On the relations between the zeros of a polynomial and its Mahler measure

M. Ounaies, G. Rhin and J.-M. Sac-Épée

March 15, 2021

Abstract

In this work, we are dealing with some properties relating the zeros of a polynomial and its Mahler measure. We provide estimates on the number of real zeros of a polynomial, lower bounds on the distance between the zeros of a polynomial and non-zeros located on the unit circle and a lower bound on the number of zeros of a polynomial in the disk \( \{ |x - 1| < 1 \} \).

1 Introduction

For a polynomial \( P \in \mathbb{C}[x] \) of the form

\[
P(x) = \sum_{j=0}^{d} a_j x^j = a_d \prod_{k=1}^{d} (x - \mu_k), \quad a_d \neq 0,
\]

its Mahler measure \( [13] \) is given by

\[
M(P) = \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \log |P(e^{it})| dt \right).
\]

Using the Jensen’s formula \( [10] \), it can be expressed as

\[
M(P) = |a_d| \prod_{|\mu_k| \geq 1} |\mu_k|.
\]

For an algebraic number \( \alpha \), its Mahler measure \( M(\alpha) \) is that of its minimal polynomial in \( \mathbb{Z}[x] \). We refer the reader to \( [23] \) for a survey of results on Mahler measure of algebraic numbers.

When \( P \in \mathbb{Z}[x] \), clearly \( M(P) \geq 1 \) and by Kronecker’s theorem \( [11] \), the only irreducible monic integer polynomials \( P \) such that \( M(P) = 1 \), apart from the monomial \( x \), are cyclotomic polynomials. In other words, the only algebraic numbers \( \alpha \) such that \( M(\alpha) = 1 \) are 0 and the roots of unity.

\*M. Ounaies, IRMA, France, myriam.ounaies@math.unistra.fr
G. Rhin, J.-M. Sac-Épée, IECL, France, georges.rhin@univ-lorraine.fr, jean-marc.sac-eppee@univ-lorraine.fr

2000 Mathematics Subject Classification : 11R06, 11C08, 12D10
In a paper from 1933 [12], Lehmer exhibited the polynomial \( P_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 + x^4 + x^3 + x + 1 \) whose Mahler measure \( M(P_L) = 1.176280 \ldots \) and raised the question whether there exists \( P \in \mathbb{Z}[x] \) such that 1 < \( M(P) \) < \( M(P_L) \). This question is still open and the existence of a constant \( \epsilon > 0 \) such that all \( P \in \mathbb{Z}[x] \) satisfy either \( M(P) = 1 \) or \( M(P) > 1 + \epsilon \) is now known as Lehmer’s conjecture.

For irreducible noncyclotomic polynomials \( P \in \mathbb{Z}[x] \) of degree \( d \), E. Dobrowolski [4] showed a lower bound of the form

\[
M(P) > 1 + C \left( \frac{\log \log d}{\log d} \right)^3,
\]

and it is the best unconditional lower bound for \( M(P) \) known thus far, up to the constant \( C \).

A polynomial \( P \) of degree \( d \) is said to be self-reciprocal if \( P(x) = x^d P \left( \frac{1}{x} \right) \). C. Smyth [22] proved an important theorem in the positive direction of Lehmer’s conjecture:

**Theorem 1.1. [Smyth]** If \( P \in \mathbb{Z}[x] \) is not self-reciprocal and satisfies \( P(0)P(1) \neq 0 \), then \( M(P) \geq \theta_0 \), where \( \theta_0 = 1.324717 \ldots \) is the real root of the polynomial \( x^3 - x - 1 \).

Let us also cite a theorem due to A. Schinzel [20]:

**Theorem 1.2. [Schinzel]** If \( P \) is a monic integer polynomial of degree \( d \) whose all roots are real and such that \( P(0)P(1)P(-1) \neq 0 \), then

\[
M(P) \geq \left( \frac{1 + \sqrt{5}}{2} \right)^{\frac{d}{2}}.
\]

We will denote by \( E_\theta \) the set of monic irreducible polynomials \( P \in \mathbb{Z}[x] \) such that 1 < \( M(P) < \theta \leq \theta_0 \), where \( \theta_0 \) is as in in Theorem 1.1. Many authors have focused on the computational search for such polynomials. D. Boyd [2, 3] gave a complete list of polynomials in \( E_{1.3} \) up to degree 20. This list was extended by Mossinghoff [15] up to degree 24, then by Flammang, Rhin and Sac-Épée [8] with exhaustive results up to degree 40 and by Mossinghoff, Rhin and Wu [18] who provided an exhaustive list up to degree 44.

M. Mossinghoff [17] maintains a website that collects all known polynomials in \( E_{1.3} \) of degree less than 180. New polynomials have been added to the list available on this website, from the works of Rhin and Sac-Épée [19], Mossinghoff, Pinner and Vaaler [16], and El Otmani, Maul, Rhin and Sac-Épée [6].

In the present work, we are dealing with estimates about the zeros of a polynomial with complex coefficients in relation with its Mahler measure. Because of their importance for Lehmer’s conjecture, we are interested in applying some of our results from Sections 4, 5 and 6 to obtain information on the zero sets of polynomials in \( E_\theta \).

The paper is organised as follows: Section 2 provides notations, preliminary results and first properties of \( E_\theta \). Section 3 deals with lower bounds on \( |\mu - \omega| \) where \( P(\mu) = 0, |\omega| = 1, P(\omega) \neq 0 \). Subsection 3.1 focuses on integer polynomials while Subsection
3.2 deals with real or complex polynomials, mainly self-reciprocal. The lower bounds involve either $M(P)$ or $L(P)$.

A. Dubickas [5] showed that for all $\varepsilon > 0$, there exists $D_0(\varepsilon)$ such that for all monic, irreducible, non-cyclotomic polynomials $P \in \mathbb{Z}[x]$ of degree $d \geq D_0(\varepsilon)$ and for all $\mu \in P^{-1}(0)$,

$$|\mu - 1| > \exp \left( -\left(\frac{\pi}{4} + \varepsilon\right) \sqrt{d \log d \log M(P)} \right).$$

(1.1)

Our Proposition 3.2 improves on (1.1) by substituting the constant $\frac{\pi}{8}$ to $\frac{\pi}{4}$ in the special case where $P$ is self-reciprocal.

For polynomials $P \in \mathbb{C}[x]$ of degree $d$, Theorem 3.3 provides the following inequality for all $\mu \in P^{-1}(0)$:

$$|\mu - 1| \geq \left| \frac{P(1)}{cdL(P)} \right|.$$ 

In particular, if all the coefficients of $P$ are positive, then

$$|\mu - 1| \geq \frac{1}{cd}.$$ 

In Section 4, we generalise Theorem 1.2 to polynomials in $\mathbb{C}[x]$ and we strengthen the dependence on $P(0)$, $P(-1)$ and $P(1)$. Our Theorem 3.3 states that for monic polynomials $P \in \mathbb{C}[x]$ of degree $d$ having $m \geq 1$ real zeros and such that $P(0)P(1)P(-1) \neq 0$, the following holds:

$$M(P) \geq \left( |P(1)P(-1)|^{\frac{1}{m}} + \left( 4^{\frac{m}{d}}|P(0)|^{\frac{1}{m}} + |P(1)P(-1)|^{\frac{1}{m}} \right)^{\frac{1}{2}} \right)^{\frac{1}{m-2}}.$$ 

In Section 5, we give upper bounds for the number of real zeros of $P(x) = \sum_{j=1}^{d} a_j x^j \in \mathbb{C}[x]$ by terms involving (besides its degree) its Mahler measure or its length $L(P) = \sum_{j=1}^{d} |a_j|$.

A. Dubickas [5] showed that for polynomials $P \in \mathbb{Z}[x]$ of degree $d$ satisfying $P(0)P(-1)P(1) \neq 0$, there is a constant $\alpha > 0$ such that

$$|P^{-1}(0) \cap \mathbb{R}| \leq \alpha \sqrt{d \log d \log M(P)}.$$ 

(1.2)

Corollary 5.4 allows us to improve (1.2) by eliminating the term $\sqrt{d \log d}$ in the special case where $\log L(P) \leq \sqrt{d \log M(P)}$.

In Section 6, we focus on integer polynomials. The main result is the following lower bound for the Mahler measure of a monic irreducible polynomial $P \in \mathbb{Z}[x]$ of degree $d \geq D_0(\varepsilon)$ and satisfying $\log M(P) = o\left( \frac{d}{\log d} \right)$:

$$\log M(P) \geq (C - \varepsilon) \frac{d}{K(P)^2 \log d} \quad \text{with} \quad C = \frac{4}{\pi^2} \log^2 \left( \frac{1 + \sqrt{5}}{2} \right)$$

and where we have denoted by $K(P) = \min \left( \left| P^{-1}(0) \cap \{|x - 1| < 1\} \right|, \left| P^{-1}(0) \cap \{|x + 1| < 1\} \right| \right)$.  

