Gonihedric 3D Ising actions

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Abstract

We investigate a generalized Ising action containing nearest neighbour, next to nearest neighbour and plaquette terms that has been suggested as a potential string worldsheet discretization on cubic lattices by Savvidy and Wegner. We use both mean field techniques and Monte-Carlo simulations to sketch out the phase diagram.

The Gonihedric (Savvidy-Wegner) model has a symmetry that allows any plane of spins to be flipped with zero energy cost, which gives a highly degenerate vacuum state. We choose boundary conditions in the simulations that eliminate this degeneracy and allow the definition of a simple ferromagnetic order parameter. This in turn allows us to extract the magnetic critical exponents of the system.

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1 Introduction

In a series of recent papers \cite{1} Savvidy, Wegner and co-workers suggested a Gonihedric random surface action which could be written as

\[
S = \frac{1}{2} \sum_{<ij>} |\vec{X}_i - \vec{X}_j| \theta(\alpha_{ij}),
\]

on triangulated surfaces, where \(\theta(\alpha_{ij}) = |\pi - \alpha_{ij}|\) with \(\zeta\) some exponent and \(\alpha_{ij}\) is the angle between the embedded neighbouring triangles with common link \(<ij>\). This action is a robust discretization of the linear size of a surface, which is a well-defined geometrical notion that may be constructed in various equivalent ways. It was intended as an alternative to gaussian plus extrinsic curvature lattice actions of the form

\[
S = \sum_{<ij>} (\vec{X}_i - \vec{X}_j)^2 + \lambda \sum_{\Delta_i, \Delta_j} (1 - \vec{n}_i \cdot \vec{n}_j)
\]

which have been much explored \cite{2} as discretizations of rigid membranes and strings \cite{3}. Although a simulation showed that the action of equ.(1) produced flat surfaces \cite{4}, potential problems arising from the failure to suppress the wanderings of vertices in the plane were pointed out in \cite{5} for the action with \(\zeta = 1\). Possible ways to cure this are to add additional Gaussian or linear terms to the action \cite{6}, or, more satisfactorily, to simply choose \(\zeta < 1\).

A study of the scaling of the string tension and mass gap in a dynamically triangulated model with an additional linear term produced inconclusive results \cite{7}, as have simulations of the gaussian plus extrinsic curvature action, because of the difficulties of simulating a complicated action on a dynamical surface. There is thus some incentive to investigate alternative approaches to such surface models where the computational costs are less onerous. One such approach for regularizing the Gonihedric action is to restrict the allowed surfaces to a (hyper)cubic lattice. This has been pursued in some detail analytically in \cite{8,9,10} and one numerical simulation carried out in three dimensions \cite{9}. Recently Pietig and Wegner \cite{11} have demonstrated with a Peierls contour argument that a transition does exist in the three-dimensional case and obtained a bound on the critical temperature that is not contradicted by the simulations. The crucial observation for this work is that the surface theory on a cubic or hypercubic lattice can be written equivalently as a generalized Ising action, where the boundaries between the spin clusters are the surfaces of the original model. The latticized Gonihedric model assigns a non-zero action only to right angled bends in the surface and self-intersections. Normalizing the weight for a right-angled bend on a link appropriately leaves a free parameter \(\kappa\) that gives the relative weight of a self-intersection of the surface on a link. The energy of a surface on a cubic lattice is thus given explicitly by \(E = n_2 + 4\kappa n_4\), where \(n_2\) is the number of links where two plaquettes meet at a right angle and \(n_4\) is the number of links where four plaquettes meet at right angles.

