Supersymmetric Gauge Theories on a Squashed Four-sphere

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Abstract

We define a squashed four-sphere by a dimensional reduction of a twisted $S^4 \times S^1$, and construct explicitly a supersymmetric Yang-Mills action on it. The action includes a non-trivial dilaton factor and a theta term with a non-constant theta. The partition function of this theory is calculated using the localization technique. The resulting partition function can be written in the form consistent with the AGT relation due to the non-constant theta term. The parameter $b$ which characterizes the partition function in this form is not restricted to be real for the squashed four-sphere.
1 Introduction and Summary

Supersymmetric gauge theories on curved compact manifolds are investigated intensively after the theories on the round four-sphere was considered in [1]. One reason for the interests is that one can calculate exactly some supersymmetric quantities, such as the partition function and the expectation value of a Wilson loop, using the localization technique. In particular, the theories on four dimensional manifolds will be important for the understanding of the non-perturbative dynamics of the gauge theories related to QCD and beyond the standard model. The 4d theories are also relevant to the check of the relations between 4d gauge theories and 2d conformal field theories, which is called the AGT relations [2].

Despite these obvious importance, such computations have been performed for the 4d supersymmetric gauge theories only on a 4d ellipsoid [3], which includes the round four-sphere, and on $S^3 \times S^1$ which gives the index [4,5]. Therefore, it will be important to find other manifolds on which we can define supersymmetric gauge theories and to

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3 Though the exact calculation of the path integral was not performed, a kind of a continuous deformation of the round four-sphere, neither included in the ellipsoids nor in the squashed four sphere we define in this paper, was also considered in [6].
compute exactly some quantities of them. There are many possibilities to find such manifolds. However, one of the simplest constructions might be a “squashed four-sphere”. In [7] it was shown that squashed \((2n-1)\)-spheres, where \(n\) is a positive integer, can be constructed by a dimensional reduction of \(S^{2n-1} \times S^1\), where the reduced circular direction is a linear combination of the \(S^1\) direction and an isometry on the \((2n-1)\)-sphere. This procedure will be called the twisted dimensional reduction. When the same procedure is applied to \(S^{2n} \times S^1\), however, one might expect that some singularity may appear since any isometry on the \((2n)\)-sphere has fixed points, i.e. the north pole and the south pole.

In this paper, we show explicitly that when we apply the above process to \(S^4 \times S^1\), we obtain a non-singular manifold which we will call a squashed four-sphere, which has two deformation parameters \(\epsilon_1, \epsilon_2\). It can be shown that the supersymmetries on it are similar to those on untwisted \(S^4 \times S^1\) [8, 9]. However, some of the supersymmetries are projected out for the compatibility with the twisting.

Since a supersymmetric Yang-Mills action on \(S^4 \times S^1\) may not be possible [9], we need to study whether there exist any supersymmetric Yang-Mills action even if the squashed four-sphere is well defined. Indeed, for \(\epsilon_1 = -\epsilon_2\), we succeed to construct explicitly a supersymmetric Yang-Mills action on the squashed four-sphere, which includes a theta term with a non-constant theta. Then, the instanton factor near the north pole and the south pole of the squashed four-sphere becomes

\[
\tau_{\text{eff}} = \frac{\theta_0}{2\pi} + \frac{4\pi i (1 + (\epsilon_1)^2 l^2)}{g_{\text{YM}}^2},
\]

due to the non-constant theta term. Then we calculate the partition function by the localization technique. The result is written as \(Z(\tau_{\text{eff}}, b, \mu)\) where \(\mu\) is an effective mass parameter for the hypermultiplet and

\[
b = \sqrt{\frac{1 - i l \epsilon_1}{1 - i l \epsilon_2}}.
\]

We find that this \(Z(\tau, b, \mu)\) is the same function which appears in [3] for the partition function of the theory on the ellipsoid and consistent with the AGT relation. Here it is non-trivial that only the three parameters out of the various parameters of the theory
appear in the partition function \( Z(\tau, b, \mu) \) due to, for example, the non-constant theta term. It should be noted that in our case, because \( \epsilon_1 \) and \( \epsilon_2 \) can be taken to be arbitrary real values, one can realize arbitrary value of \( b^2 \) in \( \mathbb{C}\backslash\mathbb{R}_+ \).

For the 4d ellipsoid, the partition function obtained in [3] has a real parameter \( b \), which is the square root of the ratio between the length of the major semi-axis and that of the minor semi-axis. In this case, however, one cannot make \( b \) complex because the metric is complex for generic \( b \) or the manifold is non-compact for pure imaginary \( b \) which implies that the path integral will be IR-divergent. Remarkably, in the case of the squashed sphere, in contrast to the case of the ellipsoid, we can take both \( \epsilon_1 \) and \( \epsilon_2 \) pure imaginary such that \(|\epsilon_i|<\frac{1}{\ell}\). Including this region, the parameter \( b \) can take any value in \( \mathbb{R}_+ \) also. The partition function with the complex \( b \) could be interesting implication for the AGT relation because naive guess for the CFT counterpart will have complex central charge [10]. We will leave this problem in future investigations.

We note that there is an analogy with the theories on the deformed three dimensional spheres for the dependence of the deformation parameters. Actually, the three dimensional ellipsoid [11] and the squashed three-sphere [12] were considered and the the partition functions are in the same form for the both cases with one parameter \( b \). However, while in the former case the parameter \( b \) is real, having similar geometrical meaning as that in the 4d case, in the latter case it is a complex number. For the three-dimensional cases, it were explained in [13, 14] why these two deformations with the different geometrical origins give the partition functions in a same form. For our four dimensional cases, we do no have a clear explanations why the partition functions take in the same form[4] although there are similar works [16, 17]. This is because we consider a deformation of \( \mathcal{N}=2 \) supersymmetric gauge field theories and the theory on the squashed four-sphere is coupled to the non-trivial “dilaton” as we will explain. The investigations for these lines will be also interesting.

From our results, we expect that there are a few different supersymmetric partition

\[ ^4 \text{A partial explanation is that because the partition function can be computed at the localized points where the Nekrasov’s omega deformations are expected to be the only possible relevant supersymmetric deformations} [15]. \text{In our case, the weights in the classical action at north pole and the south pole are modified, however, the partition function is in the same form. The AGT relation and the M5-branes are expected to be a possible origin of this property.} \]
functions on compact four dimensional manifolds. It would be interesting to have other four dimensional manifolds on which the supersymmetric partition function is different from the one considered in the paper. One of the possible directions is study of supersymmetric theories on a manifold with boundaries. Indeed, such examples have been considered for two and three dimensional manifolds with boundaries in [18, 19, 20]. We hope to report some results in this direction in near future.

The rest of this paper is organized as follows. In Section 2 we define a squashed four-sphere. To obtain the supersymmetry on it, in Section 3 we review the supersymmetry on $S^4 \times S^1$. In section 4 we confirm that there are supersymmetries on $S^4 \times S^1$ which are compatible with the twist of the periodicity. After the dimensional reduction, they turn to the supersymmetry on the squashed four-sphere. A supersymmetric Yang-Mills action on the squashed four-sphere is also constructed in section 5. In section 6 we calculate the partition function of the theory defined in the previous sections.

2 Squashed four-spheres by the twisted dimensional reduction

In this section, we will define squashed four-spheres. In [7] the squashed $(2n-1)$-spheres were constructed by a dimensional reduction of $S^{2n-1} \times S^1$, where the reduced circular direction is a linear combination of the $S^1$ direction and a $U(1)$ isometry direction on the $(2n-1)$-sphere. In this paper we call this process as twisted dimensional reduction.

We apply a similar dimensional reduction to $S^4 \times S^1$. Unlike $S^{2n-1} \times S^1$, the $U(1)$ directions on $S^4$ shrinks to a point at the North pole and the South pole, and thus one might expect that the twisted dimensional reduction would produce a singular manifold. As we will see below, however, the resulting manifold is actually non-singular, and we call it a squashed four-sphere. Since the isometry group on the round four-sphere, $SO(5)$, contains two $U(1)$’s which are consistent with the supersymmetry as we will see later, the squashed four-spheres are two parameter deformations of the round four-sphere.
First, the metric on $S^4 \times S^1$ is
\[ ds^2 = ds^2_{S^4} + dt^2, \] (2.1)
where $t$ is the coordinate of the $S^1$ direction, with $t \sim t + 2\pi \beta$. Here $\beta$ is the radius of the circle. The metric on the round four-sphere is given by
\[ ds^2_{S^4} = \ell^2 \{ d\theta^2 + \sin^2 \theta (d\phi^2 + \cos^2 \phi d\alpha_1^2 + \sin^2 \phi d\alpha_2^2) \}, \] (2.2)
where $\ell$ is the radius of the four-sphere. The domain of the coordinates are $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \frac{\pi}{2}$, $0 \leq \alpha_1 \leq 2\pi$ and $0 \leq \alpha_2 \leq 2\pi$, and $\alpha_{1,2}$ are periodic coordinates.

