Some Orbits of Free Words that are Determined by Measures on Finite Groups

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Abstract

Every word in a free group \( F \) induces a probability measure on every finite group in a natural manner. It is an open problem whether two words that induce the same measure on every finite group, necessarily belong to the same orbit of \( \text{Aut} F \). A special case of this problem, when one of the words is the primitive word \( x \), was settled positively by the third author and Parzanchevski [PP15]. Here we extend this result to the case where one of the words is \( x^d \) or \( [x,y]^d \) for an arbitrary \( d \in \mathbb{Z} \).

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1 Introduction

Let \( r \in \mathbb{Z}_{\geq 1} \), let \( F = F_r \) be the free group on \( r \) generators \( x_1, \ldots, x_r \), and let \( G \) be any finite group. We occasionally use the letters \( x \) and \( y \) to denote arbitrary distinct letters from \( \{x_1, \ldots, x_r\} \). Every word \( w \in F \) induces a map, called a word-map,

\[
  w: G \times \ldots \times G \to G,
\]

which is defined by substitutions. For example, if \( w = x_1x_3x_1x_3^{-2} \in F_3 \), then \( w(g_1,g_2,g_3) = g_1g_3g_1g_3^{-2} \). The push-forward via this word map of the uniform measure on \( G \times \ldots \times G \) is called the \( w \)-measure on \( G \). Put differently, for each \( 1 \leq i \leq r \), substitute \( x_i \) with an independent, uniformly-distributed random element of \( G \), and evaluate the product defined by \( w \) to obtain a random element in \( G \) sampled by the \( w \)-measure. We say the resulting element is a \( w \)-random element of \( G \).

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For \(w_1, w_2 \in \mathbf{F}\) write \(w_1 \overset{\text{Aut}\mathbf{F}}{\sim} w_2\) if there is an automorphism \(\theta \in \text{Aut}\mathbf{F}\) with \(\theta(w_1) = w_2\). It is easy to see, as we explain in Section 2 below, that applying an automorphism on \(w\) does not alter the resulting word-measure on finite groups. Thus,

**Fact 1.1.** If \(w_1 \overset{\text{Aut}\mathbf{F}}{\sim} w_2\) then \(w_1\) and \(w_2\) induce the same measure on every finite group.

For example, \(xyxy^{-1} \overset{\text{Aut}\mathbf{F}}{\sim} x^2y^2\) and so they both induce the same measure on every finite group. It is an open problem whether the converse is also true, namely, whether being in the same \(\text{Aut}\mathbf{F_r}\)-orbit is the sole reason for two words to induce the same measure on every finite group.

**Conjecture 1.2.** Let \(w_1, w_2 \in \mathbf{F}_r\). If \(w_1\) and \(w_2\) induce the same measure on every finite group, then \(w_1 \overset{\text{Aut}\mathbf{F}}{\sim} w_2\).

Conjecture 1.2 appears as [AV11, Question 2.2] and as [Sha13, Conjecture 4.2], and see also [PP15, Section 8]. Our focus here is on special cases, where \(w_1\) is some fixed word:

**Definition 1.3.** A word \(w \in \mathbf{F}\) is called profinitely rigid in \(\mathbf{F}\) if whenever some word \(w' \in \mathbf{F}\) induces the same measure as \(w\) on every finite group, then \(w \overset{\text{Aut}\mathbf{F}}{\sim} w'\). The word \(w \in \mathbf{F}\) is called universally profinitely rigid if it is profinitely rigid in every free extension \(\mathbf{F} \ast \mathbf{F}_\ell\) of \(\mathbf{F}\) (\(\ell \geq 1\)).

In Section 2 we extend this notion to arbitrary finitely generated groups, give equivalent definitions and justify our choice of the name “profinitely rigid”. Note that even if \(w \in \mathbf{F} = \mathbf{F}_r\) is known to be profinitely rigid in \(\mathbf{F}\), it does not automatically follow that this property extends to words written with more letters, namely, whether \(w\) is also profinitely rigid in \(\mathbf{F} \ast \mathbf{F}_\ell\) for \(\ell \geq 1\). For example, it is easy to show that the word \(x\) is profinitely rigid in \(\mathbf{F}_1 = \mathbf{F}(x) \cong \mathbb{Z}\) (the orbit of \(x\) in \(\mathbf{F}_1\) is the sole orbit which induces measures with full support on every finite group). However, it is much harder to show that \(w\) is also profinitely rigid in \(\mathbf{F}_2\) or in \(\mathbf{F}_r\) for arbitrary \(r\).

The fact that the trivial word \(w = 1\) is profinitely rigid in every free group is equivalent to the fact that free groups are residually finite. A case which attracted considerable attention was that of primitive words, namely that of \(\text{Aut}\mathbf{F} \cdot x\) – the \(\text{Aut}\mathbf{F}\)-orbit containing the free generators of \(\mathbf{F}\). This case was settled: first it was shown that \(x\) was profinitely rigid in \(\mathbf{F}_2\) [Pud14], and then that it was universally profinitely rigid [PP15]. In fact, it is shown in [PP15] that \(w \in \mathbf{F}\) induces the uniform measure on the symmetric group \(S_N\) for all \(N\) if and only if \(w\) is primitive. A completely different, geometric proof of the universal profinite rigidity of \(x\) was later found by Wilton [Wil18, Corollary E]. Our main result here extends that theorem from [PP15] and gives more special cases of Conjecture 1.2. Recall that \(x\) and \(y\) are assumed to belong to the some basis of the free group \(\mathbf{F}\).

**Theorem 1.4.** For every \(d \in \mathbb{Z}_{\geq 1}\), the words \(x^d\) and \([x,y]^d\) are universally profinitely rigid.

Namely, if \(w \in \mathbf{F}\) induces the same measure as \(x^d\) on every finite group, then \(w \overset{\text{Aut}\mathbf{F}}{\sim} x^d\), and if it induces the same measure as \([x,y]^d\), then \(w \overset{\text{Aut}\mathbf{F}}{\sim} [x,y]^d\). To the best of our knowledge, if one allows also \(d = 0\), Theorem 1.4 captures all known \(\text{Aut}\mathbf{F}\)-orbits of profinitely rigid words. We remark that although generally \(w\) and \(w^{-1}\) do not necessarily lie in the same \(\text{Aut}\mathbf{F}\)-orbit, we do have \(x^{-d} \overset{\text{Aut}\mathbf{F}}{\sim} x^d\) and \([x,y]^{-d} \overset{\text{Aut}\mathbf{F}}{\sim} [x,y]^d\), so negative powers of \(x\) and \([x,y]\) would be redundant in the statement of Theorem 1.4.

In a similar direction, one may consider word measures not only on finite groups but more generally, on any compact group, where instead of the uniform measure on \(G \times \ldots \times G\) one takes the Haar measure. Fact 1.1 remains true if the word “finite” is replaced by “compact” (e.g., [MP16, Fact 2.5]), and one can then ask a slightly weaker version of Conjecture 1.2 where one assumes that \(w_1\) and \(w_2\) induce the same measure on every compact group. Indeed, in [MP19c], Magee and the third author study this conjecture in the case that \(w_1\) is the surface word \([x_1,y_1] \cdots [x_g,y_g]\) or \(x_1^2 \cdots x_g^2\) for some \(g \in \mathbb{Z}_{\geq 1}\). They show that if
$w_2 \in F$ induces the same measure as a surface word $w_1$ on every compact group, then $w_1^{\text{Aut} F} \sim w_2$. Note that the word $[x, y]$ is a surface word which is also covered by Theorem 1.4. For this particular word, Theorem 1.4 strengthens the result from [MP19c] as it relies on measures on finite groups only.

A main tool in our proof of Theorem 1.4 is a generalization of [PP15, Theorem 1.8], which deals with word measures on the symmetric groups $S_N$. For $w \in F = F_r$, consider a $w$-random permutation in $S_N$. The number of fixed points of a permutation is equal to the trace of the corresponding permutation matrix. Thus, we denote the expected number of fixed points of a $w$-random permutation in $S_N$ by

$$\mathcal{T}_{rw}(N) \overset{\text{def}}{=} \frac{1}{(N!)^r} \sum_{\sigma_1, \ldots, \sigma_r \in S_N} \text{tr}(w(\sigma_1, \ldots, \sigma_r)).$$

(1)

A set of words $\{u_1, \ldots, u_k\} \in F$ is called free if they admit no non-trivial relation.

