Improving the Redundancy of the Knuth Balancing Scheme for Packet Transmission Systems

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1 Introduction

A binary codeword of length \( k \) is said to be balanced if the number of zeros and ones within that sequence equals \( k/2 \), for even \( k \). Balanced codes are very useful for digital recording of data on optical and magnetic storage disks. They can also be used to correct or detect errors within communication channels or to conform with a dc-free channel input constraint.

Donald Knuth proposed a simple and efficient scheme to generate balanced codewords\cite{1}. This approach stipulates that any binary unbalanced codeword, \( x \) of length \( k \) can always be encoded into a balanced one denoted as \( x' \), by inverting the first \( e \) bits of \( x \) where \( 1 \leq e \leq k \). The index \( e \) is encoded as a prefix, \( p \) that is appended to \( x' \) and send through a channel. At the receiver side, the decoder receives the codeword \( px' \), read off first the prefix and then, is able to recover the original information sequence \( x \) by inverting back the first \( e \) bits of \( x' \).

This algorithm is very suitable for long sequences as it does not make use of any lookup tables either at the encoder or the decoder.

The redundancy of Knuth’s algorithm \( p \), is approximately evaluated as

\[
p = \log_2 k \quad \text{for} \quad m \gg 1. \tag{1}
\]

Since then, some papers have been published to reduce the redundancy presented in (1).

In \cite{2}, two attempts to improve Knuth’s balancing algorithm were presented by Weber & Immink. The first one was based on the distribution of the transmitted prefix index; the basic Knuth scheme uses the first balanced point at position \( e \) to encode it as the prefix, therefore the encoder is set to choose smaller values for the position index. It has been shown that the distribution of the index for equiprobable information sequences, is not uniform and presents a redundancy of slightly less than (1). The second attempt used the multiplicity of inversion points within the information sequence, to transmit auxiliary data. These schemes both used a variable length prefix of the chosen index; this only made a minor improvement on the Knuth’s algorithm redundancy.

The second attempt from \cite{2} was employed in \cite{3}. This method was renamed bit recycling for Knuth’s algorithm (BRKA); it relies on a high probability of having more than one balance point from an information sequence candidates. The proposed scheme is suitable for various communication systems as it generates very efficient and less redundant balanced codes.
sequence; in other words, this scheme uses the multiplicity of possible encodings to reduce the gap between the lower bound redundancy and the Knuth’s one.

A major contribution in reducing the Knuth’s algorithm redundancy was shown by Immink and Weber in [4]. This new scheme does not make use of look-up tables and presents a very efficient encoding of the index prefix for both variable and fixed length prefixes. Furthermore, the distribution of the prefix length was discussed as well as the average efficiency of this construction.

In this paper, we propose a modification of a scheme described in [4] to generate efficient and less redundant balanced codes. This approach is designed for communication systems that model the data as packets and not accommodate cascading sequences as in most information theory related applications.

The rest of this paper is organized as follows: the system model of the proposed scheme is described in Section 2, then the method description based on information sequence candidates is presented in Section 3. Section 4 shows the decoding process. Section 5 and 6 present detailed analysis as well as performance and discussions on the proposed scheme redundancy. Finally, the paper is concluded in Section 7.

2 System Model

Fig. 1 presents a system system of received data for two different systems. In (a), the received data is modelled as a set of balanced cascading sequences, each of them is composed of the encoded version of the information sequence appended with a prefix. This is suitable for applications such as data storage, etc; because they represent a long stream of data as a set of cascading balanced codewords referring to various data blocks to be encoded sequentially. In this model it is crucial for the decoder to keep track of start and ending of each information sequence with length \( k_i \), in order to evaluate the number of bits reserved for every prefix for variable and fixed length prefixes. This may require inserting synchronization bits.

In Fig. 1(b), the packet conception is described whereby a single block of data is received at the time. This may occur in various communications systems such as Bluetooth/wireless communication, smart grids systems, GSM networks, power line communication (PLC), visible light communication (VLC), network communication, internet architecture, etc. This conception uses incremental communication that is, each transmitted packet demands an “ACK” message before the subsequent packet is sent. The decoder only needs to keep track of one parameter instead of two unlike in Fig. 1(a).

