Quantum-spin-Hall topological insulator in a spring-mass system

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Abstract

It is proposed that a lattice, with constituent masses and spring constants, may be considered as a model system for topological matter. For instance, a relative variation of the inter- and intra-unit cell spring constants can be used to create, tune, and invert band structure. Such an aspect is obtained while preserving time reversal symmetry, and consequently emulates the quantum spin Hall effect. The modal displacement fields of the mass-spring lattice were superposed so to yield pseudospin fields, with positive or negative group velocity. Considering that harmonic oscillators are the basis of classical and quantum excitations over a range of physical systems, the spring-mass system yields further insight into the constituents and possible utility of topological material.

1. Introduction

Inspired by the discovery of topological phases and edge states in electronic materials [1, 2], the possibility of building related devices for the control of the propagation of light [3–9] and sound [10–18] is being extensively studied. The related device building blocks may harness three major types of topological phases analogous to those in condensed matter systems: quantum Hall effect (QHE) [19, 20], quantum spin Hall effect (QSHE) [21–23], and quantum valley Hall effect (QVHE) [24–27]. The QHE has chiral edge modes, and requires an external magnetic field to break time reversal symmetry (TRS), which may be accomplished in acoustic and photonic systems by adding gyroscopic material or external circulators [3, 10–12, 28]. The QSHE is amenable to TRS, associated with a pair of spin-locked helical modes, and is obtained by introducing strong spin–orbit coupling [5, 8, 13, 17, 18]. The QVHE generates valley-locked chiral edge states, and exploits the valley degrees of freedom [6, 29].

It would of much advantage and yield insight, to consider a harmonic oscillator point of view, quite common in physics, for invoking topological phases. In this respect, a discrete spring-mass based mechanical system, may constitute a model system for topological structure as related to phononic materials. For instance, QHE based topological insulators in spring-mass lattices may be created by adding circulating gyroscopes [11, 28], Coriolis force [30] or varying spring tension [31]. QVHE has been realized in such systems by alternating the mass at A and B sites of the unit cell of a mechanical graphene-like lattice [29]: figure 1(a). QSHE-like phenomena has also been explored in spring-mass lattices, through coupled pendula [32], and a mechanical granular graphene system [33]. However, many of these systems are difficult to implement in practical applications.

In this paper, we propose a two-dimensional spring-mass system, exemplifying a QSHE topological insulator, in the acoustic domain. Various trivial and non-trivial band structures may be originated by varying the masses (m) and the relative spring constants (k) in the associated lattice. In addition to exhibiting the topological features that have now become familiar to practitioners in the field, we indicate a novel spin degree of freedom. The related pseudospins are observed, in frequency domain analysis as the polarization of modal displacement field of masses in one unit cell: figure 1(a). TRS protected edge modes, incorporating the propagation of such pseudospins, are shown to exist. This structure may be representative of different phases of...
We consider a hexagonal lattice with equal masses \( m \), \( a_1 \), and \( a_2 \) are lattice constants of the unit cell before zone folding, and \( b_1 \) and \( b_2 \) are lattice constants of the unit cell after zone folding. The spring-mass model and computational methods applied as one of the possible practical designs of photonic/matter, as the the spring constant can be view as coupling strength between unit cells in various systems. It can be applied as one of the possible practical designs of photonic/phononic topological insulators.

A basis for creating a topological material, based on a spring-mass system, to mimic the QSH effect, is to create intrinsic TRS. We consider a hexagonal lattice of masses and springs arranged in \( C_6 \) symmetry. The \( E \) and \( E' \) representations are each two-fold degenerate with the individuals being complex conjugates \(^{34}\). Consequently, a four-fold degeneracy is required to satisfy TRS and may be enabled through manifesting a double Dirac cone in the band structure. We achieve a four-fold degeneracy, in the band structure of a spring-mass constituted lattice by the zone-folding method \(^{[8]}\).

