The Kondo crossover in shot noise of a single quantum dot with orbital degeneracy

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I. INTRODUCTION

Kondo physics has been studied since the 1960s for dilute magnetic alloys and heavy fermion systems. Advancements in nanofabrication techniques in these years have made it possible to experimentally achieve the nonequilibrium Kondo state in quantum dots by applying a bias voltage, opening a new paradigm of Kondo physics. In the early days research on the nonequilibrium Kondo effect has focused on the time averaged current or linear conductance. Recently, the shot noise associated with the Kondo effect has attracted the attention of the researchers in this field.

Shot noise measurements in mesoscopic devices provide important information about the effective charge \( e^* \) of current-carrying particles. For instance, the fractional charge \( e^* = e/3 \) of a fractional quantum Hall system has been clarified through the shot noise measurement, and also the charge \( e^* = 2e \) of the Cooper-pair has been observed in normal metal/superconductor junctions. Recently, for other correlated electron systems, the shot noise has become an important probe to study the properties of the low-energy excitations. In quantum dot systems, the observation of a fractional enhancement of 5\( e/3 \) for the backscattering current in a symmetric barrier has stimulated recent studies of the shot noise in the Kondo regime. Note that the backscattering current is an essential quantity for observing the shot noise, and is defined by the deviation from the value of the linear current in the unitarity limit.

Theories for the shot noise in Kondo systems have been extended, naturally, to another class of exotic Kondo systems: the so-called orbital Kondo effect which has been observed in experiments. In the case of multiple quantum dots or a single dot with a symmetrical shape, the system can have orbital degeneracy. The orbital degrees of freedom have been expected to affect significantly the low-energy properties, which can still be described by the local Fermi-liquid theory. The shot noise for systems with orbital degeneracy has been investigated in the Kondo limit, where the local charge degrees of freedom in the dot site are quenched. These studies have provided the value of the Fano factor of the backscattering current which depends on the orbital degeneracy \( N \), and have inspired experimental noise measurements in a single-wall carbon nanotube.

There have been some qualitative discussions based on the Anderson model that capture the physics away from the Kondo limit through the local charge degrees of freedom that remain active for finite on-site Coulomb repulsions. The approximations used, however, were not applicable to low energies, and reliable results in the low-temperature Fermi-liquid regime are desired. In a previous work, Fujii has derived the expression of the Fano factor for the single orbital Anderson model \( (N = 2) \) in the particle-hole symmetric case using the renormalized perturbation theory (RPT). The RPT is an approach that starts with the Fermi-liquid ground state, and gives exact asymptotic behavior of the correlation functions at low frequencies, low temperatures, and low bias voltages. Therefore, Fujii’s result for the Fano factor \( F_b \) is asymptotically exact at low energies, and is expressed in terms of a single parameter as \( F_b = \frac{1 + 9 (R-1)^2}{1 + 9 (R-1)^2} \). The Wilson ratio that determines the universal Kondo behavior for all values of the Coulomb repulsion \( 0 \leq U < \infty \).

The exact result for the Fano factor for the orbital Kondo system with \( N \) orbitals has already been given for the Kondo limit \( (U \rightarrow \infty) \), as mentioned above. Away from the Kondo regime, however, it still has not been clarified as to how the Fano factor varies with the value...
of the Coulomb repulsion. The purpose of the present paper is to provide the exact low-energy expression of the Fano factor for arbitrary \(N\) and \(R\), on the basis of the multiorbital impurity Anderson model. To this end, following the derivation in the single orbital case, we use the RPT and consider the particle-hole symmetric case. Specifically, we start with a general formulation for the shot noise properties originating from the Kondo correlations and give the expression of the Fano factor in terms of the Wilson ratio \(R\) and the orbital degeneracy \(N\). Equations (28) and (30) are the main results of the paper. We also calculate the dependence of the Fano factor \(U\), and for intermediate \(U\) the Fano factor varies rapidly near \(U\), and for large \(U\) the Fano factor is \(\frac{U}{N+4}\) in the Kondo limit, as the crossover from the weak coupling regime to the Kondo regime takes place. Here, \(\Gamma\) is the linewidth of the dot levels owing to the coupling to the leads.

This paper is organized as follows. In Sec. II, we introduce a generalized impurity Anderson model for the multiorbital quantum dot system, and give a brief explanation for the RPT to describe the low-energy states. In Sec. III, we describe the derivation of the exact expression for the current and the shot noise at low energies. Then, we discuss the \(U\) dependence of the Fano factor calculated with the NRG. A brief discussion and summary are given in Sec. IV.

