On quantum spin glasses with finite connectivity: cavity method and applications

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Abstract. We discuss quantum spin glasses with finite connectivity by presenting an extension of the cavity method used in studies of classical spin glasses.

1. Introduction
The appearance of finite connectivity trees (eg Cayley trees, Bethe lattices) in the study of spin glasses has a long history that goes back to the first papers on the Sherrington-Kirkpatrick model [1, 2]. Thouless, Anderson and Palmer[2] wrote the mean field equations for such a tree and simplified the results in the small coupling, large connectivity regime (since in the SK model $Z = N$ and the couplings $J_{ij} \propto 1/\sqrt{N}$). Those mean field equations are known as the TAP equations and the peculiar characteristics of their solutions, in particular their large number [3], were an important indicator of the complexity of the spin glass phase that arose in parallel with the remarkable developments in the study of the replicated free energy [4, 5].

The problem of a spin glass on the Bethe lattice [8, 7, 11, 12, 10] appears naturally when studying typical complexity of difficult (NP-complete) problems in computer science [14]. A typical example of this is given by the k-SAT problem, which asks “Given a boolean expression $J$ on $N$ bits $\sigma_i$ composed of the conjunction of $M$ clauses, each of which involves exactly $k$ of the bits, is it satisfiable by some bit assignment?” This can be recast in Hamiltonian form by writing a cost function $H_J[\{\sigma_i\}]$ that evaluates the number of violated clauses. In this language, the bits naturally become Ising spins, the expression $J$ becomes a particular instance of some spin glass and the original question requires determining the ground state energy. The large $N$ limit is a problem in statistical mechanics and typical-case analysis for k-SAT becomes the disorder averaged analysis of spin glass theory. In this limit, $k$-SAT develops several phase transitions as a function of $\alpha = M/N$ the number of clauses per bit. As $\alpha$ grows, the problem goes from being...
easily solved and satisfiable, to an intermediate glassy phase with many local ground states, to a typically unsatisfiable phase where the ground state energy density is positive.

A key role in these developments has been played by the so called cavity method—a complex of analytical and numerical techniques refined recently [16, 14]—for studying classical spin glasses on tree-like graphs. Applied to the k-SAT problem, the cavity method suggests the above phase diagram and provides numerical values for its critical points. Moreover, the technique can be applied to a particular instance of k-SAT and the information so obtained about the free energy landscape now guides the search procedure in state-of-the-art k-SAT algorithms [14].

In a recent paper [9] we turned to quantum spin glasses on Bethe lattices, specifically to the problem of extending the cavity method to their analysis. As in the classical case, there are two distinct reasons to be interested in these systems. There is the intrinsic interest of the interplay between quantum mechanics and spin glass behavior, about which the difficulties of the classical case serve as both caution and enticement. A second motivation now arises from the rapid recent developments in quantum computing. In particular, the discovery of the ground state of a classical spin system $H_J$ derived from a computational problem is ideally suited to solution by the adiabatic algorithm [18], which allows a quantum computer to solve such a problem by means of an adiabatic change of the parameters in the Hamiltonian.

This discussion certainly suggests that a careful study of the phase transitions encountered on the path between $H_0$ and $H_J$ is in order. However, the necessity of understanding the “deep” quantum spin glass phase far from the phase transition is also clear. The structure of this phase, especially its nontrivial energy landscape, is probably as important as the nature of the phase transition. In this paper we take a first step toward understanding the quantum spin glass phase by generalizing the cavity method used to study the classical problem. Compared to the replica method or direct study of the quantum TAP equations, we believe the cavity method outlined in this paper provides much more physically transparent information about the SG phase. It also may be applied to many other quantum phase transitions on the Bethe lattice, whether induced by disorder or more typical symmetry breaking.

The plan of the paper is the following: we introduce the classical cavity method in Section 2; we propose its generalization to the quantum spin glass problem in 3 and we apply it to the numerical study of a Bethe lattice with connectivity three. Discussion and directions for further work will be presented in the last sections.

2. Classical Cavity Method

The cavity method is a way of finding the spin glass free energy by means of self-consistency equations for the probability distributions of quantities characterizing the statistics of the spins (the cavity fields). It has several virtues compared with the replica method, in particular if applied to spin glasses with finite connectivity.

