Variation of the spin textures of 2-species spin-1 condensates studied beyond the single spatial mode approximation and the experimental identification of these textures

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Abstract
Based on the numerical solutions of the coupled Gross–Pitaevskii equations, the spin-textures of the ground state of a Bose–Einstein condensate with two kinds of spin-1 atoms have been studied. Besides, making use of the fractional parentage coefficients, the probabilities of an atom in spin component \( \mu \) and of two correlated atoms one in \( \mu \) and one in \( \nu \) have been calculated. The previous results under the single spatial mode approximation (SMA) are found to be qualitatively consistent with the present results beyond the SMA. The variation of the probabilities against the input parameters is found to match the corresponding variation of the spin-textures excellently. In particular, once a spin-texture transition occurs, the associated probabilities will change suddenly. Since the probabilities are observable, the observation provides a way for the experimental identification of the spin-textures. Besides, since the critical points (where a transition occurs) can be accurately identified, information on the input parameters (say, the very weak spin-dependent forces) can be extracted.

Keywords: Bose–Einstein condensates, 2-species condensates with spin-1 atoms, beyond single mode approximation, identification of spin-textures, spin-component population

(Some figures may appear in colour only in the online journal)
of an atom lying at a specific spin-component and the probability of two atoms (the same or different kind(s)) lying at two specific spin-components have been derived, and their values have been numerically extracted from the solutions of the CGP. It was found that, when an input parameter varies, the variation of these probabilities matches the variation of the spin-texture very well. Since these probabilities are observables (say, via the time-of-flight images), the spin-texture can be thereby identified.

As a basic tool, the fractional parentage coefficients for spin-1 many-body systems have been extensively used in our theoretical derivation [16, 17]. With these coefficients, all the matrix elements of 1-body and 2-body spin-operators can be determined. The use of the permutation symmetry inherent in \( |\varphi\rangle \otimes |\varphi\rangle \), the Hamiltonian is equivalent to an effective Hamiltonian \( H_{\text{eff}} \), in which \( V_X \) is replaced by

\[
V_X^\text{eff} = \sum_{i<j} \delta(r_i - r_j) \left[ c x_0 + c x_2 \frac{1}{N_X (N_X - 1)} \right] (\hat{S}_X \cdot \hat{S}_X - 2N_X),
\]

and \( V_{AB} \) is replaced by

\[
V_{AB}^\text{eff} = \sum_{i,j} \delta(r_i - r_j) \left[ c a_{AB} + c a_{AB} \frac{1}{2N_A N_B} \right] \left( \hat{S}_A \cdot \hat{S}_B - \hat{S}_A \cdot \hat{S}_B \right).
\]

With these expressions we know that \( H_{\text{eff}} \) keeps \( S_A, S_B, S \), and \( M \) to be conserved. Therefore the g.s. can be rewritten as

\[
|\varphi\rangle = \prod_{i=1}^{N_A} \varphi_A(r_i) \prod_{j=1}^{N_B} \varphi_B(r_j) \Xi,
\]

where \( S_A, S_B, S \) together with \( \varphi_A \) and \( \varphi_B \) have to be determined.

Since the set \( (S_A, S_B, S) \) are good quantum numbers, they can be used to classify the spin-textures (note that \( M \) is not necessary in the classification because it manifests only the geometry).

From \( \delta(\langle \varphi|\hat{H}|\varphi\rangle)/\langle \varphi|\hat{H}|\varphi\rangle = 0 \), we obtain the CGP for \( \varphi_A \) and \( \varphi_B \) as [7]

\[
\hat{h}_A + \alpha_{11} \varphi_A^2 + \alpha_{12} \varphi_A^2 - \varepsilon_A \varphi_A = 0,
\]

\[
\hat{h}_B + \alpha_{21} \varphi_A^2 + \alpha_{22} \varphi_B^2 - \varepsilon_B \varphi_B = 0,
\]

where

\[
\alpha_{11} = D_A N_A, \quad \alpha_{12} = D_A B N_B, \quad \alpha_{21} = D_B A N_A, \quad \alpha_{22} = D_B B N_B,
\]

\[
D_X = c x_0 + \frac{S(S + 1) - S_A(S_A + 1) - S_B(S_B + 1)}{N_X(N_X - 1)} c x_2,
\]

\[
D_{AB} = c a_{AB} + \frac{S(S + 1) - S_A(S_A + 1) - S_B(S_B + 1)}{2N_A N_B} c a_{AB},
\]

and \( \varphi_A \) and \( \varphi_B \) are required to be normalized.

