STABLE FUNCTIONS OF JANOWSKI TYPE

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Abstract. A function $f \in \mathcal{A}_1$ is said to be stable with respect to $g \in \mathcal{A}_1$ if

$$
\frac{s_n(f(z))}{f(z)} < \frac{1}{g(z)}, \quad z \in \mathbb{D},
$$

holds for all $n \in \mathbb{N}$ where $\mathcal{A}_1$ denote the class of analytic functions $f$ in the unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ normalized by $f(0) = 1$. Here $s_n(f(z))$, the $n^{th}$ partial sum of $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is given by $s_n(f(z)) = \sum_{k=0}^{n} a_k z^k$, $n \in \mathbb{N} \cup \{0\}$. In this work, we consider the following function $v_{\lambda}(A, B, z) = \left( \frac{1 + Az}{1 + Bz} \right)^{\lambda}$ for $-1 \leq B < A \leq 1$ and $0 \leq \lambda \leq 1$ for our investigation. The main purpose of this paper is to prove that $v_{\lambda}(A, B, z)$ is stable with respect to $v_{\lambda}(0, B, z) = \frac{1}{(1 + Bz)^{\lambda}}$ for $0 < \lambda \leq 1$ and $-1 \leq B < A \leq 0$. Further, we prove that $v_{\lambda}(A, B, z)$ is not stable with respect to itself, when $0 < \lambda \leq 1$ and $-1 \leq B < A < 0$.

Introduction & Main Results

Let $\mathcal{A}$ denote the family of functions $f$ that are analytic in the unit disk $\mathbb{D} := \{ z : |z| < 1 \}$. Let $\mathcal{A}_1$ is the subset of $\mathcal{A}$ with the normalization $f(0) = 1$. A single valued function $f \in \mathcal{A}_1$ is said to be univalent in a domain $\Delta \subseteq \mathbb{C}$ if $f$ is one-to-one in $\Delta$. The class of all univalent functions with the normalization $f(0) = 0 = f'(0) - 1$ is denoted by $\mathcal{S}$. Let $\Omega$ be the family of functions $\omega$, regular in $\mathbb{D}$ and satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. For $f, g \in \mathcal{A}$, the function $f$ is said to be subordinate to $g$, denoted by $f \prec g$ if and only if there exists an analytic function $\omega \in \Omega$ such that $f = g \circ \omega$. In particular, if $g$ is univalent in $\mathbb{D}$ then $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$ hold.

The function $zf(z) \in \mathcal{A}_1$ is starlike of order $\lambda$ if $\text{Re} \left( \frac{zf(z)}{f(z)} \right) > \lambda$ for all $z \in \mathbb{D}$ and $0 \leq \lambda < 1$. The class of all starlike functions, denoted by $\mathcal{S}^*(\lambda)$ is a subclass of $\mathcal{S}$. The $n^{th}$ partial sum $s_n(f(z))$ of $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is given by $s_n(f(z)) = \sum_{k=0}^{n} a_k z^k$, $n = 0, 1, 2, \ldots$. For more details about the univalent functions, its subclasses and subordination properties, we refer [2, 3, 4].

The concept of stable functions was first introduced by Ruscheweyh and Salinas [6], while discussing the class of starlike functions of order $\lambda$, where $1/2 \leq \lambda < 1$. However, the class of starlike functions of order $\lambda \in [1/2, 1)$ is comparatively a much narrow class but it has many interesting properties too. Ruscheweyh and Salinas [6] proved the following result.

Theorem 1. [6] Let $\lambda \in [1/2, 1)$ and $zf \in \mathcal{S}^*(\lambda)$, then

$$
\frac{s_n(f(z))}{f(z)} \prec (1 - z)\lambda, \quad n \in \mathbb{N}, z \in \mathbb{D}.
$$

Theorem 1 has several applications in Gegenbauer polynomial sums and motivated by Theorem 1, Ruscheweyh and Salinas [6] introduced the concept of Stable functions which is stated.
as follows. For some $n \in \mathbb{N}$, a function $F$ is said to be $n$-stable function with respect to $G$ if

$$\frac{s_n(F(z))}{F(z)} \leq \frac{1}{G(z)},$$

for $F, G \in A_1$ and $z \in \mathbb{D}$.

