K-causality and domain theory

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Abstract

Using the relation $K^+$, we prove that a certain type of stably causal spacetimes is a jointly bicontinuous poset whose interval topology is the manifold topology.

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1 Introduction

It is shown by Martin and Panangaden [4] that it is possible to reconstruct globally hyperbolic spacetimes in a purely order theoretic manner using the causal relation $J^+$. These spacetimes belong to a category that is equivalent
to a special category of domains called interval domains [5]. In this paper we use the causal relation \( K^+ \) instead of \( J^+ \). The relation \( K^+ \subseteq M \times M \) is defined as the smallest transitive closed relation which contains \( I^+ \) [7]. This definition arose from the fact that the causal relation, \( J^+ \), is transitive but not necessarily closed and \( \overline{J^+} \) is closed but not necessarily transitive. The spacetime \( (M, g) \) is \( K \)-causal if \( K^+ \) is antisymmetric. Recently it is proved by Minguzzi that stable causality and \( K \)-causality are coincide. In globally hyperbolic and causally simple spacetimes \( K^+ = J^+ \). In this paper we prove that \( K \)-causal spacetimes, in which \( int(K^\pm(\cdot)) \) are inner continuous are jointly bicontinuous posets.

2 Preliminaries

A poset is a partially ordered set, i.e, a set together with a reflexive, antisymmetric and transitive relation.

In a poset \( (P, \sqsubseteq) \), a nonempty subset \( S \subseteq P \) is called directed (filtered) if \( (\forall x, y \in S)(\exists z \in S) x, y \sqsubseteq z ((\forall x, y \in S)(\exists z \in S) z \sqsubseteq x, y) \). The supremum (infimum) of \( S \) is the least of its upper bounds (greatest of all its lower bounds) provided it exists.

For a subset \( X \) of a poset \( P \), set:

\[
\uparrow X = \{ y \in P : (\exists x \in X) x \sqsubseteq y \}, \quad \downarrow X = \{ y \in P : (\exists x \in X) y \sqsubseteq x \}.
\]

A dcpo is a poset in which every directed subset has a supremum. The least element in a poset, when it exists, is the unique element \( \bot \) with \( \bot \sqsubseteq x \) for all \( x \).
A subset $U$ of a poset is scott open if:

(i) $U$ is an upper set: $x \in U$ and $x \subseteq y \Rightarrow y \in U$.

(ii) For every directed $S \subseteq P$ with supremum that $\bigcup S \in U$ implies $S \cap U \neq \emptyset$.

The collection of scott open sets on $P$ is called the scott topology.

**Definition 2.1.** For elements $x, y$ of a poset, write $x \ll y$ if and only if for all directed sets $S$ with a supremum,

$$y \subseteq \bigcup S \Rightarrow (\exists s \in S) \ x \subseteq s.$$

We set $\downarrow x = \{a \in P : a \ll x\}$ and $\uparrow x = \{a \in P : x \ll a\}$. For symbol ”$\ll$”, read ”way below”.

**Definition 2.2.** A basis for a poset $P$ is a subset $B$ such that $B \cap \downarrow x$ contains a directed set with supremum $x$ for all $x \in P$. A poset is continuous if it has a basis. A poset is $\omega$- continuous if it has a countable basis.

**Definition 2.3.** For elements $x, y$ of a poset, write $x \ll_d y$ if and only if for all filtered sets $S$ with an infimum,

$$\bigwedge S \subseteq x \Rightarrow (\exists s \in S) \ s \subseteq x.$$

We set $\downarrow_d x = \{a \in P : a \ll_d x\}$ and $\uparrow_d x = \{a \in P : x \ll_d a\}$. For symbol ”$\ll_d$”, read ”way above”.

**Definition 2.4.** A poset $P$ is dual continuous if $\uparrow_d x$ is filtered with infimum $x$ for all $x \in P$.

A poset $P$ is bicontinuous if it is both continuous and dual continuous. In addition a poset is called jointly bicontinuous if it is bicontinuous and the way below relation coincides with the way above relation. A bicontinuous
poset is called globally hyperbolic poset if all of its intervals, \([a, b] = \uparrow a \cap \downarrow b\), are compact in the interval topology.

