NON-EXISTENCE OF ORTHOGONAL COMPLEX STRUCTURES ON $S^6$ WITH A METRIC CLOSE TO THE ROUND ONE

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Abstract. I review several proofs for non-existence of orthogonal complex structures on the six-sphere, most notably by G. Bor and L. Hernández-Lamoneda, but also by K. Sekigawa and L. Vanhecke that we generalize for metrics close to the round one. Invited talk at MAM-1 workshop, 27-30 March 2017, Marburg.

1. Introduction

In 1987 LeBrun[7] proved the following restricted non-existence result for the 6-sphere. Let $(M, g)$ be a connected oriented Riemannian manifold. Denote by $J_g(M)$ the space of almost complex structures $J$ on $M$ that are compatible with the metric (i.e. $J^*g = g$) and with the orientation. This is the space of sections of an $SO(6)/U(3)$ fiber bundle, so whenever non-empty it is infinite-dimensional. Associating to $J \in J_g(M)$ the almost symplectic structure $\omega(X,Y) = g(JX,Y)$, $X,Y \in TM$, we get a bijection between $J_g(M)$ and the space of almost Hermitian triples $(g,J,\omega)$ on $M$ with fixed $g$.

Theorem 1. No $J \in J_{g_0}(S^6)$ is integrable (is a complex structure) for the standard (round) metric $g_0$. In other words, there are no Hermitian structures on $S^6$ associated to the metric $g_0$.

There are several proofs of this statement, we are going to review some of those. The method of proof of Theorem 1 by Salamon [8] uses the fact that the twistor space of $(S^6, g_0)$ is $Z(S^6) = SO(8)/U(4)$ which is a Kähler manifold (it has a complex structure because $S^6$ is conformally flat, and the metric is induced by $g_0$), and so the holomorphic embedding $s_J : S^6 \rightarrow Z(S^6)$ would induce a Kähler structure on $S^6$.

Here the symmetry of $g_0$ is used (homogeneity), so this proof is not applicable for $g \approx g_0$ (but as mentioned in [2], a modification of the...
original approach of [7], based on an isometric embedding of \((S^6, g)\) into a higher-dimensional Euclidean space, is possible).

A generalization of Theorem 1 obtained in [2] is as follows.

**Theorem 2.** Let \(g\) be a Riemannian metric on \(S^6\). Denote by \(R_g\) its Riemannian curvature, considered as a \((3, 1)\)-tensor, and by \(\tilde{R}_g : \Lambda^2 T^* S^6 \to \Lambda^2 T^* S^6\) the associated \((2, 2)\) tensor (curvature operator). Assume that its spectrum \(\{\lambda_1 \leq \cdots \leq \lambda_{15}\}\) is positive \(\lambda_{\min} > 0\) and satisfies \(5\lambda_{\max} < 7\lambda_{\min}\). Then no \(J \in \mathcal{J}_g(S^6)\) is integrable.

This theorem will be proven in Section 4 after we introduce the notations and recall the required knowledge in Sections 2 and 3. Then we will give another proof of Theorem 1 due to Sekigawa and Vanhecke [9] in Section 5. Then in Section 6 we generalize it in the spirit of Theorem 2. Section 7 will be a short summary and an outlook.

Let us start with an alternative proof of Theorem 1 following Bor and Hernández-Lamoneda [2].

**Sketch of the proof of Theorem 1.** Let \(K = \Lambda^{3,0}(S^6)\) be the canonical line bundle of the hypothetical complex structure \(J\). Equip it with the Levi-Civita connection \(\nabla\) that is induced from \(\Lambda^3_c(S^6)\) by the orthogonal projection. The curvature of \(K\) with respect to \(\nabla\) is

\[
\Omega = R_\nabla \big|_{\Lambda^{3,0}} + \Phi^* \wedge \Phi = i\tilde{R}_g(\omega) + \Phi^* \wedge \Phi,
\]

(1)

where \(\Phi\) is the second fundamental form (see §3.1). It has type \((1, 0)\) and so \(i\Phi^* \wedge \Phi \leq 0\) (see §3.2). Since for the round metric \(g = g_0\) we have \(\tilde{R}_g = \text{Id}\), so

\[
i\Omega = -\omega + i\Phi^* \wedge \Phi < 0.
\]

Thus \(-i\Omega\) is a non-degenerate (positive) scalar valued 2-form which is closed by the Bianchi identity. This implies that \(S^6\) is symplectic which is impossible due to \(H^2_{\text{DR}}(S^6) = 0\).

It is clear from the proof that for \(g \approx g_0\) the operator \(R_g \approx \text{Id}\) is still positive, so the conclusion holds for a small ball around \(g_0\) in \(\Gamma(\otimes^2_+ T^* S^6)\). It only remains to justify the quantitative claim.

2. **Background I: connections on Hermitian bundles**

Let \(M\) be a complex \(n\)-dimensional manifold. In this section we collect the facts about calculus on \(M\) important for the proof. A hurried reader should proceed to the next section returning here for reference.

