ON A SMOOTH COMPACTIFICATION OF PSL($n, \mathbb{C}$)/$T$

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ABSTRACT. Let $T$ be a maximal torus of PSL($n, \mathbb{C}$). For $n \geq 4$, we construct a smooth compactification of PSL($n, \mathbb{C}$)/$T$ as a geometric invariant theoretic quotient of the wonderful compactification $\overline{\text{PSL}}(n, \mathbb{C})$ for a suitable choice of $T$–linearized ample line bundle on $\text{PSL}(n, \mathbb{C})$. We also prove that the connected component, containing the identity element, of the automorphism group of this compactification of PSL($n, \mathbb{C}$)/$T$ is PSL($n, \mathbb{C}$) itself.

1. Introduction

Let $G$ be a semisimple group of adjoint type over the field $\mathbb{C}$ of complex numbers. De Concini and Procesi in [DP] constructed a smooth projective variety $\overline{G}$ with an action of $G \times G$ such that

- the variety $G$ equipped with the action of $G \times G$ given by the left and right translations is an open dense orbit of it, and
- the boundary $\overline{G} \setminus G$ is a union of $G \times G$ stable normal crossing divisors.

This variety $\overline{G}$ is known as the wonderful compactification of $G$.

Fix a maximal torus $T$ of $G$. Consider the right action of $T$ on $\overline{G}$, meaning the action of the subgroup $1 \times T \subset G \times G$. For a $T$–linearized ample line bundle $\mathcal{L}$ on $\overline{G}$, let $\overline{G}^s_T(\mathcal{L})$ and $\overline{G}^s_T(\mathcal{L})$ denote respectively the loci of semistable and stable points of $\overline{G}$ (see [MFK], p. 30, p. 40).

Our first main result (Proposition 3.3) says that there is a $T$–linearized ample line bundle $\mathcal{L}$ on $\overline{G}$ such that $\overline{G}^s_T(\mathcal{L}) = \overline{G}^s_T(\mathcal{L})$.

For $G = \text{PSL}(n, \mathbb{C})$, we show that there is a $T$–linearized ample line bundle $\mathcal{L}$ on $\text{PSL}(n, \mathbb{C})$ such that

- the GIT quotient $\overline{\text{PSL}}(n, \mathbb{C})^s_T(\mathcal{L})/T$ is smooth, and
- the boundary $(\overline{\text{PSL}}(n, \mathbb{C})^s_T(\mathcal{L})/T \setminus (\text{PSL}(n, \mathbb{C})/T)$ is a union of PSL($n, \mathbb{C}$) stable normal crossing divisors.

We further show that for $n \geq 4$, the connected component of the automorphism group of $\overline{\text{PSL}}(n, \mathbb{C})^s_T/\overline{T}$ containing the identity automorphism is PSL($n, \mathbb{C}$) (Theorem 4.1).

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Preliminaries and notation

In this section we recall some preliminaries and notation about Lie algebras and algebraic groups; see for example [Hu1] and [Hu2] for the details. Let $G$ be a simple group of adjoint type of rank $n$ over the field of complex numbers. Let $T$ be a maximal torus of $G$ and $B \supset T$ a Borel subgroup of $G$. Let $\hat{N}_G(T)$ denote the normalizer of $T$ in $G$. So $W := \hat{N}_G(T)/T$ is the Weyl group of $G$ with respect to $T$.

The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of $T$. The set of roots of $G$ with respect to $T$ will be denoted by $R$. Let $R^+ \subset R$ be the set of positive roots with respect to $B$. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset R^+$ be the set of simple roots with respect to $B$. The group of characters of $T$ will be denoted by $X(T)$, while the group of one-parameter subgroups of $T$ will be denoted by $Y(T)$. Let

$$\{\lambda_i \mid 1 \leq i \leq n\}$$

be the ordered set of one-parameter subgroups of $T$ satisfying the condition that $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$, where

$$\langle -, - \rangle : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{Z}$$

is the natural pairing, and $\delta_{ij}$ is the Kronecker delta function. Let $\leq$ (respectively, $\geq$) be the partial order on $X(T)$ defined as follows: $\chi_1 \leq \chi_2$, (respectively, $\chi_1 \geq \chi_2$) if $\chi_2 - \chi_1$ (respectively, $\chi_1 - \chi_2$) is a linear combination of simple roots with non-negative integers as coefficients.