2. The spring-mass model and computational methods

We consider a hexagonal lattice with equal masses \( m \) connected by linear springs \( k \), as shown in figure 1(a). The unit cell of this hexagonal lattice consists of \( 2 \) masses \( m^1 = m^2 = m \), with lattice constants \( \bar{a}_1^1 \) and \( \bar{a}_2^1 \) \((|\bar{a}_1^1| = |\bar{a}_2^1| = a)\). From Newton’s law, the governing equation \( M \ddot{u} = F(u) \), where \( M \) is a diagonal matrix with the values of the two masses on its diagonal: \( M = \text{diag}(m^1, m^1, m^2, m^2) \). \( u \) is a vector constituted from the two degrees of freedom for each mass—the \( x \) and \( y \) direction displacements for \( m^1 \) and \( m^2 \); \( u = \{u^1_x, u^1_y, u^2_x, u^2_y\} \) and \( F \) is the force. We consider a Bloch wave solution of the type \( u = U e^{i(q_1 \bar{a}_1^1 + q_2 \bar{a}_2^2 - \omega t)} \) to the governing equation of the \((q_1, q_2)\)th unit cell, where \( U = \{U^1_x, U^1_y, U^2_x, U^2_y\} \) is the modal displacement, and \( \gamma_1 \) and \( \gamma_2 \) are wave vectors. A dispersion relation is obtained by solving the eigenvalue problem \( D(\gamma_1, \gamma_2) U = \omega^2 M U \), with \( D \) as a dynamical matrix (see appendix A).

The band structure of the hexagonal lattice in figure 1(c) exhibits a single Dirac cone at the \( K(\bar{K}) \) point. The frequencies are non-dimensionalized as \( \Omega = \frac{\omega}{\sqrt{m}} \). Subsequently, we fold the first Brillouin zone (BZ) of the hexagonal lattice, twice, to form a new BZ with \( 1/3 \) of its original area, as shown in figure 1(b). Consequently, the \( K(\bar{K}) \) point is mapped to the \( \Gamma \) point at the center of the BZ, creating a double Dirac cone. The smaller BZ corresponds to an expanded unit cell in real space of \( 3 \) times of the original unit cell area, with \( 3 \times 2 = 6 \) masses, and lattice constant \( \bar{b}_1^1 \) and \( \bar{b}_2^2 \) \((|\bar{b}_1^1| = |\bar{b}_2^2| = \sqrt{3} a = b)\), as indicated in figure 1(a). The band structure based on the expanded unit cell is plotted in figure 1(d), and indicates a double Dirac cone at \( \Gamma \).

To induce a phase transition, in the topological sense, we break the spatial symmetry of the hexagonal lattice, through changing the spring constants of the connecting masses in the lattice, i.e. distinguishing the intra unit cell spring constant \( k_1 \) from the inter unit-cell spring constant \( k_2 \). Such distinction still preserves the \( C_6 \) symmetry of the original lattice, therefore one would expect a similar double Dirac cone at \( \Gamma \). However, in figure 1(d) one can see that the Dirac cone is not symmetric about \( \Gamma \) and that \( k_2 \) is the dominant coupling, yielding a single Dirac cone, a topological phase transition has occurred. This is consistent with the predictions of the theory, \(^{[8]}\), thus we confirm that our theoretical predictions hold in the computational model.
3. Results and discussions

3.1. Modal displacement fields in hexagonal spring-mass lattices: the case for pseudospins

The modal displacement and its $x$ and $y$ components, of the masses in the unit cell, at the $\Gamma$ point of the $k_1 > k_2$ lattice are shown in figures 3(a)–(d). The labeling of the modes in figures 3(a)–(d) follows the nomenclature for the lower to higher band degeneracy corresponding to figure 2(b). The modal displacements for a given mass in $p_1/\langle d_1 \rangle$ are orthogonal to $p_2/\langle d_2 \rangle$, respectively. The constituent $x$ and $y$ direction displacements are plotted successively below. Since each mass has two degrees of freedom—the displacements in the $x$- and $y$- directions, in considering the parities of modal displacements in figure 3, we consider the $x$- and the $y$- direction modal displacement fields separately. We find that the $xy$ direction displacement fields at $\Gamma$ are of odd and even spatial parities—of the $p_1/\langle p_1 \rangle$ and $d_{x^2-y^2}/\langle d_{x^2-y^2} \rangle$ variety, as inferred both from the sense of the symmetry of the displacements and stated relationships in the $C_6$ character table [34]. For instance, the $p_1/\langle p_1 \rangle$ character is antisymmetric with respect to the center, even symmetric to the $x-$ ($/y-$) axis, and odd symmetric to the $y-$ ($/x-$) axis, while the $d_{x^2-y^2}/\langle d_{x^2-y^2} \rangle$ parity is symmetric with respect to the center, and even (odd) symmetric to both the $x$ and $y$ axes.

