On Finite Elements in $f$-Algebras and in Product Algebras

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Abstract

Finite elements, which are well-known and studied in the framework of vector lattices, are investigated in $\ell$-algebras, preferably in $f$-algebras, and in product algebras. The additional structure of an associative multiplication leads to new questions and some new properties concerning the collections of finite, totally finite and self-majorizing elements. In some cases the order ideal of finite elements is a ring ideal as well. It turns out that a product of elements in an $f$-algebra is a finite element if at least one factor is finite. If the multiplicative unit exists, the latter plays an important role in the investigation of finite elements. For the product of special $f$-algebras an element is finite in the algebra if and only if its power is finite in the product algebra.

Keywords: vector lattice, $\ell$-algebra, $f$-algebra, finite element, order unit, multiplication, orthomorphisms, product algebra

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1 Introduction

Finite elements in Archimedean vector lattices were introduced 1971-1974 in the papers [20, 21, 26] as an abstract analogue of continuous functions (on a locally compact space) with compact support. Finite and totally finite elements play a very important role in the representation theory of Archimedean vector lattices by means of real valued (i.e. everywhere finite) continuous functions on a locally compact Hausdorff space, where they are required to be represented as functions with compact support. The classes of these elements in general vector lattices and Banach lattices are thoroughly studied in a number of papers, see [10, 11, 20–23, 27]. Finite elements in vector lattices of operators are dealt with in [12, 15]. A condensed short overview concerning finite and totally finite elements the reader can find in [28]. Self-majorizing elements in Archimedean vector lattices (also known as semi-order units) have been studied systematically in the recent paper [25].

In this paper we investigate finite elements in Archimedean $\ell$-, $d$- and $f$-algebras and in product algebras. It is well known that the vector lattice of all orthomorphisms on an Archimedean vector lattice is an Archimedean $f$-algebra with weak order unit, see e.g. [2, Theorem 8.24]. We use this fact in Sections 3 and 5.

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The material of the paper is organized as follows: in Section 2 we provide the notions of the theory of vector lattices and \(\ell\)-algebras which are necessary in order to present our results. In particular, we define the finite, totally finite and self-majorizing elements in an Archimedean vector lattice. Further we list some properties of \(f\)-algebras which are relevant for our purpose. In Section 3 we study finite elements in \(f\)- and \(d\)-algebras with multiplicative unit, in Section 4 we investigate them in \(f\)-algebras without multiplicative unit. For this purpose the weak factorization property is introduced and its relations to other well-known properties in \(f\)-algebras are demonstrated by examples. In Section 5 we consider finite elements in products of uniformly complete \(f\)-algebras. For details concerning vector lattices and \(\ell\)-algebras we refer to the monographs [1, 2, 24, 29, 30] as well as to the papers [3, 4, 7, 8, 13, 16]. The recent development in the theory of \(\ell\)-algebras is reflected in the survey paper [9].

2 Preliminaries

Recall some definitions and notations known in the theory of vector lattices and lattice ordered algebras. A vector lattice will be denoted by \(V\), a Banach lattice by \(E\) and, a lattice ordered algebra by \(A\). By \(V^+, E^+\) and \(A^+\) will be denoted their cones of positive elements, respectively. We consider only Archimedean vector lattices \(V\) and \(A\). This assumption, in particular, ensures the uniqueness of uniform limits (see [19]). For details see [1–3, 24, 29].

- If \(A\) is a non-empty subset of \(V\) then the smallest order ideal that contains \(A\) is denoted by \(I_A\) and is called the order ideal generated by \(A\). This order ideal is (see [2])
  \[
  I_A = \left\{ x \in V : \exists a_1, \ldots, a_n \in A \text{ and } \lambda_1, \ldots, \lambda_n \in \mathbb{R}^+ \text{ such that } |x| \leq \sum_{i=1}^{n} \lambda_i |a_i| \right\}.
  \]

- For a non-empty subset \(A \subseteq V\) by \(A^\perp\) we denote the set \(\{x \in V : \forall a \in A \ x \perp a\}\).

- The set \(A^{\perp\perp}\) is known as the band generated by \(A\), i.e. the smallest band that contains \(A\). If \(A\) consists of one element \(x\), then \(\{x\}^{\perp\perp}\) is called the principal band generated by \(x\).

- An element \(u \in V_+\) is an (strong) order unit, if for each \(x \in V\) there is a \(\lambda \in \mathbb{R}_{\geq 0}\) with \(-\lambda u \leq x \leq \lambda u\) (or equivalently, \(|x| \leq \lambda u\)).

- An element \(u \in V_+\) is a weak order unit, if \(\{u\}^{\perp\perp} = V\), i.e. \(x \in V\) and \(x \perp u\) imply \(x = 0\).

Further on an algebra is understood to be a set \(A\) equipped with several operations: beside the addition (+) and the usual scalar multiplication, which turn \(A\) into a vector space, there is also defined an associative multiplication (\(\cdot\)) satisfying the distributive laws.

- A vector lattice \(A\) is called a lattice ordered algebra, a Riesz-algebra or also an \(\ell\)-algebra, if \(A\) is equipped with an associative multiplication (\(\cdot\)) such that \(A\) becomes an algebra, where

\[
(\ell) \quad a, b \geq 0 \implies ab \geq 0 \text{ holds for all } 0 \leq a, b \in A.
\]

\[\text{It is convenient to write } ab \text{ instead of } a \cdot b \text{ for the product of } a \text{ and } b.\]
The basic notions and properties of \(\ell\)-algebras can be found in [29, Chapter 20]. Equivalent to (\(\ell\)) are the conditions:

\((\ell_1)\) if \(a, b, c \in \mathcal{A}\) satisfy \(a \leq b\) and \(c \geq 0\) then \(ac \leq bc\),

\((\ell_2)\) \(|ab| \leq |a||b|\) for all \(a, b \in \mathcal{A}\),

see [3, Sect. 1].

- An \(\ell\)-algebra is called a \(d\)-algebra (see [17]), if it satisfies the condition

\[ a \land b = 0 \implies (ac) \land (bc) = (ca) \land (cb) = 0 \quad \text{for all } c \geq 0. \]

Equivalent to (d) are the conditions:

\[(d_1)\] \(|ab| = |a||b|\) for all \(a, b \in \mathcal{A}\), and also

\[(d_2)\] \(c(a \land b) = ca \land cb\) and \((a \land b)c = ac \land bc\) for all \(a, b \in \mathcal{A}, c \in \mathcal{A}_+\),

\[(d_3)\] \(c(a \lor b) = ca \lor cb\) and \((a \lor b)c = ac \lor bc\) for all \(a, b \in \mathcal{A}, c \in \mathcal{A}_+\),

see [3, Proposition 1.2].

- An \(\ell\)-algebra is called an \(f\)-algebra, if it satisfies the condition

\[ a \land b = 0 \text{ for all } c \geq 0 \implies (ac) \land b = (ca) \land b = 0. \]

Equivalent to (f) is the condition:

\[ (f_1) \quad \{ab\}^{\perp \perp} \subset \{a\}^{\perp \perp} \cap \{b\}^{\perp \perp} \quad \text{for } 0 \leq a, b \in \mathcal{A}, \]

see [13, Proposition 3.5].

- An element \(e \in \mathcal{A}\) is called a multiplicative unit, if \(a \cdot e = e \cdot a = a\) for all \(a \in \mathcal{A}\). It is uniquely defined. An algebra with multiplicative unit is called unitary.

- An element \(a \in \mathcal{A}\) is called nilpotent, if there is \(n \in \mathbb{N}\) such that \(a^n = 0\). The set of all nilpotent elements of \(\mathcal{A}\) is denoted by \(N(\mathcal{A})\). If \(\mathcal{A}\) is an Archimedean \(f\)-algebra, then \(N(\mathcal{A}) = N_2(\mathcal{A}) := \{a \in \mathcal{A}: a^2 = 0\}\), see [13, Proposition 10.2 (i)].

- An \(\ell\)-algebra \(\mathcal{A}\) is called semiprime, if the only nilpotent element in \(\mathcal{A}\) is zero.

Remark 1 We collect here without proof the main properties of the introduced \(\ell\)-algebras and comment the relations between them. For the proofs we refer to [2, 3, 24, 29]. Let \(\mathcal{A}\) be an arbitrary \(\ell\)-algebra.

(1) It follows immediately from the definitions that each \(f\)-algebra is a \(d\)-algebra. The converse, in generally, is not true.

(2) If a \(d\)-algebra is semiprime or possesses a positive multiplicative unit, then it is an \(f\)-algebra.

(3) If in an \(f\)-algebra a multiplicative unit exists, then the latter is always positive.

(4) Even in an \(f\)-algebra the existence of a multiplicative unit is not guaranteed: The vector lattice \(c_0\) of all real zero sequences with the coordinatewise order and algebraic operations is a semiprime Archimedean \(f\)-algebra without a multiplicative unit.

(5) An Archimedean \(\ell\)-algebra with a multiplicative unit \(e > 0\) is an \(f\)-algebra if and only if \(e\) is a weak order unit.
(6) Every Archimedean $f$-algebra is commutative and every unitary Archimedean $f$-algebra is semiprime.

