On the Unboundedness of the First Eigenvalue of the Laplacian for $G$-Invariant Metrics

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July 27, 2010

Abstract

In this note we partially answer a question posed by Colbois, Dryden, and El Soufi. Consider the space of constant-volume Riemannian metrics on a connected manifold $M$ which are invariant under the action of a discrete Lie group $G$. We show that the first eigenvalue of the Laplacian is not bounded above on this space, provided $M = S^n$, $G$ acts freely, and $S^n/G$ with the round metric admits a Killing vector field of constant length, or provided $M \neq T^n$ is a compact Lie group, and $G$ is a discrete subgroup.

Let $G$ be a Lie group acting on a smooth orientable closed connected manifold $M$ of dimension $n > 2$. Given a Riemannian metric $g$ on $M$, let $\nabla$ be its Riemannian connection, and $\Delta = \nabla^i \nabla_i$ its Laplacian. This Laplacian has eigenvalues $\lambda > 0$ and associated eigenfunctions $u \in C^\infty(M)$ which are by definition solutions to

$$\Delta u + \lambda u = 0$$

with $u$ not identically zero. The set of eigenvalues is discrete, and its only limit point is $\infty$, so we can define $\lambda_1(g)$ as the lowest eigenvalue. Define a scale-invariant functional,

$$g \mapsto \Lambda_1(g) = \lambda_1(g)\text{vol}^{2/n}(M, g)$$

In [CDE] the following question is posed (Remark 3.1):

For $G$ a discrete group, can we construct $G$-invariant metrics such that the functional $\Lambda_1 = \lambda_1\text{vol}^{2/n}$ becomes arbitrarily large? Note that we are not requiring the eigenfunctions to be $G$-invariant.

Suppose $G$ is just the trivial group. If $n = 2$, Hersch showed that $\Lambda_1(g) \leq 8\pi$ on $M = S^2$. More generally, P. Yang and S. T. Yau showed that if $M$ has genus $\gamma$, then $\Lambda_1(g) \leq 8\pi(1 + \gamma)$. (See
[SY] for a discussion of these results.) On the other hand, if \( n > 2 \), it is well-known that \( \Lambda_1 \) can be arbitrarily large for any choice of \( M \) (see [CD]).

In this paper, we will consider nontrivial \( G \) actions. We will see that \( \Lambda_1 \) is unbounded for \( M = S^n \) with certain values of \( n \) and certain \( G \)-actions. We will also see that \( \Lambda_1 \) is unbounded for \( M \neq T^n \) a compact Lie group, and \( G \) a discrete subgroup \( G \leq M \) acting on the left by isometries, as a corollary to a theorem of Urakawa.

1 Spheres.

Let

\[
\bar{g}^{ij} = g^{ij} + ty^iy^j
\]  

(3)

Lemma. Suppose that \( Y^i \) is a Killing vector field (i.e. \( \nabla_i Y_j + \nabla_j Y_i = 0 \)) of constant length (\( |Y| = \text{const.} \)). Then the Laplacian \( \bar{\triangle} \) of \( \bar{g} \) satisfies

\[
\bar{\triangle}u = \triangle u - \frac{t}{1 + t|Y|^2} Y^i Y^j \nabla_i \nabla_j u
\]  

(4)

Proof. First,

\[
\bar{g}^{ij} = g^{ij} - \frac{tY^i Y^j}{1 + t|Y|^2}
\]  

(5)

Next,

\[
\bar{\Gamma}^k_{ij} = \frac{1}{2} \left( g^{kl} - \frac{tY^k Y^l}{1 + t|Y|^2} \right) \left( [g_{il,j} + g_{jl,i} - g_{ij,l}] + t \left[ (Y_i Y_l)_j + (Y_j Y_l)_i - (Y_Y)_j, l \right] \right)
\]  

(6)

\[
= \frac{1}{2} \left( g^{kl} - \frac{tY^k Y^l}{1 + t|Y|^2} \right) \left( g_{ls} \Gamma^s_{ij} + t \left[ (Y_i Y_l)_j + (Y_j Y_l)_i - (Y_Y)_j, l \right] \right)
\]  

(7)

Let \( K_{ij} = \nabla_i Y_j + \nabla_j Y_i \) and \( C_{ij} = \nabla_j Y_i - \nabla_i Y_j = Y_{i,j} - Y_{j,i} \). Then

\[
\bar{\Gamma}^k_{ij} = \frac{1}{2} \left( g^{kl} - \frac{tY^k Y^l}{1 + t|Y|^2} \right) \left( g_{ls} \Gamma^s_{ij} + t \left[ K_{ij} Y_l + 2 \Gamma^s_{ij} Y_s Y_l + Y_i C_{lj} + Y_j C_{li} \right] \right)
\]  