To obtain a squashed four-sphere, we first define the twisted $S^4 \times S^1$ which has the same metric $ds^2$ as the $S^4 \times S^1$, but the periodicities in the three circular directions $(\alpha_1, \alpha_2, t)$ are given as
\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
t
\end{pmatrix}
\sim
\begin{pmatrix}
\alpha_1 - 2\pi \beta \epsilon_1 n_3 + 2\pi n_1 \\
\alpha_2 - 2\pi \beta \epsilon_2 n_3 + 2\pi n_2 \\
t + 2\pi \beta n_3
\end{pmatrix}
\quad (n_1, n_2, n_3 \in \mathbb{Z}),
\] (2.3)
where we introduced two real parameter $\epsilon_1$ and $\epsilon_2$. Introducing new coordinates defined as
\[ \alpha_1' \equiv \alpha_1 + \epsilon_1 t, \quad \alpha_2' \equiv \alpha_2 + \epsilon_2 t, \quad t' \equiv t, \] (2.4)
(2.3) is written in an "untwisted" form:
\[
\begin{pmatrix}
\alpha_1' \\
\alpha_2' \\
t'
\end{pmatrix}
\sim
\begin{pmatrix}
\alpha_1' + 2\pi n_1 \\
\alpha_2' + 2\pi n_2 \\
t' + 2\pi \beta n_3
\end{pmatrix}.
\] (2.5)
Now it is possible to reduce the new circular direction denoted by $t'$, by taking the limit of $\beta \to 0$. To obtain the metric of resulting four dimensional manifold, we rewrite the metric on the twisted $S^4 \times S^1$ (2.1) in the new coordinates as
\[
\begin{align*}
ds^2 &= \ell^2 \{ d\theta^2 + \sin^2 \theta (d\phi^2 + \cos^2 \phi (d\alpha_1' - \epsilon_1 dt')^2 + \sin^2 \phi (d\alpha_2' - \epsilon_2 dt')^2) \} + dt'^2 \\
&= ds^2_{S^4}(\epsilon_1, \epsilon_2) + h^2 (dt' + u)^2,
\end{align*}
\] (2.6)\footnote{Here we used the symbol $\sim$ as an identification.}
where
\[
\begin{align*}
\text{d}s^2_4(e_1, e_2) &= \ell^2 \left\{ \text{d}^2 \theta + \sin^2 \theta \left( \text{d}\phi^2 + \cos^2 \phi \text{d} \alpha'^2_1 + \sin^2 \phi \text{d} \alpha'^2_2 - \frac{h^2}{\ell^2 \sin^2 \theta} u^2 \right) \right\}, \\
&= \ell^2 \left\{ \text{d}^2 \theta + \sin^2 \theta \left( \text{d}\phi^2 + \cos^2 \phi \text{d} \alpha'^2_1 + \sin^2 \phi \text{d} \alpha'^2_2 - \frac{h^2}{\ell^2 \sin^2 \theta} u^2 \right) \right\}, \quad (2.7)
\end{align*}
\]
with
\[
\begin{align*}
h &= \sqrt{1 + \ell^2 \sin^2 \theta (\cos^2 \phi e_1^2 + \sin^2 \phi e_2^2)}, \\
u &= u_\mu \text{d} x^\mu = -\frac{\ell^2 \sin^2 \theta}{h^2} (\cos^2 \phi \alpha'_1 + \sin^2 \phi \alpha'_2). \quad (2.8)
\end{align*}
\]

To see that \(\text{d}s^2_4\) is the background metric of the four dimensional theory obtained by the dimensional reduction of the five dimensional theory on twisted \(S^4 \times S^1\), we rewrite the standard kinetic term of a five dimensional scalar field in the twisted coordinate \((x', t')\):
\[
\begin{align*}
\int \text{d}^4 x \text{d} t \sqrt{g(x, t) g^{mn}(x, t) \partial_m \Phi \partial_n \Phi} &= \int \text{d}t' \text{d}^4 x' \sqrt{Gh(G^\mu\nu' \partial_\mu' \Phi \partial_\nu' \Phi + 2 G^\mu\nu' u_\mu' \partial_\nu' \Phi \partial_\nu' \Phi + (\partial_\nu' \Phi)^2)}. \quad (2.9)
\end{align*}
\]
where \(g_{mn}\) is the five dimensional metric on the twisted \(S^4 \times S^1\), i.e. the metric defined by (2.1), in \((x, t)\) coordinates and \(G^\mu\nu'\) is the metric defined by (2.7). After taking the limit of \(\beta \to 0\), all the modes with non-zero Kaluza-Klein momentum decouple, and the action of Kaluza-Klein zero modes, which can be considered as four dimensional fields, is
\[
\int \text{d}^4 x' \sqrt{GhG^\mu\nu' \partial_\mu' \Phi \partial_\nu' \Phi}. \quad (2.10)
\]

This implies that the four dimensional theory obtained by the dimensional reduction have background metric of the squashed four-sphere \(G^\mu\nu'\). Note that this theory also has the non-constant dilaton background \(h\). By considering a vector field instead of the scalar field, one finds that the theory have the graviphoton background \(u\) also.

The four dimensional manifold with the metric (2.7) is actually non-singular and thus well-defined. Indeed, the last term in (2.7) behaves as \(\ell^4 \sin^4 \theta \epsilon_i^2 \text{d} \alpha^2_j\), which vanishes appropriately at the north pole and the south pole. This metric is the one on the round \(S^4\) deformed by two parameters in a similar way to obtain a squashed \((2n - 1)\)-sphere.
from $S^{2n-1} \times S^1$ in [7]. In this paper we call the manifold with the metric (2.7) as a squashed four-sphere.

So far we have assumed that both $\epsilon_i$’s are real. Here let us take $\epsilon_i$’s pure imaginary formally in the metric (2.7). Replacing $\epsilon_i$ with $i\epsilon_i$, one obtains

$$
\ell^2 \left\{ d^2 \theta + \sin^2 \theta \left( d\phi^2 + \cos^2 \phi d\alpha_1^2 + \sin^2 \phi d\alpha_2^2 + \frac{\bar{h}^2}{\ell^2 \sin^2 \theta} u^2 \right) \right\}
$$

(2.11)

with $u(e_1, e_2)$ and

$$
\bar{h}(e_1, e_2) = \sqrt{1 - \ell^2 \sin^2 \theta (e_2^3 \cos^2 \phi + e_2^2 \sin^2 \phi)}.
$$

(2.12)

In this case the metric becomes singular at $\theta \sim \frac{\pi}{2}$ if $\max_i \{e_i^2\} \geq \ell^2$. However, for $e_i^2 < \ell^2$ one still have a non-singular manifold. Therefore, we have non-singular manifolds for $\epsilon_i$’s being both real or $\epsilon_i$’s being both imaginary with $(i\epsilon_i)^2 < \ell^2$.

3 **Supersymmetric gauge theories on $S^4 \times S^1$**

The supersymmetry transformation law on the squashed four-sphere can be obtained if we have the one on the twisted $S^4 \times S^1$ because the supersymmetry transformation are closed even if the fields are restricted to the Kaluza-Klein zero-modes. Because the difference between the $S^4 \times S^1$ and the twisted $S^4 \times S^1$ is just in the coordinate identifications, the supersymmetry transformation on the twisted $S^4 \times S^1$ can be obtained from the one on the $S^4 \times S^1$ as we will show later. Thus, in this section, we review the supersymmetry on the $S^4 \times S^1$ which were constructed in [8, 9].

Here we use the convention used in [21] unless otherwise stated. We use Greek indices $(\mu, \nu, \cdots = 1, 2, 3, 4)$ for the directions along the four-sphere and $t$ or $5$ for the circular direction. To represent all the five directions, we use $m, n, \cdots = 1, 2, 3, 4, 5$. When an index denote the corresponding local Lorentz index, a symbol “hat” is added on it (for examples, $\hat{\mu}$ and $\hat{m}$).

**Vectormultiplet**

The vectormultiplet in the five dimensional $\mathcal{N} = 1$ supersymmetry consists of a connection 1-form $A_m$, a (Hermite) scalar boson $\sigma$, two spinor fermions $\lambda_I$ and three auxiliary
scalar bosons $D_{IJ}$. Here $I, J = 1, 2$ and there is an $SU(2)_R$ symmetry acting on them. The fermions $\lambda_I$ transform as a doublet under this $SU(2)_R$. The auxiliary fields $D_{IJ}$ are symmetric with respect to the two indices and transforms as a triplet under the $SU(2)_R$.

The supersymmetry transformation law of the vectormultiplet is given by

$$\begin{align*}
\delta_\xi A_m &= i\epsilon^{IJ} \xi_I \Gamma_m \lambda_J, \\
\delta_\xi \sigma &= i\epsilon^{IJ} \xi_I \lambda_J, \\
\delta_\xi \lambda_I &= -\frac{1}{2} \Gamma^{mn} \xi_I F_{mn} + \Gamma^m \xi_I D_m \sigma + \xi_J D_K \epsilon^{JK} + 2t_I^J \Gamma_5 \xi_J \sigma, \\
\delta_\xi D_{IJ} &= -i\xi_I \Gamma^m D_m \lambda_J + [\sigma, \xi_I \lambda_J] + i\xi_K t_{IJ} \Gamma_5 \lambda_K + (I \leftrightarrow J),
\end{align*}$$

(3.1)

where $\xi_I$ are the spinors satisfying following Killing Spinor equations

$$D_\mu \xi_I = t_I^J \Gamma_\mu \Gamma_5 \xi_J, \quad D_5 \xi_I = 0. \quad (3.2)$$

We also impose the twisted $SU(2)$ Majorana condition \cite{9}, $\xi_I^\dagger = -\epsilon^{IJ} \xi_J C \Gamma_5$, where $C$ is the charge conjugation matrix. Here $t_I^J$ is a constant matrix which can be chosen as

$$t_I^J = \frac{1}{2\ell} (\sigma_3)_1^J. \quad (3.3)$$

The equations (3.2) can be explicitly solved as

$$\xi_I = \sqrt{f} \left(1 - \frac{(-1)^I}{2\ell} \Gamma^\mu x_\mu \Gamma_5 \right) \psi_I, \quad (3.4)$$

with a pair of constant spinors $(\psi_1, \psi_2)$ related to each other so that $\xi_I$ satisfy the twisted $SU(2)$ Majorana condition. Here we have chosen the coordinate $x_\mu$ as a conformal basis defined by

$$\begin{align*}
x_1 &= 2\ell \tan \frac{\theta}{2} \cos \phi \cos \alpha_1, \\
x_2 &= 2\ell \tan \frac{\theta}{2} \cos \phi \sin \alpha_1, \\
x_3 &= 2\ell \tan \frac{\theta}{2} \sin \phi \cos \alpha_2, \\
x_4 &= 2\ell \tan \frac{\theta}{2} \sin \phi \sin \alpha_2,
\end{align*}$$

(3.5)
with which the metric on the four-sphere is simply \( ds_{S^4}^2 = f^2 dx_\mu dx_\mu \), where \( f = \left(1 + \frac{x^2}{4\ell^2}\right)^{-1/2} \).