A Hamiltonian which reproduces this energy on a cubic lattice has the form

\[
H = 2\kappa \sum_{<ij>} \sigma_i \sigma_j - \kappa \sum_{<i,j>} \sigma_i \sigma_j + \frac{1}{2} \kappa \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l
\]

which is of generalized Ising form and contains nearest neighbour \(<i,j>\), next to nearest neighbour \(<<i,j>>\) and round a plaquette \([i,j,k,l]\) terms. Such actions are not new, having been investigated in some detail using both mean field methods and simulations in \cite{12}. However, the particular combination of coefficients arising in equ.(3) was not considered explicitly in this work because it corresponds to a particularly degenerate set of couplings. This degeneracy manifests itself in an extended symmetry in the model: for any value of \(\kappa\) it is possible to flip a plane of spins with no energy penalty, providing the flipped plane does not intersect any existing portion of surface. This gives a vacuum degeneracy reminiscent of a gauge theory, the difference with a true gauge theory being that here the symmetries are quasi-global, giving only the freedom to flip entire planes rather than local spins. Related surface models have also been simulated directly in \cite{13}, but again the set of coefficients appearing in equ.(3) was not explored in this work. A very rich phase structure was observed in \cite{12}, in common with other Ising models with extended interactions \cite{7} of various sorts which display first and second order phase boundaries as well as incommensurate phases. Given the generic richness of the phase diagrams for such generalized Ising models and the additional symmetries present in the Gonihedric model, the action of
equ.(3) merits investigation from purely statistical mechanical considerations as well as from the point of view of finding potential continuum string theories.

In the context of string theory one is looking for a continuous transition (or transitions) at which a sensible continuum surface theory may be defined. It is perhaps worth recalling that even this does not guarantee a good continuum surface theory. The interfaces in the standard nearest neighbour Ising model in three dimensions, which has a continuous phase transition, have been investigated in some detail recently and found to be very porous objects, decorated with lots of handles at the scale of the lattice cutoff \[13\]. Ideally one might hope that the surfaces generated by the Gonihedric action were smoother, given that it is derived from a sort of stiffness term.

Our motivation in this paper is to investigate the action of equ.(3) in order to sketch out the gross features of its phase structure. We concentrate on \( \kappa = 1 \) as in \[9\], but also discuss other values. A few cautionary words are in order before we go on to discuss the mean field approach and simulations. As we have noted the ground state of the action in equ.(3) is very degenerate as planes of spins parallel to any of the cube axes can be flipped at no energy cost. In the case \( \kappa = 0 \) the degeneracy is even larger as diagonal planes may be flipped now also. The ability to flip arbitrary spin planes makes defining a magnetic order parameter rather problematic. Even the staggered local order parameters defined in \[12\] would miss the lamellar phases with arbitrary intersheet spacings that could be generated at no cost by flips of spin planes.

The simulations for \( \kappa = 1 \) in \[9\] were restricted to measuring the energy and specific heat as the exhaustive global order parameters suggested there

\[
M^\mu = \left\langle \frac{1}{L^3} \sum_i \sigma_i^\mu (\text{vac}) \sigma_i \right\rangle
\]

(where \( \sigma_i^\mu (\text{vac}) \) is one of the possible vacuum spin configurations with \( \mu = 1, 2 \ldots 2^{3L} \) for \( \kappa = 0 \) or \( \mu = 1, 2 \ldots 3 \times 2^L \) for \( \kappa \neq 0 \) on a lattice of size \( L \)) would have been prohibitively slow to measure on even moderately sized lattices. It is possible to do rather better, however, by making use of the freedom in choosing boundary conditions on a finite lattice. It is customary to employ periodic boundary conditions in attempting to extract critical exponents from a simulation as these tend to minimize the finite size effects. It is clear that fixed boundary conditions in the Gonihedric model would penalize flipped spin planes by at least a perimeter energy, at the possible expense of greater finite size effects. A quick test simulation shows that such boundary conditions do pick out the purely ferromagnetic ground state from the many equivalent possibilities and allow the measurement of the simple ferromagnetic order parameter

\[
M = \left\langle \frac{1}{L^3} \sum_i \sigma_i \right\rangle.
\]

One can, in fact, have the best of both worlds by continuing to employ periodic boundary conditions to reduce the finite size effects whilst fixing any two perpendicular planes of spins to pick out the ferromagnetic ground state. The fixed planes of spin are, in effect, more akin to a gauge fixing of the spin flip symmetry than boundary conditions per se. In the simulations reported later in this paper we employed three fixed perpendicular planes of spins as a safety measure, with essentially identical results.

