Generalized power expansions in cosmology

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Abstract

It is given an algorithm to obtain generalized power asymptotic expansions of the solutions of the Einstein equations arising for several homogeneous cosmological models. This allows to investigate their behavior near the initial singularity or for large times. An implementation of this algorithm in the CAS system Maple V Release 4 is described and detailed calculations for three equations are shown.

1 Introduction

For several homogeneous cosmological models the Einstein equations, combined with the constitutive equations for the matter source, can be cast into nonlinear ordinary differential equations of the form

\[ D[y(t)] = \sum_{i=1}^{N} A_i y^{B_0} \left( \frac{dy}{dt} \right)^{B_1} \cdots \left( \frac{d^r y}{dt^r} \right)^{B_r} = 0 \]  

(1)

where \( y \) is usually a monotonic function of either the scale factor or the Hubble rate, \( t \) is the universal time, \( A_i \) and \( B_r \) are real constants, \( r \) is the order of the ODE and without losing generality we can assume that \( B_r^p \geq 0 \) by eventually multiplying the equation by suitable powers of \( y \) and its derivatives [1] [2] [3] [4]. Equation (1) is well defined provided we restrict to \( y(t) \geq 0 \). This restriction appears naturally in the cosmological setting because the scale factor is positive, or the Hubble rate is positive for an expanding universe.

When an exact solution of (1) is not available, one is at least interested in obtaining some information about it in the form of an expansion for the limits \( t \to 0^+ \) (usually corresponding to the behavior of the solution near the initial singularity) and \( t \to \infty \). However solutions to equations of the form (1)
frequently do not have power series solutions with integer or even rational exponents (Puiseux series) in these limits. Thus we are led to try “generalized” power series expansions of the form

\[ y(t) \sim \sum_{j=1}^{\infty} c_j t^{n_j} \]  \hspace{1cm} (2)

where \( c_j \) and \( n_j \) are real constants and \( n_1 < n_2 < \cdots \) for \( t \to 0^+ \) and \( n_1 > n_2 > \cdots \) for \( t \to \infty \). So \( c_1 t^{n_1} \) is the leading term, \( t^{n_{i+1}}/t^{n_i} \to 0 \) in either limit and the set \( \{t^{n_i}\} \) constitutes an asymptotic scale. Inserting this expansion in (1) and performing the necessary asymptotic expansions we get

\[ D \left[ \sum_{j=1}^{\infty} c_j t^{n_j} \right] \sim \sum_{k=1}^{\infty} C_k t^{e_k} \]  \hspace{1cm} (3)

where the exponents \( e_k \) are real and form an ordered set: \( e_1 < e_2 < \cdots \) for \( t \to 0^+ \) and \( e_1 > e_2 > \cdots \) for \( t \to \infty \). So, if the equation (1) admits a solution with expansion (2), each of the \( C_k \) must vanish and this set of equations fix in principle the constants \( c_j, n_j \) in (2) up to \( r - 1 \) free parameters arising from the integration constants of the general solution of (1) (for simplicity in the notation the arbitrary constant corresponding to time translation freedom is fixed to 0).

Several critical steps in the search for solutions in the form of generalized power expansions involve calculations with large number of terms and this number grows very fast with the size of the ODE making hand calculation inconvenient. However, as many of steps in these calculations have a mechanical nature, use of computer algebra systems appear ideally suited.

The algorithm of used in obtaining these approximated solutions has partial similarity in its initial steps to the algorithm used to test whether a given ODE satisfies necessary conditions to has Painlevé property, namely that its solutions are single valued around movable singularities [5]. That algorithm checks whether the behavior of the solutions near the singularities is like (2) with integer exponents. However in this paper the study of solutions is restricted to the positive real axis and no consideration will be made about their multivaluedness when extended to the complex plane.
2 The algorithm

The objective of calculations is to obtain a truncation of (2) to a finite number of terms, say $M$

$$y_M(t) = \sum_{j=1}^{M} c_j t^{n_j}$$  \hspace{1cm} (4)

The method of calculation is iterative, so that constants $c_M, n_M, M > 1$, when not free are determined by $c_1, n_1, \ldots, c_{M-1}, n_{M-1}$. For each step of the iteration in $M$, when inserted $y_M$ in (1), two critical points of the calculation are:

(i) Asymptotic expansion (to order $M$) of the terms with noninteger $B_i^p$.

(ii) Collect all the terms with the same power of $t$.

Thus we arrive at an expression of the form

$$D[y_M(t)] = \sum_{l=1}^{R} D_l t^{f_l}$$  \hspace{1cm} (5)

where $D_l$ and $f_l$ are real constants. In general the sequence of exponents $f_1, f_2, \ldots, f_R$ is not ordered. Thus, for every step the next task is:

(iii) Sort the $\{f_l\}_{1 \leq l \leq R}$.

This sorting operation is perhaps the most involved part of the whole calculation. To describe its role and a procedure to do it, only the behavior for $t \to 0^+$ will be considered in the following.

