On some Generalized rare regular Closed sets in topological spaces

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Abstract. This research introduces a new type of sets are said to be a generalized * rare regular Closed set, rare regular generalized * -closed set, generalized ** rare regular -closed set and rare regular generalized** -closed set in topological spaces. Some properties and related theorems associated with this new sets is proved in topological spaces.

Keywords. Rrg*-closed set, Rg*r –closed set, Rrg**-closed set and Rg**r –closed set.

1. Introduction
In 1900, subfields of topology was introduced such as the general topology that deals with local properties of spaces and are closely related to analysis. Many researchers have worked on topological spaces. In 1970, N. Levine [7] introduced the concept of generalized closed sets. Many researchers like Balachandran, Sundaram and Maki [4], Bhattacharyya and Lahiri [9], Dunham [13], Gnanambal [14], Malghan [12], Palaniappan and Rao [8] have worked on generalized closed sets, their generalizations and related concepts in general topology. Bhattacharya [10] introduced the concept of generalized regular closed sets. M.K.R.S. Veerakumar [5], [6] introduced several generalized closed sets namely, g*- closed sets, g#- closed sets, g* p - closed sets, Mashhour, Abd El-Monsef and El-Deeb [1], introduced a Pre-continuous and Weak Pre- continuous functions. S. Balasubramanian and *M. Lakshmi [11] introduced further properties of g p r-closed sets, A. Vadival and K. V. Manickam [2] introduced rgα-closed sets and rgα-open sets in topological spaces. This research introduces a new type of sets are said to be a Generalized rare regular Closed set in topological spaces, with study their properties. Finally, we investigate relationships between them.

2. Preliminaries
Since We require the following known definitions, notations, and some properties so we recall them in this section that led to the development of our main results.

Definition 2.1[10]: A subset \( \mu \) of \( X \) is called a rarely regular Closed set if \( \text{clint}(\mu) = X \).

Definition 2.2[10]: A subset \( \mu \) of \( X \) is called a rarely regular open set if \( \text{intcl}(\mu) = \emptyset \).

Definition 2.3[10]: 1-ClRC(\( \mu \)) = \( \bigcap \{ U : \mu \subseteq U, U \text{ is rare regular Closed subset of } X \} \)

2-IntRO(\( \mu \)) = \( \bigcup \{ U : U \subseteq \mu, U \text{ is rare regular open subset of } X \} \)
Definition 2.4[3]: A subset $A$ of a topological space $(X, \tau)$ is said to be a regular generalized* Closed set (briefly $rg^*$-Closed set) if $\text{cl}(\mu) \supseteq U$, whenever $\mu \subseteq U$ and $U$ is regular open subset of $X$.

Definition 2.5[4]: A subset $\mu$ of a space $X$ is said to be a generalized* regular Closed (briefly $g^r$-Closed) set if $\text{Rcl}(\mu) \subseteq U$, whenever $\mu \subseteq U$ and $U$ is regular-open subset of $X$.

Definition 2.6[3]: A subset $\mu$ of a space $X$ is said to be a pre-regular open (briefly, PR-Open) set if there is a pre-open set $U$ such that $U \subseteq \mu \subseteq \text{Rcl}(U)$. The family of all pre-regular-open sets of $X$ is denoted by $\text{PRO}(x)$.

Definition 2.7[3]: A subset $\psi$ of a space $X$ is said to be a regular generalized**- Closed set (briefly, $rg^{**}$ - Closed) if $U \subseteq \text{cl}(\psi)$, whenever $\psi \subseteq U$, and $U$ is pre-regular-open subset of $X$.

Definition 2.8[3]: A subset $\psi$ of a space $X$ is said to be a generalized** regular Closed set (briefly, $g^{r**}$-Closed set) if $U \subseteq \text{Rcl}(\psi)$, whenever $\psi \subseteq U$, $U$ is pre-regular-open subset of $X$.

3. Generalized rare regular Closed sets in topological spaces

In this part we introduce some definition of generalized rare regular closed sets and studied some of its properties with related theorems are proved.

Definition 3.1: A subset $\mu$ of a space $X$ is said to be rare regular generalized* Closed set (briefly $Rrg^*$-Closed) if $U \subseteq \text{cl}(\mu)$ whenever $\mu \subseteq U$, $U$ is a regular-open set and $\text{clint}(\mu)=X$.