**Proposition 2.5.** [1] If \(x \ll y\) in a continuous poset \(P\), then there is \(z \in P\) with \(x \ll z \ll y\).

**Definition 2.6.** On a bicontinuous poset \(P\), sets of the form

\[(a, b) := \{x \in P : a \ll x \ll b\}\]

form a basis for a topology called the interval topology.

A useful example of continuous domains is upper space.

**Example 2.7.** Let \(X\) be a locally compact Hausdorff space. Its upper space \(UX = \{K \neq \emptyset : K\) is compact \(\}\), with \(A \subseteq B \iff B \subseteq A\) is a continuous dcpo. For \(K, L \in UX\), \(K \ll L\) if and only if \(L \subseteq int(K)\).

### 3 Causal structure of a spacetime

In this section we suppose that \((M, g)\) is a spacetime and \(I^+\) and \(J^+\) are the chronological and causal relations [2]. The spacetime \(M\) is globally hyperbolic if it is causal and \(J^+(x) \cap J^-(y)\) is compact for every \(x, y \in M\). Martin and Panangaden defined an order on the spacetime \(M\) in the following manner:

\[p \sqsubseteq q \equiv q \in J^+(p).\]

They proved the following theorem about Globally hyperbolic spacetimes:

**Theorem 3.1.** If \(M\) is a globally hyperbolic spacetime, then \((M, \sqsubseteq)\) is a bicontinuous poset with \(I^+ = \ll\) whose interval topology is the manifold topology.
This theorem suggests a formulation of causality independently of geometry. In this paper we try to generalize theorem 3.1. We use the relation $K^+$ instead of $J^+$.

If $U$ is an open neighborhood of $M$, then we denote by $J^+_U$ the causal relation on the spacetime $U$ with the induced metric. We recall that every event $p$ of a spacetime $(M, g)$ admits arbitrary small globally hyperbolic neighborhoods.

An open set $U$ is $K$-convex if for all $p, q \in U$, $K^+(p) \cap K^-(q) \subseteq U$ [7].

The spacetime $M$ is strongly $K$-causal at $p$ if it contains arbitrary small $K$-convex neighborhoods of $p$, and it is strongly $K$-causal if it is strongly $K$-causal for all $p \in M$. $K$-causality implies strong $K$-causality [7]. The converse is trivial.

With $int(B)$ and $\overline{B}$, we denote the topological interior and closure of $B \subseteq M$, respectively. Let $F$ be a function which assigns to each point $p \in M$ an open set $F(p) \subseteq M$. We say that $F$ is inner continuous if for any $p$ and any compact set $C \subseteq F(p)$, there exists a neighborhood $U$ of $p$ with $C \subseteq F(q)$, for every $q \in U$. In globally hyperbolic spacetimes $K^+ = J^+$ and $int(K^+(.)) = I^+(.)$ that are inner continuous.

**Lemma 3.2.**[3] In a $K$-causal spacetime $(M, g)$, $int(K^+(.))$ and $int(K^-(.))$ are inner continuous if and only if for every $p, q \in M$, $p \in int(K^-(q)) \Leftrightarrow q \in int(K^+(p))$.

**Lemma 3.3.** $int(K^+(.))$ and $int(K^-(.))$ are outer continuous.

**Definition 3.3.** A $K$-causal spacetime $(M, g)$ is called $K$-causally continuous if $int(K^+(.))$ and $int(K^-(.))$ are inner continuous.

**Definition 3.4.** Let $(M, g)$ be a spacetime. Alexandrov topology on $M$ is
the one which admits as a base,

\[ B_A = \{ I^+(p) \cap I^-(q) : p, q \in M \}. \]

**Theorem 3.4.** [2] For a spacetime \((M, g)\), the following properties are equivalent:

(a) \((M, g)\) is strongly causal.

(b) Alexandrov topology is equal to the original topology on \(M\).

Using the relation \(K^+\), we define the following topology on \(M\).

\(K\)-Alexandrov topology is the one with the base,

\[ B_K = \{ \text{int}(K^+(p)) \cap \text{int}(K^-(q)) : p, q \in M \}. \]

**Theorem 3.5.** The following are equivalent:

(a) \((M, g)\) is \(K\)-causally continuous.

(b) \(K\)-Alexandrov topology is equal to the original topology on \(M\).