3
2 Notations and preliminary results

Consider a polynomial $P(x) = \sum_{j=0}^{d} a_j x^j \in \mathbb{C}[x]$ of degree $d$. In our estimates in relation with $P$, besides $M(P)$, we will use the following quantities:

$$H(P) = \max_{0 \leq j \leq d} |a_j|, \quad L(P) = \sum_{j=0}^{d} |a_j|,$$

$$||P|| = \sup_{|x|=1} |P(x)|, \quad L_2(P) = \left( \sum_{j=0}^{d} |a_j|^2 \right)^{\frac{1}{2}}.$$

The height $H(P)$ and the length $L(P)$ were introduced in [13] by K. Mahler who compared them to $M(P)$, showing the inequalities

$$|a_j| \leq M(P)^{\left(\frac{d}{j}\right)} \quad (j = 0, \ldots, d),$$

$$M(P) \leq L_2(P) \leq L(P) \leq 2^d M(P),$$

$$H(P) \leq 2^{d-1} M(P), \quad M(P) \leq (d + 1)^{\frac{d}{2}} H(P).$$

(2.1)

Using Cauchy-Schwarz inequality, we also have

$$|P(1)| \leq ||P|| \leq L(P) \leq (d + 1)^{\frac{d}{2}} L_2(P)$$

(2.2)

and by Parseval formula,

$$L_2(P) = \left( \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} \leq ||P||.$$

(2.3)

Define the reciprocal of $P$ by

$$P^*(x) = x^d P\left(\frac{1}{x}\right) = \sum_{j=0}^{d} a_{d-j} x^j.$$

Clearly, $P$ is self-reciprocal iff $P = P^*$.

Let $P(x) = \sum_{j=0}^{d} a_j x^j = \prod_{k=1}^{d} (x - \mu_k) \in \mathbb{C}[x]$, $a_d = 1$, $a_0 \neq 0$. We will denote by

$$Q(x) = a_0^{-1} P(x) P^*(x) = \prod_{k=1}^{d} (x - \mu_k) \prod_{k=1}^{d} (x - \mu_k^{-1}) = \sum_{j=0}^{2d} A_j x^j.$$

(2.4)

Then $Q$ is monic and self-reciprocal and

$$M(Q) = |a_0|^{-1} M(P)^2, \quad Q(1) = a_0^{-1} P(1)^2, \quad Q(-1) = (-1)^d a_0^{-1} P(-1)^2.$$

(2.5)

Besides, the coefficients are given by

$$A_j = a_0^{-1} \sum_{k=0}^{j} a_k a_{d-j+k} \quad (j = 0, \ldots, d)$$
and they satisfy
\[
\sum_{j=0}^{d} |A_j| \leq |a_0^{-1}| \sum_{j=0}^{d} \sum_{k=0}^{j} |a_k||a_{d-j+k}| = |a_0^{-1}| \sum_{k=0}^{d} \left( |a_k| \sum_{j=k}^{d} |a_j| \right) \leq |a_0^{-1}| L(P)^2.
\]

For a set \(A \subset \mathbb{C}\), we will denote by \(|P^{-1}(0) \cap A|\) the number of zeros of \(P\) belonging to \(A\) counted with multiplicities.

**Proposition 2.1.** Let \(r > 1\) and \(P\) be a monic polynomial in \(\mathbb{C}[x]\). Then
\[
|P^{-1}(0) \cap \{|x| > r\}| < \frac{\log M(P)}{\log r}.
\]

**Proof.** Put \(N = |P^{-1}(0) \cap \{|x| > r\}|. Then
\[
r^N < \prod_{\mu \in P^{-1}(0) \cap \{|x| > r\}} |\mu| \leq \prod_{\mu \in P^{-1}(0) \cap \{|x| \geq 1\}} |\mu| = M(P).
\]

For \(1 < \theta < \theta_0\), where \(\theta_0\) is given by Theorem 1.1 let \(E_\theta\) be the set of integer monic irreducible polynomials \(P\) such that \(1 < M(P) \leq \theta\) and satisfying conditions c1 and c2:

1. \(P\) is primitive, i.e., it cannot be written as \(Q(x^m)\) with \(m \geq 2\), \(Q \in \mathbb{Z}[x]\).
2. \(a_{j_0} > 0\), where \(j_0 = \min\{j \geq 1 \text{ such that } a_j \neq 0\}\). This condition is satisfied either by \(P(x)\) or by \(P(-x)\).

Conditions c1 and c2 are usually added by the authors looking for polynomials with small Mahler measure via computational methods in order to avoid redundancy, since
\[
M(P(x)) = M(P(-x)) \text{ and } M(P(x^m)) = M(P(x)).
\]

**Remark 2.2.**
- For any \(P \in E_\theta\) and \(\mu \in P^{-1}(0)\), \(P\) is the minimal polynomial of \(\mu\) in \(\mathbb{Z}[x]\).

- By Theorem 1.1 all polynomials of \(E_\theta\) are self-reciprocal.

- Any self-reciprocal polynomial of odd degree is divisible by \((x+1)\). Thus all polynomials in \(E_\theta\) are of even degree.

- By Kronecker’s theorem, the cyclotomic polynomials are excluded of \(E_\theta\) because their Mahler measure is equal to 1.

The first properties satisfied by the zero set \(Z_\theta = \bigcup_{P \in E_\theta} P^{-1}(0)\) are listed in the proposition below.

**Proposition 2.3.**

1. All \(P^{-1}(0), P \in E_\theta\) are disjoint and they contain only simple zeros.
2. For all \( P \in E_\theta \), \( P^{-1}(0) \) is symmetric with regard to \( \{|x| = 1\} \) and \( \{\Im x = 0\} \).

3. \( Z_\theta \subset \{ \frac{1}{\sqrt{\theta}} \leq |x| \leq \theta \} \).

4. \( Z_\theta \cap \{3x \neq 0\} \subset \{ \frac{1}{\sqrt{\theta}} \leq |x| \leq \sqrt{\theta} \} \).

5. For all \( P \in E_\theta \) of degree \( 2n \), and for all \( r > 1 \),
   \[
   |P^{-1}(0) \cap \{ \frac{1}{r} \leq |x| \leq r \}| > 2(n - \frac{\log \theta}{\log r}).
   \]

6. \( Z_\theta \) does not contain any root of unity.

7. \( Z_\theta \) does not intersect the imaginary axe.

**Proof.**

1. This follows immediately after the uniqueness of the minimal polynomial of an algebraic integer.

2. This is because \( P(x) = 0 \iff P(\bar{x}) = 0 \iff P(1/x) = 0 \).

3. Let \( P \in E_\theta \). By symmetry of \( P^{-1}(0) \) with regards to \( \{|x| = 1\} \), it suffices to note that if \( x \in P^{-1}(0) \cap \{|x| > 1\} \) then \( |x| \leq M(P) \leq \theta \).

4. Let \( P \in E_\theta \). By symmetry of \( P^{-1}(0) \) with regards to \( \{|x| = 1\} \) and \( \{3x = 0\} \), it suffices to note that if \( x \in P^{-1}(0) \cap \{|x| > 1\} \cap \{3x \neq 0\} \) then \( |x|^2 \leq M(P) \leq \theta \).

5. This is a consequence of Proposition 2.1 and 2.

6. The minimal polynomials corresponding to the roots of unity are cyclotomic polynomials, which are excluded from \( E_\theta \).

7. Assume that for some \( y \in \mathbb{R} \) and \( P = \sum_{j=0}^{2n} a_j x^j \in E_\theta \) (\( a_{2n} \neq 0 \)), \( P(iy) = 0 \). Then \( \Im(P(iy)) = \sum_{k=0}^{n-1} a_{2k+1} (iy)^{2k+1} = 0 \). But at least one of the coefficients \( a_{2k+1} \) is not zero by condition c1. This would contradict the fact that \( P \) is the minimal polynomial of \( iy \).

For illustration purposes, the graph on the left in Figure 1 shows all known roots belonging to \( Z_{1.3} \), corresponding to polynomials of degree less than 180. In particular, one can visualize the properties of symmetry of this set. The small Mahler measure polynomials from which these roots are derived are listed on Michael Mossinghoff’s website [17].
Figure 1: The known elements of $Z_{1.3}$ obtained from polynomials of degree less than 180.

3 Distance between zeros of a polynomial and non-zeros of modulus one

On Figure 1, we notice that the roots of the polynomials in $E_{1.3}$ seem to stay far from the roots of unity. The results given in this section, when specialised to polynomials in $E_\theta$, $\theta \leq \theta_0$, allow us to estimate from below the distance between the zeros of $P \in E_\theta$ and the roots of unity. Most of our results deal with polynomials which are self-reciprocal of even degree like the ones in $E_\theta$.

