Baxter equations and Deformation of Abelian Differentials.

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Abstract. In this paper the proofs are given of important properties of deformed Abelian differentials introduced earlier in connection with quantum integrable systems. The starting point of the construction is Baxter equation. In particular, we prove Riemann bilinear relation. Duality plays important role in our consideration. Classical limit is considered in details.
1 Introduction.

It is well known that spectra of quantum integrable models are defined by solutions of Baxter equations [1]. To our mind most naturally these equations appear in the method of separation of variables developed by Sklyanin [2]. In the classical case the separation of variables is closely related with algebra-geometrical methods based on spectral curves and their Jacobi varieties. One can think of Baxter equations as of definition of “quantum spectral curves”.

The averages of observables in the quantum case are written in terms of rather peculiar integrals. In this paper we shall consider the case related to $U_q(sl_2)$ in which all these integrals can be expressed in terms of one-fold deformed hyper-elliptic integrals. In classical case this statement corresponds to rather non-trivial property of cohomologies of affine hyper-elliptic Jacobian which was conjectured in [3] and proved in [4]. These deformed hyper-elliptic integrals in are considered in the paper [5], but the details of proof of their properties have never been given. This will be done in the present paper because we believe them to be interesting and instructive.

Let us be more specific in discussion of Baxter equations. If one considers a quantum integrable system with finitely many degrees of freedom related to $U_q(sl_2)$ the generating function of its integrals of motion is given by a polynomial:

$$t(z) = \sum_{k=0}^{g+1} z^k t_k$$

where $z$ is the spectral parameter and the number of degrees of freedom equals $g$ and $t_0 = 1$. If we consider a system of $U_q(sl_2(\mathbb{R}))$ type the spectrum is defined by solutions to Baxter equations:

$$Q(\zeta + i\gamma) + Q(\zeta - i\gamma) = t(z) Q(\zeta)$$

where $\gamma$ is the coupling constant (or Plank constant), $\zeta = \frac{1}{2} \log(z)$, $Q(\zeta)$ is an entire function with certain requirements on position of its zeros and the following asymptotic:

$$Q(\zeta) = e^{-\frac{g+1}{2\pi}((\pi+\gamma)\zeta \pm 2i\zeta^2)}$$

if $\zeta \to \infty$ being not too far in upper (lower) half-planes.

There is a nice argument due to Al. Zamolodchikov [6] which explains the appearance of duality in this situation. Notice that $Q(\zeta + \pi i)$ is also solution to
the same equation (2). The quantum Wronskian of these two solutions must be entire $i\gamma$-periodical function, but due to the asymptotic (3) it equals one:

$$Q\left(\zeta + \frac{\pi + \gamma}{2} i\right) Q\left(\zeta - \frac{\pi + \gamma}{2} i\right) - Q\left(\zeta + \frac{\pi - \gamma}{2} i\right) Q\left(\zeta - \frac{\pi - \gamma}{2} i\right) = 1$$

(4)

It is easy to see that (4) together with the asymptotic (3) actually imply the existence of polynomial $t(z)$ with which the Baxter equation (2) holds. Indeed, take a function $Q(\zeta)$ satisfying (4) and define

$$t(z) = \frac{Q(\zeta + i\gamma) + Q(\zeta - i\gamma)}{Q(\zeta)}$$

from (4) it is easy to see that this is indeed a function of $z$ ($\pi i$-periodical function of $\zeta$) without singularities, and (5) allows to show that $t(z)$ is in fact a polynomial of degree $g + 1$.

But two periods: $i\gamma$ and $i\pi$ enter (4) in completely symmetric way. So, the same kind of reasonings proves existence of another polynomial of degree $g + 1$ which we denote by $T(Z)$ and with which the dual Baxter equation is satisfied:

$$Q(\zeta + i\pi) + Q(\zeta - i\pi) = T(Z) Q(\zeta)$$

(5)

here $Z = \exp\left(\frac{2\pi}{\gamma} \zeta\right)$. The pair of dual equations (2) and (2) and mathematical structures related to the is subject of this paper.

The duality considered in this paper has much in common with the duality for representations of $U_q(sl_2(\mathbb{R}))$ [7]. Actually, the type of real form for integrable models that we have in mind is similar in the classical case to taking $SL_2(\mathbb{R})$ as real form of $SL_2(\mathbb{C})$.

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2 Deformed Abelian differentials.

In this section we give formal definition concerning deformed Abelian differentials. The properties of these differentials are discussed in next sections.
Consider a solution $Q(\zeta)$ to Baxter equation:

$$Q(\zeta + i\gamma) + Q(\zeta - i\gamma) = t(z)Q(\zeta) \quad (6)$$

Asymptotically it behaves as follows:

$$Q(\zeta) = Q_+(\zeta) + Q_-(\zeta),$$
$$Q_{\pm}(\zeta) = e^{\pm(g+1)\frac{\zeta^2}{i\gamma}}(zZ)^{-\frac{g+1}{2}}f_{\pm}(z)F_{\pm}(Z)$$

In this section we shall consider:

$$\varphi_{\pm}(\zeta) = e^{\pm(g+1)\frac{\zeta^2}{i\gamma}}z^{-\frac{g+1}{2}}f_{\pm}(z)$$

which are formal solutions to the equations

$$\varphi_{\pm}(\zeta + i\gamma) + \varphi_{\pm}(\zeta - i\gamma) = (-1)^{g+1}t(z)\varphi_{\pm}(\zeta)$$

These asymptotic solutions satisfy the q-Wronskian relation:

$$\varphi_+(\zeta + i\gamma)\varphi_-(\zeta) - \varphi_-(\zeta + i\gamma)\varphi_+(\zeta) = 1 \quad (7)$$

In this section we shall use the space $w$ of semi-infinite Laurent series of the form

$$a(z) = \sum_{k=-\infty}^{N} a_k z^k$$

where $N$ is arbitrary but finite. Let us give some definitions.

**Definition 1.** For Laurent series $a(z)$ define:

$$\mathcal{R}(a) = \text{res} \left( z^{-1}a(z)\varphi_{\pm}(\zeta)\varphi_{\mp}(z) \right) \quad (8)$$

The expression in RHS is Laurent series in $z$, so, the residue is well defined.

Our next goal is to define a pairing between Laurent series without residues.

**Definition 2.** Consider $a, b \in w$ such that $\mathcal{R}(a) = \mathcal{R}(a) = 0$ then

$$a \circ b = i\gamma \text{res} (a(z)z^{-1}(2\varphi_+(\zeta)\varphi_-(\zeta)\delta_{\gamma}^{-1}(b(z)\varphi_+(\zeta)\varphi_-(\zeta)) - \varphi_+(\zeta)\varphi_+(\zeta)\delta_{\gamma}^{-1}(b(z)\varphi_-(\zeta)\varphi_-(\zeta)) - \varphi_-(\zeta)\varphi_-(\zeta)\delta_{\gamma}^{-1}(b(z)\varphi_+(\zeta)\varphi_+(\zeta)))) \quad (9)$$
Where we introduced the following notation: \( \forall \eta \in \mathbb{R} \) define
\[
\delta_\eta(f(\zeta)) = f(\zeta + i\eta) - f(\zeta)
\]

Definition 2 requires some comments.

First, since \( b(z)\varphi_+(\zeta)\varphi_-(\zeta) \) is Laurent series and \( R(b) = 0 \) the expression \( \delta_{\gamma}^{-1}(b(z)\varphi_+(\zeta)\varphi_-(\zeta)) \) is well-defined as Laurent series up to a constant. The latter ambiguity does not affect the definition since \( R(a) = 0 \).

