Shape Dynamics and AdS/CFT

Flavio Mercati
Departamento de Física Teórica, Universidad de Zaragoza, Zaragoza 50009, Spain and
School of Mathematical Sciences, University of Nottingham, Nottingham, UK
E-mail: flavio.mercati@gmail.com

Abstract. I present a semi-classical correspondence between three-dimensional euclidean
conformal field theory and the large-volume dynamics of gravity with positive cosmological
constant in 3+1 dimensions. The correspondence is based on a theory of gravity called Shape
Dynamics, whose solutions are a subset of those of General Relativity, namely those that can
be foliated with constant mean extrinsic curvature. Shape Dynamics makes it possible to solve
exactly for all the local degrees of freedom, while in GR this is possible only approximately,
within a derivative expansion, due to the non-linearity of the scalar constraint.

1. Introduction
Shape Dynamics (SD)\footnote{Shape Dynamics was created by J. Barbour, Ó Murchadha and collaborators during the last decade [1, 2, 3] as a way to make explicit the hidden Machian structure of GR. It was put into complete form by H. Gomes, S. Gryb and T. Koslowski [4, 5]} is a new theory of gravity, equivalent to General Relativity (GR), but
which admits a much smaller set of solutions, i.e. those that can be foliated with constant
mean extrinsic curvature leaves. Moreover, its symmetries are different from those of GR:
invariance under 4-dimensional refoliation is traded for local spatial conformal transformations
that preserve the 3-volume of every leaf. The new symmetry is of gauge type, in the sense
that, in the Hamiltonian formulation of the theory, it generates a constraint which is linear in
the momenta. This resolves the issues associated with the quadratic constraint that appears in
the Arnowitt–Deser–Misner (ADM) formulation of GR. The price for this is having a nonlocal
Hamiltonian constraint, which however constrains only a single global degree of freedom. Also,
the volume is a physical (non-gauge) degree of freedom, and its evolution is responsible for the
expansion of the universe.

In this contribution to the Loops 11 conference I consider a perturbative construction of SD
near the infinite-volume fixed point of its dynamics. This approximation scheme is found to be
particularly suitable for the study of the AdS/CFT correspondence, and brings out an important
advantage of SD over GR that makes it possible to prove the correspondence at the semiclassical
level. This could prove crucial for further developments of the AdS/CFT idea. This advantage
is the possibility to solve all local constraints of SD, while this was not possible in GR without
a derivative expansion, and prevented the correspondence from being rigorously proven. This
note is based on the results reported in [6].

2. AdS/CFT
The AdS/CFT correspondence was originally conjectured by Maldacena [7], and then made
precise by Witten [8]. Consider the partition function of quantum gravity on a background
manifold Σ with a boundary ∂Σ; it depends on the boundary values ˜φ_i of every field φ_i that propagates in the bulk

\[ \Psi(\tilde{\phi}_i) = \int_{\phi_i|_{\partial \Sigma} = \tilde{\phi}_i} \Pi_j D\phi_j \ldots e^{iS(\phi_i)}. \]  

The conjecture states that the dependence of Ψ on ˜φ_i equals the generating functional of correlation functions of a CFT living on ∂Σ, in which the boundary values of the fields ˜φ_i act as sources for the primary operators O_j of the CFT

\[ Z_{\text{CFT}}(\tilde{\phi}_i \ldots) = \left\langle \exp \int_{\partial \sigma} O_j \phi_j \right\rangle. \]

In a semi-classical, WKB-like approximation, the wavefunctional of quantum gravity is given by the on-shell value of the gravity action: Ψ ≃ α e^{iS_{\text{on-shell}}}, where S_{\text{on-shell}} can be calculated as the solution to the Hamilton-Jacobi equation for GR. This gives, on the CFT side, the generating functional of connected graphs of composite operators.

The correspondence was initially proposed for string theory in 5-dimensional AdS space and N = 4 Super-Yang-Mills (SYM) theory in 4 dimensions at the conformal point [7]. Later a number of similar correspondences in different dimensions and for different field theories have been proposed. String theory in AdS space is still incompletely understood, but at low energy the theory reduces to supergravity with an AdS ground state coupled to Kaluza–Klein modes. On the CFT side this corresponds to the large N and strong ’t Hooft coupling regime of SYM, which is relevant for certain non-perturbative aspects of QCD. So in the AdS/CFT context the question is how can one reconstruct the bulk spacetime out of CFT data and, conversely, given a bulk spacetime, what properties of the dual CFT can one read off? Both sides of the correspondence suffer from infinities: infrared divergences on the gravity side and ultraviolet divergences on the CFT side. Thus, one needs to renormalize the theory to obtain a correspondence between finite quantities. Skenderis [9] has shown how to systematically perform this renormalization, reconstructing perturbatively the bulk metric and a solution of the other fields out of CFT data.

