On the Effectiveness of Punishments in a Repeated Epidemic Dissemination Game

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Abstract. This work uses Game Theory to study the effectiveness of punishments as an incentive for rational nodes to follow an epidemic dissemination protocol. The dissemination process is modeled as an infinite repetition of a stage game. At the end of each stage, a monitoring mechanism informs each player of the actions of other nodes. The effectiveness of a punishing strategy is measured as the range of values for the benefit-to-cost ratio that sustain cooperation. This paper studies both public and private monitoring. Under public monitoring, we show that direct reciprocity is not an effective incentive, whereas full indirect reciprocity provides a nearly optimal effectiveness. Under private monitoring, we identify necessary conditions regarding the topology of the graph in order for punishments to be effective. When punishments are coordinated, full indirect reciprocity is also effective with private monitoring.

Keywords: Epidemic Dissemination, Game Theory, Peer-to-Peer.

1 Introduction

Epidemic broadcast protocols are known to be extremely scalable and robust [2,15]. As a result, they are particularly well suited to support the dissemination of information in large-scale peer-to-peer systems, for instance, to support live streaming [17,16]. In such an environment, nodes do not belong to the same administrative domain. On the contrary, many of these systems rely on resources made available by self-interested nodes that are not necessarily obedient to the protocol. In particular, participants may be rational and aim at maximizing their utility, which is a function of the benefits obtained from receiving information and the cost of contributing to its dissemination.

Two main incentive mechanisms may be implemented to ensure that rational nodes are not interested in deviating from the protocol: one is to rely on balanced exchanges [17,16]; other is to monitor the degree of cooperation of every node and punish misbehavior [9]. When balanced exchanges are enforced, in every interaction, nodes must exchange an equivalent amount of messages of interest to each other. This approach has the main disadvantage of requiring symmetric interactions between nodes. In some cases, more efficient protocols may be achieved with asymmetric interactions [5,15,9], where balanced exchanges become infeasible. Instead, nodes are expected to forward messages without immediately receiving any benefit in return. Therefore, one must consider repeated interactions for nodes to able to collect information about the behavior of their neighbors, which may be used to detect misbehavior and trigger punishments.

Although monitoring has been used to detect and expel free-riders from epidemic dissemination protocols [9], no theoretical analysis studied the ability of punishments to sustain cooperation among rational nodes. Therefore, in this paper, we tackle this gap by using Game Theory [19]. The aim is to study the existence of equilibria in an infinitely repeated Epidemic Dissemination game. The stage game consists in a sequence of messages disseminated by the source, which are forwarded by every node $i$ to each neighbor $j$ with an independent probability $p_{i,j}$. At the end of each stage, a monitoring mechanism provides information to each node regarding $p_{i,j}$.

Following work in classical Game Theory that shows that cooperation in repeated games can be sustained using punishing strategies [8], we focus on this class of strategies. We assume that there is a
pre-defined target for the reliability of the epidemic dissemination process. To achieve this reliability, each node should forward every message to each of its neighbors with a probability higher than some threshold probability $p$, known a priori by the two neighboring nodes. We consider that a player $i$ defects from a neighbor $j$ if it uses a probability lower than $p$ when forwarding information to $j$. Each node $i$ receives a benefit $\beta_i$ per received message, but incurs a cost $\gamma_i$ of forwarding a message to a neighbor. Given this, we are particularly interested in determining the range of values of the ratio benefit-to-cost ($\beta_i/\gamma_i$) that allows punishing strategies to be equilibria. The wider is this range, the more likely it is for all nodes to cooperate.

The main contribution of this paper is a quantification of the effectiveness of different punishing strategies under two types of monitors: public and private. Public monitors inform every node of the actions of every other node with no delays. On the other hand, private monitoring inform only a subset of nodes of the actions of each node, and possibly with some delays. In addition, we study two particular types of punishing strategies: direct and full indirect reciprocity. In the former type, each node is solely responsible for monitoring and punishing each neighbor, individually. The latter type specifies that each misbehaving node should eventually be punished by every neighbor. More precisely, we make the following contributions:

- We derive a generic necessary and sufficient condition for a punishing strategy to be a Subgame Perfect Equilibrium under public monitoring. From this condition, we also derive an upper bound for the effectiveness of strategies that use direct reciprocity as an incentive. We observe that this value decreases very quickly with an increasing reliability, in many realistic scenarios. On the other hand, if full indirect reciprocity is used, then this problem can be avoided. We derive a lower bound for the effectiveness of these strategies, which is not only independent from the desired reliability, but also close to the theoretical optimum, under certain circumstances.
- Using private monitoring with delays, information collected by each node may be incomplete, even if local monitoring is perfect. We thus consider the alternative solution concept of Sequential Equilibrium, which requires the specification of a belief system that captures the belief held by each player regarding past events of which it has not been informed. For a punishing strategy to be an equilibrium, this belief must be consistent. We provide a definition of consistency that is sufficient to derive the effectiveness of punishing strategies.
- Under private monitoring with a consistent belief system, we show that certain topologies are ineffective when monitoring is fully distributed. Then, we prove that, unless full indirect reciprocity is possible, the effectiveness decreases monotonically with the reliability. To avoid this problem, punishments should be coordinated, i.e., punishments applied to a misbehaving node $i$ by every neighbor of $i$ should overlap in time. We derive a lower bound for the effectiveness of full indirect reciprocity strategies with coordinated punishments. The results indicate that the number of stages during which punishments overlap should be at least of the order of the maximum delay of the monitoring mechanism. This suggests that, when implementing a distributed monitoring mechanism, delays should be minimized.

The remainder of the paper is structured as follows. Section 2 discusses some related work. The general model is provided in Section 3. The analysis of public and private monitoring are given in Sections 4 and 5 respectively. Section 6 concludes the paper and provides directions of future work.

2 Related Work

There are examples of work that use monitoring to persuade rational nodes to engage in a dissemination protocol. In Equicast [12], the authors perform a Game Theoretical analysis of a multicast protocol where nodes monitor the rate of messages sent by their neighbors and apply punishments whenever the rate drops below a certain threshold. The protocol is shown to be a dominating
strategy. Nodes are disposed in an approximately random network, thus, the dissemination process resembles epidemic dissemination. However, given that the network is connected and nodes are expected to forward messages to every neighbor with probability 1, there is no non-determinism in the delivery of messages. Furthermore, the authors restrict the actions available to each player by assuming that they only adjust the number of neighbors with which they interact and a parameter of the protocol. This contrasts with our analysis, where we consider non-deterministic delivery of messages and a more general set of strategies available to players.

Guerraoui et al. [9] propose a mechanism that monitors the degree of cooperation of each node in epidemic dissemination protocols. The goal is to detect and expel free-riders. This mechanism performs statistical inferences on the reports provided by every node regarding its neighbors, and estimates the cooperation level of each node. If this cooperation level is lower than a minimum value, then the node is expelled from the network. The authors perform a theoretical and experimental analysis to show that this mechanism guarantees that free-riders only benefit by deviating from the protocol if the degree of deviation is not significantly high. However, no Game Theoretical analysis is performed to determine in what conditions are free-riders willing to abide to the protocol.

In [17,16], the authors rely on balanced exchanges to provide incentives for nodes to cooperate in dissemination protocols for data streaming. In BAR Gossip [17], the proposed epidemic dissemination protocol enforces strictly balanced exchanges. This requires the use of a pseudo-random number generator to determine the set of interactions in every round of exchanged updates, and occasionally nodes may have to send garbage as a payment for any unbalance in the amount of information exchanged with a neighbor. A stepwise analysis shows that nodes cannot increase their utility by deviating in any step of the protocol. In FlightPath [16], the authors remove the need for sending garbage by allowing imbalanced exchanges. By limiting the maximum allowed imbalance between every pair of nodes, the authors show that it is possible for the protocol to be an $1/10$-Nash equilibrium, while still ensuring a streaming service with high quality. Unfortunately, these results might not hold for other dissemination protocols that rely on highly imbalanced exchanges. In these cases, a better alternative might be to rely on a monitoring approach.

Other game theoretical analysis have addressed a similar problem, but in different contexts. In particular, the tit-for-tat strategy used in BitTorrent, a P2P file sharing system, has been subjected to a wide variety of Game Theoretical analysis [6,21,20,14]. These works consider a set of $n$ nodes deciding with which nodes to cooperate, given a limited number of available connections, with the intent to share content. Therefore, contrary to our analysis, there is no non-determinism in content delivery.

Closer to our goal is the trend of work that applies game theory to selfish routing [22,7,11]. In this problem, each node may be a source of messages to be routed along a fixed path of multiple relay nodes to a given destination. The benefit of a node is to have its messages delivered to the destination, while it incurs the costs of forwarding messages as a relay node. This results in a linear relationship between the actions of a player and the utility of other nodes. In our case, that relation is captured by the definition of reliability, which is non-linear. Consequently, the utility functions of an epidemic dissemination and a routing game possess an inherently different structure.

3 Model

We now describe the System and Game Theoretical models, followed by the definition of effectiveness. In Appendix A we provide a more thorough description of the considered epidemics model and include some auxiliary results that are useful for the analysis.
3.1 System Model

There is a set of nodes $\mathcal{N}$ organized into a directed graph $G$. This models a P2P overlay network with a stable membership. Each node has a set of in ($\mathcal{N}_{i}^{-1}$) and out-neighbors ($\mathcal{N}_{i}$). Communication channels are assumed to be reliable. We model the generation of messages in this network by considering the existence of a single external source $s$. Its behavior is described by a profile $p_{s}$, which defines for each node $i \in \mathcal{N}_{s}$ the probability $p_{s}[i] \in [0,1)$ of $i$ receiving a message directly from $s$, with the restriction that $p_{s}[i] > 0$ for some $i$. We consider the graph to be connected from the source $s$, i.e., there exists a path from $s$ to every node $i \in \mathcal{N}$. Conversely, every node $i$ forwards messages to every neighbor $j \in \mathcal{N}_{i}$ with an independent probability $p_{i}[j]$. Provided a profile of probabilities $p$, which includes the vector of probabilities $p_{s}$ and $p_{i}$ used by $s$ and by every node $i$, respectively, we can define the reliability of the dissemination protocol as the probability of a node receiving a message. For the analysis, it is convenient to consider the probability of a node not receiving a message, denoted by $q_{i}(p)$. The reliability of the protocol is then defined by $1 - q_{i}(p)$. The exact expression of $q_{i}$ is included in Appendix A.

3.2 Monitoring Mechanism.

The monitoring mechanism emits a signal $s \in \mathcal{S}$, where every player $i$ may observe a different private signal $s_{i} \in s$. This signal can take two values for every pair of nodes $j \in \mathcal{N}$ and $k \in \mathcal{N}_{j}$: $s_{i}[j,k] = \text{cooperate}$ notifies $i$ that $j$ forwarded messages to $k$ with a probability higher than a specified threshold, and $s_{i}[j,k] = \text{defect}$ signals the complementary action. This signal may be public, if all nodes read the same signal, or private, otherwise. Moreover, if the signal is perfectly correlated with the action taken by a node, then monitoring is perfect; otherwise, monitoring is said to be imperfect.

We consider that monitoring is performed locally by every node. A possible implementation of such monitoring mechanism in the context of P2P networks can be based on the work of [9]. A simpler and cheaper mechanism would instead consist in every out-neighbor $j$ of a given node $i$ recording the fraction of messages sent by $i$ to $j$ during the dissemination of a fixed number $M$ of messages. Then, $j$ may use this information along with an estimate of the reliability of the dissemination of messages to $i$ $(1 - q_{i})$ in order to determine whether $i$ is cooperating or defecting. When a defection is detected, $j$ is disseminates an accusation against $i$ towards other nodes. If $i$ is expected to use $p_{i}[j] < 1$ towards $j$, then monitoring is imperfect. Furthermore, accusations may be blocked, disrupted, or wrongly emitted against one node due to both malicious and rational behavior. However, in this paper, we consider only perfect monitoring, faithful propagation of accusations, and that nodes are rational. Almost perfect monitoring can be achieved with a large $M$. Faithful propagation may be reasonable to assume if the impact of punishments on the reliability of each non-punished node is small and the cost of sending accusations is not significant. We intend to relax these assumptions in future work.

In our model, an accusation emitted by a node $j$ against an in-neighbor $i$ may only be received by the nodes that are reachable from $j$ by following paths in the graph. In addition, if we consider the obvious possibility that $i$ might block any accusation emitted by one of its neighbors, then these paths cannot cross $i$. Finally, the number of nodes informed of each defection may be further reduced to minimize the monitoring costs. This restricts the set of in-neighbors of $i$ that may punish $i$ for defecting $j$. In this paper, we consider two alternative models. First, we study public monitoring, where all nodes may be informed about any defection with no delays. Then, we study the private monitoring case, taking into consideration the possible delay of the dissemination of accusations.
3.3 Game Theoretical Model

Our model considers an infinite repetition of a stage game. Each stage consists in the dissemination of a sequence of messages and is interleaved with the execution of the monitoring mechanism, which provides every node with some information regarding the actions taken by other nodes during the stage game.

Stage Game. The stage game is modeled as a strategic game. An action of a player \(i\) is a vector of probabilities \(p_i \in \mathcal{P}_i\), such that \(p_i[j] > 0\) only if \(j \in \mathcal{N}_i\). Thus, \(p_i\) represents the average probability used by \(i\) to forward messages during the stage. It is reasonable to consider that \(i\) adheres to \(p_i\) during the complete stage, since \(i\) expects to be monitored by other nodes with regard to a given \(p_i\). Hence, changing strategy is equivalent to following a different \(p_i\). Despite \(s\) not being a player, for simplicity, we consider that every profile \(p \in \mathcal{P}\) implicitly contains \(p_s\). We can also define a mixed strategy \(a_i \in \mathcal{A}_i\) as a probability distribution over \(\mathcal{P}_i\), and a profile of mixed strategies \(a \in \mathcal{A}\) as a vector containing the mixed strategies followed by every player. The utility of a player \(i\) is a function of the benefit \(\beta_i\) obtained per received message and the cost \(\gamma_i\) of forwarding a message to each neighbor. More precisely, this utility is given by the probability of receiving messages \((1 - q_i[p])\) multiplied by the difference between the benefit per message \((\beta_i)\) and the expected cost of forwarding that message to every neighbor \((\gamma_i \sum_{j \in \mathcal{N}_i} p_i[j])\):

\[
u_i[p] = (1 - q_i[p])(\beta_i - \gamma_i \sum_{j \in \mathcal{N}_i} p_i[j]).\]

If players follow a profile of mixed strategies \(a\), then the expected utility is denoted by \(\nu_i[a]\), which definition depends on the structure of every \(a_j\).

Repeated Game. The repeated game consists in the infinite interleaving between the stage game and the execution of the monitoring mechanism, where future payoffs are discounted by a factor \(\omega_i\) for every player \(i\). The game is characterized by (possibly infinite) sequences of previously observed signals, named histories. The set of finite histories observed by player \(i\) is represented by \(\mathcal{H}_i\) and \(\mathcal{H} = (\mathcal{H}_i)_{i \in \mathcal{N}}\) is the set of all histories observed by any player. A pure strategy for the repeated game \(\sigma_i \in \Sigma_i\) maps each history to an action \(p_i\), where \(\sigma \in \Sigma\) is a profile of strategies. Consequently, \(\sigma[h]\) specifies for some history \(h \in \mathcal{H}\) the profile of strategies \(p\) for the stage game to be followed by every node after history \(h\) is observed. A behavioral strategy \(\sigma_i\) differs from a pure strategy only in that \(i\) assigns a probability distribution \(a_i \in \mathcal{A}_i\) over the set of actions for the stage game. For simplicity, we will use the same notation for the two types of strategies. The expected utility of player \(i\) after having observed history \(h_i\) is given by \(\pi_i[\sigma|h_i]\). The exact definitions of equilibrium and expected utility depend on the type of monitoring being implemented. Hence, these definitions will be provided in each of the sections regarding public and private monitoring.

A Brief Note on Notation. Throughout the paper, we will conveniently simplify the notation as follows. Whenever referring to a profile of strategies \(\sigma\), followed by all nodes except \(i\), we will use the notation \(\sigma_{-i}\). Also, \((\sigma_i, \sigma_{-i})\) denotes the composite of a strategy \(\sigma_i\) and a profile \(\sigma_{-i}\). The same reasoning applies to profiles of pure and mixed strategies of the stage game. Finally, we will let \((h, s)\) denote the history that follows \(h\) after signal \(s\) is observed.

3.4 Effectiveness

We know from Game Theoretic literature that certain punishing strategies can sustain cooperation if the discount factor \(\omega_i\) is sufficiently close to 1 [8]. This minimum value is a function of the
parameters $\beta_i$ and $\gamma_i$ for every player $i$. More precisely, for larger values of the benefit-to-cost ratio $\beta_i/\gamma_i$, the minimum required value of $\omega_i$ is smaller. In addition, for certain values of the benefit-to-cost ratio, no value of $\omega_i$ can sustain cooperation. Notice that these parameters are specified by the environment and thus cannot be adjusted in the protocol. Thus, a strategy is more effective if it is an equilibrium for wider ranges of $\omega_i$, $\beta_i$, and $\gamma_i$. In this paper, we only measure the effectiveness of a profile of strategies $\sigma$ as the allowed range of values for the benefit-to-cost ratio.

**Definition 1.** The effectiveness of a profile $\sigma \in \Sigma$ is given by $\psi(\sigma) \subseteq [0, \infty)$, such that, if, for every $i \in N$, $\frac{\beta_i}{\gamma_i} \in \psi(\sigma)$, then there exists $\omega_i \in (0, 1)$ for every $i \in N$ such that $\sigma$ is an equilibrium.

### 4 Public Monitoring

In this section, we assume that the graph allows public monitoring to be implemented. That is, every node is informed about each defection at the end of the stage when the defection occurred. We can thus simplify the notation by considering only public signals $s \in \mathcal{S}$ and histories $h \in \mathcal{H}$.

With perfect monitoring, the public signal observed after players follow $p \in \mathcal{P}$ is deterministic. This type of monitoring requires accusations to be broadcast. However, since the dissemination of accusations is interleaved with the dissemination of a sequence of messages, monitoring costs may not be relevant if the size of each accusation is small, compared to the size of messages being disseminated.

The section is organized as follows. We start by providing a general definition of punishing strategies and then introduce the definition of expected utility and the solution concept for public monitoring. We then proceed to a Game Theoretical analysis, where we analyze punishing strategies that use direct reciprocity and full indirect reciprocity.

#### 4.1 Public Signal and Punishing Strategies

We study a wide variety of punishing strategies, by considering a parameter $\tau$ that specifies the duration of punishments. Of particular interest to this analysis is the case where the duration of punishments is infinite, which is known in the Game Theoretical literature as the Grim-trigger strategy. Furthermore, a punishing strategy specifies a Reaction Set $RS[i, j] \subseteq N$ of nodes that are expected to react to every defection of $i$ from $j$ during $\tau$ stages. This set always contains $i$ and $j$, but it may also contain other nodes. In particular, a third node $k \in RS[i, j]$ that is an in-neighbor of $i$ ($k \in N_i^{-1}$) is expected to stop forwarding any messages to $i$, as a punishment. If $k$ is not a neighbor of $i$, then $k$ may also adapt the probabilities used towards its out-neighbors, for instance, to keep the reliability high for every unpunished node.

In order for a node $j \in N$ to monitor an in-neighbor $i \in N_j^{-1}$, the protocol must define for every history $h \in \mathcal{H}$ a threshold probability $p_i[j|h]$ with which $i$ should forward messages to $j$. Since $h$ is public, $p_i[j|h]$ is common knowledge between $i$ and $j$, allowing for an accurate monitoring. Given this, the public signal for perfect public monitoring is defined as follows.

**Definition 2.** For every $h \in \mathcal{H}$ and $p' \in \mathcal{P}$, let $s = \text{sig}_{p'}[h]$ be the public signal observed when players follow $p'$. For every $i \in N$ and $j \in N_i$, $s[i, j] = \text{cooperate}$ if and only if $p'_i[j] \geq p_i[j|h]$.

Then, a punishing strategy becomes a set of rules specifying how every $p_i[j|h]$ should be defined. Namely, let $\sigma^* \in \Sigma$, denote a punishing strategy, which specifies that after a history $h$ every node $i$ should forward messages to a neighbor $j$ with probability $p_i[j|h]$. We will denote by $\sigma^* \in \Sigma$ the profile of punishing strategies. The restrictions imposed on every $p_i[j|h]$ can be defined as follows. Every node $i$ evaluates the set of defections observed in a history $h$ by $i$ and every neighbor $j$ to which both nodes should react. Basing on this information, $i$ uses a deterministic function
to determine the probability \( p_i[j|h] \). For convenience, we will define \( p_i[k|h] = 0 \) for every node \( k \in \mathcal{N} \setminus \mathcal{N}_i \) that is not an out-neighbor of \( i \).

For the precise definition of punishing strategy, we need an additional data structure called Defection Set (DS\(_i[j|h]\)) containing the set of defections to which both \( i \) and \( j \) are expected to react, according to RS. This information is specified in the form of tuples \((k_1, k_2, r)\) stating that both \( i \) and \( j \) are expected to react to a defection of \( k_1 \) from \( k_2 \) that occurred in the previous \( r \)-th stage. This way, \( p_i[j|h] \) is defined as a function of DS\(_i[j|h]\). Namely, if DS\(_i[j|h]\) contains some defection of \( j \) from \( k \) and \( i \) should react to it \((i \in RS[j,k])\) or if \( i \) defected from \( j \), then \( p_i[j|h] = 0 \). Otherwise, \( i \) forwards messages to \( j \) with any positive probability that is a deterministic function of DS\(_i[j|h]\).

**Definition 3.** Define DS\(_i[j|h]\) \( \subseteq \mathcal{N} \times \mathcal{N} \times \mathbb{Z} \) as follows for every \( i \in \mathcal{N}_i \), \( j \in \mathcal{N}_i \), and \( h \in \mathcal{H} \):

- DS\(_i[j|\emptyset]\) = \( \emptyset \).
- For \( h = (h', s) \), DS\(_i[j|h]\) = \( L_1 \cup L_2 \), where:
  1. \( L_1 = \{(k_1, k_2, r + 1) | (k_1, k_2, r) \in \text{DS}_{i}[j|h'] \land r + 1 < \tau\} \).
  2. \( L_2 = \{(k_1, k_2, 0) | k_1, k_2 \in \mathcal{N} \land i, j \in RS[k_1, k_2] \land s[k_1, k_2] = \text{defect}\} \).

For every \( h \in \mathcal{H} \), \( i \in \mathcal{N}_i \), and \( j \in \mathcal{N}_i \):

- If there exists \( r < \tau \) such that \((i, j, r) \in \text{DS}_{i}[j|h]\), then \( p_i[j|h] = 0 \).
- If there exist \( r < \tau \) and \( k \in \mathcal{N}_j \) such that \((j, k, r) \in \text{DS}_{i}[j|h]\), then \( p_i[j|h] = 0 \).
- Otherwise, \( p_i[j|h] \) is a positive function of DS\(_i[j|h]\).

We consider that the source \( s \) also abides to this strategy. In Section 4.3, we will show that it follows by construction that if some node \( k \) observes a defection of \( i \) from a node \( j \) and \( k \in RS[i,j] \), then \( k \) reacts to this defection during \( \tau \) stages, regardless of the ensuing actions of \( i \) and the current punishments being applied. In addition, after defecting some neighbor \( j \), \( i \) does not forward messages to any node of RS\(_i[j,j] \) in any of the following \( \tau \) stages.

### 4.2 Expected Utility and Solution Concept

The expected utility of a profile of pure strategies for every player \( i \) and history \( h \) is given by:

\[
\pi_i[\sigma|h] = u_i[p] + \omega_i \pi_i[\sigma|(h, \text{sig}[p|h])],
\]

where \( p = \sigma[h] \). Conversely, we can define the expected utility for a profile of behavioral strategies as follows:

\[
\pi_i[\sigma|h] = u_i[a] + \omega_i \sum_{s \in \mathcal{S}} \pi_i[\sigma|(h, s)] pr[s|a, h],
\]

where \( a = \sigma[h] \) and \( pr[s|a, h] \) is defined as

\[
pr[s|a, h] = \prod_{j \in \mathcal{N}} pr_j[s|a_j, h],
\]

where \( pr_j[s|a_j, h] \) is the probability of the actions of \( j \) in \( a_j \) leading to \( s \).

The considered solution concept for this model is the notion of Subgame Perfect Equilibrium (SPE) [19], which refines the solution concept of Nash Equilibrium (NE) for repeated games. In particular, a profile of strategies is a NE if no player can increase its utility by deviating, given that other players follow the specified strategies. The solution concept of NE is adequate for instance for strategic games, where players choose their actions prior to the execution of the game. However, in repeated games, players have multiple decision points, where they may adapt their actions...
Lemma 6. of nodes.  

$s_1 \exists \sigma$ such that the considered proofs are performed by induction. 

require the introduction of some auxiliary notation. The complete proofs are in Appendix B.1. All original graph. Given this, we prove some correctness properties of the punishment strategy, which respectively. The same notation will be used for profiles $a' \in A$ and $p' \in P$, namely, $\sigma^*[h|a']$ and $\sigma^*[h|p']$, respectively. The following property captures the above intuition, which is known to be true from Game Theoretic literature and can be proven in a similar fashion to [3]: 

Property 5. One-deviation. A profile of strategies $\sigma^*$ is a SPE if and only if for every player $i \in \mathcal{N}$, history $h \in \mathcal{H}$, and $a'_i \in A_i$, $\pi_i[\sigma^*_i, \sigma^*_{-i}, a'_i] \geq \pi_i[\sigma^*_i, \sigma^*_{-i}, h_i]$, where $\sigma^*_i = \sigma^*_i[h|a'_i]$. 

4.3 Evolution of the Network

After any history $h \in \mathcal{H}$, the network induced by $h$ when players follow $\sigma^*$ can be characterized by a subgraph, where a link $(i,j)$ is active iff $i$ is not punishing $j$ and $i$ has not defected from $j$ in the last $\tau$ stages. All the remaining links are inactive. When considering a profile of punishing strategies $\sigma^*$, the evolution of this subgraph over time is deterministic. That is, after a certain number of stages, inactive links become active, such that at most after $\tau$ stages we obtain the original graph. Given this, we prove some correctness properties of the punishment strategy, which require the introduction of some auxiliary notation. The complete proofs are in Appendix B.1. All the considered proofs are performed by induction.

For any profile of pure strategies $\sigma \in \Sigma$ and $h \in \mathcal{H}$, let $hist[h, r|\sigma]$ denote the history resulting from players following $\sigma$ during $r$ stages, after having observed $h$. That is: 

$$hist[h, r|\sigma] = (h, (s^r)_{r \in \{1 \ldots r\}}),$$

such that $s^1 = sig[\sigma[h]|h]$ and for every $r' \in \{1 \ldots r - 1\}$ we have $s^{r'+1} = sig[\sigma[h']|h']$, where $h' = hist[h, r'|\sigma]$. Notice that $hist[h, 0|\sigma] = h$ for every $h \in \mathcal{H}$ and $\sigma \in \Sigma$.

The following notation will be useful in the analysis, where, for every $r > 0$, $h' = hist[h, r-1|\sigma]$ and $p' = \sigma[h']$: 

- $q_i[h, r|\sigma] = q_i[p']$.
- $p_i[h, r|\sigma] = \sum_{j \in \mathcal{N}_i} p_i[j]$.
- $u_i[h, r|\sigma] = u_i[p'] = (1 - q_i[h, r|\sigma])(\beta_i - \gamma_i p_i[h, r|\sigma])$.
- $\mathcal{N}_i[h] = \{ j \in \mathcal{N}_i | p_i[j|h] > 0 \}$.

The following lemma characterizes the evolution of the punishments being applied to any pair of nodes.

Lemma 6. For every $h \in \mathcal{H}$, $r \in \{1 \ldots \tau - 1\}$, $i \in \mathcal{N}$, and $j \in \mathcal{N}_i$, 

$$DS_i[j|h^*_r] = \{(k_1, k_2, r' + r)|(k_1, k_2, r') \in DS_i[j|h] \land r' + r < \tau \},$$

where $h^*_r = hist[h, r|\sigma^*]$. 

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Consequently, this establishes an upper bound for the effectiveness of any strategy.

From this lemma, we obtain the following trivial corollary that simply states that every punishment ends after \( \tau \) stages. This is true by the fact that for every \( h \in \mathcal{H}, i \in \mathcal{N}, j \in \mathcal{N}_i \), and \((k_1,k_2,r') \in DS_i^h[j] \), we have \( r' < \tau \).

**Corollary 7.** For every \( h \in \mathcal{H}, r \geq \tau, i \in \mathcal{N}, \) and \( j \in \mathcal{N}_i \), it holds \( DS_i[j|h^*_i] = \emptyset \), where \( h^*_i = hist[h,r|\sigma^*] \).

The following lemma proves that every node \( i \) that defects from a neighbor \( j \) expects to be punished exactly during the next \( \tau \) stages, regardless of the following actions of \( i \) or the punishments already being applied to \( i \). The auxiliary notation \( CD_i[p'|h] \) is used to denote the characterization of the defections performed by \( i \) in \( p' \) after history \( h \). More precisely, for every \( i \in \mathcal{N} \),

\[
CD_i[p'|h] = \{j \in \mathcal{N}_i|p'_i[j] < p_i[j|h]\}.
\]

**Lemma 8.** For every \( h \in \mathcal{H}, p' \in \mathcal{P}, r \in \{1 \ldots \tau\}, i \in \mathcal{N}, \) and \( j \in \mathcal{N}_i \),

\[
DS_i[j|h^*_i] = DS_i[j|h^*_i] \cup \{(k_1,k_2,r-1)|k_1,k_2 \in \mathcal{N} \land k_2 \in CD_{k_1}[p'|h] \land i \in RS[k_1,k_2]\}, \tag{3}
\]

where \( h^*_i = hist[h,r|\sigma^*], h'_i = hist[h,r|\sigma'], \) and \( \sigma' = \sigma^*[h|p'] \) is the profile of strategies where all players follow \( p' \) in the first stage.

**Proof.** By induction, the base case follows from the definition of \( DS_i[j|h] \) and the fact that \( i \) registers every defection of \( k_1 \) to \( k_2 \) detected in \( p' \) by adding \((k_1,k_2,0)\) to \( DS_i[j|h] \). Inductively, after \( r \leq \tau \) stages, this pair is transformed into \((k_1,k_2,r-1)\). (Complete proof in Section B.1).

From the previous lemmas, it follows that any punishment ceases after \( \tau \) stages, which is proven in Lemma 9

**Lemma 9.** For every \( h \in \mathcal{H}, p' \in \mathcal{P}, r > \tau, i \in \mathcal{N}, \) and \( j \in \mathcal{N}_i \),

\[
DS_i[j|h^*_i] = DS_i[j|h^*_i] = \emptyset, \tag{4}
\]

where \( h^*_i = hist[r|\sigma^*], h'_i = hist[r|\sigma'], \) and \( \sigma' = \sigma^*[h|p'] \).

**Proof.** From Corollary 7 and Lemma 8 it follows that after \( \tau \) stages, every pair \((k_1,k_2)\) is removed from \( DS_i[j|h] \). (Complete proof in Section B.1).

### 4.4 Generic Results

This section provides some generic results. Namely, we first derive the theoretically optimal effectiveness, which serves as an upper bound for the effectiveness of any profile of strategies. Then, we derive a simplified generic necessary and sufficient condition for any profile of strategies to be a SPE. The complete proofs are in Appendix B.2.