Moreover, the function $F$ is said to be stable with respect to $G$, if $F$ is $n$-stable with respect to $G$ for every $n \in \mathbb{N}$. Particularly, if the function $F$ is $n$-stable with respect to itself. Then for every $n \in \mathbb{N}, F$ is stable. In the present context, for $-1 \leq B < A \leq 1$, we define a function

$$v_\lambda(A, B, z) := \left(\frac{1 + A z}{1 + B z}\right)^\lambda$$

for $z \in \mathbb{D}$ and $\lambda \in (0, 1]$.

For $\lambda = 1/2$, Ruscheweyh and Salinas [7] proved that $v_{1/2}(1, -1, z)$ is stable function with respect to itself. The stability of $v_{1/2}(1, -1, z)$ is equivalent to the simultaneous non-negativity of general class of sine and cosine sums given by Vietoris [11], the most celebrated theorem of positivity of trigonometric sums. Ruscheweyh and Salinas [7] conjectured that $v_\lambda(1, -1, z)$ is stable for $0 < \lambda < 1/2$. Using computer algebra, for $\lambda = 1/4$ it was shown in [7] that $v_{1/4}(1, -1, z)$ is $n$-stable for $n = 1, 2, 3, \ldots, 5000$. In the limiting case, the validation of stability of $v_\lambda(1, -1, z)$ for $0 < \lambda < 1/2$ interpreted in terms of positivity of trigonometric polynomials.

Further extensions of Vietoris Theorem and stable functions to Cesàro stable functions and Generalized Cesàro stable functions have been studied in [4] and [9] respectively. In this direction, conjectures are also proposed in [9] that linked Generalized Cesàro stable functions with the positivity of trigonometric sums. Chakraborty and Vasudevarao [1] considered $A = 1 - 2\alpha, B = -1$ and proved the following result.

**Theorem 2.** [1] For $0 < \lambda \leq 1$ and $1/2 \leq \alpha < 1$, $v_\lambda(1 - 2\alpha, -1, z) = \left(1 + \frac{(1 - 2\alpha)z}{1 - z}\right)^\lambda$ is stable with respect to $v_\lambda(0, -1, z) = \frac{1}{(1 - z)^\lambda}$.

Chakraborty and Vasudevarao [1] also proved that $v_\lambda(1 - 2\alpha, -1, z)$ is not stable with respect to itself when $1/2 < \alpha < 1$ and $0 < \lambda \leq 1$. For $\lambda = 1$, the function $v_1(A, B, z) = \frac{1 + A z}{1 + B z}$ have been studied widely by many researchers. The analytic functions of $A_1$ subordinate to $\frac{1 + A z}{1 + B z}$ have been studied by Janowski [3] and the class of such functions is denoted by $\mathcal{P}(A, B)$. The functions of $\mathcal{P}(A, B)$ are called Janowski functions. Moreover, the set of functions $zf \in A_1$, for which $\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}$ holds, called Janowski starlike functions and the class of such functions is denoted by $\mathcal{S}^*(A, B)$. It can be easily seen that $\mathcal{S}^*(1, -1) \equiv \mathcal{S}^*$.

In this paper, we show that $v_\lambda(A, B, z)$ is stable with respect to $v_\lambda(0, B, z) = 1/(1 + Bz)^\lambda$ for $0 < \lambda \leq 1$ and $-1 \leq B < A \leq 0$. Further, $v_\lambda(A, B, z)$ is not stable with respect to itself, when $0 < \lambda \leq 1$ and $-1 \leq B < 0$. We can write $v_\lambda(A, B, z)$ as,

$$v_\lambda(A, B, z) = \left(\frac{1 + A z}{1 + B z}\right)^\lambda$$

$$= (1 + Az)^\lambda (1 + Bz)^{-\lambda}$$

$$= \left[1 + \sum_{k=1}^{\infty} \frac{[\lambda]_k}{k!} A^k z^k \right] \left[1 + \sum_{k=1}^{\infty} \frac{\lambda}{k!} (-B)^k z^k \right]$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} \frac{[\lambda]_k}{k!} \frac{\lambda}{(n-k)!} A^k (-B)^{n-k} \right) z^n,$$

