Ameliorating the popular lepton mixings with A4 symmetry: A see-saw model for realistic neutrino masses and mixing

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Abstract

A model for neutrino masses and mixing is presented using the see-saw mechanism. The model combines Type -I and Type-II see-saw contributions of which the latter dominates. The scalars and the leptons in the model are assigned $A_4$ charges suitable to obtain the mass matrices required for the scheme. The Type -II see-saw accommodates atmospheric mass splitting and maximal mixing in the atmospheric sector ($\theta_{23} = \pi/4$). It is characterized by vanishing solar mass splitting and $\theta_{13}$ whereas the third neutrino mixing angle can acquire any value, $\theta_{12}^0$. Particular alternatives of $\theta_{12}^0$ viz. $35.3^\circ$ (tribimaximal), $45.0^\circ$ (bimaximal), $31.7^\circ$ (golden ratio) are accounted for. Another choice of $\theta_{12}^0 = 0^\circ$ (no solar mixing) is also considered. Incorporating the corrections provided by the subdominant Type-I see-saw involves degenerate perturbation theory due to vanishing solar splitting in the Type -II see-saw enabling the solar mixing angle to receive substantial corrections. Apart from amending the solar sector the Type-I see-saw also tunes all the neutrino oscillation parameters into the allowed ranges thus interrelating them all. Thus the model is testable in the light of future experimental data. As an example, $\theta_{23}$ emerges in the first (second) octant for normal (inverted) ordering. CP-violation is controlled by phases present in the right-handed Majorana neutrino mass matrix, $M_{\nu R}$. Only normal ordering is allowed if these phases are absent. If $M_{\nu R}$ is complex the Dirac CP-violating phase $\delta$, can be large, i.e., $\pm \pi/2$, and inverted ordering is also allowed. T2K and NOVA preliminary data favouring normal ordering and $\delta \sim -\pi/2$ predicts lightest neutrino mass to be 0.05 eV or more within the model framework.

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Key Words: Neutrino mixing, $\theta_{13}$, Solar splitting, A4, see-saw, Leptonic CP-violation

I Introduction

Intensive experimental investigations worldwide have determined neutrino masses and mixing to a great extent. In spite of these neutrinos retain certain mysteries including the ordering of their masses, their absolute mass scale, their Dirac or Majorana nature, the octant of the atmospheric mixing angle $\theta_{23}$ and CP-violation in lepton sector. While future experiments address these riddles, here a model of neutrino masses and mixing in concord with the experimental observations is proposed. The two small quantities $\theta_{13}$ and the ratio, $R \equiv \Delta m^2_{\text{solar}}/\Delta m^2_{\text{atmos}}$ can get interrelated when both are derived from a single perturbation \cite{1}. In \cite{2} larger mixing parameters like $\Delta m^2_{\text{atmos}}$ and $\theta_{23} = \pi/4$ were ascribed to the dominant fundamental structure of neutrino masses and mixing whereas the other oscillation parameters i.e., $\theta_{13}, \theta_{12}$, the deviation of $\theta_{23}$ from $\pi/4$, and $\Delta m^2_{\text{solar}}$ originated from a smaller see-saw \cite{3} generated perturbation\cite{4}. This induces constraints on the measured parameters. Certain symmetries

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\textsuperscript{1}Earlier attempts on neutrino mass models with some oscillation parameters much smaller than the others can be located in \cite{4}.
Table 1: $\theta^{0}_{12}$ for different popular lepton mixing patterns viz. TBM, BM, and GR mixing. NSM represents the case with vanishing solar mixing.

| Model | TBM | BM | GR | NSM |
|-------|-----|----|----|-----|
| $\theta^{0}_{12}$ | 35.3° | 45.0° | 31.7° | 0.0° |

Table: $\theta^{0}_{12}$ for different popular lepton mixing patterns viz. TBM, BM, and GR mixing. NSM represents the case with vanishing solar mixing.

can give rise to vanishing $\theta_{13}$ rather easily and new models based on perturbations of such structures are also common in literature [5, 6].

Here, a schematic outline of the current exercise is given. The following standard parametrization of the lepton mixing matrix – the Pontecorvo, Maki, Nakagawa, Sakata (PMNS) matrix – $U$ has been used

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} + s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} + s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} + c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} + c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

(1)

where $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$. Neutrino masses and mixing are generated by a two-component Lagrangian, one of the dominant Type-II see-saw kind while the subdominant contribution originates from Type-I see-saw. The larger atmospheric mass splitting, $\Delta m_{\text{atmos}}^2$ and maximal atmospheric mixing ($\theta_{23} = \pi/4$) is embedded within the Type-II see-saw structure whereas the solar splitting, $\Delta m_{\text{solar}}^2$ and $\theta_{13}$ are kept to be zero. The solar mixing angle can vary continuously and acquire any desired value of $\theta^{0}_{12}$. Needless to mention that neither $\Delta m_{\text{solar}}^2$ nor $\theta_{13}$ are vanishing [7]. Evidences of non-maximal yet large $\theta_{23}$ exist. The solar mixing angle $\theta_{12}$ is also constrained by experiments. The Type-I see-saw alleviates all these issues. Since the solar splitting is vanishing in the Type-II see-saw scenario, the first two mass eigenstates are degenerate. In order to lift this degeneracy with the help of Type-I see-saw contribution one has to use degenerate perturbation theory. As a consequence of this, corrections to the solar mixing angle can be large.

The starting structure can be of tribimaximal (TBM), bimaximal (BM), and golden ratio (GR) mixings. All of these have $\theta_{13} = 0$ and $\theta_{23} = \pi/4$, $\theta^{0}_{12}$ being the only discriminating factor as specified in Table 1. In this Table, the fourth option corresponds to no solar mixing (NSM) i.e., $\theta^{0}_{12} = 0$ which has the virtue of the mixing angles to be either maximal, i.e., $\pi/4$ ($\theta_{23}$) or vanishing ($\theta_{13}$ and $\theta^{0}_{12}$). An $A_4$-based model with identical objectives only for the NSM case was studied in [8]. This attempt along with [8] differ from the other earlier works on $A_4$ [9, 10, 11] as in most of them neutrino mass matrix was derived as an outcome of a Type-II see-saw mechanism and obtaining TBM was of chief importance. Recent activities directed towards more realistic mixing patterns [12] often leading to breaking of $A_4$ symmetry can be found in [13].

A few distinctive aspects of this model are worth noting at this point. Firstly, a combination of Type-I and Type-II see-saw is considered. Secondly, the model is constructed to accommodate many popular mixing patterns. This is the first attempt of this kind using $A_4$ flavour symmetry that amends several popular lepton mixing patterns in a single stroke in which Type-II see-saw is the dominant contribution whereas Type-I see-saw is the subdominant component. The symmetries are broken spontaneously. Further, soft symmetry breaking terms are prohibited. All symmetry conserving terms are included in the Lagrangian. Scalars and leptons involved in the model are assigned suitable $A_4$ charges to implement this feature. An analogous pursuit based on $S_3 \times Z_3$ resulted in [14].
The model comprises of scalars and leptons with specific $A_4$ charges. All terms allowed by the symmetries under consideration are included in the Lagrangian. No soft symmetry-breaking term is included.

