High-dimensional covariance matrices under
dynamic volatility models: asymptotics and
shrinkage estimation

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Abstract

We study the estimation of the high-dimensional covariance matrix and its
eigenvalues under dynamic volatility models. Data under such models have non-
linear dependency both cross-sectionally and temporally. We first investigate
the empirical spectral distribution (ESD) of the sample covariance matrix under
scalar BEKK models and establish conditions under which the limiting spectral
distribution (LSD) is either the same as or different from the i.i.d. case. We then
propose a time-variation adjusted (TV-adj) sample covariance matrix and prove
that its LSD follows the same Marčenko-Pastur law as the i.i.d. case. Based
on the asymptotics of the TV-adj sample covariance matrix, we develop a con-
sistent population spectrum estimator and an asymptotically optimal nonlinear
shrinkage estimator of the unconditional covariance matrix.

Keywords: High-dimension, dynamic volatility model, sample covariance ma-
trix, spectral distribution, nonlinear shrinkage

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1 Introduction

1.1 The Marčenko-Pastur law for the sample covariance matrix

Random matrix theory (RMT) is a powerful tool in the study of high-dimensional statistics. When the dimension and the sample size grow proportionally, for i.i.d. data, it is well known that the limiting spectral distribution (LSD) of the sample covariance matrix is connected to that of the population covariance matrix through the Marčenko-Pastur equation; see, e.g., Marčenko and Pastur (1967), Yin (1986), Silverstein and Bai (1995), and Silverstein (1995). Algorithms of recovering population spectrum based on the Marčenko-Pastur (M-P) law have been developed in El Karoui (2008), Ledoit and Wolf (2012, 2015, 2020). All these studies focus on the case where observations are i.i.d..

1.2 Dynamic volatility models

An important feature of financial returns is that their volatilities are time-varying and dependent over time. Dynamic volatility models such as the multivariate GARCH [Engle et al. (1984), Bollerslev et al. (1988)], the DCC model [Engle (2002)] and the BEKK model [Engle and Kroner (1995)] are popular in studying the dynamic variances and covariances. In particular, the widely used scalar BEKK model describes the dynamics of covariance matrix as follows:

$$\Sigma_{t+1} = (1 - a - b)\overline{\Sigma} + aR_tR_t^T + b\Sigma_t,$$

where $\overline{\Sigma}$ is the unconditional covariance matrix, $\Sigma_t$ is the conditional covariance matrix of the returns $R_t = (R_{1t}, ..., R_{pt})^T$, and $0 \leq a, b \leq 1$ with $a + b < 1$ are related parameters. The parameter $a$ is sometimes referred to as the innovation coefficient, and $a + b$ the persistence coefficient.

To estimate a dynamic volatility model, a common approach is variance/correlation targeting [Francq et al. (2011), Pedersen and Rahbek (2014), Pakel et al. (2020)]. The method requires estimating the unconditional covariance/correlation matrix. When
the dimension is high, the sample covariance/correlation matrix works poorly. For large dynamic volatility models, estimating the unconditional covariance/correlation matrix is challenging and calls for rigorous investigation.

1.3 Existing research in RMT for sample covariance matrix when there is time dependency

There exists a growing literature on the study of limiting spectral properties of the sample covariance matrix when there is time dependency. Jin et al. (2009), Yao (2012), Liu et al. (2015), and Bhattacharjee and Bose (2016) obtain the LSD of the sample covariance/autocovariance matrix of linearly dependent time series that can be transformed into data with independent columns. Banna and Merlevède (2015) and Merlevede and Peligrad (2016) investigate the LSD of the sample covariance matrix of stationary dependent processes with independent rows. Yaskov (2018) focuses on the case where data have dependence in finite lags. Zheng and Li (2011) establish the LSD, and Yang et al. (2021) derive the central limit theorem of linear spectral statistics of sample covariance matrix under elliptical models with time-varying co-volatilities.

Engle et al. (2019) propose to estimate the unconditional covariance/correlation matrix under large BEKK/DCC models using the nonlinear shrinkage (NLS) estimator developed in Ledoit and Wolf (2012, 2015). It has been documented in Ledoit and Wolf (2012, 2015) that the NLS estimator has several advantages in estimating the high-dimensional covariance matrix. For example, it is structure-free, and more importantly, for i.i.d. data, it is consistent in estimating the asymptotically optimal shrinkage estimator in the class of rotation-equivariant estimators; see Ledoit and Wolf (2012, 2015) for detailed explanations. It is worth emphasizing that the asymptotic property of the NLS relies on that the LSD of the sample covariance matrix follows the M-P law. For large dynamic volatility models, whether the NLS still enjoys the desirable asymptotic property is unclear.
1.4 Our contributions

We aim to estimate the unconditional covariance matrix under large dynamic volatility models. An important and natural question motivated by the proposal of Engle et al. (2019) is: does the NLS work under large dynamic volatility models?

To see how the dynamic volatility model can affect the spectral distribution of the sample covariance matrix, we simulate data from BEKK model (1.1) with $\Sigma = I$, $a = 0.05$ and $b = 0.9$, which is the setting used in Engle et al. (2019). The dimensions $p = 100, 500$, and the sample size $n$ satisfies $p/n = 0.8$. We compute the empirical spectral distribution (ESD) of the sample covariance matrix and compare it with the Marčenko-Pastur (M-P) distribution. The results are shown in Figure 1.

Figure 1 shows that the ESD of the sample covariance matrix under the BEKK model behaves differently from the M-P law. Therefore, it is problematic to perform NLS on the sample covariance matrix the same way as in the i.i.d. case.

In this paper, we investigate the limiting spectral properties of the sample covariance matrix under large BEKK models. We show that if $\eta(a, b, p) := (a/(1 - a - b)) \min \left( \sqrt{p(1 - a - b)}, 1 \right) \to 0$, then the LSD of the sample covariance matrix shares the same limit as the i.i.d. case; see Theorem 1. We call this case the reducible case. On the other hand, if $\eta(a, b, p)$ is bounded away from zero, then the spectral distribution of the sample covariance under the BEKK model is more heavy-tailed.
than the i.i.d. case; see Theorem 2. We call this case the non-reducible case.

Next, we address the problem of population spectrum estimation under large BEKK models. We first estimate the parameters $a$ and $b$ using a QMLE from univariate GARCH models. Next, we develop a projection matrix, $P_t$, which can track the time-variation in the conditional covariance matrices. We then define time-variation adjusted returns, $\tilde{R}_t = P_t^{-1/2} R_t$, and a time-variation adjusted (TV-adj) sample covariance matrix $\tilde{S}_n = \sum_{t=1}^{n} \tilde{R}_t \tilde{R}_t^T / n$. We prove that the TV-adj sample covariance matrix shares the same LSD as the i.i.d. case; see Theorem 3. Using the TV-adj sample covariance matrix and existing Marčenko-Pastur law reversing algorithms, we obtain a TV-adj shrinkage estimator of the population spectrum, which is shown to be consistent; see Corollary 1 for the exact statement.

Finally, we tackle the problem of unconditional covariance matrix estimation under large BEKK models. We develop a TV-adj nonlinear shrinkage (NLS) estimator and show that it consistently estimates the asymptotically optimal shrinkage estimator; see Theorem 5.

In summary, our contributions lie in the following aspects:

- First, we establish conditions under which the LSD of the sample covariance matrix under large BEKK models is the same as or different from the i.i.d. case.
- Second, we propose a TV-adj sample covariance matrix and develop an estimator that can consistently recover the population eigenvalues under large BEKK models.
- Third, we develop a TV-adj NLS estimator and prove that it is asymptotically optimal.

The rest of this paper is organized as follows. The main theoretical results are given in Section 2. Simulation studies are presented in Section 3. We conclude in Section 4. The proof of Theorem 3 is presented in Section 5. The proofs of other main results are collected in the Supplementary Material Ding and Zheng (2022).

The following notation is used throughout the paper. For any matrix $A = (A_{ij})$, its spectral norm is defined as $\|A\| = \max_{\|x\| \leq 1} \sqrt{x^T A^T A x}$, where $\|x\| = \sqrt{\sum x_i^2}$.
for any vector $\mathbf{x} = (x_i)$; the Frobenius norm is defined as $\|\mathbf{A}\|_F = \sqrt{\sum_{i,j} A_{ij}^2}$. We write $\mathbf{A} \geq 0 (> 0)$ if the matrix $\mathbf{A}$ is positive semi-definite (positive definite), and $\mathbf{A} \geq \mathbf{B} (> \mathbf{B})$ if $\mathbf{A} - \mathbf{B} \geq 0 (> 0)$. If $\mathbf{A} \geq 0$, $\mathbf{A}^{1/2}$ is defined as the matrix that satisfies $\mathbf{A}^{1/2} \geq 0$ and $(\mathbf{A}^{1/2})^2 = \mathbf{A}$. For any symmetric matrix $\mathbf{A}$ with eigenvalues $\lambda_1, ..., \lambda_p$, its empirical spectral distribution (ESD) is defined as $F^\mathbf{A}(x) = \frac{1}{p} \sum_{j=1}^{p} 1_{[\lambda_j, +\infty)}(x)$. Finally, we use $\overset{P}{\to}$ to represent convergence in probability.

2 Main Results

2.1 Setting and assumptions

Under a dynamic volatility model, returns are modeled as $\mathbf{R}_t = (\mathbf{\Sigma}_t)^{1/2} \mathbf{z}_t$, where $\mathbf{z}_t = (z_{1t}, ..., z_{pt})^T$ are i.i.d. with mean zero and covariance matrix $\mathbf{I}$. We suppose that $(\mathbf{R}_t)$ follows the scalar BEKK model (1.1). Define $\mathbf{R}_0^0 = (\mathbf{\Sigma})^{1/2} \mathbf{z}_t$, $t = 1, ..., T$, which share the same unconditional covariance matrix as $\mathbf{R}_t$ but are i.i.d.. Denote the corresponding sample covariance matrices by

$$S_n = \frac{1}{n} \sum_{t=1}^{n} \mathbf{R}_t \mathbf{R}_t^T, \quad \text{and} \quad S_0^n = \frac{1}{n} \sum_{t=1}^{n} \mathbf{R}_0^0 (\mathbf{R}_0^0)^T.$$ 

We write $\hat{\lambda}_1 \geq ... \geq \hat{\lambda}_p$ as the eigenvalues of $S_n$, and $\hat{\lambda}_1^0 \geq ... \geq \hat{\lambda}_p^0$ as the eigenvalues of $S_0^n$.

We impose the following assumptions.

Assumption 1

(i) $\mathbf{z}_t \overset{i.i.d.}{\sim} N(0, \mathbf{I})$.

(ii) $\mathbf{\Sigma}$ is nonnegative definite and its ESD, $F^\mathbf{\Sigma}$, converges in distribution to a probability distribution $H$ on $[0, \infty)$ as $p \to \infty$, and $H \neq \delta(0)$, the Dirac measure at 0.

(iii) $\|\mathbf{\Sigma}\| < C$ for some constant $C > 0$.

(iv) The dimension $p$ and the sample size $n$ satisfy that $p, n \to \infty$, and $p/n \to y > 0$.

About the parameters $a$ and $b$, we allow them to depend on $p$. Specifically, we
denote by $a_p$ and $b_p$ the coefficients in the BEKK model when the dimension is $p$.

2.2 Limiting property of ESD of sample covariance matrix under large BEKK model

2.2.1 Reducible case

Note that if $a_p = 0$, then the BEKK model reduces to the i.i.d. case with $\Sigma_t \equiv \overline{\Sigma}$. In general, if $a_p$ is close to 0, then the BEKK model will be similar to the i.i.d. case.

Recall that for any two distributions $F_1$ and $F_2$, the Levy distance between them is defined as

$$L(F_1, F_2) := \inf \{\varepsilon > 0 | F_1(x - \varepsilon) - \varepsilon \leq F_2(x) \leq F_1(x + \varepsilon) + \varepsilon \text{ for all } x \in \mathbb{R} \}.$$  \hspace{1cm} (2.1)

It is well known that convergence in Levy distance implies convergence in distribution.

Define

$$\eta(a, b, p) = \frac{a}{1 - a - b} \min \left( \sqrt{p(1 - a - b)}, 1 \right).$$

The next theorem shows convergence in Levy distance between $F^{S_n}$ and $F^{S_0}$ when $\eta(a_p, b_p, p) \to 0$.

**Theorem 1** Under model (1.1) and Assumption 1, if $\eta(a_p, b_p, p) \to 0$ as $p \to \infty$, then

$$L(F^{S_n}, F^{S_0}) = o_p(1).$$  \hspace{1cm} (2.2)

Theorem 1 implies that when $\eta(a_p, b_p, p) \to 0$ as $p \to \infty$, the ESD of sample covariance matrix under the BEKK model converges to the same M-P law as the i.i.d. case. We refer to the case when $\eta(a_p, b_p, p) \to 0$ as the reducible case. Under the reducible case, the population spectrum can be recovered by reversing the Marčenko-Pastur law.

