Uncertain Decisions Facilitate Better Preference Learning

Cassidy Laidlaw  
University of California, Berkeley  
cassidy_laidlaw@cs.berkeley.edu  
Stuart Russell  
University of California, Berkeley  
russell@cs.berkeley.edu

Abstract

Existing observational approaches for learning human preferences, such as inverse reinforcement learning, usually make strong assumptions about the observability of the human’s environment. However, in reality, people make many important decisions under uncertainty. To better understand preference learning in these cases, we study the setting of inverse decision theory (IDT), a previously proposed framework where a human is observed making non-sequential binary decisions under uncertainty. In IDT, the human’s preferences are conveyed through their loss function, which expresses a tradeoff between different types of mistakes. We give the first statistical analysis of IDT, providing conditions necessary to identify these preferences and characterizing the sample complexity—the number of decisions that must be observed to learn the tradeoff the human is making to a desired precision. Interestingly, we show that it is actually easier to identify preferences when the decision problem is more uncertain. Furthermore, uncertain decision problems allow us to relax the unrealistic assumption that the human is an optimal decision maker but still identify their exact preferences; we give sample complexities in this suboptimal case as well. Our analysis contradicts the intuition that partial observability should make preference learning more difficult. It also provides a first step towards understanding and improving preference learning methods for uncertain and suboptimal humans.

1 Introduction

The problem of inferring human preferences has been studied for decades in fields such as inverse reinforcement learning (IRL), preference elicitation, and active learning. However, there are still several shortcomings in existing methods for preference learning. Active learning methods require query access to a human; this is infeasible in many purely observational settings and may lead to inaccuracies due to the description-experience gap [1]. IRL is an alternative preference learning tool which requires only observations of human behavior. However, IRL suffers from underspecification, i.e. preferences are not precisely identifiable from observed behavior [2]. Furthermore, nearly all IRL methods require that the observed human is optimal or noisily optimal at optimizing for their preferences. However, humans are often systematically suboptimal decision makers [3], and accounting for this makes IRL even more underspecified, since it is hard to tell suboptimal behavior for one set of preferences apart from optimal behavior for another set of preferences [4].

IRL and preference learning from observational data are generally applied in situations where a human is acting under no uncertainty. Given the underspecification challenge, one might expect that adding in the possibility of uncertainty in decision making (known as partial observability) would only make preference learning more challenging. Indeed, Choi and Kim [5] and Chinaei and Chaib-Draa [6], who worked to apply IRL to partially observable Markov decision processes (POMDPs,
Decisions without uncertainty

(a) Should I quarantine a traveler with a 100% accurate negative test for a dangerous disease?

(b) Should a person with irrefutable evidence of and confession to a crime be convicted?

Decisions under uncertainty

Should I quarantine a traveler with some symptoms of a dangerous disease but no test results?

Should a person with circumstantial evidence of a crime be convicted?

Figure 1: One of our key findings is that decisions made under uncertainty can reveal more preferences than clear decisions. Here we give examples of decisions made with and without uncertainty.

(a) In the case without uncertainty, nobody would choose to quarantine the traveler, so we cannot distinguish between different people’s preferences. However, in the case with uncertainty, people might decide differently whether to quarantine the traveler depending on their preferences on the tradeoff between individual freedom and public health. This allows us to identify those preferences by observing decisions. (b) Similarly, observing decisions on whether to convict a person under uncertainty reveals preferences about the tradeoff between convicting innocent people and allowing criminals to go free.

where agents act under uncertainty), remarked that the underspecification of IRL combined with the intractability of POMDPs made for a very difficult task.

In this work, we find that, surprisingly, observing humans making decisions under uncertainty actually makes preference learning easier (see Figure 1). To show this, we analyze a simple setting, where a human decision maker observes some information and must make a binary choice. This is somewhat analogous to supervised learning, where a decision rule is chosen to minimize some loss function over a data distribution. In our formulation, the goal is to learn the human decision maker’s loss function by observing their decisions. Often, in supervised learning, the loss function is simply the 0-1 loss. However, humans may incorporate many other factors into their implicit “loss functions”; they may weight different types of mistakes unequally or incorporate fairness constraints, for instance. One might call this setting “inverse supervised learning,” but it is better described as inverse decision theory (IDT) [7, 8], since the objective is to reverse-engineer only the human’s decision rule and not any learning process used to arrive at it. IDT can be shown to be a special case of partially observable IRL (see Appendix B) but its restricted assumptions allow more analysis than would be possible for IRL in arbitrary POMDPs. However, we believe that the insights we gain from studying IDT should be applicable to POMDPs and uncertain decision making settings in general. We introduce a formal description of IDT in Section 3.

While we hope to provide insight into general reward learning, IDT is also a useful tool in its own right; even in this binary, non-sequential setting, human decisions can reveal important preferences. For example, during a deadly disease outbreak, a government might pass a law to quarantine individuals with a chance of being sick. The decision rule the government uses to choose who to quarantine depends on the relative costs of failing to quarantine a sick person versus accidentally quarantining an uninfected one. In this way, even human decisions where there is a “right” answer are revealing if they are made under uncertainty. This example could distinguish a preference for saving lives versus one for guaranteeing freedom of movement. These preferences on the tradeoff between costs of mistakes are expressed through the loss function that the decision maker optimizes.

In our main results on IDT in Section 4, we find that the identifiability of a human’s loss function is dependent on whether the decision we observe them making involves uncertainty. If the human faces sufficient uncertainty, we give tight sample complexity bounds on the number of decisions we must observe to identify their loss function, and thus preferences, to any desired precision (Theorem 4.2). On the other hand, if there is no uncertainty—i.e., the correct decision is always obvious—then we show that there is no way to identify the loss function (Theorem 4.11 and Corollary 4.12). Technically, we show that learning the loss function is equivalent to identifying a threshold function over the space of posterior probabilities for which decision is correct given an observation (Figure 2). This threshold can be determined to precision $\epsilon$ in $\Theta(1/(p_c\epsilon))$ samples, where $p_c$ is the probability density of posterior probabilities around the threshold. In the case where there is no uncertainty in the decision problem, $p_c = 0$ and we demonstrate that the loss function cannot be identified.
These results apply to optimal human decision makers—that is, those who completely minimize their expected loss. When a decision rule or policy is suboptimal, in general their loss function cannot be learned [4, 9]. However, we show that decisions made under uncertainty are also helpful in this case; under certain models of suboptimality, we can still exactly recover the human’s loss function.

We present two such models of suboptimality (see Figure 3). In both, we assume that the decision maker is restricting themselves to choosing a decision rule \( h \) in some hypothesis class \( \mathcal{H} \), which may not include the optimal decision rule. This framework is similar to that of agnostic supervised learning [10, 11], but solves the inverse problem of determining the loss function given a hypothesis class and decision samples. If the restricted hypothesis class \( \mathcal{H} \) is known, we show that the loss function can be learned similarly to the optimal case (Theorem 4.7). Our analysis makes a novel connection between Bayesian posterior probabilities and binary hypothesis classes. However, assuming that \( \mathcal{H} \) is known is a strong assumption; for instance, we might suspect that a decision maker is ignoring some data features but we may not know exactly which features. We formalize this case by assuming that the decision maker could be considering the optimal decision rule in any of a number of hypothesis classes in some family \( \mathcal{H} \). This case is more challenging because we may need to identify which hypothesis class the human is using in order to identify their loss function. We show that, assuming a smoothness condition on \( \mathcal{H} \), we can still obtain the decision maker’s loss function (Theorem 4.10).

We conclude with a discussion of our results and their implications in Section 5. We extend IDT to more complex loss functions that can depend on certain attributes of the data in addition to the chosen decision; we show that this extension can be used to test for the fairness of a decision rule under certain criteria which were previously difficult to measure. We also compare the implications of IDT for preference learning in uncertain versus clear decision problems. Our work shows that uncertainty is helpful for preference learning and suggests how to exploit this fact.

2 Related Work

Our work builds upon that of Davies [8] and Swartz et al. [7], who first introduced inverse decision theory. They describe how to apply IDT to settings in which a doctor makes treatment decisions based on a few binary test outcomes, but provide no statistical analysis. In contrast, we explore when IDT can be expected to succeed in more general cases and how many observed decisions are necessary to infer the loss function. We also analyze cases where the decision maker is suboptimal for their loss function, which are not considered by Davies or Swartz et al.

Inverse reinforcement learning (IRL) [2, 12, 13, 14, 15], also known as inverse optimal control, aims to infer the reward function for an agent acting in a Markov decision process (MDP). Our formulation of IDT can be considered as a special case of IRL in a partially observable MDP (POMDP) with two states and two actions (see Appendix B). Some prior work explored IRL in POMDPs [5, 6] by reducing the POMDP to a belief-state MDP and applying standard IRL algorithms. Our main purpose is not to present improvements to IRL algorithms; rather, we give an analysis of the difference between observable and partially observable settings for preference learning. We begin with the restricted setting of IDT but hope to extend to sequential decision making in the future. We also consider cases where the human decision maker is suboptimal, which previous work did not explore.

Performance metric elicitation (ME) aims to learn a loss function (aka performance metric) by querying a human [16, 17, 18]. ME and other active learning approaches [19, 20, 21, 22] require the ability to actively ask a user for their preference among different loss or reward functions. In contrast, IDT aims to learn the loss function purely by observing a decision maker. Active learning is valuable for some applications, but there are many cases where it is infeasible. Observed decisions are often easier to obtain than expert feedback. Also, active learning may suffer from the description-experience gap [1]; that is, it may be difficult to evaluate in the abstract the comparisons that these methods give as queries to the user, leading to biased results. In contrast, observing human decision making “in the wild” with IDT could lead to a more accurate understanding of human preferences.

Preference and risk elicitation aim to identify people’s preferences between different uncertain or certain choices. A common tool is to ask a person to choose between a lottery (i.e., uncertain payoff) and a guaranteed payoff, or between two lotteries, varying parameters and observing the resulting choices [23, 24, 25]. In our analysis of IDT, decision making under uncertainty can be
We formalize inverse decision theory using decision theory and statistical learning theory. Let the agent have chosen a decision rule (or hypothesis) \( R \in \mathcal{H} \) from some hypothesis class \( \mathcal{H} \) that assigns \( \hat{y} \in \{0, 1\} \) to each observation \( x \in \mathcal{X} \). Theorem 4.2 shows we can learn \( \hat{y} \) to precision \( \epsilon \) with \( m \geq O(1/(p_c \epsilon)) \) samples.

Fig. 2: A visualization of three settings for inverse decision theory (IDT), which aims to estimate \( \hat{y} \), the parameter of a decision maker’s loss function, given observed decisions \( \hat{y}_1, \ldots, \hat{y}_m \in \{0, 1\} \). Here, each decision \( \hat{y}_i \) is plotted against the probability \( q(x_i) = \mathbb{P}(Y = 1 \mid X = x_i) \) that the ground truth (correct) decision \( Y = 1 \) given the decision maker’s observation \( x_i \). Lemma 3.1 shows that an optimal decision rule assigns \( \hat{y}_i = 1 \{q(x_i) \geq \epsilon\} \).

(a) Uncertain decision
(b) Clear decision
(c) Suboptimal decision

Figure 2: A visualization of three settings for inverse decision theory (IDT), which aims to estimate \( \hat{y} \), the parameter of a decision maker’s loss function, given observed decisions \( \hat{y}_1, \ldots, \hat{y}_m \in \{0, 1\} \). Here, each decision \( \hat{y}_i \) is plotted against the probability \( q(x_i) = \mathbb{P}(Y = 1 \mid X = x_i) \) that the ground truth (correct) decision \( Y = 1 \) given the decision maker’s observation \( x_i \). Lemma 3.1 shows that an optimal decision rule assigns \( \hat{y}_i = 1 \{q(x_i) \geq \epsilon\} \).

CAST as a natural series of choices between lotteries. If we observe enough different lotteries, the decision maker’s preferences can be identified. On the other hand, if there is no uncertainty, then we only observe choices between guaranteed payoffs and there is little information to characterize preferences.

### 3 Problem Formulation

We formalize inverse decision theory using decision theory and statistical learning theory. Let \( \mathcal{D} \) be a distribution over observations \( X \in \mathcal{X} \) and ground truth decisions \( Y \in \{0, 1\} \). We consider an agent that receives an observation \( X \) and must make a binary decision \( \hat{Y} \in \{0, 1\} \). While many decision problems include more than two choices, we consider the binary case to simplify analysis. However, the results are applicable to decisions with larger numbers of choices; assuming irrelevance from independent alternatives (i.e., the independence axiom [26]), a decision among many choices can be reduced to binary choices between pairs of them. We generally assume that \( \mathcal{D} \) is fixed and known to both the decision maker and the IDT algorithm. Unless otherwise stated, all expectations and probabilities on \( X \) and \( Y \) are with respect to the distribution \( \mathcal{D} \).

We furthermore assume that the agent has chosen a decision rule (or hypothesis) \( h : \mathcal{X} \rightarrow \{0, 1\} \) from some hypothesis class \( \mathcal{H} \) that minimizes a loss function which depends only on the decision \( \hat{Y} = h(X) \) that was made and the correct decision \( Y \):

\[
\min_{h \in \mathcal{H}} \mathbb{E}_{(X, Y) \sim \mathcal{D}} [\ell(h(X), Y)].
\]

In general, the loss function \( \ell \) might depend on the observation \( X \) as well; we explore this extension in the context of fair decision making in Section 5.1. Assuming the formulation above, since \( Y, \hat{Y} \in \{0, 1\} \), we can write the loss function \( \ell \) as a matrix \( C \in \mathbb{R}^{2 \times 2} \) such that \( \ell(\hat{y}, y) = C_{\hat{y}y} \). We denote by \( R_C(h) = \mathbb{E}_{(X, Y) \sim \mathcal{D}} [\ell(h(X), Y)] \) the expected loss or “risk” of the hypothesis \( h \) with cost matrix \( C \). This cost matrix has four entries, but the following lemma shows that it effectively has only one degree of freedom.

**Lemma 3.1 (Equivalence of cost matrices).** Any cost matrix \( C = \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} \) is equivalent to a cost matrix \( C' = \begin{pmatrix} 0 & 1-c \\ c & 0 \end{pmatrix} \) where \( c = \frac{C_{00} - C_{01} - C_{10} - C_{11}}{C_{00} + C_{01} - C_{10} - C_{11}} \) as long as \( C_{10} + C_{01} - C_{00} - C_{11} \neq 0 \). That is, there are constants \( a, b \in \mathbb{R} \) such that \( R_C(h) = aR_{C'}(h) + b \) for all \( h \).

See Appendix A.1 for this and other proofs. Based on Lemma 3.1, from now on, we assume the cost matrix only has one parameter \( c \), which is the cost of a false positive; \( 1 - c \) is the cost of a
We aim to answer two questions about IDT. First, under what assumptions is the loss function identifiable? Second, if the loss function is identifiable, how large must the sample size be to estimate it to some precision with high probability? We adopt a framework similar to that of probably approximately correct (PAC) learning [27], and aim to calculate an estimate \( \hat{c} \) such that with probability at least \( 1 - \delta \), \( |\hat{c} - c| \leq \epsilon \). While PAC learning typically focuses on estimating parameters, we are interested in estimating the entire loss function.

4 Identifiability and Sample Complexity

We aim to answer two questions about IDT. First, under what assumptions is the loss function identifiable? Second, if the loss function is identifiable, how large must the sample size be to estimate it to some precision with high probability? We adopt a framework similar to that of probably approximately correct (PAC) learning [27], and aim to calculate an estimate \( \hat{c} \) such that with probability at least \( 1 - \delta \), \( |\hat{c} - c| \leq \epsilon \). While PAC learning typically focuses on estimating parameters, we are interested in estimating the entire loss function.

false negative. Intuitively, high values of \( c \) indicate a preference for erring towards the decision \( Y = 0 \) under uncertainty while low values indicate a preference for erring towards the decision \( Y = 1 \). Finally, we assume that making the correct decision is always better than making an incorrect decision, i.e. \( C_{00} < C_{10} \) and \( C_{11} < C_{01} \). This implies that \( 0 < c < 1 \).

We write \( \ell_c \) and \( R_c \) to denote the loss and risk functions using this loss parameter \( c \). Thus, we can formally define a binary decision problem:

**Definition 3.2 (Decision problem).** A (binary) decision problem is a pair \((D, c)\), where \( D \) is a distribution over pairs of observations and correct decisions \((X, Y) \in X \times \{0, 1\}\) and \( c \in (0, 1) \) is the loss parameter. The decision maker aims to choose a decision rule \( h : X \to \{0, 1\} \) that minimizes the risk \( R_c(h) = E_{(X,Y) \sim D} [\ell_c(h(X), Y)] \).

As a running example, we consider the decision problem where an emergency room (ER) doctor needs to decide whether to treat a patient for a heart attack. In this case, the observation \( X \) might consist of the patient’s medical records and test results; the correct decision is \( Y = 1 \) if the patient is having a heart attack and \( Y = 0 \) otherwise; and the made decision is \( \hat{Y} = 1 \) if the doctor treats the patient and \( \hat{Y} = 0 \) if not. In this case, a higher value of \( c \) indicates that the doctor places higher cost on accidentally treating a patient not having a heart attack, while a lower value of \( c \) indicates the doctor places higher cost on accidentally failing to treat a patient with a heart attack.

In inverse decision theory (IDT), our goal is to determine the loss function the agent is optimizing, which here is equivalent to the parameter \( c \). We assume access to the true distribution \( D \) of observations and labels and also a finite sample of observations and decisions \( S = \{(x_1, y_1), \ldots, (x_m, y_m)\} \) where \( x_i \sim D \) i.i.d. and the decisions are made according to the decision rule, i.e. \( y_i = h(x_i) \).

Some of our main results concern the effects on IDT of whether or not a decision is made under uncertainty. We now formally characterize such decision problems.

**Definition 3.3 (Decision problems with and without uncertainty).** A decision problem \((D, c)\) has no uncertainty if \( \mathbb{P}_{(X,Y) \sim D}(Y = 1 \mid X) \in \{0, 1\} \) almost surely. The decision problem has uncertainty otherwise.

That is, if it is always the case that, after observing and conditioning on \( X \), either \( Y = 1 \) with 100% probability or \( Y = 0 \) with 100% probability, then the decision problem has no uncertainty.
on test or prediction error, we instead focus on the estimation error for \( c \). This has multiple advantages. First, it allows for better understanding and prediction of human behavior across distribution shift or in unseen environments \[28\]. Second, there are cases where we care about the precise trade-off the decision maker is optimizing for; for instance, in the ER doctor example, there are guidelines on the tradeoff between different types of treatment errors and we may want to determine if doctors’ behavior aligns with these guidelines \[3\]. Third, if the decision maker is suboptimal for their loss function (explored in Sections 4.2 and 4.5), we may not want to simply replicate the suboptimal decisions, but find a better decision rule according to the loss function.

We consider three settings where we would like to estimate \( c \), illustrated in Figure \[9\]. First, we assume that the decision maker is perfectly optimal for their loss function. This is similar to the framework of Swartz et al. \[7\]. However, moving beyond their analysis, we present properties necessary for identifiability and sample complexity rates. Second, we relax the assumption that the decision maker is optimal, and instead assume that they only consider a restricted set of hypotheses \( \mathcal{H} \) which is known to us. Finally, we remove the assumption that we know the hypothesis class that the decision maker is considering. Instead, we consider a family of hypothesis classes; the decision maker could choose the optimal decision rule within any class, which is not necessarily the optimal decision rule across all classes.

4.1 Optimal decision maker

First, we assume that the decision maker is optimal. In this case, the form of the optimal decision rule is simply the Bayes classifier \[29\].

**Lemma 4.1 (Bayes optimal decision rule).** An optimal decision rule \( h \) for a decision problem \((D, c)\) is given by \( h(x) = 1 \{ q(x) \geq c \} \) where \( q(x) = P_{(X,Y) \sim D}(Y = 1 \mid X = x) \) is the posterior probability of class 1 given the observation \( x \).

