Anharmonic Josephson current in junctions with an interface pair breaking

Yu. S. Barash
Institute of Solid State Physics, Russian Academy of Sciences, Chernogolovka, Moscow District, 142432 Russia
(Dated: March 7, 2012)

Planar superconducting junctions with a large effective Josephson coupling constant and a pronounced interface pair breaking are shown to represent weak links with small critical currents and strongly anharmonic current-phase relations. The supercurrent near $T_c$ is described taking into account the interface pair breaking as well as the current depairing and the Josephson coupling-induced pair breaking of arbitrary strengths. A new analytical expression for the anharmonic supercurrent, which is in excellent agreement with the numerical data presented, is obtained. In junctions with a large effective Josephson coupling constant and a pronounced interface pair breaking, the current-induced depairing is substantially enhanced in the vicinity of the interface thus having a crucial influence on the current-phase relation despite a small depairing in the bulk.

PACS numbers: 74.50.+r, 74.20.De

The Josephson current is one of the remarkable manifestations of quantum coherence on the macroscopic scale in condensed matter physics. The supercurrent depends on the phase difference of the order parameters across the junction interface. The study of the current-phase relation (CPR) in the junctions makes it possible to identify physical processes, which form supercurrents under diverse conditions. It is also beneficial for junction applications. The problem attracted much attention while studying both highly transparent junctions with strongly anharmonic CPRs and tunnel junctions, where the second harmonic of the supercurrent comes into play due to the suppression of the first one. An enhancement of the current-induced depairing near the interface will be identified. The anharmonic supercurrent near $T_c$ will be obtained within the Ginzburg-Landau (GL) theory in the presence of all three types of the pair breaking processes of arbitrary strengths. Along with the numerical solution based on GL equations, a new analytical CPR will be derived and shown to be in excellent agreement with the numerical data in a wide range of parameters. For tunnel junctions the obtained results present a new description of higher harmonics of the supercurrent and extend the known expressions for the first and second harmonics to include the effects of interface pair breaking.

The CPRs obtained earlier near $T_c$ with the microscopic boundary conditions for standard dirty $s$-wave junctions, have been considered in literature solely as the particular properties of the specific systems. The anharmonic CPR obtained in this paper, and influenced by the interface pair breaking, is of general form inherent in the GL theory, and is applicable to a variety of planar junctions including those containing $d_{x^2-y^2}$-wave superconductors and/or magnetic interlayers.

The free energy functional for Josephson junctions near $T_c$ results in the GL equations and the boundary conditions (BC) for them. Consider symmetric junctions with a spatially constant width, which is much less than the Josephson penetration length, and with a plane interlayer at $x = 0$ of zero length within the GL approach. Assume the usual form of the GL free energy, which applies, for example, to $s$-wave and $d_{x^2-y^2}$-wave junctions. If the Josephson coupling $g_J|\Psi_+ - \Psi_-|^2$ is strong, not only this term but all the interface and bulk contributions to the free energy generally participate in the formation of CPRs as a consequence of the dependence of absolute values of the order parameters at the interface on the phase difference. This concerns, in particular, the gradient bulk term $K|\nabla \Psi|^2$ and the interface contribution of the form $g(|\Psi_+|^2 + |\Psi_-|^2)$.

Moving on to the order parameter $f(x)e^{i\chi(x)}$ normalized to $f = 1$ in the bulk without superflow, one gets the...
first integral of the GL equation in the presence of the supercurrent in the form of
\[
\left( \frac{df}{dx} \right)^2 + f^2 - \frac{1}{2} f^4 + \frac{4j^2}{2f^2} = 2f_\infty^2 - \frac{3}{2} f_\infty^4. \tag{1}
\]
Here \( \tilde{x} = x/\xi \), \( \xi = \xi(T) \) is the superconducting coherence length, \( j \) is the spatially constant normalized current density \( j = j/\sigma_p = -(3\sqrt{3}/2)(dx/d\tilde{x})f^2 \) and \( f_\infty \) is the asymptotic value of \( f \) in the depth of the superconducting lead.

