Abstract: A new way of constructing fusion bases (i.e., the set of inequalities governing fusion rules) out of fusion elementary couplings is presented. It relies on a polytope reinterpretation of the problem: the elementary couplings are associated to the vertices of the polytope while the inequalities defining the fusion basis are the facets. The symmetry group of the polytope associated to the lowest rank affine Lie algebras is found; it has order 24 for \( \widehat{su}(2) \), 432 for \( \widehat{su}(3) \) and quite surprisingly, it reduces to 36 for \( \widehat{su}(4) \), while it is only of order 4 for \( \widehat{sp}(4) \). This drastic reduction in the order of the symmetry group as the algebra gets more complicated is rooted in the presence of many linear relations between the elementary couplings that break most of the potential symmetries. For \( \widehat{su}(2) \) and \( \widehat{su}(3) \), it is shown that the fusion-basis defining inequalities can be generated from few (1 and 2 respectively) elementary ones. For \( \widehat{su}(3) \), new symmetries of the fusion coefficients are found.
1. Introduction

Affine fusion rules give the number of integrable representations \( \hat{\nu} \) that appear in the product of two integrable representations \( \hat{\lambda} \) and \( \hat{\mu} \) for a given affine algebra \( \hat{g} \) at fixed level \( k \) (see e.g., [1] Chapter 16). Fusions are in fact truncated finite Lie algebra tensor products, with the degree of truncation fixed solely by the level. More precisely, fusion rules are completely characterized by the tensor-product coefficients pertaining to the corresponding finite (i.e., non-affine) representations and the set of threshold levels [2]. The threshold level of a particular coupling representing one of the various copies of the representation \( \hat{\nu} \) in the product \( \hat{\lambda} \times \hat{\mu} \) is the lowest level at which this coupling appears in this fusion. (Note that only the full set of threshold levels associated to a given triple coupling is an observable; the association of a particular threshold level with a given coupling is basis dependent.)

Even for \( \hat{su}(N) \), no genuine combinatorial methods – analogous to the Littlewood-Richardson rule – have been found. The closest approach to such a goal has been obtained in [3] in which a new approach to the problem of fusion rules was introduced, centered on the notion of fusion basis. A fusion basis is simply a complete set of inequalities, formulated in terms of a complete set of variables needed to describe a tensor product, augmented with an extra variable, the level \( k \). Examples of bases have been constructed for \( \hat{su}(2) \), \( \hat{su}(3) \), \( \hat{su}(4) \) and \( \hat{sp}(4) \).

The idea of the construction in [3] is the following: one starts from the tensor-product elementary couplings, extends this set to a complete set of fusion elementary couplings (using, for instance, the conjectural completeness under outer automorphism – but there exist other possibilities) and from these, construct the inequalities in terms of the basis variables for which the elementary couplings are the elementary solutions.

In [3], the transition from the elementary couplings to the inequalities uses Farkas’ lemma. The aim of this note is to indicate another way of reconstructing the fusion basis given the fusion elementary couplings. This new construction relies on a reinterpretation of the fusion-rule computations in terms of counting points inside a polytope. A polytope can be described by its vertices or its facets. The reconstruction of the facets of a polytope from its vertices is the essential trick we want to adapt to the problem of fusion rules. In our context, the vertices are represented by the fusion elementary couplings and the facets are the inequalities for which the elementary couplings are the elementary solutions.
This reformulation is not purely cosmetic: it allows us to use powerful (e.g., computer) techniques developed for the study of polytopes, for instance for deriving the facets from its vertices.

But this conceptual shift in the description of the fusion basis has an immediate benefit: having constructed a polytope it is natural to look for its symmetries. This means looking for the symmetry group of the fusion basis and organizing the various inequalities into a number of orbits of the group. In this paper we find the symmetry group of the polytopes associated to the known fusion bases.

2. The \( \hat{su}(2) \) example

As a simple illustrative example we present the \( \hat{su}(2) \) fusion basis:

\[
\begin{align*}
  k & \geq \lambda_1 + n_{11} & n_{12} & \geq 0 & \lambda_1 & \geq n_{12} & n_{11} & \geq 0.
\end{align*}
\]  

The last three conditions define the Littlewood-Richardson (LR) basis, which is thus recovered from the fusion basis in the infinite level limit (\( k \rightarrow \infty \)). This basis describes the solution set of the fusion \( \hat{\lambda} \times \hat{\mu} \) at fixed positive integer level \( k \). The two Dynkin labels of \( \hat{\lambda} \) are \( \lambda_0 = k - \lambda_1 \) and \( \lambda_1 \), with \( \lambda_1 \) being the finite Dynkin label (and we will often use the Dynkin label notation: \( \hat{\lambda} = [\lambda_0, \lambda_1] \)). Both Dynkin labels are assumed to be non-negative integers. The LR algorithm starts by filling the boxes of the first row of the Young tableau associated to \( \mu \) with 1’s, the second row with 2’s, etc. For \( su(2) \), \( \mu \) has only one row, containing \( \mu_1 \) boxes. These boxes are inserted into the tableau representing \( \lambda \), which is a row of \( \lambda_1 \) boxes: \( n_{11} \) boxes are then added in the first row and \( n_{12} \) boxes in the second row. Therefore \( n_{11} \) and \( n_{12} \) are non-negative integers and \( n_{11} + n_{12} = \mu_1 \). Moreover, columns with two 1’s are not permitted, which forces \( \lambda_1 \geq n_{12} \). Finally, the tableau associated to the representation \( \nu \) is read off the resulting LR tableau by taking out the columns of two boxes: \( \nu_1 = \lambda_1 + n_{11} - n_{12} \). (For more details on the LR algorithm, see e.g., [4, 5, 6].)

The elementary solutions of this system of inequalities, written in terms of vectors with entries ordered as \( (k, \lambda_1, n_{11}, n_{12}) \), are

\[
\begin{align*}
  \epsilon_0 = (1, 0, 0, 0) & & \hat{E}_0 : d
  \\
  \epsilon_1 = (1, 1, 0, 1) & & \hat{E}_1 : dL_1 N_{12} : \begin{array}{|c|}
  \hline
  1
  \hline
  \end{array}
  \\
  \epsilon_2 = (1, 1, 0, 0) & & \hat{E}_2 : dL_1 : \begin{array}{|c|}
  \hline
  \hline
  \end{array}
  \\
  \epsilon_3 = (1, 0, 1, 0) & & \hat{E}_3 : dN_{11} : \begin{array}{|c|}
  \hline
  1
  \hline
  \end{array}
\end{align*}
\]  

(2.2)
We have also presented the corresponding LR tableau at the right and its ‘exponential’
description in between (where the variables \(k, \lambda_1, n_{11}, n_{12}\) appear respectively as the ex-
ponents of the dummy variables \(d, L_1, N_{11}, N_{12}\)). The problem we consider here is the
following: given \(\hat{E}_i, i = 0, 1, 2, 3\), how can we reconstruct the inequalities? In other words,
how to go from the vertices (2.2) to the facets (2.1)?

3. The polytope interpretation of Farkas’ lemma

As we just mentioned, for the present work we suppose that the complete set of fusion
elementary couplings \(\{\hat{E}_i\}\) is known. These are expressed in terms of a set of variables,
denoted collectively as \(X_j, j = 1, \cdots, n\) (which are the dummy variables \(d, L_1, N_{11}, N_{12}\)
in the previous example), that furnish a complete description of the fusion rules. These
are in fact the variables that describe the tensor products with the addition of the extra
variable \(d\) that keeps track of the level of the affine algebra under consideration.

A general coupling can always be decomposed into a product of elementary couplings
(and that this decomposition may not be unique is irrelevant at this point). This product
decomposition can be transferred into a sum decomposition by characterising an elemen-
tary coupling by the exponents \(\epsilon_{ij}\) in its decomposition in terms of the variables \(X_j\): an
elementary coupling is thereby associated to a \(n\)-dimensional vector \(\epsilon_i\).

Again, our problem is the following: what is the set of linear and homogeneous inequal-
ities for which the \(\epsilon_i\) are the elementary solutions? These inequalities will be formulated in
terms of the variables \(x_j\), whose exponential versions are the \(X_j\). The set \(\{x_i\}\) will typically
contain the finite Dynkin labels of the three affine weights entering in the fusions, together
with the missing labels appropriate for a complete description of the corresponding tensor
product, plus the level \(k\).

Any coupling \(\prod_i X_i^{x_i}\) can thus be decomposed in the form \(\prod_i \hat{E}_i^{a_i}\). With \(\hat{E}_i = \prod_j X_j^{\epsilon_{ij}}\),
reading off a particular coupling means looking for a choice set of non-negative integers
\(\{a_i\}\) fixed by

\[
\sum_i a_i \epsilon_{ij} = x_j
\]

in terms of non-negative integers \(x_j\). We are thus searching for the conditions ensuring
the existence of such a coupling. This is precisely what Farkas’ lemma can do. Quite
remarkably, the lemma gives existence conditions in the form of inequalities on the $x_j$'s and these are precisely the inequalities we are looking for.

If we write $V_{ij} = e_{ji}$ (hence, the columns of the matrix $V$ are the elementary solutions), Farkas’ lemma gives existence conditions for solutions of

$$V a = x$$

(in matrix form). For instance, for $\hat{su}(2)$, $V$ takes the form (cf. (2.2)):

$$V = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}$$

(3.3)

In fact, we are interested in the integer solutions of (3.2).

We should stress that Farkas’ lemma is true in general only in the rational case. Sufficient conditions for its application to the integer case are known (see for instance [7]), but these are not satisfied by our matrices $V$. Hence, the analysis has to be completed by a verification step, that is, to check that the elementary solutions of the inequalities obtained are the elementary solutions used at the start (cf. the discussion in sect. 3 of [3]).

The polytope interpretation of (3.2) is almost immediate (see the next section): modulo renormalization, (3.2) is the equation of a polytope whose vertices $v_i$ are the columns of the matrix $V$ without its first row, hence our elementary couplings without their first entry $d$.

Now it is a well-known result (the Weyl-Minkowski theorem) that a polytope can be described either in terms of its vertices or its facets. In the present case, we have the vertices; the question is thus: what are the facets of the polytopes whose vertices are our fusion elementary couplings? These facets are the sought for inequalities, which form the fusion basis.

