We develop a theoretical framework to investigate the two-body composite structure of a resonance as well as a bound state from its wave function. For this purpose, we introduce both a one-body bare state and two-body scattering states, and define the compositeness as the contribution from the two-body wave function to the normalization of the total wave function. We explicitly write down the wave function and the compositeness for a bound state obtained with a general separable interaction. In this formulation we can derive the Weinberg's relation for the scattering length and effective range in the weak binding limit. Our discussion on the wave function is extended to a resonance state expressed with the Gamow vector, and a relativistic formulation is also established. As the applications, we study the compositeness of the Λ(1405) resonance and the light scalar and vector mesons described with refined amplitudes in coupled-channel models with interactions up to the next to leading order in chiral perturbation theory. We find that Λ(1405) and f_0(980) are dominated by the ¯KN and K ¯K composite states, respectively, while the vector mesons ρ(770) and K^*(892) are elementary. We also briefly discuss the compositeness of N(1535) and Λ(1670) obtained in a leading-order chiral unitary approach.

1. Introduction
In hadron physics, the internal structure of an individual hadron is one of the most important subjects. Traditionally, the excellent successes of constituent quark models lead us to the interpretation that baryons consist of three quarks (qqq) and mesons of a quark-antiquark pair (q ¯q) [1]. At the same time, however, there are experimental indications that some hadrons do not fit into the classification suggested by constituent quark models. One of the classical examples is the hyperon resonance Λ(1405), which has an anomalously light mass among the negative parity baryons. In addition, the lightest scalar mesons [f_0(500) = σ, K_0^*(800) = κ, f_0(980), and a_0(980)] exhibit inverted spectrum from the naïve expectation with the q ¯q configuration. These observations motivate us to consider more exotic structure of hadrons, such as hadronic molecules and multiquarks [2–7].

It is encouraging that there have been experimental reports on the candidates of manifestly exotic hadrons such as charged quarkonium-like states by Belle collaboration [8]. Moreover, the LEPS collaboration observed the “Θ^+ signal” [9, 10], but its interpretation is still controversial [11, 12]. The accumulation of the observations of unconventional states in the heavy quark sector reinforces the existence of hadrons with exotic structure [13, 14]. In fact, recent detailed analyses of Λ(1405) in various reactions [15–18] and of the a_0(980)-f_0(980)
mixing in $J/\psi$ decay [19] are providing some clues for unusual structure of these hadrons. The exotic structure is also investigated by analyzing the theoretical models; the meson-baryon components of $\Lambda(1405)$ by using the natural renormalization scheme [20], the $N_c$ scaling behaviors of scalar and vector mesons [21, 22] and of $\Lambda(1405)$ [23, 24], spatial size of $\Lambda(1405)$ [25–27], $\sigma$ meson [28], and $f_0(980)$ [27], the nature of the $\sigma$ meson from the partial restoration of chiral symmetry [29], and the structure of $\sigma$ and $\rho(770)$ mesons studied by their Regge trajectories [30]. The possibilities to extract the hadron structure from the production yield in relativistic heavy ion collisions [31, 32] and from the high-energy exclusive productions [33, 34] are also suggested.

Among various exotic structures, hadronic molecular configurations are of special interest. These states are composed of two (or more) constituent hadrons by strong interaction between them without losing each character, in a similar way with the atomic nuclei as bound states of nucleons. The $\bar{K}N$ quasi-bound picture for $\Lambda(1405)$ is one of the examples. In contrast to the quark degrees of freedom, the masses and interactions of hadrons are defined independently of the renormalization scheme of QCD, because hadrons are color singlet states. This fact implies that the structure of hadrons may be adequately defined in terms of the hadronic degrees of freedom. This viewpoint originates in the investigations of the elementary or composite nature of particles in terms of the field renormalization constant [35–37]. Indeed, it is shown in this approach that the deuteron is dominated by the loosely bound proton-neutron component [38]. The study of the structure of hadrons from the field renormalization constant have been further developed in Refs. [39–51].

In this study, we develop a framework to investigate hadronic two-body components inside hadrons by analyzing comprehensively wave functions of resonance states. For this purpose, we explicitly introduce a one-body bare state in addition to the two-body components so as to form a complete set within them. The one-body component has not been taken into account in the preceding studies on wave functions (see Refs. [43, 52, 53]). We employ the Gamow vectors [54] for the resonance state, which enables us to have a finite normalization of the resonance wave function. In order to solve the wave equation, we make a good use of a general separable interaction. As a consequence, it is explicitly demonstrated that the compositeness corresponds to the two-body contribution as a portion of the wave function normalization, while the elementary component, to which we refer as the elementariness, arises from the bare state contribution. Here we divide the model space into the two components, the two-body component and the one-body bare component, and define the compositeness and elementariness correspondingly. In general, the model space can involve other two-body scattering states and multi-body systems. In this study such contributions may be accounted the one-body bare component. The compositeness and elementariness are further related with the quantities in the scattering equation. The generalized Ward identity found in Ref. [26] is derived as a sum rule for the normalization of the wave function. Moreover, the wave functions from a relativistically covariant wave equation are also discussed. As applications, we evaluate the compositeness of $\Lambda(1405)$ and the lightest scalar and vector mesons using the chiral coupled-channel approaches with the next-to-leading order interactions so as to discuss their internal structure from the viewpoint of hadronic two-body components.
This paper is organized as follows. In Sec. 2, we first consider two-body bound states in the nonrelativistic framework. We discuss the compositeness of the bound states, focusing on their wave functions. We then show that the compositeness can be related to the scattering length and effective range in the weak binding limit. The generalization to resonances in a relativistic covariant form is carefully presented. With an appropriate normalization of the state vectors, we obtain the expression of the compositeness of resonances. Next in Sec. 3 numerical results for the applications to physical resonances are presented. Section 4 is devoted to drawing the conclusion of this study.

2. Compositeness and elementariness from wave functions

In this section, we define the compositeness (and simultaneously elementariness) of the bound and resonance states using their wave functions and link it to the physical quantities in scattering equation. For this purpose, we consider a two-body scattering system coupled with a one-body bare state and make use of the separable type of interaction. The introduction of the one-body bare state, which has not been taken into account in the studies of wave functions, enables us to investigate the compositeness and elementariness on the same footing. We will concentrate on an s-wave scattering system, and thus the two-body wave function and the form factors are assumed to be spherical.

We first consider bound states and give an expression of the compositeness from the two-body wave functions in Sec. 2.1. In Sec. 2.2 we consider the weak binding limit to derive the Weinberg’s relation for the scattering length and the effective range [38]. Generalization to resonance states is discussed in Sec. 2.3. Finally we give a relativistic covariant formulation in Sec. 2.4.

2.1. Bound states in the nonrelativistic scattering

We consider a two-body scattering system in which there exists a discrete energy level below the scattering threshold energy. We call this energy level bound state since it is located below the threshold. At this stage we do not assume the origin and structure of the bound state at all. We take the rest frame of the center-of-mass motion, namely two scattering particles have equal and opposite momentum and the bound state is at rest with zero momentum. The system in this frame is described by Hamiltonian $\hat{H}$ which consists of the free part $\hat{H}_0$ and the interaction term $\hat{V}$

$$\hat{H} = \hat{H}_0 + \hat{V}. \quad (1)$$

We assume that the free Hamiltonian has continuum eigenstates $|q\rangle$ for the scattering state and one discrete state $|\psi_0\rangle$ for the one-body bare state. The eigenvalues are set to be

$$\hat{H}_0|q\rangle = \left(M^\text{th} + \frac{q^2}{2\mu}\right)|q\rangle, \quad \langle q|\hat{H}_0 = \left(M^\text{th} + \frac{q^2}{2\mu}\right)\langle q|, \quad (2)$$

$$\hat{H}_0|\psi_0\rangle = M_0|\psi_0\rangle, \quad \langle \psi_0|\hat{H}_0 = M_0\langle \psi_0|, \quad (3)$$

where $\mu$ is the reduced mass of the two-body system, $M_0$ is the mass of the bare state, and $q \equiv |q|$. The sum of the masses of the scattering particles $M^\text{th}$, which is just the scattering...
threshold energy, is included in the definition of the eigenenergy for later convenience. These eigenstates are normalized as
\[ \langle q' | q \rangle = (2\pi)^3 \delta^3(q' - q), \quad \langle \psi_0 | \psi_0 \rangle = 1, \quad \langle \psi_0 | q \rangle = \langle q | \psi_0 \rangle = 0. \] (4)
These states form the complete set of the free Hamiltonian\(^2\), and thus we can decompose unity in the following way
\[ 1 = |\psi_0\rangle \langle \psi_0| + \int \frac{d^3q}{(2\pi)^3} |q\rangle \langle q|. \] (5)

The bound state is realized as an eigenstate of the full Hamiltonian:
\[ \hat{H}|\psi\rangle = M_B|\psi\rangle, \quad \langle \psi|\hat{H} = M_B \langle \psi|, \] (6)
where \(M_B\) is the mass of the bound state. The bound state wave function is normalized as
\[ \langle \psi|\psi\rangle = 1. \] (7)

We define a constant \(Z\) as the probability of finding the bare state in the bound state:
\[ \langle \psi|\psi_0\rangle \langle \psi_0|\psi\rangle = Z, \] (8)
which corresponds to the field renormalization constant in the field theory. In this study we refer to \(Z\) as the elementariness. Because \(\langle \psi|\psi_0\rangle = \langle \psi_0|\psi\rangle^*, Z\) is always real and nonnegative. With Eq. (5), the normalization (7) can also be written as
\[ 1 = \langle \psi|\psi\rangle = Z + X, \quad X \equiv \int \frac{d^3q}{(2\pi)^3} \langle \psi|q\rangle \langle q|\psi\rangle. \] (9)

The second term \(X\) is the contribution from the two-body states and we call it the compositeness. Introducing the momentum space wave function
\[ \langle q|\psi\rangle = \tilde{\psi}(q), \quad \langle \psi|q\rangle = \tilde{\psi}^*(q), \] (10)
we can express the compositeness as
\[ X = \int \frac{d^3q}{(2\pi)^3} \left| \tilde{\psi}(q) \right|^2. \] (11)
Again, \(X\) is real and nonnegative. Equation (11) shows the expression of the compositeness by the momentum space wave function \(\tilde{\psi}(q)\).

