Asymptotic Dimension of Box Spaces and Odometers of Elementary Amenable Groups

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Abstract: We relate the asymptotic dimension of box spaces to the dynamic asymptotic dimension of related odometer actions and show that the DAD of odometer actions is sub-additive over extensions of groups. These results can be applied together to calculate the asymptotic dimension of box spaces of many elementary amenable groups, as we show for the Baumslag-Solitar groups BS(1, n); as well as to prove that such box spaces have dimension at most the Hirsch length of the group.

1 Introduction

The asymptotic dimension of a metric space measures its ‘large-scale’ dimension and has applications to geometric group theory. In particular, one can make groups into metric spaces in a way which is well-defined on a large scale. There are also coarse spaces called box spaces of groups which are constructed from sequences of finite quotients. It has for some time been suspected that the box spaces of many amenable (or at least elementary amenable) groups have finite asymptotic dimension. Finiteness of this dimension for box spaces of virtually nilpotent groups was established in [4], and an attempted proof for many elementary amenable groups can be found in [7].

More recently, an analogue of the asymptotic dimension was introduced for dynamical systems [10]. In this paper, we will relate the asymptotic dimension of box spaces to the dynamic asymptotic dimension of odometer actions (actions of a group on its profinite completion). We then prove an extension theorem for the dynamic asymptotic dimension of odometers.
Theorem A. Suppose $1 \to \Delta \to \Gamma \to \Lambda \to 1$ is an exact sequence of countable groups. If $(N_i)$ is a countable collection of finite index normal subgroups of $\Gamma$ directed by inclusion, it induces such collections and hence odometers of $\Delta$ and $\Lambda$. Then if $DAD_{\text{free}}(\Delta \curvearrowright \hat{\Lambda}(N_i \cap \Delta)) = m$ and $DAD_{\text{free}}(\Lambda \curvearrowright \hat{\Gamma}(N_i)) = n$, $DAD_{\text{free}}(\Gamma \curvearrowright \hat{\Gamma}(N_i)) \leq m + n$.

Using the relationship between odometers and box spaces, this allows us to prove a similar result for box spaces (a direct reapplication of the methods used to prove Theorem A in the setting of box spaces likely shows this result without assuming that $(N_i)$ is a sequence).

Theorem B. Suppose $1 \to \Delta \to \Gamma \twoheadrightarrow \Lambda \to 1$ is an exact sequence of countable groups. Suppose $(N_i)$ is a sequence of finite index normal subgroups of $\Gamma$ directed by inclusion such that $\text{diam}(\Gamma/N_i) \to \infty$. This collection induces such collections and hence box spaces of $\Delta$ and $\Lambda$. Then if $\text{asdim} \Box(N_i \cap \Delta) = m$ and $\text{asdim} \Box(\pi(N_i)) \Lambda = n$, $\text{asdim} \Box(N_i) \Gamma \leq m + n$.

This has several consequences for the asymptotic dimension of box spaces of elementary amenable groups. For instance, it allows us to show box spaces of $BS(1, n)$ along filtrations have asymptotic dimension 2 for all $n$. More generally, we can recover a conjecture in [7] that the asymptotic dimension of box spaces of many elementary amenable groups is bounded above by the Hirsch length of the group.

Theorem C. Let $\Gamma$ be a countable, residually finite, elementary amenable group. Then for any sequence $(N_i)$ which is also a filtration (i.e. for any $\gamma \in \Gamma$ with $\gamma \neq e$, there is $i$ such that $\gamma \notin N_i$), $\text{asdim} \Gamma \leq \text{asdim} \Box(N_i) \Gamma \leq h(\Gamma)$.

The reverse inequality also holds in a more restrictive setting:

Theorem D. If $\Gamma$ is countable and residually finite and $\Gamma = \Lambda \rtimes \Delta$ where $\Lambda$ is virtually polycyclic and $\Delta$ is locally finite, then for any sequence $(N_i)$ which is also a filtration of $\Gamma$, $\text{asdim} \Gamma = \text{asdim} \Box(N_i) \Gamma = h(\Gamma)$.

2 (Dynamic) Asymptotic Dimension

The following concept is originally due to Gromov, specifically [8, Section 1.E]. There are many definitions, but we will use one throughout:
Definition 2.1. Let \((X, d)\) be a metric space. If \(U \subset X\) is a subset, an \(r\)-chain in \(U\) is a sequence \(x_0, \ldots, x_n\) of points in \(U\) such that \(d(x_i, x_{i+1}) \leq r\) for all \(i\); and two points of \(U\) are in the same \(r\)-component if they are connected by an \(r\)-chain in \(U\). The subset \(U\) is said to be \(r\)-connected if any two points in \(U\) are connected by an \(r\)-chain in \(U\). Finally, \(\text{asdim} X \leq n\) means for all \(r > 0\) there is \(M > 0\) and a cover \(U = \{U_0, \ldots, U_n\}\) of \(X\) such that each \(r\)-component of each \(U_i\) has diameter at most \(M\). Such a cover is called a \((d, R, M)\)-cover of \(X\).

We now define the dynamic asymptotic dimension of a group action, originally introduced by Guentner, Willett, and Yu in [10].

Definition 2.2. Let \(\Gamma \acts X\) be an action on a compact Hausdorff space by homeomorphisms. The dynamic asymptotic dimension \(\text{DAD}(\Gamma \acts X)\), denoted \(\text{DAD}(\Gamma \acts X)\), is the smallest integer \(d\) such that for all finite \(F \subset \Gamma\) there exists an open cover \(U = \{U_0, \ldots, U_d\}\) of \(X\) such that the set

\[
\left\{ \begin{array}{l}
g \in \Gamma \\
g = f_k \cdots f_1 \text{ and } \exists x \in U_j, 0 \leq j \leq d \\
\text{such that } f_{k_0} \cdots f_1 \cdot x \in U_j \forall 1 \leq k_0 \leq k
\end{array} \right\}
\]

is finite.

If \(\Gamma \acts X\) is free, this definition is equivalent to the following:

Definition 2.3. If \(\Gamma \acts X\) is an action on a compact space by homeomorphisms, \(\text{DAD}_{\text{free}}(\Gamma \acts X)\) is the smallest integer \(d\) such that for all finite \(F \subset \Gamma\) there is an open cover \(U = U_0, \ldots, U_d\) such that

\[
\left\{ \begin{array}{l}
y \in X \\
\exists f_1, \ldots, f_k \in F, \text{ and } x \in U_j (0 \leq j \leq d) \text{ such that } \\
y = f_k \cdots f_1 \cdot x \text{ and } f_{k_0} \cdots f_1 \cdot x \in U_j \forall 1 \leq k_0 \leq k
\end{array} \right\}
\]

is uniformly finite (i.e. the same bound applies independent of \(x\)).

We now introduce similar terminology for DAD as was used for the asymptotic dimension.

Definition 2.4. Let \(\Gamma \acts X\) be an action and \(S \subset \Gamma\) a finite subset containing the identity with \(S = S^{-1}\) (such sets will always be of this form). If \(U \subset X\) is a subset, an \(S\)-chain in \(U\) is a sequence \(x_0, \ldots, x_n\) of points in \(U\) such that for all \(i\), \(\gamma \cdot x_i = x_{i+1}\) for some \(\gamma \in S\). The length of such a chain is the number of points in the chain (so \(x_0, \ldots, x_n\) has length \(n + 1\)). Two points
in $X$ are in the same $S$-component if they are connected by an $S$-chain. The subset $U$ is said to be $S$-connected if any two points in $U$ are connected by an $S$-chain in $U$. We say a cover $U = \{U_j\}_{j=0}^d$ is a $(d, S, M)$-cover for $\Gamma \act X$ (or just $X$ if unambiguous) if all $S$-components of each $U_i$ have cardinality at most $M$. Note that the dynamic asymptotic dimension of a free action $\Gamma \act X$ is then the smallest integer $d$ such that for every finite subset $S \subset \Gamma$, there is $M > 0$ and an open $(d, S, M)$-cover for $\Gamma \act X$.