Second, the meaning of \( \delta_{\gamma}^{-1}(b(z)\varphi_+(\zeta)\varphi_-(\zeta)) \) and \( \delta_{\gamma}^{-1}(b(z)\varphi_+(\zeta)\varphi_+(\zeta)) \) must be clarified. Notice that
\[
b(z)\varphi_\pm(\zeta)\varphi_\pm(\zeta) = e^{\pm 2(g+1)\frac{2}{i\gamma}} v_\pm(z)
\]
where \( v_\pm(z) \) are Laurent series. We define
\[
\delta_{\gamma}^{-1}(b(z)\varphi_\pm(\zeta)\varphi_\pm(\zeta)) = e^{\pm 2(g+1)\frac{2}{i\gamma}} u_\pm(z)
\]
where
\[
(qz)^{2(g+1)}u_+(zq^2) - u_+(z) = v_+(z), \quad (qz)^{-2(g+1)}u_-(zq^2) - u_-(z) = v_-(z)
\]
Obviously the \( u_\pm(z) \) are well-defined as Laurent series.

It is easy to see that the pairing \((\text{9})\) is skew-symmetric:
\[
a \circ b = -b \circ a
\]

Our next definition introduces "exact forms".

**Definition 3.** For any \( a(z) \in \mathfrak{w} \) define
\[
(Da)(z) = \frac{1}{i\gamma} \left( t(z)\pi_q(t(z)a(z)) + a(zq^{-2}) - a(zq^2) \right)
\]
where
\[
\pi_q \left( \sum b_k z^k \right) = \sum \frac{q^{2k} - 1}{q^{2k} + 1} b_k z^k
\]

The importance of the definition of \( D \) is due to the following fact:
\[
(Da)(z)\varphi_\epsilon(\zeta)\varphi_\epsilon'(\zeta) = \delta_{\gamma}(x_\epsilon,\epsilon'(z))
\]
for any $\epsilon, \epsilon'$ equal $+$ or $-$, the function $x_{\epsilon, \epsilon'}(z)$ is given by

$$x_{\epsilon, \epsilon'}(z) = \frac{1}{2i\gamma} \left( (-1)^g \left( \pi_q(a(z)t(z)) - a(z)t(z) \right) \right.$$

$$\times (\varphi_\epsilon(\zeta - i\gamma)\varphi_{\epsilon'}(\zeta) + \varphi_\epsilon(\zeta)\varphi_{\epsilon'}(\zeta - i\gamma)) -
\left. - 2a(zq^{-2})\varphi_\epsilon(\zeta)\varphi_{\epsilon'}(\zeta) - 2a(z)\varphi_\epsilon(\zeta - i\gamma)\varphi_{\epsilon'}(\zeta - i\gamma) \right)$$

Since $x_{+, -}(z)$ are Laurent series we have, in particular,

$$\mathcal{R}(\mathcal{D}(a)) = 0$$

Now we introduce important object: the space of deformed Abelian differentials. Consider the space $v$ of polynomials of one variable $a(z)$ such that $a(0) = 0$. These polynomials will define deformed Abelian differentials on the affine hyper-elliptic curve.

Consider a polynomial $u(z)$ which does not necessarily vanishes at $z = 0$. It is easy to see that $b = \mathcal{D}u \in v$. Let us calculate $a \circ b$ for $a \in v$. Simple calculation gives:

$$a \circ b = \text{res} \left( z^{-1}a(z)u(z) \left( \varphi_+ (\zeta)\varphi_- (\zeta - i\gamma) - \varphi_- (\zeta)\varphi_+ (\zeta - i\gamma) \right)^2 \right) =
= \text{res} \left( z^{-1}a(z)u(z) \right) = 0$$

So the pairing between $v$ and exact form vanishes.

Let us introduce the following basis in the space $v$:

$$r_k(z) = z^k, \quad \text{for } k = 1, \ldots, g + 1,$$

$$s_k(z) = \frac{1}{i\gamma} \left( t(z)\pi_q \left( [t(z)z^{-k}]_+ \right) + [z^{-k}]_+ (q^{2k} - q^{-2k}) \right), \quad \text{for } -\infty < k \leq g + 1$$

where $[\cdots]_+$ means the positive degree part of Laurent series in brackets. Obviouly,

$$\text{deg}(r_k) = k, \quad \text{deg}(s_k) = 2g + 2 - k$$

so, all the degrees are taken into account. Notice also that there is only one polynomial of degree $g + 1$ (this is $r_{g+1}(z)$) because $s_{g+1}(z) = 0$. We shall call the polynomials $r_k$ for $k = 1, \ldots, g$ first kind, $s_k$ for $k = 1, \ldots, g$ second kind, $r_{g+1}$ - third kind. The polynomials $s_k$ for $k \leq 0$ are exact forms.
Introduce the Laurent polynomial
\[ s_k^{-}(z) = \frac{1}{i\gamma} \left( t(z) \pi_q \left( [t(z)z^{-k}]_{<} \right) + [z^{-k}]_{<} (q^{2k} - q^{-2k}) \right) \] (13)
Where \([\cdots]_{<}\) means that only negative part of Laurent series is taken. Evidently,
\[ s_k(z) + s_k^{-}(z) = D(r_{-k})(z), \quad r_{-k}(z) = z^{-k} \] (14)
Consider the residues of our polynomials. It is clear that for \(z \to \infty\)
\[ r_k(z) \varphi_{\pm}(\zeta) \varphi_{\mp}(\zeta) = o(1), \quad \text{for } k = 1, \cdots, g \]
so, these \(r_k\) have no residues. It is equally clear that
\[ \mathcal{R}(r_{g+1}) = 1 \]
as it should be for the third king differential. Finally, notice that
\[ s_k^{-}(z) \varphi_{\pm}(\zeta) \varphi_{\mp}(\zeta) = o(1) \]
so, from (14) it follows that
\[ \mathcal{R}(s_k) = 0 \]
Consider the pairing between those polynomials whose residues vanish. It is easy to show that
\[ r_k \circ r_l = 0 \]
for \(k, l = 1, \cdots, g\) just because \(r_k\) do not grow sufficiently fast. Now let us calculate \(a \circ s_k\) for arbitrary \(a\). Using the formula (14) one finds
\[ a \circ s_k = \text{res} \left( a(z)z^{-k-1} \right) - a \circ s_k^{-} \] (15)
If \(a = r_l\) the second term in (15) vanishes because \(a(z)\) and \(s_k(z)\) do not grow sufficiently fast. So, one finds
\[ r_l \circ s_k = \delta_{kl} \]
Consider the case \(a = s_l\). Using
\[ s_l \circ s_k = -s_k^{-} \circ s_l \]
find
\[ s_l \circ s_k = \text{res} \left( s_l(z)z^{-k-1} + s_k^{-}(z)z^{-l-1} \right) + s_l^{-} \circ s_k^{-} \] (16)
The last term in the RHS vanishes because \(s_l^{-}, s_k^{-}\) do not grow fast enough. It is a nice exercise to show that the first term in the RHS of (16) vanishes using explicit formulae for \(s_l\) and \(s_k\).
3 Deformed Abelian integrals.

In this section we shall define deformed Abelian integrals. Let us start from the simplest case. Consider the polynomials

$$R_l(Z) = Z^l$$

for $l = 1, \cdots, g$. Take a polynomial $a(z) \in v$. Suppose that $\gamma$ is sufficiently small, more precisely, we assume that

$$\deg(a(z)) < \frac{\pi}{\gamma}$$

Then the following integral is well defined:

$$\langle a, R_l \rangle \equiv \int_{-\infty}^{\infty} a(z) R_l(Z) Q(\zeta)^2 d\zeta$$

(17)

Our first goal is to define $\langle a, R_l \rangle$ for arbitrary $\gamma$. We shall use the basis $r_k$ ($k = 0, \cdots, g$), $s_k$ ($k \leq g + 1$) introduced in the previous section.

If we substitute $a = r_k$ into (17) the integral converges for any $\gamma$, but for $a = s_k$ a regularization is needed.