Hybridizing the $p_1/\langle d_1 \rangle$ and $p_2/\langle d_2 \rangle$ modes in a symmetric and antisymmetric manner yields pseudospins [8]

$$p_\pm = (p_1 \pm ip_2)/\sqrt{2}, \quad \text{and} \quad d_\pm = (d_1 \pm id_2)/\sqrt{2}. \quad (1)$$

Figures 3(e)–(h) illustrates the related phase distribution of $p_+, p_-, d_+$, and $d_-$ in the range of $-\pi$ to $\pi$ (see appendix B). Clearly seen from the phase relationship that harmonic wave propagation in $p_1/\langle d_1 \rangle$ and $p_2/\langle d_2 \rangle$ have opposite polarizations. Taking the time harmonic component $e^{i\omega t}$ into consideration, due to the orthogonality of displacements in $p_1/\langle d_1 \rangle$ and $p_2/\langle d_2 \rangle$, each mass corresponding to the hybridized mode $p_1/\langle d_+ \rangle$ rotates in the one direction, while each mass in $p_2/\langle d_- \rangle$ rotates in the opposite direction. The incorporation of the relative motions of the six masses in the unit cell leads to rotation of the whole displacement field. Such rotation may be
considered as one manifestation of a pseudo-spin. One can follow the motion in $d_i$ during one time period $T$ to figure 4, indicating such clockwise orientability of the displacement field.

We find that for the case of $k_1 < k_2$, the modal displacement fields have exactly the same odd and even spatial parities, but $d_1$ and $d_2$ are now associated with the higher two degenerate bands, while $p_1$ and $p_2$ corresponds to the lower two bands (figure 2(c)). This demonstrates that band inversion happens at the $\Gamma$ point during the process of closing and reopening the band gap, and a change in topology of the band structure. Such a change has

Figure 3. (a) $p_1$, (b) $p_2$, and (c) $d_1$, (d) $d_2$ are total modal displacements for the two two-fold degeneracies at $\Gamma$ point when $k_1 = k_2$. $p_1$ and $p_2$ have odd parities, while $d_1$ and $d_2$ have even parities. $x$ and $y$ direction components to (a) and (b) clearly show $p_x/p_y$ symmetry, while those to (c) and (d) that have $d_{x^2-y^2}/d_{xy}$ symmetry. (e)-(h) are plots of phase relationships between the 6 masses in one unit cell for $p_x$, $p_y$, $d_x$, and $d_y$ in color map, indicating the polarization of wave propagation associated with pseudospin up and pseudospin down.

Figure 4. The spinning of modal displacement field for $d_i = (d_i + id_j)/\sqrt{2}$ as a result of time domain motion of the masses during one period $T$.
been previously quantified through the spin Chern number \([35]\). The Hamiltonian on the basis states of \([p_\uparrow, \ d_+, \ p_\downarrow, \ d_-]\) can be obtained (see appendix C) to be of the following form:

\[
H^{\text{eff}}(\gamma) = \begin{bmatrix}
M - B\gamma^2 & A\gamma_+ & 0 & 0 \\
A\gamma_- & -M + B\gamma^2 & 0 & 0 \\
0 & 0 & M - B\gamma^2 & A\gamma_- \\
0 & 0 & A\gamma_+ & -M + B\gamma^2
\end{bmatrix},
\]

where \(\gamma_\pm = \gamma_k \pm i\gamma_p\), and \(\gamma^2 = \gamma_k^2 + \gamma_p^2\). \(A = i\alpha k_0\) is imaginary \((\alpha > 0)\), and \(B < 0\). \(M = \frac{\varepsilon_d - \varepsilon_p}{2}\) indicates the relative energy of \(d\) and \(p\) bands, which is positive in the lattice of \(k_1 > k_2\), and negative in the lattice of \(k_1 < k_2\), respectively. The spin Chern number can be calculated from

\[
C_5 = \pm \frac{1}{2} (\text{sgn}(M) + \text{sgn}(B)).
\]

