Optimal global synchronization of partially forced Kuramoto oscillators

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We consider the problem of global synchronization in a large random network of Kuramoto oscillators where some of them are subject to an external periodically driven force. We explore a recently proposed dimensional reduction approach and introduce an effective two-dimensional description for the problem. From the dimensionally reduced system, we obtain analytical predictions for some critical parameters necessary for the onset of a globally synchronized state in the system. Moreover, the reduced system also allows us to introduce an optimization scheme for the problem. Our main conclusion, which has been corroborated by exhaustive numerical simulations, is that for a given large random network of Kuramoto oscillators with random natural frequencies $\omega_i$ such that a fraction of them will be subject to an external periodic force with frequency $\Omega$, the best global synchronization properties correspond to the case where the fraction of the forced oscillators is chosen to be those ones such that $|\omega_i - \Omega|$ is maximal. Our results might shed some light on the structure and evolution of natural systems for which the presence or the absence of global synchronization are desired properties. Some properties of the optimal forced networks and its relation to recent results in the literature are also discussed.

We consider here the dynamics of a large number of interacting Kuramoto oscillators. Each oscillator has its own natural frequency $\omega_i$, which is assumed to be a random variable, and the interactions among them are associated with the edges of a random network. Despite of being essentially a random system, there are plenty of robust results obtained from this kind of model which have proven to be relevant in many different areas. This is the case, for instance, of synchronization phenomena, see \cite{1} and \cite{2} for comprehensive reviews on the subject. Here, we are concerned with one of the main variations of the network of Kuramoto oscillators, the case where some of the oscillators are subject to an external periodically driven force with frequency $\Omega$. By exploring a recently proposed analytical approach, we introduce an optimization scheme for the onset of the so-called global synchronization in the system, a regime where all oscillators rigidly rotate, forming a compact swarm, in the same pace of the external force, with frequency $\Omega$. We show that the best global synchronization properties correspond to the case where the set of forced oscillators is chosen to be those ones such that the value of $|\omega_i - \Omega|$ is maximal. Our results may help to understand the evolution and structure of natural systems for which the occurrence of global synchronization is a desired property.

\textbf{I. INTRODUCTION}

Synchronization in complex networks of oscillators is a paradigmatic problem which has received huge attention recently. Its intrinsically rich dynamical properties and vast applicability range have motivated a myriad of studies in very different areas, see Refs. \cite{1} and \cite{2} for comprehensive reviews on the subject. The mostly used model for synchronization studies is still the well-known Kuramoto oscillator, introduced more than forty years ago\textsuperscript{3,4} in the context of chemical oscillations. One of the main variations of the original synchronization problem is the case of a network of Kuramoto oscillators where some of them are subject to an external periodically driven force, a problem which has also attracted considerable attention in the last years, with possible application in many different scenarios\textsuperscript{5–12}. This is namely the problem we are concerned here, which, for the case of a network with $N$ Kuramoto oscillators, corresponds to the following dynamical equations

\begin{equation}
\frac{d\theta_i}{dt} = \omega_i + I_C(i)F \sin (\Omega t - \theta_i) + \lambda \sum_{j=1}^{N} A_{ij} \sin (\theta_j - \theta_i),
\end{equation}

where $\theta_i$ stands for the dynamical state (phase) of the Kuramoto oscillator with natural frequency $\omega_i$, which is assumed to be located at the $i$-node of the underlying network. The connections among the oscillators are represented by the usual undirected adjacency matrix with components $A_{ij}$, and $\lambda$ stands for the uniform coupling strength among the oscillators. We will consider here the so-called attractive case, for which $\lambda > 0$. The subset composed of the $N_C < N$ forced nodes is denoted by $C$, and $I_C(i)$ is its indicator function, i.e., $I_C(i) = 1$ if $i \in C$, or zero otherwise. The external driven force is also assumed to have uniform intensity $F$ and frequency $\Omega$ for all nodes $i \in C$. Without loss of generality, we can assume $F \geq 0$. In fact, if $\theta_i(t)$ is a solution of (1) for some external force $F$, $\theta_i(t) + \pi$ will be a solution for the case corresponding to $-F$, and both cases have the same asymptotic dynamical properties. We will also assume $\Omega > 0$ in our analysis, but we will see that the case $\Omega < 0$ can be treated analogously.

