On the Applicability of the Ergodicity Hypothesis to Mesoscopic Fluctuations

O. Tsyplatyev, I. L. Aleiner, Vladimir I. Fal’ko, and Igor V. Lerner

1Department of Physics, Lancaster University, Lancaster LA1 4YB, United Kingdom
2Physics Department, Columbia University, New York, NY 10027
3School of Physics and Astronomy, University of Birmingham, Birmingham B15 2TT, United Kingdom

(Dated: November 3, 2018)

We evaluate a typical value of higher order cumulants (irreducible moments) of conductance fluctuations that could be extracted from magneto-conductance measurements in a single sample when an external magnetic field is swept over an interval $B_0$. We find that the $n$-th cumulant has a sample-dependent random part $\pm \langle \langle g^2 \rangle \rangle^{n/2} \sqrt{a_n B_0 / B_0}$, where $\langle \langle g^2 \rangle \rangle$ is the variance of conductance fluctuations, $B_0$ is a correlation field, and $a_n \sim n!$. This means that an apparent deviation of the conductance distribution from a Gaussian shape, manifested by non-vanishing higher cumulants, can be a spurious result of correlations of conductances at different values of the magnetic field.

The ergodicity hypothesis (EH) plays a crucial role in mesoscopic physics. It identifies mesoscopic ‘sample-to-sample’ fluctuations of a certain quantity with an apparently random but reproducible spread in the magnitude of this quantity measured in a single sample as a function of some controllable parameter, e.g., an external magnetic field or a gate voltage. The EH is based on the idea that, since the fluctuations result from the electron interference in scattering from impurities, changing the interference conditions by, e.g., sweeping the field over a wide range, one obtains a data set (usually called a magnetofingerprint) statistically equivalent to that for sample-to-sample fluctuations.

Since most of experiments are performed, by necessity, on a few samples while practically any theoretical approach uses some form of the ensemble averaging over disorder, the EH provides the framework for comparing experiment with theory in mesoscopics. Such a framework appears to work so well that it is now routinely used by default, in particular in experiments aiming to determine the shape of the conductance distribution.

Here we show that for the ergodicity hypothesis to be valid for analyzing an apparent deviation of the distribution from the normal shape in systems with large conductance, one needs the range of magnetic fields so large that it may not be reachable in modern experiments. We show that a non-Gaussian shape of the distribution extracted from a single-sample magnetofingerprint can be an entirely spurious result of the non-ergodicity due to residual correlations of conductances at different values of the magnetic field. For brevity, we consider only conductance distributions, although our method and results can be extended to any mesoscopic observable.

Let us start with a brief description of theoretical predictions on the conductance distribution. Any distribution can be described by its moments or cumulants (irreducible moments). Within the standard one-parameter scaling approach to disordered conductors the cumulants of conductance fluctuations in the metallic regime at temperature $T = 0$ (i.e. for a fully phase-coherent conductor) are given by

$$\langle \langle g^n \rangle \rangle = A_{nd} \langle g \rangle^{2-n},$$

where $A_{nd}$ are numerical coefficients dependent on the cumulant order $n$ and dimensionality $d$. Here $\langle g \rangle$ is the ensemble-averaged conductance (i.e. conductance averaged over all the realizations of disorder) measured in the units of $e^2/h$. In a good metal $\langle g \rangle \gg 1$, and the higher ($n > 2$) cumulants in Eq. (1) are small in comparison with the universal variance (the second cumulant), $\langle \langle g^2 \rangle \rangle \equiv \langle \langle g \rangle \rangle^2 - \langle g \rangle^2 = A_{2d} \sim 1$. A characteristic feature of the Gaussian (normal) distribution is that all its high cumulants vanish. Therefore, Eq. (1) means that the distribution of conductance fluctuations is expected to be almost Gaussian. An experimental detection of deviations from the Gaussian shape, characterized by the higher cumulants in Eq. (1), represents a challenging task.

This is even more so, when one realizes that at $T \neq 0$ Eq. (1) is only valid for sample sizes smaller that phase-breaking length $L_\phi$. Thus for a quasi-1D wire with length $L > L_\phi$, conductance fluctuations can be estimated as resulting from independent contributions of $N = L / L_\phi$ conductors connected in series, which lead, in accordance with the central limiting theorem, to a further suppression of higher cumulants.

$$\langle \langle g^n \rangle \rangle_{L > L_\phi} \sim \langle \langle g^n \rangle \rangle_{L = L_\phi} \left( \frac{L_\phi}{L} \right)^{2n-1}.$$

However small are the deviations from the Gaussian statistics described by Eqs. (1) and (2), they would be in principle experimentally detectable, had one been able to accumulate sufficient statistical data from uncorrelated different realizations of disorder – a task which is in principle possible for semiconductor nanostructures by applying a large number of annealing cycles. Our main point is that the failure of the EH invalidates an apparently simpler alternative (and the only one known for metallic wires) based on extracting statistical data from a magnetofingerprint $g(B)$.