2.2.2 Non-reducible case

When the reducible condition does not hold, what will the ESD of the sample covariance matrix be like? We have seen in Figure 1 that when $a = 0.05$ and $b = 0.9$, which
is a typical setting calibrated from empirical findings [Engle et al. (2019)], the ESD under the BEKK model appears to be more heavy-tailed than the i.i.d. case. We refer to the case when $\eta(a_p, b_p, p)$ is bounded away from zero as the non-reducible case. In practice, the two coefficients $a_p$ and $b_p$ learned from financial data appear to fit the non-reducible case. Therefore, the investigation of the non-reducible case is not only of theoretical interest but also practically relevant. We measure the difference in the ESD’s between the BEKK model and the i.i.d. case by the second moment of the ESD’s:

$$M_2 = M_{2}^p = \sum_{i=1}^{p} \hat{\lambda}^2_i / p = \text{tr} \left( \left( S_n \right)^2 \right) / p,$$

$$M_0 = M_{0}^p = \sum_{i=1}^{p} \left( \hat{\lambda}_0^2_i \right) / p = \text{tr} \left( \left( S_0^n \right)^2 \right) / p.$$

For the i.i.d. case, $E(M_{2}^{0,p}) = y H_1^2 + H_2 + o(1)$, where $H_1 = \lim_{p \to \infty} \text{tr} (\Sigma) / p$ and $H_2 = \lim_{p \to \infty} \text{tr} (\Sigma^2) / p$; see (4.14) of Yin (1986). The next theorem gives the property of $E(M_2^p)$ when $\eta(a_p, b_p, p)$ is bounded away from zero.

**Theorem 2** Under model (1.1) and Assumption 1, if $\eta(a_p, b_p, p) > c$ for some constant $c > 0$, then there exists $\delta > 0$ such that for all $p$ large enough,

$$E(M_2^p) \geq E(M_{2}^{0,p}) + \delta.$$

Theorem 2 implies that, under the non-reducible case, the ESD under the BEKK models is more heavy-tailed than the i.i.d. case, a feature that is suggested by Figure 1.

### 2.3 Time-variation adjusted spectrum estimator

Theorems 1 and 2 suggest that, unlike the i.i.d. case, the usual spectrum estimator based on the M-P law does not always work under the BEKK model. To recover the population spectrum under the BEKK model, a new estimator needs to be developed.

Recovering the population spectrum under large BEKK models can be done by establishing Marčenko-Pastur type equations for the sample covariance matrix. However, this is challenging due to the nonlinear dependency in returns. In the present paper, we will not pursue this direction. Instead, we provide an alternative solution
using a time-variation adjustment approach.

Zheng and Li (2011) study a similar problem under elliptical models. They propose a self-normalization approach to remove the time variation in the covariance matrices. Motivated by this idea, we aim to adjust the dynamic volatilities of the nonlinearly dependent data so that the adjusted data behave asymptotically i.i.d.. Compared with the elliptical model considered in Zheng and Li (2011), removing the time-varying dependency in BEKK models is more challenging.

Under the BEKK model (1.1), the time-variation in the covariance matrix is governed by the two coefficients, $a_p$ and $b_p$, and each $R_{it}$ follows a univariate GARCH model:

$$
\sigma_{i,t+1}^2 = (1 - a_p - b_p)\bar{\sigma}_i^2 + a_pR_{it}^2 + b_p\sigma_{i,t}^2,
$$

where $\sigma_{i,t}^2 = (\Sigma_t)_{ii}$, and $\bar{\sigma}_i^2 = (\bar{\Sigma})_{ii}$ for $1 \leq i \leq p$. As a result, $a_p$ and $b_p$ can be estimated without knowing the whole unconditional covariance matrix. Specifically, we randomly select one variable, say, $i_0$, fit a univariate GARCH model to $(R_{i_0t})$ and get QMLE $\hat{a}$ and $\hat{b}$:

$$
(\hat{a}, \hat{b}, \hat{\sigma}_{i_0}) = \arg\max_{(a,b,\sigma_{i_0})} \frac{1}{n} \sum_{t=1}^n \left( \frac{R_{i_0t}^2}{\sigma_{i_0t}^2} + \log(\sigma_{i_0t}^2) \right),
$$

where $\Omega = \{(a,b,\sigma) : 0 \leq a \leq 1, 0 \leq b \leq 1, a+b \leq 1-\delta, \delta \leq \sigma^2 < C\}$ for some positive constants $\delta$ and $C$. The QMLE of the univariate GARCH model is consistent with convergence rate $\sqrt{1/n}$; see, Theorems 2.1 and 2.2 of Francq and Zakoian (2004).

**Remark 1** In practice, we can use multiple variables to obtain a pooled estimator of $a_p$ and $b_p$. Numerical results suggest that a pooled estimator gives a more stable estimation of $a_p$ and $b_p$. However, in terms of convergence rate, a pooled estimator does not necessarily help due to the cross-sectional dependence and the error in estimating the nuisance parameter $\bar{\sigma}_{i_0}$; see Engle (2009) and Pakel et al. (2011).

We then use $\hat{a}$, $\hat{b}$ and past returns to construct a projection matrix:

$$
P_t = \frac{1 - \hat{a} - \hat{b} + \hat{a}\hat{b}M_p}{1 - \hat{b}}I + \sum_{j=1}^{M_p} \hat{a}\hat{b}^{j-1}R_{t-j}R_{t-j}^T,
$$

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where $M_p$ represents the number of lagged returns included in $P_t$, which grows with $p$ and $M_p = o(\sqrt{p})$. The intuition behind such a definition is that, $\Sigma_t = (1 - a_p - b_p)/\sum_{j=1}^{\infty} a_pb_p^{j-1}R_{t-j}R_{t-j}^T$, hence an appropriate choice of $M_p$ will make $P_t^{-1/2}\Sigma_t P_t^{-1/2}$ close to $\Sigma$.

To be more precise, using the projection matrix $P_t$, we define the time-variation adjusted returns $\tilde{R}_t = P_t^{-1/2}R_t$, and the time-variation adjusted (TV-adj) sample covariance matrix:

$$\tilde{S}_n = \frac{1}{n} \sum_{t=1}^{n} \tilde{R}_t(\tilde{R}_t)^T. \quad (2.6)$$

**Theorem 3** Under model (1.1) and Assumption 1, if, in addition, $\delta < \min(a_p, b_p) < a_p + b_p < 1 - \delta$ for some $\delta > 0$, and $M_p$ satisfies that $M_p \to \infty$ and $M_p = o(\sqrt{p})$, then

$$L(F\tilde{S}_n, F\Sigma_n) = o_p(1). \quad (2.7)$$

Theorem 3 implies that the time-variation adjusted sample covariance matrix has the same LSD as the i.i.d. case. That is, under Assumption 1(iv), $F\tilde{S}_n \xrightarrow{p} F$, and $F$ is determined by $H$ in that its Stieltjes transform

$$m_F(z) := \int_{\lambda \in \mathbb{R}} \frac{1}{\lambda - z} dF(\lambda), \quad z \in \mathbb{C}^+ := \{z \in \mathbb{C}, \text{Im}(z) > 0\} \quad (2.8)$$

solves the following equation

$$m_F(z) = \int_{\tau \in \mathbb{R}} \frac{1}{\tau(1 - y(1 + zm_F(z))) - z} dH(\tau). \quad (2.9)$$

See, e.g., Theorem 1 of Marčenko and Pastur (1967).

We can then consistently estimate the population spectrum by reversing the Marčenko-Pastur law. Specifically, we denote the eigenvalues of $\tilde{S}_n$ by $\tilde{\lambda}_1 \geq \ldots \geq \tilde{\lambda}_p$. We first regularize the eigenvalues of $\tilde{S}_n$ to be $\tilde{\lambda}^r_i = \min(\tilde{\lambda}_i, L)$ for some large constant $L$. We then apply the Quantized Eigenvalues Sampling Transform (QuEST) algorithm in Ledoit and Wolf (2015) on $\tilde{\lambda}^r_i$’s and obtain the estimated population spectrum. Denote by $\hat{\lambda}^H_1 \geq \hat{\lambda}^H_2 \geq \ldots \geq \hat{\lambda}^H_p$ the estimated eigenvalues and $\lambda^H_1 \geq \lambda^H_2 \geq \ldots \geq \lambda^H_p$ the eigenvalues of $\Sigma$. 

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Corollary 1 Under the assumptions of Theorem 3,
\[
\frac{1}{p} \sum_{i=1}^{p} (\lambda_i^H - \hat{\lambda}_i^H)^2 = o_p(1).
\]

Corollary 1 guarantees that QuEST applied to the TV-adj sample covariance matrix consistently estimates the population spectrum of the unconditional covariance matrix under large BEKK models.

2.4 Time-variation adjusted nonlinear shrinkage estimator of unconditional covariance matrix

The nonlinear shrinkage (NLS) estimator [Ledoit and Wolf (2012, 2015, 2020)] is structure-free and consistent in estimating the asymptotically optimal shrinkage estimator for i.i.d. data. In financial applications, the NLS has gained popularity in large portfolio optimization; see, e.g., Ledoit and Wolf (2017) and De Nard et al. (2021).

Motivated by the NLS developed under the i.i.d. case, to estimate the unconditional covariance matrix under large BEKK models, we make use of the TV-adj sample covariance matrix and consider rotation-equivariant shrinkage estimators in the form
\[
\hat{\Sigma} = \sum_{i=1}^{p} \hat{d}_i \tilde{u}_i \tilde{u}_i^T, \quad \text{where} \quad (\tilde{u}_i)_{1 \leq i \leq p} \text{ are eigenvectors of the TV-adj sample covariance matrix } \tilde{S}_n.
\]
The optimal rotation-equivariant estimator finds \((\hat{d}_i)_{1 \leq i \leq p}\) that minimize \(\|\hat{\Sigma} - \Sigma\|_F\). Elementary algebra shows that the optimal solution is
\[
\hat{d}_i^* = \tilde{u}_i^T \Sigma \tilde{u}_i.
\]

In search of the asymptotically optimal shrinkage formula under large BEKK models, we study the following generalized empirical spectral distribution of the TV-adj sample covariance matrix
\[
F_{\tilde{S}_n, g(\Sigma)}(x) = \frac{1}{\text{tr}(g(\Sigma))} \sum_{i=1}^{p} \left( \tilde{u}_i^T g(\Sigma) \tilde{u}_i \right) \cdot 1_{[\tilde{\lambda}_i, +\infty)}(x), \quad (2.10)
\]
which generalizes the ESD of \(\tilde{S}_n\) by replacing the weight \(1/p\) with \(\tilde{u}_i^T g(\Sigma) \tilde{u}_i / \text{tr}(g(\Sigma))\) for some bounded function \(g(\cdot)\), and \(g(\Sigma) = \sum_{i=1}^{p} g(\lambda_i^H) v_i v_i^T\), where \(v_i\)'s are the eigenvectors of \(\Sigma\). The limit of the generalized ESD of the sample covariance matrix
under the i.i.d. case is obtained in Ledoit and Péché (2011) and is used to derive the asymptotically optimal shrinkage estimator. Parallel to the i.i.d. case, we study the limiting property of the generalized ESD of the TV-adj sample covariance matrix via the following generalized Stieltjes transform:

\[
\Theta^g_n(z) = \frac{1}{p} \text{tr} \left( (\bar{S}_n - zI)^{-1} g(\bar{\Sigma}) \right).
\]  

(2.11)

The following theorem gives the limit of \( \Theta^g_n(z) \).

**Theorem 4** Under model (1.1) and Assumption 1, if, in addition, \( \delta < \min(a_p, b_p) < a_p + b_p < 1 - \delta \) for some \( \delta > 0 \), \( M_p \to \infty \), \( M_p = o(\sqrt{p}) \), the limiting distribution \( H \) is supported by \([h_1, h_2]\) for some constants \( 0 < h_1 \leq h_2 < \infty \), and \( g \) is a bounded function on \([h_1, h_2]\) with finitely many points of discontinuity, then,

\[
\Theta^g_n(z) - \Theta^g(z) = o_p(1), \text{ for all } z \in \mathbb{C}^+,
\]

where

\[
\Theta^g(z) = \int_{-\infty}^{+\infty} \left( \tau \left( 1 - y^{-1} - y^{-1} z m_F(z) \right) - z \right)^{-1} g(\tau) dH(\tau).
\]

(2.12)

The function \( \Theta^g(z) \) is the limit of the generalized Stieltjes transform under the i.i.d. case; see Theorem 2 of Ledoit and Péché (2011). Theorem 4 states that the generalized ESD based on the time-variation adjusted sample covariance matrix converges to the same limit as the i.i.d. case. Therefore, we can utilize the same nonlinear shrinkage algorithm that is developed for i.i.d. case to obtain the time-variation adjusted nonlinear shrinkage estimator under BEKK models.