(8)
Thus

\[ \Gamma^k_{ij} = \Gamma^k_{ij} - \frac{t Y^k Y_s}{1 + t|Y|^2} \Gamma^s_{ij} + t \frac{K_{ij} Y^k}{2} + t \frac{\Gamma^s_{ij} Y_s Y^k}{2} + \frac{t}{2} g^{kl} (Y_l C_{ij} + Y_j C_{li}) - \frac{t^2 K_{ij} Y^k|Y|^2}{2(1 + t|Y|^2)} \]  

(9)

\[ - \frac{t^2 \Gamma^s_{ij} Y_s Y^k|Y|^2}{1 + t|Y|^2} - \frac{t^2 Y^k Y^l (Y_l C_{ij} + Y_j C_{li})}{2(1 + t|Y|^2)} \]  

(10)

Now suppose \( Y \) is a Killing vector field. Then \( K_{ij} = 0 \), and taking its trace, \( \text{div}(Y) = 0 \). Moreover, \( C_{ij} = 2 \nabla_j Y_i \), and

\[ Y^i \nabla_i Y^j = -Y^i \nabla_j Y_i = -\frac{1}{2} \nabla_j |Y|^2 = 0 \]  

(11)

Hence \( Y^t Y_l C_{lj} = 0 \) and

\[ \Gamma^k_{ij} = \Gamma^k_{ij} - \frac{t Y^k Y_s}{1 + t|Y|^2} \Gamma^s_{ij} + t \frac{\Gamma^s_{ij} Y_s Y^k}{2} + \frac{t}{2} g^{kl} (Y_l C_{ij} + Y_j C_{li}) - \frac{t^2 \Gamma^s_{ij} Y_s Y^k|Y|^2}{1 + t|Y|^2} \]  

(12)

However,

\[ - \frac{t Y^k Y_s}{1 + t|Y|^2} \Gamma^s_{ij} + t \frac{\Gamma^s_{ij} Y_s Y^k}{2} - \frac{t^2 \Gamma^s_{ij} Y_s Y^k|Y|^2}{1 + t|Y|^2} = 0 \]  

(13)

so

\[ \Gamma^k_{ij} = \Gamma^k_{ij} + \frac{t}{2} g^{kl} (Y_l C_{ij} + Y_j C_{li}) \]  

(14)

Let

\[ A^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ij} = t g^{kl} (Y_l \nabla_j Y_i + Y_j \nabla_i Y_l) \]  

(15)

Then

\[ \bar{\Delta} u = \Delta u - \frac{t}{1 + t|Y|^2} Y^i Y^j \nabla_i \nabla_j u - A^k_{ij} g^{ij} \nabla_k u + \frac{t}{1 + t|Y|^2} A^k_{ij} Y^i Y^j \nabla_k u \]  

(16)
But since

$$A^k_{ij}g^{ij} = A^k_{ij}Y^iY^j = 0$$  \hspace{1cm} (17)$$

we obtain

$$\bar{\Delta}u = \Delta u - \frac{t}{1 + t|Y|^2}Y^iY^j\nabla_i\nabla_j u$$  \hspace{1cm} (18)$$
as desired. ■

Now rescale $Y$ so that $|Y| = 1$. Then we have

$$\bar{\Delta}u = \Delta u - \frac{t}{1 + t}Y^iY^j\nabla_i\nabla_j u$$  \hspace{1cm} (19)$$

Also, the volume transforms as $\bar{\text{vol}}^{2/n}(M,g) = (1 + t)^{1/n}\text{vol}^{2/n}(M,g)$ since

$$\frac{\partial}{\partial t}d\mu = \frac{1}{2}\bar{g}^{ij}Y_iY_j d\mu = \frac{1}{2(1 + t)}d\mu$$  \hspace{1cm} (20)$$

Now specialize to the case where $g$ is the round metric on $M = S^n$ and $Y$ is a unit-length Killing field on $(M,g)$ with the property that $Y$ is the lift of a Killing field on $M/G$. Then $\bar{g}$ is a one-parameter family of $G$-invariant metrics.

