LECTURES ON SUPERCONFORMAL QUANTUM MECHANICS AND MULTI-BLACK HOLE MODULI SPACES

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Abstract. This contribution to the proceedings of the 1999 NATO ASI on Quantum Geometry at Akureyri, Iceland, is based on notes of lectures given by A. Strominger. Topics include $N$-particle conformal quantum mechanics, extended superconformal quantum mechanics and multi-black hole moduli spaces.
1. Introduction

The problem of unifying quantum mechanics and gravity is one of the
great unsolved problems in twentieth century physics. Progress has been
slowed by our inability to carry out relevant physical experiments. Some
progress has nevertheless been possible, largely through the use of gedanken
experiments.

The quantum mechanical black hole has been a key ingredient of these
gedanken experiments, beginning with [1, 2]. It provides an arena in which
quantum mechanics and gravity meet head on. Such gedanken experiments
have led to an astonishing depth and variety of insights, not only about
the black holes themselves, but about string theory and quantum field the-
ory in general. Nevertheless many aspects of quantum black holes remain
enigmatic, and we expect they will continue to be a source of new insights.

Studies of quantum black holes have largely focused on the problem of
quantum fields or strings interacting (by scattering or evaporation) with a
single black hole. In these lectures we will address a different, less studied,
type of gedanken experiment, involving an arbitrary number $N$ of super-
symmetric black holes. Configurations of $N$ static black holes parametrize
a moduli space $\mathcal{M}_N$ [3, 4, 5]. The low-lying quantum states of the system
are governed by quantum mechanics on $\mathcal{M}_N$. As we shall see the problem
of describing these states has a number of interesting and puzzling features.
In particular $\mathcal{M}_N$ has noncompact, infinite-volume regions corresponding
to near-coincident black holes. These regions lead to infrared divergences
and presents a challenge for obtaining a unitary description of multi-black
hole scattering.

The main goal of these lectures is to describe the recent discovery of
a superconformal structure [6, 7, 8, 9] in multi-black hole quantum me-
chanics. While the appearance of scale invariance at low energies follows
simply from dimensional analysis, the appearance of the full conformal in-
variance requires particular values of the various couplings and is not a pri-
ori guaranteed. This structure is relevant both to the infrared divergences
and the scattering, which however remain to be fully understood. We begin
these lectures by developing the subject of conformal and superconformal
quantum mechanics with $N$ particles. Section 2 describes the simplest ex-
ample [10] of single-particle conformally invariant quantum mechanics. The
infrared problems endemic to conformal quantum mechanics as well as their
generic cure are discussed in this context. Section 3 contains a discussion
of conformally invariant $N$-particle quantum mechanics. Superconformal
quantum mechanics is described in section 4. In section 5 the case of a test
particle moving in a black hole geometry is discussed (following [11]) as
a warm-up to the multi-black hole problem. The related issues of confor-
mal invariance, infrared divergences and choices of time coordinate appear and are discussed in this simple context. In section 6 the five dimensional multi-black hole moduli space as well as its supersymmetric structure are described. It is shown that at low energies the supersymmetries are doubled and the $D(2, 1; 0)$ superconformal group makes an appearance. We close with a conjecture in section 7 on the possible relation to an M-brane description of the black hole and $AdS_2/CFT_1$ duality [12].

Many of the results described herein appeared recently in [13, 14].

2. A Simple Example of Conformal Quantum Mechanics

Let us consider the following Hamiltonian [10]:

$$H = \frac{p^2}{2} + \frac{g}{2x^2}. \quad (2.1)$$

In order to have an energy spectrum that is bounded from below, it turns out that we need to take $g \geq -1/4$, but otherwise $g$ is an arbitrary coupling constant, though, following [10], we will consider only $g > 0$. Next introduce the operators

$$D = \frac{1}{2}(px + xp) \quad K = \frac{1}{2}x^2. \quad (2.2)$$

$D$ is known as the generator of dilations — it generates rescalings $x \to \lambda x$ and $p \to p/\lambda$ — and $K$ is the generator of special conformal transformations. These operators obey the $SL(2, \mathbb{R})$ algebra

$$[D, H] = 2iH, \quad (2.3a)$$
$$[D, K] = -2iK, \quad (2.3b)$$
$$[H, K] = -iD. \quad (2.3c)$$

Since $D$ and $K$ do not commute with the Hamiltonian, they do not generate symmetries in the usual sense of relating degenerate states. Rather they can be used to relate states with different eigenvalues of $H$. [6, 7, 8, 9, 10].

**Exercise 1** Show that for any quantum mechanics with operators obeying the $SL(2, \mathbb{R})$ algebra (2.3), that if $|E\rangle$ is a state of energy $E$, then $e^{i\alpha D}|E\rangle$ is a state of energy $e^{2\alpha}E$. Thus, if there is a state of nonzero energy, then the spectrum is continuous.

It follows from exercise 1 that the spectrum of the Hamiltonian (2.1) is continuous, and its eigenstates are not normalizable. Hence it is awkward to describe the theory in terms of $H$ eigenstates.
This problem is easily rectified. Consider the linear combinations

\begin{align}
L_{\pm 1} &= \frac{1}{2}(aH - \frac{K}{a} \mp iD) \\
L_0 &= \frac{1}{2}(aH + \frac{K}{a}),
\end{align}

where \(a\) is a parameter with dimensions of length-squared. These obey the \(SL(2, \mathbb{R})\) algebra in the Virasoro form,

\[ [L_1, L_{-1}] = 2L_0 \quad \quad [L_0, L_{\pm 1}] = \mp L_{\pm 1}. \]

In the following, we choose our units such that \(a = 1\).

With the definitions (2.4b), (2.1) and (2.2), we have

\[ L_0 = \frac{p^2}{4} + \frac{g}{4a^2} + \frac{x^2}{4}. \]

The potential energy part of this operator achieves its minimum and asymptotes to \(\infty\) (see figure 1) and thus has a discrete spectrum with normalizable eigenstates.

**Exercise 2** Show that

\[ L^2 = L_0(L_0 - 1) - L_{-1}L_1 \]

is the \(SL(2, \mathbb{R})\) Casimir operator. Thus show that, of the eigenstates of \(L_0\), that with the smallest value of \(L_0\) is annihilated by \(L_1\). Also show that the eigenvalues of \(L_0\) form an infinite tower above the “ground state”, in integer steps.
Exercise 3 Show that for the DFF model, the Casimir operator (2.7) takes the value
\[ L^2 = \frac{g}{4} - \frac{3}{16} \]  
(2.8)
and thus that the “ground state” has \( L_0 = \frac{1}{2}(1 \pm \sqrt{g + \frac{1}{4}}) \). (It turns out that the positive root is that for which the state is normalizable.)

From exercise 2, we learn that the spectrum of \( L_0 \) is well defined, and thus has normalizable eigenstates. This motivated DFF to trade \( H \) for \( L_0 \), and use \( L_0 \) to generate the dynamics. We then have a well defined theory; this also justifies our use of the term “ground state” in exercises 2 and 3.

At this point it is a free world and one has the right to describe the theory in terms of \( L_0 \) rather than \( H \) eigenstates. Later on this issue will reappear in the context of black hole physics, and the trade of \( H \) for \( L_0 \) will take on a deeper significance.

