BRST Reduction of Quantum Algebras with *-Involutions

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Abstract: In this paper we investigate the compatibility of the BRST reduction procedure with the Hermiticity of star products. First, we introduce the generalized notion of abstract BRST algebras with corresponding involutions. In this setting we define adjoint BRST differentials and as a consequence one gets new BRST quotients. Passing to the quantum BRST setting we show that for compact Lie groups the new quantum BRST quotient and the quantum BRST cohomology are isomorphic in zero degree implying that reduction is compatible with Hermiticity.

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1. Introduction

The aim of this work is to investigate the compatibility of the BRST reduction in deformation quantization, as introduced in [4], with the Hermiticity of star products. Deformation quantization as introduced in [1] by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer relies on the idea that the quantization of a symplectic or Poisson manifold $M$...
representing the phase space of a classical mechanical system is described by a formal deformation of the commutative algebra of smooth complex-valued functions $\mathcal{C}\infty(M)$. Explicitly, one defines a **star product** $\star$ on $M$ being a $\mathbb{C}[[\hbar]]$-bilinear associative product on $\mathcal{C}\infty(M)[[\hbar]]$ of the form

$$
f \star g = f \cdot g + \sum_{r=1}^{\infty} \hbar^r C_r(f, g) \quad (1.1)
$$

for any $f, g \in \mathcal{C}\infty(M)[[\hbar]]$, where $C_1(f, g) - C_1(g, f) = i\{f, g\}$ and where all the terms $C_r$ are bidifferential operators vanishing on constants. Here the formal parameter $\hbar$ is supposed to be real. Thus the quantum observables are described by the non-commutative algebra $\mathcal{C}\infty(M)[[\hbar], \star]$. In order to get a *-algebra structure on the quantum observables we need to consider a *-involution for the star product. One calls the star product **Hermitian** if the complex conjugation is an involution, i.e. if $f \star g = g \star f$ for all $f, g \in \mathcal{C}\infty(M)[[\hbar]]$. The existence and classification of general star products on Poisson manifolds has been provided by Kontsevich’s famous formality theorem [18], and the existence of Hermitian star products on symplectic manifolds was shown in [27,28].

At the classical level it is possible to perform, under fairly general conditions, the phase space reduction which constructs from the original phase space $M$ one of a smaller dimension denoted by $M_{\text{red}}$, see e.g. [22]. More precisely, suppose that a Lie group $G$ acts by symplectomorphisms resp. Poisson diffeomorphisms and that it allows an $\text{Ad}^*$-equivariant momentum map $J: M \to \mathfrak{g}^*$ with $0 \in \mathfrak{g}^*$ as value and regular value, where $\mathfrak{g}$ denotes the Lie algebra of $G$. Then $C = J^{-1}(\{0\})$ is a closed embedded submanifold of $M$, called regular constraint surface. If the action is in addition proper and free, then the reduced manifold $M_{\text{red}}$ given by the orbit space $C/G$ is again a symplectic resp. Poisson manifold.

In the setting of deformation quantization a quantum reduction scheme has been introduced in [4], see also [11] for a more categorical approach to reduction in both the quantum and classical setting. One of the crucial ingredients is the notion of quantum momentum maps [34]. Given a star product on $M$, a quantum momentum map is a map $J = J + \sum_{r=1}^{\infty} \hbar^r J_r: M \to \mathfrak{g}^*[[\hbar]]$ into the formal series of smooth functions on $M$ such that

$$J(\xi) \star f = f \star J(\xi) = i\hbar J(\xi, f) \quad \text{and} \quad J(\xi) \star J(\eta) - J(\eta) \star J(\xi) = i\hbar J(\{\xi, \eta\}) \quad (1.2)$$

for all $\xi, \eta \in \mathfrak{g}$ and $f \in \mathcal{C}\infty(M)[[\hbar]]$. Here $J(\xi) = \langle J, \xi \rangle$ denotes the pointwise dual pairing, see [14,25]. The map $J$ is called **quantum momentum map** and the pairs $(\star, J)$ are called **equivariant star products**, see also [29–31] for a classification in the symplectic setting.

The BRST approach provides then a tool to construct a reduced star product $\star_{\text{red}}$ on $M_{\text{red}}$ that is induced by the equivariant star product $(\star, J)$ on $M$ and thus implies that the deformation quantization is compatible with the classical phase space reduction. Here the abbreviation BRST stands for the particle physicists Becchi, Rouet, Stora [2] and Tyutin [32] who investigated gauge invariances by introducing new variables, the “ghosts” and “antighosts”, see also [16] for further applications in physics. Kostant and Sternberg [19] transferred this idea to the setting of symplectic resp. Poisson geometry, introducing the classical BRST algebra $\mathcal{A}^{\bullet \bullet} = \Lambda^* \mathfrak{g}^* \otimes \Lambda^* \mathfrak{g} \otimes \mathcal{C}\infty(M)$ with ghost number grading $\mathcal{A}(n) = \bigoplus_{n = k - \ell} \mathcal{A}^{k, \ell}$ and a corresponding super Poisson structure induced by the natural pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$. The two characteristic features of the classical
BRST algebra are the classical BRST operator \( D : \mathcal{A}(\bullet) \rightarrow \mathcal{A}(\bullet^{+1}) \), satisfying \( D^2 = 0 \), and the ghost number derivation \( \text{Gh} \) inducing the ghost number grading. With these notions it was shown that one has the following isomorphism of Poisson algebras

\[
\mathcal{H}_{\text{BRST}}^{(0)}(\mathcal{A}) \cong \mathcal{C}^\infty(M_{\text{red}}), \tag{1.3}
\]

where the classical BRST cohomology \( \mathcal{H}_{\text{BRST}}^{(\bullet)}(\mathcal{A}) \) is the cohomology of \( (\mathcal{A}(\bullet), \{\cdot, \cdot\}, D) \), see also [13]. As mentioned above, Bordemann, Herbig and Waldmann [4] transferred this result to the setting of deformation quantization and constructed the standard ordered quantum BRST algebra \( (\mathcal{A}(\bullet)[[\lambda]], \star_{\text{std}}, D_{\text{std}}) \) as formal deformation of the classical BRST algebra. In particular, they proved the quantum analogue of (1.3), namely

\[
\mathcal{A}_{\text{red}} = \mathcal{H}_{\text{BRST}}^{(0)}(\mathcal{A}[[\lambda]]) \cong \mathcal{C}^\infty(M_{\text{red}})[[[\lambda]]]. \tag{1.4}
\]

Here \( \mathcal{H}_{\text{BRST}}^{(\bullet)}(\mathcal{A}[[\lambda]]) \) denotes the cohomology of the quantized BRST algebra, the so-called quantum BRST cohomology, and the ghost number zero part \( \mathcal{A}_{\text{red}} \) is called reduced quantum BRST algebra. The above construction induces a star product \( \star_{\text{red}} \) on the reduced manifold, but if the star product on \( M \) is Hermitian, the construction does not yield an involution for it. The main problem here is that in general homological algebra is not compatible with involutions and positive definite inner products. Therefore, the new question addressed in this work is whether one can modify the BRST reduction in such a way that it gives in addition an induced *-involution for the reduced star product. Note that there is also a different way to construct involutions for \( \star_{\text{red}} \) via *-representations, see [3,15]. A more general reduction scheme can also be found in [10].

To this end we introduce a notion of abstract BRST algebras and investigate various concepts of involutions that are compatible with the gradings. We show that graded *-involutions with imaginary ghost operator are the best suited involutions as they guarantee the existence of non-trivial *-representations on pre-Hilbert spaces, which is necessary from the physical point of view to encode for example the superposition principle. Applying these abstract results to the setting of deformation quantization, we construct such an involution for the quantum BRST algebra \( \mathcal{A}[[\lambda]] \) by means of a positive definite inner product on the Lie algebra as additional information. In this case, we prove that the so constructed *-algebra has sufficiently many positive linear functionals in the sense of [6–9], guaranteeing a non-trivial *-representation theory via GNS representations, see also [5]. Finally, we introduce the adjoint quantum BRST operator \( D^*_{\text{std}} \) and the quantum BRST quotient

\[
\widehat{\mathcal{H}}_{\text{BRST}}^{(\bullet)}(\mathcal{A}[[\lambda]]) = \frac{\ker D_{\text{std}} \cap \ker D^*_{\text{std}}}{\im D_{\text{std}} \cap \im D^*_{\text{std}}}. \tag{1.5}
\]

We show for compact Lie groups that its zero-th order is isomorphic to the reduced BRST algebra, i.e.

\[
\widehat{\mathcal{H}}_{\text{BRST}}^{(0)}(\mathcal{A}[[\lambda]]) \cong \mathcal{A}_{\text{red}} \cong \mathcal{C}^\infty(M_{\text{red}})[[[\lambda]]]. \tag{1.6}
\]

The crucial ingredient in the proof is a \( \mathcal{C}^\infty(\mathbb{C}[[\lambda]]) \)-valued inner product, similarly to algebra-valued inner products on Hilbert-modules [21], but over \( \mathbb{C}[[\lambda]] \) as in [15]. In particular, this isomorphism induces the complex conjugation as involution for \( \star_{\text{red}} \), hence the BRST reduction of Hermitian star products yields in this setting Hermitian reduced star products. In other words, we show that \( \mathcal{H}_{\text{BRST}}^{(\bullet)}(\mathcal{A}[[\lambda]]) \) and \( \widehat{\mathcal{H}}_{\text{BRST}}^{(\bullet)}(\mathcal{A}[[\lambda]]) \) are isomorphic in ghost number zero if the Lie group acting on \( M \) is compact, which provides a large class of examples for the physically relevant invariants.
The paper is organized as follows: In Sect. 2 we recall the basics concerning the classical BRST algebra and its counterpart in deformation quantization. Then we introduce in Sect. 3 the notion of abstract BRST algebras and look for compatible involutions and their \(^{*}\)-representation theory. Having found a suitable concept of involutions we apply this idea in Sect. 4 at first to the Grassmann part and then finally to the quantum BRST algebra. The results of this paper are partially based on the master thesis [20].

2. Preliminaries

2.1. The classical BRST complex and cohomology. In this section we recall the description of the classical Marsden-Weinstein reduction via the classical BRST cohomology in order to establish the notation. We refer to [4,13,19].

Let us consider a Hamiltonian G-space \((M, G, J)\) consisting in a symplectic or Poisson manifold \((M, \omega)\) resp. \((M, \pi)\) and a Hamiltonian action \(\Phi : G \times M \rightarrow M\) with momentum map \(J\). It is well-known that the quotient \(M_{\text{red}} = C/G\), where \(C = J^{-1}(0)\) with 0 being a regular value of \(J\), inherits a symplectic resp. Poisson structure from \(M\) if the action is free and proper, see [22]. In addition, we can identify \(\mathcal{C}^{\infty}(M_{\text{red}})\) with \(\mathcal{C}^{\infty}(C)^{G}\). From now on we call \((M, G, J, C)\) Hamiltonian G-space with regular constraint surface and we denote by \(\iota : C \rightarrow M\) the canonical embedding and by \(\mathcal{J}_{C} = \ker \iota^{*}\) the vanishing ideal of \(C\). Using a tubular neighbourhood one can construct a prolongation map

\[
\text{prol} : \mathcal{C}^{\infty}(C) \rightarrow \mathcal{C}^{\infty}(M) \tag{2.1}
\]

with \(\iota^{*}\text{prol} = \text{id}|_{\mathcal{C}^{\infty}(C)}\), see [4, Lemma 2]. This yields in particular \(\mathcal{C}^{\infty}(C) \cong \mathcal{C}^{\infty}(M)/\mathcal{J}_{C}\).

Note that if the action is proper on \(M\), then the prolongation map can even be chosen to be \(G\)-equivariant.

We aim to give another description of the Poisson algebra \(\mathcal{C}^{\infty}(M_{\text{red}})\). Let us consider the \(\mathbb{Z} \times \mathbb{Z}\)-graded vector space

\[
\mathcal{A}^{\bullet, \bullet} = \Lambda^{\bullet} g^{*} \otimes \Lambda^{\bullet} g \otimes \mathcal{C}^{\infty}(M), \tag{2.2}
\]

where the gradings are also called ghost and antighost degree. Then \(\mathcal{A}\) carries a natural \(\mathbb{Z}_{2}\)-graded vector space structure \(\mathcal{A} = \Lambda^{\text{even}}(g^{*} \oplus g) \otimes \mathcal{C}^{\infty}(M) \oplus \Lambda^{\text{odd}}(g^{*} \oplus g) \otimes \mathcal{C}^{\infty}(M)\).

