Constructive Field Theory and Applications: Perspectives and Open Problems

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Abstract
In this paper we review many interesting open problems in mathematical physics which may be attacked with the help of tools from constructive field theory. They could give work for future mathematical physicists trained with the constructive methods well within the 21st century.

I Introduction
Constructive field theory started in the 70’s as a program to study the existence and properties of non-trivial particular interacting field theories, those with simple Lagrangians. Indeed it was not obvious at that time that such structures (fulfilling a suitable set of axioms such as the Wightman axioms [Wi]) existed at all. In three decades, not only models of field theory, first superrenormalizable, then just renormalizable, have been built and to some extent analyzed, but also the methods and techniques developed in constructive field theory have been applied to a wide variety of problems outside the initial scope of the program. Constructive techniques have been applied to equilibrium statistical mechanics, particularly to the study of critical phenomena, and to disordered systems. They have been introduced successfully in the analysis of many Fermions models, such as those of condensed matter. They have also inspired and renewed studies in classical mechanics, and in time dependent problems, such as non-equilibrium phenomena. It is no longer easy to draw the contours of this nebula. However the initial group of people who pioneered constructive field theory in the early 70’s, together with
a second and now a third generation of bright students, although working in very different domains nowadays, still share in common a certain number of features. They are usually faithful to long-term programs, maybe even stubborn! Beyond the adjective “constructive”, they share in common with the “constructive” trend in mathematics, advocated for instance by Kronecker, a taste for explicit solutions, together with explicit bounds, rather than abstract existence theorems. In principle this means that when translated into algorithms, and implemented on computers, the “constructive” analysis of a physical model can lead to quantities computed with better precision and better controlled accuracy.

In this review for the special issue of JMP of year 2000, we will be brief on the successes of the past and refer to the existing books. Instead we will focus on the open problems, conjectures and challenges that lie ahead in a subject that could now perhaps be called “Constructive Physics” rather than “Constructive Field Theory”, and which remains characterized by the rigorous treatment of models issued from physics by hard analytic methods. This paper does not contain any equation; its purpose is to entice the reader to choose among the challenging problems just mentioned, and then to go for the references, where the formalisms for the corresponding problems are more precisely defined. Finally we apologize for the fact that the list of open problems emphasized inevitably reflects our personal biases and interests.

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II Constructive Field Theory

This is the historical core of the theory, and in spite of some spectacular successes, it remains largely a mine of open problems.

We recall that using the Euclidean functional integral approach, models of non-trivial interacting field theories have been built over the past thirty years, which satisfy Osterwalder-Schrader’s axioms, hence in turn have

\footnote{For classical references on constructive field theory, see [Er] [Si] [GJ1]; for reviews on constructive renormalization, and the problem of asymptotic completeness, see respectively [R] and [Ia]; and for the most recent Proceedings on constructive field theory, see [CP].}
continuation to Minkowski space that satisfy Wightman axioms [Wi][OS][Z].
Such models are unfortunately yet restricted to space-time dimensions 2 or 3 but they include now both the first wave of superrenormalizable models, such as $P(\phi)_2$ [GJS1][GJ1][Si], $\phi_4^3$ [GJ2][Fe][FO][MS] or the Yukawa model in 2 and 3 dimensions, as well as just renormalizable models such as the massive Gross-Neveu model in two dimensions, or $GN_2$ [GK1,FMRS,DR1]. Most of these models have been built in the weak coupling regime, using expansions such as the cluster and Mayer expansions; the harder models require multiscale versions of these expansions, reshuffled according to the renormalization group philosophy.

In most cases the relationship of the non-perturbative construction to the perturbative one has been elucidated, the non-perturbative Green’s functions being the Borel sum of the corresponding perturbative expansion [EMS], [MS], [FMRS].