Proposition 10 establishes a minimum necessary benefit-to-cost ratio for any profile of strategies to be a SPE of the repeated Epidemic Dissemination Game. Intuitively, the benefit-to-cost ratio must be greater than the expected costs of forwarding messages to neighbors \( (\bar{p}_i = \sum_{j \in \mathcal{N}_i} p_i[j|\emptyset]) \), since otherwise a player has incentives to not forward any messages. This is the minimum benefit-to-cost ratio that provides an enforceable utility as defined by the Folk Theorems [8], given that the utility that results from nodes following any profile \( p \in \mathcal{P} \) is feasible and the minmax utility is 0. Consequently, this establishes an upper bound for the effectiveness of any strategy.
Proposition 10. For every profile of punishing strategies $\sigma^*$, if $\sigma^*$ is a SPE, then, for every $i \in \mathcal{N}$, $\frac{\partial}{\partial \tilde{p}_i} > \tilde{p}_i$. Consequently, $\psi[\sigma^*] \subseteq (v, \infty)$, where $v = \max_{i \in \mathcal{N}} \tilde{p}_i$.

Proof. (See Section 3.2). \qed

A necessary and sufficient condition for any profile of punishing strategies to be an equilibrium is that no node has incentives to stop forwarding messages to any subset of neighbors, i.e., to drop those neighbors. This condition is named the DC Condition, which is defined as follows:

Definition 11. DC Condition. For every player $i \in \mathcal{N}$, history $h \in \mathcal{H}$, and $\mathcal{D} \subseteq \mathcal{N}_i[h]$,

$$\sum_{r=0}^{\tau} \omega_i^r(u_i[h, r|\sigma^*] - u_i[h, r|\sigma']) \geq 0,$$  

(5)

where $\sigma' = (\sigma'_i, \sigma^*_{-i})$, $\sigma'_i = \sigma_i^*[h|p'_i]$, and $p'_i$ is defined as:

- For every $j \in \mathcal{D}$, $p'_i[j] = 0$.
- For every $j \in \mathcal{N}_i \setminus \mathcal{D}$, $p'_i[j] = p_i[j|h]$.

The following Lemma shows that the DC Condition is necessary.

Lemma 12. If $\sigma^*$ is a SPE, then the DC Condition is fulfilled.

Proof. By the One-deviation property, for a profile to be a SPE, a player $i$ must not be able to increase its utility by unilaterally deviating in the first stage. In particular, this is true if $i$ deviates by dropping any subset $\mathcal{D}$ of neighbors. Furthermore, since any punishment ends after $\tau$ stages, by Lemma 9, we have that if nodes follow the deviating profile $\sigma'$, then for every $r > \tau$,

$$u_i[h, r|\sigma^*] = u_i[h, r|\sigma'].$$  

The DC Condition follows by the One-deviation property and the fact that

$$\pi_i[\sigma^*|h] - \pi_i[\sigma'|h] = \sum_{r=0}^{\infty} \omega_i^r(u_i[h, r|\sigma^*] - u_i[h, r|\sigma']).$$  

(Complete proof in Section 3.2). \qed

In Lemma 16, we also show that the DC Condition is sufficient. In order to prove this, we first need to show that every node $i$ cannot increase its utility by not following a pure strategy in every stage game and by not forwarding messages with a probability in $\{0, p_i[j|h]\}$ to every neighbor $j$. This is shown in two steps. First, Lemma 13 proves that any local best response mixed strategy only gives positive probability to an action in $\{0, p_i[j|h]\}$. Second, Lemma 14 proves that there is a pure strategy for the stage game that is a local best response.

Define the set of local best response strategies for history $h \in \mathcal{H}$ and any $i \in \mathcal{N}$ as:

$$BR[\sigma_{-i}^*|h] = \{a_i \in A_i | \forall a'_i \in A_i, \pi_i[(\sigma_i^*[h|a_i], \sigma_{-i}^*)|h] \geq \pi_i[(\sigma_i^*[h|a'_i], \sigma_{-i}^*)|h]\}.$$  

Notice that $BR[\sigma_{-i}^*|h]$ is not empty. The following lemma first proves that every player $i$ always uses probabilities in $\{0, p_i[j|h]\}$ towards a neighbor $j$.

Lemma 13. For every $i \in \mathcal{N}$, $h \in \mathcal{H}$, $a_i \in BR[\sigma_{-i}^*|h]$, and $p_i \in \mathcal{P}_i$ such that $a_i[p_i] > 0$, it is true that for every $j \in \mathcal{N}_i$ we have $p_i[j] \in \{0, p_i[j|h]\}$.
Proof. The proof is by contradiction. Namely, assume that for some \( p_i \) and \( a_i \) that is a best response and \( a_i[p_i] > 0 \), we have that \( p_i \) does not fulfill the restrictions defined above. We can find \( a'_i \) and \( p'_i \) such that:

- \( p'_i \) fulfills the restrictions defined in the lemma.
- \( a'_i[p'_i] = a_i[p_i] + a_i[p'_i] \) and \( a'_i[p_i] = 0 \).

By letting \( p^* = \sigma^*[h] \), we have that

\[
\text{sig}([p_i, p^*_i])|h| = \text{sig}([p'_i, p^*_i])|h|.
\]

Consequently,

\[
\pi_i[\sigma^*_i|h|a_i], \sigma^*_i|h| < \pi_i[\sigma^*_i|h|a'_i], \sigma^*_i|h|.
\]

Thus, \( a_i \) cannot be a best response, which is a contradiction. (Complete proof in Section B.2). \( \square \)

We now have to show that there exists a pure strategy in \( BR[\sigma^*|h] \).

Lemma 14. For every \( h \in H \) and \( i \in N \), there exists \( a_i \in BR[\sigma^*|h] \) and \( p_i \in P_i \) such that \( a_i[p_i] = 1 \).

Proof. First, notice that if only pure strategies are best-responses, then the result follows immediately. If there exists a mixed strategy \( a_i \) that is a best response, then \( i \) must be indifferent between following any profile \( p_i \) such that \( a_i[p_i] > 0 \). Otherwise, \( i \) could find a better strategy \( a'_i \). In that case, any such profile \( p_i \) is a best response. (Complete proof in Section B.2). \( \square \)

Lemma 15 is a direct consequence of Lemmas 13 and 14.

Lemma 15. For every \( h \in H \) and \( i \in N \), there exists \( p_i \in P_i \) and a pure strategy \( \sigma_i = \sigma^*_i|h|p_i \) such that:

1. For every \( j \in N_i \), \( p_i[j] \in \{0, p_i[j|h]\} \).
2. For every \( a_i \in A_i \), \( \pi_i[\sigma_i, \sigma^*_{-i}|h] \geq \pi_i[\sigma'_i, \sigma^*_{-i}|h] \), where \( \sigma'_i = \sigma^*_i|h|a_i \).

Proof. (See Section B.2). \( \square \)

It is now possible to show that the DC Condition is sufficient.

Lemma 16. If the DC Condition is fulfilled, then \( \sigma^* \) is a SPE.

Proof. If the DC Condition holds, then no player \( i \) can increase its utility by dropping any subset of neighbors. By Lemma 15, it follows that \( i \) cannot increase its utility by following any alternative strategy for the first stage game, which by the One-deviation property implies that the profile \( \sigma^* \) is a SPE. (Complete proof in Section B.2). \( \square \)

The following theorem merges the results from Lemmas 12 and 16.

Theorem 17. \( \sigma^* \) is a SPE if and only if the DC Condition holds.
4.5 Direct Reciprocity is not Effective

If \( G \) is undirected, then it is possible to use direct reciprocity only, by defining \( RS[i,j] = \{i,j\} \) for every \( i \in N \) and \( j \in N_i \). That is, if \( i \) defects from \( j \), then only \( j \) punishes \( i \). Direct reciprocity is the ideal incentive mechanism in a fully distributed environment, since it does not require accusations to be sent by any node. The goal of this section is to show that punishments that use direct reciprocity are not effective, even using public monitoring. To prove this, we first derive a generic necessary benefit-to-cost ratio and then we identify the conditions under which direct reciprocity is ineffective. The complete proofs are included in Appendix B.3.

Lemma 18 derives a minimum benefit-to-cost ratio for direct reciprocity.

**Lemma 18.** If \( \sigma^* \) is a SPE, then, for every \( i \in N \) and \( j \in N_i \), it is true that \( q_i^* > q_i^+ \) and:

\[
\frac{\beta_i}{\gamma_i} > \frac{p_i[j]}{\bar{q}_i^+} \left( 1 - q_i^+ + \frac{1 - \bar{q}_i^+}{\tau} \right),
\]

where \( p_i[j] \) is the strategy where \( i \) drops \( j \), \( \sigma' = (\sigma_i^*[\emptyset], \sigma^*_{-i}) \), \( q_i^+ = q_i[\sigma'[\emptyset]] \), and \( q_i^* = q_i[\sigma^*[\emptyset]] \).

**Proof.** By the definition of SPE and Theorem 17, the DC Condition must hold for the initial empty history and every deviation in the first stage where any player \( i \) drops an out-neighbor \( j \). After some manipulations of the DC Condition for this specific scenario, Inequality (6) is obtained. (Complete proof in Section B.3).

Lemma 19 also shows that direct reciprocity is not an effective incentive mechanism under certain circumstances. Namely, by letting \( q_i^* \) to be the probability of delivery of messages in equilibrium \( (q_i[\sigma^*[\emptyset]]) \), we find that, if \( p_i[j] + q_i^* < 1 \), then the effectiveness is of the order \( (1/q_i^*, \infty) \), which decreases to \( 0 \) very quickly with an increasing reliability. The conditions under which direct reciprocity is ineffective are easily met, for instance, when a node has more neighbors than what is strictly necessary to ensure high reliability.

**Lemma 19.** Suppose that for any \( i \in N \) and \( j \in N_i \), \( p_i[j] + q_i^* < 1 \). If \( \sigma^* \) is a SPE, then:

\[
\psi[\sigma^*] \subseteq \left( \frac{1}{q_i^*}, \infty \right),
\]

where \( q_i^* = q_i[\sigma^*[\emptyset]] \).

**Proof.** The proof follows directly from Lemma 18 and the fact that, as we show in Lemma 54 from Appendix A, we have that if \( q_i^+ \) results from exactly \( j \) punishing \( i \) for defecting from \( j \), then

\[
q_i^+ \leq q_i^* \frac{1}{1 - p_i[j]},
\]

(Complete proof in Section B.3).

4.6 Full Indirect Reciprocity is Sufficient

Unlike direct reciprocity, if full indirect reciprocity is used, then the effectiveness may be independent of the reliability of the dissemination protocol. This consists in the case where for every \( i \in N \) and \( j \in N_i \) we have

\[
N_i^{-1} \subseteq RS[i,j].
\]

The goal of this section is to show that, if full indirect reciprocity is used, then the effectiveness is independent of the reliability of the dissemination protocol under certain circumstances. To
prove this, we proceed in two steps. First, we conveniently simplify the DC Condition. Then, we
derive a sufficient benefit-to-cost ratio for $\sigma^*$ to be a SPE. The complete proofs are included in
Appendix [B.4].

The following lemma simplifies the DC Condition for this specific type of punishing strategies.

**Lemma 20.** The profile of strategies $\sigma^*$ is a SPE if and only if for every $h \in \mathcal{H}$ and $i \in \mathcal{N}$:

$$\sum_{r=1}^{\tau} (\omega_i^r u_i[h, r|\sigma^*]) - (1 - q_i[h, 0|\sigma^*])\gamma_i \tilde{p}_i[h, 0|\sigma^*] \geq 0. \quad (9)$$

**Proof.** This simplification is obtained directly from the DC Condition and the fact that, if a node
$i$ has incentives to drop some out-neighbor $j$, then the best response strategy is to drop all out-
neighbors. This is proven by defining $p''$, where $i$ drops a subset $D$ of out-neighbors, and $p'$, where
$i$ drops every out-neighbor. We can prove that:

- $\text{sig}[p'|h] = \text{sig}[p''|h]$.
- $u_i[p'] > u_i[p'']$.
- For any $r > 0$, $u_i[h, r|\sigma'] = u_i[h, r|\sigma'']$, where $\sigma'$ and $\sigma''$ differ from $\sigma^*$ exactly in that $i$ follows
$p'$ and $p''$ in the first stage, respectively.

This implies that $\pi_i[\sigma'|h] > \pi_i[\sigma''|h]$, and therefore the best response for $i$ is to drop all neighbors.
(Complete proof in Section [B.4]).

**Theorem 22** derives a lower bound for the effectiveness of a full indirect reciprocity profile of
strategies $\sigma^*$. This is done in two steps. First, it is shown that the history $h$ that minimizes the left
side of Inequality 9 results exactly in the same punishments being applied during the first $\tau - 1$
stages. This is proven in Lemma 21.

**Lemma 21.** Let $h \in \mathcal{H}$ be defined such that for every $h' \in \mathcal{H}$, the value of the left side of Inequality 9 for $h$
is lower than or equal to the value for $h'$. Then, for every $r \in \{1 \ldots \tau - 2\}$,

$$u_i[h, r|\sigma^*] = u_i[h, r + 1|\sigma^*].$$

**Proof.** The proof is performed by contradiction, where we assume that $h$ minimizes the left side of
Inequality 9 but for some $r \in \{1 \ldots \tau - 2\}$

$$u_i[h, r|\sigma^*] \neq u_i[h, r + 1|\sigma^*].$$

This implies that in $h$ a set of punishments ends at the end of stage $r$. We can find $h'$ where those
punishments are either postponed or anticipated one stage and such that the left side of Inequality 9
is lower for $h'$ than for $h$, which is a contradiction. (Complete proof in Section [B.4]).

It is now possible to derive a sufficient benefit-to-cost ratio for full indirect reciprocity to be
a SPE, which constitutes a lower bound for the effectiveness of these strategies. However, this
derivation is only valid when the following assumption holds. There must exist a constant $c \geq 1$
such that for every history $h$:

$$q_i[h, 0|\sigma^*] \geq 1 - c(1 - q_i[h, 1|\sigma^*]). \quad (10)$$

Intuitively, this states that, after some history $h$, if the value of $q_i$ varies from the first stage to
the second due to some punishments being concluded, then this variation is never too large. With
this assumption, we can derive a sufficient benefit-to-cost for $\sigma^*$ to be a SPE.
Theorem 22. If there exists a constant $c \geq 1$ such that, for every $h \in \mathcal{H}$ and $i \in \mathcal{N}$, Assumption holds, then $\psi[\sigma^*] \supseteq (v, \infty)$, where

$$v = \max_{h \in \mathcal{H}} \max_{i \in \mathcal{N}} \tilde{p}_i[h, 0|\sigma^*] \left(1 + \frac{c}{\tau}\right).$$

Proof. We consider the history $h$ that minimizes the left side of Inequality [4]. Using the result of Lemma 21, after some manipulations, we can find that if for every $i \in \mathcal{N}$, $\beta_i/\gamma_i \in (v, \infty)$, then there exist $\omega_i \in (0, 1)$ for every $i \in \mathcal{N}$ such that Inequality [4] is true for every history $h'$, which implies by Lemma [20] that $\sigma^*$ is a SPE. This allows us to conclude that $\psi[\sigma^*] \supseteq (v, \infty)$. (Complete proof in Section B.4).

We can then conclude that if $c$ is small or $\tau$ is large, and the maximum of $\tilde{p}_i[h, 0|\sigma^*]$ is never much larger than $\tilde{p}_i$ for every $i$ and $h$, then the effectiveness of full indirect reciprocity is close to the optimum derived in Proposition [10]. In particular, if Grim-trigger is used $(\tau \to \infty)$ and $\tilde{p}_i$ is maximal for every $i$, then the effectiveness is optimal. Furthermore, if for any $h$ both $q_i[h, 0]$ and $q_i[h, 1]$ are small, then the effectiveness of full indirect reciprocity differs from the theoretical optimum only by a factor $1 + 1/\tau$, which is upper bounded by 2 for any $\tau \geq 1$.

5 Private Monitoring

When using public monitoring, we make the implicit assumption that the monitoring mechanism is able to provide the same information instantly to every node, which requires the existence of a path from every out-neighbor $j$ of any node $i$ to every node of the graph, that does not cross $i$. In addition, public monitoring is only possible if accusations are broadcast to every node. We now consider private monitoring, where the dissemination of accusations may be restricted by the topology and scalability constraints. However, any node that receives an accusation may react to it. Therefore, the definition of RS is no longer necessary. In addition, accusations may be delayed.

5.1 Private Signals

In private monitoring, signals are determined by the history of previous signals $h$ and the profile $p$ followed in the last stage. Namely, sig[p|h] returns a signal $s$, such that every node $i$ observes only its private signal $s_i \in s$, indicating for every other node $j \in \mathcal{N}$ whether $j$ cooperated or defected with its out-neighbors in previous stages. The distinction between cooperation and defection is now determined by a threshold probability $p_i[j|h_i]$. If a node $i$ defects an out-neighbor $j$ in stage $r$, then $k$ is informed of this defection with a delay $d_k[i, j]$, i.e., $k$ is informed only at the end of stage $r + d_k[i, j]$. We only assume that, for every node $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$, both $i$ and $j$ are informed instantly of the action of $i$ towards $j$ in the previous stage, i.e.:

$$d_i[i, j] = d_j[i, j] = 0.$$

We consider that these delays are common knowledge among players. Moreover, we still assume that monitoring is perfect and that accusations are propagated faithfully. We intend to relax the assumptions in future work. With this in mind, it is possible to provide a precise definition of a private signal. For every player $i \in \mathcal{N}$ and history $h \in \mathcal{H}$, we denote by $h_i \in h$ the private history observed by $i$ when all players observe the history $h \in \mathcal{H}$. If $|h_i| \geq r \geq 1$, then let $h^*_i$ denote the last $r$-th signal observed by $i$, where $h^*_i$ is the last signal. A private signal is defined such that if some node $j$ observes a defection of an in-neighbor $i \in \mathcal{N}_{j}^{-1}$, every node $k \in \mathcal{N}$ such that $d_k[i, j]$ is finite ($d_k[i, j] < \infty$) observes this defection $d_k[i, j]$ stages after the end of the stage it occurred. The value $d_k[i, j]$ is infinite if and only if accusations emitted by $j$ against $i$ may never reach $k,$
either due to every path from \( j \) to \( k \) crossing \( i \) or the monitoring mechanism not disseminating the accusation to \( k \). However, if there exists a path from \( j \) to \( k \) without crossing \( i \) and \( k \in \mathcal{N}_i^{-1} \), then \( d_k[i, j] < \infty \).

Formally:

**Definition 23.** For every \( i \in \mathcal{N} \) and \( h \in \mathcal{H} \), let \( s'_i \in \text{sig}[p'_i|h] \) be the private signal observed by \( i \) when players follow \( p' \in \mathcal{P} \) after having observed \( h \). We have:

- For every \( j \in \mathcal{N} \) and \( k \in \mathcal{N}_j \) such that \( d_i[j, k] = 0 \), \( s'_i[j, k] = \text{cooperate} \) if and only if \( p'_j[k] \geq p_j[k|h_j] \), where \( h_j \in h \).
- For every \( j \in \mathcal{N} \) and \( k \in \mathcal{N}_j \) such that \( 0 < d_i[j, k] < \infty \), \( s'_i[j, k] = \text{defect} \) if and only if:
  - \(|h| \geq d_i[j, k] \).
  - For \( h_k \in h \) and \( s'_k = h^d_i[j, k] \), \( s'_i[j, k] = \text{defect} \).
- For every \( j \in \mathcal{N} \) and \( k \in \mathcal{N}_j \) such that \( d_i[j, k] = \infty \), \( s'_i[j, k] = \text{cooperate} \).

### 5.2 Private Punishments

In this context, we can define a punishing strategy \( \sigma^*_i \) for every node \( i \) as a function of the threshold probability \( p_i[j|h_i] \) determined by \( i \) for every private history \( h_i \) and out-neighbor \( j \). Notice that in the definition of private signals we assume that an accusation by \( j \) is emitted against \( i \) iff \( i \) uses \( p_i[j] < p_i[j|h_i] \) for any private history \( h_i \). This was reasonable to assume in public monitoring, where histories were public. Here, the strategy must also specify for every private history \( h_j \) the threshold probability \( p_i[j|h_j] \), since \( h_i \) may differ from \( h_j \). In order for \( j \) to accurately monitor \( i \), for every \( h \in \mathcal{H} \) and \( h_i, h_j \in h \), we must have

\[
p_i[j|h_i] = p_i[j|h_j].
\]

Therefore, both threshold probabilities must be computed as a function of the same set of signals. The only issue with this requirement is that defection signals may arrive at different stages to \( i \) and \( j \). For instance, if \( k_1 \) defects from \( k_2 \) in stage \( r \) and \( d_i[k_1, k_2] < d_j[k_1, k_2] \), then \( i \) must wait for stage \( r + d_j[k_1, k_2] \) before taking this defection into consideration in the computation of the threshold probability. Furthermore, as in public monitoring, \( i \) and \( j \) are expected to react to a given defection for a finite number of stages. However, as we will see later, this number should vary according to the delays in order for punishments to be effective. Thus, for every \( k_1 \in \mathcal{N} \) and \( k_2 \in \mathcal{N}_{k_1} \), we define \( \tau[k_1, k_2|i, j] \) to be the number of stages during which \( i \) and \( j \) react to a given defection of \( k_1 \) from \( k_2 \). Notice that \( \tau[k_1, k_2|i, j] = \tau[k_1, k_2\backslash j, i] \).

This intuition is formalized as follows. As in public monitoring, \( DS_i[j|h_i] \) denotes the set of defections observed by \( i \) and that \( j \) will eventually observe. This set also contains tuples in the form \( (k_1, k_2, r) \). The main difference is that now \( i \) may have to wait before considering this tuple in the definition of \( p_i[j|h_i] \). We signal this by allowing \( r \) to be negative and by using the tuple in the definition of \( p_i[j|h_i] \) only when \( r \geq 0 \). When \( i \) observes a defection for the first time, it adds \( (k_1, k_2, v) \) to \( DS_i[j|h_i] \), where

\[
v = \min[d_i[k_1, k_2] - d_j[k_1, k_2], 0].
\]

Then, \( i \) removes this pair when \( r = \tau[k_1, k_2|i, j] \). For simplicity, we allow \( v \) to take the value \( \infty \) when \( d_j[k_1, k_2] = \infty \), resulting in that \( i \) never takes into consideration this defection when determining \( p_i[j|h_i] \). This leads to the following definition.

**Definition 24.** For every \( i \in \mathcal{N} \), \( h_i \in \mathcal{H}_i \), and \( j \in \mathcal{N}_i \cup \mathcal{N}_i^{-1} \), define \( DS_i[j|h_i] \subseteq \mathcal{N} \times \mathbb{Z} \) as follows:

- \( DS_i[j|\emptyset] = \emptyset \).
For every $i \in \mathcal{N}$, $h_i \in \mathcal{H}_i$, and $j \in \mathcal{N}_i$, $\sigma^*_i[h_i] = p_i[j|h_i]$:

- Let $K = \{(k_1, k_2, r) \in DS_i[j|h_i] | r \geq 0\}$.
- If there exists $r \geq 0$ such that $(i, j, r) \in K$, then $p_i[j|h_i] = 0$.
- If there exist $r \geq 0$ and $k \in \mathcal{N}_i$ such that $(j, k, r) \in K$, then $p_i[j|h_i] = 0$.
- Otherwise, $p_i[j|h_i]$ is a positive function of $K$.

For every $j \in \mathcal{N}_i^{-1} \setminus \mathcal{N}_i$, $\sigma^*_i[h_i] = p_j[i|h_i]$ such that:

- Let $K = \{(k_1, k_2, r) \in DS_i[j|h_i] | r \geq 0\}$.
- If there exists $r \geq 0$ such that $(j, i, r) \in K$, then $p_j[i|h_i] = 0$.
- If there exist $r \geq 0$ and $k \in \mathcal{N}_i$ such that $(i, k, r) \in K$, then $p_j[i|h_i] = 0$.
- Otherwise, $p_j[i|h_i]$ is a positive function of $K$.

### 5.3 Expected Utility and Solution Concept

We now model the interactions as a repeated game with imperfect information and perfect recall, for which the solution concept of Sequential Equilibrium is adequate [13]. Its definition requires the specification of a belief system $\mu$. After a player $i$ observes a private history $h_i \in \mathcal{H}_i$, $i$ must form some expectation regarding the history $h \in \mathcal{H}$ observed by every player, which must include $h_i$. This is captured by a probability distribution $\mu_i[\cdot|h_i]$ over $\mathcal{H}$. By defining $\mu = (\mu_i)_{i \in \mathcal{N}}$, we call a pair $(\sigma, \mu)$ an assessment, which is assumed to be common knowledge among all players. The expected utility of a profile of strategies $\sigma$ is then defined as:

$$
\pi_i[\sigma|\mu, h_i] = \sum_{h \in \mathcal{H}} \mu_i[h|h_i] \pi_i[\sigma|h],
$$

where $\pi_i[\sigma|h]$ is defined as in the public monitoring case.

An assessment $(\sigma^*, \mu^*)$ is a Sequential Equilibrium if and only if $(\sigma^*, \mu^*)$ is Sequentially Rational and Consistent. The definition of sequential rationality is identical to that of subgame perfection:

**Definition 25.** An assessment $(\sigma^*, \mu^*)$ is Sequentially Rational if and only if for every $i \in \mathcal{N}$, $h_i \in \mathcal{H}_i$, and $\sigma'_i \in \Sigma_i$, $\pi'_i[\sigma^*|\mu^*, h_i] \geq \pi'_i[\sigma'_i, \sigma^*|\mu^*, h_i]$.

However, defining consistency for an assessment $(\sigma^*, \mu^*)$ is more intricate. The idea of defining this concept was introduced in [13], intuitively defined as follows in our context. For any profile $\sigma^*$, every private history $h_i$ that may be reached with positive probability when players follow $\sigma^*$ is said to be consistent with $\sigma^*$; otherwise, $h_i$ is inconsistent. For any consistent $h_i$, $\mu_i[h|h_i]$ must be defined using the Bayes rule. The definition of $\mu_i$ for inconsistent private histories varies with the specific definition of Consistent Assessment. It turns out that, in our case, the notion of Preconsistency introduced in [10] is sufficient.

We now provide the formal definition of Preconsistency and later provide an interpretation in the context of punishment strategies. Let $pr_i[h'|h, \sigma]$ be the probability assigned by $i$ to $h' \in \mathcal{H}$ being reached from $h \subset h'$ when players follow $\sigma \in \Sigma$. Given $h_i \in \mathcal{H}_i$, we can define

$$
pr_i[h'|\mu, h_i, \sigma] = \sum_{h \in \mathcal{H}} \mu_i[h|h_i] pr_i[h'|h, \sigma].
$$
For \( h'_i \in H_i \) such that \( h_i \subset h'_i \), let:

\[
pr_i[h'_i|\mu, h_i, \sigma] = \sum_{h' \in H: h' \in h'} pr_i[h'|\mu, h_i, \sigma].
\]

**Definition 26.** An assessment \((\sigma^*, \mu^*)\) is Preconsistent if and only if for every \( i \in N \), \( h_i \in H_i \), and \( h'_i \in H_i \) such that \( h_i \subset h'_i \), if there exists \( \sigma'_i \in \Sigma_i \) such that \( pr_i[h'_i|\mu^*, h_i, (\sigma'_i, \sigma^{*,*}_{-i})] > 0 \), then for every \( h' \in H \) such that \( h'_i \subset h' \):

\[
\mu_i[h'|h'_i] = \frac{pr_i[h'|\mu^*, h_i, (\sigma'_i, \sigma^{*,*}_{-i})]}{pr_i[h'_i|\mu^*, h_i, (\sigma'_i, \sigma^{*,*}_{-i})]}.
\]

The underlying intuition of this definition when considering a profile of punishing strategies \( \sigma^* \) is as follows. First, notice that a history \( h_i \) is consistent with \( \sigma^* \) if and only if no defections are observed in \( h_i \). For any \( \sigma'_i \), the set of strategies \( \sigma' = (\sigma'_i, \sigma^{*,*}_{-i}) \) may specify non-deterministic actions for the stage game, by only due to \( \sigma'_i \). Consider any \( h'_i \). If \( h'_i \) does not contain any defections, then the only strategies \( \sigma'_i \) such that \( h'_i \) is consistent with any such \( \sigma' \) are those where \( i \) does not defect any node. Therefore, we can set \( h_i \) to the empty history and from the above definition derive the conclusion that \( \mu_i[h'|h'_i] = 1 \) if and only if \( h' \) does not contain any defections. Similarly, if \( h'_i \) only contains defections performed by \( i \), then \( h_i \) can be the empty set and there must be only one \( h' \) such that \( \mu_i[h'|h'_i] = 1 \), which is the history where only defections performed by \( i \) are observed by any player.

If \( h'_i \) contains only defections performed by \( i \), then we can use induction on the number of defections committed by other nodes to prove that \( \mu_i[h'|h'_i] = 1 \) if and only if \( h' \) contains exactly the defections observed by \( i \) in \( h'_i \). The base case follows from the two previous scenarios. As for the induction step, there are two hypothesis. If the last defections were performed in the last stage, then there is no \( h_i \subset h'_i \) such that

\[
pr_i[h'_i|\mu, h_i, \sigma^*] > 0.
\]

Otherwise, consider that the last defections performed by other nodes occurred in the last \( r \)-th stage where \( r > 1 \), when \( i \) observed \( h_i \). In this case, it is true that \( pr_i[h'|h'_i] \) holds. Here, there is only one history \( h \) such that \( h_i \subset h \) and \( \mu_i[h|h_i] = 1 \), which is true by the induction hypothesis. This history contains exactly the defections observed by \( i \) in \( h_i \), which are also included in \( h'_i \). Thus, the only history \( h' \) that may follow \( h_i \) fulfills the condition that no other defection was performed other than what observed in \( h_i \).

In summary, \( \mu_i[h|h_i] = 1 \) if and only if \( h \) is the history containing \( h_i \) and the set of defections observed by any node \( j \in H \) in \( h_j \subset h \) is a subset of the set of defections observed in \( h_i \).

The importance of this definition of consistency is that in the authors prove that the One-deviation property also holds for Preconsistent assessments, which is sufficient for our analysis.

**Property 27.** One-deviation. A Preconsistent assessment \((\sigma^*, \mu^*)\) is Sequentially Rational if and only if for every player \( i \in N \), history \( h_i \in H_i \), and profile \( a'_i \in A_i \),

\[
\pi_i[\sigma^*_i, \sigma^{*,*}_{-i}|\mu^*, h_i] \geq \pi_i[\sigma'_i, \sigma^{*,*}_{-i}|\mu^*, h_i],
\]

where \( \sigma'_i = \sigma^*_i[h_i|a'_i] \) is defined as in public monitoring.

### 5.4 Evolution of the Network

When a player \( i \) observes a private history \( h_i \in H_i \), only the histories \( h \in H \) such that \( h_i \in h \) can be observed by other players. Given this, we use the same notation as in public monitoring, when referring to the evolution of the network after a history \( h \) is observed. Namely, \( \text{hist}[h, r|\sigma] \) is the
resulting history starting from the observation of \( h \) and when all players follow the pure strategy \( \sigma \). Therefore, we continue to use the same notation for \( q_i (q_i[h,r[\sigma]]) \), \( \bar{p}_i (\bar{p}_i[h,r[\sigma]]) \), and \( u_i (u_i[h,r[\sigma]]) \).

Now, we have

\[
N_i[h_i] = \{ j \in N_i | p_i[j|h_i] > 0 \}.
\]

The definition of \( \text{CD}_i[p|h] \subseteq N \) is almost identical to that of the public monitoring case. Namely, for every \( i \in N \), \( h \in \mathcal{H} \), and \( h_i \in h_i \),

\[
\text{CD}_i[p|h] = \{ j \in N_i | p_i[j] < p_i[j|h_i] \}.
\]

The following lemma proves that every node \( k_1 \) that defects from an out-neighbor \( k_2 \) expects \( i \) and \( j \) to react to this defection during the next \( \tau[k_1,k_2,i,j] \) stages, regardless of the following actions of \( k_1 \) or the punishments already being applied to \( k_1 \).

