Quantum Noise Theory of Exceptional Point Amplifying Sensors

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Open quantum systems can have exceptional points (EPs), degeneracies at which both eigenvalues and eigenvectors coalesce. Recently, it has been proposed and demonstrated that EPs can enhance the performance of sensors in terms of amplification of a detected signal. However, typically amplification of signals also increases the system noise, and it has not yet been shown that an EP sensor can have improved signal to noise performance. We develop a quantum noise theory to calculate the signal-to-noise performance of an EP sensor. We use the quantum Fisher information to extract a lower bound for the signal-to-noise ratio (SNR) and show that parametrically improved SNR is possible. Finally, we construct a specific experimental protocol for sensing using an EP amplifier near its lasing threshold and heterodyne signal detection to achieve the optimal scaling predicted by the Fisher bound. Our results can be generalized to higher order EPs for any bosonic non-Hermitian system with linear interactions.

A distinct feature of physical systems described by non-Hermitian operators is the possibility of exceptional points (EPs), degeneracies at which not only eigenvalues but also the eigenvectors coalesce. EPs have been extensively discussed in the context of generalizing the standard quantum theory to include non-Hermitian Hamiltonians. However, we already can study the effect of EPs in open systems within the conventional quantum framework, either by looking at resonant scattering or by treating unobserved degrees of freedom as lossy or amplifying reservoirs. Many interesting effects of EPs have been studied in electromagnetic or optomechanical systems; often these systems have parity-time symmetry, although this symmetry is not essential to the existence of EPs.

One intriguing application of EPs is to enhance the performance of sensors, which has been theoretically proposed and experimentally demonstrated using optical micro-ring resonators. Here we will focus on the most important case of resonant detectors, for which the relevant non-hermitian operator is the open system Hamiltonian (cavity wave operator), and the eigenvalues are the complex resonant frequencies of this operator. Assume that the resonances are non-degenerate initially: by tuning several parameters of this operator, two or more resonances can be brought to degeneracy without imposing any additional symmetries, thus creating an EP corresponding to only a single resonant solution at that complex frequency. One consequence of operating at a second-order EP is that a small perturbation, \( \epsilon \), which lifts the degeneracy, will split the resonant frequencies by \( \sim \sqrt{\epsilon} \), a parametrically larger sensitivity than for hermitian systems. This type of enhanced frequency sensitivity was demonstrated in the micro-ring experiments, but at present there has been no systematic analysis of the effect on the noise and the signal-to-noise ratio (SNR) of operating a sensor in the vicinity of an EP. According to quantum noise theory, the gain (and loss) introduced via non-Hermitian dynamics will unavoidably generate additional noise, making it unclear that the EP sensor will have enhanced SNR (assuming that the sensor is operating near the quantum noise limit, and is not dominated by extrinsic noise sources). Therefore, in evaluating the efficacy of EP sensors, it is crucial to calculate the behavior of both signal and noise near an EP.

In this Letter, we present such an investigation, which addresses the following three questions: (1) Can operating near an EP enhance the SNR of a sensor? (2) What is the maximal precision (in terms of SNR) of EP sensing schemes? (3) How can we design an EP sensing scheme to achieve this ultimate precision? To answer these questions, we first apply quantum noise theory to calculate the amplitude and covariance matrix associated with the output of a model of EP sensors; then we calculate the quantum Fisher information of the output state and obtain the Cramer–Rao bound for the parameter estimation. Finally, we explicitly construct an EP sensing scheme based on heterodyne detection of the output of an amplifying EP sensor to achieve this optimal scaling of the sensitivity. Our scheme is not set up to detect the frequency splitting near an EP, but rather the amplitude change of a signal amplified near the lasing frequency of our sensor. The results can be generalized to higher order EPs for any bosonic non-Hermitian system with linear interactions, i.e. involving only Gaussian processes.

Non-Hermitian dynamics and open quantum systems. We define a non-Hermitian Hamiltonian to characterize the open sensor. A generic model is shown in Fig. 1; we have coupled cavities with two resonant optical modes \( a_1 \) and \( a_2 \), with the non-Hermitian Hamiltonian (setting...
where the total loss/gain rates of modes 1,2 are \( \gamma_1, \gamma_2 \) and \( g \) is the inter-cavity coupling. To implement our EP sensing scheme we assume that we can tune the real part of the resonance frequencies such that \( \omega_1 = \omega_2 = \omega \), and that the perturbation we are sensing uniformly shifts the frequencies of both modes by \( \epsilon_1 = \epsilon_2 = \epsilon \). The resulting equation of motion is

