Abstract

A lattice calculation of the form factors that determine the “hadronization ratios”, such as \( R_{K^*} \) and \( R_{\phi} \), where \( R_{K^*} \equiv \frac{\Gamma(B \to K^*\gamma)}{\Gamma(b \to s\gamma)} \), is presented in the quenched approximation. Lattice data shows strong evidence for the scaling law suggested by heavy quark symmetry for one of the form factors (i.e. \( T_2 \)). The data also gives strong support for the simple pole ansatz for the \( q^2 \) dependence of \( T_2 \) in the range of available \( q^2 \). We thus find \( T_2(0) = .10 \pm .01 \pm .03 \) yielding \( R_{K^*} = (6.0 \pm 1.2 \pm 3.4)\% \); we also find \( R_{\phi} = (6.6 \pm 1.3 \pm 3.7)\% \).
The loop decays of the $b$-quark have long been noted for their capacity to provide important tests of the Standard Model (SM) [1]. Since many of these decays are short distance dominated at the quark level, their inclusive rates are amenable to perturbation theory. Thus for inclusive processes reliable predictions can be made. The fact that the $b$-quark has a relatively long lifetime facilitates experimental tests of the theoretical predictions. In particular, in the SM, the simple decay $b \rightarrow s + \gamma$ is predicted to have a branching ratio which varies from $\simeq 2 \times 10^{-4}$ to $\simeq 4 \times 10^{-4}$ as the top quark mass varies from 100 to 200 GeV [2]. Assuming three generation unitarity, along with $V_{cs} \simeq V_{tb} \simeq 1$, it is easy to see that $b \rightarrow s + \gamma$ is independent of CKM angles [3] to a very good approximation. Furthermore, $b \rightarrow s + \gamma$ is also noted for its sensitivity to extensions of the SM [4]. Finally, we note that $b \rightarrow s(d) + \gamma$ can also lead to a determination of $V_{ts}$ and $V_{td}$. However, the full potential of the loop decays of the $b$-quark is very difficult to capitalize upon unless reliable theoretical predictions can also be made at the exclusive level.

Indeed, the inclusive process $b \rightarrow s + \gamma$ is challenging to measure experimentally; whereas a corresponding exclusive mode ($i.e.$, $B \rightarrow K^* + \gamma$) has
a distinctive signature and is much more accessible to experiment. Thus a meaningful confrontation between experiment and the underlying electroweak theory can be facilitated through a knowledge of the “hadronization ratio”, $R_{K^*}$:

$$R_{K^*} \equiv \frac{\Gamma(B \to K^*\gamma)}{\Gamma(b \to s\gamma)}$$

which is the probability for the formation of the $K^*$. The evaluation of this ratio by continuum methods has proven to be extremely difficult. This is reflected in the wide range $\sim 1$ to $\sim 97\%$ in the value of $R_{K^*}$, as calculated by quark models, QCD sum rules, heavy quark symmetry (HQS) extended to include the $s$-quark, etc. (see Table 1) [5]. Under the circumstances, the hard earned experimental determination of the branching ratio for $B \to K^* + \gamma$ may only be used to select amongst various models of hadronization rather than to test the underlying theory. It is thus clearly important to explore the use of lattice methods for treating such exclusive decays of $B$ mesons.

At the quark level the decay is described by an effective Hamiltonian [2, 4]:

\[ \]
\[ H_{\text{eff}} = G_{g_1 g_2 g_3} (m_t, \mu) \ V_{ts} \ \bar{s}(x) \sigma_{\mu\nu} b_R(x) \ F^{\mu\nu}(x) \] (2)

where \( F^{\mu\nu} \) is the photon field strength tensor, \( b_R \equiv \frac{1+\gamma_5}{2} b \), and the \( c \) number coefficient \( G_{g_1 g_2 g_3} (m_t, \mu) \) depends on all the three gauge couplings of the Standard Model (SM), the mass (\( m_t \)) of the top quark and a renormalization point \( \mu \). The dependence on the CKM angle \( V_{ts} \) has also been factored out. \( V_{tb} \) is assumed to be 1, and the negligibly small \( u \) quark contribution is ignored.

