Joinings of W*-dynamical systems

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Abstract

We study the notion of joinings of W*-dynamical systems, building on ideas from measure theoretic ergodic theory. In particular we prove sufficient and necessary conditions for ergodicity in terms of joinings, and also briefly look at conditional expectation operators associated with joinings.

Key words: W*-dynamical systems; Joinings; Ergodicity; Conditional expectation

1 Introduction

The study of joinings (and disjointness) of measure theoretic dynamical systems was initiated by Furstenberg [4] in 1967, and Rudolph [7] in 1979. Joinings have since become a useful tool in ergodic theory. More recent treatments of joinings including further developments and some applications can be found in Glasner's book [5], Rudolph's book [8], the review [2], and the paper [6].

In this paper we study joinings of W*-dynamical systems. We will refer to W*-dynamical systems simply as “dynamical systems”; see Section 2 for the precise definition that we’ll use. In these dynamical systems one works on a von Neumann algebra rather than a measurable space, and with a state instead of a measure. The von Neumann algebra is a noncommutative generalization of the abelian algebra $L^\infty$ in the measure theoretic case. Such noncommutative dynamical systems have of course been studied extensively, and is for example a suitable framework for the mathematical study of quantum physics. Roughly a joining of two dynamical systems is a generalization of the usual product of the systems, but where the product state is replaced with a state which can in principle take into account an “overlap” or “common part” of the two systems. It is such a state that we will refer to as
a “joining” of the two dynamical systems. We focus mainly on sufficient and necessary conditions for ergodicity in terms of joinings (see Section 3), but also consider some basic aspects of conditional expectation operators associated with joinings.

The same idea has been used by Sauvageot and Thouvenot [9] to study entropy in noncommutative dynamical systems. However they consider joinings where one of the two systems is classical (i.e. its algebra is abelian). A useful reference regarding this approach to entropy is [10, Chapter 5]. Although we will not apply our results to entropy in this paper, this joining approach to entropy (together with the work on joinings in classical ergodic theory) does suggest that a general study of joinings of noncommutative dynamical systems will have uses beyond just ergodicity. We also note that in the literature on entropy the term “stationary coupling” is used, rather than “joining”, however the latter is standard in measure theoretic ergodic theory, more succinct, and appears to be older, so we will continue to use it.

For the most part we consider general group actions, but a necessary condition for ergodicity is only proved in the case of amenable countable discrete groups. For the proof of the sufficient condition it is useful to work in terms of a “factor” (essentially a subsystem) of the dynamical system, and the definition of a factor is given in Section 3. To avoid confusion, note that in this context the term does not refer to a von Neumann algebra $A$ which is a factor (i.e. $A \cap A' = \mathbb{C}1$).

Our von Neumann algebras always contain the identity operator on the underlying Hilbert space, and we will denote it by $1_A$ for a von Neumann algebra $A$, or sometimes just 1. The identity map $A \to A$ will be denoted by $\text{id}_A$ or simply id, while the group of all $*$-automorphisms of $A$ will be denoted by $\text{Aut}(A)$. We only consider dynamical systems on $\sigma$-finite von Neumann algebras, since we will be using Tomita-Takesaki theory in Section 3. Remember that a von Neumann algebra is $\sigma$-finite if and only if it has a faithful normal state. We will denote the algebraic tensor product of two von Neumann algebras $A$ and $B$ by $A \otimes B$, which is a unital $*$-algebra. For simplicity we will only consider algebraic tensor products in this paper. The von Neumann algebra of bounded linear operators $H \to H$ on a Hilbert space $H$ will be denoted by $B(H)$, and the commutant of any $S \subset B(H)$ is denoted by $S'$. Our main reference for von Neumann algebras and Tomita-Takesaki theory is [1].

Throughout this paper $G$ is an arbitrary but fixed group, except in Theorem 3.7 where we specialize.
2 Joinings

This section is devoted mainly to the basic definitions, and includes a characterization of joinings in terms of conditional expectation operators.

**Definition 2.1.** A *dynamical system* $A = (A, \mu, \alpha)$ consists of a faithful normal state $\mu$ on a $\sigma$-finite von Neumann algebra $A$, and a representation $\alpha : G \to \text{Aut}(A) : g \mapsto \alpha_g$ of $G$ as $*$-automorphisms of $A$, such that $\mu \circ \alpha_g = \mu$ for all $g$. We will call $A$ trivial if $A = C_1$. We will call $A$ an identity system if $\alpha_g = \text{id}_A$ for all $g$.

