Covariant description of the black hole entropy in 3D gravity

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Abstract

We study the entropy of the black hole with torsion using the covariant form of the partition function. The regularization of infinities appearing in the semiclassical calculation is shown to be consistent with the grand canonical boundary conditions. The correct value for the black hole entropy is obtained provided the black hole manifold has two boundaries, one at infinity and one at the horizon. However, one can construct special coordinate systems, in which the entropy is effectively associated with only one of these boundaries.

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1. Introduction

Three-dimensional (3D) gravity is an attractive model for investigating basic features of both classical and quantum gravity. In the traditional approach based on general relativity (GR), gravitational dynamics is studied in spacetime with an underlying Riemannian structure [1–9]. On the other hand, it has been well known for nearly five decades that there exists a gauge-theoretic conception of gravity, based on Riemann–Cartan geometry of spacetime (see, e.g. [10–12]). In this approach, both the curvature and the torsion are used to characterize the gravitational dynamics. The application of these ideas to 3D gravity started in the early 1990s, leading to a deeper understanding of the dynamical role of torsion [13–23].

In 1991, Mielke and Baekler proposed a topological model for 3D gravity based on Riemann–Cartan geometry [13]. The related field equations have one particularly interesting solution—the black hole with torsion [15–19], which generalizes the BTZ black hole [4]. It is well known that quantum nature of gravity is reflected in thermodynamic properties of black holes [24–28]. In the standard field-theoretic approach, these properties can be described by the Euclidean functional integral [4, 6]:

\[ Z[\beta, \Omega] = \int Db^i D\omega^i \exp(-\tilde{I}[b^i, \omega^i, \beta, \Omega]). \]  

(1.1)
Here, \( b^i \) and \( \omega^j \) are the triad field and the connection, \( \beta \) and \( \Omega \) are the Euclidean time period and the angular velocity of the black hole, and \( I \) is the Mielke–Baekler action, corrected by suitable boundary terms. The boundary conditions are chosen so that \( Z[\beta, \Omega] \) is the grand canonical partition function. Using the Hamiltonian form of the action and assuming that the black hole manifold has two boundaries, one at infinity (\( B^\infty \)) and one at the horizon (\( B^r \)), one can calculate the entropy of the black hole with torsion [22] (for a derivation based on the Cardy formula, see [23]). The result is found to be consistent with the first law of black hole thermodynamics.

One could expect that working with the Hamiltonian form of the action is of particular importance, and that transition to the covariant, Lagrangian form represents only a technical step, which cannot change the final result for the entropy. However, here we have at least two issues that deserve a careful analysis. (a) In the semiclassical calculation of the partition function (1.1), one needs the value of the covariant action at the black hole configuration, which is a divergent expression. The existence of this divergence can be taken care of by a convenient regularization procedure, but one should verify that this procedure is consistent with the adopted boundary conditions [29]. (b) Although one expects a boundary at the horizon in the Hamiltonian formalism [4, 28], it seems that its presence in the covariant formalism of GR can be ignored [29]. This situation needs a consistent explanation.

The purpose of the present work is to clarify the role of the boundaries \( B^\infty \) and \( B^r \) in the covariant description of the black hole entropy, based on the Mielke–Baekler action. In particular, we show that separate contributions stemming from \( B^\infty \) and \( B^r \) do not have an invariant meaning—they depend on the coordinate system used in the calculations, while the invariant physical content can be ascribed only to the complete boundary \( B^\infty \cup B^r \).