In what follows we shall use the operators $w, w^*$ which act as usual:

$$w Q(\zeta) = Q(\zeta - i\gamma), \quad Q(\zeta)w^* = Q(\zeta - i\gamma),$$

The following important identity holds:

$$s_k(z) Q(\zeta)^2 = Q(\zeta)\tilde{s}_k(z, w) Q(\zeta) + \delta_\gamma \left( \frac{1}{i\gamma} (i d - \pi q) ([t(z)z^{-k}] >) Q(\zeta) w Q(\zeta) \right)$$

where we have introduced new object:

$$\tilde{s}_k(z, w) = \frac{1}{i\gamma} \left( [z^{-k}] > (q^{2k} - q^{-2k}) + [z^{-k} t(z)] > (2w - t(z)) \right)$$

Notice that if $\gamma$ is sufficiently small the integral (17) can be rewritten in the following way

$$\langle s_k, R_l \rangle = \int_{-\infty}^{\infty} Q(\zeta) R_l(Z) \tilde{s}_k(z, w) Q(\zeta) d\zeta$$
because $s_k Q^2$ and $Q \tilde{s}_k Q$ differ by "total difference" and $R_l(Z)$ is $i\gamma$-periodical, the "boundary term" does not contribute for small enough $\gamma$. Let us use this form of integral in order to define the regularization for arbitrary $\gamma$.

For $l = 1, \ldots, g$ consider the integral

$$\langle s_k, R_l \rangle = \int_{-\infty}^{\Lambda} Q(\zeta) R_l(Z) \tilde{s}_k(z, w) Q(\zeta) d\zeta + \int_{\Lambda}^{\Lambda+i\gamma} Q(\zeta) R_l(Z) p_k(z, w^*, w) Q(\zeta) d\zeta - \int_{\Lambda}^{\infty} Q(\zeta) R_l(Z) \tilde{s}_k^-(z, w) Q(\zeta) d\zeta$$

where

$$\tilde{s}_k^-(z, w) = \frac{1}{i\gamma} \left( [z^{-k}]_{q^2 - q^{-2k}} + [z^{-k} t(z)]_{2w - t(z)} \right)$$

$$p_k(z, w^*, w) = \frac{1}{i\gamma} \left( z^{-k} (w^* w + q^{2k}) - t_k w \right)$$

The properties of the integrals in the RHS of (18) are:

1. The third integral in (18) converges $\forall \gamma$. Indeed, it is easy to show that the integrant decreases exponentially as $\zeta \to \infty$ for all $1 \leq l \leq g + 1$.

2. The RHS of (18) does not depend on $\Lambda$ due to the identity:

$$\delta_{\gamma} \left( Q(\zeta) p_k(z, w^*, w) Q(\zeta) \right) = -Q(\zeta) \left( \tilde{s}_k(z, w) + \tilde{s}_k^-(z, w) \right) Q(\zeta)$$

3. When $\gamma$ is sufficiently small and $1 \leq l \leq g$ one can take the limit $\Lambda \to \infty$ reproducing the original definition. In the case $l = g + 1$ the regularization is needed for any $\gamma$, so, our definition is not founded independently, however, an important evidence of self-consistency will follow from Riemann bilinear relation.

4. In the original definition $s_{g+1}(z) = 0$. Consider, however, the definition of regularized integral. It does not vanish because of contributions $\int_{\Lambda}^{\Lambda+i\gamma}$ and $\int_{\Lambda}^{\infty}$. The integrals can be easily evaluated:

$$\langle s_{g+1}, R_l \rangle = \delta_{l, g+1}$$

Thus the formula (18) provides an analytical continuation of the original definition (17) for arbitrary $\gamma$. Certainly to affirm that we have to assume that the solution to Baxter equations $Q(\zeta)$ allow analytical continuation with respect to $\gamma$. 

9
Similarly to $s_k$ consider

$$S_k(Z) = \frac{1}{i\pi} \left( T(z)\pi Q \left( [T(z)Z^{-k}]_> + [Z^{-k}]_> (Q^{2k} - Q^{-2k}) \right) \right),$$

For these dual objects define:

$$\langle r_l, S_k \rangle = \int_{-\infty}^{\Lambda} Q(\zeta) r_l(z) \tilde{S}_k(Z, W) Q(\zeta) d\zeta +$$

$$+ \int_{\Lambda}^{\Lambda+i\pi} Q(\zeta) r_l(z) P_k(Z, W^*, W) Q(\zeta) d\zeta - \int_{\Lambda}^{\Lambda'} Q(\zeta) r_l(z) \tilde{S}_k^{-}(Z, W) Q(\zeta) d\zeta$$

where

$$\tilde{S}_k(Z, W) = \frac{1}{i\pi} \left( [Z^{-k}]_> (Q^{2k} - Q^{-2k}) + [Z^{-k}T(Z)]_> (2W - T(Z)) \right),$$

$$\tilde{S}_k^{-}(Z, W) = \frac{1}{i\pi} \left( [Z^{-k}]_< (Q^{2k} - Q^{-2k}) + [Z^{-k}T(Z)]_< (2W - T(Z)) \right),$$

$$P_k(Z, W^*, W) = \frac{1}{i\pi} (Z^{-k} (W^*W + Q^{2k}) - T_kW)$$

Now we want to define the pairing $\langle s_k, S_l \rangle$. Notice that even in the region of small $\gamma$ corresponding integral is not defined in a simple way. So, we shall define this pairing by analogy, the real justification of our definition will be provided later by Riemann bilinear relation.

Define

$$\langle s_k, S_l \rangle = \int_{-\infty}^{\Lambda} Q(\zeta) \tilde{s}_k(z, w) \tilde{S}_l(Z, W) Q(\zeta) d\zeta +$$

$$+ \int_{\Lambda}^{\Lambda+i\pi} Q(\zeta) \tilde{S}_l(Z, W) p_k(z, w, \omega) Q(\zeta) d\zeta - \int_{\Lambda}^{\Lambda'} Q(\zeta) \tilde{s}_k^{-}(z, w) \tilde{S}_l(Z, W) Q(\zeta) d\zeta -$$

$$- \int_{\Lambda'}^{\Lambda'+i\pi} Q(\zeta) \tilde{s}_k^{-}(z, w) P_l(Z, W^*, W) Q(\zeta) d\zeta + \int_{\Lambda'}^{\Lambda'} Q(\zeta) \tilde{s}_k^{-}(z, w) \tilde{S}_l^{-}(Z, W) Q(\zeta) d\zeta$$
By construction this definition does not really depend on $\Lambda$, $\Lambda'$. Duality requires that (20) is equivalent to the following one

$$\langle s_k, S_l \rangle = \int_{-\infty}^{\Lambda} Q(\zeta) \tilde{s}_k(z, w) \tilde{S}_l(Z, W) Q(\zeta) d\zeta +$$

$$+ \int_{\Lambda}^{\Lambda'} Q(\zeta) \tilde{s}_k(z, w) P_l(Z, W^*, W) Q(\zeta) d\zeta - \int_{\Lambda}^{\Lambda'} Q(\zeta) \tilde{s}_k(z, w) \tilde{S}_l^{-}(Z, W) Q(\zeta) d\zeta -$$

$$- \int_{\Lambda'}^{\Lambda} Q(\zeta) \tilde{s}_k^{-}(Z, W) p_k(z, w^*, w) Q(\zeta) d\zeta + \int_{\Lambda'}^{\infty} Q(\zeta) \tilde{s}_k^{-}(z, w) \tilde{S}_l^{-}(Z, W) Q(\zeta) d\zeta$$