(1)
where \([\lambda]_k\) and \((\lambda)_k\) denote the factorial polynomials given as

\[
\begin{align*}
[\lambda]_k &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)\cdots(\lambda-k+1), \\
(\lambda)_k &= \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+k-1),
\end{align*}
\]

and \(\Gamma\) is well-known gamma function, for \(k = 1, 2, \cdots\) respectively with \([\lambda]_0 = 1 = (\lambda)_0\). So \(v_\lambda(A, B, z)\) can be written as

\[
v_\lambda(A, B, z) = 1 + \sum_{n=1}^{\infty} a_n(A, B, \lambda)z^n,
\]

where

\[
a_n := a_n(A, B, \lambda) = \sum_{k=0}^{n} \frac{[\lambda]_k (\lambda)_{n-k}}{k! (n-k)!} A^k (-B)^{n-k}.
\]

Now, we state two lemmas which will helpful to prove our main results.

**Lemma 1.** For \(0 < \lambda \leq 1\) and \(-1 \leq B < A \leq 0\), we have

\[
\sum_{k=0}^{n} \frac{[\lambda]_k (\lambda)_{n-k}}{k! (n-k)!} A^k (-B)^{n-k} > 0.
\]

**Lemma 2.** Let \(v_\lambda(A, B, z)\) be defined by (1). Then for \(\lambda \in (0, 1]\) and \(-1 \leq B < A \leq 0\),

\[
(m+1)(n+1) \left( \sum_{k=0}^{n+1} \frac{[\lambda]_k (\lambda)_{n+1-k}}{k! (n+1-k)!} A^k B^{n+1-k} \right) - mn \left( \sum_{k=0}^{n} \frac{[\lambda]_k (\lambda)_{n-k}}{k! (n-k)!} A^k B^{n-k} \right) \geq 0
\]

holds for all \(m, n \in \mathbb{N}\).

Now, we state main results of this paper which are about the stability of \(v_\lambda(A, B, z)\) with respect to \(v_\lambda(0, B, z)\) and \(v_\lambda(A, B, z)\) itself.

**Theorem 3.** For \(\lambda \in (0, 1]\) and \(-1 \leq B < A \leq 0\), \(v_\lambda(A, B, z)\) given in (1) is stable with respect to \(v_\lambda(0, B, z) = \frac{1}{(1+Bz)^\lambda}\).

If we substitute \(A = 0\) in Theorem 3, we get the following corollary which is also a generalization of the result given by Ruscheweyh and Salinas [6].

**Corollary 1.** For \(\lambda \in (0, 1]\) and \(-1 \leq B < 0\), \(v_\lambda(0, B, z) = \frac{1}{(1+Bz)^\lambda}\) is stable function.

Now for \(0 < \mu \leq \lambda \leq 1\), we have the following corollary of Theorem 3.

**Corollary 2.** For \(0 < \mu \leq \lambda \leq 1\) and for \(-1 \leq B < 0\) we have

\[
\frac{s_n(v_\mu(0, B, z))}{v_\lambda(0, B, z)} < \frac{1}{v_\lambda(0, B, z)}, \quad \text{for } z \in \mathbb{D}.
\]

Theorem 3 also generalizes result of Chakraborty and Vasudevarao [1] as if we substitute \(A = 1 - 2\alpha\) and \(B = -1\) in Theorem 3, reduces to Theorem 2. In other words, Theorem 2 is a particular case of Theorem 3.

