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The Spectrum of Fluctuations in Singularity-free Inflationary Quantum Cosmology

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We calculate the power spectrum of vacuum fluctuations of a generic scalar field in a quantum cosmological setting that is manifestly singularity-free. The power spectrum is given in terms of the usual scale invariant spectrum plus scale dependent corrections. These are induced by well-defined quantum fluctuations of de Sitter spacetime, resulting in a modified dispersion relation for the scalar field. The leading correction turns out to be proportional to the ratio of Hubble scale and Planck Mass. The maximal relative change in the spectrum is on the ten percent level and might be observable with future CMB experiments.

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A fundamental description of the physics during the Planck epoch remains elusive. Important issues like the big bang singularity, the unknown laws that govern the dynamics of all physical degrees of freedom during that period and the correct initial conditions are not yet fully understood.

The success of inflationary cosmology in explaining an overwhelming amount of data of unprecedented precision can be considered as a proof that the cosmological evolution suppresses to a large extent imprints from the Planck epoch. Nevertheless, at least in principal, some cosmological observables might have retained a memory of initial conditions and dynamics of the Planck epoch. Especially the cosmic microwave background (CMB) radiation should bear imprints from the Planck epoch, through its dependence on the spectrum of inflationary density fluctuations.

In recent years, a growing body of researchers have investigated the impact of fundamental (or unknown) physics through this cosmological window\textsuperscript{2} and the new field is often referred to as transplanckian physics\textsuperscript{11}. An important issue addressed there is the magnitude with which processes of characteristic energies around a fundamental scale $M_P$ (e.g. the string or the Planck scale) couple to known physics far below $M_P$. On dimensional grounds this coupling is expected to be proportional to $(H/M_P)^r$, where $H$ denotes the inflationary Hubble scale. The power $r$ depends on the details of the specific transplanckian model that has been chosen. Even if the inflationary Hubble scale is at the GUT scale, $r \leq 1$ is required to extract Planck imprints from the CMB, in order to disentangle them from uncertainties due to cosmic variance.

Instead of modelling physics at the Planck scale, the effective field theory approach\textsuperscript{4,5} allows a more systematic study of high-energy imprints. Physical processes characterized by energy scales above $M_P$ can be related to the dynamics below $M_P$ by only a finite number of couplings. However, the Wilson renormalization group approach presumes a Lagrange description, or an action. To pose a well-defined initial value problem, boundary conditions have to be specified that mimic the corresponding initial conditions in the Hamilton description of the field theory. This is the Boundary Effective Field Theory (BEFT) formalism. Instead of setting the boundary conditions for all modes on an equal time hypersurface, boundary conditions might be set when the physical momentum of a mode is redshifted to a physical cut-off scale. This boundary proposal is called the New Physics Hypersurface (NPH) formalism\textsuperscript{2}. Both, the BEFT and the NPH proposal allow for $r = 1$ modifications of the inflationary power spectrum, see\textsuperscript{8,9} and\textsuperscript{10}. Both boundary proposals modify the initial state of the quantum fluctuation that grow into cosmological perturbations. However, the authors of\textsuperscript{10} find an effective action based on low energy locality allowing only for $r = 2$ corrections to the power spectrum. They claim that the irrelevant operators added to Einstein gravity in the BEFT approach represent a boundary condition that presumably violates low energy locality.

Here, by contrast, we derive a modified dispersion relation from an underlying quantum theory of de Sitter spacetime, concrete models are given in\textsuperscript{11,12}. Their characteristic feature is a careful construction of the inverse scale operator $a^{-1}$, with finite eigenvalues even at the classical singularity. While these eigenvalues approach the expected value $1/a_{cl}$ in the large scale limit, they also carry quantum corrections which lead to modified evolution equations for the minimally coupled scalar field. We take these corrections into account by replacing the classical quantities $a_{cl}$ and $a_{cl}^{-1}$ by the expectation values of the corresponding operators in the evolution equations. This corresponds to a mean field approximation for the gravitational degree of freedom.

In this framework, we derive the evolution equations for the vacuum fluctuations in a general form, valid for
any quantum cosmology that is singularity-free in the sense of a bounded inverse scale operator. For the prediction of the fluctuation spectrum we then use the concrete model discussed in [12].

