Effective Theories for Quark Flavour Physics

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Abstract

The purpose of these lectures is to provide the reader with an idea of how we can probe New Physics with quark flavour observables using effective theory techniques. After giving a concise review of the quark flavour structure of the Standard Model, we introduce the effective Hamiltonian for quark weak decays. We then consider the effective Hamiltonian for $\Delta F = 2$ transitions in the Standard Model and beyond. We discuss how meson-antimeson mixing and CP violation can be described in terms of the $\Delta F = 1$ and $\Delta F = 2$ effective Hamiltonians. Finally we present the Unitarity Triangle Analysis and discuss how very stringent constraints on New Physics can be obtained from $\Delta F = 2$ processes.

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1 Introduction

Quark flavour physics is among the most powerful probes of New Physics (NP) beyond the Standard Model (SM) of electroweak and strong interactions. The sensitivity to NP in the flavour sector stems from a few peculiarities of the SM: first of all, the absence of Flavour Changing Neutral Currents (FCNC) at the tree level, which makes FCNC processes finite and therefore predictable; second, the Glashow-Iliopoulos-Maiani (GIM) suppression at the loop level \[1\]; third, the hierarchical structure of quark masses and mixing angles, resulting in the smallness of Jarlskog commutator \[2\]. Thanks to these suppression factors, NP contributions to FCNC processes generated by the exchange of heavy new particles can compete with SM amplitudes, leading to stringent bounds on the NP mass scale. As an example, in Fig. 1 we report the bounds on the NP scale $\Lambda$ obtained from $\Delta F = 2$ processes (i.e. FCNC $q_i \bar{q}_j \rightleftharpoons q_i \bar{q}_j$ transitions), assuming NP contributes at tree level with coupling equal to one in all possible chiral structures. We will return to this plot at the end of these lectures, after working out the basic ingredients of the phenomenological analysis leading to these results; we can however already see that, under the above assumptions, scales up to $\mathcal{O}(10^5)$ TeV can be probed, demonstrating the extraordinary NP sensitivity of FCNC processes.

We have stated above that FCNC processes are calculable in the SM, in the sense that a prediction can be obtained (at least in principle) once all the parameters in the SM are known. However, in practice the computation of FCNC processes in the quark sector is in general a very complicated problem, for several reasons. First of all, what can be measured are transitions between a hadronic initial state (for example a $K$, $D_{(s)}$, $B_{(s)}$ meson or a baryon) and a leptonic, semileptonic or nonleptonic final state. Thus, nonperturbative QCD effects connected to quark confinement are always involved, at least in the form of meson decay constants, form factors or other hadronic matrix elements. Furthermore, for nonleptonic final states we must include final state interactions, another very difficult task. Finally, the energy scales involved span several orders of magnitude, from the strong interaction scale $\Lambda_{\text{QCD}}$ to the weak interaction scale $M_W$ to even larger energies if NP is involved. Effective theories are then the best tool to cope with such multi-scale processes, allowing for a systematic expansion in small ratios of widely different scales, and providing the scale separation needed to disentangle perturbative and nonperturbative strong interaction effects.

The absence of tree-level FCNC in the SM implies that NP contributions to FCNC transitions must appear as higher-dimensional operators suppressed by the NP scale $\Lambda$. If the NP scale is much larger than the weak scale, then these higher-dimensional operators will be invariant under the SM gauge group, leading to the so-called Standard Model Effective Theory (SMEFT) (see the lectures by A. Manohar \[3\] and A. Pich \[4\] at this school for a detailed discussion of the SMEFT). Indeed, the bounds presented in Fig. 1 can be interpreted as bounds on the coefficients of SMEFT operators \[5\].

The impressive bounds on the NP scale reported in Fig. 1 correspond to a generic flavour structure. This suggests that any NP close to the EW scale must have a flavour structure either identical or very similar to the SM one. This observation leads to the formulation of effective theories based on the hypothesis of Minimal Flavour Violation.
Figure 1: Summary of the 95% probability lower bound on the NP scale $\Lambda$. See the text for details.
or on approximate flavour symmetries \footnote{or on approximate flavour symmetries \cite{7-11}, which can be viable at scales within the LHC reach.}

The goal of these lectures is to give the reader a basic idea of how the stringent bounds in Fig. 1 are obtained. After giving a concise review of the flavour structure of the SM in Sec. 2 we introduce in Sec. 3 the effective Hamiltonian for quark weak decays. We then consider $\Delta F = 2$ processes in Sec. 4 generalising to the case of NP. In Sec. 5 we discuss how meson-antimeson mixing and CP violation can be described in terms of the effective Hamiltonians introduced in the previous Sections. Finally, we put everything together in the context of the Unitarity Triangle Analysis in Sec. 6. Sec. 7 contains suggestions for further reading. A few useful formulæ are collected in Appendix A.

\section{The flavour structure of the Standard Model}

To set the stage for our discussion, and to fix the notation, let us quickly review the flavour structure of the SM. The SM is described by the most general renormalizable $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ gauge-invariant Lagrangian involving three generations of leptons and quarks and one Higgs doublet $^\footnote{We neglect the QCD \( \theta \) term since it is irrelevant for our discussion.}$

\begin{align}
\mathcal{L}_{\text{SM}} &= \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermionic}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}, \\
\mathcal{L}_{\text{gauge}} &= \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} + \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \\
\mathcal{L}_{\text{fermionic}} &= \sum_f \bar{\psi}_f i D^\mu \psi_f, \\
\mathcal{L}_{\text{Higgs}} &= (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2, \\
\mathcal{L}_{\text{Yukawa}} &= Y_{ij}^u \bar{Q}_i u_R^j + Y_{ij}^d \bar{Q}_i d_R^j + H.c. + \ldots
\end{align}

with $a = 1, \ldots, 8$ and $\alpha = 1, 2, 3$ indices in the adjoint representation of $SU(3)_c$ and $SU(2)_L$ respectively, $f = \{Q_L^i, u_R^i, d_R^i, L_L^i, \ell_R^i\}$, $i$ and $j$ generation indices and the ellipse in the last equation denotes lepton Yukawa couplings. $Q_L^i, u_R^i, d_R^i, L_L^i$ and $\ell_R^i$ represent left-handed $SU(2)_L$ quark doublets, right-handed up- and down-quarks, left-handed lepton $SU(2)_L$ doublets and right-handed charged leptons respectively. $\phi$ denotes the Higgs boson doublet, with $\phi^i = \epsilon^{ij} \phi_j^*$. $G_{\mu\nu}, W_{\mu\nu}$ and $B_{\mu\nu}$ represent the field strength tensors for $SU(3)_c, SU(2)_L$ and $U(1)_Y$ respectively.

Let us now focus on the flavour quantum numbers. The first three terms in eq. (1) are invariant under global $U(3)_{Q_L} \otimes U(3)_{u_R} \otimes U(3)_{d_R} \otimes U(3)_{L_L} \otimes U(3)_{\ell_R}$ transformations acting on generation indices. From now on, we concentrate on quarks. The Yukawa couplings break the $U(3)_{Q_L} \otimes U(3)_{u_R} \otimes U(3)_{d_R}$ symmetry to $U(1)_B$, corresponding to baryon number conservation, an accidental symmetry of the SM. The top Yukawa coupling provides an $\mathcal{O}(1)$ breaking of the $U(3)^3$ flavour symmetry in the quark sector, while an approximate $U(2)^3$ symmetry remains valid up to terms of $\mathcal{O}(Y_t) \sim 10^{-2}$. 
Due to the $U(3)^3$ invariance of $\mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermionic}} + \mathcal{L}_{\text{Higgs}}$, the SM Yukawa couplings are defined up to an $SU(3)^3 \otimes U(1)^2$ transformation (they are invariant under $U(1)_B$ transformations), which allows to eliminate nine real parameters and seventeen phases from $Y^{u,d}$, leaving us with nine observable real parameters and one phase. Since the Lagrangian in eq. (1) is CP-invariant only for real Yukawa couplings, we see that the observable phase in the Yukawa couplings is responsible for CP violation in weak interactions. For two generations of fermions, the Yukawa couplings would contain $8 - (9 - 1) = 0$ observable phases, leading to CP conservation. Thus, the presence of three generations of fermions is crucial to allow for CP violation in weak interactions\[12\]. This strongly restricts the number of processes in which we can observe CP violation in weak interactions: CP violation can occur only in processes where all the three generations are involved, either as interfering real states or as virtual ones.

2.1 The Cabibbo-Kobayashi-Maskawa mixing matrix

Let us now take into account electroweak symmetry breaking induced by the vacuum expectation value (vev) of the neutral component of the Higgs doublet $\phi$:

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix}. \quad (6)$$

In the SM, the Higgs vev generates masses for the $W^{\pm}$ and $Z^0$ bosons through electroweak interactions as well as for the fermions through Yukawa interactions. For the latter we obtain

$$\mathcal{L}_m = m^u_{ij} \bar{u}_L^i u_R^j + m^d_{ij} \bar{d}_L^i d_R^j + H.c. \quad (7)$$

with $m^u_{ij} \equiv Y^u_{ij} v / \sqrt{2}$. The complex mass matrices $m^u_{ij}$ can be brought to diagonal form via a biunitary transformation:

$$U_u L^m_u U_u^\dagger = U^u D, \quad (8)$$

$$U_d L^m_d U_d^\dagger = U^d D, \quad (9)$$

with $m_D$ a diagonal matrix with the masses of down, strange and bottom (up, charm and top) quarks on the diagonal. We can go to the mass eigenstate basis for quarks defining

$$u^\prime_{L,R} = U_{uL,R} u_{L,R}, \quad d^\prime_{L,R} = U_{dL,R} d_{L,R}. \quad (12)$$

Given the $U(3)_{uR} \otimes U(3)_{dR}$ invariance of $\mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermionic}} + \mathcal{L}_{\text{Higgs}}$, switching from unprimed to primed right-handed quarks has no effect. For left-handed fermions, it is convenient to rewrite the transformations in eq. (12) in the form of a transformation on $Q_L$ followed by an additional transformation on $u_L$:

$$d'_L = U_{dL} d_L, \quad u'_L = V U_{dL} u_L, \quad (13)$$
The latter angle enters the “squashed” UT corresponding to the \( (b, s) \) unitarity relation.

The CKM matrix can be parameterized in terms of three angles and one phase, as in the so-called “standard” parameterization:

\[
V = \begin{pmatrix}
 c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
 s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},
\]

where we have introduced the shorthand notation \( s_{ij} = \sin(\theta_{ij}), \ c_{ij} = \cos(\theta_{ij}) \).

Given that \( s_{13} \ll s_{23} \ll s_{12} \ll 1 \), a perturbative expansion in powers of the sine of the Cabibbo angle \( s_{12} \) can be performed \[^{14}\], defining

\[
\lambda \equiv s_{12}, \quad A \equiv s_{23}/\lambda, \quad (\rho + i\eta) \equiv s_{13}e^{i\delta}/(A\lambda^3)
\]

and imposing the unitarity constraint at the desired order \[^{15}\]. In particular, expanding all matrix elements up to \( O(\lambda^3) \), one obtains

\[
V = \begin{pmatrix}
 1 - \frac{\lambda^2}{2} - \frac{\lambda^4}{8} & \lambda & A\lambda^3 (\rho - i\eta) \\
-\lambda + A^2\lambda^5 \left( \frac{1}{2} - \rho - i\eta \right) & 1 - \frac{\lambda^2}{2} - \frac{\lambda^4(1+4\lambda^2)}{8} & A\lambda^2 \\
 A\lambda^3 \left( 1 - \rho - i\eta \right) & -A\lambda^2 \left( 1 - \frac{\lambda^2}{2} \right) & A\lambda^4 (\rho + i\eta) & 1 - \frac{\lambda^2}{2} - \frac{\lambda^4(1+4\lambda^2)}{8}
\end{pmatrix},
\]

with

\[
\bar{\rho} = \rho \left( 1 - \frac{\lambda^2}{2} \right), \quad \bar{\eta} = \eta \left( 1 - \frac{\lambda^2}{2} \right).
\]

The unitarity of the CKM matrix implies triangular relations, which however involve sides of very different lengths, except for the ones corresponding to transitions between the first and third families, namely:

\[
V_{ud}^*V_{ub} + V_{cd}^*V_{cb} + V_{td}^*V_{tb} = 0,
\]

\[
V_{ud}^*V_{ud} + V_{us}^*V_{us} + V_{ub}^*V_{ub} = 0,
\]

where all sides are of \( O(\lambda^3) \). Let us focus on the relation in eq \[^{17}\] and divide it by the last term, defining the so-called Unitarity Triangle (UT):

\[
-\frac{V_{ud}^*V_{ub}}{V_{cd}^*V_{cb}} - \frac{V_{td}^*V_{tb}}{V_{cd}^*V_{cb}} = R_b e^{i\gamma} + R_t e^{-i\beta} = \eta (\bar{\rho} + i\bar{\eta}) + (1 - \bar{\rho} - i\bar{\eta}) \equiv \gamma \equiv \arg \left( -\frac{V_{ud}^*V_{ub}}{V_{cd}^*V_{cb}} \right), \quad \beta \equiv \arg \left( -\frac{V_{td}^*V_{tb}}{V_{cd}^*V_{cb}} \right).
\]

The UT can then be represented as a triangle in the complex \( (\bar{\rho}, \bar{\eta}) \) plane, see Fig. 2.

It is useful to define also

\[
\alpha \equiv \arg \left( -\frac{V_{td}^*V_{tb}}{V_{ud}^*V_{ub}} \right), \quad \beta_s \equiv \arg \left( -\frac{V_{ts}^*V_{tb}^*}{V_{cs}^*V_{cb}^*} \right).
\]

The latter angle enters the “squashed” UT corresponding to the \( (b, s) \) unitarity relation.
2.2 Weak interactions below the EW scale

After electroweak symmetry breaking, going to the quark mass eigenstate basis through the transformations in eq. \[12\] and dropping primes for simplicity, we are left with the following couplings of gauge bosons with fermionic currents:

\[
L_{\text{int}} = -\frac{g_2}{\sqrt{2}} \left( W_\mu^+ J_\mu^+ + W_\mu^- J_\mu^- \right) - g_1 \cos \theta_W A_\mu J_\mu^\text{em} - \frac{g_2}{\cos \theta_W} Z_\mu J_\mu^Z, \tag{22}
\]

\[
J_{\text{em}}^\mu = \sum_{f=\ell,u,d} Q_f \tilde{f} \gamma^\mu f, \tag{23}
\]

\[
J_Z^\mu = \sum_{f=\ell,u,d} (I_f^3 - Q_f \sin^2 \theta_W) \tilde{f} \gamma^\mu f - Q_f \sin^2 \theta_W \tilde{f} R \gamma^\mu f, \tag{24}
\]

\[
J_{\text{ch}}^\mu = \bar{u}_i \gamma^{\mu} d^i_L + \bar{\nu}_i \gamma^{\mu} \ell^i_L, \tag{25}
\]

with \(g_1\) and \(g_2\) the \(U(1)_Y\) and \(SU(2)_L\) gauge couplings respectively, \(A_\mu\) the photon field, \(Z_\mu\) the \(Z^0\) field, \(\nu_L\) left-handed neutrinos, \(\theta_W\) the weak mixing angle, \(e = g_1 \cos \theta_W = g_2 \sin \theta_W\), \(Q_\ell = -1\), \(Q_u = 2/3\), \(Q_d = -1/3\) and \(I^3\) the third component of weak isospin. In the 't-Hooft-Feynman gauge, we have the following Feynman rules:

\[
W^\mu \sim \sim W^\nu = \frac{-ig^{\mu\nu}}{k^2 - M_W^2 + i\epsilon}, \quad \phi - \sim \sim \phi = \frac{i}{k^2 - M_W^2 + i\epsilon}, \tag{26}
\]

\[
W_\mu \sim \sim = \frac{ig_2}{\sqrt{2}} \gamma_\mu P_L V_{ui}^*, \quad W_\mu \sim \sim = \frac{ig_2}{\sqrt{2}} \gamma_\mu P_L V_{ui}^d, \tag{27}
\]
\[
\phi \rightarrow \begin{cases} 
\frac{-ig_2}{\sqrt{2}M_W} [m_d P_L - m_u P_R] V_{u,d}^*, \\
[28] 
\frac{-ig_2}{\sqrt{2}M_W} [m_d P_R - m_u P_L] V_{u,d}, 
\end{cases}
\]

(28)

\[
\phi \rightarrow \begin{cases} 
\frac{-ig_2}{\sqrt{2}M_W} [m_d P_L - m_u P_R] V_{u,d}^*, \\
[29] 
\frac{-ig_2}{\sqrt{2}M_W} [m_d P_R - m_u P_L] V_{u,d}, 
\end{cases}
\]

(29)

with \( P_{L,R} = (1 \mp \gamma_5)/2 \) the left- and right-handed chiral projectors.

Having fixed the notation, in the next Sections we introduce the effective Hamiltonians relevant for quark flavour physics.

### 3 Effective Hamiltonians for quark weak decays

Let us start by considering the amplitude for the \( u \bar{d} \rightarrow \nu \ell \bar{\ell} \) transition, which is generated at lowest order by the following Feynman diagrams:

\[
\begin{align*}
&\begin{array}{c}
u \phantom{u} \\
\downarrow \phantom{u} \\
\ell \\
\downarrow \\
u \phantom{u}
\end{array}
+ \\
&\begin{array}{c}
u \phantom{u} \\
\downarrow \phantom{u} \\
\ell \\
\downarrow \\
u \phantom{u}
\end{array}
\end{align*}
\]

(30)

The Goldstone boson exchange can be neglected here since its couplings are proportional to light fermion masses. The amplitude mediated by the \( W \) reads

\[
iA_W = \left( \frac{ig_2}{\sqrt{2}} \right)^2 V_{ud}^* (\bar{u}_{u,\nu} (p_{\nu,\ell}) \gamma_{\mu} P_L v_{\ell}(p_{\ell})) (\bar{v}_{d}(p_d) \gamma_{\mu} P_L u_{u}(p_u)) \frac{-ig^{\mu\nu}}{k^2 - M_W^2 + i\epsilon}.
\]

(31)

with \( k = (p_u + p_d) = (p_\ell + p_{\nu,\ell}) \). Now, if we are interested in low-energy processes such as pion leptonic or semileptonic decays, we should consider external momenta of the order of the pion mass, therefore much lower than \( M_W \). Thus, we can perform an expansion of the \( W \) propagator in powers of the momentum \( k \), leading to

\[
iA_W = -\frac{V_{ud}^* g_2^2}{2M_W^2} (\bar{u}_{u,\nu} (p_{\nu,\ell}) \gamma_{\mu} P_L v_{\ell}(p_{\ell})) (\bar{v}_{d}(p_d) \gamma_{\mu} P_L u_{u}(p_u)) \sum_{n=0}^{\infty} \left( \frac{k^2}{M_W^2} \right)^n \]

(32)

\[
\simeq -i \frac{4G_F}{\sqrt{2}} V_{ud}^* (\bar{u}_{u,\nu} (p_{\nu,\ell}) \gamma_{\mu} P_L v_{\ell}(p_{\ell})) (\bar{v}_{d}(p_d) \gamma_{\mu} P_L u_{u}(p_u)) + O \left( \frac{k^2}{M_W^2} \right),
\]
where we have introduced the Fermi constant
\[ \frac{G_F}{\sqrt{2}} \equiv \frac{g_2^2}{8M_W^2}, \] (33)
with \( G_F = 1.1663787(6) \times 10^{-5} \text{ GeV}^{-2} \) [16]. The dominant term in eq. (32) corresponds to the matrix element of the following local operator:
\[ Q^{\bar{d}u\nu\ell} \equiv \bar{d}_L \gamma_\mu u_L \bar{\nu}_\ell \gamma_\mu \ell_L, \] (34)
while the terms of order \( n > 0 \) in the expansion in powers of \( k^2/M_W^2 \) correspond to the matrix elements of higher dimensional local operators containing \( 2n \) derivatives.

Keeping only the dimension six operator in this Operator Product Expansion (OPE), we obtain
\[ A_W = \langle -\mathcal{H}_{\text{eff}} \rangle + O \left( \frac{k^2}{M_W^2} \right) , \quad \mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V^{\ast}_{ud} Q^{\bar{d}u\nu\ell} , \] (35)
where we have introduced the effective Hamiltonian for \( \bar{d}u \rightarrow \bar{\nu}\ell \nu \ell \) transitions. The effects of the exchange of the heavy \( W \) boson are encoded in the so-called Wilson coefficient, i.e. the coefficient in front of the local operator \( Q^{\bar{d}u\nu\ell} \) in \( \mathcal{H}_{\text{eff}} \). External momenta are irrelevant in the matching between the full and effective theory performed in eq. (35), since the dynamics at scales much lower than \( M_W \) is identical in the full and effective theory, up to the desired order in the OPE.

Of course, we should now worry about the effects of strong interactions. Given the low scale at which pion decays occur, we cannot invoke any argument to suppress strong corrections to the diagram in (30) such as the one in the first row of Fig. 3. However, such corrections are identical in the full theory and in the effective one, i.e. the diagrams in the first and second row of Fig. 3 are identical. Therefore, in this example we do not need to take strong corrections into account in the matching; all strong interactions will be captured by the matrix element of \( Q^{\bar{d}u\nu\ell} \) between the relevant initial and final states.

### 3.1 Four-quark current-current operators

The situation changes dramatically if we now turn to nonleptonic decays. Consider for example \( c\bar{s} \rightarrow u\bar{d} \) transitions. Neglecting Goldstone boson exchange and QCD corrections, in the SM these are described by diagram (a) in Fig. 4. Just as in the case of \( u\bar{d} \rightarrow \nu\bar{\ell} \nu\ell \) transitions discussed above, since the energy scale at which charm decays take place is much lower than \( M_W \), the \( W \) boson will propagate over very short distances, so we can perform an OPE and consider dimension six operators only. In this case, the amplitude we obtain expanding diagram (a) at the lowest order in \( k^2/M_W^2 \) is proportional to the one generated by diagram (c) with the insertion of operator
\[ Q_1^{\bar{c}\pi\pi d} \equiv \bar{s}_L \gamma_\mu c_L \bar{u}_L \gamma_\mu d_L . \] (36)
Imposing that the two amplitudes be equal,
\[ A_W = \langle -\mathcal{H}_{\text{eff}} \rangle + O \left( \frac{k^2}{M_W^2} \right) , \] (37)
we obtain the corresponding Wilson coefficient:

\[ \mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} V_{ud} V_{cs}^* C_1 Q_1^{c \bar{d} u} , \quad C_1 = 1 . \]  

(38)

As in the case of leptonic and semileptonic decays, the corrections to the SM amplitude generated by the exchange of gluons between the \( c \) and \( s \) quarks, such as the one in diagram (b) of Fig. 4, are identical in the full and in the effective theory, represented in this case by diagram (f) of Fig. 4 so they do not enter the matching. They will be taken into account in the evaluation of the relevant hadronic matrix element of the effective Hamiltonian. The same argument applies to the exchange of gluons between \( u \) and \( d \) quarks, which has not been explicitly reported in Fig. 4.

The situation is however totally different for the exchange of gluons between the two currents coupled to the \( W \) boson, such as in diagrams (c) and (d). In these diagrams, the \( W \) propagator has momentum \( k - \ell \), where \( \ell \) is the loop momentum; in the region \( (k - \ell)^2 \sim M_W^2 \) the \( W \) propagator opens up and it falls as \( \ell^{-2} \) for \( \ell^2 \gg M_W^2 \), making the loop integral convergent. Thus, \( M_W \) acts as an ultraviolet regulator in diagrams (c) and (d). Indeed, evaluating the amplitude explicitly, putting the quarks off-shell with \( p^2 < 0 \) to avoid infrared divergences, one finds a term proportional to \( \alpha_s \log \left( \frac{M_W^2}{-p^2} \right) \). Taken at face value, such term implies the breakdown of perturbation theory, since the effective expansion parameter \( \alpha_s \log \left( \frac{M_W^2}{-p^2} \right) \) becomes of \( \mathcal{O}(1) \) for quark momenta of \( \mathcal{O}(\Lambda_{\text{QCD}}) \), due to the large logarithm. Fortunately, the effective theory can save us from this disaster, as we shall see below.