The \(\mathbb{Z}\)-grading

\[
\mathcal{A}^{(n)} = \bigoplus_{n=k-\ell} \mathcal{A}^{k, \ell}, \tag{2.3}
\]

is called the ghost number or total degree. In particular, the ghost number grading and the \(\mathbb{Z} \times \mathbb{Z}\)-grading induce the same \(\mathbb{Z}_{2}\)-grading, so the notions of super derivations with respect to the \(\mathbb{Z}_{2}\)-grading and of graded derivations with respect to the ghost number grading coincide. With the \(\wedge\)-product of forms \((\alpha \otimes \xi) \wedge (\beta \otimes \eta) = (-1)^{k\ell}(\alpha \wedge \beta) \otimes (\xi \wedge \eta)\) for \(\alpha \in \Lambda^{k} g^{*}, \beta \in \Lambda^{\ell} g^{*}, \xi \in \Lambda^{k} g\) and \(\eta \in \Lambda^{\ell} g\) and the pointwise product of functions, \(\mathcal{A}\) becomes an associative, super-commutative algebra that is graded with respect to all the above mentioned degrees. The element \(1 \otimes 1 \otimes 1\) is a unit and one has the following differentials:

- The vertical differential is the Chevalley–Eilenberg differential

\[
\delta : \Lambda^{\bullet} g^{*} \otimes \Lambda^{\bullet} g \otimes \mathcal{C}^{\infty}(M) \rightarrow \Lambda^{\bullet+1} g^{*} \otimes \Lambda^{\bullet} g \otimes \mathcal{C}^{\infty}(M), \tag{2.4}
\]

where the representation of \(g\) on \(\Lambda^{\bullet} g \otimes \mathcal{C}^{\infty}(M)\) is defined by

\[
g \ni \xi \mapsto \rho(\xi) = \text{id}(\xi) \otimes \text{id} + \text{id} \otimes \{ J(\xi), \cdot \} \in \text{End}(\Lambda^{\bullet} g \otimes \mathcal{C}^{\infty}(M)). \tag{2.5}
\]

The corresponding cohomology is denoted by \(H_{\text{CE}}^{\bullet}(\mathcal{A})\).
• The horizontal differential \( \partial : A^{\bullet, \bullet} \rightarrow A^{\bullet, \bullet-1} \) is the extended Koszul differential, explicitly given by \( \partial(\alpha \otimes x \otimes f) = (-1)^k \alpha \otimes i(J)(x \otimes f) \) for all \( \alpha \in \Lambda^k g^*, x \in \Lambda^\bullet g \) and \( f \in \mathscr{C}^\infty(M) \). Here \( i(J) \) means the left insertion of \( J \), i.e. the standard interior product.

One can show that \((A^{\bullet, \bullet}, \partial, \delta)\) is a double complex and that the total differential

\[
D = \delta + 2\partial : A^{(\bullet)} \rightarrow A^{(\bullet + 1)}
\]

is a well-defined coboundary operator on the total complex \( A^{(\bullet)} \), the so-called classical BRST operator, see [4, Section 4]. Note that the factor 2 in front of the Koszul differential in (2.6) is just a convention and that the ghost and antighost degrees are not respected by \( D \), but the total degree is.

It turns out that \( A^{(\bullet)} \) has also a natural super Poisson structure induced by the natural pairing of \( g \) and \( g^* \). Concerning the compatibility of this super Poisson structure with the grading and the BRST operator one finds the following properties:

• Let \( 2\nu \) be the identity endomorphism of \( g \), regarded as an element \( \gamma = \frac{1}{2} e^\nu \wedge e_\alpha \in A^{1,1} \) in terms of a basis \( e_1, \ldots, e_n \) of \( g \) with dual basis \( e^1, \ldots, e^n \). Then the ghost number grading of \( A^{(\bullet)} \) is induced by the ghost number derivation \( \Phi = \{ \gamma, \cdot \} \), i.e. \( \phi \in A^{(k)} \) if and only if \( \Phi \phi = k \phi \). The element \( \gamma \in A^{(0)} \) is called ghost charge.

• The total differential \( D \) fulfills \( D = \{ \Theta, \cdot \} \) with \( \Theta = \Omega + J \) and \( \Omega = -\frac{1}{2} [\cdot, \cdot] = -\frac{1}{4} f_{jk} e^j \wedge e^k \wedge e_i \), where \( f_{jk} \) are the structure constants of \( g \). In particular, the classical BRST operator is an inner Poisson derivation of degree 1 and the odd element \( \Theta \in A^{(1)} \) is called classical BRST charge.

Summarizing, one calls the differential \( \mathbb{Z} \)-graded super Poisson algebra \((A^{(\bullet)}, D, \{ \cdot, \cdot \})\) classical BRST algebra, and the corresponding cohomology \( H^{(\bullet)}_{BRST}(A) = \ker D/\text{im} D \) classical BRST cohomology. Since the classical BRST operator is an inner Poisson derivation, it immediately follows that \( H^{(\bullet)}_{BRST}(A) \) inherits a \( \mathbb{Z} \)-graded super Poisson structure from the classical BRST algebra. Moreover, \( [1] \in H^{0}_{BRST}(A) \) is a unit with respect to the \( \wedge \)-product, see [4, Lemma 9]. It has been proved that in ghost number zero one has the isomorphism

\[
H^{(0)}_{BRST}(A) \cong H^{0}_{CE}(g, \mathscr{C}^\infty(C)) \cong \mathscr{C}^\infty(M_{\text{red}})
\]

of Poisson algebras, inducing a Poisson structure on the reduced manifold, see [4, Prop. 10].

2.2. The quantum BRST complex and cohomology. In a similar fashion, now one can perform all the above constructions in the framework of deformation quantization, where we follow again [4]. The underlying vector space of the quantum BRST algebra are the formal power series \( A^{(\bullet)}[[\lambda]] \) with values in the classical BRST algebra, inheriting all the gradings of \( A \).

Let \((M, G, J)\) be a Hamiltonian \( G \)-space with star product \( \ast \). A quantum momentum map is a formal series \( J = \sum_{r=0}^\infty \lambda^r J_r : M \rightarrow g^*[[\lambda]] \) of smooth functions \( J_r : M \rightarrow g^* \) such that \( J_0 = J \) and such that \( J \) satisfies

\[
J(\xi) \ast J(\eta) - J(\eta) \ast J(\xi) = i\lambda J([\xi, \eta]) \quad \text{and} \quad J(\xi) \ast f - f \ast J(\xi) = i\lambda [J(\xi), f]
\]

(2.8)
for all \( \xi, \eta \in \mathfrak{g} \) and \( f \in \mathcal{C}^\infty(M)[[\lambda]] \). The pair \((\star, J)\) is called \textit{equivariant star product}. The first property in (2.8) is also called \textit{quantum covariance} and ensures that the quantum momentum map is a morphism of Lie algebras. Moreover, it implies that a Lie algebra representation \( \rho_M \) is given by
\[
\rho_M(\xi) = \frac{1}{i\lambda} \text{ad}(J(\xi)),
\]
for \( \xi \in \mathfrak{g} \). The second property implies that the star product is also \textit{\( G \)-invariant}, i.e. satisfies \( \Phi^*_g(f \star h) = \Phi^*_g(f) \star \Phi^*_g(h) \) for all \( g \in G, f, h \in \mathcal{C}^\infty(M)[[\lambda]] \), and that \( \rho_M \) coincides with the classical action \( \rho_M(\xi) = -\mathcal{L}_{\xi_M} \). The quadruple \((M, \star, G, J)\) is called \textit{Hamiltonian quantum \( G \)-space} if \((M, G, J)\) is a Hamiltonian \( G \)-space with equivariant star product \((\star, J)\). Similarly, we call \((M, \star, G, J, C)\) a \textit{Hamiltonian quantum \( G \)-space with regular constraint surface} if \((M, G, J, C)\) is a Hamiltonian \( G \)-space with regular constraint surface and equivariant star product \((\star, J)\). Recall that in the symplectic case one can construct for every proper and strongly Hamiltonian group action an equivariant star product \((\star, J)\), i.e. with \( J = J \), see \cite[Sect. 5.8]{12}. Such star products are also called \textit{strongly invariant}.

Therefore, we assume from now on that \((M, \star, G, J, C)\) is a Hamiltonian quantum \( G \)-space with regular constraint surface. A quantized version of the Grassmann part \( \Lambda^*(\mathfrak{g}^* \oplus \mathfrak{g}) \) can be constructed in the following way, see \cite[Sect. 5]{4}.

Let \( \mu \) denote the \( \wedge \)-product. The \textit{standard ordered star product} \( \circ_{\text{std}} \) for \( \Lambda^*(\mathfrak{g}_C^* \oplus \mathfrak{g}_C)[[\lambda]] \) is defined by
\[
\mu \circ_{\text{std}} b = \mu \circ e^{2i\lambda P^*} a \otimes b
\]
with \( P^* = j(e^k) \otimes i(e_k) \), where \( j \) denotes the right insertion and \( i \) the left insertion. It is a formal deformation of the \( \wedge \)-product: it is a \( \mathbb{C}[[\lambda]] \)-bilinear, associative map such that for all homogeneous \( a, b \in \Lambda^*(\mathfrak{g}_C^* \oplus \mathfrak{g}_C)[[\lambda]] \) one has \( 1 \circ_{\text{std}} a = a = a \circ_{\text{std}} 1 \) and
\[
a \circ_{\text{std}} b = a \wedge b + \sum_{r=1}^{\infty} \lambda^r C_r(a, b)
\]
with \( C_1(a, b) = (\lambda | a||b|) C_1(b, a) = i[a, b] \).

Tensoring the standard ordered star product on the Grassmann part with the equivariant star product \((\star, J)\) for the functions, we obtain an associative product \( \star_{\text{std}} \) for \( \mathcal{A}[[\lambda]] \). Explicitly, we have for \( a, b \in \Lambda^*(\mathfrak{g}_C^* \oplus \mathfrak{g}_C)[[\lambda]] \) and \( f, g \in \mathcal{C}^\infty(M)[[\lambda]] \)
\[
(a \otimes f) \star_{\text{std}} (b \otimes g) = (a \circ_{\text{std}} b) \otimes (f \star g).
\]
In analogy to the classical case one defines the \textit{standard ordered quantum BRST charge} by
\[
\Theta_{\text{std}} = \Theta + J + i\lambda \chi.
\]
Here \( \chi \in \mathfrak{g}^* \subset \mathcal{A}^{(1)}[[\lambda]] \) is defined by \( \chi(\xi) = \frac{1}{i\lambda} \text{tr}(\text{ad}(\xi)) \) for all \( \xi \in \mathfrak{g} \), whence \( \Theta_{\text{std}} \) coincides in the zero-th order of \( \lambda \) with the classical \( \Theta \). One can compute \( \Theta_{\text{std}} \star_{\text{std}} \Theta_{\text{std}} = 0 \). Consequently, the \textit{standard ordered quantum BRST operator} is given by
\[
D_{\text{std}} = \frac{1}{i\lambda} \text{ad}_{\text{std}}(\Theta_{\text{std}}),
\]
where \( \text{ad}_{\text{std}} \) denotes the taking of the super commutator with respect to the standard ordered star product, and it is also a deformation of the classical BRST operator \( D \).
Then the standard ordered BRST algebra \( (A^{\bullet}[[\lambda]], \star_{\text{std}}, D_{\text{std}}) \) becomes a differential \( \mathbb{Z} \)-graded algebra with unit over \( \mathbb{C}[[\lambda]] \).

The standard ordered quantum BRST operator splits into two differentials:

- The quantized Chevalley–Eilenberg differential \( \delta : A^{\bullet,\bullet}[[\lambda]] \rightarrow A^{\bullet+1,\bullet}[[\lambda]] \), i.e. the Chevalley–Eilenberg differential on the quantum BRST complex induced by the quantum representation

\[
g \ni \xi \mapsto \rho(\xi) = \text{ad}(\xi) \otimes \text{id} + \text{id} \otimes \rho_M(\xi),
\]

where \( \rho_M \) is the representation of \( g \) on \( \mathcal{C}^\infty(M)[[\lambda]] \) as in (2.9).

- The quantized Koszul differential \( \partial : A^{\bullet,\bullet}[[\lambda]] \rightarrow A^{\bullet-1,\bullet+1}[[\lambda]] \) defined by

\[
\partial(x \otimes f) = i(e^a) x \otimes f \star J_a + \frac{i}{2} \left( f_{ab}^c (e^a) + f_{ab}^c e_c \wedge i(e^a)i(e^b) \right) (x \otimes f)
\]

for \( x \in A^{\bullet}(g_C^* \oplus g_C)[[\lambda]] \) and \( f \in \mathcal{C}^\infty(M)[[\lambda]] \). Note that the definition is independent of the basis.

By the equivariance of \( \star \) we have \( \rho = \rho \) and hence the equality \( \delta = \delta \). Moreover, as in the classical case one has the splitting

\[
D_{\text{std}} = \delta + 2\partial,
\]

see [4, Thm. 17] for further details. The corresponding cohomology ker \( D_{\text{std}}/\text{im}D_{\text{std}} \) of the standard ordered quantum BRST algebra is denoted by \( H_{\text{BRST}}(A[[\lambda]]) \) and called quantum BRST cohomology. Similarly to the classical setting, the quantum BRST cohomology is a \( \mathbb{Z} \)-graded associative algebra with \( \mathbb{C}[[\lambda]] \)-bilinear product \( \star_{\text{std}} \) induced by the associative multiplication \( \star_{\text{std}} \) of the quantum BRST algebra. One has \( [a] \star_{\text{std}} [b] = [a \star_{\text{std}} b] \) for all \( [a], [b] \in H_{\text{BRST}}(A[[\lambda]]) \) with \( D_{\text{std}}a = 0 = D_{\text{std}}b \) and \( [1] \in H_{\text{BRST}}(A[[\lambda]]) \) is a unit with respect to \( \star_{\text{std}} \).