(7) In an Archimedean commutative $d$-algebra the following frequently used formulas hold (see [4, Proposition 1], and [7, Proposition 4]) for the vector lattice operations with $p$-th powers of $a, b \in A_+$ for $p \in \mathbb{N}_{\geq 1}$:

\[(a \wedge b)^p = a^p \wedge b^p \quad \text{and} \quad (a \vee b)^p = a^p \vee b^p.\] (1)

(8) In any $\ell$-algebra the condition $a \wedge b = 0 \Rightarrow ab = 0$ and the condition $a^2 = |a|^2$ are equivalent (see [3, Proposition 1.3]). They hold in every $f$-algebra.

The following definitions are basic.

**Definition 1** Let $V$ be an Archimedean vector lattice.

1. An element $\varphi \in V$ is called finite, if there exists an element $z \in V_+$ such that the following condition holds: for any $x \in V$ there is a number $c_x > 0$ satisfying the inequality

\[|x| \wedge n |\varphi| \leq c_x z \quad \text{for all} \quad n \in \mathbb{N}.\]

*The element $z$ is called a $V$-majorant or briefly a majorant of $\varphi$."

2. An element $\varphi$ of a vector lattice $V$ is called totally finite, if it possesses a $V$-majorant which itself is a finite element.

3. An element $\varphi \in V$ is called self-majorizing, if $|\varphi|$ is a majorant of $\varphi$, i.e. for each element $x \in V$ there is a constant $c_x > 0$ such that

\[|x| \wedge n |\varphi| \leq c_x |\varphi| \quad \text{for all} \quad n \in \mathbb{N}.\] (2)

The sets of all finite elements and all totally finite elements in $V$ are denoted by $\Phi_1(V)$ and $\Phi_2(V)$, respectively. It is easy to see that $\Phi_1(V)$ and $\Phi_2(V)$ are order ideals in $V$ and $\Phi_2(V) \subseteq \Phi_1(V)$. The set of all self-majorizing elements is denoted by $S(V)$, the set of positive self-majorizing elements by $S_+(V)$, i.e. $S_+(V) = S(V) \cap V_+$. It is clear that with $\varphi$ also $|\varphi|$ is a self-majorizing element. The set $\Phi_3(V) = S_+(V) - S_+(V)$ is an order ideal in $V$ and $\Phi_3(V) \subseteq \Phi_2(V)$.

The main characterization of self-majorizing elements is contained in the following theorem. The proofs of the theorem and its corollary are provided as Theorem 1 and Corollary 3 in [25].

**Theorem 1 ([18, Corollary 7.2], and [14])** For an element $\varphi$ of a vector lattice $V$ the following statements are equivalent.

1. The element $\varphi$ is self-majorizing.

2. The order ideal $I_{\varphi}$ generated in $V$ by $\varphi$ is the projection band $\{\varphi\}^{\perp \perp}$.

**Theorem 1** yields the following corollary as an immediate consequence.

**Corollary 1** Let $V$ be a vector lattice. Then

1. any order unit in $V$ is a self-majorizing element and

2. if $V$ possesses an order unit then $\Phi_3(V) = \Phi_2(V) = \Phi_1(V) = V$.4
3 Finite elements in unitary $\ell$-algebras

The first result shows that the multiplication with elements from the order ideal generated by the positive multiplicative unit preserves the finiteness with the same majorant.

**Theorem 2** Let $A$ be an $\ell$-algebra with a positive multiplicative unit $e$ and let $a$ be an arbitrary element of $I_e = \{ a \in A : |a| \leq \lambda e \text{ for some } \lambda \in \mathbb{R}_{\geq 0}\}$. Then for $i = 1, 2$ there holds

$$\varphi \in \Phi_i(A) \text{ with the majorant } u \implies \varphi a, a \varphi \in \Phi_i(A) \text{ with the majorant } u.$$ 

*Proof.* Without loss of generality let $\varphi \geq 0$ (otherwise use $\varphi = \varphi^+ - \varphi^-$. It suffices to consider only $a \geq 0$, since by condition $(\ell_2)$ there holds $|a\varphi| \leq |a|\varphi$. For an $a \in I_e$, there is a $\lambda \in \mathbb{R}_{\geq 0}$ such that $0 \leq a \leq \lambda e$. Due to the condition $(\ell_1)$ we have for arbitrary $x \in A$ and all $n \in \mathbb{N}$ the inequality

$$|x| \wedge na\varphi \leq |x| \wedge n\lambda e \varphi = |x| \wedge n\lambda \varphi.$$ 

If now $\varphi$ is a finite element with a majorant $u$, then

$$|x| \wedge na\varphi \leq c_x u \text{ for all } n \in \mathbb{N}.$$ 

Therefore the product $a\varphi$ is also a finite element with the same majorant as $\varphi$. Analogously, the statement is proved for the product $a\varphi$.

If $\varphi$ is even totally finite, i.e. the majorant $u$ of $\varphi$ itself is a finite element, then the products $\varphi a$ and $a\varphi$ also have finite majorants, which shows that they are totally finite as well. 

The same result can be proved without the positivity of the multiplicative unit, if $\mathcal{A}$ is supposed to be a $d$-algebra. However, in contrast to the previous theorem, the majorant for the product changes and depends on the factor $a$.

**Theorem 3** Let $\mathcal{A}$ be a $d$-algebra with a (not necessarily positive) multiplicative unit and let $a \in \mathcal{A}$ be an arbitrary element. Then for $i = 1, 2$ there holds

$$\varphi \in \Phi_i(\mathcal{A}) \implies a\varphi, \varphi a \in \Phi_i(\mathcal{A}).$$

In particular, $\Phi_i(\mathcal{A})$ is a $d$-subalgebra and a ring ideal in $\mathcal{A}$.

If additionally $\mathcal{A}$ is an $f$-algebra, then $\Phi_i(\mathcal{A})$ is even an $f$-subalgebra.

*Proof.* Denote the multiplicative unit of $\mathcal{A}$ by $e$ and assume again $\varphi \geq 0$. Let first $i = 1$. Due to $\varphi \in \Phi_1(\mathcal{A})$ there are a majorant $u \in \mathcal{A}$ for $\varphi$ and, for each $x \in \mathcal{A}_+$, a number $c_x \in \mathbb{R}_{\geq 0}$ such that

$$|x| \wedge n\varphi \leq c_x u \text{ for all } n \in \mathbb{N}. \quad (3)$$

Since by condition $(d_1)$ one has $|\varphi a| = |\varphi||a|$, the elements $\varphi|a|$ and $\varphi a$ are coincidently finite, so $a \geq 0$ may be assumed.

Let $x \in \mathcal{A}$ and $n \in \mathbb{N}$ be arbitrary. Then $a \geq 0$ implies $a \vee e \geq 0$ and by means of condition $(d_3)$ one obtains from (3)

$$(a \vee e) |x| \wedge n(a \vee e) \varphi \leq c_x (a \vee e) u.$$
Since $a \leq a \lor e$ and $|x| = e|x| \leq (a \lor e)|x|$ one has for every $n \in \mathbb{N}$

$$|x| \land na \varphi \leq |x| \land n(a \lor e) \varphi \leq (a \lor e)|x| \land n(a \lor e) \varphi \leq c_x(a \lor e)u,$$

i.e. the element $a \varphi$ is finite in $A$ with the majorant $(a \lor e)u$. Analogously it can be shown that the product $\varphi a$ is finite in $A$.

The set $\Phi_1(A)$ is an order ideal in $A$, in particular a vector sublattice. According to the first part of the proof the product of two finite elements is finite and so, the set $\Phi_1(A)$ is closed under the multiplication. The properties (d) or (f) are shared by the set $\Phi_1(A)$, if $A$ is a $d$- or an $f$-algebra, respectively. Therefore, $\Phi_1(A)$ is a $d$- or an $f$-subalgebra of $A$, respectively.

It is clear from the proof that $\Phi_1(A)$ is a ring ideal.

For $i = 2$ observe that $(a \lor e)u$ is a majorant of the element $a \varphi$ as above, where $u$ as a majorant of the totally finite element $\varphi$ can be assumed to be a finite element. By what has been proved in the first part (case $i = 1$) the element $(a \lor e)u$ is finite as well, which yields the totally finiteness of $a \varphi$ in $A$. The totally finiteness of the product $\varphi a$ is proved analogously. The remaining statements for $\Phi_2(A)$ are obtained analogously to the case $i = 1$. \hfill \Box

**Remark 2** If a majorant of $\varphi$ is $u$, then a majorant of $a \varphi$, $\varphi a$ is $(a \lor e)u$, $u(a \lor e)$, respectively. In particular, the idea of this proof cannot be used to obtain an analogous result for self-majorizing elements.