Although this terminology relates specifically to the definition of DAD for free actions, we will use it even for actions which are not free. For actions on totally disconnected spaces, we can equivalently require the covers to be clopen, as shown below.

**Lemma 2.5.** Suppose $\mathcal{V} = \{V_0, \ldots, V_d\}$ is an open cover of a totally disconnected, compact space $X$. Then there is a clopen cover $\mathcal{W} = \{W_0, \ldots, W_d\}$ such that $W_i \subset V_i$ for all $0 \leq i \leq d$. In the context of the previous definition, this implies if $\mathcal{V}$ is a $(d, S, M)$-cover, then $\mathcal{W}$ is a $(d, S, M)$-cover.

**Proof.** Given $\mathcal{V}$, there is a cover $U = \{U_i\}_{i=0}^d$ by open sets such that $\overline{U_i} \subset V_i$ for $0 \leq i \leq d$ (a more general statement is shown in [14], Lemma 41.6). Then since $X$ has a basis of clopen sets, we can find for each $i$ and each $x \in U_i$ a clopen set $W_{i,x}$ containing $x$ and contained in $V_i$. For each $i$, the $W_{i,x}$’s are a cover of $\overline{U_i}$ (which is compact since $X$ is compact); and so there is a finite subcover, the union over which we denote $W_i$. This is a finite union of clopen sets and therefore clopen. Moreover, the collection $\{W_i\}_{i=0}^d$ is a cover of $X$, and $W_i \subset V_i$ for all $i$. \hfill $\square$

Two important sources of examples in coarse geometry are graphs, considered as metric spaces by equipping their vertex set with the $l^1$-path metric, and coarse disjoint unions of such spaces. Groups can be considered as metric spaces in the finitely generated case through the Cayley graph construction. More generally, any countable group has a right-invariant proper metric unique up to coarse equivalence [17, Lemma 2.3.3].

**Definition 2.6.** If $(G_n)$ is a countable collection of finite metric spaces, the coarse disjoint union $\sqcup_n G_n$ is the metric space with underlying set the disjoint union of the $G_n$ and metric $d(x, y) := d_{G_n}(x, y)$ if $x, y \in G_n$ and $d(x, y) := \text{diam}(G_n) + \text{diam}(G_m)$ if $x \in G_n$ and $y \in G_m$ with $m \neq n$. If $\Gamma$ is a group and $(N_i)$ is a collection of finite-index, normal subgroups, we define $\Box_{(N_i)} \Gamma := \sqcup_i \Gamma / N_i$ where each $\Gamma / N_i$ has the metric induced from a
fixed, right-invariant proper metric on \( \Gamma \) (i.e. the distance between cosets is the infimum over the distance between their different elements according to the original metric on \( \Gamma \)). If the metric on \( \Gamma \) is the word metric coming from some generating set, the induced metrics on quotients are the word metrics coming from the image of that generating set in the quotient. This is called a box space of \( \Gamma \). If \((N_i)\) is the collection of all finite index normal subgroups of \( \Gamma \), we call this space the full box space of \( \Gamma \) and denote it \( \square \Gamma \).

**Definition 2.7.** A filtration of \( \Gamma \) is a collection \((N_i)\) of finite index normal subgroups of \( \Gamma \) such that \( \bigcap_i N_i = \{e\} \), the trivial subgroup. If \( \Gamma \) is residually finite, the collection of all finite index normal subgroups is then a filtration.

**Definition 2.8.** If \((G_i)\) is a sequence of finite metric spaces, we say asdim\((G_i)\) \( \leq d \) uniformly if for all \( r > 0 \) there is a \((d, r, R)\)-cover of each \( G_i \).

**Lemma 2.9.** If asdim\((N_i)\) \( \Gamma \leq d \), then asdim\((\Gamma/N_i)\) \( \leq d \) uniformly. If for all \( r > 0 \) there are only finitely-many \( i \) with diam\((\Gamma/N_i)\) \( \leq r \) (for instance if the \((N_i)\) are indexed by the natural numbers and diam\((\Gamma/N_i)\) \( \to \infty \) as \( i \to \infty \)) then asdim\((\Gamma/N_i)\) \( \leq d \) uniformly implies asdim\((N_i)\) \( \Gamma \leq d \).

**Proof.** For the first inequality, notice that a \((d, r, R)\)-cover for \( \square(N_i)\) \( \Gamma \) gives rise to a \((d, r, R)\)-cover for each \( \Gamma/N_i \) (just take the intersection).

For the second inequality, let \( r > 0 \) and find a \((d, r, R)\)-cover \( U^i = \{U^i_0, \ldots, U^i_j\} \) of each \( \Gamma/N_i \). Let \( I_r = \{i \mid \text{diam}(\Gamma/N_i) > r\} \). Define \( U_j = \left( \bigcup_{i \in I_r} U^i_j \right) \cup \left( \bigcup_{i \in I_r} \Gamma/N_i \right) \). The set \( I_r \) is finite by assumption, and the \( U_j \) form a \((d, r, \max R, \#I_r \cdot r)\)-cover.

**Lemma 2.10.** Suppose \( G = \bigcup_i G_i \) is an increasing union of groups and \( X = \square_{(H_j)} G = \bigcup_i X_i \) is an increasing union of box spaces where \( X_i = \square_{(H_j \cap G_i)} G_i \) is a box space of \( G_i \) (here the metrics on \( X \) and \( G_i \) are induced from the right invariant, proper metric on \( G \), and the metric on \( X_i \) is constructed using the metric on \( G_i \)). Then asdim\(G\) = sup\(i\), asdim\(G_i\) and asdim\(X\) = sup\(i\), asdim\(X_i\).

**Proof.** Fix \( R > 0 \). Then there is \( N \) such that \( B^R_e(G) \subset G_N \). Notice that \( d_G(G_N \cdot g, G_N \cdot h) \) is either 0 or \( > R \). This follows since if \( d_G(G_N \cdot g, G_N \cdot h) \leq R \) then the two orbits have a nontrivial intersection and therefore coincide. Since different \( G_N \)-orbits are \( > R \) apart, we can cover each one separately using asdim\(G_N + 1\)-sets. For the second claim, find \( N \) as before. Then if \( x, y \in G/H_j \) are in separate \( G_N/(H_j \cap G_N) \)-orbits, their representatives in \( G \) must be in separate \( G_N \) orbits, and so \( d_G/H_j(x, y) > R \). We can therefore similarly cover each orbit separately using asdim\(\square_{(H_j \cap G_N)} G_N + 1\) sets. \( \square \)
Lemma 2.11. Suppose $\Gamma$ is a countable group with normal subgroup $N < \Gamma$ and $|\Gamma|$ is $\Gamma$ equipped with a right-invariant, proper metric. Suppose further that $|\Gamma/N|$ is a quotient of $\Gamma$ with the metric induced from $|\Gamma|$. Consider the action $\Gamma \curvearrowright \Gamma/N$ by left multiplication. Then a $B_e^r(|\Gamma|)$-connected subset $A \subset \Gamma/N$ with cardinality at most $M$ has diameter at most $rM$. Moreover, a subset $A \subset |\Gamma/N|$ with diameter at most $R$ has cardinality at most $B_e^r(|\Gamma|)$.

Proof. Observe that $x, y \in A \subset \Gamma$ are connected by a $B_e^r(|\Gamma|)$-chain in $A$ iff they are connected by an $r$-chain in $A \subset |\Gamma|$. 

3 Elementary Amenable Groups

Section 7 relies on a certain perspective on elementary amenable groups, as well as a generalization of the Hirsch length. This short section summarizes the facts we need.

Definition 3.1. Define the class of (countable) elementary amenable groups to be the smallest class of groups containing all finite and all countable abelian groups which is closed under taking subgroups, quotients, extensions, and direct limits.