Once the values of \( (S_A, S_B, S) \) has been presumed, we can solve the CGP to obtain \( \varphi_A \) and \( \varphi_B \) together with the total energy

\[
E_S = N_A \langle \varphi_A | \hat{H} | \varphi_A \rangle + N_B \langle \varphi_B | \hat{H} | \varphi_B \rangle + \frac{N_A(N_A - 1)}{2} D_A I_A + \frac{N_B(N_B - 1)}{2} D_B I_B + N_A N_B D_{AB} I_{AB}.
\]
The parameters are given as
\( N_{A} = 1000, \quad N_{B} = 1200, \quad m_{A}/m_{B} = 23/87, \quad \gamma_{A} = \gamma_{B} = 1; \)
\( c_{A0} = 10^{-4}, \quad c_{B0} = 2 \cdot c_{A0}, \quad c_{A0}/c_{B0} = 0.9; \quad c_{A2} = c_{A0}/50, \quad c_{B2} = c_{B0}/50, \quad c_{AB}/c_{A0} = 0.02 \) in (a) and 0.09 in (b). The associated spin-textures specified by \( (S_{A}, S_{B}, S) \) are marked in the panels. The units for energy and length are \( \hbar \omega \) and \( \sqrt{\hbar/2m_{\omega}} \) throughout this paper.

where \( I_{A} = \int dr |\psi_{A}(r)|^{4} \) and \( I_{AB} = \int dr |\psi_{A}(r)|^{2}|\psi_{B}(r)|^{2}. \) From a series of presumed \( (S_{A}, S_{B}, S) \), we can find out the optimal set \( (S_{A}, S_{B}, S) \) which leads to the minimum of the total energy. This set is the good quantum numbers of the g.s.

In this paper, by using the imaginary-time propagation approach [19], equations (6) and (7) are solved numerically to obtain \( \varphi_{A} \) and \( \varphi_{B}. \) An example are given in figure 1. Related input parameters are listed in the captions.

In this example, both \( c_{B0} > c_{A0} > 0 \) and \( N_{B} > N_{A} \) are given. Furthermore, \( c_{AB0} \) is given positive. Therefore, each \( B \)-atom is subjected to a stronger repulsion by the other atoms than an \( A \)-atom is subjected. However, due to the mass \( m_{B} \gg m_{A} \), \( \varphi_{B} \) is still narrower than \( \varphi_{A}. \) \( c_{AB2} \) given in figure 1(b) is much larger than the one in figure 1(a). Accordingly, the spin-textures (specified by \( (S_{A}, S_{B}, S) \)) are very different as marked in the panels. However, \( \varphi_{A} \) and \( \varphi_{B} \) plotted in figure 1(a) and those plotted in figure 1(b) are very similar (say, \( \varphi_{A}(r = 0) = 0.3147 \) in figure 1(a) and \( = 0.3156 \) in figure 1(b), \( \varphi_{B}(r = 0) = 0.4563 \) in figure 1(a) and \( = 0.4559 \) in figure 1(b)). Thus the spatial profile is nearly unaffected by the spin-dependent force.

Based on the numerical solutions we concentrate on studying the effect of the spin-dependent inter-species interaction on spin-textures. Thus \( c_{AB2} \) is considered to be variable. The other parameters are listed in the captions. They are so chosen to assure that the g.s. is a miscible state, in which the \( A \)- and \( B \)-atoms have a better overlap (otherwise, the effect of \( c_{AB2} \) is weak). The variation of the spin-texture against \( c_{AB2}/c_{AB0} \) is plotted in figure 2 (with \( c_{A2} > 0 \) and \( c_{B2} > 0 \)), figure 3 (with \( c_{A2} < 0 \) and \( c_{B2} < 0 \)), and figure 4 (with \( c_{A2} > 0 \) and \( c_{B2} < 0 \)). In these figures, the total energy \( E_{S_{A}S_{B}S} \) is also given.

\[ E_{S_{A}S_{B}S} = \int dr |\psi_{A}(r)|^{4} + \int dr |\psi_{B}(r)|^{4} \]

