Determination of catacaustics through cyclographic mapping

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Abstract. The paper presents a general algorithm of determination of catacaustics given a curvilinear source and a reflector. The algorithm is based on the method of cyclographic mapping as well as the optical property of a cyclographic projection of a spatial curve. The algorithm is consistent and universal; it is suitable for every problem where the light source is given in the form of a central, parallel or diffuse bundle of rays. Through the presented algorithm the analytic solution to the problem of determination of catacaustics is acquired. The results of the study can be utilized in applied geometric optics in design of optical systems consisting of the triad of elements: a light source, a reflector and a receiver.

1. Introduction
The focal curves known as caustics are often used in geometrical calculations in the areas of engineering that actively apply optical systems of reflection and refraction working according to the laws of geometric optics. A caustic constitutes an envelope of reflected (or refracted) rays. According to terminology introduced by Bernoulli, a caustic of reflected rays is called a catacaustic [1]. Calculation of caustics finds application in geometric optics in light field intensity studies [2,3], in astronomy in order to determine geometry of singularities (gravitational lens effect) [4], in wave front studies including the studies of density of matter in the Universe [5], in solutions to various tasks of acoustics, seismology, quantum mechanics, etc. [7]. The theory of caustics is directly connected to one of the subdisciplines of modern mathematics – catastrophe theory [6]. Geometrically, a catacaustic constitutes an evolute of wave front, while the multitude of curves modeling the wave front constitutes a multitude of involutes with respect to the catacaustic.

In studies of catacaustics in optical systems the initial light is represented, in general, by a light source that is either located in a certain point (a central light source), or is infinitely distant (a parallel light source) [8]. With technological progress in multiple areas of engineering, for example, laser technology, optics, computer graphics, there appeared a problem of finding the catacaustic given a source of curvilinear shape emitting a diffuse bundle of rays [9]. The authors of the present paper propose a method and an algorithm of determination of catacaustics given curvilinear source and reflector. This method is based on the cyclographic mapping of space $R^3$ on a plane.

2. Problem definition
The existing methods of defining catacaustics consider either a point light source generating a central bundle of rays, or an infinitely distant light source generating a parallel bundle of rays. The problem set in the present paper is to acquire catacaustics given a source and a reflector both having curvilinear shape and to develop a respective method on the basis of cyclographic mapping of space on a plane as well as the optical property of cyclographic projection of a curve.

3. Theory
The method of cyclographic mapping has appeared at the end of the XIX century [10]. The modern informational technologies allow us to apply this method in solution to a wide range of urgent scientific and technical problems [11-14].

The cyclographic method allows us to map a point of space $R^3$ on projection plane $z=0$ in the form of a directed circle known as a cycle. A cyclographic projection of a spatial curve constitutes an envelope of cycles centered at orthogonal projections of the points of the initial curve. In general, a cyclographic projection of a curve constitutes two branches of envelope and possesses a known optical property [12, 15]: if we consider one of the two branches of the envelope as a source of rays, the other as a receiver, and the orthogonal projection of the spatial curve as a reflector, then the rays of light emitted from the source curve normally and reflected from the reflector curve drop on the receiver curve also normally. Therefore, the triad of elements performing an optical transformation is generated. The common task is to define the third element of the triad given the other two [16-18]. Thus method of cyclographic mapping finds application in solution to a number of problems of geometric optics [12, 17, 18].

As it was pointed out earlier, a catacaustic constitutes a wave front evolute, in other words, a multitude of receiver curves imitating the wave front [17]. Let us consider the problem of determination of a catacaustic through the method cyclographic mapping.

Given a source and a reflector on a plane, it is required to find the catacaustic of an optical system. Cyclographically, this problem is reduced to acquiring the evolute of the receiver (or wave front) [17]. In order to solve the problem, each flat element of the system is put into correspondence with its spatial cyclographic image. For example, a point source emitting a central bundle of rays is correspondent to a projecting $\alpha$-cone. Such cone has half-angle at the vertex $\alpha=45^\circ$. A parallel bundle of rays defined as a straight line on a plane is correspondent to an $\alpha$-plane inclined to the projection plane on the same angle $\alpha=45^\circ$. A diffuse bundle defined by a certain curve is correspondent to an $\alpha$-surface. Let us consider the method of formation of the $\alpha$-surface. Consider a curve $a_0$ on a plane modeling the light source defined by equations $x_a=x_a(t), y_a=y_a(t)$. The evolute $b_0$ of the curve $a_0$ is acquired through the known formulas [19]:

\[ x_b(t) = x_a(t) + y'_a(t) \frac{(x'_a(t))^2 + (y'_a(t))^2}{x''_a(t) \cdot y'_a(t) - x'_a(t) \cdot y''_a(t)}; \]
\[ y_b(t) = y_a(t) + x'_a(t) \frac{(x'_a(t))^2 + (y'_a(t))^2}{x''_a(t) \cdot y'_a(t) - x'_a(t) \cdot y''_a(t)}. \]