### 2 Zero Temperature and Mean Field

As the Gonihedric model is a special case of the general action considered in \[12\] we can apply the methods used there for both the zero temperature phase diagram and mean field theory. For the zero temperature case this involves writing the full lattice Hamiltonian as a sum over individual cube Hamiltonians

\[
h_c = \frac{\kappa}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \frac{\kappa}{4} \sum_{\langle\langle i,j \rangle\rangle} \sigma_i \sigma_j + \frac{1 - \kappa}{4} \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l
\]

and observing that if the lattice can be tiled by a cube configuration minimizing the individual \( h_c \) then the ground state energy density is \( \epsilon_0 = \min h_c \). We list the inequivalent spin configurations on a single cube and their multiplicities in Table.1 using the same notation as \[12\] but with our choice of couplings to highlight the degeneracies that appear with
the Gonihedric action. In the list of spins the first column represents one face of the cube and the second the other. In the table two configurations are considered equivalent if one can be transformed into the other by reflections and rotations or if they are related by a global spin flip. The antiferromagnetic image of a configuration is obtained by flipping the three nearest neighbours and the spin at the other end of the cube diagonal from a given spin and is denoted by an overbar. With the Gonihedric values the couplings the freedom to flip spin planes is clear even at this level as \( \psi_0 \), which would represent a ferromagnetic state when used to tile the lattice, and \( \psi_0 \) which would represent flipped spin layers, have the same energy for any value of \( \kappa \). The higher energy configurations \( \psi_4 \) and \( \psi_3 \) are also identical. The degeneracies increase when \( \kappa = 0 \), the club of states of energy \(-3/2\) is now composed of \( \psi_0, \psi_0, \psi_6, \psi_6 \) and various extra degeneracies appear for higher energy states. The new ground states \( \psi_0, \psi_6 \) reflect the additional freedom to flip diagonal planes of spins that is present at \( \kappa = 0 \).

In the mean field approximation the spins are in effect replaced by the average site magnetizations. The calculation of the mean field free energy is an elaboration of the method used above to investigate the ground states in which the energy is decomposed into a sum of individual cube terms. The next to nearest neighbour and plaquette interactions in the Gonihedric model give the total mean field free energy as the sum of elementary cube free energies \( \phi(m_c) \), given by

\[
\phi(m_C) = -\frac{\kappa}{2} \sum_{i,j \in C} m_i m_j + \frac{\kappa}{4} \sum_{\langle i,j \rangle \in C} m_i m_j - \frac{1}{4} \sum_{\langle i,j \rangle \in C} m_i m_j m_k m_l + \frac{1}{16} \sum_{i \in C} [(1 + m_i) \ln(1 + m_i) + (1 - m_i) \ln(1 - m_i)]
\]

where \( m_C \) is the set of the eight magnetizations of the elementary cube. This gives a set of eight mean-field equations

\[
\frac{\partial \phi(m_C)}{\partial m_i} \bigg|_{(i=1...8)} = 0
\]

(one for each corner of the cube) rather than the familiar single equation for the standard nearest-neighbour Ising action. More explicitly, we have

\[
m_1 = \tanh[4\beta\kappa(m_2 + m_4 + m_5) - 2\beta\kappa(m_3 + m_6 + m_8) + 2\beta(1 - \kappa)(m_2 m_3 m_4 + m_2 m_5 m_6 + m_4 m_5 m_8)]
\]

\[\vdots\]

\[
m_8 = \tanh[4\beta\kappa(m_2 + m_5 + m_7) - 2\beta\kappa(m_1 + m_3 + m_6) + 2\beta(1 - \kappa)(m_3 m_4 m_7 + m_1 m_4 m_5 + m_5 m_6 m_7)]
\]

where we have labelled the magnetizations on a face of the cube counterclockwise \( 1 \ldots 4 \) and similarly for the opposing face \( 5 \ldots 8 \) as shown in Figure.1. If we solve these equations iteratively we arrive at zeroes for a paramagnetic phase or various combinations of \( \pm 1 \) for the magnetized phases on the eight cube vertices, and the mean field ground state is then give by gluing together the elementary cubes consistently to tile the complete lattice, in the manner of the ground state discussion.