In the remainder of this section and the next, the symbols $A$, $B$ and $F$ will denote dynamical systems $(A, \mu, \alpha)$, $(B, \nu, \beta)$ and $(F, \lambda, \varphi)$ respectively, and keep in mind that they all make use of actions of the same group $G$.

**Definition 2.2.** A *joining* of $A$ and $B$ is a state $\omega$ on $A \otimes B$ such that
\[
\omega(a \otimes 1_B) = \mu(a)
\]
\[
\omega(1_A \otimes b) = \nu(b)
\]
and
\[
\omega \circ (\alpha_g \otimes \beta_g) = \omega
\]
for all $a \in A$, $b \in B$ and $g \in G$. The set of all joinings of $A$ and $B$ is denoted by $J(A, B)$. Note that $\mu \otimes \nu \in J(A, B)$. We call $A$ disjoint from $B$ when $J(A, B) = \{\mu \otimes \nu\}$.

As part of the proof of Theorem 3.3 in Section 3, we will construct a joining other than $\mu \otimes \nu$ in the special case where $B$ is obtained in a certain way from a “factor” of $A$.

We are now going to study the conditional expectation operator associated with certain states on $A \otimes B$.

**Construction 2.3.** Let $\omega$ be any state on $A \otimes B$ such that $\omega(a \otimes 1_B) = \mu(a)$ and $\omega(1_A \otimes b) = \nu(b)$.

Consider the GNS construction $(H_\omega, \gamma_\omega)$ for $(A \otimes B, \omega)$, by which we mean $H_\omega$ is a Hilbert space and
\[
\gamma_\omega : A \otimes B \to H_\omega
\]
a linear operator such $\gamma_\omega(A \otimes B)$ is dense in $H_\omega$ and $\langle \gamma_\omega(s), \gamma_\omega(t) \rangle = \omega(s^*t)$ for all $s, t \in A \otimes B$. Then $\Omega_\omega := \gamma_\omega(1_A \otimes 1_B)$ is the corresponding cyclic vector.
Define \( \iota_A : A \to A \otimes B : a \mapsto a \otimes 1_B \) and \( \iota_B : B \to A \otimes B : b \mapsto 1_A \otimes b \) and let \( H_\mu \) and \( H_\nu \) be the closures in \( H_\omega \) of \( \gamma_\omega \circ \iota_A(A) \) and \( \gamma_\omega \circ \iota_B(B) \) respectively. Setting

\[
\gamma_\mu := \gamma_\omega \circ \iota_A : A \to H_\mu
\]

and

\[
\gamma_\nu := \gamma_\omega \circ \iota_B : B \to H_\nu
\]

we have \( \gamma_\mu(A) \) and \( \gamma_\nu(B) \) dense in \( H_\mu \) and \( H_\nu \) respectively, and \( \langle \gamma_\mu(a), \gamma_\mu(a') \rangle = \omega \left( (a \otimes 1_B)^* (a' \otimes 1_B) \right) = \mu(a^*a') \) for all \( a, a' \in A \), and similarly \( \langle \gamma_\nu(b), \gamma_\nu(b') \rangle = \nu(b^*b') \). Hence \( (H_\mu, \gamma_\mu) \) and \( (H_\nu, \gamma_\nu) \) are the GNS constructions for \( (A, \mu) \) and \( (B, \nu) \) respectively, and they both have the cyclic vector \( \Omega_\mu := \gamma_\mu(1_A) = \Omega_\omega = \gamma_\mu(1_B) =: \Omega_\nu \).

Let \( P \) be the projection of \( H_\omega \) onto the subspace \( H_\nu \) and then set

\[
P_\omega := P|_{H_\mu} : H_\mu \to H_\nu
\]

which is called the conditional expectation operator associated with \( \omega \). It is the unique mapping \( H_\mu \to H_\nu \) satisfying

\[
\langle P_\omega x, y \rangle = \langle x, y \rangle
\]

for all \( x \in H_\mu \) and \( y \in H_\nu \). The space of fixed points of \( P_\omega \) is clearly \( H_\mu \cap H_\nu \).

In particular \( P_\omega \Omega_\omega = \Omega_\omega \).