The right-handed charged leptons transform as $1(e_R)$, $1'(\mu_R)$, and $1''(\tau_R)$ under $A_4$. The left-handed lepton doublets of three flavours constitute an $A_4$ triplet, so does the right-handed neutrinos. This shows the lepton constituents of the model together with their transformation properties under $A_4$ and $SU(2)_L$. The hypercharge and lepton number assignments are also shown. The choices of $A_4$ properties of the fields are not unique. A list of all possible options can be found in [15] of which this

### Table 2: The lepton catalogue of the model. The $A_4$ quantum numbers assignments of the fields are featured together with their $SU(2)_L$ properties. The hypercharge, $Y$, and lepton number, $L$, are displayed.

| Fields                  | Notations  | $A_4$ | $SU(2)_L$ $(Y)$ | $L$ |
|-------------------------|------------|-------|----------------|-----|
| Left-handed leptons    | $(\nu_i, l_i)_L$ | 3     | 2 (-1)         | 1   |
| Right-handed charged leptons | $l_{1R}$ | 1     |                |     |
|                         | $l_{2R}$ | 1'    | 1 (-2)         | 1   |
|                         | $l_{3R}$ | 1''   |                |     |
| Right-handed neutrinos | $N_{iR}$  | 3     | 1 (0)          | -1  |

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2 The notation followed closely resembles that of [9].
3 Opposite lepton numbers are assigned to $\nu_L$ and $N_R$ in order to prohibit their coupling with $\Phi$ so that the Dirac mass matrix can remain proportional to the identity matrix.
| Purpose                                      | Notations  | A4  | SU(2) L (Y) | L | vev  |
|----------------------------------------------|------------|-----|-------------|---|------|
| Charged fermion mass                         | $\Phi = \begin{pmatrix} \phi^+ & \phi'^+ \\ \phi^- & \phi'^- \end{pmatrix}$ | 3   | 2 (1)      | 0 | $(\Phi) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ |
| Neutrino Dirac mass                          | $\eta = (\eta^0, \eta^-)$ | 1   | 2 (-1)     | 2 | $(\eta) = (u, 0)$ |
| Type-II see-saw mass                         | $\Delta_L^L = \begin{pmatrix} \Delta_{1a}^+ & \Delta_{1b}^+ \\ \Delta_{2a}^+ & \Delta_{2b}^+ \end{pmatrix}$ | 3   | 3 (2)      | -2| $(\Delta_L^L) = v_{La} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ |
| Type-II see-saw mass                         | $\Delta_L^L = \begin{pmatrix} \Delta_{1a}^+ & \Delta_{1b}^+ \\ \Delta_{2a}^+ & \Delta_{2b}^+ \end{pmatrix}$ | 3   | 3 (2)      | -2| $(\Delta_L^L) = v_{Lb} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ |
| Type-II see-saw mass                         | $\Delta_L^L = (\Delta_{1a}^+, \Delta_{1b}^+, \Delta_{2a}^+, \Delta_{2b}^+)^L$ | 1   | 3 (2)      | -2| $(\Delta_L^L) = (0, 0, u_L)$ |
| Right-handed neutrino mass                   | $\Delta_R^L = \begin{pmatrix} \Delta_{1c}^0 \\ \Delta_{1d}^0 \\ \Delta_{1e}^0 \end{pmatrix}$ | 3   | 1 (0)      | 2 | $(\Delta_R^L) = v_{Ra} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ |
| Right-handed neutrino mass                   | $\Delta_R^L = \begin{pmatrix} \Delta_{1c}^0 \\ \Delta_{2d}^0 \\ \Delta_{3e}^0 \end{pmatrix}$ | 3   | 1 (0)      | 2 | $(\Delta_R^L) = v_{Rb} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$ |
| Right-handed neutrino mass                   | $\Delta_R^L = \begin{pmatrix} \Delta_{1c}^0 \\ \Delta_{2d}^0 \\ \Delta_{3e}^0 \end{pmatrix}$ | 3   | 1 (0)      | 2 | $(\Delta_R^L) = v_{Rc} \begin{pmatrix} 1 \\ \omega^2 \end{pmatrix}$ |
| Right-handed neutrino mass                   | $\Delta_R^L = (\Delta_{1c}^0)^R$ | 1   | 1 (0)      | 2 | $(\Delta_R^L) = u_1 R$ |
| Right-handed neutrino mass                   | $\Delta_R^L = (\Delta_{2d}^0)^R$ | 1'  | 1 (0)      | 2 | $(\Delta_R^L) = u_2 R$ |
| Right-handed neutrino mass                   | $\Delta_R^L = (\Delta_{3e}^0)^R$ | 1'' | 1 (0)      | 2 | $(\Delta_R^L) = u_3 R$ |

Table 3: The scalar sector of the model. The A4 charges as well as the SU(2) L nature of the scalars are exhibited. The hypercharge, $Y$, lepton number, $L$, and the vacuum expectation value (vev) configurations of the scalars are also presented.

The model adopts class B. The model is restricted to leptons only\footnote{Quark models based on A4 have been explored in \cite{10} and \cite{17}.}

Masses of all leptons originate from A4-invariant Yukawa couplings. Several scalar fields have to be included that acquire suitable vacuum expectation values (vev). The strategy of choosing the scalar field multiplets requires some elaboration. An idea of the mass matrices of the left- and right-handed neutrinos in the flavour basis (charged lepton mass matrix diagonal) that are suitable for our avowed goal can be acquired from our previous work \cite{14}. The Lagrangian is written down in a basis which is unitarily related to the flavour basis. Consequently, the mass matrices in this defining basis have

\footnote{Models addressing this issue by separating the breaking of SU(2) L and A4 are widely studied in literature \cite{10}. The former is mediated by the usual doublet and triplet scalars of SU(2) L that are invariant under A4. The breaking of A4 is induced by the vev of ‘flavon’ scalar fields that are singlets of SU(2) L, but their transformations under A4 is non-trivial. Though such models are economic effective dimension-5 interactions comes into play in order to connect the fermions with the two types of scalar fields simultaneously leading to an interpretation as an effective theory.}
somewhat complicated structures for which the motivation is not initially obvious. These forms of the mass matrices (below) arise from a rather large set of scalars and their \( vev \)s.

The charged leptons acquire their masses through the \( SU(2)_L \) doublet scalar fields \( \Phi_i \) (\( i = 1, 2, 3 \)) forming an \( A4 \) triplet. The neutrino Dirac mass matrix is generated by an \( A4 \) invariant \( SU(2)_L \) doublet \( \eta \), having lepton number 2. \( SU(2)_L \) triplet scalars are required for the Type-II see-saw for left-handed neutrino mass matrix that include \( A4 \) triplet fields \( \hat{\Delta}_a^L \) and \( \hat{\Delta}_b^L \) along with \( \Delta_\zeta^L, \ \zeta = 1, 2, 3 \) transforming as 1, 1', \( 1'' \) of \( A4 \). These are used to construct the dominant Type-II see-saw neutrino mass matrix. Effects of the subdominant Type-I see-saw contribution is included perturbatively. \( A4 \) conserving Yukawa couplings produce the right-handed neutrino mass matrix as well. Several \( SU(2)_L \) singlet scalars are involved in generation of the Majorana masses for the right-handed neutrinos viz. \( \hat{\Delta}_p^R \) (\( p = a, b, c \)) transforming as \( A4 \) triplets and \( \Delta_\zeta^R \) (\( \zeta = 1, 2, 3 \)) transforming as 1, 1' and \( 1'' \) under \( A4 \). Table \ref{tab:scalars} evinces transformation properties of the model scalars under \( A4 \) and \( SU(2)_L \) together with their hypercharge, lepton number and \( vev \) configurations. The \( vev \)s of the \( SU(2)_L \) doublet scalars are of \( \mathcal{O}(M_W) \) while that of the \( SU(2)_L \) triplets are several orders of magnitude smaller than the doublet \( vev \)s in concord with the small neutrino masses as well as the \( \rho \) parameter of electroweak symmetry breaking. As expected, the \( vev \)s of the \( SU(2)_L \) singlets responsible for right-handed neutrino mass lies much above the electroweak scale. The mass terms of the neutrinos (both Type-I and Type-II see-saw) and that of the charged leptons are generated by a \( SU(2)_L \times U(1)_Y \) conserving Lagrangian that preserves \( A4 \) as well.