Turning loose a numerical solver on the mean field equs.\[\] gives generically a single transition to one of the phases listed in Table.1 from the high temperature paramagnetic phase. The transition temperatures and the resulting low temperature phase are listed in Table.2. We have taken the liberty of carrying out global flips where necessary to tidy up the table. Rather remarkably, we see that apart from \( \kappa = 0 \) the transition appears to be to the simple ferromagnetic phase, \( \psi_0 \). However, remembering that \( \psi_0 \) and \( \psi_0 \) have the same energy the best we can say is that we end up in a layered phase with arbitrary interlayer spacing in all directions. Although the \( \kappa = 0 \) case appears to be superficially different, the \( \psi_0 \) phase that is found at low temperature here is one of the phases that is degenerate with \( \psi_0 \) and \( \psi_0 \) when \( \kappa = 0 \). Although \( \kappa = 1 \) fits the pattern as far as a transition to \( \psi_{0,6} \) at decreasing \( \beta \) is concerned it appears to be rather atypical in that further transitions are observed at larger \( \beta \). However, this is a numerical instability that is peculiar to this particular value of \( \kappa \). It was observed in \[12\] that an iterative solution of the mean field equations written in the form

\[
m_i^{(n+1)} = f(E_i(m^n))
\]
where \( E \) is the individual cube Hamiltonian could fail to converge if an eigenvalue of \( \partial m_i^{(n+1)}/\partial m_j^n \) was less than \(-1\). Modifying the equations to

\[
m_i^{(n+1)} = \frac{f[E_i(m^n)] + \alpha m^n}{1 + \alpha}
\]  

for suitable \( \alpha \) cures this. This is precisely what happens for \( \kappa = 1 \), where introducing a non-zero \( \alpha \) suppresses the extra “transitions”.

In summary, the mean field theory suggests a rather simple phase diagram for the Gonihedric model with action equ.(3), with a single transition from a paramagnetic phase to a degenerate “layered” phase that is pushed down to \( \beta = 0 \) at large \( \kappa \). The low temperature phase is generically of the \( \psi_{0,6} \) type, apart from \( \kappa = 0 \) where we see a \( \psi_{0,6} \) phase that is degenerate with these. The degeneracy of the ground states that are indicated by these results are, as they should be, consistent with the symmetries of the original full action. We now go on to see how the zero-temperature and mean field results tally with a direct Monte-Carlo simulation.

### 3 Simulations

We carried out simulations with \( \kappa = 1 \) on lattices of size \( 10^3, 12^3, 15^3, 18^3, 20^3 \) and \( 25^3 \) and for \( \kappa = 0, 2, 5, 10 \) on lattices of size \( 10^3, 15^3, 20^3 \) and \( 25^3 \). Unless stated otherwise periodic boundary conditions were imposed in the three directions and three internal perpendicular planes of spins fixed to be \(+1\). We carried out 50K sweeps for most \( \beta \) values, increasing to 500K sweeps near the observed phase transition point. Measurements were carried out every sweep after allowing sufficient time for thermalization. A simple Metropolis update was used because of the difficulty in concocting a cluster algorithm for a Hamiltonian with such complicated interaction terms. The program was tested on the standard nearest neighbour Ising model and the some of the parameters used in the generalized Ising models of [12] to ensure it was working.