We identify and discuss several main areas for future progress:

II.1 Asymptotic Freedom, Four dimensional Models

By “Coleman’s theorem”, renormalizable asymptotically free field theories in dimension 4 must involve non-Abelian gauge fields. However these fields lead to dreadful infrared problems, e.g. confinement. Therefore no theory satisfying the flat infinite volume Wightman’s axioms in dimension 4 (the historic goal of constructive field theory) has been constructed yet. However in a finite volume Balaban succeeded in proving ultraviolet stability of the effective action for non-Abelian lattice gauge theories (after an arbitrarily large number of renormalization group steps) [Ba1]. We mention also a less advanced attempt to construct this ultraviolet limit in a particular non-standard gauge, using gauge symmetry breaking cutoffs [MRS]. This situation is not completely satisfying. We list among open problems:

II.1.1 Non-linear sigma model

Construct the ultraviolet limit of the two dimensional $O(N)$ non linear sigma model, which is a well behaved asymptotically free bosonic field theory (see e.g. [GK2] for construction of the hierarchical version of the model). It is quite irritating that we still do not have such a construction: many experts
in the field tried it without success. The infrared mass generation has been obtained recently (see [K][IT]).

II.1.2 Yang-Mills

Construct the Yang Mills 4 correlation functions in a finite volume and a standard gauge (such as the Landau gauge). This presumably implies a front attack on the Gribov problem.

Simplify and rewrite Balaban’s results on the lattice gauge theory. This is no small task since references in [Ba1] total hundreds of pages...

II.1.3 $\phi^4$

Elucidate the nature of $\phi^4$ renormalized perturbation series, proving for instance that renormalons do exist (see [DFR]). Can one prove in full generality that its ultraviolet limit is trivial [A1][Fr][R]? 

II.1.4 Supersymmetric and Topological Field theory

Develop constructive versions of the now-popular supersymmetric field theories, and topological field theories: develop a constructive understanding of Montonen-Olive duality in $N = 4$ SUSY Yang-Mills$_4$, and of Seiberg-Witten duality for $N = 2$ SUSY Yang-Mills$_4$.

Since these issues represent rather formidable challenges, it may be worth to attack first some two dimensional problems [W1]: polynomial (hence superrenormalizable) $N = 2$ supersymmetric field theories allow the construction of interesting quantities sensitive to topological change [W2]; then the more difficult Wess-Zumino-Witten model at large parameter $k$, and the Calabi-Yau models, are field theories in which the field takes value in a non-trivial manifold as target space.

According to the point of view of Witten [W1], developing constructive theory of functional field theoretic integrals for these models could attract more mathematicians to field theory and may speed up the constructive programs in other more traditional areas too.
II.2 Strong Coupling or Low Temperature Results

In the regime of strong coupling or low temperature, there are less results. Contour expansions have been used to prove the existence of the $\phi^4_2$ phase transition [GJS2]. Many results have been obtained for models with an ultra-violet cutoff, i.e. models inspired by field theory but which are truly statistical mechanics models. For instance the phase transition and non-perturbative mass generation has been proved for the $GN_2$ model with an ultraviolet cutoff [KMR], and continuous symmetry breaking (in dimensions greater or equal to 3) has been studied with renormalization group techniques [Ba2]. An interesting challenge would be to glue this non-trivial low-temperature analysis to the construction of the ultraviolet limit when it is possible. Since the weak-coupling expansion for the ultraviolet limit is somewhat in contradiction with the low-temperature expansion, this should be done first for models with an other auxiliary small parameter, such as $N$-vector models at large $N$, where the $1/N$ expansion can complete in the infrared the small coupling expansion in the ultraviolet.

Therefore the first problems to attack in that direction could be:

II.2.1 Constructive Dimensional Transmutation

Glue the ultraviolet analysis of the Gross-Neveu model [GK1][FMRS][DR1] with the non-perturbative mass generation of the same model [KMR] at large $N$, to obtain the first example of so called dimensional transmutation.

One could also glue the ultraviolet construction of $\phi^4_3$ to the infrared continuous symmetry breaking analysis of [Ba2] to control the large $N$-component $\phi^4_3$ model in the continuous symmetry broken phase without an ultraviolet cutoff.

II.2.2 Constructive conformal field theory

Develop rigorous links between conformal field theory in dimension two and constructive field theory: along this line, the first significant result should be to prove that the phase transition of $\phi^4_3$ is in the same universality class (i.e. has the same critical exponents) than the Ising 2 phase transition. This could be later extended to $P(\phi_2)$ models with more vacua and Potts models.