Since \(B\) is negative, \(C_5\) depends on the sign of \(M\), which leads to \(C_5 = 0\) when \(M > 0\), and \(C_5 = \pm 1\) when \(M < 0\). This means that for the lattice with \(k_1 > k_2\), \(C_5 = 0\), and the band gap is topologically trivial (figure 2(b)). When we decrease \(k_1\) to \(k_1 < k_2\), the band gap becomes topologically non-trivial (figure 2(c)) and \(C_5 = \pm 1\). Therefore, from the topological band theory \([1]\) it would be expected that there would exist pseudospin-dependent edge modes at the boundary between topologically trivial and topologically non-trivial lattices.

### 3.2. Propagating edge modes

The pseudospin-dependent edge modes are vividly illustrated through simulations on a ribbon-shaped lattice that is periodic in one direction and of the width of one unit cell in the other direction: figure 5(a). Such a supercell based lattice contains both topologically trivial (T) and non-trivial (NT) units. The NT lattice is constituted from one row of 20 unit cells, and cladded by two T units of 15 unit cells (we chose the number of T and NT units so that the band diagram is relatively scale invariant). Here, the masses in the T and NT units lattice are in the ratio \(\frac{m_T}{m_{NT}} = 1.315\), and spring constants are of the ratio \(k_1^T : k_2^T : k_1^{NT} : k_2^{NT} = 1.2: 1: 0.8\). The inter-cell spring constant \(k_0\) is kept the same in both the T and NT units since it connects the two different lattices. The spring constants and masses were chosen such that the T and the NT units have overlapped band gap as related to the frequency ranges indicated in figures 2(b) and (c). The band structure of the ribbon supper cell is shown in figure 5(b) (the frequencies here are non-dimensionalized as \(\Omega = \frac{\omega}{\sqrt{\frac{m_c}{k_c}}}\)). Compared to the band structures in figure 2(b) and (c), we clearly see two additional states appear within the bulk band gap connecting the lower bands to the higher bands, as illustrated by red and green lines in figure 5(b). It was noted that these two new
modes propagate with a group velocity of the same magnitude but opposite signs, and correspond to the pseudospin up and pseudospin down topological edge modes. There is a mini band gap at the point of the ribbon lattice near the boundary of the T and the NT units (see appendix D). However, this mini band gap is much smaller compared to the bulk band gap (0.003:0.08), so the pseudospins are preserved, and backscattering of edge states is suppressed as shown in the time-domain simulations below. We plotted the modal displacement corresponding to the two additional states (frequency within the bulk bands) and ω = ωb = 0.8 \sqrt{\frac{k_b}{m_{NT}}} (frequency within the bulk band gap). (d) is a spring-mass lattice that contains a topological edge with a sharp turning, and (e) simulation result on (d) with a force excitation of frequency ωb.

Figure 6. Time domain simulation of edge wave propagation. The domain is of 24 by 24 unit cells. The masses and springs are of the ratio \omegaNT = 1.14/1, and \kappa NT = 1.2: 1: 0.8, respectively. (a) Sinusoidal excitation force \( F = F_0 e^{i\omega t} \) applied on a line edge between topologically non-trivial and trivial spring-mass lattices. (b) and (c) are the simulation results with ω = ωb = 0.8 \sqrt{\frac{k_b}{m_{NT}}} (frequency within the bulk bands), and ω = ωf = 1.14 \sqrt{\frac{k_f}{m_{NT}}} (frequency within the bulk band gap). (d) is a spring-mass lattice that contains a topological edge with a sharp turning, and (e) simulation result on (d) with a force excitation of frequency ωb.

As the indicated pseudospins are symmetrized con-
was noted that the excited modes are sensitive to boundary conditions, that leads to high amplitude at the boundary.