We measured the usual thermodynamic quantities for the model: the energy \( E \), specific heat \( C \), (standard) magnetization \( M \), susceptibility \( \chi \) and various cumulants. As the large \( \beta \) limit of the energy should be determined by the zero-temperature analysis of the preceding section, we consider the energy first. The absolute value of the energy for various \( \kappa \) on lattices of size \( L = 20 \) is plotted against \( \beta \) in Figure.2, where we can see that the zero temperature prediction of \( 3(1 + \kappa)/2 \) is satisfied with good accuracy for sufficiently large \( \beta \). We can therefore observe that the zero-temperature/mean-field analysis has correctly identified the ground state(s) of the theory: \( \psi_{0,6} \) for \( \kappa \neq 0 \); or \( \psi_{0,0,6,6} \) for \( \kappa = 0 \) as these are the only states with the observed energies. Having satisfied ourselves that the simulation is finding the correct ground state energy, we can go on to consider extracting some of the critical exponents for the transition. In what follows we will, as advertized, discuss in some detail the case \( \kappa = 1 \) before commenting more briefly on the other values of \( \kappa \).

With only periodic boundary conditions the possibility of the simple ferromagnetic ordered state \( \psi_0 \) can be excluded by looking at the magnetization \( M \), which for all \( \kappa \) is either zero or fluctuates wildly as \( \beta \) is changed, reflecting the freedom to flip spin planes. Curtailing this freedom by fixing the three perpendicular spin planes picks out the transition to a simple ferromagnetic ground state and the low temperature limit of the magnetization becomes one for all \( \kappa \). The magnetization cumulant

\[
U_M = \frac{<M^4>}{<M^2>^2}
\]

is well defined once the three spin planes are fixed, but it does not show the standard behaviour of a low \( \beta \) limit of three and a high \( \beta \) limit of one, asymptoting to a value slightly larger than one at low \( \beta \), as can be seen in Figure.3. This is because the fixed planes still leave a residual magnetization at low \( \beta \), which is sufficient to make this limit look magnetized for the sizes of lattice we simulate. Nonetheless, it is still possible to apply the usual scaling analysis in the critical region, and the crossing of the cumulant plots for different lattice sizes gives an estimate of \( \beta_c = 0.44(1) \) for the critical temperature, which is in good agreement with the value reported in [11] that was extracted by looking at the change in behaviour of the spin/spin correlation as \( \beta_c \) was approached. As noted in [11] this \( \beta_c \) is very close (in our case within the error bars) to that of the standard two-dimensional Ising model on a square lattice.
The rather sharp nature of the crossing, or more accurately collapse down to a single line, makes it difficult to extract a value for \( \nu \) from the ratio of the slopes of the Binder’s cumulant curves at the critical point, so we take a different tack and consider the scaling of the maximum slope, which we would also expect to scale as \( L^{1/\nu} \). This gives an estimate of \( \nu = 1.2(1) \). We can also look at both the finite size scaling and direct fits to the susceptibility \( \chi \), namely \( \chi = AL^\alpha \) and \( \chi = \tilde{A} |\beta - \beta_c|^\gamma \), to attempt to extract \( \nu \). We choose this in preference to the specific heat fits \( C = B + DL^\beta \) and \( C = \tilde{B} + \tilde{D} |\beta - \beta_c|^{\alpha} \) because of the absence of an adjustable constant. The susceptibility data is plotted in Figure 4, and shows a clear peak. This should be contrasted with the case of no fixed planes where the \( \beta > \beta_c \) region is rendered meaningless by the lack of a well-defined magnetization for the myriad of ground states. We find the finite size scaling fit gives \( \gamma/\nu = 1.79(4) \) with a high quality, and feeding the critical value of \( \beta_c = 0.44 \) into the direct fit on the \( L = 25 \) lattice gives \( \gamma = 1.60(2) \) with rather lower quality. The deduced value for \( \nu \) is thus \( 1.10(5) \). These fits give values close to those for the standard two-dimensional Ising model with only nearest neighbour interactions, where we have \( \gamma = 1.75, \nu = 1 \).

