COMMENSURABILITY AND REPRESENTATION
EQUIVALENT ARITHMETIC LATTICES

CHANDRASHEEL BHAGWAT, SUPRIYA PISOLKAR, AND C.S.RAJAN

Abstract. Gopal Prasad and Rapinchuk defined a notion of weakly commensurable lattices in a semisimple group, and gave a classification of weakly commensurable Zariski dense subgroups. A motivation was to classify pairs of locally symmetric spaces isospectral with respect to the Laplacian on functions. For this, in higher ranks, they assume the validity of Schanuel’s conjecture.

We observe that if we use the notion of representation equivalence of lattices, then Schanuel’s conjecture can be avoided. Further, the results are applicable in a $S$-arithmetic setting.

We introduce a new relation ‘characteristic equivalence’ on the class of arithmetic lattices, stronger than weak commensurability. This simplifies the arguments used in [PR] to deduce commensurability type results.

1. Introduction

Let $M$ be a compact, connected Riemannian manifold. The spectrum of the Laplace-Beltrami operator acting on the space of smooth functions on $M$, the collection of its eigenvalues counted with (finite) multiplicity, is a discrete weighted subset of the non-negative reals. Define two compact connected Riemannian manifolds $M_1$ and $M_2$ to be isospectral on functions or just isospectral, if the spectra of the Laplace-Beltrami operator acting on the space of smooth functions on $M_1$ and $M_2$ coincide.

The inverse spectral problem is to recover the properties of the Riemannian manifold $M$ from a knowledge of the spectrum. It is known, for example, that the spectra on functions determines the dimension, volume and the scalar curvature of $M$.

Milnor constructed the first examples in the context of flat tori of non-isometric compact Riemannian manifolds which are isospectral on functions. When the spaces are compact hyperbolic surfaces, such examples were initially constructed by Vigneras [V]. In analogy with a construction in arithmetic, Sunada gave a general method for constructing pairs of isospectral spaces [S].

In many of these constructions, the manifolds are quotients by finite groups of a fixed Riemannian manifold. The question arises whether isospectral manifolds are indeed commensurable, i.e., have a common finite cover. In the context of Riemannian locally symmetric spaces this question has been studied by various authors ([R, CHLR, PR, LSV]) assuming that the spaces are isospectral for the

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Laplace-Beltrami operator acting on functions. Gopal Prasad and A. S. Rapinchuk address this question in full generality, and get commensurability type results for isospectral, compact locally symmetric spaces. For this when the locally symmetric spaces are of rank at least two, they have to assume the validity of Schanuel’s conjecture on transcendental numbers.

In this note, we consider this question assuming a stronger hypothesis that the lattices defining the locally symmetric spaces are representation equivalent rather than isospectral on functions. This allows us to obtain similar conclusions as in [PR] for representation equivalent lattices, without invoking Schanuel’s conjecture, and also extend the application to representation equivalent $S$-arithmetic lattices. In the process, we introduce a new relation of characteristic equivalence of lattices, stronger than weak commensurability. This stronger hypothesis helps in simplifying some of the arguments used in [PR].

2. Representation equivalence of lattices

The Fourier analysis for the circle group $S^1$ can be studied in two ways: either, as expanding a function in terms of the eigenfunctions of the Laplace operator, or via characters of the topological group $S^1$. In the context of Riemannian locally symmetric spaces the spectrum can also be studied in terms of representation theory of the isometry group of the universal cover.

Let $G$ be a locally compact, unimodular topological group and $\Gamma$ be a uniform lattice in $G$. Let $R_\Gamma$ denote the right regular representation of $G$ on the space $L^2(\Gamma \backslash G)$ of square integrable functions with respect to the projection of the Haar measure on the space $\Gamma \backslash G$:

$$(R_\Gamma(g)f)(x) = f(xg), \quad f \in L^2(\Gamma \backslash G), \quad g, x \in G.$$ 

As a $G$-space, $L^2(\Gamma \backslash G)$ breaks up as a (Hilbert) direct sum of irreducible unitary representations of $G$,

$$L^2(\Gamma \backslash G) \cong \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) \pi,$$

where $\hat{G}$ is the unitary dual of $G$ parametrizing isomorphism classes of irreducible, unitary representations of $G$, and $m(\pi, \Gamma)$ is the (finite) multiplicity with which an element $\pi \in \hat{G}$ occurs in $L^2(\Gamma \backslash G)$. Define the representation spectrum of a uniform lattice $\Gamma \subset G$ to be the map $\pi \mapsto m(\pi, \Gamma)$ giving the multiplicity $m(\pi, \Gamma)$ with which an irreducible unitary representation $\pi$ of $G$ occurs in $L^2(\Gamma \backslash G)$.

**Definition 2.1.** Let $G$ be a locally compact topological group and $\Gamma_1$ and $\Gamma_2$ be two uniform lattices in $G$. The lattices $\Gamma_1$ and $\Gamma_2$ are said to be representation equivalent in $G$ if

$$L^2(\Gamma_1 \backslash G) \cong L^2(\Gamma_2 \backslash G)$$

as $G$-spaces.

The relevance of this notion to spectrum is provided by the following generalization of Sunada’s criterion for isospectrality [DG]:
Theorem 2.2. Let $G$ be a locally compact topological group $G$ which acts on a Riemannian manifold $M$. Let $\Gamma_1$, $\Gamma_2$ be representation equivalent uniform lattices in $G$. Suppose $G$ acts on a Riemannian manifold $M$, such that $\Gamma_1$, $\Gamma_2$ act properly discontinuously and freely on $M$ with compact quotients. Then the Riemannian manifolds $\Gamma_1 \backslash M$ and $\Gamma_2 \backslash M$ with respect to the induced metric from $M$ are strongly isospectral; in particular, they are isospectral on $p$-forms for all $p$.

The concept of strong isospectrality is defined in [DG] as having the same spectrum for any natural (in the sense of Epstein and Stredder) elliptic differential operator with positive definite symbol. A plausible alternate definition is as follows: suppose two compact oriented Riemannian manifolds $M$, $N$ are isospectral on functions. Then it is known that their dimensions are equal, say of dimension $d$. The Riemannian metric gives a reduction of structure group of the tangent bundle to the orthogonal group $SO(d)$. Given a representation $\tau$ of $SO(d)$, this defines two metrized vector bundles on $M$ and $N$ respectively. A Laplace type operator (elliptic, self-adjoint, non-negative) can be defined on the space of smooth sections of these bundles. For strongly isospectral, we require that for any $\tau$ as above, these Laplace operators have the same spectrum. For example, one can consider the spectrum of the Hodge-deRham Laplacians acting on the space of smooth $p$-forms of a oriented compact Riemannian manifold.

