Lorentz transformations of the electric and magnetic fields according to Minkowski

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Abstract
The usual transformations (UT) of the 3-vectors $E$ and $B$ that were found by Lorentz and by Poincaré and independently by Einstein in 1905 are generally considered to be the Lorentz transformations (LT) (boosts) of $E$ and $B$. According to the UT, $E$ in one frame is ‘seen’ as $E'$ and $B'$ in a relatively moving frame. In Minkowski’s last paper, in section 11.6 of Minkowski (1908 Nachr. Ges. Wiss. Göttingen 53), he defined the vectors (4-vectors in the usual notation) of the electric $\Phi$ and magnetic $\Psi$ fields and discovered that, for example, $\Phi$ correctly transforms to $\Phi'$ by the LT. His correct LT are reinvented and generalized in Ivezic (2005 Found. Phys. Lett. 18 301). In this study, we show the essential similarity and some differences between Minkowski’s relations in section 11.6 (1908) and the results obtained in Ivezic (2005). The general expressions for the LT (boosts) and UT of vectors $E$ and $B$ are presented. They are not contained either in Minkowski (1908) or in Ivezic (2005). The low-velocity limit of the UT and LT is briefly examined.

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1. Introduction

It is generally accepted by the physics community that there is agreement between classical electromagnetism and special relativity as it is formulated in [1]. Both, in prerelativistic physics and in special relativity, the electric and magnetic fields are represented by the 3-vectors $E$ and $B$ (equations (11.148) and (11.149) in [2], i.e. $E$ and $B$ are called 3-vectors and they are designated by boldface type.) The usual transformations (UT) of the 3-vectors $E$ and $B$ (equations (11.148) and (11.149) in [2], i.e. equation (2) here) are always considered to be the relativistically correct Lorentz transformations (LT) (boosts) of $E$ and $B$. (In this paper, under the name LT we shall only consider boosts.) They were first derived by Lorentz [3] and Poincaré [4] (see also two fundamental Poincaré papers with notes by Logunov [5]) and independently by Einstein [1]. According to the UT, the $E$ in one frame is ‘seen’ as $E'$ and $B'$ in a relatively moving frame. The derivation of the UT of $E$ and $B$ will be briefly presented and discussed in section 2.

Minkowski [6, section 11.6], defined the vectors (4-vectors in the usual notation) of the electric $\Phi$ and magnetic $\Psi$ fields (equation (7) below) and discovered that, for example, $\Phi$ transforms by the correct LT (equation (9) below) again to $\Phi'$; there is no mixing with the magnetic field vector $\Psi$.

His mathematically correct LT of fields remained almost completely unknown. They are not mentioned even in Annalen der Physik, Special Topic Issue 9-10/2008: The Minkowski spacetime of special relativity—100 years after its discovery. Perhaps one of the reasons for such a systematic neglect of his important contribution was that Minkowski himself never applied these transformations of the vector fields. In all other parts of [6], he dealt with the usual 3-vectors $E$ and $B$. Recently in [7] for the first time, the importance and the relativistic correctness of section 11.6 of [6] and also the apparent similarity between the Minkowski results mentioned and recent results obtained in [8–11] were noted.

In [8–11], Minkowski’s correct LT from section 11.6 are reinvented and generalized. In this study, we show the essential similarity and some differences between Minkowski’s relations in section 11.6 and the results obtained in [8–11]. It is proved in [8–11] that the LT always transform an algebraic object defined on the 4D spacetime representing...
the electric field only to the electric field, and similarly for the magnetic field, as in equations (12)–(14) below.

Here, in section 3, it is shown that Minkowski’s relations (7)–(9) below correspond to relations (5), (4) and (12) below, obtained in, e.g., [9]. The explicit form of the LT of the electric field vector (13) was not discovered either in Minkowski’s paper [6] or in [8–11]. As can be seen from equation (12) below, the LT of vectors \( E \) and \( B \) are derived transforming both the \( F \) field and the observer \( \gamma_0 \).

