An optimized sideband cooling in the presence of initial system correlations is investigated for a standard optomechanical system coupled to a general mechanical non-Markovian reservoir. We study the evolution of phonon number by incorporating the effects of initial correlations into the time-dependent coefficients in the Heisenberg equation. We introduce the concept of cooling rate and define an average phonon reduction function to describe the sideband cooling effect in non-Markovian regime. Our results show that the instantaneous phonon number can be significantly reduced by introducing either the parametric-amplification type or the beam-splitter type initial correlations. In addition, the ground state cooling rate can be accelerated by enhancing the initial correlation of beam-splitter type. By optimizing the initial state of the system and utilizing Q-modulation technology, a stable mechanical ground state can be obtained in a very short time. Our optimized cooling protocol provides an appealing platform for phonon manipulation and quantum information processing in solid-state systems.

I. INTRODUCTION

The technique of preparing mechanical oscillator in pure states close to zero-point vibration is fundamentally important [1–4]. By suppressing the effects of stochastic driving from the thermal environment, mechanical ground-state cooling provides a critical avenue for exploring a wide range of quantum-mechanical phenomenon, including the boundary between quantum and classical mechanics [5], macroscopic quantum behavior [6, 7]. It can also be used to attain measurement precision in quantum metrology that is close to the standard quantum limit [8–10], as exemplified by the gravitational wave detection [11]. Interest in this technique has grown in the past few decades with ongoing attentions been particularly devoted to sideband cooling (back-action cooling or self-cooling) in cavity optomechanical systems. The basic process of sideband cooling is the energy exchange between thermalized mechanical oscillator and the cavity field in the vacuum bath. The mechanical oscillator is cooled via its interaction with the cavity field, and finally stabilize at a phonon number state with low excitation [3, 4]. This cooling method promises great cooling efficiency especially in the resolved-sideband limit [12–15], where the mechanical resonance frequency is greater than the decay rate of the optical cavity. Many efforts have been directed at the enhancement of cooling efficiency by changing the configuration of the optomechanical systems. For instance, cooling has been demonstrated by either dissipative [16] or quadratic coupling [17], either in hybrid systems [18, 19], or in single photonic system [17], and by using parameters modulated system [20, 21].

In sideband cooling, the steady-state cooling limit is proportional to the sideband resolution parameter $\omega_m/\kappa$ [1]. Various approaches have been proposed to break this limitation, including cooling by cavity dissipation control [16], cooling by an optomechanical heat pump [21] and cooling in unresolved-sideband regime [23]. Alternatively, sideband cooling can be achieved by environment engineering where the backflow from a non-Markovian environment to the mechanical oscillator is tailored to reduce the steady-state cooling phonon numbers. There have been several theoretical proposals on sideband cooling in the non-Markovian regime [24, 25], and non-Markovian micro-mechanical oscillator has been experimentally demonstrated [26]. In most of these investigations, the optical mode and mechanical mode are often assumed to be initially uncorrelated with each other [27]. However, in practice such as the strong coupling regime, the two systems are often closely correlated to start with [28, 29]. Various initial-correlation-induced effects have been investigated in different open quantum systems [30–32]. It has been demonstrated that the initial correlations have nontrivial differences in quantum dynamical evolutions.

In this paper, we theoretically prove that evolution of the mechanical oscillator is strongly dependent on its initial state in non-Markovian regime. Based on this observation, we propose effective approaches that could significantly enhance the cooling rate by optimizing the initial state of the system. A simplified analytical characteristic function is obtained to explore the optimal cooling condition of the optical-mechanical initial interactions in non-Markovian regime. To better understand the optimization ability of the initial correlation, we define the cooling rate $v = dN(t)/dt$, which is an important parameter in cooling. Combined with the dissipative cooling scheme [16], a stable ground state of mechanical oscillator can be
obtained in a rather short time.

The rest of the paper is structured as follows: In Sec. II in the presence of the initial correlation, the average number of mechanical oscillator is obtained. In Sec. III, an analytical approach of sideband cooling is established, and the instantaneous cooling limit and cooling rate is investigated. Then, in Sec. IV, a ultrafast optimal sideband cooling scheme is given and some numerical results are discussed. Finally, we discuss the experimental feasibility of the scheme, and conclude the paper in Sec. V.

