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A gain-scheduled observer under transmissions without delivery acknowledgment

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Abstract. This paper addresses the estimation problem for discrete-time systems where both measurements and control commands are sent to a central station through a lossy network without delivery acknowledgment. The central unit implements the estimation and control algorithms. We propose a jump observer that uses the expected value of the unknown control input at the actuator to run an open loop estimation. Then, the absence of acknowledgment in the control input transmission is dealt with through the introduction of an unknown disturbance. The observer update is performed by means of jumping gains when there are available measurements. We employ an statistic of the control error (new disturbance), i.e., the difference between the control inputs at the plant and at the observer, to schedule the observer gains in real time. The observer is designed to minimize the $H_\infty$ norm from disturbances, measurement noises and control input errors, to estimation error. The infinite-dimensional design problem is turn into an optimization problem over polynomials using sum-of-squares decomposition techniques. Benefits of the proposal are shown in a simulation example.

1. Introduction
In networked control systems (NCS) the different elements of the control architecture are not collocated. The transfer of information (control inputs and measurements) between them is performed through a communication network, which increases flexibility and maintenance [1]. However, networks offer an unreliable communication link subject, e.g., to dropouts and time delays [2, 3].

Focusing on the dropout case, communication protocols can be divided in two groups: 1) TCP-like, which implement a successful delivery acknowledgment and 2) UDP-like, where there is no reception acknowledgment. Clearly, this last protocol alleviates the network data traffic at the expense of introducing some new challenging issues.

Dealing with packet losses in the measurement channel (from sensors to controller/observer) does not depend on the use of delivery acknowledgment. Using jump observers relating their jumping modes to the measurement availability leads to better performances than implementing gain invariant approaches [4, 5]. These approaches may obtain similar performances than the ones based on Kalman filters [6] with less computational effort, and can handle a larger class of disturbances.

However, the absence of a delivery acknowledgment is a critical issue when affecting control input transmission. In [7,8] the authors showed that under UDP-like transmission the separation principle does not hold when designing LQG controllers since the estimation algorithm depends
on the control input reception, which is unknown. Several works have studied the jointly design of estimators and controller under this paradigm, e.g. [9–11]. Another approach to solve this problem is the one studied in [12, 13] where the authors proposed to estimate the transmission fate so the system fulfills the separation principle as in TCP-like communication.

In the present work, we introduce a different approach to tackle this problem in the remote state estimation problem under dropouts in UDP-like networks. We use the expected value of the received control input at the plant over a finite time window to run the open loop estimation. Then, we model the absence of control reception acknowledgment as the presence of a new unknown disturbance, which we call control error, that affects the estimation algorithm. With that, we obtain a control error statistic, calculated in real time, that can be described by a bounded time-varying parameter.

Some works as [14] started developing this idea but with invariant gains, which leads to conservative estimation performances. Here, we propose a gain-scheduled jump observer to estimate the system states. The observer gain jumping modes are related to the measurement reception scenario, modeled as a Markov chain, and depend on the control error statistic. The most significant contribution of this paper is the design of an observer whose gains are scheduled with the unknown disturbance introduced by the control input losses, i.e., with the control error statistic. Moreover, the presented results could allow to apply the separation principle when closing the loop if the controller is designed to satisfy the considered bounds on the control error statistic.

The gain-scheduled jump design is addressed by minimizing the $H_\infty$ norm from external inputs to estimation error. To handle this optimization procedure, we impose the gain-scheduling function to be polynomial so we can exploit sum-of-squares (SOS) decomposition techniques (see [15]). Similar to [16] the conceptual originality introduced with respect previous works using SOS methods (e.g. [17,18]) resides in scheduling the observer with a time-varying parameter that depends on the behavior of the network.

**Notation**: Let $\mathbb{R}$ and $\mathbb{N}$ denote the real and positive integer numbers set. Let $A$ and $B$ be some matrices. $A \preceq 0$ means that matrix $A$ is negative semidefinite. Similar applies to $\prec, \succ$ and $\succeq$. The direct sum is represented as $\oplus$, where $A \oplus B$ is a block diagonal matrix with $A$ and $B$ on its diagonal. Let $x_k \in \mathbb{R}^n$ be a stochastic process. $\mathbb{E}\{ \cdot \}$ and $\Pr\{ \cdot \}$ denotes expectation and probability. We denote the RMS norm of process $x_k$ by $\|x\|_{\text{RMS}} = \lim_{K \to \infty} \sqrt{\frac{1}{K} \sum_{k=0}^{K-1} x_k^T x_k}$.