Let \(\pi : E \to M\) be a Hermitian vector bundle, that is a holomorphic bundle over \(M\) equipped with the Riemannian structure \(\langle, \rangle\) in fibers for which the complex structure \(J\) in the fibers is orthogonal. Examples are the tangent bundle \(TM\) and the canonical line bundle \(K = \Lambda^{n,0}(M)\).
Note that a Hermitian structure is given via a \( \mathbb{C} \)-bilinear symmetric product \( \otimes^2(E \otimes \mathbb{C}) \rightarrow \mathbb{C} \) as follows: the restriction \( (,) : E' \otimes E'' \rightarrow \mathbb{C} \), where \( E \otimes \mathbb{C} = E' \oplus E'' = E_{(1,0)} \oplus E_{(0,1)} \) is the canonical decomposition into \( +i \) and \( -i \) eigenspaces of the operator \( J \), gives the Hermitian metric \( (,) : E \otimes E \rightarrow \mathbb{C} \), \( \langle \xi, \eta \rangle = \langle \xi, \eta \rangle \).

There are several canonical connections on \( E \).

2.1. The Chern connection. This is also referred to as the canonical metric \([6]\) or characteristic \([4]\) connection and is constructed as follows. Recall that the Dolbeaux complex of a holomorphic vector bundle is

\[
0 \rightarrow \Gamma(E) \xrightarrow{\bar{\partial}} \Omega^{0,1}(M; E) = \Gamma(E) \otimes \Omega^{0,1}(M) \rightarrow \Omega^{0,2}(M; E) \xrightarrow{\bar{\partial}} \ldots
\]

where the first Dolbeaux differential \( \bar{\partial} \), generating all the other differentials in the complex, is given by localization as follows (for simplicity of notations, everywhere below we keep using \( M \) for the localization).

If \( e_1, \ldots, e_m \) is a basis of holomorphic sections and \( \xi = \sum f^i e_i \in \Gamma(E) \) a general section, \( f^i \in C^\infty(M, \mathbb{C}) \), then \( \bar{\partial} \xi = \sum \bar{\partial}(f^i)e_i \). It is easy to check by passing to another holomorphic frame that this operator \( \bar{\partial} \) is well-defined, and that its extension by the Leibnitz rule \( \bar{\partial}(\xi \otimes \alpha) = \bar{\partial}(\xi) \wedge \alpha + \xi \cdot \alpha \cdot d\alpha \) yields a complex, \( \bar{\partial}^2 = 0 \).

**Theorem 3.** There exists a unique linear connection on the vector bundle \( E \), i.e. a map \( D : \Gamma(E) \rightarrow \Omega^1(M, E) = \Omega^1(M) \otimes \Gamma(E) \), that is

- compatible with the metric: \( d\langle \xi, \eta \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle \),
- compatible with the complex structure: \( D'' = \bar{\partial} \).

The first condition is \( Dg = 0 \) and the second implies \( DJ = 0 \).

Above, \( D'' \) is the \((0,1)\)-part of \( D \), i.e. the composition of \( D \) with the projection \( \Omega^1(M, E) \rightarrow \Omega^{0,1}(M, E) \).

**Proof.** The statement is local, so we can use a local holomorphic frame \( e_i \) to compute. Thus, a linear connection \( D \) is given by a connection form \( \theta = [\theta^b_a] \in \Omega^1(M; gl(n, \mathbb{C})) \): \( De_a = \theta^b_a e_b \). We use the notations \( e_a = e_a, \theta^b_a = \bar{\theta}^b_a \), etc, cf. \([6]\).

Let \( g_{ab} = \langle e_a, e_b \rangle = \langle e_a, e_b \rangle \) be the components of the Hermitian metric. The first condition on \( D \) writes

\[
dg_{ab} = \langle De_a, e_b \rangle + \langle e_a, De_b \rangle = \theta^c_a g_{cb} + \theta^c_b g_{ac}.
\]

The second condition means that all \( \theta^b_a \) are \((1,0)\)-forms, so the above formula splits: \( \partial g_{ab} = \theta^c_a g_{cb} \Leftrightarrow \bar{\partial} g_{ab} = \theta^c_b g_{ac} \).

Consequently, the connection form satisfying the two conditions is uniquely given by \( \theta = g^{-1} \cdot \partial g \), or in components \( \theta^b_a = g^{bc} \partial g_{ac} \). \( \square \)
We will denote the Chern connection, so obtained, by $\mathcal{D}$. In particular, there is a canonical connection $\mathcal{D}$ on the tangent bundle of a Hermitian manifold. Its torsion is equal to

$$T_{\mathcal{D}} = \pi_{2,0}(d^c\omega)^2,$$

where $d^c\omega(\xi, \eta, \zeta) = -d\omega(J\xi, J\eta, J\zeta)$, $\sharp : \Lambda^3 T^* M \hookrightarrow \Lambda^2 T^* M \otimes T^* M \rightarrow \Lambda^2 T^* M \otimes TM$ is the index lift operator and $\pi_{2,0} : \Lambda^2 T^* M \otimes TM \rightarrow \Lambda^{2,0} T^* M \otimes TM = \{ B : B(J\xi, \eta) = B(\xi, J\eta) = JB(\xi, \eta) \}$ is the projection, cf. [4]. This implies that the Chern connection $\mathcal{D}$ on $TM$ has a non-trivial torsion unless $(g, J, \omega)$ is Kähler.

### 2.2. The Levi-Civita connection.

A Hermitian metric induces a canonical torsionless metric connection $\nabla$ on $TM$: $\nabla g = 0$, $\nabla^2 = 0$.

