THE \( p \)-RANK OF THE REDUCTION mod \( p \) OF JACOBIANS AND JACOBI SUMS

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Abstract. Let \( Y_K \rightarrow X_K \) be a ramified cyclic covering of curves, where \( K \) is a cyclotomic field. In this work we study the \( p \)-rank of the reduction mod \( p \) of a model of the Jacobian of \( Y_K \). In this way, we obtain counterparts of the Deuring polynomial, defined for elliptic curves, for genus greater than one. To carry out this study we use the relationship between Jacobi sums and \( L \)-functions. This is established in [W] for the case of Fermat curves.

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1. INTRODUCTION AND PREVIOUS NOTATION

Previous notation: Let \( m \) be a prime integer and \( K := \mathbb{Q}(\epsilon_m) \) the cyclotomic field with \( \epsilon_m \) an \( m \)-primitive root of the unity, and let \( t \) be an integer prime to \( m \). We consider the automorphism \( \sigma_t \) of \( K \) defined by \( \sigma_t(\epsilon_m) := \epsilon_m^t \). Let \( X_K \) be a proper and geometrically irreducible curve over \( K \), of genus \( g \). Let \( x_0, \cdots, x_d \) be different points within \( X_K \).
Now let us consider $Y_K \to X_K$, a Galois ramified covering, of Galois group $G := \mathbb{Z}/m$, ramified at $x_0, \cdots, x_d$. Let $\Sigma_{Y_K}, \Sigma_{X_K}$ be the function fields of $Y_K$ and $X_K$, respectively. We have that $\Sigma_{Y_K} = \Sigma_{X_K}(\sqrt{f})$, where $f \in \Sigma_{X_K}$. We assume that $f$ is not a $m$-power and thus that $Y_K$ is connected. Let $\text{div}(f)$ be the principal divisor associated with $f \in \Sigma_{X_K}$. By Kummer theory we can choose $f$ such that $\text{div}(f) = a_0 \cdot x_0 + \cdots + a_d \cdot x_d + m \cdot D$, with $0 < a_i < m$ and $D$ is a divisor on $X_K$. Note that $a_0 + \cdots + a_d = 0 \mod m$.

We choose $n$, a positive integer, such that $X_K$ has good reduction, $X$, over $\text{Spec}(A)$ with $A := \mathbb{Z}[\epsilon_m, \frac{1}{m-n}]$ and that the points $x_0, \cdots, x_d$ do not coalesce mod $p$, for each $p \in \text{Spec}(A)$. With these conditions there exists a model $Y \to X$ over $\text{Spec}(A)$ for $Y_K \to X_K$, c.f. [Be].

We denote by $X_p$ the reduction at $p \in \text{Spec}(A)$ of $X$ and by $Y_p$ a proper, smooth model for the reduction at $p \in \text{Spec}(A)$ of $Y$. We denote by $\Sigma_{Y_p}, \Sigma_{X_p}$ the function fields of $Y_p$ and $X_p$, respectively. Let us denote $k(p)$ the residual field of $p$, which we assume to have $q = p^h$ elements. We also assume that $Y_p \to X_p$ is a ramified Galois covering of group $G$, ramified at $\bar{x}_0, \cdots, \bar{x}_d$, the reduction of $x_0, \cdots, x_d$ at $p$ and that $\Sigma_{Y_p} = \Sigma_{X_p}(\sqrt{f})$ where $\text{div}(f) = a_0 \cdot \bar{x}_0 + \cdots + a_d \cdot \bar{x}_d + m \cdot \bar{D}$, with $\bar{D}$ a divisor on $X_p$. We denote $T := \{\bar{x}_0, \cdots, \bar{x}_d\}$.

We fix an algebraic closure $F$ of $F_p$. If $Z$ is a variety over $\mathbb{F}_p$ we denote $\overline{Z} := Z \otimes_{\mathbb{F}_p} F$. In this article, we study the characteristic polynomials of the $p^h$-Frobenius morphism, $F_p$, of certain $\mathbb{Q}_l[\epsilon_m]$-modules associated with the $l$-adic cohomology group $H^1(Y_p, \mathbb{Q}_l)$. These polynomials are $L$-functions of $X_p$.

The first part of this article is devoted to proving that the constant term of these polynomials is given by Jacobi sums. This has already been proved in [W] for $g = 0, d = 2$ and in general in [D], bearing in mind that these terms are essentially the constant functions of certain
Dirichlet $L$-series. In this work, we use geometric methods to achieve these results.

The second part of this article addresses the $p$-rank of $\text{Pic}_0^0 Y_p$. We say that an abelian variety, $A$, defined over $\mathbb{F}_p$, is supersingular when $A$ is isogenous to a product of supersingular elliptic curves. Equivalently, $A$ is supersingular when the eigenvalues of the Frobenius morphism $F_p$ are $\zeta \cdot q^{1/2}$, $\zeta$ being a root of the unity.

Let us denote by $[\frac{a}{b}]$ the integer part of the fraction $\frac{a}{b}$ and $< \frac{a}{b} > := \frac{a}{b} - [\frac{a}{b}]$. By using properties of Jacobi sums we prove that if $p$ splits completely in $\mathbb{Z}[\epsilon_m, \frac{1}{m}]$ and there exists $t \in \{1, \cdots, m - 1\}$ with

$$[< \frac{a_1}{m} > + \cdots + < \frac{a_d}{m} >] \neq [< \frac{t \cdot a_1}{m} > + \cdots + < \frac{t \cdot a_d}{m} >].$$

Then the Jacobian of $Y_p$ is not a supersingular abelian variety.

Let $[p]$ be the multiplication by $p$ on $\text{Pic}_0^0 Y_p$. We say that $Y_p$ has $p$-rank 0 when $\ker[p]_{\text{red}} = \text{Spec}(k(p))$, or equivalently $\ker[p](\mathcal{F}) = \{0\}$. We prove for $X_p = \mathbb{P}^1$ that if $t \in \{1, \cdots, m - 1\}$ with

$$[< \frac{t \cdot a_1}{m} > + \cdots + < \frac{t \cdot a_d}{m} >] = 0,$

then $Y_p$ does not have $p$-rank 0.

Let $p$ be inert in $\mathbb{Z}[\epsilon_m, \frac{1}{m}]$ and let us consider the integers $a_1, \cdots, a_d$ satisfying $0 < a_i < m$ for each $1 \leq i \leq d - 2$, and $a_1 + \cdots + a_d \neq 0 \mod m$. We prove that the proper, smooth model of the curve

$$y^m - x^{a_1}(x - 1)^{a_2}(x - \alpha_1)^{a_3} \cdots (x - \alpha_{d-2})^{a_d},$$

defined on $k(p) = \mathbb{F}_{p^m - 1}$, has $p$-rank 0 if and only if $a_1, \cdots, a_{d-2}$ satisfy a system of $d-2$ algebraic equations of degrees $\leq l \cdot (a_3 + \cdots + a_d)(\frac{m^{m-1}-1}{m-1})$ with $1 \leq l \leq d - 2$, respectively. In [Bo, 5] the generalized Hasse-witt matrix it is obtained explicitly, and hence the equations to study the $p$-rank of $\text{Pic}_0^0 Y_p$.

When $g = 0, d = 3, a_1 = a_2 = a_3 = 1$ and $m = 2$, one obtains the Deuring polynomial whose roots are the values such that the elliptic
curve \( y^2 - x(x - 1)(x - \lambda) \) is supersingular,
\[
H(\lambda) = \sum_{i=0}^{n} \binom{r}{i}^2 \lambda^i, \text{ (with } r = (p - 1)/2). 
\]

We also study when \( Y_p \) has \( p \)-rank 0 in the case of \( p \) splitting completely in \( \mathbb{Z}[\epsilon_m, \frac{1}{m-n}] \).