As a consistency check on our values of \( \beta_c \) and \( \nu \), we plot the \( \beta \) values where the specific heat peaks and the \( \beta \) values where the maximum slope of the Binder’s cumulant curve occurs, both of which can serve as estimates of the pseudocritical temperature on finite size lattices, against \( L^{-\frac{1}{\nu}} \). We would expect this to be a straight line with intercept \( \beta_c \). The plot is shown in Figure 5 for the choice \( \nu = 1 \) with other values in this neighbourhood giving essentially identical results. We can see that the estimate of \( \beta_c = 0.437(7) \) coming from the two possibilities is both self-consistent and in agreement with the value obtained from cumulant crossing.

The above, apparently consistent, set of results presents us with something of a dilemma when it comes to the analysis of the specific heat peak, which is shown in Figure 6. The hyperscaling relation \( \alpha = 2 - 4d \) indicates that, if a value of \( \nu \simeq 1 \) is to be believed, the specific heat should display a cusp \((\alpha = 2 - 4d \simeq -1)\) rather than a divergence. This does not appear to be the case for the data in the figure. Setting aside the hyperscaling result for the moment and performing direct power law fits to \( C = B + DL^\beta \) and \( C = B + \tilde{D} |\beta - \beta_c|^{\alpha} \) gives poor fits. A logarithmic fit of the form \( C = B + D \log(L) \) or \( C = B + \tilde{D} \log(|\beta - \beta_c|) \) gives much better, but still not particularly good, results so the evidence is inconclusive.

Another line of attack for obtaining an estimate of \( \alpha \) is to use the finite size scaling of the energy itself \( E \simeq E_0 + E_1 L^{\frac{\alpha}{\nu}} \). With direct measurements in this form one gains nothing in general over the specific heat fits as there is still a regular term \( E_0 \) to be dealt with, although for models with higher than second order transitions this approach may be preferable. However, if one has measurements for two different sets of boundary conditions available the regular term would be expected to be the same for both and the energy difference could be used for a simple power law fit to extract \((\alpha - 1)/\nu \). We are currently measuring the string tension in the Savvidy model using two sets of fixed boundary conditions, one with all positive spins and one with half positive and half negative spins. For these measurements we would expect

\[
\Delta E = E_{++} - E_{+-} = AL^{\frac{\alpha}{\nu} - 1}
\]

where \( E_{++} \) is the energy for all positive spins and \( E_{+-} \) is the energy for half positive and half negative spins. A fit gives \((\alpha - 1)/\nu = -1.3(2)\), which is still marginally consistent with \( \alpha = 0 \).

There are two possible interpretations of the results. The first is that the value of \( \nu \) measured is simply wrong, not inconceivable as it appears either as a derived quantity from the slope of the cumulant or from the two fits to the susceptibility rather than being measured directly. However, a second possible interpretation is to accept the fits to \( \nu \) at face value and posit that the model sees an effective dimension of \( d = 2 \) in order to avoid violating the hyperscaling relation. In this case we could recover the full set of two-dimensional Ising model exponents. Although this would be highly unusual, it should be remembered that the energy in the model is essentially linear in form rather than being an area, so there is some resemblance to a two-dimensional model.

A direct fit to the magnetization exponent on the largest lattice size \( M \simeq |\beta - \beta_c|^\beta \) (with apologies for the profusion of betas!) with \( \beta_c \) fixed to be 0.44 gives \( \beta = 0.12(1) \), but the quality is low, whereas a finite size scaling fit gives a much lower value of \( \beta/\nu = 0.04(1) \). It is possible that the fixed spin planes, whose residual magnetization we have not allowed for in the fits, are biasing the finite size scaling fit, but...
intuitively one would expect their effects (of order $3/L$) to increase rather than decrease the estimated exponent by pushing up the measured magnetization on the smaller lattices.

