ON HOMOTOPY LIE BIALGEBROIDS
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ABSTRACT. Lie algebroids appear throughout geometry and mathematical physics implementing the idea of a sheaf of Lie algebras acting infinitesimally on a smooth manifold. A well-known result of A. Vaintrob characterizes Lie algebroids and their morphisms in terms of homological vector fields on supermanifolds, which might be regarded as objects of derived geometry. This leads naturally to the notions of an $L_\infty$-algebroid and an $L_\infty$-morphism. The situation with Lie bialgebroids and their morphisms is more complicated, as they combine covariant and contravariant features. We approach Lie bialgebroids in the way of odd symplectic dg-manifolds, building on D. Roytenberg's thesis. We extend Lie bialgebroids to the homotopy Lie case and introduce the notions of an $L_\infty$-bialgebroid and an $L_\infty$-morphism. The case of $L_\infty$-bialgebroids over a point coincides with the O. Kravchenko’s notion of an $L_\infty$-bialgebra, for which the notion of an $L_\infty$-morphism seems to be new.

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1. INTRODUCTION

The notion of a Lie bialgebra was introduced in the seminal works of V. Drinfeld \cite{Dri83, Dri87} on algebraic aspects of the quantum inverse scattering method. A Lie bialgebra $\mathfrak{g}$ is a Lie algebra equipped with a one-cocycle $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ (a cobracket), whose dual $\delta^*$ yields a Lie bracket on $\mathfrak{g}$. As a quintessential example, Lie bialgebras appear as infinitesimal counterparts of Poisson-Lie groups. Geometrization of this notion leads to the concept of a Lie bialgebroid that natively arises in the Poisson-geometric context. In particular, there is a canonical Lie bialgebroid associated to any Poisson manifold. The aim of this note is to introduce an extension of this concept to the case of graded manifolds and homotopy Lie structures.

We survey the basic definitions, motivating examples and results concerning Lie algebroids, $L_\infty$-algebroids, and Lie bialgebroids in sections 2 and 3. In section 4 we review the Hamiltonian approach to Lie (bi)algebroids of D. Roytenberg and then give a Hamiltonian characterization of Lie (bi)algebroid morphisms (Theorem 4.9). In the final section, we introduce the notions of an $L_\infty$-bialgebroid and an $L_\infty$-morphism of $L_\infty$-bialgebroids and list some relevant examples.

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Conventions. The ground field is $\mathbb{R}$ by default. The dual $V^*$ of a graded vector bundle $V$ is understood as the direct sum of the duals of its graded components, graded in such a way that the natural pairing $V^* \otimes V \to \mathbb{R}$ is grading-preserving. In particular, $(V[n])^* = V^*[-n]$. By default, the degree on a bigraded vector bundle, such as $S(V)$, stands for the total degree. Differentials are assumed to have degree 1.

We use the exterior algebra $\bigwedge^\bullet V$ and the symmetric algebra $S(V[-1])$ interchangeably. The former is used mostly for ungraded vector spaces $V$, whereas the latter is reserved for graded ones.

We assume vector bundles to have finite rank and graded vector bundles to have locally finite rank. Likewise, all graded manifolds will be assumed to have finite-dimensional graded components. We will work with smooth graded manifolds, which we will understand as locally ringed spaces $(M, C^\infty_V) := (M, S(V^*))$, where $V \to M$ is a graded vector bundle over a manifold $M$ and $S(V^*)$ is the graded symmetric algebra over $C^\infty_M$ on the graded dual to the sheaf $V$ of sections.

A morphism $V \to W$ of graded manifolds is a morphism of locally ringed spaces $(M, S(V^*)) \to (N, S(W^*))$. We will assume that a differential, i.e., a degree-one, $\mathbb{R}$-linear derivation $d$ of the structure sheaf satisfying $d^2 = 0$ is given on a graded manifold. Morphisms of dg-manifolds will have to respect differentials. Since we work in the $C^\infty$ category, we will routinely substitute sheaves with spaces of their global sections.

A graded manifold $V$ over $M$ comes with a morphism $V \to M$ given by the inclusion of $C^\infty_M$ into the function sheaf of $V$. There is also a relative basepoint given by the zero section of the vector bundle $V \to M$, which induces a morphism $M \to V$ of graded manifolds over $M$. A based morphism must respect zero sections of the structure vector bundles. The structure differential will also be assumed to be based, that is to say, the zero section $M \to V$ must be a dg-morphism.

We will follow a common trend and confuse the notation $V$ for a vector bundle $V$ and the sheaf $V$ of its sections, when it is clear what we mean from the context.

For a manifold $M$ with a Poisson tensor $\pi$, we will use $\pi^\#$ to denote the natural morphism $T^*M \to TM$ determined by the condition

$$\langle \alpha \wedge \beta, \pi \rangle = \langle \beta, \pi^\#(\alpha) \rangle, \quad \alpha, \beta \in T^*M.$$ 

2. Lie algebroids

2.1. Basic definitions and examples.

Definition 2.1. A Lie algebroid structure on a vector bundle $V \to M$ over a smooth manifold $M$ consists of

- an $\mathbb{R}$-bilinear Lie bracket $[,] : \Gamma(V) \otimes \mathbb{R} \Gamma(V) \to \Gamma(V)$ on the space of sections;
- a morphism of vector bundles $\rho : V \to TM$, called the anchor map,

subject to the Leibniz rule

$$[X, fY] = f[X, Y] + (\rho(X)(f))Y, \quad X, Y \in \Gamma(V), \quad f \in C^\infty(M).$$

It follows, in particular, that the anchor map is a morphism of Lie algebras:

$$\rho([X, Y]) = [\rho(X), \rho(Y)], \quad X, Y \in \Gamma(V).$$

Examples 2.2.

(1) Any Lie algebra can be regarded as a Lie algebroid over a point.
(2) The tangent bundle $TM$ taken with the standard Lie bracket of vector fields and $\rho = id : TM \to TM$ is trivially a Lie algebroid.
Definition 2.3. Then a morphism $V \to W$ of Lie algebroids is a morphism:

$$
\begin{array}{ccc}
V & \xrightarrow{\phi} & W \\
\downarrow & & \downarrow \\
M & \longrightarrow & N
\end{array}
$$

of vector bundles subject to the following conditions:

(3) More generally, any integrable distribution $V \subset TM$ is a Lie algebroid with $\rho : V \to TM$ being the inclusion. Thus, a regular foliation on a manifold gives rise to a Lie algebroid.

(4) A family of Lie algebras over a manifold $M$, i.e., a vector bundle with a Lie bracket bilinear over functions on $M$, is a Lie algebroid with zero anchor.

(5) Let $\mathfrak{g}$ be a Lie algebra acting on a manifold $M$ via an infinitesimal action map $\mathfrak{g} \to \Gamma(TM)$, $X \mapsto \xi_X$. Then $\mathfrak{g} \times M \to M$ is a Lie algebroid with $(X,m) \mapsto \xi_X(m)$ as the anchor and the bracket defined pointwise by

$$
[X,Y](m) := [X(m), Y(m)]_{\mathfrak{g}} + \xi_X Y(m) - \xi_Y X(m),
$$

where a section $X$ of $\mathfrak{g} \times M \to M$ is identified with a function $X : M \to \mathfrak{g}$.

(6) Every Lie groupoid gives rise to a Lie algebroid, see Example 3.7(5). For example the tangent Lie algebroid of Example 2 above comes from the pair groupoid of a manifold $M$. The manifold of objects of the pair groupoid is $M$ itself, the manifold of morphisms is $M \times M$, with morphism composition given by

$$(x,y) \circ (y,z) := (x,z), \quad x, y, z \in M.$$

(7) If $M$ is a Poisson manifold with the Poisson bivector $\pi \in \Gamma(\wedge^2 TM)$, then the canonical morphism $\pi^\# : T^* M \to TM$ together with the Koszul bracket

$$\{\alpha, \beta\}_\pi = L_{\pi^\#(\alpha)}(\beta) - L_{\pi^\#(\beta)}(\alpha) - d(\iota_\pi(\alpha \wedge \beta))$$

determines a Lie algebroid structure on $T^* M$.

(8) Given a vector bundle $V \to M$, the space of derivative endomorphisms $Der(V)$ is defined as the space of all linear endomorphisms $D : \Gamma(V) \to \Gamma(V)$ such that there exists $D_M \in TM$, and

$$D(f X) = f D(X) + D_M(f) X$$

for any $X \in \Gamma(V)$, $f \in C^\infty(M)$. Then $Der(V)$ equipped with the standard commutator bracket and the mapping $\rho : D \mapsto D_M$ as the anchor is a Lie algebroid.