In [Bo] the Hasse-Witt invariants of \( Y_p \) are studied and calculated. These invariants give an upper bound for the \( p \)-rank of \( Y_p \). It is showed that for \( p \) large and \( \pi_0, \ldots, \pi_d \) generic points, this upper bound is equal to the \( p \)-rank. In [E], for \( X_p = \mathbb{P}^1 \), the author gives bounds for the \( p \)-rank of \( Y_p \). In [G] the Hasse-Witt matrix is calculated to show that for the Fermat curves there exists a set of primes \( p \) with positive density such that the Fermat curves (mod \( p \)) are not supersingular but their \( p \)-rank is 0. The dimension of the moduli of hyperelliptic curves with fixed \( p \)-rank is obtained in [GIP]. For Fermat curves in [N] the supersingularity of these curves is studied. In [Y] The \( p \)-rank of a hyperelliptic curve is given by the rank of the Cartier-Manin matrix.

2. Cyclic extensions

We now give some general notation that we shall use along this work. Let \( s \) be a global section of a line bundle \( L \) on \( X_p \). We also denote by \( s \) the morphism, of \( \mathcal{O}_{X_p} \)-modules, \( \mathcal{O}_{X_p} \to L \), such that \( 1 \to s \). Here, \( \mathcal{O}_{X_p} \) denotes the sheaf of rings associated with \( X_p \).

We denote by \( E \) and \( m \) the effective divisor \( \pi_0 + \cdots + \pi_d \) on \( X_p \) and the ideal associated with \( E \) inside \( \mathcal{O}_{X_p} \), respectively.

If \( z \in k(p)^\times = \mathbb{F}_p^\times \), then we denote by \( \chi_p(z) \) the unique \( m \)-root of the unity such that \( \chi_p(z) = z^{\frac{p^m-1}{m}} \mod p \).

Let \( L \) be a line bundle over \( X_p \) and \( \iota_m : L \to \mathcal{O}_{X_p}/m \) a surjective morphism of \( \mathcal{O}_{X_p} \)-modules. Let \( \text{Pic}^0_{X_p,m} \) be the generalized Jacobian for \( m \). We have that \( \text{Pic}^0_{X_p,m} \) is a scheme over \( k(p) \) that represents
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isomorphism classes of pairs $(L, \iota_m)$ $(m$-level structures). We say that two level structures $(L, \iota_m)$ and $(L', \iota'_m)$ are equivalent when there exists an isomorphism of line bundles $u : L \to L'$ such that $\iota'_m \cdot u = \iota_m$.

Let us consider $\Sigma^m_{X_p} := \{ g \in \Sigma_{X_p} \text{ such that } g \equiv 1 \text{ mod } m \}$. The equivalence classes of level structures are in one-to-one correspondence with the $m$-equivalence classes of divisors on $X_p$ supported outside $T$; two divisors $D$ and $D'$ are $m$-equivalent when there exists a $g \in \Sigma^m_{X_p}$ with $D - D' = \text{div}(g)$.

Let $\pi$ be the natural epimorphism $\mathcal{O}_{X_p} \to \mathcal{O}_{X_p}/m$. We shall use the term (projective) space of $m$-sections of a level structure $(L, \iota_m)$, to the subspace $H^0_m((L, \iota_m)) \subset H^0(X_p, L)$, of global sections of $L$, $s : \mathcal{O}_{X_p} \to L$ such that $\iota_m \cdot s = \pi$. The effective $m$-equivalent divisors associated with $(L, \iota_m)$ are given by the zero locus of the $m$-sections $s$; Let us consider inclusions $L, L' \subset \Sigma_{X_p}$. We have that $(L', \iota'_m)$ is equivalent to $(L, \iota_m)$ if and only if there exists a $g \in \Sigma^m_{X_p}$ verifying $g \cdot L' = L$. Because the difference between two $m$-sections of $(L, \iota_m)$ is a global section of $L(-E)$, we have that if $s \in H^0_m((L, \iota_m))$, then $H^0_m((L, \iota_m)) = s + H^0(X_p, L(-E))$.

We denote by $I_T$, $I^0_T$ and $O^\times_T$ the ideles, ideles of degree 0 and integer ideles on $\Sigma_{X_p}$ outside $T$, respectively.

We consider the $p^h$-Frobenius morphism, $F_p$, and the Lang isogeny $P := F_p - Id : \text{Pic}^0_{X_p,m} \to \text{Pic}^0_{X_p,m}$. Bearing in mind a divisor of degree 1 supported on $|X_p| \setminus T$, $D_1$, we have, by translation, an immersion $X_p \setminus \text{div}(g)$.
Therefore, the cyclic extension $\Sigma_{Y_p}/\Sigma_{X_p}$ of Galois group $\mathbb{Z}/m$, where $\Sigma_{Y_p} = \Sigma_{X_p}(\sqrt[m]{f})$, gives an epimorphism of groups
\[
\frac{I^0_T}{(\Sigma_{X_p}^m)^\times \cdot O^\times_T} \to \mathbb{Z}/m.
\]
This morphism is given by the Artin map \([S\ VI, n29, Théorème 9]\)
\[
\Omega : \frac{I^0_T}{(\Sigma_{X_p}^m)^\times \cdot O^\times_T} \to \mathbb{Z}/m.
\]
If $\nu_1$ is the idele class of degree 1 associated with the divisor $D_1$, then $(\nu_1, \Sigma_{Y_p}/\Sigma_{X_p}) = 1$.

By noting that the residual fields $k(\mathfrak{p}_i), (i = 0, \ldots, d)$, are isomorphic to $k(\mathfrak{p})$, the morphism of forgetting the level structure, $(L, \iota_m) \to L$, gives the exact sequence of schemes in groups
\[
1 \to \mathbb{G}_m \times \cdots \times \mathbb{G}_m/\mathbb{G}_m \to \text{Pic}^0_{X_p, m} \to \text{Pic}^0_{X_p} \to 1
\]
and thus we have an injective morphism of groups,
\[
\eta : k(\mathfrak{p}_0)^\times \times \cdots \times k(\mathfrak{p}_d)^\times /k(\mathfrak{p})^\times \to \frac{I^0_T}{(\Sigma_{X_p}^m)^\times \cdot O^\times_T}.
\]

If we consider $\sigma \in \frac{I^0_T}{(\Sigma_{X_p}^m)^\times \cdot O^\times_T}$, then via the Artin symbol $\sigma(\sqrt[m]{f}) = z \cdot (\sqrt[m]{f})$, with $z \in k(\mathfrak{p})^\times = \mathbb{F}_p^\times$. We denote by $\chi_f$ the character of $G$ defined by $\chi_f(\sigma) := \chi_p(z)$.

For $(z_0, \ldots, z_d) \in k(\mathfrak{p}_0)^\times \times \cdots \times k(\mathfrak{p}_d)^\times /k(\mathfrak{p})^\times$ we have that
\[
\chi_f(\eta(z_0, \ldots, z_d), \Sigma_{Y_p}/\Sigma_{X_p}) = \chi_p^{-a_0}(z_0) \cdots \chi_p^{-a_d}(z_d).
\]

Note that we have an isomorphism
\[
k(\mathfrak{p}_0)^\times \times \cdots \times k(\mathfrak{p}_d)^\times /k(\mathfrak{p})^\times \simeq k(\mathfrak{p}_1)^\times \times \cdots \times k(\mathfrak{p}_d)^\times,
\]
and therefore we can assume that $z_0 = 1$.

3. L-functions

This section is devoted to studying certain incomplete $L$-functions of the curve $X_p$ over $k(\mathfrak{p})$, we follow \([A]\) and \([T]\). The action of $G$ over
$H^1(\mathcal{Y}_p, \mathbb{Q}_l)$ gives a decomposition into eigenspaces

$$H^1(\mathcal{Y}_p, \mathbb{Q}_l) = \bigoplus \chi H^1(\mathcal{Y}_p, \mathbb{Q}_l)^\chi.$$  

The sum is over all characters of $G := \mathbb{Z}/m$. In this section we shall calculate the characteristic polynomial of $F_p$ as an endomorphism of the $\mathbb{Q}_l[\epsilon_m]$-module $H^1(\mathcal{Y}_p, \mathbb{Q}_l)^\chi$.