Our analysis will be restricted to the situations where the underlying network is an undirected and unweighted large

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random network. We assume also that \(\omega_i\) is a random variable with null average and symmetric distribution \(g(\omega)\) with all moments finite. Notice that we could also have written the system (1) in an autonomous manner by introducing an extra \(i = N + 1\) node with natural frequency \(\Omega\) and connected in a directional way to the forced nodes \(i \in C\). However, since our proposed approach is indeed more appropriate to the case of undirected and unweighted large random networks, we will deal with the original non-autonomous formulation of the problem (1).

There are several dynamical behaviors for the system (1) which would deserve to be named synchronization. Since the external driven force has its own frequency \(\Omega\), we will focus here on the network dynamical states for which the oscillators rigidly rotate in the same pace of the external force, with frequency \(\Omega\), and in a phase-locked way. This situation is also called in the literature as the global synchronization for (1). Of course, it is quite natural to expect the existence of a certain threshold \(F_c > 0\) for the external driven force such that, for \(F < F_c\), the system (1) would be rather insensitive to the external excitation. In fact, this is a well-known property of the system (1) for the case of a fully connected network topology, see [6] for further details. Here, we will employ the dimensional reduction approach recently proposed by Gottwald, [13] which is based in the introduction of some collective coordinates for the Kuramoto oscillators in the same spirit of the Ott-Antonsen ansatz [14–15], to investigate the global synchronization properties of (1) for the case of large random networks. The Gottwald approach was recently extended for the case of Stuart-Landau complex oscillators in [16]. From the dimensionally reduced system, we will derive some conditions on the parameter problem which will allow (or prevent) the onset of a globally synchronized state for the system. Moreover, in the same line of the rewiring algorithm discussed in [16] and [17] we will explore the dimensional reduction approach to propose an optimization scheme for global synchronization by selecting judiciously the forced nodes subset \(C\) for a given random network. The scheme has been exhaustively tested with numerical simulations and it has always resulted in an enhancement in the global synchronization properties of the system. Our main conclusion is that the optimal subset \(C\), in the sense that will be properly define in Section III, should consist in the nodes \(i\) such that \(|\Omega - \omega_i|\) is maximal, a result which is indeed in agreement with some recent related works [18-19] and that might shed some light on the evolution and structure of natural systems for which global synchronization is a desired property. Despite our analysis will be effectively done for the \(\Omega > 0\) case with symmetric distributions \(g(\omega)\), the optimization criterion of maximizing \(|\Omega - \omega_i|\) for choosing the subset \(C\) of forced nodes is indeed valid for the general case.

The present paper is organized as follows. In the next section, the Gottwald dimensional reduction approach [13] is adapted for the global synchronization problem of the forced Kuramoto oscillators (1). A mean field analysis is performed, and we derive some conditions on the parameter space of the problem for the onset of a globally synchronized state. In Section III we introduce our optimization scheme and present the results of our numerical simulations for large random networks. The last section is devoted to some concluding remarks on the implications of our results and on the role played by possible network symmetries on the global synchronization problem of forced Kuramoto oscillators, in the context of the recently introduced asymmetry-induced synchronization (AISync) scenario [12-14].

II. THE DIMENSIONAL REDUCTION APPROACH

We will describe the global state of the system (1) by using the standard order parameters \(r\) and \(\psi\) defined as

\[
r(t) = \frac{1}{N} \sum_{j=1}^{N} e^{i\psi_j(t)},
\]

whose respective behaviors are well known: \(r \approx N^{-1/2}\) for incoherent motion, whereas \(r \approx 1\) for a fully synchronized state, up to a common rotation for all oscillators. Our global synchronization corresponds to a phase locked configuration with a rigid rotation \(\Omega\), and hence to the case \(r \approx 1\) and \(\psi \approx \Omega\). We will look for globally synchronized states of (1) by employing the Gottwald dimensional reduction approach [13] which can be adapted for the the present case as the collective ansatz

\[
\theta_i(t) = \alpha(t) + \frac{\omega_i}{\Omega} \beta(t) + \Omega t,
\]

which has been indeed extensively tested in our numerical simulations, as we will discuss in the next section. Notice that, by construction, both reduced dynamical variables \(\alpha\) and \(\beta\) are dimensionless. Using such ansatz and recalling that \(\omega_i\) is assumed to be a random variable with null average and symmetric distribution \(g(\omega)\), we will have for large random networks

\[
r(\beta) = \left\{ \cos \frac{\beta \omega_i}{\Omega} \right\} = \int d\omega \, g(\omega) \cos \left( \frac{\beta \omega}{\Omega} \right)
\]

and

\[
\psi = \alpha + \Omega t.
\]

