On the Radial Topological Constraints for Distribution System Restoration and Reconfiguration Problems

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Abstract—A radial topology is to be determined in both distribution system restoration and reconfiguration problems. When using a mathematical programming method to solve the problems, the explicit expressions of radial topological constraints are needed. However, a set of existing widely used topological constraints, i.e., the spanning tree (ST) constraints, has its limitations which are ignored by some scholars. In this letter, the limitation of ST constraints is analyzed and the effective set of constraints is presented. Furthermore, a combined set of constraints is also proposed and case studies compare different sets of constraints with the recommendation.

Index Terms—distribution system restoration, feeder reconfiguration, radial topology, mathematical programming, resilience.

I. INTRODUCTION

The distribution system is designed as a meshed network but operates radially, and a radial topology is needed in many optimization problems in the distribution system such as distribution system reconfiguration [1] and restoration, [2] [3], etc. Many researchers seek to solve the optimization problems using mathematical programming methods as it can obtain the global optimum using off-the-shelf optimization solvers [1], [4]. The radial topological constraints should be explicitly expressed in an optimization program. A set of widely used radial topological constraints is the spanning tree constraint (ST) [1] [4]. It is proven in [1] that the ST constraints can result in a radial configuration for the distribution system reconfiguration problem. However, to ensure the ST constraints resulting in radial topologies, some conditions should be clarified.

In this letter, the limitation of the spanning tree constraints is analyzed and the effective constraint set, i.e., the single-commodity flow constraint (SCF) [5] is presented. This letter proposes a combined constraint set SCF+ST. The two effective radial topological constraint sets are tested and compared, then the conclusion is made.

II. PROBLEM FORMULATION

The original distribution network is formulated as a connected undirected graph \( G = (\mathcal{N}, \mathcal{E}) \), where \( \mathcal{N} \) is the set of buses and \( \mathcal{E} \) the set of all lines including tie lines. It is assumed that \( \mathcal{S} \) is the set of all power sources including substations and distributed sources, indicating the number of power sources is \(|\mathcal{S}|\). Assume that all the lines are switchable. The decided electrical network(s) should be radial. Assume that the determined graph is \( G_{re} = (\mathcal{N}, \mathcal{E}_{re}) \), then \( G_{re} \subseteq G \) is a forest comprised with \(|\mathcal{R}|\) trees, where \( \mathcal{R} \) is the set of all roots. In trees, the number of nodes equals the number of lines plus one, so \(|\mathcal{N}| = |\mathcal{E}_{re}| + |\mathcal{R}|\).

In the distribution system reconfiguration problem, the objective is to minimize the power loss subject to power flow equations, bus voltage limits, line current limits, generation/transformer capacity constraints, and radial topology constraints [6]. The decision variables include the line status variable (0-1 integer variable) and the power output of sources. The substations would be the roots of the determined graphs, which indicates the number of substations is root number \(|\mathcal{R}|\).

In the distribution system restoration problem, the main objective is maximizing the number of weighted restored loads and the secondary objective is minimizing the power loss of the post-restoration system [7]. The constraints are similar to reconfiguration problems but the decision variables include load status variables (0-1 integer variables). The number of sub-systems \(|\mathcal{R}|\) depends on the restoration strategies. For instance, coordinating multiple sources for service restoration, the post-restoration system will be one large electric system [8], while several islands can be formed if each rooted by one capable source such as a microgrid.

III. MOTIVATION

A. ST Constraints and the Resulted Pseudo-root

The set of ST constraints [1] [4] are based on parent-child relationship and shown as follows:

\[
\begin{align*}
    b_{ij} + b_{ji} &= a_{ij}, \forall (i, j) \in \mathcal{E} \quad (1) \\
    b_{ij} &= 0, \forall i \in \mathcal{R}, (i, j) \in \mathcal{E} \quad (2) \\
    \sum_{(i,j)\in\mathcal{E}} b_{ij} &= 1, \forall i \in \mathcal{N} \setminus \mathcal{R} \quad (3)
\end{align*}
\]

where \( \mathcal{R} \) is the set of roots; \( a_{ij} \) is an undirected 0-1 integer variable indicating whether the line \((i, j)\) is connected, i.e., \( a_{ij} = 1 \) if line \((i, j)\) is connected, \( a_{ij} = 0 \) otherwise; \( b_{ij} \) is a bi-directed variable indicating the parent-child relationship, i.e., if \( i \) is the parent node of \( j, b_{ij} = 1 \); otherwise, \( b_{ij} = 0 \).
In (1), the relationship between the connection status and parent-child relationship for line \((i, j)\) is presented. \((2)\) indicates that the root node has no parent node and \((3)\) means that other nodes have only one parent node.