Lemma 3.2. If $C$ and $D$ are classes of groups, denote by $CD$ the groups which fit into an extension of the form $1 \to C \to \cdots \to D \to 1$ with $C \in C$ and $D \in D$ and denote by $LC$ the groups which are locally in $C$ (i.e. those groups whose finitely generated subgroups are all in $C$). Now define classes of groups $\Gamma_\alpha$ as follows. Begin with $\Gamma_0 = \{1\}$ and $\Gamma_1$ the class of finitely generated virtually abelian groups. Assuming the definition of $\Gamma_\alpha$ for $\alpha$ a countable ordinal, define $\Gamma_{\alpha+1}$ to be $L(\Gamma_\alpha)\Gamma_1$. If $\alpha$ is a countable limit ordinal, then $\Gamma_\alpha := \bigcup_{\beta<\alpha} \Gamma_\beta$. The class of countable elementary amenable groups is then $\bigcup_{\alpha} \Gamma_\alpha$ where $\alpha$ ranges over all countable ordinals.

Definition 3.3. The Hirsch length of a finitely generated group with finite index abelian subgroup $A$ is defined to be $\text{rank}(A)$ [12]. As in [12], we can use [3,2] to extend the definition of Hirsch length inductively by defining, for a sequence $1 \to A \to \Gamma \to \Delta \to 1$ that $h(\Gamma) = h(A) + h(\Delta)$; and defining, for
an increasing union of groups $\Gamma = \bigcup_n \Gamma_n$ that $h(\Gamma) = \sup_n h(\Gamma_n)$. It is shown in [12] that this is well defined and that Hirsch length satisfies the following

1. for $\Lambda \leq \Gamma$, $h(\Lambda) \leq h(\Gamma)$
2. for $\Lambda \preceq \Gamma$, $h(\Gamma) = h(\Lambda) + h(\Gamma/\Lambda)$.
3. $h(\Gamma) = \sup\{h(\Lambda)|\Lambda < \Gamma$ is finitely generated $\}$

4 Odrometers and Box Spaces

As we have seen, asymptotic dimension and DAD are closely related at least when the space being acted on is discrete. This relationship is encapsulated by [2,11] and, together with some topological facts about profinite completions, can be used to show the DAD (according to the definition for free actions) of an odometer formed from a family $(N_i)$ of subgroups is the same as the asymptotic dimension of the box space formed from $(N_i)$. This is spiritually similar to how the DAD of an action is bounded above by the asymptotic dimension of its warped cone (this can be shown directly, but see [15] for definitions and a more complete discussion of warped cones and DAD).

**Definition 4.1.** Let $\Gamma$ be a countable group and $(N_i)$ a countably infinite collection of finite-index normal subgroups directed by inclusion (for example the collection of all such subgroups of $\Gamma$). The profinite completion of $\Gamma$ along $(N_i)$ is the inverse limit $\lim \leftarrow_i \Gamma/N_i$ and is denoted $\hat{\Gamma}(N_i)$. If $(N_i)$ is the collection of all such subgroups, we just write $\hat{\Gamma}$. There is an action $\Gamma \actson \hat{\Gamma}(N_i)$ by left multiplication in each coordinate. The canonical quotient map $p_i : \hat{\Gamma}(N_i) \to \Gamma/N_i$ is continuous for the profinite topology and equivariant for the actions by left multiplication.

**Remark 4.2.** Some technical remarks are in order before we continue. First, notice that we assume the collection $(N_i)$ to be countable, even though a countable group may have uncountably many finite-index normal subgroups. When the collection $(N_i)$ is uncountable, the topology on the profinite completion along $(N_i)$ (and similarly the coarse structure on the box space $\square(N_i)\Gamma$) is not metrizable, requiring different language. Notwithstanding, spiritually similar proofs work in this case.
Next, it will be important for \([4.4]\) that the collection \(\{N_i\}\) is a sequence, meaning the index set is the natural numbers and \(N_{i+1} \subset N_i\), and that \(\text{diam}(\Gamma/N_i) \to \infty\) as \(i \to \infty\).

**Lemma 4.3.** Let \(\Gamma\) be a countable group and \(\{N_i\}\) a countable collection of finite index normal subgroups which is directed by inclusion. If \(\hat{\Gamma}\) is the profinite completion along \(\{N_i\}\), then a subset \(U \subset \hat{\Gamma}\) is clopen iff it is of the form \(p_i^{-1}(S)\) where \(S \subset \Gamma/N_i\) is any subset.

**Proof.** First, sets of the form \(p_i^{-1}(S)\) are clopen as \(S \subset \Gamma/N_i\) is clopen and \(p_i\) is continuous.

Fix the metric \(d((x_i, (y_i)) = \sum_{i=1}^{\infty} \frac{d_{\Gamma/N_i}(x_i,y_i)}{2^i \text{diam}(\Gamma/N_i)}\) on \(\hat{\Gamma}\). If \(U\) is clopen, then \(\hat{\Gamma} = U \cup U^c\) where \(U\) and \(U^c\) are separate connected components. In fact, since \(\hat{\Gamma}\) is a compact metric space, there is \(\epsilon\) such that the \(\epsilon\)-neighborhoods of \(U\) and \(U^c\) do not intersect.

Now, suppose \(U\) is not of the desired form. Then for all \(i\) there is \(j \geq i\) and \(x_j \in \hat{\Gamma}\) such that \(p_j(x_j) \in p_j(U)\) but \(p_{j+1}(x_j) \notin p_{j+1}(U)\). But then \(x_j \notin U\) for all such \((\text{infinitely-many})\) \(j\) and \(d(x_j, U) \leq \sum_{n=j+1}^{\infty} \frac{1}{2^n} \to 0\) as \(j \to \infty\). Hence, there are points in \(U^c\) arbitrarily close to \(U\). By what was said above, this implies \(U\) is not clopen.

**Proposition 4.4.** Let \(\Gamma\) be a countable group and \(\{N_i\}\) a sequence of finite-index normal subgroups which is directed by inclusion. Let \(\Gamma \curvearrowright \hat{\Gamma}_{(N_i)}\) be the odometer action induced by left multiplication on each finite quotient. If \(DAD_{\text{free}}(\Gamma \curvearrowright \hat{\Gamma}_{(N_i)}) \leq d\) then \(\text{asdim}(\Gamma/N_i) \leq d\) uniformly. If also \(\text{diam}(\Gamma/N_i) \to \infty\) as \(i \to \infty\), then \(\text{asdim}(\square(N_i)\Gamma) \leq d\).

**Proof.** Fix a proper, left-invariant metric on \(\Gamma\). Let \(R \geq 0\) be given and set \(F = B_{\text{e}}^R(\Gamma)\). Let \(V = \{V_0, \ldots, V_d\}\) be an open \((d, F, M_F)\)-cover for \(\Gamma \curvearrowright \hat{\Gamma}_{(N_i)}\). Observe that for each \(i\), the \(F\)-components of \(p_i(V_j)\) are \(R\)-separated according to the metric on \(\Gamma/N_i\).

Now by \([2.5]\) we can assume each \(V_j\) is clopen and hence by \([4.3]\) is the pullback of \(S \subset \Gamma/N_i\) for some \(i\). Find \(i_0\) such that all of the \(V_j\) are pullbacks of \(\Gamma/N_{i_0}\). Then if \(i \geq i_0\), \(p_i(x) \in p_i(V_j)\), and \(f \cdot p_i(x) \in p_i(V_j)\), then \(x, f \cdot x \in V_j\). Thus, the cardinality of each \(F\)-component of \(p_i(V_j)\) is bounded by \(M_F\) for all \(i \geq i_0\); so by \([2.11]\) the diameter (as a subset of \(\square(N_i)\Gamma\)) of each \(R\)-component of \(p_i(V_j) \subset |\Gamma/N_i|\) is bounded by \(RM_F\) for \(i \geq i_0\). This gives a

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\(^1\)Technically, the sum is over a rearrangement, but that doesn’t matter.
(d, R, RM_F)-cover of each \(\Gamma/N_i\) for all \(i \geq i_0\). Since there can only be finitely many finite index normal subgroups containing a given finite index normal subgroup, the set \(\{i \mid i < i_0\}\) is finite, and so the diameter of \(\Gamma/N_i\) for \(i < i_0\) is uniformly bounded. We have therefore shown \(\operatorname{asdim}(\Gamma/N_i) \leq d\) uniformly. The last part then follows from \(2.9\).