### 3.1.1 General considerations

The effective theory counterpart of diagrams (c) and (d) in Fig. 4 is given by diagrams (g) and (h). Having removed the \( W \) propagator in the effective theory, the latter
Figure 4: Diagrams relevant for $c \to su\bar{d}$ transitions in the full and effective theory, including leading order QCD corrections. See the text for details.
diagrams are divergent, so their ultraviolet behaviour is very different from the corresponding SM diagrams. This is no surprise, since we worked out the effective theory as an OPE expanding in powers of $k^2/M_W^2$, so we expect it to be valid up to a cutoff $\Lambda$ of $\mathcal{O}(M_W)$; above this cutoff the contribution of all higher dimensional operators becomes unsuppressed and the expansion breaks down. Regulating diagrams (g) and (h) with the introduction of a cutoff $\Lambda$, one would obtain terms proportional to $\alpha_s \log \left( \frac{\Lambda^2}{p^2} \right)$.

Since we have seen that $M_W$ acts as a regulator in the SM amplitude, and since the infrared logs of external momenta must be identical in the full and effective theory, it is clear that the coefficients of the log terms in the SM and in the effective theory are equal.

From the technical point of view, rather than introducing an explicit cutoff, it is much more convenient to subtract the divergences and work in the renormalized theory. In this case, the cutoff is removed but a renormalization scale $\mu$ is introduced, so that after renormalization $\log \left( \frac{\Lambda^2}{p^2} \right)$ terms are replaced by $\log \left( \frac{\mu^2}{p^2} \right)$.

We can match the amplitudes obtained from diagrams (c) and (d) with the ones obtained from diagrams (g) and (h) after subtracting the divergence in the effective theory via a renormalization constant $Z$. The infrared logs cancel and we are left with terms proportional to

$$\alpha_s \log \left( \frac{M_W^2}{p^2} \right) - \alpha_s \log \left( \frac{\mu^2}{-p^2} \right) = \alpha_s \log \left( \frac{M_W^2}{\mu^2} \right).$$

Choosing a renormalization scale $\mu_W \sim M_W$, we can therefore get rid of large logs in the matching procedure. In this way, we can go from the full theory to the effective one using ordinary perturbation theory. The Wilson coefficient obtained from the matching now carries an explicit dependence on the renormalization scale $\mu$, which cancels against the renormalization scale dependence of the matrix element of the renormalized operator, since the amplitude in the full theory does not depend on $\mu$:

$$\frac{d}{d\mu} A_{\text{full}} = 0 = \frac{d}{d\mu} \langle C(\mu) \langle Q_{\text{ren}}(\mu) \rangle \rangle = \frac{d}{d\mu} C(\mu) \langle Q_{\text{ren}}(\mu) \rangle + C(\mu) \mu \frac{d}{d\mu} \langle Q_{\text{ren}}(\mu) \rangle.$$

Performing the matching at $\mu_W \sim M_W$ we got rid of large logs in the matching procedure, but we actually just shifted them into the effective theory. Computing the matrix element in the effective theory, large logs of $(\mu_W^2/\mu^2)$ would arise again, bringing us back into trouble. However, as discussed in detail in M. Neubert’s lectures at this School, the renormalization scale dependence of a renormalized operator is governed by the Renormalization Group Equations (RGE) in terms of its anomalous dimension $\gamma_Q$ (which is nothing else but the coefficient of the divergent terms, i.e. the coefficient of the log $\left( \frac{M_W^2}{\mu^2} \right)$ terms in the full theory):

$$\mu \frac{d}{d\mu} Q_{\text{ren}} = -\gamma_Q Q_{\text{ren}}, \quad \gamma_Q = \frac{d}{d\log \mu} \log Z.$$

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Combining eqs. (40) and (41) we obtain the RGE for the renormalization scale dependence of the Wilson coefficient:

\[
\frac{d}{d\mu} C(\mu) = \gamma_Q C(\mu),
\]

which allows us to obtain the Wilson coefficient for any \( \mu \), starting from its value at \( \mu_W \):

\[
C(\mu) = U(\mu, \mu_W) C(\mu_W), \quad U(\mu, \mu_W) = e^{\int_{\mu_W}^{\mu} \frac{\gamma_Q(g_s')}{g_s'} dt},
\]

where the \( \beta \) function governs the running of the strong coupling constant with the renormalization scale:

\[
\beta(g_s) = -g_s \frac{d \log Z_{g_s}}{d \log \mu},
\]

with \( Z_{g_s} \) the renormalization constant of the \( SU(3)_c \) coupling \( g_s \). We can now run down from \( \mu_W \sim M_W \) to a low renormalization scale \( \mu_h \) close to the physical scale at which the process we are interested in computing occurs, and then compute the relevant matrix element (between an initial state \( i \) with momenta \( p_i \) and a final state \( f \) with momenta \( p_f \)) without encountering large logs, since \( \mu_h \sim p_i \sim p_f \):

\[
\langle f(p_f)|\mathcal{H}_{\text{eff}}|i(p_i)\rangle = C(\mu_h) \langle f(p_f)|Q(\mu_h)|i(p_i)\rangle.
\]

Where have the large logs gone? They have been resummed via the renormalization group evolution! Thus, the effective theory allows us to perform the matching using perturbation theory in the strong interactions and to resum large logs using the RGE. The evaluation of the relevant matrix elements of the local operators in \( \mathcal{H}_{\text{eff}} \) can then be performed, if necessary (and possible), with a nonperturbative method such as Lattice QCD.

The calculation of the \( \beta \) function and of the anomalous dimensions is particularly simple in mass-independent renormalization schemes such as modified Minimal Subtraction (\( \overline{\text{MS}} \)) \([17, 18]\). In dimensional regularization, logarithmic divergences appear as singularities as the number of space-time dimensions tends to four:

\[
\log \left( \frac{\Lambda^2}{-p^2} \right) \Leftrightarrow \frac{1}{\epsilon} + \log \left( \frac{\mu^2}{-p^2} \right),
\]

where

\[
\frac{1}{\epsilon} = \frac{2}{4-D} - \gamma_E + \log(4\pi)
\]

and \( \mu \) is the renormalization scale. In the \( \overline{\text{MS}} \) scheme we renormalize the operator by subtracting the \( \frac{1}{\epsilon} \) divergence. Dropping the bar for simplicity, and writing the renormalization constant \( Z \) as a series in inverse powers of \( \epsilon \),

\[
Z = 1 + \sum_k \frac{1}{\epsilon^k} Z_k(g_s),
\]
and
\[ \beta(g_s, \epsilon) = \frac{dg_s(\mu)}{d \log \mu} = -\epsilon g_s + \beta(g_s), \tag{49} \]

one obtains from eq. (41)
\[ \gamma_Q (1 + \frac{1}{\epsilon} Z_1(g_s) + \ldots) = \frac{1}{\epsilon} \frac{d Z_1}{d \log \mu} + \ldots \]
\[ = \frac{1}{\epsilon} \frac{d Z_1}{d g_s} \frac{dg_s}{d \log \mu} + \ldots \]
\[ = \frac{1}{\epsilon} \frac{d Z_1}{d g_s} (-\epsilon g_s + \beta(g_s)) + \ldots, \tag{50} \]

where the ellipses denote higher terms in the $1/\epsilon$ expansion. The finiteness of $\gamma_Q$ implies
\[ \gamma_Q = -2\alpha_s \frac{d Z_1}{d \alpha_s}, \tag{51} \]

so the anomalous dimension is directly obtained from the $1/\epsilon$ terms in the renormalization constant of the operator $Q$.

If we are interested in the dominant, log-enhanced QCD corrections, we can drop gluonic corrections to the matching (since no large logs arise at $\mu_W \sim M_W$) and compute the anomalous dimension of the operators in $\mathcal{H}_{\text{eff}}$ at the first order in $\alpha_s$. In general, if we expand the anomalous dimension matrix and the Wilson coefficients in a series in $\alpha_s$,
\[ C(\mu) = \sum_{n=0}^{n} \left( \frac{\alpha_s}{4\pi} \right)^n C^{(n)}(\mu), \]
\[ \gamma = \sum_{n=0}^{n} \left( \frac{\alpha_s}{4\pi} \right)^{(n+1)} \gamma_n, \tag{53} \]

we can classify the accuracy of the expansion in $\alpha_s$ as follows.

A leading order (LO) calculation resums all terms of $\mathcal{O} \left( \alpha_s \log \left( \frac{M_W^2}{-p^2} \right) \right)^n$, by computing the anomalous dimensions at $\mathcal{O}(\alpha_s)$ and the matching and matrix elements neglecting $\alpha_s$ corrections. Expanding eqs. (43), (44) and (45) we obtain
\[ \mathcal{A}_{\text{LO}} = C^{(0)}(\mu_h)(Q(\mu_h))^{(0)}, \tag{54} \]

where $(Q(\mu_h))^{(n)}$ denotes a matrix element computed at $n$-th order in strong interactions and
\[ C^{(0)}(\mu_h) = U_0(\mu_h, \mu_W) C^{(0)}(\mu_W), \quad U_0 = \left( \frac{\alpha_s(\mu_W)}{\alpha_s(\mu_h)} \right)^{\frac{\gamma_0}{2\beta_0}}. \tag{55} \]

The LO evolutor $U_0$ resums all large logs. In general, as discussed in M. Neubert’s lectures, QCD corrections induce mixing among different operators, so that in general
$H_{\text{eff}}$ comprises several operators. The equations above still apply, provided we consider $C$ and $Q$ as vectors and $\gamma$ as a matrix; in this case, $U_0 = \left( \frac{\alpha_s(\mu_W)}{\alpha_s(\mu_h)} \right) \frac{1}{\beta_0^2}$.

A (next-to-)$m$ leading order ($N^m$ LO) calculation resums all terms of $O \left( \alpha_s^{n+m} \log \left( \frac{M_W^2}{p^2} \right)^n \right)$, by computing the anomalous dimensions at $O(\alpha_s^{m+1})$ and the matching and matrix elements at $O(\alpha_s^m)$. For example, at NLO we resum all terms of $O \left( \alpha_s^{n+1} \log^n \left( \frac{M_W^2}{p^2} \right) \right)$, by computing the anomalous dimension at $O(\alpha_s^2)$ and the matching and matrix elements at $O(\alpha_s)$. Explicitly, we have

$$A_{\text{NLO}} = C^{(0)}(\mu_h) \langle Q(\mu_h) \rangle^{(1)} + \frac{\alpha_s(\mu_h)}{4\pi} C^{(1)}(\mu_h) \langle Q(\mu_h) \rangle^{(0)} ,$$

where

$$C^{(1)}(\mu_h) = U_0(\mu_h, \mu_W) C^{(1)}(\mu_W) + \left( JU_0(\mu_h, \mu_W) + \frac{\alpha_s(M_W)}{\alpha_s(\mu)} U_0(\mu_h, \mu_W) J \right) C^{(0)}(\mu_W) ,$$

where the matrix $J$ is obtained from $\gamma_1$ and $\beta_1$ as explained for example in the renowned Les Houches lectures by A.J. Buras [19], where a complete pedagogical introduction to the subtleties of NLO calculations is presented. Although in the current lectures we will confine ourselves to LO calculations, the importance of computing weak Hamiltonians to NLO (or above) cannot be overemphasized.

Finally, we warn the reader that the $\overline{\text{MS}}$ scheme, although very convenient for perturbative calculations, is not the only option. For example, matrix elements computed in Lattice QCD (LQCD) potentially take into account strong interactions to all orders in the non-perturbative regime; it is therefore possible (and desirable) to perform non-perturbative renormalization, subtracting divergences to all orders in perturbation theory [20]. To achieve this result, it is convenient to use the so-called regularization-independent renormalization schemes, that are defined by fixing the value of a given number of renormalized Green functions. For example, instead of defining the renormalized four-quark operator by subtracting the $1/\epsilon$ poles in dimensional regularization at a given perturbative order, we could define it by imposing that its matrix element on given initial and final states be equal to a given number, for example to the tree-level matrix element of the same operator. This renormalization condition can be implemented in any regularization at any perturbative order, making it possible to match the perturbative calculation of the Wilson coefficient with the nonperturbative calculation of the hadronic matrix element.

### 3.1.2 Current-current operators at LO

As we have seen above, if we are interested in capturing the dominant, log-enhanced QCD corrections only, we just need to start at $\mu_W$ with the effective Hamiltonian obtained from tree-level matching, eq. (38), and run it down to $\mu_h$ using eq. (42) with the anomalous dimension computed at $O(\alpha_s)$. To compute the latter, we need
to identify the 1/\epsilon terms generated by diagrams (f)-(h) in Fig. 4 plus the “mirror” ones reported in Fig. [3]. We can actually skip diagrams (f) and (i) since they cancel against the renormalization constants for the quark fields due to the Ward identity that protects the conserved weak current. Let us therefore start from diagram (g). Assigning momentum $p$ to the incoming $c$ and outgoing $u$ quarks, and loop momentum $k$ to the fermions in the loop, we obtain the following amplitude:

$$iA_{(g)} = \frac{4G_F V_{cd}^* V_{us}}{\sqrt{2}} \int \frac{d^D k}{(2\pi)^D} \overline{u}_i (ig_s \gamma_\mu T_{ij}^A) \frac{i}{k^\mu} \gamma_\rho P_L v_j^d \overline{u}_k \gamma_\mu P_L \frac{i}{k} (ig_s \gamma_\nu T_{kl}^B) \frac{u_i^c - ig_{\mu\nu} \delta^{AB}}{(k - p)^2}$$

where we have used the following Feynman rules for QCD in the Feynman gauge:

$$g_A^\mu \rightarrow g_A^\nu = \frac{-ig_{\mu\nu} \delta^{AB} k^2 + i\epsilon}{k^2 + i\epsilon}, \quad g_A^A = ig_s \gamma_\mu T_{ij}^A,$$

with $A, B$ colour indices in the adjoint representation, $i, j, k, l$ colour indices in the fundamental representation and $T$ the $SU(3)$ generators for the fundamental representation.

Figure 5: “Mirror” diagrams relevant for $c \rightarrow su\bar{d}$ transitions in the effective theory. See the text for details.

Pulling out the Dirac structure we can rewrite the amplitude as

$$iA_{(g)} = -i \frac{4G_F}{\sqrt{2}} V_{ud} V_{cs}^* \frac{1}{2} T_{ij}^A T_{kl}^B \overline{u}_i^c (ig_s \gamma_\mu \gamma_\rho P_L v_j^d) \overline{u}_k \gamma_\mu P_L \gamma_\nu \gamma_\mu u_i^c \mathcal{C}^\alpha \mathcal{C}^\beta,$$

(59)
with

\[ I^{\alpha\beta} = \int \frac{d^Dk}{(2\pi)^D} \frac{k^\alpha k^\beta}{(k^2)^2(k - p)^2} \]

\[ = \int_0^1 dx 2(1 - x) \int \frac{d^Dk}{(2\pi)^D} \frac{k^\alpha k^\beta}{[k^2(1 - x) + (k - p)^2x]^3} \]

\[ = \int_0^1 dx 2(1 - x) \int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell + px)^\alpha (\ell + px)^\beta}{[\ell^2 + p^2x(1 - x)]^3} \]

\[ = \int_0^1 dx 2(1 - x) \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\alpha \ell^\beta}{[\ell^2 + p^2x(1 - x)]^3} + \text{f.t.}, \]

where we have introduced the Feynman parameter \( x \) using

\[ \frac{1}{a^n b} = n \int_0^1 dx \frac{x^{n-1}}{[(1 - x)b + xa]^{n+1}} \]

(a particular case of the general parameterization in Appendix A.1), and the last equality holds up to non-divergent terms.

Computing the integral on \( \ell \) using the formulae in Appendix A.2, we obtain

\[ I^{\alpha\beta} = \int_0^1 dx 2(1 - x) \frac{g^{\alpha\beta}}{2} \frac{i \Gamma(3 - D/2 - 1)}{(4\pi)^{D/2} \Gamma(3)} \left( \frac{\mu^2}{-p^2x(1 - x)} \right)^{3-D/2-1} \]

\[ = \frac{i}{16\pi^2} \frac{g^{\alpha\beta} 1}{4} \frac{1}{\epsilon} + \text{f.t.}, \]

again up to irrelevant finite terms.

Since we are only interested in the divergent terms, we can also perform the Dirac algebra in four dimensions. This greatly simplifies things since we are then authorized to use the so-called Fierz identities [21]. Indeed, in four dimensions we can identify a complete basis for objects carrying two spinor indices. For example, we might choose

\[ \mathbb{1}, \gamma_5, \gamma^\mu, \gamma^\mu \gamma_5, \sigma^{\mu\nu}, \]

but for our purposes it is more convenient to work in a chiral basis such as

\[ P_L, P_R, \gamma^\mu P_L, \gamma^\mu P_R, \sigma^{\mu\nu}. \]

Thus, any object carrying two spinor indices can be projected on this basis. Now, looking at the Dirac string in eq. (59), we see several Dirac matrices with Lorentz indices contracted across the two different fermionic lines. Simplifying the Dirac structure would be simple if those Dirac matrices were in the same fermionic line. We can bring all those matrices together using a Fierz transformation by projecting

\[ (P_L v_j^d \bar{v}_k^c P_R)_{\alpha\beta} \]

3I am indebted to R.K. Ellis for pointing this trick out to me in an exercise session at the Parma school of theoretical physics in September 2001.
on the basis in eq. (64). Since in eq. (63) we have left-handed chirality on the left and right-handed chirality on the right, it is clear that the only structure we can project on is $\gamma^\mu P_R$. To find out the coefficient, we can act on eq. (65) with the operator $\frac{1}{2} \text{Tr} \gamma^\mu P_L$, which is a projector on $\gamma^\mu P_R$, since
\[
\frac{1}{2} \text{Tr} \gamma^\mu P_L P_L = 0, \quad \frac{1}{2} \text{Tr} \gamma^\mu P_L P_R = 0, \quad \frac{1}{2} \text{Tr} \gamma^\mu P_L \gamma^\nu P_L = 0, \quad (66)
\]
\[
\frac{1}{2} \text{Tr} \gamma^\mu P_L \gamma^\nu P_R = g^{\mu\nu}, \quad \frac{1}{2} \text{Tr} \gamma^\mu P_L \sigma^{\mu\nu} = 0. \quad (67)
\]
We obtain
\[
\frac{1}{2} \text{Tr} \gamma^\mu P_L P_L v^d_j \pi^i_k P_R = -\frac{1}{2} \pi^i_k \gamma^\mu P_L v^d_j (\gamma_\mu P_R)_{\alpha\beta}. \quad (69)
\]
Substituting eq. (69) in eq. (59), and keeping into account that the divergent part of $\mathcal{I}^\alpha_\beta$ is proportional to $g^{\alpha\beta}$, the Dirac string becomes
\[
-\frac{1}{2} \pi^i_k \gamma^\nu P_L v^d_j \pi^u_{\mu} \gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\nu P_R \gamma^\rho \gamma^\alpha \gamma^\mu u^c_i. \quad (70)
\]
Using the four-dimensional Dirac algebra rules, we move the chiral projector to the right and apply thrice the identity $\gamma^{\alpha} \gamma^{\mu} \gamma^{\alpha} = -2\gamma^{\mu}$ to obtain
\[
4\pi^i_k \gamma^\nu P_L v^d_j \pi^u_{\mu} \gamma_\mu P_L u^c_i. \quad (71)
\]
Notice that the ordering of the spinors here is different with respect to the tree-level amplitude generated by $Q_1^{a\bar{d}u}$; thus, to identify the relevant counterterms, let us use the Fierz trick again to exchange the $v^d_j$ and $u^c_i$ spinors. This is easily achieved:
\[
\pi^i_k \gamma^\nu P_L v^d_j \pi^u_{\mu} \gamma_\mu P_L u^c_i = -\frac{1}{2} \pi^u_{\mu} \gamma_\mu P_L v^d_j \pi^i_k \gamma^\nu \gamma_\nu P_R \gamma_\mu u^c_i = \pi^u_{\mu} \gamma_\mu P_L v^d_j \pi^i_k \gamma_\nu P_L u^c_i \quad (72)
\]
where in the first step we used again eq. (69). Before putting all the pieces together, let us look at the colour factor in eq. (59). We can simplify it using the SU$(N_c)$ identity
\[
T^{A}_{ij} T^{A}_{kl} = \frac{1}{2} \delta_{il} \delta_{kj} - \frac{1}{2N_c} \delta_{ij} \delta_{kl}. \quad (73)
\]
Putting everything together, we obtain the amplitude generated by diagram (g):
\[
iA_{(g)} = \frac{4G_F}{\sqrt{2}} V_{ud} V_{cs} \alpha_s \frac{1}{4\pi} \frac{1}{2c_2} \left( \pi^u_{\mu} \gamma_\mu P_L v^d_j \pi^i_k \gamma_\nu P_L u^c_i - \frac{1}{N_c} \pi^u_{\mu} \gamma_\mu P_L v^d_j \pi^i_k \gamma_\nu P_L u^c_i \right). \quad (74)
\]
We see that this diagram generates a divergence proportional to the matrix element of a new operator,
\[
Q_2^{c\bar{d}u} = \pi^i_k \gamma^\nu c^d_j \pi^u_{\mu} \gamma_\mu d^a_L. \quad (75)
\]
Thus, the divergence we obtain in the effective theory (or, equivalently, the large log present in the full theory) is proportional not only to the operator generated omitting QCD corrections, but also to a new operator. This forces us to promote the anomalous dimension to a matrix and the operator basis to a vector.

Let us now consider diagram (h) in Fig. 4. For convenience, let us assign momentum \(-p\) to the \(\bar{u}d\) quark line, so that we end up with exactly the same integral as in diagram (g). Since we are interested in the ultraviolet divergence, the choice of external momenta is irrelevant. We obtain:

\[
iA_{(h)} = -i \frac{4G_F}{\sqrt{2}} V_{ud} V^*_{cs} g_s^2 T^A_{ij} T^A_{kl} \bar{u}^i \gamma^\rho P_L \gamma_\alpha \gamma_\mu v^d_j \bar{v}^c_k \gamma^\rho P_L \gamma_\beta \gamma^\mu u^c_i (-T^{\alpha\beta})
\]

\[
= 4G_F \frac{\alpha_s}{4\pi} \frac{\alpha_s}{8\epsilon} \frac{1}{\sqrt{2}} \frac{1}{\bar{u}^i \gamma^\nu P_L v^d_j \bar{v}^c_k \gamma_\alpha \gamma_\mu i_P R \gamma_\rho \gamma_\alpha \gamma^\mu u^c_i}{P_L u^c_i}
\]

\[
= 4G_F \frac{\alpha_s}{4\pi} \frac{\alpha_s}{8\epsilon} \frac{1}{\sqrt{2}} \frac{1}{\bar{u}^i \gamma^\nu P_L v^d_j \bar{v}^c_k \gamma_\mu P_L u^c_i}{P_L u^c_i}
\]

\[
= 4G_F \frac{\alpha_s}{4\pi} \frac{\alpha_s}{8\epsilon} \frac{1}{\sqrt{2}} \frac{1}{\bar{u}^i \gamma^\nu P_L v^d_j \bar{v}^c_k \gamma_\mu P_L u^c_i}{P_L u^c_i}
\]

where, in addition to eqs. (62), (69), (72) and (73), we have used the identity

\[
\gamma^\mu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\mu = -2 \gamma_\gamma \gamma_\beta \gamma_\alpha ,
\]

always performing Dirac algebra in four dimensions.

Let us now turn to the diagrams in Fig. 5. Diagram (i) cancels against quark wave function renormalization, while diagrams (j) and (k) give the same result as diagrams (g) and (h) respectively.