Specifically, to estimate the unconditional covariance matrix, we perform the nonlinear shrinkage algorithm by Ledoit and Wolf (2015) on \( \tilde{S}_n^\tau \), where \( \tilde{S}_n^\tau = \sum_{i=1}^p \tilde{\lambda}_i^\tau \tilde{u}_i \tilde{u}_i^T \), \( \tilde{\lambda}_i^\tau = \min(\bar{\lambda}_i, L) \), and \( L \) is a large constant. The truncation is applied to ensure that the support of the ESD is bounded. We denote by \( \tilde{\Sigma} \) the resulting covariance matrix estimator, which we call the TV-adj nonlinear shrinkage estimator (TV-adj NLS).
Define $\tilde{\Sigma}^{or} = \sum_{i=1}^{p} d_{i}^{or}(\tilde{\lambda}_{i}^{T}) \tilde{\mu}_{i} \tilde{\mu}_{i}^{T}$, where

$$d_{i}^{or}(\tilde{\lambda}_{i}^{T}) = \begin{cases} \frac{1}{(y-1)\tilde{m}_{F}(0)} & \text{if } \tilde{\lambda}_{i}^{T} = 0 \text{ and } y > 1, \\ \frac{1}{(1-y-\tilde{\lambda}_{i}^{T}\tilde{m}_{F}(\tilde{\lambda}_{i}^{T}))^{2}} & \text{otherwise}, \end{cases} \quad \text{for } i = 1, \ldots, p,$$

(2.13)

$m_{F} = (y-1)/z + y m_{F}(z)$, $F'(x) = (1-y)1_{\{0,\infty]\}}(x) + y F(x)$, $\tilde{m}(\lambda) = \lim_{z \in C^{+} \to \lambda} m_{F}(z)$, $F(\cdot)$ and $m_{F}(z)$ are given in (2.8) and (2.9), respectively. By Theorem 4 and Theorem 4 of Ledoit and Péché (2011), $\tilde{\Sigma}^{or}$ is the infeasible oracle shrinkage estimator.

**Theorem 5** Under the assumptions of Theorem 4,

$$\frac{1}{\sqrt{p}} \| \tilde{\Sigma} - \tilde{\Sigma}^{or} \|_{F} = o_{p}(1).$$

Theorem 5 guarantees that the TV-adj NLS consistently estimates the oracle shrinkage estimator under large BEKK models in terms of convergence in dimension-normalized Frobenius norm. The convergence result achieved by the TV-adj NLS under BEKK models matches with that of the ordinary NLS under the i.i.d. case; see Proposition 4.3 of Ledoit and Wolf (2012) and Theorem 3.1 of Ledoit and Wolf (2015).

3 Simulation Studies

3.1 Simulation setup

We generate data from the BEKK model (1.1) with $z_{t} \sim_{\text{i.i.d.}} N(0, I)$. The unconditional covariance matrix is set to be $\tilde{\Sigma} = (\rho^{i-j})_{1 \leq i, j \leq p}$, where $\rho = 0.4$. The dimension is set to be $p = 100$ or $500$. We fix $p/n = 0.8$. About the parameters $(a, b)$:

- First, we choose four representative cases: $(a, b) \in \{(0, 0), (0.15, 0.25), (0.1, 0.65), (0.05, 0.9)\}$. The setting $(a, b) = (0, 0)$ corresponds to the i.i.d. case, which is

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1 We have also evaluated other settings of unconditional covariance matrix including $\rho = 0, 0.2, 0.6$ and $0.8$. The results are qualitatively similar.
presented as a benchmark, and the other \((a, b)\) pairs correspond to nontrivial BEKK cases sorted with increasing magnitudes of \(\eta(a, b, p)\) defined in (2.1), representing increasing levels of deviation from the i.i.d. case. The last configuration \((a, b) = (0.05, 0.9)\) is the setting used in Engle et al. (2019) and is calibrated from empirical financial returns. We simulate under each setting 100 replications and present the results in Section 3.2.1.

- Next, we examine more choices of \((a, b)\). Specifically, we consider a grid of \((a, b)\)'s in the region \(\{(a, b) : 0.05 \leq a \leq 0.5, 0.05 \leq b \leq 0.90, a + b \leq 0.95\}\) and report the average results from 100 replications in Section 3.2.2.

3.2 Simulation results

3.2.1 Four \((a, b)\) cases

In this subsection, we present the simulation results for four representative \((a, b)\) cases, \((a, b) \in \{(0, 0), (0.15, 0.25), (0.1, 0.65), (0.05, 0.9)\}\).

**Empirical spectral distribution of the sample covariance matrices**

We compute the original sample covariance matrix \(S_n = \sum_{t=1}^{n} R_t R_t^T / n\) and the TV-adj sample covariance matrix \(\tilde{S}_n = \sum_{t=1}^{n} \tilde{R}_t \tilde{R}_t^T / n\), and compare their ESDs with that of the sample covariance matrix under the i.i.d. case, \(S_n^0 = \sum_{t=1}^{n} R_t^0 (R_t^0)^T / n\).

We first illustrate the ESDs from one random realization for \(p = 500\) in Figure 2. We see that for all four cases, the ESDs of the TV-adj sample covariance matrix match remarkably well with that of the i.i.d. case. On the other hand, under nontrivial BEKK models, the ESDs of the original sample covariance matrices deviate from the M-P law, in particular, are more heavy-tailed.
Figure 2: ESDs of sample covariance matrices of the original data, the time-variation adjusted data, and the i.i.d. data for $p = 500$, $n = 625$. The unconditional covariance matrix $\Sigma = (0.4)^{|i-j|}; (a, b) \in \{(0, 0), (0.15, 0.25), (0.1, 0.65), (0.05, 0.9)\}$.

We then perform 100 replications and summarize the Euclidean distance between the eigenvalues of $\tilde{S}_n$ and $S_0^n$, and between the eigenvalues of $S_n$ and $S_0^n$ in Table 1. We see that:

- Under nontrivial BEKK settings, the distance between the ESD of the original sample covariance matrix and that under the i.i.d. case increases with increasing magnitudes of $\eta(a, b, p)$.

- The distance between the ESD of the TV-adj sample covariance matrix and that under the i.i.d. case is smaller and closer to zero under various nontrivial BEKK settings. Moreover, it decreases as $p$ gets larger.

- Comparing the performance of the TV-adj sample covariance matrix across different $(a, b)$ settings, the distance is the largest for the fourth setting $(a, b) = (0.05, 0.9)$, in which case $a + b$ is close to one, and the serial dependence is high.
Table 1: Summary of the distance \( \sqrt{\sum_{1 \leq i \leq p} (\hat{\lambda}_i - \hat{\lambda}^0_i)^2} \), where \((\hat{\lambda}^0_i)_{1 \leq i \leq p}\) are eigenvalues of \(S_n^0\), and \((\hat{\lambda}_i)_{1 \leq i \leq p}\) are eigenvalues of \(S_n\) or \(\tilde{S}_n\). The mean and standard deviation (in parenthesis) from 100 replications are reported.

| \((a,b)\)          | \((0.15,0.25)\) | \((0.1,0.65)\) | \((0.05,0.9)\) |
|---------------------|------------------|-----------------|-----------------|
| \((p,n) = (100,125)\) | \(S_n\)          | 0.277           | 0.413           | 0.907           |
|                     |                   | (0.078)         | (0.106)         | (0.195)         |
|                     | \(\tilde{S}_n\)  | 0.089           | 0.123           | 0.215           |
|                     |                   | (0.034)         | (0.033)         | (0.079)         |
| \((p,n) = (500,625)\) | \(S_n\)          | 0.279           | 0.429           | 1.046           |
|                     |                   | (0.045)         | (0.050)         | (0.120)         |
|                     | \(\tilde{S}_n\)  | 0.029           | 0.054           | 0.162           |
|                     |                   | (0.017)         | (0.029)         | (0.056)         |

Population spectrum estimation

Next, we evaluate the estimators of the population eigenvalues. The performance of the proposed time-variation adjusted NLS spectrum estimator\(^2\) (TV-adj NLS-Spectrum) is compared with that of the NLS spectrum estimator based on the original sample covariance matrix (original NLS-Spectrum). We measure the estimation error by \( \sqrt{\sum_{1 \leq i \leq p} (\lambda^H_i - \hat{\lambda}^H_i)^2} \), where \(\lambda^H_1 \geq \lambda^H_2 \geq ... \geq \lambda^H_p\) are the estimated eigenvalues, and \(\hat{\lambda}^H_1 \geq \hat{\lambda}^H_2 \geq ... \geq \hat{\lambda}^H_p\) are the eigenvalues of \(\Sigma\).

In Figure 3, we plot the distributions of the estimated eigenvalues from one random realization with \(p = 500\). We see that, under all four cases, the proposed TV-adj spectrum estimator is close to the population spectrum, and its performance is similar to the spectrum estimator based on the infeasible i.i.d. data. On the other hand, the shrinkage spectrum estimator based on the original sample covariance matrix

\(^2\)The function “tau_estimate” from R package “nlshrink” is used to compute the estimated eigenvalues.

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Figure 3: Distributions of estimated eigenvalues. They are obtained by nonlinear shrinkage estimators applied to the original data, the time-variation adjusted data, and the i.i.d. data. The dimension \( p = 500 \), \( n = 625 \). \( \Sigma = (0.4)^{|i-j|}; (a,b) \in \{(0,0), (0.15, 0.25), (0.1, 0.65), (0.05, 0.9)\} \).

significantly deviates from the population spectrum.

The Euclidean distances between the estimated population eigenvalues and the true ones from 100 replications are summarized in Table 2. We see that:

- The error of the original NLS-spectrum estimator increases with \( \eta(a,b,p) \). It also gets larger as the dimension gets higher.
- The proposed TV-adj NLS-spectrum estimator dominantly outperforms the original NLS-spectrum estimator with a substantially lower estimation error.
- As the dimension \( p \) grows, the performance of the TV-adj NLS-spectrum estimator gets closer to the performance of the shrinkage estimator under the i.i.d. case.
- The performance of the TV-adj NLS-spectrum estimator is only slightly worse
than the infeasible shrinkage estimator based on i.i.d. data for the first two nontrivial BEKK settings. For the fourth setting when \((a, b) = (0.05, 0.9)\), because \(a + b\) is close to one, the estimation error of the TV-adj NLS-spectrum estimator is larger. However, as the dimension \(p\) grows, the error decreases and becomes closer to that of the shrinkage estimator under the i.i.d. case.

Table 2: *Summary of the distance between the estimated population eigenvalues and the true ones.* The mean and standard deviation (in parenthesis) from 100 replications are reported. The pair \((a, b) = (0, 0)\) represents the i.i.d. case and is presented as the benchmark. The remaining \((a, b)\) pairs are for nontrivial BEKK cases.

\[
\begin{array}{cccccc}
& \text{i.i.d.} & & \text{BEKK} & \\
& (a,b) & (0,0) & (0.15,0.25) & (0.1,0.65) & (0.05,0.9) \\
(p, n) = (100, 125) & \text{original NLS-Spectrum} & 0.136 & 0.411 & 0.566 & 1.109 \\
& & (0.041) & (0.094) & (0.124) & (0.209) \\
& \text{TV-adj NLS-Spectrum} & 0.143 & 0.217 & 0.251 & 0.313 \\
& & (0.047) & (0.071) & (0.071) & (0.100) \\
(p, n) = (500, 625) & \text{original NLS-Spectrum} & 0.052 & 0.401 & 0.577 & 1.243 \\
& & (0.026) & (0.049) & (0.053) & (0.119) \\
& \text{TV-adj NLS-Spectrum} & 0.054 & 0.057 & 0.088 & 0.184 \\
& & (0.027) & (0.025) & (0.026) & (0.035) \\
\end{array}
\]

**Unconditional covariance matrix estimation**

Finally, we evaluate the unconditional covariance matrix estimation. We compute the NLS estimators\(^3\) based on the time-variation adjusted sample covariance matrix (TV-adj NLS) and the original sample covariance matrix (original NLS). The estima-

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\(^3\)The function “nlshrink\_cov” in R package “nlshrink” is used in computing the nonlinear shrinkage estimator of the covariance matrix.
tion error is measured by the Frobenius norm \( \sqrt{\sum_{1 \leq i,j \leq p} (\Sigma_{ij} - \hat{\Sigma}_{ij})^2} \), where \( \hat{\Sigma} \) is the estimated unconditional covariance matrix. The results are summarized in Table 3.