Here, we denote the simple average of a given variable \(h_k\) over the network as

\[
\langle h_k \rangle = \frac{1}{N} \sum_{k=1}^{N} h_k,
\]

and the continuous (mean field) approximation consisting basically in the substitution of the sum with the integral, as we have done in [1]. Notice that synchronized states with \(r \approx 1\) will demand \(\beta \approx 0\). In fact, the expression (4) can be expanded as

\[
r \approx 1 - \frac{\langle \omega^2 \rangle}{2\Omega^2} \beta^2 + \frac{\langle \omega^4 \rangle}{4\Omega^4} \beta^4 + \cdots,
\]

for any distribution \(g(\omega)\) with finite moments. It is clear that asymptotic globally synchronized states will correspond to solutions such that \(\beta \to 0\) and \(\alpha \to \) constant for \(t \to \infty\).
In order to obtain the two-dimensional reduced system in the new variable \( \alpha \) and \( \beta \), let us substitute (3) in (1), which will result in
\[
\dot{\alpha} + \frac{\omega_i \beta}{\Omega} = \omega_i - \Omega - I_c(i)F \sin \left( \alpha + \frac{\omega_i \beta}{\Omega} \right)
\]
(8)
\[+ \lambda \sum_{j=1}^{N} A_{ij} \sin \frac{\beta}{\Omega} (\omega_j - \omega_i).
\]
Averaging both sides leads to
\[
\dot{\alpha} = -\Omega - FI_1(\alpha, \beta),
\]
(9)
where the dimensionless function \( I_1(\alpha, \beta) \) is given by
\[
I_1(\alpha, \beta) = \frac{1}{N} \sum_{i=1}^{N} I_c(i) \sin \left( \alpha + \frac{\omega_i \beta}{\Omega} \right).
\]
(10)
On the other hand, multiplying Eq. (8) by \( \frac{\omega_i}{\Omega} \) and averaging both sides again, we get
\[
\dot{\beta} = \Omega - \frac{F \Omega^2}{\langle \omega^2 \rangle} I_2(\alpha, \beta) + \frac{A \Omega^2}{\langle \omega^2 \rangle} I_3(\beta),
\]
(11)
where
\[
I_2(\alpha, \beta) = \frac{1}{N} \sum_{i=1}^{N} I_c(i) \frac{\omega_i}{\Omega} \sin \left( \alpha + \frac{\omega_i \beta}{\Omega} \right)
\]
(12)
and
\[
I_3(\beta) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} \frac{\omega_i}{\Omega} \sin \frac{\beta}{\Omega} (\omega_j - \omega_i)
\]
(13)
are also dimensionless functions. The continuous (mean field) approximation in the present case corresponds to
\[
I_1(\alpha, \beta) = f r(\beta) \sin \alpha,
\]
(14)
\[
I_2(\alpha, \beta) = -f r'(\beta) \cos \alpha,
\]
(15)
\[
I_3(\beta) = \langle d_k \rangle \frac{d}{db} r^2(\beta),
\]
(16)
where \( f = N_c/N \) is the fraction of forced nodes and \( \langle d_k \rangle \) stands for the average degree of the network. In the derivation of such approximations, we employ the crucial hypothesis that the \( N_c \) forced nodes were also randomly chosen and, in particular, that their natural frequencies has the same distribution \( g(\omega) \). Equations (9) and (11) in this case read simply
\[
\dot{\alpha} = -\Omega - Fr(\beta) \sin \alpha,
\]
(17)
\[
\dot{\beta} = \Omega + r'(\beta) \frac{\Omega^2}{\langle \omega^2 \rangle} (\lambda \langle d_k \rangle r(\beta) + f F \cos \alpha).
\]
(18)
For \( r(\beta) = 1 \), equation (17) reduces to the well-known Adler equation, as in the full-connected topology case, see [6] and [11] for further details. Since \( 0 \leq r(\beta) \leq 1 \), equation (17) will never admit fixed points if
\[
\left| \frac{fF}{\Omega} \right| < 1.
\]
(19)
Thus, we have just established the critical value of the external force \( F_c \) for the onset of the global synchronization, which will always require
\[
F > F_c = \frac{\Omega}{f}.
\]
(20)
Of course, our main interest here is the stable fixed points \((\alpha_*, \beta_*)\) of the system (17) - (18). Since \( r(\beta) \) is non negative, we have that any stable fixed point in the present case is such that \( fF \cos \alpha_* > 0 \). One can explore the right-handed side of equation (18) to derive some critical value for \( \lambda \) in order to assure a globally synchronized state in the same way we did for \( F \) in (20), but it turns out that the situation is very similar to the case without the external force \( F = 0 \), which was treated in detail in some previous works [13][14]. In particular, one should not expect any stable fixed point for (17) - (18) if
\[
\frac{\lambda \langle d_k \rangle}{\Omega} + \frac{f F}{\Omega} < \zeta,
\]
(21)
where \( \zeta \) is the constant
\[
\zeta = -\frac{\langle \omega^2 \rangle}{\Omega^2 \min r'(\beta)} > 0.
\]
(22)
One can evaluate the constant \( \zeta \) for some commonly used distributions \( g(\omega) \) in the literature, namely the normal and the homogeneous distributions. For these cases, we have, respectively, the following continuous approximations for \( r \)
\[
F_n = \exp \left( -\frac{\langle \omega^2 \rangle \beta^2}{2\lambda^2} \right)
\]
(23)
and
\[
F_u = \sin \frac{\sqrt{3} \langle \omega^2 \rangle \beta}{\sqrt{3} \langle \omega^2 \rangle}.
\]
(24)
For the normal distribution, we will have \( \zeta = \sqrt{e} \approx 1.649 \), while for the uniformly distributed case we can numerically determine that \( \zeta \approx 2.293 \). However, in contrast with the critical force prediction (20), which has proved to be indeed quite accurate in our numerical simulations, the condition (21) for a possible \( \lambda_c \) seems to be a rather conservative one. Typically, one will need larger values for \( \lambda \) in order to have robust synchronized states. This situation is completely analogous to the \( F = 0 \) case discussed previously in [16] and [17].