We start with $M = 1$, that is $y_1 = c_1 t^{n_1}$. When it is inserted in the i-th term of (1) we get a term with exponent

$$g_i = \sum_{h=0}^{r} (n_1 - h) B_i^h$$  \hspace{1cm} (6)

and coefficient

$$E_i = A_i c_1 \sum_{h=0}^{r} B_i^h \prod_{h=1}^{r} (n_1 - h + 1) \sum_{j=h}^{r} B_i^j \equiv \nu_i c_1^{\mu_i}$$  \hspace{1cm} (7)

If now we take $M = 2$, crossed terms appear. Let us take any one of them, say all factors from the leading term except that of derivative $b$. Then we get
an exponent

\[ g'_i = g_i + (n_2 - n_1) B^h_i > g_i \]  \hspace{1cm} (8)

As \( \epsilon_1 \) is the minimum of the \( \{ g_i \}_{1 \leq i \leq N} \), and say that this minimum occurs for \( i \in I \), so \( C_1 = \sum_{i \in I} E_i \) is a function of \( c_1 \) and \( n_1 \), and we find that only the leading term of (2) can contribute to the leading term of (3).

The requirement that the leading term of (3) is not a constant leads to \( \mu_i > 0 \) for \( i \in I \). If \( I \) has only one element the requirement \( C_1 = 0 \) implies \( c_1 = 0 \). This implies that \( I \) must have at least two elements. In this case it is said that these terms balance. This constraint usually determines \( n_1 \). For instance if terms \( i \) and \( k \) balance and \( \mu_i \neq \mu_k \), we get

\[ n_{1ik} = \frac{\sum_{h=0}^{r} h (B^h_i - B^h_k)}{\sum_{h=0}^{r} (B^h_i - B^h_k)} \]  \hspace{1cm} (9)

For this \( n_1 \) we get \( c_1 \) provided

\[ c_1^{\mu_i - \mu_k} + \frac{\nu_k}{\nu_i} = 0 \]  \hspace{1cm} (10)

has real roots. On the other hand, if the terms \( i \) and \( k \) balance and \( \mu_i = \mu_k \), the additional constraint \( \sum_{h=0}^{r} h (B^h_i - B^h_k) = 0 \) must hold, \( c_1 \) remains arbitrary and \( n_1 \) is determined by the real roots of the equation \( \nu_i + \nu_k = 0 \).

Each pair of real numbers \((c_1, n_1)\) obtained from this analysis corresponds to the leading term of the expansion of a family of solutions. Thus, subsequent calculations must be done separately.

Both the constants \( A_i \) and \( B^h_i \) in (1) may contain free parameters of the model. Thus, considering that some of the \( B^h_i \) are free, we have to ask for the values of the parameters that make these \( n_1 \) coincide. They are critical values for the parameters as they make the behavior of the solutions change.

Repeating the argument leading to (8) order by order in \( M \) it is easy to see that \( c_M \) and \( n_M \) only appear in \( C_j \) for \( j \geq M \). This also shows that the first \( M \) terms of (5), once sorted, are equal to the first \( M \) terms of expansion (3). In particular, the first \( M - 1 \) terms are those already found in step \( M - 1 \) of the iteration. Thus the main tasks at step \( M \) are:

(iv) Find \( e_M, n_M \) (if possible) and \( C_M \).

(v) Solve \( C_M = 0 \) for either \( c_M \) or \( n_M \).
Generalizing the expressions (6) and (8) we see that the exponents \( f_i \) are linear functions of \( n_1, \ldots, n_M \)

\[
f_i = \sum_{j=1}^{M} \alpha_i^j n_j + \sum_{i=1}^{N} \sum_{h=0}^{r} \beta_i^{ih} B_i^h
\]

where the \( \alpha_i^j \) are in turn linear functions of the \( B_i^h \). To order these exponents we need to know first when they can be equal. Consider that \( f_i = f_k \), then we have the equation

\[
\sum_{j=1}^{M} (\alpha_i^j - \alpha_k^j) n_j + \sum_{i=1}^{N} \sum_{h=0}^{r} (\beta_i^{ih} - \beta_k^{ih}) B_i^h = 0
\]

This equation sets a constraint between the \( \{n_j\} \) or equivalently defines an \((M - 1)\)-dimensional hyperplane in the vector space \((n_1, \ldots, n_M)\) provided the vector \( \Delta \alpha_{ik} = (\alpha_i^1 - \alpha_k^1, \ldots, \alpha_i^M - \alpha_k^M) \neq 0 \).

Only a sector of the \( M \)-dimensional vector space is admissible. First of all, the inequalities \( n_1 < n_2 < \cdots < n_M \) must be satisfied. Besides, \( n_1, \ldots, n_{M-1} \) have been determined in the previous steps of the iteration. As they are usually functions of the parameters contained in (1), the variation of these parameters within their range further restricts the admissible sector of the \( M \)-dimensional space. Within it, if the hyperplane exist, \( n_M \) becomes a linear function of \( n_1, \ldots, n_{M-1} \), and from all pairs of exponents in \( \{f_i\}_{1 \leq i \leq R} \) the \( n_M \) must be found for which both exponents are minimum. In this way we find \( e_M = f_i = f_k \) and \( C_M = D_i + D_k \). From \( C_M = 0 \) we obtain \( c_M \). On the other hand, if \( \Delta \alpha_{ik} = 0 \), \( c_M \) remains free and \( C_M = 0 \) determines \( n_M \).

Within the admissible sector, in each subsector delimited by the hyperplanes, the \( f_i \) can be sorted. The minimum, say \( f_k \) yields \( e_M \) and provided \( D_k \) has a common factor of the form \( c_M^n \) and the equation \( D_k = 0 \) can be solved for a real \( n_M \), this case also yields the pair \((e_M, n_M)\) with \( c_M \) free. In all cases only real solutions are admissible.