Let $\langle -, - \rangle$ denote the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$. Let

$$\{\omega_j \mid 1 \leq j \leq n\}$$

be the ordered set of fundamental weights corresponding to $S$, in other words,

$$\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}, \quad 1 \leq i, j \leq n.$$ 

For $1 \leq i \leq n$, let $s_{\alpha_i}$ denote the simple reflection corresponding to $\alpha_i$.

The longest element of $W$ corresponding to $B$ will be denoted by $w_0$. Let $B^- = w_0 B w_0^{-1}$ be the Borel subgroup of $G$ opposite to $B$ with respect to $T$.

For the notion of a $G$–linearization, and the GIT quotients, we refer to [MFK] p. 30, p. 40.

Consider the flag variety $G/B$ that parametrizes all Borel subgroups of $G$. For a character $\chi$ of $B$, let

$$L_\chi = G \times_B \mathbb{C} \rightarrow G/B$$

be the $G$–linearized line bundle associated to the action of $B$ on $G \times \mathbb{C}$ given by $b.(g, z) = (gb, \chi(b^{-1})z)$ for $b \in B$ and $(g, z) \in G \times \mathbb{C}$. So, in particular, $L_\chi$ is $T$–linearized. When $L_\chi$ is
ample, we denote by \((G/B)^{ss}_{T}(L_{\chi})\) (respectively, \((G/B)^{s}_{T}(L_{\chi})\)) the semistable (respectively, stable) locus in \(G/B\) for the \(T\)–linearized ample line bundle \(L_{\chi}\).

Next we recall some facts about the wonderful compactification of \(G\). Let \(\chi\) be a regular dominant weight of \(G\) with respect to \(T\) and \(B\), and let \(V(\chi)\) be the irreducible representation of \(\hat{G}\) with highest weight \(\chi\), where \(\hat{G}\) is the simply connected covering of \(G\). By [DP, p. 16, 3.4], the wonderful compactification \(\overline{G}\), which we denote by \(X\), is the closure of the \(G\times G\)–orbit of the point \([1] \in \mathbb{P}(V(\chi) \otimes V(\chi)^*)\) corresponding to the identity element \(1\) of \(V(\chi) \otimes V(\chi)^* = \text{End}(V(\chi)^*)\). We denote by \(L_{\chi}\) the ample line bundle on \(X\) induced by this projective embedding. Since the regular dominant weights generate the weight lattice, given a weight \(\chi\), we have the line bundle \(L_{\chi}\) on \(X\) associated to \(\chi\).

By [DP, Theorem, p. 14, Section 3.1], there is a unique closed \(G\times G\)–orbit \(Z\) in \(X\). Note that \(Z = \bigcap_{i=1}^{n} D_{i}\), where \(D_{i}\) is the \(G\times G\) stable irreducible component of \(\overline{G} \setminus G\) such that \(O(D_{i}) = L_{\alpha_{i}}\) [DP, p. 29, Section 8.2, Corollary]. Further, \(Z\) is isomorphic to \(G/B \times G/B\) as a \(G\times G\) variety. By [DP, p. 26, 8.1], the pullback homomorphism

\[
i^{*} : \text{Pic}(X) \longrightarrow \text{Pic}(Z),
\]

for the inclusion map \(i : Z \hookrightarrow X\) is injective and is given by

\[
i^{*}(L_{\chi}) = p_{1}^{*}(L_{\chi}) \otimes p_{2}^{*}(L_{-\chi}),
\]

where \(L_{\chi}\) (respectively, \(L_{-\chi}\)) is the line bundle on \(G/B\) (respectively, \(G/B^{-}\)) associated \(\chi\) (respectively, \(-\chi\)) and \(p_{j}\) is the projection to the \(j\)-th factor of \(G/B \times G/B^{-}\) for \(j = 1, 2\).