(9) For a principal $G$-bundle $P$ over a manifold $M$, the quotient $TP/G$ of $TP$ by the induced action of $G$ is known as the Atiyah Lie algebroid of $P$. The bracket and the anchor map are naturally inherited from $TP$. The notion of a morphism of Lie algebroids $V \to W$ defined over the same base manifold $M$ is rather straightforward: it is a vector bundle morphism $\phi : V \to W$ such that $\phi([X,Y]) = [\phi(X), \phi(Y)]$ and $\rho_W \circ \phi = \rho_V$. In general, the definition of such a morphism in terms of brackets and anchor maps is more involved due to the fact that a morphism of vector bundles defined over different bases does not induce a morphism of sections.

To introduce relevant notation, let $\phi : V \to W$ be a morphism of vector bundles $V \to M$, $W \to N$ over $\phi : M \to N$ and $\phi_* : V \to \phi^* W$ be a canonical morphism arising from the universal property of the pullback $\phi^*$. This induces a mapping of sections

$$\Gamma(V) \to \Gamma(\phi^* W) \simeq C^\infty(M) \otimes_{C^\infty(N)} \Gamma(W)$$

that we, by a slight abuse of notation, will keep denoting by $\phi_*$. The notion of a morphism of Lie algebroids $V \to W$ is then defined as a morphism

$$
\begin{array}{ccc}
V & \xrightarrow{\phi} & W \\
\downarrow & & \downarrow \\
M & \longrightarrow & N
\end{array}
$$

of vector bundles subject to the following conditions:
• $\rho_W \circ \phi = df \circ \rho_V$;
• for $X, Y \in \Gamma(V)$, write $\phi(X) = \sum f_i \otimes X_i'$, $\phi(Y) = \sum g_j \otimes Y_j'$; then
  $$\phi([X,Y]) = \sum_{i,j} f_i g_j \otimes [X_i', Y_j'] + (\rho_V(X)(g_j)) \otimes Y_j' - (\rho_V(Y)(f_i)) \otimes X_i'.$$

It is an exercise to check that the second condition is independent of the tensor-product expansions.

**Examples 2.4.**

1. For any smooth map $f : M \to N$, the tangent map $df : TM \to TN$ is a morphism of tangent Lie algebroids as defined in Example 2.2(3).
2. Given a Lie algebroid $V \to M$, its anchor map $TM \to V$ is a morphism of Lie algebroids.
3. In Example 2.2(8) above, Lie algebroid morphisms $TM \to \text{Der}(V)$ right-inverse to the anchor map $\rho : \text{Der}(V) \to TM$ correspond to flat connection on $V$.

**2.2. The dg-manifold approach.** Given a Lie algebroid $V \to M$, the coboundary operator $d : \Gamma(\wedge^k V^*) \to \Gamma(\wedge^{k+1} V^*)$ defined by

$$d\phi(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \rho(X_i)\phi(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})$$

(1) turns $\Gamma(\wedge^* V^*)$ into a differential graded algebra.

**Examples 2.5.**

1. For a Lie algebra, this yields the standard cohomological Chevalley-Eilenberg complex with trivial coefficients.
2. If $V = TM$ is a tangent Lie algebroid, then $\Omega^\bullet(M) := \Gamma(\wedge^\bullet T^*M)$ is the standard de Rham complex of a smooth manifold $M$.
3. For the Lie algebroid associated with a Poisson manifold $M$ as in Example 2.2(7), $\Gamma(\wedge^* TM)$ is the cohomological Poisson complex. The differential standard mapping $\pi# : T^*M \to TM$ induces a morphism between the de Rham cohomology $H^*_d(M)$ and the cohomology of $(\Gamma(\wedge^* TM), d_\pi)$, which turns out to be an isomorphism in the symplectic case.

**Remark.** In fact, $\Gamma(\wedge^\bullet V^*)$ admits a slightly richer structure. Namely, the contraction $i_X : \Gamma(\wedge^k V^*) \to \Gamma(\wedge^{k+1} V^*)$ and the Lie derivative $L_X$ defined by

$$(L_X\phi)(Y_1, \ldots, Y_k) = \rho(X)(\phi(Y_1, \ldots, Y_k)) - \sum_{i=1}^k \phi(Y_1, \ldots, [X, Y_i], \ldots, Y_k)$$

satisfy all the standard rules of Cartan calculus.

Passage from Lie algebroids to the “Koszul dual” picture encoded by the corresponding dg-algebras simplifies the matters concerning Lie algebroid morphisms. This is due to the following

**Theorem 2.6 (A. Vaintrob [Vai97]).** Let $V \to M$ be a vector bundle. Then the structures of

1. a Lie algebroid on $V$,
(2) a dg-manifold, \(d^V : \mathcal{C}^\infty(V[1]) \to \mathcal{C}^\infty(V[1])\), on the graded manifold \(V[1]\) are equivalent. Furthermore, there are natural bijections between the following sets:

1. The set of morphisms of Lie algebroids \(V \to M\) and \(W \to N\);
2. The set of dg-manifold morphisms \((V[1], d^V) \to (W[1], d^W)\).

In the context of graded manifolds the differential \(d^V\) is commonly referred to as a homological vector field on \(V[1]\). The complex \(S(V^*[1]), d^V\) is often called the cohomological Chevalley-Eilenberg complex of the Lie algebroid.

Idea of proof. A derivation \(d^V : \mathcal{C}^\infty(V[1]) \to \mathcal{C}^\infty(V[1])\) of degree one is determined by its restriction to the subalgebra \(\mathcal{C}^\infty(M)\):

\[
\mathcal{C}^\infty(M) \to \Gamma(V^*[1]),
\]

which must be a derivation of the algebra \(\mathcal{C}^\infty(M)\) with values in a \(\mathcal{C}^\infty(M)\)-module, and by the restriction to the module of generators:

\[
\Gamma(V^*[1]) \to \Gamma(S^2(V^*[1])).
\]

By the universal property of Kähler differentials, (2) is equivalent to a \(\mathcal{C}^\infty(M)\)-module morphism

\[
\Omega^1(M) \to \Gamma(V^*[1]),
\]

whose dualization gives an anchor. The \(\mathbb{R}\)-dual of (3) gives a bracket. The differential property \((d^V)^2 = 0\) then translates into the Jacobi identity for the bracket and the Lie algebra morphism property for the anchor. The Leibniz rule for \(d^V\) translates into the Leibniz rule for the bracket. \(\square\)

Example 2.7. The tangent algebroid \(TM \to M\) of a (graded) manifold \(M\) corresponds to the graded manifold \(T[1]M\), a shifted tangent bundle, whose dg-algebra of smooth functions is the de Rham algebra \((\Omega^\bullet(M), d_{dR})\).

2.3. The graded case and \(L_\infty\)-algebroids. Treating the notion of a Lie algebroid via dg-manifolds leads naturally to its graded version known as an \(L_\infty\)-algebroid. The concept of an \(L_\infty\)-algebroid was conceived in the works of H. Khudaverdian and Th. Th. Voronov [KV08], H. Sati, U. Schreiber and J. Stasheff [SSS09], A. J. Bruce [Bru11], and Th. Th. Voronov [Vor10]. At the same time, \(L_\infty\)-algebras disguised as formal dg-manifolds or \(\mathbb{R}^{0,1}\)-action have been known to V. Drinfeld, V. A. Hinich, M. Kontsevich, D. Quillen, V. Schechtman, D. Sullivan, and likely some others, since the last three decades of the 20th century; see [Kon94] and [Sta16] and references therein. \(L_\infty\)-algebroids have made their way to physics, as one of the most general models of quantum field theory, the AKSZ model [AKSZ97]; it is a sigma model with an odd symplectic dg-manifold as the target space.

One may also think of an \(L_\infty\)-algebroid as a sheaf of \(L_\infty\)-algebras acting infinitesimally on a smooth manifold, see a remark after Definition 2.8. This notion is essentially K. Costello’s notion of an \(L_\infty\) space [Cos11], see [GG15] and another remark after Definition 2.8 for details.

Formal graded and differential graded manifolds. From now on we will be focusing on formal graded manifolds \((M, \mathcal{C}^\infty_{\mathbb{R}}) : = (M, S(V)^*)\), where \(M\) is a graded manifold, \(V\) a graded vector bundle over \(M\) and the sheaf of functions \(S(V)^*\) is the graded dual of the symmetric coalgebra with respect to the shuffle coproduct. One can think of the algebra \(S(V)^*\) as an algebraic version of completion of the algebra \(S(V^*)\). Any graded manifold \(M\) may be regarded as a formal manifold over itself associated with the zero vector bundle over \(M\).
A morphism \( V \to W \) of formal graded manifolds is a morphism of locally ringed spaces \((M, S(V)^*) \to (N, S(W)^*)\) induced by a vector bundle morphism

\[
\begin{array}{ccc}
S(V) & \longrightarrow & W \\
\downarrow & & \downarrow \\
M & \longrightarrow & N
\end{array}
\]

This requirement may be understood as a continuity condition with respect to our algebraic completions. A formal dg-manifold is a formal graded manifold endowed with a differential, i.e., a degree-one, \( \mathbb{R} \)-linear derivation \( d \) of the structure sheaf satisfying \( d^2 = 0 \).

