ON THE JONES POLYNOMIAL MODULO PRIMES

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Abstract. We derive an upper bound on the density of Jones polynomials of knots modulo a prime number $p$, within a sufficiently large degree range: $4/p^7$. As an application, we classify knot Jones polynomials modulo two of span up to eight.

1. Introduction

Describing the set of Jones polynomials of all knots is a difficult problem. In this note, we take a tiny step towards classifying Jones polynomials of knots with coefficients reduced modulo a prime number $p$.

Theorem 1. For all $a, b \in \mathbb{Z}$ with $b - a \geq 7$, the set of Laurent polynomials with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ within the degree range from $a$ to $b$, that are realised as Jones polynomials of knots, has density at most $4/p^7$.

As we will see, the bound $4/p^7$ is sharp in the special case $p = 2$, any $a \in \mathbb{Z}$, and $b = a + 8$.

Corollary 1. For all $a \in \mathbb{Z}$, there are exactly 16 Jones polynomials of knots modulo two with minimal degree $a$ and maximal degree $\leq a + 8$. All these Laurent polynomials are realised by finite connected sums of 54 prime knots with crossing number 12 or less.

As Jones observed in his famous publication [4], for any knot $K$, the difference between the Jones polynomial $V_K(t)$ and 1 is divisible by $(t^3 - 1)(t - 1)$. The proof of Theorem 1 rests on the following refined statement, which does not seem to appear in the literature so far.

Theorem 2. Let $h(t) = (t^3 - 1)(t - 1)(t^2 + 1)$ and

$$f(t) = (t^2 - t + 1)h(t) = t^8 - 2t^7 + 3t^6 - 4t^5 + 4t^4 - 4t^3 + 3t^2 - 2t + 1.$$  

For all knots $K$, there exists a unique polynomial $p(t) \in \mathbb{Z}[t]$ of degree at most seven, belonging to one of the four families below, so that $V_K(t) - p(t)$ is divisible by $f(t)$:

(i) $1 + nh(t)$,
(ii) $V_{31}(t) + nh(t)(2t - 1)$,
(iii) $V_{51}(t) + nh(t)$,
(iv) $V_{821}(t) + nh(t)(2t - 1)$.

All these families are parametrised by an integer $n$ satisfying $2n = \pm 1 \pm 3^l$. The symbols $3_1, 5_1, 8_{21}$ refer to knots according to Rolfsen’s notation [7].
The membership of a given knot $K$ to one of these families, as well as the value $n \in \mathbb{Z}$, is determined by the pair of values $V_K(i), V_K(\zeta_6)$. The explicit Jones polynomials appearing in Theorem 2 are

$$V_{3_1}(t) = -t^4 + t^3 + t,$$
$$V_{5_1}(t) = -t^7 + t^6 - t^5 + t^4 + t^2,$$
$$V_{8_{21}}(t) = t^7 - 2t^6 + 2t^5 - 3t^4 + 3t^3 - 2t^2 + 2t.$$

At this point, the reader might already guess that the first theorem is an easy consequence of the second. We will derive Theorems 1 and 2 in Sections 3 and 2, respectively. The corollary relies on the following curious fact: there exists a knot - 12n237 in knotinfo notation [5] - whose Jones polynomial is $t^{12}$, modulo two. This is explained in the fourth and last section.

2. Listing potential Jones polynomials

The Jones polynomial $V_K(t) \in \mathbb{Z}[t^\pm 1]$ of a knot $K \subset S^3$ satisfies the following restrictions in the roots of unity $1, i, \zeta_3, \zeta_6$:

1. $V_K(1) = 1$,
2. $V'_K(1) = 0$,
3. $V_K(\zeta_3) = 1$,
4. $V_K(i) = \pm 1$,
5. $V_K(\zeta_6) = \pm (\sqrt{-3})^m$.