\[
\mathcal{L}_{\text{mass}} = y_j \rho_{ijk} \bar{l}_{Li} l_{Rj} \Phi_k^0 \quad \text{(charged lepton mass)}
+ f \rho_{ijk} \nu_{Li} N_{Rk} \eta^0 \quad \text{(neutrino Dirac mass)}
+ \frac{1}{2} \left( \sum_{n=a, b} \bar{\nu}_n \alpha_{ijk} v_{Li}^T C^{-1} v_{Lj} \hat{\Delta}_n^L + Y_L^\gamma \rho_{ij} v_{Li}^T C^{-1} v_{Lj} \Delta_\zeta^L \right) \quad \text{(neutrino Type-II see-saw mass)}
+ \frac{1}{2} \left( \sum_{p=a, b, c} \bar{\nu}_p^{R} \alpha_{ijk} N_{Ri}^T C^{-1} N_{Rj} \hat{\Delta}_p^{R0} + Y_R^\gamma \rho_{ij} N_{Ri}^T C^{-1} N_{Rj} \Delta_\zeta^{R0} \right) \quad \text{(rh neutrino mass) + h.c.} \quad (2)
\]

The scalars acquire the following \( vev \)s (\( SU(2)_L \) part is suppressed):

\[
\langle \Phi^0 \rangle = \frac{v}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \eta^0 \rangle = u, \quad \langle \hat{\Delta}_a^{L0} \rangle = v_{La} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \langle \hat{\Delta}_b^{L0} \rangle = v_{Lb} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \langle \Delta_1^{L0} \rangle = \langle \Delta_2^{L0} \rangle = \langle \Delta_3^{L0} \rangle = u_L,
\]

\[
\langle \hat{\Delta}_a^{R0} \rangle = v_{Ra} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \hat{\Delta}_b^{R0} \rangle = v_{Rb} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \quad \langle \Delta_\zeta^{R0} \rangle = v_{Rc} \begin{pmatrix} 1 \\ \omega \\ \omega \end{pmatrix}, \quad (4)
\]

\[
\langle \Delta_1^{R0} \rangle = u_{1R}, \quad \langle \Delta_2^{R0} \rangle = u_{2R}, \quad \langle \Delta_3^{R0} \rangle = u_{3R}. \quad (5)
\]

An elaborate study of the \( A4 \) conserving scalar potential involving the fields listed in Table \ref{tab:scalars} is presented in Appendix \ref{app:scalars} of this paper. Local minimization is performed and the conditions corresponding to the particular \( vev \) structures as indicated in Eqs. \ref{eq:vevs} are obtained.

\footnote{Lepton number is also conserved for the mass terms of Dirac kind.}
The mass matrix for the charged leptons and the left-handed Majorana neutrinos so obtained are:

\[
M_{\nu愿} = \frac{v}{\sqrt{3}} \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1 & \omega y_2 & \omega^2 y_3 \\ \omega y_1 & \omega^2 y_2 & \omega y_3 \end{pmatrix}, \quad M_{\nu \bar{L}} = \begin{pmatrix} (Y^L_1 + 2Y^L_2)u_L & \frac{1}{2}Y^L_b v_{Lb} & \frac{1}{2}Y^L_b v_{Lb} \\ \frac{1}{2}Y^L_b v_{Lb} & (Y^L_1 - Y^L_2)u_L & \frac{1}{2}(Y^L_a v_{La} + Y^L_b v_{Lb}) \\ \frac{1}{2}Y^L_b v_{Lb} & \frac{1}{2}(Y^L_a v_{La} + Y^L_b v_{Lb}) & (Y^L_1 - Y^L_2)u_L \end{pmatrix},
\]

where the choice of \( Y^L_2 = Y^L_3 \) is made. The Yukawa couplings involved in the charged lepton mass matrix satisfies \( y_1 v = m_e, \ y_2 v = m_\mu, \ y_3 v = m_\tau \). The neutrino mass matrix of Dirac nature and the right-handed neutrino mass matrix of Majorana kind acquire the following structures:

\[
M_D = f u I, \quad M_{\nu \bar{R}} = m_R \begin{pmatrix} \chi_1 & \chi_6 & \chi_5 \\ \chi_6 & \chi_2 & \chi_4 \\ \chi_5 & \chi_4 & \chi_3 \end{pmatrix}.
\]

\( m_D \) sets the scale of Dirac masses of the neutrinos where one can identify \( f u = m_D \). The scale of the Type-II see-saw neutrino masses is much smaller than that of the charged leptons i.e., \( \mathcal{O}(M_{\nu \bar{L}}) \sim u_L, v_{La}, v_{Lb} \) where \( u_L, v_{La}, v_{Lb} < < v \). Such a possibility that the triplet vev is much smaller than the doublet vev can be obtained as shown in [18], albeit in a model with fewer scalars. The scale of the right-handed Majorana neutrino masses is set by \( m_R \) and \( \chi_i \) in Eq. (7) are dimensionless quantities\(^7\) of \( \mathcal{O}(1) \).

The mass matrices in Eq. (6) could be expressed in a more convenient form by applying a couple of transformations. The non-hermitian charged lepton mass matrix can be diagonalised by applying a transformation \( U_L \) (below) on the left-handed lepton doublets and no transformation on the right-handed charged leptons. The transformation matrices are expressed as:

\[
U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}.
\]

This basis in which the charged lepton mass matrix is diagonal and the entire lepton mixing is governed by the neutrino sector is termed as the flavour basis in which the mass matrices acquire the following forms:

\[
M_{\nu \bar{L}}^{\text{flavour}} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad M_{\nu \bar{L}}^{\text{ flavour}} = \frac{1}{2} \begin{pmatrix} 2m_1^{(0)} & 0 & 0 \\ 0 & m^+ & m^- \\ 0 & m^- & m^+ \end{pmatrix}.
\]

Here \( m^\pm \equiv m_3^{(0)} \pm m_1^{(0)} \). Therefore, \( m^- \) is positive (negative) for normal (inverted) ordering. As noted earlier, \( M_{\nu \bar{L}}^{\text{flavour}} \), which arises from the Type-II see-saw, is the dominant contribution to the neutrino mass.

