RESEARCH ARTICLE

Numerical analysis of a corrected Smagorinsky model

Farjana Siddiqua | Xihui Xie

Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania, USA

Correspondence
Farjana Siddiqua, Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA.
Email: fas41@pitt.edu

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Abstract
The classical Smagorinsky model’s solution is an approximation to a (resolved) mean velocity. Since it is an eddy viscosity model, it cannot represent a flow of energy from unresolved fluctuations to the (resolved) mean velocity. This model has recently been corrected to incorporate this flow and still be well-posed. Herein we first develop some basic properties of the corrected model. Next, we perform a complete numerical analysis of two algorithms for its approximation. They are tested and proven to be effective.

KEYWORDS
backscatter, complex turbulence, corrected Smagorinsky, eddy viscosity

1 | INTRODUCTION

Consider the Smagorinsky model [39], with prescribed body force \( f \), kinematic viscosity \( \nu \) in the regular and bounded flow domain \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\), which was later advanced independently by Ladyzhenskaya [22, 23]: \( \nabla \cdot w = 0 \) and

\[
\begin{align*}
    w_t + w \cdot \nabla w - \nu \Delta w + \nabla q - \nabla \cdot \left( (C_s \delta)^2 |\nabla w| \nabla w \right) &= f(x).
\end{align*}
\]

(1.1)

Here \((w, q)\) approximate an ensemble average of Navier–Stokes solutions, \((\bar{u}, \bar{p})\). This is an eddy viscosity model with turbulent viscosity, \( \nu_T = (C_s \delta)^2 |\nabla w| \), where \( C_s \approx 0.1 \), Lilly [30], \( \delta \) is a length scale (or grid scale). Like all eddy viscosity models, the Smagorinsky model represents a flow of energy from means to unresolved fluctuations \((u' = u - \bar{u})\), for a precise formula see Definition 2.11) and has errors by not representing any intermittent energy flow from fluctuations back to means. Corrections

The mechanically correct formulation is with the \( \nabla w \) instead of \( \nabla w \) in the term \(-\nabla \cdot ((C_s \delta)^2 |\nabla w| \nabla w)\) where \( \nabla' \) is the symmetric part of the gradient tensor. But since the estimates are same and analyses are simpler with \( \nabla w \) due to Korn’s inequality \( \|v\|_{L^2(\Omega)} \leq C \left( \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 \right) \), we use \( \nabla w \) throughout the paper.
have recently been made representing this flow in Jiang and Layton [16] and Rong et al. [38]. Following their ideas, we develop a corrected model in Section 3. We also analyze and test numerical algorithms for effective approximation of the resulting corrected model: \( \nabla \cdot w = 0 \) and
\[
\begin{align*}
\frac{\partial w}{\partial t} - C_s \delta^2 \mu^{-2} \Delta w + w \cdot \nabla w - \nu \Delta w + \nabla q - \nabla \cdot \left( (C_v \delta)^2 |\nabla w| \nabla w \right) &= f(x),
\end{align*}
\]
Here \( \mu \) is a constant from Kolmogorov-Prandtl relation [20, 37].

The main result of this article is the complete numerical analysis and computational testing of effective algorithms for this model. This article gives detailed numerical analyses in Sections 4 and 5. This model is able to capture the phenomenon of transferring energy from fluctuation to means, which is tested numerically in Section 6.2. There were few attempts made for extending model that represents flow at statistical equilibrium to non-equilibrium. For instance, in a previous work by Jiang and Layton [16], there was an extra fitting parameter \( \beta \) in the second term of (1.2) which is needlessly complicated. In our article, a different idea results in a simpler model with no new fitting parameters other than from the Smagorinsky model (1.1).

1.1 Previous work

For simulating turbulent flow, there are different approaches, see [12, 13, 32, 35, 42, 43]. A summary of some recent work in eddy viscosity models of turbulence is presented in [17]. One of the recent approaches is by adding a term of Kelvin–Voigt form to the equations for the mean-field [1]. Smagorinsky model is a classical model. It’s positive and negative features are well understood. There has been lot of work correcting negative features, for example Kim [19] did a different modification than ours which corrects near wall behavior. The new term in our model has similarity to the Voigt term used in Voigt/Kelvin-Voigt/Kelvin Model [41] for viscoelastic fluids. There has been lot of recent works on Voigt Model, see for example [3, 21, 24, 25]. Recently, Rong et al. [38] and Berselli et al. [4] all studied the extension of the Baldwin and Lomax model [2] to non-equilibrium \( \frac{d}{dt} \|u'\|^2 \neq 0 \), for a precise definition see equilibrium) problems. A variant of the Smagorinsky model and detailed analysis is presented in the article [8]. Jiang and Layton [16] derived a corrected eddy viscosity model for flow not at statistical equilibrium state.

2 NOTATION AND PRELIMINARIES

In this section, we introduce some of the notations and results used in this article. We denote by \( \| \cdot \| \) and \( (\cdot, \cdot) \) the \( L^2(\Omega) \) norm and inner product, respectively. We denote the \( L^p(\Omega) \) norm by \( \| \cdot \|_{L^p} \). The solution spaces \( X \) for the velocity and \( Q \) for the pressure are defined as:
\[
X := \{ v \in L^3(\Omega) : \nabla v \in L^3(\Omega) \text{ and } v = 0 \text{ on } \partial \Omega \},
Q := L^2_0(\Omega) = \left\{ q \in L^2(\Omega) : \int_{\Omega} q \ dx = 0 \right\},
\]
and
\[
V := \{ v \in X : (q \nabla \cdot v) = 0, \ \forall q \in Q \}.
\]
The space \( H^{-1}(\Omega) \) denotes the dual space of bounded linear functionals defined on \( H_0^1(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \partial \Omega \} \) and this space is equipped with the norm:
\[
\|f\|_{-1} = \sup_{0 \neq v \in X} \frac{(f, v)}{\|v\|}.
\]
The finite element method for this problem involves picking finite element spaces [27] \( X^h \subset X \) and \( Q^h \subset Q \). We assume that \( (X^h, Q^h) \) satisfies the discrete inf-sup condition:
In this article, we will need this following well-known lemma, see, for example, [11, 18, 26].

**Definition 2.2** (Trilinear form). Define the skew symmetrized trilinear form \( b^* : X \times X \times X \to \mathbb{R} \) as follows

\[
b^*(u, v, w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v).
\]

**Lemma 2.2** (Girault and Raviart [14] p. 114). For any \( u \in V \) and \( v, w \in X \),

\[
b^*(u, v, w) = (u \cdot \nabla v, w), \quad \text{and} \quad b^*(u, v, v) = 0, \quad \forall \ u, \ v \in X.
\]

**Lemma 2.3** For any \( u, v, w \in X \),

\[
\begin{align*}
\left| \int_{\Omega} u \cdot \nabla v \cdot w \ dx \right| & \leq C \|\nabla u\| \|\nabla v\| \|\nabla w\|, \\
\left| \int_{\Omega} u \cdot \nabla v \cdot w \ dx \right| & \leq C \|u\|^{1/2} \|\nabla u\|^{1/2} \|\nabla v\| \|\nabla w\|.
\end{align*}
\]

**Lemma 2.4** (Polarization identity).

\[
(u, v) = \frac{1}{2} \|u\|^2 + \frac{1}{2} \|v\|^2 - \frac{1}{2} \|u - v\|^2. \tag{2.1}
\]

**Lemma 2.5** (The Poincaré-Friedrichs’ inequality). There is a positive constant \( C_{PF} = C_{PF}(\Omega) \) such that

\[
\|u\| \leq C_{PF} \|\nabla u\|, \quad \forall u \in X. \tag{2.2}
\]

Next is a discrete Gronwall lemma see lemma 5.1 p. 369 [15].

**Lemma 2.6** Let \( \Delta t, B, a_n, b_n, c_n, d_n \) for integers \( n \geq 0 \) be nonnegative numbers such that for \( l \geq 1 \), if

\[
a_l + \Delta t \sum_{n=0}^{l} b_n \leq \Delta t \sum_{n=0}^{l-1} d_n a_n + \Delta t \sum_{n=0}^{l} c_n + B, \quad \text{for} \ l \geq 0,
\]

then for all \( \Delta t > 0 \),

\[
a_l + \Delta t \sum_{n=0}^{l} b_n \leq \exp \left( \Delta t \sum_{n=0}^{l-1} d_n \right) \left( \Delta t \sum_{n=0}^{l} c_n + B \right), \quad \text{for} \ l \geq 0.
\]

In this article, we will need this following well-known lemma, see, for example, [11, 18, 26].

**Lemma 2.7** (Strong monotonicity (SM) and local Lipschitz continuity (LLC)). There exists \( C_1, C_2 > 0 \) such that for all \( u, v, w \in L^2(\Omega), \nabla u, \nabla v, \nabla w \in L^2(\Omega) \)

**SM** \( (|\nabla u| \nabla u - |\nabla w| \nabla w, (u - w)) \geq C_1 \|\nabla (u - w)\|^2_{L^2(\Omega)}, \tag{2.3} \)

**LLC** \( (|\nabla u| \nabla u - |\nabla w| \nabla w, \nabla v) \leq C_2 r \|\nabla (u - w)\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}, \tag{2.4} \)

where \( r = \max \{ \|\nabla u\|_{L^2(\Omega)}, \|\nabla w\|_{L^2(\Omega)} \}. \)
Proposition 2.8 (See p. 173 [6]). Let $W^{m,p}(\Omega)$ denote the Sobolev space, let $p \in [1, +\infty]$ and $q \in [p, p^*]$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$ if $p < \text{dim}(\Omega) = d$. There is a $C > 0$ such that

$$
\|u\|_{L^p(\Omega)} \leq C\|u\|^{1+d/q-d/p}_{L^q(\Omega)}\|u\|^{d/p-d/q}_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega)
$$

(2.5)

The weak formulation of (3.8) is: Find $(w, p) \in (X, Q)$ such that

$$(w, v) + \frac{C}{\mu^2} \delta^2 (\nabla w, \nabla v) + (w \cdot \nabla w, v) + v(\nabla w, \nabla v)$$

$$- (p, \nabla \cdot v) + ( (1, \delta)^2 \nabla w \nabla w, \nabla v) = (f, v) \quad \text{for all } v \in X,$n

$$(q, \nabla \cdot w) = 0 \quad \text{for all } q \in Q.
$$

(2.6)

For the stationary Smagorinsky model, Du and Gunzburger [10, 11] proved that the discrete solution converges to the continuous problem under minimal regularity assumptions. The existence and uniqueness of the strong solution of the Smagorinsky model (1.1) on a periodic domain have been discussed [28, 29, 31]. Recently, the error estimates for Smagorinsky model have also been studied in [7] and it showed that both the accuracy and the stability are enhanced for flows with high Reynolds number.

The existence and uniqueness of strong solutions of the incompressible Navier–Stokes-Voigt model is studied in [3].

Here we omit the proof for the existence of a strong solution for the new CSM Model. We assume the model has a solution in the following sense.

Definition 2.9 A solution $w$ of the corrected Smagorinsky model (CSM) (3.8) is a strong solution if $w$ satisfies the following

1. $w \in L^\infty (0, T; H^1(\Omega)) \cap L^2 (0, T; W^{1,3}(\Omega)) \cap L^2 (0, T; L^6(\Omega))$,
2. $w(x, t) \rightarrow w_0(x)$ in $L^2(\Omega)$ as $t \rightarrow 0$,
3. $w$ satisfies the model’s weak form (2.6) for all $v \in L^\infty (0, T; H^1(\Omega)) \cap L^2 (0, T; W^{1,3}(\Omega)) \cap L^2 (0, T; L^6(\Omega))$.