More generally it should be nice to develop contact points between conformal field theory in two dimension [BPZ], the theory of integrable systems,
which relies more onto algebraic tools, and constructive theory which relies more on analysis. For instance there exist integrable lattice models of the ADE type which scale to conformal field theories of the [BPZ] classification at their critical point [Pa]; can we find a model which can be built with constructive methods for every such conformal theory?

II.3 Scattering, asymptotic completeness and Minkowski space

Develop phase space analysis and non-perturbative methods for field theory that work directly in Minkowski space. This should lead to first proofs of asymptotic completeness (see [Ia]) for quantum field theory models.

The easiest model in this direction may be the Gross-Neveu model since it is “purely perturbative”: although it is just-renormalizable, hence has a worse ultraviolet power counting than $\phi_2^4$, it can be written purely as a reshuffled perturbation series [DR1], so that in order to build it directly in Minkowski space one does not need to develop a theory of functional integration in Minkowski space, based on stationary phase analysis; it should be enough to simply develop a renormalization group analysis around the mass shell hyperbola, which should resemble the renormalization group around the Fermi surface of section IV.

II.4 Other problems

Complete in full detail the construction of the first non renormalizable field theory, the Gross-Neveu model in three dimensions and large number of components [dCFdVMS][dV]. Construct other models of this type, for instance the corresponding regime of the sine-Gordon model.

Develop constructive field theory in curved space-time.

Make better contact with the $C^*$-algebra approach (sometimes called axiomatic field theory). We refer to [BV] for a point of view on the renormalization group in algebraic field theory.
III  Equilibrium Statistical Mechanics

In this category we already mentioned the constructive study of continuous symmetry breaking [Ba2] and dimensional transmutation in subsection II.2.1.

III.1 Coulomb gases

After proofs of Debye screening and Kosterlitz Thouless (KT) phase transition, this area (together with the study of the sine-Gordon and Thirring model) remains very active among constructivists. For background in this subject we refer to [BM]. Here is a list of open problems for which we thank D. Brydges:

III.1.1

Find a direct proof of convergence of the Mayer expansion for dipoles at low activity (which does not use a cluster expansion). The dipole-dipole interaction should be smoothed at short distance so that it is stable.

III.1.2

Convergence of the Mayer expansion at low activity for the KT phase of the 2D Coulomb gas. This is harder than the previous problem, and involves presumably an effective analysis of that gas in terms of multipoles.

III.1.3

Prove exponential screening in the 2D Coulomb gas at not particularly small temperatures (down to the KT transition?).

III.1.4

Control the correlations at the KT transition.

III.1.5

Are the transitions between $\beta_{KT}$ and $1/2\beta_{KT}$ in the D=2 Coulomb gas visible in any correlations?
III.2 Disordered Systems

The proof by J. Imbrie that the three dimensional Random Field Ising model develops symmetry breaking at zero temperature [Im] remains a beautiful example in which rigorous constructive methods have solved a controversial physical issue. Disordered systems are common in nature (conductors or semi-conductors with structural defects or doping, spin glasses, real glasses, granular or porous media, etc...). They pose particularly challenging mathematical problems, and we review here only a few of them.

III.2.1 Anderson model of an electron in a random potential

Hereafter the main results of [FS2] and followers on the localization regime at high disorder or out of the continuous spectrum of the free Hamiltonian, we feel that the main area for open problems is the weak-coupling phase. It is expected that the Anderson 2 model $\text{And}_2$ is always localized but with exponentially small localization length when the coupling constant tends to 0. A proof of this statement through constructive field theory methods seems to require first the proof of decay of a single averaged Green’s function $<G_\pm>$ on a scale proportional to $\lambda^2$, the square of the coupling constant (this also controls the density of states of the system). A constructive analysis is under way, based on sector decomposition (as in [FMRT1]), a random matrix analogy, and Ward identities [MPR1]. Then the real study of localization involves the study of $<G_+G_->$, and requires a resummation of leading ladders plus a study of the associated “Goldstone mode”. Therefore the whole program is certainly as hard to complete as the BCS2 program for interacting fermions defined in section IV below.