The layout of the paper is as follows. In section 2, we present basic aspects of the Euclidean 3D gravity with torsion, including the form of the black hole solution. In section 3, we demonstrate the consistency of the regularization procedure with the grand canonical boundary conditions. Using the standard, ‘rotating’ coordinate system [6, 29], we calculate the covariant grand canonical partition function (1.1) and obtain the correct expression for the black hole entropy, provided the black hole manifold has two boundaries, one at infinity and one at the horizon. Both of these boundaries give nontrivial contributions to the black hole entropy. In section 4, these considerations are extended to a more general class of coordinate systems. While the entropy remains unchanged, as one expects, we find that there exists a particular coordinate system in which the contribution stemming from \( B^r \) vanishes, and consequently, \( B^r \) becomes irrelevant and can be ignored. Similar construction is then carried out for \( B^\infty \). In these coordinate systems, the complete black hole entropy is effectively associated with a single boundary—either \( B^\infty \) or \( B^r \). Section 5 is devoted to concluding remarks, while appendices contain some technical details.

Our conventions are the same as in [22]: the Latin indices \( (i, j, k, \ldots) \) refer to the local orthonormal frame, the Greek indices \( (\mu, \nu, \rho, \ldots) \) refer to the coordinate frame, and both run over \( 0, 1, 2 \); \( \eta_{ij} = (+, +, +) \) are metric components in the local frame; totally antisymmetric tensor \( \varepsilon^{ijk} \) and the related tensor density \( \tilde{\varepsilon}^{\mu\nu\rho} \) are both normalized by \( \varepsilon^{012} = +1 \).

2. Euclidean 3D gravity with torsion

Euclidean 3D gravity with torsion can be formulated as a gauge theory of the Euclidean group \( ISO(3) [22] \). In this approach, basic dynamical variables are the triad field \( b^i \) and the spin connection \( \omega^j \) (1-forms), and the corresponding field strengths are the torsion and the curvature (2-forms): \( T^i := db^i + \varepsilon^{ijk} \omega^j \wedge b^k, R^i := d\omega^j + \frac{1}{2} \varepsilon^{ijk} \omega^j \wedge \omega^k \). The geometric structure of \( ISO(3) \) gauge theory corresponds to Riemann–Cartan geometry [10–12].
2.1. The action integral

Mielke and Baekler proposed a topological model for 3D gravity in Riemann–Cartan spacetime [13], which is a natural generalization of Riemannian GR with a cosmological constant (GR$_\Lambda$). Euclidean version of the model is defined by the action $I_E$, obtained from its Minkowskian counterpart $I_M$ by the process of analytic continuation $I_M \rightarrow iI_E$ [22]. Omitting the subscript $E$ for simplicity, the Euclidean Mielke–Baekler action reads

$$I = aI_1 + \Lambda I_2 + \alpha_3 I_3 + \alpha_4 I_4 + I_m,$$  \hfill (2.1a)

where

$$I_1 = 2 \int b^i \wedge R_i,$$
$$I_2 = -\frac{1}{3} \int \varepsilon_{ijk} b^i \wedge b^j \wedge b^k,$$
$$I_3 = \int \left( \omega^j \wedge d\omega_j + \frac{1}{3} \varepsilon_{ijk} \omega^i \wedge \omega^j \wedge \omega^k \right),$$
$$I_4 = \int b^i \wedge T_i,$$  \hfill (2.1b)

and $I_m$ is a matter contribution. The first two terms are of the same form as in GR$_\Lambda$, $a = 1/16\pi G$ and $\Lambda$ is the cosmological constant, $I_3$ is the Chern-Simons action for the connection, and $I_4$ is a torsion counterpart of $I_1$.

In the sector $\alpha_3 \alpha_4 - a^2 \neq 0$, the vacuum field equations are non-degenerate:

$$2T^i = p\varepsilon^i_{\ jk} b^j \wedge b^k, \quad 2R^i = q\varepsilon^i_{\ jk} b^j \wedge b^k,$$  \hfill (2.2a)

with

$$p := \frac{\alpha_3 \Lambda + \alpha_4 a}{\alpha_3 \alpha_4 - a^2}, \quad q := -\frac{(\alpha_4)^2 + a \Lambda}{\alpha_3 \alpha_4 - a^2}.$$  \hfill (2.2b)