Finally we recall that there exists a deformed restriction map

\[
t^* = t^* \circ S = \sum_{r=0}^\infty \lambda^r t^*_r : A^{\bullet,\bullet}g^* \otimes \mathcal{C}^\infty(M)[[\lambda]] \rightarrow A^{\bullet,\bullet}g^* \otimes \mathcal{C}^\infty(C)[[\lambda]],
\]

uniquely determined by the properties

\[
t_0^* = t^*, \quad t^*_r \big|_{A^{\bullet,1}[[\lambda]]} = 0 \quad \text{and} \quad t^* \text{prol} = \text{id}_{A^{\bullet,\bullet}g^* \otimes \mathcal{C}^\infty(C)[[\lambda]]}.
\]

Here \( S = \text{id}_{\mathcal{C}^\infty(M)} + \sum_{r=1}^\infty \lambda^r S_r \) is a formal series of differential linear operators of \( \mathcal{C}^\infty(M) \) with \( S_r \) vanishing on constants. If the action of \( G \) is in addition proper on \( M \) then \( S \) can be chosen to be \( G \)-equivariant. Extending \( t^* \) by zero to the whole BRST algebra \( A^{\bullet}[[\lambda]] \) one gets the following result, see [4, Prop. 26, Thm. 29, Thm. 32] for a proof and further details.

**Proposition 2.1.** Let \( (M, \star, G, J, C) \) be a Hamiltonian quantum \( G \)-space with regular constraint surface and proper action on \( M \).
(i) There exists a G-equivariant chain homotopy $\hat{h}$ for the augmented standard ordered BRST operator

$$\hat{D}_{std} = D_{std} + \delta^c + 2t^* \in \text{End} \left( (\Lambda^* g^* \otimes \mathcal{C}^\infty(C)[[\lambda]]) \oplus \mathcal{A}[[\lambda]] \right),$$

(2.20)

with $\delta^c$ being the Chevalley–Eilenberg differential on $\Lambda^* g^* \otimes \mathcal{C}^\infty(C)[[\lambda]]$, where all maps are defined to be zero on the domains on which they were previously not defined. In particular, one has $\hat{D}_{std} \hat{h} + \hat{h}\hat{D}_{std} = 2\text{id}$ with $\hat{h} = \text{prol} + h$ and

$$h : \Lambda^* g^* \otimes \Lambda^* g \otimes \mathcal{C}^\infty(M)[[\lambda]] \longrightarrow \Lambda^* g^* \otimes \Lambda^{*+1} g \otimes \mathcal{C}^\infty(M)[[\lambda]].$$

(2.21)

(ii) The $\mathbb{C}[[\lambda]]$-linear map

$$\Psi : H_{\text{BRST}}(\mathcal{A}[[\lambda]]) \longrightarrow H^\bullet_{\text{CE}}(g, \mathcal{C}^\infty(C)[[\lambda]]) \cong H^\bullet_{\text{CE}}(g, \mathcal{C}^\infty(C)[[\lambda]], [a] \mapsto [t^* a]$$

(2.22)

is an isomorphism with inverse $\Psi^{-1}([c]) = [hc]$ for $[c] \in H^\bullet_{\text{CE}}(g, \mathcal{C}^\infty(C)[[\lambda]])$.

(iii) If the action is in addition free on $C$, then $H_{\text{BRST}}^{(0)}(\mathcal{A}[[\lambda]]) \cong \mathcal{C}^\infty(M_{\text{red}})[[\lambda]]$ and this construction induces a star product $\star_{\text{red}}$ on $M_{\text{red}}$ via

$$\pi^*(u_1 \star_{\text{red}} u_2) = t^*(\text{prol}(\pi^* u_1) \star \text{prol}(\pi^* u_2))$$

(2.23)

for all $u_1, u_2 \in \mathcal{C}^\infty(M_{\text{red}})[[\lambda]]$, the so-called reduced star product.

To shorten the notation we call $\mathcal{A}_{\text{red}} = H_{\text{BRST}}^{(0)}(\mathcal{A}[[\lambda]])$ reduced quantum BRST algebra.

3. Abstract BRST Algebras and Different Types of Involutions

3.1. Abstract BRST algebras. Let $\mathcal{R}$ be an ordered ring with $\mathbb{Q} \subseteq \mathcal{R}$ and $\mathbb{C} = \mathcal{R}(i)$ its complexification with $i^2 = -1$. The main example is $\mathcal{R} = \mathbb{R}[[\lambda]]$, see [5,7,8] for a detailed discussion on $*$-representations and the GNS construction in this abstract setting. In the following, $\mathcal{A}$ denotes a $\mathbb{Z}_2$-graded associative algebra over $\mathbb{C}$ and $\text{ad}(a) = \{a, \cdot\}$ the super commutator with respect to the $\mathbb{Z}_2$-grading.

**Definition 3.1 (BRST algebra).** Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a $\mathbb{Z}_2$-graded associative algebra over $\mathbb{C} = \mathcal{R}(i)$.

(i) An even element $\gamma \in \mathcal{A}_0$ such that the inner derivation $\mathcal{G} \gamma = \text{ad}(\gamma) = [\gamma, \cdot]$ induces a $\mathbb{Z}$-grading on $\mathcal{A}$ by

$$\mathcal{A}^{(\cdot)} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}^{(k)} \quad \text{with} \quad \mathcal{A}^{(k)} = \{a \in \mathcal{A} \mid \mathcal{G} \gamma a = ka\}$$

(3.1)

is called ghost charge. The operator $\mathcal{G} \gamma$ is called ghost number operator and the induced grading is called ghost number grading.

(ii) An odd element $\Theta$ with ghost number +1 and square zero, i.e.

$$\Theta \in \mathcal{A}^{(1)}_1 \quad \text{and} \quad \Theta^2 = 0,$$

(3.2)

is called BRST charge. The corresponding inner derivation $D = \text{ad}(\Theta)$ is called BRST operator.
The triple \((\mathcal{A}, \gamma, \Theta)\) is then called \(BRST\) algebra over \(\mathbb{C}\). A morphism \(\Phi: (\mathcal{A}, \gamma, \Theta) \rightarrow (\mathcal{A}', \gamma', \Theta')\) of \(BRST\) algebras is an even morphism of \(\mathbb{Z}_2\)-graded associative algebras \(\Phi: \mathcal{A} \rightarrow \mathcal{A}'\) with
\[
\Phi(\gamma) = \gamma' \quad \text{and} \quad \Phi(\Theta) = \Theta',
\]
(3.3)
and the category of \(BRST\) algebras is denoted by \(\text{BRST-Alg}\).

Note that the properties imply that \(\Phi\) preserves the \(\mathbb{Z}\)-grading as well. We often encounter the setting that the \(\mathbb{Z}_2\)-grading is induced by the \(\mathbb{Z}\)-grading. In addition, \(D = \text{ad}(\Theta): \mathcal{A}(\mathbb{Z}) \rightarrow \mathcal{A}(\mathbb{Z}+1)\) and \(D^2 = 0\) imply that the \(BRST\) operator is a coboundary operator, thus it defines a cohomology:

**Definition 3.2 (BRST cohomology).** Let \((\mathcal{A}, \gamma, \Theta)\) be a \(BRST\) algebra. Then
\[
H^{(*)}_{BRST}(\mathcal{A}) = \bigoplus_{k \in \mathbb{Z}} H^{(k)}_{BRST}(\mathcal{A}) \quad \text{with} \quad H^{(k)}_{BRST}(\mathcal{A}) = \frac{\ker D|_{\mathcal{A}^{(k)}}}{\text{im} D|_{\mathcal{A}^{(k-1)}}}
\]
(3.4)
is called \(BRST\) cohomology of \(\mathcal{A}\). The reduced \(BRST\) algebra is defined by
\[
\mathcal{A}_{\text{red}} = H^{(0)}_{BRST,0}(\mathcal{A}).
\]
(3.5)

Since \(D\) is an odd inner derivation, the cohomology is again a \(\mathbb{Z} \times \mathbb{Z}_2\)-graded associative algebra and \(\mathcal{A}_{\text{red}}\) is a well-defined associative subalgebra. The ghost number operator acts on \(H^{(*)}_{BRST}(\mathcal{A})\) via
\[
\text{Gh}_{BRST}[a] = [\text{Gh} a],
\]
i.e. \(H^{(k)}_{BRST}(\mathcal{A}) = \{[a] \in H^{(*)}_{BRST}(\mathcal{A}) \mid \text{Gh}_{BRST}[a] = k[a]\}\). However, it is no longer an inner derivation as \(\gamma\) is no cocycle
\[
D\gamma = [\Theta, \gamma] = -[\gamma, \Theta] = -\Theta.
\]
(3.7)

If the \(\mathbb{Z}_2\)-grading is induced by the \(\mathbb{Z}\)-grading, then we have \(\mathcal{A}_{\text{red}} = H^{(0)}_{BRST}(\mathcal{A})\) as in the case of the quantum \(BRST\) cohomology. A straightforward computation shows that the assignment of a \(BRST\) algebra \((\mathcal{A}, \gamma, \Theta)\) to its \(BRST\) cohomology \(H^{(*)}_{BRST}(\mathcal{A})\) and reduced \(BRST\) algebra \(\mathcal{A}_{\text{red}}\) is a functor from \(\text{BRST-Alg}\) into the category of \(\mathbb{Z} \times \mathbb{Z}_2\)-graded algebras resp. algebras.

Let us consider a \(*\)-involution for \(\mathcal{A}\). Since we aim to get an induced involution on \(\mathcal{A}_{\text{red}} = H^{(0)}_{BRST,0}(\mathcal{A})\), the involution on the whole of \(\mathcal{A}\) should respect the \(\mathbb{Z}_2\)-grading. We have two main possibilities for involutions on a \(\mathbb{Z}_2\)-graded algebra \(\mathcal{A}\):

- **Graded \(*\)-involutions** \(I: \mathcal{A} \rightarrow \mathcal{A}\), i.e. \(\mathbb{C}\)-antilinear involutive even maps with
\[
I(ab) = I(b)I(a)
\]
(3.8)
for all \(a, b \in \mathcal{A}\). The pair \((\mathcal{A}, I)\) is called graded \(*\)-algebra.

- **Super \(*\)-involutions** \(S: \mathcal{A} \rightarrow \mathcal{A}\), i.e. \(\mathbb{C}\)-antilinear involutive even maps with
\[
S(ab) = (-1)^{|a||b|} S(b)S(a)
\]
(3.9)
for all homogeneous elements \(a, b \in \mathcal{A}\) with degrees \(|a|, |b|\). The pair \((\mathcal{A}, S)\) is called super \(*\)-algebra.
A short computation shows that the graded resp. super *-involutions of the adjoint representations give a minus sign. This motivates the following rescaling: From now on \( \gamma \in A_0^{(0)} \) and \( \Theta \in A_1^{(1)} \) are the elements such that

\[
Gh = \text{id}(\gamma) \quad \text{and} \quad D = \text{id}(\Theta).
\]  

(3.10)

Note that the normalization does not change the cohomology of \( D \) as well as the grading induced by \( Gh \) and that in the case of the quantum BRST algebra we already have a corresponding factor \( \frac{1}{2} \) in front of the super commutator.

One can show that the notion of super and graded *-involutions can be mutually exchanged by rescaling the odd component of the involution by \( \pm i \). Thus it only remains to investigate possible compatibilities of involutions with the ghost number grading. As we ultimately want an induced *-involution on the even ghost number zero part of the BRST cohomology, the ghost number zero part should be invariant under the involution, too. There are again two main possibilities: An involution that leaves the ghost number grading invariant, or an involution that inverts the ghost number grading.

**Remark 3.3** (Involution leaving ghost number invariant). A super *-involution that leaves the ghost number degree invariant and with Hermitian BRST charge \( \Theta^* = \Theta \) induces a super *-involution on the cohomology and a *-involution on \( A_{\text{red}} \) in a functorial way. However, this kind of involution has a big disadvantage in connection with *-representations \( \pi \) on pre-Hilbert spaces over \( \mathbb{C} \), see [7]: In this case \( \pi(\Theta) = 0 \) would vanish. The induced inner product on the physical space is in general still not positive definite, which leads to so-called no ghost theorems, compare e.g. [16, Sect. 14.2].

The above remark is a consequence of a more general problem:

**Remark 3.4.** The theory of homological algebra is not compatible with star involutions resp. with the positive definiteness of inner products. In the case of three or more dimensions there are no canonically induced inner products on the cohomology as the following simple example shows: Consider \( \mathbb{R}^3 \) with Euclidean scalar product and differential

\[
d = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then \( \ker d/imd \cong \text{Span}\{(0 \ 1 \ 0)^T\} \) and the quotient map is not compatible with the inner product.

We are mainly interested in non-trivial *-representations of the reduced quantum BRST algebra with involution, where one of the motivations consists in implementing the superposition principle. Therefore, the above lack of positivity leads us to the study of other possibilities for involutions * on the BRST algebra \( A \) such that \( D \) and \( \Theta \) are not Hermitian, i.e. \( D \neq D^* \) and \( \Theta \neq \Theta^* \).