This reformulation is made more formal in the next section. The following sections are devoted to the study of the symmetry group of the fusion polytopes for the simplest affine Lie algebras.
4. Formalizing the polytope reinterpretation

To recast the problem in a general setting, let \( V \in \mathbb{N}^{n \times m} \) be such that

\[
V = (v_1 \ldots v_m) = \begin{pmatrix} w_1 & \cdots & w_m \\ v_1 & \cdots & v_m \end{pmatrix},
\]

(4.1)

with \( v_1, \ldots, v_m \) the fusion elementary couplings and \( w_1, \ldots, w_m \) the entries in the first row of \( V \). Also, let \( S \) denote the set of fusion couplings

\[
S = \left\{ x = Va = a_1 v_1 + \cdots + a_m v_m \middle| a_1, \ldots, a_m \in \mathbb{N} \right\},
\]

which, from our central hypothesis, is the set of nonnegative integral combinations of elementary couplings. It is natural to suppose that the fusion couplings are the integral points of a certain geometric object \( E \), that is, the vectors \( v_1, \ldots, v_m \) form a Hilbert basis of \( E \). (A finite set of vectors \( v_1, \ldots, v_m \) is a Hilbert basis of \( C=\text{cone}(a_1, \ldots, a_t) \) if \( C \cap \mathbb{Z}^n = \mathbb{N}v_1 + \cdots + \mathbb{N}v_m \).) One obvious choice for \( E \), and the one we will make, is the cone generated by \( v_1, \ldots, v_m \), that is

\[
E = \left\{ x = V \lambda = \lambda_1 v_1 + \cdots + \lambda_m v_m \middle| \lambda_1, \ldots, \lambda_m \in \mathbb{R}_+ \right\}.
\]

(4.2)

In general, \( v_1, \ldots, v_m \) may or may not be a Hilbert basis of this cone. However, for our examples we find that \( v_1, \ldots, v_m \) is indeed a Hilbert basis of \( E \). This checking process is equivalent to the verification step mentioned in the last section. The set \( S \) is thus given by

\[
S = E \cap \mathbb{N}^n,
\]

(4.3)

and the fusion inequalities are simply the facets of the cone \( E \). We can therefore use Farkas’ lemma (or any other method) to obtain the facets of \( E \). Now, it turns out that the fusion inequalities are also the facets of a polytope. The remainder of this section is devoted to this reinterpretation.

If we write the vectors \( x \in E \) as

\[
x = \begin{pmatrix} x_0 \\ x \end{pmatrix},
\]

(4.4)
we have that the fusion couplings at level $k$ are the integral points of the space, $P^{(k)}$, corresponding to the intersection between the hyperplane $x_0 = k$ and the cone $E$. If we drop the $x_0$ component, which has value $k$ in $P^{(k)}$, we can describe $P^{(k)}$ as

$$P^{(k)} = \left\{ x \mid \begin{pmatrix} k \\ x \end{pmatrix} \in E \right\} = \left\{ x \mid \begin{pmatrix} k \\ x \end{pmatrix} = V \lambda, \lambda \in \mathbb{R}_+^m \right\} = \left\{ x = \sum_i v_i \lambda_i \mid \lambda \in \mathbb{R}_+^m, \sum_i \lambda_i w_i = k \right\}. \quad (4.5)$$

The integral points of $P^{(k)}$ are

$$S^{(k)} = P^{(k)} \cap \mathbb{N}^{n-1}, \quad (4.6)$$

which are essentially the possible fusion couplings at level $k$ (by adding to the elements of $S^{(k)}$ an extra component $x_0$ equal to $k$, we recover the usual fusion couplings). Because $0 < w_i < \infty$, the transformation

$$\lambda_i \rightarrow \lambda_i' = \lambda_i (w_i/k), \quad v_i \rightarrow v_i' = (k/w_i)v_i, \quad w_i \rightarrow w_i' = (k/w_i)w_i = k, \quad (4.7)$$

is well defined, and if we further set

$$V' = \begin{pmatrix} w_1' & \cdots & w_m' \\ v_1' & \cdots & v_m' \end{pmatrix} = \begin{pmatrix} k & \cdots & k \\ v_1' & \cdots & v_m' \end{pmatrix}, \quad (4.8)$$

$P^{(k)}$ can now be given as

$$P^{(k)} = \left\{ x = \sum_i v_i' \lambda_i' \mid \lambda' \in \mathbb{R}_+^m, \sum_i \lambda_i' = 1 \right\}, \quad (4.9)$$

which by definition is the polytope given by the convex hull of the vertices $v_i'$, $i = 1, \ldots, m$.

The main theorem of polytope theory [8] tells us that $P$ can be equivalently described as a solution set of a finite system of $\ell$ linear inequalities (facets), that is

$$P^{(k)} = \left\{ x \mid C x \leq -k b \right\} = \left\{ x \mid C' \begin{pmatrix} k \\ x \end{pmatrix} \leq 0 \right\}, \quad (4.10)$$

where $C'$ is the concatenation $(bC)$ of $b$ and $C$, with $C \in \mathbb{R}^{\ell \times (n-1)}$ and $b \in \mathbb{R}^{\ell \times 1}$. In the last expression, we have made explicit the fact that the polytope inequalities translate into inequalities that the fusion elements at $x_0 = k$ must satisfy. We have thus obtained that the fusion inequalities are the facets of a polytope.

From a practical point of view, we stress that there exist powerful programs that give the polytope facets from its vertices (and vice-versa). The authors have used the “cdd” package of K. Fukuda [9] for computations in this article. For a description of other methods, we refer the reader to [8].
5. The symmetry group of the fusion polytopes: generalities

The fusion polytope is a geometrical object and so it is natural to look for its symmetry group. However, for a polytope there are several different kinds of symmetry we can consider. For example, consider the polytope \( E \) in \( \mathbb{R}^2 \) with vertices \((0,0), (0,2), (1,2)\) and \((1,0)\) which we label 1, 2, 3 and 4. This polytope is fixed by the reflections in the lines \( y = 1 \) and \( x = 1/2 \) and by a 180 degree rotation about the point \((1/2, 1)\). These are examples of Euclidean (length preserving) symmetries. Except for the identity transformation, there are no other Euclidean symmetries of \( E \). But there are additional symmetries if we consider affine transformations, for example \( x \to 1 - y/2, \ y \to 2 - 2x \) fixes the vertices 2 and 4, while exchanging 1 and 3.

In general, affine symmetries of a polytope \( E \) in \( \mathbb{R}^{n-1} \) are transformations of the form

\[
\tilde{\zeta} : \mathbf{x} \to A \mathbf{x} + k \mathbf{b},
\]

with \( A \in \mathbb{R}^{(n-1) \times (n-1)} \) and \( b \in \mathbb{R}^{(n-1) \times 1} \), such that

\[
\tilde{\zeta} E = E
\]

If the matrix \( A \) is orthogonal then the affine symmetry is also Euclidean.

A convenient method of finding the affine symmetries is to identify \( \mathbb{R}^{n-1} \) with the plane \( x_0 = k \) in \( \mathbb{R}^n \). Then the affine extension of \( \tilde{\zeta} \), denoted \( \zeta \), is given by

\[
\zeta : \begin{pmatrix} k \\ \mathbf{x} \end{pmatrix} \to \begin{pmatrix} k \\ A \mathbf{x} + k \mathbf{b} \end{pmatrix},
\]

or, in matrix form, as

\[
\zeta : \begin{pmatrix} k \\ \mathbf{x} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ b & A \end{pmatrix} \begin{pmatrix} k \\ \mathbf{x} \end{pmatrix}.
\]

Now, since any polytope is completely characterized by its vertices or its facets, a transformation that leaves \( E \) invariant has to leave its set of vertices and its set of facets invariant. That is, we must have from (4.9) and (4.10), that the action of \( \tilde{\zeta} \) permutes \( \mathbf{v}_1', \cdots, \mathbf{v}_m' \), so that

\[
\tilde{\zeta} (\mathbf{v}_1' \cdots \mathbf{v}_m') = (\tilde{\zeta}(\mathbf{v}_1') \cdots \tilde{\zeta}(\mathbf{v}_m')) = (\mathbf{v}_1' \cdots \mathbf{v}_m') \sigma,
\]

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for some permutation matrix $\sigma$. Using $V'$, whose columns are vectors of the type $\binom{k}{x}$, the requirement (5.5) can be put in matrix form:

$$B(\zeta) V' = V' \sigma$$

(5.6)

where $B(\zeta)$ is the matrix of the transformation $\zeta$ defined in (5.4). We shall be mainly interested in finding the affine symmetries of the fusion polytopes and methods for finding these symmetries will be described in the next section.

Note, however, that in general potential symmetries of a polytope can often be ruled out by considering the polytope’s combinatorial structure. Consider, for example, the polytope $E$ introduced at the beginning of this section. As mentioned above, a symmetry of $E$ must permute vertices and facets and so there cannot be a symmetry of any type which exchanges vertices 2 and 3 while fixing vertices 1 and 4, since vertices 1 and 2 are joined by a common edge while 1 and 3 are not.

One type of combinatorial symmetry would be to consider all permutations of the faces of the polyhedron which preserve the face lattice (see for example [10] page 128). However, a simpler type of combinatorial symmetry, which is easy to find in our examples, is to look for permutations of vertices and facets which preserve the vertex-facet incidence matrix. This matrix is easy to calculate given the vertices and the inequalities representing the facets: the incidence matrix has an entry 1 in position $(i, j)$ if the $i$th vertex saturates (i.e. satisfies with equality) the $j$th facet inequality and is zero otherwise.

If we label the edges 1-2, 2-3, 3-4 and 4-1 of our example polytope $E$ as 1, 2, 3 and 4, corresponding respectively to the inequalities $x \geq 0, y \leq 2, x \leq 1, y \geq 0$, we obtain the incidence matrix:

$$I = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

(5.7)

If permutations $\sigma$ and $\tau$ of the vertices and the edges respectively satisfy $\sigma I \tau = I$ then we say that we have an incidence symmetry. If $\sigma$ and $\tau$ are such symmetries then $\sigma I \tau^{-1} I^t \sigma^{-1} = I I^t$ and so $\sigma$ commutes with $I I^t$. Call any vertex permutation which commutes with $I I^t$ a vertex symmetry.
Note that any affine symmetry induces a combinatorial face lattice symmetry and also a combinatorial incidence symmetry and a vertex symmetry. These maps are injective since if an affine symmetry fixes all the vertices of a polytope then it fixes the whole polytope since a polytope is the convex hull of its vertices. Thus we have the inclusions: Euclidean symmetries \( \subseteq \) affine symmetries \( \subseteq \) combinatorial face lattice symmetries \( \subseteq \) combinatorial incidence symmetries \( \subseteq \) vertex symmetries. Thus finding the vertex symmetries gives a “upper bound” on the other types of symmetries.

For the polytope \( E \) we have

\[
\mathcal{I} I^t = \begin{pmatrix}
2 & 1 & 0 & 1 \\
1 & 2 & 1 & 0 \\
0 & 1 & 2 & 1 \\
1 & 0 & 1 & 2
\end{pmatrix}
\]  

(5.8)

Since any vertex symmetry \( \sigma \) commutes with \( \mathcal{I} I^t \), it must also map eigenvectors to eigenvectors and using this it is not difficult to show that every vertex symmetry of \( E \) arises from an affine symmetry.