If the multiplicative unit itself is a finite element then we get

**Theorem 4** Let $A$ be a $d$-algebra with a multiplicative unit $e$. Let $e \in \Phi_1(A)$. Then

$$\Phi_1(A) = \Phi_2(A) = A.$$

If $A$ is an $f$-algebra and $e \in \Phi_1(A)$, then $e$ is even an order unit in $A$ and

$$\Phi_1(A) = \Phi_2(A) = \Phi_3(A) = A.$$

**Proof.** First consider the case of a $d$-algebra. Since $e$ is finite, by the previous theorem the products $ae$ and $ea$ are finite elements for all $a \in A$, i.e. $A \subseteq \Phi_1(A)$. So the equalities $\Phi_1(A) = \Phi_2(A) = A$ hold.

Consider the case of an $f$-algebra $A$. We mention first that the element $e$ is positive and a weak order unit [Remark 1(3) and 1(5)]. If $A$ has a weak order unit then, according to Corollary 2.5 from [10], the equalities $\Phi_1(A) = \Phi_2(A) = A$ hold if and only if there exists an order unit in $A$. Since the equalities hold by what has been proved in the first part (here we use the fact that an $f$-algebra is also a $d$-algebra), the $f$-algebra $A$ has an order unit. From Corollary 1 it follows now that $A$ coincides also with the order ideal $\Phi_3(A)$ of all self-majorizing elements of $A$.

Now consider the weak order unit $e$ which, due to $A = \Phi_3(A)$, is a self-majorizing element, and show that $e$ is a (strong) order unit. By Theorem 1 the order ideal generated in $A$ by $e$ is a projection band and coincides with $\{e\}^{\perp \perp}$. Since $e$ is a weak order unit one has $A = \{e\}^{\perp \perp}$. Consequently, $A = \{e\}^{\perp \perp} = \{a \in A : \exists \lambda > 0 \text{ with } |a| \leq \lambda e\}$, i.e. $e$ is an order unit in $A$. \hfill \Box

**Theorem 5** Let $A$ be an $f$-algebra with a multiplicative unit $e$. Let there exist a submultiplicative Riesz norm on $A$, i.e. a Riesz norm which satisfies $\|ab\| \leq \|a\| \|b\|$ for all $a, b \in A_+$. Then
1. the multiplicative unit $e$ is an order unit and

2. $\Phi_1(\mathcal{A}) = \Phi_2(\mathcal{A}) = \Phi_3(\mathcal{A}) = \mathcal{A}$.

**Proof.** 1. We show that for each $a \in \mathcal{A}$, $0 \neq a$ there exists a $\lambda \in \mathbb{R}_{\geq 0}$ such that $-\lambda e \leq a \leq \lambda e$. Let first $a \in \mathcal{A}_+$. Further on we use the obvious decomposition $a - \lambda e = (a - \lambda e)^+ - (a - \lambda e)^-$, which holds for any $\lambda \in \mathbb{R}_{\geq 0}$. Without loss of generality $(a - \lambda e)^+ > 0$ can be assumed. Indeed, $(a - \lambda e)^+ = 0$ for some $\lambda > 0$ leads to $a - \lambda e = -(a - \lambda e)^- \leq 0$, and so to $0 \leq a \leq \lambda e$.

Now consider the element $(a - \lambda e)(a - \lambda e)^+$. Due to Remark 1(8) the product of the two positive disjoint elements $(a - \lambda e)^-$ and $(a - \lambda e)^+$ vanishes, and by taking into account the condition (\ell) we obtain the inequality

$$(a - \lambda e)(a - \lambda e)^+ = (a - \lambda e)^+(a - \lambda e)^- - (a - \lambda e)^-(a - \lambda e)^+ = ((a - \lambda e)^+)^2 \geq 0.$$  

We conclude $a(a - \lambda e)^+ - \lambda e(a - \lambda e)^+ \geq 0$, and so

$$a(a - \lambda e)^+ \geq \lambda(a - \lambda e)^+ > 0 \text{ for } \lambda > 0.$$  

Due to the norm being submultiplicative and Riesz we obtain

$$\|a\|\|(a - \lambda e)^+\| \geq \|a(a - \lambda e)^+\| \geq \lambda \|(a - \lambda e)^+\| > 0,$$

and therefore $\lambda \leq \|a\|$.

Altogether, as we have seen, the assumption $(a - \lambda e)^+ > 0$ leads to $\lambda \leq \|a\|$. Therefore $\lambda > \|a\|$ yields $(a - \lambda e)^+ = 0$, and so $a - \lambda e = -(a - \lambda e)^- \leq 0$, and again $0 \leq a \leq \lambda e$ as above.

Now let $a \in \mathcal{A}$ be an arbitrary element. In view of $\pm a \leq |a|$ we obtain the claimed result.

2. The fact that all elements in $\mathcal{A}$ are finite, totally finite and even self-majorizing follows from Corollary 1 by taking into account that $e$ is an order unit in $\mathcal{A}$.

A linear operator $T$ on an Archimedean vector lattice $V$ is called **band preserving** if $T(B) \subseteq B$ for each band $B$ in $V$. A band preserving operator which is order bounded is called an **orthomorphism**.

It is well known that the collection Orth($V$) of all orthomorphisms on an Archimedean vector lattice $V$ is an $f$-algebra with the identity as a weak order unit. Moreover, any $f$-algebra $\mathcal{A}$ with a multiplicative unit $e$ is algebraic and lattice isomorphic to Orth($\mathcal{A}$), where the image of $e$ is the identity in Orth($\mathcal{A}$) ([24, Theorems 3.1.10 and 3.1.13]).

**Corollary 2** Let $\mathcal{A}$ be a unitary $f$-algebra. Let there exist a submultiplicative Riesz norm on $\mathcal{A}$. Then

1. the identity operator $I$ is an order unit in Orth($\mathcal{A}$) and

2. Orth($\mathcal{A}$) = $\Phi_i$(Orth($\mathcal{A}$)), $i = 1, 2, 3$.

A similar result holds if there is some norm on the algebra $\mathcal{A}$ which turns it into a Banach lattice.

**Theorem 6** Let $\mathcal{A}$ be an $f$-algebra with a multiplicative unit $e$. Let there exist a norm on $\mathcal{A}$ such that $\mathcal{A}$ becomes a Banach lattice. Then

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2The main idea of the proof is due to W.A.J. Luxemburg, cf. [2, Theorem 15.5].
1. the multiplicative unit \( e \) is an order unit and

2. \( \Phi_1(A) = \Phi_2(A) = \Phi_3(A) = A \).

Proof. Since the \( f \)-algebra \( A \) and \( \text{Orth}(A) \) are algebraic and lattice isomorphic such that the image of \( e \) under the isomorphism is \( I \in \text{Orth}(A) \), then by Wickstead’s Theorem ([2, Theorem 15.5]) the identity operator \( I \) is an order unit in \( \text{Orth}(A) \) and so, \( e \) is an order unit in \( A \). By virtue of Corollary 1 all elements in \( \text{Orth}(A) \), and consequently in \( A \), are finite, totally finite and even self-majorizing.

For the \( f \)-algebra of all orthomorphisms on a vector lattice from the Theorems 3 and 6 we get the following properties which we formulate as

Corollary 3

1. Let \( V \) be a vector lattice. If \( S \in \Phi_i(\text{Orth}(V)) \) for \( i = 1, 2 \) and \( T \in \text{Orth}(V) \) then also \( S \circ T \in \Phi_i(\text{Orth}(V)) \). In particular, \( \Phi_1(\text{Orth}(V)) \) is an \( f \)-subalgebra and a ring ideal.

2. Let \( E \) be a Banach lattice. Then \( \text{Orth}(E) \) is an \( f \)-algebra and under the order unit norm \( \|T\|_I = \inf\{\lambda > 0 : |T| \leq \lambda I\} \) also an \( AM \)-space with order unit. In this case

\[
\text{Orth}(E) = \Phi_i(\text{Orth}(E)) \quad \text{for } i = 1, 2, 3.
\]

The last results throw some light also on the relations between finiteness and invertibility of elements in \( f \)-algebras.

Example 1 Consider the vector lattice \( C_b(\mathbb{R}) \) of all bounded real-valued continuous functions on \( \mathbb{R} \) equipped with the pointwise algebraic operations and partial order. Then \( C_b(\mathbb{R}) \) turns out to be an Archimedean \( f \)-algebra. It is a Banach lattice if the norm is defined by \( \|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)| \) for \( f \in C_b(\mathbb{R}) \). Since there exist (many) order units in \( C_b(\mathbb{R}) \) all elements are finite\(^3\). Observe that any function \( f \in C_b(\mathbb{R}) \) with \( \inf_{x \in \mathbb{R}} |f(x)| > 0 \) is invertible. Of course, there are non-invertible finite elements as well, e.g. functions with compact support.

Let \( A \) be a \( d \)-algebra with a multiplicative unit \( e \). If there exists at least one non-zero finite element which is invertible in \( A \), then immediately all elements of \( A \) are finite, i.e. \( A = \Phi_1(A) \). Indeed, Theorem 3 guarantees that the finiteness and the invertibility of an element \( \varphi \) imply \( e = \varphi^{-1} \varphi \) to be a finite element in \( A \). Then by Theorem 4 all elements of \( A \) are finite.