III. THE OPTIMIZATION SCHEME

Our main purpose here is to present a prescription to select judiciously the subset \( C \) of forced nodes in order to assure a better synchronization capability for the network. We will consider that a network has a better capability for a globally synchronized state if one can attain states with \( r \approx 1 \) and \( \psi \approx \Omega \) with smaller values for the external force intensity \( F \). Since the globally synchronized states require \( \beta \approx 0 \), let us linearize (9) and (11) around \( \beta = 0 \) and abandon the hypothesis that \( C \)
is a random subset of nodes of the network. We will have in this case

\[
\dot{\alpha} = -\Omega - F F \left( \sin \alpha + \frac{\langle \omega_k \rangle C}{\Omega} \beta \cos \alpha \right),
\]

\[
\dot{\beta} = \Omega \left( 1 - \frac{\langle \omega^2 \rangle C}{\langle \omega^2 \rangle} F \sin \alpha \right)
- \left( F F \langle \omega^2 \rangle C \cos \alpha + \lambda L \right) \beta,
\]

where \( \langle \cdot \rangle_C \) denotes the average in the subset \( C \) and

\[
L = \frac{\sum_{i(j)}^2 (\omega_i - \omega_j)^2}{\sum_{k=1}^N \omega_k^2},
\]

where the sum in the numerator is performed over the edges \( e(i,j) \) of the network. The quadratic quantity \( L \) is known to play a crucial role in the usual \((F = 0)\) synchronization problem\[17\]. The larger the value of \( L \), the better the synchronization properties of the underlying network. The increasing of \( L \) by means of some rewiring operations in the network is the central point of the optimization algorithm introduced in \[17\], which consists basically in changing the edges \( e(i,j) \) of the network in order to connect oscillators such that \( |\omega_i - \omega_j| \) is maximal.