That is, any optimal decision rule corresponds to a threshold function on the posterior probability \( q(x) \), where the threshold is at the loss parameter \( c \). Thus, the strategy for estimating \( c \) from a sample of observations and decisions is simple. For each observation \( x_i \), we calculate \( q(x_i) \). Then, we choose any \( \hat{c} \) such that \( q(x_i) \geq \hat{c} \Leftrightarrow \hat{y}_i = 1 \); that is, \( \hat{c} \) is consistent with the observed data. From statistical learning theory, we know that a threshold function can be PAC learned in \( O(\log(1/\delta)/\epsilon) \) samples. However, such learning only guarantees low *prediction error* of the learned hypothesis. We need stronger conditions to ensure that \( \hat{c} \) is close to the true loss function parameter \( c \). The following theorem states conditions which allow estimation of \( c \) to arbitrary precision.

**Theorem 4.2 (IDT for optimal decision maker).** Let \( \epsilon > 0 \) and \( \delta > 0 \). Say that there exists \( p_{\epsilon} > 0 \) such that \( P(q(X) \in [c, c + \epsilon]) \geq p_{\epsilon} \epsilon \) and \( P(q(X) \in [c - \epsilon, c]) \geq p_{\epsilon} \epsilon \). Let \( \hat{c} \) be chosen to be consistent with the observed decisions as stated above, i.e. \( q(x_i) \geq \hat{c} \Leftrightarrow \hat{y}_i = 1 \). Then \( |\hat{c} - c| \leq \epsilon \) with probability at least \( 1 - \delta \) as long as the number of samples \( m \geq \frac{\log(2/\delta)}{p_{\epsilon} \epsilon} \).

The parameter \( p_{\epsilon} \) can be interpreted as the approximate probability density of \( q(X) \) around the threshold \( c \). For instance, the requirements of Theorem 4.2 are satisfied if the random variable \( q(X) \) has a probability density of at least \( p_{\epsilon} \) on the interval \([c - \rho, c + \rho]\) for some \( \rho \geq \epsilon \); the requirements of Theorem 4.2 are more general to allow for cases when \( q(X) \) does not have a density. The lower the density \( p_{\epsilon} \), and thus the probability of observing decisions close to the threshold \( c \), the more difficult inference becomes. Because of this, Theorem 4.2 requires that the decision problem has uncertainty. If the decision problem has no uncertainty according to Definition 3.3, then \( q(X) \in \{0, 1\} \) always, i.e. the distribution of posterior probabilities has mass only at 0 and 1. In this case, \( p_{\epsilon} = 0 \) for small enough \( \epsilon \) and Theorem 4.2 cannot be applied. In fact, as we show in Section 4.4, it is impossible to tell what the true loss parameter \( c \) when the decision problem lacks uncertainty. Figure 2(a-b) illustrates these results.

4.2 Suboptimal decision maker with known hypothesis class

Next, we consider cases where the decision maker may not be optimal with respect to their loss function. Our model of suboptimality is that the agent only considers decision rules within some hypothesis class \( \mathcal{H} \), which may not include the optimal decision rule. This formulation is similar to that of agnostic PAC learning \[10, 11\]. It can also be considered a case of a restricted “choice set” as defined in the preference learning literature \[30, 31\]. It can encompass many types of irrationality
or suboptimality. For instance, one could assume that the decision maker is ignoring some of the features in \( x \); then \( \mathcal{H} \) would consist of only decision rules depending on the remaining features. In the ER doctor example, we might assume that \( \mathcal{H} \) consists of decision rules using only the patient’s blood pressure and heart rate; this models a suboptimal doctor who is unable to use more data to make a treatment decision.

While there are many possible models of suboptimality, this one has distinct advantages for preference learning with IDT. One alternative model is that the decision maker has small excess risk, i.e. \( \mathcal{R}_\epsilon(h) \leq \mathcal{R}_\epsilon(h^*) + \Delta \) for some small \( \Delta \) where \( h^* \) is the optimal decision rule. However, this definition precludes identifiability even in the infinite sample limit (see Appendix B). Another form of suboptimality could be that the decision maker chooses a decision rule to minimize a surrogate loss rather than the true loss. However, we show in Appendix B that for reasonable surrogate losses this is no different from minimizing the true loss. A final alternative model of suboptimality is that the human is noisily optimal; this assumption underlies models like Boltzmann rationality or the Shephard-Luce choice rule \([32, 26, 33, 14]\). However, these models assume stochastic decision making and also cannot handle systematically suboptimal humans.

In this section we begin by assuming that the restricted hypothesis class \( \mathcal{H} \) is known; this requires some novel analysis but the resulting identifiability conditions and sample complexity are very similar to the optimal case in Section 4.1. In the next section, we consider cases where we are unsure about which restricted hypothesis class the decision maker is considering.

**Definition 4.3.** A hypothesis class \( \mathcal{H} \) is monotone if for any \( h, h' \in \mathcal{H} \), either \( h(x) \geq h'(x) \) \( \forall x \in \mathcal{X} \) or \( h(x) \leq h'(x) \) \( \forall x \in \mathcal{X} \).

**Definition 4.4.** The optimal subset of a hypothesis class \( \mathcal{H} \) for a distribution \( D \) is defined as

\[
\text{opt}_D(\mathcal{H}) = \{ h \in \mathcal{H} | \exists c \text{ such that } h \in \arg \min_{h \in \mathcal{H}} \mathcal{R}_c(h) \}
\]

In this section, we consider hypothesis classes whose optimal subsets are monotone. That is, changing the parameter \( c \) has to either flip the optimal decision rule’s output for some observations from 0 to 1, or flip some decisions from 1 to 0. It cannot both change some decisions from 0 to 1 and some from 1 to 0. This assumption is mainly technical; many interesting hypothesis classes naturally have monotonic optimal subsets. Any hypothesis class formed by thresholding a function is monotone, i.e \( \mathcal{H} = \{ h(x) = 1 \{ f(x) \geq b \} \mid b \in \mathbb{R} \} \). Also, the set of decision rules based on a particular subset of the observed features satisfies this criterion, since optimal decision rules in this set are thresholds on the posterior probability that \( Y = 1 \) given the subset of features.

For hypothesis classes with monotone optimal subsets, we can prove properties that allow for similar analysis to that we introduced in Section 4.1. Let \( h_c \) denote a decision rule which is optimal for loss parameter \( c \) in hypothesis class \( \mathcal{H} \). That is, \( h_c \in \arg \min_{h \in \mathcal{H}} \mathcal{R}_c(h) \). A key lemma allows us to define a value similar to the posterior probability we used for analyzing the optimal decision maker.

**Lemma 4.5 (Induced posterior probability).** Let \( \text{opt}_D(\mathcal{H}) \) be monotone and define

\[
\overline{\mathcal{R}}_\mathcal{H}(x) \triangleq \sup \left\{ \{c \in [0, 1] \mid h_c(x) = 1\} \cup \{0\} \right\} \quad \text{and} \quad \underline{\mathcal{R}}_\mathcal{H}(x) \triangleq \inf \left\{ \{c \in [0, 1] \mid h_c(x) = 0\} \cup \{1\} \right\}.
\]

Then for all \( x \in \mathcal{X} \), \( \overline{\mathcal{R}}_\mathcal{H}(x) = \underline{\mathcal{R}}_\mathcal{H}(x) \). Define the induced posterior probability of \( \mathcal{H} \) as

\[
\overline{\mathcal{R}}_\mathcal{H}(x) = q_\mathcal{H}(x).
\]

**Corollary 4.6.** Let \( h_c \) be any optimal decision rule in \( \mathcal{H} \) for loss parameter \( c \). Then for any \( x \in \mathcal{X} \), \( h_c(x) = 1 \) if \( q_\mathcal{H}(x) > c \) and \( h_c(x) = 0 \) if \( q_\mathcal{H}(x) < c \).

Using Lemma 4.5, the problem of IDT again reduces to learning a threshold; this time, any optimal classifier in \( \mathcal{H} \) is a threshold function on the induced posterior probability \( q_\mathcal{H}(X) \), as shown in Corollary 4.6. Thus, to estimate \( \hat{c} \), we calculate an induced posterior probability \( q_\mathcal{H}(x_i) \) for each observation \( x_i \) and choose any estimate \( \hat{c} \) such that \( q_\mathcal{H}(x_i) \geq \hat{c} \Leftrightarrow \hat{y}_i = 1 \). This allows us to state a theorem equivalent to Theorem 4.2 for the suboptimal case.

**Theorem 4.7 (Known suboptimal decision maker).** Let \( \epsilon > 0 \) and \( \delta > 0 \), and let \( \text{opt}_D(\mathcal{H}) \) be monotone. Say that there exists \( p_\epsilon > 0 \) such that \( \mathbb{P}(q_\mathcal{H}(X) \in [c, c + \epsilon]) \geq p_\epsilon \) and \( \mathbb{P}(q_\mathcal{H}(X) \notin [c - \epsilon, c]) \geq p_\epsilon \). Let \( \hat{c} \) be chosen to be consistent with the observed decisions, i.e. \( q_\mathcal{H}(x_i) \geq \hat{c} \Leftrightarrow \hat{y}_i = 1 \). Then \( |\hat{c} - c| \leq \epsilon \) with probability at least \( 1 - \delta \) as long as the number of samples \( m \geq \frac{\log(2/\delta)}{p_\epsilon \epsilon} \).
4.3 Suboptimal decision maker with unknown hypothesis class

We now analyze the case when the decision maker is suboptimal but we are not sure in what manner. We model this by considering a family of hypothesis classes \( \mathcal{H} \). We assume that the decision maker considers one of these hypothesis classes \( \mathcal{H} \in \mathcal{H} \) and then chooses a rule \( h \in \arg\min_{h \in \mathcal{H}} R_c(h) \). This case is more challenging because we may need to identify \( \mathcal{H} \) to identify \( c \).

One natural family \( \mathcal{H} \) consists of hypothesis classes which depend only on some subset of the features:

\[
\mathcal{H}_{\text{feat}} \triangleq \{ \mathcal{H}_S \mid S \subseteq \{1, \ldots, n\} \} \quad \text{where} \quad \mathcal{H}_S \triangleq \left\{ h(x) = f(x_S) \mid f : \mathbb{R}^{|S|} \rightarrow \{0, 1\} \right\}
\]

(1)

where \( x_S \) denotes only the coordinates of \( x \) which are in the set \( S \). This models a situation where we believe the decision maker may be ignoring some features, but we are not sure which features are being ignored. Another possibility for \( \mathcal{H} \) is thresholded linear combinations of the features in \( x \), i.e.

\[
\mathbb{H}_{\text{linear}} \triangleq \{ \mathcal{H}_w \mid w \in \mathbb{R}^n \} \quad \text{where} \quad \mathcal{H}_w \triangleq \left\{ h(x) = 1\{w^T x \geq b\} \mid b \in \mathbb{R} \right\}.
\]

In this case, we assume that the decision maker chooses some weights \( w \) for the features arbitrarily but then thresholds the combination optimally. This could model the decision maker under- or overweighting certain features, or also ignoring some (if \( w_j = 0 \) for some \( j \)).

In the high pressure and hectic environment of the ER example, we might assume that the doctor is using only a few pieces of data to decide whether to treat a patient. Here, \( \mathcal{H}_{\text{feat}} \) would consist of a hypothesis class with decision rules that depend only on blood pressure and heart rate, a hypothesis class with decision rules that rely on these and also on an ECG, and so on. The difficulty of this setting compared to that of Section 4.2 is that the doctor could be using an optimal decision rule within any of these hypothesis classes. Thus, we may need to identify what data the doctor is using in their decision rule in order to identify their loss parameter \( c \).

Estimating the loss parameter \( c \) in the unknown hypothesis class case requires an additional assumption on the family of hypothesis classes \( \mathcal{H} \), in addition to the monotonicity assumption from Section 4.2.

**Definition 4.8.** Consider a family of hypothesis classes \( \mathcal{H} \). Let \( h \in \mathcal{H} \in \mathcal{H} \) and \( \mathcal{H} \in \mathcal{H} \). Then the minimum disagreement between \( h \) and \( \mathcal{H} \) is defined as \( MD(h, \mathcal{H}) \triangleq \inf_{\mathcal{H} \in \mathcal{H}} P(h(X) \neq h(X)) \).

**Definition 4.9.** A family of hypothesis classes \( \mathcal{H} \) and hypothesis \( h_c \in \mathcal{H} \) such that \( h_c \in \arg\min_{h \in \mathcal{H}} R_c(h) \) is \( \alpha \)-MD-smooth if \( \alpha \) is monotone for every \( \mathcal{H} \in \mathcal{H} \) and

\[
\forall \mathcal{H} \in \mathcal{H} \quad \forall \mathcal{H}' \in (0, 1) \quad MD(h_c, \alpha \cdot MD(h_c, \mathcal{H}')) \leq (1 + \alpha|c' - c|)MD(h_c, \alpha \cdot \mathcal{H}').
\]

While MD-smoothness is not particularly intuitive at first, it is necessary in some cases to ensure identifiability of the loss parameter \( c \). We present a case in Appendix D.2 where a lack of MD-smoothness precludes identifiability.

**Theorem 4.10 (Unknown suboptimal decision maker).** Let \( \epsilon > 0 \) and \( \delta > 0 \). Suppose we observe decisions from a decision rule \( h_c \), which is optimal for loss parameter \( c \) in hypothesis class \( \mathcal{H} \in \mathcal{H} \). Let \( h_c \) and \( \mathbb{H} \) be \( \alpha \)-MD-smooth. Furthermore, assume that there exists \( p_c > 0 \) such that for any \( \rho \leq c \), \( P(q_{\mathcal{H}}(X) \in (c, c + \rho)) \geq p_c \rho \) and \( P(q_{\mathcal{H}}(X) \in (c - \rho, c)) \geq p_c \rho \). Let \( d \geq \text{VCdim}(\bigcup_{\mathcal{H} \in \mathcal{H}} \mathcal{H}) \) be an upper bound on the VC-dimension of the union of all the hypothesis classes in \( \mathcal{H} \).

Let \( \hat{h}_c \in \arg\min_{h \in \mathcal{H}} R_c(h) \) be chosen to be consistent with the observed decisions, i.e. \( \hat{h}_c(x_i) = \hat{y}_i \) for \( i = 1, \ldots, m \). Then \( |\hat{c} - c| \leq \epsilon \) with probability at least \( 1 - \delta \) as long as the number of samples \( m \geq \hat{O}\left(\left(\frac{d + \log(1/\delta)}{\epsilon^2}\right) p_c^{-1}\right)\).

Theorem 4.10 requires more decision samples to guarantee low estimation error \( |\hat{c} - c| \). Unlike Theorems 4.2 and 4.7, the number of samples needed grow with the square of the desired precision \( 1/\epsilon^2 \). There is also a dependence on the VC-dimension of the hypothesis classes \( \mathcal{H}_c \in \mathcal{H} \), since we are not sure which one the decision maker is considering.

Since our results in this section are highly general, it may be difficult to see how they apply to concrete cases. In Appendix E, we explore the specific case of IDT in the unknown hypothesis class setting for \( \mathcal{H}_{\text{feat}} \) as defined in (1). We give sufficient conditions for MD-smoothness and show that the sample complexity grows only logarithmically with \( n \), the dimension of the observation space \( X \), if the decision maker is relying on a sparse set of features.
4.4 Lower bounds

Is there any algorithm which can always determine the loss parameter $c$ to precision $\epsilon$ with high probability using fewer samples than required by Theorems 4.2 and 4.7? We show that the answer is no: our previously given sample complexity rates are minimax optimal up to constant factors. We formalize this by considering any generic IDT algorithm, which we represent as a function $c : (X \times \{0, 1\})^m \rightarrow (0, 1)$. The algorithm maps the sample of observations and decisions $S$ to an estimated loss parameter $\hat{c}(S)$. The algorithm also takes as input the distribution $D$ and in the suboptimal cases the hypothesis class $H$ or family of hypothesis classes $\mathcal{H}$, but we leave this dependence implicit in our notation. First, we consider the optimal (Theorem 4.2) and known suboptimal (Theorem 4.7) cases; since these are nearly identical, we focus on the optimal case.

**Theorem 4.11 (Lower bound for optimal decision maker).** Fix $0 < \epsilon < 1/4$, $0 < \delta \leq 1/2$, and $0 < p_c \leq 1/5$. Then for any IDT algorithm $\hat{c}(\cdot)$, there exists a decision problem $(D, c)$ satisfying the conditions of Theorem 4.7 such that $m \leq \frac{\log(1/\delta)}{8p_c\epsilon}$ implies that $\mathbb{P}(|\hat{c}(S) - c| \geq \epsilon) > \delta$.

**Corollary 4.12 (Lack of uncertainty precludes identifiability).** Fix $0 < \epsilon < 1/4$ and suppose a decision problem $(D, c)$ has no uncertainty. Then for any IDT algorithm $\hat{c}(\cdot)$, there is a loss parameter $c$ and hypothesis class $H$ such that for any sample size $m$, $\mathbb{P}(|\hat{c}(S) - c| \geq \epsilon) \geq 1/2$.

Corollary 4.12 shows that a lack of uncertainty in the decision problem means that no algorithm can learn the loss parameter $c$ to a non-trivial precision with high probability. Thus, uncertainty is required for IDT to learn the loss parameter $c$. Since $c$ represents the preferences of the decision maker, decisions made under certainty do not reveal precise preference information. In Appendix D we explore lower bounds for the unknown suboptimal case (Section 4.3 and Theorem 4.10).

5 Discussion

Now that we have thoroughly analyzed IDT, we explore its applications, implications, and limitations.

5.1 IDT for fine-grained loss functions with applications to fairness

First, we discuss an extension of IDT to loss functions which depend not only on the chosen decision $\hat{Y} = h(X)$ and the ground truth $Y$, but on the observation $X$ as well. In particular, we extend the formulation of IDT from Section 3 to include loss functions which depend on the observations via a “sensitive attribute” $A \in\mathcal{A}$. We denote the value of the sensitive attribute for an observation $x$ by $a(x)$. We again assume that the decision maker chooses the optimal decision rule for this extended loss function:

$$h \in \arg\min_h \mathbb{E}_{\mathcal{D}(X,Y) \sim D}[\ell(h(X), Y, a(X))].$$

(2)

This optimal decision rule $h \in H$ is equivalent to a set of decision rules for every value of $A$, each of which is chosen to minimize the conditional risk for observations with that attribute value:

$$h(x) = h_a(x) \quad \text{where} \quad h_a \in \arg\min \mathbb{E}_{\mathcal{D}(X,Y) \sim D}[\ell(h(X), Y, a) \mid a(X) = a].$$

In this formulation, each attribute-specific decision rule $h_a$ minimizes an expected loss which only depends on the made and correct decisions $h(X)$ and $Y$ over a conditional distribution. Thus, we can split a sample of decisions into samples for each value of the sensitive attribute and perform IDT separately. This will result in a loss parameter estimate $\hat{c}_a$ for each value of $a$.

Once we have estimated loss parameters for each value of $A$, we may ask if the decision maker is applying the same loss function across all such values, i.e. if $c_a = c_{a'}$ for any $a, a' \in \mathcal{A}$. If the loss function is not identical for all values of $A$, i.e. if $c_a \neq c_{a'}$, then one might conclude that the decision maker is unfair or discriminatory against observations with certain values of $A$. For instance, in the ER example, we might be concerned if the doctor is using different loss functions for patients with and without insurance. Concepts like these have received extensive treatment in the machine learning fairness literature, which studies criteria for when a decision rule can be considered “fair.” One such fairness criterion is that of group calibration, also known as sufficiency [34, 35, 56]:

**Definition 5.1.** A decision rule $h : \mathcal{X} \rightarrow \{0, 1\}$ for a distribution $(X,Y) \sim D$ satisfies the group calibration/sufficiency fairness criterion if there is a function $r : \mathcal{X} \rightarrow \mathbb{R}$ and threshold $t \in \mathbb{R}$ such that $h(x) = I\{r(x) \geq t\}$ and $r$ satisfies $Y \perp \perp A \mid r(X)$. 


