CLASSIFICATION OF INVARIANT CONES IN LIE ALGEBRAS

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All Lie algebras in the following are finite dimensional real Lie algebras. A cone in a finite dimensional real vector space is a closed convex subset stable under the scalar multiplication by the set \( \mathbb{R}^+ \) of nonnegative real numbers; it is, therefore additively closed and may contain vector subspaces. A cone \( W \) in a Lie algebra \( \mathfrak{g} \) is called invariant if

\[
e^{\text{ad}x}(W) = W \quad \text{for all } x \in \mathfrak{g}.
\]

We shall describe invariant cones in Lie algebras completely. For simple Lie algebras see [KR82, Ol81, Pa84, and Vi80].

Some observations are simple: If \( W \) is an invariant cone in a Lie algebra \( \mathfrak{g} \), then the edge \( e = W \cap -W \) and the span \( W - W \) are ideals. Therefore, if one aims for a theory without restriction on the algebra \( \mathfrak{g} \) it is no serious loss of generality to assume that \( W \) is generating, that is, satisfies \( \mathfrak{g} = W - W \). This is tantamount to saying that \( W \) has inner points. Also, the homomorphic image \( W/e \) is an invariant cone with zero edge in the algebra \( \mathfrak{g}/e \). Therefore, nothing is lost if we assume that \( W \) is pointed, that is, has zero edge. Invariant pointed generating cones can for instance be found in \( \mathfrak{sl}(2, \mathbb{R}) \), the oscillator algebra and compact Lie algebras with nontrivial center (see [HH85b, c, HH88a, or HHL87]).

A subalgebra \( \mathfrak{h} \) of a Lie algebra \( \mathfrak{g} \) is said to be compactly embedded if the analytic group \( \text{Inn}_\mathfrak{g} \mathfrak{h} \) generated by the set \( e^{\text{ad}h} \) in Aut \( \mathfrak{g} \) has a compact closure. Even for a compactly embedded Cartan algebra \( \mathfrak{h} \) of a solvable algebra \( \mathfrak{g} \), the analytic group \( \text{Inn}_\mathfrak{g} \mathfrak{h} \) need not be closed in Aut \( \mathfrak{g} \) [HH86]. An element \( x \in \mathfrak{g} \) is called compact if \( R \cdot x \) is a compactly embedded subalgebra, and the set of all compact elements of \( \mathfrak{g} \) will be denoted \( \text{comp}_\mathfrak{g} \). It is true, although not entirely superficial that a superalgebra is compactly embedded if and only if it is contained in \( \text{comp}_\mathfrak{g} \).

1. Theorem (The Uniqueness Theorem [HH86b]). Let \( W \) be an invariant pointed generating cone in a Lie algebra \( \mathfrak{g} \). Then

(i) \( \text{int} W \subseteq \text{comp}_\mathfrak{g} \).

(ii) If \( H \) is any compactly embedded Cartan algebra, then

(a) \( H \cap \text{int} W \neq \emptyset \), and

(b) \( \text{int} W = (\text{Inn}_\mathfrak{g} \mathfrak{h}) \text{int}_\mathfrak{h}(\mathfrak{h} \cap W) \).

In particular, compactly embedded Cartan algebras exist, and if \( \mathfrak{h}_1 \) and \( \mathfrak{h}_2 \) are compactly embedded Cartan algebras and \( W_1 \) and \( W_2 \) are invariant pointed generating cones of \( \mathfrak{g} \) such that \( \mathfrak{h} \cap W_1 = \mathfrak{h} \cap W_2 \), then \( W_1 = W_2 \). □

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This result shows that we know $W$ if we know $\mathfrak{h} \cap W$ for any compactly embedded Cartan algebra $\mathfrak{h}$.