II. MODEL AND HAMILTONIAN

We consider a typical cavity optomechanical system comprised of a cavity with frequency $\omega_c$ and a mechanical resonator with frequency $\omega_m$. The mechanical oscillator is coupled to a general non-Markovian reservoir. The Hamiltonian of this system can be written as $H = H_S + H_E + H_I$, where

$$
\hat{H}_S = \hbar \omega_c \hat{a}^\dagger \hat{a} + \hbar \omega_m \hat{b}^\dagger \hat{b} - \hbar g_0 \hat{a}^\dagger \hat{a} (\hat{b}^\dagger + \hat{b})
+ i \hbar E (e^{-i \omega_m t} \hat{a}^\dagger - e^{i \omega_m t} \hat{a}),
$$

$$
\hat{H}_E = \sum_k \hbar \omega_k \hat{b}_k^\dagger \hat{b}_k,
$$

$$
\hat{H}_I = \sum_k \hbar V_k (\hat{b}^\dagger \hat{b}_k + \hat{b}_k^\dagger \hat{b}).
$$

Here $\hat{a}$ and $\hat{b}$ are annihilation operators of the optical and mechanical modes, respectively. $g_0$ is the coupling coefficient between the mechanical and the optical modes. The optical mode is driven by a coherent laser with driving strength $E$ and frequency $\omega_m$. $\omega_k$ is the reservoir frequency of the $k$-th mechanical mode, and $g_k$ denotes the system-bath coupling strength.

Without loss of generality, we transform the Hamiltonian in Eq.(1) into the displaced oscillator representation in which the steady state of a cavity mode is the vacuum state [33]. As illustrated in Fig. 1(a), the energy-level diagram is constructed under the sideband-cooling condition $\omega_c = \omega_d + \omega_m$. Kets $|n\rangle_c$, $|m\rangle_m$ and $|n_{th}\rangle_b$ are used to denote, respectively, the number states of the cavity, the mechanical oscillator, and the bath.

In our system, we have the traditional anti-Stokes cooling path (path-I i.e. A) and the additional cooling path (path-II i.e. complex combination of A and B) introduced by the non-Markovian backflow effect (which have been well discussed in Refs. [24, 25]). According to Fig. 1(a), the counter-rotating term $(\hat{a} \hat{b})$ can open a new path for the cooling of the mechanical oscillator. Under the sideband-cooling condition, the counter-rotating wave term is a high frequency oscillation term which can be ignored in stable regime. But in non-Markovian reservoir, the dynamics of the mechanical oscillator is strongly depended on the initial correlation, that is to say, both the initial parametric-amplification (AP) correlation and beam-splitter (BS) correlation can be memorized in the evolution of the system.

With the full Hamiltonian $\hat{H}$ given in Eq. (1), the Heisenberg-Langevin equations of motion for the annihilation operators of the system are given by (for convenience, we take $\hbar = 1$),

$$
\dot{\hat{a}} = -(i \Delta_c + \frac{K}{2}) \hat{a} + i g_0 \hat{a} (\hat{b}^\dagger + \hat{b}) + E + \sqrt{\kappa} \hat{a}_{in},
$$

$$
\dot{\hat{b}} = -i \omega_m \hat{b} + i g_0 \hat{b} \hat{a} - i \sum_k V_k (\hat{b}_k + \hat{b}_k^\dagger),
$$

(3b)

(3c)

(3d)

where $\Delta_c = \omega_c - \omega_d$, the vacuum noise operator of the cavity obey $\langle \hat{a}_{in}(t) \hat{a}_{in}(\tau) \rangle = \delta(t - \tau)$. To study the dynamics of our system under the strong driving condition, we make use of the linear approximation by decomposing the operators into the classical and quantum components [1], i.e., $\hat{a} \rightarrow \alpha + \delta \hat{a}$ and $\hat{b} \rightarrow \beta + \delta \hat{b}$. After formal integration of environmental degrees of freedom [25], the time evolution of the annihilation operators of the system in the Heisenberg picture is then governed by

$$
\dot{\alpha} = -(i \Delta'_c + \frac{K}{2}) \alpha + i g_0 \alpha (\beta + \beta^*) + E, \tag{3a}
$$

$$
\dot{\beta} = -i \omega_m \beta + i g_0 |\alpha|^2
+ \int_0^t d\tau f(t - \tau) \langle \beta(\tau) + \beta^*(\tau) \rangle, \tag{3b}
$$

$$
\dot{\delta a} = -(i \Delta'_d + \frac{K}{2}) \delta \alpha + i G (\delta \hat{b} + \delta \hat{b}^\dagger) + \sqrt{\kappa} \delta \hat{a}_{in}, \tag{3c}
$$

$$
\dot{\delta b} = -i \omega_m \delta \beta + i (G \delta \hat{a} + G^* \delta \hat{a})
+ \int_0^t d\tau f(t - \tau) \langle \delta \hat{b}(\tau) + \delta \hat{b}^\dagger(\tau) \rangle - \xi(t). \tag{3d}
$$