Table 3: Summary statistics of the estimation error of the estimated unconditional covariance matrix in Frobenius norm \( \sqrt{\sum_{1 \leq i,j \leq p} (\Sigma_{ij} - \hat{\Sigma}_{ij})^2} \). The mean and standard deviation (in parenthesis) from 100 replications are reported. The pair \((a, b) = (0, 0)\) corresponds to the i.i.d. case and is presented as the benchmark. The remaining \((a, b)\) pairs correspond to nontrivial BEKK cases.

| \((p, n)\)   | i.i.d. \((a, b)\) | BEKK \((a, b)\) |
|--------------|------------------|-----------------|
|              | \((0, 0)\)       | \((0.15, 0.25)\) | \((0.1, 0.65)\) | \((0.05, 0.9)\) |
| \((100, 125)\) original NLS 5.079 | 6.668 | 7.933 | 12.810 |
|              | \((0.048)\)      | \((0.558)\)     | \((0.894)\) | \((1.824)\) |
| TV-adj NLS 5.080 | 5.219 | 5.308 | 6.433 |
|              | \((0.049)\)      | \((0.096)\)     | \((0.078)\) | \((0.711)\) |
| \((500, 625)\) original NLS 11.314 | 14.850 | 17.878 | 31.384 |
|              | \((0.021)\)      | \((0.664)\)     | \((0.866)\) | \((2.405)\) |
| TV-adj NLS 11.320 | 11.362 | 11.469 | 11.970 |
|              | \((0.026)\)      | \((0.028)\)     | \((0.050)\) | \((0.196)\) |

We see from Table 3 that:

- The estimation error of the original NLS increases sharply as \( \eta(a, b, p) \) gets large and as the dimension grows.

- The proposed TV-adj NLS greatly improves over the original NLS with a substantially lower estimation error.

- The performance of the TV-adj NLS is only slightly worse than that of the NLS under the i.i.d. case for the first two nontrivial \((a, b)\) settings. For the most challenging case \((a, b) = (0.05, 0.9)\), because \(a + b\) is close to one, the error
of TV-adj NLS is larger. However, when \( p \) grows, the performance becomes closer.

### 3.2.2 Performance under more choices of \((a, b)\)

In this subsection, we present the simulation results for a grid of \((a, b)\)’s in the region \(\{(a, b) : 0.05 \leq a \leq 0.5, 0.05 \leq b \leq 0.90, a + b \leq 0.95\}\).

#### Empirical spectral distribution of the sample covariance matrices

In Figure 4, we plot the average Euclidean distance between the eigenvalues of \(S_n\) and \(S_0^n\), and between the eigenvalues of \(\tilde{S}_n\) and \(S_0^n\). We see that the Euclidean distance between the ESD of the original sample covariance matrix and that under the i.i.d. case grows substantially as \(a\) and \(a + b\) increase. In contrast, the distance of the eigenvalues of the TV-adj sample covariance matrix to that under the i.i.d. case is close to zero for various \((a, b)\) settings. The distance surface for the TV-adj sample covariance matrix is almost flat, except when \(a + b\) approaches one, but when the dimension \(p\) increases, it again becomes flatter and closer to zero.

![Figure 4: Euclidean distance between the eigenvalues of the sample covariance matrix/TV-adj sample covariance matrix under the BEKK model and the eigenvalues of the sample covariance matrix under the i.i.d. case for \(p = 100\) (left) and \(p = 500\) (right). The unconditional covariance matrix is \(\Sigma = (0.4)^{|i-j|}\). The evaluation is made for a grid of \((a, b)\)’s in the region \(\{(a, b) : 0.05 \leq a \leq 0.5, 0.05 \leq b \leq 0.90, a+b \leq 0.95\}\).](image)

#### Population spectrum estimation

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In Figure 5, we plot the average Euclidean distance between the estimated eigenvalues and the true eigenvalues. The methods under comparison are the original NLS estimator and the proposed TV-adj NLS estimator. We see that the error of the original NLS estimator increases sharply as $a$ and $a + b$ increase. It also gets larger when the dimension is higher. In contrast, the TV-adj NLS performs robustly well for various $(a, b)$ settings and it dominantly outperforms the original NLS in all cases. The error surface for the TV-adj NLS is almost flat except when $a + b$ approaches one, but it gets closer to zero when $p$ grows.

Figure 5: Estimation error of the population eigenvalues for $p = 100$ (left) and $p = 500$ (right). The unconditional covariance matrix is $\Sigma = (0.4)^{|i-j|}$. The methods under comparison are the original NLS-spectrum estimator and the TV-adj NLS-spectrum estimator. The evaluation is made for a grid of $(a, b)$’s in the region $\{(a, b) : 0.05 \leq a \leq 0.5, 0.05 \leq b \leq 0.90, a + b \leq 0.95\}$.

**Unconditional covariance matrix estimation**

Finally, in Figure 6, we plot the average Frobenius error of the NLS and TV-adj NLS in estimating the unconditional covariance matrix. We see that the original NLS estimator performs poorly when $(a, b)$ deviates from $(0, 0)$. When $p$ grows, the error also becomes bigger. The TV-adj NLS dominantly outperforms the NLS with a lower estimation error in all cases. The error surface for the TV-adj NLS is almost flat and only slightly higher near the edge when $a + b$ is close to one.
The unconditional covariance matrix is $\Sigma = (0.4)^{|i-j|}$. The methods under comparison are the original NLS estimator and the TV-adj NLS estimator. The evaluation is made for a grid of $(a, b)$’s in the region $\{(a, b) : 0.05 \leq a \leq 0.5, 0.05 \leq b \leq 0.90, a+b \leq 0.95\}$.

4 Conclusion

We investigate the limiting spectral properties of high-dimensional sample covariance matrices under dynamic volatility models. We show that under large BEKK models, the asymptotics of the spectral distribution of sample covariance matrix depend on the asymptotic size of the innovation coefficient and the persistence coefficient. In particular, we give explicit conditions under which the ESD has the same limit as or deviates from the i.i.d. case. Furthermore, we develop a consistent estimator of the spectrum of the unconditional covariance matrix under large BEKK models. The proposed approach is based on a time-variation adjusted sample covariance matrix, for which we show that the LSD shares the same Marčenko-Pastur law as the i.i.d. case. Finally, we propose a TV-adj nonlinear shrinkage estimator of the unconditional covariance matrix. The estimator is consistent in estimating the oracle shrinkage estimator under large BEKK models.
5 Proof of Theorem 3

In this section we prove Theorem 3. The proofs of other main and technical results are given in the Supplementary Material Ding and Zheng (2022).

We divide the proof of Theorem 3 into three steps. In the first step, we show that replacing $(\hat{a}, \hat{b})$ with $(a_p, b_p)$ in (2.5) does not change the LSD of the TV-adj sample covariance matrices. In the second step, we show that the problem can be reduced into proving convergence in Frobenius norm for the expected difference in the covariance matrices’ square root matrices after some orthogonal transformation. In the last step, we show (5.21). For ease of notation, we drop the subscript $p$ in $a_p$ and $b_p$ in the following proofs.

**Step One:** Define

$$
\tilde{P}_t = \frac{1 - a - b + abM_p}{1 - b} I + \sum_{j=1}^{M_p} ab^{j-1} R_{t-j} R^T_{t-j},
$$

$$
\tilde{R}_t = (\tilde{P}_t)^{-1/2} R_t, \quad \text{and}
$$

$$
\tilde{S}_n = \frac{1}{n} \sum_{t=1}^{n} \tilde{R}_t \tilde{R}_t^T.
$$

By Corollary A.42 of Bai and Silverstein (2010),

$$
L^4(F^{\tilde{S}_n}, F^{\tilde{S}_n}) \leq \frac{2}{p} \text{tr}(\tilde{S}_n + \tilde{S}_n) \cdot \frac{1}{pn} \text{tr}\left( (\tilde{R} - \tilde{R})(\tilde{R} - \til{R})^T \right),
$$

where $\tilde{R} = (\tilde{R}_1, ..., \tilde{R}_n)$, and $\tilde{R} = (\tilde{R}_1, ..., \tilde{R}_n)$. By definition, $(\hat{a}, \hat{b})$ satisfies

$$(\hat{a}, \hat{b}, \hat{\sigma}) \in \Omega = \{(a, b, \sigma), 0 \leq a \leq 1, 0 \leq b \leq 1, a + b \leq 1 - \delta, \delta \leq \sigma < C\}.
$$

By the definition of $P_t$ in (2.5), we have $P_t \geq (1 - \hat{a} - \hat{b})I \geq \delta I$. Similarly, under the assumption that $a + b < 1 - \delta$, we have $P_t \geq (1 - a - b)I \geq \delta I$. Define

$$
\tilde{\Sigma}_t = P_t^{-1/2} \Sigma_t P_t^{-1/2}, \quad \text{and} \quad \tilde{\Sigma}_t = \tilde{P}_t^{-1/2} \tilde{\Sigma}_t \tilde{P}_t^{-1/2}.
$$
Using the fact that if $A \geq 0$ and $B \geq 0$, then $\text{tr}(AB) \geq 0$, we have that, for all $t$,

$$\text{tr}(\tilde{\Sigma}_t) \leq \frac{1}{\delta} \text{tr}(\Sigma_t), \quad \text{tr}(\Sigma_t) \leq \frac{1}{\delta} \text{tr}(\Sigma_t),$$

(5.3)

and

$$\text{tr}(\tilde{R}_t\tilde{R}_t^T) \leq \frac{1}{\delta} \text{tr}(R_tR_t^T), \quad \text{tr}(\tilde{R}_t\tilde{R}_t^T) \leq \frac{1}{\delta} \text{tr}(R_tR_t^T).$$

Therefore,

$$\text{tr}(\tilde{S}_n/p) \leq \frac{1}{\delta} \cdot \text{tr}(S_n/p), \quad \text{tr}(\tilde{S}_n/p) \leq \frac{1}{\delta} \cdot \text{tr}(S_n/p).$$

By the independence between $(\Sigma_t)$ and $(z_t)$ and Assumption 1(iii), we have

$$E\left(\text{tr}(S_n/p)\right) = \frac{\text{tr}(\tilde{\Sigma})}{p} = O(1).$$

It follows that

$$\text{tr}(\tilde{S}_n/p) = O_p(1), \quad \text{and} \quad \text{tr}(\tilde{S}_n/p) = O_p(1).$$

(5.4)

Write

$$\varepsilon_n = \max \left(\max_{1 \leq j \leq M_p} \left(\left|\frac{ab^{-1}}{\hat{a}b^{-1}} - 1\right|, \left|\frac{\hat{a}b^{-1}}{ab^{-1}} - 1\right|, \left|\frac{(1 - \hat{a} - \hat{b} + \hat{a}\hat{b}^{M_p})(1 - b)}{(1 - a - b + ab^{M_p})(1 - b)} - 1\right|, \left|\frac{(1 - a - b + ab^{M_p})(1 - b)}{(1 - \hat{a} - \hat{b} + \hat{a}\hat{b}^{M_p})(1 - b)} - 1\right|\right)\right).$$

By Theorem 2.2 of Francq and Zakoian (2004), under the assumption that $\delta \leq a, b \leq a + b \leq 1 - \delta$, we have

$$\hat{a} - a = O_p(1/\sqrt{n}), \quad \text{and} \quad \hat{b} - b = O_p(1/\sqrt{n}).$$

(5.5)

By the assumptions that $M_p = o(\sqrt{p})$, $\delta \leq a, b \leq a + b \leq 1 - \delta$, $p/n \asymp 1$, and (5.5), we get

$$\varepsilon_n = o_p(1).$$

(5.6)

Note that when $\varepsilon_n < 1$, we have

$$0 < (1 - \varepsilon_n)\hat{P}_t \leq \hat{P}_t \leq (1 + \varepsilon_n)\hat{P}_t.$$
By the Löwner-Heinz inequality,
\[
\frac{1}{\sqrt{1 + \varepsilon_n}} P_t^{-1/2} \leq P_t^{-1/2} \leq \frac{1}{\sqrt{1 - \varepsilon_n}} P_t^{-1/2}.
\]

By Weyl’s theorem, we get that
\[
\left\| P_t^{-1/2} - \frac{1}{\sqrt{1 + \varepsilon_n}} P_t^{-1/2} \right\| \leq \left( \frac{1}{\sqrt{1 - \varepsilon_n}} - \frac{1}{\sqrt{1 + \varepsilon_n}} \right) \cdot \| P_t^{-1/2} \|. \tag{5.7}
\]

Recall that \( \tilde{P}_t \geq \delta I \), hence
\[
\| \tilde{P}_t^{-1/2} \| \leq \sqrt{\frac{1}{\delta}}. \tag{5.8}
\]

By the triangle inequality, (5.6), (5.7) and (5.8), we get that
\[
\begin{align*}
\| P_t^{-1/2} - \tilde{P}_t^{-1/2} \| &\leq \left\| P_t^{-1/2} - \frac{1}{\sqrt{1 + \varepsilon_n}} P_t^{-1/2} \right\| + \left( 1 - \frac{1}{\sqrt{1 + \varepsilon_n}} + 1 - \frac{1}{\sqrt{1 - \varepsilon_n}} \right) \cdot \| \tilde{P}_t^{-1/2} \| \\
&= O \left( \frac{1}{\sqrt{1 - \varepsilon_n}} - \frac{1}{\sqrt{1 + \varepsilon_n}} + 1 - \frac{1}{\sqrt{1 - \varepsilon_n}} \right) \\
&= o_p(1).
\end{align*}
\]

