h-Adic quantum vertex algebras associated with rational R-matrix in types B, C and D

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Abstract
We introduce the h-adic quantum vertex algebras associated with the rational R-matrix in types B, C and D, thus generalizing Etingof–Kazhdan’s construction in type A. Next, we construct the algebraically independent generators of the center of the h-adic quantum vertex algebra in type B at the critical level, as well as the families of central elements in types C and D. Finally, as an application, we obtain commutative subalgebras of the dual Yangian and the families of central elements of the appropriately completed double Yangian at the critical level, in types B, C and D.

Keywords Quantum vertex algebra · Double Yangian · Center at the critical level

Mathematics Subject Classification 17B37 · 17B69

Introduction
The notion of quantum vertex operator algebra, or, more briefly, quantum VOA, was introduced by Etingof and Kazhdan [3]. They constructed examples of quantum VOAs associated with the classical r-matrix on sl_N of rational, trigonometric and elliptic type. The theory of quantum VOAs was further generalized and developed by H.-S. Li; see, e.g., [16–18] and references therein. In particular, Li introduced a certain more general
notion of $h$-adic quantum vertex algebra. In comparison with quantum VOAs, $h$-adic quantum vertex algebras involve weaker constraints on the braiding operator $S$ which governs the $S$-locality, a certain quantum version of the locality property; see [17] for more details.

In this paper, following the approach in [3], we construct $h$-adic quantum vertex algebras associated with rational $R$-matrix in types $B$, $C$ and $D$. As in [3], the corresponding vertex operator map is expressed in the form of quantum currents, which were introduced by Reshetikhin and Semenov-Tian-Shansky [21]. Our construction relies on the Poincaré–Birkhoff–Witt theorem for the double Yangians of the corresponding types, which is due to Jing et al. [12]. In particular, the $h$-adic quantum vertex algebra structure is defined on the $h$-adically completed vacuum module over the double Yangian, thus resembling the type $A$ case; cf. [3,10].

Next, we study the center of the $h$-adic quantum vertex algebras in types $B$, $C$ and $D$ at the critical level. In type $B$, we construct an algebraically independent family of generators of the center. The classical limit of this family coincides with the generators of the famous Feigin–Frenkel center [4], i.e., of the center of the universal affine vertex algebra in type $B$ at the critical level, which were found by Molev [19]. Furthermore, we show that the center coincides with the $h$-adically completed polynomial algebra in infinitely many indeterminates, thus resembling the classical case. In types $C$ and $D$, we also obtain certain algebraically independent families of central elements. However, they do not exhaust the whole center, so they only generate $h$-adically completed polynomial subalgebras of the center. By taking their classical limits, we only reproduce some of the generators of the Feigin–Frenkel center constructed in [19]. Our construction of central elements relies on a particular case of the fusion procedure for the Brauer algebra, which is due to Isaev et al. [8,9]; see also [20]. It goes parallel with the corresponding construction [10], where the center of the Etingof–Kazhdan’s quantum VOA in type $A$ was determined, and which relies on the fusion procedure for the symmetric group originated in [13].

In the end, we show that the aforementioned families of central elements generate commutative subalgebras of the dual Yangians in types $B$, $C$ and $D$, as suggested by [20, Remark 11.2.5]. Moreover, by regarding their images, with respect to the vertex operator map, we find explicit formulae for the families of central elements of the appropriately completed double Yangians in types $B$, $C$ and $D$ at the critical level.

1 Preliminaries

1.1 Affine Lie algebras in types $B$, $C$ and $D$

Fix an integer $N \geq 2$. Let $\mathfrak{o}_N$ be the orthogonal and let $\mathfrak{sp}_N$ be the symplectic Lie algebra, where $N$ is even in the symplectic case. In order to consider orthogonal and symplectic case simultaneously we denote by $\mathfrak{g}_N$ any of the Lie algebras $\mathfrak{o}_N$ and $\mathfrak{sp}_N$. Introduce the scalars $\varepsilon_1, \ldots, \varepsilon_N$ by $\varepsilon_1 = \cdots = \varepsilon_N = 1$ if $\mathfrak{g}_N = \mathfrak{o}_N$ and by $\varepsilon_1 = \cdots = \varepsilon_{N/2} = 1, \varepsilon_{(N+2)/2} = \cdots = \varepsilon_N = -1$ if $\mathfrak{g}_N = \mathfrak{sp}_N$. For any $i = 1, \ldots, N$ set $i' = N - i + 1$. Define the matrix $G = (g_{ij})$ in $\text{End} \mathbb{C}^N$ by $g_{ij} = \delta_{ij} \varepsilon_i$ for all $i, j = 1, \ldots, N$. Clearly, $G$ is symmetric in the orthogonal and skew-symmetric in
the symplectic case. For any matrix $A = (a_{ij})$ in $\text{End} \mathbb{C}^N$, let $A' = GA'G^{-1}$, where $A'$ denotes the transposed matrix $A' = (a_{ji})$. We have $A' = (\epsilon_i \epsilon_j a_{j'i'})$. Set

$$
\sigma = \begin{cases} 
1, & \text{if } g_N = o_N, \\
2, & \text{if } g_N = sp_N.
\end{cases}
$$

Define the operators $P$ and $Q$ in $\text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N$ by

$$
P = \sum_{i,j=1}^N e_{ij} \otimes e_{ji} \quad \text{and} \quad Q = \sum_{i,j=1}^N \epsilon_i \epsilon_j e_{ij} \otimes e_{j'i'},
$$

where $e_{ij}$ are matrix units. One can easily verify that

$$
P'^1 = P'^2, \quad P'^i = P'^j, \quad Q'^1 = Q'^2, \quad Q'^i = Q'^j,
$$

where $^i$ and $'^i$ denote the transpositions $^i$ and $'^i$ applied on the $i$-th factor of the tensor product algebra $\text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N$ for $i = 1, 2$. Therefore, we will usually omit the index $i$ and denote the expressions in (1.1) by $P^i$, $P'^i$, $Q^i$, $Q'^i$, respectively. The operators $P$ and $Q$ possess the following properties:

$$
P' = Q, \quad Q' = P, \quad P^2 = 1, \quad Q^2 = NQ, \quad P Q = \begin{cases} 
Q, & \text{if } g_N = o_N, \\
-Q, & \text{if } g_N = sp_N.
\end{cases}
$$

Recall that the universal enveloping algebra $\text{U}(g_N)$ is the associative algebra generated by the elements $f_{ij}$, where $i, j = 1, \ldots, N$, subject to the defining relations

$$
F_1 F_2 - F_2 F_1 = (P - Q) F_2 - F_2 (P - Q) \quad \text{and} \quad F + F' = 0.
$$

The elements $F$ and $F'$ in $\text{End} \mathbb{C}^N \otimes \text{U}(g_N)$ are given by

$$
F = \sum_{i,j=1}^N e_{ij} \otimes f_{ij} \quad \text{and} \quad F' = \sum_{i,j=1}^N e_{ij} \otimes \epsilon_i \epsilon_j f_{j'i'},
$$

The copies of the matrix in the tensor product algebra $(\text{End} \mathbb{C}^N)^{\otimes 2} \otimes \text{U}(g_N)$ are indicated by the subscripts, so that in (1.3) we have

$$
F_1 = \sum_{i,j=1}^N e_{ij} \otimes 1 \otimes f_{ij} \quad \text{and} \quad F_2 = \sum_{i,j=1}^N 1 \otimes e_{ij} \otimes f_{ij}.
$$

Consider the affine Kac–Moody Lie algebra $\widehat{g}_N = g_N \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C$; see [14] for more details. Its universal enveloping algebra $\text{U}(\widehat{g}_N)$ is generated by the central
element $C$ and the elements $f_{ij}(r) = f_{ij} \otimes t^r$, where $i, j = 1, \ldots, N$ and $r \in \mathbb{Z}$, subject to the defining relations

$$F(r_1)F(s_2) - F(s_2)F(r_1) = (P - Q)F(r + s) - F(r + s)(P - Q) + \sigma r \delta_{r+s0}(P - Q)C,$$  \hspace{1cm} (1.4)

$$F(r) + F(r)' = 0. \hspace{1cm} (1.5)$$

The elements $F(r)$ and $F(r)'$ in $\text{End} \mathcal{C}^N \otimes \mathcal{U}(\mathcal{g}_N)$ are defined by

$$F(r) = \sum_{i,j=1}^{N} e_{ij} \otimes f_{ij}(r) \quad \text{and} \quad F(r)' = \sum_{i,j=1}^{N} e_{ij} \otimes \varepsilon_i \varepsilon_j f_{ij}'(r).$$

As in (1.3), the subscripts in (1.4) indicate the copy of $\text{End} \mathcal{C}^N$ in the tensor product algebra $(\text{End} \mathcal{C}^N)^{\otimes 2} \otimes \mathcal{U}(\mathcal{g}_N)$. Defining relations (1.4) can be equivalently written as

$$F_1(u)F_2(v) - F_2(v)F_1(u) = ((P - Q)F_2(v) - F_2(v)(P - Q)) \frac{1}{u} \delta \left( \frac{v}{u} \right)$$

$$- \sigma (P - Q)C \frac{1}{v} \frac{\partial}{\partial u} \delta \left( \frac{u}{v} \right), \hspace{1cm} (1.6)$$

where $\delta(z) = \sum_{r \in \mathbb{Z}} z^r \in \mathbb{C}[[z^{\pm 1}]]$ is the formal delta function and

$$F(u) = \sum_{r \in \mathbb{Z}} F(-r)u^{r-1} \in \text{End} \mathcal{C}^N \otimes \mathcal{U}(\mathcal{g}_N)[[u^{\pm 1}]].$$

Introduce the series

$$F^+(u) = \sum_{r \geq 1} F(-r)u^{r-1} \in \text{End} \mathcal{C}^N \otimes \mathcal{U}(t^{-1}\mathcal{g}_N[t^{-1}])[u],$$

$$F^-(u) = \sum_{r \leq 0} F(-r)u^{r-1} \in \text{End} \mathcal{C}^N \otimes u^{-1}\mathcal{U}(\mathcal{g}_N[t])[u^{-1}],$$

so that $F(u) = F^+(u) + F^-(u)$. Relation (1.6) implies

$$F^+_1(u)F^+_2(v) - F^+_2(v)F^+_1(u) = \frac{1}{u - v} \left( (P - Q) \left( F^+_2(v) - F^+_1(u) \right) \right)$$

$$- \left( F^+_2(v) - F^+_1(u) \right) (P - Q)$$

$$+ \frac{\sigma C}{(u - v)^2} (P - Q). \hspace{1cm} (1.7)$$

Throughout this paper, we employ the following expansion convention. For any variables $u_1, \ldots, u_k$ expressions of the form $(u_1 + \cdots + u_k)^r$ with $r < 0$ should be
expanded in nonnegative powers of the variables $u_2, \ldots, u_k$. For example, in (1.7) we have $k = 2, r = -2, -1$ and

$$(u - v)^r = \sum_{k \geq 0} \binom{r}{k} u^{r-k} v^k.$$ 

Recall that the vacuum module $V_c(\mathfrak{g}_N)$ of level $c \in \mathbb{C}$ for the algebra $U(\hat{\mathfrak{g}}_N)$ is isomorphic to the universal enveloping algebra $U(t^{-1} \mathfrak{g}_N[t^{-1}])$ as a complex vector space. The algebra $U(t^{-1} \mathfrak{g}_N[t^{-1}])$ is generated by the elements $f_{ij}(r) = f_{ij} \otimes t^r$, where $i, j = 1, \ldots, N$ and $r \in \mathbb{Z}_{<0}$, subject to the defining relations

$$F_1^+(u) F_2^+(v) - F_2^+(v) F_1^+(u) = \frac{1}{u - v} \left( (F_1^+(u) + F_2^+(v)) (P - Q) - (P - Q) (F_1^+(u) + F_2^+(v)) \right),$$ 

$$F^+(u) + F^+(u)' = 0.$$ 

(1.8) 

(1.9)

1.2 Rational $R$-matrix

Let $h$ be a formal parameter. Consider the rational $R$-matrix $R(u)$, as defined in [22], over the commutative ring $\mathbb{C}[[h]]$,

$$R(u) = 1 - \frac{hP}{u} + \frac{hQ}{u - h\kappa},$$ 

(1.10)

where

$$\kappa = \begin{cases} N/2 - 1, & \text{if } \mathfrak{g}_N = \mathfrak{sl}_N, \\ N/2 + 1, & \text{if } \mathfrak{g}_N = \mathfrak{sp}_N. \end{cases}$$

$R$-matrix (1.10) satisfies the Yang–Baxter equation

$$R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u).$$ 

(1.11)

Both sides of (1.11) are regarded as operators on the triple tensor product $(\mathbb{C}^N)^{\otimes 3}$, and the subscripts indicate the copies of $\mathbb{C}^N$ on which the $R$-matrices are applied. For example, we have $R_{23}(u) = 1 \otimes R(u)$.