4. Conclusions

In summary, we have shown that a mass-spring based lattice system may have attributes related to that of a topological insulator, in the presence of TRS. Through varying the inter- and inter-unit cell spring constants of such a lattice, for a given mass, a clear and distinct variation of the band structure was seen. A concomitant change in the modal displacement fields, corresponding to a band inversion, may be generated. The deconvolution of the fields as well as their hybridization in a symmetric and antisymmetric manner yields a basis for the creation of pseudo-spins, corresponding to clockwise/counter-clockwise rotation of the modal displacement vector. Both pseudo spin-up and pseudo spin-down modalities, corresponding to the positive or negative group velocity are proposed. The existence of polarized edge states as well as corresponding modes was demonstrated through both frequency domain analysis and time domain simulations. These edge modes are topologically protected, as they are immune to backscattering when encountering sharp edges. Considering that harmonic oscillators (which are direct manifestations of spring-mass units) form the basis for many physical systems, ranging from acoustics to electromagnetics, this work yields a general foundational framework and related methodology, i.e. modulating band structure and constituent modes through varying the respective spring constants of the physical system.

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Appendix A. Dynamical matrix for spring-mass lattice

To get the dispersion relations, we evaluate

\[ Du = \omega^2 u, \]

where \( D \) is the dynamical matrix, and \( u \) is the displacements of the masses.
For a two-mass unit cell shown in figure 8, $D$ is derived to be of form
\[
D(\gamma_1, \gamma_2) = \begin{bmatrix}
-\frac{3}{2} k & 0 & \frac{3}{4} (1 + e^{i\gamma_1}) k & \frac{\sqrt{3}}{4} (1 - e^{i\gamma_1}) k \\
0 & -\frac{3}{2} k & \frac{\sqrt{3}}{4} (1 - e^{i\gamma_2}) k & \frac{3}{2} k \\
\frac{3}{4} (1 + e^{-i\gamma_1}) k & \frac{\sqrt{3}}{4} (1 - e^{-i\gamma_1}) k & -\frac{3}{2} k & 0 \\
\frac{\sqrt{3}}{4} (1 - e^{-i\gamma_2}) k & \frac{1}{4} (1 + e^{-i\gamma_1} + e^{-i\gamma_2}) k & 0 & -\frac{3}{2} k
\end{bmatrix}.
\] (5)

The elements of $D$ were obtained through assuming a Bloch wave solution of form $u_{\bar{q}l} = U e^{i(\bar{q}k_0 + \gamma_1 \bar{l}_1 + \gamma_2 \bar{l}_2 - \omega t)}$. Here $U = \{ U_{y1}, U_{x1}, U_{y2}, U_{x2} \}$ is the modal displacement vector, and $\gamma_1$ and $\gamma_2$ are Bloch wave vectors. Take the mass $m_{q,1}$ in unit cell $(q, l)$ for example. The force balance for $m_{q,1}$ in $x$ direction can be written as
\[
m_{q,1} u_{q,1,x}^{(1)} = k \left[ (u_{q+1,x}^{(1)} - u_{q,x}^{(1)}) \cos \frac{\pi}{6} - (u_{q+1,y}^{(1)} - u_{q,y}^{(1)}) \sin \frac{\pi}{6} \right] + (u_{q+1,x}^{(1)} - u_{q,x}^{(1)}) \cos \frac{\pi}{6} + (u_{q+1,y}^{(1)} - u_{q,y}^{(1)}) \sin \frac{\pi}{6} \right].
\] (6)

Substitute the Bloch solution into equation (6) we get
\[
-\omega^2 m_{q,1} U_{x}^{(1)} = -\frac{3}{2} k U_{x}^{(1)} + 0 U_{y}^{(1)} + \frac{3}{4} (1 + e^{i\gamma_1}) k U_{x}^{(1)} + \frac{\sqrt{3}}{4} (1 - e^{i\gamma_1}) k U_{y}^{(1)},
\] (7)

which are elements of the first row of equation (5). Other entries of $D$ can be obtained in a similar manner.