In section 4, if only \( F \) is transformed by the LT, but not the observer \( \gamma_0 \), then the transformation of the vector \( E = F \gamma_0 \) is given by equations (15)–(17) below. Equation (16) was first reported in [7]. It is shown in that section that the components of the transformed \( E' \) are nothing other than the components that are obtained by the LT of the 3-vector \( E \), equation (11.148) of [2]. This result undoubtedly reveals that the UT of the 3-vectors \( E \) and \( B \) (equation (2) below) differ from the correct LT (12)–(14). Furthermore, in section 5, the low-velocity limit of the UT of the 3-vectors \( E \) and \( B \) and of the LT of vectors \( E \) and \( B \) are briefly discussed. In section 6, the conclusions are presented.

2. On the UT of the 3-vectors \( E \) and \( B \)

In the usual covariant approaches, the field-strength tensor \( F^{\alpha\beta} \) (only components) is introduced and defined in terms of the vector potential \( A^\alpha \) (the Greek indices run from 0 to 3), equation (11.136) in [2]. The six independent components of \( F^{\alpha\beta} \) are defined to be six components of the 3-vectors \( E \) and \( B \); these identifications are

\[
E^i = F^{i0}, \quad B^i = (1/2)\epsilon^{ijk}F_{jk}
\]

(indices \( i, j, k = 1, 2, 3 \)), equation (11.137) in [2].

It is worth noting that such an identification of the components of \( E \) and \( B \) with the components of \( F^{\alpha\beta} \) is synchronization dependent, as explicitly shown in [12] and discussed in [13]. There, it is shown that the mentioned identifications are meaningless in the ‘\( r \)’ (‘radio’) synchronization, as can be seen, for example, from equation (22) and the text below equation (23) in section 4 of [13].

As explained there, in the ‘\( r \)’ synchronization \( F^{i0}_r = E^i + cB^2 \). This means that if the ‘\( r \)’ synchronization is used, i.e. if the appropriate metric is used, then it is not possible to make the usual identifications. That metric is discussed in, for example, the text below equation (14) in [13]: ‘\( \ldots \) the components \( g_{\alpha\beta} \) of the metric tensor \( g_{\alpha\beta} \) are \( g_{00} = 1 \), and all other components are \( = 0 \).’ Remember that in the standard basis, \( g_{\alpha\beta} = \text{diag}(1, -1, -1, -1) \). Hence, the usual identifications (equation (11.137) in [2]) are meaningful only when the Minkowski metric, e.g. \( g_{\alpha\beta} = \text{diag}(1, -1, -1, -1) \), is used. Thus, these identifications depend on the chosen synchronization, i.e. metric. But different synchronizations are nothing but different conventions and physics must not depend on conventions.

The 3-vector \( E \) is constructed as \( E = F^{i0}i + F^{20}j + F^{30}k \).

The UT of the components of \( E \) and \( B \) are derived assuming that they transform under the LT as the components of \( F^{\alpha\beta} \) transform, equation (11.148) in [2], i.e. it is supposed that the same identifications as in (1), \( E^i = F^{i0} \), \( B^i = (1/2)\epsilon^{ijk}F_{jk} \), hold in the relatively moving inertial frame of reference \( S' \). Then, \( E' \) and \( B' \) are constructed in \( S' \) in the same way as in \( S \), i.e. multiplying the components \( E'_{\gamma'\beta'} \) and \( B'_{\gamma'\beta'} \) by the unit 3-vectors \( \vec{i}' \), \( \vec{j}' \), \( \vec{k}' \). This yields the UT of \( E \) and \( B \), equation (11.149) in [2], as such:

\[
E' = \gamma (E + \beta \times B) - (\gamma^2/(1 + \gamma))\beta (\beta \cdot E),
\]

\[
B' = \gamma (B - (1/c)\beta \times E) - (\gamma^2/(1 + \gamma))\beta (\beta \cdot B).
\]

Observe that there are no LT, or any other transformations, that transform the unit 3-vectors \( \vec{i}, \vec{j}, \vec{k} \) into the unit 3-vectors \( \vec{i}', \vec{j}', \vec{k}' \). It is seen from equations (11.148) and (11.149) of [2], i.e. from equation (2) here, that the transformed \( E' \) is expressed by the mixture of the 3-vectors \( E \) and \( B \), and similarly for \( B' \).

