Gravitational waves and the cosmological equation of state

Teviet Creighton

Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, CA 91125
(7/12/99)

As is well known, primordial gravitational waves may be amplified to detectable levels by parametric amplification during eras when their wavelengths are pushed outside the cosmological horizon; this can occur in both inflationary and “pre-big-bang” or “bounce” cosmologies. The spectrum of a gravitational wave background is expressed as a normalized spectral energy density \( \Omega(\omega) \equiv (\omega/\rho_c)(d\rho_{gw}/d\omega) \), where \( \rho_c \) is the critical energy density that makes the universe spatially flat, and \( d\rho_{gw} \) is the energy density of gravitational waves in a frequency band \( d\omega \). The logarithmic slope of \( \Omega \) is simply related to three properties of the early universe: (i) the gravitons’ mean initial quantum occupation number \( N(n) \) (\( n = 1/2 \) for a vacuum state), where \( n = a\omega \) is the (invariant) conformal frequency of the mode and \( a \) is the cosmological scale factor, and (ii) & (iii) the parameter \( \gamma \equiv p/\rho \) of the cosmological equation of state during the epoch when the waves left the horizon (\( \gamma = \gamma_s \)) and when they reentered (\( \gamma = \gamma_f \)). In the case of an inflationary cosmology, the spectral index is

\[
\frac{d\ln \Omega}{d\ln \omega} = \frac{d\ln N}{d\ln n} + 2 \left( \frac{\gamma_s + 1}{\gamma_s + 1/3} \right) + 2 \left( \frac{\gamma_f - 1/3}{\gamma_f + 1/3} \right),
\]

and for bounce cosmologies it is

\[
\frac{d\ln \Omega}{d\ln \omega} = \frac{d\ln N}{d\ln n} + 2 \left( \frac{2\gamma_i}{\gamma_i + 1/3} \right) + 2 \left( \frac{\gamma_f - 1/3}{\gamma_f + 1/3} \right).
\]

These expressions are compared against various more model-specific results given in the literature.

I. INTRODUCTION

The prospect of a detectable cosmological background of gravitational waves has opened up a new avenue for the investigation of fundamental physics and cosmology. Gravitational waves couple very weakly to matter, and are not blocked or thermalized during any cosmological epoch back to Planckian densities. The waves do, however, couple to large-scale cosmological spacetime curvature, and can therefore provide information about the evolution of this curvature, and hence about the fundamental physics of the matter fields driving this evolution.

A great deal of work has been done on this subject, deriving gravitational-wave spectra from various cosmological models, usually in hope of producing a model that would generate a detectable signature in planned gravitational-wave detectors. The most comprehensive of these is a recent paper by Gasperini [1], which presents a general prescription for constructing gravitational-wave spectra from fairly arbitrary cosmologies. The results in the present paper are consistent with Gasperini’s results, although I derive them using a different formalism and consider a more generic cosmology.

The primary intent of this paper, however, is more the reverse of this: rather than deriving a spectrum from a particular cosmological model, this paper shows how generic properties of the early universe can be immediately deduced from an observation of a gravitational-wave background in any spectral band. To this end, I have chosen as my observable the spectral characteristic that is least susceptible to the tunings of particular cosmological models — namely, the spectral index — and have expressed it in terms of quantities that are most directly related to the underlying physics of the cosmological “fluid” — namely, the equations of state \( p = \gamma \rho \). Furthermore, since the initial state of the gravitational waves (before amplification) is potentially one of the more interesting characteristics that might be deduced from observations, I have left it as a free parameter, rather than making the usual assumption of starting the waves in a quantum ground state.

A. Organization of this paper

In Sec. I, I present a skeleton of the derivation of the main formulae of this paper. Much of the derivation has been done in one form or another in the published literature, and those familiar with the field will find nothing surprising, except perhaps the consideration of non-vacuum initial conditions in Sec. II. Since the result is largely a generalization of the more model-specific formulae found in the literature, Sec. III compares this paper’s formulae with those published previously. Sec. IV presents some concluding remarks.
II. THE COSMOLOGICAL AND WAVE EQUATIONS

I consider gravitational waves that are linear perturbations on the spatially flat FRW metric:

$$ds^2 = a^2(\eta)[-d\eta^2 + dx^2 + dy^2 + dz^2] + h_{\alpha\beta}dx^\alpha dx^\beta,$$

