GENERALIZED FIBONACCI POLYNOMIALS

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The Fibonacci polynomials and their relationship to diagonals of Pascal's triangle are generalized in this paper. The generalized Q-matrix investigated by Ivie [1] occurs as a special case.

1. THE FIBONACCI POLYNOMIALS

The Fibonacci polynomials, defined by

\begin{equation}
F_0(x) = 0, \quad F_1(x) = 1, \quad F_2(x) = x, \quad F_{n+2}(x) = xF_{n+1}(x) + F_n(x),
\end{equation}

are well known to readers of this journal. That the Fibonacci polynomials are generated by a matrix \(Q_2\),

\begin{equation}
Q_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_2^n = \begin{pmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix},
\end{equation}

can be verified quite easily by mathematical induction. Also, it is apparent that, when \(x = 1\), \(F_n(1) = F_n\), the \(n\)th Fibonacci number, and when \(x = 2\), \(F_n(2) = P_n\), the \(n\)th Pell number.

Further, when Pascal's triangle is written in left-justified form, the sums of the elements along the rising diagonals give rise to the Fibonacci numbers, and, in fact, those elements are the coefficients of the Fibonacci polynomials. That is,

\begin{equation}
F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n - j - 1}{j} x^{n-2j-1},
\end{equation}

where \([x]\) is the greatest integer contained in \(x\), and

\[ \binom{n}{j} \]

is a binomial coefficient

The first few Fibonacci polynomials are displayed below as well as the array of their coefficients.
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Fibonacci Polynomials

| n | Polynomial | Coefficient Array |
|---|------------|-------------------|
| 1 | $F_1(x) = 1$ | 1 |
| 2 | $F_2(x) = x$ | 1 |
| 3 | $F_3(x) = x^2 + 1$ | 1 1 |
| 4 | $F_4(x) = x^3 + 2x$ | 1 2 |
| 5 | $F_5(x) = x^4 + 3x^2 + 1$ | 1 3 1 |
| 6 | $F_6(x) = x^5 + 4x^3 + 3x$ | 1 4 3 |
| 7 | $F_7(x) = x^6 + 5x^4 + 6x^2 + 1$ | 1 5 6 1 |
| 8 | $F_8(x) = x^7 + 6x^5 + 10x^3 + 4x$ | 1 6 10 4 |
| ... | ... | ... |

If one observes that, by rule of formation of the Fibonacci polynomials, if one writes the polynomials in descending order, to form the coefficient of the $k^{th}$ term of $F_n(x)$, one adds the coefficients of the $k^{th}$ term of $F_{n-1}(x)$ and the $(k - 1)^{st}$ term of $F_{n-2}(x)$, the array of coefficients formed has the same rule of formation as Pascal’s triangle when it is written in left-justified form, except that each column is moved one line lower, so that the coefficients formed are those elements that appear along the diagonals formed by beginning in the left-most column and preceding up one and right one throughout the left-justified Pascal triangle. Throughout this paper, this diagonal will be called the rising diagonal of such an array.

2. THE TRIBONACCI POLYNOMIALS

Define the Tribonacci polynomials by

$$T_{-1}(x) = T_0(x) = 0, \quad T_1(x) = 1, \quad T_2(x) = x^2,$$

(2.1) $$T_{n+3}(x) = x^3T_{n+2}(x) + xT_{n+1}(x) + T_n(x).$$

When $x = 1$, $T_n(1) = T_n$, the $n^{th}$ Tribonacci number 1, 1, 2, 4, 7, 13, 24, 44, 81, ···, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$. The first few Tribonacci polynomials follow.

Tribonacci Polynomials

| n | Polynomial |
|---|------------|
| 1 | $T_1(x) = 1$ |
| 2 | $T_2(x) = x^2$ |
| 3 | $T_3(x) = x^4 + x$ |
| 4 | $T_4(x) = x^6 + 2x^3 + 1$ |
| 5 | $T_5(x) = x^8 + 3x^5 + 3x^2$ |
| 6 | $T_6(x) = x^{10} + 4x^7 + 6x^4 + 2x$ |
| 7 | $T_7(x) = x^{12} + 5x^9 + 10x^6 + 7x^3 + 1$ |
| 8 | $T_8(x) = x^{14} + 6x^{11} + 15x^8 + 16x^5 + 6x^2$ |
Tribonacci Coefficient Array

1
1
1 1
1 2 1
1 3 3
1 4 6 2
1 5 10 7 1
1 6 15 16 6

... ... ... ...

