A quantization of the Hitchin hamiltonian system
and the Beilinson-Drinfeld isomorphism

Ken-ichi SUGIYAMA *
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Abstract

Let $X$ be a smooth projective curve defined over $\mathbb{C}$ whose genus is greater than one. We will generalize the Hitchin’s hamiltonian system to the cotangent bundle of the modular stack of principal $SL_2(\mathbb{C})$ bundle with parabolic reductions on $X$. Also its properties will be investigated. As an application, we will show a generalization of the Beilinson-Drinfeld isomorphism, which is a quantization of the Hitchin system.

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1 Introduction

Let $X$ be a connected smooth projective curve defined over $\mathbb{C}$. Suppose we are given a local system of rank one $\mathcal{L}$ on $X$. Then it determines a representation

$$\pi_1(X, x_0) \xrightarrow{\rho_{\mathcal{L}}} \mathbb{C}^\times,$$

which factors through the maximal abelian quotient of the fundamental group $H_1(X, \mathbb{Z})$. If we identify $H_1(X, \mathbb{Z})$ with the fundamental group of the Jacobian $\text{Jac}(X)$, $\rho_{\mathcal{L}}$ defines a local system $\mathcal{M}_{\mathcal{L}}$ of rank one on $\text{Jac}(X)$. Such a correspondence should be considered as a geometric abelian class field theory.

In order to pass to a nonabelian situation, we have to change a point of view to understand this phenomenon [8]. The local system $\mathcal{L}$ defines a flat connection

$$\nabla = d + A, \quad A \in H^0(X, \Omega),$$

where $\Omega$ is the canonical line bundle of $X$. Note that the cotangent bundle $T^*\text{Jac}(X)$ is the trivial vector bundle, which may be identified with

$$\text{Jac}(X) \times H^0(X, \Omega).$$

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*Address : Ken-ichi SUGIYAMA, Department of Mathematics and Informatics, Faculty of Science, Chiba University, 1-33 Yayoi-cho Inage-ku, Chiba 263-8522, Japan
†e-mail address : sugiyama@math.s.chiba-u.ac.jp
Let us observe that the projection
\[ T^*\text{Jac}(X) \xrightarrow{p} H^0(X, \Omega) \]
induces an isomorphism of the global coordinate rings:
\[ \Gamma(H^0(X, \Omega), \mathcal{O})^p \cong \Gamma(T^*\text{Jac}(X), \mathcal{O}). \]
On the other hand, one easily see that the ring of global differential operators \( D(\text{Jac}(X)) \) on the Jacobian is isomorphic to \( \Gamma(T^*\text{Jac}(X), \mathcal{O}) \) by the symbol map. Combining this with the isomorphism above, we obtain
\[ \Gamma(H^0(X, \Omega), \mathcal{O}) \cong D(\text{Jac}(X)). \]
Now the connection form \( A \) determines a homomorphism
\[ D(\text{Jac}(X)) \cong \Gamma(H^0(X, \Omega), \mathcal{O}) \xrightarrow{f_A} \mathbb{C}, \]
which yields a D-module \( M_A \) on \( \text{Jac}(X) \):
\[ M_A = D_{\text{Jac}(X)} \otimes_{D(\text{Jac}(X))} \mathbb{C}. \]
Here \( D_{\text{Jac}(X)} \) is the sheaf of differential operators on the Jacobian. Then \( M_A \) corresponds to the local system \( M_L \).

The latter picture has the following nonabelization due to Beilinson and Drinfeld [3]. Let \( G \) be a semisimple Lie group and let \( \text{Bun}_{G,X} \) be the classifying stack of principal \( G \)-bundles on \( X \). Hitchin has defined a map which is referred as the Hitchin's hamiltonian system [11]:
\[ T^*\text{Bun}_{G,X} \xrightarrow{h} H_G. \]
Here \( H_G \) is an affine space which depends on \( G \). Note that the map \( h \) corresponds to the projection \( p \) in the geometric abelian class field theory. Beilinson and Drinfeld have quantized the Hitchin’s system. Namely they have constructed an isomorphism between the ring of global differential operators on a half canonical of \( \text{Bun}_{G,X} \) and the global coordinate ring of \( H_G \). Using it, they have established unramified geometric nonabelian class field theory, which assigns a holonomic D-module on \( \text{Bun}_{G,X} \) to an element of \( H_G \).

We will treat a case of which a local system is allowed to be ramified at several points. We will restrict ourselves to the case that \( G \) is \( SL_2(\mathbb{C}) \).

Let \( \{ z_1, \cdots, z_N \} \) be distinct points on \( X \) and let us form a divisor
\[ D = \sum_{i=1}^{N} z_i. \]
We assume that the degree of the line bundle \( \Omega(D) \) is positive. We will consider the classifying stack \( \text{Bun}^{D\text{-fl}}_{G,X} \) of principal \( G \)-bundles with \( D \)-flags on \( X \) (see §2).
In §3, for \( \lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{Z}^N \), we will construct a sheaf of twisted differential operators \( \mathcal{D}_{D-fl, \lambda} \) on \( \text{Bun}_{G,X}^{D-fl} \). Let \( \mathcal{D}_{D-fl, \lambda} \) be the ring of its global sections. Then we will show the following theorem (see §5).

**Theorem 1.1.** There is a isomorphism of \( \mathbb{C} \)-algebras:

\[
\Gamma(H^0(X, \Omega \otimes^2 (\mathcal{D})), \mathcal{O}) \cong \mathcal{D}_{D-fl, \lambda}.
\]

When \( X \) is the projective line, Theorem 1.1 has been already established by Frenkel ([6]). As before, this will establish a geometric ramified nonabelian class field theory in the case of \( SL_2(\mathbb{C}) \), which assigns a holonomic D-module on \( \text{Bun}_{G,X}^{D-fl} \) to an element of \( H^0(X, \Omega \otimes^2 (\mathcal{D})) \) (see §5). It is conjectured that the D-module should be regular holonomic and moreover that it should be a Hecke eigensheaf. We will treat the conjecture in the near future.

Throughout the paper, we set \( G = SL_2(\mathbb{C}) \) and its Lie algebra will be denoted by \( \mathfrak{g} \).

## 2 The Hitchin system for principal \( G \) bundles with \( D \)-flags

Let \( X \) be a connected smooth projective curve defined over \( \mathbb{C} \) of genus \( g \). Let \( \{z_1, \cdots, z_N\} \) be mutually distinct points of \( X \) and we set

\[
D = \sum_{i=1}^{N} z_i \in \text{Div}(X).
\]

We will always assume that the degree of both of \( \Omega(D) \) and \( \mathcal{O}(D) \) are positive. Here \( \Omega \) is the canonical line bundle of \( X \). We will choose and fix a local coordinate \( t_i \) at each \( z_i \). In this section, we will study basic properties of the modular stack of principal \( G \) bundles with \( D \)-flags. See [12], [14] or [15] for a treatment of stacks.

### 2.1 The modular stack of principal \( G \) bundles with \( D \)-flags

Let \( P \) be a principal \( G \)-bundle on \( X \). A \( D \)-flag of \( P \) is defined to be an \( N \)-tuple

\[
\{B_1, \cdots, B_N\},
\]

where \( B_i \) is a Borel subgroup of \( P|_{z_i} \). Let \( \text{Aff}_{/\mathbb{C}, \text{et}} \) be the category of affine schemes defined over \( \mathbb{C} \) with the étale topology. We will consider the modular stack

\[
\text{Bun}_{G,X}^{D-fl}
\]

over \( \text{Aff}_{/\mathbb{C}, \text{et}} \) which classifies principal \( G \)-bundles on \( X \) endowed with \( D \)-flags. It is a smooth stack of dimension \( 3(g-1) + N \) and has the following structure.
Note that since we have assumed that degree of \( \Omega(D) \) is positive, \( 3(g-1) + N \) is nonnegative.