This type of derivation of (2) was first presented in section 3 of [6]. There, and in section 7.2 as well, Minkowski made the same identification of the components of \( F^{\alpha\beta} \) with the components of the 3-vectors \( E \) and \( B \) (his \( M \)), as in equation (11.137) in [2]. Equations (11.148) in [2] are nothing but equations (6) and (7) from section 3 in Minkowski’s paper [6]. Later, the same derivation is used in numerous textbooks.

As already mentioned, in almost the whole paper [6], except in section 11.6, Minkowski exclusively dealt with the usual 3-vectors, i.e. with their components implicitly taken in the standard basis and with their UT. His \( E, M, e \) and \( m \) are the usual 3-vectors and his Maxwell equations, e.g. equations (I–IV) in section 7.1, are nothing but the usual form of the Maxwell equations with 3-vectors (the standard basis is explained in section 3 here; it always includes the Einstein synchronization [1] of distant clocks).

Minkowski’s identifications, i.e. equation (11.137) in [2], refer only to the components implicitly taken in the standard basis, which means that they are not generally valid. Namely, components, taken alone, do not completely represent a physical quantity that is defined on four-dimensional (4D) spacetime, since a basis of the spacetime is not included. The UT (2) are derived using such synchronization-dependent identifications of components of \( F^{\alpha\beta} \) (implicitly taken in the standard basis) with components of the 3-vectors \( E \) and \( B \) in both relatively moving inertial frames of reference. This shows that both 3-vectors \( E \) and \( B \) and their UT (2) are determined in a relativistically incorrect way; the quantities entering that derivation are not properly defined in 4D spacetime.

3. The LT of vectors \( E \) and \( B \): both \( F \) and the observer are transformed

Firstly, let us give an important result regarding the usual formulation of electromagnetism (as in [2]), which is presented in [7] and mentioned in [13]. It is explained in [7] that an individual vector has no dimension; the dimension is associated with the vector space and with the manifold where this vector is tangent. Hence, what is essential for the number of components of a vector field is the number of variables on which that vector field depends, i.e. the dimension of its domain. This means that the usual time-dependent \( E(r, t) \), \( B(r, t) \) cannot be the 3-vectors, since they are defined on the spacetime. That fact shows that such vector fields, when represented in some basis, have to have four components
(some of them can be zero). Therefore, we use the term ‘vector’ for the correctly defined geometric quantity, which is defined on the spacetime. However, an incorrect expression, the 3-vector or the 3D vector, will still remain for the usual $E(r, t), B(r, t)$ from [2], see equations (2).

Our consideration will be in the geometric algebra formalism. For the exposition of the geometric algebra see [14]. The generators of the spacetime algebra are four basis vectors $\{e_{\mu}\}, \mu = 0, \ldots, 3$, satisfying $g_{\mu} \cdot g_{\nu} = \eta_{\mu\nu} = \text{diag}(+ − − −)$. This basis, the standard basis, is a right-handed orthonormal frame of vectors in the Minkowski spacetime $M^4$ with $\gamma_0$ in the forward light cone, $\gamma_0^2 = 1$ and $\gamma_k^2 = −1$ ($k = 1, 2, 3$). The standard basis $\{e_{\mu}\}$ corresponds to Einstein’s system of coordinates in which the Einstein synchronization [1] of distant clocks and Cartesian space coordinates $x^i$ are used in the chosen inertial frame of reference.

