ON ALGEBRA GENERATED BY
CHERN-BOTT 2-FORMS ON $\mathbb{SL}_n/B$

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Abstract. In this short note we give an explicit presentation of the algebra $A_n$ spanned by the curvature 2-forms of the standard Hermitian linear bundles over $\mathbb{SL}_n/B$ as the quotient of the polynomial ring. The difference between $A_n$ and $H^*(\mathbb{SL}_n/B)$ reflects the fact that $\mathbb{SL}_n/B$ is not a symmetric space. This question was raised by V. I. Arnold in [Ar].

Sur l’algèbre engendrée par les 2-formes de Chern–Bott sur $\mathbb{SL}_n/B$

Résumé: Dans cette note nous donnerons une présentation explicite, en tant que quotient de l’anneau polynomial, de l’algèbre $A_n$ engendrée par les 2-formes de courbure des fibrés en droites hermitiens standard sur $\mathbb{SL}_n/B$. La différence entre $A_n$ et $H^*(\mathbb{SL}_n/B)$ reflète le fait que $\mathbb{SL}_n/B$ n’est pas un espace symétrique. Cette question a été posée par V. I. Arnold dans [Ar].

Version française abrégée.

Considérons l’espace $\mathbb{SL}_n/B$ des drapeaux complets dans $\mathbb{C}^n$, aussi bien que les $n$ fibrés quotient $L_i = E_i/E_{i-1}$, où $E_i$ signifie le fibré tautologique standard de dimension $i$ au-dessus de $\mathbb{SL}_N/B$. Quand on fixe une métrique hermitienne dans $\mathbb{C}^n$ tous les $L_i$ deviennent des fibrés hermitiens. On notera $\omega_i$ la 2-forme de courbure de $L_i$. Il est évident que chaque $\omega_i$ est $U_n$-invariante sur $\mathbb{SL}_n/B$. Dans cette note nous allons présenter l’algèbre $A_n = A(\omega_1, \ldots, \omega_n)$, engendrée par $\omega_1, \ldots, \omega_n$, comme un quotient de l’anneau polynomial en $n$ générateurs, et on la comparera ensuite avec $H^*(\mathbb{SL}_n/B)$.

Nous étudions aussi l’algèbre $A_{k,n}$ engendrée par $k$ parmi les formes $\omega_i$. (Sa structure ne dépend pas du choix particulier de $k$-tuple.) Le résultat principal est le suivant.

Proposition. L’algèbre $A_{k,n}$ est une algèbre graduée, isomorphe à $\mathbb{C}[x_1, \ldots, x_k]/I_{k,n}$, où $I_{k,n}$ est un idéal engendré par $2^k - 1$ polynômes de la forme

$$g_{i_1, \ldots, i_j}^{(n)} = (x_{i_1} + \cdots + x_{i_j})^{j(n-j)+1}.$$

Ici $(i_1, \ldots, i_j)$ parcourt l’ensemble des sous-ensembles non vides de $\{1, \ldots, n\}$.
La conjecture suivante est un résultat des expériments faits sur l’ordinateur avec le système Macaulay.

**Conjecture.** a) La dimension de \( A_n = A_{n,n} \), en tant qu’espace vectoriel, est égale au nombre total des forêts sur \( n \) sommets marqués, et il existe une base monomiale naturelle pour \( A_n \), dont les éléments sont indexés par ces forêts.

b) La dimension totale de \( A_{k,n} \) est un polynôme unitaire en \( n \) de degré \( k \).

§1. Introduction.

