On bosonization of 2d conformal field theories

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Abstract

We show how bosonic (free field) representations for so-called degenerate conformal theories are built by singular vectors in Verma modules. Based on this construction, general expressions of conformal blocks are proposed. As an example we describe new modules for the $SL(2)$ Wess-Zumino-Witten model. They are, in fact, the simplest non-trivial modules in a full set of bosonized highest weight representations of $\hat{sl}_2$ algebra. The Verma and Wakimoto modules appear as boundary modules of this set. Our construction also yields a new kind of bosonization in 2d conformal field theories.

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1 Introduction

Conformal field theory plays a crucial role in string theory and 2d statistical mechanics [1]. In order to solve the theory it is important to find correlation functions for a set of basic conformal operators (primary fields).

The free field representation provides in principle a powerful method to obtain the correlators and to compute the operator algebra (OA) of the primary fields [2-10]. Examples of models which have been solved by this approach are given by the minimal conformal models and the $SL(2)$ Wess-Zumino-Witten (WZW) model [3,9,10]. The basic idea of the free field representation is to construct the Hilbert space of theory by means of Fock modules of free fields. However the danger of unphysical states remains present: a careful analysis of the BRST-like complex of Fock modules is needed.

At present a technique of building the free field representation (generators of symmetry, primary fields, correlators, BRST-like complexes etc.) of some conformal field theories is developed [3,4,6-8,10,11]. We shall argue that the above technique has not yet been fully realized. The missing points are the full set of representations for the primary operators as well as a proper definition for conformal blocks, BRST complexes etc.

Throughout this paper, we shall always consider only degenerate conformal theories. Note that they have important applications in physics, for example in a description of critical fluctuations in a variety of statistical models [1,12].

The outline of the paper is as follows.

The main body of this work is presented in section 2. Since Kac-Moody algebras play a central role in 2d conformal field theories (most known examples of conformal models can be understood in terms of WZW model) we discuss the construction of the full set of the representations for the primary fields in the WZW model for simple Lie algebra $g$. As a result we obtain a general representation of conformal blocks of this model on the plane.

To demonstrate the approach we shall concentrate below on the $SL(2)$ WZW model. The generalization to other models is straightforward.

In section 3, the brief review of the Wakimoto representation for the $SL(2)$ WZW model is given. In particular, the resolution of an irreducible module (Hilbert space) of the algebra $\hat{sl}_2$ in terms of Fock spaces is sketched.

Section 4 provides the construction of the first nontrivial representation, the so-called Dotsenko representation, for the $SL(2)$ WZW model. The structure of the Dotsenko modules over $\hat{sl}_2$ and the resolution of an irreducible module is also briefly discussed. As an example the two-point function of the primary fields is computed.

In section 5 we present the second nontrivial representation. We analyze its structure over $\hat{sl}_2$ and give the resolution of an irreducible module in terms of Fock spaces. Also the two-point function of the primary fields is computed.

The last, section 6, contains some conclusions and speculations.
In the appendices we give technical details which are relevant for the explicit construction of the nontrivial bosonized representations of the \( SL(2) \) WZW model.

## 2 The method

Let \( |j\rangle \) be the highest weight vector of the Kac-Moody algebra \( \hat{g} \) with the weight \( j \). Denote also by \( V_j \) the Verma module over \( \hat{g} \) generated by the vector \( |j\rangle \). The defining relations for \( V_j \) are given by

\[
\hat{n}_+ |j\rangle = 0, \quad H_i |j\rangle = j(H_i)|j\rangle,
\]

where \( \hat{n}_+ \) is the subalgebra generated by annihilation operators, and \( H_i \) are the Cartan generators.

Let \( s_\alpha \) be the full set of singular vectors of the module \( V_j \). Here \( \alpha \) is a parameter labelling these vectors. They satisfy the following conditions

\[
\hat{n}_+ s_\alpha = 0, \quad H_i s_\alpha = j_\alpha(H_i)s_\alpha.
\]

Let the generators of the algebra \( \hat{g} \) be bosonized, i.e. they are expressed in terms of free fields. In addition let us assume that one solution of the equation (2.1) is available via the free fields. Our goal is to find the full set of the solutions.

Comparing the equation (2.1) and (2.2) we see that the first differs from the second only due to the parameter \( j(H_i) \). By doing the formal transformation of the initial weight \( j \rightarrow j' \) (Weyl reflection) we reduce (2.1) to (2.2). Thus our problem becomes the one of finding the full set of singular vectors in the Verma module. The last has been solved by Malikov, Feigin and Fuchs. It is easy to see that the full number of the solutions is \( N_s + 1 \), where \( N_s \) - the number of the singular vectors in the Verma module.

From physical point of view it means that the highest weight vector has \( N_s + 1 \)-fold degeneracy. It should be noted that we have to define the solutions of Malikov-Feigin-Fuchs in the bosonized theory more carefully.

Now let us turn to the construction of conformal blocks. Since there is a one to one correspondence between the local fields in the theory and the states in the Fock space, let \( \phi_j^{(\alpha)} \) be the primary field of \( G \) WZW model corresponding to \( |j\rangle^{(\alpha)} \). The starting point is the full set of the representations (solutions) \( \{\phi_j^{(\alpha)}\} \). In principle, the representation \( \phi_j^{(\alpha)} \) generates its screening operators \( S^{(\alpha)} \) and identity operator \( 1^{(\alpha)} \). Following [4,8], one can introduce a BRST-like operator using the screening operators \( S^{(\alpha)} \). Then the

\(^1\)The module \( V_j \) has singular vectors because the theory is degenerate by assumption.

\(^2\)Due to this we denote the solutions of (2.1) as \( |j\rangle^{(\alpha)} \) below. Then the known solution is \( |j\rangle^{(0)} \).

\(^3\)In fact there are expansions of the free fields in negative (integer) powers.

\(^4\)It should be noted that there are representations without screening operators and identity operators. As we shall see in sec.5 the first takes place in the \( SL(2) \) WZW model.
irreducible representation (Hilbert space of the theory) arises as cohomology groups of this BRST operator.

Let us now construct a general N-point conformal block. First of all introduce representations for vacua

\[ |\text{vac}\rangle = 1^{(\alpha_1)}|0\rangle, \quad \langle \text{vac}| = \langle 0|1^{(\alpha_2)}\rangle, \]

where \(|0\rangle\) and \(\langle 0|\) are vacua for the free fields, \(1^{(\alpha_1)}\) is the representation of the identity operator and \(1^{(\alpha_2)}\) is the conjugate identity operator. For a set of physical operators \(\{\phi_j(z_i)\}\), one can choose the set of the representations \(\{\phi_j^{(\beta)}(z_i)\}\) (solutions of eq.(2.1)). In order to take into account the balance of charges (zero modes) we insert into the correlator a set of the screening operators \(\{S(z_i)\}\) as well as a set of the identity operators \(\{1^{(\lambda_i)}\}\). If this gives the correct balance of charges, then the conformal block is well defined.

Finally, the N-point conformal block is given by

\[
\langle \prod_{i=1}^{N} \phi_j(z_i) |^{(\alpha_1,\alpha_2,\ldots,\alpha_N)} \rangle = \langle 0|1^{(\alpha_2)}\rangle \prod_{i=1}^{N} \prod_{m=1}^{M} \prod_{l=1}^{L} \phi_j^{(\beta)}(z_i) S^{(\gamma_m)} 1^{(\lambda_l)} 1^{(\alpha_1)}|0\rangle. \tag{2.4}
\]

Here \(\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2, \ldots, \beta_N)\), etc. Notice that above we have constructed the full set of basis in the space of N-point conformal blocks.

