A NOTE ON $q$-BERNSTEIN POLYNOMIALS

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Abstract. Recently, Simsek-Acikgoz([17]) and Kim-Jang-Yi([9]) have studied the $q$-extension of Bernstein polynomials. In this paper we propose the $q$-extension of Bernstein polynomials of degree $n$, which are different $q$-Bernstein polynomials of Simsek-Acikgoz([17]) and Kim-Jang-Yi([9]). From these $q$-Bernstein polynomials, we derive some fermionic $p$-adic integral representations of several $q$-Bernstein type polynomials. Finally, we investigate some identities between $q$-Bernstein polynomials and $q$-Euler numbers.

§1. Introduction

Let $C[0, 1]$ denote the set of continuous function on $[0, 1]$. For $f \in C[0, 1]$, Bernstein introduced the following well known linear operators (see [1, 3]):

\begin{equation}
B_n(f|x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x).
\end{equation}

Here $B_n(f|x)$ is called Bernstein operator of order $n$ for $f$. For $k, n \in \mathbb{Z}_+(= \mathbb{N} \cup \{0\})$, the Bernstein polynomials of degree $n$ is defined by

\begin{equation}
B_{k,n}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \text{ (see [1, 2, 3])}.
\end{equation}

A Bernoulli trial involves performing an experiment once and noting whether a particular event $A$ occurs. The outcome of Bernoulli trial is said to be “success” if $A$ occurs and a “failure” otherwise. Let $k$ be the number of successes in $n$ independent Bernoulli trials, the probabilities of $k$ are given by the binomial probability law:

\[ p_n(k) = \binom{n}{k} p^k (1-p)^{n-k}, \text{ for } k = 0, 1, \ldots, n, \]

Key words and phrases. $q$-Bernstein polynomials, $q$-Euler numbers, $q$-Stirling numbers, fermionic $p$-adic integrals.

2000 AMS Subject Classification: 11B68, 11S80, 60C05, 05A30

The present Research has been conducted by the research Grant of Kwangwoon University in 2010

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where \( p_n(k) \) is the probability of \( k \) successes in \( n \) trials. For example, a communication system transmit binary information over channel that introduces random bit errors with probability \( \xi = 10^{-3} \). The transmitter transmits each information bit three times, an a decoder takes a majority vote of the received bits to decide on what the transmitted bit was. The receiver can correct a single error, but it will make the wrong decision if the channel introduces two or more errors. If we view each transmission as a Bernoulli trial in which a “success” corresponds to the introduction of an error, then the probability of two or more errors in three Bernoulli trial is

\[
p(k \geq 2) = \binom{3}{2}(0.001)^2(0.999) + \binom{3}{3}(0.001)^3 \approx 3(10^{-6}), \text{ see [18].}
\]

By the definition of Bernstein polynomials (see Eq.(1) and Eq.(2)), we can see that Bernstein basis is the probability mass function of binomial distribution. In the reference [15] and [16], Phillips proposed a generalization of classical Bernstein polynomials based on \( q \)-integers. In the last decade some new generalizations of well known positive linear operators based on \( q \)-integers were introduced and studied by several authors (see [1-21]). Let \( 0 < q < 1 \). Define the \( q \)-numbers of \( x \) by \( [x]_q = \frac{1-q^x}{1-q} \) (see [1-21]). Recently, Simsek-Acikgoz ([17]) and Kim-Jang-Yi ([9]) have studied the \( q \)-extension of Bernstein polynomials, which are different Phillips \( q \)-Bernstein polynomials. Let \( p \) be a fixed odd prime number. Throughout this paper \( \mathbb{Z}_p, \mathbb{Q}_p, \text{ and } \mathbb{C}_p \) denote the rings of \( p \)-adic integers, the fields of \( p \)-adic rational numbers, and the completion of algebraic closure of \( \mathbb{Q}_p \), respectively. The \( p \)-adic absolute value in \( \mathbb{C}_p \) is normalized in such way that \( |p|_p = \frac{1}{p} \). As well known definition, Euler polynomials are defined by

\[
\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \text{ (see [1-14]).}
\]

