VANISHING COHOMOLOGY ON A DOUBLE COVER OF A VERY GENERAL HYPERSURFACE

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Abstract. In this paper, we study the irreducibility of the monodromy action on the anti-invariant part of the vanishing cohomology on the double cover of a very general element in an ample hypersurface of a complex smooth projective variety branched at an ample divisor. As an application, we treat dominant rational maps from a double cover of a very general surface $S$ of degree $\geq 7$ in $\mathbb{P}^3$ branched at a very general quadric surface to smooth projective surfaces $Z$. Our method combines the classification theory of algebraic surfaces, deformation theory, and Hodge theory.

In this paper we continue to study the subfields of rational functions of complex surfaces pursued in [4], [6], and [7]. Our research has been motivated by the finiteness theorem for dominant rational maps on a variety of general type. Let $S$ be smooth complex projective variety of general type. The finiteness theorem states that dominant rational maps of finite degree $S \to Z$ to smooth projective varieties of general type, up to birational equivalence of $Z$, form a finite set. The proof follows from the approach of Maehara [8], combined with the results of Hacon and McKernan [5], of Takayama [9], and of Tsuji [10]. The main result in [6] is the following.

**Theorem 0.1.** (=$\text{Theorem 1.1 in [6]}$) Let $S \subset \mathbb{P}^3$ be a very general smooth complex surface of degree $d > 4$. Let $Z$ be a non-rational surface. Then there is no dominant rational map $f : S \to Z$ unless $f$ is a birational map.

The proof has been obtained by combining deformation theory of curves on surfaces, Hodge theoretical methods, and namely the Lefschetz theory. In [7], where the case of product of curves $C \times D$ have been considered, we also needed to take in account the $H^2$-Hodge structure of étale covering of $C \times D$ together with some improvement on the dimension counts on moduli. See also [3] for the complete intersection case.

The involved problems and the analogy with the curves theory exploiting Jacobian and Prym varieties convinced us that would be appropriate to consider the case of double coverings.

To explain our results we let $X$ be a complex smooth projective variety of dimension $n \geq 2$ with $H^0(\Omega^{n-1}_X) = 0$ where $\Omega^{n-1}_X = \wedge^{n-1}\Omega^1_X$ and $\Omega^1_X$ is the cotangent bundle of $X$. Let $B \subset X$ be a smooth divisor of $X$ and assume that the line bundle $L = \mathcal{O}_X(B)$ is two divisible $L = M^\otimes 2$. Let $\pi : Y \to X$ be the double cover branched at $B$ defined by the square of a line bundle $M$ i.e., $\mathcal{O}_X(B) = M^\otimes 2$ on $X$ and let $j$ be the induced involution. Let $H$ be

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a very ample line bundle on \( X \) and \( \tilde{H} = \pi^*H \) be its pull back. Let \( S \) be a very general element in the linear system \( |H| \). Assume that \( S \) is canonical, i.e. the rational canonical map \( k : S \to |K_S| \) is birational onto its image. Let \( f : S \to Z \) be a generically finite dominant rational map of degree \( \geq 2 \).

We obtained in (Section 10 in [4]) that \( p_g(Z) = 0 \). In fact, from Lefschetz theory and the irreducibility of the monodromy action on vanishing cycles (cf. Chapters I–III in [11]) one get that the canonical map \( k \) of \( S \) factorizes through \( f \) gives a contradiction if \( p_g(Z) \neq 0 \). This was the starting point of our previous research [6].

This paper obtains a similar result for a very general element \( \tilde{S} \) in \( \tilde{H} \) under the assumption that \( \tilde{H} \) is very ample and the ramification divisor \( R \) is ample. The \( n-1 \)-th primitive cohomology \( P_{n-1}(\tilde{S}, \mathbb{Q}) \) has a natural decomposition into the invariant part \( P^+ \) and the anti-invariant part \( P^- \), that respects the monodromy action of suitable pencils. By the ampleness of the ramification divisor \( R \), the \( P^- \) turns out to be the anti-invariant part of the vanishing cohomology. Then we show by means of the monodromy action that the anti-invariant part of vanishing cohomology is irreducible. This is the content of Theorem 1.3. Assume that \( H \) is positive enough, the precise hypothesis are given in Assumption 2.1. Then we obtain our desired application:

**Theorem 0.2.** (=Theorem 2.2) Let \( f : \tilde{S} \to Z \) be a dominant rational map where \( Z \) is a smooth projective \( n-1 \)-fold. Then \( Z \) is birational to \( \tilde{S} \), or \( S \), or we have \( p_g(Z) = 0 \).

As an application we obtain the following theorem combined with the proof of theorem 1.1 in [6] and Theorem 2.2.