II. MODEL AND CALCULATION

A. Multiorbital impurity Anderson model

Let us consider a single quantum dot system with \(N\)-degenerate orbitals. We assume the orbital conservation in the tunneling process between the dot and the leads in the manner shown in Fig. 1. This assumption has been experimentally confirmed to be reasonable for vertical quantum dot and carbon nanotube quantum dot systems. In these assumptions, our system can be described by a multiorbital version of the impurity Anderson model,

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1, \tag{1}
\]

\[
\mathcal{H}_0 = \sum_{kam} \epsilon_{kam} c_{kam}^\dagger c_{kam} + \sum_m \epsilon_m d_m^\dagger d_m + \sum_{kam} \left(V_{kam} c_{kam}^\dagger d_m + \text{h.c.}\right), \tag{2}
\]

\[
\mathcal{H}_1 = \frac{1}{2} \sum_{m \neq m'} U d_m^\dagger d_m d_{m'}^\dagger d_{m'}. \tag{3}
\]

where \(d_m(d_m^\dagger)\) annihilates (creates) an electron in the dot level \(\epsilon_m\) with state \(m\), \(c_{kam}(c_{kam}^\dagger)\) annihilates (creates) a conduction electron with momentum \(k\) and state \(m\) in the lead \(\alpha = L, R\), and \(U\) is the on-site Coulomb repulsion in the quantum dot. Here, \(m\) indicates for the \(N\)-fold orbital state \((m = 1, 2, \cdots, N)\). To investigate the particle-hole symmetric Kondo effect, the dot level is chosen to be \(\epsilon_m = -(N-1)U/2\) and the degeneracy \(N\) to be an even number. The intrinsic linewidth of the dot levels owing to tunnel coupling \(V_{kam}\) is \(\Gamma(\omega) = (\Gamma_L(\omega) + \Gamma_R(\omega))/2\) with \(\Gamma_\alpha(\omega) = 2\pi \sum_k \delta(\omega - \epsilon_{kam})|V_{kam}|^2\), which is assumed to be independent of the orbital state \(m\). For conduction electrons, without any prominent features, \(\Gamma_\alpha(\omega)\) does not have a strong dependence on \(\omega\), so it is usual to take the case of a wide conduction band with a flat density of states limit, where \(\Gamma_\alpha(\omega)\) can be taken as a constant \(\Gamma_\alpha\). It has been shown that an asymmetric barrier gives rise to a non-universal shift in the fractional enhancement for the backscattering current. To discuss the shot noise properties originating from the Kondo correlations, the lead-dot couplings are assumed to be symmetric: \(\Gamma_l = \Gamma_R\). The chemical potentials \(\mu_L = \mu_R = \pm eV/2\), satisfying \(\mu_L - \mu_R = eV\), are measured relative to the Fermi level which is defined at zero voltage \(V = 0\) such that \(\mu_L = \mu_R = 0\).

B. Fermi-liquid description for low-energy states

We make use of the renormalized perturbation theory, which has been successfully applied to the impurity Anderson model in equilibrium, and later extended to low-bias steady states. In this section, we outline the approach to the multiorbital impurity Anderson model described by Eq. (1). Furthermore, we provide explicit expressions for the renormalized parameters for small \(U\) obtained exactly up to three-loop contributions for general \(N\).

The three basic parameters that specify the model are the energy level of the dot state \(\epsilon_{dm}\), the linewidth parameter \(\Gamma\), and the Coulomb repulsion \(U\) in the dot. The low-energy properties of the system can be characterized by the quasiparticles with the renormalized parameters,
defined by
\[
\tilde{\epsilon}_{dm} = z(\epsilon_{dm} + \Sigma_{dm}^{r}\{0\}) ,
\]
\[
\bar{\Gamma} = z\Gamma ,
\]
\[
\bar{U} = z^{2}\Gamma^{(4)}_{mm}\{0,0,0,0\} \ (m \neq m') ,
\]
where \(\Sigma_{dm}^{r}(\omega)\) is the self-energy of the retarded Green function for the dot state: \(G_{dm}^{r}(\omega) = [\omega - \epsilon_{dm} + i\Gamma - \Sigma_{dm}^{r}(\omega)]^{-1}\), \(\tilde{\epsilon}_{dm} = [1 - \partial \Sigma_{dm}^{r}(\omega)/\partial \omega|_{\omega=0}]^{-1}\) is the wave function renormalization factor, and \(\Gamma^{(4)}_{mm,m'm'}(\omega_{1},\omega_{2},\omega_{3},\omega_{4})\) is the local full four-point vertex function for the scattering of the electrons with the orbital \(m\) and \(m'\). The perturbation theory in powers of \(U\) can be reorganized as an expansion with respect to the renormalized interaction \(\bar{U}\), taking the free quasiparticle Green’s function \(\tilde{g}_{dm}^{r}(\omega) = (\omega - \epsilon_{dm} + i\bar{\Gamma})^{-1}\) as the zero-order propagator. The three counter terms have to be included to prevent any further renormalization of the parameters, \(\tilde{\epsilon}_{dm}, \bar{\Gamma}, \) and \(\bar{U}\). Then, the low energy properties can be specified entirely in terms of the three renormalized parameters. These procedures are the basis of the renormalized perturbation for the impurity Anderson model, presented by Hewson.