From (3.1) one finds that a commutator of two supersymmetry transformations close into the sum of a translation generated by a linear combination of Killing vectors, a \( SU(2)_R \) rotation, a local Lorentz rotation and a gauge transformation:

\[
[\delta_\xi, \delta_\eta] A_m = -iv^m F_{nm} + D_m \gamma,
[\delta_\xi, \delta_\eta] \sigma = -iv^m D_n \sigma,
[\delta_\xi, \delta_\eta] \lambda_I = -iv^m D_n \lambda_I + i[\gamma, \lambda_I] + R_{IJ}^J \lambda_J + \frac{1}{4} \Theta^{\hat{ab}} \Gamma^{\hat{ab}} \lambda_I,
[\delta_\xi, \delta_\eta] D_{IJ} = -iv^m D_n D_{IJ} + i[\gamma, D_{IJ}] + R_{IK}^J D_{KJ} + R_{JL}^K D_{IK},
\]  

(3.6)

where

\[
v^m = 2\epsilon^{IJ} \xi_I \Gamma^m \eta_J, \quad \gamma = -2i\epsilon^{IJ} \xi_I \eta_J \sigma, \quad R_{IJ} = 4i\epsilon^{KL} \xi_K \Gamma_5 t_I J \eta_L,
\]

\[
\Theta^{\hat{ab}} = -2it^{IJ} (\xi_I \Gamma_5 \eta_J - \eta_I \Gamma_5 \eta_J \xi_J).
\]  

(3.7)

For \( S^4 \times S^1 \), it may be impossible to construct a supersymmetric Yang-Mills action which becomes the supersymmetric Yang-Mills action on the round four-sphere in the limit of \( \beta \to 0 \) \cite{9}. We will, however, construct a supersymmetric Yang-Mills action on the squashed four-sphere in section 4.

**Hypermultiplets**

The hypermultiplet consists of two bosonic scalars \( q_I \), a fermionic spinor \( \psi \) and two bosonic auxiliary scalars \( \mathcal{F}_{I'} \), where \( I' = 1, 2 \). The supersymmetry transformation law of hypermultiplet is given as

\[
\delta_\xi q_I = -2i\xi_I \psi,
\]

\[
\delta_\xi \psi = \epsilon^{IJ} \Gamma^m \xi_I D_m q_J + i\epsilon^{IJ} \xi_I \sigma q_J - 2t^{IJ} \Gamma_5 \xi_I q_J + \epsilon^{I'J'} \xi_{I'} \mathcal{F}_{J'},
\]

\[
\delta_\xi \mathcal{F}_{I'} = 2\xi_{I'} (i\Gamma^m D_m \psi + \sigma \psi + \epsilon^{KL} \lambda_K q_L),
\]  

(3.8)

where \( \xi_{I'} \) is a pair of spinors which satisfy

\[
\xi_I \xi_{I'} = 0, \quad \epsilon^{I' J'} \xi_{I'} \xi_{J'} = \epsilon^{IJ} \xi_I \xi_J.
\]  

(3.9)
Since $\xi_{I'}$ is related to $\xi_I$, the off-shell construction is possible only with respect to a single supersymmetry $\delta_\xi$, but this is enough for the localization technique to be applicable. Regarding $\delta_\xi$ as a fermionic operation, i.e. $\xi_I$ as a bosonic variable, the square of $\delta_\xi$ is given by

$$\begin{align*}
\delta_\xi^2 q_I &= iv^mD_m q_I - i\gamma q_I - R_I^J q_J, \\
\delta_\xi^2 \psi &= iv^mD_m \psi - i\gamma \psi - \frac{1}{4}\Theta^{\hat{a}\hat{b}}\Gamma^{\hat{a}\hat{b}}\psi, \\
\delta_\xi^2 \mathcal{F}_{I'} &= iv^mD_m \mathcal{F}_{I'} - i\gamma \mathcal{F}_{I'} + R_{I',J'}^J \mathcal{F}_{J'},
\end{align*}$$

(3.10)

where

$$\begin{align*}
v^m &= \epsilon^{IJ} \xi_I \Gamma^m \xi_J, \quad \gamma = -i\epsilon^{IJ} \xi_I \xi_J \sigma, \quad \Theta^{\hat{a}\hat{b}} = 2i\epsilon^{IJ} \xi_I \Gamma_5 \Gamma^{\hat{a}\hat{b}} \xi_J, \\
R_{I,J} &= 2i\epsilon_{IJ} \xi_K \Gamma^5 \xi_L, \quad R_{I',J'}^J = -2i\tilde{\xi}_{I'} \Gamma^m D_m \tilde{\xi}_{J'}. 
\end{align*}$$

(3.11)

To show (3.10), following identities are useful

$$\begin{align*}
\epsilon^{IJ} \xi_I \xi_J C + \epsilon^{I'J'} \tilde{\xi}_{I'} \tilde{\xi}_{J'} C &= -\frac{1}{2} (\xi_I \xi_I) \cdot 1, \quad \tilde{\xi}_{I'} \Gamma^m \tilde{\xi}_{J'} = -\xi_I \Gamma^m \xi_I,
\end{align*}$$

(3.12)

where $1$ is the identity matrix with spinor indices. These are derived from (3.9).

For the hypermultiplet, one can construct an action on $S^4 \times S^1$. The result is

$$\begin{align*}
S_{\text{hyp}} &= \int d^5x \sqrt{g} \text{tr} \left( \epsilon^{IJ} (D_\mu \bar{q}_I D^\mu q_J - \bar{q}_I \sigma^2 q_J) - 2i\bar{\psi}\Gamma^m D_m \psi + \bar{\psi}\sigma\psi \right) \\
&- i\bar{q}_I D_5 q_I - 4\epsilon^{I'J'} \bar{\psi}_I q_J - \epsilon^{I'J'} \bar{\mathcal{F}}_{I'} \mathcal{F}_{J'} \\
&- 2\tilde{\epsilon}^{IJ} \bar{\xi}_I D_5 \tilde{\xi}_J + \frac{4}{\ell^2} \epsilon^{IJ} \bar{\xi}_I q_J. 
\end{align*}$$

(3.13)

Above the massless case have been considered for simplicity. A mass can be introduced as follows. First add a $U(1)$ vectormultiplet $(A_m^{(m)}, \sigma^{(m)}, \lambda_I^{(m)}, D_{I,J}^{(m)})$, and assign a unit $U(1)$ charge to the hypermultiplet. Then fix the value of this vectormultiplet to a supersymmetry invariant configuration $\{A_5 = m, \text{ others} = 0\}$. As a result mass terms of hypermultiplet is induced by the covariant derivatives.
4 Supersymmetry on a squashed four-sphere

In this section, we will show how a four dimensional theory on the squashed four-sphere is obtained by the dimensional reduction of the five dimensional theory on the twisted $S^4 \times S^1$. It will be shown that a quarter of the supersymmetry survives if the periodicity of the fields in the circle is appropriately twisted.

First we consider a basis of the Killing spinors as
\[
\{\xi\} = \bigoplus_{s_1, s_2 = \pm 1} \{\xi\}_{(s_1, s_2)},
\]  
(4.1)

where $\{\xi\}_{(s_1, s_2)}$ is the set of killing spinors constructed by $\psi_2$ satisfying $\Gamma^{12}\psi_2 = i s_1 \psi_2$ and $\Gamma^{34}\psi_2 = i s_2 \psi_2$. Then we find by an explicit calculation that the killing spinors belonging to each $\{\xi\}_{(s_1, s_2)}$ satisfy the following identity
\[
O(s_1, s_2) \xi_I \equiv \mathcal{L}(\partial_t - p) \xi_I + \tilde{R}(s_1, s_2) J^J \xi_J = 0,
\]  
(4.2)

where $p = \epsilon_i \partial_{\alpha_i}$ and
\[
\tilde{R}(s_1, s_2) J^J = -i(\epsilon_1 s_1 + \epsilon_2 s_2) \ell t^J.
\]  
(4.3)

The action of the operator $O(s_1, s_2)$ is defined by the r.h.s in the first line. Here the Lie derivative of a spinor is defined by
\[
\mathcal{L}(a) \varphi \equiv \left( a^m D_m - \frac{1}{4} (\nabla_m a_n) \Gamma^{mn} \right) \varphi.
\]  
(4.4)

For later convenience, we divide $O$ into a derivative $\partial_t - p^\mu \partial_\mu = \partial_\nu$ and the other operator without the derivative, and rewrite as
\[
\partial_\nu \xi_I = -\tilde{O}(s_1, s_2) \xi_I,
\]  
(4.5)

where $\tilde{O}(s_1, s_2)$ is the non-derivative part of $O(s_1, s_2)$, i.e. $\tilde{O}(s_1, s_2) = O(s_1, s_2) - \partial_\nu$.

This equation implies that the Killing spinor satisfies
\[
\xi(x', t' + 2\pi \beta) = e^{-2\pi \beta \tilde{O}(s_1, s_2)} \xi(x', t').
\]  
(4.6)
Thus, we should modify the boundary condition of the fields in order to keep (some part of) supersymmetry on the twisted $S^4 \times S^1$.

Now we choose one of the $(s_1, s_2)$ and impose following twisted periodic boundary condition to the fields as in [12]:

$$\Phi(x', t' + 2\pi \beta) = e^{-2\pi \beta \tilde{O}(s_1, s_2)} \Phi(x', t').$$

(4.7)

Indeed, with this boundary condition, we can see that the supersymmetry transformations (3.1) are consistent for the theory on the twisted $S^4 \times S^1$ with the quarter of the supersymmetry with $\xi_I$ belonging to the $\{\xi\}_{(s_1, s_2)}$.

The Kaluza-Klein decomposition of the fields with the twisted boundary condition is

$$\Phi(x', t') = \sum_{n \in \mathbb{Z}} e^{i n t'} e^{-t' \tilde{O}(s_1, s_2)} \Phi_n(x').$$

(4.8)

Therefore, if we take the limit of $\beta \to 0$, an ordinal kinetic term give infinitely large masses in four-dimensional sense to each Kaluza-Klein modes except those with $n = 0$. As a result all the modes with $n \neq 0$ decouple and the fields are effectively restricted so that they have only the Kaluza-Klein zero modes which satisfy

$$\partial_{t'} \Phi = -\tilde{O}(s_1, s_2) \Phi.$$  

(4.9)

The identities (4.5) guarantees that this condition can be kept under the supersymmetry transformation $\delta_\xi$ with $\xi \in \{\xi\}_{(s_1, s_2)}$. Therefore these supersymmetry is preserved even in the four dimensional theory on the squashed four-sphere obtained by the dimensional reduction.

The theory on the squashed four-sphere has generically a quarter of the supersymmetry of the theory on the round four-sphere. However, for some special values of the deformation parameters $\epsilon_i$, supersymmetry enhancements occur. For generic $\epsilon_1$ and $\epsilon_2$, the conditions $\partial_{t'} \Phi = \tilde{O}(s_1, s_2)$ with different $(s_1, s_2)$s are distinct. Therefore one can

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6 By simply substituting the Kaluza-Klein decomposition into the r.h.s of the supersymmetry transformation, one finds that the modes with non-zero Kaluza-Klein momentum also contribute to the Kaluza-Klein zero mode of the l.h.s. through non-linear terms. However these contribution vanish after renormalizing the modes with $n \neq 0$ as $\Phi_n \to \frac{\beta}{n} \Phi$ and taking the limit of $\beta \to 0$. 