3. Conformally Invariant N-Particle Quantum Mechanics
In this section, we find the conditions under which a general \( N \)-particle quantum mechanics admits an \( SL(2,\mathbb{R}) \) symmetry. Specifically, we derive the conditions for the existence of operators \( D \) and \( K \) obeying the algebra (2.3). \( N \)-particle quantum mechanics can be described as a sigma model with an \( N \)-dimensional target space. The general Hamiltonian is
\[ H = \frac{1}{2} P_a g^{ab} P_b + V(X), \]  
(3.1)
where \( a, b = 1, \ldots, N \) and the metric \( g \) is a function of \( X \). The canonical momentum \( P_a \) obeys \( [P_a, X^b] = -i \delta_a^b \) and \( [P_a, P_b] = 0 \), and is given by
\[ P_a = g_{ab} \dot{X}^b = -i \partial_a. \]  
(3.2)

Exercise 4 Given the norm \( (f_1, f_2) = \int d^N X \sqrt{g} f_1^* f_2 \), show that
\[ P_a^b = \frac{1}{\sqrt{g}} P_a \sqrt{g} = P_a - i \Gamma_a^b, \]  
(3.3)
where \( \Gamma_{ab}^c \) is the Christoffel symbol built from the metric \( g_{ab} \), and the dagger denotes Hermitian conjugation. Thus, \( H \Psi = (-\nabla^2 + V) \Psi \), for all (scalar) functions \( \Psi(X) \).

\(^1\)In this and all subsequent expressions, the operator ordering is as indicated.
We first determine the conditions under which the theory, defined by equation (3.1), admits a dilational symmetry of the general form

\[ \delta_D X^a = D^a(X). \]  

(3.4)

This symmetry is generated by an operator

\[ D = \frac{1}{2} D^a P_a + \text{h.c.} \]  

(3.5)

which should obey equation (2.3a),

\[ [D, H] = 2iH. \]  

(2.3a)

From the definitions (3.5) and (3.1), one finds

\[ [D, H] = -i \frac{1}{2} P^b_a (\mathcal{L}_D g^{ab}) P_b - i \mathcal{L}_D V - \frac{i}{4} \nabla^2 \nabla_a D^a, \]  

(3.6)

where \( \mathcal{L}_D \) is the usual Lie derivative obeying

\[ \mathcal{L}_D g_{ab} = D^c g_{ab,c} + D^c_a g_{cb} + D^c_b g_{ac}. \]  

(3.7)

Comparing equations (3.7) and (2.3a) reveals that a dilational symmetry exists if and only if there exists a conformal Killing vector \( D \) obeying

\[ \mathcal{L}_D g_{ab} = 2g_{ab} \]  

(3.8a)

and

\[ \mathcal{L}_D V = -2V. \]  

(3.8b)

Note that equation (3.8a) implies the vanishing of the last term of equation (3.6). A vector field \( D \) obeying (3.8a) is known as a homothetic vector field, and the action of \( D \) is known as a homothety (pronounced h’MAWthitee).

Next we look for a special conformal symmetry generated by an operator \( K = K(X) \) obeying equations (2.3b) and (2.3c):

\[ [D, K] = -2iK, \]  

(2.3b)

\[ [H, K] = -iD. \]  

(2.3c)

With equation (3.5), equation (2.3b) is equivalent to

\[ \mathcal{L}_D K = 2K, \]  

(3.9)
while equation (2.3c) can be written
\[ D_\alpha dX^\alpha = dK. \] (3.10)

Hence the one-form \( D \) is exact. One can solve for \( K \) as the norm of \( D^\alpha \),
\[ K = \frac{1}{2} g_{ab} D^a D^b, \] (3.11)
which is globally well defined. We shall adopt the phrase “closed homothety” to refer to a homothety whose associated one-form is closed and exact.

**Exercise 5** Show that conversely, given a vector field \( D^a \) obeying equation (3.8a) and \( dD = 0 \), that \( D_\alpha dX^\alpha = dK \) where \( K \) is defined by equation (3.11). Thus, every “closed homothety” is an “exact homothety”, and there is no significance in our choice of phrase. (We have chosen to use the phrase “closed homothety” in order to avoid confusion with a discussion of, say, quantum corrections.)

**Exercise 6** Show that if a manifold admits a homothety (not necessarily closed), then the manifold is noncompact.

We should emphasize that the existence of \( D \) did not guarantee the existence of \( K \). It is not hard [14] to find examples of quantum mechanics with a \( D \) for which the corresponding unique candidate for \( K \) (by equation (3.11)) obeys neither equations (2.3c) nor (3.10). Indeed a generic homothety is not closed.²

4. **Superconformal Quantum Mechanics**

This section considers supersymmetric quantum mechanics with up to four supersymmetries and superconformal extensions with up to eight supersymmetries. In lower dimensions the Poincaré groups are smaller and hence so are the supergroups. This implies a richer class of supersymmetric structures for a given number of supercharges. In particular, in one dimension we shall encounter structures which cannot be obtained by reduction from higher dimensions.

²One can find even four dimensional theories that are dilationally, but not conformally, invariant by including higher derivative terms; for a scalar field \( \phi(x) \), the Lagrangian \( \mathcal{L} = f \left( \frac{\partial^\mu \phi \partial_\mu \phi}{\phi^4} \right) \phi^4 \) is dilationally invariant for any function \( f \), but it is conformally invariant only for \( f(y) = -\frac{1}{2} y - \Delta \). [15]
4.1. A BRIEF DIVERSION ON SUPERGROUPS

Roughly, a supergroup is a group of matrices that take the block form

\[
\begin{pmatrix}
A & F_1 \\
F_2 & B
\end{pmatrix},
\]

where \(A, B\) are ordinary matrices, and \(F_1, F_2\) are fermionic matrices. We are interested in quantum mechanics with a supersymmetry whose supergroup includes \(SL(2, \mathbb{R})\); that is, supergroups of the form

\[
\begin{pmatrix}
SL(2, \mathbb{R}) & \text{fermionic} \\
\text{fermionic} & \text{R-symmetry}
\end{pmatrix}.
\]

There are many such supergroups; these have been tabulated in table 1.

One simple series of supergroups is the \(Osp(m|n)\) series; the elements of \(Osp(m|n)\) have the form

\[
\begin{pmatrix}
Sp(n) & \text{fermionic} \\
\text{fermionic} & SO(m)
\end{pmatrix}.
\]

Since \(Sp(2) \cong SL(2, \mathbb{R})\) \(^3\) we are interested in \(Osp(m|2)\). The simplest of these is \(Osp(1|2)\), which is a subgroup of the others. We will describe the models with this symmetry group, for the supermultiplet defined in section 4.2, in section 4.3. We will skip \(Osp(2|2) \cong SU(1,1|1)\) \(^4\) — these models were described in [13] — and go directly to \(Osp(4|2)\). In fact, it will turn out that, for the supermultiplet we consider, we will naturally obtain \(D(2,1; \alpha)\) as the symmetry group, where \(\alpha\) is a parameter that depends on the target space geometry. \(Osp(4|2)\) is the special case of \(\alpha = -2\), and appears, for example, when the target space is flat. The black hole system described in section 6 will turn out to have \(D(2,1; 0)\) superconformal symmetry.\(^5\)

\(^3\)The notation is such that only \(Sp(2n)\) exist.

\(^4\)The supergroup \(U(m,n|p)\) is generated by matrices of the form (4.1), with \(A \in U(m,n)\) and \(B \in U(p)\). The subalgebra in which the matrices also obey \(\text{Tr} A = \text{Tr} B\) generates \(SU(m,n|p)\). However, with this definition, \(SU(m,n|p = m + n)\) is not even semisimple, for the identity matrix obeys \(\text{Tr} A = \text{Tr} B\) and generates a \(U(1)\) factor. The quotient \(PSU(m,n|m+n) \cong SU(m,n|m+n)/U(1)\) is simple, and is often denoted just \(SU(m,n|m+n)\), as we have done for \(SU(1,1|2)\).