In dimension 3 one expects the small coupling phase to be diffusive, hence the system should undergo an “Anderson Mott” phase transition from insulating to conducting at a certain critical coupling. To prove this one should again first control the decay on a length scale of $\lambda^{-2}$ of a single averaged Green’s function: this seems much harder than in dimension 2 essentially for the same reason that BCS3 is much harder than BCS2: the random potential viewed as a random matrix between angular directions is not of the usual type (i.e. is not independent identically distributed, see [MPR2]). After that difficulty has been solved, however, the task of controlling the square modulus of the Green’s function $<G_\pmG_->$ should be easier than in dimension...
2, since we expect diffusion rather than localization.

III.2.2 Constructive Study of Spin Glasses.

This area is not familiar to me, but it contains certainly very challenging problems which do not often belong to the culture of main stream constructivists. To “solve” in a constructive sense models like the Sherrington-Kirkpatrick model is certainly an ambitious goal for the future. One should understand the correct notion of states for the model (in particular in connexion with the ultrametric structure conjectured by the physicists). It would be fascinating to also understand in a more precise and constructive sense the replica symmetry breaking tool of Parisi. Recently an explicit formula for the partition function at low temperature has been obtained [Kou].

III.3 Polymers

Polymers and self-avoiding walks (SAW) are related to zero-component field theories and have been often studied by constructive theorists. Among the established results are the works [A2][BS][HS][L][IM] which explore the behavior of these systems in 4 dimensions or more. Scaling dimensions of SAW with specific interactions in two dimensions can also be studied rigorously through conformal invariance. See [D] for a recent result in this area using quantum gravity methods.

Here is a list of open problems:

III.3.1

Polymers with partly attractive interactions: prove that they scale to Brownian motion in \( d > 4 \), at least if the interaction is stable and small. Existence of transitions when the interaction is stable but attractive?

III.3.2

\( d = 4 \) Self avoiding walk: find new proofs that the end-to-end distance has an exponent of \( 1/2 \) with log corrections.
III.3.3
Find new proofs that random walk in random environment scales to Brownian motion in $d > 2$.

III.3.4
Prove anything at all about the expected end-to-end distance of a self avoiding walk in $d < 4$. Is it greater than that of simple random walk? Does it have an exponent? If it does, is the exponent different from $1/2$?

III.3.5
Prove that the scaling limit of True Self Avoiding Walk in $d > 2$ is Brownian Motion.

III.4 Interfaces, Wulff construction
The constructive study of functional integrals associated to interacting surfaces (Polyakov’s functional integral) is much harder than the ordinary random walk. The importance of these functional integrals (for instance in string theory) nevertheless justify that constructivists should get interested into them.

III.4.1 Wetting
The study of interfaces is more advanced for solid-on solid models than for real models such as the Ising model in the two phase regime.

An important open problem is to construct the non-trivial renormalization group fixed point for a solid-on-solid model of an interface with two competing exponentials. This should be doable at least in the regime where this fixed point is closed to a Gaussian one, thanks to a small parameter in the rate of the two exponentials [BHL].

An other important problem is to give rigorous meaning to the Wulff construction for such models [DM].

There are also perturbative results on the renormalization surfaces interacting e.g. with a single impurity [DDG] which one would like to connect to a constructive analysis.
III.5 Non Equilibrium Statistical Mechanics

The study of situations far from equilibrium made a big conceptual progress with the introduction of the SRB steady states [SRB]. A typical recent rigorous result in this domain is the fluctuation theorem of Gallavotti and Cohen [GC] on entropy production. See [G1] for a discussion of this result.

We remark that the quantum non-equilibrium statistical mechanics remain a widely open subject. An important long term goal for constructive theory should be, after many body systems are better understood and the main problems of the next section IV are solved, to develop the corresponding theory near equilibrium, namely to put on a firm mathematical microscopic analysis the Kubo formula and the Joule effect, and more generally transport theory.