Let us show that the equivalence indeed holds. To this end consider $\Lambda' \to \Lambda$. Then the equivalence in question requires:

$$\int_{\Lambda}^{\Lambda'} Q(\zeta) \tilde{S}_l(Z, W) p_k(z, w^*, w) Q(\zeta) d\zeta -$$

$$- \int_{\Lambda}^{\Lambda'} Q(\zeta) \tilde{s}_k^{-}(z, w) P_l(Z, W^*, W) Q(\zeta) d\zeta =$$

$$= \int_{\Lambda}^{\Lambda'} Q(\zeta) \tilde{s}_k(z, w) P_l(Z, W^*, W) Q(\zeta) d\zeta -$$

$$- \int_{\Lambda}^{\Lambda'} Q(\zeta) \tilde{S}_l^{-}(Z, W) p_k(z, w^*, w) Q(\zeta) d\zeta$$

or, equivalently:

$$\int_{\Lambda}^{\Lambda'} Q(\zeta) (\tilde{S}_l(Z, W) + \tilde{S}_l^{-}(Z, W)) p_k(z, w^*, w) Q(\zeta) d\zeta =$$

$$= \int_{\Lambda}^{\Lambda'} Q(\zeta) (\tilde{s}_k(z, w) + \tilde{s}_k^{-}(z, w)) P_l(Z, W^*, W) Q(\zeta) d\zeta$$

(22)
Consider the expression:

\[ X_{k,l}(\zeta) = -\mathcal{Q}(\zeta) \ p_k(z, w^*, w) \ P_l(Z, W^*, W) \ \mathcal{Q}(\zeta) \]

one has:

\[ \mathcal{Q}(\zeta) \left( \tilde{S}_l(Z, W) + \tilde{S}_l^-(Z, W) \right) p_k(z, w^*, w) \mathcal{Q}(\zeta) = \delta_{\pi} \left( X_{k,l}(\zeta) \right), \]

\[ \mathcal{Q}(\zeta) \left( \tilde{s}_k(z, w) + \tilde{s}_k^-(z, w) \right) P_l(Z, W^*, W) \mathcal{Q}(\zeta) = \delta_{\gamma} \left( X_{k,l}(\zeta) \right), \]

So, the equation (22) follows from the identity:

\[
\left( \int_{\Lambda+i\gamma}^{\Lambda+i\gamma+i\pi} - \int_{\Lambda}^{\Lambda+i\pi} \right) X_{k,l}(\zeta) = \left( \int_{\Lambda+i\gamma}^{\Lambda+i\gamma+i\pi} - \int_{\Lambda}^{\Lambda+i\pi} \right) X_{k,l}(\zeta)
\]

Recall that \( s_k \) for \( k \leq 0 \) are "exact", so, we must have

\[ \langle s_k, A \rangle = 0, \quad \text{for} \quad k \leq 0 \]

for any \( A \) which can be either \( R_l \) or \( S_l \). In the integrals \([18, 20]\) this property is transparent: corresponding \( \tilde{s}_k^- \equiv 0 \), hence one can move \( \Lambda \) to \(-\infty\), and to check that the integral \( \int_{\Lambda}^{\Lambda+i\gamma} \) vanishes. Indeed, \( p_k \) is regular at \( z \to 0 \) in this case while \( R_l \) vanishes. Similarly \( S_k \) for \( k \leq 0 \) are "exact".

Notice also that the integrals for \( \langle s_{g+1}, S_l \rangle \) vanish. So, \( \langle s_{g+1}, R_{g+1} \rangle \) remains the only non-zero pairing involving \( s_{g+1} \). Certainly, similar fact holds for \( S_{g+1} \); the only non-zero pairing involving \( S_{g+1} \) is

\[ \langle r_{g+1}, S_{g+1} \rangle = 1 \]

Finally among \( r_k \) (1 \( \leq k \leq g + 1 \)), \( s_k \) (\( -\infty < k \leq g + 1 \)) on the one hand and \( R_k \) (1 \( \leq k \leq g + 1 \)), \( S_k \) (\( -\infty < k \leq g + 1 \)) on the other we have defined all the pairings except for \( \langle r_{g+1}, R_{g+1} \rangle \). This pairing has no quasi-classical limit. Some regularization can be proposed in order to define it, but the definition is not unique. Anyway, one can avoid using this badly defined pairing in applications to integrable models.
4 Riemann bilinear relation for deformed Abelian integrals.

Riemann bilinear relation is the most important property of deformed Abelian differentials.

**Theorem.** Consider two polynomials \( a, b \in \mathfrak{v} \) such that \( \mathcal{R}(a) = \mathcal{R}(b) = 0 \). Then

\[
\sum_{l=1}^{g} \left( \langle a, S_l \rangle \langle b, R_l \rangle - \langle a, R_l \rangle \langle b, S_l \rangle \right) = a \circ b \tag{23}
\]

**Proof.** The polynomials \( s_k \) for \( k \leq 0 \) are exact forms, so, LHS of (23) for them vanishes as it must be. The polynomial \( r_{g+1} \) has residue. So, we shall chose \( a \) and \( b \) from \( r_1, \cdots r_g \) and \( s_1, \cdots, s_g \). This means, in particular, \( \deg(a) \leq 2g + 1 \), \( \deg(b) \leq 2g + 1 \). It is easy to see that for such polynomials the formula (9) can be simplified:

\[
a \circ b = i\gamma \text{res} \left( z^{-1}a(z)(2\varphi(\zeta)\varphi_-^{(\zeta)}\delta^{-1}_\gamma(b(z))\varphi_-(\zeta) + b(z)(\varphi_+(\zeta)\varphi_-^{(\zeta)})^2) \right) \tag{24}
\]

Obviously this expression is anti-symmetric with respect to \( a \leftrightarrow b \).

For simplicity we consider the range of small \( \gamma \). Namely, we shall require

\[
\gamma < \frac{\pi}{2g + 2}
\]

Considering this range of \( \gamma \) simplifies a lot the regularization for \( \langle a, S_l \rangle \). Recall that

\[
\tilde{S}^-_l(Z, W) = \frac{1}{i\pi} \left( [Z^{-\gamma}] < (Q^{2l} - Q^{-2l}) + [Z^{-l}T(Z)] < (2W - T(Z)) \right)
\]

When \( \zeta \to +\infty \) the function \( Q(\zeta)\tilde{S}^-_l(Z, W)Q(\zeta) \) decreases as \( Z^{-1} \), for small \( \gamma \) this decreasing is very fast, so, the regularization of \( \langle a, S_l \rangle \) when \( a = s_k \) can be seriously simplified. Consider the formula (21). For above reasons we can drop last two terms in (21) and put \( \Lambda' = \infty \). Also we can replace \( \tilde{s}_k \) by \( s_k \) because they differ by "exact form" and corresponding boundary term vanishes for small
γ. The result is:

\[ \langle s_k, S_l \rangle = \int_{-\infty}^{\Lambda} Q(\zeta) s_k(z) \tilde{S}_l(Z, W) Q(\zeta) d\zeta + \int_{-\infty}^{\Lambda+i\pi} Q(\zeta) s_k(z) P_l(Z, W^*, W) Q(\zeta) d\zeta - \int_{-\infty}^{\Lambda} Q(\zeta) s_k(z) \tilde{S}_l^{-}(Z, W) Q(\zeta) d\zeta \]

Hence for small γ the formula for \( \langle a, S_l \rangle \) is absolutely the same for \( a = s_k \) and for \( a = r_k \). The integral \( \int_{-\infty}^{\Lambda} \) rapidly converges, so, we can consider the following formula for in the case of small γ:

\[ \langle a, S_l \rangle = \lim_{\Lambda \to \infty} \left( \int_{-\infty}^{\Lambda} Q(\zeta) a(z) \tilde{S}_l(Z, W) Q(\zeta) d\zeta + \int_{-\infty}^{\Lambda+i\pi} Q(\zeta) a(z) P_l(Z, W^*, W) Q(\zeta) d\zeta \right) \]

Recall that \( R_l(Z) = Z^l \), and for small γ we do not need any regularization in \( \langle b, R_l \rangle \). Hence the first sum of \( a \circ b \) can be rewritten as follows:

\[ \lim_{\Lambda \to \infty} \int_{-\infty}^{\Lambda} \int_{-\infty}^{\Lambda+i\pi} d\zeta d\zeta' Q(\zeta) a(z) b(z') \sum_{l=1}^{g} \tilde{S}_l(Z, W) R_l(Z') Q(\zeta) Q(\zeta')^2 + \int_{-\infty}^{\Lambda} \int_{-\infty}^{\Lambda+i\pi} d\zeta d\zeta' Q(\zeta) a(z) b(z') \sum_{l=1}^{g} P_l(Z, W^*, W) R_l(Z') Q(\zeta) Q(\zeta')^2 \]