Then through the evolute projection $b_0$ the spatial evolute $b$ is reconstructed. Coordinate $z$ of each point of the evolute $b$ is equal to curvature radius of curve $a_0$. The formula for coordinate $z$ is of the following form:

\[ z_b(t) = \pm \sqrt{(x_a(t) - x_b(t))^2 + (y_a(t) - y_b(t))^2}. \]

Therefore, the initial curve modeling a diffuse bundle and its spatial evolute generate an $\alpha$-surface considered as a spatial cyclographic image of a diffuse bundle of rays with carrier $a_0$:

\[ X(t,l) = x_a(t) + l \left[ x_b(t) - x_a(t) \right], \]
\[ Y(t,l) = y_a(t) + l \left[ y_b(t) - y_a(t) \right], \]
\[ Z(t,l) = z_a(t) - (1-l)T_0, \quad 0 \leq t \leq T_0, \quad 0 \leq l \leq L. \]

Then the given curvilinear reflector is put into correspondence with a projecting cylindrical surface. Thus on the first stage the problem is reduced to acquiring the spatial curve $l$ of intersection of these surfaces $l:x_l(t), y_l(t), z_l(t)$. The following step is to acquire the cyclographic surface $P_{i,2}(P_{i0}, P_{i2})$ of the acquired curve of intersection $l$. In order to do that, let us utilize the formulas of cyclographic projection known in literature [11]:

\[ x_{p_{i,2}}(t) = x_l(t) + z_l(t) \frac{-x_l'(t) \cdot z_l'(t) \mp y_l'(t) \sqrt{(x_l'(t))^2 + (y_l'(t))^2 - (z_l'(t))^2}}{(x_l'(t))^2 + (y_l'(t))^2}; \]
\[ y_{p_{i,2}}(t) = y_l(t) + z_l(t) \frac{-y_l'(t) \cdot z_l'(t) \mp x_l'(t) \sqrt{(x_l'(t))^2 + (y_l'(t))^2 - (z_l'(t))^2}}{(x_l'(t))^2 + (y_l'(t))^2}. \]
One of the acquired branches of the cyclographic projection $P_{(1,2)}$, for example, the curve $P_{(1)}$, matches the source, while the other, for example, the curve $P_{(2)}$, constitutes the sought receiver. The evolute of the receiver curve $P_{(2)}$ is the sought catacaustic of the optical system.

Figure 1 represents the generalized algorithm of determination of catacaustic of an optical system.

![Algorithm Diagram]

Figure 1. The algorithm of determination of catacaustic in a «source-reflector» optical system

4. Results of experiments

Let us consider the examples. Let us determine catacaustic of an optical system given a point source emitting a central bundle of rays with coordinates $A(-3;0)$ and a reflector in the shape of an ellipse positioned at the center of coordinate system. The ellipse is defined by equations:

$$
\begin{align*}
\cos(\theta) & = x/\alpha, \\
\sin(\theta) & = y/\alpha
\end{align*}
$$

where $0 \leq t_0 \leq 2\pi$. The initial data of the “source-reflector” system is presented on figure 2.

Let us put the initial elements into correspondence with the respective cyclographic images. The cyclographic image of the central bundle of rays is an $\alpha$-cone $\Psi$ with arbitrary chosen coordinate $z$ of its vertex (5 units in this example). The equation of the $\alpha$-cone $\Psi$ is of form

$$Z = R - \sqrt{(x - x_\alpha)^2 + (y - y_\alpha)^2}.$$
Given the reflector curve, a projecting cylindrical surface $\Phi$ is reconstructed and the curve of intersection is determined $l = \Psi \cap \Phi$ (figure 3). The acquired equation of the curve of intersection $l$ is the following: 
\[ x_l = \cos(t_0); \quad y_l = 2\sin(t_0); \quad z_l = 5 - \sqrt{(\cos(t_0) + 3)^2 + 4\sin(t_0)^2}, \] where $0 \leq t_0 \leq 2\pi$.