Demanding that the neutrino Dirac mass matrix, which couples the left- and right-handed neutrinos, preserves its proportionality to the identity matrix necessitates that the transformation applied on the right-handed neutrino fields must be \( V_R = U_L \). Thus we get,

\[
M_D = f u I, \quad M_{\nu \bar{R}}^{\text{flavour}} = (V^\dagger_R M_{\nu \bar{R}} V^\dagger_R) = \frac{m_R}{4ab} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{pmatrix}.
\]

\(^7\)See Appendix C for exact expressions of \( \chi_i \) in Eq. (7).
The matrices in Eq. (10) will take part in the Type-I see-saw mechanism. Various identification of the products of the Yukawa couplings and the vevs with the neutrino mass and mixing parameters are necessary for the mass matrices to be expressed in the forms as presented in Eqs. (9) and (10). Appendix C comprises of these algebraic details.

III Modus Operandi

The four mass matrices in the flavour basis obtained from the model are given in Eq. (9) and (10). In this basis the entire lepton mixing and CP-violation is controlled solely by the neutrino sector to which we restrict our discussion now onwards. The Type-II see-saw derived $M_{\nu L}^{\text{flavour}}$ is the dominant component to which the subdominant contribution attributed by the Type-I see-saw is incorporated by perturbation theory. The flavour basis mass matrices have to undergo one more basis transformations for successful implementation of this scheme. More precisely they ought to be expressed in the mass basis of the neutrinos which by definition has the left-handed neutrino mass matrix diagonal in it. Thus,

$$M^0 = M_{\nu L}^{\text{mass}} = U^0 T M_{\nu L}^{\text{flavour}} U^0 = \begin{pmatrix} m_1^{(0)} & 0 & 0 \\ 0 & m_1^{(0)} & 0 \\ 0 & 0 & m_3^{(0)} \end{pmatrix},$$

where,

$$U^0 = \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 \\ -\sin \theta_{12} / \sqrt{2} & \cos \theta_{12} / \sqrt{2} & 1 / \sqrt{2} \\ \sin \theta_{23} / \sqrt{2} & -\cos \theta_{23} / \sqrt{2} & 1 / \sqrt{2} \end{pmatrix}.$$  

The left-handed neutrino fields in the mass basis ($|\nu_L^{\text{mass}}\rangle$) are connected to the ones in the flavour basis ($|\nu_L^{\text{flavour}}\rangle$) by this $U^0$ furnished in Eq. (12). One can obtain the $|\nu_L^{\text{mass}}\rangle$ by applying $U^0$ on $|\nu_L^{\text{flavour}}\rangle$ i.e., $|\nu_L^{\text{mass}}\rangle = U^0 |\nu_L^{\text{flavour}}\rangle$. It immediately follows from Eqs. (11), (1) and (12) that in the Type-II see-saw component solar splitting is absent, $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. The columns of $U^0$ are the unperturbed flavour basis.

Once again we demand that in the mass basis the neutrino Dirac mass matrix remains proportional to identity. In order to satisfy this the same transformation ($U^0$) has to be applied on the right-handed neutrino fields. This leads to changes in form of right-handed neutrino mass matrix given by $M_{\nu R}^{\text{mass}} = (U^0)^T M_{\nu R}^{\text{flavour}} U^0$. The matrices contributing in Type-I see-saw are as follows:

$$M_D = m_D I \quad \text{and} \quad M_{\nu R}^{\text{mass}} = \begin{pmatrix} m_R & b \\ b & \frac{a}{\sqrt{2}} \end{pmatrix}.$$ 

Here $a$ and $b$ are dimensionless quantities of $O(1)$. It is imperative to note that $a$ and $b$ can in general be complex. One can in principle trade off $a$ and $b$ in terms of complex numbers $y e^{-i\phi_2}$ and $x e^{-i\phi_1}$ respectively, where $x$ and $y$ are dimensionless real quantities of $O(1)$. The Type-I see-saw contribution

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8Explicit forms of $r_{ij}$ in Eq. (10) can be found in Appendix C.

9See Eq. (C.5) in Appendix C for details.
so obtained is given by:

\[ M' = \left[ M_D^\top (M_{\nu R})^{-1} M_D \right] = \frac{m_D^2}{m_R} \begin{pmatrix} 0 & y e^{i\phi_1} & y e^{i\phi_1} \\ y e^{i\phi_1} & \frac{x e^{i\phi_2}}{\sqrt{2}} & -\frac{x e^{i\phi_2}}{\sqrt{2}} \\ y e^{i\phi_1} & -\frac{x e^{i\phi_2}}{\sqrt{2}} & \frac{x e^{i\phi_2}}{\sqrt{2}} \end{pmatrix} . \] (14)

Here the Dirac mass matrix is proportional to identity. It was checked that the same results can follow as long as \( M_D \) is diagonal. \( M_{\nu R}^{mass} \) exhibits a \( N_2R \leftrightarrow N_3R \) discrete symmetry. The results remain intact even if that choice is relaxed. Now onwards the entire procedure is carried on in the mass basis of the neutrinos using the mass matrices expressed in Eqs. (11) and (14).

The method followed below essentially consists of the following steps. Form the Type-II see-saw a lepton mixing of the form of Eq. (12) is generated, with \( \theta_{12}^0 \) of any preferred value. At this stage, only the atmospheric mass splitting is non-zero and atmospheric mixing is maximal. Next, the Type-I see-saw is included using degenerate perturbation theory. The solar mass splitting and the desired \( \theta_{12} \) are first obtained. Then the third column of the mixing matrix is calculated and compared with Eq. (1) to extract \( \theta_{13} \), \( \theta_{23} \), and \( \delta \).

**IV Results**

The neutrino mass matrices derived from Type-I and Type-II see-saw mechanism have been discussed in the previous section, of which the former is significantly smaller than the latter. In absence of the Type-I see-saw contribution the leptonic mixing matrix is characterized by \( \theta_{13} = 0 \), \( \theta_{23} = \pi/4 \), and \( \theta_{12}^0 \) is free to vary. Consequences for four choices of the value of \( \theta_{12}^0 \) corresponding to TBM, BM, GR, and NSM cases together with vanishing solar splitting are examined. This along with the atmospheric mass splitting allowed by the data depict the Type-II see-saw structure. Inclusion of Type-I see-saw corrections perturbatively up to first order modulates the neutrino oscillation parameters into the ranges preferred by data. Owing to the vanishing solar splitting in the Type-II see-saw contribution the first two mass eigenstates are degenerate. Thus in the solar sector degenerate perturbation theory has to be applied. Hence the first order corrections to the solar mixing angle can be large. The global best-fit of the oscillation parameters are displayed in the next section.