Left-Justified Trinomial Coefficient Array

\[(1 + x + x^2)^n, \quad n = 0, 1, 2, \ldots\]

1
1 1
1 2 3 2 1
1 3 6 7 6 3 1
1 4 10 16 19 16 10 4 1
1 5 15 30 45 51 45 30 15 5 1

... ... ... ... ... ... ... ...

The Tribonacci coefficient array has the same rule of formation as the trinomial coefficient array, except that each column is placed one line lower. Thus, the sums of the rows are the same as the sums of the rising diagonals of the trinomial coefficient array, both sums yielding the Tribonacci numbers 1, 1, 2, 4, 7, 13, 24, \ldots, and the coefficients of the Tribonacci polynomials are the trinomial coefficients found on those same rising diagonals. That is,

\[
T_n(x) = \sum_{j=0}^{n} \binom{n-j-1}{j} x^{2n-3j-2},
\]

where

\[
\binom{n}{j}_3
\]

is the trinomial coefficient in the \(n^{\text{th}}\) row and \(j^{\text{th}}\) column where, as is usual, the left-most column is the zero \(0^{\text{th}}\) column and the top row the zero \(0^{\text{th}}\) row, and

\[
\binom{n}{j}_3 = 0 \quad \text{if} \quad j > n.
\]

The Tribonacci polynomials are generated by the matrix \(Q_3\),
so that

\[
Q_3^n = \begin{pmatrix}
T_{n+1}(x) & T_n(x) & T_{n-1}(x) \\
xT_{n}(x) + T_{n-1}(x) & xT_{n-1}(x) + T_{n-2}(x) & xT_{n-2}(x) + T_{n-3}(x) \\
T_n(x) & T_{n-1}(x) & T_{n-2}(x)
\end{pmatrix}.
\]

A proof could be made by mathematical induction. That \(Q_3^n\) has the given form for \(n = 1\) is apparent by inspection of \(Q_3\), element-by-element. Expansion of the matrix product \(Q_3^{n+1} = Q_3 Q_3^n\) gives the elements of \(Q_3^{n+1}\) in the required form, making use of the recursion (2.1).

Notice that \(\det Q_3^n = 1^n = 1\), analogous to the Fibonacci case. In fact, we can write an interesting determinant identity. Again using (2.1), we multiply row one of \(Q_3^n\) by \(x^2\) and add to row 2. Then we exchange rows 1 and 2 to write

\[
(-1)^n = \begin{vmatrix}
T_{n+2}(x) & T_{n+1}(x) & T_n(x) \\
T_{n+1}(x) & T_n(x) & T_{n-1}(x) \\
T_n(x) & T_{n-1}(x) & T_{n-2}(x)
\end{vmatrix}
\]

which becomes an identity for Tribonacci numbers when \(x = 1\).

3. THE QUADRANACCI POLYNOMIALS

The Quadranacci polynomials are defined by \(T^*_n(x) = T^*_n(x) = T^*_0(x) = 0, T^*_1(x) = 1, \)

\[
T^*_n(x) = x^3 T^*_n(x) + x^2 T^*_n+3(x) + x T^*_n+2(x) + T^*_n(x).
\]

The first few values are

\[
T_1(x) = 1 \\
T_2(x) = x^3 \\
T_3(x) = x^6 + x^2 \\
T_4(x) = x^9 + 2x^5 + x \\
T_5(x) = x^{12} + 3x^8 + 3x^4 + 1 \\
T_6(x) = x^{15} + 4x^{11} + 6x^7 + 4x^3 \\
T_7(x) = x^{18} + 5x^{14} + 10x^{10} + 10x^6 + 3x^2
\]

... ...
Notice that the coefficient of the \( j \)th term of \( T_n^*(x) \) is the sum of the coefficients of the \( j \)th term of \( T_{n-1}^*(x) \), \((j - 1)\)st term of \( T_{n-2}^*(x) \), \((j - 2)\)nd term of \( T_{n-3}^*(x) \), and \((j - 3)\)rd term of \( T_{n-4}^*(x) \) when the polynomials are arranged in descending order. Then, the array of coefficients, if each row were moved up one line, would have the same rule of formation as the left-justified array of quadranomial coefficients, arising from expansions of \((1 + x + x^2 + x^3)^n\), \( n = 0, 1, 2, \ldots \). Thus, the coefficients of \( T_n^*(x) \) are those found on the \( n \)th rising diagonal of the quadranomial triangle. Also, \( T_n^*(1) = T_n^* \), the \( n \)th Quadranacci number 1, 1, 2, 4, 8, 15, 29, 56, \( \ldots \), \( T_n^* = T_{n+3} + T_{n+2} + T_{n+1} + T_n^* \).