Let $F_x$ be the Frobenius element, with $x \in |X_p| \setminus T$. We consider it as an element of the Galois group, $G_m := \frac{I_0}{(\Sigma X_p)^{\times} \cdot \mathbb{Z}_m}$, of the ray class field $H_m/\Sigma X_p$. Let $t_x$ be a local parameter for $x$. We have that $F_x = (t_x, H_m/\Sigma X_p)$, where $(\cdot, H_m/\Sigma X_p)$ is the Artin symbol for the Galois extension $H_m/\Sigma X_p$. We now consider the $T$-incomplete $L$-function

$$\theta_{H_m/\Sigma X_p, \mathcal{T}}(t) := \prod_{x \in |X_p| \setminus T} \left( 1 - F_x \cdot t^{\deg(x)} \right)^{-1}.$$  

Let $N$ be a divisor supported outside $T$ and with class $[N] \in G_m$. We denote by $\sigma_N$ the element of the Galois group of the extension $H_m/\Sigma X_p$ associated with $N$ via the Artin symbol.

Let denote us by $L_m(N, i)$ the cardinal of the set of effective divisors on $X_p$ supported outside $T$ and $m$-equivalents to $N + i \cdot D_1$.

Let $D = n_1 \cdot y_1 + \cdots + n_r \cdot y_r$ be a divisor on $X_p$ with support outside $T$, with $t_{y_i}$ local parameters for $y_i$, $(i = 1, \cdots, r)$. We define

$$(D, H_m/\Sigma X_p) := \prod_{i=1}^r (t_{y_i}, H_m/\Sigma X_p)^{n_i}$$

Similar to [A] 4.1.1, we can compute this $L$-function in terms of $G_m$:

$$\theta_{H_m/\Sigma X_p, \mathcal{T}}(t) = \sum_{[N] \in G_m} \sigma_N \cdot \left( \sum_{i=0}^{2g+d-1} L_m(N, i) \cdot t^i + \sum_{j \geq 0} q^{g+j} \cdot t^{2g+d+j} \right) = \sum_{i=0}^{2g+d-1} \sum_{[N] \in G_m} (L_m(N, i) \cdot \sigma_N) \cdot t^i + \left( \sum_{[N] \in G_m} \sigma_N \right) \left( \frac{q^9 \cdot t^{2g+d}}{1 - q^9} \right) = \sum_{i=0}^{2g+d-1} \sum_{D, \deg(D) = i} (D, H_m/\Sigma X_p) \cdot t^i + \left( \sum_{[N] \in G_m} \sigma_N \right) \left( \frac{q^9 \cdot t^{2g+d}}{1 - q^9} \right),$$
where the sum is over all effective divisors \( D \) on \( X_p \) with support outside \( T \).

Note that if \( (L, \iota_m) \) is a level structure associated with \( N + i \cdot D_1 \) then \( L_m(N, i) = \#H^0_m((L, \iota_m)) \). Thus, either \( L_m(N, i) = 0 \) or \( L_m(N, i) = \#H^0(X_p, N + i \cdot D_1 - E) \). The divisor \( D_1 \) is defined in section 2.

Let \( y \) be a geometric point of \( X_p \). We denote by \( f(y) \) the value of \( f \) within the residual field \( \kappa(y) \subset \mathbb{F} \).

Let \( D = n_1 \cdot y_1 + \cdots + n_r \cdot y_r \) be a divisor on \( X_p \), we define
\[
f(D) := \prod_{i=1}^r f(y_i)^{n_i} \frac{\theta^h \deg(y_i) - 1}{\phi - 1},
\]
If \( x \in \text{sup}(\text{div}(f)) \setminus T \) then we define \( f(x) \) by considering a divisor \( D' \) linearly \( m \)-equivalent to \( x \) and \( \text{sup}(D') \cap \text{sup}(\text{div}(f)) = \emptyset \).

Let \( P_{F_p}^{\chi_f}(t) := \det(t - F_p) \in \mathbb{Z}[[t]] \) be the characteristic polynomial of the endomorphism \( F_p \) over the \( \mathbb{Q}_l(\epsilon_m) \)-vector space \( H^1(Y_p, \mathbb{Q}_l)^{\chi_f} \).

By [T, 3.5], we have that this polynomial is equal to
\[
(*) \chi_f(t^{2g+d-1}, \theta_{H_m/S_{X_p}}(\frac{1}{t})) = \sum_{i=0}^{2g+d-1} \sum_{[N] \in G_m} L_m(N, i) \cdot \chi_f(\sigma_N) \cdot t^{2g+d-1-i} \\
= \sum_{i=0}^{2g+d-1} \sum_{D, \deg(D) = i} \chi_p(f(D)) t^{2g+d-1-i},
\]
where \( \chi_f \) is defined in section 2 and the sum is over all effective divisors \( D \) on \( X_p \) with support outside \( T \).

If \( p_0(t) \) is the characteristic polynomial of \( F_p \) as an endomorphism of the \( \mathbb{Q}_l \)-vector space \( H^1(X_p, \mathbb{Q}_l) \), then the characteristic polynomial of \( F_p \), where \( F_p \) is considered as an endomorphism of the \( \mathbb{Q}_l \)-vector space \( H^1(Y_p, \mathbb{Q}_l) \), is
\[
p_0(t) \cdot \prod_{1 \leq j < m} P_{F_p}^{\chi_f}(t) \in \mathbb{Z}[t].
\]

Note that the action of \( G \) commutes with the action of the Frobenius morphism over the variety \( \text{Pic}^0_{Y_p} \), in this way, the Frobenius morphism
stabilizes the eigen spaces, \( H^1(\mathcal{Y}_p, \mathbb{Q}_l)^{\chi_f} \). In the following Remark we set the connection of the two following sections with [D].

**Remark 3.1.** The \( L \)-function, \( L(\chi_f, t) = \chi_f(\theta_{H_m/\Sigma X_p}, r(t)) \), has a functional equation

\[
L(\chi_f, t) = \epsilon(\chi_f, t) \cdot L(\chi_f^{-1}, t^{-1}).
\]

The term \( \epsilon(\chi_f, t) \) is a “constant” function studied in [D]. By the Grothendieck theory, these \( L \)-functions are given by characteristic polynomials of the Frobenius endomorphism acting on \( H^1(\mathcal{Y}_p, \mathbb{Q}_l)^{\chi_f} \). In this way the above functional equation is given, (c.f. [D, 10.3.5]), by

\[
det(Id - F_p \cdot t) = det(-F_p \cdot t) \cdot det(Id - F_p^\vee \cdot t^{-1}).
\]

Hence, we have \( \epsilon(\chi_f, t) = t^{2g+d-1} \cdot P_{F_p}^{X_f}(0) \). The term \( P_{F_p}^{X_f}(0) \) is calculated in terms of Gauss sums in [D, Section 5] and by Jacobi sums for \( g = 0 \) and \( d = 2 \) in [W]. In the two following sections, by using a different point of view to that of [D], we have that \( P_{F_p}^{X_f}(0) \) is given by Jacobi sums.

**4. The Constant Term of** \( P_{F_p}^{X_f}(t) \)

In this section, we shall calculate \( P_{F_p}^{X_f}(0) \). This means, by (*), calculating

\[
\sum_{[N] \in G_m} L_m(N, 2g + d - 1) \cdot \chi_f(\sigma_N).
\]

We shall calculate \( L_m(N, 2g + d - 1) \) for each \([N] \in G_m\). Let \( \kappa \) be a divisor of degree \( 2g - 2 \) associated with the canonical sheaf of \( X_p \). We have two cases for the cardinal

\[
\#H^0(X_p, N + (2g + d - 1) \cdot D_1 - E).
\]

It is \( = q^g \) when \( N + (2g + d - 1) \cdot D_1 - E \) is linearly equivalent to \( \kappa \) and \( = q^{(g-1)} \) in the other case.
Lemma 4.1. Let $(L, \iota_m)$ be a level structure where $\deg(L) = 2g + d - 1$. If $L$ is a line bundle not isomorphic to $\mathcal{O}_{X_p}(\kappa + E)$, then $\#H^0_m((L, \iota_m)) = q^{(g-1)}$.