### 3 Implementation

The implementation of the algorithm described in the previous section is made in Maple V Release 4. At present, this implementation involves use of some routines written in the Maple V programming language as well as interactive calculations in the worksheet environment.
For the task of collecting terms with equal power of $t$, the command `collect` of the current implementation is not suitable as it cannot collect terms with nonrational exponents. So, the first task of the implementation was to write some procedures to provide this facility. Namely through a command named `collect2`, given as input an expression, a variable (or an expression), one or more options and a procedure it yields, depending on the options,

(a) a form of the expression collected in generic powers of the variable (assumed nonnegative throughout),

(b) a list of the exponents,

(c) a list of the powers,

(d) the (collected) coefficients in the form of a table, indexed by the exponents.

For options (a) and (d) the procedure given as input is used to output the coefficients is a useful way (typically using `collect`). The package `collect2` has also some useful procedures that can be used independently like `expand1` (like `expand` but does not expand exponents), and `varsubs` (like `algsubs` but allows substitution of subexpressions raised to generic powers).

The rest of the calculations are made for each step of the iteration at the worksheet level. It is planned to automatize some of them when some more experience is obtained in solving some failures that show the system. One of the main difficulties appears in the sorting the list of exponents when a set of inequalities is imposed. Maple provides the `assume` facility by which properties about variables, in this case inequalities, are informed to the system so that its routines can make use of this information and take decisions when handling these variables. The command `is` is given to check whether a property is true provided that the necessary properties of the involved variables hold. However, in some cases, this command fails in deciding the order of two expressions, though mathematically the calculation is well defined and a hand calculation can demonstrate the inequality in a few lines. So we use procedures `ord1` and `ord2`, that use numerical comparisons, based on values that lay inside the interval of interest. These and some other auxiliary procedures are currently collected in a package named `general`.

In the following sections three examples will be shown to give a feel of how this calculations can be done with Maple.
4 Example with three terms

The system of Einstein equations for homogeneous, isotropic cosmological models with a variety of matter sources reduce to the ordinary differential equation

$$\ddot{y} + y\dot{y} + \beta y^3 = 0 \quad (13)$$

where $\beta$ is a constant. For instance, the problem of a causal viscous fluid with the bulk viscosity coefficient $\zeta$ proportional to $\rho^{1/2}$ and $y \propto H$ in the truncated Extended Irreversible Thermodynamics theory [6]. Also, the behavior near the singularity, when the relaxation term is much more important than the viscous term in the transport equation, for generic power-law relation $\zeta = \alpha \rho^\eta$ [8]. For a time decaying cosmological "constant", $\dot{\Lambda} \sim -H^3\Lambda$ [9]. In a phenomenological description of the reheating process in terms of an out-of-equilibrium mixture of two reacting fluids [11].

Equation (13) is also very interesting from the mathematical point of view as it appears in the analysis of the Painlevé equations [10], and it is the simplest case of a class of nonlinear differential equations that possess form invariance under nonlocal transformations, so that they can be linearized and its general solution can be obtained in parametric form [12] [13].

The general solution of equation (13) is

$$y(\eta) = \left(A e^{\lambda_+ \eta} + B e^{\lambda_- \eta}\right)^{1/2} \quad (14)$$

$$\Delta t(\eta) = \int \frac{d\eta}{y(\eta)} \quad (15)$$

where $\lambda_{\pm} = (1/2) \left[-1 \pm (1 - 8\beta)^{1/2}\right]$ are the roots of the characteristic polynomial of the linear equation and $A$ and $B$ are two arbitrary integration constants. The analysis of the general solution shows that it possesses, for generic $\beta$, movable singularities with asymptotic behavior

$$y(t) \sim \frac{\alpha}{\Delta t} \sum_{n=0}^{\infty} c_n(\beta) \gamma^n \Delta t^{nr} \quad (16)$$

where $r = 4 - \alpha > 0$ is the Kowalevski exponent ([14]), $\alpha = \alpha_- \quad \eta \to -\infty$ and $\alpha = \alpha_+ \quad \eta \to \infty \quad (\beta < 0)$ and $\alpha_{\pm} = -2/\lambda_{\pm}$, $c_0 = 1$ and $\gamma$ is an arbitrary integration constant. Thus (13) is a second order differential equation of the form (1) for which we can show that 2-parameter families of solutions exist.
with an expansion like (2). Then it is interesting to see how the algorithm described above works in this case.

We begin by writing equation (13) in the Maple worksheet

\[
d := \frac{\partial^2}{\partial t^2} y(t) + y(t) \frac{\partial}{\partial t} y(t) + \beta y(t)^3
\]

and then we begin the iteration.