**Lemma 28.** For every \( h \in \mathcal{H} \), \( p' \in \mathcal{P} \), \( r > 0 \), \( i \in N_i \), and \( j \in N_i \):

\[
\text{DS}_i[j[h',i,r]] = DS_i[j[h',i,r] \cup \{ (k_1,k_2,r - 1 - d_i[k_1,k_2] + v[k_1,k_2]) | k_1,k_2 \in N \land \]
\[
k_2 \in \text{CD}_i[p'|h'] \land r \in \{ d_i[k_1,k_2] + 1 \ldots d_i[k_1,k_2] + \tau[k_1,k_2,i,j] - v[k_1,k_2] \} \land \]
\[
v[k_1,k_2] = \min [d_i[k_1,k_2] - d_j[k_1,k_2], 0],
\]

where \( h'_{i,r} \in \text{hist}[h,r[\sigma^*]], h'_{i,r} \in \text{hist}[h,r[\sigma']] \), and \( \sigma' = \sigma^*[p'] \) is the profile of strategies where all players follow \( p' \) in the first stage.

**Proof.** By induction, the base case follows from the definition of \( \text{DS}_i \) and the fact that \( i \) registers every defection of \( k_1 \) to \( k_2 \) in stage \( d_i[k_1,k_2] \), adding \( (k_1,k_2,v[k_1,k_2]) \) to \( \text{DS}_i[j|h] \). Inductively, after \( r \leq d_i[k_1,k_2] + \tau[k_1,k_2,i,j] - v[k_1,k_2] \) stages, this pair is transformed into \((k_1,k_2,r - 1)\). (Complete proof in Section C.1). □

For the sake of completeness, we prove in Lemma 29 that the strategy is well defined, in terms of defining threshold probabilities that are always common knowledge between pairs of players. This supports our assumption in the definition of private signals that an accusation is emitted by \( j \) against \( i \) iff \( i \) uses \( p_i[j] < p_i[j|h_j] \) towards \( j \).

**Lemma 29.** For every \( i \in N \), \( j \in N_i \), \( h \in \mathcal{H} \), and \( h_i,h_j \in h_i \):

\[
p_i[j|h_i] = p_i[j|h_j].
\]

**Proof.** It follows from the definition of \( \text{DS} \) that a node \( i \) never includes in the set \( K \) a tuple \((k_1,k_2,r)\) such that \( r < 0 \), for any out-neighbor \( j \). This value is only negative when \( d_i[k_1,k_2] < d_j[k_1,k_2] \), in which case \( v \) is set to \(-(d_j[k_1,k_2] - d_i[k_1,k_2])\). By Lemma 28, this value only becomes 0 when \( j \) is also informed of this defection, in which case both nodes include the pair in \( K \). Consequently, \( K \) is always defined identically by \( i \) and \( j \) after any history \( h \), which implies the result. (Complete proof in Section C.1). □

### 5.5 Generic Results

Proposition 30 reestablishes the optimal effectiveness for private monitoring. The proof of this proposition is identical to that of Proposition 10. The only difference lies in the fact that now the effectiveness of a profile of strategies \( \sigma^* \) is conditional on a belief system \( \mu^* (\psi[\sigma^*|\mu^*]) \).

**Proposition 30.** For every assessment \( (\sigma^*,\mu^*) \), if \((\sigma^*,\mu^*)\) is Sequentially Rational, then, for every \( i \in N \), \( \frac{\bar{p}_i}{p_i} \geq \bar{p}_i \). Consequently, \( \psi[\sigma^*] \subseteq (v, \infty) \), where \( v = \max_{i \in N} \bar{p}_i \).
Proof. (See Section C.2).

As in the public monitoring case, we can define a necessary and sufficient condition for the defined profile of strategies to be Sequentially Rational, which we name PDC Condition. The proof of both necessity and sufficiency is almost identical to that of public monitoring.

**Definition 31. PDC Condition.** For every $i \in \mathcal{N}$, $h_i \in \mathcal{H}_i$, and $D \subseteq \mathcal{N}_i[h_i]$,

$$
\sum_{h \in \mathcal{H}} \mu_i^{*}[h|h_i] \sum_{r=0}^{\infty} \omega_r^{*}(u_i[h, r|\sigma^{*}] - u_i'[h, r|\sigma']) \geq 0,
$$

(16)

where $\sigma' = (\sigma_i^{*}[h_i|p'_i], \sigma_{-i}^{*})$ and $p'_i$ is defined as:

- For every $j \in D$, $p'_i[j] = 0$.
- For every $j \in \mathcal{N} \setminus D$, $p'_i[j] = p_i[j|h_i]$.

The following corollary captures the fact that the PDC condition is necessary, which follows directly from the One-deviation property and the definition of $\pi_i$ for private monitoring.

**Corollary 32.** If the assessment $(\sigma^{*}, \mu^{*})$ is Sequentially Rational and Preconsistent, then the PDC Condition is fulfilled.

In order to prove that the PDC Condition is sufficient, we proceed in the same fashion to public monitoring, always implicitly assuming that the considered assessment is Preconsistent. Redefine the set of local best responses for every node $i$ and private history $h_i \in \mathcal{H}_i$ as:

$$
BR(\sigma_{-i}^{*}|\mu^{*}, h_i) = \{ a_i \in A_i | \forall a'_i \in A_i, \pi_i[\sigma_i^{*}[h_i|a_i], \sigma_{-i}^{*}][\mu^{*}, h_i] \geq \pi_i[\sigma_i^{*}[h_i|a'_i], \sigma_{-i}^{*}][\mu^{*}, h_i] \}.
$$

**Lemma 33.** For every $i \in \mathcal{N}$, $h_i \in \mathcal{H}_i$, $a_i \in BR(\sigma_{-i}^{*}|\mu^{*}, h_i)$, and $p_i \in \mathcal{P}_i$ such that $a_i[p_i] > 0$, it is true that for every $j \in \mathcal{N}_i$ we have $p_i[j] \in \{0, p_i[j|h_i]\}$.

**Proof.** (See Section C.2).

**Lemma 34.** For every $i \in \mathcal{N}$ and $h_i \in \mathcal{H}_i$, there exists $a_i \in BR(\sigma_{-i}^{*}|\mu^{*}, h_i)$ and $p_i \in \mathcal{P}_i$ such that $a_i[p_i] = 1$.

**Proof.** (See Section C.2).

**Lemma 35.** For every $i \in \mathcal{N}$ and $h_i \in \mathcal{H}_i$, there exists $p_i \in \mathcal{P}_i$ and a pure strategy $\sigma_i = \sigma_i^{*}[h_i|p_i]$ such that:

1. For every $j \in \mathcal{N}_i$, $p_i[j] \in \{0, p_i[j|h_i]\}$.
2. For every $a_i \in A_i$, $\pi_i[\sigma_i, \sigma_{-i}^{*}|\mu^{*}, h_i] \geq \pi_i[\sigma_i', \sigma_{-i}^{*}|\mu^{*}, h_i]$, where $\sigma_i' = \sigma_i^{*}[h_i|a_i]$.

**Proof.** (See Section C.2).

**Lemma 36.** If the PDC Condition is fulfilled and $(\sigma^{*}, \mu^{*})$ is Preconsistent, then $(\sigma^{*}, \mu^{*})$ is Sequentially Rational.

**Proof.** (See Section C.2).

The following theorem merges the results from Corollary 32 and Lemma 36.

**Theorem 37.** If $(\sigma^{*}, \mu^{*})$ is Preconsistent, then $(\sigma^{*}, \mu^{*})$ is Sequentially Rational if and only if the PDC Condition is fulfilled.
5.6 Ineffective Topologies

An important consequence of Theorem 37 is that not every topology allows the existence of equilibria for punishing strategies. In fact, if there is some node $i$ and a neighbor $j$ such that every node $k$ that is reachable from $j$ without crossing $i$ is never in between $s$ and $i$, then the impact of the punishments applied to $i$ after defecting from $j$ is null. This intuition is formalized in Lemma 38, where $PS[i, j]$ denotes the set of paths from $i$ to $j$ in $G$.

**Lemma 38.** If the assessment $(\sigma^*, \mu^*)$ is Preconsistent and Sequentially Rational, then for every $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$, there is $k \in \mathcal{N} \setminus \{i\}$, $x \in PS[s, i]$, and $x' \in PS[j, k]$, such that $k \in x$ and $i \notin x'$.

**Proof.** Assume by contradiction the opposite. Then, nodes can follow $p'$ where $i$ drops $j$, such that, if $p''$ is the profile resulting from nodes punishing $i$, then by Lemma 53 from Appendix A, $q_i[p''] = q_i[p^*]$, where $p^* = \sigma^*[\emptyset]$. Since $i$ increases its utility in the first stage by deviating, we have 

$$u_i[p''] > u_i[p^*].$$

Moreover, by letting $\sigma' = \sigma^*[p']$ to be the profile where exactly $i$ defects $j$, it is true that for every $r > 0$

$$u_i[\emptyset, r|\sigma'] = u_i[\emptyset, r|\sigma^*].$$

This implies that $\pi_i[\sigma'|\mu^*, \emptyset] > \pi_i[\sigma^*|\mu^*, \emptyset]$, which is a contradiction. (Complete proof in Section C.2).

The main implication of this result is that many non-redundant topologies, i.e., that do not contain multiple paths between $s$ and every node, are ineffective at sustaining cooperation. This is not entirely surprising, since it was already known that cooperation cannot be sustained using punishments as incentives in non-redundant graphs such as trees [18]. But even slightly redundant structures, such as directed cycles, do not fulfill the necessary condition specified in Lemma 38. Although redundancy is desirable to fulfill the above condition, it might decrease the effectiveness of punishments unless full indirect reciprocity may be used, as shown in the following section.

5.7 Redundancy may decrease Effectiveness

In addition to the need to fulfill the necessary condition of Lemma 38, a higher redundancy increases tolerance to failures. We show in Theorem 39 that if the graph is redundant and it does not allow full indirect reciprocity to be implemented, then the effectiveness decreases monotonically with the increase of the reliability. We consider the graph to be redundant if there are multiple non-overlapping paths from the source to every node. More precisely, if for every $i \in \mathcal{N}$ and $j \in \mathcal{N} \setminus \{i\}$, there exists $x \in PS[s, i]$ such that $j \notin x$, then $G$ is redundant.

The reliability increases as the probabilities $p_i[j|\emptyset]$ approach 1 for every node $i$ and out-neighbor $j$. This is denoted by $\lim_{\sigma^* \rightarrow 1} \psi[\sigma^*|\mu^*]$. We find that the effectiveness of any punishing strategy that cannot implement full indirect reciprocity converges to $\emptyset$, i.e., no benefit-to-cost ratio can sustain cooperation. This intuition is formalized as follows:

$$ \lim_{\sigma^* \rightarrow 1} \psi[\sigma^*|\mu^*] = \psi_{i \in \mathcal{N}, j \in \mathcal{N}_i} \lim_{p_i[j|\emptyset] \rightarrow 1} \psi[\sigma^*|\mu^*] = \emptyset. \quad (17)$$

Theorem 39 proves that Equality (17) holds for any graph that does not allow full indirect reciprocity to be implemented, which in our model occurs when there is no path from some $j \in \mathcal{N}_i$ to some $k \in \mathcal{N}_i^{-1}$ that does not cross $i$.

**Theorem 39.** If $G$ is redundant and there exist $i \in \mathcal{N}$, $j \in \mathcal{N}_i$, and $k \in \mathcal{N}_i^{-1}$ such that for every $x \in PS[j, k]$ we have $i \in x$, then Equality (17) holds.
Proof. The proof defines a deviating profile of strategies $\sigma'$ where exactly $i$ drops $j$. It follows that there is a path $x$ from $s$ to $i$ such that every node $k \in x$ never reacts to this defection. By Definition 23, $k$ uses $p_k[l|\emptyset]$ towards every out-neighbor $l$; a value that converges to 1. It follows from Lemma 49 in Appendix A that
\[
\lim_{\sigma^* \to 1} (\pi_i[\sigma^*|\mu^*,0] - \pi_i[\sigma'|\mu^*,0]) < 0.
\]
Therefore, for any $\beta_i$ and $\gamma_i$, the left side of the PDC Condition converges to a negative value. By Theorem 37, this implies that $\psi[\sigma^*|\mu^*]$ converges to $\emptyset$. (Complete proof in Section C.3).

Notice that this result does not imply that only full indirect reciprocity is effective at incentivizing rational nodes to cooperate in all scenarios. In fact, in many realistic scenarios, it might suffice for a majority of the in-neighbors of a node $i$ to punish $i$ after any deviation. A more sensible analysis would take into consideration the rate of converge to $\emptyset$ as the reliability increases. However, full indirect reciprocity is necessary in order to achieve an effectiveness fully independent of the desired reliability in any redundant graph.

5.8 Coordination is Desirable

Although full indirect reciprocity is desirable for redundant graphs, we now show that for some definitions of punishing strategies, it might not be sufficient if monitoring incurs large delays. In particular, nodes also need to coordinate the punishments being applied to any node, such that these punishments overlap during at least one stage after the defection, cancelling out any benefit obtained for receiving messages along some redundant path. This intuition is formalized as follows.

**Definition 40.** An assessment $(\sigma^*,\mu^*)$ enforces coordination if and only if for every $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$, there exists $r > 0$ such that, for every $k \in \mathcal{N}_{i}^{-1}$,
\[
r \in \{d_k[i,j] + 1 \ldots d_k[i,j] + \tau[i,j,k,i]\}.
\]

We prove a similar theorem to Theorem 39 which states that for some definitions of punishing strategies, with a redundant graph, the effectiveness decreases to $\emptyset$ with the reliability.

**Theorem 41.** If the graph is redundant and $\sigma^*$ does not enforce coordination, then there is a definition of $\sigma^*$ such that Equality 17 holds.

**Proof.** Consider the punishing strategy where every node $i$ reacts only to the defections of out-neighbors or to its own defections, and uses $p_i[j|\emptyset]$ in any other case. If $\sigma^*$ does not enforce coordination, then for some $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$, and for every $r > 0$, we can find a path from $s$ to $i$ such that every node $k$ along the path uses the probability $p_k[l|\emptyset]$ towards the next node $l$ in the path. These probabilities converge to 1 as the reliability increases, which by Lemma 49 implies that
\[
\lim_{\sigma^* \to 1} (\pi_i[\sigma^*|\mu^*,0] - \pi_i[\sigma'|\mu^*,0]) < 0,
\]
where $\sigma'$ is the alternative profile of strategies where exactly $i$ drops $j$. Therefore, for any $\beta_i$ and $\gamma_i$, the left side of the PDC Condition converges to a negative value. By Theorem 37, this implies that $\psi[\sigma^*|\mu^*]$ converges to $\emptyset$. (Complete proof in Section C.4).
any deviation any node \(i\) expects to be punished by every in-neighbor during \(\tau > 0\) stages. More precisely, for every node \(i\), by letting
\[
\bar{d}_i = \max_{j \in N} \max_{k \in N} d_{k[i,j]}
\]
to be the maximum delay of accusations against any in-neighbor of \(i\), the protocol must define \(\tau[i,j|k,i]\) for every \(k \in N_i^{-1}\) and \(j \in N_i\) in order to fulfill
\[
\tau[i,j|k,i] + d_{k[i,j]} \geq \bar{d}_i + \tau.
\]
It is sufficient and convenient for the sake of simplicity to provide a definition of \(\tau[i,j|k,i]\) for every \(k \in N\) and \(l \in N_k\) where \(d_{k[i,j]} < \infty\), such that every node stops reacting to a given defection in the same stage. More precisely,

**Definition 42.** For every \(k \in N\) and \(l \in N_k\) such that \(d_{k[i,j]} < \infty\), if \(k\) and \(l\) observe the defection before \(\bar{d}_i + \tau\), i.e., \(g = \max[d_{k[i,j]}, d_{l[i,j]}] < \bar{d}_i + \tau\), then \(k\) and \(l\) react to a defection of \(i\) from \(j\):
\[
\tau[i,j|k,l] = \bar{d}_i + \tau - g.
\]
Otherwise,
\[
\tau[i,j|k,l] = 0.
\]
This ensures that no node \(k\) reacts to a defection of \(i\) from \(j\) after stage \(\bar{d}_i + \tau\).

### 5.9 Coordinated Full Indirect Reciprocity

We now study the set of punishing strategies that use full indirect reciprocity. This requires the existence of a path from every \(j \in N_i\) to every \(k \in N_i^{-1}\), which must not cross \(i\). Under some circumstances, the effectiveness of a Preconsistent assessment \((\sigma^*, \mu^*)\) that uses full indirect reciprocity does not increase with the reliability of the dissemination process. As seen in the previous section, this requires punishments to be coordinated, which we assume to be defined as in Definition 42.

The fact that accusations may be delayed has an impact on the effectiveness, which is quantified in Lemma 44. To prove this lemma, we first derive in Lemma 44 an intermediate sufficient condition for the PDC Condition to be true. The lemma simplifies the PDC Condition for the worst scenario, where all in-neighbors of a node \(i\) begin punishing \(i\) for any defection simultaneously. The proofs assume that punishing strategies are defined in a reasonable manner. More precisely, if in reaction to a defection of node \(i\) other nodes increase the probabilities used towards out-neighbors other than \(i\), then \(i\) should never expect a large increase in its reliability during the initial stages, before every in-neighbor starts punishing \(i\). This intuition is captured in Assumption 43.

**Definition 43.** *(Assumption)* There exists a constant \(\epsilon \in [0, 1)\) such that, for every \(h \in H\), \(i \in N\), \(p_i' \in P_i\), \(\sigma' = (\sigma_i^*[h|p_i'], \sigma_{-i}^*)\), and \(r > 0\),
\[
q_i[h,r|\sigma^*] - q_i[h,r|\sigma'] < \epsilon.
\]

**Lemma 44.** If \((\sigma^*, \mu^*)\) is Preconsistent, Assumption 43 holds, and Inequality 18 is fulfilled for every \(i \in N\), \(h_i \in H_i\), and \(h \in H\) such that \(\mu_i^*[h|h_i] > 0\), then \((\sigma^*, \mu^*)\) is Sequentially Rational:
\[
- \sum_{i=0}^{\bar{d}_i} \omega_i^r((1 - q_i[h,r|\sigma^*])\gamma_i\gamma_i[p_i[h,r|\sigma^*] + \epsilon\beta_i]) + \sum_{r=\bar{d}_i+1}^{\bar{d}_i+r} \omega_i^r u_i[h,r|\sigma^*] \geq 0. \tag{18}
\]
Proof. By the definition of coordinated punishments if \( i \) deviates in \( \sigma' \) by dropping some subset of neighbors such that all players follow \( \sigma' = (\sigma'_i, \sigma^*_i) \), then in the worst scenario no node punishes \( i \) in any of the first \( \tilde{d}_i \) stages. Therefore, by our assumptions, for every \( r \in \{1, \ldots \tilde{d}_i\} \),

\[
u_i[h, r|\sigma^*] - \nu_i[h, r|\sigma'] \geq - \sum_{r=0}^{\tilde{d}_i} \omega_{\omega}^r ((1 - q_i[h, r|\sigma^*])\gamma_i \tilde{p}_i[h, r|\sigma^*] + \epsilon \beta_i).\]

Also, for every \( r \in \{\tilde{d}_i + 1, \ldots \tilde{d}_i + \tau\} \), every in-neighbor of \( i \) punishes \( i \), which by Lemma 51 from Appendix A implies

\[
u_i[h, r|\sigma^*] = 0 .\]

Finally, for every \( r \geq \tilde{d}_i + \tau + 1 \), every node ends its reaction to any defection of \( i \) after \( \tilde{d}_i + \tau + 1 \) stages, implying that

\[
u_i[h, r|\sigma^*] = \nu_i[h, r|\sigma'].\]

These three facts have the consequence that if Inequality 18 is fulfilled, then the PDC Condition holds. Therefore, by Theorem 37 \((\sigma^*, \mu^*)\) is Sequentially Rational. (Complete proof in Section C.5).

We can now derive a lower bound for the effectiveness of the considered punishing strategies, in a similar fashion to Theorem 22. However, now a stronger assumption is made, defined in 45. The reasoning is similar in that after any history, if a node \( i \) defects from some out-neighbor, then the reliability \( i \) would obtain during the initial stages when it is not being punished by all in-neighbors is not significantly greater than the reliability \( i \) would obtain in the subsequent stages, had \( i \) not deviated from the specified strategy.

**Definition 45.** (Assumption). There exists a constant \( c > 0 \), such that, for every \( i \in \mathcal{N}, r \in \{0, \ldots \tilde{d}_i\}, \) and \( r' \in \{\tilde{d}_i + 1, \ldots \tilde{d}_i + \tau\} \),

\[q_i[h, r|\sigma^*] \geq 1 - c(1 - q_i[h, r'|\sigma^*]).\]

**Lemma 46.** If \((\sigma^*, \mu^*)\) is Preconsistent, Assumptions 43 and 45 hold, and Inequality 19 is fulfilled for every \( h, i \in \mathcal{N}, \) and \( r, r' \leq \tilde{d}_i + \tau \) such that \( q_i[h, r|\sigma^*] < 1 \) and \( q_i[h, r'|\sigma^*] < 1 \), then there exist \( \omega_i \in (0, 1) \) for every \( i \in \mathcal{N} \) such that \((\sigma^*, \mu^*)\) is Sequentially Rational:

\[
\frac{\beta_i}{\gamma_i} > \frac{\tilde{p}_i[h, r|\sigma^*]}{A} + \frac{\tilde{p}_i[h, r'|\sigma^*]}{B} - C , \tag{19}
\]

where

\[
A = 1 - \frac{e^{(\tilde{d}_i + 1)}}{(1 - q_i[h, r|\sigma^*]) \tau},
B = \frac{\tau}{c},
C = \frac{e^{(\tilde{d}_i + 1)}}{1 - q_i[h, r'|\sigma^*]}.
\]

Proof. The proof considers two histories \( h_1 \) and \( h_2 \) that minimize the first and the second factors of Inequality 18 respectively. Thus, if the following condition is true, then Inequality 18 is true:

\[- \sum_{r=0}^{\tilde{d}_i} \omega_{\omega}^r ((1 - q_i^*[h_1, 0|\sigma^*])\gamma_i \tilde{p}_i[h_1, 0|\sigma^*] + \epsilon \beta_i) + \sum_{r=\tilde{d}_i + 1}^{\tilde{d}_i + \tau + 1} \omega_{\omega}^r \nu_i[h_2, 0|\sigma^*] \geq 0 .\]

After some manipulations, we conclude that the above condition is fulfilled if 19 is true. This implies by Lemma 44 that if 19 holds for every \( h \), and \( r, r' \leq \tilde{d}_i + \tau \), then Inequality 18 also holds for some \( \omega_i \in (0, 1) \) and \((\sigma^*, \mu^*)\) is Sequentially Rational. (Complete proof in Section C.5).
The main conclusion that can be drawn from this lemma is that if we pick the values of \( \tau \) and \( \epsilon \) such that \( \tau \geq \bar{d} + 1 \) and \( \epsilon \ll 1 \), then we can simplify the above condition to what is expressed in Theorem 47, where \( \bar{d} = \max_{i \in \mathcal{N}} \bar{d}_i \) is the maximum delay of the monitoring mechanism.

**Theorem 47.** If \((\sigma^*, \mu^*)\) is Preconsistent, Assumptions 43 and 45 hold for \( \epsilon \ll 1 \), and \( \tau \geq \bar{d} + 1 \), then there exists a constant \( c > 0 \) such that

\[
\psi(\sigma^*, \mu^*) \supseteq (v, \infty),
\]

where

\[
v = \max_{i \in \mathcal{N}} \max_{h \in \mathcal{H}} \bar{p}_i[h, 0|\sigma^*](1 + c).
\]

**Proof.** (See Section C.5). \qed

As in public monitoring, the effectiveness is close to optimal only if the initial expected costs of forwarding messages \( \bar{p}_i \) are not significantly smaller than the expected costs incurred after any history. Provided this guarantee, if \( \epsilon \) is small and \( \tau \) is chosen to be at least of the order of the maximum delay between out and in-neighbors of any node, then for any other punishing strategy, the effectiveness differs from the optimal by a constant factor.

Notice that, although we can adjust the value of \( \tau \) to compensate for higher delays, it is desirable to have a low maximum delay. First, this is due to the fact that higher values of \( \tau \) correspond to harsher punishments, which we may want to avoid, especially when monitoring is imperfect and honest nodes may wrongly be accused of deviating. Second, a larger delay decreases the range of values of \( \omega_i \) for each benefit-to-cost ratio that ensures that punishing strategies are an equilibrium. In particular, we can derive from the proof of Lemma 46 the strict minimum \( \omega_i \) for Grim-trigger to be an equilibrium. Under our assumptions, it is approximately given by

\[
\omega_i \geq \bar{d}_i \sqrt{\frac{\gamma_i \bar{p}_i}{\beta_i}}.
\]

For larger values of \( \bar{d}_i \) and the same benefit-to-cost ratio, the minimum \( \omega_i \) is also larger, reducing the likelihood of punishments to persuade rational nodes to not deviate from the specified strategy.

### 6 Discussion and Future Work

From this analysis, we can derive several desirable properties of a fully distributed monitoring mechanism for an epidemic dissemination protocol with asymmetric interactions, which uses punishments as the main incentive. This mechanism is expected to operate on top of an overlay network that provides a stable membership to each node. The results of this paper determine that the overlay should optimally explore the tradeoff between maximal randomization and higher clustering coefficient. The former is ideal for minimizing the latency of the dissemination process and fault tolerance, whereas the latter is necessary to minimize the distances between the neighbors of each node, while maximizing the number of in-neighbors of every node \( i \) informed about any defection of \( i \). The topology of this overlay should also fulfill the necessary conditions identified in this paper. Furthermore, the analysis of private monitoring shows that each accusation may be disseminated to a subset of nodes close to the accused node, without hinder ing the effectiveness. As future work, we plan to extend this analysis by considering imperfect monitoring, unreliable dissemination of accusations, malicious behavior, and churn. One possible application of the considered monitoring mechanism would be to sustain cooperation in a P2P news recommendation system such as the one proposed in [4]. Due to the lower rate of arrival of news, a monitoring approach may be better suited in this context.
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References

1. Abbey, H.: An examination of the Reed-Frost theory of epidemics. Human Biology 24(3), 201–233 (1952)
2. Birman, K.P., Hayden, M., Ozkasap, O., Xiao, Z., Budiu, M., Minsky, Y.: Bimodal multicast. ACM Trans. Comput. Syst. 17(2), 41–88 (1999)
3. Blackwell, D.: Discounted dynamic programming. The Annals of Mathematical Statistics 36(1), 226–235 (1965)
4. Boutet, A., Frey, D., Guerraoui, R., Jegou, A., Kermarrec, A.M.: Whatsup: A decentralized instant news recommender. In: IEEE 27th International Symposium on Parallel Distributed Processing. pp. 741–752. IPDPS (2013)
5. Deshpande, M., Xing, B., Lazardis, I., Hore, B., Venkatasubramanian, N., Mehotra, S.: Crew: A gossip-based flash-dissemination system. In: Proceedings of the 26th IEEE International Conference on Distributed Computing Systems. p. 45. ICDCS (2006)
6. Feldman, M., Lai, K., Stoica, I., Chuang, J.: Robust incentive techniques for peer-to-peer networks. In: Proceedings of the 5th ACM conference on Electronic commerce. pp. 102–111. EC (2004)
7. Felegyhazi, M., Hubaux, J., Buttyan, L.: Nash equilibria of packet forwarding strategies in wireless ad hoc networks. IEEE Transactions on Mobile Computing 5(5), 463–476 (2006)
8. Fudenberg, D., Maskin, E.: The folk theorem in repeated games with discounting or with incomplete information. Econometrica 54(3), 533–554 (1986)
9. Guerraoui, R., Huguenin, K., Kermarrec, A., Monod, M., Prusty, S.: Lifting: lightweight freerider-tracking in gossip. In: Proceedings of the ACM/IFIP/USENIX 11th International Conference on Middleware. pp. 313–333. Middleware (2010)
10. Hendon, E., Jacobsen, H., Sloth, B.: The one-shot-deviation principle for sequential rationality. Games and Economic Behavior 12(2), 274–282 (1996)
11. Ji, Z., Yu, W., Liu, K.: Cooperation enforcement in autonomous manets under noise and imperfect observation. In: 2006 3rd Annual IEEE Communications Society on Sensor and Ad Hoc Communications and Networks. pp. 460–468. SECON’06 (2006)
12. Keidar, I., Melamed, R., Orda, A.: Equicast: Scalable multicast with selfish users. Comput. Netw. 53(13), 2373–2386 (2009)
13. Kreps, D., Wilson, R.: Sequential equilibria. Econometrica 50(4), 863–94 (July 1982)
14. Levin, D., LaCurts, K., Spring, N., Bhattacharjee, B.: BitTorrent is an auction: analyzing and improving bittorrent’s incentives. In: Proceedings of the ACM SIGCOMM 2008 conference on Data communication. pp. 243–254. SIGCOMM (2008)
15. Li, B., Xie, S., Qu, Y., Keung, G., Lin, C., Liu, J., Zhang, X.: Inside the new coolstreaming: Principles, measurements and performance implications. In: The IEEE 27th Conference on Computer Communications. pp. 1031–1039. INFOCOM’08 (2008)
16. Li, H.C., Clement, A., Marchetti, M., Kapritsos, M., Robison, L., Alvisi, L., Dahlin, M.: Flightpath: obedience vs. choice in cooperative services. In: Proceedings of the 8th USENIX symposium on Operating Systems Design and Implementation. pp. 355–368. OSDI (2008)
17. Li, H.C., Clement, A., Wong, E.L., Napper, J., Roy, I., Alvisi, L., Dahlin, M.: BAR gossip. In: Proceedings of the 7th symposium on Operating Systems Design and Implementation. pp. 191–204. OSDI (2006)
18. Ngan, T., Druschel, P., Wallach, D.: Incentives-Compatible Peer-to-Peer Multicast. In: 2nd Workshop on Economics of Peer-to-Peer Systems (2004)
19. Osborne, M., Rubinstein, A.: A course in game theory. The MIT Press (1994)
20. Qiu, D., Srikant, R.: Modeling and performance analysis of bittorrent-like peer-to-peer networks. SIGCOMM Comput. Commun. Rev. 34(4), 367–378 (2004)
21. Rahman, R., Vinkó, T., Hales, D., Pouwelse, J., Sips, H.: Design space analysis for modeling incentives in distributed systems. SIGCOMM Comput. Commun. Rev. 41(4), 182–193 (2011)
22. Srivastava, V., Dasilva, L.: Equilibria for node participation in ad hoc networks - an imperfect monitoring approach. In: IEEE International Conference on Communications. pp. 3850–3855. ICC (2006)
A Epidemic Model

The probability that a node $i$ does not receive a message from $s$ when all players follow $p$ can be defined recursively, as follows.

**Proposition 48.** Define $\phi$ as follows: i) $\phi[R,\emptyset|p,L] = 1$ and ii) for $I \neq \emptyset$ and $R' = R \cup I \cup L$:

$$\phi[R, I|p, L] = \sum_{H \subseteq N \setminus R'} (P[I, H|p] \cdot Q[N, R, I, H|p] \cdot \phi[R \cup I, H|p, L]),$$

(20)

where

$$P[I, H|p] = \prod_{k \in H}(1 - \prod_{l \in I}(1 - p_l[k])).$$

$$Q[N, R, I, H|p] = \prod_{k \in N \setminus (H \cup R \cup I)} \prod_{l \in I}(1 - p_l[k]).$$

Then, $\phi[\emptyset, \{s\}|p,L]$ is the probability that no node of $L$ receives a message disseminated by $s$. In particular, $q_i[p] = \phi[\emptyset, \{s\}|p, \{i\}]$.