Since \( \mu \circ \alpha_g = \mu \) and \( \nu \circ \beta_g = \nu \), we obtain well defined and unique linear operators \( U_g : H_\mu \to H_\mu \) and \( V_g : H_\nu \to H_\nu \) from \( U_g \gamma_\mu(a) := \gamma_\mu(\alpha_g(a)) \) and \( V_g \gamma_\nu(b) := \gamma_\nu(\beta_g(b)) \). For the same reason \( U_g \) and \( V_g \) are isometries. Furthermore by uniqueness, \( g \mapsto U_g \) and \( g \mapsto V_g \) are representations of \( G \), since \( \alpha \) and \( \beta \) are. In particular \( U_g \) and \( V_g \) are invertible, and hence unitary.

Note that this whole construction goes through even if we only assume that \( A \) and \( B \) are unital \( \ast \)-algebras rather than von Neumann algebras.

If we furthermore assume \( \omega \in J(A, B) \), which means we additionally have \( \omega \circ (\alpha_g \otimes \beta_g) = \omega \), then in the same way we obtain a unitary representation \( g \mapsto W_g \) of \( G \) on \( H_\omega \) such that \( W_g \gamma_\omega(t) = \gamma_\omega(\alpha_g(\beta_g(t))) \) for all \( t \in A \otimes B \). Note that \( W_g|_{H_\mu} = U_g \) and \( W_g|_{H_\nu} = V_g \).

**Proposition 2.4.** Let \( \omega \) be a state on \( A \otimes B \) such that \( \omega(a \otimes 1_B) = \mu(a) \) and \( \omega(1_A \otimes b) = \nu(b) \) for all \( a \in A \) and \( b \in B \). Then \( \omega \in J(A, B) \) if and only if

\[
P_\omega U_g = V_g P_\omega
\]

for all \( g \), in terms of Construction 2.3.
Proof. Assuming \( \omega \in J(A, B) \), then by Construction 2.3

\[
\langle V^*_g P_\omega U_g x, y \rangle = \langle P_\omega W_g x, W_g y \rangle = \langle W_g x, W_g y \rangle = \langle x, y \rangle = \langle P_\omega x, y \rangle
\]

for all \( x \in H_\mu \) and \( y \in H_\nu \), hence \( V^*_g P_\omega U_g = P_\omega \). Conversely, if \( P_\omega U_g = V^*_g P_\omega \), then for all \( a \in A \) and \( b \in B \)

\[
\omega(\alpha_g \otimes \beta_g(a \otimes b)) = \omega(\alpha_g \otimes \beta_g(a^* \otimes 1_B)^* \alpha_g \otimes \beta_g(1_A \otimes b)) \\
= \langle U_g \gamma_\mu(a^*), V_g \gamma_\nu(b) \rangle \\
= \langle P_\omega U_g \gamma_\mu(a^*), V_g \gamma_\nu(b) \rangle \\
= \langle P_\omega \gamma_\mu(a^*), \gamma_\nu(b) \rangle \\
= \langle \gamma_\mu(a^*), \gamma_\nu(b) \rangle \\
= \omega(a \otimes b)
\]

so \( \omega \circ (\alpha_g \otimes \beta_g) = \omega \) by linearity. □

3 Ergodicity

We now turn to ergodicity, in particular proving sufficient and necessary conditions for ergodicity in terms of joinings. As part of the proof of sufficiency (Theorem 3.3) we construct a special joining in terms of a factor of a dynamical system. The commutant of the algebra, and the modular conjugation operator from Tomita-Takesaki theory play a central role in this construction.

Definition 3.1. A dynamical system \( A \) is called ergodic if its fixed point algebra

\[ A_\alpha := \{ a \in A : \alpha_g(a) = a \text{ for all } g \in G \} \]

is trivial, i.e. \( A_\alpha = \mathbb{C}1_A \).

Definition 3.2. We call \( F \) a factor of \( A \) if there exists an injective unital \( * \)-homomorphism \( h \) of \( F \) onto a von Neumann subalgebra of \( A \) such that \( \mu \circ h = \lambda \) and \( \alpha_g \circ h = h \circ \varphi_g \) for all \( g \in G \). If this factor is an identity system, then we will call it an identity factor.