**Proposition 4.5.** Let \(\Gamma\) be a countable group and \((N_i)\) a countably infinite collection of finite index normal subgroups of \(\Gamma\) directed by inclusion. Let \(\Gamma \curvearrowright \widehat{\Gamma}_{(N_i)}\) be the corresponding odometer action (given by left multiplication as before). Then \(\operatorname{DAD}_{\text{free}}(\Gamma \curvearrowright \widehat{\Gamma}_{(N_i)}) \leq \operatorname{asdim}(\square_{(N_i)\Gamma})\).

**Proof.** Fix a finite subset \(F \subset \Gamma\).

Find \(R > 0\) so that \(d_{\Gamma}(\gamma \cdot x, x) < R\) for all \(\gamma \in F\) and \(x \in \Gamma\), hence \(d_{\Gamma/N_i}(\gamma \cdot x, x) < R\) for all \(\gamma \in F\) and all \(i\). Suppose \(\operatorname{asdim}\square_{(N_i)\Gamma} \leq n\). Then \(\operatorname{asdim}(\Gamma/N_i) \leq n\) uniformly. Let \(U^i = \{U^i_0, \ldots, U^i_n\}\) be a \((R, M_R)\)-cover of \(\Gamma/N_i\). Then by \(2.11\) each \(U^i\) is also a \((d, F, \#B^{M_R+1}(\Gamma/N_i))\)-cover for \(\Gamma \curvearrowright \Gamma/N_i\) (and since the cardinality of balls in finite quotients of \(\Gamma\) is bounded by the cardinality of balls in \(\Gamma\), this bound is independent of \(i\)). If \(K = \ker(\Gamma \to \widehat{\Gamma}_{(N_i)})\), then \(\Gamma/K \curvearrowright \widehat{\Gamma}_{(N_i)}\) freely and is orbit equivalent to the original \(\Gamma\)-odometer. Find \(i_F\) so that the quotient \(\Gamma/K \to \Gamma/N_{i_F}\) induces a map \(C_{\Gamma}(\Gamma/K) \to C_{\Gamma}(\Gamma/N_{i_F})\) which is an isometry on balls of radius \(\leq M_R + 2\). Then pulling back \(U^{i_F}\) by the projection \(p_{i_F} : \widehat{\Gamma}_{(N_i)} \to \Gamma/N_{i_F}\) gives a \((d, F, \#B^{M_R+1}(\Gamma/N_{i_F}))\)-cover for \(\Gamma \curvearrowright \widehat{\Gamma}_{(N_i)}\). \(\square\)

We now state both inequalities in one theorem:

**Theorem 4.6.** Let \(\Gamma\) be a countable group and \((N_i)\) a sequence of finite index, normal subgroups which is directed by inclusion. Suppose \(\Gamma \curvearrowright \widehat{\Gamma}_{(N_i)}\) by left multiplication. Then \(\operatorname{DAD}_{\text{free}}(\Gamma \curvearrowright \widehat{\Gamma}_{(N_i)}) \leq d\) if and only if \(\operatorname{asdim}(\square_{(N_i)\Gamma}) \leq d\) uniformly. If also \(\operatorname{diam}(\Gamma/N_i) \to \infty\). Then \(\operatorname{DAD}_{\text{free}}(\Gamma \curvearrowright \widehat{\Gamma}_{(N_i)}) = \operatorname{asdim}\square_{(N_i)\Gamma}\).

**Corollary 4.7.** Let \(\Gamma\) be a countable, residually finite group and \((N_i)\) a sequence which is also a filtration of \(\Gamma\) (see \(2.7\)). Then \(\operatorname{DAD}(\Gamma \curvearrowright \widehat{\Gamma}_{(N_i)}) = \operatorname{asdim}\square_{(N_i)\Gamma}\).

This can also be combined with \(3.1\): 

**Corollary 4.8.** Let \(\Gamma\) be a finitely generated, residually finite group and \((N_i)\) a sequence which is also a filtration. Then either \(\operatorname{DAD}(\Gamma \curvearrowright \widehat{\Gamma}_{(N_i)}) = \infty\) or
\[ \text{DAD}(\Gamma \curvearrowright \widehat{\Gamma}_{(N_i)}) = \text{asdim}\Gamma. \] The dimension is finite if \( \Gamma \) has polynomial growth (equivalently is virtually nilpotent).

5 Product theorem

A Hurewicz-type theorem in a dimension theory essentially says that if \( X \) and \( Y \) are two objects for this dimension theory and \( f : X \to Y \) is a morphism between them; then the dimension of \( X \) is at most the dimension of \( Y \) plus the largest dimension of a ‘fibre’ of \( f \) (a pullback of a subspace of \( Y \) with dimension 0). Such theorems exist for the dimension of a vector space (rank nullity theorem), the topological covering dimension [6, Theorem 1.12.4, page 136], and the asymptotic dimension [1] and [3]. Special cases of such theorems include subadditivity of the dimension over products and extensions, which we will investigate in this section and the next.

Most basically, one would expect DAD to be subadditive over products of groups acting on products of spaces. The work of [3] provides a way of proving this, and the proofs exhibit some of the same ideas used in the next, more technical, section.

**Definition 5.1.** Let \( \mathcal{P}_{fs}(\Gamma) \) denote the collection of all finite, symmetric subsets of \( \Gamma \) containing the identity. An \( n \)-dimensional control function for an action \( \Gamma \curvearrowright X \) on a compact space by homeomorphisms is a function \( \mathcal{D}_{\Gamma \curvearrowright X} : \mathcal{P}_{fs}(\Gamma) \to \mathbb{N} \) such that for any \( F \in \mathcal{P}_{fs}(\Gamma) \) there is an open cover \( U = \{U_0, \ldots, U_n\} \) which is a \((n, F, \mathcal{D}_{\Gamma \curvearrowright X}(F))\)-cover for \( \Gamma \curvearrowright X \).

**Definition 5.2.** If \( k \geq n+1 \geq 1 \), an \((n, k)\)-dimensional control function for \( \Gamma \curvearrowright X \) is a function \( \mathcal{D}_{\Gamma \curvearrowright X} : \mathcal{P}_{fs}(\Gamma) \to \mathbb{N} \) such that for any \( F \in \mathcal{P}_{fs}(\Gamma) \) there is an open \((k-1, F, \mathcal{D}_{\Gamma \curvearrowright X}(F))\)-cover \( U = \{U_0, \ldots, U_{k-1}\} \) and each \( x \in X \) belongs to at least \( k-n \) elements of \( U \) (equivalently \( \cup_{i \in T} U_i = X \) for every \( T \subset \{0, \ldots, k-1\} \) consisting of \( n + 1 \) elements).

**Lemma 5.3.** Suppose \( F \in \mathcal{P}_{fs}(\Gamma) \), \( r > 0 \), and \( U = \{U_0, \ldots, U_{k-1}\} \) is a \((k-1, F^r, M_r)\)-cover for \( \Gamma \curvearrowright X \) by open sets such that each \( x \) belongs to \( \geq k-n \) elements. Then there is a \((k-1, F^r, M_r)\)-cover by sets which are the closures of open sets and still have the property that each \( x \) belongs to \( \geq k-n \) elements. Moreover, if \( \Gamma \curvearrowright X \) is either free or isometric, and if \( U = \{U_0, \ldots, U_{k-1}\} \) is a \((k-1, F^r, M_r)\)-cover by sets which are each either open or closed such that each \( x \in X \) belongs to \( \geq k-n \) elements, then there
is a \((k - 1, F^r, M_r)\)-cover by open sets containing the original elements (so still having each \(x \in X\) in \(\geq k - n\) elements).

Proof. For the first part, let \(\mathcal{U} = \{U_0, \ldots, U_{k-1}\}\) be such a cover. Then \(\bigcup_{S \subseteq \{0, \ldots, k\}; |S| = k - n} \bigcap_{s \in S} U_s\) is an open cover, and so has some Lebesgue number \(\lambda\). This shows every \(x \in X\) has a \(\lambda\)-ball which is contained in at least \(k - n\) elements of \(\mathcal{U}\). Replace the \(U_i\) with the closures of their \(\lambda/2\)-interiors.