Putting everything together we obtain

\[
iA = \frac{4G_F}{\sqrt{2}} V_{ud} V^*_{cs} \frac{\alpha_s}{4\pi} \frac{3}{\epsilon} \left( \bar{u}^i \gamma_\mu P_L v^d_j \bar{v}^c_k \gamma^\mu P_L u^c_i - \frac{1}{N_c} \bar{u}^i \gamma_\mu P_L v^d_j \bar{v}^c_k \gamma^\mu P_L u^c_i \right).
\]

In order to obtain the two-by-two anomalous dimension matrix we need to compute the one-loop renormalization of \(Q^\text{bare}_{2}^{b\gamma\alpha} \), by inserting it in diagrams (f) to (k). The only difference in the calculation is given by the colour factors, which now read

\[
T^A_{ij} T^A_{kl} = \frac{1}{2} \delta_{ij} \delta_{kl} - \frac{1}{2N_c} \delta_{ij} \delta_{kl}.
\]

Defining

\[
Q^\text{bare}_i = Z_{ij} Q^\text{ren}_j
\]

we thus obtain in \(\overline{\text{MS}}\)

\[
Z = 1 + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} Z_1 = 1 + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \left( \frac{3}{N_c} - 3 \right).
\]
From this we obtain, using eq. (51),

$$\gamma_0 = \left( \begin{array}{cc} -\frac{6}{N_c} & 6 \\ 6 & -\frac{6}{N_c} \end{array} \right).$$  \hspace{1cm} \text{(82)}$$

The corresponding evoluter can be obtained from eq. (42). In practice, it can be evaluated by going to the basis in which $\gamma_0$ is diagonal, defining

$$Q_\pm = \frac{Q_1 \pm Q_2}{2}, \quad C_\pm = C_1 \pm C_2, \quad \gamma_0^\pm = \pm 6 \frac{N_c \pm 1}{N_c}, \quad U_0^\pm = \left( \frac{\alpha_s(\mu_W)}{\alpha_s(\mu_b)} \right)^{\frac{\gamma_0^\pm}{2\gamma_0}}, \hspace{1cm} \text{(83)}$$

Notice that $\beta_0$ depends on the number of active flavours, so if we want to compute the Wilson coefficients at $\mu_h = 2$ GeV we need to take into account the bottom quark threshold at a scale $\mu_b \sim m_b$:

$$C_\pm(2\text{GeV}) = \left( \frac{\alpha_s(\mu_b)}{\alpha_s(2\text{GeV})} \right)^{\frac{\gamma_0^\pm}{2\gamma_0}} C_\pm(\mu_W).$$ \hspace{1cm} \text{(84)}$$

At LO, we have $C_\pm(\mu_W) = 1$ and the evolution decreases $C_+$ and increases $C_-$, since $\gamma_0^+$ is positive and $\gamma_0^-$ is negative.

### 3.2 Penguin operators

Let us now change the flavour content of our current-current operators and consider $W$ exchange between a $\bar{s}u$ and a $\bar{u}d$ current. Following exactly the same considerations as in Sec. 3.1 we obtain the following effective Hamiltonian:

$$\mathcal{H}_{\text{eff}}^{\bar{s} \rightarrow \bar{d}} = \frac{4G_F}{\sqrt{2}} V_{ud} V_{us}^* \left( C_1 Q_1^{\bar{s}u \bar{d}} + C_2 Q_2^{\bar{s}u \bar{d}} \right).$$ \hspace{1cm} \text{(85)}$$

However, when computing the renormalization of the operators in $\mathcal{H}_{\text{eff}}^{\bar{s} \rightarrow \bar{d}}$, additional diagrams \cite{22} (called penguin diagrams) arise by contracting the $u$ and $\bar{u}$ fields in $Q_{1,2}$ and attaching a gluon, as in Fig. 6 (a).

Let us think for a second on the possible structure of the amplitude induced by these diagrams. It is clearly a FCNC $\bar{s}_i \Gamma_\mu T^A_{ij} d_j$ coupling, which as we have seen cannot arise in the SM Lagrangian. Thus, the amplitude must be proportional to the matrix element of some higher dimensional operator. Indeed, gauge invariance forces us to impose $q^\mu \bar{s} \Gamma_\mu T^A_{ij} d = 0$, so we can identify two possible structures:

$$\bar{s} \left( q^2 \gamma_\mu - q_\mu q \right) T^A d \quad \text{and} \quad \bar{s} \sigma^{\mu\nu} q_\mu T^A d.$$ \hspace{1cm} \text{(86)}$$

The second one connects quarks of different helicity, so for massless quarks it cannot be generated. The first structure corresponds to the matrix element of operator

\footnote{For an instructive recollection of how the term “penguin diagram” made its way in particle physics see ref. \cite{23}.}
\[ \bar{s} \gamma^\mu T^A d D^\nu G^{A\mu} \]. Since the equations of motion imply

\[ D^\nu G^{A\mu} = g_s \sum_f \bar{q}_f \gamma_\mu T^A q_f, \quad (87) \]

with \( f \) any active quark flavour, this is equivalent to operator

\[ \bar{s} \gamma_\mu T^A d \sum_f \bar{q}_f \gamma^\mu T^A q_f. \quad (88) \]

A diagrammatic representation of the equation of motion can be obtained by attaching the gluon line to a quark line of flavour \( f \), as in Fig. 6 (b). When contracted with the gluon propagator, the \( q^\mu \) term in the FCNC vertex acts on the fermionic current giving zero for gauge invariance. The \( q^2 \) term, instead, cancels the pole of the gluon propagator, yielding precisely the local amplitude \( \bar{v}_s \gamma_\mu T^a v_d \sum_f \bar{u}_f \gamma^\mu T^a u_f \), i.e. the matrix element of the local operator in eq. (88). By power counting, diagrams (a) or (b) are logarithmically divergent, so we will find a \( 1/\epsilon \) term which must be subtracted, forcing us to enlarge the operator basis again. Since the new operator in eq. (88) must be renormalized itself, we must insert it in all the crosses in the diagrams in Figs. 4, 5 and 6. As in the case of the insertion of operator \( Q_1 \), gluonic corrections will change the original colour structure; moreover, the Dirac algebra will be different for the left- and right-handed components of the \( q_f \) current, so we will get different renormalization constants for the two components. All in all, we need to add four “penguin” operators:

\[ Q_3^{sd} = \bar{s}_L \gamma_\mu d_L \sum_f \bar{q}_{fL} \gamma^\mu q_{fL}, \quad (89) \]
\[ Q_4^{sd} = \bar{s}_L \gamma_\mu d_L^\beta \sum_f \bar{q}_{fL}^\beta \gamma^\mu q_{fL}^\alpha, \quad (90) \]
\[ Q_5^{sd} = \bar{s}_L \gamma_\mu d_L \sum_f \bar{q}_{fR} \gamma^\mu q_{fR}, \quad (91) \]
\[ Q_6^{sd} = \bar{s}_L \gamma_\mu d_L^\beta \sum_f \bar{q}_{fR}^\beta \gamma^\mu q_{fR}^\alpha. \quad (92) \]
When we insert operators $Q_{3-6}$ in the diagram in Fig. 6, we should remember that we are using $\overline{MS}$ which is a mass-independent renormalization scheme, so we should manually decouple quark flavours at a threshold $\mu_f \sim m_f$. The anomalous dimensions (as well as the $\beta$ function) depend on the number of active flavours $n_f(\mu)$, which also determines the summation range in operators $Q_{3-6}$.

As an explicit example of how penguin operators are generated, let us evaluate diagram (b) in Fig. 6 with the insertion of operator $Q^u_{1\alpha\beta d}$. We get, omitting the prefactor $\frac{4G_F}{\sqrt{2}} V_{ud} V^*_{us}$ and setting for simplicity $p = 0$,

$$A(b) = \int \frac{d^4k}{(2\pi)^4} \gamma^\mu P_L \frac{i}{k - q} (ig_s \gamma^\nu T_{ij}^\alpha i \Gamma \gamma^\mu P_L v_j^\alpha - i \pi_f^k (ig_s \gamma^\nu T_{ki}^\alpha)) u_f^l$$

\[= -ig^2 \frac{2}{q^2} \int \frac{d^4k}{(2\pi)^4} \gamma^\mu P_L \frac{k - q}{(k - q)^2} \gamma^\nu T_{ij}^\alpha \Gamma \gamma^\mu P_L v_j^\alpha \pi_f^k \gamma^\nu T_{ki}^\alpha u_f^l \]

\[= -ig^2 \frac{2}{q^2} I_{\alpha\beta} \gamma^\mu \gamma^\nu T_{ij}^\alpha \gamma^\beta \gamma^\mu P_L v_j^\alpha \pi_f^k \gamma^\nu T_{ki}^\alpha u_f^l , \]

with

$$I_{\alpha\beta} = \int \frac{d^4k}{(2\pi)^4} \frac{(k - q)^\alpha q^\beta}{(k - q)^2 k^2}$$

\[= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{(k - q)^\alpha q^\beta}{(k - q)^2 x + k^2 (1 - x)} \]

\[= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{(k - q)^\alpha q^\beta}{k^2 - 2q \cdot kx + q^2 x^2} \]

\[= \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{l^\alpha q^\beta x (1 - x)}{l^2 + q^2 x (1 - x)} \]

\[= -\frac{i}{16\pi^2} \int_0^1 dx \left( \frac{g^\alpha q^\beta}{2} q^2 x (1 - x) + q^\alpha q^\beta x (1 - x) \right) \left( \frac{1}{\epsilon} + \text{f.t.} \right) \]

\[= -\frac{i}{16\pi^2} \left( \frac{g^\alpha q^\beta}{2} q^2 + q^\alpha q^\beta \right) \left( \frac{1}{6\epsilon} \right) , \]

where we have used the Feynman parameterization and loop integrals reported in
Appendices A.1 and A.2 Substituting eq. (94) in eq. (93) we obtain

\[ -\frac{g_s^2}{16\pi^2} \frac{1}{12\epsilon} T'_{ij} \gamma_{i\mu} \left( \gamma^\alpha \gamma^\mu \gamma_\alpha + 2 \frac{g^\mu_\nu \not q}{q^2} \right) \gamma_\mu P_L v^j s^k P_{gj}^k \gamma_\nu T^a_{kl} u^l_f \]

\[ = \frac{\alpha_s}{4\pi} \frac{1}{6} T'_{ij} \gamma_\mu \left( \gamma^\alpha \gamma^\mu \gamma_\alpha + 2 \frac{g^\mu_\nu \not q}{q^2} \right) P_L v^j s^k P_{gj}^k \gamma_\nu T^a_{kl} u^l_f \]

\[ = -\frac{\alpha_s}{4\pi} \frac{1}{3} T'_{ij} \gamma_\mu \left( \gamma^\alpha \gamma^\mu \gamma_\alpha + 2 \frac{g^\mu_\nu \not q}{q^2} \right) P_L v^j s^k \gamma_\nu T^a_{kl} u^l_f \]

From eq. (95) we see explicitly that the FCNC gluon vertex is proportional to \( q^2 \gamma^\nu - q^\mu \not q \), and that we obtain in the end a local four-quark operator as implied by the equations of motion.

Let us now notice that also current-current operators with charm quarks can mix into penguin operators via the diagrams in Fig. 6. Thus, we should add to the effective Hamiltonian in eq. (85) the corresponding operators with charm, leading to

\[ H_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \Big[ V_{ud} V_{us}^* \left( C_1 Q_1^{\pi u d} + C_2 Q_2^{\pi u d} \right) + V_{cd} V_{cs}^* \left( C_1 Q_1^{\pi c d} + C_2 Q_2^{\pi c d} \right) \Big] \]

Now, when inserted in penguin diagrams, operators \( Q_{1,2}^{\pi u d} \) and \( Q_{1,2}^{\pi c d} \) give exactly the same divergent part, since the divergence is independent on the mass of the quarks running in the loop. Thus, the penguin operators will be generated with a coefficient proportional to \( V_{ud} V_{us}^* + V_{cd} V_{cs}^* = -V_{td} V_{ts}^* \). Taking everything into account, we end up with the following effective Hamiltonian:

\[ H_{\text{eff}}' = \frac{4G_F}{\sqrt{2}} \left\{ V_{ud} V_{us}^* \left( C_1 Q_1^{\pi u d} + C_2 Q_2^{\pi u d} \right) + V_{cd} V_{cs}^* \left( C_1 Q_1^{\pi c d} + C_2 Q_2^{\pi c d} \right) \right. \]

\[ \left. -V_{td} V_{ts}^* \sum_{i=3}^{6} C_i Q_i^{\pi d} \right\} \]

\[ = \frac{G_F}{\sqrt{2}} \left\{ V_{ud} V_{us}^* \left[ C_1 \left( Q_1^{\pi u d} - Q_1^{\pi c d} \right) + C_2 \left( Q_2^{\pi u d} - Q_2^{\pi c d} \right) \right] \right. \]

\[ \left. -V_{td} V_{ts}^* \left[ C_1 Q_1^{\pi c d} + C_2 Q_2^{\pi c d} + \sum_{i=3}^{6} C_i Q_i^{\pi d} \right] \right\} , \]

where we have used CKM unitarity to eliminate the \( V_{cd} V_{cs}^* \) term. Now, the \( V_{ud} V_{us}^* \) part contains current-current operators appearing in the GIM-suppressed difference of up and charm, which does not mix into penguin operators since the divergent part of the diagrams in Fig. 6 does not depend on the mass of the quark running in the loop.
Penguin operators $Q_{3-6}$ are instead generated by the RG running with the top CKM factor, due to the insertion in the diagrams in Fig. 6 of the operators $Q_{1,2}^{sd}$ in the last line of eq. (97). Their anomalous dimension also gets contributions from the insertion of $Q_{3-6}$ in the diagrams of Figs. 4, 5 and 6.

Had we performed the matching at $\mathcal{O}(\alpha_s)$, we would have encountered the diagram (c) in Fig. 6 in the full theory, with $u$, $c$ and $t$ quarks running in the loop. Diagrams (b) and (c) are of course not identical, since diagram (b) is logarithmically divergent while diagram (c) is finite (keep in mind the two powers of external momenta required by gauge invariance, see eq. (86)). However, if we differentiate with respect to external momenta or quark masses, then also diagram (b) becomes finite, and therefore the loop integration is dominated by momenta of the order of the external momenta and quark masses. After differentiating, we are thus allowed to replace the $W$ propagator in diagram (c) with $1/M_W^2$. In this way, we obtain the following relation between the amplitudes generated by diagrams (b) and (c) with quark $i$ running in the loop:

$$A^{(i)}_{(c)} = A^{(i)}_{(b)} + O\left(\frac{p^2, m_i^2}{M_W^2}\right) + K,$$

where $p$ denotes external momenta and $K$ is a constant term, independent on quark masses or momenta, proportional to the matrix element of operators $Q_{3-6}$. Eq. (98) implies that the contribution of $u$ and $c$ quarks cancels in the matching up to the constant $K$ and to negligible corrections of $O\left(\frac{p^2, m_i^2}{M_W^2}\right)$, while the top quark contribution generates a nontrivial contribution to $C_{3-6}^1(\mu_W)$ since in the effective theory we do not have diagram (b) with top quarks running in the loop [24,25].

### 3.3 Electroweak penguins

We may wonder what happens if we replace in the diagrams of Figs. 4, 6 gluon exchange with photon exchange. Electromagnetic corrections will also get a logarithmic enhancement, making them comparable to NLO QCD corrections, since $\alpha \log(\mu^2_W/\mu^2_h) \sim \alpha_s$. While we do not need to resum these logarithmic terms, we need to include them when working at NLO in QCD [26,27]. QED corrections bring a novelty: the operator basis must be enlarged, due to the electroweak penguin operators generated by the diagrams in Fig. 6 replacing the gluon with a photon [28]. While the FCNC photon coupling emerging from diagram (a) in Fig. 6 is equivalent to the gluonic one, the equation of motion introduces a quark charge dependence, giving rise to operators with flavour structure $\bar{q}d \sum q e_q q_f$, with $e_q$ the electric charge of flavour $q$. As in the case of QCD penguins, strong interaction corrections will generate a new colour structure, and the left- and right-handed components of the quark current will renormalize differently, so we need to add four more operators to the basis:

$$Q_{7}^{sd} = \frac{3}{2} \bar{s}_L \gamma_\mu d_L \sum_f e_q \bar{f}_R \gamma^\mu q_f R,$$

$$Q_{8}^{sd} = \frac{3}{2} \bar{s}_L \gamma_\mu d_L \sum_f e_q \bar{f}_R \gamma^\mu q_f R,$$
Let us now briefly discuss the matching for operators $Q_{7-10}$. Also in this case, these operators get a contribution from the top loop in the matching from diagram (c) in Fig. 3 with the exchange of a photon. However, this is not the only contribution. Indeed, one should also consider diagram (c) with the exchange of a $Z^0$ instead of a photon, and box diagrams with the exchange of two $W$ bosons. We do not dwell into the details of the matching, but there is an important point we would like to stress. It is instructive to consider diagram (c) with the exchange of a $Z^0$ in two steps: first, the evaluation of the FC $Z$ coupling from the loop integration, and then the evaluation of the $Z$ exchange. While $SU(3)_c \otimes U(1)_{em}$ gauge invariance forbids dimension four FCNC gluon or photon couplings, this is not the case for FCNC $Z$ couplings, which can indeed arise at dimension four: a $Z \bar{q} q_L \gamma^\mu q_L$ coupling can be generated once $SU(2)_L \otimes U(1)_Y$ is broken. However, for this to happen the diagram must “feel” the EW symmetry breaking, so the FC $Z$ coupling must vanish linearly with $m^2_q/M^2_W$ for small $m_q$. Thus, the loop is dominated by the top quark \[24\]. Having obtained a top-induced FC $Z$ vertex from the loop integration, we consider the $Z$ exchange in the lower part of diagram (c). Expanding the $Z$ propagator for small momenta in the same way as for the $W$ propagator in eq. (32) gives rise to local four-fermion operators, which can be identified with the (electroweak) penguins discussed above.

3.4 Anomalous dimension

We conclude this Section reporting the LO anomalous dimension for the full set of four-quark $\Delta F = 1$ operators listed above \[29\]–\[33\]:

$$\gamma_0 = \begin{pmatrix}
\frac{-6}{N_c} & 6 & \frac{-2}{3N_c} & \frac{2}{3} & \frac{-2}{3N_c} & \frac{2}{3} & 0 & 0 & 0 & 0 \\
6 & \frac{-6}{N_c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-2n_f}{3N_c} & \frac{2n_f}{3N_c} & \frac{2n_f}{3N_c} & \frac{-1}{3N_c} & \frac{2n_f}{3N_c} & \frac{2n_f}{3N_c} & \frac{2n_f}{3N_c} & \frac{-1}{3N_c} \\
0 & 0 & \frac{6}{N_c} & \frac{-6}{N_c} & \frac{-2n_f}{3N_c} & \frac{2n_f}{3N_c} & \frac{-1}{3N_c} & \frac{2n_f}{3N_c} & \frac{2n_f}{3N_c} & \frac{-1}{3N_c} \\
0 & 0 & \frac{n_f}{N_c} & \frac{2n_f}{N_c} & \frac{2n_f}{N_c} & \frac{2n_f}{N_c} & \frac{6-11N^2}{N_c} & \frac{6+11N^2}{N_c} & \frac{6+11N^2}{N_c} & \frac{6+11N^2}{N_c} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-2(n_u-n_d/2)}{3N_c} & \frac{2(n_u-n_d/2)}{3N_c} & \frac{-2(n_u-n_d/2)}{3N_c} & \frac{2(n_u-n_d/2)}{3N_c} & \frac{-2n_f}{N_c} & \frac{2n_f}{N_c} & \frac{-2n_f}{N_c} & \frac{2n_f}{N_c} \\
0 & 0 & \frac{2}{3N_c} & \frac{-2}{3N_c} & \frac{-2}{3N_c} & \frac{2}{3N_c} & \frac{-2}{3N_c} & \frac{-2}{3N_c} & \frac{2}{3N_c} & \frac{-2}{3N_c} \\
0 & 0 & \frac{-2(n_u-n_d/2)}{3N_c} & \frac{2(n_u-n_d/2)}{3N_c} & \frac{-2(n_u-n_d/2)}{3N_c} & \frac{2(n_u-n_d/2)}{3N_c} & \frac{-2n_f}{N_c} & \frac{2n_f}{N_c} & \frac{-2n_f}{N_c} & \frac{2n_f}{N_c} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-6}{N_c} & \frac{-6}{N_c} & \frac{-6}{N_c} & \frac{-6}{N_c} & \frac{-6}{N_c} & \frac{-6}{N_c} & \frac{-6}{N_c} & \frac{-6}{N_c}
\end{pmatrix} \tag{103}$$

5This $m^2_q/M^2_W$ suppression is sometimes called “hard GIM”, as opposed to the logarithmic dependence on quark masses which arises for example when matching on the dimension six FC gluon and photon couplings, the so-called “soft GIM”. We will discuss more in detail GIM suppression in Sec. 4.
where \( n_{f,u,d} = n_{f,u,d}(\mu) \) is the number of active (up- or down-type) quarks at the scale \( \mu \).

### 3.5 The \( \Delta I = 1/2 \) rule

As an example of the applications of the \( \Delta S = 1 \) effective Hamiltonian, let us consider the \( \Delta I = 1/2 \) rule. We start by writing down the decay amplitudes for a Kaon (anti-Kaon) to decay in two pions in terms of final states with different isospin. The two-pion state in S-wave must have a symmetric isospin wave function, so it can only be in \( I = 0 \) or \( I = 2 \) states. Denoting by \(| I, I_3 \rangle \) a state with isospin \( I \) and third component \( I_3 \), we can write

\[
\langle 2, 1 \rangle = \frac{1}{\sqrt{2}} (\langle \pi^+ \pi^0 \rangle + \langle \pi^0 \pi^+ \rangle),
\]

\[
\langle 2, 0 \rangle = \frac{1}{\sqrt{6}} (\langle \pi^+ \pi^- \rangle + \langle \pi^- \pi^+ \rangle + 2\langle \pi^0 \pi^0 \rangle),
\]

\[
\langle 0, 0 \rangle = \frac{1}{\sqrt{3}} (\langle \pi^+ \pi^- \rangle + \langle \pi^- \pi^+ \rangle - \langle \pi^0 \pi^0 \rangle).
\]

The initial state Kaon is a doublet, and the \( \Delta S = 1 \) effective Hamiltonian has \( I = 1/2 \) and \( I = 3/2 \) components. Coupling the initial state and the effective Hamiltonian through the Wigner-Eckart theorem, we have:

\[
\mathcal{H}_{\text{eff}} |K^+ \rangle = (\mathcal{H}_{3/2,1/2} + \mathcal{H}_{1/2,1/2}) |1/2, 1/2 \rangle
\]

\[
= \sqrt{3} A_{3/2} |2, 1 \rangle - \frac{1}{2} A_{3/2} |1, 1 \rangle + A_{1/2} |1, 1 \rangle,
\]

\[
\mathcal{H}_{\text{eff}} |K^0 \rangle = (\mathcal{H}_{3/2,1/2} + \mathcal{H}_{1/2,1/2}) |1/2, -1/2 \rangle
\]

\[
= \frac{1}{\sqrt{2}} A_{3/2} |2, 0 \rangle + \frac{1}{\sqrt{2}} A_{3/2} |1, 0 \rangle + \frac{1}{\sqrt{2}} A_{1/2} |1, 0 \rangle + \frac{1}{\sqrt{2}} A_{1/2} |0, 0 \rangle.
\]

Finally, from eqs. (104)-(108) we obtain

\[
A(K^+ \rightarrow \pi^+ \pi^0) = \frac{\sqrt{3}}{2\sqrt{2}} A_{3/2},
\]

\[
A(K^0 \rightarrow \pi^+ \pi^-) = \frac{1}{2\sqrt{3}} (A_{3/2} + \sqrt{2} A_{1/2}),
\]

\[
A(K^0 \rightarrow \pi^0 \pi^0) = \frac{1}{\sqrt{6}} (\sqrt{2} A_{3/2} - A_{1/2})
\]

It is convenient to define \( A_{0,2} = 1/\sqrt{6} A_{1/2,3/2} \) and to extract, without loss of generality, a CP-invariant phase \( \delta_{0,2} \) from each isospin amplitude, so that \( A_{0,2} \rightarrow A_{0,2}^* \) under CP:

\[
A(K^0 \rightarrow \pi^+ \pi^-) = A_0 e^{i\delta_0} + \frac{A_2}{\sqrt{2}} e^{i\delta_2},
\]

\[
A(K^0 \rightarrow \pi^0 \pi^0) = -A_0 e^{i\delta_0} + \sqrt{2} A_2 e^{i\delta_2},
\]

\[
A(K^+ \rightarrow \pi^+ \pi^0) = \frac{3}{2} A_2 e^{i\delta_2}.
\]
We now write the relevant $S$ matrix as:

$$S = \begin{pmatrix}
K \to K & K \to (\pi\pi)_0 & K \to (\pi\pi)_2 \\
(\pi\pi)_0 \to K & (\pi\pi)_0 \to (\pi\pi)_0 & (\pi\pi)_0 \to (\pi\pi)_2 \\
(\pi\pi)_2 \to K & (\pi\pi)_2 \to (\pi\pi)_0 & (\pi\pi)_2 \to (\pi\pi)_2
\end{pmatrix} \approx \begin{pmatrix}
1 & -iT_0 & -iT_2 \\
-iT(T_0) & e^{i\Delta_0} & 0 \\
-iT(T_2) & 0 & e^{i\Delta_2}
\end{pmatrix}, \tag{111}
$$

where we have assumed for simplicity elastic, isospin-conserving $\pi\pi$ strong interaction scattering, represented by the phases $\Delta_{0,2}$ for $I = 0, 2$ $\pi\pi$ states, and we are working at lowest order in weak interactions. $T$ denotes time reversal. Unitarity of the $S$ matrix implies:

$$(S^\dagger S)_{12} = 0 = -iT_0 + e^{i\Delta}iT(T_0)^*. \tag{112}$$

Writing, as we did in eq. (110),

$$T_i = A_ie^{i\delta_i}, \quad T(T_i) = CP(T_i) = A_i^*e^{i\delta_i}, \tag{113}$$

we obtain from eq. (112)

$$-iA_0e^{i\delta_0} + e^{i\Delta_0}i(A_0^*e^{i\delta_0})^* = 0 \quad \Rightarrow \delta_0 = \Delta_0/2, \tag{114}$$

so the CP-even phase of the weak decay amplitude is just half the phase describing strong-interaction scattering of the final state. This is known as Watson theorem, and can be generalized to the case of multi-channel strong-interactions unitarity (see ref. [35] for an example of application of Watson theorem to $D$ decays).

