a}_2 \). Thus \( x = a_1 \circ a_2^{-1} \in A \circ A^{-1} \) and we get
\[
\text{Sym}(A \circ A^{-1}) \setminus \text{Sym}(B) \subseteq A \circ A^{-1}.
\]
But \( \text{Sym}(B) \subseteq B \circ B^{-1} \) and \( (B \circ B^{-1}) \cap (A \circ A^{-1}) = \{1\} \) thus \( \text{Sym}(B) \cap (A \circ A^{-1}) = \{1\} \). This completes the proof.

If \( B \) is any \( A \)-packing set of maximal size, then the appearance of the set \( \text{Sym}(B) \) in our problem of computing \( \nu(A) \) is natural in view of a trivial equality \( \nu(A \circ \text{Sym}(B)) = \nu(A) = |B| \).

5. Covering sets

Given \( A \subseteq G \), we say that \( B \subseteq G \) is an \( A \)-covering set if \( A \circ B = G \). The covering number of \( A \), denoted \( \text{cov}(A) \), is the size of the smallest \( A \)-covering set. There is a natural connection between covering and packing problems, and likewise with the problems of determining the values of \( \text{cov}(A) \) and \( \nu(A) \). In particular, it follows from Ruzsa’s Covering Lemma that
\[
\text{cov}(A \circ A^{-1}) \leq \nu(A).
\]

The problem of determining \( \text{cov}(A) \) in the case \( G = \mathbb{F}_p^* \) was studied in [4, 11, 12], where \( A = \{1, 2, \ldots, \lambda\} \). A more general study of the problem can be found in [11]; see Section 3 therein for background on this problem in the finite setting. In particular, it is proved in [11, Corollary 3.2] that for any finite group \( G \) and \( A \subseteq G \)
\[
\frac{|G|}{|A|} \leq \text{cov}(A) \leq \frac{|G|}{|A|} (\log |A| + 1).
\]

By contrast with (5.1), we showed in Section 2 of this paper that \( \nu(A) \) can essentially take any value in between \( |G|/|A|^2 \) and \( |G|/|A| \). It is interesting to note that the size of \( \text{cov}(A) \) is much more restricted than that of \( \nu(A) \).

In the special case \( G = \mathbb{F}_p^* \) and \( A = \{1, 2, \ldots, \lambda\} \) we have the improvement \( \text{cov}(A) < 2p/\lambda \) by [4, Theorem 2]. However, an interesting observation is that if we instead take \( A \) to be the middle third interval then the log factor is needed and \( \text{cov}(A) \approx \log |A| \). In particular,
this gives us a constructive example (as opposed to random choice, see, say, [1]) of a set such that upper bound in (5.1) is sharp.

**Proposition 5.1.** For a prime \( p > 3 \) put 
\[
A = \{ x \in \mathbb{F}_p^* : x \in [p/3, 2p/3] \}.
\]
Then we have 
\[
\frac{\log(p - 1)}{\log(3)} \leq \text{cov}(A) < 3(\log(p) + 1).
\]

**Proof.** Put \( T = \{ x \in \mathbb{F}_p^* : x \notin [p/3, 2p/3] \} \). For \( \lambda \in \{1, \ldots, p-1\} \) let \( \text{inv}(\lambda) \in \{1, \ldots, p-1\} \) be the unique integer with \( \text{inv}(\lambda) \lambda \equiv 1 \) mod \( p \). By the simultaneous version of the Dirichlet Approximation Theorem, see [17], for any integer \( 1 \leq k < \log(p - 1)/\log(3) \) and \( \lambda_1, \ldots, \lambda_k \in \{1, \ldots, p - 1\} \) there is an integer \( 1 \leq n < p \) and integers \( a_1, \ldots, a_k \) such that 
\[
|\text{inv}(\lambda_i)n/p - a_i| \leq 1/(p - 1)^{1/k} < 1/3
\]
for \( i = 1, \ldots, k \). In other words, for any \( \lambda_1, \ldots, \lambda_k \in \mathbb{F}_p^* \) with \( 1 \leq k < (p - 1)/\log(3) \) there is \( n \in \lambda_1T \cap \cdots \cap \lambda_kT \). Putting \( B = \{\lambda_1, \ldots, \lambda_k\} \), we see that \( n \notin AB \) and hence \( AB \neq \mathbb{F}_p^* \) for any \( B \) with \( 1 \leq |B| < \log(p - 1)/\log(3) \). By the definition this means that \( \text{cov}(A) \geq \log(p - 1)/\log(3) \). The upper bound follows from (5.1). \( \square \)

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