Bosonization out of equilibrium

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Abstract – We develop a bosonization technique for one-dimensional fermions out of equilibrium. The approach is used to study a quantum wire attached to two electrodes with arbitrary energy distributions. The non-equilibrium electron Green function is expressed in terms of functional determinants of a single-particle “counting” operator with a time-dependent scattering phase. The result reveals an intrinsic relation of dephasing and energy redistribution in Luttinger-liquids to “fractionalization” of electron-hole excitations in the tunneling process and at boundaries with leads.

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One-dimensional (1D) interacting fermionic systems are described as Luttinger liquids (LL) [1], whose experimental manifestations include carbon nanotubes, semiconductor, metallic and polymer nanowires, as well as quantum Hall edges. While equilibrium LL have been extensively explored, there is currently a growing interest in non-equilibrium phenomena on the nanoscale. The major obstacle for a theoretical study of quantum wires out of equilibrium is that bosonization, a technique most suitable for analytic treatment of LL, has been so far restricted to equilibrium situations.

In this work we develop a bosonic theory for interacting 1D fermions under generic non-equilibrium conditions. Our goal is to calculate the electron Green functions (GFs) $G^\leq(\tau)$ measurable in tunneling spectroscopy experiments [2,3]. The configuration we have in mind is shown in fig. 1: electrons with distributions $n_\eta(\epsilon)$ ($\eta = R, L$ labels right and left movers, respectively) are injected into the LL wire from two non-interacting electrodes. We assume no correlations in the incident fermionic states; such non-equilibrium states are created by external potentials acting on non-interacting fermions. Further we assume that the interaction with phonons in the wire is negligible, so that the problem is determined by incoming distributions. Our first step in addressing the non-equilibrium regime [3,4] was by means of functional bosonization approach [5], employing random phase approximation (RPA). This is controllable for leads with thermal distributions, albeit with different temperatures and chemical potentials. Here we consider general distribution functions; a particularly interesting case involves double-step distributions [4].

Our description of the problem begins with Keldysh action $S = S_0[\psi] + S_{\text{ee}}[\psi]$ in terms of fermionic fields [6],

$$S_0[\psi] = i \sum_\eta \int dt \int dx \psi^\dagger_\eta \partial_\eta \psi_\eta,$$

$$S_{\text{ee}}[\psi] = - \sum_\eta \int dt \int dx g(x)(\rho_{\eta\rho_{-\eta}} + \rho_{\eta}\rho_{\eta}).$$

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Fig. 1: (Colour on-line) Schematic view of a LL conductor connected to leads with two different incoming fermionic distributions. The LL interaction parameter $K(x)$ is also shown; the dashed line corresponds to its sharp variation at the boundaries.
Here \( \rho_0 = \psi_1^\dagger \psi_0 \) are density fields, \( \partial_\eta = \partial_t + \eta v \partial_x \), \( v \) is the Fermi velocity, and \( \eta = \pm 1 \) stands for right/left moving electrons. The spatially dependent interaction strength \( g(x) \) vanishes outside the interval \([-L/2, L/2]\); this way of modeling leads was introduced in [7] to study the conductance of a LL wire. The essence of bosonization is a reformulation of the theory in terms of bosonic (density) fields. The interacting part of the action \( S_{\text{int}} \) is already expressed in terms of density modes \( \rho_0 \). The challenge is to bosonize the free part of the action, where information concerning the state of the non-interacting fermionic system is encoded. The bosonic counterpart of the free part of the action is found by requiring that it reproduces correlation functions of a non-equilibrium free Fermi gas [8],

\[
S_0 = \sum_\eta \left( -\rho_0 \Pi_\eta^{-1} \dot{\rho}_\eta - i \ln Z_0[\chi_\eta] \right).
\]

(1)

Here we have performed a rotation in Keldysh space, decomposing fields into classical and quantum components, \( \rho, \dot{\rho} = (\rho_+ + \rho_-)/\sqrt{2} \), where the indices + and − refer to the two branches of the Keldysh contour; \( \Pi_\eta \) is the advanced component of the polarization operator, and \( Z_0[\chi_\eta] \) is a partition function of free fermions moving in the field

\[
\chi_\eta = \Pi_\eta^{-1} \dot{\rho}_\eta.
\]

(2)