Note that $p$ and $q$ satisfy the identities

$$aq + \alpha_4 p - \Lambda \equiv 0, \quad ap + \alpha_3 q + \alpha_4 \equiv 0.$$  \hfill (2.3)

Introducing the Levi-Civita connection $\tilde{\omega}^i$ by $d\tilde{\omega}^i + \varepsilon^i_{\ jk} \tilde{\omega}^j \wedge b^k = 0$, one can use the field equations to find the Riemannian piece of the curvature $R^i(\tilde{\omega})$ [22]:

$$2R^i(\tilde{\omega}) = \Lambda_{\text{eff}} \varepsilon^i_{\ jk} b^j \wedge b^k, \quad \Lambda_{\text{eff}} := q - \frac{1}{2} p^2,$$  \hfill (2.4)

where $\Lambda_{\text{eff}}$ is the effective cosmological constant. Thus, our spacetime is maximally symmetric with isometry group $SO(3, 1)$, and it is known as the hyperbolic 3D space $H^3$. In what follows, we restrict our attention to the Euclidean continuation of anti-de Sitter space, which is defined by positive $\Lambda_{\text{eff}}$. $\Lambda_{\text{eff}} := 1/\ell^2 > 0$.

2.2. The black hole with torsion

For $\Lambda_{\text{eff}} > 0$, equation (2.4) has a well-known solution for the metric, the Euclidean BTZ black hole [4, 6]. In Schwarzschild coordinates $x^\mu = (t, r, \phi)$, the metric has the form

$$\text{d}s^2 = N^2 \text{d}t^2 + N^{-2} \text{d}r^2 + r^2 (\text{d}\phi + N_\phi \text{d}t)^2,$$
$$N^2 = \left( -8Gm + r^2 \frac{16G^2 J^2}{r^2} \right), \quad N_\phi = -\frac{4GJ}{r^2}.$$  \hfill (2.5)
The zeros of \( N^2, r_+ \) and \( r_- = -i \rho_- \), are related to the black hole parameters \( m \) and \( J \) by the relations \( r_+^2 - \rho_+^2 = 8Gm^2 \), \( r_+\rho_- = 4GJ \), both \( \varphi \) and \( t \) are periodic:

\[
0 \leq \varphi < 2\pi, \quad 0 \leq t < \beta, \quad \beta = \frac{2\pi \ell^2 r_+}{r_+^2 + \rho_-^2},
\]

and the black hole manifold is topologically a solid torus \([6, 27, 28]\).

Starting with the BTZ metric (2.5), one can find the pair \( (b^i, \omega^i) \) which solves the field equations (2.2), and represents the Euclidean black hole with torsion \([22]\). Energy and angular momentum of the solution are

\[
E = m + \frac{\alpha_3}{a} \left( \frac{pm}{2} - \frac{J}{\ell^2} \right), \quad M = J + \frac{\alpha_3}{a} \left( \frac{pJ}{2} + m \right).
\] (2.6)

Instead of using the Schwarzschild coordinates, we shall go over to a new class of coordinate systems, which is suitable for exploring the geometric origin of the black hole entropy. Let us first introduce the ‘rotating’ coordinate system, denoted by \( K_0 \), by

\[
t' := t/\beta, \quad \varphi' := \varphi + \Omega t,
\]

where \( \Omega := N\varphi (r_+) = -\rho_-/\ell r_+ \) is related to the angular velocity of the black hole, and \( \varphi' \) is the usual azimuthal angle \([6, 29]\). Our new class of coordinate systems \( K_w \) is defined as a simple generalization of \( K_0 \):

\[
t'' := t' + w \varphi', \quad \varphi'' := \varphi',
\] (2.7)

where \( w = w(\beta, \Omega) \) is a parameter. Ignoring double primes for simplicity, the black hole solution \( (b^i, \omega^i) \) in \( K_w \) takes the form