**Theorem 0.3.** (=Theorem 2.7) Let \( X = \mathbb{P}^3 \) and let \( H = \mathcal{O}_{\mathbb{P}^3}(d) \) where \( d \geq 7 \). Let \( \pi : Y \to X \) be a double cover branched at a very general quadric surface. Suppose there is a dominant rational map \( f : \tilde{S} \to Z \) where \( Z \) is a smooth projective surface. Then \( Z \) is birational to \( \tilde{S} \), or \( S \), or \( \mathbb{P}^2 \).

As far as we know this gives the first examples of fields of transcendence degree 2 of non-ruled surfaces that contain only one proper non-rational subfield of transcendence degree 2.

The method of our proof combines the classification theory of algebraic surfaces, deformation theory, and Hodge theory. A careful study of Hodge theory, especially Lefschetz theory on Lefschetz pencil, Picard-Lefschetz formula, irreducibility of monodromy actions on vanishing cycles, in very ample hypersurfaces on a double cover is a main ingredient of this paper.

In this paper we work on the field of complex numbers.

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1. Irreducibility of the monodromy action

1.1. Double coverings. Let $X$ be a complex smooth projective variety of dimension $n > 1$. Let $B \subset X$ be a smooth divisor of $X$ and assume that the line bundle $L = \mathcal{O}_X(B)$ is two divisible. This means that $L = M \otimes 2$ where $M$ is a line bundle of $X$. Let $\pi : Y \rightarrow X$ be the double cover branched at $B$ defined by $M \otimes 2$ and let $j$ be the induced involution. The variety $Y$ is smooth and the ramification divisor $R$ is isomorphic to $B$. Let $H$ be a very ample line bundle on $X$ and $\tilde{H} = \pi^*H$ be its pullback.

Lemma 1.1. The line bundle $\tilde{H}$ is very ample on $Y$ if and only if $H \otimes M^{-1}$ is generated by global sections on $X$.

Proof. Assume that $H \otimes M^{-1}$ is generated by global sections. We have the identification

$$H^0(Y, \tilde{H}) = H^0(X, \pi_*\tilde{H}) = H^0(X, H) \oplus H^0(X, H \otimes M^{-1}) = W^+ \oplus W^-$$

where the sign corresponds to the positive and the negative eigenvalue induced by the involution $j^*$ induced by $j$. We show now that if $p \neq j(p)$, then points $p$ and $j(p)$ are separated by the global sections of $\tilde{H}$. In fact we can find sections $s^+ \in W^+$ and $s^- \in W^-$ such that $s^+(p) = s^-(p) = 0$. It follows that $s^-(j(p)) = -s^+(j(p))$. Therefore $s = s^+ + s^-$ vanishes on $j(p)$ but not on $p$. Next we show that $\tilde{H}$ separates the tangents at the point $p \in R$ of the ramification divisor. Since the line bundle $H$ is very ample this will complete the result. We have to show to the surjectivity of the map $\psi$ induced by derivation:

$$\psi : H^0(Y, \tilde{H}) \rightarrow \tilde{H} \otimes T_{Y,p}^*$$

For a nonzero vector $v \in T_{Y,p}$ we have to find a section $s \in H^0(Y, \tilde{H})$ such that $v \cdot \psi(s) \neq 0$. The differential of $j$ gives the eigenvector decomposition $T_{Y,p} = T^+ \oplus T^-$. Choose local coordinates $\{U, x_i\}$ such that $x_i(p) = 0$, $R \cap U = \{x_1 = 0\}$ and give the linearization of $j : (x_1, x_2, \ldots, x_n) \mapsto (-x_1, x_2, \ldots, x_n).$ The map $\pi$ in these coordinates becomes

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1^2 = y, x_2, \ldots, x_n),$$

and $y = 0$ gives the equation of $B \cap U$. We see that $\dim T^- = 1$. Write $v = v^+ + v^-$. If $v^+ \neq 0$ then we can find a section $s = s^+$ such that $v^+ \cdot s \neq 0$ and then $v \cdot s^+ = v^+ \cdot s \neq 0$. Therefore we can suppose that $v = v^-$. We need to find a section $s \in W^- \otimes \tilde{H}$ that vanishes at $p$ of order 1 in the $v$-direction. In our coordinates we may assume $v = \frac{\partial}{\partial x_1}$. Taking a trivialization of $H$ on $U$, the restriction of the section $s \in W^- \subset H^0(X, H)$ becomes a regular function $f = f(s)$ such that $f(-x_1, x_2, \ldots, x_n) = -f(x_1, x_2, \ldots, x_n)$. It follows that $f = x_1 g(x_1, x_2, \ldots, x_n)$ where $g$ is $j$-invariant. Under the identification
$W^- = H^0(X, H \otimes M^{-1})$ the local expression of $s$ in this trivialization is given by $g$. Therefore

$$v \cdot s = \frac{\partial f}{\partial x_1}(0, \ldots, 0) = g(0, \ldots, 0).$$

Now $g(0) \neq 0$ holds $\iff s$ is not in the kernel of the restriction map

$$H^0(X, H \otimes M^{-1}) \to H^0(X, H \otimes M^{-1})_p.$$ 

The existence of such an $s$ is equivalent to the surjectivity of $H^0(X, H \otimes M^{-1}) \to H^0(X, H \otimes M^{-1})_p$. The converse is clear. □

