Generalized solvability of a problem with a dynamic boundary condition for the hyperbolic equation

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Abstract. In this article, we consider a problem with a dynamic boundary condition for a one-dimensional hyperbolic equation. The conditions on coefficients and a source term providing a unique solvability of a problem are derived.

1. Introduction

Vibration problems are of great importance in engineering and nowadays have been studied by many researchers.

Construction structures and buildings are highly susceptible to both natural and man-made dynamic impacts. For example, wind and seismic impacts, loads from equipment, moving transport, pedestrians.

The energy of the oscillations of the engineering systems is gradually dissipated due to internal friction in the material and external resistance. This fact affects the vibrational process, and the decrease in the intensity of external dynamic influences leads to a fading of fluctuations. Researchers carry out dynamic calculations of structures, identify the dynamic characteristics (frequency, form of own vibrations and so on). It should be taken into account the effect of internal damping, which extinguishes fluctuations due to friction in the material and thus affects the overall vibrational process. It is rather known how to take into account the effects of external friction (external oscillation), whereas a problem to account for consideration internal friction is often more difficult.

Turning to mathematical terms, we get a problem with nonlocal conditions, which describe the model of internal friction (nonlocal damping of the material).

Nowadays various nonlocal problems for partial differential equations are actively studied. We focus our attention on nonlocal problems with integral conditions for hyperbolic equations. Systematic studies of nonlocal problems with integral conditions originated with the papers by Cannon. These and further investigations of nonlocal problems show that classical methods most widely used to prove solvability of initial-boundary problems break down when applied to nonlocal problems. Nowadays several methods have been devised for overcoming the difficulties arising because of nonlocal conditions. It appears that conditions for the existence and uniqueness of a solution to the nonlocal problem are closely related to the notion of regular boundary conditions. It is known that the system of root functions of an ordinary differential operator with strongly regular boundary conditions form a Riesz basis in $L_2(0, 1)$. This property is particularly useful for obtaining results on solvability of boundary problems.
We consider a problem with nonlocal dynamic condition in the form

\[ u_x(l, t) + \gamma u_t(l, t) + \alpha \int_0^l K(x, t) u(x, t) dx = 0. \]

This condition may be considered as perturbed dynamic condition and we begin our study starting with the case \( \alpha = 0 \).

We will introduce a definition of a generalized solution of the problem studied and prove unique solvability of the problem.

2. Main results

2.1. Setting of a problem

In interval \( Q_T = (0, l) \times (0, T) \) consider the equation

\[ Lu \equiv u_{tt} - (au_x)_x + bu_t + cu = f(x, t), \]

and set a problem: find a solution to the equation (1), satisfying initial data

\[ u(x, 0) = 0, u_t(x, 0) = 0, \]

and boundary conditions

\[ u_x(0, t) = 0, u_x(l, t) + \gamma u_t(l, t) = 0. \]

As we know [1–7], solving a problem with dynamic conditions (3) is difficult even for the equation of string vibrations. We consider an equation with arbitrary coefficients depending on both \( x, t \). It makes it pointless trying to get a solution to a problem by separating variables or taking advantage of a common solution to the equation. However, we managed to prove the unambiguous resolution of the problem in Sobolev’s space, as demonstrated in the article.

Let us assume that the coefficients of the equation (1) and its right part meet the following conditions:

\[ a, a_t, a_{tt}, b, b_t, b_{tt}, c \in C(\bar{Q}_T), \quad f \in L^2(Q_T), \quad a(x, t) \geq a_0 > 0. \]

Let’s introduce a definition of a generalized solution. To do this, we consider an equality

\[ \int_0^T \int_0^l vLu dx dt = \int_0^T \int_0^l f vdx dt, \]

where \( v(x, t) \) is an arbitrary smooth function, such that \( v(T) = 0 \), under assumption that \( u(x, t) \) is a solution to the problem. After integrating the left side by parts we get

\[ \int_0^T \int_0^l (-u_tv_t + au_x v_x + bu_t v + cv) dx dt + \gamma \int_0^T a(l, t) v(l, t) u_t(l, t) dt = \int_0^T \int_0^l f vdx dt. \]  