In [6] we have investigated the potential of SD for going beyond the approximations made in these calculations, and we have shown that with SD one avoid a derivative expansion, and can calculate the counterterms needed to regularize the effective action, solving all of the constraints simultaneously. It follows that the wavefunctional of SD is invariant under diffeomorphisms and volume preserving conformal transformations. The correspondence thus implies that the CFT partition function is also invariant under diffeomorphisms and volume-preserving conformal transformations also away from the fixed point. These results have been obtained in 3+1 dimensions, and with positive cosmological constant (dS). The extension to arbitrary dimensions, signatures and values of the cosmological constant is under way.²

3. Shape Dynamics
Shape Dynamics is an Hamiltonian theory whose canonical variables are the three-metric g_{ab}, a scalar field φ, and their conjugate momentum densities, π_{ab} and π_φ. The Hamiltonian is [4, 5]

\[ H_{SD} = \int d^3x \left( \pi^{ab} L_\xi g_{ab} + \rho(x) \left( \pi - \langle \pi \rangle \sqrt{g} \right) \right) + N H_{gl}, \quad (3) \]

where \( \pi = \pi^{ab} g_{ab}, L_\xi \) is the Lie derivative with respect to the vector field ξ_a(x), which is a Lagrange multiplier, together with ρ(x) and N. R denotes the 3-dimensional Ricci scalar, and

² The 2+1 dimensional case with the topology of the 2-torus has already been considered by T. Koslowski and T. Budd, [10] and the full quantum wavefunctional of SD has been calculated exactly, as there are no local degrees of freedom in this case.
The cosmological constant, \( \mathcal{H}_{gl} \) is a global constraint which is defined through the solution of the following equations for \( \bar{\phi} \):

\[
\mathcal{H}_{gl} = \frac{e^{-12\bar{\phi}}}{g} \left( \pi^{ab} \pi_{ab} + \frac{\langle \pi \rangle^2}{3} \right) - \frac{\langle \pi \rangle^2}{6} + 2\Lambda - e^{-4\bar{\phi}} \left( R(g) + 8 e^{-\bar{\phi}} \nabla^2 e^{\bar{\phi}} \right)
\]

where \( \langle f \rangle = \int d^3 x \sqrt{g} \frac{f(x)}{V} \), and \( V = \int d^3 x \sqrt{g} \). \( \mathcal{H}_{gl} \) is a spatial constant, and to determine it one should solve Eq. (4) for \( \bar{\phi} \) simultaneously with Eq. (5), and then plug the solution into Eq. (4). As (4) is an elliptic equation, there are conditions to be satisfied in order for its solution to be unique [11, 12]. In the case we are considering, with no matter and a cosmological constant, these conditions amount to \( \Lambda \geq 0 \). The constraint \( \pi^{ab}(x) = 0 \), imposed by the variation of the Lagrangian \( \xi_a(x) \), is the diffeomorphism constraint of the ADM formulation of GR [13]. The constraint imposed by the variation of \( \rho(x) \) enforces invariance under volume-preserving conformal transformations (VPCT), which is the symmetry that this theory has in exchange for refoliation invariance. This symmetry is much more tractable than refoliation invariance, as it is generated by a linear constraint, and for the present proposal will prove an edge that this theory has over GR. It can be shown [4, 5] that the following (partial) gauge-fixing: \( \rho(x) = 0 \), reduces the theory to GR in ADM variables, but, due to the conformal constraint \( \pi = (\pi) \sqrt{g} \), only with a particular choice of the lapse function. Recall that \( \pi \) is the trace of the extrinsic curvature \( \xi_a(x) \), is the diffeomorphism constraint of the ADM formulation of GR [13]. The solutions of SD are equivalent only to the solutions of GR which are CMC foliable.

One can isolate the 3-volume \( V \) and its conjugate momentum \( P \) from the other degrees of freedom with a canonical transformation \( (g_{ab}, \pi^{ab}) \rightarrow (\tilde{V}, \tilde{g}_{ab}; P, \tilde{\pi}^{ab}) \) and fixing a reference volume \( V_0 = \int d^3 x \sqrt{\tilde{g}} \). This allows, for large volumes \( V \gg V_0 \), to solve Eq. (4) and (5) perturbatively, through an expansion in powers of \( V_0/V \). It is convenient to express the solution in the “Yamabe” gauge, defined as that gauge in which the scalar curvature is spatially constant \( R(x) = \tilde{R} = const \) [6]. The solution is:

\[
\mathcal{H}_{gl} = \left( 2\Lambda - \frac{3}{8} P^2 \right) - \left( \frac{V_0}{V} \right)^{2/3} \tilde{R} + \left( \frac{V_0}{V} \right)^2 \frac{\langle \pi^{ab} \bar{\pi}_{ab} \rangle}{\bar{g}} + \mathcal{O} \left( \left( \frac{V}{V_0} \right)^{-8/3} \right).
\]

We see that the infinite-volume limit is an attractor of the dynamics, and when approached all the degrees of freedom “freeze out” apart from \( V \), which keeps growing at constant speed [14].