1.2. Lefschetz pencils. With the previous notation we assume that $H$ and $\tilde{H}$ are very ample and let $\ell \subset |H|$ be a pencil of global sections of $H$ i.e., $\ell = \mathbb{P}(V)$ where $V$ is a two-dimensional subspace of the vector space $H^0(X, H)$. Let $\tilde{\ell}$ be the pull back of $\ell$. We also assume

1) the pencil $\{H_t\}_{t \in \ell}$ is a Lefschetz pencil of $X$;

2) the restriction of $\ell$ to $B$: $\{B \cap H_t\}_{t \in \ell}$ is a Lefschetz pencil $\ell_B$ of $B$.

Our definition of a Lefschetz pencil is the classical one: any singular fiber has only one singular node (Chapter 2 in [11]). If the dimension of $B$ is one, we ask for a simple ramification of the map $B \to \mathbb{P}^1$.

By blowing-up the base loci of the pencils, we obtain $\tilde{B} \subset \tilde{X}$ and $\tilde{Y}$, and we get the fibrations

1) $h : \tilde{X} \to \mathbb{P}^1$

2) $h_B : \tilde{B} \to \mathbb{P}^1$.

We also have

3) $g : \tilde{Y} \to \mathbb{P}^1$

and a two-to-one map that ramifies on $\tilde{B}$:

\[
\begin{array}{ccc}
\tilde{Y} & \overset{\#}{\longrightarrow} & \tilde{X} \\
\downarrow{g} & & \downarrow{h} \\
\mathbb{P}^1 & & \\
\end{array}
\]

With the exception (that we may exclude from now on) $X = \mathbb{P}^2$, $B$ a conic and $H = \mathcal{O}_{\mathbb{P}^2}(1)$, any divisors in the pencil $\ell$ intersect transversally $B$ in at least one point, this gives that all the divisors in $\tilde{\ell}$ are irreducible. Then $g$ is a connected fibration, and all fibers are irreducible and have at most nodes. Nevertheless $\tilde{\ell}$ is not a Lefschetz pencil on $Y$.

The singular fibers of $g$ are of two types

I) the inverse image $Y_{s'} = \tilde{\pi}^{-1}(X_{s'})$ of a singular divisor $X_{s'} \in \ell$;

II) the inverse image $Y_{s''} = \tilde{\pi}^{-1}(X_{s''})$ where $B \cap X_{s''} = X_{B,s''} \in \ell_B$ is singular.

The set $S \subset \mathbb{P}^1$ of the critical values of $h$, has a disjoint decomposition $S = S' \cup S''$
where \(S'\) are the critical values of \(h\) and \(S''\) the critical values of \(h_B\). In the case I) \(\tilde{Y}_{s'}\) has two singular nodal points and in the case II) \(Y_{s''}\) is simply tangent to \(B\) in one point \(p \in S'\) and \(Y_{s''}\) has only one nodal singularity.

Set \(V = \mathbb{P}^1 \setminus S\) and \(U = g^{-1}(V)\), then by restriction we have a smooth fibration
\[
g_U : U \to V.
\]

We consider the local system \(R^{n-1}g_U^*\mathbb{Q}\) defined over \(V\). By fixing a point \(p \in V\), this local system is equivalent to the monodromy action of the fundamental group \(\pi_1(V, p)\) on \(H^{n-1}(Y_p, \mathbb{Q})\), where \(Y_p = g^{-1}(p)\). The involution \(j\) gives a decomposition
\[
R^{n-1}g_U^*\mathbb{Q} = R^+ \oplus R^-
\]
and \(H^{n-1}(Y_p, \mathbb{Q}) = H^+ \oplus H^-\) into the invariant and the anti-invariant parts. We get that \(H^+\) is isomorphic to \(H^{n-1}(X_p, \mathbb{Q})\) where \(X_p = h^{-1}(p)\) is the fiber of \(h\). Let \(P(X, \mathbb{Q}) \subset H^{n-1}(X, \mathbb{Q})\) (resp. \(P(Y, \mathbb{Q}) \subset H^{n-1}(Y, \mathbb{Q})\)) be the primitive cohomology which respect to \(P\) (resp. \(H\)). We can also define the subsystems \(P^+ \subset R^+\) and \(P^- \subset R^-\) that are given as \(\pi_1(V, p)\) modules on \(P^{n-1}(Y_p, \mathbb{Q})\). We decompose \(P^{n-1}(Y_p, \mathbb{Q})\) into the invariant and anti-invariant part: \(P^{n-1}(Y_p, \mathbb{Q}) = P^+ + P^-\).

