Thermodynamics and galactic clustering with a modified gravitational potential

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Based on thermodynamics, we study the galactic clustering of an expanding Universe by considering the logarithmic and volume (quantum) corrections to Newton’s law along with the repulsive effect of a harmonic force induced by the cosmological constant (Λ) in the formation of the large scale structure of the Universe. We derive the N-body partition function for extended-mass galaxies (galaxies with halos) analytically. For this partition function, we compute the exact equations of states, which exhibit the logarithmic, volume and cosmological constant corrections. In this setting, a modified correlation (clustering) parameter (due to these corrections) emerges naturally from the exact equations of state. We compute a corrected grand canonical distribution function for this system. Furthermore, we obtain a deviation in differential forms of the two-point correlation functions for both the point-mass and extended-mass cases. The consequences of these deviations on the correlation function’s power law are also discussed.

Keywords: Cosmology; Modified gravity; Galaxies cluster; Large scale structure of universe; Correlation function; Distribution function.

I. OVERVIEW AND MOTIVATION

The characterization of galactic clusters on very large scales under the influence of their mutual gravitational interaction is a matter of vast interest. The importance of such a process can be exaggerated as the evolution and distribution of the galaxies throughout the Universe are the main manifestations of this. The analysis of the correlation functions is one of the standard ways to study the formation of the Universe. The observation tells us that the power law of two-point correlation function scales as (intergalactic distance)$^{-1.6}$ to (intergalactic distance)$^{-1.8}$ [1], which has also been approved by N-body computer simulations [2] and by the analytic gravitational quasiequilibrium thermodynamics [3]. The calculation of the power law of the correlation function is based on the assumption that the conversion of the initial primordial matter into the observed many-body galaxies took place at the stage of evolution of the Universe and these galaxies are coupled to the expansion of the Universe. The theories of the many-body (galaxies) distribution function have been developed mainly from a thermodynamic point of view [4,5]. These theories utilize only the first two laws of thermodynamics to derive the exact equations of state of the expanding Universe (a quasiequilibrium evolution).

The relation between thermodynamics and relativity was originated in the work of Bekenstein [6], Hawking [7], and Unruh [8]. Later, Jacobson established an important connection between thermodynamics and general relativity, by which the Einstein equations themselves can be viewed as a thermodynamic equation of state under a set of minimal assumptions involving the equivalence principle and the identification of the area of a causal horizon with entropy [9,10]. Recently, Verlinde proposed a constructive idea stating that gravity is not a fundamental interaction and can be interpreted as an entropic force [11]. Although this idea fails to provide a rigorous physical explanation [12,13], it surely opens a new window into understanding gravity from first principles. For instance, the modified Newton’s law [20], Friedmann equations at the apparent horizon of the Friedmann-Robertson-Walker Universe [21,22], modified Friedmann equations [23,24], Newtonian gravity in loop quantum gravity [25], holographic dark energy [26,28], thermodynamics of black holes [29], and extension to Coulomb force [30], etc. support the entropic interpretation of gravity.

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Verlinde’s approach to get Newton’s law of gravity relies on the entropy-area relation of black holes in Einstein’s gravity, i.e., \( S = \frac{A}{4l_P^2} \), where \( S \) is the entropy of the black hole, \( A \) is the area of the horizon, and \( l_P \) is the Planck length. In order to include the quantum corrections to the area law, some modifications are required to the area law \([31]\). In the literature, the two well-studied quantum corrections to the area law are, namely, the logarithmic correction and power law correction. The logarithmic correction appears due to the thermal equilibrium fluctuations and the quantum fluctuations of loop quantum gravity \([32, 34]\). However, the power law correction appears due to the entanglement of quantum fields sitting near the horizon \([33, 34]\). Recently, combined corrections due to the logarithmic and volume terms are proposed for Newton’s law of gravitation as \([20]\)

\[
S = \frac{A}{4l_P^2} - a \log \left( \frac{A}{2l_P^2} \right) + b \left( \frac{A}{2l_P^2} \right)^{\frac{1}{2}},
\]

where \( a \) and \( b \) are the constants of the order of unity or less. Here, the second term of the rhs corresponds to the logarithmic correction and the last term of the rhs corresponds to the volume correction. Modesto and Randono \([20]\) discussed how deviations from Newton’s law caused by the logarithmic correction have the same form as the lowest-order quantum effects of perturbative quantum gravity; however, the deviations caused by the volume correction follow the form of the modified Newtonian gravity models explaining the anomalous galactic rotation curves. In fact, on the very large (cosmological) scale, it is expected or otherwise speculated that the cosmological constant, as a prime candidate for dark energy, is responsible for the expansion of the Universe through a repulsive force \([37]\). If the effect of the cosmological constant in the Newtonian limit of a metric is present in some phenomenon (such as clustering of galaxies), then the same effect should also be present in the full general relativistic treatment of the same phenomenon. Keeping the importance of the cosmological constant in mind, we want to employ the cosmological-constant-induced (harmonic-oscillator-type) modification to Newton’s law, which leads to the cosmic repulsive force. Our motivation here is to study the effects of such corrections on the characterization of the clustering of galaxies on very large scales. The clustering of galaxies within the framework of modified Newton’s gravity through the cosmological constant only has been studied very recently \([38]\).