We now discuss more briefly the other $\kappa$ values that were simulated, namely $\kappa = 0, 2, 5, 10$. Firstly we can note that in qualitative terms the transition appears similar to the $\kappa = 1$ case, a not entirely trivial result as $\kappa \neq 1$ introduces round a plaquette interactions that are missing for $\kappa = 1$. With the periodic plus fixed plane boundary conditions we still have peaks in the susceptibility and specific heat and a ferromagnetically ordered phase appearing at low temperature. The mean field analysis suggests that as $\kappa$ is increased $\beta_c$ should drop sharply. This is not observed in the simulations, the crossing of the Binder’s cumulants indicating no change within the error bars for the estimates of $\beta_c$, giving $\beta_c = 0.44(1)$ from $\kappa = 1$ to $\kappa = 10$. There is a sharper difference with the $\kappa = 0$ results which show a crossing at $\beta_c = 0.50(1)$. In general mean field theory will underestimate $\beta_c$, so the measured results are in agreement with this and do not contradict the bound obtained in [11].

The similarity of the transitions for different $\kappa$ extends beyond $\beta_c$. The measurements of $\gamma/\nu$ listed in Table 3 below for all the non-zero $\kappa$ suggest that the critical behaviour is unchanged by varying $\kappa$.

| $\kappa$ | 1   | 2   | 5   | 10  |
|----------|-----|-----|-----|-----|
| $\gamma/\nu$ | 1.79(4) | 1.6(1) | 1.9(1) | 1.75(6) |

Table 3: Fits to $\gamma/\nu$ for the non-zero $\kappa$ values.

The specific heat curves and magnetization present a similar story, with all looking similar to the $\kappa = 1$ case. From this evidence it would seem that varying $\kappa$, at least for $\kappa \geq 1$, gives little if any change in the exponents and transition temperature.

The story is slightly different for $\kappa = 0$. As we have already noted $\beta_c = 0.50(1)$, and other differences are apparent. Without the fixed spin planes (ie with only periodic boundary conditions) the susceptibility when $\beta < \beta_c$ for non-zero $\kappa$ values is similar to the fixed plane case described above and becomes meaningless when $\beta > \beta_c$ where the magnetization is ill-defined. The $\kappa = 0$ model presents qualitatively different behaviour in that the susceptibility is one for $\beta < \beta_c (\approx 0.5)$ and zero for $\beta > \beta_c$. However, this step function behaviour disappears when the fixed spin planes are introduced and we recover a divergent peak as for the other $\kappa$ values. The phase structure for $\kappa = 0$ with the fixed spin planes also appears to be broadly similar to other $\kappa$, giving a single transition to a low temperature magnetized phase. It would be interesting to examine in detail values of $\kappa$ between zero and one to see if there was a smooth change in, for example, $\beta_c$ as $\kappa \to 0$. This would give some indication of whether $\kappa = 0$ really was a special point, or joined on smoothly to the continuum of non-zero values. A test simulation at $\kappa = 0.5$ still gives very similar results to $\kappa = 1$, for example.

4 Conclusions

We have conducted zero-temperature, mean-field and Monte-Carlo investigations of the generalized Ising model action suggested in [8, 9, 10] as a cubic lattice discretization of the Gonihedric string action [1] using essentially the methods of [12]. Although the phase structure of such generalized Ising models is generically very rich [14], the one parameter family of models examined here seems to be a fairly simple “slice” of the phase diagram, with one transition to a layered ground state when periodic boundary conditions are imposed. This degenerate layered state is a consequence of the plane spin flip symmetry that is present in the model for all $\kappa$, but a judicious choice of boundary conditions - fixing enough perpendicular spin planes - allowed us to pick out an equivalent ferromagnetic ground state for the purposes of simulations. The zero-temperature/mean-field analyses are in agreement with the Monte-Carlo simulations on the nature of the ground state and its energy, but the simulations indicate a transition temperature that changes little, if at all, from its value at $\kappa = 1 (\beta_c \approx 0.44)$ for other non-zero $\kappa$ values. The mean field theory on the other hand gives a fairly sharp decline in $\beta_c$ as $\kappa$ is increased.