**Proof.** (Justification) The considered epidemics model is very similar to the Reed-Frost model [1], where dissemination is performed by having nodes forwarding messages with independent probabilities. The main difference is that the probability of forwarding each message is determined by a vector $p$, instead of being identical for every node. This implies that the dissemination process can be modeled as a sequence of steps, such that, at every step, there is a set $I$ of nodes infected in the last step, a set $R$ of nodes infected in previous steps other than the last, and a set $S$ of susceptible nodes. Given $R$ and $I$, the probability of the set $H \subseteq N \setminus (I \cup R \cup L)$ containing exactly the set of nodes infected at the current step is

$$P[I, H|p] \cdot Q[N, I, H|p] = \prod_{k \in H}(1 - \prod_{l \in I}(1 - p_l[k])).$$

(21)

That is, every node $i \in H$ is infected with a probability equal to 1 minus the probability of no node of $I$ choosing $i$, and these probabilities are all independent. Furthermore, all nodes of $I$ do not infect any node of $N \setminus (H \cup R \cup I)$. We can characterize $\phi$ with a weighted tree, where nodes correspond to a pair $(R, I)$. Moreover, for every parent node $(R, I)$, each child node corresponds to a pair $(R \cup I, H)$ for every $H \subseteq N \setminus (R \cup I \cup L)$. The root node is the pair $(\emptyset, \{s\})$ and leaf nodes are in the form $(R, \emptyset)$ for every $R \subseteq \{s\} \cup N \setminus L$. The weight of the transition from $(R, I)$ to $(R \cup I, H)$ is given by [21] which is the probability of exactly the nodes of $H$ being infected among every node of $N \setminus (R \cup I \cup L)$. The sum of these factors for any path from $(\emptyset, \{s\})$ to $(R, \emptyset)$ gives the probability of exactly the nodes of $R$ being infected in a specific order. By summing over all leaf nodes in the form $(R, \emptyset)$, we have the total probability of exactly the nodes of $R$ being infected. Finally, by summing over all possible $R \subseteq \{s\} \cup N \setminus L$, we obtain the probability of no node in $L$ being infected. In particular, $q_i[p] = \phi[\emptyset, \{s\}|p, \{i\}]$. \hfill \Box

The following are some useful axioms for the proofs, for any $p$, $R$, $I$, $L$, and $R' = R \cup I \cup L$:

$$\sum_{H \subseteq N \setminus R'} P[I, H|p] \cdot Q[N, R, I, H|p] = 1.$$ 

(22)

If $(A,B)$ is a partition of $N$, then

$$\sum_{H \subseteq N \setminus R'} P[I, H|p] \cdot Q[N, R, I, H|p] =$$

$$= \sum_{H_1 \subseteq A \setminus R'} P[I, H_1|p] \cdot Q[A, R, I, H_1|p].$$

(23)

$$\cdot \sum_{H_2 \subseteq B \setminus R'} P[I, H_2|p] \cdot Q[B, R, I, H_1|p]$$

$$j \in I \Rightarrow P[I, H|p] \leq P[I \setminus \{j\}, H|p].$$

(24)
A.1 Deterministic Delivery

Lemma 49 proves the straightforward fact that if there is a path from \( s \) to some node \( i \) where all nodes along the path forward messages with probability 1, then \( q_i = 0 \).

**Lemma 49.** For any \( p \in P \), if there exists \( i \in N \) and \( x \in PS[s,i] \) such that for every \( r \in \{0 \ldots |x| - 1\} \) we have \( p_{x_r}[x_{r+1}] = 1 \), then \( q_i[p] = 0 \).

**Proof.** The proof goes by induction on \( r \) where the induction hypothesis is that, for every \( r \in \{0 \ldots |x| - 2\} \), \( R \subseteq N \cup \{s\} \setminus \{i\} \), and \( I \subseteq N \cup \{s\} \setminus \{i\} \cup R \), such that:

- \( x_r \in I \),
- for every \( r' \in \{0 \ldots r\} \), \( x_{r'} \in R \cup I \),
- for every \( r' \in \{r + 1 \ldots |x| - 1\} \), \( x_{r'} \in N \setminus (R \cup I) \),

we have \( \phi[R,I|p,\{i\}] = 0 \).

Consider the base case for \( r = |x| - 2 \) and let \( R' = R \cup I \cup \{i\} \). In this case, for \( j = x_r \),

\[
\phi[R,I|p,\{i\}] = \sum_{H \subseteq N \setminus R'} (P[I,H|p] \cdot Q[N,R,I,H|p] \cdot \phi[R \cup I,H|p,\{i\}])
\]

\[
= \sum_{H \subseteq N \setminus R'} (P[I,H|p] \cdot Q[N \setminus \{i\},R,I,H|p]) \prod_{i \in R \setminus \{j\}} (1 - p_i[i])(1 - p_j[i]) \phi[R \cup I,H|p,\{i\}])
\]

\[
= \sum_{H \subseteq N \setminus R'} (P[I,H|p] \cdot Q[N \setminus \{i\},R,I,H|p]) 0 \cdot \phi[R \cup I,H|p,\{i\}])
\]

\[
= 0.
\]

This proves the base case. Assume now that the hypothesis is true for every \( r' \in \{0 \ldots r\} \) and some \( r \in \{1 \ldots |x| - 2\} \). Let \( a = x_{r-1} \) and \( b = x_r \), let \( R_1 = R \cup I \) and \( R_2 = R_1 \cup H \). We thus have

\[
\phi[R,I|p,\{i\}] = \sum_{H \subseteq N \setminus (R_1 \cup \{i\})} (P[I,H|p] \cdot Q[N,R,I,H|p] \cdot \phi[R_1,H|p,\{i\}])
\]

\[
= \sum_{H \subseteq N \setminus (R_1 \cup \{i\})} (P[I,H|p] \cdot Q[N \setminus \{b\},R,I,H|p] \prod_{i \in R \setminus \{a\}} (1 - p_i[b])(1 - p_a[b]) \phi[R_1,H|p,\{i\}]) + \sum_{H \subseteq N \setminus (R_1 \cup \{i,b\})} (P[I,H \cup \{b\}|p] \cdot Q[N,R,I,H \cup \{b\}|p] \cdot \phi[R_1,H \cup \{b\}|p,\{i\}])
\]

\[
= \sum_{H \subseteq N \setminus (R_1 \cup \{i,b\})} (P[I,H|p] \cdot Q[N \setminus \{b\},R,I,H|p] \cdot 0 \cdot \phi[R_1,H|p,\{i\}]) + \sum_{H \subseteq N \setminus (R_1 \cup \{i,b\})} (P[I,H \cup \{b\}|p] \cdot Q[N,R,I,H \cup \{b\}|p] \cdot 0)
\]

\[
= 0.
\]

This proves the induction step for \( r - 1 \). Consequently, since \( s = x_0 \) and \( x_r \in N \) for every \( r \in \{1 \ldots |x| - 1\} \),

\[
q_i[p] = \phi[\emptyset,\{s\}|p,\{i\}] = 0.
\]

\[\square\]
A.2 Positive Reliability

Lemma 50 shows that if every node forwards messages with a positive probability, then every node of the graph receives a message with positive probability as well.

Lemma 50. For any $p \in P$, if there exists $i \in N$ and $x \in P[5,s,i]$ such that for every $r \in \{0 \ldots |x| - 1\}$ we have $p_{xr}[x_{r+1}] > 0$, then $q_i[p] < 1$.

Proof. The proof goes by induction on $r$ where the induction hypothesis is that, for every $r \in \{0 \ldots |x| - 2\}$, $R \subseteq N \cup \{s\} \setminus \{i\}$, and $I \subseteq N \cup \{i\} \cup R$, such that:

- $x_r \in I$,
- for every $r' \in \{0 \ldots r\}$, $x_{r'} \in R \cup I$,
- for every $r' \in \{r + 1 \ldots |x| - 1\}$, $x_{r'} \in N \setminus (R \cup I),$

we have $\phi[R, I | p, \{i\}] < 1$.

Consider the base case for $r = |x| - 2$ and let $R_1 = R \cup I \cup \{i\}$. In this case, by Axiom 22 for $j = x_r$,

$$
\phi[R, I | p, \{i\}] = \sum_{H \subseteq N \setminus R_1} (P[I, H | p] \cdot Q[N, R, I, H | p] \cdot \phi[R \cup I, H | p, \{i\}])
$$

$$
= \sum_{H \subseteq N \setminus R_1} (P[I, H | p] \cdot Q[N \setminus \{i\}, R, I, H | p] \cdot \prod_{i \in I \setminus \{j\}} (1 - p_i[i])(1 - p_j[i]) \phi[R \cup I, H | p, \{i\}])
$$

$$
< \sum_{H \subseteq N \setminus R_1} P[I, H | p] \cdot Q[N, R, I, H | p] = 1.
$$

This proves the base case. Assume now that the hypothesis is true for every $r' \in \{0 \ldots r\}$ and some $r \in \{1 \ldots |x| - 2\}$. Let $a = x_{r-1}$ and $b = x_r$. Consider that $R_1 = R \cup I$ and $R_2 = R \cup H \cup I$.

We thus have by Axioms 22 and 24

$$
\phi[R, I | p, \{i\}] = \sum_{H \subseteq N \setminus (R_1 \cup \{i\})} (P[I, H | p] \cdot Q[N, R, I, H | p] \cdot \phi[R_1, H | p, \{i\}])
$$

$$
= \sum_{H \subseteq N \setminus (R_1 \cup \{i,b\})} (P[I, H | p] \cdot Q[N \setminus \{b\}, R, I, H | p] \cdot \prod_{i \in I \setminus \{a\}} (1 - p_i[k])(1 - p_i[k]) \phi[R_1, H | p, \{i\}])
$$

$$
+ \sum_{H \subseteq N \setminus (R_1 \cup \{i,b\})} (P[I, H \cup \{b\} | p] \cdot Q[N, R, I, H \cup \{b\} | p] \cdot \phi[R_1, H \cup \{b\} | p, \{i\}])
$$

$$
< \sum_{H \subseteq N \setminus (R_1 \cup \{i,b\})} (P[I, H | p] \cdot Q[N \setminus \{b\}, R, I, H | p] \cdot \prod_{i \in I \setminus \{a\}} (1 - p_i[k])(1 - p_i[k]) \cdot 1 \cdot \phi[R_1, H | p, \{i\}])
$$

$$
+ \sum_{H \subseteq N \setminus (R_1 \cup \{i,b\})} (P[I \setminus \{a\}, H \cup \{b\} | p] \cdot Q[N, R, I \setminus \{a\}, H | p])
$$

$$
= \sum_{H \subseteq N \setminus (R_1 \cup \{i,b\})} (P[I \setminus \{a\}, H | p] \cdot Q[N, R, I \setminus \{a\}, H | p])
$$

$$
= 1.
$$

This proves the induction step for $r - 1$. Consequently, since $s = x_0$ and $x_r \in N$ for every $r \in \{1 \ldots |x| - 1\}$,

$$
q_i[p] = \phi[\emptyset, \{s\} | p, \{i\}] < 1.
$$
A.3 Null Reliability

Lemma 51 shows that if every in-neighbor of a node $i$ does not forward messages to $i$, then $q_i = 0$.

**Lemma 51.** If $p \in P$ is defined such that for some $i \in N$ and for every $j \in N_i^{-1} p_j[i] = 0$, then $q_i[p] = 1$.

**Proof.** The proof goes by induction on $r$ where the induction hypothesis is that for every $r \in \{0 \ldots |N|-1\}$, $R \subseteq N \cup \{s\} \setminus \{i\}$, and $I \subseteq N \cup \{s\} \setminus \{i\} \cup R$ such that $|R| + |I| \leq |N| + 1 - r$, we have $\phi[R, I | p, \{i\}] = 1$.

The base case is for $r = 0$, where we have by the definition of $p$.

$$
\phi[R, I | p, \{i\}] = \prod_{l \in I} (1 - p_l[i])\phi[R \cup I, \emptyset | p, \{i\}] = \prod_{l \in I} (1 - 0) \cdot 1 = 1.
$$

Assume the induction hypothesis for any $r \in \{0 \ldots |N|-1\}$. Consider any two $R$ and $I$ defined as above for $r + 1$, such that $|R| + |I| = |N| - r$. Let $R_1 = R \cup I \cup \{i\}$ and $R_2 = H \cup R \cup I$. It is true by Axiom 22 that:

$$
\phi[R, I | p, \{i\}] = \sum_{H \subseteq N \setminus R_1} P[H | p]Q[N, R, I, H | p]\phi[R \cup I, H | p, \{i\}] = \sum_{H \subseteq N \setminus R_1} P[H | p]Q[N, R, I, H | p] \cdot 1 = 1.
$$

Therefore, for $r = |N|$, $q_i[p] = \phi[\emptyset, \{s\} | p, \{i\}] = 1$. 

\[\square\]
A.4 Uniform Reliability

Lemma 52 shows that for some node $i$ and for every path $x$ from $s$ to and $i$, every node of $x$ does not change its probability from $p$ to $p'$, then $q_i[p] = q_i[p']$. As an intermediate step, Lemma 52 proves that the reliability is the same whenever $I$ and $R$ contain the same set of nodes from any path from $s$ to $i$.

For any $i \in \mathcal{N}$, $p \in \mathcal{P}$, and $K \subseteq \mathcal{N}$, let

$$D_i = \{j \in \mathcal{N} \cup \{s\} \setminus \{i\} | \text{PS}[j, i] \neq \emptyset\},$$

and:

$$p[I, k] = 1 - \prod_{j \in I} (1 - p_j[k]).$$

Therefore, it is possible to write:

$$P[I, H|p] = \prod_{k \in H} p[I, k],$$

$$Q[\mathcal{N}, R, I, H|p] = \prod_{k \in \mathcal{N} \setminus (R \cup I \cup H)} (1 - p[I, k]).$$

For any $L_1, L_2 \subseteq \mathcal{N} \setminus D_i$ such that $L_1 \cap L_2 = \emptyset$, since for every $j \in L$ we have PS$[j, i] = \emptyset$ and $p_j[k] = 0$ for every $k \in D_i$, and for every $H \subseteq D_i$,

$$p[I \cup L_1, k] = p[I, k],$$

$$P[I \cup L_1, H|p] = P[I, H|p],$$

$$Q[\mathcal{N}, R \cup L_1, I \cup L_2, H|p] = Q[\mathcal{N}, I, H|p].$$

Lemma 52. For every $p \in \mathcal{P}$, $i \in \mathcal{N}$, $R \subseteq \mathcal{N} \cup \{s\} \setminus \{i\}$, $I \subseteq D_i \setminus R$, and $L_1, L_2 \subseteq \mathcal{N} \setminus (D_i \cup R)$ such that $L_1 \cap L_2 = \emptyset$,

$$\phi[R \cup L_1, I \cup L_2|p, \{i\}] = \phi[R, I|p, \{i\}].$$

Proof. Fix $p$ and $i$. First notice that for every $R \subseteq \mathcal{N}$ and $L \subseteq \mathcal{N} \setminus (D_i \cup R)$,

$$\phi[R, L|p, \{i\}] = 1. \quad (26)$$

We now prove by induction that for every $R \subseteq \mathcal{N} \cup \{s\}$, $I \subseteq D_i \setminus R$, and $L_1, L_2 \subseteq \mathcal{N} \setminus (D_i \cup R)$ such that $L_1 \cap L_2 = \emptyset$,

$$\phi[R, L_1, I \cup L_2|p, \{i\}] = \phi[R, I|p, \{i\}].$$

The induction goes on $r \in \{0 \ldots |\mathcal{N}|\}$, where $|R| + |I| + |L| = |\mathcal{N}| + 1 - r$, where $L = L_1 \cup L_2$. For $r = 0$, by Axiom 22 and 26, we can write:

$$\phi[R \cup L_1, I \cup L_2|p, \{i\}] = (1 - p[I \cup L_1, i])\phi[R \cup I \cup L, \emptyset|p, \{i\}]$$

$$= (1 - p[I, i]) \cdot (1 - p[I, i]) \cdot \sum_{H \subseteq L} \phi[R \cup I, H|p, \{i\}]$$

$$= (1 - p[I, i]) \cdot \sum_{H \subseteq \mathcal{N} \setminus (R \cup I \cup \{i\})} (P[I, H|p] \cdot Q[\mathcal{N}, R, I, H|p])$$

$$\quad \cdot \phi[R \cup I, H|p, \{i\}]$$

$$= \phi[R, I|p, \{i\}]. \quad (27)$$

This proves the base case.
Assume the induction hypothesis for every \( r' \in \{0 \ldots r\} \) and \( r \in \{0 \ldots |N| - 1\} \). By Axioms 22 and 23 and by 25 and the induction hypothesis, we can write:

\[
\phi[R \cup L_1, I \cup L_2[p, \{i\}] = \sum_{H \subseteq N \setminus (R \cup \cup L_1 \cup \{i\})} (P[I \cup L_2, H[p] \cdot Q[N, R \cup L_1, I \cup L_2, H[p] \cdot \phi[R \cup I \cup L, H[p, \{i\}])
\]

\[
= \sum_{H \subseteq D_i \setminus (R \cup \cup L_1 \cup \{i\})} (P[I \cup L_2, H_1[p] \cdot Q[D_i, R \cup L_1, I \cup L_2, H_1[p]
\]

\[
= \sum_{H \subseteq D_i \setminus (R \cup \cup L_1 \cup \{i\})} (P[I, H_1[p] \cdot Q[D_i, R \cup L_1, I \cup L_2, H_1[p]
\]

\[
= \sum_{H \subseteq D_i \setminus (R \cup \cup L_1 \cup \{i\})} (P[I, H_1[p] \cdot Q[D_i, R, I, H_1[p])
\]

\[
= \sum_{H \subseteq D_i \setminus (R \cup \cup L_1 \cup \{i\})} (P[I, H_1[p] \cdot Q[D_i, R, I, H_1[p]
\]

\[
= \sum_{H \subseteq N \setminus (R \cup \cup L_1 \cup \{i\})} (P[I, H_2[p] \cdot Q[N \setminus D_i, R, I, H_2[p] \cdot \phi[R \cup I, H_1 \cup H_2[p, \{i\}])])
\]

\[
= \sum_{H \subseteq N \setminus (R \cup \cup L_1 \cup \{i\})} (P[I, H[p] \cdot Q[N, R, I, H[p] \cdot \phi[R \cup I, H[p, \{i\})]
\]

\[
= \phi[R, I[p, \{i\}].
\]

This proves the result. \( \square \)
Lemma 53. Let $\mathbf{p}, \mathbf{p}' \in \mathcal{P}$ be any two profiles of probabilities such that for some $i \in \mathcal{N}$, for every $x \in \mathcal{P}[s, i]$, and for every $j \in x$, $\mathbf{p}_j = \mathbf{p}'_j$. Then, $q_i[\mathbf{p}] = q_i[\mathbf{p}']$.

Proof. Assume this to be the case for a fixed $i$, $\mathbf{p}$, and $\mathbf{p}'$. Then, for every $x \in \mathcal{P}[s, i]$ and $j \in x$, it is true that, for every $k \in \mathcal{N}^{-1}_j$, there exists $x' \in \mathcal{P}[s, i]$ such that

$$k \in x' \land p_k[j] = p'_k[j]. \quad (28)$$

Define $p'[I, k]$ as in (25) but for $\mathbf{p}'$. Condition (28) implies that for every $I \subseteq \mathcal{N} \cup \{s\}$ and $k \in D_i \cup \{i\}$:

$$p'[I, k] = p[I, k]. \quad (29)$$

The rest of the proof is performed by induction on $r$ where the induction hypothesis is that for every $r \in \{0 \ldots |\mathcal{N}|\}$, $R \subseteq \mathcal{N} \cup \{s\} \setminus \{i\}$, and $I \subseteq \mathcal{N} \cup \{s\} \setminus \{i\} \cup R$ such that $|R| + |I| \leq |\mathcal{N}| + 1 - r$, we have $\phi[R, I|p, \{i\}] = \phi[R, I|p', \{i\}]$.

The base case is for $r = 0$, where we have by (29) and the definition of $\mathbf{p}$ and $\mathbf{p}'$:

$$\phi[R, I|p, \{i\}] = p[I, i] \phi[R \cup I, \emptyset|p, \{i\}] = p'[I, i] \phi[R \cup I, \emptyset|p', \{i\}] = \phi[R, I|p', \{i\}].$$

This proves the base case. Now, assume the induction hypothesis for some $r \in \{0 \ldots |\mathcal{N}| - 1\}$. It is true by Lemma 52 by Axioms 22 and 23 by 29 and the induction hypothesis that:

$$\phi[R, I|p, \{i\}] = \sum_{H \subseteq \mathcal{N} \setminus (R \cup U \cup \{i\})} (p[I, H|p] \cdot Q[\mathcal{N}, R, I, H|p] \cdot \phi[R \cup I, H|p, \{i\}])$$

$$= \sum_{H \subseteq D_i \setminus (R \cup U \cup \{i\})} (p[I, H|p] \cdot Q[D_i, R, I, H|p]) \cdot \sum_{H_2 \subseteq \mathcal{N} \setminus (R \cup U \cup \{i\})} (p[I, H_2|p] \cdot Q[\mathcal{N} \setminus D_i, R, I, H_2|p]) \cdot \phi[R \cup I, H_2|p, \{i\}]$$

$$= \sum_{H_1 \subseteq D_i \setminus (R \cup U \cup \{i\})} (\phi[R \cup I, H_1|p, \{i\}] \cdot p[I, H_1|p] \cdot Q[D_i, R, I, H_1|p])$$

$$= \sum_{H_1 \subseteq D_i \setminus (R \cup U \cup \{i\})} (\phi[R \cup I, H_1|p', \{i\}] \cdot p[I, H_1|p'] \cdot Q[D_i, R, I, H_1|p'])$$

$$= \sum_{H_2 \subseteq \mathcal{N} \setminus (R \cup U \cup \{i\})} (p[I, H_2|p'] \cdot Q[\mathcal{N} \setminus D_i, R, I, H_2|p'] \cdot \phi[R \cup I, H_1 \cup H_2|p', \{i\}])$$

$$= \sum_{H_2 \subseteq \mathcal{N} \setminus (R \cup U \cup \{i\})} (p[I, H|p'] \cdot Q[\mathcal{N}, R, I, H|p'] \cdot \phi[R \cup I, H|p', \{i\}])$$

$$= \phi[R, I|p', \{i\}].$$

This concludes the proof by induction. Therefore, for $r = |\mathcal{N}|$,

$$q_i[\mathbf{p}] = \phi[\emptyset, \{s\}|\mathbf{p}, \{i\}] = \phi[\emptyset, \{s\}|\mathbf{p}', \{i\}] = q_i[\mathbf{p}'].$$

□
A.5 Single Impact

Lemma 54 provides an upper bound for the impact in the reliability when a single in-neighbor \( j \) punishes node \( i \).

**Lemma 54.** For every \( i \in \mathcal{N}, j \in \mathcal{N}_i^{j-1}, p \in \mathcal{P} \) such that \( p_j[i] < 1 \) and \( q_i[p] > 0 \), if \( p' \) is the profile where only \( j \) deviates from \( p_j[i] \) to \( p_j[i]' < p_j[i] \), then

\[
q_i[p'] \leq q_i[p] \frac{1 - p_j[i]'}{1 - p_j[i]}. 
\]

**Proof.** Fix \( i, j, p, \) and \( p' \). The proof shows by induction that for every \( r \in \{0 \ldots |\mathcal{N}|\} \), \( R \subseteq \mathcal{N} \cup \{s\} \setminus \{i\} \) and \( I \subseteq \mathcal{N} \cup \{s\} \setminus \{i\} \cup R \) such that \( |R| + |I| - r = |\mathcal{N}| - 1 \):

\[
\phi[R, I|p', \{i\}] \leq \phi[R, I|p, \{i\}] \frac{1 - p_j[i]}{1 - p_j[i]}.
\]

Notice that by the definition of \( \phi \), if \( j \in R \), then

\[
\phi[R, I|p', \{i\}] = \phi[R, I|p, \{i\}]. \tag{30}
\]

For the base case \( r = 0 \), if \( j \in R \), then the result follows immediately. Thus, consider that \( j \in I \).

We can write

\[
\phi[R, I|p', \{i\}] = p[I, i] \phi[R \cup I, \emptyset|p, \{i\}]
= p[I \setminus \{j\}, i] \cdot p'[\{j\}, i]
= p[I \setminus \{j\}, i] \cdot (1 - p_j[i]) \frac{1 - p_j'[i]}{1 - p_j[i]}
= p[I, i] \frac{1 - p_j'[i]}{1 - p_j[i]}
= \phi[R, I|p, \{i\}] \frac{1 - p_j'[i]}{1 - p_j[i]}.
\]

This proves the induction step for \( r = 0 \). Assume now that the induction hypothesis is true for every \( r' \in \{0 \ldots r\} \) and for some \( r \in \{0 \ldots |\mathcal{N}| - 1\} \).

If \( j \notin I \), then by the induction hypothesis and by \([30]\)

\[
\phi[R, I|p', \{i\}] = \sum_{H \subseteq \mathcal{N} \setminus (R \cup I \cup \{i\})} (P[I, H|p'] \cdot Q[\mathcal{N}, R, I, H|p'] \cdot \phi[R \cup I, H|p', \{i\}])
\leq \sum_{H \subseteq \mathcal{N} \setminus (R \cup I \cup \{i\})} (P[I, H|p] \cdot Q[\mathcal{N}, R, I, H|p] \cdot \phi[R \cup I, H|p', \{i\}])
\leq \phi[R, I|p, \{i\}] \frac{1 - p_j'[i]}{1 - p_j[i]}.
\]

For the final case where \( j \in I \), we have by \([30]\)

\[
\phi[R, I|p', \{i\}] = \sum_{H \subseteq \mathcal{N} \setminus (R \cup I \cup \{i\})} (P[I, H|p'] \cdot Q[\mathcal{N}, R, I, H|p'] \cdot \phi[R \cup I, H|p', \{i\}])
\leq \phi[R, I|p, \{i\}] \frac{1 - p_j'[i]}{1 - p_j[i]}.
\]

This concludes the proof. \( \square \)
Proof of Lemma 6. For every $h \in H$, $r \in \{1 \ldots \tau - 1\}$, $i \in N$, and $j \in N_i$,
\[ DS_i[j|h^*_r] = \{(k_1, k_2, r' + r)|(k_1, k_2, r') \in DS_i[j|h] \land r' + r < \tau\}, \]
where $h^*_r = \text{hist}[h, r|\sigma^*]$.

Proof. Fix $i$, $h$, and $j$. The proof goes by induction on $r$, where the induction hypothesis is that, for every $r \in \{1 \ldots \tau - 1\}$,
\[ DS_i[j|h^*_r] = \{(k_1, k_2, r' + r)|(k_1, k_2, r') \in DS_i[j|h] \land r' + r < \tau\}. \]

By Definition 3 we have that for every $r \in \{1 \ldots \tau - 1\}$, $p^* = \sigma^*[h^*_r]$, and $s^* = \text{sig}[p^*[h]$, 
\[ DS_i[j|h^*_r+1] = L_1[r + 1|\sigma^*] \cup L_2[r + 1|\sigma^*], \]
where
\[ L_1[r + 1|\sigma^*] = \{(k_1, k_2, r' + 1)|(k_1, k_2, r') \in DS_i[j|h^*_r] \land r' + 1 < \tau\}. \]
\[ L_2[r + 1|\sigma^*] = \{(k_1, k_2, 0)|k_1, k_2 \in N \land i, j \in RS[k_1, k_2] \land s^*[k_1, k_2] = \text{defect}\}. \]

First, note that by Definition 2 it holds that $s^*[k_1, k_2] = \text{cooperate}$ for every $k_1 \in N$ and $k_2 \in N_k$. Thus, by 32 for every $r \in \{1 \ldots \tau - 1\}$,
\[ L_2[r|\sigma^*] = \{(k_1, k_2, 0)|k_1, k_2 \in N \land i, j \in RS[k_1, k_2] \land s^*[k_1, k_2] = \text{defect}\} = \emptyset. \]

By 32,
\[ L_1[1|\sigma^*] = \{(k_1, k_2, r' + 1)|(k_1, k_2, r') \in DS_i[j|h] \land r' + 1 < \tau\}, \]
which, along with 33 and 31 proves the base case.

Now, consider that the induction hypothesis is valid for any $r \in \{1 \ldots \tau - 2\}$. We have by this assumption and by 32 that
\[ L_1[r + 1|\sigma^*] = \{(k_1, k_2, r' + 1)|(k_1, k_2, r') \in DS_i[j|h^*_r] \land r' + 1 < \tau\}, \]
\[ = \{(l_1, l_2, r'' + r)|(l_1, l_2, r'') \in DS_i[j|h] \land r'' + r < \tau\} \land r' + 1 < \tau\}, \]
\[ = \{(k_1, k_2, r' + (r + 1))|(k_1, k_2, r') \in DS_i[j|h] \land r' + (r + 1) < \tau\}. \]

This fact, along with 33 and 31 proves the induction step for $r + 1$. \[ \square \]
Proof of Lemma 8. For every $h \in \mathcal{H}$, $p' \in \mathcal{P}, r \in \{1\ldots \tau\}, i \in \mathcal{N}$, and $j \in \mathcal{N}_i$:

$$DS_i[j|h_i^*] = DS_i[j|h_i^*] \cup \{(k_1, k_2, r-1)|k_1, k_2 \in \mathcal{N} \land k_2 \in CD_{k_1}[p'|h] \land i, j \in RS[k_1, k_2]\},$$

where $h_i^* = \text{hist}[h, r|\sigma^*]$, $h_r^* = \text{hist}[h, r|\sigma^r]$, and $\sigma' = \sigma^*[h|p']$ is the profile of strategies where all players follow $p'$ in the first stage.

Proof. Fix $h, p', i$, and $j$. The proof goes by induction on $r$, where the induction hypothesis is that for every $r \in \{1\ldots \tau\}$, Equality 3 holds.

By Definition 3 we have that for every $r \leq \tau$, $p^r = \sigma^*[h_i^*]$, and $s^* = \text{sig}[p'|h_i^*]$:

$$DS_i[j|h_i^*+1] = L_1[r+1|\sigma^r] \cup L_2[r+1|\sigma^r],$$

where

$$L_1[r+1|\sigma^r] = \{(k_1, k_2, r'+1)|(k_1, k_2, r') \in DS_i[j|h_i^*] \land r'+1 < \tau\},$$

$$L_2[r+1|\sigma^r] = \{(k_1, k_2, 0)|k_1, k_2 \in \mathcal{N} \land i, j \in RS[k_1, k_2] \land s^*[k_1, k_2] = \text{defect}\}.\quad (35)$$

Similarly, for every $r \leq \tau$, $p^r = \sigma'[,h_i^*]$, and $s' = \text{sig}[p'|h_i^*]$,

$$DS_i[j|h_i^*+1] = L_1[r+1|\sigma'] \cup L_2[r+1|\sigma'],$$

where

$$L_1[r+1|\sigma'] = \{(k_1, k_2, r'+1)|(k_1, k_2, r') \in DS_i[j|h_i^*] \land r'+1 < \tau\},$$

$$L_2[r+1|\sigma'] = \{(k_1, k_2, 0)|k_1, k_2 \in \mathcal{N} \land i, j \in RS[k_1, k_2] \land s'[k_1, k_2] = \text{defect}\}.\quad (37)$$

First, note that for every $r \in \{1\ldots \tau\}$ and $p'' \in \mathcal{P}$, such that $\sigma'' = \sigma^*[h|p'']$, we have $\sigma''[h'] = \sigma'[h']$ for every $h' \in \mathcal{H} \setminus \{h\}$.