Proof. Let us denote $\mathcal{O}_m := H^0(X_p, \mathcal{O}_{X_p}/m)$ and let $\pi$ be the natural epimorphism $\mathcal{O}_{X_p} \to \mathcal{O}_{X_p}/m$. By taking global sections on the exact sequence

$$0 \to L(-E) \to L \xrightarrow{\iota_m} \mathcal{O}_{X_p}/m \to 0,$$

we obtain the exact sequence of vector spaces

$$0 \to H^0(X_p, L(-E)) \to H^0(X_p, L) \xrightarrow{H^0(\iota_m)} \mathcal{O}_m \to 0.$$

Thus, we have an isomorphism of $\mathbb{F}_{p^h}$-vector spaces

$$H^0(\iota_m) : H^0(X_p, L)/H^0(X_p, L(-E)) \to \mathcal{O}_m,$$

and we obtain a section $s : \mathcal{O}_{X_p} \to L$ such that $H^0(\iota_m)(s) = \pi(1)$; thus we have that $H^0_m(L, \iota_m) \neq 0$ and its cardinal is $H^0(X_p, L(-E)) = q^{(g-1)}$. □

We now study the case when $L \simeq \mathcal{O}_{X_p}(\kappa + E)$.

We denote by $\pi_{\pi_0}$ the surjective morphism of modules $\mathcal{O}_{X_p}/m \to \mathcal{O}_{X_p}/m_{\pi_0}$, with $m_{\pi_0}$ the maximal ideal associated with $\pi_0$. We denote $\iota_{\pi_0} := \pi_{\pi_0} \cdot \iota_m$. Recall that $E = \pi_0 + \cdots + \pi_d$.

We shall study when a level structure $(L, \iota_m)$, with $L$ a line bundle isomorphic to $\mathcal{O}_{X_p}(\kappa + E)$, has an $m$-section.

Bearing in mind that if $\lambda \in k(\mathfrak{p})^\times$ then $(L, \iota_m)$ and $(L, \lambda \cdot \iota_m)$ are isomorphic level structures, we can fix the morphism $\iota_{\pi_0} := \pi_{\pi_0} \cdot \iota_m : L \to \mathcal{O}_{X_p}/m \to \mathcal{O}_{X_p}/m_{\pi_0}$; For the level structures $(\mathcal{O}_{X_p}, \iota_m)$, we set $\iota_{\pi_0}(1) = 1$. Now, let us consider $\overline{D}$ an effective divisor with $\overline{\pi}_0 \notin \text{supp}(\overline{D})$. Recall that by the natural inclusion $\mathcal{O}_{X_p} \hookrightarrow \mathcal{O}_{X_p}(\overline{D})$ we have that $1 \in \mathcal{O}_{X_p}(\overline{D})$ and $1 \notin m$. We obtain an $\overline{\pi}_0$-level structures, $\iota_{\overline{\pi}_0}$, for $\mathcal{O}_{X_p}(\overline{D})$ by setting $\iota_{\overline{\pi}_0}(1) = 1$. 


Because $L$ is of degree $2g + d - 1$, we can obtain an effective divisor, $\overline{D}$, with support outside $\overline{\pi}_0$ such that $L$ is isomorphic to $\mathcal{O}_{X_p}(\overline{D})$. We consider a $\overline{D}$ linearly equivalent to $\kappa + E$, with $\overline{\pi}_0 \notin \text{sup}(\overline{D})$, and we set $\iota_{\overline{\pi}_0}$ for $L = \mathcal{O}_{X_p}(\overline{D})$.

We denote by $E'$ the divisor $\overline{\pi}_1 + \cdots + \overline{\pi}_d$, by $m'$ the ideal associated with $E'$, and $\mathcal{O}_{m'} := H^0(X_p, \mathcal{O}_{X_p}/m')$.

Because $\mathcal{O}_{m} = k(\overline{\pi}_0) \times \mathcal{O}_{m'}$, for each $m$-level structure $(L, \iota_m)$ we have that $\iota_m = \iota_{\overline{\pi}_0} \times \iota_{m'}$.

**Proposition 4.2.** We have isomorphisms of vector spaces

\[ H^0(X_p, L)/H^0(X_p, L(-E)) \cong H^0(X_p, L)/H^0(X_p, L(-E')) \cong \mathcal{O}_{m'}. \]

Moreover, we have $H^0(\iota_{\overline{\pi}_0})(H^0(X_p, L(-E))) = 0$.

**Proof.** The first isomorphism is a consequence of the equality

\[ H^0(X_p, L(-E)) = H^0(X_p, L(-E')). \]

Because of the Riemann-Roch Theorem $H^1(X_p, L(-E')) = 0$, and thus from $H^0(\iota_{m'})$ we obtain the second isomorphism. The last assertion follows because $x_0 \in \text{sup}(E)$.

We denote the composition of these two isomorphisms by $[\iota_{m'}]$.

We now choose an $m'$-level structure for $L$, $(L, \iota')$. We consider the $\mathbb{F}_{p^h}$-linear form

\[ \omega := H^0(\iota_{\overline{\pi}_0})[\iota']^{-1} : \mathcal{O}_{m'} \cong H^0(X_p, L)/H^0(X_p, L(-E)) \to k(\overline{\pi}_0) = \mathbb{F}_{p^h}. \]

By considering the standard basis for the $\mathbb{F}_{p^h}$-vector space $\mathcal{O}_{m'} = k(\overline{\pi}_1) \times \cdots \times k(\overline{\pi}_d)$, we have that $\omega(z_1, \ldots, z_d) = \lambda_1 \cdot z_1 + \cdots + \lambda_d \cdot z_d$.

Since $\deg(L) = 2g + d - 1$ and

\[ \text{Ker}(\omega) = [\iota'](H^0(X_p, L(-\overline{\pi}_0))/H^0(X_p, L(-E))), \]

we have that for $1 \leq i \leq d$,

\[ H^0(X_p, L(-\overline{\pi}_0)/H^0(X_p, L(-E)) \neq H^0(X_p, L(-\overline{\pi}_0-\overline{\pi}_i))/H^0(X_p, L(-E)). \]
Thus, $\lambda_1 \cdots \lambda_d \neq 0$. Accordingly, by considering $(-\lambda_1^{-1}, \ldots, -\lambda_d^{-1}) \cdot \iota'$ instead of $\iota'$, we can assume that $\lambda_1 = \cdots = \lambda_d = -1$.

Note that if $(L, \iota'_m)$ is an $m'$-level structure, then there exists $z \in \mathcal{O}_{m'}^\times = k(x_1)^\times \times \cdots \times k(x_d)^\times$ with $\iota'_m = z \cdot \iota'$.

**Lemma 4.3.** By using the above notations, $H^0_m((L, \iota'_0 \times z \cdot \iota')) \neq 0$ if and only if $\omega(z^{-1}) = 1$.

**Proof.** If there exists a section $s$ of $L$ such that the diagram

\[
\begin{array}{ccc}
\mathcal{O}_{X_p} & \xrightarrow{s} & L \\
\downarrow{\pi'} & & \downarrow{z \cdot \iota'} \\
\mathcal{O}_{X_p}/m' & \xrightarrow{} & \end{array}
\]

is commutative, where $\pi'$ is the natural epimorphism, then we have that the class of the section $s$ in $H^0(X_p, L)/H^0(X_p, L(-E'))$ is $[z \cdot \iota']^{-1}(1)$.