(2.1)

where the conformal time coordinate $\eta$ is related to proper time $t$ by $dt = a(\eta)d\eta$ (I have chosen units in which $c = 1$). The cosmology has a perfect fluid source ($\mu_n(\eta) = n\gamma$), where $\gamma$ need not be constant but typically evolves slowly ($\gamma' \ll a'/a$, where $' \equiv d/d\eta$). Causality considerations require that $\gamma \leq 1$; realistically, we can assume $\gamma < 1$, since not all of the energy in the universe will be in the maximally-stiff field. One can solve Einstein’s equations to zeroth order in $h$ to obtain the exact solution

$$a(\eta) = a_0 \exp\left\{\int_{\eta_0}^{\eta} \frac{d\eta_1}{a_0/a_\theta + \int_{\eta_0}^{\eta_1} d\eta_2 [1 + 3\gamma(\eta_2)/2]}\right\},$$

(2.2)

which reduces to the usual power-law evolution $a(\eta) \sim \eta^{2(1+3\gamma)}$ during epochs of (nearly) constant $\gamma$.

Following the notation in Eq. (2) of [3], I write the linear perturbations in the form:

$$h_{\alpha\beta} = \sum_{n,j} \frac{\mu_n(\eta)}{a(\eta)} U_n(x)e_n^{(1)}(\alpha\beta),$$

(2.3)

where $e_n^{(1)}$ is some basis of polarization tensors, and $U_n(x) \propto e^{in\cdot x}$ is a spatial harmonic function with a true (physical) wave number $k = n/a$ and frequency $\omega = n/a$ ($n = |\mathbf{n}|$), and $\mu_n(\eta)$ obeys the Schrödinger-like equation:

$$\mu''_n + \left(n^2 - \frac{\omega^2}{a^2}\right) \mu_n = 0.$$  

(2.4)

Physically, the relative magnitudes of $n^2$ and the effective potential $a''/a$ are related to whether the wave is inside or outside of the Hubble radius. Comparing $k$ to the Hubble radius $r_H = a/(da/dt)$, one has:

$$k^2r_H^2 = \frac{n^2}{(a'/a)^2} = \frac{n^2}{a''/a} \times \frac{1 - 3\gamma}{2}.$$  

(2.5)

Roughly speaking, a wave that is hitting the effective potential $a''/a$ increasing to meet $n^2$ is one that is exiting the Hubble radius, to order of magnitude. Similarly, a wave that is emerging from the effective potential is reentering the Hubble radius. The exception is for epochs when $\gamma \rightarrow 1/3$; see the remarks near the end of Sec. IA.

Cosmological amplification of waves occurs where initially oscillatory waves hit the effective potential during epochs of accelerated collapse ($a' < 0$, $\gamma > -1/3$) or accelerated expansion ($a' > 0$, $\gamma < -1/3$), when $a''/a$ is increasing, and then emerge into a post-inflationary universe ($a' < 0$, $\gamma > -1/3$) in which $a''/a$ is decreasing. A schematic of such an evolution is shown in Fig. 1. The periods when the wave initially hits the potential and when it finally emerges from it will be denoted by subscript $i$ and $f$, respectively. The (complex) amplification factor of the emerging waves $\beta_n \equiv \mu_n(\eta)/\mu_n(\eta_i)$ is

$$\beta_n = \frac{1}{2i} e^{in(\eta - \eta_f)} \left[ \frac{a_f}{a_i} \left(i + \frac{a'_f}{a_{f n}}\right) + \frac{a_i}{a_f} \left(i - \frac{a'_i}{a_{i n}}\right) \right] + a_i a_{f n} \left(i - \frac{a'_i}{a_{i n}}\right) \left(i + \frac{a'_f}{a_{f n}}\right) \int_{\eta_i}^{\eta_f} \frac{d\eta}{a^2};$$

(2.6)

this is Eq. (11) of [3] with a sign error corrected, and reexpressed in the current notation.

FIG. 1. Schematic of the evolution of the effective potential $|a''/a|$ (solid line) and Hubble scale $(a'/a)^2$ (dotted line) for inflation ($\eta < \eta_i$) followed by decelerating expansion ($\eta > \eta_f$). Also shown are the times $\eta_i$ and $\eta_f$ when a wave with squared conformal frequency $\eta^2$ (dashed line) hits and emerges from the potential barrier.