### 4.1 Leading term

For \( M = 1 \), inserting the leading term in (13), we get three exponents

\[
f := \text{collect2(subs(y(t)=c1*t^n1,d),t,exponents)};
\]

\[
f := [n1 - 2, 2n1 - 1, 3n1]
\]

We look for the values of \( n1 \) such that two or more terms balance

\[
> n1e := \text{balance(f,n1)};
\]

\[
n1e := [-1]
\]

This shows that all three terms balance simultaneously for \( n1 = -1 \).

\[
> f := \text{collect2(subs(y(t)=c1*t^(-1),d),t,exponents)};
\]

\[
f := [-3]
\]

Then \( e_1 = -3 \), and \( c1 \) satisfies a quadratic

\[
>_\text{coeff[-3]};
\]

\[
> c1s := \{\text{solve("},c1}\}\text{minus}\{0\};
\]

\[
c1s := \{\frac{1}{2} \frac{1 + \sqrt{1 - 8\beta}}{\beta}, \frac{1}{2} \frac{1 - \sqrt{1 - 8\beta}}{\beta}\}
\]

Thus, two families of real solutions with this leading behavior exist provided \( \beta < 1/8 \).
4.2 Second term

For $M = 2$, using $c_1, n_1$ from the previous step, we look for the exponent following $-3$.

```plaintext
> f:=collect2(subs(y(t)=c1/t+c2*t^n2,d),t,exponents);

f := [-3, n2 - 2, 2 n2 - 1, 3 n2]
```

These exponents are linear functions of $n_2$, and become equal only for $n_2 = -1$. As we require that $n_2 > n_1 = -1$, we can sort them

```plaintext
> assume(n2> -1):
> sort(f,(a,b)-> is(a<b));

[-3, n2 - 2, 2 n2 - 1, 3 n2]
```

and find that $e_2 = n_2 - 2$. Its coefficient

```plaintext
> _coeff[n2-2];

\[ c2 n2^2 - c2 n2 + c2 c1 n2 - c2 c1 + 3 \beta c1^2 c2 \]
```

is linear in $c_2$, so that this parameter remains free. Then we find $n_2 = 3 - c_1$, where the root $n_2 = -2$ is discarded as it arises from time translation invariance of one-parameter solutions. On the other hand, as $-1 < n_2$, this implies $c_1 < 4$, so that $c_1^-$ is admissible for $\beta < 1/8$, with $-1 < n_2 < 3$; while $c_1^+$ is admissible for $\beta < 0$, with $n_2 > 3$.

4.3 Third term

For $M = 3$, we look for the third exponent

```plaintext
> f:=collect2(subs(y(t)=c1/t+c2*t^n2+c3*t^n3,d),t,exponents);

f := [-3, n2 - 2, n3 - 2, 2 n2 - 1, 2 n3 - 1, 2 n2 + n3, n2 + 2 n3, n2 + n3 - 1, 3 n2, 3 n3]
```
These are linear functions of \( n3 \), where we must take into account that \( n2 < n3 \) and \(-1 < n2(\beta) < \infty\). Let us see first when these exponents become equal.

\[> \text{n3e:=balance(f,n3);}\]

\[
n3e := [-1, n2, -\frac{3}{2} - \frac{1}{2} n2, -3 - 2 n2, 1 + 2 n2, \frac{1}{2} n2 - \frac{1}{2}, -2 - n2, \frac{1}{3} n2 - \frac{2}{3}, 2 + 3 n2, \frac{2}{3} n2 - \frac{1}{3} \cdot \frac{1}{2} + \frac{3}{2} n2]\
\]

These in turn become equal when \( n2 = -1 \).

\[> \text{n2e:=balance(n3e,n2);}\]

\[n2e := [-1]\]

Hence we can sort them

\[> \text{assume(n2>}-1);}\]

\[> \text{sort(n3e,(a,b)->is(a<b))};\]

\[[-3 - 2 n2, -2 - n2, -\frac{3}{2} - \frac{1}{2} n2, -1, \frac{1}{3} n2 - \frac{2}{3}, -\frac{1}{2} + \frac{1}{2} n2, \frac{2}{3} n2 - \frac{1}{3}, n2, \frac{1}{2} + \frac{3}{2} n2, 2 n2 + 1, 3 n2 + 2]\]

We look for the third exponent within each interval.

\[> 1:=[ n2, 1/2+3/2*n2, 2*n2+1, 3*n2+2, 3*n2+2+1];\]

\[> 11:=subs(n2=.3,1):\]

\[\text{for i from 1 to nops(11)-1 do}\]

\[\text{u:=(11[i]+11[i+1])/2;}\]

\[\text{tabla[i]:=[[1[i]<n3,n3<1[i+1]],}\]

\[\text{op(3,sort(f,(a,b)->ord2(a,b,n2=.3,n3=u)))]:}\]

\[\text{print(tabla[i]);}\]

\[\text{od:}\]

\[\text{[[n2 < n3, n3 < } \frac{1}{2} + \frac{3}{2} n2, n3 - 2],}\]

\[\text{[[}\frac{1}{2} + \frac{3}{2} n2 < n3, n3 < 2 n2 + 1, n3 - 2],}\]

\[\text{[[2 n2 + 1 < n3, n3 < 3 n2 + 2], 2 n2 - 1]}\]
and we find that there is balance when \( n3 = 2n2 + 1 \) with coefficient

\[
3 \beta c1c2^2 + 36c3 + 3c2^2 + (-17c3 - c2^2)c1 + 2c3c1^2
\]

Then \(-1 < n3(\beta) < \infty\) and \( n3 = 2(n2 + 1) - 1 \) so that we recover the first three terms of (16) with

\[
c3 = -\frac{c2^2(3\beta c1 + 3 - c1)}{36 - 17c1 + 2c1^2}
\]

proportional to \( c2^2 \) as expected from (16). Also, as expected, no other solution is found.