By (1.1), we have $R(u)^{t_1} = R(u)^{t_2}$ and $R(u)^{t_1} = R(u)^{t_2}$, so we denote these transposed $R$-matrices by $R(u)^t$ and $R(u)'$, respectively. Using properties (1.2), one can easily prove

$$R(u) R(u + h\kappa)' = 1 - h^2 u^{-2} \quad \text{and} \quad R(u) R(-u) = 1 - h^2 u^{-2}.$$ 

(1.12)
Lemma 1.1 There exists a unique series $f(u) \in 1 + u^{-2} \mathbb{C}[[u^{-1}]]$ satisfying

$$f(u) f(u + \kappa) = \left(1 - u^{-2}\right)^{-1} \quad \text{and} \quad f(u) f(-u) = \left(1 - u^{-2}\right)^{-1}. \quad (1.13)$$

Proof Write the series $f$ as $f(u) = 1 + \sum_{r \geq 1} f_r u^{-r}$ for some scalars $f_r \in \mathbb{C}$. The first equality in (1.13) implies

$$\left(1 + \sum_{r \geq 1} f_r u^{-r}\right) \left(1 + \sum_{r \geq 1} \left(\sum_{s=1}^{r} \frac{\kappa^r}{r-s} \frac{f_s}{u^{-r}}\right) u^{-r}\right) = \sum_{r \geq 0} u^{-2r}. \quad (1.14)$$

One can easily see that the coefficients $f_r$ are uniquely determined by (1.14) and that the first few terms of the series $f(u)$ are found by

$$f(u) = 1 + \frac{1}{2} u^{-2} + \frac{\kappa}{2} u^{-3} + \frac{3}{8} u^{-4} + \cdots \quad (1.15)$$

It remains to prove that the series $f(u) = 1 + \sum_{r \geq 1} f_r u^{-r}$ satisfies the second equality in (1.13). Consider the series

$$F(u) = f(u) f(-u)(1 - u^{-2}) \in 1 + u^{-2} \mathbb{C}[[u^{-1}]].$$

Multiplying the first equality in (1.13) by $f(-u - \kappa) f(-u) = \left(1 - (u + \kappa)^{-2}\right)^{-1}$, we find $F(u) = F(u + \kappa)^{-1}$. This clearly implies $F(u) = F(u + 2\kappa)$. However, the only series in $1 + u^{-2} \mathbb{C}[[u^{-1}]]$ which satisfies that equality is the constant series 1, so we conclude that $f(u) f(-u)(1 - u^{-2}) = 1$, as required. \hfill \qed

As with the $R$-matrix $R(u)$, the normalized $R$-matrix $\overline{R}(u) = f(u/h) R(u)$ satisfies Yang–Baxter equation (1.11). Furthermore, by the first equalities in (1.12) and (1.13) it possesses the crossing symmetry properties

$$\overline{R}(u) \overline{R}(u + h\kappa)' = 1 \quad \text{and} \quad \overline{R}(u + h\kappa)' \overline{R}(u) = 1. \quad (1.16)$$

Using the ordered product notation, we rewrite (1.16) as

$$\overline{R}(u)'_{RL} \overline{R}(u + h\kappa) = 1 \quad \text{and} \quad \overline{R}(u)'_{LR} \overline{R}(u + h\kappa) = 1, \quad (1.17)$$

where the subscript RL (LR) in (1.17) indicates that the first tensor factor of $\overline{R}(u)'$ is applied from the right (left), while the second tensor factor of $\overline{R}(u)'$ is applied from the left (right). Note that the equations in (1.17) can be equivalently written as

$$\left((\overline{R}(u))' \overline{R}(u + h\kappa)\right)^{t_1} = 1 \quad \text{and} \quad \left((\overline{R}(u))' \overline{R}(u + h\kappa)\right)^{t_2} = 1.$$
By the second equalities in (1.12) and (1.13), the $R$-matrix $\overline{R}(u)$ possesses the unitarity property

$$\overline{R}(u)\overline{R}(-u) = \overline{R}(-u)\overline{R}(u) = 1.$$  \hspace{1cm} (1.18)

Hence, we also have

$$\overline{R}(-u)' = \overline{R}(u + h\kappa).$$  \hspace{1cm} (1.19)

### 1.3 Double Yangians in types $B$, $C$ and $D$

We follow [12] to introduce the double Yangian in types $B$, $C$ and $D$. In contrast to [12], where the (extended) Yangian double is defined as an associative algebra over $\mathbb{C}$, all algebras in this paper are defined over the commutative ring $\mathbb{C}[[h]]$. Such modification makes the structure of double Yangian suitable for the construction of $h$-adic quantum vertex algebras.

The extended Yangian double $DX(\mathfrak{g}_N)$ for $\mathfrak{g}_N$ is defined as the associative algebra over the ring $\mathbb{C}[[h]]$ generated by the central element $C$ and elements $t_{ij}^{(r)}$ and $t_{ij}^{(-r)}$, where $i, j = 1, \ldots, N$ and $r = 1, 2, \ldots, N$ subject to the defining relations

$$R(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u - v),$$  \hspace{1cm} (1.20)

$$R(u - v)T_1^+(u)T_2^+(v) = T_2^+(v)T_1^+(u)R(u - v),$$  \hspace{1cm} (1.21)

$$\overline{R}(u - v + h\sigma C/2)T_1(u)T_2^+(v) = T_2^+(v)T_1(u)\overline{R}(u - v - h\sigma C/2).$$  \hspace{1cm} (1.22)

The elements $T(u)$ and $T^+(u)$ in $\mathrm{End} \mathbb{C}^N \otimes DX(\mathfrak{g}_N)[[u^{\pm 1}]]$ are defined by

$$T(u) = \sum_{i,j=1}^N e_{ij} \otimes t_{ij}(u) \quad \text{and} \quad T^+(u) = \sum_{i,j=1}^N e_{ij} \otimes t_{ij}^+(u),$$

where the $e_{ij}$ denote the matrix units and the series $t_{ij}(u)$ and $t_{ij}^+(u)$ are given by

$$t_{ij}(u) = \delta_{ij} + h \sum_{r=1}^\infty t_{ij}^{(r)}u^{-r} \quad \text{and} \quad t_{ij}^+(u) = \delta_{ij} - h \sum_{r=1}^\infty t_{ij}^{(-r)}u^{r-1}.$$

The double Yangian $DY(\mathfrak{g}_N)$ for $\mathfrak{g}_N$ is defined as the quotient of the $h$-adically completed algebra $DX(\mathfrak{g}_N)[[h]]$ by the relations

$$T(u)T(u + h\kappa)' = 1 \quad \text{and} \quad T^+(u)T^+(u + h\kappa)' = 1.$$  \hspace{1cm} (1.23)

We now introduce certain subalgebras of the double Yangian; cf. [11,11] or [20, Chapter 11]. The Yangian $\mathcal{Y}(\mathfrak{g}_N)$ is defined as the subalgebra of $DY(\mathfrak{g}_N)$ generated by all elements $t_{ij}^{(r)}$, where $i, j = 1, \ldots, N$ and $r \in \mathbb{Z}$. By the Poincaré–Birkhoff–Witt theorem for the double Yangian [12], $\mathcal{Y}(\mathfrak{g}_N)$ is isomorphic to the associative algebra...
over \( \mathbb{C}[[ h ]] \) generated by the elements \( t_{ij}^{(r)} \), where \( i, j = 1, \ldots, N \) and \( r \in \mathbb{Z} \), subject to defining relations (1.20) and

\[
T(u)T(u + h \kappa)' = 1.
\]

The dual Yangian \( Y^+(g_N) \) is defined as the \( h \)-adically completed subalgebra of \( DY(g_N) \) generated by all elements \( t_{ij}^{(-r)} \), where \( i, j = 1, \ldots, N \) and \( r \in \mathbb{Z} \). Define the extended dual Yangian \( X^+(g_N) \) as the associative algebra over \( \mathbb{C}[[ h ]] \) generated by the elements \( t_{ij}^{(-r)} \), where \( i, j = 1, \ldots, N \) and \( r = 1, 2, \ldots \), subject to defining relations (1.21). By the Poincaré–Birkhoff–Witt theorem for the double Yangian [12], \( Y^+(g_N) \) is isomorphic to the quotient of the \( h \)-adically completed extended dual Yangian \( X^+(g_N)[[ h ]] \) by the relation

\[
T^+(u)T^+(u + h \kappa)' = 1. \tag{1.24}
\]

For any \( c \in \mathbb{C} \), define the double Yangian \( DY_c(g_N) \) at the level \( c \) as the quotient of \( DY(g_N) \) by the ideal generated by \( C - c \). The vacuum module \( \mathcal{V}_c(g_N) \) at the level \( c \) is defined as the \( h \)-adic completion of the quotient \( DY_c(g_N)/I \), where \( I \) denotes the \( h \)-adically completed left ideal in \( DY_c(g_N) \) generated by the elements \( t_{ij}^{(r)} \) with \( i, j = 1, \ldots, N \) and \( r \geq 1 \). By the Poincaré–Birkhoff–Witt theorem for the double Yangian [12], \( \mathcal{V}_c(g_N) \) and \( Y^+(g_N) \) are isomorphic as \( \mathbb{C}[[ h ]] \)-modules. Moreover, one can easily verify that the vacuum module \( \mathcal{V}_c(g_N) \) is topologically free.

As in [12] and [20, Section 11.2], set

\[
\text{deg } l = 0 \quad \text{and} \quad \text{deg } t_{ij}^{(-r)} = -r \quad \text{for all } i, j = 1, \ldots, N, \ r \geq 1 \quad (1.25)
\]

and denote by \( Y^{(-r)} \) the \( \mathbb{C}[[ h ]] \)-span of the elements of \( Y^+(g_N) \) whose degrees do not exceed \(-r\). We have the ascending filtration

\[
\cdots \subset Y^{(-r-1)} \subset Y^{(-r)} \subset \cdots \subset Y^{(-2)} \subset Y^{(-1)} \subset Y^{(0)} = Y^+(g_N). \tag{1.26}
\]

Consider the associated graded algebra

\[
\text{gr } Y^+(g_N) = \bigoplus_{r \geq 0} Y^{(-r)}/Y^{(-r-1)}.
\]

Observe that the \( \mathbb{C}[[ h ]] \)-algebra \( \text{gr } Y^+(g_N) \) is no longer \( h \)-adically complete.

By employing the notation

\[
t(u) = h^{-1} (T(u) - 1) \quad \text{and} \quad t^+(u) = h^{-1} (1 - T^+(u))
\]

we express defining relation (1.21) for the dual Yangian \( Y^+(g_N) \) as
\[ t_1^+(u)t_2^+(v) - t_2^+(v)t_1^+(u) = \frac{1}{u - v} ((t_1^+(u) + t_2^+(v))P - P(t_1^+(u) + t_2^+(v)) \]
\[ - \frac{1}{u - v - h\kappa}((t_1^+(u) + t_2^+(v))Q - Q(t_1^+(u) + t_2^+(v))) \]
\[ + \frac{h}{u - v} (Pt_2^+(v) - t_2^+(v)t_1^+(u))P \]
\[ - \frac{h}{u - v - h\kappa} (Qt_1^+(u)t_2^+(v) - t_2^+(v)t_1^+(u)Q). \]
(1.27)

Furthermore, defining relation (1.24) can be written as
\[ t^+(u) + t^+(u + h\kappa)' = h t^+(u)t^+(u + h\kappa)'. \]
(1.28)

Clearly, the relations obtained by taking the highest degree terms in (1.27) and (1.28), with respect to degree operator (1.25), coincide with relations (1.8) and (1.9), respectively. Moreover, from defining relations (1.22) we derive the following formula for the action of the Yangian generators on \( t^+(v)\mathbf{1} \in \mathcal{V}_c(g_N)[[v]] \):
\[ t_1(u)t_2^+(v)\mathbf{1} = \frac{1}{u - v} ((P - Q)t_2^+(v)\mathbf{1} - t_2^+(v)\mathbf{1})(P - Q) \]
\[ + \frac{\sigma c}{(u - v)^2} (P - Q) + \cdots, \]
(1.29)

where the ellipsis represents the summands of lower degree with respect to (1.25). The relations obtained by taking the highest degree terms in (1.29), with respect to (1.25), coincide with relations obtained by applying (1.7) to the vacuum vector \( \mathbf{1} \in \mathcal{V}_c(g_N) \).

Denote by \( \tilde{t}_{ij}^{(-r)} \) the image of the element \( t_{ij}^{(-r)} \) in the \((-r)\)-th component of \( \text{gr} \ Y^+(g_N) \). The next proposition is required in the proof of Theorem 2.8. It is a consequence of the Poincaré–Birkhoff–Witt theorem for the double Yangians of types \( B, C \) and \( D \); see [12].

**Proposition 1.2** (a) The assignments
\[ \tilde{t}_{ij}^{(-r)} \mapsto f_{ij} \otimes t^{-r} \]
(1.30)

with \( r \geq 1 \) and \( i, j = 1, \ldots, N \) define an isomorphism of \( \mathbb{C}[[h]] \)-algebras
\[ \text{gr} \ Y^+(g_N) \cong U(t^{-1}g_N[t^{-1}]) \otimes_{\mathbb{C}} \mathbb{C}[[h]]. \]

(b) For any \( c \in \mathbb{C} \) and integer \( n \geq 1 \), the image of
\[ t_0(u)\left(t_1^+(v_1) \ldots t_n^+(v_n)\right) \in (\text{End} \mathbb{C}^N)^{(n+1)} \otimes \mathcal{V}_c(g_N)[[u^{-1}, v_1, \ldots, v_n]] \] (1.31)
with respect to map (1.30) is equal to
\[
F_0^{-}(u) \left( F_1^{+}(v_1) \ldots F_n^{+}(v_n) \right) \in (\text{End } C^N)^{\otimes (n+1)} \otimes U(t^{-1} g_N [t^{-1}])[u^{-1}, v_1, \ldots, v_n].
\]

**Remark 1.3** It is worth noting that defining relation (1.22) differs from the corresponding relation in [12] because we use the normalized \( R \)-matrix \( \tilde{R}(u) \) instead of \( R(u) \). However, this does not affect the action of map (1.30) on elements (1.31). More specifically, due to the form of the normalizing function (1.15), this produces only additional terms of the lower degree with respect to (1.25), which are annihilated by map (1.30).