Suppose $M = G/K$ is a noncompact Riemannian symmetric space, where $G$ is a noncompact semisimple Lie group and $K$ is a maximal compact subgroup of $G$. Let $\Gamma$ be a uniform torsion-free lattice in $G$. To an irreducible representation $\tau$ of $K$ there is associated an automorphic vector bundle $E_\tau$ on the quotient space $\Gamma \backslash G/K$. The above theorem implies that if the lattices are representation equivalent, then the spectra of the Laplace operators on the smooth sections of $E_\tau$ are equal.

Remark 2.3. In [P], Pesce has proved that the converse of the generalized Sunada Criterion holds in the case of $G = \text{Isom}(\mathbb{H}^n)$, where $\mathbb{H}^n$ is the hyperbolic $n$-space with constant sectional curvature $-1$. However, in the general context of locally symmetric spaces, the converse to the generalized Sunada criterion is not known, i.e, whether isospectrality for all automorphic vector bundles as above yields representation equivalence.

For compact hyperbolic surfaces $X$, it is known that the spectrum on functions determines the representation equivalence of the lattice $\pi_1(X, x_0) \subset PSL(2, \mathbb{R})$. This prompts the following question:

Question 2.4. Will it be true that for compact quotients of non-compact Riemannian symmetric spaces, the spherical spectrum (the restriction of the representation spectrum to the class of spherical representations of $G$) determines the representation class of the lattice in the group of isometries ([BR])? More generally, will this be true if we just look at the spectrum of the Laplacian on functions?
2.1. Arithmetic lattices. We will have the following notations and assumptions for the rest of this paper:

- **H1**: $\mathcal{G}$ is a connected absolutely almost simple algebraic group defined over a number field $K$.
- **H2**: $S$ is a finite set of places of $K$ containing the archimedean places at which $\mathcal{G}$ is isotropic. Let $S^i$ denote the subset of places of $S$ at which $\mathcal{G}$ is isotropic.
- **H3**: There is at least one place $v \in S$ at which $\mathcal{G}$ is isotropic.

A subgroup $\Gamma$ of $\mathcal{G}(K)$ is said to be $(\mathcal{G}, K, S)$-arithmetic (or just arithmetic) subgroup, if it is commensurable with $\mathcal{G}(O_K(S)) = \mathcal{G}(K) \cap GL_n(O_K(S))$, where $O_K(S)$ is the set of $S$-integers in $K$ and we consider $\mathcal{G}$ as embedded in $GL_n$ over $K$ for some $n$.

Denote by $\mathcal{G}_S$ the locally compact group,

$$\mathcal{G}_S = \prod_{v \in S} \mathcal{G}(K_v),$$

where given a place $v$ of $K$, $K_v$ denotes the completion of $K$ at $v$.

There is an embedding of the arithmetic subgroup $\Gamma \subset \mathcal{G}_S$, which is well defined up to complex conjugation at the complex places of $K$. By results of Borel, Harishchandra, Godement and Tamagawa, this defines an arithmetic lattice $\Gamma \subset \mathcal{G}_S$, which is Zariski dense in $\mathcal{G}$.

Suppose $\mathcal{G}_1$, $\mathcal{G}_2$ are algebraic groups as above, and assume further that are anisotropic. Then the lattices $\Gamma_i$ are cocompact in $\mathcal{G}_{i,S_i}$ for $i = 1$, 2. We define two $S$-arithmetic subgroups $\Gamma_1 \subset \mathcal{G}_1(K_1)$, $\Gamma_2 \subset \mathcal{G}_2(K_2)$ to be *topologically representation equivalent* if there exists an isomorphism $\phi : \mathcal{G}_{1,S_1} \rightarrow \mathcal{G}_{2,S_2}$ of topological groups such that

$$L^2(\phi(\Gamma_1) \backslash \mathcal{G}_{2,S_2}) \cong L^2(\Gamma_2 \backslash \mathcal{G}_{2,S_2})$$

as $\mathcal{G}_{2,S_2}$-spaces.

**Remark 2.5.** By theorems of Freudenthal and Borel-Tits [BT], it is known that any abstract homomorphism of adjoint Lie groups as above is automatically continuous. Hence in the definition of representation equivalence we could have just required that there is an abstract isomorphism between the ambient groups, requiring that the image of the lattice $\Gamma_1$ is again a lattice (so that representation equivalence makes sense).

Denote by $\mathcal{G} \rightarrow \overline{\mathcal{G}}$ the isogeny to the adjoint group corresponding to $\mathcal{G}$. For a subgroup $\Gamma$ of $\mathcal{G}(K)$, $\overline{\Gamma}$ will denote the image in $\overline{\mathcal{G}}(K)$.

Define two arithmetic subgroups $\Gamma_1 \subset \mathcal{G}_1(K_1)$, $\Gamma_2 \subset \mathcal{G}_2(K_2)$ to be *commensurable*, if there are isomorphisms $\sigma : \text{Spec}K_2 \rightarrow \text{Spec}K_1$ and $\phi : \overline{\mathcal{G}}_1 \rightarrow \overline{\mathcal{G}}_2$, where the superscript $\sigma$ denotes twisting the group scheme $\mathcal{G}_1$ by $\sigma$. In particular, the image $\phi(\overline{\Gamma}_1)$ considered as a subgroup of $\overline{\mathcal{G}}_2(K_2)$ and $\overline{\Gamma}_2$ will be commensurable subgroups.
2.2. Main Theorem. Inspired by the work of Gopal Prasad and Rapinchuk, our aim now is to obtain commensurability type results for representation equivalent arithmetic lattices. Working with representation equivalence of arithmetic lattices rather than isospectrality on functions of the corresponding locally symmetric space, allows us to avoid invoking the validity of Schanuel’s conjecture (see Conjecture 2.10) on transcendental numbers:

**Theorem 2.6.** Let $G_1$ (resp. $G_2$) be anisotropic algebraic groups defined over a number field $K_1$ (resp. $K_2$). Let $S_1$ (resp. $S_2$) be a finite set of places of $K_1$ (resp. $K_2$). Assume that for $i = 1, 2$, $(K_i, G_i, S_i)$ satisfy hypothesis H1-H3.

