Charged stripes from alternating static magnetic field

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We motivate and perform a calculation of the energy of a cold fluid of charged fermions in the presence of a striped magnetic background. We find that a non-trivial value for the doping density on the walls is preferred.

The concept of “fictitious” or “statistical” gauge fields has proved quite fruitful in the study of the quantum Hall effect. Gauge fields have also figured prominently in theoretical attempts to understand other highly correlated electron systems in two dimensions, especially the doped cuprate planes famous for supporting high-temperature superconductivity. In the mean-field treatment of such theories, one is often involved in expanding around non-zero background values for the fictitious magnetic field. This background field spontaneously breaks P and T symmetry, and is therefore naturally associated with a Z2 order parameter. Unfortunately, experiments to detect macroscopic P and T violation in real materials have produced negative results.

On the other hand, recently a number of experiments have discovered evidence for inhomogeneous electronic structure in the doped cuprate planes. Existing evidence is consistent with the idea that this structure is roughly describable in terms of antiferromagnetic domains separated by narrow antiphase walls, with the dopants concentrated on the walls, at least for small doping. In view of this observed inhomogeneity, it is interesting and natural to consider the possibility of stripe structure in the context of fictitious gauge fields. Indeed, this concept has several advantages that are evident prior to any detailed calculation. The empirical difficulty of macroscopic P and T violation is ameliorated, and the Z2 nature of the order provides a raison d'être for stable domain walls.

On deeper consideration it appears that there can be, in the context of these ideas, a compelling connection between doping and stripe structure. Indeed, let us suppose that the preferred bulk phase, realized in the absence of doping, prefers values of the magnetic field ±B, with one sign chosen uniformly throughout the plane, at which the electrons just fill an integer number of Landau levels. As one dopes away from this preferred filling, two things might happen. It might be that one just has to live with an unfilled band. Alternatively, it might be preferable to produce regions with both ±B, separated by narrow walls. One could have then have the filled band in bulk, accommodating dopants on the domain walls.

To determine the viability of this concept, we have performed the following calculation.

Consider ideal fermions in 2 dimensions in uniform magnetic field B. Initially, exactly N first Landau levels (n = 0, 1, ..., N − 1) are filled. The energy density is \( \mathcal{E}_0 = \frac{1}{2} N \hbar \omega_c \rho \), where \( \rho = N eB / 2 \pi \hbar c \) is the fermion density and \( \omega_c = eB / mc \) is the cyclotron frequency.

Add or remove \( \Delta \rho \) fermions per unit area. If the magnetic field stays commensurate with density, keeping N filled Landau levels, \( \Delta \mathcal{E} = N \hbar \omega_c \Delta \rho \). Otherwise, if B remains fixed,

\[
\Delta \mathcal{E} = \begin{cases} 
(N + 1/2) \hbar \omega_c \Delta \rho & \text{if } \Delta \rho > 0, \\
(N - 1/2) \hbar \omega_c \Delta \rho & \text{if } \Delta \rho < 0.
\end{cases}
\]

In our alternative scenario, magnetic field is allowed to alternate between B and −B, thus producing magnetic domains (Fig. 1). Landau levels are distorted in the vicinity of domain walls. Midgap states induced by the domain wall accommodate newly doped fermions (or holes), so that doped charges form stripes along the domain walls. We wish to compare the energy of this state of a doped system with that in a fixed uniform magnetic field.

![Figure 1](image.png)

**FIG. 1.** Striped arrangement of magnetic field. Also shown are the semi-classical electron orbits. Notice that states on the wall have a preferred direction of momentum.

The linear density of charge on a stripe \( \nu \) and the stripe spacing \( \lambda \) are related to the average density of added charge according to \( \Delta \rho = \nu / \lambda \). For small \( \Delta \rho \), stripes are far apart compared to the width of the wave functions, and they can be considered independently of one another. In units of magnetic length \( (\hbar c/eB)^{1/2} \), this requires \( \lambda \gg (N - 1/2)^{1/2} \).
For a single domain wall of length $L$ with $N = \nu L$ added particles, the energy is $E(N) = E_0 + \epsilon(\nu)L$, where $E_0$ is the energy of the undoped system without the domain wall and $\epsilon(\nu)$ is the energy density of a domain wall (per unit length). The relative energy of the system per unit area is

$$\Delta \mathcal{E} = \frac{\epsilon(\nu)}{A} = \frac{\epsilon(\nu)}{\nu} \Delta \rho = \frac{E(N) - E_0}{N} \Delta \rho. \quad (2)$$

Thus the preferred linear charge density $\nu_0$ is found by minimizing $\epsilon(\nu)/|\nu|$, the energy per doped particle (or hole). Formation of stripes is advantageous if this energy is lower than that in a uniform field $B$:

$$\frac{\epsilon(\nu_0)}{|\nu_0|} < \begin{cases} (N + 1/2)\hbar \omega_c & \text{for electrons,} \\ -(N - 1/2)\hbar \omega_c & \text{for holes.} \end{cases} \quad (3)$$

Energy spectrum of a single domain wall: For a stripe along the $x$ axis, $B(y) = -B \text{sgn}(y)$. It is convenient to work in the Landau gauge $A_x = B|y|$, $A_y = 0$, so that translation symmetry in $x$ is manifest. Wavefunctions $\psi(y) \exp(ikx)$ then satisfy the Schrödinger equation

$$-\psi''(y) + (|y| - k)^2 \psi(y) = 2E \psi(y), \quad (4)$$
in appropriate units of length and energy.

