Fractional Logistic Map
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Abstract
A new type of an integrable mapping is presented. This map is equipped with fractional difference and possesses an exact solution, which can be regarded as a discrete analogue of the Mittag-Leffler function.

1 Introduction
Logistic map 
\[ \frac{u_{n+1} - u_n}{\varepsilon} = au_n(1 - u_{n+1}) \quad (a > 0) \quad (1.1) \]
is an integrable discretization of the well-known logistic equation.
\[ \frac{d}{dt}u = au(1 - u) \quad (a > 0) \quad (1.2) \]
Its origin was a population model in ecology. In addition, it has been reported that the logistic map is used in many other fields, for example, agriculture, life sciences and engineering.

Through dependent variable transformation,
\[ u_n = \frac{1 + a\varepsilon}{a\varepsilon g_n + 1 + a\varepsilon} \quad (1.3) \]
the logistic map (1.1) is linearized as
\[ \Delta_{-n}g_n = -ag_n, \quad (1.4) \]
where \( \Delta_{-n} \) is a backward difference operator defined by
\[ \Delta_{-n} = \varepsilon^{-1}(1 - E^{-1}), \quad E^{-1}f_n = f_{n-1}. \quad (1.5) \]
Since equation (1.4) possesses a solution,
\[ g_n = g_0(1 + a\varepsilon)^{-n} \quad (1.6) \]
a solution to the logistic map (1.1) is given by

\[ u_n = \frac{u_0}{u_0 + (1 - u_0)(1 + a\varepsilon)^{-n}} \quad (1.7) \]

The main purpose of this paper is to provide a new type of an integrable mapping, which can be regarded as an extension of the logistic map and is equipped with fractional difference. This map, which we call the fractional logistic map here, possesses an exact solution and have another parameter \( p \) which corresponds to an order of difference. In section 2, we introduce one definition of fractional difference, which is a slight modification of Hirota’s fractional difference operator. We also find an eigenfunction of this operator. Section 3 is the main consequence of this paper and presents the fractional logistic map. We also show its time evolution through numerical experiment.

2 Fractional difference

We here give a definition of fractional difference operator and its eigenfunction. Before going to its definition, let us introduce fundamental functions \( M(a; n) \) defined by

\[ M(a; n) = \frac{1}{\Gamma(a)} \varepsilon^{a-1} \frac{\Gamma(n + a - 1)}{\Gamma(n)} = \varepsilon^{a-1} \binom{n + a - 2}{n - 1} \quad (a > 0, n \in \mathbb{Z}), \quad (2.1) \]

where \( \varepsilon \) is an interval length and \( \binom{a}{n} \) \((a \in \mathbb{R}, n \in \mathbb{Z})\) stands for a binomial coefficient defined by

\[
\binom{a}{n} = \begin{cases} 
\frac{a(a-1)\cdots(a-n+1)}{n!} & (n > 0) \\
1 & (n = 0) \\
0 & (n < 0)
\end{cases}
\]

This function satisfies the following lemma.

**Lemma 2.1** The following relations hold.

\[ M(a; 0) = 0 \quad (a > 1) \quad (2.2) \]

\[ \Delta_{-n}M(a + 1; n) = \varepsilon^{-1}(M(a + 1; n) - M(a + 1; n - 1)) = M(a; n) \quad (a > 0) \quad (2.3) \]

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Proof: Equation (2.2) is obvious owing to the definition of binomial coefficient. Equation (2.3) is proved by using the relation $\Gamma(x + 1) = x\Gamma(x)$ as follows.

\[
\varepsilon^{-1}M(a + 1; n) - M(a + 1; n - 1) = \frac{\varepsilon^{a - 1}}{\Gamma(a + 1)} \left( \frac{(n + a - 1)\Gamma(n + a - 1)}{\Gamma(n)} - \frac{(n - 1)\Gamma(n + a - 1)}{\Gamma(n)} \right) = \frac{a\varepsilon^{a - 1}}{\Gamma(a + 1)} \frac{\Gamma(n + a - 1)}{\Gamma(n)} = M(a; n).
\]

\[\blacksquare\]

Next we go to the definition of fractional difference. Hirota [4] took the first $n$ terms of Taylor series of $\Delta_\alpha^n = \varepsilon^{-\alpha}(1 - E^{-1})^\alpha$ and gave the following definition.