Let \( \text{Bun}_{G,X} \) be the modular stack on \( \text{Aff}_{/C}^{et} \) which classifies principal \( G \)-bundles on \( X \). Forgetting \( D\)-flags, we have a morphism

\[
\text{Bun}_{G,X}^{D-fil} \xrightarrow{\pi} \text{Bun}_{G,X},
\]

which makes \( \text{Bun}_{G,X}^{D-fil} \) into a \((\mathbb{P}^1)^N\)-fibration over \( \text{Bun}_{G,X} \). Since \( \text{Bun}_{G,X} \) is connected and smooth, \( \text{Bun}_{G,X}^{D-fil} \) is also connected.

2.2 The cotangent bundle of \( \text{Bun}_{G,X}^{D-fil} \)

We will study the cotangent space of \( \text{Bun}_{G,X}^{D-fil} \) at \((P, \{B_i\}_i)\). Let \( \mathfrak{B}_i \) be the Lie algebra of \( B_i \) and we set \( \mathfrak{N}_i = \mathfrak{G}/\mathfrak{B}_i \), which will be regarded as a nilpotent subalgebra of \( \mathfrak{G} \). We will consider a skyscraper sheaf supported on \( \text{Supp}(D) \):

\[
\mathfrak{G}_D = \mathfrak{G} \oplus N,
\]

and its subsheaves

\[
\mathfrak{B}_D = \bigoplus_{i=1}^N \mathfrak{B}_i, \quad \mathfrak{N}_D = \bigoplus_{i=1}^N \mathfrak{N}_i.
\]

Let \( \text{ad}_P(\mathfrak{G}) \) be the adjoint \( \mathfrak{G} \)-bundle associated to \( P \). A sheaf \( \mathfrak{B}_{P,D} \) is defined to be

\[
\mathfrak{B}_{P,D} = \text{Ker}[\text{ad}_P(\mathfrak{G}) \xrightarrow{ev_D} \mathfrak{N}_D].
\]

Here \( ev_D \) is the composition of the evaluation map at \( D \):

\[
\text{ad}_P(\mathfrak{G}) \xrightarrow{} \mathfrak{G}_D
\]

with the natural projection onto \( \mathfrak{N}_D \). Note that we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{ad}_P(\mathfrak{G})(-D) & \rightarrow & \text{ad}_P(\mathfrak{G}) & \rightarrow & \mathfrak{G}_D & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathfrak{B}_{P,D} & \rightarrow & \text{ad}_P(\mathfrak{G}) & \rightarrow & \mathfrak{N}_D & \rightarrow & 0.
\end{array}
\]

(1)

such that the horizontal sequences are exact. By the deformation theory, the tangent space of \( \text{Bun}_{G,X}^{D-fil} \) at \((P, \{B_i\}_i)\) is isomorphic to \( H^1(X, \mathfrak{B}_{P,D}) \). Therefore by the Serre duality, the cotangent space is identified with

\[
H^0(X, \text{Hom}_{\mathcal{O}_X}(\mathfrak{B}_{P,D}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega_X).
\]

We want to rewrite this more convenient form.

By the snake lemma, (1) shows

\[
\text{Coker}[\text{ad}_P(\mathfrak{G})(-D) \xrightarrow{} \mathfrak{B}_{P,D}]
\]
is isomorphic to $\mathcal{B}_D$, which implies an exact sequence

$$0 \to \text{ad}_P(\mathfrak{s})(-D) \to \mathcal{B}_{P,D} \to \mathcal{B}_D \to 0. \quad (2)$$

By the Killing form, we will identify $\text{Hom}_{\mathcal{O}_X}(\text{ad}_P(\mathfrak{s})(-D), \mathcal{O}_X)$ with $\text{ad}_P(\mathfrak{s})(D)$. Then (2) implies

$$0 \to \text{Hom}_{\mathcal{O}_X}(\mathcal{B}_{P,D}, \mathcal{O}_X) \otimes \mathcal{O}_X \Omega_X \to \text{ad}_P(\mathfrak{s})(D) \otimes \mathcal{O}_X \Omega_X \to \text{Ext}^1_{\mathcal{O}_X}(\mathcal{B}_D, \mathcal{O}_X) \otimes \mathcal{O}_X \Omega_X \to 0.$$

The residue map shows us that the last term is isomorphic to $\mathcal{B}_D$ and therefore we have

$$0 \to \text{Hom}_{\mathcal{O}_X}(\mathcal{B}_{P,D}, \mathcal{O}_X) \otimes \mathcal{O}_X \Omega_X \to \text{ad}_P(\mathfrak{s})(D) \otimes \mathcal{O}_X \Omega_X \to \mathcal{B}_D \to 0.$$

Thus we have proved the following proposition.

**Proposition 2.1.** The cotangent space $T^*_1(P, \{B_i\}) \text{Bun}^{D-f}_{G,X}$ is isomorphic to

$$\ker[H^0(X, \text{ad}_P(\mathfrak{s})(D) \otimes \mathcal{O}_X \Omega_X) \xrightarrow{\text{Res}} \mathcal{B}_D]$$

**Remark 2.1.** Using the snake lemma, the commutative diagram whose the middle and all horizontal lines are exact:

$$\begin{array}{ccc}
0 & \to & \text{Hom}_{\mathcal{O}_X}(\mathcal{B}_{P,D}, \mathcal{O}_X) \otimes \mathcal{O}_X \Omega_X \\
\downarrow & & \downarrow \\
0 & \to & \mathfrak{N}_D \otimes \mathcal{O}_X \Omega_X
\end{array}$$

shows that there is an exact sequence

$$0 \to \text{ad}_P(\mathfrak{s}) \otimes \mathcal{O}_X \Omega_X \to \text{Hom}_{\mathcal{O}_X}(\mathcal{B}_{P,D}, \mathcal{O}_X) \otimes \mathcal{O}_X \Omega_X \to \mathfrak{N}_D \otimes \mathcal{O}_X \Omega_X \to 0.$$

### 2.3 The Hitchin’s Hamiltonian system

**Proposition 2.1** shows $T^*_1(P, \{B_i\}) \text{Bun}^{D-f}_{G,X}$ may be identified a subspace of

$$H^0(X, \text{ad}_P(\mathfrak{s})(D) \otimes \mathcal{O}_X \Omega_X)$$

which consists of $A$ which has a Taylor expansion at each $z_i$:

$$A = \begin{pmatrix}
a_i(t_i) & b_i(t_i) \\
c_i(t_i) & -a_i(t_i)
\end{pmatrix} \frac{dt_i}{t_i}, \quad a_i(t_i), b_i(t_i), c_i(t_i) \in \mathbb{C}[[t_i]],$$

satisfying

$$\begin{pmatrix}
a_i(0) & b_i(0) \\
c_i(0) & -a_i(0)
\end{pmatrix} \in \mathfrak{N}_i.$$
In particular for \( A \in T^*_\mathcal{P} \mathcal{B}^\mathcal{G}_\mathcal{B}X \), we have

\[
\det A \in H^0(X, \Omega^{\otimes 2}(D)),
\]
and when \((\mathcal{P}, \{\mathcal{B}_i\}_i)\) moves over \(\mathcal{B}^\mathcal{G}_\mathcal{B}X\), the determinant defines a morphism which will be referred as the Hitchin’s hamiltonian system:\n
\[
T^*\mathcal{B}^\mathcal{G}_\mathcal{B}X \xrightarrow{h} H^0(X, \Omega^{\otimes 2}(D)).
\]

**Remark 2.2.** By the Riemann-Roch theorem the dimension of \(H^0(X, \Omega^{\otimes 2}(D))\) is \(3(g - 1) + N\), which is equal to one of \(\mathcal{B}^\mathcal{G}_\mathcal{B}X\).

Note that \(T^*\mathcal{B}^\mathcal{G}_\mathcal{B}X\) possesses the canonical symplectic form. **Remark 2.2** makes us expect the Hitchin’s hamiltonian system should be an algebraically complete integrable system. We will show this is true.

The inverse image \(h^{-1}(0)\) will be referred as the Laumon’s glocal nilpotent variety and will be denoted by \(\text{Nilp}\).