The expression (6) combined with the conformal constraints allows us to make the connection with CFT. Using \( P = \frac{1}{2} \langle \pi \rangle \), solving \( \mathcal{H}_{gl} = 0 \) for \( \langle \pi \rangle \), and adding the result to \( D = 0 \), we obtain \( \pi/\sqrt{g} = \pm c \), where \( c \) is a spatial constant and commutes with all conformal constraints. Thus the VPCT constraint reduces to a conformal constraint. At infinite volume the theory has full conformal symmetry.

4. Hamilton–Jacobi equation and effective action

The Hamilton–Jacobi (HJ) method allows one to calculate, from the Hamiltonian, the value of the action evaluated on classical solutions of the theory. The HJ equation can be obtained from (6) by making the substitutions \( P \rightarrow \delta S/\delta V \), \( \pi^{ab} \rightarrow \delta S/\delta g_{ab} \), where \( S = S(g_{ab}, \alpha^{ab}) \) is the HJ functional; it depends on the metric \( g_{ab} \) and parametrically on the separation constants \( \alpha^{ab} \). These integration constants are symmetric tensor densities of weight 1. To obtain a complete integral of the HJ equation, \( S_0 \) can be taken in the form \( S_0 = \int d^3 x \bar{\alpha}^{ab} g_{ab} \). The linear constraints require \( \alpha^{ab} \) to be transverse and with covariantly constant trace. The leading
order HJ equation determines the value of the trace of $\alpha^{ab}$. This restricts the freely specifiable components of $\alpha^{ab}$ to be the freely specifiable momentum data in York’s conformal approach [15] to the initial value problem of GR.

For this paper, I will restrict to separation constants with vanishing transverse traceless part. These conditions are compatible with asymptotic (in time) dS space, which has maximally symmetric CMC slices.\(^3\) There are two solutions with opposite sign. The first three terms are

\[
S_\pm = \pm \sqrt{\frac{16}{3}} \Lambda V \mp \sqrt{\frac{3}{\Lambda}} \int d^3x \sqrt{\bar{g}} \bar{R} \sqrt[3]{\frac{V}{V_0}} \pm \left( \frac{3}{\Lambda} \right)^{3/2} \int d^3 x \sqrt{\bar{g}} \left( \frac{3}{8} \bar{R}^2 - \bar{R}^{ab} \bar{R}_{ab} \right) \frac{3}{V_0} + \ldots (7)
\]

Note that $S_{(0)}$ and $S_{(1)}$ are the only terms with positive dimension. The higher order terms can be obtained straightforwardly but become increasingly more involved because of the non-local terms appearing in the V expansion of $H_{gl}$. These solutions, at any given order, provide more information than the usual derivative expansion of $S$ in standard approaches [9, 16] to the AdS/CFT correspondence because they include non-local operators at each order in $\sqrt{V_0}/V$.

The local part of Eq. (7) reproduces the counterterm action reported in Sec. III B of [17] but, at each order, contains all of the nonlocal terms (“loop corrections”) that contribute at that order. It can be thought of as a resummation of these corrections.

The correspondence suggests an interesting possibility. The volume variable $V$ acts in the field theory as a regulator, and can be interpreted as the parameter of the RG flow out of the fixed point. On the other side, in a CMC foliation of an asymptotically dS spacetime $V$ grows monotonically with time, at an increasingly homogeneous speed. Therefore it represents a natural clock for the gravitational dynamics. This suggests that a 3-dimensional, euclidean CFT might generate the classical dynamics of gravity as the evolution under renormalization group flow. If such a CFT exists, its effective action should depend on the RG parameter $V$ through Eq. (7). From that equation, one can deduce perturbatively the classical dynamics of SD (and therefore GR) as a perturbation of an asymptotically dS spacetime. The fundamental theory would be a timeless euclidean quantum field theory, and time would emerge as evolution under RG flow. The classical dynamics of gravity would be a fundamentally quantum phenomenon.

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\(^3\) The generalization to the Euclidean AdS case is trivial, requiring the scalar constraint to be expressed as a radial, instead of a time, evolution operator.