Remark 2.10 Though existence of strong solutions is not yet proven for the new model, we believe it is reasonable to assume existence because it is known for the Smagorinsky model and the extra Voigt term is linear and regularizing.

Definition 2.11 (Mean, fluctuation, and variance). The ensembles $u(x, t; \omega_j)$, $j = 1, \ldots, J$ where $\omega$ is a random variable, mean $\overline{u}$, and fluctuation $u'$ are defined as follows:

$$
\overline{u}(x, t) = \frac{1}{J} \sum_{j=1}^{J} u(x, t; \omega_j), \quad u'(x, t; \omega_j) = u(x, t; \omega_j) - \overline{u}(x, t).
$$

The variance of $u$ and $\nabla u$ are, respectively,

$$
V(u) := \int_{\Omega} |u|^2 - |\overline{u}|^2 \, dx, \quad V(\nabla u) := \int_{\Omega} |\nabla u|^2 - |\nabla \overline{u}|^2 \, dx.
$$

Definition 2.12 (Reynolds stresses). The Reynolds stresses are

$$
R(u, u) := \overline{u} \otimes \overline{u} - \overline{u} \otimes \overline{u} = -u' \otimes u'.
$$
Ensemble averaging satisfies the following properties, for example, [33, 34, 36].

\[
\bar{u} = \bar{u}, \quad \bar{u} = 0, \quad \bar{w} \cdot \bar{v} = \bar{w} \cdot \bar{v}, \quad \bar{w} \cdot \bar{v}' = \bar{w} \cdot \bar{v}' = 0,
\]
\[
\bar{w} \otimes \bar{v} = \bar{w} \otimes \bar{v}, \quad \bar{w} \otimes \bar{v}' = \bar{w} \otimes \bar{v}' = 0, \quad \frac{\partial}{\partial t} \bar{u} = \frac{\partial}{\partial x} \bar{u}, \quad \frac{\partial}{\partial x} \bar{u} = \frac{\partial}{\partial x} \bar{u}.
\]

**Theorem 2.13** Suppose that each realization is a strong solution of the NSE. The ensemble is generated by different initial data and \( u(x, 0; \omega_j) \in L^2(\Omega), f(x, t) \in L^\infty(0, \infty; L^2(\Omega)) \). Then the following two properties are satisfied.

**Property 1** (Time averaged dissipativity).

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_\Omega R(u, u) : \nabla \bar{u} \; dx dt = \lim_{T \to \infty} \frac{1}{T} \int_\Omega v|\nabla u'|^2 \; dx dt \geq 0.
\]

**Property 2** (Equation for the evolution of variance of fluctuations).

\[
\int_\Omega R(u, u) : \nabla \bar{u} \; dx = \frac{1}{2} \frac{d}{dt} \int_\Omega |u'|^2 \; dx + \int_\Omega v|\nabla u'|^2 \; dx. \tag{2.7}
\]

**Proof.** Proof of this theorem can be found in Section 2 of [16].

**Remark 2.14** (Statistical steady state and statistical equilibrium, see [16]). Statistical steady state is \( P/\varepsilon = 1 \) where

\[
\varepsilon = \text{dissipation of turbulent kinetic energy (TKE)} = v|\nabla u'|^2,
\]

\[
P = \text{production of TKE} = \int_\Omega R(u, u) : \nabla \bar{u} dx.
\]

Hence \( P/\varepsilon = 1 \) implies \( \int_\Omega R(u, u) : \nabla \bar{u} \; dx = \int_\Omega v|\nabla u'|^2 \; dx. \)

### 3 | MODEL DERIVATION

In this section, we develop a model for turbulence not at statistical equilibrium unlike the Smagorinsky model (1.1).

Consider the Navier–Stokes equations (NSE) which govern the flow of an incompressible fluid with velocity \( u(x, t) \), pressure \( p(x, t) \), prescribed body force \( f \) and kinematic viscosity \( \nu \) in the regular and bounded flow domain \( \Omega \subset \mathbb{R}^d \):

\[
u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f(x) \quad \text{in } \Omega, \quad \nabla \cdot u = 0 \quad \text{in } \Omega. \tag{3.1}
\]

To derive the CSM, following the work in [16], we begin with an ensemble of NSE solution \( u(x, t; w_j) \) with perturbed initial data \( u(x, 0; \omega_j) = u_0(x; \omega_j), j = 1, 2, \ldots, J, x \in \Omega \).

The goal of a turbulent model solution of (1.1) and (1.2) is to approximate \( \bar{u}(x, t) \). By ensemble averaging the NSE gives a system that is not closed since \( \overline{\bar{u}} \neq \bar{u} \). Hence the Reynolds stress tensor, \( R(u, u) = \bar{u} \bar{u}' - \overline{\bar{u} \bar{u}'} \) which is accountable for all effects of the fluctuations on the mean flow must be modeled [38]. We rewrite \( \overline{\bar{u}} \bar{u}' \) as \( \bar{u} \bar{u}' = \bar{u} \bar{u} - R(u, u) \). Note that by using properties in (2.12) we get

\[
u_t + \bar{u} \cdot \nabla \bar{u} - \nu \Delta \bar{u} + \nabla p - \nabla \cdot R = f(x) \quad \text{in } \Omega, \quad \nabla \cdot \bar{u} = 0 \quad \text{in } \Omega. \tag{3.2}
\]
Take the dot product of first and second equation in (3.2) with mean flow $\bar{u}$ and $\bar{p}$ respectively and doing integration by parts, we get the energy estimate as follows [16, 38].

$$\frac{1}{2} \frac{d}{dt} \| \bar{u} \|^2 + \nu \| \nabla \bar{u} \|^2 + \int_{\Omega} R(u, u) : \nabla \bar{u} \, dx = (f, \bar{u}).$$  (3.3)

In (3.3), if the term $\int_{\Omega} R(u, u) : \nabla \bar{u} \, dx > 0$, the effect of $R(u, u)$ is dissipative while if $\int_{\Omega} R(u, u) : \nabla \bar{u} \, dx < 0$, fluctuations $u'$ transfers energy back to mean $\bar{u}$ which causes increased energy in mean flow.

Property 1 in Theorem 2.13 is consistent with the assumption of Boussinesq [5] that turbulent fluctuations are dissipative on the mean in the time averaged case. In Property 2 of b, the term $\int_{\Omega} \nu |\nabla u'|^2 \, dx$ is clearly dissipative while $\frac{d}{dt} \int_{\Omega} |u'|^2 \, dx = 0$ for flows at statistical equilibrium. The idea of any EV model is based on three assumptions [16]. First, the statistical equilibrium assumption that dissipativity holds at each instant time

$$\int_{\Omega} R(u, u) : \nabla \bar{u} \, dx \simeq \int_{\Omega} \nu |\nabla u'|^2 \, dx.$$  (3.4)

The second assumption is that $\nabla u'$ aligns with $\nabla \bar{u}$. Third, calibration [16] provides that the action of fluctuating velocities can be represented in terms of mean flow

$$\text{action } (\nabla u') \simeq a(\bar{u}) \nabla \bar{u}.$$  

Combining all these three assumptions results in the eddy viscosity closure,

$$-\nabla \cdot R(u, u) \leftrightarrow -\nabla \cdot (\nu_T(\bar{u}) \nabla \bar{u}) + \text{terms incorporated in } \nabla \bar{p}.$$  

Here $\nu_T$ denotes the turbulent viscosity. Thus we have the eddy viscosity (EV) model: $\nabla \cdot w = 0$ and

$$w_t + w \cdot \nabla w - \nu \Delta w + \nabla q - \nabla \cdot (\nu_T(w) \nabla w) = f(x).$$  (3.5)

The solution $(w, q)$ of (3.5) is an approximation of the ensemble average $(\bar{u}, \bar{p})$. In 1963, Smagorinsky [39] model $\nu_T$ by

$$\nu_T = (C_s \delta)^2 |\nabla w|,$$  (3.6)

where $C_s \approx 0.1$, Lilly [30]. Let $x$ to be the mesh size and $\delta = \Delta x << 1$ is the model length scale [40]. Thus we get the classic Smagorinsky model (1.1).

By taking the dot product with $w$, here we have the energy equality for Smagorinsky model:

$$\frac{1}{2} \frac{d}{dt} \| w \|^2 + \nabla \nu \| w \|^2 + (C_s \delta)^2 \| \nabla w \|^3_{L^2} = (f, w).$$

$(C_s \delta)^2 \| \nabla w \|^3_{L^2} \geq 0$ approximates the average energy dissipated by fluctuation. Since it is positive, it prevents the energy from returning to the mean flow. We aim to remove this flaw in the corrected model. Notice that in (3.4), the term $\frac{d}{dt} \int_{\Omega} |u'|^2 \, dx$ is omitted for flows at statistical equilibrium. This term is accountable for backscatter and other non-equilibrium effects. To model this term, we must express $u'$ in terms of $\bar{u}$. One idea in [16] is that since the Smagorinsky model is dimensionally consistent, it must conform to the Kolmogorov-Prandtl relation [20, 37].

$$\nu_T = \mu l \sqrt{k'},$$  (3.7)
where $\mu \approx 0.3$ to 0.55, $l = \text{turbulent length scale}$ and $k'$ is the turbulent kinetic energy: $k'(x,t) = \frac{1}{2}|u'(x,t)|^2$. Thus, the Smagorinsky model contains a implicit model of $l$ and $k'$. Equating (3.6) with (3.7) gives

$$\mu l \sqrt{k'} = \nu_T = (C_s \delta)^2 \|\nabla w\| = \mu \delta \left( \frac{C_s^2 \delta}{\mu} \|\nabla w\| \right).$$

Here $\delta$ is the obvious choice for $l$. With $\delta = l$, the Smagorinsky model yields the model $k' = C_s^4 \delta^2 \mu^2 \|\nabla w\|^2$. Hence, the omitted term responsible for non-equilibrium effects is modeled as

$$\frac{d}{dt} \int k' \, dx = \frac{d}{dt} C_s^4 \delta^2 \mu^{-2} \langle \nabla w, \nabla w \rangle = C_s^4 \delta^2 \mu^{-2} (-\Delta w_t, w).$$

By including $C_s^4 \delta^2 \mu^{-2} \Delta w_t$ in the model, its energy balance has a consistent representation of the term $\frac{d}{dt} \|u'\|^2$. As a result, the CSM is: $\nabla \cdot w = 0$ and

$$w_t - C_s^4 \delta^2 \mu^{-2} \Delta w + w \cdot \nabla w - \nu \Delta w + \nabla q - \nabla \cdot \left( (C_s \delta)^2 \|\nabla w\| \right) = f(x). \quad (3.8)$$

Here we impose the no-slip boundary condition, $w = 0$ on $\partial \Omega$.

## 4 | BASIC PROPERTIES OF THE MODEL

In this section, we develop some basic properties of the model which are useful in numerical analysis. In particular, we derive the basic energy estimate, we prove a stability bound and uniqueness of the solution. We also analyze the modeling error and numerical error of the model.

### 4.1 | Energy estimate for the CSM

We will identify the model’s kinetic energy and energy dissipation in Theorem 4.1.