\[
\begin{align*}
b^0 &= \beta N(dt - w \, d\varphi), \quad b^1 = N^{-1} \, dr, \\
b^2 &= r [d\varphi + \beta(N\varphi - \Omega)(dt - w \, d\varphi)], \\
\omega^i &= \tilde{\omega}^i + \frac{p}{2} b^i,
\end{align*}
\] (2.8a)

where the Levi-Civita connection \( \tilde{\omega}^i \) is

\[
\begin{align*}
\tilde{\omega}^0 &= N [d\varphi - \beta \Omega (dt - w \, d\varphi)], \\
\tilde{\omega}^1 &= -N^{-1} N\varphi \, dr, \\
\tilde{\omega}^2 &= -\beta \left( \frac{L}{\ell^2} + rN\varphi \Omega \right)(dt - w \, d\varphi) + rN\varphi \, d\varphi.
\end{align*}
\] (2.8c)

3. The black hole entropy in \( K_0 \)

The purpose of the present work is to calculate the gravitational black hole entropy, and find a mechanism by which the entropy is associated with the boundary of the black hole manifold. In this section, we focus our attention to the ‘rotating’ coordinate system \( K_0 \).

Thermodynamic properties of a black hole can be determined by the form of the partition function (1.1) \([24–29]\). The calculation of \( Z(\beta, \Omega) \) is based on the boundary conditions that define the set of allowed field configurations \( \mathcal{C}_L \), satisfying the following properties:

(i) \( \mathcal{C}_L \) contains black holes with \((m, J)\) belonging to a small region around some \((m, J)_0\),

(ii) \( \beta \) and \( \Omega \) are constant on the boundary, and

(iii) there exists a boundary term \( I_B \), such that \( \tilde{I} = I + I_B \) is differentiable on \( \mathcal{C}_L \).
In the lowest order semiclassical approximation around the black hole configuration, the logarithm of the partition function takes the form

\[ \ln Z[\beta, \Omega] = -\tilde{I}_{bh}, \]

where \( \tilde{I}_{bh} \) is the improved Mielke–Baekler action \( \tilde{I} \), evaluated at the black hole (2.8). On the other hand, using the general form of the partition function, we obtain the relation

\[ \tilde{I}_{bh} = \bar{\beta}(E - \mu M) - S, \]

(3.1)

where \( \bar{\beta} = 1/T \) is the inverse temperature, \( \mu \) is the chemical potential corresponding to the angular momentum \( M \), and \( S \) is the black hole entropy. Thus, the essential step in our calculation of the black hole entropy is to find \( \tilde{I}_{bh} \).

### 3.1. Regularization

Before we start calculating \( \tilde{I} \), let us observe that the value of the action (2.1) at the black hole configuration is divergent (appendix A):

\[ I_{bh} \approx \frac{4\pi a\beta}{\ell^2} (r_\infty^2 - r_i^2), \]

(3.2)

where \( r_\infty \to \infty \), and \( \approx \) denotes an on-shell equality. Note that this result is of the same form as in \( \text{GR}_\Lambda \) [29]. One can define a natural regularization procedure by subtracting the value of \( I_{bh} \) at the black hole vacuum, where \( m = J = 0 \) (see also [30]). The regularized action reads

\[ I_{\text{reg}} = I - \frac{4\pi a\beta}{\ell^2} r_\infty^2, \]

(3.3)

and its value at the black hole configuration is finite: \( I_{\text{reg}} \approx -4\pi a\beta r_i^2/\ell^2 \).

Our approach to the black hole thermodynamics relies on the construction of the improved action \( \tilde{I} = I + I_B \), in accordance with the adopted boundary conditions (i)–(iii). This construction is now modified by a new element—the regularization procedure. In order to be sure that the regularization does not spoil the essence of our approach, we have to verify its consistency with the structure of boundary terms.