### 3. Choice of a polarization on \(\overline{G}\)

We continue with the notation of Section 2. Let \(G\) be a simple algebraic group of adjoint type of rank \(n \geq 2\), such that its root system \(R\) is different from \(A_{2}\). Let

\[
\text{NS} := \{ \sum_{i=1}^{n} m_{i}\alpha_{i} : m_{i} \in \mathbb{N} \}.
\]

Then, we have the following:

**Lemma 3.1.** The above defined \(\text{NS}\) contains a regular dominant character \(\chi\) of \(T\) such that \(s_{\alpha_{i}}(\chi) \geq 0\) and \(\langle \chi, w(\lambda_{i}) \rangle \neq 0\) for every \(w \in W\) and \(1 \leq i \leq n\).

**Proof.** Denote by \(X(T)_{\mathbb{Q}}\) the rational vector space generated by \(X(T)\), and also denote by \(X(T)^{+}\) the semi-group of it given by the dominant characters of \(T\). Let \(\rho \in X(T)_{\mathbb{Q}}\) be the half sum of positive roots of \(R\). Then, \(2\rho = 2(\sum_{i=1}^{n} \omega_{i}) \in X(T)^{+}\) is a regular dominant character of \(T\), and we have \(2\rho \in \text{NS}\).
Since $R$ is irreducible of rank at least 2 and different from $A_2$, we see that for every simple root $\alpha_i$, there are at least 3 positive roots $\beta$ satisfying $\alpha_i \leq \beta$. Hence, the coefficient of every simple root $\alpha_j$ in the expression of $s_{\alpha_i}(2\rho) = 2\rho - 2\alpha_i$ (as a non-negative integral linear combination of simple roots) is positive. Hence, we have $s_{\alpha_i}(2\rho) \in \mathbb{NS}$. Thus, we have

$$2\rho \in X(T)^+ \cap (\bigcap_{i=1}^n s_{\alpha_i}(\mathbb{NS})).$$

Denote by $N$ the determinant of the Cartan matrix of $R$. Then we have $N\omega_i \in \mathbb{NS}$ for every $i = 1, 2, \cdots, n$. By the previous discussion, there exists $m \in \mathbb{N}$ such that $ms_{\alpha_i}(2\rho) - N\alpha_i \in \mathbb{NS}$ for every $1 \leq i \leq n$. Hence, we get

$$s_{\alpha_i}(2m\rho + N\omega_i) = ms_{\alpha_i}(2\rho) - N\alpha_i + N\omega_i \in \mathbb{NS}, \quad 1 \leq i \leq n,$$

and from this it follows that

$$2m\rho + N\omega_i \in X(T)^+ \cap (\bigcap_{j=1}^n s_{\alpha_j}(\mathbb{NS})), \quad 1 \leq i \leq n.$$

Consider the characters $2m\rho, 2m\rho + N\omega_2, \cdots, 2m\rho + N\omega_n$ of $T$. These are linearly independent in $X(T)$ and by construction they all lie in the rational cone

$$C \subset X(T)_Q^+$$

generated by the semi-group $X(T)^+ \cap (\bigcap_{i=1}^n s_{\alpha_i}(\mathbb{NS}))$. It follows that $C$ has a maximal dimension in $X(T)_Q$, hence it is not contained in any hyperplane of $X(T)_Q$. Therefore, there exists a regular dominant character $\chi \in C \cap \mathbb{NS}$ such that $\langle \chi, w(\lambda_i) \rangle \neq 0$ for all $1 \leq i \leq n$ and every $w \in W$, and hence the lemma follows. □

Lemma 3.2. Let $\chi \in \mathbb{NS}$ be a regular dominant character of $T$ satisfying the properties stated in Lemma 3.1. Then we have

1. $(G/B)^{ss}_T(L_\chi) = (G/B)_T^s(L_\chi)$, and
2. the set of all unstable points

$$(G/B) \setminus (G/B)_T^{ss}(L_\chi)$$

is contained in the union of $W$–translates of all Schubert varieties of codimension at least two.

