Boundary Lax pairs for the $A_n^{(1)}$ Toda field theories

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Abstract

Based on the recent formulation of a general scheme to construct boundary Lax pairs, we develop this systematic construction for the $A_n^{(1)}$ affine Toda field theories (ATFT). We work out explicitly the first two models of the hierarchy, i.e. the sine-Gordon ($A_1^{(1)}$) and the $A_2^{(1)}$ models. The $A_2^{(1)}$ Toda theory is the first non-trivial example of the hierarchy that exhibits two distinct types of boundary conditions. We provide here novel expressions of boundary Lax pairs associated to both types of boundary conditions.

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1 Introduction

The Lax representation of a classical dynamical system consists in the formulation of two endomorphism-valued objects (in most cases matrix- or operator-valued) $L$ and $M$, depending on the dynamical variables, such that the equations of motion are contained in the iso-spectral evolution equation:

$$\partial_t L = [L, M]$$

(1.1)

The Lax matrix $L$ therefore lives in some Lie algebra, finite-dimensional, loop algebra, or differential algebra depending on the specific system. It is collectively called “auxiliary algebra”.

The spectrum of $L$, or its associated monodromy matrix if $L$ is a differential operator, provides therefore a generating set of natural time-invariant candidates to be identified as integrable Hamiltonians. Liouville integrability of any dynamical system associated to a given function on this set follows from the Poisson-commutation of the elements of the set, guaranteed by the necessary and sufficient condition [2, 3] of the existence of a generically linear Poisson structure characterized by a so-called classical $r$-matrix, for the spectrum-generating operator (Lax matrix or monodromy matrix). A natural construction for $M$, given any specific Hamiltonian built as a function $f$ on $\text{sp}(L)$, is then available in terms of $df$, $L$ and the $r$-matrix [1, 2].

In our previous paper [4] we have examined the situation when the Poisson structure available for $L$ is expressed in terms of a non-dynamical classical $r$-matrix plus a set of non-dynamical parameters encapsulated in a “boundary” matrix $K$ obeying some purely algebraic quadratic equation in terms of this $r$-matrix. This more complicated equation (actually two distinct forms thereof) represents a classical version of the quantum Cherednik-Sklyanin reflection algebra [5, 6, 7]. The operator generating the Poisson commuting Hamiltonians then combines $L$ and $K$. We have then defined a systematic construction of the $M$ operator in terms of $r$, $L$ and $K$.

Note here that the denomination of $K$ as “boundary” matrix reflects the fact that such parameters encode indeed boundary effects in the Hamiltonians when the matrix $L$ is a coproduct of local $l_i$ matrices on a finite lattice $i = 1, \ldots, N$ or a monodromy of local differential operators on a finite line. When by contrast $L$ is a purely local Lie-algebra valued matrix, such parameters may be better characterized as coupling constants in a folding procedure (one is then considering a system on a one-site space lattice for which the notion of a “boundary” has no sense).

Two fundamental situations were described in [4]: linear Poisson structure and quadratic Poisson structure for the generating operator. The second one is relevant to describe systems
on a lattice or a continuous line. We shall here restrict ourselves to this latter case, particularizing it even more to degree-one differential operators \( L = \frac{d}{dx} + l(x) \). The generating operator is here the monodromy matrix of this differential operator computed between the two ends of the finite line for \( x \).

We consider here such operators for which the \( r \)-matrix is the one associated to the generic affine \( A_n^{(1)} \) Toda field theories. In this way we construct the associated Lax representation of ATFT with non-trivial integrable boundary conditions parametrized by the \( K \) matrix. Two “boundary” reflection equations must be considered, resp. characterized as “soliton-preserving” (SP) and “soliton non-preserving” (SNP) (see relevant studies at quantum and classical level \[8\]–\[26\]). More general reflection equation may be considered given a classical \( r \)-matrix, through the choice of some auxiliary-space anti-automorphism, but this extension shall be postponed for further studies. It will be noted that this scheme automatically yields boundary conditions compatible with integrability. Careful evaluation of the contributions to the Hamiltonians and the \( M \) matrix at the edges of the \( x \) line are required to get consistent results.

Our essential purpose here is two-fold. We shall first validate our general derivation by a comparison of the \( A_1^{(1)} \) and Soliton-non-preserving \( A_2^{(1)} \) cases with known results on low-dimension ATFT obtained by case-by-case analysis, basically for SNP boundary conditions \[9\]. We shall indeed show that both derivations yield exactly identical formulations for the boundary conditions for the dynamical fields and related Lax formulation (with a proviso, related to analyticity conditions, to be specified later). Similar consistency checks were also achieved in \[4\] within the vector non-linear Schrodinger context.

We shall then provide novel expressions for boundary Hamiltonians and the associated Lax pairs in the yet untreated case of \( A_2^{(1)} \) with SP boundary conditions, and we shall address some intriguing technical points that arise even in the simplest case, i.e. the sine-Gordon model.

## 2 The general scheme

Our analysis of classical integrable field theories with integrable boundary conditions relies on the study of an associated auxiliary linear problem (see \[27\] and references therein). Let us first recall the bulk case (no extra “boundary” parameter) to fix the notations and recall the basic structures.

The Lax pair \[28\] formulation \[29\] of classical integrable Hamiltonian systems consists in defining an auxiliary linear differential problem, reading in the simplest case (order-1
differential operators):
\[
\left( \partial_x - U(x, t, \lambda) \right) \Psi = 0 \tag{2.1}
\]
\[
\left( \partial_t - V(x, t, \lambda) \right) \Psi = 0 \tag{2.2}
\]

\( U, V \) are in general \( n \times n \) matrices with entries functions of the dynamical fields, their space derivatives, and possibly the complex spectral parameter \( \lambda \). Compatibility conditions of the two differential equations (2.1), (2.2) lead to the zero curvature condition [29, 30, 31]
\[
\partial_t U - \partial_x V + [U, V] = 0. \tag{2.3}
\]

The latter equations give rise to the corresponding classical equations of motion of the system under consideration. Natural conserved quantities are obtained from the monodromy matrix
\[
T(x, y, \lambda) = \mathcal{P} \exp \left\{ \int_x^y U(x', t, \lambda) dx' \right\} \tag{2.4}
\]

once it is assumed that \( U \) obeys the classical linear Poisson algebraic relations:
\[
\left\{ U_a(x, \lambda), U_b(y, \mu) \right\} = \left[ r_{ab}(\lambda - \mu), U_a(x, \lambda) + U_b(y, \mu) \right] \delta(x - y), \tag{2.5}
\]

and consequently \( T(x, y, \lambda) \) satisfies (see [27]):
\[
\left\{ T_a(x, y, t, \lambda), T_b(x, y, t, \mu) \right\} = \left[ r_{ab}(\lambda - \mu), T_a(x, y, t, \lambda)T_b(x, y, t, \mu) \right]. \tag{2.6}
\]

One then immediately gets:
\[
\left\{ trT(\lambda), trT(\mu) \right\} = 0 \quad \text{where} \quad t(\lambda) = trT(\lambda) \tag{2.7}
\]

and thus \( trT(\lambda) \) yields the relevant conserved quantities.

We are now interested in implementing non-trivial integrable boundary conditions. We focus here on two distinct types of integrable boundary conditions: the so-called soliton preserving (SP) and the soliton non-preserving (SNP). Formulation of the two distinct types of boundary conditions is achieved by defining two types of Poisson structures for the modified monodromy matrices \( T \). These will in fact represent the classical versions of the reflection algebra \( \mathbb{R} \), and the twisted Yangian \( \mathbb{T} \) written in the following forms (see e.g. [32, 33, 9]):

\textbf{(I) SP (reflection algebra) [5, 6]}

\[
\left\{ T_1(\lambda_1), T_2(\lambda_2) \right\} = r_{12}(\lambda_1 - \lambda_2)T_1(\lambda_1)T_2(\lambda_2) - T_1(\lambda_1)T_2(\lambda_2)r_{21}(\lambda_1 - \lambda_2)
+ T_1(\lambda_1)r_{21}(\lambda_1 + \lambda_2)T_2(\lambda_2) - T_2(\lambda_2)r_{12}(\lambda_1 + \lambda_2)T_1(\lambda_1) \tag{2.8}
\]
(II) SNP ((q)-twisted Yangian) \[34,35\]

\[
\left\{ T_1(\lambda_1), T_2(\lambda_2) \right\} = r_{12}(\lambda_1 - \lambda_2)T_1(\lambda_1)T_2(\lambda_2) - T_2(\lambda_2)r_{21}(\lambda_1 - \lambda_2)T_1(\lambda_1) + T_1(\lambda_1)r_{12}(\lambda_1 + \lambda_2)T_2(\lambda_2) - T_2(\lambda_2)r_{21}(\lambda_1 + \lambda_2)T_1(\lambda_1)
\]

In most well known physical cases, such as the $A_{N-1}^{(1)}$ $r$-matrices $r_{12}^{t_1t_2} = r_{21}$, hence all the expressions above may be written in a simpler form.

In order to construct representations of (2.8), (2.9) yielding the generating function of Poisson-commuting Hamiltonians realizing the integrals of motion for a new classical integrable system, one now introduces non-dynamical representations $(K^\pm)$ of the algebra $\mathbb{R}$ ($\mathbb{T}$). The non-dynamical condition:

\[
\left\{ K_1^\pm(\lambda_1), K_2^\pm(\lambda_2) \right\} = 0
\]

transforms (2.8), (2.9) into algebraic equations for $K^\pm$. We consider then any bulk monodromy matrix $T$ with Poisson structure (2.6) and we define in addition:

\[
\hat{T}(\lambda) = T^{-1}(-\lambda) \text{ for SP}, \quad \hat{T}(\lambda) = T^d(-\lambda) \text{ for SNP}.
\]

One expects that a more self-contained formulation may involve a more general anti-automorphism of the auxiliary algebra where $T$ lives. A corresponding reformulation of the relevant Poisson structures should also be proposed in that framework. We shall leave this generalization for later studies.