Since the spins are correlated in different ways in different spin-textures, the probability of an atom lying in a specific spin-component \( \mu \) (denoted as \( P_{\mu}^{A} \)) and the probability of two atoms with their spins in \( \mu \) and \( \nu \) (denoted as \( P_{\mu\nu}^{AB} \)) are also different in different spin-textures. Therefore, information on the spin-textures can be extracted from these probabilities. In theoretical derivation, a crucial point is to extract the spin-state of an atom and/or of two atoms from the total spin-state. By using the fractional parentage coefficients developed for spin-1 many-body systems [16, 17], the extraction can be
realized even when the details of the total spin-state have not yet been known. This is shown in the appendix.

With these coefficients, the total spin-state \((\varphi^{N_1}_{S_1}\varphi^{N_2}_{S_2})_{SM}\) can be rewritten as

\[
(\varphi^{N_1}_{S_1}\varphi^{N_2}_{S_2})_{SM} = \sum_\mu \chi_{\mu}(i) \sum_{S'M'}[T^{\mu}_{S'(\varphi^{N_1}_{S_1+1}\varphi^{N_2}_{S_2})S'M'-\mu} + Q^{\mu}_{S'}(\varphi^{N_1}_{S_1-1}\varphi^{N_2}_{S_2})S'M'-\mu],
\]

where \(\chi_{\mu}(i)\) is the spin-state of the \(i\)th \(A\)-atom in \(\mu\) component and has been extracted.

\[
T^{\mu}_{S'} = C^{SM}_{1;1+S',M'-\mu}a_{S_1}^{N_1}\sqrt{(2S_1+1)(2S'+1)}
\times W(1, S_1 + 1, S, S_1', S_1'S'),
\]

\[
Q^{\mu}_{S'} = C^{SM}_{1;1+S',M'-\mu}b_{S_2}^{N_2}\sqrt{(2S_2+1)(2S'+1)}
\times W(1, S_1 - 1, S, S_1', S_1'S'),
\]

where the Clebsch–Gordan coefficients and the \(W\)-coefficient of Racah have been introduced. Then, the normalization of the total wave function \(\Psi_{S_1S_2SM}\) can be expanded as

\[
\langle \Psi_{S_1S_2SM}|\Psi_{S_1S_2SM}\rangle = \sum_\mu P^{A}_{\mu},
\]

where

\[
P^{A}_{\mu} = \sum_{S'}[T^{\mu}_{S'}]^2 + [Q^{\mu}_{S'}]^2],
\]

is just the probability of an \(A\)-atom with its spin lying at \(\mu\). The probability of a \(B\)-atom at \(\mu\), i.e. \(P^{B}_{\mu}\), can be obtained simply by interchanging \(N_1\leftrightarrow N_2\) and \(S_1\leftrightarrow S_2\) in the above formulae.

When the coupled spin-state of two \(A\)-atoms has been extracted by using equation (22), we have the expansion

\[
\langle \Psi_{S_1S_2SM}|\Psi_{S_1S_2SM}\rangle = \sum_{\mu,\nu} P^{AA}_{\mu\nu},
\]

where

\[
P^{AA}_{\mu\nu} = \sum_{S'}(X^{\mu\nu}_{S'}S')^2,
\]

\[
X^{\mu\nu}_{S'} = \sum_{\lambda} C^{SM}_{\lambda\mu\nu\nu'}C^{SM}_{\lambda\nu\nu'\nu}W(S_1'S_1; S_2'S_2')
\times \sqrt{(2S_1 + 1)(2S_2 + 1)W(\lambda S_1'S_1'; S_2'S_2')},
\]

where \(\lambda\) runs over 0 and 2. \(P^{AA}_{\mu\nu}\) is the probability of two \(A\)-atoms lying in spin-components \(\mu\) and \(\nu\), respectively. The probability \(P^{BB}_{\mu\nu}\) can be similarly obtained by interchanging the indexes \(A\) and \(B\).