The BC introduce in the GL theory at least two characteristic lengths \( \ell = K/g_1 \), \( \delta = K/g \). The effective dimensionless Josephson \( g_\ell = g_1\xi(T)/K \) and interface \( g_s = g_s(T)/K \) coupling constants, associated with these lengths, will be used below. For symmetric junctions with \( g \) continuous through the interface, the BC for \( f \) as well as the expression for the Josephson current via \( f_0 \) and the phase difference \( \chi = \chi_0 - \chi_+ \) at the interface, are obtained from the BC for complex order parameters:
\[
\left( \frac{df}{d\tilde{x}} \right)_\pm = \pm \left( gS + 2g_\ell \sin^2 \chi/2 \right) f_0, \quad \tilde{\chi} = \frac{3\sqrt{3}}{2} g_\ell f_\infty^2 \sin \chi. \tag{2}
\]
Here the effective phase-dependent extrapolation length \( b(\chi) = (\delta^{-1} + 2\ell^{-1} \sin^2 (\chi/2))^{-1} \) controls the pair breaking produced by the phase difference and by the interface. Let’s denote \( g_S(\chi) = (g_S + 2g_\ell \sin^2 (\chi/2)) \).

Since the material parameters in the normal state \( g_1 \) and \( g \) are not assumed to depend here on \( T \) near \( T_c \), one should have \(|g_1| \gg 1 \) and/or \(|g_s| \gg 1 \) quite close to \( T_c \) due to large values of \( \xi(T) \). However, the coupling constants \( g_1 \) and \( g \) can themselves be very small and the temperature range with large \( g \) and/or \( g_1 \) be too narrow, as it occurs in standard tunnel junctions. Due to a very small surface pair breaking in conventional \( s \)-wave junctions, one parameter \( g_1 \) is usually assumed to describe the interfaces in (2) rather than both \( g \) and \( g_1 \) as is in the regular case. At \( \chi = 0 \), such symmetric junctions contain no pair breaking at all, and the BC (2) is reduced to \( (df(0)/dx) = 0 \).

If \( g_1 \) and/or \( g \) were very small, one would need to introduce into (2) the terms of the next order of smallness, in particular, in powers of the order parameter. Such terms could be of importance and bring about additional phase dependence and material-dependent parameters to the problem. Here, only the simplest conditions will be assumed, when (2) applies to a wide range of values of \( g_1 \) and \( g \). This agrees with the microscopic model results, and, for instance, takes place within the GL approach for sufficiently large values of \( g_1 \) and \( g \), which is the particular focus of this paper.

There is no need to solve differential equation (1) in order to find \( f_0 \), and, consequently, to find \( \tilde{j} \) via (2). One puts \( x = 0 \) in (1) and, using (2), eliminates the current and the first derivative of the order parameter. This results in a biquadratic relation between the self-consistent values of \( f_0^2 \) and \( f_\infty^2 \). The second relation between them follows from the current conservation and the asymptotic formulas in the bulk. The current-induced depairing in the bulk is conveniently described via the superfluid velocity \( \tilde{\chi} = (3\sqrt{3}/2)\chi_0(1 - \tilde{\chi}^2) \), \( f_\infty^2 = 1 - \tilde{\chi} \)). Equating the asymptotic expression for the current to that in (2) with \( f_0^2 = (1 - \tilde{\chi}^2)\alpha \) one obtains \( \chi_0 = \alpha g_1 \chi \). Considering that both quantities \( f_0 \) and \( f_\infty \) as well as the current itself are now expressed via the only variable \( \alpha \), the fourth-order polynomial equation for \( \alpha \) follows from the biquadratic relation between \( f_0 \) and \( f_\infty \)
\[
2g_1^2(\chi)\alpha - (1 - \alpha)^2[1 - \alpha(\alpha + 2)g_1^2 \sin^2 \chi] = 0. \tag{3}
\]
Eq. (3) is exact within the conventional GL approach with BC (2). In the particular case of standard \( s \)-wave junctions, \( g_\ell(\chi) = 2g_\ell \sin^2 (\chi/2) \). Then (3) is reduced to Eq.(8) of Ref.30,31, if one corrects a misprint \( \Gamma_B \to \Gamma_B^2 \) in (8) and identifies the parameter of the GL theory \( g_\ell^{-1} = \ell/\xi \) with the model parameter \( \Gamma_B \) entering the microscopic BC for dirty \( s \)-wave superconductors.

