A CLASSICAL ISODUAL THEORY OF ANTIMATTER

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Abstract

An inspection of the contemporary physics literature reveals that, while matter is treated at all levels of study, from Newtonian mechanics to quantum field theory, antimatter is solely treated at the level of second quantization. For the purpose of initiating the restoration of full equivalence in the treatments of matter and antimatter in due time, in this paper we present a classical representation of antimatter which begins at the primitive Newtonian level with expected images at all subsequent levels. By recalling that charge conjugation of particles into antiparticles is anti–automorphic, the proposed theory of antimatter is based on a new map, called isoduality, which is also anti–automorphic, yet it is applicable beginning at the classical level and then persists at the quantum level. As part of our study, we present novel anti–isomorphic isodual images of the Galilean, special and general relativities and show the compatibility of their representation of antimatter with all available classical experimental knowledge, that on electromagnetic interactions. We also identify the prediction of antigravity for antimatter in the field of matter (or vice–versa) without any claim on its validity, and defer its resolution to specific experiments. To avoid a prohibitive length, the paper is restricted to the classical treatment which had not been sufficiently treated until now. Studies on operator profiles, such as the equivalence of isoduality and charge conjugation and the implication of the isodual theory in particle physics, are conducted in a separate paper.

1 INTRODUCTION

After being conjectured by A. Schuster in 1898, antimatter was predicted by P. A. M. Dirac [1] in the late 1920’s in the negative–energy solutions of his celebrated equation. Dirac himself soon discovered that particles with negative–energy do not behave in a physical
way and, for this reason, he submitted his celebrated “hole theory”, which subsequently restricted the study of antimatter to the level of second quantization (for historical aspects on antimatter see, e.g., Ref. [2]).

The above occurrence created an imbalance in the physics of this century because matter is described at all levels of study, from Newtonian mechanics to quantum field theory, while antimatter is solely treated at the level of second quantization.

To initiate the study for the future removal of this imbalance in due time, in this paper we present a theory of antimatter which has been conceived to begin at the purely classical Newtonian level, and then to admit corresponding images at all subsequent levels of study.

Our guiding principle is to identify a map which possesses the main mathematical structure of charge conjugation, yet it is applicable at all levels, and not solely at the operator level.

The main characteristic of charge conjugation is that of being anti–automorphic (or, more generally, anti–isomorphic). After studying a number of possibilities, a map which is anti–isomorphic and applicable at all levels of study, is the following isodual map of any given quantity $Q$

$$Q \rightarrow Q^d = -Q^\dagger.$$  

which, for consistency, must be applied to the totality of the mathematical structure of the conventional theory of matter, including numbers, fields, spaces, geometries, algebras, etc. This results in a new mathematics, called isodual mathematics, which is at the foundation of the classical isodual theory of antimatter of this paper.

Since the isodual mathematics is virtually unknown, we shall review and expand it in Sect. 2. In Sect. 3 we shall then present, apparently for the first time, the classical isodual Galilean, special and general relativities and show that their representation of antimatter is indeed compatible with available classical experimental data, those of electromagnetic nature. In the Appendix we outline the classical isodual Lagrangian and Hamiltonian mechanics. The operator version of the isodual theory of antimatter is studied in a separate papers which also prove the equivalence between isoduality and charge conjugation.

The rather limited existing literature in isoduality is the following. The isodual map (1.1) was first proposed by Santilli in Ref.s [3] of 1985 and then remained ignored for several years. More recently, the isodual numbers characterized by map (1.1) have been studied in paper [4]. The first hypothesis on the isodual theory of antimatter appeared for the operator version in Ref. [5] of 1993 which also contains an initial study of the equivalence between isoduality and charge conjugation. The fundamental notion of this study, the isodual Poincaré symmetry, from which the entirety of the (relativistic) analysis can be uniquely derived, was submitted in Ref. [6] of 1993 also at the operator level.

The prediction of the isodual theory that antimatter in the field of matter experiences antigravity was first submitted in Ref. [7] of 1994. An experiment for the measure of the gravity of elementary antiparticles in the gravitational field of Earth was also proposed in Ref. [7]. It essentially consists in comparative measures of the gravity of collimated, low energy beams of positrons and electrons in horizontal flight on a tube with sufficient vacuum as well as protection from stray fields and of sufficient length to permit a definite result, e.g., the view by the naked eye of the displacements due to gravity of the positron and electron beams on a scintillator at the end of the flight.
The *isodual differential calculus*, which is fundamental for the correct formulation of dynamical equations all the way to those in curved spaces, was studied only recently in Ref. [8]. A review of the *operator* profile up to 1995 is available in monograph [9].

This paper is the classical counterpart of the companion paper [10] in which we study the operator profile, with particular reference to the equivalence between isoduality and charge conjugation and the predictions of the new theory in particle physics. The additional adjoining paper [11] presents a general outline of the isodualities of the broader *isotopic, genotopic and hyperstructural* formulations which are under study for antimatter in *interior* conditions (such as for the structure of an antimatter star) and they are not considered in this paper for brevity.

An important independent contribution in the field has been made by the experimentalist A.P. Mills, Jr. [12], who has confirmed the apparent feasibility with current technology of the test of the gravity of antiparticles proposed in ref. [7] via the use of eletrons and positrons with energy of the order of milli-eV in horizontal flight in a vacuum tube of approximately 100 m length with a diameter and design suitable to reduce stray fields and patch effects at its center down to acceptable levels.

Additional contributions have been made by: J. V. Kadeisvili on the *isodual functional analysis* and *isodual Lie theory* [13]; Lohmus, Paal, Sorgsepp, Sourlas, Tsagas [14]; and others.

Theoretical and experimental studies on the isodual theory of antimatter were conducted at the *International Workshop on Antimatter Gravity and Anti–Hydrogen Atom Spectroscopy*, held in Sepino, Italy, in May 1996 (see the Proceedings [15]).

## 2 RUDIMENTS OF ISODUAL MATHEMATICS

### 2.1 Isodual units, numbers, and fields.

Let $F = F(a, +, \times)$ be a conventional field of real numbers $R(n, +, \times)$, complex numbers $C(c, +, \times)$ or quaternionic numbers $Q(q, +, \times)$ with the familiar additive unit 0, multiplicative unit $I$, elements $a = n, c, q$, sum $a_1 + a_2$, $a + 0 = 0 + a = a$, and multiplication $a_1 \times a_2 = a_1 a_2$, $a \times I = I \times a = a$, $\forall a, a_1, a_2 \in F$.

The *isodual fields*, first introduced in ref. [3] and then studied in details in ref. [4], are the image $F^d = F^d(a^d, +^d, \times^d)$ of $F(a, +, \times)$ characterized by the isodual map of the unit

$$I \rightarrow I^d = -I^\dagger = -I, \quad (2.1)$$

which implies: *isodual numbers*

$$a^d = a^\dagger \times I^d = -a^\dagger \times I = -a^\dagger, \quad (2.2)$$

where $^\dagger$ is the identity for real numbers $n^\dagger = n$, complex conjugation $c^\dagger = \bar{c}$ for complex numbers $c$ and Hermitean conjugation $q^\dagger$ for quaternions $q^\dagger$; *isodual sum*

$$a_1^d +^d a_2^d = -(a_1^\dagger + a_2^\dagger); \quad (2.3)$$

and *isodual multiplication*
\[
a^d_1 \times d_a^d = a^d_1 \times I^d \times a^d_2 = = -a^\dagger_1 \times a^\dagger_2; \quad (2.4)
\]
under which \(I^d\) is the correct left and right unit of \(F^d\),
\[
I^d \times d a^d = a^d \times I^d \equiv a^d, \quad \forall a^d \in F^d,
\]
in which case (only) \(\hat{I}^d\) is called iso\-dual unit.