In general, however, the group of combinatorial symmetries will be much larger than the group of affine symmetries. For example if we take a “generic” convex \( n \)-gon in \( \mathbb{R}^2 \), its combinatorial symmetries will be a dihedral group, but there will be no affine symmetries except for the identity transformation.

Perhaps surprisingly, we find that for our examples that every vertex symmetry comes from an affine symmetry and so the most economical method of finding the affine symmetries would be to calculate first the vertex symmetries. However, this is not the usual situation and the only real advantage in our cases turns out to be that the matrices we have to calculate have smaller entries if we use vertex symmetries. So in the next section we give general methods for finding all the affine symmetries for any polytope.

In the rest of this paper we shall only be concerned with finding affine symmetries of our polytopes, and so, unless stated otherwise, by a symmetry we shall mean an affine symmetry. Similarly when we refer to a vertex permutation as being a symmetry we mean that it arises from an affine symmetry as explained at the start of the next section.
6. The symmetry group of the fusion polytopes: technical tools

In this section we will introduce some tools for finding the affine symmetries of our fusion polytopes introduced in the last section. We do this by considering vertex permutations and finding which ones arise from affine symmetries. These tools, however, apply in more generality than these applications and so we start with a more general definition of a symmetry of a matrix:

**Definition.** Let $M$ be an $n \times m$ matrix. An $m \times m$ permutation matrix $\sigma$ is a symmetry of $M$ iff $M\sigma = BM$ for some $n \times n$ matrix $B$.

The set of symmetries of $M \in \mathbb{N}^{n \times m}$ is a subset of the group of permutation matrices which is closed under multiplication, hence is a group. The corresponding set of matrices $B$ does not necessarily form a group, but it does if the rank of $M$ is $n$.

This definition is inspired by equation (5.6) since for the fusions of the affine Lie algebra $\widehat{g}$ we will take $M$ to be the matrix $V'$. So in the terminology of the last section, a symmetry of $V'$ is a vertex symmetry of our fusion polytope which comes from an affine symmetry. As explained in the last section we will call these vertex symmetries simply symmetries to avoid tedious repetition. The group of symmetries of $V'$ will be denoted by $G[\widehat{g}]$. For all our examples the rank of $V'$ turns out to be $n$, in other words the fusion polytope has “full-dimension”.

**Proposition 1.** An $m \times m$ permutation matrix $\sigma$ is a symmetry of the $n \times m$ matrix $M$ iff $\sigma N(M) = N(M)$ where $N(M) = \{z \mid Mz = 0\}$.

**Proof:** If $z \in N(M)$ then $M\sigma z = BMz = 0$, so $\sigma N(M) \subseteq N(M)$. Also $\sigma^{-1}N(M) \subseteq N(M)$, since $\sigma^{-1}$ is also a symmetry of $M$ (because the symmetries form a group) and so $\sigma N(M) = N(M)$. Conversely, if $\sigma N(M) = N(M)$ then the matrices $M$ and $M\sigma$ have the same null space. Thus their rows generate the annihilator of $N(M)$. In particular the rows of $M\sigma$ are contained in the span of the rows of $M$ and so there is a matrix $B$ such that $BM = M\sigma$. So $\sigma$ is a symmetry of $M$.

When the kernel of $M$ has dimension 0 or 1, this proposition is sufficient to classify the symmetries. As we will show in the following sections, this holds for the fusion polytopes for $\widehat{su}(2)$ and $\widehat{su}(3)$.
If the kernel has larger dimension, we will need another approach. However, to do so we make the additional assumptions that $M$ has real entries and that its rank is $n$. As noted above, the fusion polytope matrices $V'$ have these properties.

With these two assumptions, using the Gram-Schmidt procedure, we can find an invertible $n \times n$ matrix $L$ such that the rows of $W = LM$ form an orthonormal basis of the row space of $M$. Since $M \sigma = BM$ if and only if $W \sigma = LBL^{-1}W$, it is clear that $\sigma$ is a symmetry of $M$ if and only if $\sigma$ is a symmetry of $W$.

**Proposition 2.** $\sigma$ is a symmetry of $M$ if and only if $Q \sigma = \sigma Q$, where $Q = W^T W$.

**Proof:** If $\sigma$ is a symmetry of $M$ then $\sigma$ is a symmetry of $W$ and so $W \sigma = TW$ for some $n \times n$ matrix $T$. Moreover, since $\sigma$ acts as an orthogonal transformation on the rows of $W$, $T$ is an orthogonal matrix. Hence, $W \sigma = TW$ implies $\sigma^{-1}W^T = W^T T^{-1}$. So $Q \sigma = W^T W \sigma = W^T TW = \sigma W^T W$ as required. For the converse, note that the matrix $Q$ performs the orthogonal projection onto the row space of $M$. So if $\sigma$ commutes with the projection matrix $Q$ then it maps the row space of $M$ to itself and so is a symmetry of $M$.

Since $\sigma$ is a permutation, we can read off some of its properties directly from $Q$. For example a row of $Q$ can be mapped to another row of $Q$ by a symmetry only if the two rows have the same set of entries. A second simplification occurs by observing that if $\sigma$ is a symmetry and $\sigma u = \lambda u$ for some vector $u$ then $\sigma Q u = Q \sigma u = \lambda Q u$. In particular, if $u$ is fixed by the symmetry group, so is $Q u$. We will apply these two observations when we compute the symmetries for the $\tilde{sp}(4)$ fusion polytope.

7. **The symmetry group of the $\tilde{su}(2)_k$ polytope**

For $\tilde{su}(2)$, the matrix $V'$ takes the form $kV$ with $V$ given in (3.3); $N(V')$ is thus trivial. Therefore, every permutation is a symmetry. Since $n = m = 4$, this gives $S_4$ as the symmetry group $G[\tilde{su}(2)]$. $S_4$ is generated by the permutations $(1, 2, 3, 4)$ and $(1, 2)$, where $(i, j, \ldots, k)$ stands for a cyclic permutation of $i, j, \ldots, k$. Since $V'$ is invertible, we can easily find the corresponding transformation $\zeta$ acting on $x^T = (k, \lambda_1, n_{11}, n_{12})$. It reads:

\begin{equation}
(1, 2, 3, 4) : (k, \lambda_1, n_{11}, n_{12}) \mapsto (k, k - \lambda_1 - n_{11} + n_{12}, \lambda_1 - n_{12}, k - \lambda_1 - n_{11}) \quad (7.1)
\end{equation}

\begin{equation}
(1, 2) : (k, \lambda_1, n_{11}, n_{12}) \mapsto (k, k - n_{11} - n_{12}, n_{11}, k - \lambda_1 - n_{11})
\end{equation}
The fusion basis is given in (2.1). Labeling these inequalities from 1 to 4, the symmetries (1, 2, 3, 4) and (1, 2) permute the inequalities. However, the inequalities correspond to the polytope facets, which being dual to the vertices transform by the inverse of the vertex transform. Thus if $x \mapsto Bx$ is the transformation corresponding to the vertices, then $\alpha^\top \mapsto \alpha^\top B^{-1}$ is the appropriate transformation for the facets. The necessity of this can be seen from the fact that the transformations should preserve the incidence relations of the vertices and facets and that this involves quantities of the form $\alpha^\top x$.

From this it follows that the action of (1, 2) is $[[1, 2], [3], [4]]$, i.e., it permutes the first two inequalities and fixes the last two. Similarly the action of (1, 2, 3, 4) is $[[1, 2, 3], [4]]$. Thus the fusion basis is generated by any one of its inequalities under that action of the symmetry group.

Is there a simple way to understand these symmetries in terms of the symmetries of the fusion coefficients? First, notice that the fusion coefficients are described in terms of a smaller number of labels than those necessary for the complete description of the fusion basis. The complete set of labels can be split into two subsets: the Dynkin labels of the three weights under consideration and the ‘missing labels’. If some symmetries do not involve in an essential way the missing labels, they will project onto fusion-coefficient symmetries. However, if the missing labels are an essential part of the symmetry transformations, the symmetry will disappear in the projection. For instance, tensor-product coefficients can be obtained by projection of the fusion coefficients. The latter require an extra variable for their description, the level $k$, and fusion coefficients have more symmetries than the tensor-product coefficients. The extra symmetries are the outer automorphisms – see the next paragraph – and these involve the level in an essential way. Let us make the general situation more precise: Denote collectively the finite Dynkin labels $\{\lambda_i, \mu_i, \nu_i\}$ and $k$ by $D$ and the set of missing labels by $\gamma$. A facet symmetry is generically of the form $\{D, \gamma\} \rightarrow \{D'(D, \gamma), \gamma'(D, \gamma)\}$. This will be a symmetry of the fusion coefficients only when $D'$ does not depend upon $\gamma$. Therefore there is no simple relationship between the symmetries of the facets and the symmetries of the fusion coefficients. In this regard, each algebra has to be studied separately.

The symmetries of the fusion coefficients include those that are level-independent, i.e., the symmetries of the corresponding tensor-product coefficients; these are the conjugation of the three weights, $(\hat{\lambda}, \hat{\mu}, \hat{\nu}) \rightarrow (\hat{\lambda}^*, \hat{\mu}^*, \hat{\nu}^*)$ and the different permutations of $\hat{\lambda}, \hat{\mu}$ and $\hat{\nu}^*$. The remaining symmetries are intrinsically affine. These include the outer-automorphism
symmetries which take the following form: if \( A, A' \) are two arbitrary elements of the outer-automorphism group of \( \hat{g} \), the fusion coefficients satisfy
\[
N_{A\lambda, A'\hat{\mu}}^{(k)} \overset{AA'\hat{\nu}}{\longrightarrow} N_{\hat{\lambda}\hat{\nu}}^{(k)}
\]  
(7.2)

For \( \hat{su}(N) \), this group has order \( N^N = 1 \). The symmetry group can be larger than that generated by tensor product symmetries and outer automorphisms, see for example [11,12,13]. There are often symmetries which exist for some, but not all levels. The method we present here will not detect this type of symmetry and from this point we will exclude them. In other words, by symmetries of the fusion coefficients we mean symmetries which exist for all levels.

For \( \hat{su}(2) \), there are no missing labels. Hence the symmetry group of the polytope must be identical to the symmetry group of the fusion coefficients which exist for all \( k \) and that leave \( k \) fixed. This group is isomorphic to the semi-direct product \( (S_2 \times S_2) : S_3 \) . The \( S_3 \) factor comes from the permutation of the three weights, while the two factors of \( S_2 \) account for the two copies of the outer automorphism group (one acting on the weight \( \hat{\lambda} \), the other acting on \( \hat{\mu} \) ). The conjugation action of \( S_3 \) is via the outer automorphisms of \( S_2 \times S_2 \) which permute the non-identity elements. The group \( S_4 \) contains a group \( S_2 \times S_2 \) generated by the cycles of type \( 2^2 \). Any of the four \( S_3 \) subgroups act on this \( S_2 \times S_2 \) as outer automorphisms by conjugation and so the symmetry group is isomorphic to \( S_4 \).

