On the entropy of spanning trees on a large triangular lattice

M. L. Glasser
Department of Physics, Clarkson University
Potsdam, New York 13699

and

F. Y. Wu
Department of Physics, Northeastern University
Boston, Massachusetts 02115

October 24, 2018

Abstract

The double integral representing the entropy $S_{\text{tri}}$ of spanning trees on a large triangular lattice is evaluated using two different methods, one algebraic and one graphical. Both methods lead to the same result

$$S_{\text{tri}} = (4\pi^2)^{-1} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln |6 - 2\cos\theta - 2\cos\phi - 2\cos(\theta + \phi)|$$

$$= (3\sqrt{3}/\pi)(1 - 5^{-2} + 7^{-2} - 11^{-2} + 13^{-2} - \cdots).$$
1 Introduction

It is well-known that spanning trees on an $n$-vertex graph $G$ can be enumerated by computing the eigenvalues of the Laplacian matrix $Q$ associated with $G$ as

$$N_{ST}(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i ,$$  \hspace{1cm} (1)

where $\lambda_i$, $i = 1, 2, \ldots, n - 1$, are the $n - 1$ nonzero eigenvalues of $Q$. In physics one often deals with lattices. For large lattices the number of spanning trees $N_{ST}$ grows exponentially in $n$. This permits one to define the entropy of spanning trees for a class of lattices $G$ as the limiting value

$$S_G = \lim_{n \to \infty} \frac{1}{n} \ln N_{ST}(G).$$  \hspace{1cm} (2)

The resulting finite number $S_G$ is a quantity of physical interest.

For regular lattices in $d$ dimensions the eigenvalues $\lambda_i$ can be computed by standard means leading to expressions of $S_G$ in terms of $d$-dimensional definite integrals. The resulting integrals, which are independent of boundary conditions, have been given for a number of regular lattices, but very few of the definite integrals have been evaluated in closed forms.

To be sure, the entropy for the square lattice has been known and computed. It is found to be

$$S_{sq} = (4\pi^2)^{-1} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln(4 - 2 \cos \theta - 2 \cos \phi)$$

$$= \frac{4}{\pi} Cl_2\left(\frac{\pi}{2}\right)$$

$$= \frac{4}{\pi} \left[1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots\right]$$

$$= \frac{4}{\pi} G$$

$$= 1.166 243 616 \ldots \hspace{1cm} (3)$$

where

$$Cl_2(\theta) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2} \hspace{1cm} (4)$$

is the Clausen’s function and $G$ is the Catalan constant. The algebra reducing the double integral in (3) into the series given in the third line is straightforward and can be found in [5, 6].

2
The only other published closed-form numerical result on the entropy of spanning trees is that of the triangular lattice \cite{7} for which one has

\[
S_{\text{tri}} = (4 \pi^2)^{-1} \int_0^{2 \pi} d\theta \int_0^{2 \pi} d\phi \ln [6 - 2 \cos \theta - 2 \cos \phi - 2 \cos (\theta + \phi)]
\]

\[
= \frac{5}{\pi} \text{Cl}_2 \left( \frac{\pi}{3} \right)
\]

\[
= \frac{3 \sqrt{3}}{\pi} \left[ 1 - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{13^2} - \cdots \right]
\]

\[
= 1.615 \ 329 \ 736 \ 097\ldots \quad (5)
\]

However, no details leading from the double integral in (5) to the final series expression have been published. The expression in the second line is new.

The purpose of this note is to provide some details of the steps leading to the second and third lines in (5) which, as we shall see, are nontrivial. We evaluate the integral in (5) using two different methods: a direct evaluation and a graphical approach. We first evaluate the integral algebraically, and then in an alternate approach we use graphical considerations to convert the spanning tree problem to a Potts model which is in turn related to an $F$ model solved by Baxter \cite{8}. In either case, the same series expression in (5) is deduced.\footnote{The numerical values of $S_{\text{tri}}$ reported in \cite{7} and \cite{8} contain typos in the last two digits.}

\section{Algebraic approach

In this section we evaluate the integral (5) directly.