The expansion of \( Z_0[\chi_\eta] \) in \( \chi_\eta(t, x) \) generates an infinite series, \( \ln Z_0[\chi_\eta] = \sum_\eta (-1)^{n+1} \chi_\eta^n S_{\eta n}/n \), governed by irreducible fermionic density correlation functions, \( S_{\eta n}(t_1, x_1; \ldots; t_n, x_n) \equiv \langle \langle \rho_{\eta 1} \rho_{\eta 2} \cdots \rho_{\eta n} \rangle \rangle \), representing cumulants of quantum noise [9]. At equilibrium the RPA is exact [10]. Let \( S_{\eta n} = 0 \) for \( n > 2 \), and one obtains the Gaussian theory of conventional bosonization. In a generic non-equilibrium situation, the bosonic action contains terms of all orders with no small parameter; the idea to proceed analytically in a controlled manner may seem hopeless. This, however, is not the case: non-equilibrium bosonization is an efficient framework in which the functional integration can be performed exactly. Two observations are crucial here. First, \( Z_0 \) depends only on the quantum component \( \rho \), so that the action, eq. (1), is linear with respect to the classical component \( \rho \) of the density field. Hence the integration with respect to \( \rho \) can be performed exactly, generating an equation that fixes \( \rho \). As a result, the many-body field theory is mapped onto a problem which has a level of complexity of quantum mechanics. Second, all vertex functions \( S_{\eta n} \) determining \( Z_0[\rho_\eta] \) are restricted to the “mass shell”, \( \omega = \eta q v \), with respect to all of their arguments. This will render the effective “quantum mechanics” one-dimensional.

We apply this formalism first to free fermions away from equilibrium, and obtain the single-particle GFs \( G_{0,\eta}^F(x, t) = i(\psi_\eta^\dagger(0, 0)\psi_\eta(x, t)) \), \( G_{0,\eta}^F(x, t) = -i(\psi_\eta(x, t)\psi_\eta^\dagger(0, 0)) \) by means of the bosonic theory. Since \( G_{0,\eta} \) depends on \( \tau = t - \eta x/v \) only, we set \( x = 0 \) and \( \tau = t \) in the arguments of \( G_{0,\eta} \). The fermionic operators are expressed in terms of the bosonic field in the standard way, \( \hat{\psi}_\eta(x) \simeq (\Lambda/2\pi v) e^{i\eta x v} e^{i\hat{\phi}_\eta(x)} \), where \( \hat{\psi}_\eta(x) = \eta \partial_x \phi_\eta/2\pi \) and \( \Lambda \) is an ultraviolet cut-off. After integration over the classical component we find that the quantum component \( \hat{\rho} \) satisfies a continuity equation,

\[
\partial_\tau \hat{\rho}_\eta + \eta v \partial_x \hat{\rho}_\eta = j(t, x),
\]

(3)

with the source term \( j = \delta(x)(\delta(t - \tau) - \delta(t))/\sqrt{2} \). The fermionic GF is obtained as (with \( \chi_\eta \) given by eq. (2))

\[
G_{0,\eta}^\chi(\tau) = \frac{1}{2\pi v} \frac{1}{\tau + i/\Lambda} Z_0[\chi_\eta].
\]

(4)

The mass-shell nature of \( S_{\eta n} \) implies that \( Z_0[\chi_\eta] \) depends only on the integral along the world line (trajectory of a free particles moving with velocity \( v \))

\[
\delta_\eta(t) = \sqrt{2} \int_{-\infty}^{\infty} dt \chi_\eta(\tilde{t} - t, \eta v \tilde{t}).
\]

(5)

More specifically, we find [8]

\[
Z_0[\chi_\eta] = \det[1 + (e^{-i\delta_\eta} - 1)\hat{n}_\eta] \equiv \Delta_\eta[\delta_\eta(t)].
\]

(6)

For the free-fermion problem the phase \( \delta_\eta(t) = \omega_\eta(t, 0) \) where \( \omega_\eta(t, \tilde{t}) = \theta(\tilde{t} - t) - \theta(\tilde{t} - t - \tau) \) is a “window function” and \( \lambda = 2\pi \). Thus, \( Z[\chi_\eta] = \Delta_\eta(2\pi) \), where \( \Delta_\eta(\lambda) \) is the determinant (6) for a rectangular pulse.