The set of constraints \((1)-(3)\) based on the parent-child relationship is to model the radial topology and the resulted network should be radial. However, this set of constraints may result in the unconnected graph containing loops, namely the "pseudo-roots", as illustrated in Fig. 1.

![Fig. 1. A post-restoration network with pseudo-root.](image)

The original network of Fig. 1 has 2 sources and 1 loop, and 1 tree is desired whose root is node 1, indicating \(|R| = 1\). However, the determined post-restoration network \(G_{rec}\) using constraints \((1)-(3)\) is comprised of 2 sub-networks rather than a tree. In Fig. 1, the arrow indicates the parent-child relationship between nodes. Node 1, which is the real root, is the parent node of 2. However, nodes 4, 5, and 6 form a pseudo-root, which means that nodes 4, 5, and 6 are not roots and satisfy the constraint \((3)\), but the loop they form can be seen as a root satisfying the constraint \((2)\). Therefore, constraints \((1)-(3)\) result in two unconnected sub-graphs rather than a tree.

**B. Analysis**

The pseudo-root phenomenon indicates that the set of conditions \((1)-(3)\) is not the sufficient condition for a radial topology. As stated in [6], to guarantee a tree, two conditions should be satisfied: 1) the resulted graph is connected; 2) the number of nodes is equal to the number of lines plus one. To generalize the sufficient conditions for \(|R|\) radial topologies, they can be modified as follows:

- **Condition 1**: all the nodes are connected to roots;
- **Condition 2**: the number of nodes is equal to the number of lines plus \(|R|\), i.e.,
  \[
  \sum_{(i,j) \in E} a_{ij} = |N| - |R|
  \]

The constraint set \((1)-(3)\) actually satisfies Condition 2 but does not guarantee Condition 1. Therefore, the constraints \((1)-(3)\) are the necessary but insufficient conditions for radial topologies.

In some papers, the power flow equations are used to guarantee Condition 1, but the power flow equations only work when the following conditions are satisfied:

- **Condition 3**: the number of sources is equal to that of roots, i.e., \(|S| = |R|\);
- **Condition 4**: all the nodes have load demand.

These two conditions guarantee Condition 1 by power flow equations because all nodes will connect to roots (sources) as they have power demand.

As stated in [6], Condition 4 can be satisfied by assuming an additional small value of load (e.g., 0.001 pu) for nodes with no load. By doing this, when Condition 3 is satisfied, the power flow equations can guarantee the Condition 1. However, when Condition 3 is not the case, the set of conditions \((1)-(3)\) will not work. In the next section, the set of SCF constraints guaranteeing Condition 1 is introduced.

**IV. PROPOSED CONSTRAINTS**

**A. SCF Constraints for Condition 1**

In the SCF constraints, it is assumed that each node has fictitious demand and the root nodes are fictitious sources [5] [6]. The fictitious flow through lines guarantees the connectivity of the network. The SCF constraints are as follows:

\[
\begin{align*}
\sum_{j:i \rightarrow j} F_{ij} + D_i & = \sum_{k:i \rightarrow k} F_{ki}, \forall i \in N \setminus R \quad (5) \\
\sum_{j:i \rightarrow j} F_{ij} + D_i & \leq G_{i,\max}, \forall i \in R \quad (6) \\
|F_{ij}| & \leq a_{ij} M, \forall (i, j) \in E \quad (7)
\end{align*}
\]

where \(F_{ij}\) is continuous variable indicating the fictitious flow in line \((i, j)\), \(D_i\), \(G_{i,\max}\) are fictitious demand of node \(i\) and fictitious output limit of source \(i\), respectively. \(D_i\) can be set as 1 for each node and \(G_{i,\max}\) as \(|N|\) for each root source.

Constraints \((5)-(7)\) indicate the flow balance of each node, just like the Kirchhoffs current law. Constraint \((7)\) yields \(F_{ij}\) to zero when the line is disconnected.

Constraints \((3)-(7)\) can guarantee Condition 1 as nodes only be allowed and yielded to connect with roots rather than sources.

In [5] and [6], constraints \((5)-(7)\) together with \((4)\) are the constraints to guarantee the desired radial topology. We name the set of radial topological constraints as “SCF0”.

**B. The Proposed Constraints**

Based on the analysis above, to guarantee both Condition 1 and Condition 2, the combination SCF and ST constraints can also ensure radial topology, i.e., \((1)-(3)\) and \((5)-(7)\), which is named as “SCF+ST”. The radial topological constraints will introduce additional variables and constraints. The comparison of two constraint sets is shown in Table I.