\(^5\)\(D(2,1; 0)\) (and \(D(2,1; \infty)\)) is omitted from Table 1 because it is the semidirect product \(SU(1,1|2) \rtimes SU(2)\) and is therefore not simple.
| Superalgebra     | Dimension (#b,#f) | R-symmetry |
|-----------------|-------------------|------------|
| $Osp(1|2)$       | (3,2)             | 1          |
| $SU(1,1|1)$     | (4,4)             | $U(1)$     |
| $Osp(3|2)$       | (6,6)             | $SU(2)$    |
| $SU(1,1|2)$     | (6,8)             | $SU(2)$    |
| $D(2,1;\alpha), \alpha \neq -1,0,\infty$ | (9,8) | $SU(2) \times SU(2)$ |
| $Osp(5|2)$       | (13,10)           | $SO(5)$    |
| $SU(1,1|3)$     | (12,12)           | $SU(3) \times U(1)$ |
| $Osp(6|2)$       | (18,12)           | $SO(6)$    |
| $G(3)$          | (17,14)           | $G_2$      |
| $Osp(7|2)$       | (24,14)           | $SO(7)$    |
| $Osp(4^*|4)$     | (16,16)           | $SU(2) \times SO(5)$ |
| $SU(1,1|4)$     | (19,16)           | $SU(4) \times U(1)$ |
| $F(4)$          | (24,16)           | $SO(7)$    |
| $Osp(8|2)$       | (31,16)           | $SO(8)$    |
| $Osp(4^*|2n), n > 2$ | $(2n^2 + n + 6, 8n)$ | $SU(2) \times Sp(2n)$ |
| $SU(1,1|n), n > 4$ | $(n^2 + 3, 4n)$ | $SU(n) \times U(1)$ |
| $Osp(n|2), n > 8$ | $(\frac{1}{2}n^2 - \frac{1}{2}n + 3, 2n)$ | $SO(n)$ |

Table 1. The simple supergroups that contain an $SL(2,\mathbb{R})$ subgroup (see also [16]). The table is divided into those which have eight or fewer (ordinary) supersymmetries (including the exceptional supergroups) and those which have more than eight (ordinary) supersymmetries (for which there are no exceptional supergroups). The algebra of $Osp(4^*|2m)$ has bosonic part $SO^*(4) \times Usp(2m)$, where $SO^*(4) \cong SL(2,\mathbb{R}) \times SU(2)$ is a noncompact form of the $SO(4)$ algebra.

detail, in section 4.4. First, we should describe the supermultiplet under consideration.

4.2. QUANTUM MECHANICAL SUPERMULTIPLETS

There are many supermultiplets that one can construct in one dimension. In particular, unlike in higher dimensions, the smaller supersymmetry group does not require a matching of the numbers of bosonic and fermionic fields. Much of the literature — see, e.g. [17, 18, 19, 20, 21, 22] — concerns the
so-called type A multiplet, with a real boson and complex fermion \((X^a, \psi^a)\), which can be obtained by dimensional reduction of the 1 + 1 dimensional \(\mathcal{N} = (1, 1)\) multiplet.

This is not the multiplet we will consider here. For the black hole physics that we will eventually consider, each black hole will have four bosonic (translational) degrees of freedom, as well as four fermionic degrees of freedom from the breaking of one half of the minimal (8 supercharge) supersymmetry in five dimensions. Thus, we will consider the type B multiplet, consisting of a real boson and a real fermion \((X^a, \lambda^a = \lambda^a \dagger)\). The supersymmetry transformation, parametrized by a real Grassman parameter \(\epsilon\), is given by

\[
\delta_\epsilon X^a = -i\epsilon \lambda^a, \quad \delta_\epsilon \lambda^a = \epsilon \dot{X}^a, \quad (4.4)
\]

where the overdot denotes a time derivative.

**Exercise 7** In an \(\mathcal{N} = 1\) superspace formalism, the type B multiplet is given by a real supermultiplet \(X^a(t, \theta) = X^a(t, \theta)\), where \(\theta\) is the (real) fermionic coordinate, and we use the standard convention in which the lowest component of the superfield is notationally almost indistinguishable from the superfield itself. In components, we write

\[
X^a(t, \theta) = X^a(t) - i\theta \lambda^a(t). \quad (4.5)
\]

The generator of supersymmetry transformations, \(Q\) (which obeys \(Q^2 = H = i\frac{d}{dt}\)) is given by

\[
Q = \frac{\partial}{\partial \theta} + i\theta \frac{d}{dt}. \quad (4.6)
\]

Show that

\[
\delta_\epsilon X^a = [\epsilon Q, X^a], \quad (4.7)
\]

as expected. Note also that \(Q = Q^\dagger\), and thus both sides of equation \((4.7)\) are, indeed, real. For completeness, we define the superderivative

\[
D = \frac{\partial}{\partial \theta} - i\theta \frac{d}{dt}, \quad (4.8)
\]

which obeys \(D^2 = -i\frac{d}{dt}\) and \(\{D, Q\} = 0\).

\(^6\)Recently, four-dimensional black holes have been described using a multiplet with 3 bosons and 4 fermions [23].
As we have already mentioned, there are many more multiplets than just the type B one; see e.g. [24, 23].

4.3. $Osp(1|2)$-INVARIANT QUANTUM MECHANICS

We now proceed to the simplest superconformal quantum mechanics for the Type B supermultiplet defined in the previous subsection. As in section 3, we use a Hamiltonian formalism.

In general, the supercharge takes the form

$$ Q = \lambda^a \Pi_a - \frac{i}{3} c_{abc} \lambda^a \lambda^b \lambda^c, \quad (4.9) $$

where we define

$$ \Pi_a \equiv P_a - \frac{i}{2} \omega_{abc} \lambda^c + \frac{i}{2} c_{abc} \lambda^b \lambda^c \equiv P_a - \frac{i}{2} \Omega^+_{abc} \lambda^b \lambda^c, \quad (4.10) $$

where $\omega_{abc}$ is the spin connection with the last two indices contracted with the vielbein, and $c_{abc}$ is a (so-far) general 3-form. The Hamiltonian is then given by

$$ H = \frac{1}{2} \{ Q, Q \}. \quad (4.11) $$

We remark that the bosonic part of this Hamiltonian is the special case of equation (3.1) with $V = 0$.

**Exercise 8** Show that the most general, renormalizable superspace action [24]

$$ S = i \int dt d\theta \left\{ \frac{1}{2} g_{ab} D X^a \dot{X}^b + \frac{i}{6} c_{abc} D X^a D X^b D X^c \right\}, \quad (4.12) $$

is given in terms of the component fields by

$$ S = \int dt \left\{ \frac{1}{2} g_{ab} \dot{X}^a \dot{X}^b + \frac{i}{2} \lambda^a \left( g_{ab} \frac{D \lambda^b}{dt} - \dot{X}^c c_{abc} \lambda^b \right) - \frac{1}{6} \partial_d c_{abc} \lambda^d \lambda^a \lambda^b \lambda^c \right\}, \quad (4.13) $$

where

$$ \frac{D \lambda^a}{dt} = \dot{\lambda}^a + \dot{X}^b \Gamma^a_{bc} \lambda^c, \quad (4.14) $$

is the covariant time-derivative. (Note that $g_{ab} = g_{(ab)}$ and $c_{abc} = c_{[abc]}$ are arbitrary (though $g_{ab}$ should be positive definite for positivity of the kinetic
energy) functions of the superfield; e.g. \( g_{ab} = g_{ab}(X(t, \theta)) \). In terms of \( \lambda^a \equiv \lambda^a e^\alpha_a \), show that the action (4.13) is

\[
S = \int dt \left\{ \frac{1}{2} g_{ab} \dot{X}^a \dot{X}^b + \frac{i}{2} \delta_{\alpha\beta} \lambda^\alpha \frac{D\lambda^\beta}{dt} - \frac{i}{2} \dot{X}^c \epsilon_{c\alpha\beta} \lambda^\alpha \lambda^\beta - \frac{1}{6} \epsilon^d_\delta \partial_d \epsilon_{abc} \epsilon^{e}_{\alpha} \epsilon^{f}_{\beta} \epsilon^{g}_{\gamma} \lambda^\alpha \lambda^\beta \lambda^\gamma \right\},
\]