\[
\frac{d}{dt} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -i \begin{pmatrix} (\omega + \epsilon - i \frac{\gamma_1}{2} g) & \frac{g}{\sqrt{\kappa_1}} \\ \frac{g}{\sqrt{\kappa_2}} & (\omega + \epsilon + i \frac{\gamma_2}{2} g) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},
\]

characterized by a non-Hermitian effective Hamiltonian matrix which has EPs when \( (\gamma_1 + \gamma_2)^2/4 = |g|^2 \) at eigen-frequency, \( \Omega = (\omega + \epsilon + i(\gamma_2 - \gamma_1))/4 \). (Note that the EP occurs at real frequencies when \( \gamma_1 = \gamma_2 \), and coincides with the lasing threshold). EP-enhanced sensing will be achieved by monitoring the quadrature amplitudes associated with \( a_1, a_2 \) at \( \omega \) as the EP (and lasing frequency) is shifted to the frequency \( \omega + \epsilon \) via the perturbation. Due to the EP, the resonance amplitude falls off as \( \epsilon^{-2} \), leading to higher sensitivity. However Eq. 2 only describes the average behavior near the EP and not the noise properties.

The imaginary parts of the frequencies \( (\gamma_1, \gamma_2) \) arise from a combination of outcoupling to the probe channels \( A_1, A_2 \) with rate \( \kappa_1, \kappa_2 \), (for simplicity we assume \( \kappa_1 = \kappa_2 = \kappa \) and loss and gain processes in the cavities with rate \( \eta_1 (\eta_2) \)). Hence, the total loss and gain rates are \( \gamma_1 = \eta_1 + \kappa \) and \( \gamma_2 = \eta_2 - \kappa \) for the two modes, respectively. We denote \( A_{1,\text{in(out)}} \) and \( A_{1,\text{in(out)}} \) for the complex amplitudes of the two input (output) probe channels, satisfying the input-output relation

\[
\begin{pmatrix} A_{1,\text{out}} \\ A_{2,\text{out}} \end{pmatrix} = \begin{pmatrix} A_{1,\text{in}} \\ A_{2,\text{in}} \end{pmatrix} + \begin{pmatrix} \sqrt{\kappa_1} a_1 \\ \sqrt{\kappa_2} a_2 \end{pmatrix}.
\]

To fully characterize the noise properties of the output, we also need to model the fluctuations associated with the intrinsic loss \( \eta_1 \) and gain \( \eta_2 \). As shown in Fig. 1, this can be done by introducing two ancillary scattering channels \( B_1 \) and \( B_2 \) to simulate the intrinsic dissipation and amplification processes in \( a_1 \) and \( a_2 \). According to quantum noise theory \( B_1 \) is coupled to \( a_1 \) via a beam-splitter-type interaction \( \sqrt{\eta_1} \hat{a}_1 + \text{h.c.} \) with coupling strength \( \sqrt{\eta_1} \), while \( B_2 \) is coupled to \( a_2 \) via a two-mode-squeezer-type interaction \( \sqrt{\eta_2} \hat{a}_2 + \text{h.c.} \) with coupling strength \( \sqrt{\eta_2} \), and the quantum Langevin equations for both modes can be written as

\[
\frac{d}{dt} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = -i \begin{pmatrix} (\omega + \epsilon - i \frac{\gamma_1}{2} g) & \frac{g}{\sqrt{\kappa_1}} \\ \frac{g}{\sqrt{\kappa_2}} & (\omega + \epsilon + i \frac{\gamma_2}{2} g) \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} + \begin{pmatrix} \sqrt{\eta_1} \hat{a}_1 \\ \sqrt{\eta_2} \hat{a}_2 \end{pmatrix} + \sqrt{\eta_1} \hat{B}_{1,\text{in}} + \sqrt{\eta_2} \hat{B}_{2,\text{in}},
\]

which now has additional noise terms \( \hat{B}_{1,\text{in}}, \hat{B}_{2,\text{in}} \) as compared with Eq. 4 due to noise from the reservoirs.

**Amplitude vector and covariance matrix.** To model the loss and gain processes, we can simply set the channels \( B_1 \) and \( B_2 \) to zero average amplitudes, i.e. vacuum. For any pair of bosonic annihilation operator \( \hat{a} \) and creation operator \( \hat{a}^\dagger \), we define a pair of “position” quadrature operator \( \hat{q} = \hat{a} + \hat{a}^\dagger \), and “momentum” quadrature operator \( \hat{p} = -i (\hat{a} - \hat{a}^\dagger) \). Given any quantum state of \( N \) bosonic modes, the amplitude vector \( \mu \) and covariance matrix \( V \) can be defined in the quadrature basis

\[
\mu_j = \langle \hat{x}_j \rangle
\]

and

\[
V_{j,k} = \frac{1}{2} \langle \hat{x}_j \hat{x}_k + \hat{x}_k \hat{x}_j \rangle - \langle \hat{x}_j \rangle \langle \hat{x}_k \rangle,
\]
for $1 \leq j, k \leq N$, with vector $\mathbf{x} = (q_1, q_2, \ldots, q_n, \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)^T$, and $\langle \cdot \rangle$ representing the expectation value [23].