3.1 Lower bounds involving the Mahler measure

For an algebraic number $\mu \neq 1$ of degree $d$, Liouville inequality gives:

$$|\mu - 1| \geq 2^{1-d}M(\mu)^{-1}.$$ 

We prove the following Liouville-type inequalities:

**Proposition 3.1.** Assume that $P \in \mathbb{Z}[x]$ is monic, self-reciprocal of degree $2n$ and does not vanish at the roots of unity. If $P(\mu) = 0$ and $\omega^m = 1$ or $\omega^m = -1$ ($m \in \mathbb{N}^*$), then

- If $\mu \in \mathbb{R}$ or $|\mu| = 1$, $|\mu - \omega| \geq m^{-1}2^{1-n}M(P)^{-\frac{d}{2}},$

- If $\mu \notin \mathbb{R}$ and $|\mu| \neq 1$, $|\mu - \omega| \geq m^{-1}2^{1-n}M(P)^{-\frac{d}{2}}$.

**Proof.** By substituting $P(-x)$ to $P(x)$, we may assume that $\omega^m = 1$ and by substituting $\omega$ to $\omega$, we may assume that $3\mu \geq 0$. We divide $P^{-1}(0) \cap \{|x| \geq 1, 3x \geq 0\}$ into two sets:

- $\mu \in A \iff P(\mu) = 0, |\mu| = 1, 3\mu > 0$ or $|\mu| > 1, 3\mu = 0$,
- $\mu \in B \iff P(\mu) = 0, |\mu| > 1, 3\mu > 0$. 

7
Then $P^{-1}(0) = \{ \mu, \mu^{-1}, \mu \in A \} \cup \{ \mu, \mu^{-1}, \bar{\mu}, \bar{\mu}^{-1}, \mu \in B \}$ and
\[
\prod_{\mu \in P^{-1}(0)} (1 - \mu^m) = M(P)^m \prod_{\mu \in A} (1 - \mu^{-m})^2 \prod_{\mu \in B} (1 - \mu^{-m})^4.
\]

$\prod_{\mu \in P^{-1}(0)} (1 - \mu^m)$ is a symmetric polynomial in the zeros of $P$ thus it is an integer and as $P$ does not vanish at the roots of unity, it is a non-zero integer. We deduce that
\[
1 \leq M(P)^m \prod_{\mu \in A} (1 - \mu^{-m}) \prod_{\mu \in B} (1 - \mu^{-m})^2.
\]

For all $\mu \in A \cup B$, we have $|1 - \mu^{-m}| \leq 2$ thus
\[
\forall \mu \in A, \quad 1 \leq M(P)^m 2^{n-1}|1 - \mu^{-m}|,
\]
\[
\forall \mu \in B, \quad 1 \leq M(P)^m 2^{n-2}|1 - \mu^{-m}|^2.
\]

To obtain the desired estimate for all $\mu \in A \cup B$, we use the following
\[
|1 - \mu^{-m}| \leq |\bar{\omega} - \mu^{-1}| \sum_{j=0}^{m-1} |\mu^{-j}| \leq m|\bar{\omega} - \mu^{-1}| = m|\mu|^{-1} |\omega - \mu| \leq m|\omega - \mu|.
\]

If $|\mu| < 1$, we apply the former to $\bar{\mu}^{-1}$. and we use $|1 - \bar{\mu}^m| \leq m|\omega - \mu|$. ■

For integer polynomials of small Mahler measure, there are better lower bounds due to Mignotte and Waldschmidt [14]. They showed that for all $\varepsilon > 0$, there exists $D_0(\varepsilon)$ such that for $d \geq D_0(\varepsilon)$
\[
|\mu - 1| > \exp\left(-(1 + \varepsilon)\sqrt{d \log d \log M}\right).
\]

This was improved by Dubickas [5] as follows:
\[
|\mu - 1| > \exp\left(-\left(\frac{\pi}{4} + \varepsilon\right) \sqrt{d \log d \log M}\right). \tag{3.1}
\]

With a minor modification in Dubickas proof of Theorem 1 in [5], we obtain a better result than (3.1) in the special case when $P$ is self-reciprocal. This is displayed in the next proposition.

**Proposition 3.2.** Assume that $P \in \mathbb{Z}[x]$ is monic, irreducible, self-reciprocal of degree $d$ and satisfies $M(P) > 1$. Let $P(\mu) = 0$ and $\omega^m = 1$ or $\omega^m = -1$ ($m \in \mathbb{N}^*$). Then for all $\varepsilon > 0$, there exists $D_0(\varepsilon)$ such that for $d \geq D_0(\varepsilon)$
\[
|\omega - \mu| \geq \exp\left(-\left(\frac{\pi \sqrt{m}}{8} + \varepsilon\right) \sqrt{d \log d \log M(P)}\right).
\]

If $\mu \notin \mathbb{R}$ and $|\mu| \neq 1$, then
\[
|\omega - \mu| \geq \exp\left(-\left(\frac{\pi \sqrt{m}}{16} + \varepsilon\right) \sqrt{d \log d \log M(P)}\right).
\]
3.2 Lower bounds involving the coefficients

Theorem 3.3. Let \( P \in \mathbb{C}[x] \) be of degree \( d \in \mathbb{N}^* \). Let \( \mu \) be a zero of \( P \) with multiplicity \( m_\mu \), and \(|\omega| = 1\). Then,

\[
|\mu - \omega| \geq d^{-1} \left( e^{-1} \left| \frac{P(\omega)}{L(P)} \right| \right)^{\frac{1}{m_\mu}}.
\]

In particular, if all the coefficients of \( P \) are positive, then

\[
|\mu - 1| \geq d^{-1} e^{-\frac{1}{m_\mu}}.
\]

If \(|P(\omega)| = \|P\|\), then

\[
|\mu - \omega| \geq d^{-1} \left( e^{-1} (d + 1)^{-\frac{d-1}{2}} \right)^{\frac{1}{m_\mu}}.
\]

Proof. We may assume \( P(\omega) \neq 0 \). For all \( \theta \in [0, 2\pi]\),

\[
|P(\omega + d^{-1} e^{i\theta})| \leq (1 + d^{-1})^d L(P) \leq e L(P).
\]

Let \( \mu_k \) be the distinct zeros of \( P \) with respective multiplicities \( m_\mu \). Applying Jensen’s formula to \( \frac{P(x)}{P(\omega)} \) in the disk \(|x - \omega| < d^{-1}\), we find:

\[
\sum_{|\mu_k - \omega| \leq d^{-1}} m_\mu \log \frac{d^{-1}}{|\mu_k - \omega|} = \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| \frac{P(\omega + d^{-1} e^{i\theta})}{P(\omega)} \right| d\theta \leq \log \left( \frac{e L(P)}{|P(\omega)|} \right).
\]

For each \( k \), either \(|\mu_k - \omega| \geq d^{-1} \geq d^{-1} |P(\omega)|\) or, using the estimate above,

\[
(d|\mu_k - \omega|)^{m_\mu} \leq \frac{|P(\omega)|}{e L(P)}.
\]

In the case where all the coefficients of \( P \) are positive, \( P(1) = L(P) \).

In the case where \( P(\omega) = \|P\| \), we use the inequalities (2.2) and (2.3):

\[
L(P) \leq (d + 1)^{\frac{d}{2}} \|P\| = (d + 1)^{\frac{d}{2}} |P(\omega)|.
\]

\[\blacksquare\]

The rest of this subsection deals with self-reciprocal polynomials. As any self-reciprocal polynomial of odd degree is the product of \((x+1)\) and a self-reciprocal polynomial of even degree, we will only consider polynomials of degree \( d = 2n \).

Lemma 3.4. Let \( P(x) = \sum_{j=0}^{2n} a_j x^j \) be a self-reciprocal polynomial in \( \mathbb{C}[x] \) of degree \( 2n \). Let \(|\omega| = 1\), \( P(\omega) \neq 0 \) and \( \rho \in (0, 1) \). Then

\[
\sum_{\mu \in P^{-1}(0), |\mu - \omega| \leq \rho} \log \frac{\rho}{|\mu - \omega|} \leq \log \left( 1 + \frac{\rho (1 + (1 - \rho)^{-\frac{n}{2}}) n}{|P(\omega)|} \sum_{j=0}^{n-1} (n - j) |a_j| \right),
\]

(3.2)

If \( \omega \in \{-1, 1\} \),

\[
\sum_{\mu \in P^{-1}(0), |\mu - \omega| \leq \rho} \log \frac{\rho}{|\mu - \omega|} \leq \log \left( 1 + \frac{\rho^2 (1 - \rho)^{-n} n}{|P(\omega)|} \sum_{j=0}^{n-1} (n - j)^2 |a_j| \right).
\]

