Decoherence of Schrödinger cat states in a Luttinger liquid

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Decoherence of Schrödinger cat states built from quantum superpositions of left or right Luttinger fermions located at different positions in a spinless Luttinger liquid are considered. Their decoherence rates are computed within the bosonization approach using as environments the quantum electromagnetic field or two or three dimensional acoustic phonon baths. Emphasis is put on the differences between the electromagnetic and acoustic environments.

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I. INTRODUCTION AND MOTIVATIONS

Decoherence is a very important issue for mesoscopic systems since it governs the crossover between quantum and quasi-classical transport regimes. Coherence of mesoscopic conductors gives rise to various Aharonov-Bohm interference effects such as permanent currents in a mesoscopic ring, and conductance oscillations as a function of the external magnetic field.

The problem of electron decoherence in metals at zero temperature is an active area of research both from the theoretical and experimental point of view. Recent experiments claim to observe a saturation of the dephasing time \( \tau_\phi \) at very low temperatures. Strong discussions among theorists arose from these observations. The heart of the debate, summarized for example in Mohanty’s recent letter, is to determine whether the conventional theory of dephasing in Fermi liquids could explain the saturation of \( \tau_\phi \) (for example as an effect of an external microwave radiation) or whereas one should reconsider the theory completely. Although such an excitement is a strong motivation for working on decoherence in disordered mesoscopic conductors, we shall present a model for studying the decay of Schrödinger cat states in 1D ballistic conductors.

From a methodological point of view, our line of thought is very close to the one used in atomic physics, for example in theoretical works on decoherence experiments in cavity QED. It relies heavily on the use of simple exactly solvable models of decoherence, such as the Caldeira-Leggett model. We think that this point of view brings a different light on the question of decoherence by taking into account the electron fluid as a whole and studying the coupling of this many-body system to the external reservoirs. It makes a bridge between atomic physics situations, where decoherence is extremely well controlled and for which simplified decoherence models apply almost directly and mesoscopic conductors which are complex interacting systems. In particular, this point of view is well suited to the 1D case because of interactions. In a non-Fermi liquid, one cannot keep track of an individual electron because of orthogonality catastrophe effects. Moreover, electromagnetic or acoustic radiation emitted by one part of the system and absorbed by another part may have drastic effect on the strongly correlated electron state. These remarks motivated our method, based on the study of the coupling of an external environment to the low energy excitations of the 1D electron system. Following general ideas of Stern, Aharonov and Imry, we have computed the decoherence rate of Schrödinger cat states built from localized excitations (e.g Luttinger fermions) introduced at different places in the system or at the same place but moving in different directions (left and right moving components). As expected, we have found that such linear superpositions decay into statistical mixtures even at zero temperatures and computed various decoherence scenarios. This is the main result of this paper and it means that a 1D pure ballistic conductor exhibits decoherence at absolute zero in the sense of Schrödinger cat states decay.

One dimensional conductors in the ballistic regime are appropriately described by an effective interacting theory: the Luttinger liquid. In this effective theory, interactions between electrons are put by means of an electrostatic short-ranged potential. Obviously, within the approximation of a linear dispersion relation for particle-hole excitations, the Luttinger effective theory contains no source for decoherence. A coupling to an external quantum environment is necessary to introduce decoherence in the Luttinger liquid. The quantized electromagnetic field and a 2D or 3D bath of longitudinal phonons will be considered in the present paper. The combined use of bosonization and non-equilibrium techniques make it possible to solve this problem. Within the bosonization framework, the pioneering work by Martin and Loss investigates the question of equilibrium permanent currents induced by fluctuations of the quantized electromagnetic field. They have shown that coupling the Luttinger liquid to QED leads to a renormalization of the Luttinger liquid parameters. Our discussion of dynamical and therefore non-equilibrium aspects of the coupled Luttinger & QED system will show how this renormalization appears dynamically. Let us mention that coupling a Luttinger liquid to one dimensional phonons also renormalizes the Luttinger liquid parameters and drives a 1D Fermi
liquid to a non Fermi liquid fixed point. But a 1D phonon bath does not introduce any intrinsic decoherence in the Luttinger liquid since, roughly speaking, it does not have enough modes. That’s why 2D and 3D phonon baths are considered in this paper and we have shown that these baths have enough modes to kill Schrödinger cat states.

To be more precise on the results presented in this paper, we have shown that the electromagnetic decoherence time is much larger, although not infinite, than the natural time associated with the Luttinger system. This is mainly due to the weakness and the transversality of the coupling between photons and the Luttinger liquid. Therefore the coherent Luttinger liquid paradigm is not ruled out by its coupling to QED. We have also shown that the acoustic decoherence is much stronger than the electromagnetic one. This difference comes from the fact that many bosonic modes of the Luttinger liquid have an efficient acoustic radiation rate whereas only very few modes dominate the decoherence process in the electromagnetic case. In the acoustic case, decoherence takes place over a much shorter time scale than dissipation contrarily to the electromagnetic case.

This paper is organized as follows: the model is presented and the Feynman-Vernon and Keldysh basic tools are briefly recalled in section II. Section III makes contact with the Quantum Brownian Motion problem (QBM). The central problem of mutual decoherence of Schrödinger cat states is addressed in section IV. Results are summarized briefly recalled in section II. Section III makes contact with the Quantum Brownian Motion problem (QBM). The coherent Luttinger liquid paradigm is not ruled out by its coupling to QED. We have also shown that the acoustic decoherence is much stronger than the electromagnetic one. This difference comes from the fact that many bosonic modes of the Luttinger liquid have an efficient acoustic radiation rate whereas only very few modes dominate the decoherence process in the electromagnetic case. In the acoustic case, decoherence takes place over a much shorter time scale than dissipation contrarily to the electromagnetic case.

II. ELECTRON SYSTEMS COUPLED TO EXTERNAL RESERVOIRS (QED OR PHONONS)

A. The Feynman-Vernon-Keldysh method

Studying an out of equilibrium quantum system boils down to the computation of its density matrix as a function of time. A functional integral approach to this problem has been given long time ago in the context of perturbative field theory by Keldysh. Let \( \rho(t) \) denote the density operator for a closed system at time \( t \) (in Schrödinger’s picture) and \( U[t_i, t_f] \) denote the evolution operator between \( t_i \) and \( t_f \), then \( \rho(t_f) = U[t_i, t_f] \rho(t_i) U[t_i, t_f]^{-1} \). The first of these operators takes into account what will be called the “forward time branch” and the other one the “backward time branch” of the evolution. Evaluating matrix elements of \( \rho(t_f) \) can be done with an appropriate propagator which can be represented as a double functional integral (one for each evolution operator). Correlation functions are generated by introducing external sources coupled to the system’s degrees of freedom which have different values on the forward and backward time branches.

Then, one usually takes a trace at time \( t_f \). The generating functional obtained this way is called the Keldysh generating functional and can be represented as a path integral over a special contour \( K \) that goes from \( t_i \) to \( t_f \) and then back to \( t_i \).

Denoting by \( \varphi \) (respectively \( \xi \)) fields (respectively external sources) describing the dynamics of the system, and by \( \varphi_+ \) and \( \varphi_- \) their restrictions to the upper and lower branch of Keldysh’s contour (respectively \( \xi_+, \xi_- \)), we have:

\[
Z[\xi_+, \xi_-] = \int \mathcal{D}[\varphi_+, \varphi_-] e^{i(S[\varphi_+, \xi_+] - S[\varphi_-, \xi_-])} \rho[\varphi_+|t_i, \varphi_-|t_i]
\]

where \( \rho(t_i)[\varphi_+|t_i, \varphi_-|t_i] \) denotes the kernel of the initial density operator. This is the basis for Keldysh’s non-equilibrium perturbation theory. Because of the doubling of degrees of freedom, one obtains a 2 by 2 matrix Green function which can be related to well known Green functions by:

\[
G(x, y) = (G_{\varphi\varphi}(x, y), \xi\xi)_{\{1,-1\}} = \begin{pmatrix}
G_T(x, y) & G_<(x, y) \\
G_> (x, y) & G_T(x, y)
\end{pmatrix}
\]

Here \( T \) denotes the usual time ordering, \( \bar{T} \) the anti-chronological time ordering, \( < \) and \( > \) the lesser and upper time orderings. For bosonic oscillators initially in thermal equilibrium, Green’s functions are explicitly known (see appendix A).

For a matter system coupled to QED, we are interested in the evolution of the reduced density matrix for the matter system. Bosonization techniques enable us to treat the electron fluid through an effective bosonic theory. Our strategy will then be to integrate over the electromagnetic field degrees of freedom and to deal with matter degrees of freedom in a second time. It is convenient to choose the Coulomb gauge, in which dynamical degrees of freedom of the quantum electromagnetic field (transverse photons) are decoupled from the Coulomb interaction. The latter is taken into account through the effective action for the matter system which uses only instantaneous interactions such as the Luttinger liquid description. Integrating out transverse photons gives a non-local functional integral of the current density, called a Feynman-Vernon influence functional. For the sake of simplicity, we shall assume that these two
systems are independent at initial time (the initial density operator factorizes $\rho(t_i) = \rho_{\text{Matter}} \otimes \rho_{\text{QED}}$) and that the electromagnetic field is initially at equilibrium at temperature $T = 1/k_B\beta$. In the present case, using bold symbols for vectors and matrices, the influence functional is nothing but the influence functional for the current density:

$$
\mathcal{F}[\mathbf{j}_+, \mathbf{j}_-] = \int \mathcal{D}[\mathbf{A}_+, \mathbf{A}_-] e^{i(S[\mathbf{A}_+] - S[\mathbf{A}_-])} \times e^{i\int (\mathbf{A}_+ \mathbf{j}_+ - \mathbf{A}_- \mathbf{j}_-) \times \rho_B[\mathbf{A}_+(t_i), \mathbf{A}_-(t_i)]}
$$

(2)

where $\rho_B$ is the appropriate functional kernel representing the density operator for the quantum electromagnetic field at given temperature. The functional integral 2 can be evaluated in terms of Keldysh’s Green function for the electromagnetic field at finite temperature $D^\alpha_\beta(x,y) = -i(T_K A^\alpha_\beta(x) A^\beta_\alpha(y))$ and is equal to:

$$
\mathcal{F}[\mathbf{j}_+, \mathbf{j}_-] = \exp \left( -\frac{i}{2} \sum_{(\epsilon,\epsilon')} \epsilon' \int d^4x d^4y \mathbf{j}_\epsilon(x) D^\epsilon_\epsilon'(x,y) \mathbf{j}'_{\epsilon'}(y) \right)
$$

(3)

Noise and dissipation kernels are defined by:

$$
\nu^{\alpha\beta}(x,y) = -\frac{1}{2} \Im(D^\alpha_\beta(x,y))
$$

(4)

$$
\eta^{\alpha\beta}(x,y) = -\frac{1}{2} (D^\alpha_R(x,y) - D^\alpha_A(x,y))
$$

(5)

and we have:

$$
\mathcal{F}[\mathbf{j}_+, \mathbf{j}_-] = \exp \left( -\int d^4x d^4y \left( (\mathbf{j}_+(x) - \mathbf{j}_-(x), \mathbf{\nu}(x,y), (\mathbf{j}_+(y) - \mathbf{j}_-(y)) \right) \right)
$$

(6)

$$
\times \exp \left( i \int d^4x d^4y \left( (\mathbf{j}_+(x) - \mathbf{j}_-(x), \mathbf{\eta}(x,y), (\mathbf{j}_+(y) - \mathbf{j}_-(y)) \right) \right)
$$

(7)