**Theorem 1.5.** Let $w \in F_k$ be a word which is not contained in a proper free factor of $F_k$. If $u_1, \ldots, u_k \in F$ are free words in $F = F_r$ which do not generate a free factor, then for every large enough $N$,

$$\mathcal{T}_{rw}(N) < \mathcal{T}_{rw(u_1, \ldots, u_k)}(N).$$

(2)

*Remark 1.6.* Some remarks are due:

1. If $w = x$ is the single-letter word, then Theorem 1.5 states that whenever $1 \neq u \in F_r$ is non-primitive, then

$$\mathcal{T}_{rx}(N) < \mathcal{T}_{ru}(N)$$

for every large enough $N$. This yields that the word $x$ is profinitely rigid, and indeed, a more quantitative version of this inequality is the content of [PP15, Theorem 1.8]. In fact, Theorem 3.6 below gives a quantitative version of (2) which generalizes the quantitative version in [PP15, Theorem 1.8].

2. The condition on $w$ in Theorem 1.5 is necessary but “harmless”. To see it is necessary, consider the primitive word $w = xy \in F_2 = F(x, y)$. The words $u_1 = a^n$ and $u_2 = b$ are free in $F(a, b)$ and generate the subgroup $\langle a^n, b \rangle$ which is not a free factor. Yet $w(u_1, u_2) = a^n b$ is a primitive element and therefore $\mathcal{T}_{rw}(N) = \mathcal{T}_{rw(u_1, u_2)}(N) = 1$ for every $N \geq 1$. However, this condition can be easily “bypassed”: if $w$ is contained in a proper free factor of $F_k$, find some $w'$ with $w' \overset{\text{Aut} F_k}{\sim} w$ which uses the smallest possible number of letters, say $q < k$ letters, and apply the theorem with $w' \in F_q$ in the stead of $w$.

3. The statement of Theorem 1.5 is also not true without the condition that $u_1, \ldots, u_k$ be free and not generate a free factor. Consider, for example, the case $w = x^3 y^2$, $u_1 = a$ and $u_2 = a^{-1}$ with $u_1, u_2 \in F(a) \cong \mathbb{Z}$. Then $w(u_1, u_2) = a$ and for every $N \geq 3$,

$$1 + \frac{1}{N - 1} = \mathcal{T}_{rw}(N) > \mathcal{T}_{rw(u_1, u_2)}(N) = 1.$$

Moreover, with the same $w$, if we take $u_1 = w$ and $u_2 = w^{-1}$, we get $w(u_1, u_2) = w$. Finally, if $u_1, \ldots, u_k$ are free but generate a free factor of $F$ then $w^{\text{Aut} F} \sim w(u_1, \ldots, u_k)$ whence $\mathcal{T}_{rw}(N) = \mathcal{T}_{rw(u_1, \ldots, u_k)}(N)$ for all $N$.

Another ingredient of the proof of Theorem 1.4 concerns powers of general words. We prove, in fact, that profinite rigidity is preserved under taking powers:

**Theorem 1.7.** Let $w \in F$ and $d \in \mathbb{Z}$. If $w$ is profinitely rigid then so is $w^d$. 

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In particular, if \( w \) is profinitely rigid, so is \( w^{-1} \), although this case is immediate from the definition. (Note that generally, \( w \) and \( w^{-1} \) do not belong to the same \( \text{Aut}_F \)-orbit.)

As explained in Section 2 below, profinite rigidity of words yields two additional properties concerning the profinite completion of the free group and its profinite topology. We state these properties in the following corollary:

**Corollary 1.8.** Let \( w = x^d \) or \( w = [x, y]^d \) for some \( d \in \mathbb{Z}_{\geq 1} \). Then,

1. The \( \text{Aut}_F \)-orbit of \( w \) is closed in the profinite topology of \( F \).

2. Let \( \hat{F} \) be the profinite completion of \( F \). If \( w' \in F \) is in the same \( \text{Aut}_{\hat{F}} \)-orbit as \( w \), then \( \underbrace{w'}_{\text{Aut}_F} \sim F w \).

In fact, Theorem 2.2 below shows that profinite rigidity of \( w \) is equivalent to the statement of Corollary 1.8(2). Item 1 is a special case of Claim 2.5 below.

**Reducing Theorem 1.4 to Theorems 1.5 and 1.7**

Theorem 1.7 reduces Theorem 1.4 to the primitive case, which is shown in [PP15], and to the commutator word \([x, y]\). For the latter, we use a result of Khelif:

**Theorem 1.9.** [Khe04] If the image of \( w \in F \) in every finite quotient of \( F \) is a commutator, then \( w \) is a commutator, namely, \( w = [u, v] \) for some \( u, v \in F \).

Now assume that some word \( w' \in F \) induces the same measures on finite groups as the commutator \([u, v]\) for some \( u, v \in F \). If \( w' \) is not a commutator in some finite quotient \( Q \) of \( F_r \), then its image \( \overline{w'} \in Q \) is in the support of the \( w' \)-measure but not in the support of the \([u, v]\)-measure, in contradiction. Hence \( w' \) is a commutator in every finite quotient of \( F_r \), and by Khelif, \( w' = [u', v'] \) for some \( u', v' \in F \). We obtain the following:

**Corollary 1.10.** If \( w_1 \in F \) is a commutator and \( w_2 \in F \) is not, then \( w_1 \) and \( w_2 \) do not induce the same measure on all finite groups.

Now, assume \( w \in F \) induces the same measures on finite groups as the word \([x, y]\). By Khelif’s result, \( w \) is a commutator, so \( w = [u, v] \) for some \( u, v \in F \). Clearly, \( w \neq 1 \), whence \( u \) and \( v \) do not commute and are therefore free. From Theorem 1.5 applied with \([x, y]\), it immediately follows that \( \langle u, v \rangle \) is a free factor of rank 2 of \( F \) and therefore \( \langle u, v \rangle \underbrace{\text{Aut}_F}_F (x, y) \) and \( w \underbrace{\text{Aut}_F}_F [x, y] \).

We remark that the case of primitive powers follows also from the combination of Theorem 1.5 and Lubotzky’s Theorem 4.1 (see below), which yields, analogously to Corollary 1.10, that if \( w_1 \) is a \( d \)-th power and \( w_2 \) is not, then \( w_1 \) and \( w_2 \) do not induce the same measure on finite groups. Indeed, if \( w \) induces the same measures on finite groups as \( x^d \), then by Lubotzky’s theorem, \( w = u^d \) with \( u \neq 1 \) a non-power. But \( \langle T_{w^d}(N) = T_{w^d}(N) \rangle \) for all \( N \), so by Theorem 1.5, \( u \) must be primitive. See also Proposition 4.5 below.

**Paper organization**

Section 2 contains a short introduction to profinite topology and to profinite groups, generalizes the notion of profinitely rigid elements to arbitrary finitely generated groups, and gives several equivalent notions (Theorem 2.2). In Section 3 we prove Theorem 1.5 about the average number of fixed points in a \( w(u_1, \ldots, u_k) \)-random permutation. Finally, Section 4 contains the proof of Theorem 1.7, thus concluding the proof of our main theorem, Theorem 1.4.
2 Profinitely rigid elements and equivalent notions

Given a basis $x_1, \ldots, x_r$ to $F$ as above, there is a natural correspondence

$$\text{Hom}(F, G) \leftrightarrow G \times \ldots \times G,$$

where $\varphi \in \text{Hom}(F, G)$ corresponds to the $r$-tuple $(\varphi(x_1), \ldots, \varphi(x_r))$. In this language, the $w$-measure on $G$ is the distribution of $\varphi(w)$ where $\varphi \in \text{Hom}(F, G)$ is a uniformly random homomorphism. Assume that $w_1, w_2 \in F$ are in the same $\text{Aut}F$-orbit, namely, that there exists $\theta \in \text{Aut}F$ with $\theta(w_1) = w_2$. If $\varphi \in \text{Hom}(F, G)$ is uniformly random, then so is $\varphi \circ \theta \in \text{Hom}(F, G)$. Clearly, for every fixed homomorphism $\varphi$, we have $\varphi(w_2) = (\varphi \circ \theta)(w_1)$, which proves Fact 1.1: $w_1$ and $w_2$ induce the same measure on every finite group.

