HARMONIC ANALYSIS ON GRAPHS VIA BRATTELI DIAGRAMS AND PATH-SPACE MEASURES

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To the memory of Ron Graham (1935-2020) a pioneer and leader in discrete mathematics.

Abstract. The past decade has seen a flourishing of advances in harmonic analysis of graphs. They lie at the crossroads of graph theory and such analytical tools as graph Laplacians, Markov processes and associated boundaries, analysis of path-space, harmonic analysis, dynamics, and tail-invariant measures. Motivated by recent advances for the special case of Bratteli diagrams, our present focus will be on those graph systems $G$ with the property that the sets of vertices $V$ and edges $E$ admit discrete level structures. A choice of discrete levels in turn leads to new and intriguing discrete-time random-walk models.

Our main extension (which greatly expands the earlier analysis of Bratteli diagrams) is the case when the levels in the graph system $G$ under consideration are now allowed to be standard measure spaces. Hence, in the measure framework, we must deal with systems of transition probabilities, as opposed to incidence matrices (for the traditional Bratteli diagrams).

The paper is divided into two parts, (i) the special case when the levels are countable discrete systems, and (ii) the (non-atomic) measurable category, i.e., when each level is a prescribed measure space with standard Borel structure. The study of the two cases together is motivated in part by recent new results on graph-limits. Our results depend on a new analysis of certain duality systems for operators in Hilbert space; specifically, one dual system of operator for each level. We prove new results in both cases, (i) and (ii); and we further stress both similarities, and differences, between results and techniques involved in the two cases.

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1. **Introduction**

1.1. **Motivation.** A key tool in various approaches to harmonic analysis on *infinite graph networks* (sets of *vertices* $V$ and *edges* $E$) is notions of “boundary”. While in classical harmonic analysis, the Poison boundary is a favorite, in carrying over key harmonic analysis ideas to infinite graph networks, the possibilities are much wider. But the analyses introduced here are all based on the notion of *infinite paths*. The study of Markov transitions on graphs, and associated boundaries, are a case in point. Of special interest for harmonic analysis, and dynamics, on infinite graphs, is therefore the study of path-spaces, and path-space measures.

This analysis takes an especially nice form in the special case of *infinite graphs* which admits representations as *Bratteli diagrams* (including in their generalized forms, see Sections 2 - 4 and 8). For details regarding Bratteli diagrams, we refer to the literature cited below in this section, see 1.4. The main approach which we elaborate in this paper is based on an application of various ideas and methods developed in the theory of Bratteli diagrams to the case of measurable Bratteli diagrams, see Definitions 2.1 and 8.1 from Sections 2 and 8. We use this term for the case where every level of a diagram is represented by a $\sigma$-finite measure space, and a sequence *transition kernels* plays the role of *incidence matrices*.

Motivated by numerous applications, we shall offer a systematic dynamical system approach to both these extensions with countable and measure space levels.
It is worth noting that classical Bratteli diagrams (with finite levels) have been used in the solution of diverse classification problems (operator algebras, representation theory, fast Fourier transform algorithms, and dynamical systems on Cantor, Borel, and measure spaces), in modeling electrical networks, in neural networks, in harmonic analysis, and in an analysis of Cantor dynamics. We collected the corresponding references at the end of the introduction, see Subsection 1.4.

A formal definition of a generalized Bratteli diagram is given in Section 2, Definition 2.1. Here we shall adopt the view of Bratteli diagrams as a special class of graphs $G$ with specified sets of vertices $V$ and edges $E$. What sets Bratteli diagrams apart from other graph systems $G = (V, E)$ is the introduction of levels, such that edges are linking pairs of vertices only from neighboring levels of vertices. Hence, for each level, there will be a specification of a transition (incidence) matrix or a transition kernel, the term that is appropriate for measurable Bratteli diagrams. The case of discrete levels is subdivided into two principally different classes: (i) every level consists of a finite set of vertices, (ii) every level is an infinitely countable set. In case (i), the path-space is a Cantor set, and such Bratteli diagrams represent models for homeomorphisms of Cantor sets. In case (ii), the path-space is a zero-dimensional Polish space, the corresponding Bratteli diagrams are models for Borel automorphisms of a standard Borel space.

Transition (incidence) matrices of a Bratteli diagram may then constitute a discrete-time Markov process. In the traditional setting for Bratteli diagrams, this will form finite-state Markov process. But clearly, for many applications, it is natural to extend the setting in two ways: first, to consider instead a countably infinite set of vertices at each level; and secondly, to allow an even wider setting where each level is now a standard measure space. The corresponding discrete-time Markov process will then consist of a system of transition probability measures, indicating the probability distribution for transitions from one level to the next.

In more detail, a Bratteli diagram has a prescribed system of levels, see e.g., Figure 2.1, where the levels are sets of vertices, indexed by the natural numbers $\mathbb{N}_0$ including 0. For $n \in \mathbb{N}$, the level $n$ vertex set $V_n$ then only admits edges backwards to $V_{n-1}$ and forwards to $V_{n+1}$. For applications to Markov processes it is convenient to think of $n$ as discrete time, and “forward” meaning increasing $n$ to $n + 1$ for each $n$. Given a Bratteli diagram, a path is then an infinite string of edges, $(e_n)$ such that, for every $n$, the range-vertex of $e_n$ in $V_{n+1}$ matches the source of $e_{n+1}$. Our infinite paths do not admit loops. The set of all infinite paths is called the path-space. A key tool for our harmonic analysis is the study of particular path-space measures. These notions are introduced in Section 2.

A choice of path-space measure, then yields a random process. The case of measures which yield discrete time Markov processes includes the tail-invariant measures. A tail invariant measure has the following forgetful property: For each level $n$, its value on paths (cylinder sets) that meet at a fixed vertex $v \in V_n$ is independent of the variety of finite path segments from level 0 up to $v$ in $V_n$. (See Definition 2.12). An analysis of ordered Bratteli diagrams leads to one of our key tools, Kakutani-Rokhlin towers, and Kakutani-Rokhlin partitions, see Subsection 2.2.

1.2. Main results. We describe our main results proved in this paper. After defining the main concepts of discrete Bratteli diagrams $B = (V, E)$ in Section 2, we focus on tail
invariant measures. Clearly, every tail invariant measure $\mu$ is uniquely determined by its values on cylinder sets of the path-space $X_B$. It gives a sequence of non-negative vectors $\mu^{(n)} = (\mu^{(n)}_v : v \in V_n)$ where $\mu^{(n)}_v$ is the measure of a cylinder set ending at $v$. By tail invariance, $\mu^{(n)}_v$ does not depend on a cylinder set. In Theorem 2.13 we find a condition under which a sequence of non-negative vectors determines a tail invariant measure. Theorem 2.20 represents explicitly a tail invariant measure for a stationary Bratteli diagram.

In Section 3, we consider Markov measures on the path-space of a generalized Bratteli diagram defined by a sequence of probability transition matrices $(P_n)$. Theorem 3.5 states the existence of co-transition probability matrices $(Q_n)$. The properties of matrices $Q_n$ and $P_n$ are discussed in Propositions 3.6, 3.19. It is shown in Theorem 3.8 that every tail invariant measure is a Markov measure.

Section 4 contains some elements of harmonic analysis on generalized Bratteli diagrams. Starting with an initial distribution $q^{(0)}$ on the level $V_0$ and a sequence of transition kernels $(P_n)$, we define a reversible Markov process, harmonic functions, Laplace operator, and finite energy Hilbert space. Our main results are obtained in Theorem 4.5, where harmonic functions are described, and Proposition 4.8. The latter contains a criterion for a function $f : V \to \mathbb{R}$ to be in the finite energy Hilbert space.

The goal of the second part of the paper (Sections 5 - 8) is to build a theory of measurable Bratteli diagrams which is parallel to that of the purely discrete case. In Section 5 we consider the notion of a dual pair $(P, Q)$ of transition kernels associated to $\sigma$-finite measure spaces $(X_1, \nu_1)$ and $(X_2, \nu_2)$. This means that the objects satisfy the relation $d\nu_1(x)P(x, dy) = d\nu_2(y)Q(y, dx)$. A number of results are proved in this section which clarify the interplay between the notions of transition kernels $P, Q$, linear (unbounded) operators $T_P, T_Q$ generated by these kernels, and measures $\nu_1, \nu_2$. We refer to Theorems 5.15 and 5.16 as the principal statements of Section 5.

In Section 6 we develop the approach used in the previous section to the case when the operators $T_P$ and $T_Q$ are considered acting between the corresponding $L^2(\nu_i)$-spaces. Theorem 6.6 proves that $T_P$ and $T_Q$ are contractive operators whenever $P$ and $Q$ are probability kernels. In the most general case, $T_P$ and $T_Q$ are closable densely defined operators such that $T_P \subset (T_Q)^*$ and $T_Q \subset (T_P)^*$. We give also a condition when the operators $T_P$ and $T_Q$ are bounded, Proposition 6.9.

We continue our analysis of transition kernels $P, Q$ and the corresponding operators in Section 7. If $R$ is a positive definite operator which generates a symmetric measure $\lambda$ on the product $X_1 \times X_2$, then a reproducing kernel Hilbert space $\mathcal{H}(\lambda)$ can be defined by the positive definite function $(A, B) \mapsto \lambda(A \times B)$ where $A$ and $B$ are sets of finite measure. In Theorem 7.6, we give an explicit description of functions from $\mathcal{H}(\lambda)$. Theorem 7.12 states that $R$ can be factorized in the product of two operators.

In the final Section 8 we apply the method developed in Sections 5 and 6 to the case of measurable Bratteli diagrams. Recall that this name is used for a sequence of $\sigma$-finite measure spaces $(X_i, \nu_i)$ together with probability transition kernels $P_i, Q_i$. This approach is an extension of generalized Bratteli diagrams (discrete case) to the case of standard measure spaces. Our definitions and results proved in Section 8 have discrete analogues considered
in Section 4. In particular, we consider a graph Laplacian defined on a measurable Brat-teli diagram. The most important results of this section are given in Theorem 8.4 and Proposition 8.15.

1.3. Short outline of the paper. The notions of path-space, and the corresponding measures, make sense for all three types of feasible Bratteli diagrams: (i) the standard case when each \( V_n \) is assumed finite; (ii) the case when each \( V_n \) is countably infinite, Section 2; and (iii), measurable Bratteli diagrams, (the measurable category) when each \( V_n \) is an uncountable standard Borel space, see Section 8. However, as we show, the construction, and the relevant properties, of the path-space measures is quite different for the three cases.

Our analysis in the first three sections of the paper leads up to a study of graph Laplacians (Section 4) and the corresponding harmonic functions. The path-space measures which have proved most successful for our present harmonic analysis are the tail-invariant measures. For the discrete Bratteli diagrams, we characterize these tail-invariant measures (Theorem 2.13). In subsequent sections, we further demonstrate the use of tail-invariant measures in an harmonic analysis and discrete-time dynamics, for graphs with level structure (generalized Bratteli diagrams).

In Section 2, we present an algorithm (Theorem 2.20) for constructing tail-invariant measures on stationary Bratteli diagrams. It is based on a generalized Perron-Frobenius spectral analysis (Theorem 2.18).

In Sections 3, 5, and 6, we introduce the more general class of Markov measures, and their corresponding transition kernels, referring to transition between levels. These results are based on a new analysis of associated systems of operators, transition operators. An explicit spectral theory for these transition operators is presented in Section 6. In Section 7, we give a new tool for this analysis. It is a particular family of positive definite kernels, and their corresponding reproducing kernel Hilbert spaces (RKHSs). Section 8 deals with the case of measurable Bratteli diagrams, and we present results which are based on a new analysis of path-space measures, and corresponding Markov processes.

1.4. Literature. Our paper makes direct connections to various adjacent areas such as: stochastic analysis on graphs, graph-limits, dynamical systems, applied and computational harmonic analysis, weighted networks, spectral theory, Markov processes, etc. For the reader’s convenience we present the relevant references divided in several groups.

(a) Standard Borel and measure spaces. The literature where standard Borel spaces play a crucial role is very extensive. This subject is studied in ergodic theory, Borel dynamics, descriptive set theory, operator algebras, and many other fields. We refer to [Kec95, DJK94a, JKL02, CSS19, CFS82, Nad95, BDK06, KL16] where the reader can find more information.

(b) Cantor dynamics and \( C^* \)-algebras. The problems of classification of dimension groups, \( C^* \)-algebras, dynamical systems in Cantor dynamics are naturally connected with the study of invariants of the corresponding Bratteli diagrams. The list of relevant references is extremely long. We refer only to [Bra72, LT80, GPS95, BJKR01, BJKR02, Tak03, Phi05, Jor06, Zer06, Put18]. The reader can find numerous applications of Bratteli diagrams in the theory of Cantor and Borel dynamical systems, see, e.g. [HPS92, BJ15, Dur10, BDK06, BK16].
(c) Random processes. Random walks on Bratteli diagrams are discussed in [FP10, CP16] and [Ren18]. A different approach is used in the works [Ana12b, Ana12a, Ana11].

(d) Perron-Frobenius and Markov chains. The Perron-Frobenius theory has various applications in Bratteli diagrams, networks, Markov chains, and other areas such as network models and graph Laplacians: [JP11, JP13, JP19, VCH19, YE19, LZW+16, JP08]. We refer here also to several recent papers related to our study: [CD19, CN19, GTH19, GR19, CVLX19, FN18].

(e) Transition probability kernels. The notion of transition probability kernels is an important tool for the study of Markov chains and properties of random walks. The reader can find more details, for example, in the books [DMPS18, Kle14, LP16, Num84, Rev84] and recent articles [DT19, Smi19, EP19].

(f) Weighted networks, neural networks, graph limits. Engineering applications of deep neural networks realized as a generalized Bratteli diagram structure are discussed in [YD20, SSQ+20, CHL+20]; graph-limits were considered in [DSST20, Lov12, KKLS19]. Important contribution to the graph theory and weighted networks have been made in [Chu07, Chu10, Chu14, CK14, CG12].

(g) Generalized Bratteli diagram, machine learning. The literature on optimization, financial models, machine learning with deep neural networks and generalized Bratteli diagram structure includes [JP19, GS20, Ada20, HN20, XPL20, Xia20]. More information can be found in [Bau19, SL19, KS19]

2. Graph analysis via Bratteli diagrams

In this section, we briefly discuss the main definitions and facts about Bratteli diagrams. Our emphasis is on those features of Bratteli diagrams, and their generalizations, which will be important for our present analysis of general graphs which admit a system of levels as outlined above.

The literature on Bratteli diagrams and their application in dynamics is very extensive, we mention here [HPS92, GPS95, Dur10, BK16, BK20] (more references can be found therein). In contrast to most of the above sources, we focus here on the case of generalized Bratteli diagrams. This means that the set of vertices in every level is infinitely countable.

2.1. Generalized Bratteli diagrams. In the introduction, we described the notion of a Bratteli diagram considering this as an infinite graded graph. Here is a natural extension of this concept to the case of countable levels.

Definition 2.1. Let $V_0$ be a countable set (which may be identified with either $\mathbb{N}$ or $\mathbb{Z}$ for convenience). Set $V_i = V_0$ for all $i \geq 1$. A countable graded graph $B = (V,E)$ is called a generalized Bratteli diagram if it satisfies the following properties.

(i) The set of vertices $V$ of $B$ is $\bigcup_{i=0}^{\infty} V_i$.

(ii) The set of edges $E$ of $B$ is represented as $\bigcup_{i=0}^{\infty} E_i$ where $E_i$ is the set of edges between the levels $V_i$ and $V_{i+1}$.

(iii) For every $w \in V_i, v \in V_{i+1}$, the set of edges $E(w,v)$ between $w$ and $v$ is finite (or empty); set $|E(w,v)| = f^{(i)}_{vw}$. It defines a sequence of infinite (countable-by-countable) incidence matrices $(F_n; n \in \mathbb{N})$ whose entries are non-negative integers:

$$F_i = (f^{(i)}_{vw}; v \in V_{i+1}, w \in V_i), \quad f^{(i)}_{vw} \in \mathbb{N}.$$
(iv) The matrices $F_i$ have at most finitely many non-zero entries in each row.

(v) The maps $r, s : E \to V$ are defined on the diagram $B$: for every $e \in E$ there are $w, v$ such that $e \in E(w, v)$; then $s(e) = w$ and $r(e) = v$. They are called the range ($r$) and source ($s$) maps.

(vi) For every $w \in V_i$, $i \geq 0$, there exists an edge $e \in E_i$ such that $s(e) = w$ and edge $e' \in E_{i-1}$ such that $r(e') = w$. In other words, every incidence matrix $F_i$ has no zero row and zero column.

**Remark 2.2.** (1) It follows from Definition 2.1 that every generalized Bratteli diagram is uniquely determined by a sequence of matrices $(F_n)$ such that every matrix satisfies (iii) and (iv). For this, one uses the rule that the entry $f_{vw}^{(n)}$ indicates the number of edges between the vertex $w \in V_n$ and vertex $v \in V_{n+1}$. It defines the set $E(w, v)$; then one takes

$$E_n = \bigcup_{w \in V_n, v \in V_{n+1}} E(w, v)$$

(2) To emphasize that a generalized Bratteli diagram $B$ is determined by the sequence of incidence matrices $(F_n)$, we will write $B = B(F_n)$ if needed. An important particular case of a generalized Bratteli diagram is obtained when all incidence matrices $F_n$ are the same, $F_n = F$ for all $n \in \mathbb{N}_0$. Then, the generalized Bratteli diagram $B(F)$ is called stationary.

**Remark 2.3.** If $V_0$ is a singleton, and each $V_n$ is a finite set, then we obtain the standard definition of a Bratteli diagram originated in [Bra72]. Later it was used in the theory of $C^*$-algebras and dynamical systems for solving some classification problems and constructions of models (for references, see Introduction 1.4).

It is important to emphasize that we have no restriction on the entries of columns of the incidence matrices $F_n$. They may have any number of non-zero (or zero) entries.

On Figure 1, we give an example of a Bratteli diagram. As a matter of fact, this example is a small (finite) part of the diagram since every Bratteli diagram has infinitely many levels and every level is a countably infinite set.

**Definition 2.4.** A finite or infinite path in a Bratteli diagram $B = (V, E)$ is a sequence of edges $(e_i : i \geq 0)$ such that $r(e_i) = s(e_{i+1})$. Denote by $X_B$ the set of all infinite paths. Every finite path $\bar{e} = (e_0, ..., e_n)$ determines a cylinder subset $[\bar{e}]$ of $X_B$:

$$[\bar{e}] := \{x = (x_i) \in X_B : x_0 = e_0, ..., x_n = e_n\}.$$ 

The collection of all cylinder subsets forms a base of neighborhoods for a topology on $X_B$. In this topology, $X_B$ is a Polish zero-dimensional space and every cylinder set is clopen.

**Remark 2.5.** In this remark, we collected a number of simple statements about properties of generalized Bratteli diagrams.

(1) It follows from Definition 2.4 that $X_B$ is a standard Borel space whose Borel structure is generated by clopen (cylinder) sets. In contrast to the classical case, we note that $X_B$ is not compact and even not locally compact. Furthermore, clopen subsets are not compact, in general.

(2) If $x = (x_i)$ is a point in $X_B$, then it is obviously represented as follows:

$$\{x\} = \bigcap_{n \geq 0} [\bar{e}]_n$$
where \([\mathcal{E}]_n = [x_0, ..., x_n]\). But, in general, it is not true that any decreasing (nested) sequence of cylinder sets determines a point in \(X_B\) since it can be empty. If the set \(\bigcap_{n \geq 0} [\mathcal{E}]_n\) is non-empty, then it contains just a single point.

(3) The metric on \(X_B\), which is compatible with the clopen topology, can be defined as follows: for \(x = (x_i)\) and \(y = (y_i)\) from \(X_B\),

\[
\text{dist}(x, y) = \frac{1}{2^N}, \quad N = \min\{i \in \mathbb{N}_0 : x_i \neq y_i\}.
\]

(4) Considering \(X_B\) as a zero-dimensional metric space, we will assume that the diagram \(B\) is chosen so that the space \(X_B\) has no isolated points. This means that for every infinite path \((x_0, x_1, x_2, ...) \in X_B\) and every \(n \geq 1\), there exists \(m > n\) such that \(|s^{-1}(r(x_m))| > 1\). Therefore, without loss of generality, we can assume that every column of the incidence matrix \(F_n, n \in \mathbb{N}_0\), has more than one non-zero entry.

(5) We observe that if, for every vertex \(v \in V\), the set \(s^{-1}(v)\) is finite, then the path-space \(X_B\) is a locally compact Polish space. The finiteness of \(r^{-1}(v)\), which is a requirement included in the definition of a generalized Bratteli diagram, is constantly used in the next sections.

(6) In the study of Bratteli diagrams the telescoping procedure is often used. This means that, for a given generalized Bratteli diagram \(B = (V, E)\), one can take any monotone increasing sequence \((n_k : k \in \mathbb{N}_0), n_0 = 0\), and construct a new Bratteli diagram \(B' = (V', E')\) where \(V' = \bigcup_{k=0}^{\infty} V_{n_k}\), and the set of edges \(E'\) is determined by the sequence of

\[E_{n-1} \rightarrow E_1 \rightarrow E_0\]
incidence matrices $F'_k$:

$$F'_k = F_{n_k} \cdots F_{n_{k+1}-1}.$$ 

Since every matrix $F_n$ has finitely many non-zero entries in each row, the set $E(V_0, v)$ of all finite paths $\bar{e}$ with $r(\bar{e}) = v$ is finite, where $v \in V_n, n \geq 1$. Clearly,

$$|E(V_0, v)| = \sum_{w \in V_0} (F_0 \cdots F_{n-1})_{w,v}.$$ 

(7) Generalized Bratteli diagram arise in Borel dynamics as models for aperiodic Borel automorphisms of standard Borel spaces. We refer to [BDK06] where such models have been constructed.

Next, we will define the notion of an irreducible generalized Bratteli diagram. For this, it is convenient to identify all sets $V_n, n \geq 0$, i.e., we can think that the same vertex $v$ is a vertex of each level.

**Definition 2.6.** It is said that a generalized Bratteli diagram $B$ is irreducible if, for any two vertices $v$ and $w$, there exists a level $V_m$ such that $v \in V_0$ and $w \in V_m$ are connected by a finite path. This is equivalent to the property that, for any fixed $v, w$, there exists $m \in \mathbb{N}$ such that the product of matrices $F_{m-1} \cdots F_0$ has non-zero $(w, v)$-entry.

Let $B = (V, E)$ be a generalized Bratteli diagram. Define the tail equivalence relation $\mathcal{E}$ on the path space of $X_B$.

**Definition 2.7.** It is said that two paths $x = (x_i)$ and $y = (y_i)$ are tail equivalent if there exists $m \in \mathbb{N}$ such that $x_i = y_i$ for all $i \geq m$. Let $[x]_\mathcal{E} := \{y \in X_B : (x, y) \in \mathcal{E}\}$ be the set of points tail equivalent to $x$. We say that a point $x$ is periodic if $|[x]_\mathcal{E}| < \infty$. If there is no periodic points, then the tail equivalence relation is called aperiodic.

Without loss of generality, we will assume that $\mathcal{E}$ is aperiodic. Clearly, $\mathcal{E}$ is a countable Borel equivalence relation. It can be easily proved that $\mathcal{E}$ is hyperfinite in the context of Borel dynamical systems. This notion and results are discussed, for example, in [DJK94b].

2.2. *Kakutani-Rokhlin towers and ordered Bratteli diagrams.* *Kakutani-Rokhlin towers* (partitions) proved to be a very fruitful tool in dynamical systems. They have been used to construct the approximation of an aperiodic transformation by periodic ones. The idea to us a refining sequence of Kakutani-Rokhlin partition leads to the realization of aperiodic transformation as a map acting on the path-space of a Bratteli diagram [CK77, Ver81, HPS92, GPS95, BDK06, VO16].

We begin with a generalized Bratteli diagram $B = (V, E)$ defined by a sequence of matrices $(F_n)$. Let $v$ be a vertex from $V_n, n \geq 1$. Then, for every $v_0 \in V_0$, we consider the set $E(v_0, v)$ (which is non-empty only for finitely many vertices $v_0$, see Definition 2.1). Set $h_{v_0,v}^{(n)} = |E(v_0, v)|$, $v \in V_n$, and define

$$H_v^{(n)} = \sum_{v_0 \in V_0} h_{v_0,v}^{(n)},$$

It gives the sequence of vectors $H^{(n)} = (H_v^{(n)} : v \in V_n)$ which is assigned to vertices of the corresponding level $V_n$. 
Define the vector \( H^{(0)} = (H_v^{(0)} : v \in V_0) \) such that \( H_v^{(0)} = 1 \) for all \( v \). Then we see that the following relation holds.

**Lemma 2.8.** \( F_nH^{(n)} = H^{(n+1)} \) and \( F_n \cdots F_0H^{(0)} = H^{(n+1)} \), \( n \in \mathbb{N}_0 \).

The proof of this fact follows immediately from the relation
\[
H_v^{(n+1)} = \sum_{w \in V_n} f_{vw}^{(n)} H_w^{(n)}, \quad v \in V_{n+1}.
\]

Similarly, we can define a clopen subset \( X_v^{(n)} \) of \( X_B \) setting
\[
X_v^{(n)} = \bigcup_{v_0 \in V_0} \bigcup_{v \in E(v_0,v)} [\overline{v}].
\]

Recall that \( X_v^{(n)} \) is a finite union of cylinder sets. The number of cylinder sets used in the definition of \( X_v^{(n)} \) is \( x \) \( H_v^{(n)} \) because \( h_{v_0,v}^{(n)} \) gives the exact number of finite paths between \( v_0 \) and \( v \). The set \( X_v^{(n)} \) is viewed as a tower assigned to the vertex \( v \) and \( H_v^{(n)} \) is the height of this tower.

**Lemma 2.9.** Let \( B = (V, E) \) be a generalized Bratteli diagram. The sets \( (X_v^{(n)} : v \in V_n) \) constitute a partition \( \xi_n \) of \( X_B \) into disjoint clopen sets for every \( n \). The sequence of partitions \( (\xi_n) \) is a refining sequences such that the elements of all these partitions generate the topology (and Borel \( \sigma \)-algebra) on \( X_B \).

This partition can be viewed as an analogue of a Kakutani-Rokhlin partition which is widely used in the ergodic theory and Cantor dynamics. The difference is that we have not defined a transformation on \( X_B \) so far. For this, we introduce the notion of an ordered generalized Bratteli diagram. We will see that such diagrams arise naturally in Borel dynamics.

Let \( B = (V, E) \) be a generalized Bratteli diagram \((V, E)\) equipped with a partial order \( \geq \) defined on each set \( E_i, i = 0,1,..., \) such that two edges \( e, e' \) are comparable if and only if \( r(e) = r(e') \). In other words, a linear order \( \geq \) is defined on each (finite) set \( r^{-1}(v), v \in V \setminus V_0 \). For a Bratteli diagram \((V, E)\) equipped with such a partial order \( \geq \) on \( E \), one can also define a partial lexicographic order on the set \( E_k \circ \cdots \circ E_{l-1} \) of all paths from \( V_k \) to \( V_l \) \((e_k, ..., e_{l-1}) > (f_k, ..., f_{l-1}) \) if and only if for some \( i \) with \( k \leq i < l \), \( e_j = f_j \) for \( i < j < l \) and \( e_i > f_i \). Then we see that any two paths from \( E(V_0, v) \), the (finite) set of all paths connecting vertices from \( V_0 \) with \( v \), are comparable with respect to the introduced lexicographic order. We call a path \( e = (e_0, e_1, ..., e_i, ...) \) maximal (minimal) if every \( e_i \) has a maximal (minimal) number among all elements from \( r^{-1}(r(e_i)) \). Note that there are unique minimal and unique maximal finite paths in \( E(V_0, v) \) for each \( v \in V_i, i > 0 \).

**Definition 2.10.** A generalized Bratteli diagram \( B = (V, E) \) together with a partial order \( \geq \) on \( E \) is called an ordered Borel-Bratteli diagram \( B = (V, E, \geq) \) if the space \( X_B \) has no cofinal minimal and maximal paths. This means that \( X_B \) does not contain infinite paths \( e = (e_0, e_1, ..., e_i, ...) \) such that for all sufficiently large \( i \) the edges \( e_i \) have maximal (minimal) number in the set \( r^{-1}(r(e_i)) \).

For each ordered Borel-Bratteli diagram \( B = (V, E, \geq) \), define a Borel transformation \( \varphi \), which is also called the Vershik map (or automorphism), acting on the space \( X_B \) as
follows. Given \( x = (e_0, e_1, \ldots) \in X_B \), let \( k \) be the smallest number such that \( e_k \) is not a maximal edge. Let \( f_k \) be the successor of \( e_k \) in \( r^{-1}(r(e_k)) \). Then we define \( \varphi(x) = (f_0, f_1, \ldots, f_{k-1}, f_k, e_{k+1}, \ldots) \) where \((f_0, f_1, \ldots, f_{k-1})\) is the minimal path in \( E(V_0, r(f_{k-1})) \). Obviously, \( \varphi \) is a one-to-one mapping of \( X_B \) onto itself. Moreover, \( \varphi \) is a homeomorphism of \( X_B \) where the 0-dimensional topology is defined by cylinder sets.