Due to computation of the torsion $T_{\mathcal{D}}$ above, the Levi-Civita connection $\nabla$ does not preserve $J$ unless $M$ is Kähler. In other words, $\mathcal{D} = \nabla$ only in this case.

Choosing the frame $e_a = \partial_{z^a}$, $e_{\bar{a}} = \partial_{\bar{z}^a}$, for a holomorphic coordinate system $(z^a)$ on $M$, we get

$$\nabla e_a = \Gamma^c_{ab} e^b \otimes e_c + \Gamma^c_{ab\bar{b}} e^\bar{b} \otimes e_c + \Gamma^c_{a\bar{b}} e^\bar{b} \otimes e_c + \Gamma^c_{ab\bar{c}} e^\bar{c} \otimes e_c$$

and (because $g_{ab} = 0 = g_{\bar{a}\bar{b}}$) the Christoffel coefficients have the standard but shorter form, e.g.

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}(\frac{\partial g_{ad}}{\partial z^b} + \frac{\partial g_{bd}}{\partial z^a}), \quad \Gamma^c_{a\bar{b}} = \frac{1}{2} g^{cd}(\frac{\partial g_{ad}}{\partial \bar{z}^b} - \frac{\partial g_{bd}}{\partial \bar{z}^a}),$$

etc.

Introducing the 1-forms $\vartheta^c_a = \Gamma^c_{ab} e^b + \Gamma^c_{a\bar{b}} e^\bar{b}$ (not necessarily of $(1,0)$-type) we obtain the induced connection on the holomorphic bundles $T_{(1,0)} M$, $T^{(1,0)} M$ (and their conjugate):

$$\nabla e_a = \vartheta^c_a e_c, \quad \nabla^c = -\vartheta^c_a e^a, \quad \text{etc.}$$

### 2.3. The canonical connection.

Though we will almost not use it, let us mention also the canonical connection $\mathcal{D}$. [4][10]

$$\mathcal{D} = \frac{1}{2}(\nabla - J\nabla J) = \nabla - \frac{1}{2} J\nabla(J).$$

This connection is both metric $\mathcal{D}(g) = 0$ and complex $\mathcal{D}(J) = 0$. The price of this additional (second) property is the emergence of torsion: $T_{\mathcal{D}}(X, Y) = \frac{1}{2}(\nabla_X(J)JY - \nabla_Y(J)JX)$.

Clearly $\nabla$ is the canonical connection iff the structure $(g, J, \omega)$ is Kähler. Also, if the Chern connection $\mathcal{D}$ is canonical, then $(\nabla_X J)Y = (\nabla_Y J)X$, and this implies that the structure $(g, J, \omega)$ is almost Kähler. A Hermitian almost Kähler structure is necessarily Kähler [4].
2.4. Induced connections. The above connections naturally induce canonical connections on the canonical bundle $K$.

For the Chern connection $\mathbb{D}e_a = \theta^c_a e_c$ this is given via the section $\Omega = e^1 \wedge \ldots \wedge e^n \in \Gamma(K)$ by $\mathbb{D}\Omega = -\operatorname{tr}(\theta) \otimes \Omega$, where $\operatorname{tr}(\theta) = \theta^a_a$. Due to $\langle \Omega, \Omega \rangle = \det(g^{ab}) = \det g^{-1}$ we also have $\mathbb{D}\Omega = \frac{i}{2} \partial \log g \otimes \Omega$.

Similarly, for the Levi-Civita connection $\nabla e_a = \bar{v}^c_a e_c$ we get $\nabla\Omega = -\operatorname{tr}(\bar{\vartheta}) \otimes \Omega$, and in general the connection form $\operatorname{tr}(\bar{\vartheta}) = \bar{v}^a_a$ differs from that for the Chern connection (but coincides with it in the Kähler case).

2.5. Curvature and the second fundamental form. Pick a linear connection $D$ on a vector bundle $E$ over $M$. Denote $\Omega^k(M, E) = \Gamma(\Lambda^k T^* M \otimes E)$. Then $D$ can be uniquely extended to a sequence of maps $D : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ by the Leibnitz super-rule: for $\alpha \in \Omega^*(M)$ and $s \in \Gamma(E)$ let $D(\alpha \otimes s) = d\alpha \otimes s + (-1)^{|\alpha|} \alpha \wedge Ds$.

The curvature is the obstruction for $D^2$ to be a complex: identify $D^2 : \Omega^0(M, E) \to \Omega^2(M, E)$ with $R_D \in \Gamma(\Lambda^2 T^* M \otimes \operatorname{End}(E))$, $R_D(\xi, \eta) = [D_\xi, D_\eta] - D_{[\xi, \eta]} \in \Gamma(\operatorname{End}(E))$, $\xi, \eta \in \Gamma(TM)$.

Here it is important which sign convention we choose. In terms of the connection matrix $\theta_E = (\theta^b_a)$ we get:

$$D^2 e_a = D(\theta^b_a e_b) = (d\theta^b_a - \theta^c_a \wedge \theta^b_c)e_b.$$ 

This is the Maurer-Cartan form of the curvature: $\Theta^b_a = d\theta^b_a - \theta^c_a \wedge \theta^b_c$, or in coordinate-free notation $\Theta_E = d\theta_E - \frac{1}{2}[\theta_E, \theta_E]$.