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impose only one of four conditions and as a result the surviving supersymmetry is a quarter of those in the theory on $S^4 \times S^1$. On the other hand, if one of $\epsilon_i$ is zero or $\epsilon_1 \pm \epsilon_2 = 0$, two of the four conditions degenerate. Correspondingly the supersymmetry in four dimension enhances to a half of those in the theory on $S^4 \times S^1$. Moreover, if $\epsilon_1 = \epsilon_2 = 0$, all the four condition degenerate and all the supersymmetry are preserved. In this case, the four dimensional theory is just the $N = 2$ theory on the round four-sphere [1].

The supersymmetry transformation law in four dimensional theory on the squashed four-sphere is immediately obtained by replacing all the $\partial_{t'}$ in the five dimensional law (3.1) and (3.8) with $\tilde{\mathcal{O}}(s_1, s_2)$, according to (4.9). An action of the four dimensional theory also would be obtained from that of the five dimensional theory on the twisted $S^4 \times S^1$ by a similar replacement. Unfortunately, however, no such five dimensional action is known which would reproduce the four dimensional Yang-Mills term. Therefore the construction of a supersymmetry invariant Yang-Mills action on a squashed four-sphere is a non-trivial problem. In the next section we will consider this problem.

5 Supersymmetry invariant action of the vector-multiplet on a squashed four-sphere

In this section we will construct an supersymmetry invariant Yang-Mills action on the squashed four-sphere.

As pointed out in [9] it is difficult to construct a supersymmetry invariant Yang-Mills action on $S^4 \times S^1$. The reason is as follows. On the one hand, a four dimensional Yang-Mills action on the four-sphere is obtained from the one on the flat $\mathbb{R}^4$ by conformal transformation. Through this step the two scalars in vectormultiplet gain masses associated to the curvature. On the other hand, if one had a five dimensional action and dimensionally reduce the circular direction $t$, $A_t$ turned into a scalar, which must be massless because of the five dimensional gauge symmetry.

It is possible, however, to construct an action on the twisted $S^4 \times S^1$ which is both gauge invariant and supersymmetry invariant if we neglect the massive KK modes. From this action one can obtain an gauge invariant and supersymmetry invariant action
on the squashed four-sphere by the dimensional reduction.

Below, we construct such an action. Hereafter we choose \( s_1 = s_2 = 1 \) because the choice of it is just the matter of convention and can be absorbed to the definition of \( \epsilon_i \). Furthermore, for simplicity, we restrict the twisting parameters to satisfy \( \epsilon_1 = -\epsilon_2 \equiv \epsilon \). Even though it might be possible to construct an invariant action for the general \( \epsilon_i \), it would require more lengthy and tedious calculation.

We start from the following trial action:

\[
S'_\text{vec} = \int d^5x \sqrt{|g|} \text{Tr} \left( \frac{1}{2} F_{mn} F^{mn} - (D_m \sigma)(D^m \sigma) - \frac{1}{2} D_{IJ} D^{IJ} + i \lambda_I \Gamma^m D_m \lambda^I \\
- \lambda_I [\sigma, \lambda^I] + 2 A_t t^{IJ} D_{IJ} + \frac{1}{\ell^2} (3 A_t^2 - 2 \sigma^2) \right). \tag{5.1}
\]

This ansatz is motivated by modifying the one constructed in \cite{9} so that the twisted dimensional reduction produces appropriate scalar terms (written in \( A_t = A_t - p^\mu A_\mu \)). The supersymmetry transformation of this \( S'_\text{vec} \) is

\[
\delta_\xi S'_\text{vec} = \int d^5x \sqrt{|g|} \text{Tr} \left( 4 i t^{IJ} \{ \xi_I \Gamma^m \lambda_J (\mathcal{O}(1,1) A_m - \xi_I \lambda_J \mathcal{O}(1,1) \sigma) \\
- \frac{6i}{\ell^2} p^\mu A_t \xi_I \Gamma_\mu \lambda^I \\
+ 4 i p^{\mu} t^{IJ} \left( F_{mn} \xi_I \Gamma^m \lambda_J - \xi_I \lambda_J D_\mu \sigma - \frac{1}{2} D_{IJ} \xi_K \Gamma_\mu \lambda^K \right) \right), \tag{5.2}
\]

where we arranged the terms which vanishes as the limit of \( \beta \to 0 \) is taken and the fields are restricted as (4.9) into the first line. To write down the rest of terms in \( \delta_\xi S'_\text{vec} \) (the second line and the third line), we used the explicit form of \( \mathcal{O} \) acting on \( A_m \) and \( \sigma \)

\[
\mathcal{O}(1,1) A_m = \partial_\nu A_m - (\partial_m p^n) A_n, \quad \mathcal{O}(1,1) \sigma = \partial_\nu \sigma. \tag{5.3}
\]

In order to have a supersymmetry invariant action on the squashed four-sphere, we need to add some terms to the trial action \( S'_\text{vec} \) which compensate the second line in

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\footnote{Of course, we can construct an gauge and supersymmetry invariant action on the squashed four-sphere directly using the four dimensional supersymmetry transformation, however, we guess that a computation for that is harder than the five-dimensional construction we employed here.}
To find such terms, the following three identities will be useful:

\[
I^K t^J_K + t^K J s^K_J = -t^{KL} s^{KL} \xi^I_J, \quad t^I_J \xi_J = F(2) \xi_I, \quad t^I_J \xi_J C = \xi_I C F(2),
\]

(5.4)

where \(s_{IJ}\) is an arbitrary symmetric tensor and

\[
F(2) = \frac{\nabla \mu p_\nu}{4(h^2 - 1)} p^\lambda \Gamma_5 \Gamma^\mu \Gamma^\nu, \quad F'(2) = \frac{\nabla \mu p_\nu}{4(h^2 - 1)} \Gamma^\mu \Gamma^\nu p^\lambda \Gamma_5. \]

(5.5)

The last two identities in (5.4) follows from Killing spinor equation (3.2) and the supersymmetry surviving condition (4.5).

With these identities, it is shown that the following term

\[
I_1 \equiv \int d^5 x \sqrt{g} \text{tr} \left( i \lambda_I p^\mu (t^{IJ} - \epsilon^{IJ} F(2)) \Gamma^\mu \lambda_J \right)
\]

(5.6)

transforms as

\[
\delta_\xi I_1 = \int d^5 x \sqrt{g} \text{tr} \left( 2i p^\mu t^{IJ} D_{IJ} \xi_K \Gamma^\mu \lambda^K + \mathcal{O}(D_{IJ})^0 \right)
\]

(5.7)

and thus cancels the terms linear in \(D_{IJ}\) in \(\delta_\xi S'_{vec}\).

With this \(I_1\) added, the unwanted terms are

\[
\delta_\xi (S'_{vec} + I_1) - \int d^5 x \sqrt{g} \text{tr} \left( 4i t^{IJ} (\xi_I \Gamma^m \lambda_J \mathcal{O}(1,1) A_m - \xi_I \lambda_J \mathcal{O}(1,1) \sigma) \right)
\]

\[
= \int d^5 x \sqrt{g} \text{tr} \left( \frac{1}{2} (2i (\nabla^\mu p_\nu) (D_{IJ}) \xi_I \Gamma^\mu \lambda^I - i (\nabla^\mu p_\nu) F_{\mu \nu} \xi_I \lambda^I) - \frac{6i}{c^2} p^\mu A_\nu \xi_I \Gamma^\mu \lambda^I 
\right.
\]

\[
- \left. i \epsilon^\mu \nu \lambda \rho (\nabla_\mu p_\nu) F_{\lambda \rho} \xi_I \Gamma^\mu \lambda^I + \frac{i}{2} \epsilon^\mu \nu \lambda \rho (\nabla_\mu p_\nu) F_{\lambda \rho} \xi_I \Gamma^\mu \lambda^I \right),
\]

(5.8)

where we arranged all the terms like \(\xi_I (\cdots) \lambda_J\) into the form of \((\cdots)_{IJ} \xi_K (\cdots) \lambda^K\) with the help of (5.4). Now it is easy to see that this is equal to

\[
\delta_\xi \int d^5 x \sqrt{g} \text{tr} \left( \frac{1}{2} \epsilon^\mu \nu \lambda \rho (\nabla_\mu p_\nu) F_{\lambda \rho} \sigma - (\nabla^\mu p_\nu) (F_{\mu \nu} A_5 + (\partial_5 A_\mu) A_\nu) + \frac{3}{c^2} (p^\mu A_\mu)^2 \right).
\]

(5.9)
Putting above results together we finally obtain a desired action:

\[ S_{\text{vec}} = \frac{1}{g_{YM}^2} \int \frac{d^5x}{2\pi \beta} \sqrt{g} \text{tr} \left( \frac{1}{2} F_{nm}F^{mn} - \frac{1}{2} (D_m \sigma)(D^m \sigma) - \frac{1}{2} D_{IJ}D^{IJ} + i\lambda_I\Gamma^m D_m \lambda^I \right. \]

\[ \left. - \lambda_I[\sigma, \lambda^I] + 2A_\nu t_{IJ} D_{IJ} + \frac{1}{\ell^2} (3A_\nu^2 - 2\sigma^2) \right) \]

\[ + i\lambda_I \left( t^{IJ} p^\mu \Gamma_\mu + \frac{1}{4} \epsilon^{IJ} (\nabla_\mu P_\nu) \Gamma^{\mu\nu} \Gamma_5 \right) \lambda_J \]

\[ - \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} (\nabla_\mu P_\nu) F_{\lambda\rho} \]

\[ + (\nabla^\mu p^\nu) (F_{\mu\nu} A_5 + (\partial_5 A_\mu) A_\nu) - \frac{3}{\ell^2} (p^\mu A_\mu)^2 \right), \] (5.10)

which is invariant under the supersymmetry transformation \( \delta_\xi \) up to massive Kaluza-Klein modes. Here we multiplied \( \frac{1}{2\pi \beta g_{YM}} \) to the action to obtain a four dimensional action normalized in a familiar way.

The theta term

\[ \frac{i\theta_0}{32\pi^2} \int d^4 x^' \sqrt{G} \text{tr} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \] (5.11)

is also supersymmetry invariant due to its topological nature and can be added to the action.