The \( f \)-algebra \( C(\mathbb{R}) \) of all continuous functions on \( \mathbb{R} \) contains a multiplicative unit (the function \( 1 \)). However, in contrast to \( C_b(\mathbb{R}) \), there is no order unit. There is also no norm on \( C(\mathbb{R}) \) that makes it a Banach lattice. Otherwise, by Theorem 6, there would be an order unit. It is clear that the element \( 1 \) is not finite\(^4\) in \( C(\mathbb{R}) \). By what has been mentioned above no finite element can be invertible. Consequently, there exist \( f \)-algebras \( A \) with multiplicative units such that \( \Phi_1(A) \neq \{0\} \) and no finite element is invertible.

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\(^3\) The Banach algebras \( C_0(\mathbb{R}) \) and \( C(\beta \mathbb{R}) \) are lattice isomorphic, where \( \beta \mathbb{R} \) denotes the Stone-Čech compactification of \( \mathbb{R} \). So, all elements in \( C_0(\mathbb{R}) \) like in \( C(\beta \mathbb{R}) \) are finite.

\(^4\) Since \( \Phi_1(C(\mathbb{R})) = \mathcal{K}(\mathbb{R}) \), the vector lattice of all functions with compact support.
4 Finite elements in non-unitary \( f \)-algebras

In this section we consider the case of \( f \)-algebras which do not possess any multiplicative unit.

**Definition 2** Let \( \mathcal{A} \) be an \( f \)-algebra.

1. \( \mathcal{A} \) is said to be square-root closed if for any \( a \in \mathcal{A}_+ \) there exists \( b \in \mathcal{A} \) such that \( b^2 = a \), i.e. for every such element \( a \) there exists its square root.

2. \( \mathcal{A} \) is said to have the factorization property if for every \( a \in \mathcal{A} \) there exist two elements \( b, c \in \mathcal{A} \) such that \( a = bc \).

3. We say that \( \mathcal{A} \) has the weak factorization property if for every \( a \in \mathcal{A} \) there exist two elements \( b, c \in \mathcal{A} \) such that \( a \leq bc \).

In [5, Theorem 4.6] the first two properties were proved to be equivalent in uniformly complete \( f \)-algebras. The fact that the property 3. is weaker than 2. is demonstrated by the next example.

**Example 2** For the vector lattice \( \mathcal{A} := \{ f \in C[-1,1] : f(0) = 0 \} \) let the multiplication for all \( f, g \in \mathcal{A} \) be defined by

\[
(f \cdot g)(t) := \begin{cases} f(t)g(t), & t \in [0,1], \\ f(-t)g(-t), & t \in [-1,0). \end{cases}
\]

Products in \( \mathcal{A} \) are precisely the axisymmetric functions, which vanish at 0. Observe that \( \mathcal{A} \) is an \( f \)-algebra, which is not semiprime. We will show that \( \mathcal{A} \) is uniformly complete and has the weak factorization property. However, the factorization property does not hold for \( \mathcal{A} \).

To see that \( \mathcal{A} \) is uniformly complete, notice that \( \mathcal{A} \) is the kernel \( \delta_0^{-1}(0) \) of the continuous functional \( \delta_0 \) defined on the Banach lattice \( C[-1,1] \) by \( \delta_0(f) = f(0) \).

The \( f \)-algebra \( \mathcal{A} \) obviously does not have the factorization property, since an arbitrary \( g \in \mathcal{A} \), which is not axisymmetric, cannot be written as a product of two elements of \( \mathcal{A} \). Since in uniformly complete \( f \)-algebras the factorization property is equivalent to the square-root closedness, the latter does not hold in \( \mathcal{A} \) either. However, \( \mathcal{A} \) has the weak factorization property. Indeed, let \( g \in \mathcal{A} \) be an arbitrary element. Define

\[
\hat{g}(t) := \max_{t \in [-1,1]} \{ |g(t)|, |g(-t)| \} \quad \text{and} \quad \tilde{g}(t) := \sqrt{\hat{g}(t)}.
\]

Then \( \hat{g}, \tilde{g} \in \mathcal{A} \) and there holds the inequality \( g \leq \hat{g} = \tilde{g}^2 \).

**Example 3** This example shows that, in general, the weak factorization property does not hold in \( f \)-algebras. To that end, consider finite partitions \( \tau \) of the set \([0, \infty)\), i.e. \( \tau = \{ I_0, \ldots, I_n \} \) such that

\[
\bigcup_{k=0}^n I_k = [0, \infty), \quad I_k \cap I_j = \emptyset \quad \text{for } k \neq j
\]

and \( I_k \) is a subinterval of \([0, \infty)\) for any \( k \). Let \( \mathcal{P} \) be the set of all polynomials \( p \) vanishing at the point \( t = 0 \). Consider now the collection \( \mathcal{B} := \mathcal{P}([0, \infty)) \) of all continuous functions on
for each of which there exists a partition \( \tau \) such that \( f \big|_{I_k} = p_k \) with \( p_k \in \mathcal{P} \) for any \( I_k \in \tau \). The algebraic operations, the multiplication and the partial order are introduced in \( \mathfrak{P} \) pointwise. Then \( \mathfrak{P} \) is an Archimedean \( \ell \)-algebra. Moreover, it is easy to see that the disjointness\(^5\) of two functions \( f, g \in \mathfrak{P} \) is preserved also after the multiplication of one of them by a positive function \( h \in \mathfrak{P} \). Therefore \( \mathfrak{P} \) is an \( f \)-algebra. Since only the zero-element of \( \mathfrak{P} \) can satisfy the equation \( f^2 = 0 \), the \( f \)-algebra is semiprime. Observe that the restriction on \([0, \infty)\) of a polynomial \( p \) of arbitrary degree with \( p(0) = 0 \) belongs to \( \mathfrak{P} \) but the function \( 1_{[0, \infty)} \) does not. It follows that \( \mathfrak{P} \) does not contain neither an order unit nor a multiplicative unit.

The \( f \)-algebra \( \mathfrak{P} \) does not have even the weak factorization property, since the function \( f(t) = t \) can not be estimated by a product of two functions. Indeed, \( f \leq pq \) implies that both polynomials \( p, q \) take on positive values for all \( t > 0 \) and \( \deg(pq) \geq 2 \). Since \( pq(0) = 0 \) the graphs of \( f \) and \( pq \) intersect in some point. Let \( t_0 \) be the smallest number with \( 0 < t_0 \) and \( f(t_0) = pq(t_0) \). There is an interval \( I_k \) of a partition for \( pq \in \mathfrak{P} \) such that \( t_0 \in I_k \) and \( f(t) > pq(t) \) for \( t \in (0, t_0) \).

The \( p \)-fold product is used to define the \( p \)-th root of an element in an \( \ell \)-algebra: for \( g \in \mathcal{A} \) an element \( \tilde{g} \in \mathcal{A} \) is called a \( p \)-th root of \( g \), if \( \tilde{g}^p = g \). If a \( p \)-th root of \( g \) exists and is uniquely defined then we write \( \tilde{g} = g^{\frac{1}{p}} \) and call \( \tilde{g} \) the \( p \)-th root of \( g \). For details we refer to \([4, 7, 29]\) where, in particular, the following results can be found.

**Remark 3**

1. (Existence and uniqueness of the root). Let \( \mathcal{A} \) be an Archimedean uniformly complete \( f \)-algebra and \( p \in \mathbb{N}_{\geq 2} \). Then (see [7, Theorem 3]) there exists a positive \( p \)-th root for any \( p \)-fold product\(^6\) of positive elements of \( \mathcal{A} \), i.e.

\[
g_1, \ldots, g_p \in \mathcal{A}_+ \implies (g_1 \cdots g_p)^{\frac{1}{p}} \text{ exists in } \mathcal{A}_+. \tag{4}
\]

The root is uniquely defined if the algebra \( \mathcal{A} \) is semiprime.

2. (Monotonicity of the root). In every \( \ell \)-algebra \( \mathcal{A} \) for \( p \in \mathbb{N}_{\geq 2} \) and \( a, b \in \mathcal{A}_+ \) due to the property (\( \ell_1 \)) one has

\[
a \leq b \implies a^p \leq b^p.
\]

If \( \mathcal{A} \) is a semiprime \( f \)-algebra, then (see [29, Theorem 142.3] and [4, Proposition 2.(iii)]) the root is monotone, i.e.

\[
a \leq b \iff a^p \leq b^p.
\]

**Theorem 7** Let \( \mathcal{A} \) be a uniformly complete \( f \)-algebra with the weak factorization property, \( p \in \mathbb{N}_{\geq 2} \) and \( i = 1, 2 \). If \( \varphi_1, \ldots, \varphi_p \in \Phi_i(\mathcal{A}) \) with majorants \( u_1, \ldots, u_p \), respectively, then \( \varphi_1 \cdots \varphi_p \in \Phi_i(\mathcal{A}) \) with a majorant \( \left( u_1 \lor \cdots \lor u_p \right)^p \). In particular, \( \Phi_i(\mathcal{A}) \) is an \( f \)-subalgebra of \( \mathcal{A} \).