Moreover,
\[
\text{tr} \left( (\tilde{R}_t - \bar{R}_t)(\tilde{R}_t - \bar{R}_t)^T \right) \\
= R_t^T (P_t^{-1/2} - \tilde{P}_t^{-1/2})^2 R_t \\
\leq \| P_t^{-1/2} - \tilde{P}_t^{-1/2} \|^2 \cdot \| R_t \|^2 = \| P_t^{-1/2} - \tilde{P}_t^{-1/2} \|^2 \cdot (z_t^T \Sigma_t z_t).
\]

Because \( E(z_t^T \Sigma_t z_t) = E(\text{tr}(\Sigma_t)) = \text{tr}(\Sigma) = O(p) \), we have \( z_t^T \Sigma_t z_t = O_p(p) \). We then get
\[
\text{tr} \left( (\tilde{R}_t - \bar{R}_t)(\tilde{R}_t - \bar{R}_t)^T \right) = o_p(p). \tag{5.9}
\]

In addition, by \( P_t \geq \delta I \), and \( \tilde{P}_t \geq \delta I \), we have
\[
\text{tr} \left( (\tilde{R}_t - \bar{R}_t)(\tilde{R}_t - \bar{R}_t)^T / p \right) \leq 2 \text{tr} \left( \tilde{R}_t \tilde{R}_t^T / p + \bar{R}_t \bar{R}_t^T / p \right) \leq \frac{4}{\delta} R_t^T R_t / p.
\]

By the independence between \( (z_t) \) and \( (\Sigma_t) \) and Assumption 1(iii), we have \( E(R_t^T R_t / p) = \)
\[ \text{tr}(\mathbf{\Sigma})/p = O(1) \]. By (5.9) and the dominated convergence theorem,

\[
E \left( \frac{\text{tr} \left( (\mathbf{R}_t - \mathbf{\hat{R}}_t)(\mathbf{\hat{R}}_t - \mathbf{\hat{R}}_t)^T/p \right)}{p} \right) = o(1).
\]

It follows that

\[
\frac{1}{pn} E \left( \text{tr} \left( (\mathbf{R}_t - \mathbf{\hat{R}}_t)(\mathbf{\hat{R}}_t - \mathbf{\hat{R}}_t)^T \right) \right) = E \left( \frac{\text{tr} \left( (\mathbf{R}_t - \mathbf{\hat{R}}_t)(\mathbf{\hat{R}}_t - \mathbf{\hat{R}}_t)^T/p \right)}{p} \right) = o(1).
\]

By Markov’s inequality, we get

\[
\frac{1}{pn} \text{tr} \left( (\mathbf{R}_t - \mathbf{\hat{R}}_t)(\mathbf{\hat{R}}_t - \mathbf{\hat{R}}_t)^T \right) = o_p(1). \tag{5.10}
\]

By (5.1), (5.4) and (5.10), we have

\[
L(F_{\mathcal{S}_0}, F_{\mathcal{S}_0}) = o_p(1). \tag{5.11}
\]

**Step Two:** We denote by \( \mathcal{F}_t \) the \( \sigma \)-algebra generated by \( \{z_s, \infty < s \leq t\} \). For a \( p \times p \) matrix \( \mathcal{O}_t \) to be determined, which satisfies that

\[
\mathcal{O}_t \text{ is } \mathcal{F}_{t-1}\text{-measurable, and } \mathcal{O}_t \mathcal{O}_t^T = \mathcal{O}_t^T \mathcal{O}_t = I, \tag{5.12}
\]

we perform orthogonal transformation on \( z_t \) and get \( \zeta_t = \mathcal{O}_t z_t \). We then define

\[
\mathcal{R}_t^0 = \mathbf{\Sigma}^{1/2} \zeta_t, \quad \mathcal{R}^0 = (\mathcal{R}_1^0, ..., \mathcal{R}_n^0), \quad \text{and } \mathcal{S}_n^0 = \frac{1}{n} \sum_{t=1}^n \mathcal{R}_t^0 (\mathcal{R}_t^0)^T. \tag{5.13}
\]

By (5.12) and the assumption that \( z_t \overset{\text{i.i.d.}}{\sim} N(0, I) \), we have

\[
\zeta_t \overset{\text{i.i.d.}}{\sim} N(0, I). \tag{5.14}
\]

By Theorem 1 of Marčenko and Pastur (1967), \( F_{\mathcal{S}_0} \overset{p}{\rightarrow} F \), and \( F_{\mathcal{S}_0} \overset{p}{\rightarrow} F \). Hence,

\[
L(F_{\mathcal{S}_0}, F_{\mathcal{S}_0}) = o_p(1). \tag{5.15}
\]
By (5.11), (5.15) and the triangle inequality, to show Theorem 3, it suffices to show that
\[ L(F^{S_n}, F^{\hat{S}_n}) = o_p(1). \]  
(5.16)

By Corollary A.42 of Bai and Silverstein (2010) again,
\[ L^4(F^{\hat{S}_n}, F^{S_n}) \leq \frac{2}{p} \text{tr}(\hat{S}_n + S_n^0) \cdot \frac{1}{pn} \text{tr} \left( (\hat{R} - \mathcal{R}^0)(\hat{R} - \mathcal{R}^0)^T \right). \]  
(5.17)

We have \( E \left( \text{tr}(S_n^0/p) \right) = \text{tr}(\Sigma)/p = O(1) \), hence
\[ \text{tr}(S_n^0/p) = O_p(1). \]  
(5.18)

Combining (5.4) and (5.18) yields
\[ \text{tr}(S_n/p + S_n^0/p) = O_p(1). \]  
(5.19)

Define
\[ Q_t = \hat{P}_t^{-1/2}\Sigma_t^{1/2}. \]  
(5.20)

We have \( \hat{R}_t = Q_tz_t \) and \( \Sigma_t = Q_t(Q_t)^T \). We will show that for some \( \mathcal{O}_t \) satisfying (5.12),
\[ \frac{1}{p} E \left( \text{tr} \left( Q_t^T \Sigma_t^{1/2}(Q_t - \Sigma_t^{1/2}) \right) \right) = o(1). \]  
(5.21)

Then by the facts that \( \Sigma_t \) and \( \mathcal{O}_t \) are \( \mathcal{F}_{t-1} \)-measurable, we have
\[
\begin{align*}
\frac{1}{np} E \left( \text{tr} \left( (\hat{R} - \mathcal{R}^0)(\hat{R} - \mathcal{R}^0)^T \right) \right) &= \frac{1}{p} E \left( \text{tr} \left( (\hat{R}_t - \mathcal{R}_t^0)(\hat{R}_t - \mathcal{R}_t^0)^T \right) \right) \\
&= \frac{1}{p} E \left( \text{tr} \left( (Q_t - \Sigma_t^{1/2}\mathcal{O}_t)z_t^2(Q_t - \Sigma_t^{1/2}\mathcal{O}_t)^T \right) \right) \\
&= \frac{1}{p} E \left( \text{tr} \left( (Q_t\mathcal{O}_t^T - \Sigma_t^{1/2})(Q_t\mathcal{O}_t^T - \Sigma_t^{1/2})^T \right) \right) = o(1),
\end{align*}
\]  
(5.22)
which implies that
\[
\frac{1}{p^n} \text{tr} \left( (\mathbf{R} - \mathbf{R}^0)(\mathbf{R} - \mathbf{R}^0)^T \right) = o_p(1). 
\] (5.23)

The desired bound (5.16) then follows from (5.17), (5.19) and (5.23).

**Step Three:** It remains to show that there exists \( O_t \) satisfying (5.12) and (5.21).

Because \( M_p = o(\sqrt{p}) \ll p \), with probability one, for all \( p \) large enough, rank(\( \sum_{j=1}^{M_p} b^{j-1} \mathbf{R}_{t-j} \mathbf{R}_{t-j}^T \)) = \( M_p \). Write
\[
\sum_{j=1}^{M_p} b^{j-1} \mathbf{R}_{t-j} \mathbf{R}_{t-j}^T =: \mathbf{U} \Lambda \mathbf{U}^T,
\]
where \( \Lambda = \text{diag}(\lambda_1, ..., \lambda_{M_p}) \) and \( \mathbf{U} = (\mathbf{u}_1, ..., \mathbf{u}_{M_p}) \) are the nonzero eigenvalues and the corresponding eigenvectors of \( \sum_{j=1}^{M_p} b^{j-1} \mathbf{R}_{t-j} \mathbf{R}_{t-j}^T \), respectively. Recall that
\[
\tilde{\mathbf{P}}_t = \frac{1 - a - b + ab^{M_p}}{1 - b} \mathbf{I} + \sum_{j=1}^{M_p} ab^{j-1} \mathbf{R}_{t-j} \mathbf{R}_{t-j}^T = \frac{1 - a - b + ab^{M_p}}{1 - b} \mathbf{I} + a \mathbf{U} \Lambda \mathbf{U}^T.
\]

We have
\[
\tilde{\mathbf{P}}_t^{-1/2} = \mathbf{U} \left( a \Lambda + \frac{1 - a - b + ab^{M_p}}{1 - b} \mathbf{I} \right)^{-1/2} \mathbf{U}^T \]
\[
+ \sqrt{\frac{1 - b}{1 - a - b + ab^{M_p}}} (\mathbf{I} - \mathbf{U} \mathbf{U}^T).
\] (5.24)

By (1.1), we have
\[
\Sigma_t = \frac{1 - a - b}{1 - b} \Sigma + \sum_{s=1}^{\infty} ab^{s-1} \mathbf{R}_{t-s} \mathbf{R}_{t-s}^T.
\] (5.25)

Hence
\[
\Sigma_t = \left( \frac{1 - a - b + ab^{M_p}}{1 - b} \Sigma + a \mathbf{U} \Lambda \mathbf{U}^T \right) + \left( \sum_{j=M_p+1}^{\infty} ab^{j-1} \mathbf{R}_{t-j} \mathbf{R}_{t-j}^T - \frac{ab^{M_p}}{1 - b} \Sigma \right) \]
\[
=: I_t + II_t.
\] (5.26)
Recall that \( \Sigma_t = P_t^{-1/2} \Sigma_t P_t^{-1/2} \). We have
\[
\Sigma_t = P_t^{-1/2} I_t P_t^{-1/2} + P_t^{-1/2} I_t P_t^{-1/2}.
\] (5.27)

By (5.24),
\[
P_t^{-1/2} I_t P_t^{-1/2} = \Sigma - \Sigma U U^T - U U^T \Sigma + U U^T \Sigma U U^T + \mathcal{E}_t,
\] (5.28)

where
\[
\mathcal{E}_t
\]
\[
= 1 - a - b + abM_p \left( a \Lambda + \frac{1 - a - b + abM_p}{1 - b} \right) \left( I - U U^T \right) - \frac{1}{2} \left( U^T \Sigma U \right) \left( I - U U^T \right)
\]
\[
+ a U \left( a \Lambda + \frac{1 - a - b + abM_p}{1 - b} \right)^{-1/2} \left( I - U U^T \right)
\]
\[
= : \mathcal{E}_{1t} + \mathcal{E}_{2t} + \mathcal{E}_{3t} + \mathcal{E}_{4t}.
\]

Because \( U^T U = I \), we have
\[
\text{tr}(\mathcal{E}_{2t}) = \text{tr}(\mathcal{E}_{3t}) = 0.
\]

Moreover, because \( M_p \ll p \) and \( \| \Sigma \| = O(1) \),
\[
0 \leq \text{tr}(U^T \Sigma U) = \sum_{i=1}^{M_p} u_i^T \Sigma u_i \leq M_p \| \Sigma \| = o(p).
\] (5.29)

Furthermore, by (5.2), \( M_p \ll p \) and the fact that if \( A \geq 0, B \geq 0 \), then \( \text{tr}(AB) \geq 0 \), we have
\[
0 \leq \text{tr}(\mathcal{E}_{1t}) \leq \text{tr}(U^T \Sigma U) = o(p),
\]
\[
0 \leq \text{tr}(\mathcal{E}_{4t}) = \text{tr} \left( a \Lambda \left( a \Lambda + \frac{1 - a - b + abM_p}{1 - b} \right)^{-1} \right) \leq M_p = o(p).
\]
Combining the results above yields

\[ \text{tr}(\mathcal{E}_t/p) = o(1). \]

Moreover, by (5.29) and that \( U^T U = I \), we have

\[ 0 \leq \text{tr}(UU^T \Sigma UU^T) = \text{tr}(U^T \Sigma) = \text{tr}(U^T U) = o(p). \]