Using the integration formula

\[
\frac{1}{2 \pi} \int_0^{2 \pi} d\phi \ln [2A + 2B \cos \phi + 2C \sin \phi] = \ln [A + \sqrt{A^2 - B^2 - C^2}],
\]

one of the two integrations in (5) can be carried out, giving

\[
S_{\text{tri}} = \frac{1}{\pi} \int_0^\pi d\theta \ln \left[ 3 - \cos \theta + \sqrt{(7 - \cos \theta)(1 - \cos \theta)} \right]. \quad (6)
\]

Next, we use the identity

\[
3 - \cos \theta + \sqrt{(7 - \cos \theta)(1 - \cos \theta)} = \frac{2}{3} \left( v + \sqrt{3 + v^2} \right) \left( 2v + \sqrt{3 + v^2} \right) \quad (7)
\]
where \( v = \sin(\theta/2) \). Then (6) becomes

\[
S_{\text{tri}} = \ln \left( \frac{2}{3} \right) + I_1 + I_2,
\]

where

\[
I_m = \frac{2}{\pi} \int_0^{\pi/2} d\theta \ln \left( m \sin \theta + \sqrt{3 + \sin^2 \theta} \right), \quad m = 1, 2
\]

(9) after a change of variable \( \theta/2 \to \theta \).

To evaluate \( I_1 \), we consider the more general integral

\[
f(a) = \frac{2}{\pi} \int_0^{\pi/2} d\theta \ln \left( \sin \theta + \sqrt{a^2 + \sin^2 \theta} \right)
\]

such that \( f(\sqrt{3}) = I_1 \). Using

\[
2 \frac{\pi}{\pi} \int_0^\pi d\theta \ln \sin \theta = 2 \frac{\pi}{\pi} \int_0^\pi d\theta \ln \cos \theta = -\ln 2,
\]

we have

\[
f(0) = \frac{2}{\pi} \int_0^\pi d\theta \ln(2 \sin \theta) d\theta = 0,
\]

\[
f'(a) = \frac{2a}{\pi} \int_0^{\pi/2} \frac{d\theta}{(\sin \theta + \sqrt{a^2 + \sin^2 \theta})\sqrt{a^2 + \sin^2 \theta}}
\]

\[
= \frac{2}{\pi a} \tan^{-1} a
\]

The reduction of \( f'(a) \) is effected by noting that \((\sqrt{\sin^2 \theta + a^2 + \sin \theta})^{-1} = (\sqrt{\sin^2 \theta + a^2 - \sin \theta})/a^2\). Hence, \( f(a) = (2/\pi)\text{Ti}_2(a) \) and

\[
I_1 = \frac{2}{\pi} \text{Ti}_2(\sqrt{3})
\]

(11)

where

\[
\text{Ti}_2(a) = \int_0^a \frac{\tan^{-1} t}{t} dt
\]

\[
= a - \frac{a^3}{3^2} + \frac{a^5}{5^2} - \frac{a^7}{7^2} + \cdots
\]

(12)

is the inverse tangent integral function described in [3].
To evaluate $I_2$, we introduce the following identity

$$2 \sin \theta + \sqrt{3 + \sin^2 \theta} = \sqrt{3} \cos \theta \sqrt{\frac{1 + u(\theta)}{1 - u(\theta)}}, \quad 0 \leq \theta \leq \pi/2,$$

where

$$u(\theta) = \frac{2 \sin \theta}{\sqrt{3 + \sin^2 \theta}}.$$  \hspace{1cm} (13)

Substituting (13) into $I_2$ in (9) and making use of (10), we obtain

$$I_2 = \frac{1}{2} \ln 3 - \ln 2 + \frac{1}{\pi} \int_0^{\pi/2} \ln \left[ \frac{1 + u(\theta)}{1 - u(\theta)} \right] d\theta$$

$$= \frac{1}{2} \ln 3 - \ln 2 + \frac{\sqrt{3}}{2\pi} \int_0^1 \frac{1}{(1 - u^2/4) \sqrt{1 - u^2}} \ln \left( \frac{1 + u}{1 - u} \right) du \hspace{1cm} (14)$$

after an obvious change of integration variable in the last step.

Next we express the factor in the integrand as

$$\frac{1}{1 - u^2/4} = \frac{1}{2} \left[ \frac{1}{1 - u/2} + \frac{1}{1 + u/2} \right]$$

and set

$$x = \sqrt{(1 - u)/(1 + u)}.$$  \hspace{1cm} 

After a little reduction we obtain

$$I_2 = \frac{1}{2} \ln 3 - \ln 2 - \frac{2\sqrt{3}}{\pi} \int_0^1 \frac{\ln x}{3x^2 + 1} \, dx - \frac{2\sqrt{3}}{\pi} \int_0^1 \frac{\ln x}{x^2 + 3} \, dx.$$  \hspace{1cm} 