Note that all terms in (4) make sense under lesser requirements on the function of \( u(x, t), v(x, t) \), namely, all integrals exist if \( u \in W(Q_T), \quad v \in \hat{W}(Q_T) \), where we denote

\[ W(Q_T) = \{ u : u \in W_2^1(Q_T), \quad u_t(l, t) \in L_2(0, T) \}, \]

\[ \hat{W}(Q_T) = \{ v : v \in W(Q_T), \quad v(x, T) = 0 \}. \]

**Definition.**

A function \( u \in W(Q_T) \) is said to be a generalized solution to the problem (1)–(3) if \( u(x, 0) = 0 \) and an equality (4) holds for all \( v \in \hat{W}(Q_T) \).

**Theorem.**

If

\[ a, a_t, a_{tt}, b, b_t, b_{tt}, c \in C(\bar{Q}_T), \quad f \in L_2(Q_T), \quad a(x, t) \geq a_0 > 0, \]

there can not exist more than one generalized solution to the problem (1)–(3).
2.2. The uniqueness of the solution
Suppose there exist two generalized solutions, $u_1(x, t)$ and $u_2(x, t)$, to the problem (1). Then their difference $u(x, t) = u_1(x, t) - u_2(x, t)$ satisfies the condition $u(x, 0) = 0$ and identity
\[
\int_0^T \int_0^l (-u_t v_t + au_x v_x + bu_t v + cuv) dx dt + \gamma \int_0^T a(l, t)v(l, t)u_t(l, t) dt = 0
\] (5)
holds. Choose in (5)
\[
v(x, t) = \begin{cases} 
\int_0^t u(x, \eta) d\eta, x \leq t, \\
0, \tau \leq t \leq T.
\end{cases}
\]
Note that $v_t(x, t) = u(x, t)$. Integrating by parts the terms of left side in (5), we get
\[
\int_0^T \int_0^l -u_t v_t dx dt = - \int_0^T \int_0^l u_t u v dx dt = - \frac{1}{2} \int_0^l u^2(x, \tau) dx;
\]
\[
\int_0^T \int_0^l au_x v_x dx dt = - \int_0^\tau \int_0^l av_x v_x dx dt = - \frac{1}{2} \int_0^\tau \int_0^l a v^2_x dx dt - \frac{1}{2} \int_0^\tau \int_0^l av^2_x(x, 0) dx;
\]
\[
\int_0^T \int_0^l bu_t v dx dt = \int_0^T \int_0^l bu^2 dx dt + \frac{1}{2} \int_0^T \int_0^l b u v^2 dx dt + \frac{1}{2} \int_0^T \int_0^l b_t(x, 0) v^2(x, 0) dx;
\]
\[
\int_0^T a(l, t)v(l, t)u_t(l, t) dx = - \int_0^T \int_0^l a(l, t) u^2(l, t) dt - \int_0^\tau \int_0^l a v_t(l, t) v(l, t) dt = \int_0^T \int_0^l a(l, t) u^2(l, t) dt + \frac{1}{2} \int_0^T \int_0^l a u v^2(l, t) dt + \frac{1}{2} a(l, 0) v^2(l, 0).
\]
Let’s put the results in (5)
\[
\int_0^l [u^2(x, \tau) + av^2_x(x, 0)] dx + \int_0^\tau a(l, t) u^2(l, t) dt = 2 \int_0^T \int_0^l cuv dx dt - \\
- \int_0^\tau \int_0^l a v^2_x dx dt - 2 \int_0^T \int_0^l b u^2 dx dt + \int_0^T \int_0^l b u v^2 dx dt + \int_0^T \int_0^l b_t(x, 0) v^2(x, 0) dx + \\
+ \int_0^\tau a u(l, t) v^2(l, t) dt + a(l, 0) v^2(l, 0).
\] (6)
Since $a(x, t) > 0$ in the $\bar{Q}_T$, then from (6) it follows an inequality
\[
\int_0^l [u^2(x, \tau) + a(x, 0) v^2_x(x, 0)] dx + \int_0^\tau a(l, t) u^2(l, t) dt \leq 2 \int_0^T \int_0^l cuv dx dt + \\
+ \int_0^\tau \int_0^l a u v^2_x dx dt | + 2 \int_0^T \int_0^l b u^2 dx dt | + \int_0^T \int_0^l b u v^2 dx dt | + \\
+ \int_0^\tau b_t(x, 0) v^2(x, 0) dx | + \int_0^\tau a u(l, t) v^2(l, t) dt | + a(l, 0) v^2(l, 0) |.
\] (7)
Let’s estimate the right side of (7). Since $c, a, a_t, a u, b, b_t, b_t \in C(\bar{Q}_T)$, there are numbers $a_1, c_0, b_0$ such that $|a, a_t, a_u| \leq a_1, |b, b_t, b_t| \leq b_0, |c| \leq c_0$. Then
\[
2\left| \int_0^\tau \int_0^l c uv dx dt \right| \leq c_0 \int_0^\tau \int_0^l (u^2 + v^2) dx dt;
\]
\[
\left| \int_0^\tau \int_0^l a v_x^2 dx dt \right| \leq a_1 \int_0^\tau \int_0^l v_x^2 dx dt.
\]