Plugging the estimates above into (5.28) yields

\[ \text{tr}(\hat{\mathcal{P}}_t^{-1/2} I_t \hat{\mathcal{P}}_t^{-1/2}) - \text{tr}(\Sigma) = o(p). \] (5.30)

About term \( \hat{\mathcal{P}}_t^{-1/2} I_t \hat{\mathcal{P}}_t^{-1/2} \), because \( \hat{\mathcal{P}}_t \geq (1 - a - b)I \), we have

\[
\left| \text{tr} \left( \hat{\mathcal{P}}_t^{-1/2} I_t \hat{\mathcal{P}}_t^{-1/2} \right) \right| 
\leq \frac{ab^{M_p}}{(1 - a - b)(1 - b)} \text{tr}(\Sigma) + \frac{b^{M_p}}{1 - a - b} \text{tr} \left( \sum_{s=1}^{\infty} ab^{s-1} R_{t-M_p-s} R_{t-M_p-s}^T \right) 
\leq \frac{ab^{M_p}}{(1 - a - b)(1 - b)} \text{tr}(\Sigma) + \frac{b^{M_p}}{1 - a - b} \text{tr}(\Sigma_{t-M_p}).
\]

We have \( E(\text{tr}(\Sigma_{t-M_p})) = \text{tr}(\Sigma) = O(p) \), hence \( \text{tr}(\Sigma_{t-M_p}) = O_p(p) \). By the assumptions that \( M_p \to \infty \) and \( a + b < 1 - \delta \), we get

\[
\left| \text{tr} \left( \hat{\mathcal{P}}_t^{-1/2} I_t \hat{\mathcal{P}}_t^{-1/2} \right) \right| = O(\beta^{M_p}) \left( \text{tr}(\Sigma) + \text{tr}(\Sigma_{t-M_p}) \right) 
= O(\beta^{M_p}) O_p(p) = o_p(p).
\] (5.31)

Combining (5.30) and (5.31) yields

\[
\left| \frac{\text{tr}(\mathcal{E}_t)/p}{\text{tr}(\Sigma)/p} - \frac{\text{tr}(\Sigma_t)/p}{\text{tr}(\Sigma)/p} \right| = o_p(1).
\] (5.32)

We now define \( O_t \) that satisfies (5.12) and (5.21). Let

\[ G_t = \sqrt{(1 - a - b)/(1 - b)} \hat{\mathcal{P}}_t^{-1/2} \Sigma_{t-M_p}^{-1/2}. \]

We have \( G_t G_t^T = ((1 - a - b)/(1 - b)) \hat{\mathcal{P}}_t^{-1/2} \Sigma_{t-M_p}^{-1/2} \). By (5.25), \( \Sigma_t \geq ((1 - a - b)/(1 - a - b)) I_t \).

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Define
\[ Q_t = G_t \left( I + G_t^{-1} \left( \Sigma_t - G_t G_t^T \right) \left( G_t^T \right)^{-1} \right)^{1/2}, \]
and
\[ O_t = Q_t^T \left( Q_t^T \right)^{-1}, \]
where, recall that, \( Q_t \) is defined in (5.20). By definition, \( O_t \) is \( \mathcal{F}_{t-1} \)-measurable. Moreover, it is straightforward to verify that \( Q_t Q_t^T = Q_t \), \( Q_t^T = \hat{\Sigma}_t \), from which we get that \( O_t O_t^T = O_t^T O_t = I \). Therefore, \( O_t \) satisfies (5.12).

It remains to show (5.21). By (5.24),
\[ P^{-1/2} \geq \sqrt{\frac{1}{1 - a - b + ab M_p} \left( I - UU^T \right)}. \]

By (5.33) and (5.34),
\[
\begin{align*}
\text{tr} \left( Q_t^T \Sigma^{1/2} \right) &= \text{tr} \left( Q_t \Sigma^{1/2} \right) \\
&= \sqrt{\frac{1}{1 - a - b}} \text{tr} \left( I + G_t^{-1} \left( \Sigma_t - G_t G_t^T \right) \left( G_t^T \right)^{-1} \right)^{1/2} \Sigma^{1/2} P^{-1/2} \Sigma^{1/2} \\
&\geq \sqrt{\frac{1}{1 - a - b}} \text{tr} \left( P^{-1/2} \Sigma \right) \\
&\geq \sqrt{\frac{1}{1 - a - b + ab M_p}} \text{tr} \left( I - UU^T \right) \Sigma \\
&= \text{tr} (\Sigma) + o(p),
\end{align*}
\]
where the last equation holds by (5.29) and the assumptions that \( M_p \to \infty \) and
\( a + b < 1 - \delta \). By the definition of \( O_t \), (5.32) and (5.35), we get that

\[
0 \leq \frac{1}{p} \text{tr} \left( (Q_t O_t^T - \bar{\Sigma}^{1/2})(Q_t O_t^T - \bar{\Sigma}^{1/2})^T \right) = \frac{1}{p} \left( \text{tr} \left( (Q_t - \bar{\Sigma}^{1/2} O_t)(Q_t - \bar{\Sigma}^{1/2} O_t)^T \right) \right)
\]

\[
= \frac{1}{p} \left( \text{tr} \left( (Q_t - \bar{\Sigma}^{1/2})(Q_t - \bar{\Sigma}^{1/2})^T \right) \right)
\]

\[
= \left( \text{tr}(\Sigma_t)/p - \text{tr}(\bar{\Sigma})/p \right) + \left( \text{tr}(\bar{\Sigma})/p - \text{tr}(Q_t \bar{\Sigma}^{1/2}/p \right) + \left( \text{tr}(\bar{\Sigma})/p - \text{tr}(Q_t^2 \bar{\Sigma}^{1/2}/p \right)
\]

\[
= o_p(1).
\]

In addition, by (5.3), we have

\[
\text{tr} \left( (Q_t - \bar{\Sigma}^{1/2})(Q_t - \bar{\Sigma}^{1/2})^T \right)/p
\]

\[
= \sum_{1 \leq i,j \leq p} \left( (Q_t)_{ij} - (\bar{\Sigma}^{1/2})_{ij} \right)^2 / p
\]

\[
\leq 2 \sum_{1 \leq i,j \leq p} \left( (Q_t)_{ij} \right)^2 / p + 2 \sum_{1 \leq i,j \leq p} \left( (\bar{\Sigma}^{1/2})_{ij} \right)^2 / p
\]

\[
= 2 \text{tr}(\Sigma_t/p) + 2 \text{tr}(\bar{\Sigma}/p)
\]

\[
\leq \frac{2}{\delta} \left( \text{tr}(\Sigma_t)/p + \text{tr}(\bar{\Sigma}/p) \right).
\]

By Assumption 1(iii), we have

\[
E \left( \text{tr}(\Sigma_t)/p + \text{tr}(\bar{\Sigma}/p) \right) = 2 \text{tr}(\bar{\Sigma})/p \leq 2 \|\Sigma\| < 2C.
\]

By the dominated convergence theorem again, the bound (5.21) follows.

\[
\square
\]

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**Supplement Materials**

Supplement to “High-dimensional covariance matrices under dynamic volatility models: asymptotics and shrinkage estimation”.

32
This supplement contains the proofs of Theorems 1, 2, 4 and 5 and Corollary 1.

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Supplement to “High-dimensional covariance matrices under dynamic volatility models: asymptotics and shrinkage estimation”

Yi Ding and Xinghua Zheng

For ease of notation, we drop the subscript $p$ in $a_p$ and $b_p$ in the following proofs.

Proof of Theorem 1

Recall that $\eta(a,b,p) = (a/(1-a-b)) \min(\sqrt{p(1-a-b)},1)$. Under the assumption $\eta(a,b,p) \to 0$, we separately consider the following two cases:

$$\begin{cases}
\text{Case I: when } a/(1-a-b) \to 0; \\
\text{Case II: when } a/(1-a-b) > c \text{ and } a^2p/(1-a-b) \to 0.
\end{cases}$$

We first consider Case I. By (5.25) and the Löwner-Heinz inequality,

$$\Sigma_{t_0}^{1/2} \geq \sqrt{1 - a/(1-b)} \Sigma^{1/2}. \quad \text{(A.1)}$$

By Corollary A.42 of Bai and Silverstein (2010),

$$L^4(F_{S_n}, F_{S_0}) \leq \frac{2}{p^2n} \text{tr}(S_n + S_0^0) \cdot \text{tr}\left( (R - R^0)(R - R^0)^T \right), \quad \text{(A.2)}$$

where $R = (R_1,...,R_n)$ and $R^0 = (R_0^1,...,R_0^n)$. Under model (1.1), by the independence between $(\Sigma_t)$ and $(z_t)$, we have

$$E\left( \text{tr}(S_n)/p \right) = E\left( \text{tr}(S_0^0)/p \right) = \text{tr}(\Sigma)/p.$$ 

Therefore, by Assumption 1(iii), we have

$$\frac{1}{p} \text{tr}(S_n + S_0^0) = O_p(1). \quad \text{(A.3)}$$
By (A.1) and the fact that if $A \geq 0$ and $B \geq 0$, then $\text{tr}(AB) \geq 0$, we get

$$\text{tr} \left( \Sigma_t^{1/2} \Sigma_t^{1/2} \right) \geq \sqrt{1 - a/(1 - b)} \text{tr}(\Sigma).$$

Under model (1.1), we have $E(\text{tr}(\Sigma_t)) = \text{tr}(\Sigma)$. Combining these results and using the definitions of $R_t$ and $R_t^0$, we get

$$0 \leq E \left( \frac{1}{np} \text{tr} \left( (R - R^0)(R - R^0)^T \right) \right) = \frac{1}{p} E \left( \text{tr} \left( (\Sigma_t^{1/2} - \Sigma^{1/2})(\Sigma_t^{1/2} - \Sigma^{1/2})^T \right) \right) = \frac{2}{p} \left( \text{tr}(\Sigma) - E \text{tr} \left( \Sigma_t^{1/2} \Sigma_t^{1/2} \right) \right) \leq 2 \left( 1 - \sqrt{1 - a/(1 - b)} \right) \text{tr}(\Sigma/p) = o(1),$$

where the last equation holds by the assumption that $a/(1 - a - b) \to 0$, which implies that $a/(1 - b) \to 0$ as $p \to \infty$. By Markov’s inequality,

$$\frac{1}{np} \text{tr} \left( (R - R^0)(R - R^0)^T \right) = o_p(1). \quad (A.4)$$

The desired bound (2.2) follows from (A.2), (A.3) and (A.4).

Next, we consider Case II. Define

$$W_t = \Sigma_t^{1/2} \left( (1 - a - b)/(1 - b)I + \sum_{s=1}^{\infty} ab^{s-1} \Sigma_t^{-1/2} R_{t-s} R_{t-s}^T \Sigma_t^{-1/2} \right)^{1/2},$$

and

$$\Omega_t = W_t^{-1} \Sigma_t^{1/2}.$$

It is straightforward to verify that $W_t W_t^T = \Sigma_t$, from which we get that

$$\Omega_t (\Omega_t)^T = (\Omega_t)^T \Omega_t = I. \quad (A.5)$$

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Define
\[ \varepsilon_t = \mathcal{O}_t z_t, \quad \mathcal{R}_t^0 = \Sigma^{1/2} \varepsilon_t, \quad \mathcal{R}^0 = (\mathcal{R}_1^0, \ldots, \mathcal{R}_n^0), \quad \text{and} \quad \mathcal{S}_n^0 = \frac{1}{n} \sum_{t=1}^n \mathcal{R}_t^0 (\mathcal{R}_t^0)^T. \quad \text{(A.6)} \]

Note that \( \mathcal{O}_t \) is \( \mathcal{F}_{t-1} \)-measurable. By (A.5) and the assumption that \( z_t \overset{\text{i.i.d.}}{\sim} N(0, I) \), we have
\[ \varepsilon_t \overset{\text{i.i.d.}}{\sim} N(0, I). \]

It follows from Theorem 1 of Marčenko and Pastur (1967) that \( F^{\mathcal{S}_n^0} \overset{P}{\to} F \), and \( F^{\mathcal{S}_n^0} \overset{P}{\to} F \). Hence,
\[ L(F^{\mathcal{S}_n^0}, F^{\mathcal{S}_n^0}) = o_p(1). \quad \text{(A.7)} \]

By (A.7) and the triangle inequality, to show (2.2), it suffices to show that
\[ L(F^{\mathcal{S}_n^0}, F^{\mathcal{S}_n}) = o_p(1). \quad \text{(A.8)} \]

By Corollary A.42 of Bai and Silverstein (2010),
\[ L^4(F^{\mathcal{S}_n}, F^{\mathcal{S}_n^0}) \leq \frac{2}{p} \text{tr}(\mathcal{S}_n + \mathcal{S}_n^0) \cdot \frac{1}{pn} \text{tr} \left( (\mathcal{R} - \mathcal{R}^0)(\mathcal{R} - \mathcal{R}^0)^T \right). \quad \text{(A.9)} \]