At present there are really two basis in the space of conformal blocks, namely the Feigin-Fuchs representation and the Dotsenko-Fateev representation. The first contains only \(\phi_j^{(0)}(z)\) (trivial exp of the scalar fields) representations for physical operators, where as the second has one additional nontrivial representation \(\phi_j^{(1)}(z)\). Formally they can be written as

\[
\langle 0| \prod_{i=1}^{N} \prod_{m=1}^{M} \phi_j^{(0)}(z_i) S^{(0_m)} |0\rangle, \quad 1^{(0)} \equiv 1, \tag{2.5}
\]

and

\[
\langle 0|1^{(0)} \phi_j^{(1)}(z_1) \prod_{i=2}^{N} \prod_{m=1}^{M} \phi_j^{(0)}(z_i) S^{(0_m)} |0\rangle. \tag{2.6}
\]

As we shall see in sec.5 that more non-trivial representations for the physical operators are possible and their correlation functions may be calculated.

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5 In fact, it is not hard to see that we don’t define contours for the screening operators in eq.(2.4). However one can choose the Felder’s contours and take into account the structure of the corresponding BRST complex.
3 Wakimoto representation for $SL(2)$ WZW model

In order to see how our construction works in remainder of this work we shall focus our attention on the $SL(2)$ WZW model. This is, of course, a trivial case. Nevertheless, it contains all technical subtleties which appear due to bosonization.

As a preparation for a discussion of nontrivial representations in later sections, let us briefly recall the well-known Wakimoto free field description of the $SL(2)$ WZW model [5-8,10].

The model is described in terms of one free boson $\varphi$ coupled to a background charge and a first order bosonic $(\omega, \omega^\dagger)$ system of weight $(0, 1)$. In terms of mode expansions we have

$$\varphi(z) = x_0 + ia_0 \ln z - i \sum_{n=-\infty}^{+\infty} \frac{a_n}{nz^n},$$

$$\omega(z) = i \sum_{n=-\infty}^{+\infty} \frac{\omega_n}{z^n}, \quad \omega^\dagger(z) = \sum_{n=-\infty}^{+\infty} \frac{\omega_n^\dagger}{z^{n+1}}.$$  \tag{3.1}

Canonical quantization gives the following commutation relations

$$[a_0, x_0] = i, \quad [a_n, a_{-n}] = n, \quad [\omega_n, \omega_{-n}^\dagger] = 1.$$  \tag{3.2}

Now let us construct the Fock module $\mathcal{F}_j^{(0)}$. Define a vector $|j\rangle^{(0)}$ such that

$$|j\rangle^{(0)} = e^{-2i\alpha_0 jx_0}|0\rangle,$$  \tag{3.3}

where the vacuum $|0\rangle$ satisfies the following conditions

$$a_n|0\rangle = \omega_{n+1}|0\rangle = \omega_n^\dagger|0\rangle = 0, \quad n \geq 0,$$  \tag{3.4}

and $2j \in \mathbb{N}$.

The Fock space $\mathcal{F}_j^{(0)}$ is obtained by acting on the vector $|j\rangle^{(0)}$ with the mode $\omega_0$ and all the negative frequency modes of the fields $\omega, \omega^\dagger, \varphi$. The basis of $\mathcal{F}_j^{(0)}$ is given by the states

$$a_{A_1}^{A_2} \cdots a_{C_1}^{C_2} \omega_0 \omega_{B_1} \cdots \omega_{-1}^\dagger |j\rangle^{(0)}, \quad \{A_i, B_i, C_i\} \in \mathbb{N}.$$  \tag{3.5}

Note that there exists a 1-1 correspondence between states in the Fock space and fields of the theory. The correspondence is given by

$$|\phi\rangle = \lim_{z \to 0} \phi(z)|0\rangle.$$  \tag{3.6}

For example the vector (3.3) corresponds to the field

$$\phi_j^j(z) = e^{-2i\alpha_0 j\varphi(z)}.$$
The other important fields are

\[ \phi_m^j(z) = \omega^j e^{-2i\alpha_0 j \varphi(z)}, \quad -j \leq m \leq j. \]  \hspace{1cm} (3.6a)

These fields are the primary fields of the theory.

In the Fock space the \( \hat{sl}_2 \) algebra is represented by

\[ J_n^+ = \omega_n^\dagger, \]

\[ J_n^0 = \frac{1}{2\alpha_0} a_n + \sum_{m=-\infty}^{+\infty} :\omega_n - m \omega_m^\dagger : , \] \hspace{1cm} (3.7)

\[ J_n^- = kn\omega_n - \frac{1}{\alpha_0} \sum_{m=-\infty}^{+\infty} \omega_{n-m} a_m - \sum_{m=-\infty}^{+\infty} \sum_{t=-\infty}^{+\infty} :\omega_t \omega_m \omega_{n-m-t} : . \]

Here \( k \) is the level, \( 2\alpha_0^2 = 1/k + 2 \). It easy to see that the vector \( |j\rangle^{(0)} \) is the highest weight vector of the \( \hat{sl}_2 \) algebra (3.7) with the weight \( j \).

The structure of \( \mathcal{F}_j^{(0)} \) as the module over \( \hat{sl}_2 \) is shown in fig.1 (see [7,8]).

Fig.1: The structure of the Wakimoto module \( \mathcal{F}_j^{(0)} \).

Here and henceforth, arrows go from one vector to another if and only if the second vector is in the \( U_g \) submodule generated by the first one.

It should be noted that the vectors \( s_i^{(0)} \) are singular, i.e. they are annihilated by \( \hat{n}_+ \) and have a non-zero weight

\[ \hat{n}_+ s_i^{(0)} = 0, \quad J_0^0 s_i^{(0)} \neq 0. \] \hspace{1cm} (3.8)
The vectors $m^{(0)}_i$ and $c^{(0)}_i$ are not singular vectors, $c^{(0)}_i$ are the so-called cosingular vectors. The highest weight vector $|j^{(0)}⟩$ ($|j^{(0)}⟩ ≡ |ν^{(0)}⟩$) is both singular and cosingular.

In [7,8] the following complex of the Fock modules was built

$$Q^{(0)}(g + 1)Q^{(0)}(g) = 0,$$

where

$$Q^{(0)}(v, 0) \equiv F^{(0)}_j, \quad \nu = 2j + 1, \quad p = k + 2,$$

and

$$Q^{(0)}(g) = \begin{cases} Q_v, & g = 2n, \quad n \in \mathbb{Z} \\ Q_{p-v}, & g = 2n + 1, \end{cases}$$

The number $g$ is called the ghost number, in analogy to gauge theories. One can see that $Q^{(0)}(g)$ and $F^{(0)}_j$ have the ghost number 1 and 0, respectively. The BRST charge is defined by

$$Q_v = \prod_{i=1}^\nu \oint_{C_i} dz_i \omega^i(z_i) e^{2i\alpha_0 \varphi(z_i)}.$$  \hfill (3.10)

The integration contours for $z_1, \ldots, z_\nu$ are shown in fig.2.

Fig.2: Contours used in the definition of the BRST operator $Q_v$.

We denote the above complex by $F^{(0)}$.

The detail structure of this complex is indicated in fig.3. The horizontal arrows correspond to the action of the BRST charge.
Bernard, Felder and independently Feigin, Frenkel proved that the cohomology of this complex is entirely located in the middle Fock space, where it is isomorphic to the irreducible representation of the $\hat{sl}_2$ algebra. This irreducible module is the space of physical states of the model.

$$H^g = \frac{\ker Q^{(0)}(g)}{\text{Im} Q^{(0)}(g-1)} = \begin{cases} 0, & g \neq 0 \\ \mathcal{H}_\nu, & g = 0. \end{cases} \quad (3.11)$$

Here $\mathcal{H}_\nu$ is the Hilbert space or the irreducible highest weight module.

4 Dotsenko representation for $SL(2)$ WZW model

4.1 Dotsenko modules over $\hat{sl}_2$ algebra

In the previous section we discussed the Wakimoto module over the $\hat{sl}_2$ Kac-Moody algebra and the $F^{(0)}$ complex of these modules. We will now focus on the first nontrivial module over $\hat{sl}_2$, the so-called Dotsenko module, and on a resolution $\hat{sl}_2$ in terms of Dotsenko modules.