**Theorem 4.1** Let $w$ be a strong solution of the CSM (3.8), then the following energy estimate holds.

$$\frac{1}{2} \int_{\Omega} \frac{d}{dt} \left( \frac{1}{2} \|w\|^2 + \frac{C_s^4 \delta^2}{\mu} \|\nabla w\|^2 \right) + \frac{1}{|\Omega|} \|\nabla w\|^2 + \frac{1}{|\Omega|} (C_s \delta)^2 \|\nabla w\|^3 = \frac{1}{|\Omega|} \langle f, w \rangle. \quad (4.1)$$

**Proof:** First, we take dot product in (3.8) with $w$ and do integration by parts which is shown below.

$$\int_{\Omega} \left( w_t - C_s^4 \delta^2 \mu^{-2} \Delta w + w \cdot \nabla w - \nu \Delta w + \nabla q - \nabla \cdot \left( (C_s \delta)^2 \|\nabla w\| \right) \right) \cdot w \, dx = \int_{\Omega} f \cdot w \, dx.$$

Here, $\int_{\Omega} w_t \cdot w \, dx = \frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} |w|^2 \, dx \right)$. By Lemma 2.2, $\int_{\Omega} w \cdot \nabla w \cdot w \, dx = 0$. Next, $-\nabla \cdot \int_{\Omega} \Delta w \cdot w \, dx = \int_{\Omega} \nabla |\nabla w|^2 \, dx$. The next term, $\int_{\Omega} \nabla q \cdot w \, dx = \int_{\Omega} \rho w \cdot \hat{n} \, ds - \int_{\Omega} \rho \nabla \cdot w \, dx = 0$. The final term,

$$\int_{\Omega} \nabla \cdot \left( (C_s \delta)^2 \|\nabla w\| \right) w \, dx = \int_{\Omega} (C_s \delta)^2 \|\nabla w\| \nabla w \cdot w \, dx = \int_{\Omega} (C_s \delta)^2 \|\nabla w\|^3 \, dx.$$
Hence combining all these terms we get the following energy estimate per unit volume,
\[
\frac{1}{2} \frac{1}{|\Omega|} \frac{d}{dt} \left( \|w\|^2 + \frac{C^4_\delta^2}{\mu^2} \|\nabla w\|^2 \right) \\
+ \frac{1}{|\Omega|} v \|\nabla w\|^2 + \frac{1}{|\Omega|} (C_\delta \delta)^2 \|\nabla w\|_{L^3}^3 = \frac{1}{|\Omega|} \langle f, w \rangle.
\]

**Remark 4.2** In (4.1), we can identify the following quantities:

1. **Model kinetic energy of mean flow per unit volume**
   \[
   MKE := \frac{1}{2} \frac{1}{|\Omega|} \left( \|w\|^2 + \frac{C^4_\delta^2}{\mu^2} \|\nabla w\|^2 \right).
   \]

   And the second term in MKE coming from the CSM is the turbulent kinetic energy per unit volume.

2. **Rate of energy dissipation of mean flow per unit volume**
   \[
   \epsilon_{CSM}(t) := \frac{1}{|\Omega|} \left( v \|\nabla w\|^2 + (C_\delta \delta)^2 \|\nabla w\|_{L^3}^3 \right).
   \]

   This controls the time rate of change of kinetic energy. It’s always positive and it reduces the accumulation of kinetic energy.

3. **Rate of energy input to mean flow per unit volume** is \( \frac{1}{|\Omega|} \langle f, w \rangle \).

### 4.2 Stability

Next, we give the stability bound of the CSM (3.8) in Theorem 4.3. We prove the model kinetic energy is bounded uniformly in time and the time-averaged model energy dissipation rate is bounded as well in the same theorem.

**Theorem 4.3** (Stability of \( w \)). *(3.8) is unconditionally stable. The solution \( w \) of (3.8) satisfies the following inequality*

\[
\|w(T)\|^2 + \frac{C^4_\delta^2}{\mu^2} \|\nabla w(T)\|^2 \leq e^{-\alpha T} \left\{ \|w(0)\|^2 + \frac{C^4_\delta^2}{\mu^2} \|\nabla w(0)\|^2 + \frac{C}{\alpha} \left( e^{\alpha T} - 1 \right) \right\},
\]

where \( \alpha = \min \left\{ \frac{\nu}{2C_{PF}^2}, \frac{\mu^2 \nu}{C^2_\delta^2} \right\} \), and if \( f \in L^2(\Omega) \), we get

\[
\max_{0 \leq t < \infty} \left( \|w\|^2 + \frac{C^4_\delta^2}{\mu^2} \|\nabla w\|^2 \right) \leq C' < \infty.
\]

And

\[
\mathcal{O} \left( \frac{1}{T} \right) + \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \left( \frac{\nu}{2} \|\nabla w\|^2 + (C_\delta \delta)^2 \|\nabla w\|_{L^3}^3 \right) dt \leq \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \frac{C_{PF}^2}{2\nu} \|f\|^2 dt.
\]

**Proof.** Take \( L^2 \) inner product of (3.8) with \( w \), we get the following energy equality,

\[
\frac{1}{2} \frac{d}{dt} \left( \|w\|^2 + \frac{C^4_\delta^2}{\mu^2} \|\nabla w\|^2 \right) + v \|\nabla w\|^2 + (C_\delta \delta)^2 \|\nabla w\|_{L^3}^3 = \langle f, w \rangle.
\]
Consider the RHS of (4.2), \((f, w) \leq \frac{\varepsilon}{2} \| \nabla w \|^2 + \frac{1}{2\varepsilon} \| f \|^2\). Thus (4.2) implies
\[
\frac{d}{dt} \left( \| w \|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) + v \| \nabla w \|^2 + \nu \| \nabla w \|^2 + 2(C_3 \delta)^2 \| \nabla w \|_L^2 \leq \varepsilon \| w \|^2 + \frac{1}{\varepsilon} \| f \|^2.
\]
Using the inequality \(\| w \| \leq C_{PF} \| \nabla w \|\), we have
\[
\frac{d}{dt} \left( \| w \|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) + \frac{\nu}{C_{PF}^2} \| w \|^2 + \nu \| \nabla w \|^2 + 2(C_3 \delta)^2 \| \nabla w \|_L^2 \leq \varepsilon \| w \|^2 + \frac{1}{\varepsilon} \| f \|^2.
\]
Pick \(\varepsilon = \frac{\nu}{2C_{PF}^2}\) and drop the term \(2(C_3 \delta)^2 \| \nabla w \|_L^2\). We obtain
\[
\frac{d}{dt} \left( \| w \|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) + \frac{\nu}{2C_{PF}^2} \| w \|^2 + \frac{\mu^2}{C_4^4} \delta^2 \nu \left( \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) \leq \frac{2}{\nu} C_{PF}^2 \| f \|^2.
\]
Let \(\alpha = \min \left\{ \frac{\nu}{2C_{PF}^2}, \frac{\mu^2}{C_4^4} \delta^2 \nu \right\}\), resulting in
\[
\frac{d}{dt} \left( \| w \|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) + \alpha \left( \| w \|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) \leq \frac{2}{\nu} C_{PF}^2 \| f \|^2.
\]
Multiply by the integrating factor \(e^{\alpha t}\) and integrate from \(t = 0\) to \(t = T\), leading to
\[
\| w(T) \|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w(T) \|^2 \leq e^{-\alpha T} \left\{ \| w(0) \|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w(0) \|^2 + \frac{C}{\alpha} (e^{\alpha T} - 1) \right\},
\]
where \(C = \frac{2}{\nu} C_{PF}^2 \| f \|^2\).

This implies that kinetic energy is uniformly bounded, that is, if \(f \in L^2(\Omega)\), we get
\[
\max_{0 \leq t < \infty} \left( \| w \|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) \leq C' < \infty.
\]
Integrate (4.2) from \(t = 0\) to \(t = T\) and divide by \(|\Omega|\) and \(T\), we have
\[
\frac{1}{|\Omega|} \frac{1}{2T} \left\{ \left( \| w(T) \|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w(T) \|^2 \right) - \left( \| w(0) \|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w(0) \|^2 \right) \right\}
+ \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \left( v \| \nabla w \|^2 + (C_3 \delta)^2 \| \nabla w \|_L^2 \right) dt
\leq \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \frac{1}{\sqrt{\nu}} \| |C_{PF}\sqrt{\nu} \| \nabla w \| dt.
\]
Consider the term on the right. Using the Poincaré-Friedrichs’ inequality (2.2), Cauchy Schwarz and Young’s inequality gives
\[
\frac{1}{|\Omega|} \frac{1}{T} \int_0^T \frac{1}{\sqrt{\nu}} \| C_{PF} \sqrt{\nu} \| \nabla w \| dt
\leq \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \frac{\nu}{2} \| \nabla w \|^2 dt + \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \frac{C_{PF}^2}{2\nu} \| f \|^2 dt.
\]
The first term in (4.3) is bounded by the previous result. Thus,
\[
\Theta \left( \frac{1}{T} \right) + \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \left( \frac{\nu}{2} \| \nabla w \|^2 + (C_3 \delta)^2 \| \nabla w \|_L^2 \right) dt \leq \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \frac{C_{PF}^2}{2\nu} \| f \|^2 dt.
\]
The time-averaged dissipation is bounded.
4.3 Uniqueness

In this section, we prove the uniqueness of the strong solution to (3.8) in Theorem 4.4.

**Theorem 4.4** Assume $\nabla w \in L^3 \left( 0, T; L^3(\Omega) \right)$, the solution $w$ of (3.8) is unique.

**Proof.** Suppose $(w_1, q_1)$ and $(w_2, q_2)$ are two different solutions of (3.8) and let $\phi, q$ denote the difference between two solutions: $\phi = w_1 - w_2, q = q_1 - q_2, \phi$ satisfies $\cdot \phi = 0$ and

$$
\frac{\partial}{\partial t} \left( \phi - \frac{C}{\mu^2} \phi^2 \right) + w_1 \cdot \nabla \phi - w_2 \cdot \nabla \phi - C_4 \phi + \nabla q - (C_4 \phi)^2 \nabla \cdot (|\nabla w_1| \nabla w_1 - |\nabla w_2| \nabla w_2) = 0.
$$

Take the $L^2$ inner product with $\phi$ and let $\tilde{w}$ represent either $w_1$ or $w_2$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \left( \|\phi\|^2 + \frac{C}{\mu^2} \phi^2 \|\nabla \phi\|^2 \right) + \nu \|\nabla \phi\|^2
$$

$$
+ (C_4 \phi)^2 \int_{\Omega} [ |\nabla w_1| \nabla w_1 - |\nabla w_2| \nabla w_2 ] \cdot \nabla (w_1 - w_2) \, dx = - \int_{\Omega} \phi \cdot \nabla \tilde{w} \cdot \phi \, dx.
$$

Using the strong monotonicity (2.3), we get

$$
\frac{1}{2} \frac{d}{dt} \left( \|\phi\|^2 + \frac{C}{\mu^2} \phi^2 \|\nabla \phi\|^2 \right) + \nu \|\nabla \phi\|^2 + C_4(C_4 \phi)^2 \|\nabla \phi\|^2 \leq - \int_{\Omega} \phi \cdot \nabla w \cdot \phi \, dx.
$$

Consider the RHS, using (2.5) in 3D space,

$$
\left| - \int_{\Omega} \phi \cdot \nabla w \cdot \phi \, dx \right| \leq \|\nabla w\|_{L^3} \|\phi\|_{L^6}^2,
$$

$$
\leq C \|\nabla w\|_{L^3} \left( \|\phi\|^{1/2} \|\nabla \phi\|^{1/2} \right)^2,
$$

$$
\leq \frac{\epsilon}{2} \|\nabla \phi\|^2 + C(\epsilon) \|\nabla w\|^2 \|\phi\|^2.
$$