The classical Hamiltonian is

$$H = -\sum_{ij} J_{ij} \sigma_i \sigma_j$$

(1)

where $\sigma_i \in \{\pm 1\}$ are Ising spins and the sum is over bonds in the Bethe lattice. The couplings $J_{ij}$ are i.i.d. random variables drawn from some distribution $P(J)$. For simplicity and because of its connection to computational problems, we will restrict our attention to the $\pm J$ model

$$P(J_{ij}) = \frac{1}{2} \delta(J_{ij} - J) + \frac{1}{2} \delta(J_{ij} + J),$$

(2)

although much of the discussion has broader validity. To create a cavity, we pick a spin $\sigma_0$ in the Bethe lattice and imagine removing it. Each of $\sigma_0$’s neighbors is a cavity spin, connected to $q - 1$ spins and sitting at the root of a branch of the original tree. Notice that in the absence...
Consider an iteration operation, in which we join \( q-1 \) cavity branches on to an added spin \( \sigma_0 \). The added spin receives thermodynamic information regarding each of the \( q-1 \) branches only through the thermal distribution of the cavity spins \( \sigma_1, \ldots, \sigma_{q-1} \). In the absence of \( \sigma_0 \), each of these Ising variables has independent statistics characterized fully by its thermal probability distribution:

\[
\psi_i(\sigma_i) = e^{\beta h_i \sigma_i} \frac{1}{2 \cosh(\beta h_i)}.
\]

which defines the cavity field \( h_i \). Since \( \sigma_1 \in \{ \pm 1 \} \), there are only two possible configurations of the spin and only \( 2-1 = 1 \) real numbers are needed to characterize the probability distribution. In this sense, the cavity field \( h_i \) is merely a good parameterization of the distribution \( \psi_i(\sigma_i) \).

We emphasize this viewpoint because it will naturally generalize to the quantum case.

We introduce a simple graphical convention for cavity spins:

\[
\begin{align*}
\text{open circle} & \quad = \quad \circ \\
\text{wiggly line} & \quad = \quad \circ
\end{align*}
\]

The open circle indicates a spin variable and the wiggly line indicates the effective field attached to it. With this notation, an iteration operation can be represented:

\[
\begin{align*}
\psi_1(\sigma_1) & \quad = \quad \psi_1(\sigma_1) \\
\psi_0(\sigma_0) & \quad = \quad \psi_0(\sigma_0) \\
\psi_2(\sigma_2) & \quad = \quad \psi_2(\sigma_2)
\end{align*}
\]

where the filled circle indicates summing out a spin variable. More formally, the state of the spin \( \sigma_0 \) depends on the state of the \( q-1 \) spins as

\[
\psi_0(\sigma_0) = \frac{1}{Z} \sum_{\sigma_1, \ldots, \sigma_{q-1} = \pm 1} \exp \left( \beta \sum_i J_{0i} \sigma_0 \sigma_i \right) \psi_1(\sigma_1) \ldots \psi_{q-1}(\sigma_{q-1}),
\]

where \( Z \) is a normalization factor so \( \sum_{\sigma_0} \psi(\sigma_0) = 1 \). In terms of cavity fields this equation implies

\[
h_0 = \frac{1}{\beta} \sum_{i=1}^{q-1} \tanh^{-1}(\tanh(\beta J_{0i}) \tanh(\beta h_i)) \equiv U(\{h_i\}, \{J_{0i}\}).
\]

The iteration equation (6) is valid for particular realization of the \( J \)’s and since these are random variables, it defines a Markov process for the cavity fields. Throughout the graph, these fields will be site dependent random variables, but deep inside the tree, they ought to be distributed according to a probability distribution \( P(h) \) that represents a fixed point of the Markov process. This fixed point distribution will satisfy

\[
P(h) = \int \prod_{i=1}^{q-1} dh_i P(h_i) \langle \delta(h - U(\{h_i\}, \{J_{0i}\})) \rangle_J.
\]

In terms of the spin distribution, this becomes the functional equation

\[
P[\psi] = \int \left( \prod_{i=1}^{q-1} D\psi_i P[\psi_i] \right) \langle \delta[\psi(\sigma) - \psi_0(\sigma; \{\psi_i\}, \{J_{0i}\})] \rangle_J
\]

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where \( \psi_0(\sigma) \) is given by equation (6).