Thus, given an ordered Borel-Bratteli diagram \( B = (V, E, \geq) \), we have defined a Borel dynamical system \((X_B, \varphi)\). It turns out that every Borel aperiodic automorphism of a standard Borel space can be realized as a Vershik transformation acting on the space of infinite paths of an ordered Borel-Bratteli diagram.

**Theorem 2.11** ([BDK06]). Let \( T \) be an aperiodic Borel automorphism acting on a standard Borel space \((X, \mathcal{A})\). Then there exists an ordered Borel-Bratteli diagram \( B = (V, E, \geq) \) and a Vershik automorphism \( \varphi : X_B \to X_B \) such that \((X, T)\) is Borel isomorphic to \((X_B, \varphi)\).

### 2.3. Measures on the path-space of a Bratteli diagram

In this subsection, we will consider Borel probability measures which are invariant with respect to the tail equivalence relation, see Definition 2.7. In papers by Vershik and his colleagues such measures are called central measures, see e.g. [Ver15].

**Definition 2.12.** Let \( B = (V, E) \) be a generalized Bratteli diagram, and \( X_B \) the path-space of \( B \). Let \( \mu \) be a Borel measure on \( X_B \). The measure \( \mu \) is called tail invariant if, for any two finite paths \( \overline{e} \) and \( \overline{e}' \) such that \( r(\overline{e}) = r(\overline{e}') \), one has

\[
\mu([\overline{e}]) = \mu([\overline{e}']),
\]

where \([e]\) and \([e']\) denote the corresponding cylinder sets.

Given a generalized Bratteli diagram \( B = (V, E) \), let \( M_1(B, \mathcal{E}) \) denote the set of Borel positive probability measures on \( X_B \) which are invariant with respect to the tail equivalence relation \( \mathcal{E} \). Then the property of tail invariance means that the probability to arrive at a vertex \( v \in V_n \) does not depend on a starting point \( w \in V_0 \) and does not depend on the path connecting \( w \) and \( v \).

Hence, if \( \mu \) is a fixed measure from \( M_1(B, \mathcal{E}) \), relation (2.2) allows us to define a sequence of non-negative vectors \((\mu^{(n)})\) where \( \mu^{(n)} = (\mu_v^{(n)} : v \in V_n) \) and

\[
\mu_v^{(n)} = \mu([\overline{e}]), \quad \overline{e} \in E(V_0, v), \quad v \in V_n.
\]

By tail invariance of \( \mu \) the value \( \mu_v^{(n)} \) does not depend on the choice of \( \overline{e} \in E(V_0, v) \).

**Theorem 2.13.** Let \( B = (V, E) \) be a generalized Bratteli diagram defined by a sequence of incidence matrices \( F_n \). Let \( \mu \) be a Borel probability measure on the path-space \( X_B \) of \( B \) which is tail invariant. Then the corresponding sequence of vectors \( \mu^{(n)} \) (defined as in (2.3)) satisfies the property

\[
A_n \mu^{(n+1)} = \mu^{(n)},
\]

where \( A_n = F_n^T \) is the transpose of \( F_n \).

Conversely, if a sequence of vectors \( \mu^{(n)} \) satisfies (2.4), then it defines a unique tail invariant measure \( \mu \).

The theorem remains true for \( \sigma \)-finite measures \( \nu \) satisfying the property that \( \nu([\overline{e}]) < \infty \) for every cylinder set \([\overline{e}]\).
Proof. The proof is straightforward. Indeed, if \( \mu \) is a tail invariant measure, then one can define the sequence of vectors \((\mu^{(n)})\) as in (2.3). Then, for every \( n \in \mathbb{N} \), \( v \in V_n \), and \( \varepsilon \in E(V_0, v) \) with \( r(\varepsilon) = v \), we see that

\[
(2.5) \quad [\varepsilon] = \bigcup_{u \in V_{n+1}} \bigcup_{e' \in E(v, u)} [\varepsilon e'].
\]

Recall that the cardinality of the set \( E(v, u) \) equals \( f_{u,v}^{(n)} \). By tail invariance of \( \mu \), every subset of \( X_B \), which is determined by an edge from \( E(v, u) \), has the same measure, so that relation (2.5) implies that

\[
\mu^{(n)}_v = \sum_{u \in V_{n+1}} \sum_{e' \in E(v, u)} \mu([\varepsilon e']) = \sum_{u \in V_{n+1}} f_{u,v}^{(n)} \mu^{(n+1)}_u = \sum_{u \in V_{n+1}} a_{v,u}^{(n)} \mu^{(n+1)}_u,
\]

and these equalities prove (2.4).

Conversely, suppose a sequence of non-negative vectors \((\mu^{(n)})\) is given and satisfies (2.4). Then we define a measure \( \mu \) on \( X_B \) by setting

\[
(2.6) \quad \mu([\varepsilon]) = \mu^{(n)}_v
\]

for every \( \varepsilon \) with \( r(\varepsilon) = v \). Relation (2.4) can be interpreted as the Kolmogorov consistency condition, and it guarantees that the above definition of a measure on cylinder sets can be extended to all Borel subsets of \( X_B \). It follows from (2.6) that the obtained measure \( \mu \) is tail invariant.

The case of a \( \sigma \)-finite measure \( \nu \) is considered similarly if we know that all values \( \nu^{(n)}_v \) are finite. \( \square \)

Let \( B = (V, E) \) be a generalized Bratteli diagram and \((F_n)\) the sequence of incidence matrices. We define the sequence of row stochastic incidence matrices \((\hat{F}_n)\) with entries

\[
(2.7) \quad \hat{f}_{vw}^{(n)} = \frac{H_v^{(n)}}{H_v^{(n+1)}} f_{vw}^{(n)}
\]

where \( H^{(n)} = (H_v^{(n+1)}) \) is the vector of heights of Kakutani-Rokhlin towers \( X_v^{(n)} \), \( v \in V_n \).

Corollary 2.14. Let \( \mu \) be a tail invariant measure on a generalized Bratteli diagram \( B = (V, E) \). Define \( s^{(n)} = (s_v^{(n)}) \) where \( s_v^{(n)} = \mu_v^{(n)} H_v^{(n)} \) is the measure of the tower \( X_v^{(n)} \). Then

\[
s^{(n+1)} \hat{F}_n = s^{(n)}, \quad n \in \mathbb{N}_0,
\]

where \( s^{(n)} \) is considered as a row vector.

Proof. We recall that \( F_n H^{(n)} = H^{(n+1)} \) and \( \mu^{(n+1)} F_n = \mu^{(n)} \). Then, for every \( w \in V_n \),

\[
(s^{(n+1)} \hat{F}_n)_w = \sum_{v \in V_{n+1}} \mu^{(n+1)}_v H^{(n+1)}_v \frac{H^{(n)}_w}{H^{(n+1)}_v} f^{(n)}_{vw}
= H_w^{(n)} \sum_{v \in V_{n+1}} \mu^{(n+1)}_v f^{(n)}_{vw}
= \mu^{(n)}_w H^{(n)}_w
= s^{(n)}_w.
\]

\( \square \)
Remark 2.15. We recall one more method for constructing measures on the path-space $X_B$ of a generalized Bratteli diagram $B = (V,E)$. By definition, $X_B$ is a Borel subset of the product space $\Omega = E_0 \times E_1 \times \cdots$. Let $\tau_i$ be a probability distribution on the countable set $E_i$, $i \in \mathbb{N}_0$, and let $\tau$ denote the product measure $\otimes_i \tau_i$.

Consider a Borel map $\Phi$ from $\Omega$ onto $X_B$. Define a new measure $\nu$ on $X_B$ by letting

$$\nu(A) = \tau(\Phi^{-1}(A)).$$

Question. Can a tail invariant measure on $X_B$ be obtained as a pull back of a product measure? What properties have measures $\mu$ defined in this way?

2.4. Stationary processes as generalized Bratteli diagrams. We recall that a generalized Bratteli diagram $B = B(F_n)$ is called stationary if all incidence matrices $F_n$ are the same, $F_n = F$. Therefore, the set of edges $E_n$ does not depend on the level. It is convenient to enumerate the vertices of every level by integers $\mathbb{Z}$.

In the following definition, we included a few basic properties of countable nonnegative matrices, see [Kit98] for a detailed exposition of the Perron-Frobenius theory for countable matrices.

Definition 2.16. A stationary Bratteli diagram $B = B(F)$ is called irreducible if for all $i,j \in \mathbb{Z}$ there exists some $n \in \mathbb{N}_0$ such that $a_{ij}^{(n)} > 0$. (We prefer to work here with the transpose matrix $A = F^T$.) This means that there exists a finite path from $i \in V_0$ to $j \in V_{n+1}$. An irreducible matrix $A$ has period $p$ if, for all vertices $i \in \mathbb{Z}$,

$$p = \gcd\{\ell: a_{ii}^{(\ell)} > 0\}.$$ 

If $p = 1$, the matrix $A$ is called aperiodic.

An irreducible aperiodic nonnegative matrix $A$ admits a Perron-Frobenius eigenvalue $\lambda$ defined by

$$\lambda = \lim_{n \to \infty} (a_{ii}^{(n)})^{\frac{1}{n}}$$

(the limit exists and does not depend on $i$).

An irreducible aperiodic nonnegative matrix $A$ is called transient if

$$\sum_n a_{ij}^{(n)} \lambda^{-n} < \infty;$$

otherwise, $A$ is called recurrent. For a recurrent matrix $A$, define $\ell_{ij}(1) = a_{ij}$ and

$$\ell_{ij}(n+1) = \sum_{k \neq i} \ell_{ik}(n)a_{kj}.$$ 

The matrix $A$ is called null-recurrent if

$$\sum_n n\ell_{ii}(n)\lambda^{-n} < \infty;$$

otherwise $A$ is called positive recurrent.

Remark 2.17. The terminology used in Definition 2.16 comes from probability theory. If $A$ is a stochastic matrix, then $a_{ij}$ is the probability of going from state $i$ to state $j$ in one step. If one uses $n$ steps to reach $j$ from $i$, then the probability is $a_{ij}^{(n)}$. The quantity $\ell_{ij}(n)$ is the probability of going from state $i$ to state $j$ in $n$ steps without returning to $i$. The
stochastic matrix is transient if the expected number of returns to \( i \) is of a random walk beginning at \( i \) is finite and \( A \) is recurrent if it is infinite. A recurrent matrix \( A \) is null recurrent if the expected time of return to \( i \) of a walk beginning at \( i \) is infinite and positive recurrent if the expected time is finite.

The following theorem is taken from [Kit98].

**Theorem 2.18 (Generalized Perron-Frobenius theorem).** Suppose \( A \) is a countable non-negative irreducible aperiodic matrix. Suppose that \( A \) is recurrent. Then there exists a Perron-Frobenius eigenvalue

\[
\lambda = \lim_{n \to \infty} (t_{ij}^{(n)})^{\frac{1}{n}} > 0
\]

(assumed to be finite) such that:

(a) there are strictly positive left \( s \) and right \( t \) eigenvectors corresponding to \( \lambda \),
(b) the eigenvectors are unique up to constant multiples,
(c) \( s \cdot t = \sum_i s_it_i < \infty \) if and only if \( A \) is positive recurrent,
(d) if \( 0 \leq A' \leq A \) and \( \lambda' \) is the Perron-Frobenius eigenvalue for \( A' \), then \( \lambda' \leq \lambda \); the equality holds if and only if \( A' = A \),
(e) \( \lim_{n \to \infty} A^n\lambda^{-n} = 0 \) if \( A \) is null-recurrent, and \( \lim_{n \to \infty} A^n\lambda^{-n} = s \cdot t \) (normalized so that \( s \cdot t = 1 \)) if \( A \) is positive recurrent.

**Example 2.19 (Substitutions on a countable alphabet).** Let \( \mathcal{A} \) be an infinite alphabet, \( \mathcal{A} = \{a_1, a_2, \ldots\} \). Denote by \( \mathcal{A}^* \) the set of all finite words on the alphabet \( \mathcal{A} \) (including the empty word). Suppose that \( \sigma : \mathcal{A} \to \mathcal{A}^* \) is a substitution, i.e., \( \sigma(a) = a_{i_1} \cdots a_{i_k} \) where all letters are from \( \mathcal{A} \) and the word \( \sigma(a) \) may contain repeated letters. Consider the matrix \( M = (m_{ab}) \) of the substitution \( \sigma \) whose entries are defined as follows: \( m_{ab} \) is the number of occurrences of the letter \( b \) in the word \( \sigma(a) \), \( a \in \mathcal{A} \).

The matrix \( M \) can be used to construct a stationary ordered Bratteli diagram \( B = B(M) \) by the following rule. The set of vertices \( V_i \) at each level \( i \) is \( \mathcal{A} \). The set of edges \( E_i = \bigcup_{a,b \in \mathcal{A}} E(b,a) \) is the same for each \( i \) where \( E(b,a) \) is the set of \( m_{a,b} \) edges between \( a \in V_{i+1} \) and \( b \in V_i \). To define a linear order on \( r^{-1}(a) \), we identify edges from \( r^{-1}(a) \) with the letters \( a_{i_1}, \ldots, a_{i_k} \) in the word \( \sigma(a) \) and assign the order on \( r^{-1}(a) \) accordingly to the natural left-right order of letters in \( \sigma(a) \), see Figure 2.

It is known that stationary Bratteli diagrams \( B \) with finite levels are models for minimal (or aperiodic if \( B \) is non-simple) substitution dynamical systems in symbolic dynamics, see [For97], [DHS99], [BKM09], [Dur10]. We do not know whether a similar result holds for substitutions defined on a countable alphabet. The reader can find a number of interesting results in [Fer06] about some classes of such substitutions.

For stationary Bratteli diagrams, we can find explicit formulas for tail invariant measures.

**Theorem 2.20.** (1) Let \( B = B(F) \) be a stationary Bratteli diagram such that the incidence matrix \( F \) (and therefore \( A = F^T \)) is irreducible, aperiodic, and recurrent. Then there exists a tail invariant measure \( \mu \) on the path-space \( X_B \).

(2) The measure \( \mu \) is finite if and only if the right eigenvector \( t = (t_v) \) has the property \( \sum_v t_v < \infty \).
Proof. (1) Let $V = V_i$ denote the set of vertices at each level of the diagram $B$. By Theorem 2.18, find the Perron-Frobenius eigenvalue $\lambda$ for $A$. Take a right eigenvector $t = (t_v)$ such that $At = \lambda t$.

For every finite path $\bar{v}(w, v)$ that begins at $w \in V_0$ and terminates at $v \in V_n$, $n \in \mathbb{N}$, we associate the cylinder set $[\bar{v}(w, v)]$ and set

$$
(2.8) \quad \mu([\bar{v}(w, v)]) = \frac{t_v}{\lambda^{n-1}}.
$$

We need to check that this definition of the measures $\mu$ on cylinder sets is correct; in other words, it satisfies the Kolmogorov consistency condition. Indeed, we have

$$
\bar{v}(w, v) = \bigcup_{u \in V_{n+1}} f_u
$$

where $f_u = \bar{v}(w, v)e(v, u)$ is the concatenation of the path $\bar{v}(w, v)$ and the edge $e(v, u)$. Applying (2.6) and the relation $At = \lambda t$, we compute

$$
\mu\left( \bigcup_{u \in V_{n+1}} f_u \right) = \sum_{u \in V_{n+1}} a_{vu} \frac{t_u}{\lambda^{n-1}} = \frac{t_v}{\lambda^{n-1}} = \mu([\bar{v}(w, v)]).
$$

We note that the measure $\mu$ is tail invariant because, for any two finite path $\bar{v}$ and $\bar{v}'$ with the same terminal vertex,

$$
\mu([\bar{v}(w, v)]) = \mu([\bar{v}'(w', v)]),
$$

and the value depends on $v \in V_n$ only. Now we can refer to Theorem 2.13 to finish the proof of (1).

(2) Since every $t_v$ is finite and positive, the measure $\mu$ is $\sigma$-finite, in general. It follows from the definition of $\mu$ as in (2.8) that

$$
\mu(X_B) = \sum_{v \in V_0} t_v.
$$

If this sum is finite, $\mu$ can be made a probability measure. \qed
Example 2.21 (ERS and ECS Bratteli diagrams). It is said that a matrix \( A = (a_{i,j}) \) satisfies the equal row sum property (ERS property) if \( \sum_j a_{i,j} \) does not depend on \( i \). In other words, the sum of entries in each row is the same. If this sum is \( r \) we will say that \( A \) belongs to the class \( ERS(r) \). Similarly, it is said that a matrix \( A \) has the equal column sum property (ECS property) if \( \sum_i a_{i,j} \) is a constant independent of \( j \). If \( A \) is in the class \( ECS(c) \), then \( c \) is a Perron-Frobenius eigenvalue of \( A \) with the corresponding constant eigenvector.

We observe that the following fact holds. Let matrices \( A_1 \) and \( A_2 \) belong to the classes \( ERS(r_1) \) and \( ERS(r_2) \) respectively. Then the product \( A_1A_2 \) is in the class \( ERS(r_1r_2) \).

We say that a generalized Bratteli diagram \( B = B(F_n) \), defined by the sequence of incidence matrices \( (F_n) \), has the ERS (ECS) property if every matrix \( F_n \) has this property.

It follows then that the following result is true for ERS Bratteli diagrams. In particular, a substitution of constant length generates a stationary Bratteli diagram with the ECS property.

Lemma 2.22. (1) Let \( B \) be an ERS generalized Bratteli diagram such that incidence matrices \( F_n \) belong to \( ERS(r_n) \) for all \( n \in \mathbb{N} \). Then, for every tail invariant probability measure \( \mu \), we have

\[
\sum_{v \in V_{n+1}} \mu_v^{(n+1)} = \frac{1}{r_0 \cdots r_n}, \quad n \in \mathbb{N}.
\]

(2) If \( B \) is an ECS generalized Bratteli diagram and \( F_n \in ECS(c_n) \), then the measure \( \mu \) such that the vectors \( \mu_v^{(n)} \) have the entries \( \mu_v^{(n)} = (c_0 \cdots c_{n-1})^{-1}, v \in V_n \), is tail invariant.

Proof. (1) Recall that we assume that \( H_v^{(0)} = 1 \) for all \( v \in V_0 \). The fact that \( F_i \in ERS(r_i), i = 0, 1, ..., n \), means that \( H_v^{(n+1)} = H_a^{(n+1)} = r_0r_1 \cdots r_n \). Let \( \mu \) be a tail invariant probability measure. Then

\[
\sum_{v \in V_{n+1}} \mu_v^{(n+1)}H_v^{(n+1)} = r_0r_1 \cdots r_n \sum_{v \in V_{n+1}} \mu_v^{(n+1)} = 1,
\]

and relation (2.9) follows.

(2) It suffices to check that relation (2.4) of Theorem 2.13 holds. Indeed, for \( w \in V_n \),

\[
(A\mu^{(n+1)})_w = \sum_{v \in V_{n+1}} a_{w,v}^{(n)} \frac{1}{c_0 \cdots c_n} = \frac{1}{c_0 \cdots c_n}.
\]

This proves that the measure \( \mu \) is tail invariant. \( \square \)

3. Transition kernels and Markov measures on the path-space of generalized Bratteli diagrams

The central themes in the section are harmonic analysis, dynamics, and measures on graphs, but with an emphasis on the special case when the dynamics is specified by transition between levels in the graph. We use tail invariant measures, to play a central role, also in subsequent sections in the paper. We consider discrete Markov processes on the path-space of a generalized Bratteli diagram. Relative results can be found in [DH03], [Ver15], and [Ren18].
3.1. Graph induced Markov measures.

**Definition 3.1.** Let $B = (V, E)$ be a generalized Bratteli diagram constructed by a sequence of incidence matrices $(F_n)$. Let $q = (q_v)$ be a strictly positive vector, $q_v > 0, v \in V_0$, and let $(P_n)$ be a sequence of non-negative infinite matrices with entries $(p_{v,e}^{(n)})$ where $v \in V_n, e \in E_n, n = 0, 1, 2, \ldots$. To define a Markov measure $m$, we require that the sequence $(P_n)$ satisfies the following properties:

\[(3.1) \quad (a) \quad p_{v,e}^{(n)} > 0 \iff (s(e) = v); \quad (b) \quad \sum_{e : s(e) = v} p_{v,e}^{(n)} = 1.\]

Condition (3.1)(a) shows that $p_{v,e}^{(n)}$ is positive only on the edges outgoing from the vertex $v$, and therefore the matrices $P_n$ and $A_n = F_n^T$ share the same set of zero entries. For any cylinder set $[[e]] = [(e_0, e_1, \ldots, e_n)]$ generated by the path $e$ with $v = s(e_0) \in V_0$, we set

\[(3.2) \quad m([[e]]) = q_{s(e_0)} p_{s(e_0), e_0}^{(0)} \cdots p_{s(e_n), e_n}^{(n)}.\]

Relation (3.2) defines a measure $m$ of the set $[[e]]$. By (3.1)(b), this measure satisfies the Kolmogorov consistency condition and can be extended to the $\sigma$-algebra of Borel sets. To emphasize that $m$ is generated by a sequence of stochastic matrices, we will also write $m = m(P_n)$.

**Remark 3.2.** (1) More generally, we can consider a sequence of matrices $(P_n)$ such that, for every $v \in V_n$,

\[(3.3) \quad \sum_{e \in E_n : s(e) = v} p_{v,e}^{(n)} < \infty, \quad n \in \mathbb{N}_0.\]

Then, it follows from Definition 3.1 that the sequence of matrices $(P_n)$ determines finite transition kernels: given a vertex $v \in V_n$ and a set $B \subset V_{n+1}$, we define

\[(3.4) \quad P_n(v, B) := \sum_{e \in E_n : s(e) = v, r(e) \in B} p_{v,e}^{(n)}.\]

(2) If the matrices $P_n$ satisfy (3.3), then, without loss of generality, one can normalize $P_n$ such that the sum of entries in every row is one.

Define

\[\hat{p}_n(v, u) = \sum_{e \in E(v, u)} p_{v,e}^{(n)}\]

where $E(v, u) = \{ e \in E_n : s(e) = v, r(e) = u \}$. Then the entries $\hat{p}_n(v, u)$ form a matrix $\hat{P}_n$ indexed by the set $V_n \times V_{n+1}, n \in \mathbb{N}_0$.

The following lemma is a simple observation.

**Lemma 3.3.** The matrices $P_n$ and $\hat{P}_n$ have the same row sums:

\[\sum_{e : s(e) = v} p_{s(e), e}^{(n)} = \sum_{u \in V_{n+1}} \hat{p}_n(v, u), \quad v \in V_n, n \in \mathbb{N}_0.\]

Hence $\hat{P}_n$ is a row stochastic matrix if $P_n$ is a Markov matrix.
Using the initial vector \( q = q^{(0)} \) and the sequence of matrices \( (P_n) \), one can define another sequence of co-transition vectors \( (q^{(n)}) \) whose entries are assigned to the vertices of \( V_n, n \in \mathbb{N} \).

**Proposition 3.4.** (1) Let \( (P_n) \) be a sequence of matrices that defines a Markov measure \( m = m(P_n) \), and let \( q^{(0)} \) be an initial distribution on \( V_0 \). Then the formula

\[
q^{(k+1)}_v = \sum_{e : r(e) = v} q^{(k)}_{s(e)} p^{(k)}_{s(e), e}, \quad v \in V_{k+1},
\]

defines inductively a sequence of positive vectors \( q^{(k)} = (q^{(k)}_v : v \in V_k) \), \( k \geq 1 \). Equivalently, relation (3.5) is represented as \( q^{(k+1)} = q^{(k)} \hat{P}_k, k \in \mathbb{N}_0 \).

(2) If \( q^{(0)} \) is a probability vector, then all vectors \( q^{(k)} \) are probability.

**Proof.** (1) We first note that \( q^{(k+1)}_v \) is well defined because the sum in (3.5) is finite. Next, the set of all edges from \( r^{-1}(u) \) can be represented as

\[
r^{-1}(u) = \bigcup_{v \in V_n} E(v, u).
\]

Hence, for \( u \in V_{k+1} \),

\[
q^{(k+1)}_u = \sum_{e : r(e) = u} q^{(k)}_{s(e)} p^{(k)}_{s(e), e}
= \sum_{v \in V_n} \sum_{e \in E(v, u)} q^{(k)}_{s(e)} p^{(k)}_{s(e), e}
= \sum_{v \in V_n} q^{(k)}_v \hat{P}_n(v, u)
\]

which proves (1).

We now check that (2) holds. Suppose that we proved the statement for \( i = 1, \ldots, k \). We will show that \( q^{(k+1)} \) is also a probability vector.

\[
\sum_{u \in V_{k+1}} q^{(k+1)}_u = \sum_{u \in V_{k+1}} \sum_{v \in r^{-1}(u)} q^{(k)}_{s(e)} p^{(k)}_{s(e), e}
= \sum_{v \in V_k} \sum_{e \in s^{-1}(v)} q^{(k)}_v p^{(k)}_{s(e), e}
= \sum_{v \in V_k} q^{(k)}_v \sum_{e \in s^{-1}(v)} p^{(k)}_{s(e), e}
= 1.
\]

We remark that, in the above calculation, we used the summation over the set of all edges from \( E_k \) represented in two different but equivalent ways, namely,

\[
E_k = \bigcup_{u \in V_{k+1}} \bigcup_{e \in r^{-1}(u)} e = \bigcup_{v \in V_k} \bigcup_{e \in s^{-1}(v)} e.
\]

\(\square\)
Beginning with a vector $q^{(0)}$ and a sequence of probability transition kernels $(P_n)$, we constructed the vectors $q^{(n)}$. We want to find out whether there exists a transition kernel $Q_n(u, \cdot) : V_{n+1} \to V_n$ such that, for any subsets $A \subset V_n$ and $B \subset V_{n+1}$, the relation
\begin{equation}
\sum_{v \in A} q^{(n)}_v P_n(v, B) = \sum_{u \in B} q^{(n+1)}_u Q_n(u, A)
\end{equation}
holds, where $P_n(v, B)$ is defined by (3.4).

**Theorem 3.5.** For a given positive vector $q^{(0)}$ and sequence of probability transition kernels $(P_n)$, there exist a sequence of probability transition kernels $Q_n$ such that relation (3.6) holds.

**Proof.** We first find the sequence of vectors $q^{(n)}$, $n \in \mathbb{N}$, according to Proposition 3.4. For fixed vertices $v \in V_n, u \in V_{n+1}$, we set
\[
\hat{q}_n(u, v) = \frac{q^{(n)}_v}{q^{(n+1)}_u} \hat{p}_n(v, u).
\]
For $e \in E(v, u)$, we take
\[
q^{(n)}_{r(e), e} := \frac{1}{|E(v, u)|} \hat{q}_n(u, v)
\]
and define
\[
Q_n(u, A) = \sum_{v \in A} \hat{q}_n(u, v) = \sum_{v \in A} \sum_{e \in E(u, v)} q^{(n)}_{r(e), e}.
\]
The fact that $Q_n$ is probability follows from the relations:
\[
\sum_{v \in V_n} \hat{q}_n(u, v) = \sum_{e \in r^{-1}(u)} q^{(n)}_{r(e), e}
\]
and
\[
\sum_{v \in V_n} \hat{q}_n(u, v) = \frac{1}{q^{(n+1)}_u} \sum_{v \in V_n} q^{(n)}_v \hat{p}_n(v, u) = 1
\]
because of (3.5), see Proposition 3.4.

Finally, we check that $Q_n$ satisfies (3.6):
\[
\sum_{v \in A} q^{(n)}_v P_n(v, B) = \sum_{v \in A} \sum_{e : r(e) \in B, s(e) = v} q^{(n)}_v P^{(n)}_{s(e), e} = \sum_{v \in A} \sum_{u \in B} \sum_{e \in E(v, u)} q^{(n)}_v P^{(n)}_{s(e), e} = \sum_{v \in A} \sum_{u \in B} q^{(n)}_v \hat{p}_n(v, u) = \sum_{v \in A} \sum_{u \in B} q^{(n+1)}_u \hat{q}_n(u, v) = \sum_{u \in B} q^{(n+1)}_u Q(u, A).
\]

□

From the proof of Theorem 3.5 we deduce the following facts. (We use here the notation introduced above.)
Proposition 3.6. (1) The sequence of transition kernels $Q_n$ is uniquely determined by $q^{(0)}$ and the kernels $(P_n)$ if and only if the Bratteli diagram has no multiple edges, i.e., $|E(v,u)| \leq 1$ for all vertices $v$ and $u$.

(2) If $\hat{Q}_n$ is the infinite positive matrix defined by its entries $\hat{q}_n(u,v)$, then $q^{(n+1)}\hat{Q}_n = q^{(n)}$.

(3) Relation (3.6) holds also for the matrices $\hat{P}_n$ and $\hat{Q}_n$. In particular, for any $v \in V_n, u \in V_{n+1}, n \in \mathbb{N}_0$,

\begin{equation}
q^{(n)}(v)\hat{P}_n(v,u) = q^{(n+1)}(u,v).
\end{equation}

Proof. (1) Indeed, the result follows from the proof of Theorem 3.5:

\begin{equation}
|E(v,u)| = 1 \iff q^{(n)}_{r(e),e} = \hat{q}_n(u,v) \quad \forall e \in E(v,u).
\end{equation}

Moreover, if the quantities $\hat{q}_n(u,v)$ are defined as in Theorem 3.5, then there are infinitely many solutions of the equation

\begin{equation}
\sum_{e \in E(v,u)} q^{(n)}_{r(e),e} = \hat{q}_n(u,v)
\end{equation}

if and only if $|E(v,u)| > 1$.