Correlation functions can be obtained from a generating functional which takes the environment into account:

$$
\int \mathcal{D}[\varphi_+, \varphi_-] e^{i[S[\varphi_+ + \xi_+] - S[\varphi_- + \xi_-]]} \mathcal{F}[\mathbf{j}[\varphi_+], \mathbf{j}[\varphi_-]] \rho(t_i) |\varphi_+(t_i), \varphi_-(t_i)|
$$

Within the elastic approximation, coupling to acoustic phonons can be treated along the same line. Longitudinal phonons create a potential $V(\mathbf{x},t)$ such that $e V(\mathbf{x},t) = D \rho(\mathbf{x},t) \text{div}(\mathbf{u}(\mathbf{x},t))$ where $\mathbf{u}(\mathbf{x},t)$ is the elastic deformation field, $\rho(\mathbf{x},t)$ is the electric charge density and $D$ denotes the electron-phonon coupling energy. Longitudinal phonon dynamics is described by a quadratic action:

$$
S[\mathbf{u}(\mathbf{x},t)] = \frac{\rho_M}{2} \int \left( (\partial_t \mathbf{u})^2 - c_S^2 \left( \text{div}(\mathbf{u}) \right)^2 \right) d^3x dt
$$

(8)

where $c_S$ denotes the sound velocity and $\rho_M$ the volumic mass. The resulting phonon influence functional is a Gaussian in terms of $\rho_\pm(\mathbf{x})$ with kernel $D^\alpha_{\epsilon\epsilon'}(x,y) = -i(u_{\epsilon'}^\alpha(x) u_{\epsilon}^\beta(y))$.

$$
\mathcal{F}[\rho_+, \rho_-] = \exp \left( -\frac{iD^2}{2} \int d^4x d^4y \sum_{(\epsilon,\epsilon')} \rho_\epsilon(x) \rho_{\epsilon'}(y) (\partial_{\xi_\alpha} \partial_{\xi'_\beta} D^\alpha_{\epsilon\epsilon'}(x,y)) \right)
$$

(9)

In a generic electronic system, this influence functional a priori contains quartic fermion terms. The one dimensional interacting electron gas is quite interesting since it can be described by an effective free bosonic theory. Neglecting environment-induced umklapp processes, the influence functional is Gaussian (although non local) in terms of the bosonic field. Such terms arise from the introduction of $2k_F$ components of the charge or current densities in the environment’s influence functional. There is a factor $\exp(2i k_F (\sigma + \sigma'))$ in front of the term involving two $\psi^\dagger_R \psi_L$. At incommensurate filling, the usual averaging argument can be invoked to rule out these Umklapp contributions. Other terms contain both $\psi^\dagger_L \psi_L$ and $\psi^\dagger_R \psi_R$ and would imply momentum transfer of order $2k_F$ (except for special values of filling). In QED’s case, $2k_F$ photons would have a very high energy compared to the Luttinger typical energy $\hbar v_S/L$. In the acoustic case, since $k_F L >> v_S/c_S$, it would also be the case. That’s why we will not take into account $2k_F$ components of charge and current densities. Thanks to a Fourier mode analysis, this problem boils down to a set of independent harmonic oscillators, each of them being linearly coupled to a bath of quantum oscillators. This problem is known under the name of Quantum Brownian Motion (QBM) and has widely been studied. Necessary results will be recalled in section III. Elementary excitations of the Luttinger liquid can be created using vertex operators. States created this way are nothing but coherent states. Since the evolution of coherent states in QBM can be exactly computed, so can the evolution of elementary excitations in the Luttinger liquid.
B. Mode decomposition for the Luttinger liquid

Let us consider a Luttinger liquid on a circle of length $L$, with both left and right chiralities. As in \[3\], $\sigma$ denotes the coordinate along the circle. Low energy excitations of the Luttinger liquid are described by the theory of a free compactified boson, whose action is given by

$$ S[\varphi] = \frac{g}{2\pi} \int dt \, (v^{-1}_s (\partial_t \varphi)^2 - v_s (\partial_\sigma \varphi)^2) $$

The field $\varphi$ is compactified on a circle of radius $R_c$, and the Euclidean path integral representing the finite temperature equilibrium partition function contains a topological term taking into account zero mode quantization. The space of states of free bosonic theory is a representation of a $U(1)_{\alpha} \times U(1)_{l}$ algebra (two commuting copies of an oscillator algebra): $[J_k, J_l] = k \delta_{k,-l} \mathbf{1}$ and $[\tilde{J}_k, \tilde{J}_l] = \tilde{k} \delta_{k,-l} \mathbf{1}$. In Wen’s work, $J_l$ and $\tilde{J}_l$ for $l \neq 0$ are called hydrodynamic modes, whereas the $J_0$ and $\tilde{J}_0$ are called zero modes. In terms of these modes, the Hamiltonian of the Luttinger system is given by:

$$ H_{\text{tot}} = \frac{\pi v_s}{L} \sum_{l \in \mathbb{Z}} (J_l J_{-l} + \tilde{J}_l \tilde{J}_{-l}). $$

The finite temperature equilibrium partition function with a magnetic flux $\chi e/h$ is given by:

$$ Z_{\text{Lutt}}(\chi) = \frac{1}{|\eta(\tau)|^2} \sum_{(n,m) \in (1/2) \times \mathbb{Z}} q^{\frac{1}{2}R^2 m + \frac{3}{2} \pi h} \tau^{\frac{n}{2}} n^2 \tau^{\frac{m}{2}} m^2, $$

where $\alpha = g R_c^2$ encodes the interaction strength ($\alpha = 1$ for free electrons) and

$$ p_{n,m} = n \sqrt{\alpha} + \frac{m}{2 \sqrt{\alpha}}, \quad \bar{p}_{n,m} = n \sqrt{\alpha} - \frac{m}{2 \sqrt{\alpha}}. $$

States of interest will be $U(1) \times \bar{U}(1)$ highest weight states: with $J_0$ (resp. $\bar{J}_0$) eigenvalues $p_{n,m}$ (resp. $\bar{p}_{n,m}$). The charge and current densities can be expressed in terms of the $U(1)$ modes as follows:

$$ \rho(\sigma, t) = \frac{e}{L \sqrt{\alpha}} \sum_{n \in \mathbb{Z}} (J_n(t) e^{2\pi i \frac{n}{2 \alpha} \sigma} + \bar{J}_n(t) e^{-2\pi i \frac{n}{2 \alpha} \bar{\sigma}}) $$

$$ j(\sigma, t) = \frac{e v_s}{L \sqrt{\alpha}} \sum_{n \in \mathbb{Z}} (J_n(t) e^{2\pi i \frac{n}{2 \alpha} \sigma} - \bar{J}_n(t) e^{-2\pi i \frac{n}{2 \alpha} \bar{\sigma}}). $$

The $|n,m\rangle_\chi = |p_{n,m+2\chi},\bar{p}_{n,m+2\chi}\rangle$ state carries a charge $2n \in \mathbb{Z}$ and a current $(m+2\chi) v_s/L R_c$. If we assume that our Luttinger system is realized using the edge states of a FQH fluid, then the two edges of the sample are polarized with charges ($\alpha = 1/\nu$ is an odd integer):

$$ q_R = e \left( n + \frac{m}{2 \alpha} \right) \quad \text{and} \quad q_L = e \left( n - \frac{m}{2 \alpha} \right). $$

Obviously, $\bar{U}(1)$ descendants carry the same global charges and current than the highest weight states. Remember that \[1\] primaries are created using vertex operators $V_{n,m}(\sigma, t)$ which are normal ordered exponentials of bosonic modes. To complete this brief description, let us recall that the original fermionic operators get renormalized (orthogonality catastrophe historically introduced by Anderson), and that the corresponding renormalized fields are the so-called Luttinger “fermions” which correspond to $n = \pm 1/2$ and $m = \pm 1$. Within the context of fractional quantum Hall effect on a cylinder – or an annulus – edge fermions carrying a charge localized on one of the two edges, can be created or destroyed using vertex operators with $n = \pm 1/2$ and $m = \pm \nu^{-1}$. Typical experiments which may be performed on edge excitations of an annular 2DEG in an AsGaAl heterostructure lead to the following numerical values which we shall use in the rest of this paper: $v_s \simeq 3.10^3 \text{m} \text{s}^{-1}$, $L \simeq 30 \mu \text{m}$, $\alpha = 3$ and $T \simeq 0 - 10 \, K$.

C. Coupling of modes to the environments

In order to respect cylindrical geometry, the environment field is supposed to live in a cylindrical cavity with the same revolution axis than the Luttinger circle. The cavity is of radius $R$ and of height $h$ and $V = \pi R^2 h$ will denote its volume. Periodic boundary conditions are assumed in this vertical direction.
Electromagnetic modes are quantized using their momentum along z axis $k_z$, their angular momentum moment around z axis $l \in \mathbb{Z}$ and the number $n$ of radial zeroes of the electric field. The boundary condition on the cavity plays the role of polarization and accounts for the helicity degeneracy. Transverse electric (TE) modes have their (transverse) electric field orthogonal to the cavity’s edge, and transverse magnetic (TM) modes have their magnetic field orthogonal to the cavity’s edge. Normalized expressions for the modes can be found in any book on optical fibers and are recalled in appendix B. Of course, the box is considered as large meaning that electromagnetic modes form a continuum compared to the Luttinger modes.

Phonons are quantized according to the vanishing of displacement field $u$ on the cavity’s boundary. Longitudinal modes can be written as a gradient $u = \nabla (\phi)$. As shown in appendix B longitudinal modes are classified by $(k_z, l)$ and $n$ where $n$ is the number of zeroes of the $\phi$ potential.

It is convenient to introduce some kind of “phase space coordinates” to describe the $l \geq 1$ modes of the Luttinger Liquid:

$$q_l^{(+)} = \frac{1}{2\sqrt{l}}(J_l + J_{-l} - \bar{J}_l - \bar{J}_{-l}) \quad \text{and} \quad ip_l^{(+)} = \frac{1}{2\sqrt{l}}(J_l - J_{-l} - \bar{J}_l + \bar{J}_{-l})$$

(17)

$$-iq_l^{(-)} = \frac{1}{2\sqrt{l}}(J_l - J_{-l} + \bar{J}_l - \bar{J}_{-l}) \quad \text{and} \quad p_l^{(-)} = \frac{1}{2\sqrt{l}}(J_l + J_{-l} + \bar{J}_l + \bar{J}_{-l})$$

(18)

Formulating the problem in this way shows that the interaction Hamiltonian is nothing but a linear coupling between harmonic oscillators ($qx$ coupling). More precisely, for each $l \geq 1$, two harmonic oscillators are coupled to the acoustic or electromagnetic modes of angular momentum $l$. The two Luttinger modes of fixed $l$ are coupled to different sets of environment modes. The same conclusion is true for zero modes: the global current couples to acoustic modes whereas the global charge decouples from the field’s propagating modes (Coulomb gauge effect). In the acoustic case, only the total electric charge couples to $l = 0$ acoustic modes.

The influence functional for each of these environments can explicitly be obtained in a nice form. First of all, the charge and current densities can be expressed as linear combinations of the above phase space coordinates. Then, performing integration over the vector potential and spatial coordinates leads to the QED functional (assuming that $q_0^{(+)} = q_0$ and $q_0^{(-)} = 0$):

$$\mathcal{F}_{FV} = \exp \left( -\frac{i g}{2\sqrt{V}} \sum_{l \geq 0} \sum_{(\epsilon, \epsilon', \alpha) \in \{+, -\}^3} \epsilon \epsilon' \frac{c^2}{l^2} D^2 l \int dt \frac{ds}{s} q_l^{(\alpha)\epsilon}(t) G_{\epsilon'\epsilon}^{(\alpha)}(t-s) q_l^{(\alpha)\epsilon'}(s) \right)$$

(19)

where the dimensionless coupling constant only depends on the fine structure constant $\alpha_{QED}$, the Luttinger liquid interaction parameter $\alpha$ and the ratio of velocities $v_S/c$:

$$g = 4\pi \frac{\alpha_{QED}}{\alpha} \left( \frac{v_S}{c} \right)^2$$

(20)

This mode expansion is related to a multipolar expansion. Luttinger modes of momentum $l$ contribute to the electric (resp. magnetic) multipolar expansion starting at order $2l$ (resp. $2l + 1$) In particular, $l = 0$ is mainly a magnetic dipole and an electric monopole (corresponding to the global charge and current around the Luttinger ring). The $l = 1$ modes contribute first to the electric dipole and the magnetic quadrupole.