Consider again \(S' \cup S'' = S\), that we separate by open disks \(D', D''\), \(S' \subset D' \subset \mathbb{P}^1\) and \(S'' \subset D'' \subset \mathbb{P}^1\) such that \(D' \cap D'' = \emptyset\) and such that the base point \(p\) is in the closure of the disk \(p \in D' \cap D''\). Fix generators of \(\pi_1(V, p)\) corresponding to loops \(\gamma_s, s \in S\) in such a way that \(\gamma_{s'}(t) \in D'\) and \(\gamma_{s''}(t) \in D''\) for \(t \in [0, 1]\). Then we define the free groups \(G'\) and \(G''\) generated by the loops around points of \(S'\) and of \(S''\), respectively: \(G' = \langle \gamma_{s'} \rangle\) and \(G'' = \langle \gamma_{s''} \rangle\). We have that
\[
\pi_1(V, p) \equiv G' \ast G''/\alpha,
\]
where \(\ast\) stands for the free product and \(\alpha\) is the relation given by a suitable product of the loops, homotopically equivalent to the boundary of the above disks.

Let \(i : X_p \to X\) be the inclusion. Then we have by Lefschetz theory
\[
P := P^{n-1}(X_p, \mathbb{Q}) = i^* P^{n-1}(X) \oplus H^{n-1}(X_p)_{\text{van}},
\]
where \(H^{n-1}(X_p)_{\text{van}}\) is the kernel of
\[
i_* : H^{n-1}(X_p) \to H^{n+1}(X_p).
\]
We have that \(H^{n-1}(X_p)_{\text{van}}\) is an irreducible \(G'\)-module generated by the vanishing cycles \(\delta_{s'}\) where \(s' \in S'\) for the fibration \(\tilde{X} \to \mathbb{P}^1\). And the \(\delta_{s'}\) are all conjugate (see Chapter 3 in [11]) by \(G'\). We have for \(s' \in S'\)
\[
\bar{\pi}(\delta_{s'}) = \alpha_1 s' + \alpha_2 s',
\]
where \(\alpha_{1,2} i = 1, 2\) are the vanishing cycles of the two nodes over a point \(s' \in S'\). Since the \(\delta_{s'}\) are conjugated by \(G'\) it follows that, up to a sign, the cycles \(\beta_{s'} = \alpha_1 s' - \alpha_2 s' \in P^-\), are conjugated by the group \(G'\). In a similar way the cycles \(\beta_{s''} \in P^-\), where \(s'' \in S''\) the vanishing cycle corresponding to the points \(s'' \in S''\) are conjugated by the group \(G''\). Let \(H_{S'} \subset P^-\) be
the subspace generated by \( \{ \beta_{s'} \}_{s' \in S'} \), and \( H_{S''} \subset P^- \) be the subspace generated by \( \{ \beta_{s''} \}_{s'' \in S''} \).

We let \( \mathbb{G} := \text{Gr}_1(\vert H \vert) \) be the Grassmannian variety parametrized lines of the linear system \( \vert H \vert \). Let \( t \in \mathbb{G} \). Let \( W \subset \mathbb{G} \) be the subset parametrized Lefschetz pencil in \( \vert H \vert \).

It is classical (see for instance Chapter 2 in [11]) that \( W \) is a nonempty Zariski open subset of \( \mathbb{G} \). Let \( \Delta \) be the complex unit disk, we can find a curve \( \rho : \Delta \rightarrow \mathbb{G} \), such that \( \rho(0) = \hat{t} \) and \( \rho(t) = \hat{t}_t \in W \) for \( t \neq 0 \). We consider the singular divisors in the pencil \( \hat{t}_t \) and its singular points \( S(t) \).

Then for any \( s' \in S' \) we define then two curves \( s'_1(t) \) and \( s'_2(t) \) in \( S(t) \) for \( t \in \Delta \) such that \( s'_1(0) = s'_2(0) \) and for any point \( s'' \in S'' \) a curve \( s''(t) \in S(t) \) such that \( s''(0) = s'' \). We have then by continuity we may assume

1. the vanishing cycle of \( s'_1(t) \) is \( \alpha_{1s'} \);
2. the vanishing cycle of \( s'_2(t) \) is \( \alpha_{2s'} \);
3. the vanishing cycle of \( s''(t) \) is \( \beta_{s''} \).