Note that if $D$ is the Chern connection (on a Hermitian bundle), then $\Theta_E$ is a matrix of $(1, 1)$-forms, but for the Levi-Civita connection in general this is not the case.

Let $E_0 \subset E$ be a holomorphic subbundle and $E_1 = E_0^\perp$ its orthocomplement. Since $E_0$ is a Hermitian bundle in its own, we have two first order differential operators $D_{E_0} : \Gamma(E_0) \to \Omega^1(M, E)$ and $D_{E_0} : \Gamma(E_0) \to \Omega^1(M, E_0)$.

The second fundamental form of the subbundle $E_0$ in $E$ with normal bundle $E_1$ is the tensor $\Phi \in \Omega^1(M) \otimes \Gamma(\operatorname{Hom}(E_0, E_1))$ given by

$$\Phi = D_{E_1}|_{\Gamma(E_0)} - D_{E_0} : \Gamma(E_0) \to \Omega^1(M, E_1).$$

Note that for $D = \mathbb{D}$ the Chern connection, $\Phi \in \Omega^{1, 0}(M, \operatorname{Hom}(E_0, E_1))$.

The connection matrix in the splitting $E = E_0 \oplus E_1$ writes

$$\theta_E = \begin{bmatrix} \theta_{E_0} & \Phi^* \\ \Phi & \theta_{E_1} \end{bmatrix},$$

where $\Phi^* = \bar{\Phi}^t$. Hence the curvature is

$$\Theta_E = d\theta_E - \theta_E \wedge \theta_E = \begin{bmatrix} d\theta_{E_0} - \theta_{E_0} \wedge \theta_{E_0} - \Phi^* \wedge \Phi^* \\ * & * \end{bmatrix}.$$
so that
\[ \Theta_E|_{E_0} = \Theta_{E_0} - \Phi^* \land \Phi, \tag{3} \]
where for vector spaces \( V, W \) and elements \( \alpha, \beta \in V^* \otimes \text{End}(W) \), \( X, Y \in V \), we let \((\alpha \land \beta)(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)\), \( \alpha^*(X) = (\alpha(X))^* \).

3. Background II: positivity

In this section preliminary computations are made, following [2]. The first subsection is just linear algebra and so is applicable to a complex vector space \((V, J) (= T_x M)\), equipped with a complex-valued symmetric \( \mathbb{C} \)-bilinear non-degenerate form \((\cdot, \cdot)\) with \((X, X) \geq 0\). The corresponding Hermitian metric is \(\langle X, Y \rangle = (X, Y)\).

Let us call a 2-form \( \Omega \in \Lambda^2 V^* \) positive (resp. non-negative \( \Omega \geq 0 \)) if the corresponding bilinear form \( b(X, Y) = \Omega(X, JY) \) is symmetric positive definite (resp. positive semidefinite). In other words, this 2-form is \( J \)-invariant, i.e. \( \omega \in \Lambda^1 V^* \), and \( \frac{1}{i}\omega(X', X'') > 0 \) (resp. \( \geq 0 \)) for \( X \neq 0 \). Here \( X' = \frac{1}{2}(X - iJX) \) is the projection of \( X \in V \) to \( V_{1,0} \) and \( X'' = \frac{1}{2}(X + iJX) \) is the projection of \( X \) to \( V_{0,1} \).

Next, a 2-form \( \Omega \in \Lambda^2 V^* \otimes \text{End}(W) \) with values in Hermitian endomorphisms of a complex space \( W \) is positive (resp. non-negative \( \Omega \geq 0 \)) if the scalar valued 2-form \( \langle \Omega w, w \rangle \) is positive (resp. \( \geq 0 \)) \( \forall w \neq 0 \).

3.1. Projection on the canonical bundle. For an anti-symmetric endomorphism \( A : V \to V \) denote \( \hat{A} \in \Lambda^2 V^* \) the element given by lowering indices: \( \hat{A}(v \land w) = (v, Aw) = -(Av, w), v, w \in V \).

Note that \((A^*\alpha, \beta) = \langle \hat{A}, \alpha \land \beta \rangle\) for arbitrary \( \alpha, \beta \in V^* \). Indeed, using the operator \( \#: V^* \to V \) of raising indices, we get
\[(\hat{A}, \alpha \land \beta) = \hat{A}(\alpha^2 \land \beta^2) = (\alpha^2, A\beta^2) = \alpha(A\beta^2) = (A^*\alpha)(\beta^2) = (A^*\alpha, \beta).\]
Here and below star denotes the usual pull-back \( A^* : \Lambda^k V^* \to \Lambda^k V^* \).