As usual \([6]\) the lattice is used for a non-perturbative evaluation of the matrix element \( M_{\mu} \equiv \langle V(k) | J_\mu | P(p) \rangle \), where \( P \) is the initial pseudoscalar heavy-light meson, \( V \) is the final vector meson, \( J^\mu \equiv \bar{s} \sigma^{\mu\nu} q_\nu b_R = (v + a)_\mu \) is the current, with \( v_\mu \) and \( a_\mu \) the vector and axial parts, and \( q \equiv p - k \) is the 4-momentum of the photon. In general, the Euclidean matrix element can be parameterized in terms of three form factors \([5, 7]\):

\[
M_{\mu} = 2\epsilon_{\mu\nu\lambda\sigma} \eta_{\nu}(k) p^\lambda k^\sigma T_1(q^2) + [\eta_{\mu}(k)(m^2_H - m^2_V)] \\
- \eta \cdot q(p + k)_\mu T_2(q^2) + \eta \cdot q \left[ q_\mu - \frac{q^2}{m^2_H - m^2_V} (p + k)_\mu \right] T_3(q^2) \] (3)
(Our $\gamma$ matrices obey \{\(\gamma_\mu, \gamma_\nu\)\} = 2\(\delta_{\mu\nu}\), and momenta are defined by \(p_\mu = (E, i\vec{p})\), so that \(p^2 = m^2\) on shell.) For a lattice calculation it is simpler to note that the \(T_1(q^2)\) term arises purely from the vector piece of \(J_\mu\) and the \(T_2\) and \(T_3\) terms given above arise from the axial piece. The third term does not contribute when the photon is on shell. Furthermore, at the end-point, \(q^2 = q_{\max}^2 \equiv (m_H - m_V)^2\), where the final and initial mesons are both at rest, \(T_3\) term does not contribute to the axial matrix element. Since also at that kinematic point no momentum injection is required, \(T_2(q_{\max}^2)\) can be readily, and rather cleanly, evaluated on the lattice. Although \(q^2 = 0\) (the point of direct physical interest) is not exactly accessible to the lattice, in many instances the parameters used in the current simulation do allow \(q^2\) to be extremely small, \(i.e.\ q^2/m_H^2 \leq .1\). Finally we note that using the identity
\[
\sigma_{\mu\nu}\gamma_5 \equiv -\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\sigma^{\lambda\rho}
\]
one can show that
\[
T_2(0) = T_1(0) . \quad (4)
\]
Now the hadronization ratio of interest takes the simple form (for \(m_s \ll m_H\)):
\[ R_{K^*} = 4 \left( \frac{m_B}{m_b} \right)^3 \left[ 1 - \frac{m_K^2}{m_B^2} \right]^3 |T_1(0)|^2. \] (5)

On current lattices \( q^2 = 0 \) (or near that point) is inaccessible for very heavy meson masses, say \( m_H \geq 3.5 \text{ GeV} \). So at \( m_H \sim m_B \), \( T_1(0) \) is not directly calculable. However HQS \( \text{[8]} \) allows one to predict the behavior of \( T_2(q_{max}^2) \) at large \( m_H \). Indeed, when \( q^2 = q_{max}^2 \) no large momenta is transferred to the recoiling light hadron, so a straightforward argument shows that \( \sqrt{m_H T_2(q_{max}^2)} \to \text{const.} \) (up to logarithms) as \( m_H \to \infty \). This makes possible a controlled extrapolation of \( T_2(q_{max}^2) \). Our strategy on the lattice will thus be:

1) test pole dominance of \( T_2 \) at fixed \( m_H \), to the extent that the data allow, by deducing \( T_2(0) \) from \( T_2(q_{max}^2) \) using the equation:

\[ T_2(0) = T_2(q_{max}^2) \left[ 1 - \frac{q_{max}^2}{m_H^2} \right] \] (6)

and comparing to \( T_1(0) \) using eq. (4). \( T_1(0) \) is also obtained using pole dominance, but only from \( T_1 \) at small values of \( q^2 \) \((q^2/m_H^2 < 0.1)\). Pole dominance does not appear to work well for \( T_1(q^2) \) with large \( q^2 \).

2) extract \( T_2(q_{max}^2) \) at \( m_H = m_B \) by fitting the data to the form suggested
by HQS, namely:

$$\sqrt{m_H} T_2(q_{\text{max}}^2) = A_1 + A_2 \left( \frac{1}{m_H} \right).$$

(7)

3) use pole dominance for $T_2$ at $m_H = m_B$ to deduce $T_1(0) = T_2(0)$ from $T_2(q_{\text{max}}^2)$.