**Proposition 2.2.** \(\text{Nilp}\) is a Lagrangian subvariety of \(T^*\mathcal{B}^\mathcal{G}_\mathcal{B}X\).

In order to prove the proposition, we will follow the argument of Ginzburg. Let us fix a Borel subgroup \(B\) of \(G\) and let \(P\) be a principal \(B\)-bundle on \(X\). By the extension of the structure group we may associate it a principal \(G\)-bundle \(P_G\):

\[
P_G = P \times_B G.
\]

Note that the natural inclusion

\[
P \hookrightarrow P_G
\]
yields a \(D\)-flag by

\[
P|_{\mathcal{P}_i} \hookrightarrow P_G|_{\mathcal{P}_i}.
\]

Let \(\mathcal{B}^\mathcal{B}_X\) be the modular stack of principal \(B\)-bundles on \(X\). The argument above shows that we have a morphism

\[
\mathcal{B}^\mathcal{B}_X \xrightarrow{f} \mathcal{B}^\mathcal{G}_\mathcal{B}X.
\]

Now we will consider a substack

\[
Y_f = \{(P, \alpha), (Q, \beta) \in T^*\mathcal{B}^\mathcal{B}_X \times T^*\mathcal{B}^\mathcal{G}_\mathcal{B}X \mid f(P) = Q, f^*(\beta) = 0\},
\]
and let \(p(Y_f)\) be its image under the natural projection

\[
T^*\mathcal{B}^\mathcal{B}_X \times T^*\mathcal{B}^\mathcal{G}_\mathcal{B}X \xrightarrow{p} T^*\mathcal{B}^\mathcal{G}_\mathcal{B}X.
\]

**Lemma 2.1.**

\[
p(Y_f) = \text{Nilp}.
\]
Proof. Let $\mathfrak{B}$ be the Borel subalgebra corresponding to $B$ and let $\mathfrak{N}$ be its nilpotent ideal. Then it defines a normal subgroup $N$ of $B$. Let

$$Y_f \xrightarrow{\pi} \text{Bun}_{B,X}$$

be a morphism which is defined to be

$$\pi((P, \alpha), (Q, \beta)) = P.$$ 

Then the inverse image $\pi^{-1}(P)$ of $P \in \text{Bun}_{B,X}$ is isomorphic to

$$\ker[T^*_f(P)\text{Bun}_{G,X} \xrightarrow{f^*} T^*_P\text{Bun}_{B,X}].$$

We will abbreviate

$$H^i(\cdot) = H^i(X, \cdot),$$

and

$$\cdot \otimes \Omega_X = \cdot \otimes_{\mathcal{O}_X} \Omega_X.$$ 

Let us consider the diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & H^0(\text{ad}_P(\mathfrak{N})(D) \otimes \Omega_X) \\
\downarrow & & \downarrow \\
T^*_f(P)\text{Bun}_{G,X} & \xrightarrow{f^*} & T^*_P\text{Bun}_{B,X} \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^0(\text{ad}_P(\mathfrak{B})(D) \otimes \Omega_X) \\
\downarrow & & \downarrow \\
\mathfrak{B}_D & \equiv & \mathfrak{B}_D
\end{array}
$$

The rightdown vertical arrow is induced by the residue map. By the standard deformation theory, we have identified $T^*_P\text{Bun}_{B,X}$ with $H^0(\text{ad}_P(\mathfrak{B})(D) \otimes \Omega_X)$.

and the right vertical sequence is exact. Proposition 2.1 shows that the middle vertical sequence is exact. Finally the middle horizontal sequence is also exact, which is derived from the sheaf exact sequence

$$0 \rightarrow \text{ad}_P(\mathfrak{N})(D) \otimes \Omega_X \rightarrow \text{ad}_P(\mathfrak{B})(D) \otimes \Omega_X \rightarrow \text{ad}_P(\mathfrak{B})(D) \otimes \Omega_X \rightarrow 0,$$

and by the fact

$$H^1(\text{ad}_P(\mathfrak{N})(D) \otimes \Omega_X) = 0.$$ 

The diagram chasing shows that

$$T^*_f(P)\text{Bun}_{G,X} \xrightarrow{f^*} T^*_P\text{Bun}_{B,X}$$
is surjective and that $\pi^{-1}(P)$ is isomorphic to

$$H^0(\text{ad}_P(\mathfrak{g})(D) \otimes \Omega_X).$$

Hence $p(Y_f)$ is a stack of pairs $(Q, \eta)$ of principal $G$-bundles $Q$ which is induced from a principal $B$-bundle $P$ and $\eta \in H^0(\text{ad}_P(\mathfrak{g})(D) \otimes \Omega_X)$, which is nothing but $\text{Nilp}$.

△

**Proof of Proposition 2.2.** It is known by Ginzburg (cf. [10] Lemma 6.5.4) that $p(Y_f)$ is isotropic. Thus Lemma 2.1 shows the dimension of $\text{Nilp}$ is less than or equal to $3(g - 1) + N$. But Remark 2.2 implies

$$\dim \text{Nilp} \geq 3(g - 1) + N.$$ 

Therefore we have obtained the desired result.

△

The argument of (2.10.1) of [3] implies the following theorem.

**Theorem 2.1.** The Hitchin’s hamiltonian system

$$T^*\text{Bun}_{G,X}^D \xrightarrow{\text{h}} H^0(X, \Omega^{\otimes 2}(D))$$

is surjective and flat. In particular the dimension of fibres are $3(g - 1) + N$.

### 2.4 The Hitchin’s correspondence

We put

$$L = \Omega_X(D)$$

and let

$$L \xrightarrow{\pi} X$$

be the $\mathbb{A}^1$-fibration associated to $L$. Then the line bundle $\pi^*L$ possesses the tautological section $x$. An element $q$ of $H^0(X, \mathcal{L} \otimes 2)$ determines a spectral curve $\Sigma_q$, which is a closed subscheme of $L$ defined by the equation:

$$x^2 + \pi^*q = 0.$$ 

This is a double covering of $X$ and its genus $g(\Sigma_q)$ is computed by the formula of [1] Remark 3.2:

$$g(\Sigma_q) = 4g - 3 + N. \quad (3)$$

Let Prym$(\Sigma_q/X)$ be the Prym variety of the double covering $\Sigma_q \xrightarrow{\pi} X$:

$$\text{Prym}(\Sigma_q/X) = \text{Ker}[\text{Jac}(\Sigma_q) \xrightarrow{\pi_*} \text{Jac}(X)].$$

Here Jac denotes the Jacobian variety. The formula (3) shows the dimension of Prym$(\Sigma_q/X)$ is equal to $3(g - 1) + N$. Also it has the following remarkable property ([1] [11]).
Fact 2.1. Suppose $\Sigma_q$ is smooth. Then there is a bijection between

$$\text{Prym}(\Sigma_q/X)$$

and isomorphism classes of pairs $(E, \varphi)$ where $E$ is a vector bundle of rank 2 over $X$ with the trivial determinant and $\varphi$ is an element of $H^0(X, \text{End}_{\mathcal{O}_X}(E) \otimes \mathcal{L})$ such that

$$\det(t - \varphi) = t^2 + q.$$ 

Such a bijection will be referred as the Hitchin’s correspondence. We will regard $H^0(X, \Omega_X^2(D))$ as a subset of $H^0(X, \Omega_X^2(D))$. Let us take $q \in H^0(X, \Omega_X^2(D))$ so that $\Sigma_q$ is smooth and that the value of $t_i \cdot q$ at each $z_i$ is non-zero.

Proposition 2.3. There is a bijection between $h^{-1}(q)$ and $\text{Prym}(\Sigma_q/X)$.