**Proof.** Let \( i = 1 \). First we prove the claim for the \( p \)-fold power \( \varphi^p \) of a finite element \( \varphi \in \mathcal{A} \). Let \( \varphi \) be a finite element in \( \mathcal{A} \) with a majorant \( u \in \mathcal{A}_+ \). Without loss of generality, \( \varphi \) can be assumed

\(^5\)The supports of two disjoint continuous functions on \([0, \infty)\) intersect at most in one point.

\(^6\)For the product of \( p \) elements \( g_1, \ldots, g_p \) in \( \mathcal{A} \) we will use the notation \( g_1 \cdots g_p := g_1 \cdot \cdots \cdot g_p \).

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to be positive, otherwise use $|φ^p| = |φ|^p$, which holds due to property (d₁). For an arbitrary $a ∈ \mathcal{A}$ the weak factorization property of $\mathcal{A}$ yields the existence of $p$ elements $a₁, \ldots, a_p ∈ \mathcal{A}$ with $|a| ≤ a₁ \cdots a_p$. Again by property (f₂) and Remark 3(1) there follow the positivity of the elements $a₁, \ldots, a_p$ and the existence of a root $(a₁ \cdots a_p)^\frac{1}{p}$ in $\mathcal{A}$. Using the formula (1) and the finiteness of $φ$ we obtain that there is a constant $c_{φ^1 \cdots φ^p} ≥ 0$ such that for all $n ∈ \mathbb{N}$

$$|a| \wedge nφ^p ≤ (a₁ \cdots a_p) \wedge nφ^p = ((a₁ \cdots a_p)^\frac{1}{p} \wedge nφ)^p ≤ c_{φ^1 \cdots φ^p}^p n^p,$$

where the last inequality follows from Remark 3(2). Therefore the $p$-th power $φ^p$ of a finite element $φ ∈ \mathcal{A}$ is also finite.

Now let $φ₁, \ldots, φ_p$ be arbitrary finite elements in $\mathcal{A}$ with majorants $u₁, \ldots, u_p$, respectively. The modulus of the product $φ₁ \cdots φ_p$ can be estimated by

$$|φ₁ \cdots φ_p| ≤ |φ₁| \cdots |φ_p| ≤ (|φ₁| \lor \cdots \lor |φ_p|)^p = (|φ₁| \lor \cdots \lor |φ_p|)^p.$$

Since a majorant of the supremum $|φ₁| \lor \cdots \lor |φ_p|$ is given by $u₁ \lor \cdots \lor u_p$, by the first part the $p$-fold product $(|φ₁| \lor \cdots \lor |φ_p|)^p$ is finite as well with the majorant $(u₁ \lor \cdots \lor u_p)^p$.

Let $i = 2$. Then the majorants $u₁, \ldots, u_p$ can be assumed to belong to $Φ₁(\mathcal{A})$ and the element $(u₁ \lor \cdots \lor u_p)^p$ is finite due to what has been proved in the case $i = 1$.

The last theorem has been proved under stronger conditions than Theorem 9, where we drop the uniformly completeness and the factorization property of the $f$-algebra. However, in the proof of Theorem 9 the majorants are not given explicitly and so, in contrast to Theorem 7, the fate of totally finite elements remains unknown there.

For the next theorem notice that the Example 2 shows that the weak factorization property does not imply semiprimitivity, even under the additional condition of uniformly completeness. By Example 3 the converse implication is also not true. However, it is not known if uniformly completeness together with semiprimitivity imply the weak factorization property.

**Theorem 8** Let $\mathcal{A}$ be a semiprime uniformly complete $f$-algebra with the weak factorization property and $p ∈ \mathbb{N}_{≥ 2}$. If for $φ ∈ Φ₁(\mathcal{A})$ there exists the root $φ^\frac{1}{p}$ in $\mathcal{A}$, then $φ^\frac{1}{p} ∈ Φ₁(\mathcal{A})$.

**Proof.** First consider $0 < φ ∈ Φ₁(\mathcal{A})$ with a majorant $u ∈ \mathcal{A}$ for which there exists the root $φ^\frac{1}{p}$ in $\mathcal{A}$. Let $a ∈ \mathcal{A}$ be an arbitrary element. According to the formula (1) and by using the finiteness of $φ$ we get

$$(a \wedge nφ^p)^p = a^p \wedge n^p φ ≤ c_{aφ} u$$

for some constant $c_{aφ} ≥ 0$ and all $n ∈ \mathbb{N}$. Due to the weak factorization property of $\mathcal{A}$ there are $p$ elements $u₁, \ldots, u_p ∈ \mathcal{A}$ such that $u ≤ u₁ \cdots u_p$. Therefore the above inequality can be continued as follows

$$(a \wedge nφ^\frac{1}{p})^p ≤ c_{aφ} u ≤ c_{aφ} u₁ \cdots u_p.$$
Due to condition ($\ell_2$) there hold the relations $0 \leq u \leq u_1 \cdots u_p = |u_1| \cdots |u_p|$. Without loss of generality we may replace $u_i$ by $|u_i|$ and therefore assume that $u_i \geq 0$ for $i = 1, \ldots, p$.

According to Remark 3(1) there exists the root $(u_1 \cdots u_p)^{\frac{1}{p}}$ in $A$ and we obtain

$$(a \wedge n\varphi)^\frac{1}{p} \leq c_{a^p} u_1 \cdots u_p = c_{a^p} ((u_1 \cdots u_p)^{\frac{1}{p}})^p.$$ 

The monotonicity of the root allows us to extract the $p$-th root on both sides, which yields

$$a \wedge n\varphi^\frac{1}{p} \leq \sqrt[p]{c_{a^p} (u_1 \cdots u_p)^{\frac{1}{p}}},$$

and that shows that the element $\varphi^\frac{1}{p}$ is finite in $A$ with the majorant $(u_1 \cdots u_p)^{\frac{1}{p}}$.

Now let $\varphi \in \Phi_1(A)$ be arbitrary. If $\varphi$ possesses a root $\varphi^\frac{1}{p}$ then by condition (d$_1$)

$$|\varphi| = |\varphi^\frac{1}{p} \cdots \varphi^\frac{1}{p}| = |\varphi^\frac{1}{p}| \cdots |\varphi^\frac{1}{p}| \quad (p \text{ times})$$

implies that $|\varphi^\frac{1}{p}|$ is the $p$-th root of $|\varphi|$, i.e. $|\varphi^\frac{1}{p}| = |\varphi|^\frac{1}{p}$. Together with $\varphi$ the element $|\varphi|$ is also finite in $A$ and so, according to the first part of the proof, the element $|\varphi^\frac{1}{p}|$ is finite and, therefore the finiteness of $\varphi^\frac{1}{p}$ is obtained. 

In analogy to the above theorem we obtain the next result.

**Corollary 4** Let $A$ be a semiprime $f$-algebra and $p \in \mathbb{N}_{\geq 2}$. If for $\varphi \in \Phi_3(A)$ there exists the root $\varphi^\frac{1}{p}$ in $A$, then $\varphi^\frac{1}{p} \in \Phi_3(A)$.

**Proof.** First consider $0 < \varphi \in \Phi_3(A)$ for which there exists the root $\varphi^\frac{1}{p}$ in $A_+$. According to the formula (1) and since $\varphi$ is self-majorizing we get

$$(a \wedge n\varphi^\frac{1}{p})^p = a^p \wedge n^p \varphi \leq c_{a^p} \varphi$$

for some constant $c_{a^p} \geq 0$ and all $n \in \mathbb{N}$. The monotonicity of the root allows us to extract the $p$-th root on both sides, which yields

$$a \wedge n\varphi^\frac{1}{p} \leq \sqrt[p]{c_{a^p} \varphi^\frac{1}{p}},$$

and that shows that the element $\varphi^\frac{1}{p}$ is self-majorizing.

Now let $\varphi \in \Phi_3(A)$ be arbitrary such that $\varphi^\frac{1}{p}$ exists. The application of the identity $|\varphi^\frac{1}{p}| = |\varphi|^\frac{1}{p}$ analogously to the proof of the previous theorem ensures that $\varphi^\frac{1}{p}$ is self-majorizing. 

For the next result, which is similar to Theorem 3, we use the characterization of $f$-algebras given by the condition (f$_1$).

**Theorem 9** Let $A$ be an $f$-algebra, $\varphi \in \Phi_1(A)$ and $a \in A$. Then $a\varphi \in \Phi_1(A)$. In particular, $\Phi_1(A)$ is an $f$-subalgebra and a ring ideal.
\textbf{Proof.} Let first \( a \in \mathcal{A}_+ \) and \( \varphi \in \Phi_1(\mathcal{A}) \), \( \varphi \geq 0 \). Using the condition \( (f_1) \) we obtain \( \{a \varphi\}^{\perp \perp} \subseteq \{a\}^{\perp \perp} \cap \{\varphi\}^{\perp \perp} \). By [10, Theorem 2.4] for a finite element we have \( \{\varphi\}^{\perp \perp} \subseteq \Phi_1(\mathcal{A}) \) and so
\[
\{a \varphi\}^{\perp \perp} \subseteq \{a\}^{\perp \perp} \cap \{\varphi\}^{\perp \perp} \subseteq \Phi_1(\mathcal{A}).
\]
In particular, the product \( a \varphi \) is finite in \( \mathcal{A} \).