(2) We compute

\begin{equation}
\sum_{u \in V_{n+1}} q^{(n+1)}(u,v) = \sum_{u \in V_n} q^{(n)}(u) = \sum_{u \in V_n} \hat{P}_n(v,u) = q^{(n)}(v)
\end{equation}

since $\hat{P}_n$ is a row stochastic matrix.

(3) It can be checked that

\begin{equation}
\sum_{v \in A} q^{(n)}(v)\hat{P}_n(v,B) = \sum_{v \in A} q^{(n)}(v)\sum_{u \in B} \hat{P}_n(v,u)
= \sum_{u \in B} \sum_{v \in A} q^{(n+1)}(u,v)
= \sum_{u \in B} q^{(n+1)}(u)\hat{Q}_n(u,A).
\end{equation}

Relation (3.7) is a particular case of the proved property.

The proposition below clarifies the meaning of vectors $q^{(n)}$, $n \geq 1$. Recall that a measure on the path-space of a Bratteli diagram is completely determined by its values on the cylinder sets which are represented by finite paths in a Bratteli diagram. For every vertex $v \in V_n, n \in \mathbb{N}$, we defined the tower $X^{(n)}_v$ of the Kakutani-Rokhlin partition which is formed by all cylinder sets that end at $v$. It turns out that the measures of towers $X^{(n)}_v$ are exactly the entries of the vector $q^{(n)}$.

Theorem 3.7. Let $m$ be a Markov measure defined by a sequence of Markov matrices $(P_n)$. Let $(q^{(k)})$ be a sequence of probability vectors constructed accordingly to (3.5). Then

\begin{equation}
m(X^{(n)}_v) = q^{(n)}(v), \quad v \in V_n, n \in \mathbb{N}_0,
\end{equation}

where $X^{(n)}_v$ is the Kakutani-Rokhlin tower corresponding to the vertex $v$. 

Proof. The statement can be proved by induction. Indeed, it is trivial for \( n = 0 \). For \( n = k + 1 \), we see that

\[
m(X_{v}^{(k+1)}) = \sum_{e \in E_{n}: r(e) = v} m(X_{s(e)}^{(k)}) p_{s(e), e}^{(k)} = q_{v}^{(k+1)}, \quad v \in V_{k+1},
\]

since \( m(X_{s(e)}^{(k)}) = q_{v}^{(k)} \) by the induction hypothesis. \( \square \)

3.2. Tail invariant Markov measures. We will consider here Markov measures that are invariant with respect to the tail equivalence relation.

The next result proves that every tail invariant probability measure is, in fact, a Markov measure. The sequence of Markov matrices can be explicitly described. Recall that \( M_{1}(B, E) \) denotes the set of all tail invariant probability measures.

Theorem 3.8. Let \( \nu \in M_{1}(B, E) \) be a tail invariant probability measure on the path-space \( X_{B} \) of a generalized Bratteli diagram \( B = (V, E) \). Then there exists a sequence of Markov matrices \( (P_{n}) \) such that \( \nu = m(P_{n}) \).

Proof. Given an invariant probability measure \( \nu \), we have the sequence of vectors \( (\nu^{(n)}) \) that satisfies (2.4) and is uniquely determined by \( \nu \). We recall that the entry \( \nu^{(n)}_{v} \) gives the measure of a cylinder set which is determined by a finite path from \( V_{0} \) to the vertex \( v \in V_{n} \).

In the proof, we will construct inductively a sequence of Markov matrices \( (P_{n}) \) such that the corresponding Markov measure \( m = m(P_{n}) \) satisfies the property: for all \( n \in \mathbb{N}_{0} \) and \( v \in V_{n} \), \( \nu(\overline{e}) = m(\overline{e}) \) where \( \overline{e} \) is a finite path with \( r(\overline{e}) = v \).

For \( n = 0 \), we define the probability vector \( q^{(0)} \) by setting

\[
q^{(0)}_{v} = \nu(X_{v}^{(0)}) = \nu^{(0)}_{v}, \quad v \in V_{0}.
\]

To define the entries \( p_{v,e}^{(0)} \) of \( P_{0} \) where \( v \in V_{0} \), we set first \( p_{v,e}^{(0)} = 0 \) if \( v \neq s(e) \). Recall that

\[
\sum_{e' \in V_{1}} q^{(0)}_{s(e'), e} p^{(1)}_{e'} = \nu^{(0)}_{e', e}, \quad a^{(0)}_{w,v} \in A_{0}.
\]

(see (2.4)) where \( A_{n} = F_{n}^{T} \). This fact allows us to define the entries of \( P_{0} \) by

\[
P^{(0)}_{v_{0}, e} = \frac{\nu^{(1)}_{v_{1}}}{q^{(0)}_{v_{0}}} = \frac{\nu^{(1)}_{v_{1}}}{\nu^{(0)}_{v_{0}}}, \quad \forall e \in E(v_{0}, v_{1}).
\]

(3.10)

It follows from (3.10) that \( P_{0} \) is a Markov matrix because

\[
\sum_{e:s(e) = v_{0}} P^{(0)}_{v_{0}, e} = \sum_{v \in V_{1}} \sum_{e \in E(v_{0}, v)} p^{(0)}_{v_{0}, e}
\]

\[
= \sum_{v \in V_{1}} \frac{\nu^{(1)}_{v}}{q^{(0)}_{v_{0}}} a^{(0)}_{v_{0}, v}
\]

\[
= 1
\]

in virtue of (3.8) and (3.9). Relation (3.10) says that the value of the entry \( p^{(0)}_{v_{0}, e} \) does not depend on an edge \( e \in E_{0}(v_{0}, v_{1}) \). Hence we can denote it by \( p^{(0)}_{v_{0}, v_{1}} \).
Assume that the Markov matrices \( P_i \) are defined for \( i = 0, 1, \ldots, n - 1 \), and their entries satisfy the property \( p_{v_i,e}^{(i)} = p_{v_i,e'}^{(i)} = p_{v_i,v_{i+1}}^{(i)} \) for all \( e, e' \in E(v_i, v_{i+1}) \). We define the entries of \( P_n \) as follows: \( p_{v_n,e}^{(n)} = 0 \) if \( v_n \neq s(e) \), and

\[
(3.11) \quad p_{v_n,e}^{(n)} = \frac{\nu_{v_n+1}^{(n+1)}}{q_{v_0}^{(0)} p_{v_0,v_1}^{(0)} \cdots p_{v_{n-1},v_n}^{(n-1)}}, \quad \forall e \in E(v_n, v_{n+1}), \quad v_n \in V_n, v_{n+1} \in V_{n+1}.
\]

The meaning of this definition is explained by the following fact. Let \( (v_0, v_1, \ldots, v_{n+1}) \) be a finite sequence of vertices such that there exists a path \( \bar{e} \) from \( v_0 \) to \( v_{n+1} \). The \( \nu \)-measure of the corresponding cylinder set is \( \nu_{v_{n+1}}^{(n+1)} \). We want to define a Markov matrix \( P_n \) such that the Markov measure \( q_{v_0}^{(0)} p_{v_0,v_1}^{(0)} \cdots p_{v_n,v_{n+1}}^{(n)} \) of \( \bar{e} \) is exactly \( \nu_{v_{n+1}}^{(n+1)} \).

We claim that relation (3.11) is equivalent to

\[
(3.12) \quad \nu_{v_{n+1}}^{(n+1)} = q_{v_0}^{(0)} p_{v_0,v_1}^{(0)} \cdots p_{v_n,v_{n+1}}^{(n)} = q_{v_0}^{(0)} \nu_{v_1}^{(1)} \cdots \nu_{v_n}^{(n)} p_{v_n,v_{n+1}}^{(n)} = \nu_{v_{n+1}}^{(n+1)}.
\]

Indeed, by induction

\[
\nu_{v_{n+1}}^{(n+1)} = q_{v_0}^{(0)} p_{v_0,v_1}^{(0)} \cdots p_{v_n,v_{n+1}}^{(n)} = q_{v_0}^{(0)} \nu_{v_1}^{(1)} \cdots \nu_{v_n}^{(n)} p_{v_n,v_{n+1}}^{(n)} = \nu_{v_{n+1}}^{(n+1)}.
\]

It remains to check that the matrix \( P_n \) is row stochastic:

\[
\sum_{e: s(e) = w} p_{w,e}^{(n)} = \sum_{v \in V_{n+1}} \sum_{e \in E_n(w,v)} \frac{\nu_{v_n+1}^{(n+1)}}{\nu_{v_n}^{(n)}} = \sum_{v \in V_{n+1}} q_{v_n,v_{n+1}}^{(n)} \frac{\nu_{v_n+1}^{(n+1)}}{\nu_{v_n}^{(n)}} = 1.
\]

We used here relation (2.4) of Theorem 2.13. We remark that this property can be deduced also from (3.11).

\[\square\]

**Remark 3.9.** (1) We note that the Markov measure \( m \) constructed by the invariant measure \( \nu \) has the property of equal values on all edges connecting two fixed vertices: for every \( e \in E(w,v) \), \( p_{w,v}^{(n)} = p_{w,v}^{(n)} \) where \( w \in V_n, v \in V_{n+1}, n \in \mathbb{N}_0 \). This means that, for every two finite paths \( \bar{s}, \bar{r} \) such that \( s(\bar{s}) = s(\bar{r}) \), \( r(\bar{s}) = r(\bar{r}) \), and going through the same vertices \( v_0 = s(\bar{s}), v_1, \ldots, v_k = r(\bar{s}) \), the Markov measure of the corresponding cylinder sets coincide.

(2) It can be easily seen from the proof of Theorem 3.8 that the Markov measure \( m(P_n) \) is uniquely determined by the tail invariant measure \( \nu \).

Let \( \nu \in M_1(B, \mathcal{E}) \) be a tail invariant probability measure. The following result clarifies the meanings of the vectors \( q^{(n)} \) and the sequences of matrices \( \hat{P}_n \) and \( \hat{Q}_n \) generated by \( \nu \).

**Corollary 3.10.** Let \( \nu \in M_1(B, \mathcal{E}) \), and let the sequence of probability vectors \( (s^{(n)}) \) determine the measures of Kakutani-Rokhlin towers \( X^{(n)}_v \), \( v \in V_n \) as in Corollary 2.14. Then co-transition probabilities \( q^{(n)} \) coincide with \( s^{(n)} \), i.e., \( q^{(n)}_v = H^{(n)}_v s^{(n)}_v \), \( v \in V_n, n \in \mathbb{N}_0 \). Furthermore, \( \hat{Q}_n = \hat{F}_n \) for all \( n \) where \( \hat{F}_n \) is defined in (2.7).
Proof. We will prove the result by induction. We first find the matrices \( \hat{P}_n \). It follows from (3.12) that entries of \( \hat{P}_n \) are
\[
\hat{p}_n(v_{n+1}, v_n) = |E(v_n, v_{n+1})| \frac{\nu_{v_{n+1}}^{(n+1)}(n)}{\nu_{v_n}^{(n)}} = a_{v_n, v_{n+1}}^{(n)} \frac{\nu_{v_{n+1}}^{(n+1)}(n)}{\nu_{v_n}^{(n)}}.
\]

Let \( q^{(0)} = \nu^{(0)} \) as in the proof of Theorem 3.8. Since \( H_0^{(0)} = 1, v \in V_0 \), the result holds for \( n = 0 \). Define \( q^{(n)} = q^{(0)} \hat{P}_0 \cdots \hat{P}_{n-1} \). Suppose that we proved the statement for \( i \leq n \). Compute the entries of \( q^{(n+1)} = q^{(n)} \hat{P}_n \):
\[
q_{v_{n+1}}^{(n+1)} = \sum_{v_n \in V_n} q_{v_n}^{(n)} a_{v_n, v_{n+1}}^{(n+1)} \frac{\nu_{v_{n+1}}^{(n+1)}(n)}{\nu_{v_n}^{(n)}}
= \sum_{v_n \in V_n} H_{v_n}^{(n)} v_{v_n}^{(n)} a_{v_n, v_{n+1}}^{(n+1)} \nu_{v_{n+1}}^{(n+1)}(n) \nu_{v_n}^{(n)}
= \nu_{v_{n+1}}^{(n+1)} \sum_{v_n \in V_n} f^{(n)}(v_n, v_{n+1}) H_{v_n}^{(n)}
= \nu_{v_{n+1}}^{(n+1)} H_{v_{n+1}}^{(n+1)}.
\]

Having the vectors \( q^{(n)} \) determined, we can find the matrices \( \hat{Q}_n \), see Theorem 3.5 (we use here that \( F_n^T = A_n \)):
\[
\hat{q}_n(v_{n+1}, v_n) = q_{v_n}^{(n+1)} \hat{p}_n(v_n, v_{n+1})
= \frac{H_{v_n}^{(n)} v_{v_n}^{(n)}}{H_{v_{n+1}}^{(n+1)} v_{v_{n+1}}^{(n+1)}} a_{v_n, v_{n+1}}^{(n+1)} \nu_{v_{n+1}}^{(n+1)}(n) \nu_{v_n}^{(n)}
= f^{(n)}(v_{n+1}, v_n) H_{v_n}^{(n)} \nu_{v_{n+1}}^{(n+1)} H_{v_{n+1}}^{(n+1)}
= \hat{f}^{(n)}(v_{n+1}, v_n).
\]

This proves the equality \( \hat{Q}_n = \hat{F}_n \). \( \square \)

Remark 3.11. The fact proved in Corollary 3.10 says that the matrix \( \hat{Q} \) equals to the matrix \( \hat{F} \) independently of a tail invariant measure \( \nu \). In other words, \( \hat{Q} \) does not depend on \( \nu \).

Stationary Markov measure. The following observation is useful for the study of Markov measures on stationary generalized Bratteli diagrams \( B \) with the incidence matrix \( F \). It is natural to consider a special subset of Markov measures \( m = m(P) \) on such diagrams, the so called stationary Markov measures. They are determined by the property that all matrices \( P_n, n \in \mathbb{N} \), are the same and equal to a fixed matrix \( P \). Formula (3.2), which defines a Markov measure, is transformed then as follows:
\[
m((\overline{e})) = q_{s(e_0)p_s(e_0),e_0}p_{s(e_1),e_1} \cdots p_{s(e_n),e_n}.
\]

Suppose that \( B \) is a stationary Bratteli diagram and \( \mu \) is a tail invariant measure satisfying (2.8), see Theorem 2.20. In other words, we assume that the transpose \( A \) of the incidence
matrix $F$ satisfies the conditions of the Perron-Frobenius theorem: there exists a positive right eigenvector $x = (x_v)$ corresponding to the Perron-Frobenius eigenvalue $\lambda$.

**Lemma 3.12.** Let $B = B(F)$ be a stationary Bratteli diagram such that the matrix $A = F^T$ satisfies Theorem 2.18 and let $x = (x_v)$ be the right Perron-Frobenius eigenvector corresponding to the eigenvalue $\lambda$. Suppose that $\mu$ is a tail invariant measure on $X_B$ defined by (2.8). Then $\mu$ can be determined as a stationary Markov measure $m(P)$ on $X_B$ where the initial distribution $q^{(0)}$ is the vector $x$, and the Markov matrix $P$ has the entries

$$p_{s(e),e} = \frac{x_{r(e)}}{\lambda x_{s(e)}}, \quad e \in E.$$

**Proof.** It was proved in Theorem 3.8 that any tail invariant measure is a Markov measure for an appropriate choice of the matrices $(P_n)$. In conditions of the lemma, we can specify that $P_n = P$ and vectors $\mu^{(n)} = (x_v \lambda^{n-1} e_{v \in V_n})$. It follows then from (3.12) that, for every $e \in E(w, v)$, $w \in V_0, v \in V_1$,

$$p_{w,e} = \frac{\mu^{(1)}_v}{\mu^{(0)}_w} = \frac{x_v}{\lambda x_w}.$$

\[\square\]

### 3.3. Existence of finite tail invariant measures.

It is well known that every homeomorphism of a compact metric space has an invariant probability measure. Rephrasing this statement, we conclude that every (classical) Bratteli diagram has a probability tail invariant measure. This follows from the fact that every homeomorphism of a Cantor set admits its realization on the path-space of a Bratteli diagram whose orbits are essentially the same as orbits of the tail equivalence relation. We refer to [HPS92], [Med06], [BK20] where the reader can find more details.

The situation with generalized Bratteli diagrams is more difficult. First of all, there are Borel automorphisms $T$ of a standard Borel space $(X, B)$ that do not admit a probability invariant measure. Two Borel sets, $A$ and $B$, are called equivalent ($A \sim B$ in symbols) with respect to $T$ if there exists a one-to-one Borel map $f : A \to B$ such that $f(x)$ is in the $T$-orbit of $x$, i.e., $(x, f(x)) \in E_T$ where $E_T$ is the orbit equivalence relation on $X \times X$. It is said that $A \preceq B$ if $A \sim B'$ where $B' \subset B$. It can be shown that $A \sim B$ if and only if $A \preceq B$ and $B \preceq A$. A Borel aperiodic automorphism $T$ is called compressible if there is a Borel set $A$ such that $A \sim X$ and $X \setminus A$ is a complete section, that it meets every $T$-orbit. It turns out that compressible automorphisms do not admit finite invariant measures.

**Theorem 3.13** ([Nad90, Nad95]). Let $T$ be an aperiodic Borel automorphism of a standard Borel space. The following are equivalent:

(i) $T$ is not compressible,

(ii) $T$ admits an invariant probability measure.

It follows from Theorem 2.11 that we can apply Theorem 3.13 to generalized Bratteli diagrams.

**Corollary 3.14.** There exist generalized Bratteli diagrams that do not admit tail invariant probability measures.
Example 3.15 ([Fer06]). Let \( \sigma : A \to A^* \) be a substitution on a countably infinite alphabet, see Example 2.19. We say that a finite word \( w \) belongs to the language \( L(\sigma) \) of the substitution \( \sigma \) if \( w \) is a subword of \( \sigma^n(a) \) for some \( a \in A \) and \( n \in \mathbb{N} \). Let now \( X \) be a subset of one-sided sequences \( x = x_0x_1 \cdots \) from \( A^\mathbb{N} \) such that every finite word occurring in \( x \) is in the language \( L(\sigma) \).

It can be easily seen that \( X \) is a closed subset of the Polish space \( A^\mathbb{Z} \) (with respect to the product topology). Let \( T \) be the two-sided shift on \( A^\mathbb{Z} \). Then \( TX \subset X \). The pair \( (X,T) \) is a non-compact symbolic dynamical system associated to the substitution \( \sigma \).

Let now \( A = 2\mathbb{Z} \) and the substitution \( \sigma_0 \) is defined by the rule:

\[
\sigma_0 : n \to (n - 2)nn(n + 2), \quad n \in 2\mathbb{Z}.
\]

In [Fer06], \( \sigma_0 \) is called the squared drunken man substitution. The following result was proved there.

**Lemma 3.16.** The Borel dynamical system \( (X,T) \) associated to \( \sigma_0 \) has no finite invariant measure.

We finish this subsection by giving another result on the existence of finite invariant measures. We recall that a substitution \( \sigma : A \to A^* \) is called of constant length \( L \) if \( |\sigma(a)| = L \) for all \( a \in A \). It is obvious that if \( M \) is the matrix of substitution \( \sigma \) of constant length \( L \), then the vector \( (... , 1, 1, ...) \) is the right eigenvector corresponding to the eigenvalue \( L \).

**Lemma 3.17** ([Fer06]). Let \( \sigma \) be a constant length substitution on a countable alphabet \( A \). Suppose that the matrix of substitution is irreducible, aperiodic, and positive recurrent. Then the associated Borel dynamical system \( (X,T) \) admits a probability invariant measure.

An example that illustrates the above lemma is the following: \( A = \mathbb{N}_0 \) and

\[
\sigma(0) = 01, \quad \sigma(1) = 02, \quad \sigma(n) = (n - 2)(n + 1), \quad n \geq 2.
\]

### 3.4. Operators generated by transition kernels.

Let \( B = B(F_n) \) be a generalized Bratteli diagram. Suppose that \( (P_n) \) is a sequence of Markov matrices (equivalently, probability transition kernels), and \( q^{(0)} \) is an initial distribution (a discrete infinite measure on \( V_0 \)). As was shown above, the matrices \( P_n \) and vector \( q^{(0)} \) determine the stochastic matrices \( \hat{P}_n \) (Lemma 3.3), vectors \( q^{(n)} \) (Proposition 3.4) such that \( q^{(n+1)} = q^{(n)}\hat{P}_n \), and dual probability kernels \( \hat{Q}_n \) such that \( q^{(n+1)}\hat{Q}_n = q^{(n)} \) (Theorem 3.5 and Proposition 3.6). We use these objects to define operators acting in the weighted \( \ell^2 \)-spaces.

For each \( P_n \), we can define a linear operator: if \( f \) is a bounded function defined on the set of vertices \( V_{n+1} \), then setting

\[
(T_{P_n}f)(v) = \sum_{e:s(e) = v} P_{v,e}^{(n)} f(r(e)), \quad v \in V_n,
\]

we obtain a function defined on vertices of \( V_n \). For \( Q_n \), we have

\[
(T_{Q_n}g)(u) = \sum_{e:r(e) = v} Q_{u,e}^{(n)} g(s(e)), \quad u \in V_{n+1}.
\]

Similarly, \( (T_{\hat{P}_n}f)(v) = \sum_{u \in V_{n+1}} \hat{p}_n(v,u)f(u) \) and \( (T_{\hat{Q}_n}g)(u) = \sum_{v \in V_n} \hat{q}_n(u,v)g(v) \).
Lemma 3.18. For \( P_n, \hat{P}_n \) and \( Q_n, \hat{Q}_n \) as above, we have
\[
T_{P_n}(f) = T_{\hat{P}_n}(f), \quad T_{Q_n}(g) = T_{\hat{Q}_n}(g), \quad n \in \mathbb{N}_0,
\]
where \( f \) is a bounded function on \( V_{n+1} \), and \( g \) is a bounded function on \( V_n \).

Proof. We calculate
\[
T(P_n)f(v) = \sum_{e: s(e) = v} p^{(n)}_{v,e} f(r(e)) = \sum_{u \in V_{n+1}} \sum_{e \in E(v,u)} p^{(n)}_{v,e} f(r(e)) = \sum_{u \in V_{n+1}} \hat{p}_n(v,u) f(u) = T(\hat{P}_n)f(v).
\]
The other relation is proved analogously. \( \square \)

Let \( \mathcal{H}_n \) be the linear space of all sequences \( f = (f(v) : v \in V_n) \) such that
\[
||f||^2_{\mathcal{H}_n} := \sum_{v \in V_n} q^{(n)}_v f(v)^2 < \infty.
\]
Then \( \mathcal{H}_n \), equipped with this norm, is a Hilbert space with the inner product
\[
\langle \varphi, \psi \rangle_{\mathcal{H}_n} = \sum_{v \in V_n} \varphi(v)\psi(v)q^{(n)}_v.
\]

Proposition 3.19. (1) The operators \( T_{P_n} : \mathcal{H}_{n+1} \to \mathcal{H}_n \) and \( T_{Q_n} : \mathcal{H}_n \to \mathcal{H}_{n+1} \) are positive and contractive for all \( n \in \mathbb{N}_0 \).

(2) \( (T_P)^* = T_Q \) and \( (T_Q)^* = T_P \).

Proof. It is obvious that \( T_{P_n}(f) > 0 \) whenever \( f > 0 \). By Schwarz inequality, we have \( T_{P_n}(f)^2 \leq T_{P_n}(f^2) \).

It follows from Lemma 3.18 that the operator \( T_{P_n} \) is defined on vectors from \( \mathcal{H}_{n+1} \). One needs to show that \( T_{P_n}(f) \in \mathcal{H}_n \) if \( f \in \mathcal{H}_{n+1} \). Indeed, using the equality \( q^{(n+1)} = q^{(n)}\hat{P}_n \), we compute
\[
||T_{P_n}(f)||^2_{\mathcal{H}_n} = \sum_{v \in V_n} q^{(n)}_v T_{P_n}(f)^2 \leq \sum_{v \in V_n} q^{(n)}_v T_{P_n}(f^2) = \sum_{v \in V_n} q^{(n)}_v \sum_{u \in V_{n+1}} \hat{p}_n(v,u) f^2(u) = \sum_{u \in V_{n+1}} f^2(u) \sum_{v \in V_n} q^{(n)}_v \hat{p}_n(v,u) = \sum_{u \in V_{n+1}} q^{(n+1)}_{u} f^2(u) = ||f||^2_{\mathcal{H}_{n+1}}.
\]
It follows also from the proof that
\[
||T_{P_n}||_{\mathcal{H}_{n+1} \to \mathcal{H}_n} \leq 1.
\]
The same proof works for $T_{Q_n}$.

(2) To prove that $(TP)^* = T_Q$, we use the equality $q^{(n)}_v \hat{p}_n(v, u) = q^{(n+1)}_u \hat{q}_n(u, v)$, see Theorem 3.5.

$$\langle f, TP_n(g) \rangle_{H_n} = \sum_{v \in V_n} f(v) \left( \sum_{u \in V_{n+1}} \hat{p}_n(v, u) g(u) \right) q^{(n)}_v$$

$$= \sum_{v \in V_n} \sum_{u \in V_{n+1}} f(v) g(u) q^{(n)}_v \hat{p}_n(v, u)$$

$$= \sum_{u \in V_{n+1}} \sum_{v \in V_n} f(v) g(u) q^{(n+1)}_u \hat{q}_n(u, v)$$

$$= \sum_{u \in V_{n+1}} g(u) T_{Q_n}(f)(u) q^{(n+1)}_u$$

$$= \langle T_{Q_n}(f), g \rangle_{H_{n+1}}.$$

□

The following corollary is immediate.

**Corollary 3.20.** Let $T_n = T_{P_n} T_{Q_n}$. Then

1. $T_n$ is a self-adjoint operator on $H_n$ acting on functions from $H_n$ by the formula

   $$(T_n f)(v) = \sum_{u \in V_n} \sum_{u \in V_{n+1}} \hat{p}_n(v, u) \hat{q}_n(u, w) f(w),$$

2. $T_n$ is represented by a row stochastic matrix $\hat{T}_n$ with entries

   $$\hat{t}^{(n)}_{vw} = \sum_{u \in V_{n+1}} \hat{p}_n(v, u) \hat{q}_n(u, w),$$

3. $q^{(n)}$ is a left eigenvector for the matrix $\hat{T}_n$ corresponding to the eigenvalue 1. $n \in \mathbb{N}$.

**Proof.** Statement (1) follows from Proposition 3.19. To see that (2) is true, we can use that $\hat{P}_n$ and $\hat{Q}_n$ are row stochastic. For (3), we compute

$$\sum_{v \in V_n} q^{(n)}_v \hat{t}^{(n)}_{vw} = \sum_{v \in V_n} \sum_{u \in V_{n+1}} \hat{p}_n(v, u) \hat{q}_n(u, w)$$

$$= \sum_{u \in V_{n+1}} \left( \sum_{v \in V_n} \hat{q}^{(n)}_v \hat{p}_n(v, u) \right) \hat{q}_n(u, w)$$

$$= \sum_{u \in V_{n+1}} q^{(n+1)}_u \hat{q}_n(u, w)$$

$$= q^{(n)}_{w_1}.$$

□

4. **Graph Laplacians and associated harmonic functions**

In this section we consider the main concepts of the theory of weighted networks $(G, c)$ in the case when the graph $G$ is represented by a generalized Bratteli diagram. We refer to [BJ19b] where a similar approach was applied to finite Bratteli diagrams. The reader can find more information on this subject, for example, in [Ana11, Ana12b, Cho14, DJ10, Geo10, ...]}
Our goal is to show how the notions of weighted networks can be used for Bratteli diagrams. We plan to study them in details in a forthcoming paper.

We use the notation of Section 3 to fix our setting for this section: \( B = (V, E) \) is a generalized Bratteli diagram (see Definition 2.1), \((\hat{P}_n)\) and \((\hat{Q}_n)\) are the sequences of row stochastic matrices defined by Markov matrices \((P_n)\), \((q(n))\) is a sequence of positive vectors where \(q(n)\) is indexed by vertices of \(V_n\) and such that \(q(n)\hat{P}_n = q(n+1)\), \(q(n+1)\hat{Q}_n = q(n)\).

By definition, a weighted network \((G, c)\) is an undirected countable connected graph \( G = (V, E) \) without loops together with a weight function \( c : E \to [0, \infty) \). In the literature, weighted networks are also called electrical networks where \( c \) is viewed as a conductance function (see Subsection 1.4 for references). The notation \( v \sim w \) means that there exists an edge between vertices \( v \) and \( w \) (we consider the graphs with single edges between connected vertices). Define the path space \( X_G \) of \( G \) as a set of all infinite sequences \( x = (v_0, v_1, ...) \) such that \( e = (v_i, v_{i+1}) \) is an edge from \( E \), \( i \in \mathbb{N}_0 \).

We will begin with a generalized Bratteli diagram \( B = B(V, E) \) and show how to associate a weighted network \( G = G(B) = (V', E') \) to \( B \). It is important to stress that our setting includes the existence of Markov matrices \((\hat{P}_n)\) and \((\hat{Q}_n)\) defined by row stochastic matrices \((P_n)\) and the vectors \((q(n))\) (see their properties above).