Proof. $h^{-1}(q)$ consists of isomorphism classes of pairs

$$((P, \{B_i\}_i), \alpha)$$

where $(P, \{B_i\}_i)$ is a principal $G$-bundle on $X$ with a $D$-flag and

$$\alpha \in \ker[H^0(X, \text{ad}_P(G) \otimes \mathcal{L}) \xrightarrow{\text{Res}} \mathcal{B}_D].$$

We claim that $A \in H^0(X, \text{ad}_P(G) \otimes \mathcal{L})$ satisfying

$$\det A = q$$

will automatically determine Borel subgroups $\{B_i\}_i$ so that $A$ is contained in

$$\ker[H^0(X, \text{ad}_P(G) \otimes \mathcal{L}) \xrightarrow{\text{Res}} \mathcal{B}_D]. \quad (4)$$

In fact, let

$$A = \begin{pmatrix} a_i(t_i) & b_i(t_i) \\ c_i(t_i) & -a_i(t_i) \end{pmatrix} \frac{dt_i}{t_i}, \quad a_i(t_i), b_i(t_i), c_i(t_i) \in \mathbb{C}[[t_i]],$$

be the Taylor expansion of $A$ at $z_i$ and we put

$$A_i = \begin{pmatrix} a_i(0) & b_i(0) \\ c_i(0) & -a_i(0) \end{pmatrix}.$$ 

Since $\det A$ has at most single pole at $z_i$, $\det A_i$ vanishes. The assumption of residue of $q$ implies $A_i$ is a non-zero nilpotent matrix. Now using the Killing form a Borel subalgebra $\mathcal{B}_i$ is defined to be the orthogonal complement of $A_i$ and let $B_i$ be the corresponding Borel subgroup. Now it is easy to see that $A$ is contained in $(4)$. Thus we have proved there is one to one correspondence between $h^{-1}(q)$ and isomorphism classes of pairs of a principal $G$-bundle $P$ on $X$ and $A \in H^0(X, \text{ad}_P(G) \otimes \mathcal{L})$ satisfying

$$\det A = q.$$ 

Now the natural correspondence between isomorphism classes of principal $G$-bundles on $X$ and of rank 2 vector bundles on $X$ with the trivial determinant and Fact 2.1 imply the desired result.
Theorem 2.1 and Proposition 2.3 will show the following theorem.

**Theorem 2.2.** The Hitchin's hamiltonian system

\[ T^*\text{Bun}_{G,X}^{D-fl} \xrightarrow{h} H^0(X, \Omega^{\otimes 2}(D)) \]

is surjective and flat. Moreover there is a Zariski open dense subset \( U \) of \( H^0(X, \Omega^{\otimes 2}(D)) \) over which \( h \) is an abelian fibration.

**Corollary 2.1.** The Hitchin’s hamiltonian system induces an isomorphism of commutative \( \mathbb{C} \)-algebras:

\[ \Gamma(H^0(X, \Omega^{\otimes 2}(D)), \mathcal{O}) \xrightarrow{h^*} \Gamma(T^*\text{Bun}_{G,X}^{D-fl}, \mathcal{O}). \]

### 3 A uniformization of a modular stack

Throughout the paper, we will use the following notations. The *loop group* \( LG \) and the *regular loop group* \( L^+_G \) are defined to be

\[ LG = G(\mathbb{C}((t)))) , \quad L^+_G = G(\mathbb{C}[[t]]), \]

respectively. Let \( \text{ev}_0 \) be the evaluation map:

\[ L^+_G \xrightarrow{\text{ev}_0} G, \quad \text{ev}_0(A) = A(0), \]

and we set

\[ L_0G = \text{Ker}[L^+_G \xrightarrow{\text{ev}_0} G]. \]

Let \( B_+ \) be the upper Borel subgroup of \( G \) and we put

\[ \hat{B}_+ = \text{ev}_0^{-1}(B_+). \]

Also we will consider the corresponding Lie algebras:

- \( L\mathfrak{G} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t)) \) : the loop algebra.
- \( L^+_\mathfrak{G} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[t]] \) : the regular loop algebra.
- \( L_0\mathfrak{G} = \mathfrak{g} \otimes_{\mathbb{C}} t\mathbb{C}[[t]]. \)
- \( \hat{B}_+ = \mathfrak{B}_+ \oplus L_0\mathfrak{G}. \)
3.1 The vacuum module of the critical level

We will consider a central extension

\[ 0 \to \mathbb{C} K \to \hat{\mathfrak{g}}_N \to (L\mathfrak{g})^\oplus N \to 0 \]  

(5)

which is defined in the following way. Let

\[ A = (A_1 \otimes f_1, \cdots, A_N \otimes f_N), \quad B = (B_1 \otimes g_1, \cdots, B_N \otimes g_N), \]

where

\[ A_i, B_j \in \mathfrak{g}, \quad f_i, g_j \in \mathbb{C}(t) \]

be elements of \((L\mathfrak{g})^\oplus N\). Then their commutator is defined to be

\[ [A, B] = ([A_1, B_1] \otimes f_1 g_1, \cdots, [A_N, B_N] \otimes f_N g_N) - \sum_{i=1}^{N} \text{Tr}(A_i B_i) \cdot \text{Res}_{t=0}(f_i d g_i) \cdot K. \]

When \(N = 1\) we set

\[ \hat{\mathfrak{g}} = \mathfrak{g}_1. \]

Obviously (5) splits over \((L\mathfrak{g})^\oplus N\). We put

\[ Y = X \setminus \{z_1, \cdots, z_N\}, \]

and

\[ \mathfrak{g}(Y) = \mathfrak{g} \otimes_{\mathbb{C}} \Gamma(Y, \mathcal{O}), \]

which is imbedded in \((L\mathfrak{g})^\oplus N\) by the Taylor expansion. Let us observe that (5) also splits over \(\mathfrak{g}(Y)\) by the residue theorem. Hence we may consider them as Lie subalgebra of \(\hat{\mathfrak{g}}_N\).

Let \(U(\hat{\mathfrak{g}}_N)\) be the universal enveloping algebra of \(\hat{\mathfrak{g}}_N\) and its reduction at the critical level is defined to be

\[ U_{-2}(\hat{\mathfrak{g}}_N) = U(\hat{\mathfrak{g}}_N)/(K + 2). \]

Let \(\mathbb{C}_{-2}\) be a \(\mathbb{C} K \oplus (L_0\mathfrak{g})^\oplus N\)-module which is isomorphic to \(\mathbb{C}\) as an abstract vector space whose action is defined to be

\[ (L_0\mathfrak{g})^\oplus N \cdot \mathbb{C}_{-2} = 0 \]

and

\[ K = -2. \]

Then we define the vacuum module \(\text{Vac}_{-2}^0\) as

\[ \text{Vac}_{-2}^0 = U_{-2}(\hat{\mathfrak{g}}_N) \otimes U(\mathbb{C} K \oplus (L_0\mathfrak{g})^\oplus N) \mathbb{C}_{-2}, \]

and let \(v_{-2}\) be the the highest weight vector.
3.2 A uniformization of a modular stack