Generalized “monodromy” matrices, realizing the corresponding algebras $\mathbb{R}$, $\mathbb{T}$, are finally given by the following expressions [6,9]:

\[
T(x, y, t, \lambda) = T(x, y, t, \lambda) K^-(\lambda) \hat{T}(x, y, t, \lambda).
\]

The generating function of the involutive quantities is defined as

\[
t(x, y, t, \lambda) = tr\{K^+(\lambda) T(x, y, t, \lambda)\}.
\]

Indeed one shows:

\[
\left\{ t(x, y, t, \lambda_1), t(x, y, t, \lambda_2) \right\} = 0, \quad \lambda_1, \lambda_2 \in \mathbb{C}.
\]

The systematic Lax formulation in the case of open boundary conditions is described in [4]. More precisely it was shown in [4] that

\[
\left\{ \ln t(\lambda), U(x, \mu) \right\} = \frac{\partial \mathcal{V}(x, \lambda, \mu)}{\partial x} + \left[ \mathcal{V}(x, \lambda, \mu), U(x, \mu) \right]
\]

\[\text{(2.15)}\]
where we define:

(I) SP
\[
\n(\text{II}) \text{ SNP}
\]

Particular attention should be paid to the boundary points \(x = 0, -L\). Indeed, for these two points one has to take into account that \(T(x, x\lambda) = \hat{T}(x, x\lambda) = \mathbb{I}\).

(I) SP
\[
\n(\text{II}) \text{ SNP}
\]

Notice that all the boundary information, incorporated in \(K^\pm\), appears only at the boundary points \(x = 0, -L\). We shall presently see that the bulk expression has in fact no dependence whatsoever on the reflection \(K\) matrix, since it is canceled by the \(t^{-1}(\lambda)\) factor. Note finally that the expressions derived in (2.16)–(2.18) are universal, that is independent of the choice of model.
3 The $A^{(1)}_n$ ATFT: brief review

We are now in a position to systematically construct the Lax representation for any extension of the $A^{(1)}_n$ ATFT following the scheme defined above, given any solution $K$ of the algebraic boundary equations defined by the $A^{(1)}_n$ Toda classical $r$-matrix. Note that the associated boundary Hamiltonians have been extracted through the asymptotic expansion of the open transfer matrix in [36] for sine-Gordon and in [26] for the $A^{(1)}_2$ ATFT. We shall consider in the following sections two particular examples, that is the prototype model of the hierarchy, i.e. the sine-Gordon model, as well as the $A^{(1)}_2$ case. The $A^{(1)}_2$ model is indeed the first non-trivial example of this set that may exhibit both types of boundary conditions. It is worth noting that in sine-Gordon the two boundary conditions coincide due to the fact that the model is self-dual.

Recall first the classical $r$-matrix associated to the generic $A^{(1)}_n$ affine Toda field theory in particular is given by \[37\]
\[
 r(\lambda) = \frac{\cosh(\lambda)}{\sinh(\lambda)} \sum_{i=1}^{n+1} e_{ii} \otimes e_{ii} + \frac{1}{\sinh(\lambda)} \sum_{i \neq j=1}^{n+1} e^{[\text{sgn}(i-j)-(i-j)] \frac{2}{n+1}} \lambda} e_{ij} \otimes e_{ji}, \tag{3.1}
\]
with $(e_{ij})_{kl} \equiv \delta_{ik} \delta_{jl}$. Note that the classical $r$-matrix (3.1) is written in the so-called principal gradation as in [9, 13] (see details on the gauge transformation changing the principal to the homogeneous gradation in [26]). We recall the Lax pair for a generic $A^{(1)}_n$ theory [38]:

\[
 \mathbb{V}(x, t, u) = -\frac{\beta}{2} \partial_x \Phi \cdot H + \frac{m}{4} \left( u e^{\frac{2}{n+1} H} E_+ e^{-\frac{4}{n+1} H} E_+ e^{\frac{2}{n+1} H} \right) \\
 \mathbb{U}(x, t, u) = \frac{\beta}{2} \Pi \cdot H + \frac{m}{4} \left( u e^{\frac{2}{n+1} H} E_+ e^{-\frac{4}{n+1} H} E_+ e^{\frac{2}{n+1} H} \right) \tag{3.2}
\]

Φ, Π are conjugated $n$-vector fields, with components $\phi_i$, $\pi_i$, $i \in \{1, \ldots, n\}$, $u = e^{\frac{2\pi}{m}}$ is the multiplicative spectral parameter. To compare with the notation used in [9] we set $\frac{m^2}{16} = \frac{\tilde{m}}{8}$ ($\tilde{m}$ denotes the mass in [9]). Note that eventually in [9] both $\beta$, $\tilde{m}$ are set equal to unit.

We also define:

\[
 E_+ = \sum_{i=1}^{n+1} E_{\alpha_i}, \quad E_- = \sum_{i=1}^{n+1} E_{-\alpha_i} \tag{3.3}
\]

$\alpha_i$ are the simple roots plus the extended (affine) root, $H$ ($n$-vector) and $E_{\pm \alpha_i}$ are the algebra generators in the Cartan-Weyl basis, and they satisfy the Lie algebra relations:

\[
 [H, E_{\pm \alpha_i}] = \pm \alpha_i E_{\pm \alpha_i}, \\
 [E_{\alpha_i}, E_{-\alpha_i}] = \frac{2}{\alpha_i^2} \alpha_i \cdot H \tag{3.4}
\]

\[3\]Notice that the $r$-matrix employed here is in fact $r_{12}^{A_{12}}$ with $r_{12}$ being the matrix used e.g. in [11, 19]
Explicit expressions on the simple roots and the Cartan generators are presented below. Notice that the Lax pair has the following behavior:

\[ V^t(x, t, -u^{-1}) = V(x, t, u) , \quad U^t(x, t, u^{-1}) = U(x, t, u) \] (3.5)

where \(^t\) denotes usual transposition.

We provide below explicit expressions of the simple roots and the Cartan generators for \(A^{(1)}_n\) \[^{39}\]. The vectors \(\alpha_i = (\alpha_i^1, \ldots, \alpha_i^n)\) are the simple roots of the Lie algebra of rank \(n\) normalized to unity \(\alpha_i \cdot \alpha_i = 1\), i.e.

\[ \alpha_i = (0, \ldots, 0, -\sqrt{\frac{i-1}{2i}}, \sqrt{\frac{i+1}{2i}}, 0, \ldots, 0) , \quad i \in \{1, \ldots n\} \] (3.6)

The fundamental weights \(\mu_k = (\mu_k^1, \ldots, \mu_k^n)\), \(k = 1, \ldots, n\) are defined as (see, e.g., \[^{39}\]).

\[ \alpha_j \cdot \mu_k = \frac{1}{2} \delta_{j,k} . \] (3.7)

The extended (affine) root \(a_{n+1}\) is provided by the relation

\[ \sum_{i=1}^{n+1} a_i = 0 . \] (3.8)

The Cartan-Weyl generators in the defining representation are:

\[ E_{\alpha_i} = e_{i+1} , \quad E_{-\alpha_i} = e_{i+1} , \quad E_{\alpha_n} = -e_{n+1} , \quad E_{-\alpha_n} = -e_1 \]

\[ H_i = \sum_{j=1}^{n} \mu_{ij} (e_{jj} - e_{j+1, j+1}) , \quad i = 1, \ldots, n. \] (3.9)

4 The boundary \(A^{(1)}_1\) case: sine-Gordon model

Let us rewrite the Lax operator for the bulk sine Gordon model\[^{4}\],

\[ \mathbb{U}(x, t, u) = \frac{\beta}{4i} \pi(x) \sigma_3 + \frac{mu}{4i} e^{i\phi_3} \sigma_2 e^{-i\phi_3} - \frac{mu^{-1}}{4i} e^{-i\phi_3} \sigma_2 e^{i\phi_3} \] (4.2)

\(\sigma_i\) are the 2-dimensional Pauli matrices.

\[^{4}\]To recover the generic form (3.2) from (4.2) we consider the following identifications

\[ \beta \rightarrow \beta , \quad \phi \rightarrow -\phi , \quad u \rightarrow -u \] (4.1)

Here (4.2) we clearly consider the sine-Gordon, however after implementing identifications (4.1) we obtain the sinh-Gordon model.
Bearing in mind the expression for $T$ (2.12) it is clear that we need to consider the formal series expansion of $T$ and $T^{-1}(u^{-1})$. But from the following symmetry of the Lax operator:

$$U(u^{-1}, \phi, \pi) = U(-u, -\phi, \pi)$$  

we see that

$$T(u^{-1}, \phi, \pi) = T(-u, -\phi, \pi).$$  

We aim at expressing the term of order $u$ in $U$ independently of the fields, after applying a suitable gauge transformation [27]. More precisely, consider the following gauge transformation such that

$$T(x, y, u) = \Omega(x) \tilde{T}(x, y, u) \Omega^{-1}(y),$$  

$$\Omega(x) = \text{diag}\left(\Omega_1(x), \Omega_2(x)\right) = e^{\frac{i}{4} \beta \phi(x) \sigma_3}$$  

then the gauge transformed operator $\tilde{U}$ is expressed as:

$$\tilde{U}(x, t, u) = \frac{\beta}{4i} f(x) \sigma_3 + \frac{mu}{4i} \sigma_2 - \frac{mu}{4i} e^{-\frac{i}{2} \phi(x) \sigma_3} e^{\frac{i}{2} \phi(x) \sigma_3} \hat{W}(x, u)$$  

where we define

$$f(x, t) = \pi(x, t) + \phi'(x, t).$$  

Let $T'(u) = T(u^{-1})$ then we introduce the following decomposition for $\tilde{T}$, $\tilde{T}'$ as $|u| \to \infty$ [27]

$$\tilde{T}(x, y, u) = (I + W(x, u)) \exp[Z(x, y, u)] (I + W(y, u))^{-1},$$  

$$\tilde{T}'(x, y, u) = (I + \hat{W}(x, u)) \exp[\hat{Z}(x, y, u)] (I + \hat{W}(y, u))^{-1},$$  

where the hat simply denotes that $u \to -u$, $\phi \to -\phi$. $W$, $\hat{W}$ are off diagonal matrices and $Z$, $\hat{Z}$ are purely diagonal. Also

$$Z(u) = \sum_{k=-1}^{\infty} \frac{Z^{(k)}(u)}{u^k}, \quad W(u) = \sum_{k=0}^{\infty} \frac{W^{(k)}(u)}{u^k}.\quad (4.9)$$  