Thus, for $h''_r = \text{hist}[h, r|\sigma'']$, $p'' = \sigma'[h''_r]$, and $s'' = \text{sig}[p'|h''_r]$, we have by Definition 2 that $s^*[k_1, k_2] = \text{cooperate}$ for every $k_1 \in \mathcal{N}$ and $k_2 \in \mathcal{N}_k$. Thus, by Definition 3 for every $r \in \{1\ldots \tau\}$,

$$L_2[r|\sigma''] = \{(k_1, k_2, 0)|k_1, k_2 \in \mathcal{N} \land i, j \in RS[k_1, k_2] \land s^*[k_1, k_2] = \text{defect}\} = \emptyset.\quad (38)$$

It follows by 35 and 37 that, for every $r \in \{1\ldots \tau\},$ $L_2[r|\sigma^r] = L_2[r|\sigma']= \emptyset.\quad (39)$

The base case is when $r = 1$. Since $h = h'_0 = h''_0$, by 35 and 37 it is true that:

$$L_1[1|\sigma'] = L_1[1|\sigma^r].\quad (40)$$

Furthermore, if players follow $p'$, then for $s' = \text{sig}[p'|h],$ we have $s'[k_1, k_2] = \text{defect}$ iff $k_2 \in CD_{k_1}[p'|h].$ Thus:

$$L_2[1|\sigma''] = \{(k_1, k_2, 0)|k_1, k_2 \in \mathcal{N} \land i, j \in RS[k_1, k_2] \land s'[k_1, k_2] = \text{defect}\} = \{(k_1, k_2, 0)|k_1, k_2 \in \mathcal{N} \land i, j \in RS[k_1, k_2] \land s' \in CD_{k_1}[p'|h]\}.\quad (41)$$

Consequently, the base case follows from 34, 36, 39, 40, and 41.

Hence, assume the induction hypothesis for $r \in \{1\ldots \tau - 1\}$. By the induction hypothesis and by 35 since $r < \tau$, it is also true that

$$L_1[r+1|\sigma^r] = \{(k_1, k_2, r'+1)|(k_1, k_2, r') \in DS_i[j|h_i^*] \land r'+1 < \tau\}$$

$$= \{(k_1, k_2, r'+1)|(k_1, k_2, r') \in DS_i[j|h_i^*] \land r'+1 < \tau\} \cup$$

$$\cup \{(k_1, k_2, r'+1)|(k_1, k_2, r') \in \{(l_1, l_2, r-1)|l_1, l_2 \in \mathcal{N} \land l_1 \in CD_{k_1}[p'|h] \land i, j \in RS[l_1, l_2]\} \land r'+1 < \tau\}.\quad (42)$$

The induction hypothesis follows from 34, 36, 39, and 42 for $r+1$, which proves the result. \qed
Proof of Lemma 9 For every \( h \in H, p' \in P, r > \tau, i \in N, \) and \( j \in N_i, \)
\[
DS_i[j|h^*_r] = DS_i[j|h^*_r] = \emptyset,
\]
where \( h^*_r = \text{hist}[r|\sigma^*], h'_r = \text{hist}[r|\sigma'], \) and \( \sigma' = \sigma^*[h|p'] \).

Proof. Fix \( h, p', i, \) and \( j. \) The proof goes by induction on \( r, \) where the induction hypothesis is that for every \( r \in \{1 \ldots \tau \}, \) Equality 3 holds.

By Definition 3 we have that for every \( r > 0, \) \( p^r = \sigma^*[h^*_r], \) and \( s^* = \text{sig}[p^r|h^*_r]: \)
\[
DS_i[j|h^*_{r+1}] = L_1[r + 1|\sigma^*] \cup L_2[r + 1|\sigma^*], \quad \text{(43)}
\]
where
\[
L_1[r + 1|\sigma^*] = \{(k_1, k_2, r' + 1) | (k_1, k_2, r') \in DS_i[j|h^*_r] \wedge r' + 1 < \tau \},
\]
\[
L_2[r + 1|\sigma^*] = \{(k_1, k_2, 0) | k_1, k_2 \in N \wedge i, j \in RS[k_1, k_2] \wedge s^*[k_1, k_2] = \text{defect} \}. \quad \text{(44)}
\]

Similarly, for every \( r > 0, \) \( p^r = \sigma'[h'_r], \) and \( s' = \text{sig}[p^r|h'_r], \)
\[
DS_i[j|h'_{r+1}] = L_1[r + 1|\sigma'] \cup L_2[r + 1|\sigma'], \quad \text{(45)}
\]
where
\[
L_1[r + 1|\sigma'] = \{(k_1, k_2, r' + 1) | (k_1, k_2, r') \in DS_i[j|h'_r] \wedge r' + 1 < \tau \},
\]
\[
L_2[r + 1|\sigma'] = \{(k_1, k_2, 0) | k_1, k_2 \in N \wedge i, j \in RS[k_1, k_2] \wedge s'[k_1, k_2] = \text{defect} \}. \quad \text{(46)}
\]

First, note that for every \( r > 0, \) \( p'' \in P, \) and \( \sigma'' = \sigma^*[h|p''], \) we have \( \sigma''[h'] = \sigma^*[h'] \) for every \( h' \in H \setminus \{h\}. \)

Thus, for \( h'' = \text{hist}[r|\sigma''], \) \( p^* = \sigma''[h''_r], \) and \( s^* = \text{sig}[p^*|h''_r], \) we have by Definition 2 that \( s^*[k_1, k_2] = \text{cooperate} \) for every \( k_1 \in N \) and \( k_2 \in N_{k_1}. \) Thus, by Definition 3 for every \( r > 0, \)
\[
L_2[r|\sigma''] = \{(k_1, k_2, 0) | k_1, k_2 \in N \wedge i, j \in RS[k_1, k_2] \wedge s^*[k_1, k_2] = \text{defect} \} = \emptyset. \quad \text{(47)}
\]

It follows by (44) and (46) that, for every \( r > \tau, \)
\[
L_2[r|\sigma^*] = L_2[r|\sigma'] = \emptyset. \quad \text{(48)}
\]

By Lemma 8
\[
DS_i[j|h^*_r] = DS_i[j|h^*_r] \cup \{(k_1, k_2, \tau - 1) | k_1, k_2 \in N \wedge k_2 \in CD_{k_1}[p'\mid h] \wedge i, j \in RS[k_1, k_2] \}.
\]

Therefore, by (44) and (46).
\[
L_1[\tau|\sigma'] = \{(k_1, k_2, r' + 1) | (k_1, k_2, r') \in DS_i[j|h^*_r] \wedge r' + 1 < \tau \}
\]
\[
= \{(k_1, k_2, r' + 1) | (k_1, k_2, r') \in DS_i[j|h^*_r] \wedge r' + 1 < \tau \} \cup
\]
\[
\{(k_1, k_2, r' + 1) | (k_1, k_2, r') \in L_1[\tau|\sigma^*] \}\cup
\]
\[
= L_1[\tau + 1|\sigma^*].
\]

By Corollary 7
\[
L_1[\tau + 1|\sigma^*] = L_1[\tau + 1|\sigma^*] = \emptyset. \quad \text{(49)}
\]
By (43) (45) (48) (49) the base case is true.
Now, assume the induction hypothesis for some \( r \geq \tau + 1 \). By this assumption, (44) and (46):

\[
L_1[r + 1|\sigma'] = \{(k_1, k_2, r' + 1) | (k_1, k_2, r') \in DS_i[j|h'] \land r' + 1 < \tau \}
\]
\[
= \{(k_1, k_2, r' + 1) | (k_1, k_2, r') \in \emptyset \}
\]
\[
= \emptyset.
\]

By Definition 3 by (45) and (47) the induction step is true for \( r + 1 \), which proves the result. \( \square \)
B.2 Generic Results

Proof of Proposition 10. For every profile of punishing strategies $\sigma^*$, if $\sigma^*$ is a SPE, then, for every $i \in N$, $\frac{\beta_i}{\gamma_i} > \bar{p}_i$. Consequently, $\psi[\sigma^*] \subseteq (v, \infty)$, where $v = \max_{i \in N} \bar{p}_i$.

Proof. Let $p^* = \sigma^*[h]$. The equilibrium utility is

$$
\pi_i[\sigma^*|\emptyset] = \sum_{r=0}^{\infty} \omega^r_i (1 - q_i[p^*])(\beta_i - \gamma_i \bar{p}_i) = \frac{1 - q_i[p^*]}{1 - \omega_i} (\beta_i - \gamma_i \bar{p}_i).
$$

If $\frac{\beta_i}{\gamma_i} \leq \bar{p}_i$, then

$$
\pi_i[\sigma^*|\emptyset] \leq 0. \quad (51)
$$

Let $\sigma'_i \in \Sigma_i$ be a strategy such that, for every $h \in H$, $\sigma'_i[h] = 0$, and let $\sigma' = (\sigma'_i, \sigma^*_{-i})$, where $0 = (0)_{j \in N \setminus i}$. We have

$$
\pi_i[\sigma'|\emptyset] = (1 - q_i[p^*]) \beta_i + \pi_i[\sigma'|(h, \text{sig}[p'|h])] \geq (1 - q_i[p^*]) \beta_i, \quad (52)
$$

where $p' = (0, p^*_{-i})$. By Lemma 50, $q_i[p^*] < 1$. Since $\pi_i[\sigma'|(h, \text{sig}[p'|h])] \geq 0$, it is true that

$$
\pi_i[\sigma^*|\emptyset] \leq 0 < \pi_i[\sigma'|\emptyset].
$$

This contradicts the assumption that $\sigma^*$ is a SPE. \qed
Proof of Lemma 12. If $\sigma^*$ is a SPE, then the DC Condition is fulfilled.

Proof. The proof consists in assuming that $\sigma^*$ is a SPE and deriving 5. By Property 5 we must have, for every $h \in H$ and $a'_i \in A_i$,

$$\pi_i[\sigma^*|h] - \pi_i[\sigma'|h] \geq 0,$$

(53)

where $\sigma'_i = \sigma^*_i[h|a'_i]$ and $\sigma' = (\sigma'_i, \sigma^*_i)$. This is true for any $\sigma'_i$, where $a'_i[p'_i] = 1$ and $p'_i$ differs from $\sigma^*_i[h]$ exactly in that $i$ drops the nodes from any set $D \subseteq N_i[h]$:

- For every $j \in D$, $p'_i[j] = 0$.
- For every $j \in N_i \setminus D$, $p'_i[j] = p_i[j|h]$.

For any pure profile of strategies $\sigma \in \Sigma$, we can write

$$\pi_i[\sigma|h] = \sum_{r=0}^{\infty} \omega^*_i u_i[h,r|\sigma].$$

(54)

By Lemma 9 and Definition 3 for every $r > \tau$, $j \in N_i$, and $k \in N_j$,

$$DS_j[k|h'_r] = DS_j[k|h^*_r] = \emptyset,$$

$$p_j[k|h'_r] = p_j[k|h^*_r],$$

(55)

where $h'_r = \text{hist}[h,r|\sigma']$ and $h^*_r = \text{hist}[h,r|\sigma^*]$. This implies that for every $r > \tau$:

$$q_i[h,r|\sigma'] = q_i[h,r|\sigma^*],$$

$$\bar{p}_i[h,r|\sigma'] = \bar{p}_i[h,r|\sigma^*],$$

$$u_i[h,r|\sigma'] = u_i[h,r|\sigma^*].$$

Thus, 53 and 54 imply 5. $\square$
Proof of Lemma [13]. For every \( i \in \mathcal{N}, h \in \mathcal{H}, a_i \in BR[\sigma_{*i}^{|h}|], \) and \( p_i \in \mathcal{P}_i \) such that \( a_i[p_i] > 0 \), it is true that for every \( j \in \mathcal{N}_i \) we have \( p_i[j] \in \{0, p_i[j|h]\} \).

Proof. Suppose then that there exist \( h \in \mathcal{H}, i \in \mathcal{N}, a_i^1 \in BR[\sigma_{*i}^{|h}], \) and \( p_i^1 \in \mathcal{P}_i \) such that \( a_i^1[p_i^1] > 0 \) and there exists \( j \in \mathcal{N}_i \) such that \( p_i^1[j] \notin \{0, p_i[j|h]\} \). Consider an alternative \( a_i^2 \in \mathcal{A}_i \):

- Define \( p_i^2 \in \mathcal{P}_i \) such that for every \( j \in \mathcal{N}_i \), if \( p_i^1[j] \geq p_i[j|h] \), then \( p_i^2[j] = p_i[j|h] \), else, \( p_i^2[j] = 0 \).
- Set \( a_i^2[p_i^2] = a_i^1[p_i^1] + a_i^1[p_i^1] \) and \( a_i^2[p_i^1] = 0 \).
- For every \( p_i'' \in \mathcal{P}_i \setminus \{p_i^1, p_i^2\} \), set \( a_i^2[p_i''] = a_i^1[p_i''] \).

Consider the following auxiliary definitions:

- \( a^1 = (a_i^1, p_{*i}^1) \) and \( a^2 = (a_i^2, p_{*i}^2) \), where \( p^* = \sigma^{|h}| \).
- \( \sigma_i^1 = \sigma_i^{|h}[a_i^1] \) and \( \sigma_i^2 = \sigma_i^{|h}[a_i^2] \).
- \( p^1 = (p_i^1, p_{*i}^1) \) and \( p^2 = (p_i^2, p_{*i}^2) \).
- \( \sigma_i^1 = (\sigma_i^1, \sigma_{*i}^1) \) and \( \sigma_i^2 = (\sigma_i^2, \sigma_{*i}^2) \).
- \( s^1 = \text{sig}(p^1|h) \) and \( s^2 = \text{sig}(p^2|h) \).

Notice that for any \( j \in \mathcal{N}_i, p_i^1[j] \geq p_i^2[j] \) and \( p_i^1[j] \geq p_i[j|h] \) iff \( p_i^2[j] \geq p_i[j|h] \). Thus, by Definition [2] for any \( s \in \mathcal{S} \),

\[
\begin{align*}
pr_i[s | a_i^1, h] &= pr_i[s | a_i^2, h]. \\
pr[s | a^1, h] &= pr[s | a^2, h]. \tag{56}
\end{align*}
\]

Moreover, for some \( j \in \mathcal{N}_i, p_i^1[j|h] > p_i^2[j|h] \), thus, it is true that

\[
\pi_i[a^1] < \pi_i[a^2]. \tag{57}
\]

Recall that

\[
\begin{align*}
\pi_i[\sigma^1|h] &= u_i[a^1] + \omega_i \sum_{s \in \mathcal{S}} \pi_i[\sigma^1|(h, s)] pr[s | a^1, h], \\
\pi_i[\sigma^2|h] &= u_i[a^2] + \omega_i \sum_{s \in \mathcal{S}} \pi_i[\sigma^2|(h, s)] pr[s | a^2, h].
\end{align*}
\]

By [56] and the definition of \( \sigma^1 \) and \( \sigma^2 \),

\[
\sum_{s \in \mathcal{S}} \pi_i[\sigma|(h, s)] pr[s | a^1, h] = \sum_{s \in \mathcal{S}} \pi_i[\sigma|(h, s)] pr[s | a^2, h].
\]

By [57]

\[
\pi_i[\sigma^1|h] < \pi_i[\sigma^2|h].
\]

This is a contradiction, since \( a_i^1 \in BR[\sigma_{*i}^{|h}] \) by assumption, concluding the proof. \( \square \)
Proof of Lemma [14]. For every \( h \in \mathcal{H} \) and \( i \in \mathcal{N} \), there exists \( a_i \in BR[\sigma^*_i|h] \) and \( p_i \in \mathcal{P}_i \) such that \( a_i[p_i] = 1 \).

Proof.

For any \( h \in \mathcal{H} \) and \( i \in \mathcal{N} \), if \( BR[\sigma^*_i|h] \) only contains pure strategies for the stage game, since \( BR[\sigma^*_i|h] \) is not empty, the result follows. Suppose then that there exists a mixed strategy \( a_i^1 \in BR[\sigma^*_i|h] \). We know from Lemma [13] that every such \( a_i^1 \) attributes positive probability to one of two probabilities in \( \{0, p_i[j|h]\} \), for every \( j \in \mathcal{N}_i \). Let \( \sigma_i^1 = \sigma_i^*[h|a_i^1] \), \( \sigma_i^1 = (\sigma_i^1, \sigma_i^*_{-i}) \), and denote by \( \mathcal{P}^*[h] \) the finite set of profiles of probabilities that fulfill the condition of Lemma [13] i.e., for every \( p \in \mathcal{P}^*[h] \), \( j \in \mathcal{N} \), and \( k \in \mathcal{N}_j \), \( p_j[k] \in \{0, p_j[k|h]\} \). We can write

\[
\pi_i[\sigma^1|h] = \sum_{p \in \mathcal{P}^*[h]} (u_i[p] + \omega_i \pi_i[\sigma^1|(h, \text{sig}[p|h])])a_i^1[p_i],
\]

(58)

where \( p = (p_i, p_{-i}) \) and \( p^* = \sigma^*[h] \).

For any \( p_i^1 \in \mathcal{P}^*[h] \) such that \( a_i^1[p_i^1] > 0 \), let \( p^* = \sigma^*[h], p^1 = (p_i^1, p_{-i}), \sigma_i^1 = \sigma_i^*[h|p_i^1] \), and \( \sigma_i^1 = (\sigma_i^1, \sigma_i^*_{-i}) \).

There are three possibilities:

1. \( \pi_i[\sigma^1|h] = \pi_i[\sigma^0|h] \).
2. \( \pi_i[\sigma^1|h] < \pi_i[\sigma^0|h] \).
3. \( \pi_i[\sigma^1|h] > \pi_i[\sigma^0|h] \).

In possibility 1, it is true that there is \( a_i^1 \in BR[\sigma^*_{-i}|h] \) such that \( a_i^1[p_i^1] = 1 \) and the result follows. Possibility 2 contradicts the assumption that \( a_i^1 \in BR[\sigma^*_{-i}|h] \).

Finally, consider that possibility 3 is true. Recall that \( a_i^1 \) being mixed implies \( a_i^1[p_i^1] < 1 \). Thus, there must exist \( p_i^2 \in \mathcal{P}^*[h] \setminus \{p_i^1\}, \sigma_i'' = \sigma_i^*[h|p_i^2], \) and \( \sigma_i'' = (\sigma_i^1, \sigma_i^*_{-i}) \), such that \( a_i^1[p_i^2] > 0 \) and

\[
\pi_i[\sigma''|h] < \pi_i[\sigma^0|h] \tag{59}
\]

Here, we can define \( a_i^2 \in \mathcal{A}_i \) such that:

- \( a_i^2[p_i^2] = a_i^1[p_i^1] + a_i^1[p_i^2] \):
- \( a_i^2[p_i^1] = 0 \).
- For every \( p_i'' \in \mathcal{P}^*[h] \setminus \{p_i^1, p_i^2\}, a_i^2[p_i''] = a_i^1[p_i''] \).

Now, let:

- \( \sigma_i^2 = \sigma_i^*[h|a_i^2] \) and \( \sigma_i^2 = (\sigma_i^2, \sigma_i^*_{-i}) \).
- \( p_i^2 = (p_i^2, p_{-i}) \).
- \( \sigma_i'' = (\sigma_i|h|p_i^1), \sigma_i^*_{-i}) \).

By [58]

\[
\pi_i[\sigma^1|h] = l_1 + \pi_i[\sigma''|h]a_i^1[p_i^1] + \pi_i[\sigma''|h]a_i^1[p_i^2]
\]

\[
\pi_i[\sigma^2|h] = l_2 + \pi_i[\sigma''|h]a_i^2[p_i^2]
\]

where

\[
l_1 = \sum_{p'' \in \mathcal{P}^*[h] \setminus \{p_i^1, p_i^2\}} (u_i[p''] + \omega_i \pi_i[\sigma^1|h, s'']a_i^1[p_i'']),
\]

\[
l_2 = \sum_{p'' \in \mathcal{P}^*[h] \setminus \{p_i^1, p_i^2\}} (u_i[p''] + \omega_i \pi_i[\sigma^2|h, s'']a_i^2[p_i'']),
\]

and \( s'' = \text{sig}[p''|h] \).
By the definition of $a_i^2$, we have that $l_1 = l_2$. It follows that:

$$
\pi_i[\sigma^1|h] - \pi_i[\sigma^2|h] = \pi_i[\sigma'|h]a_i^1[p_i^1] + \pi_i[\sigma''|h]a_i^1[p_i^2] - \pi_i[\sigma''|h](a_i^1[p_i^2] + a_i^1[p_i^1])
$$

$$
= (\pi_i[\sigma'|h] - \pi_i[\sigma''|h])a_i^1[p_i^1].
$$

If follows from 59 that:

$$
\pi_i[\sigma^1|h] < \pi_i[\sigma^2|h],
$$
contradicting the assumption that $a_i^1 \in BR[\sigma^+|h]$. This concludes the proof. \qed
Proof of Lemma 15. For every $h \in \mathcal{H}$ and $i \in \mathcal{N}$, there exists $p_i \in \mathcal{P}_i$ and a pure strategy $\sigma_i = \sigma^*_i[h|p_i]$ such that:

1. For every $j \in \mathcal{N}_i$, $p_i[j] \in \{0, p_i[j|h]\}$.
2. For every $a_i \in \mathcal{A}_i$, $\pi_i[\sigma_i, \sigma^*_i[h] | h] \geq \pi_i[\sigma'_i, \sigma^*_i[h] | h]$, where $\sigma'_i = \sigma^*_i[h|a_i]$.

Proof. Consider any $h \in \mathcal{H}$ and $i \in \mathcal{N}$. From Lemma 14 it follows that there exists $a_i \in BR[\sigma^*_i[h]]$ and $p_i \in \mathcal{P}_i$ such that $a_i[p_i] = 1$. By Lemma 13 every such $a_i$ and $p_i$ such that $a_i[p_i] = 1$ fulfill Condition 1. Condition 2 follows from the definition of $BR[\sigma^*_i[h]]$. \hfill \square

Proof of Lemma 16. If the DC Condition is fulfilled, then $\sigma^*$ is a SPE.

Proof. Assume that Inequality 5 holds for every history $h$ and $D \subseteq \mathcal{N}_i[h]$. In particular, these assumptions imply that, for each $p_i \in \mathcal{P}_i$ such that $p_i[j] \in \{0, p_i[j|h]\}$ for every $j \in \mathcal{N}_i$, we have

$$\pi_i[\sigma^* | h] \geq \pi_i[\sigma_i, \sigma^*_i[h] | h], \quad (60)$$

where $\sigma_i = \sigma^*_i[h|p_i]$. By Lemma 15 there exists one such $p_i$ such that $\sigma_i$ is a local best response. Consequently, by (60) for every $a_i \in \mathcal{A}_i$ and $\sigma'_i = \sigma^*_i[h|a_i]$, $\pi_i[\sigma^* | h] \geq \pi_i[\sigma'_i, \sigma^*_i[h] | h]$. By Property 5 $\sigma^*$ is a SPE. \hfill \square
B.3 Direct Reciprocity is not Effective

Proof of Lemma 18. If $\sigma^*$ is a SPE, then, for every $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$, it is true that $q_i^j > q_i^*$ and:

$$\frac{\beta_i}{\gamma_i} > \tilde{p}_i + \frac{p_{i}[0]}{q_i^j - q_i^*} \left(1 - q_i^j + \frac{1 - q_i^*}{r}\right),$$

where $\mathbf{p}'_i$ is the strategy where $i$ drops $j$, $\sigma' = (\sigma_i^*[0]_{\mathbf{p}'_i}, \sigma^*_{-i})$, $q_i' = q_i[\sigma'[0]]$, and $q_i^* = q_i[\sigma^*[0]]$.

Proof. The assumption that $\sigma^*$ is a SPE implies by Theorem 17 that the DC Condition is true for the history $\emptyset$, any node $i \in \mathcal{N}$, and $D = \{j\}$, where $j \in \mathcal{N}_i$. Define $\mathbf{p}'_i \in \mathcal{P}_i$ as:

- $p'_{i}[j] = 0$.
- For every $k \in \mathcal{N}_i \setminus \{j\}$, $p'_{i}[k] = p_{i}[k|0]$.

We have that:

$$\tilde{p}_i - p_{i}[0,0|\sigma'] = \sum_{k \in \mathcal{N}_i} p_{i}[k|0] - \sum_{k \in \mathcal{N}_i \setminus \{j\}} p_{i}[k|0] = p_{i}[j|0].$$

Let $\sigma'_i = \sigma_i^*[0]_{\mathbf{p}'_i}$, $\sigma' = (\sigma'_i, \sigma^*_{-i})$, $h'_r = \text{hist}[\emptyset, r|\sigma^*]$, and $h'_r = \text{hist}[\emptyset, r|\sigma']$. It is true by the definition of $u_i$ and by (61) that

$$u_i[0,0|\sigma^*] - u_i[0,0|\sigma'] = (1 - q_i^j)(\beta_i - \gamma_i\tilde{p}_i) - (1 - q_i^*)\gamma_i\tilde{p}_i[0,0|\sigma'])
= (1 - q_i^j)\gamma_i \tilde{p}_i(0,0|\sigma') - \tilde{p}_i
= -(1 - q_i^*)\gamma_i p_{i}[j|0]
= -c.$$

where $c = (1 - q_i^*)\gamma_i p_{i}[j|0]$. Notice that for every $k \in \mathcal{N} \setminus \{i\}$

$$CD_k[\mathbf{p}'|h] = \emptyset,$$

and

$$CD_k[\mathbf{p}'|h] = \{j\}.$$  

From (63) and (64) and by Lemma 8 and Definition 3 for every $r \in \{1 \ldots \tau\}$,

$$DS_i[j|h'_r] = DS_i[j|h^*_r] \cup \{(k_1, k_2, r - 1)|k_1, k_2 \in \mathcal{N} \setminus \{i\}, j \in RS[k_1, k_2] \land k_2 \in CD_{k_1}[\mathbf{p}'|h]\}
= DS_i[j|h^*_r] \cup \{(i, j, r - 1)|i, j \in RS[i, j]\}
= \{(i, j, r - 1)\}.  \tag{65}$$

and for every $k \in \mathcal{N} \setminus \{j\}$

$$DS_i[k|h'_r] = DS_i[k|h^*_r] \cup \{(k_1, k_2, r - 1)|k_1, k_2 \in \mathcal{N} \setminus \{i\}, k \in RS[k_1, k_2] \land k \in CD_{k_1}[\mathbf{p}'|h]\}
= DS_i[k|h^*_r] \cup \{(i, k, r - 1)|i, k \in RS[i, j]\}
= \{\emptyset\}.  \tag{66}$$

By (65) and (66) Definition 3 and the definition of $\mathbf{p}'$, for every $r \in \{1 \ldots \tau\}$

$$\tilde{p}_i[0, r|\sigma'] = \sum_{k \in \mathcal{N}_i \setminus \{j\}} p_{i}[k|h'_r] = \sum_{k \in \mathcal{N}_i \setminus \{j\}} p_{i}[k|0] = \tilde{p}_i - p_{i}[j|0].  \tag{67}$$

By (65) and (67) for every $r \in \{1 \ldots \tau\}$,

$$u_i[0, r|\sigma^*] - u_i[0, r|\sigma'] = (1 - q_i[0, r|\sigma^*])(\beta_i - \gamma_i\tilde{p}_i[0, r|\sigma^*]) - (1 - q_i[0, r|\sigma^*])(\beta_i - \gamma_i\tilde{p}_i[0, r|\sigma^*])
= (1 - q_i^j)(\beta_i - \gamma_i\tilde{p}_i) - (1 - q_i^j)(\beta_i - \gamma_i\tilde{p}_i + p_{i}[j|0])
= (q_i^j - q_i^*)(\beta_i - \gamma_i\tilde{p}_i) - (1 - q_i^j)\gamma_i p_{i}[j|0]
= a - b,$$

where:
\[ a = (q_i' - q_i^*) (\beta_i - \gamma_i \bar{p}_i), \]
\[ b = (1 - q_i^*) \gamma_i p_i[j][\emptyset]. \]

By Theorem 17, the assumption that \( \sigma^* \) is a SPE, 62, and 68:

\[
\begin{align*}
\sum_{r=0}^{\tau} \omega_i^r (u_i[h, r|\sigma^*] - u_i[h, r|\sigma']) &\geq 0 \\
-c + \sum_{r=1}^{\tau} \omega_i^r (a - b) &\geq 0 \\
-c + \frac{\omega_i - \omega_i^{r+1}}{1 - \omega_i} (a - b) &\geq 0 \\
-c(1 - \omega_i) + (\omega_i - \omega_i^{r+1})(a - b) &\geq 0 \\
\omega_i (a - b + c) - \omega_i^{r+1} (a - b) - c &\geq 0.
\end{align*}
\]

(69)

This is a polynomial function of degree \( \tau + 1 \geq 2 \) that has a zero in \( \omega_i = 1 \) and is negative for \( \omega_i = 0 \), since, by Lemma 50, \( c > 0 \). For any \( \omega_i \in (0, 1) \), a solution to (69) exists only if the polynomial is strictly concave. The second derivative is

\[ -(\tau + 1) \omega_i^{\tau-1} (a - b), \]

so we must have \( a > b \) for this to be true, i.e.:

\[
(q_i' - q_i^*) (\beta_i - \gamma_i \bar{p}_i) > (1 - q_i^*) \gamma_i p_i[j][\emptyset]
\]

\[ \Rightarrow (q_i' - q_i^*) (\beta_i - \gamma_i \bar{p}_i) > 0. \tag{70} \]

By Proposition 10, we know that \( \beta_i > \gamma_i \bar{p}_i \). Thus, (70) implies \( q_i' > q_i^* \), which concludes the first part of the proof.

Furthermore, a solution to (69) exists if and only if there is a maximum of the polynomial for \( \omega_i \in (0, 1) \). We can find the zero of the first derivative in order to \( \omega_i \), obtaining

\[ \omega_i^{\tau} = \frac{a - b + c}{(a - b)(\tau + 1)}. \tag{71} \]

For a solution of (71) to exist for some \( \omega_i \in (0, 1) \), it must be true that:

1. \( a \tau > b \tau + c \).
2. \( a - b + c > 0 \).

Condition 1) yields:

\[
\begin{align*}
\tau (q_i' - q_i^*) (\beta_i - \gamma_i \bar{p}_i) \tau &> \tau (1 - q_i^*) \gamma_i p_i[j][\emptyset] + (1 - q_i^*) \gamma_i p_i[j][\emptyset] \\
\tau (q_i' - q_i^*) \beta_i &> \tau (q_i' - q_i^*) \gamma_i \bar{p}_i + \tau (1 - q_i^*) \gamma_i p_i[j][\emptyset] + (1 - q_i^*) \gamma_i p_i[j][\emptyset] \\
\beta_i &> \gamma_i \bar{p}_i + \frac{1 - q_i^*}{q_i' - q_i^*} \gamma_i p_i[j][\emptyset] + \frac{1 - q_i^*}{\tau (q_i' - q_i^*)} \gamma_i p_i[j][\emptyset] \\
\frac{\beta_i}{\gamma_i} &> \bar{p}_i + \frac{1 - q_i^*}{q_i' - q_i^*} p_i[j][\emptyset] (1 - q_i^* + \frac{1 - q_i^*}{\tau})
\end{align*}
\]

(72)

which is equivalent to (3) If Condition 1 is true, then:

\[ a > b + \frac{c}{\tau} \Rightarrow a > b - c. \]

That is, Condition 1 implies Condition 2. Therefore, if \( \sigma^* \) is a SPE for some \( \omega_i \in (0, 1) \), then (5) must hold, which proves the result.
Proof of Lemma [19]. Suppose that for any $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$, $p_i[j|\emptyset] + q_i^\ast \ll 1$. If $\sigma^\ast$ is a SPE, then:

$$\psi[\sigma^\ast] \subseteq \left( \frac{1}{q_i^\ast}, \infty \right).$$

Proof. Fix $i$ and $j$ for which the assumption holds. Let $p'_i$ be defined as:

- $p'_i[j] = 0$,
- $p'_i[k] = p_i[k|\emptyset]$ for every $k \in \mathcal{N}_i \setminus \{ j \}$.