It is easily seen that \( A_\alpha \) is itself a \( \sigma \)-finite von Neumann algebra with \( \mu|_{A_\alpha} \) a faithful normal state, and that \( A_\alpha := (A_\alpha, \mu|_{A_\alpha}, \alpha|_{A_\alpha}) \) is an identity factor of \( A \).

Theorem 3.3. If \( A \) is disjoint from all identity systems, then it is ergodic.
In order to prove this theorem, we will use a special case of the following construction:

**Construction 3.4.** Let \( F \) be any factor of \( A \) given by the \(*\)-homomorphism \( h : F \to A \) as in Definition 3.2.

Denote the cyclic representation of \((A, \mu)\), obtained using the GNS construction, by \((H, \pi, \Omega)\). For every \( g \in G \) there is a unique unitary operator \( U_g : H \to H \) such that \( U_g \Omega = \Omega \) and 

\[
U_g \pi(a) U_g^* = \pi(\alpha_g(a))
\]

for all \( a \in A \). The uniqueness ensures that \( g \mapsto U_g \) is a representation of \( G \).

Since \( \mu \) is faithful and normal, \( \Omega \) is a cyclic and separating vector for the von Neumann algebra \( M := \pi(A) \) and \( \pi : A \to M \) is a \(*\)-isomorphism. It also follows that \( \pi \) and its inverse are \( \sigma \)-weakly continuous, hence \( \pi(h(F)) \) is a von Neumann subalgebra of \( M \).

Let \( J \) be the modular conjugation associated with \((M, \Omega)\) as obtained in Tomita-Takesaki theory. Remember that \( J \) is anti-unitary, \( J^2 = 1 \) (i.e. \( J^* = J \)) and \( J \Omega = \Omega \). Define

\[
j : B(H) \to B(H) : a \mapsto J a^* J
\]

then by Tomita-Takesaki theory

\[
j(M) = M'
\]

and furthermore \( j \) is an anti-\(*\)-isomorphism, i.e. it is a linear bijection such that \( j(a^*) = j(a)^* \) and \( j(ab) = j(b) j(a) \) for all \( a, b \in B(H) \). Also, \( j^2 = \text{id} \).

From these facts it is easily seen that

\[
j(S)' = j(S')
\]

for all \( S \subset B(H) \).

Set

\[
\sigma := j \circ \pi \circ h
\]

then \( \sigma(F)' = j(\pi(h(F))') = \sigma(F) \), since \( \pi(h(F)) \) is a von Neumann algebra, hence

\[
B := \sigma(F) \subset M'
\]

is a von Neumann algebra. We can define a state \( \nu \) on \( B \) by

\[
\nu(b) := \langle \Omega, b \Omega \rangle
\]
then clearly \( \nu \) is \( \sigma \)-weakly continuous, i.e. normal. Furthermore, \( \nu \) is faithful, since

\[
0 = \nu(b^*b) = \|b\Omega\|^2
\]

implies that \( b = 0 \) because \( \Omega \) is separating for \( M' \).

Now set

\[
\beta_g(b) := j \circ \pi \circ \alpha_g \circ \pi^{-1} \circ j(b)
= JU_g Jb JU_g^* J
\]

for all \( b \in B \). Then it is clear that \( \beta \) is a representation of \( G \) as \(*\)-automorphisms of \( B \), and since \( U_g^* \Omega = \Omega \), we have

\[
\nu \circ \beta_g(b) = \langle \Omega, JU_g Jb \Omega \rangle = \langle U_g Jb \Omega, \Omega \rangle = \nu(b)
\]

for all \( b \in B \). Therefore

\[
\mathcal{B} := (B, \nu, \beta)
\]

is a dynamical system.

Note that \( \mathcal{B} \) is the "mirror image" of \( \mathcal{F} \) in \( M' \) in the sense that they can be said to be anti-isomorphic: \( \sigma : F \to B \) is an anti-\( * \)-isomorphism, since \( j \) is. Furthermore

\[
\nu \circ \sigma = \lambda
\]

and

\[
\beta_g \circ \sigma = \sigma \circ \varphi_g
\]

for all \( g \).