To study the effects of the logarithmic, volume, and the cosmological constant corrections to the clustering of galaxies, we first derive the \( N \)-body partition function by evaluating configuration integrals recursively. From the resulting partition function, we extract various thermodynamical (exact) equations of state. For instance, we compute the Helmholtz free energy, entropy, pressure, internal energy, and chemical potential, which possess deviations from their original values due to the logarithmic, volume, and cosmological constant corrections. Remarkably, a modified correlation (clustering) parameter emerges naturally from the more exact equations of state. In the limit \( a \to 0, b \to 0, \) and \( \Lambda \to 0, \) the modified correlation parameter coincides with its original value given in \([\mathbb{E}]\). By assuming that the system is in a quasiequilibrium state as described by the grand canonical ensemble, we derive the probability distribution function. The resulting distribution function depends on the modified clustering parameter. Comparative analyses are made to see the effect of corrections on the probability distribution function. In this regard, we find that the corrected distribution function first increases sharply with the number of particles (\( N \)) and gets a maximum (peak) value for the particular \( N \). As long as \( N \) increases further beyond that particular value, the value of the distribution function starts descending very fast and becomes slow later. The peak value of the corrected distribution function decreases gradually with the increasing values of logarithmic and volume corrections. Remarkably, the highly corrected distribution function starts dominating the lesser corrected distribution function after a certain value of \( N \). Due to the cosmological constant term, the peak value of the corrected distribution function decreases even further and falls rather slowly. We compute the differential form of the two-point correlation function for both the cases of point-mass and extended-mass galaxies. By solving the corresponding differential equation, we obtain a modified structure of the two-point correlation function for both the point-mass and the extended-mass galaxies. It is shown that the corrections also affect the power law of the correlation function. Although there are corrections to the power law behavior, the correlation function obeys the original result (as in Refs. \([1, 3]\)) under certain approximations.

The paper is organized as follows. In Sec. II, we derive the \( N \)-body partition function for the gravitationally interacting system with the corrected Newtonian dynamics using the logarithmic, volume,
and cosmological constant terms. The thermodynamical properties and distribution functions for such a system are discussed in Sec. III. The differential form of the two-point correlation functions for both the point-mass and extended-mass galaxies are computed in Sec. IV. Within this section, the effect of corrections on the power law behavior of the two-point correlation functions is also discussed. Finally, the discussions and conclusions are made in the last section.

II. INTERACTION OF GALAXIES THROUGH MODIFIED POTENTIAL

In this section, we consider a modified Newton’s law of gravitation due to the first-order corrections and study their effects on the partition function.

A. A modified Newton’s law of gravitation

It has been stressed \[39\] that different quantum theories of gravity may lead to different higher-order corrections to the area law of Bekenstein-Hawking entropy. These corrections may display differences and, more interestingly, relations among quantizations. In \[39\], Kaul and Majumdar computed the lowest-order corrections to the area law in a particular formulation \[40\] of a quantum geometry program. They found that the leading correction is logarithmic, with \( \Delta S \sim \log \left( \frac{A}{2l_p^2} \right) \). On the other hand, in loop quantum gravity, the entropy introduces a dependence on the number of loops \( L \) for the spin-network state dual to a region of surface \[11\]. In the limit of a larger number of loops \( L \gg n \), where \( n \) is the number of boundary edges, the entropy behaves as \( S(L \gg n) \sim n \log L \sim n^{3/2} \propto A^{3/2} \), where \( L \) has exponential growth of the type \( L \sim 2^{\sqrt{n}} \). With these types of leading-order corrections to entropy, the expression of the (modified) area law results in \(1\).

In fact, the Newtonian force \( (F) \) in terms of entropy reads \( F = -4l_p^2 \frac{GM^2}{R^2} \frac{dS}{dR} \). Therefore, corresponding to the logarithmic and the volume corrected entropy \(1\), Newton’s force law gives

\[
F = -\frac{GM^2}{R^2} \left[ 1 - a \frac{l_p^2}{\pi R^2} + b \frac{12\sqrt{\pi}}{l_p} \log \left( \frac{R}{l} \right) \right].
\]

This leads to the following corrected gravitational potential energy \( (\Phi = -\int F dR) \) \[20\]:

\[
\Phi = -GM^2 \left[ \frac{1}{R} - a \frac{l_p^2}{3\pi R^3} - b \frac{12\sqrt{\pi}}{l_p} \log \left( \frac{R}{l} \right) \right],
\]

where \( l \) is an integration constant which signifies to (an unspecified) length parameter.

However, at the cosmological scale, it is speculated that the cosmological constant \( \Lambda \) is responsible for the expansion of the Universe through a repulsive force. For example, the Schwarzschild–de Sitter spacetime in its static form is given by the following line element \[42\]:

\[
ds^2 = f(R)dt^2 - \frac{1}{f(R)}dR^2 - R^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

\[f(R) = \left( 1 - \frac{2GM}{R} - \frac{\Lambda R^2}{3} \right).\]

Here we considered velocity of light \( c = 1 \). From the line element, it is natural to include an extra \( \Lambda \)-induced harmonic-oscillator-type potential \(-\frac{1}{6} \Lambda R^2 \) \[37\] to \[33\]. Therefore, by incorporating the cosmological-constant-induced modification to Newton’s law, the potential energy finally reads

\[
\Phi = -GM^2 \left[ \frac{1}{R} - a \frac{l_p^2}{3\pi R^3} - b \frac{12\sqrt{\pi}}{l_p} \log \left( \frac{R}{l} \right) + \frac{1}{6} \frac{\Lambda R^2}{GM^2} \right].
\]

In the next subsection, we will see the effect of these modifications on the many-body partition function.
The statistical mechanics of an \(N\)-body system is primarily based on the partition function. Here we note that all our analyses are based on the assumption that our gravitational system has a statistically homogeneous distribution over large regions, which consists of an ensemble of cells having the same volume \(V\) and the same average density \(\bar{\rho}(N/V)\). In order to deal with the galactic clustering from the statistical mechanics perspective, we first need to know the partition function \(Z_N(T,V)\) of the gravitationally interacting system, which consists of \(N\) particles of equal mass \(M\), momenta \(p_i\) and average temperature \(T\). This is generally given by:

\[
Z_N(T,V) = \frac{1}{\lambda^{3N} N!} \int d^3N p \ d^{3N} R \ \exp \left[ -\frac{H}{T} \right], \tag{5}
\]

where \(N!\) corresponds to the distinguishability of classical particles and \(\lambda\) is a normalization constant which results from the integration over momentum space. Here the \(N\)-body Hamiltonian has the following form:

\[
H = \left[ \sum_{i=1}^{N} \frac{p_i^2}{2M} + \Phi(r_1, r_2, \ldots, r_N) \right].
\]