**IV.1 Data**

The current 3\( \sigma \) global fits of the neutrino oscillation parameters are: \[19, 20\]

\[ \Delta m_{21}^2 = (7.02 - 8.08) \times 10^{-5} \text{ eV}^2, \quad \theta_{12} = (31.52 - 36.18) \degree, \]
\[ |\Delta m_{31}^2| = (2.351 - 2.618) \times 10^{-3} \text{ eV}^2, \quad \theta_{23} = (38.6 - 53.1) \degree, \]
\[ \theta_{13} = (7.86 - 9.11) \degree, \quad \delta = (0 - 360) \degree . \] (15)

These numbers are taken from NuFIT2.1 of 2016 \[19\]. Needless to mention, \( \Delta m_{ij}^2 \equiv m_i^2 - m_j^2 \), such that \( \Delta m_{31}^2 > 0 \) for normal ordering (NO) and \( \Delta m_{31}^2 < 0 \) for inverted ordering (IO). Two best-fit points of \( \theta_{23} \) are evinced by the data in the first and in the second octants. Towards the end of the paper it is discussed how the model can accommodate the recent T2K and NOVA hints \[21, 22\] of \( \delta \) close to -\( \pi/2 \).
| Model ($\theta_{12}^0$) | TBM (35.3°) | BM (45.0°) | GR (31.7°) | NSM (0.0°) |
|------------------------|------------|------------|------------|------------|
| $\zeta$                | -4.0° ↔ 0.6° | -13.7° ↔ -9.1° | -0.4° ↔ 4.2° | 31.3° ↔ 35.9° |
| $\epsilon$             | -4.0° ↔ 0.6° | -14.5° ↔ -9.3° | -0.4° ↔ 4.2° | 44.0° ↔ 56.7° |
| $\epsilon - \theta_{12}^0$ | -39.2° ↔ -34.6° | -59.5° ↔ -54.4° | -39.2° ↔ -30.0° | 44.0° ↔ 56.7° |

Table 4: Data allowed 3σ ranges of $\zeta$ (Eq. (18)), $\epsilon$ (Eq. (19)), and $(\epsilon - \theta_{12}^0)$ for different popular mixing patterns are shown.

### IV.2 Real $M_{\nu_R}$ ($\phi_1 = 0$ or $\pi$, $\phi_2 = 0$ or $\pi$)

As a warm-up exercise let us consider the simpler case of $M_{\nu_R}$ real. In such a scenario there is no CP-violation as the phases $\phi_1$, $\phi_2$ of Eq. (14) are 0 or $\pi$. This leads to four different alternatives available for choosing $\phi_1$ and $\phi_2$. These are captured compactly by taking $x$ and $y$ real and allowing them to assume both signs for notational convenience. It will be soon clear how the experimental observations prefer one or the other of these four alternatives. Thus for real $M_{\nu_R}$ the Type-I see-saw contribution appears like:

$$M' = \frac{m_D^2}{m_R} \begin{pmatrix} 0 & y \\ y & x/\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{x}{\sqrt{2}} & \frac{y}{\sqrt{2}} \\ \frac{y}{\sqrt{2}} & \frac{x}{\sqrt{2}} \end{pmatrix}.$$ (16)

The degeneracy of the two neutrino masses in the Type-II see-saw ensuring the vanishing solar splitting necessitates the application of degenerate perturbation theory to obtain the corrections for the solar sector mixing parameters$^{10}$. The entire dynamics of this sector is dictated by the upper $2 \times 2$ submatrix of $M'$ given by:

$$M'^{2\times2} = \frac{m_D^2}{m_R} \begin{pmatrix} 0 & y \\ y & x/\sqrt{2} \end{pmatrix}.$$ (17)

This gives rise to:

$$\theta_{12} = \theta_{12}^0 + \zeta, \quad \tan 2\zeta = 2\sqrt{2} \left( \frac{y}{x} \right).$$ (18)

For functional ease it is useful to define a quantity, $\epsilon$ as:

$$\sin \epsilon = \frac{y}{\sqrt{y^2 + x^2/2}} \quad \text{and} \quad \cos \epsilon = \frac{x/\sqrt{2}}{\sqrt{y^2 + x^2/2}}, \quad \text{i.e.,} \quad \tan \epsilon = \frac{1}{2} \tan 2\zeta.$$ (19)

Once a mixing pattern is selected, the corresponding $\theta_{12}^0$ gets fixed and the experimental bounds of $\theta_{12}$ determines the $3\sigma$ ranges of $\zeta$ and $\epsilon$ by means of Eq. (15) and Eq. (19) as featured in Table 4. The ratio $(y/x)$ is positive (negative) when $\zeta$ is positive (negative). From Eq. (19) it is evident that the sign of $y$ is regulated by the value of $\epsilon$. Putting all these facts together it is easy to infer that $x$ is positive always, or in other words $\phi_2$ must be 0, while $y$ has to be positive, $\phi_1 = 0$ (negative, $\phi_1 = \pi$) for NSM (BM). In case of TBM and GR, both signs of $y$ are admissible. The solar splitting provided by the Type-I see-saw as extracted from Eq. (17) is:

$$\Delta m_{solar}^2 = \sqrt{2} m_D^2 m_R^0 \left( \sqrt{x^2 + 8y^2} = \sqrt{2} m_D^2 m_R^0 \frac{x}{\cos 2\zeta} \right).$$ (20)

$^{10}$Since degenerate perturbation theory is used in the solar sector, the first order correction to the solar mixing angle $\zeta$ is not constrained to be small.
For the mass basis form of the mass matrix in Eq. (11), the mixing in the leptonic sector is completely given by the $U^0$ given in Eq. (12). After including the Type-I see-saw correction to the mass matrices there is a further contribution to the mixing matrix as well, now given by:

$$U = U^0 U_\nu \quad \text{where} \quad U_\nu = \begin{pmatrix}
\cos \zeta & - \sin \zeta & \kappa_r \sin \epsilon \\
\sin \zeta & \cos \zeta & - \kappa_r \cos \epsilon \\
\kappa_r \sin (\zeta - \epsilon) & \kappa_r \cos (\zeta - \epsilon) & 1
\end{pmatrix}, \quad (21)
$$

with

$$\kappa_r \equiv \frac{m^2_D}{m_{RM}^2} \frac{\sqrt{y^2 + x^2}}{2 \cos \epsilon} = \frac{m^2_D}{m_{RM}^2} \frac{x}{\sqrt{2} \cos \epsilon}. \quad (22)$$

The third column of the lepton mixing matrix is:

$$|\psi_3\rangle = \begin{pmatrix}
\kappa_r \sin (\epsilon - \theta_{12}^0) \\
\frac{1}{\sqrt{2}} [1 - \kappa_r \cos (\epsilon - \theta_{12}^0)] \\
\frac{1}{\sqrt{2}} [1 + \kappa_r \cos (\epsilon - \theta_{12}^0)]
\end{pmatrix}. \quad (23)$$

As already pointed out, $x$ is always positive, $\kappa_r$ is positive (negative) for NO (IO).

Eq. (23) when mapped to the third column of Eq. (1) leads to:

$$\sin \theta_{13} \cos \delta = \kappa_r \sin (\epsilon - \theta_{12}^0), \quad (24)$$

and

$$\tan (\pi/4 - \theta_{23}) = \tan \omega = \kappa_r \cos (\epsilon - \theta_{12}^0). \quad (25)$$

The allowed ranges of $(\epsilon - \theta_{12}^0)$ for the different mixing patterns is given in Table 4. The CP-phase $\delta$ is 0 ($\pi$) when $\sin(\epsilon - \theta_{12}^0)$ is positive (negative) in case of normal ordering. It can be immediately concluded that $\delta = 0$ for the NSM from Table 4 and $\delta = \pi$ for the rest of the options under study. CP is conserved for both the values of $\delta$.