Proof. Set $L := L_\chi$. Since $\langle \chi, w(\lambda_i) \rangle \neq 0$ for every $w \in W$ and $1 \leq i \leq n$, by [Kα1, p. 38, Lemma 4.1] we have

$$(G/B)_T^{ss}(L) = (G/B)_T^s(L).$$

This proves (1).

To prove (2), take an unstable point $x \in G/B$ for the polarization $L$. Then, there is a one-parameter subgroup $\lambda$ of $T$ such that $\mu^L(x, \lambda) < 0$. Let $\phi \in W$ be such that $\phi(\lambda)$ is in the fundamental chamber, say

$$\phi(\lambda) = \sum_{i=1}^n c_i \lambda_i,$$
where \( \{c_i\} \) are non-negative integers. Consequently, we have

\[
\mu^L(n_\phi(x), \phi(\lambda)) = \mu^L(x, \lambda) < 0,
\]

where \( n_\phi \) is a representative of \( \phi \) in \( N_G(T) \). Now, let \( n_\phi(x) \) be in the Schubert cell \( BwB/B \) for some \( w \in W \). By \[ \text{Lemma 5.1} \], we have

\[
\mu^L(n_\phi(x), \phi(\lambda)) = (-\sum_{i=1}^{n} c_i(w(\chi), \lambda_i)) < 0.
\]

(The sign here is negative because we are using left action of \( B \) on \( G/B \) while in \[ \text{Lemma 5.1} \] the action of \( B \) on \( B\backslash G \) is on the right.) Therefore we have \( w(\chi) \not\prec 0 \). For every \( 1 \leq i \leq n \) we have \( s_{\alpha_i}(\chi) \geq 0 \), and hence \( w_0s_{\alpha_i}(\chi) \leq 0 \). Hence we have \( l(w_0) - l(w) \geq 2 \). This completes the proof of (2). \( \square \)

**Proposition 3.3.** Let \( X = \overline{G} \) be the wonderful compactification of \( G \). Let \( \chi \) be as in Lemma \[ 3.2 \], and let \( X^s_T(\mathcal{L}_\chi) \) (respectively, \( X^s_T(\mathcal{L}_\chi) \)) be the semi-stable (respectively, stable) locus of \( X \) for the action of \( 1 \times T \) and the polarization \( \mathcal{L}_\chi \) on \( X \). Then we have

1. \( X^s_T(\mathcal{L}_\chi) = X^s_T(\mathcal{L}_\chi) \), and
2. the set of unstable points \( X \setminus (X^s_T(\mathcal{L}_\chi)) \) is a union of irreducible closed subvarieties of codimension at least three.

**Proof.** Let \( Z \) be the unique closed \( G \times G \)-orbit in \( X \). Let \( Z^s_T(\mathcal{L}_\chi) \) (respectively, \( Z^s_T(\mathcal{L}_\chi) \)) be the semi-stable (respectively, stable) locus of \( Z \) for the action of \( 1 \times T \) and the polarization \( i^*(\mathcal{L}_\chi) \), where \( i : Z \hookrightarrow X \) is the inclusion map. Since \( Z \) is isomorphic to \( G/B \times G/B^{-} \) and \( i^*(\mathcal{L}_\chi) = p_1^*(L_\chi) \otimes p_2^*(L_{-\chi}) \), we see that

\[
Z^s_T(\mathcal{L}_\chi) \simeq (G/B) \times ((G/B)^{s}(L_{-\chi}))
\]

and \( Z^s_T(\mathcal{L}_\chi) \simeq (G/B) \times ((G/B)^{s}(L_{-\chi})) \). Set \( Z^s = Z^s_T(\mathcal{L}_\chi) \) and \( Z^s = Z^s_T(\mathcal{L}_\chi) \). By Lemma \[ 3.2 \] and above discussion, we have \( Z^s = Z^s \).