### 3.2. Graded *-involution with imaginary ghost operator.

Consider a BRST algebra \((A, \gamma, \Theta)\) which has an additional graded *-involution \( a \mapsto a^* \). Since super and graded *-involutions can be mutually exchanged, this is only a matter of convenience and no relevant choice.

**Definition 3.5** (*BRST *-algebra with imaginary ghost operator*). Let \( A \) be a BRST algebra together with a graded *-involution * satisfying \( Gh^* = \text{id}(\gamma^*) = -Gh \). Then one calls \((A, \gamma, \Theta, *)\) *BRST *-algebra with imaginary ghost operator*. A morphism...
\( \Phi : (\mathcal{A}, \ast) \rightarrow (\mathcal{B}, \ast) \) of BRST \( \ast \)-algebras with imaginary ghost operators is a morphism
\[
\Phi : \mathcal{A} \rightarrow \mathcal{B}
\]
of BRST algebras that fulfills
\[
\Phi(a^\ast) = \Phi(a)^\ast \quad (3.11)
\]
for all \( a \in \mathcal{A} \). The corresponding category of BRST \( \ast \)-algebras with imaginary ghost operators is denoted by \( \text{iBRST} \ast \text{-Alg} \).

Since the graded \( \ast \)-involution is compatible with the \( \mathbb{Z}_2 \)-grading and since it inverts the ghost number grading, we directly see that \( \mathcal{A}^{(0)} \) becomes a \( \mathbb{Z}_2 \)-graded \( \ast \)-subalgebra of \( \mathcal{A} \). Similarly, \( \mathcal{A}^{(0)}_0 \) becomes a \( \ast \)-subalgebra of \( \mathcal{A} \). Moreover, we obtain the following behaviour of the ghost charge and the BRST operator under the graded \( \ast \)-involution.

**Lemma 3.6.** Let \( (\mathcal{A}, \ast) \) be a BRST algebra with imaginary ghost operator.

(i) There exists a unique central Hermitian element \( c \in \mathcal{A}^{(0)}_0 \) such that one has
\[
\gamma^\ast = -\gamma + c. \quad (3.12)
\]
(ii) Define the adjoint BRST operator \( D^\ast : \mathcal{A}^{(\ast \cdot 1)} \rightarrow \mathcal{A}^{(\ast \cdot -1)} \) by \( D^\ast = \text{iad}(\Theta^\ast) \). Then one has \((D^\ast)^2 = 0\) and
\[
D^\ast a = (-1)^{|a|}(Da^\ast)^\ast \quad (3.13)
\]
for homogeneous \( a \in \mathcal{A} \) with degree \(|a|\).

**Proof.** Concerning the first point we have \(-\text{iad}(\gamma) = \text{iad}(\gamma^\ast)\) and thus \(\gamma^\ast + \gamma\) is in the center of \( \mathcal{A} \) as well as \(\gamma, \gamma^\ast \in \mathcal{A}^{(0)}_0\), hence the statement is shown. For the second point note that \(\Theta \in \mathcal{A}^{(1)}_1\), so \(\Theta^\ast \in \mathcal{A}^{(-1)}_1\), and (3.13) follows from a short computation.  

The element \( \Delta = \Theta^\ast \Theta + \Theta^\ast \Theta = \Delta^\ast \in \mathcal{A}^{(0)}_0 \) is called *Laplacian* and will play an important role in the representation theory. It follows that \(\Theta\) and \(\Theta^\ast\) are either linearly independent or both equal to zero as \(\mathcal{A}^{(1)} \cap \mathcal{A}^{(-1)} = \{0\}\). Thus the kernel of \( D \) is no \( \ast \)-subalgebra of \( \mathcal{A} \) and there is no obvious way to obtain a \( \ast \)-structure on the BRST cohomology \( \text{H}^{(\ast)}_{\text{BRST}}(\mathcal{A}) = \ker D / \text{im} D \) of \( \mathcal{A} \) or at least on the reduced algebra. Therefore, the idea is to define a new quotient \((\ker D \cap \ker D^\ast) / (\text{im} D \cap \text{im} D^\ast)\) and to show that this construction yields a well-defined \( \mathbb{Z} \times \mathbb{Z}_2 \)-graded algebra with graded \( \ast \)-involution. To this end we investigate the relation between the \( \ast \)-involution and the elements in \( \ker D \cap \ker D^\ast \) and \( \text{im} D \cap \text{im} D^\ast \).

**Lemma 3.7.** Let \( (\mathcal{A}, \ast) \) be a BRST \( \ast \)-algebra with imaginary ghost operator. Then one has for all \( a \in \mathcal{A} \):

(i) \( a \in \ker D \cap \ker D^\ast \iff a, a^\ast \in \ker D \iff a, a^\ast \in \ker D^\ast \).
(ii) \( a \in \text{im} D \cap \text{im} D^\ast \iff a, a^\ast \in \text{im} D \iff a, a^\ast \in \text{im} D^\ast \).

Consequently, the intersection \( \ker D \cap \ker D^\ast \) is a \( \mathbb{Z} \times \mathbb{Z}_2 \)-graded \( \ast \)-subalgebra of \( \mathcal{A} \) and the set \( \text{im} D \cap \text{im} D^\ast \subseteq \ker D \cap \ker D^\ast \) is a \( \mathbb{Z} \times \mathbb{Z}_2 \)-graded \( \ast \)-ideal therein.

**Proof.** The first two parts follow directly with (3.13). In addition, we have for all homogeneous elements \( a \in \ker D \cap \ker D^\ast \) and \( De = D^\ast f \in \text{im} D \cap \text{im} D^\ast \)
\[
aDe = (-1)^{|a|} D(ae) = (-1)^{|a|} D^\ast(af),
\]
thus \( \text{im} D \cap \text{im} D^\ast \) is a \( \ast \)-ideal in \( \ker D \cap \ker D^\ast \).
Hence we know that \((\ker D \cap \ker D^*)/(\im D \cap \im D^*)\) becomes a \(\mathbb{Z} \times \mathbb{Z}_2\)-graded algebra as well.

**Definition 3.8 (Reduced BRST *-algebra).** Let \((A, \ast)\) be a BRST *-algebra with imaginary ghost operator. The **BRST quotient** is defined by

\[
\widetilde{H}^{(\bullet)}_{\text{BRST}}(A) = \frac{\ker D \cap \ker D^*}{\im D \cap \im D^*},
\]

and by

\[
\widetilde{A}_{\text{red}} = \widetilde{H}^{(0)}_{\text{BRST},0}(A)
\]

one denotes the corresponding **reduced BRST *-algebra**.

Note that \(\widetilde{H}^{(\bullet)}_{\text{BRST}}(A)\) can in general not be expressed as cohomology of some cohomological chain complex since it is only a quotient of an algebra with an ideal. Nonetheless, we sometimes call it cohomology in analogy to \(H^{(\bullet)}_{\text{BRST}}(A)\) and to simplify the notation. We have the following result.

**Lemma 3.9.** The **BRST quotient** \(\widetilde{H}^{(\bullet)}_{\text{BRST}}(A)\) is a \(\mathbb{Z} \times \mathbb{Z}_2\)-graded algebra with graded *-involution \(*\) defined by

\[
[a]^* = [a^*],
\]

exchanging \(\widetilde{H}^{(k)}_{\text{BRST}}(A)\) with \(\widetilde{H}^{(-k)}_{\text{BRST}}(A)\) for all \(k \in \mathbb{Z}\). The **reduced BRST *-algebra** \(\widetilde{A}_{\text{red}}\) is a *-algebra.

**Proof.** The properties follow directly by the above results and the compatibility of the *-involution with the grading. \(\square\)

Just as for BRST algebras one shows that the passages from a BRST *-algebra with imaginary ghost operator to its BRST quotient and reduced BRST *-algebra are functorial:

**Proposition 3.10.** The assignment of a BRST *-algebra with imaginary ghost operator \((A, \gamma, \Theta, \ast)\) to the **BRST quotient** \(\widetilde{H}^{(\bullet)}_{\text{BRST}}(A)\) is a functor into the category of \(\mathbb{Z} \times \mathbb{Z}_2\)-graded algebras with graded *-involution. Similarly, the assignment to the reduced BRST *-algebra \(\widetilde{A}_{\text{red}}\) is a functor into the category of *-algebras.

Finally, we can prove that there is the following crucial relation between \(\widetilde{H}^{(\bullet)}_{\text{BRST}}(A)\) and \(H^{(\bullet)}_{\text{BRST}}(A)\).

**Proposition 3.11.** The map

\[
I_A : \widetilde{H}^{(\bullet)}_{\text{BRST}}(A) \longrightarrow H^{(\bullet)}_{\text{BRST}}(A), \quad [a] \longmapsto I_A ([a]) = [a]
\]

is a well-defined morphism of \(\mathbb{Z} \times \mathbb{Z}_2\)-graded algebras.

**Proof.** The well-definedness follows directly with the definitions of the quotients and the compatibility with the grading is clear as both \(\widetilde{H}^{(\bullet)}_{\text{BRST}}(A)\) and \(H^{(\bullet)}_{\text{BRST}}(A)\) inherit the \(\mathbb{Z} \times \mathbb{Z}_2\)-grading of \(A\). \(\square\)
Remark 3.12. The important question is if this canonical morphism $I_A$ is an isomorphism, which would justify our construction and yield a canonical involution on the BRST cohomology. In general, there seems to be no possibility to decide whether $I_A$ is injective or surjective and one has to argue which reduction scheme fits better to the respective application. In Sect. 4.2 we show that in our example of the quantum BRST algebra $I_A$ is an isomorphism if restricted to the physically most relevant zero-th degree.

Remark 3.13. The above considerations show that in some cases it might be useful to consider instead of the usual cohomology the BRST quotient from (3.14). To further justify this proposal one has to transfer the concepts of quasi-isomorphisms and chain homotopies from homological algebra to our setting. For the notion of quasi-isomorphisms there is an obvious choice: We call a morphism $\Phi : A \to B$ of BRST *-algebras quasi-isomorphism if it induces an isomorphism $\tilde{H}^{(*)}_{BRST}(A) \to \tilde{H}^{(*)}_{BRST}(B)$ on the BRST quotients. The case of chain homotopies is more subtle: One choice would be to consider a homotopy $h : A^{(*)} \to B^{(*)-1}$ between two morphisms $\Phi, \Psi : A \to B$ of BRST *-algebras with respect to the BRST operator $D$, i.e. a $\mathbb{C}$-linear map $h$ such that

$$hD_A + D_B h = \Phi - \Psi.$$  

Then the map $h^* : A^{(*)} \to B^{(*)+1}$, given on homogeneous elements $a \in A$ by $h^*(a) = \langle (-1)^{|a|} (h(a^*))^* \rangle$ turns out to be a chain homotopy with respect to $D^*$ between $\Phi$ and $\Psi$. In particular, in this case $\Phi$ and $\Psi$ induce the same maps on the BRST quotients. However, it is not yet clear to us if this is the compatibility we want to have and we plan to investigate it in a forthcoming paper.

In the remaining part of this section we want to show that *-involutions with imaginary ghost operators lead indeed to a non-trivial *-representation theory on pre-Hilbert spaces, in contrast to the involutions with Hermitian BRST charges and Hermitian ghost operators.

3.3. BRST *-representations and GNS construction. We introduce a *-representation theory of BRST *-algebras with imaginary ghost operator and show that the representations can be reduced to *-representations of the reduced BRST *-algebras. In addition, we sketch an adapted GNS construction. The notions are based on the theory of pre-Hilbert spaces as in [33, Chapter 7]. Recall that a pre-Hilbert space over $\mathbb{C}$ is a $\mathbb{C}$-module $\mathcal{H}$ with positive definite inner product $\langle \cdot, \cdot \rangle$. Note that positivity, i.e. $\langle \phi, \phi \rangle > 0$ for all $\phi \in \mathcal{H}\setminus\{0\}$, makes sense in our setting as $\mathbb{R} \subset \mathbb{C} = \mathbb{R}(i)$ is ordered. A map $A : \mathcal{H} \to \mathcal{H}$ is called adjointable if there exists a map $A^* : \mathcal{H} \to \mathcal{H}$ such that $\langle A\phi, \psi \rangle = \langle \phi, A^*\psi \rangle$ for all $\phi, \psi \in \mathcal{H}$. The set of adjointable maps is denoted by $\mathcal{B}(\mathcal{H})$. These spaces can be adapted to our setting:

Definition 3.14 (BRST pre-Hilbert space). A BRST pre-Hilbert space is a $\mathbb{Z} \times \mathbb{Z}_2$-graded $\mathbb{C}$-module $\mathcal{H}$ together with

(i) an odd endomorphism $\Theta_{\mathcal{H}} \in \text{End}^{1(1)}_1(\mathcal{H})$ of ghost number degree +1 with $\Theta_{\mathcal{H}}^2 = 0$, called BRST operator,

(ii) a ghost number operator $\gamma_{\mathcal{H}}^k$ defined by

$$i\gamma_{\mathcal{H}}^k|_{\mathcal{H}^k} = k \cdot \text{id}|_{\mathcal{H}^k} \quad (3.18)$$

and extended $\mathbb{C}$-linearly to all of $\mathcal{H}$, and
(iii) an even graded positive definite inner product \( \langle \cdot, \cdot \rangle \), such that one has the compatibilities
\[
\gamma^*_\mathcal{H} = -\gamma\mathcal{H} \quad \text{and} \quad \Theta\mathcal{H} \in \mathcal{B}(\mathcal{H}).
\] (3.19)

A morphism \( T: \mathcal{H} \rightarrow \mathcal{H}' \) of BRST pre-Hilbert spaces is an adjointable \( C \)-linear even map intertwining the BRST and ghost operators.