An analytical solution of the problem can be obtained assuming a small depairing in the bulk \( j^2 \ll 1 \) that allows to use \( f_\infty^2 \approx 1 - (4/27)\tilde{j}^2 \) to disregard the smaller terms on the right-hand side of (1). Then one gets from (1), (2) a biquadratic equation for \( f_0 \), which results in the analytical solution for the CPR:
\[
\tilde{j}(g_\ell, g_0, \chi) = \frac{3\sqrt{3}g_\ell \sin \chi}{2(1 + 2g_\ell^2 \sin^2 \chi)} \left[ 1 + g_0^2(\chi) + g_\ell^2 \sin^2 \chi - \sqrt{(g_0^2(\chi) + g_\ell^2 \sin^2 \chi)^2 + 2g_\ell^2(\chi)} \right]. \tag{4}
\]
Since only higher order terms begin with \( \sim \tilde{j}^4 \) have been neglected in its derivation, the CPR (4) turns out to describe the current behavior almost perfectly, if \( \tilde{j} < 0.7 \). For \( \tilde{j} > 0.7 \) it gives a good interpolation of the numerical solution based on (3), resulting in the deviations not exceeding 10%.

As seen in (3) and (4), the anharmonic Josephson current \( \tilde{j} \) depends, in general, on the two dimensionless effective coupling constants \( g_\ell \), \( g_0 \) and the phase difference \( \chi \). According to the simple physical arguments as well as the microscopic results, a variation of tunneling parameters principally modifies \( g_\ell \), while the surface pair breaking mostly contributes to \( g_0 \). This signifies that the junction transparency \( D \) enters the combination of microscopic parameters representing \( g_\ell \). The last statement agrees with the microscopic results for \( s \)-wave junctions.
with nonmagnetic interfaces\textsuperscript{16,19,26–28}, where the corresponding combination is sometimes identified as the effective transparency\textsuperscript{32,33}. The microscopic estimations of the effective Josephson coupling constant $g_k$ directly follow from those results. In the $s$-wave tunnel junctions ($D \ll 1$), one gets $g_k \sim D \xi(T)(l^{-1} + \xi_0^{-1})$, where $l$ is the mean free path. In dirty superconductors, the ratio $\xi(T)/l$ can easily reach 100 even at low temperatures. Hence, for small and moderate transparencies, the quantity $g_k \sim D \xi(T)/l$ can vary from vanishingly small values in the tunneling limit to those well exceeding 100 near $T_c$. In highly transparent junctions ($(1 - D) \ll 1$) the parameter $g_k \propto (1 - D)^{-1}$ can be arbitrary large\textsuperscript{34}. The quantity $g_k$ can also take on negative values, which correspond to $\pi$-junctions, as seen in [4].

The range of variation of the interface coupling $g_\delta$ can likewise be quite wide. For $s$-wave superconductor-insulator interfaces, the Josephson coupling vanishes and the extrapolation length $b$ is reduced to $\delta$. The microscopic estimations of $\delta$ in such cases show it to be very large usually resulting in a negligibly small contribution to the BC, unlike the superconductor-normal metal interfaces\textsuperscript{29}. The length $\delta$ can vary widely for $d$-wave superconductor-insulator flat surfaces, where it substantially depends on surface-to-crystal orientations\textsuperscript{35,36,37}. Although in this case $\delta$ is strongly influenced by the surface roughness, in particular, by faceting.