Example 3.2: Let $X=\{a, b, c\}$, $\tau=\{\emptyset, \{a, b\}, \{a\}, x\}$, $\tau^c=\{x, \{c\}, \{b, c\}, \phi\}$, $\text{Ro}(x)=\{\phi, x\}$, $\text{Rc}(x)=\{x, \phi\}$

$\psi=\{a, c\}$, $U=\{a, c\}$, $U \subseteq \text{cl}(\psi)$, $\psi \subseteq U$, $U$ is regular open, $\text{clint}(\psi)=X$, then $\psi$ is a $Rrg^*$-Closed set.

Definition 3.3: A subset $\mu$ of a topological space $(X, \tau)$ is said to be rare generalized* regular Closed set (briefly $Rg^r$-Closed) if $\text{Rcl}(\mu) \subseteq U$ whenever $\mu \subseteq U$, $U$ is regular-open subset of $X$ and $\text{clint}(\mu)=X$.

Example 3.4: Let $X=\{a, b, c\}$, $\tau=\{\phi, \{a, b\}, \{a\}, x\}$, $\tau^c=\{x, \{c\}, \{b, c\}, \phi\}$, $\text{Ro}(x)=\{\phi, x\}$, $\text{Rc}(x)=\{x, \phi\}$

$A=\{a, b\}$, $U=\{a, b, c\}$, $\text{Rcl}(A) \subseteq U$, $A \subseteq U$, $U$ is regular-open set and $\text{clint}(A)=X$, so $A$ is a $Rg^r$-Closed set.

Definition 3.5: A subset $\mu$ of a topological space $(X, \tau)$ is said to be rare regular generalized**- Closed set (briefly, $Rrg^{**}$-Closed set) if $U \subseteq \text{cl}(\mu)$ whenever $\mu \subseteq U$, $U$ is pre-regular-open subset of $X$ and $\text{clint}(\mu)=X$.

Example 3.6: Let $X=\{a, b, c\}$, $\tau=\{\phi, \{a, b\}, \{a\}, \{x\}\}$, $\tau^c=\{x, \{c\}, \{b, c\}, \phi\}$, $\text{Ro}(x)=\{\phi, x\}$, $\text{Rc}(x)=\{x, \phi\}$

$\text{PRO}(x)=\{\phi, x\}$, $\text{PRC}(x)=\{x, \phi\}$, $A=\{a, c\}$, $U=\{a, c\}$ is pre-regular-open set, $U \subseteq \text{cl}(A)$, $A \subseteq U$, $U$ is pre-regular-open set and $\text{clint}(A)=X$, so $A$ is a $Rrg^{**}$-Closed set.

Definition 3.7: A subset $\mu$ of a topological space $(X, \tau)$ is said to be rare generalized** regular Closed set (briefly, $Rgr^{**}$-Closed set) if $U \subseteq \text{cl}(\mu)$, whenever $\mu \subseteq U$, $U$ is pre-regular-open subset of $X$ and $\text{clint}(\mu)=X$.

Example 3.8: Let $X=\{a, b, c\}$, $\tau=\{x, \{c\}, \{b, c\}, \phi\}$, $\tau^c=\{\phi, \{a, b\}, \{a\}, X\}$, $\text{Ro}(x)=\{\phi, x\}$, $\text{Rc}(x)=\{x, \phi\}$, $\text{PRO}(x)=\{\phi, x\}$, $\text{PRC}(x)=\{\phi, x\}$, $\mathcal{B}=\{a\}$, $U=\{a, b, c\}$, $\text{Rcl}(\{a\})=X$, $U \subseteq \text{cl}(\mathcal{B})$, and $\text{clint}(\mathcal{B})=X$, then $\mathcal{B}$ is a $Rgr^{**}$-Closed set.

Proposition 3.9: rare $g^{r**}$-Closed set $\rightarrow$ rare $rg^{**}$-Closed set the reverse does not materialize.
Proof: Assume \( \mu \) is \( Rg^{**} \)-Closed set, \( \nu \) is pre-regular- open and \( cl(\mu) = X \). To prove \( \mu \) is \( Rg^{**} \)-Closed set. Since every \( \text{Rcl}(\mu) \subseteq cl(\mu) \), \( \nu \subseteq \text{Rcl}(\mu) \subseteq cl(\mu) \). \( \mu \) is \( Rg^{**} \)-Closed set.

The converse of this Proposition is not true the example is show this case.