Moreover, since the diagram

\[
\begin{array}{ccc}
\mathcal{O}_{X_p} & \xrightarrow{s} & L \\
\downarrow{\pi_{\tau_0}} & & \downarrow{\pi_0} \\
k(\tau_0) & \xrightarrow{} & \end{array}
\]

must also be commutative, we have that $H^0(\iota_{\tau_0})(s) = 1$. Thus,

$$1 = H^0(\iota_{\tau_0})([z \cdot \iota']^{-1}(1)) = H^0(\iota_{\tau_0})([\iota']^{-1}(z^{-1})) = \omega(z^{-1}).$$

Reciprocally, it suffices to consider a section $s$ of $L$ within the class $[z \cdot \iota']^{-1}(1) \in H^0(X_p, L)/H^0(X_p, L(-E'))$. \hfill \Box

We denote by $M$ a divisor of degree 0 with support outside $T$ such that the $m$-level structure associated with $M + (2g + d - 1) \cdot D_1$ is $(\mathcal{O}_{X_p}(\kappa + E), \iota_{x_0} \times \iota')$, and we recall that $\sigma_M$ is the element of the Galois group of the extension $H_m/\Sigma_{X_p}$ given, via the Artin symbol, by the class $[M] \in \frac{H^0_m}{(\Sigma_{X_p})^\times \cdot \mathcal{O}^\times_{T}}$.

We denote $z^{-1} \cdot M$ instead of $\eta(z^{-1}) \cdot [M]$. 
Lemma 4.4. We have

\[ P_{F_p}^{\chi_f}(0) = \sum_{[N] \in G_m} L_m(N, 2g + d - 1) \cdot \chi_f(\sigma_N) = \]

\[ = \sum_{z \in \mathcal{O}_m^\times} q^g \cdot \chi_f(\sigma_{z^{-1}M}) + \sum_{[N] \in G_m} q^{g-1} \cdot \chi_f(\sigma_N). \]

Proof. To prove the Lemma, we bear in mind that the set of classes of level structures for the line bundle \( \mathcal{O}_{X_p}(\kappa + E - (2g + d - 1)D_1) \) is given by \( \{ \eta(z^{-1}) \cdot [M] \}_{z \in \mathcal{O}_m^\times} \subset \prod_{\mathcal{O}_m^\times}^{\mathfrak{p}} \mathcal{O}_T^\times \) and Lemmas 4.1 and 4.3. \( \square \)

5. Jacobi sums

In the first part of this section we follow \[WW\]. Let us consider \( z_1, \ldots, z_d \in k(p)^\times = \mathbb{F}_p^h \). We consider the Jacobi sum

\[ J(a)(p) := (-1)^{d+1} \sum_{z_1 + \cdots + z_d = -1 \mod p, z_1, \ldots, z_d \mod p} \chi_p^{a_1}(z_1) \cdots \chi_p^{a_d}(z_d), \]

with \( a := (a_1, \ldots, a_d) \). The positive integers \( a_1, \ldots, a_d \) are defined in the introduction. We denote

\[ \theta(a) := \sum_{(t,m) = 1}^{d} \sum_{i=1}^{d} \left< t \cdot a_i \over m \right> \sigma^{-1}_{m}. \]

The map \( p \to J(a)(p) \) defines a Hecke character for the cyclotomic field \( K \) and the ideal generated by \( J(a)(p) \) within \( \mathbb{Z}[\epsilon_m] \) is \( p^{\theta(a)} \).

Moreover, \( |J(a)(p)|^2 = p^{h(s-2)} \), \( s \) being the cardinal of the subset of the integers \( a_1, \ldots, a_d, a_1 + \cdots + a_d \), which are \( \neq 0 \mod m \). In this case, \( s = d + 1 \).

Theorem 1. We have

\[ P_{F_p}^{\chi_f}(0) = (-1)^{d+1} \chi_f(\sigma_M) \cdot q^g \cdot J(a)(p). \]
Proof. Bearing in mind the end of Section 2, one can consider \( z := (z_0, z_1, \cdots, z_d) \in k(\overline{\mathbb{F}_0}) \times \cdots \times k(\overline{\mathbb{F}_d})^\times / k(p)^\times \) with \( z_0 = 1 \). Moreover, we denote \( \sigma_{z^{-1}, M} := \sigma_{\eta(1, z_1^{-1}, \cdots, z_d^{-1})} \cdot \sigma_M \). We have that

\[
\chi_f(\sigma_{z^{-1}, M}) = \chi_p^{a_1}(z_1) \cdots \chi_p^{a_d}(z_d) \cdot \chi_f(\sigma_M).
\]

Bearing in mind that \( \omega(z_1, \cdots, z_d) = -z_1 - \cdots - z_d \), we have that

\[
(\ast\ast)(-1)^{d+1} \chi_f(\sigma_M) \cdot q^g \cdot J_\omega(p) = \sum_{z \in \mathcal{O}_{m', \omega}^\times, \omega(z) = 1} q^g \cdot \chi_f(\sigma_{z^{-1}, M}) =
\]

\[
= \sum_{z \in \mathcal{O}_{m', \omega}^\times, \omega(z) = 1} q^g \cdot \chi_f(\sigma_{z^{-1}, M}) + \sum_{[N] \in G_m} q^{g-1} \cdot \chi_f(\sigma_N)
\]

because \( \sum_{[N] \in G_m} q^{g-1} \cdot \chi_f(\sigma_N) = 0 \).

We have that for \( \mu \in k(p)^\times \)

\[
\sum_{z \in \mathcal{O}_{m', \omega}^\times, \omega(z) = \mu} \chi_f(\sigma_{z^{-1}, M}) =
\]

\[
= \chi_p^{a_1+\cdots+a_d}(\mu) \cdot \sum_{z_1+\cdots+z_d = -1 \mod p} \chi_p^{a_1}(z_1) \cdots \chi_p^{a_d}(z_d) \cdot \chi_f(\sigma_M),
\]

therefore the sum over all \( \mu \in k(p)^\times \) of the above sum is 0, because \( a_1 + \cdots + a_d \neq 0 \mod m \), and thus, together with Lemma 4.4 we obtain

\[
P^x_{f_p}(0) = \sum_{[N] \in G_m} L_m(N, 2g + d - 1) \cdot \chi_f(\sigma_N) =
\]

\[
= \sum_{z \in \mathcal{O}_{m', \omega}^\times, \omega(z) = 1} q^g \cdot \chi_f(\sigma_{z^{-1}, M}) + \sum_{\mu \in \mathcal{O}_{m'}^\times} q^{g-1} \sum_{z \in \mathcal{O}_{m', \omega}^\times, \omega(z) = \mu} \chi_f(\sigma_{z^{-1}, M}) +
\]

\[
+ \sum_{[N] \in G_m} q^{g-1} \cdot \chi_f(\sigma_N).
\]

Since the set of classes of level structures for the line bundle associated with \( M \) is given by \( \{ \eta(z^{-1}) \cdot [M] \}_{z \in \mathcal{O}_{m'}^\times} \subset G_m \), the last sum is equal to

\[
\sum_{z \in \mathcal{O}_{m', \omega}^\times, \omega(z) = 1} q^g \cdot \chi_f(\sigma_{z^{-1}, M}) + \sum_{[N] \in G_m} q^{g-1} \cdot \chi_f(\sigma_N)
\]

and we conclude by the equality \((\ast\ast)\). \( \square \)
Remark 5.1. In this Remark we use the formula (*) written in section 3.

We have
\[ \det(F_p) = (-1)^{2g+d-1} \sum_{[N] \in G_m} L_m(N, 2g + d - 1) \cdot \chi_f(\sigma_N). \]

Thus, by [W] and Theorem 1
\[ p \to \chi_f(\sigma_M)^{-1} \det(F_p) = q^g \cdot J(a)(p), \]
with \( p \in \text{Spec}(A) \), gives a Hecke character for \( K \).

We now include a formula that relates \( \chi_f, \chi_p \), values over divisors \( X_p \) of elements of \( \Sigma X_p \) and Jacobi sums.

We have the equality,
\[ \sum_{[N] \in G_m} L_m(N, 2g + d - 1) \cdot \chi_f(\sigma_N) = \sum_{D, \deg(D)=2g+d-1} \chi_p(f(D)), \]
where the sum is over all the effective divisors on \( X_p \) with support outside \( T \). Therefore, from Theorem 1 we deduce the formula
\[ \sum_{D, \deg(D)} \chi_p(f(D)) = \chi_f(\sigma_M) \cdot q^g \cdot J(a)(p). \]

We have defined \( f(D) \) at the end of the section 3.