**Remark 3.15.** Alternatively, one could also consider isometric maps as morphisms of BRST pre-Hilbert spaces, instead of adjointable ones. The isomorphisms in this category are then unitary intertwiners, not adjointable bijective intertwiners. In our general setting these notions lead indeed to different notions of equivalent representations, and we favour adjointable ones because there might exist isometric maps not allowing for an adjoint.

Note that the definition directly implies that \( \mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) \cap \text{End}(\mathcal{H}) \) is a well-defined BRST *-algebra with imaginary ghost operator. As in the case of BRST *-algebras one can construct the usual BRST cohomology \( H^{\bullet}_{\text{BRST}}(\mathcal{H}) = \ker \Theta\mathcal{H} / \text{im} \Theta\mathcal{H} \), but there exists no canonical inner product on this quotient. Therefore, we define the BRST quotient
\[
\widetilde{H}^{\bullet}_{\text{BRST}}(\mathcal{H}) = \ker \Theta\mathcal{H} \cap \ker \Theta^*_\mathcal{H} / \text{im} \Theta\mathcal{H} \cap \ker \Theta^*_\mathcal{H},
\] (3.20)

One can directly check that this quotient is again \( \mathbb{Z} \times \mathbb{Z}_2 \)-graded and noting \( \ker \Theta^*_\mathcal{H} = (\text{im} \Theta\mathcal{H})^\perp \) we can even show more:

**Proposition 3.16.** For a BRST pre-Hilbert space \( \mathcal{H} \) one has \( \text{im} \Theta\mathcal{H} \cap (\text{im} \Theta^*_\mathcal{H})^\perp = \{0\} \) and thus
\[
\widetilde{H}^{\bullet}_{\text{BRST}}(\mathcal{H}) = \ker \Theta\mathcal{H} \cap \ker \Theta^*_\mathcal{H} = \ker \Delta\mathcal{H},
\] (3.21)

where \( \Delta\mathcal{H} = [\Theta\mathcal{H}, \Theta^*_\mathcal{H}] \) denotes the Laplacian of \( \mathcal{B}(\mathcal{H}) \). In particular, the inner product on \( \mathcal{H} \) restricts to a positive definite and non-degenerate inner product on \( \widetilde{H}^{\bullet}_{\text{BRST}}(\mathcal{H}) \).

**Proof.** One has \( \text{im} \Theta\mathcal{H} \cap \ker \Theta^*_\mathcal{H} = \text{im} \Theta\mathcal{H} \cap (\text{im} \Theta^*_\mathcal{H})^\perp = \{0\} \) and thus \( \widetilde{H}^{\bullet}_{\text{BRST}}(\mathcal{H}) = \ker \Theta\mathcal{H} \cap \ker \Theta^*_\mathcal{H} \). The positive definiteness of the inner product on \( \mathcal{H} \) implies for all \( \phi \in \mathcal{H} \)
\[
\langle \phi, \Delta\mathcal{H}\phi \rangle = \langle \Theta^*_\mathcal{H}\phi, \Theta^*_\mathcal{H}\phi \rangle + \langle \Theta\mathcal{H}\phi, \Theta\mathcal{H}\phi \rangle \geq 0,
\]
which entails \( \ker \Theta\mathcal{H} \cap \ker \Theta^*_\mathcal{H} = \ker \Delta\mathcal{H} \). Hence the inner product on \( \mathcal{H} \) restricts to \( \widetilde{H}^{\bullet}_{\text{BRST}}(\mathcal{H}) \). It is positive definite and in particular non-degenerate. \( \square \)

Elements in the kernel of \( \Delta\mathcal{H} \) are called *harmonic*. The above proposition shows that in the case of a BRST pre-Hilbert space \( \mathcal{H} \) the BRST quotient \( \widetilde{H}^{\bullet}_{\text{BRST}}(\mathcal{H}) \) and the reduced BRST pre-Hilbert space
\[
\widetilde{\mathcal{H}}_{\text{red}} = \widetilde{H}^{(0)}_{\text{BRST,0}}(\mathcal{H})
\] (3.22)

inhibit both positive definite inner products and one easily sees that all the passages are functorial. We have again a canonical inclusion of the BRST quotient into the BRST cohomology, compare Proposition 3.11. Now it is even injective by the positive definiteness:
Proposition 3.17. Let \( \mathcal{H} \) be a BRST pre-Hilbert space. Then the canonical map 
\[ I_{\mathcal{H}_c} : \tilde{\mathcal{H}}^{(\bullet)}(\mathcal{H}) \longrightarrow \mathcal{H}^{(\bullet)}_{\text{BRST}}(\mathcal{H}) \]
is injective.

Proof. Suppose \([\phi] = I_{\mathcal{H}_c}([\phi]) = I_{\mathcal{H}_c}([\psi]) = [\psi]\) for \([\phi], [\psi] \in \tilde{\mathcal{H}}^{(\bullet)}(\mathcal{H})\). This implies \(\phi - \psi = \Theta_{\mathcal{H}_c} \chi \in \text{im} \Theta_{\mathcal{H}_c} \cap \ker \Theta_{\mathcal{H}_c} \cap \ker \Theta_{\mathcal{H}_c}^*\) and we can compute
\[ \langle \Theta_{\mathcal{H}_c} \chi, \Theta_{\mathcal{H}_c} \chi \rangle = \langle \chi, \Theta_{\mathcal{H}_c}^* \Theta_{\mathcal{H}_c} \chi \rangle = 0. \]
But by the positive definiteness of the inner product this implies \(0 = \Theta_{\mathcal{H}_c} \chi = \phi - \psi\). \(\square\)

Remark 3.18. The above result is a more general version of the easy part of the Hodge theorem: the injectivity of the inclusion of the harmonic differential forms into the de Rham cohomology of a Riemannian manifold, see e.g. [24, Lemma 4.15]. The difficult part is to show the surjectivity that does not always hold in our general situation.

Now we can define BRST *-representations of BRST *-algebras with imaginary ghost operators and their reduction:

Definition 3.19 (BRST *-representation). Let \((\mathcal{A}, \gamma, \Theta, \ast)\) be a BRST *-algebra with imaginary ghost operator. A BRST *-representation of \(\mathcal{A}\) on a BRST pre-Hilbert space \(\mathcal{H}\) is a morphism
\[ \rho : \mathcal{A} \longrightarrow \mathcal{B}(\bullet)(\mathcal{H}) \] (3.23)
of BRST *-algebras with imaginary ghost operator. An intertwiner \(T\) between two such BRST *-representations \((\mathcal{H}, \rho)\) and \((\mathcal{H}', \rho')\) of \(\mathcal{A}\) is a morphism \(T : \mathcal{H} \longrightarrow \mathcal{H}'\) of BRST pre-Hilbert spaces that satisfies in addition
\[ T \circ \rho(a) = \rho'(a) \circ T \] (3.24)
for all \(a \in \mathcal{A}\).

Since all the passages from \(\mathcal{A}\) to \(\tilde{\mathcal{H}}^{(\bullet)}(\mathcal{A})\) and \(\tilde{\mathcal{A}}_{\text{red}}\) as well as from \(\mathcal{H}\) to \(\tilde{\mathcal{H}}^{(\bullet)}(\mathcal{H})\) and \(\tilde{\mathcal{H}}_{\text{red}}\) are functorial, we obtain the following behaviour of BRST *-representations under the BRST reduction:

Proposition 3.20. Consider a BRST *-algebra \((\mathcal{A}, \gamma, \Theta, \ast)\) with imaginary ghost operator and a BRST *-representation \(\rho\) of \(\mathcal{A}\) on a BRST pre-Hilbert space \(\mathcal{H}\). Then
\[ \tilde{\rho}_{\text{BRST}} : \tilde{\mathcal{H}}^{(\bullet)}(\mathcal{A}) \longrightarrow \mathcal{B}(\bullet)\left(\tilde{\mathcal{H}}^{(\bullet)}(\mathcal{H})\right), \quad \rho_{\text{BRST}}([a])[\phi] = [\rho(a)\phi] \] (3.25)
yields a *-representation of \(\tilde{\mathcal{H}}^{(\bullet)}(\mathcal{A})\) on \(\tilde{\mathcal{H}}^{(\bullet)}(\mathcal{H})\) which is compatible with all degrees. Moreover, the restriction
\[ \tilde{\rho}_{\text{red}} = \left. \rho_{\text{BRST}} \right|_{\tilde{\mathcal{H}}^{(0)}_{\text{BRST}, 0}(\mathcal{A})} \left|_{\tilde{\mathcal{H}}^{(0)}_{\text{BRST}, 0}(\mathcal{H})} \right. \] (3.26)
yields a *-representation of \(\tilde{\mathcal{A}}_{\text{red}}\) on \(\tilde{\mathcal{H}}_{\text{red}}\). All the assignments are functorial.

Finally, we apply the general formalism of the GNS construction to the case of a BRST *-algebra with imaginary ghost operator, i.e. we construct BRST *-representations out of suitable linear functionals. We recall at first the usual GNS construction from [33, Section 7.2.2]:
Remark 3.21 (GNS representation). Let $A$ be a $\ast$-algebra over $C$ and $\omega: A \rightarrow C$ a positive linear functional, i.e. $\omega(a^*a) \geq 0$ for all $a \in A$. Then one has $\omega(b^*a) = \overline{\omega(a^*b)}$ for all $a, b \in A$ as well as the Cauchy-Schwarz inequality. The subset
\[ \mathcal{J}_\omega = \{a \in A \mid \omega(a^*a) = 0\} = \{a \in A \mid \omega(b^*a) = 0 \ \forall \ b \in A\} = \{a \in A \mid \omega(a^*b) = 0 \ \forall \ b \in A\} \]
is a left ideal in $A$, the so-called Gel'fand ideal. The quotient $\mathcal{H}_\omega = A/\mathcal{J}_\omega$ becomes a left $A$-module in the canonical way by setting $\pi_\omega(a)\psi_b = \psi_{ab}$ for $a, b \in A$, where $\psi_b \in \mathcal{H}_\omega$ denotes the equivalence class of $b$. One has a positive definite inner product $\langle \psi_a, \psi_b \rangle_\omega = \omega(a^*b)$ on $\mathcal{H}_\omega$ and $\pi_\omega$ turns out to be a $\ast$-representation of $A$, the so-called GNS representation with respect to $\omega$.

The representations of BRST $\ast$-algebras $A$ should be compatible with the $\mathbb{Z} \times \mathbb{Z}_2$-grading, whence we have to consider $\mathbb{Z} \times \mathbb{Z}_2$-homogeneous positive linear functionals $\omega: A \rightarrow C$, i.e. such positive linear functionals that vanish on all degrees except $A_0^{(0)}$. In this case one easily sees that $\mathcal{J}_\omega$ and $\mathcal{H}_\omega$ are $\mathbb{Z} \times \mathbb{Z}_2$-graded and that the GNS representation is compatible with the degrees. Even more, vectors in $\mathcal{H}_\omega$ with different degrees are orthogonal.

Since the $\mathbb{Z}$-grading of the BRST algebra is induced by $\gamma$, we require in addition $\pi_\omega(\gamma) = \gamma \mathcal{J}_\omega$. Therefore, one needs a further condition on $\omega$ as a straightforward computation shows:

**Proposition 3.22.** Consider a BRST $\ast$-algebra $(A, \gamma, \Theta, \ast)$ with imaginary ghost operator and an even, positive linear functional $\omega: A \rightarrow C$. If the ghost charge satisfies $\gamma \in \mathcal{J}_\omega$, i.e. $\omega(\gamma a) = \omega(\gamma^*a) = 0$ for all $a \in A$, then $\omega$ is homogeneous with respect to the $\mathbb{Z} \times \mathbb{Z}_2$-grading and one has
\[ \pi_\omega(\gamma) = \gamma \mathcal{J}_\omega. \] (3.27)

In particular, $(\mathcal{H}_\omega = A/\mathcal{J}_\omega, \gamma_\omega = \pi_\omega(\gamma), \Theta_\omega = \pi_\omega(\Theta))$ is a BRST pre-Hilbert space and
\[ \pi_\omega: A \rightarrow \mathcal{B}(\mathcal{H}_\omega) \] (3.28)
is a BRST $\ast$-representation of $A$.

Thus we have found a way to generalize the GNS construction to BRST $\ast$-algebras with imaginary ghost operator, which gives us an explicit method to construct BRST $\ast$-representations.