Since \( \hat{n}_\eta \) is diagonal in energy space, while \( \delta_\eta \) is diagonal in time space, they do not commute, making the determinant non-trivial. Determinants of the type (6) have appeared in a theory of counting statistics [9]. Specifically, the generating function of current fluctuations \( \kappa(\lambda) = \sum_{\eta n=-\infty}^{\infty} e^{in\lambda p_\eta} \) (where \( p_\eta \) is the probability of \( n \) electrons being transferred through the system in a given time window \( \tau \) has the same structure as \( \Delta_\eta(\lambda) \). A Taylor expansion of \( \ln \kappa(\lambda) \) around \( \lambda = 0 \) defines cumulants of current fluctuations.

According to its definition, \( \kappa(\lambda) \) is \( 2\pi \)-periodic, which is a manifestation of charge quantization. Thus, \( \kappa(2\pi) = 1 \) is trivial. On the other hand, we have found that the free electron GF is determined by the value of the functional determinant exactly at \( \lambda = 2\pi \). A resolution of this apparent paradox is as follows: the determinant \( \Delta_\eta(\lambda) \) should be understood as an analytic function of \( \lambda \) increasing from 0 to \( 2\pi \). On the other hand, \( \kappa(\lambda) \) is non-analytic at the branching points \( \lambda = \pm \pi, \pm 3\pi, \ldots \). To demonstrate this, it is instructive to consider the equilibrium case (temperature \( T \)). Then the expansion of \( \ln \Delta_\eta(\lambda) \) in \( \lambda \) is restricted to the \( \lambda^2 \) term (since RPA is exact). It is easy to check that the \( \lambda = 2\pi \) point on this
and finally approaches $\epsilon$.

Continuation of the functional determinant. The value of $\Delta_{rT}(\lambda)$ at $\lambda = 2\pi$ determines the free-electron GF, while $\ln \kappa(2\pi) = 0$ in view of charge quantization. As an example, the equilibrium case is shown. Inset: contour of integration for the quasiclassical limit, eq. (7), of $\Delta_{rT}(\lambda)$ is deformed, since a singularity of the integrand crosses the real axis at $\lambda = \pi$.

Parabolic dependence correctly reproduces the fermion GF via eqs. (4), (6). As to the counting statistics $\ln \kappa(\lambda)$, it is quadratic only in the interval $[-\pi, \pi]$ and is periodically continued beyond this interval, see fig. 2.

The difference in analytical properties of $\kappa(\lambda)$ and $\Delta_{rT}(\lambda)$ becomes especially transparent if one studies the semiclassical (long-$\tau$) limit,

$$\ln \Delta_{rT}[\lambda] = \frac{\tau}{2\pi \hbar} \int dx \ln \left[ 1 + (\epsilon - i\lambda - 1)\eta_\tau(\epsilon) \right]. \quad (7)$$

For small $\lambda$ the singularity of the integrand closest to the real axis is located near $\epsilon = i\pi T$. As $\lambda$ increases, the singularity moves towards the real axis, crosses it at $\lambda = \pi$ and finally approaches $\epsilon = -i\pi T$ as $\lambda \to 2\pi$ (see inset of fig. 2). The integral for $\ln \kappa(\lambda)$ is taken along the real axis, resulting in non-analyticity at $\lambda = \pi$ and in zero value at $\lambda = 2\pi$. On the other hand, the contour for $\ln \Delta_{rT}(\lambda)$ is deformed to preserve analyticity, resulting in the long-$\tau$ asymptotics $\ln \Delta_{rT} = -\tau T \lambda^2 / 4\pi$.

The value of $\lambda = 2\pi$ appearing in the bosonic representation of the free-fermion GF $G_{0,0T}(\lambda)$ has a clear physical meaning: a fermion is a $2\pi$-soliton in the bosonic formalism [11]. A natural question to ask is whether values of $\Delta_{rT}(\lambda)$ away from $\lambda = 2\pi$ are physically important. This is indeed the case. A simple example is non-equilibrium Fermi edge singularity problem [8] (considered in ref. [12] within a fermionic approach). Functional determinants of the type $\Delta_r(\lambda)$ can be efficiently evaluated numerically [13].