To compute the amplitude vector and covariance matrix, we perform the Fourier transform of Eq. (4), and obtain the relation between the Fourier transformed operators $\hat{A}[\omega] = \int \hat{A}(t)e^{-i\omega t}dt$ associated with the input and output ports. The amplitude vector and covariance matrix of the probe output channels are [26]

$$\mu_{\text{out}} = (I - G_{\theta}) \mu_{\text{in}}$$

and

$$V_{\text{out}} = (I - G_{\theta}) V_{\text{in}} (I - G_{\theta})^T + G_{\theta} R V_{\text{in}}^T R^T G_{\theta}^T$$

where $\mu_{\text{in}}$ and $V_{\text{in}}$ are the amplitude vector and covariance matrix of the input probe channels ($A_{1,\text{in}}[\omega], A_{2,\text{in}}[\omega]$), $V_{\text{in}}$ is the covariance matrix of the ancillary input channels ($B_1[\omega], B_2[-\omega]$) that induces the additional noise associated with the loss and gain, $R = \text{diag}(\sqrt{\eta}, -\sqrt{\eta}, \sqrt{\eta}, \sqrt{\eta})$, and dimensionless linear response matrix $G_{\theta}$ rescaled by the coupling rate [26] is given by

$$G_{\theta} = -\Omega (\theta I - M)^{-1}$$

with constant matrix $\Omega = [[0, I]; [-I, 0]]$, the dimensionless perturbation strength $\theta = \epsilon/\kappa$, and $M = \begin{pmatrix} 0 & G & 0 & 0 \\ G & -\Gamma_1 & 0 & -\Gamma_2 \\ -\Gamma_1 & 0 & 0 & G \\ 0 & G & 0 & 0 \end{pmatrix}$

is the dimensionless effective Hamiltonian in quadrature basis with dimensionless parameters $\Gamma_1 = \gamma_1/(2\kappa), \Gamma_2 = \gamma_2/(2\kappa)$, and $G = g/\kappa$.

For most applications, it is sufficient to consider Gaussian states for the probe and ancillary input channels. Since the physics process involves only linear interactions between modes, the output states are also Gaussian states that are completely characterized by the amplitude vector $\mu_{\text{out}}$ and covariance matrix $V_{\text{out}}$ [23]. Hence, the above calculated amplitude vector and covariance matrix are sufficient to characterize the performance of sensing.

**EP sensing.** By choosing $\Gamma_1 = \Gamma_2 = G$ the system will be at the lasing threshold and will remain at an EP as the lasing frequency shifts by $\theta$ due to the perturbation (the situation where the system is not exactly at the lasing threshold will be discussed later). Since the system remains at an EP at frequency $\omega$, we have a non-trivial Jordan decomposition of the $M$ matrix

$$M = P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1},$$

with an invertible matrix $P$ [27].

For small perturbation $\theta \ll 1$, the linear response matrix grows as a polynomial of $\theta^{-1}$

$$G_{\theta} = -\Omega (\theta I - M)^{-1} = -\theta^{-1} \sum_{n=0}^{\infty} \theta^{-n} \Omega M^n = -\theta^{-1} \Omega - \theta^{-2} \Omega M,$$

where the second step uses the Taylor expansion and the property that $M^n = 0$ for $n \geq 2$. The first term of $G_{\theta}$ is the general feature of a sensor at the laser threshold, while the second term implies an enhanced output signal at $\omega$ associated with the amplitude vector [Eq. (7)] as $\theta \to 0$, due to the EP. However, the noise associated with the covariance matrix [Eq. (8)], which contains a $\theta^{-4}$ term because of its dependence on $G_{\theta}$ [26], also diverges as $\theta \to 0$. Therefore, we need to systematically calculate the uncertainty of the measured parameter $\theta$ in the presence of noise. In the following, we first provide a lower bound to sensitivity using the quantum Cramer–Rao bound, and then provide an EP sensing protocol which achieves the optimal scaling with $\theta$ as given by that quantum Cramer–Rao bound.