Lemma 4. Let \( n = \dim_{\mathbb{C}} V \) and \( \omega(X, Y) = \langle JX, Y \rangle \) be the symplectic form on \( V \). Then, denoting \( \pi_0 : \Lambda^n V^* \otimes \mathbb{C} \to \Lambda^{n,0} V^* \) the orthogonal projection, we have
\[ \pi_0 A^* \pi_0^* = -i(\hat{A}, \omega) \in \text{End}(\Lambda^{n,0} V^*). \]

Proof. In a unitary (holomorphic) basis \( \{e_a\} \) of \( V \) with the dual basis \( \{e^a\} \) of \( V^* \) we get \( \omega = i(e^1 \land e^1 + \ldots + e^n \land e^a) \in \Lambda^1 V^* \). The holomorphic volume form is \( \Omega = e^1 \land \ldots \land e^n \in \Lambda^{n,0} V^* \). Hence \( \pi_0 A^* \pi_0^* : \Lambda^{n,0} V^* \to \Lambda^{n,0} V^* \) is equal to
\[ \langle A^* \Omega, \Omega \rangle = \langle A^* e^1, e^1 \rangle + \ldots + \langle A^* e^n, e^n \rangle = \langle \hat{A}, e^1 \land e^1 + \ldots + e^n \land e^a \rangle \]
and the last expression is \( \langle \hat{A}, -i\omega \rangle \). \( \Box \)
For $\mathcal{R} \in \Lambda^2 V^* \otimes \text{End}(V)$ with values in anti-symmetric endomorphisms define $\tilde{\mathcal{R}} \in \text{End}((\Lambda^2 V^*)$ as the composition (where $b = \ast^{-1}$)

$$\Lambda^2 V^* \otimes \text{End}_{sbow}(V) \rightarrow ^b \Lambda^2 V^* \otimes \Lambda^2 V^* \rightarrow ^1 \otimes ^2 \Lambda^2 V^* \otimes \Lambda^2 V = \text{End}(\Lambda^2 V^*).$$

In other words, if $R = \sum \alpha_k \otimes A_k$, then for $\beta \in \Lambda^2 V^*$ the action is $\tilde{\mathcal{R}}(\beta) = (\sum (A_k, \beta) \alpha_k).$ Now the previous lemma implies

**Corollary.** Denote by $R_\nabla$ the curvature of the connection induced from the Levi-Civita connection on $\Lambda^n TM^*$. Then

$$R_\nabla|_K = i\tilde{\mathcal{R}}(\omega) \in \Omega^2(M).$$

**Proof.** If $R = \sum \alpha_k \otimes A_k$, then $R_\nabla = -\sum \alpha_k \otimes A_k^*$ and therefore $\pi_0 R_\nabla \pi_0^* = i \sum (A_k, \omega) \alpha_k = i\tilde{\mathcal{R}}(\omega).$ \qed 

This corollary and decomposition (3) yield formula (11) from the introduction.

### 3.2. Type of the second fundamental form.

Recall [6] that for the Chern connection the second fundamental form is always of type (1, 0). For the Levi-Civita connection this is not always so, however for the subbundle $K \subset \Lambda^n T^* M \otimes \mathbb{C}$ this property holds.

**Lemma 5.** The second fundamental form of the canonical bundle $K$ satisfies $\Phi \in \Omega^{1,0}(M) \otimes \Gamma(\text{Hom}(K, \Lambda^n T^* M \otimes \mathbb{C}/K)).$

**Proof.** Let us first prove the same property for the subbundle $\Lambda^{1,0}(M) \subset T^* M \otimes \mathbb{C}$. Let $e_a$ be a (local) unitary frame and $e^a$ the dual co-frame. Since $\nabla e^a = -\vartheta^a_c e^c - \vartheta^a_{\bar{c}} \bar{e}^\bar{c}$, the claim means $\vartheta^a_c \in \Omega^{1,0}(M)$.

Decompose the $(0, 1)$-part of this connection form component: $\vartheta^a_b = \sum \beta_{ab} e^a e^b.$ The Leibnitz rule applied to $d(e_a, e_b) = 0$ implies that $\beta_{abc}$ is skew-symmetric in $ab$.

On the other hand, $\nabla$ is torsion-free and so

$$d e^a = \text{alt}[\nabla e^a] = -\sum (\vartheta^a_c \wedge e^c + \vartheta^a_{\bar{c}} \wedge e^{\bar{c}}).$$

The Nijenhuis tensor of $J$ vanishes $0 = d^{-1,2} : \Omega^{1,0}(M) \rightarrow \Omega^{0,2}(M)$ and this implies $\sum \vartheta^a_{bc} \wedge e^c = 0.$ Therefore $\beta_{abc}$ is symmetric in $bc$, and the S3-lemma $(V \wedge V \otimes V) \cap (V \otimes V \otimes V) = 0$ yields $\beta_{abc} = 0$, i.e. $\vartheta^a_{\bar{c}} = 0$.

Now we pass to the subbundle $K = \Lambda^{n,0}(M) \subset \Lambda^n T^* M \otimes \mathbb{C}$. Since

$$\nabla(e^1 \wedge \ldots \wedge e^n) = -\sum \vartheta^a_c \otimes (e^1 \wedge \ldots \wedge e^{a-1} \wedge e^c \wedge e^{a+1} \wedge \ldots \wedge e^n) \mod K$$

the claim follows. \qed 

**Corollary.** We have: $-i\Phi^* \wedge \Phi \geq 0.$

**Proof.** Since $\Phi$ has type $(1, 0)$ we conclude

$$\frac{1}{2}(-i\Phi^* \wedge \Phi)(X', X'') = \Phi^*(X'')\Phi(X') = (\Phi(X'))^* \Phi(X') \geq 0. \quad \square$$
4. PROOF OF THEOREM 2

This section contains the proof of Theorem 2, following the approach of [2]. Theorem 1 is an immediate corollary.