Let us evaluate the the sum in the first integrand using

\[ \sum_{l=1}^{g} (Z')^l [Z^{-l}T(Z)] = \frac{Z'T(Z) - ZT(Z')}{Z - Z'} + T_0 \]

In writing down the result we shall use the notation:

\[ Q(\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4) = Q(\zeta + i\epsilon_1 \pi) Q(\zeta + i\epsilon_2 \pi) Q(\zeta' + i\epsilon_3 \pi) Q(\zeta' + i\epsilon_4 \pi) \]
where $\epsilon$ takes values $0, +, -$. We divide the result into three parts:

$$
\sum_{l=1}^{g} Q(\zeta) R_l(Z') \bar{S}_l(Z, W) Q(\zeta) Q(\zeta')^2 = \frac{1}{i\pi} (A(\zeta, \zeta') + B(\zeta, \zeta') + C(\zeta, \zeta'))
$$

where

$$
A(\zeta, \zeta') = \frac{Z'}{Z - Z'} \{Q(- - 00) - Q(+ + 00)\}
$$

$$
B(\zeta, \zeta') = \frac{Z}{Z - Z'} \{Q(0 + + 0) - Q(0 - 0) + Q(0 + 0) - Q(0 + 0)\}
$$

$$
C(\zeta, \zeta') = T_0 \{Q(-000) - Q(0 + 0)\}
$$

Notice that $A + B$ is not singular at $Z = Z'$ as a whole, but since we shall need to consider the items separately a self-consistent way of understanding the singularities is to be prescribed. We shall understand assume that $\zeta$ is slightly moved to the upper half-plane. Define the functions:

$$
F[b](\zeta) = \int_{-\infty}^{\infty} \frac{Z'}{Z(1 + i0) - Z'} b(z') Q(\zeta')^2 d\zeta',
$$

$$
H_{\pm}[b](\zeta) = \int_{-\infty}^{\infty} \frac{Z}{Z(1 + i0) - Z'} b(z') Q(\zeta') \mp i\pi Q(\zeta') d\zeta', \quad (28)
$$

These functions satisfy the equations:

$$
\delta_{\gamma}(F[b])(\zeta) = i\gamma b(z) Q(\zeta)^2,
$$

$$
\delta_{\gamma}(H_{\pm}[b])(\zeta) = i\gamma b(z) Q(\zeta \pm i\pi) Q(\zeta), \quad (29)
$$

Let us return to calculations. Consider, first, the integral of $A$ rewriting it as follows:

$$
\int_{-\infty}^{\Lambda} d\zeta \int_{-\infty}^{\infty} d\zeta' A(\zeta, \zeta') =
\int_{-\infty}^{\Lambda - i\pi} Q(\zeta)^2 F[b](\zeta + i\pi) a(z) d\zeta -
\int_{-\infty + i\pi}^{\Lambda + i\pi} Q(\zeta)^2 F[b](\zeta - i\pi) a(z) d\zeta
$$
We want to move the contours of integration by \( i\pi \) down in the first integral and by \( i\pi \) up in the second. In order to do that we have to understand the analytical continuation of \( F[b](\zeta) \). It is easy to find that for real \( \zeta \):

\[
F[b](\zeta + i\pi) = \int_{-\infty}^{\infty} \frac{Z'}{ZQ^2 - Z'} b(z') \mathcal{Q}(\zeta')^2 d\zeta' + i\gamma \sum_{j=1}^{[\pi]} b(zq^{-2j}) \mathcal{Q}(\zeta + i\pi - i\gamma j)^2;
\]

\[
F[b](\zeta - i\pi) = \int_{-\infty}^{\infty} \frac{Z'}{ZQ^{-2} - Z'} b(z') \mathcal{Q}(\zeta')^2 d\zeta' - i\gamma \sum_{j=0}^{[\pi]} b(zq^{2j}) \mathcal{Q}(\zeta - i\pi + i\gamma j)^2,
\]

Using these formulae we find:

\[
\int_{-\infty}^{\Lambda} d\zeta \int_{-\infty}^{\infty} d\zeta' \mathcal{A}(\zeta, \zeta') = -\int_{-\infty}^{\Lambda} \left( \mathcal{Q}(\zeta)^2 F[b](\zeta - i\pi) + \mathcal{Q}(\zeta - i\pi)^2 F[b](\zeta) \right) a(z) d\zeta - \int_{-\infty}^{\infty} \left( \mathcal{Z}' \mathcal{Q}^2 - \mathcal{Z}' \right) a(z) b(z') \mathcal{Q}(\zeta')^2 d\zeta' - \int_{-\infty}^{\Lambda} \left( \mathcal{Z}' \mathcal{Q}^2 - \mathcal{Z}' \mathcal{Q}^{-2} \right) a(z) b(z') \mathcal{Q}(\zeta) \mathcal{Q}(\zeta')^2 d\zeta d\zeta' - i\gamma \int_{-\infty}^{\Lambda} a(z) b(z) (\mathcal{Q}(\zeta) \mathcal{Q}(\zeta - i\pi))^2 d\zeta - i\gamma \sum_{j=1}^{[\pi]} \int_{-\infty}^{\Lambda} (a(z)b(zq^{-2j}) + b(z)a(zq^{-2j})) (\mathcal{Q}(\zeta) \mathcal{Q}(\zeta + i\pi - i\gamma j))^2 d\zeta + o(1)
\]

where \( o(1) \) comes from replacing the upper limits of rapidly converging integrals in the last line from \( \Lambda \) to \( \infty \). We also moved contours of integration in some
of these integrals. The formula (30) is written in such a way that the symmetry with respect to replacing \(a \leftrightarrow b\) is obvious everywhere except the first integral in the RHS. Our goal is to calculate (23) where the antisymmetrization is performed with respect to \(a\) and \(b\). So, only the first integral in the RHS is relevant.

Now let us consider the integral of \(\mathcal{B}\) which can be rewritten as follows:

\[
\int_{-\infty}^{\Lambda} d\zeta \int_{-\infty}^{\Lambda} d\zeta' \mathcal{B}(\zeta, \zeta') = \\
\int_{-\infty}^{\Lambda + i\pi} d\zeta \int_{-\infty}^{\Lambda + i\pi} d\zeta' \mathcal{B}(\zeta, \zeta') = \\
\int_{-\infty}^{\Lambda + i\pi} a(z) \mathcal{Q}(\zeta - i\pi) \mathcal{Q}(\zeta) H_+[b](\zeta - i\pi) - \\
\int_{-\infty - i\pi}^{\Lambda - i\pi} a(z) \mathcal{Q}(\zeta) \mathcal{Q}(\zeta + i\pi) H_-[b](\zeta + i\pi) d\zeta + \\
\int_{-\infty}^{\Lambda} a(z) \mathcal{Q}(\zeta) \left( \mathcal{Q}(\zeta + i\pi) H_-[b](\zeta) - \mathcal{Q}(\zeta - i\pi) H_+[b](\zeta) \right) d\zeta
\]

In the first two integrals in RHS we would like to move the contour of integration to the real axis. The result is

\[
\int_{-\infty}^{\Lambda} d\zeta \int_{-\infty}^{\Lambda} d\zeta' \mathcal{B}(\zeta, \zeta') = \\
\int_{-\infty}^{\Lambda + i\pi} \mathcal{X}[b](\zeta) a(z) d\zeta + \\
\int_{\Lambda - i\pi}^{\Lambda} a(z) \mathcal{Q}(\zeta) \left( \mathcal{Q}(\zeta - i\pi) H_+[b](\zeta - i\pi) + \mathcal{Q}(\zeta + i\pi) H_-[b](\zeta + i\pi) \right) d\zeta
\]

where

\[
\mathcal{X}[b](\zeta) = \mathcal{Q}(\zeta - i\pi) H_+[b](\zeta - i\pi) - \mathcal{Q}(\zeta) \left( \mathcal{Q}(\zeta + i\pi) H_-[b](\zeta + i\pi) + \mathcal{Q}(\zeta + i\pi) H_-[b](\zeta) - \mathcal{Q}(\zeta - i\pi) H_+[b](\zeta) \right)
\]