Experimentally, $\text{Re } A_2/\text{Re } A_0 \sim 1/22$, so $\Delta I = 1/2$ transitions happen at a much higher rate than $\Delta I = 3/2$. This is commonly denoted as the $\Delta I = 1/2$ rule. One of the most difficult problems in the study of weak decays, still lacking a complete solution, is in fact the theoretical prediction of the ratio $A_2/A_0$. Let us start from the $\Delta S = 1$ Hamiltonian in eq. (85). Considering the lowering operator for the third component of isospin, $I_-$, we have $I_-u = d, I_-d = -\bar{u}$, and $I_-d = I_-\bar{u} = 0$. Then the action of $I_-$ on the operator $Q_-$ of eq. (83) is given by

$$2I_-Q_- = \bar{s}_L\gamma^\mu(I_-u_L)\bar{u}_L\gamma_\mu d_L + \bar{s}_L\gamma^\mu u_L(I_-\bar{u}_L)\gamma_\mu d_L + \bar{s}_L\gamma^\mu u_L\bar{u}_L\gamma_\mu(I_-d_L) - \bar{s}_L\gamma^\mu(I_-u_L^3)\bar{u}_L\gamma_\mu d_L - \bar{s}_L\gamma^\mu u_L^3\bar{u}_L\gamma_\mu(I_-d_L) - \bar{s}_L\gamma^\mu u_L^3\bar{u}_L\gamma_\mu(I_-d_L^3) \tag{115}
$$

where in the last step we have fierzied the Dirac structure. Thus, $Q_-$ is the lower component of an isospin doublet. Doing the same exercise on $Q_+$ shows instead that $Q_+$ is an admixture of $I = 1/2$ and $I = 3/2$. Therefore, the enhancement of $C_-$ over $C_+$ due to RG evolution goes in the right direction to explain the $\Delta I = 1/2$ rule [30], although it can only account for about a factor of two in the amplitude ratio.

Another contribution to the $\Delta I = 1/2$ rule comes from QCD penguin operators $Q_{3,6}$ in eqs. (89)-(92) [22], since these operators are isospin doublets. Still, the effect must largely come from the matrix elements, since perturbative RG effects cannot bring the amplitude ratio close to the experimental value. Unfortunately, computing the relevant matrix elements from first principles with Lattice QCD is a tremendous
Indeed, this calculation poses all the most difficult challenges to lattice QCD calculations: final state interactions, chiral symmetry breaking, power divergences, disconnected diagrams, etc. Thus, it comes as no surprise that only very recently a pioneering lattice calculation of the matrix elements relevant for the \( \Delta I = 1/2 \) rule has been achieved \([36]\). According to this calculation, there is a large deviation from the Vacuum Insertion Approximation (VIA) in the matrix elements of current-current operators, causing a negative interference, and thus a large cancellation, in \( \Delta I = 3/2 \) matrix elements, which are therefore suppressed with respect to \( \Delta I = 1/2 \) ones. Such deviation from the VIA, with the corresponding negative interference, is also seen in \( \Delta S = 2 \) matrix elements \([37]\). However, the same calculation failed to reproduce the phase of the \( \Delta I = 1/2 \) amplitude, casting some doubts on the robustness of the estimate of final state interactions. Fortunately, with increased statistics and an improved treatment of the two-pion state, a much better agreement with the experimental value of \( \delta_0 \) was very recently obtained \([38]\). We are looking forward to the corresponding update of the results on the \( \Delta I = 1/2 \) rule, and hopefully to an independent confirmation from another lattice collaboration in the future.

4 Effective Hamiltonians for \( \Delta F = 2 \) processes

Let us now turn to the transitions that give the very stringent bounds reported in Fig. 1: \( \Delta F = 2 \) processes. In particular, let us consider \( \bar{s}d \to \bar{d}s \) transitions. Such FCNC processes cannot arise at the tree level in the SM, so we must consider one-loop contributions. These contributions must be finite, since renormalizability of the SM implies that no counterterm for FCNC amplitudes can arise. In the 't-Hooft-Feynman gauge we have the diagrams in Fig. 7.

Let us start by computing diagram (a). Neglecting external momenta, the amplitude reads

\[
i A^{(a)} = \int \bar{u}_s \left( \frac{ig_2}{\sqrt{2}} \right) \gamma_\mu P_L V_{i,s}^* \frac{i}{q - m_{u_j}} \left( \frac{ig_2}{\sqrt{2}} \right) \gamma_\nu P_L V_{u_j,d} v_d
\]

\[
\times \bar{v}_s \left( \frac{ig_2}{\sqrt{2}} \right) \gamma^\nu P_L V_{u_i,s} \frac{i}{q - m_{u_i}} \left( \frac{ig_2}{\sqrt{2}} \right) \gamma^\mu P_L V_{u_i,d} u_d \left( \frac{-i}{q^2 - M_W^2} \right)^2 \frac{d^4q}{(2\pi)^4}. \tag{116}
\]

Left-handed projectors kill the quark mass terms in the numerator of quark propagators, so we obtain

\[
i A^{(a)} = \frac{g_4^2}{4} \bar{u}_s V_{i,s}^* V_{u_i,d} V_{u_j,d} \bar{v}_s \gamma_\mu \gamma_\nu P_L v_d \bar{u}_s \gamma^\nu \gamma^\beta \gamma^\mu P_L u_d I_{ij}^{\alpha \beta}, \tag{117}
\]

with

\[
I_{ij}^{\alpha \beta} \equiv \int \frac{q_\alpha q_\beta}{(q^2 - M_W^2)^2(q^2 - m_{u_i}^2)(q^2 - m_{u_j}^2)} \frac{d^4q}{(2\pi)^4}. \tag{118}
\]

We can simplify the integral using partial fractioning in the form

\[
\frac{m_{u_j}^2 - m_{u_i}^2}{(q^2 - m_{u_i}^2)(q^2 - m_{u_j}^2)} = \frac{1}{q^2 - m_{u_i}^2} - \frac{1}{q^2 - m_{u_j}^2}, \tag{119}
\]
Figure 7: Feynman diagrams for $\bar{s}d \rightarrow \bar{d}s$ transitions in the SM.
obtaining

\[ I_{ij}^{\alpha \beta} = \frac{I_{i\alpha \beta} - I_{j\alpha \beta}}{m_{u_i}^2 - m_{u_j}^2} \]  

(120)

with

\[
I_{i\alpha \beta} = \int \frac{g_\alpha q_\beta}{(q^2 - M_W^2)(q^2 - m_{u_i}^2)} \, d^4 q
\]

\[ = \frac{g_{\alpha \beta}}{4} \int \frac{q^2 + (-m_{u_i}^2 + m_{u_i}^2)}{(q^2 - M_W^2)(q^2 - m_{u_i}^2)} \, d^4 q \]  

(121)

\[ = \frac{g_{\alpha \beta}}{4} m_{u_i}^2 \int \frac{1}{(q^2 - M_W^2)(q^2 - m_{u_i}^2)} \, d^4 q + K, \]

where \( K \) represents terms independent on \( m_{u_i}^2 \), which drop in \( I_{ij}^{\alpha \beta} \). Introducing Feynman parameters as in eq. (61), we obtain

\[
\int \frac{1}{(q^2 - M_W^2)(q^2 - m_{u_i}^2)} \, d^4 q = 2 \int_0^1 dx \int \frac{d^4 q}{(2\pi)^4} \frac{x}{[(q^2 - M_W^2)x + (q^2 - m_{u_i}^2)(1 - x)]^3}
\]

\[ = 2 \int_0^1 dx \int \frac{d^4 q}{(2\pi)^4} \frac{1}{[q^2 - M_W^2x - m_{u_i}^2(1 - x)]^3} \]

\[ = \frac{i}{16\pi^2} \int_0^1 dx \frac{x}{M_W^2x + m_{u_i}^2(1 - x)}
\]

\[ = \frac{i}{16\pi^2 M_W^2} \int_0^1 dx \frac{x}{x_i(1 - x)} + \frac{i}{16\pi^2 M_W^2} \int_0^1 dx \frac{x}{x_i + x(1 - x)}
\]

\[ = \frac{i}{16\pi^2 M_W^2} \int_0^1 dx \frac{(1 - x_i)x + x_i - x_i}{1 - x_i}
\]

\[ = \frac{i}{16\pi^2 M_W^2} \left( \frac{-x_i}{1 - x_i} \int_0^1 dx \frac{1}{1 - x_i} \frac{x_i}{x_i + (1 - x_i)} + \frac{1}{1 - x_i} \right)
\]

\[ = \frac{i}{16\pi^2 M_W^2} \left( \frac{1}{1 - x_i} + \frac{x_i \log x_i}{(1 - x_i)^2} \right), \]  

(122)

where \( x_i = m_{u_i}^2 / M_W^2 \). Thus, up to terms that do not depend on \( m_{u_i}^2 \), we have

\[ I_{i\alpha \beta} = \frac{-g_{\alpha \beta}}{4} \frac{i}{16\pi^2} J(x_i), \]  

(123)

with

\[ J(x_i) = \frac{x_i}{1 - x_i} + \frac{x_i^2 \log x_i}{(1 - x_i)^2}, \]  

(124)

and therefore

\[ I_{ij}^{\alpha \beta} = \frac{-g_{\alpha \beta}}{4 M_W^2} \frac{i}{16\pi^2} A(x_i, x_j), \]  

(125)

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with

\[ A(x_i, x_j) = \frac{J(x_i) - J(x_j)}{x_i - x_j}. \]  

(126)

We now turn to the Dirac structure

\[ \bar{u}_s \gamma_\mu \gamma_\alpha \gamma_\nu P_L v_d \bar{u}_s \gamma^\nu \gamma^\alpha \gamma^\mu P_L u_d \]  

(127)

and use again the Fierz identity in eq. (69) to obtain

\[ -\frac{1}{2} \bar{u}_s \gamma^\mu P_L v_d \bar{u}_s \gamma_\alpha \gamma_\nu \gamma_\mu P_R \gamma^\nu \gamma^\alpha \gamma^\mu u_d = 4 \bar{u}_s \gamma^\mu P_L v_d \bar{u}_s \gamma_\mu P_L u_d. \]  

(128)

Putting everything together we obtain the amplitude generated by diagram (a) as

\[
iA^{(a)} = -i \frac{4g^4}{16\pi^2 16M_W^2} \sum_{i,j=u,c,t} v_{is}^* V_{jd} V_{is} V_{jd} A(x_i, x_j) \bar{u}_s \gamma^\mu P_L v_d \bar{u}_s \gamma_\mu P_L u_d
\]

\[- \frac{iG_F^2 M_W^2}{2\pi^2} \sum_{i,j} \lambda_{id}^i \lambda_{sd}^j A(x_i, x_j) \bar{u}_s \gamma^\mu P_L v_d \bar{u}_s \gamma_\mu P_L u_d, \]

(129)

with \( \lambda_{sd}^i = V_{is}^* V_{id} \).

Diagram (b) in Fig. 7 is identical to diagram (a) if we exchange an incoming \( s \) antiquark with an outgoing \( s \) quark and vice versa:

\[
iA^{(b)} = \frac{iG_F^2 M_W^2}{2\pi^2} \sum_{i,j} \lambda_{id}^i \lambda_{sd}^j A(x_i, x_j) \bar{u}_s \gamma^\mu P_L u_d \bar{u}_s \gamma_\mu P_L v_d. \]

(130)

We now notice that the amplitudes generated by diagrams (a) and (b) can be written as the matrix element of a local operator, so we can introduce the following \( \Delta S = 2 \) effective Hamiltonian:

\[ H_{\text{eff}}^{\Delta S = 2} = C \bar{s} \gamma^\mu P_L d \bar{\sigma} \gamma_\mu P_L d, \]

(131)

with \( C \) a Wilson coefficient with mass dimension \(-2\). This effective Hamiltonian generates the following amplitude:

\[ iT^{H}(\bar{s}d \rightarrow \bar{d}s) = -iC \langle \bar{d}s | \bar{\sigma} \gamma^\mu P_L d \bar{\sigma} \gamma_\mu P_L d | \bar{s}d \rangle \]

\[ = -2iC \langle \bar{u}_s \gamma^\mu P_L v_d \bar{u}_s \gamma_\mu P_L u_d - \bar{u}_s \gamma^\mu P_L u_d \bar{u}_s \gamma_\mu P_L v_d \rangle. \]

(132)

Matching it with the amplitude in the full theory \( iA^{(a)+(b)} \) we obtain

\[ C^{(a)+(b)} = \frac{G_F^2 M_W^2}{4\pi^2} \sum_{i,j} \lambda_{id}^i \lambda_{sd}^j A(x_i, x_j). \]

(133)

Evaluating diagrams (c) to (h) in Fig. 7 we obtain

\[ C^{(c)+(d)} = C^{(e)+(f)} \]

\[ = - \frac{G_F^2 M_W^2}{4\pi^2} \sum_{i,j} \lambda_{sd}^i \lambda_{sd}^j A'(x_i, x_j)x_i x_j, \]

(134)

\[ C^{(g)+(h)} = \frac{1}{4} \frac{G_F^2 M_W^2}{4\pi^2} \sum_{i,j} \lambda_{sd}^i \lambda_{sd}^j A(x_i, x_j)x_i x_j, \]

(135)

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with
\[ A'(x_i, x_j) = \frac{J'(x_i) - J'(x_j)}{x_i - x_j}, \quad J'(x) = \frac{1}{1 - x} + \frac{x \log x}{(1 - x)^2}. \] (136)

Putting everything together we obtain
\[ C = \frac{G_F^2 M_W^2}{4\pi^2} \sum_{i,j} \lambda_{sd}^i \lambda_{sd}^j \overline{A}(x_i, x_j), \] (137)

where
\[ \overline{A}(x_i, x_j) = A(x_i, x_j) - x_i x_j A'(x_i, x_j) + \frac{1}{4} x_i x_j A(x_i, x_j). \] (138)

Next, we use CKM unitarity in the form
\[ \lambda_{sd}^u = -\lambda_{sd}^c - \lambda_{sd}^t \] (139)
to eliminate \( \lambda_{sd}^u \), and we obtain
\[ \sum_{i,j} \lambda_{sd}^i \lambda_{sd}^j \overline{A}(x_i, x_j) = (\lambda_{sd}^c + \lambda_{sd}^t)^2 \overline{A}(x_u, x_u) + 2 \lambda_{sd}^t \lambda_{sd}^c \overline{A}(x_c, x_t) \]
\[ + (\lambda_{sd}^c)^2 \overline{A}(x_c, x_c) + (\lambda_{sd}^t)^2 \overline{A}(x_t, x_t) \]
\[ - 2 \lambda_{sd}^t (\lambda_{sd}^c + \lambda_{sd}^t) \overline{A}(x_u, x_t) - 2 \lambda_{sd}^c (\lambda_{sd}^c + \lambda_{sd}^t) \overline{A}(x_c, x_u) \]
\[ = (\lambda_{sd}^t)^2 S_0(x_t) + (\lambda_{sd}^c)^2 S_0(x_c) + 2 \lambda_{sd}^t \lambda_{sd}^c S_0(x_c, x_t), \] (140)

with
\[ S_0(x_t) = \overline{A}(x_t, x_t) + \overline{A}(x_u, x_u) - 2 \overline{A}(x_u, x_t), \] (141)
\[ S_0(x_c) = \overline{A}(x_c, x_c) + \overline{A}(x_u, x_u) - 2 \overline{A}(x_u, x_c), \] (142)
\[ S_0(x_c, x_t) = \overline{A}(x_c, x_t) + \overline{A}(x_u, x_u) - \overline{A}(x_u, x_c) - \overline{A}(x_u, x_t). \] (143)

We finally obtain
\[ \mathcal{H}_{\text{eff}} = \frac{G_F^2 M_W^2}{4\pi^2} \left[ (\lambda_{sd}^t)^2 S_0(x_t) + (\lambda_{sd}^c)^2 S_0(x_c) + 2 \lambda_{sd}^t \lambda_{sd}^c S_0(x_c, x_t) \right] \bar{s} \gamma^\mu P_L d s \gamma^\mu P_L d. \] (144)

Notice that the \( S_0 \) functions are differences of \( \overline{A} \) functions with different arguments. If we Taylor-expand \( \overline{A} \) in powers of quark masses, the zeroth-order term cancels in \( S_0 \). Thus, for massless quarks no FCNC vertices arise, and the latter are suppressed by the GIM cancellation mechanism [1]. For small \( x \) the loop function \( S_0 \) vanishes linearly. Neglecting the contribution of the third family, the FCNC coupling in eq. (131) is proportional to
\[ G_F^2 M_W^2 \lambda^2 x_c = G_F^2 m_c^2 \lambda^2, \] (145)

where \( \lambda \) is the Wolfenstein parameter introduced in eq. (15). In other words, the (hard) GIM mechanism converts the effective \( \Delta S = 2 \) coupling from an \( \mathcal{O}(1/M_W^2) \) effect to an \( \mathcal{O}(m_c^2/M_W^2) \) one. Notice also that SM fermions do not decouple, since \( S_0 \) grows linearly for large \( x \); this non-decoupling explains the relevance of the top quark even in low-energy FCNC processes, and can be easily understood since the coupling to the would-be Goldstone bosons is proportional to fermion masses (or, more precisely, to Yukawa couplings).

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4.1 Locality and higher dimensional operators

The matching calculation we performed above might look as an academic exercise: what can we say on $K\bar{K}$ mixing from a matrix element between zero-momentum quarks with no strong interactions? Indeed, neglecting external momenta a local operator is generated by construction, but this is a reasonable approximation only if the dependence on external momenta is negligible. Let us now discuss this problem in some detail. First of all, we notice that diagrams containing up quarks only are in fact non-local contributions, which cannot be estimated by matching onto a local effective Hamiltonian. Indeed, up-quark contributions cancel in the matching against the matrix element of two $\Delta S = 1$ effective Hamiltonians. The latter represents the long-distance contribution to $K\bar{K}$ mixing which must be evaluated using some non-perturbative method. This point can be explicitly checked using the same argument discussed in Sec. 3.2. The diagrams in Fig. 7 become divergent when substituting the $W$ propagator with a local interaction, which corresponds to the matrix element of two $\Delta S = 1$ effective Hamiltonians. However, differentiating thrice with respect to quark masses and/or momenta, the diagram becomes convergent even when the $W$ propagator becomes local, allowing us to identify the diagrams in Fig. 7 with the matrix element of two $\Delta S = 1$ effective Hamiltonians up to a constant term, to a term proportional to $p^2/M^2_W$ and to a term proportional to $m^2_i/M^2_W$, where $p$ represents external momenta and $m_i$ the mass of the quark running in the loop.

However, if we look at the CP-odd part of the effective Hamiltonian we can drop the up-quark contribution since we can always choose a phase convention such that Im$\lambda_{sd}^u = 0$. In this convention, we have Im$\lambda_{sd}^c = -$ Im$\lambda_{sd}^t$, so that

$$\text{Im} (\lambda_{sd}^t)^2 = 2\text{Im} \lambda_{sd}^t \text{Re} \lambda_{sd}^t,$$

$$\text{Im} (\lambda_{sd}^c)^2 = -2\text{Im} \lambda_{sd}^t \text{Re} \lambda_{sd}^c,$$

$$\text{Im} \lambda_{sd}^c \lambda_{sd}^t = \text{Im} \lambda_{sd}^t (\text{Re} \lambda_{sd}^c - \text{Re} \lambda_{sd}^t),$$

leading to the following Wilson coefficient:

$$\frac{G^2_F M_W^2}{2\pi^2} \text{Im} \lambda_{sd}^t \left[ \text{Re} \lambda_{sd}^t (S_0(x_t) - S_0(x_t, x_c)) - \text{Re} \lambda_{sd}^c (S_0(x_c) - S_0(x_t, x_c)) \right].$$

We can indeed check that loops of up quarks drop in the differences of $S_0$ functions in eq. (149), leaving us with the following expressions:

$$S_0(x_t) - S_0(x_c, x_t) = \overline{A}(x_t, x_t) - \overline{A}(x_t, x_c) - \overline{A}(x_t, x_u) + \overline{A}(x_c, x_u),$$

$$S_0(x_c) - S_0(x_c, x_t) = \overline{A}(x_t, x_u) - \overline{A}(x_t, x_c) - \overline{A}(x_c, x_u) + \overline{A}(x_c, x_c).$$

Now,

$$S_0(x_t) - S_0(x_c, x_t) = \frac{m_t^2}{M_W^2} S_t(x_t, x_c),$$

$$S_0(x_c) - S_0(x_c, x_t) = \frac{m_c^2}{M_W^2} S_c(x_t, x_c).$$
with $S_{c,t}(x_t, x_c)$ non-vanishing in the limit $x_c \to 0$. Had we kept the dependence on external momenta $p$ in the evaluation of the loop functions, we would have obtained terms of $\mathcal{O}(p^2/m_W^2)$ (or higher) in $S_0$, corresponding to a correction of $\mathcal{O}(p^2/m_t^2)$ to eq. (152) and to an $\mathcal{O}(p^2/m_c^2)$ correction to eq. (153). The first one is fully negligible, but the second one is potentially relevant, since $m^2_K/m_c^2 \sim 10\%$. Fortunately, we have a systematic way to keep these corrections into account, since a contribution of $\mathcal{O}(p^2/M_W^2)$ to the amplitude can be described by the matrix element of a local operator of dimension eight. To perform the matching of the full amplitude onto the effective theory including dimension eight operators, we need to expand the diagrams we computed above at $\mathcal{O}(p^2/m_c^2)$. However, this is not enough since at dimension eight the operator basis includes an operator involving the commutator of two covariant derivatives,

$$g_s \bar{s} \gamma_\mu P_L \tilde{G}^{\mu\nu} d \bar{s} \gamma_\nu d,$$

which has vanishing matrix element on four-quark states. Therefore, we need to consider external states with four quarks and a gluon to complete the matching at dimension eight. In this way we can estimate the corrections of $\mathcal{O}(p^2/m_c^2)$, which turn out to be at the few percent level \cite{39,40}.

To summarize, the expansion in local operators is safe and systematically improvable by going to dimension eight operators for the CP violating part of the Hamiltonian, while the CP conserving one is dominated by long distance contributions, which must be evaluated as a long distance matrix element of two $\Delta S = 1$ effective Hamiltonians.

### 4.2 QCD corrections

The inclusion of LO QCD corrections goes exactly along the lines of Sec. 3.1.2, the only difference being that in the $\Delta F = 2$ case we do not need to introduce the second operator with a different colour structure since we can Fierz Dirac indices to fall back on the original operator in eq. (144):

$$\bar{s}^\alpha \gamma_\mu P_L d^3 \bar{s}^\beta \gamma_\mu P_L d^\alpha = \bar{s} \gamma_\mu P_L d \bar{s} \gamma_\mu P_L d. \tag{155}$$

The relevant anomalous dimension can then be obtained by a straightforward combination of the results in Sec. 3.1.2 yielding the same result as for $Q_+$ in eq. (83), namely

$$\gamma_0 = 6 \frac{N_c - 1}{N_c}, \tag{156}$$

leading to a suppression of $C(\mu_h)$ with respect to $C(M_W)$.

