Toward a construction of scalar-flat Kähler metrics on affine algebraic manifolds

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Abstract

Let $(X, L_X)$ be an $n$-dimensional polarized manifold. Let $D$ be a smooth hypersurface defined by a holomorphic section of $L_X$. In this paper, we study the existence of a complete scalar-flat Kähler metric on $X \setminus D$ on the assumption that $D$ has a constant positive scalar curvature Kähler metric.

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1 Introduction

The existence of constant scalar curvature Kähler (cscK) metrics on complex manifolds is a fundamental problem in Kähler geometry. If a complex manifold is noncompact, there are many positive results in this problem. In 1979, Calabi [6] showed that if a Fano manifold has a Kähler Einstein metric, then there is a complete Ricci-flat Kähler metric on the total space of the canonical line bundle. In addition, there exist following generalizations. In 1990, Bando-Kobayashi [5] showed that if a Fano manifold admits an anti-canonical smooth divisor which has a Ricci-positive Kähler Einstein metric, then there exists a complete Ricci-flat Kähler metric on the complement. In 1991, Tian-Yau [13] showed that if a Fano manifold admits an anti-canonical smooth divisor which has a Ricci-flat Kähler metric, then there is a complete Ricci-flat Kähler metric on the complement. In 2002, on the other hand, as a scalar curvature version of Calabi’s result [6], Hwang-Singer [9] showed that if a polarized manifold has a nonnegative cscK metric, then the total space of the dual line bundle admits a complete scalar-flat Kähler metric. However, a similar generalization of Hwang-Singer [9] like Bando-Kobayashi [5] and Tian-Yau [13] is unknown since it is hard to solve a forth order nonlinear partial differential equation.

In this paper, assuming the existence of a smooth hypersurface which admits a constant positive scalar curvature Kähler metric, we study the existence of a complete scalar-flat Kähler metric on the complement of this hypersurface.

Let $(X, L_X)$ be a polarized manifold of dimension $n$, i.e., $X$ is an $n$-dimensional compact complex manifold and $L_X$ is an ample line bundle over $X$. Assume that there is a smooth hypersurface $D \subset X$ with $D \in |L_X|$. Set an ample line bundle $L_D := \mathcal{O}(D)|_D = L_X|_D$ over $D$. Since $L_X$ is ample, there exists a Hermitian metric $h_X$ on $L_X$ which defines a Kähler metric $\theta_X$ on $X$, i.e., the curvature form of $h_X$ multiplied by $\sqrt{-1}$ is positive definite. Then, the restriction of $h_X$ to $L_D$ defines also a Kähler metric $\theta_D$ on $D$. Let $\hat{S}_D$ be the average of the scalar curvature $S(\theta_D)$ of $\theta_D$ defined by

$$\hat{S}_D := \frac{\int_D S(\theta_D)\theta_D^{n-1}}{\int_D \theta_D^{n-1}} = \frac{(n-1)c_1(K^{-1}_D) \cup c_1(L_D)^{n-2}}{c_1(L_D)^{n-1}},$$

where $K^{-1}_D$ is the anti-canonical line bundle of $D$. Note that $\hat{S}_D$ is a topological invariant in the sense that it is representable in terms of Chern classes of the line bundles $K^{-1}_D$ and $L_D$. In this paper, we treat the following case:

$$\hat{S}_D > 0. \tag{1.1}$$

Let $\sigma_D \in H^0(X, L_X)$ be a defining section of $D$ and set $t := \log ||\sigma_D||_{h_X}^2$. Following [5], we can define a complete Kähler metric $\omega_0$ by

$$\omega_0 := \frac{n(n-1)}{\hat{S}_D} \sqrt{-1} \partial \bar{\partial} \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right).$$
on the noncompact complex manifold \( X \setminus D \). In addition, since \((X \setminus D, \omega_0)\) is of asymptotically conical geometry (see [5] or Section 4 of this paper), we can define the weighted Banach space \( C^{k,\alpha}_\delta = C^{k,\alpha}_\delta(X \setminus D) \) for \( k \in \mathbb{Z}_{\geq 0}, \alpha \in (0, 1) \) and with a weight \( \delta \in \mathbb{R} \) with respect to the distance function \( r \) defined by \( \omega_0 \) from some fixed point in \( X \setminus D \). It follows from the construction of \( \omega_0 \) that \( S(\omega_0) = O(r^{-2}) \) near \( D \).

The cscK condition implies the following stronger decay property.

**Theorem 1.1.** If \( \theta_D \) is a constant positive scalar curvature Kähler metric on \( D \), i.e., \( S(\theta_D) = \hat{S}_D > 0 \), we have

\[
S(\omega_0) = O\left(r^{-2-2(n-1)/\hat{S}_D}\right)
\]

as \( r \to \infty \).

Thus, the cscK condition implies that \( S(\omega_0) \in C^{k,\alpha}_\delta \) for some \( \delta > 2 \) and any \( k, \alpha \).

In order to construct a complete scalar-flat Kähler metric on \( X \setminus D \), the linearization of the scalar curvature operator plays an important role:

\[
L_{\omega_0} = -D^{*}_{\omega_0} D_{\omega_0} + (\nabla^{1,0}, \nabla^{0,1} S(\omega_0))_{\omega_0}.
\]

Here, \( D_{\omega_0} = \bar{\partial} \circ \nabla^{1,0} \). We will show that if \( 4 < \delta < 2n \) and there is no nonzero holomorphic vector field on \( X \) which vanishes on \( D \), then \( D^{*}_{\omega_0} D_{\omega_0} : C^{4,\alpha}_{\delta-4} \to C^{0,\alpha}_{\delta} \) is isomorphic. For such operators, we consider the following:

**Condition 1.2.** Assume that \( n \geq 3 \) and there is no nonzero holomorphic vector field on \( X \) which vanishes on \( D \). For \( 4 < \delta < 2n \), the operator

\[
L_{\omega_0} : C^{4,\alpha}_{\delta-4} \to C^{0,\alpha}_{\delta}
\]

is isomorphic, i.e., we can find a constant \( \hat{K} > 0 \) such that

\[
||L_{\omega_0}\phi||_{C^{0,\alpha}_{\delta}} \geq \hat{K} ||\phi||_{C^{4,\alpha}_{\delta-4}}
\]

for any \( \phi \in C^{4,\alpha}_{\delta-4} \).

In addition, we consider

**Condition 1.3.**

\[
||S(\omega_0)||_{C^{0,\alpha}_{\delta}} < c_0 \hat{K}/2.
\]

Here, the constant \( c_0 \) will be defined in Lemma 6.2 later. Under these conditions, Theorem 1.1 implies the following result:

**Theorem 1.4.** Assume that \( n \geq 3 \) and there is no nonzero holomorphic vector field on \( X \) which vanishes on \( D \). Assume that \( \theta_D \) is a constant scalar curvature Kähler metric satisfying

\[
0 < \hat{S}_D < n(n-1).
\]

Assume moreover that Condition 1.2 and Condition 1.3 hold, then \( X \setminus D \) admits a complete scalar-flat Kähler metric.
In fact, we can show the existence of a complete scalar-flat Kähler metric on \( X \setminus D \) under the following assumptions: (i) \( n \geq 3 \) and there is no nonzero holomorphic vector field on \( X \) which vanishes on \( D \), (ii) there exists a complete Kähler metric on \( X \setminus D \) which is of asymptotically conical geometry, such that its scalar curvature is sufficiently small and decays at a higher order. So, if there exists a complete Kähler metric on \( X \setminus D \) which is sufficiently close to \( \omega_0 \) at infinity, satisfying Condition 1.2 and Condition 1.3, we can show the existence of a complete scalar-flat Kähler metric on \( X \setminus D \). Theorem 1.4 is proved by the fixed point theorem on the weighted Banach space \( C^{4,\alpha}_{\delta-4}(X \setminus D) \) by following Arezzo-Pacard [3], [4] (see also [12]). In general, constants \( c_0, \hat{K} \) which arise in Condition 1.2 and Condition 1.3 depend on the background Kähler metric \( \omega_0 \). In addition, to construct such a Kähler metric, we have to find a complete Kähler metric \( X \setminus D \) whose scalar curvature is arbitrarily small. We will handle this problem in next papers [1] and [2].

This paper is organized as follows. In Section 2, recalling the result due to Hwang-Singer [9], we give the volume growth of a geodesic ball with respect to the Kähler metric obtained in [9]. This case is a toy-model of our problem. In Section 3, we prove Theorem 1.1. To prove this, we use fundamental results in matrix analysis. In Section 4, we will introduce the asymptotically conicalness of open Riemannian manifolds and weighted Banach spaces by following Bando-Kobayashi [5]. In Section 5, we study the linearization of the scalar curvature operator between some weighted Banach spaces. In section 6, we prove Theorem 1.4 by following Arezzo-Pacard [3], [4] (see also [12]).