Appendix B. Phase relationship for masses in one unit cell for pseudo spin modes

Modal displacement for each mass in the unit cell in $p_k = p_1 \pm ip_2$ and $d_k = d_1 \pm id_2$ are complex numbers. We take the phase angle of the displacement for each mass and plot the phase relation of the unit cell as shown in figures 3(e)–(h).

Take $d_+ = d_1 + id_2$ for example. From the eigenvalue problem of the dynamical matrix in equation (8), when $k_1 = 0.8$ N m$^{-1}$, $k_2 = 1$ N m$^{-1}$, and $m = 1$ kg, we have the values (in meter) of $x$- and $y$- direction modal displacements for each mass in $d_1$ and $d_2$ shown as figure 9. The modal displacements and phase relation for $d_+$ can be calculated accordingly. As shown in figure 9, phase plots for $x$- and $y$- direction modal displacement fields show same polarization, and both have the change of $4\pi$ in one unit cell, indicative of even parity/quadruple symmetries.
Appendix C. Effective Hamiltonian, Berry curvature, spin Chern number, and $Z_2$ invariant

The dynamical matrix $D$ for 6 masses with 12 constituent modal displacements (i.e. $U = [U_{x^1}, U_{x^2}, U_{x^3}, U_{y^1}, U_{y^2}, U_{y^3}, U_{y^4}, U_{y^5}, U_{y^6}, U_{y^7}, U_{y^8}, U_{y^9}]$) is of the form:

\[
D = \begin{bmatrix}
\frac{3 k_1}{2 m} & 0 & \frac{\sqrt{3} k_1}{2 m} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3 k_1}{2 m} & \frac{\sqrt{3} k_1}{2 m} & 0 & 0 & \frac{3 k_1}{2 m} & \frac{\sqrt{3} k_1}{2 m} \\
0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} \\
\frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & 0 & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} \\
0 & 0 & 0 & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} \\
0 & 0 & 0 & 0 & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} \\
\frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & 0 & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} \\
0 & 0 & 0 & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} \\
0 & 0 & 0 & 0 & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} \\
0 & 0 & 0 & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} \\
0 & 0 & 0 & 0 & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} & 0 & 0 & \frac{3 k_1}{4 m} & \frac{\sqrt{3} k_1}{4 m} \\
\end{bmatrix}
\]  

(8)

There are 12 bands corresponding to the 12 by 12 matrix $D$. To investigate the spin–Chern number, we derive the effective Hamiltonian [36] assuming that the other 8 bands have negligible influence. Modal displacement vector $U$ can be rewritten as the superposition of $p_1, p_2, d_1,$ and $d_2$: $U' = c_1 p_1 + c_2 p_2 + c_3 d_1 + c_4 d_2,$ where $c_1, c_2, c_3,$ and $c_4$ are coefficients. Based on these assumptions, equation (4) gives

\[
DU' = \begin{bmatrix}
\omega_p^2 & 0 & 0 & 0 \\
0 & \omega_p^2 & 0 & 0 \\
0 & 0 & \omega_d^2 & 0 \\
0 & 0 & 0 & \omega_d^2 \\
\end{bmatrix}
U'.
\]  

(9)

From this we get the 4 by 4 effective Hamiltonian on the basis of $[p_1, p_2, d_1, d_2]$ as

\[
H = [p_1, p_2, d_1, d_2]^T D [p_1, p_2, d_1, d_2].
\]  

(10)
And the eigenvalue problem can be written as
\[
H = \begin{bmatrix}
    c_1 & c_2 \\
    c_3 & c_4 \\
\end{bmatrix} = \begin{bmatrix}
    \omega_p^2 & 0 & 0 & 0 \\
    0 & \omega_p^2 & 0 & 0 \\
    0 & 0 & \omega_p^2 & 0 \\
    0 & 0 & 0 & \omega_p^2 \\
\end{bmatrix} \begin{bmatrix}
    c_1 \\
    c_2 \\
    c_3 \\
    c_4 \\
\end{bmatrix}.
\] (11)

(Since \(p_1, p_2, d_1, \text{ and } d_2\) are normalized and orthogonal vectors, \([p_1, p_2, d_1, d_2] = [p_1, d_1, d_2] = 1\).) Each element in \(H\) can be approximated to the second order using Taylor expansion.