**Theorem 4.** For \(\lambda \in (0, 1]\) and \(-1 \leq B < A < 0\), \(v_\lambda(A, B, z) = \left(\frac{1+Az}{1+Bz}\right)^\lambda\) is not stable with respect to itself.
Proof of Main Results

Proof of Lemma 1. Consider,
\[
1 = (1 - z)^{\lambda}(1 - z)^{-\lambda}
\]
\[
= 1 + \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n} \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k}}{(n-k)!} (-1)^k \right) z^n
\]
Comparing the coefficients of \(z^n\) on both the sides we have
\[
\sum_{k=0}^{n} \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k}}{(n-k)!} (-1)^k = 0,
\]
which can be expanded as
\[
\frac{(\lambda)(\lambda+1)\cdots(\lambda+n-1)}{(n)!} \lambda + \frac{(\lambda)(\lambda+1)\cdots(\lambda+n-2)}{(n-1)!} \left(\frac{\lambda}{1!}\right) (-1) \\
+ \frac{(\lambda)(\lambda+1)\cdots(\lambda+n-3)\lambda(\lambda-1)}{(n-2)!} \left(-\frac{\alpha}{\beta}\right)^2 + \cdots + \frac{\lambda(\lambda)(\lambda-1)\cdots(\lambda+n-2)}{(n-1)!} \left(-\frac{\alpha}{\beta}\right)^{n-1} \\
+ \frac{(\lambda)(\lambda-1)\cdots(\lambda-n+1)}{n!} \left(-\frac{\alpha}{\beta}\right)^n \geq 0.
\]
After multiplying by \(\beta^n\) we obtain
\[
\beta^n \sum_{k=0}^{n} \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k}}{(n-k)!} (-1)^k \left(\frac{\alpha}{\beta}\right)^k \geq 0
\]
By substituting \(\alpha = -A\), \(\beta = -B\) in (3) so that for \(-1 \leq B < A \leq 0\), the lemma is proved. \(\square\)

Proof of Lemma 3. Let \(v_\lambda(A, B, z)\) be defined by (1). Then,
\[
v_\lambda(A, B, z) = \left(\frac{1 + A z}{1 + B z}\right)^{\lambda} = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots
\]
\[
v_\lambda'(A, B, z) = \lambda \left(\frac{1 + A z}{1 + B z}\right)^{\lambda-1} \left(\frac{(1 + B z)A - (1 + A z)B}{(1 + B z)^2}\right)
\]
\[
= \frac{\lambda(A - B)(1 + A z)^{\lambda-1}}{(1 + B z)^{\lambda+1}}
\]
\[
(1 + B z)v_\lambda'(A, B, z) = \lambda(A - B)(1 + A z)^{\lambda-1}(1 + B z)^{-\lambda}
\]
Since \(0 > A > B, 0 < \lambda \leq 1\), \((1 + A z)^{\lambda-1} = 1 + (\lambda - 1)A z + \frac{(\lambda-1)(\lambda-2)}{2!} A^2 z^2 + \cdots\) and \((1 + B z)^{-\lambda} = 1 - \lambda B z + \frac{\lambda(\lambda+1)}{2!} B^2 z^2 + \cdots\) have positive Taylor series coefficients. A simple
computation yields that
\[(1 + Bz)v'_{\lambda}(A, B, z) = (a_1 + 2a_2z + 3a_3z^2 + \cdots)(1 + Bz)\]
\[= a_1 + \sum_{n=1}^{\infty}((n + 1)a_{n+1} + Bna_n)z^n.\] (5)

Since right hand side of (4) has positive Taylor coefficients, from (4) and (5) we conclude that
\[(n + 1)a_{n+1} + Bna_n > 0, \quad n \in \mathbb{N}.\] (6)

The left hand side of the expression given in (2) can be rewritten as
\[(m + 1)(n + 1)a_{n+1} + mnBa_n.\] (7)

Equivalently, (7) can be written as
\[m((n + 1)a_{n+1} + Bna_n) + (n + 1)a_{n+1}.\]

Using (6) and the fact that \(a_n \geq 0\) for \(m, n \in \mathbb{N}\), the lemma is proved for \(\lambda \in (0, 1]\) and \(-1 \leq B < A \leq 0\). □

Before going to proceed further for the proof of Theorem 3, it is easy to verify the following relations.

\[s'_{n}(v_{\lambda}(A, B, z), z) = s_{n-1}(v'_{\lambda}(A, B, z), z),\]
\[zs'_{n}(v_{\lambda}(A, B, z), z) = s_{n}(zv'_{\lambda}(A, B, z), z),\]
\[z^2s'_{n}(v_{\lambda}(A, B, z), z) = s_{n}(z^2v'_{\lambda}(A, B, z), z).\] (8)

Now, we are ready to give the proof of Theorem 3.