If the evolution of $a(\eta)$ between $a_i$ and $a_f$ is monotonic, one can follow [3] and use the equation of state to reexpress the integral over $\eta$ as an integral over $a$, eventually obtaining, for $a_f > a_i$:

$$|\beta_n| \approx \frac{1}{2} \frac{a_f}{a_i} \frac{a_{f n}}{a_{i n}} \left[1 + \frac{2}{3\gamma} \left(1 - \frac{\gamma}{\gamma} \right) \left(1 - \frac{i a_{i n}}{a'_i} \right)\right],$$

(2.7)

where $\langle \gamma \rangle$ is the average of $\gamma$ from $\eta_i$ to $\eta_f$ with weighting factor $1/a^2$.

The derivation of Eq. (2.7) breaks down for the case of a bounce cosmology, where $a(\eta)$ is not monotonic. In this case, however, Eq. (2.6) is almost entirely dominated by the contribution of the integral $\int d\eta/\sigma^2$ near the time of the bounce. Modeling the bounce as smooth with some finite concavity $a''_0$ around the minimum $a_0$, we obtain:

$$|\beta_n| \approx \frac{1}{2} \frac{a_i a_{f n}}{a_{i n}} \left[i + \frac{a'_f}{a_{f n}}\right] \left(i - \frac{a'_i}{a_{i n}}\right) \left|\frac{1}{\sqrt{a_0 a''_0}}\right|.$$  

(2.8)

A. Dependence on $n$

The amplification factor $\beta_n$ is at least implicitly dependent on $n$, since waves of different $n$ will hit and
emerge from the effective potential at different times $\eta_i$ and $\eta_f$, and hence with different $a_i, a_f$. For periods of constant $\gamma$, we have $a \sim e^{\gamma f}$. The condition $n^2 = a''/a$ for hitting or emerging from the effective potential implies that $a_i f \sim n^{-2/(1+3\gamma_f)}$, and that $a_i f/(a_i f) = \sqrt{2/(1-3\gamma_f)} \sim$ constant. If $\gamma$ is changing gradually near $\eta_f$, these formulae remain true up to corrections of order $(\gamma/\gamma_i)(a''/a)^{-1}$. Furthermore, the weighted average $\langle \gamma \rangle$ in Eq. (2.7) depends only very weakly on $\eta_i, f$, and hence on $n$, except for values of $n^2$ near the peak of the effective potential. These considerations, combined with Eqs. (2.7) and (2.8), give

$$|\beta_n| \sim n^{2/(1+3\gamma_i)} - 2/(1+3\gamma_f)$$

for inflationary cosmologies, and

$$|\beta_n| \sim n^{1+2/(1+3\gamma_i)} + 2/(1+3\gamma_f)$$

for bounce cosmologies.

A special case is when $\gamma_f \approx 1/3$ (the equation of state for a relativistic or radiation-dominated fluid), for which the effective potential $a''/a \approx 0$. This is depicted in Fig. 2. If the transition to radiation dominance is rapid, then $\eta_f$, and hence $a_f$ and $a'_f$, are independent of $n$ for a wide range of $n$. However, waves that are well outside the Hubble radius at the moment of transition will have $a'_f/a_f \gg n$, as is clear from Eq. (2.3) and Fig. 2. So the appropriate terms in Eqs. (2.3) and (2.8) have the form

$$a f \bigg|_{i} + a'_f/a_i f \sim \frac{1}{n}.$$  

(2.11)

This is the same dependence as one would get by naively plugging $\gamma_f = 1/3$ into Eqs. (2.3) and (2.10), so these equations are correct even as $\gamma_f \to 1/3$. (This argument also applies for waves that hit the effective potential during radiation-dominated collapse.)

A similar situation occurs for waves that remain outside the Hubble radius throughout an intermediate period of $\gamma \approx 1/3$, when formally they have emerged from the effective potential and are oscillatory. This is depicted in Fig. 2. The effect of an intermediate period $\Delta \eta$ of zero effective potential is a term $\sim \sin(n\Delta \eta)/n$ in the final amplitude. If the waves remain well outside the Hubble radius throughout this era, then $n\Delta \eta \ll 1$, and this term is a constant with respect to $n$. Once again, Eqs. (2.9) and (2.10) remain unchanged. Both this and the preceding cases confirm one’s physical intuition, that the final spectrum cannot resolve the details of sudden phase transitions that occur when the waves are much larger than the Hubble radius.

FIG. 2. Schematic of the evolution of the effective potential and Hubble scale for inflation ($\eta < \eta_i$) followed by a radiation-dominated equation of state ($\eta > \eta_i$). Note that $\eta_f = \eta_i$ independent of $n$ for a broad range of frequencies.

