Resolvability in $E_\gamma$ with Applications to Lossy Compression and Wiretap Channels

Jingbo Liu  Paul Cuff  Sergio Verdú
Dept. of Electrical Eng., Princeton University, NJ 08544
{jingbo,cuff,verdu}@princeton.edu

Abstract—We study the amount of randomness needed for an input process to approximate a given output distribution of a channel in the $E_\gamma$ distance. A general one-shot achievability bound for the precision of such an approximation is developed. In the i.i.d. setting where $\gamma = \exp(\alpha E)$, a (nonnegative) randomness rate above $\inf_{Q_U} D(Q_U|\pi_X) \leq E\{D(Q_X\|\pi_X) + I(Q_U, Q_X|U) - E\}$ is necessary and sufficient to asymptotically approximate the output distribution $\pi_X^n$ using the channel $Q_{X^n|U}$, where $Q_U \rightarrow Q_{X|U} \rightarrow Q_X$. The new resolvability result is then used to derive a one-shot upper bound on the error probability in the rate distortion problem; and a lower bound on the size of the eavesdropper list to include the actual message in the wiretap channel problem. Both bounds are asymptotically tight in i.i.d. settings.

I. INTRODUCTION

Approximation of a target output distribution with a given channel has proved to be the key technical step in the solution of many problems in information theory. In 1975 Wyner first studied such an approximation task to establish the achievability part for Wyner’s common information [1], where he used the normalized relative entropy to quantify the distance between the synthesized output distribution and the target distribution. Later Han and Verdú coined the term resolvability for the minimum rate of the randomness needed for the input [2]. Motivated by the strong converse of the identification coding theorem, [3] considered resolvability in the total variation distance (TV), as well as relative entropy. The achievable part of resolvability (also known as the soft-covering lemma [4]) is particularly useful, e.g. in secrecy [5][6][7], channel synthesis [4] and lossless and lossy source coding [2][8][9]. Under both the normalized relative entropy measure and TV, the resolvability can be shown to be the minimum mutual information over all input distributions inducing the target output distribution, and this is also known to be true for unnormalized relative entropy as well (see for example [3]).

In this paper we propose two new measures for approximation of output statistics. The first one, excess information, gives a straightforward upper bound on the second metric\(^1\), the $E_\gamma$ metric. The $E_\gamma$ metric was, to our knowledge, originally introduced in [10] to simplify the formula of the DT bound therein. The latter metric has clear operational significance and reduces to the TV in the special case of $\gamma = 1$, whereas the former is easier to upperbound. Asymptotically, however, the two metrics behave in the same way. We derive a one-shot upperbound on the first (hence also the second) metric in the resolvability problem. Bounding the new metrics requires more care than the traditional TV to achieve asymptotic tightness.

Particularly interesting is the case where the channel is stationary memoryless and $\gamma$ grows exponentially as the number of channel uses tends to infinity. In this case a single letter formula of the rate of randomness needed to approximate a tensor power output distribution in $E_\gamma$ can be obtained from the aforementioned one-shot bound. Here a peculiar feature of approximation in $E_\gamma$ emerges: the distribution of each codeword in the generation of the random codebook need not induce the target output distribution through the stationary memoryless channel, and in fact the optimal choice of such a distribution (in the sense of requiring the minimum rate of randomness) generally does not induce the target distribution. This is in stark contrast to the case of TV measure, where the codeword distribution must induce the target distribution to ensure that the total variation between the output distribution and the target distribution does not converge to its maximum value, 2, asymptotically.

Two applications of the new channel resolvability results are presented. First, the simplest application to lossy source coding yields a new achievability bound on the probability that the distortion lies below a certain number, which in the asymptotic setting recovers the exponent of this probability previously obtained using the method of types (c.f. [11]). The advantage of the new derivation is its applicability beyond the discrete memoryless framework.

The second application is in the achievable part of wiretap channels, where we propose a novel interpretation of secrecy in terms of the eavesdropper’s ability to perform list decoding. In contrast to the previous proofs for wiretap channels using TV-resolvability [6][7] which only applies when the rate is below the perfect secrecy capacity, the new resolvability in $E_\gamma$ yields lower bounds on the required size of the eavesdropper list for all possible rates. This interpretation of security in terms of list size is reminiscent of equivocation [12], and indeed we obtain the same formula in the asymptotic setting, even though it is not immediate to prove a correspondence between the two. We also consider the case where the eavesdropper wishes to detect that no message is sent with high probability. This is a practical setup because “no message” may be a special piece of information which the eavesdropper wants to know with high certainty. We obtain single letter

\(^1\)Here “metric” or “distance” are used informally since they do not satisfy either symmetry or the triangle inequality.
expressions of the tradeoff between the transmission rate, eavesdropper list, and the exponent of the probability that the eavesdropper fails to detect non-message. Those bounds are asymptotically tight for random codes.