We are now prepared to address the problem formulated in the beginning of the paper: an interacting quantum wire out of equilibrium, fig. 1. We consider $G_{0T}(\tau)$ for the tunneling point ($x = 0$) located inside the interacting part; generalization to tunneling into a non-interacting region is straightforward. The bosonized action is a sum of $S_b$, (as given by eqs. (1), (2)) and $S_{sc}$. Integrating over the classical components $\rho_\eta$ as explained above, we obtain equations satisfied by the quantum components $\tilde{\rho}_\eta$ of the density fields,

$$\begin{align*}
\partial_t \tilde{\rho}_R + \partial_x \left[ \left( v + \frac{g}{2\pi} \right) \tilde{\rho}_R + \frac{g}{2\pi} \tilde{\rho}_L \right] &= j, \\
\partial_t \tilde{\rho}_L - \partial_x \left[ \left( v + \frac{g}{2\pi} \right) \tilde{\rho}_L + \frac{g}{2\pi} \tilde{\rho}_R \right] &= 0.
\end{align*} \quad (8)$$

The solution of eq. (8) determines the phases $\delta_\eta(t)$ according to eqs. (5), (2), which can be cast in the form

$$\delta_\eta(t) = -2\pi \sqrt{2} \eta T \lim_{\tau \to -\infty} \int_0^{\tau} dx \tilde{\rho}_\eta(x, \tilde{t}) \quad (9)$$

Remarkably, eq. (9) expresses the phase $\delta_\eta(t)$ affected by the electron-electron interaction, through the asymptotic behavior of $\tilde{\rho}(x, t)$ in the non-interacting parts of the wire (regions I and III in fig. 1). The phases $\delta_\eta(t)$ determine the GFs via

$$G_{0T}^\infty(\tau) = -\frac{\Delta R[\delta_\eta(t)]}{\Delta L[\delta_\eta(t)]} \quad (10)$$

where $\gamma = (1 - K)^2 / 2K$ and $K = (1 + g / \pi v)^{-1 / 2}$ is the standard LL parameter in the interacting region.

To explicitly evaluate $\delta_\eta(t)$ for the structure of fig. 1, it is convenient to rewrite eqs. (8) as a second-order differential equation for the current $J = v(\tilde{\rho}_R - \tilde{\rho}_L)$,

$$\omega^2 + \partial_{\tau^2}(x) \partial_x J(\omega, x) = 0, \quad x \neq 0. \quad (11)$$

where $u(x) = v(1 + g(x) / \pi v)^{1 / 2}$ is a spatially dependent plasmon velocity. Reflection and transmission of plasmons on both boundaries is characterized by the coefficients $r_\eta$, $t_\eta$ ($r_\eta^2 + t_\eta^2 = 1$); for simplicity, we assume them to be constant over a characteristic frequency range $\omega \sim \tau^{-1}$.

Subsequently $\delta_\eta(t)$ is found to be a superposition of rectangular pulses, $\delta_\eta(t) = \sum_{n=0}^{\infty} \delta_\eta, n \omega_n(t, t_n)$, where $t_n = (n + 1 / 2 - 1 / 2K) / L / u$, and

$$\begin{align*}
\delta_\eta, 2m &= \pi t_\eta \eta^m R(1 + \eta K) \sqrt{K}, \\
\delta_\eta, 2m + 1 &= \pi t_\eta \eta^m R(1 - \eta K) \sqrt{K}. \quad (12)
\end{align*}$$

The phases $\delta_\eta(t)$ are shown in fig. 3 for two limits of adiabatic ($r_\eta = 0$) and sharp $[r_\eta = (1 - K) / (1 + K)]$ boundaries.

In physical terms, $\delta_\eta(t)$ characterizes phase fluctuations in the leads that arrive at the time interval $[0, \tau]$ at the measurement point $x = 0$, governing the dephasing and the energy distribution of electrons encoded in the GFs $G_{0T}^\infty(\tau)$. Up to inversion of time, one can think of $\delta_\eta(t)$ as describing the fractionalization of a phase pulse (electron-hole pair) injected into the wire at point $x$ during
the time interval \([0, \tau]\). This is closely related to the
physics of charge fractionalization discussed earlier \([7,14]\).
At the first step, the pulse splits into two with relative
amplitudes \((1+K)/2\) and \((1-K)/2\) carried by plasmons
in opposite directions. As each of these pulses reaches
the corresponding boundary, another fractionalization
process takes place: a part of it is transmitted into a lead,
while the rest is reflected. The reflected pulse reaches
the other boundary, is again fractionalized there, etc.
Let us stress an important difference between boundary
fractionalization of transmitted charge \([7]\) and that of
dipole pulses. While in the former case the boundaries can
always be thought as sharp, in the present problem the
way \(K(x)\) is turned on is crucially important for reflection
coefficients \(\tau_\eta\) at \(\omega \sim \tau^{-1}\).