Using Eqs. (20), (22), and (24) it can be found:

$$\Delta m_{solar}^2 = 2 m^0 - m_{1}^{(0)} \frac{\sin \theta_{13} \cos \delta \cos \epsilon}{\cos 2\zeta \sin (\epsilon - \theta_{12}^0)}. \quad (26)$$

For real $M_{\nu R}$ inverted ordering is forbidden as can be seen from Eq. (26). In order to justify this one can define:

$$z = m^0 - m_{1}^{(0)} / \Delta m_{atmos}^2 \quad \text{and} \quad \tan \xi = m_0 / \sqrt{|\Delta m_{atmos}^2|}, \quad (27)$$

where $z$ is positive for both the orderings of neutrino masses. With the help of Eq. (26) it can be written as:

$$z = \left( \frac{\Delta m_{solar}^2}{|\Delta m_{atmos}^2|} \right) \left( \frac{\cos 2\zeta \sin (\epsilon - \theta_{12}^0)}{2 \sin \theta_{13} \cos \delta \cos \epsilon} \right). \quad (28)$$

From Eq. (27) it is straightforward to show that:

$$z = \sin \xi / (1 + \sin \xi) \quad \text{i.e.,} \quad 0 \leq z \leq \frac{1}{2} \quad \text{(for normal ordering)},$$

$$z = 1/(1 + \sin \xi) \quad \text{i.e.,} \quad \frac{1}{2} \leq z \leq 1 \quad \text{(for inverted ordering)}.$$

\(^{11}\)Inverted ordering is prohibited for real $M_{\nu R}$.  

10
Figure 1: $\omega = (\pi/4 - \theta_{23})$ -vs- $\theta_{12}$ plot for normal ordering. The 3$\sigma$ allowed range of $\sin \theta_{13}$ is marked by the solid lines whereas the dashed line indicates the best-fit value. Thin pink (thick green) lines denote the BM (NSM) case. The horizontal and vertical lines represent the data allowed 3$\sigma$ range. The first octant of $\theta_{23}$ is preferred since $\omega$ is positive always. Although $\omega$ is positive for TBM and GR mixing patterns its value lies beyond the 3$\sigma$ range. Best-fit values of atmospheric and solar mass splittings are taken. Inverted ordering is disallowed for $M_{\nu R}$ real.

The lightest neutrino mass $m_0$ has a one-to-one correspondence with $z$. In the quasi-degenerate limit, i.e., $m_0 \to$ large, $z \to \frac{1}{2}$ for both orderings. For real $M_{\nu R}$, $|\cos \delta| = 1$ in Eq. (28). It simply follows from the global fit mass splittings and mixing angles in Sec. IV.1 and Table 4 that $z \sim 10^{-2}$ or smaller for all four popular mixing alternatives. Thus inverted ordering is forbidden for real $M_{\nu R}$.

Using Eqs. (24) and (25) the deviation of the atmospheric mixing angle from maximality is found to be:

$$\tan \omega = \frac{\sin \theta_{13} \cos \delta}{\tan(\epsilon - \theta_{12}^0)}.$$  

(30)

Eq. (25) implies that $\omega$ is positive always for normal ordering irrespective of the mixing pattern. Thus $\theta_{23}$ is confined only to the first octant for real $M_{\nu R}$. $\epsilon$ can be expressed in terms of $\theta_{12}$ using Eqs. (18) and (19). Thus $\omega$ in Eq. (30) can be expressed as a function of $\theta_{13}$ and $\theta_{12}$ only. Fig. 1 exhibits $\omega$ as a function of $\theta_{12}$ for BM (thin pink lines) and NSM (thick green lines) alternatives. $\theta_{12}$ and $\omega$ varied within 3$\sigma$ allowed ranges as shown in Sec. IV.1. The TBM and GR cases are excluded owing as for the allowed values of $\theta_{12}$ they predict $\theta_{23}$ beyond the 3$\sigma$ range. The 3$\sigma$ limiting values of $\theta_{13}$ are marked by the solid lines whereas the dashed lines indicate its best-fit value. The vertical and horizontal blue dot-dashed lines denote the 3$\sigma$ experimental limits of $\theta_{12}$ and $\theta_{23}$.

With the help of Eq. (28), one can translate any allowed point in the $\omega - \theta_{12}$ plane and the $\theta_{13}$ associated with it to a value of $z$ or equivalently $m_0$, when the solar and the atmospheric mass splittings are provided. For both the allowed mixing patterns $m_0$ varies over a very small range. This range is found to be $2.13$ meV $\leq m_0 \leq 3.10$ meV ($3.20$ meV $\leq m_0 \leq 4.42$ meV) for NSM (BM) when both mass splittings and all the three mixing angles are allowed to vary over their entire 3$\sigma$ ranges.

The salient features of real $M_{\nu R}$ case are:
1. Only the normal ordering of neutrino masses is allowed.
2. Only the first octant of $\theta_{23}$ is admissible.
3. Type-I see-saw corrections is unable to make the TBM and GR mixing patterns consistent with the allowed ranges of the mixing angles.
4. NSM and BM alternatives can produce solutions in agreement with the observed neutrino masses and mixing. The allowed ranges of lightest neutrino mass is very narrow.

IV.3 Complex $M_{\nu R}$

Real $M_{\nu R}$ has several limitations viz. inverted ordering and CP-violation is forbidden. Moreover TBM and GR mixing patterns cannot be included within the ambit of the model when $M_{\nu R}$ is real. In order to overcome these constraints the general complex form of $M_{\nu R}$ leading to Type-I see-saw contribution $M'$ furnished in Eq. (14) has to be considered. It is worth reminding ourselves that this choice introduces the complex phases $\phi_{1,2}$ while $x$ and $y$ can only be positive.

Thus, $M'$ is no longer hermitian. To retain the hermitian nature the combination $(M^0 + M')^\dagger (M^0 + M')$ is considered among which $(M'^0 M^0)$ and $(M'^1 M^0)$ are treated as the leading term and the perturbation at the lowest order respectively. The unperturbed eigenvalues are given by $(m_i^{(0)})^2$ and perturbation matrix is:

$$
(M'^0 M^0 + M'^1 M^0) = \frac{m_2^2}{m_R} \begin{pmatrix}
0 & 2ym_1^{(0)} \cos \phi_1 & y f(\phi_1) \\
2ym_1^{(0)} \cos \phi_1 & \sqrt{2}xm_1^{(0)} \cos \phi_2 & -x \sqrt{2} f(\phi_2) \\
y f^*(\phi_1) & -x \sqrt{2} f^*(\phi_2) & \sqrt{2}xm_3^{(0)} \cos \phi_2
\end{pmatrix},
$$

(31)

where,

$$
f(\varphi) \equiv m^+ \cos \varphi - im^- \sin \varphi.
$$

(32)

The rest of the procedure is analogous to what was done in case of real $M_{\nu R}$ keeping in mind the discriminating factors of Eq. (31). Now, instead of Eqs. (18) and (19) of the real $M_{\nu R}$ case, the solar mixing obtained from Eq. (31) is given by

$$
\theta_{12} = \theta_{12}^{(0)} + \zeta, \quad \tan 2\zeta = 2\sqrt{2} \frac{y \cos \phi_1}{x \cos \phi_2},
$$