Let \( \mathbb{S}L_n/B \) denote the space of complete flags in \( \mathbb{C}^n \). One has the obvious sequence of tautological bundles \( 0 \subset E_1 \subset \ldots \subset E_n = E \) (where \( E \) is the trivial \( \mathbb{C}^n \)-bundle over \( \mathbb{S}L_n/B \)) and the corresponding \( n \)-tuple of quotient line bundles \( L_i = E_i/E_{i-1} \). Fixing some Hermitian metric on the original \( \mathbb{C}^n \) one equips every bundle \( E_i \) and \( L_i \) with the induced Hermitian metric. Let \( w_i \) denote the curvature form of the above Hermitian metric on \( L_i \). Each \( w_i \) is a \( \mathcal{U}_n \)-invariant 2-form on \( \mathbb{S}L_n/B \) representing \( c_1(L_i) \) in \( H^2(\mathbb{S}L_n/B) \) (the \( \mathcal{U}_n \)-action is provided by the chosen metric). Denoting \( c_1[L_i] \) by \( x_i \) one has that \( H^*(\mathbb{S}L_n/B) = \frac{\mathbb{C}[x_1,\ldots,x_n]}{(s_1,s_2,\ldots,s_n)} \) where the denominator is the ideal generated by all elementary symmetric functions in variables \( x_1, \ldots, x_n \), see e.g. [Bo]. Since we have the standard representative \( w_i \) for each \( x_i = c_1(L_i) \) it seems natural to study the algebra \( A_n = \mathcal{A}(w_1,\ldots,w_n) \) generated by all \( w_i \)’s and compare it to \( H^*(\mathbb{S}L_n/B) \). We will also discuss the algebra \( A_{k,n} = \mathcal{A}(w_{i_1},\ldots,w_{i_k}) \) generated by any \( k \) of \( w_i \)’s (the structure of \( A_{k,n} \) does not depend on a particular choice of a \( k \)-tuple). The main result of this short note is as follows.

1.1. **Proposition.** \( A_{k,n} \) is a graded algebra isomorphic to \( \frac{\mathbb{C}[x_1,\ldots,x_k]}{I_{k,n}} \) where the ideal \( I_{k,n} \) is generated by the set of \( 2^k - 1 \) polynomials of the form

\[
g_{i_1,\ldots,i_j}^{(n)} = (x_{i_1} + \ldots + x_{i_j})^{j(n-j)+1}
\]

where \( \{i_1,\ldots,i_j\} \) runs over the set of all nonempty subsets of the set \( \{1,\ldots,n\} \).

1.2. **Example.** The algebra \( A_3 = \mathcal{A}_{3,3} \) is isomorphic to \( \frac{\mathbb{C}[x_1,x_2,x_3]}{I_{3,3}} \) where \( I_{3,3} \) is generated by \( x_1^3, x_2^3, x_3^3, (x_1 + x_2)^3, (x_1 + x_3)^3, (x_2 + x_3)^3, x_1 + x_2 + x_3 \). The Hilbert series of \( A_3 \) is \( (1,2,3,1) \). (For comparison, the Hilbert series for \( H^*(\mathbb{S}L_3/B) \) is \( (1,2,2,1) \).)

1.3. **Remark.** One has the standard surjective map of algebras \( \pi : A_n \to H^*(\mathbb{S}L_n/B) \).

The following conjecture is the result of calculation of the Hilbert series of \( A_n \) for \( n \leq 7 \), see [SS].

1.4. **Conjecture.** 1) The total dimension of \( A_n \) (as a vector space) equals the number of forests on \( n \) labeled vertices and there exists a natural monomial basis for \( A_n \) whose monomials are enumerated by the above forests.

2) The total dimension of \( A_{k,n} \) is a monic polynomial in \( n \) of degree \( k \).

Besides purely aesthetic reasons the study of \( A_n \) is motivated by the fact that many known results for \( H^*(\mathbb{S}L_n/B) \) (such as e.g. the existence of a good monomial basis, \( S_n \)-module structure, the action of divided difference operators etc) have natural counterparts for \( A_n \). One might hope to develop...
a theory of characteristic classes with values in $A_n$ for $U_n$-invariant Hermitian bundles over $\mathbb{S}L_n/B$. Finally, the study of $A_n$ seems to be important for the understanding of structure for the algebra $A(\mathbb{S}L_n/B)$ of all $U_n$-invariant forms on $\mathbb{S}L_n/B$ which recently appeared in the arithmetic intersection theory for flag varieties, comp. [Ta1], [Ta2].

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§2. PROOFS.

On algebra $A(\mathbb{S}L_n/B)$ of $U_n$-invariant forms on $\mathbb{S}L_n/B$.

One knows that $\mathbb{S}L_n/B$ has another presentation as a homogeneous space, namely $U_n/T^n$ where $T^n \subset U_n$ is the usual torus of diagonal matrices. Let us recall an old result from the general theory of homogeneous spaces, see e.g. [DNF].