For the second part, let \(\mathcal{U} = \{U_0, \ldots, U_{k-1}\}\) be such a cover. For \(x \in U_i\), denote by \(Y_x\) the set \(\{y \mid y\) is connected to \(x\) by an \(F^r\)-chain in \(U_i\}\). Then for each closed set \(U_i\) in \(\mathcal{U}\) there is \(\epsilon > 0\) such that

\[
\inf_{x \in U_i} \inf_{Y_x} \inf_{\gamma \in F^r | \gamma y \notin U_i} d(\gamma \cdot y, U_i) > \epsilon
\]

where we interpret the innermost infimum as \(\infty\) when \(\{\gamma \in F^r \mid \gamma y \notin U_i\}\) is empty. This follows from a compactness argument and the properties of \(\mathcal{U}\).

Now assume \(\Gamma \curvearrowright X\) is free. Let \(S \subseteq \Gamma\) be the union over all \(x \in U_i\) of the sets \(\{\gamma \in \Gamma \mid \gamma x = y \in Y_x\}\). Since the \(Y_x\) are uniformly finite and \(U_i\) is compact, \(S\) is finite. Find \(\delta < \epsilon/2\) such that \(d(\gamma \cdot x, \gamma y) < \epsilon/2\) whenever \(d(x, y) < \delta\) for all \(\gamma \in F^r S\). We can then replace each closed \(U_i\) with its open \(\delta\)-neighborhood.

If instead \(\Gamma \curvearrowright X\) is isometric, we can simply take \(\delta = \epsilon/2\).

Lemma 5.4. If \(D_{\Gamma \curvearrowright X}^{(n+1)}\) is an \(n\)-dimensional control function for \(\Gamma \curvearrowright X\) and we define \(\{D_{\Gamma \curvearrowright X}^{(i)}\}_{i \geq n+1}\) inductively on powers of \(F\) by \(D_{\Gamma \curvearrowright X}^{(i+1)}(F^r) = D_{\Gamma \curvearrowright X}^{(i)}(F^r) \cdot |B_{\epsilon'}(C_F(\Gamma))|\), then for each \(k\), there is an open \((k-1, F^r, D_{\Gamma \curvearrowright X}^{(k)})\)-cover such that each \(x \in X\) is covered by at least \(k - n\) elements of the cover (we say it this way since the functions \(D_{\Gamma \curvearrowright X}^{(k)}\) are not defined on all finite subsets of \(\Gamma\) if \(\Gamma\) is not finitely generated, so we can’t quite call them control functions).

Proof. The case \(k = n+1\) is obvious. Suppose the result holds for some larger \(k\). Let \(\mathcal{U} = \{U_1, \ldots, U_k\}\) be a \((k-1, F^{3r}, D_{\Gamma \curvearrowright X}^{(k)})\)-cover for \(\Gamma \curvearrowright X\) with all \(x \in X\) in at least \(k - n\) elements of \(\mathcal{U}\). By the first part of the previous lemma, we can change the sets \(U_i\) to be the closures of open sets \(V_i\) which form a cover with the same properties. Define \(U'_i = F^r \cdot U_i\) (these are closed sets). Then \(\mathcal{U}' = \{U'_1, \ldots, U'_k\}\) is a closed \((k-1, F^r, D_{\Gamma \curvearrowright X}^{(k)} \cdot |B_{\epsilon'}(C_F(\Gamma))|)\)-cover for \(\Gamma \curvearrowright X\). Define \(U'_{k+1}\) to be the union of all the (disjoint, open)
sets $W_S := (\cap_{s \in S} V_i) \setminus \cup_{i \notin S} U_i$ where $S \subseteq \{1, \ldots, k\}$ has $k - n$ elements. We claim that for $S \neq T$ (where $S, T \subseteq \{1, \ldots, k\}$ each have $k - n$ elements) that the sets $W_S$ and $W_T$ are $F^r$-disjoint (i.e. not connected by an $F^r$-chain contained in their union).

Suppose $a, b \in U_{k+1}'$ are such that $a \in \cap_{i \in T} V_i \setminus \cup_{i \notin T} U_i$ and $b \in \cap_{s \in S} V_s \setminus \cup_{i \notin S} U_i$ for $T \neq S$ and $|S| = |T| = k - n$. Then there is $t \in T \setminus S$ such that $a \in V_t$, so if $f \cdot a = b$ for some $f \in F^r$, then $b \in U_i \supseteq F^r \cdot V_t$, a contradiction.

Now suppose $x \in X$ belongs to exactly $k - n$ sets $U_i', i \leq k$. Since the sets $\cap_{s \in S} V_s$ form a cover as $S$ ranges over subsets of $\{1, \ldots, k\}$ with $|S| = k - n$; $x \in \cap_{s \in S} V_s$ for some $s \in S$. But then if $x \notin U_i', k \leq k$, we must have $x \in U_j'$ for some $j \notin S$, contradicting that $x$ is only contained in $k - n$ sets $U_i'$.

Thus, $\{U_{k+1}' \cup U_k', \ldots, U_1'\}$ is a $(k, F^r, D^{(k)}_{\cap X}(3r) \cdot |B^{+}_{\cap}(C_{F}(\Gamma)))$-cover with each $x \in X$ contained in at least $k - n + 1$ sets. We can then use the second part of the previous lemma to produce a cover by open sets with the same properties.

**Theorem 5.5.** If $\Gamma \triangleleft X$ and $\Lambda \triangleleft Y$ are free, then $DAD(\Gamma \times \Lambda \triangleleft X \times Y) \leq DAD(\Gamma \triangleleft X) + DAD(\Lambda \triangleleft Y)$.

**Proof.** Fix finite, symmetric subsets $F \subseteq \Gamma$ and $H \subseteq \Lambda$ each containing the identity. Suppose $\Gamma \triangleleft X$ and $\Lambda \triangleleft Y$ have dimension at most $m$ and at most $n$, respectively. Let $k = m + n + 1$. Use this assumption and 3.4 to produce the functions $D^{(k)}_{\Gamma \triangleleft X}$ and $D^{(k)}_{\Lambda \triangleleft Y}$. So there is an open $(k-1, F, D^{(k)}_{\Gamma \triangleleft X}(F))$-cover $U = \{U_i\}_{i=0}^{k-1}$ of $X$ and an open $(k-1, H, D^{(k)}_{\Lambda \triangleleft Y}(H))$-cover $V = \{V_i\}_{i=0}^{k-1}$ of $Y$ such that each $x \in X$ is in at least $k - m$ elements of $U$ and each $y \in Y$ is in at least $k - n$ elements of $V$. Since $k - m + k - n = k + 1$, the family $\{U_i \times V_i\}_{i=0}^{k-1}$ is a cover of $X \times Y$. Moreover, it is a $(F \times H, D^{(k)}_{\Gamma \triangleleft X}(F) \cdot D^{(k)}_{\Lambda \triangleleft Y}(H))$-cover for $\Gamma \times \Lambda \triangleleft X \times Y$. 

**6 Extension theorem for odometers**

By showing a Hurewicz-type theorem for actions on finite quotients, we can follow closely the content of [3] to show that the DAD of full algebraic odometers is subadditive when taking extensions of the acting groups. In fact, much of the content of that paper can be transferred straightforwardly to the dynamical context by assuming actions to be minimal and working with a dense orbit. However, the author encountered difficulties proving a dynamical analogue of [3] Proposition 4.8, and so some deviation was necessary. The result
for asymptotic dimension also cannot be directly applied at the level of finite quotients since the relation between the asymptotic dimension of a coarse map and its control function given by [3, 4.5] is not quantitative on the level of control functions, and we need a bound independent of the finite quotient. We therefore do not define the DAD of a map, although there is an obvious choice for such a definition.

This extension theorem will provide new examples of odometers with finite DAD. Moreover, in conjunction with [4.6] it also shows new examples of groups whose box spaces are finite dimensional. We first fix some notation.