As a function of momentum $k$ along the stripe, there are 3 simple regimes. If $k > 0$ and large one has two well-separated parabolic wells. Then there are doublets near the levels of the harmonic oscillator, with an exponentially small splitting between the symmetric and antisymmetric states. If $k < 0$ and large one has a single wedge-like well. Energy levels start (at least) at $|k|$ and thus do not contribute to low Landau levels. Finally if $k = 0$, one has the simple harmonic oscillator.

Thus as $k$ varies from $+\infty$ to $0$, the $n$-th Landau level splits into symmetric and antisymmetric branches. The energy of the symmetric branch $E_+(k)$ is lower than $n + 1/2$ for large $k$. At $k = 0$, $E_+ = 2n + 1/2$ and $E_- = 2n + 3/2$.

Solutions of the Schrödinger equation \(4\) regular at $y = \infty$ are parabolic cylinder functions \[3\]

$$\psi(y) = \begin{cases} D E^{-1/2} (|y - k| \sqrt{2}) & \text{if } y \geq 0, \\ \pm \psi(|y|) & \text{if } y < 0. \end{cases} \quad (5)$$

The spectra $E_\pm(k)$ are determined by a boundary condition at $y = 0$: $\psi'(0) = 0$ for symmetric states and $\psi(+0) = 0$ for antisymmetric ones. All the necessary calculations are now fast computations for MAPLE. The results are shown in Fig. \[3\].

![FIG. 2. Energies of the symmetric and antisymmetric eigenstates of a stripe, as functions of $k$. Straight lines are the Fermi levels for 0 and optimal dopant density on a stripe.](image)

Energy of a single stripe: Further analysis of the energy competition is greatly simplified by the existence of a one-to-one correspondence between states with and without a domain wall. Since $k$ is quantized in units $2\pi$ in both cases, there are no phase shifts to worry about. We will consider in detail the case of $N = 1$ filled Landau level.

In uniform magnetic field, all states of the lowest Landau level ($n = 0, -\infty < k < +\infty$) are occupied. To obtain the same number of particles in the system with a single domain wall in the most economical way, we must partially fill the two lowest branches ($n = 0$), as follows. First fill all the symmetric states with $k \geq 0$ and all antisymmetric states with $k > 0$; then slightly rearrange occupied and empty states within the two bands to achieve the state with lowest energy. The Fermi energy and momentum are determined by the equation $E_+(-k_F) = E_-(k_F) = E_F$. Note that the Fermi level lies below the bottom of the symmetric $n = 1$ band (Figure 2).

Without doping, a stripe has a positive linear density of energy. At large doping levels, however, $E(N) - E_0 \sim -N/2 + C$, as holes are doped into unperturbed Landau states away from the domain wall. The constant $C$ is negative. This becomes evident from Figure 2 if one considers $E_F$ just above 1/2, for then most of the remaining electrons are in the symmetric band, i.e., below 1/2. The energy per doped hole is

$$\frac{\epsilon(\nu)}{|\nu|} = \frac{E(N) - E_0}{|N|} \sim -\frac{1}{2} + \frac{c}{|\nu|}, \quad c = C/L < 0. \quad (6)$$

Because $\epsilon(\nu)/|\nu| > 0$ for $\nu \to -0$, this function has a minimum at a finite $\nu_0$. Moreover, condition \[3\] is satisfied. Thus for $N = 1$ the stripe phase has lower energy than the phase with a fixed uniform magnetic field, at least for sufficiently low doping levels, such that stripe wavefunctions do not overlap.
FIG. 3. Energy per added hole in the stipe phase, for \( N = 1 \) and 2 filled Landau levels. Both curves display shallow minima at values below the corresponding no-stripe values.

In Figure 3, we have displayed the energy per added hole \( \epsilon(\nu)/|\nu| \) for \( N = 1 \) and 2 filled Landau levels in bulk. For \( N = 1 \), we find \( |\nu_0| \approx 0.13 \) particles per magnetic length.

On a lattice with flux \( \phi \) piercing an elementary plaquette, the lattice constant is \( a = \phi^2 \). The preferred doping density in lattice units is therefore

\[
|\nu_0| \approx \frac{0.13 \text{ particles}}{\text{magn. length}} = \frac{0.13 \phi^{1/2} \text{ particles}}{\text{lattice constant}}. \tag{7}
\]

If \( \phi = \pi \), as suggested theoretically \cite{9, 11}, we find \( |\nu_0| \approx 0.23 \) particles per lattice constant.

Discussion: In our picture, the domain walls resemble quantum Hall edges. They support currents, alternating in direction from wall to wall, in the ground state.

Our calculation has been highly idealized, of course. For one thing, we have ignored any possibility of energy intrinsically associated with the fictitious field (other than the implicit constraint enforcing values \( \pm B \)). While this approximation is in the spirit of fictitious Chern-Simons fields, or of fields implementing constraints, which have no independent dynamics, in the absence of a detailed microscopic model we cannot assess its accuracy. Similarly, we have been shamelessly opportunistic in maneuvering back and forth between lattice and continuum. With all due reserve, it nevertheless seems appropriate to mention that the sort of model discussed here is remarkable and virtually unique, as far as we are aware, in predicting a non-trivial preferred dopant density along the stripes. Moreover, the preferred numerical value, which has emerged from a dynamical calculation containing no disposable continuous parameters, is consistent with the observed one.

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