II. PRELIMINARIES

A. EXCESS INFORMATION METRIC

One natural measure of the discrepancy between two distributions $P$ and $Q$ on the same alphabet may be called the excess information metric with threshold $\gamma$:

$$\mathbb{P}[\gamma_P||Q(X) > \log \gamma]$$

where $X \sim P$ and

$$\gamma_{P||Q}(x) := \log \frac{dP}{dQ}(x).$$

Notice that in additional to being more suitable for a one-shot approach, (1) provides richer information than the relative entropy measure since

$$D(P||Q) = \int_{[0, +\infty)} \mathbb{P}[\gamma_{P||Q}(X) > \tau] d\tau$$

$$-\int_{(-\infty, 0)} (1 - \mathbb{P}[\gamma_{P||Q}(X) > \tau]) d\tau.$$ (3)

We note that the excess information metric does not satisfy a data processing property. More precisely, suppose $P_X \rightarrow P_{Y|X} \rightarrow P_Y$, $Q_X \rightarrow P_{Y|X} \rightarrow P_Y$, then it is not always true that

$$\mathbb{P}[\gamma_{P_X||Q_X}(X) > \tau] < \mathbb{P}[\gamma_{P_Y||Q_Y}(Y) > \tau]$$

where $(X, Y) \sim P_{XY}$.

B. THE $E\gamma(P||Q)$ METRIC

Next we consider another metric which does satisfy the data processing inequality and has a clearer operational meaning. Given probability distributions $P$, $Q$ and a constant $\gamma \geq 1$, define an $f$-divergence [13]

$$E\gamma(P||Q) := \mathbb{P}[\gamma_{P||Q}(X) > \log \gamma] - \gamma \mathbb{P}[\gamma_{P||Q}(Y) > \log \gamma]$$

where $X \sim P$ and $Y \sim Q$. This quantity was introduced in [10] to simplify the expression of DT bound. From the Neyman-Pearson lemma we have the alternative formula for the above quantity:

$$E\gamma(P||Q) = \max_A (P(A) - \gamma Q(A)),$$ (6)

which becomes half of the total variation distance (the $L_1$ distance) between $P$ and $Q$ when $\gamma = 1$. Some basic properties of $E\gamma$ are in order:

**Proposition 1.** 1) For any event $A$,

$$Q(A) \geq \frac{1}{\gamma} (P(A) - E\gamma(P||Q)).$$ (7)

where conditioned on $c$, $\hat{X} \sim P_{X(c)}$, and $(U, X) \sim Q_U Q_{X|U}$.

2) If $P_X P_{Y|X}$ and $Q_X Q_{Y|X}$ are joint distributions on $X \times Y$, then

$$E\gamma(P_X||Q_X) \leq E\gamma(P_X P_{Y|X}||Q_X Q_{Y|X})$$

where equality holds when $P_{Y|X} = Q_{Y|X}$. In the latter case we obtain the data processing inequality:

$$E\gamma(P_Y||Q_Y) \leq E\gamma(P_X||Q_X)$$

3) Given $P_X$, $P_{Y|X}$ and $Q_{Y|X}$, define

$$E\gamma(P_{Y|X}||Q_{Y|X}|P_X) := \mathbb{E}[E\gamma(P_{Y|X}(|X)||Q_{Y|X}(|X))]$$

where the expectation is w.r.t. $X \sim P_X$. Then

$$E\gamma(P_X P_{Y|X}||P_X Q_{Y|X}) = E\gamma(P_X||Q_X) - E\gamma(P_{Y|X}||Q_{Y|X}) = E\gamma(P_X||Q_X) - E\gamma(P_Y||Q_Y).$$ (11)

