ON ONE FRACTAL PROPERTY OF THE MINKOWSKI FUNCTION

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Abstract. The article is devoted to answer the question about preserving the Hausdorff-Besicovitch dimension by the singular Minkowski function. It is proved that the function is not the DP-transformation, i.e., the Minkowski function does not preserve the Hausdorff-Besicovitch dimension.

It is well-known that the main problem of the fractals theory is a problem of calculating of dimension. In particular, the dimension is the Hausdorff-Besicovitch dimension (a fractal dimension). But, since some classes of sets have complicated determination, a calculation of a value of the dimension is a difficult and labour-consuming problem for these sets. Therefore, to simplify of fractal dimension calculation, a problem of searching of auxiliary facilities appears. Transformations preserving the Hausdorff-Besicovitch dimension (DP-transformations) are such facilities. The transformations help to simplify of sets determinations and to investigate of belonging to the class of DP-transformations of other transformations.

Monotone singular distribution functions as transformations of the segment $[0; 1]$ are very interesting for studying of DP-transformations. The present article is devoted to considering of fractal properties of one example of such functions. In the article a preserving of the Hausdorff-Besicovitch dimension by the Minkowski function is investigated. Results of the present article were presented by the author of the article on Second Interuniversity Scientific Conference on Mathematics and Physics for Young Scientists in April, 2011.

The following function

$$G(x) = 2^{1-a_1} - 2^{1-(a_1+a_2)} + ... + (-1)^{n-1}2^{1-(a_1+a_2+...+a_n)} + ...,$$

is called the Minkowski function. An argument $x$ of the function determined in terms of representation of real numbers by continued fractions,

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\[ [0; 1] \ni x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} \equiv [0; a_1, a_2, \ldots, a_n, \ldots], \ a_n \in \mathbb{N}. \]

To establish the one-to-one correspondence between rational numbers and quadratic irrationalities, the function was introduced by Minkowski in [5]. A difficulty of investigation of preserving the Hausdorff-Besicovitch dimension by the Minkowski function is caused by singularity and a complexity of the argument determination of the function.

Let us consider the following set
\[ E_9 \equiv \{ x : x = [0; a_1, a_2, \ldots, a_i, \ldots], a_i \in \{1, 2, 3, \ldots, 9\} \ \forall i \in \mathbb{N} \}. \]

"The geometry of continued fractions" does not have a property of "classical self-similarity". It is the reason of rather greater difficulties for obtaining exact results of fractal properties of continued fractions sets.

The main researches, in which fractal properties of some types of sets of continued fractions are studied, there are the articles of Jarnik [4] (the sets $E_n$, that representation of its elements by continued fractions contains symbols that do not exceed $n$), Good [1] (the sets of continued fractions, whose elements $a_n(x)$ quickly tend to infinity), Hirst [3] (the sets of such continued fractions, whose elements belong to infinite sequence of positive integers and increase indefinitely) and in [2] Hensley clarified estimations for Hausdorff-Besicovitch dimension of $E_n$ such that were obtained by Jarnik and Good:
\[ 1 - \frac{1}{n \lg 2} \leq \alpha_0(E_n) \leq 1 - \frac{1}{8n \lg n}, n > 8. \]

Whence, for $E_9$ we obtain that
\[ 0.6308969 \leq \alpha_0(E_9) \leq 0.985445112. \quad (1) \]

The answer to the main question of this article follows from the next statement about a value of the Hausdorff-Besicovitch dimension of the set $G(E_9)$.

**Theorem 1.** A value of the Hausdorff-Besicovitch dimension of the following set
\[ \left\{ y : y = G(x) = \frac{2}{2a_1} - \frac{2}{2a_1 + a_2} + \ldots + \frac{2(-1)^{n-1}}{2a_1 + a_2 + \ldots + a_n} + \ldots, \ x = [0; a_1, a_2, \ldots, a_n, \ldots] \right\}, \]
where $\forall i \in \mathbb{N} : a_i \in \{k_1, k_2, \ldots, k_3\}$ and $\{k_1, k_2, \ldots, k_3\}$ is a fixed tuple of positive integers for a fixed positive integer number $S > 1$, can be calculate by the formula:

$$\left(\frac{1}{2^{k_1}}\right)^{a_0} + \left(\frac{1}{2^{k_2}}\right)^{a_0} + \left(\frac{1}{2^{k_3}}\right)^{a_0} + \ldots + \left(\frac{1}{2^{k_3}}\right)^{a_0} = 1.$$