At low bias voltages described by the Fermi-liquid theory, there are explicit relations between the renormalized parameters and the susceptibilities,
\[
\bar{U}/\pi\bar{\Gamma} = \frac{R - 1}{\sin^{2}(\pi n_{dm})} ,
\]
\[
\bar{\Gamma} = \frac{N}{(N-1)\chi_{d}^{*} + \chi_{c,d}^{*}}\bar{\Gamma} ,
\]
\[
\tilde{\epsilon}_{d} = \bar{\Gamma} \cot(\pi n_{dm}) .
\]

Here, \(n_{dm}\) is the average number of electrons in the dot-site, \(R\) is the Wilson ratio defined by
\[
R \equiv \frac{\chi_{c,d}^{*}}{\gamma_{d}^{*}} = \frac{N}{(N-1) + \chi_{c,d}^{*}/\chi_{d}^{*}} ,
\]
where \(\chi_{c,d}^{*} = 1 - \partial \Sigma_{dm}^{r}(\omega)/\partial h|_{\omega=0,h=0}\) with the Zeeman energy at the dot-site \(h\) is the enhancement factor for the dot susceptibility, \(\gamma_{d}^{*} = 1 + \partial \Sigma_{dm}^{r}(\omega)/\partial \epsilon_{d}|_{\omega=0}\) is the enhancement factor for the charge susceptibility, and \(\gamma_{d} = z^{-1}\) is the enhancement factor for the \(T\)-linear specific heat coefficient.

In the particle-hole symmetric case, the electron filling is given by \(n_{dm} = 1/2\), and the relations above can be simplified as,
\[
\frac{\bar{U}}{\pi\bar{\Gamma}} = R - 1 , \quad \tilde{\epsilon}_{dm} = 0 .
\]

In the Kondo limit \((U \to \infty)\), the charge susceptibility is suppressed \(\chi_{c,d}^{*} \to 0\), and \(\bar{\Gamma}\) can be considered as the Kondo temperature as, \(T_{K} = \pi\bar{\Gamma}/4\). Therefore, Eq. (10) shows that the Wilson ratio converges to \(R \to N/(N-1)\) in the Kondo limit. In the noninteracting case \((U = 0)\), the parameters take the value \(\chi_{d}^{*} = \chi_{c,d}^{*} = \gamma_{d}^{*} = 1\) by definition.

The behavior of the renormalized parameters for small \(U\) can be clarified with the perturbation approach. We have calculated the enhancement factor \(\gamma_{d}^{*}\) and the four-point vertex for \(m \neq m'\) in the particle-hole symmetric case, extending the calculations of Yamada-Yosida for \(N = 245-49\) to general \(N\),

\[
\gamma_{d}^{*} = 1 + \left(3 - \frac{\pi^{2}}{4}\right) u^{2} - \frac{21}{2}\zeta(3) - 7 - \frac{\pi^{2}}{2} \right) (N-1)(N-2) u^{3} + \mathcal{O}(u^{4}) ,
\]
\[
\frac{1}{\pi\Gamma^{(4)}_{mm,m'm'}}(0,0,0,0) = u - (N-2)u^{2} + \left[N^{2} + \left(1 - \frac{\pi^{2}}{2}\right) \right] N - \frac{\pi^{2}}{2} + 9 \right] u^{3}
\]
\[-(N-2) \left[ N^{2} + \left(21\zeta(3) - 7\pi^{2}/4 - 12\right) N + \frac{133}{2}\zeta(3) - 71/12\pi^{2} - 17 \right] u^{4} + \mathcal{O}(u^{5}) ,
\]
where \(u \equiv U/(\pi\bar{\Gamma})\) and \(\zeta(x)\) is the Riemann zeta function. Note that \(\gamma_{d}^{*}\) captures the terms of odd order in \(U\) for \(N > 2\). Correspondingly, \(\Gamma_{mm,m'm'}^{(4)}(0,0,0,0)\) have finite contributions from the terms of even order for \(N > 2\). The appearance of the zeta function, which is absent in the case of \(N = 2\), in the coefficients of the perturbation series is also caused by the orbital degeneracy. Substituting Eqs. (12) and (13) into Eqs. (5) and (6), and then through Eq. (11), we have obtained the Wilson ratio exactly up to terms of order \(U^{4}\),
\[
R = 1 + u - (N-2)u^{2} + \left[N^{2} - \left(2 + \frac{\pi^{2}}{4}\right) N - \frac{3}{4}\pi^{2} + 12 \right] u^{3}
\]
\[-(N-2) \left[ N^{2} + \left(21\zeta(3) - 8\pi^{2} - 3\right) N + 77\zeta(3) - \frac{20}{3}\pi^{2} - 21 \right] u^{4} + \mathcal{O}(u^{5}) .
\]