Testing for group calibration is known to be difficult because of the problem of infra-marginality [37]. While complex Bayesian models have previously been used to perform a “threshold test” for group calibration, we can use IDT to directly test this criterion in an observed decision maker:

**Lemma 5.2 (Equal loss parameters imply group calibration).** Let \( h \) be chosen as in (2) where 
\[
\ell(\hat{y}, y, a) = \begin{cases} 
    c_a & \text{if } \hat{y} = 1 \text{ and } y = 0, \\
    1 - c_a & \text{if } \hat{y} = 0 \text{ and } y = 1, \\
    0 & \text{otherwise}. 
\end{cases}
\]

Then \( h \) satisfies group calibration (sufficiency) if \( c_a = c_{a'} \) for every \( a, a' \in A \).

Conversely, if there exist \( a, a' \in A \) such that \( c_a \neq c_{a'} \) and \( \mathbb{P}(q(X) \in (c_a, c_{a'})) > 0 \), then \( h \) does not satisfy group calibration.

If we can estimate \( c_a \) for a decision rule \( h \) for each \( a \in A \), then Lemma 5.2 allows us to immediately determine if \( h \) satisfies sufficiency. The minimax guarantees on the accuracy of IDT may make this approach more attractive than the Bayesian threshold test in many scenarios.

### 5.2 Suboptimal decision making with and without uncertainty

We have so far compared the effect of decisions made with and without uncertainty on the identifiability of preferences; here, we argue that uncertainty also allows for much more expressive models of suboptimality in decision making. In decisions made with certainty, suboptimality can generally only take two forms: either the decision maker is noisy and sometimes randomly makes incorrect decisions, or the decision maker is systematically suboptimal and always makes the wrong decision. Neither seems realistic in the ER doctor example: we would not expect the doctor to randomly choose not to treat some patients who are clearly having heart attacks, and certainly not expect them to never treat patients having heart attacks. In contrast, the models of suboptimality we have presented for uncertain decisions allow for much more rich and realistic forms of suboptimal decision making, like ignoring certain data or over-/under-weighting evidence. We expect that there are similarly more rich forms of suboptimality for uncertain sequential decision problems.

### 5.3 Limitations and future work

While this study sheds significant light on preference learning for uncertain humans, there are some limitations that may be addressed by future work. First, while we assume the data distribution \( D \) of observations \( X \) and ground truth decisions \( Y \) is known, this is rarely satisfied in practice. However, statistics is replete with methods for estimating properties of a data distribution given samples from it. Such methods are beyond the scope of this work, which focuses on the less-studied problem of inferring a decision maker’s loss function. Our work also lacks computational analysis of algorithms for performing IDT. However, such algorithms are likely straightforward; we decide to focus on the statistical properties of IDT, which are more relevant for preference learning in general. Finally, we assume in this work that the decision maker is maximizing expected utility (EU), or equivalently minimizing expected loss. In reality, human decision making may not agree with EU theory; alternative models of decision making under uncertainty such as prospect theory are discussed in the behavioral economics literature [38]. Some work has applied these models to statistical learning [39], but we leave their implications for IDT to future work.

### 6 Conclusion and Societal Impact

We have presented an analysis of preference learning for uncertain humans through the setting of inverse decision theory. Our principle findings are that decisions made under uncertainty can reveal more preference information than obvious ones; and, that uncertainty can alleviate underspecification in preference learning, even in the case of suboptimal decision making. We hope that this and other work on preference learning will lead to AI systems which better understand human preferences and can thus better fulfill them. However, improved understanding of humans could also be applied by malicious actors to manipulate people or invade their privacy. Additionally, building AI systems which learn from human decisions could reproduce racism, sexism, and other harmful biases which are widespread in human decision-making. Despite these concerns, understanding human preferences is important for the long-term positive impact of AI systems. Our work shows that uncertain decisions can be a valuable source of such preference information.
Acknowledgments and Disclosure of Funding

We would like to thank Kush Bhatia for valuable discussions, Meena Jagadeesan, Sam Toyer, and Alex Turner for feedback on drafts, and the NeurIPS reviewers for helping us improve the clarity of the paper. This research was supported by the Open Philanthropy Foundation. Cassidy Laidlaw is also supported by a National Defense Science and Engineering Graduate (NDSEG) Fellowship.

References

[1] Ralph Hertwig and Ido Erev. The Description–Experience Gap in Risky Choice. Trends in Cognitive Sciences, 13(12):517–523, December 2009. ISSN 1364-6613. doi: 10.1016/j.tics.2009.09.004. URL https://www.sciencedirect.com/science/article/pii/S1364661309002125

[2] Andrew Y. Ng and Stuart J. Russell. Algorithms for Inverse Reinforcement Learning. In ICML, volume 1, page 2, 2000.

[3] Sendhil Mullainathan and Ziad Obermeyer. A Machine Learning Approach to Low-Value Health Care: Wasted Tests, Missed Heart Attacks and Mis-predictions. Technical report, National Bureau of Economic Research, 2019.

[4] Stuart Armstrong and Sören Mindermann. Occam’s Razor is Insufficient to Infer the Preferences of Irrational Agents. Advances in Neural Information Processing Systems, 31, 2018. URL https://proceedings.neurips.cc/paper/2018/hash/d89a66c7c80a29b1bdbab0f2a1a944af8-Abstract.html

[5] Jaedeug Choi and Kee-Eung Kim. Inverse Reinforcement Learning in Partially Observable Environments. Journal of Machine Learning Research, 12(21):691–730, 2011. ISSN 1533-7928. URL http://jmlr.org/papers/v12/choi11a.html

[6] Hamid R. Chinaei and Brahim Chaib-Draa. An Inverse Reinforcement Learning Algorithm for Partially Observable Domains with Application on Healthcare Dialogue Management. volume 1, pages 144–149, December 2012. doi: 10.1109/ICMLA.2012.31.

[7] Richard J Swartz, Dennis D Cox, Scott B Cantor, Kalatu Davies, and Michele Follen. Inverse Decision Theory. Journal of the American Statistical Association, 101(473): 1–8, March 2006. ISSN 0162-1459. doi: 10.1198/016214506000000998. URL https://amstat.tandfonline.com/doi/abs/10.1198/016214505000000998 Publisher: Taylor & Francis.

[8] Kalatu Davies. Inverse Decision Theory with Medical Applications. PhD thesis, Rice University, Houston, Texas, May 2005. URL https://scholarship.rice.edu/handle/1911/18756

[9] Rohin Shah, Noah Gundotra, Pieter Abbeel, and Anca Dragan. On the Feasibility of Learning, Rather than Assuming, Human Biases for Reward Inference. In International Conference on Machine Learning, pages 5670–5679. PMLR, 2019.

[10] David Haussler. Decision Theoretic Generalizations of the PAC Model for Neural Net and Other Learning Applications. Information and Computation, 100(1):78–150, September 1992. ISSN 0890-5401. doi: 10.1016/0890-5401(92)90010-D. URL https://www.sciencedirect.com/science/article/pii/089054019290010D

[11] Michael J. Kearns, Robert E. Schapire, and Linda M. Sellie. Toward Efficient Agnostic Learning. Machine Learning, 17(2):115–141, November 1994. ISSN 1573-0565. doi: 10.1007/BF00993468. URL https://doi.org/10.1007/BF00993468

[12] Pieter Abbeel and Andrew Y. Ng. Apprenticeship Learning via Inverse Reinforcement Learning. In Proceedings of the twenty-first international conference on Machine learning, page 1, 2004.

[13] Deepak Ramachandran and Eyal Amir. Bayesian Inverse Reinforcement Learning. In IJCAI, volume 7, pages 2586–2591, 2007.

[14] Brian D. Ziebart, Andrew L. Maas, J. Andrew Bagnell, and Anind K. Dey. Maximum Entropy Inverse Reinforcement Learning. In Aaai, volume 8, pages 1433–1438. Chicago, IL, USA, 2008.

[15] Justin Fu, Katie Luo, and Sergey Levine. Learning Robust Rewards with Adversarial Inverse Reinforcement Learning. arXiv preprint arXiv:1710.11248, 2017.
[16] Gaurush Hiranandani, Shant Boodaghians, Ruta Mehta, and Oluwasanmi Koyejo. Performance Metric Elicitation from Pairwise Classifier Comparisons. arXiv:1806.01827 [cs, stat], January 2019. URL http://arxiv.org/abs/1806.01827.

[17] Gaurush Hiranandani, Shant Boodaghians, Ruta Mehta, and Oluwasanmi Koyejo. Multiclass Performance Metric Elicitation. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d’Alché-Buc, E. Fox, and R. Garnett, editors, Advances in Neural Information Processing Systems 32, pages 9356–9365. Curran Associates, Inc., 2019. URL http://papers.nips.cc/paper/9133-multiclass-performance-metric-elicitation.pdf.

[18] Gaurush Hiranandani, Harikrishna Narasimhan, and Oluwasanmi Koyejo. Fair Performance Metric Elicitation. arXiv:2006.12732 [cs, stat], November 2020. URL http://arxiv.org/abs/2006.12732.

[19] Erdem Biyik and Dorsa Sadigh. Batch Active Preference-Based Learning of Reward Functions. In Proceedings of The 2nd Conference on Robot Learning, pages 519–528. PMLR, October 2018. URL https://proceedings.mlr.press/v87/biyik18a.html.

[20] Sören Mindermann, Rohin Shah, Adam Gleave, and Dylan Hadfield-Menell. Active Inverse Reward Design. arXiv:1809.03060 [cs, stat], November 2019. URL http://arxiv.org/abs/1809.03060.

[21] Erdem Biyik, Malayandi Palan, Nicholas C. Landolfi, Dylan P. Losey, and Dorsa Sadigh. Asking Easy Questions: A User-Friendly Approach to Active Reward Learning. arXiv:1910.04365 [cs], October 2019. URL http://arxiv.org/abs/1910.04365.

[22] Kush Bhatia, Peter L. Bartlett, Anca D. Dragan, and Jacob Steinhardt. Agnostic Learning with Unknown Utilities. arXiv:2104.08482 [cs, stat], April 2021. URL http://arxiv.org/abs/2104.08482.

[23] Michele Cohen, Jean-Yves Jaffray, and Tanios Said. Experimental Comparison of Individual Behavior Under Risk and Under Uncertainty for Gains and for Losses. Organizational Behavior and Human Decision Processes, 39(1):1–22, February 1987. ISSN 0749-5978. doi: 10.1016/0749-5978(87)90043-4. URL https://www.sciencedirect.com/science/article/pii/0749597887900434.

[24] Charles A. Holt and Susan K. Laury. Risk Aversion and Incentive Effects. The American Economic Review, 92(5):1644–1655, 2002. ISSN 0002-8282. URL https://www.jstor.org/stable/3083270. Publisher: American Economic Association.

[25] Tamás Csermely and Alexander Rabas. How to Reveal People’s Preferences: Comparing Time Consistency and Predictive Power of Multiple Price List Risk Elicitation Methods. Journal of Risk and Uncertainty, 53(2):107–136, 2016. ISSN 0895-5646. doi: 10.1007/s11166-016-9247-6.

[26] R. Duncan Luce. The Choice Axiom After Twenty Years. Journal of Mathematical Psychology, 15(3):215–233, June 1977. ISSN 0022-2496. doi: 10.1016/0022-2496(77)90032-3. URL https://www.sciencedirect.com/science/article/pii/0022249677900323.

[27] Leslie G. Valiant. A Theory of the Learnable. Communications of the ACM, 27(11):1134–1142, 1984. Publisher: ACM New York, NY, USA.

[28] Adam Gleave, Michael Dennis, Shane Legg, Stuart Russell, and Jan Leike. Quantifying Differences in Reward Functions. In International Conference on Learning Representations, 2021. URL https://openreview.net/forum?id=LwEQnp6CYev.

[29] Luc Devroye, László Györfi, and Gábor Lugosi. A Probabilistic Theory of Pattern Recognition, volume 31. Springer Science & Business Media, 2013.

[30] Hong Jun Jeon, Smitha Milli, and Anca D. Dragan. Reward-Rational (Implicit) Choice: A Unifying Formalism for Reward Learning. arXiv:2002.04833 [cs], December 2020. URL http://arxiv.org/abs/2002.04833.

[31] Rachel Freedman, Rohin Shah, and Anca Dragan. Choice Set Misspecification in Reward Inference. arXiv:2101.07691 [cs], January 2021. URL http://arxiv.org/abs/2101.07691.

[32] Roger N. Shepard. Stimulus and Response Generalization: A Stochastic Model Relating Generalization to Distance in Psychological Space. Psychometrika, 22(4):325–345, December 1957. ISSN 1860-0980. doi: 10.1007/BF02288967. URL https://doi.org/10.1007/BF02288967.
[33] Chris L. Baker, Joshua B. Tenenbaum, and Rebecca R. Saxe. Goal Inference as Inverse Planning. In *Proceedings of the Annual Meeting of the Cognitive Science Society*, volume 29, 2007. Issue: 29.

[34] Jon Kleinberg, Sendhil Mullainathan, and Manish Raghavan. Inherent Trade-Offs in the Fair Determination of Risk Scores. arXiv:1609.05807 [cs, stat], November 2016. URL http://arxiv.org/abs/1609.05807 arXiv: 1609.05807.

[35] Lydia T. Liu, Max Simchowitz, and Moritz Hardt. The Implicit Fairness Criterion of Unconstrained Learning. In *International Conference on Machine Learning*, pages 4051–4060. PMLR, May 2019. URL http://proceedings.mlr.press/v97/liu19f.html ISSN: 2640-3498.

[36] Solon Barocas, Moritz Hardt, and Arvind Narayanan. *Fairness and Machine Learning*. fairmlbook.org, 2019.

[37] Camelia Simoiu, Sam Corbett-Davies, and Sharad Goel. The Implicit Fairness Criterion of Unconstrained Learning. In *International Conference on Machine Learning*, pages 4051–4060. PMLR, May 2019. URL http://proceedings.mlr.press/v97/liu19f.html ISSN: 2640-3498.

[38] Daniel Kahneman and Amos Tversky. Prospect Theory: An Analysis of Decision under Risk. *Econometrica*, 47(2):263–291, 1979. ISSN 0012-9682. doi: 10.2307/1914185. URL https://www.jstor.org/stable/1914185 Publisher: [Wiley, Econometric Society].

[39] Liu Leqi, Adarsh Prasad, and Pradeep K Ravikumar. On Human-Aligned Risk Minimization. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d’Alché-Buc, E. Fox, and R. Garnett, editors, *Advances in Neural Information Processing Systems 32*, pages 15055–15064. Curran Associates, Inc., 2019. URL http://papers.nips.cc/paper/9642-on-human-aligned-risk-minimization.pdf

[40] V. Vapnik. *Estimation of Dependences Based on Empirical Data*. Information Science and Statistics. Springer-Verlag, New York, 2006. ISBN 978-0-387-30865-4. doi: 10.1007/0-387-34239-7. URL https://www.springer.com/gp/book/9780387308654

[41] Anselm Blumer, A. Ehrenfeucht, David Haussler, and Manfred K. Warmuth. Learnability and the Vapnik-Chervonenkis Dimension. *Journal of the ACM*, 36(4):929–965, October 1989. ISSN 0004-5411, 1557-735X. doi: 10.1145/76359.76371. URL https://dl.acm.org/doi/10.1145/76359.76371

[42] Jiří Matoušek and Jan Vondrák. The Probabilistic Method. Lecture Notes, Charles University, Prague, Czech Republic, March 2008.

[43] D. Angluin and L. G. Valiant. Fast Probabilistic Algorithms for Hamiltonian Circuits and Matchings. *Journal of Computer and System Sciences*, 18(2):155–193, April 1979. ISSN 0022-0000. doi: 10.1016/0022-0000(79)90045-X. URL https://www.sciencedirect.com/science/article/pii/002200007990045X

[44] Andrzej Ehrenfeucht, David Haussler, Michael Kearns, and Leslie Valiant. A General Lower Bound on the Number of Examples Needed for Learning. *Information and Computation*, 82(3):247–261, 1989. Publisher: Elsevier.

[45] V. Vapnik. Principles of Risk Minimization for Learning Theory. *Advances in Neural Information Processing Systems*, 4, 1991. URL https://proceedings.neurips.cc/paper/1991/hash/ff4d5fbbaf7976cfdc032e8bde78de5-Abstract.html

[46] Lorenzo Rosasco, Ernesto De Vito, Andrea Caponnetto, Michele Piana, and Alessandro Verri. Are loss functions all the same? *Neural Computation*, 16(5):1063–1076, May 2004. ISSN 0899-7667. doi: 10.1162/089976604773135104. URL https://doi.org/10.1162/089976604773135104
Appendix

A Proofs

A.1 Proof of Lemma 3.1

Lemma 3.1 (Equivalence of cost matrices). Any cost matrix $C = (C_{00} C_{01})$ is equivalent to a cost matrix $C' = (0^{1-c} \ 1^c)$ where $c = \frac{C_{10} - C_{00}}{C_{10} + C_{01} - C_{00} - C_{11}}$ as long as $C_{10} + C_{01} - C_{00} - C_{11} \neq 0$. That is, there are constants $a, b \in \mathbb{R}$ such that $\mathcal{R}_C(h) = a\mathcal{R}_{C'}(h) + b$ for all $h$.

Proof. Let $a = C_{10} + C_{01} - C_{00} - C_{11}$ and $b = P(Y = 0)C_{00} + P(Y = 1)C_{11}$. Then

$$\mathcal{R}_C(h) = P(h(X) = 0 \land Y = 0)C_{00} + P(h(X) = 1 \land Y = 0)C_{10} + P(h(X) = 0 \land Y = 1)(C_{01} - C_{11}) + P(Y = 1)C_{11}$$

$$= (C_{10} + C_{01} - C_{00} - C_{11}) \left( P(h(X) = 1 \land Y = 0) \frac{C_{10} - C_{00}}{C_{10} + C_{01} - C_{00} - C_{11}} \right) + b$$

$$= a(P(h(X) = 1 \land Y = 0) + P(h(X) = 0 \land Y = 1)(1 - c)) + b$$

$$= a\mathcal{R}_{C'}(h) + b. \quad \square$$

A.2 Proof of Lemma 4.1

Lemma 4.1 (Bayes optimal decision rule). An optimal decision rule $h$ for a decision problem $(\mathcal{D}, c)$ is given by $h(x) = 1\{q(x) \geq c\}$ where $q(x) = P(Y \sim \mathcal{D} \mid h = 1 \mid X = x)$ is the posterior probability of class $1$ given the observation $x$.

This result is well-known [29] but we include a proof here for completeness.

Proof. Let $h(x) = 1\{q(x) \geq c\}$ and let $\hat{h} : \mathcal{X} \to \{0, 1\}$ be any other decision rule. We will show that not only is $\hat{h}$ an optimal decision rule, but in fact that if $P(h(X) \neq \hat{h}(X) \land q(X) \neq c) > 0$, then $\mathcal{R}_c(h) > \mathcal{R}_c(\hat{h})$; that is, $\hat{h}$ is strictly suboptimal. Thus, any optimal decision rule $h^*$ must satisfy $h(x) = h^*(x)$ almost surely except where $q(x) = c$.