We have \( E\left( \text{tr}(\mathcal{S}_n^0/p) \right) = \text{tr}(\mathcal{S})/p = O(1) \). By Markov’s inequality,
\[ \text{tr}(\mathcal{S}_n^0/p) = O_p(1). \quad \text{(A.10)} \]

By the independence between \( (\Sigma_t) \) and \( (z_t) \), \( E\left( \text{tr}(\mathcal{S}_n) \right) = E\left( \text{tr}(\Sigma_t) \right) = \text{tr}(\mathcal{S}). \) Therefore, (A.3) holds. Combining (A.3) and (A.10) yields
\[ \text{tr}(\mathcal{S}_n/p + \mathcal{S}_n^0/p) = O_p(1). \quad \text{(A.11)} \]

We will show that
\[ \frac{1}{p} E \left( \text{tr} \left( (\mathcal{W}_t - \Sigma^{1/2})(\mathcal{W}_t - \Sigma^{1/2})^T \right) \right) = o(1). \quad \text{(A.12)} \]
Then by the facts that $\Sigma_t$ and $\Omega_t$ are $\mathcal{F}_{t-1}$-measurable, we have

$$
\frac{1}{np} E \left( \text{tr} \left( (R - \mathcal{R}^0)(R - \mathcal{R}^0)^T \right) \right) \\
= \frac{1}{p} E \left( \text{tr} \left( (R_t - \mathcal{R}^0_t)(R_t - \mathcal{R}^0_t)^T \right) \right) \\
= \frac{1}{p} E \left( \text{tr} \left( (\Sigma_t^{1/2} - \Sigma_t^{1/2} \Omega_t)z_t z_t^T (\Sigma_t^{1/2} - \Sigma_t^{1/2} \Omega_t)^T \right) \right) \\
= \frac{1}{p} E \left( \text{tr} \left( (W_t - \Sigma_t^{1/2})(W_t - \Sigma_t^{1/2})^T \right) \right) = o(1),
$$

which implies that

$$
\frac{1}{np} \text{tr} \left( (R - \mathcal{R}^0)(R - \mathcal{R}^0)^T \right) = o_p(1). \quad (A.14)
$$

The desired bound (A.8) then follows from (A.9), (A.11) and (A.14).

It remains to show (A.12). Define

$$
y_t = \Sigma_t^{-1/2} R_t,
$$

$$
\Sigma_{y,t} = \Sigma_t^{-1/2} \Sigma_t \Sigma_t^{-1/2} = (1 - a - b)/(a - b)I + \sum_{s=1}^{\infty} ab^{s-1} y_{t-s} y_{t-s}^T,
$$

$$
z_t^y = (\Sigma_{y,t})^{-1/2} y_t = (\Sigma_{y,t})^{-1/2} \Sigma_t^{-1/2} \Sigma_t^{1/2} z_t,
$$

$$
\Omega_t^y = (\Sigma_{y,t})^{-1/2} \Sigma_t^{-1/2} \Sigma_t^{1/2}.
$$

Note that $\Omega_t^y$ is $\mathcal{F}_{t-1}$-measurable, and $\Omega_t^y (\Omega_t^y)^T = (\Omega_t^y)^T \Omega_t^y = I$. By the assumption that $z_t \sim_{\text{i.i.d.}} N(0, I)$, we have that $z_t^y$ is independent with $\Sigma_{y,t}$, and $z_t^y \sim_{\text{i.i.d.}} N(0, I)$. We then get that $y_t = (\Sigma_{y,t})^{1/2} z_t^y$ follows model (1.1) with $E(y_t y_t^T) = E(\Sigma_{y,t}) = I$. We have

$$
\text{tr}(W_t \Sigma_t^{1/2}) = \text{tr}(\Sigma_t^{1/2} \Sigma_{y,t}^{1/2} \Sigma_t^{1/2}) \\
= \text{tr}(\Sigma) + \text{tr}(\Sigma_t^{1/2} (\Sigma_{y,t}^{1/2} - I) \Sigma_t^{1/2}). \quad (A.15)
$$

We denote by $x_1, ..., x_p$, and $u_1, ..., u_p$ the eigenvalues of $\Sigma_{y,t}$ and the corresponding eigenvectors. By Assumption 1(iii) and the fact that $|\sqrt{x} - 1| \leq |x - 1|$ for all $x \geq 0,$
we have
\[
\frac{1}{p} \left| \operatorname{tr} \left( \Sigma^{1/2} (\Sigma^{1/2}_{y,t} - I) \Sigma^{1/2} \right) \right| \\
= \frac{1}{p} \left| \sum_{i=1}^{p} (\sqrt{x_i} - 1) \left( u_i^T \Sigma u_i \right) \right| \\
\leq C \sum_{i=1}^{p} |\sqrt{x_i} - 1|/p \\
\leq C \sqrt{\sum_{i=1}^{p} (x_i - 1)^2/p} = C \sqrt{\operatorname{tr}(\Sigma^{2}_{y,t})/p - 2 \operatorname{tr}(\Sigma_{y,t})/p + 1}.
\]

By Jensen’s inequality and noting that \( E(y_t y_t^T) = I \), we get
\[
\frac{1}{p} E \left( \left| \operatorname{tr} \left( \Sigma^{1/2} (\Sigma^{1/2}_{y,t} - I) \Sigma^{1/2} \right) \right| \right) \\
\leq C \sqrt{E \left( \operatorname{tr}(\Sigma^{2}_{y,t})/p - 2 \operatorname{tr}(\Sigma_{y,t})/p + 1 \right)} \\
= C \sqrt{E \left( \operatorname{tr}(\Sigma^{2}_{y,t}/p) \right)} - 1. \tag{A.16}
\]

Next, we compute \( E(\operatorname{tr}(\Sigma^{2}_{y,t})) \). Note that for any fixed matrix \( A \geq 0 \) and \( z \sim N(0, I) \), we have
\[
E \left( (z^T A z)^2 \right) = (\operatorname{tr}(A))^2 + 2 \operatorname{tr}(A^2). \tag{A.17}
\]

Because \( \Sigma_{y,t+1} = (1 - a - b)I + a y_t y_t^T + b \Sigma_{y,t} \), we have
\[
\operatorname{tr}(\Sigma^{2}_{y,t+1}) = (1 - a - b)^2 p + a^2 (y_t^T y_t)^2 + b^2 \operatorname{tr}(\Sigma^{2}_{y,t}) \\
+ 2(1 - a - b) a y_t^T y_t \\
+ 2(1 - a - b) b \operatorname{tr}(\Sigma_{y,t}) \\
+ 2aby_t^T \Sigma_{y,t} y_t. \tag{A.18}
\]

By normality of \( z^T_t \), the independence between \( (\Sigma_{y,t}) \) and \( (z_t^y) \), (A.17) and conditional
on $\Sigma_{y,t}$, we have

$$E\left( (y_t^T y_t)^2 \right) = E\left( ((z_t^y)^T \Sigma_{y,t} z_t^y)^2 \right)$$

$$= E\left( \left( \text{tr}(\Sigma_{y,t}) \right)^2 \right) + 2E\left( \text{tr}(\Sigma_{y,t}^2) \right).$$

Moreover, $E(\Sigma_{y,t}) = I$. Taking expectations on both sides of (A.18) yields

$$E\left( \text{tr}(\Sigma_{y,t}^2) \right) = \frac{1 - (a + b)^2}{1 - a^2 - (a + b)^2} p + \frac{a^2}{1 - a^2 - (a + b)^2} E\left( \left( \text{tr}(\Sigma_{y,t}) \right)^2 \right). \quad \text{(A.19)}$$

About $\left( \text{tr}(\Sigma_{y,t}) \right)^2$, note that

$$\left( \text{tr}(\Sigma_{y,t+1}) \right)^2 = (1 - a - b)^2 p^2 + a^2(y_t^T y_t)^2 + b^2 \left( \text{tr}(\Sigma_{y,t}) \right)^2$$

$$+ 2(1 - a - b)ap(y_t^T y_t)$$

$$+ 2(1 - a - b)bp \text{tr}(\Sigma_{y,t})$$

$$+ 2aby_t^T y_t \text{tr}(\Sigma_{y,t}). \quad \text{(A.20)}$$

Taking expectations on both sides of (A.20) yields

$$E\left( \left( \text{tr}(\Sigma_{y,t}) \right)^2 \right) = p^2 + \frac{2a^2}{1 - (a + b)^2} E\left( \text{tr}(\Sigma_{y,t}^2) \right). \quad \text{(A.21)}$$

Combining (A.19) and (A.21), we get

$$E\left( \text{tr}(\Sigma_{y,t}^2) \right) = C_p C^{(2)}_p \left( \frac{1 - (a + b)^2}{a^2} p + p^2 \right), \text{ and}$$

$$E\left( \left( \text{tr}(\Sigma_{y,t}) \right)^2 \right) = C_p \left( p^2 + 2C^{(2)}_p p \right). \quad \text{(A.22)}$$
where
\[ C_p = \left( 1 - \frac{2a^4}{(1 - a^2 - (a + b)^2)(1 - (a + b)^2)} \right)^{-1}, \quad \text{and} \]
\[ C_p^{(2)} = \frac{a^2}{1 - a^2 - (a + b)^2}, \]
and (A.22) holds when \( C_p \) and \( C_p^{(2)} \) are both positive. Under Case II, we have
\[ \frac{1 - (a + b)^2}{1 - a^2 - (a + b)^2} \to 1, \quad C_p = 1 + o(1), \quad \text{and} \quad C_p^{(2)} = \frac{a^2}{1 - (a + b)^2} \cdot (1 + o(1)). \]
It follows that
\[ E\left( \text{tr}\left( \Sigma^{2}_{y,t} \right) \right) = p + O\left( p^2 a^2 / \left( 1 - a - b \right) \right) = p + o(p). \quad (A.23) \]
Combining (A.15), (A.16) and (A.23) yields
\[ E\left( \text{tr}(W_t \Sigma^{1/2}) \right) = \text{tr}(\Sigma) + o(p). \]
We then get
\[
\frac{1}{p} E\left( \text{tr}\left( (W_t - \Sigma^{1/2})(W_t - \Sigma^{1/2})^T \right) \right) \\
= \frac{1}{p} \left( E(\text{tr}(\Sigma_t)) - \text{tr}(\Sigma) \right) + \frac{2}{p} \left( \text{tr}(\Sigma) - E(\text{tr}(W_t \Sigma^{1/2})) \right) \\
= o(1),
\]
namely, (A.12) holds. \( \square \)

**Proof of Theorem 2**

Write
\[ M_p^2 = \frac{1}{p} \text{tr} \left( (S_n)^2 \right) = \frac{1}{n^2 p} \sum_{1 \leq t_1, t_2 \leq n} z_{t_1}^T \Sigma_t^{1/2} \Sigma_t^{1/2} z_{t_2} z_{t_2}^T \Sigma_t^{1/2} \Sigma_t^{1/2} z_{t_1}. \]
We have
\[
E(M_p^2) = \frac{1}{n^2 p} \sum_{t=1}^{n} E\left((z_t^T \Sigma_t z_t)^2\right) + \frac{2}{n^2 p} \sum_{t_1=1}^{n-1} \sum_{t_2=t_1+1}^{n} E\left(z_{t_1}^T \Sigma_{t_1}^{1/2} \Sigma_{t_2} \Sigma_{t_1}^{1/2} z_{t_1}\right)
\]
\[
=: I + II.
\]

(A.24)

Under Assumption 1(i)--(iv), by (4.14) of Yin (1986), we have
\[
E(M_0^2) = \text{tr}\left(\Sigma^2 / p\right) + y\left(\text{tr}(\Sigma)/p\right)^2 + o(1).
\]

(A.25)

We will show
\[
I \geq y\left(\text{tr}(\Sigma)/p\right)^2 (1 + o(1)), \ \text{and} \ II \geq \text{tr}\left(\Sigma^2 / p\right) (1 + o(1)) + \delta.
\]

(A.26)

The desired bound (2.3) then follows from (A.25) and (A.26).