Let $\Gamma_1 \subset G_1(K_1)$ (resp. $\Gamma_2 \subset G_2(K_2)$) be $S_1$ (resp. $S_2$)-arithmetic subgroup of $G_1$ (resp. $G_2$).

Suppose that the lattices $\Gamma_1 \subset G_1, S_1$, $\Gamma_2 \subset G_2, S_2$ are topologically representation equivalent.

Then the following hold:

1. The groups $G_1$ and $G_2$ are of the same geometric type, i.e., $\overline{G_1} \times \overline{K} \simeq \overline{G_2} \times \overline{K}$.
2. The fields $K_1$ and $K_2$ are Galois conjugate.
3. There exists an isomorphism $\sigma : K_1 \rightarrow K_2$ such that the set of isotropic places coincide: $S'_1 = \sigma(S'_2)$.
4. If $G_1$ is not of type $A_n$, $D_{2n+1}$, $E_6$ ($n > 1$), then the lattices $\Gamma_1$ and $\Gamma_2$ are commensurable, i.e., $\overline{G_1} \simeq \overline{G_2}$ over $K$.
5. In any topologically representation equivalence class of arithmetic lattices, there are only finitely many commensurability classes of arithmetic lattices.

Part (1) of the above theorem, follows immediately from the definition of topologically representation equivalent lattices. The existence of an isomorphism between $G_{1,S_1}$ and $G_{2,S_2}$ gives an isomorphism at the level of Lie algebras. By assumption, at any place $v_1 \in S_1$ (resp. $v_2 \in S_2$), the Lie algebra of $G_1(K_1,v_1)$ (resp. $G_2(K_2,v_2)$) is simple. Hence (1) follows.

**Remark 2.7.** Since the lattices $\Gamma_i$ are uniform for $i = 1, 2$, any element belonging to $\Gamma_i$ is semisimple.

**Remark 2.8.** The first instance of this theorem was established by A. Reid [R], who showed that the spectrum of the Laplacian on functions of an arithmetic compact hyperbolic surface associated to a quaternion division algebra defined over a totally real number field determines the underlying number field and the division algebra.

For a compact Riemannian manifold $M$, denote by $L(M)$ the subset of $\mathbb{R}$ consisting of lengths of closed geodesics in $M$. Two Riemannian manifolds $M_1$ and $M_2$ are said to be length commensurable (resp. length isospectral) if $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$ (resp. $L(M_1) = L(M_2)$). The starting point of the proof of
Reid’s theorem is to use the Selberg trace formula to conclude that two compact hyperbolic surfaces are isospectral if and only if their length spectrums coincide.

Reid also proved that the complex length spectrum (length together with the holonomy of the closed geodesic) of a compact, arithmetic hyperbolic three manifold determines the commensurability class of the manifold. It can be seen from the trace formula that the complex length spectrum determines the representation equivalence class of the lattice. Working with only the length spectrum, Chirurg, Hamilton, Long and Reid showed in [CHLR] that length commensurable hyperbolic three manifolds are commensurable.

These results were vastly generalized by Gopal Prasad and A. Rapinchuk ([PR]). First, using results of Duistermaat, Guillemin, Kolk and Varadarajan ([DG, DKV]), Prasad-Rapinchuk-Uribe-Zelditch show that if two compact, Riemannian locally symmetric spaces of nonpositive sectional curvature are isospectral for the Laplace-Beltrami operator on functions then they are length commensurable (see Theorem 10.1 in [PR]).

Prasad and Rapinchuk define a notion of weak commensurability of lattices:

**Definition 2.9.** Let $G_1$ and $G_2$ be two semi-simple groups defined over a field $F$ of characteristic zero. Two Zariski dense subgroups $\Gamma_i$ of $G_i(F)$, for $i = 1, 2$ are said to be weakly commensurable if given any element of infinite order $\gamma_1 \in \Gamma_1$ (resp. $\gamma_2 \in \Gamma_2$) there exists an element of infinite order $\gamma_2 \in \Gamma_2$ (resp. $\gamma_1 \in \Gamma_1$) such that the subgroup of $\bar{F}^*$ generated by the eigenvalues of $\gamma_1$ (resp. $\gamma_2$) (in a faithful representation of $G_1$) intersects nontrivially the subgroup generated by the eigenvalues of an element $\gamma_2$ (resp. $\gamma_1$).

Prasad and Rapinchuk show ([PR Section 10]) that length commensurable arithmetic lattices are weakly commensurable. For this, when the locally symmetric spaces are of rank greater than one, they assume the validity of Schanuel’s conjecture:

**Conjecture 2.10 (Schanuel).** If $z_1, \cdots, z_n$ are $\mathbb{Q}$-linearly independent complex numbers, then the transcendence degree over $\mathbb{Q}$ of the field generated by

$$z_1, \cdots, z_n, e^{z_1}, \cdots, e^{z_n}$$

is at least $n$.

From the notion of weak commensurability of lattice, using methods from arithmetic theory of algebraic groups, they obtain results on commensurability, in particular the conclusions of Theorem 2.6

The use of representation equivalence instead of isospectrality on functions allows us to bypass the use of Schanuel’s conjecture in the higher ranks. The proof of Theorem 2.6 is an application of the Selberg trace formula and the ideas and methods given in [PR].

**Remark 2.11.** An initial motivation for this paper was to extend the results of A. Reid [R] to the context of $S$-arithmetic groups. An advantage of working
with the representation theoretic spectrum, is that the notion applies even when there is no Riemannian geometric interpretation. This allows us to consider $S$-arithmetic lattices.

Remark 2.12. Examples of representation equivalent lattices which are not commensurable have been given by Lubotzky, Samuels and Vishne [LSV]. It would be interesting to know whether such examples can be constructed in the exceptional cases given in [PR, Section 9], where commensurability fails.

3. Element-wise conjugate lattices

Definition 3.1. Let $G$ be a locally compact group and $\Gamma_1, \Gamma_2$ be lattices in $G$. The lattices $\Gamma_1$ and $\Gamma_2$ are said to be elementwise conjugate in $G$ if for any element $\gamma_1 \in \Gamma_1$ (resp. $\gamma_2 \in \Gamma_2$) there exists an element $\gamma_2 \in \Gamma_2$ (resp. $\gamma_1 \in \Gamma_1$) such that $\gamma_1$ and $\gamma_2$ are conjugate in $G$.