This distribution is the order parameter for the spin glass. It is a \( \delta \) function at \( h = 0 \) in the high temperature phase. As the temperature is lowered, \( P(h) \) broadens to have finite support below some mean field-like phase transition. Defining \( \tau = T/J \), this phase transition is located at \( \tau_c = 1/\tanh(\sqrt{q-1}) \). It is possible to write the free energy per site in terms of \( P(h) \) as

\[
F = \int \prod_{i=1}^{q} dh_i P(h_i) F_{q+1} - \frac{q}{2} \int \prod_{i=1}^{2} dh_i P(h_i) F_2,
\]

(10)

\[
F_{q+1} = -\frac{1}{\beta} \left\langle \ln \sum_{\sigma_0,\sigma_1,\ldots,\sigma_q} \exp(\beta \sum_{i=1}^{q} J_{0i}\sigma_0\sigma_i) \prod_{i=1}^{q} \psi_i(\sigma_i) \right\rangle_{J,\psi},
\]

(11)

\[
F_2 = -\frac{1}{\beta} \left\langle \ln \sum_{\sigma_1,\sigma_2} \exp(\beta J_{12}\sigma_1\sigma_2) \psi_1(\sigma_1)\psi_2(\sigma_2) \right\rangle_{J,\psi}.
\]

(12)

This rather complicated looking expression is actually simply the average change in free energy due to a merge operation minus \( \frac{q^2}{2} \) times the average change in free energy due to a link addition. Graphically,

\[
F = \left\langle \mathcal{F} \left( \begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_3 \end{array} \right) \right\rangle - \frac{q}{2} \left\langle \mathcal{F} \left( \begin{array}{c} \psi_1 \\ \overline{\psi}_2 \end{array} \right) \right\rangle_{J,\psi}
\]

(13)

where \( \mathcal{F} \) of a diagram is the free energy of a system with spin variables given by the unfilled circles.

It is possible to see that if \( P \) satisfies (8) then \( \delta F/\delta P(h) = 0 \). Other expressions for the free energy \( F \) have appeared in the literature but it has been shown that they are all equivalent to each other, if the consistency equation (8) is verified. The role of the free energy in the cavity method is secondary as one does not solve the variational problem, as in the replica method, by working on the free energy directly. Rather, one finds the probability distribution \( P \) by analytical or numerical methods and then derives all of the statistical observables from \( P \).

The equivalence of the two formulations has been put forward in Ref. [16, 12] and in many other works.

### 3. Quantum Cavity Method

#### 3.1. Exact Framework

We consider now the modification of the Hamiltonian (1) due to the introduction of a transverse magnetic field

\[
H = -\sum_{\langle ij \rangle} J_{ij}\sigma_i^x\sigma_j^x - B_t \sum_i \sigma_i^x.
\]

(14)

This is called the transverse field Ising spin glass in the literature. The Ising variables of the previous section have been replaced by Pauli matrices \( \sigma^x \) and the magnetic field couples to the matrices \( \sigma^z \). The fact that \( \sigma^z \) and \( \sigma^x \) do not commute gives rise to a host of interesting new features due to the interplay of quantum mechanics and disorder [23].

The usual Suzuki-Trotter decomposition allows us to rewrite the problem in terms of \( N_t \) Ising spins per quantum spin, where the number \( N_t \) needs to be sent to infinity eventually. The

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1 Readers familiar with the literature may note that this allows one to bypass the parametrization of the order parameter on the replica indices, a particularly thankless task on the Bethe lattice since an infinite sequence of order parameters \( q_{a,b}, q_{a,b,c}, q_{a,b,c,d}, \ldots \) is necessary [24].
additional dimension which is introduced in this way is the usual imaginary time. The $\sigma^x_i \sigma^x_j$ interactions are time-translation invariant (the disorder is correlated in the time direction) while the $\sigma^z$ terms give a ferromagnetic nearest-neighbor interaction in the time direction. Before writing the Hamiltonian let us introduce some notation.

For any finite $N_t$ we will refer to the Ising spin configuration at a given site $i$ as a “rod” of spins. The rod at site $i$ is described by $N_t$ spins $\sigma_i(t)$ where $t$ takes values from 0 to $\beta$ in steps of $\Delta t = \beta/N_t$, with periodic boundary conditions $\sigma(0) = \sigma(\beta)$. This notation is convenient if the limit $N_t \to \infty$ is eventually performed, since the rod is represented by a function $\sigma(t) : [0, \beta] \to \{-1, 1\}$ with $\sigma_i(t) = \sigma_i(\beta)$. The rod statistics are described by a probability distribution $\psi[\sigma(t)]$, a functional of $\sigma(t)$, that gives a positive real number for every configuration $\sigma(t)$. The normalization condition reads $\sum_{\{\sigma_i(t)\}} \psi[\sigma(t)] = 1$.