Considering that \( v(x, 0) = -\int_0^\tau u(x, t) dt \) and using Cauchy-Bunyakovsky inequality we get
\[
v^2(x, 0) = (\int_0^\tau ud t)^2 \leq \int_0^\tau u^2 dt.
\]

Then
\[
\left| \int_0^l b_t(x, 0)v^2(x, 0) dx \right| \leq b_0 \int_0^\tau \int_0^l u^2 dx dt;
\]

Applying the inequality (3.9) [1] we get:
\[
v^2(l, t) \leq 2l \int_0^l v^2_x(x, t) dx + \frac{2}{l} \int_0^l v^2(x, t) dx.
\]

Then
\[
\left| \int_0^\tau a_t(l, t) v^2(l, t) dt \right| \leq 2a_1 l \int_0^\tau \int_0^l v_x^2 dx dt + \frac{2a_1}{l} \int_0^\tau \int_0^l v^2(x, t) dx dt.
\]

To estimate the terms containing the value of \( v(l, t) \), apply Cauchy’s inequality “with \( \varepsilon \)” [8]:
\[
|a_1(l, 0)v^2(l, 0)| \leq a_1 \varepsilon \int_0^l v^2_x(x, 0) dx + a_1 c(\varepsilon) \int_0^\tau v^2(x, 0) dx \leq
\]
\[
\leq a_1 \varepsilon \int_0^l v^2_x(x, 0) dx + a_1 c(\varepsilon) \tau \int_0^\tau v^2 dx dt.
\]

We’ll choose \( \varepsilon \) so that \( \mu = a_0 - a_1 \varepsilon > 0 \) and transfer the integral from right side to left:
\[
\int_0^\tau [u^2(x, \tau) + \mu v_x^2(x, 0)] dx \leq M \int_0^\tau \int_0^l (u^2 + v_x^2 + \tau^2) dx dt,
\]
where \( M = max\{a_1 c(\varepsilon), a_1 c(\varepsilon) \tau\} \).

We’re going to do a number of calculations. Because of
\[
v^2(x, t) = (\int_\tau^l u d\eta)^2 \leq (\tau - t) \int_0^\tau u^2 dt,
\]
we have
\[
M \int_0^\tau \int_0^l (u^2 + v_x^2 + v^2) dx dt \leq M \int_0^\tau \int_0^l (u^2 + v_x^2 + \tau (\int_0^\tau u^2)) dx dt \leq
\]
\[
\leq M_1 \int_0^\tau \int_0^l (u^2 + v_x^2) dx dt,
\]
where \( M_1 = M + \tau^2 \).