We remark that in testing pole dominance, we have simply used the pseudoscalar mass in eq.(6) because in the limit of large $m_H$, HQS implies that resonances of different spin parities become degenerate \footnote{8}. Note also that step 3) uses pole dominance over a wider range of $q^2 (q_{\text{max}}^2/m_B^2 \approx 0.65)$ than can be explicitly checked in step 1) $(q_{\text{max}}^2/m_H^2 \lesssim 0.3)$. We attempt to estimate below the systematic error associated with this step.

We mention the following technical points, regarding the lattice calculations, in brief \footnote{9}:

1) The recently proposed normalization of the Wilson quarks on the lattice \footnote{10, 11, 12}:

$$\psi_{\text{continuum}} = \sqrt{2\kappa} \exp(a\tilde{m}) \psi_{\text{lattice}}$$

(8)

where
\[ a\tilde{m} = \ln \left[ \frac{1}{2\tilde{\kappa}} - 3 \right] \]

and \( \tilde{\kappa} = \kappa/8\kappa_c \) (\( \kappa_c \) is the critical hopping parameter) is used. Thus the leading corrections that become important as \( am \) gets large are automatically included.

2) For the renormalization of the tensor current we incorporate the correction calculated in lattice weak coupling perturbation theory to one loop order [13]. However, following Lepage and MacKenzie [11], the tadpole contribution is removed from the correction (it is already included in eq. (8)), and a “boosted” value of \( g_3 = g_V(1/a) \) is used.

We have done the calculation of \( T_1 \) and \( T_2 \) on four different sets of lattices:

A) \( \beta = 6.3, \ 24^3 \times 61 \) (20 configurations, \( a^{-1} = 3.01 \text{ GeV} \)),
B1) \( \beta = 6.0, \ 24^3 \times 39 \) (8 configurations, \( a^{-1} = 2.29 \text{ GeV} \)),
B2) \( \beta = 6.0, \ 24^3 \times 39 \) (a second set of 8 configurations, \( a^{-1} = 2.29 \text{ GeV} \)),
C) \( \beta = 6.0, \ 16^3 \times 39 \) (19 configurations, \( a^{-1} = 2.10 \text{ GeV} \)).

The lattice spacings given above are determined through a calculation of \( f_\pi \) with the same point sources that are used here [12]. The “B” is always taken at rest; the “\( K^* \)” is given lattice
momentum \((0,0,0), (1,0,0), (1,1,0)\) or \((2,0,0)\), with \((2,0,0)\) used only on B1 and B2. Preliminary results of this computation have been presented previously. \([14]\)

We first work in the case when the masses of the two light quarks are held equal. Experimentally this situation corresponds to the decay, for example, \(B_s \rightarrow \phi + \gamma\). For the light quark we use \(\kappa = .148\) at \(\beta = 6.3\) and \(\kappa = .152\) at \(\beta = 6.0\), yielding a vector meson in the final state with mass \(\approx 1.3\) GeV. The dependence of the amplitude on the heavy quark mass is then studied. Specifically, for \(\beta = 6.3\), we use \(\kappa = 0.140, 0.125, 0.110\) and 0.100 for the heavy quark. Results are given in Table 2; the last column shows that \(\sqrt{m_H} T_2(q_{max}^2)\) is approximately constant. We then fit the data to the two parameter form (eq. (6)) suggested by HQS taking the correlations in the data into account through covariant fits. For the \(\beta = 6.3\) data we find \(A_1 = .806 \pm .069\) (GeV)\(^{1/2}\), \(A_2 = -.545 \pm .082\) (GeV)\(^{3/2}\), \((\chi^2/dof \approx 2.3/2)\).

Thus

\[ T_2(m_H = m_{B_s}, q_{max}^2) = .304 \pm .030 . \]

We now discuss the systematic errors on \(T_2\), first considering those relevant to \(B_s \rightarrow \phi + \gamma\). To correct for the physical s-quark we also study the
matrix elements with $\kappa = 0.149$ (corresponding to vector meson of mass about 1.1 GeV), at $\beta = 6.3$. We find a shift in $T_2$, from its value at $\kappa = 0.148$ of about $-7\%$. Extrapolating to the physical $s$-quark would give a shift of $-10\%$. In passing we mention that a similar study of our lattices at $\beta = 6.0$ indicates a smaller error than the $10\%$ seen at $\beta = 6.3$.