For lattice with \(k_1 < k_2\), take \(k_1 = 0.8, k_2 = 1\) and \(m = 1\). Neglect second-order off-diagonal terms, the effective Hamiltonian is (here \(\gamma_\chi = \gamma_1 \pm \frac{1}{2} \gamma_2\) and \(\gamma_p = \sqrt{\frac{3}{2}} \gamma_2\))
\[
H_{NT} = \begin{bmatrix}
    \omega_p^2 - 0.168k_1 \left(\gamma_\chi^2 + \frac{1}{3} \gamma_p^2\right) & 0 & 0.2387i k_2 \gamma_p & 0.2387i k_2 \gamma_\chi \\
    0 & \omega_p^2 - 0.168k_1 \left(\gamma_\chi^2 \gamma_p + \gamma_\chi^2 \gamma_\chi + \frac{1}{3} \gamma_p^2\right) & -0.2387i k_2 \gamma_\chi & 0.2387i k_2 \gamma_p \\
    -0.2387i k_2 \gamma_p & 0.2387i k_2 \gamma_\chi & \omega_p^2 + 0.2387k_2 \left(\gamma_\chi^2 + \frac{1}{3} \gamma_p^2\right) & 0 \\
    -0.2387i k_2 \gamma_\chi & -0.2387i k_2 \gamma_p & 0 & \omega_p^2 + 0.2387k_2 \left(\gamma_\chi^2 + \frac{1}{3} \gamma_p^2\right) \\
\end{bmatrix}.
\] (12)

Since \([p_\chi, d_1, p_\bar{d}, p_\bar{d}] = [p_1, p_2, d_1, d_2]Q\), where \(Q = \begin{bmatrix}
    1 & 0 & \frac{1}{\sqrt{2}} & 0 \\
    0 & i & 0 & \frac{1}{\sqrt{2}} \\
    \frac{1}{\sqrt{2}} & 0 & 1 & 0 \\
    0 & -\frac{i}{\sqrt{2}} & 0 & 1 \\
\end{bmatrix}\), \(H_{NT}\) can be rewritten on the basis of \([p_\chi, d_1, p_\bar{d}, p_\bar{d}]\),
\[
H_{NT}' = Q'H_{NT}Q.
\] (13)

Analogous to equation (11), on the \([p_\chi, d_1, p_\bar{d}, p_\bar{d}]\) basis
\[
H_{NT}' = \begin{bmatrix}
    c_\chi^+ & c_d^+ & c_p^+ & c_\bar{p}^+ \\
\end{bmatrix} = \begin{bmatrix}
    \omega_{\chi} & 0 & 0 & 0 \\
    0 & \omega_d & 0 & 0 \\
    0 & 0 & \omega_p & 0 \\
    0 & 0 & 0 & \omega_\bar{p} \\
\end{bmatrix} \begin{bmatrix}
    c_\chi^+ \\
    c_d^+ \\
    c_p^+ \\
    c_\bar{p}^+ \\
\end{bmatrix}.
\] (14)

We obtain
\[
H_{NT}' = \begin{bmatrix}
    \omega_{\chi} + F(\gamma_\chi^2 + \gamma_p^2) & A\gamma_\chi & 0 & 0 \\
    A^*\gamma_\chi & \omega_d + E(\gamma_\chi^2 + \gamma_p^2) & 0 & 0 \\
    0 & 0 & \omega_p & F(\gamma_\chi^2 + \gamma_p^2) \\
    0 & 0 & A^*\gamma_\chi & \omega_\bar{p} + E(\gamma_\chi^2 + \gamma_p^2) \\
\end{bmatrix}.
\] (15)

where \(\gamma_\chi = \gamma_p = \pm \frac{1}{3}\gamma_\chi, A = 0.2387i k_2, E = \frac{k_2}{b}\), and \(F = -0.1120k_2\).