(4.15)

where

\[
\frac{D\lambda^\alpha}{dt} = \dot{\lambda}^\alpha + \dot{X}^a \omega^\alpha_{a\beta} \lambda^\beta.
\]

(4.16)

*Finally, show that equation (4.9) follows from equation (4.4) (or (4.7)).*

We note that, from equation (4.15), the fermions \( \lambda^\alpha \) obey the canonical anticommutation relation

\[
\{ \lambda^\alpha, \lambda^\beta \} = \delta^{\alpha\beta},
\]

(4.17)

and commute with \( X^a \) and \( P_a \).\(^7\) It follows from equation (4.17), that the fermions can be represented on the Hilbert space by \( \lambda^\alpha = \gamma^\alpha / \sqrt{2} \), where \( \gamma^\alpha \) are the \( SO(n) \) \gamma-matrices (\( n \) is the dimension of the target space), and that the wavefunction is an \( SO(n) \) spinor. Thus \( \Pi_a \) is just the covariant derivative (with torsion \( c \) — see appendix A for a brief summary of calculus with torsion) on the Hilbert space.\(^8\)

So far we have only discussed \( \mathcal{N} = 1 \) supersymmetric quantum mechanics, whereas we would like to discuss superconformal quantum mechanics. We have already shown in section 3 that in order to have conformal quantum mechanics, the metric \( g_{ab} \) must admit a closed homothety \( D^a \), out of which were built the operators \( D \) and \( K \). The supersymmetric extensions of the expressions (3.5) and (3.11) for \( D \) and \( K \) — that is including fermions — are given by replacing the \( P_a \) in equation (3.5) (which is a covariant derivative on the scalar wavefunction of the bosonic theory) with the covariant derivative \( \Pi_a \):\(^9\)

\[
D = \frac{1}{2} D^a \Pi_a + \text{h.c.}
\]

(4.18)

and

\[
K = \frac{1}{2} D^a D_a.
\]

(4.19)

\(^7\)Note that this implies that (generically) \( \lambda^\alpha \) does not commute with \( P_b \), but rather,

\[
[P_a, \lambda^b] = -i(\omega^b_{a\ c} - \Gamma^b_{a\ c})(\lambda^c).
\]

\(^8\)It also follows [25, 26] that, for these theories, the Witten index, \( \text{Tr}(-1)^F \) [17, 18], is equivalent to the Atiyah-Singer index.

\(^9\)But the reader should not extrapolate too far, for \( H \neq \frac{1}{2} \Pi_a g^{ab} \Pi_b \).
However, it turns out that closure of the superalgebra places two constraints on the torsion $c_{abc}$:

$$D^a c_{abc} = 0$$  \hfill (4.20a)

and

$$L_D c_{abc} = 2c_{abc}.$$  \hfill (4.20b)

The final operator that appears in an $Osp(1|2)$-invariant theory is

$$S = i [Q, K] = \lambda^a D_a.$$  \hfill (4.21)

*Exercise 9* Verify that (with equations (4.20)) the operators $H$, $D$, $K$, $Q$ and $S$, defined by equations (4.11), (4.18), (4.19), (4.9) and (4.21) satisfy the $Osp(1|2)$ algebra

$$[H, K] = i D, \quad [H, D] = -2i H, \quad [K, D] = 2i K,$$

$$\{Q, Q\} = 2H, \quad [Q, D] = -i Q, \quad [Q, K] = -i S,$$

$$\{S, S\} = 2K, \quad [S, D] = i S, \quad [S, H] = i Q,$$

$$\{S, Q\} = D, \quad [Q, H] = 0, \quad [S, K] = 0.$$  \hfill (4.22)

4.4. $D(2,1;\alpha)$–INVARIANT QUANTUM MECHANICS

The $D(2,1;\alpha)$ algebra is an $\mathcal{N} = 4$ (actually $\mathcal{N} = 4B$, since we use the type B supermultiplet) superconformal algebra, and thus contains four supercharges $Q^m$, $m = 1, \ldots, 4$, and their superconformal partners $S^m$. Of course, for fixed $m$, $Q^m$, $S^m$, $H$, $K$, $D$ should satisfy the $OSp(1|2)$ algebra (4.22). In addition, as is evident from table 1, there are two (commuting) sets of $SU(2)$ R-symmetry generators $R^r_\pm$, $r = 1, 2, 3$, under which the supercharges $Q^m$ and $S^m$ transform as $(2,2)$. There are no other generators, and the complete set of (anti)commutation relations, which define the algebra, are \cite{27}

$$[H, K] = i D, \quad [H, D] = -2i H, \quad [K, D] = 2i K,$$

$$\{Q^m, Q^n\} = 2H \delta^{mn}, \quad [Q^m, D] = -i Q^m, \quad [Q^m, K] = -i S^m,$$

$$\{S^m, S^n\} = 2K \delta^{mn}, \quad [S^m, D] = i S^m, \quad [S^m, H] = i Q^m,$$

$$[R^r_\pm, Q^m] = iT^r_{mn}Q^n, \quad [R^r_\pm, S^m] = iT^r_{mn}S^n, \quad [R^r_\pm, R^s_\pm'] = i \delta_{\pm \pm'} \epsilon^{rst} R^t_\pm,$$

$$[R^r_\pm, H] = 0, \quad [R^r_\pm, D] = 0, \quad [R^r_\pm, K] = 0,$$

$$[Q^m, H] = 0, \quad [S^m, K] = 0,$$

$$\{S^m, Q^n\} = D \delta^{mn} - \frac{4\alpha}{1 + \alpha} t^+_m R^r_+ - \frac{4}{1 + \alpha} t^-_m R^r_-,$$  \hfill (4.23)
where
\[ t_{mn}^{\pm} = \mp \delta[m]^{\gamma} \delta[n]^{\gamma} + \frac{1}{2} \epsilon_{mn}. \] (4.24)

Clearly the \( D(2, 1; \alpha) \) algebra is not defined for \( \alpha = -1 \); for \( \alpha = 0 \) (\( \infty \)), \( R_+^2 \) (\( R_-^2 \)) does not appear on the right-hand side of the commutation relations (4.23), and thus the group is the semidirect product of \( SU(1, 1|2) \) \(^{10}\) (the unique group in table 1 with the correct number of generators and bosonic subalgebra) and \( SU(2) \).

Before we discuss the conditions under which the action (4.15) admits a \( D(2, 1; \alpha) \) superconformal symmetry, we should first discuss the conditions for \( N = 4B \) supersymmetry.

4.4.1. \( N = 4B \) Supersymmetric Quantum Mechanics.