Let $\text{Bun}_{D,fr}^{G,X}$ be the modular stack over $\text{Aff}_{/C,et}$ which classifies isomorphism classes of pairs

$$(P, \{\alpha_i\}),$$

where $P$ is a principal $G$-bundle on $X$ and $\alpha_i$ is its trivialization $z_i$:

$$G^{\alpha_i} \simeq P|_{z_i}.$$  

(Such an $N$-tuple of trivializations will be called as a $D$-framing of $P$.) This becomes a principal $(B_+)^N$-bundle over $\text{Bun}_{D,fl}^{G,X}$ by the morphism

$$\text{Bun}_{D,fr}^{G,X} \xrightarrow{\pi} \text{Bun}_{D,fl}^{G,X},$$

where

$$\pi((P, \{\alpha_i\})) = (P, \{\alpha_i(B_+)\}).$$

By the Taylor expansion

$$G(Y) = G(\Gamma(Y, \mathcal{O})), $$

may be considered as a subgroup of $(LG)^N$ and we set

$$Z = (LG)^N/G(Y).$$

Then each of $\text{Bun}_{D,fr}^{G,X}$, $\text{Bun}_{D,fl}^{G,X}$ and $\text{Bun}_{G,X}$ has the following uniformization as a stack in terms of $Z$([12] [15]):

$$\text{Bun}_{D,fr}^{G,X} \simeq (L_0G)^N \setminus Z, \quad \text{Bun}_{D,fl}^{G,X} \simeq (\hat{B}_+)^N \setminus Z,$$

and

$$\text{Bun}_{G,X} \simeq (L_+G)^N \setminus Z.$$

Let $\Theta_Z$ be the tangent sheaf of $Z$. Then the left action of $(LG)^N$ on $Z$ induces a surjection

$$(L\Theta)^\oplus N \otimes \mathcal{O}_Z \longrightarrow \Theta_Z.$$  

Combining with this the projection

$$\Theta_N \longrightarrow (L\Theta)^\oplus N,$$

we obtain a surjective map

$$\Theta_N \otimes \mathcal{O}_Z \xrightarrow{\alpha} \Theta_Z,$$

which will referred as the anchor map([2] [9] Appendix 3). We will consider a $\mathcal{O}_Z$-module

$$\mathcal{J} = \Theta_N \otimes \mathcal{O}_Z/\ker \alpha,$$

which is a central extension of $\Theta_Z$:

$$0 \to \mathcal{O}_Z \to \mathcal{J} \to \Theta_Z \to 0.$$
Now we set
\[ D'_Z = U(J)/(K + 2). \]
This is a sheaf of $\mathbb{C}$-algebra on $Z$ and the anchor map defines an algebra homomorphism
\[ U_{-2}(\mathfrak{g}_N) \xrightarrow{U(\alpha)} D'_Z, \] (6)
which is compatible with the action of $(L_+ \mathfrak{g})^{\otimes N}$.

### 3.3 A sheaf of twisted differential operators on $\text{Bun}_{G,X}^{D-fl}$

Let us fix a theta characteristic $\kappa$ of $X$. Namely we will fix a line bundle on $X$ so that $\kappa^{\otimes 2} \simeq \Omega_X$. Then it will determine a half canonical $\omega^{\otimes \frac{1}{2}}$ of $\text{Bun}_{G,X}$ (15).

Let
\[ Z \xrightarrow{\tau} \text{Bun}_{G,X} \]
be the projection. Then it is known that $(\tau_*(D'_Z))^{(L_+ G)^N}$ is isomorphic to the sheaf of differential operators $D'$ on $\omega^{\otimes \frac{1}{2}}$ (30).

For
\[ \lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{Z}^N, \]
let $\mathcal{O}(\lambda)$ be the line bundle on $(\mathbb{P}^1)^N$ which is defined to be
\[ \mathcal{O}(\lambda) = p_1^* \mathcal{O}_{\mathbb{P}^1}(\lambda_1) \otimes \cdots \otimes p_N^* \mathcal{O}_{\mathbb{P}^1}(\lambda_N), \]
where $p_i$ is the projection to the $i$-th factor. Let $S$ be an affine scheme defined over $\mathbb{C}$ and let
\[ X \times S \xrightarrow{p_2} S \]
be the projection. Then a principal $G$-bundle $P$ on $X \times S$ induces a Cartesian diagram
\[ \begin{array}{ccc} (\mathbb{P}^1)^N \times S & \longrightarrow & \text{Bun}_{G,X}^{D-fl} \\ \downarrow & & \downarrow \\ S & \xrightarrow{P} & \text{Bun}_{G,X}. \end{array} \]

By definition, the restriction $\omega|_S$ of the canonical sheaf $\omega$ of $\text{Bun}_{G,X}$ to $S$ is given by
\[ \omega|_S = \det(Rp_*\text{ad}_P(\mathfrak{g}))^{\otimes (-1)}, \]
and in particular we have
\[ \omega^{\otimes \frac{1}{2}}|_S = \det(Rp_*\text{ad}_P(\mathfrak{g}))^{\otimes (-\frac{1}{2})}. \]
(Note that in order to define the right hand side, it is necessary to choose a theta characteristic.)

Now we define a line bundle $\mathcal{L}_\lambda$ on $\text{Bun}_{G,X}^{D-fl}$ so that its restriction to $(\mathbb{P}^1)^N \times S$ is equal to
\[ p^* \mathcal{O}(\lambda) \otimes q^*(\omega^{\frac{1}{2}}|_S), \]
where \( p \) (resp. \( q \)) be the projection onto \((\mathbb{P}^1)^N \) (resp. \( S \)). The sheaf of differential operators on \( L_\lambda \) will be denoted by \( \mathcal{D}'_{D-\text{fl},\lambda} \) and let \( D'_{D-\text{fl},\lambda} \) be its global sections. As before it has the following description in terms of the uniformization.

Let

\[
Z \xrightarrow{\tau_{D-\text{fr}}} \text{Bun}_{G,X}^{D-\text{fr}}
\]

be the projection and we put (\[9\]) \( D'_{D-\text{fr}} = (\tau_{D-\text{fr}}^* \mathcal{D}'_Z)^N \).

This is a sheaf of differential operators on \( \text{Bun}_{G,X}^{D-\text{fr}} \) and let \( D'_{D-\text{fr}} \) be the space of its global sections. Since \( U(\mathfrak{g})^\otimes N \) is a subalgebra of \( U_{\text{-2}}(\hat{\mathfrak{g}}) \), (6) induces a homomorphism:

\[
U(\mathfrak{g})^\otimes N \longrightarrow D'_Z.
\]

Note that \( U(\mathfrak{g})^\otimes N \) is invariant under the action of \( (L_0G)^N \) and we have a homomorphism

\[
U(\mathfrak{g})^\otimes N \longrightarrow D'_{D-\text{fr}}.
\]

For an integer \( k \), let \( M_k \) be the Verma module of \( G \) of the highest weight \( k \) with the highest weight vector \( \mu_k \). We will consider a \( U(\mathfrak{g})^\otimes N \)-module

\[
M_\lambda = M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_N},
\]

and let \( \mu_\lambda \) be its highest weight vector:

\[
\mu_\lambda = \mu_{\lambda_1} \otimes \cdots \otimes \mu_{\lambda_N}.
\]

Using the map (7), one has the sheaf of twisted differential operators on \( \text{Bun}_{G,X}^{D-\text{fl}} \)

\[
[\pi_* (\mathcal{D}'_{D-\text{fr}} \otimes U(\mathfrak{g})^\otimes N M_\lambda)]^N
\]

which is easily seen to be isomorphic to \( D'_{D-\text{fl},\lambda} \) (cf. \([6]\) \(5.2\)).

4 The Beilinson-Drinfeld homomorphism

4.1 The Feigin-Frenkel isomorphism

We put

\[
I = (L_0\mathfrak{g})^\otimes N,
\]

which will be considered as a Lie subalgebra of \( \hat{\mathfrak{g}}_N \). Let \( \mathcal{D}_{\mathfrak{g},N} \) be the left normalizer of \( U_{-2}(\hat{\mathfrak{g}}_N) \cdot I \) in \( U_{-2}(\hat{\mathfrak{g}}_N) \) and we set

\[
\mathcal{D}_{\mathfrak{g},N} = \mathcal{D}_{\mathfrak{g},N}/U_{-2}(\hat{\mathfrak{g}}_N) \cdot I.
\]
Note that $G^N$ preserves both of $U_{-2}(\hat{\mathfrak{g}}_N) \cdot I$ and $\mathfrak{D}_{\mathfrak{g},N}$ and therefore it acts on $\mathfrak{D}_{\mathfrak{g},N}$. The map (6) induces a homomorphism (cf. [3] 1.1.3)

$$\mathfrak{D}_{\mathfrak{g},N} \longrightarrow D'_{D_{-fr}},$$

which is compatible with the $G^N$ action. Putting

$$\deg X = 1$$

for any $X \in (L\mathfrak{g})^{\oplus N}$, $U_{-2}(\hat{\mathfrak{g}}_N)$ becomes a filtered algebra and $\mathfrak{D}_{\mathfrak{g},N}$ has the induced filtration. On the other hand $D'_{D_{-fr}}$ possesses a filtration by the order of a differential operator and (8) preserves these filtrations.