First, let’s define the conditional risk of $h$ at $x$, denoted by $\mathcal{R}_c(h \mid X = x)$:

$$\mathcal{R}_c(h \mid X = x) = cP(h(X) = 1 \land Y = 0 \mid X = x) + (1 - c)P(h(X) = 0 \land Y = 1 \mid X = x).$$

Note that one of the two terms is always zero, depending on whether $h(X) = 0$ or $1$, since $h(X)$ is deterministic given $X$. The risk of $h$ is the expectation of the conditional risk:

$$\mathcal{R}_c(h) = \mathbb{E}_{X \sim \mathcal{D}}[\mathcal{R}_c(h \mid X = x)].$$

We can bound the conditional risk for the optimal decision rule $h$:

$$\mathcal{R}_c(h \mid x) = \begin{cases} 
    cP(Y = 0 \mid X = x) & q(x) \geq c \\
    (1 - c)P(Y = 1 \mid X = x) & q(x) < c 
\end{cases}$$

$$= \begin{cases} 
    c(1 - q(x)) & q(x) \geq c \\
    (1 - c)q(x) & q(x) < c 
\end{cases}$$

$$\leq c(1 - c). \quad (3)$$

Now, consider the conditional risk for the other decision rule $\hat{h}$ at $x$. First, suppose $\hat{h}(x) = h(x)$; that is, the decision rule agrees with the optimal one. Then clearly $\mathcal{R}_c(h \mid X = x) = \mathcal{R}_c(\hat{h} \mid X = x) \leq c(1 - c)$. Next,
suppose \( q(x) \neq c \) and \( \tilde{h}(x) \neq h(x) \). Then

\[
\mathcal{R}_c(\tilde{h} | X = x) = \begin{cases} 
  \epsilon \mathbb{P}(Y = 0 | X = x) & q(x) < c \\
  (1 - \epsilon) \mathbb{P}(Y = 1 | X = x) & q(x) > c 
\end{cases}
\]

\[
= \begin{cases} 
  c(1 - q(x)) & q(x) < c \\
  (1 - c)q(x) & q(x) > c 
\end{cases}
\]

> \( c(1 - c) \). \hspace{1cm} (4)

Finally, suppose \( q(x) = c \); in this case, it is clear that \( \mathcal{R}_c(\tilde{h} | X = x) = c(1 - c) \) regardless of what \( \tilde{h}(x) \) is. Putting this together, we can break down the risk of \( \tilde{h} \) by conditioning on whether \( \tilde{h}(x) = h(x) \) or \( q(x) = c \):

\[
\mathcal{R}_c(\tilde{h}) = \mathbb{E}[\mathcal{R}_c(\tilde{h} | X = x)]
\]

\[
= \mathbb{E}[\mathcal{R}_c(\tilde{h} | X = x) | \tilde{h}(X) = h(X) \lor q(X) = c] \mathbb{P}(\tilde{h}(X) = h(X) \lor q(X) = c)
\]

\[
+ \mathbb{E}[\mathcal{R}_c(\tilde{h} | X = x) | \tilde{h}(X) \neq h(X) \land q(X) \neq c] \mathbb{P}(\tilde{h}(X) \neq h(X) \land q(X) \neq c)
\]

\[
\geq (i) \mathbb{E}[\mathcal{R}_c(\tilde{h} | X = x) | \tilde{h}(X) = q(X) = c] \mathbb{P}(\tilde{h}(X) = q(X) = c)
\]

\[
+ \mathbb{E}[\mathcal{R}_c(\tilde{h} | X = x) | \tilde{h}(X) \neq h(X) \land q(X) \neq c] \mathbb{P}(\tilde{h}(X) \neq h(X) \land q(X) \neq c)
\]

\[
= \mathcal{R}_c(\tilde{h}) \geq \mathcal{R}_c(h).
\]

(i) uses 4 and (ii) uses 3. The above shows that \( \mathcal{R}_c(\tilde{h}) \geq \mathcal{R}_c(h) \) for any decision rule \( \tilde{h} \), demonstrating that \( h \) must have the lowest risk achievable. Note that (i) is strictly greater as long as \( \mathbb{P}(\tilde{h}(X) \neq h(X) \land q(X) \neq c) > 0 \), validating the claim above that any optimal decision rule must agree with \( h \) almost surely except when \( q(X) = c \).

\[\Box\]

A.3 Proof of Theorem 4.2

**Theorem 4.2 (IDT for optimal decision maker).** Let \( \epsilon > 0 \) and \( \delta > 0 \). Say that there exists \( p_c > 0 \) such that \( \mathbb{P}(q(X) \in (c, c + \epsilon)) \geq p_c \) and \( \mathbb{P}(q(X) \in [c - \epsilon, c]) \geq p_c \). Let \( \tilde{h} \) be chosen to be consistent with the observed decisions as stated above, i.e., \( \{q(x_i) \geq \tilde{h} \land y_i = 1\} \). Then \( |\tilde{h} - c| \leq \epsilon \) with probability at least \( 1 - \delta \) as long as the number of samples \( m \geq \frac{\log(2/\delta)}{p_c} \).

**Proof.** Let \( h \) denote the decision maker’s decision rule. From the proof of Lemma 4.1, we know that the optimality of \( h \) means that \( h(X) = 1 \{q(X) \geq c\} \) almost surely as long as \( q(X) \neq c \).

Let \( E \) denote the event that we observe \( x_i \) and \( x_j \) in the sample such that \( q(x_i) \in (c, c + \epsilon) \) and \( q(x_j) \in [c - \epsilon, c) \):

\[
E = \exists x_i \ q(x_i) \in (c, c + \epsilon) \land \exists x_j \ q(x_j) \in [c - \epsilon, c).
\]

First, we will lower bound the probability of \( E_1 \):

\[
\mathbb{P}(E_1) = 1 - \mathbb{P}(\forall x_i \ q(x_i) \notin (c, c + \epsilon))
\]

\[
= 1 - (\mathbb{P}(q(X) \notin (c, c + \epsilon)) ^ m)
\]

\[
= 1 - (1 - \epsilon \mathbb{P}(q(X) \notin (c, c + \epsilon)) ) ^ m
\]

\[
\geq 1 - (1 - c p_c)^m
\]

\[
\geq 1 - e^{-m p_c}
\]

\[
\geq 1 - e^{-\log(2/\delta)}
\]

\[
= 1 - \delta / 2.
\]

Second, we will lower bound the probability of \( E_2 \):

\[
\mathbb{P}(E_2) = 1 - \mathbb{P}(\forall j \ q(x_j) \notin [c - \epsilon, c))
\]

\[
= 1 - (\mathbb{P}(q(X) \notin [c - \epsilon, c)) ^ m)
\]

\[
= 1 - (1 - \mathbb{P}(q(X) \notin [c - \epsilon, c)) ) ^ m
\]

\[
\geq 1 - (1 - c p_c)^m
\]

\[
\geq 1 - e^{-m p_c}
\]

\[
\geq 1 - e^{-\log(2/\delta)}
\]

\[= 1 - \delta / 2.
\]
\[= 1 - (1 - \mathbb{P}(q(X) \in [c - \epsilon, c]))^n \]
\[\geq 1 - (1 - ep)^n \]
\[\geq 1 - e^{-np}\]
\[\geq 1 - e^{-\log(2/\delta)} \]
\[= 1 - \delta/2. \]

Putting the above together, we can lower bound the probability of \(E\):
\[\mathbb{P}(E) = \mathbb{P}(E_1 \land E_2) \]
\[= 1 - \mathbb{P}(\neg E_1 \lor \neg E_2) \]
\[\geq 1 - \mathbb{P}(\neg E_1) - \mathbb{P}(\neg E_2) \]
\[\geq 1 - \delta. \]

Finally, we will show that \(E\) implies \(|\hat{c} - c| \leq \epsilon\). Suppose \(E\) occurs. Then \(q(x_i) > c\), so \(h(x_i) = \hat{y}_i = 1\). This means that \(\hat{c} \leq q(x_i) \leq c + \epsilon\). Also, \(q(x_j) < c\), so \(h(x_j) = \hat{y}_j = 0\). This means that \(\hat{c} > q(x_j) \geq c - \epsilon\). Thus
\[c - \epsilon < \hat{c} \leq c + \epsilon \]
\[|\hat{c} - c| \leq \epsilon. \]

So with probability at least \(1 - \delta\), \(|\hat{c} - c| \leq \epsilon\).

### A.4 Proof of Lemma 4.5

The proof of Lemma 4.5 depends on another lemma, which will also be useful in the unknown hypothesis setting. This lemma bounds the conditional probability that the correct decision \(Y = 1\) for observations \(x\) between the decision boundaries of two optimal decision rules.

**Lemma A.1.** Suppose \(\text{opt}_{\mathcal{P}}(\mathcal{H})\) is monotone and let \(h_c, h_{c'} \in \mathcal{H}\) be optimal decision rules for loss parameters \(c\) and \(c'\), respectively, where \(c < c'\). Then for every \(x \in \mathcal{X}\), \(h_{c'}(x) \leq h_c(x)\). Furthermore, assuming \(\mathbb{P}(h_c(X) \neq h_{c'}(X)) = \mathbb{P}(h_c(X) = 1 \land h_{c'}(X) = 0) > 0\),
\[c \leq \mathbb{P}(Y = 1 \mid h_c(X) = 1 \land h_{c'}(X) = 0) \leq c'. \]

**Proof.** We can write the risk of a decision rule \(h\) for cost \(c\) as
\[R_c(h) = c\mathbb{P}(h(X) = 1 \land Y = 0) + (1 - c)\mathbb{P}(h(X) = 0 \land Y = 1) \]
\[= c\mathbb{P}(Y = 0) - \mathbb{P}(h(X) = 0) + (1 - c)\mathbb{P}(h(X) = 0) - \mathbb{P}(h(X) = 1) \]
\[= c\mathbb{P}(Y = 0) - c\mathbb{P}(h(X) = 0) + c\mathbb{P}(h(X) = 0) + \mathbb{P}(h(X) = 1) \]
\[= c\mathbb{P}(Y = 0) - c\mathbb{P}(h(X) = 0) + \mathbb{P}(h(X) = 0) \]
\[= c\mathbb{P}(Y = 0) - c\mathbb{P}(h(X) = 0) + \mathbb{P}(h(X) = 0) \land Y = 1\].

(5)

Since \(h_c\) is optimal for \(c\), we have
\[R_c(h_{c'}) - R_c(h_c) \geq 0. \]

(6)

Applying (5) to (6) gives
\[\mathbb{P}(h_{c'}(X) = 0 \land Y = 1) - \mathbb{P}(h_c(X) = 0 \land Y = 1) - c\left[\mathbb{P}(h_c(X) = 0) - \mathbb{P}(h_c(X) = 0)\right] \geq 0. \]

(7)

Now, suppose the lemma does not hold; that is, there is some \(x \in \mathcal{X}\) such that \(h_{c'}(x) > h_c(x)\). Since \(\text{opt}_{\mathcal{P}}(\mathcal{H})\) is monotone, this implies
\[\forall x \in \mathcal{X} \quad h_c(x) \leq h_{c'}(x). \]

(8)

Assuming (2) we have the following two identities:
\[\mathbb{P}(h_c(X) = 0) - \mathbb{P}(h_{c'}(X) = 0) = \mathbb{P}(h_c(X) = 0) \land h_{c'}(X) = 1 \]
\[\mathbb{P}(h_c(X) = 0 \land Y = 1) - \mathbb{P}(h_{c'}(X) = 0 \land Y = 1) = \mathbb{P}(h_c(X) = 0) \land h_{c'}(X) = 1 \land Y = 1. \]

Plugging these in to (5) gives
\[c\mathbb{P}(h_c(X) = 0) \land h_{c'}(X) = 1) - \mathbb{P}(h_c(X) = 0) \land h_{c'}(X) = 1 \land Y = 1 \geq 0 \]
\[\mathbb{P}(h_c(X) = 0) \land h_{c'}(X) = 1 \land Y = 1 \leq c\mathbb{P}(h_c(X) = 0) \land h_{c'}(X) = 1 \]
\[\frac{\mathbb{P}(h_c(X) = 0) \land h_{c'}(X) = 1 \land Y = 1}{\mathbb{P}(h_c(X) = 0) \land h_{c'}(X) = 1} \leq c \]
This is the first claim of the lemma. Now, we can apply the same set of steps to \( R_c(h_c) - R_c'(h_c) \geq 0 \) (i.e., using (5) and the above identities) to obtain

\[ c' \leq \mathbb{P}(Y = 1 \mid h_c(X) = 0 \wedge h_c'(X) = 1). \]

Combining these two equations implies \( c' \leq c \), but we assumed that \( c < c' \), so this is a contradiction. Thus, (6) must be false!

Since \( \text{opt}_D(H) \) is monotone, the falsity of (6) implies that actually,

\[
\forall x \in X \quad h_c'(x) \leq h_c(x). \tag{8}
\]

Now, we can complete the proof by repeating the above steps using (8) instead of (5) to obtain

\[ c \leq \mathbb{P}(Y = 1 \mid h_c(X) = 1 \wedge h_c'(X) = 0) \leq c'. \]

\[ \square \]

**Lemma 4.5 (Induced posterior probability).** Let \( \text{opt}_D(H) \) be monotone and define

\[
\overline{\beta}_H(x) \triangleq \sup \left( \{ c \in [0, 1] \mid h_c(x) = 1 \} \cup \{0\} \right) \quad \text{and} \quad \underline{\beta}_H(x) \triangleq \inf \left( \{ c \in [0, 1] \mid h_c(x) = 0 \} \cup \{1\} \right).
\]

Then for all \( x \in X \), \( \overline{\beta}_H(x) = q_{\beta_H}(x) \). Define the induced posterior probability of \( H \) as

\[
q_H(x) \triangleq \overline{\beta}_H(x).
\]

**Proof.** Fix \( x \in X \). Using Lemma (A.1), we have that

\[ c < c' \quad \Rightarrow \quad h_c(x) \geq h_c'(x). \]

That is, \( h_c(x) \) is monotone non-increasing in \( c \). This is enough to show that \( q_H(x) \) is well-defined. Consider three cases:

1. \( \forall c, h_c(x) = 1 \). In this case, \( \overline{\beta}_H(x) = \sup \{ c \in [0, 1] \mid h_c(x) = 1 \} \cup \{0\} = 1 \) and \( q_{\beta_H}(x) = \inf \{ c \in [0, 1] \mid h_c(x) = 0 \} \cup \{1\} = 1 \) so \( q_H(x) = 1 \).

2. \( \forall c, h_c(x) = 0 \). In this case, \( \overline{\beta}_H(x) = \sup \{ c \in [0, 1] \mid h_c(x) = 1 \} \cup \{0\} = \sup \{0\} = 0 \) and \( q_{\beta_H}(x) = \inf \{ c \in [0, 1] \mid h_c(x) = 0 \} \cup \{1\} = 0 \) so \( q_H(x) = 0 \).

3. \( \exists c_0, c_1 \) such that \( h_{c_0}(x) = 0 \) and \( h_{c_1}(x) = 1 \). In this case, neither \( \{ c \in [0, 1] \mid h_c(x) = 1 \} \) nor \( \{ c \in [0, 1] \mid h_c(x) = 0 \} \) is empty so we have

\[
\overline{\beta}_H(x) = \sup \{ c \in [0, 1] \mid h_c(x) = 1 \}
\]

\[
q_{\beta_H}(x) = \inf \{ c \in [0, 1] \mid h_c(x) = 0 \}.
\]

Say \( q_H(x) \) is not well-defined; that is,

\[ \sup \{ c \in [0, 1] \mid h_c(x) = 1 \} \neq \inf \{ c \in [0, 1] \mid h_c(x) = 0 \}. \]

First, suppose \( \sup \{ c \in [0, 1] \mid h_c(x) = 1 \} < \inf \{ c \in [0, 1] \mid h_c(x) = 0 \} \). Then there exists some \( c \) for which \( h_c(x) \notin \{0, 1\} \), which is impossible. So \( \sup \{ c \in [0, 1] \mid h_c(x) = 1 \} > \inf \{ c \in [0, 1] \mid h_c(x) = 0 \} \). However, this implies that \( \exists c_1 \geq c_0 \) such that \( h_{c_1}(x) = 1 \) but \( h_{c_0}(x) = 0 \). Since \( h_c(x) \) is nonincreasing in \( c \), this is a contradiction. Thus \( q_H(x) = \overline{\beta}_H(x) = q_{\beta_H}(x) \) is well-defined.

\[ \square \]

**Corollary 4.6.** Let \( h_c \) be any optimal decision rule in \( H \) for loss parameter \( c \). Then for any \( x \in X \), \( h_c(x) = 1 \) if \( q_H(x) > c \) and \( h_c(x) = 0 \) if \( q_H(x) < c \).

**Proof.** Let

\[ h_c \in \arg \min_{h \in \mathcal{H}} R_c(h) \]

be an optimal decision rule in \( H \) for loss parameter \( c \).

Fix any \( x \in X \). If \( q_H(x) = c \), we don’t need to prove anything. If \( q_H(x) > c \), then suppose \( h_c(x) \neq 1 \), i.e. \( h_c(x) = 0 \). Then

\[ q_{\beta_H}(x) = \inf \{ c' \in [0, 1] \mid h_{c'}(x) = 0 \} \leq c \]

since \( h_c(x) = 0 \). However, this is a contradiction since we assumed \( q_H(x) > c \). Thus \( h_c(x) = 1 \).

Now, if \( q_H(x) < c \), suppose \( h_c(x) \neq 0 \), i.e. \( h_c(x) = 1 \). Then

\[ \overline{\beta}_H(x) = \sup \{ c' \in [0, 1] \mid h_{c'}(x) = 1 \} \geq c. \]

This is also a contradiction since we assumed \( q_H(x) < c \), so \( h_c(x) = 0 \).

\[ \square \]
A.5 Proof of Theorem 4.10

Theorem 4.10 (Unknown suboptimal decision maker). Let $\epsilon > 0$ and $\delta > 0$, and let $\text{opt}_p(\mathcal{H})$ be monotone. Say that there exists $p_c > 0$ such that $\mathbb{P}(q_H(x) \in \{c, c + \epsilon\}) \geq p_c \epsilon$ and $\mathbb{P}(q_H(x) \in \{c - \epsilon, c\}) \geq p_c \epsilon$. Let $\hat{c}$ be chosen to be consistent with the observed decisions, i.e. $q_H(x_i) \geq \hat{c} \iff y_i = 1$. Then $|\hat{c} - c| \leq \epsilon$ with probability at least $1 - \delta$ as long as the number of samples $m \geq \frac{\log(2/\delta)}{p_c \epsilon}$.

Proof. Let $h \in \mathcal{H}$ denote the decision maker’s decision rule. From Corollary 4.6 we know that $h(x) = 1\{q_H(x) \geq c\}$ as long as $q_H(x) \neq c$.

Let $E$ denote the event that we observe $x_i$ and $x_j$ in the sample such that $q_H(x_i) \in \{c, c + \epsilon\}$ and $q_H(x_j) \in \{c - \epsilon, c\}$. An analogous computation to the proof of Theorem 4.2 (Section A.3) shows that if $m \geq \frac{\log(2/\delta)}{p_c \epsilon}$, then $\mathbb{P}(E) \geq 1 - \delta$.

If $E$ occurs, then $h(x_i) = 1$ and so $\hat{c} \leq c + \epsilon$. Also, $h(x_j) = 0$ so $\hat{c} \geq c - \epsilon$. Thus, we have

$$\mathbb{P}(|\hat{c} - c| \leq \epsilon) \geq \mathbb{P}(E) \geq 1 - \delta.$$ 

\[ \square \]

A.6 Proof of Theorem 4.10

Theorem 4.10 (Unknown suboptimal decision maker). Let $\epsilon > 0$ and $\delta > 0$. Suppose we observe decisions from a decision rule $h_c$ which is optimal for loss parameter $c$ in hypothesis class $\mathcal{H} \in \mathbb{H}$. Let $h_c$ and $\mathcal{H}$ be $\alpha$-MD-smooth. Furthermore, assume that there exists $p_c > 0$ such that for any $\rho \leq c$, $\mathbb{P}(q_H(x) \in \{c, c + \rho\}) \geq p_c \rho$ and $\mathbb{P}(q_H(x) \in \{c - \rho, c\}) \geq p_c \rho$. Let $d \geq \text{VCdim}(\cup_{\mathcal{H} \in \mathbb{H}} \mathcal{H})$ be an upper bound on the VC-dimension of the union of all the hypothesis classes in $\mathbb{H}$.

Let $\hat{h}_c$ be $\arg \min_{h \in \mathcal{H}} R_c(h)$ be chosen to be consistent with the observed decisions, i.e. $\hat{h}_c(x_i) = y_i$ for $i = 1, \ldots, m$. Then $|\hat{c} - c| \leq \epsilon$ with probability at least $1 - \delta$ as long as the number of samples $m \geq O\left(\frac{\alpha}{\epsilon} + \frac{1}{\epsilon^2}\right) \left(\frac{d \log(1/\delta)}{p_c \epsilon} + \log(1/\delta)\right)$.