To explain how this occurs, let us formalize a procedure of extracting the moments $\langle g^n \rangle_B$ of conductance distribution from $g(B)$. It is done by sampling it into a histogram, so that $\langle g^n \rangle_B$ are determined by the following averages (where a smoothed histogram $f_B(g)$ is the...
distribution function:

\[ \langle g^n \rangle_B = \int_B^{B+B_0} g^n(B) \frac{dB}{B_0} = \int g^n(B) f_B(g) \frac{dB}{B_0}. \]  

The cumulants \( \langle g^n \rangle_B \) are then found from these (reducible) moments using the standard relations, see Eq. (6) below.

It is well known that for \( B, B' \gg B_c \) values of \( g(B) \) are correlated for different \( B \) with the co-variance

\[ \kappa(B - B') = \langle g(B)g(B') \rangle = \langle g^2 \rangle^2 K \left( \frac{B - B'}{B_c} \right), \]  

where \( K(x) \) decays slowly (as a power-law, see Eq. (15) below) from \( K(0) = 1 \) to 0. Its form and the value of the correlation field \( B_c \) (which corresponds to one magnetic flux quantum through the region of coherence) depend on the dimensionality, temperature regime, size and geometry of the system. The ergodicity has formally been proved in the limit \( B_0/B_c \rightarrow \infty \). For a finite \( B_0 \), one needs to evaluate to what extent the correlations contribute to \( \langle g^n \rangle_B \). To estimate a typical value of \( \langle g^n \rangle_B \) in a given sample, we calculate its disorder-averaged mean square, \( \langle \langle g^n \rangle_B^2 \rangle \).

In the leading order in \( B_c/B_0 \ll 1 \), we find

\[ \langle \langle g^n \rangle_B^2 \rangle = (\langle g^2 \rangle)^n + a_n \frac{B_c}{B_0} \langle g^2 \rangle^n \]

\[ a_n = n! \int_{-\infty}^{\infty} [K(x)]^n dx \]  

This equation is the main result of the paper. It shows that the ergodic hypothesis works only for the variance itself, as \( \langle \langle g^2 \rangle_B \rangle^2 = A \langle g^2 \rangle^2 \) with the coefficient of proportionality \( A \) being very close to 1 for \( B_c/B_0 \ll 1 \). For \( n \geq 3 \), the second term in Eq. (4) represents a systematic error in extracting a true value of the cumulant from the magnetofingerprint. No sampling procedure in the statistical analysis of a magnetofingerprint would produce such a route would be very cumbersome due to a large number of cancellations of non-leading contributions.

Thus even a perfectly Gaussian sample-so-sample distribution function defined for any well-convergent distribution \( f(g) \) by

\[ F(t) = \int_{-\infty}^{\infty} e^{gt} f(g) \, dg. \]  

The moments are given by coefficients of Taylor’s expansion of \( F(t) \), while the cumulants are generated by

\[ \ln F(t) = \sum \frac{\langle g^n \rangle_B}{n!} \Rightarrow \langle g^n \rangle_B = \partial^n \ln F(t) \bigg|_{t=0}. \]  

We use this expression to obtain directly the disorder-averaged variance of \( \langle g^n \rangle_B \). It is more convenient to deal with central moments of the distribution, i.e. with moments of \( \delta g(B) = g(B) - g_0(B) \), for which the disorder-averaged \( \langle \delta g(B) \rangle = 0 \). Such a shift of the distribution center does not affect the cumulants. To simplify notations, we will still write \( \langle g^n \rangle_B \) instead of \( \langle \delta g^n \rangle_B \).