A regular situation is characterized by a local suppression of the order parameter at the interface. For this condition to hold, the effective extrapolation length $b(\chi)$ should be positive at any phase difference and, hence, $g_\delta/(g_\delta + 2g_k) > 0$. A superconducting state occurs locally near the interface above the bulk $T_c$ under the opposite condition $b(\chi) < 0$ with $\chi$ ensuring the free energy minimum\textsuperscript{28}. Only the simplest conditions $g_\delta > 0$ will be analyzed in detail in this paper, although the main results obtained here apply to substantially more general circumstances. Other conditions, including magnetic field effects and/or negative $g_\delta$, will be studied elsewhere.

Figs. 4a, b show the critical current $\tilde{j}_c$ as a function of coupling constants $g_k$ and $g_\delta$. Solid curves have been calculated based on [3]. Dashed curves correspond to the analytical expression [4]. Only for a small interface pair breaking ($g_\delta \lesssim 1$) and for $g_k \gtrsim 1$, the current $\tilde{j}_c$ becomes comparable with 1, i.e., with the depairing current in the bulk. Thus the condition $g_k \gtrsim 1$ is the hallmark of a strong Josephson coupling. Comparatively small deviations of dashed curves from the solid ones are discernible only when the current exceeds about 0.7. With increasing $g_\delta$, the growing interface pair breaking suppresses the critical current. For $g_k \gtrsim 4$, the critical current remains quite small $\tilde{j}_c \ll 1$ at any $g_k$, which would normally occur in conventional tunnel junctions with small effective transparencies. In other words, in the regime of strong interface pair breaking $g_\delta > 4$, the junctions represent weak links at any $g_k$, including $g_k \gtrsim 1$.

Though [4] is a combined result of all depairing effects, the origin of its characteristic anharmonic features is traced back unambiguously. The whole of the phase-dependence in [3], except for that contained in $g_k(\chi)$, is generated by the current via $j_c$, on the right hand side or by the last term on the left hand side in [4]. Such dependence would retain the CPR [4] unchanged under the transformation $\chi \rightarrow \pi - \chi$. The symmetry is destroyed by the phase dependence of $g_k^2(\chi)$, which originates from the BC [2] and can become pronounced, if $|g_k| \lesssim 2|g_k|$. Whereas the CPR [4] is derived by assuming small depairing effects in the bulk, the depairing can be of crucial importance in [4] within its domain of applicability. This is the case in the presence of a pronounced interface pair breaking, where an enhancement of the current-induced depairing, unlike the bulk, occurs near interfaces of junctions with $g_k \gg 1$. In particular, the phase-dependent term in the denominator of (3), which is directly induced by the depairing, plays a key role in the case $g_k \gg 1$ in restricting the normalized current value. The bracketed expression in the denominator originates from the coefficient before $f_0^4$ in the biquadratic equation for $f_0$. The relative depairing correction coming from the bulk is $(8/27)j_c^2 = 2g_k^2f_0^6\sin^2\chi$ and its smallness signifies $2g_k^2\sin^2\chi f_0^4 \ll 1$. As seen, the term $2g_k^2\sin^2\chi$ in the denominator is allowed to exceed the unit considerably, when the condition $2g_k^2\sin^2\chi f_0^4 \ll 1$ holds at the expense of a strongly suppressed order parameter at the interface $f_0^4 \ll 1$. Numerical results corroborate that, if $g_k \gtrsim 4$, the condition is satisfied at any $g_k$ including $g_k \gg 1$ (see also Figs. 4a, b). This validates keeping [4] without its expanding in powers of $g_k^2\sin^2\chi$ and explains the quantitative applicability of [4] to junctions with the pronounced interface pair breaking at arbitrary $g_k$.