We set \( V' = V = \bigcup_n V_n \) so that all vertices are partitioned into levels. Define the set of edges \( E' \) by the following rule: \( E' = \bigcup_n E'_n \) where \( E'_n \) is the set of edges between vertices of \( V_n \) and \( V_{n+1} \). Two vertices, \( v \in V_n \) and \( u \in V_{n+1} \), are connected by an edge from \( E' \) if and only if the set \( E(v, u) \neq \emptyset \) in the Bratteli diagram \( B \). In other words, we replace the set \( E(v, u) \) with a single edge if this set is not empty.

Define now weight function \( c : E' \to [0, \infty) \). Let \( e' = (v, u) \) where \( v \in V_n \), \( u \in V_{n+1} \). For this, we fix a vertex \( v \in V_n \) and assign a weight \( c^{(n)}_{vu} \) for all edges \((v, u), u \in V_{n+1} \) and \( c^{(n-1)}_{vw} \) for all edges \((w, v), w \in V_{n+1} \):

\[
(4.1) \quad c^{(n)}_{vu} = \frac{1}{2} q^{(n)}_v \hat{P}_n(v, u), \quad c^{(n-1)}_{vw} = \frac{1}{2} q^{(n)}_v \hat{q}_{n-1}(v, w).
\]

Lemma 4.1. (1) \( c^{(n)}_{vu} = c^{(n)}_{uv} \), i.e., the conductance function \( c \) is correctly defined on edges from \( E' \).

(2) \( c_n(v) = q^{(n)}_v, \quad v \in V_n, \quad n \in \mathbb{N}_0 \).

Proof. (1) Formula (4.1) defines the value of the function \( c \) on an edge connecting \( w \in V_{n-1} \) and \( v \in V_n \) in two (formally different) ways. In fact, we use (3.7) to show that

\[
(4.1) \quad c^{(n-1)}_{vw} = \frac{1}{2} q^{(n-1)}_v \hat{P}_{n-1}(w, v) = \frac{1}{2} q^{(n)}_v \hat{q}_{n-1}(v, w) = c^{(n-1)}_{vw}.
\]

(2) We calculate the sum using statement (1):

\[
c_n(v) = \sum_{u \in V_{n+1}} c^{(n)}_{vu} + \sum_{w \in V_{n-1}} c^{(n-1)}_{vw}
= \frac{1}{2} \sum_{u \in V_{n+1}} q^{(n)}_v \hat{P}_n(v, u) + \frac{1}{2} \sum_{w \in V_{n-1}} q^{(n)}_v \hat{q}_{n-1}(v, w)
= q^{(n)}_v.
\]

\[\square\]
Lemma 4.1 shows that \( c(v) = (c_n(v)) \) is finite for every \( v \in V \). We will omit the index \( n \) in \( c_n(v) \) if it is clear that \( v \) is taken from \( V_n \).

**Definition 4.2.** For \( G = G(B) \) and the conductance function \( c \) as above, we define a *reversible Markov kernel* \( M = \{m(v, u) : v, u \in V \} \) by setting

\[
m(v, u) = \begin{cases} 
\frac{c(vu)}{c_n(v)} = \frac{1}{2} \bar{p}_n(v, u), & v \in V_n, u \in V_{n+1}, \\
\frac{c(vu)}{c_n(v)} = \frac{1}{2} \bar{q}_{n-1}(v, u), & v \in V_n, u \in V_{n-1}.
\end{cases}
\]

**Remark 4.3.** It follows from Lemma 4.1 that \( m(v, u) \) can be viewed as the probability to get to \( u \) from \( v \) because \( \sum_{u \sim v} m(v, u) = 1 \).

The Markov kernel \( M \) is reversible, i.e.,

\[
c(v)m(v, u) = c(u)m(u, v), \quad \forall v, u, v \sim u.
\]

This fact follows from (3.7).

**Definition 4.4.** Let \( f = (f_n) \) be a function defined on vertices of the graph \( G(B) \) (or the Bratteli diagram). Suppose that a Markov kernel \( M \) is defined as in 4.2. The operator

\[
(Mf)(v) = \sum_{u \sim v} m(v, u)f(u), \quad v \in V.
\]

is called a *Markov operator* acting on the weighted network \((G, c)\).

Define a *Laplacian operator* \( \Delta \):

\[
(\Delta f)(v) = \sum_{u \sim v} c_{vu}(f(v) - f(u)) = c(v)[f(v) - (Mf)(v)], \quad v \in V.
\]

A function \( f \) is called *harmonic*, if \( M(f) = f \). Equivalently, \( f \) is harmonic if \( \Delta f = 0 \).

**Theorem 4.5.** Let \( f = (f_n) \) be a function on the vertex set of \( G(B) \). Then \( f \) is harmonic if and only if

\[
2f_n = \sum_{u \in V_{n+1}} \bar{p}_n(v, u)f_{n+1}(u) + \sum_{w \in V_{n-1}} \bar{q}_{n-1}(v, w)f_{n-1}(w), \quad \forall n \in \mathbb{N},
\]

or, equivalently, \( 2f_n = \bar{P}_n f_{n+1} + \bar{Q}_{n-1} f_{n-1} \).

Similarly, \( f \) is harmonic if and only if \( D_n f_n = \bar{P}_n f_{n+1} + \bar{Q}_{n-1} f_{n-1} \) where \( D_n \) is the diagonal matrix with entries \( 2q^{(n)}_{vu}, v \in V_n \), on the main diagonal, \( n \geq 1 \).

In particular,

\[
q^{(n)} M = \frac{1}{2} (q^{(n+1)} + q^{(n-1)}), \quad n \in \mathbb{N}.
\]

**Proof.** It follows from Definition 4.2 that the action of a Markov operator \( M \) on a function \( f \) can be represented in the following form:

\[
(Mf_n)(v) = \frac{1}{2} \left( \sum_{u \in V_{n+1}} \bar{p}_n(v, u)f_{n+1}(u) + \sum_{w \in V_{n-1}} \bar{q}_{n-1}(v, w)f_{n-1}(w) \right).
\]

Then we use Definition 4.2 to deduce the first statement.
For the Laplacian operator $\Delta = c(\text{Id} - M)$, we obtain that $\Delta(f) = 0$ if and only if

$$c_n(v)(f_n(v) - \frac{1}{2} \left( \hat{P}_n(f_{n+1})(v) + \hat{Q}_{n-1}(f_{n-1})(v) \right)) = 0.$$  

By Lemma 4.1 we have $2c_n(v)f_n(v) = (Df_n)_v$ and the result follows.

To prove (4.2), we note that $q^{(n+1)} = q^{(n)}\hat{P}_n$ and $q^{(n-1)} = q^{(n)}\hat{Q}_{n-1}$. \(\square\)

Our next focus is on the path-space of a weighted network $G(B)$ defined by a generalized Bratteli diagram $B$ and measures on this space. We recall a few well known notions related to random walks on graphs. Let $\Omega$ be a subset of $V_0 \times V_1 \times \cdots$ of infinite paths $\omega = (v_i)_{i \in \mathbb{N}_0}$ such that $(v_i, v_{i+1})$ is an edge for every $i$. By $\Omega_v$ we denote the subset of $\Omega$ formed by all paths beginning at $v \in V_0$. Let $\xi_n$ be a random variable $\xi_n : \Omega \to V_n$ defined on the path-space $\Omega$: for $\omega = (v_i)$ we set $\xi_n(\omega) = v_n$, $n \in \mathbb{N}_0$.

The Markov kernel $M$ defines a probability measure $m_v$ on cylinder sets of $\Omega_v$ for every $v \in V_0$ as follows:

$$m_v(\omega : \xi_1(\omega) = v_1, \ldots, \xi_n(\omega) = v_n \mid \xi_0 = v) = m(v, v_1) \cdots m(v_{n-1}, v_n).$$

Then it is extended to all Borel sets. The sequence of random variables $(\xi_n)$ defines a Markov chain on $(\Omega_v, m_v)$ such that $m_v(\xi_{n+1} = y \mid \xi_n = z) = m(z, y)$. Let $\lambda = (\lambda_v)$ be a positive probability vector on $V_0$. If $\lambda M = \lambda$, then $\mathbb{P} = \sum_v \lambda_v m_v$ is a Markov measure on $\Omega$.

We will assume that the transition probability kernel $M$ is irreducible. The kernel $M$ generates the random walk on the graph $G(B)$. It is said that $M$ is recurrent if for every vertex $v \in V$ the random walk returns to $v$ infinitely often with probability one. Otherwise it is called transient.

Remark 4.6. We point out the difference between the path-spaces of a generalized Bratteli diagram $B$ and that of the graph $G(B)$. For a Bratteli diagram $B$, the path-space $X_B$ is formed by concatenation of consecutive edges $(e_0, e_1, \ldots)$ such that $e_i$ is an edge between the levels $V_i$ and $V_{i+1}$. The path-space of $G(B)$ is formed by the sequences of vertices of $B$, moreover a vertex $v_i$ is not necessarily from $V_i$ because the transition probability kernel $M$ is defined on both incoming and outgoing edges.

Our next small topic is the finite energy space $\mathcal{H}_E$ for a weighted network $(G(V, E), c)$. Consider functions on vertices $V$ of the graph $G$. It is said that two functions $f$ and $g$ are equivalent if $f - g = \text{const}$.

Definition 4.7. Define the finite energy space $\mathcal{H}_E$ as the set of equivalence classes of functions $f$ on $V$ such that

$$||f||^2_{\mathcal{H}_E} := \frac{1}{2} \sum_{(v,u) \in E} c_{vu}(f(v) - f(u))^2 < \infty.$$  

The space $\mathcal{H}_E$ equipped with this norm is a Hilbert space.

We refer to the papers [Jor12, JP16, BJa] where the reader will find more details about the finite energy space.

Now we adapt this definition to the case of weighted networks $G(B)$ defined by a generalized Bratteli diagram $B$. We note that the coefficient $1/2$ in (4.3) is used because the
The reader can find the references in Subsection 1.4.

We first recall the definition of a transition kernel and introduce linear operators generated by such a kernel.

We apply formula (4.4) and compute the norm of \( T \):

\[
\|f\|^2_{\mathcal{H}_E} = \sum_{n \in \mathbb{N}_0} \sum_{v \in V_n, u \in V_{n+1}} q_v^{(n)}(v, u) (f_n(v) - f_{n+1}(u))^2
\]

where \( f \) is a function from \( \mathcal{H}_E \).

Proposition 4.8. Suppose that a function \( f = (f_n) \) is such that every \( f_n \) belongs to \( \mathcal{H}_n, n \geq 0 \), where the Hilbert space \( \mathcal{H}_n \) is defined in (3.14). Then \( f \in \mathcal{H}_E \) if and only if

\[
\sum_{n \geq 0} \left( \|f_n\|^2_{\mathcal{H}_n} - 2 \langle f_n, T_{\hat{P}_n} (f_{n+1}) \rangle_{\mathcal{H}_n} + \|f_{n+1}\|^2_{\mathcal{H}_{n+1}} \right) < \infty
\]

where the operator \( T_{\hat{P}_n} : \mathcal{H}_{n+1} \to \mathcal{H}_n \) is defined in Proposition 3.19.

Proof. We apply formula (4.4) and compute the norm of \( f \):

\[
\|f\|^2_{\mathcal{H}_E} = \frac{1}{2} \sum_{n \in \mathbb{N}_0} \sum_{v \in V_n, u \in V_{n+1}} q_v^{(n)}(v, u) (f_n(v) - f_{n+1}(u))^2
\]

where \( f = (f_n) \) is a function from \( \mathcal{H}_E \).

5. Graph transition kernels and measures

In this section, we discuss the interplay between transition (probability) kernels and the corresponding measures. While our focus below is on the operator theory of transition kernels, we stress that their application (in subsequent sections) will be to our study of the class of graph dynamical systems which are specified by transition between levels in the graph.

5.1. Definitions of transition kernels and associated measures. We first recall the definition of a transition kernel and introduce linear operators generated by such a kernel. The reader can find the references in Subsection 1.4.
Let \((X_i, \mathcal{A}_i), i = 1, 2,\) be two standard (uncountable) Borel spaces. Without loss of
generality, one can assume that these spaces are two copies of the same standard Borel
space \((X, \mathcal{A})\).

**Definition 5.1.** A map \(R : X_1 \times \mathcal{A}_2 \to [0, \infty)\) is called a **transition kernel** if it satisfies the
following conditions:

(i) for every set \(C \in \mathcal{A}_2\), the function \(x \mapsto R(x, C), x \in X_1,\) is Borel;

(ii) for every \(x \in X_1\), the map \(C \mapsto R(x, C)\) is a \(\sigma\)-finite Borel measure.

If \(R(x, \cdot)\) is a finite measure for every \(x \in X_1\), i.e., \(0 \leq R(x, x_2) < \infty\), then the kernel
\(R\) is called **finite**. If the measure \(R(x, \cdot)\) is probability for every \(x \in X_1\), i.e., \(R(x, X_2) = 1\),
then \(R\) is called a **transition probability kernel**.

Denote by \(\mathcal{F}(X, \mathcal{A}) = \mathcal{F}(X)\) the linear space of bounded Borel functions and by \(M(X, \mathcal{A}) = M(X)\) the set of Borel positive \(\sigma\)-finite measures. We define actions of \(R\) on these sets (with
some abuse of notation, we will use the same letter \(R\) for these actions). For \(f \in \mathcal{F}(X_2, \mathcal{A}_2)\),
we set

\[
(Rf)(x) = \int_{X_2} R(x, dy) f(y).
\]

Then, relation (5.1) determines a positive linear operator \(T_R : \mathcal{F}(X_2, \mathcal{A}_2) \to \mathcal{F}(X_1, \mathcal{A}_1)\).

Applying (5.1) to characteristic functions, we obtain

\[
\text{In particular, } R \text{ is a probability kernel if } R(1) = 1 \text{ where } 1 \text{ is a constant function taking the value } 1.
\]

In contrast to (5.1), the action of \(R\) on measures generates a map from \(M(X_1, \mathcal{A}_1)\) to
\(M(X_2, \mathcal{A}_2)\):

\[
(\mu R)(A) = \int_{X_1} d\mu(x) R(x, A) = \int_{X_1} d\mu(x) \left( \int_A R(x, dy) \right), \quad \mu \in M(X_1, \mathcal{A}_1).
\]

Writing \(R(f)\) and \(\mu R\) as in (5.1) and (5.3), we stress on the similarity with multiplication
of a matrix and row and column vectors.

In the following remark, we provide several examples of transition probability kernels.

**Remark 5.2.** (1) Let \((X_1, \mathcal{A}_i, \nu_i), i = 1, 2,\) be standard probability measure spaces. Then,
defining \(R_1(x, B) = \nu_2(B), B \in \mathcal{A}_2,\) we obtain a constant transition probability kernel. We will see below that this kernel determines the product measure \(\rho = \nu_1 \times \nu_2\) on \(X_1 \times X_2\).

(2) For standard Borel spaces \((X, \mathcal{A}_i),\) take \(\nu_1 = \delta_{x_0}, x_0 \in X_1,\) the Dirac measure on
\((X_1, \mathcal{A}_1)\). Then, setting \(R_0(x_0, \cdot) = \nu_2,\) where \(\nu_2\) is a \(\sigma\)-finite (or probability)
measure on \((X_2, \mathcal{A}_2)\), we obtain a \(\sigma\)-finite (or probability) transition kernel \(R_0 : X_1 \times \mathcal{A}_2 \to [0, 1]\).

(3) Let \(T\) be a positive operator acting on the set of bounded Borel functions \(\mathcal{F}(X, \mathcal{A})\)
on a standard Borel space \((X, \mathcal{A})\). This means that \(T(f) \geq 0\) whenever \(f \geq 0\). It is said
that \(T\) has the **Riesz property** if, for every \(x \in X,\) there exists a Borel measure \(\mu_x\) such that

\[
T(f)(x) = \int_X f(y) d\mu_x(y), \quad f \in \mathcal{F}(X, \mathcal{A}).
\]

The set \((\mu_x : x \in X)\) is called a **Riesz family** of measures corresponding to \(T\).
If $T$ is normalized ($T(1) = 1$), then every measure $\mu_x$ is probability. Observe that the field of measures $x \mapsto \mu_x$ is Borel in the sense that the function $x \mapsto \mu_x(f)$ is Borel for every $f \in \mathcal{F}(X, A)$. As a conclusion, we see that the Riesz family $(\mu_x)$ determines a transition probability kernel $R = R_T : X \times A \to [0, 1]$ by setting $R(x, B) = \mu_x(B)$, $B \in A$.

In what follows, we will consider an interaction between transition kernels and measures. Let $(X, A, \mu)$ be a $\sigma$-finite measures space. Denote by $D(\mu)$ the collection of Borel sets $C \in A$ such that $\mu(C) < \infty$. Then $D(\mu)$ generates the $\sigma$-algebra of Borel sets $A$. By $\mathcal{F}(\mu)$ we denote the linear space of functions $\varphi$ spanned by characteristic functions of the sets from $D(\mu)$, i.e.,

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \quad \mu(A_i) < \infty, \quad \alpha_i \in \mathbb{R}.$$ 

**Definition 5.3.** For given $\sigma$-finite measure spaces $(X_i, A_i, \nu_i), i = 1, 2$, suppose that there are transition kernels

$$P : X_1 \times A_2 \to [0, \infty) \quad \text{and} \quad Q : X_2 \times A_1 \to [0, \infty)$$

such that, for any bounded Borel function $f$,

$$\int_{X_1 \times X_2} d\nu_1(x)P(x, dy) f(x, y) = \int_{X_1 \times X_2} d\nu_2(y)Q(y, dx) f(x, y),$$

or in a short form,

$$(5.4) \quad d\nu_1(x)P(x, dy) = d\nu_2(y)Q(y, dx), \quad (x, y) \in X_1 \times X_2.$$ 

Then these kernels $P$ and $Q$ are called *associated to the measures* $\nu_1$ and $\nu_2$. We will also say that $P$ and $Q$ satisfying (5.4) form a *dual pair* of transition kernels, see Figure (3) for illustration.

For $f(x, y) = \chi_{A}(x)\chi_{B}(y)$, relation (5.4) defines a Borel $\sigma$-finite measure $\rho$ on $(X_1 \times X_2, A_1 \times A_2)$ by the formula

$$(5.5) \quad \rho(A \times B) = \int_A d\nu_1(x)P(x, B) = \int_B d\nu_2(y)Q(y, A)$$

where $A \in A_1, B \in A_2$. More precisely, the measure $\rho$ is defined by (5.4) for Borel sets of finite measure, i.e., $A \in D(\nu_1), B \in D(\nu_2)$, and then it is extended to all Borel sets. Obviously, the measure $\rho$ is probability if the kernel $P$ (or $Q$) and measure $\nu_1$ (or $\nu_2$) are probability. The converse is not true, in general.

**Remark 5.4.** (1) If $P(x, B) = \nu_2(B)$ and $Q(y, A) = \nu_1(A)$ are constant transition kernels (see Remark 5.2 (1)), then the corresponding measure $\rho$ is the product measure $\nu_1 \times \nu_2$.

(2) Suppose that the kernels $P$ and $Q$ are finite. Let $c_1(x) = P(x, X_2)$ and $c_2(y) = Q(y, X_1)$. The functions $c_1, c_2$ are Borel and take finite values for all $x$ and $y$. In general, they may be unbounded. In a number of statements below, we will assume that these function are locally integrable, i.e.,

$$c_i \in L^1_{\text{loc}}(\nu_i) \iff \int_B c_i(\cdot)\,d\nu_i(\cdot) < \infty, \quad B \in \mathcal{D}(\nu_i), \quad i = 1, 2.$$ 

(3) In case of need, we will assume the property $c_i \in L^2_{\text{loc}}(\nu_i)$. This requirement will be used for the study of unbounded operators in $L^2$-spaces.
Consider the product space \( Z := X_1 \times X_2 \) equipped with the product Borel structure \( \mathcal{C} \). Denote by \( \pi_i \) the natural projection from \((Z, \mathcal{C}, \rho)\) onto \((X_i, \mathcal{A}_i, \nu_i)\). Let \( E \subset Z \) be the essential support of the measure \( \rho \). This set is defined up to a null set. There are two measurable partitions of \( E \), \( \xi_1 \) and \( \xi_2 \), where \( \xi_1 \) is formed by the subsets \( \{x, y\} \in E \), \( x \in X_1 \), and \( \xi_2 \) consists of subsets \( \{x, y\} \in E \), \( y \in X_2 \) (measurable partitions are discussed in many books and articles on descriptive set theory and ergodic theory, e.g., [Roh49, Kec95, CFS82]). If needed, we will identify the sections \( E_x \) and \( E_y \) with the corresponding projections \( \pi_1(E_y) \) and \( \pi_2(E_x) \) onto \( X_1 \) and \( X_2 \).

The next result follows directly from Definition 5.3 and Remark 5.4.

**Lemma 5.5.** Let the kernels \( P \) and \( Q \) be as Definition 5.3 with locally integrable functions \( c_1(x) = P(x, X_2) \) and \( c_2(y) = Q(y, X_1) \). Then

\[
\rho(A \times X_2) = \rho \circ \pi_1^{-1}(A) < \infty, \quad \rho(X_1 \times B) = \rho \circ \pi_2^{-1}(B) < \infty
\]

for all \( A \in \mathcal{D}(\nu_1), B \in \mathcal{D}(\nu_2) \). In particular, \( \rho(A \times B) < \infty \) for all \( A \in \mathcal{D}(\nu_1), B \in \mathcal{D}(\nu_2) \).

Moreover, the kernels \( P \) and \( Q \) define linear operators \( T_P \) and \( T_Q \) such that

\[
T_P : \mathcal{F}(\nu_2) \to L^1_{\text{loc}}(\nu_1), \quad T_Q : \mathcal{F}(\nu_1) \to L^1_{\text{loc}}(\nu_2).
\]

**Proof.** Indeed, if \( A \in \mathcal{D}(\nu_1) \), then

\[
\rho \circ \pi_1(A) = \rho(A \times X_2) = \int_A c_1(x) \, d\nu_1(x) < \infty.
\]

Similarly, we have \( \rho \circ \pi_2(B) < \infty \) for \( B \in \mathcal{D}(\nu_2) \).

For the second statement, it suffices to note that the condition \( \rho(A \times B) < \infty \), where \( A \in \mathcal{D}(\nu_1), B \in \mathcal{D}(\nu_2) \), implies that \( P(\chi_B)(x) = P(x, B) \in L^1_{\text{loc}}(\nu_1) \) and \( Q(\chi_A)(y) = Q(y, B) \in L^1_{\text{loc}}(\nu_2) \). These properties are extended to \( \mathcal{F}(\nu_i) \) by linearity. \( \square \)
We remark that (5.6) can be extended by continuity to the set of functions $g$ such that $suppg \subset A$ where $A$ has a finite measure.

**Lemma 5.6.** In conditions of Lemma 5.5,
\[ T_P : L^1(c_2\nu_2) \rightarrow L^1(\nu_1), \quad T_Q : L^1(c_1\nu_1) \rightarrow L^1(\nu_2). \]

**Proof.** For the proof we use (5.4) and (5.5): let \( f \in L^1(c_2\nu_2) \), then
\[
\int_{X_1} |T_P(f)|(x) \, dv_1(x) = \int_{X_1} \left| \int_{X_2} P(x, dy)f(y) \right| \, dv_1(x) \leq \int_{X_1} \int_{X_2} |f(y)| \, P(x, dy) \, dv_1(x) = \int_{X_1} \int_{X_2} |f(y)| \, Q(y, dx) \, dv_2(y) = \int_{X_2} |f(y)|Q(y, X_1) \, dv_2(y) \leq \int_{X_2} |f(y)|c_2(y) \, dv_2(y)
\]
A similar proof works for \( T_Q \).

We recall the following definition and result proved in [Sim12] (we give here a modified statement adapted to our purposes).

**Definition 5.7.** Let \((Z, C, \mu)\) and \((Y, D, \nu)\) be standard \(\sigma\)-finite measure spaces, and let \( \pi : Z \rightarrow Y \) be a measurable function. A system of conditional measures of \( \mu \) with respect to \( \pi \) is a collection of measures \( \{\mu_y : y \in Y\} \), such that
(i) \( \mu_y \) is a Borel measure on \( \pi^{-1}(y) \),
(ii) for every \( B \in C \), \( \mu(B) = \int_Y \mu_y(B) \, dv(y) \), i.e., \( \mu \) is disintegrated by the conditional measures.

**Theorem 5.8** ([Sim12]). Let \((Z, C, \mu)\) and \((Y, D, \nu)\) be as above. For any measurable function \( \pi : Z \rightarrow Y \) such that \( \mu \circ \pi^{-1} \ll \nu \) there exists a uniquely determined system of conditional measures \( \{(\mu_y)_{y \in Y}\} \) which disintegrates the measure \( \mu \).

Under conditions of Theorem 5.8, we will write down
\[ \mu = \int_Y \mu_y \, dv(y). \]
In particular, Theorem 5.8 determines a system of conditional measures for any measure \( \rho \) which is defined as in (5.5).

**Corollary 5.9.** Let \((X, A_i, \nu_i), i = 1, 2, \) be \(\sigma\)-finite standard measure spaces, and let \( \rho \) be defined by (5.5). Then \( \rho \circ \pi_i^{-1} \ll \nu_i \) and the measure \( \rho \) admits its disintegration with respect to \( \pi_1 \) and \( \pi_2 \) the families \((\delta_x \times P(x, \cdot))_{x \in X_1}\) and \((\delta_y \times Q(y, \cdot))_{y \in X_2}\) are the corresponding systems of conditional measures.

We observe that the measures \( P(x, \cdot) \) and \( Q(y, \cdot) \) are defined on \( X_2 \) and \( X_1 \), respectively. So that we need to identify the spaces \( \{x\} \times X_2 \) and \( X_1 \times \{y\} \) with \( X_2 \) and \( X_1 \) if we want to treat these families of measures as conditional measures.
Proof. We first note that if a measure $\rho$ is defined as in (5.5), then the condition $\nu_i(C) = 0$ implies that $\rho \circ \pi_i^{-1}(C) = 0$ for $i = 1, 2$. Hence, we can apply Theorem 5.8. It claims the disintegrating system of conditional measures is unique and therefore $P(x, dy)$ is a measure on $\pi_1^{-1}(x)$ and $Q(y, dx)$ is a measure on $\pi_2^{-1}(y)$. Since $P$ and $Q$ are transition kernels, they satisfy the properties (i) and (ii) of Definition 5.7. □

It follows from Corollary 5.9 that the measure $\rho$ on the product space $X_1 \times X_2$ admits disintegrations $(\rho_x)$ and $(\rho^y)$ with respect to the projection maps $\pi_1 : Z \to X_1$ and $\pi_2 : Z \to X_2$:

$$\rho = \int_{X_1} \rho_x \, d\nu_1(x) = \int_{X_2} \rho^y \, d\nu_2(y).$$

5.2. Three sets. In the next definition we discuss relations between transition kernels $P, Q$ and measures $\nu_1, \nu_2$, and $\rho$.

Definition 5.10. Let $(X_i, A_i), i = 1, 2$, be standard Borel spaces.

(i) For a Borel $\sigma$-finite measure $\rho$ on $(X_1 \times X_2, A_1 \times A_2)$, we define the set

$$F(\rho) = \{ (\nu_1, \nu_2) : \text{there are finite transition kernels } P \text{ and } Q \text{ such that } d\rho(x, y) = d\nu_1(x)P(x, dy) = d\nu_2(y)Q(y, dx) \}.$$

(ii) If $\nu_1$ and $\nu_2$ are $\sigma$-finite measures on Borel spaces $(X_1, A_1)$ and $(X_2, A_2)$, respectively, then define the set

$$L(\nu_1, \nu_2) = \{ \rho : \text{there are finite transition kernels } P \text{ and } Q \text{ such that } (5.5) \text{ holds} \}$$

where $\rho$ is a measure on the product space $Z = X_1 \times X_2$.

(iii) If $P$ and $Q$ are two transition kernels as in Definition 5.1, then we set

$$M(P, Q) = \{ (\nu_1, \nu_2) : \text{such that } d\nu_1(x)P(x, dy) = d\nu_2(y)Q(y, dx) \}.$$

One of our purposes is to characterize the elements of these sets. It will be done in this and next sections.

A few obvious facts are included in the following remark.

Remark 5.11. (1) For a given measure $\rho$ on $Z = X_1 \times X_2$, every pair $(\nu_1, \nu_2) \in F(\rho)$ generates a subset $F_{\nu_1, \nu_2}(\rho)$ of $F(\rho)$ whose elements $(\nu'_1, \nu'_2)$ are the measures absolutely continuous with respect to $\nu_1$ and $\nu_2$ (see also Theorem 5.12 below).

It is important to observe that if one additionally requires that the kernels $P$ and $Q$ in the definition of $F(\rho)$ must be finite, then these kernels are determined uniquely by the measures $\nu_1$ and $\nu_2$. This fact follows, in general, from Theorem 5.8 (though it can be proved directly).

(2) It follows from Theorem 5.8 that the set $F(\rho)$ is not empty for any measure $\rho$ as the pair $(\rho \circ \pi_1^{-1}, \rho \circ \pi_2^{-1})$ is always in $F(\rho)$.