IV Condensed Matter

In the constructive theory of condensed matter, the main event of the past was the adaptation of renormalization group techniques to models with a Fermi surface [BG][FT1-2][FMRT1-5].

IV.1 Interacting fermions in 2 dimensions

- In two dimensions there is a well-defined strategy which should lead ultimately to the complete construction of the BCS2 model, namely the control of the BCS phase at zero temperature [FMRT5]. There exists already a control of the model until a scale where the coupling constant becomes small but of order unity, which proves that any transition temperature has to be exponentially small in the coupling [DR2-3]. Then the zone where the coupling constant is of order unity should be under control through some kind of $1/N$ expansion, where here $N$ is no longer an ad hoc parameter but is the effective number of angular directions on the Fermi surface at the scale considered [FMRT2]; this expansion is not easy to write, and one may start with a simpler model which has only quartic interaction at the BCS scale, like in [KMR]; then one has to glue this analysis to the previous one, hence treat the corrections to the quartic effective action. Finally one has to control the distance scales much longer that the BCS scale, where the physics is governed by the infrared singularity of the Goldstone boson. Here the
key tool should be a multiscale renormalization group analysis that relies on Ward identities [FMRT4] like in [Ba2]. This is a long and difficult program (even by the constructive standards!).

IV.2 Interacting fermions in 3 dimensions (BCS3)

In dimension 3 the BCS program is less advanced. Although perturbative power counting for the Fermi liquid is independent of dimension, and the Goldstone boson problem is easier in 3+1 than in 2+1 dimensions, the initial regime (the equivalent of [FMRT1]-[DR2]) is harder to control for BCS3 because the momentum conservation laws are not as restrictive in 3 than in 2 dimensions: vertices can be non-planar, or “twisted” in 3 dimensions [FMRT3]. The only rigorous result so far is that the radius of convergence of perturbation theory in a slice around the Fermi surface is independent of the distance of that slice to the singularity [MR]). To find the analog of [FMRT1][DR2], namely that the sum of all “convergent contributions” to the theory is analytic in the coupling constant remains in our opinion a major challenge of constructive theory.

IV.3 Bose-Einstein condensation

Develop the theory of Bose-Einstein condensation. This can be viewed as a piece of the previous BCS program where the bosons are Cooper pairs, i.e. bound states of Fermions, or as an independent program if the bosons are given from the start (see [Be]).

IV.4 Non-Spherical Surfaces; Hubbard Model

IV.4.1

Treat non-spherical surfaces. After the work on the renormalization of convex surfaces [FKLT][FST], treat surfaces with flat pieces and/or singular points: the regular Hubbard model at half-filling on a square lattice has both these features. Until now, this has to rely for at least some part on numerical rather than analytical tools.
IV.4.2

Develop a rigorous non-perturbative mean-field theory for condensed matter, i.e. develop the non-perturbative version of the dynamical mean-field or $d = \infty$ limit of models such as the Hubbard model: this dynamical mean-field model is really a one dimensional theory with a self-consistent condition, but without an explicit action [GKKR].

IV.5 Quasi-periodic potentials, quasi-crystals

Develop the mathematical theory of conduction in quasi-crystals.

In one dimension it is believed that fermions develop a Charge Density Wave instability at small temperature with period equal to the inverse of the density. An interesting goal is to prove the generation of such CDW in a system of interacting Fermions. In this direction an expansion for interacting Fermions with an incommensurate external potential satisfying a proper diophantine conditions was shown to converge in [Ma]; this is a first bridge on the gap between solid state physics and classical mechanics (the KAM theorem below), since it amounts to solve a small denominator problem “with loops”. It would be nice to extend this bridge to other models, in particular in greater dimensions.

V Classical Mechanics

Again this is an area I do not fell too competent to review and my remarks will be brief. Contributions from constructive theorists have been devoted in particular to the area of the KAM theorem, where in particular the Italian school around G. Gallavotti has developed the renormalization group approach to the KAM theorem, but also to classical and quantum chaos, and to classical mechanics in random environment.