The calculation of NLO (and of NNLO) QCD corrections is more involved and goes beyond the scope of these lectures; the interested reader can find all the details in refs. \cite{19,41,47}.

### 4.3 $\Delta B = 2$ effective Hamiltonian

In the previous paragraphs we introduced the $\Delta S = 2$ effective Hamiltonian. If we consider instead $\bar{b} q \to \bar{q} b$ transitions, with $q = d, s$, we see that in this case at the scale
\(\mu_h \sim m_b\) up and charm quarks remain dynamical and thus their contribution cancels in the matching, leaving us with the top-quark contribution only. Recalling the relative size of the relevant CKM factors,

\[
\lambda^t_{bs} \sim \lambda^c_{bs} \gg \lambda^u_{bs}, \quad \lambda^t_{bd} \sim \lambda^c_{bd} \sim \lambda^u_{bd},
\]
and the relative size of the loop functions \(S_0(x_t)\) and \(S_0(x_c, x_t)\), we immediately realize that the top-charm contribution enters at \(\mathcal{O}(m_c^2/m_t^2)\) and is therefore fully negligible, leaving us with the effective Hamiltonian

\[
\mathcal{H} \Delta B = 2^{\text{eff}} = \frac{G^2 F M_W^2}{4\pi^2} (\lambda^t_{bs})^2 S_0(x_t) \bar{b}_c \gamma^\mu P_L \bar{b} \gamma^\mu P_L q.
\]

QCD corrections can be included up to NLO following the same line as for the top-top contribution to \(\Delta S = 2\). Electroweak corrections have been computed in ref. [48].

### 4.4 \(\Delta C = 2\) effective Hamiltonian

One could think of applying the same procedure as for \(\Delta S = 2\) processes to obtain an effective Hamiltonian for \(\Delta C = 2\) transitions. However, in this case the role played by the charm in \(\Delta S = 2\) goes to the strange quark, which is still dynamical at the charm scale. One would then be in a situation similar to \(\Delta B = 2\), except that for \(\Delta C = 2\) one has

\[
\lambda^d_{cu} \sim \lambda^s_{cu} \gg \lambda^b_{cu},
\]
and \(\lambda^b_{cu} / \lambda^s_{cu} \ll m_b/m_s\)\(^6\) so that the process is dominated by the matrix element of two \(\Delta C = 1\) effective Hamiltonians. Indeed, to an excellent approximation GIM cancellation in \(\Delta C = 2\) processes coincides with flavour SU(3).

### 4.5 \(\Delta F = 2\) Hamiltonians beyond the SM

While generalizing \(\Delta F = 1\) Hamiltonians beyond the SM, \(i.e.\) writing down the most general \(\Delta F = 1\) Hamiltonian including all dimension six, \(SU(3) \otimes U(1)_{em}\) gauge-invariant operators, increases the number of operators up to \(\sim 120\) [49,50], the number of independent operators that may arise is much smaller for \(\Delta F = 2\) transitions, so let us discuss this as an illustrative example of going beyond the SM.

There is a large degree of arbitrariness in the choice of the operator basis, since Fierz transformations can be used to get rid of a Dirac and colour structure in favour of a different one. As an example, let us choose the basis in ref. [51]:

\[
\begin{align*}
Q^{sdsd}_{1} &= \bar{s}_L \gamma^\mu d_L s_L \gamma^\nu d_L \\
Q^{sdsd}_{2} &= \bar{s}_L d_R \bar{s}_L d_R \\
Q^{sdsd}_{3} &= \bar{s}_L d_L \bar{s}_R \gamma^\nu d_R \\
Q^{sdsd}_{4} &= \bar{s}_L d_R \bar{s}_R \gamma^\nu d_L \\
Q^{sdsd}_{5} &= \bar{s}_L d_L \bar{s}_R \gamma^\nu d_L \\
\end{align*}
\]

\(6\)The strange quark mass in the denominator should actually be replaced by a suitable hadronic scale, making the ratio even smaller.
With respect to the SM, where only $Q_1^{sdsd}$ is present, we need to add two new Dirac structures, each one with two different colour structures, plus the operators obtained by the $L \leftrightarrow R$ transformation. As originally pointed out in ref. [52] and confirmed at NLO in refs. [49, 53, 54], the additional operators have large anomalous dimensions (especially $Q_4^{sdsd}$) which strongly enhance their coefficients at the hadronic scale with respect to the high scale, making them very important in phenomenological studies of $\Delta F = 2$ processes beyond the SM.

4.6 $\Delta F = 2$ matrix elements in the VIA

Before closing this Section on $\Delta F = 2$ effective Hamiltonians, let us briefly discuss how their matrix elements can be estimated in the VIA. VIA matrix elements are useful not only because they give a first (and not too rough) estimate of the matrix elements, but also because it is often easier and more accurate to compute the ratio of the full matrix element normalized to the VIA than the absolute matrix element. For this reason, matrix elements are often expressed in terms of VIA results times the so-called $B$-parameters, which in fact parameterize the ratio of the full matrix element with respect to the VIA one.

For the sake of concreteness, let us consider $\Delta S = 2$ processes. The SM effective Hamiltonian only contains $Q_1^{sdsd}$, whose VIA matrix element is given by

$$\langle K^0|\bar{s}\gamma^\mu P_L d\bar{s}\gamma_\mu P_L d|^K^0\rangle_{\text{VIA}} = 2\langle K^0|\bar{s}\gamma^\mu P_L d|0\rangle\langle 0|\bar{s}\gamma_\mu P_L d|^K^0\rangle$$

(161)

where the second term corresponds to Fierzed contractions with respect to the first term. Using

$$\langle 0|\bar{s}\gamma^\mu \gamma_5 d|^K^0\rangle = -i\gamma^\mu \frac{F_K}{\sqrt{2m_K}}$$

(162)

we obtain for the first term

$$\frac{1}{4} \frac{F_K^2 m_K^2}{2m_K} = \frac{1}{8} m_K F_K^2 .$$

(163)

For the second term we perform a colour Fierz transformation:

$$\delta_{\alpha\beta} \delta_{\gamma\delta} = 2T_{\alpha\delta}^a T_{\gamma\beta}^a + \frac{1}{3} \delta_{\alpha\delta} \delta_{\gamma\beta}$$

(164)

getting

$$\langle K^0|\bar{s}^a \gamma^\mu P_L d^a|0\rangle\langle 0|\bar{s}^a \gamma_\mu P_L d^a|^K^0\rangle = \frac{1}{3} \langle K^0|\bar{s}\gamma^\mu P_L d|0\rangle\langle 0|\bar{s} \gamma_\mu P_L d|^K^0\rangle$$

(165)

$$+ 2\langle K^0|\bar{s}^a T^a \gamma^\mu P_L d^a|0\rangle\langle 0|\bar{s}^a T^a \gamma_\mu P_L d^a|^K^0\rangle .$$

The second term vanishes and the first one reduces to eq. (163), so that in the end we obtain

$$\langle K^0|\bar{s}\gamma^\mu P_L d\bar{s}\gamma_\mu P_L d|^K^0\rangle_{\text{VIA}} = 2(1 + \frac{1}{3}) \frac{1}{8} F_K^2 m_K = \frac{1}{3} F_K^2 m_K .$$

(166)
In general we can therefore write

\[
\langle K^0 | \bar{\psi} \gamma^\mu P_L \bar{\psi} \gamma_\mu P_L | K^0 \rangle = \langle K^0 | \bar{\psi} \gamma^\mu P_L \bar{\psi} \gamma_\mu P_L | K^0 \rangle_{\text{VIA}} B_K
\]

\[
= \frac{1}{3} F_K^2 m_K B_K .
\]

It is now interesting to look at the VIA matrix elements of the additional operators that arise beyond the SM. From eq. (160) we see that the operators \( Q_i^{sdsd} \) for \( i > 1 \) are built by products of scalar/pseudoscalar densities. We have

\[
\partial_\mu \langle 0 | \bar{\psi} \gamma^\mu \gamma_5 d | K^0 \rangle = \frac{-i}{m^2 F_K} \langle 0 | \bar{\psi} \gamma_5 d | K^0 \rangle
\]

where in the first case we applied the derivative to the quark bilinear while in the second case we applied it to the whole matrix element. Using eq. (168) we obtain

\[
\langle K^0 | \bar{\psi} \gamma_5 d | 0 \rangle \langle 0 | \bar{\psi} \gamma_5 d | K^0 \rangle = - \left( \frac{m_K}{m_s + m_d} \right)^2 \frac{m_K F_K^2}{2} m^2 K
\]

which for Kaons is chirally enhanced by one order of magnitude since \( m_K \sim 3(m_s + m_d) \). Combining this chiral enhancement with the RG enhancement one sees that these operators play a crucial role in \( \Delta S = 2 \) processes beyond the SM. We will return to this point later. For completeness, we write down the matrix elements for all the operators in eq. (160) in the VIA:

\[
\langle K^0 | Q_1 | K^0 \rangle = \frac{1}{3} m_K F_K^2 ,
\]

\[
\langle K^0 | Q_2 | K^0 \rangle = - \frac{5}{24} \left( \frac{m_K}{m_s + m_d} \right)^2 m_K F_K^2 ,
\]

\[
\langle K^0 | Q_3 | K^0 \rangle = \frac{1}{24} \left( \frac{m_K}{m_s + m_d} \right)^2 m_K F_K^2 ,
\]

\[
\langle K^0 | Q_4 | K^0 \rangle = \left[ \frac{1}{24} + \frac{1}{4} \left( \frac{m_K}{m_s + m_d} \right)^2 \right] m_K F_K^2 ,
\]

\[
\langle K^0 | Q_5 | K^0 \rangle = \left[ \frac{1}{8} + \frac{1}{12} \left( \frac{m_K}{m_s + m_d} \right)^2 \right] m_K F_K^2 .
\]

A word of caution is necessary at this point. Operators \( Q_{4,5} \) have VIA matrix elements that contain two contributions, one from pseudoscalar density matrix elements and one from axial vector currents. To define the corresponding \( B \)-parameters it is convenient to choose as normalization just the pseudoscalar density contributions. However, this corresponds to having \( B \neq 1 \) in the VIA [55].
5 Effective Hamiltonians at work: meson-antimeson mixing and CP violation

Having discussed the basics of $\Delta F = 1$ and $\Delta F = 2$ effective Hamiltonians, let us now use them to study meson-antimeson mixing and CP violation.

5.1 Meson-antimeson mixing

There are four neutral mesons which differ from their antiparticles just because of their flavour quantum numbers: $K^0$, $D^0$, $B_d$ and $B_s$ mesons. While strong and electromagnetic interactions preserve flavour, the full Hamiltonian does not, due to flavour-changing weak interactions. Therefore, its eigenstates will be superpositions of mesons and antimesons, giving rise to the phenomenon of meson-antimeson oscillations, which entails a difference of mass and width of the two eigenstates [56]. Let us first write down the formalism for a generic neutral meson, which we denote by $M^0$, and then specialize to the four cases above, in which different simplifying assumptions can be made.

Notice that a CP transformation takes a neutral meson into its antiparticle with an arbitrary phase shift $\xi$:

\[
\text{CP} |M^0\rangle = e^{i\xi} |\overline{M}^0\rangle,
\]

\[
\text{CP} |\overline{M}^0\rangle = e^{-i\xi} |M^0\rangle.
\]

The matrix elements of the full Hamiltonian between $M^0$ and $\overline{M}^0$ states can be written as a two-by-two complex matrix:

\[
\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \equiv \begin{pmatrix} \langle M^0 | H | M^0 \rangle & \langle M^0 | H | \overline{M}^0 \rangle \\ \langle \overline{M}^0 | H | M^0 \rangle & \langle \overline{M}^0 | H | \overline{M}^0 \rangle \end{pmatrix} \equiv \hat{M} - \frac{i}{2} \hat{\Gamma},
\]

where in the last equality we have split the complex matrix $\hat{H}$ in its Hermitian ($\hat{M}$) and anti-Hermitian ($-i/2\hat{\Gamma}$) parts.

CPT invariance requires $M_{11} = M_{22}$ and $\Gamma_{11} = \Gamma_{22}$, while it does not constrain the off-diagonal matrix elements. CP invariance instead connects off-diagonal elements among themselves:

\[
H_{21} = \langle \overline{M}^0 | H | M^0 \rangle \xrightarrow{\text{CP}} e^{i\xi} \langle M^0 | H^{\text{CP}} | \overline{M}^0 \rangle e^{i\xi},
\]

so that CP conservation ($H^{\text{CP}} = H$) implies

\[
H_{21} = e^{2i\xi} H_{12} \Rightarrow |H_{21}| = |H_{12}| \Rightarrow \text{Im} \left( M_{12}^* \Gamma_{12} \right) = 0 \Rightarrow \text{Im} \left( \frac{\Gamma_{12}}{M_{12}} \right) = 0.
\]

The eigenvalue equation reads, assuming CPT invariance,

\[
\det \left( \hat{H} - \lambda \mathbb{1} \right) = 0 = (H_{11} - \lambda)^2 - H_{12} H_{21} \Rightarrow \lambda = H_{11} \pm \sqrt{H_{12} H_{21}}.
\]
Defining
\[ \lambda_{1,2} = m_{1,2} - i/2\Gamma_{1,2}, \quad m = \frac{m_1 + m_2}{2}, \quad \Gamma = \frac{\Gamma_1 + \Gamma_2}{2} , \tag{180} \]
\[ \Delta m = m_1 - m_2, \quad \Delta \Gamma = \Gamma_1 - \Gamma_2, \quad x = \frac{\Delta m}{\Gamma}, \quad y = \frac{\Delta \Gamma}{2\Gamma}, \]
one can alternatively label the two eigenstates by their mass, i.e. defining \( \Delta m = m_H - m_L \) to be positive (here \( H \) stands for heavy and \( L \) for light), or by their width, i.e. defining \( \Delta \Gamma = \Gamma_S - \Gamma_L \) to be positive (here \( S \) stands for short-lived and \( L \) for long-lived).

We have
\[ \Delta \lambda = \lambda_1 - \lambda_2 = \Delta m - \frac{i}{2} \Delta \Gamma = 2\sqrt{H_{12}H_{21}} , \tag{181} \]
\[ (\Delta m)^2 - \frac{1}{4}(\Delta \Gamma)^2 - i\Delta m \Delta \Gamma = 4H_{12}H_{21} = \]
\[ = 4 \left( |M_{12}^2| - \frac{1}{4} |\Gamma_{12}|^2 \right) - 4i \text{Re}(M_{12}\Gamma_{12}^*) . \tag{182} \]

Taking real and imaginary parts we obtain
\[ (\Delta m)^2 - \frac{1}{4}(\Delta \Gamma)^2 = 4 \left( |M_{12}|^2 - \frac{1}{4} |\Gamma_{12}|^2 \right) , \tag{183} \]
\[ \Delta m \Delta \Gamma = 4 \text{Re}(M_{12}\Gamma_{12}^*) . \tag{184} \]

Notice that
\[ (\Delta m)^2 = 4(\text{Re}\sqrt{H_{12}H_{21}})^2 = 2|H_{12}H_{21}| + 2\text{Re}H_{12}H_{21} , \tag{185} \]
so that
\[ (|H_{12}| + |H_{21}|)^2 = |H_{12}|^2 + |H_{21}|^2 + 2|H_{12}H_{21}| \]
\[ = |H_{12}|^2 + |H_{21}|^2 - 2\text{Re}(H_{12}H_{21}) + (\Delta m)^2 \]
\[ = |H_{12} - H_{21}^*|^2 + (\Delta m)^2 \]
\[ = |\Gamma_{12}|^2 + (\Delta m)^2 . \tag{186} \]

Let us write the eigenstates as
\[ |M_{1,2}⟩ = p|M^0⟩ ± q|M^0⟩ , \quad \text{with} \quad |p|^2 + |q|^2 = 1 . \tag{187} \]

Then we have
\[ H_{11}p ± H_{12}q = \lambda_{1,2}p \Rightarrow H_{11} ± \frac{q}{p} H_{12} = H_{11} ± \sqrt{H_{12}H_{21}} \]
\[ \Rightarrow \frac{q}{p} = \sqrt{\frac{H_{21}}{H_{12}}} = \frac{2M_{12}^* - i\Gamma_{12}^*}{\Delta m - \frac{i}{2} \Delta \Gamma} = \frac{\Delta m - \frac{i}{2} \Delta \Gamma}{2M_{12} - i\Gamma_{12}} . \tag{188} \]

Using eq. (178) we see that CP conservation implies
\[ \frac{q}{p} = \sqrt{\frac{H_{21}}{H_{12}}} = \sqrt{\frac{H_{12}e^{2i\xi}}{H_{12}}} = e^{i\xi} , \quad \text{so that} \quad \left| \frac{q}{p} \right| = 1 . \tag{189} \]
Thus, \[ \left| \frac{q}{p} \right| \neq 1 \] implies CP violation, \hspace{1cm} (190)

usually denoted as CP violation in mixing.

It is useful to define
\[ x_{12} = \frac{2|M_{12}|}{\Gamma} \quad \text{and} \quad \Phi_{12} = \arg \left( \frac{\Gamma_{12}}{M_{12}} \right), \] \hspace{1cm} (191)

so that
\[ \left| \frac{q}{p} \right| = \frac{|2M_{12}^* - i\Gamma_{12}^*|}{\Gamma |x - iy|} = \frac{\sqrt{x_{12}^2 + y_{12}^2 - 2x_{12}y_{12} \sin \Phi_{12}}}{\sqrt{x^2 + y^2}}, \] \hspace{1cm} (192)

implying that
\[ \left| \frac{q}{p} \right| \neq 1 \iff \sin \Phi_{12} \neq 0. \] \hspace{1cm} (193)

Finally, CP violation in meson-antimeson mixing can also be expressed in terms of the parameter \( \delta \) defined as
\[ \delta \equiv \frac{|H_{12} - H_{21}|}{|H_{12}| + |H_{21}|} = \langle M_1 M_2 \rangle = |p|^2 - |q|^2 = \frac{1 - |\frac{q}{p}|^2}{1 + |\frac{q}{p}|^2}. \] \hspace{1cm} (194)

One has
\[ 1 + \delta^2 = 1 + \frac{|H_{12}|^2 + |H_{21}|^2 - 2|H_{12}| |H_{21}|}{|H_{12}|^2 + |H_{21}|^2 + 2|H_{12}| |H_{21}|} = \frac{2|H_{12}|^2 + |H_{21}|^2}{(|H_{12}| + |H_{21}|)^2}, \] \hspace{1cm} (195)

so that
\[ \delta = \frac{1}{2} \frac{|H_{12}|^2 - |H_{21}|^2}{|H_{12}|^2 + |H_{21}|^2} = \frac{2|M_{12}\Gamma_{12}| \sin \Phi_{12}}{4|M_{12}|^2 + |\Gamma_{12}|^2}, \] \hspace{1cm} (197)

and
\[ \delta = \frac{2|M_{12}\Gamma_{12}| \sin \Phi_{12}}{(\Delta m)^2 + |\Gamma_{12}|^2}. \] \hspace{1cm} (198)

Let us also write down the expressions for \( |M_{12}| \), \( |\Gamma_{12}| \) and \( \Phi_{12} \) in terms of \( \Delta m \), \( \Delta \Gamma \) and \( \delta \):
\[ |M_{12}| = \sqrt{\frac{4(\Delta m)^2 + \delta^2(\Delta \Gamma)^2}{16(1 - \delta^2)}} \sim \frac{\Delta m}{2} + O(\delta^2), \] \hspace{1cm} (199)

\[ |\Gamma_{12}| = \sqrt{\frac{(\Delta \Gamma)^2 + 4\delta^2(\Delta m)^2}{4(1 - \delta^2)}} \sim \frac{\Delta \Gamma}{2} + O(\delta^2), \] \hspace{1cm} (200)

\[ \sin \Phi_{12} = \frac{4|\Gamma_{12}|^2 + 16|M_{12}|^2 - (4(\Delta m)^2 + (\Delta \Gamma)^2)|q/p|^2}{16|M_{12}\Gamma_{12}|} \sim \frac{4(\Delta m)^2 + (\Delta \Gamma)^2}{2\Delta m \Delta \Gamma} \delta + O(\delta^2). \] \hspace{1cm} (201)
and viceversa:

\[(\Delta m)^2 = \frac{4|M_{12}|^2 - \delta^2|\Gamma_{12}|^2}{1 + \delta^2} \sim 4|M_{12}|^2 + \mathcal{O}(\delta^2), \quad (202)\]

\[(\Delta \Gamma)^2 = \frac{4|\Gamma_{12}|^2 - 16\delta^2|M_{12}|^2}{1 + \delta^2} \sim 4|\Gamma_{12}|^2 + \mathcal{O}(\delta^2). \quad (203)\]

Notice that the transformation in eq. (187) is unitary only if \(|p|^2 - |q|^2 = \delta = 0\), i.e. if CP is conserved in the mixing. Therefore, if CP is violated, one must be careful in defining outgoing \(M_{12}\) states using the so-called reciprocal basis \([57–64]\):

\[
\langle \tilde{M}_{1,2} | = \frac{q\langle M^0 | \pm p\langle \bar{M}^0 |}{2pq}. \quad (204)\]

### 5.2 Time evolution of mixed meson states

Having obtained the eigenstates of the Hamiltonian in eqs. (179) and (180), we can write down the time evolution of a state initially produced as an \(M^0\) or as an \(\bar{M}^0\). We start from the time evolution of the eigenstates,

\[
|M_{1,2}(t)\rangle = e^{-i\lambda_{1,2}t} |M_{1,2}(0)\rangle, \quad (205)\]

and use eq. (187) to rotate back to the \(\tilde{M}^0\):

\[
|M^0(t)\rangle = \frac{1}{2p} (M_1(t) + M_2(t)) = g_+(t)|M^0\rangle + \frac{q}{p} g_-(t)|\bar{M}^0\rangle, \quad (206)\]

\[
|\bar{M}^0(t)\rangle = \frac{1}{2q} (M_1(t) - M_2(t)) = \frac{p}{q} g_-(t)|M^0\rangle + g_+(t)|\bar{M}^0\rangle, \quad (207)\]

with

\[
g_\pm(t) = \frac{e^{-i\lambda_1t} \pm e^{-i\lambda_2t}}{2}. \quad (208)\]

The probability that a meson initially produced as a \(\tilde{M}^0\) remains such at time \(t\) is given by \(|g_+(t)|^2\), while the probabilities of an \(M^0\) becoming an \(\bar{M}^0\) and viceversa are not equal to each other if CP is violated:

\[
\mathcal{P}(M^0(0) \rightarrow \bar{M}^0(t))) = \left|\frac{q}{p}\right|^2 |g_-(t)|^2, \quad (209)\]

\[
\mathcal{P}(\bar{M}^0(0) \rightarrow M^0(t))) = \left|\frac{p}{q}\right|^2 |g_-(t)|^2. \quad (210)\]
We have
\[
|g_\pm(t)|^2 = \frac{e^{-\Gamma_1 t} + e^{-\Gamma_2 t} \pm 2e^{-\Gamma_1 t} \cos(\Delta m t)}{4} 
\]
(211)
\[
= \frac{e^{-\Gamma_1 t}}{2} (\cosh(\Delta t/2) \pm \cos(\Delta m t)) ,
\]
\[
g_+(t)g^*_-(t) = \frac{e^{-\Gamma_1 t} - e^{-\Gamma_2 t} - 2e^{-\Gamma_1 t} \sin(\Delta m t)}{4}
\]
(212)
\[
= -\frac{e^{-\Gamma_1 t}}{2} (\sinh(\Delta t/2) - i \sin(\Delta m t)) .
\]

5.3 Observables for meson-antimeson mixing

Since the $M^0$ mesons are unstable, we must consider their weak decays in building observables related to meson mixing. The probabilities in eqs. (209) and (210) can be directly accessed using semileptonic decays, since for decays of down-type quarks one has $M^0 \not\to \ell \bar{\nu}_\ell X$ and $\bar{M}^0 \not\to \bar{\ell} \nu_\ell X$, where $X$ represents an unspecified hadronic final state. Therefore, those decays can only happen through mixing, and one can define the semileptonic CP asymmetry as the difference of the number of semileptonic decays to wrong sign leptons in $\bar{M}^0$ decays normalized to the total number of such decays:
\[
a_{SL} \equiv \frac{N(M^0 \to \ell \bar{\nu}_\ell X) - N(M^0 \to \ell \nu_\ell X)}{N(M^0 \to \ell \bar{\nu}_\ell X) + N(M^0 \to \ell \nu_\ell X)} .
\]
(213)
Assuming that $M^0$ and $\bar{M}^0$ are produced in equal number $N_0$ and CP invariance of the semileptonic decay amplitude $A$, one has
\[
N(M^0 \to \ell \bar{\nu}_\ell X) = N_0 |A|^2 \left| \frac{q}{p} \right|^2 \int_0^\infty |g_-(t)|^2 dt ,
\]
(214)
\[
N(\bar{M}^0 \to \bar{\ell} \nu_\ell X) = N_0 |A|^2 \left| \frac{p}{q} \right|^2 \int_0^\infty |g_-(t)|^2 dt .
\]
(215)
All factors except for the mixing parameters drop in the ratio, leading to
\[
a_{SL} = \left| \frac{\frac{q}{p}}{\frac{p}{q}} \right|^2 = \frac{1 - \left| \frac{q}{p} \right|^4}{1 + \left| \frac{q}{p} \right|^4} .
\]
(216)

In general, the decay amplitude into a given final state $|f\rangle$ and its CP-conjugate $\bar{f}\rangle = e^{-i\xi f} \mathcal{CP}|f\rangle$ in the SM can be always written in the following form\footnote{We prefer to keep the equations in a symmetric form, keeping in mind that an overall phase could be dropped since it is physically irrelevant.}
\[
A(M \to f) \equiv A_f = A_f e^{i\phi_f} e^{i\delta_f} (1 + r_f e^{i\phi_f} e^{i\delta_f}) ,
\]
(217)
\[
A(\bar{M} \to \bar{f}) \equiv A_{\bar{f}} = e^{-i\xi_f} e^{i\xi_f} A_f e^{-i\phi_f} e^{i\delta_f} (1 + r_f e^{-i\phi_f} e^{i\delta_f}) ,
\]
(218)
\[
A(M \to \bar{f}) \equiv A_{\bar{f}} = A_f e^{-i\phi_f} e^{i\delta_f} (1 + r_f e^{i\phi_f} e^{i\delta_f}) ,
\]
(219)
\[
A(\bar{M} \to f) \equiv A_f = e^{-i\xi_f} e^{i\xi_f} A_f e^{-i\phi_f} e^{i\delta_f} (1 + r_f e^{-i\phi_f} e^{i\delta_f}) ,
\]
(220)
with $A_f, A_f, r_f$ and $r_f$ real, and $\mathcal{CP}|M| = e^{i\xi}|M|$. Indeed, using CKM unitarity where needed one can have at most two independent CKM factors in each decay amplitude, corresponding for example in eq. (217) to $A_f$ and $A_f r_f$. For later convenience, we have written the second amplitude as a multiplicative factor, since in several cases one has $|r_f| \ll 1$ and an expansion in $r_f$ can be performed. We have written explicitly the CP-odd weak phases $\phi_i$ and the CP-even strong phases $\delta_i$.