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## 2 The case of line bundles

Before considering the general case, we consider the existence of a complete scalar-flat Kähler metrics on line bundles and compute the volume growth. Let \( (X, L) \) be an \( n \)-dimensional polarized manifold and \( \theta \in 2\pi c_1(L) \) be a cscK metric. In this case, the value of the scalar curvature is equal to the following average value given by

\[
\hat{S}_X := \frac{nc_1(X) \cup c_1(L)^{n-1}}{c_1(L)^n},
\]

where \((2\pi)^n c_1(L)^n = \int_X \theta^n \) and \((2\pi)^n c_1(X) \cup c_1(L)^{n-1} = \int_X \text{Ric}(\theta) \land \theta^{n-1}\). Note that \( \hat{S}_X \) is a topological invariant. In 2002, Hwang-Singer [9] showed the following:

**Theorem 2.1.** Let \( (X, L) \) be an \( n \)-dimensional polarized manifold. Suppose that there exists a constant scalar curvature Kähler metric \( \theta \in 2\pi c_1(L) \) and the value of the scalar curvature of \( \theta \) is nonnegative:

\[
\hat{S}_X \geq 0.
\]

Then, there exists a complete scalar-flat Kähler metric \( \omega \) on the total space of the dual line bundle \( L^{-1} \).
Remark 2.2. In [9], they treat more general cases which contain the existence of a complete scalar-flat Kähler metric on the disc bundle in $L^{-1}$. In this paper, it is enough for us to consider Theorem 2.1.

To compute the volume growth of the Kähler metric $\omega$ above, we need to recall the proof of Theorem 2.1 by following [9] (see also [12]).

2.1 LeBrun-Simanca metrics

In this subsection, we construct the LeBrun-Simanca metric for an ample line bundle. This metric for the dual of the tautological line bundle over $\mathbb{CP}^{n-1}$ was found by LeBrun and Simanca [10],[11] (see also [12]).

Fix an $n$-dimensional polarized manifold $(X,L)$. Consider a Hermitian metric $h$ on $L$ which defines the Kähler metric $\theta \in 2\pi c_1(L)$. Let $p : L^{-1} \to X$ be the projection map and $(L^{-1})^*$ be the complement of the zero section of $L^{-1}$. Define a smooth function $s$ by

$$s : (L^{-1})^* \to \mathbb{R}, \quad \xi \mapsto \log h^{-1}(\xi,\xi),$$

where $h^{-1}$ is a Hermitian metric on $L^{-1}$ induced by $h$.

Definition 2.3. The LeBrun-Simanca metric $\omega$ on $(L^{-1})^*$ is defined by

$$\omega := \sqrt{-1} \partial \bar{\partial} f(s),$$

where $f$ is a smooth, increasing and strictly convex function on $\mathbb{R}$.

To extend $\omega$ to the whole space $L^{-1}$, we start to compute $\omega$ in local coordinates. Fix a point $z_0$ in the base space $X$. Then we can find a local holomorphic coordinate chart $U$ around $z_0$ with a local holomorphic trivialization of $L^{-1}$ around $z_0$:

$$(L^{-1})|_U \cong U \times \mathbb{C}, \quad \xi \mapsto (z,w),$$

where $w$ is a fiber coordinate. Then, in these local coordinates, we can write as

$$s(z,w) = \log |w|^2 - \log h(z),$$

where $h(z)$ is a positive function defined by some local non-vanishing holomorphic section of $L$ on $U$. For any point $z_0 \in X$, we can choose the above trivialization so that:

$$d \log h(z_0) = 0.$$

Let us compute at a point $(z_0,w)$ with $w \neq 0$;

$$\sqrt{-1} \partial \bar{\partial} f(s) = \dot{f}(s) \sqrt{-1} \frac{dw \wedge d\bar{w}}{|w|^2} + \dot{f}(s)p^*\theta,$$

where the symbol $\dot{f}$ denotes the differential of $f$ with respect to the variable $s$.

To simplify the construction of a scalar-flat Kähler metric on $L^{-1}$, following [12], we introduce Legendre transforms and momentum profiles and recall fundamental facts of them. By using them, we can give a condition on the extension of $\omega$ to the whole space $L^{-1}$ and compute the scalar curvature not as a nonlinear PDE in forth order but as an ODE in second order by following [9].
**Definition 2.4.** Let $f$ be a strictly convex and smooth function on $\mathbb{R}$. Set $\tau := \dot{f}(s)$ and $I$ be an image of $\tau$. The Legendre transform $F$ on $I$ of $f$ with variable $\tau$ is defined by

$$s\tau = f(s) + F(\tau).$$

Note that there are following relations:

$$F'(\tau) = s, \quad F''(\tau) = \frac{1}{\dot{f}(s)},$$

where we use the symbol $F'(\tau)$ as the differential of $F$ with respect to the variable $\tau$.

**Definition 2.5.** Let $I \subset \mathbb{R}$ be an image of $\tau$. The momentum profile $\varphi$ of the metric $\omega = \sqrt{-1} \partial \bar{\partial} f(s)$ is defined by the following:

$$\varphi : I \to \mathbb{R}, \quad \varphi(\tau) = \frac{1}{F''(\tau)},$$

(2.9)

where $F$ is the Legendre transform of $f$ defined above.

Clearly, there are following relations:

$$\varphi(\tau) = \dot{\tau}, \quad \frac{d\tau}{ds} = \varphi(\tau).$$

(2.10)

The following proposition is the converse of the above construction:

**Proposition 2.6.** Let $I \subset \mathbb{R}$ be any interval and $\varphi$ be a smooth positive function defined on $I$. Then we can find a smooth and strictly convex function $f$ on some interval $J$ of $\mathbb{R}$ such that

$$\tau = \dot{f}(s), \quad \varphi(\tau) = \ddot{f}(s).$$

(2.11)

**Proof.** Let $G = G(\tau)$ be a function on $I$ with $G'(\tau) = 1/\varphi(\tau)$. Since $G$ is strictly monotone increasing, we have $\tau = G^{-1}(s)$. Set $J := G(I)$. Proposition 2.5 is proved by setting

$$f(s) := \int_c^s G^{-1}(t) dt$$

for some $c \in J$. 

\[\square\]

### 2.2 The extension to the total space

In this subsection, we give a condition such that the LeBrun-Simanca metric $\omega$ can be extended to the whole space $L^{-1}$ as a Kähler metric by following [9] (see also [12]). By using the momentum profile $\varphi$, $\omega$ can be rewritten as follows:

$$\omega = \varphi(\tau) \sqrt{-1} \frac{dw \wedge d\bar{w}}{|w|^2} + \tau p^* \theta.$$  

(2.12)

First, we set a momentum profile $\varphi$ defined on $I := (1, N)$ for some $N \in (1, \infty]$ so that a function $f$ is defined on $J = \mathbb{R}$ in the way in the proof of Proposition 2.6. Then, the formula (2.12) implies that the LeBrun-Simanca metric $\omega$ is positive in the base direction at any point in the zero section. Following [9], to obtain the positivity in the fiber direction and the smoothness of $\omega$ on the whole space $L^{-1}$, we pose the boundary condition on $\varphi$: 

Proposition 2.7. Suppose that \( \varphi \) satisfies the following boundary condition:
\[
\varphi(1) = 0, \quad \varphi'(1) = 1,
\]
and can be extended smoothly in a neighborhood of 1. Then \( \omega \) can be extended to \( L^{-1} \) as a Kähler metric.

For simplicity, we will denote the extended metric on \( L^{-1} \) by the same symbol \( \omega \).

2.3 The Ricci form and the scalar curvature

In this subsection, we compute the Ricci form and the scalar curvature of the LeBrun-Simanca metric \( \omega \).

Proposition 2.8. The Ricci form \( \text{Ric}(\omega) \) and the scalar curvature \( S(\omega) \) of \( \omega \) are given by
\[
\text{Ric}(\omega) = -\varphi \left( \varphi' + n \frac{\varphi}{\tau} \right)' \sqrt{-1}dw \wedge d\bar{w} \quad \text{and}
\]
\[
S(\omega) = \frac{p^* S(\theta)}{\tau} - \frac{1}{\tau^n} \frac{d^2}{d\tau^2} (\tau^n \varphi(\tau)),
\]
where \( \text{Ric}(\omega) \) is a pointwise formula.

Proof. First, the Ricci form \( \text{Ric}(\omega) \) is locally given by the following:
\[
\text{Ric}(\omega) = -\varphi \left( \varphi' + n \frac{\varphi}{\tau} \right)' \sqrt{-1}dw \wedge d\bar{w} \quad \text{with}
\]
\[
S(\omega) = \frac{p^* S(\theta)}{\tau} - \frac{1}{\tau^n} \frac{d^2}{d\tau^2} (\tau^n \varphi(\tau)).
\]
If we choose another trivialization \((z, \hat{w})\) of \( L^{-1} \), there exists a holomorphic transform function \( g \) such that \( \hat{w} = g(z)w \). The differential of \( g(z) \) does not affect the above formula because \( p^* \theta^n \) is the top wedge product in the base direction. Therefore, \( \omega^{n+1} \) is invariant under the choice of the local coordinates \((z, w)\) and we can use the formula above globally.