### 3.2. Boundary terms

Now, we wish to improve the form of \( I \) on the set of allowed field configurations \( C_{\text{L}} \), so that the improved action corresponds to the grand canonical ensemble. The boundary terms in \( \tilde{I} = I + I_B \), are constructed to cancel the unwanted surface terms in \( \delta I \), arising from integrations by parts. In other words, \( I_B \) is defined by the requirement \( \delta (I + I_B) \approx 0 \).

The general variation of the action (2.1) at fixed \( r \) has the following form:

\[ \delta I|_{r} = -\left[ 2\alpha \int b' \wedge \delta \omega_i + \alpha_3 \int \omega' \wedge \delta \omega_i + \alpha_4 \int b' \wedge \delta b_i \right]. \]

(3.4)

The black hole manifold is taken to be a solid torus with two boundaries: one at infinity, and one at the horizon. After completing the calculation, we find out, in contrast to the Riemannian \( \text{GR}_\Lambda \) [29], that the boundary at the horizon \( B^\infty \) is absolutely necessary, otherwise the result for the black hole entropy would be incorrect.

**Spatial infinity.** On the boundary \( B^\infty \) located at spatial infinity, the fields \( b' \) and \( \omega' \) are restricted to the family of black hole configurations (2.8) with \( w = 0 \), \( \beta \) and \( \Omega \) are treated as independent parameters, but their ‘on-shell’ values \( \beta = 2\pi \ell^2 r_i/(r_i^2 + \rho_i^2) \) and \( \Omega = N_\Lambda(r_i) \) are used at the end of calculation, in order to avoid conical singularities [6, 29]. The variation
of the action $I$ at infinity is calculated in appendix B. The first term in (B.1) is just the term needed in the regularization procedure, so that the complete result can be rewritten as

$$\delta I_{\text{reg}}|_{r\to\infty} = -\delta(\beta m) + E\delta\beta - M\delta(\beta\Omega).$$  \hspace{1cm} (3.5)

Consequently,

(a) the regularization procedure is consistent with the structure of boundary terms at infinity.

The horizon. Looking at the black hole solution at $r = r_+$, we find

$$b^0 = 0, \quad b^2 = r_+ d\varphi,$$

$$\bar{a}^0 = 0, \quad \bar{a}^2 = -2\pi dt + r_+ N_\varphi(r_+) d\varphi,$$

where we used $\beta(r_+\ell^2 + r_+\Omega N_\varphi(r_+)) = 2\pi$. These relations imply

$$b^\alpha_0 = 0, \quad a^\alpha_0 = -2\pi \delta^\alpha_2 \quad (a = 0, 2).$$  \hspace{1cm} (3.6a)

After going back to the Schwarzschild coordinates, one finds that the above conditions coincide with those given in equation (5.5) of [22].

By using relations (3.6) as the boundary conditions at the horizon, the variation of the regularized action at the horizon has the form (B.2):

$$\delta I_{\text{reg}}|_{r = r_+} = -2\pi^2 \alpha_3 \delta \left( pr_+ - 2\frac{\rho_+}{\ell} \right).$$  \hspace{1cm} (3.6b)

Thus, the total variation of the regularized action is

$$\delta I_{\text{reg}} = \delta I_{\text{reg}}|_{r\to\infty} - \delta I_{\text{reg}}|_{r = r_+} = -\delta(\beta m) + E\delta\beta - M\delta(\beta\Omega) + 2\pi^2 \alpha_3 \delta \left( pr_+ - 2\frac{\rho_+}{\ell} \right).$$  \hspace{1cm} (3.7)

We see that $I_{\text{reg}}$ is not differentiable, but this can be easily corrected.

Grand canonical action. Consider the improved action

$$\tilde{I} = I_{\text{reg}} + \beta m - 2\pi^2 \alpha_3 \left( pr_+ - 2\frac{\rho_+}{\ell} \right),$$  \hspace{1cm} (3.9)

the variation of which has the form

$$\delta \tilde{I} = E\delta\beta - M\delta(\beta\Omega).$$  \hspace{1cm} (3.10)

Since $\delta \tilde{I}$ vanishes when $\beta$ and $\Omega$ are fixed, $\tilde{I}$ is differentiable, and moreover, it represents the grand canonical action.