It is shown, particularly in [15], that the bivector $F = F(x)$, which represents the electromagnetic field, can be taken as the primary quantity for electromagnetism, and the field equation for $F$,

$$\partial F = j/\varepsilon_0 c, \quad \partial \cdot F + \partial \wedge F = j/\varepsilon_0 c, \quad (3)$$

is the basic equation; see, for example, equation (4) in [15] or equation (1) in [10]. As shown in [15], the bivector field $\Psi$ yields the complete description of the electromagnetic field and, in fact, there is no need to introduce either the field vectors or the potentials. For the given sources, the Clifford algebra formalism enables one to find in a simple way the electromagnetic field $\Psi$, see equations (7) and (8) in [15]. However, if one introduces the electric and magnetic fields, then they can be represented by different algebraic objects. These fields are not determined by the usual identifications of the components, equation (1), but are derived in a mathematically correct way from $F$, as in equations (4) and (5) below.

In equation (23) in [9], or equation (38) in [10], the electric and magnetic fields are represented by vectors $E(x)$ and $B(x)$. We deal with such representations of the electric and magnetic fields because they are simple and much closer to the classical representation of the electric and magnetic fields by the 3D vectors $E$ and $B$ than, for example, the representations by bivectors which are used in [14]. Also, they will enable us to perform a simple comparison with Minkowski’s results from section 11.6 in [6]. In the above-mentioned equations from [9, 10] the decomposition of $F$ in terms of vectors $E, B$ and $v$ is given as

$$F = (1/c)E \wedge v + (IB) \cdot v, \quad (4)$$

and $E$ and $B$ are determined as

$$E = (1/c)F \cdot v, \quad B = -(1/c^2)I(F \wedge v), \quad (5)$$

where $v$ denotes the velocity vector of a family of observers who measure $E$ and $B$ fields. From (4) and (5), it also holds that $E \cdot v = B \cdot v = 0$. Note that $E$ and $B$ in (5) depend not only on $F$ but also on $v$. The unit pseudoscalar $I$ is defined algebraically without introducing any reference frame, as in section 1.2 in the second reference of [14]. We choose $I$ in such a way that when $I$ is represented in the $\{n_\mu\}$ basis it becomes $I = n_0 \wedge n_1 \wedge n_2 \wedge n_3$. With such choice for $I$, $\{n_0, n_1, n_2\}$ forms a right-handed orthonormal set, as is usual for a 3D Cartesian frame. The LT do not change the orientation for spacetime.

Inserting (4) into the field equation (3) for $F$, one finds the field equation in terms of $E$ and $B$:

$$\partial [E \wedge (v/c) + (IB) \cdot v] = j/\varepsilon_0 c. \quad (6)$$

In sections 2.1 and 2.2 in [10], it is shown in detail that the definitions (5), together with the field equation (39) in [10], represent the Lorentz invariant generalization of the usual Maxwell equations. If the observers who measure fields are at rest and the standard basis is chosen in that frame, $v = c\gamma_0$, then the temporal components of $E$ and $B$ are zero and only the spatial components remain, which are the same as the components of the usual 3D $E$ and $B$. In that case, the field equation with $E$ and $B$ (6) leads to the usual Maxwell equations for the components of the 3D $E$ and $B$. Thus, the principle of correspondence is simply satisfied with the definitions (5), including the choice of $I$, and with the field equation for $E$ and $B$, equation (6).

Furthermore, the equations that correspond to equations (5) and (4), but in the tensor formalism, are $F^\mu = (1/c)F^{\alpha \beta} e_{\beta} v_{\alpha}, B^\mu = (1/2c^2) \varepsilon^{\alpha \beta \gamma \delta} F_{\beta \gamma} v_{\alpha}$ and $F^{\alpha \beta} = (1/c) \times (E^\alpha v^\beta - E^\beta v^\alpha) + \varepsilon^{\alpha \beta \gamma \delta} v_{\gamma} B_{\delta}$, see for example equation (1) in [13]. They are based on the theorem (see reference [6] in [13]) that any second rank antisymmetric tensor can be decomposed into two vectors and a unit time-like vector ((the velocity vector)/c). These equations show that in the tensor formalism too, both the electric and magnetic fields can be represented by vectors.