**TABLE I**

| Quantities of Additional Variables and Constraints | SCF0 | SCF+ST |
|---|---|---|
| \(|S| = |R|\) | \(|E|\) | \(|E|\) + \(|R|\) |
| \(|S| \neq |R|\) | \(|E|\) | \(|E|\) + \(|R|\) + 1 |

In the scenario \(|S| = |R|\), the power flow equations can replace the SCF constraints by assuming small value of load
demand for nodes, so SCF0 will only introduce 1 constraint (4) and additional constraints and variables are introduced by (1)–(3) in SCF+ST. However, in the scenario |S| ≠ |R|, the SCF cannot be replaced by power flow equations, so extra variables and constraints will be introduced by (5)–(7).

It can be seen that the proposed constraints will introduce more scalar variables, inequalities, and equalities due to ST constraints. However, more constraints can narrow down the feasible region of integer variables to accelerate the computation speed in some cases. Case studies validate this point.

V. Case Studies

In this paper, the modified 32-node system in [2] and IEEE 123-node system [9] are used to compare the performance of ST, SCF0, and the proposed SCF+ST. The quantities of lines for 32 and 123-node systems are 36 and 124, respectively, indicating the numbers of line status 0-1 variables are 34 and 124. The numbers of loads (load status 0-1 variables) are 32 and 85, respectively. Feeder reconfiguration and distribution system restoration using distributed power sources are tested. The generation capacity is sufficient in feeder reconfiguration and insufficient in restoration problems. In both systems, the number of sources is not equal to the number of roots. The radial topological constraints are incorporated with the mixed-integer second-order cone program (MISOCP) presented in [2]. The MISOCPs are modeled in the CVXPY toolbox in Python 3 [9], and solved using the optimization solver in MOSEK. The tests are carried out on an Intel Core i7 CPU at 3.6GHz with 32GB of RAM.

Based on different locations of DGs and critical loads, and different load levels, 300 scenarios for 32-node and IEEE 123-node systems are generated to test the performance of different sets of topological constraints. The maximum computation time of optimization is set as 100s, and the relative optimality gap tolerance for the 32-node system is 1e-8 and 1e-4 for 123-node system [8] are used to compare the performance of ST, SCF0, and the proposed SCF+ST. The quantities of lines for 32 and 123-node systems are 36 and 124, respectively, indicating the numbers of line status 0-1 variables are 34 and 124. The numbers of loads (load status 0-1 variables) are 32 and 85, respectively. Feeder reconfiguration and distribution system restoration using distributed power sources are tested. The generation capacity is sufficient in feeder reconfiguration and insufficient in restoration problems. In both systems, the number of sources is not equal to the number of roots. The radial topological constraints are incorporated with the mixed-integer second-order cone program (MISOCP) presented in [2]. The MISOCPs are modeled in the CVXPY toolbox in Python 3 [9], and solved using the optimization solver in MOSEK. The tests are carried out on an Intel Core i7 CPU at 3.6GHz with 32GB of RAM.

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Table I

| Prob. | Case | No. of failed scenarios | Average time (s) |
|-------|------|-------------------------|------------------|
|       |      | SCF0 & SCF+ST | ST | SCF0 | SCF+ST |
| Res.  | 32-Node | 0 | 68 | 21.12 | 23.03 |
|       | 123-Node | 0 | 39 | 78.22 | 84.87 |
| Rec.  | 32-Node | 0 | 33 | 1.69  | 1.48  |
|       | 123-Node | 0 | 24 | 2.19  | 1.88  |

The entries “Res.” and “Rec.” mean restoration problem and reconfiguration problem, respectively. The entry “No. of failed scenarios” indicates the number of scenarios that the resulted topology has pseudo root(s).

It can be seen that the constraints SCF0 and the proposed SCF+ST can ensure the radiality of the determined network, while the ST constraints themselves cannot guarantee radiality. As to the computation time, for reconfiguration problem, the average computation time is significantly shorter than the restoration problem due to the extra 0-1 integer variables indicating load status in the restoration problem. For reconfiguration problem, more constraints on binary variables can help the optimization algorithm converge to the optimum as the feasible region is narrowed down. However, for the restoration problem with extra binary variables, when the problem scale is large, the computation efficiency will be improved when extra variables and constraints are less. Therefore, for the reconfiguration problem only containing binary line status variables, the method SCF+ST is recommended. For the restoration problem with other binary variables, when the scale is large, the constraint SCF0 is recommended.

VI. Conclusion

This letter shows that the widely used radial topological ST constraints are the necessary but insufficient conditions for the radial topology and analyzes the additional conditions to ensure ST’s validity. The constraints guaranteeing the radial topology SCF0 are presented and SCF+ST, which is a combination of SCF and ST, is proposed. Case studies show that the proposed SCF+ST can guarantee desired radiality and is faster in feeder reconfiguration problems.

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