The problem we will address in this section arises quite naturally in the free field approach to conformal field theories, namely how to construct all families of modules.
over $\hat{sl}_2$ (3.7) in terms of the free fields (3.1) and complexes whose cohomologies are isomorphic to $\mathcal{H}$.

A first step in this direction was done by Dotsenko [10]. He found the conjugate representation for the highest weight vector using the corresponding Operator Product Algebra. Explicitly

$$|\tilde{j}\rangle = (\omega^+_{-1})^{s+2j} e^{2i\alpha_0(j+s)x_0} |0\rangle,$$

with $s = -k - 1$.

Let us say a few words about the Dotsenko representation. It is clear from the equation (4.1) that the power of $\omega^+_{-1}$ is a negative integer. It means that analytic continuation in $s$ is necessary. However the continuation becomes a problem in the case when the correlator contains two or more operators in the Dotsenko representation.

It easy to see that the vector (4.1) corresponds to the first singular vector in the Verma module over $\hat{sl}_2$. Using the technique developed in sec.2 one arrives, after a straightforward calculation, at the following formula for the first non-trivial representation of the highest weight vector

$$s_1 = \left( J_{-1}^+ \right)^{p-\nu} |j\rangle^{(0)} \rightarrow |j\rangle^{(1)} = (\omega^+_{-1})^{2j-k-1} e^{2i\alpha_0(j+k-1)x_0} |0\rangle,$$

In the above, we also used the formulas (3.3) and (3.7).

Now let us introduce a vector $|j\rangle^{(1)}(s)$ as

$$|j\rangle^{(1)}(s) = -\frac{1}{2i\pi} \Gamma(2j+1+s) \int_C dt(-t)^{-2j-1-s} e^{-tw^+_{-1}} e^{2i\alpha_0(j+k-1)x_0} |0\rangle,$$

where the integration contour is indicated in fig.4.

![Contour C](image-url)

Fig.4: Contour $C$ used in the construction of the Dotsenko representation as well as the 2-representation.

In (4.3) $s \in C, s \neq -1, -2, \ldots$. Our goal is to define the vector (4.3) as well as a Fock space (Dotsenko module) in the case of $s = -k - 1$.

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*We consider the case of integrable representations when $k$ is positive integer and $0 \leq j \leq k/2$, $2j \in \mathbb{N}$.*
First of all compute the norm of the vector (4.3). Using the formula (A.2) of the
appendix A, one finds

$$\langle j|j \rangle^{(1)}(s) = \Gamma(2j + 1 + s).$$  \hspace{1cm} (4.4)

Let us now define a normalized vector $|\nu\rangle^{(1)}$ as

$$|\nu\rangle^{(1)}(s) = |j\rangle^{(1)}(s)/(\langle j|j \rangle^{(1)}(s))^{1/2}, \quad \nu = 2j + 1.$$  \hspace{1cm} (4.5)

The vector $|\nu\rangle^{(1)}(s)$ corresponds to the operator

$$\phi_j^{(1)}(z, s) = -\frac{1}{2i\pi} \Gamma^{1/2}(\nu + s) \int_C \, dt (-t)^{-\nu-s} e^{-t\omega^\dagger(z)} e^{2i\alpha_0(j-k-1)\varphi(z)}.$$  \hspace{1cm} (4.6)

It has the following OP expansions with the currents (3.7)

$$J^+(z_1)\phi_j^{(1)}(z_2, s) = RT,$$
$$J^0(z_1)\phi_j^{(1)}(z_2, s) = \frac{1}{z_{12}} (j + s + k + 1) \phi_j^{(1)}(z_2, s) + RT,$$
$$J^-(z_1)\phi_j^{(1)}(z_2, s) = \frac{1}{z_{12}} J^-_1 \phi_j^{(1)}(z_2, s) + \frac{1}{z_{12}} J^-_0 \phi_j^{(1)}(z_2, s) + RT,$$

where

$$J^-_1 \phi_j^{(1)}(z, s) = \frac{s + k + 1}{2i\pi} \Gamma^{1/2}(\nu + s) \int_C \, dt (-t)^{-2j-s} e^{-t\omega^\dagger(z)} e^{2i\alpha_0(j-k-1)\varphi(z)},$$

$$J^-_0 \phi_j^{(1)}(z, s) = \frac{1}{\pi} \Gamma^{1/2}(\nu + s) \int_C \, dt (-t)^{-2j-s} e^{-t\omega^\dagger(z)} e^{2i\alpha_0(j-k-1)\varphi(z)}$$

$$\times \left( -\frac{t}{2i} \partial \omega^\dagger(z) + (j + s + k + 1) t^{-1} \omega(z) - \frac{1}{2\alpha_0} \partial \varphi(z) \right),$$

$RT$ means terms which are regular as $z_1 \rightarrow z_2; z_{12} = z_1 - z_2$.

Set

$$|J^-_1\rangle(s) = \lim_{z \rightarrow 0} J^-_1 \phi_j^{(1)}(z, s)|0\rangle,$$
$$|J^-_0\rangle(s) = \lim_{z \rightarrow 0} J^-_0 \phi_j^{(1)}(z, s)|0\rangle.$$

Using (A.2) one can easily obtain the norms of these vectors

$$\langle J^-_1|J^-_1\rangle(s) = (2j + s)(k + 1 + s)^2,$$
$$\langle J^-_0|J^-_0\rangle(s) = 4j^2 + (2j + s)(2j + s + k/2).$$  \hspace{1cm} (4.8)
After taking the limit $s \to -k - 1$, eq. (4.8) becomes

$$\langle J^-_1 | J^-_1 \rangle = 0, \quad \langle J^-_0 | J^-_0 \rangle = 8j^2 - j(3k + 4) + (k + 1)(k/2 + 1). \tag{4.9}$$

From (4.7) and (4.9) we get the following OP relations

$$J^+(z_1)\phi^{j(1)}_j(z_2) = RT,$$

$$J^0(z_1)\phi^{j(1)}_j(z_2) = \frac{1}{z_{12}} j \phi^{j(1)}_j(z_2) + RT,$$  

$$J^-(z_1)\phi^{j(1)}_j(z_2) = \frac{1}{z_{12}} J^-_0 \phi^{j(1)}_j(z_2) + RT,$$  

with $\phi^{j(1)}_j(z) = \lim_{s \to -k - 1} \phi^{j(1)}_j(z, s)$.

As can easily be seen from (4.10), $\phi^{j(1)}_j(z)$ is the primary field corresponding to the highest weight vector $|\nu\rangle^{(1)} = \lim_{s \to -k - 1} |\nu\rangle^{(1)}(s) = \lim_{z \to 0} \phi^{j(1)}_j(z)|0\rangle$.

The screening operator in the Dotsenko representation is given by

$$S^{(1)} = \lim_{s \to -k - 1} S^{(1)}(s),$$

with

$$S^{(1)}(s) = \oint_{C_z} dz : J^+(z)\phi^{-1}_{-1}(z, s) :=$$

$$-\frac{1}{2i\pi}(s - 1)^{1/2} \Gamma^{1/2}(s) \oint_{C_z} dz dt (-t)^{-s} e^{-t\omega(z)} e^{-2\alpha p \phi(z)}.$$

Let $V^{(1)}_j(s)$ be a space generated by the vector $|\nu\rangle^{(1)}(s)$ and the bosonized generators (3.7).

**Proposition 4.1**

Let $v \in V^{(1)}_j(s)$ and $w \in \mathcal{F}^{(0)}(\nu, -1)$. Then $\langle w | v \rangle(s) = 0$, $\forall v, w$.

**Proof.** The proof of this proposition follows from the definitions of the spaces $V^{(1)}_j(s)$ and $\mathcal{F}^{(0)}(\nu, -1)$.

The scalar product of these vectors is given by

$$\langle w | v \rangle(s) = \text{const} \int_C dt (-t)^{n-s}, \quad n \in \mathbb{Z}.$$

Choosing $s$ as $s > 0$, $s \in R$, $s >> n$ one can close the integration contour as shown in fig.5.
From this it follows that
\[ \int_C dt(-t)^{n-s} = \oint_C dt(-t)^{n-s} = 0. \]

Note that the above procedure corresponds to the integration by parts in the derivation of the OP expansions (4.7). From this point of view, closing the integration contour seems natural.