From Lefschetz theorem (apply to \( \hat{t}_t \)) we see that \( \{ \alpha_{1s'}, \alpha_{2s'} \}_{s' \in S'} \cup \{ \beta_{s''} \}_{s'' \in S''} \) generates the cohomology of \( H^{-1}(Y_p)_{\text{van}} \). And \( H^{-1}(Y_p, \mathbb{Q}) \) is

\[
H^{-1}(Y_p)_{\text{van}} \oplus i^* H^{-1} Y
\]

where \( i^* : Y_p \rightarrow Y \) is the inclusion. It follows that \( \{ \alpha_{1s'}, \alpha_{2s'} \}_{s' \in S'} \cup \{ \beta_{s''} \}_{s'' \in S''} \) generates the vanishing part of the anti-variant part of the primitive cohomology

\[
H_{S'} + H_{S''} = H^{-1}(Y_p)_{\text{van}}^-.
\]

That is, we have:

**Lemma 1.2.** \( H_{S'} + H_{S''} = H_{\text{van}}^- \) where \( H_{\text{van}}^- = H^{-1}(Y_p)_{\text{van}}^- \).

Another application of the Lefschetz theory gives:

**Theorem 1.3.** If \( \tilde{H} \) is very ample then the action of the monodromy of \( \pi_1(V, p) \) on \( H_{\text{van}}^- \) is irreducible.

*Proof.* Let \( F \subset H_{\text{van}}^- \) be a sub-local system. We have to show that either \( F = 0 \) or \( F = H_{\text{van}}^- \). Let \( F' \) be a sub-local system orthogonal to \( F \):

\[
F' = \{ v \in H_{\text{van}}^- : < v, w > = 0, \forall w \in F \}.
\]

Note that \( F' = 0 \) \( \iff \) \( F = H_{\text{van}}^- \) and \( F = 0 \) \( \iff \) \( F' = H_{\text{van}}^- \) since the polarization is non-degenerate on \( H_{\text{van}}^- \).

Now we have that for all \( s \in S \) either \( \beta_s \in F \) or \( \beta_s \in F' \). To see this, we take, for instance, \( s' \in S' \) and any \( v \in F \). Let \( T \) be the monodromy around \( s' \), then we must have \( T(v) \in F \). The monodromy map can be computed by means of the Picard-Lefschetz formula, and it gives (cf. Theorem 3.16, Chapter 3 in [11])

\[
T(v) = v + < v, \alpha_{1s'} > \alpha_{1s'} + < v, \alpha_{2s'} > \alpha_{2s'}.
\]

As \( v \in F \subset P^- \) then we have also

\[
0 = < v, \alpha_{1s'} + \alpha_{2s'} >= < v, \alpha_{1s'} > + < v, \alpha_{2s'} >,
\]

therefore

\[
T(v) = v + < v, \alpha_{1s'} > \alpha_{1s'} - < v, \alpha_{1s'} > \alpha_{2s'} = v + < v, \alpha_{1s'} > (\alpha_{1s'} - \alpha_{2s'})
\]
\[ v + \frac{1}{2} < v, \alpha_{1s'} - \alpha_{2s'} > (\alpha_{1s'} - \alpha_{2s'}) = v + \frac{1}{2} < v, \beta_{s'} > \beta_{s'}. \]

Now \( v \in F \) and \( T(v) \in F \) gives that \( T(v) - v \in F \), that is
\[ < v, \beta_{s'} > \beta_{s'} \in F. \]

Then either \( \beta_{s'} \in F \) or \( < v, \beta_{s'} >= 0 \), for all \( v \in F \), that is \( \beta_{s'} \in F' \). A similar computation applies to the \( \beta_{s''}, s'' \in S'' \).

Interchanging \( F \) with \( F' \) if necessary we may assume that there is \( s' \in S' \) such that \( \beta_{s'} \in F \). Since all the \( \beta_{s'} \) are conjugate by \( G' \), we obtain then \( H_{S'} \subset F \). If \( F \) contains an element \( \beta_{s''} s'' \in S'' \) the same argument shows that \( F' \supset H_{S''} \) therefore \( F' \supset H_{S'} + H_{S''} = H_{\text{van}} \) and the proof is complete. If we assume by contradiction that is not the case we will have \( F = H_{S'} \) and \( F' \supset H_{S''} \). In particular this implies for all \( s' \in S' \) and \( s'' \in S'' \):
\[ < \beta_{s'}, \beta_{s''} >= < \alpha_{1s'} - \alpha_{2s'}, \beta_{s''} >= 0. \]

We have also
\[ < \alpha_{1s'} + \alpha_{2s'}, \beta_{s''} >= 0 \]
and \( \beta_{s'} \in P^- \) and \( \alpha_{1s'} + \alpha_{2s'} \in P^+ \).