(31)

This is the place to discuss asymptotic of \(H_\pm[b]\) and \(\mathcal{F}[b]\). Due to (29) these asymptotic must be related to \(\delta^{-1}\left( b(z) \varphi_\epsilon(\zeta) \right)\) as they were defined in the Section 2. There is, however, a subtlety: the asymptotic may differ by quasiconstants: \(i\gamma\)-periodical functions. For \(H_\pm[b]\) the situation is simple, one finds that
for $0 \leq \text{Im}(\zeta) \leq \pi$

$$H_-(b)(\zeta) \simeq H_+(b)(\zeta - i\pi) = i\gamma (\delta^{-1}(b(z)\varphi_-(-\zeta)\varphi_+(-\zeta)) - b(z)\varphi_-(\zeta)^2) (1 + O(e^{-2(g+1)}))$$

(32)

The asymptotics of $F[b](\zeta)$ in the strip $-\pi \leq \text{Im}(\zeta) \leq \pi$ can be found:

$$F[b](\zeta) = \sum_{k=1}^{g} Z^{-k} \langle b, R_k \rangle +$$

$$+ i\gamma Z^{-g} (2b(z)\varphi_-(\zeta)\varphi_+(\zeta)) - b(z)\varphi_-(-\zeta)^2 (1 + O(e^{-2(g+1)}))$$

(33)

Using the asymptotical formulae $\mathcal{A}$ we find that

$$\mathcal{X}[b](\zeta) = -i\gamma b(z) (Q(\zeta)Q(\zeta - i\pi))^2 + O(z^{-1})$$

Recall that we have finally to anti-symmetrize. The above formula guaranties rapid convergence of the integral:

$$\int_{-\infty}^{\Lambda} (a(z)\mathcal{X}[b](\zeta) - b(z)\mathcal{X}[a](\zeta)) d\zeta$$

So, the limit $\Lambda \to \infty$ can be taken for this integral separately. It can be shown further that

$$\int_{-\infty}^{\infty} (a(z)\mathcal{X}[b](\zeta) - b(z)\mathcal{X}[a](\zeta)) d\zeta = 0$$

Let us summarize taking into account the integrals containing $P_l$ in (26):

$$\sum_{l=1}^{g} (\langle a, S_l \rangle \langle b, R_l \rangle - \langle a, R_l \rangle \langle b, S_l \rangle) = \lim_{\Lambda \to \infty} \frac{1}{\pi i} \int_{\Lambda}^{\Lambda + i\pi} a(z) \times$$

$$\times \left\{ Q(\zeta)Q(\zeta - i\pi)(H_-[b](\zeta) + H_+[b](\zeta - i\pi)) - \sum_{k=0}^{g} T_k \langle b, R_k \rangle \right\}$$

$$- Q(\zeta - i\pi)^2 (F[b](\zeta) - \sum_{k=1}^{g} Z^{-k} \langle b, R_k \rangle) -$$

$$- Q(\zeta - i\pi) (F[b](\zeta) - \sum_{k=1}^{g} Z^{-k} Q^{2k} \langle b, R_k \rangle) \right\} d\zeta - (a \leftrightarrow b)$$

(34)
Using the asymptotics and the fact that $R(a) = R(b) = 0$ one finds

$$\sum_{l=1}^{g} \left( \langle a, S_l \rangle \langle b, R_l \rangle - \langle a, R_l \rangle \langle b, S_l \rangle \right) = \lim_{\Lambda \to \infty} \frac{\gamma}{\pi} \int_{\Lambda} a(z) \times$$

$$\times \left( 2\varphi(\zeta) \varphi_-(\zeta) \delta_\gamma^{-1}(b(z)\varphi_+(\zeta)\varphi_-(\zeta)) + b(z)(\varphi_+(\zeta)\varphi_-(\zeta))^2 \right) d\zeta = a \circ b$$

where the formula (24) for pairing has been used.

QED

The Riemann bilinear identity can be reformulated in the following fashion. Corollary 1. Consider the $2g \times 2g$-matrix

$$P_{2g} = \begin{pmatrix} \langle r, R \rangle, & \langle r, S \rangle \\ \langle s, R \rangle, & \langle s, S \rangle \end{pmatrix}$$

then

$$P_{2g} \in Sp(2g)$$

(36)

Until now we have considered only first and second king differentials, but actually the third kind differential $r_{g+1}$ can also be taken into account. In the proof of the Theorem we did not use the requirement $R(a) = 0$ up to the very last transformation from (34) to (35). Suppose $a = r_{g+1}$ and $b = r_k, k = 1, \ldots, g$. In the formula (34) we have the expression

$$H_- [b](\zeta) + H_+ [b](\zeta - i\pi) - \sum_{k=0}^{g} T_k \langle b, R_k \rangle \simeq 2i\gamma \delta_\gamma^{-1} \left( b(z)\varphi_+(\zeta)\varphi_-(\zeta) \right)$$

(37)

Usually $\delta_\gamma^{-1}$ is defined up to a constant which was until now irrelevant. However, in the formula (37) this constant can be calculated:

$$C = \langle b, R_{g+1} \rangle$$

where the requirement $b = r_k$ was important for convergence of integrals. Obviously this is the constant $C$ which is important for calculation of the integral containing (37) because $a = r_{g+1}$ has residue. The remaining integrals as well as the term $(a \leftrightarrow b)$ do not contribute. So, we find for $b = r_k$:

$$\sum_{l=1}^{g} \left( \langle r_{g+1}, S_l \rangle \langle b, R_l \rangle - \langle r_{g+1}, R_l \rangle \langle b, S_l \rangle \right) = \langle b, R_{g+1} \rangle$$

(38)

With somewhat more elaborate calculations we find the same result for $b = s_k$ for which $\langle b, R_{g+1} \rangle$ is given by usual formula (18).
5 Classical limit.

In this section we show that in the classical limit the deformed Abelian differentials and integrals reproduce their classical counterparts. But we have to start with some more explicit formulae concerning the classical limit of solutions to Baxter equation.

Consider the hyper-elliptic algebraic curve $X$:

$$w^2 - t(z)w + 1 = 0$$

where the real polynomial $t(z)$ with real positive zeros:

$$t(z) = \prod_{k=1}^{g+1} (z - x_k)$$

We assume that $t(0) > 2$ and zeros of the polynomial $t(z)^2 - 4$ are real positive. We shall denote by $y_j$ the branch points:

$$t(z)^2 - 4 = \prod_{j=1}^{2g+2} (z - y_j)$$

Obviously in the above conditions we have

$$y_{2k-1} < x_k < y_{2k}$$

Introduce the notation

$$\xi_j = \frac{1}{2} \log y_j$$

The real axis appears to be divided into intervals:

$$I_k = [\xi_{2k}, \xi_{2k+1}], \quad k = 0, \cdots g + 1,$$

where $\xi_0 = -\infty$, $\xi_{2g+3} = \infty$.

$$J_k = [\xi_{2k-1}, \xi_{2k}], \quad k = 1, \cdots g + 1$$

Let us realize the Riemann surface as follows. In the plane of $\zeta$ consider the strip $-\pi < \text{Im}(\zeta) < \pi$ with cuts along the intervals $I_k$. We identify the following segments:

$$I_k + i0 = I_k + i\pi - i0, \quad I_k - i0 = I_k - i\pi + i0,$$

$$J_k + i\pi - i0 = J_k - i\pi + i0,$$
The segments $[\infty, \infty + i\pi], [\infty - i\pi, \infty]$ are contracted to points $\infty^+, \infty^-$, the segments $[-\infty, \infty + i\pi], [-\infty - i\pi, \infty]$ are contracted to the points $0^+$ and $0^-$ (they correspond to two points over $z = 0$ if the surface is realized as covering of $z$-plane), the latter two points do not play any special role in our considerations.