Then the cyclographic projection $P_{(1,2)}(P_{(0)}, P_{(2)})$ of curve $l$ is constructed through formulas (3). The curve $P_{(1,2)}(P_{(0)}, P_{(2)})$ consists of two branches defined by equations of the following form:
\[
\begin{align*}
x_{P_{(1,2)}} &= \cos(t_0) + \frac{1}{\sin(t_0)^2 + 4\cos(t_0)^2} (-\frac{3\sin(t_0)^2(\cos(t_0) - 1)}{\sqrt{M}} + \cos(t_0)^2 \sqrt{N - \frac{K}{M}}(-5 + \sqrt{M})); \\
y_{P_{(1,2)}} &= 2\sin(t_0) + \frac{1}{8\cos(t_0)^2 + 2\sin(t_0)^2} (\sin(t_0)(-5 + \sqrt{M})(12\cos(t_0)^2 - \cos(t_0)^2) + \sqrt{M} \sqrt{N - \frac{K}{M}})).
\end{align*}
\]
where $M = (\cos(t_0) + 3)^2 + 4\sin(t_0)^2$, $N = 4\sin(t_0)^2 + 16\cos(t_0)^2$, $K = (-2(\cos(t_0) + 3)\sin(t_0) + 8\sin(t_0)\cos(t_0))^2$, $0 \leq t_0 \leq 2\pi$.

The next stage is to calculate the evolute of the cyclographic projection $P_{(1,2)}(P_{(0)}, P_{(2)})$ through the formulas (1). The calculated evolute constitutes a catacaustic consisting of two branches (figure 4).

Figure 2. The elements of the optical system: a point source ($A$) and an elliptic reflector

Figure 3. The curve of intersection $l$ between the projecting cylindrical surface and the $\alpha$-cone
Let us consider a more complex example: an optical system that includes a diffuse source. The carrier of the diffuse source is represented by a curve. In particular, let us define the light source through a two-segment Bezier spline of the second order. The equations of the segments are given in the following form:

\[
x_1 = (1-t)^2 + 1.375(1-t) + t^2, \quad x_2 = (1-t)^2 + 2.625(1-t) + 2.25t^2, \\
y_1 = 3(1-t)^2 + 7.75(1-t) + 5t^2; \quad y_2 = 5(1-t)^2 + 12.25(1-t) + 7.5t^2,
\]

where \(0 \leq t \leq 1\).

The reflector is given in the form of parabola:

\[
x = 3 - \frac{t_y^2}{4}; \quad y = 5 + 3t_y, \quad \text{where} \quad -1 \leq t_y \leq 1.
\]

The subsequent calculations are performed according to the algorithm presented on figure 1. The segments of the spline that model the light source are put into correspondence with \(\alpha\)-surfaces \(\Psi_1\) and \(\Psi_2\) respectively, while the reflector is put into correspondence with a projecting cylindrical surface \(\Phi\). Let us determine the curve of intersection \(l = (\Psi_1 \cup \Psi_2) \cap \Phi\) (figure 5). Through the formulas (3) we acquire the cyclographic projection \(P_{(1,2)}(P_{(1)}, P_{(2)})\) of the curve of intersection \(l\). One of the branches of the cyclographic projection completely matches the initial curve of the source, while the other constitutes an imaginary receiver due to its exceptional position with respect to the reflector (figure 5).

The evolute of the imaginary source curve acquired through the equations (1) is the sought catacaustic. The final result is depicted on figure 6.

5. Consideration of results

The proposed algorithm of analytic solution to the problem of determination of catacaustic allows us to acquire the catacaustic not only in optical systems with central or infinitely distant source, but also in more complex optical systems with diffuse source. However, it is worth mentioning that applying the proposed algorithm to the problems featuring curves of higher orders requires significant computational resources. The numerical methods of calculation might be better suited for these problems.
Figure 5. The curve $l$ of intersection between the projecting cylindrical surface and the $\alpha$-surface

Figure 6. Catacaustic in the optical system with a diffuse light source

6. Conclusion
The paper presents the capability of analytic solution to the problem of catacaustic determination in planar optical systems. The proposed algorithm is based on the method of cyclographic mapping of space. The advantage of the proposed algorithm is the capability of catacaustic determination in
optical systems featuring a diffuse light source previously not studied in the scientific literature. This algorithm can be utilized in studies of optical systems in applied geometric optics.

7. References

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