Let us compute \( \text{Ric}(\omega) \) at a point \((z_0, w_0)\):
\[
\text{Ric}(\omega) = -\sqrt{-1} \partial \bar{\partial} \log \varphi(\tau(s)) - n \sqrt{-1} \partial \bar{\partial} \log \tau(s) + p^* \text{Ric}(\theta)
\]
\[
= -\varphi \left( \varphi' + n \frac{\varphi}{\tau} \right)' \sqrt{-1}dw \wedge d\bar{w} \quad \text{with}
\]
\[
S(\omega) = -\frac{1}{\tau^n} \frac{d^2}{d\tau^2} (\tau^n \varphi(\tau)) + p^* S(\theta).\]

Note that the equation of \( \text{Ric}(\omega) \) is completely divided into the base direction and fiber direction. Taking a trace of the Ricci form by the metric \( \omega = \varphi(\tau) \sqrt{-1}dw \wedge d\bar{w} / |w|^2 + \tau p^* \theta \), we have the following:
\[
S(\omega) = -\frac{1}{\tau^n} \frac{d^2}{d\tau^2} (\tau^n \varphi(\tau)) + p^* S(\theta).\]
Thus, the proof of Proposition 2.8 is finished.
2.4 ODE

In this subsection, we prove Theorem 2.1 by using Proposition 2.8. The key of the proof is that we can consider the scalar-flat condition as the case of ordinary differential equations (ODE) in second order on the assumption that $\omega$ is cscK.

Proof of Theorem 2.1. If the Kähler metric $\theta$ has a constant scalar curvature, the value of the scalar curvature $S(\theta)$ is equal to the average of the scalar curvature:

$$\hat{S}_X = \frac{nc_1(X) \cup c_1(L)^{n-1}}{c_1(L)^n}.$$  \hspace{1cm} (2.14)

By the formula in Proposition 2.8 to make $\omega$ scalar-flat, it is enough to solve the following ODE with the boundary condition:

$$\frac{d^2}{d\tau^2}(\tau^n \varphi(\tau)) = \hat{S}_X \tau^{n-1}, \quad \varphi(1) = 0, \quad \varphi'(1) = 1.$$  \hspace{1cm} (2.15)

In fact, a solution of this is easily given by

$$\varphi(\tau) = \frac{\hat{S}_X}{n(n+1)} \tau - \left(\frac{\hat{S}_X}{n} - 1\right) \tau^{1-n} + \left(\frac{\hat{S}_X}{n+1} - 1\right) \tau^{-n}.$$  \hspace{1cm} (2.16)

If $N = \infty$, $\varphi = O(\tau)$ as $\tau \to \infty$. If $N < \infty$, $\varphi$ vanishes like a polynomial. Recall that $s = \int \varphi^{-1} d\tau$. In both cases, applying Proposition 2.6 for these $\varphi$, we can obtain an increasing, strictly convex and smooth function $f$ defined on $\mathbb{R}$ by setting an interval $I := \{\tau \in \mathbb{R} \mid \varphi(\tau) \text{ is positive}\}$. Then, we have finished the proof of Theorem 2.1. \hfill $\square$

2.5 Volume growth

In this subsection, we compute the volume growth of the Kähler metric $\omega$.

Proposition 2.9. Fix a point $\xi \in L^{-1}$. Suppose that $\omega$ be the LeBrun-Simanca metric as above. Let $B(\xi, r)$ be a geodesic ball with respect to $\omega$ in $L^{-1}$ of radius $r$ centered at $\xi$.

1. If $\hat{S}_X > 0$, we have

$$\int_{B(\xi, r)} \omega^{n+1} = O(r^{2(n+1)}) \text{ as } r \to \infty.$$  \hspace{1cm} (2.16)

2. If $\hat{S}_X = 0$, we have

$$\int_{B(\xi, r)} \omega^{n+1} = O(r^2) \text{ as } r \to \infty.$$  \hspace{1cm} (2.17)

For a point $y \in X$, a symbol $\xi_y$ denotes an element of the fiber $L_y^{-1}$. In particular, a symbol $0_y$ denotes the zero element of the fiber $L_y^{-1}$. In this section, we use the LeBrun-Simanca metric $\omega$ given by the solution (2.15) in the both cases of $\hat{S}_X > 0$ and $\hat{S}_X = 0$. First, we compute a relation between the geodesic distance for the metric $\omega$ and the Hermitian norm for $h^{-1}$. For simplicity, $|\xi_y|$ denotes a square root of the hermitian norm $h^{-1}(\xi_y, \xi_y)$.
Lemma 2.10.

(a) If $\hat{S}_X > 0$, we have

$$d(0, \xi_y) = O(|\xi_y|^{\frac{\hat{S}_X}{n(n+1)}}) \quad \text{as} \quad |\xi_y| \to \infty$$

and

$$\tau = O(d(0, \xi_y)^2) \quad \text{as} \quad |\xi_y| \to \infty.$$  \hfill (2.19)

(b) If $\hat{S}_X = 0$, we have

$$d(0, \xi_y) = O((\log |\xi_y|)^{\frac{\hat{S}_X}{2n}}) \quad \text{as} \quad |\xi_y| \to \infty$$

and

$$\tau = O(d(0, \xi_y)^{\frac{\hat{S}_X}{n(n+1)}}) \quad \text{as} \quad |\xi_y| \to \infty.$$  \hfill (2.21)

Proof. First, we prove the statement (a). By the completeness of $\omega$, there exists a length minimizing geodesic connecting any pair of two points in $L^{-1}$. Fix the length minimizing geodesic $\gamma(t), t \in [0, 1]$ from $0_y$ to $\xi_y$. Clearly, for fixed $y \in X$, the image of the geodesic $\gamma(t)$ is in the fiber $L^{-1}_y$. For simplicity, we assume that $h^{-1}(\xi_y, \xi_y) = |w|^2$ in the trivialization \[2.5\]. Set $v = |w|$.  

(1) Recall that the positive function $\varphi(\tau)$ in the case (a) is written as

$$\varphi(\tau) = \frac{\hat{S}_X}{n(n+1)} \tau - \left(\frac{\hat{S}_X}{n} - 1\right) \tau^{1-n} + \left(\frac{\hat{S}_X}{n+1} - 1\right) \tau^{-n}.$$  

Since $\tau \to \infty$ as $v \to \infty$, the second and third terms above are very small as $v \to \infty$. Then,

$$\frac{d\tau}{dv} = \varphi(\tau) \frac{2}{v}$$

$$= \frac{2\hat{S}}{n(n+1)} \frac{\tau}{v} + \text{l.o.t.}$$ \hfill (2.22)

Here the symbol l.o.t. denotes lower order terms as $v \to \infty$. We have

$$\frac{1}{\tau} \frac{d\tau}{dv} = \frac{2\hat{S}}{n(n+1)} \frac{1}{v} + \text{l.o.t.}$$

Thus, we have

$$\tau = O(|\xi_y|^\frac{\hat{S}_X}{n(n+1)}),$$ \hfill (2.23)

as $|\xi_y| \to \infty$.

Denote $\gamma(t) = (y, w_t)$. Since the LeBrun-Simanca metric $\omega$ is $S^1$-invariant, we can write

$$w_t = \theta_t w_1,$$

where $\theta_t$ is some real nonnegative function such that $\theta_0 = 0$ and $\theta_1 = 1$. Since it is enough to compute for sufficiently large $|\xi_y|$, we have


\begin{align*}
  d(0_y, \xi_y) &= \int_0^1 \sqrt{\omega(\gamma(t), \dot{\gamma}(t))} \, dt \\
  &= \int_0^1 \sqrt{\frac{|\dot{w}_t|^2}{|w_t|^2} \varphi(\tau)} \, dt \\
  &= \int_0^1 \hat{\theta}_t \sqrt{\varphi(\tau)} \, dt \\
  &= \int_c^1 |w_1|^{\frac{s}{n(n+1)}} \hat{\theta}_t \hat{\theta}_t^{\frac{s}{n(n+1)-1}} \, dt + \text{l.o.t.} \\
  &= O(\frac{1}{\xi_y^{n+1}}) \\
  \end{align*}

as \(|\xi_y| \to \infty\), where \(c \in (0,1)\) is some fixed constant. Thus, the statement (a) follows.

(b) Recall that the positive function \(\varphi(\tau)\) in the case (b) is written as

\[ \varphi(\tau) = \tau^{1-n} - \tau^{-n}. \]

Similarly, we have

\[ \frac{d\tau}{dv} = \frac{2\tau^{1-n}}{v} + \text{l.o.t.} \]

Then,

\[ \tau^n = O(\log |\xi_y|), \]

as \(|\xi_y| \to \infty\).

Denote \(\gamma(t) = (y, w_t)\). Similarly, we have

\begin{align*}
  d(0_x, \xi_y) &= \int_0^1 \sqrt{\omega(\xi_y, \gamma(t))} \, dt \\
  &= \int_c^1 \hat{\theta}_t \tau^{\frac{1-n}{2}} \, dt + \text{l.o.t.} \\
  &= A \int_c^1 \hat{\theta}_t (\log |w_1| + \log \theta_t)^{\frac{1-n}{2m}} \, dt + \text{l.o.t.} \\
  &= O((\log |\xi_y|)^{\frac{n+1}{2m}}) \tag{2.25}
\end{align*}

as \(|\xi_y| \to \infty\), where \(c \in (0,1)\) and \(A > 0\) are some fixed constants. Thus, the statement (b) follows.

Using Lemma 2.10, we prove Proposition 2.9.