We first prove a quantitative assertion that a small perturbation of a positive form is positive. Let \((g, J, \omega)\) be a (linear) Hermitian structure on a vector space \(V\), \(n = \dim_{\mathbb{C}} V = \frac{1}{2} \dim_{\mathbb{R}} V\). The Euclidean structure on \(V\) induces the following norm on \(\text{End}(V)\): \(\|A\|_E = \sqrt{\sum |A_{ij}|^2}\), where \(A = [A_{ij}]\) is the matrix representation in some unitary basis. It also yields the norm \(\sigma \mapsto \|\sigma\|_{\Lambda^2}^2\) on \(\Lambda^2 V^*\).

Note that the embedding \(\Lambda^2 V^* \cong \text{End}_{\text{skew}}(V) \subset \text{End}(V)\), \(\hat{A} \mapsto A\), scales the norm: \(\|\sigma\|^2_E = 2\|\sigma\|^2_{\Lambda^2}\). Below we identify \(\hat{A}\) with \(A\).

**Lemma 6.** Let \(\sigma_0\) be a real \((1,1)\)-form and \(\sigma_0 \geq \omega\). Then any real \((1,1)\)-form \(\sigma\), such that \(\|\sigma - \sigma_0\|_{\Lambda^2} \leq \frac{1}{2\sqrt{n}}\), is nondegenerate.

**Proof.** Recall that if \(\|A\|_E < 1\) for \(A \in \text{End}(V)\), then \(1 - A \in \text{End}(V)\) is invertible. Indeed, \((1 - A)^{-1} = \sum_{k=0}^{\infty} A^k\).

Diagonalize \(\omega\) and \(\sigma\) simultaneously: in some unitary co-frame \(e^a\)

\[
\omega = i \sum e^a \wedge e^a, \quad \sigma = i \sum \lambda_a e^a \wedge e^\bar{a},
\]

and \(\lambda_a \geq 1\) by the assumptions. Then \(\sigma_0^{-1} = -i \sum \lambda_a^{-1} e^a \wedge e^\bar{a}\) and \(\|\sigma_0^{-1}\|^2_E = 2 \sum \lambda_a^{-2} \leq 2n\).

Decompose \(\sigma = \sigma_0 + (\sigma - \sigma_0) = \sigma_0 \cdot (1 + \sigma_0^{-1}(\sigma - \sigma_0))\). The claim follows from \(\|\sigma_0^{-1}(\sigma - \sigma_0)\|_E \leq \|\sigma_0^{-1}\|_E \cdot \|\sigma - \sigma_0\|_E < \sqrt{2n} \cdot \frac{1}{2\sqrt{n}} = 1\). \(\square\)

Now the proof of Theorem 2 is concluded as follows. Let \(n = 3\). Normalize \(g\) by the requirement \(\text{Sp}(\tilde{R}) \in \left(\frac{5}{6}, \frac{7}{6}\right)\). Then \(\text{Sp}(1 - \tilde{R}) \in (-\frac{1}{6}, \frac{1}{6})\), and so \(\|\tilde{R}(\omega) - \omega\| < \frac{1}{6}\|\omega\| = \frac{1}{6} \sqrt{3} = \frac{1}{2\sqrt{3}}\). By (11) we get

\[-i\Omega = \tilde{R}(\omega) - i\Phi^* \wedge \Phi = (\tilde{R}(\omega) - \omega) + (\omega - \Phi^* \wedge \Phi)\]

Since \(\sigma_0 = \omega - \Phi^* \wedge \Phi \geq \omega\), then by Lemma 6 we conclude that \(i\Omega\), and hence \(\Omega\) are nondegenerate. Since \(\Omega\) is closed by Bianchi’s identity, it is symplectic on \(\mathbb{S}^6\), which is a contradiction. \(\square\)

5. ANOTHER APPROACH

In this section we give yet another proof of Theorem 1 due to K. Sekigawa and L. Vanhecke [9]. We should warn the reader of some unspecified sign choices in their paper, which we amend here.

Our sign conventions in this respect are in agreement with [3, 4], though in these sources the curvature is defined as minus that of ours. Since there are several differences in sign agreements, for instance in
passing from \((g, J)\) to \(\omega\), in Ricci contraction etc, this will be reflected in sign differences of our formulae, which otherwise are fully equivalent.

5.1. The first Chern class. Given a connection \(D\) and its curvature tensor \(R_D\) on an almost Hermitian manifold \((M, g, J)\) of dimension \(2n\) define its holomorphic Ricci curvature by

\[
\text{Ric}_D^e(X, Y) = -\text{Tr}
\left( \right.
R(X, J\cdot Y)\left. \right) = \frac{1}{2} \sum_{i=1}^{2n} R_D(X, JY, e_i, Je_i),
\]

where \(e_1, e_2 = Je_1, \ldots, e_{2n-1}, e_{2n} = Je_{2n-1}\) is a \(J\)-adapted orthonormal basis. For the characteristic (Chern) connection \(D\) the 2-form

\[
\gamma_1(X, Y) = -\frac{1}{2\pi} \text{Ric}_D^e(X, JY) = \frac{1}{2\pi} \text{Ric}_D^e(JX, Y)
\]

represents the first Chern class \(c_1 = [\gamma_1]\), see [4]. When passing to the Levi-Civita connection \(\nabla\), this simple formula is modified.