In the special case, \( x = 0 \), \( E_n(0) = E_n \) are called the \( n \)-th Euler numbers. By (3), we see that the recurrence formula of Euler numbers is given by

\[
E_0 = 1, \text{ and } (E + 1)^n + E_n = 0 \text{ if } n > 0, \text{ (see [12]),}
\]

with the usual convention of replacing \( E^n \) by \( E_n \). When one talks of \( q \)-analogue, \( q \) is variously considered as an indeterminate, a complex number \( q \in \mathbb{C} \), or a \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \), we normally assume \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we normally always assume that \( |1 - q|_p < 1 \). As the \( q \)-extension of (4), author defined the \( q \)-Euler numbers as follows:

\[
E_{0,q} = 1, \text{ and } (qE + 1)^n + E_{n,q} = 0 \text{ if } n > 0, \text{ (see [21]),}
\]

with the usual convention of replacing \( E_q^n \) by \( E_{n,q} \). Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable function on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), the fermionic \( p \)-adic \( q \)-integral was defined by

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [12]).}
\]
In the special case, $q = 1$, $I_1(f)$ is called the fermionic $p$-adic integral on $\mathbb{Z}_p$ (see [12, 21]). By (6) and the definition of $I_1(f)$, we see that

$$I_1(f_1) + I_1(f) = 2f(0), \text{ where } f_1(x) = f(x + 1).$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x + n)$. Then we can also see that

$$I_1(f_n) + (-1)^{n-1}I_1(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-l-1}f(l), \text{ (see [21]).}$$

From (5), (7) and (8), we note that

$$\int_{\mathbb{Z}_p} e^{[x]q^t} d\mu_{-1}(x) = \sum_{n=0}^{\infty} E_{n,q,t} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1+q^l} \right) \frac{t^n}{n!}.$$ 

Thus we have

$$E_{n,q} = \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1+q^l}, \text{ (see [21]).}$$

In [21], the $q$-Euler polynomials are defined by

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [y + x]^n d\mu_{-1}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{lx}}{1+q^l}.$$ 

By (9) and (10), we get

$$E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} E_{l,q} = (q^x E_q + 1)^n,$$

with the usual convention of replacing $E^n_q$ by $E_{n,q}$. In this paper we firstly consider the $q$-Bernstein polynomials of degree $n$ in $\mathbb{R}$, which are different $q$-Bernstein polynomials of Simsek-Acikgoz([17]) and Kim-Jang-Yi([9]). From these $q$-Bernstein polynomials, we try to study for the fermionic $p$-adic integral representations of the several $q$-Bernstein type polynomials on $\mathbb{Z}_p$. Finally, we give some interesting identities between $q$-Bernstein polynomials and $q$-Euler numbers.

§2. $q$-Bernstein Polynomials

For $n, k \in \mathbb{Z}_+$, the generating function for $B_{k,n}(x)$ is introduced by Acikgoz and Araci as follows:

$$F^{(k)}(t, x) = \frac{te^{(1-x)t}x^k}{k!} = \sum_{n=0}^{\infty} B_{k,n}(x) \frac{t^n}{n!}, \text{ (see [1, 9, 10, 17]).}$$
For \( k, n \in \mathbb{Z}_+ \), \( 0 < q < 1 \) and \( x \in [0, 1] \), consider the \( q \)-extension of (12) as follows:

\[
F_q^{(k)}(t, x) = \frac{(t[x]_q)_k e^{[1-x]_q t}}{k!} = \frac{[x]_q^k}{k!} \sum_{n=0}^\infty \frac{[1-x]_q^n}{n!} t^{n+k} = \sum_{n=k}^\infty \frac{n! [x]_q^k [1-x]_q^{n-k}}{(n-k)! k!} \frac{t^n}{n!} = \sum_{n=k}^\infty B_{k,n}(x, q) \frac{t^n}{n!}.
\]

Because \( B_{k,0}(x, q) = B_{k,1}(x, q) = \cdots = B_{k,k-1}(x, q) = 0 \), we obtain the following generating function for \( B_{k,n}(x, q) \):

\[
F_q^{(k)}(t, x) = \frac{(t[x]_q)_k e^{[1-x]_q t}}{k!} = \sum_{n=0}^\infty B_{k,n}(x, q) \frac{t^n}{n!}, \text{ where } k \in \mathbb{Z}_+ \text{ and } x \in [0, 1].
\]

Thus, for \( k, n \in \mathbb{Z}_+ \), we note that

\[
B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}, \text{ if } n \geq k,
\]

\[
= 0, \text{ if } k < n.
\]