In general, the gravitational potential energy, \(\Phi(r_1, r_2, \ldots, r_N)\), depends on the relative position vector of the \(i\)th and \(j\)th particles (i.e., \(R = |r_i - r_j|\)) and, hence, describes the sum of the potential energies of all pairs. Therefore, \(\Phi(r_1, r_2, \ldots, r_N)\) can be expressed as

\[
\Phi(r_1, r_2, \ldots, r_N) = \sum_{1 \leq i < j \leq N} \Phi_{ij}(R) = -T \sum_{1 \leq i < j \leq N} \log(1 + f_{ij}). \tag{6}
\]

The two-point function \(f_{ij}\) is introduced here to simplify the partition function elegantly. The \(f_{ij}\) takes nonzero values only if there are interactions present in the system. This function becomes negligibly small for the system with asymptotically high temperature as well.

Upon integration over momentum space, the expression for the partition function given in (5) reduces to the following,

\[
Z_N(T,V) = \frac{1}{N!} \left( \frac{2\pi MT}{\lambda^2} \right)^{3N/2} \Omega_N(T,V), \tag{7}
\]

where the configurational integral \(\Omega_N(T,V)\) has the following form:

\[
\Omega_N(T,V) = \int \ldots \int \prod_{1 \leq i < j \leq N} (1 + f_{ij}) d^{3N} R. \tag{8}
\]

In this work, we will study the clustering of galaxies interacting through the modified Newtonian dynamics due to the logarithmic, volume, and \(\Lambda\)-induced corrections.

It is evident from (1) that for the point-mass (galaxies\(^1\)) particles (i.e., \(R = 0\)), the potential energy diverges. This leads to an ill-defined Hamiltonian and, thus, the partition function. In order to remove this divergence, we consider the extended nature of galaxies (galaxies with halos) by introducing a softening parameter \(\epsilon\), which assures that the galaxies are of finite size. The softening parameter takes a typical value \(0.01 \leq \epsilon \leq 0.05\) in units of the constant cell. Thus, the effective potential energy modified by the logarithmic, volume, and \(\Lambda\) terms, for the extended (real) mass galaxies in an expanding Universe, is given by

\[
\Phi_{ij}(R) = -GM^2 \left[ \frac{1}{(R^2 + \epsilon^2)^{1/2}} - \frac{l_p^2}{3\pi (R^2 + \epsilon^2)^{3/2}} - b \frac{6\sqrt{\pi}}{l_p} \log \left( \frac{R^2}{l_p^2} \right) + \frac{1}{6} \frac{\Lambda R^2}{GM^2} \right]. \tag{9}
\]

By definition, we note that the systems (that are still clustering) are not virialized on all scales, which implies that the two-particle function \(f_{ij}\) of the system will be dominating up to linear order only. Thus, the two-particle function for the potential energy (1) is given by

\[
f_{ij} = \frac{GM^2}{T} \left[ \frac{1}{(R^2 + \epsilon^2)^{1/2}} - \frac{l_p^2}{3\pi (R^2 + \epsilon^2)^{3/2}} - b \frac{6\sqrt{\pi}}{l_p} \log \left( \frac{R^2}{l_p^2} \right) + \frac{1}{6} \frac{\Lambda R^2}{GM^2} \right]. \tag{10}
\]
Here the higher-order terms are neglected. Exploiting relation (8), the configuration integral over a spherical volume of radius \( R_1 \) for \( N = 1 \) (i.e., \( f_{ij} = 0 \)) reads

\[
\Omega_1(T, V) = \int_0^{R_1} d^3R = V. \tag{11}
\]

Now, for \( N = 2 \), the configuration integral \( \Omega_2(T, V) \) (8) is calculated as

\[
\Omega_2(T, V) = 4\pi V \int_0^{R_1} dR \frac{R^2}{2} \left[ 1 + \frac{GM^2}{T} \left( \frac{1}{(R^2 + \epsilon^2)^{1/2}} - \frac{\ell_p^2}{3\pi (R^2 + \epsilon^2)^{3/2}} \right) - \frac{6\sqrt{\pi}}{l_p} \log \left( \frac{R^2}{T^2} \right) + \frac{1}{6GM^2} \right].
\]

Here, in order to convert the double integral into the single integral, we consider the position of one particle (galaxy) to be fixed. The expression of the configuration integral \( \Omega_2(T, V) \) further simplifies to

\[
\Omega_2(T, V) = V^2 \left[ 1 + \frac{3GM^2}{2R_1T} \left( \frac{1}{1 + \frac{\epsilon^2}{R_1^2}} + \frac{\epsilon^2}{R_1^2} \log \frac{\epsilon/R_1}{1 + \sqrt{1 + \frac{\epsilon^2}{R_1^2}}} \right) + \frac{\ell_p^2}{3\pi} \left( \frac{1}{R_1^2 \sqrt{1 + \frac{\epsilon^2}{R_1^2}}} - \frac{1}{R_1^2 \epsilon} \log \frac{\epsilon/R_1}{1 + \sqrt{1 + \frac{\epsilon^2}{R_1^2}}} \right) + \frac{6\sqrt{\pi}}{l_p} \left( \frac{R_1}{3} + R_1 \log \frac{l}{R_1} \right) + \frac{\Lambda R_1^3}{10T} \right]. \tag{12}
\]