Let \(i\) denotes the \(i\)th \(A\)-atom and \(i'\) denotes the \(i'\)th \(B\)-atom. When \(\chi_{\mu}(i)\) and \(\chi_{\nu}(i')\) have been simultaneously extracted, by using equation (20) we have the expansion

\[
\langle \Psi_{S_1S_2SM}|\Psi_{S_1S_2SM}\rangle = \sum_{\mu\nu} P^{AB}_{\mu\nu},
\]

where

\[
P^{AB}_{\mu\nu} = \sum_{S_2} [Z_{S_2}^{L\mu\nu} + Z_{S_2}^{R\mu\nu} + Z_{S_2}^{W\mu\nu} + Z_{S_2}^{IV\mu\nu}],
\]

\[
Z_{S_2}^{L\mu\nu} = \sum_{\lambda} C^{SM}_{\lambda\mu\nu\nu'}C^{SM}_{\lambda\nu\nu'\nu}a^{N_1}_{S_1}a^{N_2}_{S_2}
\times \sqrt{(2\lambda + 1)(2S_2 + 1)(2S_1 + 1)(2S_1 + 1)}
\times \left(\begin{array}{c} 1 \cr S_1 + 1 \cr S_2 + 1 \cr S_1 S_2 \cr S \end{array}\right),
\]

where \(P^{AB}_{\mu\nu}\) is the probability of an \(A\)-atom and a \(B\)-atom lying in \(\mu\) and \(\nu\), respectively. The 9- \(j\) symbol has been introduced, \(\lambda\) runs over 0, 1, and 2. \(Z_{S_2}^{L\mu\nu}\) is similar to \(Z_{S_2}^{L\mu\nu}\) but with \(a^{N_2}_{S_2}\) being changed to \(b^{N_2}_{S_2}\) and the index \(S_2 + 1\) alone in the 9- \(j\) symbol being changed to \(S_2 - 1\). \(Z_{S_2}^{W\mu\nu}\) is similar to \(Z_{S_2}^{L\mu\nu}\) but with \(a^{N_1}_{S_1}a^{N_2}_{S_2}\) changed to \(b^{N_1}_{S_1}b^{N_2}_{S_2}\), \(S_1 + 1\rightarrow S_1 - 1\), and \(S_2 + 1\rightarrow S_2 - 1\).

Numerical examples of these probabilities are shown in figures 5–9.

4. Variation of the spin-texture against the interspecies interaction

Remind that, when the spin–spin force is repulsive (attractive), a two-atom system with two anti-parallel (parallel) spins will be lower in energy. For convenience, the spins of the \(X\)-atoms is called \(X\)-spins. In the following figures the unit for \(c_{AB2}\) is \(c_{AB1}\). \(M = S\) is given, i.e. the \(Z\)-axis is chosen lying along the orientation of \(S\).

4.1. The case \(c_{AB2} > 0\) and \(c_{B2} > 0\)

From figure 2 we found the following.

(i) In the domain marked by zone III, \((S_1, S_2, S) = (0, 0, 0)\), Thus both the \(A\)- and \(B\)-atoms are in the p phase, while the mixture is in the p–p phase.
On the other hand, figure 5 demonstrates that all the $P_{μA} = P_{μB} = 1/3$ in zone III. Thus the texture $(0, 0, 0)$ can be identified via the observation of the isotropism of $P_{μX}$.

(ii) When $c_{AB2}$ enters into zone IV ($p_3 < c_{AB2} < p_4$), $S_A = N_A$ (the $A$-spins are fully aligned), $S_A > S_B$, and $S = S_A - S_B$. Since $S$ is given lying along the $Z$-axis, $S_A (S_B)$ must be lying along $+Z$-axis ($−Z$-axis) to assure $S = S_A - S_B$. Due to the sudden change of the good quantum numbers at $p_3$, obviously a spin-texture transition (spin-transition) occurs. On the other hand, figure 5(a) demonstrates that the probabilities undergo also a sudden change at $p_3$, and they become $P_{μA} = 1$, $P_{μB} = 0$ and $P_{−μB} = 0$. Thus the spin-transition can be identified by the sudden change in the probabilities.

Note that the effect of $c_{AB2} > 0$ alone is to have all the $A$-spins and $B$-spins to align along opposite directions. Thus, when $c_{AB2}$ is sufficiently large, the alignment of spins is inevitable. The fact that $P_{0B} = 0$ when $c_{AB2}$ enters zone IV demonstrates a complete breakdown of all the $s$-pairs of the $A$-spins (if a few $s$-pairs survived, $P_{0A}$ can not be exactly zero). Meanwhile all the $B$-spins are also suddenly free from the $s$-pairs and re-align along the $±Z$-axis as shown in figure 5(b) where $P_{0A} = 0$ and $P_{−1A} > 1/2 > P_{1A} > 0$ in zone IV. Accordingly, $S_B = N_B(P_{−1B} - P_{1B})$ lying along the $−Z$-axis. Remind that, for $c_{B2} > 0$, the case with...
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Figure 8. Selected $P_{\mu\nu}^{AB}$ against $c_{AB2}/c_{AB0}$. The parameters are the same as in figure 3.