The action (5.10) should be also invariant under the gauge transformation \( G(\epsilon_g) \) with the gauge transformation parameter satisfying \( O(1, 1) \epsilon_g = \partial_5 \epsilon_g = 0 \) in order to obtain a gauge invariant action on the squashed four-sphere after the dimensional reduction. We can see that this is indeed the case. Actually, with some computations, we can show that the terms in the last line in (5.10) can be rewritten as

\[ \frac{1}{g_{YM}^2} \int \frac{d^5x}{2\pi \beta} \sqrt{g} \text{tr} \left( \frac{1}{2} (\nabla^\mu p^\nu) F_{\mu\nu} A_{5'} - \frac{\epsilon^2 \ell^2}{4} \frac{1 - \frac{x^2}{4\ell^2}}{1 + \frac{x^2}{4\ell^2}} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \right). \] (5.12)

The second term looks like the theta term, but it has the non-constant pre-factor. As we will see in the next section, only the values of the action around the north pole
and the south pole are relevant to the partition function. There, the second term in \((5.12)\) behaves like the ordinal theta term. Note, however, that the non-constant pre-factor \(\frac{1 - \frac{r^2}{4\ell^2}}{1 + \frac{r^2}{4\ell^2}}\) flip its sign as one goes from the north pole to the south pole. Therefore the contribution of this term cannot be absorbed to the addition of the theta term \((5.11)\). Rather, it changes the Yang-Mills coupling constants for the instantons which are point-wisely localized at the north pole and the south pole.

6 Localization

In this section we will calculate the partition function of the four dimensional theory on the squashed four-sphere defined in the previous sections by using the localization technique.

The localization technique can be applied if the theory has a continuous symmetry with a fermionic generator \(Q\), for which there exist a fermionic potential \(V\) such that the bosonic part of \(S_r \equiv QV\) is positive semi definite and \(QS_r = 0\). Below, we will call \(V\) as regulator potential and \(S_r\) as regulator action.

To explain the localization technique, we consider following quantity

\[
Z'(\tau) = \int \mathcal{D}\Phi \exp \left[ -S - \tau S_r \right],
\]

where \(\tau\) is a real parameter. We denote all the fields by \(\Phi\). This \(Z'(\tau)\) includes the original partition function as \(Z = Z'(0)\). If \(\tau\) is large, all configurations but those around the zeroes of \(S_r\) is suppressed, and in the limit of \(\tau \to \infty\) the saddle point approximation becomes exact:

\[
Z'(\infty) = \sum_{\Phi_0} \exp \left[ -S[\Phi_0] \right] \times Z_{\text{1-loop}}(\Phi_0).
\]

where \(\Phi_0\) is a zero point configuration of \(S_r\) and \(Z_{\text{1-loop}}(\Phi_0)\) is the perturbative one loop determinant of the theory described by the action \(S_r\), around \(\Phi = \Phi_0\). On the other
hand, by differentiating \( Z' \) by \( \tau \) one obtains

\[
\frac{dZ'(\tau)}{d\tau} = \int D\Phi(-S_t) \exp[-S - \tau S_t]
= \int D\Phi Q (-V \exp[-S - \tau S_t])
= 0,
\]

(6.3)

which shows that \( Z'(\tau) \) is actually independent of the value of \( \tau \). Therefore the partition function is

\[
Z = Z'(\infty) = \sum_{\Phi_0} \exp[-S[\Phi_0]] \times Z_{\text{1-loop}}(\Phi_0).
\]

(6.4)

The 1-loop factor \( Z_{\text{1-loop}}(\Phi_0) \) is a non-trivial, but, an exactly calculable quantity. We can decompose \( \Phi \) into its bosonic part \( \Phi_b \) and fermionic part \( \Phi_f \), and then can chose \( V \) as

\[
V = ((Q\Phi_f)^\dagger, \Phi_f),
\]

(6.5)

where \( ^\dagger \) is an Hermitian conjugate and \( (\cdot, \cdot) \) is a \( Q^2 \) invariant positive definite inner product of functions. \( S_r = QV \) with this \( V \) indeed satisfies the required properties for the localization. If we decompose \( \Phi_b \) and \( \Phi_f \) further as \( \Phi_b = (X, Q\Xi) \) and \( \Phi_f = (QX, \Xi) \), fix Hermicity of fields appropriately and write the regulator potential \( V \) as

\[
V = [QX \ \Xi] \begin{bmatrix} D_{0,0} & D_{0,1} \\ D_{1,0} & D_{1,1} \end{bmatrix} \begin{bmatrix} X \\ Q\Xi \end{bmatrix},
\]

(6.6)

then the regulator action \( S_r \) is

\[
S_r = [X \ Q\Xi] \begin{bmatrix} \mathcal{H} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} D_{0,0} & D_{0,1} \\ D_{1,0} & D_{1,1} \end{bmatrix} \begin{bmatrix} X \\ Q\Xi \end{bmatrix}
+ [QX \ \Xi] \begin{bmatrix} D_{0,0} & D_{0,1} \\ D_{1,0} & D_{1,1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} QX \\ \Xi \end{bmatrix},
\]

(6.7)

where

\[
\mathcal{H} \equiv Q^2.
\]

(6.8)

From (6.7) it immediately follows that

\[
Z_{\text{1-loop}} = \left( \frac{\det\text{coKer}(D_{1,0})}{\det\text{Ker}(D_{1,0}) \mathcal{H}} \right)^{\frac{2}{2}}.
\]

(6.9)
If the eigenvalues of $\mathcal{H}$ are $h_i$ and their degeneracies in $\text{Ker}(D_{1,0})$ and in $\text{coKer}(D_{1,0})$ are $n_{bi}$ and $n_{fi}$ respectively, the 1-loop determinant can be written as

$$Z_{1\text{-loop}} = \prod_i h_i^{n_{fi} - n_{bi}}. \quad (6.10)$$

Here $n_{bi}$ and $n_{fi}$ can be read off from the index of $D_{1,0}$ defined by

$$\text{ind}(D_{1,0}, \mathcal{H}; q) \equiv \text{tr}_{\text{coKer}(D_{1,0})} e^{q\mathcal{H}} - \text{tr}_{\text{Ker}(D_{1,0})} e^{q\mathcal{H}}. \quad (6.11)$$

Note that this index is well defined only when both of $n_{bi}$ and $n_{fi}$ are finite for every eigenvalue. This is satisfied when $D_{1,0}$ is transversally elliptic, which have been checked in many cases [3, 1].

In this section we first consider the 5d theory on the twisted $S^4 \times S^1$ with the twisted periodicity (4.7) and obtain the saddle point configurations and 1-loop determinants. Then we consider only the Kaluza-Klein zero modes which satisfy (4.9) and obtain the results on the squashed four-sphere. After this dimensional reduction, there are the supersymmetry invariant actions (3.13) and (5.10), thus we obtain the partition function by evaluating them at the saddle point configurations.

Although the saddle point configurations and 1-loop determinants are already obtained in the 5d theory on the untwisted $S^4 \times S^1$ in [8, 9], it is not straightforward to obtain our result from them. First, the saddle point configurations obtained in the untwisted theory may be inconsistent to the twisted periodicity. Second, since the periodicity in our theory is different from that in the untwisted theory and they give different eigenvalues for the same differential operator, the 1-loop determinant must be changed even if the saddle point configurations are same.

As we will see in the following, the saddle point configurations are the same. On the other hand, the 1-loop determinants and the values of the action at the saddle point configurations are different from those in the untwisted theory, which makes the partition function non-trivial.

---

8 One may also worry that the transversal ellipticity of $D_{1,0}$ may be violated because of the twisted periodicity. However, it is not the case, since the notion of transversal ellipticity is invariant under any continuous (small) deformation of the differential operator.
The vectormultiplet

In our case we can choose $Q$ as supersymmetry $\delta_\xi$ with some fixed killing spinor $\xi_I$. We take $\xi_I$ bosonic to have $\delta_\xi$ fermionic. In this section, we denote this fermionic $\delta_\xi$ as $\delta$.

We chose the $\xi_I$ or, equivalently, $\psi_I$ as

$$
\Gamma^{\hat{1}\hat{2}} \psi_2 = \Gamma^{\hat{3}\hat{4}} \psi_2 = i \psi_2,
$$

(6.12)

so that the supersymmetry generated by this $\xi_I$ remains in the four dimensional limit, and normalize them as

$$
s = \xi_I \xi^I = \cos \theta,
$$

$$
v = \xi_I \Gamma^m \xi^I \partial_m = \frac{i}{\ell} \left( -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} \right) + \frac{\partial}{\partial t}.
$$

(6.13)

As discussed in the beginning of this section, $V_{\text{vec}}$, regulator potential for the vectormultiplet, can be chosen as

$$
V_{\text{vec}} = \int d^5 x \sqrt{\det g} \delta \left( \text{tr}(\lambda_I)^\dagger \lambda_I \right),
$$

(6.14)

where

$$
(\delta \lambda_I)^\dagger = \frac{1}{2} \xi_I \Gamma^{mn} F_{mn} - \xi_I \Gamma^m D_m \sigma + \xi_J D_J^I - 2 \xi^K \Gamma^5 t_K \sigma.
$$

(6.15)

(6.15) follows from (3.11) and Hermicity condition of fields

$$
\sigma^\dagger = -\sigma, \quad (D_{IJ})^\dagger = -D^{IJ}
$$

(6.16)

which are required for the convergence of the original path integral $Z$.