The conditions on the geometry for an \( N = 4B \) theory have been given in [24, 28]. We will repeat them in the simplified form given in [13]. The object is to find supercharges \( Q^m \), such that
\[ \{Q^m, Q^n\} = 2 \delta^{mn} H. \] (4.25)

We take \( Q^4 \) to be the \( Q \) of equation (4.9) — i.e. the Noether charge associated with the symmetry generated by equation (4.4). We now look for three more symmetry transformations such that
\[ [\delta^{(m)}_\epsilon, \delta^{(n)}_{\eta}] = -2i \delta_{mn} \eta \epsilon \frac{d}{dt}, \] (4.26)

where \( \delta^{(m)}_\epsilon \) is the \( m^{th} \) supersymmetry transformation, generated by the Grassmann variable \( \epsilon \). It is standard (see e.g. [29, 24, 28]) to give these transformations according to the following rather tedious exercise.

**Exercise 10** Define
\[ \delta^{(r)}_\epsilon X^a(t, \theta) = \epsilon I^a(X) DX^b, \] (4.27)

where \( I^a_b(X(t, \theta)) \) is some tensor-valued function on superspace. Then, show that the supersymmetry algebra (4.26) is obeyed iff \( I^a_b \) are almost complex structures obeying
\[ I^c_r I^b_s + I^c_s I^b_r = -2 \delta^{rs} \delta^b_c, \] (4.28a)

with vanishing Nijenhuis concomitants,
\[ N(r, s)_{ab}^c \equiv \left\{ 2 I^d_r c_d \partial_r I^b_s + 2 I^d_r c_d \partial_r I^b_s \right\} + (r \leftrightarrow s) = 0. \] (4.28b)

\(^{10}\)See footnote 4 (page 8) for the definition of \( SU(m, n|p) \).
From exercise 10, we learn that supersymmetry requires a complex target space, with three anticommuting complex structures. $N = 4B$ supersymmetry is defined to have a quaternionic target space,

$$I^{rc}_a I^{sb}_c = -\delta^{rs}_a \delta^b_a + \epsilon^{rst}_a I^{tb}_a. \quad (4.29)$$

This provides a natural $SU(2)$ structure, which will give rise to self-dual rotations in the black hole context. Given such a manifold, it is convenient to define the three exterior derivatives $d^r$ by

$$d^r \omega = (-1)^{p+1} I^r dI^r \omega; I^r \omega \equiv \frac{(-1)^p}{p!} I^r_{a_1} \cdots I^r_{a_p} \omega_{b_1 \cdots b_p} dX^{a_1} \wedge \cdots \wedge dX^{a_p},$$

$$= \frac{1}{p!} \left[ I^r_a \partial_a \omega_{c_1 \cdots c_p} - p(\partial_a I^r_{c_1}) \omega_{b_2 \cdots c_p} \right] dX^a \wedge dX^{c_1} \wedge \cdots \wedge dX^{c_p},$$

$$d^r = i(\partial - \bar{\partial}). \quad (4.30)$$

where $\omega = \frac{1}{p!} \omega_{a_1 \cdots a_p} dX^{a_1} \wedge \cdots \wedge dX^{a_p}$ is a $p$-form.

**Exercise 11** Show that, in complex coordinates adapted to $I^r_a$, $d^r = i(\partial - \bar{\partial})$.

Having defined the supersymmetry transformations using the quaternionic structure of the manifold, we should now check that the action is invariant. We will simply quote the result [13]. The action is invariant provided that

$$\{d^r, d^s\} = 0, \quad (4.31a)$$

$$I^r_a I^{sb}_c = -\delta^{rs}_a \delta^b_a + \epsilon^{rst}_a I^{tb}_a, \quad (4.31b)$$

$$g_{ab} = I^r_c g_{cd} I^r_d (\forall r), \quad (4.31c)$$

$$c = \frac{1}{6} c_{abc} dX^a \wedge dX^b \wedge dX^c = \frac{1}{2} d^r J^r (\forall r); J^r \equiv \frac{1}{2} I^r_a g_{cd} dX^a \wedge dX^b. \quad (4.31d)$$

Equation (4.31a) is secretly a restatement of equation (4.28b) and equation (4.31b) was our demand (4.29). The new conditions are equations (4.31c) and (4.31d). Equation (4.31c) states that the metric is Hermitian with respect to each complex structure. Equation (4.31d) is a highly nontrivial differential constraint between the complex structures, which generalizes the hyperkähler condition, and from which the torsion $c_{abc}$ is uniquely determined. It is equivalent to the condition that the quaternionic structure be covariantly constant:

$$\nabla^+_a I^r_c \equiv \nabla_a I^r_c + c^c \partial_d I^r_d - c^d a I^r_d = 0. \quad (4.32)$$

$^{11}$For the more general case, with a Clifford, but not a quaternionic, structure see [24, 28, 30].
A manifold that satisfies the conditions (4.31) is known as a hyperkähler with torsion (HKT) (or sometimes weak HKT) manifold.

4.4.2. \( \mathcal{N} = 4B \) Superconformal Quantum Mechanics

We must now find the further restrictions to a \( D(2,1;\alpha) \)-invariant superconformal quantum mechanics. Clearly, this will include equations (3.8a), (3.10) and (4.20). The additional restrictions are obtained by demanding the proper behaviour of the R-symmetries, and are most easily phrased by defining the vector fields

\[
D^{rb} \equiv D^a_I r^b_a. \tag{4.33}
\]

\( D(2,1;\alpha) \)-invariance then forces these to be Killing vectors

\[
\mathcal{L}_{D^r} g_{ab} = 0, \tag{4.34}
\]

which also obey the \( SU(2) \) algebra for some normalization

\[
[\mathcal{L}_{D^r}, \mathcal{L}_{D^s}] = -\frac{2}{\alpha + 1} \epsilon^{rst} \mathcal{L}_{D^t}. \tag{4.35}
\]

Equation (4.35) gives a geometric definition of \( \alpha \). Note that because the normalization of \( D^a \) is specified by equations (3.8a) and (4.33), \( \alpha \) is unambiguous. In fact, equation (4.35) is not a sufficiently strong condition for the proper closure of the algebra; we must have

\[
\mathcal{L}_{D^r} I^s_{a b} = -\frac{2}{\alpha + 1} \epsilon^{rst} I^b_{a}. \tag{4.36}
\]

Equation (4.36) implies equation (4.35).

These are the necessary and sufficient conditions for the quantum mechanics defined by (4.13) to be \( D(2,1;\alpha) \) superconformal. They imply that

\[
\alpha J^r = (\alpha + 1)(d^r dK - \frac{1}{2} \epsilon^{rst} d^s d^t K); \tag{4.37}
\]

i.e. that (at least for \( \alpha \neq 0 \)) the HKT metric is described by a potential which is proportional to \( K \). Actually, as discussed in more detail in [13], when the three complex structures are \textit{simultaneously} integrable, there is always a potential, but a general HKT manifold admits a potential only under the conditions given in [31].

A general (but not most general!) set of models can be obtained from a function \( L(X) \), where \( X^a \) are coordinates on \( \mathbb{R}^{4n} \), and the \( I^r_a \) are given by the self-dual complex structures on \( \mathbb{R}^4 \) tensored with the \( n \)-dimensional
identity matrix.\textsuperscript{12} If (but not iff — in particular, this is not true of the system described in section 6.3) $L(X)$ also obeys
\begin{equation}
X^a \partial_a L(X) = h L(X), \quad X^a \Gamma^b_a \partial_b L(X) = 0,
\end{equation}
then we obtain a $D(2,1;\alpha = -\frac{h+2}{2})$-invariant model, with
\begin{align}
g_{ab} &= \left( \delta_a^c \delta_b^d + \Gamma_a^c \Gamma_b^d \right) \partial_c \partial_d L(X), \\
D^a &= \frac{2}{h} X^a, \\
K &= \frac{h+2}{2h} L,
\end{align}
and $c_{abc}$ given by equation (4.31d). In an $\mathcal{N} = 2$ superspace formalism, $X^a$ is a superfield obeying certain constraints [29, 33] and the potential $L$ is the superspace integrand [13, 32, 30].