**Proof.** Assume that \((M, g)\) is \(K\)-causal. \(K\)-causality implies strong \(K\)-causality. Definition of strong \(K\)-causality implies that each point has arbitrary small \(K\)-convex neighborhoods. If \(V\) be an open neighborhood of \(p\) in the manifold topology, then there exists a causally \(K\)-convex neighborhood \(U\) of \(p\), \(U \subseteq V\), that is contained in a globally hyperbolic neighborhood. Indeed, \(K^+_U(p) = J^+_U(p)\) and \(\text{int}(K^+_U(p)) = I^+_U(p)\). Thus \(K\)-Alexandrov topology on \((U, g|_U)\) agrees with Alexandrov topology. Using theorem 3.4 and the fact that a strongly \(K\)-causal spacetime is strongly causal demonstrate that the manifold topology on \(U\) agrees with \(K\)-Alexandrov topology. Hence \(K\)-Alexandrov topology agrees with the manifold topology.
Conversely, suppose that $M$ is not strongly $K$- causal at $p$. There is a neighborhood $V$ of $p$ that for every neighborhood $W \subseteq V$ of $p$ there exist points $p', q' \in W$ such that $K^+(p') \cap K^-(q')$ is not a subset of $W$. Thus there isn’t any open set in $K$- Alexandrov topology that is contained in $V$. Indeed, if for $c, c' \in M$, $W = \text{int}(K^+(c)) \cap \text{int}(K^-(c')) \subseteq V$ then by assumption, there is a point $d \in K^+(p') \cap K^-(q')$ that $d \notin W$. But since $\text{int}(K^\pm(\cdot))$ are inner continuous, $d \in W$ that is a contradiction. As a consequence, $K$- Alexandrov topology is different from the given manifold topology.

4 Spacetime and domain theory

Let $M$ be a $K$- causal spacetime. We write the relation $K^+$ as:

$$p \sqsubseteq q \equiv (p, q) \in K^+.$$

**Example 4.2.** Let $M$ be a globally hyperbolic spacetime. In a globally hyperbolic spacetime, $K^+ = J^+$. Let $S$ be a directed set with supremum, then $\bigsqcup S = \bigcap_{s \in S} [s, \bigsqcup S]$. Let $V$ be an arbitrary small neighborhood of $\bigsqcup S$. Using the approximation on the upper space of $M$, $\overline{V} \ll \bigsqcup S = \bigcap_{s \in S} [s, \bigsqcup S]$ where the intersection is a directed collection of nonempty compact sets by directedness of $S$ and global hyperbolicity of $M$. Thus for some $s \in S$, $[s, \bigsqcup S] \subseteq \overline{V}$.

**Lemma 4.3.** Let $p$, $q$ and $r \in M$. Then:

i) $p \sqsubseteq q$ and $r \in \text{int}(K^+(q)) \Rightarrow r \in \text{int}(K^+(p))$.

ii) $p \in \text{int}(K^-(q))$ and $q \sqsubseteq r \Rightarrow p \in \text{int}(K^-(r))$. 


Lemma 4.4. Let \( y_n \) be a sequence in \( M \) with \( y_n \sqsubseteq y \) \((y_n \sqsubseteq y)\) for all \( n \) and \( \lim_{n \to \infty} y_n = y \); then \( \bigsqcup y_n = y \) \((\bigsqcup y_n = y)\). 

Proof. Let \( y_n \sqsubseteq x \) for every \( n \in \mathbb{N} \). Since \( K^+ \) is closed and \( y_n \in K^-(x) \), \( y = \lim_{n \to \infty} y_n \in K^-(x) \). Thus \( y \sqsubseteq x \) and this proves \( y = \bigsqcup y_n \). The proof for the dual part is similar to this.

Note that the above lemma is true for every causal closed relation.

Lemma 4.5.\([4]\) For any \( x \in M \), \( I^-(x) \) \((I^+(x))\) contains an increasing \((\text{decreasing})\) sequence with supremum \((\text{infimum})\) \( x \).

Lemma 4.6. Let \( S \) be a directed set in \((M, g)\) with supremum \( \bigsqcup S \). Then there is an increasing sequence \( \{s_n\} \) in \( S \) such that \( \lim_{n \to \infty} s_n = \bigsqcup S \).