V.1 KAM theory

Invariant tori in Hamiltonian systems analytically close to integrable systems can be written as perturbative Lindstedt series in the perturbation parameter. A direct proof of the convergence of such series, done by Eliasson [E], can be also obtained in the quantum field theory language, using multiscale
analysis as in the renormalization group, and cancellations \([G2]\). (These cancellations can be interpreted as Ward identities related to translation invariance \([BGK1]\)).

An interesting open question is what happens to Lindstedt series in the non-analytic case. Moser, by using Nash theorem, proved that KAM tori exist also in this case, with suitable conditions, but in general they are not analytic. In \([BGGM]\) analyticity was nevertheless proved for a class of non-analytic pertubations by direct analysis of new cancellations in the Lindstedt series. In more general cases where the summability of Lindstedt series may fail it is an open question to know if some extended notion of summability, like Borel summability (quite frequent in quantum field theory), may still hold.

Another set of problems concern Arnold’s diffusion. In a priori unstable systems, a key quantity is the splitting which is the determinant of a certain matrix whose elements are series whose first order is exponentially small, but the others are not. However the determinant itself is exponentially small due to cancellations. Using Dyson equation for classical mechanics, Arnold diffusion can be proved in certain a priori stable systems \([GGM]\), but the same question is open in general a priori stable systems such as those arising from celestial mechanics.

## V.2 Classical Chaos, Turbulence

Of course the solution of Navier-Stokes equation and their scaling laws remain a challenge, pretty much as it was at the beginning of the century. In the fully developed turbulence, experimentally observed deviations from Kolmogorov’s scaling of the velocity correlators signal a non-Gaussian character of the velocity distributions at short distances, called intermittency. Such intermittency, or deviations from Kolmogorov’s scaling, has been more or less understood in the particular case of the passive advection of a scalar quantity (temperature, or density of a pollutant) by a random velocity field \([GK3][BGK2]\). However an explanation of the origin of intermittency in the general case of developed turbulence remains one of the main open problems of theoretical hydrodynamics.
V.3 Quantum Chaos

Here let us mention the results [CRR] on the Gutzwiller trace formula, that one would like to extend to longer time evolution. A main challenge is to put the heuristic connection between quantum chaos and the spectra of random matrices on a mathematically rigorous footing.

V.4 Partial Differential Equations and Renormalization

I would like to cite the work of Bricmont and Kupiainen on random walks in a random environment [BK1], and more generally the application of renormalization group methods to partial differential equations [BK2].

The major open problems listed in [BK2] are the study of stability of fronts in dissipative equations; extension of renormalization group methods to hyperbolic equations; the study of invariant measures for dynamical systems called Coupled Map Lattices [Ka], and of nonequilibrium “phase transitions” in which these invariant measures change as the coupling is varied.

VI Improving Constructive Techniques

In this section we would like to gather some list of mathematical techniques which are quite general, so that they ought to be useful not only for a single problem but for many models in different branches of physics.

VI.1 Renormalization Group

A central problem in constructive theory is to simplify and further rationalize the various techniques which allow to perform rigorous Renormalization Group computations. The inductive version of the renormalization group itself has been better formalized by Brydges and coworkers [Br]; the multiscale phase space expansions which are some kind of expanded solution of the renormalization group induction have been also recently formalized more explicitly [AR], and also recast using wavelets [Bat]. These efforts should be continued if we want the rigorous approach to become part of the regular cursus of field theory. An open problem which could be mentioned along
these lines is to find an inductive rigorous constructive renormalization which would be as simple as Polchinski’s induction for perturbative renormalization [P]: even for Fermions, this remains an open problem [S2][DR1].

VI.2 Gluing together various expansions

Techniques to glue together different expansions or different regimes of the renormalization group (e.g. small coupling/1/N coupling) should be developed. This is a condition to treat many interesting models with “non-perturbative” phases. Somewhat like the geometric description of non-trivial manifolds requires to glue several local charts together, the construction of non-trivial models with non-perturbative effects requires to develop some experience in such gluing operations (see II.2.1).

In a similar vein it is interesting to combine together several expansion techniques which are usually treated separately. For instance one can study the Many Body Models of section IV with the additional complication of random or quasi-periodic environment (see [Ma] for a one-dimensional example).