Morphisms of (formal) dg-manifolds will have to respect differentials. Since we work in the \( C^\infty \) category, we will routinely substitute sheaves with spaces of their global sections.

A graded manifold \( V \) over \( M \) of any of the above flavors comes with a morphism \( V \to M \) given by the inclusion of \( C_M^\infty \) into the function sheaf of \( V \). There is also a relative basepoint given by the zero section of the vector bundle \( V \to M \), which induces a morphism \( M \to V \) of (formal) graded manifolds over \( M \). A based morphism must respect zero sections of the structure vector bundles. The structure differential in the dg case will also be assumed to be based, that is to say, the zero section \( M \to V \) must be a dg-morphism.

Let \( V \to M \) be a graded vector bundle over a (possibly, graded) manifold \( M \). We will think of its total space as a pointed formal graded manifold \( V[1] \), fibered over \( M \). The term “pointed” is understood in a fiberwise (relative) sense and refers to the fact that the fiber bundle \( V[1] \to M \) has a canonical zero section, given by a standard augmentation \( \hat{\mathbb{C}}^\infty(V[1]) \to C^\infty(M) \). A dg structure on the pointed formal graded manifold \( V[1] \) over \( M \) is a choice of a square-zero, degree-one derivation \( d \) of the \( \mathbb{R} \)-algebra \( \hat{\mathbb{C}}^\infty(V[1]) \) such that the zero section \( \hat{\mathbb{C}}^\infty(V[1]) \to C^\infty(M) \) respects the differentials. Here the differential on \( C^\infty(M) \) is assumed to be zero.

**Definition 2.8.**

1. An \( L_\infty \)-algebroid is a graded vector bundle \( V \to M \) with the structure of a dg-manifold on the pointed formal graded manifold \( V[1] \) over \( M \). That is, it is the structure of a dg-algebra over \( C^\infty(M) \) on the graded commutative algebra \( \Gamma(M, S(V[1])^*) \) such that the differential is compatible with the augmentation \( \Gamma(M, S(V[1])^*) \to C^\infty(M) \).

2. An \( L_\infty \)-morphism of \( L_\infty \)-algebroids \( V \to M \) and \( W \to N \) is a formal pointed dg-manifold morphism \((V[1], d^V) \to (W[1], d^W)\). Equivalently, it is an augmented dg-algebra morphism \( \Gamma(N, S(W[1])^*) \to \Gamma(M, S(V[1])^*) \) over a graded algebra morphism \( C^\infty(N) \to C^\infty(M) \).

**Example 2.9.** When \( M \) is a point, an \( L_\infty \)-algebroid is nothing but an \( L_\infty \)-algebra, and the notion of an \( L_\infty \)-morphism of Lie algebroids over a point reproduces the standard notion of an \( L_\infty \)-morphism of \( L_\infty \)-algebras.

**Remark.** There is a generalized \( L_\infty \)-anchor map associated to an \( L_\infty \)-algebroid \( V \to M \). Indeed, the composition of its structure differential with the unit map \( C^\infty(M) \to \Gamma(M, S(V[1])^*) \) gives a degree-one derivation with values in \( \Gamma(M, S(V[1])^*) \). This gives rise to a \( C^\infty(M) \)-module morphism \( \Omega^1(M) \to \Gamma(M, S(V[1])^*) \) by the universal property of Kähler differentials. It extends uniquely to a dg-algebra morphism \( \Omega^\bullet(M) \to \Gamma(M, S(V[1])^*) \). This, in its turn, induces a morphism of formal pointed dg-manifolds \( V[1] \to T[1]M \), or an \( L_\infty \)-morphism of \( L_\infty \)-algebroids,
generalizing the anchor. One can also think of it as an \(L_\infty\)-action of the \(L_\infty\)-algebroid \(V\) on the base graded manifold \(M\).

2.4. **The Poisson manifold approach.** The data of a Lie algebroid on \(V \to M\) can also be cast in the form of a Poisson structure on the linear dual bundle \(V^* \to M\) generalizing the well-known Kostant-Kirillov (also known as the Lie-Poisson) bracket defined on the linear dual of a Lie algebra. More specifically, identifying smooth functions on \(V^*\) constant along the fibers with functions on \(M\) and identifying functions on \(V^*\) linear along the fibers with sections of \(V\), we set

\[
(f,g)_{V^*} = \begin{cases} 
[f,g], & f, g \in \Gamma(V) \\
\rho(f)g, & f \in \Gamma(V), g \in C^\infty(M) \\
0, & f, g \in C^\infty(M)
\end{cases}
\]

Extending this bracket further to the polynomial and smooth functions via the Leibniz rule and completion endows \(V^*\) with a well-defined Poisson structure. Note that the corresponding Poisson tensor will be linear along the fibers of \(V^* \to M\).

**Theorem 2.10** (T. J. Courant [Con90]). Let \(V \to M\) be a vector bundle. Then the following structures are equivalent:

1. A Lie algebroid structure on \(V \to M\);
2. A Poisson structure on the total space of the vector bundle \(V^* \to M\) such that the Poisson structure is linear along the fibers.

Two Poisson structures arising in this fashion on the linear duals of Lie algebroids \(V \to M, W \to N\) can be related by means of Lie algebroid comorphisms rather than morphisms. Namely, a Lie algebroid comorphism from \(V \to M\) to \(W \to N\) is a (base-preserving) morphism of vector bundles \(\phi: f^*W \to V\) over \(M\) such that

\[
\phi^\#([X,Y]) = [\phi^\#(X), \phi^\#(Y)]
\]

for all \(X,Y \in \Gamma(W)\) and \(df \circ \rho_V \circ \phi^\# = \rho_W\). Here, \(\phi^\#\) is the natural composition \(\Gamma(W) \xrightarrow{f^*} \Gamma(f^*W) \xrightarrow{\phi^\#} \Gamma(V)\). Such a comorphism yields a vector bundle morphism

\[
V^* \to (f^*W)^* \xrightarrow{\sim} f^*W^* \to W^*.
\]

**Theorem 2.11** (P. Higgins, K. Mackenzie [HM93]). Lie algebroid comorphisms from \(V \to M\) to \(W \to N\) are in one-to-one correspondence with vector bundle morphisms \(V^* \to W^*\) that are Poisson.

2.5. **Lie coalgebroids.** Keeping in mind our main objective of studying Lie bialgebroids and their homotopy generalization, we briefly describe the notion of a Lie coalgebroid.

**Definition 2.12.** A **Lie coalgebroid** structure on a vector bundle \(V \to M\) over a smooth manifold \(M\) consists of

- an \(\mathbb{R}\)-linear mapping \(\delta: \Gamma(V) \to \Gamma(V) \wedge \mathbb{R} \Gamma(V)\) (a Lie cobracket) satisfying the co-Jacobi identity

\[
\bigcirc (\delta \otimes \text{id})\delta = 0,
\]

where

\[
\bigcirc (x \otimes y \otimes z) = x \otimes y \otimes z + y \otimes z \otimes x + z \otimes x \otimes y;
\]

- a vector-bundle morphism \(\sigma: T^*M \to V\), called the **coanchor**, subject to the co-Leibniz rule

\[
\delta(fX) = f\delta(X) + \sigma(df) \wedge X, \quad X \in \Gamma(V), f \in C^\infty(M).
\]
This implies, in particular, that
\[ \delta(\sigma(\omega)) = (\sigma \wedge \sigma)(d\omega), \quad \omega \in \Gamma(T^*M). \]
That is, \( \sigma \) induces a morphism \( \Gamma(T^*M) \to \Gamma(V) \) of Lie coalgebras.

**Theorem 2.13.** Let \( V \to M \) be a vector bundle. Then the structures of
1. a Lie coalgebroid on \( V \),
2. a dg-manifold on the graded manifold \( V^*[1] \)
are equivalent.

The proof of this theorem is a rather straightforward exercise on the definitions; it is similar to the proof of Theorem 2.6.

**Examples 2.14.**

1. The cotangent bundle \( T^*M \to M \) is trivially a Lie coalgebroid with the cobracket being the restriction of the de Rham differential \( d_{dR} \) onto \( T^*M \) and the coanchor \( \sigma = id_{T^*M} \).
2. Any Lie algebroid structure on \( V \to M \) gives rise to a Lie coalgebroid structure on the linear dual bundle \( V^* \to M \) [BD04, Section 1.4.14].

In particular, taking \( V \) to be the standard tangent Lie algebroid \( TM \to M \) recovers the previous example. Conversely, a Lie coalgebroid structure on \( V \to M \) induces a Lie algebroid structure on \( V^* \to M \).