(3) On the other hand, in the definition of the set $L(\nu_1, \nu_2)$ one can take different pairs of kernels $P, Q$ and $P', Q'$ such that the corresponding measures $\rho$ and $\rho'$ are in the same set $L(\nu_1, \nu_2)$. Moreover, these pairs of kernels can be chosen such that $\rho$ and $\rho'$ are mutually singular measure.
(4) It follows from the definitions of the sets \( F(\rho) \) and \( L(\nu_1, \nu_2) \) that if \((\mu_1, \mu_2)\) is in \( F(\rho) \), then the set \( L(\mu_1, \mu_2) \) will contain the measure \( \rho \). Conversely, if \( \rho \in L(\nu_1, \nu_2) \), the \((\nu_1, \nu_2) \in F(\rho) \).

(5) The following question is interesting: When does the set \( M(P, Q) \) contain a pair of positive measures?

It can be shown that, even in the case of matrices, there are matrices \( P \) and \( Q \) such that \( M(P, Q) \) contains only the pair \((0, 0)\). For this, one can take \( P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

Suppose that \((\nu'_1, \nu'_2)\) is a pair of measures on \( X_1 \) and \( X_2 \) that defines the same measure \( \rho \) as the pair \((\nu_1, \nu_2)\), i.e., \((\nu'_1, \nu'_2) \in F(\rho) \). What can be said about relations between these two pairs \((\nu_1, \nu_2)\) and \((\nu'_1, \nu'_2)\)?

**Theorem 5.12.** Suppose that a measure \( \rho \) is defined on \((X_1, A_1, \nu_1) \times (X_2, A_2, \nu_2) \) as in (5.5). Suppose that some measures \( \nu'_1 \) and \( \nu'_2 \) form a pair from \( F(\rho) \). Then the corresponding measures \( \nu_1, \nu'_1 \) and \( \nu_2, \nu'_2 \) are pairwise equivalent, i.e., there are positive functions \( f_1 \) and \( f_2 \) such that

\[
\frac{d\nu'_i}{d\nu_i} = f_1 \frac{d\nu_i}{d\nu_i}, \quad i = 1, 2.
\]

**Proof.** Since \((\nu_1, \nu_2)\) and \((\nu'_1, \nu'_2)\) are in \( F(\rho) \), there are finite transition kernels \( P, Q \) and \( P', Q' \), respectively, such that (5.4) holds. Then

\[
\frac{d\nu'_i}{d\nu_i}(x) = \frac{P(x, dy)}{P'(x, dy)}, \quad \frac{d\nu'_2}{d\nu_2}(y) = \frac{Q(y, dx)}{Q'(y, dx)}.
\]

We see that the quotients of measures in the right-hand sides of the both equations do not depend on \( dy \) and \( dx \), respectively. Hence one can take apply these relations to any sets. We consider the positive Borel functions \( c_1(x) = P(x, X_2) \) and \( c'_1(x) = P'(x, X_2) \). Then relation (5.5) leads to

\[
\rho(A \times X_2) = \rho \circ \pi_1^{-1}(A) = \int_A c_1(x) \, d\nu_1(x)
\]

or

\[
c_1(x) = \frac{d\rho \circ \pi_1^{-1}}{d\nu_1}(x).
\]

In the same way we obtain similar relations for \( \nu'_1 \) and \( P' \) and deduce that

\[
c'_1(x) = \frac{d\rho \circ \pi_1^{-1}}{d\nu'_1}(x).
\]

Hence, setting \( f_1(x) = c_1(x)(c'_1(x))^{-1} \), we see that \( f_1(x) \) is the Radon-Nikodym derivative for the measures \( \nu'_1 \) and \( \nu_1 \).

Analogously, we can define \( c_2(y) = Q(y, X_1) \), \( c'_2(y) = Q'(y, X_1) \) and show that

\[
c_2(y) = \frac{d\rho \circ \pi_2^{-1}}{d\nu_2}(y).
\]

Therefore \( f_2(y) = c_2(y)(c'_2(y))^{-1} \) is the Radon-Nikodym derivative for \( \nu_2 \) and \( \nu'_2 \). Also, the above argument proves that the measures \( P(x, \cdot), P'(x, \cdot) \) and \( Q(y, \cdot), Q'(y, \cdot) \) are equivalent. Moreover, the functions \( f_1(x) \) and \( f_2(y) \) give the Radon-Nikodym derivatives of these measures. This proves the result. \( \square \)
**Corollary 5.13.** Let \( \rho \) be a \( \sigma \)-finite measure on \((X_1 \times X_2, A_1 \times A_2)\), and a pair of measures \((\nu_1, \nu_2)\) is in \( F(\rho) \). Then there exists a pair \((\nu_1', \nu_2') \in F(\rho)\) such that \( \nu_1' \sim \nu_1 \), and the corresponding kernels \( P' \) and \( Q' \) are probability.

**Proof.** It follows from \((\nu_1, \nu_2) \in F(\rho)\) that there exist finite transition kernels \( P(x, \cdot) \) and \( Q(y, \cdot) \) with the property \( d\nu_1(x)P(x, dy) = d\nu_2(y)Q(y, dx) \). For \( c_1(x) = P(x, X_2) \) and \( c_2(y) = Q(y, X_1) \), we define \( d\nu_1'(x) = c_1(x)d\nu_1(x) \) and \( d\nu_2'(y) = c_2(y)d\nu_2(y) \). Then \((\nu_1', \nu_2') \in F(\rho)\) where the corresponding kernels

\[
P'(x, dy) = \frac{1}{c_1(x)}P(x, dy), \quad Q'(y, dx) = \frac{1}{c_2(y)}Q(y, dx)
\]

are probability kernels. \( \square \)

**Lemma 5.14.** For given measure spaces \((X_i, A_i, \nu_i), i = 1, 2, \) let \( \rho \) be a measure from the set \( L(\nu_1, \nu_2) \). Then, for \( \nu_1 \)-a.e. \( x \in X_1 \), the measure \( P(x, \cdot) \) is absolutely continuous with respect to \( \nu_2 \). Similarly, \( Q(y, \cdot) \ll \nu_1 \) for \( \nu_2 \)-a.e \( y \in X_2 \).

**Proof.** Let \( C \) be a subset of \( X_2 \) with \( \nu_2(C) = 0 \). Then

\[
\rho(X_1 \times C) = \int_C Q(y, X_1) \, d\nu_2(y) = 0.
\]

Therefore \( \int_{X_1} P(x, C) \, d\nu_1(x) = 0 \). This means that \( P(x, C) = 0 \) for \( \nu_1 \)-a.e. \( x \in X_1 \), hence \( P(x, \cdot) \ll \nu_2 \).

The other result is proved analogously. \( \square \)

For a \( \sigma \)-finite measure \( \rho \) on \( X_1 \times X_2 \) and a fixed set \( B \in A_2 \) such that \( \rho(X_1 \times B) > 0 \), we define a \( \sigma \)-finite measure \( \tau_B \) on \((X_1, A_1)\) by setting \( \tau_B(C) = \rho(C \times B) \). Similarly, we define a measure \( \tau^A \) on \((X_2, A_2)\) when a set \( A \in A_1 \) is fixed. It follows from these definitions that \( \tau_B(A) = \tau^A(B) \).

**Theorem 5.15.** Let \( \rho \) be a measure on the product space \((X_1 \times X_2, A_1 \times A_2)\). Then a pair of measures \((\nu_1, \nu_2)\) defined on \( X_1 \) and \( X_2 \), respectively, is in \( F(\rho) \) if and only if

\begin{equation}
(5.7) \quad \rho \circ \pi_i^{-1} \ll \nu_1, \quad \rho \circ \pi_2^{-1} \ll \nu_2.
\end{equation}

**Proof.** If \((\nu_1, \nu_2) \in F(\rho)\), then there are finite transition kernels \( P \) and \( Q \) such that \( d\rho(x, y) = d\nu_1(x)P(x, dy) = d\nu_2(y)Q(y, dx) \). Then the measures \( \rho \circ \pi_i^{-1} \) are defined by

\[
\rho \circ \pi_1^{-1}(A) = \int_A P(x, X_2) \, d\nu_1(x), \quad \rho \circ \pi_2^{-1}(B) = \int_B Q(y, X_1) \, d\nu_2(y).
\]

These equations show that \( \rho \circ \pi_i^{-1} \ll \nu_i, i = 1, 2 \). In particular, we can use the kernels \( P \) and \( Q \) to represent the measures

\[
\tau_B(A) = \int_A P(x, B) \, d\nu_1(x), \quad \tau^A(B) = \int_B Q(y, A) \, d\nu_2(y)
\]

To prove that the converse holds for a given \( \rho \), we take the measures \( \tau_B(\cdot) = \rho(\cdot \times B) \) and \( \tau^A(\cdot) = \rho(A \times \cdot) \) where the sets \( B \in A_2 \) and \( A \in A_1 \) are of finite measure. Then, it is clear from \((5.7)\) that

\[
\tau_B \ll \rho \circ \pi_1^{-1} \ll \nu_1, \quad \tau^A \ll \rho \circ \pi_2^{-1} \ll \nu_2.
\]
Hence, we can define the Radon-Nikodym derivatives

\[
P(x, B) = \frac{d\tau_B}{d\nu_1}(x), \quad Q(y, A) = \frac{d\tau_A}{d\nu_2}(y),
\]

where \( B \in \mathcal{A}_2 \) and \( A \in \mathcal{A}_1 \) are fixed subsets. To see that \( P \) and \( Q \) are finite transition kernels, one needs to check that they define finite measures \( P(x, \cdot) \) and \( Q(y, \cdot) \) on \( X_2 \) and \( X_1 \) respectively. We remark that \( \tau_{B_1 \cup B_2}(A) = \rho(A \times (B_1 \cup B_2)) = \tau_{B_1}(A) + \tau_{B_2}(A) \) where \( B_1 \cap B_2 = \emptyset \). This property can be extended to sigma-additivity using the fact that \( \rho \) is a measure on the product space. The functions \( P(x, X_2) \) and \( Q(y, X_1) \) are finite a.e. because they are the Radon-Nikodym derivatives. Hence, using the kernels \( P \) and \( Q \), we see that \((\nu_1, \nu_2) \in F(\rho)\).

\[\square\]

**Theorem 5.16.** Let \((X_1, \mathcal{A}_1)\) and \((X_2, \mathcal{A}_2)\) be standard Borel spaces, and let \( P : X_1 \times \mathcal{A}_2 \to [0, 1] \) be a probability transition kernel. Suppose that \( \nu_1 \) is a Borel \( \sigma \)-finite measure on \((X_1, \mathcal{A}_1)\). Then there exist a Borel \( \sigma \)-finite measure \( \rho \) on \( Z = X_1 \times X_2 \), a Borel \( \sigma \)-finite measure \( \nu_2 \) on \((X_2, \mathcal{A}_2)\), and a transition kernel \( Q : X_2 \times \mathcal{A}_1 \to [0, 1] \) such that \( \rho \circ \pi_i^{-1} = \nu_i, i = 1, 2, \)

\[
d\nu_1(x)P(x, dy) = d\nu_2(y)Q(y, dx),
\]

and

\[
\nu_1 P = \nu_2, \quad \nu_2 Q = \nu_1.
\]

The objects \( \rho, \nu_2, \) and \( Q \) are uniquely defined by \( P \) and \( \nu_1 \).

**Proof.** We first define a measure \( \rho \) on rectangles \( A \times B \in \mathcal{A}_1 \times \mathcal{A}_2 \) from \( Z \) by setting

\[
\rho(A \times B) := \int_A (\{\delta_x\} \times P(x, B)) \ d\nu_1(x)
\]

where \( A \) is a set of finite measure \( \nu_1 \), and \( B \in \mathcal{A}_2 \) (we recall that \( P(x, B) \leq P(x, X_2) = 1 \)). Then \( \rho \) can be extended to a Borel \( \sigma \)-finite measure on the the product \( \sigma \)-algebra \( \mathcal{A}_1 \times \mathcal{A}_2 \). By construction, we have \( \rho \circ \pi_i^{-1} = \nu_1 \) (see also Proposition 5.25 for more details).

Let \( E \) be the essential support of the measure \( \rho \). Define \( \nu_2 := \rho \circ \pi_2^{-1} \). We can disintegrate \( \rho \) with respect to the measurable partitions \( \{E_x : x \in X_1\} \) and \( \{E^y : y \in X_2\} \) and measures \( \nu_1 \) and \( \nu_2 \). By Theorem 5.8,

\[
\int_{X_1} \rho_x \ d\nu_1(x) = \rho = \int_{X_2} \rho^y \ d\nu_2(y).
\]

By uniqueness of disintegrating family of measures, \( \rho_x(\cdot) \) is supported by \( E_x \) and can be identified with \( P(x, \cdot) \). The measure \( \rho^y \) is supported by the set \( E^y \). We use the identification of \( E^y \) with its projection onto \( X_1 \) to define a transition kernel \( Q(y, A) \). It follows from (5.12) that we can set \( Q : X_2 \times \mathcal{A}_1 \to [0, 1] \) by letting

\[
\{\delta_y\} \times Q(y, A) = \rho^y(A), \quad A \in \mathcal{A}_1.
\]

In general, the kernel \( Q : X_2 \times \mathcal{A}_1 \to [0, 1] \) need not to be finite.

Since the pairs \((\nu_1, P)\) and \((\nu_2, Q)\) generate the same measure \( \rho \), we see that equality (5.9) holds.
It remains to check that (5.10) holds:
\[
\nu_2(B) = \rho(X_1 \times B)
\]
\[
= \int_{X_1} P(x, B) \, d\nu_1(x)
\]
\[
= \int_B d(\nu_1 P)
\]
\[
= (\nu_1 P)(B),
\]
and
\[
(\nu_2 Q)(A) = \int_{X_2} Q(x A) \, d\nu_2(y)
\]
\[
= \int_{X_2} Q(y, A) \, d\nu_2(y)
\]
\[
= \int_A P(x, X_2) \, d\nu_1(x)
\]
\[
= \nu_1(A).
\]

We used (5.2) here.

The fact that the measures \(\rho, \nu_2\), and the kernel \(Q\) are uniquely defined by \(P\) and \(\nu_1\) is based on the uniqueness of disintegration and follows directly from (5.11) and (5.13). □

Using a similar approach, we can show that the following corollaries hold. The proofs are straightforward and follows from the statements considered above in this section.

**Corollary 5.17.** In conditions of Theorem 5.16, suppose that the measure \(\nu\) and kernel \(P\) are probability. Then the measures \(\rho\) and \(\nu_2\), and the transition kernel \(Q\) are also probability.

If \(c_1(x) = P(x, X_2)\) is in \(L^1(\nu_1)\), then the kernel \(Q(y, \cdot)\) is finite. Moreover, the function \(c_2(y) = Q(y, X_1)\) is locally integrable with respect to \(\nu_2\).

**Corollary 5.18.** Suppose that \(\rho\) is a \(\sigma\)-finite Borel measure on the direct product \(Z\) of standard Borel spaces \((X_i, A_i)\), \(i = 1, 2\). Let \(\nu_i := \rho \circ \pi_i^{-1}\) be measures on \((X_i, A_i)\). Suppose that \(\rho\) is disintegrated, \(\rho = \int_{X_1} \rho_x \, d\nu_1(x) = \int_{X_2} \rho^y \, d\nu_2(y)\) with respect to the projections \(\pi_1\) and \(\pi_2\) such that the fiber measures \(\rho_x\) and \(\rho^y\) are finite. Then there are finite transition kernels \(P : X_1 \times A_2 \to \mathbb{R}_+\) and \(Q : X_2 \times A_1 \to \mathbb{R}_+\) such that (5.10) holds and
\[
d\rho(x, y) = P(x, dy) \, d\nu_1(x) = Q(y, dx) \, d\nu_2(y).
\]

**Corollary 5.19.** (1) Let \((X_i, A_i, \nu_i), i = 1, 2\) be two \(\sigma\)-finite measure spaces and let \(P : X_1 \times A_2\) be a transition kernel such that \(P(x, \cdot) \ll \nu_2\) for \(\nu_1\)-a.e. \(x\). Denote
\[
\varphi(x, y) = \frac{P(x, dy)}{d\nu_2(y)}.
\]

Then the formula
\[
Q(y, A) = \int_A \varphi(x, y) \, d\nu_1(x), \quad A \in A_1,
\]
determines a dual transition kernel associated to \((\nu_1, \nu_2)\).
(2) For a given pair of kernels $P$ and $Q$ such that $P(x, \cdot) \ll \nu_1$ and $Q(y, \cdot) \ll \nu_2$, we have that $(\nu_1, \nu_2) \in M(P, Q)$ if and only if

$$\frac{P(x, dy)}{d\nu_2(y)} = \frac{Q(y, dx)}{d\nu_1(x)}.$$ 

Proof. (1) Since all objects are well defined, we need to check only that, for any Borel bounded function $f(x, y)$,

$$\iint_{X \times Y} d\nu_1(x)P(x, dy)f(x, y) = \iint_{X \times Y} d\nu_2(y)Q(y, dx)f(x, y).$$

It suffices to show that this relation holds for $f(x, y) = \chi_A(x)\chi_B(y)$:

$$\int_B d\nu_2(y)Q(y, A) = \int_B d\nu_2(y)\int_A \varphi(x, y) d\nu_1(x) = \int_B \int_A P(x, dy)\frac{d\nu_1(x)}{d\nu_2(y)} = \int_B \int_A P(x, dy)d\nu_1(x) = \int_A d\nu_1(x)P(x, B).$$

Statement (2) follows from (1). □

**Corollary 5.20.** Let $(X_i, A_i, \nu_i)$ be standard $\sigma$-finite measure spaces, and let $P : X_1 \times A_2 \to [0, \infty)$ and $Q : X_2 \times A_1 \to [0, \infty)$ be transition kernels, $i = 1, 2$. The following are equivalent:

(i) there exists a measure $\rho$ on $X_1 \times X_2$ such that

$$\rho(A \times B) = \int_A P(x, B) d\nu_1(x) = \int_B Q(y, A) d\nu_2(y), \quad A \in A_1, B \in A_2,$$

(ii) $d\nu_1(x)P(x, dy) = d\nu_2(y)Q(y, dx)$,

(iii) there exists a measure $\rho$ on $X_1 \times X_2$ such that, for any $B \in A_2$ and $A \in A_1$, the kernels $P$ and $Q$ are Radon-Nikodym derivatives:

$$P(x, B) = \frac{\rho(dx, B)}{d\nu_1(x)}, \quad Q(y, A) = \frac{\rho(A, dy)}{d\nu_2(y)}.$$

**Proposition 5.21.** Let $(X_i, A_i, \nu_i)$, $i = 1, 2$, be two $\sigma$-finite measure spaces, and let $P$ and $Q$ be a pair of dual transition kernels, i.e., $d\nu_1(x)P(x, dy) = d\nu_2(y)Q(y, dx)$. Suppose that $P(\chi_B) \in L^1(\nu_1)$ for every $B \in D(\nu_2)$. Then $c_2(y) = Q(y, X_1)$ is finite a.e. and locally integrable.

Proof. The condition that $P$ and $Q$ form a dual pair can be written in the form

$$\int_{X_1} \int_{X_2} d\nu_1(x)P(x, dy)f(x, y) = \int_{X_1} \int_{X_2} d\nu_2(y)Q(y, dx)f(x, y)$$

for every bounded Borel function. Take a set $B \in D(\nu_2)$. Then for $f(x, y) = \chi_B(y)$, we get

$$\int_{X_1} d\nu_1(x) \int_{X_2} P(x, dy)\chi_B(y) = \int_{X_2} d\nu_2(y)\chi_B(y) \int_{X_1} Q(y, dx).$$
or
\[ \int_{X_1} d\nu_1(x) \, P(x, B) = \int_B d\nu_2(y) \, c_2(y). \]
Since the left hand side here is finite, we conclude that \( c_2 \) is finite a.e. and locally integrable.

\[ \square \]

**Remark 5.22.** Obviously, the condition \( P(\chi_B) \in L^1(\nu_1) \) for every \( B \in \mathcal{D}(\nu_2) \) from Proposition 5.21 holds for a probability transition kernel \( P \). It follows that the dual transition kernel \( Q \) is finite. By normalizing \( Q \) and replacing the measure \( \nu_2 \) by an equivalent measure, we can always assume that the two kernels \( P \) and \( Q \) are probability.

5.3. **More results on kernels and measures; examples.** Let \( \rho \) and \( \rho' \) are two equivalent measures on \( Z = X \times X_2 \). What can be said about the relation between the transition kernels \( P, Q \) and \( P', Q' \) generated by these measures? In the following lemma, we consider a partial case when the Radon-Nikodym derivative for the measures \( \rho \) and \( \rho' \) has some special form.

**Lemma 5.23.** (1) Let \( \rho \) and \( \rho' \) are two equivalent measures on \( Z = X_1 \times X_2 \). Suppose that there exist two Borel functions \( f : X_1 \to (0, \infty) \) and \( g : X_2 \to (0, \infty) \) such that
\[ \frac{d\rho'}{d\rho}(x,y) = f(x)g(y). \]
Let \( P(x, \cdot) = \rho_x(\cdot) \) and \( P'(x, \cdot) = \rho'_x(\cdot) \) be the corresponding transition kernels, see Corollary 5.18. Then
\[ P'(x, dy) = k_x g(y) \]
where the coefficient \( k_x = f(x) \frac{d\nu_1}{d\nu'_1}(x) \) depends on \( x \) only.

In particular, if \( \rho' \) and \( \rho \) are probability measures such that \( d\rho'(x,y) = f(x)d\rho(x,y) \), then \( P'(x, dy) = P(x, dy) \) and \( f(x) \) is the Radon-Nikodym derivative \( \frac{d\nu'}{d\nu}(x) \).

(2) If \( d\nu'_1(x) := \varphi(x)d\nu_1(x) \) where \( \varphi \) is a Borel positive function, then, in notation of Corollary 5.18, we obtain
\[ P(x, dy)d\nu'_1(x) = Q'(y, dx)d\nu_2(y) \]
where \( Q'(y, dx) := \varphi(x)Q(y, dx) \).

**Proof.** It follows from Corollary 5.20 (ii) that
\[ P'(x, dy)d\nu'_1(x) = f(x)g(y)P(x, dy)d\nu_1(x). \]
This equality can be written as in (1). Assuming that \( g(y) = 1 \) and using the fact that the measures \( \rho \) and \( \rho' \) are probability, we obtain the other statement from (1).

Relation (2) follows from Corollary 5.20. \[ \square \]

**Lemma 5.24.** Suppose that \( P \) and \( P' \) are finite transition kernels on \( X_1 \times A_2 \). Let \((\nu, P)\) and \((\nu', P')\) be pairs that determine measures \( \rho = \int P(x, \cdot) \, d\nu \) and \( \rho' = \int P'(x, \cdot) \, d\nu' \) on \( X_1 \times X_2 \). Then \( \rho = \rho' \) if and only if \( \int_A c \, d\nu = \int_A c' \, d\nu' \) where \( c(x) = P(x, X_2) \), \( c'(x) = P'(x, X_2) \). In particular, if \( P \) and \( P' \) are probability, then \( \rho = \rho' \) if and only if \( \nu = \nu' \).

**Proof.** We leave the proof to the reader. \[ \square \]
Proposition 5.25. Let \((X_i, \mathcal{A}_i, \nu_i), i = 1, 2,\) be standard measure spaces with \(\sigma\)-finite measures \(\nu_1\) and \(\nu_2\), and let \(P : X_1 \times \mathcal{A}_2 \to [0, 1]\) and \(Q : X_2 \times \mathcal{A}_1 \to [0, 1]\) be transition kernels such that

\[
P(x, dy) d\nu_1(x) = Q(y, dx) d\nu_2(y).
\]

Then, for the measure \(d\rho(x, y) = P(x, dy) d\nu_1(x) = Q(y, dx) d\nu_2(y)\) on \(X_1 \times X_2\), we have

(i) \(\nu_1P = \nu_2 \iff Q(y, X_1) = 1\) for \(\nu_2\)-a.e. \(y\),

\[
\nu_2Q = \nu_1 \iff P(x, X_2) = 1\) for \(\nu_1\)-a.e. \(x\);
\]

(ii) \(\nu_1 = \rho \circ \pi_1^{-1} \iff P(x, X_2) = 1\) for \(\nu_1\)-a.e. \(x\),

\[
\nu_2 = \rho \circ \pi_2^{-1} \iff Q(y, X_1) = 1\) for \(\nu_2\)-a.e. \(y\);
\]

(iii) \(\nu_1P = \nu_2 \iff \nu_1 = \rho \circ \pi_1^{-1}\),

\[
\nu_2Q = \nu_1 \iff \nu_2 = \rho \circ \pi_2^{-1}.
\]

Proof. In the proof, we use the measure \(\rho\) on \(X_1 \times X_2\) defined by \(\nu_1, P\) or \(\nu_2, Q\):

\[
\rho(A \times B) = \int_A P(x, B) d\nu_1(x) = \int_B Q(y, A) d\nu_2(y)
\]

where \(B \in \mathcal{A}_2\) and \(A \in \mathcal{A}_1\) are any Borel sets.

For (ii),

\[
\rho \circ \pi_1^{-1}(A) = \rho(A \times X_2) = \int_A P(x, X_2) d\nu_1(x).
\]

If \(P(x, X_2 = 1\) a.e., then \(\rho \circ \pi_1^{-1} = \nu_1\). Conversely, if for any \(A \in X_1\), \(\rho \circ \pi_1^{-1}(A) = \nu_1(A)\), then \(P(x, X_2 = 1\) a.e. (otherwise, equation (5.16) immediately leads to a contradiction). The other statement in (ii) is proved similarly.

For (iii), suppose that \(\nu_i = \rho \circ \pi_i^{-1}, i = 1, 2\). Then

\[
(\nu_1P)(B) = \int_{X_1} P(x, \chi_B) d\nu_1 = \rho(X_1 \times B) = \rho \circ \pi_2^{-1}(B) = \nu_2(B).
\]

Conversely, if \(\nu_1P = \nu_2\), the above relation proves that \(\nu_2 = \rho \circ \pi_2^{-1}\). The other statement of (iii) can be checked analogously.

Statement (i) follows automatically from (ii) and (iii). \(\Box\)

We consider now the notion of the \textit{product of transition kernels}. Let \((X_i, \mathcal{A}_i)\) be standard Borel spaces, \(i = 1, 2, 3\). Suppose that \(P_i : X_i \times \mathcal{A}_{i+1} \to [0, 1], i = 1, 2,\) is a \(\sigma\)-finite transition kernel. For \(x_1 \in X_1, C \in \mathcal{A}_3\), we define

\[
(P_1P_2)(x_1, C) := \int_{X_2} P_1(x_1, dx_2)P_2(x_2, C).
\]

We define the action of \(P_1P_2\) on Borel functions and measures: for \(f \in \mathcal{F}(X_3, \mathcal{A}_3)\), we set

\[
P_1P_2(f)(x_1) = \int_{X_2} \int_{X_3} P_1(x_1, dx_2)P_2(x_2, dx_3)f(x_3)
\]

is a Borel function on \((X_1, \mathcal{A}_1)\). If \(\nu_1\) is a Borel measure on \((X_1, \mathcal{A}_1)\), then one can consequently define \(\nu_2 = \nu_1P_1\) and \(\nu_3 = \nu_2P_2 = \nu_1P_1P_2\).

Hence, we have two measures \(\rho_1\) and \(\rho_2\) on \(X_1 \times X_2\) and \(X_2 \times X_3\), respectively, such that

\[
d\rho_1(x_1, x_2) = P_1(x_1, dx_2)d\nu_1(x_1), \quad d\rho_2(x_2, x_3) = P_2(x_2, dx_3)d\nu_2(x_2).
\]
On the other hand, the transition probability kernel \( P = P_1 P_2 \) and the initial measure \( \nu_1 \) can be used to define a probability measure \( \rho \) on \((X_1 \times X_3, A_1 \times A_3)\).

**Lemma 5.26.** In the above notation,
\[
d\rho(x_1, x_3) = \int_{X_2} d\rho_1(x_1, x_2) P_2(x_2, dx_3).
\]

**Proof.** Indeed, it follows from the relations
\[
P(x_1, dx_3) = \int_{X_2} P_1(x_1, dx_2) P_2(x_2, dx_3).
\]
and
\[
d\rho(x_1, x_3) = d\nu_1(x_1) P(x_1, dx_3).
\]
\[\Box\]

**Example 5.27.** We consider a particular case of atomic measures on a standard Borel space. Suppose that we have two copies, \((X_i, A_i)\), \(i = 1, 2\), of a standard Borel space \((X, A)\). According to Theorem 5.16, we begin with a probability kernel \( P(x, B) \) and a measure \( \nu_1 \) and define other objects.

**Lemma 5.28.** Let \( P(x, \cdot) \) be a finite transition kernel and \( \nu_1 = \delta_{x_0}, x_0 \in X_1 \). Then the pair \((\delta_{x_0}, P)\) defines the measure \( \rho = \delta_{x_0} \otimes P \) on \(X_1 \times X_2\) by the formula
\[
\rho(A \times B) = \chi_A(x_0) P(x_0, B).
\]
Moreover, the projection of \( \rho \) onto \(X_2\) is the measure \( \nu_2(B) = P(x_0, B) \), and the dual kernel \( Q(y, \cdot) \) is \( \delta_{x_0}(\cdot) \).