We now assess the systematic error due to the use of heavy quarks with $am \gtrsim 1$. For that purpose, we fit to the two parameter form using the two lightest quarks from our heavy set (of four) at $\beta = 6.3$; i.e. we retain only $\kappa = 0.140$ and $\kappa = 0.125$. We find a shift in the value of $T_2$ of $3.1\%$.

To estimate scale breaking errors we compare the fit for the $\beta = 6.3$ data with the heavy quarks at $\kappa = .140$ and .125 to the fit for the $\beta = 6.0$ data with the corresponding heavy quarks at $\kappa = .135$ and .118. We attribute the difference of $12.2\%$ to scaling violations.

The systematic errors due to finite size effects are deduced by comparing the value of $T_2$ on our $16^3$ lattice with the one on the $24^3$ lattice, both at $\beta = 6.0$. We find a difference of $9.4\%$.

Adding in quadrature the errors due to the four sources mentioned above we find a total systematic error of $19\%$. In passing we note, however, that
the systematic error due to each of these four sources is actually smaller than the statistical errors in the appropriate subset of data. It is, therefore, quite likely that the estimate of 19% is a conservative one. Thus, we arrive at:

\[ T_2(m_H = m_{B_s}, q^2_{\text{max}}) = 0.304 \pm 0.030 \pm 0.057 . \] (9)

Table 3 summarizes our test of the pole dominance for \( T_2 \). By examining the agreement between \( T_2(0) \) and \( T_1(0) \) we see that, within the available range of \( q^2_{\text{max}}/m^2_H \leq 0.3 \), the pole-model seems to work very well. We must note, however, that in the actual physical reactions of interest \( q^2_{\text{max}}/m^2_H \) approaches about 0.65. To estimate the error involved, we note that the biggest difference between \( T_1(0) \) and \( T_2(0) \) is \( \sim 9\% \) (for lattice C). Scaling by the increased range in \( q^2 \) for the physical reaction (0.3 → 0.65), we arrive at an error of 20%. Since the data points with higher \( q^2_{\text{max}}/m^2_H \) in Table 3 seem to support pole ansatz just as well as those with lower values of \( q^2_{\text{max}}/m^2_H \), this is likely to be an overestimate, but we wish to be conservative. Using eqs.(4), (5), (6) and (9) we thus find:

\[ T_{1, B_s \to \phi}(0) = T_{2, B_s \to \phi}(0) = 0.104 \pm 0.010 \pm 0.028 . \] (10)
\[ R_\phi = (6.6 \pm 1.3 \pm 3.7)\% , \]  

which is the hadronization ratio for \( B_s \to \phi + \gamma \). Note that in this calculation we have taken \( m_b = 4.5 \text{ GeV} \), so that we may use the result for \( \text{BR}(b \to s\gamma) \) given by Misiak \[4\]. A 13\% uncertainty is added in quadrature to the systematic errors on \( R \) corresponding to an assumed 200 MeV uncertainty in \( m_b \).

Next we turn our attention to \( B \to K^* + \gamma \). For this purpose we study matrix elements with unequal masses for the light quarks. For example, at \( \beta = 6.0 \) we take the spectator quark with \( \kappa = 0.154 \) and the “s” quark (corresponding to the light quark that results from the weak decay of the \( b \)-quark) with \( \kappa = 0.152 \). Furthermore, we have to extrapolate in the masses of these two quarks. In particular, the spectator quark requires extrapolation to the chiral limit (\textit{i.e.} \( \kappa_c = 0.157 \) at \( \beta = 6.0 \)). For this study we use the \( \beta = 6.0, 24^3 \) lattice as it has the largest physical volume. This lattice has two independent sets of configurations with eight configurations in each sample. As a result we find that extrapolation to the physical limit causes a shift in \( T_2 \) from its value calculated with degenerate light quarks (\( \kappa = 0.152 \) at
\( \beta = 6.0 \) of about \(+7\%\). We shift the central value of \( T_2 \) accordingly and also include this additional \( 7\% \) in the systematic errors. Consequently we arrive at:

\[
T_2(m_H = m_B, q_{\text{max}}^2) = 0.325 \pm 0.033 \pm 0.065 .
\] (12)

