A Note on Some Recent Results for the Bernoulli Numbers of the Second Kind

Iaroslav V. Blagouchine

University of Toulon, France
&
Steklov Institute of Mathematics at St.-Petersburg, Russia.

Abstract

In a recent issue of the Bulletin of the Korean Mathematical Society, Qi and Zhang discovered an interesting integral representation for the Bernoulli numbers of the second kind (also known as Gregory’s coefficients, Cauchy numbers of the first kind, and the reciprocal logarithmic numbers). The same representation also appears in many other sources, either with no references to its author, or with references to various modern researchers. In this short note, we show that this representation is a rediscovery of an old result obtained in the 19th century by Ernst Schröder. We also demonstrate that the same integral representation may be readily derived by means of complex integration. Moreover, we discovered that the asymptotics of these numbers were also the subject of several rediscoveries, including very recent ones. In particular, the first-order asymptotics, which are usually (and erroneously) credited to Johan F. Steffensen, actually date back to the mid-19th century, and probably were known even earlier.

Keywords: Bernoulli number of the second kind, Gregory coefficient, Cauchy number, logarithmic number, Schröder, rediscovery, state of art, complex analysis, theory of complex variable, contour integration, residue theorem.

I. Rediscovery of Schröder’s integral formula

In a recent article in the Bulletin of the Korean Mathematical Society [10], several results concerning the Bernoulli numbers of the second kind were presented.

We recall that these numbers (OEIS A002206 and A002207), which we denote below by \( G_n \), are

---

Email address: iaroslav.blagouchine@univ-tln.fr, iaroslav.blagouchine@centrale-marseille.fr, iaroslav.blagouchine@pdmi.ras.ru. (Iaroslav V. Blagouchine)

---

Note to the readers of the 3rd arXiv version: this version is a copy of the journal version of the article, which has been published in the Journal of Integer Sequences, vol. 20, no. 3, Article 17.3.8, pp. 1-7, 2017. URL: https://cs.uwaterloo.ca/journals/JIS/VOL20/Blagouchine/blag5.html

Article history: submitted 20 December 2016, accepted 26 January 2017, published 27 January 2017. The layout of the present version and its page numbering differ from the journal version, but the content, the numbering of equations and the numbering of references are the same. For any further reference to the material published here, please, use the journal version of the paper, which you can always get for free from the journal site (Journal of Integer Sequences is an open access journal).
Schröder’s\(^{-1}\) are exactly our \(G_n\). and were introduced by the Scottish mathematician and astronomer James Gregory in 1670 in the context of area’s interpolation formula. Subsequently, they were rediscovered by many famous mathematicians, including Gregorio Fontana, Lorenzo Mascheroni, Pierre-Simon Laplace, Augustin-Louis Cauchy, Jacques Binet, Ernst Schröder, Oskar Schlömilch, Charles Hermite and many others. Because of numerous rediscoveries these numbers do not have a standard name, and in the literature they are also referred to as Gregory’s coefficients, (reciprocal) logarithmic numbers, Bernoulli numbers of the second kind, normalized generalized Bernoulli numbers \(B_n^{(n-1)}\) and normalized Cauchy numbers of the first kind \(C_{1,n}\). Usually, these numbers are defined either via their generating function

\[
\frac{u}{\ln(1+u)} = 1 + \sum_{n=1}^{\infty} G_n u^n, \quad |u| < 1, \tag{1}
\]

or explicitly

\[
G_n = \frac{C_{1,n}}{n!} = \lim_{s \to n} \frac{-B_n^{(s-1)}}{(s-1)!} = \frac{1}{n!} \int_0^1 x(x-1)(x-2) \cdots (x-n+1) \, dx, \quad n \in \mathbb{N}.
\]

It is well known that \(G_n\) are alternating \(G_n = (-1)^{n-1}|G_n|\) and decreasing in absolute value; they behave as \((n \ln^2 n)^{-1}\) at \(n \to \infty\) and may be bounded from below and from above accordingly to formulas (55)–(56) from [3]. For more information about these important numbers, see [3, pp. 410–415], [2, p. 379], and the literature given therein (nearly 50 references).