2.1. PROPOSITION. The ring of $G$-invariant differential forms on a homogeneous space $G/H$ is isomorphic to the exterior algebra $\Lambda_{inv}(\mathfrak{g}^*)$, i.e. to the algebra of skew-symmetric polylinear functions on $\mathfrak{g}$ which a) vanish on $\mathfrak{h}$ and b) are invariant under the action of internal automorphisms by the elements of $H$. (Here $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $G$ and $H$ respectively.)

Take the $\binom{n}{2}$-dimensional vector space $V_n$ of all skew-hermitian matrices of the form

$$V_n = \begin{pmatrix}
0 & a_{1,2} & \cdots & \cdots & \cdots & a_{1,n} \\
-a_{1,2} & 0 & a_{2,3} & \cdots & \cdots & a_{2,n} \\
-a_{1,3} & -a_{2,3} & 0 & a_{3,4} & \cdots & a_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
-a_{1,n} & -a_{2,n} & \cdots & \cdots & 0
\end{pmatrix}.$$  \hfill (2)

Its matrix entries form the linear space $(\mathfrak{u}_n/\mathfrak{f}^*)^*$ where $\mathfrak{u}_n$ (resp. $\mathfrak{f}^*$) is the Lie algebra of $U_n$ (resp. of $T^n$). Let us denote by $(e^{i\lambda_1}, e^{i\lambda_2}, \ldots, e^{i\lambda_n})$ the diagonal entries of elements in $T^n$ acting by conjugation on $V_n$. Under this action each entry $a_{i,j}$ above the main diagonal is multiplied by $e^{i(\lambda_i - \lambda_j)}$ and each entry $-\bar{a}_{i,j}$ below the main diagonal is multiplied by $e^{i(\lambda_j - \lambda_i)}$.

Introducing the principal weights $\alpha_i = \lambda_i - \lambda_{i+1}, i = 1, \ldots, n - 1$ we get that $a_{i,j}$ is multiplied by $e^{i(\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1})}$. The expression $\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}$ is called the multiweight of the entry $a_{i,j}$. (Under this convention the entry $-\bar{a}_{i,j}$ has the opposite multiweight $-\alpha_i - \alpha_{i+1} - \ldots - \alpha_{j-1}$.) The multiweight of an exterior monomial $\bar{a}_{i_1,j_1} \wedge \bar{a}_{i_2,j_2} \ldots \wedge \bar{a}_{i_r,j_r}$ where each $\bar{a}_{i_t,j_t}$ is either $a_{i_t,j_t}$ or $-\bar{a}_{i_t,j_t}$ equals to the sum of multiweights of its factors.

2.2. COROLLARY. The ring $A(\mathbb{S}L_n/B)$ is the linear span of all exterior monomials $\bar{a}_{i_1,j_1} \wedge \bar{a}_{i_2,j_2} \ldots \bar{a}_{i_r,j_r}$ having vanishing multiweight. (In particular, $A(\mathbb{S}L_n/B)$ has no degree 1 elements.)

2.3. EXAMPLE. The Hilbert series for $A(\mathbb{S}L_3/B)$ is $(1, 0, 3, 2, 3, 0, 1)$. (We assume that $A(\mathbb{S}L_n/B)$ contains constants.) Its degree 2 component is
spanned by \( a_{1,2} \wedge \bar{a}_{1,2}, a_{1,3} \wedge \bar{a}_{1,3}, a_{2,3} \wedge \bar{a}_{2,3} \); degree 3 component is spanned by \( a_{1,2} \wedge a_{2,3} \wedge \bar{a}_{1,3}, \bar{a}_{1,2} \wedge a_{2,3} \wedge \bar{a}_{1,3} \); degree 4 component is spanned by \( a_{1,2} \wedge a_{1,3} \wedge a_{1,3}, a_{1,2} \wedge a_{2,3} \wedge a_{2,3}, a_{1,3} \wedge a_{2,3} \wedge a_{2,3} \) and, finally, its degree 6 component is spanned by \( a_{1,2} \wedge a_{1,3} \wedge a_{1,3} \wedge a_{2,3} \wedge a_{2,3} \). The Hilbert series for \( \mathfrak{A}(\mathbb{S}L_{4}/B) \) is \((1, 0, 6, 4, 18, 12, 26, 12, 18, 4, 6, 0, 1)\).