**Remark 3.23.** The next question one could ask is if for such a linear functional $\omega$ the reduced representations $\overline{\pi_\omega}_{\text{BRST}}$ and $\overline{\pi_\omega}_{\text{red}}$ are again GNS representations of some linear functionals on the quotients. In particular, it is interesting if there is a canonical way to construct the corresponding linear functionals if they exist. It turns out that there is a positive answer to both questions if one requires in addition $\omega(\Delta) = 0$. This reflects the compatibility of $\omega$ with the BRST charge $\Theta$ and its adjoint $\Theta^\ast$ that are responsible for the reduction.

**Remark 3.24.** Note that $\ast$-involutions and analogues of GNS representations are also studied in the more general context of involutive categories, see e.g. [17]. One can interpret the BRST $\ast$-algebras as special involutive monoids in the involutive monoidal category of mixed complexes: Recall that the objects $(K^\ast, d, D)$ in this category are $\mathbb{Z}$-graded $C$-modules $K^\ast$ that are both chain complex $(K^\ast, d)$ and cochain complex
(K^\bullet, D), where dD + Dd may be non-zero. One can check that the functor K^\bullet \mapsto \overline{K}^\bullet given by \overline{K}^n = K^{-n} turns the category into an involutive monoidal category. BRST *-algebras are then special involutive monoids, where the chain and cochain map as well as the grading are given by inner derivations. The above GNS construction turns out to be a special case of [17, Thm. 7.1]. However, we want to stress that in our setting positivity plays a crucial role since we are interested in *-representations on pre-Hilbert spaces. In the general framework we are not aware of such an implementation of positivity.

4. *-Involutions for the Quantum BRST Algebra

In this section we apply the above results and construct a *-involution for the quantum BRST algebra A^\bullet(\lambda) = \Lambda^\bullet C^* \otimes \Lambda C^* \otimes \mathcal{C}^\infty (M)[[\lambda]] corresponding to a Hamiltonian quantum G-space (M, *, G, J).

4.1. Graded *-involutions on the Grassmann algebra. Let us assume that we have an equivariant and Hermitian star product on M, so that the complex conjugation is an involution on \mathcal{C}^\infty (M)[[\lambda]]. Thus we only need to find a suitable *-involution for the Grassmann algebra leading to a quantum BRST algebra having sufficiently many positive functionals.

A first possibility for an involution is the complex conjugation. Unfortunately, we can check that it is neither a graded nor a super *-involution with respect to the standard ordered star product.

We define a standard ordered representation in analogy with the case of cotangent bundles [26]. Let \nu^* be the restriction

\nu^* : \Lambda^\bullet C^* (\nu) \rightarrow \Lambda^\bullet C^* (\nu), \quad (4.1)

\nu^* sets all forms with a nontrivial g-part to zero. Moreover, we denote by

\nu^* pr^* : \Lambda^\bullet g^*_C (\nu) \rightarrow \Lambda^\bullet (g^*_C \otimes g_C)[[\lambda]] \quad (4.2)

the inclusion map. It immediately follows \nu^* pr^* = id_{\Lambda^\bullet g^*_C (\nu)} and we can define the following representation.

Definition 4.1 (Standard ordered representation). The standard ordered representation

\nu_{std} : (\Lambda^\bullet g^*_C \otimes g_C)[[\lambda]], \circ_{std} \rightarrow End(\Lambda^\bullet g^*_C (\nu)) \quad (4.3)

of (\Lambda^\bullet g^*_C \otimes g_C)[[\lambda]], \circ_{std}) is defined by

\nu_{std}(a) \alpha = \nu^* (a \circ_{std} pr^* \alpha) \quad (4.4)

for a \in \Lambda^\bullet (g^*_C \otimes g_C)[[\lambda]] and \alpha \in \Lambda^\bullet g^*_C (\nu).

Then we directly see that \nu_{std} is \mathbb{C}[[\lambda]]-linear and satisfies \nu_{std}(1) = id_{\Lambda^\bullet g^*_C (\nu)}.

Remark 4.2. The idea comes from the theory of Clifford algebras, from which we know

\Lambda^\bullet (g^*_C \otimes g_C) \cong Cl(g^*_C \otimes g_C; (\cdot, \cdot)) \cong Cl(g^*_C \otimes g_C) \cong End(\Lambda^\bullet g^*_C), \quad (4.5)

since all non-degenerate bilinear symmetric inner products are equivalent on \mathbb{C}^{2n}, see e.g. [23, Prop. 2.4]. Note that here the first isomorphism is an isomorphism of vector spaces, whereas the other two are isomorphisms of Clifford algebras. We transferred this idea to the quantized setting.
Proposition 4.3. The standard ordered representation \( \rho_{\text{std}} \) defined by (4.4) is a faithful representation of \( (\Lambda^* (g^*_C \oplus g_C)[[\lambda]], \circ_{\text{std}}) \) on \( \Lambda^* g^*_C[[\lambda]] \), i.e. we have

\[
\rho_{\text{std}}(a) \rho_{\text{std}}(\bar{a}) \alpha = \rho_{\text{std}}(a \circ_{\text{std}} \bar{a}) \alpha
\]

for all \( a, \bar{a} \in \Lambda^* (g^*_C \oplus g_C)[[\lambda]] \) and \( \alpha \in \Lambda^* g^*_C[[\lambda]] \).

Proof. The properties follow from lengthy but straightforward computations. \( \square \)

Using the definition of \( \rho_{\text{std}} \) we can immediately compute

\[
\rho_{\text{std}}(1) = \text{id}, \quad \rho_{\text{std}}(e_i) = 2i\lambda i(e_i) \quad \text{and} \quad \rho_{\text{std}}(e^i) = e^i \wedge
\]

for the elements \( 1, e_1, \ldots, e_n, e^1, \ldots, e^n \) that generate \( (\Lambda^* (g^*_C \oplus g_C)[[\lambda]], \circ_{\text{std}}) \).

It is known that \( \Lambda^* g^*_C \) has a structure of a pre-Hilbert space over \( \mathbb{C} \), which extends to \( \mathbb{C}[[\lambda]] \).

Lemma 4.4. Let \( g \) be a positive definite symmetric bilinear inner product on \( g \). Then it induces a positive definite sesquilinear product \( \langle \cdot, \cdot \rangle_{*} \) on \( \Lambda^* g^*_C[[\lambda]] \) via

\[
\langle a_1 \wedge \cdots \wedge a_k, b_1 \wedge \cdots \wedge b_k \rangle_{*} = \det(g^{-1}(a_i, b_j))
\]

for all \( a_1, \ldots, a_k, b_1, \ldots, b_k \in g^*_C \). In particular, \( (\Lambda^* g^*_C[[\lambda]], \langle \cdot, \cdot \rangle_{*}) \) is a pre-Hilbert space over \( \mathbb{C}[[\lambda]] \).

In order to get an involution on \( \Lambda^* (g^*_C \oplus g_C)[[\lambda]] \) which is independent of \( \lambda \), we define the rescaled inner product \( \langle \cdot, \cdot \rangle \) for each \( \Lambda^k g^*_C[[\lambda]] \) by

\[
\langle a, b \rangle = (2\lambda)^k \langle a, b \rangle_{*}, \quad \text{where} \quad a, b \in \Lambda^k g^*_C[[\lambda]].
\]

Note that for \( \lambda = 0 \), the inner product \( \langle \cdot, \cdot \rangle \) on \( \Lambda^* g^*_C[[\lambda]] \) is degenerate, but the corresponding *-involution on \( \Lambda^* (g^*_C \oplus g_C)[[\lambda]] \) resp. \( \Lambda^* g^*_C \) is still well-defined.

Proposition 4.5. The standard ordered representation \( \rho_{\text{std}} \) of \( \Lambda^* (g^*_C \oplus g_C)[[\lambda]] \) from Definition 4.1 is a *-representation with respect to the graded *-involution induced by

\[
\xi^* = -ig^b(\xi) \quad \text{and} \quad \alpha^* = -ig^a(\alpha) \quad \forall \xi \in g_C, \alpha \in g^*_C.
\]

Moreover, \( \gamma = \frac{1}{2} e^k \wedge e_k \) fulfills \( \gamma^* = -\gamma \).

Proof. For \( c \in \mathbb{C} \) we have

\[
\langle \rho_{\text{std}}(e_i)^* c, e^j \rangle = \langle c, \rho_{\text{std}}(e_i)e^j \rangle = 2i\lambda \langle c, i(e_i)e^j \rangle = 2i\lambda \bar{c}b^j_i = ig_{ik}\bar{c}(e^k, e^j) = \langle \rho_{\text{std}}(-ig_{ik}e^k)c, e^j \rangle,
\]

and analogously \( \langle \rho_{\text{std}}(e^j)^* e^j, c \rangle = \langle \rho_{\text{std}}(g_{ik}e^i)e^j, c \rangle \). In other words, we get \( \rho_{\text{std}}(e_i)^* = \rho_{\text{std}}(-ig^b(e_i)) \) and \( \rho_{\text{std}}(e^j)^* = \rho_{\text{std}}(-ig^a(e^j)) \). Furthermore we have \( \rho_{\text{std}}(c)^* = \rho_{\text{std}}(c) \), inducing the involution from (4.10). Finally we compute

\[
2\gamma^* = (e^k \rho_{\text{std}} e_k)^* = -g^{km}g_{kn}e^m \rho_{\text{std}} e_m = -2\gamma.
\]

\( \square \)
The above result shows that this graded \(*\)-involution is in some sense a “natural” one since it is induced by the above representation \(\rho_{\text{std}}\). Moreover, one can show that the standard ordered representation \(\rho_{\text{std}}\) from Definition 4.1 is even unitarily equivalent to a GNS representation.

Finally, we want to show that we have sufficiently many positive linear functionals. We recall [7, Def. 2.7]: Let \((\mathcal{A}, \ast)\) be a \(*\)-algebra over \(\mathbb{C} = \mathbb{R}(i)\). Then \(\mathcal{A}\) has sufficiently many positive linear functionals if for any non-zero Hermitian element \(h = h^\ast \in \mathcal{A}\setminus\{0\}\) there exists a positive linear functional \(\omega: \mathcal{A} \longrightarrow \mathbb{C}\) with \(\omega(h) \neq 0\).

The non-deformed Grassmann algebra has obviously not sufficiently many positive linear functionals as \(a^\ast \wedge a = 0\) for all \(a \in \Lambda^k \mathfrak{g}^\ast \otimes \Lambda^\ell \mathfrak{g}\) with \(k + \ell > n\), hence the Cauchy-Schwarz inequality implies \(\omega(a) = 0\) for all such \(a\) and all positive linear functionals \(\omega\), in particular for the Hermitian ones.

**Proposition 4.6.** Let \(g\) be a positive definite and symmetric bilinear inner product, inducing the involution \(*\) via (4.10). Then the deformed Grassmann algebra \((\Lambda^\bullet(\mathfrak{g}_C^\ast \oplus \mathfrak{g}_C)[[\lambda]], \circ_{\text{std}}, \ast)\) has sufficiently many positive linear functionals.

**Proof.** At first, note that if \(\omega: \Lambda^\bullet(\mathfrak{g}_C^\ast \oplus \mathfrak{g}_C)[[\lambda]] \longrightarrow \mathbb{C}[[\lambda]]\) is a positive linear functional, then

\[
\omega_b: \Lambda^\bullet(\mathfrak{g}_C^\ast \oplus \mathfrak{g}_C)[[\lambda]] \ni a \longmapsto \omega_b(a) = \omega(b^\ast \circ_{\text{std}} a \circ_{\text{std}} b) \in \mathbb{C}[[\lambda]]
\]

is also positive and linear. Consider now the projection \(\delta: \Lambda^\bullet(\mathfrak{g}_C^\ast \oplus \mathfrak{g}_C)[[\lambda]] \longrightarrow \mathbb{C}[[\lambda]]\) that is a positive linear functional. For elements \(c \in \mathbb{R}[[\lambda]]\setminus\{0\}\) we have \(\delta(c) \neq 0\), hence we can restrict ourselves to elements of non-trivial \(\mathbb{Z} \times \mathbb{Z}\)-degree. Take an orthonormal basis \(e_1, \ldots, e_n\) of \(g\) with respect to \(g\) with orthonormal dual basis \(e^1, \ldots, e^n\) of \(g^\ast\) with respect to \(g^{-1}\). In particular, we have \(e^i_j = -ie^j_i\). Then every non-zero Hermitian element \(h\) has to be the sum of elements of the form \(a = (-i)^{i+j} c \otimes e^{k_1} \wedge \cdots \wedge e^{k_j} \wedge e_{\ell_1} \wedge \cdots \wedge e_{\ell_1} + c e^{\ell_1} \wedge \cdots \wedge e^{\ell_i} \wedge e_{\ell_i} \wedge \cdots \wedge e_{\ell_i+1}\) with \(i, j = 0, \ldots, n\) not both equal to zero and with \(c \in \mathbb{C}[[\lambda]]\setminus\{0\}\). Choose now \(b = c_1 e^{k_1} \wedge \cdots \wedge e^{k_j} + c_2 e^{\ell_1} \wedge \cdots \wedge e^{\ell_i}\) with \(c_1, c_2 \in \mathbb{C}[[\lambda]]\setminus\{0\}\). Since \(a \circ_{\text{std}} b = \mu \circ e^{2i\lambda j(e^k \otimes e_{\ell})}(a \otimes b)\) we get

\[
\delta_b(a) = \delta(b^\ast \circ_{\text{std}} a \circ_{\text{std}} b) = (2i\lambda)^{i+j} (-i)^j \left( -1 \right)^j \frac{cc_1 c_2 + c c_1 c_2}{cc_1 c_2 + c c_1 c_2}.
\]

Choosing for example \(c_1 = \overline{c}\) as well as \(c_2\) such that \((-1)^j c_2 = \overline{c_2}\) yields \(\delta_b(a) \neq 0\). The above procedure can be easily extended to a general Hermitian element. □

**Remark 4.7.** Even though the complex conjugation yields no involution for the standard ordered star product, one can check that it is a well-defined super \(*\)-involution for the Weyl ordered one, see [4] for a definition. However, in this setting one can show that there are no non-trivial positive linear functionals.