Proof. Specifically, we will prove that $\mathbb{P}(|\hat{c} - c| \leq \epsilon) \geq 1 - \delta$ as long as

$$m \geq O\left(\frac{\alpha}{\epsilon} + \frac{1}{\epsilon^2}\right) \left(\frac{d \log(1/\delta)}{p_c \epsilon} + \log(1/\delta)\right).$$

(9)

Throughout the proof, let $h_c \in \arg \min_{h \in \mathcal{H}} R_c(h)$ be the true decision rule and let $\hat{h}_c \in \arg \min_{h \in \mathcal{H}} R_c(\hat{h})$ be the estimated decision rule, i.e. one that agrees with the decisions in the sample of observations $\mathcal{S}$.

First, we use a standard result from PAC learning theory to upper bound the disagreement between the estimated decision rule $\hat{h}_c$ and the true decision rule $h_c$. In particular, since this is a case of realizable PAC learning, i.e. the true decision rule $h_c$ is in one of the hypothesis classes $\mathcal{H} \in \mathbb{H}$, we have that

$$\mathbb{P}(h_c(X) \neq \hat{h}_c(X)) \leq O\left(\frac{1}{p_c \epsilon} \frac{1}{p_c \epsilon^2}\right) = O\left(\min\left(\frac{p_c \epsilon}{\alpha}, \frac{p_c \epsilon^2}{\alpha}\right)\right)$$

with probability at least $1 - \delta$ over the drawn sample. This bound follows from Vapnik [40] and Blumer et al. [41] since the set of all possible hypotheses $\cup_{\mathcal{H} \in \mathbb{H}} \mathcal{H}$ has VC-dimension at most $d$, and we observe a sample of $m$ observations $x_i$ and decisions $y_i = h_c(x_i)$ where $m$ satisfies (9). In particular, denote

$$r = \mathbb{P}(h_c(X) \neq \hat{h}_c(X)) \leq \min\left(\frac{p_c \epsilon}{\alpha}, \frac{p_c \epsilon^2}{\alpha}\right).$$

(10)

Next, we show that (10) implies that $|\hat{c} - c| \leq \epsilon$; since (10) holds with probability at least $1 - \delta$, this is enough to complete the proof of Theorem 4.10. We will prove that $\hat{c} - c \leq \epsilon$ given (10). The proof that $c - \hat{c} \leq \epsilon$ is analogous. We require a technical lemma on probability theory:

Lemma A.2. Let $A$, $B$, and $C$ be events in a probability space with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Then

$$\left|\mathbb{P}(C \ | A) - \mathbb{P}(C \ | B)\right| \leq \frac{\mathbb{P}(A \wedge \neg B) + \mathbb{P}(\neg A \wedge B)}{\min(\mathbb{P}(A), \mathbb{P}(B))}.$$  

Proof of Lemma A.2. To simply the proof of this lemma, we adopt the boolean algebra notation that $AB$ is equivalent to $A \wedge B$ and $\neg A$ is equivalent to $\neg A$. Then we have 

$$\left|\mathbb{P}(C \ | A) - \mathbb{P}(C \ | B)\right|$$
We will establish bounds on each term in (13).

\[
\begin{align*}
\frac{\Pr(AC)}{\Pr(A)} & \quad \frac{\Pr(BC)}{\Pr(B)} \\
\frac{\Pr(ABC) + \Pr(\bar{A}BC)}{\Pr(AB) + \Pr(\bar{A}B)} & \quad \frac{\Pr(ABC) + \Pr(\bar{A}BC)}{\Pr(AB) + \Pr(\bar{A}B)} \\
\frac{\Pr(ABC)\Pr(\bar{A}B) + \Pr(\bar{A}BC)\Pr(B) - \Pr(ABC)\Pr(\bar{A}B) - \Pr(\bar{A}BC)\Pr(A)}{\Pr(A)\Pr(\bar{A}B)} \\
& \leq \max(i) \frac{\Pr(ABC)\Pr(\bar{A}B) + \Pr(\bar{A}BC)\Pr(B)}{\Pr(A)\Pr(\bar{A}B)}, \\
& \leq \max(ii) \frac{\Pr(ABC)\Pr(\bar{A}B) + \Pr(\bar{A}BC)\Pr(B)}{\Pr(A)\Pr(\bar{A}B)} \\
& = \max \left( \frac{\Pr(\bar{A}B) + \Pr(\bar{A}B)}{\Pr(A)}, \frac{\Pr(\bar{A}B) + \Pr(\bar{A}B)}{\Pr(A)} \right) \\
& = \max \left( \frac{\Pr(\bar{A}B)}{\min(\Pr(A), \Pr(B))} \right). \\
\end{align*}
\]

(i) uses the fact that for positive \( u \) and \( v \), \(|u - v| \leq \max(u, v)\). (ii) uses the fact that \( \Pr(E_1E_2) \leq \Pr(E_1) \) for any events \( E_1 \) and \( E_2 \). \( \square \)

Essentially, Lemma [X,2] says that if events \( A \) and \( B \) have high “overlap,” then the conditional probabilities of another event \( C \) given \( A \) and \( B \) should be close. We next carefully construct two such events with high overlap.

First, let \( c' = c + \epsilon/2 \) and let \( h_{c'} \in \arg \min_{h \in \mathcal{H}} \mathcal{R}_{\epsilon'}(h) \). Since \( h_c \) and \( \mathcal{H} \) are \( \alpha\)-MD-smooth, we have that

\[
\text{MD}(h_{c'}, \text{opt}_{\Pr}(\hat{H})) \leq (1 + \alpha c' - \epsilon) \text{MD}(h_c, \text{opt}_{\Pr}(\hat{H})) \\
\leq (1 + \alpha \epsilon/2)\Pr(h_c(X) \neq \hat{h}_c(X)) \\
\leq (1 + \alpha \epsilon/2)r. \tag{12}
\]

Since \( \text{MD}(h, \text{opt}_{\Pr}(\hat{H})) = \inf_{\hat{h} \in \text{opt}_{\Pr}(\hat{H})} \Pr(h(X) \neq \hat{h}(X)) \), there must be some hypothesis \( \hat{h}_{c'} \in \arg \min_{h \in \mathcal{H}} \mathcal{R}_{\epsilon'}(\hat{h}) \) that matches the minimum disagreement with \( h_{c'} \) plus a small positive number (in case the infimum is not achieved):

\[\Pr(\hat{h}_{c'}(X) \neq h_{c'}(X)) \leq \text{MD}(h_{c'}, \text{opt}_{\Pr}(\hat{c})) + r \leq (2 + \alpha \epsilon/2)r.\]

Now, let the events \( A, B, \) and \( C \) be defined as follows:

\[
\begin{align*}
A & : h_c(X) = 1 \land h_{c'}(X) = 0, \\
B & : \hat{h}_c(X) = 1 \land \hat{h}_{c'}(X) = 0, \\
C & : Y = 1.
\end{align*}
\]

Using Lemma [X,2], we can write the bound

\[
\Pr(Y = 1 \mid B) \leq \Pr(Y = 1 \mid A) + \frac{\Pr(\bar{A} \land \bar{B} \lor \bar{\bar{A}} \land B)}{\min(\Pr(A), \Pr(B))}. \tag{13}
\]

We will establish bounds on each term in (13).

**Upper bound on** \( \Pr(A \land \bar{B} \lor \bar{A} \land B) \)  
It is easy to see that

\[
A \land \bar{B} \lor \bar{A} \land B \Rightarrow h_c(X) \neq \hat{h}_c(X) \lor h_{c'}(X) \neq \hat{h}_{c'}(X).
\]

Given this implication, it must be that

\[
\begin{align*}
\Pr(A \land \bar{B} \lor \bar{A} \land B) & \leq \Pr(h_c(X) \neq \hat{h}_c(X) \lor h_{c'}(X) \neq \hat{h}_{c'}(X)) \\
& \leq (3 + \alpha \epsilon/2)r \\
& \leq p_c \epsilon^2/12 + p_{c'} \epsilon^2/12 = p_c \epsilon^2/6.
\end{align*}
\]

where the inequalities follow from (10) and (12).

**Lower bound on** \( \min(\Pr(A), \Pr(B)) \)  
Since \( h_c \) is optimal within \( \mathcal{H} \) for loss parameter \( \epsilon \), Corollary [X,6] gives that \( h_c(x) = 1 \) if \( q_{\mathcal{H}}(x) > c \). Similarly, \( h_c(x) = 0 \) if \( q_{\mathcal{H}}(x) < c \). Therefore,

\[
q_{\mathcal{H}}(X) \in (c, c') \Rightarrow h_c(X) = 1 \land h_{c'}(X) = 0 \iff A.
\]
This implication allows us to lower bound \( \mathbb{P}(A) \):
\[
\mathbb{P}(A) \geq \mathbb{P}(q_H(X) \in (c, c')) = \mathbb{P}(q_H(X) \in (c, c + \epsilon/2)) \geq p_c \epsilon/2
\]
where the final inequality is by assumption. We also need to lower bound \( \mathbb{P}(B) \) in order to lower bound \( \min(\mathbb{P}(A), \mathbb{P}(B)) \):
\[
\mathbb{P}(B) = \mathbb{P}(A \land B) + \mathbb{P}(\neg A \land B) = \mathbb{P}(A) - \mathbb{P}(A \land \neg B) + \mathbb{P}(\neg A \land B) \geq \mathbb{P}(A) - (\mathbb{P}(A \land \neg B) + \mathbb{P}(\neg A \land B)) \geq p_c \epsilon/2 - p_c \epsilon^2/6 \geq p_c \epsilon/3.
\]
We assume that \( \epsilon \leq 1 \) to lower bound \( \epsilon \geq \epsilon^2 \), but this is fine since if \( \epsilon > 1 \) then Theorem 4.10 holds trivially. Thus we have \( \min(\mathbb{P}(A), \mathbb{P}(B)) \geq p_c \epsilon/3 \).

**Lower bound on** \( \mathbb{P}(Y = 1 \mid B) \) **By Lemma A.1**, we have that, since \( \mathbb{P}(B) > 0 \),
\[
\mathbb{P}(Y = 1 \mid B) = \mathbb{P}(Y = 1 \mid \hat{h}_{c'}(X) = 1 \land \hat{h}_{c'}(X) = 0) \geq \hat{c}.
\]

**Upper bound on** \( \mathbb{P}(Y = 1 \mid A) \) **Similarly**, by Lemma A.1 we have that, since \( c' > c \) and \( \mathbb{P}(A) > 0 \),
\[
\mathbb{P}(Y = 1 \mid A) = \mathbb{P}(Y = 1 \mid h_c(X) = 1 \land h_c(X) = 0) \leq c' = c + \epsilon/2.
\]

**Concluding the proof** Given all these bounds, we can rewrite (12) as
\[
\hat{c} \leq \mathbb{P}(Y = 1 \mid B) \leq \mathbb{P}(Y = 1 \mid A) + \frac{\mathbb{P}(A \land \neg B \lor \neg A \land B)}{\min(\mathbb{P}(A), \mathbb{P}(B))} \leq c + \epsilon/2 + \frac{p_c \epsilon^2/6}{p_c \epsilon/3} \leq c + \epsilon/2 + \epsilon/2 = c + \epsilon
\]
\[
\hat{c} - c \leq \epsilon.
\]
This completes the proof that \( \hat{c} - c \leq \epsilon \) with probability at least \( 1 - \delta \); the proof that \( c - \hat{c} \leq \epsilon \) is analogous. ■

### A.7 Proof of Theorem 4.11

**Theorem 4.11 (Lower bound for optimal decision maker).** Fix \( 0 < \epsilon < 1/4 \), \( 0 < \delta < 1/2 \), and \( 0 < p_c \leq 1/8 \). Then for any IDT algorithm \( \hat{c}(\cdot) \), there exists a decision problem \( (D, c) \) satisfying the conditions of Theorem 4.7 such that \( m < \frac{\log(1/2\delta)}{8p_c \epsilon} \) implies that \( \mathbb{P}(|\hat{c}(S) - c| \geq \epsilon) > \delta \).

**Proof.** Consider a distribution over \( X \in \mathcal{X} = [0, 1] \) where
\[
q(x) = \mathbb{P}(Y = 1 \mid X = x) = x.
\]
Let the distribution \( D_X \) over \( X \) have density \( p_c \) on the interval \((1/2 - 2\epsilon, 1/2 + 2\epsilon)\) and let \( \mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 1/2 - 2p_c \epsilon \).

Let \( c_1 = 1/2 - \epsilon \) and \( c_2 = 1/2 + \epsilon \). Then clearly, for \( c \in \{c_1, c_2\} \), the conditions of Theorem 4.2 are satisfied:
\[
\mathbb{P}(q(X) \in [c - \epsilon, c]) = \mathbb{P}(q(X) \in (c, c + \epsilon)) = p_c \epsilon.
\]
By Lemma 4.1, the optimal decision rule for loss parameter \( c_1 \) is \( h_{c_1}(x) = 1\{x \geq c_1\} \) and for \( c_2 \) it is \( h_{c_2}(x) = 1\{x \geq c_2\} \).

Now suppose
\[
m < \frac{\log(1/2\delta)}{8p_c \epsilon}
\]
as stated in the theorem. We can bound the probability of the following event \( E \):
\[
\mathbb{P}(\forall x \in S \ q(x) \in \{0, 1\}) = \left[\mathbb{P}(X \in \{0, 1\})\right]^m = (1 - 4p_c \epsilon)^m \geq (e^{-8p_c \epsilon})^m.
\]

20
\[ e^{-\log(1/2\epsilon)} = 2\epsilon. \]

(i) uses the fact that \[ 1 - u \geq e^{-2u} \text{ for } u \in [0, 1/2]. \] Now, suppose \( E \) occurs. In this case, \( h_{c_1}(x) \equiv h_{c_2}(x) \) for all \( x \in S \). That is, regardless of which loss parameter \( c \in \{c_1, c_2\} \) is used, the distribution of samples will be the same. Let \( S_1 \) denote the random variable for a sample taken from a decision maker using \( h_{c_1} \) and \( S_2 \) a sample taken from \( h_{c_2} \). Since these have the same distribution under \( E \), they must induce the same probabilities when the IDT algorithm \( \hat{c} \) is applied to them:

\[
\begin{align*}
P_1 &= \Pr(\hat{c}(S_1) \leq 1/2 \mid E) = \Pr(\hat{c}(S_2) \leq 1/2 \mid E), \\
P_2 &= \Pr(\hat{c}(S_1) > 1/2 \mid E) = \Pr(\hat{c}(S_2) > 1/2 \mid E).
\end{align*}
\]

Since \( P_1 + P_2 = 1 \), at least one of \( P_1, P_2 \geq 1/2 \). Suppose WLOG that \( P_1 \geq 1/2 \). Then

\[
\Pr(|\hat{c}(S_2) - c_2| \geq \epsilon) \geq \Pr(\hat{c}(S_2) \leq 1/2) = \Pr(\hat{c}(S_2) \leq 1/2 \mid E) \Pr(E) \geq 1/2(2\epsilon) = \delta.
\]

Thus there is a decision problem \((D, c_2)\) for which the IDT algorithm \( \hat{c} \) must make an error of at least size \( \epsilon \) with at least probability \( \delta \). This concludes the proof.

**Corollary 4.12 (Lack of uncertainty precludes identifiability).** Fix \( 0 < \epsilon < 1/4 \) and suppose a decision problem \((D, c)\) has no uncertainty. Then for any IDT algorithm \( \hat{c}(\cdot) \), there is a loss parameter \( c \) and hypothesis class \( H \) such that for any sample size \( n \), \( \Pr(\hat{c}(S) - c \geq \epsilon) \geq 1/2 \).

**Proof.** Let the loss parameters \( c_1 = 1/2 - \epsilon \) and \( c_2 = 1/2 + \epsilon \) be defined as in the proof of Theorem 4.11 above. By Lemma 4.1, the optimal decision rule for loss parameter \( c_1 \) is \( h_{c_1}(x) = 1\{q(x) \geq c_1\} \) and for \( c_2 \) it is \( h_{c_2}(x) = 1\{q(x) \geq c_2\} \). Since \( \Pr(q(x) \in [0, 1]) = 1 \), it is clear that the decision rules make the same decision rules almost surely, i.e. \( \Pr(h_{c_1}(X) = h_{c_2}(X)) = 1 \). Thus, letting \( S_1 \) and \( S_2 \) denote samples drawn from decision rules \( h_{c_1} \) and \( h_{c_2} \), respectively, as above, we have that the distributions of \( S_1 \) and \( S_2 \) are indistinguishable. Thus by the same argument as above we can show that (WLOG)

\[
\Pr(|\hat{c}(S_2) - c_2| \geq \epsilon) \geq \Pr(\hat{c}(S_2) \leq 1/2) \geq 1/2.
\]

**A.8 Proof of Lemma 5.2.**

**Definition 5.1.** A decision rule \( h : X \to \{0, 1\} \) for a distribution \((X, Y) \sim D\) satisfies the group calibration/sufficiency fairness criterion if there is a function \( r : X \to \mathbb{R} \) and threshold \( t \in \mathbb{R} \) such that \( h(x) = 1\{r(x) \geq t\} \) and \( r \) satisfies \( Y \perp A \mid r(X) \).

**Lemma 5.2 (Equal loss parameters imply group calibration).** Let \( h \) be chosen as in (2) where \( \ell(\hat{y}, y, a) = c_a \) if \( \hat{y} = 1 \) and \( y = 0 \), \( \ell(\hat{y}, y, a) = 1 - c_a \) if \( \hat{y} = 0 \) and \( y = 1 \), and \( \ell(\hat{y}, y, a) = 0 \) otherwise. Then \( h \) satisfies group calibration (sufficiency) if \( c_a = c_a' \) for every \( a, a' \in A \).

Conversely, if there exist \( a, a' \in \mathcal{A} \) such that \( c_a \neq c_a' \) and \( \Pr(q(X) \in (c_a, c_a')) > 0 \), then \( h \) does not satisfy group calibration.

**Proof that equal \( c_a \) imply group calibration.** Assume \( c_a = c_a' \) for every \( a, a' \in \mathcal{A} \). Then define

\[
r(x) = q(x) + h(x).
\]

That is, \( r(x) \) is the posterior probability \( q(x) = \Pr(Y = 1 \mid X = x) \) plus one if the decision rule outputs the decision \( h(x) = 1 \). From the proof of Lemma 4.1 we know that \( h(x) = 1 \) if \( q(x) > c \) and \( h(x) = 0 \) if \( q(x) < c \). From this and (1) we can write

\[
h(x) = 1\{r(x) \geq c + 1\}.
\]

Now we need to show that \( Y \perp A \mid r(X) \). Note that \( r(X) \in [0, c] \cup [c + 1, 2] \). First, we consider \( r(X) \in [0, c] \).

In this case, for any \( a \in \mathcal{A} \), we have

\[
\Pr(Y = 1 \mid A = a, r(X) = r) = \Pr(Y = 1 \mid A = a, q(X) = r) = \Pr(Y = 1 \mid r(X) = r).
\]

Next, say \( r(X) \in [c + 1, 2] \). Then

\[
\Pr(Y = 1 \mid A = a, r(X) = r) = \Pr(Y = 1 \mid A = a, q(X) = r - 1) = \Pr(Y = 1 \mid r(X) = r).
\]

So in either case, \( \Pr(Y = 1 \mid A = a, r(X) = r) = \Pr(Y = 1 \mid r(X) = r) \). Thus \( Y \perp A \mid r(X) \).
A POMDP is a tuple consisting of seven elements. For an IDT decision problem, as mentioned in the main text, IDT can be seen as a special case of inverse reinforcement learning (IRL) in a partially observable Markov decision process (POMDP). Here, we present the equivalent POMDP and discuss connections to our results.