We note that \(R\) is an alternating series up to order \(U^{4}\) for \(N \geq 4\).
For intermediate values of $U$, the renormalized parameters can be calculated with the NRG approach in the case of $N = 2$ and 4, and the Bethe ansatz exact solution (BAE) in the case of $N = 2^{50-53}$.

III. RENORMALIZED PERTURBATION THEORY FOR NONEQUILIBRIUM TRANSPORT

Here, we derive the expression for the current and shot noise of the multiorbital impurity Anderson model using the second-order perturbation in the renormalized interaction $U$. The obtained expression covers not only the strong-coupling $^8,14,15,32,33$ and weak-coupling $^9,47$ but also the whole range of the Coulomb repulsion $U$, which is an extension of the previous SU(2) result $^{38,41}$ to general $N$. One of the significant advantages of the RPT is that the current and shot noise calculated up to the second order in $U$ are asymptotically exact at low energies, up to the third order in the bias voltage. From the results for the shot noise, we also derive the exact Fano factor of the backscattering current for low bias voltages.

A. Current

The symmetrized current operator $J$ across the quantum dot system described by Eq. (3) is written as,

$$J = \frac{J_L - J_R}{2},  \quad (15)$$

where,

$$J_\alpha = -\frac{e}{h} \sum_{km} \left( V_{km} c_{\alpha m}^\dagger d_{km} - V_{km}^* d_{km}^\dagger c_{\alpha m} \right),  \quad (16)$$

is the current operator for the electrons tunneling from lead $\alpha$ to the quantum dot with number operator for electrons in lead $\alpha$, $N_\alpha = \sum_{km} c_{\alpha m}^\dagger c_{\alpha m}$. A general formula for the time-averaged current $I$ owing to the applied voltage through a quantum dot has been given by Hershfield et al. $^{55,56}$ and Meir and Wingreen $^{58}$. The formula can be specialized for the symmetric lead-dot coupling case $^{42}$.

B. Shot noise

The current noise $S$ is defined by the correlation function for the current fluctuation,

$$S \equiv \int dt \left\langle \{\delta J(t), \delta J(0)\} \right\rangle,  \quad (20)$$

where the fluctuation operator for a quantity $A$ is given by $\delta A \equiv A - \langle A \rangle$, and $\{A, B\} \equiv AB + BA$ is anticommutator. Conventionally, the shot noise has been defined by the value of $S$ at zero temperature $T = 0$, where the thermal noise is completely suppressed. This definition of the shot noise has been successfully exploited for studying the properties at low temperatures $^{59}$. It is difficult, however, for interacting electron systems to separate the shot noise and thermal noise at finite temperatures.

On the basis of a generalized Kubo formalism, Fujii has shown that the shot noise can be related to a correlation function between the current fluctuation and the charge fluctuation,

$$S_h \equiv -\left\langle \{\delta J, e(\delta N_L - \delta N_R)\} \right\rangle,  \quad (21)$$

and has suggested this correlation function as a finite-temperature shot noise $^{57}$. The correlation function $S_h$ has been shown to satisfy the identity,

$$S_h = S - 4k_B T G,  \quad (22)$$

with the differential conductance $G$ and $S$ defined in Eq. (20). The expression (22) is identical to an empirical formula that has been applied as an estimation of the
demonstrate two different strategies to achieve our goal. It has also been confirmed that a number of the properties of the shot noise can be redervied from the expression of \( S_h \) in Eq. \((21)\). First, at zero temperature, Eq. \((22)\) is simplified as \( S_h = S \). Therefore, \( S_h \) defined in Eq. \((21)\) clearly agrees with the conventional shot noise defined at \( T = 0 \) by \( S = \). This means that the Nyquist-Johnson relation is also reproduced. These observations show that Eq. \((21)\) can be regarded as an extension of the shot noise to all temperatures.