We have in this way the iso\-dual real field \(R^d(n^d, +^d, \times^d)\) with iso\-dual real numbers
\[
n^d = -n^\dagger \times I \equiv -n, \quad n \in R, \quad n^d \in R^d; \quad (2.6)
\]
the iso\-dual complex field \(C^d(c^d, +^d, \times^d)\) with iso\-dual complex numbers
\[
c^d = -\bar{c} = -(n_1 - i \times n_2) = -n_1 + i \times n_2,
\]
\[
n_1, n_2 \in R, \quad c \in C, \quad c^d \in C^d; \quad (2.7)
\]
and the iso\-dual quaternionic field which is not used in this paper for brevity.

Under the above assumptions, \(F^d(a^d, +^d, \times^d)\) verifies all the axioms of a field [loc. cit.], although \(F^d\) and \(F\) are anti\-isomorphic, as desired. This establishes that the field of numbers can be equally defined either with respect to the traditional unit +1 or with respect to its negative image -1. The key point is the preservation of the axiomatic character of the latter via the isoduality of the multiplication. In other words, the set iso\-dual numbers \(a^d\) with unit -1 and conventional product does not constitute a field because \(I^d \times a^d \neq a^d\).

It is also evident that all operations of numbers implying multiplications must be subjected for consistency to iso\-duality. This implies the iso\-dual square root
\[
a^{d\frac{1}{2}} = -\sqrt{-a^d}, \quad a^{d\frac{1}{2}} \times d a^{d\frac{1}{2}} = a^d, \quad 1^{d\frac{1}{2}} = -i; \quad (2.8)
\]
the iso\-dual quotient
\[
a^d/b^d = -(a^d/b^d) = -(a^\dagger/b^\dagger =) = c^d, \quad b^d \times d c^d = a^d; \quad (2.9)
\]
and so on.

A property of iso\-dual fields of fundamental relevance for our characterization of antimatter is that they have negative\-definite norm, called iso\-dual norm [3,4]
\[
|a^d|^d = |a^\dagger| \times I^d = -(aa^\dagger)^{1/2} < 0. \quad (2.10)
\]
where \(|...|\) denotes the conventional norm. For iso\-dual real numbers \(n^d\) we therefore have the iso\-dual isonorm
\[
|n^d|^d = -|n| < 0, \quad (2.11)
\]
and for iso\-dual complex numbers we have
\[
|c^d|^d = -|\bar{c}| = -(\bar{c}c)^{1/2} = -(n_1^2 + n_2^2)^{1/2}. \quad (2.12)
\]
Lemma 2.1: All quantities which are positive–definite when referred to fields (such as mass, energy, angular momentum, density, temperature, time, etc.) became negative–definite when referred to isodual fields.

As recalled in Sect. 1, antiparticles have been discovered in the negative–energy solutions of Dirac’s equation [1] and they were originally thought to evolve backward in time (Stueckelberg, and others, see [2]). The possibility of representing antimatter and antiparticles via isodual methods is therefore visible already from these introductory notions.

The main novelty is that the conventional treatment of negative–definite energy and time was (and still is) referred to the conventional contemporary unit +1, which leads to a number of contradictions in the physical behavior of antiparticles whose solution forced the transition to second quantization.

By comparison, the negative-definite physical quantities of isodual methods are referred to a negative–definite unit $I^d < 0$. As we shall see, this implies a mathematical and physical equivalence between positive–definite quantities referred to positive–definite units, characterizing matter, and negative–definite quantities referred to negative–definite units, characterizing antimatter. These foundations then permit a novel characterization of antimatter beginning at the Newtonian level, and then persisting at all subsequent levels.

Definition 2.1: A quantity is called isoselfdual when it is invariant under isoduality.

The above notion is particularly important for this paper because it introduces a new invariance, the invariance under isoduality. We shall encounter several isoselfdual quantities. At this introductory stage we indicate that the imaginary number $i$ is isoselfdual,

$$i^d = -i^\dagger = -\bar{i} = -(-i) = i.$$  \hspace{1cm} (2.13)

This property permits to understand better the isoduality of complex numbers which can be written explicitly [4]

$$c^d = (n_1 + i \times n_2)^d = n_1^d + i^d \times n_2^d = -n_1 + i \times n_2 = -\bar{c}.$$  \hspace{1cm} (2.14)

We assume the reader is aware of the emergence here of basically new numbers, those with a negative unit, which have no connection with ordinary negative numbers and which are the true foundations of the proposed isodual theory of antimatter.

2.2 Isodual functional analysis.

All conventional and special functions and transforms, as well as functional analysis at large must be subjected to isoduality for consistent applications of isodual theories, resulting in a simple, yet unique and significant isodual functional analysis, whose study was initiated by Kadeisvili [13].

We here mention the isodual trigonometric functions

$$\sin^d \theta^d = -\sin(-\theta), \quad \cos^d = \theta^d = -\cos(-\theta),$$  \hspace{1cm} (2.15)

with related basic property
\[
\cos^d \theta^d + \sin^d \theta^d = 1^d = -1, \quad (2.16)
\]
the *isodual hyperbolic functions*

\[
sinh^d w^d = -\sinh(-w), \quad \cosh^d w^d = -\cosh(-w), \quad (2.17)
\]
with related basic property

\[
cosh^d w^d - \sinh^d w^d = 1^d = -1, \quad (2.18)
\]
the *isodual logarithm*

\[
\log^d n^d = -\log(-n), \quad (2.19)
\]

etc. Interested readers can then easily construct the isodual image of special functions, transforms, distributions, etc.

### 2.3 Isodual differential calculus.

The conventional differential calculus is indeed dependent on the assumed unit. This property is not so transparent in the conventional formulation because the basic unit is the trivial number +1, thus having null differential. However, the dependence of the unit emerges rather forceful under its generalization.

The *isodual differential calculus*, first introduced in ref. [8], is characterized by the *isodual differentials*

\[
d^d x^k = I^d \times dx^k = -dx^k, \quad d^d x_k = -dx_k, \quad (2.20)
\]
with corresponding *isodual derivatives*

\[
\partial^d / \partial^d x^k = -\partial / \partial x^k, \quad \partial^d / \partial^d x_k = -\partial / \partial x, \quad (2.21)
\]
and other isodual properties.

Note that *conventional differentials are isoselfdual*, i.e.,

\[
(dx^k)^d = d^d x^kd \equiv d x^k, \quad (2.22)
\]
but *derivatives are not in general isoselfdual*,

\[
(\partial f(x)/\partial x^k)^d = \partial^d f^d / \partial^d x^kd = -\partial f / \partial x^k. \quad (2.23)
\]

Other properties can be easily derived and shall be hereon assumed.
2.4 Isodual Lie theory.

Let $L$ be an $n$-dimensional Lie algebra with universal enveloping associative algebra $\xi(L)$, $[\xi(L)]^{-}\approx L$, $n$-dimensional unit $I = \text{diag}(1, 1, \ldots, 1)$ for the regular representation, ordered set of Hermitean generators $X = X^\dagger = \{X_k\}$, conventional associative product $X_i \times X_j$, and familiar Lie’s Theorems over a field $F(a, +, \times)$.

The isodual Lie theory was first submitted in ref. [3] and then studied in Ref. [9] as well as by other authors [13,14]. The isodual universal associative algebra $[\xi(L)]^d$ is characterized by the isodual unit $I^d$, isodual generators $X^d = -X$, and isodual associative product

$$X^d_i \times^d X^d_j = -X_i \times X_j,$$

with corresponding infinite-dimensional basis (isodual version of the conventional Poincaré–Birkhoff–Witt theorem [3]) characterizing the isodual exponentiation of a generic quantity $A$

$$e^{dA} = I^d + A^d/d^1 + A^d \times^d A^d/d^2 + \ldots = -e^{A^\dagger},$$

where $e$ is the conventional exponentiation.

The attached isodual Lie algebra $L^d \approx (\xi^d)^-$ over the isodual field $F^d(a^d, +^d, \times^d)$ is characterized by the isodual commutators [loc. cit.]

$$[X^d_i, X^d_j]^d = -[X_i, X_j] = C_{ij}^k X^d_k,$$

with a classical realization given in Appendix A.