Once again we use pole dominance to get:

\[
T_1^{B \rightarrow K^*}(0) = T_2^{B \rightarrow K^*}(0) = 0.101 \pm 0.010 \pm 0.028 ,
\] (13)

\[
R_{K^*} = (6.0 \pm 1.2 \pm 3.4)\% .
\] (14)

Now, as mentioned earlier, the inclusive branching ratio for \( b \rightarrow s \gamma \) is predicted to lie in the range of about \((2 - 4) \times 10^{-4}\) depending on \( m_t \). Thus, in the SM, there is a bound, \( \text{BR}(b \rightarrow s + \gamma) \leq 4 \times 10^{-4} \), corresponding to \( m_t \approx 200 \text{ GeV} \). Combining this upper bound with the above lattice result one gets:

\[
\text{BR}(B \rightarrow K^* \gamma) \lesssim (2.4 \pm 0.5 \pm 1.4) \times 10^{-5} .
\] (15)
We recall now the recent CLEO result \cite{15}:

\[ BR(B \to K^*\gamma) = (4.5 \pm 1.5 \pm 0.9) \times 10^{-5}. \]  

Given the size of the errors in the lattice calculation, as well as in the experiment, the CLEO result is certainly \textit{not inconsistent} with the expectations based on the lattice. However, we note that the numbers seem to mildly favor a rather heavy top quark.

In an attempt to quantify the statement about \( m_t \) we note that the experimental result along with the lattice implies:

\[ BR(b \to s\gamma) = (4.5\pm1.5\pm0.9)\times10^{-5}/(6.0\pm1.2\pm3.4)\times10^{-2} \approx (7.5\pm5.4)\times10^{-4} \]

where we have assigned a \( \sim 70\% \) combined error to the lattice plus the experimental result. At the 1-\( \sigma \) level one then finds \( m_t \gtrsim 100\text{GeV} \). However modest improvements in the the lattice and/or experimental results could produce a rather stringent bound.

To summarize, we have used lattice methods to evaluate the form factors for the radiative \( B \) transitions. We emphasize that the heavy quark limit of QCD \cite{8} enters in an important way in making this calculation feasible on current lattices. We also want to highlight two drawbacks of the present
effort. First, numerical limitations did not allow us to check pole dominance for the specific value of the momentum transfer relevant to the experiment. However, lattice parameters did allow us to check the pole model for $T_2(0)$ for an appreciable range of momenta, giving strong support to its validity. We have included what we believe is a conservative estimate of 20% systematic error due to the use of pole dominance. The second limitation is, of course, the quenched approximation. It is generally believed that with the use of a physical quantity (e.g. $f_\pi$ in our work [12]) to set the scale for the lattice calculations, errors due to quenching are likely to be quite small, perhaps $\leq 10\%$, in the form factors of interest here. It is, therefore, unlikely that the present limitations would seriously affect our results, given the $\sim 28\%$ error in amplitude. Quenched simulations are now in progress that should allow us to improve the calculations to the 10–15% level. At that stage errors due to quenching may also start to become relevant.

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Table 1: A sample compilation of the predictions for $R_{K^*} \equiv \frac{\Gamma(B - K^*\gamma)/\Gamma(b \to s\gamma)}{\Gamma(B - K^*\gamma)/\Gamma(b \to s\gamma)}$. See Ref. 5.

| Author(s)                | $R_{K^*}$   |
|--------------------------|-------------|
| O’Donnell (1986)         | 97%         |
| Deshpande et al. (1988)  | 6%          |
| Domingues et al. (1988)  | 28 ± 11%    |
| Altomari (1988)          | 4.5%        |
| Deshpande et al. (1989)  | 6–14%       |
| Aliev et al. (1990)      | 39%         |
| Ali et al. (1991)        | 28–40%      |
| Du et al. (1992)         | 69%         |
| Faustov et al. (1992)    | 6.5%        |
| El-Hassan et al. (1992)  | $\sim 0.7\% - 12\%$ |
| O’Donnell et al. (1993)  | $\sim 10\%$ |
| Ali et al. (1993)        | 13 ± 3%     |
| Ball (1994)              | 20 ± 6%     |
| This work                | (6.0 ± 1.2 ± 3.4)% |
Table 2: Lattice data on four sets of lattices. $\kappa_1$ represents the initial heavy quark undergoing weak decay, $\kappa_2$ the light quark emerging from the weak decay. The spectator quark is taken to have $\kappa_2$ as well. $m_H$ and $m_V$ are the masses of the initial and the final $0^-$ and $1^-$ mesons respectively and $r_{\text{max}} \equiv [q_{\text{max}}^2/m_H^2]$.