Since under the integral in the left part there is a \( v_x(x, 0) \), and in the right is not, we will introduce a function
\begin{equation}
    w(x,t) = \int_0^\tau u_x d\eta,
\end{equation}
then
\begin{equation}
    v_x(x,t) = w(x,t) - w(x,\tau), => v_x(x,0) = -w(x,\tau).
\end{equation}

\begin{equation}
    \int_0^l [u^2(x,\tau) + \mu w^2(x,\tau)]dx \leq M_1 \int_0^\tau \int_0^l u^2 dxdt + \\
    + 2M_1 \int_0^\tau \int_0^l w^2(x,t)dxdt + 2M_1 \int_0^\tau \int_0^l w^2(x,\tau)dxdt.
\end{equation}

Note that
\begin{equation}
    \int_0^\tau \int_0^l w^2(x,\tau)dxdt = \tau \int_0^l w^2(x,\tau)dx.
\end{equation}

We will choose \( \tau \) by the arbitrariness so that \( \mu - 2M_1\tau > 0 \). Let for definiteness \( \mu - 2M_1\tau \geq \frac{\mu}{2} \),
then for \( \tau \in [0, M/4M_1] \)
\begin{equation}
    m_0 \int_0^l [u^2(x,\tau)]dx \leq 2M_1 \int_0^\tau \int_0^l [u^2(x,\tau) + w^2(x,\tau)]dxdt,
\end{equation}
where \( m_0 = min\{1, \frac{M}{4M_1}\} \).

Now from Gronowall’s lemma we obtain
\begin{equation}
    u(x,\tau) = 0, \tau \in [0, M/4M_1].
\end{equation}
Continuing this procedure for the \( \tau \in [\frac{M}{4M_1}, \frac{M}{2M_1}] \) using algorithm [8], we get zeros again, which follows the singularity of the generalized solution to the problem.

2.3. The existence of the solution
Let the functions \( w_0(x) \in C^2(0,l) \) linearly independent and form a complete system in the \( W^1_2(0,l) \). We will look for a solution to the problem (1) in the form of
\begin{equation}
    u^m(x,t) = \sum_{k=1}^m c_k(t)w_k(x).
\end{equation}
from the relations
\begin{equation}
    \int_0^l \left(w_{tt}^m w_j + aw_x^m w_j + bw^m w_j + cu^m w_j \right)dx + \gamma \left[a(l,t)u_l^m(l,t)w_j(l)\right] = \int_0^l f w_jdx.
\end{equation}
By substituting (10) in (11) we come to a system of ordinary differential equations:
\begin{align}
    \sum_{k=1}^m \left[c_k''(t) \int_0^l w_k w_j dx + c_k(t) \int_0^l (aw_{kx} w_j + bw_{k t} w_j + cw_k w_j) dx + \\
    \right. \\
    + \gamma aw_{k t} w_j \right] = \int_0^l f w_j dx,
\end{align}
by attaching to which the initial conditions
\begin{equation}
    c_k(0) = 0, c_k'(0) = 0,
\end{equation}
Let's leave only positive terms in the left side:

\[ \int_0^\tau \int_0^l (u_t^m u_x^m + au_x^m u_{xt}^m + b(u_t^m)^2 + cu_x^m u_t^m) dx dt + \gamma \int_0^\tau a(u_t^m(l,t))^2 dt = \int_0^\tau \int_0^l f u_t^m dx dt. \]  