Let us first reexpress all the basis symmetries in terms of the Dynkin labels:
\[
(1, 2, 3, 4) : (k, \lambda_1, \mu_1, \nu_1) \rightarrow (k, k - \nu_1, k - \mu_1, \lambda_1) \\
(1, 2) : (k, \lambda_1, \mu_1, \nu_1) \rightarrow (k, k - \mu_1, k - \lambda_1, \nu_1) \\
(2, 3) : (k, \lambda_1, \mu_1, \nu_1) \rightarrow (k, \lambda_1, \nu_1, \mu_1) \\
(3, 4) : (k, \lambda_1, \mu_1, \nu_1) \rightarrow (k, \mu_1, \lambda_1, \nu_1)
\]  
(7.3)

(Clearly, the last two symmetries can be obtained from the first two.)

Let us make explicit the correspondence between these symmetries and symmetries of the fusion coefficients. For this, notice first that the multiplicity of the \( \hat{su}(2)_k \) product \( \hat{\lambda} \times \hat{\mu} \supset \hat{\nu} \) is the same as that of \( \hat{\lambda} \times \hat{\mu} \times \hat{\nu} \). Let \( P_{12} \) be the operator that permutes the first two weights and similarly for the other permutation operators. Moreover, let \( a \) be the \( \hat{su}(2) \) automorphism that interchanges the two simple roots, hence the two Dynkin labels:
\[
a[k - \lambda_1, \lambda_1] = [\lambda_1, k - \lambda_1].
\]
Therefore, the nontrivial actions on \( \hat{\lambda} \times \hat{\mu} \times \hat{\nu} \) are simply:
\[
a\hat{\lambda} \times a\hat{\mu} \times \hat{\nu} \supset 0 \quad a\hat{\lambda} \times \hat{\mu} \times a\hat{\nu} \supset 0 \quad \hat{\lambda} \times a\hat{\mu} \times a\hat{\nu} \supset 0
\]  
(7.4)
Denote these actions respectively as \((a, a, 1), (a, 1, a), (1, a, a)\). The fusion basis symmetries can then be related directly to \(a\) and \(P\) actions as follows:

\[
\begin{align*}
(1, 2, 3, 4) &\equiv (a, a, 1)P_{13} \\
(1, 2) &\equiv (a, a, 1)P_{12} \\
(2, 3) &\equiv P_{23} \\
(3, 4) &\equiv P_{12}
\end{align*}
\]  

(7.5)

‘Pure’ finite or affine symmetries can be obtained by composition, e.g.,

\[
\begin{align*}
(2, 3, 4) &\equiv P_{321} \\
(1, 2)(3, 4) &\equiv (a, a, 1)
\end{align*}
\]  

(7.6)

8. The symmetry group of the \(\widehat{su}(3)_k\) polytope

The situation for \(\widehat{su}(2)\) is not typical in two ways. First, there are no missing labels; hence any permutation of the vertices is bound to be a symmetry of the fusion coefficients. In addition, there are no linear relations between the elementary solutions. Such relations will induce severe constraints on the possible lifts of the fusion-coefficient symmetries to polytope symmetries.

For \(\widehat{su}(3)\), the elementary couplings are (using the notation \(\hat{\lambda} = [\lambda_0, \lambda_1, \lambda_2]\) with \(\lambda_0 + \lambda_1 + \lambda_2 = k\) and the LR variables - cf. [3]. Note that \(\lambda_1\) and \(\lambda_2\) are Dynkin labels and not the partition labels to which they usually refer in the Littlewood-Richardson rule as in [4] chapter 1 for example):

\[
\begin{align*}
&\widehat{E}_0: [1, 0, 0] \times [1, 0, 0] \supset [1, 0, 0]: \quad d \quad (1, 0, 0, 0, 0, 0, 0, 0) \quad (1, 1) \\
&\widehat{E}_1: [0, 1, 0] \times [0, 0, 1] \supset [1, 0, 0]: \quad dL_1N_{12}N_{23} \quad (1, 1, 0, 0, 1, 0, 0, 1) \quad (a, a^2) \\
&\widehat{E}_2: [0, 1, 0] \times [1, 0, 0] \supset [0, 1, 0]: \quad dL_1 \quad (1, 1, 0, 0, 0, 0, 0, 0) \quad (a, 1) \\
&\widehat{E}_3: [1, 0, 0] \times [0, 1, 0] \supset [0, 1, 0]: \quad dN_{11} \quad (1, 0, 0, 1, 0, 0, 0, 0) \quad (1, a) \\
&\widehat{E}_4: [0, 0, 1] \times [0, 1, 0] \supset [1, 0, 0]: \quad dL_2N_{13} \quad (1, 0, 1, 0, 0, 1, 0, 0) \quad (a^2, a) \quad (8.1) \\
&\widehat{E}_5: [0, 0, 1] \times [1, 0, 0] \supset [0, 0, 1]: \quad dL_2 \quad (1, 0, 1, 0, 0, 0, 0, 0) \quad (a^2, 1) \\
&\widehat{E}_6: [1, 0, 0] \times [0, 0, 1] \supset [0, 0, 1]: \quad dN_{11}N_{22} \quad (1, 0, 0, 1, 0, 0, 1, 0) \quad (1, a^2) \\
&\widehat{E}_7: [0, 1, 0] \times [0, 1, 0] \supset [0, 0, 1]: \quad dL_1N_{12} \quad (1, 1, 0, 0, 1, 0, 0, 0) \quad (a, a) \\
&\widehat{E}_8: [0, 0, 1] \times [0, 0, 1] \supset [0, 1, 0]: \quad dL_2N_{11}N_{23} \quad (1, 0, 1, 1, 0, 0, 0, 1) \quad (a^2, a^2)
\end{align*}
\]
Besides each coupling, we have written the ‘exponential description’, the corresponding vector $\epsilon_i$ with entries in the order $(k, \lambda_1, \lambda_2, n_{11}, n_{12}, n_{13}, n_{22}, n_{23})$, as well as the action of the outer automorphism on the first two weights of $\hat{E}_0$ (in the form $(a^n, a^m)$) that yields the coupling under consideration.

The corresponding facets (the fusion basis) are found to be

\begin{align*}
    n_{12} &\geq 0 & \lambda_2 + n_{12} - n_{13} - n_{23} &\geq 0 \\
    n_{13} &\geq 0 & n_{11} - n_{22} &\geq 0 \\
    n_{22} &\geq 0 & n_{11} + n_{12} - n_{22} - n_{23} &\geq 0 \\
    n_{23} &\geq 0 & k - \lambda_1 - \lambda_2 - n_{22} &\geq 0 \\
    \lambda_1 - n_{12} &\geq 0 & k - \lambda_1 - \lambda_2 - n_{11} + n_{23} &\geq 0 \\
    \lambda_2 - n_{13} &\geq 0 & k - \lambda_1 - n_{13} - n_{11} &\geq 0
\end{align*}

This agrees with the system of inequalities obtained in [3].

For $\widehat{su}(3)$, the matrix $V'$ is $8 \times 9$:

\[
V' = k \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and since the rank of $V'$ is 8, $N(V')$ is one-dimensional and is spanned by

\[
w^\top = (1 \ -1 \ 0 \ -1 \ 0 \ -1 \ 0 \ 1 \ 1)
\]

The condition on $\sigma$ is thus $\sigma w = \lambda w$ for some scalar $\lambda$ and, since the eigenvalues of $\sigma$ are roots of unity, $\lambda$ is either 1 or $-1$. Thus $G[\widehat{su}(3)]$ is isomorphic to $S_3 \times (S_3 \wr S_2)$. The $S_3$ permutes the 0’s of $w$ and $S_3 \wr S_2$ (the wreath product of $S_3$ and $S_2$ i.e., the direct product of two $S_3$’s with $S_2$ acting by exchanging them) permutes and exchanges the 1’s and $-1$’s.
Let \( x^T = (k, \lambda_1, \lambda_2, n_{11}, n_{12}, n_{13}, n_{22}, n_{23}) \). The action of a set of generators of the symmetry group on the variables is:

\[
(3, 5, 7) : x^T \mapsto (k, n_{12} + n_{22}, \lambda_1 + \lambda_2 - n_{12} - n_{13}, n_{11} + n_{13} - n_{22}, n_{12}, \lambda_1 - n_{12}, n_{13}, n_{23})
\]

\[
(3, 5) : x^T \mapsto (k, n_{12} + n_{13}, \lambda_1 + \lambda_2 - n_{12} - n_{13}, n_{11}, n_{12}, \lambda_1 - n_{12}, n_{22}, n_{23})
\]

\[
(1, 8, 9) : x^T \mapsto (k, k - \lambda_2 - n_{11} - n_{12} + n_{23}, \lambda_2 + n_{12} - n_{23}, n_{11} + n_{12} - n_{23}, k - \lambda_1 - \lambda_2 - n_{11} + n_{23}, n_{13}, n_{22}, n_{23})
\]

\[
(1, 8) : x^T \mapsto (k, k - \lambda_2 - n_{11} - n_{12} + n_{23}, \lambda_2, n_{11}, k - \lambda_1 - \lambda_2 - n_{11} + n_{23}, n_{13}, n_{22}, n_{23})
\]

(8.5)

Together with

\[
(1, 2)(4, 8)(6, 9) : x^T \mapsto (k, k - \lambda_2 - n_{12} - n_{22}, \lambda_2, \lambda_2 + n_{12} - n_{13} + n_{22} - n_{23}, k - \lambda_1 - \lambda_2 - n_{22}, n_{13}, n_{22}, k - \lambda_1 - n_{13} - n_{11})
\]

(8.6)

The fusion basis is given by (8.2). Labeling these inequalities successively, column by column, from 1 to 12 (i.e., 8 corresponds to \( n_{11} - n_{22} \geq 0 \)), we find that the action of the symmetry group on the inequalities considered as polytope facets is generated by:

\[
(3, 5, 7) : [[1], [2, 3, 5], [4], [6], [7], [8], [9], [10], [11], [12]]
\]

\[
(3, 5) : [[1], [2, 5], [3], [4], [6], [7], [8], [9], [10], [11], [12]]
\]

\[
(1, 8, 9) : [[1, 4, 11], [2], [3], [5], [6, 12, 7], [8], [10], 9]
\]

(8.7)

\[
(1, 8) : [[1, 11], [2], [3], [4], [5], [6], [7, 12], [8], [9], 10]
\]

\[
(1, 2)(4, 8)(6, 9) : [[1, 10], [2], [3], [4, 12], [5], [6], [7, 8], [9], [11]]
\]

where as before \([i, j, \cdots, k] \) stands for a cyclic permutation of the inequalities \( i, j, \cdots, k \). Thus there are two orbits under the symmetry group. One consisting of the inequalities 2, 3 and 5 and the other consisting of the inequalities 1, 4, 6, 7, 8, 9, 10, 11 and 12. So the fusion basis is generated by the two inequalities \( n_{12} \geq 0 \) and \( n_{13} \geq 0 \) under the action of \( G[\widehat{su}(3)] \).