Therefore \( S_h \) given in Eq. \((21)\) enables one to study finite-temperature effects on the shot noise directly, without calculating \( S \) and \( G \) separately. In the present paper, however, we focus on the shot noise at zero temperature. Specifically, we calculate \( S_h \) exactly up to \( V^3 \) in the particle-hole symmetric case where the occupation number of each orbital is given by \( n_{dm} = 1/2 \). We demonstrate two different strategies to achieve our goal.

The first starts from \( S_h \) defined in Eq. \((21)\) and reaches the result given in Eq. \((25)\). The second from Eq. \((20)\) yields the final expression given in Eq. \((28)\). The results obtained in these two ways, Eq. \((25)\) and Eq. \((28)\), agree with each other naturally, although the contributions of each Feynman diagram for \( S_h \) and that for \( S \) do not have one-to-one correspondence as summarized in Table I.

1. Shot noise \( S_h \) at absolute zero

We now calculate the shot noise from the expression given in Eq. \((21)\) and derive the leading asymptotic dependence of applied bias voltage at \( T = 0 \) in the particle-hole symmetric case.

First, substituting the current operator Eq. \((15)\) and charge fluctuation operator into Eq. \((21)\), the shot noise is readily expanded in the Keldysh formalism as,

\[
S_h = [(F_{hLL} - F_{hLR}) + (L \leftrightarrow R)] + (c.c.) ,
\]

with,

\[
F_{h\alpha\alpha'} = -ie^2 \hbar \sum_{k,k',m,m'} V_{km} \left< T_c, S_c c_{km}^\dagger(0^+) d_{m'}(0^+) c_{k'm'}^\dagger(0^-) c_{k'\alpha'} m'(0^-) \right>_{connected} .
\]

The second order perturbation in \( \bar{U} \) can be classified using the diagrams shown in Fig. 2. Note that for \( S_h \) defined in Eq. \((21)\) the fluctuation operator assigned for the left end of each diagram and that for the right end are different, i.e., one of the two is the current operator and the other is the charge fluctuation operator. The calculated contributions of each diagram are summarized in Table II. In the particle-hole symmetry case the contributions from diagram (d) and (e) vanish. The contribution from diagram (g) and that from (f) are the same for arbitrary parameters. Furthermore the contribution from (h) coincides with that of (f) and (g) in the particle-hole symmetric case for the symmetric lead-dot coupling.

Collecting all these contributions, we obtain the total shot noise at \( T = 0 \) up to order \( V^3 \),

\[
S_h = S_h^{(a)} + S_h^{(b)} + S_h^{(c)} + 3S_h^{(f)} = \frac{2Ne^3}{h} |V| \left( \frac{eV}{T} \right)^2 \left[ \frac{1}{12} + \frac{3}{4} (N - 1)(R - 1)^2 \right].
\]
This result agrees with Eq. (25) which has been deduced in the particle-hole symmetric case for the symmetric lead-dot coupling. In the Kondo limit where \( R \to N/(N-1) \), our result agrees with Mora et al.’s result in the particle-hole symmetric case for the symmetric lead-dot coupling.

The contributions of the second order perturbation in \( \hat{U} \) can be classified using the diagrams shown in Fig. 2 for each diagram, are given by the current fluctuation operator \( \hat{S} \) defined in Eq. (20). Because the thermal noise is completely suppressed at zero temperature, we can extract the pure shot noise from the current noise \( \hat{S} \). We derive the asymptotic form of the shot noise along a similar line, as that described above.

Substituting the current operator Eq. (19) into Eq. (20) readily leads to the current noise expression in the Keldysh formalism as,

\[
S = \frac{2e^2}{\hbar} \int dt \left[ (F_{LL}^+(t) - F_{RR}^+(t)) + (L \leftrightarrow R) \right] + \text{(c.c.)},
\]

with,

\[
F_{\alpha\alpha'}(t,t') = i^2 \sum_{kk'mm'} \left[ V_{ak} V_{a'k'}^* \left\langle T_c S_c c_{\alpha km}^\dagger(t) d_{\alpha'km'}^\dagger(t') d_{\alpha km}(t') \right\rangle_{\text{connected}} - V_{ak} V_{a'k'}^* \left\langle T_c S_c c_{\alpha km}^\dagger(t) d_{\alpha'km'}^\dagger(t') c_{\alpha'km'}(t') \right\rangle_{\text{connected}} \right].
\]