**Sensitivity lower bound.** In the presence of noise, the standard deviation of an estimation of the parameter $\theta$, calculated from data obtained from some measurement on a quantum state, is bounded by inverse of the quantum Fisher information $I(\theta)$ of the state through quantum Cramer–Rao inequality [25]

$$\delta\theta \geq I(\theta)^{-1/2}.$$ \hspace{1cm} (13)

For Gaussian processes, such as our scheme, the quantum Fisher information takes the form [26, 30]

$$I(\theta) = I_0(\theta) + I_1(\theta),$$

where $I_0(\theta)$ is always positive and only depends on $V_{\text{out}}$, and not on $\mu_{\text{out}}$, and

$$I_1(\theta) = \left( \frac{d\mu_{\text{out}}}{d\theta} \right)^T V_{\text{out}}^{-1} \frac{d\mu_{\text{out}}}{d\theta}.$$ \hspace{1cm} (14)

Since $I_0(\theta)$ only contains information on fluctuations in the absence of the probe, the quantum Fisher information is dominated by $I_1(\theta)$, which can be regarded as a generalization of the squared SNR associated with the covariance matrix.

We plug Eqs. (7, 8, 12) into Eq. (14) and obtain [26]

$$I_1(\theta) = \theta^{-4} (c_0 + O[\theta]),$$

which implies that the leading contribution to the quantum Fisher information scales at least as $\theta^{-4}$ (orange curve in Fig. 2) and the sensitivity, $\delta\theta \geq c_0 \times \theta^2$, where the constant $c_0$ is determined by the choice of input probe signals and $c_0 > 0$ in generic situations [26]. This EP-enhanced behavior is compared with a non-EP sensing
scheme which shows no enhancement as \( \theta \to 0 \) (green curves in Fig. 2). Hence, EP sensing has a more favorable lower bound than the conventional sensing protocols.

One can intuitively understand the improvement by EP sensing as a result of correlations between different eigenmodes of the output state. Because of the special EP structure of the linear response matrix \( \mathbf{G}_\theta \), different linear combinations of the quadratures of the output state will accumulate noise with different dependence of \( \theta \). For our system, as shown in the Supplementary Material [26], only two orthogonal directions in the four-dimensional input space leads to noise being amplified \( \delta^{-1} \); thus there is a large subspace of inputs for which the SNR is enhanced by operating at or near an EP.

Heterodyne detection to achieve optimized EP sensing scaling. The Cramer–Rao bound applies to all possible sensing schemes; now we provide a specific EP sensing scheme that achieves the same scaling with \( \theta \) as predicted by the Cramer–Rao bound. The idea is to use heterodyne measurement to extract the output amplitude vector \( \mu_{\text{out}} \). The covariance matrix associated with the heterodyne detection is \( \mathbf{V}_{\text{out}} + i \mathbf{I} \) [26 51], which includes the additional quantum noise inherent in the simultaneous measurement of both position and momentum quadratures. Fortunately, the additional quantum noise does not depend on \( \theta \), and becomes negligibly small as compared with \( \mathbf{V}_{\text{out}} \) for \( \theta \to 0 \). Hence, we have \( (\mathbf{V}_{\text{out}} + i \mathbf{I})^{-1} = \mathbf{V}_{\text{out}}^{-1} (\mathbf{I} + O(\theta)) \approx \mathbf{V}_{\text{out}}^{-1} \).

For example, by injecting coherent state probe input with \( \mu_{\text{in}} = \mathbf{P} \cdot (0,1,0,0)^T \), the heterodyne detection can measure the output amplitude vector \( \mu_{\text{out}} = (\mathbf{I} - \mathbf{G}_\theta) \mu_{\text{in}} = \mu_{\text{in}} - \mathbf{\Omega} \cdot (\theta^{-2}, -\theta^{-1}, 0, 0)^T \). We can obtain uncertainty \( \delta \theta \approx \left[ \left( \frac{d\mu_{\text{out}}}{d\theta} \right)^T \mathbf{V}_{\text{out}}^{-1} \left( \frac{d\mu_{\text{out}}}{d\theta} \right) \right]^{-1/2} \sim \theta^2 \), which has the same scaling as the lower bound obtained from Eqs. [13 14 15].