It is easy to see that the relations (5) correspond to Minkowski’s relations from section 11.6 in [6]:

$$\Phi = -wF, \quad \Psi = iwF^*. \quad (7)$$

In the vacuum $f = F$ one could write the second equation in (7) as $\Psi = iwF^*$, where $F^*$ is the dual field-strength tensor, $aF^\beta = (1/2)\varepsilon^{\alpha \beta} F_{\alpha \delta}$. Observe that (5) are coordinate-free relations, which hold for any observer. If geometric quantities from (5) are represented in some basis, then they contain both components and basis vectors. In contrast, Minkowski considered that $w, \Phi$ and $\Psi$ are $1 \times 4$ matrices and $F$ is a $4 \times 4$ matrix. Their components are implicitly determined in the standard basis.

The relation that corresponds to (4) is equation (55) in [6]:

$$F = [w, \Phi] + i\mu[w, \Psi]. \quad (8)$$

In section 11.6 in [6], the paragraph below equation (44), Minkowski described how $w$ and $F$ separately transform under the LT $A$ (the matrix of the LT is denoted as $A$ in [6]) and then how the product $wF$ transforms. Thus, he wrote $w' = wA$ for the LT of the velocity vector $w$ and $F' = A^{-1}FA$ for the LT of the field-strength tensor. So the mathematically correct LT of $wF$ are

$$\Phi = wF \rightarrow \Phi' = wAA^{-1}FA = (wF)A = \Phi A, \quad (9)$$

which means that under the LT both terms, the velocity $w$ and $F$, are transformed and their product transforms as any
other vector (i.e. in [6], a $1 \times 4$ matrix) transforms. The most important thing is that the electric field vector $\Phi$ transforms by the LT again to the electric field vector $\Psi'$; there is no mixing with the magnetic field $\Psi$.

These correct LT of the electric and magnetic fields are reinvented in [8–11]. Let us choose the frame in which the observers who measure $E$ and $B$ from (5) are at rest. For them $v = c \gamma_0$. In the geometric algebra, the LT are described with rotors $R$, $\bar{R} \bar{R} = 1$, where the reverse $\bar{R}$ is defined by the operation of reversion according to $AB = B\bar{A}$, for any multivectors $A$ and $B$, $\bar{a} = a$, for any vector $a$, and it reverses the order of vectors in any given expression. For boosts in an arbitrary direction, the rotor $R$ is given by equation (8) in [9, 11] as

$$R = (1 + \gamma + \gamma_0 \beta y)/(2(1 + \gamma)^{1/2}),$$

(10)

where $\gamma = (1 - \beta^2)^{-1/2}$, the vector $\beta$ is $\beta = \beta n$, $\beta$ on the right-hand side of that equation is the scalar velocity in units of $c$ and $n$ is not the basis vector but any unit space-like vector orthogonal to $\gamma_0$. Then, any multivector $M$ transforms by active LT in the same way, i.e. as in equation (9) in [11],

$$M \to M' = RM\bar{R}.$$  

(11)

Hence, vector $E$ is transformed by the LT $R$ as $E \to E' = R E \bar{R}$. If $v = c \gamma_0$ is taken in (5), then $E$ becomes $E = F \cdot \gamma_0$ and it transforms under the LT in the same manner as in (9), i.e. both $F$ and $v$ are transformed by the LT $R$ as

$$E = F \cdot \gamma_0 \to E' = (R F \bar{R}) \cdot (R \gamma_0 \bar{R}) = R(F \cdot \gamma_0) \bar{R}. $$

(12)

These correct LT yield

$$E' = E + \gamma(E \cdot \beta) [\gamma_0 - (\gamma/(1 + \gamma)) \beta] .$$

(13)

In the same way, every vector is transformed, i.e. the vector $B$ as well. For boosts in the direction $\gamma_1$, one has to take $\beta = \beta \gamma_1$ (on the left-hand side $\beta$ is a vector and on the right-hand side $\beta$ is a scalar) in the above expression for the rotor $R$ (all in the standard basis). Hence, in the $\{\gamma_1\}$ basis and when $\beta = \beta \gamma_1$, equation (13) becomes

$$E'' \gamma_0 = -\beta \gamma E^1 \gamma_0 + \gamma E^1 \gamma_1 + E^2 \gamma_2 + E^3 \gamma_3 ,$$

(14)

which is equation (9) in [9]. The same components would be obtained for $\Phi' = \Phi A$ in Minkowski’s relation (9) if the components of $w$ are $(0, 0, 0, ic)$ in his notation, which corresponds to $v = c \gamma_0$ in our formulation.