| $\beta(a^{-1}/\text{GeV})$ | Lattice Set | $\kappa_1,\kappa_2$ | $am_H$ | $am_V$ | $r_{\text{max}}$ | $T_2(q_{\text{max}}^2)$ | $\sqrt{m_H T_2(q_{\text{max}}^2)}$ (GeV)² |
|-----------------------------|-------------|----------------------|--------|--------|-----------------|-----------------|-----------------------------|
| 6.3(3.01) $\{0.151\}$     | A           | 140,148              | .590   | .422   | .081            | .406 ± .046     | .54 ± .06         |
|                             | A           | 125,148              | .934   | .422   | .301            | .384 ± .044     | .64 ± .08         |
|                             | A           | 110,148              | 1.248  | .422   | .443            | .364 ± .048     | .71 ± .10         |
|                             | A           | 100,148              | 1.465  | .422   | .508            | .346 ± .055     | .73 ± .12         |
| 6.0(2.29) $\{0.157\}$     | B1          | 135,152              | .894   | .561   | .139            | .409 ± .090     | .58 ± .13         |
|                             | B1          | 118,152              | 1.244  | .561   | .301            | .371 ± .105     | .63 ± .18         |
| 6.0(2.29)                   | B2          | 135,152              | .891   | .566   | .139            | .478 ± .090     | .69 ± .13         |
|                             | B2          | 118,152              | 1.245  | .566   | .301            | .415 ± .105     | .71 ± .17         |
| 6.0(2.10)                   | C           | 142,152              | .734   | .564   | .053            | .470 ± .062     | .58 ± .08         |
|                             | C           | 135,152              | .888   | .564   | .133            | .459 ± .065     | .63 ± .09         |
|                             | C           | 118,152              | 1.241  | .564   | .298            | .414 ± .089     | .67 ± .14         |
Table 3: Test of the pole model for the $q^2$ dependence of the form factors; in particular, that of $T_2$. $T_2(0)$ and $T_1(0)$ are deduced, from $T_1(q^2)$ and $T_2(q^2)$ seen on the lattice, by using pole dominance, i.e. eq. (6). Note $r \equiv q^2/m_H^2$.

| Lattice Set | $\kappa_1, \kappa_2$ | $r$  | $r_{\text{max}}$ | $T_1(q^2)$  | $T_2(q^2_{\text{max}})$ | $T_1(0)$  | $T_2(0)$  |
|-------------|-------------------|-----|----------------|--------------|--------------------------|------------|------------|
| A           | 125,148           | 0.02| 0.300         | 0.259 $\pm$ 0.035 | 0.384 $\pm$ 0.044 | 0.259 $\pm$ 0.035 | 0.269 $\pm$ 0.032 |
| B1          | 135,152           | 0.09| 0.139         | 0.391 $\pm$ 0.069 | 0.409 $\pm$ 0.090 | 0.388 $\pm$ 0.068 | 0.352 $\pm$ 0.077 |
| B2          | 135,152           | 0.09| 0.139         | 0.436 $\pm$ 0.092 | 0.478 $\pm$ 0.090 | 0.432 $\pm$ 0.091 | 0.411 $\pm$ 0.078 |
| B1          | 118,152           | −0.34| 0.301       | 0.264 $\pm$ 0.050 | 0.371 $\pm$ 0.105 | 0.272 $\pm$ 0.051 | 0.260 $\pm$ 0.073 |
| B2          | 118,152           | −0.34| 0.301       | 0.316 $\pm$ 0.110 | 0.415 $\pm$ 0.100 | 0.327 $\pm$ 0.113 | 0.290 $\pm$ 0.070 |
| C           | 118,152           | −0.069| 0.298      | 0.300 $\pm$ 0.039 | 0.414 $\pm$ 0.089 | 0.321 $\pm$ 0.042 | 0.291 $\pm$ 0.062 |