An application of the Selberg trace formula for compact quotients yields the following theorem:

Theorem 3.2. Let $G$ be a locally compact groups and $\Gamma_1, \Gamma_2$ be uniform lattices in $G$. Suppose the lattices $\Gamma_1$ and $\Gamma_2$ are representation equivalent. Then they are elementwise conjugate.

Corollary 3.3. With notation as in Theorem 2.6, suppose that the lattices $\Gamma_1 \subset G_1, S_1, \Gamma_2 \subset G_2, S_2$ are topologically representation equivalent by an isomorphism $\phi : G_1, S_1 \to G_2, S_2$. Then $\phi(\Gamma_1)$ and $\Gamma_2$ are elementwise conjugate in $G_2, S_2$.

3.1. Selberg trace formula. We recall the Selberg trace formula for uniform lattices [W]. Let $f$ be a continuous, compactly supported function on $G$. The convolution operator $R_G(f)$ on $L^2(\Gamma \backslash G)$ is defined by,

$$R_G(f)(\phi)(x) = \int_G f(y) R_G(y)(\phi)(x) \, d\mu(y),$$

where $\mu$ is an invariant Haar measure on $G$. It is known that $R_G(f)$ is of trace class.

Let $[\gamma]_G$ (resp. $[\gamma]_\Gamma$) be the conjugacy class of $\gamma$ in $G$ (resp. in $\Gamma$). Let $[\Gamma]$ (resp. $[\Gamma]_G$) be the set of conjugacy classes in $\Gamma$ (resp. the $G$-conjugacy classes of elements in $G$). For $\gamma \in \Gamma$, let $G_\gamma$ be the centralizer of $\gamma$ in $G$. Put $\Gamma_\gamma = \Gamma \cap G_\gamma$. It can be seen that $\Gamma_\gamma$ is a lattice in $G_\gamma$ and the quotient $\Gamma_\gamma \backslash G$ is compact. Since $G_\gamma$ is unimodular, there exists a $G$-invariant measure on $G_\gamma \backslash G$, denoted by $d_\gamma x$. After normalizing the measures on $G_\gamma$ and $G_\gamma \backslash G$ appropriately and rearranging the terms on the right hand side of above equation, we get :

$$\text{tr}(R_G(f)) = \sum_{[\gamma] \in [\Gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) \, d_\gamma x$$
\[
= \sum_{[\gamma] \in [\Gamma]_G} a(\gamma, \Gamma) O_\gamma(f)
\]
where \(O_\gamma(f)\) is the orbital integral of \(f\) at \(\gamma\) defined by,
\[
O_\gamma(f) = \int_{G_\gamma \backslash G} f(x^{-1} \gamma x) \, d_\gamma x.
\]
Here
\[
a(\gamma, \Gamma) = \sum_{[\gamma'] \subseteq [\gamma]_G} \text{vol} (\Gamma_{\gamma'} \backslash G_{\gamma'}).
\]
If \(\gamma\) is not conjugate to an element in \(\Gamma\), we define \(a(\gamma, \Gamma) = 0\).

Let \(\pi\) be an irreducible unitary representation of \(G\). Denote by \(\chi_\pi(f)\) the distributional character of \(\pi\) given by,
\[
\chi_\pi(f) = \text{Trace}(\pi(f)).
\]
The trace of \(R_\Gamma(f)\) on the spectral side can be written as an absolutely convergent series as,
\[
(2) \quad \text{tr}(R_\Gamma(f)) = \sum_{\pi \in \hat{G}} m(\pi, \Gamma) \chi_\pi(f)
\]
Hence from (1) and (2), we obtain the Selberg trace formula:
\[
(3) \quad \sum_{\pi \in \hat{G}} m(\pi, \Gamma) \chi_\pi(f) = \sum_{[\gamma] \in [\Gamma]_G} a(\gamma, \Gamma) O_\gamma(f).
\]

3.2. Proof of Theorem 3.2. We prove a few lemmas before giving the proof of Theorem 3.2

Lemma 3.4. Let \(G\) be a locally compact topological group and \(\Gamma\) be a uniform lattice in \(G\). Let \(U\) be a relatively compact subset of \(G\). Then the set
\[
A_U = \{ [\gamma]_G : \gamma \in \Gamma \text{ and } [\gamma]_G \cap U \neq \emptyset \}
\]
is finite.

Proof. Since the quotient \(\Gamma \backslash G\) is compact, there exists a relatively compact subset \(D\) of \(G\) such that \(G = \Gamma D\). Let \(x \in G\) be such that \(x^{-1} \gamma x \in U\) for some \(\gamma \in \Gamma\). Write \(x = \gamma' \delta\) where \(\gamma' \in \Gamma\) and \(\delta \in D\). Hence \(\gamma'^{-1} \gamma \gamma' \in D UD^{-1}\) which is relatively compact in \(G\). Hence \(\gamma'^{-1} \gamma \gamma' \in D UD^{-1} \cap \Gamma\) which is a finite set.
\(\square\)
Lemma 3.5. Let $\Gamma_1$ and $\Gamma_2$ be uniform lattices in $G$. Let $\gamma_1 \in \Gamma_1$. Then there exists a relatively compact open set $U$ containing $\gamma_1$ such that

$$U \cap [\gamma]_G = \emptyset$$

whenever $\gamma \in \Gamma_1 \cup \Gamma_2$ and $[\gamma_1]_G \neq [\gamma]_G$.

Proof. Easily follows from Lemma 3.4. □

Proof of Theorem 3.2. By comparing the Selberg trace formula (3) for the lattices $\Gamma_1$ and $\Gamma_2$ in $G$, we get for any compactly supported continuous function $f$ on $G$, 

$$\sum_{\pi \in \hat{G}} [m(\pi, \Gamma_1) - m(\pi, \Gamma_2)] \chi_\pi(f) = \sum_{[\gamma] \in [\Gamma_1]_G \cup [\Gamma_2]_G} [a(\gamma, \Gamma_1) - a(\gamma, \Gamma_2)] O_{\gamma}(f).$$

Since the lattices $\Gamma_1$, $\Gamma_2$ are representation equivalent in $G$, the left side is identically zero in the above equation.

Suppose $\gamma_1 \in \Gamma_1$ is not conjugate to any element of $\Gamma_2$ in $G$. Choose $U$ as in Lemma 3.5 and a positive function $f$ supported on $U$. For such $f$, we have that the orbital integral $O_{\gamma_1}(f)$ vanishes whenever $[\gamma]_G \neq [\gamma_1]_G$. Further $O_{\gamma_1}(f)$ is non-zero.