Let \( \mathcal{F}_j^{(1)}(s) = V_j^{(1)}(s) \oplus \mathcal{F}(0)(\nu, -1) \). Then define the Dotsenko module as
\[ \mathcal{F}_j^{(1)} = \lim_{s \to -k-1} \mathcal{F}_j^{(1)}(s). \]

Now let us turn to the structure of the Dotsenko module as a \( \hat{sl}_2 \) module. Consider the following set of vectors
\[
\begin{align*}
  s_1^{(1)} &= (J_{-1}^+)^{p-\nu} |\nu\rangle^{(1)}(s), \\
  s_2^{(1)} &= (J_0^-)^{\nu} |\nu\rangle^{(1)}(s), \\
  s_3^{(1)} &= (J_{-1}^+)^{p+\nu} (J_0^-)^{\nu} |\nu\rangle^{(1)}(s), \\
  s_4^{(1)} &= (J_0^-)^{2p-\nu} (J_{-1}^+)^{p-\nu} |\nu\rangle^{(1)}(s), \\
  s_5^{(1)} &= \ldots
\end{align*}
\]

It is straightforward to see that, under \( s \to -k - 1 \{ s_i^{(1)} \} \to \) the singular vectors of the Verma module.

In terms of the free fields we have the following expressions for \( s_1^{(1)} \) and \( s_2^{(1)} \)
\[
\begin{align*}
  s_1^{(1)} &= \frac{(-)^{p-2j}}{2i\pi} (\nu + s)^{1/2} \ldots (p + s - 1)^{1/2} \Gamma^{1/2}(p + s) \int_C dt(-t)^{-p-s} e^{-\omega_1 t} e^{2i\alpha_0 (j+s)x_0} |0\rangle, \\
  s_2^{(1)} &= -\frac{1}{2i\pi} \Gamma^{1/2}(1 + s) \int_C dt(-t)^{-1-s} \left( \frac{1}{\alpha_0} t^{-1} a_{-1} - t^2 \omega_{-2}^+ \right) e^{-\omega_{-1}^t} e^{2i\alpha_0 (j+s)x_0} |0\rangle.
\end{align*}
\]

Their norms are given by
\[ \langle s_1|s_1 \rangle^{(1)}(s) = \Gamma(s + p)/\Gamma(\nu + s), \quad \langle s_2|s_2 \rangle^{(1)}(s) = s(2p + s - 1). \] (4.11)

In the above, we consider the \( j = 0 \) case for the vector \( s_2^{(1)} \) because its explicit expression in terms of the free fields has a rather complicated form. However the generalization to a general \( j \) is straightforward. After taking the limit \( s \to -k - 1 \), eq.(4.11) becomes

\[ \langle s_1|s_1 \rangle^{(1)} = 0, \quad \langle s_2|s_2 \rangle^{(1)} \neq 0. \] (4.12)

In an analogous way, one can calculate the norms of \( s_3 \) and \( s_4 \)

\[ \langle s_3|s_3 \rangle^{(1)} = \langle s_4|s_4 \rangle^{(1)} = 0. \] (4.13)

From (4.12) and (4.13) it follows that the structure of \( V_j^{(1)} \) is described by the diagram shown in fig.6.

Fig.6: The submodule structure of \( V_j^{(1)} \).

Finally, the structure of the Dotsenko module is given by the diagram presented in fig.7.

Fig.7: The submodule structure of the Dotsenko module \( F_j^{(1)} \).
An important remark is that the Dotsenko module contains the Verma module part at the top and the Wakimoto module part at the bottom.

4.2 BRST-LIKE COMPLEX OF DOTSENKO MODULES

Let us now look at a BRST-like complex of the Dotsenko modules. We first define the modules with the negative ghost numbers. They have the same structure as shown in fig. 7. Set

\[ \mathcal{F}^{(1)}(\nu, q) = \begin{cases} \mathcal{F}^{(1)}_{j-pn}, & q = -2n, \quad n \in \mathbb{Z}_+ \\ \mathcal{F}^{(1)}_{j-1-pn}, & q = -2n - 1 \end{cases} \quad (4.14) \]

where \( \mathcal{F}^{(1)}(\nu, 0) \equiv \mathcal{F}^{(1)}_j \). The number \( q \) is called the ghost number, in analogy to gauge theories.

Now let us turn to the construction of the Dotsenko modules with the positive ghost numbers. The starting point is the Wakimoto modules \( \mathcal{F}^{(0)}(\nu, q-1) \) with \( q > 0 \). Let \( \{c_i, i > 1\} \) be a subset of cosingular vectors of the Wakimoto module \( \mathcal{F}^{(0)}(\nu, q-1) \) (see fig. 1). These cosingular vectors generate a space \( \mathcal{F}^{(1)}(\nu, q) \). By construction of \( \mathcal{F}^{(1)}(\nu, q) \) one has the arrows of the diagram shown in fig. 8.

Fig. 8: The submodule structure of the Dotsenko modules \( \mathcal{F}^{(1)}(\nu, q) \), \( q > 0 \).
Proposition 4.2

The following infinite sequence

\[ Q^{(-2)} \rightarrow F^{(1)}(\nu, -1) \rightarrow F^{(1)}(\nu, 0) \rightarrow F^{(1)}(\nu, 1) \rightarrow F^{(1)}(\nu, 2) \rightarrow F^{(1)}(\nu, -1) \rightarrow F^{(1)}(\nu, 0) \rightarrow F^{(1)}(\nu, 1) \rightarrow F^{(1)}(\nu, 2) \rightarrow \]

\[ Q^{(1)}(q + 1)Q^{(1)}(q) = 0, \]

\[ Q^{(1)}(g) = \begin{cases} Q_{p-\nu}, & q = 2n, \quad n \in \mathbb{Z} \\ Q_{\nu}, & q = 2n + 1 \end{cases} \]

is a complex.

We denote this complex by \( F^{(1)} \). Notice that one has a relation between the ghost numbers of \( F^{(0)} \) and \( F^{(1)} \) complexes.

\[ g = q - 1. \quad (4.15) \]

Proof. The proof of this proposition follows from the structure of the \( F^{(0)} \) complex, our construction and an additional result presented below. The detailed structure of \( F^{(1)} \) can be pictured as shown in fig.9.

Fig.9: The complex \( F^{(1)} \). Horizontal arrows indicate where special vectors are mapped to under the BRST operator. The other arrows indicate the submodule structure of the Dotsenko modules.

By construction of \( F^{(1)} \), the horizontal arrows of its bottom part are ones of the \( F^{(0)} \) complex (see fig.3).
Proposition 4.3

There are a sequence of horizontal arrows \((-i)\).

**Proof.** Let us consider the arrow \((-1)\). In this case it is rewritten as (see fig.9)

\[
|v\rangle \sim Q\nu|\nu\rangle^{(1)} , \quad |v\rangle = \left(J_0^{\dagger}\right)^{\nu}|\nu\rangle^{(1)} .
\]

For simplicity we set \(j = 0, \nu = 1\).

Next define vectors

\[
|w\rangle(s) = -\frac{1}{2i\pi} \Gamma^{1/2}(s-1) \int_C dt(-t)^{1-s} e^{-sw^1_1} e^{2ia_0px_0} |0\rangle ,
\]

\[
|v\rangle(s) = -\frac{1}{2i\pi} \Gamma^{1/2}(s+1) \int_C dt(-t)^{-s} \left(-\frac{1}{\alpha_0} a_{-1} + tw_{-2}^{\dagger}\right) e^{-sw^1_{-1}} e^{2ia_0(p-1)x_0} |0\rangle .
\]

It is easy to see that \(|w\rangle(s) \rightarrow |\!-1\rangle^{(1)} , |v\rangle(s) \rightarrow J_0^{-1}|1\rangle^{(1)} \) under \(s \rightarrow -k - 1\).