A number of specific CPRs follow from [4] under a variety of particular conditions. Consider here two basic examples. The tunneling limit shows up in [4] under the condition $|g_k| \ll 1$. Developing [4] as series in $g_k$ at any value of $g_\delta$, one obtains numerous harmonics whose weight is determined by $g_k$ and $g_\delta$ rather than by the transparency itself. The first and the second order terms result in

$$\tilde{j}_c \approx j_{c1}^{(4)} \left[ \sin \chi - \frac{2g_k \text{sgn}(g_\delta)}{\sqrt{2 + g_\delta^2}} \left( \sin \chi - \frac{1}{2} \sin 2\chi \right) \right].$$ (5)
Here \( \tilde{j}_{c1}^{(1)} = (3\sqrt{3}/4)g_T \left( \sqrt{2 + g_3^2} - |g_3| \right)^2 \) is the main contribution to the first harmonic \( \tilde{j}_1 = \tilde{j}_{c1} \sin \chi \) that is applicable at any \( g_3 \). Under the condition \( |g_3| \ll 1 \) it is reduced to the well-known result for tunnel junctions \( \tilde{j}_{c0} = (3\sqrt{3}/2)g_T \), which is only justified when disregarding the interface pair breaking. In the opposite limit \( g_3^2 \ll 1 \) the pair breaking strongly suppresses the current and \( \tilde{j}_{c1}^{(1)} \approx \tilde{j}_{c0}/(2g_3^2) \ll \tilde{j}_{c0} \), as is also known.\(^1\)\(^2\)\(^3\)\(^4\)\(^5\)\(^6\)\(^7\)\(^8\)\(^9\)\(^10\)\(^11\) In particular, the original current \( j_{c1}^{(1)} = \tilde{j}_{c1}^{(1)} \propto (T_c - T)^2 \) for \( |g_3| \ll 1 \) and \( j_{c1}^{(1)} \propto (T_c - T)^2 \) for \( g_3^2 \gg 1 \) near \( T_c \). The second order terms in \( g_T \) bring about the main contribution to the second harmonic \( \tilde{j}_2 = \tilde{j}_{c2} \sin 2\chi \) as well as corrections to the first one. The relative weight of the second harmonic in \( \tilde{j}_{c2} \) diminishes with increasing \( g_3^2 \). The sign of \( \tilde{j}_{c2} \) coincides with the sign of \( g_T \), while the sign of \( \tilde{j}_{c0} \) is determined by the sign of \( g_3 \). For small pair breaking \( 0 < g_3 < 1 \) the second order term \( \propto g_3^2 \) is simplified to the following correction to the current \( -\sqrt{2}j_{c0}g_3/(\sin \chi - (1/2) \sin 2\chi) \), in agreement with the corresponding microscopic results\(^1\)\(^2\)\(^3\)\(^4\)\(^5\)\(^6\)\(^7\)\(^8\)\(^9\)\(^10\)\(^11\) for dirty and pure s-wave junctions. Note that the phase dependence generated by the current depairing shows up in \( \tilde{j}_{c2} \) beginning with the third order terms in \( g_T \).

The second example reveals the strongly anharmonic features contained in \( \tilde{j}_{c2} \). Consider junctions with the strong interface pair breaking \( g_3^2 \ll 1 \). Then a comparatively simple approximate expression follows from \( \tilde{j}_{c2} \):

\[
\tilde{j} = \frac{3\sqrt{3}g_T}{4|g_3^2 + 4(g_3 + g_T)g_T \sin \frac{\chi}{2}|} \sin \chi.
\]

The corresponding critical current \( j_c = 3\sqrt{3}g_T/(4|g_3(g_3 + 2g_T)|) \ll 1 \) is always small. The associated phase difference is determined by the relation \( \sin \chi_c = |g_3(g_3 + 2g_T)|/(g_3 + g_T)^2 + g_T^2 \). It varies widely: \( \chi_c \) is small \( \approx (g_3/g_T) \), if \( g_T \gg g_3 \), and approaches \( \pi/2 \) in the opposite limit \( g_3 \gg g_T \). Strongly anharmonic CPRs show up in \( \tilde{j}_{c2} \) under the conditions \( g_3^2 \gg g_3^2 \gg 1 \). Also one has \( j_c \propto (T_c - T)^2 \). Thus, at finite \( g_T \), the temperature dependence \( j_c(T) \) is quadratic quite close to \( T_c \), where \( g_3 \gg 1 \). With increasing \( T_c - T \), a crossover to the linear dependence on the temperature takes place in the region \( T_c - T < T_c \), for sufficiently small \( g_T \).