(3.3)
Proof. Jensen’s formula applied to the function \( \frac{x^n P(x)}{P(\omega)} \) in the disk \( \{|x - \omega| < \rho\} \) gives
\[
\sum_{\mu \in P^{-1}(0), |\mu - \omega| \leq \rho} \log \frac{\rho}{|\mu - \omega|} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{(\omega + \rho e^{i\theta})^{-n} P(\omega + \rho e^{i\theta})}{P(\omega)} \right| d\theta. \tag{3.4}
\]
Let \( x \) be a non-zero complex number. A straightforward computation shows that
\[
x^{-n} P(x) = \omega^n P(\omega) - \sum_{j=0}^{n-1} a_j \omega^{n-j} (x^{j-n} - \omega^{j-n}) (x^n - \omega^n).
\]
We use
\[
\left| (x^{j-n} - \omega^{j-n}) (x^n - \omega^n) \right| = |x - \omega| |x^{j-n} - \omega^{j-n}| \sum_{k=0}^{n-j-1} x^k \omega^{-1-k}
\]
\[
= |x - \omega| \sum_{k=0}^{n-j-1} (x^{k+j-n} \omega^{-1-k} - x^k \omega^{n-j-1-k})
\]
\[
\leq |x - \omega| (n - j) (1 + \max(|x|^{-n}, |x|^n))
\]
and we readily obtain
\[
\left| \frac{x^{-n} P(x)}{P(\omega)} \right| \leq 1 + \frac{|x - \omega| (1 + \max(|x|^{-n}, |x|^n)) \sum_{j=0}^{n-1} (n - j)|a_j|}{|P(\omega)|}. \tag{3.5}
\]
\((3.2)\) follows from \((3.1)\), \((3.5)\) and the fact that for all \( \theta \in [0, 2\pi] \),
\[
1 - \rho \leq |\omega + \rho e^{i\theta}| \leq 1 + \rho \leq (1 - \rho)^{-1}. \tag{3.6}
\]
In the case where \( \omega \in \{1, -1\} \), we write
\[
\left| (x^{j-n} - \omega^{j-n}) (x^n - \omega^n) \right| = |x|^{-1} |x - \omega|^2 \sum_{k=0}^{n-j-1} x^k \omega^{-1-k} \sum_{k=0}^{n-j-1} x^{-k} \omega^{1-k}
\]
\[
\leq |x - \omega|^2 (n - j)^2 \max(|x|^{-n}, |x|^n)
\]
which gives the inequality
\[
\left| \frac{x^{-n} P(x)}{P(\omega)} \right| \leq 1 + \frac{|x - \omega|^2 \max(|x|^{-n}, |x|^n) \sum_{j=0}^{n-1} (n - j)^2 |a_j|}{|P(\omega)|}. \tag{3.7}
\]
\((3.3)\) follows from \((3.1)\), \((3.6)\) and \((3.7)\). \(\blacksquare\)

**Theorem 3.5.** Let \( P(x) = \sum_{j=0}^{2n} a_j x^j \) be a self-reciprocal polynomial in \( \mathbb{R}[x] \) of degree \( 2n \). Assume \( P(\mu) = 0, |\mu| = 1, P(\omega) \neq 0 \) and \( \delta > 1 \). Then
\[
\frac{1}{|\mu - \omega|} \leq \frac{n}{\log \delta} + 1 + \frac{1 + \delta}{|P(\omega)|} \sum_{j=0}^{n-1} (n - j)|a_j|, \tag{3.8}
\]
Besides, if \(|\mu| \neq 1\) then
\[
\frac{|\mu|}{|\mu - \omega|^2} \leq \left( \frac{n}{\log \delta} + 1 \right)^2 + \frac{\delta}{|P(\omega)|} \sum_{j=0}^{n-1} (n-j)|a_j|. \tag{3.9}
\]

For \(\omega \in \{-1, 1\}\),
\[
\frac{|\mu|}{|\mu - \omega|^2} \leq \left( \frac{n}{\log \delta} + 1 \right)^2 + \frac{\delta}{|P(\omega)|} \sum_{j=0}^{n-1} (n-j)^2|a_j|. \tag{3.10}
\]

Besides, if \(\mu \notin \mathbb{R}\) and \(|\mu| \neq 1\) then
\[
\frac{\rho^2|\mu|}{|\mu - \omega|^2} \leq \prod_{\alpha \in P^{-1}(0)} \frac{\rho}{|\alpha - \omega|} \leq 1 + \frac{\rho(1 + \rho)^{n-1}}{|P(\omega)|} \sum_{j=0}^{n-1} (n-j)|a_j|.
\]

Proof. We apply Lemma 3.4 with \(\rho = \frac{\log \delta}{n+\log \delta}\). Let \(\mu \in \mathbb{R}\) and \(P(\omega) \neq 0\). We may assume that \(|\mu - \omega| \leq \rho\), otherwise the inequalities are trivial. Then, applying (3.2),
\[
\frac{\rho}{|\mu - \omega|} \leq \prod_{\alpha \in P^{-1}(0) \cup \{\omega\}} \frac{\rho}{|\alpha - \omega|} \leq 1 + \frac{\rho(1 + (1-\rho)^n)}{|P(\omega)|} \sum_{j=0}^{n-1} (n-j)|a_j|.
\]

Now we use the inequality \((1-\rho)^{-n} = \left(1 + \frac{\log \delta}{n+\log \delta}\right)^n \leq \delta\) and we obtain (3.3). In the rest of this proof, since \(P(\mu^{-1}) = 0\) and \(\frac{|\mu|}{|\mu - \omega|}\) is invariant by substituting \(\mu^{-1}\) to \(\mu\), we may assume that \(|\mu| \geq 1\).

If \(|\mu| > 1\), then \(\mu^{-1} \neq \mu\) and
\[
|\mu^{-1} - \omega| = \frac{|\mu - \omega|}{|\mu|} \leq |\mu - \omega| \leq \rho.
\]

Thus
\[
\frac{\rho^2|\mu|}{|\mu - \omega|^2} \leq \prod_{\alpha \in P^{-1}(0) \cup \{\omega\}} \frac{\rho}{|\alpha - \omega|} \leq 1 + \frac{\rho(1 + \delta)}{|P(\omega)|} \sum_{j=0}^{n-1} (n-j)|a_j|
\]

and this proves (3.9).

Let us assume now that \(\omega \in \{-1, 1\}\). Since \(P(\mu^{-1}) = 0\) and \(|\mu^{-1} - \omega| \leq |\mu - \omega| \leq \rho\), applying (3.3) we obtain
\[
\frac{\rho^2|\mu|}{|\mu - \omega|^2} \leq \prod_{\alpha \in P^{-1}(0) \cup \{\omega\}} \frac{\rho}{|\alpha - \omega|} \leq 1 + \frac{\rho^2 \delta}{|P(\omega)|} \sum_{j=0}^{n-1} (n-j)^2|a_j|,
\]

which proves (3.10).

Finally, to prove (3.11), we note that if \(|\mu| > 1\) and \(\mu \notin \mathbb{R}\), then \(\mu, \bar{\mu}, \mu^{-1}\) and \(\bar{\mu}^{-1}\) are distinct zeros of \(P\) and all of them are located in the disk \(|x - \omega| \leq \rho\).

Thus
\[
\frac{\rho^4|\mu|}{|\mu - \omega|^2} \leq \prod_{\alpha \in P^{-1}(0) \cup \{\omega\}} \frac{\rho}{|\alpha - \omega|} \leq 1 + \frac{\rho^4 \delta}{|P(\omega)|} \sum_{j=0}^{n-1} (n-j)^2|a_j|.
\]
In the remainder of this section, we will use the following notation: for a polynomial \( P \) of degree \( 2n \), we will write \( f(P) \gtrsim n \to \infty g(n) \) if there exists an expression \( h(n) \) only depending on \( n \) such that \( f(P) \geq h(n) \) and \( \frac{h(n)}{g(n)} \to n \to \infty 1 \).

**Corollary 3.6.** Let \( P(x) = \sum_{j=0}^{2n} a_j x^j \) be a self-reciprocal polynomial in \( \mathbb{R}[x] \) of degree \( 2n \). Assume \( H(P) \leq H \). Let \( P(\mu) = 0 \), \( |\omega| = 1 \), \( |P(\omega)| \geq 1 \). Then

\[
|\mu - \omega| \gtrsim n \to \infty H^{-n^{-2}}.
\]

Besides, if \( |\mu| \neq 1 \) then

\[
|\mu|^{-\frac{1}{2}} |\mu - \omega| \gtrsim n \to \infty (2c^{-1}H^{-1})^{\frac{3}{2}} n^{-\frac{1}{2}},
\]

where \( c > 1 \) is such that \( c + 1 = c \log c \) (\( c = 3.594... \)).

For \( \omega \in \{-1, 1\} \):

\[
|\mu|^{-\frac{1}{2}} |\mu - \omega| \gtrsim n \to \infty (3H^{-1})^{\frac{3}{2}} n^{-\frac{1}{2}}.
\]

If \( |\mu| \neq 1 \) and \( \mu \notin \mathbb{R} \), then

\[
|\mu|^{-\frac{1}{2}} |\mu - \omega| \gtrsim n \to \infty (2c^{-1})^{\frac{3}{2}} (3H^{-1})^{\frac{3}{2}} n^{-\frac{1}{2}}.
\]

**Proof.** We observe that

\[
\sum_{j=0}^{n-1} (n - j)|a_j| \leq \frac{n(n + 1)}{2} H,
\]

\[
\sum_{j=0}^{n-1} (n - j)^2|a_j| \leq \frac{n(n + 1)(2n + 1)}{6} H.
\]

Then we apply Theorem 3.5 with different values of \( \delta \) in order to get the best upper bound: \( \delta = 1 + \frac{1}{\sqrt{n}} \) in (3.8), \( c \) in (3.9), \( \delta = 1 + \frac{1}{n^{\frac{1}{4}}} \) in (3.10) and finally \( \delta = e^2 \) in (3.11).