**Proof of inverse.** Now, assume $\exists a, a' \in A$ such that $c_a \neq c_{a'}$. WLOG, suppose that $c_a < c_{a'}$. Let $r : X \rightarrow \mathbb{R}$ be any function satisfying $h(x) = 1 \{ r(x) \geq t \}$. WLOG we can also assume $t = 0$. From Lemma 4.1, we know that if $a(x) = a$, then $q(x) < c_a$ implies $h(x) = 0$ and $q(x) > c_a$ implies $h(x) = 1$. Also, if $a(x) = a'$, then $q(x) < c_{a'}$ implies $h(x) = 0$ and $q(x) > c_{a'}$ implies $h(x) = 1$. Therefore,

$$\begin{align*}
P(Y = 1 | A = a, r(X) > 0) &= P(Y = 1 | A = a, q(X) > c_a) \\
&= P(Y = 1 | q(X) > c_a) \\
&= P(Y = 1 | q(X) \in (c_a, c_{a'}) \wedge \frac{P(q(X) \in (c_a, c_{a'}))}{P(q(X) > c_a)} \\
&+ P(Y = 1 | q(X) \geq c_{a'}) \frac{P(q(X) \geq c_{a'})}{P(q(X) > c_a)} \\
&\leq \frac{P(Y = 1 | q(X) \geq c_{a'})}{P(q(X) > c_a)} \\
&\leq P(Y = 1 | q(X) > c_{a'}) \\
&= P(Y = 1 | A = a', q(X) > c_{a'}) \\
&= P(Y = 1 | A = a', r(X) > 0).
\end{align*}$$

(i) and (ii) make use of the fact that

$$P(Y = 1 | q(X) \in (c_a, c_{a'})) = E[q(X) | q(X) \in (c_a, c_{a'})]$$

$$< c_{a'} \leq E[q(X) | q(X) \geq c_{a}] = P(Y = 1 | q(X) \geq c_{a'}).$$

(i) also uses the assumption that $P(q(X) \in (c_a, c_{a'})) > 0$.

Thus, we have that $P(Y = 1 | A = a, r(X) > 0) \neq P(Y = 1 | A = a', r(X) > 0)$; therefore, $Y$ and $A$ are not independent given $r(X)$, so group calibration is not satisfied.

**B POMDP Formulation of IDT**

As mentioned in the main text, IDT can be seen as a special case of inverse reinforcement learning (IRL) in a partially observable Markov decision process (POMDP) (or equivalently, belief state MDP). Here, we present the equivalent POMDP and discuss connections to our results.

A POMDP is a tuple consisting of seven elements. For an IDT decision process $(\mathcal{D}, c)$ they are:

- The state space consists of two states, each corresponding to a value of $Y$, the ground truth/correct decision. We call them $s^0$ for $Y = 0$ and $s^1$ for $Y = 1$.
- The action space consists of two actions, each corresponding to one of the decisions $\hat{Y}$. We equivalently call them $a^0$ for $\hat{Y} = 0$ and $a^1$ for $\hat{Y} = 1$.
A graphical depiction of this belief state reduction is shown in Figure 5.

A graphical depiction of this POMDP is shown in Figure 4. Any decision rule

Thus, observing decision

Since the POMDP is non-sequential, these beliefs only depend on the most recent observation

The above POMDP can be equivalently formulated as a belief state MDP. The belief states correspond to values of the posterior probability

\[ q(x) = \begin{cases} 0 & \text{if } x = \text{false} \\ p & \text{if } x = \text{true} \\ 1 & \text{if } x = \text{sufficient} \end{cases} \]

\[ \mathbb{E}[R] = \begin{cases} 0 & \text{if } x = \text{false} \\ -p(1-c) & \text{if } x = \text{true} \\ -(1-c) & \text{if } x = \text{sufficient} \end{cases} \]

\[ a_0 \rightarrow a_1 \]

\[ \mathbb{E}[R] = c \]

\[ \mathbb{E}[R] = (1-c) \]

Figure 5: A graphical depiction of the belief state MDP formulation of IDT. There is a belief state for each posterior probability \( q(x) = \mathbb{P}(Y = 1 \mid X = x) \in [0, 1] \). Observing the agent at a belief state gives a constraint on their reward function [2]. Thus, if \( q(X) \) has support on [0, 1], i.e. if there is a significant range of uncertainty in the decision problem, then there can be arbitrarily many such constraints, allowing the loss parameter \( c \) to be learned to arbitrary precision.

- The transition probabilities do not depend on the previous state or action; rather, \( s^0 \) or \( s^1 \) is randomly selected based on their probabilities under the distribution \( D \):

\[
p(s_{t+1} = s^0 \mid s_t, a_t) = \mathbb{P}_{X,Y \sim D}(Y = 0),
\]

\[
p(s_{t+1} = s^1 \mid s_t, a_t) = \mathbb{P}_{X,Y \sim D}(Y = 1).
\]

- The reward function is the negative of the loss function described in Section [3]

\[
R(s^0, a^0) = 0 \\
R(s^1, a^0) = -(1-c),
\]

\[
R(s^0, a^1) = -c \\
R(s^1, a^1) = 0.
\]

- The observation space includes elements for each \( X \in \mathcal{X} \). We denote by \( o^x \) the POMDP observation for \( x \in \mathcal{X} \).

- The observation probabilities are

\[
p(o_t = o^x \mid s_t = s^0) = \mathbb{P}(X = x \mid Y = y).
\]

- The discount factor \( \gamma \) is basically irrelevant to IDT, since the decisions are non-sequential. Thus any \( \gamma \) will produce the same behavior.

A graphical depiction of this POMDP is shown in Figure [4]. Any decision rule \( h : \mathcal{X} \rightarrow \{0, 1\} \) corresponds to a policy \( \pi \) in this POMDP:

\[
\pi(a_t = a^0 \mid o_t = o^x) = 1\{h(x) = \hat{y}\}.
\]

**Belief state MDP** The above POMDP can be equivalently formulated as a belief state MDP. The belief states correspond to values of the posterior probability

\[
\mathbb{P}(s = s^1 \mid o = o^x) = \mathbb{P}(Y = 1 \mid X = x) = q(x).
\]

A graphical depiction of this belief state reduction is shown in Figure [5].

Since the POMDP is non-sequential, these beliefs only depend on the most recent observation \( o^x \). The expected reward for action \( a^0 \) at belief state with posterior probability \( q(x) \) is

\[
R(q(x), a^0) = \mathbb{P}(s = s^0 \mid q(x))R(s^0, a^0) + \mathbb{P}(s = s^1 \mid q(x))R(s^1, a^0) = -q(x)(1-c),
\]

\[
R(q(x), a^1) = \mathbb{P}(s = s^0 \mid q(x))R(s^0, a^1) + \mathbb{P}(s = s^1 \mid q(x))R(s^1, a^1) = -(1-q(x))c.
\]

Thus, observing decision \( a^0 \) at a belief state \( q(x) \) indicates that

\[
R(q(x), a^0) \geq R(q(x), a^1)
\]

\[
-q(x)(1-c) \geq -(1-q(x))c
\]

\[
c \geq q(x).
\]

23
Similarly, observing decision $a^1$ at a belief state $q(x)$ indicates that

$$R(q(x), a^0) \leq R(q(x), a^1)$$
$$-q(x)(1 - c) \leq -(1 - q(x))c$$
$$c \leq q(x).$$

Thus, as described in Section 4.1, IDT in this (optimal) case consists of determining the threshold on $q(x)$ where the action switches from $a$ to $a^1$ for observations $a'$. This formulation gives some additional insight into why uncertainty is helpful for IDT. If $q(x) \in \{0, 1\}$ always, then there are only two belief states corresponding to $q(x) = 0$ and $q(x) = 1$. Thus, we only obtain two constraints on the value of $c$, i.e. $0 \leq c \leq 1$. However, if $q(X)$ has support on all of $[0, 1]$, then we have belief states corresponding to every $q(x) \in [0, 1]$. Thus we can obtain infinite constraints on the value of $c$, allowing learning to arbitrary precision as shown in Section 4.1.

C Alternative Suboptimality Model

As mentioned in Section 4.2, there are many ways to model suboptimal decision making. One possibility is to only require that the decision rule $h$ is close to optimal, i.e.

$$R_c(h) \leq R_c^{opt} + \Delta \quad \text{where} \quad R_c^{opt} = \inf_{h^*} R_c(h^*).$$

However, as we show in the following lemma, this assumption can preclude identifiability of $c$. The models of suboptimality we present in Sections 4.2 and 4.3, in contrast, still allow exact identifiability of the loss parameter.

Lemma C.1 (Loss cannot always be identified for close-to-optimal decision rules). Fix $0 < \Delta \leq 1$ and $0 < \epsilon < 1/\Delta$. Then for any IDT algorithm $\hat{c}(\cdot)$, there is a decision problem $(D, c)$ and a decision rule $h$ which is $\Delta$-close to optimal as in \ref{eq:close_optimal} such that

$$\Pr(|\hat{c}(S) - c| \geq \epsilon) \geq 1/2,$$

where the sample $S$ of any size is observed from the decision rule $h$. Furthermore, the distribution $D$ and loss parameter $c$ satisfy the requirements of Theorem 4.2 or when the decision maker is optimal.

Proof. Consider a distribution over $X \in [0, 1]$ where

$$q(x) = \Pr(Y = 1 \mid X = x) = x.$$

Let the distribution $D_X$ have density $\Delta$ on the interval $(1/2 - 2\epsilon, 1/2 + 2\epsilon)$ and let $\Pr(X = 0) = \Pr(X = 1) = 1/2 - \Delta\epsilon$. Let $c_1 = 1/2 - \epsilon$ and $c_2 = 1/2 + \epsilon$. Then clearly $\Pr(q(X) \in [c - \epsilon, c]) = \Pr(q(X) \in (c, c + \epsilon)) = \epsilon \Delta$ for $c \in \{c_1, c_2\}$. Thus either $c_1$ or $c_2$ satisfies the conditions of Theorem 4.2.

Now define identical decision rules

$$h_1(x) = h_2(x) = 1\{x \geq 1/2 - \epsilon\}.$$

From Lemma 4.1 we know that $h_1$ is optimal for $c_1$, so it is certainly $\Delta$-close to optimal. We can show that $h_2$ is $\Delta$-close to optimal for $c_2$ as well:

$$R_{c_2}(h_2) - R_{c_2}(x \mapsto 1\{x \geq 1/2 + \epsilon\})$$
$$= \mathbb{E}\left[\ell(1\{X \geq 1/2 - \epsilon\}, Y) - \ell(1\{X \geq 1/2 + \epsilon\}, Y)\right]$$
$$= \mathbb{E}\left[\ell(1\{X \geq 1/2 - \epsilon\}, Y) - \ell(1\{X \geq 1/2 + \epsilon\}, Y) \mid X \in [1/2 - \epsilon, 1/2 + \epsilon]\right] \Pr(X \in [1/2 - \epsilon, 1/2 + \epsilon])$$
$$\leq 2\Pr(X \in [1/2 - \epsilon, 1/2 + \epsilon])$$
$$= 4\epsilon \Delta \leq \Delta.$$

Since $h_1$ and $h_2$ are identical, we must have that for a sample $S$ chosen according to either, at least one of $\Pr(\hat{c}(S) \geq 1/2) \geq 1/2$ or $\Pr(\hat{c}(S) < 1/2) \geq 1/2$. Thus for some $c \in \{c_1, c_2\}$,

$$\Pr(|\hat{c}(S) - c| \geq \epsilon) \geq 1/2.$$
We give two lower bounds for the sample complexity in the unknown hypothesis class case from Section 4.3.

Theorem D.1 (First lower bound for suboptimal decision maker).

This can also be shown to be increasing in \( s \) satisfying the conditions of Theorem 4.10 with the above parameters such that \( p, \epsilon \) and that in some suboptimal cases a number of samples proportional to \( 1/\epsilon^2 \) is needed to estimate \( c \) to precision \( \epsilon \) more than the \( 1/\epsilon \) needed for an optimal decision maker.

Theorem D.1 (First lower bound for suboptimal decision maker). Fix \( 0 < \epsilon \leq 1/8 \), \( 0 < \delta \leq 1/2 \), and \( p_c \leq 1/10 \). Then there is a decision problem \((D, c)\), hypothesis class family \( \mathcal{H} \), and hypothesis class \( H \in \mathcal{H} \) satisfying the conditions of Theorem 4.10 with the above parameters such that

\[
m < \Omega \left( \frac{\log(1/\delta)}{p_c \epsilon^2} \right)
\]

implies that \( \mathbb{P}(|\hat{c}(S) - c| \geq \epsilon) \geq \delta \).

Proof. Specifically, let the sample size

\[
m = \frac{\log(1/(2\delta))}{40p_c \epsilon^2}.
\]

Defining the distribution First, we define a joint distribution \( D \) over \( X = (X_1, X_2) \in \mathcal{X} = \mathbb{R}^2 \) and \( Y \in \{0, 1\} \). The distribution of \( X \) has support on 2 line segments in \( \mathbb{R}^2 \) and at a point. It can be summarized as follows:

1. \( D_X \) has density \( \frac{2\pi}{\sqrt{2}} \) on the line segment from \((-1, 0)\) to \((1, 0)\), \( \mathbb{P}(Y = 1 \mid X = (x_1, 0)) = \frac{1+x_1}{1+x_2} \).
2. \( D_X \) has density \( 10p_c x_1 \) at points \((x_1, 1)\) on the line segment from \((0, 1)\) to \((1, 1)\), \( \mathbb{P}(Y = 1 \mid X = (x_1, 1)) = 1 \).
3. \( D_X \) has point mass \( \mathbb{P}(X = (-1, 0)) = 1 - 10p_c \), \( \mathbb{P}(Y = 1 \mid X = (-1, 0)) = 0 \).

Defining the family of hypothesis classes Now, we define a family of two hypothesis classes:

\[
\mathcal{H}_1 \triangleq \{ h(x) = \mathbf{1}\{x_1 \geq b\} \mid b \in [3/8, 5/8] \}
\]
\[
\mathcal{H}_2 \triangleq \{ h(x) = \mathbf{1}\{x_1 \geq b + 2c x_2\} \mid b \in [1/2, 3/4] \}
\]
\[\mathbb{H} \triangleq \{ \mathcal{H}_1, \mathcal{H}_2 \} \]

Let’s analyze \( \mathcal{H}_1 \) first. The posterior probability that \( Y = 1 \) given that \( X_1 = x_1 \) is

\[
\mathbb{P}(Y = 1 \mid X_1 = x_1) = \begin{cases} \frac{1+x_1}{1+x_2}, & x_1 < 0 \\ \frac{1+x_2}{1+x_1}, & x_1 \geq 0. \end{cases}
\]

It is simple to show that this is increasing in \( x_1 \); thus, the Bayes optimal decision rule based on \( X_1 \) for \( c \) is

\[
h_1^c(x_1) = \begin{cases} \mathbf{1}\{x_1 \geq 2c - 1\}, & c \leq 1/2 \\ \mathbf{1}\{x_1 \geq \frac{2c - 1}{2c - 1 - 6c}\}, & c > 1/2. \end{cases}
\]

Now, let’s analyze \( \mathcal{H}_2 \). The posterior probability that \( Y = 1 \) given that \( X_1 - 2c x_2 = b \) for \( b \geq -2c \) is

\[
\mathbb{P}(Y = 1 \mid X_1 - 2c x_2 = b) = \frac{1 + 9b + 16c}{2 + 8b + 16c}.
\]

This can also be shown to be increasing in \( b \), so the Bayes optimal decision rule based on \( X_1 - 2c x_2 \) for \( c \geq 1/2 \) is

\[
h_2^c(x_1) = \mathbf{1}\left\{ x_1 - 2c x_2 \geq \frac{2c - 1 - 16c + 16c}{9 - 8c} \right\}.
\]

For this proof, we consider two hypothesis class and loss parameter pairs: \( c_1 = 1/2 \) for \( \mathcal{H}_1 \) and \( c_2 = \frac{1+16c}{2+16c} \) for \( \mathcal{H}_2 \). These correspond to the decision rules

\[
h_1^c(x_1) = \mathbf{1}\{x_1 \geq 0\}.
\]
\[ h^2(x) = 1\{x_1 - 2\epsilon x_2 \geq 0\} = \begin{cases} x_1 \geq 0 & x_2 = 0 \\ x_1 \geq 2\epsilon & x_2 = 1. \end{cases} \]

It should be clear that these decision rules agree except when \(x_2 = 1\) and \(x_1 \in [0, 2\epsilon)\).

Another important fact is that
\[ c_2 = \frac{1 + 16\epsilon}{2 + 16\epsilon} = \frac{1}{2} + \frac{4\epsilon}{1 + 8\epsilon} \geq \frac{1}{2} + 2\epsilon \]

since \(\epsilon \leq 1/8\).

We defer to the end of the proof to show that these hypotheses and distribution satisfy the conditions of Theorem 4.10.

**Deriving the lower bound** Similarly to the proof of Theorem 4.11 we can bound the probability of an event \(E\):
\[
P(\exists x_i \in S \quad x_{i,1} \in [0, 2\epsilon) \land x_{i,2} = 1) = [1 - P(X_1 \in [0, 2\epsilon) \land X_2 = 1)]^m
\]
\[
= (1 - 20p_c \epsilon^2)^m \\
\geq \left(e^{-40p_c \epsilon^2}\right)^m \\
= e^{-\log(2/\delta)} = 2\delta.
\]

Conditional on \(E\), the distributions of samples \(S_1\) and \(S_2\) for decision rules \(h^1\) and \(h^2\) are identical:
\[
p_1 = P(\hat{c}(S_1) \leq 1/2 + \epsilon \mid E) = P(\hat{c}(S_2) \leq 1/2 + \epsilon \mid E),
\]
\[
p_2 = P(\hat{c}(S_1) > 1/2 + \epsilon \mid E) = P(\hat{c}(S_2) > 1/2 + \epsilon \mid E).
\]

Since \(p_1 + p_2 = 1\), at least one of \(p_1, p_2 \geq 1/2\). Suppose WLOG that \(p_1 \geq 1/2\). Then
\[
P(|\hat{c}(S_2) - c_2| \geq \epsilon) \geq \mathbb{P}(\hat{c}(S_2) \leq 1/2 + \epsilon)
\]
\[
= \mathbb{P}(\hat{c}(S_2) \leq 1/2 + \epsilon \mid E) \mathbb{P}(E)
\]
\[
\geq 1/2(2\delta) = \delta.
\]

(i) uses the fact shown earlier in (20). Thus, there is a decision problem \((D, c_2)\) for which the IDT algorithm \(\hat{c}\) must make an error of at least size \(\epsilon\) with at least probability \(\delta\). This concludes the main proof.