We consider a compactly embedded Cartan algebra $\mathfrak{h}$ and denote by $\Gamma$ the torus $\text{Inn}_g \mathfrak{h}$. Then we obtain the linear projection operator $P : \mathfrak{g} \rightarrow \mathfrak{g}$ by $P(x) = \int_{\Gamma} g(x) \, dg$ with normalized Haar measure on $\Gamma$. Then $\mathfrak{h} = P(\mathfrak{g})$ and $\mathfrak{g}$ decomposes into a direct sum of $\mathfrak{h}$-modules $\mathfrak{h} \oplus \mathfrak{h}^+$ with $\mathfrak{h}^+ \overset{\text{def}}{=} \ker P$. For an invariant cone $W$ and any compactly embedded Cartan algebra $\mathfrak{h}$ the meet $\mathfrak{h} \cap W$ and the projection $P(W)$ are related by

$$P(W) = \mathfrak{h} \cap W. \tag{2}$$

If $C$ is a pointed cone in a compactly embedded Cartan algebra $\mathfrak{h}$ we define a cone in $\mathfrak{g}$ by

$$\tilde{C} = \bigcap_{g \in \text{Inn}_g \mathfrak{g}} gP^{-1}(C). \tag{3}$$

Then $\tilde{C} = \{ x \in \mathfrak{g} | P((\text{Inn}_g \mathfrak{g})x) \subseteq C \}$ and $\tilde{C}$ is an invariant cone in $\mathfrak{g}$. Its edge is the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{h}^+$. It is not a seriously restrictive assumption that $H^+$ should not contain nonzero ideals. Under these circumstances, unfortunately, $\tilde{C}$ may be zero. However, the following theorem uses the device $\tilde{C}$ to reconstruct $W$ from $\mathfrak{h} \cap W$:

2. Theorem (The Reconstructions Theorem [HH86b]). Suppose that $\mathfrak{h}$ is a compactly embedded Cartan algebra $\mathfrak{h}$ such that $\mathfrak{h}^+$ contains no nonzero ideal of $\mathfrak{g}$. If $C$ is a pointed generating cone in $\mathfrak{h}$ then the following statements are equivalent:

(A) There exists an invariant pointed cone $W$ in $L$ such that $C = \mathfrak{h} \cap W$.

(B) $C = \mathfrak{h} \cap \tilde{C}$.

(C) Each conjugacy class of an element $c \in C$ projects into $C$ under $P$.

Moreover, if these conditions are satisfied, then $W = \tilde{C}. \Box$

The problem is now to determine which cones $C$ satisfy condition (C) of Theorem 1 and in which Lie algebras they can occur.

3. Proposition [HH86]. Every compactly embedded Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is contained in a unique maximal compactly embedded subalgebra $\mathfrak{k}(\mathfrak{h})$. A subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is maximal compactly embedded if and only if $\text{Inn}_g \mathfrak{h} \overset{\text{def}}{=} \text{Inn}_g \mathfrak{k}$ is a maximal compact subgroup of $\text{Inn}_g \mathfrak{g}$. \Box

Under the circumstances of Proposition 3, the normalizer $N(\mathfrak{h})$ of the maximal torus $\Gamma = \text{Inn}_g \mathfrak{h}$ in $\text{Inn}_g \mathfrak{g}$ is contained in the compact subgroup $K(\mathfrak{h}) = \text{Inn}_g \mathfrak{k}(\mathfrak{h})$. Thus $N(\mathfrak{h})/\Gamma$ is a finite group, called the Weyl group $\mathcal{W}$ of the pair $(\mathfrak{g}, \mathfrak{h})$. The space $\mathfrak{h}^+$ is a $\Gamma$-module for the torus $\Gamma$ and thus decomposes into isotypic components. The search for an appropriate natural indexing for such an isotypic component $\mathfrak{v}$ leads to a real linear form $\omega : \mathfrak{h} \rightarrow \mathbb{R}$ and a complex structure $I_\omega : \mathfrak{h}^+ \rightarrow \mathfrak{h}^+$ (that is, a vector space automorphism with $I_\omega^2 = -1$) such that the $\mathfrak{h}$-module structure of $\mathfrak{v}$ is given by

$$[h, x] = \omega(h) \cdot I_\omega(x).$$
We define
\[ g^\omega = \{ x \in g | (\exists I_\omega) I_\omega^2 = -1 \text{ and } (\forall h \in h) [h, x] = \omega(h) \cdot Ix \}. \]