The exponent $m$ in condition (5) coincides with the rank of the first homology of the double branched cover $M_2(K)$ with coefficients in $\mathbb{Z}/3\mathbb{Z}$, and can also be interpreted as the dimension of the 3-coloring invariant of $K$, as described by Przytycki [6]. The sign in condition (4) is determined by the Arf invariant of $K$: $V_K(i) = (-1)^{\text{Arf}(K)}$. The first four conditions were already derived by Jones [4]. In terms of Vassiliev invariants, the first two conditions reflect the fact that knots admit no non-constant finite type invariants of order zero and one [1]. Interestingly, this implies that no monomial other than 1 is the Jones polynomial of a knot [3]. Here is a remarkable consequence of the first three conditions together: $V_K(t) - 1$ is divisible by $(t - 1)^2(t^2 + t + 1) = (t^3 - 1)(t - 1)$.

Even better, suppose $p(t) \in \mathbb{Z}[t^\pm 1]$ admits the same values as $V_K(t)$, for $t = 1, i, \zeta_3, \zeta_6$, and satisfies $p'(1) = 0$. Then the difference $V_K(t) - p(t)$ is divisible by the product of cyclotomic polynomials

$$f(t) = (t - 1)^2(t^2 + t + 1)(t^2 - t + 1)$$
$$= t^8 - 2t^7 + 3t^6 - 4t^5 + 4t^4 - 4t^3 + 3t^2 - 2t + 1.$$

Therefore, all we need in order to derive Theorem 2 is finding a suitable set of reference polynomials $p(t)$, with $p'(1) = 0$, covering all the possible values of knot Jones polynomials at $t = 1, i, \zeta_3, \zeta_6$. This is easy enough.

First, we observe that all the four families of polynomials listed in Theorem 2 satisfy $p(1) = 1, p'(1) = 0, and p(\zeta_3) = 1$. Here we use the fact that
h(t) = (t^3 - 1)(t - 1)(t^2 + 1) has a double root at t = 1, and a single root at t = \zeta_6.

Next, we observe that all the polynomials of families (i) and (iv) listed in Theorem 2 satisfy p(i) = 1, and all the polynomials of families (ii) and (iii) satisfy p(i) = -1. Here we use that h(t) also has a single root at t = i.

Last, we take care of the value p(\zeta_6), which should cover all the complex numbers of the form ±(\sqrt{-3})m. The values of

\[ p(t) = 1, V_{31}(t), V_{53}(t), V_{821}(t) \]

at t = \zeta_6 are 1, \sqrt{3i}, -1, \sqrt{3i}, respectively. Furthermore, we have h(\zeta_6) = 2 and \( h(\zeta_6)(2\zeta_6 - 1) = 2\sqrt{3}i \). This implies that the polynomials of families (i) and (iii) cover all the odd integers at t = \zeta_6, while the polynomials of families (ii) and (iv) cover all the odd multiples of \sqrt{3}i at t = \zeta_6. Altogether, the four families listed in Theorem 2 cover all the possible combinations of values of knot Jones polynomial at \( t = 1, i, \zeta_3, \zeta_6 \), including the double root at \( t = 1 \). This finishes the proof of Theorem 2.

3. Jones polynomial modulo primes

The goal of this section is to derive Theorem 1 by reducing Theorem 2 modulo a prime number \( p \). We use the notation \( f(t) \in \mathbb{F}_p[t^{\pm 1}] \) for the reduction of \( f(t) \in \mathbb{Z}[t^{\pm 1}] \) modulo \( p \). Theorem 2 remains valid modulo \( p \), with the additional feature that the parameter \( n \) is in \( \mathbb{F}_p \). From this, we deduce that the number of Jones polynomials of knots modulo \( p \) in the degree range \([0, 7]\) is at most \( 4p \). This is in accordance with the ratio \( 4/p^7 \), since there are exactly \( p^8 \) polynomials modulo \( p \) in the degree range \([0, 7]\). We will refer to these \( 4p \) potential Jones polynomials as reference polynomials \( f_1, f_2, \ldots, f_{4p} \in \mathbb{F}_p[t^{\pm 1}] \).