Inserting the latter expressions (4.9) in (2.11) one may identify the matrices $W^{(k)}$ and $Z^{(k)}$. Indeed, from equation (2.11) we conclude that the gauge transformed operators satisfy:

$$\frac{dZ}{dx} = \tilde{U}^{(D)} + \tilde{U}^{(O)} W$$  

$$\frac{dW}{dx} + W\tilde{U}^{(D)} - \tilde{U}^{(D)} W + W\tilde{U}^{(O)} W - \tilde{U}^{(O)} = 0$$  

(4.10)
where $\tilde{U}^{(D)}$, $\tilde{U}^{(O)}$ are the diagonal and off diagonal parts of $\tilde{U}$ respectively. By solving the latter equations we may identify the matrices $Z$, $W$. It is sufficient for our purposes here to identify only the first couple of terms of the expansions. Indeed based on equation (4.10) we conclude (see also [27]):

$$W^{(0)} = i\sigma_1, \quad W^{(1)} = -\frac{i\beta}{m} f(x)\sigma_1$$  \hspace{1cm} (4.11)

Note that the leading contribution as $iu \to \infty$ comes from the $e^{Z_{22}}$ term. We assumed here for simplicity, but without losing generality, Schwartz boundary conditions at the end point $x = -L$, that is $\pi(-L) = \phi(-L) = 0$ and $K^{-} \propto \mathbb{I}$. We may rewrite the expression for the boundary operator $V$ (let $\hat{r}_{ab} = r_{ba}$) as:

$$V(x, t, u, v) = t^{-1}(u)e^{Z_{22} - \hat{Z}_{22}} \left( (1 + \hat{W}(0))^{-1} \Omega(0) K^+ (u) \Omega(0) (1 + W(0)) \right)_{22}$$  \hspace{1cm} (4.13)

but it is easy to show for the transfer matrix (2.13) as $|u| \to \infty$:

$$t(u) = e^{Z_{22} - \hat{Z}_{22}} \left( (1 + \hat{W}(0))^{-1} \Omega(0) K^+ (u) \Omega(0) (1 + W(0)) \right)_{22}$$  \hspace{1cm} (4.14)

and finally

$$V (x, t, u, v) = \left[ (1 + W(x))^{-1} \Omega^{-1} (x) r(uv^{-1}) \Omega(x) (1 + W(x)) \right]_{22}$$  \hspace{1cm} (4.15)

Again using the ansatz for the monodromy matrix we obtain from (2.18) for the end point $x = 0$:

$$V(0, t, u, v) = \left[ (1 + \hat{W}(0))^{-1} \Omega(0) K^+ (u) \Omega(0) (1 + W(0)) \right]^{-1}_{22}$$  \hspace{1cm} (4.16)

The $r$-matrix is given in (3.1) and we consider below two cases with non-diagonal and diagonal $K$-matrix respectively.

\footnote{Note that in this particular case the SP and SNP boundary conditions coincide because:

$$r_{12}(\lambda) = V_1 r_{12}^{\dagger}(-\lambda) V_1, \quad V = \text{antidiag}(1, 1)$$  \hspace{1cm} (4.12)}
4.1 Non-diagonal $K$-matrix

We shall first examine the case with the generic non-diagonal $K$-matrix \[40, 41\]

\[
K^+(\lambda) = \frac{1}{\kappa} \sinh(\lambda + i\xi)e_{11} + \frac{1}{\kappa} \sinh(-\lambda + i\xi)e_{22} + x^+ \sinh(2\lambda)e_{12} + x^- \sinh(2\lambda)e_{21}
\]

(4.17)

$x^\pm$ are a priori free independent boundary parameters. The next step is to expand expressions (4.15), (4.16) in powers of $u^{-1}$, and identify the first order term of the expansion. Taking into account the expansion of $W$ as well as \(|u| \to \infty\)

\[K^+(u) \sim K^{+(0)} + u^{-1}K^{+(1)} + O(u^{-2}), \quad r(uv^{-1}) \sim r^{(0)} + u^{-1}r^{(1)} + O(u^{-2})\]

(4.18)

where we define:

\[
K^{+(0)} = x^+ e_{12} + x^- e_{21}, \quad K^{+(1)} = \frac{e^{i\xi}}{\kappa} e_{11} - \frac{e^{-i\xi}}{\kappa} e_{22},
\]

\[
r^{(0)} = \sum_{i=1}^{2} e_{ii} \otimes e_{ii}, \quad r^{(1)} = 2v(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}), \quad \dot{r}^{(1)} = 2v^{-1}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12})
\]

(4.19)

we may expand $V(u, v)$ in powers of $u^{-1}$. Multiplying the resulting expression by a factor $\frac{m}{4i}$ we obtain at first order:

\[
V(x, t, v) = \frac{\beta}{4i} \phi'(x, t)\sigma_3 + \frac{vm}{4i} \Omega(x, t) \sigma_2 \Omega^{-1}(x, t) + \frac{v^{-1}m}{4i} \Omega^{-1}(x, t) \sigma_2 \Omega(x, t)
\]

(4.20)

We see that the operator $V(x, t, v)$ at any point $x \neq 0$ coincides with the bulk operator \[3.2\], consistently with the fact that $\mathcal{H}$ coincides with the bulk sine-Gordon boundary Hamiltonian \[40, 36\] except for $x = 0$:

\[
\mathcal{H} = \int_{-L}^{0} dx \left[ \frac{1}{2}(\pi^2 + \phi'^2) + \frac{m^2}{\beta^2}(1 - \cos \beta \phi) \right] + \frac{4Pm}{\beta^2} \cos \frac{\beta \phi(0)}{2} - \frac{4Qm}{\beta^2} \sin \frac{\beta \phi(0)}{2}
\]

(4.21)

Recall that in \[36\] the latter Hamiltonian was obtained as the first order term from the expansion of the generating function $t(u)$ as $|u| \to \infty$, assuming Schwartz boundary conditions at $x = -L$. The boundary parameters $P$, $Q$ are related to the parameters $\xi$, $\kappa$ of the $K$ matrix as:

\[
P = \frac{e^{i\xi} - e^{-i\xi}}{4\kappa}, \quad Q = \frac{e^{i\xi} + e^{-i\xi}}{4\kappa i}
\]

(4.22)

We should stress that the constraint $x^+ = -x^-$ here arises by requiring that the expansions of the transfer matrix as $iu \to \infty$ and $iu \to -\infty$ provide the same Hamiltonians (again,
an analyticity condition at infinity). Such a requirement leads also to the cancelation of boundary terms proportional to $\phi'(0)$.

It is clear that the bulk $V$-operator is independent of the choice of $K$-matrix. Expanding carefully the boundary expression (2.19) and multiplying the result with a factor $\frac{m}{4i}$ we obtain at the boundary point:

$$V^{(b)}(0, t, v) = V(0, t, v) + \Delta V(0, t, v),$$

where

$$\Delta V(0, t, v) = -\frac{\beta}{4i} \phi'(0) \sigma_3 - \frac{m}{8} \left( e^{i\xi} e^{i\frac{\phi(0)}{2\beta}} + e^{-i\xi} e^{-i\frac{\phi(0)}{2\beta}} \right) \sigma_3$$

(4.23)

$V(0, t, v)$ is provided by the bulk expression (4.20). Note that all the boundary information is incorporated at the boundary point $x = 0$. The equations of motion and the corresponding boundary conditions emerge in this Lax formulation from the zero curvature condition. The zero curvature condition for the ‘bulk’ Lax pair yields the familiar equations of motion for the sine-Gordon model. We should note that analyticity requirements on the boundary Lax pair leads to extra constraints among the boundary parameters, i.e. $x^+ = -x^-$ (see also [36]). The boundary operator found here is associated to the boundary Hamiltonian (4.21). Note that a particular choice of diagonal $K^+$ matrix leads to discrepancies between the two descriptions (Hamiltonian vs Lax pair). This suggests that one has to consider as a starting point a generic solution of the reflection equation with several boundary parameters, which may then satisfy further constraints dictated by certain consistency requirements.

The relevant boundary conditions are obtained by considering the zero curvature condition (2.3) at the point $x = 0$:

$$\hat{U}(0, t, v) - \lim_{\delta \to 0} \frac{V(\delta, t, v) - V(0, t, v) - \Delta V(0, t, v)}{\delta} + \left[ \hat{U}(0, t, v), V(0, t, v) + \Delta V(0, t, v) \right] = 0.$$  

(4.25)

Explicit expression of the derivative of $V$ at $x = 0$ in (4.25) indicates

$$\Delta V(0, t, v) = 0,$$

(4.26)

in order to eliminate a potential uncompensated divergence due to $\Delta V$.