We let \( \Omega \) denote the set of all \( \omega \) for which \( g^\omega \neq \{0\} \) and call these linear forms on \( h \) the real roots of the pair \((g, h)\). We note \( g^0 = h \). Any choice of a closed half space \( E \) in the dual \( \hat{h} \) of \( h \) whose boundary hyperplane meets the finite set \( \Omega \) only in 0 allows us to represent \( \Omega \) as a union \( \Omega = \Omega^+ \cup -\Omega^+ \) with \( \Omega^+ = \Omega \cap E \). We shall call \( \Omega^+ \) a selection of positive roots and find the real roots decomposition

\[ g = h \oplus h^+, \quad h^+ = \sum_{0 \neq \omega \in \Omega^+} g^\omega, \]

of \( g \) with respect to \( h \). The family of complex structures \( I_\omega \) on \( g^\omega \) then, once a selection of positive roots has been made, gives a complex structure \( I \) on \( h^+ \) with which the bracketing of elements from \( h \) with those from any \( g^\omega \) is described by

\[ [h, x] = \omega(h) \cdot Ix \quad \text{for all } x \in g^\omega. \]

At a later point it is important to have available certain special selections of positive roots.

The complex structure \( I \) on \( g^+ \) allows us to define a quadratic function

\[ Q: h^+ \to h, \quad Q(x) = P([Ix, x]). \]

For \( 0 \neq \omega \in \Omega^+ \) and \( x \in g^\omega \) we have

\[ Q(x) = [Ix, x] = -[x, Ix]. \]

Keep in mind that \( Q \) depends on the selection of a set of positive roots via \( I \). Changing such a selection may change \( Q(x) \) by a sign.

4. PROPOSITION [HH86b, HHL87]. If \( g \) accommodates an invariant pointed generating cone and \( h \) is a compactly embedded Cartan algebra, then \( Q(x) = 0 \) and \( x \in g^\omega \) imply \( x = 0 \). \( \square \)

This motivates the following definition.

5. DEFINITION. A Lie algebra \( g \) is said to have cone potential if it has a compact embedded Cartan algebra \( h \) and \( 0 \neq x \in g^\omega \) for any positive real root \( \omega \) implies \( Q(x) \neq 0 \).
The structure of Lie algebras with cone potential is special:

6. THEOREM. Let \( \mathfrak{g} \) be a Lie algebra with cone potential, \( \mathfrak{h} \) a compactly embedded Cartan algebra, \( \mathfrak{r} \) its radical, \( \mathfrak{n} \) is nilradical, \( \mathfrak{z} \) its center. Let \( \Omega^+ \) be any selection of positive real roots with respect to \( \mathfrak{h} \). For any \( \mathfrak{h} \)-submodule \( \mathfrak{v} \) of \( \mathfrak{g} \) we write \( \mathfrak{v}^\omega = \mathfrak{v} \cap \mathfrak{g}^\omega \). Then the following conclusions hold:

(i) \( \mathfrak{z} \) is the center of \( \mathfrak{n} \) and \( \mathfrak{n}/\mathfrak{z} \) is abelian.

(ii) \[
\{ [\mathfrak{n}^\omega, \mathfrak{n}^\omega'] \neq \{0\}, \text{ if } \omega = \omega'; \\
\{ \{0\}, \text{ if } \omega \neq \omega'.
\]

(iii) \( \mathfrak{r}^\omega = \mathfrak{n}^\omega \) for \( 0 \neq \omega \in \Omega^+ \).

(iv) There is a Levi complement \( \mathfrak{s} \) such that \( \mathfrak{z} = \mathfrak{n} \cup \mathfrak{z} \) for \( \omega \in \Omega^+ \).

(v) \( [\mathfrak{z}, \mathfrak{s}] \subseteq \mathfrak{s} \) and \( \mathfrak{h} \cap \mathfrak{s} = (\mathfrak{h} \cap \mathfrak{r}) \oplus \mathfrak{s} \) is a reductive subalgebra.