Compute the matrix element \(\langle v|Q^{(1)}(-1)|w\rangle(s)\), here the BRST operator \(Q^{(1)}(-1) = Q_1\) is defined in (3.10). Using the integral (A.2) one gets

\[
\langle v|Q_1|w\rangle(s) = (s(s-1))^{1/2} .
\]

After taking the limit \(s \rightarrow -k - 1\), eq.(4.16) becomes

\[
\langle 1|J_0^{-1}, Q_1|\!-1\rangle^{(1)} = (p(p-1))^{1/2} .
\]

The same calculation can be repeated for a general \(j\) as well as \(i\).

Finally, one can compute the cohomology of this complex

\[
H^q = \frac{\text{Ker} Q^{(1)}(q)}{\text{Im} Q^{(1)}(q-1)} = \begin{cases} 0, & q \neq 0 \\ \mathcal{H}_\nu, & q = 0 . \end{cases}
\]

where \(\mathcal{H}_\nu\) is the irreducible module of \(\tilde{sl}_2\) (Hilbert space of the model).

4.3 CONFORMAL BLOCKS ON THE PLANE

In this section we use the Dotsenko representation together with the Wakimoto representation to give an explicit calculations for conformal blocks on the plane.

As an example we compute the two-point function of the primary fields. This is a trivial case, but it contains all technical subtleties which appear due to using the Dotsenko representation.

Set \(|\text{vac}\rangle = |0\rangle , \langle\text{vac}| = \langle 0|1^{(1)}\rangle^{\dagger}\), where \(1^{(1)}\) is the Dotsenko representation for the identity operator. The simplest two-point function is given by
\[ \langle \phi^j_{-j}(z_1)\phi^j_j(z_2) \rangle . \]

Choose the following representation for the primary fields: \( \phi^j_{-j}(0)(z_1) \) and \( \phi^j_j(1)(z_2) \). It is easy to see that one does not need additional screening operators as well as identity operators. Finally, the two-point function is written as

\[ \langle \phi^j_{-j}(z_1)\phi^j_j(z_2) \rangle (\vec{\alpha},\vec{\beta},\vec{\gamma},\vec{\lambda}) = \langle 0|1(1)^{\dagger}\phi^j_{-j}(0)(z_1)\phi^j_j(1)(z_2)|0 \rangle . \] (4.19)

Here \( \vec{\alpha} = (0,1), \vec{\beta} = (0,1), \vec{\gamma} = \vec{\lambda} = 0 \).

Then using the following two-point functions

\[ \langle \omega(z_1)\omega^\dagger(z_2) \rangle = i/z_{12}, \quad \langle \varphi(z_1)\varphi(z_2) \rangle = -\ln(z_{12}) , \] (4.20)
as well as the explicit formulas (3.6a), (4.6) for the primary fields and taking the limit \( s \to -k-1 \), one finds

\[ \langle \phi^j_{-j}(z_1)\phi^j_j(z_2) \rangle (\vec{\alpha},\vec{\beta},\vec{\gamma},\vec{\lambda}) = C_1/z_{12}^{2\Delta} , \quad C_1 = i^{4j+1}(k...(k-2j))^\frac{1}{2} \] (4.21)

Here \( \Delta = j(j+1)/k+2 \). \( C_1 \) can be absorbed into a proper normalization of the primary fields.

In order to complete the representation of the conformal blocks on the plane let us in the remaining part of this section sketch the BRST invariant chiral primary fields for the Dotsenko representation.

Following [4,8], we introduce a chiral primary field \( m\Phi_{\nu\rho}^{(1)} \) as

\[ m\Phi_{\nu\rho}^{(1)} : \mathcal{H}_{\nu} \to \mathcal{H}_{\mu} , \] (4.22)

where \( \nu = 2j+1, \rho = 2j_1+1, \mu = 2j_2+1 \).

In terms of the free fields one has

\[ m\Phi_{\nu\rho}^{(1)}(z) = \phi^j_m^{(1)}(z) \prod_{i=1}^r \oint_{C_i} dz_i \omega^\dagger(z_i) e^{2i\alpha_0\varphi(z_i)} , \] (4.23)

where the field \( \phi^j_m^{(1)}(z) \) corresponds to the state \( (J_0^-)^{j-m}\varphi^{(1)} \) of the Dotsenko module. A number of screening operators \( r \) is determined by the balance of charges in a three-point function (matrix element of the operator (4.23))\footnote{Note that \( r \) depends on a representation for the three-point conformal block, namely \( \{\vec{\beta},\vec{\gamma},\vec{\lambda}\} \)} \( C_i \) are the integration contours of the Felder’s type.

Following the method used by Bernard and Felder in [4,8], one can prove that the chiral field (4.23) commutes with the BRST operator (3.10). In particular, one finds

\[ Q_{\mu m}\Phi_{\nu\rho}^{(1)}(z) = e^{i\pi\alpha_{\nu}(z)} m\Phi_{\nu\rho}^{(1)}(z) Q_\rho . \] (4.24)
It should be noted that a phase factor in (4.24) is due to the free scalar field only. This is so, because for \( s = -k - 1 \) the \( \omega, \omega^† \) fields do not give rise to nontrivial monodromy. This completes the representation of the conformal blocks on the plane.

5 One more representation for \( SL(2) \) WZW model

5.1 NEW MODULES OVER \( \hat{sl}_2 \) ALGEBRA

In this section we want to derive another nontrivial representation for the \( SL(2) \) WZW model.

Our starting point is the second singular vector of the Verma module\(^8\). Explicitly

\[
s_2 = (J^-)^\nu |j\rangle^{(0)}. \tag{5.1}
\]

After a straightforward calculation one arrives at the following formula for the second nontrivial representation of the highest weight vector

\[
|j\rangle^{(2)} = (J^-)^{-2j-1} |j - 1\rangle^{(0)}. \tag{5.2}
\]

Following the method used in sec.4 for the Dotsenko representation, one can define a vector \(|j\rangle^{(2)}(s)\) as

\[
|j\rangle^{(2)}(s) = -\frac{1}{2i\pi} \Gamma(1 + s - 2j) \int_C dt (-t)^{-s-1} (1 + t\omega_0)^{-2j-2} e^{2i\alpha_0(j+1)\omega} |0\rangle, \tag{5.3}
\]

where the integration contour C is shown in fig.4, and where \( s \) is an arbitrary noninteger parameter. Also, we used the formulas (3.5) and (3.7) in the above.

Note that, in comparison with the Dotsenko representation, the 2-representation has simpler expressions for the primary fields and the singular vector. Namely

\[
|j, m\rangle^{(2)}(s) = -\frac{1}{2i\pi} \Gamma(1 + s - j - m) \int_C dt (-t)^{j+m-s-1} (1 + t\omega_0)^{-2j-2} e^{2i\alpha_0(j+1)\omega} |0\rangle, \tag{5.4}
\]

\[
s_2 = -\frac{1}{2i\pi} \Gamma(2 + s) \int_C dt (-t)^{-s-2} (1 + t\omega_0)^{-2j-2} e^{2i\alpha_0(j+1)\omega} |0\rangle. \tag{5.5}
\]

Here \(|j\rangle^{(2)}(s) \equiv |j, j\rangle^{(2)}(s)|.\)

\(^8\)We shall call the representation \( \varphi_j^{(\alpha)} \) as the \( \alpha \)-representation here and below. Then the Wakimoto representation is the 0-representation, the Dotsenko - the 1-representation, etc.
The normalized vector $|\nu\rangle^{(2)}(s) = |j\rangle^{(2)}/(|j|j\rangle^{(2)}(s))^{1/2}$ corresponds to the operator

$$\phi_j^{(2)}(z, s) = -\frac{1}{2i\pi} N_j \int_C dt (-t)^{2j-s-1}(1-it\omega(z))^{-2j-2}e^{2i\alpha_0(j+1)\varphi(z)}, \quad (5.6)$$

where $N_j = \Gamma^{1/2}(1 + s - 2j)\Gamma(2 + 2j)/\Gamma(s + 3)$.