Now let \( a \in \mathcal{A} \) be arbitrary and \( \varphi \) positive. The first part of the proof yields \( a^+ \varphi, a^- \varphi \in \Phi_1(\mathcal{A}) \) and so we obtain the finiteness of \( a \varphi = a^+ \varphi - a^- \varphi \) in \( \mathcal{A} \). Finally, assume \( \varphi \) to be arbitrary. Since \( \Phi_1(\mathcal{A}) \) is an order ideal, we obtain the finiteness of \( \varphi^+ \) and \( \varphi^- \) and therefore also the finiteness of \( a \varphi = a \varphi^+ - a \varphi^- \) in \( \mathcal{A} \).

Notice that the product \( a_1 \cdots a_p \) belongs to \( \Phi_1(\mathcal{A}) \) if at least one of the elements \( a_1, \ldots, a_p \in \mathcal{A} \) belongs to \( \Phi_1(\mathcal{A}) \).

The next theorem generalizes Theorem 6, since, as was already mentioned in Remark 1(6), a unital \( f \)-algebra \( \mathcal{A} \) is automatically semiprime. For its proof we need the following result, which we obtain by resuming and restricting [6, Theorem 12.3.8.].

First we introduce the following notation. Let \( \mathcal{A} \) be an \( \ell \)-algebra and \( c \in \mathcal{A} \). Denote by \( c \pi \) and \( \pi_c \) the left and right multiplications by \( c \), respectively, i.e. \( c \pi, \pi_c : \mathcal{A} \to \mathcal{A} \), defined by
\[
c \pi (a) = ca \quad \text{and} \quad \pi_c (a) = ac \quad \text{for all} \quad a \in \mathcal{A}.
\]
It is clear that every multiplicative operator \( c \pi, \pi_c \) is order bounded. If \( \mathcal{A} \) additionally satisfies the condition \( (f) \), then for \( c \geq 0 \) the operators \( c \pi \) and \( \pi_c \) are band preserving (and hence orthomorphisms), since then one has \( \pi_c (a) \wedge b = c \pi (a) \wedge b = 0 \) whenever \( a \wedge b = 0 \) (see [2, Theorem 8.2]).

Notice that the map \( h : a \mapsto \pi_a \) from an \( f \)-algebra \( \mathcal{A} \) into \( \text{Orth}(\mathcal{A}) \) is a homomorphism. Indeed the condition \( (d_2) \) implies
\[
\pi_{a \wedge b} (c) = (a \wedge b)c = ac \wedge bc = \pi_a (c) \wedge \pi_b (c) = (\pi_a \wedge \pi_b) (c)
\]
and thus \( h(a \wedge b) = h(a) \wedge h(b) \). The other properties of \( h \) follow analogously.

\textbf{Lemma 1} \quad \text{For an Archimedean } f \text{-algebra } \mathcal{A} \text{ the following conditions are equivalent:}

1. The algebra \( \mathcal{A} \) is semiprime.

2. The map \( h \) is an injective homomorphism from \( \mathcal{A} \) into \( \text{Orth}(\mathcal{A}) \). In particular, \( \mathcal{A} \) is embeddable as an \( f \)-subalgebra into the Archimedean unitary \( f \)-algebra \( \text{Orth}(\mathcal{A}) \).

\textbf{Proof.} \( \Rightarrow \): Since \( \mathcal{A} \) is semiprime, one has \( \pi_a \neq 0 \) for all \( a \in \mathcal{A}, 0 \neq a \). Therefore \( \ker(h) = \{0\} \), i.e. \( h \) is injective.

\( \Leftarrow \): Since \( \mathcal{A} \) is embeddable into \( \text{Orth}(\mathcal{A}) \) by means of \( h \), we can identify \( \mathcal{A} \) with a sublattice of \( \text{Orth}(\mathcal{A}) \). Let \( a \) be a nilpotent element in \( \mathcal{A} \). Then the element \( a \) is also nilpotent in \( \text{Orth}(\mathcal{A}) \).

But the unitary \( f \)-algebra \( \text{Orth}(\mathcal{A}) \) is semiprime, i.e. \( a \) is the zero element in \( \text{Orth}(\mathcal{A}) \) and also in \( \mathcal{A} \).

\textbf{Remark 4} \quad \text{Let } \mathcal{A} \text{ be a semiprime } f \text{-algebra. Then}
\[
\pi_\varphi \in \Phi_3(\text{Orth}(\mathcal{A})) \quad \Rightarrow \quad \varphi \in \Phi_3(\mathcal{A}).
\]
Indeed, by Lemma 1 the \(f\)-algebra \(A\) is a sublattice of \(\text{Orth}(A)\), and so for each \(x \in A\) we obtain

\[
|x| \wedge n|\varphi| = |\pi_x| \wedge n|\pi \varphi| \leq c_{|\pi_x|}|\pi \varphi| = c_{|\pi_x|}|\varphi|
\]

for any \(n \in \mathbb{N}\) and some constant \(c_{|\pi_x|} \in \mathbb{R}_{\geq 0}\). Notice that the same statement for finite and totally finite elements, in general, is not true, since in these cases the majorants might not belong to \(A\).

The inverse implication, in general, is not true because for \(\varphi \in \Phi_3(A)\) the element \(\pi \varphi \in \text{Orth}(A)\) may not be a majorant for itself. Indeed, if \(A\) does not possess a multiplicative unit then for \(x \in \text{Orth}(A) \setminus A\) a corresponding constant \(c_x\) might not exist.

**Theorem 10** Let \(A\) be a semiprime \(f\)-algebra and let there exist a norm on \(A\), under which \(A\) is a Banach lattice. Then

\[
\Phi_1(A) = \Phi_2(A) = \Phi_3(A) = A.
\]

**Proof.** Since \(A\) is semiprime, according to Lemma 1 the \(f\)-algebra \(A\) can be embedded as a subalgebra into \(\text{Orth}(A)\). We write \(A \subseteq \text{Orth}(A)\) after identifying \(A\) with its image \(h(A)\) in \(\text{Orth}(A)\). According to [2, Theorem 15.5] the identity \(I\) is an order unit in \(\text{Orth}(A)\). By Corollary 1 we obtain

\[
\Phi_3(\text{Orth}(A)) = \text{Orth}(A) \supseteq A.
\]

It follows for two arbitrary elements \(a, \varphi \in A_+\) that

\[
a \wedge n \varphi \leq c_a \varphi,
\]

i.e. all positive elements in \(A\) are self-majorizing. Therefore, as the cone \(A_+\) is reproducing in \(A\), each element \(x \in A\) is self-majorizing, and we get \(\Phi_1(A) = \Phi_2(A) = \Phi_3(A) = A\).

\[
\square
\]

## 5 Finite elements in product algebras

Let \(A\) be an \(\ell\)-algebra and \(p \in \mathbb{N}_{>2}\). The following construction is well-known. For details the reader is referred to [4, 7, 8, 19]. By

\[
\Pi_p(A) := \{g_1 \cdots g_p: g_i \in A \text{ for } i = 1, \ldots, p\} \subseteq A
\]

we denote the set of all \(p\)-fold products in \(A\). Clearly, \(\Pi_p(A) \subseteq A\). In general, this inclusion is proper, e.g. define \(A\) to be as in Example 2. Even if \(A\) is a semiprime uniformly complete \(f\)-algebra, then, in general, still \(\Pi_p(A) \neq A\), see e.g. [5], page 136, where an example for a semiprime uniformly complete and not square-root closed \(f\)-algebra is provided.

If the set \(\Pi_p(A)\), equipped with the order and algebraic operations induced from \(A\), turns out to be an algebra, then it is called the **product algebra of order \(p\) of \(A\)**. Denote by

\[
\Sigma_p(A) := \{g^p: g \in A_+\}
\]

the set of all \(p\)-fold powers of positive elements of \(A\).

For completeness we provide without proofs some important properties of \(\Pi_p(A)\).
Let $\mathcal{A}$ be a uniformly complete $f$-algebra and $p \in \mathbb{N}_{\geq 2}$.

- The set $\Pi_p(\mathcal{A})$ is a semiprime uniformly complete $f$-subalgebra of $\mathcal{A}$ (see [8, Corollary 5.3(iv)] [7, Corollary 3] and [9, Corollary 4]).

- The set $\Pi_p(\mathcal{A})$ is a vector lattice under the ordering inherited from $\mathcal{A}$, where

$$\Pi_p^+(\mathcal{A}) = \Sigma_p(\mathcal{A}). \quad (5)$$

Additionally, for the supremum $\vee_p$ and infimum $\wedge_p$ in $\Pi_p(\mathcal{A})$ the following formulas hold:

$$f^p \wedge_p g^p = (f \wedge g)^p \quad \text{and} \quad f^p \vee_p g^p = (f \vee g)^p \quad \text{for} \quad f, g \in \mathcal{A}_+.$$

(6)

- If $f_1, \ldots, f_p \in \mathcal{A}$ are arbitrary elements, then for the modulus of the product $f_1 \cdots f_p$ in $\Pi_p(\mathcal{A})$ the following formula is true

$$|f_1 \cdots f_p| = |f_1| \cdots |f_p|.$$

(7)

(see [4, Proposition 1] and [8, Corollary 5.3(i), (iv)]).