It follows that all terms on the right hand side of the above equation vanish except that corresponding to $[\gamma_1]_G$. Consequently, $a(\gamma_1, \Gamma_1) O_{\gamma_1}(f) = 0$. Since both these quantities are non-zero by definition, we arrive at a contradiction. Hence the lattices $\Gamma_1$ and $\Gamma_2$ are elementwise conjugate in $G$. □

4. Characteristic equivalence of lattices

Corollary 3.3 assures us that the elements in two topologically representation equivalent arithmetic lattices are elementwise conjugate (upto an isomorphism) in some large group, for instance in the group of complex points of the algebraic group. In particular this implies that the lattices $\Gamma_1$ and $\Gamma_2$ are weakly commensurable. Theorem 2.6 follows now from the results proved by Gopal Prasad and A. Rapinchuk ([PR][Theorems 1 to 5]).

The conclusion of Corollary 3.3 is stronger than the notion of weak commensurability. However, it does not seem easy to go directly from elementwise conjugacy to commensurability results, for example, to obtain Theorem 5.4. This leads us to define a new relation on the class of arithmetic lattices, stronger than weak commensurability, which we call as characteristic equivalence. We rephrase the property of elementwise conjugacy in terms of characteristic polynomials. This notion allows us to directly invoke results from the arithmetic theory of algebraic groups and simplify the arguments deducing commensurability type results from weak commensurability given in [PR].

Let $G$ be an algebraic group defined over a number field $K$. Consider the adjoint action $Ad$ of $G$ on its Lie algebra $L(G)$. Given a semisimple element $\gamma \in G(L)$ for an extension field $L$ of $K$, and any field $M$ containing $L$, denote by $P(Ad_G(\gamma), x)$ the characteristic polynomial of $Ad(\gamma)$ acting on $L(G) \otimes_K M$. 


The characteristic polynomial is independent of the extension field \( M \), and has coefficients in \( L \). In particular, if \( \Gamma \subset \mathcal{G}(K) \) is an arithmetic lattice, and if \( v \) is any place of \( K \), then the characteristic polynomials coincide,

\[
P(\text{Ad}_{\mathcal{G}}(\gamma), x) = P(\text{Ad}_{\mathcal{G}(K_v)}(\gamma_v), x),
\]

where \( \gamma \in \Gamma \) and by \( \gamma_v \) we denote its image in \( \mathcal{G}(K_v) \).

Note that the characteristic polynomial is also independent of the isogeny class of \( \mathcal{G} \): given \( \gamma \in \mathcal{G}(K) \), then

\[
P(\text{Ad}_{\mathcal{G}}(\gamma), x) = P(\text{Ad}_{\mathcal{G}(\overline{\gamma})}, x),
\]

where \( \overline{\gamma} \) denotes the image of \( \gamma \) in the adjoint group \( \mathcal{G}(K) \).

The characteristic polynomial is also independent up to isomorphisms:

**Lemma 4.1.** Let \( \mathcal{G}_1, \mathcal{G}_2 \) be simple algebraic groups defined over an algebraically closed field \( F \), and let \( \theta : \mathcal{G}_1 \to \mathcal{G}_2 \) be an isomorphism defined over \( F \). Suppose \( t \) is a semisimple element in \( \mathcal{G}_1(F) \). Then

\[
P(\text{Ad}_{\mathcal{G}_1}(t, x)) = P(\text{Ad}_{\mathcal{G}_2}(\theta(t), x)).
\]

**Proof.** Let \( T_1 \) be a maximal torus in \( \mathcal{G}_1 \) containing \( t \). The eigenvalues of \( \text{Ad}_{\mathcal{G}_1}(t) \) are 0 with multiplicity equal to the rank of \( \mathcal{G}_1 \) and \( \alpha(t) \) where \( \alpha \) runs over the roots of \( L(\mathcal{G}_1) \) with respect to \( T_1 \). If \( X_\alpha \) is a root vector corresponding to the root \( \alpha \), then

\[
\text{Ad}(\theta(t))(d\theta(X_\alpha)) = d\theta(\text{Ad}(t)X_\alpha) = \alpha(t)X_\alpha.
\]

Hence the eigenvalues of \( \theta(t) \) are the same as \( t \), and this proves the lemma.

\( \square \)

The topological elementwise conjugacy of the lattices \( \Gamma_1 \) and \( \Gamma_2 \) given by Corollary \ref{cor:elementwise} yields the following key proposition stating an equality of characteristic polynomials with respect to the adjoint representation:

**Proposition 4.2.** With assumptions as in Theorem \ref{thm:main}, there exists a locally compact field \( F \) and embeddings \( \iota_1 : K_1 \to F, \iota_2 : K_2 \to F \), and a topological automorphism \( \sigma \) of \( F \) such that given any element \( \gamma_1 \in \Gamma_1 \) (resp. \( \gamma_2 \in \Gamma_2 \)) there exists an element \( \gamma_2 \in \Gamma_2 \) (resp. \( \gamma_1 \in \Gamma_1 \)) such that the characteristic polynomials coincide,

\[
\sigma(P(\text{Ad}_{\mathcal{G}_1}(\gamma_1), x)) = P(\text{Ad}_{\mathcal{G}_2}(\gamma_2), x).
\]

**Proof.** Let \( v_1 \in S_i^1 \) be an isotropic place of \( \mathcal{G}_1 \). The group \( \mathcal{G}_1(K_{1,v_1}) \) is a non-compact normal subgroup of \( \mathcal{G}_{1,S_i^1} \). Hence there exists an isotropic place \( v_2 \in S_i^2 \) for \( \mathcal{G}_2 \), such that the projection to \( \mathcal{G}_2(K_{2,v_2}) \) of the image \( \phi(\mathcal{G}_1(K_{1,v_1})) \) is a non-compact normal subgroup \( N \) of \( \mathcal{G}_2(K_{2,v_2}) \). Since \( \mathcal{G}_2 \) is absolutely almost simple, \( N \) is Zariski dense in \( \mathcal{G}_2 \).

By Theorem A of Borel-Tits (\cite{BT}), there is a continuous homomorphism \( \sigma : K_{1,v_1} \to K_{2,v_2} \) such that the map \( \mathcal{G}_1(K_{1,v_1}) \to \mathcal{G}_2(K_{2,v_2}) \) is induced by an algebraic
morphism between the base changed group schemes,
\[ \sigma(\mathcal{G}_1 \times K_{1,v_1}) \rightarrow \mathcal{G}_2 \times K_{2,v_2}, \]
where the superscript \( \sigma \) denotes twisting the group scheme \( \mathcal{G}_1 \) by \( \sigma \). This map yields an isomorphism of algebraic groups at the adjoint level.