**Definition 6.1.** From here through 6.6, we consider a group $\Gamma$ and a quotient $f : \Gamma \to \Lambda$; along with the actions $\Gamma \curvearrowright \Gamma$ and $\Lambda \curvearrowright \Lambda$ by left multiplication. However, we will use $X$ to refer to $\Gamma$ when it is the set being acted on, and $Y$ to refer to $\Lambda$ when it is the set being acted on. Then $f : X \to Y$ as well and we have $f(\gamma \cdot x) = f(\gamma) \cdot f(x)$. Suppose that $F \subseteq \Gamma$ and $S \subseteq \Lambda$ are symmetric, finite subsets containing the identity and $U \subseteq X$. An $(F, S)$-chain in $U$ is an $F$-chain whose image under $f$ is an $S$-chain. Two points $x_1, x_2 \in U$ are in the same $(F, S)$-component if they are connected by an $(F, S)$-chain. We say $U$ has cardinality at most $(R_X, R_Y)$ if $U$ has cardinality at most $R_X$ and $f(U)$ is has cardinality at most $R_Y$.

The idea for the following lemma comes from [3, 3.5].

**Lemma 6.2.** With the same setup as [6.1], let $A, B \subseteq X$. Assume that the $(F^r_X, f(F)^r_Y)$-components of $A$ have cardinality at most $(R^A_X, R^A_Y)$, that the $(F^r_B, f(F)^r_Y)$-components of $B$ have cardinality at most $(R^B_X, R^B_Y)$, and that $r^B_X R^B_X + 2r^B_X < r^A_X$ and $r^B_Y R^B_Y + 2r^B_Y < r^A_Y$. Then the $(F^r_X, f(F)^r_Y)$-components of $A \cup B$ have cardinality at most

$$\#B^0_{r^X_{(R^X_X, R^X_Y) + 1}}(C^0_{F^r_X(Y)}(\Gamma)), \#B^0_{r^Y_{(R^F_Y, R^Y_Y) + 1}}(C^0_{f(F)^r_Y(\Lambda)}).$$

**Proof.** Let $x_0, \ldots, x_n$ be an $(F^r_X, f(F)^r_Y)$-chain in $A \cup B$ with no repeated points. Suppose $x_j$ and $x_k$ are two consecutive points contained only in $A$. Then $x_{j+1}, \ldots, x_{k-1}$ is an $(F^r_B, f(F)^r_Y)$-chain in $B$ and therefore (as a subset of $B$) has cardinality at most $(R^B_X, R^B_Y)$, hence length at most $(R^B_X, R^B_Y)$. We therefore have that $x_{j+1}$ and $x_{k-1}$ are in the same $(F^r_X R^B_X, f(F)^r_Y)$-component, so $x_j$ and $x_j$ are in the same $(F^r_X R^B_X + 2r^B_X, f(F)^r_Y)$-component of $A$. Since $r^B_X R^B_X + 2r^B_X < r^A_X$ and $r^B_Y R^B_Y + 2r^B_Y < r^A_Y$, the points in the
original chain which are contained only in $A$ form an $(r^A_X, r^A_Y)$-chain in $A$ and so (considered as a subset of $A$) have cardinality at most $(R^A_X, R^A_Y)$, hence length at most $(R^A_X - 1)(R^B_X + 1) + 1, (R^A_Y - 1)(R^B_Y + 1) + 1$ and so the $(r^B_X, r^B_Y)$-components of $A \cup B$ have cardinality at most

$$(\#B_e^{(R^A_X - 1)(R^B_X + 1) + 1}(C_{F^r_X}^B(\Gamma)), \#B_e^{(R^A_Y - 1)(R^B_Y + 1) + 1}(C_{f(F^r)}}^r_B(\Lambda))).$$

\[\square\]

A little more notation before we continue.

**Definition 6.3.** Let $f : X \rightarrow Y$ be as in 6.1. An $m$-dimensional control function for $f$ is a function $D_f : \mathcal{P}_{fs}(\Gamma) \times \mathcal{P}_{fs}(\Lambda) \rightarrow \mathbb{R}_+$ such that for every $F \in \mathcal{P}_{fs}(\Gamma), S \in \mathcal{P}_{fs}(\Lambda)$, and every $y \in Y$, $f^{-1}(S \cdot y)$ can be covered by $m + 1$-sets whose $F$-components have cardinality at most $D_f(F, S)$. An $(m, k)$-dimensional control function is a $k - 1$-dimensional control function with the additional property that each point in $X$ is covered by at least $k - m$ elements of the cover.

Several of the results below are proved in similar ways to their counterparts in [3] and so we make note at the beginning of each statement of what prior result is being analogized.

**Proposition 6.4.** (compare to [3, Proposition 4.7]) Let $f : X \rightarrow Y$ be as in 6.1. Further let $m \geq 0$, and $F \in \mathcal{P}_{fs}(\Gamma)$. Suppose $D^{(m+1)}_f : \mathcal{P}_{fs}(\Gamma) \times \mathcal{P}_{fs}(\Lambda) \rightarrow \mathbb{R}_+$ is an $m$-dimensional control function for $f$. If one defines inductively for $i \geq m + 1$ and for $r > 0$ and $T \in \mathcal{P}_{fs}(\Lambda)$ that $D^{(i+1)}_f(F^r, T) = D^{(i)}_f(F^{3r}, T).|B^r_e(C_{F^r}(\Gamma))|$, then for each $k$ and $T$, $f^{-1}(T \cdot y)$ can be covered by $k$ open sets whose $F^r$-components have cardinality at most $D^{(k)}_f(F^r, T)$ and each point is covered by at least $k - m$ elements of the cover.

**Proof.** Having an $(m, k)$-dimensional control function for $f$ is equivalent to having (for every $y \in Y$ and $S \in \mathcal{P}_{fs}(\Lambda)$) an $(m, k)$-dimensional control function for every $f^{-1}(S \cdot y)$ in the sense of 5.2 depending only on $S$. We can therefore repeat the proof of 5.4 \[\square\]

**Lemma 6.5.** As always, $X$, $Y$, and $f$ are as in 6.1. Denote the kernel of $f : \Gamma \rightarrow \Lambda$ by $K \hookrightarrow \Gamma$ and suppose $D_K$ is an $m$-dimensional control function for $K \sim X$. Then there is an $m$-dimensional control function for $f$ which depends only on $D_K$.  

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Proof. Fix a finite set \( F = F_0 \cup F_1 \) of \( \Gamma \) where \( F_1 \subset K \) and \( F_0 \cap K = \emptyset \). Let \( y \in Y \). Observe that \( f^{-1}(\{y\}) \) is the \( K \)-orbit of some \( x \in X \). We therefore have a cover of \( f^{-1}(\{y\}) \) by \( m+1 \) sets whose \( F_1 \)-components have cardinality at most \( D_K(F_1) \). Moreover, applying an element of \( F_0 \) always moves a point in the \( K \)-orbit of \( x \) outside of the \( K \)-orbit of \( x \), so in fact \( f^{-1}(\{y\}) \) can be covered by \( m+1 \) sets whose \( F \)-components have cardinality at most \( D_K(F_1) \).

Now suppose \( B \subset Y \) is a finite subset of cardinality \( N \). Inductively define natural numbers \( r_N, \ldots, r_1 \) and \( R_N, \ldots, R_1 \) by starting with \( r_N = 1 \) and then defining \( R_N = D_K(F_1), r_\cdot \cdot = R_i + 2 \), and \( R_\cdot \cdot \cdot = (R_i - 1)(D_K(F^{r_{i-1}}_1) + 1) + 1 \). Choose some enumeration of \( B \) and construct a \((F^r_1, D_K(F^{r_1}_i))\)-cover of \( f^{-1}(\{b_{r_1}\}) \). As \( f \) is a homomorphism, the \((F^r_1, f(F_1)^r)\)-components of subsets of \( X \) coincide with the \( F^r \) components (for any \( r \)). We can therefore apply \( 6.2 \) repeatedly with \( r_X^B = r_Y^B = 1 \) at every step and \( r_X^A = r_Y^A = r_{N-s} \) for the \( s \)-th step \( (s = 0, \ldots, N-1) \). This produces a \((m, F, R_1)\)-cover of \( f^{-1}(B) \). Hence, defining \( D_f(F, S) = R_1 \) (using the cardinality of \( S \) as \( N \) above) gives an \( m \)-dimensional control function for \( f \) which depends only on \( D_K \). \[ \square \]

Lemma 6.6. (compare to [3 Proposition 4.7]) Once again let \( X, Y \) and \( f : X \rightarrow Y \) be as in [6.1]. If \( D_f \) is an \((m, k)\)-dimensional control function for \( f \) and \( F \in \mathcal{P}_{fs}(\Lambda) \) is finite, then for any \( T \in \mathcal{P}_{fs}(\Lambda) \) and any \( B \subset Y \) whose \( T \)-components have cardinality at most \( R_Y \), \( f^{-1}(B) \) can be covered by \( k \) sets whose \((F, T)\)-components have cardinality at most \( D_f(F, TR^Y) \); and each element of \( f^{-1}(B) \) belongs to at least \( k - m \) elements of that covering.