Longitudinal phonons can be treated in the same way. In this case, one needs to perform a canonical transformation on the Luttinger phase space coordinates, exchanging $p_l^{(\pm)}$ and $q_l^{(\pm)}$ in order to obtain a Gaussian expression similar to (19). The dimensionless coupling constant corresponding to longitudinal phonons in dimension $d = 2, 3$ is given by:

$$g_{ph}(L) = \frac{D^2}{\alpha \rho_M L^{d-1} \hbar c_S^3}$$

(21)

where $c_S$ is the sound velocity, $\rho_M$ the volumic $(d = 3)$ or surfacic $(d = 2)$ mass and $D$ the typical electron/phonon coupling energy. With typical values $D \simeq 7 \text{ eV}$, $c_S \simeq 3 \times 10^3 \text{ m s}^{-1}$ and $\rho_M \simeq 3 \times 10^3 \text{ kg m}^{-3}$, we get, in dimension three $g_{ph}(L) \simeq 4 \times 10^{-6}$. 
III. RELATION TO THE QUANTUM BROWNIAN MOTION

A. The Quantum Brownian Motion model

This basic model consists in a single quantum harmonic oscillator (called “the system”) coupled to a bath of oscillators (called the “reservoir”):

\[ L = \frac{m}{2} \left( \dot{q}^2 - \Omega^2 q^2 \right) + \sum_l C_l q q_l + \sum_l \frac{M_l}{2} \left( \dot{q}_l^2 - \omega_l^2 q_l^2 \right) \]  \hspace{1cm} (22)

We assume that these two systems are initially independent and that the oscillator bath is at equilibrium with inverse temperature \( \beta \). One could also assume that the whole system is initially at equilibrium but it makes things more involved without illuminating the discussion. Interested readers are referred to Grabert’s review or to Romero and Paz. The question addressed here is to compute the time evolution of the system. The physics depends on the reservoir’s influence at each frequency, encoded in appropriate spectral densities. Therefore, several natural time scales will appear in the Luttinger & environment problem:

- The cut-off time : above a certain UV cutoff, the effective description by a Luttinger liquid is no longer valid. Another natural cutoff could also be provided by the spectral distribution itself (Debye frequency for phonons). For this reason, an UV cutoff is present in the model. The associated time will be denoted by \( 1/\Lambda \). Below this time scale, our simple model cannot be considered as valid.

- The environment time scale \( \tau_E \) is the time needed by the light (\( \tau_{EM} = L/c \)) or by phonons (\( \tau_S = L/c_S \)) to circle around the system. Spectral densities and influence functionals can be normalized with respect to this characteristic time (they only depend on \( \omega \tau_E \)). The low frequency regime is defined by \( \omega \tau_E << 1 \) and the high frequency regime by \( \omega \tau_E >> 1 \).

- The Luttinger time scale \( \tau_L = L/v_S \) is the time needed by one excitation of the Luttinger liquid to circle around the system. This is the natural time scale from the Luttinger liquid point of view.

- The thermal time scale \( \tau_{Th} = \hbar/k_BT \) is the inverse frequency associated with temperature \( T \). The corresponding thermal length is \( l(\beta) = c \tau_{Th} \) for photons and \( l(\beta) = c_S \tau_{Th} \) for phonons. Let us notice that at \( T \approx 1 \text{K} \), \( l(\beta) = 1 \text{mm} \) for photons whereas it is of order 10 \( \mu \text{m} \) for phonons.

The general solution to the Quantum Brownian Motion problem is due to Hu, Paz and Zhang (HPZ) who computed the evolution kernel for the system’s reduced density matrix. Caldeira and Leggett’s work is concerned with a special case of reservoir, corresponding to the so-called Ohmic spectral density \( \mathcal{J}(\omega) \propto \omega \). As we have already noticed, the Luttinger liquid can be seen as a collection of harmonic oscillators and some zero modes. As we shall see, each harmonic mode coupled to the environment is a QBM problem. Each reservoir is characterized by a set of spectral functions (one for each \( l \geq 0 \)), which are made dimensionless for simplicity.

In the electromagnetic case, using expressions \( (26) \) and \( (27) \) in appendix B, we obtain:

\[ \mathcal{J}_I(\omega) = \frac{g L^3}{V} \sum_{l \neq l_i} \delta_{\log} \frac{D^2_I}{2 \omega_I \tau_{EM}} \delta(\tau_{EM}(\omega - \omega_I)) \] \hspace{1cm} (23)

The \( l = 0 \) case needs a slight modification:

\[ \mathcal{J}_0(\omega) = \frac{g L^3}{V} \sum_{l \in \mathcal{E}} \frac{D^2_I}{2 \omega_I \tau_{EM}} \delta(\tau_{EM}(\omega - \omega_I)) \] \hspace{1cm} (24)

The electromagnetic influence functional can be rewritten as:

\[ \mathcal{F}_{FV}[\mathbf{j}_{+}, \mathbf{j}_{-}] = \exp \left( \int \frac{c^2}{L^2} \sum_{l=0}^{\infty} \sum_{\alpha=+,-} \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} ds \left( q_l^{(\alpha)+} - q_l^{(\alpha)-} \right)(t) \eta_l(t - s) \left( q_l^{(\alpha)+} + q_l^{(\alpha)-} \right)(s) \right) \times \exp \left( -\frac{c^2}{L^2} \sum_{l=0}^{\infty} \sum_{\alpha=+,-} \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} ds \left( q_l^{(\alpha)+} - q_l^{(\alpha)-} \right)(t) \nu_l(t - s) \left( q_l^{(\alpha)+} - q_l^{(\alpha)-} \right)(s) \right) \] \hspace{1cm} (25)
Dimensionless dissipation and noise kernels for each mode are expressed in terms of the spectral density by:

\[ \eta(s) = -\frac{L}{c} \int_0^\infty d\omega \, J_l(\omega) \sin(\omega s) \] (26)

\[ \nu_l(s) = \frac{L}{c} \int_0^\infty d\omega \, J_l(\omega) \coth\left(\frac{\beta \omega}{2}\right) \cos(\omega s) \] (27)

An acoustic reservoir leads to similar expressions for spectral functions (using expressions [C1] and [C2]):

\[ J_l(\omega) = g_{ph}(L) \frac{L^3}{V} \sum_{l', l_l = l} l_l N_l^2 (\tau_S \omega_l)^3 \delta(\tau_S (\omega - \omega_l)) \] (28)

The \( l = 0 \) case needs a slight modification:

\[ J_0(\omega) = g_{ph}(L) \frac{L^3}{V} \sum_{l_l = 0} N_l^2 (\tau_S \omega_l)^3 \delta(\tau_S (\omega - \omega_l)) \] (29)

Numerical computations of spectral densities as well as analytic estimates of their asymptotics are available (see appendix D). The electromagnetic case is illustrated on figure 3. In this case, all \( l \neq 1 \) modes are supraohmic \( (J_l(\omega) \) decreases faster than \( \omega) \) at low frequency. The \( l = 1 \) modes show an ohmic behavior \( (J_1(\omega) \propto \omega) \). In this case, the dissipation kernel is local in time and, as we shall see in section IVF, an effective Caldeira-Leggett model can be used to perform analytic computations.

In the acoustic case \( J_l(\omega) \) goes as \( (\omega \tau_S)^{2l+d} \) in the low frequency regime and as \( (\omega \tau_S)^{d-1} \) at higher frequencies. The main difference between the electromagnetic and acoustic reservoirs is that the natural Luttinger frequency \( 2\pi/\tau_L \) falls in the low frequency domain for QED, whereas it does not for phonons since \( \tau_S/\tau_L >> 1 \).

### B. Phase space evolution using Wigner functions

The time evolution can be computed using the Wigner function associated with the system’s density operator. This form is especially adapted to the study of decoherence of Gaussian wave packets. Moreover, it provides a nice quasi-classical insight on the evolution of the system since in the Luttinger liquid, charge and current density fluctuations play the role of “phase space coordinates” for the hydrodynamic modes and within the FQH effect framework, encode the shape of the incompressible quantum Hall fluid droplet. The Wigner function associated with an operator \( B \) is defined by:

\[ W_B(p, q) = \int dy \, e^{i py/\hbar} \langle q - \frac{y}{2} | B | q + \frac{y}{2} \rangle \] (30)

We use the following notation for phase space: \( \phi = \left( \begin{array}{c} p \\ q \end{array} \right) \). The evolution kernel for the Wigner function can be computed from the density operator evolution kernel and is given by:

\[ J_W(p, q, t|p_0, q_0, 0) = \mathcal{N}(t) \exp \left( -\frac{1}{4} \left( \begin{array}{cc} N \cdot \phi + N_0 \cdot \phi_0 \end{array} \right) A(t) \cdot \left( \begin{array}{cc} N \cdot \phi + N_0 \cdot \phi_0 \end{array} \right) \right) \] (31)

where the \( N_0 \) and \( N \) time dependent matrices are given by:

\[ N = \left( \begin{array}{cc} -1 & \dot{u}_2(t) \\ 0 & -\dot{u}_2(0) \end{array} \right) \quad \text{and} \quad N_0 = \left( \begin{array}{cc} 0 & \dot{u}_1(t) \\ 1 & -\dot{u}_1(0) \end{array} \right) \] (32)

The \( u_i(t) \) \((i \in \{1, 2\})\) functions are defined in Hu, Paz and Zhang’s paper as solutions to the classical equations of motion with dissipation and boundary conditions \((u_1(0), u_1(t)) = (1, 0)\) and \((u_2(0), u_2(t)) = (0, 1)\). The \( A(t) = (a_{i,j}(t))(i,j) \in \{1, 2\}^2 \) matrix is defined by

\[ a_{i,j}(t) = \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2 \, u_i(s_1) \nu(s_1 - s_2) \, u_j(s_2). \]
The reader can check that, when the reservoir decouples, the evolution kernel reduces to a delta function, giving back the classical evolution in phase space. Henceforth, we shall call $-N^{-1} N_0$ the “Wigner evolution operator” and denote it by $U_i$. When turning on the coupling to the environment, the classical delta distribution in phase space spreads into a Gaussian, the center of which moves according to $U_i$. As we shall now see, the evolution kernel encodes all effects of dissipation and decoherence.