FIG. 3. Schematic of the evolution of the effective potential and Hubble scale for inflation ($\eta < \eta_i$), followed by radiation dominance ($\eta_i < \eta < \eta_f$), followed by generic decelerated expansion ($\eta_f > \eta_f$). Waves emerging from the potential barrier after $\eta_f$ will have been outside the Hubble radius throughout the period of radiation dominance.

Eqs. (2.9), (2.10) also break down when $\gamma_i f \approx -1/3$, but this condition will never arise. When $\gamma \approx -1/3$, one has $a \sim e^{3\eta} n$, so $a''/a \approx$ constant, and waves will neither hit nor leave the effective potential.

B. The initial and final spectra

Most analyses of primordial gravitational waves assume that the initial state of the metric perturbations (during the Planck era or during pre-Big-Bang collapse) is a quantum-mechanical vacuum. While this is a reasonable assumption, one does not know the initial state in advance of observation. In fact, primordial gravitational waves are a potential means of observing the initial state of the cosmos. It is instructive, therefore, to leave the initial graviton spectrum as a free parameter.

I parameterize the initial spectrum using the mean quantum occupation number of graviton modes as a function of mode frequency, $N(n)$. In the semiclassical approach, the creation of gravitons can be treated as the classical amplification of the vacuum energy $\hbar \omega/2$ in each mode, so I normalize $N(n)$ to be $1/2$ for a mode in the ground state (no real gravitons), $3/2$ for the first excited state (one real graviton), and so on. This approach is valid provided the final state is classical ($|\beta_n| \gg 1$).

Cosmological perturbation spectra are normally expressed as a normalized energy density per logarithmic frequency interval, $\Omega \equiv (dE/dVd\ln \omega)/\rho_c$ (where $\rho_c$ is the energy density required to make the universe spatially flat). There are $2\omega^2 d\omega dV$ modes in a frequency interval $d\omega$ in a volume $dV$, each with energy $\hbar \omega N(n)$ (where $n = a \omega$), so the initial energy spectrum is:

$$\Omega_{\text{initial}}(n) = \frac{2n^4 N(n)}{a_{\text{initial}}^4 \rho_c}.$$  

(2.12)
Note that $a_{\text{initial}}$ is the (constant) scale factor of the universe at the time that one is evaluating the initial spectrum; it is not the same as $a_i$, which depends on $n$.

In considering the logarithmic slope of the spectrum I ignore overall amplitude factors, which are highly model-specific. The energy spectrum at the present day is quadratic in $|\beta_n|$, so its logarithmic slope is:

$$\frac{d \ln \Omega}{d \ln \omega} = \frac{d \ln N}{d \ln n} + 2 \left( \frac{\gamma_f + 1}{\gamma_f + 1/3} \right) = 3 \left( \frac{\gamma_f + 1}{\gamma_f + 1/3} \right)$$

where the evaluation is at $n = \omega/a_{\text{present}}$. Combining this with Eqs. (2.13), (2.9), and (2.10) gives the principal result of this paper:

$$\frac{d \ln \Omega}{d \ln \omega} = \frac{d \ln N}{d \ln n} + 2 \left( \frac{\gamma_i + 1}{\gamma_i + 1/3} \right) + 2 \left( \frac{\gamma_f - 1/3}{\gamma_f + 1/3} \right)$$

for an inflationary cosmology, and:

$$\frac{d \ln \Omega}{d \ln \omega} = \frac{d \ln N}{d \ln n} + 2 \left( \frac{2 \gamma_i}{\gamma_i + 1/3} \right) + 2 \left( \frac{\gamma_f - 1/3}{\gamma_f + 1/3} \right)$$

for a bounce-type cosmology.

### III. COMPARISON WITH SPECIAL CASES

The formulae given above are consistent with results established previously in the literature. I first show this for the two “standard” cases of deSitter inflation and of cosmological rebound from a state of weakly coupled strings. I then compare with recent papers that consider more generic pre- and post-inflationary evolutions.