### 4.2. Comparison of the reduced quantum BRST algebras

Throughout this section assume that \((M, \star, G, J, C)\) is a Hamiltonian quantum G-space with regular constraint surface, proper action on \(M\) and Hermitian star product. Moreover, choose a positive definite inner product \(g\) on the Lie algebra \(g\) with induced involution \(*\) on \((\Lambda^\bullet(\mathfrak{g}_C^\ast \oplus \mathfrak{g}_C)[[\lambda]], \circ_{\text{std}})\) as in (4.10).

**Lemma 4.8.** The triple \((\mathcal{A}[[\lambda]], \circ_{\text{std}}, \ast)\) is a BRST \(*\)-algebra with imaginary ghost operator and it has sufficiently many positive linear functionals.
Proof. We immediately get a graded *-involution on the whole quantum BRST algebra \(A^*(\Lambda\lambda),\) again denoted by *, via \((\alpha \otimes f)^* = \alpha^* \otimes \overline{f}\). In particular, it follows that
\[
((\alpha \otimes f)^* \otimes (\beta \otimes g))^* = (\alpha \circ_{\text{std}} (\beta^*)^* \otimes \overline{f} \ast g = (\beta^* \circ_{\text{std}} \alpha^*) \otimes (\overline{g} \ast \overline{f})) = \beta \otimes g \ast_{\text{std}} (\alpha \otimes f)^*
\]
for all \(\alpha, \beta \in \Lambda^*(g^*_C \oplus g_C)\) and \(f, g \in \mathcal{C}^\infty(M)[\lambda]\). For the BRST charge \(\gamma = \frac{1}{2} e^k \otimes e_k\) we have already seen \(\gamma^* = -\gamma\) in Proposition 4.5, thus the ghost number derivation \(G \gamma = \frac{1}{2} \text{ad}(\gamma)\) fulfills \(G \gamma^* = -G \gamma\). Therefore, we have constructed a graded *-involution with imaginary ghost operator. The only thing remaining to be shown is that the quantum BRST algebra has sufficiently many positive linear functionals. By Proposition 4.6 the Grassmann part has sufficiently many positive linear functionals and \((\mathcal{C}^\infty(M)[\lambda], \ast)\) with complex conjugation as involution has sufficiently many positive linear functionals, see [6, Prop. 5.3]. Moreover, [7, Prop. 2.8] states that a unital *-algebra has sufficiently many positive linear functionals if and only if it has a faithful *-representation on a pre-Hilbert space. Both the Grassmann algebra \((\Lambda^*(g^*_C \oplus g_C)[\lambda], \circ_{\text{std}})\) and the functions are unital *-algebras and the Grassmann part has already such a *-representation \(\rho_{\text{std}}\) as discussed in Proposition 4.5. In particular, we know
\[
\rho_{\text{std}}(x_{ij}) : \Lambda^k g^*_C[\lambda] \longrightarrow \begin{cases} \Lambda^{k+i-j} g^*_C[\lambda] & \text{for } k \geq j \\ 0 & \text{else} \end{cases} \quad (\ast)
\]
for all \(x_{ij} \in \Lambda^i g^*_C[\lambda] \otimes \Lambda^j g_C[\lambda].\) Let \(\pi : \mathcal{C}^\infty(M)[\lambda] \longrightarrow \mathcal{H}\) be a faithful *-representation of the functions on a pre-Hilbert space \(\mathcal{H}\). It remains to show that
\[
\rho = \rho_{\text{std}} \otimes \pi : A[\lambda] \longrightarrow \mathcal{B}(\Lambda^* g^*_C[\lambda] \otimes \mathcal{H})
\]
is injective. Consider a general element \(\sum x_{ij} \alpha_{ij} \otimes F^{ij}_{\alpha ij} \in A[\lambda],\) where \(\{x_{ij}\}_{\alpha ij}\) is a basis of \(\Lambda^i g^*_C \otimes \Lambda^j g_C,\) and where \(F^{ij}_{\alpha ij} \in \mathcal{C}^\infty(M)[\lambda].\) Let \(k\) be the minimal index such that \(\sum x_{k\alpha_{ik}} \otimes F^{ik}_{\alpha ik} \neq 0.\) Using \((\ast)\) we have \(\rho_{\text{std}}(x_{ik\alpha_{ik}})z \in \Lambda^i g^*_C,\) for \(z \in \Lambda^k g^*_C[\lambda]\), so the images for different \(i = 0, 1, \ldots, n\) are either zero or linearly independent, allowing us to fix the index \(i.\) A straightforward computation shows
\[
\rho_{\text{std}}(e^{j_1} \wedge \cdots \wedge e^{j_i} \wedge e^{\ell_1} \wedge \cdots \wedge e^{\ell_1})z = (2i\lambda)^{k} e^{j_1} \wedge \cdots \wedge e^{j_i} \wedge i(e^{\ell_1} \wedge \cdots \wedge i(e^{\ell_1}))z.
\]
Choosing now \(\phi \in \mathcal{H}\) such that \(\pi \left(F^{j_k \cdots j_i}_{p_1 \cdots p_i}\right) \phi \neq 0\) for some sets of indices \(\{r_1, \ldots, r_k\}\) and \(\{p_1, \ldots, p_i\}\) yields
\[
\rho \left(e^{j_1} \wedge \cdots \wedge e^{j_i} \wedge e^{\ell_1} \wedge \cdots \wedge e^{\ell_1} \otimes F^{\ell_k \cdots \ell_1}_{j_k \cdots j_i}\right) \left(\pi \left(F^{r_k \cdots r_i}_{j_k \cdots j_i}\right) \phi \right) = k! (2i\lambda)^{k} e^{j_1} \wedge \cdots \wedge e^{j_i} \otimes \pi \left(F^{r_k \cdots r_i}_{j_k \cdots j_i}\right) \phi \neq 0.
\]
Now we can define as in the general setting from Sect. 3 an adjoint standard ordered BRST operator \(D^*_{\text{std}}\) by
\[
D^*_{\text{std}} = \frac{1}{i\lambda} \text{ad}_{\text{std}}(\Theta^*_{\text{std}}).
\]
We get two different quantum BRST cohomologies: on one hand, the usual quantum BRST cohomology $H_{\text{BRST}}^\bullet([\lambda])$ from Proposition 2.1 with corresponding reduced quantum BRST algebra

$$A_{\text{red}} = H_{\text{BRST}}^0([\lambda]) = \frac{\ker D_{\text{std}} \cap \mathcal{A}^0[[\lambda]]}{\text{im}^\lambda D_{\text{std}} \cap \mathcal{A}^0[[\lambda]]}. \quad (4.12)$$

On the other hand, we have the BRST quotient $\tilde{H}_{\text{BRST}}^\bullet([\lambda])$ from (3.14) with corresponding reduced quantum BRST $^\star$-algebra

$$\tilde{A}_{\text{red}} = \tilde{H}_{\text{BRST}}^0([\lambda]) = \frac{\ker D_{\text{std}} \cap \ker D_{\text{std}}^\star \cap \mathcal{A}^0[[\lambda]]}{\text{im}^\lambda D_{\text{std}} \cap \text{im}^\lambda D_{\text{std}}^\star \cap \mathcal{A}^0[[\lambda]]} \quad (4.13)$$

that is indeed a $^\star$-algebra by Lemma 3.9. Therefore, the natural question is whether we can compare $H_{\text{BRST}}^\bullet([\lambda])$ with $\tilde{H}_{\text{BRST}}^\bullet([\lambda])$, in particular in ghost number zero. The rest of the section consists in the proof of the following main result.

**Theorem 4.9.** Let $(M, ^\star, G, J, C)$ be a Hamiltonian quantum $G$-space with regular constraint surface, compact Lie group and Hermitian star product $^\star$. Moreover, choose a positive definite inner product on the corresponding Lie algebra $\mathfrak{g}$, inducing the involution $^\star$. Then one has

$$\tilde{A}_{\text{red}} \cong C^\infty(C)^G[[\lambda]] \cong A_{\text{red}} \quad (4.14)$$

with isomorphism $\iota^\star: \tilde{A}_{\text{red}} \longrightarrow C^\infty(C)^G[[\lambda]]$ and inverse prol.

We already know from Proposition 2.1 that there is an isomorphism

$$\iota^\star: A_{\text{red}} = \frac{\ker D_{\text{std}} \cap \mathcal{A}^0[[\lambda]]}{\text{im}^\lambda D_{\text{std}} \cap \mathcal{A}^0[[\lambda]]} \longrightarrow C^\infty(C)^G[[\lambda]]$$

with inverse $\tilde{h}|_{C^\infty(C)^G[[\lambda]]} = \text{prol}$, where we understand $\iota^\star$ to act on the representatives of the equivalence classes and prol to map into the corresponding equivalence class. We directly observe the following:

**Lemma 4.10.** The map

$$\iota^\star: \tilde{A}_{\text{red}} \longrightarrow C^\infty(C)^G[[\lambda]]$$

is well-defined with right inverse prol. Moreover, prol is also a left inverse if

$$\sum_{i, \alpha_i} \text{prol} \iota^\star \biggl( x_{i\alpha_i} \otimes F^{i\alpha_i} \biggr) - \sum_{i, \alpha_i} x_{i\alpha_i} \otimes F^{i\alpha_i} \in \text{im} D_{\text{std}} \cap \text{im} D_{\text{std}}^\star, \quad (4.15)$$

holds for any element $\sum_{i, \alpha_i} x_{i\alpha_i} \otimes F^{i\alpha_i} \in \ker D_{\text{std}} \cap \ker D_{\text{std}}^\star \cap \mathcal{A}^0[[\lambda]]$, where $\{x_{i\alpha_i}\}_{\alpha_i}$ denotes a basis of $\Lambda^i \mathfrak{g}^* \otimes \Lambda^i \mathfrak{g}$, and $F^{i\alpha_i} \in C^\infty(M)[[\lambda]]$ for any $i = 0, 1, \ldots, n$ and $\alpha_i$.

**Proof.** The map (4.15) is well-defined as $\iota^\star$ vanishes on $\text{im} D_{\text{std}} \cap \text{im} D_{\text{std}}^\star \cap \mathcal{A}^0[[\lambda]]$. Moreover, since $\phi \in C^\infty(C)^G[[\lambda]]$ implies $\bar{\phi} \in C^\infty(C)^G[[\lambda]]$ and since $\delta = \delta$, we have $\text{prol} \iota^\star = \text{id}_{C^\infty(C)^G[[\lambda]]}$ and

$$D_{\text{std}}(\text{prol}(\phi)) = (\delta + 2\delta)(\text{prol}(\phi)) = \text{prol} \delta^c \phi = 0,$$

$$D_{\text{std}}^\star(\text{prol}(\phi)) = ((\delta + 2\delta)(\text{prol}(\phi)))^* = (\text{prol} \delta^c \bar{\phi})^* = 0.$$

So prol is still a well-defined right inverse of $\iota^\star$. $\square$
These conditions can be further simplified by exploiting the chain homotopy from Proposition 2.1.

**Proposition 4.11.** Let \((M, \star, G, J, \mathcal{C})\) be a Hamiltonian quantum G-space with regular constraint surface, proper action on \(M\) and Hermitian star product \(\star\). Moreover, let \(g\) be a positive definite inner product on \(g\) inducing the involution \(\ast\) via \((4.10)\). Then \(\iota^* : \mathcal{A}_{\text{red}} \to \mathcal{C}^\infty(C)^G[[\lambda]]\) is an isomorphism with inverse prol if

\[
\iota^* F^0 = \iota^* F^0
\]  

(4.17)

for \(\sum_{i, \alpha_i} x_{i \alpha_i} \otimes F^{i \alpha_i} \in \ker D_{\text{std}} \cap \ker D_{\text{std}} \cap \mathcal{A}^{(0)}[[\lambda]]\) with \(F^0 = x_{0 \alpha_0} \otimes F^{0 \alpha_0} \in \mathcal{C}^\infty(M)[[\lambda]]\) as above.