For the explicit calculation, we assume the separable form of the matrix elements of \(\hat{V}\) in the momentum space. The matrix elements are given by
\[ \langle q'|\hat{V}|q\rangle = v f^*(q'^2)f(q^2), \quad \langle q'|\hat{V}|\psi_0\rangle = g_0 f^*(q^2), \quad \langle \psi_0|\hat{V}|\psi_0\rangle = 0, \] (12)
where \(v\) and \(g_0\) are the constants which determine the strength of the interaction. The matrix element \(\langle \psi_0|\hat{V}|\psi_0\rangle\) is taken to be zero since it can be absorbed into \(\hat{H}_0\) without loss of generality. The form factor \(f(q^2)\) is responsible for the off-shell momentum dependence of the interaction and suppresses the high momentum contribution to tame the ultraviolet

\(^2\) This framework can be straightforwardly generalized for the cases with multiple bare states. Inclusion of several scattering states will be explicitly discussed later.
divergence. The normalization is chosen to be \( f(0) = 1 \). The hermiticity of the Hamiltonian ensures that \( v \) is real and

\[
\langle \psi_0 | \hat{V} | q \rangle = g_0^* f(q^2). \tag{13}
\]

In this study we further assume the time-reversal invariance of the scattering process, which constraints the interaction, with an appropriate choice of phases of the states, as

\[
\langle q' | \hat{V} | q \rangle = \langle q | \hat{V} | q' \rangle = v f(q'^2) f(q^2), \quad \langle q | \hat{V} | \psi_0 \rangle = \langle \psi_0 | \hat{V} | q \rangle = g_0 f(q^2), \quad \langle \psi_0 | \hat{V} | \psi_0 \rangle = 0. \tag{14}
\]

Thus all of the quantities \( v, g_0, \) and \( f(q^2) \) are now real. We emphasize that the assumptions made in the present framework are just the factorization of the momentum dependence and the time-reversal invariance of the interaction. With the interaction (14), we obtain the exact solution of this system without introducing any further assumptions.

For the separable interaction, the wave function \( \tilde{\psi}(q) \) can be analytically obtained [55]. To this end, we multiply \( \langle q \rangle \) and \( \langle \psi_0 \rangle \) to Eq. (6):

\[
\langle q | \hat{H} | \psi \rangle = \left( M^{th} + \frac{q^2}{2\mu} \right) \tilde{\psi}(q) + v f(q^2) \int \frac{d^3q'}{(2\pi)^3} f(q'^2) \tilde{\psi}(q') + g_0 f(q^2) \langle \psi_0 | \psi \rangle = M_B \tilde{\psi}(q), \tag{15}
\]

\[
\langle \psi_0 | \hat{H} | \psi \rangle = M_0 \langle \psi_0 | \psi \rangle + g_0 \int \frac{d^3q}{(2\pi)^3} f(q^2) \tilde{\psi}(q) = M_B \langle \psi_0 | \psi \rangle, \tag{16}
\]

where we have inserted Eq. (5) between \( \hat{V} \) and \( | \psi \rangle \). Eliminating \( \langle \psi_0 | \psi \rangle \), we obtain the integral equation for \( \tilde{\psi}(q) \)

\[
\left( M^{th} + \frac{q^2}{2\mu} \right) \tilde{\psi}(q) + v^{tot}(M_B) f(q^2) \int \frac{d^3q'}{(2\pi)^3} f(q'^2) \tilde{\psi}(q') = M_B \tilde{\psi}(q), \tag{17}
\]

where we have defined the energy-dependent interaction \( v^{tot} \) as

\[
v^{tot}(E) \equiv v + \frac{(g_0)^2}{E - M_0}. \tag{18}
\]

Equation (17) is equivalent to the single-channel Schrödinger equation for the relative motion of the scattering particles under the presence of the bare state interacting with them by \( \hat{V} \). The effect of the bare state is incorporated into the energy dependent interaction \( v^{tot}(E) \).

The solution of Eq. (17) can be obtained as

\[
\tilde{\psi}(q) = \frac{-c f(q^2)}{B + q^2/(2\mu)}, \tag{19}
\]

where we have defined the binding energy \( B \equiv M^{th} - M_B > 0 \) and the normalization constant

\[
c \equiv v^{tot}(M_B) \int \frac{d^3q'}{(2\pi)^3} f(q'^2) \tilde{\psi}(q'). \tag{20}
\]

In general, Eq. (17) is an integral equation to determine the wave function \( \tilde{\psi}(q) \). For the separable interaction, however, the integral in Eq. (20) (and hence, the constant \( c \)) is independent of \( q \). In this way, the wave function \( \tilde{\psi}(q) \) is analytically determined by the form
factor $f(q^2)$ and the constant $c$, which will be determined through the comparison with the scattering amplitude. Substituting the wave function (19) into Eq. (20), we obtain

$$c = -v^{\text{tot}}(M_B) \int \frac{d^3q}{(2\pi)^3} \frac{[f(q^2)]^2}{B + q^2/(2\mu)} c.$$  

(21)

For the existence of the bound state at $E = M_B$, Eq. (21) should be satisfied with nonzero $c$. The nontrivial solution can be obtained by

$$1 = v^{\text{tot}}(M_B)G(M_B),$$  

(22)

where we have introduced a function

$$G(E) = \int \frac{d^3q}{(2\pi)^3} \frac{[f(q^2)]^2}{E - M^\text{th} - q^2/(2\mu)},$$  

(23)

which plays an important role in the following discussion and is called the loop function. We note that here and in the following the energy in the denominator of the loop function is considered to have an infinitesimal positive imaginary part $i\epsilon$: $E \rightarrow E + i\epsilon$.

The normalization constant $c$ is equal to the square root of the residue of the scattering amplitude at the pole position of the bound state. To prove this, we first represent the compositeness $X$ and elementariness $Z$ using $c$. With the explicit form of the wave function (19) and the loop function (23), the compositeness for the separable interaction can be expressed with the derivative of the loop function as

$$X = \int \frac{d^3q}{(2\pi)^3} |\tilde{\psi}(q)|^2 = -|c|^2 \left[ \frac{dG}{dE} \right]_{E=M_B}.$$  

(24)

We note that both the wave function $\tilde{\psi}(q)$ and the loop function have the same structure of $1/(E - H_0)$ at $E = M_B$. Substituting the wave function into Eq. (16), we obtain

$$\langle \psi_0 | \psi \rangle = \frac{c g_0}{M_B - M_0} G(M_B),$$  

(25)

and hence

$$Z = \langle \psi | \psi_0 \rangle \langle \psi_0 | \psi \rangle = |c|^2 G(M_B) \frac{(g_0)^2}{(M_B - M_0)^2} G(M_B) = -|c|^2 \left[ G \frac{dv^{\text{tot}}}{dE} G \right]_{E=M_B}.$$  

(26)

where we have used the derivative of Eq. (18). We note that Eqs. (24) and (26) provide a sum rule

$$1 = -|c|^2 \left[ \frac{dG}{dE} + G \frac{dv^{\text{tot}}}{dE} G \right]_{E=M_B}.$$  

(27)

Next the scattering amplitude $t(E)$ is obtained by taking the matrix element of the $T$-operator for the scattering states $|q\rangle$ with the on-shell condition as $\langle q' | \hat{T} | q \rangle = t(E)f(q^2)f(q^2)$ for the separable interaction. The $T$-operator satisfies the Lippmann-Schwinger equation

$$\hat{T} = \hat{V}^{\text{tot}}(E) + \hat{V}^{\text{tot}}(E) \frac{1}{E - H_0} \hat{T},$$  

(28)

where we have defined $\hat{V}^{\text{tot}}(E) \equiv \hat{V} + \hat{V} |\psi_0\rangle \langle \psi_0| \hat{V} / (E - M_0)$, which leads to $\langle q' | \hat{V}^{\text{tot}}(E) | q \rangle = v^{\text{tot}}(E)f(q^2)f(q^2)$. Actually this equation can be obtained by removing the degrees of freedom of the bare state in the Lippmann-Schwinger equation, hence the $\hat{T}$ and $\hat{V}^{\text{tot}}$ operators

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act only on the two-body states. The amplitude \( t(E) \) can be algebraically obtained as

\[
t(E) = v^\text{tot}(E) + v^\text{tot}(E)G(E)t(E) = \frac{v^\text{tot}(E)}{1 - v^\text{tot}(E)G(E)}.
\]  

(29)

The bound state condition (22) ensures that the amplitude \( t(E) \) has a pole at \( E = M_B \). The residue of the amplitude \( t(E) \) at the pole reflects the properties of the bound state. The residue turns out to be real and positive, so we represent the residue as \(|g|^2\):

\[
|g|^2 \equiv \lim_{E \to M_B} (E - M_B)t(E) = \frac{1}{\left[\frac{dG}{dE} + \frac{1}{(s\text{tot})^2}\right]_{E=M_B}}.
\]

(30)

We can interpret \( g \) as the coupling constant of the bound state to the two-body state. Using the bound state condition (22), we obtain the relation

\[
1 = -|g|^2 \left[ \frac{dG}{dE} + G\frac{dv^\text{tot}}{dE} \right]_{E=M_B}.
\]

(31)

Comparing this with Eq. (27), we find \( c = g \) with an appropriate choice of the phase.

The equality \( c = g \) is also confirmed by the following form of the \( T \)-operator:

\[
\hat{T} = \hat{V}^\text{tot}(E) + \hat{V}^\text{tot}(E)\frac{1}{E - H_0 - \hat{V}^\text{tot}(E)}\hat{V}^\text{tot}(E).
\]

(32)

As we have seen before, the operator \( \hat{H}_0 + \hat{V}^\text{tot} \) corresponds to the full Hamiltonian for the two-body system with the implicit bare state. Near the bound state pole, the amplitude is dominated by the pole term in the expansion by the eigenstates of the full Hamiltonian as

\[
\lim_{E \to M_B} \hat{T}(E) \sim \hat{V}^\text{tot}(M_B)|\psi\rangle \frac{1}{E - M_B} \langle \psi|\hat{V}^\text{tot}(M_B),
\]

(33)

and hence, taking the matrix element of the scattering states, we have

\[
\lim_{E \to M_B} t(E) \sim \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} v^\text{tot}(M_B)f(q^2)\langle q|\psi\rangle\langle q|p\rangle E - M_B f(p^2)v^\text{tot}(M_B) \rightarrow \frac{|c|^2}{E - M_B},
\]

(34)

where we have used Eq. (20). From the definition of the residue (30), this verifies \( c = g \).