We see that

$$\phi_{r_f} \neq 0 \text{ and } \delta_{r_f} \neq 0 \Rightarrow A_f \neq \overline{A_f}. \quad (221)$$

This is usually denoted as “direct” CP violation or CP violation in the decay. The corresponding CP asymmetry can be written as

$$A_{\mathcal{CP}}(f) = \frac{|A(M \rightarrow \overline{f})|^2 - |A(M \rightarrow f)|^2}{|A(M \rightarrow \overline{f})|^2 + |A(M \rightarrow f)|^2} = \frac{2r_f \sin \phi_{r_f} \sin \delta_{r_f}}{1 + r_f^2 + 2r_f \cos \phi_{r_f} \cos \delta_{r_f}}. \quad (222)$$

For neutral meson decays, one can consider the case of a final state which is a CP eigenstate with eigenvalue $\eta_f$:

$$A(M \rightarrow f_{\mathcal{CP}}) = A_f e^{i\phi_f} e^{i\delta_f} (1 + r_f e^{i\phi_f} e^{i\delta_f}), \quad (223)$$

$$A(M \rightarrow \overline{f}_{\mathcal{CP}}) = e^{-i\xi} \eta_f A_f e^{-i\phi_f} e^{i\delta_f} (1 + r_f e^{-i\phi_f} e^{i\delta_f}). \quad (224)$$

It is useful to introduce

$$\lambda_f = \frac{q}{p} \overline{A_f} = \frac{q}{p} \frac{e^{-i\xi} e^{-i\xi_f} A_f e^{-i\phi_f} e^{i\delta_f} (1 + r_f e^{-i\phi_f} e^{i\delta_f})}{A_f e^{i\phi_f} e^{i\delta_f} (1 + r_f e^{i\phi_f} e^{i\delta_f})}, \quad (225)$$

which is manifestly rephasing invariant since for $|M^0 \rangle \rightarrow e^{i\Xi}|M^0 \rangle$ and $|\overline{M}^0 \rangle \rightarrow e^{i\Xi}|\overline{M}^0 \rangle$ we have

$$\frac{q}{p} \rightarrow e^{i(\Xi - \Xi_f)} \frac{q}{p}, \quad \overline{A_f} \rightarrow e^{i(\Xi - \Xi_f)} \overline{A_f} \overline{A_f} \quad (226)$$

(or equivalently $\xi \rightarrow \xi + \Xi - \Xi$), so that $\lambda_f \rightarrow \lambda_f$.

For decays to a CP eigenstate this simplifies to

$$\lambda_{f_{\mathcal{CP}}} = \frac{q}{p} \frac{e^{-i\xi} \eta_f e^{-2i\phi_f} (1 + r_f e^{-i\phi_f} e^{i\delta_f})}{(1 + r_f e^{i\phi_f} e^{i\delta_f})} \quad (227)$$

$$= \frac{q}{p} e^{-i\xi} \eta_f e^{-2i\phi_f} (1 - 2i r_f e^{i\delta_f} \sin \phi_{r_f} + \mathcal{O}(r_f^2)).$$

CP conservation implies $q/p = e^{i\xi}$ (see eq. (178)) and $\phi_{f, r_f} = 0$, i.e. $\text{Im} \lambda_{f_{\mathcal{CP}}} = 0$. Therefore,

$$\text{Im} \lambda_{f_{\mathcal{CP}}} \neq 0 \text{ implies CP violation.} \quad (228)$$

This form of CP violation requires the interference between mixing and decay. Indeed, $\lambda_{f_{\mathcal{CP}}}$ determines the time evolution of $\hat{M}^0$ decays into $f_{\mathcal{CP}}$. Using eqs. (217), (220) and
we can write

\[
\Gamma(M^0(t) \to f_{CP}) = |g_+(t)|^2 |A_{f_{CP}}|^2 + \left| \frac{q}{p} \right|^2 |g_-(t)|^2 |\overline{A}_{f_{CP}}|^2 \\
+ 2\text{Re} \left( g_+^*(t)g_-(t) \frac{q}{p} A_{f_{CP}}^* \overline{A}_{f_{CP}} \right) \\
= |A_{f_{CP}}|^2 \left[ |g_+(t)|^2 + |\lambda_{f_{CP}}|^2 |g_-(t)|^2 + 2\text{Re} (g_+^*(t)g_-(t)\lambda_{f_{CP}}) \right] \\
= |A_{f_{CP}}|^2 e^{-\Gamma t} \left[ (1 + |\lambda_{f_{CP}}|^2) \cosh(\Delta m t) \right. \\
+ \left. (1 - |\lambda_{f_{CP}}|^2) \cos(\Delta m t) \right. \\
- \left. 2\text{Re} \lambda_{f_{CP}} \sinh(\Delta t/2) + 2\text{Im} \lambda_{f_{CP}} \sin(\Delta m t) \right],
\]

\[
\Gamma(M^0(t) \to f_{CP}) = \left| \frac{p}{q} \right|^2 |g_-(t)|^2 |A_{f_{CP}}|^2 + |g_+(t)|^2 |\overline{A}_{f_{CP}}|^2 \\
+ 2\text{Re} \left( g_+^*(t)g_-(t) \frac{p}{q} A_{f_{CP}} \overline{A}_{f_{CP}}^* \right) \\
= |\overline{A}_{f_{CP}}|^2 \left[ |g_+(t)|^2 + |\lambda_{f_{CP}}|^{-2} |g_-(t)|^2 + 2\text{Re} (g_+^*(t)g_-(t)\lambda_{f_{CP}}^{-1}) \right] \\
= |\overline{A}_{f_{CP}}|^2 e^{-\Gamma t} \left[ (1 + |\lambda_{f_{CP}}|^{-2}) \cosh(\Delta m t) \right. \\
+ \left. (1 - |\lambda_{f_{CP}}|^{-2}) \cos(\Delta m t) \right. \\
- \left. 2\text{Re} \lambda_{f_{CP}}^{-1} \sinh(\Delta t/2) + 2\text{Im} \lambda_{f_{CP}}^{-1} \sin(\Delta m t) \right],
\]

where in the last step we have used eqs. (211) and (212).

Using the expressions above we can find an explicit form for the so-called “time-
dependent CP asymmetry”, defined as follows:

\[
A_{CP}(t) \equiv \frac{\Gamma(M^0(t) \to f_{CP}) - \Gamma(M^0(t) \to f_{CP}^-)}{\Gamma(M^0(t) \to f_{CP}) + \Gamma(M^0(t) \to f_{CP}^-)} \\
= \frac{(1 - |\lambda_{f_{CP}}|^2) \cos(\Delta m t) - 2\text{Im} \lambda_{f_{CP}} \sin(\Delta m t)}{(1 + |\lambda_{f_{CP}}|^2) \cosh(\Delta t/2) - 2\text{Re} \lambda_{f_{CP}} \sinh(\Delta t/2)} + O\left(1 - \left| \frac{p}{q} \right|^2 \right).
\]

Neglecting CP violation in mixing, i.e. assuming, according to eq. (190), |q/p| = 1, the
coefficient of the \cos(\Delta m t) term is nonvanishing in the presence of direct CP violation
in the \( M \to f_{CP} \) decay (see eq. (221)), while the coefficient of the \( \sin(\Delta m t) \) term
signals CP violation in the interference between mixing and decay (see eq. (228)).

Let us now discuss in turn \( K, D, B_d \) and \( B_s \) mixing. As we shall see, different
simplifying assumptions can be made in each sector.

### 5.4 Kaon mixing and \( \epsilon_K \)

If CP were conserved, the CP-odd eigenstate would not decay in a two-pion final state,
resulting in a much longer lifetime. Allowing for small CP violation, it remains true
that one eigenstate has a much longer lifetime, so it is convenient to label the eigenstate by the lifetime as Long- and Short-lived. Thus, we have

\[ K_{S,L} = p_K |K^0\rangle \pm q_K |K^\ast\rangle. \] (231)

We can simplify the general expressions in eqs. (181)-(188) using two peculiarities of the Kaon system. First of all, the $\Delta I = 1/2$ rule implies that

\[ \Gamma_{12} \approx A_0^\ast \overline{A}_0. \] (232)

Furthermore, one has

\[ \Delta \Gamma_K \sim -2\Delta m_K. \] (233)

From eq. (198), using eqs. (200), (233) and (232), we obtain

\[ \delta = \frac{2 \text{Im} (M_{12}^\ast \Gamma_{12})}{(\Delta m)^2 + |\Gamma_{12}|^2} \approx \frac{-2 \text{Im} (M_{12}^\ast \Gamma_{12})}{-2(\Delta m)|\Gamma_{12}|} \approx \frac{\text{Im} (M_{12} A_0 \overline{A}_0^\ast)}{(\Delta m)|A_0 \overline{A}_0|}. \] (234)

\[ \simeq \frac{1}{\Delta m} \left( \text{Im} M_{12} + 2 \frac{\text{Im} A_0}{\text{Re} A_0} \text{Re} M_{12} \right) \simeq \frac{1}{\Delta m} \text{Im} M_{12} + \frac{\text{Im} A_0}{\text{Re} A_0}, \]

where we have taken into account that in the standard phase convention

\[ \text{Im} A_0 \ll \text{Re} A_0. \] (235)

Neglecting $\text{Im} A_0$, dimension eight operators in $\text{Im} M_{12}$ and nonlocal matrix elements of two insertions of $\Delta S = 1$ Hamiltonians, one has

\[ \delta \approx \frac{\text{Im} M_{12}^{SD:D=6}}{\Delta m_K}. \] (236)

This approximation has an accuracy of $\sim 5\%$; going beyond it requires evaluating $\text{Im} A_0$, long-distance contributions to $\text{Im} M_{12}$ and the contribution of dimension eight operators to $\text{Im} M_{12}$, a formidable task [39,40,65].

To evaluate eq. (236) we make use of the results obtained in Sec. 4, and in particular of eqs. (149) and (167):

\[ \text{Im} M_{12}^{SD:D=6} = \frac{G_F^2 F_K^2 m_K B_K (\mu) \text{Im} \lambda_{sd}^t \left[ \text{Re} \lambda_{sd}^c (\eta_c (\mu) S_0 (x_c) \right. \\
- \eta_c (\mu) S_0 (x_t, x_c) \left. \right) - \text{Re} \lambda_{sd}^t (\eta_c (\mu) S_0 (x_t) - \eta_c (\mu) S_0 (x_t, x_c)) \right], \] (237)

where the QCD corrections from the matching and from the RG evolution have been lumped in the factors $\eta_{t,c,te} (\mu)$. Notice that in the literature it is customary to define the scale-invariant parameters $\bar{B}_K$ and $\eta_{1,2,3}$ by combining the $\mu$-dependent part of $\eta_{t,c,te} (\mu)$ with $B_K (\mu)$, thereby cancelling explicitly the $\mu$ dependence at the given order, see for example ref. [66].
5.4.1 Phenomenology of CP violation in the Kaon system

In the CP-invariant case, $K_{S,L}$ would correspond to CP eigenstates and $K_L$ decays to two pions would be forbidden. To test CP conservation, one can therefore measure

$$\eta_{00} = \frac{\langle \pi^0 \pi^0 | H | K_L \rangle}{\langle \pi^0 \pi^0 | H | K_S \rangle} \quad \text{and} \quad \eta_{+-} = \frac{\langle \pi^+ \pi^- | H | K_L \rangle}{\langle \pi^+ \pi^- | H | K_S \rangle}. \quad (238)$$

Defining

$$A_f = \langle f | H | K^0 \rangle, \quad \bar{A}_f = \langle f | H | \bar{K}^0 \rangle \quad \text{and} \quad \lambda_f = \left( \frac{q}{p} \right)_K \frac{A_f}{\bar{A}_f}, \quad (239)$$

we have

$$\eta_f = \frac{1 - \lambda_f}{1 + \lambda_f}. \quad (240)$$

Writing $\pi \pi$ decay amplitudes in terms of final states with fixed isospin as in eq. (110), we take the combination

$$\epsilon_K = \frac{1}{3} (\eta_{00} + 2\eta_{+-}) = \frac{1 - \lambda_0}{1 + \lambda_0} + O \left( \frac{A_2^3}{A_0^4} \right), \quad (241)$$

selecting a pure $I = 0$ state up to $2\%$. Since there is only one final state strong phase, the conditions of eq. (221) are not met and there is no direct CP violation in $\epsilon_K$. We have

$$\operatorname{Re} \epsilon_K = \frac{1 - |\lambda_0|^2}{1 + 2\operatorname{Re} \lambda_0 + |\lambda_0|^2}, \quad (242)$$

so that

$$\operatorname{Re} \epsilon_K \neq 0 \quad \Rightarrow \quad |\lambda_0| \neq 1 \quad \Rightarrow \quad \frac{|q_K|}{|p_K|} \neq 1 \quad (243)$$

implies CP violation in $K - \bar{K}$ mixing (see eq. (190)), while

$$\operatorname{Im} \epsilon_K = \frac{-2\operatorname{Im} \lambda_0}{1 + 2\operatorname{Re} \lambda_0 + |\lambda_0|^2}, \quad (244)$$

so that

$$\operatorname{Im} \epsilon_K \neq 0 \quad \Rightarrow \quad \operatorname{Im} \lambda_0 \neq 0 \quad (245)$$

implies CP violation in the interference between mixing and decay (see eq. (228)). Experimentally, $\operatorname{arg} \epsilon_K \approx \pi/4$, so the two CP-violating effects are comparable.

From eqs. (242), (239), (194), (236) and (237) we obtain

$$\operatorname{Re} \epsilon_K \approx \frac{11 - |\lambda_0|^2}{21 + |\lambda_0|^2} = \frac{11 - |q_K|}{21 + |q_K|^2} = \frac{\delta}{2}$$

$$= \frac{G_F^2 M_W^2}{12 \Delta m_K \pi^2} F_K^2 m_K B_K(\mu) \operatorname{Im} \lambda_{s d}^c \left[ \operatorname{Re} \lambda_{s d}^c (\eta_{c}(\mu)S_0(x_e) - \eta_{c}(\mu)S_0(x_t,x_e)) - \operatorname{Re} \lambda_{s d}^c (\eta_{c}(\mu)S_0(x_t) - \eta_{c}(\mu)S_0(x_t,x_e)) \right]. \quad (246)$$
The expression above is valid up to corrections from dimension eight operators, from nonlocal matrix elements of two \( \Delta S = 1 \) effective Hamiltonians and from the deviation from \( \pi/4 \) of the phase of \( \epsilon_K \). These effects have been partially estimated in ref. \([65]\), leading to a correction factor of \( 0.94 \pm 0.02 \). At NLO, the SM prediction from ref. \([67]\)

\[
|\epsilon_K| = (1.97 \pm 0.18) \cdot 10^{-3}
\]

compares very well with the experimental value

\[
|\epsilon_K| = (2.228 \pm 0.011) \cdot 10^{-3}.
\]

We will come back again to \( \epsilon_K \) when discussing the UTA in the SM and beyond in Section 6.

We can form another interesting combination of \( \eta_{+-} \) and \( \eta_{00} \):

\[
\epsilon' \equiv \frac{1}{3} (\eta_{+-} - \eta_{00})
\]

\[
\simeq \frac{\langle (\pi\pi)_{I=0} | H | K_L \rangle \langle (\pi\pi)_{I=2} | H | K_S \rangle - \langle (\pi\pi)_{I=0} | H | K_S \rangle \langle (\pi\pi)_{I=2} | H | K_L \rangle}{\sqrt{2} \langle (\pi\pi)_{I=0} | H | K_S \rangle^2}
\]

\[
\simeq \frac{ie^{i(\delta_2 - \delta_0)}}{\sqrt{2}} \text{Im} \left( \frac{A_2}{A_0} \right) \simeq \frac{ie^{i(\delta_2 - \delta_0)}}{\sqrt{2}} \left( \frac{\text{Im} A_2}{\text{Re} A_0} - \frac{\text{Im} A_0}{\text{Re} A_0} \right),
\]

where \( \omega = \text{Re} A_2 / \text{Re} A_0 \) and the equalities are valid up to corrections of relative order \( \mathcal{O}(\omega, \epsilon_K, \text{Im} A_0 / \text{Re} A_0) \). For \( \delta_2 \neq \delta_0 \) and \( \text{Im} (A_2 / A_0) \neq 0 \) the conditions for CP violation in the decay are satisfied and we have \( \text{Re} \epsilon' \neq 0 \).

Obtaining a solid estimate of \( \epsilon' \) is an extremely difficult task: it contains all the difficulties of the \( \Delta I = 1/2 \) rule and it is also affected by the cancellation between the two terms in the right-hand side of eq. (249). Indeed, in the SM the CP-violating effects from QCD penguins in \( A_0 \) and from electroweak penguins in \( A_2 \) cancel to a large extent, leading typically to predictions for \( \text{Re} \epsilon' / \epsilon \) in the \( 10^{-4} \) range \([68, 70]\), below the world average of \( \text{Re} \epsilon' / \epsilon = (16.6 \pm 2.3) \times 10^{-4} \) \([71, 73]\). Very recently, a first estimate of \( \text{Re} \epsilon' / \epsilon \) in Lattice QCD has been obtained in the same framework of the first estimate of the \( \Delta I = 1/2 \) rule, pointing to a value in the lower \( 10^{-4} \) range, but with a large uncertainty \([74, 75]\). This result has triggered a reanalysis of the SM prediction combining lattice QCD results with phenomenological considerations and/or arguments based on Dual QCD \([76, 80]\), leading to a claimed discrepancy of \( \sim 3 \sigma \) with the experimental value. On the other hand, the lattice calculation underestimates the \( I = 0 \) strong interaction phase, and underestimating final state interactions could bring to an underestimate of \( \epsilon' / \epsilon \), as noted in \([81, 87]\) and more recently stressed in \([88, 90]\).

Further progress in the evaluation of the relevant matrix elements is needed to assess the compatibility of the SM prediction with the experimental value, keeping in mind that \( \epsilon' / \epsilon \) is one of the observables with higher sensitivity to NP.
5.5 D – D̄ mixing and CP violation

In complete analogy with ∆S = 2 transitions, M_{12} and Γ_{12} for D – D̄ mixing have the following structure:

\[(\lambda_{cu}^s)^2 (f_{dd} + f_{ss} - 2 f_{ds}) + 2 \lambda_{cu}^s \lambda_{cu}^b (f_{dd} - f_{ds} - f_{db} + f_{sb}) + (\lambda_{cu}^b)^2 (f_{dd} + f_{bb} - 2 f_{db})\],

where \(\lambda_{cu}^s = V_{cq} V_{qg}^{*}\), \(f_{qi}\), and \(f_{qj}\), and intermediate states containing a b quark only appear in \(M_{12}\). We see that the third generation here plays a very minor role with respect to \(K – K\) mixing, since its contribution is suppressed by \(m_b^2/m_t^2\) with respect to ∆S = 2 amplitudes. Indeed, we can safely neglect the term proportional to \((\lambda_{cu}^b)^2\). Then, the GIM mechanism essentially coincides with the U-spin subgroup of the flavour SU(3) symmetry of strong interactions. Repeating the arguments of Sec. 4.1 we see that in this case the mixing amplitudes are dominated by non-local contributions, making even a rough estimate of \(M_{12}\) and \(Γ_{12}\) a tremendous task. While we may hope that in the future the pioneering studies of \(\Delta m_K\) on the lattice [91] may be extended to \(D – D̄\) mixing, it turns out that CP violation in \(D – D̄\) mixing is already today a very powerful probe of NP. Indeed, the approximate decoupling of the third generation implies a strong suppression of CP-violating effects. We can quantify this suppression by looking at the relevant combination of CKM elements:

\[r = \text{Im} \frac{\lambda_{cu}^b}{\lambda_{cu}^s} \approx 6.5 \cdot 10^{-4} .\] (251)

The long-distance contributions to \(M_{12}\) and \(Γ_{12}\) can be parameterized in terms of their U-spin quantum numbers:

\[(\lambda_{cu}^s)^2 (\Delta U = 2) + 2 \lambda_{cu}^s \lambda_{cu}^b (\Delta U = 1 + \Delta U = 2) + \mathcal{O}((\lambda_{cu}^b)^2) \approx (\lambda_{cu}^s)^2 \epsilon^2 + 2 \lambda_{cu}^s \lambda_{cu}^b \epsilon ,\] (252)

so that we expect CP violation to arise at the level of \(r \epsilon / \epsilon^2 \approx 2 \cdot 10^{-3} \approx 0.1\) for an U-spin breaking of the order of 30%. Given the current experimental errors, it is therefore adequate to assume all SM amplitudes to be real, and interpret the (non)-observation of CP violation in \(D – D̄\) mixing as an effect of (a constraint on) NP. In fact, heavy NP could generate a short-distance contribution to Im \(M_{12}\), which could be observable either via \(|q_D/p_D| \neq 1\) or equivalently via \(\phi \equiv \text{arg}(q/p)_{D} \neq 0\) (the two are not independent if all decay amplitudes are real [92]). Allowing for NP-induced CP violation in \(M_{12}\) only, and keeping all decay amplitudes real, a global combination of \(D\)-mixing related decays can be performed, leading to stringent constraints on NP. For example, the Summer 2018 update of the analysis of refs. [93] finds the distributions for \(|q_D/p_D|\), \(\phi\), \(|M_{12}|\) and its phase \(Φ_{12}\) reported in Fig. 8 corresponding to a bound on \(|Φ_{12}| < 3.5^\circ @ 95\% probability.