Let $q'_i = q_i[(p'_i, p^\ast_{-i})]$ and $q_i^\ast = q_i[\sigma^\ast[\emptyset]]$, where $p^\ast = \sigma[\emptyset]$. By Lemma [18] we must have

$$\frac{\beta_i}{\gamma_i} > \bar{p}_i + \frac{p_i[j|\emptyset]}{q_i^i - q_i^\ast} \left( 1 - q'_i + \frac{1 - q_i^\ast}{\tau} \right). \quad (73)$$

By Lemma [54] from Appendix A it is true that $q'_i \leq \frac{q_i^\ast}{1 - p_i[j|\emptyset]}$. By including this fact in (73) we obtain:

$$\frac{\beta_i}{\gamma_i} > \bar{p}_i + \frac{p_i[j|\emptyset]}{q_i^i - q_i^\ast} \left( 1 - q'_i + \frac{1 - q_i^\ast}{\tau} \right) > \bar{p}_i + \frac{p_i[j|\emptyset]}{q_i^i - q_i^\ast} \left( 1 - q'_i \right)$$

$$\geq \bar{p}_i + \frac{p_i[j|\emptyset]}{\bar{q}_i^i - q_i^\ast \left( 1 - p_i[j|\emptyset] \right)} \left( 1 - p_i[j|\emptyset] - q_i^\ast \right)$$

$$= \bar{p}_i + \frac{1}{q_i^\ast} \left( 1 - p_i[j|\emptyset] - q_i^\ast \right)$$

$$\approx \bar{p}_i + \frac{1}{q_i^\ast} \left( 1 - q_i^\ast \right) > \frac{1}{q_i^\ast}.$$

The result follows from the fact that if for every $i$ there exists $\omega_i \in (0, 1)$ such that $\sigma^\ast$ is a SPE, then $[0, \frac{1}{q_i^\ast}] \cap \psi[\sigma^\ast] = \emptyset$. 

\[\Box\]
B.4 Full Indirect Reciprocity is Sufficient

Proof of Lemma 20. The profile of strategies $\sigma^*$ is a SPE if and only if for every $h \in H$ and $i \in N$:

$$
\sum_{r=1}^{\tau} (\omega^r u_i[h, r|\sigma^*]) - (1 - q_i[h, 0|\sigma^*])\gamma_i \tilde{p}_i[h, 0|\sigma^*] \geq 0.
$$

Proof. Using the result from Theorem 17 it is true that $\sigma^*$ is a SPE if and only if the DC Condition is true for every $i \in N$, $h \in H$, and $D \subseteq N\{h\}$.

Consider any strategy $\sigma'_i = \sigma^*_i[h|p']$ where $p' = (0, p^{*,}_i)$ and $p' = \sigma^*[h]$. Alternatively, define $\sigma''_i = \sigma^*_i[h|p'']$ where $p'' = (p''_i, p^{*,}_i)$ such that for some $D \subset N\{i\}$:

- For every $j \in D$, $p''_i[j] = 0$.
- For every $j \in N_i \setminus D$, $p''_i[j] = p_i[j|h]$.

Define $h^*_i = \text{hist}[r|\sigma^*]$. Let $\sigma' = (\sigma'_i, \sigma^*_{-i})$, $h'_i = \text{hist}[r|\sigma']$, $\sigma'' = (\sigma''_i, \sigma^*_{-i})$, and $h''_i = \text{hist}[r|\sigma'']$. By Lemma 8 and the definition of full indirect reciprocity, for every $r \in \{1 \ldots \tau\}$ and $j \in N_i^{-1}$, since for every $k \in N \setminus \{i\}$,

$$
\begin{align*}
\text{CD}_i[p'|h] &= N_i, \\
\text{CD}_i[p''|h] &= D, \\
\text{CD}_k[p'|h] &= \text{CD}_k[p''|h] = \emptyset,
\end{align*}
$$

it holds that

$$
\begin{align*}
\text{DS}_j[i|h'_i] &= \text{DS}_j[i|h^*_i] \cup \{(k_1, k_2, r-1)|k_1, k_2 \in N \land k_2 \in \text{CD}_{k_1}[p'|h] \land j, i \in \text{RS}[k_1, k_2]\} \\
&= \text{DS}_j[i|h^*_i] \cup \{(i, k, r-1)|k \in N_i \land j, i \in \text{RS}[i, k]\}. \quad (75)
\end{align*}
$$

and

$$
\begin{align*}
\text{DS}_j[i|h''_i] &= \text{DS}_j[i|h^*_i] \cup \{(k_1, k_2, r)|k_1, k_2 \in N \land k_2 \in \text{CD}_{k_1}[p''|h] \land j, i \in \text{RS}[k_1, k_2]\} \\
&= \text{DS}_j[i|h^*_i] \cup \{(i, k, r-1)|k \in D \land j, i \in \text{RS}[i, k]\}. \quad (76)
\end{align*}
$$

By Definition 3 and by 75 and 76 for every $r \in \{1 \ldots \tau\}$ and $j \in N_i^{-1}$,

$$
p_j[i|h'_i] = p_j[i|h''_i] = 0. \quad (77)
$$

It follows from 77 and Lemma 31 that for every $r \in \{1 \ldots \tau\}$

$$
\begin{align*}
q_i[h, r|\sigma'] &= q_i[h, r|\sigma''] = 1. \\
u_i[h, r|\sigma'] &= u_i[h, r|\sigma''] = 0. \quad (78)
\end{align*}
$$

Furthermore, we have

$$
\tilde{p}_i[h, 0|\sigma'] \leq \tilde{p}_i[h, 0|\sigma''],
$$

which implies that

$$
u_i[h, 0|\sigma'] > u_i[h, 0|\sigma'']. \quad (79)
$$

By 78 and 79

$$
\sum_{r=0}^{\tau} \omega^r (u_i[h, r|\sigma^*] - u_i[h, r|\sigma']) < \sum_{r=0}^{\tau} \omega^r (u_i[h, r|\sigma^*] - u_i[h, r|\sigma'']). \quad (80)
$$

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Finally, we have
\[
\sum_{r=0}^{r} \omega^r (u_i[h,r|\sigma^*] - u_i[h,r|\sigma']) \\
(1 - q_i[h,0|\sigma^*])(\beta_i - \bar{p}_i[h,0|\sigma^*]) - (1 - q_i[h,0|\sigma^*])\beta_i + \sum_{r=1}^{r} \omega^r u_i[h,r|\sigma^*] \geq 0 \\
\sum_{r=1}^{r} \omega^r u_i[h,r|\sigma^*] - (1 - q_i[h,0|\sigma^*])\bar{p}_i[h,0|\sigma^*] \geq 0.
\]

(81)

It is direct to conclude by Theorem 17 that if \(\sigma^*\) is a SPE, then DC Condition is fulfilled for \(D = N_i\). By 80, if DC Condition is fulfilled for \(D = N_i\), then it is also fulfilled for every \(D \subset N_i\), and by Theorem 17, \(\sigma^*\) is a SPE. Thus, \(\sigma^*\) is a SPE iff 81 holds. This concludes the proof. \(\square\)
Proof of Lemma \[21\] Let \( h \in \mathcal{H} \) be defined such that for every \( h' \in \mathcal{H} \), the left side of Inequality \[22\] for \( h \) is lower than or equal to the value for \( h' \). Then, for every \( r \in \{1 \ldots \tau - 2\} \),

\[
u_i[h, r|\sigma^*] = u_i[h, r+1|\sigma^*].\]

**Proof.** The proof goes by contradiction. First, assume that \( h \) minimizes the left side of Inequality \[22\]

\[
\sum_{r=1}^{\tau} (\omega^i_r u_i[h, r|\sigma^*]) - (1 - q_i[h, 0|\sigma^*])\gamma_i\bar{p}_i[h, 0|\sigma^*] \geq 0.
\]

This implies that for every \( h' \in \mathcal{H} \), we have:

\[
\sum_{r=1}^{\tau} (\omega^i_r u_i[h, r|\sigma^*]) - (1 - q_i[h, 0|\sigma^*])\gamma_i\bar{p}_i[h, 0|\sigma^*] \\
- \sum_{r=1}^{\tau} (\omega^i_r u_i[h', r|\sigma^*]) - (1 - q_i[h', 0|\sigma^*])\gamma_i\bar{p}_i[h', 0|\sigma^*] \leq 0
\]

\[
\sum_{r=1}^{\tau} \omega^i_r (u_i[h, r|\sigma^*] - u_i[h', r|\sigma^*]) \\
+ (1 - q_i[h, 0|\sigma^*])\gamma_i(\gamma_i\bar{p}_i[h', 0|\sigma^*] - \bar{p}_i[h, 0|\sigma^*]) \leq 0
\]

(82)

Assume by contradiction that for every \( h \) that minimizes the above condition, there is some \( r \in \{1 \ldots \tau - 3\} \),

\[
u_i[h, r|\sigma^*] \neq u_i[h, r+1|\sigma^*].
\]

Fix \( h \). Without loss of generality, suppose

\[
u_i[h, r|\sigma^*] < u_i[h, r+1|\sigma^*].
\]

(83)

By Lemma \[6\] and Definition \[3\] this implies the existence of \( D \subset \mathcal{N}^2 \), such that, for every \( j \in \mathcal{N} \) and \( k \in \mathcal{N}_j \),

\[
\begin{align*}
DS_j[k|h^*_r] &= DS_j[k|h^*_r] \setminus \{(k_1, k_2, \tau - 1)|(1_1, k_2) \in D \land j, k \in RS[k_1, k_2]\}, \\
DS_j[k|h^*_r] &= DS_j[k|h^*_r] \cup \{(k_1, k_2, \tau - 1)|(1_1, k_2) \in D \land j, k \in RS[k_1, k_2]\}, \\
(k_1, k_2, \tau - r - 1) &\in DS_j[k|h],
\end{align*}
\]

(84)

where \( h^*_r = \text{hist}[h, r|\sigma^*] \), and there exist \( j \) and \( (k_1, k_2) \in D \) such that \( i, j \in RS[k_1, k_2] \), for instance, \( k_1 \) and \( k_2 \).

Notice that, if \[S3\] is true, then \( \tau - r - 1 \geq 1 \). Thus, we can define \( h' \) such that:

- \( |h'| = |h| \).
- \( DS_j[k|h'] = DS_j[k|h] \setminus \{(k_1, k_2, \tau - r - 1)|(k_1, k_2) \in D\} \cup \{(k_1, k_2, \tau - r - 2)|(k_1, k_2) \in D\} \).
Let $h'_r = \text{hist}[h', r|\sigma^*]$. By Lemma 3 for every $r' \in \{0 \ldots r - 1\}$, $j \in \mathcal{N}$, and $k \in \mathcal{N}_j$,

$$DS_j[k|h'_{r+1}] = \{(l_1, l_2, r'' + 1)|(l_1, l_2, r'') \in DS_j[k|h'_r] \land r'' + 1 < \tau\}$$

$$= \{(l_1, l_2, r'' + r'))|(l_1, l_2, r'') \in (DS_j[k|h] \setminus \{(k_1, k_2, \tau - r - 1)|(k_1, k_2) \in D\} \cup \{(k_1, k_2, \tau - r - 2)|(k_1, k_2) \in D \land j, k \in \text{RS}[k_1, k_2]\}) \land r'' + r' < \tau\}$$

$$= \{(k_1, k_2, r'' + r')|(k_1, k_2, r'') \in DS_j[k|h] \land (k_1, k_2) \notin D \land r'' + r' < \tau\} \cup \{(k_1, k_2, \tau - r + r' - 2)|(k_1, k_2) \in D \land j, k \in \text{RS}[k_1, k_2] \land \tau - r - 2 + r' < \tau\}$$

$$= DS_j[k|h^*_{r+1}] \setminus \{(k_1, k_2, \tau - r + r' - 1)|(k_1, k_2) \in D \land j, k \in \text{RS}[k_1, k_2]\} \cup \{(k_1, k_2, \tau - r + r' - 2)|(k_1, k_2) \in D \land j, k \in \text{RS}[k_1, k_2] \land \tau - 2 + r' < \tau\}$$

By the definition of $DS_j[k|h]$ and $DS_j[k|h']$ and by Definition 3 for every $r' \in \{0 \ldots r\}$, $j \in \mathcal{N}$, and $k \in \mathcal{N}_j$,

$$p_j[k|h'_r] = p_j[k|h^*_r].$$

$$\bar{p}_i[h', r'|\sigma^*] = \bar{p}_i[h, r|\sigma^*].$$

$$q_i[h', r'|\sigma^*] = q_i[h, r|\sigma^*].$$

$$u_i[h', r'|\sigma^*] = u_i[h, r|\sigma^*].$$

Furthermore, by 84

$$DS_j[k|h'_{r+1}] = \{(l_1, l_2, r' + 1)|(l_1, l_2, r') \in DS_j[k|h'_r] \land r' + 1 < \tau\}$$

$$= \{(l_1, l_2, r' + 1)|(l_1, l_2, r') \in (DS_j[k|h^*_r] \cup \{(k_1, k_2, \tau - 2)|(k_1, k_2) \in D \land j, k \in \text{RS}[k_1, k_2]\}) \land r' + 1 < \tau\}$$

$$= \{(k_1, k_2, r' + 1)|(k_1, k_2, r') \in DS_j[k|h^*_r]\} \cup \{(k_1, k_2, \tau - 1)|(k_1, k_2) \in D \land j, k \in \text{RS}[k_1, k_2] \land \tau - 1 < \tau\}$$

By Definition 3 and 87

$$p_j[k|h'_{r+1}^*] = p_j[k|h^*_r].$$

$$\bar{p}_i[h', r + 1|\sigma^*] = \bar{p}_i[h, r|\sigma^*].$$

$$q_i[h', r + 1|\sigma^*] = q_i[h, r|\sigma^*].$$

$$u_i[h', r + 1|\sigma^*] = u_i[h, r|\sigma^*].$$

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Finally, by Lemma 82 for every $r' > r$, $j \in \mathcal{N}$, and $k \in \mathcal{N}_j$,

\[
\begin{align*}
DS_j[k|h_{r+1}'] &= \{(l_1, l_2, r'' + 1)|(l_1, l_2, r'') \in DS_j[k|h_r'] \land r'' + 1 < \tau \} \\
&= \{(l_1, l_2, r'' + 1)|(l_1, l_2, r'') \in (DS_j[k|h] \setminus \{(k_1, k_2, \tau - r - 1)|(k_1, k_2) \in D \land j, k \in RS[k_1, k_2] \} \cup \{(k_1, k_2, \tau - r - 2)|(k_1, k_2) \in D \land j, k \in RS[k_1, k_2] \}) \land r'' + 1 < \tau \}
\end{align*}
\]

\[
\begin{align*}
&= \{(l_1, l_2, r'' + r' + 1)|(l_1, l_2, r'') \in DS_j[k|h] \land r'' + r' + 1 < \tau \} \\
&\setminus \{(k_1, k_2, \tau - r + r' - 1)|(k_1, k_2) \in D \land j, k \in RS[k_1, k_2] \land r' < r \} \\
&\cup \{(k_1, k_2, \tau - r + r' - 2)|(k_1, k_2) \in D \land j, k \in RS[k_1, k_2] \land r' < r + 1 \} \tag{89}
\end{align*}
\]

\[
\begin{align*}
&= \{(l_1, l_2, r'' - r')|(l_1, l_2, r'') \in DS_j[k|h] \land r'' > r' \} \\
&= DS_j[k|h_{r+1}'].
\end{align*}
\]

Therefore, by Definition 83 for every $r' > r + 1$, $j \in \mathcal{N}$, and $k \in \mathcal{N}_j$,

\[
\begin{align*}
p_j[k|h_r'] &= p_j[k|h_r'], \\
\bar{p}_i[h', r'|\sigma^*] &= \bar{p}_i[h, r'|\sigma^*], \\
q_i[h', r'|\sigma^*] &= q_i[h, r'|\sigma^*], \\
u_i[h', r'|\sigma^*] &= u_i[h, r'|\sigma^*]. \tag{90}
\end{align*}
\]

By 83 84 88 and 90,

\[
\begin{align*}
\sum_{r'=1}^r \omega^r_i (u_i[h, r'|\sigma^*] - u_i[h', r'|\sigma^*]) + (1 - q_i[h, 0|\sigma^*])\gamma_i(\gamma_i\bar{p}_i[h', 0|\sigma^*] - \bar{p}_i[h, 0|\sigma^*]) \\
= u_i[h, r + 1|\sigma^*] - u_i[h', r + 1|\sigma^*] = u_i[h, r + 1|\sigma^*] - u_i[h, r|\sigma^*] > 0.
\end{align*}
\]

This is a contradiction to 82 which concludes the proof. For the case where

\[
u_i[h, r|\sigma^*] > u_i[h, r + 1|\sigma^*],
\]

the proof is identical, except that $h'$ is defined such that the punishments that end at stage $r$ are anticipated, i.e.:

\[
DS_j[k|h'] = DS_j[k|h] \setminus \{(k_1, k_2, \tau - r - 1)|(k_1, k_2) \in D \} \cup \{(k_1, k_2, \tau - r)|(k_1, k_2) \in D \}.
\]
**Proof of Theorem 22.** If there exists a constant $c \geq 1$ such that, for every $h \in H$ and $i \in N$, Assumption [10] holds, then $\psi[\sigma^*] \supseteq (v, \infty)$, where

$$v = \max_{h \in H} \max_{i \in N} \bar{p}_i[h, 0|\sigma^*] \left(1 + \frac{c}{\tau}\right).$$

Proof. Assume by contradiction that, for any player $i \in N$, $\sigma^*$ is not a SPE for any $\omega_i \in (0, 1)$ and that

$$\frac{\beta_i}{\gamma_i} > \max_{h \in H} \bar{p}_i[h, 0|\sigma^*] \left(1 + \frac{c}{\tau}\right).$$

Fix $i$. The proof considers a history $h$ that minimizes the left side of Inequality [3]. The reason for this is that, if Inequality [3] is true for $h$, then it is also true for every other history $h'$. If $\bar{p}_i[h, 0|\sigma^*] = 0$, then the inequality is trivially fulfilled. Hence, consider that $\bar{p}_i[h, 0|\sigma^*] > 0$. By Lemma 21 for every $r \in \{1, \ldots, \tau - 1\}$,

$$u_i[h, r|\sigma^*] = u_i[h, r + 1|\sigma^*].$$

We are left with stage $\tau$. Let $u_i^h = u_i[h, 1|\sigma^*]$. If for every $k \in N$ and $l \in N_k$ we have $DS_k[l|h_i^*] = DS_k[l|h_{i-1}^*]$, where $h_i^* = \text{hist}[h, r|\sigma^*]$, then

$$u_i^h = u_i[h, \tau|\sigma^*].$$

Otherwise, by Lemma 21

$$u_i^h \leq u_i[h, \tau|\sigma^*].$$

Either way, by Lemma 21 for every $h' \in H$:

$$(1 - q_i[h, 0|\sigma^*]) \gamma_i \bar{p}_i[h, 0|\sigma^*] + \sum_{r=1}^{\tau} \omega_i^r u_i^h \leq (1 - q_i[h', 0|\sigma^*]) \gamma_i \bar{p}_i[h, 0|\sigma^*] + \sum_{r=1}^{\tau} \omega_i^r u_i[h', r|\sigma^*].$$

We can write:

$$-(1 - q_i[h, 0|\sigma^*]) \gamma_i \bar{p}_i[h, 0|\sigma^*] + \sum_{r=1}^{\tau} \omega_i^r u_i^h \leq -a + \frac{\omega_i - \omega_i^{r+1}}{1 - \omega_i^h} u_i^h \geq 0 \quad (92)$$

where

$$a = (1 - q_i[h, 0|\sigma^*]) \gamma_i \bar{p}_i[h, 0|\sigma^*].$$

Again, this Inequality corresponds to a polynomial with degree $\tau + 1$. If $q_i[h, 1|\sigma^*] = 1$, then by our assumptions $q_i[h, 0|\sigma^*] = 1$, $a = 0$, and the Inequality holds. Suppose then that

$$q_i[h, 1|\sigma^*], q_i[h, 0|\sigma^*] < 1.$$

The polynomial has a zero in $\omega_i = 1$. If $\bar{p}_i[h, 0|\sigma^*] = 0$, then $a = 0$ and the Inequality holds. Consider, then, that $\bar{p}_i[h, 0|\sigma^*] > 0$, which implies that $a > 0$. In these circumstances, a solution to (92) exists for $\omega_i \in (0, 1)$ iff the polynomial is strictly concave and has another zero in $(0, 1)$. This is true iff the polynomial has a maximum in $(0, 1)$. The derivatives yield the following conditions:

1. $\exists_{\omega_i \in (0, 1)} u_i^h + a - (\tau + 1) \omega_i u_i^h = 0 \Rightarrow \exists_{\omega_i \in (0, 1)} \omega_i = \frac{u_i^h + a}{(\tau + 1) u_i^h}.$
2. $- (\tau + 1) \tau u_i^h < 0 \Rightarrow u_i^h > 0.$

Condition 1 implies:

$$\frac{u_i^h \tau}{(1 - q_i[h, 1|\sigma^*]) \beta_i \tau} > a \quad (93)$$

$$\frac{\bar{p}_i[h, 1|\sigma^*]}{\gamma_i} > \bar{p}_i[h, 1|\sigma^*] + \bar{p}_i[h, 0|\sigma^*] \left(1 - \frac{q_i[h, 0|\sigma^*]}{1 - q_i[h, 1|\sigma^*]} \right).$$
By the assumption that \( q_i[h, 0|\sigma^*] \leq 1 - c(1 - q_i[h, 1|\sigma^*]) \), if Inequality 11 is true, then so is 93. Furthermore, it also holds that
\[
\beta_i > \gamma_i p_i[h, 1|\sigma^*] \Rightarrow u_i^h > 0,
\]
thus, Condition 2 and 92 are also true for some \( \omega_i \in (0, 1) \). By 91 for every \( h' \in H \) and \( i \in N \), Inequality 9 is true, implying by Lemma 20 that \( \sigma^* \) is a SPE for some value \( \omega_i \in (0, 1) \). This is a contradiction, proving that if for every \( i \in N \) we have \( \frac{\beta_i}{\gamma_i} \in (v, \infty) \), then \( \sigma^* \) is a SPE. By the definition of \( \psi, \psi[\sigma^*] \supset (v, \infty) \). \( \square \)
C Private Monitoring

C.1 Evolution of the Network

Auxiliary Lemma.

Lemma 55. For every $h \in \mathcal{H}$, $p' \in \mathcal{P}$, $i \in \mathcal{N}$, $j \in \mathcal{N}_i$, $k_1, k_2 \in \mathcal{N}$, and $r \in \{0 \ldots d_i[k_1, k_2] + \tau[k_1, k_2|i, j] - v[k_1, k_2] - 1\}$, where $v[k_1, k_2] = \min[d_i[k_1, k_2] - d_j[k_1, k_2], 0]$, let $s'_1 \in \text{sig}[\sigma'[h'_r]|h'_r]$ and $s''_2 \in \text{sig}[\sigma^*[h^*_r]|h^*_r]$, where $h^*_r = \text{hist}[h, r|\sigma^*]$, $h'_r = \text{hist}[h, r|\sigma']$, and $\sigma' = \sigma^*[h|p']$. Then, we have

1. If $k_2 \notin \text{CD}_{k_1}[p'|h]$ or $k_2 \in \text{CD}_{k_1}[p'|h]$ and $d_i[k_1, k_2] > r$, we have $s^*_1[k_1, k_2] = s'_1[k_1, k_2]$.
2. If $k_2 \in \text{CD}_{k_1}[p'|h]$ and $d_i[k_1, k_2] = r$, then $s^*_1[k_1, k_2] = \text{cooperate}$ and $s'_1[k_1, k_2] = \text{defect}$.
3. Else, $s^*_1[k_1, k_2] = s'_1[k_1, k_2] = \text{cooperate}$.

Proof. Fix $i, j, k_1, k_2, h$, and $p'$.

Let $s'_{k_2} \in \text{sig}[\sigma'[h]|h]$ and $s^*_{k_2} \in \text{sig}[\sigma^*[h]|h]$. For $k_2 \notin \text{CD}_{k_1}[p'|h]$ and $h_{k_1} \in h$,

$$p'_{k_1}[k_2] = p_{k_1}[k_2|h_{k_1}],$$

and, by Definition 23,

$$s^*_1[k_1, k_2] = s'_1[k_1, k_2].$$

If $k_2 \in \text{CD}_{k_1}[p'|h]$ and $r = d_i[k_1, k_2]$, then, by the definition of $\sigma'$ and $\sigma^*$,

$$s'_{k_2}[k_1, k_2] = \text{defect},$$

$$s^*_{k_2}[k_1, k_2] = \text{cooperate},$$

and, by Definition 23,

$$s'_1[k_1, k_2] = \text{defect},$$

$$s^*_1[k_1, k_2] = \text{cooperate}.$$

If $r > d_i[k_1, k_2]$, then

$$s'_1[k_1, k_2] = s^*_1[k_1, k_2] = \text{cooperate}.$$

To see this, assume first that $s'_1[k_1, k_2] = \text{defect}$. Then, by Definition 23 there must exist a round $r' > 0$ and history $\text{hist}[h, r'|\sigma']$ after which $k_2$ defects $k_1$. That is, define $r' = r - d_i[k_1, k_2]$. For $s''_{k_2} \in \text{sig}[\sigma'[h'_r]|h'_r]$, $s^*_{k_2}[k_1, k_2] = \text{defect}$, which is true iff $p''_{k_1}[k_2] < p_{k_1}[k_2|h'_r]$. Since $r' > 0$, this contradicts the definition of $\sigma'$. Hence, $s^*_1[k_1, k_2] = \text{cooperate}$.

If we assume that $s'_1[k_1, k_2] = \text{defect}$, then by Definition 23 there must exist a round $r' > 0$ and history $\text{hist}[h, r'|\sigma']$ after which $k_2$ defects $k_1$. As before, since $r' > 0$, another contradiction is reached and we can conclude that we must have $s^*_1[k_1, k_2] = \text{cooperate}$.

For $r < d_i[k_1, k_2]$ the result holds immediately. This is because by Definition 23 $s'_1[k_1, k_2] = \text{defect}$ iff $|h| + r \geq d_i[k_1, k_2]$ and for $s''_{k_2} = h''_{k_2}[k_1, k_2] - r$, $s''_{k_2}[k_1, k_2] = \text{defect}$. This implies that

$$s^*_1[k_1, k_2] = \text{defect}.$$

Similarly, if $s'_1[k_1, k_2] = \text{cooperate}$, then $s''_{k_2}[k_1, k_2] = \text{cooperate}$, implying that

$$s^*_1[k_1, k_2] = \text{cooperate}.$$

This proves the result. 

\[ \square \]
Proof of Lemma 28. For every $h \in \mathcal{H}$, $p' \in \mathcal{P}$, $r > 0$, $i \in \mathcal{N}$, and $j \in \mathcal{N}_i$:

$$DS_i[j|h^*_{i,r}] = DS_i[j|h^*_{i,r}] \cup \{(k_1, k_2, r - 1 - d_i[k_1, k_2] + v[k_1, k_2]) | k_1, k_2 \in \mathcal{N} \land \exists k_2 \in CD_{k_1}[p'|h] \land r \in \{d_i[k_1, k_2] + 1 \ldots d_i[k_1, k_2] + \tau[k_1, k_2, i, j] - v[k_1, k_2]\} \land v[k_1, k_2] = \min[d_i[k_1, k_2] - d_j[k_1, k_2], 0]\},$$

where $h^*_{i,r} \in hist[h, r|\sigma^*]$, $h^*_{i,r} \in hist[h, r|\sigma']$, and $\sigma' = \sigma^*[h|p']$ is the profile of strategies where all players follow $p'$ in the first stage.

Proof. Fix $h$, $p'$, $i$, and $j$. The proof goes by induction on $r$, where the induction hypothesis is that for every $r \geq 0$, Equality 15 holds for $\sigma^*[h^*_{i,r}] | h^*_{i,r}$, and $\sigma' = \sigma^*[h|p']$.

We will simplify the notation by first dropping the factor $v[k_1, k_2] = \min[d_i[k_1, k_2] - d_j[k_1, k_2], 0]$ whenever possible, and by removing the redundant indexes $k_1, k_2, i, j$, except when distinguishing between different delays. We will also remove the factor $k_1, k_2 \in \mathcal{N}$. Namely:

- $d_i[k_1, k_2] = d_i$ and $d_j[k_1, k_2] = d_j$.
- $v[k_1, k_2] = v$.
- $\tau[k_1, k_2, i, j] = \tau$.

By Definition 24 we have that for every $r \geq 0$, $h^*_{i,r} = hist[h, r|\sigma^*]$, and $s^*_i \in sig[\sigma^*[h^*_{i,r}] | h^*_{i,r}]$.

$$DS_i[j|h^*_{i,r+1}] = L_1[r + 1|\sigma^*] \cup L_2[r + 1|\sigma^*],$$

where

$$L_1[r + 1|\sigma^*] = \{(k_1, k_2, r' + 1)(k_1, k_2, r') \in DS_i[j|h^*_{i,r}] \land \tau' + 1 < \tau\},$$

$$L_2[r + 1|\sigma^*] = \{(k_1, k_2, v) | s^*_i[k_1, k_2] = defect\}.$$  

Similarly, for every $r \geq 0$, $h'_{i,r} = hist[h, r|\sigma']$, and $s'_i \in sig[\sigma'[h^*_{i,r}] | h'_{i,r}]$.

$$DS_i[j|h^*_{i,r+1}] = L_1[r + 1|\sigma'] \cup L_2[r + 1|\sigma'],$$

where

$$L_1[r + 1|\sigma'] = \{(k_1, k_2, r' + 1)(k_1, k_2, r') \in DS_i[j|h^*_{i,r}] \land \tau' + 1 < \tau\},$$

$$L_2[r + 1|\sigma'] = \{(k_1, k_2, v) | s'_i[k_1, k_2] = defect\}.$$  

For any $r \geq 0$, let $s'_i \in sig[\sigma'[h'_r]] | h'_r]$ and $s^*_i \in sig[\sigma^*[h'_r]] | h'_r]$, where $h^*_{r} = hist[h, r|\sigma^*]$, $h'_{r} = hist[h, r|\sigma']$, and $\sigma' = \sigma^*[h|p']$.

By Lemma 53 we have:

1. $s^*_i[k_1, k_2] = s'_i[k_1, k_2]$ for $k_2 \notin CD_{k_1}[p'|h]$ and $k_2 \notin CD_{k_1}[p'|h]$ such that $d_i > r$.
2. $s'_i[k_1, k_2] = \text{defect}$ and $s^*_i[k_1, k_2] = \text{cooperate}$ for $k_2 \notin CD_{k_1}[p'|h]$ such that $d_i = r$.
3. $s'_i[k_1, k_2] = s^*_i[k_1, k_2] = \text{cooperate}$ for $k_2 \notin CD_{k_1}[p'|h]$ such that $d_i < r$.