Recall that an Eulerian digraph is a digraph such that the numbers of coming and leaving edges at each vertex are equal. (We do not allow loops.)

The following proposition is a relatively simple reformulation of 2.2.

2.4. PROPOSITION, see [SS]. The dimension of \( \mathfrak{A}(\mathbb{S}L_{n}/B) \) equals \( 2^{|S|} + 2\#eul(n) \) where \( \#eul(n) \) is the number of Eulerian digraphs on \( n \) labeled vertices.

On curvature forms.

Using the above description of \( \mathcal{U}_{n} \)-invariant forms on \( \mathbb{S}L_{n}/B \) one can be easily derive the following proposition from the results of [GS], see also [Ta2].

2.5. PROPOSITION. The curvature 2-form \( w_{i}, i = 1, ..., n \) of the tautologic line bundle \( L_{i} = E_{i}/E_{i-1} \) over \( \mathbb{S}L_{n}/B \) equals to the sum of all entries in the \( i \)th row of the following matrix of 2-forms

\[
\begin{pmatrix}
0 & \gamma_{1,2} & \gamma_{1,3} & \cdots & \gamma_{1,n} \\
-\gamma_{1,2} & 0 & \gamma_{2,3} & \cdots & \gamma_{2,n} \\
-\gamma_{1,3} & -\gamma_{2,3} & 0 & \cdots & \gamma_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\gamma_{1,n} & -\gamma_{2,n} & \cdots & \cdots & -\gamma_{n-1,n} & 0
\end{pmatrix}
\]

where \( \gamma_{i,j} = \alpha_{i,j} \wedge \bar{\alpha}_{i,j} \).

Now we are ready to start proving the main proposition 1.1. Consider the algebra \( \mathfrak{A}(w_{1}, ..., w_{k}) \) generated by \( w_{1}, ..., w_{k} \). Since all \( w_{j} \) are \( \mathcal{U}_{n} \)-invariant one has that \( \mathfrak{A}(w_{i}, ..., w_{i}) \) is a subalgebra of the exterior algebra \( \Lambda(\mathbb{C}^{(2)}) \).

2.6. LEMMA. There exists a natural \( S_{n} \)-action on the set of all permutations \((w_{1}, ..., w_{n})\) of \( w_{1}, ..., w_{n} \).

PROOF. The elementary transposition \( \tau_{i}, i = 1, ..., n-1 \) acts on the matrix (3) of 2-forms by the simultaneous interchanging of 1) the \( i \)th row with the \((i + 1)\)st row, 2) the \( i \)th column with the \((i + 1)\)st column and \( 3) \) changing sign of \( \gamma_{i,i+1} \). One can easily check that this determines the required \( S_{n} \)-action. \( \square \)

2.7. COROLLARY. All algebras \( \mathfrak{A}(w_{1}, ..., w_{k}) \) are isomorphic to each other and, in particular, to \( \mathfrak{A}(w_{1}, ..., w_{k}) \).

We denote this class of isomorphic algebras by \( \mathfrak{A}_{k,n} \).

2.8. LEMMA. \( \mathfrak{A}_{n-1,n} \) is isomorphic to \( \mathfrak{A}_{n} = \mathfrak{A}_{n,n} \).

PROOF. Indeed, one has \( w_{1} + w_{2} + ... + w_{n} = 0 \). \( \square \)

DEFINITION. Let us call by the vanishing ideal \( I_{k,n} \) of \( \mathfrak{A}_{k,n} \) the set of all polynomials \( p \in \mathbb{C}[x_{1}, ..., x_{k}] \) which vanish if we substitute the variables \( x_{1}, ..., x_{k} \) by the curvature forms \( w_{1}, ..., w_{k} \) resp.

2.9. LEMMA. The vanishing ideal \( I_{k,n} \) consists of all \( p \in \mathbb{C}[x_{1}, ..., x_{k}] \) such that \( p \) derivatives \( \frac{\partial p}{\partial x_{1}}, \frac{\partial p}{\partial x_{2}}, ..., \frac{\partial p}{\partial x_{k}}, \frac{\partial^{2} p}{\partial x_{1} \partial x_{2}}, ..., \frac{\partial^{k} p}{\partial x_{1} \partial x_{2} ... \partial x_{k}} \) belong to \( I_{k,n-1} \). (Derivatives are taken with respect to all subsets \( \{i_{1}, ..., i_{l}\} \subset \{1, ..., n\} \) of indices (no repetitions of indices are allowed) including the empty set.)
Proof. Notice that $A_{k,n}$ has a natural module structure over $A_{k,n-1}$.