Proof. Let us write by $\Delta_0 = \{y : y = 2 \frac{a_1}{2a_1 + a_2} + \ldots + 2\frac{(-1)^n}{2a_1+a_2+\ldots+a_n} + \ldots + \frac{2}{2a_1+a_2+\ldots+a_n} \frac{(-1)^n}{2a_{n+1} + \frac{(-1)^{n+1}}{2a_{n+1}+a_{n+2}+\ldots}}\}$ the set

$$\Delta_0 = \left\{y : y = G(x) = \frac{2}{2a_1} - \frac{2}{2a_1 + a_2} + \ldots + \frac{2}{2a_1 + a_2 + \ldots + a_n} + \ldots + \frac{2(-1)^{n-1}}{2a_1 + a_2 + \ldots + a_n} \right\}.$$

Call the set $D_n = \left\{y : y = \frac{(-1)^n}{2a_{n+1}} + \frac{(-1)^{n+1}}{2a_{n+1}+a_{n+2}} + \ldots + \frac{(-1)^n}{2a_{n+1} + a_{n+2} + a_{n+3}} + \ldots \right\}$

by an "indicative set" of rank $n$.

Let $S = 2$ and $k_1 < k_2$, then $2 \frac{a_1}{2a_1} > \frac{2}{2a_1}$,

$$\sup \Delta_0 = \frac{2}{2^{k_1}} - \frac{2}{2^{k_1} + k_2} - \frac{2}{2^{k_1} + k_2 + k_1} + \ldots = \frac{2}{2^{k_1}} - \frac{2}{2^{k_1} + k_2} - \ldots + \frac{2}{2^{k_1} + k_2 - 1}.$$}

$$\inf \Delta_0 = \frac{2}{2^{k_2}} + \frac{2}{2^{k_2} + k_1} + \frac{2}{2^{k_2} + k_1 + k_2} + \ldots = \frac{2}{2^{k_2}} + \frac{2}{2^{k_2} + k_1} + \ldots + \frac{2}{2^{k_2} + k_1 - 1}.$$}

So,

$$\lambda(\Delta_0) = \frac{2(2^{k_2} - 2^{k_1})}{2^{k_1} + k_2 - 1},$$

where $\lambda(\cdot)$ is a diameter of a set.

Consider the following two sets

$$\Delta_{k_1} = \left\{y : y = \frac{2}{2^{k_1}} - \frac{2}{2^{k_1} + a_2} + \ldots + \frac{2(-1)^{n-1}}{2^{k_1} + a_2 + \ldots + a_n} + \ldots \right\}$$

where $j_1 \in \{1, 2\}$ and $\{k_1, k_2\} \ni j_1$ is a fixed for $\Delta_{k_1}$. That is

$$\Delta_{k_1} = \left\{y : y = \frac{2}{2^{k_1}} \left(1 - \frac{1}{2^{a_2}} + \frac{1}{2^{a_2+a_3}} - \frac{1}{2^{a_2+a_3+a_4}} + \ldots \right) = \frac{2}{2^{k_1}} - \frac{2}{2^{k_1}} D_1 \right\}.$$
\[ \Delta_{k_2} = \left\{ y : y = \frac{2}{k_2} \left( 1 - \frac{1}{2^{a_2}} + \frac{1}{2^{a_2+a_3}} - \frac{1}{2^{a_2+a_3+a_4}} + \ldots \right) \right\} = \left\{ \frac{2}{k_2} - \frac{2}{k_2} D_1 \right\}. \]

It is easy to see that
\[ \lambda(\Delta_{k_1}) = \frac{2}{k_1} \lambda(D_1) = \frac{1}{k_1} \lambda(\Delta_0) \]
and
\[ \lambda(\Delta_{k_2}) = \frac{2}{k_2} \lambda(D_1) = \frac{1}{k_2} \lambda(\Delta_0). \]

In the second step we obtain the following four sets: \( \Delta_{k_1 k_1}, \Delta_{k_1 k_2}, \Delta_{k_2 k_1}, \Delta_{k_2 k_2} \).
In the \( n \)th step we shall have \( 2^n \) sets
\[ \Delta_{k_1 k_2 \ldots k_n} \equiv \left\{ y : y = \sum_{m=1}^{n} \frac{2(-1)^{m-1}}{2^{k_1+k_2+\ldots+k_m}} + \sum_{t=n+1}^{\infty} \frac{2(-1)^{t-1}}{2^{k_1+k_2+\ldots+k_m+a_n+\ldots+a_t}} \right\}, \]
where \( k_1, k_2, \ldots, k_n \) is a fixed tuple of numbers from \( \{k_1, k_2\} \), and the following expression is true for all \( n \in \mathbb{N} \).
\[ \frac{\lambda(\Delta_{k_1 k_2 \ldots k_n})}{\lambda(\Delta_{k_1 k_2 \ldots k_{n-1}})} = \frac{1}{2^{k_n}} \in \left\{ \frac{1}{2^{k_1}}, \frac{1}{2^{k_2}} \right\}, \]

The last-mentioned fact follows from the next proposition.