2. Problem setup

We consider the remote state estimation problem of a linear time-invariant discrete-time system where measurements and control inputs are transmitted through a communication network without successful delivery acknowledgment of received packets (e.g. UDP-like networks) subject to dropouts. We describe the system in its state-space form as

\[
\begin{align*}
x_{k+1} &= A x_k + B_u u_k + B_w w_k, \\
y_k &= C x_k + v_k
\end{align*}
\tag{1a}
\tag{1b}
\]

where $x_k \in \mathbb{R}^n$ is the state at instant $k \in \mathbb{N}$, $u_k \in \mathbb{R}^{n_u}$ is the applied control input, $w_k \in \mathbb{R}^{n_w}$ is the state disturbance, $y_k \in \mathbb{R}^{n_y}$ is the measured output and $v_k \in \mathbb{R}^{n_v}$ is the measurement noise. We assume that both state disturbances and measurement noise are wide-sense stationary stochastic processes\footnote{If $x_k$ is wide sense stationary its RMS norm becomes $\|x\|_{\text{RMS}} = \sqrt{\mathbb{E}\{x_k^4\}}$.} with bounded variances. Then their RMS norms are such as

\[
\|w\|_{\text{RMS}} \leq \bar{w}_{\text{rms}}, \quad \|v\|_{\text{RMS}} \leq \bar{v}_{\text{rms}}.
\]

Let us also assume that the sent control input at each instant is known as the observer and controller are collocated in the central station.
2.1. Measurement reception description
Each sensor $s$ samples at instant $k$ its output, synchronously with the control input update, and sends independently to the central station a time-tagged message with the correspondent measurement $y_{s,k}$. We characterize the individual measurement reception from sensor $s$ at instant $k$ with

$$\alpha_{s,k} = \begin{cases} 
1 & \text{if } y_{s,k} \text{ is acquired at instant } k, \\
0 & \text{otherwise}.
\end{cases}$$ (2)

Aggregating the above binary variables such as $\alpha_k = \bigoplus_{k=1}^{nm} \alpha_{s,k}$, we model the full measurement reception scenario at instant $k$. Let us consider that $\alpha_k$ follows a finite Markov chain having its states in the set $\alpha_k \in \Xi = \{\eta_0, \eta_1, \ldots, \eta_r\}$, $r = 2^{nm} - 1$, (3)

where $\eta_0$ represents the case when $\alpha_k = 0$. We assume that the transition probability matrix $\Lambda = [p_{i,j}]$ with

$$p_{i,j} = \Pr\{\alpha_{k+1} = \eta_j | \alpha_k = \eta_i\}$$

is known.

**Remark 1.** Considering mutually independent Markovian dropouts (see [19]), i.e.,

$$\Pr\{\alpha_{s,k} = 0 | \alpha_{s,k-1} = 0\} = q_s^y, \quad \Pr\{\alpha_{s,k} = 1 | \alpha_{s,k-1} = 0\} = 1 - q_s,$$

$$\Pr\{\alpha_{s,k} = 1 | \alpha_{s,k-1} = 1\} = p_s^y, \quad \Pr\{\alpha_{s,k} = 0 | \alpha_{s,k-1} = 1\} = 1 - p_s,$$

for all $s = 1, \ldots, n_y$, the transition probabilities of $\alpha_k$ are calculated as

$$p_{i,j} = \prod_{s=1}^{nm} \Pr\{\alpha_{s,k} = \eta_{s,j} | \alpha_{s,k-1} = \eta_{s,i}\}$$

where $\eta_{s,i}$ is the $s$-th diagonal element of $\eta_i$.

2.2. Control update description
The controller transmits (through the network) at each instant $k - 1$ a single packet that aggregates all the control commands to be used at instant $k$ (see [20]). Let us denote by $u^t_k$ the known sent control input at $k - 1$ to be applied at $k$. We describe the control input reception at instant $k - 1$ with

$$\theta_{k-1} = \begin{cases} 
1 & \text{if } u^t_k \text{ is received at instant } k - 1, \\
0 & \text{otherwise}.
\end{cases}$$ (4)

We consider that the actuators implement a zero order hold strategy, i.e., the last received control input is applied if no new one is acquired.