The quantum theory of the scalar field in the canonical approach is formulated in the dynamical variables \((\Phi, \pi)\) on a de Sitter background. The Hilbert space for the coupled gravity-scalar field system is \(\mathcal{H} = \mathcal{H}_g \otimes \mathcal{H}_s\) where \(\mathcal{H}_g\) denotes the geometrical Hilbert space and \(\mathcal{H}_s\) is the Hilbert space of the scalar field. As discussed above, the quantum dynamics of the combined system is reduced to dynamics for the scalar field operator on a de Sitter background including quantum corrections in the mean field approximation. The mean field approximation is characterized by expectation values \(\langle a \rangle\) and \(\langle a^{-1} \rangle\) of the operators corresponding to the scale factor and the inverse scale factor with respect to appropriate quantum cosmological states. As we aim at deriving the effective evolution equation in as general a form as possible, we will leave this state unspecified until we specialize to the concrete framework of [12].

The effective Hamilton operator for the scalar field theory on this background is then given by

\[
\langle H_s \rangle = \int \langle \text{vol } \rho_s \rangle, \tag{1}
\]

with the energy density

\[
\langle \rho_s \rangle = \frac{1}{2} \left( \langle a^{-3} \rangle \right)^2 + \frac{1}{2} \left( \langle a^{-1} \rangle \right)^2 \nabla \Phi^2 + V(\Phi). \tag{2}
\]

The dynamics of the field operators on the FRW background is given by Heisenberg’s equations of motion

\[
\dot{\Phi} = \langle a^3 \rangle \left( \langle a^{-3} \rangle \right)^2 \pi, \tag{3}
\]

\[
\dot{\pi} = \langle a^3 \rangle \left( \langle a^{-3} \rangle \right)^2 \Phi - \frac{dV}{d\Phi}, \tag{4}
\]

where dots denote derivatives with respect to cosmic time. Heisenberg’s equations are equivalent to the second order differential equation for the scalar field

\[
\ddot{\Phi} = \left[ 3 \langle a \rangle \langle a^{-3} \rangle + 2 \langle a^{-3} \rangle \right] \dot{\Phi} + \left( \langle a^3 \rangle \langle a^{-3} \rangle \right)^2 \left( \langle a^{-1} \rangle \right)^2 \Phi - \frac{dV}{d\Phi}. \tag{5}
\]

Following the standard procedure, we expand the scalar field around its homogeneous expectation value with respect to an arbitrary but fixed vacuum state and Fourier transform the quantum fluctuations around the homogeneous state. The fluctuations are conveniently expressed in terms of time dependent oscillators \(A(k, t)\) and \(A^\dagger(-k, t)\), where \(k\) denotes the comoving wavenumber characterizing the fluctuation. The initial conditions imposed on these solutions correspond to a choice of the vacuum state \(|\Omega, t_i\rangle\) at the initial time \(t_i\). \(A(k, t_i)\) annihilates \(|\Omega, t_i\rangle\). The vacuum state defined this way is often referred to as the lowest energy state, the minimal uncertainty state with respect to \(\Delta \Phi \Delta \pi\) or the instantaneous Minkowski vacuum. For \(t_i \to -\infty\) this vacuum proposal includes the Bunch-Davies vacuum state.

The time evolution given by Bogolubov transformations mixes annihilation and creation operators. In terms of annihilation and creation operators the fluctuations are given by

\[
\Phi(k, t) = \phi(k, t) A(k, t_i) + \phi^*(k, t) A^\dagger(k, t_i), \tag{6}
\]

with \(\phi\) denoting the corresponding mode functions, satisfying the reality condition \(\phi(k, t) = \phi^*(-k, t)\).

Close to the classical de Sitter limit we can expand the expectation values in the gravitational sector around their classical values: \(\langle a \rangle = a_{cl}\) and \(\langle a^{-1} \rangle = a_{cl}^{-1} (1 + \langle a^{-1} \rangle)\), where \(q\) denotes the quantum corrections to de Sitter spacetime and \(\langle a^{-1} \rangle \ll 1\). The quantum cosmological model developed in [12], \(\langle a^{-1} \rangle = \sqrt{2\pi a_{cl}^{-1}} + O(1/a_{cl}^2)\).

Let us introduce the dimensionless variable \(x = a/a_i\), with \(a_i \equiv a(t_i)\), the value of \(a\) at a fixed initial time \(t_i\). In the quantum cosmological models [11][12] the initial scale factor can be fixed very close to the Planck scale, \(a_i = \beta \sqrt{8\pi} L_P\), with \(\beta\) being a parameter constrained by the requirement of having a consistent perturbation analysis. We find \(\beta = O(1 - 10)\), since the quantum corrected spectrum of the inverse scale factor approaches the classical value very fast for \(a > \sqrt{8\pi} L_P\). As we are interested in the semiclassical regime only, we focus on the case \(x > 1\). Then, the fluctuation modes \(\phi(k, x)\) obey

\[
\frac{d^2 \phi}{dx^2} = - (4 + 3f_H(x)) \frac{1}{x} \frac{d\phi}{dx} - (1 + f_k(x)) \frac{k/a_i}{H} \phi, \tag{7}
\]

with the leading quantum correction \(f_H(x) = (\sqrt{8\pi} L_P/a_i) 1/x\) to the Hubble friction and \(f_k(x) = 4f_H(x)\) to the redshift term.