Let us now try to understand these results. As already pointed out, these symmetries must be compatible with the symmetries of the fusion coefficients. But there is a further constraint on the fusion symmetries that has not been spelled out yet: in general there are linear relations among the elementary couplings and the symmetries of the facets must preserve these relations. In the \( \widehat{su}(3) \) case, there is only one such linear relation, which is:

\[
\widehat{E}_0 \widehat{E}_7 \widehat{E}_8 = \widehat{E}_1 \widehat{E}_3 \widehat{E}_5
\]

(8.8)
This explains the existence of the three $S_3$ blocks: permutations among the sets \{${\tilde{E}_0, \tilde{E}_7, \tilde{E}_8}$\} and \{${\tilde{E}_1, \tilde{E}_3, \tilde{E}_5}$\} are symmetries that preserve each sides of the relation. The third $S_3$ factor is bound to relate the three remaining vertices. Moreover, the $S_2$ group generated by $(1, 2)(4, 8)(6, 9)$ is another transformation that leaves the relation unchanged: but instead of leaving each side invariant, it interchanges the two sides of the relation: ${\tilde{E}_0} \leftrightarrow \tilde{E}_1$, ${\tilde{E}_3} \leftrightarrow \tilde{E}_7$, ${\tilde{E}_5} \leftrightarrow \tilde{E}_8$. The occurrence of a linear relation is thus responsible for the fact that the symmetry group is not $S_9$ but only a subgroup thereof.

Of course, the existence of relations is intimately connected with the matrix $V'$. In fact, $N(V')$ is precisely the set of generating relations (in the sense of [3]) out of which all the relations can be obtained.

To complete the analysis of the $\tilde{su}(3)_k$ polytope symmetries, let us investigate their explicit relation with symmetries of the fusion coefficients. For this, we first reexpress the symmetry transformations in terms of the Dynkin labels of the three weights plus $n_{23}$. Thus $n_{23}$ is the missing label. We also reformulate the results in terms of the symmetrized product: $\hat{\lambda} \times \hat{\mu} \supset \hat{\nu} \geq 0$. Recall that the multiplicity of $\hat{\lambda} \times \hat{\mu} \supset \hat{\nu}^*$ is the same as that of $\hat{\lambda} \times \hat{\mu} \supset \hat{\nu} \geq 0$, where $^*$ denotes the finite weight conjugation; for $\tilde{su}(3)$, it amounts to interchanging the two finite Dynkin labels. The precise transformation relations are as follows:

$$
\begin{align*}
n_{11} &= L_2 - \lambda_1 - \lambda_2 & n_{12} &= L_1 - L_2 + n_{23} \\
n_{13} &= L_2 - \nu_1 - \nu_2 - n_{23} & n_{22} &= \mu_2 - n_{23}
\end{align*}
$$

(8.9)

With the vector $y^\top$ defined as $y^\top = (k, \lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2, n_{23})$, we can rewrite the symmetries of the fusion basis as:

\begin{align*}
(3, 5, 7): & \quad y^\top \to (k, L_2 - \lambda_2 - \nu_2, L_2 - \mu_1 - \mu_2, L_2 - \lambda_2 - \mu_2, L_2 - \nu_1 - \nu_2, L_2 - \mu_2 - \nu_2, \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad L_2 - \lambda_1 - \lambda_2, n_{23}) \\
(3, 5): & \quad y^\top \to (k, L_1 - \nu_1 - \nu_2, L_2 - \mu_1 - \mu_2, L_2 - \lambda_2 - \mu_2, \mu_2, \nu_1, L_1 - \lambda_1 - \nu_1, n_{23}) \\
(1, 8, 9): & \quad y^\top \to (k, k - L_1 + \lambda_1, L_2 - \mu_2 - \nu_2, k - L_1 + \mu_1, L_2 - \lambda_2 - \nu_2, \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad k - L_1 + \nu_1, L_2 - \lambda_2 - \mu_2, L_1 - L_2 + n_{23}) \\
(1, 8): & \quad y^\top \to (k, k - L_1 + \lambda_1, \lambda_2, k - L_1 + \mu_1, \mu_2, k - L_1 + \nu_1, \nu_2, n_{23}) \\
(1, 2)(4, 8)(6, 9): & \quad y^\top \to (k, k - L_2 + \nu_2, \lambda_2, \mu_1, k - L_1 + \nu_1, L_2 - \lambda_1 - \lambda_2, L_1 - \mu_1 - \mu_2, \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad n_{23} + k - L_1 - \mu_2 + \nu_1)
\end{align*}

(8.10)
where \( L_i = (\lambda + \mu + \nu, \omega_i) \) with \( \omega_i \) the \( i \)-th fundamental weight. The remarkable feature of these symmetry transformations is that they send \( y^\top \rightarrow y'^\top \) such that none of the prime variables except \( n'_{23} \) depends upon \( n_{23} \). In other words, the new Dynkin labels, collectively denoted by \( D' \), do not depend upon \( n_{23} \). Therefore, these symmetries map a state (i.e., a tableau) of a given fusion to another state of another fusion. The same is necessarily true for the inverted transformations. There is thus a one-to-one correspondence between the two fusions, i.e., they have the same multiplicity! In other words, the symmetries of the fusion basis are symmetries of the fusion coefficients. For instance,

\[
(3, 5, 7) : \quad N^{(k)}_{(\lambda_1, \lambda_2)(\mu_1, \mu_2)(\nu_1, \nu_2)} = N^{(k)}_{(L_2-\lambda_2-\nu_2, L_2-\mu_1-\mu_2)(L_2-\lambda_2-\mu_2, L_2-\nu_1-\nu_2)(L_2-\mu_2-\nu_2, L_2-\lambda_1-\lambda_2)}
\]

\[
(3, 5) : \quad N^{(k)}_{(\lambda_1, \lambda_2)(\mu_1, \mu_2)(\nu_1, \nu_2)} = N^{(k)}_{(L_1-\nu_2, L_2-\mu_1-\mu_2)(L_2-\lambda_2-\mu_2, \nu_1-\lambda_1-\nu_1)}
\]

\[
(1, 8, 9) : \quad N^{(k)}_{(\lambda_1, \lambda_2)(\mu_1, \mu_2)(\nu_1, \nu_2)} = N^{(k)}_{(k-L_1+\lambda_1, L_2-\mu_2-\nu_2)(k-L_1+\mu_1, L_2-\lambda_2-\nu_2)(k-L_1+\nu_1, L_2-\lambda_2-\nu_2)}
\]

\[
(1, 8) : \quad N^{(k)}_{(\lambda_1, \lambda_2)(\mu_1, \mu_2)(\nu_1, \nu_2)} = N^{(k)}_{(k-L_1+\lambda_1, \lambda_2)(k-L_1+\mu_1, \mu_2)(k-L_1+\nu_1, \nu_2)}
\]

\[
(1, 2)(4, 8)(6, 9) : \quad N^{(k)}_{(\lambda_1, \lambda_2)(\mu_1, \mu_2)(\nu_1, \nu_2)} = N^{(k)}_{(k-L_2+\nu_2, \lambda_2)(\mu_1, k-L_1+\nu_1)(L_2-\lambda_1-\lambda_2, L_1-\mu_1-\mu_2)}
\]

(8.11)

In this rewriting, the dependence upon the \( n_{23} \) variable is dropped.

These are clearly new fusion symmetries as they mix the labels of the three representations. It is simple to verify that they leave the explicit expression of the \( \widehat{su}(3)_k \) fusion coefficients (given in [14]) invariant. Here is a numerical illustration: \( (1, 8) \) maps

\[
[4, 10, 2] \times [1, 4, 11] \times [9, 3, 4] \supset 0 \leftrightarrow E_0 E_1^8 E_3^2 E_6^2 E_7^2 E_8^2, \quad E_1^8 E_2 E_3^2 E_4 E_6^2 E_7 E_8 \quad (8.12)
\]

onto

\[
[5, 9, 2] \times [2, 3, 11] \times [10, 2, 4] \supset 0 \leftrightarrow E_7 E_1^8 E_3^2 E_6 E_8^2, \quad E_1^8 E_2 E_3^2 E_4 E_6^2 E_0 E_8 \quad (8.13)
\]

Both fusions have two decompositions in terms of elementary couplings, meaning that they have the same multiplicity. This symmetry is manifestly distinct from those already known. It is not a trivial symmetry in that, in terms of the elementary-coupling decompositions, it corresponds to the interchange of \( E_0 \) and \( E_7 \) in each decomposition.

The usual fusion symmetries can be obtained from various combinations of the above symmetries of the fusion basis. For instance,
\[ (2, 4, 6)(3, 5, 7) \equiv P_{123} \]
\[ (8, 9)(5, 7)(2, 4) \equiv C \; P_{13} \]
\[ (2, 8)(1, 4)(6, 9) \equiv (1, a, a^2) \; P_{23} \]

where \( C \) is the conjugation:
\[ C(\hat{\lambda} \times \hat{\mu} \times \hat{\nu} \ni 0) = \hat{\lambda}^* \times \hat{\mu}^* \times \hat{\nu}^* \ni 0 \] 

We thus recover the known symmetries and additional ones.

9. The symmetry group of the \( \hat{sp}(4)_k \) polytope

The \( \hat{sp}(4) \) fusion rules are most conveniently described by means of the variables introduced in [15] and used in [3, 6] (with a slight change of notation), namely \( \{k, \lambda_1, \lambda_2, \mu_1, \mu_2, r_1, r_2, p, q\} \), with \( p, q \) and \( r_i/2 \) nonnegative integers. The Dynkin labels of the third weight are
\[ \nu_1 = r_2 - r_1 - 2p + \lambda_1 + \mu_1 \]
\[ \nu_2 = p - q - r_2 + \lambda_2 + \mu_2 \] 

In terms of the exponentiated variables, the elementary couplings are \( \hat{E}_0 = d \), together with
\[ \hat{A}_1 = dM_1 \]
\[ \hat{A}_2 = dL_1 \]
\[ \hat{A}_3 = dL_1 M_1 P Q \]
\[ \hat{B}_1 = dM_2 \]
\[ \hat{B}_2 = dL_2 \]
\[ \hat{B}_3 = dL_2 M_2 R_1^2 R_2^2 \]
\[ \hat{C}_1 = dL_2 M_1 Q \]
\[ \hat{C}_2 = dL_1 M_2 R_2^2 P \]
\[ \hat{C}_3 = dL_1 M_1 P \]
\[ \hat{D}_1 = d^2 L_1^2 M_2 R_2^2 P^2 \]
\[ \hat{D}_2 = d^2 L_2^2 M_1^2 R_1^2 \]
\[ \hat{D}_3 = d^2 L_2 M_2 R_2^2 \]

Taking these to be the vertices of a polytope, the corresponding facets are found to be:
\[ k - \lambda_1 - \lambda_2 - \mu_2 - r_1/2 + r_2 \geq 0 \]
\[ \mu_1 - q \geq 0 \]
\[ k - \lambda_1 - \lambda_2 - \mu_2 + r_2/2 \geq 0 \]
\[ \mu_1 - q - r_1 + r_2 \geq 0 \]
\[ k - \lambda_1 - \mu_1 - \mu_2 + p \geq 0 \]
\[ \mu_1 - p - r_1 + r_2 \geq 0 \]
\[ k - \lambda_1 - \lambda_2 - \mu_2 - \mu_1 + p + q + r_1/2 \geq 0 \]
\[ \mu_2 - r_2/2 \geq 0 \]
\[ \lambda_1 - p \geq 0 \]
\[ r_1 \geq 0 \]
\[ \lambda_2 - r_1/2 \geq 0 \]
\[ r_2 \geq 0 \]
\[ \lambda_2 - r_1/2 - q + p \geq 0 \]
\[ p \geq 0 \]
\[ \lambda_2 - r_2/2 - q + p \geq 0 \]
\[ q \geq 0 \]
in agreement with the results obtained in [3].