5 Equation with a variable exponent

In order to treat dissipative processes in cosmology which are not close to equilibrium a nonlinear phenomenological generalization of the Israel-Stewart theory was developed recently [15]. Processes for which this kind of processes may have occurred are inflation driven by a viscous stress [15,16], and the reheating era at the end of inflation [11].

In a spatially flat FLRW universe, Einstein’s equations together with state and transport equations of the fluid give the evolution equation for the Hubble rate [15]

\[
\left[ 1 - \frac{k^2}{v^2} - \frac{2k^2}{3\gamma v^2} \frac{\dot{H}}{H^2} \right] \left\{ \ddot{H} + 3H\dot{H} + \left( \frac{1 - 2\gamma}{\gamma} \right) \frac{\dot{H}^2}{H} + \frac{9}{4\gamma}H^3 \right\} + \frac{3\gamma v^2}{2\alpha} \left[ 1 + \left( \frac{\alpha k^2}{\gamma v^2} \right)^{H^3-1} \right] H^{2-q} (2\dot{H} + 3\gamma H^2) - \frac{9}{2} \gamma v^2 H^3 = 0 \quad (17)
\]

where \( \gamma, \alpha, v, q \) and \( k \) are parameters describing the thermodynamical properties of the fluid. Note that \( q \) appears in the exponent.

Let us insert equation (17) in a new worksheet.

\[
d:=(1-k^2/v^2-2*k^2/(3*gamma*v^2)*diff(H(t),t)/H(t)^2)*
\]
(diff(H(t),t$2)+3*H(t)*diff(H(t),t)+
(1-2*gamma)/gamma*diff(H(t),t)^2/H(t)+9/4*gamma*H(t)^3)+
(3*gamma*v^2/(2*alpha)*(1+alpha*k^2/(gamma*v^2))*H(t)^2/(q-1)*H(t)^2/(2-q)*(2*diff(H(t),t)+
3*gamma*H(t)^2)-9/2*gamma*v^2*H(t)^3):

Multiplication by a factor $H^3$ makes all powers of $H$ positive.

> d2:=map(simplify,expand1(H(t)^3*d),power,symbolic):
> dview(d2,t,H);

\[ H^3 H'' + 3 H^4 H' + \frac{H^2 (H')^2}{\gamma} - 2 H^2 (H')^2 + \frac{9}{4} H^6 \gamma - \frac{H^3 k^2 H''}{v^2} - \frac{9}{2} \frac{H^4 k^2 H'}{v^2} - 3 \frac{H^2 k^2 (H')^2}{\gamma v^2} + 2 \frac{H^2 k^2 (H')^2}{v^2} - \frac{9}{4} \frac{H^6 k^2 \gamma}{v^2} - \frac{2}{3} \frac{H k^2 H' H''}{\gamma v^2} - \frac{2}{3} \frac{k^2 (H')^3}{\gamma v^2} + \frac{4}{3} \frac{k^2 (H')^3}{\gamma v^2} + 3 \frac{H^{5-q} \gamma v^2 H'}{\alpha} + \frac{9}{2} \frac{H^{(7-q)} \gamma^2 v^2}{\alpha} + 3 H^4 k^2 H' + \frac{9}{2} H^6 \gamma k^2 - \frac{9}{2} H^6 \gamma v^2 \]

5.1 Leading term

> f:=collect2(subs(H(t)=c1*t^n1,d2),t,exponents,
x->collect(x,[k,c1,v]));

\[ f := [6 n1, 4 n1 - 2, 5 n1 - 1, 3 n1 - 3, -n1 (-7 + q), 6 n1 - n1 q - 1] \]

The values of $n1$ that make these terms balance are

> nie:=balance(f,n1);

\[ n1e := [0, -1, \frac{2}{-3 + q}, \frac{1}{-2 + q}, \frac{3}{-4 + q}, -\frac{1}{q}] \]

These in turn are function of $q$, so we look whether there is a critical value of $q$

> qe:=balance(nie,q);

\[ qe := [1] \]

This shows that $q = 1$ delimits different behaviors of the solutions as it was
already known from investigation of equation (17) \[17\]. To be concise, in the
following only the case $q < 1$ will be considered. Further, sorting must be
done separately for each interval $(-\infty, 0), (0, 1), (1, 2), (2, 3), (3, 4), (4, \infty)$, so
we will restrict to the case $0 < q < 1$.

```maple
> n1ea:=sort(n1e,(a,b)->ord1(a,b,q=.5));

\[
n1ea := [\frac{-1}{q}, -1, \frac{3}{-4 + q}, \frac{2}{-3 + q}, \frac{1}{-2 + q}, 0]
\]
```

Now we look for the leading exponent within each interval.

```maple
l:=[n1ea[1]-1,op(n1ea),n1ea[-1]+1]:
l1:=subs(q=.5,l):
for i from 1 to nops(l1)-1 do
  u:=(l1[i]+l1[i+1])/2;
  tabla[i]:=[[l[i]<n1,n1<l[i+1]],
            op(1,sort(f,(a,b)->ord2(a,b,q=.5,n1=u)))]:
  print(tabla[i]);
od:
```

Thus we find that the leading exponent switch at $n1 = -1$ and $n1 = 2/(-3 + q)$, where terms balance. Let us start with $n1 = -1$.

```maple
> f:=collect2(subs(H(t)=c1*t^(-1),d2),t,exponents,
x->collect(x,[k,c1,v])):
> assume(q<1):
```

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Thus, $n_1 = -1$ and $c_1 = 2/(3\gamma)$ gives a leading behavior. Let us see the other case.