Proof of Proposition 2.9. It is enough to compute the volume growth of a geodesic ball in \(L^{-1}\) of radius \(r\) centered at 0. Since the restriction of \(\omega\) to the zero section is \(\theta\), we have

\[ d(0_x, \xi_y) = d(0_x, 0_y) + d(0_y, \xi_y) \leq \text{diam}(X, \theta) + d(0_y, \xi_y). \]
Thus,

\[-\text{diam}(X, \theta) + d(0_x, \xi_y) \leq d(0_y, \xi_y) \leq \text{diam}(X, \theta) + d(0_y, \xi_y).\] (2.26)

By Stokes’ theorem, we have

\[
\int_{B(0,x,r)} \omega^{n+1} = \int_{B(0,x,r)} \tau^n \varphi(\tau) \sqrt{-1} \partial s \wedge \bar{\partial} s \wedge p^* \theta^n \\
= \frac{1}{n+1} \int_{B(0,x,r)} \sqrt{-1} \partial s \wedge \bar{\partial}(\tau^{n+1}) \wedge p^* \theta^n \\
= -\frac{1}{n+1} \left( \int_{\partial B(0,x,r)} \tau^{n+1} \sqrt{-1} \partial s \wedge p^* \theta^n + (2\pi)^n c_1(L^n) \right) \\
= -C_1 \int_{\partial B(0,x,r)} \tau^{n+1} \sqrt{-1} \frac{dw}{w} \wedge p^* \theta^n - C_2, \] (2.27)

where \(C_1\) and \(C_2\) are positive constants depending only on \(n\) and \(L\).

In the case (1), previous computations (2.19) and (2.26) imply

\[\tau^{n+1} = O(r^{2(n+1)}).\] (2.28)

For \(r > \text{diam}(X, \theta)\), the residue theorem holds for each \(y \in X\). Thus, we have

\[-\int_{\partial B(0,x,r)} \sqrt{-1} \frac{dw}{w} \wedge p^* \theta^n = (2\pi)^n c_1(L^n).\] (2.29)

The formula (2.27) implies the first statement in Proposition 2.9.

In the case (2), previous computations (2.21) and (2.26) imply

\[\tau^{n+1} = O(r^2).\] (2.30)

Similarly, (2.29) and the formula (2.27) imply the second statement in Proposition 2.9. \(\square\)

3 The higher order decay

In this section, we prove Theorem 1.1. Let \((X, L_X)\) be an \(n\)-dimensional polarized manifold. Let \(h_X\) be a Hermitian metric on the line bundle \(L_X\) which defines a Kähler metric \(\theta_X\) on \(X\). Then, the restriction \(h_D\) of \(h_X\) to a line bundle \(L_D := L_X|_D\) over \(D\) defines a Kähler metric \(\theta_D\) on \(D\). Let \(\sigma_D \in H^0(X, L_X)\) be a defining section of \(D\). Set \(t := \log ||\sigma_D||^{-2}\), where \(||\sigma_D||^2 = h_X(\sigma_D, \sigma_D)\). From the construction of the complete Kähler metrics in Theorem 2.1 and \([5]\), we can define a complete Kähler metric \(\omega_0\) on \(X \setminus D\) by

\[
\omega_0 := \frac{n(n-1)}{\hat{S}_D} \sqrt{-1} \partial \bar{\partial} \exp \left( \frac{\hat{S}_D}{n(n-1)} \right) \\
= \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right) \left( \theta_X + \frac{\hat{S}_D}{n(n-1)} \sqrt{-1} \partial t \wedge \bar{\partial} t \right),
\]
where $\hat{S}_D > 0$ is the average value of the scalar curvature $S(\theta_D)$:

$$\hat{S}_D := \frac{\int_D S(\theta_D)\theta_D^{n-1}}{\int_D \theta_D^{n-1}} = \frac{(n-1)c_1(K_D^{-1}) \cup c_1(L_D)^{n-2}}{c_1(L_D)^{n-1}}.$$ 

By similar ways in Section 2, we have the followings

**Lemma 3.1.** Let $r$ be a distance function defined by $\omega_0$ from a fixed point $x_0 \in X \setminus D$. Then,

$$r(x) = O(||\sigma_D||^{-\frac{\hat{S}_D}{n(n-1)}}(x))$$

as $x \to D$.

**Lemma 3.2.** The volume growth of $\omega_0$ is given by

$$\text{Vol}_{\omega_0}(B(x_0, r)) = O(r^{2n})$$

as $r \to \infty$.

Thus, Lemma 3.1 implies that it is enough to show that

$$S(\omega_0) = O(||\sigma_D||^{2 + 2\hat{S}_D/n(n-1)})$$

as $\sigma_D \to 0$.

To show Theorem 1.1 we have to compute $\text{Ric}(\theta_X)$ and $\text{Ric}(\omega_0)$. Unfortunately, we can’t compute the scalar curvature of $\omega_0$ in the same way in the proof of Proposition 2.8. So, we study the determinant of $\theta_X$ and the inverse matrix of $\omega_0$. First, we recall fundamental results in matrix analysis.

### 3.1 Matrix analysis

To compute Ricci forms of Kähler metrics $\theta_X, \omega_0$, we need the following lemma:

**Lemma 3.3.** Consider the following matrix

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A$ is an invertible matrix. Then, the determinant of $T$ is given by

$$\det T = \det A \det(D - CA^{-1}B).$$

**Proof.** The result immediately follows from the following formula:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & O \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & O \\ O & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ O & I \end{bmatrix}$$

where $I$ and $I$ denote suitable identity matrices. 

$\blacksquare$
To take a trace with respect to the Kähler metric $\omega_0$, we need the following inverse matrix formula:

**Lemma 3.4.** Consider the following matrix

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$ 

Assume that $A$ and $S := D - CA^{-1}B$ are invertible. Then, $T$ is invertible and the inverse matrix of $T$ can be written as

$$T^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}. $$

**Proof.** From the proof of the previous lemma, we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} I & A^{-1}B \\ O & I \end{bmatrix}^{-1} \begin{bmatrix} A & O \\ O & S \end{bmatrix}^{-1} \begin{bmatrix} I & O \\ CA^{-1} & i \end{bmatrix}^{-1} = \begin{bmatrix} I & -A^{-1}B \\ O & i \end{bmatrix} \begin{bmatrix} A^{-1} & O \\ O & S^{-1} \end{bmatrix} \begin{bmatrix} I & O \\ -CA^{-1} & i \end{bmatrix} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}. $$

\[\square\]

### 3.2 Local trivialization and normal coordinates

Before studying the scalar curvature $S(\omega_0)$ near $D$, we choose a local trivialization and normal coordinates around a point of $D$.

First, fix a point $p \in D$. Since $D$ is the smooth hypersurface of $X$, there exist local holomorphic coordinates $(z^1, z^2, \ldots, z^{n-1}, w)$ centered at $p$ where $D$ is defined by $\{w = 0\}$ locally and $(z^1, z^2, \ldots, z^{n-1})$ are local holomorphic coordinates of $D$. Then, there exists a local trivialization of $L_X$ such that we can write as $||(\sigma_D)||^2 = |w|^2 e^{-\varphi}$ for a smooth function $\varphi$ near $p$ satisfying

$$d\varphi(0) = 0.$$ 

We may assume that if $(z^1, z^2, \ldots, z^{n-1}, w) = (0, 0, \ldots, 0, w)$, we have

$$\varphi = O(|w|^2). $$

(3.1)

Second, we consider the existence of normal coordinates with respect to the Kähler metric $\theta_X$ around $p$ preserving the condition (3.1). Since $\theta_X = \sqrt{-1}\partial\bar{\partial}t = \sqrt{-1}\partial\bar{\partial}\log ||\sigma_D||^{-2}$ is the Kähler metric on $X$, in coordinates above, we can write locally as

$$\theta_X = \sqrt{-1} \left( \sum_{i,j=1}^{n-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j + \sum_{a=1}^{n-1} (g_{a\bar{a}} dz^a \wedge d\bar{w} + g_{a\bar{w}} dw \wedge d\bar{z}^a) + g_{w\bar{w}} dw \wedge d\bar{w} \right).$$
For simplicity, write \((z^1, ..., z^{n-1}, w) = (z; w)\). Consider another holomorphic coordinate chart \((\hat{z}^1, ..., \hat{z}^{n-1}, w) = (\hat{z}; w)\) around \(p \in D\). Directly, we have
\[
\frac{\partial}{\partial w} \left( g_{i\overline{j}} \right) (0; 0) = \frac{\partial}{\partial w} \left( \frac{\partial z^k}{\partial \hat{z}^i} \frac{\partial \overline{z}^l}{\partial \hat{z}^j} g_{k\overline{l}} \right) (0; 0) + \frac{\partial z^k}{\partial \hat{z}^i} \frac{\partial \overline{g}_{k\overline{l}}}{\partial \hat{z}^j} \frac{\partial g_{i\overline{j}}}{\partial w} (0; 0).
\]

Set the condition
\[
\frac{\partial z^k}{\partial \hat{z}^i}(0; 0) = \delta_{k,i}.
\]

So, we have
\[
\frac{\partial}{\partial w} \left( g_{i\overline{j}} \right) (0; 0) = \frac{\partial z^k}{\partial \hat{z}^i} g_{k\overline{j}}(0; 0) + \frac{\partial g_{i\overline{j}}}{\partial w}(0; 0).
\]

Considering the equation \(\partial g_{i\overline{j}} / \partial w(0; 0) = 0\), we have
\[
\frac{\partial}{\partial w} \frac{\partial z^k}{\partial \hat{z}^i}(0; 0) = - \sum_j g^{k\overline{j}}(0; 0) \frac{\partial g_{i\overline{j}}}{\partial w}(0; 0).
\]

Thus, we have

**Lemma 3.5.** By the change of holomorphic coordinates \((\hat{z}; w)\) around \(p \in D\) defined by
\[
z^\alpha = \sum_{i=1}^{n-1} \hat{z}^i \left( \delta_{i,\alpha} - w \sum_{j=1}^{n-1} g^{\alpha\overline{j}}(0; 0) \frac{\partial g_{i\overline{j}}}{\partial w}(0; 0) \right) \quad (\alpha = 1, 2, ..., n - 1),
\]
we have
\[
\frac{\partial g_{i\overline{j}}}{\partial w}(0; 0) = 0. \quad (3.2)
\]

In particular, at \((\hat{z}; w) = (0; w)\), we have
\[
g_{i\overline{j}}(0; w) = g_{i\overline{j}}(0; 0) + O(|w|^2). \quad (3.3)
\]

Consequently, we obtain

**Proposition 3.6.** We can find a local trivialization of \(L_X\) and local holomorphic coordinates so that
\[
\varphi = O(|w|^2), \quad g_{i\overline{j}}(0; w) = g_{i\overline{j}}(0; 0) + O(|w|^2).
\]

at \((\hat{z}^1, ..., \hat{z}^{n-1}, w) = (0, ..., 0, w)\).