The $a$-cycle $\alpha_k$ goes around the cuts $I_k$ ($k = 1, \ldots, g$), the $b$-cycle $\beta_k$ goes in the upper half-strip form a point on $I_k$ to identified point on $I_k + i\pi$.

Consider $\log w$. It is not a single-valued function on the surface. Actually $d\log w$ is quite special Abelian differential: it is of the third kind with simple poles at $\infty^\pm$ and with vanishing $a$-periods. So, $\log w$ is single-valued on the plane with cuts, its imaginary part is constant along the cuts:

\[
\begin{align*}
\text{Im}(\log w) &= -\pi k \quad \text{at} \quad I_k \pm i0, \\
\text{Im}(\log w) &= \pi k \quad \text{at} \quad I_k \pm (i\pi - i0),
\end{align*}
\]

Away from the cuts the quasi-classical solution to the Baxter equation is given by

\[
Q^c_+ (\zeta) = (t(z)^2 - 4)^{-\frac{1}{4}} \exp \left\{ \frac{1}{i\gamma} \int_{-\xi_l}^{\xi} \log w \, d\zeta + \frac{\pi i}{4} \right\}
\]

(39)

where $(t(z)^2 - 4)^{\frac{1}{4}}$ is defined to be positive on $I_k + i0$. The function (39) is
Actually we have two solutions: one of them is (39) and another with minus in the exponent. However, this second solution is small everywhere except the intervals $I_k$ where oscillations take place. Inside these intervals we have to take for $Q^o(\zeta)$ the sum of values of (39) on the upper and lower banks:

$$Q^o(\zeta) = Q^o_+(\zeta + i0) + Q^o_+(\zeta - i0), \quad \zeta \in I_k$$  \hspace{1cm} (40)

The definition of $Q^o(\zeta)$ in $I_0$ is obvious, for the rest of $I_k$ a simple calculation gives:

for $\zeta \in I_k$, $k = 1, \ldots, g + 1$:

$$Q^o(\zeta) = C_k e^{-\pi j \zeta} |t(z)^2 - 4|^{-1/2} 2 \cos \left\{ \frac{1}{\gamma} \int_{\xi_{2k}}^{\zeta} \log |w| d\zeta + \frac{\pi i}{4} \right\}$$  \hspace{1cm} (41)

where

$$C_k = \exp \left\{ \frac{\pi}{\gamma} \sum_{l=1}^{k} \sigma_l \right\}$$

where

$$\sigma_l = \xi_{2l-1} + \frac{1}{\pi i} \int_{\xi_{2l-1}}^{\xi_{2l}} (\log w + \pi i(l + 1)) d\zeta$$  \hspace{1cm} (42)

By simple contour integration one finds:

$$\sum_{l=1}^{g+1} \sigma_l = 0$$  \hspace{1cm} (43)

Inside the interval $J_l$ the function $\log w$ is pure imaginary. Moreover, the real in this interval function

$$\frac{1}{\pi i} (\log w + \pi il)$$

decreases monotonically from 1 to 0. So, we have an important inequality:

$$\xi_{2l-1} < \sigma_l < \xi_{2l}$$  \hspace{1cm} (44)
Let us understand the meaning of $\sigma_l$. Recall that in the quantum case duality holds which implies that

$$Q(\zeta + i\pi) + Q(\zeta - i\pi) = T(Z)Q(\zeta)$$  \hspace{1cm} (45)

where $T(Z)$ is a polynomial

$$T(Z) = \prod_{l=1}^{g+1} (Z - X_l)$$

Let us show that quasi-classically

$$X_l \simeq e^{\frac{2\pi}{\gamma} \sigma_l}$$  \hspace{1cm} (46)

Indeed, take $\zeta \in I_k$ and consider $Q^{cl}(\zeta + i\pi)$ and $Q^{cl}(\zeta - i\pi)$. They are understood as analytical continuations respectively of $Q^{cl+}(\zeta + i0)$ and $Q^{cl+}(\zeta - i0)$. After some calculation we find:

$$Q^{cl}(\zeta + i\pi) = (-1)^{g+1-k} Z^k C_{k-2}^{cl} Q^{cl+}(\zeta + i0),$$
$$Q^{cl}(\zeta - i\pi) = (-1)^{g+1-k} Z^k C_{k-2}^{cl} Q^{cl+}(\zeta - i0), \quad \zeta \in I_k$$  \hspace{1cm} (47)

the only contribution which needs explanations is $(-1)^{g+1-k}$ which comes from branch points of $(t^2 - t)^{\frac{1}{4}}$. Quasi-classically inside $I_k$ one has

$$X_i \ll Z \ll X_j, \quad i \leq k < j$$

So, putting together (47) and (41) we prove that the equation (45) holds in $I_k$ quasi-classically.

Let us consider the classical limit of deformed Abelian differentials. Recall that the deformed Abelian differentials are defined by polynomials $a \in v$. For corresponding Abelian differential we want to take

$$\omega = \frac{a(z)}{z \sqrt{t(z)^2 - 4}} dz$$

but one should take limit $\gamma \to 0$ of $a(z)$ if the latter depends explicitly upon $\gamma$.

The polynomials $r_k(z) = z^k$ do not depend on $\gamma$ while for $s_k$ one finds:

$$\lim_{\gamma \to 0} s_k(z) = t(z) z \frac{d}{dz} \left[ z^{-k} t(z) \right]$$
The differentials
\[ \omega_k = \frac{z^k}{z \sqrt{t(z)^2 - 4}} \, dz \]
for \( k = 1, \ldots, g \) are of the first kind. They vanish at \( \infty + i0 \) as follows:
\[ \omega_k = \frac{z^{k-1}}{t(z)} \left( 1 + O(z^{-2(g+1)}) \right) \, dz \quad (48) \]

The differential
\[ \omega_{g+1} = \frac{z^g}{\sqrt{t(z)^2 - 4}} \, dz \]
is of the third kind, it behaves at \( \infty + i0 \) as follows
\[ \omega_{g+1} = \frac{dz}{z} \left( 1 + O(z^{-1}) \right) \]
The differentials
\[ \tilde{\omega}_k = t(z) \frac{d}{dz} \left[ z^{-k}t(z) \right] > \frac{1}{\sqrt{t(z)^2 - 4}} \, dz \]
are of the second kind with the following singularities at \( \infty \pm \):
\[ \tilde{\omega}_k = d \left[ z^{-k}t(z) \right] > \left( 1 + O(z^{-2(g+1)}) \right) \quad (49) \]
The differentials \( \omega_k, \tilde{\omega}_k \) do not have residue on the surface. For such differentials we define pairing:
\[ \omega \circ \omega' = \sum \text{res} \left( \omega d^{-1} \omega' \right) \]
The only singularities of \( \omega_k, \tilde{\omega}_k \) are situated at the points \( \infty \pm \). From (48) and (49) it is clear that \( \omega_k, \tilde{\omega}_k \) constitute a canonical basis:
\[ \omega_k \circ \omega_l = \tilde{\omega}_k \circ \tilde{\omega}_l = 0, \quad \omega_k \circ \tilde{\omega}_l = \delta_{k,l} \]
All these formulae actually explain the terminology used before. Now we have to do some analysis in order to calculate classical limit of deformed Abelian integrals.
Consider, first, the pairing
\[ \langle a, R_k \rangle = \int_{-\infty}^{\infty} a(z) \, Z^k \, Q(\zeta)^2 d\zeta \]
for $k = 1, \ldots g$. For small $\gamma$ the integral rapidly converges. What we need to know is the quasi-classical behaviour of $Q(\zeta)^2$. We have seen already that in the intervals $I_k$ the function $Q^{cl}(\zeta)$ rapidly oscillates. In the intervals $J_k$ the function $Q^{cl}(\zeta)$ is real positive. Up to rapidly oscillating part which is denoted by $\cdots$ we can present $Q^{cl}(\zeta)^2$ for real $\zeta$ in universal way:

$$Q^{cl}(\zeta)^2 = \frac{1}{\sqrt{t(z)^2 - 4}} \exp \left\{ \frac{2}{\gamma} \int_{-\infty}^{\zeta} \arg w \, d\zeta \right\} + \cdots$$  (50)