III. ACHIEVABILITY BOUNDS ON EXCESS INFORMATION

We present a one-shot information spectrum achievability bound for resolvability under the excess information metric, which then automatically implies a bound under the $E\gamma$ metric. Consider the setting of Figure 1. The input to the channel $Q_{X|U}$ is equiprobably selected from a codebook $(c_i)_{i=1}^L \in \mathcal{U}^L$. It turns out that codewords are i.i.d. codewords are usually good enough, and the expected distance from the synthesized distribution $P_{X(e)}$ to the target distribution $\pi_X$ under the excess information metric is gauged as follows:

$$l \rightarrow (c_i)_{i=1}^L \rightarrow Q_{X|U} \rightarrow P_X \approx \pi_X$$

**Figure 1:** Synthesizing a target distribution $\pi_X$ using a random number generator and a codebook $(c_i)_{i=1}^L$.

**Theorem 2.** Fix $\pi_X$ and $Q_{U|X} = Q_U Q_{X|U}$. Let $c = [c_1, \ldots, c_L]$ be i.i.d. according to $Q_X$. Define

$$P_{X(e)} := \frac{1}{L} \sum_{i=1}^L Q_{X|U=c_i}.$$ (12)

Then for any $\tau, \gamma, \epsilon, \sigma > 0$ satisfying $\gamma > \epsilon + \sigma$ and $0 < \delta < 1$, it holds that

$$\mathbb{P} \left[ \frac{dP_{X(e)}(\hat{X})}{d\pi_X} > \gamma \right] \leq \mathbb{P} \left[ \frac{dQ_X}{d\pi_X}(X) > \gamma - \sigma - \epsilon \right]$$

$$+ \mathbb{P} \left[ \frac{dQ_{X|U}}{d\pi_X}(X|U) > \delta L \sigma \right]$$

$$+ \exp(\tau) (\gamma - \sigma - \epsilon)^2 L (1 - \delta)^2 \sigma^2$$

$$+ \frac{\gamma - \sigma - \epsilon}{\epsilon} \mathbb{P} [U; X(U; X) > \tau].$$ (13)
Remark 3. By setting $\tau \leftarrow -\infty$ and letting $\delta \uparrow 1$, the bound in Theorem 2 can be weakened in the following slightly simpler form:

$$\mathbb{P}\left[ \frac{dP_X}{d\pi_X}(\hat{X}) > \gamma \right] \leq \mathbb{P}\left[ \frac{dQ_X}{d\pi_X}(X) > \gamma - \sigma - \epsilon \right] + \mathbb{P}\left[ \frac{dQ_X|U}{d\pi_X}(X|U) \geq L\sigma \right] + \frac{\gamma - \sigma - \epsilon}{\epsilon} \quad (14)$$

The weakened bound (14) is still asymptotically tight provided that the exponent with which the threshold $\gamma$ grows is positive; see Corollary 4 below. However, when the exponent is zero (corresponding to the total variation case), we do need $\tau$ in the bound for asymptotic tightness.

The proof of Theorem 2 is omitted due to space limitations.

Corollary 4. Fix per-letter distributions $\pi_X$ and $Q_{UX} = Q_U Q_{X|U}$. Let $c = [c_1, \ldots, c_L]$ be i.i.d. according to $Q_X^\otimes n$. Define

$$P_{X^n}(c) := \frac{1}{L} \sum_{l=1}^L Q_X^n|U^n=c_l.$$  \(15\)

Suppose $\gamma = \exp(nE)$ and $L = \exp(nR)$. Then

$$\lim_{n \to \infty} \mathbb{E}[E(X^n|\pi_X^n)] = \lim_{n \to \infty} \mathbb{P}\left[ \frac{dP_{X^n}(c)}{d\pi_X}(\hat{X}^n) > \gamma \right] = 0 \quad (16)$$

provided that

$$E > D(Q_X||\pi_X) + [I(Q_U, Q_{X|U}) - R]^{+}, \quad (17)$$

where conditioned on $c$, the vector $\hat{X}^n \sim P_{X^n}(c)$. Moreover, the bound in (17) is tight.