In more compact form, this reads

\[
\Omega_2(T, V) = V^2 \left[ 1 + \frac{3}{2} (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \frac{GM^2}{R_1T} \right]. \tag{13}
\]

which utilizes the following definitions:

\[
\alpha_1(\epsilon) = \sqrt{1 + \frac{\epsilon^2}{R_1^2}} + \frac{\epsilon^2}{R_1^2} \log \frac{\epsilon/R_1}{1 + \sqrt{1 + \frac{\epsilon^2}{R_1^2}}}, \quad \beta_1 = \frac{8\sqrt{\pi}}{l_p} \left( \frac{R_1}{3} + R_1 \log \frac{l}{R_1} \right),
\]

\[
\alpha_2(\epsilon) = \frac{\ell_p^2}{3\pi} \left[ \frac{1}{R_1^2 \sqrt{1 + \frac{\epsilon^2}{R_1^2}}} - \frac{1}{R_1^2 \epsilon} \log \frac{\epsilon/R_1}{1 + \sqrt{1 + \frac{\epsilon^2}{R_1^2}}} \right], \quad \beta_2 = \frac{\Lambda R_1^3}{15GM^2}. \tag{14}
\]

Since \( \frac{GM^2}{R_1T} \) is a dimensionless quantity and remains invariant under the scale transformations \( T \to \eta^{-1}T \) and \( R_1 \to \eta R_1 \), then according to the scaling property (see Ref. [43] for details), we scale the quantity \( \frac{GM^2}{R_1T} \) to \( \left( \frac{GM^2}{R_1T} \right)^3 \) without the loss of generality. Following this scaling, the expression (13) reduces to

\[
\Omega_2(T, V) = V^2 \left[ 1 + \frac{3}{2} \left( \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \right) \left( \frac{GM^2}{R_1T} \right)^3 \right]. \tag{15}
\]

Since the radius of the cell \( R_1 \) and the average number density per unit volume \( [\bar{\rho} \sim (N/V)] \) are related through \( R_1 \sim (\bar{\rho})^{-1/3} \), then

\[
\frac{3}{2} \left( \frac{GM^2}{R_1T} \right)^3 \approx \frac{3}{2} \left( \frac{GM^2}{T} \right)^3 \bar{\rho} := \omega. \tag{16}
\]
In terms of $\omega$, the configuration integral (15) is expressed by

$$\Omega_2(T, V) = V^2 \left[ 1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega \right].$$

(17)

Following the procedure mentioned above for $N$ particles recursively, we get the most general configuration integral as follows:

$$\Omega_N(T, V) = V^N \left[ 1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega \right]^{N-1}.$$ 

(18)

Plugging the value of $\Omega_N(T, V)$ (18) simply into the general partition function (7), we get the following (explicit) expression for gravitational partition function:

$$Z_N(T, V) = \frac{1}{N!} \left( \frac{2\pi M T}{\lambda^2} \right)^{3N/2} V^N \left[ 1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega \right]^{N-1}.$$ 

(19)

This is a (canonical) partition function for a gravitational system of $N$ particles interacting through the modified Newton’s law. The corrections due to the logarithmic, volume, and cosmological constant terms are inherent in the parameters $\alpha_2$, $\beta_1$, and $\beta_2$ respectively. Once the expression of the partition function is known, it is matter of calculating the exact equations of state. The expressions of the exact equations of state are important because these serve as the primary ingredients to evaluate all the thermodynamical properties rigorously.

### III. THERMODYNAMICS OF GALAXIES UNDER MODIFIED POTENTIAL

In this section, we first derive the various exact equations of state for the gravitating system under the modified Newtonian dynamics. Later, we emphasize the gravitational quasiequilibrium distribution function which originates from the canonical partition function given in (19).

#### A. Exact equations of state

In order to derive the various equations of state, let us begin with the Helmholtz free energy. The Helmholtz free energy is related to the general partition function by $F = -T \log Z_N(T, V)$ (here Boltzmann’s constant is set to 1). Therefore, for the partition function given in (19), the Helmholtz free energy takes the following particular form:

$$F = -T \log \left[ \frac{1}{N!} \left( \frac{2\pi M T}{\lambda^2} \right)^{3N/2} V^N \left[ 1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega \right]^{N-1} \right].$$

(20)

which further simplifies to

$$F = NT \log \left( \frac{N}{V} T^{-3/2} \right) - NT - NT \log \left[ 1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega \right] - \frac{3}{2} NT \log \left( \frac{2\pi M}{\lambda^2} \right).$$

(21)

Here, keeping the large value of $N$ in mind, the approximation $N - 1 \approx N$ is assumed. Once the expression for the Helmholtz free energy is known, it is straightforward to calculate various important thermodynamical entities, like pressure, entropy, and chemical potential, which are directly related to the Helmholtz free energy. For instance, the entropy is related to the Helmholtz free energy in the following sense: $S = -\left( \frac{\partial F}{\partial T} \right)_{N,V}$. Therefore, corresponding to the Helmholtz free energy (21), the entropy reads

$$S = N \log \left( \frac{N}{V} T^{-3/2} \right) + N \log \left[ 1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega \right] - 3N \frac{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega}{1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega} + S_0,$$

(22)
where the fiducial entropy \( S_0 = \frac{5}{2} N + \frac{3}{2} N \log \left( \frac{2\pi M}{\lambda^2} \right) \). Now, the specific entropy \( \left( \frac{S}{N} \right) \) is evident from the above as follows:

\[
\frac{S}{N} = \log \left( \frac{V}{N} T^{3/2} \right) - \log \left[ 1 - \frac{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega}{1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega} \right] - \frac{3}{2} \frac{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega}{1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega} + s_0,
\]

where \( s_0 = \frac{5}{2} + \frac{3}{2} \log \left( \frac{2\pi M}{\lambda^2} \right) \) is an arbitrary constant. The entropy and free energy are related to the internal energy \( U \) through the relation \( U = F + TS \). Therefore, exploiting the expressions of the free energy \([21]\) and entropy \([22]\), the internal energy of the system is calculated as

\[
U = \frac{3}{2} NT \left[ 1 - \frac{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega}{1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega} \right].
\]