The fixed points of \[25\] and \[26\] are such that

\[
\frac{\beta_*}{\Omega} = \frac{1}{F F \left( \frac{\langle \omega^2 \rangle C}{\langle \omega^2 \rangle} - \frac{\langle \omega \rangle C}{\Omega} \right) \cos \alpha_* + \lambda L},
\]

from where we can see that one effectively attains better values for \( r \) (close to 1, requiring smaller \( \beta \)) if \( \langle \omega^2 \rangle_C \) is maximal and \( \langle \omega \rangle_C \) is minimal. This is equivalent to select the subset \( C \) of forced nodes as those ones with the minimal values of their frequencies \( \omega_i \) in the network. Since we are assuming \( \Omega > 0 \), one can say that the optimal subset \( C \) is formed by the oscillator such that \( |\omega_i - \Omega| \) is maximal. As we will see, this criterion is also valid for the \( \Omega < 0 \) case and even for more general distributions \( g(\omega) \). Notice that, from \[28\], we have that \( \lambda L \) also plays an important role here. As in the \( F = 0 \) case, the larger the value of \( \lambda L \), the larger the value of \( r \) (smaller \( \beta_* \)). On the other hand, the system is rather insensitive to the precise value of \( \alpha_* \). Our predictions obtained from the dimensionally reduced model, including the optimization criterion, were exhaustively tested in numerical simulations, whose main details will be presented in the following subsection.

### A. Numerical results

For our simulations, we have numerically solved the equations \[1\] for large random networks and construct several synchronization diagrams. We have made extensive use of the 3.7 Python packages NetworkX\[23\] which allows us to build many types of random networks with some prescribed topological and statistical properties, and Scipy\[24\] in particular its integrate.solve_ivp function. For our purposes here, it is more convenient to introduce the dimensionless evolution parameter \( \tau = \Omega \tau \) in \[1\], leading to the following system of ordinary differential equations

\[
\dot{\theta_i} = \frac{\omega_i}{\Omega} + 1 c(\tau) \frac{F}{\Omega} \sin(\tau - \theta_i) + \frac{\lambda}{\Omega} \sum_{j=1}^N A_{ij} \sin(\theta_j - \theta_i),
\]

where the prime denotes differentiation with respect to \( \tau \). As we can see, for a given random network with adjacency matrix \( A_{ij} \) and oscillator frequencies \( \omega_i \), our parameter space is effectively bi-dimensional and spanned by \( (F/\Omega, \lambda/\Omega) \). In this case, \( \Omega^{-1} \) simply plays the role of the unit of time for the problem. The equations \[29\] are the base of all our numerical analysis. We perform the simulations for different network topologies and frequencies distributions \( g(\omega) \) with null average, and we have not detected any appreciable dependence of our results on the network topology or the employed distribution \( g(\omega) \). All results presented here correspond to the case of an Erdos-Renyi random network with 1500 nodes, average degree \( d_k \approx 7.7 \) (the maximal node degree is 18, the minimal is 1), and with 151 nodes subjected to the external force. We employ a normal distribution \( g(\omega) \) with \( \sigma_\omega/\Omega = 1 \) for the oscillators natural frequencies. None of these parameters has demonstrated to have strong influence on our results. The initial conditions \( \theta_i(0) \) for the numerical solutions of \[29\] were randomly chosen, with uniform distribution, in the circle \([0, 2\pi]\).

Our first results, depicted in Fig. 1, correspond to the verification that the ansatz \[3\] is indeed valid for the study of global synchronization in the system \[29\]. The figure shows the \( (\omega_i, \theta_i(\tau) \mod 2\pi) \) graphics, for a fixed \( \tau \) in a globally synchronized regime with \( \lambda/\Omega = 2 \) and \( f F/\Omega = 3 \). Each point in the graphics corresponds to a Kuramoto oscillator in the network. The depicted line is the simple linear regression line of the data. The linear correlation between \( \omega_i \) and \( \theta_i \), as