We next analyze the symmetries of the $\hat{s}p(4)$ fusion polytope. The matrix $V'$, with the column ordering $(\hat{E}_0, \hat{A}_1, \ldots, \hat{D}_3)$, reads

$$V' = k \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1/2 & 1/2 \ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \ \end{pmatrix} (9.4)$$

Unfortunately for this case the kernel of $V'$ is not one-dimensional. In fact, $N(V')$ is four-dimensional, which we understand from the fact that there are four generating relations – cf. [3] eq. (5.18). As a result, we must study the symmetries of $V'$ via the commutant of the matrix $Q$ that performs the orthogonal projection onto the row space of $V'$ (cf. proposition 2 of sect. 6). This matrix reads

$$Q = \frac{1}{108} \begin{pmatrix} 82 & 16 & 0 & 18 & 0 & 10 & -2 & -18 & -14 & -20 & 16 & 4 & 16 \ 16 & 67 & 0 & -9 & 0 & -20 & -14 & 9 & 10 & 13 & -14 & 28 & 22 \ 0 & 0 & 108 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 18 & -9 & 0 & 81 & 0 & -18 & 0 & 27 & 0 & 9 & 18 & 0 & -18 \ 0 & 0 & 0 & 0 & 108 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 10 & -20 & 0 & -18 & 0 & 82 & -2 & 18 & -14 & 16 & 16 & 4 & 16 \ -2 & -14 & 0 & 0 & 0 & -2 & 94 & 0 & 10 & -14 & 4 & 28 & 4 \ -18 & 9 & 0 & 27 & 0 & 18 & 0 & 81 & 0 & -9 & -18 & 0 & 18 \ -14 & 10 & 0 & 0 & 0 & -14 & 10 & 0 & 70 & 10 & 28 & -20 & 28 \ -20 & 13 & 0 & 9 & 0 & 16 & -14 & -9 & 10 & 67 & 22 & 28 & -14 \ 16 & -14 & 0 & 18 & 0 & 16 & 4 & -18 & 28 & 22 & 40 & -8 & 4 \ 4 & 28 & 0 & 0 & 0 & 4 & 28 & 0 & -20 & 28 & -8 & 52 & -8 \ 16 & 22 & 0 & -18 & 0 & 16 & 4 & 18 & 28 & -14 & 4 & -8 & 40 \ \end{pmatrix} (9.5)$$

As already pointed out in sect. 6, a row of $Q$ can be mapped to another row by a symmetry transformation $\sigma$ only if the two rows have the same entries. Hence, from
the explicit form of (9.5), we see that any symmetry fixes the sets \{1, 6\}, \{2, 10\}, \{3, 5\}, \{4, 8\}, \{11, 13\}, \{7\}, \{9\} and \{12\}. It is not difficult to see that the permutation \((3, 5)\) is a symmetry.

We can again examine \(Q\) to see that a symmetry exchanges rows 1 and 6 iff it exchanges columns 2 and 10. It exchanges rows 2 and 10 iff it exchanges columns 11 and 13 and exchanges rows 11 and 13 iff it exchanges columns 4 and 8.

It is easy to verify that \((1, 6)(2, 10)(11, 13)(4, 8)\) is a symmetry. Thus, the symmetries of \(V'\) are

\[
G[\hat{sp}(4)] = \{(1, 6)(2, 10)(11, 13)(4, 8), (3, 5)(1, 6)(2, 10)(11, 13)(4, 8)\}
\]

(9.6) isomorphic to \(S_2 \times S_2\).

With \(x^\top = (k, \lambda_1, \lambda_2, \mu_1, \mu_2, r_1, r_2, p, q)\), the changes of variables corresponding to the generators of the symmetry group are:

\[
(3, 5) : x^\top \mapsto (k, \mu_2 - r_2/2 + p, \lambda_2, \mu_1, \lambda_1 + r_2/2 - p, r_1, r_2, p, q)
\]

\( (1, 6)(2, 10)(11, 13)(4, 8) : x^\top \mapsto (k, \lambda_1 + \mu_1 - r_1 + r_2 - 2p, k - \lambda_1 - \lambda_2 - \mu_1 - \mu_2 + r_1 + p + q, \mu_1, \mu_2, r_1, r_2, r_1 - p - r_1 + r_2, q) \) (9.7)

The orbits of the symmetries on the inequalities of the fusion basis are (with the inequalities (9.3) labelled consecutively, column by column, from 1 to 16):

\[
(3, 5) : [[5, 12]]
\]

(1, 6)(2, 10)(11, 13)(4, 8) : [[1, 7], [2, 8], [4, 6], [11, 15]]

(9.8) where orbits of length 1 have been omitted.

As indicated in the context of the \(\hat{su}(3)\) analysis, the most severe constraints on the symmetries come from the linear relations. There is indeed a large number of relations for \(\hat{sp}(4)\) [3]:

\[
\begin{align*}
\hat{E}_0\hat{C}_1\hat{C}_2 &= \hat{A}_3\hat{D}_3 & \hat{E}_0\hat{C}_2\hat{C}_3 &= \hat{A}_1\hat{D}_1 & \hat{E}_0\hat{C}_3\hat{C}_1 &= \hat{A}_1\hat{A}_3\hat{B}_2 \\
\hat{D}_1\hat{D}_2 &= \hat{E}_0\hat{B}_3\hat{C}_3^2 & \hat{D}_2\hat{D}_3 &= \hat{E}_0\hat{A}_1^2\hat{B}_2\hat{B}_3 & \hat{D}_1\hat{D}_3 &= \hat{E}_0\hat{B}_2\hat{C}_3^2 \\
\hat{C}_1\hat{D}_1 &= \hat{A}_3\hat{B}_2\hat{C}_2 & \hat{C}_2\hat{D}_2 &= \hat{A}_1\hat{B}_3\hat{C}_3 & \hat{C}_3\hat{D}_3 &= \hat{A}_1\hat{B}_2\hat{C}_2
\end{align*}
\]

(9.9)
It is not difficult to check that the symmetries leave these relations invariant. In fact, \((3,5) = (\hat{A}_2, \hat{B}_1)\) (which means the interchange of \(\hat{A}_2\) and \(\hat{B}_1\)) and these two couplings do not appear in the relations. The other symmetry reads \((\hat{E}_0,\hat{B}_2) (\hat{A}_1,\hat{C}_3) (\hat{D}_1,\hat{D}_3)(\hat{A}_3,\hat{C}_1)\).

With the vector \(y^\top\) defined as \(y^\top = (k, \lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2, p, q)\), we can rewrite the symmetries of the fusion basis as:

\[
(3,5) : y^\top \rightarrow (k, -1/2\lambda_2 + 1/2\mu_2 + 1/2\nu_2 + 1/2p + 1/2q, \lambda_2, \mu_1, \\
\lambda_1 + 1/2\lambda_2 + 1/2\mu_2 + 1/2\nu_2 - 1/2p - 1/2q, -\lambda_1 - 1/2\lambda_2 + 1/2\mu_2 + \nu_1 + 1/2\nu_2 + 1/2p + 1/2q, \\
\lambda_1 + 1/2\lambda_2 - 1/2\mu_2 + 1/2\nu_2 - 1/2p - 1/2q, p, q)
\]

\[(1,6)(2,10)(11,13)(4,8) : y^\top \rightarrow (k, \nu_1, k - \nu_1 - \nu_2, \mu_1, \mu_2, \lambda_1, k - \lambda_1 - \lambda_2, -\lambda_1 + \nu_1 + p, q)\]

(9,10)

The first polytope symmetry does not correspond to a fusion-coefficient symmetry. However, the second one is a combination of an outer automorphism and a permutation of two weights: \((1,6)(2,10)(11,13)(4,8) = (a, 1, a)F_{13}\). This is not a new symmetry of fusion coefficients.

10. The symmetry group of the \(\text{\tilde{su}}(4)_k\) polytope

The whole set of \(\text{\tilde{su}}(4)_k\) fusion elementary couplings can all be generated from two couplings that have no elementary finite relative:

\[
\hat{E}_0 = [1,0,0,0] \times [1,0,0,0] \supset [1,0,0,0] \quad \hat{F} = [0,1,0,1] \times [0,1,0,1] \supset [0,1,0,1]
\]

(10.1)

by means of the outer-automorphism group. We can thus characterize a coupling by a pair \((a^n, a^m)_i\) where \(a^n\) and \(a^m\) act on the first and the second weight respectively, understanding that the action on the third weight is \(a^{n+m}\). Here \(a\) permutes the Dynkin labels of an affine \(\text{\tilde{su}}(4)\) weight as \(a[\lambda_0, \lambda_1, \lambda_2, \lambda_3] = [\lambda_3, \lambda_0, \lambda_1, \lambda_2]\) so that \(a^4 = 1\). The subindex \(i\) refers to the elementary coupling \(\hat{E}_0\) or \(\hat{F}\) from which it is obtained; these are labelled respectively as \(i = 0, 1\). The remaining elementary coupling are thus

\[
\hat{A}_1 = (a^0, a^3)_0 \quad \hat{A}_2 = (a^3, a^1)_0 \quad \hat{A}_3 = (a^1, a^0)_0 \\
\hat{B}_1 = (a^0, a^2)_0 \quad \hat{B}_2 = (a^2, a^2)_0 \quad \hat{B}_3 = (a^2, a^0)_0 \\
\hat{C}_1 = (a^0, a^1)_0 \quad \hat{C}_2 = (a^1, a^3)_0 \quad \hat{C}_3 = (a^3, a^0)_0 \\
\hat{D}_1 = (a^2, a^1)_0 \quad \hat{D}_2 = (a^1, a^1)_0 \quad \hat{D}_3 = (a^1, a^2)_0 \\
\hat{E}_1 = (a^0, a^1)_1 \quad \hat{E}_2 = (a^1, a^1)_1 \quad \hat{E}_3 = (a^1, a^0)_1
\]

(10.2)
(The explicit reexpression of the elementary couplings in terms of the LR variables can be found in [3].)