#### A. Standard inflation

The “standard” model for an inflationary cosmology consists of a period of deSitter expansion ($\gamma = -1$) due to the universe being in a false vacuum with nonzero energy density. In the post-inflationary era, the universe quickly reheats to a radiation-dominated equation of state ($\gamma = 1/3$, followed by cooling to a matter-dominated equation of state ($\gamma = 0$). If we assume the initial graviton state from the Planck era was a vacuum, then for waves that emerged into the radiation-dominated universe we have $N \equiv 1/2$, $\gamma_i = -1$, $\gamma_f = 1/3$, and

$$\frac{d \ln \Omega}{d \ln \omega} = 0$$

This is, of course, the flat spectrum predicted by Harrison and Zel’dovich.

The last two decades of the spectrum ($10^{-16}$–$10^{-18}$ Hz) consist of waves that emerged more recently, when the universe was matter-dominated ($\gamma_f = 0$). In this band the spectrum is tilted towards low frequencies, with a spectral index of $-2$.

#### B. Standard string-motivated rebound cosmologies

It has been shown that string theories can lead to a gravitational-wave background that is increasing with frequency. These models are consistent with the current derivation provided one recognizes that the current formalism applies to the Einstein frame (where the metric represents intervals measured by physical rulers and clocks), rather than the string frame (where the metric geodesics describe the trajectories of weakly coupled strings). In the Einstein frame, a typical evolution consists of an epoch of decreasing lengthscales $a$ and weakly coupled strings, followed by a period of strongly coupled strings that reverse the collapse, followed by relaxation of the string dilaton and an expanding radiation-dominated universe, eventually cooling to matter dominance. As usual, a vacuum initial state ($N \equiv 1/2$) is normally assumed. Most modes hit the effective potential during the epoch of weakly interacting strings, when the kinetic term of the dilaton dominates the energy background, giving a nearly maximally stiff equation of state ($\gamma_i \approx 1$).

For waves that emerge during the radiation-dominated phase ($\gamma_f \approx 1/3$), Eq. (2.13) gives:

$$\frac{d \ln \Omega}{d \ln \omega} = 3$$

Eq. (3.3) of [7] and Eq. (5.7) of [6] also give a spectral index of $3$ for low frequencies. These papers also predict a logarithmic cutoff at high frequencies, $\Omega \sim \omega^3 \ln^2 (\omega_s/\omega)$, where $\omega_s$ is the maximum frequency of waves that encountered the effective potential during the epoch of maximal stiffness (typically of order $10^{11}$ Hz, depending on the details of the strongly coupled string epoch). This effect occurs only for $\gamma_i$ exactly equal to 1, which the current analysis doesn’t consider, and only affects the spectral index within a decade of $\omega_s$ in any case.

The low-frequency tail ($\omega < 10^{-16}$ Hz) of waves that emerge when $\gamma_f = 0$ will also be tilted towards higher frequencies, but with a spectral index of $1$.

#### C. Nonstandard equations of state

A recent paper has explored the effects of an intermediate phase between the inflationary and radiation-dominated eras, in which the dominant cosmological fluid had an equation of state stiffer than radiation ($1/3 < \gamma_f < 1$). One consequence is that the background gravitational-wave spectrum would be tilted towards high frequencies for those wavelengths that emerged during this era. Assuming an initial vacuum ($N \equiv 1/2$) and deSitter inflation, Eq. (2.13) gives us:

$$\frac{d \ln \Omega}{d \ln \omega} = 2 \left( \frac{\gamma_f - 1/3}{\gamma_f + 1/3} \right)$$

Eq. (3.32) of [7] gives the same dependence on $\gamma$. 

4
Eq. (3.31) of [7] extends the analysis to the case of a maximally-stiff equation of state \((\gamma_f = 1)\), which this paper does not consider. This gives a high-frequency logarithmic cutoff \(\Omega \sim \omega \ln^2(\omega_1/\omega)\), where \(\omega_1\) is the highest frequency that encountered the amplifying potential. The cutoff does not affect the spectral index at frequencies more than a decade or so below \(\omega_1\), which is likely in the MHz to GHz range (well above the pass-bands of proposed gravitational-wave detectors). This is analogous to the situation mentioned earlier, when \(\gamma_i \to 1\) during collapse.