It has the following OP expansions with the currents (3.7)

$$J^+(z_1)\phi_j^{(2)}(z_2, s) = \frac{(1 + s)N_j}{2i\pi z_{12}} \int_C dt (-t)^{2j-s}(1-it\omega(z_2))^{-2j-2}e^{2i\alpha_0(j+1)\varphi(z_2)} + RT,$$

$$J_0(z_1)\phi_j^{(2)}(z_2, s) = -\frac{(j - s - 1)N_j}{2i\pi z_{12}} \int_C dt (-t)^{2j-s-1}(1-it\omega(z_2))^{-2j-2}e^{2i\alpha_0(j+1)\varphi(z_2)} + RT,$$

$$J^-(z_1)\phi_j^{(2)}(z_2, s) = \frac{1 + s - 2j)N_j}{2i\pi z_{12}} \int_C dt (-t)^{2j-s-2}(1-it\omega(z_2))^{-2j-2}e^{2i\alpha_0(j+1)\varphi(z_2)} + RT. \quad (5.7)$$

In terms of states the equation (5.7) is given by

$$J_n^\alpha|\nu\rangle^{(2)}(s) = 0, \quad \alpha = \{+, -, 0\}, \quad n > 0. \quad (5.7a)$$

Set

$$|i\rangle(s) = \begin{cases} (J_0^+)^i |\nu\rangle^{(2)}(s), & i > 0, \\ |\nu\rangle^{(2)}(s), & i = 0, \\ (J_0^-)^i |\nu\rangle^{(2)}(s), & i < 0. \end{cases} \quad (5.8)$$

Using (3.7) and (B.4) one can easily get the norms of these vectors

$$\langle i|j\rangle(s) = \begin{cases} (s - 2j)...(s - 2j - i + 1), & i > 0, \\ 1, & i = 0, \\ (s + 1 - 2j)...(s - i - 2j)/(s + 2)...(s + 1 - i)^2, & i < 0. \end{cases} \quad (5.9)$$

The connections between these vectors can be illustrated by the following diagram

$$\cdots \longleftrightarrow |2\rangle(s) \longleftrightarrow |1\rangle(s) \longleftrightarrow |0\rangle(s) \longleftrightarrow | - 1\rangle(s) \longleftrightarrow | - 2\rangle(s) \longleftrightarrow \cdots$$

Here the arrows denote an action of $J_0^+$ and $J_0^-$. After taking the limit $s \to -1$, eq.(5.9) reduces to

$$\langle i|j\rangle = \begin{cases} (-i)(2j + 1)...(2j + i), & i > 0, \\ 1, & i = 0, \\ (-i)2j...(2j + 1 + i)/(i!)^2, & -2j \leq i < 0. \end{cases} \quad (5.10)$$
In fact, it is not hard to see that the previous diagram becomes

$$\cdots \leftrightarrow |2] \leftrightarrow |1] \leftrightarrow |0] \leftrightarrow |-1] \leftrightarrow |-2] \leftrightarrow \cdots \leftrightarrow |-2j]$$

As a consequence of the eq. (5.10) the vector $|\nu\rangle^{(2)} = \lim_{s \to -1} |\nu\rangle^{(2)}(s)$ is not the highest weight vector. One has

$$J^\alpha_n |j\rangle^{(2)} = 0, \quad \alpha = \{+, -, 0\}, \quad n > 0,$$

$$J^0_+ |\nu\rangle^{(2)} \neq 0, \quad J^0_0 |\nu\rangle^{(2)} = j|\nu\rangle^{(2)}, \quad J^-_0 |\nu\rangle^{(2)} \neq 0. \quad (5.11)$$

Let $\hat{n}'$ be a subalgebra on the creation operators of $\hat{sl}_2$ without $J^-_0$. Let $V_i(s)$ be a space generated by the vector $|i\rangle(s)$ and the bosonized generators of $\hat{n}'$. Define a space $V_j(s)$ as $V_j(s) = \sum_{i=-\infty}^{+\infty} V_i(s)$. The algebra $sl_2$ acts on $V_i(s)$ as follows

$$J^+_0 : V_i(s) \to V_{i+1}(s) \oplus V_i, \quad J^0_0 : V_i(s) \to V_i, \quad J^-_0 : V_i(s) \to V_{i-1}(s) \oplus V_i.$$

It is not hard to prove that in the case of $s = -1$ the space $V = \lim_{s \to -1} V_j(s)$ has the same structure as given in the above diagram.

Now one can define a space $V_j^{(2)}$ as $V_j^{(2)} = V/\text{SV}$, where $SV = \sum_{i=1}^{+\infty} \oplus V_i$, $V_i = \lim_{s \to -1} V_i(s)$. As a consequence of the above, the vector $|\nu\rangle^{(2)}$ becomes the highest weight vector (with respect to $V_j^{(2)}$).

We now proceed in complete accordance with the construction of the Dotsenko module.

**Proposition 5.1**

Let $v \in V_j^{(2)}(s)$ and $w \in F^{(0)}(\nu, 1)$. Then $\langle w|v \rangle = 0, \quad \forall v, w$.

**Proof.** The proof is the same as one of the Proposition 4.1.

Define the 2-module as

$$F_j^{(2)} = V_j^{(2)} \oplus F^{(0)}(\nu, 1).$$

Now let us turn to the structure of this module as a $\hat{sl}_2$ module.

Consider the following set of vectors

$$s_1^{(2)} = (J^+_1)^{p-\nu} |\nu\rangle^{(2)}(s), \quad s_2^{(2)} = (J^-_0)^{\nu} |\nu\rangle^{(2)}(s),$$

$$s_3^{(2)} = (J^-_1)^{p+\nu} (J^-_0)^{\nu} |\nu\rangle^{(2)}(s), \quad s_4^{(2)} = (J^-_0)^{2p-\nu} (J^+_1)^{p-\nu} |\nu\rangle^{(2)}(s),$$

$$s_5^{(2)} = ...$$

It is straightforward to see that, under $s \to -1 \{s_i^{(2)}\} \to$ the singular vectors of the Verma module. In terms of the free fields they are given by
\[
\begin{align*}
s_1^{(2)} &= -\frac{1}{2i\pi} N_j \int_C dt (-t)^{2j-s-1} (\omega_{-1}^\dagger)^{p-\nu} (1 + t\omega_0)^{-2j-2} e^{2i\alpha(j+1)x_0} |0\rangle, \\
s_2^{(2)} &= -\frac{1}{2i\pi} N'_j \int_C dt (-t)^{-s-2} (1 + t\omega_0)^{-2j-2} e^{2i\alpha(j+1)x_0} |0\rangle, \\
s_3^{(2)} &= -\frac{1}{2i\pi} N'_j \int_C dt (-t)^{-s-2} (\omega_{-1}^\dagger)^{p+\nu} (1 + t\omega_0)^{-2j-2} e^{2i\alpha(j+1)x_0} |0\rangle, \\
s_4^{(2)} &= -\frac{1}{2i\pi} N_j \int_C dt (-t)^{2j-s-1} \left(J_0^{-}(\omega_n, \omega_{-n}^\dagger, a_n)\right)^{2p-\nu} (1 + t\omega_0)^{-2j-2} e^{2i\alpha(j+1)x_0} |0\rangle. 
\end{align*}
\]

where \(N'_j = \Gamma(2j+2)/(s+2)\Gamma^{1/2}(1+s-2j)\).

Their norms in the case \(s=-1\) are given by

\[
\begin{align*}
\langle s_1|s_1\rangle^{(2)} &= (p - \nu + 1), \\
\langle s_2|s_2\rangle^{(2)} &= \langle s_3|s_3\rangle^{(2)} = \langle s_4|s_4\rangle^{(2)} = 0. 
\end{align*}
\]

(5.12)

In the above, we consider the \(j = 0\) case for the vector \(s_4^{(2)}\) because its explicit expression in terms of the free fields has a rather complicated form. Recall that the same is in the case of the vector \(s_2^{(1)}\) for the Dotsenko representation (see sec.4.1). In this sense the 2-representation has simpler expressions for the primary fields and singular vectors than the 1-representation. However the generalization to a general \(j\) is straightforward.