From integration by parts one has

$$\int_0^1 \frac{\ln x}{x^2 + a^2} \, dx = -\left( \frac{1}{a} \right) \int_0^1 \frac{dx}{x} \tan^{-1} \left( \frac{x}{a} \right)$$

$$= -\left( \frac{1}{a} \right) T_2 \left( \frac{1}{a} \right). \hspace{1cm} (15)$$

This yields

$$I_2 = \frac{1}{2} \ln 3 - \ln 2 + \frac{2}{\pi} \left[ T_2(\sqrt{3}) + T_2 \left( \frac{1}{\sqrt{3}} \right) \right]. \hspace{1cm} (16)$$
To express $I_1$ and $I_2$ in terms of the Clausen’s series \textbf{(4)}, we have from Eqs. (2.6), (4.31) and (4.18) of \textbf{[9]} the identities

\begin{equation}
T_i \left( y^{-1} \right) = T_i \left( y \right) - \frac{\pi}{2} \ln y, \quad y > 0
\end{equation}

\begin{equation}
T_i \left( \tan \theta \right) = \theta \ln \left( \tan \theta \right) + \frac{1}{2} \left[ \text{Cl}_2 (2\theta) + \text{Cl}_2 (\pi - 2\theta) \right], \quad \theta > 0
\end{equation}

\begin{equation}
\text{Cl}_2 \left( \frac{2\pi}{3} \right) = \frac{2}{3} \text{Cl}_2 \left( \frac{\pi}{3} \right).
\end{equation}

Setting $y = \sqrt{3}$ in \textbf{(17)} and $\theta = \pi/3$ in \textbf{(18)}, we obtain after making use of \textbf{(19)}

\begin{align*}
T_i \left( \sqrt{3} \right) &= \text{Cl}_2 \left( \frac{\pi}{3} \right) + \frac{\pi}{6} \ln 3 \\
T_i \left( \frac{1}{\sqrt{3}} \right) &= \text{Cl}_2 \left( \frac{\pi}{3} \right) - \frac{\pi}{12} \ln 3.
\end{align*}

(20)

Thus, combining \textbf{(11)} and \textbf{(16)} with \textbf{(8)}, we obtain

\begin{equation}
S_{\text{tri}} = \frac{5}{\pi} \text{Cl}_2 \left( \frac{\pi}{3} \right)
\end{equation}

as given in the second line in \textbf{(5)}.

To express $S_{\text{tri}}$ in the form of the series given in \textbf{(5)}, we note that

\begin{equation}
\text{Cl}_2 \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2} \ S
\end{equation}

where

\begin{align*}
S &= \sum_{m=0}^{\infty} \left[ \frac{1}{(6m+1)^2} + \frac{1}{(6m+2)^2} - \frac{1}{(6m+4)^2} - \frac{1}{(6m+5)^2} \right] \\
&= S_1 + \frac{1}{2^2} S_2
\end{align*}

with

\begin{align*}
S_1 &= \sum_{m=0}^{\infty} \left[ \frac{1}{(6m+1)^2} - \frac{1}{(6m+5)^2} \right], \\
S_2 &= \sum_{m=0}^{\infty} \left[ \frac{1}{(3m+1)^2} - \frac{1}{(3m+2)^2} \right].
\end{align*}
Separating terms in $S_2$ with odd and even denominators, we have

$$S_2 = S_1 - \frac{1}{2^2} S_2 .$$

It follows that $S_2 = (4/5)S_1$ hence $S = (6/5)S_1$, and from (21) and (22) we obtain

$$S_{\text{tri}} = \frac{3\sqrt{3}}{\pi} S_1$$

$$= \frac{3\sqrt{3}}{\pi} \left[ 1 - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{13^2} - \cdots \right].$$

This is the result given in (5).

### 3 Graphical approach

In this section we evaluate the integral (5) with the help of a graphical mapping.

The medial graph (or the surrounding lattice) $\mathcal{L}'$ of a triangular lattice $\mathcal{L}$ is the kagome lattice shown in Fig. 1, where we have shaded its triangular faces for convenience. We start from a well-known equivalence between the $q$-state Potts model on $\mathcal{L}$ and an ice-rule vertex model on $\mathcal{L}'$ [10] - [12]. Denote the respective partition functions by $Z_{\text{Potts}}$ and $Z_{\text{kag-ice}}$. Then the equivalence reads

$$Z_{\text{Potts}}(q, \sqrt{q} x) = q^{n/2} Z_{\text{kag-ice}}(1, 1, x, x, t^{-1} + xt^2, t + xt^{-2}) \quad (23)$$

where

$$x = (e^K - 1)/\sqrt{q} ,$$

$$t^3 + t^{-3} = \sqrt{q} , \quad (24)$$

$K$ being the nearest-neighbor interaction of the Potts model, and the arguments of $Z_{\text{kag-ice}}$ are weights\(^2\) of the ice-rule model as depicted in Fig. 2.