By Corollary 3.3, given any element \( \gamma_1 \in \Gamma_1 \) (resp. \( \gamma_2 \in \Gamma_2 \)) there exists an element \( \gamma_2 \in \Gamma_2 \) (resp. \( \gamma_1 \in \Gamma_1 \)) such that the element \( \phi(\gamma_1) \) (resp. \( \phi(\gamma_2) \)) is conjugate in \( \mathcal{G}_2(K_{2,v_2}) \) to \( \phi(\gamma_1) \) (resp. \( \phi(\gamma_2) \)).

Let \( F = K_{2,v_2} \) and \( \iota_2 : K_2 \rightarrow F \) be the natural embedding. The restriction of \( \sigma \) to \( K_1 \) gives an embedding \( \iota_1 \) of \( K_1 \) into \( F \). By Lemma 4.1 and the remarks preceding it, we have
\[ P(Ad_{\mathcal{G}_2}(\gamma_2), x) = P(Ad_{\mathcal{G}_2}(\phi(\gamma_1), x) = \sigma(P(Ad_{\mathcal{G}_1}(\gamma_1), x). \]
This proves the proposition. \( \square \)

We now show Part (2) of Theorem 2.6 that the fields of definition of the arithmetic lattices are conjugate:

**Proof of Part (2) of Theorem 2.6** In the notation of the proof of the foregoing proposition, let \( K'_1 = \iota_1(K_1) \). Consider the group \( \Gamma'_1 := \iota_1(\Gamma_1) \) as an arithmetic lattice of the group \( \mathcal{G}'_1 = \iota_1 \mathcal{G}_1 \) defined over the number field \( K'_1 \). We have an algebraic isomorphism \( \theta : \mathcal{G}'_1 \times F \rightarrow \mathcal{G}_2 \times F \) defined over \( F \) of the groups base changed to \( F \). Further \( \theta(\Gamma'_1) \) and \( \Gamma_2 \) are elementwise conjugate in \( \mathcal{G}_2(F) \).

By a theorem of Vinberg as given in Lemma 2.6 of [PR], it follows that the fields generated by \( \text{Trace}(\text{Ad}(\gamma)) \) for \( \gamma \) belonging to \( \Gamma'_1 \) (resp. \( \Gamma_2 \)) generate the field of definition \( K'_1 \) (resp. \( K_2 \)) of the ambient group \( \mathcal{G}'_1 \) (resp. \( \mathcal{G}_2 \)). Hence \( K'_1 = K_2 \) and this proves Part (2) of Theorem 2.6. \( \square \)

Henceforth, we will assume up to twisting the group scheme \( \mathcal{G}_1 \) by a field automorphism \( \sigma : K_1 \rightarrow K'_1 \), that \( K := K_1 = K_2 \) and both the group schemes \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are defined over the same number field \( K \).

We now define a notion of characteristic equivalence of lattices:

**Definition 4.3.** Let \( \mathcal{G}_1 \) (resp. \( \mathcal{G}_2 \)) be algebraic groups defined respectively over a number field \( K \). Let \( S_1 \) (resp. \( S_2 \)) be a finite set of places respectively of \( K \). Assume that for \( i = 1, 2 \), \((K, \mathcal{G}_i, S_i)\) satisfy hypothesis \( \text{H1-H3} \).

Let \( \Gamma_1 \subset \mathcal{G}_1(K) \) (resp. \( \Gamma_2 \subset \mathcal{G}_2(K) \)) be \( S_1 \) (resp. \( S_2 \))-arithmetic subgroup of \( \mathcal{G}_1 \) (resp. \( \mathcal{G}_2 \)).

We say that \( \Gamma_1 \) and \( \Gamma_2 \) are **characteristically equivalent** if given any semisimple element \( \gamma_1 \in \Gamma_1 \) (resp. \( \gamma_2 \in \Gamma_2 \)) there exists a semisimple element \( \gamma_2 \in \Gamma_2 \) (resp. \( \gamma_1 \in \Gamma_1 \)) such that the characteristic polynomials coincide,
\[ P(Ad_{\mathcal{G}_1}(\gamma_1), x) = P(Ad_{\mathcal{G}_2}(\gamma_2), x). \]
Lemma 4.4. In the definition of characteristic equivalence, we can further assume that the tori given by the identity component of the algebraic subgroup generated by $\gamma_i$ in $G_i$ ($i = 1, 2$) are isogenous.

Proof. Since the algebraic groups are absolutely almost simple, the isogeny class of the tori given by the identity component of the algebraic subgroup generated by $\gamma_i$ is determined by the element $Ad_{G_i}(\gamma_i)$ for $i = 1, 2$. Identifying the Lie algebras with $K^N$ as vector spaces over $K$, the lemma follows. $\square$

We now establish Theorems 1 to 5 of [PR] under this stronger hypothesis of characteristic equivalence of lattices:

Theorem 4.5. Let $G_1$ (resp. $G_2$) be algebraic groups defined respectively over a number field $K$. Let $S_1$ (resp. $S_2$) be a finite set of places respectively of $K$. Assume that for $i = 1, 2$, $(K, G_i, S_i)$ satisfy hypothesis H1-H3.

Let $\Gamma_1 \subset G_1(K)$ (resp. $\Gamma_2 \subset G_2(K)$) be $S_1$ (resp. $S_2$)-arithmetic subgroup of $G_1$ (resp. $G_2$).

Suppose that $\Gamma_1$ and $\Gamma_2$ are characteristically equivalent lattices. Then the following holds:

1. The groups $G_1$ and $G_2$ are of the same geometric type, or one of them is of type $B_n$ and the other is of type $C_n$.
2. The set of isotropic places $S_1^i$ and $S_2^i$ coincide.
3. Assume further that $G_1$ and $G_2$ are of the same geometric type. If $G_1$ is not of type $A_n$, $D_{2n+1}$, $(n > 1)$ $E_6$, then the lattices $\Gamma_1$ and $\Gamma_2$ are commensurable.
4. In any characteristic equivalence class of arithmetic lattices, there are only finitely many commensurability classes of arithmetic lattices.

Theorem 4.5 combined with Proposition 1.2 gives a proof of Theorem 2.6.