Pick $\epsilon = 2 \frac{C}{\mu^2} \phi^2$, leading to

$$
\frac{1}{2} \frac{d}{dt} \left( \|\phi\|^2 + \frac{C}{\mu^2} \phi^2 \|\nabla \phi\|^2 \right) + \nu \|\nabla \phi\|^2 + C_4(C_4 \phi)^2 \|\nabla \phi\|^2
$$

$$
\leq \frac{C_4 \phi^2}{\mu^2} \|\nabla \phi\|^2 + C(\epsilon) \|\nabla w\|^2 \|\phi\|^2,
$$

$$
\leq \max \{ 1, C(\epsilon) \|\nabla w\|^2 \} \left( \frac{C_4 \phi^2}{\mu^2} \|\nabla \phi\|^2 + \|\phi\|^2 \right).
$$

Here $a(t) := \max \{ 1, C(\epsilon) \|\nabla w_1\|^2 \} \in L^1(0, T)$, because

$$
\int_0^T \|\nabla w_1\|^2 \, dt \leq \left( \int_0^T 1^3 \, dt \right)^{1/3} \left( \int_0^T \|\nabla w_1\|^2 \, dt \right)^{2/3}
$$

$$
= \left( \int_0^T 1^3 \, dt \right)^{1/3} \left( \int_0^T \|\nabla w_1\|^3 \, dt \right)^{2/3} < \infty.
$$

Then we can form its antiderivative

$$
A(T) := \int_0^T a(t) \, dt < \infty, \text{ for } \nabla w \in L^3 \left( 0, T; L^3(\Omega) \right).
$$
Multiplying through by the integrating factor $e^{-A(t)}$ gives
\[
\frac{d}{dt} \left[ \frac{1}{2} e^{-A(t)} \left( \| \phi \|^2 + \frac{C_4}{\mu^2} \| \nabla \phi \|^2 \right) \right] = -e^{-A(t)} \left[ v \| \nabla \phi \|^2 + C_1 (C_\delta)^2 \| \nabla \phi \|^3_{L_3} \right] \leq 0.
\]

Then, integrating over $[0, T]$ and multiplying through by $e^{A(t)}$ gives
\[
\frac{1}{2} \left( \| \phi(T) \|^2 + \frac{C_4}{\mu^2} \| \nabla \phi(T) \|^2 \right) + \int_0^T (v \| \nabla \phi \|^2 + C_1 (C_\delta)^2 \| \nabla \phi \|^3_{L_3}) \, dt \\
\leq e^{A(T)} \frac{1}{2} \left( \| \phi(0) \|^2 + \frac{C_4}{\mu^2} \| \nabla \phi(0) \|^2 \right).
\]

### 4.4 Modeling error

In this section, we analyze the error between the solution to the NSE (3.1) and the corrected Smagorinsky model (3.8) in Theorem 4.5.

**Theorem 4.5** Assume $\nabla u_t \in L^2(\Omega)$ and $\nabla w \in L^2 \left( 0, T; L^3 \right)$, let $\phi = u_{NSE} - w_{Smag}$ be the modeling error of the corrected Smagorinsky, then $\phi$ satisfies the following:

\[
\frac{1}{2} \left( \| \phi(T) \|^2 + \frac{C_4}{\mu^2} \| \nabla \phi(T) \|^2 \right) + \int_0^T v \| \nabla \phi \|^2 + \frac{C_1}{\mu^2} (C_\delta)^2 \| \nabla \phi \|^3_{L_3} \, dt \\
\leq C^* \left\{ \frac{1}{2} \left( \| \phi(0) \|^2 + \frac{C_4}{\mu^2} \| \nabla \phi(0) \|^2 \right) + \int_0^T (C_\delta)^2 \| \nabla u_t \|^3_{L_3} + \frac{C_4}{\mu^2} \| \nabla u_t \|^2 \, dt \right\}.
\]

Here $C^*$ depends on $v$, $T$, $\int_0^T \| \nabla w \|^2_{L_3} \, dt$.

**Proof.** $u_{NSE}$ satisfies $\nabla \cdot u = 0$ and the following equation

\[
u u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = (C_\delta)^2 \nabla \cdot (|\nabla u| \nabla u) - \frac{C_4}{\mu^2} \delta^2 \Delta u_t
\]

\[
= f - (C_\delta)^2 \nabla \cdot (|\nabla u| \nabla u) - \frac{C_4}{\mu^2} \delta^2 \Delta u_t. \quad (4.4)
\]

Subtract (3.8) from (4.4). We obtain, $\nabla \cdot \phi = 0$ and

\[
\phi_t - \frac{C_4}{\mu^2} \delta^2 \Delta \phi_t + u \cdot \nabla u - w \cdot \nabla w + \nu \Delta \phi + \nabla (p - q)
\]

\[
- (C_\delta)^2 \nabla \cdot (|\nabla u| \nabla u - |\nabla w| \nabla w) = -(C_\delta)^2 \nabla \cdot (|\nabla u| \nabla u) - \frac{C_4}{\mu^2} \delta^2 \Delta u_t.
\]

Here, $u \cdot \nabla u - w \cdot \nabla w = u \cdot \nabla u - u \cdot \nabla u + u \cdot \nabla w - w \cdot \nabla w = u \cdot \nabla \phi + \phi \cdot \nabla w$.

Take $L^2$ inner product with $\phi$ gives

\[
\frac{1}{2} \frac{d}{dt} \left( \| \phi \|^2 + \frac{C_4}{\mu^2} \| \nabla \phi \|^2 \right) + \int_\Omega \phi \cdot \nabla w \cdot \phi \, dx + \nu \| \nabla \phi \|^2 \, dx
\]

\[
+ (C_\delta)^2 \int_\Omega (|\nabla u| \nabla u - |\nabla w| \nabla w) \cdot \nabla (u - w) \, dx
\]

\[
= (C_\delta)^2 \int_\Omega |\nabla u| \nabla u : \phi \, dx + \frac{C_4}{\mu^2} \delta^2 \int_\Omega \nabla u_t : \nabla \phi \, dx.
\]
Using strong monotonicity (2.7), we have

\[ \frac{1}{2} \frac{d}{dt} \left( \| \phi(t) \|^2 + \frac{C_4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) + v \| \nabla \phi \|^2 + C_1(C_\delta)^2 \| \nabla \phi \|^3_{L^3} \]

\[ \leq - \int_\Omega \phi \cdot \nabla w \cdot \phi \ dx + (C_\delta)^2 \int_\Omega \nabla u \cdot \nabla \phi \ dx + \frac{C_4}{\mu^2} \delta^2 \int_\Omega \nabla u_t : \nabla \phi \ dx. \]  

(4.5)

Consider the first term in the RHS, similar to the previous steps

\[ \left| - \int_\Omega \phi \cdot \nabla w \cdot \phi \ dx \right| \leq \frac{\epsilon_1}{2} \| \nabla \phi \|^2 + C(\epsilon_1) \| \nabla w \|^2_{L^3} \| \phi \|^2. \]

The second term in the RHS is

\[ \left| (C_\delta)^2 \int_\Omega \nabla u \cdot \nabla \phi \ dx \right| \leq (C_\delta)^2 \| \nabla \phi \|^2_{L^3} \| \nabla u \|^2_{L^3}, \]

\[ \leq \frac{\epsilon_2}{3} (C_\delta)^2 \| \nabla \phi \|^3_{L^3} + C(\epsilon_2) (C_\delta)^2 \| \nabla u \|^3_{L^3}, \]

\[ = \frac{\epsilon_2}{3} (C_\delta)^2 \| \nabla \phi \|^3_{L^3} + C(\epsilon_2) (C_\delta)^2 \| \nabla u \|^3_{L^3}. \]

The third term in the RHS satisfies

\[ \left| \frac{C_4}{\mu^2} \delta^2 \int_\Omega \nabla u_t : \nabla \phi \ dx \right| \leq \frac{C_4}{\mu^2} \delta^2 \| \nabla u_t \| \| \nabla \phi \|, \]

\[ \leq \frac{\epsilon_3}{2} \frac{C_4}{\mu^2} \delta^2 \| \nabla \phi \|^2 + C(\epsilon_3) \frac{C_4}{\mu^2} \delta^2 \| \nabla u_t \|^2. \]

Pick \( \epsilon_1 = v, \epsilon_2 = \frac{3C_1}{2}, \) collect all terms, (4.5) becomes

\[ \frac{1}{2} \frac{d}{dt} \left( \| \phi \|^2 + \frac{C_4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) + v \| \nabla \phi \|^2 + \frac{C_1}{2} (C_\delta)^2 \| \nabla \phi \|^3_{L^3} \]

\[ \leq \max \left\{ C(\epsilon_1) \| \nabla w \|^2_{L^3}, \frac{\epsilon_1}{2} \right\} \left( \| \phi \|^2 + \frac{C_4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) \]

\[ + C(\epsilon_2) (C_\delta)^2 \| \nabla u \|^3_{L^3} + C(\epsilon_3) \frac{C_4}{\mu^2} \delta^2 \| \nabla u_t \|^2. \]

Denote \( a(t) := \max \left\{ C(\epsilon_1) \| \nabla w \|^2_{L^3}, \frac{\epsilon_1}{2} \right\} \) and its antiderivative is given by

\[ A(T) := \int_0^T a(t) \ dt < \infty \text{ for } \nabla w \in L^2(0, T; L^3). \]

Multiplying through by the integrating factor \( e^{-A(t)} \) gives

\[ \frac{d}{dt} \left[ \frac{1}{2} e^{-A(t)} \left( \| \phi \|^2 + \frac{C_4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) \right] + e^{-A(t)} \left[ \frac{v}{2} \| \nabla \phi \|^2 + \frac{C_1}{2} (C_\delta)^2 \| \nabla \phi \|^3_{L^3} \right] \]

\[ \leq e^{-A(t)} \left\{ C(\epsilon_2) (C_\delta)^2 \| \nabla u \|^3_{L^3} + C(\epsilon_3) \frac{C_4}{\mu^2} \delta^2 \| \nabla u_t \|^2 \right\}. \]

Then, integrating over \([0, T]\) and multiplying through by \( e^{A(t)} \) gives

\[ \frac{1}{2} \left( \| \phi(T) \|^2 + \frac{C_4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) + \int_0^T \frac{v}{2} \| \nabla \phi \|^2 + \frac{C_1}{2} (C_\delta)^2 \| \nabla \phi \|^3_{L^3} \ dt \]
5 | NUMERICAL ERROR

Consider the semi-discrete approximation of the CSM (3.8) with grad-div stabilization. Suppose \(w^h(x, 0)\) is approximation of \(w(x, 0)\). The approximate velocity and pressure are maps

\[w^h : [0, T] \rightarrow X^h, \quad p^h : (0, T] \rightarrow Q^h\]

satisfying

\[
\begin{align*}
(w^h_i, v^h) + \frac{C^4}{\mu^2} \delta^2 (\nabla w^h_i, \nabla v^h) + b^* (w^h, w^h, v^h) + \nu (\nabla w^h, \nabla v^h) + \gamma (\nabla \cdot w^h, \nabla \cdot v^h) \\
\quad - (p^h, \nabla \cdot v^h) + ((C_0 \delta)^2 |\nabla w^h| \nabla w^h, \nabla v^h) = (f, v^h) \quad \text{for all } v^h \in X^h,
\end{align*}
\]