\footnote{One should consider the regulator potential for the hypermultiplet simultaneously because the original action is invariant only when both the vectormultiplet and the hypermultiplet are simultaneously transformed by (3.1) and (3.8). As we will see later, however, the regulator potential for the hypermultiplet neither affects the saddle point condition of the vectormultiplet nor includes fluctuations in the vectormultiplet around the saddle point configurations. Therefore one can forget the hypermultiplet completely when one calculate the contribution to the 1-loop determinant from the vectormultiplet.}
The bosonic part of $\delta V_{\text{vec}}$ is obviously positive. Concretely, it is

$$\delta V_{\text{vec}}|_{\text{bos}} = \int d^5x \sqrt{g} \text{tr} \left( \frac{1}{2} F_{mn} F^{mn} - (D_m \sigma)(D^m \sigma) - \frac{1}{2} D_{IJ} D^{IJ} - \frac{1}{\ell^2} \sigma^2 ight.$$

$$\left. - \frac{s}{4} \varepsilon_{\mu
onumber
\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} - \frac{s}{4} \varepsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} - \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} F_{\mu\nu} (D_{\lambda \rho} \sigma) \right)$$

$$= \int d^5x \sqrt{g} \text{tr} \left( \frac{1 - s}{2} \left( (F_+)^{\mu\nu} - \frac{\sigma}{2(1 - s)} ((dv)_+)^{\mu\nu} \right)^2 
+ \frac{1 + s}{2} \left( (F_-)^{\mu\nu} + \frac{\sigma}{2(1 + s)} ((dv)_-)^{\mu\nu} \right)^2 
+ F_{\mu\nu} F^{\mu\nu} 
- (D_m \sigma)(D^m \sigma) - \frac{1}{2} D_{IJ} D^{IJ} \right), \quad (6.17)$$

where $((dv)^{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ and the subscript $\pm$ denotes the sd/asd decomposition of a two-form defined as

$$(F_\pm)^{\mu\nu} \equiv \frac{1}{2} \left( F^{\mu\nu} \pm \frac{1}{2} \varepsilon_{\mu
onumber
\nu\lambda\rho} F_{\lambda\rho} \right). \quad (6.18)$$

Now we investigate the zero configurations of (6.17). When $s \neq \pm 1$, all the squares must vanish at a saddle point:

$$F_{\mu\nu} = 0, \quad D_m \sigma = 0, \quad D_{IJ} = 0$$

$$F_- - \frac{\sigma}{2(1 - s)} ((dv)_-) = 0, \quad F_+ + \frac{\sigma}{2(1 + s)} ((dv)_+) = 0, \quad (6.19)$$

which implies

$$A_\mu = 0, \quad A_{t'} = a_{t'}, \quad \sigma = 0, \quad D_{IJ} = 0 \quad (6.20)$$

up to a gauge transformation. Here $a_{t'}$ is a constant in the Lie algebra, which is periodic because of the large gauge transformations in $t'$ unfixed by (6.20):

$$a_{t'} \sim a_{t'} + \frac{n_i H_i}{\beta} \quad (n_i \in \mathbb{Z}), \quad (6.21)$$

where $H_i$ are the basis of the Cartan subalgebra of the gauge group.
At the point $s = \pm 1$ (north/south pole), on the other hand, one of the last two conditions in (6.19) is not imposed since the coefficient of corresponding term in (6.17) vanishes. Thus instanton configurations which are pointwisely localized at the two poles are also allowed as saddle point configurations.

Evaluating the Yang-Mills action (5.10) with the theta term (5.11) at these saddle points, we have

$$S_{vec} = \frac{8\pi^2}{g_{YM}^2} \text{tr} a^2 + \frac{16\pi^2}{g_{YM}^2} (-\nu_{np} + \nu_{sp}) + 2i\theta_0 (\nu_{np} + \nu_{sp}),$$

(6.22)

where

$$\frac{1}{g_{YM}'} = 1 + \epsilon^2 \ell^2 \frac{g_{YM}}{g_{YM}^2}. \quad (6.23)$$

The first term comes from the mass term of $A_\mu$ in the second line of (5.10). The second term and the third term are the contributions of the instantons which are pointwisely localized at the north pole or the south pole. At the north pole $(s = -1)$, $F_-$ are unconstrained and thus there are anti-instantons. We denoted the instanton number as $\nu_{np}$, which is non-positive integer. At the south pole $(s = 1)$, on the other hand, the self-dual instanton solutions are allowed, the instanton number of which we denote as $\nu_{sp} \geq 0$. Because of the second term in (5.12), the Yang-Mills coupling constant in the instanton contributions is effectively changed.

To evaluate the saddle point value of the action, we restricted the two squashing parameters such that $\epsilon_1 = -\epsilon_2 \equiv \epsilon$. It may be hard task, but interesting to find the Yang-Mills action even for $\epsilon_1 \neq -\epsilon_2$ although there would not exist such term.

**BRST complex**

To calculate the 1-loop determinant of the vectormultiplet, here we introduce ghosts which consist of the fermionic ones $c, \bar{c}, c_0, \bar{c}_0$, and the bosonic ones $B, B_0, a_0, \bar{a}_0$. We

---

10. Here the Yang-Mills coupling constant itself is not changed because, for example, the local high energy scattering of the gluons are described by $g_{YM}$, not $g_{YM}'$. Only the weights for the instanton contributions are changed by the term.
define the BRST symmetry $\delta_B$, under which the ghosts transform as

$$
\begin{align*}
\delta_B c &= ic + a_0, \quad \delta_B a_0 = 0, \quad \delta_B \bar{c} = B, \quad \delta_B B = i[a_0, \bar{c}], \\
\delta_B \bar{a}_0 &= \bar{c}_0, \quad \delta_B \bar{c}_0 = i[a_0, \bar{a}_0], \quad \delta_B B_0 = c_0, \quad \delta_B c_0 = i[a_0, B_0]
\end{align*}
$$

(6.24)

and all the other fields in the vectormultiplet and the hypermultiplet transform as

$$
\delta_B(\text{boson}) = G(c)(\text{boson}), \quad \delta_B(\text{fermion}) = iv^m\partial_m\bar{c}, \quad \delta(\text{other ghosts}) = 0
$$

(6.25)

We define the supersymmetry transformation law of ghosts as

$$
\begin{align*}
\delta c &= -i(s\sigma - v^m A_m), \quad \delta B = iv^m\partial_m\bar{c}, \quad \delta(\text{other ghosts}) = 0
\end{align*}
$$

(6.26)

so that the square of the new fermionic charge defined by $Q = \delta + \delta_B$ is the same for all the fields:

$$
\mathcal{H} \equiv Q^2 = i\mathcal{L}(v) - 2it_I J^I + G(a_0).
$$

(6.27)

To fix the gauge, we introduce a $Q$-exact term $S_{GF} = QV_{GF}$ with

$$
V_{GF} = \int d^5x \sqrt{g} \text{tr}((\overline{c}G + B_0) + c\bar{a}_0),
$$

(6.28)

where

$$
G = i\nabla^m A_m + i\mathcal{L}(v)(\Phi - A_5) = i\nabla^\mu A_\mu + iv^\mu\partial_\mu(v^\nu A_\nu - s\sigma) + i\partial_5 \Phi
$$

(6.29)

with $\Phi = v^m A_m - s\sigma$. One can show by explicit calculation that this term play the same role as $\delta_B V_{GF}$ and fix the gauge properly after the ghosts being integrated out.

Now $\Phi = \{A_m, \sigma, D_{IJ}, \lambda_I\} \oplus \{\text{ghosts}\}$. Then we replace $V_{vec}$ with

$$
V'_{vec} = V_{vec} + V_{gho} + V_{GF},
$$

(6.30)

where $V_{gho} = \int d^5x \sqrt{g} \text{tr}((Qc)^1c + (Q\bar{c})^1\bar{c})$. The saddle point equations are the previous ones (6.19) from $QV_{vec|\text{bos}} = 0$ and

$$
-i(s\sigma - v^m A_m) + a_0 = 0, \quad B = 0
$$

(6.31)
For systematic calculation, it is convenient to replace the spinor $\lambda_I$ with the fields with integral spins [9]:

$$\Psi \equiv \delta \sigma = i \xi_I \lambda^I, \quad \Psi_\mu \equiv \delta A_\mu = i \xi_I \Gamma_\mu \lambda^I, \quad \Xi_{IJ} \equiv \xi_I \Gamma_5 \lambda_J + \xi_J \Gamma_5 \lambda_I.$$  (6.32)

These can be solved for $\lambda_I$ as

$$\lambda_I = -i \Gamma_5 \xi_I \Psi - i \Gamma^5 \xi_I \Psi_\mu + \xi^I \Xi_{II}. \quad (6.33)$$

With these, we divide the set of fields $\Phi$ into

$$X = (\sigma, A_\mu, \bar{a}_0, B_0), \quad \Xi = (\Xi_{IJ}, \bar{c}, c) \quad (6.34)$$

and their supersymmetry partners $QX$ and $Q\Xi$.

Below we will calculate the index (6.11). Since the gauge transformation in $\mathcal{H}$ acts uniformly to all the fields, its contribution is factored out in the index:

$$\text{ind}(D_{1,0}, \mathcal{H}; q) = \sum_{\alpha \in \mathfrak{g}} e^{iq\alpha \cdot \alpha} (-2 + \text{ind}'(D_{1,0}, \mathcal{H}(a_0 = 0); q)), \quad (6.35)$$

where $\mathfrak{g}$ is the Lie algebra of the gauge group. $-2$ is the contribution from two bosonic zero modes $\bar{a}_0$ and $B_0$. Here the second term in the parentheses is defined for the remaining fields, $X' = (A_\mu, \sigma)$ and $\Xi$, which can be further decomposed into the sum of the contributions from the modes who have the same Kaluza-Klein momentum $\frac{k}{\beta}$, $X'(k)$ and $\Xi(k)$. With these, we find

$$\text{ind}'(D_{1,0}, \mathcal{H}(a_0 = 0); q) = \sum_{k \in \mathbb{Z}} \text{ind}'(D_{1,0}^{(k)}, \mathcal{H}^{(k)}(a_0 = 0); q), \quad (6.36)$$

where $D_{1,0}^{(k)}$ and $\mathcal{H}^{(k)}$ are defined by replacing $\partial_t$ with $\frac{ik}{\beta} + \tilde{O}(1,1)$ in $D_{1,0}$ and $\mathcal{H}$, respectively (see (4.8)). More explicitly, $\mathcal{H}^{(k)}$ is given by

$$\mathcal{H}^{(k)} = -\mathcal{L}(v_{\mathcal{H}}) - i(2t_I^J + \tilde{R}(1,1)_I^J) - \frac{k}{\beta} + \mathcal{G}(a_0), \quad (6.37)$$

11 The 1-loop determinant with the same twisted periodicity was already calculated in [8] to compute the superconformal index in five dimension, although our main concern in this paper is the theory on the squashed four-sphere. The computations in [8] are essentially same as the ones in this paper, however, we use the pure spinor like method for the hypermultiplets as in [21] and construct explicitly the BRST complex.
where

\[ v_H = -i(v^\mu \partial_\mu + p) = \frac{\gamma_i}{\ell} \partial_{\alpha_i} \]  \hspace{1cm} (6.38)

with

\[ \gamma_i = 1 - i \ell \epsilon_i, \]  \hspace{1cm} (6.39)

and \( \hat{R} \) was defined in (4.3). Again, since \( -\frac{k}{\beta} \) in \( \mathcal{H}^{(k)} \) acts uniformly to all the fields in \( X'(k) \) and \( \Xi^{(k)} \), its contribution is factored out and thus the index is

\[ \text{ind}(D_{1,0}, \mathcal{H}; q) = \sum_{\alpha \in g} e^{i q_{a_0} \cdot \alpha} \left( -2 + \sum_{k \in \mathbb{Z}} e^{-\frac{q \beta}{k}} \text{ind}^{(0)}(D_{1,0}, \mathcal{H}^{(0)}(a_0 = 0); q) \right). \]  \hspace{1cm} (6.40)

In order for the index \( \text{ind}^{(0)} \) to be well defined, the degeneracy of each eigenvalue of \( \mathcal{H}^{(0)}(a_0 = 0) \) must be finite both in \( \text{Ker}(D_{1,0}) \) and \( \text{coKer}(D_{1,0}) \). The sufficient condition for this is that \( D_{1,0} \) is a transversally elliptic operator with respect to \( v_H \). In our case this is satisfied as discussed in the beginning of this section.