The evolution of a density operator built from Gaussian wave packets can be computed exactly. The generic form of a Gaussian Wigner function is, up to some normalization:

$$W[\bar{\phi}, K, Q](\phi) = \exp \left(-\frac{1}{2}(\phi - \bar{\phi}).Q.(\phi - \bar{\phi}) + i\phi.K.\phi\right) \quad (33)$$

The 2 by 2 matrix $Q$ encodes the spreading of the packet, $\bar{\phi}$ is the center of the packet and $K$ a phase modulation. Gaussian wave packets and in particular coherent states lead to Gaussian Wigner functions. A Gaussian Wigner function remains Gaussian. If $W(t = 0) = W[\bar{\phi}_0, K_0, Q_0]$ then, after time $t$,

$$W(t) = e^{-i\left[K.\phi + \int d\omega.\Theta(t, \omega)\right]} \times W[U_i.\phi_0, K(t), Q(t)] \quad (34)$$

where the basic parameters at time $t$ are given by $d[K_0, Q_0](t) = i\alpha, D(t), K_0$ and:

$$D(t) = (2Q_0 + \alpha A(t)^{-1} N_0)^{-1}$$

$$K(t) = \alpha U_i^{-1}(1 - 2Q_0D(t)).K_0$$

$$Q(t) = \alpha U_i^{-1}(Q_0 - 2Q_0D(t)Q_0).U_i^{-1} \quad (35)$$

The $D(t)$ matrix contains the decoherence effect. To understand this, let us start with a coherent superposition of two Gaussian wave packets $(\psi_1 + \psi_2)/\sqrt{2}$. The initial Wigner function is given by:

$$W(0) = \frac{1}{2}(W_{11} + W_{12} + W_{21} + W_{22}) \quad \text{where} \ W_{\alpha\beta}(0) = W[\bar{\phi}_\alpha, \bar{\phi}_\beta, Q_0] = W[\bar{\phi}_\alpha + \bar{\phi}_\beta, -i\sigma_y.(\bar{\phi}_\alpha - \bar{\phi}_\beta), Q_0] \quad (36)$$

Under time evolution, the form of the wave packet is preserved up to a global factor and a phase modulation:

$$W_{\alpha\beta}(t) = e^{-i\left[\sigma_y.(\bar{\phi}_\alpha - \bar{\phi}_\beta)\right]} W[U_i.\bar{\phi}_\alpha, U_i.\tilde{\phi}_\beta, Q(t)] \quad (37)$$

where:

$$d_{\alpha\beta}(t) = d[\sigma_y.(\bar{\phi}_\alpha - \bar{\phi}_\beta), Q_0] \quad (38)$$

$$\Theta_{\alpha\beta}(t, \phi) = \alpha U_i^{-1} \frac{d\bar{\phi}}{dt}_1(t) \quad (39)$$

$$K_q(t) = \alpha \left[K(t) - \frac{\bar{u}_1(t)}{\bar{u}_2(t)}U_i^{-1}K_0\right] \quad (40)$$

Each density operator $|\psi_\alpha\rangle\langle\psi_\alpha|$ has its own evolution, described by $\bar{\phi}_\alpha(t) = U_i.\bar{\phi}_\alpha$ and $Q(t)$. Note that $d_{11}(t) = d_{22}(t) = 0$. The coherence part is contained in $W_{12}$, and evolves according to $\bar{\phi}_{12}$ and $Q(t)$ plus an exponential factor $e^{-d_{12}(t) + i\Theta_{12}(t, \phi)}$. The $d_{12}(t)$ factor gives the attenuation of off-diagonal correlations and should therefore be interpreted as the decoherence factor between the two wave packets.

Unfortunately, as noticed before, explicit and closed expressions for $U_i$ and $D(t)$ are not known for a general supraphysical environment. Therefore, we shall perform a perturbative expansion in the coupling constant. In the case of the decoherence factor $d_{12}(t)$, it is enough to start from standard harmonic oscillators expressions for the $u_1$ and $u_2$ functions since $A(t)$ is of first order in the coupling constant. The decoherence matrix can then be expressed as an integral over the spectral density:

$$D(t) = \int_0^{+\infty} J(\omega) D(\Omega, \omega, t) \cosh \left(\frac{\beta \hbar \omega}{2}\right) d\omega. \quad (41)$$

where $D(\Omega, \omega, t)$ has an explicit but involved expression. Let us discuss its asymptotics in various physically relevant limits:
• At short times $\Omega t \ll 1$, $\Omega$ can be neglected provided the spectral density of the bath contains modes with frequencies much higher than $\Omega$. Then, this regime is dominated by the high frequencies of the bath and:

$$D(\Omega, \omega, t) \simeq \left( \frac{\omega^2}{4\omega t} + \frac{\omega t^4}{4} \right)$$

The last expression is only valid at very short times and contains non-Markovian effects. Even in the Caldeira-Leggett model, at very short times, all frequencies of the bath take part in the evolution of the system, leading to these non-Markovian effects.

• When the condition $\Omega t \ll 1$ is no longer valid, dissipation effects with exponential relaxation will appear through linear terms in $t$. We must of course assume that $t$ is much smaller than typical relaxation times. At growing times, linear terms will dominate oscillating resonant ones and provide a “Golden rule” estimate for the relaxation and decoherence times. This approximation will be used in the next section in order to evaluate the typical decoherence time of Schrödinger cat states in the Luttinger model.

When $t$ reaches the dissipation time scale, this perturbative treatment breaks down. The long time regime of decoherence can however be computed in some cases using the Caldeira-Leggett model.

IV. MUTUAL DECOHERENCE OF ELEMENTARY EXCITATIONS IN A LUTTINGER LIQUID

A. Statement of the problem

As recalled in section II B, elementary excitations of the Luttinger liquid are created by vertex operators $V_{n,m}(\sigma)$. Such a state is not an eigenstate of the Hamiltonian but rather corresponds to the introduction of a “localized” excitation at point $\sigma$ around the circle. Let us consider a Schrödinger cat state build as a superposition of a right moving Luttinger fermion ($n = 1/2$ and $m = 1$) at different places around the circle:

$$|\psi_{RR}(0)\rangle = \frac{1}{\sqrt{2}} \left( \psi_R^\dagger(\sigma_1)|0\rangle + \psi_R^\dagger(\sigma_2)|0\rangle \right)$$

In an isolated system, such a state will evolve according to:

$$|\psi_{RR}(t)\rangle = \frac{1}{\sqrt{2}} \left( \psi_R^\dagger(\sigma_1,t)|0\rangle + \psi_R^\dagger(\sigma_2,t)|0\rangle \right)$$

Therefore, this coherent superposition will remain coherent, as it should in any isolated quantum system. Switching on the coupling to the quantum electromagnetic field or phonons changes the situation: according to general works on decoherence, we expect this Schrödinger cat to decohere into a statistical mixing of two states: one excitation at one position, or the excitation at the other position. Here, two questions will be addressed: what is the strength of the decoherence process and on which time scale does it take place?

In the following, two cases will be considered: the $R/R$ Schrödinger cat, already presented in equation 44 and the $R/L$ Schrödinger cat, defined as:

$$|\psi_{RL}(0)\rangle = \frac{1}{\sqrt{2}} \left( \psi_R^\dagger(\sigma_1)|0\rangle + \psi_L^\dagger(\sigma_2)|0\rangle \right)$$

Practically, the case of zero modes is simpler and will be considered in the next paragraph. We shall then turn to the $l \neq 0$ modes in section IV C. Decoherence time estimates for electromagnetic (resp. acoustic) reservoirs are given in IV D (resp. IV E). An effective Caldeira-Leggett model will be used to deal with the long time behavior in sections IV F to IV H.

B. Evolution of zero modes

For the zero modes, the zero coupling evolution corresponds to a free particle (and not to an harmonic oscillator). The Ambegaokar and Hakim method (exact diagonalization for the coupled system) can be used to compute the
evolution of the density matrix for the zero modes. The corresponding explicit formula in the Luttinger liquid case is, at finite temperature:

\[ \langle n, m|\rho(t)|n', m'\rangle = \langle n, m|\rho(0)|n', m'\rangle \times \exp \left( -i(\omega_{n,m}(t) - \omega_{n,m}(t)) t \right) \times \exp \left( -\frac{d(t)}{\alpha}(m - m')^2 \right) \]

(47)

where \( h\omega_{n,m}(0) \) denotes the energy of \(|n, m\rangle\) in the isolated Luttinger system and:

\[ \omega_{n,m}(t) = \omega_{n,m}(0) + \frac{\pi v_S}{2L\alpha} n^2 \int_0^{+\infty} \frac{c}{\pi v_S} \mathcal{J}_0(\omega) \left( \frac{\sin(\omega t)}{\omega t} - 1 \right) \frac{d\omega}{\omega} \]

(48)

\[ d(t) = \frac{t^2}{\tau_{EM}} \int_0^{+\infty} d\omega \mathcal{J}_0(\omega) \left( \frac{\beta h\omega}{2} \right) \frac{1 - \cos(\omega t)}{(\omega t)^2} \]

(49)

The first term in (47) can be interpreted as a dynamical renormalization of \( v_S/\alpha \). Only \( v_S/\alpha \) is renormalized since the charge density does not couple to the transverse degrees of freedom of the electromagnetic field. To be precise, at \( t \to +\infty \), the velocity and interacting parameters of the Luttinger liquid get renormalized as \( v'_S = v_S \sqrt{\alpha'/\alpha} \) where the renormalization constant \( \alpha'/\alpha \) is equal to the \( t \to +\infty \) limit of:

\[ \left( \frac{\alpha'}{\alpha} \right) = 1 + \frac{1}{2} \int_0^{+\infty} \frac{c}{\pi v_S} \mathcal{J}_0(\omega) \left( \frac{\sin(\omega t)}{\omega t} - 1 \right) \frac{d\omega}{\omega} \]

(50)

The renormalization effect is of course strongly cut-off dependent. The dimensionless coupling constant appearing here is \( \frac{\alpha'_{EM}}{\alpha} \), \( \frac{\alpha'}{\alpha} \simeq 10^{-5} \).

The second term is the decoherence coefficient between two different highest weight states \(|n, m\rangle\) and \(|n', m'\rangle\) of the LL. Decoherence takes place in a time of the order of the cutoff time \( \Lambda^{-1} \) and then reaches saturation. Figure 4 summarizes \( d(t) \) and \( \zeta(t) \)’s behavior.

The typical value of the \( t \to +\infty \) value of the decoherence exponent is of typical order \( g \):

\[ d(+\infty) = \int_0^{+\infty} \frac{c}{\lambda^2} \mathcal{J}_0(\omega) \coth \left( \frac{\beta h\omega}{2} \right) d\omega \]

(51)

In the acoustic case, computations can be performed in the same way. Since acoustic zero modes couple to the total charge of the Luttinger system, the decoherence factor between states \(|n, m\rangle\) and \(|n', m'\rangle\) is found to be \exp \left( -2\alpha d(t)(n^2 - (n')^2) \right) where \( d(t) \) is obtained from (49) by using the acoustic spectral density and \( \tau_S \) instead of their electromagnetic counterparts. The Luttinger parameters \( \alpha \) and \( v_S \) also get renormalized. In the acoustic case, only \( v_S/\alpha \) is renormalized. Then, \( \alpha' = \alpha\zeta_{ph} \) and \( v'_S = v_S\zeta_{ph} \) where \( \zeta_{ph} \) is the \( t \to +\infty \) limit of \( \zeta_{ph}(t) \), obtained by using the acoustic spectral density and the speed of sound in formula (49).

In both cases, the final decoherence exponent is proportional to the square of the difference between the total current (resp. charge), quantities which measure the "distance" between the two quantum states. Such a result is expected since, as explained in C. Cohen’s lectures, such a dependence is common in the case of a linear coupling with a conserved quantity. In particular, a Schrödinger cat obtained by superposing the same elementary excitation of the Luttinger liquid at two different positions along the ring has all its decoherence due to hydrodynamic modes! We also notice that zero mode decoherence has a weak dependence in the cut off and temperature.