The partition function is written as

$$Z = \sum_{\{\sigma_i(t)\}} e^{-\beta H[\sigma]}$$

where the Hamiltonian is:

$$\beta H = -\sum_i \sum_{(ij)} \Delta t J_{ij} \sigma_i(t) \sigma_j(t) - \Gamma \sum_i \sum_t \sigma_i(t) \sigma_i(t + \Delta t).$$

We can also write this as a sum over links of the energy of a link:

$$\beta H_{ij} = -\sum_t \Delta t J_{ij} \sigma_i(t) \sigma_j(t) - \frac{1}{q} \Gamma \sum_t \sigma_i(t) \sigma_i(t + \Delta t)$$

where a fraction $1/q$ of the imaginary-time interaction is associated to each link (there are $q$ links per spin). Here $\Gamma = \frac{1}{2} \ln \text{coth}(\Delta t B_t)$.

Notice that $\Gamma > 0$ so the system is ferromagnetic in the time direction, and moreover when $\beta B_t/N_t \ll 1$ then $\Gamma \gg 1$ and the coupling along the time direction is strong. In particular, for $B_t = 0$ the spins in any given rod are locked together as $N_t$ useless copies of a single Ising spin. Thus, the results reduce to the classical case smoothly.

The spatial tree-like structure of the original problem is reflected in the tree-like structure of the interaction between rods. We can therefore imagine an iteration process with rods replacing the spins, in which we have $q - 1$ cavity rods $\psi_i[\sigma_i(t)]$ which are merged and determine the state of the rod $\psi_0[\sigma_0(t)]$ (see Fig. 1). This corresponds to a recursion relation for the calculation of the partition function of the branches, analogous to the classical equation (6):

$$\psi_0[\sigma_0(t)] = \frac{1}{Z} \sum_{\{\sigma_i(t)\}_{i \geq 1}} e^{-\sum_{i=1}^{q-1} \beta H_{0i} \prod_{i=1}^{q-1} \psi_i[\sigma_i(t)]}.$$  (18)

Returning to the case of the spin glass, we can write down the quantum cavity fixed point equation analogous to the classical equation (8) immediately:

$$P_{FP} [\psi[\sigma(t)]] = \langle \delta [\psi[\sigma(t)] - \psi_0[\sigma(t); \{J_{0i}, \psi_i\}_{i=1}^{q-1}]] \rangle_{J_{0i}, \psi_i}$$

$$= \int \left( \prod_{i=1}^{q-1} D\psi_i P_{FP} [\psi_i] dJ_{0i} P(J_{0i}) \right) \delta \left[ \psi[\sigma(t)] - \psi_0[\sigma(t); \{J_{0i}, \psi_i\}_{i=1}^{q-1}] \right]$$

where the iterated rod action $\psi_0[\sigma(t); \{J_{0i}, \psi_i\}_{i=1}^{q-1}]$ is given by (18) and $P(J_{0i})$ is the fixed prior distribution for couplings (2). This is a functional equation for $P_{FP} [\psi[\sigma(t)]]$, the fixed point
probability distribution of the effective distribution describing iterated cavity rods. In the limit $N_t \to \infty$, it is exact but difficult to solve in closed form. It is certainly possible that analytic progress can be made, but we have not succeeded thus far. However, it is amenable to numerical study at finite $N_t$ under certain approximations and also perhaps by continuous time Monte Carlo for $N_t \to \infty$. In the remainder of this section we will explore the finite $N_t$ approach.

We must first parametrize our generic vector $\psi$ in the $2^{N_t}$-dimensional space of the configurations of the rods. In principle it is described by $2^{N_t} - 1$ real numbers, which can be reduced by a factor $O(N_t)$ by exploiting time-translation symmetry and the periodic boundary conditions. A natural way to parametrize it is in term of the effective action

$$S[\sigma] = - \log Z - h \Delta t \sum_t \sigma(t) - \sum_{t,t'} \Delta t^2 C^{(2)}(t' - t)\sigma(t)\sigma(t') - \sum_{t,t',t''} \Delta t^3 C^{(3)}(t' - t, t'' - t')\sigma(t)\sigma(t')\sigma(t'') + ...$$

where we expand $S$ in a series of increasing clusters of interacting spins.