That is \( < \alpha_{1s'}, \beta_{s''} >= < \alpha_{2s'}, \beta_{s''} >= 0 \), but in this case we will have that \( H_{S''} = F' \) is invariant by the monodromy around all the critical points \( s_1'(t), s_2'(t), \) and \( s''(t) \) of the pencil \( \ell_t, \) fort \( \neq 0 \). This gives a contradiction with the Lefschetz irreducibility theorem since \( \ell_t \in W \) is a Lefschetz pencil. \( \square \)

Corollary 1.4. With the previous notation we assume that the ramification divisor \( R \) is ample. Then \( P^- = H_{\text{van}} \) and therefore it is irreducible.

Proof. As \( R \) is ample the map \( H^{n-1}(Y) \rightarrow H^{n-1}(R) \) is injective. Since the cohomology \( H^{n-1}(R) \) is \( j \) invariant it follows that \( H^{n-1}(Y)^- = 0 \). Then it follows that \( PH^{n-1}(Y_p)^- = H^{n-1}(Y_p)^{-\text{van}}. \) \( \square \)

We note that the ampleness of \( B \) is equivalent to the ampleness of \( R \) by Lemma 1.1.

2. Applications

2.1. Geometric genus of very general double coverings. We recall our notation. Let \( X \) be a smooth projective \( n \)-fold and \( H \) be a very ample divisor on \( X \). Let \( \pi : Y \rightarrow X \) be a double cover ramified over \( R \subset Y \), branched at \( B \subset X \), and \( M^{\otimes 2} = O_X(B) \). Let \( S \) be a very general element of \( |H| \). Let \( K_S = O_S(K_X + H) \) be the canonical bundle of \( S \) and set \( K_S(M) = O_S(K_X + H + M) \). We consider the canonical rational map \( k: S \dasharrow |K_S| \) and \( k': S \dasharrow |K_S(M)| \). We finally set \( \tilde{S} = \pi^{-1}(S) \).

Assumption 2.1. We assume additionally:

1. \( h^0(O_X^{n-1}) = 0; \)
2. \( B \) is smooth and very ample (or ample and base points free);
3. \( k \) and \( k' \) are birational onto its image.
It follows immediately that $\tilde{k}: \tilde{S} \dashrightarrow |K_{\tilde{S}}|$ is birational onto its image (if it factorizes through $\tilde{S} \rightarrow S \dashrightarrow |K_S|$ then the anti-invariant part must be trivial).

Moreover $h^0(\mathcal{O}_Y^n) = 0$: In fact from Corollary 1.4 $H^{n-1}(Y, \mathbb{Q}) = 0$. And we have
$$H^0(\mathcal{O}_Y^n) = H^0(\mathcal{O}_Y^n)^+ \oplus H^0(\mathcal{O}_Y^n)^- = H^0(\Omega_X^n) \oplus H^0(\Omega_Y^n)^- = 0$$
since $H^0(\Omega_X^n) = 0$ and by the Hodge decomposition
$$H^0(\mathcal{O}_Y^n)^- \subset H^{n-1}(Y, \mathbb{Q})^- \cong H^{n-1}(Y, \mathbb{Q})^- \otimes \mathbb{C} = 0.$$

**Theorem 2.2.** Let $f: \tilde{S} \dashrightarrow Z$ be a dominant rational map where $Z$ is a smooth projective $n - 1$-fold. Assume $\deg f > 1$. Then either $Z$ is birational to $S$ and $f \cong \pi$ up to birational automorphisms, or $p_g(Z) = 0$.

Before to give the proof we recall a standard notion. Let $V$ be a smooth projective $n$-fold with $n \geq 2$. Let $H_Y^n$ be the Hodge structure to $H^n(V, \mathbb{Z})$. The transcendental Hodge structure, $T_V$, of $V$ is the smallest sub-hodge structure of $H^n_Y$ such that $T^n_Y = H^{n,0}(V)$. We recall that $T_V$ is a birational invariant and moreover if $f: V \dashrightarrow W$ is a dominant rational of finite degree we have an injective Hodge-structure map $f^*: T_W \rightarrow T_V$ (this can be seen by resolving the indeterminacy of $f$). Let $\tilde{f}: V \rightarrow W$ be the map after resolving $f$. Let $H$ be a very ample divisor of $V$ and $S$ be a very general element of $|H|$. Suppose $h^0(\mathcal{O}_V^n) = 0$. It follows then that $H^{n-1}(S)_{\text{van}} = T_S$: it contains $H^{n-1,0}(S)$ since $H^{n-1,0}(V) = 0$ and it is irreducible by Lefschetz theory.