Let $\Sigma(V\text{ac}^0_{-2})$ be the space of singular vectors:

$$\Sigma(V\text{ac}^0_{-2}) = \{ x \in V\text{ac}^0_{-2} \mid \gamma \cdot x = 0 \text{ for } \gamma \in I \}.$$ 

Then the map

$$U_{-2}(\hat{\mathfrak{g}}_N) \longrightarrow V\text{ac}^0_{-2}, \quad \epsilon(X) = X \cdot v_{-2},$$

induces an isomorphism of vector spaces:

$$\mathfrak{D}_{\mathfrak{g},N} \xrightarrow{\epsilon'} \Sigma(V\text{ac}^0_{-2}).$$

On the other hand, let $Z_{-2}(\hat{\mathfrak{g}})$ be the center of the local completion of $U_{-2}(\hat{\mathfrak{g}}_N)(\mathbb{C})$. Then $Z_{-2}(\hat{\mathfrak{g}})^{\otimes N}$ acts on $V\text{ac}^0_{-2}$ and since $Z_{-2}(\hat{\mathfrak{g}})$ commutes with $U_{-2}(\hat{\mathfrak{g}}_N)$ we have a map

$$Z_{-2}(\hat{\mathfrak{g}})^{\otimes N} \longrightarrow \Sigma(V\text{ac}^0_{-2}),$$

which is defined to be

$$\rho(Z) = Z \cdot v_{-2}, \quad Z \in Z_{-2}(\hat{\mathfrak{g}})^{\otimes N}.$$ 

Thus we have a linear map:

$$Z_{-2}(\hat{\mathfrak{g}})^{\otimes N} \xrightarrow{\epsilon'^{-1} \circ \rho} \mathfrak{D}_{\mathfrak{g},N},$$

which is checked to be a homomorphism of $\mathbb{C}$-algebras. Note that the map (9) is compatible with the action of $G^N$. In particular since $Z_{-2}(\hat{\mathfrak{g}})$ is the center, its image is contained in

$$\mathfrak{D}_{\mathfrak{g},N}^{BN} = \{ x \in \mathfrak{D}_{\mathfrak{g},N} \mid b \cdot x = x \text{ for } b \in B_N \}.$$ 

By [5] (see also [7] §12), it is known that $Z_{-2}(\hat{\mathfrak{g}})$ is isomorphic to a topological completion of a polynomial algebra $\mathbb{C}\{\{S_m\}_{m\in\mathbb{Z}}\}$ generated by the Sugawara operators $\{S_m\}_{m\in\mathbb{Z}}$. Thus we have obtained a homomorphism

$$\mathbb{C}\{\{S_m\}_{m\in\mathbb{Z}}\}^{\otimes N} \longrightarrow \mathfrak{D}_{\mathfrak{g},N}^{BN}.$$
Since each $S_m$ is an infinite sum of normally ordered quadratics of $L\mathfrak{G}(\mathfrak{B})$ \ref{2.5.10}, if we put
\[ \text{deg } S_m = 2, \]
and give a filtration $\mathbb{C}[\{S_m\}_{m \in \mathbb{Z}}]$ by the degree, \ref{10} becomes a filtered algebra homomorphism.

We will use the following notations:
- $D = \text{Spec } \mathbb{C}[[t]]$: a formal disc,
- $D^\times = \text{Spec } \mathbb{C}((t))$: a punctured formal disc.

Let $c_m$ be a linear functional on $H^0(D^\times, \Omega^\otimes 2)$ which is defined to be $c_m(q) = q_m$, for $q = \sum_{n \in \mathbb{Z}} q_n t^{-n-2}(dt)^2 \in H^0(D^\times, \Omega^\otimes 2)$.

Then $\mathbb{C}[\{S_m\}_{m \in \mathbb{Z}}]$ is isomorphic to $\Gamma(H^0(D^\times, \Omega^\otimes 2), \mathcal{O}) \simeq \mathbb{C}[\{c_m\}_{m \in \mathbb{Z}}]$, by a map $\phi(S_m) = c_m$.

Thus \ref{10} turns out to be a homomorphism:
\[ \Gamma(H^0(D^\times, \Omega^\otimes 2), \mathcal{O}) \otimes \mathbb{C} \xrightarrow{\psi} \mathcal{D}_{\mathfrak{B}, N}. \]

For $S \in Z_{-2}(\mathfrak{B})$, we put
\[ S^{(i)} = 1 \otimes \cdots \otimes S \otimes \cdots \otimes 1 \in Z_{-2}(\mathfrak{B})^\otimes N, \]
where $S$ appears in the $i$-th place. Then by the reason of degree, we will find
\[ S_m^{(i)} \cdot v_{-2} = 0, \]
for any $1 \leq i \leq N$ and $m > 0$. Thus the homomorphism above factors through
\[ \Gamma(H^0(D, \Omega^\otimes 2(2\mathcal{O})), \mathcal{O}) \otimes \mathbb{C} \xrightarrow{\psi} \mathcal{D}_{\mathfrak{B}, N}. \]

Here, in general for an integer $k$, $H^0(D, \Omega^\otimes 2(k\mathcal{O}))$ will stand for the space of quadratic differentials on the formal disc which has a pole at the origin $\mathcal{O}$ at most $k$-th order. Note that $\Gamma(H^0(D, \Omega^\otimes 2(2\mathcal{O})), \mathcal{O})$ is isomorphic to $\mathbb{C}[\{c_m\}_{m \leq 0}]$. Putting
\[ \text{deg } c_m = 2, \]
for any $m$, we will introduce a filtration on $\Gamma(H^0(D, \Omega^\otimes 2(2\mathcal{O})), \mathcal{O}) \simeq \mathbb{C}[\{c_m\}_{m \leq 0}]$. Then \ref{11} preserves the filtrations and combining it with \ref{8}, we will obtain a filtered algebra homomorphism
\[ \Gamma(H^0(D, \Omega^\otimes 2(2\mathcal{O})), \mathcal{O}) \otimes \mathbb{C} \xrightarrow{\beta} (D_{D-f_r}^N)^{B_N}. \]
4.2 The Beilinson-Drinfeld homomorphism

By definition of the Verma module, there is a surjection as $U(\mathfrak{g})^\otimes N$-modules:

$$U(\mathfrak{g})^\otimes N \xrightarrow{\rho_\lambda} M_\lambda,$$

which is defined to be

$$\rho_\lambda(X_1 \otimes \cdots \otimes X_N) = (X_1 \otimes \cdots \otimes X_N) \cdot \mu_\lambda.$$

The map (13) induces a homomorphism

$$D'_{D-f} \simeq D'_{D-f} \otimes_{U(\mathfrak{g})^\otimes N} U(\mathfrak{g})^\otimes N \longrightarrow D'_{D-f} \otimes_{U(\mathfrak{g})^\otimes N} M_\lambda,$$

which implies (cf. 3.3)

$$(\pi_*(D'_{D-f}))^N \longrightarrow [\pi_*(D'_{D-f} \otimes_{U(\mathfrak{g})^\otimes N} M_\lambda)]^N \simeq D'_{D-fl,\lambda}.$$

Taking global sections on $\text{Bun}_{G,X}^{D-fl}$, we have a homomorphism

$$(D'_{D-f})^N \longrightarrow D'_{D-fl,\lambda}. \quad (14)$$

Note that, by the construction, this preserves the filtrations. Combining (14) with (12), we will obtain the following proposition.