Let \( m \geq 1 \) be an arbitrary integer, \( v = (v_1, \ldots, v_m) \) an \( m \)-tuple of variables and \( z \) a single variable. Label the tensor factors of \((\text{End } C^N)^{\otimes (n+m)}\) as follows,

\[
\underbrace{(\text{End } C^N)^{\otimes n}}_{1} \otimes \underbrace{(\text{End } C^N)^{\otimes m}}_{2}.
\] (1.32)

Introduce the functions with values in (1.32) by

\[
\tilde{R}_{nm}^{12}(u|v|z) = \prod_{i=1, \ldots, n} \prod_{j=n+1, \ldots, n+m} \tilde{R}_{ij}(z + u_i - v_{j-n}),
\] (1.33)

\[
\tilde{\tilde{R}}_{nm}^{12}(u|v|z) = \prod_{i=1, \ldots, n} \prod_{j=n+1, \ldots, n+m} \tilde{R}_{ij}(z + u_i - v_{j-n}).
\] (1.34)

In (1.33) and (1.34), the superscripts 1 and 2 indicate the tensor factors in (1.32), while the arrows indicate the order of the factors. For example, if \( n = 3, m = 2 \) and \( \tilde{R}_{ij} = \tilde{R}_{ij}(z + u_i - v_{j-n}) \), we have

\[
\tilde{R}_{32}^{12}(u|v|z) = \tilde{R}_{15}^{12} \tilde{R}_{14}^{12} \tilde{R}_{25}^{12} \tilde{R}_{24}^{12} \tilde{R}_{35}^{12} \tilde{R}_{34}^{12} \text{ and } \tilde{\tilde{R}}_{32}^{12}(u|v|z) = \tilde{R}_{34}^{12} \tilde{R}_{35}^{12} \tilde{R}_{24}^{12} \tilde{R}_{25}^{12} \tilde{R}_{14}^{12} \tilde{R}_{15}^{12}.
\] (1.35)

Observe that, due to the expansion convention introduced in Sect. 1.1, the expressions of the form \((z + u_i - v_{j-n})^r\) with \( r < 0 \) are expanded in negative powers of the variable \( z \), so that (1.33) and (1.34) contain only nonnegative powers of the variables \( u_1, \ldots, u_n \) and \( v_1, \ldots, v_m \). In order to simplify the notation, we write

\[
\tilde{R}_{nm}^{12}(u|v) = \tilde{R}_{nm}^{12}(u|v|0) \text{ and } \tilde{\tilde{R}}_{nm}^{12}(u|v) = \tilde{\tilde{R}}_{nm}^{12}(u|v|0),
\] (1.36)

where, due to the aforementioned expansion convention, expressions of the form \((u_i - v_{j-n})^r\) with \( r < 0 \) are expanded in negative powers of the variable \( u_i \), so that they contain only nonnegative powers of \( v_1, \ldots, v_m \). The functions \( R_{nm}^{12}(u|v|z) \), \( R_{nm}^{12}(u|v) \) and \( \tilde{R}_{nm}^{12}(u|v) \) corresponding to \( R \)-matrix (1.10) can be defined analogously.
We will often combine the ordered product notation, as introduced in Sect. 1.2, with the products of the form as in (1.33) and (1.34). For example, in the expressions such as (1.22) one obtains the more general form of the relations.

As in [3], by employing Yang–Baxter equation (1.11) and defining relations (1.20)–(1.22), we have

\[
R^{12}_{nm}(u|v|z) \cdot X, \quad \text{where} \quad X \in (\text{End } \mathbb{C}^N)^{\otimes (n+m)},
\]

the tensor factors 1, \ldots, \(n\) of \(R^{12}_{nm}(u|v|z)\) are applied from the left, while the tensor factors \(n+1, \ldots, n+m\) are applied from the right, e.g., for \(n = 3\) and \(m = 2\) we have

\[
\overline{R}^{12}_{32}(u|v|z) \cdot X = \overline{R}^{35}_{LR} \left( \overline{R}^{34}_{LR} \left( \overline{R}^{25}_{LR} \left( \overline{R}^{24}_{LR} \left( \overline{R}^{15}_{LR} \left( \overline{R}^{14}_{LR} \cdot X \right) \right) \right) \right) \right).
\]

For any integer \(n \geq 1\), the variables \(u = (u_1, \ldots, u_n)\) and the single variable \(z\) define

\[
T_{[n]}(u|z) = T_1(z + u_1) \cdots T_n(z + u_n) \quad \text{and} \quad T_{[n]}^+(u|z) = T_1^+(z + u_1) \cdots T_n^+(z + u_n).
\]

In particular, we write

\[
T_{[n]}(u) = T_1(u_1) \cdots T_n(u_n) \quad \text{and} \quad T_{[n]}^+(u) = T_1^+(u_1) \cdots T_n^+(u_n).
\]

We regard the coefficients of the series in (1.37) as operators on the vacuum module \(\mathcal{V}_c(\mathfrak{g}_N)\). Hence, the series \(T_{[n]}^+(u|z)\) belongs to \((\text{End } \mathbb{C}^N)^{\otimes n} \otimes \text{End } \mathcal{V}_c(\mathfrak{g}_N)[[u_1, \ldots, u_n, z]]\). Moreover, due to the expansion convention introduced in Sect. 1.1 and relations (1.22), the series \(T_{[n]}(u|z)\) belongs to \((\text{End } \mathbb{C}^N)^{\otimes n} \otimes \text{Hom } (\mathcal{V}_c(\mathfrak{g}_N), \mathcal{V}_c(\mathfrak{g}_N)[z^{-1}])[[u_1, \ldots, u_n, h]]\). As before, we use the arrows to indicate the opposite order of factors. For example,

\[
\overline{T}_{[n]}(u|z) = T_n(z + u_n) \cdots T_1(z + u_1) \quad \text{and} \quad \overline{T}_{[n]}^+(u|z) = T_n^+(z + u_n) \cdots T_1^+(z + u_1).
\]

As in [3], by employing Yang–Baxter equation (1.11) and defining relations (1.20)–(1.22) one obtains the more general form of the RTT relations.

**Proposition 1.4** For any \(c \in \mathbb{C}\) and integers \(n, m \geq 1\), the equalities

\[
R^{12}_{nm}(u|v|z - w) \cdot T_{[n]}^{13}(u|z)T_{[m]}^{23}(v|w) = T_{[m]}^{23}(v|w)T_{[n]}^{13}(u|z)R^{12}_{nm}(u|v|z - w),
\]

\[
R^{12}_{nm}(u|v|z - w) \cdot T_{[n]}^{+13}(u|z)T_{[m]}^{23}(v|w) = T_{[m]}^{23}(v|w)T_{[n]}^{+13}(u|z)R^{12}_{nm}(u|v|z - w),
\]

\[
\overline{R}^{12}_{nm}(u|v|z - w - h\sigma c/2) \cdot T_{[n]}^{13}(u|z)T_{[m]}^{+23}(v|w) = T_{[m]}^{+23}(v|w)T_{[n]}^{13}(u|z)\overline{R}^{12}_{nm}(u|v|z - w - h\sigma c/2)
\]

hold for operators on \((\text{End } \mathbb{C}^N)^{\otimes n} \otimes (\text{End } \mathbb{C}^N)^{\otimes m} \otimes \mathcal{V}_c(\mathfrak{g}_N)\).
2 $h$-Adic quantum vertex algebras in types $B$, $C$ and $D$

2.1 Vacuum module as an $h$-adic quantum vertex algebra

The following notion of $h$-adic quantum vertex algebra was introduced by Li [17]. As explained therein, it presents a slight generalization of the notion of quantum VOA, which was introduced by Etingof and Kazhdan [3]. From now on, the tensor products are understood as $h$-adically completed.

**Definition 2.1** An $h$-adic quantum vertex algebra is a quadruple $(V, Y, 1, S)$ which satisfies the following axioms:

1. $V$ is a topologically free $\mathbb{C}[[h]]$-module.
2. $Y$ is the vertex operator map, a $\mathbb{C}[[h]]$-module map

\[
Y : V \otimes V \to V((z))[[h]]
\]

\[
u \otimes v \mapsto Y(z)(u \otimes v) = Y(u, z)v = \sum_{r \in \mathbb{Z}} u_r v z^{-r-1}
\]

which satisfies the weak associativity: for any $u, v, w \in V$ and $n \in \mathbb{Z}_{\geq 0}$ there exists $r \in \mathbb{Z}_{\geq 0}$ such that

\[
(z_0 + z_2)^r Y(u, z_0 + z_2)Y(v, z_2)w = -(z_0 + z_2)^r Y(Y(u, z_0)v, z_2)w \in h^n V[[z_0^{\pm 1}, z_2^{\pm 1}]].
\] (2.1)

3. $1$ is the vacuum vector, an element of $V$ which satisfies

\[
Y(1, z)v = v \quad \text{for all } v \in V,
\] (2.2)

and for any $v \in V$ the series $Y(v, z)1$ is a Taylor series in $z$ with the property

\[
\lim_{z \to 0} Y(v, z)1 = v.
\] (2.3)

4. $S = S(z)$ is a $\mathbb{C}[[h]]$-module map $V \otimes V \to V \otimes V \otimes \mathbb{C}((z))$ which satisfies the shift condition

\[
[D \otimes 1, S(z)] = -\frac{d}{dz} S(z) \quad \text{for } D \in \text{End } V \text{ defined by } Dv = v_{-2}1,
\] (2.4)

the Yang–Baxter equation

\[
S_{12}(z_1)S_{13}(z_1 + z_2)S_{23}(z_2) = S_{23}(z_2)S_{13}(z_1 + z_2)S_{12}(z_1),
\] (2.5)

the unitarity condition

\[
S_{21}(z) = S^{-1}(-z),
\] (2.6)
the $S$-locality: for any $u, v \in V$ and $n \in \mathbb{Z}_{\geq 0}$ there exists $r \in \mathbb{Z}_{\geq 0}$ such that

$$(z_1 - z_2)^r Y(z_1)(1 \otimes Y(z_2)) (S(z_1 - z_2)(u \otimes v) \otimes w) - (z_1 - z_2)^r Y(z_2)(1 \otimes Y(z_1)) (v \otimes u \otimes w) \in h^n V[[z_1^{\pm 1}, z_2^{\pm 1}]] \quad \text{for all } w \in V, \quad (2.7)$$

and the hexagon identity:

$$S(z_1)(Y(z_2) \otimes 1) = (Y(z_2) \otimes 1) S_{23}(z_1) S_{13}(z_1 + z_2). \quad (2.8)$$

Denote by $\mathbf{1}$ the image of the unit $1 \in DY_c(g_N)$ in the vacuum module $V_c(g_N)$. The next theorem is a generalization of Etingof–Kazhdan’s construction of quantum VOAs in type $A$; see [3, Theorem 2.3].

**Theorem 2.2** For any $c \in \mathbb{C}$, there exists a unique structure of $h$-adic quantum vertex algebra on $V_c(g_N)$, where $g_N = o_N, sp_N$, such that the vacuum vector is $\mathbf{1} \in V_c(g_N)$, the vertex operator map is defined by

$$Y(T_{[n]}^+(u) \mathbf{1}, z) = T_{[n]}^+(u|z) T_{[n]}^-(u|z + h\sigma c/2)^{-1} \quad (2.9)$$

and the map $S(z)$ is defined by

$$S(z) \left( \overline{R}_{nm}^{12}(u|v|z)^{-1} T_{[n]}^+(v) \overline{R}_{nm}^{12}(u|v|z - h\sigma c) T_{[n]}^+(u) \mathbf{1} \right) = T_{[m]}^+(u) \overline{R}_{nm}^{12}(u|v|z + h\sigma c)^{-1} T_{[m]}^+(v) \overline{R}_{nm}^{12}(u|v|z) \mathbf{1} \quad (2.10)$$

for operators on $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes (\text{End } \mathbb{C}^N)^{\otimes m} \otimes V_c(g_N) \otimes V_c(g_N)$.

**Proof** Recall that the $\mathbb{C}[[h]]$-module $V_c(g_N)$ is topologically free. Let us prove that the map $Y(z)$ is well defined by (2.9). As the coefficients of matrix entries of all $T_{[n]}^+(u) \mathbf{1}$ span an $h$-adically dense subset of $V_c(g_N)$, it is sufficient to show that $Y(z)$ maps the ideal of defining relations (1.21) and (1.24) for the dual Yangian $Y^+(g_N)$ to itself. We only verify this for defining relations (1.24). As for (1.21), this follows by employing Proposition 1.4, Yang–Baxter equation (1.11) and arguing as in the proof of [3, Lemma 2.1].