\(^2\)In comparing vertex weights with those in [12], it should be noted that for the triangular lattice which we are considering, roles of shaded and unshaded faces are reversed from those indicated in [12].
On the other hand, it is also known that by taking an appropriate \( q \to 0 \) limit the Potts partition function generates spanning trees [7, 13]. Write the spanning tree generating function in a slightly more general form by attaching a weight \( x \) to each line segment of the tree. Since the number of lines in a spanning tree on an \( n \)-vertex graph is fixed at \( n - 1 \), this merely adds an overall factor \( x^{n-1} \). Then, the equivalence reads \[ x^{n-1} N_{ST} = \lim_{q \to 0} q^{-(n+1)/2} Z_{\text{Potts}}(q, \sqrt{q} x). \] (25)

The equivalences (23) and (25) hold quite generally for finite lattices, provided that appropriate vertex weights are defined for the boundary [11].

We next combine (23) and (25) and take the thermodynamic limit \( n \to \infty \). Assuming the two limits on \( n \) and \( q \) commute, this leads to

\[
S_{\text{tri}} = -\ln x + z_{\text{kag-ice}}(1, 1, x, x, t^{-1} + xt^2, t + xt^{-2}), \quad t = e^{i\pi/6}
\] (26)

where

\[
z_{\text{kag-ice}} = \lim_{n \to \infty} n^{-1} \ln Z_{\text{kag-ice}}.
\]

We note in passing that the vertex weights in (26) now satisfy the free-fermion condition [14]

\[
1 \cdot 1 + x \cdot x = (t^{-1} + xt^2) \cdot (t + xt^{-2})
\]

so \( z_{\text{kag-ice}} \) can be evaluated by standard means such as using Pfaffians [7]. This verifies the integral expression of \( S_{\text{tri}} \) given in (5).

Alternately, using a mapping introduced by Lin [15] (see also [16]), the kagome ice-rule model can be mapped into an ice-rule model on the triangular lattice. The trick is to ‘shrink’ each shaded face of the kagome lattice into a point as shown in Fig. 1. This results in a 20-vertex ice-rule model on the triangular lattice which has 3 arrows in and 3 arrows out at every vertex.

There are 4 different vertex types in the 20-vertex model: (i) six vertices in which the 3 incoming arrows are adjacent, (ii) two vertices in which the incoming and outgoing arrows alternate around the vertex, (iii) six vertices in which 2 incoming arrows are adjacent while the third in-arrow points in a direction opposite to one of the 2 adjacent incoming arrows, and (iv) six vertices in which 2 incoming arrows are adjacent while the third in-arrow
points opposite to the other adjacent incoming arrow. These 4 configurations are shown in the left-side column in Fig. 3 to which we assign respective energies $\epsilon_i, \ i = 1, 2, 3, 4$. The other 16 ice-rule configurations are obtained by rotations.

The shrinking processes from which these ice-rule configurations are deduced are also shown in Fig. 3. Write the 4 weights as $a, b, c, c'$, respectively. We read off from Fig. 3 to obtain

$$
a = e^{-\epsilon_1/kT} = 1 \cdot 1 \cdot x
b = e^{-\epsilon_2/kT} = (t + xt^{-2})^3 + x^3
c = e^{-\epsilon_3/kT} = x(t + xt^{-2})^2 + x^2(t^{-1} + xt^2)
c' = e^{-\epsilon_4/kT} = x^2(t + xt^{-2}) + x(t^{-1} + xt^2)^2.
$$

(27)

Thus, we have established the equivalence

$$z_{\kag-ice}(1, 1, 1, x, x, t^{-1} + xt^2, t + xt^{-2}) = z_{\tri-ice}(a, b, c, c'),
$$

(28)

where $z_{\tri-ice}$ is defined similar to $z_{\kag-ice}$.