Proof of Achievability: Choose $E'$ such that

$$E > E' > D(Q_X||\pi_X) + [I(Q_U, Q_{X|U}) - R]^{+}. \quad (18)$$

Set $\delta = \frac{1}{2}$, $\gamma = \exp(nE')$, $L = \exp(nR)$, $\epsilon = \exp(nE') - \exp(nE'')$ and $\sigma = \frac{1}{2}(\gamma - \epsilon) = \frac{1}{2} \exp(nE'')$, and apply (14). Notice that

$$\mathbb{E} \left[ \frac{dQ_X}{d\pi_X}(X|U) \right] = n[I(Q_U, Q_{X|U}) + D(Q_X||\pi_X)] \quad (19)$$

where $(X, U) \sim Q_X^\otimes n$. By the law of large numbers, the first and second terms in (14) vanish because

$$D(Q_X||\pi_X) < E'; \quad (20)$$

$$I(Q_U, Q_{X|U}) + D(Q_X||\pi_X) < E' + R \quad (21)$$

are satisfied.

Remark 6. In the i.i.d. setting, let $R(\pi_X, d)$ be the rate-distortion function when the source has per-letter distribution $\pi_X$. The distortion function for the block is derived from the per-letter distortion by

$$d^{(n)}(u^n, x^n) := \frac{1}{n} \sum_{i=1}^n d(u_i, x_i). \quad (28)$$
Let \((X^n, \bar{U}^n)\) be the source-reconstruction pair distributed according to \(\pi_{X^n,U^n}\). If \(0 \leq R < R(\pi_X, d)\), the maximal probability that the distortion does not exceed \(d\) converges to zero with the exponent
\[
\lim_{n \to \infty} \frac{1}{n} \log \frac{1}{P[d(n)(\bar{U}^n, X^n) \leq d]} = G(R, d)
\]
where
\[
G(R, d) := \min_{Q} \left[ D(Q) | P \right] + \left[ R(Q, d) - R \right]^{+}.
\]
A weaker achievability result than (30) was proved in [14, p.168], whereas the final form (30) is given in [11, p.158, Ex6] based on method of types. Here we can easily prove the achievability part of (30) using Theorem 5 and Corollary 4 by setting \(Q_X\) to be the minimizer of (30) and \(Q_{U|X}\) to be such that
\[
E_{d}(U, X) \leq d,
\]
\[
I(Q_X, Q_{U|X}) \leq R.
\]
Then \(\gamma_n = \exp(nE)\) with
\[
E > D(Q_X || \pi_X) + [I(U; X)_Q - R]^{+},
\]
ensures that
\[
P[d(n)(\bar{U}^n, X^n) \leq d] \geq \frac{1}{2} \exp(-nE)
\]
for \(n\) large enough, by the law of large numbers.

**Remark 7.** Since the \(E_{\gamma}\) metric reduces to TV when \(\gamma = 1\), Theorem 5 generalizes the likelihood source encoder based on the standard soft-covering/coverability lemma [8]. In [8], the error exponent for the likelihood source encoder at rates above the rate-distortion function is analyzed using the exponential decay of TV in the approximation of output statistics, and the exponent does not match the optimal exponent in [13]. It is also possible to upperbound the success exponent of the TV-based likelihood encoder at rates below the rate-distortion function by analyzing the exponential convergence to 2 of TV in the approximation of output statistics; however that does not yield the optimal exponent (30) either. The power of \(E_{\gamma}\)-resolvability lies in the ability to convert a large deviation analysis into an ex-cercise of the law of large numbers, that is, we only care about whether \(E_{\gamma}\) converges to 0, but not the speed, even when dealing with error exponent problems.

**V. APPLICATION TO WIRETAP CHANNELS**

Next we apply the \(E_{\gamma}\)-resolvability to the wiretap channel \(P_{Y|X}\) as depicted in Figure 2. The receiver and the eavesdropper observe \(Y\) and \(Z\), respectively. Given a codebook \((c_{wl})\), the input to the channel is \(c_{wl}\) where \(w \in \{1, \ldots, M\}\) is the message to be sent and \(l\) is equiprobably chosen from \(\{1, \ldots, L\}\) to randomize the eavesdropper’s observation. Moreover, the eavesdropper’s observation has the distribution \(\pi_Z\) when no message is sent. For general wiretap channels the performance may be enhanced by appending a conditioning channel \(Q_{X|U}\) at the output of the encoder [6]. But in that case the same analysis can be carried out for the new wiretap channel \(Q_{Y|Z|U}\). Thus the model in Figure 2 entails no loss of generality. We need the following definitions to quantify the eavesdropper’s knowledge.