Then the variance \( \var[\langle g^n \rangle_B] \equiv \langle \langle g^n \rangle_B^2 \rangle - \langle \langle g^n \rangle_B \rangle^2 \) of sample-to-sample fluctuations of \( \langle g^n \rangle_B \) is given by

\[ \var[\langle g^n \rangle_B] = \partial^n \partial^n \left( \ln F(t) \times \ln F(\tau) \right) \bigg|_{t=0}. \]  

where the moment-generating function for the distribution \( F(t) \) is given by

\[ F(t) = \int_B^{B+B_0} e^{g(B) t} \frac{dB}{B_0}. \]
We keep small non-Gaussian corrections only in the l.h.s. of Eq. (9), representing the disorder averaging in the r.h.s. as that over the normal distribution of \( g(B) \):

\[
\langle A[g(B)] \rangle = \frac{1}{Z} \int Dg(B) A[g(B)] e^{-S},
\]

\[
S = \frac{1}{2} \int dB dB' g(B) \kappa^{-1} g(B') , \quad Z = \int Dg(B) e^{-S},
\]

where \( \kappa^{-1} \) is the resolvent of the correlation function \( \kappa \).

Equivalently, it can be expressed in terms of the Fourier transforms \( g_\omega \) and \( \kappa_\omega \) of a random \( g(B) \) and the correlation function \( \kappa(B-B') \), with \( g_\omega = g_\omega^* \) and \( \kappa_\omega = \kappa_\omega^* \):

\[
(A) = \frac{1}{Z} \prod_{\omega} \int d g_\omega \ A[g_\omega] \ e^{- \frac{1}{2} \int d \omega |g_\omega|^2 \kappa_\omega^{-1}},
\]

(11)

To calculate the logarithm of the moment-generating function we use the standard replica trick:

\[
\ln F = \lim_{N \to 0} \frac{F_N - 1}{N}.
\]

Using also Eq. (10), we represent the variance \( \langle (g^n) \rangle_B \) as

\[
\mathrm{var} [\langle (g^n) \rangle_B] = \lim_{N,M \to 0} \frac{X_{NM}}{NM}.
\]

(12)

where

\[
X_{NM}^{(n)} = \partial^n_{\tau} \partial^n_{t} \int \frac{Dg(B)}{Z} \ e^{-S} \int_B e^{\sum_k g(B_k)} + \tau \sum_k \kappa_k g(B_k') \bigg|_{t=0, \tau=0} \int_B = \int \frac{B+B_0}{\tilde{B}+B_0} \prod_{i=1}^{B} \frac{dB_i}{B_0} \prod_{k=1}^{M} \frac{dB_k'}{B_0}.
\]

(13)

To calculate \( X_{NM}^{(n)} \) we perform the integration over \( Dg \) first, turning it into the Gaussian integral over the Fourier components of \( g \), Eq. (11):

\[
\frac{1}{Z} \prod_{\omega} \int d g_\omega e^{- \frac{1}{2} \int d \omega |g_\omega|^2 \kappa_\omega^{-1} - \int d \omega \sum_k \kappa_k g_\omega (\sum \kappa_k g_\omega + \tau \sum \kappa_k g_\omega')}
\]

\[
= \exp \left\{ \int d \omega \left[ \frac{\kappa_\omega}{2} \left( \int t \Sigma e^{i\omega B_i} + \tau \Sigma e^{i\omega B_i'} \right)^2 \right] \right\}.
\]

After the inverse Fourier transform for \( \kappa_\omega \), we arrive at

\[
X_{NM}^{(n)} = \partial^n_{\tau} \partial^n_{t} \int_B \exp \left[ t^2 V_{BB} + \tau^2 V_{BB'} + t \tau V_{BB} \right] \bigg|_{t=0, \tau=0}.
\]

where

\[
V_{BB} = \frac{N}{2} \kappa(0) + \sum_{i=1}^{N} \sum_{j=i+1}^{N} \kappa(B_i - B_j)
\]

(14a)

\[
V_{BB'} = \frac{M}{2} \kappa(0) + \sum_{k=1}^{M} \sum_{m=k+1}^{M} \kappa(B_k' - B_m')
\]

(14b)

\[
V_{BB'} = \sum_{i=1}^{N} \sum_{k=1}^{M} \kappa(B_i - B_k')
\]

(14c)

The derivatives with respect to \( t \) and \( \tau \) in the limit \( t, \tau \to 0 \) in the expression for \( X_{NM}^{(n)} \) above allows one to keep only the terms proportional to \( t^m \tau^n \) in the Taylor expansion of the exponent, which leads to

\[
X_{NM}^{(n)} = (n!)^2 \int_B \sum_{l} \frac{V_{BB'}^{l-2l}}{(n-2l)!l!} \frac{V_{BB}^{l}}{(l!)}^{2l}.
\]

(15)

The expression above is entirely equivalent to one that can be obtained directly from diagrammatic techniques. The replica trick, as always in perturbative calculations, is an exact tool that serves to eliminate automatically spurious diagrams, in the present case those that contribute to reducible moments but not to cumulants.