Finally from the ‘bulk’ zero curvature condition and from the later expression the following equations of motion and mixed boundary conditions are entailed:

$$\ddot{\phi}(x, t) - \phi''(x, t) = -\frac{m^2}{\beta} \sin(\beta \phi(x, t))$$

$$\beta \phi'(0) = \frac{m}{2i\kappa} \cos(\xi + \frac{\beta}{2} \phi(0)),$$

(4.27)
which of course coincide with the equations of motion found in \([40, 36]\) (recall also the identification of boundary parameters \((4.22)\)). In a consistent way the boundary conditions are obtained exactly from the Hamiltonian through

\[
\frac{\partial \phi(x, t)}{\partial t} = \{H, \phi(x, t)\}, \quad \frac{\partial \pi(x, t)}{\partial t} = \{H, \pi(x, t)\}, \quad x \in [-L, 0]
\] (4.28)

by noticing that the contribution containing the term \(\phi(0)\) in the Hamiltonian yields a \(\delta(0)\) term in the equations of motion for \(\pi(x)\) since \(\{\phi(x), \pi(y)\} = \delta(x - y)\). Elimination of this term yields exactly the boundary conditions \((4.27)\).

Requiring cancelation of the \(\Delta V(0)\) term is equivalent to requiring that the formal series expansion in \(u^{-1}\) coincides for \(V(0, t, u, v)\) and \(V(x, t, u, v)\) at \(x \to 0\). Indeed the technical origin of \(\Delta V(x = 0)\) is the non-commutation of limits \(x \to 0\) and \(u \to \infty\), in particular in \(e^Z\). If these limits are required to commute then \(V(x, t, u)\) has its analytic behavior in \(x, u\) continued to the limit \(x = 0\), which may suggest that it can be analytically continued “beyond” the boundary. This may in turn be a relevant consistency condition in implementing the notion of “gluing” different boundary systems. We have thus established a straightforward and elegant way to extract the associated boundary conditions from the zero curvature condition.

### 4.2 Diagonal \(K\)-matrix

We shall now consider the diagonal \(K\)-matrix \([40, 41]\)

\[
K^+(\lambda) = \sinh(\lambda + i\xi)e_{11} + \sinh(-\lambda + i\xi)e_{22}.
\] (4.29)

In particular we shall be mostly interested in the degenerate limit where \(i\xi \to \infty\).

The relevant boundary Hamiltonian, obtained from the first order term of the expansion of \(\ln t(\lambda)\), is given by:

\[
\mathcal{H} = \int_{-L}^{0} dx \left[ \frac{1}{2}(\pi^2 + \phi'^2) + \frac{m^2}{\beta^2}(1 - \cos \beta \phi) \right] + \frac{2}{\beta} \phi'(0) \frac{\cos(\xi + \frac{\beta}{2} \phi(0))}{\sin(\xi + \frac{\beta}{2} \phi(0))}.
\] (4.30)

The boundary contribution in \((4.30)\) is not identical with the \(\kappa = 0\) limit of the boundary conditions in \((4.21)\). One needs to normalize the \(K\)-matrix as \(uK(\kappa \to 0)\) to get a consistent \(u\)-expansion, hence \(\mathcal{H}\) in \((4.30)\) picks boundary contribution from higher orders in \((4.18)\).

By requiring the boundary term, proportional to \(\phi'(0)\) to disappear we obtain the following constraint

\[
\cos(\xi + \frac{\beta}{2} \phi(0)) = 0,
\] (4.31)
which of course may be seen as the boundary condition to the associated equations of motion, as we shall see below.

The next step as in the previous case is to expand expressions (4.15), (4.16) in powers of \( u^{-1} \), and identify the associated \( V \)-operator from the first order term of the expansion. As we have seen the bulk \( V(u, v) \)-operator is independent of the choice of boundary conditions, i.e. the \( K \)-matrix and is given at any point \( x \neq 0 \) by (3.2). Expanding carefully the boundary expression (2.19) and multiplying the result with a factor \( -\frac{m^2}{2\Delta} \) we obtain at the boundary point:

\[
V(b)(0, t, v) = \frac{\beta}{i\Delta^2} y^+ y^- \Omega_1^2(0) \Omega_2^2(0) \phi'(0) \sigma_3 + \frac{m}{2\Delta} \Omega_1(0) \Omega_2(0) \left( v(y^+ e_{21} - y^- e_{12}) + v^{-1}(y^- e_{21} - y^+ e_{12}) \right)
\]

where \( \Delta = y^+ \Omega_1^2(0) + y^- \Omega_2^2(0) \), and \( y^\pm = \pm e^\pm i\xi \).

By requiring \( V(0) = V(b)(0) \) we obtain the corresponding boundary conditions. Finally from the ‘bulk’ zero curvature condition and from the later expression the following equations of motion and mixed boundary conditions are entailed:

\[
\ddot{\phi}(x, t) - \frac{m^2}{\beta^2} \sin(\beta \phi(x, t)) = -\frac{m^2}{\beta} \sin(\beta \phi(x, t)) \sin(\frac{\beta}{2} \phi(0)) = 0,
\]

which of course coincide with the equations of motion found in [40, 36] (recall also the identification of boundary parameters (4.22) and the boundary conditions found earlier. Notice that the obtained boundary conditions are easily obtained from the generic situation described in (4.27) by simply setting the non diagonal contributions to zero.

We are however mostly interested in the case where the \( K \)-matrix is degenerate (in the homogeneous gradation). Consider for instance the situation where \( K(\lambda) = \text{diag}(e^\lambda, e^{-\lambda}) \). The Hamiltonian and boundary \( V \)-operator in this case are given respectively by:

\[
\mathcal{H} = \int_0^L dx \left[ \frac{1}{2} (\pi^2 + \phi'^2) + \frac{m^2}{\beta^2} (1 - \cos \beta \phi) \right] + \frac{2}{\beta} \phi'(0)
\]

and

\[
V(b)(0, v) = \frac{m}{4} \Omega_1(0) \Omega_2^{-1}(0) \left( ve_{21} - v^{-1} e_{12} \right).
\]

The boundary conditions emerging from the Hamiltonian are: \( \phi'(0) = 0 \) whereas requiring \( V(b)(0, v) = V(0, v) \) in addition to the field space derivative being zero one more constraint is obtained:

\[
e^{i\frac{\beta}{2} \phi(0)} = 0
\]
which of course is also automatically obtained from the boundary conditions found previously in the full diagonal case at $e^{-i \xi} = 0$. Note that there is no way to trace the extra constraint (4.36) from Hamiltonian point of view although we have to note that such a constraint is not incompatible with the Hamiltonian.

To conclude we note that in the degenerate case some important information is automatically lost when considering the Hamiltonian description. More precisely, in the degenerate case there is no $\xi$ dependence so constraints of the type (4.36) disappear when examining the boundary conditions from the Hamiltonian viewpoint. Whenever we pass from the most general situation to some special situation some information is lost and inconsistencies between the two descriptions arise. This of course happens only when the $K$-matrix possesses several boundary parameter and some of them are set to zero or to infinity. We shall examine in the following section a similar situation for the next model of the hierarchy, the $A_2^{(1)}$ theory, and we shall see that the arising inconsistencies may be explained in the same spirit.

5 The boundary $A_2^{(1)}$ case

We come now to the second member of the hierarchy and the first model of this class exhibiting both types of distinct boundary conditions SP and SNP, i.e. the $A_2^{(1)}$ model. In this case we have:

$$\alpha_1 = (1, 0), \quad \alpha_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), \quad \alpha_3 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$$ (5.1)

define also the following $3 \times 3$ generators

$$E_1 = E_{12}^t = e_{12}, \quad E_2 = E_{23}^t = e_{23}, \quad E_3 = E_{31}^t = -e_{31}. \quad (5.2)$$

The diagonal Cartan generators $H_{1,2}$ are then:

$$H_1 = \frac{1}{2}(e_{11} - e_{22}), \quad H_2 = \frac{1}{2\sqrt{3}}(e_{11} + e_{22} - 2e_{33}) \quad (5.3)$$

Let $T'(x, y, u) = T(x, y, u^{-1})$ and $U'(x, u) = U(x, u^{-1})$. Following the logic described previously (see also [27]) for the sine-Gordon model, we aim at expressing the part associated to $E_+, E_-$ in $U, U'$ respectively independently of the fields, thus we consider the following gauge transformation:

$$T(x, y, u) = \Omega(x) \bar{T}(x, y, u) \Omega^{-1}(y),$$

$$T'(x, y, u) = \Omega^{-1}(x) \bar{T}'(x, y, u) \Omega(y) \quad (5.4)$$
where we define
\[
\Omega(x) = \text{diag} \left( \Omega_1(x), \Omega_2(x), \Omega_3(x) \right) = e^{\beta \Phi(x) \cdot H}.
\] (5.5)

From equation (2.1) the gauge transformed operators \( \tilde{U}, \tilde{U}' \) are expressed as:
\[
\tilde{U}(x, t, u) = \Omega^{-1}(x) U(x, t, u) \Omega(x) - \Omega^{-1}(x) \frac{d\Omega(x)}{dx} \right]
\tilde{U}'(x, t, u) = \Omega(x) U'(x, t, u) \Omega^{-1}(x) - \Omega(x) \frac{d\Omega^{-1}(x)}{dx}.
\] (5.6)

After implementing the gauge transformations \( \tilde{U}, \tilde{U}' \) take the following simple forms:
\[
\tilde{U}(x, t, u) = \frac{\beta}{2} \tilde{S} \cdot H + \frac{m}{4} \left( uE_+ + \frac{1}{u} X_- \right), \quad \tilde{U}'(x, t, u) = \frac{\beta}{2} \hat{S} \cdot H + \frac{m}{4} \left( uE_+ + \frac{1}{u} X_+ \right)
\] (5.7)

where we define:
\[
\tilde{S} = \Pi - \partial_x \Phi, \quad \hat{S} = \Pi + \partial_x \Phi, \quad X_- = e^{-\beta \Phi \cdot H} E_- e^{\beta \Phi \cdot H}, \quad X_+ = e^{\beta \Phi \cdot H} E_+ e^{-\beta \Phi \cdot H}
\] (5.8)

\( \tilde{T}, \tilde{U} \) also satisfy (2.1), and \( \tilde{S}, \hat{S} \) are vectors with two components \( f_i, \hat{f}_i, i \in \{1, 2\} \) respectively.