**Proof.** In new local coordinates above, we have
\[
\frac{\partial \varphi}{\partial \hat{z}^i} = \frac{\partial \varphi}{\partial z^i} \frac{\partial z^i}{\partial \hat{z}^i} + \frac{\partial \varphi}{\partial w} \frac{\partial w}{\partial \hat{z}^i},
\]
and
\[
\frac{\partial z^i}{\partial \hat{z}^i}(0; 0) = \delta_{i,j}, \quad \frac{\partial \varphi}{\partial z^i}(0; 0) = \frac{\partial \varphi}{\partial w}(0; 0) = 0.
\]

Thus, the proposition follows. \(\square\)

For simplicity, we write new local coordinates \((\hat{z}^1, ..., \hat{z}^{n-1}, w)\) by the same symbol \((z^1, ..., z^{n-1}, w)\).
3.3 Proof of Theorem 1.1

Recall that \( \omega_0 \) is written as

\[
\omega_0 = \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right) \left( \theta_X + \frac{\hat{S}_D}{n(n-1)} \sqrt{-1} \partial t \wedge \bar{\partial} t \right)
\]

and it is enough to show that

\[
S(\omega_0) = O(\|\sigma_D\|^2 + 2\hat{S}_D/n(n-1))
\]

as \( \sigma_D \to 0 \). First, we show Lemma 3.7.

The Ricci form of \( \omega_0 \) is given by

\[
\text{Ric}(\omega_0) = \text{Ric}(\theta_X) - \frac{\hat{S}_D}{n-1} \theta_X - \sqrt{-1} \partial \bar{\partial} \log \left( 1 + \frac{\hat{S}_D}{n(n-1)} \|\partial t\|_{\theta_X}^2 \right).
\]

Proof. To prove this lemma, it is enough to see the volume form of \( \omega_0 \). From the definition of \( \omega_0 \), we have

\[
\omega_0 = \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right) \left( \theta_X + \frac{\hat{S}_D}{n(n-1)} \sqrt{-1} \partial t \wedge \bar{\partial} t \right).
\]

So, the following identity

\[
\sqrt{-1} \partial t \wedge \bar{\partial} t \wedge \theta_X^{-1} = \frac{1}{n} \|\partial t\|_{\theta_X}^2 \theta_X^n
\]

implies that the volume form of \( \omega_0 \) is given by

\[
\omega_0^n = \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right) \left( 1 + \frac{\hat{S}_D}{n(n-1)} \|\partial t\|_{\theta_X}^2 \right) \theta_X^n.
\]

Recall that the Ricci form is given by \( \text{Ric}(\omega_0) = -\sqrt{-1} \partial \bar{\partial} \log \omega_0^n \). Thus, the lemma follows.

Thus, we easily have \( S(\omega_0) = O(\|\sigma_D\|^2 + 2\hat{S}_D/n(n-1)) \) as \( \sigma_D \to 0 \). Next, we compute the Ricci form of \( \theta_X \). Write

\[
\theta_X = \begin{bmatrix}
g_{1,1} & \cdots & g_{1,n-1} & g_{1,n} \\
\vdots & \ddots & \vdots & \vdots \\
g_{n-1,1} & \cdots & g_{n-1,n-1} & g_{n-1,n} \\
g_{w,1} & \cdots & g_{w,n-1} & g_{w,n}
\end{bmatrix} = \begin{bmatrix}
B & R \\
\bar{R} & W
\end{bmatrix}
\]

in the previous local holomorphic coordinates. Since Lemma 3.3 implies that \( \det \theta_X = \det B \det (W - \bar{R} B^{-1} R) \), we have

\[
\text{Ric}(\theta_X) = -\sqrt{-1} \partial \bar{\partial} \log \det B - \sqrt{-1} \partial \bar{\partial} \log (W - \bar{R} B^{-1} R).
\]
Recall the notation \((z^1, \ldots, z^{n-1}, w) = (z; w)\). Consider the expansion at \(w = 0\):
\[
\det B(z; w) = \det B(z; 0) + w \frac{\partial \det B}{\partial w} + w^2 \frac{\partial \det B}{\partial w} + O(|w|^2).
\]
Since the condition \((3.3)\) implies that
\[
\frac{\partial \det B}{\partial w} (0; w) = O(w),
\]
we have
\[
\det B(0; w) = \det B(0; 0) + O(|w|^2). \tag{3.4}
\]
Recall that
\[
\text{Ric}(\theta_D) = -\sqrt{-1} \sum_{i,j=1}^{n-1} \frac{\partial^2 \log \det B(z; 0)}{\partial z^i \partial \overline{z}^j} dz^i \wedge d\overline{z}^j,
\]
and \(S(\theta_D) = \text{tr}_{\theta_D} \text{Ric}(\theta_D) = \hat{S}_D\). By \((3.4)\),
\[
\text{Ric}(\theta_X) = \text{Ric}(\theta_D) + O(|w|^2) dz \wedge d\overline{z} - \sqrt{-1} \partial \overline{\partial} \log(W - \overline{R} B^{-1} R) + O(1) dw \wedge d\overline{z} + O(1) dz \wedge d\overline{w} + O(1) dw \wedge d\overline{w}
\]
at \((0; w)\). Here \(dz\) denote differential 1-forms in directions of \(D\). To prove Theorem 1.1, it is clearly enough to take the trace with respect to the metric
\[
\theta_X + \frac{\hat{S}_D}{n(n-1)} \sqrt{-1} \partial t \wedge \overline{\partial} t.
\]
For simplicity, set \(a := \frac{\hat{S}_D}{n(n-1)} > 0\). Since \(\partial t = \partial \varphi - dw/w\), the metric \(\theta_X + a \sqrt{-1} \partial t \wedge \overline{\partial} t\) can be written as
\[
\begin{bmatrix}
g_{1,\overline{1}} + a \varphi_1 \varphi_{\overline{1}} & \cdots & g_{1,\overline{n-1}} + a \varphi_1 \varphi_{n-1} & g_{1,\overline{n}} + a \varphi_1 (\varphi_{\overline{n}} - 1/w) \\
\vdots & \ddots & \vdots & \vdots \\
g_{n-1,\overline{1}} + a \varphi_{n-1} \varphi_{\overline{1}} & \cdots & g_{n-1,\overline{n-1}} + a \varphi_{n-1} \varphi_{n-1} & g_{n-1,\overline{n}} + a \varphi_{n-1} (\varphi_{\overline{n}} - 1/w) \\
g_{w,\overline{1}} + a (\varphi_w - 1/w) \varphi_{\overline{1}} & \cdots & g_{w,\overline{n-1}} + a (\varphi_w - 1/w) \varphi_{n-1} & g_{w,\overline{n}} + a (\varphi_w - 1/w) (\varphi_{\overline{n}} - 1/w)
\end{bmatrix},
\]
where \(\varphi_i\) denotes \(\partial \varphi / \partial z^i\). For simplicity, write the matrix above as
\[
\theta_X + a \sqrt{-1} \partial t \wedge \overline{\partial} t = \begin{bmatrix} E & F \\ G & H \end{bmatrix}.
\]
In order to take the trace of \(\text{Ric}(\omega_0)\) with respect to the metric \(\theta_X + a \sqrt{-1} \partial t \wedge \overline{\partial} t\), we compute the inverse matrix of this. Since we only consider \(S(\omega_0)\) near \(D\), \(H = O(|w|^{-2})\) as \(w \to 0\). By Lemma 3.4, we have
\[
\begin{bmatrix} E^{-1} + E^{-1} F S^{-1} G E^{-1} & -E^{-1} F S^{-1} \\ -S^{-1} G E^{-1} & S^{-1} \end{bmatrix},
\]
where $S := H - GE^{-1}F$. Since $S = O(|w|^2)$ as $w \to 0$, we get
\[
E^{-1}FS^{-1}GE^{-1}, E^{-1}FS^{-1}, S^{-1}GE^{-1}, S^{-1} = O(|w|^2).
\]
Thus, to compute the scalar curvature $S(\omega_0)$, it is enough to study the block $E^{-1} + E^{-1}FS^{-1}GE^{-1}$. In this case, by considering the expansion at $w = 0$, we can write
\[
E = B(0; 0) + J,
\]
where $J = O(|w|^2)$. So we have
\[
E^{-1} = (B(0; 0) + J)^{-1} = B(0; 0)^{-1}(I + JB(0; 0)^{-1})^{-1} = B(0; 0)^{-1}(I + \sum_{i>0}(-JB(0; 0)^{-1})^i) = B(0; 0)^{-1} + O(|w|^2).
\]
Consider the term
\[
-\sqrt{-1}\partial\bar{\partial} \log \left( 1 + \frac{\hat{S}_D}{n(n-1)} ||\partial t||^2_{\theta_X} \right),
\]
where
\[
||\partial t||^2_{\theta_X} = \sum_{i,j} g^{ij}\varphi_i\varphi_j + \sum_{a=1}^{n-1} \left( g^{a\bar{a}}\varphi_a(\varphi_a - 1/\bar{w}) + g^{w\bar{w}}(\varphi_w - 1/w)\varphi_w \right)
+ g^{w\bar{w}}(\varphi_w - 1/w)(\varphi_w - 1/\bar{w}).
\]
Note that $g^{w\bar{w}} = (W - \overline{R}B^{-1}R)^{-1}$. Thus, we have
\[
-\sqrt{-1}\partial\bar{\partial} \log(W - \overline{R}B^{-1}R) - \sqrt{-1}\partial\bar{\partial} \log \left( 1 + \frac{\hat{S}_D}{n(n-1)} ||\partial t||^2_{\theta_X} \right)
= -\sqrt{-1}\partial\bar{\partial} \log(1 + O(|w|^2)).
\]
Thus,
\[
||\sigma_D||^{-2a} S(\omega_0) = \text{tr}_{\theta_X + a\sqrt{-1}\partial\bar{\partial} \theta_D} \text{Ric}(\omega_0)
= \text{tr}_{\theta_X + a\sqrt{-1}\partial\bar{\partial} \theta_D} \left( \text{Ric}(\theta_D) - \frac{\hat{S}_D}{(n-1)}\theta_D \right) + O(|w|^2)
\]
as $w \to 0$. Therefore, Theorem 1.1 is proved.
\[\square\]