Indeed, one has $w_i^{(n)} = w_i^{(n-1)} + \gamma_{i,n}$ where the upper index shows the dimension of the initial space, see (3). Therefore for any polynomial $p \in \mathbb{C}[x_1, ..., x_k]$ one has after substitution of the curvature forms

\[ p(w_i^{(n)}),..., w_k^{(n)}) = p(w_i^{(n-1)} + \gamma_{i,n}, ..., w_k^{(n-1)} + \gamma_{k,n}) = p(w_i^{(n-1)}, ..., w_k^{(n-1)}) + p_x_1(w_1^{(n-1)}, ..., w_k^{(n-1)})\gamma_{1,n} + ... + p_x_k(w_1^{(n-1)}, ..., w_k^{(n-1)})\gamma_{k,n} + ... + p_x_1, x_2, ..., x_l(w_1^{(n-1)}, ..., w_k^{(n-1)})\gamma_{1,n} \gamma_{2,n} ... \gamma_{k,n}. \]

(4)

Since $\gamma_{i,n} = 0$ the occurring monomials in the r.h.s. are square-free and the coefficients at the product $\gamma_{i_1,n} ... \gamma_{i_l,n}$ equals $\frac{\partial^j p}{\partial x_{i_1} ... \partial x_{i_l}}$. Finally, the condition $p \in I_{k,n}$, i.e. $p(w_1^{(n)}, ..., w_k^{(n)}) = 0$ is equivalent to vanishing of all polynomial coefficients $p_{x_1, ..., x_l}(w_1^{(n-1)}, ..., w_k^{(n-1)})$ in the r.h.s of (4). By definition this means that $p_{x_1, ..., x_l}(x_1, ..., x_k) \in I_{k,n-1}$. □

Denote $D_{i_1, ..., i_l} = \frac{\partial^j p}{\partial x_{i_1} ... \partial x_{i_l}}$ and let $V_{i_1, ..., i_l, r, m, k}$ be the space of all homogeneous polynomials of degree $m + r$ in $k$ variables of the form $p = (x_{i_1} + ... + x_{i_l})^r f$ where $deg(f) = m$.

2.10. Lemma. The linear map $D_{i_1, ..., i_l} : V_{i_1, ..., i_l, r, m+k-l, k} \to V_{i_1, ..., i_l, r, m, k}$ is a surjection.

Proof. Simple linear algebra. □

Proof of the main proposition 1.1.

We use a double induction on $k \leq n$, i.e. for a given $k$ we apply induction on $n$ and then cover the first case $A_{k+1, k+1}$ for $k + 1$ using lemma 2.8.

Base of induction. $A_{1,n} = \mathbb{C}[x_1]$. Indeed, $w_1 = \gamma_{1,1} + \gamma_{1,3} + ... + \gamma_{1,n}$ where $\gamma_{1,i}$ are independent Grassmann variables, i.e. $\gamma_{1,1}^2 = 0$. Statement follows.