Consider again the ansatz (4.8) for \( \tilde{T}, \tilde{T}' \) as \( |u| \to \infty \). As in the previous section inserting expressions (4.9) in (2.1) one then identifies the coefficients \( W^{(k)}_{ij} \) and \( Z^{(k)}_{ii} \). Indeed from (2.1) we obtain the following fundamental relations:
\[
\frac{dZ}{dx} = \tilde{U}^{(D)} + (\tilde{U}^{(O)} W)^{(D)}
\]
\[
\frac{dW}{dx} + W \tilde{U}^{(D)} W + W(\tilde{U}^{(O)} W)^{(D)} - \tilde{U}^{(O)} - (\tilde{U}^{(O)} W)^{(O)} = 0
\] (5.9)

where the superscripts \( O, D \) denote off-diagonal and diagonal part respectively. Similar relations may be obtained for \( \tilde{Z}, \tilde{W} \), in this case \( \tilde{U} \to \tilde{U}' \). We omit writing these equations here for brevity.

It will be useful in what follows to introduce some compact notation:
\[
\frac{\beta}{2} \tilde{S} \cdot H = \text{diag}(a, b, c), \quad \frac{\beta}{2} \hat{S} \cdot H = \text{diag}(\hat{a}, \hat{b}, \hat{c}), \quad e^{\beta \alpha_i \cdot \Phi} = \gamma_i.
\] (5.10)

Explicit expressions of \( a, b, c \) and \( \gamma_i \) are given by:
\[
a = \frac{\beta}{2} \left( \frac{f_1}{2} + \frac{f_2}{2\sqrt{3}} \right), \quad b = \frac{\beta}{2} \left( \frac{f_1}{2} + \frac{f_2}{2\sqrt{3}} \right), \quad c = -\frac{\beta}{2} \frac{f_2}{\sqrt{3}}
\]
\[
\gamma_1 = e^{\beta \phi_1}, \quad \gamma_2 = e^{\beta(-\frac{1}{2} \phi_1 + \frac{\sqrt{3}}{2} \phi_2)}, \quad \gamma_3 = e^{\beta(-\frac{1}{2} \phi_1 - \frac{\sqrt{3}}{2} \phi_2)}.
\] (5.11)

apparently \( \hat{a}, \hat{b}, \hat{c} \) are defined in the same way as \( a, b, c \) but with \( f_i \to \hat{f}_i \).
The computation of \( W, \dot{W} \) is essential for what follows. First it is important to discuss the leading contribution of the above quantities as \( |u| \to \infty \). To achieve this we shall need the explicit form of \( Z^{(-)} \), \( \dot{Z}^{(-)} \):

\[
Z^{(-)}(x, y) = \frac{m(x - y)}{4} \begin{pmatrix}
 e^{i\pi \frac{m}{4}} & e^{i\pi \frac{m}{4}} \\
 e^{i\pi \frac{m}{4}} & e^{i\pi \frac{m}{4}} \\
-1 & -1
\end{pmatrix}, \quad \dot{Z}^{(-)}(x, y) = \frac{m(x - y)}{4} \begin{pmatrix}
 e^{-i\pi \frac{m}{4}} & e^{-i\pi \frac{m}{4}} \\
 e^{-i\pi \frac{m}{4}} & e^{-i\pi \frac{m}{4}} \\
-1 & -1
\end{pmatrix}.
\]

(5.12)

From the formulas (5.9) the matrices \( W^{(k)}, \dot{W}^{(k)}, Z^{(k)}, \dot{Z}^{(k)} \) may be determined. In particular, we write below explicit expressions of these matrices for the first orders, which will be necessary in the subsequent sections (see also [26]):

\[
W^{(0)} = \dot{W}^{(0)} = \begin{pmatrix}
 0 & e^{i\pi \frac{m}{4}} & 1 \\
 e^{i\pi \frac{m}{4}} & e^{i\pi \frac{m}{4}} & 0 \\
e^{i\pi \frac{m}{4}} & e^{-i\pi \frac{m}{4}} & 0
\end{pmatrix}, \quad \frac{m}{4}W^{(1)} = \begin{pmatrix}
 0 & e^{2i\pi \frac{m}{4}}a & c \\
-a & 0 & b \\
e^{i\pi \frac{m}{4}}c & -b & 0
\end{pmatrix}, \quad \frac{m}{4}\dot{W}^{(1)} = \begin{pmatrix}
 0 & -\hat{b} & -\hat{a} \\
-\hat{a} & 0 & -\hat{c} \\
-\hat{c} & -e^{i\pi \frac{m}{4}}\hat{c} & 0
\end{pmatrix}.
\]

(5.13)

For computing the boundary conserved quantities, energy and momentum, we shall in addition need the following expressions:

\[
\begin{align*}
\frac{dZ^{(1)}_{11}}{dx} &= e^{-i\pi \frac{m}{3}} \frac{m}{4} (\gamma_1 + \gamma_2 + \gamma_3) + 4e^{-i\pi \frac{m}{3m}} (a' - c') + 4e^{-i\pi \frac{m}{6m}} (a^2 + b^2 + c^2) \\
\frac{dZ^{(1)}_{22}}{dx} &= e^{i\pi \frac{m}{3}} \frac{m}{4} (\gamma_1 + \gamma_2 + \gamma_3) + 4e^{i\pi \frac{m}{3m}} (b' - a') + 4e^{i\pi \frac{m}{6m}} (a^2 + b^2 + c^2) \\
\frac{d\dot{Z}^{(1)}_{11}}{dx} &= 4e^{i\pi \frac{m}{3}} \frac{m}{4} (\gamma_1 + \gamma_2 + \gamma_3) - 4e^{i\pi \frac{m}{3m}} (b' - \hat{a}') + 4e^{i\pi \frac{m}{6m}} (a^2 + \hat{b}^2 + \hat{c}^2) \\
\frac{d\dot{Z}^{(1)}_{22}}{dx} &= 4e^{-i\pi \frac{m}{3}} \frac{m}{4} (\gamma_1 + \gamma_2 + \gamma_3) - 4e^{-i\pi \frac{m}{3m}} (b' - \hat{c}') + 4e^{-i\pi \frac{m}{6m}} (\hat{a}^2 + \hat{b}^2 + \hat{c}^2).
\end{align*}
\]

(5.14)

### 5.1 SNP boundary conditions

We shall focus in this section on the analysis of the SNP integrable boundary conditions in \( A^{(1)}_2 \) ATFT. Comparison with some already known results [9] will validate our approach, which then presents the advantage of being systematically implementable once a non dynamical “boundary” matrix is chosen. The boundary \( \mathcal{V} \)-operator in this case is given by (2.17), (2.19). We assume here for simplicity, but without losing generality, Schwartz boundary conditions at \( x = -L \) and \( K^- \propto \mathbb{I} \) (see also [26]). Taking into account the ansatz for the
monodromy matrix \((4.8)\) as well as bearing in mind that as \(u \to \infty\) the main contribution for the diagonal terms comes from \(e^{Z_33}, \hat{e}^{Z_33}\) (see also [26]), we conclude

\[
\mathbb{V}(x, t, u, v) = \left[ (1 + W(x, u))^{-1} \Omega^{-1}(x) r(uv^{-1}) \Omega(x)(1 + W(x)) \right]_{33} + \left[ (1 + \hat{W}(x, u)) \Omega^{-1}(x) r^{t_1}(u^{-1}v^{-1}) \Omega(x)((1 + \hat{W}(x))^{-1})^{t_1} \right]_{33} \quad (5.15)
\]

We recall that the superscript \(^{t_1}\) denotes transposition in the first space. Also, in the expressions with ‘hat’ we simply consider \(\Phi \to -\Phi\). For further technical details we refer the interested reader to [26]. Note that in this case the limit \(u \to \infty\) is easier to consider due to the expressions (5.12). In any case, although technically more involved, one can show that the \(u \to -\infty\) limit provides the same conserved quantities and Lax pairs. In the following we shall expand expression (5.15), so we need expansions of all the involved quantities:

\[
r(uv^{-1}) \sim r^{(0)} + u^{-1}r^{(1)} + \mathcal{O}(u^{-2}), \quad r^{t_1}(u^{-1}v^{-1}) \sim -r^{(0)} - u^{-1}\hat{r}^{(1)} + \mathcal{O}(u^{-2}) \quad (5.16)
\]

where we define:

\[
r^{(0)} = \sum_{i=1}^{3} e_{ii}, \quad r^{(1)} = 2v(e_{21} \otimes e_{12} + e_{32} \otimes e_{23} + e_{13} \otimes e_{31})
\]

\[
\hat{r}^{(1)} = 2v^{-1}(e_{21} \otimes e_{21} + e_{32} \otimes e_{32} + e_{13} \otimes e_{13}) \quad (5.17)
\]

From the first order of the expansion of the \(\mathbb{V}\)-operator and after multiplying with a factor of \(-\frac{3m}{8}\) we obtain:

\[
\mathbb{V}(x, t, v) = -\frac{\beta}{2} \Phi^t(x, t) \cdot H + \frac{mv}{4} \Omega(x, t) E_+ \Omega^{-1}(x, t) - \frac{mv}{4} \Omega^{-1}(x, t) E_- \Omega(x, t) \quad (5.18)
\]

which as anticipated coincides with the bulk \(\mathbb{V}\)-operator of \(A_2^{(1)} (3.2)\).