Let $G$ be the conventional, connected, $n$-dimensional Lie transformation group on $S(x, g, F)$ admitting $L$ as the Lie algebra in the neighborhood of the identity, with generators $X_k$ and parameters $w = \{w_k\}$. The isodual Lie group $G^d$ [3] admitting the isodual Lie algebra $L^d$ in the neighborhood of the isodual identity $I^d$ is the $n$-dimensional group with generators $X^d = \{-X_k\}$ and parameters $w^d = \{-w_k\}$ over the isodual field $F^d$ with generic element

$$U^d(w^d) = e^{dA^d \times^d w^d \times^d X^d} = -e^{\dagger \times (-w) \times X} = -U(-w).$$

The isodual symmetries are then defined accordingly via the use of the isodual groups $G^d$ and they are anti–isomorphic to the corresponding conventional symmetries, as desired. For additional details, one may consult Ref. [9]. In this paper we shall therefore use:

1) **Conventional Lie symmetries**, for the characterization of matter; and

2) **Isodual Lie symmetries**, for the characterization of antimatter.

2.5 Isodual Euclidean geometry.

Conventional (vector and) metric spaces are defined over conventional fields. It is evident that the isoduality of fields requires, for consistency, a corresponding isoduality of (vector and) metric spaces. The need for the isodualities of all quantities acting on a metric space
(e.g., conventional and special functions and transforms, differential calculus, etc.) becomes then evident.

Let $S = S(x, g, R)$ be a conventional N–dimensional metric space with local coordinates $x = \{x^k\}, \ k = 1, 2, ..., N$, nowhere degenerate, sufficiently smooth, real–valued and symmetric metric $g(x, ...)$ and related invariant

$$x^2 = x^i g_{ij} x^j, \quad (2.28)$$

to the reals R.

The isodual spaces, first introduced Ref. [3], are the spaces $S^d(x^d, g^d, R^d)$ of $S(x, g, R)$ with isodual coordinates $x^d = x \times I^d$, isodual metric

$$g^d(x^d, ...) = -g^t(-x, ...) = -g(-x, ...), \quad (2.29)$$

and isodual interval

$$(x - y)^{2d}^d = [(x - y)^{id} \times g^{d}_{ij} \times (x - y)^{jd}] \times I^d = [(x - y)^i \times g^d_{ij} \times (x - y)^{jd}] \times I^d, \quad (2.30)$$
defined over the isodual field $R^d = R^d(n^d, +^d, \times^d)$ with the same isodual isounit $I^d$.

The basic space of our analysis is the three–dimensional isodual Euclidean space,

$$E^d(r^d, \delta^d, R^d) : r^d = \{r^{kd}\} = \{-r^k\} = \{-x, -y, -z\},$$

$$\delta^d = -\delta = \text{diag.}(-1, -1, -1), \ I^d = -I = \text{diag.}(-1, -1, -1). \quad (2.31)$$

The isodual Euclidean geometry is then the geometry of the isodual space $E^d$ over $R^d$ and it is given by a step–by–step isoduality of all the various aspects of the conventional geometry.

We only mention for brevity the notion of isodual line on $E^d$ over $R^d$ given by the isodual image of the conventional notion of line on E over R. As such, its coordinates are isodual numbers $x^d = x \times 1^d$ with unit $1^d = -1$. By recalling that the norm on $R^d$ is negative–definite, the isodual distance among two points on an isodual line is also negative definite and it is given by $D^d = D \times 1^d = -D$, where D is the conventional distance. Similar isodualities apply to all remaining notions, including the notions of parallel and intersecting isodual lines, the Euclidean axioms, etc. The following property is of evident proof:

**Lemma 2.2:** The isoeuclidean geometry on $E^d$ over $R^d$ is anti–isomorphic to the conventional geometry on E over R.

The isodual sphere is the perfect sphere in $E^d$ over $R^d$ and, as such, it has negative radius,

$$R^{d2d} = [x^{d2d} + y^{d2d} + z^{d2d}] \times I^d. \quad (2.32)$$

A similar characterization holds for other isodual shapes which characterize the shape of antimatter in our isodual theory.

The group of isometries of $E^d$ over $R^d$ is the isodual euclidean group studied in Ref. [9].
2.6 Isodual Minkowskian geometry.

The isodual Minkowski space, first introduced in Refs [3], is given by

\[ M^d(x^d, \eta^d, R^d) : x^d = \{ x^{id} \} = \{ x^\mu \times I^d \} = \{ -r, -c_0 t \} \times I, \]

\[ \eta^d = -\eta = \text{diag.}(-1,-1,-1,+1), \quad I^d = \text{diag.}(-1,-1,-1,-1). \quad (2.33) \]

The isodual Minkowskian geometry [6] is the geometry of isodual spaces \( M^d \) over \( R^d \). It is also characterized by a simple isoduality of the conventional Minkowskian geometry and its explicit presentation is omitted for brevity.

We here merely mention the isodual light cone

\[ x^{d2d} = (x^{id} \times \eta^{d\mu} \times x^{\nu d}) \times I^d = (-x^1 x^2 y^2 y^3 z^1 + tc_0^2 t^1) \times (-I) = 0. \quad (2.34) \]

As one can see, the above cone formally coincides with the conventional light cone, although the two cones belong to different spaces. The isodual light cone is used in these studies as the cone of light emitted by antimatter in empty space (exterior problem).

The group of isometries of \( M^d \) over \( R^d \) is the isodual Poincaré symmetry \( P^d(3.1) = L^d(3.1) \times T^d(3.1) \) [6] and constitutes the fundamental symmetry of this paper.

2.7 Isodual Riemannian geometry.

Consider a Riemannian space \( \mathcal{R}(x, g, R) \) in \((3 + 1)\) dimensions with basic unit \( I = \text{diag.}(1, 1, 1, 1) \) and related Riemannian geometry in local formulation. The isodual Riemannian spaces are given by

\[ \mathcal{R}^d(x^d, g^d, R^d) : x^d = \{ \dot{x}^\mu \}, \]

\[ g^d = -g = (x), \quad g \in \mathcal{R}(x, g, R), \]

\[ I^d = \text{diag.}(-1,-1,-1,-1) \quad (2.35) \]

with interval \( x^{2d} = [x^t \times g^d(x^d) \times x^d] \times I^d = [x^t \times g^d(x^d)] \times I \) on \( R^d \), where \( t \) stands for transposed.

The isodual Riemannian geometry is the geometry of spaces \( \mathcal{R}^d \) over \( R^d \), and it is also given by step–by–step isodualities of the conventional geometry, including, most importantly, the isoduality of the differential and exterior calculus.

As an example, an isodual vector field \( X^d(x^d) \) on \( \mathcal{R}^d \) is given by \( X^d(x^d) = -X(-x) \). The isodual exterior differential of \( X^d(x^d) \) is given by

\[ D^d X^{kd}(x^d) = d^d X^{kd}(x^d) + \Gamma^{dk}_{ij} \times d^d X^{id} \times d^d x^d = D X^k(-x), \quad (2.36) \]

where the \( \Gamma^{dk}_s \)’s are the components of the isodual connection. The isodual covariant derivative is then given by

\[ X^{id}(x^d)|_{\dot{x}^{\mu}} = \partial^{d} X^{id}(x^d)/\partial^{d} \times x^{kd} + \Gamma^{dk}_{ij} \times \times d^d X^{id}(x^d) = -X^i(-x)|_{\dot{x}^{\mu}}. \quad (2.37) \]
The interested reader can then easily derive the isoduality of the remaining notions of the conventional geometry.