5. The Quantum Mechanics of a Test Particle in a Reissner-Nordström Background

Our goal is to apply the results on superconformal quantum mechanics to the quantum mechanics of a collection of supersymmetric black holes. As a warm-up in this section we consider the problem of a quantum test particle moving in the black hole geometry. The four-dimensional case was treated in [11], which will be followed and adapted to five dimensions in this section.

Consider a five-dimensional extremal Reissner-Nordström black hole of charge $Q$. The geometry of such a black hole is described by the metric
\begin{equation}
 ds^2 = -\frac{dt^2}{\psi^2} + \psi d\vec{x}^2, 
\end{equation}
and the gauge field
\begin{equation}
 A = \psi^{-1} dt, 
\end{equation}
where $\vec{x}$ is the $\mathbb{R}^4$ coordinate, and $\psi = 1 + \frac{Q}{|\vec{x}|}$. We have set $M_p = L_p = 1$.

The horizon in these coordinates is at $|\vec{x}| = 0$.

Introduce a test particle with mass $m$ and charge $q$. The particle action is
\begin{equation}
 S = -m \int d\tau + q \int A. 
\end{equation}

\textsuperscript{12}This implies that the three complex structures are simultaneously integrable. Hellerman and Polchinski [32] have recently shown how to relax this limitation by generalizing the $\mathcal{N} = 2$ superfield constraints of [29, 33].
Parametrize the particle’s trajectory as $\vec{x} = \vec{x}(t)$. Eventually we will require the test particle to be supersymmetric (by imposing $q = m$). A supersymmetric test particle at rest at a fixed distance from the black hole, remains at rest, so it is sensible to consider a test particle that moves slowly. Accordingly we shall assume $|\vec{x}| \ll 1$. In this parametrization, we can make the following substitution:

$$d\vec{x} = \dot{\vec{x}} dt,$$

which allows us to rewrite (5.1) to obtain the metric

$$ds^2 = -\frac{dt^2}{\psi^2} + \psi |\dot{\vec{x}}|^2 dt^2.$$

Now we solve the equation $ds^2 = -d\tau^2$ and find

$$d\tau = \frac{dt}{\psi} - \frac{1}{2} \psi^2 |\dot{\vec{x}}|^2 dt + O(\dot{x}^4),$$

which is substituted into (5.2) to obtain the action,

$$S = -m \int \left( \frac{dt}{\psi} - \frac{1}{2} \psi^2 |\dot{\vec{x}}|^2 dt \right) + q \int \frac{dt}{\psi}.$$  

(5.5)

For a supersymmetric test particle, $m = q$, this action reduces to

$$S = \frac{m}{2} \int \psi^2 |\dot{\vec{x}}|^2 dt.$$  

(5.6)

If the particle is near the horizon, at distances $r \ll \sqrt{Q}$, then we can approximate $\psi = \frac{Q}{|x|^2}$, so that

$$S_p = \frac{mQ^2}{2} \int dt \frac{|\dot{\vec{x}}|^2}{|\vec{x}|^4},$$

(5.8)

or, if we define a new quantity $\vec{y} = \frac{\vec{x}}{|\vec{x}|^2}$, then we see that we are actually in flat space:

$$S_p = \frac{mQ^2}{2} \int dt |\dot{\vec{y}}|^2.$$  

(5.9)

Far from the black hole, spacetime and the moduli space look flat once again. Thus the moduli space can be described as two asymptotically flat regions connected by a wormhole whose radius scales as $\sqrt{Q}$. At low energies (relative to $M_p/\sqrt{Q}$) the wavefunctions spread out and do not fit into
the wormhole. Hence the quantum mechanics is described by near and far superselection sectors that decouple completely at low energies.

This geometry leads to a problem. Consider the near horizon quantum theory. Given any fixed energy level \( E \), there are infinitely many states of energy less than \( E \). This suggests that there are infinitely many states of a test particle localized near the horizon of a black hole, which appears problematic for black hole thermodynamics. The possibility of such states arises from the large redshift factors near the horizon of a black hole. Similar problems have been encountered in studies of ordinary quantum fields in a black hole geometry.

The new observation of \[11\] is that this problem is in fact equivalent to the problem encountered by DFF \[10\] in their analysis of conformal quantum mechanics. To see this equivalence let \( \rho \) denote the radial coordinate \( |\vec{y}| \). The Hamiltonian corresponding to (5.9) is

\[
H = \frac{1}{2mQ^2}(p_\rho^2 + \frac{4}{\rho^2}J^2).
\]

(5.10)

This is the DFF Hamiltonian of (2.1) with \( g = \frac{4}{mQ^2}J^2 \). The coordinate \( \rho \) grows infinite at the horizon. Thus this potential pushes a particle to the horizon whenever \( J^2 \) is nonzero. Our problem of infinitely many states at low energies is just the problem discussed by DFF.

Applying the DFF trick, as discussed in section 2, provides the solution to this problem. We work in terms of \( H + K \) rather than \( H \), since the former has a discrete spectrum of normalizable eigenstates. There is an \( SL(2, \mathbb{R}) \) symmetry generated by \( H, D \) and \( K \), where \( D \) and \( K \) are defined to be

\[
D = \frac{1}{2}(\rho p_\rho + p_\rho \rho);
\]

(5.11a)

\[
K = \frac{1}{2}mQ^2\rho^2.
\]

(5.11b)

These generators satisfy equations (2.3).

The appearance of the \( SL(2, \mathbb{R}) \) symmetry was not an accident. It arises from the geometry of our spacetime. Near the horizon, we find that

\[
ds^2 \to -\frac{r^4}{Q^2}dt^2 + \frac{Q}{r^2}dr^2 + Qd\Omega_3^2.
\]

(5.12)

We recognize this metric as that of \( AdS_2 \times S^3 \). Introduce new coordinates \( t^\pm = t \pm \frac{Q}{4\pi^2} \) on \( AdS_2 \). Now the metric can be written in the form

\[
ds_2^2 = -\frac{Qdt^+dt^-}{(t^+ - t^-)^2}.
\]

(5.13)
The $SL(2, \mathbb{R})$ isometry generators are then
\[
    h = \frac{\partial}{\partial t^+} + \frac{\partial}{\partial t^-}, \quad (5.14a)
\]
\[
    d = t^+ \frac{\partial}{\partial t^+} + t^- \frac{\partial}{\partial t^-}, \quad (5.14b)
\]
\[
    k = (t^+)^2 \frac{\partial}{\partial t^+} + (t^-)^2 \frac{\partial}{\partial t^-}. \quad (5.14c)
\]

Here $h$ shifts the time coordinate, and $d$ rescales all coordinates.

The $SL(2, \mathbb{R})$ symmetry of the near-horizon particle action reflects the $SL(2, \mathbb{R})$ isometry group of the near-horizon $AdS_2$ geometry. As pointed out in [11, 34], the trick of DFF to replace $H$ by $H + K$ has a nice interpretation in $AdS_2$. To understand it, we must first review $AdS_2$ geometry.