(5.1)

(5.1) \(\text{Numerical error of semi-discrete case.}\) Let \(w\) be the strong solution of the CSM (3.8) (in particular \(\|w\| \in L^\infty(0, T), \|\nabla w\|_{L^2} \in L^2(0, T), w \in L^2 (0, T; W^{1,3}(\Omega)) \cap L^2 (0, T; L^6(\Omega))\)) and \(w^h\) be a solution to the semi-discrete problem (5.1). Let

\[
a(t) := C(\nu) \|\nabla w\|_{L^2}^2 + \frac{1}{4} \|w\|_{L^3}^2.
\]

Then, for \(T > 0\) the error \(w - w^h\) satisfies

\[
\begin{align*}
\| (w - w^h) (T) \|^2 + & \frac{C^4 \delta^2}{\mu^2} \|\nabla (w - w^h) (T)\|^2 \\
\quad + \int_0^T \left\{ \frac{\nu}{4} \|\nabla (w - w^h)\|^2 + \gamma \|\nabla \cdot (w - w^h)\|^2 + \frac{2}{3} C_1 (C_0 \delta)^2 \|\nabla (w - w^h)\|^3_{L^2} \right\} \, dt \\
\leq & \exp \left( \int_0^T a(t) \, dt \right) \left\{ \| (w - w^h) (0)\|^2 \\
\quad + \frac{C^4 \delta^2}{\mu^2} \|\nabla (w - w^h) (0)\|^2 + \inf_{\nu^h \in V^h} \| (w - \nu^h) (T)\|^2 \\
\quad + \int_0^T [C(\nu) \inf_{\nu^h \in V^h} \|\nabla w_i - \nu^h_i\|^2_{L^2} + \left( \frac{C^4 \delta^2}{\mu^2} \right)^2 \|\nabla (w_i - \nu^h_i)\|^2 + \|\nabla (w - \nu^h)\|^2] \\
\quad + C \inf_{\nu^h \in V^h} (C_0 \delta)^2 \|\nabla (w - \nu^h)\|^3_{L^2} + \delta^{-1} \|w - \nu^h\|^3_{L^6} + \gamma \|\nabla \cdot (w - \nu^h)\|^2 \\
\quad + C \left( \gamma^{-1} \right) \inf_{q^h \in Q^h} \|\nabla q^h\|_{L^2}^2 \right) \, dt.
\end{align*}
\]

Proof. Consider the variational problem of the CSM (3.8): Find \(w : [0, T) \rightarrow X = L^\infty (0, T; L^2(\Omega)) \cap L^3 (0, T; W^{1,3}(\Omega))\) satisfying (2.6). Let \(\nu^h \in V^h =
\( \{ v^h \in X^h : (\nabla \cdot v^h, q^h) = 0 \ \forall q^h \in Q^h \} \). Since \( v \in X \) and \( v^h \in V^h \subset X^h \subset X \), we restrict \( v = v^h \) in continuous variational problem. Then subtract semi-discrete problem (5.1) from continuous problem (2.6). Let \( e = \text{error} = w - w^h \). This gives,

\[
(e, v^h) + (C_2 \delta^2 \mu^{-2} \nabla e, v^h) + b^s (w, w, v^h) - b^s (w^h, w^h, v^h) \\
+ v (\nabla e, \nabla v^h) + \gamma (\nabla \cdot e, \nabla \cdot v^h) + (C_3 \delta)^2 \int_\Omega \left( |\nabla w| |\nabla w - |\nabla w^h| \nabla w^h| : \nabla v^h \right) \cdot \nabla v^h \ dx \\
- (p - p^h, \nabla \cdot v^h) = 0.
\]  

(5.2)

We can write,

\[
b^s (w, w, v^h) - b^s (w^h, w^h, v^h) \\
= b^s (w, w, v^h) - b^s (w^h, w, v^h) + b^s (w^h, w, v^h) - b^s (w^h, w^h, v^h),
\]

\[
= b^s (e, w, v^h) + b^s (w^h, e, v^h).
\]

Also,

\[
\int_\Omega \left( |\nabla w| |\nabla w - |\nabla w^h| \nabla w^h| : \nabla v^h \right) \cdot \nabla v^h \ dx \\
= \int_\Omega \left( |\nabla w| |\nabla w - |\nabla w^h| \nabla w^h| + |\nabla w^h| \nabla w^h - |\nabla w^h| \nabla w^h \right) \cdot \nabla v^h \ dx.
\]

Pick \( \widetilde{v} \in V^h \). Let \( \eta = w - \widetilde{w}, \phi^b = w^h - \widetilde{w}, \phi^h \in V^h \). This implies \( e = (w - \widetilde{w}) - (w^h - \widetilde{w}) = \eta - \phi^h \). Then (5.2) becomes

\[
(\phi^b, v^h) + (C_2 \delta^2 \mu^{-2} \nabla \phi^b, \nabla v^h) + b^s (e, w, v^h) + b^s (w^h, e, v^h) \\
+ v (\nabla \phi^b, \nabla v^h) + \gamma (\nabla \cdot \phi^b, \nabla \cdot v^h) \\
+ (C_3 \delta)^2 \int_\Omega \left( |\nabla w| |\nabla w - |\nabla w^h| \nabla w^h| : (\nabla v^h) \right) \ dx - (p - p^h, \nabla \cdot v^h) = (\eta, v^h) \\
+ (C_4 \delta^2 \mu^{-2} \nabla \phi^b, \nabla v^h) + v (\nabla \eta, \nabla v^h) + \gamma (\nabla \cdot \eta, \nabla \cdot v^h) \\
+ (C_3 \delta)^2 \int_\Omega \left( |\nabla w| |\nabla w - |\nabla w^h| \nabla w^h| : (\nabla v^h) \right) \ dx.
\]

Take \( v^h = \phi^b \) and \( \lambda^h \in Q^h \). Here \( b^s (w^h, e, \phi^h) = b^s (w^h, \eta - \phi^h, \phi^h) = b^s (w^h, \eta, \phi^h) \)

since \( b^s (w^h, \phi^h, \phi^h) = 0 \). Using strong monotonicity (2.7), we get

\[
(C_3 \delta)^2 \int_\Omega \left( |\nabla w| |\nabla w - |\nabla w^h| \nabla w^h| : (\nabla v^h) \right) \ dx \geq C_1 (C_3 \delta)^2 \| \nabla \phi^b \|_{L^2}^2.
\]

Using LLC (2.7), we get

\[
(C_3 \delta)^2 \int_\Omega \left( |\nabla w| |\nabla w - |\nabla w^h| \nabla w^h| : (\nabla v^h) \right) \ dx \leq (C_3 \delta)^2 C_2 r \| \nabla \eta \|_{L^1} \| \nabla \phi^b \|_{L^2},
\]

where \( r = \max \{ \| \nabla w \|_{L^1}, \| \nabla \widetilde{w} \|_{L^1} \} \).

Hence,

\[
\frac{1}{2} \frac{d}{dt} \left\{ \| \phi^b \|^2 + \frac{C_1 \delta^2}{\mu^2} \| \nabla \phi^b \|_{L^2}^2 \right\} + v \| \nabla \phi^b \|_{L^2}^2 + \gamma \| \nabla \cdot \phi^b \|_{L^2}^2 \\
+ b^s (\eta, \phi^h, w^h, \phi^h) + b^s (w^h, \eta, \phi^h) + C_1 (C_3 \delta)^2 \| \nabla \phi^b \|_{L^2}^2 \\
\leq (\eta, \phi^h) + (C_2 \delta^2 \mu^{-2} \nabla \phi^b, \nabla \phi^h) + v (\nabla \eta, \nabla \phi^h) + \gamma (\nabla \cdot \eta, \nabla \cdot \phi^h) \\
+ (C_3 \delta)^2 C_2 r \| \nabla \eta \|_{L^1} \| \nabla \phi^b \|_{L^2} + (p - \lambda^h, \nabla \cdot \phi^h).
\]
We can rewrite it as
\[
\frac{1}{2} \frac{d}{dt} \left\{ \| \phi^h \|_2^2 + \frac{C_4}{\mu^2} \| \phi^h \|_2^2 \| \nabla \phi^h \|_2^2 \right\} + \nu \| \nabla \phi^h \|_2^2 + \gamma \| \nabla \phi^h \|_2^2 + C_1(C, \delta)^2 \| \nabla \phi^h \|_{L^3}^3 \]
\leq (\eta, \phi^h) + \frac{C_4}{\mu^2} \delta^2 (\nabla \eta, \nabla \phi^h) + \nu (\nabla \eta, \nabla \phi^h) + (p - \lambda h, \nabla \phi^h) + \gamma (\nabla \cdot \eta, \nabla \cdot \phi^h) + (C, \delta)^2 C_2 r \| \nabla \eta \|_{L^3} \| \nabla \phi^h \|_{L^3} - b^* (\eta, w, \phi^h) + b^* (\phi^h, w, \phi^h) - b^* (w^h, \eta, \phi^h).
\]

Next we find the bounds for the terms in the RHS. For the first five terms on the right, use the Cauchy Schwarz and Young’s inequality,
\[
| (\eta, \phi^h) | \leq \| \eta \|_{-1} \| \nabla \phi^h \| \leq \frac{\nu}{2} \| \nabla \phi^h \|_2^2 + C(\nu) \| \eta \|_{L^2}^2.
\]
\[
\frac{C_4}{\mu^2} \delta^2 | (\nabla \eta, \nabla \phi^h) | \leq \| \nabla \phi^h \|_2 \frac{C_4 \delta^2}{\mu^2} \| \nabla \eta \|_2.
\]
\[
\leq \frac{\nu}{4} \| \Delta \phi^h \|_2^2 + C(\nu) \left( \frac{C_4 \delta^2}{\mu^2} \right)^2 \| \nabla \eta \|_2^2.
\]
\[
\nu | (\nabla \eta, \nabla \phi^h) | \leq \nu \| \nabla \eta \|_2 \| \nabla \phi^h \| \leq \frac{\nu}{16} \| \nabla \phi^h \|_2^2 + C(\nu) \| \nabla \eta \|_2^2.
\]
\[
| (p - \lambda h, \nabla \cdot \phi^h) | \leq \| p - \lambda h \|_2 \| \nabla \cdot \phi^h \| \leq \frac{\gamma}{4} \| \nabla \cdot \phi^h \|_2^2 + C(\nu) \| \nabla \cdot \phi^h \|_2^2.
\]
\[
| \gamma (\nabla \cdot \eta, \nabla \cdot \phi^h) | \leq \gamma \| \nabla \cdot \eta \|_2 \| \nabla \cdot \phi^h \| \leq \frac{\gamma}{4} \| \nabla \cdot \phi^h \|_2^2 + C(\nu) \| \nabla \cdot \eta \|_2^2.
\]

For the fifth term on the right, use the Hölder’s inequality,
\[
(C, \delta)^2 C_2 r \| \nabla \eta \|_{L^3} \| \nabla \phi^h \|_{L^3} \leq (C, \delta)^2 \left\{ \frac{C_1}{3} \| \nabla \phi^h \|_{L^3}^3 + \frac{2}{3} (C, \delta)^2 r \| \nabla \eta \|_{L^3}^{3/2} \right\}.
\]