**Example 3.10:** Let \( X = \{a, b, c\} \), \( \tau = \{\phi, \{a, b\}, \{a\}, x\} \), \( \tau^c = \{x, \{c\}, \{b, c\}, \phi\} \), \( \text{Ro}(x) = \{\phi, x\} \)

\( \text{Rc}(x) = \{x, \phi\} \), \( \text{PRO}(x) = \{\phi, x, \{a\}, \{b, c\}, \phi\} \), \( \text{PRC}(x) = \{x, \phi, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}\} \)

\( \omega = \{a\} \)

\( U = \{a\}, c(x) = X \), \( U \subseteq cl(\omega) \) and \( cl(\omega) = X \), then \( \omega \) is \( Rg^{**} \)-Closed set but \( \omega \) is not \( Rg^{*} \)-Closed set, because

\( \text{Rcl}(\{a\}) = X \), \( \omega \neq \{a\} \) then \( x = \text{cl}(\omega) \) but \( \omega \) is not \( \text{Rcl}(\omega) \).

**Proposition 3.11:** The relation between rare \( rg^{*} \)-Closed set with rare \( g^{*} \)-Closed set is Separately.

**Example 3.12:** Let \( X = \{a, b, c\} \), \( \tau = \{\phi, \{a, b\}, \{a\}, x\} \), \( \tau^c = \{x, \{c\}, \{b, c\}, \phi\} \), \( \text{Ro}(x) = \{\phi, x\} \)

\( \text{Rc}(x) = \{x, \phi\} \), \( \text{PRO}(x) = \{\phi, x, \{a\}, \{b, c\}, \phi\} \), \( \text{PRC}(x) = \{x, \phi, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}\} \)

\( U = \{a\}, c(x) = X \), \( U \subseteq \text{cl}(\omega) \), \( U = \{a\} \), \( \text{Rcl}(\{a\}) = X \), \( \omega = \{a\} \).

\( \omega \neq \{a\} \) then \( x = \text{cl}(\omega) \), \( \omega = \{a\} \).

Recall example 3.2 : \( \psi = \{a, c\}, U = \{a, c\} \), \( U \subseteq cl(\psi) \), \( \psi \subseteq U \), \( U \) is regular open, \( cl(\psi) = X \). So \( \psi \) is \( Rg^* \)-Closed set but \( \psi \) is not \( Rg^{*} \) -Closed set because \( Rcl(\psi) = \phi \), \( U \subseteq \text{Rcl}(\psi) \), then \( \psi \) is not \( Rg^{*} \)-Closed set.

**Proposition 3.13:** rare \( g^{*} \)-Closed set \( \rightarrow \) rare \( rg^{*} \)-Closed set but the converse is not true.

Proof: Assume \( \mu \) is \( Rg^{*} \)-Closed set, \( U \subseteq \text{Rcl}(\mu) \), \( U \) is pre-regular -open and \( cl(\mu) = X \), to prove \( \mu \) is

\( Rg^{*} \)-Closed set.

Since every \( \text{Rcl}(\mu) \subseteq \text{cl}(\mu) \), \( U \subseteq \text{Rcl}(\mu) \subseteq \text{cl}(\mu) \). Hence \( \mu \) is \( Rg^{*} \)-Closed set.

Recall example 3.2 : \( \psi = \{a, c\}, U = \{a, c\} \), \( U \subseteq cl(\psi) \), \( \psi \subseteq U \), \( U \) is regular open, \( cl(\psi) = X \). So \( \psi \) is \( Rg^{*} \)-Closed set but \( \psi \) is not \( Rg^{*} \)-Closed set because \( Rcl(\psi) = \phi \), \( U \subseteq \text{Rcl}(\psi) \), then \( \psi \) is not \( Rg^{*} \)-Closed set.

**Proposition 3.14:** rare \( g^{*} \)-Closed set \( \rightarrow \) rare \( g^{**} \)-Closed set but the reverse does not materialize.

Proof: Assume \( \mu \) is \( Rg^{*} \)-Closed set, \( U \subseteq \text{Rcl}(\mu) \), \( U \) is regular -open, \( cl(\mu) = X \). To prove \( \mu \) is \( Rg^{*} \)-Closed set.

Since every \( \text{Rcl}(\mu) \subseteq \text{cl}(\mu) \), \( U \subseteq \text{Rcl}(\mu) \subseteq \text{cl}(\mu) \). Then \( U \subseteq \text{cl}(\mu) \).