(vi) \( \mathfrak{g}^\omega = \mathfrak{r}^\omega \oplus \mathfrak{s}^\omega \) for \( \omega \in \Omega^+ \).

However, Lie algebras supporting invariant cones are even more special.

7. PROPOSITION [HH86b]. Let \( W \) be an invariant pointed generating cone in \( \mathfrak{g} \) and let \( \mathfrak{h} \) be a compactly embedded Cartan algebra. Then the center \( \mathfrak{c} \) of the unique maximal compactly embedded subalgebra \( \mathfrak{t}(\mathfrak{h}) \) containing \( \mathfrak{h} \) contains inner points of \( \text{comp} \mathfrak{g} \). Moreover, the centralizer of \( \mathfrak{c} \) in \( \mathfrak{g} \) is \( \mathfrak{t}(\mathfrak{h}) \).

Such phenomena occur in the context of hermitean symmetric spaces inside semisimple Lie algebras. This motivates the following notation:

8. DEFINITION. A Lie algebra \( \mathfrak{g} \) is called quasihermitean if it contains a compactly embedded Cartan algebra \( \mathfrak{h} \) such that the center \( \mathfrak{c} \) of \( \mathfrak{t}(\mathfrak{h}) \) satisfies

\[
\mathfrak{c} \cap \text{int}(\text{comp } \mathfrak{g}) \neq \emptyset.
\]

Recalling that \( \mathfrak{z}(x) = \ker \text{ad } x \) is the centralizer of \( x \) in \( \mathfrak{g} \), one shows that

\[
\mathfrak{c} \cap \text{int}(\text{comp } \mathfrak{g}) = \{ x \in \mathfrak{g} \mid \mathfrak{z}(x) = \mathfrak{t}(\mathfrak{h}) \}.
\]

9. DEFINITION. Let \( \Omega \) be the set of real roots of a quasihermitean Lie algebra \( \mathfrak{g} \) with respect to a compactly embedded Cartan algebra \( \mathfrak{h} \). Then \( \omega \in \Omega \) is said to be a compact root if \( \mathfrak{g}^\omega \subseteq \mathfrak{t}(\mathfrak{h}) \). All other roots are noncompact.

The set of compact roots is denoted \( \Omega^+_k \), the complement is \( \Omega_p \). For any selection of positive roots \( \Omega^+ \) we set \( \Omega^+_k = \Omega^+ \cap \Omega_k \) and \( \Omega^+_p = \Omega^+ \cap \Omega_p \). Finally, we set

\[
\mathfrak{p}(\mathfrak{h}) = \bigoplus_{\omega \in \Omega^+_p} \mathfrak{g}^\omega.
\]

For any choice of an element \( c \in \mathfrak{c} \cap \text{int}(\text{comp } \mathfrak{g}) \) there is a selection \( \Omega^+ \) of positive roots such that \( \omega(c) > 0 \) for all noncompact roots \( \omega \).

10. THEOREM. Let \( \mathfrak{g} \) denote a quasihermitean Lie algebra and fix a compactly embedded Cartan algebra \( \mathfrak{h} \). Let \( \mathfrak{r} \) denote the radical. Then the following
conclusions hold:

(i) \( \mathfrak{t}(\mathfrak{h}) = \mathfrak{h} \oplus \bigoplus_{\omega \in \Omega_+^+} \mathfrak{g}^{\omega} \).

(ii) \( \mathfrak{g} = \mathfrak{t}(\mathfrak{h}) \oplus \mathfrak{p}(\mathfrak{h}) \) and \( [\mathfrak{t}(\mathfrak{h}), \mathfrak{p}(\mathfrak{h})] \subseteq \mathfrak{p}(\mathfrak{h}) \).

(iii) The unique largest ideal of \( \mathfrak{g} \) contained in \( \mathfrak{p}(\mathfrak{h}) \) contains all ideals \( \mathfrak{i} \) with \( \mathfrak{h} \cap \mathfrak{i} = \{0\} \).

(iv) \( r \subseteq \mathfrak{h} \oplus \mathfrak{p}(\mathfrak{h}) \).