**Corollary 3.7.** Let \( P(x) = \sum_{j=0}^{2n} a_j x^j \) be a self-reciprocal polynomial in \( \mathbb{R}[x] \) of degree \( 2n \). Assume \( a_j \geq 0 \) (\( j = 0, \ldots, n \)). Let \( P(\mu) = 0 \). Then

\[
|\mu|^{-\frac{1}{2}} |\mu - 1| \gtrsim n \to \infty \frac{A}{n}.
\]

Besides, if \( |\mu| \neq 1 \) and \( \mu \notin \mathbb{R} \), then

\[
|\mu|^{-\frac{1}{2}} |\mu - 1| \gtrsim n \to \infty \frac{B}{n},
\]

where \( A \) and \( B \) are given by

\[
A^2 = 4 (a(2 + \log a))^{-1}, a > 1, a(\log a)^3 = 4 \quad (A = 0.655...).
\]

\[
B^4 = 8 \log b^2 (b(\log b + 2))^{-1}, b > 1, b(\log b)^2(\log b - 2) = 8 \quad (B = 0.984...).
\]
Proof. \[ n^{-1} \sum_{j=0}^{n-1} (n-j)^{|a_j|} \leq \frac{n^2}{2} L(P) \]

and \( L(P) = P(1) = \|P\| \). Then we apply Theorem 3.5 with \( \delta = a \) in 3.10 and with \( \delta = b \) in 3.11. \( \Box \)

**Corollary 3.8.** Let \( P(x) = \sum_{j=0}^{2n} a_j x^j \) be a self-reciprocal polynomial in \( \mathbb{R}[x] \) of degree \( 2n \). Let \( P(\mu) = 0, |\omega| = 1, \|P\| = |P(\omega)| \). Then

\[ |\mu - \omega| \gtrsim n^{-\frac{3}{2} - \frac{1}{2n} - \frac{3}{2} - \frac{1}{n}}. \]

Besides, if \( |\mu| \neq 1 \), then

\[ |\mu|^{-\frac{3}{4}} |\mu - \omega| \gtrsim n^{-\frac{3}{4} - \frac{1}{4} - \frac{3}{2} - \frac{1}{n}}. \]

where \( c > 1 \) is such that \( c + 1 = c \log c \) (\( c = 3.594... \)). For \( \omega \in \{-1, 1\} \):

\[ |\mu|^{-\frac{7}{8}} |\mu - \omega| \gtrsim n^{-\frac{7}{8} - \frac{3}{2} - \frac{1}{n}}. \]

If \( |\mu| \neq 1 \) and \( \mu \notin \mathbb{R} \), then

\[ |\mu|^{-\frac{3}{2}} |\mu - \omega| \gtrsim n^{-\frac{3}{2} - \frac{1}{8} - \frac{3}{2} - \frac{1}{n}}. \]

**Proof.** By Cauchy-Schwarz inequality,

\[ \sum_{j=0}^{n-1} (n-j)|a_j| \leq \frac{1}{\sqrt{12}} (n(n+1)(2n+1))^{\frac{1}{2}} L_2(P), \]

\[ \sum_{j=0}^{n-1} (n-j)^2|a_j| \leq \frac{1}{\sqrt{60}} (n(n+1)(2n+1)(3n^2 + 3n - 1))^{\frac{1}{2}} L_2(P) \]

and \( L_2(P) \leq \|P\| = |P(\omega)| \). Then we apply Theorem 3.5 with \( \delta = 1 + \frac{1}{n^4} \) in 3.8, \( c > 1 \) in 3.9, \( \delta = 1 + \frac{1}{n^8} \) in 3.10 and finally \( \delta = e^2 \) in 3.11. \( \Box \)

4 A lower bound for the Mahler measure of complex polynomials

A result due to Schinzel [20] states that any monic polynomial \( P \in \mathbb{Z}[x] \) of degree \( d \) whose all zeros are real (resp. positive) and such that \( P(-1)P(1) \neq 0 \) and \( |P(0)| = 1 \) satisfies

\[ M(P) \geq \left( \frac{1 + \sqrt{5}}{2} \right)^d \] (resp. \( M(P) \geq \left( \frac{1 + \sqrt{5}}{2} \right)^d \)).

13
V. Flammang [7] generalised Schinzel’s theorem by proving that any monic $P \in \mathbb{R}[x]$ of degree $d$ whose all zeros are real (resp. positive) and such that $P(0) \neq 0$, $|P(1)| \geq 1$, $|P(-1)| \geq 1$ satisfies

$$M(P) \geq \left(1 + \frac{4|P(0)|\frac{1}{d} + 1}{2\frac{1}{d}}\right)^{\frac{d}{2}}$$

(resp. $M(P) \geq \left(1 + \frac{4|P(0)|\frac{1}{d} + 1}{2\frac{1}{d}}\right)^{\frac{d}{2}}$).

J. Garza [9] showed that for monic polynomials $P \in \mathbb{Z}[x]$ satisfying $P(0)P(-1)P(1) \neq 0$ and having $m \geq 1$ real zeros, the following holds:

$$M(P) \geq \left(1 + \frac{4a + 1}{2a}\right)^{m}.$$ 

Our next theorem generalises the three above results to polynomials in $\mathbb{C}[x]$. It also strengthens the dependence on $P(0)$, $P(1)$ and $P(-1)$.

**Theorem 4.1.** Let $P \in \mathbb{C}[x]$ be a monic polynomial of degree $d \geq 2$. Assume that $P$ has $m \geq 1$ real zeros and that $P(0)P(1)P(-1) \neq 0$, then

$$M(P) \geq \left(|P(1)P(-1)|\frac{d}{2} + \frac{4|P(0)|\frac{1}{d} + |P(1)P(-1)|\frac{d}{2}}{2\frac{1}{d}}\right)^{\frac{1}{m}}$$

(4.1)

with equality if and only if $P$ is of the form

$$P = (z - i)^{d_1}(z + i)^{d_2}(z - a)^{d_3}(z + a)^{d_4}(z - a^{-1})^{d_5}(z + a^{-1})^{d_6}, \ a > 1. \ (4.2)$$

If $P$ has $n \geq 1$ positive zeros and $P(0)P(1) \neq 0$, then

$$M(P) \geq \left(|P(1)|\frac{d}{2} + \frac{4|P(0)|\frac{1}{d} + |P(1)|\frac{d}{2}}{2\frac{1}{d}}\right)^{n}$$

(4.3)

with equality if and only if $P$ is of the form

$$P = (z - a)^{d_1}(z - a^{-1})^{d_2}, \ a > 1.$$

**Proof.** Let $Q = P(0)^{-1}PP^*$ defined by (2.4). Let $\{\beta_k\}_{k=1}^{m}$ be the real zeros of $Q$ outside $[-1, 1]$ and $\{\delta_k\}_{k=1}^{f}$ its non-real zeros such that either $|\delta_k| > 1$ or $|\delta_k| = 1$, $\exists \delta_k > 0$. Then $d = m + f$ and

$$Q(1) = \prod_{k=1}^{f} (1 - \delta_k)(1 - \delta_k^{-1}) \prod_{k=1}^{m} (1 - \beta_k)(1 - \beta_k^{-1})$$

$$Q(-1) = \prod_{k=1}^{f} (1 + \delta_k)(1 + \delta_k^{-1}) \prod_{k=1}^{m} (1 + \beta_k)(1 + \beta_k^{-1})$$

$$|Q(1)Q(-1)|^{\frac{1}{2}} = M(Q) \prod_{k=1}^{f} |1 - \delta_k^{2}| \prod_{k=1}^{m} (1 - \beta_k^{-2})$$

14
We find which gives

\[ \prod_{k=1}^{m} (1 - \beta_k^{-2}) \leq \left( 1 - \left( \prod_{k=1}^{m} \beta_k^{-2} \right)^\frac{1}{m} \right)^m \leq \left( 1 - M(Q)^{-\frac{2}{m}} \right)^m, \]

which follows from the concavity of the function \( t \mapsto \log(1 - e^{-t}), t > 0 \). We obtain

\[ |Q(1)Q(-1)|^\frac{1}{2} \leq M(Q)2^{d-m} \left( 1 - M(Q)^{-\frac{2}{m}} \right)^m. \] (4.4)

We deduce that

\[ M(Q)^{-\frac{2}{m}} - 2^{1 - \frac{2}{m}} |Q(1)Q(-1)|^{\frac{1}{2m}} M(Q)^{-\frac{2}{m}} - 1 \geq 0 \]

and solving the inequality \( t^2 - 2^{1 - \frac{2}{m}} |Q(1)Q(-1)|^{\frac{1}{2m}} t - 1 \geq 0 \), we find

\[ 2^{\frac{2}{m}} M(Q)^{\frac{1}{m}} \geq |Q(1)Q(-1)|^{\frac{1}{2m}} + \left( 4^{\frac{1}{m}} + |Q(1)Q(-1)|^{\frac{1}{2m}} \right)^{\frac{1}{2}}. \]

We conclude the proof of (4.1) by using (2.5).