It is an instructive exercise for the interested reader to work out in detail the proof of the following:

**Lemma 2.3:** The isoduality of the Riemannian space $\mathbb{R}(x, g, R)$ to its anti–automorphic image $\mathbb{R}^d(x^d, g^d, R^d)$ is characterized by the following isodual quantities:

- **Basicunit**: $I \rightarrow I^d = -I,$
- **Metric**: $g \rightarrow g^d = -g,$
- **Connection coefficients**: $\Gamma_{klh} \rightarrow \Gamma_{klh}^d = -\Gamma_{klh},$
- **Curvature tensor**: $R_{ij} \rightarrow R_{ij}^d = -R_{ij},$
- **Ricci tensor**: $R_{\mu\nu} \rightarrow R_{\mu\nu}^d = -R_{\mu\nu},$
- **Ricci scalar**: $R \rightarrow R^d = R,$
- **Einstein tensor**: $G_{\mu\nu} \rightarrow G_{\mu\nu}^d = -G_{\mu\nu},$
- **Electromagnetic potentials**: $A_{\mu} \rightarrow A_{\mu}^d = -A_{\mu},$
- **Electromagnetic field**: $F_{\mu\nu} \rightarrow F_{\mu\nu}^d = -F_{\mu\nu},$
- **Einstein–mom. tensor**: $T_{\mu\nu} \rightarrow T_{\mu\nu}^d = -T_{\mu\nu},$

The reader should be aware that recent studies have identified the universal symmetry of conventional gravitation with Riemannian metric $g(x)$, the so–called *isospin symmetry* $\hat{P}(3.1) = \hat{L}(3.1) \times \hat{T}(3.1)$ [6]. The latter symmetry is the image of the conventional symmetry constructed with respect to the generalized unit

$$\hat{I}(x) = [T(x)]^{-1},$$

where $T(x)$ is a 4 × 4 matrix originating from the factorization of the Riemannian metric into the Minkowskian one,

$$g(x) = T(x) \times \eta.$$

In particular, since $T(x)$ is always positive–definite, we have the local isomorphism $\hat{P}(3.1) \approx P(3.1)$.

The same Ref. [6] has constructed the operator version of the *isodual isoPoincaré symmetry* $\hat{P}^d(3.1) \approx P^d(3.1)$, whose classical realization is the universal symmetry of the isodual Riemannian spaces $\mathbb{R}^d$ over $R^d$.

In summary, the geometries significant in this paper are:

1) **The conventional Euclidean, Minkowskian and Riemannian geometries**, which are used for the characterization of matter; and

2) **The isodual Euclidean, Minkowskian and Riemannian geometries**, which are used for the characterization of antimatter.
3 CLASSICAL ISODUAL THEORY OF ANTIMATTER

3.1 Fundamental assumption.

As it is well known, the contemporary treatment of matter is characterized by conventional mathematics, here referred to conventional numbers, fields, spaces, etc. with positive unit and norm, thus having conventional positive characteristics of mass, energy, time, etc.

In this paper we study the following:

**Hypothesis 3.1:** Antimatter is characterized by the isodual mathematics, that with isodual numbers, fields, spaces, etc., thus having negative-definite units and norm. All characteristics of matter therefore change sign for antimatter represented via isoduality.

The above hypothesis evidently provides the correct conjugation of the charge at the desired classical level. However, by no means, the sole change of the sign of the charge is sufficient to ensure a consistent classical representation of antimatter. To achieve consistency, the theory must resolve the main problematic aspect of current classical treatments of antimatter, the fact that their operator image is not the correct charge conjugation of that of matter, as evident from the existence of a single quantization procedure.

It appears that the above problematic aspect is indeed resolved by the isodual theory. The main reason is that, jointly with the conjugation of the charge, isoduality also conjugates all other physical characteristics of matter. This implies two channels of quantization, the conventional one for matter and a new isodual quantization for antimatter (see App. A) such that its operator image is indeed the charge conjugate of that of matter.

In this section we shall study the physical consistency of the theory in its classical formulation. The novel isodual quantization, the equivalence of isoduality and charge conjugation and related operator issues are studied in papers [5,10].

To begin our analysis, we note that Hypothesis 3.1 removes the traditional obstacles against negative energies and masses. In fact, particles with negative masses and energies referred to negative units are fully equivalent to particles with positive energy and masses referred to positive units. Moreover, as we shall see shortly, particles with negative energy referred to negative units behave in a fully physical way. This has permitted the study in ref. [10] of the possible elimination of necessary use of second quantization for the quantum characterization of antiparticles, as the reader should expect because our main objective is the achievement of equivalent treatments for particles and antiparticles at all levels, thus including first quantization.

Hypothesis 3.1 also resolves the additional, well known, problematic aspects of motion backward in time. In fact, time moving backward referred to a negative unit is fully equivalent to time moving forward referred to a positive unit. This confirms the plausibility of the first conception of antiparticles by Stueckelberg and others as moving backward in time (see the historical analysis of Ref. [2]), and creates new possibilities for the ongoing research on the so-called ”time machine” to be studied in separate works.

In this section we construct the classical isodual theory of antimatter at the Galilean, relativistic and gravitational levels, prove its axiomatic consistency and verify its compat-
ibility with available classical experimental evidence (that on electromagnetic interactions only). We also identify the prediction of the isodual theory that antimatter in the field of matter experiences gravitational repulsion (antigravity), and point out the ongoing efforts for its future experimental resolutions [12,15]. For completeness, the classical isodual Lagrangian and Hamiltonian mechanics are provided in the Appendix as the foundation of the isoquantization of the joining paper [10].

3.2 Representation of antimatter via the classical isodual Galilean relativity.

We now introduce the isodual Galilean relativity as the most effective way for the classical nonrelativistic characterization of antimatter according to Hypothesis 3.1.

The study can be initiated with the isodual representation of antimatter at the most primitive dynamical level, that of Newton’s equation. Once a complete symmetry between the treatment of matter and antimatter is reached at the Newtonian level, it is expected to persist at all subsequent levels.

The conventional Newton’s equations for a system of N point-like particles with (non-null) masses $m_a$, $a = 1, 2, ..., N$, in exterior conditions in vacuum are given by the familiar expression

$$m_a x d v_{ka} / d t = F_{ka}(t, r, v), \quad r = \{x, y, z\}, \quad a = 1, 2, ..., N, \quad v = dr/dt, \quad (3.1)$$
defined on the 7-dimensional Euclidean space $E_{Tot}(t, r, v) = E(t, R_t) \times E(r, \delta, R_r) \times E(v, \delta, R_v)$ with corresponding 7-dimensional total unit $I_{Tot} = I_t \times I_r \times I_v$, where one usually assumes $R_t = R_r = R_v$, $I_t = 1$, $I_r = I_v = \text{Diag.(1,1,1)}$.

The isodual Newton equations here submitted for the representation of n point-like antiparticles in vacuum are defined on the isodual space

$$E^d(t^d, r^d, v^d) = E^d(t^d, R_t^d) \times E^d(r^d, \delta^d, R_r^d) \times E^d(v^d, \delta^d, R_v^d), \quad (3.2)$$
with total isodual unit $I_{Tot}^d = I_t^d \times I_r^d \times I_v^d$, $I^d_t = -1$, $I^d_r = I^d_v = = -\text{Diag.(1,1,1)}$, and can be written for (non-null) isodual masses $m^d_a = -m_a$

$$m^d_a x d v_{ka} / d t^d = F_{ka}^d(t^d, r^d, v^d), \quad k = x, y, z, \quad a = 1, 2, ..., N. \quad (3.3)$$

It is easy to see that, when projected in the original space $S(t, r, v)$, isoduality changes the sign of all physical characteristics, as expected. It is also easy to see that the above isodual equations are anti-isomorphic to the conventional forms, as desired.