C. Spatial dependence of decoherence

Using the explicit form of vertex operators, one easily finds the relevant parameters to be used for the decoherence of each mode. Of course, these parameters depend on positions of each of vertex operator. Here, we shall only present the results for \( R/R \) and \( R/L \) Schrödinger cats \((\sigma_{12} = \sigma_1 - \sigma_2)\):

\[ d_{RR}(t, \sigma_1, \sigma_2) = \frac{4}{\alpha} \sum_{l=1}^{+\infty} \frac{1}{l} \left( m^2 D_{11}^{(l)}(t) + 4n^2 \alpha^2 D_{22}^{(l)}(t) \right) \sin^2 \left( \frac{\pi l\sigma_{12}}{L} \right) \]

(52)

\[ d_{RL}(t, \sigma_1, \sigma_2) = \frac{4}{\alpha} \sum_{l=1}^{+\infty} \frac{1}{l} \left( m^2 D_{11}^{(l)}(t) \left( 1 - \sin^2 \left( \frac{\pi l\sigma_{12}}{L} \right) \right) \right) \]

(53)

\[ + 4n^2 \alpha^2 D_{22}^{(l)}(t) \sin^2 \left( \frac{\pi l\sigma_{12}}{L} \right) + 2nm \alpha D_{12}^{(l)}(t) \sin \left( \frac{2\pi l\sigma_{12}}{L} \right) \]

(54)
Here $D_j^{(l)}(t)$ denotes the decoherence matrix for the $q_l^{(k)}$ modes computed along the lines of section [11B]. The main change from the HPZ computations arises from our normalization choice for the $q$s. The effective spectral density to be used in the HPZ formulas is given by:

$$2\pi l \frac{v_S}{c} \frac{\mathcal{J}_l(\omega)}{\tau_E^2}.$$ 

This rescaling takes into account the ratio of the Luttinger mode $l$ time scale (i.e. $\tau_L/l$) and of the environment time scale $\tau_E = L/c$.

The appearance of an odd dependence – in term of $\sigma_{12}$ – in the $d_{RL}(t, \sigma_1, \sigma_2)$ coefficient is understood by noticing that an appropriate parity operation transforms the $R/L$ Schrödinger cat into an $L/R$ one. Therefore $d_{RL}(t, \sigma_1, \sigma_2)$ is invariant into simultaneous changes $\sigma_1 \leftrightarrow \sigma_2$ and $nm \rightarrow -nm$.

A first estimate is obtained using a perturbative approach. We perform a secular approximation and retain only terms linear in time. The corresponding decoherence rates are given by:

$$\frac{\tau_{L}}{\tau_l^{(R/R)}(\sigma_1, \sigma_2)} = 8\pi^2 \Delta_{n,m} \cdot \sin^2 \left( \frac{l\pi \sigma_{12}}{L} \right) \frac{\mathcal{J}_l(\omega_l)}{\omega_l \tau_E}$$ (55)

$$\frac{\tau_{L}}{\tau_l^{(R/L)}(\sigma_1, \sigma_2)} = = 8\pi^2 \Delta_{n,m} \cdot \left(1 + \frac{m^2 - 4n^2 \alpha^2}{m^2 + 4n^2 \alpha^2} \cos \left( \frac{2\pi l \sigma_{12}}{L} \right) \right) \frac{\mathcal{J}_l(\omega_l)}{\omega_l \tau_E}$$ (56)

Here $\Delta_{n,m}$ is the conformal dimension of the vertex operator $V_{n,m}(\sigma)$. Not surprisingly, the decoherence time of a $R/R$ Schrödinger cat diverges when $\sigma_{12} \rightarrow 0$. This result is obvious since in this limit, the initial state is a pure state. For $L/R$ cats, the decoherence time shows a slow variation in term of the differences of positions.

### D. Decoherence time estimations: QED’s case

Using asymptotics of spectral densities (see appendix [D]), one obtains the decoherence time of the $l$th modes in the $R/R$ case:

$$\frac{\tau_{L}}{\tau_l^{(R/R)}(\sigma_1, \sigma_2)} = 4g \Delta_{n,m} \left( \frac{2\pi v_S}{c} \right)^{2(l-1)} \frac{l^2 (l + 1)}{(2l + 1)!, \sin^2 \left( \frac{\pi l \sigma_{12}}{L} \right)}$$ (57)

Similarly, for the $R/L$ case:

$$\frac{\tau_{L}}{\tau_l^{(R/L)}(\sigma_1, \sigma_2)} = 4g \Delta_{n,m} \left( \frac{2\pi v_S}{c} \right)^{2(l-1)} \frac{l^2 (l + 1)}{(2l + 1)!, \left(1 + \frac{m^2 - 4n^2 \alpha^2}{m^2 + 4n^2 \alpha^2} \cos \left( \frac{2\pi l \sigma_{12}}{L} \right) \right)}$$ (58)

Since $v_S/c \approx 10^{-3}$, decoherence times for the $l$ and $l + 1$ modes are related by a typical factor of $10^6$. This argument shows that the $l = 1$ modes dominate the decoherence process. Physically, higher Luttinger modes contribute to higher electric and magnetic multipoles, for which radiative dissipation is known to be weaker. Since dissipation governs decoherence, this is the physical reason for the predominance of the $l = 1$ Luttinger mode in the decoherence process. The decoherence time of the $l = 1$ mode is nothing but the electromagnetic relaxation time:

$$\tau^{-1} \approx \frac{16\pi}{3} \frac{\alpha_{\text{QED}}}{\alpha} \left( \frac{v_S}{c} \right)^2 \cdot \tau_L \approx 10^{-8} \tau_L^{-1}$$

Numerical results for the decoherence times are shown on figure 5 for Luttinger fermions.

The temperature dependence can be found easily since coth $(\beta \hbar \omega_l/2)$ varies slowly around $\omega_l$ in a scale $g\omega_l$. Therefore:

$$\frac{\tau_l^{(R/R)}(\sigma_1, \sigma_2, T)}{\tau_l^{(R/R)}(\sigma_1, \sigma_2, T = 0)} = \tanh \left( \frac{\hbar \omega_l}{2 k_B T} \right)$$ (59)
E. Decoherence time estimations: acoustic case

Since the sound velocity \( c_S \) is much smaller than \( v_S \), the condition \( \omega_l \tau_E << 1 \) does not hold for the coupling to phonons. Explicit computations show that the behavior of spectral densities for the coupling to longitudinal phonons differs from the electronic one. Luttinger modes frequencies fall into a range of frequencies where the acoustic spectral densities are proportional to \( (\omega_T S)^{d-1} \). Remember also that the natural cut-off frequency for the phonon bath is given by \( \omega_D = c_S/a \) where \( a \) is a typical microscopic length. The zero temperature acoustic decoherence rate of Luttinger modes is given by formulas \( 55 \) where the \( \Gamma_l^{(d)} = J_l(\omega_l)/\omega_l \tau_S \) factor is given by:

\[
\Gamma_l^{(3)} = \frac{g_{ph}(L) l^2 v_S}{2 c_S} \\
\Gamma_l^{(2)} = \frac{g_{ph}(L) l}{\pi} 
\]

In opposition to QED’s case, these damping rates do not decrease with increasing \( l \). In the QED case, higher \( l \) modes are bad antennas for the microwave radiation emitted by the system. In the case of phonons the situation goes the other way because of the longitudinal coupling. Although in some cases the coupling constant \( g_{ph}(L) \) is very small, “decoherence repartition” effects between modes plays a much more important role here than in QED’s case since one has to sum up over many mode contributions to decoherence. For Luttinger modes of index \( l \), the perturbative expansion is governed by the relative damping rate \( \gamma_l/\omega_l \), an upper value of which is given by

\[
g_{ph}(a) \frac{\omega_l \tau_S}{(\omega_D \tau_S)^2} \quad \text{for } d = 3 \quad \text{and} \quad \frac{g_{ph}(a)}{\pi (\omega_D \tau_S)} \quad \text{for } d = 2
\]

where \( g_{ph}(a) \) is the rescaled coupling constant for the length \( a \) (typically of order 1). Assuming that \( \omega_D \tau_S = L/a \) is much greater than one, we see that all \( l \neq 0 \) modes can be considered as weakly damped.

The total decoherence exponent in the linear regime is obtained by summing over all the modes up to the Debye frequency. For \( R/R \) Schrödinger cat states, one finds:

\[
\tau_L \cdot \gamma^{(R/R)}(\sigma_1, \sigma_2, T) = 8\pi^2 \Delta_{n,m} \sum_{l=1}^{l_{max}} \Gamma_l^{(d)} \coth \left( \frac{\beta \hbar \omega_l}{2} \right) \sin^2 \left( \frac{\pi l \sigma_{12}}{L} \right)
\]

(63)

Since we sum over a large number of modes, the decoherence time rapidly decreases when \( \sigma_{12} >> av_S/c_S \), a spectacular effect due to the ratio \( c_S/v_S << 1 \). Roughly speaking, the Luttinger fermion has the time to circle many times around the loop before emitted phonons escape whereas it barely has the time to move in the electromagnetic case. This “averaging effect” explains why the dependence in the initial relative position is much weaker for acoustic than for electromagnetic decoherence. The maximal inverse decoherence rate can be expressed as an integral in the limit \( L >> av_S/c_S \) (\( k_B \Theta_D = \hbar \omega_D \)):

\[
\tau_L \cdot \gamma^{(R/R)}(T) = \Delta_{n,m} \frac{g_{ph}^{(d)}(a)}{4^{d-2}2\pi} \left( \frac{c_S}{v_S} \right)^2 \frac{L}{a} \int_0^1 x^{d-1} \coth \left( \frac{x \Theta_D}{2T} \right) dx
\]

(64)

The temperature dependence is very weak (remember we are typically working in situations where \( \omega_D \tau_L > 10^5 \) and \( k_B T \simeq \hbar \omega_L \)). To be precise, it goes like:

\[
\frac{\gamma^{(R/R)}(T) - \gamma^{(R/R)}(0)}{\gamma^{(R/R)}(0)} \simeq \left( \frac{T}{\Theta_D} \right)^d
\]

(65)

In opposition with the photon bath case, the total acoustic decoherence time scales as \( L^{-1} \) in units of \( \tau_L \).

F. Caldeira-Leggett computations

The Caldeira-Leggett model corresponds to the Ohmic spectral density: at low frequencies, \( J(\omega) = \frac{M \omega}{\alpha} \). In this case, for time scales large compared to the cut-off time, noise and dissipation kernels are local in time. The solution of the model is then much simpler. In the HPZ approach, the equation of motion defining the \( (u_i)_{i=1,2} \) functions
can be solved exactly, taking into account non perturbatively all effects of dissipation. For general spectral densities, solutions of the equation of motions are given in full generality by a Laplace transform of the form:

\[ \tilde{u}(p) = \frac{\tilde{u}(0) + pu(0)}{1 + p^2 + 2\eta(p)} \quad \text{and} \quad \tilde{\eta}(p) = \int_0^\infty d\omega \frac{\omega J(\omega)}{\omega^2 + p^2}. \]

This expression has clearly a cut along the \( p \)-imaginary axis for \(|p| < \omega_c\) (\( \omega_c \) is an UV cutoff such that \( J(\omega) = 0 \) for \( \omega \geq \omega_c \)). It has no poles in the physical sheet. The Bromwich contour that encircles the cut is used to find the inverse Laplace transform \[4\]:

\[ u(t) = 2 \int_0^{\omega_c} S(\omega) \sin(\omega t) d\omega \]

\[ S(\omega) = \frac{\Sigma_I(\omega)}{(\omega^2 - \omega_R^2 - \Sigma_R(\omega)) (\omega^2 - \omega_c^2) + \Sigma_I^2(\omega)} \]

The self-energy due to the bath is given by:

\[ \Sigma_R(\omega) + i \Sigma_I(\omega) = 2PP \int \frac{\omega' J(\omega')}{\omega^2 - \omega'^2} d\omega' + i \pi \text{sgn}(\omega) J(|\omega|). \quad (66) \]

In the weak coupling regime (\( \gamma \tau_L << 1 \)), a Breit-Wigner approximation can be performed since \( \Sigma_I(\omega_R) << \omega_R^2 \). Within this approximation, the \( u_I \) functions can be approximated by standard damped oscillator solutions. In fact, one could then imagine to use an effective Caldeira-Leggett model in order to estimate the decoherence properties of all Luttinger modes.