Let us assume that we know the probability of employing at instant $k$ the control input sent $\tau + 1$ instants before, i.e.

$$\varphi_\tau = P\{u_k = u^t_{k-\tau}\}, \quad \tau = 0, \ldots, N^u, \quad \sum_{\tau=0}^{N^u} \varphi_\tau = 1,$$ (5)

where $N^u$ is the maximum number of consecutive control packet dropouts.
Remark 2. Considering again mutually independent Markovian dropouts characterized by
\[ \Pr\{\theta_k = 0|\theta_{k-1} = 0\} = q^u, \quad \Pr\{\theta_k = 1|\theta_{k-1} = 0\} = 1 - q^u, \]
\[ \Pr\{\theta_k = 1|\theta_{k-1} = 1\} = p^u, \quad \Pr\{\theta_k = 0|\theta_{k-1} = 1\} = 1 - p^u. \]
and assuming that the control link can assure that the maximum number of consecutive dropouts is below \( N^u \), the probabilities (5) can be computed as
\[ \varphi_0 = \frac{1}{\xi} \pi^u_1, \quad \varphi_{\tau > 0} = \frac{1}{\xi} (q^u)^{\tau - 1} (1 - p^u) \pi^u_1 \]
where \( \pi^u_1 = \Pr\{\theta_k = 1\} \),
\[ \xi = \pi^u_1 + \sum_{\tau = 1}^{N^u} (q^u)^{\tau - 1} (1 - p^u) \pi^u_1, \quad \pi^u = \pi^u \Lambda^u, \]
\[ \pi^u = [\pi^u_0 \quad \pi^u_1] \] (being \( \pi^u_0 = \Pr\{\theta_k = 0\} \)) and \( \Lambda^u \) is the associated transition probability matrix of \( \theta_k \).

Since the current control input value acting on the plant is unknown, we propose employing its expectation \( \mathbb{E}\{u_k\} \) to run the open loop estimation. We denote \( \mathbb{E}\{u_k\} \) by \( u^c_k \) where
\[ u^c_k = \sum_{d=0}^{N^u} \varphi_d u^l_{k-d}. \] (6)

The next lemma defines some statistics of the involved control error \( \tilde{u}_k = u_k - u^c_k \) (between the applied control input in the plant and the one assumed in the observer).

Lemma 1. The control error \( \tilde{u}_k \) has a zero expected value and
\[ \mathbb{E}\{\tilde{u}_k^T \tilde{u}_k\} = \sum_{d=0}^{N^u} \varphi_d \left( u^l_{k-d} - \sum_{d=0}^{N^u} \varphi_d u^l_{k-d} \right)^T \left( u^l_{k-d} - \sum_{d=0}^{N^u} \varphi_d u^l_{k-d} \right). \] (7)
Proof. First, the expected value of \( \tilde{u}_k \) is zero by definition of \( u^c_k \), see (6). Second, using the total probability law we obtain (7). □

Let us denote \( \mathbb{E}\{\tilde{u}_k^T \tilde{u}_k\} \) by \( \delta_k \). The value of \( \delta_k \) can be computed at each time instant with (7), and therefore is a known value. Let us assume that \( \delta_k \) is a bounded time-varying parameter such that \( \delta_k \in S \) with
\[ S = \{ \delta_k : 0 \leq \delta_k \leq \bar{\delta}, \forall k \} \] (8)
where \( \bar{\delta} \) is a known value or, if not, can be used as a tuning parameter as we will see later.

2.3. Estimation algorithm
Considering the previous description, the estimation algorithm is:
\[ \hat{x}_{k-} = A \hat{x}_{k-1} + B_u u^c_{k-1}, \] (9a)
\[ \hat{x}_k = \hat{x}_{k-} + L_k \alpha_k (y_k - C \hat{x}_{k-}). \] (9b)
At instant \( k \), we compute an open loop estimation (9a) employing the expected value of the control command at the plant \( u^c_{k-1} \). Then, when possible, we amend the estimation with the received measurements (nonzero diagonal elements of \( \alpha_k \)) through the gain matrix \( L_k \), see (9b).
The evolution on the state estimation error $\tilde{x}_k = x_k - \hat{x}_k$ is

$$\tilde{x}_k = (I - L_k \alpha_k C) (A \tilde{x}_{k-1} + B_w W_{k-1}) - L_k \alpha_k v_k$$  \hspace{1cm} (10)

where

$$B_w = [B_w \ B_u], \quad W_{k-1} = [w_{k-1}^T \ \hat{u}_{k-1}^T]^T.$$  

Note that the control error is viewed by the observer as a new unknown disturbance.