For example, let \( X \) be a proper, smooth model for the elliptic curve \( y^2 = x \cdot (x-1) \cdot (x-\lambda) \), defined over \( A := \mathbb{Z}[\epsilon_m, \frac{1}{2m}], m \neq 3 \). We have that \( \text{div}(y) = -3 \cdot \infty + (0,0) + (1,0) + (\lambda,0) \) and \( a = (-3,1,1,1) \), thus
\[ \sum_{D, \deg(D)=4} \chi_p(y(D)) = \chi_y(\sigma_M) \cdot p \cdot J(a)(p), \]
where the sum is over all effective divisors \( D \) on the elliptic curve, \( X_p \), with support outside \( T := \{ \infty, (0,0), (1,0), (\lambda,0) \} \).

6. The \( p \)-rank of Jacobians

Let \( W, S \) be proper, smooth and geometrically irreducible curves over \( \mathbb{F}_q \).
Definition 6.1. The curve $W$ has p-rank 0 when, for the morphism $[p] : \text{Pic}_W^0 \rightarrow \text{Pic}_W^0$, $\text{Ker}[p]_{\text{red}} = \text{Spec}(\mathbb{F}_q)$ or, equivalently, when $[p]$ is purely inseparable. Here, $\text{Ker}[p]$ is considered as a scheme and $\text{Ker}[p]_{\text{red}}$ is the reduced scheme. Note that $\text{Ker}[p]_{\text{red}} = \text{Spec}(\mathbb{F}_q)$ if and only if $\text{Ker}[p](\mathbb{F}) = \{0\}$.

In the following Proposition we remember a known result. We consider $F_q$, the $q$-Frobenius morphism, as a $\mathbb{Q}_l$-linear application over the $\mathbb{Q}_l$-vector space $H^1(W, \mathbb{Q}_l)$.

Proposition 6.2. We have that $W$ has p-rank 0 if and only if the characteristic polynomial of $F_q$, $t^{2\pi} + a_1t^{2\pi-1} + \cdots + a_{2\pi-1}t + a_{2\pi} \in \mathbb{Z}[t]$, satisfies $p|a_1, \cdots, p|a_{2\pi}$. Here, $\pi$ denotes the genus of $W$.

Proof. If $p|a_1, \cdots, p|a_{2\pi}$ then by considering $F_q$ as an endomorphism of $\text{Pic}_W^0$, we have that $F_q^{2\pi} = [p] \cdot \Phi$, where $\Phi$ is also an isogeny of $\text{Pic}_W^0$. Thus $[p]$ is purely inseparable because $F_q$ is purely inseparable.

Conversely, since $\text{Ker}[p]_{\text{red}} = \text{Spec}(\mathbb{F}_q)$ there exists $s \in \mathbb{N}$ such that $F_q^s(\text{Ker}[p]) = \text{Spec}(\mathbb{F}_q)$. In this way, $F_q^s = [p] \cdot \Phi$ with $\Phi$ an isogeny of $\text{Pic}_Z^0$. Thus, the characteristic polynomial of $F_q$ is $t^{2\pi} \mod p$. $\square$

Let $W \rightarrow S$ be a ramified abelian covering, of group $G := \mathbb{Z}/m$. Let $\chi$ be a non-trivial character of $G$. We denote by $p_1(t) \in \mathbb{Z}[\epsilon_m][t]$ the characteristic polynomial of $F_q$ as $\mathbb{Q}_l(\epsilon_m)$-endomorphism over the $\mathbb{Q}_l(\epsilon_m)$-vector space $H^1(W, \mathbb{Q}_l)^\chi$, and by $p_0(t)$ the characteristic polynomial of $F_q$ over the $\mathbb{Q}_l$-vector space $H^1(S, \mathbb{Q}_l)$.

Lemma 6.3. If $W$ has p-rank 0 then $p_1(0) = u \cdot p$, with $u \in \mathbb{Z}[\epsilon_m]$.

Proof. By Proposition 6.2, $F_q^s = [p] \cdot \Phi$, where $\Phi$ is an isogeny of the abelian variety $\text{Pic}_W^0$. The characteristic polynomial of $\Phi$ as a $\mathbb{Q}_l(\epsilon_m)$-endomorphism of $H^1(W, \mathbb{Q}_l)^\chi$ is $t^r + b_{r-1}t^{r-1} + \cdots + b_0 \in \mathbb{Z}[\epsilon_m]$. Note that $F_q^s$ and $[p]$ are $\mathbb{Q}_l(\epsilon_m)$-endomorphisms and thus $\Phi$. Therefore, the
characteristic polynomial of \( F_q^s \) is 
\[ t^r + c_{r-1}t^{r-1} + \cdots + b_0 \cdot p^r \in \mathbb{Z}[\epsilon_m] \]
and we have \( p_1(0)^s = b_0 \cdot p^r \). We conclude because the primary ideal decomposition of \( p \) in \( \mathbb{Z}[\epsilon_m] \) is a product of different prime ideals. \( \square \)

We consider the notations of sections 1 and 2. For \( W = Y_p, S = X_p \) and \( \chi = \chi_f \), we have \( p_1(t) = P_{F_p}^{\chi_f}(t) \).

Let \( h \) be the least positive integer with \( m|(p^h - 1) \). The ideal \( p \cdot \mathbb{Z}[\epsilon_m, \frac{1}{m-1}] \) decomposes into a product of \( b := \frac{m-1}{h} \) prime ideals \( p_1 \cdots p_b \). The action of \( (\mathbb{Z}/m)^\times \) on \( \{p_1, \cdots, p_b\} \) has as isotropy \( C_h \), the unique \( h \)-cyclic subgroup of \( (\mathbb{Z}/m)^\times \).

By choosing the representative \( i \) within the class \([i]\), we obtain the identification, as sets, \( \{1, \cdots, m - 1\} = (\mathbb{Z}/m)^\times \). We denote by \( c_h \subseteq \{1, \cdots, m - 1\} \) the subset associated, by the above identification, with \( C_h \). We consider the group quotient \( (\mathbb{Z}/m)^\times /C_h \), which acts transitively and without isotropy on \( \{p_1, \cdots, p_b\} \). Let \( e_h := \{i_1, \cdots, i_b\} \subseteq \{1, \cdots, m - 1\} \) be representatives of the classes of \( (\mathbb{Z}/m)^\times /C_h \).

We use \( < \frac{a}{m} > \) to denote the fractional part of \( \frac{a}{m} \) and \( d_u := \sum_{1 \leq u < m} d_u \cdot \sigma_{-u} \)
are the dimensions of the eigenspaces of \( H^1(Y_p, \mathcal{O}_{Y_p}) \) for the different characters of \( G \), c.f. \([B3]\). We set \( O_t := \sum_{u \in c_h} d_{t-u} \), which is the dimension of the orbit for the action of \( C_h \) on the eigenspaces \( H^1(Y_p, \mathcal{O}_{Y_p})^{\chi_f} \).

**Theorem 2.** 1) If there exists \( t \in e_h \) with \( O_1 \neq O_t \) then \( \text{Pic}^0_{Y_p} \) is not supersingular.

2) If \( X_p = \mathbb{P}^1 \) and there exists \( t \in e_h \) with \( O_t = 0 \), then \( Y_p \) does not have \( p \)-rank 0.

**Proof.** 1) By Theorem [1] the ideal within \( \mathbb{Z}[\epsilon_m, \frac{1}{m-1}] \) generated by the term constant of \( P_{F_p}^{\chi_f}(t) \) is \((q^a \cdot J_{(a)}(p))\). Thus, by using [W] (8), (9) this ideal is \((q^a \cdot p^{\theta(a)})\) with \( \theta(a) := \sum_{1 \leq u < m} d_u \cdot \sigma_{-u}^{-1} \).
Because \( \sum_{u \in h_r} d_u / \sum_{u \in h_r} d_{u + t} \) for some \( t \in \mathfrak{e}_r \), we have that the ideal primary decomposition of \((q^g \cdot J(a)(p))\) is \( p_1^{n_1} \cdots p_b^{n_b} \), with \( n_1 \neq n_v \), for some \( v \) and we conclude. Bear in mind that Pic\(^0_{Y_p} \) is supersingular if and only if the eigenvalues of the Frobenius morphism are \( \zeta \cdot q^{1/2} \), \( \zeta \) being a root of the unity.