We now introduce the isodual Galilean symmetry $G^d(3.1)$ as the step-by-step isodual image of the conventional symmetry $G(3.1)$ (see, e.g., Ref. [16]). By using conventional symbols for the Galilean symmetry of a system of N particles with non-null masses $m_a$, $a = 1, 2, ..., N, G^d(3.1)$ is characterized by isodual parameters and generators

$$w^d = (\theta^d_k, r^d_0, v^d_0, t^d_0) = -w, \quad J^d_k = \sum_{a} \delta^{d} a j_{k}^{d} x^{d} p_{j_{k}}^{d} = -J_k, \quad P^d_k = \sum_{a} a p_{k}^{d} = -P_k, \quad (3.4)$$
\[ G^d_k = \sum_a (m_a^d \times d r^d_{ak} - t^d \times p^d_{ak}), \quad H^d = \frac{1}{2} \times d \sum_a p^d_{ak} \times d p^{kd}_{a} + V^d(r^d) = -H, \quad (3.5) \]
equipped with the isodual commutator \((A.11)\), i.e.,

\[(A^d, B^d)^d = \sum_{a,k} [(\partial^d A^d / \partial r^d_{a}) \times d (\partial^d B^d / \partial r^d_{a}) -
-(\partial^d B^d / \partial r^d_{a}) \times d (\partial^d A^d / \partial r^d_{a})] = -[A, B]. \quad (3.6)\]

In accordance with rule \((2.27)\), the structure constants and Casimir invariants of the isodual Lie algebra \(G^d(3.1)\) are negative-definite. From rule \((2.27)\), if \(g(w)\) is an element of the (connected component) of the Galilei group \(G(3.1)\), its isodual is characterized by

\[ g^d(w^d) = e^{-d \times d w^d \times d x^d} = -e^{i \times (-w) \times x} = -g(-w) \in G^d(3.1). \quad (3.7) \]

The isodual Galilean transformations are then given by

\[ t^d \rightarrow t'^d = t^d + t^d_0 = -t', \quad r^d \rightarrow r'^d = r^d + r^d_0 = -r' \quad (3.8) \]
\[ r^d \rightarrow r'^d = r^d + v^d_0 \times d t^d_0 = -r', \quad r^d \rightarrow r'^d = R^d(\theta^d) \times d r^d = -R(-\theta). \quad (3.9) \]

where \(R^d(\theta^d)\) is an element of the isodual rotational symmetry first studied in the original proposal \([3]\).

The desired classical nonrelativistic characterization of antimatter is therefore given by imposing the \(G^d(3.1)\) invariance of isodual equations \((3.3)\). This implies, in particular, that the equations admit a representation via the isodual Lagrangian and Hamiltonian mechanics outlined in Appendix A.

We now verify that the above isodual representation of antimatter is indeed consistent with available classical experimental knowledge for antimatter, that under electromagnetic interactions. Once this property is established at the primitive Newtonian level, its verification at all subsequent levels of study is expected from mere compatibility arguments.

Consider a conventional, classical, massive particle and its antiparticle in exterior conditions in vacuum. Suppose that the particle and antiparticle have charge \(-e\) and \(+e\), respectively (say, an electron and a positron), and that they enter into the gap of a magnet with constant magnetic field \(B\).

As it is well known, visual experimental observation establishes that particles and antiparticles have spiral trajectories of opposite orientation. But this behavior occurs for the representation of both the particle and its antiparticle in the same Euclidean space. The situation under isoduality is different, as described by the following:

**Lemma 3.1:** The trajectory of a charged particle in Euclidean space under a magnetic field and the trajectory of the corresponding antiparticle in isodual Euclidean space coincide.

**Proof:** Suppose that the particle has negative charge \(-e\) in Euclidean space \(E(r, \delta, R)\), that is, the value \(-e\) is defined with respect to the positive unit \(+1\) of the underlying field of real numbers \(R = R(n, +, \times)\). Suppose that the particle is under the influence of the magnetic field \(B\). The characterization of the corresponding antiparticle via isoduality implies the reversal of the sign of all physical quantities, thus yielding the charge \((-e)^d = +e\)
in the isodual Euclidean space $E^d(r^d, \delta^d, R^d)$, as well as the reversal of the magnetic field $B^d = -B$, although now defined with respect to the negative unit $(+1)^d = -1$. It is then evident that the trajectory of a particle with charge $-e$ in the field $B$ defined with respect to the unit +1 in Euclidean space and that for the antiparticle of charge $+e$ in the field $-B$ defined with respect to the unit -1 in isodual Euclidean space coincide. q.e.d.

An aspect of Theorem 3.1 which is particularly important for this paper is given by the following

**Corollary 3.1.A:** Antiparticles reverse their trajectories when projected from their isodual space into the conventional space.

Lemma 3.1 assures that isodualities permit the representation of the correct trajectories of antiparticles as physically observed, despite their negative energy, thus providing the foundations for a consistent representation of antiparticles at the level of first quantization studied in paper [10]. Moreover, Lemma 3.1 tells us that the trajectories of antiparticles may appear to exist in our space while in reality they may belong to an independent space, the isodual Euclidean space, coexisting with our own space.

To verify the validity of the isodual theory at the level of Newtonian laws of electromagnetic phenomenology, let us consider the repulsive Coulomb force among two particles of negative charges $-q_1$ and $-q_2$ in $E(r, \delta, R)$,

$$F = K \times (-q_1) \times (-q_2)/r \times r > 0,$$

where the operations of multiplication $\times$ and division $/$ are the conventional ones of the underlying field $R(n, +, \times)$. Under isoduality to $E^d(r^d, \delta^d, R^d)$ we have

$$F^d = K^d \times^d (-q_1)^d \times^d (-q_2)^d /^d r^d \times^d r^d = -F < 0,$$

where $\times^d = -\times$ and $/^d = -/$ are the isodual operations of the underlying field $R^d(n^d, +, \times^d)$.

But the isodual force $F^d = -F$ occurs in the isodual Euclidean space and it is therefore defined with respect to the unit -1. As a result, isoduality correctly represents the repulsive character of the Coulomb force for two antiparticles with positive charges.

The Coulomb force between a particle and an antiparticle can only be computed by projecting the antiparticle in the conventional space of the particle or vice-versa. In the former case we have

$$F = K \times (-q_1) \times (-q_2)^d/r \times r < 0,$$

thus yielding an attractive force, as experimentally established. In the projection of the particle in the isodual space of the antiparticle we have

$$F^d = K^d \times^d (-q_1) \times^d (-q_2)^d /^d r^d \times^d r^d > 0.$$

But this force is now referred to the unit -1, thus resulting to be again attractive.

In conclusion, the isodual Galilean relativity correctly represent the electromagnetic interactions of antimatter at the classical Newtonian level.
3.3 Representation of antimatter via the isodual special relativity.

We now introduce the isodual special relativity as the best way to represent classical relativistic antimatter according to Hypothesis 3.1.

In essence, the conventional special relativity (see, e.g., Pauli’s historical account [17]) is constructed on the fundamental 4-dimensional unit of the Minkowski space $I = \text{Diag.} \{1,1,1,1\}$, which represents the dimensionless units of space $\{+1,+1,+1\}$, and the dimensionless unit of time $+1$, and is the unit of the Poincaré symmetry $P(3.1)$. The isodual special relativity is characterized by the map

$$I = \text{diag.}(\{1,1,1,1\},1) > 0 \rightarrow I^d = -\text{diag.}(\{1,1,1\},1) < 0.$$  \hfill (3.14)

namely, it is based on negative units of space and time. The isodual special relativity is then expressed by the isodual image of all mathematical and physical aspects of the conventional relativity in such a way to admit the negative-definite quantity $I^d$ as the correct left and right unit.

This implies the reconstruction of the entire mathematics of the special relativity with respect to the single, common, 4-dimensional unit $I^d$, including: the isodual field $R^d = R^d(n^d, +^d, \times^d)$ of isodual numbers $n^d = n \times I^d = -n \times I$ with fundamental unit $I^d = -\text{Diag}(1,1,1,1 = 1)$; the isodual Minkowski space $M^d(x^d, \eta^d, R^d)$ with isodual coordinates $x^d = x \times I^d$, isodual metric $\eta^d = -\eta$ and basic invariant over $R^d$

$$(x - y)^{2d} = \left((x^\mu - y^\mu)d \times (x^\nu - y^\nu)d \times I^d \in R^d\right);$$ \hfill (3.15)

the fundamental isodual Poincaré symmetry [6]

$$P^d(3.1) = L^d(3.1) \times^d T^d(3.1),$$ \hfill (3.16)

where $L^d(3.1)$ is the isodual Lorentz symmetry, $\times^d$ is the isodual direct product and $T^d(3.1)$ represents the isodual translations, whose classical formulation is given by a simple relativistic extension of the isodual Galilean symmetry of the preceding section.