For any nonnegative integers $n, m$ and the variables $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_m)$ and $w$, we apply $Y(z)$ on the expression

$$a := T_{[n]}^+(u) T_{[1]}^+(w) T_{[1]}^+(w + h\kappa)^{-1} T_{[m]}^+(v) \mathbf{1}.$$ 

The coefficients of $a$ belong to the tensor product

$$\left( \text{End } \mathbb{C}^N \right)^{\otimes n} \otimes \text{End } \mathbb{C}^N \otimes \left( \text{End } \mathbb{C}^N \right)^{\otimes m} \otimes V_c(g_N) \otimes V_c(g_N).$$
and its superscripts indicate the tensor copies as indicated above. Using (2.9), we get
\[ Y(a, z) = T^{+14}_{[n]}(u|z) T^{+24}_{[1]}(w|z) A T^{+14}_{[n]}(u|z + h\sigma c/2)^{-1}, \] where
\[ A = B^{24}_{LR} \left( T^{+24}_{[1]}(w|z + h\kappa)' T^{+34}_{[m]}(v|z) T^{+34}_{[m]}(v|z + h\sigma c/2)^{-1} \right), \]
\[ B = T^{+14}_{[1]}(z + w + h\sigma c/2)^{-1}_{LR} \left( T^{+14}_{[1]}(z + w + h\kappa + h\sigma c/2)^{-1} \right)'. \]

Note that
\[ B' = T^{+14}_{[1]}(z + w + h\kappa + h\sigma c/2)^{-1} \left( T^{+14}_{[1]}(z + w + h\sigma c/2)^{-1} \right)' = 1 \]
so, consequently, \( B = 1 \). Indeed, the second equality in (2.13) follows directly from defining relations (1.20). Therefore, by combining (2.11) and (2.12) we find
\[ Y(a, z) = T^{+14}_{[n]}(u|z) T^{+24}_{[1]}(w|z) T^{+24}_{[1]}(w|z + h\kappa)' T^{+34}_{[m]}(v|z) \]
\[ \cdot T^{+34}_{[m]}(v|z + h\sigma c/2)^{-1} T^{+14}_{[n]}(u|z + h\sigma c/2)^{-1}. \]

Finally, due to (1.24) we have \( T^{+24}_{[1]}(w|z) T^{+24}_{[1]}(w|z + h\kappa)' = 1 \), so that
\[ Y(a, z) = T^{+14}_{[n]}(u|z) T^{+34}_{[m]}(v|z) T^{+34}_{[m]}(v|z + h\sigma c/2)^{-1} T^{+14}_{[n]}(u|z + h\sigma c/2)^{-1}. \]

It is now clear that the expression \( Y(a, z) \) coincides with the image of \( T^{+14}_{[n]}(u) T^{+34}_{[m]}(v) \) with respect to (2.9), so we conclude that the map \( Y(z) \) is well defined.

Next, we prove that the map \( S(z) \) is well defined. Due to crossing symmetry property (1.17), we express (2.10) as
\[ S(z) \left( T^{+13}_{[n]}(u) T^{+24}_{[m]}(v)(1 \otimes 1) \right) = \frac{\nabla_{12} R_{nm}(u|v|z - h\kappa - h\sigma c) \cdot (\nabla_{12} R_{nm}(u|v|z) T^{+13}_{[n]}(u)}{\nabla_{12} R_{nm}(u|v|z + h\sigma c)^{-1} T^{+24}_{[m]}(v) \nabla_{12} R_{nm}(u|v|z) (1 \otimes 1)}, \]

where, in consistency with notation introduced in (1.33) and (1.34), we write
\[ \nabla_{12} R_{nm}(u|v|z) = \prod_{i=1,...,n}^{\rightarrow} \prod_{j=n+1,...,n+m}^{\leftarrow} R_{ij}(z + u_i - v_j - n)', \]
\[ \nabla_{12} R_{nm}(u|v|z) = \prod_{i=1,...,n}^{\rightarrow} \prod_{j=n+1,...,n+m}^{\leftarrow} R_{ij}(z + u_i - v_j - n)'. \]
Therefore, the map $S(z)$ coincides with the composition $S^{(1)}(z) \circ S^{(2)}(z) \circ S^{(3)}(z) \circ S^{(4)}(z)$ of the following maps:

\[
\begin{align*}
&T_{[n]}^{+13}(u)T_{[m]}^{+24}(v)(1 \otimes 1) \xrightarrow{S^{(1)}(z)} T_{[n]}^{+13}(u)\overline{R}_{nm}^{12}(u|v|z + h\sigma c)^{-1}T_{[m]}^{+24}(v)(1 \otimes 1), \\
&T_{[n]}^{+13}(u)T_{[m]}^{+24}(v)(1 \otimes 1) \xrightarrow{S^{(2)}(z)} T_{[n]}^{+13}(u)T_{[m]}^{+24}(v)\overline{R}_{nm}^{12}(u|v|z)(1 \otimes 1), \\
&T_{[n]}^{+13}(u)T_{[m]}^{+24}(v)(1 \otimes 1) \xrightarrow{S^{(3)}(z)} \overline{R}_{nm}^{12}(u|v|z)T_{[n]}^{+13}(u)T_{[m]}^{+24}(v)(1 \otimes 1), \\
&T_{[n]}^{+13}(u)T_{[m]}^{+24}(v)(1 \otimes 1) \xrightarrow{S^{(4)}(z)} T_{[m]}^{+24}(v)\overline{R}_{nm}^{12}(u|v|z - h\kappa - h\sigma c)T_{[n]}^{+13}(u)(1 \otimes 1). 
\end{align*}
\]

Hence, it is sufficient to check that all $S^{(i)}(z)$ are well-defined maps on $\mathcal{Y}_c(g_N)$, i.e., that they map the ideal of defining relations (1.21) and (1.24) for the dual Yangian $Y^+(g_N)$ to itself. The fact that the given maps preserve relation (1.21) can be proved by using Yang–Baxter equation (1.11) and arguing as in the proof of [3, Lemma 2.1]. As for relation (1.24), this follows by employing crossing symmetry properties (1.16) and (1.17).

The weak associativity (2.1), Yang–Baxter equation (2.5), unitarity property (2.6) and $S$-locality property (2.7) can be verified by straightforward calculations which rely on the properties of the $R$-matrix $\overline{R}(u)$ provided in Sect. 1.2 and on Proposition 1.4. They closely follow the corresponding proofs in type A, as given in [10, Theorem 4.1]. Regarding the axioms concerning the vacuum vector $1$, (2.2) is clear and (2.3) is a consequence of the identity $T(u)1 = 1$.

Let us prove shift condition (2.4). Let $n, m \geq 0$ be arbitrary integers. By applying (2.9) on $1$ and then taking the coefficient of the variable $z$, we obtain

\[
\mathcal{D}T_{[n]}^{+}(u)1 = \left(\sum_{k=1}^{n} \frac{\partial}{\partial u_k}\right)T_{[n]}^{+}(u)1.
\] (2.15)

In particular, note that (2.2) implies $\mathcal{D}1 = 0$. Therefore, by applying $\mathcal{D} \otimes 1$ on (2.14) we find that $\mathcal{D}T_{[n]}^{+13}(u)T_{[m]}^{+24}(v)(1 \otimes 1)$ is equal to

\[
\begin{align*}
&\overline{R}_{nm}^{12}(u|v|z - h\kappa - h\sigma c)\overline{R}_{nm}^{12}(u|v|z)\left(\left(\sum_{k=1}^{n} \frac{\partial}{\partial u_k}\right)T_{[n]}^{+13}(u)\right) \\
&\cdot \overline{R}_{nm}^{12}(u|v|z + h\sigma c)^{-1}T_{[m]}^{+24}(v)\overline{R}_{nm}^{12}(u|v|z)(1 \otimes 1).
\end{align*}
\] (2.16)

On the other hand, by applying the map $S(z)$ on

\[
(\mathcal{D} \otimes 1)\left(T_{[n]}^{+13}(u)T_{[m]}^{+24}(v)(1 \otimes 1)\right) = \left(\left(\sum_{k=1}^{n} \frac{\partial}{\partial u_k}\right)T_{[n]}^{+13}(u)\right)T_{[m]}^{+24}(v)(1 \otimes 1)
\]

we get

\[\square\]
\[
\left( \sum_{k=1}^{n} \frac{\partial}{\partial u_{k}} \right) \left( -\frac{R_{nm}^{12}(u|v|z + h\kappa - h\sigma c)}{LR} \left( R_{nm}^{12}(u|v|z) T_{[n]}^{+13}(u) \right. \right.
\]
\[
\left. \cdot R_{nm}^{12}(u|v|z + h\sigma c)^{-1} T_{[m]}^{+24}(v) \left( R_{nm}^{12}(u|v|z) (1 \otimes 1) \right) \right) \right). \tag{2.17}
\]

Finally, using the observation \( \frac{\partial}{\partial u_k} R(u_k + z) = \frac{\partial}{\partial z} R(u_k + z) \) we see that the difference of expressions (2.16) and (2.17) coincides with
\[
- \frac{\partial}{\partial z} S(z) \left( T_{[n]}^{+13}(u) T_{[m]}^{+24}(v) (1 \otimes 1) \right),
\]
so that shift condition (2.4) follows.

It remains to verify hexagon identity (2.8). Its proof goes similarly as the proof in type A; see proof of [7, Theorem 2.3.8]. Let \( n, m, k \) be arbitrary nonnegative integers. Label the tensor copies as follows
\[
\frac{1}{(\text{End} \mathbb{C}^n)^{\otimes n}} \otimes \frac{2}{(\text{End} \mathbb{C}^n)^{\otimes m}} \otimes \frac{3}{(\text{End} \mathbb{C}^n)^{\otimes k}} \otimes \frac{4}{\mathcal{V}_c(\mathfrak{g}_N)} \otimes \frac{5}{\mathcal{V}_c(\mathfrak{g}_N)} \otimes \frac{6}{\mathcal{V}_c(\mathfrak{g}_N)}. \tag{2.18}
\]

First, we apply the left-hand side of hexagon identity (2.8) on the expression
\[
T_{[n]}^{+14}(u) T_{[m]}^{+25}(v) T_{[k]}^{+36}(w) (1 \otimes 1 \otimes 1), \tag{2.19}
\]
whose coefficients, with respect to the variables \( u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_m) \) and \( w = (w_1, \ldots, w_k) \), belong to (2.18). By applying \( Y(z_2) \otimes 1 \) on (2.19), we obtain the expression
\[
T_{[n]}^{+14}(u|z_2) T_{[n]}^{+14}(u|z_2 + h\sigma c/2)^{-1} T_{[m]}^{+24}(v) T_{[k]}^{+35}(w) (1 \otimes 1). \tag{2.20}
\]

By combining relation (1.40) with crossing symmetry properties (1.17) and (1.19), we rewrite (2.20) as
\[
A \cdot \left( T_{[n]}^{+14}(u|z_2) T_{[m]}^{+24}(v) T_{[k]}^{+35}(w) B \right) (1 \otimes 1), \quad \text{where}
\]
\[
A = \frac{R_{nm}^{12}(u|v|z_2 + h\sigma c + 2h\kappa)}{R_{nm}^{12}(u|v|z_2)} \quad \text{and} \quad B = \frac{R_{nm}^{12}(u|v|z_2)}{R_{nm}^{12}(u|v|z_2)^{-1}}. \tag{2.21}
\]

Due to (2.14), by applying the map \( S(z_1) \) on (2.21) we get
\[
A \cdot \left( H \cdot \left( K T_{[n]}^{+14}(u|z_2) T_{[m]}^{+24}(v) L T_{[k]}^{+35}(w) K B \right) \right) (1 \otimes 1), \tag{2.22}
\]
where
\[
H = H_{[2]} H_{[1]}, \quad H_{[1]} = \frac{R_{nm}^{13}(u|w|z_1 + z_2 - h\sigma c - h\kappa)}{R_{nm}^{13}(u|w|z_1 + z_2 - h\sigma c - h\kappa)}, \quad H_{[2]} = \frac{R_{nk}^{35}(v|w|z_1 - h\sigma c - h\kappa)}{R_{mk}^{35}(v|w|z_1 - h\sigma c - h\kappa)},
\]
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\[K = K_1|K_2, \quad K_1 = R_{\text{H}}^{13}(u|w|z_1 + z_2), \quad K_2 = R_{\text{H}}^{23}(v|w|z_1),\]
\[L = L_2|L_1, \quad L_1 = R_{\text{H}}^{13}(u|w|z_1 + z_2 + h\sigma c)^{-1}, \quad L_2 = R_{\text{H}}^{23}(v|w|z_1 + h\sigma c)^{-1}.\]

We now apply the right-hand side of hexagon identity (2.8) on (2.19) and show that the result coincides with (2.22). The tensor copies are again labeled as in (2.18). First, applying the map \(S_{46}(z_1 + z_2)\) on (2.19) we obtain

\[H_{1L} \cdot (K_1 T_{[n]}^{+14} (u) L_1 T_{[m]}^{+25} (v) T_{[k]}^{+36} (w) K_1) (1 \otimes 1 \otimes 1).\]

(2.23)

Next, applying the map \(S_{56}(z_1)\) on (2.23) we get

\[H_{1L} \cdot \left(H_{1L} \cdot K_1 T_{[n]}^{+14} (u) L_1 \left(H_{1L} \cdot K_2 T_{[m]}^{+25} (v) L_2 T_{[k]}^{+36} (w) K_2 \right) \right) K_1) (1 \otimes 1 \otimes 1)
\]

\[= H_{1L} \cdot \left(H_{1L} \cdot K_1 T_{[n]}^{+14} (u) L_1 K_2 T_{[m]}^{+25} (v) L_2 T_{[k]}^{+36} (w) K_2 \right) K_1) (1 \otimes 1 \otimes 1).\]

Finally, by applying \(Y(z_2) \otimes 1\) we get

\[H_{1L} \cdot \left(H_{1L} \cdot K_1 X L_2 T_{[k]}^{+35} (w) K_2 \right) K_1) (1 \otimes 1),\]