**Remark 3.8.** Roughly, we have proved that
\[
S(\omega_0) = C||\sigma_D||^{-2\frac{\hat{S}_D}{(n-1)} S(\theta_D) - \hat{S}_D + O(||\sigma_D||^2)}
\]
near $D$. Thus, in fact, $\theta_D$ is cscK if and only if $S(\omega_0)$ has a zero along $D$ of order $2 + 2\hat{S}_D/(n-1)$ in our construction.

**Remark 3.9.** In [5, p.176], if $\theta_D$ is a Ricci-positive Kähler-Einstein metric, the background Kähler metric $\omega_0$ can be chosen so that the Ricci potential of $\omega_0$ decays at a higher order by altering the Hermitian metric $h_X$ on $K_X^{-1/\alpha}$. In Theorem 1.1 we can prove that $S(\omega_0)$ decays at a higher order without altering the Hermitian metric $h_X$. 

4 Asymptotically conical geometry

Recall that the Kähler metric defined by

\[
\omega_0 = \frac{n(n-1)}{S_D} \sqrt{-1} \partial \bar{\partial} \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right)
\]

is complete on \( X \setminus D \). Set \( r(x) := d(x, x_0) \), where \( d \) is the distance function from some fixed point \( x_0 \in X \setminus D \) defined by \( \omega_0 \). Following [5], the Riemannian manifold \( (X \setminus D, \omega_0) \) is of asymptotically conical geometry which is the analytic framework in this paper.

**Definition 4.1.** A complete Riemannian metric \( g \) on an open manifold \( M \) of dimension \( m \) is said to be of \( C^{k,\alpha} \)-asymptotically conical geometry if for each point \( p \in M \) with distance \( r \) from a fixed point \( o \in M \), there exists a harmonic coordinate system \( x = (x^1, x^2, \cdots, x^m) \) centered at \( p \) which satisfies the following conditions:

- The coordinate \( x \) runs over a unit ball \( B^m_p \subset \mathbb{R}^m \).
- If we write \( g = \sum g_{i,j}(x) dx^i dx^j \), then the matrix \( (r^2 + 1)^{-1} g_{i,j}(x) \) is bounded from below by a constant positive matrix independent of \( p \).
- The \( C^{k,\alpha} \)-norms of \( (r^2 + 1)^{-1} g_{i,j}(x) \) are uniformly bounded.

In particular, we simply say that \( (M, g) \) is of asymptotically conical geometry if \( (M, g) \) is of \( C^{k,\alpha} \)-asymptotically conical geometry for any \( k \in \mathbb{Z}_{\geq 0} \) and \( \alpha \in (0, 1) \).

**Definition 4.2.** Assume that a Riemannian manifold \( (M, g) \) is of asymptotically conical geometry. The \( C^{k,\alpha} \)-norm of a function \( u \) of weight \( \delta \in \mathbb{R} \) is defined by

\[
||u||_{C^{k,\alpha}_\delta} := \sup_{p \in M} (r(p)^2 + 1)^{\delta/2} ||u||_{C^{k,\alpha}(B^m_p)}.
\]

The Banach space \( C^{k,\alpha}_\delta \) is defined by the set of functions \( u \) such that \( ||u||_{C^{k,\alpha}_\delta} < \infty \). In the above definition, we use the coordinates \( x \in B^m_p \) centered at \( p \) with \( d(o, p) = r \) in the definition of the asymptotically conicalness.

5 Forth order elliptic linear operators

To prove Theorem 1.3, we study the linearization of the scalar curvature operator. For a smooth function \( \varphi \) on \( X \setminus D \), set \( \omega_t := \omega_0 + t \sqrt{-1} \partial \bar{\partial} \varphi \). Recall that \( S(\omega_t) = g^{ij} R_{t,ij} \). Thus, the linearization of the scalar curvature operator is defined by

\[
L_{\omega_0}(\varphi) := \left. \frac{d}{dt} \right|_{t=0} S(\omega_t) = -\Delta^2_{\omega_0} \varphi - g^{ij} \varphi_{i,j} R_{j,i} = -\Delta^2_{\omega_0} \varphi - R_{j}^{i} \varphi_{i,j}.
\]

Set \( M := X \setminus D \). The following operator plays an important role in this paper.
Definition 5.1. The operator $D_{\omega_0}$ is defined by

$$D_{\omega_0} : C^{k,\alpha}_{\delta}(M, \mathbb{C}) \to C^{k-2,\alpha}_{\delta+2}(M, \Omega^{0,1}_M \otimes T^{1,0}M)$$

$\varphi \mapsto \overline{\partial}(\nabla^{1,0}\varphi)$

Here $\overline{\partial}$ is the (0,1)-part of the Levi-Civita connection and $\nabla^{1,0}$ is the (1,0)-gradient with respect to $\omega_0$. We call $D_{\omega_0}^*D_{\omega_0}$ the Lichnerowicz operator and we have

Lemma 5.2. The Lichnerowicz operator $D_{\omega_0}^*D_{\omega_0}$ satisfies

$$D_{\omega_0}^*D_{\omega_0}\varphi = \Delta^2_{\omega_0}\varphi + R^{i\bar{j}}\varphi_{i\bar{j}} + (\nabla^{1,0}\varphi, \nabla^{0,1}S(\omega_0))_{\omega_0}. \quad (5.1)$$

Thus, we have

$$L_{\omega_0} = -D_{\omega_0}^*D_{\omega_0} + (\nabla^{1,0}, \nabla^{0,1}S(\omega_0))_{\omega_0}.$$

The idea of proving Theorem 1.4 follows from Arezzo-Pacard [3] and [4] (see also [12]).

Consider the following expansion:

$$S(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi) = S(\omega_0) + L_{\omega_0}(\phi) + Q_{\omega_0}(\phi).$$

To solve the following equation:

$$S(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi) = 0,$$

we will find a following fixed point:

$$\phi = -L_{\omega_0}^{-1}(S(\omega_0) + Q_{\omega_0}(\phi)).$$

When we prove Theorem 1.4, we assume that $L_{\omega_0}$ is invertible. Therefore we need to prove that the operator

$$N(\phi) := -L_{\omega_0}^{-1}(S(\omega_0) + Q_{\omega_0}(\phi)) \quad (5.2)$$

is a contraction on some Banach space.

In particular, we mainly use the weighted Banach spaces $C_{\delta-4}^{4,\alpha}(X \setminus D)$ and $C_{\delta}^{0,\alpha}(X \setminus D)$. From the definition of the weighted Banach space and local formulae of these operators, we easily have

Lemma 5.3. Following three operators

$$L_{\omega_0}, \ D_{\omega_0}^*D_{\omega_0}, \ \Delta^2_{\omega_0} : C_{\delta-4}^{4,\alpha}(X \setminus D) \to C_{\delta}^{0,\alpha}(X \setminus D)$$

are bounded.