**Proposition 4.1.** There is a homomorphism of algebras which preserves the filtrations

$$\Gamma(H^0(D, \Omega^\otimes 2(\mathcal{O})), \mathcal{O})^\otimes N \xrightarrow{\hat{\beta}_\lambda} D'_{D-fl,\lambda}.$$

For $1 \leq i \leq N$, we set

$$c^{(i)}_m = 1 \otimes \cdots \otimes 1 \otimes c_m \otimes 1 \otimes \cdots \otimes 1 \in \Gamma(H^0(D, \Omega^\otimes 2(\mathcal{O})), \mathcal{O})^\otimes N,$$

where $c_m$ appears in the $i$-th place. Then it is easily checked ([3] 3.6 Example) that

$$\hat{\beta}_\lambda(c^{(i)}_m) = \frac{\lambda_i(\lambda_i + 2)}{4}.$$

Thus Proposition 4.1 implies the following theorem.

**Theorem 4.1.** There is a homomorphism of algebras which preserves the filtrations

$$\Gamma(H^0(D, \Omega^\otimes 2(\mathcal{O})), \mathcal{O})^\otimes N \xrightarrow{\beta_\lambda} D'_{D-fl,\lambda}.$$

The homomorphism $\beta_\lambda$ will be referred as the Beilinson-Drinfeld homomorphism.
5 A quantization of the Hitchin system

5.1 A quantization of the Hitchin’s hamiltonian system

Let 
\[ \{f_1, \cdots, f_{3(g-1)+N} \} \]
be a basis of \( H^0(X, \Omega^{\otimes 2}(D)) \) and for each \( i \) we put 
\[ \deg f_i = 2. \]
Then \( \Gamma(H^0(X, \Omega^{\otimes 2}(D)), \mathcal{O}) \) is isomorphic to a weighted polynomial ring 
\[ \mathbb{C}[f_1, \cdots, f_{3(g-1)+N}]. \]
Let \( D_i \) be the formal disc at \( z_i \):
\[ D_i = \text{Spec} \mathbb{C}[[t_i]]. \]
The restriction map 
\[ H^0(X, \Omega^{\otimes 2}(D)) \xrightarrow{r} \oplus_{i=1}^N H^0(D_i, \Omega^{\otimes 2}(\mathcal{O})) \]
induces a surjective homomorphism 
\[ \otimes_{i=1}^N \Gamma(H^0(D_i, \Omega^{\otimes 2}(\mathcal{O})), \mathcal{O}) \xrightarrow{r^*} \Gamma(H^0(X, \Omega^{\otimes 2}(D)), \mathcal{O}), \]
which preserves the degree. As \[ \text{Corollary 2.1} \] one may see that the Beilinson-Drinfeld homomorphism yields a commutative diagram:
\[ \otimes_{i=1}^N \Gamma(H^0(D_i, \Omega^{\otimes 2}(\mathcal{O})), \mathcal{O}) \xrightarrow{\text{Gr}(\beta)} \Gamma(T^* \text{Bun}_{\mathcal{D}-\text{fl},G,X}, \mathcal{O}) \]
\[ \xrightarrow{h^*} \Gamma(H^0(X, \Omega^{\otimes 2}(D)), \mathcal{O}), \]
Here the right vertical arrow \( h^* \) is induced by the Hitchin’s hamiltonian system, which is an isomorphism as we have seen at \[ \text{Corollary 2.1} \]. Since \( r^* \) is surjective, so is \( \text{Gr}(\beta) \). An easy induction argument with respect to the filtration will show that \( \beta_\lambda \) is surjective. In particular, we have obtained the following proposition.

**Proposition 5.1.** \( D'_{D-\text{fl},\lambda} \) is a commutative ring.

Let \( F \) be the filtration. For each \( f_i \), we choose \( \delta_i \in F_2(D'_{D-\text{fl},\lambda}) \) so that 
\[ [\delta_i] = h^* f_i \in \text{Gr}_2^F(D'_{D-\text{fl},\lambda}) \simeq \Gamma(\text{Bun}^{D_{-\text{fl}}}_{G,X}, \text{Sym}^2(T^* \text{Bun}_{G,X}^{D-\text{fl}})), \]
where \( \text{Sym}^2(T^* \text{Bun}_{G,X}^{D-\text{fl}}) \) is the symmetric square of the cotangent bundle of \( \text{Bun}_{G,X}^{D-\text{fl}} \). This defines a homomorphism 
\[ \Gamma(H^0(X, \Omega^{\otimes 2}(D)), \mathcal{O}) \xrightarrow{h_\lambda} D'_{D-\text{fl},\lambda}, \quad h_\lambda(f_i) = \delta_i. \]
which preserves the filtrations and that
\[ \text{Gr}(h_\lambda) = h^*. \]

As before, by the induction argument with respect to the filtration, one can see that \( h_\lambda \) is surjective.

**Lemma 5.1.** \( h_\lambda \) is injective.

**Proof.** If \( h_\lambda \) were not injective, there is
\[ x \neq 0 \in \Gamma(H^0(X, \Omega^{\otimes 2}(D)), \mathcal{O}) \]
such that \( h_\lambda(x) = 0 \). We take the minimal nonnegative integer \( i \) so that
\[ x \in F_i \setminus F_{i-1}. \]
Here \( F_i \) denotes \( F_i(\Gamma(H^0(X, \Omega^{\otimes 2}(D)), \mathcal{O})) \). Let \([x]\) be the element of \( \text{Gr}^F_i((\Gamma(H^0(X, \Omega^{\otimes 2}(D)), \mathcal{O}))) \) determined by \( x \). By our choice of \( i \), we have
\[ [x] \neq 0. \quad (15) \]

Now \( h_\lambda(x) = 0 \) implies
\[ h^*([x]) = \text{Gr}(h_\lambda)([x]) = 0. \]

Since \( h^* \) is an isomorphism, \([x]\) must be 0. But this contradicts to (15).

\[ \triangle \]

Thus we have proved the following theorem.

**Theorem 5.1.** There is an isomorphism of filtered algebras:
\[ \Gamma(H^0(X, \Omega^{\otimes 2}(D)), \mathcal{O}) \overset{h_\lambda}{\cong} D'_{D-fi,\lambda}, \]

such that
\[ \text{Gr}(h_\lambda) = h^*. \]

Let
\[ \Omega^{\otimes 2}(2D) \overset{R_{-2}}{\longrightarrow} \mathbb{C}^N \]
be a map which is defined to be
\[ R_{-2}(\omega) = (\text{Res}_{t_1=0}(t_1 \cdot \omega), \ldots, \text{Res}_{t_N=0}(t_N \cdot \omega)), \quad \omega \in \Omega^{\otimes 2}(2D). \]

Then we have an exact sequence
\[ 0 \rightarrow \Omega^{\otimes 2}(D) \rightarrow \Omega^{\otimes 2}(2D) \overset{R_{-2}}{\longrightarrow} \mathbb{C}^N \rightarrow 0. \quad (16) \]

Note that the assumption for \( \Omega(D) \) implies
\[ H^1(X, \Omega^{\otimes 2}(D)) = 0. \]
Thus (16) induces an exact sequence

\[ 0 \to H^0(X, \Omega^{\otimes 2}(D)) \to H^0(X, \Omega^{\otimes 2}(2D)) \xrightarrow{R_{-2}} \mathbb{C}^N \to 0. \]

Now we put

\[ \Delta(\lambda_i) = \frac{\lambda_i(\lambda_i + 2)}{4}, \]

and

\[ \Delta(\lambda) = (\Delta(\lambda_1), \ldots, \Delta(\lambda_N)) \in \mathbb{C}^N. \]

Let us consider an affine variety

\[ H_{\Delta(\lambda)} = (R_{-2})^{-1}(\Delta(\lambda)). \]

This is a homogeneous space of \( H^0(X, \Omega^{\otimes 2}(D)) \) and the translation by \( \Delta(\lambda) \) gives an isomorphism between \( \Gamma(H_{\Delta(\lambda)}, \mathcal{O}) \) and \( \Gamma(H^0(X, \Omega^{\otimes 2}(D)), \mathcal{O}) \) as filtered algebras. Now Theorem 5.1 implies

**Theorem 5.2.** There is an isomorphism of \( \mathbb{C} \)-algebras:

\[ \Gamma(H_{\Delta(\lambda)}, \mathcal{O}) \xrightarrow{h^+_\lambda} D'_{D_{-\text{fl}}}, \]

which preserves the filtrations and that

\[ \text{Gr}(h^+_\lambda) = h^*. \]

The isomorphism \( h^+_\lambda \) will be referred as a quantization of the Hitchin’s hamiltonian system.