The algebra of the connected component $P^d_{+}(3.1)$ of $P^d(3.1)$ can be constructed in terms of the isodual parameters $w^d = \{-w_k\} = \{-\theta, -v, -a\}$ and isodual generators $X^d = -X = \{-X_k\} = \{-M_{\mu\nu}, -P_{\mu}\}$, where the factorization by the four-dimensional unit $I$ is understood. The isodual commutator rules are given by

$$[M^d_{\mu\nu}, M^d_{\alpha\beta}]^d = c^d \times^d (\eta^d_{\nu\alpha} \times^d M^d_{\mu\beta} - \eta^d_{\nu\beta} \times^d M^d_{\mu\alpha} + \eta^d_{\mu\beta} \times^d M^d_{\nu\alpha} + \eta^d_{\mu\alpha} \times^d M^d_{\nu\beta}),$$ \hfill (3.17)

$$[M^d_{\mu\nu}, P^d_{\alpha}]^d = c^d \times^d (\eta^d_{\nu\alpha} \times^d P^d_{\mu} - \eta^d_{\nu\alpha} \times^d P^d_{\mu} - \eta^d_{\mu\alpha} \times^d P^d_{\nu} - \eta^d_{\mu\alpha} \times^d P^d_{\nu} + 0),$$ \hfill (3.18)

The isodual group $P^d_{+}(3.1)$ has a structure similar to that of Eq.s (3.7). These results then yield the following

**Lemma 3.2:** The classical isodual Poincaré transforms are given by

$$x^{1d} = x^{1d}, \quad x^{2d} = x^{2d} = -x^{2d},$$

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\[ x^{3d} = \gamma^d \times^d (x^{3d} - \beta^d \times^d x^{4d}) = -x^{3d}, \quad x^{4d} = \gamma^d \times^d (x^{4d} - \beta^d \times^d x^{3d}) = -x^{4d}, \]
\[ x^{d\mu} = x^{d\mu} + a^{d\mu} = -x^{d\mu}, \]
\[ x^{d\nu} = x^{d\nu} = -\pi \times x = (-r, x^4), \quad \tau^d \times^d x^d = -\tau \times x = -(r, -x^4), \quad (3.19) \]

where

\[ \beta^d = v^d c_o = -\beta, \quad \beta^{2d} = -\beta^2, \quad \gamma^d = -(1 - \beta^2)^{-1/2}. \quad (3.20) \]

and the use of the isodual operations (quotient, square roots, etc.), is implied.

The isodual spinorial covering of the Poincaré symmetry \( \mathcal{P}^\dagger(3.1) = \text{SL}^d(2.\mathbb{C}_d) \times^d T^d(3.1) \) can then be constructed via the same methods.

The basic postulates of the isodual special relativity are also a simple isodual image of the conventional postulates. For instance, the maximal isodual causal speed is the speed of light in \( M^d \), i.e.,

\[ V_{\text{max}} = c^d_o = -c_o, \quad (3.21) \]

with the understanding that it is referred to a negative-definite unit, thus being fully equivalent to the conventional maximal speed \( c_o \) referred to a positive unit. A similar situation occurs for all other postulates.

A fundamental property of the isodual theory is the following:

**Theorem 3.1:** The line elements of metric or pseudo-metric spaces are isoselfdual (Definition 2.1), i.e., they coincide with their isodual images. In particular, isoduality leaves invariant the fundamental space-time interval of the special relativity,

\[ x^{d2d} = (x^{d\mu} \times^d \eta^{d\mu \nu} \times^d x^{d\nu}) = \]
\[ = (-x^1 x^1 - x^2 x^2 - x^3 x^3 - x^4 x^4) \times (-I) \equiv (x^1 x^1 + x^2 x^2 + x^3 x^3 - x^4 x^4) \times I = x^2. \quad (3.22) \]

The above novel property evidently assures that conventional relativistic laws for matter are also valid for antimatter represented via isoduality, since they share the same fundamental space-time interval.

The above property illustrates that the isodual map is so natural to creep in un-noticed. The reason why, after about a century of studies, the isoduals of the Galilean, special and general relativities escaped detection is that their identification required the prior knowledge of new numbers, those with a negative unit.

Note that the use of the two Minkowskian metrics \( \eta \) and \( \eta^d = -\eta \) has been popular since Minkowski’s times. The point is that both metrics are referred to the same unit \( I \), while in the isodual theory one metric is referred to the unit \( I \) on the field \( R(\mathbb{N}, +, \times) \) of conventional numbers, and the other metric is referred to the new unit \( I^d = -I \) on the new field \( R^d(\mathbb{N}^d, +, \times) \) of isodual numbers \( n^d = n \times I^d \).

The novelty of the isodual relativities is illustrated by the following

**Lemma 3.3.** Isodual maps and space-time inversions are inequivalent.
In fact, space-time inversions are characterized by the change of sign $x \rightarrow -x$ by always preserving the original metric referred to positive units, while isoduality implies the map $x \rightarrow x^d = -x$ but now referred to an isodual metric $\eta^d = -\eta$ with negative units $I^d = -I$. Thus, space-time inversions occur in the same space while isoduality implies the map to a different space. Moreover, as shown by lemma 3.2 isodualities interchange the space and time inversions.

The interested reader is encouraged to verify that the physical consistency in the representation of electromagnetic interactions by the isodual Galilean relativity carries over in its entirety at the level of the isodual special relativity, thus confirming the plausibility of the isodual theory of antimatter also at the classical relativistic level.

### 3.4 Representation of antimatter via the isodual general relativity.

We finally introduce the isodual general relativity as the most effective gravitational characterization of antimatter according to Hypothesis 3.1. The new image is also characterized by the isodual map of all aspects of the conventional relativity (see, e.g., [18]), now defined on the isodual Riemannian spaces $\mathbb{R}^d(x^d, g^d, R^d)$ of Sect. 2.7.

The primary motivation warranting the study of the above new image of general relativity is the following. A problematic aspect in the use of the Riemannian geometry for the representation of antimatter is the positive–definite energy-momentum tensor.

In fact, such a gravitational representation of antimatter has an operator image which is not the charge conjugate of that of matter, does not admit the negative-energy solutions as needed for operator treatments of antiparticles, and may be one of the reasons for the lack of achievement until now of a consistent grand unification inclusive of gravitation. After all, gauge theories are bona-fide field theories which, as such, admit both positive- and negative-energy solutions, while the contemporary formulation of gravity admits only positive-energy states, with an evident structural incompatibility.

Isoduality offers a new possibility for a future resolution of these shortcomings. In fact, the isodual Riemannian geometry is defined on the isodual field of real numbers $R^d(n^d, +^d, \times^d)$ for which the norm is negative–definite, Eq. (2.11). As a result, all quantities which are positive in Riemannian geometry become negative under isoduality, thus including the energy-momentum tensor.

Explicitly, the electromagnetic field follows the simple rule under isoduality

$$F^d_{\mu\nu} = \partial^d A^d_{\mu/} \partial^d x^\nu d - \partial^d A^d_{\nu/} \partial^d x^\mu d = -F_{\mu\nu},$$

and for the energy-momentum tensor we have the corresponding law

$$T^d_{\mu\nu} = (4m)^{-1d} \times^d (F^d_{\mu\alpha} \times^d F^d_{\alpha\nu} + (1/4)^{-1d} \times^d g^d_{\mu\nu} \times^d F^d_{\alpha\beta} \times^d F^d_{\alpha\beta}) = -T_{\mu\nu}. \quad (3.24)$$

As such, antimatter represented in isodual Riemannian geometry has negative–definite energy-momentum tensor and other physical quantities, as desired, thus offering new possibilities for attempting a grand unified theory.