In order to obtain the explicit form of the boundary \(\mathbb{V}\)-operator we should also review known results on the solutions of the reflection equation for SNP boundary conditions. The generic solution for the \(A_n^{(1)}\) case in the principal gradation are given by [13, 18]:

\[
K(\lambda) = (ge^\lambda + \bar{g}e^{-\lambda}) \sum_{i=1}^{n+1} e_{ii} + \sum_{i>j} f_{ij}e^{\lambda - \frac{n+1}{n+1}(i-j)}e_{ij} + \sum_{i<j} f_{ij}e^{-\lambda - \frac{n+1}{n+1}(i-j)}e_{ij} 
\]

\[
g = q^{\frac{1}{2} + \frac{n+1}{2}}, \quad \bar{g} = \pm q^{\frac{1}{2} - \frac{n+1}{2}}, \quad f_{ij} = \pm q^{\frac{n+1}{2}}, \quad f_{ji} = q^{-\frac{n+1}{2}}, \quad i < j. \quad (5.19)
\]

In order to effectively compare with the results of [13] as well as being compatible with [9] we always express both \(r\) and \(K\) matrices in the principal gradation (see also [26]). The parameter \(q\) in the solution (5.19) is the parameter of the underlying quantum algebra (quantum case) \(U_q(\hat{sl}_3)\). It is clear that we are dealing here with the classical limit \(q \to 1\) of
the later solution (5.19) compatible with the classical quadratic algebra (2.9). In this limit: \( g \to 1, \bar{g} \to \pm 1, f_{ij} \to \pm 1, f_{ji} \to 1, i < j \).

We come back now to the \( A^{(1)}_2 \) case. Recall that \( K^+(u) = K^t(u^{-1}) \) then \( K^+ \) is a \( 3 \times 3 \) matrix written explicitly as:

\[
K^+(u) = u \frac{\partial}{\partial u} G + u \frac{\partial}{\partial u} F + u^{-\frac{\partial}{\partial u}} G + u^{-\frac{\partial}{\partial u}} F
\]

where

\[
G = g I, \quad \bar{G} = \bar{g} I,
\]

\[
\bar{F} = f_{12} e_{21} + f_{23} e_{32} + f_{31} e_{13},
\]

\[
F = f_{21} e_{12} + f_{32} e_{23} + f_{13} e_{31}
\]

and the coefficients \( g, \bar{g}, f_{ij} \) are given in (5.19) with \( n = 2 \) and \( q \to 1 \). Note, as in the previous case for the sine-Gordon model, that the boundary case for \( x = 0 \) has to be treated separately. Indeed, in this case the operator \( V(0, u) \) takes the form:

\[
\bar{V}(x = 0) = \left[(1 + \bar{W}^t(0))\Omega^{-1}(0)K^+(u)\Omega(1 + W(0))\right]_{33}
\]

\[
\times \left\{\left[(1 + \bar{W}^t(0))\Omega^{-1}(0)K^+(u)r(uv^{-1})\Omega(0)(1 + W(0))\right]_{33}
\]

\[
+ \left[(1 + \bar{W}^t(0))\Omega^{-1}(0)r^{11}(u^{-1}v^{-1})K^+(u)\Omega(0)(1 + W(0))\right]_{33}\right\}
\]

(5.21)

again using the standard procedure we obtain from the first order of the expansion and after multiplying with an overall factor \( -\frac{3m}{8} \):

\[
V^{(b)}(0, t, u) = V(0, t, u) + \Delta V(0, t, u), \quad \text{where}
\]

\[
\Delta V(0, t, u) = \frac{\beta}{2} \Phi'(0) \cdot H + \frac{m}{8g} \left(\Omega_1(0)\Omega_2^{-1}(0)f_{12} + \Omega_1^{-1}(0)\Omega_3(0)f_{31}\right)e_{11}
\]

\[
+ \left(\Omega_2(0)\Omega_3^{-1}(0)f_{23} - \Omega_1(0)\Omega_2^{-1}(0)f_{12}\right)e_{22}
\]

\[
- \left(\Omega_2(0)\Omega_3^{-1}(0)f_{23} + \Omega_1^{-1}(0)\Omega_3(0)f_{31}\right)e_{33}\right)
\]

(5.22)

\( V \) is the bulk \( V \)-operator given in (5.18). The equations of motion are again obtained from the ‘bulk’ zero curvature condition:

\[
- \ddot{\Phi}(x, t) + \Phi''(x, t) = \frac{m^2}{2\beta} \sum_{i=1}^{3} \alpha_i e^{\beta \alpha_i \cdot \Phi(x, t)}.
\]

(5.23)

Using the same argument as in the previous section we conclude:

\[
\Delta V(0) = 0 \Rightarrow \phi_1'(0) = \frac{m}{2g\beta} \left(2f_{12} e^{\frac{\beta}{2} \alpha_1 \cdot \Phi(0)} - f_{23} e^{\frac{\beta}{2} \alpha_2 \cdot \Phi(0)} + f_{31} e^{\frac{\beta}{2} \alpha_3 \cdot \Phi(0)}\right)
\]

\[
\phi_2'(0) = \frac{m}{2g\beta} \left(f_{23} e^{\frac{\beta}{2} \alpha_2 \cdot \Phi(0)} + f_{31} e^{\frac{\beta}{2} \alpha_3 \cdot \Phi(0)}\right)
\]

(5.24)
The latter boundary conditions agree as expected with those analyzed in [9].

The quantities found above are associated to the first non-trivial boundary integral of motion:

\[ H_1^{(b)} = \int_{-L}^{0} dx \left( \sum_{i=1}^{2} (\pi_i^2 + \phi_i^2) + \frac{m^2}{\beta^2} \sum_{i=1}^{3} e^{\beta \alpha_i} \Phi \right) + \frac{2m}{\beta^2} \left( f_{12} e^{\beta \alpha_1} \Phi(0) + f_{23} e^{\beta \alpha_2} \Phi(0) - f_{34} e^{\beta \alpha_3} \Phi(0) \right). \]

(5.25)

It is quite easy to check the consistency of the Lax formulation procedure in [4], i.e. that the Hamiltonian above leads exactly to the same equations of motion and boundary conditions as the zero curvature conditions. More precisely, the equations of motion from the Hamiltonian are obtained through:

\[ \frac{\partial \phi_i(x,t)}{\partial t} = \{ H, \phi_i(x,t) \}, \quad \frac{\partial \pi_i(x,t)}{\partial t} = \{ H, \pi_i(x,t) \} \]

\[ x \in [-L, 0] \]

(5.26)

The boundary Lax pair in the SNP case was also constructed in [9]. The boundary conditions and the Hamiltonian extracted in [26] coincide with the ones of [9]; comparison between our Lax pair and that derived in [9] shows that the relevant Lax pairs are also the same. Although we have to note that in [9] everything is expressed in terms of \( \theta \) and \( \delta \) functions given that the Lax pair is derived taking into account some overlapping boundary regime, which in the present framework is unnecessary. Moreover in [9] the Lax pair construction requires a priori knowledge of the Lagrangian, given that the boundary Lax pair explicitly contains the Lagrangian boundary contribution. In our formulation on the other hand the Lax pair is derived from first principles from the expansion of (2.17), (2.19), and requires no a priori knowledge of the boundary terms in the associated Hamiltonian, although the relevant Hamiltonian may be simultaneously obtained from the expansion of \( \ln t(\lambda) \).

This ends our discussion on the sine-Gordon and SNP \( A_2^{(1)} \) ATFT. We have been able to systematically derive consistent explicit expressions from first principle derivation for their Lax equations by evaluating exactly the boundary contribution. Comparison with the case-by-case derivation in [9] shows the same results. Our method is therefore validated and can be now extended to a much broader set of models beyond the ATFT. The next example will be the yet untreated case of \( A_2^{(1)} \) ATFT under the “soliton-preserving” reflection algebra (see also [28] on SP boundary conditions within ATFT).
5.2 SP boundary conditions

We shall now examine the SP boundary conditions (see [26]). We shall consider below two cases with the K-matrix being non-diagonal and diagonal respectively.

5.2.1 Non-diagonal K-matrix

We choose for $K$ the non-diagonal solution of the reflection equation found in [44, 45], and demonstrate how this particular choice of K-matrix contributes to the integrals of motion, and the relevant boundary Lax pair. We consider for simplicity Schwartz boundary conditions at the end point $x = -L$, whereas the right boundary is described by this K-matrix [44, 45]:

$$K^+(u) = V K(u) V \quad \text{with} \quad V = \text{diag}(1, e^{\frac{\lambda}{4}}, e^{\frac{\lambda}{4}})$$

$$K(u) = (e^{4\lambda} - 1)g^2 + (e^{4\lambda} - 1)(\xi e^{2\lambda} - \alpha - \beta)g - (\xi e^{4\lambda} - (\alpha + \beta)\xi e^{2\lambda} + \alpha\beta)I(5.27)$$

where we define the $3 \times 3$ matrix [44, 45]:

$$g = (\alpha + \beta)e_{11} + x^+ e_{12} + x^- e_{21}, \quad x^+ x^- = -\alpha\beta \quad (5.28)$$

$\alpha$, $\beta$, $\xi$ are free independent boundary parameters, and $x^\pm$ satisfy (5.28). To proceed with the expansion of $\ln t(\lambda)$ and $V$ in powers of $u^{-1}$ we shall need the expansions of $r$, $\hat{r}$, $K^+$, (as $|u| \to \infty$):

$$\hat{r}(uv) \sim r^{(0)} + u^{-1}\hat{r}^{(1)} + O(u^{-1})$$

$$K^+(u) \sim K^{(0)} + u^{-1}K^{(1)} + O(u^{-2}). \quad (5.29)$$

where $r^{(0)}$ is defined in (5.17) and

$$\hat{r}^{(1)} = 2v^{-1}(e_{12} \otimes e_{21} + e_{23} \otimes e_{32} + e_{31} \otimes e_{13}),$$

$$K^{(0)} = -\xi^2 e_{33} + \xi x^+ e_{12} + \xi x^- e_{21}, \quad K^{(1)} = \xi (\alpha + \beta)e_{11} \quad (5.30)$$

We shall consider henceforth for simplicity $x^+ = x^-$, and also set $y = x^\pm \xi$. The integrals of motion follow from the asymptotic expansion of the logarithm of the open transfer matrix as $iu \to \pm \infty$, i.e.