From (5.12) it follows that the structure of \(V_j^{(2)}\) is described by the diagram shown in fig.10. Finally, the structure of the 2-module is given by the diagram presented in fig.11.

Fig.10: The submodule structure of \(V_j^{(2)}\).

Fig.11: The submodule structure of the 2-module \(F_j^{(2)}\).
Note that the 2-module contains the Verma module part at the top and the Wakimoto part at the bottom.

5.2 BRST-LIKE COMPLEX OF 2-MODULES

Let us now construct BRST-like complex of 2-modules.

We first define the modules with the positive ghost numbers. They have the same structure as shown in fig.11.

Set

\[
\mathcal{F}^{(2)}(\nu, b) = \begin{cases} 
\mathcal{F}^{(2)}_{j+pn}, & b = 2n, \quad n \in \mathbb{Z}_+ \\
\mathcal{F}^{(2)}_{j-1+pn}, & b = 2n + 1,
\end{cases}
\]

where \(\mathcal{F}^{(2)}(\nu,0) \equiv \mathcal{F}^{(2)}_j\). The number \(b\) is called the ghost number, in analogy to gauge theories.

Now let us turn to the construction of the 2-modules with the negative ghost numbers. The starting point is the Wakimoto modules \(\mathcal{F}^{(0)}(\nu, g)\) with \(g < 1\). Define a space \(\mathcal{F}^{(2)}(\nu, b)\) as \(\mathcal{F}^{(2)}(\nu, b) = \mathcal{F}^{(0)}(\nu, b+1)/S\mathcal{F}^{(0)}(\nu, b+1)\), where \(S\mathcal{F}^{(0)}(\nu, b+1)\) is a submodule generated by the vector \(|\nu + pb\rangle^{(0)}\), if \(b = 2n\), or by \(|\nu + p(b - 1)\rangle^{(0)}\), if \(b = 2n - 1\). By construction of \(\mathcal{F}^{(2)}(\nu, b)\) one has the arrows of the diagram shown in fig.12.

Fig.12: The submodule structure of the 2-module \(\mathcal{F}^{(2)}(\nu, b)\), \(b < 0\).
Proposition 5.2

The following infinite sequence

\[ Q^{(2)(-2)}(\nu, -1) \rightarrow F^{(2)(\nu, 0)} \rightarrow F^{(2)(\nu, 1)} \rightarrow F^{(2)(\nu, 2)} \rightarrow 0, \]

\[ Q^{(2)}(b + 1)Q^{(2)}(b) = 0, \]

\[ Q^{(2)}(b) = \begin{cases} Q_{p-\nu} : & b = 2n, \quad n \in \mathbb{Z} \\ Q_{\nu} : & b = 2n + 1, \end{cases} \]

is a complex\footnote{In fact $Q^{(2)}(b)$ is the BRST operator with respect to the Wakimoto part of this complex (see below).}

We denote this complex by $F^{(2)}$. Notice that one has a relation between the ghost numbers of $F^{(0)}$ and $F^{(2)}$ complexes.

\[ q = b + 1 \quad (5.14) \]

Proof. The proof of this proposition follows from the structures of the $F^{(0)}$ complex of the Wakimoto modules, and of the Feigin-Fuchs complex of the Verma modules as well as from the above construction.

The detailed structure of $F^{(2)}$ can be pictured as in fig.13.

Fig.13: The complex $F^{(2)}$. Horizontal arrows indicate where special vectors are mapped to under the BRST operator. The other arrows indicate the submodule structure of the 2-modules.
By construction of $F^{(2)}$, the horizontal arrows of its bottom part are the ones of the $F^{(0)}$ complex (see fig.3). The arrows (i) of its top part are due to the Feigin-Fuchs complex of the Verma modules [15]. It is easy to see that the ghost number $b$ is one with respect to the Wakimoto part of the complex $F^{(2)}$. The ghost number for the Verma part is equal to $-b$. It should be noted that the BRST operator for the Verma part is not expressed in terms of the free fields.

Finally, one can compute the cohomology of this complex

$$H^b = \frac{\text{Ker } Q^{(2)}(b)}{\text{Im } Q^{(2)}(b-1)} = \begin{cases} 0, & b \neq 0 \\ \mathcal{H}_b, & b = 0. \end{cases}$$

(5.15)

where $\mathcal{H}_b$ is the irreducible $\hat{sl}_2$-module (Hilbert space of the model).

It is expected that there is an infinite set of modules. The Verma and Wakimoto modules are the boundary modules of this set. Repeating the procedure used in sec.4 and 5 one can step by step build an arbitrary module from this set. In fact we have the following diagram

$$\bullet \infty \ldots \bullet^2 \leftarrow \bullet^1 \leftarrow \bullet^0$$

where 0 corresponds to the Wakimoto module, 1 to the Dotsenko module,..., $\infty$ to the Verma module.

### 5.3 CONFORMAL BLOCKS ON THE PLANE

In this section we use the 2-representation together with the Wakimoto representation to give an explicit calculation for conformal blocks on the plane. As an example we compute the two-point function of the primary fields.

Set $|\text{vac}\rangle = |0\rangle, \langle \text{vac}| = \langle 0|1^{(2)\dagger}$, where $1^{(2)}$ is the 2-representation for the identity operator. The simplest two-point function is given by

$$\langle \phi^j_{-m}(z_1)\phi^j_m(z_2) \rangle.$$

Chose the following representations for the primary fields $\phi^j_{-m}^{(0)}(z_1)$ and $\phi^j_{m}^{(2)}(z_2)$. It is easy to see that one does not need either additional screening operators or identity operators. In order to compare the calculation with the one of sec.4.3 we set $m = j$.

Then

\[\text{In this representation as well as in the Wakimoto there are simple explicit expressions for the fields } \phi^j_{m}(z) \text{ (see (3.6a),(5.4)).}\]
\[
\langle \phi_{-j}^j(z_1) \phi_j^j(z_2) \rangle^{(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\lambda})} = \langle 0 | 1^{(2)\dagger} \phi_{-j}^j(0) (z_1) \phi_j^j(2) (z_2) | 0 \rangle .
\] (5.16)

Here $\bar{\alpha} = (0, 2), \bar{\beta} = (0, 2), \bar{\gamma} = \bar{\lambda} = 0$.

Using (4.10) as well as the formulas (3.6a),(5.4) for the primary fields and taking the limit $s \to -1$, one arrives at the result

\[
\langle \phi_{-j}^j(z_1) \phi_j^j(z_2) \rangle^{(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\lambda})} = C_2 / z_{12}^{2\Delta} , \quad C_2 = i^{4j} (2j!) .
\] (5.17)

$C_2$ can be absorbed into a proper normalization of the primary fields.

Let us conclude this section with some remarks.

(i) In contrast to the previous case (4.21) the $z_{12}$ dependence in (5.17) is only due to the scalar field $\varphi$. The fields $(\omega, \omega^\dagger)$ don’t give rise to the dependence on $z_{12}$. It is the same as in the Dotsenko-Fateev representation of a two-point function for the minimal model.

(ii) An interesting result is that the 2-representation does not have the screening operator. Explicitly

\[
S^{(2)} = \oint_{C_1} dz : J^+(z) \phi_{-1}^{-1}(2)(z) : \sim \oint_C dt (-t)^{-2} = 0 .
\] (5.18)

(iii) One has to be careful using the 2-representation to compute the conformal blocks, since the vector $|\nu\rangle^{(2)}$ is the highest weight vector for the factor space (see sect.5.1). The problem is to take this fact into account in explicit calculations of the conformal blocks.

(iii) Following [4,8], one can try to introduce a chiral primary field as

\[
m_{\Phi^{(2)}_{\nu\rho}} : \mathcal{H}_\nu \to \mathcal{H}_\mu ,
\] (5.19)

where $\nu, \mu, \rho$ are given by (4.22).