The pressure and the Helmholtz free energy are related through the identity \( P = -\left( \frac{\partial F}{\partial V} \right)_{N,T} \). This leads to the following pressure equation of state:

\[
P = \frac{NT}{V} \left[ 1 - \frac{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega}{1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega} \right].
\]

The measure of the exchange of particles (chemical potential \( \mu \)) for a given Helmholtz free energy \([21]\) is calculated by the following relation \( \mu = \left( \frac{\partial F}{\partial N} \right)_{V,T} \):

\[
\mu = T \log \left( \frac{N}{V T^{3/2}} \right) + T \log \left[ 1 - \frac{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega}{1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega} \right] - \frac{3}{2} T \log \left( \frac{2\pi M}{\lambda^2} \right)
\]

\[- \frac{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega}{1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega} + (\alpha_2 + \beta_1 + \beta_2)\omega \right].
\]

By drawing a comparison between the exact equations of state obtained here to their standard expressions (given in \([14]\)), we can classify the amount of corrections to these equations of the state. These expressions of Helmholtz free energy, entropy, free energy, pressure, and chemical potential yield the form of the modified clustering parameter \( (B_i) \) naturally. This is given by

\[
B_i = \frac{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega}{1 + (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)\omega}.
\]

It is worth evaluating the clustering (correlation) parameter, as this provides the information regarding the clustering of galaxies. The key feature of the clustering parameter is as follows: in the typical limit of vanishing gravitational interaction, the clustering parameter \( B_i \) tends to zero and, therefore, the system behaves as a perfect gas. However, as long as \( B_i \) increases towards unity, the system becomes more and more (strongly) bounded into clusters. The corrected clustering parameter for the extended-mass structure \([27]\), in terms of the original clustering parameter (when there is no correction) \( b_c = \alpha_1 \omega / [1 + \alpha_1 \omega] \) \([2]\), is given by

\[
B_i = \frac{b_c (1 - \alpha_2 \omega - \beta_1 \omega - \beta_2 \omega) + (\alpha_2 + \beta_1 + \beta_2)\omega}{1 + (\alpha_2 + \beta_1 + \beta_2)\omega - b_c (\alpha_2 \omega + \beta_1 \omega + \beta_2 \omega)}.
\]

Here we note that, as seen from \([14]\), the point-mass approximation (i.e., \( \epsilon = 0 \)) is possible for the volume and cosmological constant corrections only (when \( a = 0 \)), but not for the logarithmic case (when \( a \neq 0 \)) as the parameter \( \alpha_2 \) diverges in this approximation.

### B. Gravitational quasiequilibrium distribution

To study the distribution of voids, we assume that the system follows a quasiequilibrium state, which is described by the equilibrium thermodynamics at least as a first approximation \([2]\). This requirement is met
by considering a grand canonical ensemble, where the number of galaxies and their mutual gravitational energy vary among the members of the system. The grand partition function \(Z_G\) and the canonical partition function \(Z_N\) are related by

\[
Z_G(T, V, z) = \sum_{N=0}^{\infty} z^N Z_N(V, T),
\]

where \(z\) is an arbitrary variable. The grand partition function for the gravitationally interacting system can be expressed in terms of the thermodynamic variables as

\[
\log Z_G = \frac{PV}{T}.
\]

Exploiting relation (25), this further simplifies to

\[
\log Z_G = \tilde{N}(1 - B_i),
\]

where \(\tilde{N}\) refers to the average number of particles (which indicates a grand canonical system).

The distribution function \(F(N)\) for finding \(N\) particles in the energy state \(U(N, V)\) (of the grand canonical ensemble) is the sum over all of the energy states. It is given by

\[
F(N) = \frac{\sum_i e^{\frac{N\mu}{T} - U_i} e^{\frac{N\mu}{T} Z_N(V, T)}}{Z_G(T, V, z)} = \frac{e^{\frac{N\mu}{T} Z_N(V, T)}}{Z_G(T, V, z)}.
\]

Exploiting the values of \(Z_N\) and \(Z_G\), the distribution function for finding \(N\) extended-mass particles (galaxies) in the energy state \(U(N, V)\) is computed as

\[
F(N, \epsilon) = \frac{\tilde{N}}{N!} (\tilde{N}(1 - B_i) + NB_i)^{N-1} (1 - B_i) \exp [-NB_i - \tilde{N}(1 - B_i)].
\]

Here, we observe that the structure of \(F(N, \epsilon)\) is the same except for the value of the clustering parameter \(B_i\). Though the structure of the distribution function is similar to that of the unmodified Newtonian gravity case, the corrections due to the logarithmic, volume, and cosmological constant terms are inherent in the clustering parameter \(B_i\). Now, we would like to compare our results to those of the Ref. [5]. The comparative study, as given in Figs. 1 and 2, shows that \(F(N)\) takes the maximum (peak) value for \(N = \tilde{N}\). As long as the values of corrections increase, the peak value of \(F(N)\) decreases. But as long as the value of \(N\) increases, after a particular \(N\) the higher corrections of \(F(N)\) dominate their lower corrections. From these figures, it can be seen that \(F(N)\) decreases faster as the correction parameter increases.

### IV. THE MODIFIED CORRELATION FUNCTIONS

In this section, we compute the two-point correlation functions under the modified Newton’s law for the point-mass and extended-mass galaxies. This is done by solving the differential forms of the two-point correlation function. We analyze the power law behavior of the corrected correlation functions also.