5.6 B_d – B̄_d mixing

Let us consider the structure of \(M_{12}\) and \(Γ_{12}\) for \(B_d – B̄_d\) mixing:

\[(\lambda_{bd}^s)^2 (f_{uu} + f_{cc} - 2 f_{uc}) + 2 \lambda_{bd}^s \lambda_{bd}^b (f_{uu} - f_{uc} - f_{ut} + f_{ct}) + (\lambda_{bd}^b)^2 (f_{uu} + f_{tt} - 2 f_{ut}) ,\] (253)
Figure 8: Probability density functions for $\phi$ vs $|q_D/p_D| - 1$ (left panel) and for $\Phi_{12}$ vs $|M_{12}|$ (right panel). The darker (lighter) regions correspond to 68% (95%) probability.

where $\lambda_{bd}^q = V_{qb}^* V_{qd}$, $f_{q_i q_j}$ represents an intermediate state with flavours $q_i$ and $q_j$, and again intermediate states containing a $t$ quark only appear in $M_{12}$. A few remarks are in order:

- contrary to the case of $\Delta S = 2$ transitions, $\lambda_{bd}^q \sim \lambda_{bd}^t$, so there is no CKM enhancement of light quark contributions and $M_{12}$ is dominated by top quark exchange, *i.e.*

$$M_{12} \simeq (\lambda_{bd}^{t*})^2 f_{tt}.$$  

(254)

Following the reasoning in Sec. 4.1, we see that corrections to the leading contribution from nonlocal matrix elements of two $\Delta B = 1$ effective Hamiltonians and from higher dimensional operators arise at $\mathcal{O}(m_b^2/M_W^2, m_b^2/m_t^2)$ and are thus fully negligible;

- also at variance with $K - \bar{K}$ mixing, $\Gamma_{12}$ is suppressed with respect to $M_{12}$ since the top quark does not contribute there, so that

$$\left| \frac{\Gamma_{12}}{M_{12}} \right| \sim \mathcal{O} \left( \frac{m_b^2}{m_t^2} \right) \ll 1;$$  

(255)

- CP violation in mixing is even further suppressed, since the dominant contribution to both $M_{12}$ and $\Gamma_{12}$ is proportional to $(\lambda_{bd}^{t*})^2$, so that the CKM phase drops in the ratio $\Gamma_{12}/M_{12}$. CP violation is then induced solely by the other GIM-suppressed contributions to $\Gamma_{12}$;
last but not least, since the number of channels contributing to $\Gamma_{12}$ is large, and the momentum of intermediate states is of $O(m_b)$, we can advocate quark-hadron duality and perform an operator product expansion for $\Gamma_{12}$ as well. While a detailed discussion of this subject goes well beyond the scope of these lectures, the interested reader will find all the details in refs. [94–96].

Let us now work out the expressions of Sec. 5.1 with the approximation $|\Gamma_{12}| \ll |M_{12}|$:

\begin{align}
\Delta m_{B_d} &= 2|M_{12}|, \\
\frac{\Delta \Gamma_{B_d}}{\Delta m_{B_d}} &= \text{Re} \frac{\Gamma_{12}}{M_{12}}, \\
\left(\frac{q}{p}\right)_{B_d} &= \frac{M_{12}^*}{|M_{12}|} \left(1 - \frac{1}{2} \text{Im} \frac{\Gamma_{12}}{M_{12}}\right), \\
\left|\frac{q}{p}\right|_{B_d} - 1 &= -\frac{1}{2} \text{Im} \frac{\Gamma_{12}}{M_{12}}.
\end{align}

The mass difference is obtained taking the matrix element of the $\Delta B = 2$ effective Hamiltonian as

\begin{equation}
\Delta m_{B_d} = \frac{G_F^2 m_W^2}{2\pi^2} |V_{tb} V_{td}|^2 S_0(x_t) \eta_b m_{B_d} f_{B_d}^2 B_{B_d},
\end{equation}

where the QCD corrections [26][27][97] have been absorbed in $\eta_b$ and $B_{B_d}$ is the $B$-parameter computed in the same scheme and at the same scale as $\eta_b$. The Summer 2018 SM prediction by the UTfit collaboration is

\begin{equation}
\Delta m_{B_d}^{\text{SM}} = (0.54 \pm 0.03) \text{ ps}^{-1}
\end{equation}

which compares very well with the experimental average

\begin{equation}
\Delta m_{B_d}^{\text{exp}} = (0.5064 \pm 0.0019) \text{ ps}^{-1}.
\end{equation}

The experimental sensitivity to $\Delta \Gamma_{B_d}$ is still well above the SM prediction, and the same is true for the semileptonic asymmetry $A_{SL}^{\text{B}_d}$ defined in eq. (213), which measures CP violation in mixing.

From the phenomenological point of view, $B_d$ mesons have three peculiarities that make them a golden system to study meson-antimeson oscillations and CP violation [98,99]:

- since CKM angles involving the third generation are small, the $B_d$ lifetime is of $O(\text{ps}^{-1})$, so that a relatively small boost is enough to allow for a $B_d$ meson to fly a measurable distance before it decays;
- the $B_d - \bar{B}_d$ mass difference is comparable to the $B_d$ lifetime, opening the possibility to measure the time dependence of the oscillations;
- the time-dependent CP asymmetry defined in eq. (230) allows to measure the CP-violating $\text{Im} \lambda_f$ for a variety of final states $f$, allowing for an extensive test of the CKM mechanism and of possible NP contributions.
For these reasons, the idea of an asymmetric $B$-factory, where entangled pairs of $B_d - \bar{B}_d$ mesons could be produced with a boost sufficient to observe the time oscillation, was put forward \cite{100} and developed, leading to the extraordinary success of the BaBar and Belle experiments at SLAC and KEK \cite{101}.

5.6.1 Time-dependent CP asymmetry in $B_d \rightarrow J/\Psi K_S$

Let us now discuss the time-dependent CP asymmetries for a series of final states, starting from the famous “golden channel” $B_d \rightarrow J/\Psi K_S$. The underlying weak decay is $\bar{b} \rightarrow \bar{c} c \bar{s}$, which is generated by the following piece of the $\Delta B = 1$ effective Hamiltonian (see eqs. (97) and (99)-(102)):

$$H_{\bar{b} \rightarrow \bar{c} c \bar{s}}^{\text{eff}} = \frac{4G_F}{\sqrt{2}} \left\{ \lambda_c^{\bar{b} s} \left( C_1 Q_{1 \bar{s}}^{\bar{b} c} + C_2 Q_{2 \bar{s}}^{\bar{b} c} + \sum_{i=3}^{10} C_i Q_{i \bar{s}}^{\bar{b} c} \right) 
+ \lambda_u^{\bar{b} s} \left( C_1 Q_{1 \bar{s}}^{\bar{u} \bar{c}} + C_2 Q_{2 \bar{s}}^{\bar{u} \bar{c}} + \sum_{i=3}^{10} C_i Q_{i \bar{s}}^{\bar{u} \bar{c}} \right) \right\}. \quad (263)$$

To obtain the $\langle J/\Psi K^0 | H_{\bar{b} \rightarrow \bar{c} c \bar{s}}^{\text{eff}} | B_d \rangle$ matrix element we need to consider all possible Wick contractions of the fields in $H_{\bar{b} \rightarrow \bar{c} c \bar{s}}^{\text{eff}}$ with the initial and final states. Following refs. \cite{102, 103}, where the interested reader can find all details, we can classify the different Wick contraction topologies as in Figures 9 and 10 where the left (right) panels contain “disconnected” (“connected”) topologies. In the infinite $m_b$ limit, the case in which the “emitted” meson (i.e. $M_1$ in Figure 9) is light becomes computable in terms of form factors, decay constants and perturbative QCD corrections, as argued in ref. \cite{104} and carefully demonstrated in refs. \cite{105, 106}. The basic idea is that the emitted light meson flies away too fast for soft gluons to be exchanged with the $B$ meson and with the other final state meson, the so-called “colour transparency” argument. In spite of this tremendous theoretical progress, however, a full-fledged computation of the diagrams in Figures 9 and 10 for realistic values of the $b$-quark mass remains well beyond our capabilities. Indeed, long-distance contributions and rescattering effects arising at $O(\Lambda/m_b)$ are not systematically computable and have a strong phenomenological impact in two-body nonleptonic $B$ decays, as emphasized in refs. \cite{102, 107, 108}. A particularly dangerous class of long-distance contributions are the so-called “charming penguins”, namely penguin matrix elements as in Figure 10 with a charm quark running in the loop, which are affected by $D_{(s)}^{(s)} - \bar{D}_{(s)}^{(s)}$ rescattering into light mesons.

To be able to obtain robust phenomenological results, one must therefore seek observables where the dangerous long-distance contributions are either absent or strongly suppressed. To this aim, it is convenient to consider renormalization-group invariant combinations of Wilson coefficients times Wick contractions, as detailed in ref. \cite{103}, and to express the decay amplitudes in terms of these parameters. In the case of $B_d \rightarrow J/\Psi K_S$ we obtain, using eq. (204) for the $K_S$ in the final state:

$$A_{B_d \rightarrow J/\Psi K_S} = \left[ \lambda_{bs}^c (E_2 + P_2) + \lambda_{bs}^u (P_2 - P_2^{\text{GIM}}) \right] / (2p_K), \quad (264)$$

50
Figure 9: Emission, annihilation and emission-annihilation topologies of Wick contractions in the matrix elements of operators $Q_i$. From ref. [103].
Figure 10: Penguin, penguin-emission, penguin-annihilation and double-penguin-annihilation topologies of Wick contractions in the matrix elements of operators $Q_i$. From ref. [103].
where $E_2$ contains emission matrix elements of $Q_{1,2}^{E\pi d}$ in the colour-suppressed combination $C_1 CE + C_2 DE$; $P_2$ contains penguin-emission matrix elements of $Q_{1,2}^{E\pi d}$ together with emission, emission-annihilation and penguin emission matrix elements of $Q_{3-10}^{E\pi d}$. $Q_2^{GIM}$ contains penguin-emission matrix elements of the GIM-suppressed combinations $Q_{1,2}^{E\pi d} - Q_{1,2}^{EM}$, $p_K$ appears to project the $K^0$ onto the $K_S$ final state. We expect the dominant contribution to come from $E_2$, since $P_2$ is suppressed either by small Wilson coefficients or by penguin matrix elements, and $P_2^{GIM}$ is suppressed by penguin matrix elements and by the GIM mechanism.

Thus, we can use the expansion of eqs. (223)-(224) with

$$r_{J/\Psi K_S} = \left| \frac{\lambda_{bs}}{\lambda_{cb}} \right| \left| \frac{P_2 - P_2^{GIM}}{E_2 + P_2} \right| \lesssim \left| \frac{V_{ub} V_{us}}{V_{cb}} \right| \sim \mathcal{O}(10^{-2}) .$$

(265)

Let us first be bold and put $r_{J/\Psi K_S}$ to zero. Then we obtain from eqs. (227) and (264)

$$\lambda_{J/\Psi K_S} = \left( \frac{q}{p} \right)_{B_d} \frac{\lambda_{bs}^{c,s}}{\lambda_{cb}^{c,s}} \left( \frac{p}{q} \right)_{K} = \frac{(\lambda_{bd}^*)^2 \lambda_{bs}^{c,s} \lambda_{cb}^{c,s}}{(\lambda_{bd}^*)^2 \lambda_{bs}^{c,s} \lambda_{cb}^{c,s}} = \frac{V_{tb} V_{td} V_{cb} V_{cs} V_{cs} V_{cd}}{V_{tb} V_{td} V_{cb} V_{cs} V_{cs} V_{cd}} \frac{V_{cb} V_{cd} V_{tb} V_{td}}{V_{tb} V_{td} V_{cb} V_{cd}} = e^{-2i\beta} ,$$

(266)

where the angle $\beta$ of the Unitarity Triangle is defined in eq. (20). Plugging eq. (266) in eq. (230) we obtain

$$\mathcal{A}_{CP}^{B_d \to J/\Psi K_S} (t) = -\sin 2\beta \sin(\Delta m_{B_d} t) ,$$

(267)

taking into account that the final state is CP odd.

Thus, in the approximation $r_{J/\Psi K_S} = 0$, measuring the time-dependent asymmetry in this channel we should find a vanishing coefficient of the $\cos(\Delta m_{B_d} t)$ term, and the coefficient of the $\sin(\Delta m_{B_d} t)$ measures $\sin 2\beta$. The current world average of $\mathcal{A}_{CP}^{B_d \to J/\Psi K_S, L} (t)$ gives [109] [113]

$$\sin 2\beta = 0.690 \pm 0.018 ,$$

(268)

corresponding to $\beta \approx 21.8^\circ$.

Let us now go back to the assumption $r_{J/\Psi K_S} = 0$, under which we obtained eq. (267), and investigate if we can get any theoretical or experimental handle on the actual value of $r_{J/\Psi K_S}$, or at least an upper bound on its value. A theoretical calculation of $r_{J/\Psi K_S}$ from first principles is currently impossible even in the infinite $m_b$ limit, since the emitted meson is heavy. The direct CP asymmetry, i.e. the coefficient of the $\cos(\Delta m_{B_d} t)$ term in the time-dependent asymmetry, according to eq. (222) is sensitive to

$$r_{J/\Psi K_S} \sin \phi_{r_{J/\Psi K_S}} \sin \delta_{r_{J/\Psi K_S}} \approx \lambda^2 \mathcal{R}_b \left| \frac{P_2 - P_2^{GIM}}{E_2 + P_2} \right| \sin \gamma \sin \arg \left( \frac{P_2 - P_2^{GIM}}{E_2 + P_2} \right) ,$$

(269)

with the UT parameters $\lambda$, $\mathcal{R}_b$ and $\gamma$ defined in eqs. (15), (20) and (21). The last term in eq. (269), i.e. the sine of the strong phase difference between the two amplitudes,
prevents us from using directly the direct CP asymmetry as a bound on \( r_{J/\Psi K_S} \). Even if we ignore this problem, bounding \( r_{J/\Psi K_S} \) at the level of the direct CP asymmetry would anyway give a theoretical error comparable to the experimental one. A way out can be found using the SU(3)-related decay channel \( B_d \to J/\Psi \pi^0 \). The effective Hamiltonian governing this decay is given by

\[
H_{\text{eff}}^{B_d \to \bar{c}c\bar{d}} = \frac{4G_F}{\sqrt{2}} \left\{ \lambda_{bd}^c \left( C_1 \tilde{Q}_{1d}^c + C_2 \tilde{Q}_{2d}^c + \sum_{i=3}^{10} C_i \tilde{Q}_{id}^c \right) + \lambda_{bd}^u \left( C_1 \tilde{Q}_{1d}^u + C_2 \tilde{Q}_{2d}^u + \sum_{i=3}^{10} C_i \tilde{Q}_{id}^u \right) \right\},
\]

(270)

and the decay amplitude is given in the SU(3) limit by

\[
A_{B_d \to J/\Psi \pi^0} = \lambda_{bd}^c (E_2 + P_2) + \lambda_{bd}^u (P_2 - P_2^{\text{GIM}}),
\]

(271)

neglecting a small, colour suppressed emission-annihilation contribution. In eq. (271) the second term is not doubly-Cabibbo-suppressed anymore, so that this channel is much more sensitive to \( P_2^{\text{GIM}} - P_2 \). Using the information from \( B_d \to J/\Psi \pi^0 \) one can constrain the theoretical error in the extraction of \( \sin 2\beta \) from \( B_d \to J/\Psi K_S \) to be subdominant even allowing for an SU(3) breaking of 100%. It is however crucial that in the future the experimental progress on \( B_d \to J/\Psi \pi^0 \) parallels the one on \( B_d \to J/\Psi K_S \), so that the theory uncertainty remains subdominant.

5.6.2 Time-dependent CP asymmetry in \( B \to \pi\pi \)

Thanks to isospin symmetry, \( B \to \pi\pi \) decays have the unique property that all decay amplitudes can be determined experimentally, allowing for a measurement of the CKM angle \( \alpha \) with essentially no theoretical input other than isospin [116]. Effects of isospin breaking due to electromagnetic interactions and to quark masses are negligible with respect to current experimental uncertainties, so we will not discuss them here [117].

Using the isospin decomposition of eq. (110) for \( B \) decays, we see that the independent parameters are the relative strong phase of \( I = 0 \) and \( I = 2 \) amplitudes, the weak phases of \( I = 0 \) and \( I = 2 \) amplitudes, and their absolute values, so five independent parameters. If we consider time-dependent CP asymmetries, we should add \((q/p)_{B_d}\); neglecting CP violation in the mixing, this amounts to another parameter, \( \arg(q/p)_{B_d} \).

The observables are three CP-averaged branching ratios (\( B_{+-}, B_{+0} \) and \( B_{00} \)) and four CP asymmetries (the coefficients \( S_{+-,00} \) of \( \sin \Delta m_{B_d} t \) and \( C_{+-,00} \) of \( \cos \Delta m_{B_d} t \) terms in \( A_{CP}^{B_d \to \pi^+\pi^-\pi^0} \)), so the system is over-determined. In practice, however, the measurement of the time-dependent CP asymmetry in \( B_d \to \pi^0\pi^0 \) is very difficult, but there are enough observables to determine all parameters even if only \( C_{00} \) is used.

It is convenient to write the decay amplitudes separating terms with different weak phases rather than with different strong phases as in eq. (110). In particular, using CKM unitarity, we separate the amplitudes in terms proportional to \( \lambda_{bd}^c \) and \( \lambda_{bd}^u \). Taking into account that \((q/p)_{B_d} \simeq (\lambda_{bd}^c/\lambda_{bd}^u)^2 \) and that \( \lambda_f = q/p\bar{A}_f/A_f \), it is convenient
to absorb a factor of $\lambda_{bd}^f (\lambda_{bd}^f)$ in $A_f (\tilde{A}_f)$. In this way we obtain

$$A(B_d \to \pi^+ \pi^-) = e^{-i\alpha} T^{+-} + P,$$  \hspace{1cm} (272)

$$A(B_d \to \pi^0 \pi^0) = (e^{-i\alpha} T^{00} - P),$$  \hspace{1cm} (273)

$$A(B^- \to \pi^- \pi^0) = \frac{1}{\sqrt{2}} e^{-i\alpha} (T^{+-} + T^{00}).$$  \hspace{1cm} (274)

One can then extract $\alpha$, together with $T^{+-}$, $T^{00}$, $P$ and their relative phases, up to an eight-fold ambiguity (explicit formulæ can be found in refs. [118,119]). It is however clear that the degeneracies in $\alpha$ correspond to different values of the parameters $T^{+-}$, $T^{00}$ and $P$. One can then follow the same argument used in Sec. 5.6.1 and relate $B \to \pi^+ \pi^-$ decays to $B_s \to K^+ K^-$ via a U-spin transformation. Since $B_s \to K^+ K^-$ is a $\bar{b} \to \bar{s}u\bar{u}$ transition, the $T$ and $P$ terms in the amplitudes are weighted by a different CKM factor, breaking the degeneracy between different solutions of the $B \to \pi\pi$ system. Thus, the isospin analysis of $B \to \pi\pi$ supplemented by $B_s \to K^+ K^-$ is more efficient [120].

Notice that the isospin analysis of $B \to \pi\pi$ can be generalized beyond the SM as long as new physics does not enhance electroweak penguins by orders of magnitude, and as long as it does not contribute sizeably to current-current operators. Then, one can still extract $\alpha$ even allowing for a NP weak phase to be present in $P$ [121,122], although with a slightly larger uncertainty.

Finally, the same analysis presented for $B \to \pi\pi$ can be carried out for each polarization of the $B \to \rho \pi$ decays; it turns out that the latter profits from larger branching ratios, making it more sensitive than the $\pi\pi$ channel.

### 5.6.3 Extracting $\alpha$ from $B \to \rho \pi$ decays

In general, decays to final states including vector mesons can be analyzed with a very powerful tool, the Dalitz plot, which allows in principle to extract the absolute values of all amplitudes contributing to a given final state, and all their relative phases, provided that they interfere among each other in a non-negligible region of phase space. Although the isospin structure of $B \to \rho \pi$ decays is richer than the one of $\pi\pi$, since the final state can also have isospin one, this just turns the triangular relation for $B \to \pi\pi$, $A(B_d \to \pi^+ \pi^-) + A(B_d \to \pi^0 \pi^0) = \sqrt{2} A(B^+ \to \pi^+ \pi^0)$, into a pentagonal relation, $A(B_d \to \pi^+ \rho^-) + A(B_d \to \pi^- \rho^+) + 2A(B_d \to \pi^0 \rho^0) = \sqrt{2} (A(B^+ \to \rho^+ \pi^0) + A(B^+ \to \rho^0 \pi^+))$. Again, this allows to determine the relative phase of the $I = 3/2$ amplitudes for $B$ and $\bar{B}$ decays, which corresponds to $2\alpha$ [121,123,124]. While a detailed discussion of Dalitz analyses of three-body heavy meson decays goes beyond the scope of these lectures, we refer the interested reader to chapter 13 of ref. [101] for a review of several Dalitz analysis techniques.

### 5.7 $B_s - \bar{B}_s$ mixing

The structure of $M_{12}$ and $\Gamma_{12}$ for $B_s - \bar{B}_s$ mixing is analogous to the one for $B_d - \bar{B}_d$ mixing given in eq. (253), with the substitution $\lambda_{bd}^f \to \lambda_{bs}^f$. However, while in the case
of $B_d - \bar{B}_d$ mixing one has $|\lambda_{bd}^{u,c,t}| \sim \lambda^2$, so all three factors arise at third order in the CKM parameter $\lambda$, for $b \to s$ transitions the relative weight of the three CKM factors is instead hierarchical:

$$|\lambda_{bs}^{t,c}| \sim \lambda^2 \gg \lambda_{bs}^u \sim \lambda^4.$$  \hspace{1cm} (275)

This has three very important phenomenological consequences:

1. CP violation in $B_s - \bar{B}_s$ mixing is tiny, since the $O(\lambda^2)$ decoupling of the first generation is reflected in the smallness of the angle $\beta_s \sim O(\lambda^2)$ defined in eq. (21). This suppression acts on top of the mechanism already discussed for $B_d - \bar{B}_d$ mixing, leading to $\text{Im}(\Gamma_{12}/M_{12}) \sim O(10^{-5})$;

2. since $\Delta m_{B_s}/\Delta m_{B_d}$ goes approximately like the ratio $V_{ts}/V_{td} \sim 1/\lambda$ while $\Gamma_{B_s} \sim \Gamma_{B_d}$, one has $\Delta m_{B_s}/\Gamma_{B_s} \sim 25$, making it much more difficult to resolve experimentally the time-dependence of the mixing;

3. the enhancement factor $\Delta m_{B_s}/\Gamma_{B_s}$ brings $\Delta \Gamma_{B_s}/\Gamma_{B_s} \sim 25 \Delta \Gamma_{B_s}/\Delta m_{B_s}$ to the observable level of $O(10\%)$.

Therefore, in studying $B_s - \bar{B}_s$ mixing we should keep the terms proportional to $\Delta \Gamma_{B_s}$ in the expressions of Sec. 5.3, in particular in eq. (230).

The Summer 2018 prediction for $\Delta m_{B_s}$ in the SM by the UTfit collaboration is

$$\Delta m_{B_s}^{SM} = (17.25 \pm 0.85)\text{ps}^{-1},$$  \hspace{1cm} (276)

which compares very well with the experimental average

$$\Delta m_{B_s}^{exp} = (17.757 \pm 0.0021)\text{ps}^{-1},$$  \hspace{1cm} (277)

while the prediction for $\Delta \Gamma_{B_s}$ yields

$$(\Delta \Gamma_{B_s}/\Gamma_{B_s})^{SM} = 0.15 \pm 0.01,$$  \hspace{1cm} (278)

well compatible with the experimental average

$$(\Delta \Gamma_{B_s}/\Gamma_{B_s})^{exp} = 0.132 \pm 0.008.$$  \hspace{1cm} (279)

### 5.7.1 Time-dependent CP asymmetry in $B_s \to J/\Psi \phi$

If we apply the same arguments presented in Sec. 5.6.1 and consider a $\bar{b} \to \bar{c}c\bar{s}$ transition for $B_s$ decays, we are led to $B_s \to J/\Psi \phi$ as the golden channel for the measurement of the CKM angle $\beta_s$:

$$\lambda_{J/\Psi \phi} = \left(\frac{q}{p}\right)_{B_s} \frac{\lambda_{bs}^{t,c}}{\lambda_{bs}^u} = \frac{(\lambda_{bs}^u)^2 \lambda_{bs}^{t,c}}{(\lambda_{bs}^u)^2 \lambda_{bs}^u} = V_{tb}^* V_{ts} V_{cb}^* V_{cs} V_{tb} V_{ts} V_{cb} V_{cs} = e^{2i\beta_s},$$  \hspace{1cm} (280)
where we have assumed $r_{J/Ψφ} = 0$ and for simplicity we have omitted the CP parity of
the final state, to be determined with an angular analysis of the decay products of the
$J/Ψφ$ intermediate state. In the case of the $B_s$ meson, one cannot neglect the terms
proportional to $ΔΓ_{B_s}$ in eq. (230), so the result of the measurement is a combined fit
of $ΔΓ_{B_s}$ and $Imλ_{J/Ψφ}$.