By 95 and items 1), 2), and 3) above,

$$L_2[r + 1|\sigma^*] = \{(k_1, k_2, v) | s^*_i[k_1, k_2] = defect\} \cup \{(k_1, k_2, v) | k_2 \notin CD_{k_1}[p'|h] \land s^*_i[k_1, k_2] = defect\} \cup \{(k_1, k_2, v) | k_2 \in CD_{k_1}[p'|h] \land d_i > r \land s^*_i[k_1, k_2] = defect\},$$

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Equation (88), 1), 2), and 3) allows us to write:

\[ L_2[r + 1 | \sigma^*] = \{(k_1, k_2, v) | s_i^r[k_1, k_2] = \text{defect} \}
\]

\[ = \{(k_1, k_2, v) | k_2 \notin CD_k_1[p', h] \land s_i^r[k_1, k_2] = \text{defect} \}
\cup \{(k_1, k_2, v) | k_2 \in CD_k_1[p'|h] \land s_i^r[k_1, k_2] = \text{defect} \}
\]

\[ = \{(k_1, k_2, v) | k_2 \notin CD_k_1[p'|h] \land s_i^r[k_1, k_2] = \text{defect} \}
\cup \{(k_1, k_2, v) | k_2 \in CD_k_1[p'|h] \land d_i > r \land s_i^r[k_1, k_2] = \text{defect} \}
\cup \{(k_1, k_2, v) | k_2 \in CD_k_1[p'|h] \land d_i = r \}
\]

\[ = L_2[r + 1 | \sigma^*] \cup \{(k_1, k_2, v) | k_2 \in CD_k_1[p'|h] \land d_i = r \}.
\]

Now proceed to the base case, for \( r = 0 \). Since \( h = \text{hist}[h, 0 | \sigma^*] = \text{hist}[h, 0 | \sigma^*] \), by (95) and (97) it is true that:

\[ L_1[1 | \sigma^*] = L_1[1 | \sigma^*]. \]

Furthermore, by (99)

\[ L_2[r + 1 | \sigma^*] = L_2[r + 1 | \sigma^*] \cup \{(k_1, k_2, v) | k_2 \in CD_k_1[p'|h] \land d_i = 0 \}
\]

\[ = L_2[r + 1 | \sigma^*] \cup \{(k_1, k_2, v) | k_2 \in CD_k_1[p'|h] \land 1 \geq d_i + 1 \land d_i \leq \tau - v \}
\]

\[ = L_2[r + 1 | \sigma^*] \cup \{(k_1, k_2, v) | k_2 \in CD_k_1[p'|h] \land r + 1 \in \{d_i + 1 \ldots d_i + \tau - v\} \}.
\]

The base case is true by (94) (96) (100) and (101).

Hence, assume the induction hypothesis for some \( r \geq 0 \) and consider the induction step for \( r + 1 \), which consists in determining the value of \( DS_i[j|h_i^r, r+2] \).

By the induction hypothesis and by (95)

\[ L_1[r + 2 | \sigma^*] = \{(k_1, k_2, v) | (k_1, k_2, v') \in (A \cup B) \land r' + 1 < \tau \}
\]

\[ = \{(k_1, k_2, v) | (k_1, k_2, v') \in A \land r' + 1 < \tau \} \cup \{(k_1, k_2, v) | (k_1, k_2, v') \in B \land r' + 1 < \tau \}.
\]

where

\[ A = DS_i[j|h_i^r, r+1] \]

\[ B = \{(k_1, k_2, v) | k_2 \in CD_k_1[p'|h] \land r + 1 \in \{d_i + 1 \ldots d_i + \tau - v\} \}.
\]

We have by (95)

\[ \{(k_1, k_2, v) | (k_1, k_2, v') \in A \land r' + 1 < \tau \}
\]

\[ = \{(k_1, k_2, v) | (k_1, k_2, v') \in DS_i[j|h_i^r, r+1] \land r' + 1 < \tau \} \]

(103)

It is also true that

\[ \{(k_1, k_2, v) | (k_1, k_2, v') \in B \land r' + 1 < \tau \}
\]

\[ = \{(l_1, l_2, v') \in (l_1, l_2, v') \in \{(k_1, k_2, v - d_i + v) | k_2 \in CD_k_1[p'|h] \land r + 1 \in \{d_i + 1 \ldots d_i + \tau - v\} \} \land r' + 1 < \tau \}
\]

\[ = \{(k_1, k_2, v + 1 - d_i + v) | k_2 \in CD_k_1[p'|h] \land r + 1 \in \{d_i + 1 \ldots d_i + \tau - v\} \land r + 1 - d_i + v < \tau \}
\]

\[ = \{(k_1, k_2, v + 1 - d_i + v) | k_2 \in CD_k_1[p'|h] \land r + 1 \in \{d_i + 1 \ldots d_i + \tau - v - 1\} \}
\]

\[ = \{(k_1, k_2, v + 1 - d_i + v) | k_2 \in CD_k_1[p'|h] \land r + 2 \in \{d_i + 2 \ldots d_i + \tau - v\} \}
\]

(104)
By 94, 96, 101, 102, 103, and 104

\[
\begin{align*}
\text{DS}_i[j|^{h_i^*, r+2}] &= L_1[r + 2|\sigma^*] \cup \\
& \quad \{(k_1, k_2, r + 1 - d_i + v) | k_2 \in \text{CD}_{k_1}[p'|h] \wedge r + 2 \in \{d_i + 2 \ldots d_i + \tau - v \} \cup \\
& \quad L_2[r + 2|\sigma^*] \cup \{(k_1, k_2, r + 1 - d_i + v) | k_2 \in \text{CD}_{k_1}[p'|h] \wedge d_i = r + 1 \}
\end{align*}
\]

\[
= L_1[r + 2|\sigma^*] \cup L_2[r + 2|\sigma^*] \cup \\
\{(k_1, k_2, r + 1 - d_i + v) | k_2 \in \text{CD}_{k_1}[p'|h] \wedge r + 2 \in \{d_i + 2 \ldots d_i + \tau - v \} \cup \\
\{(k_1, k_2, r + 1 - d_i + v) | k_2 \in \text{CD}_{k_1}[p'|h] \wedge d_i + 1 = r + 2 \}
\]

\[
= \text{DS}_i[j|^{h_i^*, r+2}] \cup \\
\{(k_1, k_2, r + 1 - d_i + v) | k_2 \in \text{CD}_{k_1}[p'|h] \wedge r + 2 \in \{d_i + 1 \ldots d_i + \tau - v \}.
\]

This proves the induction step and concludes the proof. \qed
Proof of Lemma 29. For every $i \in \mathcal{N}$, $j \in \mathcal{N}_i$, $h \in \mathcal{H}$, and $h_i, h_j \in h$:

$$p_i[j|h_i] = p_j[j|h_j].$$

Proof. Fix $i$, $j$, $h$, and $h_i, h_j \in h$. Consider any tuple $(k_1, k_2, r) \in DS_i[j|h_i]$ and let $K_i$ and $K_j$ represent the sets used by $i$ and $j$ to compute $p_i[j|h_i]$ and $p_j[j|h_j]$, respectively. By Lemma 23 and Definitions 23 and 24, it is true that for some $r' \geq d_i[k_1, k_2]$:

$$r = r' - d_i[k_1, k_2] + v_i,$$  \hspace{1cm} (105)

where $v_i = \min[d_i[k_1, k_2] - d_j[k_1, k_2], 0]$. 

By Definition 23, this implies that

$$|h_i| \geq r' + d_i[k_1, k_2] + 1,$$

and for $h_{k_2} \in h$ and $s_{k_2} = h_{k_2}' + d_i[k_1, k_2] + 1$:

$$s_{k_2}[k_1, k_2] = \text{defect}.$$  \hspace{1cm} (107)

Since $r' \geq d_i[k_1, k_2]$, if $r < 0$, then we have by 105

$$v_i < 0 \Rightarrow d_i[k_1, k_2] < d_j[k_1, k_2],$$

which implies by Definition 23 that $j$ has yet to observe the defection that caused $i$ to add $(k_1, k_2, v_i)$ to $DS_i[j|h_i]$. Consequently, $j$ has not included this tuple in $DS_j[i|h_j]$ or in $K_j$. Also, by Definition 24, $i$ does not include the tuple in $K_i$, since $r < 0$.

Consider, now, that $r \geq 0$, where by Definition 24 $i$ adds the tuple to $K_i$. By 105

$$r' \geq d_i[k_1, k_2] - v_i \geq d_i[k_1, k_2] - d_i[k_1, k_2] + d_j[k_1, k_2] = d_j[k_1, k_2],$$  \hspace{1cm} (108)

Furthermore, since by Definition 24 $r < \tau[k_1, k_2|i, j]$, we also have by 105

$$r' < \tau[k_1, k_2|i, j] + d_i[k_1, k_2] - v_i.$$  \hspace{1cm} (109)

If $d_i[k_1, k_2] \leq d_j[k_1, k_2]$, then $v_i < 0$, $v_j = 0$, and by 108 and 109 we have

$$r' < \tau[k_1, k_2|i, j] + d_j[k_1, k_2] - v_j,$$

$$r' + 1 \in \{d_j[k_1, k_2] + 1 \ldots d_j[k_1, k_2] + \tau[k_1, k_2|i, j] - v_j\},$$

where $v_j = \min[d_j[k_1, k_2] - d_i[k_1, k_2], 0]$. By Lemma 29, $j$ adds $(k_1, k_2, r' - d_j[k_1, k_2] + v_j)$ to $DS_j[i|h_j]$, such that $v_j = 0$ and by 108

$$r' - d_j[k_1, k_2] + v_j \geq 0.$$  \hspace{1cm} (110)

If $d_i[k_1, k_2] > d_j[k_1, k_2]$, then $v_i = 0$,

$$v_j = -(d_i[k_1, k_2] - d_j[k_1, k_2]),$$

hence, by 109

$$d_j[k_1, k_2] + \tau[k_1, k_2|i, j] - v_j = \tau[k_1, k_2|i, j] + d_i[k_1, k_2] - v_i > r'.$$
Therefore, by 108
\[ r' + 1 \in \{ d_j[k_1, k_2] + 1 \ldots d_j[k_1, k_2] + \tau[k_1, k_2] - v_j \}, \]
and by Lemma 28 j adds \((k_1, k_2, r' - d_j[k_1, k_2] + v_j)\) to DS\(j[i|\mathbf{h}_j]\). Again, by 105 and the assumption that \(r \geq 0\),
\[ r' - d_j[k_1, k_2] + v_j = r' - d_i[k_1, k_2] = r + d_i[k_1, k_2] - d_i[k_1, k_2] - v_i = r \geq 0. \quad (111) \]
In any case, by 110 and 111 j adds the tuple to \(K_j\). This proves that \(i\) adds the tuple to \(K_i\) iff \(j\) adds the tuple to \(K_j\), implying that \(K_i = K_j\). Since \(p_i[j|\mathbf{h}_i]\) and \(p_i[j|\mathbf{h}_j]\) are obtained by applying the same deterministic functions to \(K_i\) and \(K_j\), respectively, we have
\[ p_i[j|\mathbf{h}_i] = p_i[j|\mathbf{h}_j]. \]
C.2 Generic Results

Proof of Proposition 30. For every assessment \((\sigma^*, \mu^*)\), if \((\sigma^*, \mu^*)\) is Sequentially Rational, then, for every \(i \in N\), \(\frac{\beta_i}{\gamma_i} \geq \bar{p}_i\). Consequently, \(\psi[\sigma^*] \subseteq (v, \infty)\), where \(v = \max_{i \in N} \bar{p}_i\).

Proof. Let \(p^* = \sigma^*[\emptyset]\). For \(h_i = \emptyset\), the only history \(h\) that fulfills \(\mu_i^*[h|h_i] > 0\) is \(h = \emptyset\). Therefore, the equilibrium utility is also

\[
\pi_i[\sigma^*|\mu^*, \emptyset] = \sum_{r=0}^{\infty} \omega_i^r (1 - q_i[p^*]) (\beta_i - \gamma_i \bar{p}_i) = \frac{1 - q_i[p^*]}{1 - \omega_i} (\beta_i - \gamma_i \bar{p}_i).
\]

If \(\frac{\beta_i}{\gamma_i} \leq \bar{p}_i\), then

\[
\pi_i[\sigma^*|\mu^*, \emptyset] \leq 0. \tag{112}
\]

Let \(\sigma'_i \in \Sigma_i\) be a strategy such that, for every \(h_i \in \mathcal{H}_i\), \(\sigma'_i[h_i] = \emptyset\), and let \(\sigma' = (\sigma'_i, \sigma^*_{-i})\), where \(0 = (0)_{j \in \mathcal{N}_i}\). We have

\[
\pi_i[\sigma'|\mu^*, \emptyset] = (1 - q_i[p^*]) \beta_i + \pi_i[\sigma'|\text{sig}[p'|\emptyset]] \geq (1 - q_i[p^*]) \beta_i, \tag{113}
\]

where \(p' = (0, p^*_{-i})\). By Lemma 50, \(q_i[p^*] < 1\). Since \(\pi_i[\sigma'|\text{sig}[p'|\emptyset]] \geq 0\), it is true that

\[
\pi_i[\sigma^*|\mu^*, \emptyset] \leq 0 < \pi_i[\sigma'|\mu^*, \emptyset].
\]

This contradicts the assumption that \(\sigma^*\) is a SPE. \qed

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Proof of Lemma 33. For every $i \in \mathcal{N}$, $h_i \in \mathcal{H}_i$, $a_i \in BR[\sigma^*_i|\mu^*,h_i]$, and $p_i \in \mathcal{P}_i$ such that $a_i[p_i] > 0$, it is true that for every $j \in \mathcal{N}_i$ we have $p_i[j] \in \{0,p_i[j|h_i]\}$.

Proof. Suppose then that there exist $i \in \mathcal{N}$, $h_i \in \mathcal{H}_i$, $a_i \in BR[\sigma^*_i|\mu^*,h_i]$, and $p_i \in \mathcal{P}_i$ such that $a_i[p_i] > 0$ and there exists $j \in \mathcal{N}_i$ such that $p_i[j] \notin \{0,p_i[j|h_i]\}$. Fix any $h$ such that $h_i \in h$ and define an alternative $a_i^2 \in A_i$:

- $a_i^1 = (a_i^2,p_{-i}^*)$ and $a_i^2 = (a_i^2,p_{-i}^*)$, where $p_{-i}^* = \sigma^*[h]$.
- Define $p_i^2 \in \mathcal{P}_i$ such that for every $j \in \mathcal{N}_i$, if $p_i^1[j] \geq p_i[j|h_i]$, then $p_i^2[j] = p_i[j|h_i]$, else, $p_i^2[j] = 0$.
- Set $a_i^2[p_i^2] = a_i^1[p_i^1] + a_i^2[p_i^1]$ and $a_i^2(p_i^1) = 0$.
- For every $p_i'' \in \mathcal{P}_i \setminus \{p_i^1,p_i^2\}$, set $a_i^2[p_i''] = a_i^1[p_i''].$
- Define $\sigma_i^1 = \sigma_i^*[h_i|p_i^1]$ and $\sigma_i^2 = \sigma_i^*[h_i|p_i^2].$
- Set $\sigma_i = (\sigma_i^1,\sigma_i^2)$ and $\sigma_i = (\sigma_i^1,\sigma_i^2,i)$. 

Notice that for any $j \in \mathcal{N}_i$, $p_i^1[j] \geq p_i^2[j]$ and $p_i^1[j] \geq p_i[j|h_i]$ iff $p_i^2[j] \geq p_i[j|h_i]$. Thus, by Definition 2, for every $s \in S$,

\[
pr_i[s|a_i^1,h] = pr_i[s|a_i^2,h]. \\
pr(s|a_i^1,h) = pr(s|a_i^2,h).
\]  \hspace{1cm} (114)

Moreover, for some $j \in \mathcal{N}_i$, $p_i^1[j|h_i] > p_i^2[j|h_i]$, thus, it is true that

\[
\pi_i[a_i^1] < \pi_i[a_i^2]. \hspace{1cm} (115)
\]

Recall that

\[
\pi_i[\sigma_i^1|h] = u_i[a_i^1] + \omega_i \sum_{s \in S} \pi_i[\sigma_i^1|(h,s)]pr[s|a_i^1,h],
\]

\[
\pi_i[\sigma_i^2|h] = u_i[a_i^2] + \omega_i \sum_{s \in S} \pi_i[\sigma_i^2|(h,s)]pr[s|a_i^2,h].
\]

By (114) and the definition of $\sigma_i^1$ and $\sigma_i^2$,

\[
\sum_{s \in S} \pi_i[\sigma_i|(h,s)]pr[s|a_i^1,h] = \sum_{s \in S} \pi_i[\sigma_i|(h,s)]pr[s|a_i^2,h].
\]

It follows from (115) that, for every $h \in \mathcal{H}$ such that $h_i \in h$,

\[
\pi_i[\sigma_i^1|h] < \pi_i[\sigma_i^2|h].
\]

Consequently,

\[
\pi_i[\sigma_i^1|h_i] < \pi_i[\sigma_i^2|h_i].
\]

This is a contradiction, since $a_i^1 \in BR[\sigma^*_i|h_i]$ by assumption, concluding the proof. \hfill \Box
Proof of Lemma 34. For every $i \in \mathcal{N}$ and $h_i \in \mathcal{H}_i$, there exists $a_i \in BR(\sigma_{i-1}^*, \mu^*, h_i)$ and $p_i \in \mathcal{P}_i$ such that $a_i[p_i] = 1$.

Proof. Fix $i$ and $h_i$. If $BR(\sigma_{i-1}^*, \mu^*, h_i)$ only contains pure strategies for the stage game, since $BR(\sigma_{i-1}^*, \mu^*, h_i)$ is not empty, the result follows. Suppose then that there exists a mixed strategy $a_i^1 \in BR(\sigma_{i-1}^*, \mu^*, h_i)$. We know from Lemma 33 that every such $a_i$ attributes positive probability to one of two probabilities in $\{0, p_i[j/h_i]\}$, for every $j \in \mathcal{N}_i$. Denote by $\mathcal{P}_i^*[h_i]$ the finite set of profiles of probabilities that fulfill the condition of Lemma 33 i.e., for every $p_i \in \mathcal{P}_i^*[h_i]$ and $j \in \mathcal{N}_i$, $p_i[j] \in \{0, p_i[j/h_i]\}$. Define $\mathcal{P}^*[h]$ similarly for any $h \in \mathcal{H}$.

For any $h \in \mathcal{H}$ such that $h_i \in h$, we can write

$$\pi_i[\sigma^1|h] = \sum_{p_i \in \mathcal{P}_i^*[h_i]} (u_i[p] + \omega_i \pi_i[\sigma^1|(h, \text{sig}(p|h_i))]a_i[p_i],$$

where $p = (p_i, p_{i-1})$ and $p^* = \sigma^*[h]$.

For any $p_i^1 \in \mathcal{P}_i^*[h_i]$ such that $a_i[p_i^1] > 0$, let $\sigma_i^1 = \sigma_i^1[h_i[a_i^1]]$, $\sigma_i^1 = (\sigma_i^1, \sigma_{i-1}^1)$, $\sigma_i^1 = \sigma_i^1[h_i|p_i^1]$, and $\sigma'' = (\sigma_i'', \sigma_{i-1}'', h_i)$.

There are three possibilities:

1. $\pi_i[\sigma^1|\mu^*, h_i] = \pi_i[\sigma''|\mu^*, h_i]$.
2. $\pi_i[\sigma^1|\mu^*, h_i] < \pi_i[\sigma''|\mu^*, h_i]$.
3. $\pi_i[\sigma^1|\mu^*, h_i] > \pi_i[\sigma''|\mu^*, h_i]$.

In possibility 1, it is true that there exists $a_i^2 \in BR(\sigma_{i-1}^*, \mu^*, h_i)$ such that $a_i^2[p_i^1] = 1$ and the result follows. Possibility 2 contradicts the assumption that $a_i^1 \in BR(\sigma_{i-1}^*, \mu^*, h_i)$.

Finally, consider that possibility 3 is true. Recall that $a_i^1$ being mixed implies $a_i^1[p_i^1] < 1$. Thus, there must exist $p_i^2 \in \mathcal{P}_i^*[h_i]$, $\sigma_i^2 = \sigma_i^2[h_i[p_i^2]]$, and $\sigma'' = (\sigma_i'', \sigma_{i-1}'')$, such that $a_i^2[p_i^2] > 0$ and

$$\pi_i[\sigma''|\mu^*, h_i] < \pi_i[\sigma'', h_i].$$

Here, we can define $a_i^2 \in \mathcal{A}_i$ such that:

- $a_i^2[p_i^2] = a_i^1[p_i^1] + a_i^1[p_i^2]$;
- $a_i^2[p_i^1] = 0$.
- For every $p_i'' \in \mathcal{P}_i^*[h_i] \setminus \{p_i^1, p_i^2\}$, $a_i^2[p_i''] = a_i^1[p_i'']$.

Now, let $\sigma_i^2 = \sigma_i^2[h_i|a_i^2]$, and $\sigma'' = (\sigma_i'', \sigma_{i-1}'')$.

By (116) it holds that for every $h \in \mathcal{H}$ such that $h_i \in h$:

$$\pi_i[\sigma^1|h] = l_1 + \pi_i[\sigma'|h]a_i^1[p_i^1] + \pi_i[\sigma''|h]a_i^1[p_i^2],$$

$$\pi_i[\sigma^2|h] = l_2 + \pi_i[\sigma''|h]a_i^2[p_i^2],$$

where

$$l_1 = \sum_{p_i'' \in \mathcal{P}_i^*[h_i] \setminus \{p_i^1, p_i^2\}} (u_i[p_i''] + \omega_i \pi_i[\sigma^1|h, s''])a_i^1[p_i''],$$

$$l_2 = \sum_{p_i'' \in \mathcal{P}_i^*[h_i] \setminus \{p_i^1, p_i^2\}} (u_i[p_i''] + \omega_i \pi_i[\sigma^2|h, s''])a_i^2[p_i''],$$

and $s'' = \text{sig}(p_i'|h)$. By the definition of $a_i^2$, we have that $l_1 = l_2$. It follows that

$$\pi_i[\sigma^1|h] - \pi_i[\sigma^2|h] = \pi_i[\sigma'|h]a_i^1[p_i^1] + \pi_i[\sigma''|h]a_i^1[p_i^2] - \pi_i[\sigma''|h](a_i^1[p_i^2] + a_i^1[p_i^1])$$

$$= (\pi_i[\sigma'|h] - \pi_i[\sigma''|h])a_i^1[p_i^1].$$
Consequently, by [117]

\[
\pi_i[\sigma^1|\mu^*,h_i] - \pi_i[\sigma^2|\mu^*,h_i] = \sum_{h \in \mathcal{H}} \mu_i^*[h|h_i](\pi_i[\sigma^1|h] - \pi_i[\sigma^2|h]) \\
= \sum_{h \in \mathcal{H}} \mu_i^*[h|h_i](\pi_i[\sigma'|h] - \pi_i[\sigma''|h])a_i^1[p_i^1] \\
= (\pi_i[\sigma'|\mu^*,h_i] - \pi_i[\sigma''|\mu^*,h_i])a_i^1[p_i^1] \\
< 0.
\]

Thus,

\[
\pi_i[\sigma^1|\mu^*,h_i] < \pi_i[\sigma^2|\mu^*,h_i],
\]

contradicting the assumption that \(a_i^1 \in BR[\sigma^*|\mu^*,h_i]\). This concludes the proof. \(\Box\)
Proof of Lemma 35. For every $i \in \mathcal{N}$ and $h_i \in \mathcal{H}_i$, there exists $p_i \in \mathcal{P}_i$ and a pure strategy $\sigma_i = \sigma_i^*[h_i|p_i]$ such that:

1. For every $j \in \mathcal{N}_i$, $p_i[j] \in \{0, p_i[j|h_i]\}$.
2. For every $a_i \in \mathcal{A}_i$, $\pi_i[\sigma_i, \sigma_i^*|\mu^*, h_i] \geq \pi_i[\sigma'_i, \sigma_i^*|\mu^*, h_i]$, where $\sigma'_i = \sigma_i^*[h|a_i]$.

Proof. Consider any $i \in \mathcal{N}$ and $h_i \in \mathcal{H}$. From Lemma 34 it follows that there exists $a_i \in BR[\sigma_i^*, \mu^*, h_i]$ and $p_i \in \mathcal{P}_i$ such that $a_i[p_i] = 1$. By Lemma 33 every such $a_i$ and $p_i$ such that $a_i[p_i] = 1$ fulfill Condition 1. Condition 2 follows from the definition of $BR[\sigma_i^*, \mu^*, h_i]$. $\square$

Proof of Lemma 36. If the PDC Condition is fulfilled and $(\sigma^*, \mu^*)$ is Preconsistent, then $(\sigma^*, \mu^*)$ is Sequentially Rational.

Proof. Assume that Inequality 16 holds for every player $i$, history $h_i$ and $D \subseteq \mathcal{N}_i[h_i]$. In particular, these assumptions imply that, for each $p_i \in \mathcal{P}_i$ such that $p_i[j] \in \{0, p_i[j|h_i]\}$ for every $j \in \mathcal{N}_i$, we have

$$\pi_i[\sigma^*|h_i] \geq \pi_i[\sigma_i, \sigma_i^*|h_i],$$

where $\sigma_i = \sigma_i^*[h_i|p_i]$. By Lemma 35 there exists one such $p_i$ such that $\sigma_i$ is a local best response. Consequently, by 118 for every $a_i \in \mathcal{A}_i$ and $\sigma'_i = \sigma_i^*[h|a_i]$,

$$\pi_i[\sigma^*|h_i] \geq \pi_i[\sigma'_i, \sigma_i^*|h_i].$$

By Property 27, $(\sigma^*, \mu^*)$ is Sequentially Rational. $\square$
Proof of Lemma 38. If the assessment \((\sigma^*, \mu^*)\) is Preconsistent and Sequentially Rational and \(G\) is redundant, then for every \(i \in N\) and \(j \in N_i\), there exists \(k \in N \setminus \{i\}\), \(x \in PS[s, i]\), and \(x' \in PS[j, k]\), such that \(k \in x\) and \(i \notin x'\).

Proof. Suppose that there exists a player \(i \in N\), and a neighbor \(j \in N_i\) such that for every \(k \in N \setminus \{i\}\), \(x' \in PS[j, k]\), and \(x \in PS[s, i]\), we have \(k \notin x\) or \(i \notin x'\). Define \(D_j \subseteq N\) and \(D \subseteq N_i\) as:

\[
D_j = \{k \in N \setminus \{i\} | \exists x \in PS[j, k] : k \notin x\}.
D = \{j \in N_i | \forall k \in D_j \forall x \in PS[s, i] \forall x' \in PS[j, k] k \notin x \vee i \notin x'\}.
\]

(119)

Let \(RS_D = \bigcup_{j \in D} D_j\). By our assumptions, \(D\) is not empty. Define \(\sigma'_i = \sigma_i^*[h_i[p'_i]]\) for every \(h_i \in H\) such that:

- For every \(j \in D\), \(p'_i[j] = 0\).
- For every \(j \in N_i \setminus D\), \(p'_i[j] = p_i[j|\emptyset]\).

Let \(\sigma' = (\sigma'_i, \sigma^*_{-i})\).

Notice that, for every \(k \in N \setminus \{i\}\),

\[
CD_i[p'_i|h] = D.
CD_k[p'_i|h] = \emptyset.
\]

(120)

For every \(j \in D\) and \(k \in N \setminus (RS_D \cup \{i\})\), we have that \(d_k[i, j] = \infty\). Therefore, by Lemma 28 and by 120 for every \(l \in N_k, r \geq 0\),

\[
DS_k[l|h^*_k,r] = DS_k[l|\emptyset],
\]

(121)

where \(h^*_k,r \in hist[\emptyset, r|\sigma']\) and \(h^*_k,r \in hist[\emptyset, r|\sigma']\).

By Definition 24 and 121 we have that for every \(r \geq 0\),

\[
p_k[l|h^*_k,r] = p_k[l|\emptyset].
\]

(122)

Consequently, by 119 and 122 for every \(r > 0\), there exist \(p^* = \sigma^*[hist[\emptyset, r - 1|\sigma^*]]\) and \(p' = \sigma'[hist[\emptyset, r - 1|\sigma']]\) such that for every \(x \in PS[s, i]\) and \(k \in x \setminus \{i\}\) we have \(p^*_k = p'_k\).

It follows from Lemma 53 of Appendix A that for every \(r \geq 0\)

\[
q_i[\emptyset, r|\sigma^*] = q_i[\emptyset, r|\sigma'].
\]

(123)

By the definition of \(p'_i\), for every \(r \geq 0\),

\[
\bar{p}_i[\emptyset, r|\sigma'] < \bar{p}_i[\emptyset, r|\sigma^*].
\]

(124)

By the definition of \(u_i[h, r|\sigma]\), from 123 and 124 we have for every \(r \geq 0\)

\[
u_i[\emptyset, r|\sigma^*] < u_i[\emptyset, r|\sigma'].
\]

This implies that

\[
\sum_{r=0}^{\infty} \omega^r(u_i[\emptyset, r|\sigma^*] - u_i[\emptyset, r|\sigma']) < 0.
\]

Since \(h = \emptyset\) is the only history such that \(\mu_i[h|\emptyset] = 1\), by Theorem 37 \((\sigma^*, \mu^*)\) cannot be Sequentially Rational, which is a contradiction. \(\square\)
C.3 Redundancy may Reduce Effectiveness

Proof of Theorem 39. If \( G \) is redundant and there exist \( i \in \mathcal{N}, j \in \mathcal{N}_i, \) and \( k \in \mathcal{N}_{i-1} \) such that for every \( x \in \text{PS}[j, k] \) we have \( i \in x \), then Equality 17 holds.

\( \text{Proof.} \) Assume that there exist \( i \in \mathcal{N}, j \in \mathcal{N}_i, \) and \( k \in \mathcal{N}_{i-1} \) such that for every \( x \in \text{PS}[j, k] \) we have \( i \in x \). This implies by Definition 23 that

\[
d_k[i, j] = \infty. \tag{125}
\]

Define \( \sigma'_i = \sigma^*_i[h_i|p'_i] \) for every \( h_i \in \mathcal{H}_i \), such that \( p'_i[j] = 0 \) and \( p'_i[l] = p_i[l|h_i] \) for every \( l \in \mathcal{N}_i \setminus \{j\} \) and let \( \sigma' = (\sigma'_i, \sigma^*_i) \).