(v) Let \( c \in C \cap \text{int} \left( \text{comp} \mathfrak{g} \right) \) and let \( \Omega^+ \) be a selection of positive roots such that \( \omega(c) > 0 \) for all \( \omega \in \Omega_+^+ \). Then, with respect to the complex structure \( I|\mathfrak{p}(\mathfrak{h}) \), the vector space \( \mathfrak{p}(\mathfrak{h}) \) is a complex \( k(\mathfrak{h}) \)-module, i.e., \( [k, I\mathfrak{p}] = I[k, \mathfrak{p}] \).

It is not hard to record some necessary conditions for a pointed generating cone \( C \) in \( \mathfrak{h} \) to be of the form \( W \cap \mathfrak{h} \). The first is immediate from the definitions

\[ (\text{WEYL}) \quad W C = C. \]

A detailed analysis of the orbits of an element \( h \in \mathfrak{h} \) under a one-parameter group of inner automorphisms \( e^{R \cdot \text{ad} x} \) for a root vector \( x \in \mathfrak{g}^{\omega} \) reveals another necessary condition.

For each nonzero real root \( \omega \in \Omega \) we define a function \( Q_\omega : \mathfrak{h} \times \mathfrak{g}^{\omega} \to \mathfrak{h} \) by \( Q_\omega(h, x) = \omega(h) \cdot Q(x) = \omega(h) \cdot [I\omega, x, x] = \omega(h) \cdot [L^\omega x, x] \). While \( I \) and \( Q \) depend on a selection of positive roots, the functions \( Q_\omega \) do not. If \( C = \mathfrak{h} \cap W \) for an invariant pointed generating cone \( W \), then we find \( Q_\omega(C \times \mathfrak{g}^{\omega}) \subseteq C \) for all \( \omega \in \Omega_\mathfrak{p} \).

This condition is equivalent to

\[ (\text{ROOT}) \quad (\text{ad} x)^2 C \subseteq C \quad \text{for all} \quad x \in L^\omega, \quad \omega \in \Omega_\mathfrak{p}. \]

The main result is that the two conditions (WEYL) and (ROOT) are also sufficient for \( C \) to be of the form \( \mathfrak{h} \cap W \).

11. Theorem (The Main Characterisation Theorem). Let \( \mathfrak{g} \) denote a quasihermitean Lie algebra with cone potential, and let \( \mathfrak{h} \) be a compactly embedded Cartan algebra. Let \( C \) be a pointed generating cone in the vector space \( \mathfrak{h} \). Then there exists a unique invariant pointed generating cone \( W \) in \( \mathfrak{g} \) if and only if conditions (WEYL) and (ROOT) are satisfied.

References

[HH85a] J. Hilgert and K. H. Hofmann, Lorentzian cones in real Lie algebras, Monatsh. Math. 100 (1985), 183–210.

[HH85b] ———, Old and new on \( \text{Sl}(2) \), Manuscripta Math. 54 (1985), 17–52.

[HH85c] ———, Lie semialgebras are real phenomena, Math. Ann. 270 (1985), 97–103.

[HH86a] ———, On the automorphism group of cones and wedges, Geom. Dedicata 21 (1986), 205–217.

[HH86b] ———, Compactely embedded Cartan algebras and invariant cones in Lie algebras, THD preprint (1986), Adv. in Math. (to appear).

[HHL87] J. Hilgert, K. H. Hofmann and J. D. Lawson, Lie groups, convex cones, and semigroups, Oxford Univ. Press (to appear).

[KR82] S. Kumaresan and A. Ranjan, On invariant convex cones in simple Lie algebras, Proc. Indian Acad. Sci. Math. 91 (1982), 167–182.

[O81] G. I. Ol’shanskii, Invariant cones in Lie algebras, Lie semigroups and the holomorphic discrete series, Funct. Anal. Appl. 15 (1981), 275–285.
S. Paneitz, *Determination of invariant convex cones in simple Lie algebras*, Ark. Mat. 21 (1984), 217–228.

E. B. Vinberg, *Invariant cones and orderings in Lie groups*, Funct. Anal. Appl. 14 (1980), 1–13.

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