**Proof of Theorem 3.** To show that \(v_{\lambda}(A, B, z)\) is stable with respect to \(v_{\lambda}(0, B, z)\), it is enough to show that
\[\frac{s_{n}(v_{\lambda}(A, B, z), z)}{v_{\lambda}(A, B, z), z} \prec \frac{1}{v_{\lambda}(0, B, z)}, \quad z \in \mathbb{D}\]
for all \(n \in \mathbb{N}\), i.e., to prove that
\[\frac{(1 + Bz)^\lambda s_{n}(v_{\lambda}(A, B, z), z)}{(1 + Az)^\lambda} \prec (1 + Bz)^\lambda, \quad z \in \mathbb{D},\]
which can be equivalently written as
\[\frac{(1 + Bz)s_{n}(v_{\lambda}(A, B, z), z)^{\frac{1}{\lambda}}}{(1 + Az)^{\frac{1}{\lambda}}} \prec (1 + Bz).\]

To show that , it is enough to prove that
\[\left|\frac{(1 + Bz)s_{n}(v_{\lambda}(A, B, z), z)^{\frac{1}{\lambda}}}{(1 + Az)^{\frac{1}{\lambda}}} - 1\right| \leq |B| \leq 1, \quad z \in \mathbb{D}.\]

For fixed \(n\) and \(\lambda\), we consider the following function
\[h(z) = 1 - \frac{(1 + Bz)s_{n}(v_{\lambda}(A, B, z), z)^{\frac{1}{\lambda}}}{(1 + Az)^{\frac{1}{\lambda}}}, \quad z \in \mathbb{D}.\]