**Definition 1** Let $\alpha \in \mathbb{R}$. Then difference operator of order $\alpha$ is defined by

\[
\Delta_\alpha^nu_n = \begin{cases} 
\varepsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} (-1)^j u_{n-j} & \alpha \neq 1, 2, \ldots \\
\varepsilon^{-m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j u_{n-j} & \alpha = m \in \mathbb{Z}_{>0}
\end{cases}
\]

It should be noted that Diaz, Osler [2] gave another definition of fractional difference,

\[
\Delta_\alpha^nu(t) = \varepsilon^{-\alpha} \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j u_{n-j}
\]

We here adopt another difference operator $\Delta_{s,-n}^\alpha$ by modifying Hirota’s operator.

**Definition 2** Let $\alpha \in \mathbb{R}$ and $m$ be an integer such that $m - 1 < \alpha \leq m$. We define difference operator of order $\alpha$, $\Delta_{s,-n}^\alpha$, by

\[
\Delta_{s,-n}^\alpha u_n \equiv \Delta_{-n}^{\alpha-m} \Delta_{-n}^mu_n = \varepsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{\alpha - m}{j} (-1)^j \Delta_{-k}^m u_k |_{k=n-j}
\]

We define a new function,

\[
F_\alpha(\lambda, n) = \sum_{j=0}^{\infty} \lambda^j M(aj + 1; n) = \sum_{j=0}^{\infty} \lambda^j \varepsilon^{aj} \frac{\Gamma(n + aj)}{\Gamma(aj + 1)\Gamma(n)}
\]
Remark 1 Putting $a = 1$ in the above definition, we have

$$F_1(\lambda, n) = \sum_{j=0}^{\infty} \lambda^j \varepsilon^j \frac{\Gamma(n+j)}{\Gamma(j+1)\Gamma(n)}$$

$$= \sum_{j=0}^{\infty} (\lambda \varepsilon)^j \binom{n+j-1}{j}$$

$$= \sum_{j=0}^{\infty} (-\lambda \varepsilon)^j \binom{-n}{j} = (1 - \lambda \varepsilon)^{-n}. \quad (2.8)$$

Remark 2 In the limit of $\varepsilon \to 0, n \to \infty$ with $t = n \varepsilon$ fixed, the function $F_a(\lambda, n)$ converges to

$$F_a(\lambda, n) \to \sum_{j=0}^{\infty} \frac{\lambda^j t^a_j}{\Gamma(a_j + 1)} = E_a(\lambda t^a). \quad (2.9)$$

$E_a(x)$ is the well-known Mittag-Leffler function and and its asymptotic behavior is studied in detail in [7].

Theorem 2.1 If $a > 0$, the function $F_a(\lambda, n)$ is an eigen-function of the fractional difference operator (2.6). That is,

$$\Delta^{a}_{a-n} F_a(\lambda, n) = \lambda F_a(\lambda, n) \quad (2.10)$$

Proof of Theorem 2.1: Let $m$ be an integer such that $m - 1 < a \leq m$. Then we have

$$\Delta^{a}_{a-n} F_a(\lambda, n) = \Delta^{a}_{a-n} \left( 1 + \sum_{j=1}^{\infty} \lambda^j M(a_j + 1; n) \right)$$

$$= \Delta^{a}_{a-n} \sum_{j=1}^{\infty} \lambda^j M(a_j + 1; n)$$

$$= \Delta^{a-m}_{a-n} \sum_{j=1}^{\infty} \lambda^j \Delta^{m}_{a-n} M(a_j + 1; n)$$

$$= \sum_{j=1}^{\infty} \lambda^j \Delta^{a-m}_{a-n} M(a_j + 1 - m; n) \quad (2.11)$$
Each summand in the above equation is given by

\[
\Delta^{-m}_{aj} M(aj + 1 - m; n) = \sum_{k=0}^{n-1} \binom{a - m}{k} (-1)^k M(aj + 1 - m; n - k)
\]

\[
= \sum_{k=0}^{n-1} \binom{a - m}{k} (-1)^k \binom{aj - m + n - k - 1}{n - k - 1}
\]

\[
= \sum_{k=0}^{n-1} \binom{a - m}{k} (-1)^{n-1} \binom{-aj + m - 1}{n - k - 1}
\]

\[
= (-1)^{n-1} \binom{a - aj - 1}{n - 1} = \binom{aj - a - 1}{n - 1} = M(aj - a + 1; n).
\]  \tag{2.12}