Integrating by parts, taking into account \( u^m(x,0) = u_t^m(x,0) = 0 \) we get

\[ \int_0^\tau \int_0^l u_t^m u_t^m dx dt = \frac{1}{2} \int_0^l (u_t^m(x,\tau))^2 dx; \]

\[ \int_0^\tau \int_0^l au_x^m u_{xt}^m dx dt - \int_0^\tau \int_0^l au_x^m u_x^m dx dt - \int_0^\tau \int_0^l a_t(u_x^m(x,\tau))^2 dx dt = > \]

\[ \int_0^\tau \int_0^l au_x^m u_{xt}^m dx dt = -\frac{1}{2} a_t(u_x^m)^2 dx dt + \frac{1}{2} a(u_x^m(x,\tau))^2 dx. \]

Hence

\[ \frac{1}{2} \int_0^l [(u_t^m(x,\tau))^2] dx + \int_0^\tau \int_0^l b(u_t^m)^2 dx dt + \int_0^\tau \int_0^l cu_x^m u_t^m dx dt - \frac{1}{2} \int_0^\tau \int_0^l a_t(u_x^m)^2 dx dt + \gamma \int_0^\tau a(u_t^m(l,t))^2 dt = \int_0^\tau \int_0^l f u_t^m dx dt. \]  

Let's leave only positive terms in the left side:

\[ \int_0^l [(u_t^m(x,\tau))^2] dx + 2 \gamma \int_0^\tau a(u_t^m(l,t))^2 dt = \int_0^\tau \int_0^l a_t(u_x^m)^2 dx dt - 2 \int_0^\tau \int_0^l b(u_t^m)^2 dx dt - 2 \int_0^\tau \int_0^l cu_x^m u_t^m dx dt + 2 \int_0^\tau \int_0^l f u_t^m dx dt. \]  

From (14) an inequality follows

\[ \int_0^\tau \int_0^l [(u_t^m(x,\tau))^2] dx + 2 \gamma \int_0^\tau a(u_t^m(l,t))^2 dt \leq \]

\[ \leq \int_0^\tau \int_0^l |a_t|(u_x^m)^2 dx dt - 2 \int_0^\tau \int_0^l |b|(u_t^m)^2 dx dt - 2 \int_0^\tau \int_0^l cu_x^m u_t^m dx dt | + 2 | \int_0^\tau \int_0^l f u_t^m dx dt |. \]  

(15)

Let \( a, a_t, b, c \in C^Q \), consequently, there are \( a_1, a_2, b_1, c_0 \) positive and such that

\[ |a| \leq a_1, |b| \leq b_1, |c| \leq c_0. \]

Let's estimate the right part (15) using Cauchy’s inequality:

\[ 2 | \int_0^\tau \int_0^l cu_x^m u_t^m dx dt | \leq c_0 \int_0^\tau \int_0^l [(u_t^m)^2 + (u_x^m)^2] dx dt, \]
\[ 2\int_0^\tau \int_0^l f u^m_t dxdt \leq \int_0^\tau \int_0^l (u^m_t)^2 dxdt + \int_0^\tau \int_0^l f^2 dxdt. \]

So, from (15):

\[ \int_0^l [(u^m_t(x, \tau))^2 + a(u^m_x(x, \tau))^2] dx + 2\gamma \int_0^\tau a(u^m_t(l, t))^2 dt \leq \int_0^\tau \int_0^l [a_2(u^m_x)^2 + (2b_1 + c_0 + 1)(u^m_t)^2 + c_0(u^m)^2] dxdt + \int_0^\tau \int_0^l f^2 dxdt. \]

As

\[ u^m(x, \tau) = \int_0^\tau u^m_t(x, t) dt, \]

then

\[ \int_0^l (u^m(x, \tau))^2 dx \leq \tau \int_0^\tau \int_0^l (u^m_t)^2 dxdt. \]