(2.24)

where

\[X = T_{[n]}^{+14} (u|z_2) T_{[n]}^{14} (u|z_2 + h\sigma c/2)^{-1} L_1 K_2 T_{[m]}^{+24} (v)
\]

\[= L_{1R} \cdot \left(K_2 T_{[n]}^{+14} (u|z_2) T_{[n]}^{14} (u|z_2 + h\sigma c/2)^{-1} T_{[m]}^{+24} (v) \right).\]

As before, we combine relation (1.40) with crossing symmetry properties (1.17) and (1.19) to write \(X\) as

\[X = L_{1R} \cdot \left(K_2 T_{[n]}^{+14} (u|z_2) \left(A_{RL} \cdot \left(T_{[m]}^{+24} (v) B T_{[n]}^{14} (u|z_2 + h\sigma c/2)^{-1} \right) \right) \right).\]

(2.25)

Identities \(T(z)1 = 1\) and (2.25) imply that (2.24) is equal to

\[H_{1L} \cdot \left(H_{1L} \cdot K_1 Z L_2 T_{[k]}^{+35} (w) K_2 \right) K_1) (1 \otimes 1),\]

(2.26)

where

\[Z = L_{1R} \cdot \left(K_2 T_{[n]}^{+14} (u|z_2) A_{RL} \cdot \left(T_{[m]}^{+24} (v) B \right) \right).\]

(2.27)

The following consequences of Yang–Baxter equation (1.11) and crossing symmetry property (1.19) can be verified by a straightforward calculation:

\[L_{1R} \cdot (K_2 A) = A_{RL} \cdot (K_2 L_{1}), \quad H_{1L} A K_{[1]} = K_{[1]} A H_{2L},\]

(2.28)
By using the first equality in (2.28), we can write (2.27) as

\[ Z = T_{[n]}^{+14}(u|z_2) \left( A \cdot_{RL} \left( K_{[2]} T_{[m]}^{+24} (v) B L_{[1]} \right) \right). \]

Therefore, the original expression in (2.26) is equal to

\[
H_{[1]} \cdot_{LR} \left( H_{[2]} \cdot_{LR} \left( \left( K_{[1]} T_{[n]}^{+14} (u|z_2) K_{[2]} T_{[m]}^{+24} (v) B L_{[1]} L_{[2]} T_{[k]}^{+35} (w) K_{[2]} \right) \right) K_{[1]} \right) (1 \otimes 1).
\]

By moving the element \( A \) to the left and using \( K = K_{[1]} K_{[2]} \), we obtain

\[
H_{[1]} \cdot_{LR} \left( H_{[2]} \cdot_{LR} \left( A \cdot_{RL} \left( K_{[1]} T_{[n]}^{+14} (u|z_2) K_{[2]} T_{[m]}^{+24} (v) B L_{[1]} L_{[2]} T_{[k]}^{+35} (w) K_{[2]} \right) \right) \right) K_{[1]} (1 \otimes 1).
\]

Next, we employ the second equality in (2.28) and \( H = H_{[2]} H_{[1]} \) to write the given expression as

\[
A \cdot_{RL} \left( H_{[1]} \cdot_{LR} \left( H_{[2]} \cdot_{LR} \left( K T_{[n]}^{+14} (u|z_2) T_{[m]}^{+24} (v) B L_{[1]} L_{[2]} T_{[k]}^{+35} (w) K_{[2]} K_{[1]} \right) \right) \right) (1 \otimes 1)
\]

\[
= A \cdot_{RL} \left( H \cdot_{LR} \left( K T_{[n]}^{+14} (u|z_2) T_{[m]}^{+24} (v) B L_{[1]} L_{[2]} T_{[k]}^{+35} (w) K_{[2]} K_{[1]} \right) \right) (1 \otimes 1).
\]

Finally, we use both equalities in (2.29) to move the element \( B \) in (2.30) to the right, thus getting (2.22), as required. Therefore, we conclude that hexagon identity (2.8) holds, so the proof of the theorem is over. \( \square \)

2.2 Center of the \( h \)-adic quantum vertex algebra at the critical level

In this section, we consider the \( h \)-adic quantum vertex algebra \( \mathcal{V}_{\text{crit}}(g_N) = \mathcal{V}_{\text{crit}}(g_N) \) at the critical level

\[
c_{\text{crit}} = -\frac{2\kappa}{\sigma} = \begin{cases} 
-N + 2, & \text{if } g_N = \mathfrak{so}_N, \\
-N - 1, & \text{if } g_N = \mathfrak{sp}_N.
\end{cases}
\]

First, we follow the exposition in [19] to recall a particular case of the fusion procedure for the Brauer algebra [2]; see also [20, Section 1.2]. Let \( \omega \) be an indeterminate, and let \( B_m(\omega) \) be the Brauer algebra over the field \( \mathbb{C}(\omega) \) generated by the elements
\(s_1, \ldots, s_{m-1} \text{ and } \epsilon_1, \ldots, \epsilon_{m-1} \) subject to the defining relations as in [19, Section 3]. Its complex subalgebra generated by the elements \(s_1, \ldots, s_{m-1} \) is isomorphic to the group algebra of the symmetric group \(S_m\), so that the elements \(s_i\) are identified with the transpositions \((i, i + 1)\). Let \(s_{ij} \in B_m(\omega), \ i < j\), be the element corresponding to the transposition \((i, j)\) with respect to that isomorphism. Introduce \(\epsilon_{ij} \in B_m(\omega)\) by \(\epsilon_{j-1} = \epsilon_{j-1} \epsilon_{j-1} s_{i} s_{i-1}\) for \(i < j - 1\). Let \(s^{(m)} \in B_m(\omega)\) be the idempotent corresponding to the one-dimensional representation of the Brauer algebra which maps all \(s_{ij}\) to the identity operator and all \(\epsilon_{ij}\) to the zero operator. It satisfies

\[
s_{ij}s^{(m)} = s^{(m)}s_{ij} = s^{(m)} \quad \text{and} \quad \epsilon_{ij}s^{(m)} = s^{(m)}\epsilon_{ij} = 0. \tag{2.31}
\]

Consider the expression

\[
R(u_1, \ldots, u_m) = \frac{1}{m!} \prod_{1 \leq i < j \leq m} R_{ij}(u_i - u_j),
\]

where the products are taken in the lexicographical order on the pairs \((i, j)\). Define

\[
u_m = \begin{cases} (u - (m - 1)h, \ldots, u - h, u) & \text{for } g_N = o_N \text{ and } m = 1, \ldots, N, \\ (u, u - h, \ldots, u - (m - 1)h) & \text{for } g_N = sp_N \text{ and } m = 1, \ldots, N/2. \end{cases} \tag{2.32}
\]

In the orthogonal case, let \(S_{[m]}\) with \(m = 1, \ldots, N\) denote the action of the idempotent \(s^{(m)} \in B_m(N)\) on the tensor product space \((\mathbb{C}^N)^{\otimes m}\) with respect to the representation defined by \(s_{ij} \mapsto P_{ij}\) and \(\epsilon_{ij} \mapsto Q_{ij}\) for \(i < j\). In the symplectic case, let \(S_{[m]}\) with \(m = 1, \ldots, N/2\) denote the action of the idempotent \(s^{(m)} \in B_m(-N)\) on the tensor product space \((\mathbb{C}^N)^{\otimes m}\) with respect to the representation defined by \(s_{ij} \mapsto -P_{ij}\) and \(\epsilon_{ij} \mapsto -Q_{ij}\) for \(i < j\). Due to a particular case of the fusion procedure for the Brauer algebra \(B_m(\omega)\), see \([8,9]\), we have

\[
S_{[m]} = R(u_m). \tag{2.33}
\]

Consider the tensor product

\[
\text{End } \mathbb{C}^N \otimes (\text{End } \mathbb{C}^N)^{\otimes m} \otimes \mathcal{V}_{\text{crit}}(g_N). \tag{2.34}
\]

We use the following consequences of the fusion procedure in the proof of Theorem 2.4.

**Lemma 2.3** For any \(m = 1, \ldots, N\) in the orthogonal case, \(m = 1, \ldots, N/2\) in the symplectic case and \(\alpha \in \mathbb{C}\), we have

\[
S_{[m]}^{1} R_{1m}^{01} (v + h\alpha|u_m) = \overline{R_{1m}}^{01} (v + h\alpha|u_m) S_{[m]}^{1}, \tag{2.35}
\]

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\[ S_{[m]}^{1} T_{[m]}^{12}(u_{[m]} + h\alpha) = \tilde{T}_{[m]}^{12}(u_{[m]} + h\alpha) S_{[m]}^{1}, \]  
(2.36)
\[ S_{[m]}^{1} T_{[m]}^{-12}(u_{[m]}) = \tilde{T}_{[m]}^{-12}(u_{[m]}) S_{[m]}^{1}, \]  
(2.37)

where the superscripts indicate the tensor factors in (2.34).

**Proof** Yang–Baxter equation (1.11) implies
\[ R(u_{1}, \ldots, u_{m}) R_{1m}^{01}(v + h\alpha | u) = \tilde{R}_{1m}^{01}(v + h\alpha | u) R(u_{1}, \ldots, u_{m}) \]  
(2.38)
and defining relations (1.20) and (1.21) imply
\[ R(u_{1}, \ldots, u_{m}) T_{[m]}^{12}(u + h\alpha) = T_{[m]}^{12}(u + h\alpha) R(u_{1}, \ldots, u_{m}) \]  
(2.39)
\[ R(u_{1}, \ldots, u_{m}) T_{[m]}^{-12}(u) = \tilde{T}_{[m]}^{-12}(u) R(u_{1}, \ldots, u_{m}) \]  
(2.40)

for the variables \( u = (u_{1}, \ldots, u_{m}) \), where \( R(u_{1}, \ldots, u_{m}) \) is applied on the tensor factor \((\End \mathbb{C}^{N})^{\otimes m}\) of (2.34) and \( u + h\alpha = (u_{1} + h\alpha, \ldots, u_{m} + h\alpha) \). By evaluating the variables \( u = (u_{1}, \ldots, u_{m}) \) at \( u_{[m]} \) in equalities (2.38)–(2.40) and then applying fusion procedure (2.33), we obtain (2.35)–(2.37), as required. □

Let \( V \) be an \( h \)-adic quantum vertex algebra. As in [10], we define the center of \( V \) in analogy with vertex algebra theory, see, e.g., [6, Chapter 3.3], as a \( \mathbb{C}[[[h]]] \)-submodule
\[ \mathcal{Z}(V) = \{ v \in V : Y(w, z)v \in V[[z]] \text{ for all } w \in V \}. \]

Due to (2.9), the center of \( \mathcal{V}_c(\mathfrak{g}_N) \) coincides with the \( Y(\mathfrak{g}_N) \)-invariants, i.e.,
\[ \mathcal{Z}(\mathcal{V}_c(\mathfrak{g}_N)) = \{ v \in \mathcal{V}_c(\mathfrak{g}_N) : t_{ij}(u)v = \delta_{ij}v \text{ for all } i, j = 1, \ldots, N \}. \]  
(2.41)

As in [20, Chapter 11.2], define the series
\[ T_{[m]}^{m+}(u) = \text{tr}_{1, \ldots, m} S_{[m]}^{1} T_{[m]}^{1+}(u_{[m]}) \mathbf{1} = \text{tr}_{1, \ldots, m} S_{[m]}^{1} T_{1}^{1+}(u_{1}) \ldots T_{m}^{1+}(u_{m}) \mathbf{1} \]  
(2.42)
in \( \mathcal{V}_\text{crit}(\mathfrak{g}_N)[[u]] \), where \( m = 1, \ldots, N \) in the orthogonal case, \( m = 1, \ldots, N/2 \) in the symplectic case and the trace is taken over all \( m \) copies of \End \mathbb{C}^{N}.

**Theorem 2.4** All coefficients of \( T_{[m]}^{m+}(u) \) belong to the center of the \( h \)-adic quantum vertex algebra \( \mathcal{V}_\text{crit}(\mathfrak{g}_N) \).