We now construct a joining of \( \mathcal{A} \) and \( \mathcal{B} \). Consider the bilinear mapping

\[
A \times B \to B(H) : (a, b) \mapsto \pi(a)b
\]

and extend it to the linear mapping \( \delta : A \odot B \to B(H) \), which is a unital \(*\)-homomorphism, since \( \pi(A) = M \) while \( B \subset M' \). Thus we can define a state \( \omega \) on \( A \odot B \) by

\[
\omega(t) := \langle \Omega, \delta(t) \Omega \rangle
\]

for all \( t \in A \odot B \). Then

\[
\omega(a \otimes 1_B) = \langle \Omega, \pi(a) \Omega \rangle = \mu(a)
\]

and

\[
\omega(1_A \otimes b) = \langle \Omega, \pi(1_A)b \Omega \rangle = \nu(b)
\]

for all \( a \in A \) and \( b \in B \). The theory of self-dual cones and standard forms in Tomita-Takesaki theory provides (see [II, Corollary 2.5.32]) a unitary representation \( \Aut(M) \ni \theta \mapsto u(\theta) \) of the group \( \Aut(M) \) on the Hilbert space \( H \) such that (among other properties) \( u(\theta)a u(\theta)^* = \theta(a) \) and \( u(\theta)J = J u(\theta) \).
for all \( a \in M \) and \( \theta \in \text{Aut}(M) \), while \( u(\theta)\Omega = \Omega \) for all \( \theta \in \text{Aut}(M) \) for which \( \langle \Omega, \theta(a)\Omega \rangle = \langle \Omega, a\Omega \rangle \) for all \( a \in M \). Since \( U_g \) is the unique unitary operator on \( H \) satisfying \( U_g\pi(a)U^*_g = \pi(\alpha_g(a)) \) and \( U_g\Omega = \Omega \), we must have \( u(\pi \circ \alpha_g \circ \pi^{-1}) = U_g \) and therefore

\[
U_g J = J U_g
\]

for all \( g \). Hence

\[
\omega \circ (\alpha_g \otimes \beta_g)(a \otimes b) = \langle \Omega, \pi(\alpha_g(a)) \beta_g(b)\Omega \rangle
\]

\[
= \langle \Omega, U_g\pi(a)U^*_g J U_g J b\Omega \rangle
\]

\[
= \langle U^*_g\Omega, \pi(a)U^*_g J J b\Omega \rangle
\]

\[
= \langle \Omega, \pi(a)b\Omega \rangle
\]

\[
= \omega(a \otimes b)
\]

and therefore by linearity \( \omega \circ (\alpha_g \otimes \beta_g) = \omega \). So \( \omega \) is indeed a joining of \( A \) and \( B \). \( \Box \)

**Lemma 3.5.** In Construction 3.4 we have \( \omega = \mu \otimes \nu \) if and only if \( F \) is trivial.

**Proof.** First note that \( F \) is trivial if and only if \( B \) is. Now, if \( B \) is trivial, i.e. \( B = \mathbb{C}1 \), then \( \omega(a \otimes b) = \langle \Omega, \pi(a)b\Omega \rangle = \langle \Omega, \pi(a)\Omega \rangle b = \mu(a)\nu(b) = \mu \otimes \nu(a \otimes b) \), since we can view \( b \in B \) as an element of \( \mathbb{C} \). By linearity it follows that \( \omega = \mu \otimes \nu \).

Conversely, suppose \( \omega = \mu \otimes \nu \). Then for any \( a \in A \) and \( b \in B \)

\[
\langle \pi(a)\Omega, b\Omega \rangle = \langle \Omega, \pi(a^*)b\Omega \rangle
\]

\[
= \omega (a^* \otimes b)
\]

\[
= \mu(a^*)\nu(b)
\]

\[
= \langle \pi(a)\Omega, \langle \Omega, b\Omega \rangle \Omega \}
\]

but \( \pi(A)\Omega \) is dense in \( H \), and \( \Omega \) is separating for \( B \), hence \( b = \langle \Omega, b\Omega \rangle 1 \in \mathbb{C}1 \). \( \Box \)

**Proof of Theorem 3.3.** Let \( F = A_\alpha \) in Construction 3.4, then \( F \) is an identity factor of \( A \) as mentioned previously, and so \( B \) is an identity system. If \( A \) is not ergodic, then by definition \( F \) is not trivial, hence \( J(A, B) \neq \{\mu \otimes \nu\} \) by Lemma 3.5 and Construction 3.4. This means that \( A \) is not disjoint from \( B \). \( \Box \)
Before we proceed to necessary conditions for ergodicity, which require additional assumptions on the group and the allowed joinings, we briefly return to the conditional expectation operator of Construction 2.3 for a related but independent result:

**Proposition 3.6.** Let $P_\omega$ be as in Construction 2.3, with $\omega \in J(A, B)$, and assume that $A$ is ergodic and $B$ an identity system. Then the fixed point space of $P_\omega$ is $C_\Omega_\omega$.