where $\Delta'_c(t) = \Delta_c - g_0 |\beta(t) + \beta^*(t)|$ is the detuning modified by the optomechanical coupling and $G(t) = \alpha(t) g_0$ describes the field enhanced optomechanical coupling strength. The memory kernel $f(t)$ characterizes the non-Markovian dynamics of the reservoir, where $f(t) = 2i \sum_k V_k^2 \sin(\omega_k t) = 2i \int_0^\infty d\omega F(\omega) \sin(\omega t)$; $F(\omega)$ refers to the spectral density of the reservoir. The noise opera-
tor \( \dot{\xi}(t) = i \sum_k V_k \hat{b}_k(0)e^{-i\omega_k t} + \hat{b}_k^\dagger(0)e^{i\omega_k t} \) is a non-local time correlation function for a non-Markovian environment. We adopt the commonly used spectral density expression \( J(\omega) = \eta \omega (\frac{\omega}{\omega_1})^{s-1}e^{-\omega/\omega_1} \) [34], where \( \eta \) is the strength of system-bath coupling and \( \omega_1 \) is the cut-off frequency. The exponent \( s \) is a real number that determines the \( \omega \) dependence of \( J(\omega) \) in the low-frequency region. The baths with \( 0 < s < 1 \), \( s = 1 \), and \( s > 1 \) are termed “sub-Ohmic”, “Ohmic” and “super-Ohmic” baths, respectively.

We now consider the non-Markovian effect in the sideband cooling with \( \Delta'_c \approx \omega_m \). Under this condition, the dynamics of the phonon number can be obtained by using iterative method of Eqs. (3). We assume that the initial conditions of system and bath are given by \( \langle \delta \hat{b}^\dagger(0)\delta \hat{b}(0) \rangle = m_0, \langle \delta \hat{a}^\dagger(0)\delta \hat{a}(0) \rangle = n_0, \langle \hat{a}_{in}(t)\hat{a}^\dagger_{in}(\tau) \rangle = \delta(t - \tau), \) and \( \langle \hat{b}_k(0)\hat{b}_k(0) \rangle = m_k \) with \( m_k = 1/(e^{\hbar\omega_k/\kappa_B T} - 1) \), representing the phonon distribution function of the reservoir. The initial system correlations are given by \( \langle \delta \hat{b}^\dagger(0)\delta \hat{a}(0) \rangle = c_1 \) and \( \langle \delta \hat{b}(0)\delta \hat{a}(0) \rangle = c_2 \). We set the mirror to be initially in thermal equilibrium with the environment with \( n_0 = 1/(e^{\hbar\omega_m/\kappa_B T} - 1) \). The time evolution of the fluctuation of phonon number \( N_b(t) = \langle \delta \hat{b}^\dagger(t)\delta \hat{b}(t) \rangle \) is then given by (see appendix for more details)

\[
N_b(t) = |M(t)|^2 + |L(t)|^2 m_0 + |L(t)|^2 \int_0^t d\tau_1 d\tau_2 [L(t - \tau_1) + M^\ast(t - \tau_1) L^\ast(t - \tau_2) + M(t - \tau_2)] \\
\times f_1(\tau_1, \tau_2) + f_2(\tau_1, \tau_2) + f_{ih}(\tau_1, \tau_2) + \text{Re} \left[ M^\ast(t) \int_0^t d\tau i [M(t - \tau) + L^\ast(t - \tau)] f_{im}(\tau) \right] \\
+ \text{Re} \left[ L(t) \int_0^t d\tau [M^\ast(t - \tau) + L(t - \tau)] f_{im}(\tau) \right], \tag{4}
\]

where

\[
f_1(\tau_1, \tau_2) = G(\tau_1) G^\ast(\tau_2) e^{-u(\tau_1, \tau_2)} n_0 + G^\ast(\tau_1) G(\tau_2) e^{-u(\tau_1, \tau_2)} (n_0 + 1), \tag{5}
\]

\[
f_2(\tau_1, \tau_2) = -\kappa \int_0^{\tau_1} \int_0^{\tau_2} d\tau [G^\ast(\tau_1) G(\tau_2) e^{-u(\tau_1, \tau_2)} + G(\tau_1) G^\ast(\tau_2) e^{-u(\tau_2, \tau_1)} - G(\tau_1) G^\ast(\tau_2) e^{-u(\tau_2, \tau_1)}],
\]

\[
f_{ih}(\tau_1, \tau_2) = \int_0^\infty J(\omega) d\omega e^{-i\omega(\tau_1 - \tau_2)} + 2 \cos \omega(\tau_1 - \tau_2) \left( e^{\frac{k_B T}{\hbar\omega}} - 1 \right)^{-1},
\]

\[
f_{im}(\tau) = [G^\ast(\tau) e^{u(\tau, 0)} c_1 + G(\tau) e^{-u(\tau, 0)} c_2],
\]