(33)

and

$$
\sin \epsilon = \frac{y \cos \phi_1}{\sqrt{y^2 \cos^2 \phi_1 + x^2 \cos^2 \phi_2/2}}, \quad \cos \epsilon = \frac{x \cos \phi_2/\sqrt{2}}{\sqrt{y^2 \cos^2 \phi_1 + x^2 \cos^2 \phi_2/2}}, \quad \tan \epsilon = \frac{1}{2} \tan 2\zeta.
$$

(34)

Table 4 shows the allowed ranges of $\zeta$ and $\epsilon$ which depend on the mixing patterns. For all mixing alternatives $\cos \epsilon$ is found to be positive. Thus from Eq. (34) $\phi_2$ must always lie in the first or fourth quadrants. For the different mixing patterns the ranges of $\phi_1$ are also given by that of $\epsilon$. When $\epsilon$ is positive (negative) then from the first relation contained in Eq. (34), it is evident that $\phi_1$ has to be in the first or fourth (second or third) quadrants. Using the results displayed in Table 4 one can infer that the first (second) option holds for the NSM (BM) patterns. In case of TBM and GR, $\epsilon$ varies over positive and negative values making both options equally admissible.
Table 5: The octant of $\theta_{23}$ and the quadrants of the CP-phase $\delta$ for different mixing patterns for both orderings of neutrino masses are exhibited.

Applying degenerate perturbation theory the solar mass splitting attributed completely to the Type-I see-saw contribution can be obtained from Eq. (31):

$$\Delta m^2_{\text{solar}} = \sqrt{2}m^2_1 \frac{m_D^2}{m_R} \sqrt{x^2 \cos^2 \phi_2 + 8y^2 \cos^2 \phi_1} = \sqrt{2}m^2_1 \frac{m_D^2}{m_R} x \cos \phi_2 \cos 2\zeta = \sqrt{2}m^2_1 \frac{m_D^2}{m_R} 2\sqrt{2}y \cos \phi_1.$$

In place of Eq. (23) one gets:

$$|\psi_3\rangle = \left(\begin{array}{c} \frac{\sin \epsilon}{\cos \phi_1} f(\phi_1) \cos \theta_{12}^0 - \frac{\cos \epsilon}{\cos \phi_2} f(\phi_2) \sin \theta_{12}^0 \mbox{/}m^+ \\ \frac{1}{\sqrt{2}} \{1 - \kappa_c \left[ \frac{\sin \epsilon}{\cos \phi_1} f(\phi_1) \sin \theta_{12}^0 + \frac{\cos \epsilon}{\cos \phi_2} f(\phi_2) \cos \theta_{12}^0 \mbox{/}m^+ \right] \} \\ \frac{1}{\sqrt{2}} \{1 + \kappa_c \left[ \frac{\sin \epsilon}{\cos \phi_1} f(\phi_1) \sin \theta_{12}^0 + \frac{\cos \epsilon}{\cos \phi_2} f(\phi_2) \cos \theta_{12}^0 \mbox{/}m^+ \right] \} \end{array}\right),$$

where,

$$\kappa_c = \frac{m_D^2}{m_R m_\nu^2} \sqrt{y^2 \cos^2 \phi_1 + x^2 \cos^2 \phi_2} / 2,$$

Here Eq. (34) and the complex function $f(\phi_1, 2)$ defined in Eq. (32) have been used. $\kappa_c$ is positive (negative) for NO (IO). Comparing Eq. (36) with the third column of Eq. (11) leads to:

$$\sin \theta_{13} \cos \delta = \kappa_c \sin(\epsilon - \theta_{12}^0),$$

$$\sin \theta_{13} \sin \delta = \kappa_c \frac{m^+}{m \cos \phi_1 \cos \phi_2} \left[ \sin \epsilon \sin \phi_1 \cos \phi_2 \cos \theta_{12}^0 - \cos \epsilon \cos \phi_1 \sin \phi_2 \sin \theta_{12}^0 \right].$$

From Table 4 it is obvious that $(\epsilon - \theta_{12}^0)$ exists in the first (fourth) quadrant for the NSM (BM, TBM, and GR) mixing pattern. From Eq. (38) one can immediately conclude that for NSM (BM, TBM, and GR) case(s) $\delta$ remains in the first or fourth (second or third) quadrants in case of normal ordering. $\kappa_c$ changes sign for inverted ordering. Thus the quadrants get modified accordingly. The different alternatives are furnished in Table 5. There are two allowed quadrants of $\delta$ having $\sin \delta$ of opposite sign for any mixing option and ordering of neutrino masses. The sign of the right-hand-side of Eq. (39) governs the phases $\phi_{1,2}$ which in its turn decides the quadrants CP-phase $\delta$ out of the two allowed options. As already discussed, $\phi_2$ can be in either the first or fourth quadrants. The quadrant of $\phi_1$ depends on the mixing pattern in such a manner that $\sin \phi_1$ can be of either sign. Therefore, the phases $\phi_1$ and $\phi_2$ can be chosen in a way such that $\sin \delta$ can acquire any particular sign. Thus the two alternate quadrants of $\delta$ for every case in Table 5 are equally allowed in the model.

The Type-I see-saw perturbative contribution to the atmospheric mixing angle can be obtained from Eq. (36) as:

$$\tan \omega = \frac{\sin \theta_{13} \cos \delta}{\tan(\epsilon - \theta_{12}^0)}.$$
Figure 2: In the left (right) panel the CP-phase $\delta (\theta_{23})$ predicted by this model is plotted as a function of the lightest neutrino mass $m_0$ for all the four mixing patterns when the best-fit values of the data are taken as input. The NSM, BM, TBM and GR mixing alternatives are represented by the green solid, pink dashed, red dot-dashed, and violet dotted curves respectively. Thick (thin) curves of each kind denote NO (IO).

Let us recall, Eq. (38) relates $\delta$ and $(\epsilon - \theta_{12}^{(0)})$ through $\kappa_c$. Thus for all mixing alternatives $\theta_{23}$ always remains in first (second) octant for NO (IO). This is one of the most important results of the model as shown in Table 5.

In the solar splitting expressed in Eq. (35), the factor of $m_D^2/m_R$ can be replaced in terms of $\kappa_c$. This together with Eq. (38) gives,

$$\Delta m_{solar}^2 = \frac{2m_1^{(0)}m_{13} \sin \theta_{13} \cos \delta \cos \epsilon}{\sin(\epsilon - \theta_{12}^{(0)}) \cos 2\zeta}.$$  

Predictions of the model can be extracted from Eqs. (40) and (41). The three mixing angles $\theta_{13}$, $\theta_{12}$, and $\theta_{23}$ are taken as inputs. Eq. (40) determines a value of the CP-violating phase $\delta$. With the help of these and the experimentally observed solar splitting the combination $m_1^{(0)}m^-$, or equivalently the variable $z$ can be calculated using Eq. (41) that fixes the lightest neutrino mass $m_0$. It may seem that arbitrarily large values of $m_0$, and hence $m_1^{(0)}m^-$, may be accounted for by tuning $\cos \delta$ to smaller and smaller values. However, this certainly is not the case. Experimental data necessitate $\omega = (\pi/4 - \theta_{23})$ to be restricted within observed limits. As all other factors have ranges determined experimentally, Eq. (40) also puts lower and upper bounds on $\delta$. Subsequently, $m_0$ lies within a fixed range for any mixing pattern.