General approach and higher-order EP sensing. We summarize our general approach to achieve EP sensing with the scaling as obtained from the Cramer–Rao bound. For a given EP sensing scheme based on a Gaussian process, we calculate the corresponding matrices that can help us track the change of amplitude and covariance matrix. Then we can calculate the quantum Fisher information and obtain the precision bound. For a general EP sensing scheme, \( \mathbf{G}_\theta = -\mathbf{\Omega} (\theta \mathbf{P} - \mathbf{M})^{-1} \) (in our previous discussion, \( \mathbf{P} = \mathbf{I} \), but in general it can be any invertible matrix), and \( \mathbf{M}^{-1} = \mathbf{P} \mathbf{A} \mathbf{P}^{-1} \), with \( \mathbf{P} \) invertible , and \( \mathbf{A} \) known as the Jordan normal form of \( \mathbf{M}^{-1} \) consisting of diagonal blocks of size \( N_i \) (for the \( r \)th block), each with eigenvalue zero. When \( N_i = 1 \), the corresponding block is just a scalar, which is not an EP. To have EP enhanced sensing, we need at least one non-trivial Jordan block (\( N_i \geq 2 \)) with eigenvalue zero.

Let \( N = \max_i N_i \) be the size of the largest zero-eigenvalue Jordan block, which corresponds to the \((N - 1)\)-th order EP. Then it is easy to show that \( \mathbf{G}_\theta = \theta^{-N} (\mathbf{\Omega} \mathbf{C}_0 + O(\theta)) + \cdots + \theta \to 0 \) and \( \mathbf{C}_0 \) a constant matrix. This divergence near \( \theta = 0 \) leads to \( \theta^{-N} \) amplification of the amplitude and \( \theta^{-2N} \) amplification of the covariance matrix. One might be tempted to argue that \( \mathbf{I}_1 (\theta) \) is then proportional to \( \theta^{2} \) since the scaling of amplification with \( N \) can be perfectly canceled by covariance matrix. However, a more rigorous calculation shows this is over pessimistic. As \( d\mathbf{G}_\theta / d\theta = \mathbf{G}_\theta \mathbf{P} \mathbf{A} \mathbf{P}^{-1} \mathbf{G}_\theta \), only one of the \( \mathbf{G}_\theta \) cancel with the amplification in the covariance matrix. We have \( \mathbf{I}_1 (\theta) = \mu_{\text{in}}^T \mathbf{G}_\theta^T \mathbf{G}_\theta \mu_{\text{in}} \), where \( \mathbf{C}_1 \) is a positive definite matrix [26]. So one can conclude that

\[
\mathbf{I}_1 (\theta) \approx \theta^{-2N}
\]  

(16)

for \((N - 1)\)-th order EP. Since \( \mathbf{I}_0 (\theta) \) is always positive, \( \mathbf{I}_1 (\theta) \) gives a lower bound on quantum Fisher information. We have then the quantum Cramer–Rao bound \( \delta \theta \gtrsim \theta^N [32] \), the scaling of which can be achieved by performing heterodyne measurement on all of the outputs even in this general situation.

Robustness of enhanced sensing The main assumptions used to realize enhanced sensing is precise tuning to an EP and operating the sensor at the lasing threshold. As noted above, if this is realized, there is a large family of input states which generate enhanced sensitivity for appropriately chosen outputs, so the sensor will be robust to the input state used. While operation at the
lasing threshold condition is assumed in order to derive the results above, we can actually relax this condition, and still achieve sensitivity enhancement over some parameter range. A small detuning $\delta$ from the lasing threshold will simply cut off the enhanced sensing for small $\theta$. As shown in Fig. 2 (purple curve) when we introduce additional loss $\delta$ to both cavities, the quantum Fisher information is upper bounded by $I^{UB} \approx \|G_{\theta=0}\|^2 \approx \delta^{-4}$, where $\|\cdot\|$ represents the trace norm [20]. When the system is sufficiently close to the lasing threshold ($\delta \ll \theta$), the quantum Fisher information does not exceed the upper bound ($I \approx \theta^{-4} \ll I^{UB}$), so that we will have EP enhanced sensing. Generally, for higher order EPs, we will have EP enhanced sensing near the lasing threshold as long as $\|G_{\theta}\| \gg \theta^{-N}$ [20].

In conclusion, we have established a theoretical framework using quantum noise theory to calculate systematically both the signal and noise of EP sensors, operating near the lasing threshold. Using the quantum Fisher information, we have obtained the lower bound of ultimate sensitivity of EP-sensors. Moreover, we provide a heterodyne detection scheme to achieve the optimal scaling of sensitivity of EP-sensors. Moreover, we provide a heterodyne detection scheme to achieve the optimal scaling of sensitivity predicted by this bound. Since these EP sensors are described by Gaussian processes with linear interactions, EP sensing near the laser threshold coupled with heterodyne detection should be feasible with the current experimental techniques.

Note added: During the completion of this work, we became aware of a related study by Lau and Clerk [33].

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