The framework is straightforwardly generalized to the coupled-channel scattering. The eigenstates of the free Hamiltonian \( \hat{H}_0 \) now include several two-body states \( |q_j\rangle \) in channel \( j \). We assume that the bound state is located below the lowest threshold of the two-body channels. The normalization and the completeness relation are given by

\[
\langle q'_j|q_k\rangle = (2\pi)^3\delta_{j,k}\delta^3(q' - q), \quad \langle \psi_0|q_j\rangle = \langle q_j|\psi_0\rangle = 0,
\]

(35)

\[
1 = |\psi_0\rangle\langle \psi_0| + \sum_j \int \frac{d^3q}{(2\pi)^3} |q_j\rangle\langle q|.
\]

(36)

The matrix elements of the interaction are

\[
\langle q'_j|\hat{V}|q_k\rangle = v_{jk}f_j(q'^2)f_k(q^2), \quad \langle q_j|\hat{V}|\psi_0\rangle = \langle \psi_0|\hat{V}|q_j\rangle = g_{0,j}f_j(q^2), \quad \langle \psi_0|\hat{V}|\psi_0\rangle = 0,
\]

(37)

where, due to the time-reversal invariance, \( v_{jk} \) is a real symmetric matrix and \( g_{0,j} \) and \( f_j(q^2) \) are real with an appropriate choice of phases of states. The total normalization of the bound
state wave function now leads to

\[ 1 = Z + \sum_j X_j, \]  

(38)

with the compositeness and the wave function for each channel

\[ X_j \equiv \int \frac{d^3q}{(2\pi)^3} \left| \tilde{\psi}_j(q) \right|^2, \quad \langle q_j | \psi \rangle = \tilde{\psi}_j(q), \quad \langle \psi | q_j \rangle = \tilde{\psi}_j^*(q). \]  

(39)

Following the same procedure, the bound state wave function in channel \( j \) is determined by the Schrödinger equation as

\[ \tilde{\psi}_j(q) = \frac{-c_j f_j(q^2)}{B_j + q^2/(2\mu_j)}, \]  

(40)

where \( \mu_j \) and \( B_j = M_j^{th} - M_B \) are respectively the reduced mass and the binding energy measured from the threshold \( M_j^{th} \) in channel \( j \). The normalization constant is given by

\[ c_j \equiv \sum_k v_{jk}^{tot}(M_B) \int \frac{d^3q}{(2\pi)^3} f_k(q^2) \tilde{\psi}_k(q), \quad v_{jk}^{tot}(E) \equiv v_{jk} + \frac{g_{0,j}g_{0,k}}{E - M_0}, \]  

(41)

where \( v_{jk}^{tot} \) is a real symmetric matrix for a real energy. The bound state condition for nonzero \( c_j \) can be summarized as

\[ \det[1 - v^{tot}(M_B)G(M_B)] = 0, \]  

(42)

with the loop function

\[ G_j(E) = \int \frac{d^3q}{(2\pi)^3} \frac{[f_j(q^2)]^2}{E - M_j^{th} - q^2/(2\mu_j)}. \]  

(43)

which is diagonal with respect to the channel indices.

The coupled-channel scattering equation is in the matrix form

\[ t(E) = [1 - v^{tot}(E)G(E)]^{-1} v^{tot}(E). \]  

(44)

Equation (42) ensures the existence of the bound state pole at \( E = M_B \), and the residue of the amplitude at the pole, which is real for the bound state, is interpreted as the product of the coupling constants\(^3\)

\[ g_j g_k = \lim_{E \to M_B} (E - M_B) t_{jk}(E). \]  

(45)

On the other hand, with Eq. (32), the amplitude near the bound state pole is given by

\[ \lim_{E \to M_B} t_{jk}(E) \sim \sum_{l,m} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} v_{jl}^{tot}(M_B) f_l(q^2) \frac{\langle q_l | \psi \rangle \langle \psi | p_m \rangle}{E - M_B} f_m(p^2) v_{mk}^{tot}(M_B) \rightarrow \frac{c_j c_k^*}{E - M_B}, \]  

(46)

which shows that \( c_j = g_j \) with an appropriate choice of the phase.

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\(^3\)Since an interaction of a symmetric matrix \( v_{jk}^{tot} \) leads to a symmetric \( t \)-matrix, \( t_{jk} = t_{kj} \), the residue of the \( t \)-matrix is also symmetric and can be factorized as \( g_j g_k \).
Now the compositeness in channel $j$ can be expressed as

$$X_j = \int \frac{d^3q}{(2\pi)^3} |\tilde{\psi}_j(q)|^2 = -|c_j|^2 \left[ \frac{dG_j}{dE} \right]_{E=M_B} = -|g_j|^2 \left[ \frac{dG_j}{dE} \right]_{E=M_B}. \quad (47)$$

The overlap of the bound state wave function with the bare state is given by

$$\langle \psi_0 | \psi \rangle = \sum_j c_j g_{0,j} M_B - M_0 G_j(M_B), \quad (48)$$

and the elementariness is

$$Z = \sum_{j,k} c_k c_j^* G_j(M_B) \frac{g_{0,j} g_{0,k}}{(M_B - M_0)^2} G_k(M_B) = - \sum_{j,k} g_k g_j \left[ G_j \frac{dv_{\text{tot}}}{dE} G_k \right]_{E=M_B}. \quad (49)$$

From the normalization (38), we obtain the sum rule

$$- \sum_{j,k} g_k g_j \left[ \delta_{jk} \frac{dG_j}{dE} + G_j \frac{dv_{\text{tot}}}{dE} G_k \right]_{E=M_B} = 1. \quad (50)$$

This corresponds to the nonrelativistic counterpart of the generalized Ward identity derived in Ref. [26]. We note that the sum rule (50) as the normalization of the wave function can be obtained by the explicit treatment of both the two-body states and the one-body bare state, which complements the discussion of the bound-state wave function with an energy-independent separable interaction done in Ref. [52].

Here we mention that, as seen in Eqs. (26) and (49), the elementariness $Z$ is proportional to the energy derivative of the interaction $dv_{\text{tot}}/dE$ at the bound state energy. This is instructive to interpret the origin of the elementariness $Z$. In quantum mechanics, the two-body interaction should not depend on the energy. In the present case, the energy dependence of $v_{\text{tot}}$ stems from the implicit bare state channel $|\psi_0\rangle$. In the same way, when one of the coupled channels is removed by the Feshbach method [56, 57], such contribution also appears as the energy dependence of the effective interaction which acts on the reduced model space (see also Ref. [48]). The strong energy dependence of the interaction emerges when the removed channel lies close to the physical bound state. In this sense, $Z \approx 1$ means that the energy dependence of the interaction is strong and the effects other than the scattering channels of interest are responsible for the formation of the bound state. The weak energy dependence for $Z \approx 0$ can be understood that the removed channels exist far away from the pole position of the physical bound state. In this case, the bound state is dominated by the scattering channels considered.

It is important to note that the structure of the bound state with a separable interaction is determined only by the quantities at the bound-state pole position. Namely, the compositeness and elementariness of the bound state are determined by the values of $v_{\text{tot}}(E)$, $G(E)$, and their derivatives at the pole position. This means that not global but only local behavior of the interaction is relevant to the structure of the bound state. Because the global information of $v_{\text{tot}}(E)$ is not relevant, we can apply the formulae of the compositeness and elementariness to interactions with an arbitrary energy dependence, as the analysis of physical hadronic resonances in Sec. 3. However, in general, the energy dependence of the interaction $v(E)$ is not always attributed to the explicit bare pole contribution. For instance, the interactions used in Sec. 3 are energy dependent as a consequence of the chiral
low energy theorem. In the present formulation, we regard that these also contribute to the elementariness, as its derivative with respect to the energy $E$ is finite. This shares viewpoints with Ref. [20], where it was discussed that the energy-dependent Weinberg-Tomozawa term can provide the effect of the CDD pole [58].

2.2. Weak binding limit and threshold parameters

In this subsection, we consider the weak binding limit to derive the Weinberg’s compositeness condition [38] on the scattering length $a$ and the effective range $r_e$. This ensures that the expression for the compositeness in this paper correctly reproduces the model-independent result of Ref. [38] in the weak binding limit.

In the single-channel problem, the elastic scattering amplitude $F(E)$ is written with the $t$-matrix $t(E)$ given in Eq. (29) as

$$F(E) = \frac{1}{(2\pi)\lambda} (2\pi)^2 |\mu t(E)|^2 f(k^2)^2,$$

with $k \equiv \sqrt{2\mu(E - M_{th})}$. The scattering length $a$ is defined as the value of the scattering amplitude at the threshold:

$$a \equiv -F(M_{th}) = \frac{\mu}{2\pi} t(M_{th}) = \frac{\mu}{2\pi} v^{-1}(M_{th}) - G(M_{th}),$$

where we have abbreviated $v^\text{tot}$ as $v$ for simplicity. Now we perform the expansion in terms of the energy $E$ around $M_B$ by considering $B = M_{th} - M_B$ to be small. To expand the denominator, we write

$$v^{-1}(M_{th}) = v^{-1}(M_B) + B \left[ \frac{dv^{-1}}{dE} \right]_{E=M_B} + \Delta v^{-1},$$

$$G(M_{th}) = G(M_B) + B \left[ \frac{dG}{dE} \right]_{E=M_B} + \Delta G,$$

where we have defined

$$\Delta v^{-1} \equiv \sum_{n=2}^{\infty} \frac{B^n}{n!} \left[ \frac{d^n v^{-1}}{dE^n} \right]_{E=M_B}, \quad \Delta G \equiv \sum_{n=2}^{\infty} \frac{B^n}{n!} \left[ \frac{d^n G}{dE^n} \right]_{E=M_B}.$$  

Here we allow arbitrary energy dependence for $v$ but assume that the effective range expansion is valid up to the energy of the bound state, which is a precondition for the formula in Ref. [38]. In this case there should exist no singularity of $v^{-1}(E)$ between $E = M_B$ and $M_{th}$ and expansion (53) is safely performed up to the threshold, and hence $\Delta v^{-1} = O(B^2)$. Otherwise the singularity of $v^{-1}(E)$ below the threshold spoils the effective range expansion, as the divergence of $v^{-1}$ leads to the existence of the CDD pole. As a result, with the bound state condition (22), the scattering length is now given by

$$a = \frac{\mu}{2\pi} \left( B \left[ \frac{dv^{-1}}{dE} - \frac{dG}{dE} \right]_{E=M_B} - \Delta G + O(B^2) \right)^{-1}.$$  

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The first term in the parenthesis in Eq. (56) is calculated as

\[
B \left[ \frac{dv^{-1}}{dE} - \frac{dG}{dE} \right]_{E=M_B} = -B \left[ G^2 \frac{dv}{dE} + \frac{dG}{dE} \right]_{E=M_B} = \frac{B}{|g|^2} E = \frac{B}{X} \left[ \frac{dG}{dE} \right]_{E=M_B} = \frac{B}{X} \int \frac{d^3q}{(2\pi)^3} \frac{[f(q^2)]^2 + O(q^2)}{(B + q^2/(2\mu))^n+1}
\]

where we have used Eqs. (22), (31), (24), and the normalization \( f(0) = 1 \), and we have defined \( R \equiv 1/\sqrt{2\mu B} \) in the last line. To evaluate \( \Delta G \), we first note that

\[
\left[ \frac{d^nG}{dE^n} \right]_{E=M_B} = \int \frac{d^3q}{(2\pi)^3} \frac{(-1)^n n! [f(q^2)]^2}{M_B - M_{th} - q^2/(2\mu)^{n+1}}
\]

\[
= -n! \int \frac{d^3q}{(2\pi)^3} \frac{[f(q^2)]^2}{(B + q^2/(2\mu))^{n+1}}
\]

\[
= -\frac{n!}{B^n} \int \frac{d^3q'}{(2\pi)^3} \frac{[f(2\mu B q'^2)]^2}{(q'^2 + 1)^{n+1}},
\]

where \( q' \equiv Rq \). Thus summing up all contributions we have

\[
\Delta G = -\sum_{n=2}^{\infty} \frac{2\mu}{R} \int \frac{d^3q'}{(2\pi)^3} \frac{[f(0)]^2}{(q'^2 + 1)^{n+1}} + O(B)
\]

\[
= -\frac{\mu}{R^2} \int_0^{\infty} dx x^2 \sum_{n=2}^{\infty} \frac{1}{(x^2 + 1)^{n+1}} + O(B)
\]

\[
= -\frac{\mu}{4\pi} \frac{1}{R} + O(B),
\]

where we have used the summation relation

\[
\sum_{n=2}^{\infty} \frac{1}{(x^2 + 1)^{n+1}} = \frac{1}{x^2(x^2 + 1)^2} \quad (x \neq 0).
\]

As a consequence, we obtain the expression of the scattering length in terms of the compositeness \( X \) from Eqs. (56), (57) and (59):

\[
a = \frac{\mu}{2\pi} \left( \frac{\mu}{4\pi X} \frac{1}{R} + \frac{\mu}{4\pi} \frac{1}{R} + O(B) \right)^{-1} = R \frac{2X}{1 + X} + O(B^0),
\]

which agrees with the result in Ref. [38] with \( X = 1 - Z \). It is important that in the weak binding limit details of the form factor \( f(q^2) \) are irrelevant to the determination of the compositeness of the bound state from the scattering length of two constituents. In contrast, the correction terms of \( O(B^0) \) depend on the explicit form of the function \( f(q^2) \).