Then \( \text{ind}^{(0)} \) can be calculated by applying Atiyah-Bott formula [22] to the complex \( E_{X'(0)} \rightarrow E_{\Xi^{(0)}} \) (where \( \Gamma(E_{X'(0)}) = \{ X'(0) \} \) and \( \Gamma(E_{\Xi^{(0)})} = \{ \Xi^{(0)} \} \)):

\[ \text{ind}^{(0)}(D_{1,0}, \mathcal{H}^{(0)}(a_0 = 0); q) = \sum_{x_p \in F} \frac{\text{tr}_{E_{X'(0)} e^{q \hat{\mathcal{H}}^{(0)}(a_0 = 0)}} - \text{tr}_{E_{X'(0)} e^{q \hat{\mathcal{H}}^{(0)}(a_0 = 0)}}}{\det(1 - \frac{\partial x'}{\partial x})} \bigg|_{x_p}, \]  \hspace{1cm} (6.41)

where \( F \) is the set of the fixed points under \( x^\mu \rightarrow x'^\mu = e^{-q \mathcal{L}(v_H)} x^\mu \), i.e. the north pole and the south pole. The determinant in the denominator is

\[ \det \left( 1 - \frac{\partial x'}{\partial x} \right) = (1 - e^{i \gamma_1 / \ell})(1 - e^{-i \gamma_1 / \ell})(1 - e^{i \gamma_2 / \ell})(1 - e^{-i \gamma_2 / \ell}). \]  \hspace{1cm} (6.42)

\( \hat{\mathcal{H}}^{(0)} \) is the vector bundle homomorphism naturally induced from \( \mathcal{H}^{(0)} \). The eigenvalues of \( \mathcal{H}^{(0)}(a_0 = 0) \) are

\[ e^{i \gamma_1 / \ell}, e^{-i \gamma_1 / \ell}, e^{i \gamma_2 / \ell}, e^{-i \gamma_2 / \ell}, 1 \]  \hspace{1cm} (6.43)

in \( E_{X'(0)} \) and

\[ 1, e^{i \gamma_1 + \gamma_2 / \ell}, e^{-i \gamma_1 + \gamma_2 / \ell}, 1, 1 \]  \hspace{1cm} (6.44)
in \( E_{\Xi(0)} \). Thus

\[
\text{ind}^{(0)}(D_{1,0}^{(0)}, \mathcal{H}^{(0)}(a_0 = 0); q) = \frac{1 + e^{-i\frac{q}{2}(\gamma_1 + \gamma_2)}}{(1 - e^{-i\frac{q}{2}\gamma_1})(1 - e^{-i\frac{q}{2}\gamma_2})} + \frac{1 + e^{-i\frac{q}{2}(\gamma_1 + \gamma_2)}}{(1 - e^{-i\frac{q}{2}\gamma_1})(1 - e^{-i\frac{q}{2}\gamma_2})} = \sum_{n_1, n_2 \geq 0} (e^{-i\frac{q}{2} n_1 \gamma_1 + e^{-i\frac{q}{2} (n_1 + 1) \gamma_1} + e^{i\frac{q}{2} n_1 \gamma_1 + e^{i\frac{q}{2} (n_1 + 1) \gamma_1}}),
\]

(6.45)

where we expanded the denominator of the terms coming from the north pole in positive power of \( e^{-i\frac{q}{2} \gamma_1} \) and those from the south pole in negative power in the second line, to obtain the correct index [1].

From (6.40) and (6.45), we find that the 1-loop determinant is

\[
Z^{\text{vec,5d}}_{\text{1-loop}} = \prod_{\alpha \in \Delta} \left( i\alpha_0 \cdot \alpha \right)^{-1} \times \prod_{k \in \mathbb{Z}, n_1, n_2 \geq 0} \left( \frac{k}{\beta} - \frac{in_i \gamma_i}{\ell} + ia_0 \cdot \alpha \right)^{\frac{1}{2}} \left( \frac{k}{\beta} - \frac{i(n_i + 1) \gamma_i}{\ell} + ia_0 \cdot \alpha \right)^{\frac{1}{2}}
\]

\[
\left( \frac{k}{\beta} + \frac{in_i \gamma_i}{\ell} + ia_0 \cdot \alpha \right)^{\frac{1}{2}} \left( \frac{k}{\beta} + \frac{i(n_i + 1) \gamma_i}{\ell} + ia_0 \cdot \alpha \right)^{\frac{1}{2}}
\]

(6.46)

where \( \Delta \) is the set of roots and \( \Delta_+ \) is the set of positive roots in \( g \). We neglected the contribution from the modes in Cartan subalgebra, which gives only an \( a_0 \) independent overall factor.

Omitting the contributions from the massive Kaluza-Klein modes, we obtain the
four dimensional result

\[
Z_{1\text{-loop}}^{\text{vec}} = \prod_{\alpha \in \Delta_+} (i\alpha_0 \cdot \alpha)^{-2} \prod_{n_1, n_2 \geq 0} \left( -\frac{in_i \gamma_i}{\ell} + i\alpha_0 \cdot \alpha \right) \left( -\frac{i(n_i + 1) \gamma_i}{\ell} + i\alpha_0 \cdot \alpha \right) \\
\times \left( \frac{in_i \gamma_i}{\ell} + i\alpha_0 \cdot \alpha \right) \left( \frac{i(n_i + 1) \gamma_i}{\ell} + i\alpha_0 \cdot \alpha \right)
\]

\[
= \prod_{\alpha \in \Delta_+} \frac{\Upsilon_b(i\hat{a}_0 \cdot \alpha) \Upsilon_b(-i\hat{a}_0 \cdot \alpha)}{(\hat{a}_0 \cdot \alpha)^2}, \quad (6.47)
\]

where

\[
b = \sqrt{\frac{\gamma_1}{\gamma_2}} = \sqrt{\frac{1 - i\ell \epsilon_1}{1 - i\ell \epsilon_2}}, \quad (6.48)
\]

and

\[
\hat{a}_0 = \frac{i\ell}{\sqrt{\gamma_1 \gamma_2}} a_0. \quad (6.49)
\]

In the last step in (6.47) we again neglected some \( a_0 \) independent overall factors. Here we defined the Upsilon function by the following infinite product

\[
\Upsilon_b(x) = \prod_{n_1, n_2 \geq 0} \left( bn_1 + \frac{n_2}{b} + x \right) \left( bn_1 + \frac{n_2}{b} + b + \frac{1}{b} - x \right). \quad (6.50)
\]

\( \Upsilon_b(x) \) is also characterized by following relations

\[
\Upsilon_b(x) = \Upsilon_b \left( b + \frac{1}{b} - x \right), \quad \Upsilon_b \left( \frac{1}{2} \left( b + \frac{1}{b} \right) \right) = 1,
\]

\[
\Upsilon_b(x + b) = \Upsilon_b(x) \frac{\Gamma(bx)}{\Gamma(1 - bx)} b^{1-2bx}, \quad \Upsilon_b \left( x + \frac{1}{b} \right) = \Upsilon_b(x) \frac{\Gamma(\frac{x}{b})}{\Gamma(1 - \frac{x}{b})} \left( \frac{1}{b} \right)^{1-\frac{2x}{b}}. \quad (6.51)
\]

The 1-loop determinant (6.47) is identical to the result obtained for the ellipsoid, except that the parameter \( b \), which is in the case of the ellipsoid the square root of the ratio between the length of the major semi-axis and that of the minor semi-axis and thus must be real, is replaced by (6.48). \( b \) can take arbitrary value in \( \mathbb{C} \setminus \mathbb{R}_+ \) with \( \text{Re}b > 0 \). Moreover, when both of \( \epsilon_i \) are pure imaginary and \( |\epsilon_i| < \ell \), the value of \( b \) becomes real positive and the parameter region obtained in [3] is reproduced.
The hypermultiplet

Now we consider the contribution from the hypermultiplet. The regulator potential for the hypermultiplet can be taken as

$$V_{\text{hyp}} = \int d^5x \sqrt{g} \text{tr}((\delta \psi) \dagger \psi).$$ (6.52)

Here we divide $\psi$ into the components proportional to $\xi$, that is, $\delta q_I$, and the components proportional to $\bar{\xi}$ as

$$\psi = -\frac{i}{s}(\xi_I \psi^I_+ + \bar{\xi}_I \psi^I_-),$$ (6.53)

where

$$\psi^I_+ = -2i\xi_I \psi = \delta q_I, \quad \psi^{-I}_- = \bar{\xi}_I \psi.$$ (6.54)

This field redefinition eliminates spinor indices and thus simplify the form of $\delta^2$. Since (6.53) is an orthogonal decomposition, the saddle point condition of $S_r|_{\text{bos}} = QV_{\text{hyp}}|_{\text{bos}}$, that is, $\delta \psi = 0$, is equivalent to $\delta \psi^I_+ = \delta \psi^{-I}_- = 0$, which implies

$$q_I = F_{-\nu} = 0,$$ (6.55)

with an appropriate Hermicity conditions for $q, F$ as shown in [9]. Note that we do not obtain any further constraint on the saddle point configurations of the vectormultiplet, as commented in the footnote when we calculated the contribution of the vectormultiplet.