Figure 9. $P_{\mu\nu}^{A}$ against $c_{AB2}/c_{AB0}$. The parameters are the same as in figure 4. In this case $P_{\mu\nu}^{A} = 1$ holds disregarding $c_{AB2}$.

$P_{-1}^{B} = P_{1}^{B} = 1/2$ would minimize the repulsion from $c_{B2}$, while for $c_{AB2}$ greater than $0$ and when $P_{1}^{A} = 1$, the optimized case is to have $P_{-1}^{A} = 1$. Thus the competing effects of $c_{B2}$ and $c_{AB2}$ lead to a balance so that $P_{-1}^{B}$ should be larger but not much larger than 1/2. It results in $P_{-1}^{B} > 1/2 > P_{1}^{B} > 0$ as found in figure 5(b). In particular, when $c_{AB2}$ increases further, the reduction of the repulsion caused by $c_{AB2}$ becomes more and more important. Therefore, more and more $B$-spins would align along the opposite direction of $S_{A}$. This is confirmed by the linear increase of $P_{1}^{B}$ as shown in figure 5(b), and accordingly an increase of $S_{B}$ as shown in figure 2.

Recall that, at $p_{3}$, $S_{B} = N_{A} > S_{A}$ and $S_{A}$ is lying along the +Z-axis while $S_{B}$ is lying along the −Z-axis. Thus the increase of $S_{B}$ leads to a decrease of $S$. When $c_{AB2} = p_{4}$, $S = 0$ and the whole system suddenly becomes isotropic. Accordingly, we found in figures 5(a) and (b) that all the six $P_{\mu\nu}^{A}$ are suddenly changed to 1/3 at $p_{4}$. However, if the parameters are so given that, at $p_{3}$, $S_{A} = N_{A} < S_{B}$, the above isotropism will not appear.

(iii) In zone V, $S_{B}$ becomes larger than $S_{A}$ and, accordingly, $S$ has reversed its direction. Accordingly, since $M = S$ is adopted, the Z-axis in zone V has been reversed. This results in an interchange of $P_{\mu\nu}^{A}$ (say, $P_{\mu}^{A} = \delta_{\mu,1}$ in zone IV but changed to $\delta_{\mu,-1}$ in zone V). Note that $P_{\mu}^{A}$ keeps monotonically increasing and will arrive at 1 when $c_{AB2} = p_{5}$. When $c_{AB2} > p_{5}$, $P_{\mu}^{A} = 1$ remains unchanged. In zone VI each B-spin is anti-parallel to every A-spins so that the repulsion from $c_{AB2}$ is minimized. Therefore the texture is inert against the further increase of $c_{AB2}$. This texture is denoted as $(N_{A}, N_{B}, |N_{A} - N_{B}|)$ and named antiparallel f–f phase. $p_{5}$ marks the boundary of this phase.

(iv) For the case $c_{AB2} < 0$ and enters into zone II, the breakdown of the s-pairs and the alignment of all the spins occur again. The A-spins align along a direction and the B-spins align along two directions as before. But the larger part of B-spins are now parallel to the A-spins. Accordingly, $S_{A} = N_{A}$ and $S = N_{A} + N_{B}$. In zone II, $P_{\mu\nu}^{A}$ increases with the decrease of $c_{AB2}$. When $c_{AB2} < p_{1}$ and enters into zone I, $P_{\mu\nu}^{A} = 1$ and each B-spin is parallel to every A-spins. Accordingly, $S_{B} = N_{B}$, $S = N_{A} + N_{B}$, the associated texture is denoted as $(N_{A}, N_{B}, N_{A} + N_{B})$ and named parallel f–f phase. In this phase the attraction raised by $c_{AB2}$ is maximized so that it is inert to a more negative $c_{AB2}$. $p_{1}$ marks the boundary of this phase.