In the electromagnetic case, the decoherence process for the \( l > 1 \) modes takes place on a typical time scale of order \( \tau_l \sim \tau_L (c/\nu_S)^{2l-2} \) which is much longer than for the Ohmic \( l = 1 \) modes. In our particular problem, using asymptotics \[47\] electromagnetic dissipation for the dominant \( l = 1 \) modes is given by \( \gamma = g \tau_L^{-1}/3 \). An effective Caldeira-Leggett model for this mode can then be used. In the acoustic case, for a two dimensional phonon bath and within the relevant range of frequencies \( \omega_L < \omega < \omega_D \), the spectral density \( J_I(\omega) \) is ohmic for any \( l \), the Caldeira-Leggett model can be used to describe the long time dependence of decoherence.

Although the HPZ method can be used to perform explicit computations, an approximate master equation can often be used to obtain the time evolution of the density matrix of a small subsystem \( S \) coupled to a “reservoir” \( R \). The usual approximation consists first in assuming that the state of the reservoir is unaltered by the coupling to the system, and to forget correlations between \( S \) at time \( t' \leq t \) and \( R \) at time \( t \). The second approximation usually done consists in neglecting non-Markovian terms which can occur for example in the noise kernel (the dissipation kernel is always local in the CL model). The condition for these two approximations to be valid is \[43\]

\[ \tau^2 \cdot \text{Tr} \left( H_{SR}^2 \rho_{SR}(0) \right) << \hbar^2 \quad (67) \]

where \( H_{SR} \) denotes the coupling between the system and the reservoir, \( \rho_{SR}(0) \) the initial total density operator and \( \tau_c \) the correlation time of the reservoir. In the present context, this condition can be expressed as:

\[ \xi_l \int_0^{+\infty} J_l(x/\tau_L) \coth \left( \frac{\ell(\beta)}{2L} x \right) dx << \left( \frac{\tau_E}{\tau_c} \right)^2 \quad (68) \]

where \( \xi_l \) is a dimensionless coefficient that characterizes the spreading of the \( l \)-th mode initial state : \( \xi_l = 1 \) for any coherent state and \( \xi_l = \coth (\pi \hbar \nu_S / \beta / L) \) for thermal equilibrium.

Within the temperature range used here \( T/T_L \sim 0 \rightarrow 10 \) (\( T_L = 2\pi \hbar \nu_S / k_B L \)), we have \( \ell(\beta) >> L \) for the electromagnetic bath. Therefore, an estimate of the correlation time for the QEM field is given by the thermal time. High frequency asymptotics \[49\] show that temperature dependence has negligible effect on the l.h.s. of \( 68 \) (the effect could be important only for the \( l = 1 \) mode). Then, within our temperature range, the l.h.s is approximately bounded by \( g/l/a \). The r.h.s is bounded by \( L/a \) at high temperatures and \( (\nu_S/c) (T/T_L) \) at lower temperatures. Therefore, since \( g \approx 10^{-8} \), the validity condition is valid for the photon bath at temperatures above \( g (L/a) (c/\nu_S) \times T_L \). In the acoustic case, the l.h.s. temperature dependence is only relevant for values of \( x \) below \( (c S/\nu_S) \times (T/T_L) \). But the main contribution to the l.h.s comes from higher frequencies which is of order \( g_{ph}(L) (L/a)^{d} \approx g_{ph}(L/a) \). At low temperatures, the r.h.s can be bounded by \( (c S/\nu_S) (T/T_L) \) using \( \hbar \beta c_S \) as an upper value for \( \tau_c \). Condition \( 68 \) can therefore be rewritten as \( (L S/c S a) g_{ph}(L/a) << T/T_L \). Subsequent computations will assume that the acoustic UV cutoff is much higher than the typical Luttinger liquid frequency \( 2 \pi v_S / L \), i.e that \( L S/c S a >> 1 \). The master equation can therefore only be used for quite high temperatures.
In the case of an underdamped oscillator, as pointed out by C. Cohen-Tannoudji, the approximate Markovian description can be used provided the noise kernel is local compared to the observation time scale. In the present case, this means $\gamma << \omega_0$ and $k_BT >> \hbar\gamma$. In the limit of very low temperatures $k_BT << \hbar\gamma$, this description is still valid but one could expect very long time algebraic tails which precisely take into account non Markovian effects induced by the divergence of the bath’s correlation time $\hbar\beta$. This possibility is discussed in appendix where these corrections are computed in a weakly damped regime and shown to be extremely weak.

G. Long time decoherence (Markovian master equation treatment)

Within the secular approximation, the master equation can be translated in the following evolution equation for the characteristic function. In the case of a single oscillator of renormalized eigenfrequency $\Omega$ coupled to an Ohmic bath with dissipation rate $\gamma$: $Z_t(\lambda, \bar{\lambda}) = \text{Tr}(\rho_S(t) e^{i\lambda t} e^{\lambda \sigma})$ is given by:

$$
\left[ \frac{\partial}{\partial t} - \left( i\Omega - \frac{\gamma}{2} \right) \lambda \frac{\partial}{\partial \lambda} + \left( i\Omega + \frac{\gamma}{2} \right) \bar{\lambda} \frac{\partial}{\partial \bar{\lambda}} \right] Z_t(\lambda, \bar{\lambda}) = \frac{\gamma}{e^{\beta\hbar\Omega} - 1} Z_t(\lambda, \bar{\lambda})
$$

(69)

The general solution of this equation is given by:

$$
Z_t(\lambda, \bar{\lambda}) = Z_0 \left( \lambda e^{-\frac{\Omega}{2} - i\Omega t}, \bar{\lambda} e^{-\frac{\Omega}{2} - i\Omega t} \right) \exp \left( \frac{1 - e^{-\gamma t}}{e^{\beta\hbar\Omega} - 1} \lambda \bar{\lambda} \right)
$$

(70)

Such a formula immediately shows that, at zero temperature, coherent states remain coherent but their parameters evolve according to $\alpha(t) = \alpha(0) e^{-\frac{\gamma}{2} - i\Omega t}$. A coherent superposition of two coherent states decohere as follows:

$$
|\alpha\rangle\langle\beta| \rightarrow |\alpha(t)\rangle\langle\beta(t)| e^{\theta(t)} e^{-d(t)}
$$

(71)

$$
\theta(t) = (1 - e^{-\gamma t}) \Im(\alpha \bar{\beta})
$$

(72)

$$
d(t) = \frac{|\alpha - \beta|^2}{2} (1 - e^{-\gamma t})
$$

(73)

The decoherence time is therefore given by a very simple formula:

$$
\tau_{\text{Dec}}^{-1} = \gamma \cdot \frac{|\alpha - \beta|^2}{2}
$$

(74)

Using formula in the Caldeira-Leggett case provides the same result and the temperature dependence is given, in the limit $\gamma/\Omega_1 \rightarrow 0$ ($\Omega_1 = \sqrt{\Omega^2 - \gamma^2/4}$) by:

$$
D_T(t) = \frac{\Theta(T)(1 - e^{-\gamma t})}{1 + (\Theta(T) - 1)(1 - e^{-\gamma t})} \times 1 \quad \text{with} \quad \Theta(T) = \coth \left( \frac{\beta\hbar\Omega_1}{2} \right)
$$

(75)

Therefore, the $t \rightarrow +\infty$ limit of decoherence is independent of temperature but the decoherence time will scale with temperature according to the $\Theta(T)$ factor:

$$
\frac{\tau_{\text{Dec}}(T)}{\tau_{\text{Dec}}(0)} \approx \tanh \left( \frac{\hbar\Omega_1}{2k_BT} \right)
$$

(76)

H. Decoherence at long time

Applying previous results to the electromagnetic decoherence ($l = 1$ modes), we obtain:

$$
d_{\infty}^{(R/R)}(\sigma_1, \sigma_2) = 4\Delta_{n,m} \cdot \sin^2 \left( \frac{\pi \sigma_{12}}{L} \right)
$$

(77)

$$
d_{\infty}^{(L/R)}(\sigma_1, \sigma_2) = 2\Delta_{n,m} \cdot \left( 1 + \frac{m^2 - 4\pi^2 \sigma^2}{m^2 + 4\pi^2 \sigma^2} \cos \left( \frac{2\pi \sigma_{12}}{L} \right) \right)
$$

(78)

As expected, the $R/R$ limiting decoherence vanishes for $\sigma_1 = \sigma_2$ whereas the $R/L$ one does not. The typical asymptotic value is proportional to $\Delta_{n,m}$. In fact, this number can be viewed as measuring the “distance” between
the two quantum states which built our Schrödinger cat. Using vertex operator with small values of $m$ and $n$ in Schrödinger cats \[44 \text{ and } 46\] produces mesoscopically separated coherent states in each mode.

Figure 6 summarizes the electromagnetic decoherence exponent as a function of time for $\gamma t \simeq 0 - 5$ and for various temperatures.

In the acoustic case (two dimensional phonon bath), contributions of all relevant modes should be summed. As before, the $\sigma_{12}$ dependence disappears as soon as $\sigma_{12} > a v_S/c_S$. Introducing $T = 2\pi^2 \tau_L v_S/(c_S g_{ac}(a))$, the sum over all Luttinger modes up to the cut-off frequency can be evaluated:

\[
\frac{d^{(R/L)}(t)}{d^{(R/L)}(\infty)} = 1 + e^{-t/T} - 1 \left(\frac{t}{T}\right)
\]

where $d^{(R/L)}(\infty) = d^{(R/L)}(\infty) = \Delta_{n,m} c_S L/(v_S a)$. For the continuum approximation to be valid, we have assumed that $L/a > v_S/c_S$ and therefore $d^{(R/L)}(\infty) > 1$. This also implies that most of the decoherence process is accomplished within the previously computed acoustic decoherence time $2T/d^{(R/L)}(\infty) < T$.

V. CONCLUSION AND DISCUSSION

Within the bosonization framework, we have shown how the coupling to an external quantum electromagnetic field or to a two or three dimensional bath of longitudinal phonons can lead to decoherence of a Schrödinger cat state formed of localized elementary excitations in a Luttinger liquid. Three different phases build the decoherence scenario for Schrödinger cat states of the Luttinger liquid (see 7 for the electromagnetic case):

- At a very short time, because of the time/energy uncertainty relation, energy exchanges between the environment and the Luttinger system are not conservative. In this regime, high frequencies take part in the decoherence process and non-Markovian effects are important. The precise time evolution of the system is strongly cut-off dependent.

- After a transitory regime, the decoherence exponent becomes linear in time. This is the “Golden Rule” regime: energy conservation between the environment field and the Luttinger system is satisfied with a spectral width going down to its natural value (spontaneous photon or phonon emission). Only frequencies in resonance with the Luttinger eigenfrequencies contribute to dissipation.

The structure of spectral densities for the environmental modes leads to a hierarchy of decoherence times corresponding to the multipolar expansion of the “radiation” emitted by the Luttinger system. In the acoustic case, the various decoherence times do not increase as much with increasing $l$. In the electromagnetic case, they increase with $l$. The total decoherence time is therefore much smaller in the acoustic than in the electromagnetic case.

- At longer times, decoherence tends to saturation. In this regime, an effective Caldeira-Leggett model can be used to describe the dominating decoherence processes: in the electromagnetic case, one can keep only $l = 1$ modes, corresponding to dipolar electromagnetic radiation. In the 2D acoustic case, one can use an effective Caldeira-Leggett model for all modes. Caldeira-Leggett computations are non-perturbative since they take into account all orders of the coupling between the quantum environment and the oscillator.