For \(\tau \ll L/u\) the coherence of plasmon scattering may be
neglected and the result splits into a product
\[
\Delta_\eta[\delta_\eta(t)] \approx \prod_{n=0}^{\infty} \Delta_{\eta n}(\delta_{\eta n}),
\]
with each factor representing a contribution of a single
phase pulse \(\delta_{\eta n}(t) = \delta_{\eta n}(w/\tau, 0)\). For the “partial
equilibrium” state (when \(n_R(t)\) and \(n_L(t)\) are of Fermi-Dirac form
but with different temperatures) the functional determinants
are Gaussian functions of phases, reproducing earlier
results of functional bosonization \([3]\).

We now apply our general result (13) to the “full
non-equilibrium” case, when \(n_\eta(\epsilon)\) have a double-step
form, \(n_\eta(\epsilon) = a_\eta \theta(V_R - \epsilon) + (1 - a_\eta) \theta(-\epsilon)\). The split
electron-bias-anomaly dips are then broadened by non-equilibrium
dephasing rate, \(1/\tau_\eta^R\) \([3,4]\). The latter can be found from
the long-time asymptotics of \(G_\eta(\tau)\), employing eq. (7).
We obtain \(1/\tau_\phi^R = 1/\tau_\phi^{RR} + 1/\tau_\phi^{RL}\), where \(1/\tau_\phi^{RR}\) is the
contribution to dephasing of \(\eta\) fermions governed by the
distribution of \(\eta\) fermions. Focussing on the adiabatic
limit, we find these dephasing rates to be
\[
1/\tau_\phi^{RR} = -\frac{eV_R}{2\pi} \ln \left( 1 - 4a_\eta(1-a_\eta) \sin^2 \frac{\pi(1+\eta K)}{2\sqrt{R}} \right),
\]
see fig. 4. Two remarkable features of this result should be
pointed out. First, for certain values of the interaction parameter \(K\)
(different for \(\eta = R,L\)) the dephasing rates \(1/\tau_\phi^{RR}\) vanish. Second, consider the weak-interaction
regime, \(\gamma \ll 1\). We then obtain \(1/\tau_\phi^{RL} \approx \pi \gamma eV_L a_L(1-a_L)\) and \(1/\tau_\phi^{RR} \approx \pi (\gamma^2/8) eV_R a_R(1-a_R)\). This should
be contrasted with RPA which predicts equal \(1/\tau_\phi^{RL}\) and
\(1/\tau_\phi^{RR}\) \([4]\). While \(1/\tau_\phi^{RL}\) agrees with the RPA result,
\(1/\tau_\phi^{RR}\) is parametrically smaller (suppressed by an extra
factor of \(\gamma\)). The reason for this failure of RPA is clear from
our analysis. For a weak interaction the contributions of \(R\)
and \(L\) movers to \(G_R\) are given by the functional determinants
\(\Delta_{\eta R}(\delta_{\eta R})\) with phases (for an adiabatic barrier) \(\delta_{\eta R} \approx
(1-K)\pi\) and \(\delta_R \approx \pi(1+K)\). While the contribution of
the small phase \(\delta_L\) is captured correctly by RPA, a small-\(\delta\)
expansion of \(\ln \Delta_{\eta R}(\delta_{\eta R})\) fails for large \(\delta_{\eta R}\) (apart from equi-
and “partial equilibrium” where \(\ln \Delta_{\eta R}(\delta) \propto \delta^2\).)

To conclude, we have developed a non-equilibrium
bosonization technique and used it to find GFs for tunnel-
ning spectroscopy of a LL conductor. Our solution is
presented in terms of functional determinants of single-
particle operators. It elucidates the relation between the
charge fractionalization on the one hand, and energy redis-
tribution and non-equilibrium dephasing, on the other
hand. We have calculated dephasing broadening of the
ZBA dips for double-step distributions. Our approach has
various further applications, including spinful electrons,
interferometry, noise, and many-body entanglement \([8]\).
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