In fact, this last observation is relevant not only to finitely generated free groups, but to arbitrary finitely generated groups. Let $\Gamma$ be a finitely generated group and $G$ some finite group. The set of homomorphisms $\text{Hom}(\Gamma, G)$ is finite, and so every element $\gamma \in \Gamma$ induces a measure on $G$ defined by the random element $\varphi(\gamma)$ where $\varphi: \Gamma \to G$ is a uniformly random homomorphism. The previous paragraph yields the following generalization of Fact 1.1:

Claim 2.1. Let $\Gamma$ be a finitely generated group and $\gamma_1, \gamma_2 \in \Gamma$. If $\gamma_1 \sim^\text{Aut} \gamma_2$ then $\gamma_1$ and $\gamma_2$ induce the same measure on every finite group.

The study of measures induced on finite groups by words, and more generally by elements of a finitely generated group $\Gamma$, is closely related to some aspects of the profinite topology on $\Gamma$ and of its profinite completion. The standard references to the theory of profinite groups are the books [Wil98, RZ10]. Let us give here some basic definitions and facts.

The profinite topology on $\Gamma$ is defined by the basis of (say, left) cosets of subgroups of finite index. The profinite completion of $\Gamma$, denoted $\hat{\Gamma}$, is the inverse limit

$$\lim_{\overset{\rightarrow}{N \subseteq \Gamma}} \Gamma/N,$$

where $N$ runs over all normal subgroups of finite index in $\Gamma$. This is a (Hausdorff, compact, totally disconnected) topological group. There is a natural homomorphism $\iota: \Gamma \to \hat{\Gamma}$ defined by mapping $\gamma \in \Gamma$ to the element $(\gamma, N)_{N \subseteq \Gamma}$ in the inverse limit (3). The homomorphism $\iota$ is injective if and only if $\Gamma$ is residually finite. The image of $\Gamma$ is dense in $\hat{\Gamma}$ [RZ10, Lemma 3.2.1].

By definition, a homomorphism from $\hat{\Gamma}$ to a finite group is assumed to be continuous, and the sets $\text{Hom}(\hat{\Gamma}, G)$ and $\text{Epi}(\hat{\Gamma}, G)$ are the sets of continuous homomorphisms and epimorphisms, respectively, from $\hat{\Gamma}$ to the finite group $G$. For every finite group $G$, there is a one-to-one correspondence between $\text{Hom}(\hat{\Gamma}, G)$ and $\text{Hom}(\Gamma, G)$, given by $\psi \mapsto \psi \circ \iota$: this is due to the universal property of the profinite completion of a group, namely, for every $\varphi \in \text{Hom}(\Gamma, G)$ there is a unique $\hat{\varphi} \in \text{Hom}(\hat{\Gamma}, G)$ such that $\varphi = \hat{\varphi} \circ \iota$ (see [RZ10, Lemma 3.2.1]). Similarly, there is a one-to-one correspondence between $\text{Epi}(\hat{\Gamma}, G)$ and $\text{Epi}(\Gamma, G)$.

Let $\gamma \in \Gamma$ and $g \in G$ where $\Gamma$ is finitely generated and $G$ finite. Define

$$\text{Hom}_{\gamma,g}(\Gamma, G) \overset{\text{def}}{=} \{ \varphi \in \text{Hom}(\Gamma, G) \mid \varphi(\gamma) = g \},$$
$$\text{Epi}_{\gamma,g}(\Gamma, G) \overset{\text{def}}{=} \{ \varphi \in \text{Epi}(\Gamma, G) \mid \varphi(\gamma) = g \},$$
$$\text{EpiIm}_{\gamma}(\Gamma, G) \overset{\text{def}}{=} \{ \varphi(\gamma) \mid \varphi \in \text{Epi}(\Gamma, G) \},$$
$$K_{\Gamma}(G) \overset{\text{def}}{=} \bigcap_{N \subseteq \Gamma, \Gamma/N \cong G} N,$$

(4)
(if $G$ is not a quotient of $\Gamma$, define $K_\Gamma (G) \overset{\text{def}}{=} \Gamma$). Note that $K_\Gamma (G)$ is a characteristic finite index subgroup of $\Gamma$, as there are finitely many normal subgroups $N \trianglelefteq \Gamma$ with quotient $G$, and as every automorphism of $\Gamma$ permutes these normal subgroups.

The following theorem gives a number of equivalent definitions to a relation between different elements of $\Gamma$. The automorphism group $\text{Aut}\hat{\Gamma}$ is the group of all continuous automorphisms of $\hat{\Gamma}$.

**Theorem 2.2.** Let $\Gamma$ be a finitely generated group, and $\gamma_1, \gamma_2 \in \Gamma$. Then the following six properties are equivalent:

1. $\gamma_1 \overset{\text{Aut}\hat{\Gamma}}{\sim} \gamma_2$, namely, there exists an automorphism $\theta \in \text{Aut}\hat{\Gamma}$ with $\theta (\gamma_1) = \gamma_2$.

2. $|\text{Hom}_{\gamma_1,g} (\Gamma, G)| = |\text{Hom}_{\gamma_2,g} (\Gamma, G)|$ for every finite group $G$ and every $g \in G$, namely, $\gamma_1$ and $\gamma_2$ induce the same measure on every finite group.

3. $|\text{Epi}_{\gamma_1,g} (\Gamma, G)| = |\text{Epi}_{\gamma_2,g} (\Gamma, G)|$ for every finite group $G$ and every $g \in G$.

4. $\text{Epi}\text{Im}_{\gamma_1} (\Gamma, G) = \text{Epi}\text{Im}_{\gamma_2} (\Gamma, G)$ for every finite group $G$, namely, $\gamma_1$ and $\gamma_2$ have the same possible images under epimorphisms to finite groups.

5. $\gamma_1 K \overset{\text{Aut}(f/K)}{\sim} \gamma_2 K$ with $K = K_\Gamma (G)$ for every finite group $G$.

6. For every $N \trianglelefteq f.l. \Gamma$ there exists $K \trianglelefteq f.l. \Gamma$ with $K \leq N$ such that $\gamma_1 K \overset{\text{Aut}(f/K)}{\sim} \gamma_2 K$.

This theorem explains why the following definition generalizes Definition 1.3.

**Definition 2.3.** Let $\Gamma$ be a finitely generated group. An element $\gamma \in \Gamma$ is called **profinitely rigid** if whenever $\gamma' \in \Gamma$ satisfies $\gamma \overset{\text{Aut}\hat{\Gamma}}{\sim} \gamma'$ then $\gamma \overset{\text{Aut}\Gamma}{\sim} \gamma'$.

Note that by Claim 2.1, the six equivalences in Theorem 2.2, including $\gamma_1 \overset{\text{Aut}\hat{\Gamma}}{\sim} \gamma_2$, are all a consequence of $\gamma_1 \overset{\text{Aut}\Gamma}{\sim} \gamma_2$. It is natural to generalize Conjecture 1.2 and ask which finitely generated groups have the property that each of their elements is profinitely rigid. This property holds trivially for finite groups and easily for finitely generated abelian groups (see, for instance, [CMP19, Theorem 5.2], for the case of $\mathbb{Z}^r$).

As mentioned above, this property is (much) stronger than residually finiteness.

The proof of Theorem 2.2 relies on the following lemma, which is a cousin of [RZ10, Proposition 4.4.3]:

**Lemma 2.4.** Let $\Gamma$ be finitely generated. Then

$$\text{Aut}\hat{\Gamma} \cong \varprojlim_{K} \text{Aut} (\Gamma/K),$$

where the inverse limit is taken over all subgroups $K \leq \Gamma$ such that $K = K_\Gamma (G)$ (defined in (4) above) for some finite group $G$, with arrows $K_1 \rightarrow K_2$ whenever $K_1 \leq K_2$.