Because we have assumed that the bound state pole lies within the valid region of the effective range approximation, the relation between the scattering length and the effective
range is given by\(^4\)

\[ r_e = 2R \left( 1 - \frac{R}{a} \right) \]  

(62)

Comparing it with Eq. (61), we find

\[ r_e = RX - 1 + O(B^0). \]  

(63)

This again corresponds to the expression in Ref. [38].

In this way, the structure of the bound state is related to \(a\) and \(r_e\). This means that, in principle, the bound state can have an arbitrary structure by tuning \(a\) and \(r_e\). It is however shown in Ref. [59] that the bound state with \(Z \sim 0\) naturally appears near the threshold, and a significant fine tuning is required to realize \(Z \sim 1\). This is related to the value of \(Z\) in the exact \(B \rightarrow 0\) limit. The value of \(Z\) is shown to vanish in the \(B \rightarrow 0\) limit, as far as the bound state pole exist in the scattering amplitude [60]. It is therefore natural to realize the bound state with \(Z \sim 0\) in the small binding region.

2.3. Generalization to resonances

Now we generalize our argument to a resonance state. We first introduce the Gamow state [54] denoted as \(|\psi\rangle\) to express the resonance state. The eigenvalue of the Hamiltonian is allowed to be complex for the Gamow state:

\[ \hat{H}|\psi\rangle = \left( M_R - i \frac{\Gamma_R}{2} \right) |\psi\rangle. \]  

(64)

Here \(M_R\) and \(\Gamma_R\) are the mass and width of the resonance state, respectively. The state with a complex eigenvalue cannot be normalized in the ordinary sense. To establish the normalization, we define the corresponding bra-state as the complex conjugate of the Dirac bra-state:

\[ (|\psi\rangle)^\ast \equiv \langle \psi^* |, \]  

(65)

which was firstly introduced to describe unstable nuclei [61–63]. As a consequence, the eigenvalue of the Hamiltonian is the same with the ket vector:\(^5\)

\[ (|\psi\rangle \hat{H} = \left( M_R - i \frac{\Gamma_R}{2} \right) |\psi\rangle. \]  

(66)

These eigenvectors can be normalized as

\[ (|\psi\rangle |\psi\rangle = 1. \]  

(67)

With the same eigenstates of the free Hamiltonian (36), we can decompose this normalization as

\[ 1 = (|\psi\rangle |\psi\rangle_0 \langle \psi_0 |\psi\rangle + \sum_j \int \frac{d^3q}{(2\pi)^3} (|\psi\rangle |q_j\rangle \langle q_j| |\psi\rangle) = Z + \sum_j X_j, \]  

(68)

---

\(^4\)The relation (62) can be obtained from the condition \(F^{-1}(k) = -1/a - ik + r_e k^2/2 = 0\) at \(k = i/R\).

\(^5\)The eigenvectors \(|\psi^*\rangle\) and \(\langle \psi^* | = \langle \psi |\) have the eigenvalue \(M_R + i\Gamma_R/2\).
where we have defined the elementariness $Z$ and compositeness $X_j$ as

$$Z \equiv \langle \psi | \psi_0 \rangle \langle \psi_0 | \psi \rangle, \quad X_j \equiv \int \frac{d^3q}{(2\pi)^3} \langle \psi | q_j \rangle \langle q_j | \psi \rangle.$$  \hspace{1cm} (69)

In addition, we define the momentum space wave function $\tilde{\psi}_j(q) \equiv \langle q_j | \psi \rangle$. It follows from Eqs. (65) and (66) that

$$\langle \psi | q_j \rangle = \langle q_j | \psi \rangle = \tilde{\psi}_j(q).$$  \hspace{1cm} (70)

The compositeness is then given by

$$X_j = \int \frac{d^3q}{(2\pi)^3} [\tilde{\psi}_j(q)]^2.$$  \hspace{1cm} (71)

In contrast to Eq. (39), where $X_j$ is given by the absolute value squared, the compositeness of the resonance is given by the complex number squared. This is also the case for $Z$, because $\langle \psi | \psi_0 \rangle = \langle \psi_0 | \psi \rangle \neq \langle \psi_0 | \psi \rangle^*$. In this way, $Z$ and $X_j$ are in general complex, and the probabilistic interpretation of $Z$ and $X_j$ is not guaranteed.

To determine the wave function, we solve the Schrödinger equation

$$\left( M_R - i \frac{\Gamma_R}{2} \right) \tilde{\psi}_j(q) = \left( M_j^{th} + \frac{q^2}{2\mu_j} \right) \tilde{\psi}_j(q) + \sum_k v_{jk} f_j(q^2) \int \frac{d^3q'}{(2\pi)^3} f_k(q'^2) \tilde{\psi}_k(q'),$$  \hspace{1cm} (72)

and

$$\left( M_R - i \frac{\Gamma_R}{2} \right) \langle \psi_0 | \psi \rangle = M_0 \langle \psi_0 | \psi \rangle + \sum_k g_{0,k} \int \frac{d^3q}{(2\pi)^3} f_k(q^2) \tilde{\psi}_k(q).$$  \hspace{1cm} (73)

Eliminating $\langle \psi_0 | \psi \rangle$, we obtain

$$\tilde{\psi}_j(q) = \frac{-c_j f_j(q^2)}{M_j^{th} - M_R + i\Gamma_R/2 + q^2/(2\mu_j)},$$  \hspace{1cm} (74)

with the normalization constant

$$c_j \equiv \sum_k v_{jk}^{tot} (M_R - i\Gamma_R/2) \int \frac{d^3q}{(2\pi)^3} f_k(q^2) \tilde{\psi}_k(q),$$  \hspace{1cm} (75)

where $v_{jk}^{tot}(E)$ is defined in the same way with Eq. (41). The condition for nonzero $c_j$ is

$$\det [1 - v^{tot}(M_R - i\Gamma_R/2)G(M_R - i\Gamma_R/2)] = 0.$$  \hspace{1cm} (76)

This is the condition for the resonance pole at $E = M_R - i\Gamma_R/2$. We note that the loop function in the complex energy plane is defined on the $2^n$-sheeted Riemann surface for an $n$-channel problem. The resonance pole can exist in any sheets, except for the one which is reached by choosing the first sheet for all channels. The most relevant Riemann sheet for the scattering amplitude at a given energy is reached by choosing the first sheet for the closed channels and the second sheet for the open channels. In the following, we concentrate on the poles in this Riemann sheet, while the framework is in principle applicable to the complex poles in the other Riemann sheets.
Also for the resonance pole, the residue of the scattering amplitude is interpreted as product of the coupling constants $g_j g_k$:

$$g_j g_k = \lim_{E \to M_R - i\Gamma_R/2} (E - M_R + i\Gamma_R/2)t_{jk}(E),$$

(77)

where the complex conjugate should not be taken for the coupling constant $g_k$ since $t_{jk}$ is symmetric: $t_{jk} = t_{kj}$. In contrast to the bound states, the coupling constant $g_j$ is in general complex. The amplitude near the resonance pole is also given by

$$\lim_{E \to M_R - i\Gamma_R/2} t_{jk}(E) \sim \sum_{l,m} \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} v_{tot}^{\rm l}(E) f_l(q^2) \frac{\langle q|\psi\rangle\langle\psi|p_m\rangle}{E - M_R + i\Gamma_R/2} f_m(p^2) v_{tot}^{\rm k}(E),$$

(78)

thus we find $c_j = g_j$. The compositeness in channel $j$ is then given by

$$X_j = \int \frac{d^3 q}{(2\pi)^3} \left[ \psi_j(q) \right]^2 = -g_j^2 \left[ \frac{dG_j}{dE} \right]_{E=M_R - i\Gamma_R/2}. $$

(79)

The loop function in the complex energy plane should be evaluated by choosing the Riemann sheets consistently with the choice to obtain the pole condition (76).

From Eq. (73) and its counterpart coming from Eq. (66), we obtain

$$\langle \psi_0|\psi \rangle = (\psi|\psi_0\rangle = \sum_j \frac{c_j g_{0,j}}{M_R - i\Gamma_R/2 - M_0} G_j(M_R - i\Gamma_R/2),$$

(80)

so the elementariness is obtained as

$$Z = - \sum_{j,k} g_j g_k \left[ G_j \frac{dG_j}{dE} G_k \right]_{E=M_R - i\Gamma_R/2}. $$

(81)

Using Eqs. (68), we obtain

$$- \sum_{j,k} g_j g_k \left[ \delta_{jk} \frac{dG_j}{dE} + G_j \frac{dG_k}{dE} G_k \right]_{E=M_R - i\Gamma_R/2} = 1. $$

(82)

This corresponds to the nonrelativistic counterpart of the generalized Ward identity for resonance states derived in Ref. [26]. The special case of $Z = 0$ of Eq. (82) is obtained in Ref. [53] by using an energy-independent separable interaction without the bare-state contribution. Here we mention that we should obtain the same results in appropriate ways to treat resonance states such as the complex scaling method [64].