For convenience, we will denote \( X^s_T(\mathcal{L}_\chi) \) and \( X^s_T(\mathcal{L}_\chi) \) by \( X^s \) and \( X^s \) respectively. If \( X^s \neq X^s \), then the complement \( X^s \setminus X^s \) is a non-empty \( G \times T \) invariant closed subset of \( X^s \). Hence, the complement \( (X^s/T) \setminus (X^s/T) \) is a non-empty \( G \times \{1\} \)-invariant closed subset of \( X^s/T \). In particular, \( (X^s/T) \setminus (X^s/T) \) is a finite union of non-empty \( G \times \{1\} \)-invariant projective varieties. Therefore, there is a \( B \times \{1\} \)-fixed point in \( (X^s/T) \setminus (X^s/T) \).

Let

\[
p \in (X^s/T) \setminus (X^s/T)
\]

be a \( B \times \{1\} \)-fixed point. Let \( Y \) be the closed \( \{1\} \times T \)-orbit in the fiber \( \pi^{-1}(\{p\}) \) over \( p \) for the geometric invariant theoretic quotient map \( \pi : X^s \rightarrow X^s/T \). Since this map \( \pi \) is \( G \times \{1\} \) equivariant, we conclude that \( \pi^{-1}(\{p\}) \) is \( B \times \{1\} \)-invariant. Hence, for any \( b \in B \), the translation \( (b, 1) \cdot Y \) lies in \( \pi^{-1}(\{p\}) \). Since the actions of \( B \times \{1\} \) and \( \{1\} \times T \) on \( X \) commute with each other, we see that \( (b, 1) \cdot Y \) is also a closed \( \{1\} \times T \)-orbit in \( \pi^{-1}(\{p\}) \). By the uniqueness of the closed \( \{1\} \times T \)-orbit in \( \pi^{-1}(\{p\}) \), we conclude that \( (b, 1) \cdot Y = Y \). Hence \( Y \) is preserved by the action of \( B \times \{1\} \). In particular, \( Y \) is \( U \times \{1\} \)-invariant, where \( U \subset B \) is the unipotent radical. The action of \( U \times \{1\} \) on
Y induces a homomorphism from $U$ to $T/S$ of algebraic groups, where $\{1\} \times S$ is the stabilizer in $\{1\} \times T$ of some point $q$ in $Y$. Since there is no nontrivial homomorphism from an unipotent group to a torus, we conclude that $U \times \{1\}$ fixes the point $q$.

By [DP, p. 32, Proposition], for any regular dominant character $\chi$ of $T$ with respect to $B$, the morphism $X \hookrightarrow \mathbb{P}(V(\chi) \otimes V(\chi^*))$ is a $G \times G$ equivariant embedding, where $V(\chi)$ is the irreducible representation of $G$ with highest weight $\chi$, and $V(\chi^*)$ is its dual. Hence, the $U \times 1$–fixed point set of $X$ is equal to $X \cap \mathbb{P}(\mathbb{C}_\chi \otimes V(\chi^*))$, where $C_\chi$ is the one dimensional $B$–module associated to the character $\chi$. Therefore, by the above discussion, we have $q \in X \cap \mathbb{P}(\mathbb{C}_\chi \otimes V(\chi^*))$.

Further, by [DP, Theorem, p. 30] we have $H^0(X, L_\chi) = \bigoplus_{\nu \leq \chi} V(\nu)^* \otimes V(\nu)$, where the sum runs over all dominant characters $\nu$ of $T$ satisfying $\nu \leq \chi$. By [DP] p. 29, Corollary and [DP] p. 30, Theorem], the zero locus of $\bigoplus_{\nu < \chi} V(\nu)^* \otimes V(\nu) \subset H^0(X, L_\chi)$ in $X$ is the unique closed $G \times G$–orbit $Z = G/B \times G/B^-$. Hence, by the discussion in the previous paragraph, we have $q \in Z$. This contradicts the choice of the polarization $L_\chi$. Therefore, the proof of (1) is complete.