The reconstruction of the polytope facets out of these vertices reproduce the inequalities obtained in [3]. These are

\[
\begin{align*}
  k - \lambda_1 - \lambda_2 - \lambda_3 - n_{33} &\geq 0 \\
  k - \lambda_1 - \lambda_2 - n_{11} - n_{14} + n_{34} &\geq 0 \\
  k - \lambda_1 - n_{11} - n_{13} - n_{14} &\geq 0 \\
  k - \lambda_1 - \lambda_2 - \lambda_3 - n_{22} + n_{34} &\geq 0 \\
  k - \lambda_1 - \lambda_2 - n_{14} - n_{22} &\geq 0 \\
  k - \lambda_1 - \lambda_2 - \lambda_3 - n_{11} + n_{24} + n_{34} &\geq 0 \\
  k - \lambda_1 - \lambda_2 - \lambda_3 + n_{12} - n_{22} - n_{23} + n_{34} &\geq 0 \\
  k - \lambda_1 - \lambda_2 - n_{14} + n_{13} - n_{22} - n_{24} &\geq 0 \\
  k - \lambda_1 - \lambda_2 - n_{11} - n_{14} + n_{23} &\geq 0 \\
  2k - 2\lambda_1 - 2\lambda_2 - \lambda_3 - n_{14} - n_{22} - n_{11} + n_{34} &\geq 0 \\
  \lambda_1 - n_{12} &\geq 0 \\
  \lambda_2 - n_{13} &\geq 0 \\
  \lambda_2 + n_{12} - n_{13} - n_{23} &\geq 0 \\
  \lambda_3 - n_{14} &\geq 0 \\
  \lambda_3 + n_{13} - n_{14} - n_{24} &\geq 0 \\
  \lambda_3 + n_{13} + n_{23} - n_{14} - n_{24} - n_{34} &\geq 0 \\
  n_{11} - n_{22} &\geq 0 \\
  n_{11} + n_{12} - n_{22} - n_{23} &\geq 0 \\
  n_{11} + n_{12} + n_{13} - n_{22} - n_{23} - n_{24} &\geq 0 \\
  n_{22} - n_{33} &\geq 0 \\
  n_{22} + n_{23} - n_{33} - n_{34} &\geq 0 \\
  n_{12} &\geq 0 \\
  n_{13} &\geq 0 \\
  n_{14} &\geq 0 \\
  n_{23} &\geq 0 \\
  n_{24} &\geq 0 \\
  n_{33} &\geq 0 \\
  n_{34} &\geq 0
\end{align*}
\]

(10.3)

We then look for the symmetry group of the resulting polytope using the commutant of \(Q\). The matrix \(V'\), with the column ordering \((\hat{E}_0, \hat{A}_1, \ldots, \hat{D}_1', \ldots, \hat{D}_3, \hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{F})\),
is

\[ V' = k \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \] (10.4)
The matrix $Q' = 3240Q$ reads

\[
\begin{pmatrix}
2088 & 0 & 0 & 0 & 168 & 168 & 600 & 600 & -360 & -360 & -360 & -360 & 384 & 384 & 384 & -144

0 & 3240 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 3240 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 3240 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

168 & 0 & 0 & 0 & 2338 & 43 & 43 & 250 & -425 & -425 & -15 & 660 & -285 & 660 & -15 & -56 & 484 & 484 & -360

168 & 0 & 0 & 0 & 43 & 2338 & 43 & -425 & 250 & -15 & 660 & -285 & 660 & -15 & -285 & 484 & 484 & -360

168 & 0 & 0 & 0 & 43 & 43 & 2338 & -425 & -425 & 250 & -15 & 660 & -285 & 660 & -15 & -285 & 484 & 484 & -360

600 & 0 & 0 & 0 & 250 & -425 & -425 & 2050 & -245 & -245 & 255 & 525 & 525 & 525 & -420 & 525 & -420 & 340 & 340 & 484

600 & 0 & 0 & 0 & -425 & 250 & -425 & -245 & 2050 & -245 & -420 & 255 & 525 & 525 & 525 & -420 & 525 & -420 & 340 & 340 & 484

600 & 0 & 0 & 0 & -425 & -425 & 250 & -245 & -245 & 2050 & 525 & -420 & 255 & 525 & 525 & 525 & -420 & 525 & -420 & 340 & 340 & 484

-360 & 0 & 0 & 0 & -285 & 660 & -15 & 255 & -420 & 525 & 225 & 225 & 225 & -45 & -45 & -450 & -420 & 120 & 660 & 360

-360 & 0 & 0 & 0 & -285 & 660 & -15 & 255 & -420 & 525 & 225 & 225 & 225 & -45 & -45 & -450 & -420 & 120 & 660 & 360

-360 & 0 & 0 & 0 & 660 & -15 & -285 & 525 & 255 & 225 & 225 & 225 & -45 & -45 & -450 & -420 & 120 & 660 & 360

-360 & 0 & 0 & 0 & 660 & -15 & -285 & 525 & 255 & 225 & 225 & 225 & -45 & -45 & -450 & -420 & 120 & 660 & 360

-360 & 0 & 0 & 0 & 660 & -285 & -15 & -420 & 525 & 255 & 225 & 225 & 225 & -45 & -45 & -450 & -420 & 120 & 660 & 360

-360 & 0 & 0 & 0 & 660 & -285 & -15 & -420 & 525 & 255 & 225 & 225 & 225 & -45 & -45 & -450 & -420 & 120 & 660 & 360

384 & 0 & 0 & 0 & -56 & 484 & 484 & -200 & 340 & 340 & -420 & 660 & 120 & -420 & 120 & 660 & 952 & -128 & -128 & 48

384 & 0 & 0 & 0 & 484 & -56 & 484 & 340 & -200 & 340 & 120 & -420 & 660 & 660 & -420 & 120 & -128 & -128 & 952 & 4

384 & 0 & 0 & 0 & 484 & 484 & -56 & 340 & 340 & -200 & 660 & 120 & -420 & 120 & 660 & -420 & -128 & -128 & 952 & 4

-144 & 0 & 0 & 0 & -384 & -384 & -384 & 480 & 480 & 480 & 360 & 360 & 360 & 360 & 48 & 48 & 48 & 792
\end{pmatrix}

(10.5)

Again, considering the sets of entries in each row of $Q$ we find that the symmetry group fixes the sets $\{1\}, \{2, 3, 4\}, \{5, 6, 7\}, \{8, 9, 10\}, \{11, 12, 13, 14, 15, 16\}, \{17, 18, 19\}$ and $\{20\}$. We notice immediately that the permutations of 2, 3 and 4 give an $S_3$ symmetry that commutes with all the other symmetries.

Let us suppose that $\sigma$ is a symmetry and let $\tau$ be the induced permutation of $\{11, 12, 13, 14, 15, 16\}$. The submatrix of $Q$ corresponding to rows $\{11, 12, 13, 14, 15, 16\}$
and columns \( \{5, 6, 7\} \) is

\[
U_1 = \frac{1}{3240} \begin{pmatrix}
-285 & 660 & -15 \\
-15 & -285 & 660 \\
660 & -15 & -285 \\
-285 & -15 & 660 \\
660 & -285 & -15 \\
-15 & 660 & -285
\end{pmatrix}
\]  \tag{10.6}

while the submatrix for rows \( \{11, 12, 13, 14, 15, 16\} \) and columns \( \{8, 9, 10\} \) is

\[
U_2 = \frac{1}{3240} \begin{pmatrix}
255 & -420 & 525 \\
-420 & 525 & 255 \\
525 & 525 & -420 \\
-420 & 255 & 525 \\
255 & -420 & 255 \\
-420 & 255 & 255
\end{pmatrix}
\]  \tag{10.7}

and for rows \( \{11, 12, 13, 14, 15, 16\} \) and columns \( \{17, 18, 19\} \) it is

\[
U_3 = \frac{1}{3240} \begin{pmatrix}
-420 & 120 & 660 \\
120 & -420 & 660 \\
660 & -420 & 120 \\
-420 & 660 & 120 \\
120 & -420 & 660 \\
660 & 120 & -420
\end{pmatrix}
\]  \tag{10.8}

Since each of these matrices have rows with distinct entries, we can deduce the action of \( \sigma \) on each of the sets \( \{5, 6, 7\}, \{8, 9, 10\} \) and \( \{17, 18, 19\} \) from \( \tau \). Since \( \sigma \) fixes 1 and 20 this determines \( \sigma \) except for its action on 2, 3 and 4, which, as noted above, is arbitrary.

Thus, to find all symmetries, it suffices to find all possible \( \tau \). The submatrix of \( Q \) corresponding to rows \( \{11, 12, 13, 14, 15, 16\} \) and columns \( \{11, 12, 13, 14, 15, 16\} \) is

\[
K = \frac{1}{3240} \begin{pmatrix}
2250 & 225 & 225 & -45 & -45 & -450 \\
225 & 2250 & 225 & -450 & -45 & -45 \\
225 & 225 & 2250 & -45 & -450 & -45 \\
-45 & -450 & -45 & 2250 & 225 & 225 \\
-45 & -45 & -450 & 225 & 2250 & 225 \\
-450 & -45 & -45 & 225 & 225 & 2250
\end{pmatrix}
\]  \tag{10.9}
The permutation $\tau$ commutes with $K$ and $K$ has an eigenvector $w^\top = (-1, -1, -1, 1, 1, 1)$, the corresponding eigenspace being 1-dimensional. $\tau$ fixes this eigenspace and so either $\tau w = w$ or $\tau w = -w$. The group of all such permutations is $S_3 \wr S_2$ with the two $S_3$ groups permuting $\{11, 12, 13\}$ and $\{14, 15, 16\}$ and the $S_2$ interchanging them. Thus, every $\sigma$ gives a $\tau$ in $S_3 \wr S_2$, but it is not necessarily true that every element of $S_3 \wr S_2$ extends to a symmetry of $V'$. In fact, only a subgroup can be extended, as we now show.

By trial and error, we can find two elements $\alpha = (11, 14)(12, 16)(13, 15)$ and $\beta = (11, 12, 13)(14, 15, 16)$ of $S_3 \wr S_2$ which can be extended to the two symmetries 

$(6,7)(9,10)(11,14)(12,16)(13,15)(18,19)$ and $(5,6,7)(8,9,10)(11,12,13)(14,15,16)(17,18,19)$. 

The group generated by $\alpha$ and $\beta$ turns out to be isomorphic to $S_3$ (which does not have the standard action).