Let $u$ be a first eigenfunction of $\Delta$ on $S^n$. Any such $u$ is just a coordinate projection $x^1, x^2, x^3, \ldots$ or $x^{n+1}$ restricted to $S^n$ (or a linear combination thereof), and is characterized by

$$\nabla_i\nabla_j u = -g_{ij}u$$  \hspace{1cm} (21)$$

Then

$$\bar{\Delta}u = \left(-n + \frac{t}{1 + t}\right)u$$  \hspace{1cm} (22)$$

It follows that $n - t/(1 + t)$ is an eigenvalue of $\bar{\Delta}_t$. But that's not all. Let $\lambda_k(t)$ be the $k$th smallest eigenvalue of $\bar{g}(t)$, counted with multiplicity. Since $\lambda_k(t)$ is continuous in $t$ (see [BU]), it follows that $n - t/(1 + t)$ is the first eigenvalue of $\bar{\Delta}_t$. 
To see this, let $S$ be the set of $t \in (-1, \infty$) such that $\lambda_1(t) = n - t/(1 + t)$. We will show that $S$ is open and closed in $(-1, \infty)$, as well as nonempty. Obviously, $0 \in S$. Next, by continuity of $\lambda_k(t)$, $S$ is closed. Finally, let $\tau \in S$, and suppose that $\lambda_1(t) < n-t/(1+t)$ for $t$ close to $\tau$. Then there exists a maximal natural number $m > n + 1$ such that $n - t/(1 + t) = \lambda_m(t)$. But $\lambda_m(\tau) > n - \tau/(1 + \tau)$ which contradicts the continuity of $\lambda_m(t)$ at $\tau$. So $n - t/(1 + t)$ is indeed the first eigenvalue of $\Delta_t$ for all $t \in (-1, \infty)$.

Thus

$$\Lambda_1(t) = \left(n - \frac{t}{1 + t}\right) (1 + t)^{1/n} \text{vol}^{2/n}(M, g(0))$$  \hspace{1cm} (23)$$

Since $n > 1$, we see that $\Lambda_1 \to \infty$ as $t \to \infty$. This proves unboundedness.

Now it only remains to find examples of $S^n$ and $G$ for which $S^n/G$ admits a nonvanishing Killing field. The Euler characteristic obstructs the existence of a nonvanishing vector field. Even-dimensional spheres, then, are at once ruled out, since the Euler characteristic $\chi(S^{2m})$ is nonzero. The Euler characteristic of odd closed manifolds, however, vanishes by Poincaré Duality.

Let us make the following definition. A Riemannian manifold $(M, g)$ is called a Sasakian manifold if $M$ has a unit length Killing vector field $Y$ such that, for any vector fields $A$ and $B$ on $M$ we have

$$R(A, Y)B = g(Y, B)A - g(A, B)Y$$  \hspace{1cm} (24)$$

It is a classical result of Sasaki that all 3-dimensional spherical space forms are Sasaki. Thus examples of $G$ include any cyclic group $\mathbb{Z}_m$ (whose actions result in the Lens spaces). More generally, $G$ can be any finite subgroup of $SO(4)$ acting freely by rotations on $S^3$. These $G$ give rise to the so-called spherical 3-manifolds. By Grigori Perelman’s proof of the elliptization conjecture, these are all the possible fundamental groups of discrete quotients of $S^3$. Such a $G$ is either cyclic, or a central extension of a dihedral, tetrahedral, octahedral, or icosahedral group by a cyclic group of even order. For example, if $G$ is

$$\langle a, b | (ab)^2 = a^3 = b^5 \rangle$$ \hspace{1cm} (25)$$

then $S^3/G$ is the Poincaré homology sphere.

Further higher-dimensional examples include $G = \mathbb{Z}_2$ acting on $S^{4m-1}$ since

$$\mathbb{R}P^{4m-1} = \frac{Sp(m)}{Sp(m - 1) \times \mathbb{Z}_2}$$ \hspace{1cm} (26)$$
is Sasakian. For these and more results on Sasakian manifolds, refer to [BGM].

2 Compact Lie Groups.

In [U] we have the following (Theorem 4):

Let $M$ be a compact connected Lie group. We assume $M$ has nontrivial commutator subgroup. That is, the commutator Lie subalgebra $m_1$ of $m$ is not zero. Then there exists a family of left-invariant Riemannian metrics $g(t)$ ($0 < t < \infty$) on $M$ such that

$$\lim_{t \to \infty} \lambda_1(g(t)) = \infty$$

(27)

$$\lim_{t \to 0} \lambda_1(g(t)) = 0$$

(28)

and $\text{vol}(M, g(t))$ is constant in $t$.

To say that $m_1 = 0$ is to say that $m$ is abelian, and so $M$ is abelian, i.e. $M$ is a torus $T^n$, being compact. Therefore, we have

Corollary. If $M \neq T^n$ is a compact connected Lie group, and $G$ is a discrete subgroup, then $\Lambda_1$ is unbounded among the $G$-invariant metrics on $M$, where $G$ acts by left-multiplication.

Proof. Since $M \neq T^n$, the existence of the family $g(t)$ from Theorem 4 is guaranteed to us. Furthermore, all those metrics are $G$-invariant, since they are left-invariant. □

3 References.

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