5. Proof of Theorem 4.5

Let $G$ be a connected, absolutely almost simple algebraic group defined over $K$. Let $T$ be a maximal $K$-torus in $G$. Denote by $\Phi_T$ the root system of $G$ with respect to $T$, and by $W(\Phi_T)$ the Weyl group of $\Phi_T$. Let $L$ be the splitting field of $T$. There exists a natural injective homomorphism $\theta_T : Gal(L/K) \rightarrow Aut(\Phi_T)$.

For the proof of Theorem 4.5, we need the following theorem on the existence of irreducible tori ([PR2][Theorem 1]):

Theorem 5.1. Let $G$ be a connected, absolutely almost simple algebraic group defined over a number field $K$. Suppose $v$ is a place of $K$ and $T_v$ is a maximal $K_v$-torus of $G$. Then there exists a $K$-torus $T$ of $G$ such that it is conjugate to $T_v$ by an element of $G(K_v)$. Further, the image of $\theta_T$ contains the Weyl group $W(\Phi_T)$. In particular, $T$ is an irreducible, anisotropic maximal $K$-tori of $G$.

The proof of this theorem is based on a theorem of A. Grothendieck that the variety of maximal tori is rational, and based on this a theorem of V. E.
Voskresenskii showing that the Galois group of the splitting field of the generic maximal tori contains the Weyl group.

**Corollary 5.2.** With notation as in Theorem 5.1, let $\Gamma$ be a $S$-arithmetic lattice in $G$ and $v \in S^1$. Assume further that $T_v$ is an isotropic torus. Then there exists an element $\gamma \in \Gamma$ which generates $T$ over $K$.

**Proof.** By [PR, Theorem 5.12], there exists non-torsion elements in $T_1(O_K(S))$. Since $\Gamma \cap T(O_K(S))$ is of finite index in $T(O_K(S))$, there exists a non-torsion element $\gamma \in \Gamma \cap T(O_K(S))$. Since $T$ is irreducible, $\gamma$ will generate $T$ over $K$. □

**Proof of Part (1) of Theorem 4.5.** The equality of the characteristic polynomials with respect to the adjoint representation implies that the dimensions of the Lie algebras are equal. If the algebraic groups involved are not of type $B_6$, $C_6$ or $E_6$, then the geometric type is determined by the dimension of the Lie algebra.

For the proof of Part (1) in this exceptional case, we argue as in proof of Theorem 1 in [PR] page 130: by Corollary 5.2, choose a torus $T_1$ and an element $\gamma_1 \in \Gamma_1$ which generates $T_1$ over $K$. By characteristic equivalence, there exists an element $\gamma_2 \in \Gamma_2$ having the same characteristic polynomial as $\gamma_1$. By Lemma 4.4, we can further assume that the tori $T_1$ generated by $\gamma_1$, and the tori $T_2$ given by the identity component of the diagonalizable subgroup generated by $\gamma_2$ are isogenous over $K$.

Let $L$ be the splitting field of $T_1$ (equivalently of $T_2$). By Theorem 5.1, the image of $\theta_{T_1}$ contains the Weyl group $W(\Phi_{T_1})$. We can assume that the geometric type of $G_i$ is of type either $B_6$ or $C_6$. In this case, all automorphisms of $\Phi_{T_1}$ are inner, and the cardinality of $\text{Gal}(L/K)$ is thus equal to $|W(\Phi_{T_1})|$. From the injectivity of the map $\theta_{T_2}$, we see that $|W(\Phi_{T_1})|$ divides the cardinality of $\text{Aut}(\Phi_{T_2})$. But the cardinality of $W(B_6)$ is $2^{10}3^{25}$, whereas the cardinality of $\text{Aut}(E_6)$ is given by $2^73^45$. This implies that $G_2$ cannot be of type $E_6$. □

**Proof of Part (2) of Theorem 4.5.** This is the analogue of Theorem 3 of [PR], and we follow the proof as given in [PR] page 139 of this theorem. If $v \in S^1$ is a place where $G_1$ is isotropic, choose a maximal split $K_v$-torus $T_1,v$ of $G_1$. By Theorem 5.1, there exists a $K$-irreducible anisotropic maximal $K$-torus $T_1$ of $G_1$ such that it is conjugate to $T_1,v$ by an element of $G_1(K_v)$. Since $T_1$ is anisotropic the quotient $T_{1,S_1}/T_1(O_K(S_1))$ is compact where $T_{1,S_1} = \prod_{v \in S_1} T_1(K_v)$. This implies that the quotient $T_1(K_v)/C$ is also compact, where $C$ is the closure of $T_1(O_K(S_1))$ in $T_1(K_v)$. Since $T_1$ is $K_v$-isotropic, $C$ is noncompact. The closure of $\text{Ad}(T_1(O_K(S_1)))$ inside $GL_N(K_v)$ will also be noncompact. Since $T_1(O_K(S_1))$ is a finitely generated abelian group consisting of semisimple elements, it can be simultaneously diagonalised over $K_v$. If the eigenvalues of every element in $T_1(O_K(S_1))$ is a $v$-adic unit, then this implies that the closure of $T_1(O_K(S_1))$ is compact, contradicting our earlier conclusion. Since $\Gamma_1 \cap T_1(O_K(S_1))$ is of finite index in $T_1(O_K(S_1))$, there exists an element $\gamma_1 \in \Gamma_1$ such that at least one
eigenvalue of $Ad_{G_1}(\gamma_1) \in GL_N$ is not a $v$-adic unit. By assumption there exists an element $\gamma_2 \in \Gamma_2$ which is characteristic equivalent to $\gamma_1$.

If $v \notin S_i^1$, then the closure of the subgroup $G_2(\mathcal{O}_K(S_2))$ in $G_2(K_v)$ is compact. But this implies that all the eigenvalues of $Ad_{G_2}(\gamma_2)$ are $v$-adic units. This yields a contradiction and hence $S_i^1 \subset S_i^2$. By symmetry we get $S_i^1 = S_i^2$. □

**Remark 5.3.** It is known that weak approximation holds for the tori constructed in Theorem 5.1 (see [PR2]). One could have also used this fact to give a slight variation of the above argument.

We now prove Theorem 6.2 of [PR]. It is the basic input needed to prove Parts (3) and (4) of Theorem 4.5.