![Flow chart of the encoding method.](image-url)

Fig. 1 System conception (a) Cascading sequence vs. (b) Packet

For the rest of this work, the packet conception is considered. The advantage of a single tracking parameter is exploited to derive an efficient and less redundant scheme for generating balanced codewords.

3 Encoding method based on information sequence candidates

Fig. 2 presents the flow chart of the encoding method for a VL prefix. The idea of this scheme is to associate every information sequence of length \( k \) to a balanced codeword using the Knuth’s inversion rule as described in [4]. This leads to \( \binom{k+1}{2} \) distinct set of information sequence candidates. Furthermore each of this set is compressed to decrease the redundancy appended to every information sequence.

![Flow chart of the encoding method.](image-url)
Given a random binary sequence \( \mathbf{x} \) to be encoded, if \( \mathbf{x} \) is already balanced, a protocol is adopted between transmitter and receiver to have a prefix-less codeword (zero-length prefix); otherwise, \( \mathbf{x} \) is balanced following the Knuth’s algorithm, then the associated balanced codeword is obtained and denoted as \( \mathbf{x'} \) from the least inversion point index. All other information sequence candidates associated to \( \mathbf{x'} \) are recorded and listed following the lexicographic order. The prefix of \( \mathbf{x} \) corresponds to its rank amongst the information sequence candidates.

**Example 1** Let us consider all sequences of length \( k = 4 \). All information sequence candidates are associated to a balanced codeword.

| \( \mathbf{x'} \) | 0011 | 0101 | 0110 | 1001 | 1010 | 1100 |
|---|---|---|---|---|---|---|
| \( \mathbf{x} \) | 1011 | 1101 | 1000 | 0001 | 0010 | 0000 |
| | 1111 | 1010 | 1110 | 0111 | 0101 | 0100 |
| | 1100 | 1001 | 0110 | 0011 | | |

(2) shows the encoding process described in [4], whereby balanced codewords (marked in bold) are part of the information sequence candidates.

| \( \mathbf{x'} \) | 0011 | 0101 | 0110 | 1001 | 1010 | 1100 |
|---|---|---|---|---|---|---|
| \( \mathbf{x} \) | 1011 | 1101 | 1000 | 0001 | 0010 | 0000 |
| | 1111 | 1100 | 1110 | 0011 | 0101 | 0100 |

However, in our scheme, balanced codewords are excluded from every set of information sequence candidates as shown in [3] based on Theorem 1.

**Theorem 1** Any balanced codeword of length \( k \) is always associated to another balanced one.

**Proof** By applying Knuth’s inversion algorithm on any already balanced codeword, another balanced codeword is generated; at the worst case scenario, it is found by inverting all bits.

Let us denote by \( c(\mathbf{x'}) \), the cardinality of information sequence candidates associated to the balanced codeword \( \mathbf{x'} \). In [2], \( 2 \leq c(\mathbf{x'}) \leq 3 \), while in [3], \( 1 \leq c(\mathbf{x'}) \leq 2 \).

The inclusion of balanced sequences within the set of information sequence candidates as presented in [4], adds an extra rank in the ranking process which is required to keep track of start and end of encoded information sequence in the cascading sequence conception. However, as we described in Theorem 1 a balanced sequence is always associated to another one. This important observation leads to a reduction in \( c(\mathbf{x'}) \) and it is ideal for packet transmission systems.

Let us denote by \( \max\{d_l\} \) and \( \min\{d_l\} \), the maximum and minimum of running digital sum respectively performed on the information sequence \( \mathbf{x} \) of length \( k \). \( d_l \) is expressed as follow:

\[
d_l = \sum_{i=1}^{l} x_i, \text{ with } l \leq k.
\]

**Theorem 2**

\[
c(\mathbf{x'}) = \max\{d_l\} - \min\{d_l\}.
\]

**Proof** It was proved in [4] that \( c(\mathbf{x'}) = \max\{d_l\} - \min\{d_l\} + 1 \); the balanced codeword was removed out of every set of information sequence candidates. Therefore the new \( c(\mathbf{x'}) \) is subtracted by 1, that is \( c(\mathbf{x'}) = \max\{d_l\} - \min\{d_l\} \).