**Proof.** We compute the measure \( \rho \) as in Theorem 5.16:
\[
\rho(A \times B) = \int_A P(x, B) \delta_{x_0}(dx)
= \begin{cases} 
P(x_0, B), & x_0 \in A \\
0, & x_0 \notin A \end{cases}
= \chi_A(x_0) P(x_0, B)
= \int_B \chi_A(x_0) P(x_0, dy).
\]
(5.17)

Therefore, we can take the projection of \( \rho \) onto \(X_2\) and find
\[
\nu_2(B) = \rho(X_1 \times B) = \chi_{X_1}(x_0) P(x_0, B) = P(x_0, B).
\]
On the other hand, when we use the dual kernel \( Q(y, A) \) to calculate \( \rho(A \times B) \), we have
\[
\rho(A \times B) = \int_B Q(y, A) \nu_2(dy) = \int_B Q(y, A) P(x_0, dy).
\]
(5.18)

By uniqueness of disintegration, it follows from (5.17) and (5.18) that \( Q(y, A) = \delta_{x_0}(A) \). \[\Box\]

The probability transition kernels \( P \) and \( Q \), which are defined as in Lemma 5.28, act on functions as follows:
\[
(Pf)(x) = \int_{X_2} P(x, dy) f(y), \quad f \in \mathcal{F}(X_2, A_2),
\]
\[(Qg)(y) = \int_{X_1} Q(y, dx)g(x) = g(x_0), \quad g \in \mathcal{F}(X_1, A_1).\]

### 6. Actions of transition kernels in \(L^2\)-spaces

We introduce a new duality framework for transition operators. While the setting below is that of pairs of measure spaces in duality, our applications (in section 8 below) will be to path-space measures and the corresponding Markov processes.

In this section, we will focus on the actions of operators \(T_P\) and \(T_Q\) in \(L^2\)-spaces. It turns out that they produce the self-adjoint operators \(T_{PQ}\) and \(T_{QP}\) acting in \(L^2(\nu_2)\) and \(L^2(\nu_1)\). Our analysis of such dual pairs of operators is in the framework of Hilbert space. This holds both for the results in the present Section 6, and Section 7 below. Indeed, these duality results will serve as key tools in our subsequent introduction of non-stationary Markov processes, and their harmonic analysis; see Section 8 below. In more detail, the present duality results will be used at each level in discrete-time Markov process dynamics.

#### 6.1. Symmetric measures and symmetric operators.

In this subsection we give a few definitions and results about symmetric measures and associated with them operators. We refer to our previous works [BJb, BJa, BJ19a] where the reader can find more details.

Let \(\lambda\) be a measure on the Cartesian product \((X \times X, A \times A)\) of a standard Borel space \((X, A)\) such that

\[\lambda(A \times B) = \lambda(B \times A), \quad A, B \in A.\]

Then \(\lambda\) is called a symmetric measure.

A positive linear operator \(R, \mathcal{F}(X, A) \to \mathcal{F}(X, A)\) is called symmetric if it satisfies the following relation:

\[\int_X fR(g) \, d\mu = \int_X R(f)g \, d\mu,\]

where \(\mu\) is a \(\sigma\)-finite measure on \((X, A)\) and \(f, g \in \mathcal{F}(X, A)\).

**Remark 6.1.** (1) It can be easily seen that the operator \(R\) is symmetric if and only if the measure

\[\lambda(A \times B) = \int_X \chi_A R(\chi_B) \, d\mu\]

is symmetric.

(2) When \(R\) is considered as an operator (in general, unbounded) acting in \(L^2(\nu)\), then we can use the methods of operators in Hilbert spaces for the study of its properties, see Theorem 6.3 and Theorem 7.6 below for further details.

(3) If \(\lambda\) is a symmetric measure on the product space \(X \times X\), then the operator \(R\) is not uniquely determined by \(\lambda\), see e.g., Theorem 7.6.

Let \((X, A, \mu)\) be a \(\sigma\)-finite measure space, and let \(\lambda\) be a symmetric measure on \(X \times X\) supported by a symmetric set \(E\). Let \(x \mapsto \lambda_x\) be the measurable family of conditional measures on \((X, A)\) that disintegrates \(\lambda\). We assume that the function \(c(x) = \lambda_x(X)\) is finite for \(\mu\)-a.e. \(x\).

**Definition 6.2.** For a symmetric measure \(\lambda\) on \((X \times X, A \times A)\), we define two linear operators \(R\) and \(R_1\) acting on the space of bounded Borel functions \(\mathcal{F}(X, A)\).
(i) The symmetric operator $R$:

\[
R(f)(x) := \int_V f(y) \, d\lambda_x(y) = \lambda_x(f).
\]

(ii) The Markov operator $R_1$:

\[
R_1(f)(x) = \frac{1}{c(x)} R(f)(x)
\]

or

\[
R_1(f)(x) := \frac{1}{c(x)} \int_V f(y) \, d\rho_x(y) = \int_V f(y) \, P_1(x, dy)
\]

where $P_1(x, dy)$ is the probability measure obtained by normalization of $d\lambda_x(y)$, i.e.

\[
P_1(x, dy) := \frac{1}{c(x)} d\lambda_x(y).
\]

In the case when the operator $R$ is considered in a $L^2$-space, we will have to deal with self-adjoint operators which are unbounded, in general. We refer to the well known books [DS88, Kat95, Sch12] on unbounded operators and their self-adjoint extensions.

**Theorem 6.3 ([BJa]).** For a standard measure space $(X, \mathcal{A}, \mu)$, let $\lambda$ be a symmetric measure on $X \times X$ such that the function $c(x) = \lambda_x(X)$ is locally integrable. Let $d\nu(x) = c(x) d\mu(x)$ be the $\sigma$-finite measure on $(X, \mathcal{A})$ equivalent to $\mu$, and let the operators $R$ and $R_1$ be defined as in Definition 6.2.

1. Suppose that the function $x \mapsto \rho_x(A) \in L^2(\mu)$ for every $A \in \mathcal{D}(\mu)^2$. Then $R$ is a symmetric unbounded operator in $L^2(\mu)$, i.e.,

\[
\langle g, R(f) \rangle_{L^2(\mu)} = \langle R(g), f \rangle_{L^2(\mu)}.
\]

If $c \in L^\infty(\mu)$, then $R : L^2(\mu) \to L^2(\mu)$ is a bounded operator, and

\[
||R||_{L^2(\mu) \to L^2(\mu)} \leq ||c||_{\infty}.
\]

Relation (6.3) is equivalent to the symmetry of the measure $\lambda$.

2. The operator $R : L^1(\nu) \to L^1(\mu)$ is contractive, i.e.,

\[
||R(f)||_{L^1(\mu)} \leq ||f||_{L^1(\nu)}, \quad f \in L^1(\nu).
\]

Moreover, for any function $f \in L^1(\nu)$, the formula

\[
\int_V R(f) \, d\mu(x) = \int_V f(x) c(x) \, d\mu(x)
\]

holds. In other words, $\nu = \mu R$, and

\[
\frac{d(\mu R)}{d\mu}(x) = c(x).
\]

3. The bounded operator $R_1 : L^2(\nu) \to L^2(\nu)$ is self-adjoint. Moreover, $\nu R_1 = \nu$.

---

1The more natural notation $P$ has been already reserved for a finite transition kernel between two standard spaces.
2This means that the operator $R$ is densely defined on functions from $\mathcal{D}(\mu)$; in particular, this property holds if $c \in L^2(\mu)$
Definition 6.4. Suppose that \( c \) is a measurable locally integrable function. Suppose that \( \nu \)'s measure is symmetric. The following are equivalent:

\( \int_B c(x)P_1(x, A) \, d\mu(x) = \int_A c(x)P_1(x, B) \, d\mu(x). \)

It turns out that the notion of reversibility is equivalent to the following properties.

Theorem 6.5 ([BJ19a, BJb]). Let \( (X, A, \mu) \) be a standard \( \sigma \)-finite measure space, \( x \mapsto c(x) \in (0, \infty) \) a measurable locally integrable function. Suppose that \( x \mapsto P_1(x, \cdot) \) is a probability kernel. The following are equivalent:

(i) \( x \mapsto P_1(x, \cdot) \) is reversible (i.e., it satisfies (6.5));

(ii) the Markov operator \( R_1 \) defined by \( x \mapsto P_1(x, \cdot) \) is self-adjoint on \( L^2(\nu) \) and \( \nu R_1 = \nu \)

where \( d\nu(x) = c(x)d\mu(x) \);

(iii) \( c(x)P_1(x, dy)d\mu(x) = c(y)P_1(y, dx)d\mu(y) \);

(iv) the operator \( R \) defined by the relation \( R(f)(x) = c(x)R_1(f)(x) \) is symmetric;

(v) the measure \( \lambda \) on \( (X \times X, A \times A) \) defined by

\[ \lambda(A \times B) = \int_X \chi_A R_1(\chi_B) \, d\mu \]

is symmetric.

6.2. Transition kernels and corresponding operators \( T_P \) and \( T_Q \). As we seen in Section 5, the transition kernels \( P \) and \( Q \) generate linear operators acting on the space of bounded Borel functions. To emphasize the difference between the kernel \( P \) and the corresponding operator, we will denote the latter by \( T_P \). Let \( \mathcal{F}(X, A) = \mathcal{F}(X) \) denote the space of bounded Borel functions. Define \( T_P : \mathcal{F}(X_2) \to \mathcal{F}(X_1) \) and \( T_Q : \mathcal{F}(X_1) \to \mathcal{F}(X_2) \) by setting

\( T_P(g)(x) = \int_{X_2} P(x, dy)g(y), \quad T_Q(f)(x) = \int_{X_1} Q(y, dx)f(x). \)

We observe that the kernels \( P \) and \( Q \) admit the following representations: for \( A \in \mathcal{A}_1, B \in \mathcal{A}_2, \)

\( P(x, B) = T_P(\chi_B)(x), \quad Q(y, A) = T_Q(\chi_A)(y). \)

Recall that any finite transition kernel \( R : X \times A \to [0, 1] \) acts also on the set of \( \sigma \)-finite measures \( M(X) \): for \( \mu \in M(X), \)

\( (\mu R)(f) = \int_X R(f) \, d\mu = \int_X f \, d\mu R. \)

In this section, we focus on actions of the operators \( T_P \) and \( T_Q \) in the corresponding \( L^2 \)-spaces. In general, these operators are unbounded with dense domain. The following
Theorem 6.6. (1) Let \((X_i, A_i, \nu_i), i = 1, 2,\) be standard \(\sigma\)-finite measure spaces. Let \(P\) and \(Q\) form a dual pair of probability kernels associated to the measures \(\nu_1\) and \(\nu_2\). Then \(P : X_1 \times A_2 \to [0,1]\) defines a contractive operator \(T_P : L^2(\nu_2) \to L^2(\nu_1)\). Similarly, \(T_Q\) is a contractive operator from \(L^2(\nu_1)\) to \(L^2(\nu_2)\).

(2) Suppose that \((P,Q)\) is a pair of dual probability transition kernels associated to the measures \(\nu_1\) and \(\nu_2\) as in (1). Then

\[
\langle f, T_P(g) \rangle_{L^2(\nu_1)} = \langle T_Q(f), g \rangle_{L^2(\nu_2)}, \quad f \in L^2(\nu_1), g \in L^2(\nu_2),
\]
i.e., \((T_P)^* = T_Q\) and \((T_Q)^* = T_P\).

(3) Let \(\nu_1\) and \(\nu_2\) be Borel \(\sigma\)-finite measures on \((X_i, A_i), i = 1, 2,\) satisfying Corollary 5.20 (ii). Suppose that the finite transition kernels \(P(x,\cdot)\) and \(Q(y,\cdot)\) have the property: \(P(\chi_B) \in L^2(\nu_1)\) and \(Q(\chi_A) \in L^2(\nu_2)\) for any Borel sets \(A \in A_1\) and \(B \in A_2\) of finite measure. Then the operators \(T_P\) and \(T_Q\)

\[
T_P : L^2(\nu_2) \to L^2(\nu_1) : f \mapsto \int_{X_2} P(x,dy)f(y),
\]

\[
T_Q : L^2(\nu_1) \to L^2(\nu_2) : f \mapsto \int_{X_1} Q(y,dx)f(x)
\]

are densely defined, and they satisfy the relations \(T_P \subseteq (T_Q)^*\), \(T_Q \subseteq (T_P)^*\). The operators \(T_P\) and \(T_Q\) are closable.

Remark 6.7. The containment in the statement of Theorem 6.6 (3) refers to containment of unbounded densely defined linear operators, i.e., containment of the respective graphs. A pair of operators, with each one contained in the adjoint of the other, is called a symmetric pair. Such pairs arise naturally in many areas of pure and applied analysis, see e.g., [Jor12, JP08, JP16, JP17, JPT18, Kat95, Sch12], and the papers cited there.

Proof. For (1), let \(\rho\) be the measure on \(X_1 \times X_2\) defined by the dual pair \(P,Q\) and the measures \(\nu_1,\nu_2\) as in (5.5). We first show that for any function \(g \in L^2(\nu_2)\) the function \(T_P(g)\) belongs to \(L^2(\nu_1)\). For this, we compute

\[
\int_{X_1} |T_P(g)|^2 \, d\nu_1(x) = \int_{X_1} \left| \int_{X_2} P(x,dy)g(y) \right|^2 \, d\nu_1(x)
\]

\[
\leq \int_{X_1} \int_{X_2} g(y)^2 P(x,dy) \, d\nu_1(x)
\]

\[
= \int_{X_1 \times X_2} g(y)^2 \, d\rho(x,y)
\]

\[
= \int_{X_1 \times X_2} g(y)^2 Q(y,dy) \, d\nu_2(y)
\]

\[
= \int_{X_2} g(y)^2 \, d\nu_2(y)
\]

\[
= \|g\|^2_{L^2(\nu_2)}.
\]
We used here the fact that $P$ and $Q$ are probability kernels. Thus, we proved that $\|T_P\|_{L^2(\nu_2) \to L^2(\nu_1)} \leq 1$. Similarly, one can show that $T_Q$ is contractive. This proves (1).

For (2), let $\rho$ be the measure defined by $(\nu_1, P)$ (or $\nu_2, Q$). If $A \times B \in \mathcal{A}_1 \times \mathcal{A}_2$ (where $A, B$ are of finite measure), then we can represent $\rho(A \times B)$ in two ways:

$$\rho(A \times B) = \int_A P(x, B) \, d\nu_1(x) = \int_{X_1} \chi_A(x) P(\chi_B)(x) \, d\nu_1(x) = \langle f, T_P(g) \rangle_{L^2(\nu_1)},$$

and

$$\rho(A \times B) = \int_B Q(y, A) \, d\nu_2(y) = \int_{X_2} \chi_B(y) Q(\chi_A)(y) \, d\nu_2(y) = \langle g, T_Q(f) \rangle_{L^2(\nu_2)}.$$

Then (2) follows.

To show (3), we first observe that the linear operators $T_P$ and $T_Q$, considered in the $L^2$-spaces, are in general unbounded in the case when the measures $\nu_1$ and $\nu_2$ are $\sigma$-finite and the transition kernels $P, Q$ are finite. It is not hard to see that the operators $T_P$ and $T_Q$ are densely defined on a set $\mathcal{F}(\nu_i)$ which is spanned by characteristic functions of subsets with finite measure $\nu_i$. Moreover, it follows from the relation

$$\int_{X_2} (T_Qf) g \, d\nu_2 = \int_{X_1} (T_Pg) f \, d\nu_1$$

(where $f$ and $g$ are functions from $\mathcal{D}$) that $T_P \subset (T_Q)^*$. This proves that $T_P$ is closable. Similarly, one can show that $T_Q$ is closable.

\begin{corollary}
Let $(X_i, \mathcal{A}_i, \nu_i)$ be $\sigma$-finite standard measure spaces, $i = 1, 2$. Suppose that $P$ and $Q$ are transition kernels which define the operators $T_P$ and $T_Q$ such that

$$T_P(g) = \int_{X_2} P(x, dy)g(y) : L^2(\nu_2) \to L^2(\nu_1)$$

$$T_Q(f) = \int_{X_1} Q(y, dx)f(x) : L^2(\nu_1) \to L^2(\nu_2).$$

Then the following are equivalent:

(i) $T_P \subset (T_Q)^*$,

(ii) $T_Q \subset (T_P)^*$,

(iii) $d\nu_1(x) P(x, dy) = d\nu_2(y) Q(y, dx)$, and this relation defines a measure $\rho$ on $X_1 \times X_2$ which belongs to the set $L(\nu_1, \nu_2)$,

(iv) for all sets $A \in \mathcal{D}(\nu_1), B \in \mathcal{D}(\nu_2)$,

$$\int_A P(x, B) \, d\nu_1(x) = \int_{X_2} Q(y, A) \, d\nu_2(y),$$

\end{corollary}

\begin{proof}
These results follow from Theorem 6.6. We leave the details for the reader. \qed

As we seen, the operators $T_P$ and $T_Q$ are, in general, unbounded. Here we address the question under what conditions these operators are bounded in the $L^2$-spaces.

\begin{proposition}
(1) Let $T_P$ and $T_Q$ be as above and $c_1(x) = P(x, X_2)$. If

$$M = \text{ess sup}_{y \in X_2} \int_{X_1} Q(y, dx) c_1(x) < \infty,$$

then $T_P : L^2(\nu_1) \to L^2(\nu_2)$ is bounded.

(2) If $\mathcal{F}(\nu_1)$, $\mathbb{P}(\nu_1)$, and $\mathcal{G}(\nu_2)$, $\mathbb{Q}(\nu_2)$ are standard measure spaces, then $T_P : \mathbb{P}(\nu_1) \to \mathbb{Q}(\nu_2)$ is measurable.

(3) If $\mathcal{F}(\nu_1)$, $\mathbb{P}(\nu_1)$, and $\mathcal{G}(\nu_2)$, $\mathbb{Q}(\nu_2)$ are standard measure spaces, then $T_P : \mathbb{P}(\nu_1) \to \mathbb{Q}(\nu_2)$ is measurable.

\end{proposition}

\begin{proof}
These results follow from Corollary 6.8. We leave the details for the reader. \qed

As we seen, the operators $T_P$ and $T_Q$ are, in general, unbounded. Here we address the question under what conditions these operators are bounded in the $L^2$-spaces.
then the operator $T_P : L^2(\nu_2) \to L^2(\nu_1)$ is bounded, \( \|T_P\|_{L^2(\nu_2) \to L^2(\nu_1)} \leq M \). A similar statement holds for the operator $T_Q$ when the roles of $T_P$ and $T_Q$ are interchanged.

(2) Suppose there are functions $a \in L^2(\nu_1)$ and $b \in L^2(\nu_2)$ such that
\[
\frac{P(x,dy)}{d\nu_2(y)} \leq a(x)b(y).
\]
Then
\[
\|T_P\|_{L^2(\nu_2) \to L^2(\nu_1)} \leq \|a\|_{L^2(\nu_1)} \|b\|_{L^2(\nu_2)}.
\]
A similar estimate holds for the norm of $T_Q$ if there are functions $a_1 \in L^2(\nu_1)$ and $b_1 \in L^2(\nu_2)$ such that
\[
\frac{Q(y, dx)}{d\nu_1(x)} \leq a_1(x)b_1(y).
\]

Proof. (1) To prove the first assertion, we use the Fubini-Tonelli theorem: in the computation below the right-hand side is well defined, so that $P(f) \in L^2(\nu_1)$:
\[
\int_{X_2} P(f)^2 \, d\nu_1 = \int_{X_2} \left( \int_{X_2} P(x,dy) f(y) \right)^2 \, d\nu_1
\leq \int_{X_2} \left( \int_{X_2} P(x,dy) \right) \left( \int_{X_2} P(x,dy) f^2(y) \right) \, d\nu_1
= \int_{X_2} \left( \int_{X_2} P(x,dy) f^2(y) \right) c_1(x) \, d\nu_1(x)
= \int_{X_2} f^2(y) \, d\nu_2(y) \left( \int_{X_1} Q(y, dx) c_1(x) \right)
= \|f\|_{L^2(\nu_2)}^2 \int_{X_1} Q(y, dx) c_1(x).
\]
If
\[
M = \operatorname{ess \, sup}_{y \in X_2} \int_{X_1} Q(y, dx) c_1(x) < \infty,
\]
then the norm of $T_P$ is bounded by $M$.

(2) Recall that it was proved above that, for a.e. $x_1 \in X_1$, the measures $P(x, \cdot)$ is absolutely continuous with respect to $\nu_2$. Hence, the result follows from the following computation: for $f \in L^2(\nu_2)$,
\[
\|T_P(f)\|_{L^2(\nu_2) \to L^2(\nu_1)}^2 = \int_{X_1} \left( \int_{X_2} P(x,dy) f(y) \right)^2 \, d\nu_1(x)
= \int_{X_1} a^2(x) \left( \int_{X_2} f(y)b(y) \, d\nu_2 \right)^2 \, d\nu_1(x)
\leq \|f\|_{L^2(\nu_2)}^2 \|b\|_{L^2(\nu_2)}^2 \int_{X_1} a^2(x) \, d\nu_1(x)
= \|a\|_{L^2(\nu_1)}^2 \|b\|_{L^2(\nu_2)}^2 \|f\|_{L^2(\nu_2)}^2.
\]
The proved Theorem 6.6 has a few important consequences. We consider some of them in the following statements. As follows from Theorem 6.6, one can define the product of operators $T_P : L^2(\nu_2) \rightarrow L^2(\nu_1)$ and $T_Q : L^2(\nu_1) \rightarrow L^2(\nu_2)$.

**Lemma 6.10.** Let $P$ and $Q$ be finite transition kernels defined on the measure spaces $(X_i, \mathcal{A}_i, \nu_i)$ as in Definition 5.3. Suppose that $d\nu_1(x)P(x, dy) = d\nu_2(y)Q(y, dx)$. Then they generate the finite transition kernels $PQ : X_1 \times \mathcal{A}_1 \rightarrow [0, \infty)$ and $QP : X_2 \times \mathcal{A}_2 \rightarrow [0, \infty)$:

$$PQ(x, A) = \int_{X_2} P(x, dy)Q(y, A), \quad A \in \mathcal{A}_1,$$

$$QP(y, B) = \int_{X_1} Q(y, dx)P(x, B), \quad B \in \mathcal{A}_2.$$

**Proof.** These formulas can be proved directly. □

Consider now the corresponding linear operators $T_{PQ}$ and $T_{QP}$ acting in the corresponding $L^2$-spaces.

**Lemma 6.11.** Suppose that finite transition kernels $P$ and $Q$ satisfy the property:

$$P(\chi_B) \in L^2(\nu_1), \quad \forall B \in \mathcal{A}_2, \quad \nu_2(B) < \infty,$$

and

$$Q(\chi_A) \in L^2(\nu_2), \quad \forall A \in \mathcal{A}_1, \quad \nu_1(A) < \infty.$$

Then the operators $T_{PQ} : L^2(\nu_1) \rightarrow L^2(\nu_1)$ and $T_{QP} : L^2(\nu_2) \rightarrow L^2(\nu_2)$ are self-adjoint densely defined linear operators acting by the formulas:

$$T_{PQ}(\chi_A) = T_P(Q(\cdot, A)(x), \quad A \in \mathcal{A}_1, \quad \nu_1(A) < \infty,$$

$$T_{QP}(\chi_B) = T_Q(P(\cdot, B)(y), \quad B \in \mathcal{A}_2, \quad \nu_2(B) < \infty.$$

**Proof.** It suffices to notice that, by the condition of the lemma, we can consequently apply the operators $T_Q$ and $T_P$ to get $T_{PQ}$ because the function $T_Q(\chi_A)$ is in $L^2(\nu_2)$. It gives the operator acting in $L^2(\nu_1)$. The same is true for $T_{QP}$ as an operator in $L^2(\nu_2)$.

Since $T_P \subset (T_Q)^*$ and $T_Q \subset (T_P)^*$, we see that $T_{PQ}$ and $T_{QP}$ are self-adjoint. □

Apply the approach used in Section 5 to the kernels $PQ$ and $QP$ considered in Lemma 6.10. This means that we can define two measures $\lambda_1$ and $\lambda_2$ on the product spaces $X_1 \times X_1$ and $X_2 \times X_2$, respectively:

$$\lambda_1(A_1 \times A_2) = \int_{A_1} d\nu_1(x)(PQ)(x, A_2)$$

and

$$\lambda_2(B_1 \times B_2) = \int_{B_1} d\nu_2(y)(QP)(y, B_2).$$

**Theorem 6.12.** For any sets $A_1$ and $A_2$ of finite measure $\nu_1$,

$$\lambda_1(A_1 \times A_2) = \int_{X_2} Q(y, A_1)Q(y, A_2) d\nu_2(y)$$

Similarly,

$$\lambda_2(B_1 \times B_2) = \int_{X_1} P(x, B_1)P(x, B_2) d\nu_1(x).$$
The measures $\lambda_1$ and $\lambda_2$ are symmetric

$$\lambda_1(A_1 \times A_2) = \lambda_1(A_2 \times A_1), \quad \lambda_2(B_1 \times B_2) = \lambda_2(B_2 \times B_1).$$

Proof. In the following equality we use the fact that $T_P$ and $T_Q$ form a symmetric pair of operators (see Theorem 6.6).

$$\lambda_1(A_1 \times A_2) = \int_{A_1} d\nu_1(x)(PQ)(x, A_2)$$
$$= \int_{A_1} d\nu_1(x)TP(Q(\cdot, A_2))(x)$$
$$= \int_{X_1} d\nu_1(x)\chi_{A_1}(x)TPQ(\chi_{A_2})(x)$$
$$= \int_{X_2} d\nu_2(y)T_Q(\chi_{A_1})(y)TPQ(\chi_{A_2})(y)$$
$$= \int_{X_2} Q(y, A_1)Q(y, A_2) d\nu_2(y).$$

The other equality is proved analogously.

The fact that the measures $\lambda_1$ and $\lambda_2$ are symmetric obviously follows from the proved formulas.

$\square$

7. RKHSs, symmetric measures, and transfer operators $R$

In this section, we give a description of measures $\rho$ that belong to the set $L(\nu_1, \nu_2)$. As a preliminary part, we remind first the notion of a reproducing kernel Hilbert space (RKHS). The well known references to the theory of RKHS are [Aro50, AS57, AFMP94, PR16, SS16], see also more recent results and various applications in [BTA04, AJ14, AJ15, JT15, JT16, JT19b, JT19a].

7.1. RKHS and symmetric measures. Let $S$ be an arbitrary set, and let $K : S \times S \to \mathbb{R}$ be a positive definite function, i.e., the function $K(s, t)$ has the property

$$\sum_{i,j=1}^{N} \alpha_i \alpha_j K(s_i, s_j) \geq 0$$

which holds for any $N \in \mathbb{N}$ and for any $s_i \in S, \alpha_i \in \mathbb{R}$, $i = 1, \ldots, N$. We consider here real-valued functions. (For a complex-valued function $K$ some obvious changes must be made.)

Definition 7.1. Fix $s \in S$ and denote by $K_s$ the function $K_s(t) = K(s, t)$ of one variable $t \in S$. Let $\mathcal{K} := \text{span}\{K_s : s \in S\}$. The reproducing kernel Hilbert space (RKHS) $\mathcal{H}(K)$ is the Hilbert space obtained by completion of $\mathcal{K}$ with respect to the inner product defined on $\mathcal{K}$ by

$$\left( \sum_{i} \alpha_i K_{s_i}, \sum_{j} \beta_j K_{s_j} \right)_{\mathcal{H}(K)} := \sum_{i,j=1}^{N} \alpha_i \beta_j K(s_i, s_j)$$
It immediately follows from Definition 7.1 that
\[(K(\cdot, s), K(\cdot, t))_{\mathcal{H}(K)} = K(s, t)\]
(here we use the notation \(K(\cdot, s) = K_s(\cdot)\)). More generally, this result can be extended to the following property that characterizes functions from the RKHS \(\mathcal{H}(K)\). For any \(f \in \mathcal{H}(K)\) and any \(s \in S\), one has
\[(7.1) \quad f(s) = \langle f(\cdot), K(\cdot, s) \rangle_{\mathcal{H}(K)}.
\]
It suffices to check that (7.1) holds for any function from \(\mathcal{K}\) and then extend it by continuity.

One can prove that the following property determines functions from the reproducing kernel Hilbert space \(\mathcal{H}(K)\) constructed by a positive definite function \(K\) on the set \(S\). We formulate it as a statement for further references.

**Lemma 7.2.** A function \(f\) is in \(\mathcal{H}(K)\) if and only if there exists a constant \(C = C(f)\) such that for any \(n \in \mathbb{N}\), any \(\{s_1, ..., s_n\} \subset S\), and any \(\{\alpha_1, ..., \alpha_n\} \subset \mathbb{R}\), one has
\[(7.2) \quad \left(\sum_{i=1}^{n} \alpha_i f(s_i)\right)^2 \leq C(f) \sum_{i,j=1}^{n} \alpha_i \alpha_j K(s_i, s_j).
\]

We will need the following result.

**Lemma 7.3.** Let \(K(s, t)\) be a positive definite function, and let \(\mathcal{H}(K)\) be the corresponding RKHS. Take an orthonormal basis \(\{g_n(\cdot)\}\) in \(\mathcal{H}(K)\). Then
\[(7.3) \quad K(s, t) = \sum_n g_n(s)g_n(t).
\]

For every \(s\), \((g_n(s))\) is in \(\ell^2\). Moreover, every vector from \(\mathcal{H}\) can be represented uniquely as \(\sum_c c_n g_n(\cdot)\) where \((c_n) \in \ell^2\).