**Proof.** Assume that $K_\Gamma (G_1) = K_1 \leq K_2 = K_\Gamma (G_2)$, and denote $Q_i = \Gamma/K_i$ for $i = 1, 2$. The image of $K_2$ in $Q_1$ is equal to $K_\Gamma (G_2)$, whence this image is characteristic in $Q_1$. Therefore, there is a well-defined homomorphism $\text{Aut} (Q_1) \rightarrow \text{Aut} (Q_2)$. In addition, for every $K_1 = K_\Gamma (G_1)$ and $K_2 = K_\Gamma (G_2)$, define $G_3 = \Gamma/K_1 \cap K_2$ and then $K_3 = K_\Gamma (G_3) \leq K_1, K_2$. Therefore the right hand side of (5) is a well-defined inverse system.

By [RZ10, Proposition 3.2.2], for $K \trianglelefteq f.l. \Gamma$ we have $\Gamma/K \cong \hat{\Gamma}/\iota (K)$, and $\iota (K_\Gamma (G)) = K_\Gamma (G)$. It is therefore enough to show that

$$\text{Aut}\hat{\Gamma} \cong \varprojlim_{K} \text{Aut} (\Gamma/K),$$

(6)
where the inverse limit runs over all subgroups \( K \leq \hat{\Gamma} \) such that \( K = K_\Gamma (G) \) for some finite group \( G \). As \( K = K_\Gamma (G) \) is characteristic in \( \hat{\Gamma} \), every automorphism of \( \hat{\Gamma} \) induces an automorphism of \( \Gamma/K \) which agrees with the inverse system, so there is a natural continuous homomorphism

\[
\omega : \text{Aut} \hat{\Gamma} \to \lim_{\longleftarrow K} \text{Aut} (\Gamma/K).
\]

The map \( \omega \) is injective because \( \bigcap_{G \text{ finite}} K_\Gamma (G) = \{ e_\Gamma \} \). The map \( \omega \) is surjective because every element of the inverse system \( \lim_{\longleftarrow K} \text{Aut} (\Gamma/K) \) defines a continuous automorphism of \( \lim_{\longleftarrow K} \Gamma/K \cong \hat{\Gamma} \).

**Proof of Theorem 2.2.**

**The implication 1\(\implies\)2:** Notice that by the one-to-one correspondence mentioned above between \( \text{Hom} (\Gamma, G) \) and \( \text{Hom} (\hat{\Gamma}, G) \), we have \( |\text{Hom}_{\gamma,g} (\Gamma, G)| = |\text{Hom}_{\hat{\gamma},g} (\hat{\Gamma}, G)| \). Hence

\[
|\text{Hom}_{\gamma,g} (\Gamma, G)| = |\text{Hom}_{\hat{\gamma}_1,g} (\hat{\Gamma}, G)| = |\text{Hom}_{\hat{\gamma}_2,g} (\hat{\Gamma}, G)| = |\text{Hom}_{\hat{\gamma}_2,g} (\Gamma, G)|,
\]

where the proof of the middle equality is identical to the proof of Claim 2.1.

**The equivalence 2\(\iff\)3** goes by induction on the cardinality of the finite group \( G \), as the case \( |G| = 1 \) is trivial, and

\[
\text{Hom}_{\gamma,g} (\Gamma, G) = \bigcup_{H \subseteq G : g \in H} \text{Epi}_{\gamma,g} (\Gamma, H).
\]

**The implication 3\(\implies\)4** is evident: \( g \in \text{EpiIm}_{\gamma} (\Gamma, G) \) if and only if \( |\text{Epi}_{\hat{\gamma},g} (\Gamma, G)| > 0 \).

**The implication 4\(\implies\)5:** Let \( K = K_\Gamma (G) \) for some finite group \( G \). As \( \text{Epi} (\Gamma, \Gamma/K) \neq \emptyset \), also \( \text{EpiIm}_{\gamma_1} (\Gamma, \Gamma/K) = \text{EpiIm}_{\gamma_2} (\Gamma, \Gamma/K) \neq \emptyset \). Choose an arbitrary \( q \in \text{EpiIm}_{\gamma_1} (\Gamma, \Gamma/K) \), and for \( i = 1, 2 \) let \( f_i \in \text{Epi}_{\gamma_i,g} (\Gamma, \Gamma/K) \). Clearly, \( \Gamma/\text{ker} f_i \cong \Gamma/K \), but we claim that \( \text{ker} f_i \leq K \). Indeed, the number of normal subgroups in \( \Gamma/\text{ker} f_i \) with quotient \( G \) is the same as in \( \Gamma/K \) and thus the same as in \( \Gamma \). Hence \( \text{ker} f_i \) is contained in every \( N \leq \Gamma \) with \( \Gamma/N \cong G \), and so \( \text{ker} f_i \leq K \), but \( \Gamma : \text{ker} f_i = |\Gamma : K| \), whence \( \text{ker} f_i = K \). We deduce that \( f_i \) induces an automorphism \( \gamma_1 K \to \gamma_2 K \).

**The equivalence 5\(\iff\)6:** Assume first that \( \gamma_1 K \cong \gamma_2 K \) for every \( K = K_\Gamma (G) \). If \( N \leq \Gamma \), then \( K = K_\Gamma (\Gamma/N) \) would work for 6. Conversely, assume item 6 holds. Let \( K = K_\Gamma (G) \) for some finite group \( G \). By assumption, there exists a subgroup \( K' \leq K \) such that \( K' \leq \Gamma \) and \( \gamma_1 K' \cong \gamma_2 K' \). But the image of \( K \) in \( Q' = \Gamma/K' \) is precisely \( K_\Gamma (G) \), so this image is characteristic, and so every automorphism of \( Q' \cong \Gamma/K' \) induces an automorphism of \( \Gamma/K \). Hence \( \gamma_1 K \cong \gamma_2 K \).

**The implication 5\(\implies\)1:** By assumption, for every \( K \) in the inverse system in (5), the subset \( \text{Aut}_{\gamma_1,\gamma_2} (\Gamma/K) \subseteq \text{Aut} (\Gamma/K) \) of automorphisms mapping \( \gamma_1 \) to \( \gamma_2 \) is not empty. By a standard compactness argument, there is an element in \( \lim_{\longleftarrow K} \text{Aut} (\Gamma/K) \) mapping \( \gamma_1 K \) to \( \gamma_2 K \) for every \( K = K_\Gamma (G) \). We are done by Lemma 2.4.

The following property of profinitely rigid elements is a generalization of Corollary 1.8(1):

**Claim 2.5.** Let \( \Gamma \) be a finitely generated group and \( \gamma \in \Gamma \) a profinitely rigid element. Then the orbit \( \text{Aut} \Gamma (\gamma) \) is closed in the profinite topology on \( \Gamma \).

**Proof.** We give two proofs for this claim. First, the automorphism group \( \text{Aut} \hat{\Gamma} \) is a profinite group ([RZ10, Proposition 4.4.3] or Lemma 2.4 above), and in particular compact, so \( \text{Aut} \hat{\Gamma} (\gamma) \) is closed in \( \hat{\Gamma} \). As \( \gamma \) is profinitely rigid, \( \text{Aut} \Gamma (\gamma) = \text{Aut} \Gamma (\gamma) \cap \Gamma \), which shows that \( \text{Aut} \Gamma (\gamma) \) is closed in the profinite topology on \( \Gamma \).

The second proof uses item 5 in Theorem 2.2: if \( \gamma \) is profinitely rigid, then for every \( \delta \in \Gamma \setminus \text{Aut} \Gamma (\gamma) \), there is some finite group \( G \) so that \( \gamma K \not\sim \delta K \) with \( K = K_\Gamma (G) \). Then \( \delta K \) in an open neighborhood of \( \delta \) in \( \Gamma \) which is disjoint from \( \text{Aut} \Gamma (\gamma) \).
Remark 2.6. We remark on several relations between words in free groups that lie between \( \text{Aut} F \) and \( \hat{\text{Aut}} F \). The implications between them are described in the following diagram.