**Theorem 5.4.** With hypothesis as in Theorem 4.5, for any place $v$ of $K$,

$$\text{rk}_{K_v} G_1 = \text{rk}_{K_v} G_2$$

**Proof.** Let $T_{1,v}$ be a maximal $K_v$-split torus of $G_1$, and choose a $K$-torus $T_1$ and an element $\gamma_1 \in \Gamma_1$ as in Corollary 5.2.

By the characteristic equivalence of $\Gamma_1$ and $\Gamma_2$, there exists an element $\gamma_2 \in \Gamma_2$ for which there is an equality of characteristic polynomials

$$P(Ad_{G_1}(\gamma_1), x) = P(Ad_{G_2}(\gamma_2), x).$$

This implies that the elements $Ad_{G_1}(\gamma_1)$ and $Ad_{G_2}(\gamma_2)$ considered as elements in $GL_N/K$ are conjugate, and hence generate isomorphic diagonalizable subgroups over $K$. Let $T_2$ be the tori given by the identity component of the subgroup generated by $\gamma_2$. We have,

$$\text{rk}_{K_v} G_1 = \text{rk}_{K_v} T_1 = \text{rk}_{K_v} T_2 \leq \text{rk}_{K_v} G_2$$

By symmetry, this proves the theorem. □

The proofs of Part (3) and (4) of Theorem 4.5 follow as in page 147-148 of [PR]. For the sake of completeness, we give a brief outline of the proof.

**Proof of Part (3) of Theorem 4.5** If the geometric type is of type $D_{2n}$, $(n > 2)$ (resp. $D_4$) this is proved in [PR3] (resp. [G]).

If the geometric type is not of $A$, $D$ or $E_6$ type, the equality of local ranks implies that $\overline{G}_{1,v} \simeq \overline{G}_{2,v}$ for any place $v$ of $K$. For archimedean places, this follows from classification results [T]. For a non-archimedean place, this follows from the fact that there can be at most two possible forms for the adjoint group. To see the latter fact, we observe that the centre $Z$ of the simply connected cover of $G$ is a subgroup of $\mu_2$, where $G$ is not of type $A$, $D$, $E_6$. From the equality of the Galois cohomology groups, we get that $H^1(K_v, \overline{G}) \simeq H^2(K_v, Z)$, which can be identified with a subgroup of the 2-torsion in the Brauer group of $K_v$. Since this is of cardinality two, and the outer automorphism group is trivial, this implies that there are at most two forms of $\overline{G}$ for any non-archimedean place $v$. Hence an equality of ranks over $K_v$ implies that the forms are isomorphic.
Now Part (3) of Theorem 4.5 follows from the Hasse principle, viz., the injectivity of the localization map,
\[ H^1(K, \mathcal{G}) \to \bigoplus_v H^1(K_v, \mathcal{G}) \]
where \( v \) runs over all places of \( K \).

**Proof of Part (4) of Theorem 4.5.** From Theorem 5.4, it can be seen by a Chebotarev density argument ([PR, Theorem 6.3] that the minimal splitting field \( L_i \) over which \( \mathcal{G}_i \) becomes the inner form of a split group for \( i = 1, 2 \) coincide. Moreover, the set of places \( V_i \) at which \( \mathcal{G}_i \) is not quasi-split coincide.

Fixing the geometric type, say a split form \( \mathcal{G}_0 \) over \( K \) of adjoint type, the groups are parametrized by cocycles \( c \in H^1(K, \text{Aut}(\mathcal{G}_0)) \). Consider the exact sequence,
\[ 1 \to \mathcal{G}_0 \to \text{Aut}(\mathcal{G}_0) \to \text{Out}(\mathcal{G}_0) \to 1. \]
This yields an exact sequence,
\[ H^1(K, \mathcal{G}_0) \to H^1(K, \text{Aut}(\mathcal{G}_0)) \to H^1(K, \text{Out}(\mathcal{G}_0)). \]
The condition that the group becomes an inner form over \( L \) implies that this cocycle lies in the image of \( H^1(G(L/K), \text{Out}(\mathcal{G}_0)) \) which is a finite group. Now the required finiteness follows from the finiteness of the Hasse principle, i.e., the kernel of the localization map,
\[ H^1(K, \mathcal{G}_0) \to \bigoplus_{v \not\in V} H^1(K_v, \mathcal{G}_0), \]
where \( V = V_i \) is a fixed finite set of places of \( K \). \( \square \)

**Remark 5.5.** It is further deduced ([PR, Theorem 6], that if \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) have the same geometric type satisfying the hypothesis of Theorem 5.4 then the Tits indices are equal at all places of \( K \).

**Remark 5.6.** It is possibly more appropriate to call the notion of characteristic equivalence given out here as *weakly characteristic equivalence*, since we are not taking into account multiplicities. A notion of multiplicity will be to count the number upto \( \Gamma \)-conjugacy of the set of elements in \( \Gamma \) which have the same characteristic polynomial with respect to the adjoint representation. It would then be interesting to know whether characteristically equivalent lattices (counted with multiplicities) are (topologically) representation equivalent.

It is also clear that stably conjugate elements i.e., conjugate in \( G(K) \) will have the same characteristic polynomials. This yields another possible definition of characteristic equivalence, but it is in the conjugacy in the Lie algebra version that we can directly relate to the underlying arithmetic of the ambient group \( G \).

It is possible to consider modifications of the concept of characteristic equivalence, say more generally on the class of subgroups not necessarily arithmetic
lattices: for example one can consider the equality of the characteristic polynomials on ‘big’ subsets, like subgroups of finite index, or Zariski open, or even some kind of Hilbertian sets.

Yet another relation that can be imposed is to define two lattices to be trace equivalent if the set of traces of elements with respect to the adjoint representation coincide for the two lattices.

It would be interesting to know whether these properties would imply commensurability results. To conclude commensurability type results will require analogues of Theorems 5.1 or 5.4.

Remark 5.7. It is clear that characteristically equivalent lattices are weakly commensurable. Examples have been given in [PR, Sections 6 and 9] of weakly commensurable lattices which are not commensurable. It would be interesting to know whether these examples give characteristically equivalent lattices.

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Indian Institute of Science Education and Research, Pune- 411021, India
E-mail address: cbhagwat@iiserpune.ac.in

School of Mathematics, Tata Institute Of fundamental research, Homi Bhabha Road, Mumbai- 400005, India
E-mail address: supriya@math.tifr.res.in

School of mathematics, Tata Institute of fundamental research, Homi Bhabha road, Mumbai- 400005, India
E-mail address: rajan@math.tifr.res.in