The last example combined with the Poisson bracket construction outlined in the previous section implies, in particular, that the structure of a Lie coalgebroid on \( V \to M \) induces a fiberwise linear Poisson structure on \( V \). More concretely, identifying, as before, fiberwise linear functions on \( V \) with the sections of \( V^* \) and fiberwise constant functions on \( V \) with the elements of \( C^\infty(M) \), we get
\[
\{\alpha, \beta\}_V = \begin{cases} 
(\alpha \otimes \beta)\delta, & \alpha, \beta \in \Gamma(V^*) \\
\alpha(\sigma(d\beta)), & \alpha \in \Gamma(V^*), \beta \in C^\infty(M) \\
0, & \alpha, \beta \in C^\infty(M)
\end{cases}
\]
This is to be extended further via the Leibniz rule and completion. As an upshot, it enables us to give a concise definition of a Lie coalgebroid morphism.

**Definition 2.15.** A morphism \( V \to W \) of Lie coalgebroids is a vector bundle morphism \( (V \to M) \to (W \to N) \) such that the map of total spaces \( V \to W \) is Poisson.

**Example 2.16.** The coanchor map \( \sigma : T^*M \to V \) of a Lie coalgebroid \( V \to M \) is a Lie coalgebroid morphism, where \( T^*M \) is given the standard cotangent Lie coalgebroid structure as in Example 2.14(1). In this case, the Poisson structure on \( T^*M \) corresponding to the Lie coalgebroid structure is the standard Poisson structure on the cotangent bundle.

2.6. **The odd Poisson manifold approach.** Yet another algebraic structure naturally associated with a Lie algebroid \( V \to M \) is defined on \( \Gamma(\wedge^*V) \). Namely, there is a canonical extension of the Lie bracket on \( \Gamma(V) \) to a bracket
\[
[,] : \Gamma(\wedge^kV) \otimes \Gamma(\wedge^lV) \to \Gamma(\wedge^{k+l-1}V), \quad k, l \geq 1
\]
of degree \( -1 \) satisfying the graded Jacobi and Leibniz rules. For \( X \in \Gamma(V) \), \( f \in C^\infty(M) \), we set
\[
[X, f] = \rho(X)(f).
\]

1We remind the reader that we assume all vector bundles to be of finite rank.
Altogether, this turns $\Gamma(\wedge V)$ into a Gerstenhaber (or an odd Poisson) algebra. The converse is also true:

**Theorem 2.17** (A. Vaintrob [Vai97]). Let $V \to M$ be a vector bundle. The following structures are equivalent:

1. A Lie algebroid structure on $V \to M$;
2. A Gerstenhaber algebra structure on $\Gamma(\wedge V)$ (taken with the standard multiplication);
3. A graded Poisson structure of degree $-1$ on $V^*[1]$.

This theorem can be regarded as an odd analogue of Courant’s Theorem 2.10. Likewise, the following statement is an odd analogue of Higgins-Mackenzie’s Theorem 2.11.

**Proposition 2.18.** There are natural bijections between the following sets:

1. The set of Lie algebroid comorphisms from $V \to M$ to $W \to N$;
2. The set of Gerstenhaber algebra morphisms $\Gamma(\wedge W) \to \Gamma(\wedge V)$;
3. The set of graded Poisson manifold morphisms $V^*[1] \to W^*[1]$.

**Examples 2.19.**

1. For a Lie algebra $V$, $\Gamma(\wedge V)$ is the underlying space of the homological Chevalley-Eilenberg complex with trivial coefficients. The odd Poisson bracket on $\Gamma(\wedge V)$ is a derived bracket generated by the homological Chevalley-Eilenberg differential.
2. For a tangent Lie algebroid $TM$, $\Gamma(\wedge TM)$ is the Schouten-Nijenhuis algebra of multivector fields.
3. For a Lie algebroid associated with a Poisson manifold $M$ as in Example 2.2(7), $\Gamma(\wedge T^* M) = \Omega^*(M)$ is the underlying space of the homological Poisson complex. The differential in this case (known as the Brylinski differential) is $d = [i_\pi, d_{dR}]$.

**Remark.** The Poisson manifold $V^*$, the dg-manifold $V[1]$ and the odd Poisson manifold $V^*[1]$ determined by a Lie algebroid $V \to M$ are known as $P$-, $Q$- and $S$-manifolds, respectively, associated to $V$ [Vor92]. In that regard, Lie bialgebroids (see Section 2.5) manifest themselves in the form of $QP$- or $QS$-manifolds, comprising a pair of such structures in a compatible way.

We also have analogous statements for Lie coalgebroids.

**Theorem 2.20.** Let $V \to M$ be a vector bundle. Then the structures of

1. a Lie coalgebroid on $V$,
2. a graded Poisson structure of degree $-1$ on $V[1]

are equivalent. Furthermore, there are natural bijections between the following sets:

1. The set of morphisms of Lie coalgebroids $V \to M$ and $W \to N$;
2. The set of graded Poisson morphisms $V[1] \to W[1]$.

### 2.7. Connections and associated BV algebras

This section follows the paper [Xu99] by Ping Xu. For a Lie algebroid $V \to M$, endowing the associated Gerstenhaber algebra $\Gamma(\wedge V)$ with some extra data in the form of a differential operator of order one or two, subject to certain compatibility conditions, determines an additional structure on $V \to M$. The former case will be addressed in Section 3 to handle the latter case, we need the following

**Definition 2.21.** Let $V \to M$ be a Lie algebroid and $E \to M$ be a vector bundle. A linear mapping

$$\nabla : \Gamma(V) \otimes \Gamma(E) \to \Gamma(E), \quad X \otimes s \mapsto \nabla_X(s)$$

**Theorem 2.22.** Let $V \to M$ be a Lie algebroid and $E \to M$ be a vector bundle.
is called a $V$-connection if

1. $\nabla f_X(s) = f\nabla_X(s)$;
2. $\nabla_X(fs) = (\rho(X)f)s + f\nabla_X(s)$

for all $f \in C^\infty(M)$, $X \in \Gamma(V)$, $s \in \Gamma(E)$.

The curvature of a $V$-connection $\nabla$ on $E \to M$ is an element $R \in \Gamma(\wedge^2 V^*) \otimes \text{End}(E)$ defined by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \quad X,Y \in \Gamma(V),$$

and the torsion is

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad X,Y \in \Gamma(V).$$

A $V$-connection is said to be flat if $R \equiv 0$.

If a Lie algebroid $V \to M$ is of rank $n$ as a vector bundle, then any $V$-connection of the canonical line bundle $E = \wedge^n V$ determines an operator $\Delta : \Gamma(\wedge^n V) \to \Gamma(\wedge^{n-1} V)$ on the Gerstenhaber algebra $\Gamma(\wedge V)$:

$$\Delta \omega(X_1, \ldots, X_{p+1}) := \sum_i (-1)^{i-1} \nabla_X \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}),$$

where $\omega \in \Gamma(\wedge^{n-p} V)$ is identified with a section of $\text{Hom}(\wedge^p V, \wedge^n V)$.

**Lemma 2.22.** $\Delta$ is a differential operator of order two.

**Theorem 2.23** (P. Xu [Xu99]). There is a one-to-one correspondence between $V$-connections on $E = \wedge^n V$ and linear operators $\Delta$ generating the bracket on $\Gamma(\wedge^* V)$. Under this correspondence, a flat $V$-connection induces a square-zero differential operator $\Delta$ of order two, thus turning $\Gamma(\wedge^* V)$ into a Batalin-Vilkovisky algebra.

Note that a flat $V$-connection on $E = \wedge^n V$ always exists.

**Examples 2.24.**

1. Given a Lie algebra $V$ of dimension $n$, the line bundle $\wedge^n V$ has a (flat) $V$-connection corresponding to the adjoint action of $V$ on $\wedge^n V$. The corresponding operator $\Delta$ is the homological Chevalley-Eilenberg operator.
2. The Brylinski differential $d$ on the homological Poisson complex $\Gamma(\wedge^* T^* M) = \Omega^*(M)$ of an $n$-dimensional Poisson manifold $M$ generates the corresponding bracket. The flat connection on $\Omega^*_M \to M$ associated to $d$ by means of Theorem 2.23 is given by $\nabla_\omega = \theta \wedge d\omega$, $\theta \in \Gamma(T^* M)$, $\omega \in \Omega^n(M)$, [ELW99].