Thus $n_1 = 2/(-3 + q)$ and

$$c_1 = (-\frac{9}{4} \frac{\gamma^3 v^4 (9 - 6 q + q^2)}{k^2 \alpha (-\gamma + \gamma q - 2)^{3/2}})^{1/3}$$

give the leading behavior of another family of solutions that, for the sake of brevity, we will not pursue further in the step $M = 2$. No solution is found when $n_1$ is between the balancing values.

### 5.2 Second term for $n_1 = -1$

We start by expanding the terms with non integer exponents.
\[ f := \{-6, -2 + 4 n2, -4 + 2 n2, -7 + q, -3 + 3 n2, 6 n2, \]
\[-5 + q + 2 n2, -1 + 5 n2, -5 + n2, -6 + q + n2\]  

These are the balancing values of \( n_2 \)

\[ > \text{n2e} := \text{balance}(f, n2): \]
\[ > \text{assume}(q < 1): \]
\[ > \text{sort(n2e, (a, b) -> is(a < b))}: \]
\[ \{-2 + q, -\frac{3}{2} + \frac{q}{2}, -\frac{4}{3} + \frac{q}{3}, -\frac{5}{4} + \frac{q}{4}, -\frac{6}{5} + \frac{q}{5}, -\frac{7}{6} + \frac{q}{6}, -1, -\frac{1}{2} - \frac{q}{2}, -q\} \]

As we require \( n_2 > -1 \), only two balancing values are admissible \( n_2 = -q \) and \( n_2 = -\frac{1}{2} - \frac{q}{2} \). In the case \( n_2 = -q \)

\[ > f := \text{collect2(subs(n2=-q,d3), t, exponents,} \]
\[ > x -> \text{collect(x, [k, c1, c2, v, q]))}: \]
\[ > \text{assume}(q < 1): \]
\[ > \text{sort(f, (a, b) -> is(a < b))}: \]
\[ \{-7 + q, -6, -5 - q, -4 - 2 q, -3 - 3 q, -2 - 4 q, -1 - 5 q, -6 q\} \]

it turns out that \( e_2 = -6 \). From its coefficient

\[ > \_\text{coeff}[-6]: \]
\[ \text{expand}("/c1^3"): \]
\[ \text{simplify(subs(c1=2/(3*gamma), ")")}; \]
\[ \frac{4 v^2 (\alpha + 3^q \gamma^{(q+1)} c2 2^{(-q)} q - 6 3^{(q-1)} \gamma^{(q+1)} c2 2^{(-q)})}{\gamma^2 \alpha} \]

we find \( c2 = \frac{4\gamma^2}{\gamma(2-q)(3\gamma)^q} \). In the case \( n_2 = -\frac{1}{2} - \frac{q}{2} \)

\[ > f := \text{collect2(subs(n=1/2+q/2,d3), t, exponents,} \]
\[ > x -> \text{collect(x, [k, c1, c2, v, q]))}: \]
\[ > \text{assume}(q < 1): \]
\[ > \text{sort(f, (a, b) -> is(a < b))}: \]
\[ \{-7 + q, -\frac{13}{2} + \frac{q}{2}, -6, -\frac{11}{2} - \frac{q}{2}, -5 - q, -\frac{9}{2} - \frac{3}{2} q, -4 - 2 q, \]
\[ -\frac{7}{2} - \frac{5}{2} q, -3 - 3 q\} \]

we find \( e_2 = -13/2 + q/2 \). Its coefficient however only vanishes for \( c2 = 0 \).
5.3 Solutions

We collect here the solutions of eq. (17) found thus far, including those for \( q > 1 \) whose calculation has not been given in detail in this section. We give first the solutions with \( n_1 = -1 \).

Case 2 \(-q < n_2 < 1\):

\[
c_1 = \frac{2}{3\gamma} \frac{1}{\sqrt{2}v + 1}, \quad n_2 = \frac{\sqrt{2}(\gamma - 2)v + \gamma}{\gamma(\sqrt{2}v + 1)}
\]  

(18)

In this case, \((1/3) \left( 1/ \left( 1 + \sqrt{2} \right) \right) \leq c_1 < 2/3\), and \((1 - \sqrt{2}) / (1 + \sqrt{2}) \leq n < 1\). We are assuming that \( c_2 \neq 0 \).

Case \( n_2 = q < 1\):

\[
c_1 = \frac{2}{3\gamma}, \quad c_2 = \frac{\alpha}{(2 - q)\gamma} \left( \frac{2}{3\gamma} \right)^q
\]  

(19)

Case \( n_2 = 2 - q < 1\):

There are three subcases.

a)

\[
c_1 = \frac{2}{3\gamma} \frac{k^2}{k^2 - v^2}, \quad c_2 = -\frac{8k^{6-2q}(k^2 - v^2)^{q-3}v^4}{9\gamma\alpha q(2k^2 - v^2)} \left( \frac{3\gamma}{2} \right)^q
\]  

(20)

with \( k > v \).

b)

\[
c_1 = \frac{2}{3\gamma} \frac{1}{1 + \sqrt{2}v}
\]  