It is easy to see that
\[v'_{\lambda}(A, B, z) = \lambda(A - B)\frac{(1 + Az)^{\lambda - 1}}{(1 + Bz)^{\lambda + 1}} = \lambda(A - B)\frac{v_{\lambda}(A, B, z)}{(1 + Bz)(1 + Az)},\]
which can be rewritten in the following form
\[v_{\lambda}(A, B, z) - \frac{(1 + (A + B)z + ABz^2)}{\lambda(A - B)}v'_{\lambda}(A, B, z) = 0 \quad \text{for} \quad z \in \mathbb{D}.\] (9)
A simple calculations gives that
\[
h'(z) = \frac{A - B}{(1 + Az)^2} s_n(v_\lambda(A, B, z), z)^{1/\lambda} - \frac{(1 + Bz)}{(1 + Az)^\lambda} s_n(v_\lambda(A, B, z), z)^{1/\lambda - 1} s_n'(v_\lambda(A, B, z), z)
\]
\[
= \frac{(A - B) s_n(v_\lambda(A, B, z), z)^{1/\lambda}}{(1 + Az)^2} \left( s_n(v_\lambda(A, B, z), z) - \frac{(1 + Az)(1 + Bz)}{(A - B)\lambda} s_n'(v_\lambda(A, B, z), z) \right)
\]
Using relations (8) in (10), we get
\[
h'(z) = \frac{(A - B) s_n(v_\lambda(A, B, z), z)^{1/\lambda - 1}}{(1 + Az)^2} \left[ s_n \left( v_\lambda(A, B, z), z \right) - \frac{(1 + Az)(1 + Bz)}{(A - B)\lambda} v_n'(A, B, z, z) \right]
\]
\[\quad + \frac{(n + 1)}{\lambda(A - B) \sum_{k=0}^{n+1} \frac{[\lambda]_k (\lambda)_{n-k+1}}{k! (n-k+1)!} A^k (-B)^{n-k+1} z^n} \]
\[\quad - \frac{nAB}{\lambda(A - B) \sum_{k=0}^{n} \frac{[\lambda]_k (\lambda)_{n-k}}{k! (n-k)!} A^k (-B)^{n-k} z^{n+1}}.\]
Substituting (8) in (11) and using definition of \(a_n\), the following form of \(h'(z)\) can be obtained.
\[
h'(z) = \frac{z^n s_n(v_\lambda(A, B, z), z)^{1/\lambda - 1}}{(1 + Az)^2} \left( (n + 1)a_n + ABzn_n \right)
\]
\[= \frac{z^n s_n(v_\lambda(A, B, z), z)^{1/\lambda - 1}}{\lambda} \left( (n + 1)a_n + ABzn_n \right) (1 + Az)^{-2}
\]
\[= \frac{z^n s_n(v_\lambda(A, B, z), z)^{1/\lambda - 1}}{\lambda} \left( (n + 1)a_n + ABzn_n \right) (1 - 2Az + 3A^2z^2 - 4A^3z^3 + \cdots)
\]
\[= \frac{z^n s_n(v_\lambda(A, B, z), z)^{1/\lambda - 1}}{\lambda} \left( (n + 1)a_n + \sum_{m=1}^{\infty} (m + 1)(n + 1)a_n + mnBn_n(1 - A)^m z^m \right).
\]
Since \(A \in (-1, 0]\), we have \(-A \geq 0\). Therefore in view of Lemma 1, we obtain \(a_n > 0\) for all \(n \in \mathbb{N}\). Further, from Lemma 2 we obtain \((m + 1)(n + 1)a_n + Bn_n > 0\) for all \(n, m \in \mathbb{N}\). Thus
\[
(n + 1)a_n + \sum_{m=0}^{\infty} [(m + 1)(n + 1)a_n + Bn_n](-A)^m z^m
\]
represents a series of positive Taylor’s coefficients. Since \(a_n > 0\) for all \(n \in \mathbb{N}\), the function \(v_\lambda(A, B, z)\) has a series representation with positive Taylor coefficients. Hence,
\[
|s_n(v_\lambda(A, B, z), z)| \leq s_n(v_\lambda(A, B, z), |z|)
\]
holds and consequently
\[
|h'(z)| \leq h'(|z|), \quad \text{for all } z \in \mathbb{D}\text{ holds.} \tag{12}
\]
Since \(h(0) = 0\) and \(h(-B) = 1\), using (12) we obtain
\[
|h(z)| = \left| \int_0^{z} h'(t)dt \right| \leq \int_0^{B} h'(t) \left( -\frac{t^2}{B} \right) dt \leq \int_0^{B} h'(t) dt = 1, \quad z \in \mathbb{D}.
\]
Therefore,
\[
\left| \frac{(1 + Bz)(s_n(v, A, B, z), z)}{(1 + Az)^{1/\lambda} - 1} \right| < 1, \quad z \in \mathbb{D}.
\]
which implies that
\[
\frac{s_n(v_\lambda(A, B, z), z)}{v_\lambda(A, B, z)} < \frac{1}{v_\lambda(0, B, z)}.
\]
Therefore, \( v_\lambda(A, B, z) \) is \( n \)-stable with respect to \( v_\lambda(0, B, z) \) for all \( n \in \mathbb{N} \). Hence \( v_\lambda(A, B, z) \) is stable with respect to \( v_\lambda(0, B, z) \) for all \( 0 < \lambda \leq 1 \) and \( -1 \leq B < A \leq 0 \).

For the proof of Corollary 2 we need the following proposition which follows the same procedure as given in [5].

**Proposition 1.** Let \( \alpha, \beta > 0 \) and \( B \in [-1, 0) \). If \( F \prec (1 + Bz)^\alpha \) and \( G \prec (1 + Bz)^\beta \) then \( FG \prec (1 + Bz)^{\alpha + \beta} \) for \( z \in \mathbb{D} \).