$$\ln t(iu \to \infty) = \sum \frac{\mathbb{I}_n}{u^n} = \sum \frac{Z_{11}^{(n)} - \hat{Z}_{11}^{(1)}}{u^n} + \sum \frac{h_n}{u^n}$$

$$\ln t(iu \to -\infty) = \sum \frac{\mathbb{I}_n}{u^n} = \sum \frac{Z_{22}^{(n)} - \hat{Z}_{22}^{(1)}}{u^n} + \sum \frac{\bar{h}_n}{u^n} \quad (5.31)$$
More technical details on such expansions will be found by the interested reader in [26]. The integrals of motion obtained from the first order of the expansion are given by:

\[ I_1 = -\frac{\beta^2}{12m}(P^{(b)} + i\sqrt{3}H^{(b)}), \]
\[ \tilde{I}_1 = -\frac{\beta^2}{12m}(P^{(b)} - i\sqrt{3}H^{(b)}). \] (5.32)

The momentum and energy are directly obtained from the above conserved quantities and defined as:

\[ P_i = \int_{-L}^0 dx \left( \pi_i \dot{\phi}_i - \pi_i \dot{\phi}_i \right) + \sum_{i=1}^2 \pi_i(0) \phi_i(0) - \frac{12m}{\beta^2 \Delta} \xi(\alpha + \beta) \Omega^2(0) \]
\[ -\frac{8}{\beta^2 \Delta} \left\{ y \Omega_1(0) \Omega_2(0) \left( -4a(0) + b(0) - 4\dot{a}(0) + \dot{b}(0) \right) + \xi^2 \Omega_3^2(0) \left( c(0) - b(0) + \dot{c}(0) - \dot{b}(0) \right) \right\} \]

\[ H^{(b)} = \int_{-L}^0 dx \left( \sum_{i=1}^2 \left( \pi_i^2 + \phi_i^2 \right) + \frac{m^2}{\beta^2} \sum_{i=1}^3 e^{3\alpha \cdot \Phi} \right) \]
\[ -\frac{8}{\beta^2 \Delta} \left\{ y \Omega_1(0) \Omega_2(0) \left( c(0) - a(0) - \dot{c}(0) + \dot{a}(0) \right) + \xi^2 \Omega_3^2 \left( c(0) - b(0) + \dot{c}(0) + \dot{b}(0) \right) \right\}. \] (5.33)

where \( \Delta = y \Omega_1(0) \Omega_2(0) - \xi^2 \Omega_3^2(0) \), and \( \Omega \) and \( a, b, c, \dot{a}, \dot{b}, \dot{c} \) have been previously defined. As already pointed out in [26] the two boundary cases exhibit essential differences: in SNP the c-number \( K \)-matrix contains no free parameters, and consequently no such parameters occur in the deduced integrals of motion. In the SP case on the other hand the \( K \)-matrix contains free parameters, which explicitly appear in the boundary integrals of motion. Moreover, in the SNP case, only the boundary Hamiltonian belongs to the family of commuting quantities, whereas now both boundary Hamiltonian and momentum turn out to be conserved quantities. Note that we are here in a situation where the terminology “boundary effects” is misleading, since it suggests that no conserved momentum could exist due to the presence of a physical boundary. As we have shown here this is not true, indicating that the physical interpretation of this field theory is more subtle and has to be further explored.

We shall now derive the associated boundary \( V \)-operator. From the two different limits \((iu \to \pm \infty)\) we obtain essentially two quantities from the first order of each expansion:

\[ \mathbb{V}_1^{(1)}(x, t, v) = \frac{4}{3m} \left( U(x, t, v) - i\sqrt{3}V(x, t, v) \right) \text{ from } \ u \to \infty \]
\[ \mathbb{V}_2^{(1)}(x, t, v) = \frac{4}{3m} \left( U(x, t, v) + i\sqrt{3}V(x, t, v) \right) \text{ from } \ u \to -\infty \] (5.34)

where \( V, U \) is the Lax pair of the \( A_2^{(1)} \) theory defined in [32]. The bulk operator as already mentioned is independent of the choice of \( K \)-matrix.
The expressions of the $\mathbb{V}$-operator for each end point are given below:

\[
\mathbb{V}(0, t, u, v) = \left[ (1 + \hat{W}(0))^{-1} \Omega(0) K^+(u) \Omega(0)(1 + W(0)) \right]^{-1}_{jj} \\
\times \left\{ \left[ (1 + \hat{W}(0))^{-1} \Omega(0) K^+(u) r(uv^{-1}) \Omega(0)(1 + W(0)) \right]_{jj} \right\}, \quad j \in \{1, 2\}.
\]

(5.35)

At the boundary point we get:

\[
\mathbb{V}_1^{(b)}(0, v) = \frac{2}{m} \left( \mathbb{U}^{(b)}(0, v) - i\sqrt{3} \mathbb{V}^{(b)}(0, v) \right), \quad iu \to \infty
\]

\[
\mathbb{V}_2^{(b)}(0, v) = \frac{2}{m} \left( \mathbb{U}^{(b)}(0, v) + i\sqrt{3} \mathbb{V}^{(b)}(0, v) \right), \quad iu \to -\infty.
\]

(5.36)

Let us focus on $\mathbb{V}^{(b)}$ (5.36), which is associated to the Hamiltonian of the system (5.33) as will become transparent in the following. Note that $\mathbb{U}^{(b)}$ similarly plays the role of the $\mathbb{V}$-operator associated to the momentum of the system. We then define:

\[
\mathbb{V}^{(b)}(0, v) = \frac{1}{\Delta^2} \xi^2 \Omega_3^2(0) y \Omega_1(0) \Omega_2(0) \left( c(0) - \dot{c}(0) \right) \left( e_{11} + e_{22} - 2e_{33} \right)
\]

\[
+ \frac{mv}{2\Delta} \left( -\xi^2 \Omega_2(0) \Omega_3(0) e_{23} - y \Omega_2(0) \Omega_3(0) e_{31} \right)
\]

\[
- \frac{mv}{2\Delta} \left( -\xi^2 \Omega_2(0) \Omega_3(0) e_{32} - y \Omega_2(0) \Omega_3(0) e_{13} \right)
\]

\[
\mathbb{U}^{(b)}(0, v) = \frac{1}{\Delta^2} \xi^2 \Omega_3^2(0) y \Omega_1(0) \Omega_2(0) \left( a(0) - \dot{a}(0) + \dot{b}(0) \right) \left( e_{11} + e_{22} - 2e_{33} \right)
\]

\[
+ \frac{\Omega_1(0) m \xi}{2\Delta} \left( \alpha(0) + \beta(0) \right) \left( y \Omega_1(0) \Omega_2(0)(e_{11} - e_{22}) - 2\xi^2 \Omega_3^2(0)(e_{11} - e_{33}) \right)
\]

\[
+ \frac{mv}{2\Delta} \left( 2y \Omega_1^2(0) e_{21} - \xi^2 \Omega_2(0) \Omega_3(0) e_{32} + y \Omega_2(0) \Omega_3(0) e_{31} \right)
\]

\[
+ \frac{mv}{2\Delta} \left( 2y \Omega_1^2(0) e_{21} - \xi^2 \Omega_2(0) \Omega_3(0) e_{32} + y \Omega_2(0) \Omega_3(0) e_{13} \right).
\]

We shall focus on the equations of motion emerging from the Hamiltonian $\mathcal{H}^{(b)}$ via (5.26). We expect that identical equations arise from the zero curvature condition. Indeed the bulk part gives rise to the equations of motion that coincide with the familiar ones (see equations (5.23)).

From the Hamiltonian derivation the boundary conditions arise by requiring the boundary terms (proportional to $\phi_i'(0)$) to vanish, yielding:

\[
\Phi'(0) = 0 \quad y \Omega_1(0) \Omega_2(0) = -\xi^2 \Omega_3^2(0).
\]

(5.37)

From the zero curvature condition however, as analyzed in section 2, we require $\mathbb{V}(0) = \mathbb{V}^{(b)}(0)$, and we end up with an extra constraint in addition to the ones (5.37) emerging from
the Hamiltonian, i.e.
\[ \Omega_1(0)\Omega_2^{-1}(0) = 0. \] (5.38)

Although this extra constraint is compatible with the Hamiltonian description it is nevertheless missing when analyzing the boundary conditions from the Hamiltonian point of view. This ‘missing’ information may be associated to the fact that the \( K \)-matrix we have chosen is not the most general matrix to start with, and exhibit some degenerate or at least non-generic behavior, as we shall comment in the final section. Finally one may derive the equations of motion from the boundary \( V \)-operator associated to the momentum \( P \) along the lines described above.

To conclude we have here also been able to explicitly derive exact expressions of the boundary Hamiltonian and momentum –both conserved quantities for the particular boundary conditions– as well as the associated boundary Lax pairs.

### 5.2.2 Diagonal \( K \)-matrix

We shall now focus on diagonal degenerate solutions of the reflection equation given by the following expressions (in the principal gradation):

\[
K_{(l)}(\lambda, \xi) = \sinh(\lambda + i\xi)e^{-\lambda} \sum_{j=1}^{l} e^{-\frac{4\lambda}{n+1}(j-1)}e_{jj} + \sinh(-\lambda + i\xi)e^{\lambda} \sum_{j=l+1}^{n} e^{-\frac{4\lambda}{n+1}(j-1)}e_{jj} \] (5.39)

(recall \( u = e^{\frac{2\lambda}{n+1}} \)). As we have seen from the previous section on the sine-Gordon degenerate \( K \)-matrices give rise to inconsistencies. Some information is usually lost from the Hamiltonian point view when taking the limit from the generic non-degenerate \( K \) matrix to the degenerate one, which is a special case of the generic solution. To obtain the \( K \)-matrix in the homogeneous gradation one implements a simple gauge transformation (see e.g. [26]).