#### A. Differential form for two-point correlation function

Let us start the analysis by writing the internal energy for the homogeneous particles system as

\[
U = \frac{3}{2} N \tilde{\rho} T + \frac{N \tilde{\rho}}{2} \int_V \Phi(R) \xi_2(\tilde{\rho}, R, T) dV,
\]

\[
(34)
\]
FIG. 1: The distribution function $F(N)$ (perpendicular axis) versus $N$ (horizontal axis) for an extended-mass structure for $b_1 = 0.5$ and $\bar{N} = 10$. Left: For $\beta_2 \omega = 0$, $(\alpha_2 + \beta_1) \omega = 0.5$ corresponds to the green line, and $(\alpha_2 + \beta_1) \omega = 1$ corresponds to the red line. Right: For $\beta_2 \omega = 1$, $(\alpha_2 + \beta_1) \omega = 0$ corresponds to the violet line, $(\alpha_2 + \beta_1) \omega = 0.5$ corresponds to the green line, and $(\alpha_2 + \beta_1) \omega = 1$ corresponds to the red line.

FIG. 2: The distribution function (perpendicular axis) versus $N$ (horizontal axis) for an extended-mass structure for $b_1 = 0.5$ and $\bar{N} = 50$. Left: For $\beta_2 \omega = 0$, $(\alpha_2 + \beta_1) \omega = 0$ corresponds to the violet line, $(\alpha_2 + \beta_1) \omega = 0.5$ corresponds to the green line, and $(\alpha_2 + \beta_1) \omega = 1$ corresponds to the red line. Right: For $\beta_2 \omega = 1$, $(\alpha_2 + \beta_1) \omega = 0$ corresponds to the violet line, $(\alpha_2 + \beta_1) \omega = 0.5$ corresponds to the green line, and $(\alpha_2 + \beta_1) \omega = 1$ corresponds to the red line.

where Boltzmann’s constant is set to unit. Here, temperature $T$ represents the random velocity; $\Phi(R)$ is a general interaction potential; and $\xi_2(\bar{\rho}, R, T)$ is the two-point correlation function. The pressure equation of state for the gravitating system is given by

$$P = \frac{NT}{V} - \frac{N\bar{\rho}}{6V} \int_V R \frac{d\Phi}{dR} \xi_2(\bar{\rho}, R, T) dV. \quad (35)$$

Here the dynamical conditions for quasiequilibrium evolution are taken into account.

Corresponding to the specific potential energy given in (4), the internal energy ($U_0$) and the pressure ($P_0$) equations of state in the case of point-mass galaxies read, respectively,

$$U_0 = \frac{3}{2} NT - \frac{GM^2 N\bar{\rho}}{2} \int_V \left[ \frac{1}{R} - a \frac{l_p^2}{3\pi R^3} - b \frac{12\sqrt{\pi}}{l_p} \log \frac{R}{l} + \frac{1}{6} \frac{L R^2}{GM^2} \right] \xi_2(\bar{\rho}, R, T) dV, \quad (36)$$

and

$$P_0 = \frac{NT}{V} - \frac{GM^2 N\bar{\rho}}{6V} \int_V \left[ \frac{1}{R} - a \frac{l_p^2}{\pi R^3} + b \frac{12\sqrt{\pi}}{l_p} - \frac{1}{3} \frac{L R^2}{GM^2} \right] \xi_2(\bar{\rho}, R, T) dV. \quad (37)$$

Here we note that, for the grand canonical ensemble, the two-point correlation function $\xi_2$ depends on the variables $\bar{\rho}, R$, and $T$. So, one can write the differential form of the two-point correlation function as

$$d\xi_2 = \frac{\partial \xi_2}{\partial \bar{\rho}} d\bar{\rho} + \frac{\partial \xi_2}{\partial T} dT + \frac{\partial \xi_2}{\partial R} dR. \quad (38)$$
This further leads to
\[
\frac{d\xi_2}{dV} = -\frac{\dot{\rho}}{V} \frac{\partial \xi_2}{\partial \rho} + \frac{1}{4\pi R^2} \frac{\partial \xi_2}{\partial R},
\]
where \(dT/dV = 0\) is employed. The Maxwell thermodynamic equation can be given in terms of internal energy \((U)\) and pressure \((P)\) by
\[
\left(\frac{\partial U}{\partial V}\right)_{T,N} = T \left(\frac{\partial P}{\partial T}\right)_{N,V} - P.
\]
Plugging the specific values of the internal energy \((36)\) and pressure \((37)\), obtained in the case of the point-mass galaxies, to the above Maxwell thermodynamic equation and then differentiating the resulting equation with respect to \(V\) leads to
\[
V \frac{d\xi_2}{dV} = \left[\frac{6\pi R^2 l_p GM^2 - 6al_p^3 GM^2 + 72\pi^{3/2} b R^3 GM^2 - 2\pi Al_p R^5}{6\pi R^2 l_p GM^2 - 2al_p^3 GM^2 - 72\pi^{3/2} b R^3 GM^2 \log \frac{2}{7} + \pi Al_p R^5}\right] T \frac{\partial \xi_2}{\partial T},
\]
A first-order partial differential equation for the two-point correlation function in the case of a point-mass structure has the following form:
\[
3\rho \frac{\partial \xi_2}{\partial \rho} - R \frac{\partial \xi_2}{\partial R} + 3 \left[\frac{6\pi R^2 l_p GM^2 - 6al_p^3 GM^2 + 72\pi^{3/2} b R^3 GM^2 - 2\pi Al_p R^5}{6\pi R^2 l_p GM^2 - 2al_p^3 GM^2 - 72\pi^{3/2} b R^3 GM^2 \log \frac{2}{7} + \pi Al_p R^5}\right] T \frac{\partial \xi_2}{\partial T} = 0,
\]
where we have utilized expressions \((39)\) and \((41)\).