(21)

and \( c_2 \) a long expression of the form \( N/D \), with

\[
N = 8\sqrt{2}v^4 \left( \sqrt{2}v - 1 \right)^2 \left( 3\gamma/2 \right)^q \left( \sqrt{2}v + 1 \right)^{q-3}
\]
\[ D = 9\gamma q\alpha \left\{ 2\sqrt{2} [(q - 1) \gamma - 2] v^4 + \left[ \gamma (2k^2 - 1) (q - 1) + 4 (1 - k^2) \right] 2v^3 \\
+ \left[ -\sqrt{2} (1 + 2k^2) (q - 1) + 2\sqrt{2} (4k^2 - 1) \right] v^2 \\
+ \left[ \gamma (1 - 2k^2) (q - 1) - 4k^2 \right] v + \sqrt{2} (q - 1) k^2 \gamma \right\} \]

c) \[ c_1 = \frac{2}{3\gamma} \frac{1}{1 - \sqrt{2}v} \] (22)

with \( v < \sqrt{2} \) and \( c_2 \) a very long expression that is omitted here.

On the other hand, for \( n_1 = 2/(q - 3) \) we get

\[ c_1 = \left[ \frac{9\gamma^3 v^4 (9 - 6q + q^2)^{\frac{1}{q - 3}}}{4k^2\alpha (2 + \gamma - \gamma q)} \right] \]

where \( q < 1 \).

6 Anisotropic universe with a scalar field

Homogeneous anisotropic cosmological models with a self-interacting scalar field has been investigated recently to verify the generality of inflationary solutions. For this purpose it has been found convenient to cast the metric in the semiconformal form [4]. In these coordinates the Bianchi VI\(_0\) metric becomes

\[ ds^2 = e^{f(t)} (-dt^2 + dz^2) + G(t) \left( e^z dx^2 + e^{-z} dy^2 \right) \] (23)

A scalar field with exponential potential \( V = V_0 e^{k\phi} \) is interesting from the physical point of view because it arises in several particle theories, in the effective four-dimensional theories induced by Kaluza-Klein theories, including various higher-dimensional supergravity [18] and superstring [19] models [20–22]. It is also interesting from the mathematical point of view because it allows decoupling of the geometric and matter degrees of freedom [23]. Thus the whole evolution of the model is obtained from the solution of a third order ODE for \( G(t) \)

\[ \ddot{G}^2 - KG^2 - \dot{G} \dot{G} + \frac{1}{2} \ddot{G}G^2 + m^2 \dot{G} = 0; \] (24)
where $K = \frac{k^2}{4} - \frac{1}{2}$ and $m$ is an arbitrary integration constant. We will investigate here the singular behavior of this model, corresponding to $G(t) \to 0$ for $t \to t_0$. Then we write eq. (24) in a new worksheet.

\[ d := \text{diff}(G(t), t^2)^2 * G(t) - K * \text{diff}(G(t), t^2) * \text{diff}(G(t), t)^2 - \text{diff}(G(t), t^3) * \text{diff}(G(t), t) * G(t) + 1/2 * \text{diff}(G(t), t^2) * G(t)^2 + m^2 * \text{diff}(G(t), t^2) : \]

\[ \text{dview}(d, t, G); \]

\[(G''')^2 G - K G''' (G')^2 - G''' G' G + \frac{1}{2} G'' G^2 + m^2 G''\]

### 6.1 Leading term

Inserting the leading term in (24) we obtain the exponents

\[ f := \text{collect2} \left( \text{subs}(G(t) = c1 * t^n1, d), t, \text{exponents} \right) ; \]

\[ f := [3 n1 - 4, 3 n1 - 2, n1 - 2] \]

The values of $n1$ that make these terms balance are

\[ n1e := \text{balance}(f, n1) ; \]

\[ n1e := [0, 1] \]

and the leading exponent inside each interval is

\[ l1 := [-\infty, \text{op}(n1e), \infty] ; \]

for $i$ from 1 to $\text{nops}(n1e)+1$ do assume($l1[i]<n1,n1<l1[i+1]$);

\[ \text{tabla}[i] := \left( \left[ l1[i]<n1,n1<l1[i+1] \right], \text{op}(1, \text{sort}(f, (a, b) -> \text{is}(a<b))) \right) ; \]

\[ \text{print}(\text{tabla}[i]) ; \]

od:

\[ [[-\infty < n1, n1 < 0], 3 n1 - 4] \]

\[ [[0 < n1, n1 < 1], 3 n1 - 4] \]

\[ [[1 < n1, n1 < \infty], n1 - 2] \]

Thus the leading behavior changes at $n = 1$. Let us consider first that $n1 < 1$.

\[ \_\text{coeff}[3* n1 - 4] ; \]
We obtain \( n_1 = 1/K \) and \( c_1 \) remains free. This implies \( 1/K < 1 \), that is either \( 1/2 < K < 0 \) or \( K > 1 \). On the other hand, the coefficient for \( n_1 > 1 \)

> \texttt{coeff[n1-2];}

\[
m^2 c_1 n_1^2 - m^2 c_1 n_1
\]

shows that this case can occur only for \( m = 0 \). In such a case \( n_1 \) remains free. We reject the balancing value \( n_1 = 0 \) because it does not yield a singular behavior. We note however that \( G = c_1 \) is an exact solution. The other balancing value \( n_1 = 1 \) is quite special as \( G = c_1 t \) is also an exact solution with \( c_1 \) arbitrary.