To prove (2), note that $X \setminus X^{ss}$ is a closed subset of $X$, and

$$Z \setminus Z^{ss} = (X \setminus X^{ss}) \cap Z.$$  

Also, by Lemma [3.2], the complement $Z \setminus Z^{ss} \subset Z$ is of codimension at least two. Since we have $Z = \bigcap_{i=1}^n D_i$, the complement $D_i \setminus D_i^{ss}$ is of codimension at least two for all $1 \leq i \leq n$. Further, every point in the open subset $G \subset X$ is semistable. Hence, $X \setminus X^{ss}$ is of codimension at least three. \hfill \Box

The following lemma will be used in the proof of Corollary 3.5.

**Lemma 3.4.** Let $H$ be a reductive algebraic group acting linearly on a polarized projective variety $V$. Assume that $V^{ss} = V^s$, where $V^{ss}$ (respectively, $V^s$) is the set of semi-stable (respectively, stable) points of $V$ for the action of $H$. Then the set of all points in $V^{ss}$ whose stabilizer in $H$ is trivial is actually a Zariski open subset (it may be possibly empty).

**Proof.** Consider the morphism

$$f : H \times V^{ss} \longrightarrow V^{ss} \times V^{ss}, \quad (h, v) \longmapsto (h \cdot v, v).$$

Since $V^{ss} = V^s$, this map $f$ is proper [MFK, p. 55, Corollary 2.5]. Hence the image

$$M := f(H \times V^{ss}) \subset V^{ss} \times V^{ss}$$

is a closed subvariety. Now, let

$$U' \subset V^{ss}$$
be the locus of points with trivial stabilizer (for the action of $H$). Take any

$$v_0 \in U',$$

and set $z_0 := f((1,v_0)) = (v_0,v_0)$. Then, $(f_*O_{H \times \emptyset})_{z_0}$ is a free $O_{M,z_0}$-module of rank one. Hence by [Mu, p. 152, Souped-up version II of Nakayama’s Lemma], the locus of points $x \in M$ such that $(f_*O_{H \times \emptyset})_x$ is a free $O_{M,x}$-module of rank at most one is a non-empty Zariski open subset of $M$; this Zariski open subset of $M$ will be denoted by $U$. Note that

$$f^{-1}(U) = p_2^{-1}(U'),$$

where $p_2 : H \times V^{ss} \rightarrow V^{ss}$ is the second projection. Since $p_2$ is flat of finite type over $\mathbb{C}$, it is an open map (see [Ha, p. 266, Exercise 9.1]). Hence $U' = p_2(f^{-1}(U))$ is a Zariski open subset. This finishes the proof of the lemma. □

**Corollary 3.5.** Let $X = \overline{\text{PSL}(n+1,\mathbb{C})}$ be the wonderful compactification of $\text{PSL}(n+1,\mathbb{C})$, $n \geq 3$. For the choice of the regular dominant character $\chi$ of $T$ as in Proposition 3.3

1. the action of $\{1\} \times T$ on $X^{ss}_T(\mathcal{L}_\chi)/T$ is free,
2. $X^{ss}_T(\mathcal{L}_\chi)/T$ is a smooth projective embedding of $G/T$, and
3. the set of unstable points $X \setminus (X^{ss}_T(\mathcal{L}_\chi))$ is a union of irreducible closed subvarieties of codimension at least three.