Equality in (4.1) is equivalent to equality in (4.4). It is attained iff

\[ \prod_{k=1}^{f} \frac{2}{|1 - \delta_k^{-2}|} = \left( 1 - M(Q)^{-\frac{2}{m}} \right)^{-m} \prod_{k=1}^{m} (1 - \beta_k^{-2}) = 1, \]

which gives

\[ \delta_1^2 = \ldots = \delta_f^2 = -1, \quad \beta_1^2 = \ldots = \beta_m^2 = a^2 \quad \text{with} \quad a > 1. \]

For the proof of (4.3), let \( \{\beta_k\}_{k=1}^{n} \) be the positive zeros of \( Q \) and \( \{\beta_k\}_{k=n+1}^{m} \) be the negative zeros of \( Q \). Using once again the concavity of the function \( t \mapsto \log(1 - e^{-t}), t > 0 \), we have

\[ |Q(1)|^{\frac{1}{2}} = M(Q)^{\frac{1}{2}} \prod_{k=1}^{f} |1 - \delta_k^{-1}| \prod_{k=1}^{n} (1 - \beta_k^{-1}) \prod_{k=n+1}^{m} (1 - \beta_k^{-1}) \]

\[ \leq M(Q)^{\frac{1}{2}} 2^{d-n} \prod_{k=1}^{n} (1 - \beta_k^{-1}) \] (4.5)

\[ \leq M(Q)^{\frac{1}{2}} 2^{d-n} \left( 1 - M(Q)^{-\frac{1}{m}} \right)^n. \]

We find

\[ M(Q)^{\frac{1}{2}} - 2^{1 - \frac{2}{m}} |Q(1)|^{\frac{1}{2m}} M(Q)^{\frac{1}{2}} - 1 \geq 0. \]

We proceed as in the end of the proof of (4.1) by substituting \( M(Q)^{\frac{1}{2m}} \) to \( M(Q)^{\frac{1}{2}} \) and \( |Q(1)|^{\frac{1}{2m}} \) to \( |Q(1)Q(-1)|^{\frac{1}{2m}} \).

Equality in (4.3) is equivalent to equalities in (4.5). It is attained iff

\[ \prod_{k=1}^{f} \frac{2}{|1 - \delta_k^{-1}|} \prod_{k=n+1}^{m} \frac{2}{|1 - \beta_k^{-1}|} = \left( 1 - M(Q)^{-\frac{1}{m}} \right)^{-n} \prod_{k=1}^{m} (1 - \beta_k^{-1}) = 1, \]

which gives

\[ f = 0, \quad n = m = d, \quad \beta_1 = \ldots = \beta_m = a \quad \text{with} \quad a > 1. \]

\[ \blacksquare \]
5 The number of real zeros

5.1 Bounds involving the Mahler measure

A. Dubickas [5] showed that for polynomials \( P \in \mathbb{Z}[x] \) of degree \( d \) such that \( P(0)P(-1)P(1) \neq 0 \), there is a constant \( D_0 \) such that for all \( d \geq D_0 \),

\[
|P^{-1}(0) \cap \mathbb{R}| \leq 1.085 \sqrt{d \log d \log M(P)}.
\]  

(5.1)

A bound of this size does not apply to polynomials in \( \mathbb{R}[x] \). For example, the polynomials of the form \( P = (x - d)^d \) satisfy \( M(P) = d \),

\[
\sqrt{d \log d \log M(P)} = \sqrt{d \log d} \quad \text{while} \quad |P^{-1}(0) \cap \mathbb{R}| = d = \frac{d \log M(P)}{\log d}.
\]

Our next theorem shows that the bound \( d \log 2 + \log |P(0)| - \log |P(1)P(-1)| \) holds, up to a small multiplicative constant, for polynomials in \( P \in \mathbb{C}[x] \) such that \( |P(1)P(-1)| \geq 1 \) and \( |P(0)| \geq 1 \).

**Theorem 5.1.** Let \( P \in \mathbb{C}[x] \) be a monic polynomial of degree \( d \geq 2 \). Assume \( P(0)P(1)P(-1) \neq 0 \). Then

\[
|P^{-1}(0) \cap \mathbb{R}| \leq \max \left( \frac{d \log 2 + \log |P(0)| - \log |P(1)P(-1)|}{-\log \left( \sinh \left( \log \left( \frac{\log d}{d} \right) \right) \right)}, \frac{d \log (M(P)^2|P(0)|^{-1})}{\log d} \right).
\]  

(5.2)

We note that \(- \log \left( \sinh \left( \frac{\log d}{d} \right) \right) \sim \log d\).

**Remark 5.2.** The equality in (5.2) is attained by polynomials of the form (4.2) with \( a = d^{\frac{1}{2}} \). If we denote by \( m = |P^{-1}(0) \cap \mathbb{R}| \), they satisfy

\[
M(P)^2|P(0)|^{-1} = d^m, \quad |P(0)|^{-1}|P(1)P(-1)| = 2^d \left( \sinh \left( \frac{\log d}{d} \right) \right)^m.
\]

Thus

\[
\frac{d \log 2 + \log |P(0)| - \log |P(1)P(-1)|}{-\log \left( \sinh \left( \frac{\log d}{d} \right) \right)} = \frac{d \log (M(P)^2|P(0)|^{-1})}{\log d} = m.
\]

**Proof of Theorem 5.1**

Let us repeat the beginning of the proof of Theorem 4.1 with the same notations. We assume that \( m \geq \frac{d \log M(Q)}{\log d} \) and we use inequality (4.4):

\[
|Q(1)Q(-1)|^\frac{1}{2} \leq M(Q)2^{d-m} \left( 1 - M(Q)^{-\frac{1}{2}} \right)^m
\]

\[
= 2^d \left( \sinh \left( \frac{\log M(Q)}{m} \right) \right)^m
\]

\[
\leq 2^d \left( \sinh \left( \frac{\log d}{d} \right) \right)^m
\]
We deduce that
\[
m \leq d \log 2 - \frac{1}{2} \log |Q(1)Q(-1)| - \log \left( \sinh \left( \frac{\log d}{2} \right) \right).
\]
The result follows by using (2.5). ■

5.2 Bounds involving the Mahler measure and the length

A theorem announced by E. Schmidt in 1932 states that for polynomials of the form
\[
P(x) = \sum_{j=0}^{d} a_j x^j, \quad a_0 a_d \neq 0, \quad a_j \in \mathbb{C},
\]
such that
\[
|P^{-1}(0) \cap \mathbb{R}|^2 \leq C d \log \left( \frac{L(P)}{\sqrt{|a_0 a_d|}} \right), \quad (5.3)
\]
I. Schur [21] gave an elementary proof of (5.3) and showed that the best possible constant \( C \) is 4.

P. Borwein, T. Erdélyi and G. Kós [1] showed that whenever \( |a_0| = |a_d| = 1, |a_j| \leq 1 \), there is an absolute constant \( C > 0 \) such that
\[
|P^{-1}(0) \cap \mathbb{R}| \leq C \sqrt{d}.
\]

Our next result gives an upper bound depending both on the Mahler measure and on the length of the polynomial.

**Theorem 5.3.** Let \( P \in \mathbb{C}[x] \) be a monic polynomial of degree \( d \). Assume \( P(0)P(1)P(-1) \neq 0 \). Then
\[
|P^{-1}(0) \cap \mathbb{R}| \leq 2 \sqrt{c} \left( \frac{d \log M(P)^2}{|P(0)|} \right)^{\frac{1}{2}} + (c + 1) \log \left( \frac{M(P)^2}{|P(0)|} \right),
\]
\[
+ (\log c)^{-1} \log(1 + d^2) + (\log c)^{-1} \log \left( \frac{L(P)^2}{2|P(1)|^2} + \frac{L(P)^2}{2|P(-1)|^2} \right), \quad (5.4)
\]
where \( c > 1 \) is such that \( 1 + c = e (c = 3.594...) \).