\[
u(t_1, t_2) = -\int_{t_2}^{t_1} d\tau i \Delta_c(\tau) + \kappa/2,
\]

in which \( f_1 \) describes the contribution from the cavity photons which depends on the initial photon number \( n_0 \), \( f_2 \) results from the cavity input noise, \( f_{ih} \) represents the effect from the oscillator bath which depends strongly on the spectral density \( J(\omega) \), and \( f_{im} \) results from the system initial correlations. The time-depended function \( L(t) \) and \( M(t) \) are governed by

\[
M(t) = -i\omega_m M(t) + \int_0^t d\tau F(t - \tau) [M(\tau) + L(\tau)],
\]

\[
L(t) = i\omega_m L(t) + \int_0^t d\tau F^\ast(t - \tau) [M(\tau) + L(\tau)], \tag{6}
\]

where \( F(t - \tau) = f(t - \tau) - \{G^\ast(t) G(\tau) \exp[u(t, \tau)] - H.c. \} \) denotes the memory kernel term that contains the effect of radiation pressure.

### III. Instantaneous Cooling Limit and Cooling Rate

We now focus on the mechanical cooling in non-Markovian regime. Eq. (4) fully describe the fluctuation characteristics of the mechanical oscillator. According to Eq. (4), the last two terms denote the contribution of initial correlation to cooling. According to the definition of \( f_{im} \) in Eq. (5), the rotating wave (RW) term \( c_1 \) and the counter-rotating wave (CRW) term \( c_2 \) are formally symmetric. Therefore, the effects of these two terms on mechanical dynamics are also symmetric.

To better investigate the evolution of mechanical oscillator in sideband cooling, we introduce cooling rate \( \nu(t) = dN_b(t)/dt \) as an important parameter to evaluate cooling performances [19, 35]. It is difficult to directly obtain the analytical expression of \( \nu(t) \) by solving the differential equation of Eq. (4). Back to Eqs. (3), according to the Heisenberg-Langevin equations, cooling rate
can be solved through

$$\nu(t) = -iG(\hat{a}^\dagger \hat{b}) - G^*(\hat{a}^\dagger \hat{b})^* + G(\hat{a} \hat{b})^* - G^*(\hat{a} \hat{b})$$

$$+ i \sum_k V_k (\langle \hat{b} \hat{b}_k \rangle - \langle \hat{b} \hat{b}_k \rangle^* + \langle \hat{b}_k \hat{b}_k \rangle^* - \langle \hat{b}^\dagger \hat{b}_k \rangle),$$

the first four terms denotes the interaction between optical mode and mechanical mode. The last four terms denotes the interaction between mechanical mode and its environment. Solving a set of differential equations for the mean values of the second-order moments $N_a, N_b, \langle \delta a \delta b \rangle, \langle \delta a \delta b \rangle^*, \langle \delta b \delta b \rangle, \langle \delta \delta^2 \rangle, \langle \delta \delta^2 \rangle, \langle \delta b^2 \rangle, \langle \delta b^2 \rangle$, one can obtain the cooling rate [25],

$$\nu(t) = -iG(t)[c_1 e^{\mu_1(t)} + c_2 e^{\mu_2(t)}] + H.c. - X_1(t),$$

where $u_1(t) = -\int_0^t dt \{i[\omega_m + \Delta_s^c(\tau)] + \kappa/2 \}$ and $u_2(t) = -\int_0^t dt \{-i[\omega_m - \Delta_s^c(\tau)] + \kappa/2 \},$

$$X_1(t) = -i \int_0^t d\tau \{G(t)G^*(\tau)[(N_a - N_b)e^{u_1(t-\tau)} + (N_b + (aa^\dagger))e^{u_2(t-\tau)}] + \sum_k G(t)V_k [(\hat{a}^\dagger \hat{b}_k)^* e^{u_1(t-\tau)} - \langle a^\dagger b_k \rangle e^{u_2(t-\tau)}] - \langle a^\dagger b_k \rangle e^{u_2(t-\tau)}\} - i \int_0^t d\tau G(t) \sum_k [G(\tau)\langle a^\dagger b_k \rangle - V_k (N_k - N_b)] + H.c..$$

The function $X_1(t)$ is not directly dependent on the initial system correlation. We notice from Eq. (8) that the effect of initial correlation on cooling rate $\nu$ is embodied in the imaginary terms: $-G[c_1 e^{\mu_1(t)} + c_2 e^{\mu_2(t)}]$, i.e. $\nu(t) = -\text{Im}[Gc_1 e^{\mu_1(t)} + Gc_2 e^{\mu_2(t)}]$. When $\nu(t) > 0$, the system's initial correlation is capable of enhancing the cooling effect, whilst when $\nu(t) = 0$, the initial correlation has no effect on the cooling, and the phonon number will reach the peak. In contrast, $\nu(t) < 0$ indicates the initial correlation of the system can enhance the heating effect.