3. **Lie bialgebroids**

Let $V$ be a vector bundle over $M$ such that $V \to M$ is both a Lie algebra and a Lie coalgebroid. Denote by $d$ the coboundary operator on $\Gamma(\wedge^* V) = \Gamma(S(V[-1]))$ induced by the Lie coalgebroid structure, as in Theorem 2.13

**Definition 3.1.** A vector bundle $V \to M$ with the structures of a Lie algebroid and a Lie coalgebroid is a Lie bialgebroid if $d$ is a derivation of bracket:

$$d([X,Y]) = [dX,Y] + [X,dY], \quad X,Y \in \Gamma(V).$$

Note that in the equation above, $dX,dY \in \Gamma(\wedge^2 V)$ and the Lie bracket is extended from $V$ to $\wedge^* V$ as a coderivation of degree $-1$, see Section 2.6.
Remark. In view of the equivalence between Lie coalgebroid structures on $V \rightarrow M$ and Lie algebroid structures on $V^* \rightarrow M$, see Example 2.14(2), one may think of a Lie bialgebroid as a pair $(V, V^*)$ of Lie algebroids satisfying the compatibility condition (4). This is a more common point of view. However, we prefer to think of a Lie bialgebroid as two compatible structures on $V$ in this section for functoriality reasons. We will return to the former viewpoint later, when we discuss the Hamiltonian approach.

Theorem 3.2. If $V$ is a Lie bialgebroid, then so is $V^*$.

Theorem 3.3. Let $V \rightarrow M$ be a vector bundle. The structures of

1. a Lie bialgebroid on $V$,
2. a dg-Gerstenhaber algebra on $\Gamma(\wedge V^*)$,
3. a dg-Poisson manifold on $V[1]$ with Poisson bracket of degree $-1$

are equivalent.

Remark. The multiplication on $\Gamma(\wedge V^*)$ and the differential are to be related via the Leibniz rule. Sometimes, dg Gerstenhaber algebras with this property are called strong, [Xu99].

Definition 3.4. A morphism $V \rightarrow W$ of Lie bialgebroids is a vector bundle morphism $(V \rightarrow M) \rightarrow (W \rightarrow N)$ which is a morphism of Lie algebroids and a morphism of Lie coalgebroids.

Taking into account Definition 2.15, the above definition may immediately be reworded as follows.

Proposition 3.5. Let $V$ and $W$ be Lie bialgebroids. A Lie algebroid morphism $V \rightarrow W$ is a morphism of Lie bialgebroids iff it is a Poisson map with respect to the Poisson structures on $V$ and $W$ induced by the Lie coalgebroid structures on $V$ and $W$, respectively, as in Section 2.5.

In the vein of Theorem 3.3, one obtains a characterization of Lie bialgebroid morphisms:

Theorem 3.6. There are natural bijections between the following sets:

1. The set of Lie bialgebroid morphisms from $V$ to $W$;
2. The set of dg-Gerstenhaber algebra morphisms $\Gamma(\wedge V^*) \rightarrow \Gamma(\wedge W^*)$;
3. The set of dg-Poisson manifold morphisms $V[1] \rightarrow W[1]$.

Examples 3.7.

1. If $M$ is a point, a Lie bialgebroid over $M$ is a Lie bialgebra in the sense of Drinfeld.
2. Let $M$ be a Poisson manifold with a Poisson bivector $\pi$, $T^*M \rightarrow M$ be the cotangent Lie algebroid associated to $\pi$ as in Example 2.14(1) with the canonical Lie coalgebroid structure as in Example 2.14(1). The Lichnerowicz differential $d_\pi = [\pi, -]$ is a derivation of the Schouten-Nijenhuis bracket on $\Gamma(\wedge T^*M)$, thus giving the cotangent bundle $T^*M$ a Lie bialgebroid structure.

Conversely, let $V$ be a Lie bialgebroid over $M$. Then $\pi_V := \rho \circ \sigma : T^*M \rightarrow T^*M$, where $\rho, \sigma$ are the anchor and the coanchor maps of $V \rightarrow M$, respectively, defines a Poisson structure on $M$.
3. Suppose $V \rightarrow M$ is a Lie algebroid with an anchor $\rho$ and a structure differential $d^V$ on $\Gamma(\wedge^2 V^*)$. Let $r \in \Gamma(\wedge^2 V)$ be such that $[r, r] = 0$. Denote
by $r^\#$ the associated bundle map $V^* \to V$. One can show that $\rho^* := \rho \circ r^\# : V^* \to TM$ and
\[
[\phi, \psi] = L_{r^\pi(x)}(\psi) - L_{r^\pi(\phi)}(\phi) - dV((\phi \wedge \psi)r), \quad \phi, \psi \in \Gamma(V^*),
\]
where
\[
L_X := [dV, \iota_X], \quad X \in \Gamma(V),
\]
determine a Lie algebroid structure on $V^* \to M$. Furthermore, a simple check confirms that the pair $(V, V^*)$ is actually a Lie bialgebroid. In particular, taking $V = TM$ recovers the previous example.

(4) A Nijenhuis structure on a smooth manifold $M$ is a vector bundle endomorphism $N : TM \to TM$ such that its Nijenhuis torsion
\[
[N(X), N(Y)] - N([N(X), Y] + [X, N(Y)]) + N^2([X, Y])
\]
vanishes for any $X, Y \in \Gamma(TM)$. A prototypical example of a Nijenhuis structure arises in the form of a recursion operator of an integrable bi-Hamiltonian system [Mag78, Olv90]. Namely, given a pair $\pi_0, \pi_1$ of Poisson tensors on a manifold $M$ such that any linear combination $\lambda \pi_0 + \mu \pi_1$ is Poisson as well and $\pi_0$ is symplectic, then $N := \pi_1^\# \circ (\pi_0^\#)^{-1}$ is a Nijenhuis structure on $M$.

A Nijenhuis structure on $M$ induces [KS96] a Lie algebroid structure on $TM$ with the bracket
\[
[X, Y]_N := [N(X), Y] + [X, N(Y)] - N([X, Y])
\]
and $N : TM \to TM$ being an anchor map. Now, if $M$ is a Poisson manifold, then $T^*M$ can be given the Lie algebroid structure of Example 2.2(7). This produces a Lie coalgebroid structure on $TM$ by Example 2.14(2). It turns out [KS96] that these two structures on $TM$ are compatible, thereby making $TM$ a Lie bialgebroid, provided
\[
N \circ \pi^\# = \pi^\# \circ N^*,
\]
and
\[
\{\alpha, \beta\}_N = \{N^*(\alpha), \beta\}_\pi + \{\alpha, N^*(\beta)\}_\pi - N^*(\{\alpha, \beta\}_\pi)
\]
for all $\alpha, \beta \in \Gamma(T^*M)$.

(5) Recall that a Lie groupoid is a (small) groupoid $s, t : \mathcal{G} \rightrightarrows M$ such that its set of objects $M$ and the set morphisms $\mathcal{G}$ are smooth manifolds, and the source and the target maps $s, t$ along with the composition $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, the unit $M \to \mathcal{G}$ and the inverse map $\mathcal{G} \to \mathcal{G}$ are smooth. The source and target maps are also assumed to be submersions.

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, we define the associated Lie algebroid $V \to M$ as follows. As a vector bundle, $V = ker(ds)|_M$, where the restriction is taken along the unit map $M \to \mathcal{G}, x \mapsto 1_x$. A Lie bracket on $\Gamma(V)$ is obtained by identifying the sections of $V$ with the right-invariant vector fields on $\mathcal{G}$ and the anchor map is $dt : T\mathcal{G} \to TM$ restricted onto $V \subset T\mathcal{G}$.

A Poisson groupoid is a Lie groupoid $\mathcal{G} \rightrightarrows M$ with a Poisson structure $\pi$ such that the graph of the composition $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a coisotropic submanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$, where $\mathcal{G}$ denotes $\mathcal{G}$ with the opposite Poisson tensor $-\pi$. One can show [MX94] that a Poisson structure on $\mathcal{G}$ induces a Lie algebroid structure on $V^*$. Furthermore, it is compatible with the Lie algebroid structure on $V \to M$, giving rise to a Lie bialgebroid over $M$. This generalizes the well-known construction of Lie bialgebras arising as infinitesimal counterparts of Poisson-Lie groups.
4. The Hamiltonian approach

Let $Q$ be a (graded) vector field on a graded manifold $V$. The cotangent (or Hamiltonian) lift

$$\Gamma(V, TV) \to C^\infty(T^*V),$$

$$Q \mapsto \mu_Q,$$

is defined by setting

$$\mu_Q(x, p) = p(Q_x), \quad p \in T^*_x V.$$  

In local Darboux coordinates $(x^i, x^*_i, \xi^a)$ on $T^*V$, if $Q = \sum_i Q^i(x) \partial / \partial x^i$, then $\mu_Q = \sum_i Q^i(x) x^*_i$, a function linear along the fibers of $\pi : T^*V \to V$.

Remark. This construction has been rediscovered in the case where $M$ is a point by C. Braun and A. Lazarev [BL15] under the name of doubling, see also [Kra07].