![Fig. 1. The \((\omega_i, \theta_i(\tau) \mod 2\pi)\) graphics for a fixed \( \tau \) in a globally synchronized regime, for an Erdos-Renyi random network with 1500 nodes, with 151 of them subjected to the external periodic force. Each point in the graphics corresponds to a Kuramoto oscillator in the network. The drawn line is the simple linear regression of the data. The linear correlation between \( \omega_i \) and \( \theta_i \), as incorporated in the ansatz \( 3 \), is evident. Moreover, we can clearly identify two displaced oscillator populations which similar linear regression slopes. See the last section for further details on this curious dynamical behavior.](image)
FIG. 2. Synchronization diagrams for the Erdos-Renyi network described in Section III A. The blue circles, red crosses, and green stars correspond, respectively, to the optimal, the random, and the worst subset $C$ of forced nodes, see the text for further details. Top panels: $⟨r⟩$ and $⟨a⟩$ as functions of $fF/Ω$, for $λ/Ω = 2$. The critical external force $F_c$, see (20), corresponds precisely to $fF/Ω = 1$ in this case. As one can see, for $F > F_c$, all the oscillators follow the same pace of the external force since $a ≈ 1$. One can also appreciate that $⟨r⟩$ is indeed enhanced according to our optimization procedure for $F > F_c$. Middle panels: $⟨r⟩$ and $⟨a⟩$ as functions of $λ/Ω$, for $fF/Ω = 2$. Although one sees that synchronization ($r ≈ 1$) can occur for some small values of $λ/Ω$, the global synchronization ($a ≈ 1$) does require a larger value for the coupling constant. However, we can see that the threshold values of $λ/Ω$ for the onset of global synchronization are also compatible with our optimization procedure, in the sense that the smallest threshold corresponds to the optimal set, and the largest to the worst one. Bottom panels: $⟨r⟩$ and $⟨a⟩$ as functions of $λ/Ω$, for $fF/Ω = 0.2$. This is a situation where the system is rather intensive to the external force. The synchronization diagrams are similar for the three cases. Notice, in particular, that $a = 0$, meaning that the synchronized state does not follow the pace of the external force. This situation is essentially the same one of the $F = 0$ case discussed in [17].

incorporated in the ansatz (3), is evident. Moreover, we can clearly identify two displaced oscillator populations with similar linear regression slopes. We will return to this interesting dynamical behavior in the last section.

In order to test our optimization scheme, we have considered several synchronization diagrams of the type $r(τ)$ and $ψ(τ)$ versus $λ/Ω$ and $F/Ω$, for $τ$ sufficiently large to assure the relaxation of any transient regime. The order parameters $r(τ)$ and $ψ(τ)$ are evaluated accordingly their definition (2) for the numerical solutions of (29), taking into account the usual statistical precautions, see [17] for further details. Our results are summarized in Fig. 2 from where one can appreciate that the selection of the subset of forced nodes according to our optimization scheme effectively gives origin to networks
with better global synchronization capabilities. The diagrams 
\( r \times f F/\Omega \) and \( r \times \lambda/\Omega \) have evident meaning and can be 
easily understood. The case for the order parameter \( \psi \) is more 
involved. Its asymptotic behavior for large values of \( \tau \) in a 
synchronized regime is expected to be \( \psi(\tau) \sim \alpha \tau \). For a glob-
ally synchronized state, we will have \( \alpha = 1 \), or \( \psi \sim \Omega \), in 
terms of the original time variable \( t \) of \([1]\). However, different 
values for \( \alpha \) are also associated with synchronized states, but 
for which the oscillators do not follow the same pace of the ex-
ternal force. In particular, the usual synchronization state for 
the Kuramoto network with \( F = 0 \) is known to be such that 
\( \alpha = \langle \omega \rangle \). We have explicitly compared three cases for the sub-
set \( C \), namely the optimal, the random, and the worst cases. 
The optimal and the worst cases correspond, respectively, to 
selection of the forced nodes as those ones with maximal and 
minimal values of \( |\omega_i - \Omega| \). The results are in complete agree-
ment with the expectations for the optimization scheme: the 
optimal subset always implies better global synchronization 
capabilities than the random one, while the worst subset al-
ways exhibits worse capabilities when compared with the ran-
dom case.