By (14), we easily get \( \lim_{q \to 1} B_{k,n}(x, q) = B_{k,n}(x) \). For \( 0 \leq k \leq n \), we have

\[
[1-x]_q^\frac{1}{q} B_{k,n-1}(x, q) + [x]_q B_{k-1,n-1}(x, q)
= [1-x]_q^\frac{1}{q} \binom{n-1}{k} [x]_q^k [1-x]_q^{n-k-1} + [x]_q \binom{n-1}{k-1} [x]_q^{k-1} [1-x]_q^{n-k}
= \binom{n-1}{k} [x]_q^k [1-x]_q^{n-k} + \binom{n-1}{k-1} [x]_q^{k-1} [1-x]_q^{n-k} = \binom{n}{k} [x]_q^k [1-x]_q^{n-k},
\]

and the derivative of the \( q \)-Bernstein polynomials of degree \( n \) are also polynomials of degree \( n-1 \).

\[
\frac{d}{dx} B_{k-1,n}(x, q)
= k \binom{n}{k} [x]_q^{k-1} [1-x]_q^{n-k} \left( \frac{\log q}{q-1} \right) q^x + \binom{n}{k} [x]_q^k (n-k) [1-x]_q^{n-k-1} \left( \frac{\log q}{1-q} \right) q^x
= \frac{\log q}{q-1} q^x \left( n \binom{n-1}{k-1} [x]_q^{k-1} [1-x]_q^{n-k} - n \binom{n-1}{k} [x]_q^{k} [1-x]_q^{n-k-1} \right)
= n (B_{k-1,n-1}(x, q) - B_{k,n-1}(x, q)) \frac{\log q}{q-1} q^x.
\]

Therefore, we obtain the following theorem.
Theorem 1. For \( k, n \in \mathbb{Z}_+ \) and \( x \in [0, 1] \), we have

\[
[1 - x]_q^q B_{k,n-1}(x, q) + [x]_q B_{k-1,n-1}(x, q) = B_{k,n}(x, q),
\]

and

\[
\frac{d}{dx} B_{k,n}(x, q) = n (B_{k-1,n-1}(x, q) - B_{k,n-1}(x, q)) \frac{\log_q q - 1}{q - 1} q^x.
\]

Let \( f \) be a continuous function on \([0, 1]\). Then the \( q \)-Bernstein operator of order \( n \) for \( f \) is defined by

\[
\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x, q), \quad \text{where } 0 \leq x \leq 1 \text{ and } n \in \mathbb{Z}_+.
\]

By (14) and (15), we see that

\[
\mathbb{B}_{n,q}(1|x) = \sum_{k=0}^{n} B_{k,n}(x, q) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) [x]_q^k [1 - x]_{\frac{1}{q}}^{n-k} = \left( [x]_q + [1 - x]_{\frac{1}{q}} \right)^n = 1.
\]

Also, we get from (15) that for \( f(x) = x \),

\[
\mathbb{B}_{n,q}(x|x) = \sum_{k=0}^{n} \frac{k}{n} \left( \begin{array}{c} n \\ k \end{array} \right) [x]_q^k [1 - x]_{\frac{1}{q}}^{n-k} = \sum_{k=0}^{n-1} \left( \begin{array}{c} n-1 \\ k \end{array} \right) [x]_q^{k+1} [1 - x]_{\frac{1}{q}}^{n-k-1} = [x]_q.
\]

The \( q \)-Bernstein polynomials are symmetric polynomials in the following sense:

\[
B_{n-k,n}(1 - x, \frac{1}{q}) = \left( \begin{array}{c} n \\ n-k \end{array} \right) [1 - x]_{\frac{1}{q}}^{n-k} [x]_q^k = B_{k,n}(x, q).
\]

Thus, we obtain the following theorem.

Theorem 2. For \( n, k \in \mathbb{Z}_+ \) and \( x \in [0, 1] \), we have

\[
B_{n-k,n}(1 - x, \frac{1}{q}) = B_{k,n}(x, q).
\]

Moreover, \( \mathbb{B}_{n,q}(1|x) = 1 \) and \( \mathbb{B}_{n,q}(x|x) = [x]_q \).