**Proof** Recall (2.32). Consider the expressions
\[ A = \tilde{R}_{1m}^{01}(v - h\kappa | u_{[m]})^{-1} \text{ and } B = \tilde{R}_{1m}^{01}(v + h\kappa | u_{[m]}). \]  
(2.43)

Their coefficients with respect to the variables \( v \) and \( u \) belong to tensor product (2.34) and their superscripts indicate the tensor copies therein. By (2.35), we have
\[ S_{[m]}^{1} A \tilde{S}_{[m]}^{1} \text{ and } S_{[m]}^{1} B \tilde{S}_{[m]}^{1}, \]  
(2.44)
where the superscript 1 indicates that the idempotent $S_{[m]}$ acts on the tensor copy $(\text{End } \mathbb{C}^N)^{\otimes m}$ of (2.34). Crossing symmetry property (1.16) implies

$$A = \left( R^{01}_{1m}(v|u_{[m]})) \right)^0 = R^{01}_{1m}(v|u_{[m]}),$$

where the transposition $'$ in the middle term is applied on the tensor copy $\text{End } \mathbb{C}^N$ of (2.34). Therefore, by crossing symmetry property (1.17) we have

$$A \cdot B = 1.$$  \hspace{1cm} (2.45)

In order to prove the theorem, it is sufficient to verify that the coefficients of $T^+_m(u)$ belong to the $\mathbb{C}[[h]]$-submodule of $Y_{(g_N)}$-invariants (2.41), i.e.,

$$T(v) T^+_m(u) = T^+_m(u).$$  \hspace{1cm} (2.46)

By using (2.42), we write the left-hand side in (2.46) as

$$\text{tr}_{1,...,m} T^{02}_{[1]}(v) S^1_{[m]} T^{+12}_{[m]}(u_{[m]}),$$

where the superscripts indicate the tensor factors in (2.34). Next, by (1.22) and $T(v)1 = 1$ we conclude that (2.47) is equal to

$$\text{tr}_{1,...,m} S^1_{[m]} A T^{+12}_{[m]}(u_{[m]})) B.$$ \hspace{1cm} (2.48)

We now combine the cyclic property of the trace and $(S_{[m]})^2 = S_{[m]}$ with equalities (2.37) and (2.44) to rewrite (2.48) as follows:

$$\text{tr}_{1,...,m} S^1_{[m]} A T^{+12}_{[m]}(u_{[m]}) B = \text{tr}_{1,...,m} \tilde{A} S^1_{[m]} T^{+12}_{[m]}(u_{[m]}) B$$

$$= \text{tr}_{1,...,m} \tilde{A} (S^1_{[m]})^2 T^{+12}_{[m]}(u_{[m]}) B = \text{tr}_{1,...,m} S^1_{[m]} \tilde{A} \tilde{T}^{+12}_{[m]}(u_{[m]}) \tilde{B} S^1_{[m]}$$

$$= \text{tr}_{1,...,m} A \tilde{T}^{+12}_{[m]}(u_{[m]}) \tilde{B} (S^1_{[m]})^2 = \text{tr}_{1,...,m} A \tilde{T}^{+12}_{[m]}(u_{[m]}) \tilde{B} S^1_{[m]}$$

$$= \text{tr}_{1,...,m} A S^1_{[m]} T^{+12}_{[m]}(u_{[m]}) B.$$ \hspace{1cm} (2.49)

Finally, by the cyclic property of the trace, expression (2.49) equals

$$\text{tr}_{1,...,m} A \cdot B \left( S^1_{[m]} T^{+12}_{[m]}(u_{[m]}) B \right) = \text{tr}_{1,...,m} S^1_{[m]} T^{+12}_{[m]}(u_{[m]} \left( A \cdot B \right).$$

By (2.45) this is equal to $T^+_m(u)$, so equality (2.46) follows and the proof is over. \hspace{1cm} \Box

In the orthogonal case, the series $T^+_m(u)$ can be also written in the form

$$T^+_m(u) = \text{tr}_{1,...,m} S_{[m]} T^+_1(u) T^+_2(u - h) \ldots T^+_m(u - (m - 1)h).$$  \hspace{1cm} (2.50)
Indeed, by the first equality in (2.31) we have \( P_{ij} S_{[m]} P_{ij} = S_{[m]} \) for any \( i, j = 1, \ldots, m \). Hence, conjugating the expression under the trace by a suitable element of the symmetric group \( S_m \) we rewrite (2.50) as

\[
\text{tr}_{1,...,m} S_{[m]} T_m^+(u) T_{m-1}^+(u - h) \ldots T_1^+(u - (m - 1)h).
\]

Next, using the cyclic property of the trace we move the idempotent \( S_{[m]} \) to the right, thus getting

\[
\text{tr}_{1,...,m} T_m^+(u) T_{m-1}^+(u - h) \ldots T_1^+(u - (m - 1)h) S_{[m]}. 
\]

Finally, fusion procedure (2.37) implies that this equals \( T_m^+(u) \), as required.

**Theorem 2.5** In \( V_{\text{crit}}(g \mathcal{N}) \), we have

\[
S(z)(T_k^+(u) \otimes T_m^+(v)) = T_k^+(u) \otimes T_m^+(v). \quad (2.51)
\]

**Proof** Label the tensor copies as follows:

\[
\begin{array}{c}
\text{(End } \mathbb{C}^N)^{\otimes k} \otimes (\text{End } \mathbb{C}^N)^{\otimes m} \otimes V_{\text{crit}}(g \mathcal{N}) \otimes V_{\text{crit}}(g \mathcal{N}).
\end{array}
\]

(2.52)

Let

\[
A = \overrightarrow{R}_{km}^{12}(u_{[k]}|v_{[m]}|z + h\kappa), \quad B = \overrightarrow{R}_{km}^{12}(u_{[k]}|v_{[m]}|z),
\]

\[
C = \overrightarrow{R}_{km}^{12}(u_{[k]}|v_{[m]}|z - 2h\kappa)^{-1},
\]

where \( u_{[k]} = (u_1, \ldots, u_k) \) and \( v_{[m]} = (v_1, \ldots, v_m) \) are defined by (2.32). Using crossing symmetry properties (1.16), (1.17) and (1.19), we find

\[
B A = 1 \quad \text{and} \quad B \cdot C = 1. \quad (2.53)
\]

Set

\[
\begin{align*}
\hat{A} &= \prod_{i=1,...,k} \prod_{j=k+1,...,k+m} \overrightarrow{R}_{ij}(z + \overline{u}_i - \overline{v}_{j-k} + h\kappa)', \\
\hat{B} &= \prod_{i=1,...,k} \prod_{j=k+1,...,k+m} \overrightarrow{R}_{ij}(z + \overline{u}_i - \overline{v}_{j-k}), \\
\hat{C} &= \prod_{i=1,...,k} \prod_{j=k+1,...,k+m} \overrightarrow{R}_{ij}(z + \overline{u}_i - \overline{v}_{j-k} - 2h\kappa)^{-1}.
\end{align*}
\]
By using fusion procedure (2.33) and arguing as in the proof of Lemma 2.3, one can prove

$$S^1_{[k]} A = \hat{A} S^1_{[k]}, \quad S^1_{[k]} B = \hat{B} S^1_{[k]}, \quad \text{and} \quad S^1_{[k]} C = \hat{C} S^1_{[k]}, \quad (2.54)$$

Explicit formula (2.14) for the operator $S(z)$ at the critical level implies

$$S(z)\left(\mathbb{T}^+_k(u) \otimes \mathbb{T}^+_m(v)\right) = \text{tr}_{1, \ldots, m+k} S^1_{[k]} S^2_{[m]} A \cdot (B T^+_k (u)) C T^+_m (v) B (1 \otimes 1).$$

Note that $S^2_{[m]} S^1_{[k]} = S^1_{[k]} S^2_{[m]}$, so we can use (2.54) to move $S^1_{[k]}$ to the right, thus getting

$$\text{tr}_{1, \ldots, m+k} S^2_{[m]} S^1_{[k]} A \cdot (\hat{B} S^1_{[k]} T^+_k (u)) C T^+_m (v) B (1 \otimes 1).$$

Since $S^1_{[k]} = (S^1_{[k]})^2$, this equals

$$\text{tr}_{1, \ldots, m+k} S^2_{[m]} S^1_{[k]} A \cdot (\hat{B} (S^1_{[k]})^2 T^+_k (u)) C T^+_m (v) B (1 \otimes 1).$$

Using (2.37) and (2.54), we move one copy of $S^1_{[k]}$ to the left and another copy to the right:

$$\text{tr}_{1, \ldots, m+k} S^1_{[k]} S^2_{[m]} A \cdot (B \hat{T}^+_k (u)) \hat{C} T^+_m (v) \hat{B} S^1_{[k]} (1 \otimes 1).$$

By employing the cyclic property of the trace and then $(S^1_{[k]})^2 = S^1_{[k]}$, we get

$$\text{tr}_{1, \ldots, m+k} S^2_{[m]} A \cdot (B \hat{T}^+_k (u)) \hat{C} T^+_m (v) \hat{B} (S^1_{[k]})^2 (1 \otimes 1) = \text{tr}_{1, \ldots, m+k} S^2_{[m]} A \cdot (B \hat{T}^+_k (u)) \hat{C} T^+_m (v) \hat{B} S^1_{[k]} (1 \otimes 1).$$

Next, we use equalities (2.37) and (2.54) to move $S^1_{[k]}$ to the left:

$$\text{tr}_{1, \ldots, m+k} S^2_{[m]} A \cdot (B S^1_{[k]} T^+_k (u)) C T^+_m (v) B (1 \otimes 1).$$

Note that the tensor copies 1, \ldots, $k$ of $A$ commute with $S^2_{[m]}$, so this is equal to

$$\text{tr}_{1, \ldots, m+k} A \cdot (S^2_{[m]} B S^1_{[k]} T^+_k (u)) C T^+_m (v) B (1 \otimes 1).$$

By using the cyclic property of the trace and then the first equality in (2.53), we get
\[
\text{tr}_{1,\ldots,m+k} S_{m}^{2} B S_{k}^{1} T_{k}^{+13} (u[k]) C T_{m}^{+24} (v[m]) B A (I \otimes I) = \text{tr}_{1,\ldots,m+k} S_{m}^{2} B S_{k}^{1} T_{k}^{+13} (u[k]) C T_{m}^{+24} (v[m]) (I \otimes I).
\]

Again, by the cyclic property of the trace, this equals
\[
\text{tr}_{1,\ldots,m+k} S_{m}^{2} S_{k}^{1} T_{k}^{+13} (u[k]) (B \cdot C) T_{m}^{+24} (v[m]) (I \otimes I).
\]

Finally, as \( S_{m}^{2} S_{k}^{1} = S_{k}^{1} S_{m}^{2} \), by employing the second equality in (2.53) we obtain
\[
\text{tr}_{1,\ldots,m+k} S_{m}^{1} T_{k}^{+13} (u[k]) S_{m}^{2} T_{m}^{+24} (v[m]) (I \otimes I) = \mathbb{T}^{+}_{k} (u) \otimes \mathbb{T}^{+}_{m} (v),
\]
as required. \( \square \)

Let us consider the classical limit of the series \( \mathbb{T}^{+}_{m} (u) \). Recall (1.25) and define \( \deg u = 1 \) and \( \deg \partial u = -1 \). Clearly, this extends (1.26) to the ascending filtration of the algebra \( Y^{+}(g_{N})[[u, \partial u]]_{\text{fin}} \) which consists of all finite degree elements in \( Y^{+}(g_{N})[[u, \partial u]] \). By Proposition 1.2, the corresponding graded algebra is \( \left( U(t^{-1} g_{N}[t^{-1}]) \otimes_{\mathbb{C}} \mathbb{C}[[h]] \right) [[u, \partial u]]_{\text{fin}} \) which consists of all finite degree elements in \( \left( U(t^{-1} g_{N}[t^{-1}]) \otimes_{\mathbb{C}} \mathbb{C}[[h]] \right) [[u, \partial u]] \). Consider the element
\[
\text{tr}_{1,\ldots,m} S_{m} \left( 1 - T_{1}^{+} (u)e^{-h \partial u} \right) \left( 1 - T_{2}^{+} (u)e^{-h \partial u} \right) \ldots \left( 1 - T_{m}^{+} (u)e^{-h \partial u} \right).
\]

Its degree equals \(-m\), so it belongs to the algebra \( Y^{+}(g_{N})[[u, \partial u]]_{\text{fin}} \). By Proposition 1.2, its image in the corresponding graded algebra equals
\[
h^{m} \text{tr}_{1,\ldots,m} S_{m} \left( \partial u + F_{1}^{+} (u) \right) \left( \partial u + F_{2}^{+} (u) \right) \ldots \left( \partial u + F_{m}^{+} (u) \right).
\]

On the other hand, by arguing as in [10], we can express (2.55) as follows. First, using \( e^{-h \partial u} u^{r} = (u - h)^{r} e^{-h \partial u} \) with \( r \in \mathbb{Z} \) we observe that the given element is equal to
\[
\text{tr}_{1,\ldots,m} S_{m} \sum_{k=0}^{m} \sum_{1 \leq i_{1} < \ldots < i_{k} \leq m} (-1)^{k} T_{i_{1}}^{+} (u) T_{i_{2}}^{+} (u - h) \ldots T_{i_{k}}^{+} (u - (k - 1)h) e^{-kh \partial u}.
\]

Moreover, due to [19, Lemma 4.1], we have
\[
\text{tr}_{m} S_{m} = a_{m} S_{m-1} \quad \text{for} \quad a_{m} = \pm \frac{(\pm N + m - 3)(\pm N + 2m - 2)}{m(\pm N + 2m - 4)},
\]
where the upper sign corresponds to the orthogonal and the lower sign corresponds to the symplectic case. Conjugating the summands under the trace by suitable elements.
of the symmetric group $S_m$ and using the first equality in (2.31) and the cyclic property of the trace, we write (2.57) as

$$
\sum_{k=0}^{m} b_k \text{tr}_{1,...,k} S[S_k] T_1^+ (u) T_2^+ (u-h) \ldots T_k^+ (u-(k-1)h) e^{-kh\partial_u} = \sum_{k=0}^{m} b_k \mathbb{T}_k^+(u) e^{-kh\partial_u},
$$

where $b_k = (-1)^k a_{k+1} \ldots a_m (\frac{m}{k})$. In the above equality, we used alternative expression (2.50) for $T_k^+ (u)$ in the orthogonal case, so that our calculation treats both cases simultaneously. Introduce the series

$$
\Phi_m (u) = \sum_{r \geq 0} \Phi_{mr} u^r = h^{-m} \sum_{k=0}^{m} b_k \mathbb{T}_k^+(u) \in \mathcal{V}_{\text{crit}}(g_N)[[u]].
$$

Theorem 2.5 implies

**Corollary 2.6** All coefficients of $\Phi_m (u)$ belong to the center of the $h$-adic quantum vertex algebra $\mathcal{V}_{\text{crit}}(g_N)$.