The contributions of the second order perturbation in \( \hat{U} \) can be classified using the diagrams shown in Fig. 2 for this correlation function. Note that for the current noise \( \hat{S} \) both of the two vertices, at the left and right ends of each diagram, are given by the current fluctuation operator \( \hat{S} \). The calculated contributions of each diagram are summarized in Table I. For this reason, the contributions from the diagram (b) and those from (c) are the same for \( \hat{S} \), in contrast to the case for \( S_h \). Nevertheless, there are some similarities between the two cases. The contributions from (d) and (e) vanish in the particle-hole symmetric case. The contribution from diagram (g) and that from (f) are the same for arbitrary parameters, and the contribution from (h) coincides with those of (g) or (f) in the particle-hole symmetric case for the symmetric lead-dot coupling. In the Kondo limit the contribution of each diagram agrees with that of the corresponding diagram used in Mora’s calculation.

Collecting all these contributions, we obtain the total current noise up to order \( V^3 \) as,

\[
S = S^{(a)} + 2S^{(b)} + 3S^{(f)} = \frac{2Ne^2}{\hbar} \left| V \right| \left( \frac{eV}{T} \right)^2 \left[ \frac{1}{12} + \frac{3}{4} (N-1)(R-1)^2 \right].
\]

This result agrees with Eq. (27) which has been deduced from \( S_h \).

### C. Fano factor

We consider the backscattering current \( I_b \) which contains all effects of the quantum and thermal fluctuations. It is defined via the deviation of the nonequilibrium current \( I \) from the value in the unitary limit \( Ne^2V/\hbar \), as,

\[
I_b = \frac{Ne^2}{\hbar} V - I
\]

\[
= \frac{Ne^2}{\hbar} V \left( \frac{eV}{T} \right)^2 \left[ \frac{1}{12} + \frac{5(N-1)}{12} (R-1)^2 \right].
\]

Here, we have used the result for \( I \) given in Eq. (19). Generally, the current noise for the forward current and that for the backscattering current are equivalent in the case where the systems is coupled to two leads. Therefore, the Fano factor of the backscattering current can be expressed in the form,

\[
F_b = \frac{S}{2eI_b} = \frac{1 + 9(N-1)(R-1)^2}{1 + 5(N-1)(R-1)^2},
\]

and this is one of the main results of the present work.

In the noninteracting case \( (U = 0) \), the Wilson ratio takes the value \( R = 1 \), which leads to the Poisson noise.
The Fano factor \( F_b \) in the Kondo limit \( (U \to \infty) \), for several choices of orbital degeneracy \( N \).

| \( N \) | 2  | 4  | 6  | 8  | \( \to \infty \) |
|------|----|----|----|----|----------------|
| \( F_b \) | 5/3 | 3/2 | 7/5 | 4/3 | \( \to 1 \) |

This expression obviously shows that a larger orbital degeneracy makes the initial rise of \( F_b \) steeper, as the coefficient of the order \( u^2 \) term increases with \( N \).

In the opposite limit, namely, the Kondo limit \( (U \to \infty) \), the Wilson ratio approaches to the universal value \( R \to N/(N-1) \), and the Fano factor takes the form,

\[
F_b \to \frac{N+8}{N+4}.
\]

The explicit values for several \( N \) are given in the Table above. Specifically in the limit of large orbital degeneracy \( N \to \infty \), the shot noise takes the Poisson value \( S \to 2eI_b \), and the Fano factor approaches to the noninteracting value \( F_b \to 1 \) even though the Coulomb repulsion has been taken first to be \( U \to \infty \). This originates from the fact that renormalization is weakened in the limit of large \( N \).

In practice, renormalized parameters takes \( \bar{U} \to 0 \) and \( \bar{\Gamma} \to \Gamma \) in the large \( N \) limit \( (R \to 1) \) of Eqs. 7 and 8. For \( N = 2 \), i.e., in the SU(2) case, our result is consistent with the previous results obtained by Gogolin and Komnik, Sela et al. and Fujii 14,15,38.

In order to clarify the behavior of \( F_b \) in the intermediate values of \( U \), we have carried out the NRG calculations for \( N = 2 \) and 4. Specifically, we have deduced the Wilson ratio \( R \) from the low-energy NRG fixed point63. This method has also been applied to the two-channel Anderson model in the recent work of Nishikawa et al. In the case of \( N = 2 \), the NRG and Bethe ansatz results for the renormalized parameters have been shown to agree well52,53,63 as seen in Fig. 3 (a). There is no Bethe ansatz solutions for \( N > 2 \) in the particle-hole symmetric case, still the NRG is applicable for \( N = 4 \). We can see, in Fig. 3 (a), that the NRG results for the Wilson ratio for \( N = 4 \) are in good agreement with the perturbation result up to a term of order \( U^4 \) given in Eq. 13 for a small Coulomb repulsion \( u \lesssim 0.3 \). Furthermore, the NRG results approach the correct universal value \( R \to 4/3 \) for larger \( U \). Therefore, the \( U \) dependence of the Fano factor for \( N = 4 \) can also be deduced, using expression 30 with the nonperturbative NRG approach, which is discussed below.