### 5.2 A generalization of the Beilinson-Drinfeld correspondence

An element \( q \) of \( H_{\Delta(\lambda)} \) defines a homomorphism of algebra:

\[ \Gamma(H_{\Delta(\lambda)}, \mathcal{O}) \xrightarrow{f_q} \mathbb{C}. \]

By Theorem 5.2, \( \text{Ker} f_q \) may be considered as a kernel of \( D'_{D_{-\text{fl}}}, \). We will define a D-module \( \mathcal{M}_q \) on \( \text{Bun}_{G, X}^{D_{-\text{fl}}} \) to be

\[ \mathcal{M}_q = (D'_{D_{-\text{fl}}}/\text{Ker} f_q) \otimes_{\mathcal{O}} L^{-1}_\lambda. \]

One may see that the characteristic variety of \( \text{char}(\mathcal{M}_q) \) of \( \mathcal{M}_q \) coinsides with \( \text{Nilp}(\text{cf. Proposition 5.1.2}) \). Proposition 2.2 implies the following theorem.

**Theorem 5.3.** \( \mathcal{M}_q \) is a holonomic D-module.

The correspondence which assigns the D-module \( \mathcal{M}_q \) to \( q \in H_{\Delta(\lambda)} \) will be referred as a generalized Beilinson-Drinfeld correspondence.
6 A localization functor

Let

\[ Z \overset{\tau_{D-fl}}{\longrightarrow} \text{Bun}_{G,N}^{D-fl} \cong \hat{B}_N^+ \setminus Z \]

be the projection. Using the map (6), one can associate a D-module \( \Delta_{D-fl}(M) \) on \( \text{Bun}_{G,N}^{D-fl} \) to a \( U_2(\hat{\mathfrak{g}}_N) \)-module \( M \) by

\[ \Delta_{D-fl}(M) = [\tau_{D-fl}^* (DZ' \otimes U_2(\hat{\mathfrak{g}}_N) M)]^{\hat{B}_N^+} \]

The functor \( \Delta_{D-fl} \) will be referred as a localization functor ([16.2]). Let us remember that \( \mathfrak{g}_N \) is a Lie subalgebra of \( \hat{\mathfrak{g}}_N \). Therefore we have a homomorphism

\[ U(\mathfrak{g})^\otimes N \longrightarrow U_2(\hat{\mathfrak{g}}_N), \]

and for \( \lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{Z}^N \), we may form \( M_\lambda \) into \( U_2(\hat{\mathfrak{g}}_N) \)-module \( M_{-2,\lambda} \):

\[ M_{-2,\lambda} = U_2(\hat{\mathfrak{g}}_N) \otimes U(\mathfrak{g})^\otimes N M_\lambda. \]

As we have seen in 3.3, \( \Delta_{D-fl}(M_{-2,\lambda}) \) is isomorphic to \( \mathcal{D}_{D-fl,\lambda} \).

**Definition 6.1.** We will call

\[ q = (q_1, \cdots, q_N) \in \oplus_{i=1}^N H^0(D_i, \Omega^\otimes 2(\mathcal{O})) \]

is global if it is contained in the image of the Taylor expansion

\[ H^0(X, \Omega^\otimes 2(D)) \longrightarrow \oplus_{i=1}^N H^0(D_i, \Omega^\otimes 2(\mathcal{O})). \]

Also it will be called \( \lambda \)-admissible if it has a Taylor expansion

\[ q_i = q_i(t_i) dt^\otimes 2 \]

such that

\[ q_i(t_i) = \frac{\Delta(\lambda_i)}{t_i^2} + \cdots, \]

for each \( i \).

Let

\[ q = (q_1, \cdots, q_N) \in \oplus_{i=1}^N H^0(D_i, \Omega^\otimes 2(\mathcal{O})) \]

be \( \lambda \)-admissible. It defines a homomorphism

\[ \otimes_{i=1}^N \Gamma(H^0(D_i, \Omega^\otimes 2(\mathcal{O})), \mathcal{O}) \xrightarrow{\Delta} \mathbb{C}. \]

By the isomorphism of 4.2:

\[ \otimes_{i=1}^N \Gamma(H^0(D_i, \Omega^\otimes 2(\mathcal{O})), \mathcal{O}) \cong \mathbb{C}[[s_m]_{m \leq 0}]^\otimes N, \]

(17)
it may be considered as a homomorphism from \( C[[S_m]]_{m \leq 0} \otimes N \). Now let us recall that each \( S_m \) is contained in the center of the local completion of \( U_{-2}(\hat{\mathfrak{g}}_N) \) and that it acts on \( M_{-2,\lambda} \). Hence we can consider a \( U_{-2}(\hat{\mathfrak{g}}_N) \)-module \( M_{-2,\lambda}^q \):

\[
M_{-2,\lambda}^q = M_{-2,\lambda} \otimes C[[S_m]]_{m \leq 0} \otimes N \mathbb{C}.
\]

Here we have regarded \( \mathbb{C} \) as \( C[[S_m]]_{m \leq 0} \otimes N \)-module via \( f_q \).

On the other hand, by the homomorphism \( \hat{\beta}_\lambda \) of Proposition 4.1, we have a \( U_{-2}(\hat{\mathfrak{g}}_N) \)-module:

\[
D'_{D_{-f\lambda}} \otimes_{\bigotimes_{i=1}^N \Gamma(H^0(D_i, \Omega^{\otimes 2}(2D)))} \mathcal{O} \mathbb{C}.
\]

Proposition 6.1.

\[
\Delta_{D-f\lambda}(M_{-2,\lambda}^q) \simeq D'_{D_{-f\lambda}} \otimes_{\bigotimes_{i=1}^N \Gamma(H^0(D_i, \Omega^{\otimes 2}(2D)))} \mathcal{O} \mathbb{C}.
\]

Proof. By definition, we have

\[
D'_{Z} \otimes_{U_{-2}(\hat{\mathfrak{g}}_N)} M_{-2,\lambda}^q = (D'_{Z} \otimes_{U_{-2}(\hat{\mathfrak{g}}_N)} M_{-2,\lambda}) \otimes_{C[[S_m]]_{m \leq 0} \otimes N} \mathbb{C}.
\]

The commutativity of \( S_m \) with the action of \( \mathcal{B}_+^N \) implies

\[
[\tau_{D-f\lambda}^*(D'_{Z} \otimes_{U_{-2}(\hat{\mathfrak{g}}_N)} M_{-2,\lambda}^q)]^{\mathcal{B}_+^N} \simeq \tau_{D-f\lambda}^*(D'_{Z} \otimes_{U_{-2}(\hat{\mathfrak{g}}_N)} M_{-2,\lambda}) \otimes_{C[[S_m]]_{m \leq 0} \otimes N} \mathbb{C}.
\]

Now the isomorphism (17) implies the desired result.

\[\triangle\]

By Theorem 5.2, Spec \( D'_{D_{-f\lambda}} \) may be identified with \( H_\Delta(\lambda) \) which is a closed subvariety of \( \bigotimes_{i=1}^N H^0(D_i, \Omega^{\otimes 2}(2D)) \) by the restriction map. Now the identity

\[
\hat{\beta}_\lambda(\tau_0^{(i)}) = \Delta(\lambda_i).
\]

and Proposition 6.1 will imply the following theorem.

Theorem 6.1. \( \Delta_{D-f\lambda}(M_{-2,\lambda}^q) \) is nonzero if and only if \( q \) is global and \( \lambda \)-admissible.

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Address : Department of Mathematics and Informatics
Faculty of Science
Chiba University
1-33 Yayoi-cho Inage-ku
Chiba 263-8522, Japan

e-mail address : sugiyama@math.s.chiba-u.ac.jp