Now, the first main result of [10, p. 987] is Theorem 1.\(^2\) It states: the Bernoulli numbers of the second

\[^2\text{Our } G_n \text{ are exactly } b_n \text{ from [10] and } c_n^{(1)} \text{ from [4, Sect. 5]. Despite a venerable history, these numbers still lack a standard notation and various authors may use different notation for them.}\]
kind may be represented as follows

\[ G_n = (-1)^{n+1} \int_1^\infty \frac{dt}{(\ln^2(t-1) + \pi^2)^n}, \quad n \in \mathbb{N}. \]  

(2)

The same representation appears in a slightly different form\(^3\)

\[ G_n = (-1)^{n+1} \int_0^\infty \frac{du}{(\ln^2 u + \pi^2)(u+1)^n}, \quad n \in \mathbb{N}, \]  

(3)

in [5, pp. 473–474] and [4, Sect. 5], and is called Knessl’s representation and the Qi integral representation respectively. Furthermore, various internet sources provide the same (or equivalent) formula, either with no references to its author or with references to different modern writers and/or their papers. However, the integral representation in question is not novel and is not due to Knessl nor to Qi and Zhang; in fact, this representation is a rediscovery of an old result. In a little-known paper of the German mathematician Ernst Schröder [11], written in 1879, one may easily find exactly the same integral representation on p. 112; see Fig. 1. Moreover, since this result is not difficult to obtain, it is possible that the same integral representation was obtained even earlier.

II. Simple derivation of Schröder’s integral formula by means of the complex integration

Schröder’s integral formula [11, p. 112] may, of course, be derived in various ways. Below, we propose a simple derivation of this formula based on the method of contour integration.

If we set \( u = -z - 1 \), then equality (1) may be written as

\[ \frac{z + 1}{\ln z - \pi i} = -1 + \sum_{n=1}^\infty |G_n| (z + 1)^n, \quad |z + 1| < 1. \]

Now considering the following line integral along a contour \( C \) (see Fig. 2), where \( n \in \mathbb{N} \), and then letting \( R \to \infty, r \to 0 \), we have by the residue theorem

\[ \oint_C \frac{dz}{(1+z)^n (\ln z - \pi i)} = \left( \int_r^R \ldots \right) dz + \left( \int_R^r \ldots \right) dz + \left( \int_r^R \ldots \right) dz + \left( \int_R \ldots \right) dz \]

\[ = \int_0^\infty \left\{ \frac{1}{\ln x - \pi i} - \frac{1}{\ln x + \pi i} \right\} \frac{dx}{(1+x)^n} = 2\pi i \int_0^1 \frac{1}{(1+x)^n} \cdot \frac{dx}{\ln^2 x + \pi^2} =
\]

\[ = 2\pi i \text{ res}_{z=-1} \frac{1}{(1+z)^n (\ln z - \pi i)} = 2\pi i \frac{d^n}{dz^n} \frac{z + 1}{\ln z - \pi i} \bigg|_{z=-1} = 2\pi i |G_n|, \]

\(^3\)Put \( t = 1 + u.\)
since
\[
\left| \int_{C_R} \frac{dz}{(1+z)^n (\ln z - \pi i)} \right| = O\left( \frac{1}{R^{n-1} \ln R} \right) = o(1), \quad R \to \infty, \quad n \geq 1,
\]
and because at \( z = -1 \) the integrand of the contour integral has a pole of the \((n+1)\)th order. This completes the proof. Note that above derivations are valid only for \( n \geq 1 \), and so is Schröder’s integral formula, which may also be regarded as one of the generalizations of \( G_n \) to the continuous values of \( n \).