From (15), we note that

\[
\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x, q) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \left( \begin{array}{c} n \\ k \end{array} \right) [x]_q^k [1 - x]_{\frac{1}{q}}^{n-k}
\]

\[
= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \left( \begin{array}{c} n \\ k \end{array} \right) [x]_q^k \sum_{j=0}^{n-k} \left( \begin{array}{c} n-k \\ j \end{array} \right) (-1)^j [x]_q^j.
\]

(16)
By the definition of binomial coefficient, we easily get
\[
\binom{n}{k} \binom{n-k}{j} = \binom{n}{k+j} \binom{k+j}{k}.
\]
Let \(k+j=m\). Then we have
\[
(17) \quad \binom{n}{k} \binom{n-k}{j} = \binom{n}{m} \binom{m}{k}.
\]
From (16) and (17), we have
\[
(18) \quad \mathbb{B}_{n,q}(f|x) = \sum_{m=0}^{n} \binom{n}{m} [x]_q^m \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f\left(\frac{k}{n}\right).
\]
Therefore, we obtain the following theorem.

**Theorem 3.** For \(f \in C[0,1]\) and \(n \in \mathbb{Z}_+\), we have
\[
\mathbb{B}_{n,q}(f|x) = \sum_{m=0}^{n} \binom{n}{m} [x]_q^m \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f\left(\frac{k}{n}\right).
\]

It is well known that the second kind stirling numbers are defined by
\[
(19) \quad \frac{(e^t-1)^k}{k!} = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e^{lt} = \sum_{n=0}^{\infty} s(n,k) \frac{t^n}{n!}, \text{ for } k \in \mathbb{N}, \text{ (see [12, 21]).}
\]
Let \(\Delta\) be the shift difference operator with \(\Delta f(x) = f(x+1) - f(x)\). By iterative process, we easily get
\[
(20) \quad \Delta^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k).
\]
From (19) and (20), we can easily derive the following equation (21).
\[
(21) \quad \frac{1}{k!} \Delta^k 0^n = s(n,k).
\]
By (18) and (20) we obtain the following theorem.

**Theorem 4.** For \(f \in C[0,1]\) and \(n \in \mathbb{Z}_+\), we have
\[
\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^k \Delta^k f\left(\frac{0}{n}\right).
\]
In the special case, \(f(x) = x^m (m \in \mathbb{Z}_+)\), we have the following corollary.
Corollary 5. For \( x \in [0, 1] \) and \( m, n \in \mathbb{Z}_+ \), we have

\[
n^m B_{n,q}(x^m | x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^k \Delta^k 0^m,
\]

and

\[
n^m B_{n,q}(x^m | x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^k k! s(m, k).
\]

For \( x, t \in \mathbb{C} \) and \( n \in \mathbb{Z}_+ \) with \( n \geq k \), consider

(22) \[
\frac{n!}{2\pi i} \int_C \frac{([x]_q t)^k}{k!} e^{(\frac{1-x}{q} t)} \frac{dt}{t^{n+1}},
\]

where \( C \) is a circle around the origin and integration is in the positive direction. We see from the definition of the \( q \)-Bernstein polynomials and the basic theory of complex analysis including Laurent series that

(23) \[
\int_C \frac{([x]_q t)^k}{k!} e^{(\frac{1-x}{q} t)} \frac{dt}{t^{n+1}} = \sum_{m=0}^{\infty} \int_C B_{m,k}(x, q) t^m \frac{dt}{m! t^{n+1}} = 2\pi i \left( \frac{B_{k,n}(x, q)}{n!} \right).
\]

We get from (22) and (23) that

(24) \[
\frac{n!}{2\pi i} \int_C \frac{([x]_q t)^k}{k!} e^{(\frac{1-x}{q} t)} \frac{dt}{t^{n+1}} = B_{k,n}(x, q),
\]

and

(25) \[
\int_C \frac{([x]_q t)^k}{k!} e^{(\frac{1-x}{q} t)} \frac{dt}{t^{n+1}} = \frac{[x]_q^k}{k!} \sum_{m=0}^{\infty} \left( \frac{1-x}{m! q} t^{m} \int_C t^{m-n-k} dt \right)
\]

\[
= 2\pi i \left( \frac{[x]_q^k [1-x]^{n-k}}{k!(n-k)!} \right) = \frac{2\pi i}{n!} \left( \binom{n}{k} [x]_q^k [1-x]^{n-k} \right).
\]