Here, \( w_1 \sim \text{CharQuot} \sim w_2 \) mean \( w_1 K \overset{\text{Aut}(F/K)}{\sim} w_2 K \) for every characteristic subgroup \( K \text{char} F \). The relation \( w_1 \sim \text{PosDef} \sim w_2 \), introduced in [CMP19], means that \( \tau(w_1) = \tau(w_2) \) for all \( \text{Aut} F \)-invariant positive definite functions \( \tau \) on \( F \). We write \( w_1 \sim \text{CptGrp} \sim w_2 \) to mean that \( w_1 \) and \( w_2 \) induce the same measure on every compact group, and write \( w_1 \sim \text{Aut} F \sim w_2 \) to mean that there is an automorphism \( \theta \) of \( \hat{F} \) which lies in the closure of \( \text{Aut} F \) in \( \hat{\text{Aut}} F \) so that \( \theta(w_1) = \theta(w_2) \). The one implication that is not immediate, \( w_1 \sim \text{PosDef} \Rightarrow w_1 \sim \text{CptGrp} \sim w_2 \) is explained in [CMP19, Lemma 1.12]. Of course, Conjecture 1.2 implies that all these relations are equivalent.

All these relations, except for \( w_1 \sim \text{CptGrp} \sim w_2 \), can be immediately generalized for elements of every finitely generated group.

3 Fixed points of random permutations and the proof of Theorem 1.5

The proof of Theorem 1.5 relies on the partial order defined by “algebraic extensions” on the set of (finitely generated) subgroups of the free group \( \hat{F} \). We begin with a short presentation of this notion.

3.1 Algebraic extensions

Let \( F \) be a free group as above and \( H, J \leq F \) two subgroups. We call \( J \) an algebraic extension of \( H \), denoted \( H \leq_{\text{alg}} J \), if and only if \( H \leq J \) and there is no intermediate proper free factor of \( J \), namely, if \( H \leq_{\text{alg}} J \) whenever \( H \preceq M \preceq J \), we must have \( M = J \) (here \( M \preceq J \) means that \( M \) is a free factor of \( J \)). We collect some of the properties of this notion in the following proposition. For proofs and more details consult the survey [MVW07] or Section 4 in [PP15].

Proposition 3.1. Let \( F \) be a finitely generated free group.

1. Algebraic extensions form a partial order on the set of subgroups of \( F \). In particular, \( H \leq_{\text{alg}} H \) for all \( H \), and \( H \leq_{\text{alg}} K \) whenever \( H \leq_{\text{alg}} J \) and \( J \leq_{\text{alg}} K \).

2. If \( H \leq J \leq K \) and \( H \leq_{\text{alg}} K \) then \( J \leq_{\text{alg}} K \).

3. For every extension of free groups \( H \leq J \) there is a unique intermediate subgroup \( A \) satisfying \( H \leq_{\text{alg}} A \preceq_{\text{alg}} J \). Moreover, every algebraic extension of \( H \) which is contained in \( J \), is also contained in \( A \).

4. Every finitely generated subgroup of \( F \) has finitely many algebraic extensions.
3.2 Fixed points of random permutations and Möbius inversions

In [PP15], the main object of study is $T_{w}(N)$, the expected number of fixed point in a $w$-random permutation in $S_{N}$. We discuss here parts of the analysis in [PP15] which are relevant for the proof of Theorem 1.5.

First, as explained at the beginning of Section 2, a $w$-random permutation can be obtained as $\varphi(w)$, where $\varphi \in \text{Hom}(F, S_{N})$ is a uniformly random homomorphism. In a similar manner, for every subgroup $H \leq F$, one can define an $H$-random subgroup of $S_{N}$ as $\varphi(H)$, the image of $H$ through a random homomorphism. Denote by $\Phi_{H,F}(N)$ the expected number of elements in $\{1, \ldots, N\}$ which are fixed by all permutations in $\varphi(H)$. In particular, $T_{w}(N) = \Phi_{\langle w \rangle,F}(N)$. This notion can be then defined for every pair of finitely generated subgroups:

**Definition 3.2.** Let $H,J \leq F$ be finitely generated with $H \leq J$. Denote by $\Phi_{H,J}(N)$ the expected number of joint fixed points of all permutations in $\varphi(H)$ where $\varphi \in \text{Hom}(J,F_{S})$ is uniformly random.

Clearly, if $A \leq F$ is a free factor and $\varphi \in \text{Hom}(F,S_{n})$ is uniformly random, then $\varphi|_{A} \in \text{Hom}(A,S_{n})$ is also uniformly random. In this case, therefore,

$$
\Phi_{A,F}(N) = \Phi_{A,A}(N) = N^{1-\text{rank}(A)}.
$$

Likewise, if $H \leq_{\text{alg}} A \leq F$ is the unique factorization of the extension $H \leq F$ to an algebraic extension and a free extension, then

$$
\Phi_{H,F}(N) = \Phi_{H,A}(N).
$$

Next, one can define a “Möbius inversion” of the function $\Phi$ based on the partial order “$\leq_{\text{alg}}$” defined above. Assume that $H \leq_{\text{alg}} J$. Because every finitely generated subgroup $H \leq F$ has only finitely many algebraic extensions, we have, in particular, that there are only finitely many intermediate subgroups $M$ with $H \leq_{\text{alg}} M \leq_{\text{alg}} J$. This allows us to define the “right inversion” (or derivation) $R$ of the function $\Phi$, as follows:

$$
\Phi_{H,J}(N) = \sum_{M: H \leq_{\text{alg}} M \leq_{\text{alg}} J} R_{H,M}(N).
$$

Indeed, this well-defines $R_{H,J}(N)$ by induction on the number of intermediate subgroups in the poset defined by “$\leq_{\text{alg}}$”:

$$
R_{H,J}(N) = \Phi_{H,J}(N) - \sum_{M: H \leq_{\text{alg}} M \leq_{\text{alg}} J} R_{H,M}(N).
$$

**Remark 3.3.** The initial definition of $R$ in [PP15] is slightly different. It is based on a different partial order “$\leq_{X}$” on the finitely generated groups of $F$, a partial order based on Stallings core graphs and which is basis-dependent (here $X$ marks a given basis). This order is “finer” then $\leq_{\text{alg}}$, in the sense that $H \leq_{X} J$ whenever $H \leq_{\text{alg}} J$. The resulting function is denoted there $R^{X}$. However, it is then shown [PP15, Proposition 5.1] that $R^{X}$ is supported on algebraic extensions, that the value is independent of the basis $X$, and that, in fact, it is equal to the function defined in (8) [PP15, Equation (5.3)].

The main result of [PP15] easily follows from the following more technical statements about the function $R$.

**Theorem 3.4.** Assume that $H \leq_{\text{alg}} J \leq F$ are all finitely generated groups.

1. [PP15, immediate corollary of Lemma 6.4] For large enough $N$, the function $R_{H,J}(N)$ is equal to a rational expression in $N$.

2. [PP15, Proposition 7.2]

$$
R_{H,J}(N) = N^{1-\text{rank}(J)} + O\left(N^{-\text{rank}(J)}\right).
$$
3. By definition,

\[ R_{H,H} (N) = \Phi_{H,H} (N) = N^{1-\text{rank}(H)}. \]

Let \( H \leq F \) be finitely generated. Let \( H \leq_{\text{alg}} A \leq F \) be the unique factorization into an algebraic and a free extensions. Using items 2 and 3 from Theorem 3.4, we obtain that

\[
\Phi_{H,F} (N) \overset{(7)}{=} \Phi_{H,A} (N) \overset{(8)}{=} \sum_{J: H \leq_{\text{alg}} J \leq_{\text{alg}} A} R_{H,J} (N)
\]

Proposition \( \overset{3.1}{=} \sum_{J \leq F: H \leq_{\text{alg}} J} R_{H,J} (N) \)

Theorem \( \overset{3.4}{=} N^{1-\text{rank}(H)} + \sum_{J \leq F: H \leq_{\text{alg}} J} \left[ N^{1-\text{rank}(J)} + O \left( N^{-\text{rank}(J)} \right) \right]. \quad (10) \)

The equation (10) leads immediately to the following theorem, which is the main result of [PP15] with regards to \( T_{r,w} (N) \).