Some of the CPRs \( \tilde{j}(\chi) \) are shown in Figs. 2a, b. Except for the first curve in Fig. 2b, the strongly anharmonic CPRs in junctions with large Josephson couplings are displayed. As seen in Fig. 2a, the heights of the anharmonic peaks diminish considerably and the peak positions change weakly, when the interface pair breaking goes up. Although the anharmonicity can be well pronounced even in the presence of quite a large pair breaking. This concerns, in particular, the curve 1 in Fig. 2a, which is identical to the curve 3 in Fig. 2b shown there in a different scale. Eq. 4 describes the CPR almost perfectly and the corresponding dashed curves can be distinguished from the exact solid ones only near the high peak of curve 5 in Fig. 2a. All curves in Fig. 2b are also well approximated by a simple formula \( \tilde{j}(\chi) \) with deviations (not shown) approaching only several percent. However, in contrast to \( \tilde{j}(\chi) \), Eq. 4 does not apply to describing upper three curves in Fig. 2b. The CPR similar to \( \tilde{j}_{c2} \) was found earlier within the microscopic description of the dirty s-wave junctions with metallic interlayers.\(^1\)\(^2\) The strong pair breaking can take place in those junctions, if the interlayer conductivity considerably exceeds the normal conductivity of the superconducting metal.

In conclusion, the paper reveals the qualitative features and develops the quantitative description of the anharmonic Josephson current near \( T_c \). The interface pair breaking as well as the current depairing and the Josephson coupling-induced pair breaking have been taken into account and shown to play an important part in forming the CPR. The results obtained, in particular, concern the junctions involving d-wave superconductors and/or magnetic interlayers.

ACKNOWLEDGMENTS

The support of RFBR grant 11-02-00398 is acknowledged.

---

1. A. A. Golubov, M. Y. Kupriyanov, and E. Il’ichev, Rev. Mod. Phys. 76, 411 (2004).

2. C. C. Tsuei and J. R. Kirtley, Rev. Mod. Phys. 72, 969 (2000).

3. A. I. Buzdin, Rev. Mod. Phys. 77, 935 (2005).

4. V. B. Geshkenbein and A. I. Larkin, Pis’ma Zh. Eksp. Teor. Fiz. 43, 306 (1986), [JETP Lett. 43, 395 (1986)].

5. Y. Tanaka, Phys. Rev. Letters 72, 3871 (1994).

6. S. Yip, Phys. Rev. B 52, 3087 (1995).

7. E. Il’ichev, V. Zakosarenko, R. P. J. IJsselsteijn, H. E.
Hoenig, V. Schultze, H.-G. Meyer, M. Grajcar, and R. Hlubina, Phys. Rev. B 60, 3096 (1999).

8 H. Hilgenkamp and J. Mannhart, Rev. Mod. Phys. 74, 485 (2002).

9 T. Lindström, S. A. Charlebois, A. Y. Tzalenchuk, Z. Ivanov, M. H. S. Amin, and A. M. Zagoskin, Phys. Rev. Letters 90, 117002 (2003).

10 T. Lindström, J. Johansson, T. Bauch, E. Stepantsov, F. Lombardi, and S. A. Charlebois, Phys. Rev. B 74, 014503 (2006).