**Theorem 3**

\[
1 \leq c(\mathbf{x'}) \leq \frac{k}{2}.
\]

**Proof** It was established in [4] that \( 2 \leq c(\mathbf{x'}) \leq \frac{k}{2} + 1 \); then after removing the balanced codeword out of every set of information sequences candidates, it follows that \( 1 \leq c(\mathbf{x'}) \leq \frac{k}{2} \).

Therefore, the required fixed prefix length for this scheme is \( \log_2 \frac{k}{2} \); this is a slight improvement on the Knuth’s scheme that has a redundancy of \( \log_2 k \) as well as on the scheme in [4] where it equals \( \log_2 \frac{k}{2} + 1 \).

Finally, the prefix is obtained from ranking the information sequence candidates associated to the balanced codeword from 0 to \( \frac{k}{2} - 1 \).

### 4 Decoding

The decoding process is illustrated in Fig. 3. The flow is as follow: The prefix is extracted from the overall received codeword of length \( n = k + p \) as the first \( \log_2 (\frac{k}{2}) \) bits; then all information sequence candidates associated to \( \mathbf{x'} \) are listed and ordered lexicographically from 0 to \( \frac{k}{2} - 1 \). Finally, the prefix is mapped to the rank of the initial information sequence.

**Example 2** We want to decode the received codeword, 1111000011, where the bold and underline word represents the prefix.

| Info. seq. candidates | Prefix rank |
|-----------------------|-------------|
| 01000011 | 0 (00) |
| 00000011 | 1 (01) |
| 00110011 | \( \times \) (not ranked because balanced) |
| 00111011 | 2 (10) |
| 00111111 | 3 (11) |
Theorem 4 The number of balanced codewords \( x' \) of length \( k \) and \( c(x') = \lambda \), \( N(\lambda, k) \) for \( 1 \leq \lambda \leq \frac{k}{2} \), is such that

\[
N(\lambda, k) = \sum_{i=1}^{\lambda+1} M^k_{\lambda+i}(i,i) - 2 \sum_{i=1}^{\lambda} M^k_{\lambda+i}(i,i) + \sum_{i=1}^{\lambda-1} M^k_{\lambda-i}(i,i).
\]

Proof The number of balanced codewords such that \( c(x') = \lambda \) for \( 2 \leq \lambda \leq \log_2 \frac{k}{2} + 1 \) in [4] was as follow:

\[
N(\lambda, k) = \sum_{i=1}^{\lambda} M^k_{\lambda}(i,i) - 2 \sum_{i=1}^{\lambda-1} M^k_{\lambda-i}(i,i) + \sum_{i=1}^{\lambda-2} M^k_{\lambda-2-i}(i,i).
\]

Furthermore, for \( 1 \leq \lambda \leq \frac{k}{2} \), we just shifted the set of all random walks between bounds \( B2 \) and \( B1 \) by one step down. This leads to \( N(\lambda', k) \) balanced codewords such that \( c(x') = \lambda \) where \( \lambda' = \lambda + 1 \).

This leads to the following

\[
N(\lambda', k) = \sum_{i=1}^{\lambda+1} M^k_{\lambda+i}(i,i) - 2 \sum_{i=1}^{\lambda} M^k_{\lambda+i}(i,i) + \sum_{i=1}^{\lambda-1} M^k_{\lambda-i}(i,i).
\]

A simplified expression of \( M_B \) was provided in [4] based on a formula to compute powers of \( M_B \) derived by Salkuyeh [5] as follow:

\[
\sum_{i=1}^{B} M^k_{B}(i,i) = 2^k \sum_{i=1}^{B} \cos^{k} \frac{\pi i}{B+1}.
\]

This makes the computation of \( N(\lambda', k) \) much simpler as follow:

\[
N(\lambda, k) = 2^k \left( \sum_{i=1}^{\lambda+1} \cos^{k} \frac{\pi i}{\lambda+2} - 2 \sum_{i=1}^{\lambda} \cos^{k} \frac{\pi i}{\lambda+1} + \sum_{i=1}^{\lambda-1} \cos^{k} \frac{\pi i}{\lambda} \right).
\]

The computation of \( N(\lambda, k) \) as presented in [8] becomes obvious for special values of \( \lambda \) as shown in [9].