**Proof.** Once again we use \( Q = P(0)^{-1}PP^* \) and we may assume that \( M(Q) > 1 \) otherwise there are no real zeros at all. We apply Lemma 3.4 to \( Q \) with \( \rho = \frac{\gamma}{\gamma + 1} \),
\[
\gamma = \sqrt{(c+1) \log c \log M(Q)}
\]
and first with \( \omega = 1 \):
\[
2 \log c |Q^{-1}(0) \cap \{ 1 < x \leq 1 + \frac{\rho}{c} \}| \leq \sum_{\substack{Q(\mu) = 0 \\mu \in \mathbb{C}^{\gamma+\frac{1}{2}}}} \left( \log \frac{\rho}{|\mu - 1|} + \log \frac{\rho}{|\mu^{-1} - 1|} \right)
\]
\[
\leq \sum_{\substack{Q(\mu) = 0 \\mu^{-1} \leq \rho}} \log \frac{\rho}{|\mu - 1|}
\]
\[
\leq \log \left( 1 + \frac{\rho^2 (1 - \rho)^{-d} d^2}{|Q(1)|} \sum_{k=0}^{d-1} |A_k| \right). \quad (5.5)
\]

17
Repeating the argument with \( \omega = -1 \),
\[
2 \log c |Q^{-1}(0) \cap \{ -1 - \frac{\rho}{c} \leq x < -1 \} | \leq \log \left( 1 + \frac{\rho^2 (1 - \rho)^{-d} d^2}{|Q(1)|} \sum_{k=0}^{d-1} |A_k| \right).
\]

Adding (5.5) and (5.6) and using the inequality \((1 + u)(1 + v) \leq (1 + \frac{u + v}{2})^2\),
\[
\log c |Q^{-1}(0) \cap \{ 1 < |x| \leq 1 + \frac{\rho}{c} \} | \leq \log \left( 1 + d^2 \rho^2 (1 - \rho)^{-d} \left( \frac{1}{2|Q(1)|} + \frac{1}{2|Q(-1)|} \right) \sum_{k=0}^{d-1} |A_k| \right) 
\leq \log \left( \left( \frac{1}{2|Q(1)|} + \frac{1}{2|Q(-1)|} \right) \sum_{k=0}^{d} |A_k| \right) + \log(1 + d^2 \rho^2 - d \log(1 - \rho))
\]

Now we use the inequality \(- \log(1 - \rho) \leq \frac{\rho}{1 - \rho} = \gamma,\) \((2.5)\) and \((2.6)\) to obtain
\[
\log c |Q^{-1}(0) \cap \{ 1 < |x| \leq 1 + \frac{\rho}{c} \} | \leq \log \frac{M(Q)}{\log(1 + \frac{\rho}{c})} \leq \log M(Q)(c + 1) \rho^{-1} = \log M(Q)(c + 1)(1 + \gamma^{-1}) 
\leq (c + 1) \log M(Q) + \sqrt{c} \log M(Q)d.
\]

Finally,
\[
|P^{-1}(0) \cap \mathbb{R}| = |Q^{-1}(0) \cap \{ |x| > 1 \} | 
\leq (\log c)^{-1} \log \left( \frac{L(P)^2}{2|P(1)|^2} + \frac{L(P)^2}{2|P(-1)|^2} \right) + 
+ (\log c)^{-1} \log(1 + d^2) + (c + 1) \log M(Q) + 2 \sqrt{c} \log M(Q)d.
\]

We conclude the proof of (5.4) by applying (2.5). \[\blacksquare\]

**Corollary 5.4.** Let \( P \in \mathbb{Z}[x] \) be of degree \( d \). Assume \( P(0)P(1)P(-1) \neq 0 \).
Then
\[
|P^{-1}(0) \cap \mathbb{R}| \leq 2 \sqrt{2cd} \log M(P) + 2(c + 1) \log M(P)
+ (\log c)^{-1} \log(1 + d^2) + 2(\log c)^{-1} \log L(P),
\]
where \( c > 1 \) is such that \( c \log c = 1 + c \) \( (c = 3.594...) \).
If moreover, \( \log L(P) \leq c_1 \sqrt{d} \log M(P) \) for some constant \( c_1 > 0 \), then for all \( \varepsilon > 0 \), there exists \( D_0(\varepsilon) \) such that for \( d \geq D_0(\varepsilon) \),
\[
|P^{-1}(0) \cap \mathbb{R}| \leq (c_2 + \varepsilon) \sqrt{d} \log M(P),
\]
where \( c_2 = 2 \left( \sqrt{2c} + c + 1 + (\log c)^{-1}c_1 \right) \).

**Remark 5.5.** Inequality (5.7) improves the bound (5.1) given by Dubickas in the particular case where \( L(P) \leq c_1 \sqrt{d} \log M(P) \).
Proof. Let \( P = a_dx^d + \ldots + a_0 \) with \( a_0, \ldots, a_d \in \mathbb{Z}, a_d \neq 0 \) and \( P(0)P(1)P(-1) \neq 0 \). We apply Theorem 5.3 to the polynomial \( \tilde{P} = a_d^{-1}P \). Inequality (5.7) follows after observing that

\[
\frac{M(\tilde{P})^2}{|\tilde{P}(0)|} = \frac{M(P)^2}{a_d|P(0)|} \leq M(P)^2
\]

\[
\frac{L(\tilde{P})^2}{2|\tilde{P}(1)|^2} + \frac{L(\tilde{P})^2}{2|\tilde{P}(-1)|^2} = \frac{L(P)^2}{2|P(1)|^2} + \frac{L(P)^2}{2|P(-1)|^2} \leq L(P)^2.
\]

For the proof of (5.8), we may assume that \( M(P) > 1 \), otherwise by Kronecker’s theorem \( P \) has no real zeros. Then, we know from [4] that for some constant \( \sigma > 0 \),

\[
\log M(P) \geq \sigma \left( \frac{\log \log d}{\log d} \right)^3.
\]  (5.9)

Thus for all \( \varepsilon > 0 \), there exists \( D_0(\varepsilon) \) such that for \( d \geq D_0(\varepsilon) \),

\[
\left( \log c \right)^{-1} \log(1 + d^2) \leq \varepsilon \sqrt{\log M(P)}.
\]

On the other hand, we may also assume that \( \log M(p) \leq d \) because otherwise \( \sqrt{d \log M(P)} \geq d \) and (5.8) is again trivially satisfied. Since \( \log M(p) \leq \sqrt{d \log M(p)} \), (5.8) follows immediately from (5.7) and the assumption that \( \log L(P) \leq c_1 \sqrt{d \log M(P)} \).  

6 The number of zeros in \( \{ |x - 1| < 1 \} \).

In [5], A. Dubickas used a Vandermonde determinant which, in its simplified form is defined by

\[
d(x) = \det(x^{(j-1)l})_{1 \leq j, l \leq N} = \prod_{1 \leq u < v \leq N} (x^u - x^v) \quad (N \in \mathbb{N}).
\]

Following the lines of his work, we put

\[
R(x) = |d(x)| \prod_{1 \leq u < v \leq N} |x|^v = \prod_{j=1}^{N-1} |x^j - 1|^{N-j}.
\]

In [5], the author applies Hadamard’s inequality to obtain:

\[
R(x) \leq \max(1, |x|)^\frac{(N-1)N(N+1)}{6} N^N. \quad (6.1)
\]

Lemma 6.1. Assume \( P \in \mathbb{Z}[x] \) is monic, has degree \( d \) and does not vanish at the roots of unity. Then for all integer \( N \geq 2 \),

\[
|P(1)| \leq N^{\frac{d+1}{d+2}} M(P)^{\frac{d}{4}}.
\]

Remark 6.2.

- By Kronecker’s theorem, either \( P = x^d \) or \( M(P) > 1 \).
- By (2.1) and (2.3), we already had \( |P(1)| \leq 2^d M(P) \). We retrieve this inequality if we put \( N = 2 \) in Lemma 6.1.
Proof. Let $\mu_1, \ldots, \mu_d$ be the zeros of $P$. Then

$$R(\mu_k) = |\mu_k - 1| \left( \frac{(N-1)N}{2} \prod_{j=1}^{N-1} \sum_{l=0}^{j-1} |\mu_k^l| \right)^{N-j}$$

and

$$\prod_{k=1}^d R(\mu_k) = |P(1)| \left( \frac{(N-1)N}{2} \prod_{k=1}^d \prod_{j=1}^{N-1} \sum_{l=0}^{j-1} |\mu_k^l| \right)^{N-j}.$$

Observing that

$$\prod_{k=1}^d \prod_{j=1}^{N-1} \left( \sum_{l=0}^{j-1} |\mu_k^l| \right)^{N-j} \in \mathbb{Z}^*,$$

and using the inequality (6.1), we deduce that

$$|P(1)| \left( \frac{(N-1)N}{2} \prod_{k=1}^d R(\mu_k) \right)^{N-j} \leq \prod_{k=1}^d R(\mu_k) \leq N^{2N} M(P)^{\frac{(N-1)(N+1)}{2}}.$$

The result follows easily.