Without lose of generality, we assume $c_1$ and $c_2$ in Eq. (8) are real numbers. Neglecting the indirect terms of the initial system correlation, the average phonon re-duction of mechanical oscillator can be approximately simplified as $N_{cl}(t) = \int_0^t d\tau \nu(t)$. The system will eventually tend to steady-state cooling. When $N_{cl}(t) > 0$, the decrease of the phonon number is accelerated, that is to say that, the mechanical cooling is advanced in the presence of initial correlations. When $N_{cl}(t) < 0$, the increase of the phonon number is accelerated, that is to say that, the mechanical cooling is prolonged in the presence of initial correlations.

As shown in Fig. 2(a), it is obviously that $N_{cl} > 0$, thus the cooling rate and the instantaneous cooling time can be enhanced by initial correlation $c_1$. When the time $t$ (units of $\omega_m^{-1}$) is around 20, $N_{cl}$ has a significant maximum value, which means that the cooling rate can be significant increased. The evolution of $N_{cl}$ tends to stabilize due to dissipation in the long time scale. As shown in Fig. 2(b), it is obviously that $N_{cl} < 0$, thus the cooling rate and the instantaneous cooling time can be delayed by initial correlation $c_2$. When $t < 0$, $N_{cl}$ is slightly greater than 0. Thus, the overall cooling effect of the oscillator will be delayed in the presence of $c_2$. When $t$ around 45, $N_{cl}$ has a significant minimum value, which means that the cooling rate can be significant delayed in this time region. The evolution of $N_{cl}$ also tends to be stable due to dissipation in the long time scale. Thus, the instantaneous cooling rate can be accelerated by introducing initial correlation $c_1$.

The full numerical simulation of $N_{cl}(t)$ with different values of initial correlations is displayed in Fig. 3 (the initial states under interrogation are legal two-mode Gaussian states [36]). Figure. 3(a) investigates the cooling effect in the presence of BS-type initial correlation $c_1$. It is shown that, a rather low instantaneous phonon number state is achievable in a short time scale: the larger the initial correlation $c_1$ is, the shorter the time is required to reach the minimum phonon number state. It
is noteworthy that, when the value of $c_1$ is weak enough, the acceleration is negligible. When $c_1 = 0$, the instantaneous minimum state appears at $t = 22.5$ and $N_b \approx 0.4$. When $c_1 = 50$, the time reduces $t = 19.15$ and $N_b \approx 0.2$. As we further increase $c_1$ to 100, the minimum phonon number state is attained at $t = 17.15$ and $N_b \approx 0.07$. In addition, the instantaneous minimum phonon state is split into two due to the polarization effect caused by the initial correlation, which is similar to the mode splitting in optomechanical system.

Figure 3(b) showcases the cooling effect in the presence of AP-type initial correlation $c_2$. We observe similar acceleration shown in Fig. 3(a): a rather low instantaneous phonon state is attainable in the short time region. By increasing the strength of the initial correlation $c_2$, the attainment of the minimum phonon number state is delayed. For $c_2 = 50$, the delayed time is $t = 23.25$ and $N_b \approx 0.2$, while for $c_2 = 100$, the delayed time is $t = 23.85$ and $N_b \approx 0.02$. Similar splitting of the instantaneous minimum phonon number state arises from the polarization effect induced by the initial correlation [37].

As shown in the results in Fig. 3, both the RW term and the CRW term have a positive effect on cooling in non-Markovian regime. This is different but not contrary to the mechanism of steady-state cooling, in which BS-type interaction (RW terms) lead cooling and AP-type interaction (CRW terms) lead heating. A driving laser in red-detuning can makes the BS interaction dominant in the dynamics of the system and finally cool the mechanical oscillator to its ground state. The memory effect of non-Markovian environment can retain the initial correlation even in red-detuning regime. Therefore, we are able to maintain the cooling effect of BS-type interaction and meanwhile benefit from the cooling optimization effect of AP-type interaction by introducing initial correlation of the system.

According to the expression of $f_{in}$ in Eq. (5), both RW and CRW components each contains a time-dependent exponential term $\exp(\alpha(t))$ and $\exp(\alpha^*(t))$, respectively. The real and imaginary part of the exponential terms will oscillate from negative to positive with the evolution of time, and the amplitude of the oscillation is directly proportional to the absolute value of $c_1$ and $c_2$. When the oscillation makes the last two terms of negative, the cooling effect will be enhanced. Therefore, increasing the value of $c_1$ and $c_2$ can enhance the cooling effect.