Let $D : \mathcal{V}_{\text{crit}}(g_N) \rightarrow \mathcal{V}_{\text{crit}}(g_N)$ be the translation operator defined on the universal affine vertex algebra $\mathcal{V}_{\text{crit}}(g_N) = \mathcal{V}_{\text{crit}}(g_N)_{\text{crit}}$ at the critical level $c_{\text{crit}}$ by

$$
D1 = 0 \quad \text{and} \quad [D, f_{ij}(-r)] = rf_{ij}(-r-1) \quad \text{for all} \quad i, j = 1, \ldots, N, \quad r \geq 1,
$$

where $1$ is the vacuum vector in $\mathcal{V}_{\text{crit}}(g_N)$. By combining explicit formula (2.15) for the operator $D : \mathcal{V}_{\text{crit}}(g_N) \rightarrow \mathcal{V}_{\text{crit}}(g_N)$ and Proposition 1.2, one can easily verify that map (1.30) acts as follows

$$
\mathcal{V}_{\text{crit}}(g_N) \ni \mathcal{D} r^{i_{1 \rangle j_{1}}} \ldots r^{i_{s \rangle j_s}} 1 \mapsto \mathcal{D} f_{i_{1 \rangle j_{1}}}(-r_{1}) \ldots f_{i_{s \rangle j_s}}(-r_{s}) 1 \in \mathcal{V}_{\text{crit}}(g_N)
$$

(2.58)

for all $i_1, \ldots, j_s = 1, \ldots, N, r_1, \ldots, r_s \geq 1$ and $s \geq 0$, thus justifying our notation.

Denote by $\mathbb{Z}(\mathcal{V}_{\text{crit}}(g_N))$ the Feigin–Frenkel center [4], i.e., the center of the vertex algebra $\mathcal{V}_{\text{crit}}(g_N)$. The complete sets $\{ \phi_{22}, \phi_{44}, \ldots, \phi_{2n \cdot 2n} \} \subset \mathbb{Z}(\mathcal{V}_{\text{crit}}(g_N))$ of Segal–Sugawara vectors for $\mathcal{V}_{\text{crit}}(g_N)$, where $g_N = \mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$, i.e., the sets such that

$$
\mathbb{Z}(\mathcal{V}_{\text{crit}}(g_N)) = \mathbb{C}[D^{i} \phi_{2i \cdot 2i} : i = 1, \ldots, n, k \geq 0]
$$

(2.59)

were constructed by Molev [19]. Extend the affine Lie algebra $\widehat{g}_N$ with the element $\tau$ so that the following commutation relations hold on $\widehat{g}_N \oplus \mathbb{C}\tau$:

$$
[\tau, C] = 0 \quad \text{and} \quad [\tau, f_{ij}(-r)] = rf_{ij}(-r-1) \quad \text{for all} \quad i, j = 1, \ldots, N, r \in \mathbb{Z}.
$$
The Segal–Sugawara vectors \( \phi_{2i, 2i} \in V_{\text{crit}}(g_N) = U(t^{-1} g_N[t^{-1}]) \) for \( i = 1, 2, \ldots, p_{g_N} \), where

\[
p_{g_N} = \begin{cases} 
  n, & \text{if } g_N = \mathfrak{o}_{2n+1}, \\
  [n/2], & \text{if } g_N = \mathfrak{sp}_{2n}, \\
  n - 1, & \text{if } g_N = \mathfrak{o}_{2n},
\end{cases}
\]

are found in [19] as constant terms of the polynomials

\[
\text{tr}_{1, \ldots, m} S_{[m]} (\partial_u + F_1^+(u)) (\partial_u + F_2^+(u)) \ldots (\partial_u + F_m^+(u)) = \phi_{m0} \tau^m + \phi_{m1} \tau^{m-1} + \ldots + \phi_{mm}
\]

(2.60)

with \( m = 2, 4, \ldots, 2p_{g_N} \). We now recover these vectors by taking the classical limit of the constant terms of the series \( \Phi_m(u) \).

**Proposition 2.7** The images of the elements \( \Phi_{2i, 0} \in \mathfrak{z}(V_{\text{crit}}(g_N)) \) with respect to map (1.30) coincide with the Segal–Sugawara vectors \( \phi_{2i, 2i} \in \mathfrak{z}(V_{\text{crit}}(g_N)) \) for all \( i = 1, \ldots, p_{g_N} \). In particular, the images of the elements \( \Phi_{20}, \Phi_{40}, \ldots, \Phi_{2n0} \in \mathfrak{z}(V_{\text{crit}}(o_{2n+1})) \) form a complete set of Segal–Sugawara vectors for the universal affine vertex algebra \( V_{\text{crit}}(o_{2n+1}) \).

**Proof** Recall (2.56). The image of the element \( \Phi_{m0} \in \mathfrak{z}(V_{\text{crit}}(g_N)) \) with respect to map (1.30) is found by moving all \( \partial_u \) in

\[
\text{tr}_{1, \ldots, m} S_{[m]} (\partial_u + F_1^+(u)) (\partial_u + F_2^+(u)) \ldots (\partial_u + F_m^+(u))
\]

to the right and then taking the constant term with respect to \( u \) and \( \partial_u \). However, since

\[
[\partial_u, F^+(u)] = \sum_{r \geq 1} r F(-r - 1) u^{r-1} = \sum_{r \geq 1} [\tau, F(-r)] u^{r-1} = [\tau, F^+(u)],
\]

it is clear from (2.60) that this image coincides with \( \phi_{mm} \), as required. \( \square \)

Let \( V \) be an \( h \)-adic quantum vertex algebra with vacuum vector \( \mathbf{1} \). The product

\[
a \cdot b = a_{-1} b \quad \text{for all } a, b \in \mathfrak{z}(V)
\]

(2.61)

defines the structure of an associative algebra with unit \( \mathbf{1} \) on \( \mathfrak{z}(V) \). Moreover, this algebra is equipped with a derivation defined as the restriction of the operator \( \mathcal{D} \); see [10, Proposition 3.7]. We now explicitly describe the center of the \( h \)-adic quantum vertex algebra \( V_{\text{crit}}(o_{2n+1}) \).

**Theorem 2.8** The algebra \( \mathfrak{z}(V_{\text{crit}}(o_{2n+1})) \) coincides with the \( h \)-adically completed polynomial algebra in infinitely many variables,

\[
\mathfrak{z}(V_{\text{crit}}(o_{2n+1})) = \mathbb{C}[\mathcal{D}^k \Phi_{2i, 0} : i = 1, \ldots, n, k \geq 0][[h]].
\]

(2.62)
The $\mathbb{C}[[h]]$-module $V_{\text{crit}}(o_{2n+1})$ is topologically free, so it can be written as $V_0[[h]]$ for some complex vector space $V_0$. Let $v$ be an arbitrary element of the center $\mathfrak{z}(V_{\text{crit}}(o_{2n+1}))$. We will prove by induction that for every nonnegative integer $r$ there exists an element $p \in \mathcal{P}$ such that $v - p$ belongs to $h^r V_{\text{crit}}(o_{2n+1})$.

First, note that the aforementioned statement holds trivially for $r = 0$. Suppose that for some integer $r \geq 0$ we have

$$v - p = h^r v_r + h^{r+1} v_{r+1} + \cdots$$

for some $p \in \mathcal{P}$ and $v_z \in V_0$.

Clearly, both $v - p$ and $v_r + h v_{r+1} + h^2 v_{r+2} + \ldots$ belong to the center $\mathfrak{z}(V_{\text{crit}}(o_{2n+1}))$. Write $v_r$ as the sum $v_r = v_{r,1} + \cdots + v_{r,l_r}$, so that all $v_{r,j}$ are homogeneous with respect to degree operator (1.25). Set $d_r = \min \{ \deg v_{r,j} : j = 1, \ldots, l_r \}$.

Choose any homogeneous element

$$w = h^{s_1} v_{s_1} + h^{s_2} v_{s_2} + \cdots$$

with $s_{j+1} > s_j \geq r$ for all $j$ such that $\deg(v - p - w) < \deg(v - p) = \deg w = \deg v_{s_j}$ for all $j$. By combining (2.41) and Proposition 1.2, we conclude that the images $\bar{v}_{s_j}$ of $v_{s_j}$ with respect to map (1.30) belong to the center of the vertex algebra $V_{\text{crit}}(o_{2n+1})$ for all $j \geq 0$. Hence, due to (2.59), there exist polynomials $\bar{p}_j$ in the variables $D^k \Phi_{2i}$ such that $\bar{v}_{s_j} = \bar{p}_j$ for all $j$. Let $p_j$ be the polynomials obtained from $\bar{p}_j$ by replacing the variables $D^k \Phi_{2i}$ with the respective variables $D^k \Phi_{2i}$. The element $v - p - \sum_j h^{s_j} p_j$ belongs to the center $\mathfrak{z}(V_{\text{crit}}(o_{2n+1}))$ and its degree is strictly less than $\deg(v - p) = \deg w$.

Let

$$v - p - \sum_j h^{s_j} p_j = h^{r'} v_{r'} + h^{r'+1} v_{r'+1} + \cdots$$

for some $v' \in V_0$. (2.63)

Write $v'$ as a sum $v' = v'_{r,1} + \cdots + v'_{l_r}$, so that all elements $v'_{r,j}$ are homogeneous with respect to degree operator (1.25). Set $d'_r = \min \{ \deg v'_{r,j} : j = 1, \ldots, l'_r \}$. Observe that $d'_r \geq d_r$ because lower degree terms, with respect to (1.25), in all elements $D^k \Phi_{2i}$ come up multiplied by a positive power of $h$. We can continue to repeat the same procedure, now starting with element (2.63), for an appropriate number of times. As we demonstrated, in each step the degree of the left-hand side is reduced, while the degree of the lowest degree term of the coefficient of $h^r$ on the right-hand side does not decrease. Therefore, after finitely many steps we end up with the expression of the form
\[ v - p - \sum_j h^j q_j = h^{r+1} v''_{r+1} + h^{r+2} v''_{r+2} + \cdots \] for some \( v''_r \in \mathcal{V}_0 \) and \( q_j \in \mathcal{P} \),

thus finishing the inductive step. Note that the sequence \((t_j)\) is strictly increasing, so that the sum \( \sum_j h^j q_j \) is indeed a well-defined element of \( \mathcal{P} \). Therefore, we proved that for any \( v \in \mathfrak{z}(\mathcal{V}_{\text{crit}}(\mathfrak{g}_{2n+1})) \) and for any integer \( r \geq 0 \) there exists \( p \in \mathcal{P} \) such that \( v - p \) belongs to \( h^r \mathcal{V}_{\text{crit}}(\mathfrak{g}_{2n+1}) \). This implies \( \mathcal{P} = \mathcal{V}_{\text{crit}}(\mathfrak{g}_{2n+1}) \).

It remains to prove that the algebra \( \mathcal{P} \) is commutative. By [10, Proposition 3.8], the center of every \( h \)-adic quantum vertex algebra is \( \mathcal{S} \)-commutative, i.e., we have

\[ Y(z_1)(1 \otimes Y(z_2))s(z_1 - z_2)(a \otimes b) = Y(b, z_2)Y(a, z_1) \text{ for all } a, b \in \mathfrak{z}(\mathcal{V}_{\text{crit}}(\mathfrak{g}_{2n+1})). \]

By setting \( a = \Phi_{2i0}, b = \Phi_{2j0} \) and using Theorem 2.5, we find

\[ Y(\Phi_{2i0}, z_1)Y(\Phi_{2j0}, z_2) = Y(\Phi_{2j0}, z_2)Y(\Phi_{2i0}, z_1). \quad (2.64) \]

Due to [17, Lemma 2.13], we have

\[ Y(Dv, z) = \frac{d}{dz} Y(v, z) \text{ for all } v \in \mathcal{V}_{\text{crit}}(\mathfrak{g}_{2n+1}), \]

so by applying the partial derivatives \( \partial^k / \partial z_1^k \) and \( \partial^m / \partial z_1^m \) to (2.64), we get

\[ Y(D^k \Phi_{2i0}, z_1)Y(D^m \Phi_{2j0}, z_2) = Y(D^m \Phi_{2j0}, z_2)Y(D^k \Phi_{2i0}, z_1). \]

By arguing as in the proof of [15, Proposition 3.4], one can prove that this implies

\[ Y(a, z_1)Y(b, z_2) = Y(b, z_2)Y(a, z_1) \text{ for all } a, b \in \mathfrak{z}(\mathcal{V}_{\text{crit}}(\mathfrak{g}_{2n+1})). \]

Finally, applying this equality to the vacuum vector \( 1 \) and then taking the constant terms with respect to the variables \( z_1 \) and \( z_2 \) we find \( a \cdot b = b \cdot a \), as required. \( \square \)

It is worth to single out the commutativity property of the restriction of vertex operator map (2.9) on the center, which was obtained in the proof of Theorem 2.8.

**Corollary 2.9** For any \( a, b \in \mathfrak{z}(\mathcal{V}_{\text{crit}}(\mathfrak{g}_{2n+1})) \), we have

\[ Y(a, z_1)Y(b, z_2) = Y(b, z_2)Y(a, z_1). \]

By arguing as in the proof of Theorem 2.8, one obtains the following partial result on the quantum center in types \( C \) and \( D \).