The next step of the calculation consists in selecting contributions of the lowest power in \( B_c/B_0 \) from Eq. (15). Each term in the sum contains a product of \( n \) correlation functions \( \kappa \), each being dependent on a difference between two \( B \)’s from the multiple integral \( \Sigma \). The integration will produce a factor of \( (B_c/B_0)^m \), where \( m \) is the number of different pairs of \( B \)’s in the product.

To illustrate this point, let us first consider the term with \( l = 0 \) in Eq. (15), proportional to the \( n \)-th power of the sum of \( NM \) different \( \kappa \)'s in Eq. (14c). The leading contribution comes from the sum of the \( n \)-th power of each term in Eq. (14c), as each contains only one difference between different \( B \)’s; as they are all equivalent, re-labelling reduces the \( l = 0 \) contribution in Eq. (15) to

\[
X_{NM}^{(n)} \to n! \int_B \kappa^n (B-B') = \frac{B_c}{B_0} a_n \langle g^n \rangle^2,
\]

(16)

with \( a_n \) given in Eq. (13), since the multiple integral \( \Sigma \) is reduced to a single integral over \( x = (B-B')/B_0 \), proportional to \( B_c/B_0 \). All other contributions from the \( l = 0 \) term are proportional to \( NM P_{N,M} \) (where \( P_{N,M} \) is regular in the replica limit), but they contain more integrals over powers of \( K \), each adding an extra small factor of \( B_c/B_0 \).

Now we show that the terms with \( l \neq 0 \) in Eq. (15) do not contribute in the leading order in \( B_c/B_0 \). Schematically, their contributions may be represented as

\[
\Sigma_{NM}^{n-2l} (\alpha N + \Sigma_{N(M-1)})^l (\alpha M + \Sigma_{M(M-1)})^l,
\]

(17)

where \( \Sigma \)'s with appropriate indices stand for one of the sums in Eqs. (14). The number of independent integrations over \( \kappa \)'s is not smaller than the number of different \( \Sigma \)'s: therefore, contributions from Eq. (17) that contain different \( \Sigma \)'s will be small in powers of \( B_c/B_0 \). On the other hand, those which contain only one \( \Sigma \), like \( \Sigma_{NM}^{n-1} (\alpha^2 N M)^l \) (or \( \alpha N )^{n/2} \Sigma_{NM}^{n/2} (M-1) \), would vanish in the replica limit in Eq. (15), the latter term vanishing due to the fact that each \( \Sigma \) is proportional to (at least) the number of the terms in it, indicated by its index.

Therefore, Eq. (15) gives the only non-vanishing in the replica limit term proportional to the first power of
for the three types of the conductors with different values of the exponent γ and the correlation field $B_c$:

(a) $γ = 1/2$ and $B_c \sim \phi_0 / [w \times \min(L_\varphi, L_s)]$ for a long wire of width $w \ll \min(L_\varphi, L_s) < L$, where $\phi_0 = \hbar e / c$

(b) $γ = 1$ and $B_c \sim \sqrt{\frac{N_\varphi}{\pi^2}} \phi_0 / L^2$ for a dot of size $L$ connected to the reservoirs via leads with $N$ channels and mean level spacing $\Delta$ at temperature $T > N\Delta$

(c) $γ = 2$ and $B_c \sim \phi_0 / L^2$ for a dot or a short wire (with $L < \min(L_\varphi, L_s)$) at $T = 0$.

For these cases, coefficients $a_n$ in Eq. (6) are given by

$$a_n = \begin{cases} \frac{n^n}{\Gamma(n+1)} \left( \frac{\pi}{2} \right)^n & (a) \\ \frac{n!}{\Gamma(n+1)} \left( \frac{\pi}{2} \right)^n & (b) \\ \frac{n!}{\Gamma(n+1)} \left( \frac{\pi}{2} \right)^n & (c) \end{cases}$$

The appropriate power of the variance in the r.h.s. of Eq. (6) represents an inevitable systematic error in extracting the $n^{\text{th}}$ cumulant from a single magnetofingerprint. This can be rather large. The widest range of magnetic fields achievable experimentally is limited to $B_0 \lesssim 20T$ by the necessity to use noiseless superconducting magnets. For the lowest temperatures in experiments with metallic wires, $T \sim 10 \div 100mK$, the correlations field $B_c \sim 10 \div 100mK$, in agreement with the estimates for type (a) above. A meaningful deviation of, say, $\langle \langle g^3 \rangle \rangle$ from the combined prediction of Eqs. (1) and (2) (not speaking of a more accurate estimate of the third cumulant) that shows it to be of order $1/g^2$ rather than $1/g$ which would manifest a breakdown of the one-parameter scaling, or inadequacy of the model of noninteracting electrons in disordered potential should therefore comfortably exceed the value of $0.15\langle \langle g^2 \rangle \rangle$. So far, reliable experimental data for the higher cumulants in the metallic regime fall well below this limit.