Therefore, substitution of eq. \((2.12)\) into eq. \((2.11)\) gives

\[
\Delta_{\epsilon, n}^a F_a(\lambda, n) = \sum_{j=1}^{\infty} \lambda^j M(aj - a + 1; n) = \sum_{j=0}^{\infty} \lambda^{j+1} M(aj + 1; n) = \lambda F_a(\lambda, n)
\]  \tag{2.13}

which completes the proof.  \hfill \blacksquare

3 Fractional Logistic map

This section presents a fractional integrable mapping, which we call fractional logistic map here. We start with a linear equation

\[
\Delta_{\epsilon, n}^p g_n = -ag_n \quad (0 < p \leq 1, a > 0)
\]  \tag{3.1}

instead of linear eq. \((1.4)\). The above equation is rewritten as

\[
g_n = (1 + a\epsilon^p)^{-1} \left\{ g_{n-1} - \sum_{j=1}^{n-1} \binom{p - 1}{j} (-1)^j (g_{n-j} - g_{n-j-1}) \right\}
\]  \tag{3.2}

Through the similar dependent variable transformation as logistic map,

\[
u_n = \frac{1 + a\epsilon^p}{a\epsilon^p g_n + 1 + a\epsilon^p}
\]  \tag{3.3}

we finally obtain a fractional logistic map

\[
u_n = \frac{1}{1 + \frac{1}{1 + a\epsilon^p} \left\{ \frac{1}{u_{n-1}} - \sum_{j=1}^{n-1} \binom{p - 1}{j} (-1)^j \left( \frac{1}{u_{n-j}} - \frac{1}{u_{n-j-1}} \right) \right\}}.
\]  \tag{3.4}
Due to the Theorem 2.1, eq. (3.1) has a solution,

\[ g_n = g_0 F_p(-a; n). \] (3.5)

Therefore, a solution to the fractional logistic map (3.4) is given by

\[ u_n = \frac{u_0}{u_0 + (1 - u_0) F_p(-a; n)} \] (3.6)

Putting \( p = 1 \) in eq. (3.4), we have

\[ u_n = \frac{(1 + a \varepsilon) u_{n-1}}{1 + (1 + a \varepsilon) u_{n-1}}, \] (3.7)

which recovers the logistic map. The following figure illustrates time evolutions of fractional logistic map with order parameter \( p = n/4 (n = 1, 2, 3, 4) \). We have put \( u_0 = 0.1, a = 1.0 \) and \( \varepsilon = 0.1 \).

![Figure 1: Time evolutions of fractional logistic map](image)

Considering the fact that the Mittag-Leffler function has an asymptotic behavior \([7]\),

\[ E_a(\lambda t^a) = - \sum_{n=1}^{N-1} \frac{\lambda^{-n} t^{-an}}{\Gamma(1 - an)} + O(t^{-aN}), \quad t \to \infty, \lambda < 0, \] (3.8)
we can observe that \( u_n \) converges to 1 at the order of \( O(1/n^p) \) if \( 0 < p < 1 \). The following table illustrates a numerical result in which we apply the \( \rho \)-algorithm

\[
\rho_{k+1}^n = \rho_{k-1}^n + \frac{(k+n)^p - n^p}{\rho_{k+1}^n - \rho_k^n}
\]

\( \rho_{-1}^0 = 0, \rho_0^n = u_n \)

to the sequence \( \{u_n\} \) in the case \( p = 1/4 \). This table shows that \( u_n \) converges to the value near 1.0 at the order \( O(1/n^{1/4}) \) as \( n \) tends to \( +\infty \).

Table 1: The \( \rho \)-algorithm applied to the sequence \( \{u_n\} \) in the case \( p = 1/4 \)

|   | \( \rho_1^n \) | \( \rho_3^n \) | \( \rho_5^n \) | \( \rho_7^n \) | \( \cdots \) | \( \rho_{21}^n \) |
|---|----------------|----------------|----------------|----------------|----------|----------------|
| 1 | 0.1            | 0.19745        | 0.84609        | 0.84288        | 1.00145  |
| 2 | 0.14791        | 0.23921        | 0.86006        | 1.30904        | 0.99915  |
| 3 | 0.16019        | 0.27802        | 0.99118        | 1.21129        | 1.00131  |
| 4 | 0.16764        | 0.31406        | 1.07379        | 1.17824        | 1.00139  |
| 5 | 0.17313        | 0.34778        | 1.11866        | 1.16134        | 1.00112  |
| 6 | 0.17752        | 0.37941        | 1.14064        | 1.15284        | 1.00120  |
| 7 | 0.18121        | 0.40908        | 1.14968        | 1.15045        | 1.00120  |
| 8 | 0.18442        | 0.43692        | 1.15161        | 1.15375        | 1.00121  |
| 9 | 0.18726        | 0.46304        | 1.14970        | 1.16376        | 1.00122  |
| ... | ...            | ...            | ...            | ...            | ...      |

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