Baxter [8] has considered the evaluation of $z_{\tri-ice}$ in the subspace of $c = c'$. Now the right-hand side of (28) is independent of the value of $x$ so we have the freedom to choose a value of $x$ for which $z_{\kag-ice}$ can be evaluated. It is readily verified that by taking $x = 1/\sqrt{3}$ and $t = e^{i\pi/6}$ we have

$$a = 1/\sqrt{3}, \ b = 3a, \ c = c' = 2a$$

which is a case solved by Baxter. In this case Baxter obtained

$$z_{\tri-ice}(a, 3a, 2a, 2a) = \ln a + P \int_{-\infty}^{\infty} \frac{dx}{x} \frac{1 + e^{-x}}{(e^x - 1 - e^{-x})(1 + e^{-2x} + e^{-4x})}
= \ln a + \frac{3\sqrt{3}}{\pi} \left[ 1 - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{13^2} - \cdots \right],
$$

(29)

where $P$ denotes the principal value. Combining (26) and (28) with (29) and $a = 1/\sqrt{3}$, we obtain

$$S_{\tri} = \frac{3\sqrt{3}}{\pi} \left[ 1 - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{13^2} - \cdots \right].$$

This is the result given in [5] as first reported in [7].

9
Acknowledgment

Work has been supported in part by NSF Grants DMR-0121146 (MLG) and DMR-9980440 (FYW). We thank W. T. Lu for assistance in preparing the graphs.
References

[1] N. L. Biggs, *Algebraic Graph Theory*, Cambridge University Press (second edition), Cambridge, 1993.

[2] W. J. Tzeng and F. Y. Wu, “Spanning trees on hypercubic lattices and nonorientable surfaces,” *Lett. Appl. Math.* 13 (2000), 19-25.

[3] R. Shrock and F. Y. Wu, Spanning trees on graphs and lattices in $d$ dimensions, *J. Phys. A: Math. Gen.* 33 (2000), 3881-3902.

[4] H. N. V. Temperley, in *Combinatorics: Proc. Oxford Conference on Combinatorial Mathematics* (D. J. A. Welsh and D. R. Woodall, eds.), Institute of Mathematics and Its Applications, Oxford, 1972, pp. 356-357.

[5] P. W. Kasteleyn, “The statistics of dimers on a lattice,” *Physica* 27 (1961), 1209-1225.

[6] M. E. Fisher, “Statistical mechanics of dimers on a plane lattice,” *Phys. Rev.* 124 (1961), 1664-1672.

[7] F. Y. Wu, “Number of spanning trees on a lattice,” *J. Phys. A: Math. Gen.* 10 (1977), L113-115.

[8] R. J. Baxter, “$F$ model on a triangular lattice,” *J. Math. Phys.* 10 (1969), 1211-1216.

[9] L. Lewin, *Dilogarithms and associated functions*, Macdonald, London, 1958.

[10] H. N. V. Temperley and E. H. Lieb, “Relations between the ‘Percolation’ and ‘Colouring’ problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the ‘Percolation’ problem,” *Proc. Roy. Soc. London* A. 322 (1971), 251-280.

[11] R. J.Baxter, S. B. Kelland, and F. Y. Wu, “Equivalence of the Potts model or Whitney polynomial with an ice-type model,” *J. Phys. A: Math. Gen.* 9 (1976), 397-406.

[12] F. Y. Wu, “The Potts model,” *Rev. Mod. Phys.* 54 (1982), 235-268.
[13] C. M. Fortuin and P. W. Kasteleyn, “On the random-cluster model, I. Introduction and relation to other models,” Physica 57 (1972), 536-406.

[14] C. Fan and F. Y. Wu, “General Lattice Model of Phase Transitions,” Phys. Rev. B 2 (1970), 723-733.

[15] C. K. Lin, “Equivalence of the 8-vertex model on a kagome lattice with the 32-vertex model on a triangular lattice,” J. Phys. A: Math. Gen. 9 (1976), L183-186.

[16] R. J. Baxter, H. N. V. Temperley, and S. E. Ashley, “Triangular Potts model at its transition temperature, and related models,” Proc. Roy. Soc. London A. 358 (1978), 535-559.
Figure captions

Fig. 1. The shrinking of a kagome lattice into a triangular lattice. Each shaded triangular face is shrunk into a point.

Fig. 2. The six ice-rule configurations of the kagome lattice and the associated vertex weights.

Fig. 3. The four types of ice-rule configurations of the triangular lattice and the associated shrinking processes.
Fig. 2
\[ \varepsilon_1 = \begin{array}{c}
\end{array} \]

\[ \varepsilon_2 = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \]

\[ \varepsilon_3 = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \]

\[ \varepsilon_4 = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \]

Fig. 3