The enumeration of sequences corresponding to these values of \( \lambda \) as well as the pseudo code for computing
c(x'), for generating the ordered set of information sequence candidates and for determining the prefix index were provided in [4].

\[
\lambda \quad N(\lambda, k)
\]

| \( \lambda \) | \( N(\lambda, k) \) |
|-----|--------|
| 1   | 2      |
| 2   | 2(2^{k/2} - 1) |
| \( k \) - 1 | \( k(k-4), k > 4 \) |
| \( k/2 \) | \( k \) |

(9)

6 Analysis and discussions

In this section, the average number of bits denoted as \( H(k) \) required to encode the prefix index of a sequence of length \( k \) is computed. The number of information sequence candidates associated to a balanced codeword \( x' \) out of the \( 2^k - \left( \frac{k}{2} \right) \) possible information sequence candidates.

\[
\sum_{\lambda=1}^{k/2} \lambda N(\lambda, k) = 2^k - \left( \frac{k}{2} \right).
\]

(10)

It follows that

\[
H(k) = \frac{\sum_{\lambda=1}^{k/2} \lambda N(\lambda, k) \log_2 \lambda}{2^k - \left( \frac{k}{2} \right)}
\]

(11)

The minimum redundancy for the full set of balanced codewords is given in [2] by:

\[
H_0(k) = k - \log_2 \left( \frac{k}{2} \right) \approx \frac{1}{2} \log_2 k + 0.326
\]

(12)

The average number of bits for the construction in [4] is as follow:

\[
H_1(k) = 2^{-k} \sum_{\lambda=2}^{k+1} \lambda N(\lambda, k) \log_2 \lambda.
\]

(13)

The average number of bits for the method in [4] is given by

\[
H_2(k) = \sum_{c=1}^{k/2} P(c)AV(c),
\]

(14)

where

\[
P(c) = 2^{c+1-k} \left( \frac{k-1-c}{2} \right), 1 \leq c \leq \frac{k}{2}, d = c - 2^{\lfloor \log_2 c \rfloor}, \text{ and}
\]

\[
AV(c) = (c - 2d). [\log_2 c]. \frac{1}{2^{\lfloor \log_2 c \rfloor}} + 2d. \frac{1}{2^{\lfloor \log_2 c \rfloor}}. [\log_2 c].
\]

Table 1 presents the comparison of the average number of bits necessary to encode the prefix from various schemes. Let \( d_{H, H_0}, d_{H, H_1} \) be the difference between the average prefix length \( H_0 \) and \( H_1 \); we observed that \( d_{H, H_0} \leq 0.61, d_{H, H_1} \leq 0.64 \) and \( d_{H_2, H} \leq 1.23 \).

| \( k \) | \( H_0 \) | \( H \) | \( H_1 \) | \( H_2 \) |
|-----|------|------|------|------|
| 4   | 1.4150 | 0.8090 | 1.4387 | 0.5000 |
| 8   | 1.8707 | 1.4632 | 1.8985 | 0.9375 |
| 16  | 2.3483 | 2.0806 | 2.3790 | 1.3706 |
| 32  | 2.8370 | 2.6629 | 2.8691 | 1.8082 |
| 64  | 3.3314 | 3.2207 | 3.3641 | 2.2516 |
| 128 | 3.8286 | 3.7615 | 3.8616 | 2.7039 |
| 256 | 4.3272 | 4.2902 | 4.3603 | 3.1647 |
| 512 | 4.8265 | 4.8104 | 4.8597 | 3.6330 |
| 1024| 5.3261 | 5.3246 | 5.3594 | 4.1082 |