**Proof.** We have obvious equalities:
\[K(\cdot, s) = \sum_n \langle K(\cdot, s), g_n \rangle_{\mathcal{H}(K)} g_n, \quad K(\cdot, t) = \sum_n \langle K(\cdot, t), g_n \rangle_{\mathcal{H}(K)} g_n.
\]

Then we use (7.1) in the following computation:
\[
K(s, t) = \langle K(\cdot, s), K(\cdot, t) \rangle_{\mathcal{H}(K)} = \left(\sum_n \langle K(\cdot, s), g_n \rangle_{\mathcal{H}(K)} g_n, \sum_m \langle K(\cdot, t), g_m \rangle_{\mathcal{H}(K)} g_m \right)_{\mathcal{H}(K)} \]
\[= \sum_n \langle K(\cdot, s), g_n \rangle_{\mathcal{H}(K)} \langle K(\cdot, t), g_n \rangle_{\mathcal{H}(K)} = \sum_n g_n(s)g_n(t).
\]

The second statement of the lemma is obvious. □

**Example 7.4.** For a standard measure space \((X, \mathcal{A}, \mu)\), define a symmetric measure \(\lambda\) on \(X \times X\) by setting \(\lambda(A \times B) = \mu(A \cap B)\) where \(A, B \in \mathcal{D}(\mu)\). Then the corresponding symmetric operator \(R\) must be the identity operator, \(R(f) = f\), because of the relations
\[
\lambda(A \times B) = \int_X \chi_A R(\chi_B) \, d\mu, \quad \mu(A \cap B) = \int_X \chi_A \chi_B \, d\mu.
\]
The function \( k_\mu : (A, B) \mapsto \mu(A \cap B) \), \( A, B \in \mathcal{D}(\mu) \), is positive definite and defines a RKHS \( \mathcal{H}_\mu \). It follows from Lemma 7.2 that a function \( F \) on \( \mathcal{D}(\mu) \) is in \( \mathcal{H}_\mu \) if and only if there exists a function \( f \in L^2_{\text{loc}}(\mu) \) such that

\[
F(A) = \int_A f \, d\mu, \quad A \in \mathcal{D}(\mu).
\]

Moreover, the correspondence between the functions \( F_A = (B \mapsto k_\mu(A, B)) \) and \( \chi_A \) can be extended to an isometry such that \( ||F||_{\mathcal{H}_\mu} = ||f||_{L^2(\mu)} \). More details about this example and other examples can be found in [BJa, JT19a, JT19b].

In what follows, we will work with an unbounded operator \( R \) acting in \( L^2(\nu) \) where \( (X, \mathcal{B}, \nu) \) is a \( \sigma \)-finite standard measures space. Let \( \mathcal{F}(\nu) \) be the linear space spanned by the characteristic functions \( \chi_A \) with \( A \in \mathcal{D}(\nu) \). We need some assumptions about \( R \).

**Assumption.** In this section the operator \( R \) is assumed to have the following properties:

(i) \( R \) is a densely defined unbounded (in general) operator in \( L^2(\nu) \) such that \( \mathcal{F}(\nu) \subset \text{Dom}(R) \) and \( R(\mathcal{F}(\nu)) \subset L^2(\nu) \),

(ii) \( R \) is a symmetric positive operator, i.e., \( R(f) \geq 0 \) whenever \( f \geq 0 \),

(iii) \( R \) is a positive definite operator, i.e., the quadratic form \( f \mapsto \langle R(f), f \rangle_{L^2(\nu)} \) is non-negative,

(iv) the kernel of \( R \) is trivial.

Using Subsection 6.1, we can associate a symmetric measure \( \lambda \) on \( (X \times X, \mathcal{A} \times \mathcal{A}) \) to the operator \( R \):

\[
\lambda(A \times B) = \int_X \chi_A R(\chi_B) \, d\nu = \lambda(B \times A).
\]

It follows from Assumption that \( R \) is a symmetric operator, and therefore \( R \) has a self-adjoint extension (Friedrichs extension). We denote it by \( \hat{R} \). Note that \( \hat{R} \) is also positive. It is obvious that \( \hat{R} = R \) on the subset \( \mathcal{F}(\nu) \).

It can be seen that the above assumptions about the operator \( R \) and its extension \( \hat{R} \) lead to the following statement.

**Lemma 7.5.** For \( R \) and \( \hat{R} \) as above, the subset \( \hat{R}^{1/2}(\text{Dom}(\hat{R}^{1/2})) \) of \( L^2(\nu) \) is a Hilbert space with the inner product \( \langle \hat{R}^{1/2}f, \hat{R}^{1/2}g \rangle_{L^2(\nu)} \).

Our main result of this section is formulated as follows.

**Theorem 7.6.** Let \( \hat{R} \) be the Friedrichs extension of a symmetric positive operator \( R \) acting in \( L^2(\nu) \). Denote by \( \lambda = \lambda(R) \) the symmetric measure defined by (7.5). Then

\[
K(A, B) := \lambda(A \times B), \quad A, B \in \mathcal{D}(\nu)
\]

is a positive definite function defined on the set \( \mathcal{D}(\nu) \) and generates the RKHS \( \mathcal{H}(\lambda) \). The elements \( \hat{f}(A) \) of \( \mathcal{H}(\lambda) \) are signed measures on \( \mathcal{D}(\nu) \) which are defined by the formula

\[
\hat{f}(A) = \int_X R(x, A) f(x) \, d\nu(x) = \langle R(\chi_A), f \rangle_{L^2(\nu)}
\]

where \( f \in \text{Dom}(\hat{R}^{1/2}) \). They are absolutely continuous with respect to \( \nu \) and

\[
\frac{d\hat{f}}{d\nu} = R(f).
\]
Moreover,

\begin{equation}
\|\hat{R}^{1/2}(f)\|_{L^2(\nu)} = \|\hat{f}\|_{\mathcal{H}(\lambda)},
\end{equation}

and therefore the map \( \hat{R}^{1/2}(\text{Dom}(\hat{R}^{1/2})) \ni \hat{R}^{1/2}(f) \mapsto \hat{f} \in \mathcal{H}(\lambda), \) is an isometric isomorphism between these Hilbert spaces.

\textbf{Proof.} We first show that \( K : (A, B) \mapsto \lambda(A \times B) \) is a positive definite function, \( A, B \in \mathcal{D}(\nu). \) Indeed, for any collection of sets \( A_1, \ldots, A_k \) from \( \mathcal{D}(\nu) \) and any real numbers \( \alpha_1, \ldots, \alpha_k, \) we consider two families of functions: \( \mathcal{G}(\nu) = \text{span}\{K(\cdot, A) : A \in \mathcal{D}(\nu)\} \) and \( \mathcal{F}(\nu) = \text{span}\{\chi_A : A \in \mathcal{D}(\nu)\}. \) Then, for

\[ \varphi(x) = \sum_{i=1}^{k} \alpha_i \chi_{A_i}(x), \]

we compute

\begin{equation}
\sum_{i,j}^{k} \alpha_i \alpha_j \lambda(A_i \times A_j) = \sum_{i,j}^{k} \alpha_i \alpha_j \int_X \chi_{A_i} R(\chi_{A_j}) \, d\nu \\
= \int_X \left( \sum_{i=1}^{k} \alpha_i \chi_{A_i} \right) R \left( \sum_{j=1}^{k} \alpha_j \chi_{A_j} \right) \, d\nu \\
= \langle \varphi, R(\varphi) \rangle_{L^2(\nu)}
\end{equation}

because \( R \) is a positive defined operator. Therefore, the function \( K : (A, B) \mapsto \lambda(A \times B) \) is positive definite and defines a RKHS \( \mathcal{H}(\lambda) = \mathcal{H} \) as in Definition 7.1.

It follows from the definition of functions \( \hat{f} \), see (7.6), that \( \hat{f}(A) = \int_A R(f) \, d\nu, \quad f \in \text{Dom}(\hat{R}^{1/2}). \)

This means that, for fixed \( f \), the signed measure \( \hat{f}(A) \) is absolutely continuous with respect to \( \nu \) and \( R(f) \) is the Radon-Nikodym derivative \( \frac{df}{d\nu}. \)

The computation used in (7.8) shows that the following equalities are true: for the function \( \varphi \in \mathcal{F}(\nu) \) as above,

\begin{equation}
\|\varphi\|_{\mathcal{H}(\lambda)}^2 = \sum_{i,j}^{k} \alpha_i \alpha_j \lambda(A_i \times A_j) = \langle \varphi, R(\varphi) \rangle_{L^2(\nu)} = \|\hat{R}^{1/2}(\varphi)\|_{L^2(\nu)}^2.
\end{equation}

Next, we observe that \( \hat{\chi}_A(\cdot) = K(\cdot, A) \) and \( K(A, B) = \langle \chi_A, \chi_B \rangle_{\mathcal{H}(\lambda)} \). By definition, the family \( \mathcal{G}(\nu) \) is dense in the RKHS \( \mathcal{H}(\lambda) \), and \( \mathcal{F}(\nu) \) is dense in \( L^2(\nu) \). These two families are in the one-to-one correspondence \( K(\cdot, A) \leftrightarrow \hat{R}^{1/2}(\chi_A) \) and satisfy the property

\begin{equation}
\|K(\cdot, A)\|_{\mathcal{H}(\lambda)} = \|\hat{R}^{1/2}(\chi_A)\|_{L^2(\nu)}, \quad A \in \mathcal{D}(\nu).
\end{equation}

Our goal is to extend this isometry to the closures of \( \mathcal{G}(\nu) \) and \( \hat{R}^{1/2}(\mathcal{F}(\nu)) \) and show that the Hilbert spaces \( \mathcal{H}(\lambda) \) and \( \text{Dom}(\hat{R}^{1/2}) \) are isometrically isomorphic.
Prove that $\hat{f}(A)$ is a function from $\mathcal{H}(\lambda)$. Let $f$ be the corresponding function from $\text{Dom}(\hat{R}^{1/2})$ which defines $\hat{f}$. By Lemma 7.2, the function

$$\hat{f}(A) = \langle R(\chi_A), f \rangle_{L^2(\nu)}$$

belongs to $\mathcal{H}(\lambda)$ if and only if, for $A_i, \nu \in \mathcal{D}(\nu)$ and $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, k$,

$$\left( \sum_{i=1}^{k} \alpha_i \hat{f}(A_i) \right)^2 = \left( \sum_{i=1}^{k} \alpha_i \langle R(\chi_{A_i}), f \rangle_{L^2(\nu)} \right)^2$$

$$= \langle R(\varphi), f \rangle_{L^2(\nu)}^2$$

$$= \langle \hat{R}^{1/2}(\varphi), \hat{R}^{1/2}(f) \rangle_{L^2(\nu)}^2$$

$$\leq ||\hat{R}^{1/2}(f)||_{L^2(\nu)}^2 \cdot ||\hat{R}^{1/2}(\varphi)||_{L^2(\nu)}^2$$

$$= ||\hat{R}^{1/2}(f)||_{L^2(\nu)}^2 \sum_{i,j} \alpha_i \alpha_j \lambda(A_i \times A_j)$$

where $\varphi(x) = \sum_{i=1}^{k} \alpha_i \chi_{A_i}(x)$. Denoting $||\hat{R}^{1/2}(f)||_{L^2(\nu)}^2$ by $C(f)$, we can use the criterion formulated in Lemma 7.2. Hence, $\hat{f} \in \mathcal{H}(\lambda)$.

Furthermore, we note that, as the operator $\hat{R}^{1/2}$ is closed, relation (7.10) can be extended by completion to the equality

$$||\hat{f}||_{\mathcal{H}(\lambda)} = ||\hat{R}^{1/2}(f)||_{L^2(\nu)},$$

which holds for all functions $\hat{f}$ from $\mathcal{H}(\lambda)$.

We show that $\hat{f}$, defined by (7.6), satisfies the reproducing property for the kernel $K(A, B)$, i.e., $\hat{f}(A) = \langle K(\cdot, A), \hat{f} \rangle_{\mathcal{H}(\lambda)}$ where the inner product in $\mathcal{H}(\lambda)$ is defined as in Definition 7.1. For this, we use (7.7) and the facts that $R$ is symmetric and that $\text{Dom}(R) \subseteq \text{Dom}(\hat{R}^{1/2})$:

$$\hat{f}(A) = \int_A R(f)(x) \, d\nu(x)$$

$$= \langle R(f), \chi_A \rangle_{L^2(\nu)}$$

$$= \langle \hat{R}^{1/2}(f), \hat{R}^{1/2}(\chi_A) \rangle_{L^2(\nu)}$$

$$= \langle K(\cdot, A), \hat{f} \rangle_{\mathcal{H}(\lambda)},$$

where $A \in \mathcal{D}(\nu)$.

It remains to prove that the map $f \mapsto \hat{f}$ from $\text{Dom}(\hat{R}^{1/2})$ to $\mathcal{H}(\lambda)$ is onto, i.e., for every $F \in \mathcal{H}(\lambda)$, there exists a unique function $f \in \text{Dom}(\hat{R}^{1/2})$ such that $f = \hat{f}$. For this, we note first that

$$\left( \sum_{i=1}^{n} \alpha_i F(A_i) \right)^2 \leq C \sum_{i,j=1}^{n} \alpha_i \alpha_j K(A_i, A_j) = C \langle \varphi, R(\varphi) \rangle_{L^2(\nu)}$$

where $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \in \text{Dom}(\hat{R}^{1/2})$. Therefore, $F$ defines a linear continuous functional $\tau$ on the space $\mathcal{F}(\nu)$:

$$\tau_F(\varphi) = \sum_{i=1}^{n} \alpha_i F(A_i),$$
and

\[ |\tau_F(\varphi)|^2 \leq C\langle \varphi, R(\varphi) \rangle_{L^2(\nu)}. \]

By the definition of the Friedrichs extension, the latter can be extended to functions from 
\( \text{Dom}(\hat{R}^{1/2}) \), so that

\[ |\tau_F(\varphi)|^2 \leq C\|\hat{R}^{1/2}\|^2_{L^2(\nu)}. \]

By the Riesz theorem, there exists a unique function \( f \in \text{Dom}(\hat{R}^{1/2}) \) such that

\[ \tau_F(g) = \langle \hat{R}^{1/2}(g), \hat{R}^{1/2}(f) \rangle_{L^2(\nu)}. \]

Show that \( F = \hat{f} \) considered as functions from \( \mathcal{H}(\lambda) \). Indeed, we use the reproducing property of the generating family of functions \( (\mathcal{X}, \mathcal{A}) \) and the fact that \( R \) is symmetric to deduce that

\[
F(A) = \langle F, K(\cdot, A) \rangle_{\mathcal{H}(\lambda)} \\
= \langle \hat{R}^{1/2}f, \hat{R}^{1/2}\mathcal{X}A \rangle_{L^2(\nu)} \\
= \int_{X_1} R(f)\mathcal{X}A \, d\nu \\
= \hat{f}(A),
\]

where \( A \in \mathcal{D}(\nu) \). As was mentioned above, the isometric property \( \|F\|_{\mathcal{H}(\lambda)} = \|\hat{R}^{1/2}(f)\|_{L^2(\nu)} \) can be proved by taking the closure of dense subsets in \( \mathcal{H}(\lambda) \) and \( \text{Dom}(\hat{R}^{1/2}) \).

This proves that \( \mathcal{H}(\lambda) \) and \( \hat{R}(\text{Dom}(\hat{R}^{1/2})) \) are isometric Hilbert spaces.

\[ \Box \]

**Remark 7.7.** (1) We note that the definition of functions \( \hat{f} \) from \( \mathcal{H}(\lambda) \) as in (7.6) is analogous to the formula (7.4) where the operator \( R \) was the identity operator. On the other hand, Theorem 7.6 states the RKHS \( \mathcal{H}(\lambda) \) can be defined as the set \( \{ \hat{f}(A) = \langle \mathcal{X}A, R(f) \rangle_{L^2(\nu)} : f \in \text{Dom}(\hat{R}^{1/2}) \} \).

(2) Let \( R \) be a symmetric positive definite unbounded operator in \( L^2(\nu) \) which determines a symmetric measure \( \lambda \) (see Assumption). Let \( \hat{R} \) be the Friedrichs extension of \( R \). Then \( \hat{R} \) is also a positive definite self-adjoint operator. Hence, in its turn, it defines a symmetric measure \( \hat{\lambda} \) on \( X \times X \). It is not hard to see that, in fact, \( \lambda = \hat{\lambda} \).

### 7.2. Spectral properties and factorization of \( R \)

Suppose that we have now two \( \sigma \)-finite measure spaces \( (X_i, A_i, \nu_i), i = 1, 2 \). We will use Theorem 7.6 to describe measures \( \rho \) that belong to the set \( L(\nu_1, \nu_2) \). We recall that \( \rho \) is in \( L(\nu_1, \nu_2) \) if there is a pair of finite transition kernels \( P \) and \( Q \) such that (5.5) holds. Let \( R \) be a positive symmetric operator acting on functions from \( L^2(\nu_1) \) and satisfying Assumption formulated before Theorem 7.6. Beginning with \( (X_1, A_1, \nu_1, R) \), we define the symmetric measure \( \lambda_1 \) on \( (X_1 \times X_1, A_1 \times A_1) \) by setting

\[
\lambda_1(A \times B) = \langle \mathcal{X}A, R(\mathcal{X}B) \rangle_{L^2(\nu_1)}, \quad A, B \in \mathcal{D}(\nu_1),
\]

and then extend it to all Borel sets of \( X_1 \times X_1 \).

Using the symmetric measure \( \lambda_1 \), construct the RKHS \( \mathcal{H}(\lambda_1) \) as in Theorem 7.6. Fix an orthonormal basis (ONB) \( \{ \hat{k}_n \} \) in the RKHS \( \mathcal{H}(\lambda_1) \). As follows from (7.6), there are functions \( \{ f_n \} \in \text{Dom}(\hat{R}^{1/2}) \subset L^2(\nu_1) \) such that

\[
\hat{k}_n(A) = \langle R(\mathcal{X}A), f_n \rangle_{L^2(\nu_1)} = \langle \mathcal{X}A, R(f_n) \rangle_{L^2(\nu_1)}, \quad A \in \mathcal{D}(\nu_1).
\]
Choose an ONB \{\varphi_n\} in the Hilbert space \(L^2(\nu_2)\). We will consider the two bases in the Hilbert spaces \(H(\lambda_1)\) and \(L^2(\nu_2)\) which are formed by non-negative functions, i.e., \(f_n \geq 0\) and \(\varphi_n \geq 0\).

**Definition 7.8.** We use the objects \(R, (\widehat{k}_n), (\varphi_n)\) to define two transition kernels \(P : X_1 \times A_2 \to [0, \infty)\) and \(Q : X_2 \times A_1 \to [0, \infty)\). For this, we set

\[
Q(y, A) := \sum_{n=1}^{\infty} \widehat{k}_n(A)\varphi_n(y) = \sum_{n=1}^{\infty} \langle \chi_A, R(f_n) \rangle_{L^2(\nu_1)} \varphi_n(y), \quad A \in \mathcal{D}(\nu_1),
\]

and

\[
P(x, B) := \sum_{n=1}^{\infty} \left( \int_B \varphi_n \, d\nu_2 \right) R(f_n)(x)
\]

\[
= \sum_{n=1}^{\infty} \langle \chi_B, \varphi_n \rangle_{L^2(\nu_2)} R(f_n)(x), \quad B \in \mathcal{D}(\nu_2).
\]

**Remark 7.9.** (1) We first note that Theorem 7.6 establishes a correspondence between orthonormal bases in \(H(\lambda)\) and \(\widehat{R}^{1/2}(\text{Dom} \widehat{R}^{1/2})\). This means that one can take the functions \((f_n)\) such that \((\widehat{R}^{1/2}(f_n))\) is an ONB. Moreover, since \(\text{Dom}(\widehat{R}) \subset \text{Dom}(\widehat{R}^{1/2})\) and \(R(g) = \widehat{R}(g)\) on a dense subset, we can take \(f_n\) such that \(R(f_n)\) is defined.

(2) As mentioned in Lemma 7.3, the sequence \((\widehat{k}_n(A))\) is in \(l^2\) for every set \(A \in \mathcal{D}(\nu_1)\). Hence the kernels \(Q\) and \(P\) take finite values on the sets \(A\) and \(B\) of finite measure.

(3) One can show that the operators \(T_P\) and \(T_Q\) defined by the kernels \(P\) and \(Q\) do not depend on the choice of bases \((\widehat{k}_n)\) and \((\varphi_n)\).

(4) It can be shown that \(P(x, B)\) and \(Q(y, A)\), which are defined on sets of finite measure, can be extended to the measures \(P(x, \cdot)\) and \(Q(y, \cdot)\) on \((X_2, \mathcal{A}_2)\) and \((X_1, \mathcal{A}_1)\). Obviously, these measures are absolutely continuous with respect to \(\nu_2\) and \(\nu_1\), respectively. We leave the details for the reader.

Recall that the transition kernels \(P\) and \(Q\) generate a pair of linear operators \(T_P\) and \(T_Q\) acting on bounded Borel functions by

\[
T_Q(f)(y) = \int_{X_1} Q(y, dx) f(x), \quad T_P(g)(y) = \int_{X_2} P(x, dy) g(y).
\]

We will consider these operators in the corresponding \(L^2\)-spaces. It follows from our previous results that \(T_P\) and \(T_Q\) are unbounded densely defined operators such that

\[
T_P : L^2(\nu_2) \to L^2(\nu_1), \quad T_Q : L^2(\nu_1) \to L^2(\nu_2),
\]

see Theorem 6.6. In Proposition 7.10, we give exact formulas for the operators \(T_P\) and \(T_Q\) in the case when the kernels are taken from Definition 7.8.

Recall that the subspace \(\mathcal{F}(\nu_1)\) of simple functions from \(L^2(\nu_1)\) belongs to the domain of \(T_Q\), and \(\mathcal{F}(\nu_2) \subset L^2(\nu_2)\) is in the domain of \(T_P\).

**Proposition 7.10.** (1) The kernels \(Q\) and \(P\), defined by (7.13) and (7.14), form a dual pair of kernels associated to the measures \(\nu_1\) and \(\nu_2\) (see Definition 5.3).

(2) The dual pair of transition kernels defines a measure \(\rho\) on \(X_1 \times X_2\) such that, for \(A \in \mathcal{D}(\nu_1)\) and \(B \in \mathcal{D}(\nu_2)\),
\[ \rho(A \times B) = \int_B Q(y, A) \, d\nu_2(y) = \sum_{n=1}^{\infty} \tilde{k}_n(A) \langle \chi_B, \varphi_n \rangle_{L^2(\nu_2)}. \]

(3) The transition kernels \( P \) and \( Q \) generate the operators \( T_P \) and \( T_Q \) that form a symmetric pair of operators, i.e.,

\[ \langle g, T_Q(f) \rangle_{L^2(\nu_2)} = \langle T_P(g), f \rangle_{L^2(\nu_1)}, \]

and \( T_P \subset (T_Q)^* \), \( T_Q \subset (T_P)^* \) on the corresponding domains.

Proof. (1) To prove the result, we need to show that, for any sets \( A \in \mathcal{D}(\nu_1) \) and \( B \in \mathcal{D}(\nu_2) \), the property

\[ \int_A P(x, B) \, d\nu_1(x) = \int_B Q(y, A) \, d\nu_2(y) \]

holds. In the computation below, we use relations (7.12), (7.13), (7.14), and the fact that \( R \) is symmetric:

\[
\begin{align*}
\int_B Q(y, A) \, d\nu_2(y) &= \int_X \chi_B(y) \sum_{n=1}^{\infty} k_n(A) \varphi_n(y) \, d\nu_2(y) \\
&= \int_X \chi_B(y) \sum_{n=1}^{\infty} \langle R(\chi_A), f_n \rangle_{L^2(\nu_1)} \varphi_n(y) \, d\nu_2(y) \\
&= \sum_{n=1}^{\infty} \langle \chi_B, \varphi_n \rangle_{L^2(\nu_2)} \int_X R(\chi_A) f_n \, d\nu_1 \\
&= \sum_{n=1}^{\infty} \langle \chi_B, \varphi_n \rangle_{L^2(\nu_2)} \int_A R(f_n) \, d\nu_1 \\
&= \int_A \left( \sum_{n=1}^{\infty} \langle \chi_B, \varphi_n \rangle_{L^2(\nu_2)} R(f_n) \right) \, d\nu_1 \\
&= \int_A P(x, B) \, d\nu_1(x).
\end{align*}
\]

(2) It follows from (1) that the measure \( \rho \) can be defined as we did before:

\[ \rho(A \times B) = \int_A P(x, B) \, d\nu_1(x) = \int_B Q(y, A) \, d\nu_2(y). \]

To prove the result we compute
\[
\int_B Q(y, A) \, dv_2(y) = \int_{X_2} \chi_B(y) \left( \sum_{n=1}^{\infty} \widehat{k}_n(A) \varphi_n(y) \right) \, dv_2(y)
\]
\[
= \int_{X_2} \chi_B(y) \left( \sum_{n=1}^{\infty} \langle R(\chi A), f_n \rangle_{L^2(\nu_1)} \varphi_n(y) \right) \, dv_2(y)
\]
\[
= \sum_{n=1}^{\infty} \left( \int_B \langle R(\chi A), f_n \rangle_{L^2(\nu_1)} \varphi_n(y) \right) \, dv_2(y)
\]
\[
= \sum_{n=1}^{\infty} \langle R(\chi A), f_n \rangle_{L^2(\nu_1)} \langle \chi_B, \varphi_n \rangle_{L^2(\nu_2)}
\]
\[
= \sum_{n=1}^{\infty} \widehat{k}_n(A) \langle \chi_B, \varphi_n \rangle_{L^2(\nu_2)}.
\]

(3) The proof of this statement is similar to that given in (1). We sketch it here. Since

\[
T_Q(f)(y) = \int_{X_1} Q(y, dx) f(x) = \sum_{n=1}^{\infty} \langle f, R(f_n) \rangle_{L^2(\nu_1)} \varphi_n(y),
\]

and

\[
T_P(g)(y) = \int_{X_2} P(x, dy) g(y) = \sum_{n=1}^{\infty} \langle g, \varphi_n \rangle_{L^2(\nu_2)} R(f_n)(x)
\]

we have

\[
\langle g, T_Q(f) \rangle_{L^2(\nu_2)} = \int_{X_2} g(y) \left( \sum_{n=1}^{\infty} \varphi_n(y) \int_{X_1} R(f_n)(x) f(x) \, d\nu_1(x) \right) \, dv_2(y)
\]
\[
= \sum_{n=1}^{\infty} \langle g, \varphi_n \rangle_{L^2(\nu_2)} \langle R(f_n), f \rangle_{L^2(\nu_1)}
\]
\[
= \int_{X_1} \left( \sum_{n=1}^{\infty} \langle g, \varphi_n \rangle_{L^2(\nu_2)} R(f_n)(x) \right) f(x) \, d\nu_1(x)
\]
\[
= \langle T_P(g), (f) \rangle_{L^2(\nu_1)}.
\]

\[\square\]

**Remark 7.11.** We proved in Proposition 7.10 that the measures \( P(x, \cdot) \) and \( Q(y, \cdot) \) have the property

\[
\frac{P(x, dy)}{dv_2(y)} = \sum_{n=1}^{\infty} (Rf_n)(x) \varphi_n(y) = \frac{Q(y, dx)}{d\nu_1(x)}.
\]

**Theorem 7.12.** Let the kernels \( P \) and \( Q \) be defined as in Definition 7.8. Then \( \widehat{R} = T_P T_Q \), i.e., the corresponding operators \( T_P \) and \( T_Q \) factorize the operator \( \widehat{R} : L^2(\nu_1) \to L^2(\nu_1) \).

**Proof.** To obtain the result, it suffices to show that, for any sets \( A_1, A_2 \in D(\nu_1) \),

(7.15)

\[
\langle \chi_{A_1}, T_P T_Q(\chi_{A_2}) \rangle_{L^2(\nu_1)} = \langle \chi_{A_1}, R(\chi_{A_2}) \rangle_{L^2(\nu_1)}.
\]
Then relation (7.15) can be extended to all functions from dense sets $F$ and $\mathcal{F}(\nu_1)$, $\mathcal{F}(\nu_2)$. Since the operator $T_p T_Q$ is self-adjoint, we obtain the equality $\hat{R} = T_p T_Q$.