The function $\arg w$ decreases monotonously from 0 to $-\infty$ when $\zeta$ goes from $-\infty$ to $\infty$. It is constant equal to $-\pi l$ in the interval $I_l$. Under the integral we have the function $Q^{cl}(\zeta)^2 Z^k$. The two multipliers compete: $Q^{cl}(\zeta)$ decreases and $Z^k$ grows. The result is that in quasi-classical limit the integrand is everywhere exponentially small with $1/\gamma$ in exponent with respect to its values in $I_k$. Now using explicit formula (41) we find:

$$\langle a, R_k \rangle \simeq C_k^2 \int_{\alpha_k} a(z) \overline{z} \sqrt{t(z)^2 - 4} \, dz$$  (51)

Now we turn to the pairings $\langle a, S_k \rangle$. They look much more complicated, but the final calculation will be even simpler. In order to simplify the situation let us use the following trick. Up to now we always considered $\gamma$ in generic position, when $\gamma$ is a rational multiple of $\pi$ everything becomes much more complicated, literally some of our formulae loose meaning. It is not so bad, but some additional work is needed. However, the limit $\gamma \to 0$ can be taken along some special direction, for example, we can put

$$\gamma = \frac{\pi}{n}, \quad n \in \mathbb{N}$$

and take the limit $n \to \infty$. For sufficiently large $n$ the problems mentioned above disappear in every particular formula. We insist on considering this values of $\gamma$ because for them the formulae for pairings simplify a lot. There are two important simplifications for these values of $\gamma$: first, $Q^2 = 1$, second, $Z$ becomes $\pi$-periodic. Recall the regularizations (19) and (21). For $n \gg 1$ (small $\gamma$) we can use the formula (25) for $\langle s_l, S_k \rangle$. In other words we can take the same formula for $\langle a, S_k \rangle$. 25
if \( a = r_l \) or \( a = s_l \)

\[
\langle a, S_k \rangle = \int_{-\infty}^{\Lambda} Q(\zeta) a(z) S_k(Z, W) Q(\zeta) d\zeta + \int_{\Lambda}^{\Lambda+i\pi} Q(\zeta) a(z) P_k(Z, W^*, W) Q(\zeta) d\zeta - \int_{\Lambda}^{\infty} Q(\zeta) a(z) S_k^{-}(Z, W) Q(\zeta) d\zeta
\]

Notice that for the values of \( \gamma \) under consideration

\[
Q(\zeta) S_k(Z, W) Q(\zeta) = \delta_\pi (\left[ Z^{-k} T(Z) \right]_Q (\zeta - i\pi) Q(\zeta))
\]

\[
Q(\zeta) S_k^{-}(Z, W) Q(\zeta) = \delta_\pi (\left[ Z^{-k} T(Z) \right]_Q (\zeta - i\pi) Q(\zeta))
\]

So, we find the following beautiful result:

\[
\langle a, S_k \rangle = \int_{\Lambda}^{\Lambda+i\pi} a(z) Z^{-k} (Q(\zeta) Q(\zeta) - Q(\zeta - i\pi) Q(\zeta + i\pi)) d\zeta \tag{52}
\]

This is exact result for \( \gamma = \frac{\pi}{n} \). It is a nice exercise to show that for these values of \( \gamma \) the RHS of (52) does not depend on \( \Lambda \)

Now we are ready to take the classical limit. First, notice that quasi-classically

\[
|Q^{cl}(\zeta)|^2 \ll |Q^{cl}(\zeta - i\pi) Q^{cl}(\zeta + i\pi)|
\]

for \( 0 \leq \text{Im}(\zeta) \leq \pi \), so, we can neglect the first part of the integrand. Now let us use the freedom in definition of \( \Lambda \) putting it inside the interval \( I_k \). Use the formulae (47) in order to estimate \( Q^{cl}(\zeta - i\pi) Q^{cl}(\zeta + i\pi) \) for \( \zeta \in I_k \):

\[
Q^{cl}(\zeta - i\pi) Q^{cl}(\zeta + i\pi) = Z^{2k} C_k^{-4} Q^{cl}_-(\zeta - i0) Q^{cl}_+(\zeta + i0) = Z^{2k} C_k^{-2} \tag{53}
\]

The leading term of asymptotics (53) continues analytically to the strip \( 0 \leq \text{Im}(\zeta) \leq \pi \). Since the interval \([\Lambda, \Lambda+i\pi]\) for \( \Lambda \in I_k \) is the b-cycle \( \beta_k \), thus

\[
\langle a, S_k \rangle \simeq C_k^{-2} \int_{\beta_k} \frac{a(z)}{z \sqrt{f(z)^2 - 4}} dz \tag{54}
\]

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Now it is clear that the Riemann bilinear relation for deformed Abelian differential turn in the classical limit into usual Riemann bilinear relation: for two differentials $\omega, \omega'$ which can be singular only at $\infty^{\pm}$ and do not have residues the relation holds:

$$\sum_{l=1}^{g} \left( \int_{\alpha_l} \omega \int_{\beta_l} \omega' - \int_{\alpha_l} \omega' \int_{\beta_l} \omega \right) = \omega \circ \omega'$$  \hspace{1cm} (55)$$

On the affine curve

$$X_{\text{aff}} = X - \infty^{\pm}$$

one can consider two additional cycles: the first one ($\sigma$) goes around $\infty^+$, the second one ($\rho$) is non-compact: it goes from $\infty^- t 0$ $\infty^+$. Obviously

$$\sigma \circ \rho = 1$$

The integrals over $\sigma$ are well defined:

$$\int_{\sigma} \omega = \text{res}_{\infty^+}(\omega)$$  \hspace{1cm} (56)$$

The second cycle is not included into usual homology theory, but we can define integrals over $\rho$ of all first and second kind differentials. For the first kind differential the integral is well defined from the very beginning. For the second kind one we shall do the following:

$$\int_{\rho} \tilde{\omega}_k \equiv - \int_{\rho} t(z) \left( \frac{d}{dz} \left[ z^{-k} t(z) \right] + 4k z^{-k-1} \right) \frac{dz}{\sqrt{t(z)^2 - 4}}$$  \hspace{1cm} (57)$$

which means that we have subtracted from $\tilde{\omega}_k$ the exact form $d \left( z^{-k} \sqrt{t(z)^2 - 4} \right)$. We consider $\sigma$ and $\rho$ as classical limit of respectively $S_{g+1}$ and $R_{g+1}$. The formulae (56), (57) reproduce classical limit of corresponding quantum formulae. Let us consider the classical limit of the relation (58):

$$\sum_{l=1}^{g} \left( \int_{\alpha_l} \omega_{g+1} \int_{\beta_l} \omega - \int_{\alpha_l} \omega \int_{\beta_l} \omega_{g+1} \right) = \int_{\rho} \omega$$  \hspace{1cm} (58)$$
If $\omega$ is of the first kind this is obvious form of Riemann bilinear relation with third kind differential $\omega_{g+1}$: in the RHS we have difference of values of primitive function of $\omega$ at the points where $\omega_{g+1}$ has simple poles. For second kind differentials we subtract the exact form as it is explained above, this does not change the periods of $\omega$ in the LHS. In the RHS we have the integral over $\rho$ defined by (57) plus contribution from residue at $z = 0$ where $d(z^{-k}\sqrt{f(z)^2 - 4})$ is singular. However $r_{g+1}$ vanishes as $z^{g+1}$ at $z = 0$ so the second contribution disappears.

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