A simple proof of tail–polynomial bounds on the diameter of polyhedra

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Abstract

Let $\Delta(d, n)$ denote the maximum diameter of a $d$-dimensional polyhedron with $n$ facets. In this paper, we propose a unified analysis of a recursive inequality about $\Delta(d, n)$ established by Kalai and Kleitman in 1992. This yields much simpler proofs of a tail–polynomial and tail–almost–linear bounds on $\Delta(d, n)$ which are recently discussed by Gallagher and Kim.

1 Introduction

The 1-skeleton of a polyhedron $P$ is an undirected graph $G = (V, E)$ which represents the vertex-vertex adjacency defined by the edges of $P$. More precisely, $V$ is a set of vertices of $P$, and $E$ is a set defined in such a way that $\{u, v\} \in E$ if and only if $\{(1 - \lambda)u + \lambda v : 0 \leq \lambda \leq 1\}$ forms a edge of $P$. The diameter $\delta(P)$ of $P$ is the diameter of its 1-skeleton. Formally, if we let $\rho_P(u, v)$ denote the shortest path length from $u$ to $v$, i.e., the number of edges required for joining $u$ and $v$ in $G$, then $\delta(P) = \max\{\rho_P(u, v) : u, v \in V\}$.

Let $\Delta(d, n)$ denote the maximum diameter of a $d$-dimensional polyhedron with $n$ facets. In 1957, Hirsch conjectured that $\Delta(d, n) \leq n - d$, which is disproved for unbounded polyhedra by Klee and Walkup [6], and for even bounded polyhedra, i.e., polytopes, by Santos [9]. An outstanding open problem in polyhedral combinatorics is to determine the behavior of $\Delta(d, n)$. In particular, the

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existence of a polynomial bound on $\Delta(d, n)$ is a major question. This question arose from its relation to the complexity of the simplex method; $\Delta(d, n)$ is a lower bound on the number of pivots required by the simplex method to solve a linear programming problem with $d$ variables and $n$ constraints.

In this paper, we focus on the behavior of $\Delta(d, n)$ when $n$ is sufficiently large. Recently, in [1], Gallagher and Kim showed that

**Theorem 1** (Tail–polynomial bound [1]). For $d \geq 3$ and $n \geq 2^d - 1$,

$$\Delta(d, n) \leq \frac{1}{16} \sqrt[3]{3 \log(n) - 5}.$$  

Their proof is based on two existing results:

- $\Delta(d, n) \leq \Delta(d - 1, n - 1) + 2\Delta(d, \lfloor \frac{n}{d} \rfloor) + 2$ for $\lfloor \frac{n}{d} \rfloor \geq d \geq 2$,
- $\Delta(d, n) \leq \Delta(d, n + 1) - 1$ for $n > d > 1$.

The former is due to Kalai and Kleitman [2], which has been used for proving bounds on $\Delta(d, n)$ for years, see, e.g., [2, 12, 10, 11, 1]. The latter is by Klee and Walkup [6]. It should be noted that Gallagher and Kim also show a tail–almost–linear bound $n^{1+\epsilon}$ for sufficiently large $n$.

We propose a unified analysis of the former recursive inequality, Kalai–Kleitman inequality. As a corollary, we readily obtain tail–polynomial and tail–almost–linear bounds which are analogous to but slightly better than those discussed by Gallagher and Kim. Our proofs are much simpler and use Kalai–Kleitman inequality only.

## 2 Main results

Using Kalai–Kleitman inequality, we show the following.

**Theorem 2.** Let $g(d, n) = \alpha^{d-3} (n-d)^k$ be a function with parameters $\alpha, k > 1$. If $\alpha$ and $k$ satisfy

$$\frac{1}{\alpha} + \frac{1}{2^{k-1}} \leq 1,$$

then $\Delta(d, n) \leq g(d, n)$ holds for $n \geq d \geq 3$.

**Proof.** See Section 3.

This bound $g(d, n)$ is motivated by Todd [12]. He derived a bound $(n-d)^{\log_2(d)}$ by refining the proof of the sub-exponential bound $n^{\log_2(d)+2}$ shown by Kalai and Kleitman. As with Todd bound, our bound $g(d, n)$ takes 0 when $n = d$, and hence coincides with the fact that $\Delta(d, d) = 0$.

**Corollary 1.** From Theorem 2 we obtain

a) for $d \geq 3$ and $n - d \geq 2^{d-3}$, $\Delta(d, n) \leq (n-d)^3$,
b) for \( d \geq 3 \) and \( n \geq 2^{2(d+2)} \),
\[
\Delta(d, n) \leq \frac{1}{16} \frac{n(n-d)^2}{\sqrt{3 \log(n) - 5}}.
\]

c) for \( \epsilon > 0 \), \( d \geq 3 \), and \( n - d \geq \left\lceil \frac{2^{d/2}}{2^{d-1} - 1} \right\rceil \), \( \Delta(d, n) \leq (n-d)^{1+\epsilon} \).

Proof. Setting \( \alpha, k := 2 \) in Theorem 2, we have \( \Delta(d, n) \leq 2^{d-3}(n-d)^k \), which implies a). For b) and c), see Section 4. \( \square \)

In particular, the bound in b) is analogous to but slightly better than that of Theorem 1.

3 Proofs of Theorem 2

We always assume that \( n \geq d \) since if otherwise the polyhedron has no vertex.

Now, the goal is to show that

\[
P(d): \Delta(d, n) \leq g(d, n) = \alpha^{d-3}(n-d)^k \quad \text{for } n \geq d
\]

is true for each \( d \geq 3 \). We prove this by induction on \( d \). First, we can observe that \( P(3) \) is true for any \( \alpha, k > 1 \). This is because we have \( \Delta(3, n) \leq n - 3 \), see, e.g., [3, 4, 5].