**Proof.** We consider at first the augmented standard ordered BRST operator \(\hat{D}_{\text{std}} = D_{\text{std}} + \delta^c + 2\iota^*\). We know from Proposition 2.1 that \(\hat{D}_{\text{std}} \hat{h} + \hat{h} D_{\text{std}} = 2\iota^*\), which entails

\[
\sum_{i, \alpha_i} \text{prolt}^* \left( x_{i \alpha_i} \otimes F^{i \alpha_i} \right) = \frac{1}{2} \sum_{i, \alpha_i} \left( \hat{h} D_{\text{std}} \left( x_{i \alpha_i} \otimes F^{i \alpha_i} \right) \right) = \frac{1}{2} \sum_{i, \alpha_i} D_{\text{std}}^* \left( \hat{h} \left( x_{i \alpha_i} \otimes F^{i \alpha_i} \right) \right)
\]

as \(\hat{D}_{\text{std}} \hat{h} \left( \sum_{i, \alpha_i} x_{i \alpha_i} \otimes F^{i \alpha_i} \right) = D_{\text{std}} \hat{h} \left( \sum_{i, \alpha_i} x_{i \alpha_i} \otimes F^{i \alpha_i} \right)\). Applying the \(\ast\)-involution yields

\[
\sum_{i, \alpha_i} \text{prolt}^* \left( x_{i \alpha_i} \otimes F^{i \alpha_i} \right) = \sum_{i, \alpha_i} \left( \chi_{i \alpha_i} \otimes F^{i \alpha_i} \right)^* + \frac{1}{2} \sum_{i, \alpha_i} D_{\text{std}}^* \left( \hat{h} \left( x_{i \alpha_i} \otimes F^{i \alpha_i} \right) \right)^*
\]

by the definition of \(D_{\text{std}}^*\). Because of \(\left( \sum_{i, \alpha_i} x_{i \alpha_i} \otimes F^{i \alpha_i} \right)^* \in \ker D_{\text{std}} \cap \ker D_{\text{std}} \cap \mathcal{A}^{(0)}[[\lambda]]\) we also get

\[
\sum_{i, \alpha_i} \left( x_{i \alpha_i} \otimes F^{i \alpha_i} \right)^* = \sum_{i, \alpha_i} \text{prolt}^* \left( x_{i \alpha_i} \otimes F^{i \alpha_i} \right)^* + \frac{1}{2} \sum_{i, \alpha_i} D_{\text{std}}^* \hat{h} \left( x_{i \alpha_i} \otimes F^{i \alpha_i} \right)^*.
\]

Thus to prove the desired \((4.16)\) it suffices to show

\[
\sum_{i, \alpha_i} \text{prolt}^* \left( x_{i \alpha_i} \otimes F^{i \alpha_i} \right) - \sum_{i, \alpha_i} \text{prolt}^* \left( x_{i \alpha_i} \otimes F^{i \alpha_i} \right)^* \in \text{im} D_{\text{std}}^*,
\]

which is fulfilled if \(\iota^* F^0 = \iota^* F^0\). \(\square\)

In general we do not know if \(F^0\) is G-invariant, i.e. \(\delta F^0 = \delta F^0 = 0\), as the higher orders of \(\sum_{i, \alpha_i} x_{i \alpha_i} \otimes F^{i \alpha_i}\) could cancel this term under \(D_{\text{std}}\). Hence we can not apply [15, Cor. 4.6], giving exactly the property \(\iota^* f = \iota^* f\) for invariant functions \(f \in \mathcal{C}^\infty(M)[[\lambda]]\). The idea to check \((4.17)\) is to construct an inner product on \(\mathcal{C}^\infty(C)^G[[\lambda]]\) with values in \(\mathcal{C}^\infty(C)^G[[\lambda]]\) and a corresponding \(\ast\)-representation of \((\mathcal{C}^\infty(M)[[\lambda]], \ast)\), see [15, Def. 5.4].

**Definition 4.12 (Algebra-valued inner product).** Let \(G\) be a compact Lie group. The \(\mathcal{C}^\infty(C)^G[[\lambda]]\)-valued inner product on \(\mathcal{C}^\infty(C)[[\lambda]]\) is pointwise defined by

\[
(\phi, \psi)_{\text{red}}(c) = \int_G \left( \iota^* \left( \text{prolt}(\phi) \ast \text{prolt}(\psi) \right) \right) (\Phi_{g^{-1}}(c)) d\text{left} g \quad (4.18)
\]

for all \(\phi, \psi \in \mathcal{C}^\infty(C)[[\lambda]]\) and \(c \in C\), where \(d\text{left} g\) denotes the left invariant Haar measure.
Then \((\cdot, \cdot)_{\text{red}}\) is well-defined, \(\mathbb{C}[[\lambda]]\)-sesquilinear and can be rewritten in the following way, see [15, Lemmas 5.6, 5.8 and 5.9].

**Proposition 4.13.** The map \((\cdot, \cdot)_{\text{red}}\) defines a non-degenerate inner product on \(\mathcal{C}^\infty(C)[[\lambda]]\) with values in the invariant functions \(\mathcal{C}^\infty(C)^G[[\lambda]]\). One can rewrite it in an alternative way
\[
(\phi, \psi)_{\text{red}} = t^* \int_G \frac{\Phi^{*-1}}{g^*} \left( \text{prol}(\phi) \star \text{prol}(\psi) \right) \, d^\text{left} g = t^* \int_G \frac{\text{prol}(\Phi^{*-1}) \star \text{prol}(\Phi^{*-1}) \psi}{g^*} \, d^\text{left} g.
\]

(4.19)

*In particular, one has for all \(\phi, \psi \in \mathcal{C}^\infty(C)[[\lambda]]\)*
\[
(\phi, \psi)_{\text{red}} = (\psi, \phi)_{\text{red}}.
\]

(4.20)

Recall the left action \(\bullet\) of \((\mathcal{C}^\infty(M)[[\lambda]], \star)\) on \(\mathcal{C}^\infty(C)[[\lambda]]\) from [15, Def. 3.7] that is given by
\[
\mathcal{C}^\infty(M)[[\lambda]] \times \mathcal{C}^\infty(C)[[\lambda]] \ni (f, \phi) \mapsto f \bullet \phi = t^*(f \star \text{prol}(\phi)) \in \mathcal{C}^\infty(C)[[\lambda]].
\]

(4.21)

The key point is now that this action yields a \(^*\)-representation, see [15, Prop. 5.11].

**Proposition 4.14.** The action \(\bullet\) is a \(^*\)-representation of \((\mathcal{C}^\infty(M)[[\lambda]], \star)\) on \(\mathcal{C}^\infty(C)[[\lambda]]\) with respect to the inner product \((\cdot, \cdot)_{\text{red}}, \star\), i.e. we have for all \(\phi, \psi \in \mathcal{C}^\infty(C)[[\lambda]]\) and \(f \in \mathcal{C}^\infty(M)[[\lambda]]\)
\[
(\phi, f \bullet \psi)_{\text{red}} = (\mathcal{F}, \phi, \psi)_{\text{red}}.
\]

(4.22)

Now we can finally prove Theorem 4.9.

**Proof.** (of Theorem 4.9) By Proposition 4.11 it suffices to show
\[
\overline{t^* F^0} = t^* F^0,
\]
where \(F^0 = 1 \otimes F^0\) is again the lowest order of some \(\sum_{i,a_i} x_{ai} \otimes F_{ai} \in \ker D_{\text{std}} \cap \ker D^*_{\text{std}} \cap A(0)[[\lambda]]\). Just as above, \(\{ x_{ai} \}_{ai} \) is a basis of \(\Lambda^i g^* \otimes \Lambda^i g\) and \(F_{ai} \in \mathcal{C}^\infty(M)[[\lambda]]\) for all \(i = 0, 1, \ldots, n\) and all \(a_i\). By the construction of prol and assuming \(M = M_{\text{nice}}\) in the notation of [15, Sec. 2.2] we know prol(1) = 1 \(\in \mathcal{C}^\infty(M)[[\lambda]]\) for the constant function \(1 \in \mathcal{C}^\infty(C)[[\lambda]]\). Thus we get
\[
F^0 \bullet 1 = t^*(F^0 \star 1) = t^* F^0 \in \mathcal{C}^\infty(C)^G[[\lambda]]
\]
as well as \(\overline{F^0} \bullet 1 = t^* F^0 \in \mathcal{C}^\infty(C)^G[[\lambda]]\), which implies \(\Phi_{g-1} t^* F^0 = t^* F^0\) and \(\Phi_{g-1} t^* F^0 = t^* F^0\). Using (4.19) we can compute
\[
(1, F^0 \bullet 1)_{\text{red}} = \int_G t^* \left( \text{prol} \left( \Phi_{g-1} \right) \star \text{prol} \left( \Phi_{g-1} t^* F^0 \right) \right) \, d^\text{left} g = t^* F^0 \int_G d^\text{left} g,
\]
and analogously
\[
(\overline{F^0} \bullet 1, 1)_{\text{red}} = \int_G t^* \left( \text{prol} \left( \Phi_{g-1} t^* F^0 \right) \star \text{prol} \left( \Phi_{g-1} \right) \right) \, d^\text{left} g = t^* F^0 \int_G d^\text{left} g,
\]
which together with (4.22) implies the desired \(t^* F^0 = \overline{t^* F^0}\). □
If the action is in addition free on \( C \), we know that \( M_{\text{red}} = C/G \) is a smooth manifold and with Proposition 2.1 we have an induced star product \( \ast_{\text{red}} \) on \( \mathcal{C}^\infty(M_{\text{red}})[[\lambda]] \cong \mathcal{C}^\infty(C)^G[[\lambda]] \) given by

\[
\pi^*(u_1 \ast_{\text{red}} u_2) = t^*(\text{prol}(\pi^*u_1) \ast \text{prol}(\pi^*u_2))
\]

(4.23)

for \( u_1, u_2 \in \mathcal{C}^\infty(M_{\text{red}})[[\lambda]] \). Thus we immediately get:

**Corollary 4.15.** Let \((M, \ast, G, J, C)\) be a Hamiltonian quantum \( G \)-space with regular constraint surface, positive definite inner product on \( g \) and Hermitian star product \( \ast \). In addition, let the compact Lie group \( G \) act freely on \( C \). Then one has

\[
\tilde{A}_{\text{red}} \cong \mathcal{C}^\infty(M_{\text{red}})[[\lambda]] \cong A_{\text{red}}.
\]

(4.24)

Moreover, \( t^\ast \) induces the complex conjugation as involution on \((\mathcal{C}^\infty(M_{\text{red}})[[\lambda]], \ast_{\text{red}})\).

**Proof.** The fact that this construction induces the complex conjugation as involution for the reduced star product follows as in [15, Prop. 4.7]. Explicitly, we have

\[
\pi^*(u_1 \ast_{\text{red}} u_2) = t^*(\text{prol}(\pi^*u_1) \ast \text{prol}(\pi^*u_2)) = t^*(\text{prol}(\pi^*u_1) \ast \text{prol}(\pi^*u_2)) = t^*(\text{prol}(\pi^*u_2) \ast \text{prol}(\pi^*u_1)) = \pi^*(u_2 \ast_{\text{red}} u_1)
\]

for all \( u_1, u_2 \in \mathcal{C}^\infty(M_{\text{red}})[[\lambda]] \). \( \square \)

There exists another construction of a \( \ast \)-involution for \( \ast_{\text{red}} \) via the GNS representation for a suitably chosen positive functional depending on a density, see [15, Thm. 4.17]. In comparison to this one we get always the same \( \ast \)-involution for the reduced star product, independently on the inner product on \( g \). This is due to the fact that the choice of a density for the positive functional is a non-canonical one, whereas all inner products on the Lie algebra lead to isomorphic reduced \( \ast \)-algebras. Also, in the only order where \( t^* |_{\mathcal{A}^{(0)}[[\lambda]]} \) does not vanish identically, i.e. on the functions \( \mathcal{A}^{(0)}[[\lambda]] = \mathcal{C}^\infty(M)[[\lambda]] \), the induced involution is always just the complex conjugation.

**Remark 4.16.** From [15, Cor. 4.6] we know that one has \( t^* f = \overline{t^* f} \) for \( G \)-invariant functions \( f \in \mathcal{C}^\infty(M)[[\lambda]] \), which implies that the complex conjugation is an involution for \( \ast_{\text{red}} \). Furthermore, as one needs this identity to show that \((\cdot, \cdot)_{\text{red}}\) satisfies for all \( \phi, \psi \in \mathcal{C}^\infty(C)[[\lambda]] \)

\[
(\phi, \psi)_{\text{red}} = (\overline{\psi}, \phi)_{\text{red}} \quad \text{and} \quad (\phi, f \ast \psi)_{\text{red}} = (\overline{f} \ast \phi, \psi)_{\text{red}}.
\]

it is not surprising that the construction via the algebra-valued inner product yields again the complex conjugation as involution on \((\mathcal{C}^\infty(M_{\text{red}})[[\lambda]], \ast_{\text{red}})\). Therefore, it is important to remark that the complex conjugation is induced by an isomorphism with the reduced quantum BRST \( \ast \)-algebra \( \tilde{A}_{\text{red}} \cong \mathcal{C}^\infty(M_{\text{red}})[[\lambda]] \) if \( M_{\text{red}} \) exists as smooth manifold.

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