Next, for the first and the third nonlinear terms, here we follow the estimates in [18 p. 1007–1008, equations (4.5) and (4.6)] and we omit the details.
\[
| b^* (\eta, w, \phi^h) | \leq \frac{1}{4} \| \nabla w \|_{L^2}^2 \| \phi^h \|_2^2 + \frac{1}{4} \| \eta \|_{L^2}^2 + \varepsilon_1 \| \nabla \phi^h \|_{L^2}^3 + C \varepsilon_1^{-1/2} \| w \|_{L^2}^{3/2} \| \eta \|_{L^2}^{3/2}.
\]
\[
| b^* (\phi^h, w, \phi^h) | \leq \| \nabla w \|_{L^2} \| \nabla \phi^h \|_{L^3}^2,
\]
\[
\leq \| \nabla w \|_{L^2} \left( \| \phi^h \|_{L^2}^{1/2} \| \nabla \phi^h \|_{L^3}^{1/2} \right)^2,
\]
\[
\leq \frac{\nu}{16} \| \nabla \phi^h \|_2^2 + C(\nu) \| \nabla w \|_{L^2}^2 \| \phi^h \|_2^2.
\]
\[
| b^* (w^h, \eta, \phi^h) | \leq \frac{1}{4} \| \eta \|_{L^2} \| \phi^h \|_2^2 + \frac{1}{4} \| \nabla \eta \|_{L^2}^2 + \varepsilon_2 \| \nabla \phi^h \|_{L^2}^3 + C \varepsilon_2^{-1/2} \| w^h \|_{L^2}^{3/2} \| \eta \|_{L^2}^{3/2}.
\]

Setting \( \varepsilon_1 = \varepsilon_2 = \frac{1}{6} C_1(C, \delta)^2 \) and collecting all the terms gives
\[
\frac{1}{2} \frac{d}{dt} \left\{ \| \phi^h \|_2^2 + \frac{C_4 \mu^2}{\delta^2} \| \phi^h \|_2^2 \right\} + \nu \| \nabla \phi^h \|_2^2 + \gamma \| \nabla \phi^h \|_2^2 + \frac{2}{3} C_1(C, \delta)^2 \| \nabla \phi^h \|_{L^3}^3
\]
\leq C(\nu) \left[ \| \phi^h \|_{L^2}^2 + \frac{1}{4} \| \nabla \phi^h \|_{L^2}^2 + \frac{1}{4} \| w \|_{L^2}^2 \right] \| \phi^h \|_2^2
\]
\[
+ \left( \frac{C_4 \delta^2}{\mu^2} \right)^2 \| \nabla \eta \|_2^2 + C(\nu) \| \nabla \eta \|_2^2 + \gamma^{-1} \| p - \lambda h \|_2^2 + C(\nu) \| \nabla \cdot \phi^h \|_2^2 + (C, \delta)^2 r \| \nabla \eta \|_{L^3}^{3/2} \\
+ \frac{1}{4} \| \eta \|_{L^2}^2 + \delta^{-1} \| w \|_{L^2}^{3/2} \| \eta \|_{L^2}^{3/2} + \frac{1}{4} \| \nabla \eta \|_{L^2}^2 + \delta^{-1} \| w^h \|_{L^2}^{3/2} \| \eta \|_{L^2}^{3/2} \right\}. \]
Denote \( a(t) := C(\nu)||\nabla w||_{L^2}^2 + \frac{1}{4}||\nabla w||_{L^2}^2 + \frac{1}{4}||w||_{L^2}^2 \) and its antiderivative is

\[
A(t) := \int_0^T a(t) \ dt < \infty \text{ for } w \in L^2 \left( 0, T; L^1(\Omega); L^2 \left( 0, T; L^6(\Omega) \right) \right).
\]

Multiplying through by the integrating factor \( e^{-A(t)} \) gives

\[
\frac{d}{dt} \left[ \frac{1}{2} e^{-A(t)} \left( ||\phi h||^2 + \frac{C_1^2 \delta^2}{\mu^3} ||\nabla \phi h||^2 \right) \right] + e^{-A(t)} \left[ \frac{V}{8} ||\nabla \phi h||^2 + \frac{\gamma}{2} ||\nabla \cdot \phi h||^2 + \frac{1}{3} C_1(C_\delta \delta^2) ||\nabla \phi h||^3_{L^3} \right] \\
\leq e^{-A(t)} \left\{ C(\nu) \left[ ||\eta||_{L^1}^2 + \left( \frac{C_1^2 \delta^2}{\mu^3} \right) ||\nabla \eta||^2 + ||\nabla \eta||^2 \right] \\
+ C \left( \gamma^{-1} \right) ||\rho - \lambda^h||^2 + C(\gamma)||\nabla \cdot \eta||^2 + (C_\delta \delta^2)^2 ||\nabla \eta||^4_{L^4} + \frac{1}{4} ||\eta||^4_{L^6} \\
+ \delta^{-1} ||w||^3_{L^6} ||\eta||^3_{L^6} + \frac{1}{4} ||\nabla \eta||^2_{L^3} + \delta^{-1} ||w^h||^3_{L^6} ||\eta||^3_{L^6} \right\}.
\]

Integrating over \([0, T]\) and multiplying through by \( e^{A(t)} \) gives

\[
\frac{1}{2} \left\{ ||\phi^h(T)||^2 + \frac{C_1^2 \delta^2}{\mu^3} ||\nabla \phi^h(T)||^2 \right\} \\
+ \int_0^T \left( \frac{V}{8} ||\nabla \phi^h||^2 + \frac{\gamma}{2} ||\nabla \cdot \phi^h||^2 + \frac{1}{3} C_1(C_\delta \delta^2) ||\nabla \phi^h||^3_{L^3} \right) \ dt \\
\leq \exp \left( \int_0^T a(t) \ dt \right) \left\{ \frac{1}{2} \left( ||\phi^h(0)||^2 + \frac{C_1^2 \delta^2}{\mu^3} ||\nabla \phi^h(0)||^2 \right) \\
+ \int_0^T C(\nu) \left[ ||\eta||_{L^1}^2 + \left( \frac{C_1^2 \delta^2}{\mu^3} \right) ||\nabla \eta||^2 + ||\nabla \eta||^2 \right] \\
+ C \left( \gamma^{-1} \right) ||\rho - \lambda^h||^2 + C(\gamma)||\nabla \cdot \eta||^2 + (C_\delta \delta^2)^2 ||\nabla \eta||^4_{L^4} + \frac{1}{4} ||\eta||^4_{L^6} \\
+ \delta^{-1} ||w||^3_{L^6} ||\eta||^3_{L^6} + \frac{1}{4} ||\nabla \eta||^2_{L^3} + \delta^{-1} ||w^h||^3_{L^6} ||\eta||^3_{L^6} \right\} \ dt \right\}.
\]

Apply the Hölder’s inequality gives

\[
\int_0^T r^{3/2} ||\nabla \eta||_{L^3}^{3/2} \ dt \leq \left( \int_0^T r^3 \ dt \right)^{1/2} ||\nabla \eta||_{L^3(0,T;L^3)}^{3/2},
\]

\[
\int_0^T ||w||_{L^6}^{3/2} ||\eta||_{L^6}^{3/2} \ dt \leq ||w||_{L^3(0,T;L^3)}^{3/2} ||\eta||_{L^6(0,T;L^6)}^{3/2},
\]

\[
\int_0^T ||w^h||_{L^6}^{3/2} ||\eta||_{L^6}^{3/2} \ dt \leq ||w^h||_{L^3(0,T;L^3)}^{3/2} ||\eta||_{L^6(0,T;L^6)}^{3/2}.
\]

\(|w||_{L^3(0,T;L^3)}^{3/2} \) and \(|w^h||_{L^3(0,T;L^3)}^{3/2} \) are bounded by problem data by stability bound.

Here \( r = \max \left\{ ||\nabla w||_{L^3}, ||\nabla w||_{L^3} \right\} \) and \( \left( \int_0^T r^3 \ dt \right)^{1/2} = ||\nabla w||_{L^3(0,T;L^3)}^{3/2} \) or.

\(|\nabla \tilde{w}||_{L^3(0,T;L^3)}^{3/2} \) also bounded. Using triangle inequality: \( ||\varepsilon|| \leq ||\phi^h|| + ||\eta|| \), we obtain

the desired result.
Remark 5.2 If \( \tilde{w} \) is taken to be the Stokes projection, then \( \| \nabla \eta \|^2 \) does not occur at the RHS.

Remark 5.3 Considering the nonlinear terms (5.3) and (5.4), alternatively we have
\[
|b^*(\eta, w, \phi^h)| \leq M\|\nabla \eta\|\|\nabla w\|\|\nabla \phi^h\| \leq \epsilon\|\phi^h\|^2 + \frac{1}{4\epsilon}M^2\|\nabla w\|^2\|\nabla \eta\|^2.
\]
\[
|b^*(w^h, \eta, \phi^h)| \leq C\|w^h\|^{1/2}\|w^h\|^{1/2}\|\nabla \eta\|\|\nabla \phi^h\|.
\]
\[
\leq \epsilon\|\nabla \phi^h\|^2 + C\left(\frac{\epsilon^{-1}}{\epsilon}\right)\|w^h\|\|\nabla w^h\|\|\nabla \eta\|^2.
\]

By taking \( \epsilon = \nu/32 \), we can avoid the term \( \delta^{-1}\|\eta\|^{3/2}_h \) at the RHS but instead we have \( \nu^{-1}\|\nabla \eta\|^2 \).

Combining Theorems 5.1 and 6.1, we get the error estimate between \( u_{NSE} \) and \( w_{Smag}^h \)
\[
\| (u_{NSE} - w_{Smag}^h)(T) \|^2 + \frac{C_4^2}{\mu^2}\delta^2\| (u_{NSE} - w_{Smag}^h)(T) \|^2
\]
\[
+ \int_0^T \frac{\nu}{2}\| (u_{NSE} - w_{Smag}^h) \|^2 + \frac{C_4}{2}(C_3\delta)^2\| (u_{NSE} - w_{Smag}^h) \|_L^3 \| \, dt
\]
\[
\leq C(\nu, T) \left\{ \| (u_{NSE} - w_{Smag}^h)(0) \|^2 + \frac{C_4^2}{\mu^2}\delta^2\| (u_{NSE} - w_{Smag}^h)(0) \|^2
\]
\[
+ \int_0^T (C_3\delta)^2\|u\|_L^3 + \frac{C_4^2}{\mu^2}\delta^2\|u_t\|^2 \, dt + \inf_{\tilde{w} \in V_h}\| (w - \tilde{w})(T) \|^2
\]
\[
+ \int_0^T C(\nu) \left[ \|w_t - \tilde{w}_t\|_{L^2}^2 + \left( \frac{C_4^2\delta^2}{\mu^2} \right) \| (w_t - \tilde{w}_t) \|^2 + \| (w - w^h) \|^2 + \inf_{\lambda \in Q^h} \| \lambda - \lambda^h \|^2 \right]
\]
\[
+ (C_3\delta)^2\|w - \tilde{w}\|_{L^2}^3 + \delta^{-1}\left( \|w\|_{L^2}^3 + \|w^h\|_{L^2}^3 \right) \|w - \tilde{w}\|_{L^2}^3 \, dt \right\}
\]

5.1 Time discretization of the corrected Smagorinsky model

This section presents the unconditionally stable, linearly implicit, full discretization of (3.8). Let the time-step and other quantities be denoted by
\[
time-step = k, \quad n = nk, \quad f_n(x) = f(x, t_n),
\]
\[
w_n^h(x) = approximation to w(x, t_n),
\]
\[
p_n^h(x) = approximation to p(x, t_n).
\]