Now suppose we are given a degree range \([a, b] \) with \( b - a \geq 7 \) and a knot \( K \) with Jones polynomial \( V_K(t) \) in that degree range. By Theorem 2, there exists a reference polynomial \( f_i \), so that \( V_K(t) - f_i \) is divisible by

\[ \bar{f}(t) = t^8 - 2t^7 + 3t^6 - 4t^5 + 4t^4 - 4t^3 + 3t^2 - 2t + 1 \in \mathbb{F}_p[t^{\pm 1}] \]

Denote the minimal and maximal degree of \( V_K(t) - f_i \) by \( \alpha \) and \( \beta \), respectively. Then there exist unique coefficients

\[ c_\alpha, c_{\alpha+1}, \ldots, c_{\beta-8} \in \mathbb{F}_p, \]

satisfying the following equation:

\[ V_K(t) - f_i = \bar{f}(t)(c_\alpha t^\alpha + c_{\alpha+1} t^{\alpha+1} + \ldots + c_{\beta-8} t^{\beta-8}) \]

The polynomial \( V_K(t) \) is therefore determined by \( \beta - \alpha - 7 \) parameters in \( \mathbb{F}_p \). However, since \( V_K(t) \) is in the degree range \([a, b] \), all the coefficients \( c_\gamma \) with \( \gamma \not\in [a, b-8] \) are determined by \( f_i \) alone. In other words, only the coefficients \( c_a, c_{a+1}, \ldots, c_{b-8} \) change if we vary \( V_K(t) \) in the given degree range. Since there are \( 4p \) reference polynomials \( f_i \), this allows for a maximum of \( 4p \) times \( p^{ b-a - 7 } \) potential Jones polynomials, out of a total of \( p^{b-a+1} \) polynomials.
with coefficients in \( \mathbb{F}_p \) in the degree range \([a, b]\). The resulting ratio is again \( 4/p^7 \), as claimed.

For odd primes \( p \geq 5 \), the bound \( 4/p^7 \) is never sharp, since the parameter \( n \) appearing in Theorem 2, case (i), satisfies \( 1 + 2n = \pm 3^l \). In particular, \( 2n \) cannot be \(-1 \pmod{p} \), since \( 3^l \) cannot be zero \( \pmod{p} \). The knot table at our disposition (knotinfo, up to 12 crossings [5]), is too small to draw any conclusion about the sharpness of the bound \( 4/p^7 \) for \( p = 3 \). This leaves us with the case \( p = 2 \), which is most interesting and deserves its own section.

4. Jones polynomial modulo two

The list of \( 4p \) potential Jones polynomials in the degree range \([0, 7]\), called reference polynomials in the previous section, boils down to eight polynomials for \( p = 2 \). These are in fact realised by the following knots: the trivial knot \( O \), \( 3_1 \), \( 5_1 \), \( 5_2 \), \( 8_21 \), \( 9_43 \), \( 10_{140} \), \( 10_{160} \). The corresponding Jones polynomials (mod 2) are

\[
\begin{align*}
1 & , t + t^3 + t^4 , t^2 + t^4 + t^5 + t^6 + t^7 , t + t^2 + t^4 + t^5 + t^6 , \\
& t^3 + t^4 + t^7 , 1 + t + t^7 , 1 + t + t^2 + t^3 + t^5 + t^6 + t^7 , 1 + t^2 + t^3 + t^5 + t^6 .
\end{align*}
\]

In order to prove Corollary 1, we need to find 16 knot Jones polynomials in the degree range \([a, a+8]\), for all \( a \in \mathbb{Z} \), which appears rather difficult. Luckily, a single knot comes at our rescue: \( 12n237 \).