**Proof.** The function \( \log(1 + Bz) \) is convex univalent for \( z \in \mathbb{D} \) and \( B \in [-1, 0) \). Our claim follows from

\[
\frac{1}{\alpha + \beta} \log(F(z)G(z)) = \frac{\alpha}{\alpha + \beta} \log(1 + Bu(z)) + \frac{\beta}{\alpha + \beta} \log(1 + Bv(z)) \prec \log(1 + Bz),
\]

where \( u, v \) are analytic functions such that \(|u(z)| \leq |z| \) and \(|v(z)| \leq |z| \) for \( z \in \mathbb{D} \).

Now we are ready to give proof of Corollary 2.

**Proof of Corollary 2.** For \( 0 < \mu \leq \lambda \leq 1 \) we have,

\[
\frac{1}{\lambda} \log \left( (1 + Bz)^\lambda s_n(v_\mu(0, B, z), z) \right)
= \frac{1}{\lambda} \log \left( (1 + Bz)^{\lambda - \mu} (1 + Bz)^\mu s_n(v_\mu(0, B, z), z) \right)
= \frac{1}{\lambda} \log(1 + Bz)^{\lambda - \mu} + \frac{1}{\lambda} \log \left( (1 + Bz)^\mu s_n(v_\mu(0, B, z), z) \right)
= \frac{1}{\lambda} \log(1 + Bu(z))^{\lambda - \mu} + \frac{1}{\lambda} \log(1 + Bw(z))^\mu
\]

\( \prec (1 + Bz)^\lambda \)

for \(|u(z)| \leq |z| \) and \(|w(z)| \leq |z| \). Therefore, \((1 + Bz)^\lambda s_n(v_\mu(0, B, z), z) \prec (1 + Bz)^\lambda \) holds for all \( z \in \mathbb{D} \) and \( 0 < \mu \leq \lambda \leq 1 \).

Now we prove that \( v_\lambda(A, B, z) \) is not stable with respect to itself for \( \lambda \in (0, 1] \) and \(-1 \leq B < A \leq 0 \).

**Proof of Theorem 3.** For \(-1 \leq B < A \leq 0 \), to prove that \( v_\lambda(A, B, z) \) is stable with respect to itself, we need to show that

\[
\frac{s_n(v_\lambda(A, B, z), z)}{v_\lambda(A, B, z)} \prec \frac{1}{v_\lambda(A, B, z)}, \quad z \in \mathbb{D}.
\]

Equivalently \( G(z) \prec H(z) \) where

\[
G(z) := \frac{(1 + Bz)s_n(v_\lambda(A, B, z), z)}{1 + Az} \quad \text{and} \quad H(z) := \frac{1 + Bz}{1 + Az}.
\]

Since \( G(z) \) and \( H(z) \) are analytic in \( \mathbb{D} \) for \(-1 \leq B < A \leq 0 \) and \( H(z) \) is univalent in \( \mathbb{D} \). In the point of view of the subordination, we have \( G(z) \prec H(z) \) if and only if \( G(0) = H(0) \) and \( G(\mathbb{D}) \subseteq H(\mathbb{D}) \) and \( G = H \circ \omega \), where \( \omega \in \Omega \) analytic in \( \mathbb{D} \) satisfying \( \omega(0) = 0 \) and \( |\omega'(0)| < 1 \).

In view of the Schwartz Lemma, we have \(|\omega'(z)| \leq |z| \) for \( z \in \mathbb{D} \) and \(|\omega'(0)| \leq 1 \). If \( G \prec H \), it follows that \(|G'(0)| \leq |H'(0)| \) and \( G(|z| \leq r) \subseteq H(|z| \leq r), 0 \leq r \leq 1 \).