**Theorem 3.5.** [PP15, Theorem 1.8] Let \( H \leq F \) be finitely generated free groups. Denote by \( \pi(H) \) the smallest rank of a proper algebraic extension of \( H \), or \( \pi(H) = \infty \) if there are no proper algebraic extensions, namely, if \( H \leq F \). Then,

\[ \Phi_{H,F} (N) = N^{1-\text{rank}(H)} + C \cdot N^{1-\pi(H)} + O \left( N^{-\pi(H)} \right), \]

where \( C \) is the number of proper algebraic extensions of \( H \) of rank \( \pi(H) \).

In particular, for a word \( w \in F \), denote by \( \pi(w) \) the smallest rank of a proper algebraic extension of \( \langle w \rangle \), or \( \pi(w) = \infty \) if \( w \) is primitive in \( F \). Then,

\[ T_{r,w} (N) = \Phi_{\langle w \rangle,F} (N) = 1 + C \cdot N^{1-\pi(w)} + O \left( N^{-\pi(w)} \right), \]

where \( C \) is the number of proper algebraic extensions of \( \langle w \rangle \) of rank \( \pi(w) \).

The following theorem is a generalization of Theorem 3.5, which can also be seen as a quantitative version of Theorem 1.5.

**Theorem 3.6.** Let \( H \leq_{\text{alg}} J \) be an algebraic extension of finitely generated free groups. Let \( : J \hookrightarrow F = F_r \) be an embedding of \( J \) in \( F \). Denote by \( \pi_\ast(H) \) the smallest rank of an algebraic extension of \( \ast(H) \) in \( F \) \( \pi_\ast(H) \) which is not contained in \( \ast(J) \), or \( \pi_\ast(H) = \infty \) if \( \ast(J) \leq F \). Then,

\[ \Phi_{\ast(H),F} (N) = \begin{cases} \Phi_{H,J} (N) + C \cdot N^{1-\pi_\ast(H)} + O \left( N^{-\pi_\ast(H)} \right) & \text{if } \pi_\ast(H) < \infty \\ \Phi_{H,J} (N) & \text{if } \pi_\ast(H) = \infty \end{cases}, \]

where \( C \) is the number of algebraic extensions of \( \ast(H) \) of rank \( \pi_\ast(H) \) inside \( F \) not contained in \( \ast(J) \). In particular, if \( \pi_\ast(H) < \infty \) then for every large enough \( N \), we have

\[ \Phi_{\ast(H),F} (N) \geq \Phi_{H,J} (N). \]

If \( H \) is a subgroup of \( F \) and we let \( J = H \) and \( : H \hookrightarrow F \) be the embedding, then Theorem 3.6 reduces to Theorem 3.5.
**Proof of Theorem 3.6.** Let \( \iota(H) \leq_{\text{alg}} A \leq_{\text{F}} \mathbf{F} \) be the unique factorization of \( \iota(H) \leq \mathbf{F} \) to an algebraic extension and a free extension. By (7), (8) and Proposition 3.1,

\[
\Phi_{\iota(H),\mathbf{F}}(N) = \Phi_{\iota(H),A}(N) = \sum_{M : \iota(H) \leq_{\text{alg}} M \leq_{\text{alg}} A} R_{\iota(H),M}(N) = \sum_{M \leq_{\text{F}} \iota(H) \leq_{\text{alg}} M} R_{\iota(H),M}(N)
\]

\[
= \sum_{M \leq_{\text{F}} \iota(H) \leq_{\text{alg}} M \leq \iota(J)} R_{\iota(H),M}(N) + \sum_{M \leq_{\text{F}} \iota(H) \leq_{\text{alg}} M \leq \iota(J)} R_{\iota(H),M}(N).
\]

(11)

By Theorem 3.4(2), the second summand in (11) is precisely \( C \cdot N^{1-\pi_{\iota}(H)} + O\left(N^{-\pi_{\iota}(H)}\right) \) where \( C \) is as in the statement of the theorem. It remains to show that the first summand is equal to \( \Phi_{H,J}(N) \).

Indeed, as \( \iota \) is an isomorphism between \( J \) and \( \iota(J) \), the algebraic extensions of \( H \) in \( J \) are precisely \( \{ \iota^{-1}(M) \mid \iota(H) \leq_{\text{alg}} M \leq \iota(J) \} \). It is thus enough to show that for every \( M \) with \( \iota(H) \leq_{\text{alg}} M \leq \iota(J) \) (and every \( N \)) we have

\[
R_{H,\iota^{-1}(M)}(N) = R_{\iota(H),M}(N).
\]

But by the definition of the derivation \( R \) in (8), the values \( \{ R_{H,K}(N) \}_{H \leq_{\text{alg}} K \leq J} \) are given by the values \( \{ \Phi_{H,K}(N) \}_{H \leq_{\text{alg}} K \leq J} \), and it is clear that if \( H \leq_{\text{alg}} K \leq J \) then \( \Phi_{H,K}(N) = \Phi_{\iota(H),\iota(K)}(N) \).

**Proof of Theorem 1.5.** Recall the assumptions of Theorem 1.5: \( w \in \mathbf{F}_{k} \) is not contained in a proper free factor, and \( u_{1}, \ldots, u_{k} \in \mathbf{F} \) are free and do not generate a free factor of \( \mathbf{F} \). The assumption that \( w \) is not contained in a proper free factor is equivalent to that \( \langle w \rangle \leq_{\text{alg}} \mathbf{F}_{k} \). If \( y_{1}, \ldots, y_{k} \) is a basis of \( \mathbf{F}_{k} \), define a map \( \iota : \mathbf{F}_{k} \to \mathbf{F} \) by \( \iota(y_{i}) = u_{i} \in \mathbf{F} \) for \( 1 \leq i \leq k \). The assumption that \( u_{1}, \ldots, u_{k} \) are free is equivalent to that \( \iota : \mathbf{F}_{k} \to \mathbf{F} \) is an embedding. Finally, \( u_{1}, \ldots, u_{k} \) generate a free factor of \( \mathbf{F} \) if and only if \( \pi_{i}(\langle w \rangle) = \infty \).

So if \( u_{1}, \ldots, u_{k} \) do not generate a free factor then \( \pi_{i}(\langle w \rangle) < \infty \) and, by Theorem 3.6, we obtain

\[
\begin{aligned}
\mathcal{T}_{w_{u_{1},\ldots,u_{k}}}(N) &= \Phi_{\iota(\langle w \rangle),\mathbf{F}}(N) = \Phi_{\langle w \rangle,\mathbf{F}_{k}}(N) = N^{1-\pi_{\iota}(\langle w \rangle)} + O(1).
\end{aligned}
\]

which is strictly larger than \( \Phi_{\langle w \rangle,\mathbf{F}_{k}}(N) = \mathcal{T}_{w_{u}}(N) \) for every large enough \( N \).

**Remark 3.7.** As explained towards the end of Section 1, Theorem 1.5 can be used to show that if the word \( [u,v] \) (for some \( u, v \in \mathbf{F} \)) induces the same measures on finite groups as \( [x,y] \), then \( \{ u,v \} \) can be extended to a basis of \( \mathbf{F} \), and that if \( u^{n} \) induces the same measures on finite groups as \( x^{m} \), then \( u \) is primitive.