For completeness, we mention here the isodual Einstein equations for the exterior gravitational problem of antimatter in vacuum.
\[ G^d_{\mu\nu} = R^d_{\mu\nu} - \frac{1}{2} \times^d \delta^d_{\mu\nu} \times^d R^d = k^d \times^d T^d_{\mu\nu}, \] (3.25)

We also mention the field equations characterized by the Freud identity [19] of the Riemannian geometry (reviewed by Pauli [17] and then generally forgotten)

\[ R^\alpha_\beta - \frac{1}{2} \times^\alpha_\beta \times R - \frac{1}{2} \times^\delta_\beta \times \Theta = U^\alpha_\beta + \partial V^\alpha_\rho / \partial x^\rho = k \times (t^\alpha_\beta - \tau^\alpha_\beta) \] (3.26)

where

\[ \Theta = g^{\alpha\beta} g^{\gamma\delta} (\Gamma^\rho_{\alpha\beta} \Gamma^\rho_{\gamma\delta} - \Gamma^\rho_{\rho\beta} \Gamma^\rho_{\gamma\delta}), \] (3.27)

\[ U^\alpha_\beta = -\frac{1}{2} \frac{\partial \Theta}{\partial g_{\alpha\beta}} \delta^\alpha_\beta \Gamma_\beta, \] (3.28)

\[ V^\alpha_\rho_\beta = \frac{1}{2} [g^{\gamma\delta} (\delta^\alpha_\beta \Gamma^\rho_{\alpha\gamma} - \delta^\rho_\gamma \Gamma^\alpha_{\beta\gamma}) + \] \[ + (\delta^\rho_\beta g^{\alpha\gamma} - \delta^\alpha_\beta g^{\rho\gamma}) \Gamma^\delta_{\gamma\delta} + g^{\rho\gamma} \Gamma^\alpha_{\beta\gamma} - g^{\alpha\gamma} \Gamma^\beta_{\gamma\delta}], \] (3.29)

which are currently under study for the interior gravitational problem of matter.

The isodual version of Eq.s (3.26)

\[ R^d_{\beta} - \frac{1}{2} \times^d \delta^d_\beta \times^d R^d - \frac{1}{2} \times^d \delta^d_\beta \times^d \Theta^d = k^d \times^d (t^\alpha_\beta - \tau^\alpha_\beta) \] (3.30)

are then suggested for the study of interior gravitational problems of antimatter.

It is instructive for the interested reader to verify that the preceding physical consistency of the isodual theory carries over at the above gravitational level, including the attractive character of antimatter-antimatter systems and their correct behavior under electromagnetic interactions.

Note in the latter respect that curvature in isodual Riemannian spaces is negative–definite (Sect. 2.7). Nevertheless, such negative value for antimatter-antimatter systems is referred to a negative unit, thus resulting in attraction.

The universal symmetry of the isodual general relativity, the isodual isoPoincaré symmetry \(\tilde{P}^d(3.1) \approx P^d(3.1)\), has been introduced at the operator level in Ref. [6]. The construction of its classical counterpart is straightforward, although it cannot be reviewed here because it requires the broader isotopic mathematics, that based on generalized unit (2.39).

### 3.5 The prediction of antigravity.

We close this paper with the indication that the isodual theory of antimatter predicts the existence of antigravity (here defined as the reversal of the sign of the curvature tensor in our space–time) for antimatter in the field of matter, or vice-versa.

The prediction originates at the primitive Newtonian level, persists at all subsequent levels of study [10], and it is here identified as a consequence of the theory without any
claim on its possible validity due to the lack of experimental knowledge at this writing on the gravitational behavior of antiparticles.

In essence, antigravity is predicted by the interplay between conventional geometries and their isoduals and, in particular, by Corollary 3.1.A according to which the trajectories we observe for antiparticles are the projection in our space–time of the actual trajectories in isodual space. The use of the same principle for the case of the gravitational field then yields antigravity.

Consider the Newtonian gravitational force of two conventional (thus positive) masses \( m_1 \) and \( m_2 \)

\[
F = -G \times m_1 \times m_2 / r \times r < 0,
\]

where the minus sign has been added for similarity with law \((3.10)\).

Within the context of contemporary theories, the masses \( m_1 \) and \( m_2 \) remain positive irrespective of whether referred to a particle or an antiparticle. This yields the well known Newtonian gravitational attraction among any pair of masses, whether for particle–particle, antiparticle–antiparticle or particle–antiparticle.

Under isoduality the situation is different. First, the particle–particle gravitational force yields exactly the same law \((3.6)\). The case of antiparticle–antiparticle under isoduality yields the different law

\[
F^d = -G^d \times d_m^d \times d_m^d \times d_r^d \times d_r^d > 0.
\]

But this force is defined with respect to the negative unit -1. The isoduality therefore correctly represents the attractive character of the gravitational force among two antiparticles.

The case of particle–antiparticle under isoduality requires the projection of the antiparticle in the space of the particle, as it is the case for the electromagnetic interactions of Corollary 2.1.A

\[
F = -G \times m_1 \times m_2^d / r \times r > 0,
\]

which is now repulsive, thus illustrating the prediction of antigravity. Similarly, if we project the particle in the space of the antiparticle we have

\[
F^d = -G^d \times d_m^d \times m_2^d / d_r^d \times d_r^d \times d_r^d < 0,
\]

which is also repulsive because referred to the unit -1.

We can summarize the above results by saying that the classical representation of antiparticles via isoduality renders gravitational interactions equivalent to the electromagnetic ones, in the sense that the Newtonian gravitational law becomes equivalent to the Coulomb law. Note the impossibility of achieving these results without isoduality.

The interested reader can verify the persistence of the above results at the relativistic and gravitational levels.

We should indicate that the electroweak behavior of antiparticles is experimentally established nowadays, while there no final experimental knowledge on the gravitational behavior of antiparticles is available at this writing.
A first experiment on the gravity of antiparticles was done by Fairbank and Witteborn [20] via low energy positrons in vertical motion, although the measurements were not conclusive because of interferences from stray fields and other reasons.

Additional data on the gravity of antiparticles are those from the LEAR machine on antiprotons at CERN [21], although these data too are inconclusive because of the excessive energy of the antiprotons as compared to the low magnitude of gravitational effects, the sensitivity of the measures and other factors.

Other information on the gravity of antiparticles is of theoretical character, as reviewed, e.g., in Ref. [22], which outlines various arguments against antigravity, such as those by Morrison, Schiff and Good and others. The latter arguments do not apply under isodualities owing to their essential dependence on positive units, as one can verify.

The argument against antigravity based on the positronium [22] also do not apply under isoduality, because systems of elementary particle–antiparticle are attracted in both fields of matter and antimatter under the isodual theory, as studies in the joint paper [10].

Review [22] also indicates models in which the gravity of antimatter in the field of matter is weakened, yet it remains attractive.

We can therefore state that the gravitational behavior of antiparticles is theoretically and experimentally unsettled at this writing.

The true scientific resolution is evidently that via experiments, such as that proposed by Santilli [7] via the use of a suitably collimated beam of very low energy positrons in horizontal flight in a vacuum tube of sufficient length and diameter to yield a resolutory answer, that is, a displacement under gravity at the end of the flight up or down which is visible by the naked eye.

According to Mills [12], the above experiment appears to be feasible with current technology via the use of µeV positrons in a horizontal vacuum tube of about 100 m in length and 1 m in diameter for which stray fields and patch effects should be small as compared to the gravitational deflection. A number of additional experimental proposals to measure the gravity of antiparticles are available in Proceedings [15], although their measures are more sophisticated and not "visible by the naked eyes" as test [7,12].