If we set the reference energy level to be \(\frac{1}{2}\omega_p^2 + \omega_\bar{p}^2 + (E + F)(\gamma_\chi^2 + \gamma_p^2)\), equation (15) becomes
\[
H_{NT}' = \begin{bmatrix}
    H_+ & 0 & 0 \\
    0 & 0 & H_+ \\
\end{bmatrix}.
\] (16)

with \(H_+ = \begin{bmatrix}
    -M + B\gamma_\chi^2 & A\gamma_\chi \\
    A^*\gamma_\chi & M + B\gamma_\chi^2 \\
\end{bmatrix}\), where \(M = \frac{\omega_\chi - \omega_\bar{p}}{2}\), which is negative when \(k_1 < k_2\), and \(B = \frac{E - F}{2}\), which is also negative. Since \(H_{NT}'\) has a similar formula as the Bernevig–Hughes–Zhang model [35], the spin Chern number can be calculated from equation (3). Since \(M\) and \(B\) are both negative, the spin Chern number for lattice with \(k_1 < k_2\) is \pm 1, which indicates it is topologically non-trivial.
The projections of pseudo spin eigenvectors on \([p, d_+, p, d_-]\) are plotted in figure 10. From figure 10 we can see that for the degenerate bands below the band gap, eigenvectors on most of the BZ are \(p\)-like, except for near the \(\Gamma\) point, where the eigenvectors are \(d\)-like. On the other hand, eigenvectors for the higher bands are more \(d\)-like near the \(\Gamma\) point and \(p\)-like elsewhere. The Berry curvature \(\mathcal{F}_{12}(\gamma_x, \gamma_y)\) [37] for each of the pseudo spin channels are plotted in figure 11. By integrating the Berry curvature over the BZ [11]

\[
C_s = \frac{1}{2\pi i} \sum \gamma_x \sum \gamma_y \mathcal{F}_{12}(\gamma_x, \gamma_y),
\]

with values consistent with those previously obtained.
The $Z_2$ invariant is defined as $Z_2 = n_s \pmod{2}$, where $n_s = \frac{C_s^+ - C_s^-}{2}$ is the quantum spin Hall conductivity \[1\]. The calculated spin Chern numbers $C_s^+$ and $C_s^-$ give $n_s = 1$, implying $Z_2$ is unity.

Similarly, for a lattice with $k_1 > k_2$, the effective spin Hamiltonian takes the same form as equation (16), but with $M > 0$, and $B < 0$. According to equation (3), $C_s = 0$, which proves that the lattice with $k_1 > k_2$ is topologically trivial. The projections of pseudo spin eigenvectors with $k_1 = 1.2$, $k_2 = 1$ and $m = 1$ are plotted in figure 12, which shows eigenvectors of the lower bands are more $p$-like, while eigenvectors to the higher bands tend to be $d$-like, as expected for an ordinary/trivial insulator.

**Appendix D. Mini band gap due to $C_6$ symmetry breaking at the T–NT boundary**

There would indeed be level repulsion/band anti-crossings (mini bandgap) when levels/bands of similar symmetry intersect, as would be relevant to the slight perturbation from $C_6$ symmetry at the boundary between the T and the NT regions. The magnitude of the gap could be related to the extent of asymmetry and could, in principle, be reduced, e.g. through minimizing the effect of $C_6$ symmetry breaking at the T–NT interface \[37\].

We indicate such influences in figure 13. The figures have differing relative mass ratios, and ratio of the inter-cell spring constant (T); intra-cell spring constant; inter-cell spring constant (NT). It can then be seen that as the asymmetry in mass and spring constants between the T and NT lattices increases, the mini band gap at the ‘crossing’ becomes larger as well.
Figure 13. Mini bandgap at the crossing of the two helical modes for spring-mass lattices with (a) $\frac{m_T}{m_0} = 1.2$: 1: 0.8, and (b) $\frac{m_T}{m_0} = 1.3$: 0.8, and $k_x^\uparrow$: $k_y$: $k_x^\uparrow$: $k_y^\uparrow$: 1.2: 1: 0.8, and $k_x^\uparrow$: $k_y$. and $k_x^\uparrow$: $k_y^\uparrow$: 1.55: 1: 0.4.

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