6.2 Second term for \( n_1 = 1 \)

We obtain the exponents

> \texttt{f:=collect2(subs(G(t)=c1*t+c2*t^n2,d),t,exponents);}  
> \texttt{assume(1<n2);}  
> \texttt{sort(f,(a,b)->is(a<b));}

\[
[n_2 - 2, 2 n_2 - 3, 3 n_2 - 4, n_2, 2 n_2 - 1, 3 n_2 - 2]
\]

and we find that \( e_2 = n_2 - 2 \) for \( n_2 > 1 \), with a coefficient

> \texttt{coeff[n2-2];}

\[
m^2 c_2 n_2^2 - m^2 c_2 n_2 - c_2 n_2^3 c_1^2 - K c_2 n_2^2 c_1^2 + K c_2 n_2 c_1^2 - 2 c_2 n_2 c_1^2 + 3 c_2 n_2^2 c_1^2
\]

Thus we find \( n_2 = \frac{m^2}{c_2} + 2 - K \), \( c_2 \) remains free and the constraint \( \frac{m^2}{c_2} + 2 - K > 1 \) must be satisfied.

6.3 Second term for \( n_1 = 1/K \)

In this case we obtain the exponents

> \texttt{f:=collect2(subs(G(t)=c1*t^(1/K)+c2*t^n2,d),t,exponents);}
\[
f := \left[ \frac{2 - 2K + n^2K}{K}, \frac{1 - 2K + n^2K}{K}, n^2 - 2, \frac{1 - 4K + 2n^2K}{K}, \frac{3n^2 - 2}{K}, \frac{2 - 4K + n^2K}{K}, \frac{-3 + 2K}{K}, \frac{-1 + 2K}{K} \right]
\]

The values of \( n^2 \) that make them balance

> \( n^2e := \text{balance}(f, n^2); \)

\[
n^2e := \left[ \frac{1}{3K}, -\frac{K - 1}{K}, 0, 1, \frac{1 + 2K}{3K}, \frac{1}{K}, 1, 2 + 3K, -\frac{1 + 2K}{K}, 1 - 3^2K, \frac{3}{K}, -\frac{1 + 2K}{K}, \frac{1}{K}, 1 + 2K, -\frac{1}{K} \right]
\]

depend on \( K \), and in turn the values of \( K \) that make them balance are

> \( Ke := \text{balance}(n^2e, K); \)

> \( \text{sort}(Ke, (a, b) \rightarrow \text{is}(a < b)); \)

\[
[-3, -2, -\frac{3}{2}, -1, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}, 2, 3, 4]
\]

Of these, only the values in \([-1/2, 0)\) or \((1, \infty)\) are allowed. Now we could sort the balancing values of \( n^2 \) within each allowed interval of \( K \) where it must be taken into account that \( n^2 > 1/K \). Clearly this analysis is quite branched and we will not pursue it here. We just consider the case of the balancing value \( n^2 = 1 \).

> \( f := \text{collect2}(\text{subs}(G(t) = c1*t^{1/K} + c2*t, d), t, \text{exponents}); \)

\[
f := \left[ -\frac{2 + K}{K}, \frac{1}{K}, -\frac{3 + 2K}{K}, -\frac{1 + 2K}{K} \right]
\]

In this case the exponents depend on \( K \) with balancing values

> \( \text{balance}(f, K); \)

\([-1, 1]\)

Now we sort the exponents within each allowed interval

> \( \text{assume}(\text{-1/2} < K, K < 0); \)
> \( \text{sort}(f, (a, b) \rightarrow \text{is}(a < b)) \)
\[-\frac{3 + 2K}{K}, -\frac{2 + K}{K}, -\frac{1 + 2K}{K}, \frac{1}{K}\]

> assume(K>1):
> sort(f,(a,b)->is(a<b));

\[-\frac{-1 + 2K}{K}, -\frac{-3 + 2K}{K}, -\frac{-2 + K}{K}, \frac{1}{K}\]

For \(K > 1\) the coefficient is

> _coeff[\(-(1+2*K)/K]\);

\[-\frac{c2^2 c1}{K^3} + \frac{m^2 c1}{K^2} - \frac{m^2 c1}{K} + 3 \frac{c1 c2^2}{K^2} - 3 \frac{c1 c2^2}{K} + c2^2 c1\]

and we get a solution: \(n2 = 1\) and \(c2 = \pm \frac{\sqrt{K m}}{K-1}\).

7 Conclusions

We have shown that approximations of solutions of nonlinear ordinary differential equations relevant to cosmology can be obtained with the help of CAS in the form of generalized power asymptotic expansions, without much effort. For this purpose we have sketched an algorithm to make these calculations by an iterative process.

The implementation of these calculations were made in Maple V Release 4. This work environment is quite productive as it allows interactive exploration as well as a powerful programming language. The set already written of procedures to collect terms, obtain exponents and coefficients work in satisfactory way. The next step that remains to be coded is sorting of exponents. This task is more complex as it involves branching of options and has the further difficulty that arises in the weakness of the current version of the \texttt{assume} facility.

From a more theoretical point of view, it deserves to be investigated how the complexity of the algorithm increases with the order of iteration, and whether this growth puts an effective limit to practical calculations. In such a case it would be interesting to know whether more efficient algorithms can be devised.

Also it would be interesting to know whether expansions of solutions as shown in this paper, can give information about the integrability of the equation.
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