Using \(\phi \to x^{-2} \exp(3f_H(x)/2) \phi\), [16] can be transformed into

\[
\left[ \frac{d^2}{dx^2} + \omega^2(x) \right] \phi(k, x) = 0, \tag{8}
\]

with \(\omega^2 \equiv \omega_\beta^2 + \omega_\phi^2\), see [1]. Here, \(\omega_\beta^2 \equiv (1/x)^4 (k/a_i/H)^2 - 2/x^2\) is the frequency of an oscillator on classical de Sitter spacetime. The time dependence in the classical dispersion relation is due to the space expansion, causing redshift and damping of fluctuations. The leading quantum corrections to de Sitter spacetime modify the dispersion
of the fluctuations:

\[
\omega_q^2 \equiv \frac{1}{x^2} \left[ \frac{1}{x^2} \left( \frac{k/a_i}{H} \right)^2 f_k(x) \right.
\]

\[
- \frac{3}{2} \left( 3 + x \frac{d}{dx} + \frac{3}{2} \right) f_H(x) \right].
\]  

(9)

Deep inside the Hubble radius \((k/aH \gg 1)\), the redshift term dominates the dispersion relation. In this regime, the leading modification of the dispersion relation decays like \(1/\sqrt{x}\) relative to the classical redshift. In the super-Hubble \((k/aH \ll 1)\) regime, the friction term generated by the Hubble expansion dominates and the leading modification has the same asymptotic behavior like the redshift correction for \(x \gg 1\).

The dynamics of fluctuation modes on subhorizon scales is altered mainly due to the redshift modification induced by quantum corrections of de Sitter spacetime. More precisely, the redshift correction modifies the oscillation frequency, while it has only minor influence on the oscillation amplitude. The amplitude is influenced by quantum corrections to the friction term induced by the Hubble expansion. The asymptotics of modes on sub-Hubble scales can be extracted for \(x \gg 1\) from

\[
\phi_{\text{sub}}(x) = \frac{1}{a(2k)^{1/2}} \frac{\exp(3f_H(x)/2)}{(1 + f_k(x))^{1/4}}
\]

\[
\times \exp \left( \frac{k/a_i}{H} \left( 1 + f_k(x) \right)^{3/2} \right). \]  

(10)

The integration constants have been determined by the requirement that on sub-Hubble scales (but still on energy scales below \(M_P\)), the vacuum is populated by quantum fluctuations with positive frequencies only, normalized as in the standard theory of scalar fields on Minkowski spacetime.

Fig. 1 shows the subhorizon evolution of vacuum fluctuations on the standard de Sitter and on the quantum corrected de Sitter spacetime. It can be seen that the main effect of the quantum corrected spacetime is to roughly quadruple the oscillation frequencies. As usual, the oscillation decays like \(1/\sqrt{k}\) for constant \(x\) and increasing wavenumber, and decreases like \(1/x\) for \(x \gg 1\) and constant \(k\).

Vacuum fluctuations on super-Hubble scales are mostly affected by quantum corrections to the Hubble expansion. However, there is a mild dependence on the redshift corrections through the matching of the dynamics on sub- and super-Hubble scales at horizon crossing. The asymptotic behavior of the vacuum fluctuations for \(x \gg 1\) can be extracted from

\[
\phi_{\text{sup}}(x) = \frac{1}{a\sqrt{2k}} \frac{\exp(3f_H(x)/2)}{(1 + f_k(x))^{1/4}}
\]

\[
\times \frac{W_M(1, 3/2, 3f_H(x))}{W_M(1, 3/2, 3f_H(x))}. \]  

(11)

Here, \(W_M\) denotes the Whittaker function that is related to Kummer’s function \(M\), see [14]. \(x_c \equiv k/(a_i H)\) is the value of the scale factor when the vacuum fluctuation characterized by comoving wavenumber \(k\) crossed the Hubble radius, normalized to the initial scale factor.