In the computation below we will use Proposition 7.10 (3), relation (7.13), Lemma 7.3, and Theorem 7.6. Recall that $(\varphi_n)$ is an ONB in $L^2(\nu_2)$.

$$
\langle \chi_{A_1}, T_p T_Q(\chi_{A_2}) \rangle_{L^2(\nu_1)} = \langle \chi_{A_1}, (T_Q)^* T_Q(\chi_{A_2}) \rangle_{L^2(\nu_1)} \\
= \langle T_Q(\chi_{A_1}), T_Q(\chi_{A_2}) \rangle_{L^2(\nu_2)} \\
= \int_{X_2} Q(y, A_1) Q(y, A_2) \, d\nu_2(y) \\
= \int_{X_2} \left( \sum_n \hat{k}_n(A_1) \varphi_n(y) \right) \left( \sum_m \hat{k}_m(A_2) \varphi_m(y) \right) \, d\nu_2(y) \\
= \sum_n \hat{k}_n(A_1) \hat{k}_n(A_2) \\
= K(A_1, A_2) \\
= \lambda_1(A_1 \times A_2) \\
= \langle \chi_{A_1}, R(\chi_{A_2}) \rangle_{L^2(\nu_1)}. 
$$

**Remark 7.13.** In this remark, we show another proof of the Theorem 7.12. It is based on the direct computation of $T_p(T_Q(f))$ using the formulas given in (7.13) and (7.14) and Remark 7.11. As in the proof of Theorem 7.12 we take an arbitrary function $f \in \mathcal{F}(\nu_1)$ and note that $\hat{R}$ and $R$ coincide on this set.

$$
T_p(T_Q(f))(x) = \int_{X_2} P(x, dy) T_Q(f)(y) \\
= \int_{X_2} P(x, dy) \int_{X_1} Q(y, dx) f(x) \\
= \int_{X_2} P(x, dy) \left( \sum_{n=1}^{\infty} \langle f, R(f_n) \rangle_{L^2(\nu_1)} \varphi_n(y) \right) \\
= \int_{X_2} \left( \sum_{n=1}^{\infty} (R f_n)(x) \varphi_n(y) \right) \left( \sum_{n=1}^{\infty} \langle f, R(f_n) \rangle_{L^2(\nu_1)} \varphi_n(y) \right) \, d\nu_2(y) \\
= \sum_{n=1}^{\infty} \langle f, R(f_n) \rangle_{L^2(\nu_1)} (R f_n)(x) \\
= \hat{R}^{1/2} \left( \sum_{n=1}^{\infty} \langle \hat{R}^{1/2}(f), \hat{R}^{1/2}(f_n) \rangle_{L^2(\nu_1)} \hat{R}^{1/2}(f_n) \right) \\
= \hat{R}(f).
$$

**Lemma 7.14.** Let $\lambda_1$ on $X_1 \times X_1$ be the symmetric measure defined by the operator $R : L^2(\nu_1) \to L^2(\nu_1)$ as in (7.11). Then

$$
\lambda_1(A_1 \times A_2) = \int_{X_2} Q(y, A_1) Q(y, A_2) \, d\nu_2(y), \quad A_1, A_2 \in \mathcal{D}(\nu_1).
$$
In particular,

\[ \lambda(A \times A) < \infty \iff Q(\cdot, A) \in L^2(\nu_2), \]

for \( A \in \mathcal{D}(\nu_1) \).

Proof. It follows from Theorem 7.12 that \( R(x, A) = QP(x, A) = \int_{X_2} P(x, dy)Q(y, A) \). We use it in the following computation:

\[
\begin{align*}
\lambda_1(A_1 \times A_2) &= \int_{X_1} \chi_{A_1} R(\chi_{A_2}) \, d\nu_1 \\
&= \int_{X_1} \chi_{A_1} \left( \int_{X_2} P(x, dy)Q(y, A_2) \right) \, d\nu_1 \\
&= \int_{X_2} \int_{A_1} Q(y, A_2)Q(y, dx) \, d\nu_2(y) \\
&= \int_{X_2} Q(y, A_1)Q(y, A_2) \, d\nu_2(y).
\end{align*}
\]

The particular case \( A_1 = A_2 \) gives the second statement of the lemma.

\[ \square \]

8. Measurable Bratteli diagrams

In this section, we consider a measurable version of generalized Bratteli diagrams. The main differences are: (i) the levels \( V_n \) of a measurable Bratteli diagram are formed by standard Borel spaces \((X_n, \mathcal{A}_n)\), and (ii) the sets of edges \( E_n \) are Borel subsets of \( X_n \times X_{n+1}, n \in \mathbb{N}_0 \). An important subclass of measurable Bratteli diagrams is obtained when \( X_n = X \) and the sets \( E_n \) are represented by equivalence relations.

Our goal in this final section is to show that the main notions, and some results, from the case of discrete levels carry over to the more general case when instead the levels are measure spaces. This entails new developments in the analysis of path-space measures and the associated Markov processes. Our treatment is brief, and full details are planned for a future paper.

8.1. Definitions, dynamics, and applications. We begin this section with definitions of main objects.

Definition 8.1. Let \((X, \mathcal{A})\) be an uncountable standard Borel space. Let \((X_n, \mathcal{A}_n) : n \geq 0\) be a sequence of standard Borel spaces (each of them is Borel isomorphic to \((X, \mathcal{A}))\). Suppose that \((E_n)\) is a sequence of Borel subsets of \( X_n \times X_{n+1} \) such that the projections \( s_n : E_n \to X_n \) and \( r_n : E_n \to X_{n+1} \) are onto Borel maps where \( s_n(e) = s_n(x, y) = x \) and \( r_n(e) = r_n(x, y) = y \) for every \( e = (x, y) \in E_n \). Then we call \( B = (X_n, E_n) \) a measurable Bratteli diagram. The pair \((X_n, E_n)\) is called the \( n\)-th level of the measurable Bratteli diagram \( B \).

Remark 8.2. (1) We emphasize that all levels \( X_n \) are formed by the same Borel space \( X \), so that we can (if necessary) treat a subset \( A \) of \( X \) as a subset of every \( X_n \). Recall that this identification is used for stationary Bratteli diagrams (standard and generalized diagrams). On the other hand, the sets \( E_n \) are not identified.

(2) An important particular case of a measurable Bratteli diagram is formed by countable Borel equivalence relations (CBER) \( E_n \), see e.g. [JKL02], [DJK94a] for details on CBER.
Briefly speaking, this means that (i) \( E_n \) is a symmetric Borel subset of \( X_n \times X_{n+1} \) containing the diagonal, (ii) \( (x, y) \in E_n \) and \( (y, z) \in E_n \) implies \( (x, z) \in E_n \), and (iii) for any \( x \in X_n \), the set \( \{ y \in X_{n+1} : (x, y) \in E_n \} \) is countable. We note that the requirement that \( s_n, r_n \) are Borel onto maps is included in the definition of a measurable Bratteli diagram. This property is automatically true when \( E_n \) is a CBER.

(3) Generalizing the definition of \( s_n \) and \( r_n \), one can define two maps \( r, s : \bigcup E_n \to \bigcup X_n \) by setting \( s(e) = s_n(e) \) and \( r(e) = r_n(e) \) where \( e \in E_n \). Clearly, \( r \) and \( s \) are Borel maps since they coincide with \( r_n \) and \( s_n \) on each \( E_n \). This observation allows us to simplify our notation and omit the subindex \( n \) working with the maps \( r, s \).

(4) In some cases, it is useful to represent the set \( E_n, n \in \mathbb{N}_0 \), as the union of sections:

\[
\bigcup_{x \in X_n} s^{-1}(x) = E_n = \bigcup_{y \in X_{n+1}} r^{-1}(y).
\]

They are analogous to “vertical” and “horizontal” sections of a subset in the product of two spaces. The fact that and \( r_n \) in Definition 8.1 is an onto map means that \( \forall y \in X_{n+1} \exists x \in X_n \text{ such that } e = (x, y) \in E_n \). A similar property holds for \( s_n \).

Suppose now that the standard Borel space \((X_0, \mathcal{A}_0)\), the initial level of a measurable Bratteli diagram \( \mathcal{B} \), is endowed with a \( \sigma \)-finite atomless Borel positive measure \( \nu_0 \) so that \((X_0, \mathcal{A}_0, \nu_0)\) is a standard measure space. We will consider a sequence of probability transition kernels \((P_n)\) (they are called \textit{Markov kernels}) on a measurable Bratteli diagram \( \mathcal{B} \). Recall that a map \( R : X \times \mathcal{A} \to [0, 1] \) is called a probability transition kernel if: (a) \( x \mapsto R(x, A) \) is a measurable map for every set \( A \in \mathcal{A} \), (b) for every \( x \in X \), the function \( A \mapsto R(x, A) \) is a probability measure.

\textbf{Definition 8.3.} Let \( \mathcal{B} = (X_n, E_n) \) be a measurable Bratteli diagram as in Definition 8.1. Let \((P_n)\) be a sequence of probability Markov kernels such that \( P_n : X_n \times \mathcal{A}_{n+1} \to [0, 1] \), \( n \in \mathbb{N}_0 \). We say that the sequence \((P_n)\) is \textit{consistent} with the diagram \( \mathcal{B} \), if, for every \( x \in X_n \) and \( n \in \mathbb{N}_0 \), the map \( x \mapsto P_n(x, \cdot) \) determines a probability measure on the set \( r(s^{-1}(x_n)) \subset X_{n+1} \) such that \( P(x_n, \cdot) \ll \nu_{n+1} \), i.e., \( P(x_n, r(s^{-1}(x_n))) = 1 \) for all \( x_n \in X_n \) and \( n \in \mathbb{N}_0 \).

If all \((X_n, \mathcal{A}_n) = (X, \mathcal{A})\) and \( P_n = P, n \in \mathbb{N}_0 \), then the corresponding measurable Bratteli diagram \( \mathcal{B} \) is called \textit{stationary}.

We are now in the setting that we used in Section 5. The only difference is that we have to work with a sequence of Markov kernels \((P_n)\) which generates a sequence of linear operators

\[
T_{P_n}(f) = \int_{X_{n+1}} P_n(x, dx_{n+1})f(y), \quad x_n \in X_n.
\]

Every \( T_{P_n} \) sends bounded Borel functions \( f \in \mathcal{F}(X_{n+1}, \mathcal{A}_{n+1}) \) into the set \( \mathcal{F}(X_n, \mathcal{A}_n) \).

Similarly, we can define an action of operators \( P_n \) on the set of measures \( M(X_n, \mathcal{A}_n) \). More precisely, we set

\[
(\mu P_n)(f) = \int_{X_n} P_n(f) \, d\mu, \quad f \in \mathcal{F}(X_{n+1}, \mathcal{A}_{n+1}), \quad \mu \in M(X_{n+1}, \mathcal{A}_{n+1}).
\]

Then \( P_n : M(X_n, \mathcal{A}_n) \to M(X_{n+1}, \mathcal{A}_{n+1}) \). Note that \( \mu P_n \) is a probability measure if and only if \( \mu \) is probability.
Theorem 8.4. Let $\nu_0$ be a $\sigma$-finite measure on the initial level $(X_0, A_0)$ of a measurable Bratteli diagram $B = (X_n, E_n)$. Suppose that $(P_n)$ is a sequence of Markov kernels consistent with $B$. Then:

(a) there exists a sequence of $\sigma$-finite measures $(\nu_n)$, where $\nu_n \in M(X_n, A_n)$, such that

$$\nu_n = \nu_{n+1} Q_n, \quad \nu_{n+1} = \nu_n P_n, \quad n \in \mathbb{N}_0,$$

(b) there exists a sequence of transition kernels $(Q_n)$, where $Q_n : X_{n+1} \times A_n \to [0, \infty)$, such that $(P_n, Q_n)$ form a dual pair for all $n$, i.e.,

$$\nu_n(x_n) P_n(x_n, dx_{n+1}) = \nu_{n+1}(x_{n+1}) Q(x_{n+1}, dx_n),$$

(c) there exists a sequence of measures $(\rho_n)$, which supported by the sets $E_n$, such that

$$\rho_n(A \times B) = \int_A \nu_n(x_n) P(x_n, B) = \int_B \nu_{n+1}(x_{n+1}) Q_n(x_{n+1}, A),$$

where $A \times B \in A_n \times A_{n+1}$,

(d) for all $n \geq 0$, $P_n(x_n, \cdot) \ll \nu_{n+1}$ $\nu_n$-a.e. and $Q_n(x_{n+1}, \cdot) \ll \nu_n \nu_{n+1}$-a.e.

Proof. The proposition follows from the results proved in Section 5, see Lemma 5.14, Theorem 5.16, Corollary 5.17, and Corollary 5.18. We mention here a few moments that are needed for further developments.

Beginning with the initial measure $\nu_0$ and using the sequence of Markov kernels $(P_n)$ we define first a measure $\nu_1 = \nu_0 P_0$, and then inductively $\nu_{n+1} = \nu_n P_n$ for all $n$.

Next, we define the measures $\rho_n$ by setting

$$(8.2) \quad d\rho_n(x_n, x_{n+1}) = \nu_n(x_0) P_n(x_n, dx_{n+1}).$$

Since $(P_n)$ is consistent with the measurable Bratteli diagram and $E_n = \bigcup_{x_n} E_n(x_n) = \bigcup_{x \in X_n} s^{-1}(x_n)$, we see from (8.2) that $\rho_n$ is defined on $E_n$. Moreover, the measures $\nu_n$ and $\nu_{n+1}$ are projections of $\rho_n$ on $X_n$ and $X_{n+1}$, i.e., $s(\rho_n) = \nu_n$ and $r(\rho_n) = \nu_{n+1}$.

In order to define the dual transition kernel $Q_n$, we use the formula

$$Q_n(x_{n+1}, A) = \frac{\rho_n(A, dx_{n+1})}{\nu_{n+1}(x_{n+1})}, \quad A \in A_n,$$

(see also Theorem 5.16 for the definition of $Q_n$).

Remark 8.5. The results formulated in Theorem 8.4 can be obtained from another equivalent setting. Namely, let $B$ be a measurable Bratteli diagram with levels $(X_n, A_n)$ and $E_n$. Suppose that a sequence of $\sigma$-finite measures $(\nu_n)$ is chosen on the spaces $(X_n, A_n)$. Take measures $\rho_n$, supported by $E_n$, such that $\rho_n \in L(\nu_n, \nu_{n+1})$. Then there are probability transition kernels $P_n$ that satisfy Theorem 8.4, see Corollaries 5.13 and 5.18.

Using the objects defined in Theorem 8.4, we can determine the path-space measure $\mathbb{P}$ on the space $X_B$ of paths of a measurable Bratteli diagram $B$.

Definition 8.6. Let $B$ be a measurable Bratteli diagram such as in Definitions 8.1 and 8.3. Define the path-space of $X_B$ to be a subset of $E_0 \times E_1 \times \cdots$ formed by sequences $(e_i)$ such that $s(e_{i+1}) = r(e_i), i \geq 0$. The space $X_B$ can be equivalently described as a set of sequences $(x_i), i \geq 0$ where $x_i = s(e_i)$. Denote by $X_B(x)$ the subset of $X_B$ consisting of all paths that begin at $x \in X_0$.
Let $\nu_0$ be a $\sigma$-finite measure on $(X_0, \mathcal{A}_0)$ and $(P_n)$ the sequence of probability transition kernels. Define a probability measure $\mathbb{P}_x$ for $x \in X_0$. Let $f = f(x_1, \cdots, f_n)$ be a Borel bounded function on $\mathcal{X}_B(x)$ depending on the first $n$ coordinates (cylinder function). We set

$$
(8.3) \int f \ d\mathbb{P}_x = \int_{X_1} \cdots \int_{X_n} P_0(x, dx_1) P_1(x_1, dx_2) \cdots P_{n-1}(x_{n-1}, dx_n) f(x_1, \cdots, x_n).
$$

Then $\mathbb{P}_x$ is extended to a probability measure on the set $\mathcal{X}_B(x)$.

In more details, the measure $\mathbb{P}$ is computed as follows.

$$
\int_{\mathcal{X}_B} g \ d\mathbb{P} = \int_{\mathcal{X}_0} \mathbb{P}_x(g) \ d\nu_0(x_0)
= \int_{X_0} \cdots \int_{X_n} d\nu_0(x_0) P_0(x_0, dx_1) \cdots P_{n-1}(x_{n-1}, dx_n) g(x_0, \cdots, x_n)
$$

where $g$ is a cylinder function. Extending $\mathbb{P}$ to all Borel bounded functions, we obtain a measure which is called a path-space measure on $\mathcal{X}_B$.

**Remark 8.7.** Let $\xi_n : \mathcal{X}_B \to (X_n, \mathcal{A}_n)$ be a random variable such that $\xi_n(x_i) = x_n$, $n \in \mathbb{N}$. The values of the path-space measure $\mathbb{P}$ on the cylinder sets can be also found by the formula:

$$
(8.5) \mathbb{P}_x(\xi_1 \in A_1, \ldots, \xi_n \in A_n) = \int_{A_n} \cdots \int_{A_1} P_0(x, dx_1) \cdots P_{n-1}(x_{n-1}, dx_n)
$$

In particular, $\mathbb{P}_x(\xi_1 \in B) = P(x, B)$.

As a consequence of (5.2), we have also the following formula:

$$
(8.6) \mathbb{P}_x(\xi_1 \in A_1, \ldots, \xi_n \in A_n) = P_0(\chi_{A_1} P_1(\chi_{A_2} P_2(\cdots P_{n-1}(\chi_{A_n}) \cdots)))(x).
$$

**Lemma 8.8.** For a measurable Bratteli diagram $B$ as above, let $\xi_n : \mathcal{X}_B \to X_n$ be the random variable defined by $\xi_n(x) = x_n$ where $x = (x_i)$. Then the transition probability kernels $P_n$ and $Q_n$ can be restored as follows:

$$
\mathbb{E}(\xi_{n+1} \in B \mid \xi_n = x) = P_n(x, B), \quad \mathbb{E}(\xi_{n+1} = y \mid \xi_n = y) = Q_n(y, A)
$$

where $\mathbb{E}$ is the mathematical expectation with respect to the path-space measure $\mathbb{P}$, $A \in \mathcal{A}_n$, $B \in \mathcal{A}_{n+1}$, and $n \in \mathbb{N}_0$.

As in the case of discrete Bratteli diagrams, we can define the notion of tail equivalence relation $\mathcal{R}$: two paths $x = (x_n)$ and $y = (y_n)$ from the path-space $\mathcal{X}_B$ are called tail equivalent if there exists some $m$ such that $x_n = y_n$ for all $n \geq m$. Clearly, $\mathcal{R}$ is a Borel equivalence relation with uncountable classes.

Now we define an analogue of the tail invariant measure in the case of measurable Bratteli diagrams.

**Definition 8.9.** Let $\mathbb{P}$ be a path-space measure on the set $\mathcal{X}_B$ where $B$ is a measurable Bratteli diagram, see Definition 8.6 for notation. The measure $\mathbb{P}$ is called *tail invariant* if
for every \( n \in \mathbb{N} \) and every \( A \subset X_n \), \( \nu_n(A) > 0 \) the function \( \mathbb{P}(\xi_n \in A \mid \xi_0 = x) = \mathbb{P}_x(\xi_n \in A) \) does not depend on \( x \in X_0 \) \( \nu_0 \)-a.e.

The existence of tail invariant measures follows from the following observation.

**Lemma 8.10.** Let \( \mathcal{B} \) be a measurable Bratteli diagram with levels represented by probability measure spaces \((X_n, \mathcal{A}_n, \nu_n)\). Suppose that the probability transition kernels are defined by \( P_n(x, B) = \nu_{n+1}(B) \), \( B \in \mathcal{A}_{n+1} \). Define the measures \( \mathbb{P}_x \) and \( \mathbb{P} \) as in (8.3) and (8.4). Then \( \mathbb{P} \) is a tail equivalent measure.

The proof is obvious because \( \mathbb{P}_x(\xi_n \in B) = \nu_{n+1}(B) \).

### 8.2. Harmonic functions and graph Laplacians on measurable Bratteli diagrams

Let \( \mathcal{B} \) be a measurable Bratteli diagram defined in Subsection 8.1. Denote the \( n \)-th level of \( \mathcal{B} \) by \((X_n, \mathcal{A}_n, \nu_n)\), a \( \sigma \)-finite measure space. Suppose that we have two sequences of finite transition kernels \( P = (P_n) \) and \( Q = (Q_n) \) compatible with the diagram \( \mathcal{B} \) and such that \( P_n : X_n \times \mathcal{A}_{n+1} \to [0, \infty) \) and \( Q_n : X_{n+1} \times \mathcal{A}_n \to [0, \infty) \). Assume also that they satisfy the following conditions:

(i) \((P_n, Q_n)\) is a dual pair, i.e., they satisfy the equality

\[
\int_{X_n} c_n(x_n) \, d\nu_n(x_n) = \int_{X_{n+1}} d_{n+1}(x_{n+1}) \, d\nu_{n+1}(x_{n+1}), \quad n \in \mathbb{N}_0,
\]

which determines a measure \( \rho_n \) on \( X_n \times X_{n+1} \).

(ii) the Borel function \( c_n(x_n) = P_n(x_n, X_{n+1}) \) and \( d_{n+1}(x_{n+1}) = Q_n(x_{n+1}, X_n) \) are locally integrable.

It follows from the definition of \((P_n, Q_n)\) that

\[
\int_{X_n} c_n(x_n) \, d\nu_n(x_n) = \int_{X_{n+1}} d_{n+1}(x_{n+1}) \, d\nu_{n+1}(x_{n+1}), \quad n \in \mathbb{N}_0.
\]

Let \( F = (F_n) \) be an arbitrary Borel function such that every \( F_n \in L^2(\nu_n) \). Define the actions of \( P \) and \( Q \) on the functions \( F \):

\[
P(F) = (T_{P_n}(F_{n+1})), \quad T_{P_n}(F_{n+1})(x_n) = \int_{X_{n+1}} P_n(x_n, dy)F_{n+1}(y),
\]

\[
Q(F) = (T_{Q_n}(F_n)), \quad T_{Q_n}(F_n)(x_{n+1}) = \int_{X_n} Q_n(x_{n+1}, dy)F_n(y).
\]

Recall that by Theorem 6.6 we have the following properties: (a) if \( P_n \) and \( Q_n \) are probability kernels, then \( T_{P_n} \) and \( T_{Q_n} \) are contractive operators such that \( T_{P_n} : L^2(\nu_{n+1}) \to L^2(\nu_n) \), \( T_{Q_n} : L^2(\nu_n) \to L^2(\nu_{n+1}) \), and \( T_{P_n} = T_{Q_n} \); (b) if \( P_n \) and \( Q_n \) are finite transition kernels, then \( T_{P_n} \) and \( T_{Q_n} \) constitute a dual pair of densely defined operators between the two \( L^2 \)-spaces, provided the conditions \( T_{P_n} \mathcal{D}(\nu_{n+1}) \subset L^2(\nu_n) \), \( T_{Q_n} \mathcal{D}(\nu_n) \subset L^2(\nu_{n+1}) \) hold. Moreover, \( T_{P_n} \subset (T_{Q_n})^* \) and \( T_{Q_n} = (T_{P_n})^* \).

In the next definition, we apply the approach used in Section 4. It is worth noting that every finite transition kernel can be normalized, if necessary.

**Definition 8.11.** Let \((P_n)\) and \((Q_n)\) be two sequences of finite transition kernels satisfying the properties (i) and (ii) described above. Let \( F = (F_n) \) be a sequence of bounded Borel
functions. Define a new probability transition kernel $M = (M_n)$ by the formula

$$M(F) = (M(F)_n), \quad M(F)_n = \frac{1}{2} \left( \frac{1}{c_n} P_n(F_{n+1}) + \frac{1}{d_n} Q_{n-1}(F_{n-1}) \right), \tag{8.7}$$

or $M(F)_n = \tilde{P}_n(F_{n+1}) + \tilde{Q}_{n-1}(F_{n-1})$ where $2c_n \tilde{P}_n = P_n$ and $2d_n \tilde{Q}_{n-1} = Q_{n-1}$.

The operator $\Delta$ such that

$$\Delta F)_n = (c_n + d_n) F_n - P_n(F_{n+1}) - Q_{n-1}(F_{n-1}) \tag{8.8}$$

is called a Laplace operator.

A function $H = (H_n)$ such that $M(H)_n = H_n$ (or, equivalently, $\Delta(H)_n = 0$ is called a harmonic function.

From Definition 8.11, we see that $M = (M_n)$ has the property $M_n(1) = 1$.

**Lemma 8.12.** Formulas (8.7) and (8.8) can be expanded as follows:

$$M(F)_n(x_n) = \frac{1}{2} \left( \frac{1}{c_n(x_n)} \int_{X_{n+1}} P_n(x_n, dy) F_{n+1}(y) + \frac{1}{d_n(x_n)} \int_{X_{n-1}} Q_{n-1}(x_n, dy) F_{n-1}(y) \right)$$

and

$$\Delta(F)_n(x_n) = \int_{X_{n+1}} P_n(x_n, dy) (F_n(x_n) - F_{n+1}(y)) + \int_{X_{n-1}} Q_{n-1}(x_n, dy) (F_n(x_n) - F_{n-1}(y)).$$

If $c_n = d_n$, then

$$\Delta(F)_n = 2(c_n F_n - (MF)_n) = 2c_n(Id - M)(F)_n.$$

In what follows we show another approach for the definition of a Laplace operator. Our main references here are [BJ19a, JP19, BJa].

**Remark 8.13.** Let $(X_B, \mathbb{P})$ be the path-space of a measurable Bratteli diagram equipped with a path-space measure $\mathbb{P}$. It is a standard measure space with a $\sigma$-finite measure, in general. Consider the Cartesian product $X_B \times X_B$ and take a symmetric measure $\lambda$ on $X_B \times X_B$. The support of $\lambda$ is a symmetric subset $Z(\lambda) \subset X_B \times X_B$. We discussed symmetric measures in Sections 6 and 7. Then $\lambda$ admits a disintegration

$$\lambda = \int_{X_B} \lambda_x \ d\mathbb{P}(x).$$

Our choice of $\lambda$ is restricted by the property that $c(x) = \lambda_x(X_B)$ must be finite for a.e. $x$.

Define a linear operator $\overline{\Delta}$ acting on the space of bounded Borel functions on $(X_B, \mathbb{P})$:

$$\overline{\Delta}(f)(x) = \int_{X_B} (f(x) - f(y)) d\lambda_x(y) = c(x) \left( f(x) - \int_{X_B} f(y) \ d\tau_x(y) \right) \tag{8.9}$$

where $d\tau_x(y) = \frac{1}{c(x)} d\lambda_x(y)$.

The operator $T : f \mapsto \int_{X_B} f \ d\tau_x$ is a Markov operator.

We define now the finite energy space $\mathcal{H}_E$. Suppose that a measurable Bratteli diagram $B$ is defined by an initial distribution $\nu_0$ and a sequence $(P_n)$ of Markov kernels. These objects generate a sequence of dual kernels $(Q_n)$ and measures $(\rho_n)$. Recall that $\rho_n$ is the measure which is defined by the equality $\rho(dx, dy) = d\nu_n(x) P_n(x, dy) = d\nu_{n+1}(y) Q(y, dx)$, $x \in X_n, y \in X_{n+1}$. 
Definition 8.14. For a measurable Bratteli diagram, let \( F = (F_n) \) be a Borel function on the path-space \( X_B \) of \( B \). It is said that a function \( F \) has finite energy and belongs to the finite energy space \( \mathcal{H}_E \) if

\[
(8.10) \quad ||F||^2_{\mathcal{H}_E} = \sum_{n=0}^{\infty} \int_{X_n \times X_{n+1}} (F_n(x) - F_{n+1}(y))^2 \, d\rho_n(x,y) < \infty \quad x \in X_n, y \in X_{n+1}.
\]

As a matter of fact, the elements of \( \mathcal{H}_E \) are the classes of functions: two functions \( F \) and \( F' \) are identified if \( F - F' = \text{const} \). It can be checked that \( \mathcal{H}_E \), equipped with the norm as in (8.10), is a Hilbert space.

Proposition 8.15. Let \( B \) be a measurable Bratteli diagram with levels represented by measure spaces \( (X_n, \mathcal{A}_n, \nu_n) \). Suppose that \( (P_n) \) and \( (Q_n) \) are sequences of dual probability transition kernels compatible with \( B \). Then a function \( F = (F_n) \) on the path-space of \( B \) belongs to the finite energy space \( \mathcal{H}_E \) if and only if

\[
\sum_{n \geq 0} \left( ||F_n||^2_{L^2(\nu_n)} - 2 \langle F_n, T_{P_n}(F_{n+1}) \rangle_{L^2(\nu_n)} + ||F_{n+1}||^2_{L^2(\nu_{n+1})} \right) < \infty.
\]

Proof. We first remind that if \( P_n \) is a probability transition kernel, then \( T_{P_n} : L^2(\nu_n) \to L^2(\nu_{n+1}) \) is a contractive operator, see Theorem 6.6. Expanding the expression in (8.10), we obtain

\[
||F||^2_{\mathcal{H}_E} = \sum_{n=0}^{\infty} \int_{X_n \times X_{n+1}} (F_n^2(x) - 2F_n(x)F_{n+1}(y) + F_{n+1}^2(y)) \, P_n(x,dy)\nu_n(x)
\]

\[
= \sum_{n=0}^{\infty} \left[ \int_{X_n} \int_{X_{n+1}} F_n^2(x) \, P_n(x,dy)\nu_n(x) - 2 \int_{X_n} \int_{X_{n+1}} F_n(x)F_{n+1}(y) \, P_n(x,dy)\nu_n(x) \right.
\]

\[
+ \left. \int_{X_n} \int_{X_{n+1}} F_{n+1}^2(y) \, P_n(x,dy)\nu_n(x) \right]
\]

\[
= \sum_{n=0}^{\infty} \left( \int_{X_n} F_n^2(x) \, d\nu_n(x) - \int_{X_n} F_n(x)T_{P_n}(F_{n+1})(x) \, d\nu_n(x) \right.
\]

\[
+ \left. \int_{X_n} \int_{X_{n+1}} F_{n+1}^2(y) \, Q_n(y,dx)\nu_{n+1}(y) \right)
\]

\[
= \sum_{n=0}^{\infty} \left( ||F_n||^2_{L^2(\nu_n)} - 2 \langle F_n, T_{P_n}(F_{n+1}) \rangle_{L^2(\nu_n)} + ||F_{n+1}||^2_{L^2(\nu_{n+1})} \right)
\]

\[\square\]

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