3.3. Entropy

Once the grand canonical action is constructed, we can easily find the black hole entropy. The value of the action (3.9) at the black hole is

$$\tilde{I}_{\text{bh}} = -\pi r_+ \frac{4G}{2\pi^2 \alpha_3} \left( pr_+ - 2\frac{\rho_+}{\ell} \right).$$  \hspace{1cm} (3.11)

The last term in this expression represents the contribution from the boundary at the horizon. Using the generalized Smarr formula in Riemann–Cartan spacetime

$$\beta(E - \Omega M) \approx \frac{\pi r_+}{4G} + 2\pi^2 \alpha_3 \left( pr_+ - 2\frac{\rho_+}{\ell} \right),$$
we easily find that the above result can be rewritten in the form
\[ \tilde{I}_{\text{bh}} = \beta(E - \Omega M) - S, \] (3.12)
where \( \beta \) and \( \Omega \) take their ‘on-shell’ values, and
\[ S = \frac{2\pi r_s}{4G} + 4\pi^2 \alpha_3 \left( pr_+ - \frac{\rho}{\ell} \right). \] (3.13)
Comparing this result with the expected form of \( \tilde{I}_{\text{bh}} \), given by equation (3.1), we come to the following thermodynamic interpretation: \( \beta \) is the inverse temperature, \( \Omega \) is the thermodynamic potential corresponding to \( M \), and \( S \) is the black hole entropy. The above formula for the black hole entropy coincides with the results obtained in [22, 23]; in particular, it is in perfect agreement with the first law of black hole thermodynamics.

(b) The boundary at the horizon produces the last term in (3.11), and consequently, its contribution to the black hole entropy is essential.

In GR\( \Lambda \), where \( \alpha_3 = 0 \), the last term in (3.11) vanishes and the boundary at the horizon can be safely ignored, as has been observed in [29]. More generally, this is true whenever the Chern-Simons term in the action is absent \( (\alpha_3 = 0) \). On the other hand, whenever \( \alpha_3 \neq 0 \), even in Riemannian theory \( (\rho = 0) \) [31], the boundary at the horizon yields a nontrivial contribution, and its presence cannot be disregarded.

Let us stress that these results hold in the ‘rotating’ coordinate system \( K_0 \). In the next section, we will extend our discussion to \( K_w \).

### 4. The black hole entropy in \( K_w \)

In this section, we show that the black hole entropy remains unchanged when we generalize our considerations to an arbitrary coordinate system of the type \( K_w \). In order to clarify the dynamical role of boundaries, we construct two specific coordinate systems, in which the complete contribution to the black hole entropy comes from a single boundary, \( B^\infty \) or \( B^r \).

#### 4.1. Boundary terms and entropy

**Spatial infinity.** Using the general result (3.4), the variation of \( I_{\text{reg}} \) at infinity around the black hole solution (2.8) has the form
\[ \delta I_{\text{reg}}|_{r \to \infty} = -\delta(\beta m) + E \delta \beta - M \delta (\beta \Omega) - \beta^2 \left[ M \left( \frac{1}{\ell^2} - \Omega^2 \right) + 2\Omega E \right] \delta w. \]

The last term can be simplified by using the ‘on-shell’ equality
\[ \beta^2 \left[ M \left( \frac{1}{\ell^2} - \Omega^2 \right) + 2\Omega E \right] = 8\pi^2 \alpha_3, \]
which leads to
\[ \delta I_{\text{reg}}|_{r \to \infty} \approx -\delta(\beta m) + E \delta \beta - M \delta (\beta \Omega) - 8\pi^2 \alpha_3 \delta w. \] (4.1)
Comparing this expression with the result (3.5) valid in \( K_0 \), we see that the only difference comes from the last term in (4.1).