**Definition 8.** For a fixed codebook we say the eavesdropper can perform \((A, T, \bar{c})\)-decoding if when no message is sent, it detects no message with probability at least \(1 - A^{-1}\); and when a message \(m\) is sent, it can produce a list of \(T\) messages containing \(m\) with probability at least \(\epsilon_m\) such that
\[
\bar{c} = \frac{1}{M} \sum_{m=1}^{M} \epsilon_m.
\]

For stationary memoryless channels, the quantities \(P_{ZY|X}, M\) and \(L\) in Figure 2 are identified as \(P_{Z|Y=1|X}, \exp(nR)\) and \(\exp(nR_L)\).

We consider an \((M, L, Q_X)\)-random code, which is defined as the ensemble of the codebook \((c_{wl})\), \(w \in \{1, \ldots, M\}\), \(l \in \{1, \ldots, L\}\) where each codeword is i.i.d. chosen according to \(Q_X\). The following definition captures the asymptotic performance of the eavesdropper.

**Definition 9.** Fix \((R, R_L)\). The rate pair \((\alpha, \tau)\) is \(\bar{c}\)-achievable by the eavesdropper if there exist sequences \(\{A_n\}\) and \(\{T_n\}\) with
\[
\lim_{n \to \infty} \frac{1}{n} \log A_n = \alpha
\]
\[
\lim_{n \to \infty} \frac{1}{n} \log T_n = \tau
\]
such that for sufficiently large \(n\), the eavesdropper can achieve \((A_n, T_n, \bar{c})\)-decoding with high probability when the codebook is the \((\exp(nR), \exp(nR_L), Q_X^{\otimes n})\)-random code.

Then we have the following result:

**Theorem 10.** For any \(Q_X, R, R_L\) and \(0 < \bar{c} < 1\), the pair \((\alpha, \tau)\) is \(\bar{c}\)-achievable by the eavesdropper in the sense of Definition 9 if
\[
\{ \alpha \leq D(Q_Z || \pi_Z) + [I(Q_X, P_{Z|X}) - R - R_L]^{+} \}
\]
\[
\{ \tau \geq R - [I(Q_X, P_{Z|X}) - R - R_L]^{+} \}
\]
where \(Q_X \to P_{Z|X} \to Q_Z\).

**Remark 11.** From the noisy channel coding theorem, the supremum randomization rate \(R_L\) such that the sender can reliably transmit messages at the rate \(R\) is \(I(Q_X, P_{Y|X}) - R\). The larger \(R_L\) the less reliably the eavesdropper can decode, so the optimal encoder chooses \(R_L\) as close to this supremum as possible. Thus Theorem 10 implies that to reliably transmit...
messages at the rate $R$, codebooks can be selected such that the eavesdropper cannot perform $(\exp(n\alpha), \exp(n\tau), \bar{\epsilon})$ for large $n$ if there exists some $Q_X$ such that

$$\alpha > D(Q_Z || \pi_Z) + [I(Q_X, P_{Z|X}) - I(Q_X, P_{Y|X})]^+ \quad (39)$$

or

$$\tau < R - [I(Q_X, P_{Z|X}) - I(Q_X, P_{Y|X}) + R]^+ \quad (40).$$

**Remark 12.** In general the sender-receiver want to minimize $\alpha$ and maximize $\tau$ obeying the tradeoffs (39), (40) by selecting $Q_X$. In the special case where $\alpha$ has no importance and $R$ is larger than the secrecy capacity $C := \sup_{Q_X} \{ I(Q_X, P_{Y|X}) - I(Q_X, P_{Z|X}) \}$, we see from (40) that the supremum $\tau$ is $C$. The formula is the same as the equivocation measure defined as $\frac{1}{n} H(W|Z^n)$ [12], but technically our result does not follow directly from the lower bound on equivocation, since it may be possible that the a posterior distribution of $W$ is concentrated on a small list but has a tail spread over an exponentially large set, resulting a large equivocation.

The (eavesdropper) achievability part of Theorem 10 follows by analyzing the eavesdropper decoding ability for different cases of the rates $(R, R_L)$. The (eavesdropper) converse part of Theorem 10 follows by applying the following non-asymptotic bounds to different cases of $(R, R_L)$ and invoking Corollary 4.