### IV. ULTRAFAST OPTIMAL SIDEBAND COOLING

According to the results in Sec. III, a time accelerated and fluctuation decreased instantaneous minimum phonon number state can be obtained by introducing BS-type initial correlations. Once this optimized state is sustains, a fast steady-state cooling with low mechanical fluctuations can be achieved. As depicted in Fig. 1(b), regardless of the intermediate process, energy of the mechanical oscillator always depletes till through the dissipation of the cavity. Therefore, the state of the mechanical oscillator can be stabilized quickly by enhancing the dissipation of the cavity, which can be easily obtained according to Eqs. (3) (the cavity dissipation of the cavity will introduce an exponential decay term $\exp(-\alpha t)$ to accelerate the stability of the equations). This mechanism has been adopted to optimize the optomechanical sideband cooling in Markovian regime in Ref. [16].

Thus, by combining with dissipate cooling scheme and acceleration effect caused by appropriate initial correlations, we obtain a fast steady-state cooling scheme with low mechanical fluctuations, as illustrated in Fig. 4. An effective dissipative modulation, as shown in Fig. 4(a) is used to stabilize the instantaneous cooling state. After applying the dissipative modulation at a appropriate time, a stable ground state is achieved in a rather short time. The dynamic of sideband cooling of our scheme in non-Markovian regime is explored in Fig. 4(b) with initial correlation $c_1 = 100$ and $c_2 = 0$. As shown in

![Figure 3](image-url)  
Fig. 3. (a) $N_b$ as a function of $t$ with different value of $c_1$.  
(b) $N_b$ as a function of $t$ with different value of $c_2$. Values of other parameters are the same as in Fig. 2.

![Figure 4](image-url)  
Fig. 4. (a) Modulation scheme of the cavity dissipation rate $\kappa$ for rapid stability. (b) Dynamics of ultrafast optimized sideband cooling without (blue-solid line) and with (red-dashed line) dissipation modulation. Inset: Long-time scale of sideband cooling without dissipation modulation. Values of other parameters are the same as in Fig. 2.
Fig. 4(b), the fluctuation of mechanical oscillator decreases with time evolution, and it can be cooled into a low-excitation steady-state with \( N_b \approx 0.11 \) in the long time scale. In consideration of practicability and feasibility, it is always desired to speed up the cooling process with lower fluctuation. As shown in Fig. 4(b) (solid blue line), a low-excitation level (\( N_b \approx 0.07 \)) in a non-steady state is achieved at \( t = 17.15 \). At this moment, a Q-switch technology [16] is utilized to increase the loss rate of the cavity, i.e., increase \( \kappa \) from 0.05\( \omega_m \) to \( \omega_m \). Thus, the stability of the low-excitation state can be accelerated after Q-switch manipulation. The modulation results are represented in Fig. 4(b) by the dashed red line. The phonon number reaches a low and stable value, i.e., \( N_b \approx 0.096 \) within time \( t = 70 \). It is worthwhile noting that by enhancing the BS-type initial correlations, one can further shorten the cooling time and reduce the mechanical fluctuation.

We now consider the feasibility of our scheme. The sideband cooling of non-Markovian micro-mechanical oscillator can be implemented in existing miro-optomechanical systems. Such an experimental device consists of a thick layer of Si$_3$N$_4$ with a high-reflectivity mirror pad in its centre as a mechanically moving end mirror in a Fabry-Pérot cavity. The corresponding mechanical resonance frequency is \( \omega_m = 2\pi \times 914 kHz \) and Ohmic-type mechanical environment is reported in Ref. [26]. The required initial correlations can be introduced by selecting a strong laser drive under red-detuning condition before the cooling dynamics begins. And it has been experimentally demonstrated that a sideband resolution parameter \( \omega_m / \kappa \) close to and even exceed 10 is achievable in miro-optomechanical domain [38–40]. This provides a promising platform for implementing our cooling optimization scheme. Thus, with existing experimental parameters and a mechanical resonance frequency of \( \omega_m = 2\pi \times 914 kHz \), the cooling time of our scheme can reach \( t \approx 10^{-5} s \) and the resultant mean value of steady-state mechanical fluctuation is \( N_b \approx 0.096 \).

V. CONCLUSIONS

In conclusion, we have investigated the effect of initial correlations on sideband cooling in the non-Markovian regime. The results show that both the BS-type and AP-type initial correlations both have positive effect on the sideband cooling. By increasing the initial correlations, the fluctuation of mechanical oscillator is significantly reduced in instantaneous regime. The instantaneous cooling limit can be reduced by one order of magnitude with initial correlations \( c_{1,2} = 100 \). In addition, the BS-type initial correlation is effective for accelerating the process of sideband cooling. When \( c_1 = 100 \), the instantaneous cooling time can be reduced to \( 17.15 \omega_m^{-1} \). By combining the conventional dissipative cooling method with the present optimization of initial correlations, we present a stable cooling scheme that is ultra-fast and has ultra-low fluctuation. We believe this scheme is useful in exploring the quantum properties of mechanical oscillator and solid-state quantum information processing.