Gasperini [1] has performed an even more general analysis of the primordial gravitational wave spectrum, considering generic evolution of \(a(\eta)\) both in the early and late cosmological epochs. As usual, the initial state of the waves was taken to be a vacuum \((N \equiv 1/2)\). Eqs. (2.14), (2.15) can then be written as:
\[
\frac{d \ln \Omega}{d \ln \omega} = 4 + \frac{4}{1 + 3\gamma_i} - \frac{4}{1 + 3\gamma_f} \tag{3.4}
\]

or
\[
\frac{d \ln \Omega}{d \ln \omega} = 6 - \frac{4}{1 + 3\gamma_i} - \frac{4}{1 + 3\gamma_f} \tag{3.5}
\]

for monotonic or bounce cosmologies, respectively. By comparison, Eq. (4.26) of [1] gives the spectral index as \(4 - 2\alpha - 2\alpha_i + 1\), where \(\nu = [\alpha - 1/2]\) and \(\alpha\) is the exponent of the power law \(a \sim \eta^\alpha\) during the epoch when the waves first strike the effective potential (for \(\nu_i + 1\)) and when they leave the effective potential (for \(\nu_f\)). Gasperini’s results are identical to Eqs. (3.4) and (3.5), provided one recognizes two things.

First, a phase of slowly varying \(\alpha\) corresponds to a phase of slowly varying \(\gamma\), with \(\alpha = 2/(1 + 3\gamma)\). Inflationary cosmologies have \(\gamma_i < -1/3\), hence \(\alpha_i + 1 < 0\). Bounce cosmologies have \(-1/3 < \gamma_i \leq 1\) and \(\alpha_i + 1 \geq 1/2\). Either cosmology has \(-1/3 < \gamma_f \leq 1\) and \(\alpha_f \geq 1/2\). (The condition \(0 < \alpha < 1/2\) corresponds to the physically unrealistic case of \(\gamma > 1\).)

Second, Gasperini assumes in Eq. (4.26) of [1] that all waves leave the effective potential at the same cosmological epoch, so that \(\nu_f\) is a constant across all frequencies (normally \(\nu_f = 1/2\) for radiation domination). In fact, the spectral index (but not necessarily the normalization) given in Eq. (4.26) of [1] is valid for \(\nu_f\) varying across frequencies.

IV. CONCLUSIONS

Eqs. (2.14) and (2.15) show how much can be determined about early cosmology from an observation of the spectral index of primordial gravitational waves. Although these formulae are quite simple, they still involve three independent parameters: \(\gamma_i\), \(\gamma_f\), and \(N(n)\). To make definite statements about any one of them, one must make assumptions about the others. The usual procedure is to assume that we know \(N(n)\) and \(\gamma_f\), leaving \(\gamma_i\) to be deduced; however, it is quite possible that future advances in theory will fix both \(\gamma_i\) and \(\gamma_f\), allowing gravitational-wave observations to probe directly the Planck-scale structure of the universe.

The formulae are also useful in showing which cosmological theories produce equivalent gravitational signatures. For instance, a monotonically inflating cosmology with \(\gamma_i \leq -1/3\) will yield the same spectral index as a bounce-type cosmology with \(\gamma_i = (\gamma_i + 1)/(3\gamma_i - 1)\) in the range \((-1/3, 1/3)\). Since these are physically valid ranges for \(\gamma_i\) in each case, it follows that a monotonic cosmology cannot be distinguished from a bounce cosmology given only the gravitational-wave spectral index.

The purpose of this paper was to present the simplest possible expressions relating a generic cosmological gravitational-wave signature to the physical processes that produced it, with a minimum of assumptions about the underlying cosmological model. I have deliberately chosen an approach of “maximum ignorance”, allowing, as much as possible, for the observations to dictate the cosmological parameters. Formulae such as these may prove valuable in an era when gravitational-wave observations begin to explore a new realm of physics, possibly revealing things beyond our most informed predictions.

ACKNOWLEDGEMENTS

This work was supported by NSF grant AST-9731698 and NASA grant NAG5-6840. I thank Kip Thorne for his help and encouragement.

[1] M. Gasperini, “Relic Gravitons from the Pre-Big Bang: What we Know and What we Do Not Know,” hep-th/9607146.
[2] L. P. Grishchuk and M. Solokhin, Phys. Rev. D 43, 2566 (1991).
[3] E. R. Harrison, Phys. Rev. D 1, 2726 (1970).
[4] Ya. B. Zel’dovich, Mon. Not. Roy. Astron. Soc. 160, 1p (1972).
[5] R. Brustein, M. Gasperini, M. Giovannini, and G. Veneziano, Phys. Lett. B 361, 45 (1995).
[6] M. Gasperini, “Amplification of vacuum fluctuations in string cosmology backgrounds,” hep-th/9506140.
[7] M. Giovannini, Phys. Rev. D 58, 083504 (1998).