Altogether we obtain for a uniformly complete $f$-algebra $\mathcal{A}$ and $p \in \mathbb{N}_{\geq 2}$ that $\Pi_p(\mathcal{A})$ is a semiprime uniformly complete $f$-subalgebra of $\mathcal{A}$, where the formulas (1), (5) – (7) hold.

We study now the finite elements in $\Pi_p(\mathcal{A})$.

**Theorem 11** Let $\mathcal{A}$ be a uniformly complete $f$-algebra and let $p \in \mathbb{N}_{\geq 2}$. Then

$$g \in \Phi_1(\mathcal{A}) \text{ with a majorant } u \implies g^p \in \Phi_1(\Pi_p(\mathcal{A})) \text{ with the majorant } u^p.$$  

If, in addition, $\mathcal{A}$ is semiprime, then

$$g \in \Phi_1(\mathcal{A}) \text{ with a majorant } u \iff g^p \in \Phi_1(\Pi_p(\mathcal{A})) \text{ with the majorant } u^p.$$  

**Proof.** Without loss of generality we assume $0 < g \in \Phi_1(\mathcal{A})$. Otherwise consider $|g|$ and apply (d1). If $u \in \mathcal{A}_+$ is a majorant of $g$, then for each $f \in \mathcal{A}_+$ there is a constant $c_f \geq 0$ with $f \wedge ng \leq c_f u$ for all $n \in \mathbb{N}$. Then by means of formula (6) for the $p$-th power we get

$$f^p \wedge_p n^p g^p = f^p \wedge_p (ng)^p = (f \wedge ng)^p \leq (c_f u)^p = c_f^p u^p,$$

where the last inequality follows\(^7\) due to the condition ($\ell_1$).

Let $f_1 \cdots f_p \in \Pi_p^+(\mathcal{A})$ be an arbitrary element. By (5) we have $\Pi_p^+(\mathcal{A}) = \Sigma_p(\mathcal{A})$, therefore there exists an $h \in \mathcal{A}_+$ with $f_1 \cdots f_p = h^p$. Using (8) we get

$$h^p \wedge_p n^p g^p \leq c_h^p u^p \quad \text{for all } n \in \mathbb{N}$$

\(^7\)The twofold application of ($\ell_1$) on $0 \leq a \leq b$ yields $a^2 \leq b^2$. Indeed, by multiplying the inequality $0 \leq a \leq b$ with $a$, resp. $b$, we obtain $a^2 \leq ab$, resp. $ab \leq b^2$. 

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and 
\[(f_1 \cdots f_p) \wedge_p n^p g^p = h^p \wedge_p n^p g^p \leq c_h^p u^p \quad \text{for all } n \in \mathbb{N}.\]

This shows that \(g^p \in \Phi_1(\Pi_p(\mathcal{A}))\) with the majorant \(u^p\).

\(\Leftarrow\): Let \(g^p\) be a positive finite element in \(\Pi_p(\mathcal{A})\). There exist elements \(u_1, \ldots, u_p \in \mathcal{A}_+\), such that for arbitrary \(a_1, \ldots, a_p \in \mathcal{A}_+\) the inequality
\[(a_1 \cdots a_p) \wedge_p n g^p \leq c_{a_1 \cdots a_p} (u_1 \cdots u_p)\]  
holds for all \(n \in \mathbb{N}\) and some number \(0 < c_{a_1 \cdots a_p}\). Since in \(\mathcal{A}\) there exists the element \(u = (u_1 \cdots u_p)^\frac{1}{p}\) the inequality (9) can be rewritten as
\[(a_1 \cdots a_p) \wedge_p n g^p \leq c_{a_1 \cdots a_p} u^p.\]  
(10)

Now let be \(a \in \mathcal{A}_+.\) By taking into consideration the relation (6) and the inequality (10) we get then
\[(a \wedge \sqrt[p]{n} g)^p = a^p \wedge_p (\sqrt[p]{n} g)^p = a^p \wedge_p n g^p \leq c_a u^p \quad \text{for all } n \in \mathbb{N}.\]

Due to the semiprimitivity the root is monotone and there holds
\[\left((a \wedge \sqrt[p]{n} g)^p\right)^\frac{1}{p} \leq \sqrt[p]{c_a u}.\]

Therefore for all \(n \in \mathbb{N}\) there follows the inequality\(^8\) \(a \wedge \sqrt[p]{n} g \leq \tilde{c}_a u\) with \(\tilde{c}_a = \sqrt[p]{c_a u}.\) This shows \(g \in \Phi_1(\mathcal{A})\) with \(u\) as one of its majorants.

**Corollary 5** Let \(\mathcal{A}\) be a uniformly complete \(f\)-algebra and let \(p \in \mathbb{N}_{\geq 2}\). Then

1. \(g_1, \ldots, g_p\) are finite in \(\mathcal{A}\) \(\implies\) \(g_1 \cdots g_p\) is finite in \(\Pi_p(\mathcal{A})\).

If, in addition, \(\mathcal{A}\) is semiprime, then

2. \(g_1 \cdots g_p\) is finite in \(\Pi_p(\mathcal{A})\) \(\implies\) \((g_1 \cdots g_p)^\frac{1}{p}\) is finite in \(\mathcal{A}\),

3. \(g_1, \ldots, g_p\) are finite in \(\mathcal{A}\) \(\implies\) \((g_1 \cdots g_p)^\frac{1}{p}\) is finite in \(\mathcal{A}\),

4. \(\Phi_1(\Pi_p(\mathcal{A})) \subseteq \Phi_1(\mathcal{A})\),

5. \(\Phi_1(\Pi_p(\mathcal{A})) = \Phi_1(\mathcal{A}) \cap \Pi_p(\mathcal{A})\), provided \(\mathcal{A}\) has the weak factorization property.

**Proof.** 1. Let \(g_1, \ldots, g_p\) be positive finite elements in \(\mathcal{A}\). Then the element \(g = g_1 \lor \cdots \lor g_p\) is also finite in \(\mathcal{A}\) and, by the previous theorem the element \(g^p\) is finite in \(\Pi_p(\mathcal{A})\). Since \(0 \leq g_i \leq g\) for all \(i = 1, \ldots, p\) by condition \((\ell_1)\) we have
\[g_1 \cdots g_p \leq g_1 \cdots g_p - 1 g \leq g_1 \cdots g_p - 2 g^2 \leq \ldots \leq g_1 g^{p-1} \leq g^p.\]

The element \(g_1 \cdots g_p\) is finite in \(\Pi_p(\mathcal{A})\) since \(\Phi_1(\Pi_p(\mathcal{A}))\) is an order ideal in \(\Pi_p(\mathcal{A})\).

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\(^8\)For each \(m \in \mathbb{N}\) there exists \(n \in \mathbb{N}\) such that \(m < \sqrt[p]{n}\), so that \(a \wedge mg \leq a \wedge \sqrt[p]{n} g \leq \tilde{c}_a u\) for all \(m \in \mathbb{N}\).
Let \( g_1, \ldots, g_p \) be arbitrary finite elements in \( \mathcal{A} \). By the first part of the proof the element \( |g_1| \cdots |g_p| \) is finite in \( \Pi_p(\mathcal{A}) \). Due to (7) we have \( |g_1| \cdots |g_p| = |g_1 \cdots g_p| \) and so \( g_1 \cdots g_p \) is a finite element in \( \Pi_p(\mathcal{A}) \).

2. Without loss of generality we may assume that \( g_1, \ldots, g_p \) are positive elements in \( \mathcal{A} \), otherwise consider \( |g_1|, \ldots, |g_p| \) and apply (7). According to Remark 3(1) there exists an element \( \tilde{g} = (g_1 \cdots g_p)^{\frac{1}{p}} \in \mathcal{A} \). The equality \( g_1 \cdots g_p = ((g_1 \cdots g_p)^{\frac{1}{p}})^p = \tilde{g}^p \) shows that the finiteness of \( g_1 \cdots g_p \) in \( \Pi_p(\mathcal{A}) \) implies that \( \tilde{g}^p \) is finite. By the theorem one has \( \tilde{g} \in \Phi_1(\mathcal{A}) \).

3. Follows directly from 1. and 2.

4. Let \( g_1 \cdots g_p \in \Phi_1(\Pi_p(\mathcal{A})) \). Then by part 2. we get \( (g_1 \cdots g_p)^{\frac{1}{p}} \in \Phi_1(\mathcal{A}) \), which according to Theorem 11 yields \( g_1 \cdots g_p \in \Phi_1(\mathcal{A}) \).