The Quadranacci polynomials are generated by the matrix \( Q_4 \),

\[
Q_4 = \begin{pmatrix}
x^3 & 1 & 0 & 0 \\
x^2 & 0 & 1 & 0 \\
x & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

so that \( Q_4^n = (a_{ij}) \) has its \( j \)th column given by \( a_{ij} = T_{n+2-j}^*(x) \), \( a_{2j} = x^2T_{n+1-j}^*(x) + xT_{n-j}^*(x) + T_{n-j}^*(x) \), \( a_{3j} = xT_{n+1-j}^*(x) + T_{n-j}^*(x) \), and \( a_{4j} = T_{n+1-j}^*(x) \), \( j = 1, 2, 3, 4 \).

That \( Q_4^n \) has the form claimed above can be established by mathematical induction. That \( Q_4 \) has the stated form follows by inspection. Let \( Q_4^{n+1} = (b_{ij}) \) and \( Q_4 = (a_{ij}) \). Then we expand \( Q_4^{n+1} = Q_4 Q_4^n \). The first row of \( Q_4^{n+1} \) has the required form, for

\[
b_{1j} = q_{11}a_{1j} + q_{12}a_{2j} + q_{13}a_{3j} + q_{14}a_{4j} = \left[ x^3T_{n+2-j}^*(x) \right] + \left[ x^2T_{n+1-j}^*(x) + xT_{n-j}^*(x) + T_{n-j}^*(x) \right] + 0 + 0 = T_{(n+1)+2-j}^*
\]

where we make use of (3.1). Computation of \( b_{2j} \) and \( b_{3j} \) is similar, and shows that \( Q_4^{n+1} \) has the required form, which would complete the proof.

We derive a determinant identity for Quadranacci polynomials from \( Q_4^n \) by forming the matrix \( Q_4^* \) as follows. Add \( x^3 \) times row 1 to row 2, making \( a_{2j} = T_{n+3-j}^*(x) \). Add \( x^2 \) times row 1 and \( x^3 \) times row 2 to row 3, producing \( a_{3j} = T_{n+4-j}^*(x) \). Exchange rows 1 and 3. Then matrix \( Q_4^* \) has

\[
a_{1j}^* = T_{n+4-j}^*(x), \quad a_{2j}^* = T_{n+3-j}^*(x), \quad a_{3j}^* = T_{n+2-j}^*(x),
\]

and

\[
a_{4j}^* = T_{n+1-j}^*(x), \quad j = 1, 2, 3, 4, \quad \text{and} \quad \det Q_4^n = (-1)^{n+1}
\]

because there was one row exchange. That is, for example, when \( x = 1 \),
\[ (-1)^{n+1} = \begin{vmatrix} T^*_n & T^*_n+2 & T^*_n+1 & T^*_n \\ T^*_n & T^*_n+1 & T^*_n & T^*_n-2 \\ T^*_n+1 & T^*_n & T^*_n-1 & T^*_n-3 \\ T^n & T^*_n-1 & T^*_n-2 & T^*_n-3 \end{vmatrix} \]

where \( T^*_n \) is the \( n^{th} \) Quordanacci number.

4. THE R-BONACCI POLYNOMIALS

Define the \( r \)-bonacci polynomials by

\[ R_{-(r-2)}(x) = R_{-(r-1)}(x) = \ldots = R_1(x) = x^r - 1, \quad R_0(x) = 0, \quad R_1(x) = 1, \quad R_2(x) = x^r - 1, \]

\[ R_{n+r}(x) = x^{r-1}R_{n+r-1}(x) + x^{r-2}R_{n+r-2}(x) + \ldots + R_n(x). \]