**Proposition 4.1.** Let $Q_1, Q_2, Q$ be vector fields on $V$. Then

1. $\{\mu_{Q_1}, \mu_{Q_2}\} = \mu_{[Q_1, Q_2]}$
2. $\pi_*([\mu_Q, -]) = Q$.

Note that if $[Q, Q] = 0$, the proposition implies $\{\mu_{Q_1}, \mu_{Q_2}\} = 0$.

The case we are interested in corresponds to the graded manifold $V$ associated with a vector bundle $V \to M$. By Theorem 2.6, a vector field $Q$ of degree $+1$ and such that $[Q, Q] = 0$ determines a Lie algebroid structure on $V \to M$. This leads to the following chain of correspondences:

$\begin{align*}
\text{Lie algebroid structures} & \quad \text{Homological vector fields} \\
on V \to M & \quad \text{of degree $+1$} \\
on V[1] & \quad \text{on $V[1]$} \\
\text{“Integrable odd Hamiltonians”} & \quad \{\mu_Q, \mu_Q\} = 0
\end{align*}$

**Theorem 4.2** (D. Roytenberg [Roy99]). Lie algebroid structures on $V \to M$ are in one-to-one correspondence with functions $\mu$ on $T^*V[1]$ which are linear along the fibers of $T^*V[1] \to V[1]$, of total degree one in the natural $\mathbb{Z}$-grading on $C^\infty(T^*V[1])$ and such that $\{\mu, \mu\} = 0$.

In terms of local Darboux coordinates $(x^i, \xi^a, x^*_i, \xi^*_a)$, where

- $\{x^i\}$ are coordinates on $U \subset M$,
- $\{e_a\}$ is a basis of sections of $V$ over $U$,
- $\{\xi^a\}$ are the corresponding generators of $\Gamma(U, \Lambda^*V^*)$,
- $\{x^*_i\}$ are coordinates along $T^*U \to U$,
- $\{\xi^*_a\}$ are coordinates along $T^*V[1]|_U \to V[1]|_U$,

the above correspondence takes the following form:

$$\begin{align*}
\rho(e_a) = A^i_a(x) \frac{\partial}{\partial x^i} \\
\rho(e_a, e_b) = C^c_{ab}(x)e_c
\end{align*}$$

$$\begin{align*}
d & = \xi^a A^i_a(x) \frac{\partial}{\partial x^i} + \frac{1}{2} C^c_{ab}(x) \xi^a \xi^b \frac{\partial}{\partial x^c} \\
\mu & = \xi^a A^i_a(x) x^*_i + \frac{1}{2} C^c_{ab}(x) \xi^a \xi^b \xi^*_c
\end{align*}$$

**Examples 4.3.**

1. The Hamiltonian of a tangent Lie algebroid $TM \to M$ is $\mu = \xi^i x^*_i$.
2. A Hamiltonian $\mu = \xi^a A^i_a(x) + \frac{1}{2} C^c_{ab}(x) \xi^a \xi^b \xi^*_c$ with coordinate-independent structure coefficients $C^c_{ab}$ corresponds to an action Lie algebroid, cf. Example 2.2.4. Namely, in that case, $C^c_{ab}$ are the structure constants of the Lie algebra $g$ acting on $M$ and $A^i_a(x)$ are the coefficients of the anchor map $\rho(e_a) = A^i_a \frac{\partial}{\partial x^i}$. 
(3) For a Poisson manifold \( M \) with a Poisson bivector \( \pi = \pi^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \), the Hamiltonian of the corresponding Lie algebroid \( T^*M \to M \) is
\[
\mu = \sum_{i,j} \xi^i \pi^{ij} x_j^* + \sum_{i,j,k} \frac{1}{2} \frac{\partial}{\partial x_k} (\pi^{ij}) \xi^i \xi^j \xi^k.
\]

We would like to give a Hamiltonian characterization of Lie algebroid morphisms. To this end, consider Lie algebroids \( V \to M, W \to N \). We let \( F : f^*(T^*W[1]) \to T^*W[1] \) be the pullback of \( f : V[1] \to W[1] \) along the projection \( T^*W[1] \to W[1] \), and \( \Phi : f^*(T^*W[1]) \to T^*V[1] \) be the morphism of graded vector bundles over \( V[1] \) given as the dual of the bundle map \( TV[1] \to f^*(TW[1]) \) induced by the differential \( df : TV[1] \to TW[1] \).

**Proposition 4.4.** Let \( \mu \in C^\infty(T^*V[1]), \nu \in C^\infty(T^*W[1]) \) be the Hamiltonians corresponding to the Lie algebroid structures on \( V \to M, W \to N \). Then Lie algebroid morphisms \( V \to W \) are in one-to-one correspondence with graded manifold morphisms \( f : V[1] \to W[1] \) such that
\[
F^*(\nu) = \Phi^*(\mu).
\]

**Proof.** We have to show that, given Hamiltonians \( \mu, \nu \), the condition above is equivalent to the homological vector fields \( Q^V = dp(\{\mu, -\}) \) and \( Q^W = dp(\{\nu, -\}) \) being \( f \)-related. That is,
\[
df(Q^V) = Q^W_{f(v)}
\]
for any \( v \in V[1] \).

Indeed, for any point \((v, \zeta)\) in \( f^*(T^*W[1]) \), we have
\[
(F^*(\nu))(v, \zeta) = (\nu \circ F)(v, \zeta) = \nu(f(v), \zeta) = \zeta(Q^W_{f(v)}).
\]

On the other hand,
\[
(\Phi^*(\mu))(v, \zeta) = (\mu \circ \Phi)(v, \zeta) = \mu(v, f^*(\zeta)) = (f^*(\zeta))(Q^V_v) = \zeta(df(Q^V_v)).
\]

Thus, the equation \( F^*(\nu) = \Phi^*(\mu) \) is equivalent to \( \zeta(Q^W_{f(v)}) = \zeta(df(Q^V_v)) \) for all \( v \) and \( \zeta \).

Given a pair \((V, V^*)\) of Lie algebroids over \( M \), we get the corresponding pair of Hamiltonians
\[
\mu \in C^\infty(T^*V[1]) \quad \text{and} \quad \mu_* \in C^\infty(T^*V^*[1])
\]
degree one in the \( Z \)-grading. We can bring them together in \( C^\infty(T^*V[1]) \) by means of a canonical symplectomorphism
\[
L : T^*[2]V[1] \to T^*[2]V^*[1],
\]
known as the Legendre transform. In local coordinates, it reads
\[
(x, \xi, x^*, \xi^*) \mapsto (x, \xi^*, x^*, \xi)
\]
in our \( Z \)-graded setting, see [Roy99 Section 3.4] for the \( \mathbb{Z}/2\mathbb{Z} \)-graded case. Note that \( V^*[1] = (V[1])^*[2] \) and the shift along the cotangent directions is needed to make sure that \( L \) respects grading. After the shift, the Poisson bracket \( \{,\} \) acquires degree \(-2\), and both \( \mu \) and \( \mu_* \) become elements of degree 3 in \( C^\infty(T^*[2]V[1]) \) and \( C^\infty(T^*[2]V^*[1]) \), respectively.

**Lemma 4.5.** Let \( \mu_* \in C^\infty(T^*[2]V^*[1]) \) be a Hamiltonian corresponding to a Lie algebroid structure on \( V^* \). Then
\[
L^* \mu_*(v, \zeta) = \pi_{V^*, v}(\zeta, \zeta), \quad (v, \zeta) \in T^*[2]V[1]
\]
where \( \pi_{V^*, v} \) is the graded Poisson tensor induced by the Lie algebroid structure on \( V^* \) and \( \pi_{V^*, v} \) is its value at \( v \in V[1] \).
Proof. In local coordinates we may write \((v, \zeta) = (x^i, \xi^a, x_i^+, \xi^c)\), and \(L(v, \zeta) = (x^i, \xi^a, x_i^+, \xi^c)\). Then

\[
(L^*\mu_*)(v, \zeta) = \mu_*(L(v, \zeta)) = \xi^a A^i_a(x)x_i^+ + \frac{1}{2}C^{ab}_c(x)\xi^b_\alpha \xi^c_\beta,
\]

where \(A^i_a(x)\) and \(C^{ab}_c(x)\) are the structure functions of \(\mu_*\).