Comparing figures 2 and 5, we see that the spin-transition are sensitively reflected in $P_{\mu\nu}^{A}$. In particular, the five critical or turning points $p_{1}$ to $p_{5}$ are clearly shown in $P_{\mu\nu}^{A}$. Besides, selected $P_{\mu\nu}^{AB}$ ($\mu$ is for an A-atom and $\nu$ is for a B-atom) are shown in figure 6. Information on the spin-textures can also be extracted from this figure. Say, all the $P_{\mu\nu}^{AB} = 1/2$ in zone III and at the boundary separating Zone IV and V due to $S = 0$. Besides, $P_{\mu\nu}^{AB} = 1$ in zone VI demonstrates explicitly the anti-parallel f–f phase. $P_{\mu\nu}^{AB} = 1$ in zone I demonstrates explicitly the parallel f–f phase. It was shown that $P_{\mu\nu}^{AB} > 1$ in zone II, while $P_{\mu\nu}^{AB} > 1$ in zone IV. These demonstrate explicitly how the spins are aligned.

4.2. The case $c_{A2} < 0$ and $c_{B2} < 0$

From figure 3 we found that both species will keep the f phase disregarding how $c_{AB2}$ is. Nonetheless, the two groups of
aligned spins will point to the same direction if \( c_{AB2} < 0 \) (i.e. \( S = |N_A + N_B| \) and in parallel f-f phase), or point to opposite directions if \( c_{AB2} > 0 \) (i.e. \( S = |N_A - N_B| \) and in antiparallel f-f phase). This is confirmed by \( P_X^\mu \) and \( P_{\mu \nu}^{AB} \) as shown in figures 7 and 8. Obviously, \( c_{AB2} = 0 \) is a critical point where the transition \( (N_h, N_g, N_A + N_B) \leftrightarrow (N_h, N_g, |N_A - N_B|) \) occurs.

4.3. The case \( c_{A2} > 0 \) and \( c_{B2} < 0 \)

From figure 4 we found that \( S_h = N_B \) and \( P_X^\mu = \delta_{\mu,1} \) disregarding how \( c_{AB2} \) is. Thus the B-spins remain in the f phase and lying along the +Z-axis. Whereas, the A-spins are in the p phase with \( P_X^A = 1/3 \) when \( c_{AB2} = 0 \) (refer to figure 9). However, the polar phase is extremely fragile against \( c_{AB2} \). Once \( |c_{AB2}| \) deviates from zero, \( P_X^A \) falls sharply from 1/3 to zero (refer to the sub-figure inside figure 9). It implies the swift breakdown of all the s-pairs. When \( c_{AB2} \) increases from 0 to \( p_2, P_X^A \) increases from 1/3 to 1 and, accordingly, \( S_A \) increases from 0 to \( N_A + N_B \) and is lying along the \(-Z\)-axis. Thus, \( S = S_B - S_A \). When \( c_{AB2} > p_2 \), all the A-spins remain to align along the \(-Z\)-axis so that each A-spin is antiparallel to every B-spin. Accordingly, the mixture is in the antiparallel f-f phase. In this phase the repulsion from \( c_{AB2} \) has been minimized and therefore the phase is inert to the further increase of \( c_{AB2} \). \( p_2 \) marks the boundary of the antiparallel f-f phase. Similarly, \( p_1 \) marks the boundary of the parallel f-f phase.

5. Summary

We have solved the CGP numerically to obtain the solutions for 2-species spin-1 BEC, and we have derived and calculated the probabilities \( P_X^\mu \) and \( P_{\mu \nu}^{AB} \) extracted from the solutions. We have confirmed that the previous results under the SMA are qualitatively correct [8, 14], and we have demonstrated that the spin-textures and the spin-transition can be identified via the observation of the probabilities. The following points are reminded.

(i) The p-p phase exists only if \( c_{A2} > 0 \), \( c_{B2} > 0 \), and \( |c_{AB2}| \) is smaller than a critical value.
(ii) When the A-atoms are in p phase and accompanied by the B-atoms in f phase, the p phase will become extremely fragile. The aligned B-spins attract the A-spins to align with them along the same or opposite direction (refer to figures 4 and 9). Consequently, all the s-pairs of the A-atoms will be swiftly broken when \( |c_{AB2}| \) deviates from zero. When \( |c_{AB2}| \) is smaller than a critical value, the A-spins split into two parts lying along two opposite directions, the associated phase of the A-atoms is named quasi-ferromagnetic (qf) phase. Meanwhile the mixture is in qf-f phase. The appearance of the qf phase is a distinguished feature of 2-species condensates. Whereas the presumable p-f phase does not exist (unless \( c_{AB2} = 0 \)).