As mentioned above, no monomial other than 1 is the Jones polynomial of a knot. Indeed, no polynomial of the form \( p(t) = at^n \), except 1, satisfies \( p(1) = 1 \) and \( p'(1) = 0 \). In contrast, the Jones polynomial of the knot \( 12n237 \) is a non-trivial monomial modulo 2:

\[ V_{12n237}(t) = t^{12} \pmod{2}. \]

**Remark.** The connected sum of the knot \( 12n237 \) with its mirror image has trivial Jones polynomial modulo 2. The existence of non-trivial knots with that property, even prime ones, was known before [2]. Likewise, for odd primes \( p \), the monomial \( t^{12p} \) is a potential Jones polynomial modulo \( p \), since \( t^{12p} - 1 \) is divisible by \( f(t) = (t^2 - t + 1)(t^3 - 1)(t - 1)(t^2 + 1) \) in \( \mathbb{F}_p[t^{\pm 1}] \). We do not know whether \( t^{12p} \) (modulo \( p \)) is the Jones polynomial of an actual knot.

Back to \( p = 2 \), suppose we find 16 Jones polynomials in a fixed degree range \([a, a+8]\), realised by the knots \( K_1, K_2, \ldots, K_{16} \). Then, by adding \( k \) copies of the knot \( 12n237 \) to the knots \( K_i \), we obtain 16 Jones polynomials in the degree range \([a + 12k, a + 12k + 8]\). This also works for negative integers \( k \), by adding \( |k| \) copies of the mirror image of the knot \( 12n237 \) to the \( K_i \). Hence, in order to cover all degree ranges, it is sufficient to consider the cases \(-9 \leq a \leq 2 \). In fact, it is even enough to consider the cases \(-4 \leq a \leq 2 \), by the symmetry \( V_K(t) = V_{K^*}(t^{-1}) \) between the Jones polynomial of a knot \( K \) and its mirror image \( K^* \). Based on Rolfsen’s table [7] and knotinfo [5], we found 53 prime knots, plus the trivial knot \( O \), which
provide 16 Jones polynomials in all degree ranges of the form \([a, a + 8]\),
a \(\in \{-4, -3, -2, -1, 0, 1, 2\}\). These knots include all knots with crossing
number \(\leq 8\), except the knots \(8_9, 8_{13}, 8_{16}, 8_{18}\) (whose Jones polynomials
modulo 2 coincide with the ones of \(4_1\#4_1, 8_4, 8_{10}, 8_{12}\), in this order), as well
as the following knots:

\[9_{42}, 9_{43}, 9_{44}, 10_{124}, 10_{126}, 10_{127}, 10_{128}, 10_{133}, 10_{136}, 10_{140}, 10_{143}, 10_{145},
10_{146}, 10_{147}, 10_{160}, 10_{163}, 10_{165}, 11n63, 11n99, 11n118, 11n173, 11n18, 41\#8_{21}\]

The table below indicates the degree range of their corresponding Jones
polynomials modulo 2. Our convention here is chosen so that \(K\) has higher
maximal degree than \(K^\ast\). By taking suitable connected sums of these knots,
together with the knot 12n237 (making it a total of 54 prime knots), and
all their mirror images, we find 16 Jones polynomials in every degree range
of the form \([a, a + 8]\), as stated in Corollary \([\text{I}]\). We do not know to what
extent the latter can be generalised. For example, we found 64 knot Jones
polynomials modulo two in the degree ranges \([-5, 5]\) and \([0, 10]\), all realised
by knots with 12 or fewer crossings.

We invite the reader to answer the following concluding questions.

**Question 1.** Let \(p\) be an odd prime. Is there a knot \(K \subset S^3\) with

\[\hat{V}_K(t) = t^{12p} \pmod{p}\]?

**Question 2.** Does every degree range \([a, b]\) with \(b - a \geq 7\) contain \(2^{b-a-4}\)
Jones polynomials modulo 2, as predicted by Theorem \([\text{I}]\)?

**Question 3.** Is every Laurent polynomial \(p(t) \in \mathbb{Z}[t^{\pm 1}]\) satisfying conditions
(1)-(5) the Jones polynomial of a knot?
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