In what follows, assuming that \( P(d-1) \) is true, we show that \( P(d) \) is true. This is done by induction on \( n \) while \( d \) is fixed. When \( n < 2d \), it is known that \( \Delta(d, n) \leq \Delta(d-1, n-1) \), see, e.g., [12]. Hence, by the definition of \( g \) and the validity of \( P(d-1) \), we have

\[
\Delta(d, n) \leq \Delta(d-1, n-1) \leq g(d-1, n-1) \leq g(d, n)
\]

for \( n < 2d \). Then, let us consider the case when \( n \geq 2d \). In this case, we employ the following result.

Lemma 1 (Kalai–Kleitman inequality [2]). For \( \left\lceil \frac{n}{2} \right\rceil \geq d \geq 2 \),
\[
\Delta(d, n) \leq \Delta(d-1, n-1) + 2\Delta\left(d, \left\lceil \frac{n}{2} \right\rceil \right) + 2.
\]

Proof. See, e.g., [2, 12]. \( \square \)

In addition to the validity of \( P(d-1) \), now, we can assume that \( P(d) \) is true for
\[ \Delta(d, n) \leq \Delta(d - 1, n - 1) + 2\Delta \left( d, \frac{n}{2} \right) + 2 \]
\[ = g(d - 1, n - 1) + 2g \left( d, \frac{n}{2} \right) + 2 \]
\[ \leq \alpha^{d-4}(n-d)^k + 2\alpha^{d-3} \left( \frac{n}{2} - d \right)^k + 2 \]
\[ = \alpha^{d-3}(n-d)^k \left[ \frac{1}{\alpha} + \frac{1}{2^{k-1}} \left( 1 - \frac{d}{n-d} \right)^k + \frac{2}{\alpha^{d-3}(n-d)^k} \right] \]
\[ = g(d, n) \left[ \frac{1}{\alpha} + \frac{1}{2^{k-1}} \left( 1 - \frac{d}{n-d} \right)^k + \frac{2}{\alpha^{d-3}(n-d)^k} \right] \]
\[ \leq g(d, n) \left[ \frac{1}{\alpha} + \frac{1}{2^{k-1}} \left( 1 - \frac{d}{n-d} + \frac{2}{n-d} \right) \right] \]

where the last inequality follows from
\[ k > 1, \quad \frac{1}{\alpha} < 1, \quad d \geq 3, \quad \text{and} \quad \frac{2}{n-d} \leq 1. \]

Since \( d \geq 3 \), we have
\[ \Delta(d, n) \leq g(d, n) \left[ \frac{1}{\alpha} + \frac{1}{2^{k-1}} \right] \leq g(d, n). \]

4 Comparison with Theorem 1 and a tail–almost–linear bound

We add annotations to Corollary 1. First, let us observe b). Set \( \alpha, k := 2 \) as in a). Then, we have \( \Delta(d, n) \leq 2^{d-3}(n-d)^2 \). Hence, it suffices to identify a lower bound \( n_L \) such that \( n \geq n_L \) implies
\[ 2^{d-3} \leq \frac{1}{16 \sqrt{3 \log(n) - 5}}. \]

A sufficient condition for the inequality above is \( 2^{d+2} \leq n/\sqrt{\log(n)} \). As \( \sqrt{\log(n)} \leq \sqrt{n} \) for \( n \geq 1 \), the condition can be further simplified to \( 2^{d+2} \leq \sqrt{n} \). This means that we can set \( n_L = 2^{2(d+2)} \), which implies b).

Then, let us observe c). For a given \( \epsilon > 0 \), set \( k = 1 + \epsilon/2 \). It is easy to see that \( \alpha = \frac{2^{\epsilon/2}}{2^{\epsilon/2} - 1} \) satisfies the condition \( \frac{1}{\alpha} + \frac{1}{2^{k-1}} \leq 1 \). Hence, we have
\[ \Delta(d, n) \leq \left[ \frac{2^{\epsilon/2}}{2^{\epsilon/2} - 1} \right]^{d-3} (n-d)^{1+\epsilon/2}. \]

Then, for \( n \) satisfying \( (n-d)^{\frac{\epsilon}{2}} \geq \left[ \frac{2^{\epsilon/2}}{2^{\epsilon/2} - 1} \right]^{d-3} \), i.e., \( n - d \geq \left[ \frac{2^{\epsilon/2}}{2^{\epsilon/2} - 1} \right]^{\frac{2(d-3)}{\epsilon}} \), we have \( \Delta(d, n) \leq (n-d)^{1+\epsilon} \), which implies c).
5 Concluding remarks

We finally point out that Larman bound \( n^{2d-3} \) also derive tail–polynomial and tail–almost–linear bounds shown in Theorem \([1]\). Note that however, Larman bound does not imply our bounds with the line of \( n - d \).

**Observation 1.** Since \( \Delta(d, n) \leq n^{2d-3} \) for \( n \geq d \geq 3 \) \([8]\), we have

- for \( d \geq 3 \) and \( n \geq 2^{d-3} \), \( \Delta(d, n) \leq n^2 \),
- for \( \epsilon > 0 \), \( d \geq 3 \), and \( n \geq 2^{\frac{d}{d-3}} \), \( \Delta(d, n) \leq n^{1+\epsilon} \).

Larman bound was originally proven for only bounded polyhedra. Recently, in \([8]\), Labbé, Manneville, and Santos proved it for simplicial complexes, which include general polyhedra.

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