**Proof.** Let $\chi$ be a regular dominant character of $T$ as in Proposition 3.3. As in the proof of Proposition 3.3, let $Z$ denote the unique closed $G \times G$ orbit in $X$. Also, let $X^{ss}$, $X^s$, $Z^{ss}$ and $Z^s$ be as in the proof of Proposition 3.3. By Proposition 3.3, we have $X^{ss} = X^s$. Hence by Lemma 3.4, the locus $V$ of points in $X^{ss}$ with trivial stabilizer (for the action of $\{1\} \times T$) is a Zariski open subset of $X^{ss}$. Therefore, $X^{ss} \setminus V$ is a $G \times \{1\}$ stable closed subvariety of $X^{ss}$. By using the arguments in the the proof of Proposition 3.3 we see  that the set of $B \times \{1\}$-fixed points in $Z \setminus (X^{ss} \setminus V)$ is non-empty. But on the other hand by the proof of [Ka2, p. 194, Example 3.3] we see that given any point $z \in Z^{ss}$, its stabilizer subgroup in $\{1\} \times T$ is trivial. This is a contradiction. Hence we conclude that the action of $\{1\} \times T$ on $X^{ss}$ is free. This proves Part (1) and Part (2).

Part (3) follows immediately from the corresponding statement in Proposition 3.3. □

4. **Automorphism group of $\overline{\text{PSL}(n+1,\mathbb{C})}^{ss}_T(\mathcal{L})/T$**

Let $G = \text{PSL}(n+1,\mathbb{C})$, with $n \geq 3$, and define

$$Y := \overline{\text{PSL}(n+1,\mathbb{C})}^{ss}_T(\mathcal{L}_\chi)/T,$$

where $\chi$ is as in Proposition 3.3.

**Theorem 4.1.** Let $A$ denote the connected component, containing the identity element, of the group of holomorphic (= algebraic) automorphisms of $Y$. Then
(1) A is isomorphic to G, and
(2) the Picard group of Y is a free Abelian group of rank 2n.

Proof. Let TY denote the algebraic tangent bundle of Y. From [MO, Theorem 3.7] we know that A is an algebraic group. The Lie algebra of A is $H^0(Y, TY)$ equipped with the Lie bracket operation of vector fields.

The Lie algebra of G will be denoted by $\mathfrak{g}$. Define $X := \text{PSL}(n+1, \mathbb{C})$ and $U := \text{PSL}(n+1, \mathbb{C}) \cap \{L_X\}$. The connected component, containing the identity element, of the automorphism group of X is $G \times G$ [Br, Example 2.4.5]. From this, and the fact that the complement $X \setminus U \subset X$ is of codimension at least three (see Corollary 3.5), we conclude that $H^0(U, T_U) = H^0(X, TX) = \mathfrak{g} \oplus \mathfrak{g}$.

Let $\phi : U \to Y$ be the geometric invariant theoretic quotient map. Let $T_U \supset T_\phi \to U$ be the relative tangent bundle for $\phi$. Since $\phi$ makes $X$ a principal $T$–bundle over $Y$ (see Corollary 3.5(1)), we have the following short exact sequence of vector bundles on $U$

$$0 \to T_\phi \to T_U \to \phi^*(TY) \to 0,$$

and the relative tangent bundle $T_\phi$ is identified with the trivial vector bundle $\mathcal{O}_U \otimes \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of $T$.

Set $Z = X \setminus U$. Since $\text{codim}(Z) \geq 3$ (Corollary 3.5), we have $H^0(U, T_\phi) = H^0(X, \mathcal{O}_X \otimes \mathfrak{h}) = \mathfrak{h}$.

Note that $H^1(U, T_\phi) = H^2_Z(X, \mathcal{O}_X \otimes \mathfrak{h})$. Indeed, this follows from the following cohomology exact sequence (see [Gr, Corollary 1.9])

$$H^1(X, \mathcal{O}_X \otimes \mathfrak{h}) \to H^1(U, \mathcal{O}_X \otimes \mathfrak{h}) \to H^2_Z(X, \mathcal{O}_X \otimes \mathfrak{h}) \to H^2(X, \mathcal{O}_X \otimes \mathfrak{h})$$

combined with the fact that $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ [DP, p. 30, Theorem]. As $X$ is smooth and $\text{codim}(Z) \geq 3$, it follows from [Gr, Theorem 3.8 and Proposition 1.4] that $H^2_Z(X, \mathcal{O}_X) = 0$,

and hence $H^1(U, T_\phi) = 0$. Now, using this fact in the long exact sequence of cohomologies corresponding to the short exact sequence in (4.1) we obtain the following short exact sequence:

$$0 \to 0 \oplus \mathfrak{h} \to \mathfrak{g} \oplus \mathfrak{g} \to H^0(U, \phi^*TY) \to 0.$$