In any cases, when \( |c_{AB2}| \) is sufficiently large, both species are in the f phase, and accordingly the mixture is in the parallel or antiparallel f-f phase depending on \( c_{AB2} < 0 \) or \( > 0 \).

(iii) the probabilities \( P_X^\mu \) and \( P_{\mu \nu}^{XY} \) of each spin-texture have their own feature. Thus the theoretical calculation of \( P_X^\mu \) and \( P_{\mu \nu}^{XY} \) as presented in this paper together with related experimental observation provides a way to discriminate the spin-textures. In particular, the critical points of the spin-transition are sensitive to the very weak spin-dependent forces. Since these points can be clearly identified via the probabilities, the strengths of these very weak forces could be thereby determined.

(iv) The results from 2-species BEC give a hint for understanding multi-species BEC. For 3-species BEC, it is interesting to find out the conditions for the emergence of the presupposable p-p-p phase, qf-f-f phase, qf-qf-f phase, f-f-f phase, and so on.

Note that the effect of the traps and the central forces is embodied by the three quantities \( I_A \), \( I_B \), and \( I_{AB} \). In the previous related literatures under the SMA, they are simply considered as three constants. Their details and how they relate to experimental input parameters are not taken into account. Thus the previous theory can not relate directly to a specific experiment. Whereas in our paper these three can be exactly obtained via the numerical solution of the CGP. Therefore, our paper is an improvement of the SMA because the theory now can relate to each specific experiment.

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Appendix

Let \( \vartheta_{SM}^N \) be the normalized all-symmetric total spin-state of \( N \) spin-1 atoms of the same kind, all the spins are coupled to \( S \) and \( M \). Then, \( \chi(i) \) (the spin-state of the \( i \)th particle) can be extracted as [16, 17]

\[
\vartheta_{SM}^N = a_S^N \chi(i) \vartheta_{S+1}^{N-1} \vartheta_{SM} + b_S^N \chi(i) \vartheta_{S-1}^{N-1} \vartheta_{SM}.
\]

where \( a_S^N \) and \( b_S^N \) are the fractional parentage coefficients for extracting one particle. They appear as

\[
a_S^N = \sqrt{\frac{[1 + (-1)^{N-S}](N-S)(S+1)}{2N(2S+1)}},
\]

\[
b_S^N = \sqrt{\frac{[1 + (-1)^{N-S}](N+S+1)}{2N(2S+1)}}.
\]
where $\vartheta_{S \pm 1}^N$ are also normalized and all-symmetric but for $N - 1$ spins coupled to $S \pm 1$.

Let the $i$th and $j$th spins be coupled to $\lambda$, and the coupled spin-state is denoted by $[\chi(i) \chi(j)]_\lambda$. Then this pair can be extracted as

$$\vartheta_{SM}^N = \sum_{S'} h_{N,S'}^N [\chi(i) \chi(j)]_\lambda \vartheta_{S'M}^{N-1}$$

(22)

where the coefficients $h_{N,S'}^N$ are the fractional parentage coefficients for extracting two particles. They are

$$h_{0,SS}^N = \left[ \frac{(N + S + 1)(N - S)}{3N(N - 1)} \right]^{1/2},$$

$$h_{2S+2,2S}^N = \left[ \frac{(S + 1)(S + 2)(N - S)(N - S - 2)}{(2S + 1)(2S + 3)N(N - 1)} \right]^{1/2},$$

$$h_{2S,2S+2}^N = \left[ \frac{S(2S + 2)(N - S)(N + S + 1)}{3(2S - 1)(2S + 3)N(N - 1)} \right]^{1/2},$$

$$h_{2S-2,2S}^N = \left[ \frac{S(S - 1)(N + S + 1)(N - S - 1)}{(2S - 1)(2S + 1)N(N - 1)} \right]^{1/2}.

(23)

All the other $h_{X,SS}^N$ are zero. Furthermore, when $S = 0$, $h_{2S,2S}^N$ and $h_{2S-2,2S}^N$ should be zero.

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