By definition, the compositeness for the resonance state becomes complex. Therefore, strictly speaking, it cannot be interpreted as a probability of finding the two-body component. Nevertheless, because it represents the contribution of the channel wave function to the total normalization, the compositeness $X_j$ will be an important piece of information on the structure of the resonance. For instance, consider a resonance such that the real part of a single $X_j$ is close to unity with small imaginary part, and all the other components have small absolute values. In this case, the resonance wave function is considered to be...
similar to that of the bound state dominated by the $j$-th channel. It is therefore natural to interpret the resonance state in this case is dominated by the component of the channel $j$. In general, however, all $X_j$ and $Z$ can be arbitrary complex numbers constrained by Eq. (68). The interpretation of the structure of such a state from $X_j$ and $Z$ is not straightforward.

2.4. Relativistic covariant formulation

Finally we consider the coupled-channel two-body scattering in a relativistic form. Here we do not consider the intermediate states with more than two particles but simply solve the two-body wave equation. To describe the wave function of the resonances, we extract the relative motion of the two-body system from a relativistic scattering equation with a three-dimensional reduction [65, 66].

According to Appendix A, we introduce the state $|q_j^{\text{co}}\rangle$ as the two-body scattering state of the particles with mass $m_j$ and $M_j$ and the relative momentum $q_j$, and its normalization is fixed as

$$\langle q_j^{\text{co}}|q_k^{\text{co}}\rangle = \frac{2\omega_j(q)\Omega_j(q)}{\sqrt{s_{qj}}}(2\pi)^3\delta_{jk}\delta^3(q' - q), \quad (83)$$

which is chosen so that the expression of the relativistic wave equation (89) becomes a natural extension of the nonrelativistic Schrödinger equation (See Appendix A). Furthermore, we also introduce the bare state $\Psi_0$, which satisfies the following orthonormal conditions:

$$\langle \Psi_0|\Psi_0\rangle = 1, \quad \langle q_j^{\text{co}}|\Psi_0\rangle = \langle \Psi_0|q_j^{\text{co}}\rangle = 0. \quad (84)$$

We note that with the normalization (83) and (84) the complete set of the system is given by

$$1 = |\Psi_0\rangle\langle \Psi_0| + \sum_j \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{s_{qj}}}{2\omega_j(q)\Omega_j(q)}|q_j^{\text{co}}\rangle\langle q_j^{\text{co}}|. \quad (85)$$

The scattering state $|q_j^{\text{co}}\rangle$ and the bare state $|\Psi_0\rangle$ span the space of the eigenstates of the kinetic energy operator $\hat{K}$ which extracts the total energy squared of the state. Namely, for the two-body scattering state $|q_j^{\text{co}}\rangle$ we have

$$\hat{K}|q_j^{\text{co}}\rangle = s_{qj}|q_j^{\text{co}}\rangle, \quad \langle q_j^{\text{co}}|\hat{K} = \langle q_j^{\text{co}}|s_{qj}, \quad (86)$$

where $s_{qj} \equiv [\omega_j(q) + \Omega_j(q)]^2$ with the on-shell energies $\omega_j(q) \equiv \sqrt{q^2 + m_j^2}$ and $\Omega_j(q) \equiv \sqrt{q^2 + M_j^2}$. For the bare state, the eigenvalue of $\hat{K}$ is the mass squared of the bare state $\Psi_0$, $M_0^2$:

$$\hat{K}|\Psi_0\rangle = M_0^2|\Psi_0\rangle, \quad \langle \Psi_0|\hat{K} = \langle \Psi_0|M_0^2. \quad (87)$$

The dynamics of the system is determined by the interaction operator $\hat{V}$. We again adopt the separable form as

$$\langle q_j^{\text{co}}|\hat{V}|q_k^{\text{co}}\rangle = V_{jk}f_j(q^2)f_k(q^2), \quad \langle q_j^{\text{co}}|\hat{V}|\Psi_0\rangle = \langle \Psi_0|\hat{V}|q_j^{\text{co}}\rangle = g_{0,j}f_j(q^2), \quad \langle \Psi_0|\hat{V}|\Psi_0\rangle = 0, \quad (88)$$

---

7 In general relativistic field theory, there are infinitely many diagrams which contribute to the scattering amplitude. The present formulation picks up the summation of the $s$-channel two-body loop diagrams, which is the most dominant contribution in the nonrelativistic limit.
where $V_{jk}$ is a real symmetric matrix and $g_{0,j}$ and $f_j(q^2)$ are real with an appropriate choice of phases of the states.\(^8\) In order to make a three-dimensional reduction of the scattering equation, we assume that the form factor $f_j(q^2)$ depends only on the magnitude of the three momentum. We consider that the wave equation with the operator $\hat{K} + \hat{V}$ contains a resonance $|\Psi\rangle$ with mass $M_R$ and width $\Gamma_R$ as an eigenstate [65, 66]

$$[\hat{K} + \hat{V}] |\Psi\rangle = s_R |\Psi\rangle, \quad (\Psi| \hat{K} + \hat{V} = (\Psi| s_R, \quad (89)$$

where $(\Psi| = (\Psi^*)^T$ and $s_R = (M_R - i\Gamma_R/2)^2$. By using Eq. (85) we can decompose the normalization of the resonance vector $(\Psi| \Psi) = 1$ as

$$1 = (\Psi| \Psi) = Z + \sum_j X_j, \quad (90)$$

where we have defined the elementariness $Z$ and compositeness $X_j$ as:

$$Z \equiv (\Psi| \Psi_0)(\Psi_0| \Psi), \quad X_j \equiv \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{s_{qj}}}{2\omega_j(q)\Omega_j(q)} \left[\tilde{\Psi}_j(q)\right]^2, \quad (91)$$

with the momentum space wave function

$$(q_j^\text{co}| \Psi) = \tilde{\Psi}_j(q) = (\Psi| q_j^\text{co}). \quad (92)$$

In the same way with Sec. 2.3, the wave function is determined as

$$\tilde{\Psi}_j(q) = \frac{C_j f_j(q^2)}{s_R - s_{qj}}, \quad (93)$$

$$C_j = \sum_k V_{jk}^\text{tot}(s_R) \int \frac{d^3q'}{(2\pi)^3} \frac{\sqrt{s_{q'k}}}{2\omega_k(q')\Omega_k(q')^2} f_k(q'^2)\tilde{\Psi}_k(q'), \quad (94)$$

$$V_{jk}^\text{tot}(s) = V_{jk} + \frac{g_{0,j}g_{0,k}}{s - M_{qj}^2} \quad (95)$$

The consistency condition for nonzero $C_j$ is given by

$$\det[1 - V^\text{tot}(s_R)G(s_R)] = 0, \quad (96)$$

where the loop function $G$ is diagonal with respect to the channel indices and is expressed as

$$G_j(s) = \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{s_{qj}}}{2\omega_j(q)\Omega_j(q)} \left[f_j(q^2)\right]^2 = \int \frac{d^4q}{(2\pi)^4} \frac{i[f_j(q^2)]^2}{[(P/2 + q)^2 - m_j^2][(P/2 - q)^2 - M_j^2]}. \quad (97)$$

with the energy squared $P^2 = s$. The energy squared $s$ in the denominator of the loop function is considered to have an infinitesimal positive imaginary part $i\epsilon$: $s \rightarrow s + i\epsilon$. We note that the dimensional regularization of the loop function, which is used for the application in the next section, is achieved by setting $f_j(q^2) = 1$ and modifying the integration variable as $d^4k \rightarrow \mu_{\text{reg}}^{4-d}d^dk$ with the regularization scale $\mu_{\text{reg}}$.

\(^8\)In relativistic field theory, the coupling $g_0$ can have an energy dependence from the derivative coupling. We do not consider the energy dependence of the coupling, in order to ensure a smooth reduction to the results in the previous section in the nonrelativistic limit.
In Appendix A we confirm that the wave equation (89) indeed describes a two-body system governed by the relativistic scattering equation. Namely, with the energy-dependent two-body interaction $V_{jk}^{\text{tot}}(s)$ (95) and the loop function $G_j(s)$ (97), the scattering amplitude $T_{jk}(s)$ can be calculated as

$$T_{jk}(s) = V_{jk}^{\text{tot}}(s) + \sum_l V_{jl}^{\text{tot}}(s)G_l(s)T_{lk}(s). \quad (98)$$

Therefore, Eq. (96) ensures that the scattering amplitude $T_{jk}(s)$ has a pole at $s = s_R$.

By comparing the residue of the resonance pole as in Eq. (78), we find

$$g_j g_k = \lim_{s \to s_R} (s - s_R)T_{jk}(s), \quad (99)$$

Then, in the same way with Sec. 2.3, we obtain

$$\langle \Psi_0 | \Psi \rangle = (\Psi | \Psi_0 \rangle = \sum_j g_j g_{0,j} \frac{G_j(s_R)}{s_R - s_0}, \quad (100)$$

Therefore, we obtain the elementariness and compositeness as

$$Z = -\sum_{j,k} g_k g_j \left[ G_j \frac{dV_{jk}^{\text{tot}}}{ds} G_k \right]_{s = s_R}, \quad X_j = -g_j^2 \left[ \frac{dG_j}{ds} \right]_{s = s_R}, \quad (101)$$

and the sum rule is derived from the normalization (90) as

$$-\sum_{j,k} g_k g_j \left[ \delta_{jk} \frac{dG_j}{ds} + G_j \frac{dV_{jk}^{\text{tot}}}{ds} G_k \right]_{s = s_R} = 1. \quad (102)$$

This is another derivation of the generalized Ward identity in Ref. [26]. In Ref. [26], Eq. (102) is obtained by attaching one probe current to the meson-baryon scattering amplitude. The derivative of the loop function corresponds to the diagrams in Fig. 1 (a) in the soft limit of the probe current. It is therefore consistent to interpret the first term of Eq. (102) as compositeness which reflects the contribution from the two-body molecule component. On the other hand, the derivative of the contact interaction corresponds to the attachment of the probe current to the interaction vertex [Fig. 1 (b)], which represents something other than the compositeness and thus is understood as the elementariness.

We note that, although both the compositeness $X_j$ and elementariness $Z$ are complex for resonances, their sum should be unity, provided that the proper normalization of the wave function is adopted. As in the nonrelativistic case, the compositeness (elementariness) is
expressed with the derivative of the loop function (interaction), and they can be determined by the local behavior of the interaction and loop function. Finally we mention that the expression of the elementariness $Z$ in Eq. (101) coincides with that derived by matching with the Yukawa theory in Ref. [42]. In this work, we derive $Z$ and $X_j$ without specifying the explicit form of the vertex and relate them with the wave function of the bound and resonance states.

3. Applications — structure of dynamically generated hadrons

3.1. Compositeness and elementariness in chiral dynamics

Having established the elementariness and compositeness in Eq. (101), we now turn to the analysis of physical hadronic resonances by theoretical models with hadronic degrees of freedom. One of the most prominent models is the coupled-channel approach with the chiral perturbation theory. In this model the nonperturbative summation of the chiral interaction makes it possible to generate hadronic resonances dynamically, and hence these hadronic resonances are often called dynamically generated hadrons. This framework has been successfully applied to the description of the low energy hadron scatterings with resonance states. Among others, the $\Lambda(1405)$ resonance in the strangeness $S = -1$ meson-baryon scattering [67–75] and the lightest scalar and vector mesons in the meson-meson scattering [76–85] have been extensively studied in this approach.