**Proposition 1.** The condition
\[ \frac{\lambda(D_{n+1})}{\lambda(D_n)} = 1 \]
holds for all \( n \in \mathbb{N} \).

Really, let \( n = 2l, l \in \mathbb{N} \). Then
\[ D_{n+1} = D_{2l+1} \equiv \left\{ y : y = -\frac{1}{2^{a_2+l}} + \frac{1}{2^{a_2+l+2}} - \frac{1}{2^{a_2+l+2}+a_2+l+3} + \ldots \right\}, \]
\[ D_n = D_{2l} \equiv \left\{ y : y = -\frac{1}{2^{a_2+l}} - \frac{1}{2^{a_2+l+2}} + \frac{1}{2^{a_2+l+2}+a_2+l+3} - \frac{1}{2^{a_2+l+2}+a_2+l+4} + \ldots \right\}, \]
\[ \sup(D_{2l+1}) = -\frac{1}{2^{k_2}} + \frac{1}{2^{k_2+k_2}} = \frac{1-2^{k_1}}{2^{k_1+k_2}-1}, \]
\[ \inf(D_{2l+1}) = -\frac{1}{2^{k_2}} - \frac{1}{2^{k_2+k_2}} = \frac{1-2^{k_2}}{2^{k_1+k_2}-1}, \]
\[ \lambda(D_{2l+1}) = \frac{2^{k_2}-2^{k_1}}{2^{k_1+k_2}-1}. \]
Similarly,

\[
\sup(D_{2l}) = \frac{1}{2^{k_1}} - \frac{1}{2^{k_1+k_2}} = \frac{2^{k_2} - 1}{2^{k_1+k_2} - 1},
\]

\[
\inf(D_{2l}) = \frac{1}{2^{k_1}} - \frac{1}{2^{k_1+k_2}} = \frac{2^{k_1} - 1}{2^{k_1+k_2} - 1},
\]

\[
\lambda(D_{2l}) = \frac{2^{k_2} - 2^{k_1}}{2^{k_1+k_2} - 1}.
\]

Let \(I_{k_1,k_2,...,k_{1+n}}\) be a segment, whose endpoints coincide with endpoints of the corresponding set \(\Delta_{k_1,k_2,...,k_{1+n}}\). Since \(\Delta_0 \subset I_0\) and \(\Delta_0\) is a perfect set and

\[
\Delta_0 = [I_{k_1} \cap \Delta_0] \cup [I_{k_2} \cap \Delta_0],
\]

\[
[I_{k_1} \cap \Delta_0] \sim^{2^{k_1}} \Delta_0, \quad [I_{k_2} \cap \Delta_0] \sim^{2^{k_2}} \Delta_0,
\]

the theorem proved for the case of \(S = 2\).

Let \(S > 2\) and \(k_1 < k_2 < ... < k_S\). Similarly, in the \(n\)th step we shall have the \(S^n\) sets \(\Delta_{k_1,k_2,...,k_{1+n}}\). It is easy to see that

\[
\frac{\lambda(\Delta_{k_1,k_2,...,k_{1+n}})}{\lambda(\Delta_{k_1,k_2,...,k_{1+n-1}})} = \frac{1}{2^{k_n}},
\]

where \(k_n \in \{k_1, k_2, ..., k_S\}, S \in \mathbb{N}, S > 1\).

Since the set \(\Delta_0\) is a compact self-similar set of the space \(\mathbb{R}^1\), the Hausdorff-Besicovitch dimension \(\alpha_0(\Delta_0)\) of the set \(\Delta_0\) is calculating by the formula:

\[
\left(\frac{1}{2^{k_1}}\right)^{\alpha_0} + \left(\frac{1}{2^{k_2}}\right)^{\alpha_0} + \left(\frac{1}{2^{k_3}}\right)^{\alpha_0} + ... + \left(\frac{1}{2^{k_S}}\right)^{\alpha_0} = 1.
\]

The theorem is proved. \(\square\)

So, for \(k_1 = 1, k_2 = 2, ..., k_9 = 9\) we obtain that

\[
\alpha_0(G(E_9)) \approx 0.9985778625536.
\]  \(2\)

From (1) and (2) it follows that \(\alpha_0(E_9) \neq \alpha_0(G(E_9))\). So, the Minkowski function does not preserve the Hausdorff-Besicovitch dimension.
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