Proof. Given a \( T \)-component, \( C \), of \( B \), express \( f^{-1}(C) \) as \( A^C_1 \cup \ldots \cup A^C_k \) such that the \( F \)-components of \( A^C_i \) have cardinality at most \( D_f(F, TR^Y) \) and every element of \( f^{-1}(C) \) belongs to at least \( k - m \) elements of that covering. Notice that each \( C \) is a subset of some set of the form \( TR^Y \cdot y \) for some \( y \in C \). Put \( A_i = \bigcup C A^C_i \) and notice each \((F, T)\)-component of \( A_i \) is contained in an \( F \)-component of some \( A^C_i \). \[ \square \]

Theorem 6.7. (compare to [3 Theorem 4.9]) Let \( k = m + n + 1 \), where \( m, n \geq 0 \). Let \( \Gamma \curvearrowright X, \Lambda \curvearrowright Y \) and \( f : X \rightarrow Y \) be as in [6.1]. Suppose \( D_{\Lambda \curvearrowright Y} \) is an \( n \)-dimensional control function for \( \Lambda \curvearrowright Y \) and \( D_f \) is an \( m \)-dimensional control function for \( \Gamma \curvearrowright X \) depending only on \( D_{\Lambda \curvearrowright Y}, D_f, m, \) and \( n \).

Proof. Fix \( F \in \mathcal{P}_{fs}(\Gamma) \).
Apply 6.4 to $D_{\Lambda \cap Y}$ to produce the function $D^{(k)}_{\Lambda \cap Y}$. Define inductively a sequence $r_Y^{(n+1)} < R_Y^{(n+1)} < r_Y^{(n)} < R_Y^{(n)} < \cdots < r_Y^{(1)} < R_Y^{(1)} < r_Y^{(0)}$ of numbers beginning with $r_Y^{(n+1)} = 1$, $R_Y^{(n+1)} = D^{(k)}_{\Lambda \cap Y}(h(F))$, and then $r_Y^{(i)} = R_Y^{(i+1)} + 2$ ($0 \leq i \leq n$), and $R_Y^{(i)} = (R_Y^{(i+1)} - 1)(D^{(k)}_{\Lambda \cap Y}(h(F)r_Y^{(i)}) + 1) + 1$.

Express $Y$ as the union of $n + 1$ sets $\{A_i\}_{i=1}^{n+1}$ such that all $h(F)r_Y^{(i)}$-components of each $A_i$ have cardinality at most $R_Y^{(i)}$ for every $i$ (this is possible since $R_Y^{(0)}$ is at least $D^{(k)}_{\Lambda \cap Y}(h(r_Y^{(0)})))$. Since $A_i \subset Y$ (and by definition of the $r_Y^{(i)}$ and $R_Y^{(i)}$), we can further express $A_i$ as the union of $k$ sets $\{U^j_i\}_{j=1}^k$ such that all $h(F)r_Y^{(i)}$-components of each $U^j_i$ have cardinality at most $R_Y^{(i)}$ and every point $y \in A_i$ belongs to at least $k - n = m + 1$ sets.

Apply 6.4 to $D_f$ to produce the function $D^{(k)}_f$. Define inductively a sequence of numbers $r_X^{(n+1)} < R_X^{(n+1)} < r_X^{(n)} < R_X^{(n)} < \cdots < r_X^{(1)} < R_X^{(1)}$ starting with $r_X^{(n+1)} = 1$, $R_X^{(n+1)} = D^{(k)}_f(f(F)r_X^{(n+1)})$ and then $r_X^{(i)} = R_X^{(i+1)} + 2$ and $R_X^{(i)} = (R_X^{(i+1)} - 1)(D^{(k)}_f(f(F)r_Y^{(i)}r_X^{(i)}) + 1) + 1$.

For every $i$, we can use the properties of $A_i$ and 6.6 to express $f^{-1}(A_i)$ as the union of $k$ sets $\{B^j_i\}_{j=1}^k$ such that all $(F_X^{(j)}h(F)r_Y^{(i)})$-components of $B^j_i$ have cardinality at most $R_X^{(i)}$ for every $j$ and every point $x \in f^{-1}(A_i)$ belongs to at least $k - m = n + 1$ sets.

Put $D^j_i = B^j_i \cap f^{-1}(U^j_i)$ and let $D^j_1$ be the union of all $D^j_i$ for $i \in \{1, \ldots, n+1\}$. The collection of the $D^j_i$ is a cover by Kolmogorov’s argument: given $x \in X$ there is $i$ so that $f(x) \in A_i$. The set of $j$’s such that $x \in B^j_i$ has at least $k - m$-elements and the set of $j$’s such that $f(x) \in U^j_i$ has at least $k - n = m + 1$ elements, so they cannot be disjoint since $k - m + (k - n) = n + 1 + m + 1 = k + 1 > k$ and there are $k$ possible $j$’s.

Notice all $(F_X^{(j)}f(F)r_Y^{(i)})$-components of the sets $D^j_i$ have cardinality at most $(R_X^{(i)}r_Y^{(i)})$. By repeated application of 6.2 with $r_X^B = r_Y^B = 1$ at every step and $r_X^A = r_X^{(n-k)}$ and $r_Y^A = r_Y^{(n-k)}$ at the $k$-th step ($k = 0, \ldots, n$), that is, for the union of $D^j_1$ through $D^j_1$; we can show all $(F_X^{(n+1)}f(F)r_Y^{(n+1)})$-components (that is, the $(F,f(F))$-components) of the set $D^j_1$ have cardinality at most $(R_X^{(1)}r_Y^{(1)})$. By definition of $f$, $F$-components and $(F,f(F))$-components coincide, so the $F$-components of $D^j_1$ have cardinality at most $R_X^{(1)}$ (hence uniformly finite). Notice that $R_X^{(1)}$ depends only on the control functions $D_{\Lambda \cap Y}$ and $D_f$. □

Theorem 6.8. Suppose $1 \rightarrow \Delta \rightarrow \Gamma \rightarrow \Lambda \rightarrow 1$ is an exact sequence of
countable groups. If \((N_i)\) is a countable collection of finite index normal subgroups of \(\Gamma\) directed by inclusion, it induces collections \(N_i \cap \Delta\) and \(N_i/(\Delta \cap N_i)\) and hence odometers of \(\Delta\) and \(\Lambda\). Then if \(DAD_{free}(\Delta \curvearrowright \hat{\Delta}(N_i \cap \Delta)) = m\) and \(DAD_{free}(\Lambda \curvearrowright \hat{\Lambda}(N_i/(\Delta \cap N_i))) = n\), \(DAD_{free}(\Gamma \curvearrowright \hat{\Gamma}(N_i)) \leq m + n\).