In principle, the sum includes up to $N_t$-spin interaction terms (the normalization factor has been included as a spin-independent term in the effective action). In practice, we truncate the action expansion at second order to keep the numerical requirements manageable. Notice that $C^{(2)}(t)$ is the kernel for 2-point in time interactions, not the dynamical two-point correlation function, often denoted $c^{(2)}(t) = \langle \sigma(t)\sigma(0) \rangle$.

The functions $h, C^{(i)}$ are random quantities characterized by the Markov process defined by the iteration procedure. By writing the representations of the vectors $\psi$ in terms of the effective action (21) we can rewrite the iteration equation as

$$e^{-S[\sigma, \{h_0, C_0\}]} = \sum_{\{\sigma_1(t), ..., \sigma_{q-1}(t)\}} e^{-\sum_{i=1}^{q-1} \beta H_0} \prod_{j=1}^{q-1} e^{-S[\sigma, \{h_j, C_j\}]}.$$  

This gives an implicit update map from the ‘old’ $q - 1$ parameters $h_j, C_j^{(2)}, C_j^{(3)}, ...$ and the couplings $J_{0j}, B_t$ to the ‘new’ parameters $h_0, C_0^{(2)}, C_0^{(3)}, ...$. The statistics generated by this Markov process and in particular its fixed point distribution

$$P(h, C^{(2)}, C^{(3)}, ...)$$

are the solution of the problem.
3.2. Numerical Results

3.2.1. Phase Diagram

We present numerical results for an investigation of the $q = 3$ connectivity model using a naive (exact) approach to the exponential summation involved in the cavity iteration and merging operations. Fig. 2(a) shows the phase diagram calculated at $N_t = 10$, $N_{rods} = 2500$, $N_{iter} \sim 1000N_{rods}$ and suggestively fit to $N_t \to \infty$ using asymptotic expansions in $1/N_t^2$. Qualitatively, as expected, the phase transition curve predicts a $B_t = 0$ critical temperature in agreement with the analytic prediction of $\tau_c = 1/(\text{tanh}^{-1}(1/\sqrt{q-1}) \approx 1.13$. The upturn in the $N_t = 10$ phase boundary at low temperature is due to the finite discretization of time. By fitting the finite $N_t$ results and extrapolating to the $N_t \to \infty$ we predict that $B_t = 1.77 \pm 0.03$ at $T = 0$.

Figure 2(b) shows the instance averaged single site von Neumann entropy $S_{\text{vonN}}$ which has a remarkably clear maximum near the phase transition curve above the classical line. This reflects the strength of quantum correlations even at the finite temperature phase transition. See Ref. [27] for discussion of local measures of entanglement at finite temperature.

Zooming in on the horizontal stripe at $B_t = 1$ indicated on the phase diagram, we find that $q_{EA}$ vanishes linearly at the critical temperature (Fig. 3(a)). This reflects the underlying broadening transition in $P[\psi]$, which can be seen sharply in the variances of each of the effective action coefficients (Figs. 3(b,c)). We use this behaviour to estimate sharp transition points despite softening due to critical slowing in the convergence of our procedure.

Finally, we note that much of the phase diagram is surprisingly stable to variation in $N_t$ in the range explored, $N_t = 6, ..., 11$ (the classical line $B_t = 0$ is stable down to $N_t = 1$).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{phase_diagram.png}
\caption{(a) Phase diagram at $q = 3$ calculated at $N_t = 10$, $N_{rods} = 2500$, $N_{iter} \sim 1000N_{rods}$. The vertical dotted line is the asymptotic critical line at $N_t = 10$. The points marked x with error bars indicate $N_t \to \infty$ fits. The dashed transition curve is a weighted quadratic fit through the estimated low temperature points which leads to an estimated $B_t^c = 1.77 \pm 0.03$. The stars and stripes indicate points in the phas space which we have investigated in more detail below. (b) The average von Neumann entropy $S_{\text{vonN}}$ (in bits) of a central spin as a function of $(\tau, B_t)$ at $N_t = 8$. The dashed line indicates the estimated region of validity of the discretization approximation ($B_t \leq N_t\tau/2$).}
\end{figure}

3.2.2. Structure of the Glassy Phase

Since the order parameter is a probability distribution of an action, there is a rich structure to be investigated at even a single point $(\tau, B_t)$ within the glassy phase. In Fig. 4, the marginal probability distribution of the field term $h$ in the cavity
action is shown at the four points indicated on Fig. 2(a). The two lower distributions lie on the classical line \((B_t = 0)\), one deep within the glassy phase and one near the transition. It is clear that the distinctive features of the classical solution are reproduced here: a Gaussian-like structure around \(h = 0\) near the phase transition with the appearance of delta function spikes on the integer fields deep within the phase. At \(B_t = 1\), the qualitative picture of spread from narrow Gaussian near the phase transition to broader, bumpier distribution remains. It is less clear whether the sharply defined spikes on integer fields would remain at \(\tau = 0\) with large \(B_t\).