However there are two problems. The first problem is to write the field $m_{\Phi^{(2)}_{\nu\rho}}$ in terms of the free fields. The second is to prove the BRST invariance of this field.

6 Conclusions

First let us say a few words about the results.

The free field representation provides in principle a powerful way to obtain the correlators and compute the Operator Algebra of the primary fields. In this paper we have reworked and completed the free field representation for so-called degenerate conformal field theories, relying on the singular vectors in the Verma modules. Based on this construction, the full set of representations for the primary operators as well as a proper definition for conformal blocks have been proposed. As an example we have considered

\footnote{A possible solution of the first problem is $m_{\Phi^{(2)}_{\nu\rho}}(z) = \phi_{m(2)}^L(z) \prod_{i=1}^2 \oint_{C_i} dz_i \omega^1(z_i) e^{2z_{12}z_i(\varphi(z_i))}$ (see eq.(4.23)). The second problem is more complicated because an explicit expression for a BRST operator in the Verma part of $F^{(2)}$ is not available in terms of the free fields.}
the new modules for the $SL(2)$ WZW model. They are, in fact, the simplest non-trivial modules in the full set of bosonized highest weight representations of $\hat{sl}_2$ algebra. The Wakimoto and Verma modules appear as boundary modules of this set. We have constructed the BRST-like complexes of these modules. We have also computed the two-point function of the primary fields by using the new nontrivial representations.

Let us now conclude by mentioning some open problems.

(i) We have seen that the bosonized theory has the additional quantum number. This means that there is a hidden symmetry in the theory. The problem is to understand what underlies this symmetry.

Going one step further, one can consider, as an example, the $SL(2)$ WZW model. One interesting observation is that in this case the screened vertex operators (chiral primary field) define the representation of the quantum group $SU(2)_q$ [16]. In fact they are a subset of the full set of the representations for the screened vertex operators. In the bosonized theory there is an infinite representation of a quantum group. The problem is to understand what is a quantum group as well as an algebra of the screening operators\textsuperscript{12}. Note that the quantum group appears due to the bosonization.

(ii) The second problem is impressive too. It concerns a new type of the bosonization. The point is that in the well-known Friedan-Martinec-Shenker bosonization when one writes an initial set of the free fields in terms of a new set then the structure of the Fock space changes automatically [17]. In our bosonization we have the same set of the free fields but we change the structure of the Fock space by using a different representation\textsuperscript{13}. Hence, we will call this bosonization as the bosonization II.

It would be very interesting to obtain identities between the special functions by using our bosonization. It seems that the computation of the characters (partition functions of the irreducible representation $\mathcal{H}_\nu$ for $\hat{sl}_2$) is the simplest example. More complicated examples are the correlation functions on the higher genus Riemann surfaces.

(iii) As we have seen in sec.4,5 the $SL(2)$ WZW model yields an example of a $\beta\gamma$ system with a background charge. It seems that such systems appear naturally in non-trivial representations and it would be usefully to develop a theory for these systems. Some steps in this direction have been taken in works [10,18].

(iii) One can use the different representations for the primary fields in order to obtain solutions of the Knizhnik-Zamolodchikov equation via the Ward identities [14]\textsuperscript{14}. The generalization is to use more than one field in nontrivial representation. Then we may find a set of solutions via the Ward identities. Now the problem is to separate the basis (linear-independent solutions) from this set.

(iii) One has seen in the example of the $SL(2)$ WZW model that the BRST-like operators are built via the $S^{(0)}$ screening operator. It is expected that these operators

\textsuperscript{12}As we have seen in above there is an infinite set of the screening operators $S^{(\gamma)}$ in this theory.

\textsuperscript{13}These representations correspond to the different values of the new quantum number.

\textsuperscript{14}In fact, one does not need the bosonized theory in this case. We can use the expressions for the primary fields in terms of currents (Malikov-Feigin-Fuchs solutions).
can also be constructed in terms of an arbitrary screening operator $S^{(\gamma)}$. From this point of view one has an infinite set of the BRST-like operators as well as complexes corresponding to them.

(iiiii) In conclusion, we would like to stress that one can apply the technique developed in sec. 2 to the minimal models, $W$ algebras, $d = 2$ gravity etc. Now the problem is to find explicit expressions for the singular vectors in these theories. However one can try to use a such effective tool as the Drinfeld-Sokolov reduction, which relates $W$ algebras to the Kac-Moody algebras, in order to overcome this obstacle.

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Appendix A

In this appendix we will compute an integral used in sec. 4 in order to construct the Dotsenko representation.

Let us consider an integral

$$I(a) = \int_{C_t} \int_{C_v} dt dv (tv)^a e^{tv},$$

(A.1)

where the integration contours $C_t$ and $C_v$ are shown in fig. 4, and where $a$ is a parameter.

Using the formal formula

$$\int_{C_t} dt (-t)^{-a} e^{-bt} = -2i\pi b^{a-1}/\Gamma(a)$$

one easily arrives at the result

$$I(a) = (2i\pi)^2/\Gamma(-a).$$

(A.2)

The right hand side of (A.2) can be defined for other values of $a$ by the analytic continuation.
Appendix B

The purpose of this appendix is to compute an integral relevant for the 2-representation.

Let

\[ I(a, b) = \left( \frac{\Gamma(a)}{2i\pi} \right)^2 \int_{C_t} \int_{C_v} dtdv(tv)^{-a}(0|(1 - v\omega_0^\dagger)^b(1 + t\omega_0)^b|0) , \]  

where the integration contours \( C_t, C_v \) are represented in fig.4. \( a, b \) are parameters and \( \omega_0, \omega_0^\dagger \) are zero modes of \( \omega(z) \) and \( \omega^\dagger(z) \).

Taking into account the commutation relation (3.2), one finds

\[ I(a, b) = \left( \frac{\Gamma(a)}{2i\pi} \right)^2 \int_{C_t} \int_{C_v} dtdv(tv)^{-a} 2F_0(-b, -b; tv) , \]

where \( 2F_0(-b, -b; tv) \) is the degenerate hypergeometric function [19].

Using the integral representation for \( 2F_0(-b, -b; tv) \), we arrive at the following formula

\[ I(a, b) = \left( \frac{\Gamma(a)}{2i\pi} \right)^2 \lim_{k \to \infty} \frac{1}{B(-b, k + b)} \int_{C_t} \int_{C_v} \int_0^1 dtdvdx(tv)^{-a} x^{-b-1} (1 - x)^{k+b-1} (1 - x(1 + ktv))^{-b} . \]

Finally, it follows from (B.7), the definition of the \( B \) function and the relation \( \lim_{|z| \to \infty} z^x \Gamma(z)/\Gamma(z + x) = 1 \), that the integral \( I(a, b) \) is given by

\[ I(a, b) = \Gamma(a)\Gamma^2(a - b - 1)/\Gamma^2(-b) . \]

Now let us calculate one more integral.

Set

\[ I(a, b, k, x) = \int_{C_t} \int_{C_v} dtdv(tv)^a(1 - x(1 + ktv))^b . \]

Using the hypergeometric function and its integral representation, one obtains

\[ I(a, b, k, x) = \frac{(1 - x)^b}{B(-b, 1 + b)} \int_{C_t} \int_{C_v} \int_0^1 dtdvdy(tv)^{y-b-1}(1 - y)^b (1 + kvxy/(1 - x)) . \]

Choose the parameter \( a \), so that \( \text{Re} \ a < 0 \). This suggests that the contour \( C_t \) becomes the one shown in fig.5. Now the point \( (1 - x)/kvxy \) is inside of the contour. By using the Cauchy theorem as well as the definition of the \( B \) function, we finally arrive at the result

\[ I(a, b, k, x) = (2i\pi)^2(-xk)^{-a-1}(1 - x)^{a+b+1} \frac{\Gamma(-a - b - 1)}{\Gamma(-a)\Gamma(-b)} . \]
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