**Proof.** For every finite index normal subgroup \(N_k\) of \(\Gamma\), we have an induced sequence \(1 \rightarrow \Delta_k \rightarrow \Gamma_k \rightarrow \Lambda_k \rightarrow 1\) of finite quotients. The map \(\Gamma_k \rightarrow \Lambda_k\) (which is the quotient map by \(\Delta_k/(\Delta \cap N_k)\)) plays the role of \(f\) in the above results for the actions \(\Gamma \curvearrowright \Gamma_k\) and \(\Lambda \curvearrowright \Lambda_k\). Further note that these actions can be thought of as \(\Gamma\) actions using \(h\), and the quotients by \(N_k\) and \(J_k\) (see the diagram below). Even so, we can apply \ref{6.5} and \ref{6.7}, which would really apply to \(\Gamma_k \curvearrowright \Gamma_k\) and \(\Lambda_k \curvearrowright \Lambda_k\). This is ok since the image of a finite subset \(S\) of \(\Lambda\) under \(\pi\) or \(\pi\) is a finite subset \(F\) of \(\Gamma_k\) or \(\Lambda_k\) (so we can apply \ref{6.5} and \ref{6.7} using this set), and a \((d, F, M)\)-cover for \(\Gamma_k \curvearrowright \Gamma_k\) or \(\Lambda_k \curvearrowright \Lambda_k\) is also a \((d, S, M)\)-cover for \(\Gamma \curvearrowright \Gamma_k\) or \(\Lambda \curvearrowright \Lambda_k\).

To see this more explicitly, we have the commutative diagram:

\[
\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1 & K \cap N_k & \rightarrow & K & \rightarrow & K/(K \cap N_k) & \rightarrow & 1 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1 & \rightarrow & N_k & \rightarrow & \Gamma & \rightarrow & \Gamma/N_k & \rightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1 & \rightarrow & J_k & \rightarrow & \Lambda & \rightarrow & \Lambda/J_k & \rightarrow & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

and in particular we have \(J = N_k/(K \cap N_k)\) and (by a well known isomorphism theorem) \(K/(K \cap N_k) \cong (K \cdot N_k)/N_k\).

By assumption, the actions \(\Lambda \curvearrowright \Lambda_k\) and \(\Delta \curvearrowright \Delta_k\) have dynamic asymptotic dimension uniformly at most \(n\) and \(m\) respectively, that is, the same control function works independent of \(k\). We can now apply \ref{6.5} (as the pullback of a point in \(\Lambda_k\) looks like the \(K\)-orbit of a point in \(\Gamma_k\)) to get a control function for \(f\) (again independent of \(k\)) and \ref{6.7} to construct a control function for \(\Gamma \curvearrowright \Gamma_k\) which works independent of \(k\). Thus, \(DAD_{free}(\Gamma \curvearrowright \hat{\Gamma}) \leq m + n\). □

This theorem can be combined with \ref{4.6} to show the the asymptotic di-
Corollary 6.9. Suppose $1 \to \Delta \to \Gamma \xrightarrow{\pi} \Lambda \to 1$ is an exact sequence of countable groups. Suppose $(N_i)$ is a sequence of finite index normal subgroups of $\Gamma$ directed by inclusion such that $\text{diam}(\Gamma/N_i) \to \infty$. This collection induces such collections and hence box spaces of $\Delta$ and $\Lambda$. Then if $\text{asdim}_{(N_i \cap \Delta)} \Delta = m$ and $\text{asdim}_{(\pi(N_i))} \Lambda = n$, $\text{asdim}_{(N_i)} \Gamma \leq m + n$.

Proof. Apply 4.5, then 6.8, then 4.4.

This is essentially the desired extension theorem described in [7, Theorem A] – the proof in that paper is incomplete. It should be noted that we only prove this extension theorem for countable groups. Despite this, we are still able to recover the essential content of the rest of [7, Section 4]. For instance, the corollary above immediately implies that the asymptotic dimension of the full box space of a polycyclic group is the Hirsch length of the group. This works since, for finitely generated groups, there are only finitely many finite index normal subgroups which intersect the ball of radius $R$ about the identity (for any given $R$).

Remark 6.10. It is worth remarking that similar methods as used this section can also be applied to the asymptotic dimension, which would directly prove an extension theorem for the asymptotic dimension of box spaces without assuming $(N_i)$ to be a sequence.

7 Box spaces of elementary amenable groups

The extension theorem proved in the previous section shows that box spaces (and odometers) of many elementary amenable groups are finite dimensional. This includes groups which are not virtually nilpotent, as we now show with an example.

Definition 7.1. The Baumslag-Solitar group $BS(1,n)$ is the group given by the presentation $\langle a,b \mid bab^{-1} = a^n \rangle$.

Example 7.2. The group $BS(1,n)$ is not virtually nilpotent for $n \geq 3$ and has $\text{asdim}_{(N_i)} BS(1,n) = \text{asdim} BS(1,n) = 2$ for any sequence $(N_i)$ which is a filtration.
Proof. Since $BS(1, n)$ is finitely generated and residually finite, the collection of all its finite index normal subgroups is countable and the diameters of the quotients tend to infinity. We can therefore apply the extension theorem.

The group $BS(1, n)$ fits into the exact sequence $1 \to \mathbb{Z}[1/n] \to BS(1, n) \to \mathbb{Z} \to 1$ (the first map takes $1/n$ to $a$, and the second is the quotient by the subgroup generated by $a$). Moreover, $\mathbb{Z}[1/n]$ is locally $\mathbb{Z}$ and so $\square_{(Z\cap N_i)Z[1/n]}$ has asymptotic dimension 1 by 2.10. By 6.9, $\text{asdim} \square_{(N_i)}BS(1, n) \leq 2$. Since $BS(1, n)$ is residually finite by [13, Theorem C], $\text{asdim}BS(1, n) \leq \text{asdim}\square_{(N_i)}BS(1, n)$, and so $\text{asdim}\square_{(N_i)}BS(1, n) = 2$. It is shown in [16] that, for $n \geq 3$, $BS(1, n)$ has exponential growth and is therefore not virtually nilpotent.

We conclude with some more general applications of the extension theorem for box spaces. The main ideas for these applications come from the end of [7, Section 4].

**Theorem 7.3.** Let $\Gamma$ be a countable, residually finite, elementary amenable group. Then for any sequence $(N_i)$ which is also a filtration, $\text{asdim} \Gamma \leq \text{asdim} \square_{(N_i)} \Gamma \leq h(\Gamma)$.

**Proof.** Since $\Gamma$ is residually finite by [13, Theorem C], $\text{asdim} \Gamma \leq \text{asdim} \square_{(N_i)} \Gamma$ by [4, Proposition 3.1].

Using the description of elementary amenable groups given in 3.2, we can assume $\Gamma$ is in the class $\Gamma_{\alpha}$ for some countable ordinal $\alpha$. The case where $\alpha = 1$ follows from [5, Theorem 3.5] and [4, Corollary 4.2]. Now suppose $\alpha > 1$ and assume the result holds for groups in the class $\Gamma_{\alpha-1}$. Then 2.10 shows the result holds for groups in $L(\Gamma_{\alpha-1})$; and 6.9 shows the result holds for groups in the class $\Gamma_{\alpha}$. If $\beta$ is a limit ordinal, $\Gamma_\beta = \bigcup_{\alpha<\beta} \Gamma_\alpha$, and we can apply 2.10 again.

**Theorem 7.4.** If $\Gamma$ is countable and residually finite and $\Gamma = \Lambda \rtimes \Delta$ where $\Lambda$ is virtually polycyclic and $\Delta$ is locally finite, then for any sequence $(N_i)$ which is also a filtration of $\Gamma$, $\text{asdim} \Gamma = \text{asdim} \square_{(N_i)} \Gamma = h(\Gamma)$.

**Proof.** That $\text{asdim} \Gamma \leq h(\Gamma)$ comes from the previous theorem. The reverse inequality comes from [5, 3.5] and [2, 58], as these show $h(\Gamma) = h(\Lambda) = \text{asdim} \Lambda \leq \text{asdim} \Gamma$.

**Corollary 7.5.** Let $F$ be a finite abelian group and $H$ a virtually polycyclic group. Denote by $F \wr H$ the reduced wreath product. Then for any sequence
\((N_i)\) which is also a filtration of \(\Gamma\), 
\[ \text{asdim}(F \wr H) = \text{asdim}(\square_{(N_i)}(F \wr H)) = h(F \wr H) = h(H). \]

\textbf{Proof.} The wreath product \(F \wr H\) is residually finite by [9, 3.2]. Furthermore, [12, Theorem 1] shows that 
\[ h(F \wr H) = h(H). \] The corollary then follows by applying the previous theorem. \(\square\)

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