Add this inequality to (16):

\[ \int_0^l [(u^m(x, \tau))^2 + (u^m_t(x, \tau))^2 + a(u^m_x(x, \tau))^2] dx + 2\gamma \int_0^\tau a(u^m_t(l, t))^2 dt \leq \int_0^\tau \int_0^l [a_2(u^m_x)^2 + (2b_1 + c_0 + 1 + \tau)(u^m_t)^2 + c_0(u^m)^2] dxdt + 2\int_0^\tau \int_0^l f^2 dxdt, \]

where \( m_0 = \min\{1, a_0\}, M = \max\{a_2, 2b_1 + c_0 + 1 + \tau\}, \) and get

\[ m_0 \int_0^l [(u^m(x, \tau))^2 + (u^m_t(x, \tau))^2 + (u^m_x(x, \tau))^2] dx + 2\gamma a_0 \int_0^\tau (u^m_t(l, t))^2 dt \leq M \int_0^\tau \int_0^l [(u^m_x)^2 + (u^m_t)^2 + (u^m)^2] dxdt + \int_0^\tau \int_0^l f^2 dxdt. \]

Apply Gronwall’s lemma:

\[ \int_0^l [(u^m(x, \tau))^2 + (u^m_t(x, \tau))^2 + (u^m_x(x, \tau))^2] dx \leq e^{\frac{m_0}{m_0}} \frac{1}{m_0} \int_0^\tau \int_0^l f^2 dxdt. \]

Integrating over \((0, T)\), we get

\[ ||u^m||_{W^2_2(Q_T)}^2 \leq C ||f||_{L^2_2(Q_T)}^2, \]

where

\[ C = \frac{M}{M}(e^{m_0} - 1). \]

From (17) we obtain

\[ \int_0^\tau (u^m_t(l, t))^2 dt \leq N ||f||_{L^2_2(Q_T)}^2 \]

\[(\forall \tau[0, T] = \Rightarrow ||u^m_t(l, t)||_{L^2_2(Q_T)}^2, \Gamma = \{(x, t) : x = l, t \in [0, T]\}.

Hence,

\[ ||u^m||_{W_2(Q_T)} \leq K, \]

where \( K > 0 \) and does not depend on \( m \).
3. Results and discussion

So we get that \( \{u^m\} \) is bounded in \( W(Q_T) \). Since \( W(Q_T) \) is Hilbert space, there exists a subsequence \( \{u^\mu\} \) which converges weakly to some \( u(x,t) \in W(Q_T) \). We need only to show that this limit function is a required solution to the problem.

The initial condition will be met due to the weak convergence of \( W(Q_T)u^\mu(x,t) \) to \( u(x,t) \) and that \( u^\mu(x,0) = 0 \), in \( L_2(0,l) \). Let’s show that this limit satisfies (4). Multiply (11) by the function of \( h_j(t) \in C^1(Q_T) \) such that \( h_j(T) = 0 \), sum from \( j = 1 \) to \( j = m \), and integrate over \((0,T)\).

Denote

\[
\eta(x,t) = \sum_{j=1}^{m} h_j(t) w_j(x).
\]

Then

\[
\int_0^T \int_0^l \left( u^m_t \eta + au^m_x \eta_t + bu^m_t \eta + cu^m \eta \right) dxdt + \gamma \int_0^T a(l,t) u^m_t \eta dt = \int_0^T \int_0^l f(x,t) \eta(x,t) dxdt.
\]

Integrate by parts, taking into account that \( \eta(x,t) = 0 \):

\[
\int_0^T \int_0^l \left( -u^m_t \eta_t + au^m_x \eta_t + bu^m_t \eta + cu^m \eta \right) dxdt + \gamma \int_0^T au^m_t \eta dt - \int_0^l u^m_t(x,0) \eta(x,0) dx = \int_0^T \int_0^l f(x,t) \eta(x,t) dxdt.
\]

Identity (19) is fulfilled for any function \( \eta(x,t) \). Note that the totality of such functions \( \eta(x,t) = \sum_{j=1}^{m} h_j(t) w_j(x) \) is dense in \( \hat{W}(Q_T) \). Hence, (19) is fulfilled for all \( v \in \hat{W}(Q_T) \).

The proof is completed.

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