Step of induction. Assume that 1.1. is proven for all pairs $(k', n')$ where $k' < k$ and for all $(k, n')$ where $n' < n$. Let us show that it holds for the pair $(k, n)$. Notice that all polynomials (1) are contained in $I_{k,n}$. This can be either checked directly or by induction using 2.9. Let us temporarily denote by $\tilde{I}_{k,n}$ the ideal generated by all polynomials in (1). We have to show that $I_{k,n} = \tilde{I}_{k,n}$. By the above remark $\tilde{I}_{k,n} \subset I_{k,n}$. Take any $p \in I_{k,n}$. Using lemma 2.10 we will step by step add to $p$ certain polynomials from $\tilde{I}_{k,n}$ in such a way that the resulting polynomial $p_{fin}$ will have derivatives from lemma 2.9 vanishing. Consider $D_{i_1, ..., i_l}(p)$. By 2.9. it belongs to $I_{k,n-1}$. By the inductive assumption $I_{k,n-1}$ is generated by $g_{i_1, ..., i_l}^{(n-1)}$, see (1). Therefore one has $D_{i_1, ..., i_l}(p) = \sum i_1, ..., i_l (x_{i_1} + ... + x_{i_l})^{j(n-1)+1} h_{i_1, ..., i_l}$. By lemma 2.10 for each $\phi = (x_{i_1} + ... + x_{i_l})^{j(n-1)+1} h_{i_1, ..., i_l}$ there exists (but nonunique) $\psi = (x_{i_1} + ... + x_{i_l})^{j(n-1)+1} H_{i_1, ..., i_l}$ such that $D_{i_1, ..., i_l}(\psi) = \phi$. Notice that $\psi \in \tilde{I}_{k,n}$. Therefore subtracting from $p$ an appropriate polynomial belonging to $\tilde{I}_{k,n}$ we get $\tilde{p} \in I_{k,n}$ such that $D_{i_1, ..., i_l}(\tilde{p}) = 0$. Consider now any derivative $D_{i_1, ..., i_l, k} = \frac{\partial^k}{\partial x_{i_1} ... \partial x_{i_l} \partial x_k}$ applied to $\tilde{p}$. One has that $D_{i_1, ..., i_l, k}(\tilde{p})$ does not depend on $x_k$ since $D_{i_1, ..., i_l, k}(\tilde{p}) = 0$. Therefore using the same argument as above we can subtract from $\tilde{p}$ some polynomial belonging to $\tilde{I}_{k,n}$ which does not depend on $x_k$ and such that $D_{i_1, ..., i_l, k}$ applied to the resulting difference vanish. Notice that if we apply this procedure for each $D_{i_1, ..., i_l, k}$ consecutively to the polynomial obtained on the previous step we will not change
vanishing of all previous $\mathcal{D}_{1,\ldots,\hat{j},\ldots,k}$, $\hat{j} < i$ applied to the polynomial obtained on the current step since we add a function which does not depend on $x_i$.

Having obtained vanishing of all $\mathcal{D}_{1,\ldots,\hat{i},\ldots,k}$, $i \leq k$ applied to our polynomial we can proceed with all $\mathcal{D}_{1,\ldots,\hat{i},\ldots,\hat{j},\ldots,k}$ (ordering them e.g. lexicographically). Our assumptions imply that $\mathcal{D}_{1,\ldots,\hat{i},\ldots,\hat{j},\ldots,k}$ applied to our polynomial does not depend on $x_i$ and $x_j$. Therefore we can subtract a polynomial from $\tilde{I}_{k,n}$ which does not depend on $x_i$ and $x_j$ either and such that $\mathcal{D}_{1,\ldots,\hat{i},\ldots,\hat{j},\ldots,k}$ applied to the resulting difference vanish. Using this procedure consecutively we do not spoil vanishing of the previous derivatives since the polynomials we subtract depend on different groups of variables. Continuing in the same manner we get some polynomial $p_{\text{fin}}$ all derivatives of which mentioned in 2.9 vanish. This means that $p_{\text{fin}}$ is a constant. But since constants different from zero do not lie in $I_{k,n}$ we have that $p_{\text{fin}}$ should also vanish. The statement follows. □

§3. SOME RELATED PROBLEMS.

PROBLEM 1. Give a presentation of the obvious analog of $\mathcal{A}_n$ for any $\mathbb{SL}_n/P$ generated by the standard $\mathcal{U}_n$-invariant forms representing the Chern classes of the tautological (quotient) bundles.

REMARK. Notice that in the case of Grassmanian $G_{k,n}$ the analogous algebra coincides with $H^*(G_{k,n})$ since $G_{k,n}$ is a symmetric space and therefore has no nontrivial left-invariant forms homologous to 0, see e.g. [St].

PROBLEM 2. Prove conjecture 1.4 and determine the $S_n$-module structure for $\mathcal{A}_n$ and its $\mathbb{SL}_n/P$-analogues.

PROBLEM 3. Calculate the Poincare series for the algebra $\mathfrak{A}(\mathbb{SL}_n/P)$ of $\mathcal{U}_n$-invariant forms on $\mathbb{SL}_n/P$.

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