In the \( A_2^{(1)} \) case we end up with two types of diagonal boundary matrices corresponding to the two possible values \( l = 1, 2 \). We shall consider an example here to demonstrate how the particular choice of boundary \( K \)-matrix contributes to the integrals of motion. It is practical for the following to consider a non-trivial left boundary described by \( K_{(1)} \), and a right boundary described by the \( K_{(2)} \)-matrix, i.e.

\[
K^+(u, \xi^+) = K_{(1)}(u^{-1}, \xi^+), \quad K^-(u, \xi^-) = K_{(2)}(u, \xi^-). \] (5.40)

We now proceed with the expansion of \( V \) in powers of \( u^{-1} \). The bulk part, for \( x \neq 0, -L \), is given in (5.34).
To obtain the generic results with the least effort it is practical to consider the two different types of $K$-matrices to each end of the theory. The expressions of the $V$-operator for each end point $x_b = 0, -L$ are given by

\[
\mathbb{V}_1^{(b)}(x_b, t, v) = \frac{4}{m} (\mathbb{U}^{(b)}(x_b, t, v) - i\sqrt{3} \mathbb{Y}^{(b)}(x_b, t, v) ) \quad \text{for} \quad iu \to \infty \\
\mathbb{V}_2^{(b)}(x_b, t, v) = \frac{4}{m} (\mathbb{U}^{(b)}(x_b, t, v) + i\sqrt{3} \mathbb{Y}^{(b)}(x_b, t, v) ) \quad \text{for} \quad iu \to -\infty.
\]

We are mostly interested in $\mathbb{V}^{(b)}$, which we shall associate to the Hamiltonian of the system as will become transparent in the following. Note that $\mathbb{U}^{(b)}$ plays the role of the $V$-operator associated to the momentum of the system. We then have

\[
\mathbb{V}^{(b)}(0, v) = \frac{m}{4} \Omega_2^2(0) \Omega_3^{-2}(0) (v e_{23} - v^{-1} e_{32}) \\
\mathbb{V}^{(b)}(-L, v) = \frac{m}{4} \Omega_2^2(-L) \Omega_3^{-2}(-L) (v e_{12} + v^{-1} e_{21}).
\]

Just for the record we also give:

\[
\mathbb{U}^{(b)}(0, v) = \frac{m}{2} e^{-2i\xi^+} \Omega_1^2(0) \Omega_3^{-2}(0) (e_{33} - e_{11}) + \frac{m}{4} \Omega_2^2(0) \Omega_3^{-2}(0) (v e_{23} + v^{-1} e_{32}) \\
\mathbb{U}^{(b)}(-L, v) = \frac{m}{2} e^{-2i\xi^+} \Omega_1^2(-L) \Omega_3^{-2}(-L) (e_{33} - e_{11}) + \frac{m}{4} \Omega_2^2(-L) \Omega_3^{-2}(-L) (v e_{12} + v^{-1} e_{21}).
\]

The integrals of motion emerging from the first order of the asymptotics of the transfer matrix as $iu \to \pm \infty$ are given by:

\[
\mathbb{I}_1 = -\frac{\beta^2}{12m} (\mathcal{P}^{(b)} + i\sqrt{3} \mathcal{H}^{(b)}), \\
\mathbb{I}_1 = -\frac{\beta^2}{12m} (\mathcal{P}^{(b)} - i\sqrt{3} \mathcal{H}^{(b)}).
\]

The momentum and energy are directly obtained from the above conserved quantities and defined as:

\[
\mathcal{P}^{(b)} = \int_{-L}^{0} dx \sum_{i=1}^{2} (\pi_i \phi_i' - \pi_i' \phi_i) + \sum_{i=1}^{2} \pi_i(0) \phi_i(0) + \frac{8}{\beta} \alpha_2 \cdot \Pi(0) + \frac{12m}{\beta^2} e^{-2i\xi^+} \Omega_1^2(0) \Omega_3^{-2}(0) \\
- \sum_{i=1}^{2} \pi_i(-L) \phi_i(-L) - \frac{8}{\beta} \alpha_1 \cdot \Pi(-L) + \frac{12m}{\beta^2} e^{-2i\xi^+} \Omega_1^2(-L) \Omega_3^{-2}(-L) \\
\mathcal{H}^{(b)} = \int_{-L}^{0} dx \left( \sum_{i=1}^{2} (\pi_i^2 + \phi_i'^2) + \frac{m^2}{\beta^2} \sum_{i=1}^{3} e^{\beta \alpha_i \phi} \right) + \frac{8}{\beta} \alpha_2 \cdot \Phi(0) - \frac{8}{\beta} \alpha_1 \cdot \Phi'(-L).
\]

Naturally the two boundary cases are distinguished; in SNP the $c$-number $K$-matrix contains no free parameters, and consequently no free parameters occur in the entailed integrals of motion. In the SP case on the other hand the $K$-matrix contains free parameters, which explicitly appear in the boundary integrals of motion.
As mentioned earlier upon the ‘boundary’ quantity $V^{(b)}$ is associated to the Hamiltonian $\mathcal{H}^{(b)}$. We shall focus on the equations of motion emerging from the Hamiltonian via (5.26). The same equations should emerge from the zero curvature condition. Indeed the bulk part gives rise to the equations of motion that coincide with the familiar ones. Note however that from the Hamiltonian the boundary condition are:

$$\Phi'(0) = \Phi'(-L) = 0$$

(5.45)

whereas from the condition $V^{(b)}(x_b) = V(x_b)$ in addition to (5.45) extra constraints involving the boundary field appear, more precisely:

$$\Omega^{-2}_2(0) \Omega^2_3(0) \rightarrow 0 \quad \Omega^{-2}_1(-L) \Omega^2_2(-L) \rightarrow 0.$$  

(5.46)

Again as in the non-diagonal case a discrepancy between the two descriptions is apparent due to the degenerate nature of the $K$-matrix.

6 Comments

We have analyzed via the boundary Lax pair formulation two distinct types of boundary conditions. The SP case presents a particular interest given that certain subtle technical points arise requiring further clarification. Let us briefly comment on the difficulties emerging in this context.

We see that when the $K$-matrix possesses several free boundary parameters, discrepancies between the Hamiltonian and Lax description, which however are expected to be equivalent, are seen to emerge when some parameters are equal to zero (e.g. for SP boundary conditions in $A^{(1)}_2$ case and even in sine-Gordon). This is an intriguing point and its resolution is probably associated to defining a notion of “appropriate choice” of the $c$-number representation ($K$-matrix) of the reflection equation. A generic non-diagonal solution of the reflection equation is required as a starting point. For instance in sine-Gordon model, amongst all non-diagonal solutions (4.17) the only “legitimate” solution is the one with $x^+ = -x^-$. Such a restriction is here dictated by the requirement of a consistent asymptotic behavior of the generating function.

It is worth noting that diagonal $K$-matrices we have considered here as well as non-diagonal ones of the type [42] in $A_2^{(1)}$ (SP) have only two distinct eigenvalues (homogeneous gradation) hence their spectra are doubly degenerate. Similar discrepancies occur as we have seen in the context of sine-Gordon model when choosing to consider $K \propto I$ (homogeneous gradation). Such a matrix has just one distinct eigenvalue so it is doubly degenerate, leading again to inconsistencies. In the example we considered here in the $A_2^{(1)}$ case even though the
non-diagonal matrix $K$ matrix is not degenerate (homogeneous gradation), the matrix $g$ in (5.27) has a zero eigenvalue, and presumably this is the point that creates the problem in this case.

In a more algebraic framework, all solutions of the reflection equation –the solutions of the quantum reflection equation satisfy the classical reflection equation as well– considered in the present work are representations of the cyclotomic Hecke algebra (see e.g. [43, 44, 45]). The generators $g_l, g_0, l = 1, \ldots, N - 1$ of the cyclotomic algebra $C_N^{(n)}$ satisfy the following set of constraints:

\[
\begin{align*}
g_l g_{l+1} g_l &= g_{l+1} g_l g_{l+1}, \\
(g_l - q)(g_l + q^{-1}) &= 0, \\
\prod_{\alpha=1}^{n} (g_0 - \xi_\alpha) &= 0 \\
[ g_l, g_m ] &= 0, \quad |l - m| > 1, \\
[ g_0, g_l ] &= 0, \quad l > 1
\end{align*}
\] (6.1)

One expects that the most generic solution for the $A_{n-1}^{(1)}$ case should be expressed in terms of representations of the generator $g_0$ with all $\xi_\alpha \neq 0$ and $\xi_\alpha \neq \xi_\beta \forall l \neq k$, i.e. (see also (5.27))

\[
K(\lambda) = \sum_{\alpha=0}^{n-1} c_\alpha(\lambda) g_0^\alpha
\] (6.2)

the general solution should be thus dictated by the rank of the algebra. Note in particular that for the sine-Gordon model $n = 2$ one recovers the boundary Temperley-Lieb algebra (see e.g. [43]). In this context all solutions that give rise to inconsistencies are special in the sense that are either degenerate $\xi_\alpha = \xi_\beta$ or correspond to a case with at least one zero eigenvalue $\xi_\alpha = 0$.

To summarize: special cases of $K$-matrices give rise to inconsistencies, hence one needs to consider the most general possible solutions of the reflection equation with distinct independent boundary parameters. In the SNP case no extra free boundary parameters appear and no extra constraints among the boundary fields occur. In the $A_2^{(1)}$ SP case we conjecture that any generic (non-degenerate) non-diagonal solution with free boundary parameters will be appropriate. For the moment we have no such a generic matrix at our disposal, but the inconsistencies arising give us a strong hint that there should exist solutions with more boundary parameters. We thus conjecture that the $K$ matrix (5.27) will turn out to follow from some yet-to-be found general solution via a limit process. This procedure is causing loss of information in the Hamiltonian analysis (possibly through a subtlety in the formulation of the exchange between this $K$-matrix limit and the asymptotic expansion limit which yields the boundary contributions), giving rise to the observed inconsistencies. We have thus a strong motivation to systematically search for more general solutions in the SP $A_2^{(1)}$ case. We shall further pursue this significant issue in a separate publication.
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