Further structure can be found in the nontrivial probability distribution for the interaction terms that develop in the spin glass phase:

- The two-body interactions are ferromagnetic and the effect of coupling to neighboring rods is only to enhance the ferromagnetic interaction from the nearest neighbor only \(\Gamma\). Indeed, this \(\Gamma\) sets the minimum strength of \(C(\delta T)\).
- The strength of two-body interactions are strongly anticorrelated with the strength of the cavity field \(h\). Large cavity fields on a central spin come from large fields biasing neighboring rods. These fields pin the neighboring spins more strongly and reduce the ability of those spins to mediate interactions in time between the central rod spins, reducing the effective two-body interaction.
- The multimodal spikiness in these distributions reflect the “classical” spikiness in the low temperature cavity field distributions through the field-interaction correlation.

Finally, we emphasize that the phase transition is signaled by a singular broadening of \(P\) rather than any singularity in its first moments or in the structure of the typical imaginary time action.

4. Discussion and further work
The cavity method has a long and illustrious history in the study of statistical systems – from Bethe’s early work on the Ising ferromagnet to modern studies of random constraint satisfaction problems in computer science. In this paper, we have introduced a new variant for studying disordered quantum systems within an imaginary time formalism. This represents an intuitively appealing, natural synthesis of the classical disordered model techniques with the quantum homogeneous models studied in DMFT. We note that our framework can be simply adapted to study many other transverse field Ising systems on trees with fixed or fluctuating connectivity – such as the ferromagnet, diluted ferromagnet or biased glass.
We have shown that the transverse field Ising glass on a Bethe lattice of connectivity three has a phase transition line all the way from the classical $B_t = 0, T = 1.13$ to the quantum $B_t \approx 1.75, T = 0$. At finite temperature, the transition is classical and mean field like. Inside the frozen phase the picture is similar to the classical case: when a randomly chosen spin is extracted from the graph, its effective action (analogous to the cavity field) is well-defined in the paramagnetic region but is a random functional in the spin-glass phase. In this phase all of the local observables, classical and quantum, are therefore also random variables. In principle one could also study the properties of the entanglement of distant spins, which cannot be done in the better known but fully connected SK model.

The reader familiar with classical spin glass theory will have noticed our avoidance of the important issue of replica symmetry breaking (RSB), which is widely believed to be a feature of a correct treatment of the random graph Bethe lattice. In this connection, we note that our simpler treatment is indeed correct for models on Cayley trees with fixed boundary conditions. Quantum RSB phenomena certainly deserve further study: the conceptual difficulties of the RSB ansatz become even thornier in the presence of quantum tunneling. We note that formally breaking replica symmetry at the one step level (1RSB) should be straightforward in our framework. In analogy with Ref. [16], one should introduce populations of populations of effective actions and weigh them according to their free energy (as one does for different solutions of the TAP equations). This straightforward modification of the algorithm makes it computationally considerably more time consuming. For this first pass, we decided not to embark on such a project.

Unfortunately, we are not aware of Quantum Monte Carlo or other numerical studies on the transverse Ising model on the Bethe lattice with which to compare our results. The primary difficulty that has prevented the direct simulation of this system is that it requires a very large number of spins to approximate the infinite system effectively. Since a random graph with fixed connectivity has an extensive number of loops of lengths $\ell \sim O(\ln N/\ln(q-1))$, $N$ needs to be
exponentially larger than any statistically relevant length scale (e.g. the coherence length).

Another direction for further development would be to find a spin glass (or otherwise) model amenable to analytic treatment within the quantum cavity method. A “soft spin” Gaussian model would do, since the path integrals to be performed in an iteration could be computed exactly. However, we do not believe this model has a spin glass phase in the absence of higher order couplings. Whether one could treat such a coupling perturbatively is a question worthy of further exploration.

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