We perform the finite element spatial discretization and the first-order Backward Euler scheme for time discretization to get the following full discretization: given \( (w_n^h, p_n^h) \in (X^h, Q^h) \), find \( (w_{n+1}^h, p_{n+1}^h) \in (X^h, Q^h) \) satisfying
\[
\left( \frac{w_{n+1}^h - w_n^h}{k}, \psi^h \right) + \frac{C_4^2\delta^2}{\mu^2} \left( \frac{\nabla w_{n+1}^h - \nabla w_n^h}{k}, \psi^h \right) + b^*(w_n^h, w_{n+1}^h, \psi^h)
\]
\[
+ \nu \left( \nabla w_{n+1}^h, \nabla \psi^h \right) + (C_3\delta)^2 \left( |\nabla w_n^h| \nabla w_{n+1}^h, \nabla \psi^h \right)
\]
\[
+ \gamma \left( \nabla \cdot w_{n+1}^h, \Delta \cdot \psi^h \right) - (p_{n+1}^h, \Delta \cdot \psi^h) = (f_{n+1}(x), \psi^h) \quad \forall \ \psi^h \in X^h,
\]
\[
\left( \nabla \cdot w_{n+1}^h, q^h \right) = 0 \quad \forall \ \ q^h \in Q^h.
\]

This method is semi-implicit. We shall prove it is unconditionally stable in in Theorem 5.4.
Theorem 5.4  \((5.5)\) is unconditionally energy stable. For any \(N \geq 1\),

\[
\left(\frac{1}{2}\|w_n^h\|^2 + \frac{1}{2} C_i^2 \frac{\delta^2}{\mu^2}\|\nabla w_n^h\|^2\right) + \sum_{n=0}^{N-1} \frac{1}{2} \left(\|w_{n+1}^h - w_n^h\|^2\right)
\]

\[
+ C_i^2 \frac{\delta^2}{\mu^2}\|\nabla w_{n+1}^h - \nabla w_n^h\|^2 + k \sum_{n=0}^{N-1} \int_{\Omega} \left[ v \left( C_i \delta \right)^2 \left| \nabla w_n^h \right| \left| \nabla w_{n+1}^h \right| \right]^2 dx
\]

\[
+ \gamma \| \nabla \cdot w_{n+1}^h \|^2 = \left(\frac{1}{2}\|w_0^h\|^2 + \frac{1}{2} C_i^2 \frac{\delta^2}{\mu^2}\|w_0^h\|^2\right) + k \sum_{n=0}^{N-1} (f_{n+1}, w_{n+1}^h).
\]

(5.6)

Proof. Multiply \((5.5)\) by \(k\) and take \(v^h = w_{n+1}^h\). Use Lemma 2.2 to get.

\[
b^* (w_n^h, w_{n+1}^h, w_{n+1}^h) = 0.
\]

Hence,

\[
\|w_{n+1}^h\|^2 - \left(\|w_{n+1}^h\|^2 + \frac{C_i^2 \delta^2}{\mu^2}\|\nabla w_{n+1}^h\|^2\right) - \frac{C_i^2 \delta^2}{\mu^2} \left(\|w_{n+1}^h - w_n^h\|^2\right)
\]

\[
+ \gamma \| \nabla \cdot w_{n+1}^h \|^2 + k \int_{\Omega} \left[ v \left( C_i \delta \right)^2 \left| \nabla w_n^h \right| \left| \nabla w_{n+1}^h \right| \right]^2 dx = k (f_{n+1}, w_{n+1}^h).
\]

For the second and fourth term, apply the polarization identity (2.1),

\[
(w_{n+1}^h, w_n^h) = \frac{1}{2} \left(\|w_{n+1}^h\|^2 + \frac{1}{2} \|w_n^h\|^2\right) - \frac{1}{2} \|w_{n+1}^h - w_n^h\|^2,
\]

\[
(\nabla w_{n+1}^h, \nabla w_n^h) = \frac{1}{2} \left(\|\nabla w_{n+1}^h\|^2 + \frac{1}{2} \|\nabla w_n^h\|^2\right) - \frac{1}{2} \|\nabla w_{n+1}^h - \nabla w_n^h\|^2.
\]

Collecting terms and summing from \(n = 0\) to \(N - 1\), we get the result. \(\blacksquare\)

Remark 5.5  \((5.6)\) is an energy equality, we can identify the following quantities:

1 Model kinetic energy = \(\frac{1}{2} \|w_n^h\|^2 + \frac{1}{2} C_i^2 \frac{\delta^2}{\mu^2}\|\nabla w_n^h\|^2\).

2 Eddy viscosity dissipation = \(\int_{\Omega} \left( C_i \delta \right)^2 \left| \nabla w_n^h \right| \left| \nabla w_{n+1}^h \right| \right]^2 dx\).

3 Numerical diffusion = \(\frac{1}{2} \left(\|w_{n+1}^h - w_n^h\|^2 + \frac{C_i^2 \delta^2}{\mu^2}\|\nabla w_{n+1}^h - \nabla w_n^h\|^2\right)\). This numerical diffusion arises due to the Backward Euler scheme.

Remark 5.6  The energy equality \((5.6)\) can be also written as

\[
\frac{1}{2k} \left(\|w_{n+1}^h\|^2 - \|w_n^h\|^2\right) + \frac{1}{2k} \|w_{n+1}^h - w_n^h\|^2 + v \|\nabla w_{n+1}^h\|^2 + \gamma \| \nabla \cdot w_{n+1}^h \|^2
\]

\[
+ \left\{ \frac{C_i^2 \delta^2}{2k \mu^2} \left(\|\nabla w_{n+1}^h\|^2 - \|\nabla w_n^h\|^2\right) + \frac{C_i^2 \delta^2}{2k \mu^2} \|\nabla w_{n+1}^h - \nabla w_n^h\|^2\right\}
\]

\[
+ \int_{\Omega} (C_i \delta)^2 \left| \nabla w_n^h \right| \left| \nabla w_{n+1}^h \right| dx = (f_{n+1}, w_{n+1}^h).
\]

Line one and the RHS are from the backward Euler discretization of usual NSE. The bracketed term is a discretized form of model dissipation at \(t = t_{n+1}\). Here the term model dissipation in the article can be positive or negative. When it is positive, it aggregates energy from mean to fluctuations. And when it is negative, energy is being transferred from fluctuations back to mean.
Remark 5.7 For (5.5), the model dissipation is
\[ MD^{n+1} = \frac{C_4^2 \delta^2}{2k} \left( ||w_{n+1}^h||^2 - ||w_n^h||^2 \right) + \frac{C_4^2 \delta^2}{2k} \|\nabla w_{n+1}^h - \Delta w_n^h\|^2 \]
\[ + \int_{\Omega} (C_4 \delta)^2 \|\nabla w_n^h\| \|\nabla w_{n+1}^h\|^2 \, dx. \]

In this Test 8.2, we test use both Backward Euler and Crank–Nicolson to see the difference.
We perform the finite element spatial discretization and the linearly implicit Crank–Nicolson (also
called CNLE-CN with Linear Extrapolation) scheme for time discretization to get the following full
discretization: for function \( w \), we denote
\[ w_{n+\frac{1}{2}}^h = \frac{w_{n}^h + w_{n+1}^h}{2}, \quad \bar{w}_{n+\frac{1}{2}}^h = \frac{3w_{n}^h - w_{n-1}^h}{2}. \]
Given \((w_n^h, p_n^h) \in (X^h, Q^h)\), find \((w_{n+1}^h, p_{n+1}^h) \in (X^h, Q^h)\) satisfying
\[
\left( \frac{w_{n+1}^h - w_n^h}{k} , v^h \right) + \frac{C_4^2 \delta^2}{\mu^2} \left( \frac{\nabla w_{n+1}^h - \nabla w_n^h}{k} , v^h \right) + b^s \left( \bar{w}_{n+\frac{1}{2}}^h , w_{n+\frac{1}{2}}^h , v^h \right)
+ \nu \left( \nabla w_{n+\frac{1}{2}}^h , \nabla v^h \right) + (C_4 \delta)^2 \left( \nabla w_{n+\frac{1}{2}}^h , |\nabla w_{n+\frac{1}{2}}^h| , \nabla v^h \right)
+ \gamma \left( \nabla \cdot w_{n+\frac{1}{2}}^h , \nabla \cdot v^h \right) - (p_{n+\frac{1}{2}}^h , \nabla \cdot v^h) = (f_{n+\frac{1}{2}}(x) , v^h) \quad \forall \, v^h \in X^h,
\]
\[
\left( \nabla \cdot w_{n+\frac{1}{2}}^h , q^h \right) = 0 \quad \forall \, q^h \in Q^h. \tag{5.7}
\]
We will prove it is unconditionally stable in in Theorem 5.8.

Theorem 5.8 (5.7) is unconditionally energy stable. For any \( N \geq 1 \),
\[
\left( \frac{1}{2} ||w_N^h||^2 + \frac{1}{2} \frac{C_4^2 \delta^2}{\mu^2} ||w_N^h||^2 \right) + k \sum_{n=0}^{N-1} \int_{\Omega} \left[ \nu \left( \nabla w_{n+\frac{1}{2}}^h , |\nabla w_{n+\frac{1}{2}}^h| , \nabla v^h \right) \right] \|\nabla w_{n+\frac{1}{2}}^h\|^2 \, dx
+ \gamma \|w_{n+\frac{1}{2}}^h\|^2 = \left( \frac{1}{2} ||w_0^h||^2 + \frac{1}{2} \frac{C_4^2 \delta^2}{\mu^2} ||w_0^h||^2 \right) + k \sum_{n=0}^{N-1} \left( f_{n+\frac{1}{2}}(x) , w_{n+\frac{1}{2}}^h \right). \tag{5.8}
\]

Proof. Multiply (5.7) by \( k \) and take \( v^h = w_{n+\frac{1}{2}}^h \). Use Lemma 2.2 to get
\[ b^s \left( \bar{w}_{n+\frac{1}{2}}^h , w_{n+\frac{1}{2}}^h , w_{n+\frac{1}{2}}^h \right) = 0. \]
Hence,
\[
\frac{1}{2} ||w_{n+1}^h||^2 - \frac{1}{2} ||w_n^h||^2 + \frac{1}{2} \frac{C_4^2 \delta^2}{\mu^2} ||\nabla w_{n+1}^h||^2 - \frac{1}{2} \frac{C_4^2 \delta^2}{\mu^2} ||\nabla w_n^h||^2
+ \gamma \|\nabla \cdot w_{n+\frac{1}{2}}^h\|^2 + k \int_{\Omega} \left[ \nu \left( \nabla w_{n+\frac{1}{2}}^h , |\nabla w_{n+\frac{1}{2}}^h| , \nabla v^h \right) \right] \|w_{n+\frac{1}{2}}^h\|^2 \, dx = k \left( f_{n+\frac{1}{2}}(x) , w_{n+\frac{1}{2}}^h \right).
\]
Collecting terms and summing from \( n = 0 \) to \( N - 1 \), we get the result. \( \blacksquare \)

Remark 5.9 (5.8) is an energy equality, we can identify the following quantities:
1. Model kinetic energy \( = \frac{1}{2} ||w_N^h||^2 + \frac{1}{2} \frac{C_4^2 \delta^2}{\mu^2} ||\nabla w_N^h||^2 \).
2. Eddy viscosity dissipation \( = \int_{\Omega} (C_4 \delta)^2 \|\nabla \bar{w}_{n+\frac{1}{2}}^h\| \|\Delta w_{n+\frac{1}{2}}^h\|^2 \, dx \).
3. No numerical diffusion.
Remark 5.10 The energy equality can be also written as
\[
\frac{1}{2k} \left( ||\Delta w^n_{n+1}||^2 - ||\Delta w^n||^2 \right) + v ||\nabla w^n_{n+1}||^2 + \gamma ||\nabla \cdot w^n_{n+1}||^2 \\
+ \left\{ \frac{C_4 \delta^2}{2k \mu^2} \left( ||\nabla w^n_{n+1}||^2 - ||\nabla w^n||^2 \right) + \int_\Omega (C_s \delta)^2 \left| \nabla \tilde{w}^n_{n+1} \right| \left| \nabla w^n_{n+1} \right|^2 \, dx \right\} \\
= \left( f_{n+1}, w^n_{n+1} \right).
\]

Line one and line three are from the CNLE discretization of usual NSE. The bracketed term in the second line is a discretized form of model dissipation at \( t = t_{n+1} \).