As already stated, equations (12) and (14) are presented in [8–11], whereas equation (13) is first reported here. Minkowski wrote (7)–(9) in section 11.6 in [6], but, as already stated, in the rest of [6] he exclusively dealt with the usual 3-vectors $E$ and $B$ and not with correctly defined vectors $\Phi$ and $\Psi$.

If one represents the relation $E = (1/c)F \cdot v$ from (5) in the standard basis $\{\gamma_0\}$, then $E = E^i \gamma_0^i$, where $E^i = F^{i \mu} v_\mu$ (e.g. $E^1 = F^{10} v_0 + F^{12} v_2 + F^{13} v_3$ and $E^0 = F^{01} v_1 + F^{02} v_2 + F^{03} v_3$). These relations for components exactly correspond to Minkowski’s expressions for the relation $\Phi = -w F$ in components, when the components $\Phi_1, \ldots, \Phi_4$ are expressed in terms of the components $u_1, \ldots, u_4$ and the components $F_{lk}$ (in [6], $h$, $k = 1, 2, 3, 4$ and $h = 4$ denotes the imaginary time component). Thus, for example, $\Phi_1 = u_1 F_{14} + u_2 F_{21} + u_3 F_{31}$ and $\Phi_4 = u_1 F_{14} + u_2 F_{22} + u_3 F_{34}$; see the equations just before equation (47) in [6]. Minkowski was a very good mathematician and he completely understood that mathematically correct LT of fields are those of his $\Phi$, equation (9). But probably due to the generally accepted belief and the authorities in physics (Maxwell, Lorentz, Einstein, etc), he also believed that physical quantities are the usual 3-vectors $E$, $B$ and $D$, $H$. Therefore, he expressed in equations (47), (48) and (51), (52) in [6] the components of his mathematically and physically correct fields (in our opinion) $\Phi$ and $\Psi$ in terms of the usual 3-vectors $E$, $B$ and $D$, $H$. He wrote for equation (47) in [6] that the first three components $\Phi_1$, $\Phi_2$, $\Phi_3$ are the components of the 3-vector $(E + w \times M)/(1 - w^2)^{-1/2}$, whereas $\Phi_4 = i(w \cdot E)/(1 - w^2)^{-1/2}$ (his $M$ is our $B$). He called $\Phi$ the electric field at rest and similarly $\Psi$ the magnetic field at rest because it follows from equations (47) and (48) that for his $w = (0, 0, 0, ic)$ the temporal component $\Phi_4 = 0$ and the spatial components $\Phi_1$, $\Phi_2$, $\Phi_3$ are the same as the components of the usual electric field 3-vector $E$ and similarly for $\Psi$. Hence, he believed that only when $w = (0, 0, 0, ic)$ his fields $\Phi$ and $\Psi$ are the electric and magnetic fields since then they coincide with ‘physical’ $E$ and $B$. However, regardless of the problem with physical interpretation or, better to say, because of that problem, Minkowski’s section 11.6 is very important for all physicists.

4. The UT of vectors $E$ and $B$: $F$ is transformed but not the observer

Now, let us see what will be obtained if, in the transformation of $E = F \cdot \gamma_0$, only $F$ is transformed by the LT $R$, but not the velocity of the observer $v = c \gamma_0$. Of course, it will not be the LT of $E = F \cdot \gamma_0$, since they are given by (12). Thus

$$E = F \cdot \gamma_0 \to E'_F = (R F \bar{R}) \cdot \gamma_0 $$

(15)

This yields

$$E'_F = \gamma (E + (\beta \gamma_0 \wedge c B)I) + (\gamma^2/(1 + \gamma)) \beta (\beta \cdot E),$$