By solving this first-order differential equation, we get the explicit form for the two-point correlation function as
\[
\xi_2(\rho, R, T) = c(\rho) T^+ R^{X+3\zeta} \left(2al_p^3 GM^2 - 6l_p \pi GM^2 R^2 + \pi Al_p R^5 + 72b\pi^{3/2} GM^2 R^3 \log \frac{2}{7}\right)^{-\zeta},
\]
where \(c\) is an integration constant and \(X\) and \(\zeta\) are some arbitrary constant parameters.

In order to get the differential equation for the two-point correlation in the case of an extended-mass structure, we write the expressions of internal energy \(U_{\text{ext}}\) and pressure \(P_{\text{ext}}\) as follows:
\[
U_{\text{ext}} = \frac{3}{2} NT - \frac{GM^2 N\dot{\rho}}{2} \int_V \left[\frac{1}{(R^2 + \epsilon^2)^{3/2}} - a\frac{l_p^2}{3\pi(R^2 + \epsilon^2)^3/2} \right] T \frac{\partial \xi_2}{\partial T} dV.
\]
\[
P_{\text{ext}} = \frac{NT}{V} - \frac{GM^2 N\dot{\rho}}{6V} \int_V \left[\frac{R^2}{(R^2 + \epsilon^2)^{3/2}} - a\frac{l_p^2 R^2}{\pi(R^2 + \epsilon^2)^{3/2}} \right] T \frac{\partial \xi_2}{\partial T} dV,
\]
which can be obtained simply by exploiting the relations \((31)\) and \((38)\) together with potential energy \((39)\).

Following the similar steps mentioned above for the case of point masses, we get a first-order partial differential equation for the two-point correlation function for the extended-mass galaxies as follows:
\[
\frac{6\pi GM^2 R^2(R^2 + \epsilon^2)l_p - 6al_p^3 GM^2 R^2 + 72\pi^{3/2} b GM^2(R^2 + \epsilon^2)^{3/2} - 2\pi AGM^2 l_p R^2(R^2 + \epsilon^2)^{3/2}}{6\pi GM^2(R^2 + \epsilon^2)^{3/2}l_p - 2al_p^3 GM^2(R^2 + \epsilon^2) - 72\pi^{3/2} b GM^2(R^2 + \epsilon^2)^{3/2} \log \frac{2}{7} + \pi AGM^2 l_p R^2(R^2 + \epsilon^2)^{3/2}} T \frac{\partial \xi_2}{\partial T}
= \frac{R}{3} \frac{\partial \xi_2}{\partial R} - \rho \frac{\partial \xi_2}{\partial \rho}.
\]
By solving this differential equation, we obtain an explicit expression of the two-point correlation function in the case of extended-mass galaxies:
\[
\xi_2(\rho, R, T) = C\rho T^+ R^X(R^2 + \epsilon^2)^{3\zeta/2} \left[2al_p^3 GM^2 - 6l_p \pi GM^2(R^2 + \epsilon^2) + \pi AGM^2 l_p R^2(R^2 + \epsilon^2)^{3/2} + 72b\pi^{3/2} GM^2(R^2 + \epsilon^2)^{3/2} \log \frac{2}{7}\right]^{-\zeta}.
\]

Here, we notice that the above expression for the two-point correlation function for an extended-mass structure in the limit $\epsilon \to 0$ reduces to the point-mass two-point correlation function \[43\].

### B. Power law for correlation function

Peebles’s assumption that the two-point correlation function in a gravitational (galaxy) clustering obeys a power law \[1\] is in agreement with both the N-body computer simulations \[2\] and analytic gravitational quasiequilibrium thermodynamics \[3\]. In order to see the effects of the logarithmic, volume, and the cosmological constant deviated power law of the two-point correlation function, we write the correlation parameter as

$$ B_i = \frac{GM^2 \bar{\rho}}{6T} \int_V \left[ \frac{1}{(R^2 + \epsilon^2)^{1/2}} - a \frac{l_p^2}{3\pi (R^2 + \epsilon^2)^{3/2}} - b \frac{6\sqrt{\pi}}{l_p} \log \left( \frac{R^2}{l_p^2} \right) + \frac{1}{6} \frac{\Delta R^2}{GM^2} \right] \xi_2(\bar{\rho}, R, T) dV, $$

where $\Delta R = R_0 - \bar{R}$. This further simplifies to

$$ B_i = \frac{2\pi GM^2 \bar{\rho}}{3T} \int_V \xi_2 \left[ \frac{R^2}{(R^2 + \epsilon^2)^{1/2}} - a \frac{l_p^2 R^2}{3\pi (R^2 + \epsilon^2)^{3/2}} - b \frac{6\sqrt{\pi}}{l_p} R^2 \log \left( \frac{R^2}{l_p^2} \right) + \frac{1}{6} \frac{\Delta R^4}{GM^2} \right] dR. \quad (47) $$

This form of correlation parameter is obvious due to the expressions \[27\] and \[44\].

By performing a differentiation with respect to $V$, this yields

$$ \frac{\bar{\rho}}{V} \frac{\partial B_i}{\partial \bar{\rho}} = \frac{B_i}{V} - \frac{GM^2 \bar{\rho}}{6T} \left[ \frac{1}{(R^2 + \epsilon^2)^{1/2}} - a \frac{l_p^2}{3\pi (R^2 + \epsilon^2)^{3/2}} - b \frac{6\sqrt{\pi}}{l_p} \log \left( \frac{R^2}{l_p^2} \right) \right] + \frac{1}{6} \frac{\Delta R^2}{GM^2} \xi_2(\bar{\rho}, R, T), \quad (48) $$

where relation $\frac{\partial \bar{\rho}}{\partial V} = -\frac{\bar{\rho}}{V}$ is utilized. This further simplifies to

$$ \frac{\bar{\rho}}{V} \left[ \frac{\partial B_i}{\partial \bar{\rho}} - \frac{B_i}{\bar{\rho}} \right] = -\frac{GM^2 \bar{\rho}}{6T} \left[ \frac{1}{(R^2 + \epsilon^2)^{1/2}} - a \frac{l_p^2}{3\pi (R^2 + \epsilon^2)^{3/2}} - b \frac{6\sqrt{\pi}}{l_p} \log \left( \frac{R^2}{l_p^2} \right) \right] + \frac{1}{6} \frac{\Delta R^2}{GM^2} \xi_2. \quad (49) $$