Let \( \omega = H(z) = \frac{1 + Bz}{1 + Az} \), then \( z = \frac{1 + B\omega}{1 + A\omega} \). Therefore, the image of \(|z| \leq r \) under \( H(z) \) is \(|\frac{1 + B\omega}{1 + A\omega}| \leq r \) which after simplification is equivalent to \(|w - C(r, A, B)| \leq R(r, A, B)| \) where

\[
C(r, A, B) := \frac{r^2A - B}{B^2 - r^2A^2} \quad \text{and} \quad R(r, A, B) := \frac{r(A - B)}{B^2 - r^2A^2}.
\]
To show that $G \not\prec H$, it is enough to show that $G(|z| \leq r) \not\subset H(|z| \leq r)$. To prove that $G(|z| \leq r) \not\subset H(|z| \leq r)$, it is enough to choose a point $z_0$ with $|z_0| \leq r_0$ such that $G(z_0)$ does not lie in the disk $|\omega - C(r, A, B)| \leq R(r, A, B)$ for some $-1 \leq B < A \leq 0$.

Choose $z_0 = 0.915282 - 0.357037i$, $A = -0.679$, $B = -0.97$, and $\lambda = 0.3$. Then $G(z_0) = 0.8697 + 0.5845i$, $C(r_0, A, B) = 0.634444$ and $R(r_0, A, B) = 0.576521$. Clearly $G(z_0)$ does not lie in the disk $|\omega - C(r_0, A, B)| \leq R(r_0, A, B)$. Therefore $G \not\prec H$ i.e., $\lambda \not\in [1/2, 1)$.

The graphical illustration of these values is also given here in Figure 1. Hence $v_\lambda(A, B, z)$ is not stable with respect to itself.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{G(z_0, A, B) \not\subset H(z_0, A, B) for z_0 = 0.915282 - 0.357037i, A = -0.679, B = -0.97, and \lambda = 0.3.}
\end{figure}

Choose $z_0 = 0.915282 - 0.357037i$, $r_0 = 0.98$, $A = -0.679$, $B = -0.97$, $\lambda = 0.3$ and $n = 1$. Then $G(z_0) = 0.8697 + 0.5845i$, $C(r_0, A, B) = 0.634444$ and $R(r_0, A, B) = 0.576521$. Clearly $G(z_0)$ does not lie in the disk $|\omega - C(r_0, A, B)| \leq R(r_0, A, B)$. Therefore $G \not\prec H$ i.e., $\lambda \not\in [1/2, 1)$ does not hold. The graphical illustration of these values is also given here in Figure 1. Hence $v_\lambda(A, B, z)$ is not stable with respect to itself.

\begin{thebibliography}{9}
1. S. Chakraborty, A. Vasudevarao, \textit{On stable Functions}, Comput. Methods Funct. Theory \textbf{18} (2018) 677-688.
2. P.L. Duren, \textit{Univalent Functions}, Springer–Verlag, Berlin, 1983.
3. W. Janowski, Some extremal problems for certain families of analytic functions. I, Ann. Polon. Math. \textbf{28} (1973), 297–326.
4. S. R. Mondal and A. Swaminathan, \textit{Stable functions and extension of Vietoris’ theorem}, Results Math. \textbf{62} (2012), no. 1-2, 33–51.
5. S. Ruscheweyh, \textit{Convolutions in geometric function theory}, Séminaire de Mathématiques Supérieures, 83, Presses Univ. Montréal, Montreal, QC, 1982.
6. S. Ruscheweyh and L. Salinas, \textit{On starlike functions of order $\lambda \in [1/2, 1)$}, Ann. Univ. Mariae Curie-Skłodowska Sect. A \textbf{54} (2000), 117–123.
7. S. Ruscheweyh and L. Salinas, \textit{Stable functions and Vietoris’ theorem}, J. Math. Anal. Appl. \textbf{291} (2004), no. 2, 596–604.
8. S. Koumandos and S. Ruscheweyh, \textit{On a conjecture for trigonometric sums and starlike functions}, J. Approx. Theory \textbf{149} (2007), no. 1, 42–58.
9. P. Sangal and A. Swaminathan, \textit{On generalized Cesàro Stable functions}, Mathematical Inequalities and Applications, \textbf{22} (1), (2019). 227247.
\end{thebibliography}
10. Wolfram Research, Inc. Mathematica. Version 10.0
11. L. Vietoris, Über das Vorzeichen gewisser trigonometrischer Summen, Sitzungsber, Oest. Akad. Wiss. 167, 1958, 125–135.

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