By (22) and (25), we see that

(26) \[
\frac{n!}{2\pi i} \int_C \frac{([x]_q t)^k}{k!} e^{(\frac{1-x}{q} t)} \frac{dt}{t^{n+1}} = \binom{n}{k} [x]_q^k [1-x]^{n-k}.
\]

From (24) and (26), we note that

\[
B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1-x]^{n-k}.
\]
By the definition of $q$-Bernstein polynomials, we easily get

\[
\frac{n-k}{n} B_{k,n}(x, q) + \frac{k+1}{n} B_{k+1,n}(x, q) \\
= \left( \frac{(n-1)!}{k!(n-k-1)!} \right) \left[ x \right]_q^k [1-x]^{\frac{n-k}{q}} + \left( \frac{(n-1)!}{k!(n-k-1)!} \right) \left[ x \right]_q^{k+1} [1-x]^{\frac{n-k-1}{q}} \\
= \left( \left[ 1-x \right]_q^{\frac{k}{q}} + \left[ x \right]_q \right) B_{k,n-1}(x, q) = B_{k,n-1}(x, q).
\]

Therefore, we can write $q$-Bernstein polynomials as a linear combination of polynomials of higher order.

**Theorem 6.** For $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$, we have

\[
\frac{n+1-k}{n+1} B_{k,n+1}(x, q) + \frac{k+1}{n+1} B_{k+1,n+1}(x, q) = B_{k,n}(x, q).
\]

We easily get from (14) that for $n, k \in \mathbb{N},$

\[
\frac{n-k+1}{n} \left( \frac{[x]_q}{[1-x]^{\frac{k}{q}}} \right) B_{k-1,n}(x, q) \\
= \left( \frac{n-k+1}{k} \right) \left( \frac{[x]_q}{[1-x]^{\frac{k}{q}}} \right) \left( \frac{n}{k-1} \right) \left[ x \right]_q^{k-1} [1-x]^{\frac{n-k+1}{q}} \\
= \left( \frac{n!}{k!(n-k)!} \right) \left[ x \right]_q^k [1-x]^{\frac{n-k}{q}} = B_{k,n}(x, q).
\]

Therefore, we obtain the following corollary.

**Corollary 7.** For $k, n \in \mathbb{N}$ and $x \in [0, 1]$, we have

\[
\frac{n-k+1}{k} \left( \frac{[x]_q}{[1-x]^{\frac{k}{q}}} \right) B_{k-1,n}(x, q) = B_{k,n}(x, q).
\]

By (14) and binomial theorem, we easily see that

\[
B_{k,n}(x, q) = \binom{n}{k} \left[ x \right]_q^k \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \left[ x \right]_q^l = \sum_{l=k}^{n} \binom{n}{l} \binom{n}{k} (-1)^{l-k} \left[ x \right]_q^l.
\]

Therefore, we obtain the following theorem.
Theorem 8. For \( k, n \in \mathbb{Z}_+ \) and \( x \in [0, 1] \), we have

\[
B_{k,n}(x, q) = \sum_{l=k}^{n} \binom{l}{k} \binom{n}{l} (-1)^{l-k} [x]^l q^l.
\]

It is possible to write \([x]^k\) as a linear combination of the \(q\)-Bernstein polynomials by using the degree evaluation formulae and mathematical induction. We easily see from the property of the \(q\)-Bernstein polynomials that

\[
\sum_{k=1}^{n} \frac{\binom{k}{n}}{\binom{n}{k}} B_{k,n}(x, q) = \sum_{k=0}^{n-1} \binom{n-1}{k} [x]_{q}^{k+1} [1-x]_{q}^{n-k-1} = [x]_q,
\]

and that

\[
\sum_{k=2}^{n} \frac{\binom{2k}{n}}{\binom{n}{2}} B_{k,n}(x, q) = \sum_{k=0}^{n-2} \binom{n-2}{k} [x]_{q}^{k+2} [1-x]_{q}^{n-2-k} = [x]_q^2.
\]

Continuing this process, we get

\[
\sum_{k=j}^{n} \frac{\binom{k}{n}}{\binom{n}{j}} B_{k,n}(x, q) = [x]_q^j, \text{ for } j \in \mathbb{Z}_+.
\]

Therefore, we obtain the following theorem.