VI. ACKNOWLEDGMENTS

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Appendix: DYNAMICS OF MECHANICAL OSCILLATOR

According to Eq. (3c), we can obtain the formal solution of \( \delta \hat{a} \)

\[
\delta \hat{a}(t) = \delta \hat{a}(0) e^{u(t,0)} + \int_0^t dt e^{u(t-\tau,0)} \{ iG[\delta \hat{b}^\dagger(\tau) + \delta \hat{b}(\tau)] + \sqrt{\kappa} \hat{a}_{in}(\tau) \},
\]

where \( u(t_1, t_2) = - \int_{t_1}^{t_2} d\tau [i\Delta'_c + \kappa / 2] \). Substituting Eq. (3c) into Eq. (3d), we have

\[
\dot{\delta \hat{b}} = -i \omega_m \delta \hat{b} + \int_0^t d\tau F(t - \tau)[\delta \hat{b}^\dagger(\tau) + \delta \hat{b}(\tau)] + \hat{A}_0(t) + \hat{A}_{in}(t) - \hat{\xi}(t),
\]

where,

\[
F(t - \tau) = f(t - \tau) - [e^{u(t-\tau,0)}G(t)G(\tau) - H.c.],
\]

\[
\hat{A}_0(t) = i[G^\dagger(t)\delta \hat{a}(0)e^{u(t,0)} + H.c.],
\]

\[
\hat{A}_{in}(t) = i \int_0^t d\tau [G^\dagger(t)e^{u(t-\tau,0)} \sqrt{\kappa} \hat{a}_{in}(\tau) + H.c.].
\]
In consideration of the linearity of Eq. (A.2), we can assume that the solution of the operator $\hat{\delta}b(t \geq 0)$ is of the form

$$\hat{\delta}b(t) = M(t)\hat{\delta}b(0) + L^*(t)\hat{\delta}^\dagger(t) + \hat{S}(t), \quad (A.6)$$

with the initial conditions $M(0) = 1$ and $L(0) = 0$. The equations for the time-dependent coefficients $M(t)$, $L(t)$ and $S(t)$ can be found by substituting Eq. (A.6) into Eq. (A.2) and then comparing the coefficients. We have

$$\dot{M}(t) = -i\omega_m M(t) + \int_0^t d\tau F(t - \tau)[M(\tau) + L(\tau)], \quad (A.7)$$

$$\dot{L}(t) = -i\omega_m L(t) + \int_0^t d\tau F^*(t - \tau)[M(\tau) + L(\tau)], \quad (A.8)$$

$$\dot{S}(t) = -i\omega_m S(t) + \int_0^t d\tau F(t - \tau)U[\hat{S}(\tau) + \hat{S}^\dagger(\tau)] + \hat{A}_0(t) + \hat{A}_m(t) - \hat{\xi}(t). \quad (A.9)$$

If $M(t)$ and $L(t)$ are known, the operator $\hat{S}(t)$ can be completely determined through

$$\hat{S}(t) = \int_0^t d\tau [M(t - \tau) + L^*(t - \tau)][\hat{A}_0(\tau) + \hat{A}_m(\tau) - \hat{\xi}(\tau)]. \quad (A.10)$$

To obtain the time evolution of the mean phonon number of the quantum part with initial system-reservoir correlations and without system-bath correlations. The time evolution of the mean phonon number $N_b(t) = \langle \hat{\delta}b^\dagger(t)\hat{\delta}b(t) \rangle$ is given by

$$N_b(t) = |M(t)|^2\langle \hat{\delta}b^\dagger(0)\hat{\delta}b(0) \rangle + |L(t)|^2\langle \hat{\delta}b(0)\hat{\delta}b(0) \rangle + M^*(t)\langle \hat{\delta}b^\dagger(0)\hat{S}(t) \rangle + M(t)\langle \hat{S}^\dagger(t)\hat{\delta}b(0) \rangle$$

$$+ L(t)\langle \hat{\delta}b(0)\hat{S}(t) \rangle + L^*(t)\langle \hat{S}^\dagger(t)\hat{\delta}b(0) \rangle + \langle \hat{S}^\dagger(t)\hat{S}(t) \rangle, \quad (A.11)$$

where

$$\langle \hat{\delta}b^\dagger(0)\hat{S}(t) \rangle = \int_0^t d\tau [M(t - \tau) + L^*(t - \tau)]\langle \hat{\delta}b^\dagger(0)[\hat{A}_0(\tau) - \hat{\xi}(\tau)] \rangle, \quad (A.12)$$