We thank Boris Altshuler and George Pickett for useful discussions. This work, which was initiated during workshop INT-02-2 'Chaos and Interactions: from Nuclei to Quantum Dots’ in the Institute for Nuclear Physics at the University of Washington. It has further been supported by EPSRC (VIF and IVL) and Packard Foundation (ILA). VIF also acknowledges Royal Society for travel support.

1. B. L. Altshuler, *JETP Letters* 41, 648 (1985).
2. B. L. Altshuler and D. E. Khmel’nitskii, *ibid* 42, 359 (1985).
3. P. A. Lee and A. D. Stone, *Phys. Rev. Lett.* 55, 1622 (1985); P. A. Lee, A. D. Stone, and H. Fukuyama, *Phys. Rev. B* 35, 1039 (1987).
4. For earlier reviews of both theory and experiment, see *Mesoscopic Phenomena in Solids*, ed. by B. L. Altshuler, P. A. Lee, and R. A. Webb, Elsevier, Amsterdam (1991).
5. A. M. Chang et al., *Phys. Rev. Lett.* 76, 1695 (1996).
6. J. A. Folk et al., *Phys. Rev. Lett.* 76, 1699 (1996).
7. P. Mohanty and R. A. Webb, *Phys. Rev. Lett.* 88, 146601 (2002).
8. S. Kar, A.K.Raychaudhuri, and A. Ghosh, cond-mat/0212165
9. M. G. Kendall and A. Stuart, *The advanced theory of statistics*, 4th ed., Vol.1, Griffin, London (1976).
10. E. Abrahams et al., *Phys. Rev. Lett.* 42, 673 (1979).
11. B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, Zh. Eksp. Teor. Fiz. 91, 2276 (1986); in Ref. 3, p.449.
12. K. A. Muttalib, J. L. Pichard, and A. D. Stone, *Phys. Rev. Lett.* 59, 2475 (1987).
13. There exist contributions to higher cumulants that lead to lognormal tails of the distribution function and are not described within the one-parameter scaling approach; they become larger than those in Eq. (12) only for $n > \langle g \rangle$ so that they are irrelevant for considerations in this paper.
14. For the conductance distribution in quasi-1D wires, a theoretically expected deviation from the Gaussian shape is even smaller than predicted by Eq. (12), since $A_{31} = 0$ so that the third cumulant turns out to be of order $1/g^2$ rather than $1/g$.
15. We keep only the leading terms in Eq. (2); assuming independent contributions of $N$ connected in series conductors making the wire of length $L = NL_\varphi$ and $\langle \langle g^m \rangle \rangle \ll \langle g \rangle^m$, one obtains more detailed expressions, e.g. for $n = 2, 3$,

$$\langle \langle g^2 \rangle \rangle_{L = NL_\varphi} = \frac{2(N-1)}{N^2} \langle g \rangle^2 + \frac{3(N-1)^2}{N^3} \langle g \rangle^3$$

As this is irrelevant for further considerations, we also disregard here the averaging over spectra which would lead to a substitution of the thermal length $L_T = \sqrt{D/T}$ for some of the phase breaking lengths $L_\varphi$ in the above expansion.
16. D. Mailly and M. Sanquer, *J. Phys. I (France)* 2, 357 (1992).
17. To exclude effects of weak localization (or antilocalization), in determining the conductance distribution from $g(B)$ one disregards small magnetic fields, choosing $B \gg B_c$.
18. V. I. Fal’ko, *Phys. Rev. B* 51, 5227 (1995).
19. In the 1D case, Eq. (13) for $\gamma = 1/2$ is a numerically accurate approximation to the analytic result of I. L. Aleiner and Y. M. Blanter, *Phys. Rev. B* 65, 115317 (2002).
20. B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, *JETP Letters* 43, 441 (1986).
21. S. Edwards and P. W. Anderson, *J. Phys. F* 5, 965 (1975).
22 V. I. Fal’ko and K. B. Efetov, Europhys. Lett. 32, 627 (1995); Phys. Rev. B 52, 17413 (1995).
23 A. M. S. Macêdo, Phys. Rev. B 49, 1858 (1994); M. C. W. van Rossum et al., Phys. Rev. B 55, 4710 (1997).