INTERLUDE: $AdS_2$ GEOMETRY

On $AdS_2$, introduce global coordinates $u^\pm$ defined in terms of the coordinates of (5.13) by the relation
\[
    t^\pm = \tan u^\pm. \quad (5.15)
\]

Then the $AdS_2$ metric takes the form
\[
    ds^2 = -\frac{Q}{4} \frac{du^+ du^-}{\sin^2(u^+ - u^-)}. \quad (5.16)
\]

In these coordinates, the global time generator is
\[
    h + k = \frac{\partial}{\partial u^+} + \frac{\partial}{\partial u^-}. \quad (5.17)
\]

In figure 2 it is seen that the time coordinate conjugate to $h$ is not a good global time coordinate on $AdS_2$, but the time coordinate conjugate to $h + k$ is. In fact, the generators $h$ and $d$ preserve the horizon, while $h + k$ preserves the boundary $u^+ = u^- + \pi$ (the right boundary in figure 2).

So in conclusion the DFF trick has a beautiful geometric interpretation in the black hole context. It is simply a coordinate transformation to “good” coordinates on $AdS_2$.

6. Quantum Mechanics on the Black Hole Moduli Space

6.1. THE BLACK HOLE MODULI SPACE METRIC

In this section we will consider five-dimensional $\mathcal{N} = 1$ supergravity with a single $U(1)$ charge coupled to the graviphoton and no vector multiplets.\(^{13}\)

\(^{13}\)Adding neutral hypermultiplets would not affect the discussion, since they decouple. Since these lectures were given, the case with additional vector multiplets was solved
We will use units with $M_p = L_p = 1$. The action is

$$S = \int d^5x \sqrt{g} \left[ R - \frac{3}{4} F^2 \right] + \frac{1}{2} \int A \wedge F \wedge F + \text{fermions}. \quad (6.1)$$

We can also get this from M-theory compactified on a Calabi-Yau with $b_2 = 1$ (the simplest example of such a threefold is the quintic). The black holes are then M2-branes wrapping Calabi-Yau two-cycles.

This system has a solution describing $N$ static extremal black holes

$$ds^2 = -\psi^{-2} dt^2 + \psi d\vec{x}^2, \quad (6.2a)$$

$$A = \psi^{-1} dt, \quad (6.2b)$$

(cf. equation (5.1)) where $\psi$ is the harmonic function on $\mathbb{R}^4$

$$\psi = 1 + \sum_{A=1}^{N} \frac{Q_A}{|\vec{x} - \vec{x}_A|^2}, \quad (6.2c)$$

and $\vec{x}_A$ is the $\mathbb{R}^4$ coordinate of the $A^{th}$ black hole, whose charge is $Q_A$. Another picture of these holes is M2-branes wrapping Calabi-Yau cycles. The space of solutions is called the moduli space, which is parametrized by in [35], and the four-dimensional case was solved in [23]. The supersymmetry of cases with more than eight initial supersymmetries [36, 37] has not been worked out.
the $4N$ collective coordinates $\vec{x}_A$. The slow motion of such black holes is governed by the moduli space metric $G_{AB}$, so that the low energy effective action takes the form

$$ S = \frac{1}{2} \int dt \dot{\vec{x}}^A \dot{\vec{x}}^B G_{AB}. \quad (6.3) $$

Note that due to the no-force condition there is no potential term in the action, and since $|\dot{\vec{x}}_A| \ll 1$, the higher order corrections can be neglected.

The first calculation of the moduli space metric of the four-dimensional Reissner-Nordström black holes was performed in [3, 4] and was generalized to dilaton black holes in [38]. The metric on the moduli space for the five-dimensional black holes (6.2) was derived in [14]. In order to find this metric, one starts with the following ansatz describing the linear order perturbation of the black hole solution (6.2)

$$ ds^2 = -\psi^{-2} dt^2 + \psi d\vec{x}^2 + 2\psi^{-2} \vec{R} \cdot d\vec{x} dt, \quad (6.4a) $$

$$ A = \psi^{-1} dt + (\vec{P} - \psi^{-1} \vec{R}) \cdot d\vec{x}, \quad (6.4b) $$

where $\vec{P}$ and $\vec{R}$ are quantities that are first order in velocities. In equation (6.2c), $\vec{x}_A$ is replaced with $\vec{x}_A + \vec{v}_A t$. This is the most general Galilean-invariant ansatz to linear order. Then (roughly) one uses the equations of motion to solve for $\vec{P}$ and $\vec{R}$. Inserting this into the five-dimensional supergravity action gives the following result [14] for the action:

$$ S = \frac{1}{2} \int dt \dot{\vec{x}}^A \dot{\vec{x}}^B G_{AB} = \frac{1}{4} \int dt \dot{\vec{x}}^A \dot{\vec{x}}^B (\delta^i_k \delta^j_l + I^r_i I^s_j) \partial A_i \partial B_j L, \quad (6.5) $$

where

$$ L = -\int d^4 x \psi \bar{\psi}, \quad (6.6) $$

with

$$ \psi = 1 + \sum_{A=1}^N \frac{Q_A}{|\vec{x} - \vec{x}_A|^2}, \quad (6.7) $$

and $I^r$ is a triplet of self-dual complex structures on $\mathbb{R}^4$ obeying equation (4.29). This Lagrangian has $\mathcal{N} = 4$ supersymmetry when Hermitian fermions $\lambda^{Ai} = \lambda^{Ai\dagger}$ are added.

6.2. THE NEAR-HORIZON LIMIT

6.2.1. Spacetime geometry

Taking the near-horizon limit of (6.2a) corresponds to neglecting the constant term in (6.2c). In figure 3 we have illustrated the resultant spatial
geometry at a moment of fixed time for three black holes. Before the limit is taken (figure 3a), the geometry has an asymptotically flat region at large $|\vec{x}|$. Near the limit (figure 3b), as the origin is approached along a spatial trajectory, a single “throat” approximating that of a charge $\sum Q_A$ black hole is encountered. This throat region is an $AdS_2 \times S^3$ geometry with radii of order $\sqrt{\sum Q_A}$. As one moves deeper inside the throat towards the horizon, the throat branches into smaller throats, each of which has smaller charge and correspondingly smaller radii. Eventually there are $N$ branches with charge $Q_A$. At the end of each of these branches is an event horizon. When the limit is achieved (figure 3c), the asymptotically flat region moves off to infinity. Only the charge $\sum Q_A$ “trunk” and the many branches remain.

6.2.2. Moduli space geometry

It is also interesting to consider the near-horizon limit of the moduli space geometry. The metric is again given by (6.5), where one should neglect the constant term in the harmonic function (6.7). This is illustrated in figure 4 for the case of two black holes. Near the limit there is an asymptotically flat $\mathbb{R}^{4N}$ region corresponding to all $N$ black holes being widely separated. This is connected to the near-horizon region where the black holes are strongly interacting, by tubelike regions which become longer and thinner as the limit is approached. When the limit is achieved, the near-horizon region is severed from the tubes and the asymptotically flat region.
6.3. CONFORMAL SYMMETRY

The near-horizon quantum mechanics has an \( SL(2, \mathbb{R}) \) conformal symmetry. The dilations \( D \) and special conformal transformations \( K \) are generated by

\[
D = -\frac{1}{2} (x^{Ai} P_{Ai} + \text{h.c.}), \tag{6.8}
\]

\[
K = 6\pi^2 \sum_{A \neq B}^N \frac{Q_A^2 Q_B}{|\vec{x}_A - \vec{x}_B|^2}. \tag{6.9}
\]

By splitting the potential \( L \) appearing in the metric (6.5) into pieces representing the 1-body, 2-body and 3-body interactions, one can show [14] that the conditions (4.34) and (4.36) are satisfied. Thus the \( SL(2, \mathbb{R}) \) symmetry can be extended to the full \( D(2, 1; 0) \) superconformal symmetry as was described in section 4.4. This group is the special case of the \( D(2, 1; \alpha) \) superconformal groups for which there is an \( SU(1, 1|2) \) subgroup (in fact, \( D(2, 1; 0) \cong SU(1, 1|2) \rtimes SU(2) \)), in agreement with [39].