Fig. 4 presents the average number of bits for prefix encoding for various schemes. The proposed scheme’s average redundancy given by (11), performed better than the average minimum redundancy for the full set of balanced codewords as in (12) and the Immink & Weber average redundancy as in (13). However the difference in length between the proposed scheme and the Al-rababa’s et al average redundancy as in (14) is less than 1.23.

\[
\text{Fig. 4: } H_0(k), H(k), H_1(k) \text{ and } H_2(k) \text{ vs } \log_2 k.
\]

Fig. 5 shows the comparison between the average redundancy for balanced prefixes for \( H(k) \) and \( H_1(k) \), denoted as \( H'(k) \) and \( H'_1(k) \) respectively as well as \( \log_2 k \) and \( [\log_2 k] \). \( H'(k) \) is obtained from a simple modification of \( H(k) \) provided in (11) as follow

\[
H'(k) = \sum_{\lambda=1}^{k/2} \lambda N(\lambda, k) \Delta(\lambda) \frac{1}{2^k - \left( \frac{k}{2} \right)}
\]

(15)

Similarly, \( H'_1(k) \) is derived from \( H_1(k) \) given in (13) as follow:

\[
H'_1(k) = 2^{-k} \sum_{\lambda=2}^{k+1} \lambda N(\lambda, k) \Delta(\lambda).
\]

(16)
Where \( \Delta(\lambda) \) corresponds to the smallest value of length \( k \) such that \( \left( \frac{k}{2} \right) \geq \lambda \).

![Fig. 5 H'(k), H_1'(k), log_2(k) and \lceil log_2(k) \rceil vs log_2 k.](image)

The graphs of \( \log_2(k) \) and \( \lceil \log_2(k) \rceil \) represents the minimum redundancy and that of integer valued redundancy of the traditional Knuth’s construction. We observe that, it is only from \( k > 64 \) that the average redundancy of the scheme presented in [4] is less than that of the Knuth scheme; whereas for the proposed construction, the average redundancy becomes advantageous as soon as \( k > 16 \). Furthermore, the proposed scheme performed much lower than the scheme in [4] for \( k < 1024 \).

![Fig. 6 Fixed length schemes](image)

According to Theorem 3, the two coding schemes are applicable for the proposed scheme. For the fixed length prefix construction, the encoding of the prefix requires exactly \( \log_2\left( \frac{k}{2} \right) \) bits representing the balanced index \( e \) ranging from 0 to \( \frac{k}{2} - 1 \); whereas for the VL scheme, the prefix length varies between 0 and \( \log_2\left( \frac{k}{2} \right) \) depending on the nature of the information to be encoded. A zero-prefix is used when the information sequence is already balanced. However, the VL scheme is more efficient than the fixed length one on the average basis.

![Fig. 7 Rounded up fixed length schemes](image)

Fig. 5 presents the fixed length performance, we observed that the proposed scheme is more efficient than the classic Knuth scheme for any length and it performs better than the fixed length construction presented in [4] for \( k < 512 \).

For practical systems purpose, a redundancy can only be a positive integer value. Fig 7 presents the rounded up fixed length schemes. This confirmed the previous assumption that the proposed fixed length scheme is more efficient than that of [4] for \( k < 512 \). This improvement on short length is a great advantage for most communication systems as they make use of short lengths data to communicate through a various channels.

7 Conclusion

We have presented a modification of the construction given in [6], for encoding and decoding of binary code-words suitable for communication systems. The proposed scheme requires exactly \( \log_2\left( \frac{k}{2} \right) \) bits for the fixed length prefix and a prefix length between 0 and \( \log_2\left( \frac{k}{2} \right) \) bits for VL scheme. The sparseness of the prefix length was analysed and the average efficiency of this scheme was discussed and compared to existing ones. The proposed construction is advantageous compared to some
prior schemes as look-up tables are not used and it is
less redundant.
Furthermore, this scheme can be featured with the
construction provided in [7,8,9] to achieve the overall
codeword balancing (encoded information with prefix).
As future work, the proposed scheme could also be ap-
plied on the overall codeword length to close the re-
main ing gap from the lower redundancy bound.

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