**III. Several remarks on the asymptotics for the Bernoulli numbers of the second kind**

The first-order asymptotics \( |G_n| \sim (n \ln^2 n)^{-1} \) at \( n \to \infty \) are usually credited to Johan F. Steffensen [12, pp. 2–4], [13, pp. 106–107], [9, p. 29], [7, p. 14, Eq. (14)], [8], who found it in 1924. However, in our recent work [3, p. 415] we noted that exactly the same result appears in Schröder’s work written 45 years earlier, see Fig. 3, and the order of the magnitude of the closely related numbers is contained in a work of Jacques Binet dating back to 1839 [1]. In 1957 Davis [7, p. 14, Eq. (14)] improved this first-order approximation slightly by showing that \( |G_n| \sim \Gamma(1 + \zeta) (n \ln^2 n + n \pi^2)^{-1} \) at \( n \to \infty \) for some \( \zeta \in [0, 1] \), without noticing that 7 years earlier S. C. Van Veen had already obtained the complete asymptotics for them [14, p. 336], [9, p. 29]. Equivalent complete asymptotics were re-

---

4The same first-order asymptotics also appear in [6, p. 294], but without the source of the formula.

5By the “closely related numbers” we mean the so-called Cauchy numbers of the second kind (OEIS A002657 and A002790), and numbers \( I'(k) \), see [3, pp. 410–415, 428–429]. The comment going just after Eq. (93) [3, p. 429] is based on the statements from [1, pp. 231, 339]. The Cauchy numbers of the second kind \( C_{2,n} \) and Gregory’s coefficients \( G_n \) are related to each other via the relationship \( nC_{2,n-1} - C_{2,n} = n! |G_n| \), see [3, p. 430].
recently rediscovered in slightly different forms by Charles Knessl [5, p. 473], and later by Gergő Nemes [8]. An alternative demonstration of the same result was also presented by the author [3, p. 414].

IV. Acknowledgments

The author is grateful to Jacqueline Lorfanfant from the University of Strasbourg for sending a scanned version of [11].

References

[1] J. Binet, Mémoire sur les intégrales définies euleriennes et sur leur application à la théorie des suites, ainsi qu’à l’évaluation des fonctions des grands nombres, J. Éc. Roy. Polytech., 16 (27) (1839), 123–343.

[2] Ia. V. Blagouchine, Expansions of generalized Euler’s constants into the series of polynomials in $\pi^{-2}$ and into the formal enveloping series with rational coefficients only, J. Number Theory, 158 (2016), 365–396. Corrigendum: 173 (2017), 631–632.

[3] Ia. V. Blagouchine, Two series expansions for the logarithm of the gamma function involving Stirling numbers and containing only rational coefficients for certain arguments related to $\pi^{-1}$, J. Math. Anal. Appl., 442 (2016), 404–434.

[4] J. Chikhi, Integral representation for some generalized Poly-Cauchy numbers, preprint, 2016, https://hal.archives-ouvertes.fr/hal-01370757v1.

[5] M. W. Coffey, Series representations for the Stieltjes constants, Rocky Mountain J. Math., 44 (2014), 443–477.

[6] L. Comtet, Advanced Combinatorics. The Art of Finite and Infinite Expansions (revised and enlarged edition), D. Reidel Publishing Company, 1974.

[7] H. T. Davis, The approximation of logarithmic numbers, Amer. Math. Monthly, 64 (1957), 11–18.

[8] G. Nemes, An asymptotic expansion for the Bernoulli numbers of the second kind, J. Integer Seq., 14 (2011), Article 11.4.8.

[9] N. E. Nörlund, Sur les valeurs asymptotiques des nombres et des polynômes de Bernoulli, Rend. Circ. Mat. Palermo, 10 (1) (1961), 27–44.
[10] F. Qi and X.-J. Zhang, An integral representation, some inequalities, and complete monotonicity of Bernoulli numbers of the second kind, *Bull. Korean Math. Soc.*, **52** (3) (2015), 987–998.

[11] E. Schröder, Bestimmung des infinitären Werthes des Integrals $\int_0^1 (u)^n du$, *Z. Math. Phys.*, **25** (1880), 106–117.

[12] J. F. Steffensen, On Laplace’s and Gauss’ summation-formulas, *Skandinavisk Aktuarietidskrift (Scand. Actuar. J.)*, **1** (1924), 1–15.

[13] J. F. Steffensen, *Interpolation*, 2nd edition, Chelsea Publishing, 1950.

[14] S. C. Van Veen, Asymptotic expansion of the generalized Bernoulli numbers $B^{(n-1)}_n$ for large values of $n$ ($n$ integer), *Indag. Math.*, **13** (1951), 335–341.