\( \text{Rc}(x) = \{x, \phi\} \), \( \text{PRO}(x) = \{\phi, x, \{a\}, \{b, c\}, \phi\} \), \( \text{PRC}(x) = \{x, \phi, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}\} \)

\( \rho = \{a\} \), \( U = \{a\} \), \( cl(\rho) = \rho \), \( U \subseteq \text{cl}(\rho) \) and \( cl(\rho) = X \) then \( \rho \) is \( Rg^{**} \)-Closed set but \( \rho \) is not \( Rg^{*} \)-Closed set, because \( Rcl(\{a\}) = X \), \( X \neq \{a\} \) then \( Rcl(\{a\}) = X \), \( X \neq \{a\} \), then \( \rho = cl(\rho) \) and \( \rho \) is not \( Rg^{*} \)-Closed set.

**Theorem 3.16:** Arbitrary union of \( Rg^{*} \)-Closed (resp.\( Rg^{**} \)-Closed) set is \( Rg^{*} \)-Closed (resp.\( Rg^{**} \)-Closed) set.

Proof: Assume \( U \) is a regular -open [ resp. pre-regular -open] set such that \( \{A_{\alpha}\} \subseteq U \) By definition for \( Rg^{*} \)-Closed set we get \( U \subseteq \text{cl}(\bigcup_{\alpha} A_{\alpha}) \subseteq U \)

**Theorem 3.17:** Arbitrary union of \( Rg^{*} \)-Closed set is \( Rg^{**} \)-Closed set.

Proof: Suppose \( U \) is a pre-regular -open set such that \( \{A_{\alpha}\} \subseteq U \) By definition for
**Theorem 3.18.** Arbitrary union of $Rg^*$-Closed set is $Rg^*$-Closed set.

Proof: Suppose $U$ is a regular -open set such that $\cup \{A_a\}_{a \in A} \subseteq U$, by definition for $Rg^*$-Closed we get $Rcl \{A_a\}_{a \in A} \subseteq U$. Then $Rcl (\cup \{A_a\}_{a \in A}) \subseteq U$.

**Remark 3.19.** Finite intersection of $Rrg^*$ - Closed [resp.$Rrg^{**}$ -Closed ] set needs not be so which follows example show this case.

Recall example 3.10 Let $X=\{a,b,c\}$, $\tau=\{\phi,\{a, b\},\{a\},\{b, c,\},\phi\}$, $\mathcal{N}=\{a\}, U=\{a,b\}$ $cl(\mathcal{N})=X$, $U \subseteq Cl (\mathcal{N})$ and $clint \mathcal{N} =X$, then $\mathcal{N}$ is is $Rg^*$ - Closed [resp. $Rg^{**}$ -Closed ] set. And $\mathcal{M}=\{b\}, cl(\{b\})=X$ $U \subseteq cl (\mathcal{M})$, $clint(\mathcal{M})=X$, then $\mathcal{M}$ is $Rg^*$ - Closed [resp. $Rg^{**}$ -Closed ] set. And $\mathcal{N} \cap \mathcal{M} =\emptyset$, $cl(\emptyset) =\emptyset$ but $\{a, b\} \notin \emptyset$. Then $\mathcal{N} \cap \mathcal{M}$ is not $Rg^*$ - Closed [resp. $Rg^{**}$ -Closed ] set.

**Remark 3.20** :Finite intersection of $Rgr^{**}$ - Closed set needs not be so which follows example show this case.

Recall example 3.8: Let $X=\{a,b,c\}$, $\tau=\{\phi,\{a, b, c\},\{a\},\{b, c\},\phi\}$, $\tau^c=\{\phi,\{a, b\},\{a\},\{a\},\{b\},\{c\}\}$ $Ro(x)=\{\phi, x\}$, $Rc(x)=\{x, \phi\}$, $PRO(x)=\{\phi, x\}$, $PRC(x)=\{x, \phi\}$, $\mathcal{Z}=\{a\}, U=\{a,b,c\}$, $Rcl(\mathcal{Z}) = X$ $U \subseteq Rcl(\mathcal{Z})$, and $clint(\mathcal{Z})=X$, then $\mathcal{Z}$ is a $Rgr^{**}$ - Closed set, and $\mathcal{Y}=\{c\}, cl(int(\mathcal{Y})=X, U \subseteq Rcl(\mathcal{Y})$, $clint(\mathcal{Y})=X$, then $\mathcal{Y}$ is a $Rgr^{**}$ - Closed set.Then $\mathcal{Z} \cap \mathcal{Y} =\emptyset$, $cl(int(\emptyset) =\emptyset$ but $\{a, b, c\} \notin \emptyset$. Then $\mathcal{Z} \cap \mathcal{Y}$ is not $Rgr^{**}$ - Closed set.

Fig. 1. follows from the above results.

**Figure 1.**

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