In this work we aim to compute gain matrices $L_k$ to minimize the estimation error under state disturbance, measurement noises and the uncertainty due to the control input dropouts. Previous works have shown that jump estimators whose jumping modes depend on the measurement reception scenario improve estimation performance [4]. Following that idea and in order to avoid conservativeness with respect to control errors, we propose a jump observer whose gains are scheduled in real time with $\delta_k$. The gain-scheduling law is

$$L_k = \begin{cases} L_i(\Delta_{k-1}) & \text{if } \alpha_k = \eta_i \text{ for } i = 1, \ldots, r, \\ 0 & \text{if } \alpha_k = \eta_0. \end{cases} \hspace{1cm} (11)$$

where

$$\Delta_{k-1} = [\delta_{k-1} \ \delta_{k-2}]^T.$$  \hspace{1cm} (12)

Note that $L_k$ is scheduled with both the current control error statistic $\delta_{k-1}$ (at instant $k$) and the previous one $\delta_{k-2}$ (at instant $k - 1$). This allows us to include in the estimation algorithm the present control uncertainty $\delta_{k-1}$ and also its variation from $\delta_{k-2}$ to $\delta_{k-1}$.

3. Observer design

Let us address in this section the design of the gain-scheduled jump observer $L(\alpha_k, \Delta_{k-1})$ using an $H_\infty$-based method. First, we derive a sufficient condition for its existence based on parameter-dependent matrix inequalities (PDMI) related to $\Delta_k$. Second, constraining those dependences to be polynomial, we present a SOS-based procedure to solve the design problem in polynomial time.

The next theorem shows the $H_\infty$ observer design based on PDMI.

**Theorem 1.** Consider the estimation algorithm (9) over system (1). If there exist positive definite symmetric matrices $P_j(\Delta_k) \in \mathbb{R}^{n \times n}$, full matrices $G_j(\Delta_k) \in \mathbb{R}^{n \times n}$ and $X_j(\Delta_k) \in \mathbb{R}^{n \times ny}$, and positive real scalar functions $\gamma_\omega(\Delta_k)$, $\gamma_u(\Delta_k)$ and $\gamma_v(\Delta_k)$ for all $i, j = 0, \ldots, r$ and $\delta_k \in \mathcal{S}$ fulfilling

$$\Upsilon_i(\Delta_k, \Delta_{k-1}) = \begin{bmatrix} \Omega \\ M_{A,i}^T \end{bmatrix} (\Delta_{k-1}) = I - \Gamma \tilde{M}_{B,i}$$  \hspace{1cm} (13)

with

$$\Omega = \bigoplus_{j=0}^r G_j(\Delta_k) + G_j(\Delta_k)^T - P_j(\Delta_k),$$

$$\tilde{M}_{A,i} = \begin{bmatrix} \sqrt{\eta_i} M_{A,0}^T \\ \vdots \\ \sqrt{\eta_i} M_{A,r}^T \end{bmatrix},$$

$$M_{A,i} = (G_j(\Delta_k) - X_j(\Delta_k) \eta_j C) A,$$

$$\tilde{M}_{B,i} = \begin{bmatrix} \sqrt{\eta_i} M_{B,0}^T \\ \vdots \\ \sqrt{\eta_i} M_{B,r}^T \end{bmatrix},$$

$$M_{B,i} = (G_j(\Delta_k) - X_j(\Delta_k) \eta_j C) B_w,$$

$$\Gamma = \gamma_\omega(\Delta_k) I_{n_w} \oplus \gamma_u(\Delta_k) I_{n_u} \oplus \gamma_v(\Delta_k) I_{n_m}$$.
and $B_W = [B_w \ B_k]$, then, defining the observer gains as $L_i(\Delta_k) = G_i(\Delta_k)^{-1}X_i(\Delta_k)$, the next assertions hold for all $\alpha_k \in \Xi$, $\delta_k \in \mathcal{S}$, $\|w\|_{RMS} \leq \bar{w}_{RMS}$ and $\|v\|_{RMS} \leq \bar{v}_{RMS}$: (i) under no disturbances, measurement noises and control errors, system (10) is asymptotically mean square stable; (ii) under zero initial conditions, the estimation error is bounded by
\[
\mathbb{E}\{\|\hat{x}\|^2_{RMS}\} < \bar{\gamma}_w \bar{w}_{RMS}^2 + \bar{\gamma}_v \bar{v}_{RMS}^2 + \bar{\gamma}_u, \tag{14}
\]
where
\[
\bar{\gamma}_u = \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} \gamma_u(\Delta_k) \delta_k, \quad \bar{\gamma}_w = \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} \gamma_w(\Delta_k), \quad \bar{\gamma}_v = \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} \gamma_v(\Delta_k). \tag{15}
\]