On the other hand,

\[
\pi_{V^*,\nu}(\zeta, \xi) = \sum_{a,b} \pi_{V^*,\nu}(d\xi^a, d\xi^b) + \sum_{i,j} \pi_{V^*,\nu}(dx^i, dx^j) + \sum_{a,i} \pi_{V^*,\nu}(dx^a, dx^i),
\]

where the first and the last summands contribute \(\frac{1}{2}C^{ab}_c(x)\xi^b_\alpha \xi^c_\beta\) and \(\xi^a A^i_a(x)x_i^+\), respectively, and the second one is identically zero. \(\square\)

**Theorem 4.6** (D. Roytenberg [Roy99]). A pair \((V, V^*)\) of Lie algebroids is a Lie bialgebroid if and only if

\[
\{\mu + L^*\mu_*, \mu + L^*\mu_*\} = 0.
\]

**Corollary 4.7.** A structure of a Lie bialgebroid on a vector bundle \(V \to M\) is equivalent to a Hamiltonian \(\chi\) on \(T^*\to V[1]\), which is linear-quadratic along the fibers of \(T^*\to V[1]\) and is of degree three in the natural \(\mathbb{Z}\)-grading on functions on \(T^*\to V[1]\), and such that \(\{\chi, \chi\} = 0\).

**Example 4.8.** For a Lie bialgebroid \((T^*M, TM)\) associated to a Poisson manifold \(M\) as in Example 3.7(2), the Hamiltonian \(\chi\) on \(T^*\to T^*M[1]\) is given by

\[
\chi = \sum_i \xi^a_i x_i^+ + \sum_{i,j} \xi_i^j \pi^i_\nu x_i^+ + \sum_{i,j,k} \frac{1}{2} \frac{\partial}{\partial x^k} (\pi^i_\nu) \xi^i_j \xi^c_k.
\]

**Theorem 4.9.** Lie bialgebroid morphisms \(V \to W\) are in one-to-one correspondence with formal graded manifold morphisms \(f : V[1] \to W[1]\) such that

\[
F^*(\psi) = \Phi^*(\chi),
\]

where and \(\chi, \psi\) are the Hamiltonians on \(T^*\to V[1]\) and \(T^*\to W[1]\) corresponding, respectively, to the given Lie bialgebroid structures and \(F, \Phi\) are as in Proposition 4.4.

**Proof.** As in the proof of Proposition 4.4 at any point \((v, \zeta)\) in \(f^*(T^*\to W[1])\), the value \((F^*(\psi))(v, \zeta) = \psi(f(v), \zeta)\) is given by

1. evaluating the linear in \(\zeta \in T^*_{f(v)}[2]W[1]\) part \(\psi = \nu + L^*\epsilon\) given by the vector field \(Q^W_{f(v)}\) on \(W[1]\) as a linear functional on the cotangent bundle \(T^*\to W[1]\), see Equation 5,

2. evaluating the quadratic part \(L^*\epsilon\), which by Lemma 4.5 is given by the Poisson tensor on \(W[1]\) corresponding to the Lie algebroid structure on \(W^*\), and

3. adding the results together.

Likewise, the value \((\Phi^*(\chi))(v, \zeta) = \chi(v, f^*(\zeta))\) at \((v, \zeta)\) is the sum of the values at \(f^*(\zeta)\) of the linear part \(\mu = \mu + L^*\mu_*\) given by the vector field \(Q^W_V\) and the quadratic part \(L^*\mu_*\) given by the Poisson tensor on \(V[1]\). Since equality of polynomial functions is equivalent to equality of their homogeneous parts, the agreement of the functions \(F^*(\psi)\) and \(\Phi^*(\chi)\) implies that \(f\) is a morphism of Lie algebroids respecting the graded Poisson structures. \(\square\)
5. $L_\infty$-bialgebroids

Corollary 4.7 and Theorem 4.9 motivate the following $L_\infty$ generalizations of the notions of a Lie bialgebroid and a Lie-bialgebroid morphism.

**Definition 5.1.** An $L_\infty$-bialgebroid over a (graded) manifold $M$ is a graded vector bundle $V \to M$ along with a degree-three function $\chi$ on the pointed formal graded manifold $T^*[2]V[1]$ such that

- $\{\chi, \chi\} = 0$, i.e., $\chi$ is an integrable Hamiltonian;
- $\chi$ vanishes on the zero section $V[1] \subset T^*[2]V[1]$ of the vector bundle $T^*[2]V[1] \to V[1]$ as well as on the restriction $T^*[2]V[1]_M$ of this bundle to the zero section $M \subset V[1]$ of the vector bundle $V[1] \to M$.

Removing the second condition leads to an $L_\infty$ generalization of the notion of a quasi-Lie bialgebroid, [Roy99], also known as a curved Lie bialgebroid, [GG15].

**Remark.** A seemingly natural attempt to define the notion of an $L_\infty$-bialgebroid in a way similar to Definition 3.1 as a pair of $L_\infty$-algebroids and $L_\infty$-coalgebroid structures on $V$ subject to some compatibility conditions, would be too restrictive, as such a structure would fail to comprise higher $L_\infty$ operations with multiple inputs and multiple outputs, cf. Examples 5.2 and 5.5. However, defining the notion of an $L_\infty$-bialgebroid via Manin $L_\infty$-triples, as an $L_\infty$-algebroid structure on $V \oplus V^*$ under some finite-rank conditions, should be possible, see [Kra07], where this is done for $L_\infty$-bialgebras, i.e., when $M$ is a point.

**Example 5.2.** This example generalizes triangular Lie bialgebras in the sense of Drinfeld [Dri87].

A generalized (or higher) Poisson structure on a graded manifold $M$ is a (total) degree-two multivector field $P \in \Gamma(S(T[-1]M))$ such that $[P, P] = 0$, where the bracket is the standard Schouten bracket. As shown by H. Khudaverdian and Th. Th. Voronov [KV08], such a structure induces $L_\infty$ brackets on the algebra of smooth functions $C^\infty(M)$ and on the de Rham complex $\Gamma(S(T^*[−1]M))$ of $M$. These higher brackets are known as the higher Poisson and higher Koszul brackets, respectively. The former generalizes the standard Poisson bracket construction, while the latter generalizes Example 2.2[7], see also [Bru11].

Pursuing these ideas in the direction of Example 3.7[9], we start with a graded manifold $M$ and a graded Lie algebroid $V \to M$, determined by a Hamiltonian $\mu$. Let $r \in \Gamma(S(V[-1])) \subset C^\infty(V^*[1])$ be a degree-two element such that $[r, r] = 0$, where the bracket is the degree-(-1) Poisson bracket on $V^*[1]$ induced by the Lie algebroid structure on $V$, as described in Section 2.6. Then the following sequence of maps takes place:

$$\alpha : C^\infty(V^*[1]) \to \Gamma(V^*[1], TV^*[1]) \to C^\infty(T^*V^*[1]).$$

Here, the first mapping associates the Hamiltonian vector field $[f, −]$ to a function $f$, using the odd Poisson bracket on $V^*[1]$, and the second one is the cotangent lift. Each of these morphisms respects the brackets, thus letting $r$ pass to a degree-one element $\alpha(r)$ such that $\{\alpha(r), \alpha(r)\} = 0$ on $T^*V^*[1]$. After the degree shift to $T^*[2]V^*[1]$, the element $\alpha(r)$ acquires degree 3. Altogether, as a Hamiltonian, the sum $\mu + L^*(\alpha(r))$ determines an $L_\infty$-bialgebroid structure on $V$.

Compatibility of the coalgebroid component $L^*(\alpha(r))$ with $\mu$ becomes more apparent upon recognizing that $L^*(\alpha(r)) = \{\mu, L^*(r)\}$, where $\{,\}$ is the canonical bracket on $T^*[2]V[1]$; see [KSR10]. The $L_\infty$-algebroid part on $V$ is a just a graded Lie algebroid, and there are no higher mixed operations. A construction giving nontrivial higher brackets, higher cobrackets and higher mixed operations along these lines in the case of $M$ being a point can be found in [BV15].
A semistrict $L_\infty$-morphism $V \to W$ of $L_\infty$-bialgebroids $V \to M$ and $W \to N$ is a morphism $f : V[1] \to W[1]$ of pointed formal graded manifolds relating the Hamiltonians on the shifted cotangent bundles $T^*[2]V[1]$ and $T^*[2]W[1]$ in the sense of Proposition 4.4. The “pointed” condition means that $f$ maps the zero section of $V[1] \to M$ to the zero section of $W[1] \to N$.

Even though this definition looks like a direct generalization of Theorem 4.9, which gives a Hamiltonian characterization of morphisms of Lie bialgebroids, we have chosen to use the word semistrict, because the morphism $V \to W$ of $L_\infty$-coalgebroids under this definition is strict, while the morphism $V \to W$ of $L_\infty$-algebroids may have “higher,” i.e., $L_\infty$, components. See also Example 5.5 below.

The definition of a full-fledged $L_\infty$-morphism beautifully overlaps with the idea of deformation quantization, whence we need to introduce a formal quantization parameter $h$, to which it would be convenient to assign a degree 2 in our graded context. It is common in deformation quantization to consider functions on the cotangent bundle polynomial in the momenta, see e.g. [BNW98], but the presence of a formal parameter will allow us to consider formal series in the momenta.