The compositeness and elementariness have been evaluated in the chiral model with the simple leading order chiral interaction for $\Lambda(1405)$ and the scalar mesons in Ref. [27]. The compositeness of the $\rho(770)$ meson [43] and $K^*(892)$ [44] are also studied in phenomenological models. Here we aim at more quantitative discussion by using refined chiral models constrained by the recent experimental data. For this purpose, we employ the next-to-leading order calculations for $\Lambda(1405)$ [73, 74] and for scalar and vector mesons [85].

As we will show below, the scattering amplitudes in Refs. [73, 74, 85] can be reduced to the form of the coupled-channel algebraic equation

$$T_{jk}(s) = V_{jk}(s) + \sum_l V_{jl}(s)G_l(s)T_{lk}(s).$$

Here the separable interaction kernel $V_{jk}$ is a symmetric matrix with respect to the channel indices and depends on the Mandelstam variable $s$, and $G_j$ is the two-body loop function. The explicit forms of $V_{jk}$ and $G_j$ will be given for each model. The resonances are identified by the poles of the scattering amplitude $T_{jk}$, and the scattering amplitude can be written in the vicinity of one of the resonance poles as:

$$T_{jk}(s) = \frac{g_j g_k}{s - s_R} + T_{jk}^{BG}(s),$$

where $g_j$ and $s_R$ are the coupling constant and the pole position for the resonance, respectively, and $T_{jk}^{BG}$ is a background term which is regular at $s \rightarrow s_R$.

As discussed in the end of Sec. 2.4, the compositeness and elementariness of Eq. (101) depend only on the local behavior of the interaction and the loop function. We thus apply Eq. (101) to the general energy dependence of the interaction $V_{jk}(s)$ in Eq. (103). The expression for the $j$-channel compositeness is given by

$$X_j = -g_j^2 \left[ \frac{dG_j}{ds} \right]_{s = s_R},$$

where $dG_j/ds$ is the derivative of the loop function with respect to the Mandelstam variable $s$. This expression shows that the compositeness depends only on the local behavior of the loop function and the residue of the resonance. The elementariness $Z$ is defined as the overlap of the wave function with the resonance state.

In this work, we derive $Z$ and $X_j$ without specifying the explicit form of the vertex and relate them with the wave function of the bound and resonance states.
while the elementariness $Z$ is given by
\[
Z = -\sum_{j,k} g_k g_j \left[ G_j \frac{dV_{jk}}{ds} G_k \right]_{s=s_R}.
\] (106)

Sum of the compositeness $X_j$ and elementariness $Z$ coincides with the normalization relation of the total wave function for the hadronic resonances as
\[
\sum_j X_j + Z = (\Psi | \Psi) = 1.
\] (107)

The compositeness $X_j$ represents the contribution of the $j$-channel two-body wave function to the total normalization.

Let us summarize the interpretation of the compositeness and elementariness for resonances. As shown in Sec. 2.3, $X_j$ and $Z$ for resonances are in general complex. This fact spoils the probabilistic interpretation in a strict sense. It is however possible to interpret the structure of the resonance when one of the real parts of $X_j$ or $Z$ is close to unity and all the other numbers have small absolute values. In this case, we interpret that the resonance is dominated by the $j$-th channel component or something other than the two-body channels, respectively, on the basis of the similarity of the wave function of the stable bound state.

### 3.2. Structure of $\Lambda(1405)$

In Refs. [73, 74] the low-energy meson-baryon interaction in the strangeness $S = -1$ sector has been constructed in chiral perturbation theory up to the next-to-leading order, which consists of the Weinberg-Tomozawa contact term, the $s$- and $u$-channel Born terms, and the next-to-leading order contact terms. After the $s$-wave projection, the interaction kernel $V_{jk}$ depends only on the Mandelstam variable $s$ as a real symmetric separable interaction. The explicit form of $V_{jk}$ can be found in Refs. [74, 75]. The loop function is regularized by the dimensional regularization:
\[
G_{jk}(s) = i\mu_{\text{reg}}^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{[(P/2 + q)^2 - m_j^2]((P/2 - q)^2 - M_j^2)}
\]
\[
= a_j(\mu_{\text{reg}}) + \frac{1}{16\pi^2} \left[ 1 + \ln \left( \frac{M_j^2}{\mu_{\text{reg}}^2} \right) \right] + \frac{s + M_j^2 - m_j^2}{2s} \ln \left( \frac{M_j^2}{m_j^2} \right)
\]
\[
- \frac{2\lambda^{1/2}(s, m_j^2, M_j^2)}{s} \text{artanh} \left( \frac{\lambda^{1/2}(s, m_j^2, M_j^2)}{m_j^2 + M_j^2 - s} \right),
\] (108)

with the Källen function $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$. The finite part is specified by the subtraction constant $a_j(\mu_{\text{reg}})$ at the regularization scale $\mu_{\text{reg}}$. Because the meson-baryon one loop is counted as next-to-next-to-leading order in the baryon chiral perturbation theory, the amplitude is not renormalizable and hence it depends on the subtraction constants in this framework. The low-energy constants in the next-to-leading order contact interaction terms and the subtraction constants of the loop function have been determined by fitting to the low-energy total cross sections of $K^-p$ scattering to elastic and inelastic channels, the threshold branching ratios, and the recent measurement of the $1s$ shift and width of the kaonic hydrogen [86, 87]. In this approach, the $\Lambda(1405)$ resonance is associated with two poles of the scattering amplitude in the complex energy plane [71].
For convenience we refer to the pole which has higher (lower) mass $M_R = \text{Re} \left( \sqrt{s_R} \right)$ as the higher (lower) pole. It is expected from the structure of the Weinberg-Tomozawa interaction that the higher pole originates in a bound state caused by the $\bar{K}N$ attraction [72].

With the formulae in Sec. 3.1 we calculate the pole positions, compositeness $X_j$ and the elementariness $Z$ of the $\Lambda(1405)$ resonance in this model. Results are summarized in Table 1. In Refs. [73, 74], the isospin symmetry is slightly broken by the physical hadron masses. Therefore, we evaluate the compositeness in the charge basis and define the compositeness in the isospin basis by summing up all the channels in the charge basis, i.e., $X_{\bar{K}N} = X_{K^-p} + X_{\bar{K}^*n}$, and so on. Although there are nonzero contributions from the $I = 1$ channels, $X_{\pi^0\Lambda}$ and $X_{\eta\Sigma^0}$, to the total normalization, these are negligible and hence not listed in Table 1. As one can see from Table 1, it is remarkable that the real part of the $X_{\bar{K}N}$ component of the higher $\Lambda(1405)$ pole is close to unity and its imaginary part is very small. In addition, the magnitude of real and imaginary parts of all the other components is also small ($\lesssim 0.2$).

This indicates that the wave function of the higher $\Lambda(1405)$ pole is similar to that of the pure $\bar{K}N$ bound state which has $X_{\bar{K}N} = 1$, $X_i = 0$ ($i \neq \bar{K}N$) and $Z = 0$. It is therefore natural to interpret that the higher $\Lambda(1405)$ pole is dominated by the $\bar{K}N$ composite component. This is consistent with the non-$qqq$ nature of this pole from the $N_c$ scaling analysis [23, 24].

On the other hand, for the lower pole, there is a certain amount of cancellation ($\sim 0.4$) in the real part of the sum rule (107), and the absolute values of the imaginary parts are as large as $\sim 0.5$. Although one may observe relatively large contributions in $X_{\pi\Sigma}$ and $Z$, the dominance of these components is comparable with the magnitude of the imaginary part. Therefore, it is not possible to clearly conclude the structure of the lower pole from the present analysis.

The compositeness and elementariness of $\Lambda(1405)$ were calculated in Ref. [27] using the simple chiral model with the leading order Weinberg-Tomozawa interaction. The qualitative features of $X_j$ and $Z$ are not changed very much, so we confirm the earlier results in the present refined model. At the quantitative level, the results of the lower pole shows relatively larger model dependence. This model dependence also implies the difficulty of the clear interpretation of the structure of the lower pole.

Before closing this subsection, we mention that the structure of $\Lambda(1405)$ was investigated in the complex scaling method in Refs. [88, 89]. In a $\bar{K}N-\pi\Sigma$ two-channel model, the norm of each component is evaluated from the wave function. It is found that the norm of the $\bar{K}N$ component of the higher $\Lambda(1405)$ pole is close to unity with a small imaginary part. Thus, the result for the higher pole is qualitatively consistent with ours. On the other hand, the result

---

**Table 1** Compositeness $X_j$ and elementariness $Z$ of $\Lambda(1405)$ in the isospin basis.

| $\Lambda(1405)$, higher pole | $\Lambda(1405)$, lower pole |
|-----------------------------|-----------------------------|
| $\sqrt{s_R}$ [MeV]          |                             |
| $X_{\bar{K}N}$              | $X_{\bar{K}N}$              |
| $1424 - 26i$                 | $1381 - 81i$                |
| $1.14 + 0.01i$               | $-0.39 - 0.07i$             |
| $-0.19 - 0.22i$              | $0.66 + 0.52i$              |
| $0.13 + 0.02i$               | $-0.04 + 0.01i$             |
| $0.00 + 0.00i$               | $-0.00 + 0.00i$             |
| $Z$                         | $Z$                         |
| $-0.08 + 0.19i$              | $0.77 - 0.46i$              |
for the lower pole in Ref. [89] shows the dominance of the \( \pi \Sigma \) component. This is because the complete set to decompose the resonance wave function in Ref. [89] does not contain the elementary component. Namely, the application of our formula to their amplitude would indicate a certain amount of the elementary component \( Z \), as we have found here, since the interaction in Ref. [89] has an energy dependence. In fact, this is in accordance with the observation in Ref. [89] that the lower pole disappears when the energy dependence of the interaction is switched off.

### 3.3. Structure of the lightest scalar and vector mesons

The lowest lying scalar and vector mesons in the meson-meson scattering have been studied in Ref. [85] using the Inverse Amplitude Method (IAM) with the chiral interaction up to the next-to-leading order. The scattering amplitude in the coupled-channel IAM is given by

\[
T(s) = T_2(s) \left[ T_2(s) - T_4(s) \right]^{-1} T_2(s),
\]

where \( T_2 \) and \( T_4 \) are respectively the leading and next-to-leading order amplitudes in the matrix form with channel indices from chiral perturbation theory and have been projected to the orbital angular momentum \( L = 0 \) (scalar) and \( L = 1 \) (vector). In contrast to the model in the previous subsection, meson-meson one loop is in the next-to-leading order, and hence the amplitude does not depend on the renormalization scale. Therefore, the parameters in this model are the renormalized low energy constants in the next-to-leading order chiral Lagrangians. These constants are determined by fitting the experimental meson-meson scattering data such as the \( \pi \pi \) scattering up to \( \sqrt{s} = 1.2 \text{ GeV} \) [85]. The lightest scalar mesons \( \sigma, f_0(980), \) and \( \kappa \) are found as poles of the \( s \)-wave amplitude, while the \( a_0(980) \) resonance appears as a cusp at the \( K \bar{K} \) threshold but the corresponding resonance pole is not found. The vector mesons \( \rho(770) \) and \( K^*(892) \) are also dynamically generated as poles of the \( p \)-wave amplitude.