**Proof.** If (13) holds, then $G_j(\Delta_k) + G_j(\Delta_k)^T - P_j(\Delta_k) > 0$ and therefore, $G_j(\Delta_k)$ is nonsingular. Since $P_j(\Delta_k)$ is a symmetric positive definite matrix, then
\[
(P_j(\Delta_k) - G_j(\Delta_k)) P_j(\Delta_k)^{-1} (P_j(\Delta_k) - G_j(\Delta_k))^T \geq 0,
\]
which means that
\[
G_j(\Delta_k) + G_j(\Delta_k)^T - P_j(\Delta_k) \leq G_j(\Delta_k) P_j(\Delta_k)^{-1} G_j(\Delta_k)^T.
\]
Now, let us define the Lyapunov function as
\[
V_k = V(\bar{x}_k, \alpha_k, \Delta_k-1) = \bar{x}_k^T P_k(\Delta_k-1) \bar{x}_k
\]
for $\alpha_k = \eta_i$ and $i = 0, \ldots, r$. Remembering that $X_j(\Delta_k) = G_j(\Delta_k) L_j(\Delta_k)$, applying a congruence transformation on (13) by matrix $\left( \bigoplus_{j=0}^r G_j(\Delta_k)^{-1} \right) \oplus I \oplus I \oplus I$, taking Schur's complements and premultiplying the result by $[\bar{x}_k^T \ w_k^T \ u_k^T \ \bar{v}_{k+1}^T]$ and postmultiplying by its transpose, we obtain that
\[
\mathbb{E}\{V_{k+1}|\alpha_k = \eta_i\} - V_k + \bar{x}_k^T \bar{x}_k - \gamma_w(\Delta_k) w_k^T w_k - \gamma_u(\Delta_k) u_k^T u_k - \gamma_v(\Delta_k) \bar{v}_{k+1}^T \bar{v}_{k+1} < 0 \tag{16}
\]
for all $i = 0, \ldots, r$ and $\delta_k \in \mathcal{S}$.

(1) Under no disturbances ($w_k = 0$), control errors ($u_k = 0$) and measurement noises ($v_k = 0$), (16) leads to $\mathbb{E}\{V_{k+1}|\alpha_k = \eta_i\} - V_k < 0$ for all $i = 0, \ldots, r$ and $\delta_k \in \mathcal{S}$, which guarantees asymptotically mean square stability. This proves the first statement.

(ii) Let us write $\mathbb{E}\{V_{k+1}|\alpha_k = \eta_i\}$ as $\mathbb{E}\{V_{k+1}|\alpha_k\}$. Taking conditional expectation given $\alpha_{k-1}$ over (16), remembering that $\alpha_k$ is known at instant $k$, we get
\[
\mathbb{E}\{V_{k+1}|\alpha_k\} - \mathbb{E}\{V_k\} + \mathbb{E}\{\bar{x}_k^T \bar{x}_k\} - \gamma_w(\Delta_k) w_k^T w_k - \gamma_u(\Delta_k) u_k^T u_k - \gamma_v(\Delta_k) \bar{v}_{k+1}^T \bar{v}_{k+1} < 0 \tag{17}
\]
for all $\alpha_k \in \Xi$ and $\delta_k \in \mathcal{S}$, where we have considered the assumptions on $w_k$ and $v_k$, that $\delta_k = \mathbb{E}\{\bar{u}_k^T \bar{u}_k\}$ and that $\|w\|_{RMS}^2 \leq \bar{w}_{RMS}^2$, $\|v\|_{RMS}^2 \leq \bar{v}_{RMS}^2$.

For brevity reasons, let us omit in the following the fact that the inequalities are fulfilled for all $\alpha_k \in \Xi$ and $\delta_k \in \mathcal{S}$. Under zero initial conditions ($V_0 = 0$), adding the above expression from $k = 0$ to $K - 1$, dividing the result by $K$ and taking the limit when $K$ tends to infinity we obtain that
\[
\lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}\{\bar{x}_k^T \bar{x}_k\} < \bar{\gamma}_w \bar{w}_{RMS}^2 + \bar{\gamma}_u + \bar{\gamma}_v \bar{v}_{RMS}^2 \tag{18}
\]
where we have considered that $E\{V_{K+1}\alpha_K\} > 0$, that $\tilde{\gamma}_\cdot$ is as given in (15) and that
\[
\lim_{K \to \infty} \frac{1}{K} \sum_{t=0}^{K-1} (E\{V_k|\alpha_{k-1}\} - E\{V_k\}) = 0.
\]

Then, expression (18) leads to (14), which ends this proof.

Note that gains $\gamma_\cdot(\Delta_k)$ (with $\gamma_\cdot = \{\gamma_w, \gamma_u, \gamma_v, \gamma_f\}$) depend on $\Delta_k$ to improve the characterization of the propagation of the state disturbances, control errors and measurements noises to the estate estimation error for all $\delta_k \in S$.

Remark 3. The proposed observer design method would allow to apply the separation principle when closing the loop with an observer-based controller whenever the controller is designed to assure a control error $\delta_k \leq \delta$.