**Theorem 13.** In the wiretap channel, fix an arbitrary distribution $\mu_Z$ and a measurable subset $D_0 \subseteq Z$. Suppose the eavesdropper can either detect that no message is sent upon observing $z \in D_0$ with

$$\mu_Z(D_0) \geq 1 - A^{-1} \quad (41)$$

or outputs a list of $T(z)$ messages upon observing $z \notin D_0$ that contains the actual message $m \in \{1, \ldots, M\}$ with probability at least $1 - \epsilon_m$. Define the average quantities

$$T := \frac{1}{\mu_Z(D_0^n)} \int_{D_0^n} T(z) d\mu_Z(z), \quad (42)$$

$$\bar{\epsilon} := \frac{1}{M} \sum_{m=1}^M \epsilon_m. \quad (43)$$

Then,

$$\frac{1}{A} \geq \frac{1}{\gamma} \left( 1 - \bar{\epsilon} - E_{\gamma}(P_Z || \pi_Z) \right), \quad (44)$$

where we recall that $\pi_Z$ is the non-message distribution, and

$$\frac{T}{MA} \geq \frac{1}{\gamma} \left( 1 - \bar{\epsilon} - \frac{1}{M} \sum_{m=1}^M E_{\gamma}(P_{Z|W=m} || \mu_Z) \right). \quad (45)$$

From the eavesdropper viewpoint, a larger $A$ and a smaller $T$ is more desirable since it will then be able to find out that no message is sent with smaller error probability or narrow down to a smaller list when a message is sent. This observation agrees with (44) and (45): a smaller $\gamma$ implies a higher degree of approximation, and hence higher indistinguishability of output distributions which is to the eavesdropper disadvantage.

**VI. DISCUSSION**

As we have demonstrated, the achievability part of resolvability in $E_{\gamma}$ has various applications in information theory, especially for bounding rare event probabilities. (c.f. (22)(44) and (45)). However the asymmetry of $E_{\gamma}$ (when $\gamma > 1$) places a limitation on $E_{\gamma}$-resolvability in certain problems. In particular, there is no counterpart of Theorem 2 for $E_{\gamma}(\pi_X || P_X)$.

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**REFERENCES**

[1] A. D. Wyner, “The common information of two dependent random variables,” *IEEE Transactions on Information Theory*, vol. 21, no. 2, pp. 163–179, 1975.

[2] T. Han and S. Verdú, “Approximation theory of output statistics,” *IEEE Transactions on Information Theory*, vol. 39, no. 3, pp. 752–772, 1993.

[3] T. S. Han, H. Endo, and M. Sasaki, “Reliability and security functions of the wiretap channel under cost constraint,” *IEEE Transactions on Information Theory*, vol. 60, no. 11, pp. 6819-6843, 2014.

[4] P. Cuff, “Distributed channel synthesis,” *IEEE Transactions on Information Theory*, vol. 59, pp. 7071–7096, Nov. 2013.

[5] I. Csiszár, “Almost independence and secrecy capacity,” *Problems Inf. Transmission*, vol. 32, no. 1, pp. 40–47, 1996.

[6] M. Hayashi, “General nonasymptotic and asymptotic formulations in channel resolvability and identification capacity and their application to the wiretap channel,” *IEEE Transactions on Information Theory*, vol. 52, pp. 1562–1575, Apr. 2006.

[7] M. Bloch and N. Laneman, “Strong secrecy from channel resolvability,” *IEEE Transactions on Information Theory*, vol. 59, pp. 8077–8098, Dec. 2013.

[8] E. C. Song, P. Cuff, and H. V. Poor, “The likelihood encoder for lossy source compression,” *arXiv:1408.4522*, Aug. 2014.

[9] Y. Steinberg and S. Verdú, “Simulation of random processes and rate-distortion theory,” *IEEE Transactions on Information Theory*, vol. 42, no. 1, pp. 63–86, 1996.

[10] Y. Polyanskiy, H. V. Poor, and S. Verdú, “Channel coding rate in the finite blocklength regime,” *IEEE Transactions on Information Theory*, vol. 56, no. 5, pp. 2307–2359, 2010.

[11] I. Csiszár and J. Körner, *Information theory: coding theorems for discrete memoryless systems*. Cambridge University Press, 2011.

[12] A. D. Wyner, “The wire-tap channel,” *The Bell System Technical Journal*, vol. 54, no. 8, pp. 1355–1387, 1975.

[13] I. Csiszár, “Information-type measures of difference of probability distributions and indirect observation,” Studia Sci. Math. Hungar., vol. 2, pp. 229-318, 1967.

[14] J. K. Omura, “A lower bounding method for channel and source coding probabilities,” *Information and Control*, vol. 27, no. 2, pp. 148–177, 1975.