A relation between the two connections is given by [4, (6.2)] that, in the case of integrable \(J\), states

\[
g(D_X Y, Z) = g(D_X Y, Z) - \frac{1}{2} g(JX, \nabla_Y (J) Z - \nabla_Z (J) Y),
\]

with the canonical connection \(D\) given by (2). This allows to express the first Chern form in terms of \(\nabla\) (the curvature of \(D\) is expressed through that of \(\nabla\) in [10], and the curvature of \(D\) – in [4]).

Define the 2-forms

\[
\psi(X, Y) = -2 \text{Ric}_D^e(X, JY) = \sum R_D(X, Y, e_i, Je_i)
\]

and

\[
\varphi(X, Y) = \text{Tr}
\left( J(J\nabla_X J)(\nabla_Y J) \right) = -\sum (\nabla_X J)^a_b (\nabla_J Y)^b_a.
\]

With these choices (cf. [4, 9]) the first Chern form is given by

\[
8\pi \gamma_1 = 2\psi + \varphi.
\]

5.2. Alternative proof of Theorem [11]. Now suppose that \(g\) has constant sectional curvature \(k > 0\), i.e.

\[
R_\nabla(X, Y, Z, T) = g(R_\nabla(X, Y) Z, T) = k \cdot (g \otimes g)(X, Y, Z, T),
\]

where \((g \otimes g)(X, Y, Z, T) = g(X, Z)g(Y, T) - g(X, T)g(Y, Z)\) is the Kulkarni-Nomizu product, whence

\[
\text{Ric}_\nabla(X, Y) = \sum R_\nabla(X, Y, e_i, e_i) = (2n - 1) k g(X, Y),
\]

\[
\text{Ric}_\nabla^e(X, Y) = \sum R_\nabla(X, Y, Je_i, Je_i) = k g(X, Y).
\]

In other words, this metric \(g\) is both Einstein and \(*\)-Einstein, and the scalar and \(*\)-scalar curvatures are both positive.

Thus both \(\text{Ric}_\nabla\) and \(\text{Ric}_\nabla^e\) are positive definite, and hence \(\psi > 0\). Now since \(\varphi(X, JX) = \|\nabla_J X J\|^2 = \|\nabla_X J\|^2 \geq 0\), we have \(\varphi \geq 0\), and consequently \(\gamma_1 > 0\). Integrating \(\gamma_1^n\) yields \(c_1^n(M) \neq 0\).
Returning to the case $M = S^6$, $n = 3$, and the standard round metric $g = g_0$ of constant sectional curvature 1, we obtain a contradiction because $c_1 \in H^2(S^6) = 0$, and so $c_1^3 = 0$ as well. \hfill \Box

6. Generalization of the idea of Section 5

If we perturb the metric $g$ starting from $g_0$, it is no longer $\ast$-Einstein, and the argument of the previous section literally fails.

However, since the space of $g$-orthogonal complex structures $\mathcal{J}_g \simeq O(2n)/U(n)$ is compact, the image of the map

$$\mathcal{J}_g \ni J \mapsto \text{sym}[\psi(\cdot, J \cdot)] \in \Gamma(\bar{\otimes}^2 T^* M)$$

is close to the one-point set $\{g_0\}$ (because $g_0$ is $\ast$-Einstein) and so is positive for $g$ sufficiently close to $g_0$. Thus we still get the inequality $8\pi \gamma_1 = 2\psi + \varphi > 0$ as in the previous section, and so conclude non-existence of $g$-orthogonal complex structures $J$ on $S^6$ for an open set of metrics $g \in \Gamma(\bar{\otimes}^2 T^* S^6)$ in $C^2$-topology.

A quantitative version of this idea is a novel result given below.

6.1. Bounds in the space of curvature tensors. Fix a Euclidean space $V$ of even dimension $2n$ with metric $g = \langle \cdot, \cdot \rangle$, and consider the space $\mathcal{R}$ of algebraic curvature tensors on it. Identifying $(3,1)$ and $(4,0)$ tensors via the metric, $\mathcal{R} = \text{Ker}[\wedge : \otimes^2 \Lambda^2 V^* \to \Lambda^4 V^*].$

In this subsection we restrict to linear tensors in $V$. Denote by $\mathcal{P}$ the space $\{R \in \mathcal{R} : \text{Ric}^R(X, X) \geq 0 \ \forall X \in V, \forall J \in \mathcal{J}_g\}$, where Ric$^R$ is computed via $R$ and $J$ as in the previous section.

This can be exposed in index terms as follows. Denote by $\mathcal{F}_g$ the space of $g$-orthonormal frames $e = \{e_1, \ldots, e_{2n}\}$ on $V$. Each such frame yields an orthogonal complex structure on $V$ by $J e_i = (-1)^i e_i^\#$, where $i^\# = i - (-1)^i$. For every $e \in \mathcal{F}_g$ and $R \in \mathcal{R}$ compute $\alpha_{ij} = \sum_{k=1}^{2n} R(e_i, e_k, e_{j^\#}, e_{k^\#}) = (-1)^{i+j} \alpha_{j^\# i^\#}$ and form the symmetric matrix $A$ with entries $a_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji})$. Then $R \in \mathcal{P}$ iff $A$ is positive semidefinite for every $e \in \mathcal{F}_g$, and this can be determined by finite-dimensional optimization via the Silvester criterion.