The dependence on \(x_c\) is chosen in order to match the solutions on sub-Hubble scales [10] at horizon crossing. In this way the choice of a vacuum proposal determines the otherwise unknown integration constants in the super-Hubble solutions. On super-Hubble scales, the general solution is actually a linear combination of both Whittaker functions \(W_M\) and \(W_W\). For \(x \gg 1\), \(W_M \propto 1/x^2\) while \(W_W \propto x\). Hence, only \(W_M\) has the correct asymptotic behavior.

We are now ready to present the main result of this letter: the linear power spectrum (defined as \(P_k \equiv \)
\((k^3/2\pi^2)\langle \Omega | (\Phi^4)(k, x_c) | \Omega \rangle\) of quantum fluctuations of a generic scalar field around its vacuum state:

\[
P_\Phi(k, x_c) = \left( \frac{H}{2\pi} \right)^2 \left[ 1 + 3f_H(x_c) - \frac{1}{2} f_k(x_c) \right] + \mathcal{O} \left[ (H/M_P)^2 \right]. \quad (12)
\]

The leading quantum correction to the Hubble friction at horizon crossing is given by \(f_H(x_c) = f_H(1)/x_c\), with \(f_H(1) \equiv \sqrt{8\pi}L_P/a_i\) and \(f_k(x_c) = 4f_H(x_c)\).

Note that quantum corrections to the redshift evolution lower the statistical power of large wavelength modes, while corrections of the Hubble friction enhance the statistical power on large scales. However, both the quantum correction to the Hubble friction and the correction to the cosmological redshift decay like \(1/x_c\). So vacuum fluctuations characterized by large wavenumbers have less statistical support compared to modes with smaller wavenumbers.

We finally find for the linear power spectrum

\[
P_\Phi(k, x_c) = \left( \frac{H}{2\pi} \right)^2 \left[ 1 + f_H(1) \left( \frac{k/a_i}{H} \right)^{-1} \right] \frac{1}{x_c} + \mathcal{O} \left[ (H/M_P)^2 \right]. \quad (13)
\]

In order to estimate the size of the leading modification to the classical power spectrum \(P_\Phi^{cl}\), we note that \(\delta P_\Phi/P_\Phi^{cl} \equiv P_\Phi/P_\Phi^{cl} - 1 = (\sqrt{8\pi}L_P/a_i)(1/x_c(k))\). Let us estimate \(a_c\) relative to the initial scale factor \(a_i\) for the CMB quadrupole:

\[
\frac{1}{x_c} \bigg|_{l=2} = \frac{H}{x_0 H_0} \cdot (14)
\]

Here, \(H\) and \(H_0\) denote the Hubble radius during inflation and today, respectively. Now, \(x_0 = (a_0/a_c)(a_c/a_i)\), with \(a_c\) denoting the scale factor at the end of inflation. Assuming instantaneous reheating, \(a_0/a_c \approx T_{ri}/T_0 \approx \sqrt{H/M_P}/T_0\), where \(T_0\) is the current temperature of the CMB photons and \(T_{ri}\) is the reheating temperature. We parameterize the ratio of the scale factor at the end of inflation and the initial scale factor by the number of e-folds before the end of inflation, \(a_c/a_i \equiv \exp(N)\). Then (14) becomes

\[
\frac{1}{x_c} \bigg|_{l=2} \approx \exp(-N) \frac{T_0}{H_0} H \sqrt{\frac{H}{M_P}} \approx 10^{29} \exp(-N) \sqrt{\frac{H}{M_P}}. \quad (15)
\]

For the scale corresponding to the CMB quadrupole we find \(x_c = 1\) for \(N \approx 62\), assuming \(H/M_P \approx 10^{-4}\).

As mentioned earlier, in the quantum cosmological model we use here, \[12\], the initial scale factor can be fixed close to the Planck scale, \(a_{ri} = \mathcal{O}(1 - 10)\sqrt{\pi}L_P\). For the scale corresponding to the CMB quadrupole we therefore find

\[
\frac{\delta P_\Phi}{P_\Phi^{cl}} \bigg|_{l=2} \approx \mathcal{O}(10)^\% . \quad (16)
\]

In summary, we have derived the modified dispersion relation for a generic scalar field on de Sitter space subject to quantum gravitational fluctuations in the framework of a singularity-free quantum cosmology. We have calculated the corresponding corrections to the power spectrum and find the leading correction term to be of order \(H/M_P\), which would be promising for future CMB experiments. For the quadrupole scale, we expect an \(\mathcal{O}(10)^\%\) modification of the classical power spectrum.

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