IV. FINAL REMARKS

Although all our analyses have effectively done for the case 
with \( \Omega > 0 \) and for symmetric distributions \( g(\omega) \), the optimi-
ization criterion of selecting the subset \( C \) as those oscilla-
tors with maximal value of \( |\omega_i - \Omega| \) is also valid for the gen-
eral case. Let us consider, first, the case with \( \Omega < 0 \). From 
\([1]\), supposing that \( \theta_i(t) \) is a solution for the case with \( \Omega > 0 \) 
for a network of Kuramoto oscillators with natural frequencies 
\( \omega_i \), we have that \( -\theta_i(t) \) will be the solution for the case corre-
sponding to \( \Omega < 0 \), but for a network with natural frequencies 
\(-\omega_i \). Both situations will have the same fixed points for the 
equations equivalent to \(25\) and \(26\), but now the criterion of 
minimal \( \langle \omega \rangle_C \) for \( \Omega < 0 \) will effectively select those nodes 
with larger natural frequencies, which indeed corresponds to 
maximize \( |\omega_i - \Omega| \). The case of non-symmetric distributions is 
a bit more involved since one needs to keep track of the terms 
proportional to \( \langle \omega \rangle \) in the derivation of \(9\) and \(11\) and all the 
subsequent manipulations. For this case, Eq. \(9\) and \(11\) will read

\[
M \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} \langle \omega \rangle - \Omega - F I_1 \\ \langle \omega \rangle - \langle \omega \rangle - F I_2 + \lambda I_3 \end{pmatrix},
\]

where

\[
M = \begin{pmatrix} 1 & \langle \omega \rangle \\ \frac{\langle \omega \rangle}{\Omega} & \frac{\langle \omega \rangle}{\Omega} \end{pmatrix}.
\]

Since \( \det M = \frac{\langle \omega \rangle^2}{\Omega} \), we see that, unless all \( \omega_i \) be equal, a situ-
ation where our optimization procedure obviously does not apply, 
the fixed points \((\alpha, \beta_*)\) in this case correspond to the 
zeros of the right-handed side of \(30\). However, the lineariza-
tion of the right-handed side of \(30\) around \( \beta = 0 \) gives ori-
gin to exactly the same condition \(28\), and hence the same 
analysis of the symmetric case does apply here. Hence, our 
opimization criterion is indeed valid for the case of general \( \Omega \) 
and \( g(\omega) \).

The role played by possible symmetries of the network in 
its synchronization capability was recently discussed in \[16\] 
for the case of multilayer networks of Stuart-Landau complex 
oscillators, in the context of the so-called asymmetry-induced 
synchronization (AI Sync) scenario.\[17\][21]. The main conclusion 
was that the presence of certain regularities in the interlayer 
connection pattern tends to diminish the synchronization ca-
pability of the network or, in other words, asymmetries in the 
network tend to enhance its synchronization properties. 
The key point here is the quadratic quantity \(L\) given by \([27]\), which 
always decreases if interlayer symmetries are present, see \[16\] 
for the details. It is important to stress that the same conclu-
sions hold here: the presence of possible symmetries as those 
ones discussed in \[16\] tends to diminish the global synchronization 
capabilities of the forced network. It is also interesting to 
notice that, in the autonomous formulation of the problem \([1]\), 
in which an extra extra \( i = N + 1 \) node with natural frequency 
\( \Omega \) is connected in a directional way to the forced nodes \( i \in C \), 
one can interpret our optimization scheme in the same way of 
the optimal synchronization for the \( F = 0 \) case\[22\], in particu-
lar that anti-correlation between the frequencies of neighbor 
nodes always favors synchronization, see also \[24\]. The in-
vestigation of such phenomena might shed some light on the 
evolution and structure of natural systems for which global 
synchronization is a desired property. For a recent study of 
this type involving external stimuli in the \( C. \ elegans \) neural 
network, see \[12\].

We finish recalling the dynamical behavior involving the 
two oscillator populations we could identify in the globally 
synchronized regime, see Fig. \[1\]. Typically, the displaced 
oscillators are the forced ones, giving origin to a curious dynam-
ical configuration. When the global synchronization is com-
pletely attained, the forced nodes clump together and move 
ahead of the remaining swarm of oscillators. The dynamics in 
this case consist in a large group of oscillators (the free ones) 
chasing the smaller group of the forced ones. The existence of 
the two disjoint populations prevents the globally synchro-
nized state to have an order parameter \( r \) very close to \( 1 \), which 
can be clearly seen in the top left panel in Fig. \[2\] where one 
can observe a sudden decrease in \( r \) when global synchroniza-
tion is attained. It is certainly worth to incorporate such two 
populations behavior in a second order approximation for the 
collective ansatz \(5\). These points are now under investiga-

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