**Proof.** Since $A$ is ergodic, the fixed point space of $U_\mu$ is $C_\Omega_\omega$; see for example [1, Theorem 4.3.20]. But $V_g = id$, since $B$ is an identity system, so for any $x \in H_\mu \cap H_\nu$ one has $U_g x = W_g x = V_g x = x$, since $\omega$ is a joining. Therefore $H_\mu \cap H_\nu = C_\Omega_\omega$. □

Thus far we haven’t required joinings to be $\sigma$-weakly continuous, but $\sigma$-weak continuity is of course a natural assumption in the von Neumann algebra context, and in the next result we indeed need it.

**Theorem 3.7.** Let $G$ be amenable, countable and discrete. Assume $\omega \in J(A, B)$ is $\sigma$-weakly continuous. If $A$ is ergodic and $B$ an identity system, then $\omega = \mu \otimes \nu$.

**Proof.** We follow a standard plan from measure theoretic ergodic theory as can be found in [2, Proposition 2.2]. Let $(\Lambda_n)$ be a (right) Følner sequence in $G$, i.e. every $\Lambda_n$ is a compact (in other words, finite) subset of $G$ with $|\Lambda_n| > 0$ such that

$$\lim_{n \to \infty} \frac{|\Lambda_n \triangle (\Lambda_n g)|}{|\Lambda_n|} = 0$$

for all $g \in G$ (see for example [3, Theorems 1 and 2] for the general theory). For any $a \in A$ and $b \in B$ we then have

$$\omega(a \otimes b) = \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} \omega(a \otimes b)$$

$$= \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} \omega \circ (\alpha_g \otimes \beta_g)(a \otimes b)$$

$$= \omega \left( \left( \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} \alpha_g(a) \right) \otimes b \right).$$

Let $(H, \pi, \Omega)$ be the cyclic representation of $(A, \mu)$ obtained from the GNS construction, and $g \mapsto U_g$ the corresponding unitary representation of $G$ on
by the mean ergodic theorem, since $A$ is ergodic and hence the fixed point space of $U_G$ is $C\Omega$, which corresponds to the projection $\Omega \otimes \Omega$. Since $\Omega$ is cyclic for $\pi(A)'$, i.e. $\pi(A)' \Omega$ is dense in $H$, while $\pi \left( \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} \alpha_g(a) - \mu(a)1_A \right)$ is a bounded sequence, it follows that this sequence converges strongly and hence weakly to 0. However the weak and $\sigma$-weak topologies are the same on bounded norm closed balls, hence the sequence converges $\sigma$-weakly to 0. But $\pi^{-1}$ is a $*$-isomorphism between von Neumann algebras, and hence $\sigma$-weakly continuous, therefore

$$e_n := \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} \alpha_g(a) - \mu(a)1_A$$

converges $\sigma$-weakly (and hence weakly) to 0. If the Hilbert spaces on which $A$ and $B$ are defined are denoted $H_A$ and $H_B$ respectively, then we therefore have $\langle x, e_n y \rangle \to 0$ for all $x, y \in H_A$ hence $\langle x_1 \otimes x_2, e_n \otimes b(y_1 \otimes y_2) \rangle = \langle x_1, e_n y_1 \rangle \langle x_2, b y_2 \rangle \to 0$ for all $x_1, y_1 \in H_A$ and $x_2, y_2 \in H_B$. Since $(e_n)$ is bounded, and the finite linear combinations of elementary tensors are dense in $H_A \otimes H_B$, it follows that $e_n \otimes b$ converges weakly, and hence $\sigma$-weakly because of boundedness, to 0. This means $\omega(e_n \otimes b) \to 0$, from which we conclude that $\omega(a \otimes b) = \omega(\mu(a)1_A \otimes b) = \mu(a)\omega(1_A \otimes b) = \mu(a)\nu(b)$, so $\omega = \mu \otimes \nu$. □

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