To evaluate the compositeness and elementariness, we rewrite the amplitude (109) in the form of Eq. (103). To this end, we first notice that \( T_4 \) can be decomposed as the \( s \)-channel loop part and the rest:

\[
T_4 = T_2G T_2 + T_{4,\text{non-G}},
\]

where \( T_{4,\text{non-G}}(s) \) consists of the next-to-leading order tree-level amplitudes, tadpoles, and \( t \)- and \( u \)-channel loop contributions. The loop function \( G_j(s) \) in Eq. (110) is given by

\[
G_j(s) = \frac{1}{16\pi^2} \left[ -1 + \left( \frac{M_j^2 - m_j^2}{2s} + \frac{m_j^2 + M_j^2}{2(m_j^2 - M_j^2)} \right) \ln \left( \frac{M_j^2}{m_j^2} \right) \right. \\
- \frac{2\lambda^{1/2}(s, m_j^2, M_j^2)}{s} \left. \text{artanh} \left( \frac{\lambda^{1/2}(s, m_j^2, M_j^2)}{m_j^2 + M_j^2 - s} \right) \right].
\]

Note that there is no degrees of freedom of the subtraction constant; the finite part is determined by the low energy constants included in \( T_{4,\text{non-G}} \). We then define

\[
V \equiv T_2(T_2 - T_{4,\text{non-G}})^{-1} T_2.
\]

It is easily checked that the amplitude in IAM (109) is formally equivalent to Eq. (103) with the interaction (112) and the loop function (111). We thus interpret Eq. (112) as the effective interaction kernel used in IAM. Physically, this interaction kernel contains not only
Table 2  Compositeness $X_j$ and elementariness $Z$ of scalar mesons in the isospin basis.

| $\sqrt{s_R}$ [MeV] | $f_0(500) = \sigma$ | $f_0(980)$ | $K_0^+(800) = \kappa$ |
|---------------------|----------------------|------------|----------------------|
| $\rho(770)$ : 746 – 11i MeV, $K^+(892)$ : 890 – 0i MeV. | $\kappa$ | $\sigma$ | $\kappa$ |

Before evaluating the compositeness, let us focus on the structure of the interaction kernel (112). Because of the $(T_2 - T_{4,non-G})^{-1}$ factor, the interaction kernel can have a pole when $\text{det}(T_2(s) - T_{4,non-G}(s)) = 0$ is satisfied. Thus, even though the IAM is constructed from chiral perturbation theory without bare fields of the scalar and vector mesons, there can be a bare pole contribution in the effective interaction $V$. In fact, we find bare poles in the vector channel near the physical resonances as

$$\rho(770) : 746 - 11i \text{ MeV}, \quad K^+(892) : 890 - 0i \text{ MeV}. \quad (113)$$
Table 3  Compositeness $X_j$ and elementariness $Z$ of vector mesons in the isospin basis.

|       | $\rho(770)$           | $K^*(892)$       |
|-------|-----------------------|------------------|
| $\sqrt{s_R}$ [MeV] | $760 - 84i$          | $885 - 22i$      |
| $X_{\pi\pi}$  | $-0.08 + 0.03i$      | $-0.03 + 0.04i$  |
| $X_{K\bar{K}}$ | $-0.02 + 0.00i$      | $-0.03 + 0.00i$  |
| $X_{\pi K}$   | $-0.03 + 0.04i$      | $-0.03 + 0.00i$  |
| $X_{\eta K}$  | $-0.03 + 0.00i$      | $-0.03 + 0.00i$  |
| $Z$            | $1.10 - 0.04i$       | $1.06 - 0.04i$   |

In the vector channels\(^9\), we find that, for both the $\rho(770)$ and $K^*(892)$ mesons, the real part of the elementariness $Z$ is close to unity and the magnitude of the imaginary part is less than 0.1. This indicates that the structure originates in the elementary component. This is consistent with the finding of the bare pole contribution in the interaction kernel $V$ for the vector channels. In fact, the physical pole position in Table 3 is very close to that in the bare interaction (113). We thus conclude that these vector mesons are not dominated by the two-meson composite structure. This is consistent with the $N_c$ scaling analysis in Refs. [21, 22], which indicates the $q\bar{q}$ structure of vector mesons.

The compositeness of scalar mesons [$\sigma$, $f_0(980)$, and $a_0(980)$] has been studied in Ref. [27] using the leading order chiral interaction. The qualitative tendency of the results of the $\sigma$ and $f_0(980)$ is similar with the present calculation, while the dominance of the $K\bar{K}$ component of $f_0(980)$ is much clear in the present results. Also for the vector mesons, the present calculation in IAM with the next-to-leading order chiral interaction is consistent with the previous phenomenological ones in Refs. [43, 44], which suggest that $\rho(770)$ and $K^*(892)$ are elementary.

3.4. Structure of other hadrons

In the preceding subsections we have evaluated the compositeness and elementariness of $\Lambda(1405)$, light scalar mesons, and light vector mesons using the scattering amplitudes calculated in chiral dynamics with systematic improvements by higher order contributions. In this subsection we also discuss the compositeness and elementariness of $N(1535)$ and $\Lambda(1670)$ in a simplified model with the lowest order Weinberg-Tomozawa interaction. Although a systematic analysis is not performed for these resonances, the model with appropriate subtraction constants [90, 91] describes $N(1535)$ and $\Lambda(1670)$ reasonably well.

Using $V(s)$ and $G(s)$ with the subtraction parameters given in Ref. [90] for $N(1535)$ and Ref. [91] for $\Lambda(1670)$, we calculate the compositeness and elementariness of $N(1535)$ and $\Lambda(1670)$.

\(^9\)In the framework of IAM, the loop function in $p$ wave is identical to that in $s$ wave. On the other hand, with a nonrelativistic separable interaction, the loop function in the $l$-th partial wave should contain the $q^{2l}$ factor in the integrand [43, 44]. This is to ensure the correct low energy behavior of the amplitude $F_l(q) \sim q^{2l}$. The difference of the loop function may be regarded as the difference of the definition of $Z$ and $X_j$ (basis to form the complete set). We however note that the present definition leads to $Z = 0$ in the $B \to 0$ limit even in $p$ wave, while the definition in Refs. [43, 44] does not constrain the value of $Z$ at threshold for nonzero $l$. The general threshold behavior is consistent with the latter [60], so the present definition would lead to a special behavior near the threshold. In practice, the $\rho(770)$ and $K^*(892)$ mesons locate away from the threshold energies of meson-meson channels, so the special nature of the definition would not cause a problem in the present analysis.
Table 4  Compositeness $X_j$ and elementariness $Z$ of $N(1535)$ and $\Lambda(1670)$ in the isospin basis.

|                | $N(1535)$ | $\Lambda(1670)$ |
|----------------|-----------|-----------------|
| $\sqrt{s_R}$ [MeV] | 1529 − 37$i$ | 1678 − 21$i$ |
| $X_{\pi N}$     | −0.02 − 0.01$i$ | $X_{KN}$       | 0.03 + 0.00$i$ |
| $X_{\eta N}$    | −0.03 + 0.23$i$ | $X_{\pi \Sigma}$ | 0.00 + 0.00$i$ |
| $X_{K \Lambda}$ | 0.09 − 0.04$i$  | $X_{\eta \Lambda}$ | −0.09 + 0.16$i$ |
| $X_{K \Sigma}$  | 0.26 − 0.09$i$  | $X_{K \Xi}$    | 0.53 − 0.10$i$ |
| $Z$             | 0.70 − 0.09$i$  | $Z$            | 0.53 − 0.06$i$ |

$\Lambda(1670)$. The results are listed in Table 4. First of all, interestingly, for both resonances the imaginary parts of the values of the compositeness $X_j$ and elementariness $Z$ are relatively small. This may allow us to interpret $X_j$ and $Z$ as the components of the resonance state. For $N(1535)$, $Z$ is a dominant piece with a relatively small imaginary part. This suggests that $N(1535)$ in the present model has a large component originating from contribution other than the pseudoscalar meson-baryon dynamics considered, in accordance with Ref. [20]. In contrast, for $\Lambda(1670)$ the $K \Xi$ compositeness $X_{K \Xi}$ and the elementariness $Z$ share unity half-and-half. This implies that in the present model the $K \Xi$ composite state plays a substantial role for the $\Lambda(1670)$ pole together with a bare state coming from components other than meson-baryon systems. This conclusion on $\Lambda(1670)$ is consistent with the discussion with the natural renormalization scheme in Ref. [92].

Here we emphasize that both $N(1535)$ and $\Lambda(1670)$ discussed in this subsection are described by scattering amplitudes which do not fully reproduce the experimental data in relevant energies [93, 94]. For more realistic discussion, it is desirable to improve the theoretical models so as to reproduce the experimental data well, for instance, by including vector meson-baryon channels as in Ref. [95] as well as by implementing higher order terms.

4. Conclusion

In this study we have developed a framework to investigate the internal structure of bound and resonance states with their compositeness and elementariness by using their wave functions. For this purpose we have explicitly taken into account both a one-body bare state and two-body scattering states. Compositeness and elementariness are respectively defined as the two-body and the bare state contributions to the normalization of the total wave function. Here we take the two-body scattering and the one-body bare states as our model space. In general, other channels, such as multi-body scatterings, can be included into the model space and the elementariness can be defined accordingly within that model space. After reviewing the formulation for the bound state, we have discussed the extension to the resonance state.

Because the wave function is analytically obtained for a separable interaction, we have explicitly written down the wave function for a bound state in a general separable interaction and obtained the expressions of the compositeness and elementariness. The existence of the bare-state degree of freedom induces the energy dependence of the effective two-body interaction. We have demonstrated that the compositeness (elementariness) is determined by the energy dependence of the loop function (interaction) at the bound state pole. Because
the structure of the bound state is determined only by the local behavior at the pole position, the formulae of the compositeness and elementariness can be applied to the interaction with an arbitrary energy dependence. Of particular value is the derivation of the Weinberg’s relation for the scattering length and effective range in the weak binding limit. In the present formulation, thanks to the separable interaction, the scattering amplitude is analytically obtained. With this fact we have explicitly performed the expansion of the amplitude around the threshold to derive the Weinberg’s relation. In this derivation, the higher order corrections are related to the explicit expression of the form factor as well as higher order derivatives in the expansion. The limitation of the formula due to the existence of the CDD pole is clearly linked to the breakdown of the effective range expansion.