As the Coulomb repulsion increases further, the charge fluctuation in the dot is suppressed and the Wilson ratio increases rapidly from the noninteracting value \( R = 1 \) to the universal value in the Kondo limit. This crossover from the weak-coupling regime to the Kondo regime is also observed in the Fano factor, shown in Fig. 3 (b), as a rapid convergence of the Fano factor to the universal value \( F_b = 5/3 \) and 3/2 for \( N = 2 \) and 4, respectively. Note that the Wilson ratio is a decreasing function of \( \chi_{c,d}^{*}/\chi_d \), namely, the ratio of the charge susceptibility to the susceptibility as shown in Eq. 10. Therefore, the charge fluctuation at the dot-site suppresses the Fano factor for small Coulomb repulsion \( U \lesssim \pi T \). The NRG results for \( N = 4 \) have successfully revealed the precise feature of the crossover for the system with the orbital degeneracy.

The perturbation series is applicable also for large orbital degeneracies \( N \) where the NRG is not feasible. Therefore, it is worthwhile to discuss the convergence of the series expansion. For this purpose, the Fano factor for \( N = 4 \) is plotted in Fig. 4 (a), keeping the first few terms of the series given in Eq. 30. We can see that the values of \( F_b \) deduced from Eq. 21 agree with the NRG result for a relatively narrow range of the Coulomb interaction \( u \lesssim 0.1 \), while the series expansion for the Wilson ratio quantitatively works in a wider range \( u \lesssim 0.3 \). This discrepancy is caused by the expansion of the denominator on the right-hand side of Eq. 30 with respect to \( (R-1) \), the convergence radius for which becomes small as shown in the Appendix. This can be resolved, however, using the expression of the Fano factor Eq. 30, as it is, without expanding the denominator, and then substituting the series expansion for the Wilson ratio given in Eq. 11. The results obtained in this way agree with the NRG data for \( u \lesssim 0.3 \), as shown by the dashed line in Fig. 4 (a). Specifically, for \( N = 4 \), both the coefficient of the order \( u^3 \) term and that for the order \( u^4 \) term in Eq. 31 become negative, while that for the order \( u^5 \) term is positive. For this reason, in Fig. 4 (a), the fourth order curve deviates from the NRG result earlier than the third order curve, and then the fifth order curve again approaches the NRG result. Finally,
FIG. 3: (Color online) (a) The Wilson ratio $R$, and (b) the Fano factor $F_b$ as functions of the on-site Coulomb repulsion $U$. The dashed-dotted line denotes the results deduced from the BAE for $N = 2$. The NRG results are plotted for $N = 2$ (▲), and for $N = 4$ with the solid circle (●). The dashed line in (a) is the Wilson ratio for $N = 4$, obtained from the perturbative expansion up to terms of order $U^4$ given in Eq. (14).

we examine the $N = 6$ case, for which nonperturbative approaches are not available at present. The value of the Fano factor for $N = 6$ deduced from the perturbation series are shown in Fig. 4 (b). We see that the results are quantitatively valid for $u \lesssim 0.1$, although the convergence of the series expansion for $R$ and that for $F_b$ given in Eqs. (14) and (31) become worse with an increase of $N$. Nevertheless, the results show clearly that the initial rise of $F_b$ near $u \approx 0.1$ is steeper for $N = 6$ than that for $N = 4$.

FIG. 4: (Color online) Perturbation results for the Fano factor for (a) $N = 4$, and (b) $N = 6$. The dashed line is obtained just by substituting Eq. (14) into Eq. (30). The dotted, dashed-dotted, and dashed-dot-dotted lines denote the values from the series expansion up to terms of order $U^3$, $U^4$ and $U^5$, respectively, given in Eq. (31). The solid line with the solid circle (●) is the NRG results for $N = 4$.

which is readily obtained by inverting Eq. (30). This scheme for estimating $R$ from $F_b$ does not need the value of the $g$-factor for the electrons in the dot. Only the observation of $F_b$ determines the accuracy of observed $R$. Alternatively, one can estimate the Wilson ratio from the measurements of the susceptibilities $\chi_{c,d}^*/\chi_d^*$. In this case, the $g$ factor is also needed to determine them accurately. Therefore, Eq. (33) may enable us to experimentally observe the Wilson ratio with a more reasonable accuracy.