Remark 5.11 For (5.7), the model dissipation is
\[
MD^{n+1} = \frac{C_4 \delta^2}{2k \mu^2} \left( ||\nabla w^n_{n+1}||^2 - ||\nabla w^n||^2 \right) + \int_\Omega (C_s \delta)^2 \left| \nabla \tilde{w}^n_{n+1} \right| \left| \nabla w^n_{n+1} \right|^2 \, dx.
\]

6 | NUMERICAL TESTS

In this section, we perform two numerical tests. In the first test, we show the numerical error and the rate of convergence of the Backward Euler scheme. In the second test, we show among Backward Euler (BE) and Crank–Nicolson with Linear Extrapolation (CNLE), CNLE exhibits intermittent backscatter.

6.1 | A test with exact solution

(Taken from V. DeCaria, W. J. Layton and M. McLaughlin [9]) The first experiment tests the accuracy of the CSM (3.8) and convergence rate of (5.5). The following test has an exact solution for the 2D Navier Stokes problem.

Let the domain \( \Omega = (-1, 1) \times (-1, 1) \). The exact solution is as follows:
\[
\begin{align*}
    u(x, y, t) &= \pi \sin t \left( \sin 2\pi y \sin^2 \pi x - \sin 2\pi x \sin^2 \pi y \right), \\
    p(x, y, t) &= \sin t \cos \pi x \sin \pi y.
\end{align*}
\]

This is inserted into the CSM and the body force \( f(x, t) \) is calculated.

Uniform meshes were used with 270 nodes per side on the boundary and the degrees of freedom for the velocity space is 292,681 and for the pressure space is 73,441. The mesh is fine enough compared to the time-step so that the main error from time-steps is only considered here. Taylor-Hood elements (P2-P1) were used in this test. We ran the test up to \( T = 10 \). We take \( C_s = 0.1, \mu = 0.4, \delta \) is taken to be the shortest edge of all triangles. The norms used in the table heading are defined as follows,
\[
||w||_{\infty,0} := \text{ess sup}_{0 < t < T} ||w||_{L^2(0)} \quad \text{and} \quad ||w||_{0,0} := \left( \int_0^T ||w(\cdot, t)||_{L^2(0)}^2 \, dt \right)^{1/2}.
\]

From the Table 1, we see the temporal convergence rate is 1 which is expected from Backward Euler (5.5) discretization.

Using Taylor-Hood elements, Theorem 5.1 predicts a convergence rate in space of \( O \left( h^{1.75} \right) \), with a moderate constant, for \( ||w - w^h||_{\infty,0} \) and \( \|\nabla (w - w^h)\|_{0,0} \). But with the estimates in in Remark 5.3, the order of convergence is \( O \left( h^{2} \right) \), with a large constant \( \frac{1}{\nu} \). In Table 2, third and fifth column show rates \( O \left( h^{1.78} \right) \) until the error plateaus (last line) at the error in the time discretization (last line in Table 1).
| $\Delta t$ | $||w - w^h||_{\infty,0}$ | Rate | $||\nabla (w - w^h) ||_{0,0}$ | Rate | $||p - p^h||_{0,0}$ | Rate |
|------------|------------------|------|------------------|------|------------------|------|
| 0.05       | 3.27068          | -    | 5.25129          | -    | 0.640537         | -    |
| 0.02       | 0.823036         | 1.506| 1.59313          | 1.302| 0.235862         | 1.091|
| 0.01       | 0.348629         | 1.239| 0.739145         | 1.108| 0.108216         | 1.124|
| 0.005      | 0.169429         | 1.041| 0.39714          | 0.89621| 0.0470406 | 1.202|

| $h = \delta$ | $||w - w^h||_{\infty,0}$ | Rate | $||w - w^h||_{0,0}$ | Rate | $||p - p^h||_{0,0}$ | Rate |
|---------------|------------------|------|------------------|------|------------------|------|
| 0.08571       | 57.9769          | -    | 88.1677          | -    | 14.5602          | -    |
| 0.04221       | 1.41386          | 5.244| 3.30974          | 4.635| 0.313994         | 5.418|
| 0.02095       | 0.407421         | 1.776| 0.95483          | 1.774| 0.0562327        | 2.455|
| 0.01048       | 0.169429         | 1.266| 0.39714          | 1.266| 0.0470406        | 0.258|

There is still some gap between the theoretical convergence rate and the actual convergence rate we get in Table 2. The behavior of the pressure error for this test problem is unclear as well in Table 2.

### 6.2 Test 2. Flow between offset cylinder

(Taken from N. Jiang and W. J. Layton [16]). This flow problem is tested to show the transfer of energy from fluctuations back to means in the turbulent flow using the corrected Smagorinsky model (3.8).

The domain is a disk with a smaller off center obstacle inside. Let $r_1 = 1, r_2 = 0.1, c = (c_1, c_2) = (1/2, 0)$, then the domain is given by

$$\Omega = \{(x, y) : x^2 + y^2 < r_1^2 \text{ and } (x - c_1)^2 = (y - c_2)^2 >, r_2^2\}.$$

The flow is driven by a counterclockwise rotational body force

$$f(x, y, t) = \left(-4y \star (1 - x^2 - y^2), 4x \star (1 - x^2 - y^2)\right)^T,$$

with no-slip boundary conditions on both circles. We discretize in space using Taylor-Hood elements. There are 80 mesh points around the outer circle and 60 mesh points around the inner circle. The flow is driven by a counterclockwise force ($f = 0$ on the outer circle). Thus, the flow rotates about the origin and interacts with the immersed circle.

We start the initial condition by solving the Stokes problem. We compute up to final time $T_{\text{final}} = 3$.

Take $C_s = 0.1, \mu = 0.3, \delta$ is taken to be the shortest edge of all triangles $\approx 0.0112927$, $Re = 10,000$.

For Backward Euler (5.5), we compute the following quantities:

**Model dissipation**

$$MD = \int_{\Omega} \left( \frac{C_s^4 \delta^2 \nabla w_{n+1}^h - \nabla w_n^h}{\mu^2} \cdot \nabla w_{n+1}^h \right. + \left. (C_s \delta)^2 \|\nabla w_n^h\| \|\nabla w_{n+1}^h\|^2 \right) dx.$$

**Effect of new term from CSM**

$$CSMD = \int_{\Omega} \frac{C_s^4 \delta^2 \nabla w_{n+1}^h - \nabla w_n^h}{\mu^2} \cdot \nabla w_{n+1}^h dx.$$
FIGURE 1  Comparison of Backward Euler (34) and linearized Crank–Nicolson (37) with \( \Delta t = 0.01, Re = 10,000, T_{final} = 3, C_s = 0.1, \mu = 0.4, \delta = 0.0112927. \)

Eddy viscosity dissipation \( EVD = \int_{\Omega} (C_s\delta)^2 \left| \nabla W_h^n \right| \left| \nabla W_h^{n+1} \right|^2 \, dx. \)

Viscous dissipation \( VD = \nu \| \nabla W_h^{n+1} \|^2. \)

For Crank–Nicolson CNLE (5.7), we compute the following quantities:

Model dissipation \( MD = \int_{\Omega} \left( \frac{C_s^4 \delta^2}{\mu^2} \frac{\nabla W_h^{n+1} - \nabla W_h^n}{k} \cdot \nabla W_h_n^{n+\frac{1}{2}} \right) \left| \nabla W_h^{n+1} \right|^2 \, dx. \)

Effect of new term from CSM, \( CSMD = \int_{\Omega} \frac{C_s^4 \delta^2}{\mu^2} \frac{\nabla W_h^{n+1} - \nabla W_h^n}{k} \cdot W_h^{n+\frac{1}{2}} \, dx. \)

Eddy viscosity dissipation \( EVD = \int_{\Omega} (C_s\delta)^2 \left| \nabla W_h_n^{n+\frac{1}{2}} \right| \left| \nabla W_h^{n+1} \right|^2 \, dx. \)

Viscous dissipation \( VD = \nu \| \nabla W_h^{n+1} \|_2^2. \)

It can be seen from the Figure 1, model dissipation MD becomes negative sometimes for linearized Crank–Nicolson (5.7) and MD are all positive for Backward Euler (5.5). Only CNLE for the Corrected Smagorinsky has backscatter, which is consistent with the purpose of this model. Backward Euler has too much numerical diffusion, which makes it harder to see the backscatter from BE.
In the Figure 2, streamline, we notice the flow becomes smoother as it approaches statistical equilibrium.

### 6.2.1 Comparison with NSE and standard Smagorinsky

Here we compare the CSM (3.8) with Navier Stokes (3.1) and the standard Smagorinsky (1.1). The Taylor microscale \[ \lambda_T := \frac{||u||}{||\nabla u||} \], which represents an average length scale for the flow. We use the same setting but with \( Re = 100,000 \) to compare the Taylor microscale of each model. All numerical tests are calculated using Crank–Nicolson with grad-div stabilization \( \gamma = 1 \).

To further see the difference between these three models, here we focus on time-interval [7, 10] and see the relative length-scale \( \lambda_T/h \) with \( h \) being the meshsize.

From Figure 3, notice the CSM has larger Taylor microscale. Since the CSM models backscatter, more energy is expected in velocity means. Consistent with this, the averaged length-scale of CSM is larger than Smagorinsky and NSE. And from Figure 4, the relative length-scale of the CSM at the final time is almost twice as large as the relative length-scale calculated with NSE and standard Smagorinsky.
FIGURE 3  Taylor microscale comparison between CSM, NSE, and the standard Smagorinsky with Δt = 0.01, Re = 100,000, T_{final} = 10, C_s = 0.1, μ = 0.4, δ = 0.0112927.

FIGURE 4  Relative length-scale (λ_T/h) comparison between CSM, NSE, and standard Smagorinsky with t = 0.01, Re = 100,000, C_s = 0.1, μ = 0.4, δ = 0.0112927, time-interval shown as [7, 10].
CONCLUSION AND FUTURE PROSPECTS

It was demonstrated that the Smagorinsky model could be extended to non-equilibrium turbulence. In addition to that, we were able to show statistical backscatter without using negative turbulent viscosities. The stability of the model, uniqueness of the model’s solution, modeling error, and numerical error were analyzed in the article. Since BE has numerical diffusion while CNLE does not, we can clearly observe backscatter from CNLE in the second numerical test.

In the next step, we can incorporate the penalty method with this model to get the desired result more efficiently.

AUTHOR CONTRIBUTIONS

Farjana Siddiqua: Writing – original draft; writing – review and editing. Xihui Xie: Writing – original draft; writing – review and editing.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID

Farjana Siddiqua https://orcid.org/0000-0002-7724-7250

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