Hence, we have

$$H^0(U, \phi^*TY) = \mathfrak{g} \oplus (\mathfrak{g}/\mathfrak{h}).$$

By using geometric invariant theory, $H^0(Y, TY)$ is the invariant part

$$H^0(Y, TY) = H^0(U, \phi^*TY)^{(1) \times T} \subset H^0(U, \phi^*TY).$$

Thus we have $H^0(Y, TY) = \mathfrak{g}$. This proves (1).
To prove (2), let \( \{D_i \mid 1 \leq i \leq n\} \) be the \((G \times G)\)–stable irreducible closed subvarieties of \( G \) of codimension one such that
\[
G = \overline{G} \setminus \bigcup_{i=1}^{n} D_i.
\]
Let \( D_i^{ss} = D_i \cap X^{ss} \subset D_i \) be the semistable locus of \( D_i \). Set \( Z := Y \setminus (G/T) \), and write it as a union
\[
Z = \bigcup_{i=1}^{n} Z_i,
\]
where each \( Z_i = D_i^{ss}/T \) is an irreducible closed subvariety of \( Y \) of codimension one. As \( Y \) is smooth, each \( Z_i \) produces a line bundle \( L_i \longrightarrow Y \) whose pullback to \( X^{ss} \) is \( \mathcal{O}_{X^{ss}}(D_i^{ss}) \).

Since \( \text{Pic}(X^{ss}) = \text{Pic}(X) \) and \( \{\mathcal{O}_X(D_i)\}_{1 \leq i \leq n} \) are linearly independent in \( \text{Pic}(X) \) (see [DP, p. 26, 8.1]), we get that \( L_i, 1 \leq i \leq n, \) are linearly independent in \( \text{Pic}(Y) \). The Picard group of \( G/T \) is isomorphic to the group of characters of the inverse image \( \hat{T} \) of \( T \) inside the simply connected covering \( \hat{G} \) of \( G \) (see [KKV]). Now it follows from the exact sequence in [Fu, Proposition 1.8] that \( \text{Pic}(Y) \) is a free Abelian group of rank \( 2n \), thus completing the proof of (2). \( \square \)

**Remark 4.2.** The compactification \( Y \) of \( G/T \) constructed here is an example of a non-spherical variety for the action of \( G \) whose connected component of the automorphism group is \( G \).

**Remark 4.3.** Note that both \( Y \) and \( G/B \times G/B \) are smooth compactifications of \( G/T \) with isomorphic Picard groups. Further both are Fano varieties, i.e., the anti-canonical line bundle is ample. The fact that \( G/B \times G/B \) is Fano is well known. That the variety \( Y \) is Fano follows as a consequence of the exact sequence in (4.1) together with the facts that \( X \) is Fano (see [DP]) and the codimension of \( X \setminus U \) is greater than or equal to 3, where \( X \) and \( U \) are as in the proof of Theorem 4.1. But \( Y \) and \( G/B \times G/B \) are not isomorphic, as \( \text{Aut}^0(Y) \simeq G \) and \( \text{Aut}^0(G/B \times G/B) \simeq G \times G \), where \( \text{Aut}^0(M) \) denotes the connected component of the group of algebraic automorphisms of a smooth projective variety \( M \) containing the identity element.

**Remark 4.4.** In [St], Strickland extended the construction of \( \overline{G} \) to any arbitrary algebraically closed field. Also, \( \overline{G} \) is a Frobenius split variety in positive characteristic [St, p. 169, Theorem 3.1] (see [MR] for the definition of Frobenius splitting). Since \( T \) is linearly reductive, using Reynolds operator, one can see that the geometric invariant theoretic quotient of \( \overline{G} \) for the action of \( T \) is also Frobenius split for any polarization on \( \overline{G} \).

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