(16)

which, in the standard basis and when $\beta = \beta \gamma_1$, becomes

$$E'_F \gamma_0 = E^1 \gamma_1 + \gamma (E^2 - c \beta B^3) \gamma_2 + \gamma (E^3 + c \beta B^2) \gamma_3 .$$

(17)

The transformation (16) can be compared with the UT for the 3-vector $E$ that are given, e.g. by equation (11.149) in [2], i.e. with (2) here, and equation (17) with equation (11.148) in [2]. Remember that in (2), $E$, $E'$, $B$, $B'$ and $B$ are all the usual 3-vectors. It can be seen from a comparison of equation (17) with equation (11.148) in [2] that the transformations of components (taken in the standard basis) of $E'_F$ are exactly the same as the transformations of $E_{x,y,z}$ from equation (11.148) in [2]. The UT for $B$ are given by the second equation in equation (11.149) in [2], i.e. with the second equation in (2) here. The result that the components in (17) are the same as the components of $E'$ from (2) is completely understandable.
Namely, (16) and (17) are obtained by the application of the LT only to \( F \). On the other hand, it has already been stated in section 2 that the UT of the components of \( E \) and \( B \) are derived assuming that they transform under the LT as the components of \( F^\beta \) transform, equation (11.148) in [2].

The transformations (15) and (17) are first discussed in detail in [8–11] and compared with the UT (11.148) and (11.149) from [2], whereas the general form of \( E' \), equation (16), is first given in [7].

Here, it is appropriate to point out an important difference between the LT and UT. If, instead of the active LT, we consider the passive LT, then, for example, the vector \( E' = E' \gamma_v = E' \gamma_v' \) will remain unchanged because the components \( E' \) transform by the LT and the basis vectors \( \gamma_v \) by the inverse LT leaving the whole \( E \) invariant under the LT of the passive LT. Of course, the same holds for all bases including those with nonstandard synchronizations, as shown, for example, in [12, 13]. This invariance of \( E \) under the LT means that the electric field \( E \) is the same physical quantity for all relatively moving observers. It is not so with the 3-vector \( E \) and its UT. Namely, \( E = E_i \hat{e}_i + E_j \hat{e}_j + E_k \hat{e}_k \) is completely different from \( E' \) by (2); see the discussion in section 2. This means that although \( E \) and \( E' \) are measured by different observers they are not the same quantity for such relatively moving observers. The observers are not looking at the same physical object, here the electric field vector, but at different objects. Every observer makes measurements of its own 3-vector field, \( E \) and \( E' \), and the LT cannot connect such quantities. Different relatively moving inertial 4D observers can compare only 4D quantities, here \( E' \gamma_v \) and \( E' \gamma_v' \), because the LT connects such quantities. The experimentalists have to measure all components of 4D quantities, here of \( E \), in both frames \( S' \) and \( S \). The observers in \( S' \) and \( S \) are able to compare only such a complete set of data that corresponds to the same 4D geometric quantity.

5. The low-velocity limit of the UT of the 3-vectors \( E \) and \( B \) and of the LT of vectors \( E \) and \( B \)

If the low-velocity limit \( \beta \ll 1 \), or \( \gamma \approx 1 \), is taken in (2), then the following relations with 3-vectors are obtained \( E' = E + \beta \times cB \) and \( B' = B - (1/c)\beta \times E \). They are commonly used in the literature. However, it is argued in [16] that these transformations have to be replaced by two well-defined Galilean limits, the magnetic and electric limits, i.e. with two sets of low-velocity formulae. These two limits are obtained from the UT (2). In a vacuum, the magnetic limit is obtained taking in the UT (2) that not only \( \beta \ll 1 \), but \( |E| \ll c|B| \) as well. Hence, the UT in the magnetic limit are \( E' = E + \beta \times cB \) and \( B' = B \). Conversely, the electric limit is obtained taking in the UT (2) that \( \beta \ll 1 \) and \( |E| \gg c|B| \). Hence, the UT in the electric limit are \( E' = E \) and \( B' = B - (1/c)\beta \times E \). The results from [16] are used, developed and applied to different problems in a series of papers in [17]. Observe that in all papers in [16, 17] the UT of \( E \) and \( B \) are considered to be the relativistically correct LT.