First, we study the square of the Laplacian operator $\Delta^2_{\omega_0}$. Define a barrier function $\rho$ on $X \setminus D$ by

$$\rho := \exp \left( \frac{-\hat{S}_D}{2n(n-1)} t \right) = ||\sigma_D||^{-\hat{S}_D/n(n-1)}.$$
Note that for $\delta > 0$, $\rho$ satisfies
\[
\Delta \omega_0 \rho^{-\delta} = \text{tr} \omega_0 \sqrt{-1} \partial \bar{\partial} \exp \left( \frac{-\delta \hat{S}_D}{2n(n-1)} t \right)
\]
\[
= \frac{-\delta \hat{S}_D}{2n(n-1)} \text{tr} \omega_0 \left( \exp \left( \frac{-\delta \hat{S}_D}{2n(n-1)} t \right) \left( \sqrt{-1} \partial \bar{\partial} t + \frac{-\delta \hat{S}_D}{2n(n-1)} \sqrt{-1} dt \wedge \partial t \right) \right)
\]
\[
= \frac{-\delta \hat{S}_D}{2n(n-1)} \rho^{-\delta-2} \text{tr} \omega_0 \left( \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right) \left( \sqrt{-1} \partial \bar{\partial} t + \frac{-\delta \hat{S}_D}{2n(n-1)} \sqrt{-1} dt \wedge \partial t \right) \right)
\]
\[
= \frac{-\delta \hat{S}_D}{2n(n-1)} \rho^{-\delta-2} \left( \omega_0 + \frac{-(\delta + 2) \hat{S}_D}{2n(n-1)} \text{exp} \left( \frac{\hat{S}_D}{n(n-1)} t \right) \sqrt{-1} dt \wedge \partial t \right)
\]
\[
= \frac{-\delta \hat{S}_D}{2n(n-1)} \rho^{-\delta-2} \left( n + \text{tr} \omega_0 \left( \frac{-(\delta + 2) \hat{S}_D}{2n(n-1)} \text{exp} \left( \frac{\hat{S}_D}{n(n-1)} t \right) \sqrt{-1} dt \wedge \partial t \right) \right)
\]
\[
\leq \frac{-\delta \hat{S}_D}{2n(n-1)} \rho^{-\delta-2} \left( n - \frac{\delta + 2}{2} \right)
\]

Here we have used the following inequality:
\[
\omega_0 \geq \frac{\hat{S}_D}{n(n-1)} \text{exp} \left( \frac{\hat{S}_D}{n(n-1)} t \right) \sqrt{-1} dt \wedge \partial t.
\]

From Lemma 3.2 we have known that the volume growth of $\omega_0$ is given by
\[
\operatorname{Vol}_{\omega_0}(B(x_0, r)) = O(r^{2n}).
\]

In addition, we have known that $||\text{Ric}(\omega_0)||_{\omega_0} = O(r^{-2})$ as $r \to \infty$. From [8 Theorem 1.2], we have

**Lemma 5.4.** Set $\gamma := n/(n-1)$. Then the following Sobolev inequality holds, i.e., there exists a constant $C > 0$ such that
\[
\left( \int_{X \setminus D} |v|^2 \gamma \omega_0^n \right)^{1/\gamma} \leq C \int_{X \setminus D} |\partial v|^2 \omega_0^n
\]
for any compactly supported smooth function $v$ on $X \setminus D$.

Then, we can apply the Moser’s iteration to obtain the $C^0$-estimate. Following [5 p.178], we have

**Lemma 5.5.** If $2 < \delta < 2n$, the Laplacian $\Delta_{\omega_0} : C^{k-2,\alpha}_{\delta}(X \setminus D) \to C^{k-2,\alpha}_{\delta}(X \setminus D)$ is isomorphic.

Recall that the standard theorem on Banach spaces (see [14 p.77]).

**Theorem 5.6.** Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Assume that $L : \mathcal{X} \to \mathcal{Y}$ is a bounded and isomorphic linear operator. Then, the inverse $L^{-1}$ is also bounded.
Thus, the inverse of the Laplacian $\Delta^{-1}_{\omega_0}$ is bounded. In addition, recall the definition of Fredholm operators (see [7, Chapter 1, §1.4]):

**Definition 5.7.** We say that a bounded linear operator $L : \mathcal{X} \to \mathcal{Y}$ between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is a **Fredholm operator** if the $\dim(\text{Ker}L)$ and $\dim(\text{Coker}L)$ are finite and $\text{Im}L$ is a closed linear subspace of $\mathcal{Y}$. For such an operator $L$, we define an **index** of $L$ by

$$\text{ind}(L) := \dim(\text{Ker}L) - \dim(\text{Coker}L).$$

Thus, immediately we obtain

**Lemma 5.8.** If $2 < \delta < 2n$, the Laplacian $\Delta^{-1}_{\omega_0} : C^{k,\alpha}_{\delta-2}(X \setminus D) \to C^{k-2,\alpha}_{\delta}(X \setminus D)$ is a Fredholm operator whose index $\text{ind}(\Delta^{-1}_{\omega_0})$ is zero. Moreover, there exists a bounded inverse $\Delta^{-1}_{\omega_0}$ which is also a Fredholm operator whose index is zero.

Next, we study the operator $D^*_{\omega_0} D_{\omega_0}$.

**Lemma 5.9.** Assume that $\delta > 4$ and there is no nonzero holomorphic vector field on $X$ which vanishes on $D$. Then, the operator $D^*_{\omega_0} D_{\omega_0} : C^{4,\alpha}_{\delta-4}(X \setminus D) \to C^{0,\alpha}_{\delta}(X \setminus D)$ is injective.

**Proof.** Assume that $\phi \in C^{4,\alpha}_{\delta-4}(X \setminus D)$ satisfies $D^*_{\omega_0} D_{\omega_0} \phi = 0$. Integrating by parts, we have

$$0 = \int_{X \setminus D} \phi D^*_{\omega_0} D_{\omega_0} \phi \omega_0^n = \int_{X \setminus D} |D_{\omega_0} \phi|^2 \omega_0^n.$$  

Since $\overline{\partial} \nabla^{1,0} \phi = D_{\omega_0} \phi = 0$, $\nabla^{1,0} \phi$ is a holomorphic vector field on $X \setminus D$. By writing locally $\omega_0 = \sqrt{-1} g_{i\overline{j}} dz^i \wedge d\overline{z}^j$, the $(1,0)$-gradient of $\phi$ can be written as

$$\nabla^{1,0} \phi = g^{i\overline{j}} \frac{\partial \phi}{\partial \overline{z}^j} \frac{\partial}{\partial z^i}.$$  

So, all coefficients $g^{i\overline{j}} \partial \phi/\partial \overline{z}^j$ are holomorphic. Moreover, the definition of $\phi$ and the asymptotically conicalness imply differentials of $\phi$ and factors $g^{i\overline{j}}$ decay near $D$. Thus, $\nabla^{1,0} \phi$ can be extended holomorphically to $X$ and vanishes on $D$. The hypothesis implies that $\phi$ is constant. Since $\phi$ decays near $D$, we have $\phi = 0$ and conclude that $D^*_{\omega_0} D_{\omega_0}$ is injective. \qed

Recall the following fundamental fact (see [7, Chapter 1, §1.4])

**Theorem 5.10.** Let $L : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. Then, $L$ is Fredholm if and only if there exists a bounded linear operator $H : \mathcal{Y} \to \mathcal{X}$ such that operators $I_\mathcal{X} - H \circ L$ and $I_\mathcal{Y} - L \circ H$ are compact. Moreover, $H$ is also Fredholm and satisfies

$$\text{ind}(L) = -\text{ind}(H).$$

Then, we can show the following.
Lemma 5.11. If $4 < \delta < 2n$, the operator $D_{\omega_0}^* D_{\omega_0} : C^{k,\alpha}_{\delta-4}(X \setminus D) \to C^{k-4,\alpha}_{\delta}(X \setminus D)$ is a Fredholm operator whose index $\text{ind}(D_{\omega_0}^* D_{\omega_0})$ is zero.

Proof. Recall the equation

$$D_{\omega_0}^* D_{\omega_0} \phi = \Delta_{\omega_0}^2 \phi + R^{i\overline{j}} \phi_{i\overline{j}} + (\nabla^{1,0} \phi, \nabla^{0,1} S(\omega_0))_{\omega_0}.$$ 

Since $(\Delta_{\omega_0}^2)^{-1} : C^{k-4,\alpha}_{\delta}(X \setminus D) \to C^{k,\alpha}_{\delta-4}(X \setminus D)$ is bounded, it is continuous. Consider the linear operator

$$I_{C^{k,\alpha}_{\delta-4}(X \setminus D)} - (\Delta_{\omega_0}^2)^{-1} \circ D_{\omega_0}^* D_{\omega_0}.$$ 

From the equation above, we obtain

$$\left( I_{C^{k,\alpha}_{\delta-4}(X \setminus D)} - (\Delta_{\omega_0}^2)^{-1} \circ D_{\omega_0}^* D_{\omega_0} \right) \phi = (\Delta_{\omega_0}^2)^{-1} (R^{i\overline{j}} \phi_{i\overline{j}} + (\nabla^{1,0} \phi, \nabla^{0,1} S(\omega_0))_{\omega_0})$$

for any $\phi \in C^{k,\alpha}_{\delta-4}$. Since $\phi \in C^{k,\alpha}_{\delta-4}(X \setminus D)$, the Arzela-Ascoli theorem implies that the operator

$$\phi \to R^{i\overline{j}} \phi_{i\overline{j}} + (\nabla^{1,0} \phi, \nabla^{0,1} S(\omega_0))_{\omega_0}$$

is compact. The fact that $(\Delta_{\omega_0}^2)^{-1}$ is continuous implies that $I_{C^{k,\alpha}_{\delta-4}(X \setminus D)} - (\Delta_{\omega_0}^2)^{-1} \circ D_{\omega_0}^* D_{\omega_0}$ is also compact. Similarly, we obtain the compactness of the operator

$$I_{C^{k,\alpha}_{\delta-4}(X \setminus D)} - D_{\omega_0}^* D_{\omega_0} \circ (\Delta_{\omega_0}^2)^{-1}.$$ 

From Theorem 5.10, we have finished the proof. \hfill \Box

Then, Lemma 5.9 and Lemma 5.11 imply that

Proposition 5.12. If $4 < \delta < 2n$ and there is no nonzero holomorphic vector field on $X$ which vanishes on $D$, the operator $D_{\omega_0}^* D_{\omega_0} : C^{k,\alpha}_{\delta-4}(X \setminus D) \to C^{k-4,\alpha}_{\delta}(X \setminus D)$ is isomorphic and has a bounded inverse.