Theorem 9. For \( n, j \in \mathbb{Z}_+ \) and \( x \in [0, 1] \), we have

\[
\sum_{k=j}^{n} \frac{\binom{k}{n}}{\binom{n}{j}} B_{k,n}(x, q) = [x]_q^j.
\]

In [7], the \(q\)-stirling numbers of the second kind are defined by

\[
s_q(n, k) = \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \binom{k}{j} [k-j]_q^n,
\]

where \( \binom{k}{j}_q = \frac{[k]_q!}{[j]_q! [k-j]_q!} \) and \([k]_q! = \prod_{i=1}^{k} [i]_q\). For \( n \in \mathbb{Z}_+ \), it is known that

\[
[x]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{x}{k} [k]_q! s_q(n, k), \text{ (see [7, 21]).}
\]

By (27), (28) and Theorem 7, we obtain the following corollary.
Corollary 10. For \( n, j \in \mathbb{Z}_+ \) and \( x \in [0, 1] \), we have
\[
\sum_{k=j}^{n} \binom{k}{j} B_{k,n}(x, q) = \sum_{k=0}^{j} q^{\binom{k}{2}} \binom{x}{k}_q [k]_q ! s_q(j, k).
\]

§3. On fermionic \( p \)-adic integral representations of \( q \)-Bernstein polynomials

In this section we assume that \( q \in \mathbb{C}_p \) with \( |1 - q|_p < 1 \). From (10) we note that
\[
E_{n, \frac{1}{q}}(1 - x) = \int_{\mathbb{Z}_p} [1 - x + x_1]^n \frac{1}{q} d\mu_{-1}(x_1) = (-1)^n q^n \int_{\mathbb{Z}_p} [x + x_1]^n d\mu_{-1}(x_1), \quad \text{(see [21]).}
\]
From (29) we have
\[
\int_{\mathbb{Z}_p} [1 - x]^n \frac{1}{q} d\mu_{-1}(x) = q^n (-1)^n \int_{\mathbb{Z}_p} [x - 1]^n \frac{1}{q} d\mu_{-1}(x)
\]
\[
= \int_{\mathbb{Z}_p} (1 - [x]_q)^n d\mu_{-1}(x) = (-1)^n q^n E_{n, \frac{1}{q}}(-1) = E_{n, q}(2).
\]
By (5) and (10), we easily get
\[
E_{n, q}(2) = 2 + E_{n, q}, \text{ if } n > 0.
\]
Thus, we obtain the following theorem.

Theorem 11. For \( n \in \mathbb{N} \), we have
\[
\int_{\mathbb{Z}_p} [1 - x]_q^n d\mu_{-1}(x) = \int_{\mathbb{Z}_p} (1 - [x]_q)^n d\mu_{-1}(x) = 2 + \int_{\mathbb{Z}_p} [x]^n d\mu_{-1}(x).
\]
By using Theorem 11, we derive our main results in this section. Taking the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) for one \( q \)-Bernstein polynomials in (14), we get
\[
\int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_{-1}(x) = \binom{n}{k} \int_{\mathbb{Z}_p} [x]^k [1 - x]_q^{n-k} d\mu_{-1}(x)
\]
\[
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]^{k+l} d\mu_{-1}(x)
\]
\[
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l E_{k+l, q}.
\]
From (14) and Theorem 2, we note that

\[ \int_{\mathbb{Z}_p} B_{k,n}(x,q) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} B_{n-k,n}(1-x,\frac{1}{q}) d\mu_{-1}(x) \]

\[ = \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k+j} \int_{\mathbb{Z}_p} [1-x]^{n-j} d\mu_{-1}(x). \]

(31)

For \( n > k \), by (31) and Theorem 11, we get

\[ \int_{\mathbb{Z}_p} B_{k,n}(x,q) d\mu_{-1}(x) = \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k+j} \left( 2 + \int_{\mathbb{Z}_p} [x]^{n-j} d\mu_{-1}(x) \right) \]

\[ = 2 + E_{n,q}, \text{ if } k = 0 \]

\[ = \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k+j} E_{n-j,q}, \text{ if } k > 0. \]

(32)