In closing we indicate that the most forceful argument favoring the existence of antigravity is given by studies on the origin of the gravitational field. In fact, the mass of all particles constituting a body has a primary electromagnetic origin, with second–order contributions from weak and strong interactions. By using this established physical evidence, Ref. [23] proposed the identification (rather than the "unification") of the gravitational field with the fields originating mass. This can be done by identifying the \( \tau \)–tensor in Eqs. (3.23) with the electromagnetic field originating mass, and the \( \tau \)–tensor with the weak and strong contributions. Since the latter are short range, in the exterior problem in vacuum we would only have the identification of the gravitational and electromagnetic fields which would evidently imply the equivalence of the respective phenomenologies, thus including the capability for attractive and repulsive forces for both fields.

Note that this implies the existence of a first–order nowhere null electromagnetic source also for bodies with null total charge (see Ref. [23] for details). Note also the full compatibility of the above argument with the isodual representation of antimatter, because both approaches imply the equivalence of Coulomb’s law for charges with Newton’s law for masses.

The forceful nature of the above argument is due to the fact that the lack of antigravity
would imply the lack of identity of gravitational and electromagnetic interactions. In turn, this would require a revision of the contemporary theory of elementary particles in such a way to avoid the primary electromagnetic origin of their mass.

APPENDIX A.

A.1: Isodual Lagrangian mechanics.

After having verified the isodual theory of antimatter at the primitive Newtonian level, it may be of some value to outline its analytic representation because it constitutes the foundations of the novel quantization for antimatter studied in the joint paper [10].

A conventional (first–order) Lagrangian $L(t,x,v) = \frac{1}{2}mv^2 + V(t,x,v)$ on the configuration space $E(t,x,v) = E(t,R_t) \times E(r,\delta,R) \times E(v,\delta,R_v)$ of Newton’s equations is mapped under isoduality into the negative value $L^d(t^d,r^d,v^d) = -L$ defined on isodual space $E^d(t^d,r^d,v^d)$ of Eq. (3.2). The isodual Lagrange equations are then given by

$$\frac{d}{dt} \frac{\partial L^d(t^d,r^d,v^d)}{\partial v^k} - \frac{\partial L^d(t^d,r^d,v^d)}{\partial r^k} = 0,$$

(A.35)

All various aspects of the isodual Lagrangian mechanics can then be readily derived.

It is easy to see that Lagrange’s equations change sign under isoduality and can therefore provide a direct representation (i.e., a representation without integrating factors) of the isodual Newton’s equations,

$$\frac{d}{dt} \frac{\partial L^d(t^d,r^d,v^d)}{\partial v^k} - \frac{\partial L^d(t^d,r^d,v^d)}{\partial r^k} = m^d_i \times d^d_j d^d_k d^d t^d - F^d_{SA}(t,r,v) = 0.$$

(A.36)

Where SA stands for variational selfadjointness, i.e., verification of the conditions to be derivable from a potential. The compatibility of the isodual Lagrangian mechanics with the primitive Newtonian results then follows.

A.2: Isodual Hamiltonian mechanics.

The isodual Hamiltonian is evidently given by

$$H^d = p^d_k \times d^d_j d^d k^d (2m)^d + V^d(t^d,r^d,v^d) = -H.$$

(A.37)

It can be derived from (nondegenerate) isodual Lagrangians via a simple isoduality of the Legendre transforms and it is defined on the 7–dimensional carrier space (for one particle)

$$E^d(t^d,r^d,p^d) = E^d(t^d,R^d) \times E^d(r^d,\delta^d,R^d) \times E^d(p^d,\delta^d,R^d).$$

(A.38)

The isodual canonical action is given by

$$A^d = \int_{t_1}^{t_2} (p^d_k \times d^d_j d^d k^d - H^d \times d^d_j d^d t^d) =$$
Conventional variational techniques under simple isoduality then yield the *isodual Hamilton equations* which can be written in disjoint form

\[
\frac{d^d x^{kd}}{dt^d} = \frac{\partial^d H^d(t^d, x^d, p^d)}{\partial^d p_k^d}, \quad \frac{d^d p_k^d}{dt^d} = -\frac{\partial^d H^d(t^d, x^d, p^d)}{\partial^d x^{kd}},
\]  

or in the unified notation

\[
\omega^d_{\mu \nu} \times d^d b^{\mu \nu} \frac{d^d t^d}{dt^d} = \frac{\partial^d H^d(t^d, b^d)}{\partial^d b^d_{\mu}},
\]

where \(\omega_{\mu \nu}^d\) is the *isodual canonical symplectic tensor*

\[
(\omega_{\mu \nu}^d) = \left( \partial^d R_{\nu}^d / \partial^d b^{\mu} - \partial^d R_{\mu}^d / \partial^d b^{\nu} \right) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = -(\omega_{\mu \nu}).
\]

Note that in matrix form the canonical symplectic tensor is mapped into the canonical Lie tensor.

The isodual *Hamilton–Jacobi equations* are then given by

\[
\partial^d A^d / \partial^d t^d + H^d = 0, \quad \partial^d A^d / \partial^d x^d - \hat{p}_k = 0, \quad \partial^d A^d / \partial^d p^d_k \equiv 0.
\]

The *isodual Lie brackets* among two isodual functions \(A^d\) and \(B^d\) on \(S^d(t^d, x^d, p^d)\) then become

\[
[A^d, B^d] = \frac{\partial^d A^d}{\partial^d b^d_{\mu}} \times d^d \omega_{\mu \nu}^d \times d^d \frac{\partial^d B^d}{\partial^d b^d_{\nu}} = -[A, B]
\]

where

\[
\omega_{\mu \nu}^d = [\omega_{\alpha \beta}^d]^{-1} \mu \nu,
\]

is the *isodual Lie tensor*. The direct representation of the isodual Newton equations in first–order form is self–evident.

In summary, all properties of the isodual theory at the Newtonian level carry over at the level of isodual Hamiltonian mechanics. In so doing, there is the emergence of a fundamental notion of these studies, the characterization of antimatter via *isodual space–time symmetries*, i.e., the isodual Galilean symmetry \(G^d(3.1)\) for nonrelativistic treatments, the isodual Poincarè symmetry \(P^d(3.1)\) for relativistic treatments and the isodual isoPoincarè symmetry for gravitational treatments [6].

**Isodual naive quantization.** The isodual Hamiltonian mechanics and its underlying isodual symplectic geometry permit the identification of the novel *naive isodual quantization*

\[
A^d \rightarrow -i^d \times d H^d \times d \ln^d \psi^d(t^d, r^d),
\]

\[
\partial^d A^d / \partial^d t^d + H^d = 0 \rightarrow i^d \times d \partial^d \psi^d / \partial^d t^d = H^d \times d \psi^d = E^d \times d \psi^d,
\]

\[
\partial^d A^d / \partial^d x^d - \hat{p}_k = 0 \rightarrow p_k^d \times d \psi^d = -i^d \times d \partial^d \psi^d,
\]
or more refined isodualities of symplectic quantization (see, e.g., Ref. [24] for the conventional case), which characterize a novel image of quantum mechanics for antiparticles, called *isodual quantum mechanics*, studied in the joint paper [10].

**Acknowledgements.**

The author would like to express his appreciation to all participants of the *International Workshop on Antimatter Gravity and Anti-Hydrogen Atom Spectroscopy* held in Sepino, Molise, Italy, in May 1996, for invaluable critical comments. Particular thanks are also due to Professors A. K. Aringazin, P. Bandyopadhyay, S. Kalla, J. V. Kadeisvili, N. Kamiya, A. U. Klimyk, R. Miron, R. Oehmke, G. Sardanashvily, H. M. Srivastava, T. Gill, Gr. Tsagas, N. Tsagas, C. Udriste and others for penetrating comments. Special thanks are finally due to M. Holzscheiter and J. P. Mills, jr., for invaluable critical remarks and to H. E. Wilhelm for a detailed critical reading of this manuscript.
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