Thus, there exists $K > 0$ such that

$$\|(D_{\omega_0}^* D_{\omega_0})^{-1}\|_{C^{k,\alpha}_{\delta-4} \to C^{k-4,\alpha}_{\delta}} < K^{-1}.$$ 

In the next section, we study the operator $L_{\omega_0} = -D_{\omega_0}^* D_{\omega_0} + (\nabla^{1,0}, \nabla^{0,1} S(\omega_0))_{\omega_0}$.

Remark 5.13. We can show that if the $C^{1,\alpha}_2$-norm of $S(\omega_0)$ is sufficiently small, there exists the bounded inverse of $L_{\omega_0}$ satisfying

$$\|L_{\omega_0}^{-1}\|_{C^{0,\alpha}_{\delta} \to C^{4,\alpha}_{\delta}} < \hat{K}^{-1}$$

for some $\hat{K} > 0$ (Condition 1.2).
6 Proof of Theorem 1.4

This section also follows from Arezzo-Pacard [3], [4] (see also [12]). Since we assume that

\[ 0 < \hat{S}_D < n(n-1), \]

we can choose a weight \( \delta \) so that

\[ \delta \in (4, \min\{2n, 2 + 2n(n-1)/\hat{S}_D\}) \quad (6.1) \]

Note that if \( \theta_D \) is cscK, Theorem 1.1 implies that

\[ S(\omega_0) = O(r^{-\delta}). \]

In addition, from Lemma 3.1, we can choose \( \delta \) sufficiently close to \( \min\{2n, 2 + 2n(n-1)/\hat{S}_D\} \) so that a function

\[ \phi \] is integrable for \( \phi \in C^{4,\alpha}_{\delta-4} \) with respect to the volume form \( \omega_0^n \). Hereafter, we fix a weight \( \delta \) satisfying (6.1) and (6.2).

Remark 6.1. If \((D, L_D) = (\mathbb{P}^{n-1}, O(1))\), the equality above holds, i.e., \( \hat{S}_D = n(n-1) \).

We will show that the operator \( \mathcal{N} : C^{4,\alpha}_{\delta-4}(X \setminus D) \to C^{4,\alpha}_{\delta-4}(X \setminus D) \) defined in (5.2) has a fixed point under Condition 1.2 and Condition 1.3. First, we have

Lemma 6.2. There exists \( c_0 > 0 \) depending only on \( \omega_0 \) such that if \( ||\phi||_{C^{4,\alpha}_{\delta-4}(X \setminus D)} \leq c_0 \), we have

\[ ||L_{\omega_\phi} - L_{\omega_0}||_{C^{4,\alpha}_{\delta-4} \to C^{4,\alpha}_{\delta-4}} \leq \hat{K}/2 \]

and \( \omega_\phi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi \) is positive.

Proof. Note that

\[ g_\phi^{-1} - g^{-1} = g_\phi^{-1}(g - g_\phi)g^{-1} \quad (6.3) \]

for \( \phi \) such that \( \omega_\phi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi \) is positive.

Locally, we can write as

\[ L_{\omega_\phi} \psi = -\Delta^2_{\omega_\phi} \psi - R^\bar{\nu}_{\omega_\phi} \psi_{\bar{\nu}j}. \quad (6.4) \]

For instance, we have

\[ ||(r^2 + 1)^{\delta/2}(g_\phi^{i\bar{j}} g_\phi^{k\bar{l}} - g^{i\bar{j}} g^{k\bar{l}})\psi_{i\bar{j}k\bar{l}}||_{C^{0,\alpha}} \leq ||(r^2 + 1)^{4/2}(g_\phi^{i\bar{j}} g_\phi^{k\bar{l}} - g^{i\bar{j}} g^{k\bar{l}})||_{C^{0,\alpha}} ||\psi||_{C^{4,\alpha}_{\delta-4}(X \setminus D)} \]

\[ = ||(r^2 + 1)^{4/2}(g_\phi^{i\bar{j}} g_\phi^{k\bar{l}} - g^{i\bar{j}} g^{k\bar{l}}) + (g_\phi^{i\bar{j}} - g^{i\bar{j}})g^{k\bar{l}}||_{C^{0,\alpha}} ||\psi||_{C^{4,\alpha}_{\delta-4}(X \setminus D)}. \]

Note that \( g_\phi^{i\bar{j}} = O(r^{-2}) \) as \( r \to \infty \), if \( c_0 \) is sufficiently small. Thus, if \( c_0 \) is sufficiently small, the equation (6.3) implies that the term above can be made small arbitrarily. Applying the same argument to remainders in (6.4), we can use the asymptotically conicalness to obtain the desired result.
To show that the operator $N: C^{4,\alpha}_{\delta-4} \to C^{4,\alpha}_{\delta-4}$ is a contraction, we need the following lemma:

**Lemma 6.3.** Assume that

$$||\phi||_{C^{4,\alpha}_{\delta-4}}, ||\psi||_{C^{4,\alpha}_{\delta-4}} \leq c_0.$$  

Then, we have

$$||N(\phi) - N(\psi)||_{C^{4,\alpha}_{\delta-4}} \leq \frac{1}{2} ||\phi - \psi||_{C^{4,\alpha}_{\delta-4}}.$$  

**Proof.** Since the operator $N$ is defined by

$$N(\phi) := -L^{-1}_{\omega_0}(S(\omega_0) + Q_{\omega_0}(\phi)),$$

we have

$$N(\phi) - N(\psi) = -L^{-1}_{\omega_0}(Q_{\omega_0}(\phi) - Q_{\omega_0}(\psi)).$$

The mean value theorem implies that there exists $\chi = t\phi + (1-t)\psi$ for $t \in [0,1]$ such that

$$DQ_{\omega_0,\chi}(\phi - \psi) = Q_{\omega_0}(\phi) - Q_{\omega_0}(\psi),$$

and the direct computation implies that

$$DQ_{\omega_0,\chi} = L_{\omega_\chi} - L_{\omega_0}.$$  

We know that $||\phi||_{C^{4,\alpha}_{\delta-2}} \leq ||\phi||_{C^{4,\alpha}_{\delta-4}} \leq c_0$. Using Lemma 6.2, we finish the proof.  

The following Proposition implies that the existence of a complete scalar-flat Kähler metric.

**Proposition 6.4.** Set

$$U := \left\{ \phi \in C^{4,\alpha}_{\delta-4} : ||\phi||_{C^{4,\alpha}_{\delta-4}} \leq c_0 \right\}.$$  

If Condition 1.2 and Condition 1.3 hold, the operator $N$ is a contraction on $U$ and $N(U) \subset U$.

**Proof.** By the condition $||\phi||_{C^{4,\alpha}_{\delta-2}} \leq ||\phi||_{C^{4,\alpha}_{\delta-4}} \leq c_0$, Lemma 6.2 and Lemma 6.3 obviously imply that $||\phi||_{C^{4,\alpha}_{\delta-2}} \leq c_0, N$ is a contraction on $U$. Immediately, we have

$$||N(\phi)||_{C^{4,\alpha}_{\delta-4}} \leq ||N(\phi) - N(0)||_{C^{4,\alpha}_{\delta-4}} + ||N(0)||_{C^{4,\alpha}_{\delta-4}}.$$  

From the previous lemma, we obtain:

$$||N(\phi) - N(0)||_{C^{4,\alpha}_{\delta-4}} \leq \frac{1}{2} c_0.$$  

The hypothesis in this proposition implies that

$$||N(0)||_{C^{4,\alpha}_{\delta-4}} \leq K^{-1} ||S(\omega_0)||_{C^{4,\alpha}_{\delta}} \leq \frac{1}{2} c_0.$$  

Thus, $N(\phi) \in U$.  

**Proof of Theorem 1.4.** If Condition 1.2 and Condition 1.3 hold, Proposition 6.4 implies that there is a unique $\phi_\infty := \lim_{i \to \infty} N^i(\phi)$ for any $\phi \in U \subset C^{4,\alpha}_{\delta-4}$ satisfying $\phi_\infty = N(\phi_\infty)$. Therefore, $\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_\infty$ is a complete scalar-flat Kähler metric on $X \setminus D$.  

□
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