**Verifying the requirements of Theorem 4.10** First, we need to show that \(q_{h^1}(X)\) has density at least \(p_c\) on \([c_1 - \epsilon, 1/2 + 1/2 + \epsilon] = [1/2 - \epsilon, 1/2 + \epsilon]\). From (16) and (17), it is clear that
\[
q_{h^1}(x) = g_1(x_1) = \begin{cases} \frac{1 + \epsilon}{1 + 2\epsilon} x_1 < 0 \\ \frac{1 + 2\epsilon}{1 + 9\epsilon} x_1 \geq 0. \end{cases}
\]

We can write the density of \(q_{h^1}(X)\) as the density of \(X_1\) multiplied by the derivative of the inverse of \(g_1\):
\[
p(x_1) \frac{d}{dc} g_1^{-1}(c) \geq \frac{5p_c}{2} \frac{d}{dc} \begin{cases} 2c - 1 & c \leq 1/2 \\ 2 - 8c & c > 1/2 \end{cases}
\]
\[
= \frac{5p_c}{2} \begin{cases} 2 & c \leq 1/2 \\ \frac{16}{(9 - 8c)^2} & c > 1/2 \end{cases}
\]
\[
\geq p_c.
\]

Next, we need to show that \(q_{h^2}(X)\) has density at least \(p_c\) on \([c_2 - \epsilon, c_2 + \epsilon] \subseteq [1/2, 1]\). From (18) and (19), we know that
\[
q_{h^2}(x) = g_2(x_1 - 2\epsilon x_2) = \frac{1 + 9(x_1 - 2\epsilon x_2) + 16\epsilon}{2 + 8(x_1 - 2\epsilon x_2) + 16\epsilon}.
\]

Using the same method as for \(q_{h^1}(X)\) and the fact that the density of \(X_1 - 2\epsilon X_2\) is at least the density of \(X_1\) (i.e., \(\frac{2}{9}\)), we have that the density of \(q_{h^2}(X)\) is at least
\[
\frac{5p_c}{2} \frac{d}{dc} g_2^{-1}(c) = \frac{5p_c}{2} \frac{d}{dc} \begin{cases} 2c - 1 - 16\epsilon + 16\epsilon x & c \leq 1/2 \\ 9 - 8c & c > 1/2 \end{cases}
\]
\[
= \frac{5p_c}{2} \begin{cases} 10 + 16\epsilon & c \leq 1/2 \\ \frac{10 + 16\epsilon}{(9 - 8c)^2} & c > 1/2 \end{cases}
\]
\[
\geq \frac{5p_c}{2} \geq p_c.
\]
The only remaining condition of Theorem 4.10 to prove is MD-smoothness. Again, consider $\mathcal{H}^1$ first:

$$\text{MD}(h^1_{b_1}, \mathcal{H}^2) = \min_{b_2 \in [1/2, 3/4]} P(h^1_{b_2}(X) \neq h^2_{b_2}(X))$$

$$= \min_{b_2 \in [1/2, 3/4]} \frac{5p_c}{2} |b_1 - b_2| + 5p_c |b_1^2 - (b_2 + 2\epsilon)^2|$$

$$= \frac{5p_c}{2} |b_1 - b_2| + 5p_c |b_1^2 - (b_1 + 2\epsilon)^2|$$

$$= 20p_c |\epsilon(b_2 + \epsilon)|.$$

From (17), we know that $b_1 - b_1' \leq 10(c_1 - c_1')$ where $b_1$ and $b_1'$ are the optimal thresholds for loss parameters $c_1$ and $c_1'$, respectively. So we have that

$$\text{MD}(h^1_{b_1'}, \mathcal{H}^2) - \text{MD}(h^1_{b_1}, \mathcal{H}^2) = 20p_c |\epsilon(b_1' + \epsilon) - |b_1 + \epsilon||$$

$$\leq 20p_c |\epsilon b_1' - b_1|$$

$$\leq 200p_c |\epsilon c_1' - c_1|.$$

Thus $h^1$ and $\mathcal{H}$ are $\alpha$-MD-smooth with $\alpha = 200p_c\epsilon$.

Similarly, for $\mathcal{H}^2$,

$$\text{MD}(h^2_{b_2}, \mathcal{H}^1) = \min_{b_1 \in [1/2, 3/4]} P(h^2_{b_1}(X) \neq h^1_{b_1}(X))$$

$$= \min_{b_1 \in [1/2, 3/4]} \frac{5p_c}{2} |b_2 - b_1| + 5p_c |b_2^2 - (b_1 + 2\epsilon)^2|$$

$$= \frac{5p_c}{2} |b_2 - b_1| + 5p_c |b_2^2 - (b_2 + 2\epsilon)^2|$$

$$= 20p_c |\epsilon(b_2 + \epsilon)|.$$

So we have that

$$\text{MD}(h^2_{b_2'}, \mathcal{H}^1) - \text{MD}(h^2_{b_2}, \mathcal{H}^1) = 20p_c |\epsilon(b_2' + \epsilon) - |b_2 + \epsilon||$$

$$\leq 20p_c |b_2' - b_2|$$

$$\leq 200p_c |\epsilon c_2' - c_2|,$$

and thus $h^2$ and $\mathcal{H}$ are also $200p_c\epsilon$-MD-smooth.

Theorem D.2 (Second lower bound for suboptimal decision maker). Let $d \geq 6$ such that $d \equiv 2 \pmod{4}$. Let $\epsilon \in (0, \frac{\sqrt{d} - 2}{64p_c})$ and $p_c \in (0, 1)$. Then for any IDT algorithm $\hat{c}(\cdot)$, there is a decision problem $(\mathcal{D}, c)$, hypothesis class family $\mathcal{H}$, and hypothesis class $\mathcal{H} \in \mathcal{H}$ satisfying the conditions of Theorem 2.10 with the above parameters such that

$$m < \Omega \left( \frac{\sqrt{d}}{p_c \epsilon} \right) \quad \text{implies that} \quad P(|\hat{c}(\mathcal{S}) - c| \geq \epsilon) \geq \frac{1}{100}.$$

Proof. Specifically, let

$$m = \frac{\sqrt{d} - 2}{64p_c \epsilon}.$$

Defining the distribution Let $n = d - 2 \geq 1$; $n$ is divisible by four. First, we define a joint distribution $\mathcal{D}$ over $X \in X = \mathbb{R}^{n+1}$ and $Y \in \{0, 1\}$. Let $X_j$ refer to the $j$th coordinate of the random vector $X$ and let $x_{ij}$ refer to the $j$th coordinate of the $i$th sample $x_i$. Furthermore, let $X_{1:n}$ refer to the first $n$ components of $X$.

The distribution of $X$ has support on $n$ line segments in $\mathbb{R}^{n+1}$ and at the origin. In particular, it has density $p_c/n$ on each line segment from $(0, \ldots, X_j = 1, \ldots, 0, 0)$ to $(0, \ldots, X_j = 1, \ldots, 0, 1)$, where the density is with respect to the Lebesgue measure on the line. There is additionally a point mass of probability $1 - p_c$ at the origin. Everywhere on the support of $\mathcal{D}$,

$$P(Y = 1 \mid X_{1:n} = x_{1:n}, X_{n+1} = x_{n+1}) = x_{n+1}.$$

Defining the family of hypothesis classes Next, we define a family of hypothesis classes. Let $\sigma \in \{-1, 1\}^n$ and define

$$f^\sigma(x) = x_{n+1} - 8\epsilon \sqrt{n}\sigma^T x_{1:n}.$$
Then we define $2^n$ hypothesis classes, one for each value of $\sigma$:

$$H^\sigma \triangleq \{ h(x) = 1 \{ f^\sigma (x) \geq b \} \mid b \in [1/4, 3/4] \},$$

$$\mathcal{H} \triangleq \{ H^\sigma \mid \sigma \in \{0, 1\}^n \}.$$

Now, we can derive the optimal decision rule in hypothesis class $H^\sigma$ for loss parameter $c$. Let $[f^\sigma (X)]_{1/4}^{3/4} = \max(1/4, \min(3/4, f^\sigma (X)))$ denote the value $f^\sigma (X)$ clamped to the interval $[1/4, 3/4]$. Then for $b \in (1/4, 3/4)$,

$$P \left( Y = 1 \mid [f^\sigma (X)]_{1/4}^{3/4} = b \right) = P \left( Y = 1 \mid X_{n+1} - 8\sqrt{n}\sigma^T X_{1:n} = b \right)$$

$$= \frac{1}{n} \sum_{j=1}^{n} P \left( Y = 1 \mid X_j = 1 \land X_{n+1} = b + 8\sqrt{n}\sigma_j \right)$$

$$= b + 8\sqrt{n}\frac{1}{n} \sigma^T.$$

where $1$ is the all-ones vector. Thus, the Bayes optimal decision rule based on $[f^\sigma (X)]_{1/4}^{3/4}$ is

$$h^\sigma_n (x) = 1 \left\{ f^\sigma (x) + 8\sqrt{n}\frac{1}{n} \sigma^T \geq c \right\}$$

$$= 1 \left\{ f^\sigma (x) \geq c - 8\sqrt{n}\frac{1}{n} \sigma^T \right\}$$

for $c - 8\sqrt{n}\frac{1}{n} \sigma^T \in (1/4, 3/4)$. The induced posterior probability for $H^\sigma$ is

$$q_{H^\sigma} (x) = f^\sigma (x) + 8\sqrt{n}\frac{1}{n} \sigma^T.$$

We consider one hypothesis from each hypothesis class $H^\sigma \in \mathcal{H}$. Specifically, we consider the optimal decision rule for

$$c^\sigma = \frac{1}{2} + 8\sqrt{n}\frac{1}{n} \sigma^T,$$

which, as shown above is,

$$h^\sigma (x) = 1 \left\{ f^\sigma (x) \geq \frac{1}{2} \right\}. \quad (21)$$

We leave until the end of the proof to show that each of these decision rules $h_\sigma$ for $\sigma \in \{ -1, 1 \}^n$ satisfies the requirements of Theorem 4.10.

**Deriving the lower bound** Now, we are ready to derive the lower bound that there is some $h^\sigma$ such that $P([c(S) - c^\sigma] \geq \epsilon) \geq \frac{1}{100}$. First, we can rewrite $h^\sigma$ from (21) as

$$h^\sigma ((0, x_j = 1, 0, x_{n+1})) = 1 \{ x_{n+1} - 8\sqrt{n}\sigma_j \geq 1/2 \}$$

$$= 1 \{ x_{n+1} \geq 1/2 + 8\sqrt{n}\sigma_j \}.\]$$

Thus, only decisions made on points where $x_{n+1} \in [1/2 - 8\sqrt{n}, 1/2 + 8\sqrt{n}]$ are dependent on $\sigma_j$. Denote by $E_j$ the event that there is an observed sample that depends on $\sigma_j$:

$$E_j \triangleq \exists x_i \in S \text{ such that } x_{ij} = 1 \land x_{i,n+1} \in [1/2 - 8\sqrt{n}, 1/2 + 8\sqrt{n}].$$

Suppose we let $\sigma_j$ be independently Rademacher distributed, i.e. we assign equal probability $1/2^n$ to each $\sigma \in \{ -1, 1 \}$. Then if $E_j$ does not occur, the sample of decisions $S$ is independent from $\sigma_j$, i.e.

$$S \perp \sigma_j \mid \neg E_j.$$

Now let $F$ denote the event that more than $n/2$ of the $E_j$ events occur:

$$F \triangleq \left| \{ j \in 1, \ldots, n \mid E_j \} \right| > n/2.$$

We will start by proving a lower bound on $P([c(S) - c^\sigma] \geq \epsilon \mid \neg F)$. If $F$ does not occur, then at least half of the $E_j$ do not occur. Thus at least half of the elements of $\sigma$ are independent from the sample $S$. Let $I$ be the set of indices $j$ for which $E_j$ does not occur; thus, $\sigma_I \perp \perp S$, and given $\neg F$, $|I| \geq n/2$.

We can decompose $c^\sigma$ into part that depends on $\sigma_I$ and part that depends on $\sigma_{Ic}$:

$$c^\sigma = \frac{1}{2} + 8\sqrt{n}\frac{1}{n} \sigma_I^T + 8\sqrt{n}\frac{1}{n} \sigma_{Ic}^T.\quad (22)$$
Note that for each $j \in I$, $\frac{c_j + 1}{n}$ is $\frac{1}{2}$-Bernoulli distributed. Thus
\[
Z = \frac{1}{2} \sum_{j \in I} \frac{c_j + 1}{n} = \sum_{j \in I} \frac{c_j + 1}{2} \sim \text{Binom}\left(|\mathcal{I}| \cdot \frac{1}{2}\right).
\]
We can establish lower bounds on the tails of this given that $F$ occurs:
\[
P\left(Z - \frac{|\mathcal{I}|}{2} \geq t \mid \neg F\right) = \left(Z - \frac{|\mathcal{I}|}{2} \leq -t \mid \neg F\right) \geq \frac{1}{15} e^{-32t^2/n}.
\]
This lower bound is from Matoušek and Vondrák [42]. Plugging in $t = \frac{1}{4} \sqrt{n}$, we obtain
\[
P\left(Z - \frac{|\mathcal{I}|}{2} \geq \frac{1}{8} \sqrt{n} \mid \neg F\right) = \left(Z - \frac{|\mathcal{I}|}{2} \leq -\frac{1}{8} \sqrt{n} \mid \neg F\right) \geq \frac{1}{20}
\]
\[
P\left(1^\top \sigma_j \geq \frac{1}{4} \sqrt{n} \mid \neg F\right) = \left(1^\top \sigma_j \leq -\frac{1}{4} \sqrt{n} \mid \neg F\right) \geq \frac{1}{20}.
\]
(23)

Given $S$, $\sigma_j | F$ is completely known (since $E_j$ occurs for each $j \in I^c$, revealing $\sigma_j$). So plugging (23) into (22) gives
\[
P\left(c' - \frac{1}{2} - 8\epsilon \sqrt{n} \frac{1^\top \sigma_j | F}{n} \geq 2\epsilon \mid \neg F, S\right) = \left(c' - \frac{1}{2} - 8\epsilon \sqrt{n} \frac{1^\top \sigma_j | F}{n} \leq -2\epsilon \mid \neg F, S\right) = \frac{1}{20}
\]
\[
P\left(c' - c'j | \neg F, S\right) = \left(c' - c'j \leq -2\epsilon \mid \neg F, S\right) \geq \frac{1}{20}.
\]
That is, there is at least probability $\frac{1}{20}$ that $c'$ is more than $2\epsilon$ above and below $c'j | \neg F$, given $\neg F$ and the observed sample $S$.

This is enough to show that $P(|\hat{c}S - c' | \geq \epsilon | \neg F, S) \geq \frac{1}{40}$. First, observe that
\[
P(\hat{c}S \geq c'j | \neg F, S) + P(\hat{c}S < c'j | \neg F, S) = 1,
\]
so one of these probabilities must be at least $\frac{1}{2}$. Say WLOG that it is the first. Then
\[
P(\hat{c}S - c' | \geq \epsilon | \neg F, S) \geq P(\hat{c}S - c'j \leq -2\epsilon | \neg F, S)
\]
\[
\overset{(i)}{=} P(c' - c'j \leq -2\epsilon | \neg F, S) \cdot P(\hat{c}S \geq c'j | \neg F, S)
\]
\[
\geq \left(\frac{1}{20}\right) \left(\frac{1}{2}\right) = \frac{1}{40}.
\]
Here, (i) makes use of the fact that $S \perp \sigma_j | - F$. Given this, we can finally derive the lower bound on the unconditional probability that $P(\hat{c}S - c' | \geq \epsilon)$:
\[
P(\hat{c}S - c' | \geq \epsilon) = P(\hat{c}S - c' | \geq \epsilon | F)P(F) + P(\hat{c}S - c' | \geq \epsilon | \neg F)P(\neg F)
\]
\[
\geq P(\hat{c}S - c' | \geq \epsilon | \neg F)P(\neg F)
\]
\[
\geq \frac{P(\neg F)}{40}.
\]
(24)

So we need to derive a lower bound on $P(\neg F)$. We can do so by noting that in order for $F$ to occur, there must be at least $n/2$ samples $x_i$ with $x_{i,n+1} \in [1/2 - 8\epsilon \sqrt{n}, 1/2 + 8\epsilon \sqrt{n}]$. The probability of this event for a particular sample is
\[
P\left(X_{n+1} \in [1/2 - 8\epsilon \sqrt{n}, 1/2 + 8\epsilon \sqrt{n}]\right) = 16p_0 \epsilon \sqrt{n}.
\]
So at least $n/2$ of the $m$ samples must have the event with probability $16p_0 \epsilon \sqrt{n}$ occur for $F$ to occur. Let $\text{GE}(p, m, r)$ denote the probability of at least $r$ successes of probability $p$ in $m$ independent trials. Then there is the following fact from probability theory [43]:
\[
\text{GE}(p, m, (1 + \gamma)mp) \leq e^{-\gamma^2 mp/3}.
\]
Then
\[
P(F) \leq \text{GE}(16p_0 \epsilon \sqrt{n}, m, n/2)
\]
\[
= \text{GE}\left(16p_0 \epsilon \sqrt{n}, \sqrt{n}_{64p_0 \epsilon}, 2 \left(\sqrt{n}_{64p_0 \epsilon}\right) (16p_0 \epsilon \sqrt{n})\right)
\]
29
Verifying the requirements of Theorem 4.10

Now we show that the distribution and hypothesis class family $A$.

A similar result can be shown for $P$.

The lower bounds given in Section D.1 do not depend on the $\alpha$.

Thus overall $h^\sigma$ is $H^\beta$ which has the same tails on $c(S)$. Thus we conclude the proof.

**Verifying the requirements of Theorem 4.10**

Now we show that the distribution and hypothesis class family satisfy the conditions of Theorem 4.10. First, note that all $h \in H \in \mathbb{H}$ are thresholds on linear functions of the observation $x$. Thus, $\cup_{H \in \mathbb{H}} H$ is a subset of the halfspaces in $\mathbb{R}^{n+1}$ and so it has VC-dimension at most $n + 2 = d$

Next, it is clear that for $\rho \leq \epsilon$,

$$P(q_{H^\sigma}(X) \in (c, c + \rho)) = \mathbb{P}\left(f^\sigma(X) + 8\epsilon\sqrt{n}\frac{1}{\sqrt{n}} \in (c, c + \rho)\right)$$

$$= \sum_{j=1}^{n} P\left(X_j = 1 \land X_{n+1} - 8\epsilon\sqrt{n}\sigma_j + 8\epsilon\sqrt{n}\frac{1}{\sqrt{n}} \in (c, c + \rho)\right)$$

$$= \sum_{j=1}^{n} \frac{p_j\rho}{n} = p_r\rho.$$  

A similar result can be shown for $P(q_{H^\sigma}(X) \in (c - \rho, c))$.

Finally, we need to show that MD-smoothness holds. Take any $h^\sigma$ and any $H^\beta$. Then the disagreement between $h^\sigma$ and a hypothesis in $H^\beta$ with threshold $b$ is

$$P(h^\sigma(X) \neq h^\beta_b(X)) = \frac{p_r}{n} \sum_{j=1}^{n} \left| \frac{1}{2} + 8\epsilon\sqrt{n}\sigma_j - b - 8\epsilon\sqrt{n}\sigma_j \right|$$

This is minimized when $b$ is the median of $(\frac{1}{2} + 8\epsilon\sqrt{n}(\sigma_j - \bar{\sigma}))$ for $j = 1, \ldots, n$. Thus $b \in [\frac{1}{2} - 8\epsilon\sqrt{n}, \frac{1}{2} + 8\epsilon\sqrt{n}]$; since $\epsilon \leq \frac{1}{160\sqrt{n}}$, this implies $b \in [3/8, 5/8]$. Suppose now we let $c' \in [c^\sigma - 1/8, c^\sigma + 1/8]$. Then we can let $b' = b + (c' - c^\sigma)$ and

$$\text{MD}(h^\sigma_c, H^\beta) \leq \mathbb{P}\left(h^\sigma_c(X) \neq h^\beta_b(X)\right) = \mathbb{P}\left(h^\sigma(X) \neq h^\beta_b(X)\right) = \text{MD}(h^\sigma, H^\beta).$$

Thus for $|c' - c^\sigma| \leq 1/8$, $h^\sigma$ and $H^\beta$ are 0-MD-smooth. If $|c' - c^\sigma| > 1/8$, then we have

$$\text{MD}(h^\sigma_c, H^\beta) \leq 1 < \frac{8}{\text{MD}(h^\sigma, H^\beta)} |c' - c^\sigma| \text{MD}(h^\sigma, H^\beta).$$

Thus overall $h^\sigma$ and $H^\beta$ are $\alpha$-MD-smooth with

$$\alpha = \max_{\delta \neq \sigma} \frac{8}{\text{MD}(h^\sigma, H^\beta)}.$$  

**Bibliographic note:** we establish dependence on the VC dimension $d$ in Theorem 4.12 using a technique similar to that used by Ehrenfeucht et al. [44].

**D.2 Necessity of MD-smoothness**

The lower bounds given in Section D.1 do not depend on the $\alpha$ parameter from the MD-smoothness assumption made in Theorem 4.3, thus, one may wonder if this assumption is necessary. In the following lemma, we show that it is necessary in some cases by giving an example of an IDT problem where a lack of MD-smoothness precludes identifiability of the loss parameter.

**Lemma D.3 (No MD-smoothness can prevent identifiability).** Let $\epsilon \in (0, 1/10)$. Then for any IDT algorithm $\hat{c}(\cdot)$, there is a decision problem $(D, c)$, hypothesis class family $H$, and hypothesis class $H \in \mathbb{H}$ satisfying the conditions of Theorem 4.10 except for MD-smoothness such that

$$P(|\hat{c}(S) - c| \geq \epsilon) \geq \frac{1}{2}$$

for a sample $S$ of any size $m$.
We consider the optimal decision rules for clearly, the family of hypothesis classes defined above verifying the other requirements of Theorem 4.10.