The \( r \)-bonacci polynomials, by their recursive definition will have the coefficients of \( R_n(x) \), written in descending order, given by the coefficients on the \( n^{th} \) rising diagonal of the left-justified \( r \)-nomial coefficient array, the coefficients arising from expansions of

\[ (1 + x + x^2 + \ldots + x^{r-1})^n, \quad n = 0, 1, 2, \ldots. \]

That is,

\[ R_n(x) = \sum_{j=0}^{n} \binom{n-j-1}{j} x^{(r-1)(n-1)-r}, \]

where

\[ \binom{n}{j}_r \]

is the element in the \( n^{th} \) row and \( j^{th} \) column of the left-justified \( r \)-nomial triangle, and

\[ \binom{n}{j}_r = 0 \quad \text{when} \quad j > n. \]

The \( r \)-bonacci polynomials are generated by the \( r \times r \) matrix \( Q_r \),

\[
Q_r = \begin{pmatrix}
x^{r-1} & 1 & 0 & 0 & \ldots & 0 \\
x^{r-2} & 0 & 1 & 0 & \ldots & 0 \\
x^{r-3} & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x & 0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]
which is an identity matrix of order \((r - 1)\) bordered on the left by a column of descending powers of \(x\) and followed by a bottom row of zeros. The matrix \(Q_r^n\) has \(R_{n+1}(x)\) as the element in the upper left and \(R_n(x)\) in the lower left, with general element \(a_{ij}\) given by

\[
a_{ij} = \sum_{k=1}^{r-1+1} x^{r+1-k+i} R_{n+1-j-k}(x) .
\]

Proof of (4.3) is by mathematical induction. Let \(Q_r = (b_{ij})\). Then \(b_{ii} = x^{r-i}\), \(i = 1, 2, \cdots, r; b_{ij} = 1, j = i + 1, i = 1, 2, \cdots, r; \) and \(b_{ij} = 0\) whenever \(j \neq 1\) and \(j \neq i + 1\). Let

\[
Q_r^{n+1} = Q_r Q_r^n = (c_{ij}) .
\]

Then

\[
c_{ij} = \sum_{k=1}^{r} b_{ik} a_{kj} = b_{11} a_{1k} + \sum_{k=2}^{r} b_{ik} a_{kj}
\]

\[
= x^{r-i} R_{n+1-j}(x) + a_{1+1,j} + 0
\]

\[
= x^{r-i} R_{n+1-j}(x) + \sum_{k=1}^{r-i} x^{r-k+i} R_{n+1-j-k}(x)
\]

\[
= \sum_{k=1}^{r-i+1} x^{r+1-k+i} R_{(n+1)+1-j-k}(x) ,
\]

which is the required form for the general element of \(Q_r^{n+1}\), completing the proof.

If we operate upon \(Q_r^n\) as before, we can again make a determinant identity. Repeatedly add \(x^{r-1}\) times the \((i - 1)^{th}\) row, \(x^{r-2}\) times the \((i - 2)^{nd}\) row, \(\cdots\), \(x^{r+1-i}\) times the first row to the \(i^{th}\) row, to produce a new \(i^{th}\) row with \(R_{n+i-1}\) in its first column, for \(i = 2, 3, \cdots, r - 1\). Then make \((r - 1)/2\) row exchanges to put the elements in the columns in descending order. The matrix \(R\) formed has its general element given by

\[
r_{ij} = R_{n+r+1-i-j}(x) ,
\]

and its determinant has value \((-1)^{(r-1)n+[r-1]/2}\). That is, when \(x = 1\), the \(r\)-bonacci numbers

\[
\cdots, 0, 1, 1, 2, \cdots, R_{n+r} = R_{n+r-1} + R_{n+r-2} + \cdots + R_n ,
\]

have the determinant identity
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\[
\begin{vmatrix}
R_{n+r-1} & R_{n+r-2} & \cdots & R_{n+1} & R_n \\
R_{n+r-2} & R_{n+r-3} & \cdots & R_n & R_{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
R_{n+1} & R_n & \cdots & R_{n-r+3} & R_{n-r+2} \\
R_n & R_{n-1} & \cdots & R_{n-r+2} & R_{n-r+1}
\end{vmatrix} = (-1)^{(r-1)(2n+1)/2}
\]

Notice that Eq. (1.2) gives
\[
\det Q^n_r = x^{n+1}F_{n+1}(x)F_{n-1}(x) - F_n^2(x) = (-1)^n = \det R
\]
for \( r = 2 \). Since we recognize
\[
\begin{vmatrix}
F_{n+1}(x)F_{n-1}(x) - F_n^2(x)
\end{vmatrix} = 1
\]
as the characteristic value [2], [3], [4] of sequences arising from the Fibonacci polynomials, we define \( \det R \) as the characteristic value of the sequences arising from the generalized Fibonacci polynomials, \( r \geq 2 \). Then, for example, (2.4) gives the characteristic value \( |(-1)| = 1 \) for the sequences arising from the Tribonacci polynomials, while (3.2) is the array giving the characteristic value \( |(-1)^{n+1}| = 1 \) for the Quadranacci numbers.