So we have seen that there are noncompact regions of the near-horizon moduli space corresponding to coincident black holes. These regions are eliminated by the potential \( K \) in the modified Hamiltonian \( L_0 = \frac{1}{2} (H + K) \), which is singular at the boundary of the noncompact regions. \( L_0 \) has a well defined spectrum with discrete eigenstates. A detailed description of the quantum states of this system remains to be found [40].

7. Discussion

Let us recapitulate. We have found that at low energies the quantum mechanics of \( N \) black holes divides into superselection sectors. One sector
describes the dynamics of widely separated, non-interacting black holes. The other “near horizon” sector describes highly redshifted, near-coincident black holes and has an enhanced superconformal symmetry. Since they completely decouple from widely separated black holes, states of the near horizon theory are multi-black hole bound states.

It is instructive to compare this to an M-theoretic description of these black holes. In Calabi-Yau compactification of M theory to five dimensions, the black holes are described by M2-branes multiply wrapped around holomorphic cycles of the Calabi-Yau. In principle all the black hole microstates are described by quantum mechanics on the M2-brane moduli space, which at low energies should be the dual CFT$_1$ living on the boundary of AdS$_2$ [12]. In practice so far this problem has not been tractable. This moduli space has what could be called (in a slight abuse of terminology) a Higgs branch and a Coulomb branch. This Higgs branch is a sigma model whose target is the moduli space of a single multiply wrapped M2-brane worldvolume in the Calabi-Yau. In the Coulomb branch the M2-brane has fragmented into multiple pieces, and the branch is parametrized by the M2-brane locations. At finite energy the Coulomb branch connects to the Higgs branch at singular points where the M2-brane worldvolume degenerates.

At first one might think that the considerations of this paper correspond to the Coulomb branch, since the multi-black hole moduli space is parametrized by the black hole locations. However it is not so simple. The fact that the near horizon sector decouples from the sector describing non-interacting black holes strongly suggests that it is joined to the Higgs branch. Indeed in the D1/D5 black hole, there is a similar near-horizon region of the Coulomb branch which is not only joined to but is in fact a dual description of the singular regions of the Higgs branch [41, 42, 43, 44, 45]. We conjecture there is a similar story here: the near-horizon, multi-black hole quantum mechanics is dual to at least part of the Higgs branch of multiply wrapped M2-branes. Near-horizon microstates should therefore account for at least some of the internal black hole microstates. Exactly how much of the black hole microstructure is accounted for in this way remains to be understood.

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A. Differential Geometry with Torsion

In this appendix, we give a brief summary of differential calculus with torsion, for the reader who is frustrated by the usual absence of such a discussion in most general relativity books. Recall [47] that the covariant derivative of a tensor is given in terms of the (not necessarily symmetric) connection $C_{c}^{\alpha\beta}$. The torsion $c_{ab}^{\alpha}$ is just the antisymmetric part of the connection:

$$c_{ab}^{\alpha} \equiv C_{[ab]}^{\alpha} = \frac{1}{2}(C_{ab}^{\alpha} - C_{ba}^{\alpha}).$$

Either by direct computation, or by recalling that the difference between two connections is a tensor, one finds that the torsion is a true tensor. Of course, the torsion does contribute to the curvature tensor, and we remind the reader that many of the familiar symmetries of the curvature tensor are not obeyed in the presence of torsion. Also, if the symmetric part of the connection is given by the Levi-Civita connection, then the full connection annihilates the metric iff the fully covariant torsion tensor $c_{abc} = g_{ad}C_{bdc}^{d}$ is completely antisymmetric.

Hopefully, the preceding paragraph was familiar. We now discuss the torsion in a tangent space formalism. As usual, the first step is to define the vielbein $e_{a}^{\alpha}$, which is a basis of cotangent space vectors, labelled by $\alpha = 1, \ldots, n$, where $n$ is the dimension of the manifold, obeying

$$\delta_{\alpha\beta}e_{a}^{\alpha}e_{b}^{\beta} = g_{ab}.$$

The vielbein $e_{a}^{\alpha}$, and the inverse vielbein $e_{a}^{\alpha}$ which obeys

$$e_{a}^{\alpha}e_{b}^{\beta} = \delta_{\alpha}^{\beta},$$

can then be used to map tensors into the tangent space; e.g. $V^{\alpha} \equiv V^{\alpha}e_{a}^{\alpha}$.

The connection one-form $\Omega_{a}^{\alpha\beta}$ is defined by demanding that the vielbein is covariantly constant:

$$\nabla_{a}e_{b}^{\alpha} \equiv \partial_{a}e_{b}^{\alpha} + \Omega_{a}^{\alpha\beta}e_{b}^{\beta} - C_{ab}^{c}e_{c}^{\alpha} = 0.$$

Note that equation (A.4) is valid for any choice of connection, and does not imply that the metric is covariantly constant. The metric is covariantly constant iff $\delta_{\alpha\beta}$ is covariantly constant, which in turn holds iff the connection one-form $\Omega_{aa\beta}$ is antisymmetric in the tangent space indices, where

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14 One excellent reference for physicists is [46].
we have lowered the middle index using the tangent space metric $\delta_{a\beta}$. In other words, the familiar antisymmetry of the connection one-form \[ \alpha \beta \] exists if and only if the metric is covariantly constant, whether or not there is torsion.

Equation (A.4) is easily solved for the connection one-form, giving
\[ \Omega_a^{\alpha \beta} = e_b^{\alpha} \partial_a e_{\beta}^{b} + C_{ab}^{c} e_c^{\alpha} e_{\beta}^{b}. \] (A.5)

An immediate corollary of this, and the fact that the difference of two connections $C_{ab}^{c}$ and $C_{ab}^{c}$ is a tensor, is that the difference between two connection one-forms is a tensor, and is, in fact, the same tensor as $C_{ab}^{c} = C_{ab}^{c} - C_{ab}^{c}$, but with the $b$ and $c$ indices lifted to the tangent bundle.\(^\text{15}\)

The unique torsion-free connection one-form which annihilates the metric (i.e. that obtained from equation (A.4) using the Levi-Civita connection) is known as the spin connection, and is usually denoted $\omega_a^{\alpha \beta}$. Given a completely antisymmetric torsion $e_{abc} = e_{[abc]}$, as in the first paragraph of this appendix, we define the connection one-form
\[ \Omega_a^{+\alpha \beta} = \omega_a^{\alpha \beta} + c_{a\beta}, \] (A.6)

where, of course, any required mapping between the tangent bundle and the spacetime is achieved by contracting with the vielbein.

As usual, spinors $\psi$ are defined on the tangent bundle, and their covariant derivative is given by
\[ \nabla_a \psi = \partial_a \psi - \frac{1}{4} \Omega_{a\alpha\beta} \gamma^{\alpha \beta} \psi, \] (A.7)

where $\gamma^{\alpha \beta} \equiv \frac{1}{2} [\gamma^\alpha, \gamma^\beta]$ is a commutator of $SO(n)$ $\gamma$-matrices, which satisfy $\{ \gamma^\alpha, \gamma^\beta \} = 2\delta^\alpha\beta$.

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\(^\text{15}\)In this discussion, we are assuming that a vielbein has been chosen once and for all; we do not consider the effect of changing frames or coordinates.
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