**The horizon.** The black hole solution (3.8) at the horizon \( r = r_+ \) is given by
\[
\begin{align*}
b^0 &= 0, & b^2 &= r_+ \, d\varphi, \\
\tilde{\omega}^0 &= 0, & \tilde{\omega}^2 &= -2\pi \, dt + [r_+ N_\varphi(r_+) + 2\pi w] \, d\varphi.
\end{align*}
\]
As a consequence, we find that relations (3.6b) hold for every $w$. The variation of $I_{\text{reg}}$ at the horizon yields

$$
\delta I_{\text{reg}}|_r = -2\pi^2\alpha_3\delta\left(pr_r - 2\frac{\rho_r}{\ell}\right) - 8\pi^3\alpha_3\delta w. \tag{4.2}
$$

Since the last, $w$-dependent terms in (4.1) and (4.2) are equal, their contribution to $\delta I_{\text{reg}}$ is cancelled. Consequently, $\delta I_{\text{reg}}$ is the same as in (3.8), the improved (grand canonical) action is of the form (3.9), and we end up with the same formula (3.13) for the black hole entropy, as expected.

(c) The black hole entropy remains the same in every coordinate system in $K_w$.

### 4.2. The analysis of two particular cases

Now, we wish to analyse the isolated contributions coming from the boundaries $B^{\infty}$ and $B^r$, in two particular coordinate systems.

1. If we choose the parameter $w$ so that

$$
\left(pr_r - 2\frac{\rho_r}{\ell}\right) + 4\pi w = 0, \tag{4.3}
$$

the variation of $I_{\text{reg}}$ at the horizon vanishes, and the complete variation is determined by the boundary at infinity. This implies that both the improved action and the entropy are completely determined by the contributions from $B^{\infty}$.

(d) In the coordinate system defined by the condition (4.3), the complete contribution to the black hole entropy is determined by the boundary at infinity. In this sense, the boundary $B^r$ is superfluous and can be ignored.

This result explains the mechanism used in [29] for GR/\Lambda (in the ‘rotating’ coordinate system), and extends it to the more general Mielke–Baekler model. It tells us that the contribution of the complete boundary can be effectively reduced just to $B^{\infty}$, which is an effect inseparably connected with the specific coordinate system.

The effect just described may help us to better understand the relation between (i) the present approach based on the gravitational partition function, and (ii) the approach based on the Cardy formula [8, 23]. Namely, it seems that the latter approach needs only one boundary, the boundary at infinity, where all elements of the Cardy formula are calculated. However, such an assumption would lead to problems with physical interpretation (see, for instance, the last reference in [1]). It is not clear that the existence of one boundary in (ii) is a genuine geometric fact. It might be the result of an effective description, related, for instance, to the specific choice of coordinates. Without having a deeper geometric and physical understanding of the Cardy formula, we cannot properly compare the geometric content of (i) and (ii).

2. Alternatively, we can choose $w$ so that the variation of $I_{\text{reg}}$ vanishes at infinity, for fixed $\beta$ and $\Omega$:

$$
\beta m + 8\pi^3\alpha_3 w = 0. \tag{4.4}
$$

The complete variation of $I_{\text{reg}}$ is now determined by the boundary at the horizon, while $B^{\infty}$ can be effectively ignored.

(e) In the coordinate system defined by the condition (4.4), the complete contribution to the black hole entropy is determined by the boundary at the horizon.

It should be noted that the condition (4.4) cannot be realized in GR/\Lambda, where $\alpha_3 = 0$.

There are arguments that the most natural location for the dynamical degrees of freedom of the black hole is the horizon [32]. Clearly, one should ensure that any realization of such an idea is based on genuine geometric considerations.
5. Concluding remarks

We investigated thermodynamic properties of the black hole with torsion using the covariant form of the action in the grand canonical partition function.

(1) The regularization procedure, needed for a consistent treatment of the divergent value of action at the black hole configuration, is shown to be consistent with the boundary conditions corresponding to the grand canonical partition function.