The matrix \( R \) just defined has the interesting property that multiplication by \( Q_n^r \) produces a matrix of the same form. To clarify, let \( R = R_{r,n} = (r_{ij}) \) be the \( r \times r \) matrix with \( R_n(x) \) appearing in the lower left corner, \( r_{ij} = R_{n+r+1-i-j}(x) \). Then

\[(4.4) \quad R_{r,0}Q^n_r = R_{r,n} \]

which is proved by mathematical induction as follows. Consider the matrix product \( R_{r,n}Q_r \) = \( (p_{ij}) \) for any \( n \), where we observe that the first column of \( Q_r \) contains the multipliers for the recursion relation for the polynomials \( R_n(x) \). The \( i \)th row of \( R_{r,n} \) multiplied by the first column of \( Q_r \) produces

\[p_{i1} = \sum_{k=1}^r R_{n+r+1-i-k}(x)x^{r+1-k} = R_{n+r+1-i}(x) = R_{(n+1)+r+1-i-1}(x),\]

while, when \( j \neq 1 \), since the only non-zero elements of \( Q_r \) occur when \( i = j - 1 \), the \( i \)th row of \( R_{r,n} \) times the \( j \)th column of \( Q_r \) produces

\[p_{ij} = R_{n+r+1-i-j-1} = R_{(n+1)+r+1-i-j}, \quad j = 2, 3, \ldots, r,\]

so that \( R_{r,n}Q_r = R_{r,n+1} \) for any \( n \). Then, we must have that \( R_{n,0}Q_r = R_{r,1} \), and if

\[R_{r,0}Q_r^{k-1} = R_{r,k-1},\]

then

\[R_{r,0}Q_r^k = (R_{r,0}Q_r^{k-1})Q_r = R_{r,k-1}Q_r = R_{r,k}.\]
which completes a proof of (4.4) by mathematical induction.

If we equate the elements in the upper left corner of $R_{r,n}$ and $R_{r,0}Q^n_r$ we obtain the identity

$$R_{n+r-1}(x) = R_{r-1}(x)R_{n+1}(x) + R_{r-2}(x)[x^{r-2}R_n(x) + x^{r-3}R_{n-1}(x) + \ldots + R_{n-r+2}(x)] + R_{r-3}(x)[x^{r-3}R_n(x) + x^{r-4}R_{n-1}(x) + \ldots + R_{n-r+3}(x)] + \ldots + R_1(x)[xR_n(x) + R_{n-1}(x)] + Q_n(x)R_n(x).$$

(4.5)

Notice that the matrix $Q^n_r$ provides the multipliers for the recursion relation for the polynomials $R_n(x)$ but does not depend upon the original values of the polynomials in the proof of (4.4). Let $H_n(x)$ be any sequence of polynomials with $r$ arbitrary starting values $H_0(x), H_1(x), \ldots, H_{r-1}(x)$, and with the same recursion relation as the polynomials $R_n(x)$. Form the matrix $R^*_r = (r^*_{ij}), r^*_0 = H_{n+r+1-i-j}(x)$. By the arguments used earlier, we can derive $R^*_r, 0 Q^n_r = R^*_r, n$ and thus obtain

$$H_{n+r-1} = H_{r-1}(x)R_{n+1}(x) + H_{r-2}(x)[x^{r-2}R_n(x) + x^{r-3}R_{n-1}(x) + \ldots + R_{n-r+2}(x)] + H_{r-3}(x)[x^{r-3}R_n(x) + x^{r-4}R_{n-1}(x) + \ldots + R_{n-r+3}(x)] + \ldots + H_1(x)[xR_n(x) + R_{n-1}(x)] + Q_n(x)R_n(x).$$

(4.6)

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