Thus, we get a group of symmetries isomorphic to $S_3 \times S_3$:

$$G[\hat{su}(4)] = < (2, 3), (2, 3, 4), (6, 7)(9, 10)(11, 14)(12, 16)(13, 15)(18, 19),$$

$$(5, 6, 7)(8, 9, 10)(11, 12, 13)(14, 15, 16)(17, 18, 19) >$$

(10.10)

There are no other symmetries for the following reason. Suppose that $\sigma$ is a symmetry and $\tau$ is the induced permutation on $\{11, 12, 13, 14, 15, 16\}$. Since the group generated by $\alpha$ and $\beta$ is transitive on $\{11, 12, 13, 14, 15, 16\}$, there is some $\tau'$ such that $\tau \tau'$ fixes 11. Then, by considering the first row of $U_1$, $\tau \tau'$ also fixes $\{5, 6, 7\}$. Hence $\tau \tau'$ must fix $U_1$. But this implies $\tau \tau'$ is the identity since $U_1$ has distinct rows. Thus $\tau^{-1} = \tau'$ and so $\sigma$ is in the group $G$.

With $x^\top = (k, \lambda_1, \lambda_2, \lambda_3, n_{11}, n_{12}, n_{13}, n_{14}, n_{22}, n_{23}, n_{24}, n_{33}, n_{34})$, the changes of vari-
ables corresponding to the generating symmetries are:

\[(2, 3) : x^\top \mapsto (k, \lambda_1, \lambda_2, \lambda_3 - n_{14} + n_{33}, n_{11} + n_{14} - n_{33}, n_{12}, n_{13}, n_{33},
    n_{14} + n_{22} - n_{33}, n_{23}, n_{24}, n_{14}, n_{34})\]

\[(2, 3, 4) : x^\top \mapsto (k, n_{12} + n_{14}, \lambda_2, \lambda_3 - n_{14} + n_{33}, \lambda_1 + n_{11} - n_{12} - n_{33}, n_{12}, n_{13},
    n_{33}, \lambda_1 - n_{12} + n_{22} - n_{33}, n_{23}, n_{24}, \lambda_1 - n_{12}, n_{34})\]

\[(11, 14)(12, 16)(13, 15)(6, 7)(18, 19)(9, 10) : x^\top \mapsto (k, \lambda_1 + \lambda_3 - n_{12} - n_{14}, \lambda_2, n_{12} + n_{14},
    n_{11} + n_{12} + n_{13} - n_{24} - n_{34}, \lambda_3 - n_{14}, \lambda_2 - n_{13}, n_{14}, n_{22} + n_{23} - n_{34},
    \lambda_3 + n_{13} - n_{14} - n_{24}, \lambda_2 + n_{12} - n_{13} - n_{23}, n_{33}, \lambda_3 + n_{13} + n_{23} - n_{14} - n_{24} - n_{34})\]

\[(5, 6, 7)(8, 9, 10)(11, 12, 13)(14, 15, 16)(17, 18, 19) : x^\top \mapsto
    (k, \lambda_1 + n_{11} + n_{13} - n_{22} - n_{23} - n_{24}, n_{22} + n_{23} + n_{24} - n_{33} - n_{34},
    n_{14} + n_{34}, \lambda_2 + \lambda_3 - n_{13} - n_{14} + n_{33}, n_{11} + n_{12} + n_{13} - n_{22} - n_{23} - n_{24},
    n_{22} + n_{23} - n_{33} - n_{34}, n_{14}, \lambda_2 - n_{13} + n_{33}, n_{11} + n_{12} - n_{22} - n_{23}, n_{22} - n_{33}, n_{33}, n_{11} - n_{22})\]

\[(10.11)\]

Labelling the \(\widehat{su}(4)\) fusion inequalities of (10.3) from 1 to 28 (row by row), the orbits on the fusion basis are:

\[(2, 3) : [[24, 27]]\]

\[(2, 3, 4) : [[11, 27, 24]]\]

\[(11, 14)(12, 16)(13, 15)(6, 7)(18, 19)(9, 10) : [[2, 9], [3, 6], [4, 8], [5, 7], [12, 23], [13, 26], [14, 22], [15, 25], [16, 28], [17, 19], [20, 21]]\]

\[(5, 6, 7)(8, 9, 10)(11, 12, 13)(14, 15, 16)(17, 18, 19) : [[1, 6, 3], [2, 5, 4], [7, 9, 8], [12, 20, 26], [13, 21, 23], [14, 17, 28], [15, 18, 25], [16, 19, 22]]\]

\[(10.12)\]

where the orbits of length 1 have been omitted.

As already indicated, severe constraints on the symmetries come from the linear relations. And in fact there are many such relations in the \(\widehat{su}(4)\) case. The full list is [3]:

\[
\widehat{E}_0 \widehat{D}_j \widehat{D}_k = \widehat{C}_i \widehat{E}_i \quad \widehat{E}_0 \widehat{D}_j \widehat{D}'_k = \widehat{B}_i \widehat{C}_j \widehat{C}_k \quad \widehat{E}_i \widehat{E}_j = \widehat{E}_0 \widehat{B}_k \widehat{D}_k \widehat{D}'_k
\]

\[
\widehat{D}_i \widehat{E}_i = \widehat{C}_j \widehat{B}_k \widehat{D}_k \quad \widehat{D}'_i \widehat{E}_i = \widehat{B}_j \widehat{D}'_j \widehat{C}_k \quad \widehat{E}_0 \widehat{F} = \widehat{C}_1 \widehat{C}_2 \widehat{C}_3
\]

\[(10.13)\]
with \( i, j, k \) a cyclic permutation of 1, 2, 3. The large number of relations, and more precisely, the fact that they mix elementary couplings with different threshold levels, is responsible for the absence of symmetries involving the level. All the symmetries found above leave this set of relations invariant. For instance, the \( S_3 \) group generated by (2,3) and (2,3,4) is the permutation group of the three \( \hat{A}_i \)'s, which do not appear in the relations.

With the vector \( y^\top \) defined as \( y^\top = (k, \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3, n_{12}, n_{14}, n_{33}) \), we can rewrite the symmetries of the fusion basis as:

\[
(2, 3) : y^\top \rightarrow (k, \lambda_1, \lambda_2, \lambda_3 - n_{14} + n_{33}, \mu_1 - n_{14} + n_{33}, \mu_2, \mu_3 + n_{14} - n_{33}, \nu_1 + n_{14} - n_{33}, \\
\nu_2, \nu_3, n_{12}, n_{14}, n_{33})
\]

\[
(2, 3, 4) : y^\top \nu_1 + \lambda_1 - n_{12} - n_{33}, \nu_2, \nu_3 - \lambda_1 + n_{12} + n_{14}, \rightarrow (k, n_{12} + n_{33}, \lambda_2, \lambda_1 + \lambda_3 - n_{12} - n_{14}, \lambda_1 + \mu_1, \\
\nu_1 + n_{14} - n_{33}, \nu_2, -\lambda_1 + \nu_3 + n_{12} + n_{33}, n_{12}, \lambda_1 - n_{12}, n_{14})
\]

\[
(11, 14)(12, 16)(13, 15)(6, 7)(18, 19)(9, 10) : y^\top \rightarrow (k, \lambda_1 + \lambda_3 - n_{12} - n_{14}, \lambda_2, n_{12} + n_{14}, \\
\nu_3 - \lambda_1 + n_{12} + n_{14}, \nu_2, \nu_1, \mu_3, \mu_2, \lambda_1 + \mu_1 - n_{12} - n_{14}, \lambda_3 - n_{14}, n_{14}, n_{33})
\]

\[
(5, 6, 7)(8, 9, 10)(11, 12, 13)(14, 15, 16)(17, 18, 19) : y^\top \rightarrow (k, \lambda_1 + \nu_1 - n_{12} - n_{33}, \nu_2, \\
-\lambda_1 + \nu_3 + n_{12} + n_{14}, n_{12} + n_{14}, \lambda_2, \lambda_3 - n_{14} + n_{33}, \mu_1 - n_{14} + n_{33}, \mu_2, \\
\lambda_1 + \mu_3 - n_{12} - n_{33}, \nu_1 - n_{33}, n_{14}, n_{33})
\]

These symmetries do not correspond to a fusion-coefficient symmetries.

11. Conclusion

In this paper we presented the concept of fusion bases, first introduced in [3], from a novel point of view as the set of facets of a polytope. This reformulation gives access to the powerful computer programs that have been developed for generating facets out of the vertices. Moreover, by reformulating the problem in a geometrical way, we were led to the study of the affine symmetry group of the fusion polytope, introduced here for the first time. We developed simple tools for studying this group analytically for the lowest rank affine Lie algebras. We also defined the vertex symmetry group of a polytope and noted that for fusion polytopes the affine and vertex symmetry groups appear to be the same – a property which does not hold for general polytopes.
The order of the vertex symmetry group of the fusion polytope of the lowest rank affine Lie algebras was found to be 24 for \( \widehat{\mathfrak{su}}(2) \), 432 for \( \widehat{\mathfrak{su}}(3) \), 36 for \( \widehat{\mathfrak{su}}(4) \), and 4 for \( \widehat{\mathfrak{sp}}(4) \). Comparing \( \widehat{\mathfrak{su}}(2) \) and \( \widehat{\mathfrak{su}}(3) \), it is natural to see an increase of the order of the group with the rank since the number of vertices increases rapidly. However, it might be surprising to observe this drastic reduction in the order when passing from \( \widehat{\mathfrak{su}}(3) \) to \( \widehat{\mathfrak{su}}(4) \). The reason is that the number of linear relations, which have to be preserved by the symmetry transformations, also increases rapidly with the rank.

We should stress that a fusion polytope is rather special type of polytope in that its vertices are the elementary solutions of the facets, which is far from being a generic property of polytopes.

It is an interesting open problem to try to generate the full set of fusion inequalities, or equivalently, to give a generic description of the fusion polytope, from a general Lie algebraic point of view.

On the other hand, we could ask whether the polytope we have obtained, whose description relies heavily on the LR variables, is the ‘genuine’ fusion polytope or whether it is just one among a variety of polytopes. In that vein, we note that Rasmussen and Walton [16] have recently also developed a polytope interpretation of \( \widehat{\mathfrak{su}}(3) \) and \( \widehat{\mathfrak{su}}(4) \) fusion coefficients using a different approach from ours. Prompted by the referee we have investigated the relationship with the polytopes found in this paper. We find, perhaps surprisingly, after removing redundant variables and making a suitable change of coordinates that the two sets of polytopes coincide. This suggests that the polytopes might be in some sense unique.

Finally, concerning the symmetry analysis, we stress that we have restricted our analysis to a special class of symmetries, namely those which exist at all level. Whether symmetries at particular levels or even symmetries that relate fusion polytopes at different levels can be unraveled by our method remains to be studied.

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