From \( m, n, k \in \mathbb{Z}_+ \) with \( m + n > 2k \), the fermionic \( p \)-adic integral for multiplication of two \( q \)-Bernstein polynomials on \( \mathbb{Z}_p \) can be given by the following relation:

\[ \int_{\mathbb{Z}_p} B_{k,n}(x,q)B_{k,m}(x,q) d\mu_{-1}(x) = \binom{n}{k} \binom{m}{k} \int_{\mathbb{Z}_p} [x]^{2k} [1-x]^{n+m-2k} d\mu_{-1}(x) \]

\[ = \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \int_{\mathbb{Z}_p} [1-x]^{n+m-j} \frac{1}{q} d\mu_{-1}(x) \]

\[ = \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} (-1)^{j+2k} \left( 2 + \int_{\mathbb{Z}_p} [x]^{n+m-j} d\mu_{-1}(x) \right). \]

(33)

From (33), we have

\[ \int_{\mathbb{Z}_p} B_{k,n}(x,q)B_{k,m}(x,q) d\mu_{-1}(x) = 2 + E_{n+m,q}, \text{ if } k = 0 \]

\[ = \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j,q}, \text{ if } k > 0. \]

(34)

For \( m, k \in \mathbb{Z}_+ \), it is difficult to show that

\[ \int_{\mathbb{Z}_p} B_{k,n}(x,q)B_{k,m}(x,q) d\mu_{-1}(x) = \binom{n}{k} \binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k,q}. \]

Continuing this process we obtain the following theorem.
Theorem 12. (I). For $n_1, \ldots, n_s, k \in \mathbb{Z}_+ \ (s \in \mathbb{N})$ with $n_1 + \cdots + n_s > sk$, we have
\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x,q) \right) d\mu_{-1}(x) = 2 + E_{n_1+\cdots+n_s,q}, \text{ if } k = 0,
\]
and
\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x,q) \right) d\mu_{-1}(x) = \prod_{i=1}^{s} \left( \begin{array}{c} n_i \\ k \end{array} \right) \sum_{j=0}^{sk} (-1)^{sk-j} E_{n_1+\cdots+n_s-j,q}, \text{ if } k > 0.
\]

(II). Let $k, n_1, \ldots, n_s \in \mathbb{Z}_+ \ (s \in \mathbb{N})$. Then we have
\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x,q) \right) d\mu_{-1}(x) = \left( \prod_{i=1}^{s} \left( \begin{array}{c} n_i \\ k \end{array} \right) \right) \sum_{j=0}^{\sum_{i=1}^{s} n_i - sk} \left( \sum_{i=1}^{s} n_i - sk \right) (-1)^j E_{j+sk,q}.
\]

By Theorem 12, we obtain the following corollary.

Corollary 13. For $n_1, \ldots, n_s, k \in \mathbb{Z}_+ \ (s \in \mathbb{N})$ with $n_1 + \cdots + n_s > sk$, we have
\[
\sum_{j=0}^{\sum_{i=1}^{s} n_i - sk} \left( \sum_{i=1}^{s} n_i - sk \right) (-1)^j E_{j+sk,q} = 2 + E_{n_1+\cdots+n_s,q}, \text{ if } k = 0,
\]
and
\[
\sum_{j=0}^{sk} \left( \sum_{i=1}^{s} n_i - sk \right) (-1)^{sk-j} E_{n_1+\cdots+n_s-j,q}, \text{ if } k > 0.
\]

Let $m_1, \ldots, m_s, n_1, \ldots, n_s, k \in \mathbb{Z}_+ \ (s \in \mathbb{N})$ with $m_1 n_1 + \cdots + m_s n_s > (m_1 + \cdots + m_s)k$. By the definition of $B_{k,n_i}^{m_s}(x,q)$, we can also easily see that
\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}^{m_i}(x,q) \right) d\mu_{-1}(x)
\]
\[
= \prod_{i=1}^{s} \left( \begin{array}{c} n_i \\ k \end{array} \right) \sum_{j=0}^{m_i} \left( k \sum_{i=1}^{s} m_i \right) (-1)^{k \sum_{i=1}^{s} m_i - j} \int_{\mathbb{Z}_p} \left[ 1 - x \right] \frac{1}{\eta} \sum_{i=1}^{s} n_i m_i - j d\mu_{-1}(x)
\]
\[
= \prod_{i=1}^{s} \left( \begin{array}{c} n_i \\ k \end{array} \right) \sum_{j=0}^{m_i} \left( k \sum_{i=1}^{s} m_i \right) (-1)^{k \sum_{i=1}^{s} m_i - j} \left( 2 + E_{\sum_{i=1}^{s} m_i n_i - j,q} \right).
\]
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