3.1. SOS decomposition

Considering conditions of Theorem 1 leads to an infinite-dimensional problem. To work with a finite-dimensional one, we impose matrices and scalar functions to be polynomial functions of $\delta_k$ of fixed degree. Then, SOS decompositions offer a computationally tractable approach.

The following lemmas extracted from [15] define SOS polynomials and show that checking whether a polynomial matrix is nonnegative over a domain characterized by polynomial constraints can be restated with sufficient linear matrix inequality (LMI) conditions.

Lemma 2. Let $p(x)$ be a 2d-order polynomial in $x \in \mathbb{R}^n$. Let $x^{(d)}$ be a vector with all the monomials in $x$ of degree $\leq d$. Then, $p(x)$ is said to be SOS if and only if there is a positive semidefinite matrix $Q$ fulfilling $p(x) = (x^{(d)})^T Q x^{(d)}$. The set of SOS polynomials in $x$ is denoted by $\Sigma(x)$.

Lemma 3. Let $p(x)$ be a polynomial in $x \in \mathbb{R}^n$, and let $X = \{x \in \mathbb{R}^n : g_j(x) \geq 0, j = 1, \ldots, m\}$. Suppose there exist SOS polynomials $s_j(x) \in \Sigma(x)$ ($j = 1, \ldots, n, x \in \mathbb{R}^n$) fulfilling $p(x) - \sum_{j=1}^m s_j(x) g_j(x) \in \Sigma(x)$, then, the following condition holds: $p(x) \geq 0, \forall x \in X$.

Lemma 4. Let $P(x) \in \mathbb{R}^{N \times N}$ be a symmetric polynomial matrix in $x \in \mathbb{R}^n$ and let $X = \{x \in \mathbb{R}^n : g_j(x) \geq 0, j = 1, \ldots, m\}$. Suppose there exist SOS polynomials $s_j(x, v) \in \Sigma(x, v)$ ($j = 1, \ldots, m$) fulfilling $v^T P(x) v - \sum_{j=1}^m s_j(x, v) g_j(x) \in \Sigma(x, v)$ with $v \in \mathbb{R}^N$, then, the following condition holds: $P(x) \geq 0, \forall x \in X$.

The next theorem derives sufficient conditions to obtain the parametric matrices and functions that guarantee the properties stated in Theorem 1. We use $\delta_1, \delta_2, \delta_3, \Delta_1 = [\delta_1 \delta_2]^T$ and $\Delta_2 = [\delta_2 \delta_3]^T$ to denote independent SOS variables representing the possible values of $\delta_k$, $\delta_{k-1}$, $\delta_{k-2}$, $\Delta_k$ and $\Delta_{k-1}$ respectively where $\delta_k \in S$ for all $k$.

Theorem 2. If there exist symmetric polynomial matrices
\[
P_1(\Delta_1) = \left(\Delta_1^{(dp)} \otimes I_n\right)^T P_1 \left(\Delta_1^{(dp)} \otimes I_n\right),
\]
\[
P_1(\Delta_2) = \left(\Delta_2^{(dp)} \otimes I_n\right)^T P_1 \left(\Delta_2^{(dp)} \otimes I_n\right)
\]
with real symmetric matrices $P_i$ for $i = 0, \ldots, r$, polynomial matrices
\[
G_i(\Delta_1) = G_i \left(\Delta_1^{(dg)} \otimes I_n\right), \quad X_i(\Delta_1) = X_i \left(\Delta_1^{(dp)} \otimes I_{nm}\right)
\]
with real matrices $G_i$ and $X_i$, and polynomial functions

$$\gamma_w(\Delta_i) = \gamma_w^T \Delta_i^{[d_w]} , \quad \gamma_u(\Delta_i) = \gamma_u^T \Delta_i^{[d_u]} , \quad \gamma_v(\Delta_i) = \gamma_v^T \Delta_i^{[d_v]}$$

with real vectors $\gamma_w$, $\gamma_u$ and $\gamma_v$, where $2d_p$, $d_G$, $d_X$, $d_v$ and $d_f$ are the degrees of the involved polynomials, that fulfill the following constraints