**Corollary 2.10** The algebra \( \mathfrak{z}(\mathcal{V}_{\text{crit}}(\mathfrak{g}_N)) \) with \( \mathfrak{g}_N = \mathfrak{sp}_{2n}, \mathfrak{o}_{2n} \) contains the \( h \)-adically completed polynomial algebra in infinitely many variables

\[ \mathbb{C}[D^k \Phi_{2i0} : i = 1, \ldots, p_{\mathfrak{g}_N}, k \geq 0] [[h]]. \quad (2.65) \]
In particular, the commutativity of the restriction of the vertex operator map \( Y(\cdot) \) on (2.65) is established by arguing as in the last part of the proof of Theorem 2.8. Recall that by (2.41) the center \( z(V_{\text{crit}}(\mathfrak{g}_N)) \) consists of \( Y(\mathfrak{g}_N) \)-invariants, so that we have

\[
Y\left( T_{[n]}^+(u) 1, z \right) v = T_{[n]}^+(u|z) v \quad \text{for all } v \in z(V_{\text{crit}}(\mathfrak{g}_N)).
\]

By combining these two observations with Corollaries 2.9 and 2.10, we obtain commutative subalgebras of the dual Yangian in types \( B, C \) and \( D \), as suggested by [20, Remark 11.2.5].

**Corollary 2.11** The coefficients of the series \( T_m^+(u) \), where \( m = 1, \ldots, N \) in the orthogonal case and \( m = 1, \ldots, N/2 \) in the symplectic case, generate a commutative subalgebra of the dual Yangian \( Y^+(\mathfrak{g}_N) \).

### 3 Central elements of the completed double Yangian at the critical level

We now employ (2.42) to obtain explicit formulae for families of central elements of the appropriately completed double Yangian at the critical level. Introduce the completion of the double Yangian \( \tilde{\text{DY}}_c(\mathfrak{g}_N) \) at the level \( c \in \mathbb{C} \) as the inverse limit

\[
\tilde{\text{DY}}_c(\mathfrak{g}_N) = \lim_{\leftarrow} \text{DY}_c(\mathfrak{g}_N)/I_p,
\]

where \( p \geq 1 \) and \( I_p \) denote the \( h \)-adically completed left ideal of \( \text{DY}_c(\mathfrak{g}_N) \) generated by all elements \( t_{ij}^{(r)} \) with \( r \geq p \).

From now on, we consider the completed double Yangian \( \tilde{\text{DY}}_{\text{crit}}(\mathfrak{g}_N) \) at the critical level \( c_{\text{crit}} = -2\kappa/\sigma \). Introduce the series in \( \tilde{\text{DY}}_{\text{crit}}(\mathfrak{g}_N)[[u^{\pm 1}]] \) by

\[
\mathbb{T}_m(u) = \text{tr}_{1,...,m} S_{[m]} T_{[m]}^+(u_{[m]}) T_{[m]}(u_{[m]} - h\kappa)^{-1} = \text{tr}_{1,...,m} S_{[m]} T_1^+(\vec{u}_1) \cdots T_m^+(\vec{u}_m) T_m(\vec{u}_m - h\kappa)^{-1} \cdots T_1(\vec{u}_1 - h\kappa)^{-1}, \quad (3.1)
\]

where \( m = 1, \ldots, N \) in the orthogonal case, \( m = 1, \ldots, N/2 \) in the symplectic case, the trace is taken over all \( m \) copies of \( \text{End} \mathbb{C}^N \) and \( u_{[m]} = (\vec{u}_1, \ldots, \vec{u}_m) \) is given by (2.32). The proof of the next theorem is analogous to the proof of [5, Theorem 3.2].

**Theorem 3.1** All coefficients of \( \mathbb{T}_m(u) \) belong to the center of the completed double Yangian \( \tilde{\text{DY}}_{\text{crit}}(\mathfrak{g}_N) \).

**Proof** It is sufficient to verify the equalities

\[
T(v_0)\mathbb{T}_m(u) = \mathbb{T}_m(u)T(v_0) \quad \text{and} \quad T^+(v_{m+1})\mathbb{T}_m(u) = \mathbb{T}_m(u)T^+(v_{m+1}). \quad (3.2)
\]
Let us prove the first equality in (3.2). Label the tensor copies as follows:

\[
\begin{array}{c}
\vcenter{\hbox{\includegraphics[width=0.5\textwidth]{fig1.png}}}
\end{array}
\]

The elements

\[A = R_{1m}^{01}(v_0 - h\kappa|u_{[m]}|)^{-1} \quad \text{and} \quad B = R_{1m}^{01}(v_0 + h\kappa|u_{[m]}|)\]

satisfy

\[S_{[m]}^1A = \tilde{A} S_{[m]}^1, \quad S_{[m]}^1B = \tilde{B} S_{[m]}^1 \quad \text{and} \quad A \cdot B = 1. \quad (3.3)\]

Indeed, the first two equalities follow from (2.35), while the third equality is a consequence of crossing symmetry properties (1.16) and (1.17).

By applying \(T(v_0)\) on \(\mathcal{T}_m(u)\) and using defining relations (1.20) and (1.22), we find

\[
T(v_0)\mathcal{T}_m(u) = \mathcal{T}_{[1]}^0(v_0) \text{tr}_{1,...,m} S_{[m]}^1 T_{[m]}^{+12}(u_{[m]}) T_{[m]}^{12}(u_{[m]} - h\kappa)^{-1}
\]

\[
= \text{tr}_{1,...,m} S_{[m]}^1 T_{[m]}^{02}^0(v_0) T_{[m]}^{+12}(u_{[m]}) T_{[m]}^{12}(u_{[m]} - h\kappa)^{-1}
\]

\[
= \text{tr}_{1,...,m} S_{[m]}^1 T_{[m]}^{+12}(u_{[m]}) T_{[m]}^{02}^0(v_0) B T_{[m]}^{12}(u_{[m]} - h\kappa)^{-1}
\]

\[
= \text{tr}_{1,...,m} S_{[m]}^1 T_{[m]}^{+12}(u_{[m]}) T_{[m]}^{12}(u_{[m]} - h\kappa)^{-1} B T_{[1]}^{02}(v_0).
\]

Therefore, by employing the first equality in (3.3) we obtain

\[T(v_0)\mathcal{T}_m(u) = \text{tr}_{1,...,m} \tilde{A} S_{[m]}^1 T_{[m]}^{+12}(u_{[m]}) T_{[m]}^{12}(u_{[m]} - h\kappa)^{-1} B T_{[1]}^{02}(v_0). \quad (3.4)\]

Since \(S_{[m]} = (S_{[m]})^2\), we can write

\[
\text{tr}_{1,...,m} \tilde{A} S_{[m]}^1 T_{[m]}^{+12}(u_{[m]}) T_{[m]}^{12}(u_{[m]} - h\kappa)^{-1} B
\]

as

\[
\text{tr}_{1,...,m} \tilde{A} (S_{[m]}^1)^2 T_{[m]}^{+12}(u_{[m]}) T_{[m]}^{12}(u_{[m]} - h\kappa)^{-1} B.
\]

We now use (2.36), (2.37) and the first two equalities in (3.3) to move one copy of the symmetrizer \(S_{[m]}^1\) to the left and another copy to the right, thus getting

\[
\text{tr}_{1,...,m} S_{[m]}^1 A \tilde{T}_{[m]}^{+12}(u_{[m]}) \tilde{T}_{[m]}^{12}(u_{[m]} - h\kappa)^{-1} \tilde{B} S_{[m]}^1.
\]

By the cyclic property of the trace and \((S_{[m]})^2 = S_{[m]}\), this equals

\[
\text{tr}_{1,...,m} A \tilde{T}_{[m]}^{+12}(u_{[m]}) \tilde{T}_{[m]}^{12}(u_{[m]} - h\kappa)^{-1} \tilde{B} S_{[m]}^1.
\]
Next, we employ (2.36), (2.37) and the second equality in (3.3) to move the symmetrizer to the left, thus getting

$$\text{tr}_{1,\ldots,m} A S^{1}_{[m]} T^{+12}_{[m]} (u_{[m]}) T^{12}_{[m]} (u_{[m]} - h\kappa)^{-1} B.$$ 

Finally, by the cyclic property of the trace and the third equality in (3.3) this is equal to

$$\text{tr}_{1,\ldots,m} S^{1}_{[m]} T^{+12}_{[m]} (u_{[m]}) T^{12}_{[m]} (u_{[m]} - h\kappa)^{-1} = \mathbb{T}_m(u).$$

Therefore, we conclude that the right-hand side in (3.4) equals $\mathbb{T}_m(u) T(v_0)$, as required.

Let us prove the second equality in (3.2). Label the tensor copies as follows:

$$\underbrace{1}_{(\text{End } \mathbb{C}^N)^{\otimes m}} \otimes \underbrace{2}_{\text{End } \mathbb{C}^N} \otimes \underbrace{3}_{\text{DYcrit}(\mathbb{B}_N)}.$$ 

The elements

$$X = \overline{R}^{12}_{m1} (u_{[m]}|v_{m+1}) \quad \text{and} \quad Z = \overline{R}^{12}_{m1} (u_{[m]}|v_{m+1} + 2h\kappa)^{-1}$$

satisfy

$$S^{1}_{[m]} X = \tilde{X} S^{1}_{[m]} , \quad S^{1}_{[m]} Z = \tilde{Z} S^{1}_{[m]} \quad \text{and} \quad X \cdot Z \underset{\text{RL}}{=} 1.$$ (3.5)

As with (2.35), the first two equalities in (3.5) can be proved by using Yang–Baxter equation (1.11) and fusion procedure (2.33). The third equality is a consequence of crossing symmetry properties (1.16) and (1.17).

By applying $T^{+}_+ (v_{m+1})$ on $\mathbb{T}_m(u)$ and using defining relations (1.21) and (1.22), we find

$$T^{+}_+ (v_{m+1}) \mathbb{T}_m(u) = T^{+13}_{[1]} (v_{m+1}) \text{tr}_{1,\ldots,m} S^{1}_{[m]} T^{+13}_{[m]} (u_{[m]}) T^{13}_{[m]} (u_{[m]} - h\kappa)^{-1}$$

$$= \text{tr}_{1,\ldots,m} S^{1}_{[m]} T^{+13}_{[1]} (v_{m+1}) T^{+13}_{[m]} (u_{[m]}) T^{13}_{[m]} (u_{[m]} - h\kappa)^{-1}$$

$$= \text{tr}_{1,\ldots,m} S^{1}_{[m]} X T^{+13}_{[m]} (u_{[m]}) T^{+13}_{[m]} (v_{m+1}) X^{-1} T^{13}_{[m]} (u_{[m]} - h\kappa)^{-1}$$

$$= \text{tr}_{1,\ldots,m} S^{1}_{[m]} X T^{+13}_{[m]} (u_{[m]}) T^{13}_{[m]} (u_{[m]} - h\kappa)^{-1} Z T^{+13}_{[1]} (v_{m+1}).$$

Therefore, by employing the first equality in (3.5) we obtain

$$T^{+}_+ (v_{m+1}) \mathbb{T}_m(u) = \text{tr}_{1,\ldots,m} \tilde{X} S^{1}_{[m]} T^{+13}_{[m]} (u_{[m]}) T^{13}_{[m]} (u_{[m]} - h\kappa)^{-1} Z T^{+13}_{[1]} (v_{m+1}).$$ (3.6)

Since $S_{[m]} = (S_{[m]})^2$, we can write

$$\text{tr}_{1,\ldots,m} \tilde{X} S^{1}_{[m]} T^{+13}_{[m]} (u_{[m]}) T^{13}_{[m]} (u_{[m]} - h\kappa)^{-1} Z$$

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as
\[ \text{tr}_{1,\ldots,m} \tilde{X} (S^1_{[m]})^2 T^{+13}_{[m]}(u_{[m]}) T^{13}_{[m]}(u_{[m]} - h\kappa)^{-1} Z. \]

We now use (2.36), (2.37) and the first two equalities in (3.5) to move one copy of the symmetrizer \( S_{[m]} \) to the left and another copy to the right, thus getting
\[ \text{tr}_{1,\ldots,m} S^1_{[m]} X T^{+13}_{[m]}(u_{[m]}) T^{13}_{[m]}(u_{[m]} - h\kappa)^{-1} \tilde{Z} S^1_{[m]}. \]
By the cyclic property of the trace and \((S_{[m]})^2 = S_{[m]}\), this equals
\[ \text{tr}_{1,\ldots,m} X T^{+13}_{[m]}(u_{[m]}) T^{13}_{[m]}(u_{[m]} - h\kappa)^{-1} \tilde{Z} S^1_{[m]}. \]
Next, we employ (2.36), (2.37) and the second equality in (3.5) to move the symmetrizer to the left, thus getting
\[ \text{tr}_{1,\ldots,m} S^1_{[m]} T^{+13}_{[m]}(u_{[m]}) T^{13}_{[m]}(u_{[m]} - h\kappa)^{-1} Z. \]
Finally, by the cyclic property of the trace and the third equality in (3.5) this is equal to
\[ \text{tr}_{1,\ldots,m} S^1_{[m]} T^{+13}_{[m]}(u_{[m]}) T^{13}_{[m]}(u_{[m]} - h\kappa)^{-1} = T^m(u). \]
This implies that the right-hand side in (3.6) is equal to \( T^m(u) T^+(v_{m+1}) \), as required. Therefore, we proved both equalities in (3.2), so the theorem follows.

In the end, it is worth noting the following equality
\[ T^m(u) = Y(T^+_m(0)1, u) \]
for operators on \( \mathcal{V}_{\text{crit}}(\mathfrak{g}_N) \), which suggests the form of formulae (3.1), as well as the close connection between double Yangians and the corresponding \( h \)-adic quantum vertex algebras which is yet to be investigated.

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**Compliance with ethical standards**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.
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