2) Since \( g = 0 \), the ideal \( P_{F_p}(0) \cdot \mathbb{Z}[\epsilon, \frac{1}{m-n}] \) is \( (J(a)(p)) \). Because \( \sum d_{u-t} = 0 \), the ideal primary decomposition of this ideal is \( p_1^{n_1} \cdots p_b^{n_b} \) with \( n_v = 0 \) for some \( v \). We conclude because if \( Y_p \) has \( p \)-rank 0, then \( \sum_{x \in h_r} f(x) = 0 \) (c.f Lemma 6.3).

Lemma 6.4. We have that \( Y_p \) has \( p \)-rank 0 if and only if \( X_p \) has \( p \)-rank 0 and \( P_{F_p}(t) = t^{r_p} + p \cdot Q_j(t) \), where \( Q_j(t) \in \mathbb{Z}[\epsilon, t] \) and \( \deg(Q_j(t)) < r \), for each \( j, 1 \leq j \leq m-1 \).

Proof. This follows from Proposition 6.2, bearing in mind that the characteristic polynomial of the \( p^h \)-Frobenius morphism, considered as a \( \mathbb{Q}_l \)-linear application of \( H^1(Y_p, \mathbb{Q}_l) \), is \( p(t) \cdot \prod_{(j,m)=1} P_{F_p}(t) \) and that the primary ideal decomposition of \( p \) in \( \mathbb{Z}[\epsilon, \frac{1}{m-n}] \) is a product of different prime ideals. \( \square \)

In the following theorem we assume the notation of Theorem 2 and that the sums are over all effective divisors \( D \) on \( X_p \) with support outside \( T \).

Theorem 3. 1) Let \( p \) be a integer prime inert in \( \mathbb{Z}[\epsilon, \frac{1}{m-n}] \). We have that \( Y_p \) has \( p \)-rank 0 if and only if \( X_p \) has \( p \)-rank 0 and

\[
\sum_{D, \deg(D) = l} \chi_p(f(D)) = 0 \mod p,
\]

for each \( l, 1 \leq l \leq 2g + d - 2 \). Note that \( k(p) = \mathbb{F}_p^{m-1} \).
2) We now assume \( g = 0 \) and that \( \sum_{u \in c} d_{u \cdot t} \neq 0 \) for each \( t \in e_h \). We have that \( Y_p \) has \( p \)-rank \( 0 \) if and only if
\[
\sum_{D, \deg(D) = l} \chi_p(f(D)) = 0 \mod p
\]
for each \( j \in e_h \) and each \( l, 1 \leq l \leq d - 2 \). Note that in this case \( k(p) = \mathbb{F}_p^h \).

**Proof.** 1) From \((*)\) section 3, we have
\[
P_{F_p}^{\chi_j}(t) = \sum_{i=0}^{2g+d-1} \sum_{D, \deg(D) = i} \chi_p(f(D)) t^{2g+d-1-i}.
\]
We conclude by using the Lemma 6.4 and the fact that by Theorem 1,
\[
P_{F_p}^{\chi_j}(0) \cdot Z[\epsilon_m, \frac{1}{m \cdot n}] = (q^g \cdot J(a)(p)).
\]
Note that in the case of \( p \) being inert in \( Z[\epsilon_m, \frac{1}{m \cdot n}] \), \( p = p \cdot Z[\epsilon_m, \frac{1}{m \cdot n}] \) and \( q^g J(a)(p) = \epsilon_m^j \cdot p^u \) for some \( 1 \leq u, j \in \mathbb{N} \). Moreover, \( P_{F_p}^{\chi_j}(t) = \sigma^j(P_{F_p}^{\chi_j}(t)) = 0 \mod p \), for each \( j \) with \( 1 \leq j \leq m - 1 \), if and only if \( P_{F_p}^{\chi_j}(t) = 0 \mod p \).

2) One proceeds in the same way as 1). Bearing in mind that \( g = 0 \) and that \( \sum_{u \in c} d_{u \cdot t} \neq 0 \), for each \( t \in e_h \), we have that, \( (J(a)(p)) = (p^{\theta(a)}) \subseteq p \cdot Z[\epsilon_m, \frac{1}{m \cdot n}] \). Note that \( P_{F_p}^{\chi_j}(t) = \sigma_j(P_{F_p}^{\chi_j}(t)), p \cdot Z[\epsilon_m, \frac{1}{m \cdot n}] = p_1 \cdots p_b \), and \( \{\sigma_j\}_{1 \leq j \leq m-1} \) operates transitively on \( p_1, \cdots, p_b \). \( \Box \)

In the next Corollary, by using part 1) of this Theorem we obtain counterparts to the Deuring polynomial for genus greater than 1. In [Bo, 5], it is obtained, explicitly, the generalized Hasse-Witt matrix for \( Y_p \) and the author thus obtained equations to study the \( p \)-rank of \( Y_p \).

We consider the curves defined over \( Z[\epsilon_m, \frac{1}{m \cdot n}] \), with \( n \) the product of integer primes \( p \) with \( \#k(p) = p^r < d \), \( X = \mathbb{P}^1 \) and \( Y \) the curve associated with the plane curve
\[
y^m = x^{a_1}(x-1)^{a_2}(x-\alpha_1)^{a_3} \cdots (x-\alpha_{d-2})^{a_d},
\]
where the \(a_1, \ldots, a_d, a_1 + \cdots + a_d\) are integers \(\neq 0 \mod m\), with \(1 \leq a_j < m\) and \(\alpha_1, \ldots, \alpha_{d-2}\) are different elements of \(k(p) \setminus \{0,1\}\) and \(f = x^{a_1}(x-1)^{a_2}(x-\alpha_1)^{a_3}\cdots(x-\alpha_{d-2})^{a_d}\).

**Corollary 6.5.** Let \(Y_p\) be the proper, smooth model of the reduction at \(p\) of \(Y\), with \(p\) inert in \(\mathbb{Z}[\epsilon_m, \frac{1}{m^n}]\). Accordingly, \(Y_p\) has \(p\)-rank 0 if and only if
\[
\sum_{q(x) \atop \deg(q(x)) = l} [f(\text{div}(q(x)))]^{(p^n-1)/m} = 0
\]
or equivalently,
\[
\sum_{q(x) \atop \deg(q(x)) = l} [q(0)^{a_1}q(1)^{a_2}q(\alpha_1)^{a_3}\cdots q(\alpha_{d-2})^{a_d}]^{(p^n-1)/m} = 0
\]
for each \(l\), with \(1 \leq l \leq d-2\). Here, the sums are over all monic polynomials \(q(x) \in \mathbb{F}_p[x]\).

**Proof.** It suffices to consider in Theorem 3 that \(X_p = \mathbb{P}^1, T := (x \cdot (x-1) \cdot (x-\alpha_1) \cdots (x-\alpha_{d-2}))_0 \cup \{\infty\}\) and also to consider that the effective divisors of degree \(l\) on \(\text{Spec}(\mathbb{F}_p[x])\) are given by the zero locus of monic polynomials \(q(x)\) of degree \(l\). To conclude it suffices bear in mind that if \(\text{div}(q(x)) = \sum_i n_i \cdot y_i - l \cdot \infty\), then \(f(\text{div}(q(x))) = \prod_i f(y_i)^{n_i} \cdot f(\infty)^{-l}\). By the Weil’s reciprocity law, we have
\[
f(\sum_i n_i \cdot y_i) = (-1)^{(a_0 + \cdots + a_d)} q(0)^{a_1} q(1)^{a_2} q(\alpha_1)^{a_3} \cdots q(\alpha_{d-2})^{a_d}.
\]
Note that \(q(x)\) and \(f(x)\) are monic polynomials and thus \(q^{\deg(q(x))} f^{-l}(\infty) = 1\). Moreover, it is not necessary to impose to \(q(x)\) that \(q(0) \cdot q(1) \cdot q(\alpha_1) \cdots q(\alpha_{d-2}) \neq 0\) because in the above sum these terms are 0.