To continue, we take $\bar{\xi}_I$ explicitly as

$$\bar{\xi}_I = \frac{M \frac{1}{2}}{2} \left(1 - \frac{1}{s} v^n \Gamma_n\right) \eta_I,$$ (6.56)

where $\eta_1$ and $\eta_2$ are the solutions of (3.4) with the constant spinor chosen as $\Gamma^{12} \psi_2 = -\Gamma^{34} \psi_2 = i \psi_2$ and normalized as $\eta_I \eta_I' = -s$. $M$ is a scalar function to normalize $\bar{\xi}$ as (3.9) which we take as independent of $t$. This $\bar{\xi}_I$ satisfy the following differential equation

$$\mathcal{L}(\partial_v)\bar{\xi}_I = i(\gamma_1 - \gamma_2)t_{+J'} \bar{\xi}_J,$$ (6.57)
where \( t_{\mu',\nu'} \) have the same components as \( t_{\mu}^j \). This means that one have to twist the periodicity of the fields further with the rotation of \( I' = 1, 2 \):

\[
\Phi(x', t' + 2\pi \beta) = e^{-2\pi \beta \left(G(1, 1) - i(\gamma_1 - \gamma_2) t_{\mu', \nu'}^j \right)} \Phi(x', t')
\]

(6.58)

to preserve the supersymmetry on the squashed four-sphere.

To calculate the 1-loop determinant, we choose \( X \) as \( q_I \) and \( \Xi \) as \( \psi_I' \). Then the index (6.11) can be calculated just in the same way as in the case of the vectormultiplet:

\[
\text{ind}(D_{1,0}, \mathcal{H}; q) = \sum_{r \in R} e^{iqa_0 \cdot r} \text{ind}^{(k)}(D_{1,0}^{(k)}, \mathcal{H}^{(k)}(a_0 = 0); q) = \sum_{r \in R} e^{iqa_0 \cdot r} \sum_{k \in \mathbb{Z}} e^{-\frac{ak}{\beta}} \text{ind}^{(0)}(D_{1,0}^{(0)}, \mathcal{H}^{(0)}(a_0 = 0); q).
\]

(6.59)

where \( \mathcal{H}^{(k)} \) is defined by the action of \( \mathcal{H} \) on the fields satisfying the twisted periodic boundary condition (6.58) with Kaluza-Klein momentum \( k/\beta \):

\[
\mathcal{H}^{(k)} = -\mathcal{L}(v_{\mathcal{H}}) - i(\gamma_1 + \gamma_2) t_{\mu}^j + (R'_{\mu'} j' - i(\gamma_1 - \gamma_2) t_{\mu' j'}) - \frac{k}{\beta} + \mathcal{G}(a_0) + m.
\]

(6.60)

We can show that \( R' = 0 \) at the north and the south poles. Here we also added a mass term in the manner as explained in Section 2. \( D_{1,0}^{(k)} \) is defined by the similar restriction, and it is guaranteed by the discussion in the beginning of this section that it is transversally elliptic. We can also compute \( \text{ind}^{(0)} \) in the same way as for the vector multiplets. The result is

\[
\text{ind}^{(0)} = \sum_{n_1, n_2 \geq 0} \left( e^{-i\frac{\pi}{4}(n_1 + \frac{1}{2}) \gamma_1} + e^{i\frac{\pi}{4}(n_1 + \frac{1}{2}) \gamma_1} \right),
\]

(6.61)

which gives the 1-loop determinant for the hypermultiplet on the twisted \( S^4 \times S^1 \).

The 1-loop determinant for the hypermultiplet on the squashed four-sphere, up to irrelevant overall factors, is obtained by dropping the contributions from massive
Kaluza-Klein modes as

\[
Z_{1\text{-loop}}^{\text{hyp}} = \prod_{r \in R_{n_1,n_2 \geq 0}} \left( \frac{i\gamma_i}{\ell} \left( n_i + \frac{1}{2} \right) + ia_0 \cdot r + m \right)^{-\frac{1}{2}} \\
\quad \times \left( -\frac{i\gamma_i}{\ell} \left( n_i + \frac{1}{2} \right) + ia_0 \cdot r + m \right)^{-\frac{1}{2}} \\
= \prod_{r \in R} \Upsilon_b \left( \frac{1}{2} \left( b + \frac{1}{b} \right) + i\hat{a}_0 \cdot r + \frac{i\ell m}{\sqrt{\gamma_1 \gamma_2}} \right)^{-\frac{1}{2}}. 
\tag{6.62}
\]

where \( R \) is the weights of the representation of the hypermultiplet in the gauge group.

**Partition function and Wilson loop**

So far we have considered vector saddle point configurations of \( F_{\mu\nu} = 0 \) only. Relaxing this restriction, however, as we saw in the subsection of the vector multiplet, we have additional saddle point configurations, in which \( F_- \neq 0 \) on the north pole and \( F_+ \neq 0 \) on the south pole. They are weighted by (6.22).

The supersymmetry algebras (3.6) at the poles are identical to those of the 4d \( \mathcal{N} = 2 \) multiplets in the \( \Omega \)-background, with the parameters of the each theory appropriately identified. Therefore we have only to quote the result \( Z_{\text{inst}}(ia, \varepsilon_1, \varepsilon_2, q) \) in that case, where \( a \) is the vev of the Higgs scalar, \( (\varepsilon_1, \varepsilon_2) \) are the \( \Omega \)-deformation parameter and \( q = e^{2\pi i \tau} \) with

\[
\tau = \tau(g, \theta) = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \tag{6.63}
\]

the complexified coupling constant [23]. Here we used the notation adopted in \[ ] for the parameters in \( Z_{\text{inst}} \). With this, the contributions in our case can be written as \( |Z_{\text{inst}} \left( -a_0, \frac{\gamma_1}{\ell}, \frac{\gamma_2}{\ell}, e^{2\pi i r(g'_{YM}, \theta_0)} \right)|^2 \).

Putting all together, the partition function on the squashed four-sphere is

\[
Z = \int d\hat{a}_0 \exp \left[ -\frac{8\pi^2 \gamma_1 \gamma_2}{g_{YM}^2} \text{tr} \hat{a}_0^2 \right] |Z_{\text{inst}} \left( -a_0, \frac{\gamma_1}{\ell}, \frac{\gamma_2}{\ell}, e^{2\pi i r(g'_{YM}, \theta_0)} \right)|^2 \\
\prod_{\alpha \in \Delta_+} \Upsilon_b(i\hat{a}_0 \cdot \alpha) \Upsilon_b(-i\hat{a}_0 \cdot \alpha) \times \prod_{r \in R} \Upsilon_b \left( \frac{1}{2} \left( b + \frac{1}{b} \right) + i\hat{a}_0 \cdot r + \frac{i\ell m}{\sqrt{\gamma_1 \gamma_2}} \right)^{-\frac{1}{2}}. \tag{6.64}
\]

30
where $g'_{\text{YM}}$ is defined by (6.23).

In the similar way, one can also calculate the expectation value of any gauge invariant operator which is invariant under $\delta \xi$. One important example is the Wilson loop. Since the combination

$$v^mA_m - s\sigma = i\gamma_1A_{\alpha_1'} + i\gamma_2A_{\alpha_2'} + A_\nu - s\sigma$$  \hfill (6.65)

is supersymmetry invariant, one can construct Wilson loops which are both gauge invariant and supersymmetry invariant. For the generic values of the squashing parameters $\epsilon_i$, we have two kinds of the supersymmetric Wilson loops which are defined by

$$W_i(\theta') = \text{trP} \exp \left[ \ell \int_{L_i} d\lambda \left( iA_{\alpha_i'} + \frac{1}{\gamma_i} (A_\nu - s\sigma) \right) \right],$$  \hfill (6.66)

where $i = 1, 2$ and the two loops $L_1$ and $L_2$ are

$$L_1 = \{ (\theta', \phi', \alpha_1', \alpha_2') = (c_1, 0, \lambda, c_2) | 0 \leq \lambda \leq 2\pi \},$$  \hfill (6.67)

$$L_2 = \{ (\theta', \phi', \alpha_1', \alpha_2') = \left( c_1, \frac{\pi}{2}, c_2, \lambda \right) | 0 \leq \lambda \leq 2\pi \},$$  \hfill (6.68)

where $c_1, c_2$ are constants. The expectation values of these Wilson loops are immediately obtained by inserting the following saddle point values

$$W_1(\theta') = \text{tr} \exp \left[ \frac{2\pi i\hat{a}_0}{b} \right], \quad W_2(\theta') = \text{tr} \exp [2\pi ib\hat{a}_0]$$  \hfill (6.69)

into the integrand in (6.64).

Though the 1-loop determinant is calculated for general $(\epsilon_1, \epsilon_2)$, the full result (6.64) is valid only for the special case of $\epsilon_1 = -\epsilon_2 = \epsilon$ where we constructed the action of the vectormultiplet. In this case, the partition function (6.64) takes the following form

$$Z(\tau, b, \mu) = \int da \exp \left[ -\frac{8\pi^2}{g^2} \text{tr} a^2 \right] |Z_{\text{inst}} \left( ia, b, \frac{1}{b}, e^{2\pi i r(g, \theta)} \right)|^2 \times \prod_{\alpha \in \Delta_+} \Upsilon_b(i\alpha \cdot \alpha) \Upsilon_b(-i\alpha \cdot \alpha) \times \prod_{r \in R} \Upsilon_b \left( \frac{1}{2} \left( b + \frac{1}{b} \right) + i\alpha \cdot r + i\mu \right)^{-\frac{1}{2}},$$  \hfill (6.70)

with the substitutions of $g = \frac{g_{\text{YM}}}{\sqrt{\gamma_1 \gamma_2}}, \theta = \theta_0, b = \sqrt{\frac{\pi}{\gamma_2}}$ and $\mu = \frac{\hat{m}}{\sqrt{\gamma_1 \gamma_2}}$. On the other hand, the partition function in the case of the ellipsoid is also in this form, with $g = g_{\text{YM}}, b = \sqrt{\frac{r}{\gamma_2}}$. 31
In this sense the partition function on the squashed four sphere coincide with that on the ellipsoid. Moreover, \( b = \sqrt{\frac{\gamma_1}{\gamma_2}} \) also can be complex with \(|b| = 1\) as well as real, unlike in the case of the ellipsoid where \( b \) is given as the square root of the ratio between the length of the major semi-axis and that of the minor semi-axis and thus always real. That is, by considering the squashed four-spheres we can realize more general parameter region than in the case of the ellipsoids.

For \( \epsilon_1 \neq -\epsilon_2 \), if there is a corresponding CFT, the central charge would be complex, thus, there could not be an AGT-like relation and the partition function could be different from the function (6.70).

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