The simulations at $\kappa = 1$ indicate that the fitted exponents, with the exception of the finite size scaling fit to $\beta/\nu$, and even the critical temperature are all in the vicinity of those for the two-dimensional Ising model, though given our modest statistics it would be foolhardy to claim they were identical on the basis of the current fits. Comparison with the other non-zero $\kappa$ values also gives similar exponents and critical temperatures. There is some evidence that the $\kappa = 0$ model is a special case: - in the zero
temperature and mean field analyses more ground states are allowed and in the simulations a different 
critical temperature is observed and the behaviour of the susceptibility is radically different when no spin 
planes are fixed.

An immediate extension of the current work, given the closeness of the fitted exponents to the two-
dimensional Ising model, is to carry out a higher statistics simulation near the transition point in order 
to pin down the various exponents and \( \beta_c \) more accurately. A further test of the critical behaviour 
of the model would be to investigate the scaling of the string tension as one approached the critical point, 
along the lines of [17]. The various higher dimensional generalizations that were formulated in [8, 9, 10] 
also merit investigation.

If we return to our original stringy motivation it would be useful to characterize the surfaces generated 
by the Gonihedric action in the style of [15] to see whether they were any less “spongy” than those in the 
standard 3D Ising model. As a playground for exploring plaquette discretizations of string and gravity 
inspired models the generalized Ising models clearly have some interesting quirks that are worthy of 
further exploration. It would certainly be amusing to show that a candidate discretized string model was 
a close relation of the two-dimensional Ising model.

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Table.1: The inequivalent spin configurations of a single cube and the associated energies and degeneracies

| State | Top | Bottom | Energy         | Multiplicity |
|-------|-----|--------|----------------|--------------|
| $\psi_0$ | ++ | ++ | $-3/2 - 3\kappa/2$ | 2            |
|       | ++ | + + | $-3/2 + 21\kappa/2$ | 2            |
| $\psi_1$ | + + | + + | $-3\kappa/2$ | 16           |
|       | ++ | + + | $9\kappa/2$ | 16           |
| $\psi_2,3$ | + + | ++ | $1/2 + \kappa/2$ | 24           |
| $\psi_4$ | + + | + + | $3/2 - 3\kappa/2$ | 8            |
| $\psi_5$ | + + | ++ | $-3\kappa/2$ | 48           |
| $\psi_6$ | + + | ++ | $\kappa/2$ | 48           |
| $\psi_7,8$ | + + | ++ | $1/2 - 3\kappa/2$ | 24           |
Table 2: The ground state configurations and transition temperatures for various $\kappa$.

The states shown appear above the quoted temperature.

| $\kappa$ | $\beta_c$ | Configuration |
|----------|-----------|---------------|
| 0.0      | 0.325     | + - + +      |
| 0.25     | 0.31      | ++ ++        |
| 0.5      | 0.278     | ++ ++        |
| 1.0      | 0.167     | ++ ++        |
| 2.0      | 0.09      | ++ ++        |
| 5.0      | 0.0335    | ++ ++        |
| 10.0     | 0.02      | ++ ++        |
| 15.0     | <0.02     | ++ ++        |
Figure 1: The labelling of the cube vertices for the mean field equations.
Figure 2: The energies for various $\kappa$, all on lattices of size $L = 20$. 
Figure 3: Binder’s cumulant for $\kappa = 1$
Figure 4: The susceptibility for $\kappa = 1$
Figure 5: The pseudocritical temperature estimated from the specific heat and Binder’s cumulant vs $L^{-1/\nu}$ at $\kappa = 1$
Figure 6: The specific heat for $\kappa = 1$. 

Specific Heat

$\Phi$ L=25
$\Box$ L=20
$\times$ L=18
$\Diamond$ L=15