(2) According to the calculations in the standard coordinate system $K_0$, the expression for the black hole entropy has the correct value provided the black hole manifold has not only the boundary at infinity, but also the boundary at the horizon. The value of the black hole entropy remains the same in every coordinate system $K_w$.

(3) In the specific coordinate system (4.3), the complete contribution to the black hole entropy stems from the boundary at infinity. This mechanism explains the nature of the corresponding result in GR$_{\Lambda}$ [29]. Moreover, it suggests that a similar analysis of the Cardy formula could help us to properly understand the underlying boundary geometry, and verify its consistency with the present approach.

(4) Similarly, the complete contribution in the coordinate system (4.4) stems from the boundary at the horizon. Such a coordinate system cannot be realized in GR$_{\Lambda}$, where $\alpha_3 = 0$.

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Appendix A. The value of the action

Using the field equations (2.2), the values of the four pieces of the action (2.1) at the black hole configuration (2.8) are given as follows:

\[ I_1 = q\varepsilon_{ijk} \int b^i \wedge b^j \wedge b^k = 6q\pi\beta r^2 |_{r_+} \]

\[ I_2 = -\frac{1}{3} \varepsilon_{ijk} \int b^i \wedge b^j \wedge b^k = -2\pi\beta r^3 |_{r_+} \]

\[ I_3 = \frac{1}{2} q\varepsilon_{ijk} \int \omega^i \wedge b^j \wedge b^k - \frac{1}{6} \varepsilon_{ijk} \int \omega^i \wedge \omega^j \wedge \omega^k = pq\pi\beta r^2 |_{r_+} \]

\[ I_4 = \frac{1}{2} p\varepsilon_{ijk} \int b^i \wedge b^j \wedge b^k = 3p\pi\beta r^2 |_{r_+} \]

The value of the complete action reads

\[ I_{bh} = (6aq - 2\Lambda + \alpha_3 pq + 3\alpha_4 p)\pi\beta r^2 |_{r_+} = \frac{4a\pi\beta}{\ell^2} \left( r_+^2 - r_+^2 \right) \]

where we used the identities (2.3) and the relation $q - p^2/4 = 1/\ell^2$. The result is the same for every coordinate system in $K_w$.

Appendix B. Variation of the action

The variation of the action (2.1) around the black hole configuration (2.8) produces two boundary terms, one at infinity and one at the horizon. The calculation is carried out in the ‘rotating’ coordinate system $K_0$. 
The contribution from the boundary at infinity is determined by the relations

\[-2a \int b^i \wedge \delta \alpha_i |_{r \to \infty} = \frac{4\pi a}{\ell^2} r_\infty^2 \delta \beta - \beta \delta m - J \delta (\beta \Omega) + apY,\]

\[-\alpha_3 \int \omega^i \wedge \delta \alpha_i |_{r \to \infty} = (E - m) \delta \beta - (M - J) \delta (\beta \Omega) + \alpha_3 qY,
\]

\[-\alpha_4 \int b^i \wedge \delta b_i |_{r \to \infty} = \alpha_4 Y,\]

where \( Y = \frac{2\pi r_\infty^2}{\ell^2} \delta (\beta \Omega) + \delta (\beta J)/2a. \) Using the second identity in (2.3), the sum of these contributions yields

\[\delta I |_{r \to \infty} = \frac{4\pi a}{\ell^2} r_\infty^2 \delta \beta - \delta (\beta m) + E \delta \beta - M \delta (\beta \Omega),\]  

(B.1)

which is equivalent to (3.5).

The contribution from the boundary at the horizon has the form (3.7):

\[\delta I |_{r = r_+} = -\alpha_3 \int \omega^i \wedge \delta \alpha_i |_{r = r_+} = -2\pi^2 \alpha_3 \delta \left( pr_+ - \frac{\rho_+}{\ell} \right).\]  

(B.2)

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