The infinite time decoherence depends on two factors. The first one is, as expected, the distance between the two quantum states which depends on the dimension of the operators used in these states. The second one reflects the relative weight of each hydrodynamic mode in the decoherence process. In the electromagnetic case, since $l = 1$ modes dominate, it gives a geometrical factor depending on the relative position $\sigma_{12}$. In the acoustic case, spatial dependence is almost always lost and we are left with an important mode number factor. That’s why, although $\psi^{\dagger}_{R}(\sigma_{1})|0\rangle$ and $\psi^{\dagger}_{R}(\sigma_{1})|0\rangle$ can be considered as “mesoscopically close” with respect to their “distance”, decoherence is much faster than dissipation in the acoustic case. This is a major difference with single mode decoherence studies\[2\] where the ratio between decoherence and dissipation times is only due the distance between states entering the Schrödinger cat.

Non-linearities in the spectrum of low energy excitations may also contribute to decoherence. As showed by Haldane\[32\] and Scott, non-linearities in the spectrum couple the bosonic modes of the theory, turning the simple free model used in bosonization into an interacting theory. Coupling between modes should also play an important role in the decoherence properties of Schrödinger cat states. Indeed, the model presented here provides an upper limit for
decoherence times. Non-linearities should be also taken into account when investigating the resistive behavior of small metallic loops induced by inelastic collisions.\cite{48}

In the present work, photons or longitudinal phonons initially at equilibrium were used as the thermal bath for the system, but of course, one could imagine various extensions. One could use another description for quantum fluctuations of the environment. For example, one may think about changing the state of the environment, taking for example into account an external microwave radiation. Increasing the incoming radiation power within the range of resonant frequencies should increase decoherence of Schrödinger cat states (enhancement of dissipation by stimulated emission of radiation). With such environmental states, one expects to meet also the problem of “decoherence repartition” between all the modes of the Luttinger system (even in the electromagnetic case). Although this makes computations much harder to control, it may lead to more interesting behaviors. Finally, we are also investigating the extension of these ideas to two or three dimensional systems.

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APPENDIX A: KELDYSH’S GREEN’S FUNCTIONS

Let us recall the well known expressions for a single harmonic mode of frequency $\omega$ at temperature $\beta$ (unit mass)\cite{28}:

\[
G_>(t,s) = -\frac{i}{2\omega} \left( \coth \left( \frac{\beta \hbar \omega}{2} \right) \cos (\omega (t-s)) - i \sin (\omega (t-s)) \right) \tag{A1}
\]

\[
G_< (t,s) = -\frac{i}{2\omega} \left( \coth \left( \frac{\beta \hbar \omega}{2} \right) \cos (\omega (t-s)) + i \sin (\omega (t-s)) \right) \tag{A2}
\]

\[
G_T (t,s) = -\frac{i}{2\omega} \left( \coth \left( \frac{\beta \hbar \omega}{2} \right) \cos (\omega (t-s)) - i \sin (\omega |t-s|) \right) \tag{A3}
\]

\[
G_\tilde{T} (t,s) = -\frac{i}{2\omega} \left( \coth \left( \frac{\beta \hbar \omega}{2} \right) \cos (\omega (t-s)) + i \sin (\omega |t-s|) \right) \tag{A4}
\]

The retarded and advanced Green functions can be related to these expressions by

\[
G_R = G_T - G_< \quad \text{and} \quad G_A = G_T - G_> \tag{A6}
\]

For the electromagnetic field, a mode decomposition can be used. Using results of appendix\footnote{B}, we obtain:

\[
D^{\alpha \beta}((x,t);(y,s)) = \sum_I A_I^{\alpha}(x) A_I^{\beta}(y) G_{\omega_I} (t,s) \tag{A7}
\]

The noise and dissipation kernels\footnote{3} can be expressed as:

\[
\nu^{\alpha \beta} (x,y) = \sum_I A_I^{\alpha}(x) A_I^{\beta}(y) \coth \left( \frac{\beta \hbar \omega_I}{2} \right) \frac{\cos (\omega_I (t-s))}{2 \omega_I} \tag{A8}
\]

\[
\eta^{\alpha \beta} (x,y) = \sum_I A_I^{\alpha}(x) A_I^{\beta}(y) \frac{\sin (\omega_I (t-s))}{2 \omega_I} \tag{A9}
\]
APPENDIX B: ELECTROMAGNETIC MODES IN A CYLINDRICAL CAVITY

We shall decompose the transverse vector potential in orthonormal modes:

\[ \mathbf{A}(\mathbf{r},t) = \sum_I \psi_I(t) \mathbf{A}_I(\mathbf{r}) \]  

(B1)

These modes can be real, or complex. In the latter case, we assume the existence of an involution \( I \mapsto \hat{I} \) over the set of indices implementing complex conjugation: \( (\mathbf{A}_I(\mathbf{r}))^* = \mathbf{A}_{\hat{I}}(\mathbf{r}) \). Modes are normalized with respect of the volume \( V \) of the cavity by imposing an orthogonality condition:

\[ \int d^3r \; \mathbf{A}_I(\mathbf{r}) \cdot \mathbf{A}_J(\mathbf{r}) = \delta_{I,J}. \]

The \( \psi_I(t) \) are coordinates for a set of independent harmonic oscillators of frequency \( \omega_I \).

Expressions are given in cylindrical coordinates \( (r, \vartheta, z) \). \( R \) denotes the radius of the cavity. It is useful to introduce:

\[ \mathbf{A}(\mathbf{r}) = \mathbf{A}(\mathbf{r})/\sqrt{V}. \]

TE modes Let \( l \) be the angular momentum of the mode around the \( Oz \) axis and \( k_z \) its momentum along this axis. Let \( u'_{l,n} \) denote the \( n \)-th zero of the derivative of the \( l \)-th order Bessel function \( J_l \). Then, the orthoradial component of the TE \( (l, k_z, n) \) complex mode is given by:

\[ \mathbf{A}_\vartheta = - \frac{u'_{l,n}}{\sqrt{(u'_{l,n})^2 - l^2}} \frac{J'_l(\frac{\pi}{R} u'_{l,n})}{J_l(u'_{l,n})} e^{i(l \vartheta - k_z z)} \]  

(B2)

We have \( \omega = c \sqrt{k_z^2 + (u'_{l,n}/R)^2} \).

TM modes Here \( u_{l,n} \) denotes the \( n \)-th zero of the \( l \)-th order Bessel function \( J_l \). Then, the orthoradial component is given by:

\[ \mathbf{A}_\vartheta = - \frac{k_z R}{\sqrt{u_{l,n}^2 + (k_z R)^2}} \frac{l}{J_l(u_{l,n})} \frac{J_l(\frac{\pi}{R} u_{l,n})}{J'_l(u_{l,n})} e^{i(l \vartheta - k_z z)} \]  

(B3)

We have \( \omega = c \sqrt{k_z^2 + (u_{l,n}/R)^2} \).

Normalized real modes are inferred from these complex modes by:

\[ \mathbf{A}^{(+)}_I = \frac{1}{\sqrt{2}} (\mathbf{A}_I + \mathbf{A}_{\hat{I}}) \quad \text{and} \quad \mathbf{A}^{(-)}_I = \frac{i}{\sqrt{2}} (\mathbf{A}_I - \mathbf{A}_{\hat{I}}) \]  

(B4)

Expressions (B2) and (B3) show that:

\[ \begin{cases} 
\mathbf{A}^{(+)\vartheta}_I = \mathcal{D}_I \cos (l \vartheta - k_z z) \\
\mathbf{A}^{(-)\vartheta}_I = -\mathcal{D}_I \sin (l \vartheta - k_z z) 
\end{cases} \]  

(B5)

where \( \mathcal{D}_I \) contains the \( r \) dependence and depends on the mode characteristics \( TE/TM \) and \( (l, k_z, n) \). More precisely we have

\[ \begin{align*}
\text{TE modes} : \quad \mathcal{D}_I &= - \frac{u'_{l,n} \sqrt{2}}{\sqrt{(u'_{l,n})^2 - l^2}} \frac{J'_l(\frac{\pi}{R} u'_{l,n})}{J_l(u'_{l,n})} \\
\text{TM modes} : \quad \mathcal{D}_I &= \frac{R k_z \sqrt{2}}{\sqrt{u_{l,n}^2 + (k_z R)^2}} \frac{l}{u_{l,n}} \frac{J_l(\frac{\pi}{R} u_{l,n})}{J'_l(u_{l,n})}
\end{align*} \]  

(B6) (B7)
APPENDIX C: LONGITUDINAL ACOUSTIC MODES IN A CYLINDRICAL CAVITY

The displacement field \( u(x, t) \) contains a gradient part:

\[
\mathbf{u}(x, t) = \sum_I \frac{Q_I(t)}{\sqrt{V}} \nabla \phi_I(x)
\]

where the \( \phi_I \) functions are eigenvalues of the Laplacian with von Neumann’s boundary conditions. The modes \( \mathbf{u}_I = \nabla \phi_I \) are normalized so that

\[
\int d^3r \mathbf{u}_I(r) \cdot \mathbf{u}_J(r) = \delta_{I,J} V.
\]

The \( Q_I(t) \) are coordinates for a set of independent harmonic oscillators with given frequency \( \omega_I \). Longitudinal acoustic modes are indexed by \( I = (l, k, n) \) for \( d = 3 \) and \( I = (l, n) \) for \( d = 2 \) and are typically of the form \( e^{i l \theta - k_n z} N_I \) where

\[
\begin{align*}
\text{for } d = 2: & & N_I &= \frac{2 u'_{l,n}}{(\omega_I \tau)^{3/2}} \sqrt{2} \left( \frac{\omega_I \tau}{2\pi} \right)^{3/2} J_l(u'_{l,n}) \cos(z - l\pi/2 - \pi/4) \\
\text{for } d = 3: & & N_I &= \frac{u'_{l,n} \sqrt{2}}{(\omega_I \tau)^{3/2}} \sqrt{2} \left( \frac{\omega_I \tau}{2\pi} \right)^{3/2} J_l(u'_{l,n}) \cos(z - l\pi/2 - \pi/4)
\end{align*}
\]

APPENDIX D: ASYMPTOTIC BEHAVIOR OF SPECTRAL DENSITIES

Asymptotic expressions of the spectral densities corresponding to the infinite-cavity limit are useful to study the long time behavior of the decoherence and dissipation processes. The radial quantization is of \( \Delta \omega_\perp = \Delta \omega_{l,n}^2 c/R \simeq \pi c/R \) and the longitudinal one \( \Delta \omega_\parallel = 2\pi c/h \). We shall therefore use the following expressions for the extrema of the Bessel functions:

\[
J_l(u'_{l,n}) \simeq \sqrt{\frac{2}{\pi u'_{l,n}}} \quad \text{and} \quad J'_l(u_{l,n}) \simeq \sqrt{\frac{2}{\pi u_{l,n}}}
\]

and the following low- and high- frequency expansions

\[
J_l(z) \simeq \frac{z^l}{2^l l!} \quad \text{and} \quad J_l(z) \simeq \sqrt{\frac{2}{\pi z}} \cos(z - l\pi/2 - \pi/4)
\]

A straightforward algebra gives us:

1. In the low-frequency regime \( (\omega \tau_{EM} \ll 1) \)

\[
\begin{align*}
\mathcal{J}_{l \neq 0}^{TE}(\omega) & \simeq \frac{g}{\pi} \frac{l}{(2l - 1)!} \left( \frac{\omega \tau_{EM}}{2\pi} \right)^{2l-1} \\
\mathcal{J}_0^{TE}(\omega) & \simeq \frac{1}{3\pi} \frac{v_s}{c} \frac{\alpha_{QED}}{\alpha} \left( \frac{\omega \tau_{EM}}{2\pi} \right)^3 \\
\mathcal{J}_{l \neq 0}^{TM}(\omega) & \simeq \frac{g}{\pi} \frac{l^2}{(2l + 1)!} \left( \frac{\omega \tau_{EM}}{2\pi} \right)^{2l-1} \\
\mathcal{J}_0^{TM}(\omega) & = 0
\end{align*}
\]

Henceforth, the total spectral densities behave as:

\[
\begin{align*}
\mathcal{J}_{l \neq 0}(\omega) & \simeq \frac{g}{\pi} \frac{l^2}{(2l + 1)!} \left( \frac{\omega \tau_{EM}}{2\pi} \right)^{2l-1} \\
\mathcal{J}_0(\omega) & \simeq \frac{1}{3\pi} \frac{v_s}{c} \frac{\alpha_{QED}}{\alpha} \left( \frac{\omega \tau_{EM}}{2\pi} \right)^3
\end{align*}
\]
2. In the high-frequency regime \((\omega \tau_{EM} >> 1)\)

\[
\mathcal{J}_l^{TE}(\omega) \simeq \frac{g}{2\pi} \quad (D9)
\]

\[
\mathcal{J}_l^{TM}(\omega) \simeq \omega^{-1} \quad (D10)
\]

These analytic results agree with the numerics, which are represented in figure 3. The main result is that all the modes are supraohmic at low frequency, except the mode \(l = 1\) which shows ohmic behavior.