$$\mu^T P_1(\Delta_1) \mu - s_{P1,i} h_1 - s_{P2,i} h_2 \in \Sigma(\Delta_1, \mu), \quad (19a)$$
$$s_{P1,i} \in \Sigma(\delta_1, \mu), \quad s_{P2,i} \in \Sigma(\delta_2, \mu), \quad (19b)$$
$$\nu^T \Upsilon_i(\Delta_1, \Delta_2) \nu - s_{\Upsilon1,i} h_1 - s_{\Upsilon2,i} h_2 - s_{\Upsilon3,i} h_3 \in \Sigma(\Delta_1, \Delta_2, \nu), \quad (19c)$$
$$s_{\Upsilon1,i} \in \Sigma(\delta_1, \nu), \quad s_{\Upsilon2,i} \in \Sigma(\delta_2, \nu), \quad s_{\Upsilon3,i} \in \Sigma(\delta_3, \nu), \quad (19d)$$
$$\gamma_v(\Delta_i) - s_{j1} h_1 - s_{j2} h_2 \in \Sigma(\Delta_i), \quad (19e)$$
$$s_{j1} \in \Sigma(\delta_1), \quad s_{j2} \in \Sigma(\delta_2), \quad j = \{w, u, f, v\}, \quad (19f)$$

for $i = 0, \ldots, r$ with $\Upsilon_i(\cdot)$ as in (13), $\mu$ and $\nu$ real vectors of corresponding dimensions, and

$$h_m = \delta_m (\bar{\delta} - \delta_m), \quad m \in \{1, 2, 3\}, \quad (20)$$

then constraints of Theorem 1 are fulfilled.

**Proof.** Let us note first that the set $\mathcal{S}$ is reformulated with its corresponding polynomial $h$ as $\mathcal{S} = \{\delta_m : h_m \geq 0\}$, see (20). Second, by Lemma 3 and Lemma 4 constraints (19c) and (19d) assure the positive definiteness of $\Upsilon_i(\Delta_1, \Delta_2)$ for any $\delta_1, \delta_2, \delta_3 \in \mathcal{S}$, which guarantee (13) in Theorem 1. Similar apply to constraints (19a), (19b), (19e) and (19f) which guarantee the positive definiteness of $P_1(\Delta_1), \gamma_w(\Delta_1), \gamma_u(\Delta_1)$, and $\gamma_v(\Delta_i)$ for any $\delta_1, \delta_2 \in \mathcal{S}$, as required in Theorem 1.

In the previous feasibility SOS problem $\mu$ and $\nu$ are scalarization vectors employed to turn polynomial matrices into polynomials (see Lemma 4). The decision variables are matrices $P_1$, $G_i$, $X_i$, $\gamma_w$, $\gamma_u$ and $\gamma_v$; as well as the coefficients of SOS polynomials $s_j$ in (19). We select the degree of the SOS polynomials to have the same degree on all the variables in the corresponding SOS expression. Choosing $d_p$, $d_G$, $d_X$, and $d_v$, this can be obtained by imposing for all $j = 1, 2$ and $i = 0, \ldots, r$

$$\deg s_{P,j,i} = \deg \left\{ \delta_j^{\max\{2d_p-2,0\}}, \mu^2 \right\},$$
$$\deg s_{\Upsilon,j,i} = \deg \left\{ \delta_j^{\max\{2d_p-2,2d_G-2,d_X-2,0\}}, \nu^2 \right\},$$
$$\deg s_{\Upsilon3,j,i} = \deg \left\{ \delta_j^{\max\{2d_p-2,0\}}, \nu^2 \right\},$$
$$\deg s_{w,j} = \deg s_{v,j} = \deg \delta_j^{\max\{d_v-2,0\}},$$
$$\deg s_{u,j} = \deg \delta_j^{\max\{d_u-3,0\}}$$

where $\deg$ gives the maximum degree for each variable.

### 3.2. Optimization-based design

In order to minimize the estimation error, one possibility could be to minimize the right hand side of expression (14) for any $k$ and $\delta_k \in \mathcal{S}$. This would lead to the worst case estimation performance. In order to obtain less conservative results, we propose to introduce some weighting function $g(\Delta_1)$ such that

$$\tilde{\Gamma} = \int_{\mathcal{S} \times \mathcal{S}} g(\Delta_1) \Gamma \chi d\Delta_1 \quad (21)$$
\[ \Gamma_X = \gamma_w(\Delta_1)\bar{w}_{\text{rms}}^2 + \gamma_v(\Delta_1)v_{\text{rms}}^2 + \gamma_u(\Delta_1)\delta_1. \]

Then, the minimization problem

\[
\begin{align*}
\text{minimize} & \quad J \\
\text{subject to} & \quad (19), \\
& \quad \bar{\Gamma} < J,
\end{align*}
\]

minimizes the RMS norm of the state estimation error of the resulting observer under the weighting function \( g(\Delta_1) \).