Another case where Theorem 1.5 is handy is the case of surface words analyzed in [MP19c]. Consider first a word \( w \) which induces the same measures on compact groups as the orientable surface word \( s_{g} = [x_{1}, y_{1}] \cdots [x_{g}, y_{g}] \). The proof in [MP19c] that \( w \overset{\text{Aut}\mathbf{F}}{\sim} s_{g} \) consists of three steps. First, using measures on unitary groups, it is shown that \( w \) is a product of at most \( g \) commutators. Then, using measures on a generalized symmetric group \( S^{1} \times S_{N} \), it is shown that \( w \) is in fact a product of \( g \) commutators, so \( w = [u_{1}, v_{1}] \cdots [u_{g}, v_{g}] \) such that \( u_{1}, v_{1}, \ldots, u_{g}, v_{g} \) are free (see the section “Overview of the proof” in [MP19c, Page 5]). Then, the more involved [MP19c, Theorem 3.6] is used to finish the proof. However, for this last step, one can also use Theorem 1.5, from which it follows that if \( w = [u_{1}, v_{1}] \cdots [u_{g}, v_{g}] \) with \( u_{1}, v_{1}, \ldots, u_{g}, v_{g} \) free and \( w \) induces the same measures on \( S_{N} \) as \( s_{g} \), then \( w \overset{\text{Aut}\mathbf{F}}{\sim} s_{g} \).

Theorem 1.5 can be similarly used for the other type of words studied in [MP19c]: that of non-orientable surface words \( x_{1}^{2} \cdots x_{g}^{2} \).

## 4 Measures induced by powers

### 4.1 Proof of Theorem 1.7

In the language of Profinite topology, Khelif’s Theorem 1.9 says that the set of commutators in \( \mathbf{F} \) is closed in the profinite topology. There is a similar result, due to Lubotzky, concerning the set of \( d \)th powers in \( \mathbf{F} \). As explained in the paragraph following Theorem 1.9, this immediately implies that \( d \)th powers are distinguishable by measures from non-\( d \)th powers:
Theorem 4.1 (Lubotzky, see [Tho97, Page 252]). The set \( \{ u^d \mid u \in F \} \) of \( d \)th powers is closed in the profinite topology on \( F \). In particular, if \( w_1 \) is an \( d \)th power and \( w_2 \) is not, then \( w_1 \) and \( w_2 \) do not induce the same measure on all finite groups.

We include below (Theorem 4.8) another proof of Theorem 4.1 which relies on homomorphisms from the free group to \( S_N \).

Another ingredient in our proof of Theorem 1.7 is the following theorem due to Herfort and Ribes. The free profinite product of two profinite groups \( A \) and \( B \) is denoted \( A \sqcup B \) – see [RZ10, Section 9.1] for a discussion on free profinite products. We denote the centralizer of \( g \) in the group \( G \) by \( C_G(g) \).

Theorem 4.2. [HR85, Theorem B] Let \( A \) and \( B \) be profinite groups and let \( A \sqcup B \) be their free profinite product. If \( a \in A \) then the centralizer of \( a \) in \( A \sqcup B \) is contained in \( A \).

For a subset \( S \) of \( F \) we denote by \( \overline{S}^F \) the closure of \( \iota(S) \) in the profinite completion \( \hat{F} \).

Lemma 4.3. For every \( 1 \neq w \in F \), \( C_F(w) = \overline{\langle w \rangle}^F \). In particular, \( C_F(w) \cong Z \).

Proof. Let \( u \in F \) be a root of \( w \) which is a non-power. So \( C_F(w) = \langle u \rangle \cong Z \). By Hall’s theorem (see [LS77, Proposition I.3.10]), \( u \) can be extended to a basis \( \{ u_1, u_2, \ldots, u_l \} \) of a finite index subgroup \( H \leq F \). If \( A \) and \( B \) are abstract groups, then \( A \sqcup B = A \ast B \) [RZ10, Exercise 9.1.1]. So if we denote by \( \overline{\Pi} = \overline{\Pi}^F \) the closure of \( H \) in \( \hat{F} \), then \( \overline{\Pi} = \overline{\langle u \rangle}^F \sqcup \overline{\langle u_2, \ldots, u_l \rangle}^F \), and by Theorem 4.2, \( C_{\overline{\Pi}}(w) \leq \overline{\langle u \rangle}^F \cong Z \).

But \( \hat{Z} \) is abelian, so \( C_{\overline{\Pi}}(w) = \overline{\langle u \rangle}^F = \overline{C_F(w)}^F \cong Z \).

It remains to show that for every \( b \in \hat{F} \setminus \overline{\Pi} \), \( b \) does not commute with \( w \). Pick \( e = b_1, b_2, \ldots, b_s \in F \) representatives for the left cosets of \( H \) in \( F \), which are also representatives for the left cosets of \( \Pi \) in \( \hat{F} \). Assume that \( b \in b_i \Pi \) for some \( 2 \leq i \leq s \). If \( b_i^{-1} wb_i \notin \Pi \), then \( b^{-1} wb \notin \Pi \), so we may assume that \( b_i^{-1} wb_i \in \Pi \) and thus also \( b^{-1} wb \in \Pi \). As \( C_F(w) = \langle u \rangle \leq H \) and \( b_i \notin H \), \( b_i^{-1} wb_i \) is not conjugate to \( w \) in \( H \). Because free groups are conjugacy-separable (e.g. [LS77, Proposition I.4.8]), \( b_i^{-1} wb_i \) is not conjugate to \( w \) also in \( \Pi \). So \( b^{-1} wb \neq w \).

Corollary 4.4. Every root of \( w \in F \) in \( \hat{F} \) belongs to \( F \).

Proof. Since \( \hat{F} \) is torsion free, we may assume \( w \neq 1 \). Assume that \( w = x^m \) with \( x \in \hat{F} \) and \( m \in Z_{\geq 1} \). By Theorem 4.1, the set of \( m \)th powers in \( F \) is closed in the profinite topology, which means that \( w = v^m \) for some \( v \in F \). Let \( u \in F \) be a non-power such that \( v = u^\ell \) for some \( \ell \in Z_{\geq 1} \). By Lemma 4.3, \( C_F(w) = \overline{\langle u \rangle}^F \cong Z \). As \( x, v \in C_F(w) \) and \( \hat{Z} \) is abelian, \( x \) and \( v \) commute. So \( (x^{-1} v)^m = x^{-m} v^m = 1 \). But \( \hat{Z} \) is torsion-free and thus \( x = v \).

We now have the tools to prove Theorem 1.7.

Proof of Theorem 1.7. The theorem is clear for \( d = 0 \). Now let \( d \in Z \setminus \{ 0 \} \) and assume that \( w \in F \) is profinitely rigid and that \( w_2 \in F \) induces the same measures on finite groups as \( w^d \). From Theorem 4.1 it follows that \( w_2 \) is a \( d \)th power, so \( w_2 = v^d \) for some \( v \in F \). By Theorem 2.2 there is an automorphism \( \theta \in \text{Aut} \hat{F} \) with \( \theta(w^d) = v^d \). So \( \theta(w) \) is a \( d \)th root of \( v^d \) in \( \hat{F} \), and by Corollary 4.4, \( \theta(w) = v \), namely, \( w \cong^F v \). But \( w \) is profinitely rigid so \( w \cong^F v \). Thus also \( w^d \cong^F v^d = w_2 \).

4.2 Powers in symmetric groups

For completeness, we also include a proof of the fact that it is enough to consider measures on the symmetric groups \( S_N \) (for all \( N \)) to obtain the case of primitive powers in Theorem 1.4. Note that this is not true for the words \( [x, y]^d \). Indeed, the word \( [x, y] = xyx^{-1}y^{-1} \) and the word \( xyxy^{-1} \), which lie in different \( \text{Aut} F \)-orbits, induce the exact same measure on \( S_N \) for all \( N \): in both words one takes a uniformly
random permutation (the image of $x$) and multiplies it with a uniform random conjugate (the image of $yx^{-1}y^{-1}$ or of $yx y^{-1}$). It is plausible that measures on the alternating groups $\operatorname{Alt}(N)$ do distinguish the orbit of $[x,y]$ from all other orbits, but we do not know whether this is true or not.

**Proposition 4.4.** If $w$ induces the same measures as $x^d$ on the symmetric group $S_N$ for all $N$, then $w \overset{\operatorname{Aut}F}{\sim} x^d$.