$$\langle \hat{\delta}b(0)\hat{S}(t) \rangle = \int_0^t d\tau [M(t - \tau) + L^*(t - \tau)]\langle \hat{\delta}b(0)[\hat{A}_0(\tau) - \hat{\xi}(\tau)] \rangle, \quad (A.13)$$

$$\langle \hat{S}^\dagger(t)\hat{\delta}b(0) \rangle = \int_0^t d\tau [L(t - \tau) + M^*(t - \tau)]\langle \hat{\delta}b(0)[\hat{A}_0(\tau) - \hat{\xi}(\tau)] \rangle, \quad (A.14)$$

$$\langle \hat{S}^\dagger(t)\hat{\delta}^\dagger(0) \rangle = \int_0^t d\tau [L(t - \tau) + M^*(t - \tau)]\langle \hat{\delta}^\dagger(0)[\hat{A}_0(\tau) - \hat{\xi}(\tau)] \rangle, \quad (A.15)$$

$$\langle \hat{S}^\dagger(t)\hat{S}(t) \rangle = \int_0^t \int_0^t d\tau d\tau' [L(t - \tau) + M^*(t - \tau)][M(t - \tau_2) + L^*(t - \tau_2)]$$

$$\times \langle [\hat{A}_0(\tau_1),\hat{A}_0(\tau_2)] + \langle \hat{A}_m(\tau_1),\hat{A}_m(\tau_2) \rangle + \hat{\xi}(\tau_1)\hat{\xi}(\tau_2) \rangle, \quad (A.16)$$

in which the autocorrelation functions are given by

$$\langle \hat{A}_0(\tau_1)\hat{A}_0(\tau_2) \rangle = -[G(\tau_1)G^*(\tau_2)e^{-\mu_t(\tau_1,\tau_2)}\langle \delta\hat{a}^\dagger(0)\delta\hat{a}(0) \rangle + G^*(\tau_1)G(\tau_2)e^{\mu_t(\tau_1,\tau_2)}\langle \delta\hat{a}(0)\delta\hat{a}(0) \rangle], \quad (A.17)$$

$$\langle \hat{A}_m(\tau_1)\hat{A}_m(\tau_2) \rangle = -\kappa \int_0^{\tau_1} \int_0^{\tau_2} d\tau_1 d\tau_2 [G_1^2(\tau_1)G_2^2(\tau_2)e^{\mu_t(\tau_2,\tau_1) - \mu_t(\tau_1,\tau_2)} + G_1(\tau_1)G_2^*(\tau_2)e^{\mu_t(\tau_2,\tau_1) - \mu_t(\tau_1,\tau_2)}], \quad (A.18)$$

$$\langle \hat{\xi}(\tau_1)\hat{\xi}(\tau_2) \rangle = -\sum_k V_k^2 \langle e^{-i\omega_k(\tau_1 - \tau_2)}\hat{b}_k(0)\hat{b}_k^\dagger(0) \rangle + \langle e^{i\omega_k(\tau_1 - \tau_2)}\hat{b}_k^\dagger(0)\hat{b}_k(0) \rangle. \quad (A.19)$$
The cross-correlation function are given by

\[
\langle \delta \hat{b}(0) \hat{A}(t) \rangle = i \langle G^*(t) e^{u(t,0)} (\delta \hat{b}(0) \delta \hat{a}(0)) \rangle + G(t) e^{u(t,0)} \langle \delta \hat{a}(0) \delta \hat{b}(0) \rangle^*,
\]  \hspace{1cm} (A.20)

\[
\langle \delta \hat{a}(0) \hat{A}(t) \rangle = i \langle G^*(t) e^{u(t,0)} (\delta \hat{a}(0) \delta \hat{b}(0)) \rangle + G(t) e^{u(t,0)} \langle \delta \hat{b}(0) \delta \hat{a}(0) \rangle^*,
\]  \hspace{1cm} (A.21)

\[
\langle \hat{A}(0) \delta \hat{b}(0) \rangle = i \langle G^*(t) e^{u(t,0)} (\delta \hat{a}(0) \delta \hat{b}(0)) \rangle + G(t) e^{u(t,0)} \langle \delta \hat{b}(0) \delta \hat{a}(0) \rangle^*,
\]  \hspace{1cm} (A.22)

\[
\langle \hat{A}(0) \delta \hat{a}(0) \rangle = i \langle G^*(t) e^{u(t,0)} (\delta \hat{a}(0) \delta \hat{b}(0)) \rangle + G(t) e^{u(t,0)} \langle \delta \hat{b}(0) \delta \hat{a}(0) \rangle^*.
\]  \hspace{1cm} (A.23)