**Remark 4.** A thoughtful choice for \( g(\Delta_1) \) is the one that considers the scenario when \( \delta_k \) is time invariant (constant or ramp-like transmitted control inputs, see (7)), i.e., \( \delta_1 = \delta_2 \), and none of the possible values of \( 0 \leq \delta_k \leq \bar{\delta} \) dominates over the others (i.e., assuming no knowledge about which value is more likely). Thus,

\[
g(\Delta_1) = \begin{cases} 
\frac{1}{\delta} & \text{if } \delta_2 = \delta_1 \text{ and } 0 \leq \delta_1 \leq \bar{\delta}, \\
0 & \text{otherwise.}
\end{cases}
\]

### 4. Examples

For ease of analysis, let us study a linear time invariant discrete time system described by matrices

\[
A = \begin{bmatrix} 0.48 & 0.11 \\ 0.11 & 0.97 \end{bmatrix}, \quad B_u = \begin{bmatrix} -0.5 \\ 0.7 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.2 \\ -0.6 \end{bmatrix}, \quad C = [0.18 \ 0.8],
\]

where there is only one measurable output. The state disturbances and measurement noises are Gaussian noises with zero mean and bounded RMS norms given by

\[
w_{\text{rms}} = 0.05, \quad v_{\text{rms}} = 0.01.
\]

The control input is assumed to be defined by

\[
u_k = 2.7 \sin(k/11).
\]

The transition probability matrix of the possible measurement reception scenario is characterized by \( q^y = 0.8 \) and \( p^y = 0.3 \) (see Remark 1). The control input dropouts follow a modified version of a Markov chain with parameters \( q^u = 0.5 \) and \( p^u = 0.4 \) with a maximum number of consecutive dropouts of \( N^u = 6 \) (see Remark 2). Taking that into account, the probabilities of applying at the plant the control input sent \( \tau - 1 \) instants before are

\[
\varphi = \begin{bmatrix} 0.668 & 0.2 & 0.08 & 0.032 & 0.0128 & 0.005 & 0.002 \end{bmatrix},
\]

where \( \varphi = [\varphi_0 \ \varphi_1 \ \varphi_2 \ \varphi_3 \ \varphi_4 \ \varphi_5 \ \varphi_6] \). Let us consider that in the worst case the control error statistic \( \delta_k \) is bounded by \( 0 \leq \delta_k \leq 0.1 \) for all \( k \), i.e. \( \bar{\delta} = 0.1 \).

In this example, we analyze the performance of the proposed gain-scheduled jump observer obtained using second order polynomials \( (d_P = 1, \ d_G = d_X = d_\gamma = 2) \) with the weighting function in (23) (see Section 3) and the resulting one from imposing zero order polynomials \( (d_P = d_G = d_X = d_\gamma = 0) \), i.e. gains that do not depend on \( \delta_k \). Both observers are calculated using YALMIP parser [21].

Figure 1 shows how the observer gains depend on \( \delta_k \) for the case when the control error statistic is time invariant, i.e., when \( \delta_k = \delta_{k-1} = \delta_1 \). Note that thanks to the scheduling approach the gains change up to a 30% while \( \delta_1 \) changes from 0 to 0.1.
Simulating the designed observers under the proposed scenario we get Figure 2 showing that at the instants when the control error is low, the gain-scheduling approach of degree 2 performs better than the one with gains not depending on $\delta_k$. When the control error is near to its maximum both approaches have similar behavior. This could have been predicted from Figure 1 since the gains have similar values for control errors close to the upper bound.

![Figure 1](image)

**Figure 1.** Observer gains as a function of $\delta_1$, with polynomials of second order (continuous line) and zero order (dashed line).

5. Conclusion
In this work, we have studied the remote estimation problem under measurement and control input dropouts without successful delivery acknowledgment. We used a Markov chain to characterize the measurement reception scenario and described the control error between the control input at the plant and the one at the observer. We handle this control error as a new unknown disturbance input. Then, we related the observer gain to the measurement reception scenario and scheduled it with a control error statistic. We proved by simulation that our gain scheduling approach performs as well as the design with invariant gains (zero order polynomials) for the worst control error case, but improves the estimation performance whenever the control error statistic is not near its upper bound.

Further research directions may include extensions of the current results to observer-based control, where the separation principle holds if the controller can be designed to produce a control error that fulfills the imposed bounds, and uncertain transmission probabilities.

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\[ \| \tilde{x}_k \|_2^2 - \| \tilde{x}_0 \|_2^2 / \| \tilde{x}_0 \|_2^2 \]

\[ \tilde{u}_k^2 \]

\[ \delta_k \]

\[ \theta_k \]

\[ \alpha_k \]

\[ \text{Control errors} \]

\[ \text{simulation instants, } k \]

Figure 2. State estimation performances in simulation under control input and measurement dropouts.

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