Proof. Let \( A = \{\{s_n\} : s_n \in S, s_n \sqsubseteq s_{n+1} \ \forall n \in \mathbb{N}\} \). We define an equivalence relation on \( A \) in the following manner:

\( \{s_n\} \sim \{s'_n\} \iff \exists m \in \mathbb{N} : s_n = s'_n \ \forall n > m. \)

Now we define a partial order on \( A/\sim \).

\( [{\{s_n\}}] \sqsubseteq_{1} [{\{s'_n\}}] \iff \exists m \in \mathbb{N} : s_n \sqsubseteq s'_n, \forall n \geq m. \)

Suppose that \( \{a_m\}_{m \in \mathbb{N}} = \{\{s_{m,n} \}_{n \in \mathbb{N}} : m \in \mathbb{N}\} \) is a chain in \( A/\sim \). We show that it has an upper bound. We define the sequence \( \{b_m\} \) in the following manner:

\[
b_1 = s_{1,n_1} : s_{1,n} \sqsubseteq s_{2,n} \ \forall n > n_1,
\]

\[
b_i = s_{i,n_i} : s_{i,n} \sqsubseteq s_{i+1,n} \ \forall n > m \text{ and } n_i = \max\{m, n_1, \ldots, n_{i-1}\}.\]

It is easy to show that \( [{\{b_m\}}] \) is an upper bound of \( \{a_m\} \). Hence by zorn’s lemma \( A/\sim \) has a maximum element \( c = [{\{c_m\}}]. \) Suppose by contradiction...
that there is a neighborhood $U$ of $\bigcup S$ with compact closure such that $S \cap U = \emptyset$. Let $\{c_m\}$ be a representation of $[\{c_m\}]$. Since $c_m \sqsubseteq \bigcup S$, there is $d_m \in \partial U$, such that $c_m \sqsubseteq d_m$ and $d_m \sqsubseteq \bigcup S$. $\{d_m\}$ has an accumulation point like $d$ since $\partial U$ is compact. There is $m \in \mathbb{N}$ such that $c_n \sqsubseteq c_{n+1}, \forall n > m$ and $K^+$ is closed. Hence $c_i \sqsubseteq d_j, \forall i, j > m$ and consequently $c_i \sqsubseteq d, \forall i > m$. But $[\{c_m\}]$ is a maximal element of $A/\sim$ and this implies that $d$ is an upper bound of $S$ which is a contradiction to the fact that $d \sqsubseteq \bigcup S$ and $d \neq \bigcup S$.

**Theorem 4.6.** Let $M$ be a $K$- causally continuous spacetime. Then

$$x \ll y \iff y \in \text{int}(K^+(x)) \iff x \ll_d y.$$  

**Proof.** Let $y \in \text{int}(K^+(x))$. If for the directed set $S$ $y \sqsubseteq \bigcup S$, then by assumption and lemma 3.2, $\bigcup S \in \text{int}(K^+(x))$. By lemma and the fact that $\text{int}(K^+(x))$ is open, there exists $s \in S$ such that $s \in \text{int}(K^+(x))$. Consequently, $x \ll y$.

If $x \ll y$, by lemma 4.5 there exists an increasing sequence $y_n$ in $I^-(y)$ such that $\bigcup y_n = y$. Thus $x \sqsubseteq y_n$, for some $n$. Since $I^+$ is an open relation, $x \in \text{int}(K^-(y))$. The proof of the other part is similar to this.

**Theorem 4.7.** If $M$ is a $K$- causally continuous spacetime, then $(M, \sqsubseteq)$ is a jointly bicontinuous poset with $\ll = \text{int}(K^-(\cdot))$ whose interval topology is equal to the manifold topology.

**Proof.** By lemma 4.6, $\Downarrow x = \text{int}(K^-(x))$. In addition, by lemma 4.5, for every $x \in M$ there is an increasing sequence $x_n \subseteq I^-(x) \subseteq \text{int}(K^-(x)) = \Downarrow x$ with $\bigcup x_n = x$. Hence $M$ is continuous. In a similar way we can prove that it is dually continuous. In addition, by theorem 4.6 and 3.4, interval topology is equal to the manifold topology.

9
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