The same kind of expansion can be performed for the acoustic spectral densities. In this case, one obtains \((d \in \{2,3\})\):

- In the low frequency regime:

\[
\mathcal{J}_l(\omega) \propto g_{ph}(L) (\omega \tau_S)^{2l+d} \quad (D11)
\]

- In the high frequency regime:

\[
\mathcal{J}_l(\omega) \simeq g_{ph}(L) \frac{l}{2(l-d+2)\pi} (\omega \tau_S)^{d-1} \quad (D12)
\]

Only the latter will be used to compute decoherence properties since \(\omega_l\) falls into the high frequency regime since \(v_S >> c_S\).

**APPENDIX E: DECOHERENCE MATRIX COMPUTATIONS FOR A CALDEIRA-LEGGETT SOLUTION**

The appendix presents details of the computation of the decoherence matrix \(D\) in the case of a damped oscillator solution. This has direct relevance for the Caldeira-Leggett model but also within the framework of the Breit-Wigner approximation in more general environments. These results have been discussed in \(32,50\) but their derivation is recalled here in a simpler way. Strictly speaking the approximate master equation approach described in section IV F is not valid for temperatures below the \(\hbar \gamma/k_B\). Therefore, this appendix aims at finding ultra-low temperature corrections to decoherence arising from the non-Markovian effects arising from the \(T \to 0\) behavior of the reservoir symmetric two point correlation function.

Let \(\Omega_R\) and \(\gamma\) denote the renormalized frequency of the oscillator and \(\gamma\) the damping coefficient. We shall work in the weakly damped case, defined by \(\Omega^2_1 = \Omega^2_R - \gamma^2/4 \geq 0\), and measure the strength of dissipation by \(\phi\) such that \(\tan(\phi) = \gamma/2\Omega_R\). The decoherence matrix can then be expressed in terms of the following auxiliary functions:

\[
Z_{\pm}(t, \omega) = \int_0^t e^{-\gamma s/2+i(\omega \pm \Omega_1)s} \, ds, \quad (E1)
\]

\[
S(t) = \int_0^\infty (|Z_+|^2 + |Z_-|^2)(t, \omega) \mathcal{J}(\omega) \coth\left(\frac{\beta\hbar\omega}{2}\right) \, d\omega, \quad (E2)
\]

\[
P(t) = \int_0^\infty (Z_+ Z_-^*)(t, \omega) \mathcal{J}(\omega) \coth\left(\frac{\beta\hbar\omega}{2}\right) \, d\omega, \quad (E3)
\]

We have:

\[
D = \frac{1}{d} \begin{pmatrix}
D_{11} & D_{12} \\
D_{12} & D_{22}
\end{pmatrix}, \quad (E4)
\]

\[
d = 4 \left(1 + \frac{2e^{-\gamma t}}{\Omega_R(S^2 - 4PP)} \left(S\left(\frac{\Omega_R}{\Omega} + \frac{\Omega}{\Omega_R}\right) + 2 \Re\left(\frac{P e^{-2\Omega_1 t}}{\Omega} e^{-2i\phi} \left(\frac{\Omega_R}{\Omega} \Omega_R e^{-2i\phi} - \frac{\Omega R}{\Omega}\right)\right)\right) + \frac{16 \cos^2(\phi) e^{-2\gamma t}}{\Omega^2_R(S^2 - 4PP)} \right), \quad (E5)
\]
and the coefficients are given by:

\[ D_{11} = \Omega + \frac{2e^{-\gamma t}}{S^2 - 4PP}(S + 2 \Re(Pe^{-2i(\Omega_1 t + \phi)})), \]

\[ D_{12} = D_{21} = \frac{-2e^{-\gamma t}}{\Omega_R(S^2 - 4PP)}(S \sin(\phi) + 2 \Im(Pe^{-i(\Omega_1 t + \phi)})), \]

\[ D_{22} = \frac{1}{\Omega + \frac{2e^{-\gamma t}}{\Omega_R(S^2 - 4PP)}(S + 2 \Re(Pe^{-2i\Omega_1 t}))}. \]

The \( S \) and \( P \) functions can be computed by the residue theorem:

\[ S = \frac{1}{2} \int_{-\infty}^{\infty} \left( \frac{1 + e^{-\gamma t} - 2e^{-\gamma t/2}e^{i(\omega + \Omega_1 t)/2}}{(\omega + \Omega_1)^2 + \gamma^2/4} + \frac{1 + e^{-\gamma t} - 2e^{-\gamma t/2}e^{i(\omega - \Omega_1 t)/2}}{(\omega - \Omega_1)^2 + \gamma^2/4} \right) J(\omega) \coth \left( \frac{\beta \hbar \omega}{2} \right) d\omega, \quad (E10) \]

\[ P = \frac{1}{2} \int_{-\infty}^{\infty} \left( 1 + e^{-\gamma t}e^{2i\Omega_1 t} - 2e^{-\gamma t/2}e^{i\Omega_1 t}e^{i\omega t} \right) J(\omega) \coth \left( \frac{\beta \hbar \omega}{2} \right) d\omega. \quad (E11) \]

The main contribution to \( S \) and \( P \) is due to the poles \( \pm \Omega_1 + i\gamma/2 \):

\[ S^{(Main)} = \frac{2\pi}{\gamma} (1 - e^{-\gamma t}) \Re \left( J(\Omega_1 + i\gamma/2) \coth \left( \frac{\beta \hbar \Omega_1}{2} \right) \right), \quad (E12) \]

\[ P^{(Main)} = \frac{i\pi}{2} (1 - e^{-\gamma t}e^{2i\Omega_1 t}) \frac{J(\Omega_1 + i\gamma/2)}{\Omega_1 + i\gamma/2} \coth \left( \frac{\beta \hbar \Omega_1}{2} \right), \quad (E13) \]

In this expression cut-off dependent quantities have been discarded since they can be shown to be of order \((\gamma/2\Omega_1) \log(\Lambda/\Omega_R)\), i.e., much smaller than the oscillator eigenfrequency’s renormalization. The temperature dependent part also contain poles which give a series of exponentially decreasing terms of the form \(\exp(-2\pi n t/\tau_h)\) \((n \geq 1)\). At vanishing temperature, this series can be resummed into an algebraically decreasing correction as follows (here \(J(\omega) = \gamma/\Omega_1^2\)):

\[ S^{(Corr.)} = -\frac{2e^{-\gamma t/2}}{\pi \Omega_1^2} \Re(e^{i\Omega_1 t}(z_i S(z_i) + \bar{z}_i S(-\bar{z}_i)) + e^{-i\Omega_1 t}(z S(-z_i) + \bar{z}_i S(z_i))), \quad (E14) \]

\[ P^{(Corr.)} = -\frac{\phi e^{-\gamma t/2}e^{i\Omega_1 t}}{\pi \Omega_1} \left( S(z_i) + S(-z_i) - \overline{S(\bar{z}_i)} - \overline{S(-\bar{z}_i)} \right), \quad (E15) \]

where \(S(z) = e^z \text{Ei}(1, z)\) and \(z = (\gamma/2 - i\Omega_1) t\). \(S(z_i)\) has an asymptotic expansion in \(1/t\) for \(t \to +\infty\).

At zero temperature, these corrective terms \[E14\] and \[E15\] dominate \[E12\] and \[E13\] for \(\gamma t >> \log(2\Omega_1/\gamma)\). The very long time asymptotic of \(S\) is therefore given by:

\[ S \simeq \frac{2}{\Omega_1} \left( 1 + \frac{4\phi e^{-\gamma t/2} \cos(\Omega_1 t)}{\pi (\Omega_1 t)^2} \right), \quad (E16) \]

In case of weak damping \(\gamma << \Omega_R\), the following approximations can be made: \(\phi << 1, \Omega \simeq \Omega_R \simeq \Omega_1\) and \(f(\Omega_1 + i\gamma/2) \simeq f(\Omega_1)\). We will neglect \(P\) since, in full generality, \(P \simeq \phi S\). Plugging everything in \(D\)’s expression, one finally ends up with a very long time asymptotics:

\[ D(T = 0) \simeq \left( 1 - e^{-\gamma t} + \frac{4\phi}{\pi} e^{-3\gamma t/2} \cos(\Omega_1 t) \right) \left( \begin{array}{cc} \Omega_1/4 & 0 \\ 0 & 1/4\Omega_1 \end{array} \right), \quad (E17) \]

Henceforth, in the very weak damping situation, master equation results for the decoherence matrix can safely be extrapolated down to \(T = 0\), even if strictly speaking non-Markovian effects should be taken into account.

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1 P. Mohanty, E. Jariwala, and R. Webb, Phys. Rev. Lett. 77, 3366 (1997).
One would naturally think of $\omega_D$ as the Debye frequency but a linear dispersion relation for phonons has been assumed, an assumption which certainly fails near the Debye frequency. The $a$ length scale should rather be considered as small compared to the mesoscopic size of the system but large compared to the atomic length scale.
Figure 1: The Keldysh contour consisting in a forward time path (upper branch) and a backward time bath (lower branch)
Figure 2:
An annular 1D Luttinger liquid embedded in a cylindrical electromagnetic cavity.
Figure 3: Normalized electromagnetic spectral densities for the $l = 1$, $l = 2$ and $l = 3$ Luttinger modes (no cut off introduced here).
Figure 4: Time dependence of the decoherence exponent $d(t)$ and the renormalization coefficient $\eta(t)$ for zero modes.
Figure 5: Spatial dependence of the electromagnetic decoherence time. The two top curves correspond to $T = 1\;K$ whereas bottom ones correspond to $T = 10\;K$. Here $\theta = 2\pi \sigma_{12}/L$ and $k_B T_L = 2\pi v \hbar /L$. 

\[\text{Decoherence time} \; \tau_{\text{Dec}} (s)\] 

\[\text{Relative angle} \; \theta \; (\text{rad})\]
Figure 6: Effect of temperature in the $\gamma t \geq 1$ regimes.
Figure 7: Cross over between transitory regimes in decoherence: $\gamma t << 1$ regimes. We have also represented off-diagonal terms in the decoherence matrix (35).