**Proof.** We set $T = T_S^{n-1}$. Then $T$ is decomposed into $T^+ \oplus T^-$. Since $h^0(\mathcal{O}_X^n) = h^0(\mathcal{O}_Y^n) = 0$ we get that $T^+ = H^{n-1}(S)_{\text{van}}$ and $T^- = H^{n-1}((S)_{\text{van}}$. Then under our hypothesis they are both irreducible (1.4). Assume by contradiction that $p_g(Z) \neq 0$ then the transcendental Hodge structure $T_Z$ is not zero. We get $f^* T_Z \subset T^+ \oplus T^-$ and $f$ then factorizes through $k$ or $k'$, or $k$ accordingly $f^* T_Z = T^+$, $f^* T_Z = T^-$ or $f^* T_Z = T^+ \oplus T^-$. Since the maps are all birational it proves our theorem. \hfill \square

We can continue this double covering construction. For each $i = 0, 1, 2, \ldots,$ we recall the double cover construction, $X^i$ is smooth projective $n$-fold, $\pi_i: X^{i+1} \rightarrow X^i$ a two-to-one map ramified on $B^i \subset X^{i+1}$ and branched at $B^i \subset X^i$. Let $M_i^{(2)} = \mathcal{O}_{X^i}(B^i)$, $H^i$ be a very ample divisor on $X^i$, and $S^i$ be a very general element of $|H^i|$. Let $K_{S^i} = \mathcal{O}_{S^i}(K_{X^i} + H^i)$ be the canonical bundle of $S^i$ and set $K_{S^i}(M_i) = \mathcal{O}_{S^i}(K_{X^i} + H^i + M_i)$. We consider the canonical rational map $k_i: S^i \dashrightarrow |K_{S^i}|$ and $k'_i: S^i \dashrightarrow |K_{S^i}(M_i)|$. We set $S^{i+1} = \pi_i^{-1}(S^i)$.

Let $X_0 = X, X_1 = Y, M_0 = M, H^0 = H, B^0 = B, R^0 = R, S^0 = S, S^1 = \tilde{S}$ in Assumption 2.1

**Assumption 2.3.** We assume additionally:

1. $h^0(\mathcal{O}_{X^0}^{n-1}) = 0$;
2. $B^i$ is smooth and very ample (or ample and base points free) for all $i$;
3. $k_i$ and $k'_i$ are birational onto its image for all $i$. 

By the same argument in Theorem 2.7, we get the following.

**Corollary 2.4.** Let \( f^i : S^i \dashrightarrow Z \) be a dominant rational map where \( Z \) is a smooth projective \( n - 1 \)-fold. Assume \( \deg f^i > 1 \). Then either \( Z \) is birational to one of \( S^j \) for \( j = 0, 1, \ldots, i - 1 \), or \( p_g(Z) = 0 \).

**Corollary 2.5.** Let \( X = \mathbb{P}^2 \) and let \( H = \mathcal{O}_{\mathbb{P}^2}(d) \) where \( d \geq 4 \). Set \( M = \mathcal{O}_{\mathbb{P}^2}(a) \) with \( 1 \leq a \leq d - 1 \). Let \( \pi : Y \rightarrow X \) be a double cover branched at a very general element \( B \in |M^{\oplus 2}| \). Let \( C \) be a very general element in \( |H| \) and let \( \tilde{C} = \pi^{-1}(C) \). Suppose there is a finite map \( f : \tilde{C} \rightarrow Z \) where \( Z \) is a smooth projective curve. Then \( Z \) is isomorphic to either \( C \), or \( \tilde{C} \), or \( Z = \mathbb{P}^1 \).

**Proof.** Under our hypothesis, \( H \otimes M^{-1} = \mathcal{O}_{\mathbb{P}^2}(d - a) \), \( K_C = \mathcal{O}_C(d - 3) \), and \( K_C + M = \mathcal{O}_C(d + a - 3) \) are very ample. Moreover, we have \( H^1(\mathbb{P}^2, \mathbb{Q}) = 0 \).

**Corollary 2.6.** Let \( X = \mathbb{P}^3 \) and let \( H = \mathcal{O}_{\mathbb{P}^3}(d) \) where \( d \geq 5 \). Let \( M = \mathcal{O}_{\mathbb{P}^3}(a) \) with \( 1 \leq a \leq d - 1 \). Let \( \pi : Y \rightarrow X \) be a double cover branched at a very general element \( B \in |M^{\oplus 2}| \). Let \( S \) be a very general element in \( |H| \) and let \( \tilde{S} = \pi^{-1}(S) \). Suppose there is a dominant rational map \( f : \tilde{S} \rightarrow Z \) where \( Z \) is a smooth projective surface. Then \( Z \) is birational to \( \tilde{S} \), or \( S \), or \( Z = \mathbb{P}^2 \).

**Proof.** Under our hypothesis, \( H \otimes M^{-1} = \mathcal{O}_{\mathbb{P}^3}(d - a) \), \( K_S = \mathcal{O}_S(d - 4) \), and \( K_S + M = \mathcal{O}_S(d + a - 4) \) are very ample. And \( h^0(\mathbb{Omega}_{\mathbb{P}^3}^2) = 0 \).