We start with term I. By the independence between \((\Sigma_t)\) and \((z_t)\) and Jensen’s inequality, we have
\[
I = \frac{1}{n^2 p} \sum_{t=1}^{n} E\left((z_t^T \Sigma_t z_t)^2\right)
\]
\[
\geq \frac{1}{n^2 p} \sum_{t=1}^{n} \left(E(z_t^T \Sigma_t z_t)\right)^2
\]
\[
= \frac{1}{np} \left(\text{tr}(\Sigma)\right)^2 = y\left(\text{tr}(\Sigma)/p\right)^2 (1 + o(1)).
\]

(A.27)

About term II, by (A.17), the independence between \((\Sigma_t)\) and \((z_t)\), \(z_t \sim \text{i.i.d.} \ N(0, I)\), taking conditional expectations recursively and using the fact that
\[
\Sigma_{t+j} = (1 - (a + b)^j) \Sigma
\]
\[
+ \sum_{i=1}^{j} a(a + b)^{j-i} (\Sigma_{t+i-1}^{1/2} z_{t+i-1} z_{t+i-1}^T \Sigma_{t+i-1}^{1/2} - \Sigma_{t+i-1})
\]
\[
+ (a + b)^j \Sigma_t,
\]
we have

$$II = \frac{2}{n^2p} \sum_{t_1=1}^{n-1} \left( \sum_{t_2=t_1+1}^{n} (1 - (a + b)^{t_2-t_1}) \right) E \left( z_{t_1}^T \Sigma_{t_1}^{1/2} \Sigma \Sigma_{t_1}^{1/2} z_{t_1} \right)$$

$$+ \frac{2a}{n^2p} \sum_{t_1=1}^{n-1} \left( \sum_{t_2=t_1+1}^{n} (a + b)^{t_2-t_1-1} \right) E \left( \left( z_{t_1}^T \Sigma_{t_1} z_{t_1} \right)^2 - E \left( z_{t_1}^T \Sigma_{t_1} z_{t_1} \right) \right)$$

$$+ \frac{2}{n^2p} \sum_{t_1=1}^{n-1} \left( \sum_{t_2=t_1+1}^{n} (a + b)^{t_2-t_1} \right) E \left( z_{t_1}^T \Sigma_{t_1}^2 z_{t_1} \right)$$

$$= \frac{2}{n^2p} \sum_{t_1=1}^{n-1} \left( \sum_{t_2=t_1+1}^{n} (1 - (a + b)^{t_2-t_1}) \right) \text{tr} (\Sigma^2)$$

$$+ \frac{2}{n^2p} \cdot \frac{a}{a+b} \sum_{t_1=1}^{n-1} \left( \sum_{t_2=t_1+1}^{n} (a + b)^{t_2-t_1} \right) E \left( \text{tr} (\Sigma_{t_1})^2 \right)$$

$$+ \frac{2}{n^2p} \cdot \left( 1 + \frac{a}{a+b} \right) \sum_{t_1=1}^{n-1} \left( \sum_{t_2=t_1+1}^{n} (a + b)^{t_2-t_1} \right) E \left( \text{tr} (\Sigma_{t_1}^2) \right).$$

Define

$$\gamma_n = \sum_{t_1=1}^{n-1} \left( \sum_{t_2=t_1+1}^{n} (a + b)^{t_2-t_1} \right)$$

$$= \frac{(a + b)((n - 1)(1 - (a + b)) - (a + b) + (a + b)^n)}{(1 - (a + b))^2}.$$

It is easy to show that

$$\gamma_n \approx \frac{n}{1 - a - b} \min \left( n(1 - a - b), 1 \right). \quad (A.28)$$

We then get

$$II = \left( \frac{n-1}{n} \right) \text{tr} (\Sigma^2) / p + \frac{2\gamma_n}{n^2p} \left( E \left( \text{tr} (\Sigma^2) \right) - \text{tr} (\Sigma^2) \right)$$

$$+ \frac{2a\gamma_n}{(a + b)n^2p} E \left( \text{tr} (\Sigma) \right)^2 \right) + \frac{2\gamma_n}{n^2p} \left( \frac{a}{a + b} \right) E \left( \text{tr} (\Sigma^2) \right)$$

$$\geq \frac{1}{p} \text{tr} (\Sigma^2) (1 + o(1)) + \frac{2\gamma_n}{n^2p} \left( E \left( \text{tr} (\Sigma^2) \right) - \text{tr} (\Sigma^2) \right).$$

(A.29)
About $E\left( \text{tr}(\Sigma^2_i) \right)$, by $\Sigma_{t+1} = (1-a-b)\overline{\Sigma} + a\Sigma^{1/2}_t z_t z^T_t \Sigma^{1/2}_t + b\Sigma_t$ and using an argument similar to (A.17), (A.18) and (A.19), one can show that when $a^2 + (a+b)^2 < 1$,

$$E\left( \text{tr}(\Sigma^2_i) \right) = \frac{1 - (a + b)^2}{1 - a^2 - (a + b)^2} \text{tr}(\overline{\Sigma}^2) + \frac{a^2}{1 - a^2 - (a + b)^2} E\left( \left( \text{tr}(\Sigma_i) \right)^2 \right). \quad (A.30)$$

About $E\left( \left( \text{tr}(\Sigma_i) \right)^2 \right)$, by Jensen’s inequality, we have

$$E\left( \left( \text{tr}(\Sigma_i) \right)^2 \right) \geq \left( E\left( \text{tr}(\Sigma_i) \right) \right)^2 = \left( \text{tr}(\overline{\Sigma}) \right)^2. \quad (A.31)$$

Noting that

$$\frac{a^2}{1 - a^2 - (a + b)^2} \geq \frac{a^2}{1 - (a + b)^2} \geq \frac{a^2}{2(1 - a - b)},$$

(A.30) and (A.31) imply that

$$E\left( \text{tr}(\Sigma^2_i) \right) \geq \text{tr}(\overline{\Sigma}^2) + \frac{a^2}{2(1 - a - b)} \left( \text{tr}(\overline{\Sigma}) \right)^2. \quad (A.32)$$

By Assumption 1(ii), $\text{tr}(\overline{\Sigma}) \approx p$. By Assumption 1(iv), (A.28), (A.29) and (A.32),

$$II \geq \text{tr} \left( \Sigma^2 / p \right) \left( 1 + o(1) \right) + C \left( \eta(a,b,p) \right)^2. \quad (A.33)$$

The bound (A.26) follows from (A.27), (A.33) and the assumption that $\eta(a,b,p) > c$. □

**Proof of Corollary 1**

Theorem 3 implies that $F_{\tilde{S}^n} - F_{S^0} \overset{p}{\rightarrow} 0$ as $p,n \rightarrow \infty$. By Theorem 1 of Marčenko and Pastur (1967), $F_{S^0} \overset{p}{\rightarrow} F$. Denote the empirical distribution of the truncated eigenvalues $\overline{\lambda_i}$ by $F_{\tilde{S}^n}$. Then as long as the truncation level is greater than the upper bound of the support of $F$, we have

$$F_{\tilde{S}^n} \overset{p}{\rightarrow} F. \quad (A.34)$$
The conclusion $\sum_{i=1}^{p}(\lambda_i^H - \hat{\lambda}_i^H)^2/p = o_p(1)$ then follows from (A.34) and the proof of Theorem 2.2 of Ledoit and Wolf (2015). \hfill \Box$

**Proof of Theorem 4**

Note that
\[
\frac{1}{p} \text{tr} \left( (\tilde{S}_n - zI)^{-1}g(\Sigma) \right) - \frac{1}{p} \text{tr} \left( (S^0_n - zI)^{-1}g(\Sigma) \right)
\]
\[
= \frac{1}{p} \text{tr} \left( (S^0_n - zI)^{-1}(S^0_n - \tilde{S}_n)(\tilde{S}_n - zI)^{-1}g(\Sigma) \right),
\]
where, recall that $S^0_n = R^0(R^0)^T/n$ for $R^0$ defined in (5.13). We have $\|g(\Sigma)\| < c$, $\|(S^0_n - zI)^{-1}\| < 1/v$ and $\|\tilde{S}_n - zI\| < 1/v$, where $v$ is the imaginary part of $z$. By Corollary A.12 and Theorem A.14 of Bai and Silverstein (2010), for some $C > 0,$

\[
\left| \frac{1}{p} \text{tr} \left( (\tilde{S}_n - zI)^{-1}g(\Sigma) \right) - \frac{1}{p} \text{tr} \left( (S^0_n - zI)^{-1}g(\Sigma) \right) \right|
\]
\[
\leq \frac{1}{p} \sum_{i=1}^{p} s_i(\tilde{S}_n - S^0_n) \cdot s_i\left( (\tilde{S}_n - zI)^{-1}g(\Sigma)(S^0_n - \tilde{S}_n)(\tilde{S}_n - zI)^{-1} \right)
\]
\[
\leq C \frac{1}{p} \sum_{i=1}^{p} s_i(\tilde{S}_n - S^0_n),
\]
where for any matrix $A$, $s_i(A)$ represents its $i$th singular value. Note that

\[
\tilde{S}_n - S^0_n = \frac{1}{n}(\tilde{R} - R^0)(\tilde{R} - R^0)^T + \frac{1}{n}(\tilde{R} - R^0)(R^0)^T + \frac{1}{n}R^0(\tilde{R} - R^0)^T.
\]

By Theorem A.8 of Bai and Silverstein (2010),

\[
\sum_{i=1}^{p} s_i(\tilde{S}_n - S^0_n) \leq 2\sum_{i=1}^{p} \left( \frac{1}{n}(\tilde{R} - R^0)(\tilde{R} - R^0)^T \right) + 4 \sum_{i=1}^{p} \left( \frac{1}{n}(\tilde{R} - R^0)(R^0)^T \right).
\]
\[
(A.36)
\]
By (5.10) and (5.23),

\[
\frac{1}{pn} \text{tr} \left( (\tilde{R} - \mathcal{R}^0)(\tilde{R} - \mathcal{R}^0)^T \right) \\
\leq \frac{2}{pn} \text{tr} \left( (\hat{R} - \mathcal{R}^0)(\hat{R} - \mathcal{R}^0)^T \right) + \frac{2}{pn} \text{tr} \left( (\tilde{R} - \hat{R})(\tilde{R} - \hat{R})^T \right) \\
= o_p(1). \tag{A.37}
\]

Therefore,

\[
\sum_{i=1}^{p} s_i \left( \frac{1}{n} (\tilde{R} - \mathcal{R}^0)(\tilde{R} - \mathcal{R}^0)^T \right) = \text{tr} \left( (\tilde{R} - \mathcal{R}^0)(\tilde{R} - \mathcal{R}^0)^T / n \right) = o_p(p). \tag{A.38}
\]

By the Cauchy-Schwarz inequality, we have

\[
\sum_{i=1}^{p} s_i \left( \frac{1}{n} (\tilde{R} - \mathcal{R}^0)(\mathcal{R}^0)^T \right) \leq \sqrt{p} \sqrt{\sum_{i=1}^{p} \left( s_i \left( \frac{1}{n} (\tilde{R} - \mathcal{R}^0)(\mathcal{R}^0)^T \right) \right)^2} \\
= \sqrt{p} \sqrt{\text{tr} \left( \frac{1}{n^2} (\tilde{R} - \mathcal{R}^0)(\mathcal{R}^0)^T \mathcal{R}^0(\tilde{R} - \mathcal{R}^0)^T \right)}. 
\]

By (5.14) and Theorem 3.1 of Yin et al. (1988), \( \| \sum_{t=1}^{n} \zeta_t \zeta_t^T / n \| = O_p(1) \). It follows that

\[
\| (\mathcal{R}^0)^T \mathcal{R}^0 / n \| = \| \mathcal{R}^0(\mathcal{R}^0)^T / n \| \leq \| \Sigma \| \cdot \left( \sum_{t=1}^{n} \zeta_t \zeta_t^T / n \right) = O_p(1). \tag{A.39}
\]

By (A.37), (A.39), Corollary A.12 and Theorem A.14 of Bai and Silverstein (2010) again,

\[
\text{tr} \left( \frac{1}{n^2} (\tilde{R} - \mathcal{R}^0)(\mathcal{R}^0)^T \mathcal{R}^0(\tilde{R} - \mathcal{R}^0)^T \right) \\
\leq \| (\mathcal{R}^0)^T \mathcal{R}^0 / n \| \cdot \text{tr} \left( (\tilde{R} - \mathcal{R}^0)(\tilde{R} - \mathcal{R}^0)^T / n \right) \\
= o_p(p). \tag{A.40}
\]

Combining (A.35), (A.36), (A.38) and (A.40) yields

\[
\left| \frac{1}{p} \text{tr} \left( (\tilde{S}_n - zI)^{-1} g(\Sigma) \right) - \frac{1}{p} \text{tr} \left( (\mathcal{S}^0_n - zI)^{-1} g(\Sigma) \right) \right| = o_p(1). \tag{A.41}
\]
By Theorem 2 of Ledoit and Péché (2011), we have

\[
\frac{1}{p} \text{tr} \left( (S_n^{0} - zI)^{-1} g(\Sigma) \right) - \Theta^g(z) = o_p(1).
\] (A.42)

The conclusion follows. \(\Box\)

**Proof of Theorem 5**

By Corollary 1, the estimated population eigenvalues satisfy that \(\sum_{i=1}^{p} (\hat{\lambda}_i - \lambda_i^H)^2 / p \xrightarrow{p} 0\). By Theorem 4 and Theorem 4 of Ledoit and Péché (2011), \(\tilde{\Sigma}^{\text{or}}\) is the oracle shrinkage estimator. The desired result then follows from the proof of Theorem 3.1 of Ledoit and Wolf (2015). \(\Box\)