We will need the following result due to Zhang [25] and for which Zagier gave an elementary proof in [24]:

**Theorem 6.3.** [Zhang, Zagier] Let $\omega$ be a primitive 6th root of unity. Let $P \in \mathbb{Z}[x]$ be a polynomial of degree $d$ such that $P(0)P(1)P(\omega) \neq 0$. Denote $P_*(x) = P(1-x)$. Then

$$M(P)M(P_*) \geq \left( \frac{1 + \sqrt{5}}{2} \right)^d.$$

**Theorem 6.4.** Let $P \in \mathbb{Z}[x]$ be monic, irreducible, of degree $d$ and satisfying $0 < \log M(P) \leq \phi(d)$ where $\phi(d) = o\left( \frac{d}{\log d} \right)$. Then for all $\varepsilon > 0$, there exists $D_0(\varepsilon)$ such that for $d \geq D_0(\varepsilon)$,

$$|P^{-1}(0) \cap \{|x-1| < 1\}| \geq \left( \frac{2}{\pi} \log \frac{1 + \sqrt{5}}{2} - \varepsilon \right) \sqrt{\frac{d}{\log d \log M(P)}}.$$

This leads to the lower bound

$$\log M(P) \geq \left( \frac{2}{\pi} \log \frac{1 + \sqrt{5}}{2} - \varepsilon \right)^2 \frac{d}{K(P)^2 \log d}, \quad (6.2)$$

where $K(P) = \min\left( |P^{-1}(0) \cap \{|x-1| < 1\}|, |P^{-1}(0) \cap \{|x+1| < 1\}| \right)$. 20
Proof. We apply Lemma 6.1 with \( N = \left\lfloor \frac{d \log d}{\log M(P)} \right\rfloor + 2 \). We obtain

\[
\frac{N + 1}{3} \log M(P) \leq \log M(P) + \frac{1}{3} \sqrt{d \log d \log M(P)},
\]

\[
d \frac{\log N}{N - 1} \leq \sqrt{\frac{d \log M(P)}{\log d}} \log \left( \frac{2}{\sqrt{d \log d \log M(P)}} \right)
\]

\[
= \sqrt{d \log d \log M(P)} \left( \frac{1}{2} + \frac{\log (4 \log d)}{2 \log d} \right) - \sqrt{\frac{d \log M(P) \log \log M(P)}{\log d}}
\]

\[
\leq \sqrt{d \log d \log M(P)} \left( \frac{1}{2} + \frac{\log (4 \log d)}{2 \log d} \right) + e^{-1} \sqrt{\frac{d \log d}{\log d}}.
\]

\[
|P(1)| \leq M(P) \exp \left( \sqrt{d \log d \log M(P)} \left( \frac{5}{6} + \frac{\log (4 \log d)}{2 \log d} \right) + \sqrt{\frac{d \log d}{\log d}} \right).
\]

Denote by \( \mu_1, \ldots, \mu_d \) the zeros of \( P \). By Theorem 6.3, we have

\[
\prod_{|1 - \mu_k| \geq 1} |1 - \mu_k| \geq \left( \frac{1 + \sqrt{5}}{2} \right)^d M(P)^{-1}.
\]

We immediately deduce that

\[
\prod_{|1 - \mu_k| < 1} |1 - \mu_k| \leq |P(1)| \left( \frac{1 + \sqrt{5}}{2} \right)^{-d} M(P)
\]

\[
\leq M(P)^2 \exp \left( \sqrt{d \log d \log M(P)} \left( \frac{5}{6} + \frac{\log (4 \log d)}{2 \log d} \right) + \sqrt{\frac{d \log d}{\log d}} - \frac{d \log \left( \frac{1 + \sqrt{5}}{2} \right)}{2} \right) \tag{6.3}
\]

Denote by \( J = |P^{-1}(0) \cap \{|x - 1| < 1\}| \) and let \( \varepsilon > 0 \). Using the lower bound given by A. Dubickas (see [5] or [81]), there exists \( D_0(\varepsilon) \) such that for all \( d \geq D_0(\varepsilon) \)

\[
\prod_{|1 - \mu_k| \leq 1} |1 - \mu_k| \geq \exp \left( -J \left( \frac{\pi}{4} + \varepsilon' \right) \sqrt{d \log d \log M(P)} \right) \tag{6.4}
\]

where \( \varepsilon' > 0 \) is chosen small enough with regards to \( \varepsilon \). Using (6.3), (6.4) and \( \log M(P) \leq \phi(d) = o\left( \frac{d}{\log d} \right) \), we find for \( d \geq D_0(\varepsilon) \)

\[
J \geq \left( \frac{2}{\pi} \log \frac{1 + \sqrt{5}}{2} - \varepsilon \right) \sqrt{\frac{d}{\log d \log M(P)}}.
\]

As the same apply to \( P(-x) \), we have the same estimate for \( |P^{-1}(0) \cap \{|x + 1| < 1\}| \) and we obtain (6.2). \( \blacksquare \)
7 Acknowledgments

We sincerely thank the anonymous reviewer, whose comments and suggestions helped us to improve and clarify our manuscript.

References

[1] P. Borwein, T. Erdélyi, G. Kós, Littlewood-type problems on [0,1]. Proc. London Math. Soc. (3) 79 (1999), no. 1, 22-46.
[2] D. W. Boyd, Reciprocal polynomials having small measure, Math. Comp. 35 (1980), no. 152, 1361–1377.
[3] D. W. Boyd, Reciprocal polynomials having small measure, II, Math. Comp. 53 (1989), no. 187, 355–357, S1–S5.
[4] E. Dobrowolski, On a question of Lehmer and the number of irreducible factors of a polynomial, Acta Arithmetica 34.4 (1979), 391-401.
[5] A. Dubickas, On algebraic numbers of small measure. Liet. Mat. Rink. 35 (1995), no. 4, 421–431; reprinted in Lithuanian Math. J. 35 (1995), no. 4, 333–342 (1996)
[6] El Otmani, S.; Maul, A.; Rhin, G.; Sac-Épée, J.-M., Finding new small degree polynomials with small Mahler measure by genetic algorithms, Rocky Mountain J. Math. 47 (2017), no. 8, 2619-2626
[7] V. Flammang, Inégalités sur la mesure de Mahler d’un polynôme. (French) [Inequalities for the Mahler measure of a polynomial] J. Théor. Nombres Bordeaux 9 (1997), no. 1, 69-74.
[8] V. Flammang, G. Rhin and J.-M. Sac-Épée, Integer transfinite diameter and polynomials with small Mahler measure, Math. Comp. 75 (2006), no. 255, 1527–1540.
[9] J. Garza, On the height of algebraic numbers with real conjugates, Acta Arith. 128 (2007), no. 4, 385-389.
[10] J. L. W. V. Jensen, Sur un nouvel et important théorème de la théorie des fonctions, Acta Math. 22 (1899), 359–364.
[11] L. Kronecker, Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten, J. Reine Angew. Math. 53 (1857), 173-175.
[12] D. H. Lehmer, Factorization of certain cyclotomic functions, Ann. of Math. (2) 34 (1933), no. 3, 461–479.
[13] K. Mahler, On some inequalities for polynomials in several variables, J. London Math. Soc. 37 (1962), 341-344.
[14] M. Mignotte, M. Waldschmidt, On algebraic numbers of small height: linear forms in one logarithm, J. Number Theory 4 (1994), 43-62.
[15] M. J. Mossinghoff. Polynomials with Small Mahler Measure. Math. Comp. 67:224 (1998), 1697–1705, S11–S14.
[16] M. J. Mossinghoff, C. G. Pinner and J. D. Vaaler. Perturbing Polynomials with All Their Roots on the Unit Circle. Math. Comp. 67 (1998), no. 224, 1707-1726.

[17] M. J. Mossinghoff. Lehmer’s Problem. Available online (http://www.cecm.sfu.ca/~mjm/Lehmer), 2007.

[18] M. J. Mossinghoff, Georges Rhin and Qiang Wu, Minimal Mahler Measures, Experimental Mathematics, (2008), 17:4, 451–458

[19] G. Rhin and J.-M. Sac-Épée. New Methods Providing High Degree Polynomials with Small Mahler Measure. Experiment. Math. 12:4 (2003), 457–461.

[20] A. Schinzel, On the product of the conjugates outside the unit circle of an algebraic integer, Acta Arith. 24 (1973), 385-399.

[21] I. Schur, Untersuchungen über algebraische Gleichungen, Akad. Wiss, Phys-Math. K1. (1933), 403-428.

[22] C. J. Smyth. On the Product of the Conjugates outside the Unit Circle of an Algebraic Integer. Bull. London Math. Soc. 3 (1971), 169–175.

[23] C. J. Smyth, The Mahler Measure of Algebraic Numbers: A Survey. In Number Theory and Polynomials, edited by J. McKee and C. Smyth, pp. 322–349, London Math. Society Lecture Note Series 352. Cambridge, UK: Cambridge University Press, 2008.

[24] D. Zagier, Algebraic numbers close to both 0 and 1. Math. Comp. 61 (1993), 485-491

[25] S. Zhang, Positive line bundles on arithmetic surfaces, Annals of Math. A36 (1992), 569-587