A simple sufficient criterion for this is the following. Introduce the following $L^\infty$-norm on $\mathcal{R}$: $\|R\|_\infty = \max_{\|\nu\| = 1} |R(\nu_1, \nu_2, \nu_3, \nu_4)|$.

Lemma 7. If $\|R - g \otimes g\|_\infty \leq \frac{1}{2\pi}$, then $R \in \mathcal{P}$.

Proof. Denote $\dot{R} = R - g \otimes g$. Then for any $J \in \mathcal{J}_g$ and $X \in V$ with $\|X\| = 1$ we get

$$\text{Ric}^R(X, X) = \sum R(X, e_i, JX, Je_i) = \|X\|^2 + \sum \dot{R}(X, e_i, JX, Je_i)$$

$$\geq \|X\|^2 - 2n \|X\|^2 \max_{\|u\| = \|v\| = 1} |\dot{R}(u, v, Ju, Jv)| \geq 0.$$
Thus $R \in \mathcal{P}$. \hfill \Box

6.2. **A non-existence alternative to Theorem 2**. Write $g \in \mathcal{P}$ if the curvature tensor of $g$ satisfies this positivity property on every tangent space $V = T_x M$, $x \in M$. The set $\mathcal{P}$ is a neighborhood of the round metric $g_0$ on $M = S^6$ in the space of all metrics in $C^2$-topology.

**Theorem 8.** $S^6$ possesses no Hermitian structure $(g, J, \omega)$ with $g \in \mathcal{P}$.

**Proof.** In formula (11) $\varphi \geq 0$, and if $g \in \mathcal{P}$ then $\psi \geq 0$ as well. Thus $\gamma_1 \geq 0$ and we conclude $0 = c_3^1[S^6] = \int_{S^6} \gamma_1^3 \geq 0$. This integral is the sum of several non-negative summands, the last of which is $\int_{S^6} \varphi^3$. Since all of these summands have to vanish, we conclude $\varphi = 0$ implying $\|\nabla J\|^2 = 0$. Thus $\nabla J = 0$ meaning that $(g, J, \omega)$ is a Kähler structure on $S^6$ and this is a contradiction. \hfill \Box

**Corollary.** If $\|R_{\nabla} - g \otimes g\|_\infty \leq \frac{1}{6}$ for the curvature $R_{\nabla}$ of a metric $g$ on $S^6$, then no $g$-orthogonal almost complex structure $J$ is integrable.

Note that this can be again considered as a perturbation result for the metric $g_0$, for which the curvature tensor is $R_{\nabla_0} = g_0 \otimes g_0$. If $\|R_{\nabla} - R_{\nabla_0}\|_\infty \leq \epsilon_1$, $\|g - g_0\|_\infty \leq \epsilon_2$ and $\epsilon_1 + 4\epsilon_2 + 2\epsilon_2^2 \leq \frac{1}{6}$ (one can check that this follows from the linear constraint $\epsilon_1 + (2 + \sqrt{13}/3)\epsilon_2 \leq \frac{1}{6}$), the claim follows. Indeed (note that $\|v\|^2 = g_0(v, v)$) we have:

$$\frac{1}{2}\|g \otimes g - g_0 \otimes g_0\|_\infty \leq \max_{\|v_i\| = 1} |g(v_1, v_3)g(v_2, v_4) - g_0(v_1, v_3)g_0(v_2, v_4)| \leq \max_{\|v_i\| = 1} |g(v_1, v_3) - g_0(v_1, v_3)| |g(v_2, v_4)| + |g(v_2, v_4) - g_0(v_2, v_4)||g_0(v_1, v_3)| \leq \|g - g_0\|_\infty (2 + \|g - g_0\|_\infty).$$

Thus $\|R_{\nabla} - g \otimes g\|_\infty \leq \|R_{\nabla} - g_0 \otimes g_0\|_\infty + \|g_0 \otimes g_0 - g \otimes g\|_\infty \leq \epsilon_1 + 2\epsilon_2 (2 + \epsilon_2)$.

7. **Concluding remarks**

The non-existence results of this paper are not sharp. Indeed, both Theorems 2 and Corollary of Theorem 8 deal with rough upper bound and could be further improved.

Note also that the property of an almost complex structure $J$ being $g$-orthogonal depends only on the conformal class of the metric $g$, while the Riemann and holomorphic Ricci tensors, used in the proofs, are not conformally invariant. It is a challenge to further elaborate the results to get better bounds, implying non-existence of orthogonal complex structures in a larger neighborhood of the round metric on $S^6$.

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2This has the following corollary generalizing Theorem 4: If $g$ is conformally equivalent to $g_0$ on $S^6$, then the space $J \in \mathcal{J}_g(S^6)$ contains no complex structures.
Every almost complex structure $J$ on $\mathbb{S}^6$ is orthogonal with respect to some metric $g$, but as this (or any conformally equivalent metric) can be far from $g_0$, the positivity argument will not work.

Note that all known proofs of non-existence of Hermitian structures for certain $g$ use only one property of the 6-sphere, namely that $H^2(\mathbb{S}^6) = 0$. It would be interesting to find a proof of non-existence of orthogonal complex structures based on some other ideas.

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