The distribution and decision rules are visualized in Figure 6.

**Proof.** Defining the distribution First, we define a distribution \( \mathcal{D} \) over \( X \in \mathbb{R}^2 \) and \( Y \in \{0, 1\} \). \( \mathcal{D}_X \) has density \( \frac{1}{2} \) on two squares \([-1, 0] \times [-1, 0] \) and \([0, 1] \times [0, 1] \), and the distribution of \( Y \mid X \) is defined as follows:

\[
P(Y = 1 \mid X = x) = \begin{cases} \frac{2}{5} + \frac{2}{15}x_1 + \frac{8}{15}x_2 & x \in [-1, 0] \times [-1, 0] \\ \frac{1}{5} + \frac{8}{15}x_1 + \frac{2}{15}x_2 & x \in [0, 1] \times [0, 1]. \end{cases}
\]

**Defining the family of hypothesis classes** We consider the two hypothesis classes which are thresholds on one component of the observation \( x \):

\[
\mathcal{H}_1 = \{ h(x) = 1 \{ x_1 \geq b \} \mid b \in [-1, 1] \}, \\
\mathcal{H}_2 = \{ h(x) = 1 \{ x_2 \geq b \} \mid b \in [-1, 1] \}.
\]

That is, \( \mathbb{H} = \{ \mathcal{H}_1, \mathcal{H}_2 \} \). The conditional probabilities for \( Y = 1 \) given just one of the observation components are

\[
q_{\mathcal{H}_1}(x) = \mathbb{P}(Y = 1 \mid X_1 = x_1) = \frac{2}{5} + \frac{2}{15}x_1 + \frac{8}{15}x_1 \{ x_1 \geq 0 \}, \\
q_{\mathcal{H}_2}(x) = \mathbb{P}(Y = 1 \mid X_2 = x_2) = \frac{3}{5} + \frac{2}{15}x_2 + \frac{2}{15}x_2 \{ x_2 \leq 0 \}.
\]

We consider the optimal decision rules for \( c_1 = 2/5 \) and \( c_2 = 3/5 \) in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively, which from the above can be calculated as

\[
h_1(x) = 1 \{ x_1 \geq 0 \}, \\
h_2(x) = 1 \{ x_2 \geq 0 \}.
\]

The distribution and decision rules are visualized in Figure 6.

**Lack of identifiability** Note that since \( X \) only has support where \( \text{sgn}(X_1) = \text{sgn}(X_2) \), the above decision rules are indistinguishable. Thus, we use the same techniques from Corollary 4.12 and Lemma C.1 to show that for at least one of \( c \in \{ c_1, c_2 \} \)

\[
P(|\hat{c}(S) - c| \geq \frac{1}{2} | c_2 - c_1 |) \geq \frac{1}{10} \geq \frac{1}{2}.
\]

**Hypothesis classes are not MD-smooth** Although this is not required for the proof of the lemma, we will demonstrate that the defined hypothesis classes are not \( \alpha \)-MD-smooth for any \( \alpha \). By way of contradiction, assume that there is some \( \alpha \) such that \( h_1 \) and \( \mathbb{H} \) are MD-smooth. Then for any \( c' \in [0, 1] \),

\[
\text{MD}(h_1, \mathcal{H}_2) \leq (1 + \alpha |c' - c_1|) \text{MD}(h_1, \mathcal{H}_2) = 0.
\]

Here, \( \text{MD}(h_1, \mathcal{H}_2) \) since \( \mathbb{P}(h_1(X) \neq h_2(X)) = 0 \), i.e. \( h_1 \) and \( h_2 \) do not disagree at all. However, there are clearly values of \( c' \) such that \( \text{MD}(h_1', \mathcal{H}_2) > 0 \), so we have a contradiction.

**Verifying the other requirements of Theorem 4.10** Clearly, the family of hypothesis classes defined above have finite VC-dimension.

31
The densities of \( q_{H_1}(X) \) and \( q_{H_2}(X) \) can be calculated as the density of \( X_1 \) or \( X_2 \) multiplied by the derivative of the inverse of the posterior probability functions. The densities of \( X_1 \) and \( X_2 \) are both \( 1/2 \) on the interval \([-1, 1]\), and the derivative of the inverse of the equations in (25) is at least \( 15/8 \). So the distribution satisfies the requirements of Theorem 4.10 other than \( \alpha \)-MD-smoothness with \( p_c \geq 15/16 \).

\[ \blacksquare \]

### E Feature Subset Hypothesis Class Family

In this section, we work through the application of Theorem 4.10 to a practical example. Theorem 4.10 concerns the case of IDT when the decision maker could be restricting themselves to any suboptimal hypothesis class \( \mathcal{H} \in \mathcal{H} \) for some family of hypothesis classes \( \mathcal{H} \). In this example, we consider \( \mathcal{H}_{\text{feat}} \) as defined in (1) and repeated here:

\[ \mathcal{H}_{\text{feat}} \triangleq \{ \mathcal{H}_S \mid S \subseteq \{1, \ldots, n\} \} \quad \text{where} \quad \mathcal{H}_S \triangleq \left\{ h(x) = f(x_S) \mid f : \mathbb{R}^{|S|} \to \{0, 1\} \right\}. \]

This family can model decision makers that have bounded computational capacity and may only be able to reason based on a few features of the data. An application of structural risk minimization [45] from learning theory shows that the sample complexity of IDT in this case may scale only linearly in the number of features considered and logarithmically in the total feature count:

**Lemma E.1.** Let a decision maker use a hypothesis class \( \mathcal{H}_S \in \mathcal{H}_{\text{feat}} \) as defined in (1) which consists of decision rules depending only on the subset of the features in \( S \). Let \( s = |S| \) be the number of such features; neither \( s \) nor \( S \) is known. Suppose \( \mathcal{X} = \mathbb{R}^d \), i.e. \( d \) is the total number of features. Let assumptions on \( \epsilon, \delta, \alpha, \) and \( p_c \) be as in Theorem 4.10.

Let \( \hat{h}_c \in \arg\min_{h \in \mathcal{H}_S} R_{\epsilon}(\hat{h}) \) be chosen to be consistent with the observed decisions, i.e. \( \hat{h}_c(x_i) = \hat{y}_i \), and such that \( |\hat{S}| \) is as small as possible. Then \( |\hat{c} - c| \leq \epsilon \) with probability at least \( 1 - \delta \) as long as the number of samples \( m \) satisfies

\[ m \geq O \left( \left( \frac{\alpha}{\epsilon} + \frac{1}{\epsilon} \right) \left( \frac{s \log d + \log(1/\delta)}{p_c} \right) \right). \]

**Proof.** We prove Lemma E.1 by bounding the VC-dimension of the union of all optimal decision rules in all \( \mathcal{H}_S \in \mathcal{H}_{\text{feat}} \) where \( |S| \leq s \). An optimal decision rule for loss parameter \( c \) in \( \mathcal{H}_S \) is given by the Bayes optimal classifier:

\[ h^*_c(x) = 1 \{ \mathbb{P}(Y = 1 \mid X_S = x_S) \geq c \}. \]

Now consider a set of observations \( x_1, \ldots, x_d \in \mathcal{X} \). We will show that for \( d > 1 + 2s \log_2(n + 1) \), this set cannot be shattered by \( d \). To see why, note that decision rules in any particular class \( \mathcal{H}_S \) threshold the posterior probability \( \mathbb{P}(Y = 1 \mid X_S = x_S) \). Thus, each hypothesis class can only produce \( d + 1 \) distinct labelings of the set of observations. The number of hypothesis classes \( \mathcal{H}_S \) with \( |S| \leq s \) is.

\[ \sum_{k=0}^{s} \left( \begin{array}{c} n \\end{array} \right) k \leq \sum_{k=0}^{s} n^k \leq (n + 1)^{s}. \]

So the number of distinct labelings assigned by hypotheses in \( \mathcal{H} \) to the observations must be at most \((d + 1)(n + 1)^s < 2^d\) if \( d > 1 + 2s \log_2(n + 1) \). Thus this set cannot be shattered, so

\[ \text{VCDim} \left( \bigcup_{|S| \leq s} \mathcal{H}_S \right) \leq 1 + 2s \log_2(n + 1) = O(s \log n). \]

Applying Theorem 4.10 with \( d = O(s \log n) \) completes the proof.

\[ \blacksquare \]

The following lemma states conditions under which \( \alpha \)-MD-smoothness holds for \( \mathcal{H}_{\text{feat}} \).

**Lemma E.2.** Let \( \mathcal{H}_{\text{feat}} \) and \( \mathcal{H}_S \) be defined as in (1). Let \( h \in \mathcal{H}_S \). Suppose that there is a \( \zeta > 0 \) such that for any \( S \subseteq \{1, \ldots, n\} \), one of the following holds: either (a) \( \mathbb{P}(Y = 1 \mid X = x_S) = \mathbb{P}(Y = 1 \mid X = x_S) \) for all \( x \in \mathbb{R}^d \), or (b) \( MD(h, \mathcal{H}_S) \geq \zeta \). Furthermore, suppose that the distribution of \( q_{H_1}(X) \) is absolutely continuous with respect to the Lebesque measure and that its density is bounded above by \( M < \infty \). Then \( h \) and \( \mathcal{H}_{\text{feat}} \) are \( \alpha \)-MD-smooth with \( \alpha = M / \zeta \).

Since \( \alpha \)-MD-smoothness is a sufficient condition for identification of the loss function parameter \( c \), Lemma E.2 gives conditions under which IDT can be performed. The main requirement is that considering different subsets of the features either gives identical decision rules (case (a)) or decision rules which disagree by some minimum amount (case (b)). If decision rules using a different subset of the features can be arbitrarily close to the true one, it may not be possible to apply IDT.
Proof. Consider any $\tilde{S} \subseteq \{1, \ldots, n\}$. If (a) holds for $\tilde{S}$, then $h_{\tilde{S}}^S(x) = h_{\tilde{S}}^S(x)$ for any $c \in [0, 1]$ and $x \in X$. Thus

$$\text{MD}(h_{\tilde{S}}^S, H_{\tilde{S}}) = 0 \leq (1 + \alpha|c' - c|)\text{MD}(h_{\tilde{S}}^S, H_{\tilde{S}}) = 0$$

so $\alpha$-MD-smoothness holds in this case for any $\alpha$.

If (b) holds, then let $\hat{h} \in \arg\min_{h \in H_{\tilde{S}}} \mathbb{P}(h(X) \neq \hat{h}(X))$. Let $c' \in [0, 1]$; without loss of generality, we may assume that $c' > c$. Denote $q_S(x) = \mathbb{P}(Y = 1 \mid X_S = x)$. Then

$$\text{MD}(h_{\tilde{S}}^S, H_{\tilde{S}})$$

$$\leq \mathbb{P}(h_{\tilde{S}}^S(X) \neq \hat{h}(X))$$

$$= \mathbb{P}(q_S(X) < c' \land \hat{h}(X) = 1) + \mathbb{P}(q_S(X) > c' \land \hat{h}(X) = 0)$$

$$\leq \mathbb{P}(q_S(X) \in [c, c') \land \hat{h}(X) = 1) + \mathbb{P}(q_S(X) > c \land \hat{h}(X) = 1) + \mathbb{P}(q_S(X) > c \land \hat{h}(X) = 0)$$

$$= \mathbb{P}(q_S(X) \in [c, c') \land \hat{h}(X) = 1) + \text{MD}(h, H_{\tilde{S}})$$

$$\leq M(c' - c) + \text{MD}(h, H_{\tilde{S}})$$

$$\leq \left[1 + \frac{M}{\zeta}(c' - c)\right] \text{MD}(h, H_{\tilde{S}}).$$

So $h$ and $H$ satisfy $\alpha$-MD-smoothness with $\alpha = M/\zeta$. ■

F Surrogate Loss Functions

Here, we explore using IDT when the decision maker minimizes a surrogate loss instead of the true loss. So far, as formulated in Section 3, we have assumed that the decision maker chooses a decision rule $h$ which minimizes the expected loss $E[\ell_c(h(X), Y)]$, where the loss function is defined as

$$\ell_c(y, y) = \begin{cases} 0 & \hat{y} = y \\ c & \hat{y} = 1 \land y = 0 \\ 1 - c & \hat{y} = 0 \land y = 1 \end{cases}$$

$$= \begin{cases} c1\{\hat{y} = 1\} & y = 0 \\ (1 - c)1\{\hat{y} = 0\} & y = 1. \end{cases} \quad (26)$$

However, this loss function is not convex or continuous, so it is difficult to optimize. Thus, we might expect the decision maker to choose their decision rule using a surrogate loss which is convex. In particular, suppose that the decision rule $h(\cdot)$ is calculated by thresholding a function $f : X \to \mathbb{R}$:

$$h(x) = 1\{f(x) \geq 0\}.$$

Then, we can replace the indicator functions in (26) with a surrogate loss $V : \mathbb{R} \to \mathbb{R}$:

$$\tilde{\ell}_c(w, y) = \begin{cases} cV(w) & y = 0 \\ (1 - c)V(-w) & y = 1. \end{cases} \quad (27)$$

Say that the decision maker minimizes this loss $\tilde{\ell}_c$ instead of the true loss $\ell$:

$$f^* \in \arg\min_f \mathbb{E}[\tilde{\ell}_c(f(X), Y)]. \quad (28)$$

The following lemma shows that, for reasonable surrogate losses, if the decision maker is optimal then minimizing the surrogate loss is equivalent to minimizing the true loss. The proof is adapted from Section 4.2 of Rosasco et al. [46]; they show that the hinge loss, squared loss, and logistic loss all satisfy the necessary conditions.

**Lemma F.1.** Suppose $V : \mathbb{R} \to \mathbb{R}$ is convex and that it is strictly increasing in a neighborhood of 0. Let $f^*$ be chosen as in (25), and let $h(x) = 1\{f^*(x) \geq 0\}$. Then $h \in \arg\min_h \mathbb{E}[\ell_c(h(X), Y)]$; that is, the threshold of $f^*$ is an optimal decision rule for the true cost function.

**Proof.** We prove the lemma by contradiction; assume that $h$ is not an optimal decision rule for the true loss function. Then by Lemma [46],

$$\mathbb{P}(h(X) \neq 1\{q(X) \geq c\} \land q(X) \neq c) > 0.$$
This implies that either
\[ P(h(X) = 0 \land q(X) > c) > 0 \quad \text{or} \quad P(h(X) = 1 \land q(X) < c) > 0, \]
or equivalently,
\[ P(f^*(X) < 0 \land q(X) > c) > 0 \quad \text{or} \quad P(f^*(X) \geq 0 \land q(X) < c) > 0. \]  \tag{29}

Without loss of generality, assume the former. Define
\[ \tilde{f}(x) = \begin{cases} 
0 & \text{if } f^*(x) < 0 \land q(x) > c \\
f^*(x) & \text{otherwise.}
\end{cases} \]

Consider any \( x \) which satisfies \( f^*(x) < 0 \) and \( q(x) > c \). We can write
\[
\mathbb{E}\left[ \tilde{\ell}_c(f^*(X), Y) - \tilde{\ell}_c(\tilde{f}(X), Y) \mid X = x \right] \\
= P(Y = 0 \mid X = x) c \left( V(f^*(x)) - V(\tilde{f}(x)) \right) + P(Y = 1 \mid X = x) (1 - c) \left( V(-f^*(x)) - V(-\tilde{f}(x)) \right) \\
= (1 - q(x)) c (V(f^*(x)) - V(0)) + q(x) (1 - c) (V(-f^*(x)) - V(0)) \\
= \tilde{\ell}_c(f^*(x) \mid x) - \tilde{\ell}_c(0 \mid x),
\]
where we define
\[ \tilde{\ell}_c(w \mid x) = (1 - q(x)) c V(w) + q(x) (1 - c) V(-w). \]

\( \tilde{\ell}_c(w \mid x) \) satisfies two properties:

1. It is convex in \( w \), since it is a sum of two convex functions.
2. It is strictly decreasing in \( w \) in a neighborhood of 0. To see why, note that we assumed \( q(x) > c \), so
\[ (1 - q(x)) c < (1 - c) c < q(x) (1 - c). \]

Thus, since the weight on \( V(-w) \) is greater than the weight on \( V(w) \), and \( V(w) \) is strictly increasing about 0, \( \tilde{\ell}_c(w \mid x) \) must be strictly decreasing about 0.

Together, these properties imply that
\[ \tilde{\ell}_c(f^*(x) \mid x) - \tilde{\ell}_c(0 \mid x) > 0 \]
since we assumed that \( f^*(x) < 0 \). Thus we have that
\[
\mathbb{E}\left[ \tilde{\ell}_c(f^*(X), Y) - \tilde{\ell}_c(\tilde{f}(X), Y) \mid X = x \right] > 0 \tag{30}
\]
for any \( x \) where \( f^*(x) < 0 \) and \( q(x) > c \).

Now, we analyze the difference in expected loss for \( f^* \) and \( \tilde{f} \). Since these agree on all points except when \( f^*(x) < 0 \) and \( q(x) > c \), we have that
\[
\mathbb{E}[\tilde{\ell}(f^*(X), Y)] - \mathbb{E}[\tilde{\ell}(\tilde{f}(X), Y)] \\
= \mathbb{E}\left[ \tilde{\ell}(f^*(X), Y) - \tilde{\ell}(\tilde{f}(X), Y) \mid f^*(X) < 0 \land q(X) > c \right] \mathbb{P}(f^*(X) < 0 \land q(X) > c) > 0 \tag{31}
\]
Here, (i) is due to the combination of (30), which implies the first term is positive, and the first case of (29), which implies the second term is positive.

(31) implies that \( \tilde{f} \) has lower expected surrogate loss than \( f^* \). However, we assumed that \( f^* \) minimized the expected surrogate loss; thus we have a contradiction. \( \blacksquare \)

Lemma 4.1 means that all the results for an optimal decision maker (e.g., Theorem 4.2) apply immediately to a decision maker minimizing a reasonable surrogate loss. In the case of decision problems without uncertainty, the decision rule will encounter zero loss and thus must be optimal, so Lemma 4.1 also applies in this case for an optimal or suboptimal decision maker (e.g., Corollary 4.12). In the case of a suboptimal decision maker facing uncertainty, different loss functions may lead to different decision rules, so we cannot extend the results in that case to surrogate losses. Table 4 summarizes which results hold equivalently for decision makers minimizing an expected surrogate loss.
### Further Comparison to Prior Work

In this section, we compare two prior papers on preference learning to our results. Mindermann et al. [20] and Bıyık et al. [21] both propose methods for active preference learning, i.e. querying a person to learn their preferences. In each method, queries are prioritized which minimize the uncertainty of the person. The authors argue that such queries are easier to answer and thus lead to more effective preference learning. At first, these results may seem to contradict our findings that uncertain decisions make preference learning easier. However, we argue that their results are not in conflict with ours. Decisions with more uncertainty are probably more difficult for people to make, and those close to the decision boundary are probably the most difficult. However, our results show that it is necessary to observe such decisions in order to recover the person’s preferences. If we cannot observe decisions made arbitrarily close to the person’s decision boundary, we cannot exactly characterize the loss function they are optimizing. Thus, combining the results of Mindermann et al. [20] and Bıyık et al. [21] with ours suggests that there is a tradeoff between the ease of the decision problem for the human and the identifiability of their preferences. That is, uncertainty may make the human’s decision problem more difficult but our problem of identifying preferences easier.

### Table 1: An overview of which of our results apply in the setting when the decision maker is minimizing a surrogate loss rather than the true loss.

| Setting                                                                 | True loss | Surrogate loss |
|------------------------------------------------------------------------|-----------|----------------|
| IDT for optimal decision maker (Theorem 4.2)                           | ✓         | ✓              |
| IDT for suboptimal decision maker (Theorems 4.7 and 4.10)              | ✓         | ✗              |
| No identifiability for decisions without uncertainty (Corollary 4.12)  | ✓         | ✓              |