VI.3 Symmetries and Ward identities

Many difficult constructive problems involve symmetries which are crucial to their understanding (gauge symmetries, supersymmetry, replica symmetry). One would like to have more general methods to quotient out or break these symmetries, and develop a more general theory of non-perturbative Ward identities.

VI.4 Non-integer dimensions

Non-integer dimensions is an interesting perturbative tool (e.g. for the renormalization of non-Abelian gauge fields or for the $\epsilon$ expansion in statistical mechanics) that has no constructive analog. One should understand why and build the non-perturbative theory of functional integration in non-integer dimensions. This is a long-term difficult goal, perhaps related to non-commutative geometry, where ordinary space is lost and the ordinary algebra of functions is replaced by a non-commutative algebra.
VI.5 Random Matrices

Random matrices is a powerful tool for a wide range of physical problems, from nuclear physics to quantum chaos, localization, quantized gravity and M-theory. The classical theory is the theory of independent identically distributed random matrices, and relates them to orthogonal polynomials and integrable PDE’s [M]. An important progress may come from the understanding of random matrices with non-independent coefficients. In the point of view of Voiculescu [V], the Wigner law for independent identically distributed matrices model is the non-commutative analog of Gaussian integration. Constructivists, just as they developed the theory of non-Gaussian functional integration, may therefore try to develop a more general theory of random matrices, including in particular those with constraints of geometric origin (see e.g. [MPR2] for an example). This could presumably be very useful for the physics in spatial dimensions higher than 2 (condensed matter, scattering, phase transitions).

VII String Theory and Conclusion

When the constructive field theory program began in the 60’s, field theory was the prominent candidate for a fundamental theory of nature at the microscopic level (although it did not include quantization of gravity). Today the main stream of theoretical physics holds the view that field theory is only an effective theory and that superstring or M-theory is the best candidate for a fundamental global theory of nature, a “theory of everything”. Even if on a philosophical level the very existence of such a final theory is dubious, it is certainly a fascinating dream. So in order to remain faithful to its initial quest, one could ask whether constructivists should not join the efforts to find and build this TOE?

I would be tempted to adopt a rather cautious answer to this question, namely “Perhaps, but not yet”. There are three reasons for this cautious attitude.

- String theory or M-theory are mathematically very difficult: even the perturbative theory of superstring amplitudes contain enormous difficulties: a proof of finiteness e.g. of the 10-dimensional $E_8 \times E_8$ heterotic superstring amplitudes is a very difficult program in itself.
The theory is in such a state of rapid evolution that it is not clear what should really be built. In the recent years, the different models had a rather short life time before they were absorbed in a more general formalism. Under such circumstances, to launch a major constructive effort could be premature, since the model might be outdated well before the rigorous construction is completed.

The theory has not yet received direct experimental confirmation. We can at best hope for indirect hints, which may come in the next decades (spatial experiments such as those probing the background cosmic radiation, large cosmic rays detectors, new accelerators such as the LHC, etc... may select along various cosmological or high energy scenarii, and give indirect support to such or such models).

For these three reasons I do not think that time is ripe to launch today “constructive string theory”, as “constructive field theory” was launched by A. Wightman and followers in the 60’s.

To soften slightly these remarks, let me add that of course I consider string theory extremely important for the future of mathematical physics. Indeed string theory has not only been a very successful motivation to attract some of the best minds to theoretical physics and to lead them to brilliant insights; it has also opened up a new interface with mathematicians, mostly centered around geometry (differential, symplectic and algebraic geometry, mirror symmetry, quantum cohomology, knot theory, ...). However this rapidly growing interface is very different from the one opened in the past by constructive theory. Algebra and geometry dominate over analysis, and there are no longer precise programs centered around axioms; but various pieces of the theory and various cross-consistent results emerge progressively from this interaction between mathematicians and theoretical physicists.

In conclusion, although at the present stage I would still rather personally favor the applications of constructive field theory methods to well established physics, I would be happy, when some of the dust has settled, to see new generations of mathematical physicists attack in the constructive spirit the problem of building rigorously the high energy models that will emerge and survive in the coming century.

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