11 I. O. Kulik and A. N. Omelyanchuk, Pis'ma Zh. Eksp. Teor. Fiz. 21, 216 (1975), [JETP Lett. 21, 96 (1975)].

12 I. O. Kulik and A. N. Omelyanchuk, Fiz. Nizk. Temp. 3, 945 (1977), [Sov. J. Low Temp. Phys. 3, 459 (1977)].

13 W. Habekorn, H. Knauer, and J. Richter, Phys. Status Solidi A 47, K161 (1978).

14 K. K. Likharev, Rev. Mod. Phys. 51, 101 (1979).

15 A. V. Zaitsev, Zh. Eksp. Teor. Fiz. 86, 1742 (1984), [Sov. Phys. JETP 59, 1015 (1984)].

16 M. Y. Kupriyanov, Pis'ma Zh. Eksp. Teor. Fiz. 56, 414 (1992), [JETP Lett. 56, 399 (1992)].

17 F. Sols and J. Ferrer, Phys. Rev. B 49, 15913 (1994).

18 J. K. Freericks, B. K. Nikolić, and P. Miller, Int. J. Mod. Phys. B 16, 531 (2002).

19 Z. G. Ivanov, M. Y. Kupriyanov, K. K. Likharev, S. V. Meriakri, and O. V. Snigirev, Fiz. Nizk. Temp. 7, 560 (1981), [Sov. J. Low Temp. Phys. 7, 274 (1981)].

20 P. G. de Gennes, Superconductivity of Metals and Alloys (Addison Wesley Publishing Co., Inc., Reading, MA, 1966).

21 S.-K. Yip, O. F. D. A. Bonfim, and P. Kumar, Phys. Rev. B 41, 11214 (1990).

22 M. Sigrist and K. Ueda, Rev. Mod. Phys. 63, 239 (1991).

23 E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics. Part 2. Theory of the Condensed State (Butterworth-Heinemann, Oxford, 1995).

24 V. P. Mineev and K. Samokhin, Introduction to Unconventional Superconductivity (Gordon & Breach Science Publishers, New York, 1999).

25 J. S. Langer and V. Ambegaokar, Phys. Rev. 164, 498 (1967).

26 V. P. Galaiko, A. V. Svidzinskii, and V. A. Slyusarev, Zh. Eksp. Teor. Fiz. 56, 835 (1969), [Sov. Phys. JETP 29, 454 (1969)].

27 E. N. Bratus’ and A. V. Svidzinskii, Teor. Mat. Fiz. 30, 239 (1977), [Theor. Math. Phys. 30, 153 (1977)].

28 A. V. Svidzinskii, Spatially Inhomogeneous Problems in the Theory of Superconductivity (Nauka, Moscow, 1982).

29 V. B. Geshkenbein, Zh. Eksp. Teor. Fiz. 94, 368 (1988), [Sov. Phys. JETP 67, 2166 (1988)].

30 J. Bardeen, Rev. Mod. Phys. 34, 667 (1962).

31 M. Tinkham, Introduction to Superconductivity, 2nd ed. (McGraw-Hill, Inc., New York, 1996).

32 E. V. Bezuglyi, E. N. Bratus’, and V. P. Galaiko, Low Temp. Phys. 25, 167 (1999).

33 E. V. Bezuglyi, A. S. Vasenko, V. S. Shumeiko, and G. Wendin, Phys. Rev. B 72, 014501 (2005).

34 The solutions based on (3) or (4) satisfy the relation $|g_b|/f_0 \lesssim 1$, in particular, at large values of $|g_\ell|$ and/or $|g_\delta|$. In view of (2), this agrees with the condition that a strong suppression of the order parameter on each side of the interface takes place on a scale comparable with $\xi(T)$.

35 Y. S. Barash, A. V. Galaktionov, and A. D. Zaikin, Phys. Rev. B 52, 665 (1995).

36 M. Alber, B. Bäuml, R. Ernst, D. Kienle, A. Kopf, and M. Rouchal, Phys. Rev. B 53, 5863 (1996).

37 D. F. Agterberg, J. Phys. Condens. Matter 9, 7435 (1997).

38 A. F. Andreev, Pis’ma Zh. Eksp. Teor. Fiz. 46, 463 (1987), [JETP Lett. 46, 584 (1987)].

39 G. Deutscher and K. A. Müller, Phys. Rev. Letters 59, 1745 (1987).

40 T. N. Antsygina and A. V. Svidzinskii, Teor. Mat. Fiz. 14, 412 (1973), [Theor. Math. Phys. 14, 306 (1973)].