Let $V$ be a graded vector bundle over a graded manifold $M$ and $\chi \in \hat{C}^\infty(T^*[2]V[1])$ an integrable Hamiltonian defining an $\chi$-algebroid on functions. Using the natural pairing between both the tangent and the cotangent bundles of $V$, the corresponding tangent bundle will be reflexive. Using the natural paring between the tangent and the cotangent bundles of $V$, we can pair $(d_{vert}\chi)|_{V[1]}$ and $dg$, producing a function on $V[1]$, denoted

$$\chi(g) := ((d_{vert}\chi)|_{V[1]}, dg).$$

Note that this pairing will shift the degree by $-2$, given that $(d_{vert}\chi)|_{V[1]}$ is a section of $T[-2]V[1]$.

Extend this pairing to pairings of degree $-2k$ between iterated differentials $(d_{vert}^k\chi)|_{V[1]}$ and $d^k g$, viewed as sections of the $k$th symmetric powers $S^k(T[-2]V[1])$ and $S^kT^*V[1]$, respectively, for all $k \geq 1$:

$$\chi(g)_k := ((d_{vert}^k\chi)|_{V[1]}, d^k g).$$

Note that because of the vanishing condition $\chi|_{V[1]} = 0$, the $k = 0$ term $\chi(g)_0$ will automatically be zero. Finally, define a differential operator

$$C^\infty(V[1]) \xrightarrow{\Delta} C^\infty(V[1])[[M]],$$

$$g \mapsto \chi(g) := \sum_{k=1}^{\infty} h^{k-1} \chi(g)_k.$$

Since the degree of $\chi$ as a function on $T^*[2]V[1]$ is three and $|h| = 2$, the degree of $\chi$ as an operator on functions on $V[1]$ is one, i.e.,

$$|\chi(g)| = |g| + 1.$$

Note that the vanishing condition $\chi|_{T^*[2]V[1]} = 0$ on the Hamiltonian implies that $\chi(g)|_M = 0$, whereas the integrability, $\{\chi, \chi\} = 0$, yields $\chi(\chi(g)) = 0$ or $\chi \circ \chi = 0$.

This action has a more familiar form in coordinates. Let $q'$s denote coordinates on $V[1]$ and $p_i$'s denote the conjugate momenta, resulting in Darboux coordinates $(p_i, q')$ on $T^*[2]V[1]$. Under stronger assumptions of finite dimensionality, such as
the finiteness of the total rank of $V$ and the total dimension of $M$, so as the index $i$ takes finitely many values, we would have

$$\chi(g) = \sum_{k=1}^{\infty} \frac{h^{k-1}}{k!} \sum_{i_1, \ldots, i_k} \pm \frac{\partial^k \chi}{\partial p_{i_1} \cdots \partial p_{i_k}}|_{V[1]} \frac{\partial^k g}{\partial q^{i_1} \cdots \partial q^{i_k}},$$

where $\pm$ is a suitable Koszul sign.

Compare this with the star product of standard type:

$$\chi \ast g := \sum_{k=0}^{\infty} \frac{h^k}{k!} \sum_{i_1, \ldots, i_k} \pm \frac{\partial^k \chi}{\partial p_{i_1} \cdots \partial p_{i_k}} \frac{\partial^k g}{\partial q^{i_1} \cdots \partial q^{i_k}},$$

which in our case, when $g$ is a function on the base $V[1]$ of the cotangent bundle, coincides with the star product of Moyal-Weyl type, see [BNW98]. Since $h$ has degree 2, the above star product is homogeneous.

We will be considering smooth maps

$$f : V[1] \to S(W[1]),$$

with which we would like to associate certain linear maps

$$C^\infty(W[1]) \to C^\infty(V[1]).$$

A map $f : V[1] \to S(W[1])$ induces a morphism

$$f^* : C^\infty(S(W[1])) \to C^\infty(V[1])$$

of algebras of smooth functions. On the other hand, starting from a smooth function $g \in C^\infty(W[1])$, we can build a linear (and thereby, smooth) function on $S(W[1])$ by taking a formal Taylor series

$$T(g) := \sum_{k=0}^{\infty} \frac{1}{k!} d^k_{\text{vert}} g|_M, \quad d^k_{\text{vert}} g|_M \in \Gamma(M, S^k(W[1])^*).$$

This produces a linear map

$$T : C^\infty(W[1]) \to \Gamma(N, S(W[1])^*) \subset C^\infty(S(W[1])).$$

**Definition 5.4.** An $L_\infty$-morphism $V \to W$ of $L_\infty$-bialgebroids $V \to M$ and $W \to N$ is a morphism

$$f : V[1] \to S(W[1])$$

of pointed formal graded manifolds relating the Hamiltonians $\chi_V$ and $\chi_W$ on the shifted cotangent bundles $T^*[2]V[1]$ and $T^*[2]W[1]$, respectively, as follows:

$$\chi_V \circ f^* \circ T = f^* \circ T \circ \chi_W,$$

where the structure Hamiltonians $\chi_V$ and $\chi_W$ are regarded as operators on functions via the action $\partial$ and the condition above is understood as an equation on linear operators

$$C^\infty(W[1]) \to C^\infty(V[1])[\hbar].$$

**Example 5.5.** If $M$ is a point, $T^*[2]V[1]$ can be identified, as a graded manifold, with $V^*[1] \oplus V[1]$ equipped with a Poisson bracket of degree $-2$ known as the big bracket, [KS04, Kra07]. The algebra of smooth functions on $V^*[1] \oplus V[1]$ may be thought of as a completion of $S(V[-1] \oplus V^*[-1])$, and it encodes various Lie and co-Lie operations on $V$. Schematically, the big bracket of two homogeneous tensors $f$ and $g$, interpreted as linear maps between graded symmetric powers of $V[\pm 1]$, may be depicted as follows:
Here, the summation is done over all possible ways to form an input-output pair for \( f \) and \( g \); the relevant signs are suppressed.

Now, an integrable Hamiltonian on \( T^* V[1] \) is a degree-three function \( \chi \) on \( V^* \oplus V[1] \) satisfying \( \{ \chi, \chi \} = 0 \). The condition that \( \chi \) vanishes on \( V[1] \) and \( V^* \) implies that \( \chi \) belongs to a functional completion of \( S^\geq 0(V[-1]) \otimes S^\geq 0(V^*) \), thereby resulting in a notion of an \( L_\infty \)-bialgebra structure on \( V \), equivalent, up to completion, to Kravchenko’s notion \([Kra07]\) of an \( L_\infty \)-bialgebra. We will adopt the algebraic version \([BV15]\) and assume \( \chi \in \bigoplus_{m,n \geq 1} \text{Hom}(S^m(V[1]), S^n(V[-1])) \) of degree three and satisfying the Maurer-Cartan equation \( \{ \chi, \chi \} = 0 \).

The notion of an \( L_\infty \)-morphism for two \( L_\infty \)-bialgebroids, see Definition \([5.4]\), leads to the following version of the notion of an \( L_\infty \)-morphism of two \( L_\infty \)-bialgebras \( V \) and \( W \), regarded as \( L_\infty \)-algebroids over a point. An \( L_\infty \)-morphism \( V \to W \) of \( L_\infty \)-bialgebras is a linear map \( \Phi : S(V[1]) \to S(W[1]) \) commuting with the solutions \( \chi_V \in \bigoplus_{m,n \geq 1} \text{Hom}(S^m(V[1]), S^n(V[-1])) \) and \( \chi_W \in \bigoplus_{m,n \geq 1} \text{Hom}(S^m(W[1]), S^n(W[-1])) \) of the Maurer-Cartan equation \( \{ \chi, \chi \} = 0 \) defining the \( L_\infty \)-bialgebra structures:

\[
\Phi \circ \chi_V = \chi_W \circ \Phi.
\]

Here, as in the definition of an \( L_\infty \)-morphism of \( L_\infty \)-algebroids, \( \chi_V \) and \( \chi_W \) are understood as operators. For example, \( \chi_V \in \bigoplus_{m,n \geq 1} h^{-1} \text{Hom}(S^m(V[1]), S^n(V[-1])) \cong \bigoplus_{m,n \geq 1} \text{Hom}(S^m(V[1]), S^n(V[-1]))[-2] \).

C. Bai, Y. Sheng, and C. Zhu \([BSZ13]\) have considered an interesting truncated version of \( L_\infty \)-bialgebras, namely 2-Lie bialgebras, as well as morphisms of them. Since their notion of a 2-Lie bialgebra is not a straightforward truncation of the notion of an \( L_\infty \)-bialgebra, it is not clear how our theory compares to theirs. This could be an exciting topic of further study, especially taking into account that Bai, Sheng, and Zhu present several examples of 2-Lie bialgebras.

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