In section VIII.A of the third paper in [17], the electric limit approximation of the UT is used in a comparison with the Trouton–Noble experiment. There, it is argued that there is a Trouton–Noble paradox (there is a 3D torque in one frame but no 3D torque in a relatively moving frame) for larger velocities, but the Trouton–Noble paradox is not there when the electric limit of the low-velocity approximation is used. Hence, the principle of relativity is violated for larger velocities, but not violated in the low-velocity approximation. Such a result clearly indicates that both the approach with the UT, equations (2), and its two low-velocity limits from [16, 17] are not relativistically correct. Namely, the principle of relativity has to be satisfied for all velocities less than the velocity of light.

As shown in [8–11], and also here, the UT are not the LT; the LT are given by equations (13) and (14), and the same for \( B' \). This means that neither the commonly used set of low-velocity transformations nor the two mentioned limits are the low-velocity approximations of the LT. In the UT, equations (2), the components of the electric and magnetic fields are mixed together and therefore it is possible to compare their moduli and to obtain two different limits. For the LT (13) and (14) there is only one low-velocity approximation, which is simply obtained, taking the limit \( \beta \ll 1 \) or \( \gamma \approx 1 \). In that approximation the LT (13) become \( E' = E + (E \cdot \beta)\gamma_v \), and the same for the vector \( B \). It can be easily shown that to \( \bigcirc (\beta^2) \) this low-velocity approximation of the vector \( E \) is invariant under the passive LT.

Regarding the Trouton–Noble paradox, it is shown in [15, 18] that in the geometric approach with 4D quantities the 4D torques will not appear for the moving capacitor if they do not exist for the stationary capacitor, which means that with 4D geometric quantities the principle of relativity is naturally satisfied and the Trouton–Noble paradox is not there. Of course, the same conclusion will hold in the low-velocity approximation \( \beta \ll 1 \), or \( \gamma \approx 1 \). One very similar paradox to the Trouton–Noble paradox is Jackson’s paradox. It is discussed in detail in [19]; the second paper is a more pedagogical version of the first one.

6. Conclusions

From the result that the transformations (15), (16) and (17) are not the LT, it can be concluded that, contrary to general opinion, neither of the transformations (2), i.e. the UT of the 3-vectors \( E(r, t) \) and \( B(r, t) \), equations (11.148) and (11.149) of [2], are the LT. Furthermore, comparisons with experiments (the Trouton–Noble experiment [15, 18], the motional emf [9, 20] and the Faraday disc [10]), show that the approach with multivectors always agrees with the principle of relativity and is in true agreement (independent of the chosen inertial reference frame and of the chosen system of coordinates in it) with experiments in electromagnetism. As shown in [9, 10, 15, 18, 20] and here, this is not the case with the usual approach, as in [2], in which the electric and magnetic fields are represented by 3-vectors \( E(r, t) \) and \( B(r, t) \) that transform according to the UT (2) or according to their two low-velocity limits from [16, 17].

Minkowski’s great discovery of the correct LT (9), section 11.6 in [6], their explicit forms (12), (14) and (13) that are found in [8–11] and here, respectively, and also the mathematical argument from [7] that space- and time-dependent electric and magnetic fields cannot be the usual 3-vectors strongly suggest the need for
further critical examination of the usual formulation of electromagnetism with 3-vectors $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ and their UT (2). It also suggests the possibility of a complete and relativistically correct formulation of classical and quantum electromagnetism with multivector fields (as physically real fields), which are defined on spacetime and which transform according to the correct LT (11), i.e. (12)–(14). The advantages of such a formulation with multivector fields are already revealed in the cases of the interaction between the dipole moment tensor $D^{ab}$ and the electromagnetic field $F^{ab}$ in the first paper in [21] and in much more detail in [13], in the discussion of quantum phase shifts in the second and third papers in [21] and in the formulation of the Majorana form of the Dirac-like equation for the free photon [22].

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