However, if we now allow for a nonvanishing $r_{J/Ψφ}$, which again we can estimate,
following eq. (265), as

$$r_{J/Ψφ} = \left| \frac{λ_{bs}^u}{λ_{bs}^c} \right| \left| \frac{P_2 - P_{2GIM}}{E_2 + P_2} \right| \lesssim \left| \frac{V_{ub}V_{us}}{V_{cb}} \right| \sim O(10^{-2}) ,$$

we immediately see that the correction to $Imλ_{J/Ψφ}$ is of the same order of $sin 2β_s$:

$$Imλ_{J/Ψφ} = sin 2β_s - 2r_{J/Ψφ} sin γ cos δ_{r_{J/Ψφ}} + O(r_{J/Ψφ}^2, r_{J/Ψφ}λ^2) .$$

In other words, both $B_d → J/ΨK_S$ and $B_s → J/Ψφ$ suffer from doubly-Cabibbo
suppressed corrections, but the leading term is of $O(1)$ for $B_d$ and doubly Cabibbo
suppressed for $B_s$. Still, the time-dependent CP asymmetry in $B_s → J/Ψφ$ remains a
most precious tool to constrain possible NP contributions to CP violation in $B_s$ mixing,
at least down to the level of $r_{J/Ψφ}$. One could of course envisage a strategy to keep
the corrections due to $r_{J/Ψφ}$ under control, using $SU(3)$ as was discussed in Sec. 5.6.1.
However, this approach is complicated by the mixed singlet-octet flavour structure of
the $φ$ meson, requiring a detailed analysis of several final states. We refer the interested
reader to the discussion in ref. [125].

6 The Unitarity Triangle Analysis in the SM and
beyond

Let us now very quickly review how we can combine a large amount of theoretical and
experimental information using the Unitarity Triangle introduced in Sec. 2.1. Since
the CKM matrix is governing all flavour and CP violation in weak interactions, we can
translate virtually any flavour- or CP-violating process into a constraint on the UT.
Let us start from charged-current processes arising at the tree level in the SM, before
turning to FCNC transitions.

6.1 The UT from tree-level decays

The CKM matrix elements $|V_{ud}|$ and $|V_{us}|$ can be measured from super-allowed $β$
decays [126,127] and from semileptonic/leptonic kaon decays [16,128,129] respectively,
providing an accurate determination of the sine of the Cabibbo angle. Similarly, $|V_{cb}|$
and $|V_{ub}|$ can be determined using (semi-)leptonic $B$ decays. In this case, one can use
either exclusive or inclusive decays, which have different theoretical and experimental
systematic errors. For $b → c$ transitions, the analysis of inclusive semileptonic decays
relies on heavy quark symmetry and on global quark-hadron duality, while the study of
inclusive semileptonic $b \to u$ transitions requires local quark-hadron duality, as well as some model-dependent regularization of singularities that are absent in $b \to c$ decays. In exclusive decays, an estimate of the relevant form factors, as well as of their momentum dependence, is needed to extract CKM factors. Unfortunately, determinations of $|V_{cb}|$ and $|V_{ub}|$ from inclusive and exclusive semileptonic $B$ decays have been displaying a $\sim 3\sigma$ discrepancy for quite a while \cite{16}, although it was recently noticed that for $|V_{cb}|$ the situation improves considerably if one relaxes some assumptions on the momentum dependence of the form factors based on the heavy quark limit \cite{130,132}. Hopefully more precise data and improved lattice calculations will bring to a resolution of this long-standing puzzle.

The measurements discussed above provide us with the normalization of the UT and with the length of one of the non-unit sides, $R_b$. Fortunately, we can complete the determination of the UT using only tree-level decays by measuring the angle $\gamma$, defined in eq. \eqref{eq:gamma}. The measurement of $\gamma$ can be achieved by exploiting the interference between $\bar{b} \to \bar{u}q \to f$ and $\bar{b} \to \bar{u}q \to f$ transitions, where $q = d, s$ and $f$ is a generic final state accessible through both decay chains \cite{133,136}. The theoretical uncertainty in the extraction of $\gamma$ can be always kept subdominant \cite{137}, so future experimental progress will have a strong impact on the UT analysis.

Figure 11 shows the current status of the UT determined through tree level decays only. Notice that two regions in the $\rho - \eta$ planes are selected, since we can determine $\gamma$ only up to $\pm 180^\circ$. We will discuss below how this ambiguity can be lifted using measurements of CP violation in $B - \bar{B}$ mixing \cite{138}.

### 6.2 Adding FCNC to the UT Analysis in the Standard Model and Beyond

We are now ready to add to the processes used in Sec. 6.1 meson-antimeson mixing in $K$, $B_d$ and $B_s$ sectors, using eq. \eqref{eq:epsilonK} for $\epsilon_K$, eq. \eqref{eq:deltaMB} for $\Delta m_{B_d,s}$, eq. \eqref{eq:sin2beta} for $\sin 2\beta$ and the results of Sec. 5.6.2 for $\alpha$. This allows to break the degeneracy between the first and third quadrant. The global fit displays a very good consistency of all observables within the SM, as can be seen from Fig. 12.

The consistency of the SM fit can be translated into constraints on NP contributions to meson-antimeson mixing. Let us proceed in two steps. First, we generalize the UT analysis by parameterizing the relevant NP contributions. Second, we translate the constraints on NP contributions into bounds on the scale of NP.

Following refs. \cite{122,139}, we introduce the following parameters to account for possible NP contributions to meson-antimeson mixing:

\begin{equation}
C_{B_q} \equiv \frac{M_{B_q}^{\text{full}}}{M_{B_q}^{\text{SM}}} \frac{e^{2i\phi_q}}{\rho_q}, \quad (q = d, s)
\end{equation}

\begin{equation}
C_{\epsilon_K} = \frac{\text{Im} M_{K}^{\text{full}}}{\text{Im} M_{K}^{\text{SM}}}. \quad (284)
\end{equation}
Figure 11: Current status of the UT determination from tree-level decays, from the UTfit Collaboration.

We can then immediately see how the observables entering the UT analysis are affected:

\[ \Delta m_{B_q} = C_{B_q} (\Delta m_{B_q})_{\text{SM}} \]
\[ \lambda_{J/ΨK_S} = e^{-2i(β + φ_{B_q})} \]
\[ \lambda_{J/Ψφ} = e^{2i(β_s - φ_{B_s})} \]
\[ α_{\text{exp}} = α - φ_{B_d} \]

where \( α_{\text{exp}} \) denotes the value of \( α \) extracted from \( B_d \to ππ, ρπ \) and \( ρρ \) decays.\(^8\)

As pointed out in ref. \cite{138}, the presence of \( φ_{B_q} \) can have a large impact on the semileptonic asymmetries defined in eq. (213). As we have seen in Sec. 5.6, in the SM the dominant contributions to \( M_{12} \) and \( Γ_{12} \) have the same CKM phase which drops in the ratio \( Γ_{12}/M_{12} \), so that \( \text{Im}(Γ_{12}/M_{12}) \) only arises from subdominant GIM-suppressed contributions to \( Γ_{12} \). However, if the mixing amplitude is affected by NP so that it gets an additional phase \( φ_{B_q} \), the phase cancellation between \( M_{12} \) and \( Γ_{12} \) is spoiled and one gets a contribution to \( \text{Im}(Γ_{12}/M_{12}) \) from the dominant term, proportional to \( \text{Re}(Γ_{12}/M_{12})_{\text{SM}}/C_{B_q} \cos 2φ_{B_q} \). It is then evident that the region in the third quadrant in Fig. 11 allowed at the tree-level, requires a large value of \( φ_{B_d} \) which is ruled out at

\(^8\)In the presence of NP contributions to loop-mediated SM processes, in the isospin or amplitude analysis one should allow the penguin contribution to have a phase different from the SM one \cite{139}.
more than 95\% probability by the experimental value of $A_{SL}^{B_d}$.  

We can therefore perform a simultaneous determination of the UT and of the NP parameters introduced in eqs. (283) and (284). The Summer 18 update from the UTfit collaboration is reported in Fig. 13. It is instructive to extract from the $C_{B_q}$ and $\phi_{B_q}$ parameters the absolute value and phase of the NP contributions relative to the SM:

$$C_{B_q} e^{2i\phi_{B_q}} = 1 + \frac{A_{q}^{NP} e^{2i\phi_{q}^{NP}}}{A_{q}^{SM}}. \quad (289)$$

\footnote{Also in this case when allowing for NP to be present in loop-mediated SM processes one should allow for penguin contributions to $\Gamma_{12}$ to have a phase different from the SM one \cite{139}.}
The current constraints on $A^{\text{NP}}_q$ and $\phi^{\text{NP}}_q$ are reported in Fig. 14. We see that our knowledge of the UT in the presence of NP is roughly a factor of two worse than in the SM, and that NP contributions to SM mixing amplitudes at the level of $\approx 30 - 40\%$ are still allowed at 95\% probability, especially if their phase does not differ too much from the SM one. This shows that ample room is left for improvements, both from the experimental and theoretical point of view, until we will be sensitive to NP contributions in the flavour sector at the percent or sub-percent level. However, given the combined loop and GIM suppression of these observables in the SM, and given the hierarchical structure of quark masses and mixings, already this relatively rough sensitivity to NP contributions is able to provide us with the most stringent constraints on the NP scale, as we will see below.

6.3 Constraining the NP scale with $\Delta F = 2$ amplitudes

We now combine the results on the $\Delta F = 2$ effective Hamiltonian beyond the SM in Secs. 4.5 and 4.6 with the constraints on NP contributions obtained in Secs. 5.5 and 6.2 to learn more on NP.

Assuming, as we did above, that NP has a negligible impact on processes that arise at the tree-level in the SM, we write for meson-antimeson mixing in all sectors

$$M_{12} = M^{\text{SM}}_{12} + \frac{F_i L_i}{\Lambda^2} \langle M^0 | Q_i | M^0 \rangle, \quad \Gamma_{12} \approx \Gamma^{\text{SM}}_{12},$$

(290)

where $F_i$ is a function of the (complex) NP flavour couplings, $L_i$ is a loop factor that is present in models with no tree-level Flavour Changing Neutral Currents (FCNC), and $\Lambda$ is the scale of NP, i.e. the typical mass of the new particles mediating the $\Delta F = 2$ transition. For a generic strongly-interacting theory with arbitrary flavour structure, we expect $F_i \sim L_i \sim O(1)$ so that the allowed range for each of the NP contributions can be immediately translated into a lower bound on $\Lambda$. Specific assumptions on the flavour structure of NP, for example Minimal or Next-to-Minimal Flavour Violation (MFV or NMFV), correspond to particular choices of the $F_i$ functions, as detailed below. Notice that in eq. (290) the SM contribution $M^{\text{SM}}_{12}$ should be computed using for the CKM parameters the results of the UT analysis in the presence of NP.

Switching on one operator at a time, assuming that $F_i \sim L_i \sim O(1)$, running its coefficient down from the NP scale $\Lambda$ to the hadronic scale $\mu$ at which the relevant matrix elements have been computed (see refs. [140–151] for computations of the matrix elements for the full set of relevant operators), computing its contribution according to eq. (290) and comparing it to the results presented in Secs. 5.3 and 6.2, we obtain the 95\% probability lower bounds on $\Lambda$ presented in Fig. 1. The bounds are dominated by CP violation in $K - \bar{K}$ and $D - \bar{D}$ mixing, as expected from the extreme suppression of these processes in the SM, and by the contributions of the chirality-violating operators, which are enhanced both by the RG evolution and by the matrix elements. These bounds are clearly beyond the reach of any direct detection experiment, and strongly suggest us that any NP close to the EW scale must have a hierarchical flavour structure analogous to the SM one. One can then envisage the so-called NMFV scenario, in which one has
Figure 13: From left to right and from top to bottom: probability density functions for \((\bar{\rho}, \bar{\eta}), C_{cK}, (C_{Bd}, \phi_{Bd}), (C_{Bs}, \phi_{Bs})\). Darker (lighter) regions correspond to smallest 68% (95%) probability regions.

\(F_i \simeq F^{SM}\), where \(F^{SM}\) is the CKM factor of the relevant SM amplitude. The bounds on the NP scale in NMFV for \(L_i = 1\) are reported in Fig. 15. We see that the chiral and RG enhancement of \(Q_4\) pushes the NP scale to \(O(100)\) TeV; to keep \(\Lambda\) below 10 TeV one must not only enforce the same flavour structure of the SM, but also the same chiral structure.

Requiring the same flavour and chiral structure of the SM corresponds to the so-called MFV framework, initially formulated as the requirement of NP contributions to
Figure 14: Probability density functions for $A_{q}^{\text{NP}}, \phi_{q}^{\text{NP}}$. Darker (lighter) regions correspond to smallest 68% (95%) probability regions.

FCNC observables being just a redefinition of the loop function associated to the top-quark contribution, also known as Constrained MFV (CMFV) [152, 153]. The CMFV hypothesis allows for an improved determination of the UT in the presence of NP and for several tests of consistency, since it implies the independence on NP of ratios of observables in which the top-mediated loop function drops, such as for example $\Delta m_{B_d}/\Delta m_{B_s}$ [154, 155]. More generally, one can observe that the requirement that NP has the same flavour and chiral structure of the SM can be formulated in terms of the flavour symmetry of the SM Lagrangian when the Yukawa couplings are put to zero: the MFV hypothesis then amounts to the requirement that Yukawa couplings are the only source of violation of the flavour symmetry [6]. This automatically leads to small deviations from the SM for NP scales close to the EW one, provided that Yukawa couplings are close to their “SM” value (i.e., to the value they would take in the SM), while for example in two Higgs doublets models with a large $v_2/v_1$ ratio of the two vacuum expectation values one could face larger deviations enhanced by this ratio.

While assuming that MFV holds exactly amounts to assuming that Yukawa couplings are fundamental, thus giving up the hope of finding a dynamical explanation of their hierarchical structure, it is certainly true that a NP scale close to the EW one implies a flavour structure close to MFV. This is clearly possible if the NP responsible for the origin of the Yukawa couplings structure is much heavier than the EW scale.

7 Conclusion and further reading

The goal of these lectures is to allow the reader to get a first idea of how we can probe NP with flavour observables. The level of refinement of current forefront analyses in this field is clearly way beyond the few basic elements here presented. Fortunately, several excellent reviews are available on most of the topics sketched in the previous
Figure 15: Summary of the 95% probability lower bound on the NP scale $\Lambda$ for NMFV. See the text for details.
sections. First of all, there are several other lectures on the same topics which are
more detailed, more general and more inspired than ours, starting from the classic Les
Houches lectures by A.J. Buras [19], SLAC and Trieste lectures by Y. Nir [156] and
from the excellent book by Branco, Lavoura and Silva [63], continuing to more recent
lectures [64, 155, 157, 177]. For what concerns instead review articles, the NLO classic is
ref. [66], while among the many NP-oriented reviews I find refs. [178, 179] particularly
inspiring. Ref. [180] contains a remarkably complete discussion of meson-antimeson
mixing in the charm and bottom sectors. Finally, while it was impossible to collect
here all original references for the topics we discussed, the reader is strongly encouraged
to read the original papers where all details can be found.

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A Loops in Dimensional Regularization

We collect here a few useful formulæ for loop calculations in dimensional regularization.

A.1 Feynman parameters

Feynman parameters are useful to group denominators in loop amplitudes. The basic
formula is the following:

\[
\frac{1}{AB} = \int_0^1 dx \, dy \, \frac{\delta(x + y - 1)}{[xA + yB]^2} = \int_0^1 dx \, \frac{1}{[xA + (1-x)B]^2} .
\] (291)

It can be easily verified explicitly:

\[
\int_0^1 dx \, \frac{1}{[xA + (1-x)B]^2} = - \frac{1}{A-B} \left[ \int_0^1 \frac{1}{xA + (1-x)B} \right] = \frac{1}{A-B} \left( \frac{1}{A} - \frac{1}{B} \right) = \frac{1}{AB} .
\] (292)
We can raise the powers in the denominator by differentiating:

\[
\frac{1}{AB^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial B^{n-1}} \frac{1}{AB} = \frac{(-1)^{n-1}}{(n-1)!} \int_0^1 dx \, dy \, \frac{\delta(x + y - 1) y^{n-1} (1)^{n-1}}{(xA + yB)^{n+1}}.
\]  

We can add further terms in the denominator by iterating with eqs. (291) and (293):

\[
\frac{1}{ABC} = \frac{1}{AB} \frac{1}{C} = \frac{1}{C} \int_0^1 dx \, dy \, \frac{\delta(x + y - 1)}{(xA + yB)^2}
\]

\[
= \int_0^1 dw \, dz \, dx \, dy \, 2w \frac{\delta(w + z - 1) \delta(x + y - 1)}{(zC + w(xA + yB))^3} \frac{x' = wx}{y' = wy} \int_0^1 dw \, dz \, \delta(w + z - 1) \int_0^w dw' \, dy' \frac{2 \delta(x' + y' - w)}{(zC + x'A + y'B)^3}.
\]

We thus obtain the general formula

\[
\frac{1}{A_1^{m_1} A_2^{m_2} \ldots A_n^{m_n}} = \int_0^1 dx_1 \, dx_2 \ldots dx_n \frac{\delta(\sum_i x_i - 1)}{[\sum_i x_i A_i]^{\sum_i m_i} \prod_i \Gamma(m_i)}.
\]  

A.2 Loop integrals

A.2.1 Momentum shift

After grouping the denominators with Feynman parameters using eq. (295), the denominator will contain in general not only the square of the loop momentum and constant terms, but also terms linear in the loop momentum (from dot products with external momenta). We get rid of linear terms in the denominator by performing a shift of the loop momentum, which brings us to the general form

\[
\int \frac{d^d k \, \delta(k^{m_1} \ldots k^{m_n})}{(2\pi)^d (k^2 - D + i\epsilon)^m}.
\]  

Then integrals with odd powers of \(k\) in the numerator vanish by symmetry, and we are left with even powers of \(k\) only.
A.2.2 Wick rotation

The $i\epsilon$ term in eq. (296) is there to remind us that we should be careful about the poles of the propagators entering the diagram we are calculating. We can form a closed contour in the complex $K^0$ plane by going from $-\infty$ to $+\infty$ on the real axis, from $+\infty$ to $-\infty$ on the imaginary axis, and closing the contour with two arcs at infinity from the real to the imaginary axis. Noting that the $i\epsilon$ prescription moves the poles to the second and fourth quadrant, so that they are not inside the contour, and neglecting the contribution at infinity, we see that the integral on the real axis is equal to the integral on the imaginary axis. We then go to the Euclidean with $k^0 = i k_E^0$, and obtain

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - D)^m} = i(-1)^m \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2 + D)^m}$$

(297)

A.2.3 Angular integration

Let us now split the integration pulling out the angular one:

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2 + D)^m} = \int \frac{d\Omega_d}{(2\pi)^d} \int k_E^{d-1} \frac{1}{(k_E^2 + D)^m}.$$  

(298)

We can obtain the angular term using a Gaussian integral:

$$\left(\sqrt{\pi}\right)^d = \left(\int_{-\infty}^{\infty} dx e^{-x^2}\right)^d = \int_{-\infty}^{\infty} d^d x e^{-x^2} = \int d\Omega_d \int_0^{\infty} dx x^{d-1} e^{-x^2}$$

$$= \int d\Omega_d \frac{1}{2} \int_0^{\infty} dx x^2 (x^2)^{d-1} e^{-x^2} = \int d\Omega_d \frac{1}{2} \Gamma\left(\frac{d}{2}\right)$$

(299)

so that

$$\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}.$$  

(300)

A.2.4 Momentum integration

We now turn to the integral over the absolute value of the Euclidean momentum:

$$\int_0^{\infty} k_E^{d-1} dk_E \frac{1}{(k_E^2 + D)^m} = \frac{1}{2} \int_0^{\infty} (k_E^2)^{\frac{d}{2}-1} dk_E \frac{1}{(k_E^2 + D)^m} = \frac{1}{2} \int_0^{x=x^D / (k_E^2 + D)^m} \frac{x^\frac{d}{2} - 1 D x^{-2} dx (D/x)^{\frac{d}{2}-1} (1-x)^{\frac{d}{2}-1}}{(D/x)^m}$$

$$= \frac{1}{2} D^\frac{d}{2} - m \int_0^1 dx x^{m-\frac{d}{2}-1} (1-x)^{\frac{d}{2}-1}$$

$$= \frac{1}{2} \left(\frac{1}{D}\right)^{m-\frac{d}{2}} B(m - \frac{d}{2}, \frac{d}{2})$$

(301)

where

$$\int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$  

(302)
A.2.5 Expansion for $d = 4 - 2\epsilon$

We now put together the results in eqs. (297), (298), (300) and (301):

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - D)^m} = \frac{i}{(4\pi)^{d/2}} (-1)^m \left( \frac{1}{D} \right)^{m-d/2} \frac{\Gamma(m-d/2)}{\Gamma(m)}
\]  

(303)

and expand for $d$ close to four in the small parameter $\epsilon$:

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - D)^m} = \frac{i}{16\pi^2} \left( -1 \right)^{m-2} \left( \frac{1}{4\pi D} \right)^\epsilon \frac{\Gamma(m-2+\epsilon)}{\Gamma(m)}.
\]  

(304)

Using the expansion of Euler $\Gamma$

\[
\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon),
\]  

(305)

\[
\Gamma(-n + \epsilon) = \frac{(-1)^m}{n!} \left( \frac{1}{\epsilon} - \gamma_E + 1 + \ldots + \frac{1}{n} + \mathcal{O}(\epsilon) \right),
\]

one obtains for example

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - D)^2} = \frac{i}{16\pi^2} \left( \frac{1}{\epsilon} - \gamma_E - \ln 4\pi - \ln D + \mathcal{O}(\epsilon) \right) \equiv \frac{i}{16\pi^2} \left( \frac{1}{\tilde{\epsilon}} - \ln D + \mathcal{O}(\tilde{\epsilon}) \right),
\]  

(306)

where we introduced for convenience the parameter $\tilde{\epsilon}$ defined in eq. (47).

A.2.6 Some useful integrals

The reader may find the following list of integrals useful:

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - D)^m} = \frac{i}{(4\pi)^{d/2}} (-1)^m \left( \frac{1}{D} \right)^{m-d/2} \frac{\Gamma(m-d/2)}{\Gamma(m)}
\]  

(307)

\[
\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 - D)^m} = \frac{i}{(4\pi)^{d/2}} (-1)^m \frac{d}{2} \left( \frac{1}{D} \right)^{m-d/2-1} \frac{\Gamma(m-d/2-1)}{\Gamma(m)}
\]  

(308)

\[
\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - D)^m} = \frac{i}{(4\pi)^{d/2}} (-1)^m \frac{1}{2} \frac{g^{\mu\nu}}{2} \left( \frac{1}{D} \right)^{m-d/2-1} \frac{\Gamma(m-d/2-1)}{\Gamma(m)}
\]  

(309)

\[
\int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^2}{(k^2 - D)^m} = \frac{i}{(4\pi)^{d/2}} (-1)^m \frac{d(d+2)}{4} \left( \frac{1}{D} \right)^{m-d/2-2} \frac{\Gamma(m-d/2-2)}{\Gamma(m)}
\]  

(310)

\[
\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2 - D)^m} = \frac{i}{(4\pi)^{d/2}} (-1)^m \frac{g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}}{4} \left( \frac{1}{D} \right)^{m-d/2-2} \frac{\Gamma(m-d/2-2)}{\Gamma(m)}
\]  

(311)
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