We begin with a lemma which identifies powers in $S_N$. Denote by $c_t(\sigma)$ the number of $t$-cycles in the cycle decomposition of $\sigma \in S_N$, and for a prime $p$ and a positive integer $n$, denote

$$\nu_p(n) = \max \left\{ e \in \mathbb{Z}_{\geq 0} \mid p^e \mid n \right\}.$$ 

**Lemma 4.6.** A permutation $\sigma \in S_N$ is a $d$th power if and only if for all $t \in \mathbb{Z}_{\geq 1}$ we have

$$\left( \prod_{p \mid t} p^{\nu_p(d)} \right) \mid c_t(\sigma),$$

the product being over all prime divisors of $t$. In particular, when $d \mid t$, the condition on $c_t$ is that $d \mid c_t(\sigma)$.

**Proof.** First, assume that $\sigma$ is a $d$th power. The $d$th power of an $\ell$-cycle is the union of $\gcd(d,\ell)$ cycles, each of length $\frac{\ell}{\gcd(d,\ell)}$. We need to show that if $p \mid t$ and $t = \frac{t}{\gcd(d,\ell)}$ then $p^{\nu_p(d)} \mid \gcd(d,\ell)$. But if $p \mid \frac{t}{\gcd(d,\ell)}$ then $\nu_p(\ell) > \nu_p(d)$ and so $\nu_p(\gcd(d,\ell)) = \nu_p(d)$.

For the converse implication, it is enough to prove the claim in the case where $\sigma$ is simply the product of $\left( \prod_{p \mid t} p^{\nu_p(d)} \right)$ disjoint cycles of length $t$. In this case, there is a cycle of length $t \cdot \left( \prod_{p \mid t} p^{\nu_p(d)} \right)$ whose $d$th power is $\sigma$. □

**Lemma 4.7.** Assume that $b, t \in \mathbb{Z}_{\geq 1}$ with $b \mid t$. Let $N \geq 2bt$ and let $\sigma \in S_N$ be a uniform random permutation. Then

$$\mathbb{E} \left[ c_t(\sigma^b) \right] = \frac{1}{t}, \quad \text{and} \quad \mathbb{E} \left[ c_t^2(\sigma^b) \right] = \frac{b}{t} + \frac{1}{t^2}.$$ 

**Proof.** As $b \mid t$, a $t$-cycle in $\sigma^b$ must come from a $bt$-cycle in $\sigma$, and each $bt$-cycle in $\sigma$ gives rise to $b$ disjoint cycles of length $t$ in $\sigma^b$. Hence $c_t(\sigma^b) = bc_{bt}(\sigma)$. When $N \geq bt$, the expected number of $bt$-cycles in $\sigma$ is $\frac{1}{bt}$, and so $\mathbb{E} \left[ c_t(\sigma^b) \right] = b \mathbb{E} \left[ c_{bt}(\sigma) \right] = \frac{1}{t}$. Likewise, when $N \geq 2bt$, $\mathbb{E} \left[ c_{bt}^2(\sigma) \right] = \frac{1}{bt} + \frac{1}{bt^2}$, so

$$\mathbb{E} \left[ c_t^2(\sigma^b) \right] = b^2 \mathbb{E} \left[ c_{bt}^2(\sigma) \right] = b^2 \left( \frac{1}{bt} + \frac{1}{bt^2} \right) = \frac{b}{t} + \frac{1}{t^2}.$$ □

We can now give a proof of Lubotzky’s Theorem 4.1 using homomorphic images in symmetric groups.

**Theorem 4.8.** Let $w \in F$. If $\varphi(w) \in S_N$ is a $d$th power for every $N \in \mathbb{Z}_{\geq 1}$ and every $\varphi \in \operatorname{Hom}(F, S_N)$, then $w$ is a $d$th power.

**Proof.** The statement is trivial for $d = 1$ or $w = 1$, so assume $d \geq 2$ and $w \neq 1$. Let $w = u^b$ with $u \neq 1$ a non-power. Denote $t = \operatorname{lcm}(b,d)$. Every image $\varphi(w)$ of $w$ in $S_N$ is a $b$th power, and by Lemma 4.6, as $b \mid t$,

$$b \mid c_t(\varphi(w)).$$

By assumption, every image $\varphi(w)$ of $w$ in $S_N$ is also a $d$th power, and, as $d \mid t$,

$$d \mid c_t(\varphi(w)).$$

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We deduce that \( t = \text{lcm} (b, d) \mid c_t (\varphi (w)) \). Thus
\[
c_t^2 (\varphi (w)) \geq t \cdot c_t (\varphi (w)).
\]
Taking expectations and then taking the limit as \( N \to \infty \), we see that
\[
\lim_{N \to \infty} E [c_t^2 (\varphi (w))] \geq t \cdot \lim_{N \to \infty} E [c_t (\varphi (w))].
\]
(12)

Recall that \( w = u^b \) with \( u \) a non-power. It is a theorem of Nica [Nic94, Theorem 1.1] that the random variables \( c_t (\varphi (w)) \) (where \( \varphi \in \text{Hom} (\mathcal{F}, S_N) \) is uniformly random) have a limit distribution which depends only on \( b \) and not on\(^1\) \( u \). In particular, the limits in (12) remain unchanged when \( w \) is replaced with \( x^b \). By Lemma 4.7 this gives
\[
\frac{b}{t} + \frac{1}{t^2} \geq t \cdot \frac{1}{t} = 1.
\]
Since \( t = \text{lcm} (b, d) \) and \( d \geq 2 \) we must have \( b = t \), so \( d \mid b \).

\[\square\]

**Proof of Proposition 4.5.** Assume that for all \( N, w \) induces the same measures on \( S_N \) as \( x^d \). In particular, every image of \( w \) through an homomorphism to \( S_N \) is a \( d \)th power, so by Theorem 4.8, \( w = v^d \) for some \( v \in \mathcal{F} \). We are now done by Theorem 1.5 applied with the word \( x^d \).

\[\square\]

**Remark 4.9.** The phenomenon observed by Nica that if \( u \) is a non-power then (moments of) \( u^d \)-measures converge to the same limits as \( x^d \)-measures in \( S_N \), is true in many families of groups. It is true in the wreath products \( C_m \wr S_N \) (as illustrated in [MP19c]) and for general linear groups over finite fields [PW19]. It is also true for families of infinite compact groups, such as unitary groups ([MSS07] or [MP19a, Corollary 1.13]) and Orthogonal and compact Symplectic groups [MP19b, Corollary 1.17].

**Remark 4.10.** Assume \( 1 \neq u \in \mathcal{F} \) is a non-power. In the notation of Section 3, \( \mathcal{T}_{u^d} (N) = \Phi (u^d),_F (N) \), and to use Theorem 3.6 for this case, we apply it with \( H = \langle x^d \rangle \) and \( J = \langle x \rangle \). Then,
\[
\mathcal{T}_{u^d} (N) = \Phi (u^d),_F (N) = \Phi (x^d),_x (N) + C \cdot N^{1-\pi_x (\langle x^d \rangle)} + O \left( N^{-\pi_x (\langle x^d \rangle)} \right),
\]
with \( u : J \to \mathcal{F} \) defined by \( x \mapsto u, \pi_x (\langle x^d \rangle) \) the smallest rank of an algebraic extension of \( \langle u^d \rangle \) not contained in \( \langle u \rangle \), and \( C \) the number of such algebraic extensions of rank \( \pi_x (\langle x^d \rangle) \). In [HP19] the first and last author study more generalizations of the results in [PP15], and, in particular, show that in this case \( \pi_x (\langle x^d \rangle) = \pi (u) \), and all algebraic extensions of \( \langle u^d \rangle \) of rank \( \pi (u) \) are also algebraic extensions of \( \langle u \rangle \). If we denote \( f_u (N) = \mathcal{T}_{u^d} (N) - \mathcal{T}_u (N) \), we get that
\[
f_u (N) = f_x (N) + O \left( N^{-\pi (u)} \right) = \delta (d) - 1 + O \left( N^{-\pi (u)} \right),
\]
where \( \delta (d) \) is the number of positive divisors of \( d \).

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\(^1\)It is more generally true that \( c_1 (\varphi (w)), c_2 (\varphi (w)), \ldots, c_t (\varphi (w)) \) have a limit joint distribution which only depends on \( a \) – see [LP10, Remark 31].
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