\(\square\)

We set \(\pi\) the genus of \(Y_p\). By [A] 4.1.1, we have that the zeta function for \(Y_p\) is:
\[ H \cdot \left( \frac{T^{2\pi}}{(1-T)(1-qT)} \right) + \sum_{i=0}^{\pi-1} A_i \cdot T^i + q \cdot \sum_{i=\pi}^{2\pi-2} B_i \cdot T^i, \]
with \( A_i, B_i, H \in \mathbb{Z} \). Thus, the characteristic polynomial \( \det(t - F_p) \in \mathbb{Z}[t] \) is \( \sum_{i=0}^{2\pi} a_i \cdot t^i + q \cdot \sum_{i=0}^{\pi-1} b_i \cdot t^i \), with \( a_i, b_i \in \mathbb{Z} \). Therefore, in the above corollary the last \( \pi - 1 \) equations are 0 mod \( p \).

Now, the system of equations of this corollary is a system on \( \mathbb{F}_p^{m-1} \) with \( d - 2 \) variables and \( d - 2 - (g - 1) \) equations. In [GlP] it is shown that the moduli space of hyperelliptic curves with \( p \)-rank 0 is \( \pi - 1 \).

The hyperelliptic curve defined over \( \mathbb{F}_p \)
\[ y^2 - x(x - 1)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \]
has \( p \)-rank 0 with \( p \geq 5 \), if and only if
\[ (1) \sum_{a \in \mathbb{F}_p} [a(1 + a) \prod_{i=1}^{3}(\alpha_i + a)]^{(p-1)/2} = 0 \]
\[ (2) \sum_{a,b \in \mathbb{F}_p} [a(1 + b + a) \prod_{i=1}^{3}(\alpha_i^2 + \alpha_i \cdot b + a)]^{(p-1)/2} = 0 \]

Let \( A \) be the Cartier-Manin matrix, which is the \( 2 \times 2 \)-matrix whose entry \( a_{ij} \) is the coefficient \( c_{ip-j} \) of \( x^{ip-j} \) in the polynomial
\[ [x(x - 1)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)]^{(p-1)/2}. \]

In [Y, Theorem 3.1] the conditions for this hyperelliptic curve to be of \( p \)-rank 0 are given by the four algebraic equations obtained from \( A^2 = 0 \). Therefore, this hyperelliptic curve has \( p \)-rank 0 if and only if
\[ \text{Trace}(A) = c_{p-1} + c_{2p-2} = 0 \]
and
\[ \det(A) = c_{p-1}c_{2p-2} - c_{p-2}c_{2p-1} = 0. \]

We are now going to prove that these equations match the equations below (1) and (2). Equation (1) is equal to \( \sum_{a \in \mathbb{F}_p} f(a)^{(p-1)/2} \). Bearing in mind that \( \sum_{a \in \mathbb{F}_p} a^s = 0 \) if and only if \( s \) is a multiple of \( p - 1 \), we have
\[ (1) = -c_{p-1} - c_{2p-2}. \]
Equation (2) is equal to
\[
1/2(\sum_{\alpha \in \mathbb{F}_p} [f(\alpha)f(\alpha^p)]^{p-1} + \sum_{(\alpha, \beta) \in \mathbb{F}_p \times \mathbb{F}_p} [f(\alpha)f(\beta)]^{p-1}).
\]

Now, the only terms that we must consider in the first summand are the coefficients of \(x^{p^2-1}\) and \(x^{2(p^2-1)}\) in \([f(x)f(x^p)]^{p-1}\). We consider \(x^i x^{jp}, 0 < i, j < \frac{5p-1}{2}\) with \(i + jp\) equal to either \(p^2 - 1\) or \(2(p^2 - 1)\).

We have the following possibilities:
\[
i = p - 2 \quad j = 2p - 1 \quad i = 2p - 1 \quad j = p - 2,
\]
\[
i = j = p - 1 \quad \text{and} \quad i = j = 2p - 2.
\]
Thus, \(\sum_{\alpha \in \mathbb{F}_p} [f(\alpha)f(\alpha^p)]^{p-1} = -2c_{2p-1}c_{p-2} - c_{p-1}^2 - c_{2p-2}^2\).

For the sum \(\sum_{(\alpha, \beta) \in \mathbb{F}_p \times \mathbb{F}_p} [f(\alpha)f(\beta)]^{p-1}\) we must consider in \([f(x)f(y)]^{p-1}\) the coefficients of \(x^i y^j\) with \(i, j \in \{p - 1, 2p - 2\}\). In this way, this sum is equal to \(2c_{2p-2}c_{p-1} + c_{p-1}^2 + c_{2p-2}^2\). Thus (2) = \(-c_{2p-1}c_{p-2} + c_{2p-2}c_{p-1}\) and we conclude.

The next Corollary is an application of part 2) of Theorem 3 when \(r = 1\). Let \(Y\) be the above plane curve.

**Corollary 6.6.** Let us consider
\[
[< \frac{t \cdot a_1}{m} > + \cdots + < \frac{t \cdot a_d}{m} >] \neq 0
\]
for each \(t, 1 \leq t \leq m - 1\). We have that \(Y_p\) has \(p\)-rank 0 if and only if
\[
\sum_{\deg(q(x)) = l} [f(\text{div}(q(x)))]^{j(p^{m-1} - 1)/m} = 0
\]
or equivalently,
\[
\sum_{\deg(q(x)) = l} [q(0)^{a_1} q(1)^{a_2} q(\alpha_1)^{a_3} \cdots q(\alpha_{d-2})^{a_d}]^{j(p-1)/m} = 0 \mod p,
\]
for each \(l\) and \(j\) with \(1 \leq l \leq d - 2\) and \(1 \leq j \leq m - 1\), respectively.
The sums are over all monic polynomials \(q(x) \in \mathbb{F}_p[x]\).
Proof. This is an application of part 2) of Theorem 3. With the values $f(D)$ one proceeds in the same way as in the above Corollary by using Weil’s reciprocity law.

As an example of these corollaries we take $Y \equiv y^3 - x \cdot (x - 1) \cdot (x - \alpha)^2 = 0$.

Let $p$ be an integer prime $\geq 5$. We have that $p$ in $\mathbb{Z}[\epsilon_3, \frac{1}{3}]$ either splits completely when $p \equiv 1 \text{ mod } 3$ or is inert when $p \equiv -1 \text{ mod } 3$.

We have that for all $t \in \{1, 2\}$, $[< \frac{t}{3}> + < \frac{t}{3}> + < \frac{2t}{3}>] \neq 0$. Hence, the desingularization of this curve has $p$-rank 0 over $\mathbb{F}_p$, with $p \equiv 1 \text{ mod } 3$, if and only if

$$\sum_{a \in \mathbb{F}_p} [a \cdot (a - 1) \cdot (a - \alpha)^2]^{(p-1)/3} = 0, \quad \sum_{a \in \mathbb{F}_p} [a^2 \cdot (a - 1)^2 \cdot (a - \alpha)^4]^{(p-1)/3} = 0.$$

For $p \equiv -1 \text{ mod } 3$ this curve has $p$-rank 0 over $\mathbb{F}_{p^2}$ if and only if

$$\sum_{a \in \mathbb{F}_{p^2}} [a \cdot (a - 1) \cdot (a - \alpha)^2]^{(p^2-1)/3} = 0.$$

As an example of this corollary one proves that the desingularization of the above curve, defined over $\mathbb{F}_7$, has 7-rank $\neq 0$. Moreover, by using the upper bounds of [E, Theorem 1.1], this curve has $p$-rank $\leq 2$.

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