Finally, we comment on a physical interpretation for the values of the Fano factor for the orbital Kondo systems. The expression for the Fano factor presented in Eq. (30) shows that effects of the Coulomb repulsion enter through the Wilson ratio $R$, which characterizes the low-energy Fermi liquid state. The terms having $R$ in the coefficient come from the vertex corrections and self-energy appearing in the RPT approach to the shot noise defined by Eqs. (20) or (21), and the nonequilibrium current $I$. In this aspect, the role of the Fano factor for the shot noise resembles the role of the Stoner factor for the magnetic susceptibility. These factors quantitatively enhance the response of the system against external fields, but do not change the qualitative feature of the low-energy states. Therefore, the deviations of the Fano factor from the noninteracting value are caused by

IV. DISCUSSION AND SUMMARY

At the end of the previous section, we have calculated the value of the Fano factor from the data for the Wilson ratio. Conversely, it may be possible to determine the value of the Wilson ratio from the measurements of the Fano factor, using the expression,

$$R = 1 + \sqrt{\frac{F_b - 1}{(N - 1)(9 - 5F_b)}} \ ,$$  

(33)
the scattering between the renormalized quasiparticles of the Fermi liquid, rather than an effective charge as with other kinds of elementary excitations.

In summary, we have studied the current and shot noise through the single quantum dot described by the multiorbital impurity Anderson model, using the nonequilibrium Kubo formalism. Employing the renormalized perturbation theory, the current and the shot noise have been calculated exactly in the particle-hole symmetric case up to order $V^3$ at $T = 0$. The result for the shot noise is given by Eq. (25), and then the Fano factor $F_b$ of the backscattering current is determined by two parameters, i.e., the Wilson ratio $R$ and the orbital degeneracy $N$, as shown in Eq. (30). We have also presented the explicit form of the Fano factor for small Coulomb repulsions up to terms of order $U^5$. Furthermore we have deduced the dependence of $F_b$ for $N = 2$ and $N = 4$ on the Coulomb repulsion, using the Wilson ratio obtained with the NRG approach. The Fano factor varies monotonically with an increase of $U$ from the noninteracting value $F_b = 1$ until almost saturating for $U \gtrsim \pi \Gamma$ to the value in the Kondo limit, $F_b = (N + 8)/(N + 4)$. For small $U$, the charge fluctuation at the dot-site suppresses the Fano factor.

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Appendix A: The convergence of the Fano factor

We discuss the convergence of the Fano factor as a power series of $U$ given in (31). For $0 \leq U < \infty$, $F_b$ can be expanded as a power series in $R - 1$ around $R - 1 = 0$, as,

$$F_b = \left[1 + 9(N - 1)(R - 1)^2\right] \sum_{n=0}^{\infty} C_n(N)(R - 1)^{2n} \quad (A1)$$

with $C_n(N) = [-5(N - 1)]^n$. and the radius of convergence,

$$|R - 1| < \sqrt{\frac{1}{5(N - 1)}} \quad (A2)$$

It follows from Eq. (A1) that for large $N$ the power series slowly converges. In addition, the Wilson ratio as a power series in $U$ given in Eq. (14) slowly converges for large $N$. Therefore, it can be concluded that the convergence of the Fano factor as a power series in $U$ given in Eq. (31) is slower for larger degeneracy $N$.

Second, we consider the radius of convergence of Eq. (31). As mentioned in the text, the value of the Wilson ratio is bounded in a range,

$$0 \leq R - 1 < \frac{1}{N - 1} \quad (A3)$$

for $0 \leq U < \infty$. For $N = 2, 4$, it follows from Eqs. (A2) and (A3) that the radius of convergence of the Fano factor as a power series of $U$ is determined by,

$$0 \leq R - 1 < \sqrt{\frac{1}{5(N - 1)}} \quad (A4)$$

Therefore, the radius of convergence can be obtained as $u \sim 0.48$ for $N = 2$ and $u \sim 0.72$ for $N = 4$ from the Wilson ratio data shown in Fig. 3 (a). However, for $N \geq 6$, the range of $R - 1$ given in Eq. (A3) is narrower than that of Eq. (A2). Therefore, the Fano factor for $N \geq 6$ can be written as a power series of $U$ for arbitrarily positive $U$, similarly to the universal quantities of the Kondo effect.
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