Notice that

\[
\bar{p}_i[0, r|\sigma'] = \bar{p}_i[0, r|\sigma^*] - p_i[j|0].
\]

By Theorem 37 if \( (\sigma^*, \mu^*) \) is Sequentially Rational, then, for every \( r \geq 0 \) and the empty history:

\[
\sum_{r=0}^{\infty} \omega_i(u_i[0, r|\sigma^*] - u_i[0, r|\sigma']) \geq 0
\]

\[
\sum_{r=0}^{\infty} \omega_i((1 - q_i[0, r|\sigma^*])(\beta_i - \gamma_i \bar{p}_i[0, r|\sigma^*]) -
(1 - q_i[0, r|\sigma'])((\beta_i - \gamma_i \bar{p}_i[0, r|\sigma'])) \geq 0 \tag{126}
\]

\[
\sum_{r=0}^{\infty} \omega_i((q_i[0, r|\sigma'] - q_i[0, r|\sigma^*])(\beta_i - \gamma_i \bar{p}_i[0, r|\sigma^*])
- (1 - q_i[0, r|\sigma'])\gamma_i p_i[j|0]) \geq 0.
\]

By Lemma 40, since \( G \) is connected from \( s \) and \( q_i \) is continuous in \([0,1]\), for every \( r \geq 0 \):

\[
\lim_{\sigma^* \to 1} q_i[0, r|\sigma^*] = q_i[1] = 0. \tag{127}
\]

Now, let \( p^* = \sigma^*[0], p' = (p'_i, p^*_i), h'_{k,r} \in \text{hist}[0, r|\sigma'], \) and \( h^*_{k,r} \in \text{hist}[0, r|\sigma^*] \). We have that for every \( l \in \mathcal{N} \setminus \{i\} \):

\[
\text{CD}_l[p'|0] = \emptyset, \tag{128}
\]

and

\[
\text{CD}_s[p'|0] = \{j\}. \tag{129}
\]

It follows immediately by Definition 23 and Lemma 28 that, for every \( k \in \mathcal{N} \) and \( l \in \mathcal{N}_k \) such that \( d_k[i, j] = \infty \), and \( r \geq 0 \),

\[
\text{DS}_k[l|h'_{k,r}] = \text{DS}_k[l|h^*_{k,r}],
\]

\[
p_k[l|h'_{k,r}] = p_k[l|h^*_{k,r}]. \tag{130}
\]

For any \( r \geq 0 \), let

\[
p'' = \lim_{\sigma^* \to 1} \sigma'[h'_i].
\]

Since \( G \) is redundant, there is a path \( x \in \text{PS}[s, k] \) such that \( i \notin x \). Furthermore, by 125, for every \( l \in x \setminus \{i\}, d_l[i, j] = \infty \). By 130, this implies that \( p''_l[a] = 1 \) for every \( a \in \mathcal{N}_l \).

Therefore, by Lemma 49 and, since \( G \) is connected from \( s \) and \( q_i \) is continuous in \([0,1]\), for every \( r \geq 0 \),

\[
\lim_{\sigma^* \to 1} q_i[0, r|\sigma'] = q_i[p''] = 0. \tag{131}
\]
By [126, 127, and 131]

\[ \lim_{\sigma^* \to 1} \sum_{r=0}^{\infty} \omega_i (u_i[0, r|\sigma^*] - u_i[0, r|\sigma']) = \]

\[ \lim_{\sigma^* \to 1} \sum_{r=0}^{\infty} \omega_i ((q_i[0, r|\sigma'] - q_i[0, r|\sigma^*])(\beta_i - \gamma_i\bar{p}_i[0, r|\sigma^*])
-(1 - q_i[0, r|\sigma'])(\gamma_i\bar{p}_i[0, r|\sigma^*])) = \]

\[ \lim_{\sigma^* \to 1} -(1 - q_i[0, r|\sigma'])(\gamma_i\bar{p}_i[0, r|\sigma^*])) = \]

\[ -\gamma_i < 0. \]  

(132)

Therefore, in the limit, the PDC Condition is never fulfilled for any values of \( \beta_i, \gamma_i, \) and \( \omega_i \in (0, 1) \), which implies by Theorem 37 that \((\sigma^*, \mu^*)\) is not Sequentially Rational for an arbitrarily large reliability and

\[ \lim_{\sigma^* \to 1} \psi[\sigma^*|\mu^*] = \emptyset. \]
C.4 Coordination is Desirable

Proof of Theorem 41. If the graph is redundant and $\sigma^*$ does not enforce coordination, then there is a definition of $\sigma^*$ such that:

$$\lim_{\sigma^* \to 1} \psi[\sigma^* | \mu^*] = \emptyset.$$  

Proof. In the aforementioned circumstances, define $\sigma^*$ such that for every $i \in \mathcal{N}$, $j \in \mathcal{N}_i$, and $h_i \in \mathcal{H}_i$,

$$p_i[j|h_i] > 0 \equiv p_i[j|h_i] = p_i[j|\emptyset].$$  

By the assumption that $\sigma^*$ does not enforce coordination, there exists $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$ such that for every $r > 0$ there is $k \in \mathcal{N}_i^{-1}$ for which

$$r \leq d_k[i,j] \lor r \geq d_k[i,j] + \tau[i,j|k,i] + 1.$$  

Fix $r$. Let $\sigma'_i = \sigma^*_i[0|p'_i]$ and $\sigma' = (\sigma'_i, \sigma^*_{-i})$, such that

- $p'_i[j] = 0$.
- For every $k \in \mathcal{N}_i \setminus \{j\}$, $p'_i[k] = p_i[k|\emptyset]$.

For $\sigma' = \sigma^*[0|\mu^*]$, we have

$$CD_i[p'|0] = \{j\},$$  

and for every $k \in \mathcal{N} \setminus \{i\}$

$$CD_k[p'|0] = \emptyset.$$  

By Lemma 28 and Definition 24 and by (135) and (136) we have for every $a \in \mathcal{N}$ and $b \in \mathcal{N}_a$, $h'_{a,r} \in \text{hist}[0, r|\sigma'^*]$, and $h'_{a,r} \in \text{hist}[0, r|\sigma^*]$:  

$$DS_a[b|h'_{a,r}] = DS_a[b|h^*_{a,r}] \cup \{(k_1, k_2, r - 1 - d_a[k_1, k_2] + v[k_1, k_2])|k_1, k_2 \in \mathcal{N} \land \ k_2 \in CD_{k_1}[p'|0] \land r \in \{d_a[k_1, k_2] + 1 \ldots d_a[k_1, k_2] + \tau[k_1, k_2|a, b] - v[k_1, k_2]\} \land \ v[k_1, k_2] = \min[d_a[k_1, k_2] - d_b[k_1, k_2], 0]\}$$  

$$= \emptyset \cup \{(i, j, r - 1 - d_a[i, j] + v[i, j])|v[i, j] = \min[d_a[i, j] - d_b[i, j], 0]\}.$$  

For $DS_k[i|h'_{k,r}]$, we have $v[i, j] = 0$, since $d_i[i, j] = 0$. Therefore, by (137)

$$p_a[b|h_{a,r}] = p_a[b|\emptyset].$$  

Also, for every $a \in \mathcal{N} \setminus \{N_i \cup \mathcal{N}^{-1}_i \cup \{k, i\}\}$ and $b \in \mathcal{N}_a$, by Definition 24 and the definition of $\sigma^*$ in this context, $a$ does not react to a defection of $i$ from $j$, which implies that:

$$DS_a[b|h'_{a,r}] = \{(i, j, r - 1 - d_a[i, j] + v[i, j])|v[i, j] = \min[d_a[i, j] - d_b[i, j], 0]\}. \quad (139)$$

Since the graph is redundant, there exists $x \in PS[s,k]$ such that, for every $a \in \{N_i \cup \mathcal{N}^{-1}_i \cup \{i\}\} \setminus \{k\}$, $a \notin x$. Thus, by (138) and (139) for

$$p' = \lim_{\sigma^* \to 1} \sigma'[\text{hist}[0, r|\sigma'^*]],$$

there exists a path $x \in PS[s,i]$ such that for every $a \in x \setminus \{i\}$ and $b \in \mathcal{N}_a$ we have

$$p'_a[b] = 1.$$
By Lemma 49, given that $q_i$ is continuous in $[0, 1]$, for every $r > 0$,
\[
\lim_{\sigma^* \to 1} (q_i[0, r|\sigma^*] - q_i[0, r|\sigma']) = 0.
\]
\[
\lim_{\sigma^* \to 1} (u_i[0, r|\sigma^*] - u_i[0, r|\sigma']) = \lim_{\sigma^* \to 1} (1 - q_i[0, r|\sigma^*]) (\beta_i - (\gamma_i \bar{p}_i[0, r|\sigma^*]) - (1 - q_i[0, r|\sigma']) (\beta_i - (\gamma_i \bar{p}_i[0, r|\sigma^*] + p_i[j|h^*_i])) < 0.
\]
Therefore, in the limit, the PDC Condition is never fulfilled for any values of $\beta_i$ and $\gamma_i$, which implies by Theorem 37 that $(\sigma^*, \mu^*)$ is never Sequentially Rational and:
\[
\lim_{\sigma^* \to 1} \psi[\sigma^*|\mu^*] = \emptyset.
\]
C.5 Impact of Delay

**Auxiliary Lemmas.** Lemma 56 shows that the utility obtained by \( i \) during the \( \tau \) stages that follow stage \( \bar{d}_i \) after any defection by \( i \) is null. This is because during that period \( i \) is necessarily punished by every in-neighbor.

**Lemma 56.** For every \( i \in \mathcal{N}, h \in \mathcal{H} \) and \( h_i \in h, D \subseteq \mathcal{N}_i[h_i] \), and \( r \in \{d_i + 1 \ldots \bar{d}_i + \tau\} \), \( u_i[h, r|\sigma'] = 0 \), where \( \sigma' = (\sigma^*_i[h_i|p'_i], \sigma^*_{-i}) \) and \( i \) drops every node from \( D \) in \( p'_i \).

**Proof.** By Definition 42 since \( d_i[i, j] = 0 \) for every \( i \), then, for every \( k \in \mathcal{N}_i^{-1}, \max[d_k[i, j], d_i[i, j]] = d_k[i, j] \) and:

\[
\tau[i, j|k, i] \leq \bar{d}_i.
\]

\[
\tau[i, j|k, i] = d_i - d_k[i, j] + \tau.
\]

Notice that for \( p' = \sigma'[h] \)

\[
CD_i[p'|h] = D,
\]

and for every \( j \in \mathcal{N} \setminus \{i\} \)

\[
CD_j[p'|h] = \emptyset.
\]

By Lemma 28 and by (141) for every \( k \in \mathcal{N}_i^{-1} \) and \( r \in \{d_i + 1 \ldots \bar{d}_i + \tau\} : \)

\[
\text{DS}_k[i|h_{k,r}^*] = \text{DS}_k[i|\bar{h}_{k,r}^*] \cup \{(k_1, k_2, r - 1 - d_k[k_1, k_2] + v[k_1, k_2])|k_1, k_2 \in \mathcal{N} \land \]
\[
k_2 \in CD_k[p'|h] \land r \in \{d_k[i, j] + 1 \ldots d_k[i, j] + \tau[i, j|k, i] - v[i, j]\} \land \]
\[
v[k_1, k_2] = \min[d_k[i, j] - d_i[i, j], v[i, j], 0]\}
\]

\[
\subseteq \text{DS}_k[i|h_{k,r}^*] \cup \{(i, j, r - 1 - d_k[i, j] + v[i, j])|j \in D \land \]
\[
r \in \{d_k[i, j] + 1 \ldots d_k[i, j] + \tau[i, j|k, i] - v[i, j]\} \land v[i, j] = 0\}
\]

where \( h_{k,r}^* \in \text{hist}[h, r|\sigma^*] \) and \( h'_{k,r} \in \text{hist}[h, r|\sigma'] \).

By (144) and Definition 24 it follows that, for every \( r \in \{\bar{d}_i + 1 \ldots \bar{d}_i + \tau\} \),

\[
p_k[i|h'_{k,r}] = 0,
\]

which by Lemma 51 leads to

\[
q_i[h, r|\sigma'] = 1.
\]

\[
u_i[h, r|\sigma'] = 0.
\]

This concludes the proof. \( \square \)
Lemma 57 proves that all punishments of a node $i$ are concluded after stage $\bar{d}_i + \tau$ that follows any defection of $i$.

**Lemma 57.** For every $i \in \mathcal{N}$, $h \in \mathcal{H}$ and $h_i \in h$, $D \subseteq N_i[h_i]$, and $r > \bar{d}_i + \tau$,

$$u_i[h, r|\sigma'] = u_i[h, r|\sigma^*],$$

where $\sigma' = (\sigma_i^*[h_i|\rho'_j], \sigma^*_{-i})$ and $i$ drops every node from $D$ in $\rho'_i$.

**Proof.** Notice that for $\rho' = \sigma'[h]$ and $j \in \mathcal{N} \setminus \{i\}$

$$CD_i[\rho'|h] = D.$$  \hspace{1cm} (146)

By Lemma 28 and by (146) for every $k \in \mathcal{N}$, $l \in N_k$, and $r > \bar{d}_i + \tau$,

$$DS_k[l|h^*_{k,r}] = DS_k[l|h^*_{k,r}] \cup \{ (k_1, k_2, r - 1 - d_{k_1}[k_1, k_2] + v[k_1, k_2]) | k_1, k_2 \in \mathcal{N} \setminus k_2 \in CD_k[\rho'|h] \cap r \in \{ d_{k_1}[k_1, k_2] + 1 \ldots d_{k_1}[k_1, k_2] + \tau[k_1, k_2] - v[k_1, k_2] \} \cup$$

$$v[k_1, k_2] = \min[d_{k_1}[k_1, k_2] - d_{k_1}[k_1, k_2], 0]\},$$

$$= DS_k[l|h^*_{k,r}] \cup \{ (i, j, r - 1 - d_{k_1}[i, j] + v[i, j]) | j \in D \setminus r \in \{ d_{k_1}[i, j] + 1 \ldots d_{k_1}[i, j] + \tau[i, j] - v[i, j] \} \cap$$

$$v[i, j] = \min[d_{k_1}[i, j] - d_{l_1}[i, j], 0]\},$$

$$= DS_k[l|h^*_{k,r}] \cup A$$

where $h^*_{k,r} \in \text{hist}[h, r|\sigma^*]$ and $h^*_{k,r} \in \text{hist}[h, r|\sigma']$.

The goal now is to show that for every $(i, j, r') \in A$, we have $r' < 0$. Fix any $j$.

By Definition 42 for every $k \in \mathcal{N}$ and $l \in N_k$ such that

$$g = \max[d_{k_1}[i, j], d_{l_1}[i, j]] > \bar{d}_i + \tau,$$  \hspace{1cm} (148)

we have

$$\tau[i, j] = 0.$$  \hspace{1cm} 

Thus, by (148) if $g = d_{k_1}[i, j]$, then we have $v[i, j] = 0$ and

$$d_{k_1}[i, j] + \tau[i, j] - v[i, j] = d_{k_1}[i, j].$$

Thus, there is no $r$ such that

$$r \in \{ d_{k_2}[i, j] + 1 \ldots d_{k_2}[i, j] + \tau[i, j] - v[i, j] \},$$

which implies that $A = \emptyset$ and the result is true.

If $g = d_{l_1}[i, j]$, then we have $v[i, j] = d_{l_1}[i, j] - d_{l_1}[i, j]$ and

$$d_{k_1}[i, j] + \tau[i, j] - v[i, j] = d_{l_1}[i, j].$$

$$r' = r - 1 - d_{k_1}[i, j] + v[i, j] = r - 1 - d_{l_1}[i, j].$$

Here, if $r' \geq 0$, then $d_{l_1}[i, j] \leq r - 1$ and there is no $r > \bar{d}_i + \tau$ such that

$$r \in \{ d_{k_2}[i, j] + 1 \ldots d_{k_2}[i, j] + \tau[i, j] - v[i, j] \},$$

which implies that $A = \emptyset$. So, either $A = \emptyset$ or $r' < 0$, which concludes the step for any $k \in \mathcal{N}$ and $l \in N_k$ that fulfill (148)

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Consider now that
\[
g = \max\{d_k[i, j], d_l[i, j]\} < \bar{d}_i + \tau,  \tag{149}\]
which results by Definition 42 in
\[
\tau[i, j|k, l] = \bar{d}_i + \tau - g.\]
Notice that
\[
d_k[i, j] - v[i, j] = g,
\]
which implies that
\[
d_k[i, j] + \tau[i, j|k, l] - v[i, j] = g + \bar{d}_i + \tau - g = \bar{d}_i + \tau.
\]
Thus, by [148] there is no \( r > \bar{d}_i + \tau \) such that
\[
r \in \{d_k[i, j] + 1 \ldots d_k[i, j] + \tau[i, j|k, l] - v[i, j]\},
\]
which implies that \( A = \emptyset \).

This allows us to conclude that for every \( k \in \mathcal{N}, \ l \in \mathcal{N}_k \), and \( r > \bar{d}_i + \tau \),
\[
\text{DS}_k[l|h'_{k,r}] = \text{DS}_k[l|h^*_{k,r}] \cup A,
\]
where for every \((i, j, r') \in A\) we have \( r' < 0 \). By Definition 24 \( i \) adds \((k_1, k_2, r'') \in \text{DS}_k[l|h^*_{k,r}]\) to \( K \) if and only if \( i \) adds \((k_1, k_2, r'')\) to \( \text{DS}_k[l|h'_{k,r}] \). Therefore,
\[
p_k[l|h'_{k,r}] = p_k[l|h^*_{k,r}],
q_i[h, r|\sigma^\prime] = q_i[h, r|\sigma^*],
u_i[h, r|\sigma^\prime] = u_i[h, r|\sigma^*].  \tag{150}\]

This concludes the proof. \[\square\]
Proof of Lemma 44. If $(\sigma^*, \mu^*)$ is Preconsistent, Assumption 43 holds, and Inequality 18 is fulfilled for every $i \in \mathcal{N}$, $h_i \in \mathcal{H}_i$, and $h \in \mathcal{H}$ such that $\mu^*_i[h|h_i] > 0$, then $(\sigma^*, \mu^*)$ is Sequentially Rational:

$$- \sum_{r=0}^{d_i} \omega_r^i((1 - q_i[h, r|\sigma^*])\gamma_i\bar{p}_i[h, r|\sigma^*] + \epsilon \beta_i) + \sum_{r=d_i+1}^{d_i+\tau} \omega_r^i u_i[h, r|\sigma^*] \geq 0.$$

Proof. Fix $i$, $h_i$, and $h$. Define $\sigma' = (\sigma^*_i[h_i|p_i^*], \sigma^*_h)$ for any $D \subseteq \mathcal{N}_i[h_i]$, where:

- For every $j \in D$, $p'_i[j] = 0$.
- For every $j \in \mathcal{N}_i \setminus D$, $p'_i[j] = p_i[j|h_i]$.

By Assumption 43, for every $r \in \{0 \ldots d_i\}$,

$$\bar{p}_i[h, r|\sigma'] \geq 0.$$  
$$q_i[h, r|\sigma^*] - q_i[h, r|\sigma'] \geq \epsilon.$$  
$$u_i[h, r|\sigma^*] - u_i[h, r|\sigma'] = (1 - q_i[h, r|\sigma^*])(\beta_i - \gamma_i\bar{p}_i[h, r|\sigma^*]) - (1 - q_i[h, r|\sigma^*])(\beta_i - \gamma_i\bar{p}_i[h, r|\sigma^*]) - (1 - q_i[h, r|\sigma^*])\beta_i \geq -(1 - q_i[h, r|\sigma^*])\beta_i - (q_i[h, r|\sigma^*] - q_i[h, r|\sigma^*])\beta_i \geq -(1 - q_i[h, r|\sigma^*])\beta_i - (1 - q_i[h, r|\sigma^*])\beta_i - \epsilon \beta_i.$$

By Lemma 56 for every $r \in \{d_i + 1 \ldots d_i + \tau\}$,

$$u_i[h, r|\sigma'] = 0.$$  

(151)

Finally, by Lemma 57 for every $r \geq d_i + \tau + 1$,

$$u_i[h, r|\sigma^*] = u_i[h, r|\sigma^*].$$  

(153)

It follows from (151), (152) and (153) that:

$$\sum_{r=0}^{\infty} \omega_r^i(u_i[h, r|\sigma^*] - u_i[h, r|\sigma^*]) \geq \sum_{r=0}^{d_i} \omega_r^i((1 - q_i[h, r|\sigma^*])\gamma_i\bar{p}_i[h, r|\sigma^*] + \epsilon \beta_i) + \sum_{r=d_i+1}^{d_i+\tau} \omega_r^i u_i[h, r|\sigma^*].$$

Therefore, if Inequality 18 is fulfilled for every $i \in \mathcal{N}$, $h_i \in \mathcal{H}_i$, and $h \in \mathcal{H}$ such that $\mu^*_i[h|h_i] > 0$, then the PDC Condition holds. Consequently, by Theorem 57 $(\sigma^*, \mu^*)$ is Sequentially Rational. □
Proof of Lemma 46. If \((\sigma^*, \mu^*)\) is Preconsistent, Assumptions 43 and 45 hold, and Inequality 19 is fulfilled for every \(h, i \in \mathcal{N}\), and \(r, r' \leq d_i + \tau\) such that \(q_i[h, r'|\sigma^*] < 1\), then there exist \(\omega_i \in (0, 1)\) for every \(i \in \mathcal{N}\) such that \((\sigma^*, \mu^*)\) is Sequentially Rational:

\[
\frac{\beta_i}{\gamma_i} > \bar{p}_i[h, r|\sigma^*] \frac{1}{A} + \bar{p}_i[h, r'|\sigma^*] \frac{1}{B - C},
\]

where

- \(A = 1 - \frac{\epsilon(d_i + 1)}{1 - q_i[h, r|\sigma^*]/\gamma_i} \tau\).
- \(B = \frac{\tau}{\epsilon}\).
- \(C = \frac{\epsilon(d_i + 1)}{1 - q_i[h, r'|\sigma^*]}\).

Proof. Consider the above assumptions and assume by contradiction that \((\sigma^*, \mu^*)\) is not Sequentially Rational.

The proof considers history \(h_1\) that minimizes the first component of Inequality 18 and \(h_2\) that minimizes the second component, for any history \(h \in \mathcal{H}\). More precisely, fix \(h\) and \(i\):

\[
\begin{align*}
    h_1 &= \arg\min_{\text{hist}[h, r|\sigma^*]} \gamma_i \bar{p}_i[h, r|\sigma^*] + \epsilon \beta_i, \\
    h_2 &= \arg\min_{\text{hist}[h, r|\sigma^*]} \gamma_i \bar{p}_i[h, r'|\sigma^*] + \epsilon \beta_i.
\end{align*}
\]

(154)

Let \(u_i^{h_2} = u_i[h_2, 0|\sigma^*]\). We can write:

\[
\begin{align*}
- \sum_{r=0}^{d_i} \omega_i^r ((1 - q_i[h_1, 0|\sigma^*]) &\gamma_i \bar{p}_i[h_1, 0|\sigma^*] + \epsilon \beta_i) + \sum_{r=0}^{d_i + \tau} \omega_i^r u_i^{h_2}[h, r|\sigma^*] \\
- \sum_{r=0}^{d_i} \omega_i^r ((1 - q_i[h_1, 0|\sigma^*]) &\gamma_i \bar{p}_i[h_1, 0|\sigma^*] + \epsilon \beta_i) + \sum_{r=1}^{d_i + \tau} \omega_i^r u_i^{h_2} \\
- a \frac{1 - \omega_i^{d_i + 1}}{1 - \omega_i} + \frac{\omega_i^{d_i + 1} - \omega_i^{d_i + \tau + 1}}{1 - \omega_i} u_i^{h_2} &
\end{align*}
\]

(155)

where

\[
a = (1 - q_i[h_1, 0|\sigma^*]) \gamma_i \bar{p}_i[h_1, 0|\sigma^*] + \epsilon \beta_i.
\]

We want to fulfill

\[
- a \frac{1 - \omega_i^{d_i + 1}}{1 - \omega_i} + \frac{\omega_i^{d_i + 1} - \omega_i^{d_i + \tau + 1}}{1 - \omega_i} u_i^{h_2} \geq 0 \\
- a + \omega_i^{d_i + 1} (u_i^{h_2} + a) - \omega_i^{d_i + \tau + 1} u_i^{h_2} \geq 0.
\]

(156)

Again, this inequality corresponds to a polynomial with degree \(\tau + 1\). If \(q_i[h_1, 0|\sigma^*] = 1\), then by our assumptions \(q_i[h_2, 0|\sigma^*] = 1, a = 0\), and the Inequality holds. Suppose then that

\[
q_i[h_1, 0|\sigma^*], q_i[h_2, 0|\sigma^*] < 1.
\]

The polynomial has a zero in \(\omega_i = 1\). If \(a = 0\), then the Inequality holds. Consider, then, that \(a > 0\). In these circumstances, a solution to (156) exists for \(\omega_i \in (0, 1)\) iff the polynomial is strictly concave and has another zero in \((0, 1)\). This is true iff the polynomial has a maximum in \((0, 1)\). The derivatives yield the following conditions:

1. \(\exists_{\omega_i \in (0, 1)} (d_i + 1)(u_i^{h_2} + a) - (\tau + d_i + 1) \omega_i u_i^{h_2} = 0 \Rightarrow \exists_{\omega_i \in (0, 1)} \omega_i = (d_i + 1)(u_i^{h_2} + a) / (\tau + d_i + 1) u_i^{h_2}.
2. \(- (\tau + d_i + 1) \tau u_i^{h_2} < 0 \Rightarrow u_i^{h_2} > 0.

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By our assumptions, Condition 1 implies that:

\[ u_i^h \tau > (\bar{d}_i + 1) \alpha \]

\[ (1 - q_i[h_2, 0|\sigma^*])\beta_i \tau > (1 - q_i[h_1, 0|\sigma^*])\gamma_i \beta_i[h_1, 0|\sigma^*] + \epsilon \beta_i \]

\[ ((1 - q_i[h_2, 0|\sigma^*])\tau - \epsilon(\bar{d}_i + 1))\beta_i > (1 - q_i[h_2, 0|\sigma^*])\gamma_i \beta_i[h_2, 0|\sigma^*] + (1 - q_i[h_1, 0|\sigma^*])\gamma_i \beta_i[h_1, 0|\sigma^*] \]

\[ ((1 - q_i[h_2, 0|\sigma^*])\tau - \epsilon(\bar{d}_i + 1)) > (1 - q_i[h_1, 0|\sigma^*])\beta_i[h_2, 0|\sigma^*] + (1 - q_i[h_1, 0|\sigma^*])\beta_i[h_1, 0|\sigma^*]. \]

Solving in order to the benefit-to-cost ratio,

\[ \frac{(1 - q_i[h_2, 0|\sigma^*])\beta_i[h_2, 0|\sigma^*]}{(1 - q_i[h_1, 0|\sigma^*])} = \frac{1}{1 - q_i[h_2, 0|\sigma^*]} = \frac{1}{\frac{1}{1 - q_i[h_2, 0|\sigma^*]}.} \]

where \( A = 1 - \frac{\epsilon(\bar{d}_i + 1)}{(1 - q_i[h_2, 0|\sigma^*])}. \)

Continuing, by Assumption \([15]\), it is true that

\[ \frac{(1 - q_i[h_2, 0|\sigma^*])}{(1 - q_i[h_1, 0|\sigma^*])} \geq \frac{1}{c}. \]

Therefore,

\[ \frac{(1 - q_i[h_1, 0|\sigma^*])\beta_i[h_1, 0|\sigma^*]}{(1 - q_i[h_2, 0|\sigma^*])} = \frac{1}{1 - q_i[h_1, 0|\sigma^*]} \frac{1}{\frac{\epsilon(\bar{d}_i + 1)}{(1 - q_i[h_2, 0|\sigma^*])}} \]

\[ \leq \frac{1}{\frac{\epsilon(\bar{d}_i + 1)}{(1 - q_i[h_2, 0|\sigma^*])}} \frac{\beta_i[h_1, 0|\sigma^*] \frac{1}{\frac{\epsilon(\bar{d}_i + 1)}{(1 - q_i[h_2, 0|\sigma^*])}}}{\frac{\epsilon(\bar{d}_i + 1)}{(1 - q_i[h_1, 0|\sigma^*])}}. \]

where:

\[ - \quad B = \frac{\tau}{c}. \]

\[ - \quad C = \frac{\epsilon(\bar{d}_i + 1)}{1 - q_i[h_1, 0|\sigma^*]}. \]

In summary, we have

\[ \frac{\beta_i[h_1, 0|\sigma^*]}{\gamma_i} > \beta_i[h_2, 0|\sigma^*] \frac{1}{1 - \frac{\beta_i[h_1, 0|\sigma^*]}{1 - \frac{\beta_i[h_2, 0|\sigma^*]}{B - C}} \Rightarrow \]

\[ \frac{\beta_i[h_2, 0|\sigma^*]}{\gamma_i} ((1 - q_i[h_2, 0|\sigma^*])\tau - \epsilon(\bar{d}_i + 1)) > (1 - q_i[h_2, 0|\sigma^*])\beta_i[h_2, 0|\sigma^*] + (1 - q_i[h_1, 0|\sigma^*])\beta_i[h_1, 0|\sigma^*]. \]

Consequently, if Inequality \([19]\) is true, then so is \([157]\). Furthermore, it also holds that

\[ \beta_i > \gamma_i \beta_i[h_1, 0|\sigma^*] \Rightarrow u_i^h > 0. \]

That is, Inequality \([19]\) implies Conditions 1 and 2 of the polynomial for any \( h \) and some \( \omega_i \in (0, 1) \), which by transitivity implies that \([156]\) is true. By \([155]\) Inequality \([18]\) is fulfilled for every history \( h \). Lemma \([44]\) allows us to conclude that \((\sigma^*, \mu^*)\) is Sequentially Rational. This is a contradiction, proving the result. \( \square \)
Proof of Theorem 47. If \((\sigma^*, \mu^*)\) is Preconsistent, Assumptions 43 and 45 hold for \(\epsilon \ll 1\), and \(\tau \geq d + 1\), then there exists a constant \(c > 0\) such that \(\psi(\sigma^*, \mu^*) \supseteq (v, \infty)\), where

\[
v = \max_{i \in N} \max_{h \in H} \bar{p}_i[h, 0|\sigma^*](1 + c).
\]

Proof. The idea is to simplify Inequality 19 for \(\epsilon \ll 1\) and \(\tau \geq d + 1\).

Recall that

\[
A = 1 - \frac{\epsilon (d_i + 1)}{(1 - q_i[h, r|\sigma^*]) \tau}.
\]

Thus, this yields

\[
\frac{1}{A} = \frac{(1 - q_i[h, r|\sigma^*]) \tau}{(1 - q_i[h, r|\sigma^*]) \tau (1 - \epsilon (d_i + 1))} \leq \frac{(1 - q_i[h, r|\sigma^*]) \tau}{(1 - q_i[h, r|\sigma^*]) \tau (1 - \epsilon)} \approx 1.
\]

Moreover, by Assumption 45

\[
\frac{1}{B - C} = \frac{(1 - q_i[h, r|\sigma^*]) \tau}{(1 - q_i[h, r|\sigma^*]) \tau - \epsilon (d_i + 1)} = \frac{(1 - q_i[h, r|\sigma^*]) \tau}{(1 - q_i[h, r + \tau|\sigma^*]) \tau - \epsilon (d_i + 1)} \leq \frac{(1 - q_i[h, r|\sigma^*]) \tau}{(1 - q_i[h, r + \tau|\sigma^*]) (d_i + 1)} \approx \frac{(1 - q_i[h, r|\sigma^*]) \tau}{(1 - q_i[h, r + \tau|\sigma^*]) (d_i + 1)} \leq \frac{c}{d_i + 1}.
\]

Thus, for any \(r, r' \geq 0\),

\[
\bar{p}_i[h, r|\sigma^*] \frac{1}{A} + \bar{p}_i[h, r'|\sigma^*] \frac{1}{B - C} \leq \bar{p}_i[h, r|\sigma^*] + \bar{p}_i[h, r'|\sigma^*] \frac{c}{d_i + 1}.
\]

Thus, there exists a constant \(c' = \frac{c}{d_i + 1}\) such that if for every \(i\) we have

\[
\beta_i \gamma_i \max_{h \in H} \bar{p}_i[h|\sigma^*](1 + c') \geq \bar{p}_i[h, r|\sigma^*] + \bar{p}_i[h, r'|\sigma^*] \frac{c}{d_i + 1},
\]

then Inequality 19 is fulfilled for every \(h, r, r', \) and \(h, r\), for some \(\omega_i \in (0, 1)\). By Lemma 46 this implies \((\sigma^*, \mu^*)\) is Sequentially Rational and the result follows. \(\square\)