The relation \[27\] yields

$$ \frac{\partial B_i}{\partial \bar{\rho}} - \frac{B_i}{\bar{\rho}} = -\frac{B_i^2}{\bar{\rho}}. \quad (50) $$

Now, by substituting the value of \[50\] to Eq. \[49\], we obtain

$$ \frac{B_i^2}{V} = \frac{GM^2 \bar{\rho}}{6T} \left[ \frac{1}{(R^2 + \epsilon^2)^{1/2}} - a \frac{l_p^2}{3\pi (R^2 + \epsilon^2)^{3/2}} - b \frac{6\sqrt{\pi}}{l_p} \log \left( \frac{R^2}{l_p^2} \right) \right] + \frac{1}{6} \frac{\Delta R^2}{GM^2} \xi_2. \quad (51) $$

Consequently, we get the power law behavior of the two-point correlation function for the extended-mass galaxies as follows:

$$ \xi_2 = \frac{9TB_i^2/GM^2 \bar{\rho}}{\left[ \frac{2\pi R_0^2}{(R^2 + \epsilon^2)^{1/2}} - a \frac{2R^2 l_p^2}{3\pi (R^2 + \epsilon^2)^{3/2}} - 24\pi^{3/2} b \frac{R^2}{l_p} \log \frac{R}{l_p} + \frac{4\Delta R^4}{3GM^2} \right]}. \quad (52) $$

In the limit of point masses, this reduces to

$$ \xi_2 = \frac{9TB_i^2/GM^2 \bar{\rho}}{\left[ 2\pi R^2 - 4aR^2 - 24\pi^{3/2} b R^2 \log \frac{R}{l_p} + \frac{4\Delta R^4}{3GM^2} \right]}, \quad (53) $$

which, eventually, characterizes the modification to the power law equation given in \[3\]. These modifications are due to the logarithmic, volume, and $\Lambda$ terms. From observation, the correlation function has a simple power law \[1\],

$$ \xi_2 = \left( \frac{R_0}{R} \right)^\gamma, \quad (54) $$

where $\gamma$ is a constant determined by the parameters of the model.
where $\gamma \sim 1.77$ and $R_0 \sim 5.4h^{-1}\text{Mpc}$. This apparent simplicity has led many investigators to describe theoretical results and numerical simulations. However, $\xi_2$ possesses very limited information. Later observational analyses give a considerable range for $\gamma$ and $R_0$, which suggests that $\xi_2$ may not have a simple power law as given in (54); rather it requires the higher-order corrections to the correlation function. In this regard, the corrected correlation function having expression (53) may be of interest. The quantitative discussions, like the particular scale on which these modifications will be important, are the subject of further investigation. Remarkably, we note here that the power law still behaves as $\xi_2 \sim R^{-2}$ at leading order in the $l_P \sim R$ approximation.

V. DISCUSSION AND CONCLUSIONS

In order to study the clustering of galaxies in an expanding Universe under the modified Newton’s law, we have considered a logarithmic, volume, and the cosmological constant modified Newtonian potential. Corresponding to this modified Newtonian potential, we have calculated an explicit expression for the (canonical) partition function, which describes $N$ extended-mass galaxies, with the help of configuration integrals. Our whole calculation is based on the assumption that the system (ensemble), which is made of cells of the same volume $V$ and average density $\bar{\rho}$, has statistically homogeneous distribution over large regions. With the help of the resulting partition function, we have derived various thermodynamical equations of state, namely, the Helmholtz free energy, internal energy, pressure, entropy, and chemical potential equations of state. As a consequence of these exact equations of state, a corrected correlation (clustering) parameter emerges naturally for the clusters of the galaxies with halos. Here, we have found that the point-mass approximation is possible only for the volume correction but not for the logarithmic and $\Lambda$-induced corrections as the parameter $\alpha_2$ only diverges in the point-mass approximation. Nevertheless, this is not a serious threat as all the (real) galaxies are of finite size as described by the softening parameter $\epsilon$. Moreover, we have derived a modified version of the distribution function $F(N)$ (probability of finding $N$ gravitating bodies in volume $V$) for the gravitating system which follows the quasiequilibrium and, thus, resembles the grand canonical ensemble. We note that the corrections on the distribution function are inherent in the clustering parameter. The behavior of these corrections on the distribution function $F(N)$ is discussed through the plot (see, e.g., Figs. 1 and 2 above), where we observe that the peak (maximum) values of $F(N)$ (at $N = \bar{N}$) decrease as the correction dominates. As long as $N$ increases, after a certain value of $N$ the higher value of the correction starts dominating the smaller one because the values of $F(N)$ decrease faster for smaller corrections. The presence of the cosmological constant, which is responsible for the expansion of the Universe through a repulsive force, reduces the peak value (at $N = \bar{N}$) of the corrected distribution function even further, but makes the descent of the distribution function slower.

We have also derived the two-point correlation function ($\xi_2$) for the gravitating system under this modified Newton’s law for the cases of both point masses and extended masses. In this regard, we have first calculated the differential form of the modified two-point correlation function. The solution of the differential equation leads to the exact form of the two-point correlation function, where deviations from the original value are evident. In the limit $\epsilon \to 0$, the extended-mass two-point correlation function coincides to that of the point-mass case. The effect of corrections on the power law of the correlation function is also discussed and it has been found that it behaves as $\xi_2 \sim R^{-2}$ at leading order in a certain approximation, which is consistent with the result obtained in Ref. [3].

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