Our discussion on the wave function has been extended to resonance states with the Gamow vectors. The use of the Gamow vector enables us to have finite normalization of the resonance wave function. For resonance states, by definition both the compositeness and elementariness become complex, which are difficult to interpret. Nevertheless, utilizing the fact that the compositeness and elementariness are defined by the wave functions, we have proposed the interpretation of the structure of certain class of resonance states, on the basis of the similarity of the wave function of the bound state. Namely, if the compositeness (elementariness) in a channel is close to unity with small imaginary part and all the other components have small absolute values, this resonance state can be considered to be a composite state (an elementary state) in the channel. Finally we have given the expressions of the compositeness and elementariness with a general separable interaction in a relativistic covariant form by considering a relativistic scattering with a three-dimensional reduction.

As applications, the expression of the compositeness in a relativistic form has been used to investigate internal structure of hadronic resonances. By employing the chiral coupled-channel scattering models with interactions up to the next to leading order, we have observed that the higher pole of Λ(1405) and f_0(980) are dominated by the $\bar{K}N$ and $K\bar{K}$ composite states, respectively, while the vector mesons $\rho(770)$ and $K^*(892)$ are elementary.

Finally we emphasize that the fact that constituent hadrons are observable as asymptotic states in QCD is essential to construct the two-body wave functions and to determine the compositeness for hadronic resonances.

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A. Conventions of relativistic two-body state and two-body equation
In this Appendix we summarize our conventions of the two-body state in the relativistic kinematics and confirm that the wave equation (89) indeed describes a two-body system whose motion is governed by the Klein-Gordon equation. In the following we concentrate
on single-channel kinematics of the two-body system, but generalization to multi-channel kinematics is straightforward.

A.1. Normalization of states

First we consider an on-shell one-body state of a scalar field of mass $m$ with definite three-dimensional momentum $p$, $|p\rangle$, whose normalization is defined as follows:

$$\langle p'|p\rangle = 2\sqrt{p^2 + m^2}(2\pi)^3\delta^3(p' - p). \quad (A1)$$

Since we do not explicitly treat spin components of scattering baryons in this paper, we use the above normalization also for baryons.

Next we construct a two-body state, in which both two particles are on the mass shell and the relative momentum is denoted as $q$ in the center-of-mass frame, used in Sec. 2.4. In this kinematical condition, the momenta of two particles are given by

$$p^\mu_1 = (\omega(q), q) \quad \text{and} \quad p^\mu_2 = (\Omega(q), -q),$$

where $\omega(q) \equiv \sqrt{q^2 + m^2}$ and $\Omega(q) \equiv \sqrt{q^2 + M^2}$ are the on-shell energies of two particles with $m$ ($M$) being the mass of the first (second) particle, and the total momentum becomes $P^\mu \equiv p^\mu_1 + p^\mu_2 = (\sqrt{s q}, 0)$, with $s = \omega(q) + \Omega(q)$. Then the two-body state with relative momentum $q$, $|q^{\text{co}}\rangle$, can be defined by using product of two one-body states, $|p_1\rangle \otimes |p_2\rangle$. In this study we adopt the following normalization of $|q^{\text{co}}\rangle$:

$$|q^{\text{co}}\rangle \equiv N_{sq}|q_1\rangle \otimes |q_{-2}\rangle, \quad \langle q^{\text{co}}| \equiv N_{sq}^*|q_1\rangle \otimes \langle q_{-2}|, \quad |N_{sq}|^2 \equiv \frac{1}{2V_3\sqrt{s q}}. \quad (A2)$$

In the normalization factor $N_{sq}$, $V_3$ is the total spatial volume and is related to the delta function for the momentum as $V_3 = (2\pi)^3\delta^3(0)$. The advantage to adopt this normalization factor is that the expression of the relativistic two-body wave equation becomes a natural extension of the nonrelativistic Schrödinger equation, as we will see in the next subsection.

With the definition of the two-body state $|q^{\text{co}}\rangle$ in Eq. (A2) and the normalization of the one-body state in Eq. (A1), we can calculate the normalization for $|q^{\text{co}}\rangle$ in a straightforward way as

$$\langle q'^{\text{co}}|q^{\text{co}}\rangle = \frac{2\omega(q)\Omega(q)}{\sqrt{s q}}(2\pi)^3\delta^3(q' - q). \quad (A3)$$

This normalization leads to the projection operator to the two-body state:

$$\hat{P}_{\text{two}} = \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{s q}}{2\omega(q)\Omega(q)} |q^{\text{co}}\rangle \langle q^{\text{co}}|,$$

which corresponds to a part of the completeness condition.

A.2. Relativistic wave equation and scattering equation

Now we would like to confirm that the wave equation (89) indeed describes a two-body system whose motion is governed by the Klein-Gordon equation, by deriving the scattering equation from the operators in the wave equation. Here, in the same manner as in Sec. 2, we introduce a one-body bare state and a two-body scattering state, and assume that the bare state contribution is effectively contained in the two-body interaction $V^{\text{tot}}$. In this sense, the relation in Eq. (A4) coincides with the completeness condition; $\hat{P}_{\text{two}} = 1$.

In general, the wave equation can be composed of the free two-body Green’s operator $\hat{G}(s)$ and the two-body interaction operator $\hat{V}^{\text{tot}}(s)$. The two-body Green’s operator $\hat{G}(s)$ is
defined as \( \hat{G}(s) \equiv 1/(s - \hat{\mathcal{K}}) \) with the kinetic energy operator \( \hat{\mathcal{K}} \) so that\(^{10}\)

\[
\hat{G}(s)|q'^{\text{co}}\rangle = \frac{1}{s - s_q}|q'^{\text{co}}\rangle, \quad \langle q'^{\text{co}}|\hat{G}(s) = \frac{1}{s - s_q}\langle q'^{\text{co}}|.
\] (A5)

On the other hand, two-body interaction operator \( \hat{V}^{\text{tot}}(s) \) has a general separable interaction as in Eq. (88), thus we have

\[
\langle q'^{\text{co}}|\hat{V}^{\text{tot}}(s)|q'^{\text{co}}\rangle = V^{\text{tot}}(s)f(q^2)f(q'^2),
\] (A6)

where \( V^{\text{tot}}(s) \) corresponds to the interaction in Eq. (95), which contains the implicit contribution from the bare state.\(^ {11}\) Here we also assume that the form factor \( f(q^2) \) depends only on the three momentum so as to make a three-dimensional reduction of the scattering equation. Then, by using \( \hat{G} \) and \( \hat{V}^{\text{tot}} \), we can express the wave equation for a relativistic resonance state \(|\Psi\rangle\), whose mass and width are described by an eigenvalue \( s_R \), as

\[
\hat{G}^{-1}(s_R)|\Psi\rangle = \hat{V}^{\text{tot}}(s_R)|\Psi\rangle, \quad (\Psi|\hat{G}^{-1}(s_R) = (\Psi|\hat{V}^{\text{tot}}(s_R),
\] (A7)

which is equivalent to the wave equation in Eq. (89) with the implicit bare-state degree of freedom.

Let us now derive the scattering equation with the above normalizations. To this end, we define the \( T \)-operator \( \hat{T} \) by the interaction \( \hat{V}^{\text{tot}} \) and two-body Green’s operator \( \hat{G} \) as:

\[
\hat{T} = \hat{V}^{\text{tot}} + \hat{V}^{\text{tot}}\hat{G}\hat{T}.
\] (A8)

This corresponds to the two-body scattering equation in an operator form. For the separable interaction (A6), the matrix element of the \( T \)-operator is given in the form \( \langle q'^{\text{co}}|\hat{T}|q'^{\text{co}}\rangle = T(s)f(q^2)f(q'^2) \). The scattering equation is then obtained from Eq. (A8) as

\[
T(s) = V^{\text{tot}}(s) + V^{\text{tot}}(s)G(s)T(s),
\] (A9)

where \( G(s) \) corresponds to the loop function and is defined as

\[
G(s) = \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{s_q}}{2\omega(q)\Omega(q)} \frac{|f(q^2)|^2}{s - s_q} = \int \frac{d^3q}{(2\pi)^3} \frac{[f(q^2)]^2}{[(P/2 + q)^2 - m^2][(P/2 - q)^2 - M^2]},
\] (A10)

with \( P^\mu \equiv (\sqrt{s}, 0) \). The second term of the right-hand side in Eq. (A9) can be obtained by inserting the operator \( \hat{P}^{\text{two}} = 1 \) (A4) between \( \hat{V}^{\text{tot}} \) and \( \hat{G} \) as

\[
\langle q'^{\text{co}}|\hat{V}^{\text{tot}}\hat{G}|q'^{\text{co}}\rangle = \int \frac{d^3q''}{(2\pi)^3} \frac{\sqrt{s_q''}}{2\omega(q'')\Omega(q'')} \frac{\langle q'^{\text{co}}\hat{V}^{\text{tot}}|q''^{\text{co}}\rangle (q''^{\text{co}}|\hat{T}|q'^{\text{co}})}{s - s_q''}
\]

\[
= \int \frac{d^3q''}{(2\pi)^3} \frac{\sqrt{s_q''}}{2\omega(q'')\Omega(q'')} \frac{V^{\text{tot}}(s)f(q''^2)f(q'^2)\times T(s)f(q^2)f(q''^2)}{s - s_q''}
\]

\[
= V^{\text{tot}}(s)G(s)T(s)f(q^2)f(q'^2).
\] (A11)

As seen in the last expression of the loop function \( G(s) \) in Eq. (A10), Eq. (A9) is nothing but the scattering equation with the Klein-Gordon propagators, and hence the

\(^{10}\) In the nonrelativistic framework the two-body Green’s operator is \( \hat{G}(E) = 1/(E - \hat{H}_0) \) with \( \hat{H}_0 \) being the free Hamiltonian and Eq. (A7) is reduced to the Schrödinger equation.

\(^{11}\) By using the notations in Sec. 2.4, the two-body interaction operator \( \hat{V}^{\text{tot}}(s) \) can be defined as \( \hat{V}^{\text{tot}}(s) \equiv \hat{V} + \hat{V}|\Psi_0\rangle\langle \Psi_0|\hat{V}/(s - M^2) \) in a similar manner to the operator \( \hat{V}^{\text{tot}}(E) \) in Sec. 2.1.
wave equation (89) indeed describes a two-body system whose motion is governed by the Klein-Gordon equation.

At last we emphasize that the normalization (A3) is consistent with the two-body Green’s operator \( \hat{G}(s) = 1/(s - \hat{K}) \), which is a natural extension of the nonrelativistic Green’s operator \( \hat{G}(E) = 1/(E - \hat{H}_0) \). Otherwise, we should redefine \( \hat{G}(s) \) so as to absorb a kinematical factor coming from \( \sqrt{q}/[2\omega(q)\Omega(q)] \) in the loop integral (A10). This allows us to determine the coefficient of the relativistic two-body wave function in Sec. 2.4 in a straightforward way as in the nonrelativistic case.

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