5. The relation \( \subseteq \) follows from 4. For the converse relation \( \supseteq \) let \( \varphi \in \Phi_1(\mathcal{A}) \cap \Pi_p(\mathcal{A}) \). Then the element \( \varphi \) can be written as a \( p \)-fold product \( \varphi = \varphi_1 \cdots \varphi_p \) and therefore possesses the root \( \varphi^{\frac{1}{p}} \in \mathcal{A} \). By Theorem 8 we have \( \varphi \in \Phi_1(\mathcal{A}) \) and by means of 1. then \( \varphi \in \Phi_1(\Pi_p(\mathcal{A})) \). \( \Box \)

In the next corollary we obtain some information on the totally finite and the self-majorizing elements in an \( f \)-algebra. For its proof we need the following

**Lemma 2** Let \( \mathcal{A} \) be a uniformly complete \( f \)-algebra and let \( p \in \mathbb{N}_{\geq 2} \). Then for all \( g \in \mathcal{A} \) the following implication holds:

\[
\text{If, in addition, } \mathcal{A} \text{ is semiprime, then} \\
g \in S(\mathcal{A}) \implies g^p \in S(\Pi_p(\mathcal{A})).
\]

(11)

**Proof.** Let \( g \) be a self-majorizing element in \( \mathcal{A} \), i.e. \( |g| \) is a majorant of \( g \) in \( \mathcal{A} \). By Theorem 11 this implies that \( |g|^p \) is a majorant of \( g^p \) in \( \Pi_p(\mathcal{A}) \). Formula (7) yields the equality \( |g|^p = |g^p|^p \), so \( |g^p|^p \) is a majorant of \( g^p \) in \( \Pi_p(\mathcal{A}) \). Therefore \( g^p \in S(\Pi_p(\mathcal{A})) \).

Conversely, let \( g^p \in S(\Pi_p(\mathcal{A})) \), i.e. \( |g^p|^p \) is a majorant of \( g^p \) in \( \Pi_p(\mathcal{A}) \). The equality \( |g^p|^p = |g|^p \) and Theorem 11 imply that \( |g| \) is a majorant of \( g \) in \( \mathcal{A} \) and therefore, \( g \in S(\mathcal{A}) \). \( \Box \)

**Corollary 6** Let \( \mathcal{A} \) be a uniformly complete \( f \)-algebra and let \( p \in \mathbb{N}_{\geq 2} \). Then for all \( g \in \mathcal{A} \) the following implications hold:

1. \( g \in \Phi_2(\mathcal{A}) \implies g^p \in \Phi_2(\Pi_p(\mathcal{A})) \).

2. \( g \in \Phi_3(\mathcal{A}) \implies g^p \in \Phi_3(\Pi_p(\mathcal{A})) \).

**Proof.** 1. \( \Rightarrow \): Let \( g \in \Phi_2(\mathcal{A}) \) have a finite majorant \( u \in \mathcal{A} \). By the first part of Theorem 11 we obtain \( g^p \in \Phi_1(\Pi_p(\mathcal{A})) \) with majorant \( u^p \), and the same theorem guarantees the finiteness of the majorant \( u^p \) in \( \Pi_p(\mathcal{A}) \), i.e. \( g^p \in \Phi_2(\Pi_p(\mathcal{A})) \).

\( \Leftarrow \): Let \( g^p \in \Phi_2(\Pi_p(\mathcal{A})) \) with a finite majorant \( u_1 \cdots u_p \). By Remark 3(1) we can write this majorant as a \( p \)-fold product \( u_1 \cdots u_p = ((u_1 \cdots u_p)^{\frac{1}{p}})^p = u^p \) of the element \( u = (u_1 \cdots u_p)^{\frac{1}{p}} \). Then the semiprimity of \( \mathcal{A} \) and Theorem 11 yield \( g \in \Phi_1(\mathcal{A}) \) with the majorant \( u \) and also the finiteness of the majorant \( u \) in \( \mathcal{A} \). Therefore \( g \in \Phi_2(\mathcal{A}) \).

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2. The set $\Phi_3(A)$ coincides with the order ideal generated by the set $S(A)$ (see [25, Corollary 2]), i.e.

$$\Phi_3(A) = \{a \in A : \exists s_1, \ldots, s_n \in S(A) \text{ and } \lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0} \text{ with } |a| \leq \sum_{i=1}^{n} \lambda_i |s_i| \}.$$ 

Since $\sum_{i=1}^{n} \lambda_i |s_i|$ is a positive self-majorizing element (see [25, Proposition 1]), the order ideal $\Phi_3(A)$ can be written as $\Phi_3(A) = \{a \in A : \exists s \in S_+(A) : |a| \leq s\}$. 

$\Rightarrow$: Let $g \in \Phi_3(A)$. There is an $s \in S_+(A)$ such that $|g| \leq s$. Since $s$ is a majorant of $s$ in $A$, the element $s$ is also a majorant of $|g|$. By the first part of Theorem 11 we obtain that the element $|g|^p$ is finite in $\Pi_p(A)$ with a majorant $s^p$. Due to (11) the element $s^p$ is self-majorizing in $\Pi_p(A)$. The formula (7) yields that the element $|g|^p|_p$ belongs to the order ideal generated by $S_+(\Pi_p(A))$, i.e. $g^p \in \Phi_3(\Pi_p(A))$. 

$\Leftarrow$: Conversely, let $g^p \in \Phi_3(\Pi_p(A))$. Since $\Phi_3(\Pi_p(A))$ is the order ideal generated by $S_+(\Pi_p(A))$ in $\Pi_p(A)$, there is an element $s \in S_+(\Pi_p(A))$ such that $|g|^p|_p \leq s$. Using Remark 3(1) we can write the majorant $s$ as $s = s_1 \cdots s_p = \tilde{s}^p$, where $\tilde{s} := (s_1 \cdots s_p)^{\frac{1}{p}}$. Notice that $\tilde{s}^p$ has itself as a majorant in $\Pi_p(A)$. Due to (12) and the second part of Theorem 11 the element $\tilde{s}$ is self-majorizing in $A$ and is a majorant of $g$ in $A$. Therefore we obtain $g \in \Phi_3(A)$.

By summing up the results obtained in Theorem 11, Corollaries 5 and 6 we may write

**Corollary 7** Let $A$ be a semiprime uniformly complete $f$-algebra and $p \in \mathbb{N}_{\geq 2}$. Then for $i = 1, 2, 3$ there holds

$$(\Phi_i(A))^p = \Phi_i(\Pi_p(A)),$$

where $(\Phi_i(A))^p = \{g_1 \cdots g_p \in \Pi_p(A) : g_1, \ldots, g_p \in \Phi_i(A)\}$.

**Proof:** Let $i = 1$. Indeed, the relation ”$\subseteq$” follows from Corollary 5.1. The relation ”$\supseteq$” is obtained as follows: Let $g \in \Phi_i(\Pi_p(A))$, i.e. $g = g_1 \cdots g_p$ with $g_j \in A$, $j = 1, \ldots, p$. Then by Corollary 5.2. the element $g^{\frac{1}{p}}$ is finite in $A$. From $g = (g^{\frac{1}{p}})^p$ it is clear that $g$ is a product consisting of $p$ finite elements of $A$, i.e. $g \in (\Phi_1(A))^p$.

The cases $i = 2, 3$ are proved similarly using Corollary 6. 

The proof of the second inclusion of Corollary 7 (for $i = 1$) shows that each finite element of $\Pi_p(A)$ has a representation as the $p$-th power of a single finite element of $A$. In general, $g = g_1 \cdots g_p \in \Phi_1(\Pi_p(A))$ does not imply $g_1, \ldots, g_p \in \Phi_1(A)$, what is demonstrated by the next example.

**Example 4** Let $A = C([0, \infty))$ be the vector lattice of all continuous functions on the interval $[0, \infty]$ and equip $A$ with the pointwise order and the algebraic operations. Then $A$ is an Archimedean unitary semiprime uniformly complete $f$-algebra. For $p = 3$ consider $\Pi_3(A) = \{f_1 f_2 f_3 : f_1, f_2, f_3 \in A\}$. Since the function $1_{[0, \infty)}$ is the multiplicative unit in $A$, all functions of $A$ belong to $\Pi_3(A)$. This means $A$ and $\Pi_3(A)$ coincide.

The finite elements in $A$ are exactly the functions with compact support. Consider the following three functions of $A$:

$$f_1(t) = t, \quad f_2(t) = 1_{[0, \infty)} \quad \text{and} \quad f_3(t) = \begin{cases} \sin t & \text{for } t \in [0, \pi], \\ 0 & \text{for } t \in (\pi, \infty). \end{cases}$$

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The only finite element among them is $f_3$. The product $f_1 f_2 f_3$, i.e. the function

$$
\varphi(t) = \begin{cases} 
    t \sin t & \text{for } t \in [0, \pi], \\
    0 & \text{for } t \in (\pi, \infty),
\end{cases}
$$

is a finite element in $\mathcal{A} = \Pi_3(\mathcal{A})$, however not all of its factors are finite elements.

In view of Corollary 7 we know that there exists a finite function $\tilde{\varphi}$ in $\mathcal{A}$ such that $\tilde{\varphi}^3 = f_1 f_2 f_3$. In our case this is the function

$$
\tilde{\varphi}(t) = \begin{cases} 
    (t \sin t)^{\frac{1}{3}} & \text{for } t \in [0, \pi], \\
    0 & \text{for } t \in (\pi, \infty).
\end{cases}
$$

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