### 2.2. Rational maps

By using deformation of curves and similar arguments in Section 2 in [6], we can show:

**Theorem 2.7.** Let \( \pi : Y \rightarrow \mathbb{P}^3 \) be a double cover branched at a very general element in the linear system \( |\mathcal{O}_{\mathbb{P}^3}(2)| \). Let \( H = \mathcal{O}_{\mathbb{P}^3}(d) \) where \( d \geq 7 \) and let \( S \) be a very general element in \( |H| \). Set \( \tilde{S} = \pi^{-1}(S) \). Suppose there is a dominant rational map \( f : \tilde{S} \rightarrow Z \) where \( Z \) is a smooth projective surface. Then \( Z \) is birational to \( \tilde{S} \), or \( S \), or \( \mathbb{P}^3 \).

**Proof.** Suppose that \( Z \) is not birational to \( \tilde{S} \) and \( S \). By Corollary 2.6, it is enough to treat the case that \( p_g(Z) = 0 \). We note that \( Y \) is a quadric hypersurface in \( \mathbb{P}^4 \). Let \( D \) be a very general curve of \((d, d)\) type in a smooth quadric surface \( Q \). Then we claim that there is no birational immersion \( \kappa : D \rightarrow Z \) into any smooth projective surface \( Z \) with \( p_g(Z) = q(Z) = 0, \pi_1(Z) = 1 \), and non-negative Kodaira dimension if \( d \geq 7 \). The proof is similar to the argument in Section 2 in [6].

We can assume that \( Z \) is minimal because \( D \) is a very general curve of \((d, d)\) type in \( Q \). Let \( U \) be the Kuranishi space of deformation of \( \kappa \). Since \( \kappa \) is a very general birational immersion, a basic result (see Corollary 6.11 in [1], and Chapter XXI in [2]) gives that

\[
\dim U \leq h^0(\mathcal{O}_D(N_{D|Q})) - 6 = d^2 + 2d - 6
\]

by Riemann-Roch theorem and \( h^1(\mathcal{O}_D(N_{D|Q}) = 0 \). And \( g(D) = d^2 - 2d + 1 \).

Suppose that \( D \) can be birationally immersed in \( Z \) of general type with \( p_g(Z) = q(Z) = 0, \pi_1(Z) = 1 \). Then by the same argument in Proposition
2.3 in [6]
\[ d^2 + 2d - 6 - 19 \leq g(D) - \frac{\deg \kappa^*(K_Z)}{2}, \]
since minimal surfaces of general type with \( p_g = q = 0 \) depends on 19 parameters (Corollary in [4]). It implies that
\[ 4d - 26 + \frac{\deg \kappa^*(K_Z)}{2} \leq 0. \]
So we get a contradiction if \( d \geq 7 \) because \( \deg \kappa^*(K_Z) > 0 \).

Now, suppose \( D \) can be birationally immersed in \( Z \) with \( p_g(Z) = q(Z) = 0, \pi_1(Z) = 1, \) and of Kodaira dimension one. Then
\[ d^2 + 2d - 6 - 10 \leq g(D) - \frac{\deg \kappa^*(K_Z)}{2}, \]
because minimal surfaces of Kodaira dimension one with \( p_g(Z) = q(Z) = 0, \) and \( \pi_1(Z) = 1 \) depend also on 10 parameters (cf. Proof of Proposition 3.5 in [6]). It implies that
\[ 4d - 17 + \frac{\deg \kappa^*(K_Z)}{2} \leq 0. \]
So we get a contradiction if \( d \geq 5 \) because \( \deg \kappa^*(K_Z) \geq 0 \). We prove the claim.

Suppose there is a dominant rational map \( p \) from \( \tilde{S} \) to a smooth projective surface \( Z \). Let \( Z \) be a non-rational surface. By Corollary 2.6 we have \( p_g(Z) = 0 \), and by the argument in [4] we have \( \pi_1(Z) = 1 \). Since the intersection of \( \tilde{S} \) and a very general hyperplane section of \( Y \) is \( D \), we may assume that a general point of \( Z \) belongs \( \tilde{f}(D) \). By the above claim, \( \tilde{f}(D) \) cannot be birational. Therefore, we have two possible cases. The normalization of \( \tilde{f}(D) \) is a very general plane curve of degree \( d \) or rational. If the normalization of \( \tilde{f}(D) \) is a very general plane curve of degree \( d \) then we have a birational immersion from a very general plane curve of degree \( d \) to \( Z \). Then we get a contradiction by the result (proof of Theorem 1.1) in [6]. If the normalization of \( \tilde{f}(D) \) is rational then \( Z \) is a ruled surface. It follows that \( f \) is not dominant. Therefore we get a contradiction. \( \square \)

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