Fig. 2 contains the CP-phase $\delta (\theta_{23})$ as a function of the lightest neutrino mass $m_0$ for different mixing patterns as predicted by this model in the left (right) panel while the best-fit values of the various measured angles and mass splittings are used. The NSM, BM, TBM and GR are depicted by green solid, pink dashed, red dot-dashed, and violet dotted curves respectively. The thick (thin) curves of each kind indicate NO (IO). Normal and inverted orderings are always associated with the first and second octants of the atmospheric mixing angle $\theta_{23}$ respectively. For NSM case $\delta$ lies in the first (second) quadrant for normal (inverted) ordering, while for the rest of the mixing options it is in the second (first) quadrant. For inverted neutrino mass ordering $|\delta|$ remains close to $\pi/2$ for the complete range.
of $m_0$. The CP-phase $\delta$ lies near $\pi/2$ for normal ordering for $m_0$ larger than around 0.05 eV.

From Table 5 it is evident that if $\delta$ is a solution for some $m_0$ then by properly choosing alternate values of the phases $\phi_{1,2}$ appearing in $M_{\nu R}$ one can also obtain a second solution with the phase $-\delta$. This mirror set of solutions are not shown in Fig. 2. The preliminary data presented by the T2K [21] and NOVA [22] collaborations can be considered as primary hint of normal ordering associated with $\delta \sim -\pi/2$. The consistency of this model with these observations is clearly visible from Fig. 2 with $\delta \sim -\pi/2$ favouring $m_0$ in the quasi-degenerate regime, i.e., $m_0 \geq \mathcal{O}(0.05 \text{ eV})$, for normal ordering. If this result is determined with better accuracy in the future analysis then the model will predict neutrino masses to be in a range that ongoing experiments are capable of probing [23, 24].

These interrelationships between the octant of $\theta_{23}$, the quadrant of the CP-violating phase $\delta$, and the neutrino mass ordering provide a clear set of correlations characteristic of this $A_4$ based model. In the model the corrections to the three neutrino mixing angles and $\Delta m^2_{\text{solar}}$ all have a common origin – the Type-I see-saw. As a result these parameters get correlated. Such interrelationships are specific to this model. Although the model has a large number of parameters, only this correlated region of the parameter space allowed by neutrino mass and mixing data leads to testable predictions in Table 5.

V Conclusions

In this paper an $A_4$ based see-saw model for neutrino masses and mixing has been proposed. The flavour quantum numbers suitable for the model are assigned to the leptons and the scalars. The Lagrangian is inclusive of all the symmetry conserving terms. No soft breaking of symmetry is entertained. The Yukawa couplings induce the charged lepton masses, Dirac and Majorana masses for the left- and right-handed neutrinos after the symmetry is broken spontaneously. Neutrino masses are produced by a combined effect of both Type-I and Type-II see-saw terms present in the Lagrangian of which the former can be thought of to be a small correction. The Type-II see-saw dominant contribution is associated with the atmospheric mass splitting, no solar splitting, keeps $\theta_{23} = \pi/4$, and $\theta_{13} = 0$ and $\theta_{12}$ can be given any preferred value. In particular, this model is scrutinized in context of tribimaximal, bimaximal, golden ratio, and ‘no solar mixing’ patterns. The contribution of Type-I see-saw can be treated as a perturbation that generates the solar splitting and tunes the mixing angles to values in agreement with the global fits. As a corollary a correlation between the octants of $\theta_{23}$ and neutrino mass ordering followed – first (second) octant is allowed for normal (inverted) ordering of neutrino mass. The model has several testable predictions including that of the CP-phase $\delta$, relationships between mixing angles and mass splittings. Moreover, inverted ordering got associated with near-maximal CP-phase $\delta$ and arbitrarily small neutrino masses are allowed. In case of normal ordering $\delta$ can vary over a larger range and maximality is accomplished in the quasi-degenerate regime. The lightest neutrino mass has to be at least a few meV for this case.

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A Appendix: The group $A_4$

$A_4$ is the even permutation group of four objects having 12 elements and two generators $S$ and $T$ satisfying the property $S^2 = T^3 = (ST)^3 = I$. It has four inequivalent irreducible representations viz. one 3 dimensional representation and three 1 dimensional representations namely, 1, 1' and 1''. These three dimension-1 representations are singlets under $S$ whereas they transform as 1, $\omega$, and $\omega^2$ respectively under the action of $T$, $\omega$ being a cube root of unity. Therefore it is apparent that $1' \times 1'' = 1$. The pertinent form of the generators $S$ and $T$ acting on the 3 dimensional representations are given by\(^{12}\)

\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

It is imperative to note the product rule for the three dimensional representation is:

\[
3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1
\]

When two triplets of $A_4$ given by $3_a \equiv a_i$ and $3_b \equiv b_i$, with $i = 1, 2, 3$; are combined according to Eq. (A.2), then the resultant triplets can be represented by $3_c \equiv c_i$ and $3_d \equiv d_i$ where,

\[
c_i = \left( \frac{a_2 b_3 + a_3 b_2}{2}, \frac{a_3 b_1 + a_1 b_3}{2}, \frac{a_1 b_2 + a_2 b_1}{2} \right), \quad \text{or}, \quad c_i \equiv \alpha_{ijk} a_j b_k,
\]

\[
d_i = \left( \frac{a_2 b_3 - a_3 b_2}{2}, \frac{a_3 b_1 - a_1 b_3}{2}, \frac{a_1 b_2 - a_2 b_1}{2} \right), \quad \text{or}, \quad d_i \equiv \beta_{ijk} a_j b_k, \quad (i, j, k, \text{are cyclic}).
\]

and the 1, 1' and 1'' so obtained can be scripted as:

\[
1 = a_1 b_1 + a_2 b_2 + a_3 b_3 \equiv \rho_{1ij} a_i b_j,
\]

\[
1' = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \equiv \rho_{3ij} a_i b_j,
\]

\[
1'' = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \equiv \rho_{2ij} a_i b_j.
\]

The group is studied in extensive details in \([9, 10]\).

B Appendix: Minimization of the scalar potential

Some detailed analysis of the nature of the scalar potential is presented in this Appendix. The conditions that have to be satisfied by the parameters of the potential so that the vevs acquire the values considered in the model are extracted. The conditions so obtained guarantee the potential is locally minimized by those choices. To confirm if those choices are in concurrence with the global minimum is beyond the scope of this work\(^{13}\).

\(^{12}\)This choice of basis has the generator $S$ diagonal. One can equivalently perform an analogous analysis in a basis in which the generator $T$ is diagonal. Needless to mention that the two bases are related by some unitary basis transformation.

\(^{13}\)As an example one can take a look at \([26]\) where a comparatively simpler scenario consisting of an $A_4$ triplet composed of three $SU(2)_L$ doublet scalar or in other words an $A_4$ symmetric three Higgs doublet model (3HDM) was analyzed in terms of the global minimization of the scalar potential. In \([25]\), it is shown that alignment follows as a natural consequence when the vevs acquire the configurations corresponding to those global minima. Three Higgs doublets symmetric under $A_4$ group has been vividly discussed in \([27]\). A model for leptons using an $A_4$ symmetric 3HDM can be found in \([28]\).
The fields catalogued in Table 3 comprise of scalars having lepton numbers as well as $SU(3)$ and $U(1)_Y$ charges. The scalar potential must be of the most general quartic nature conserving all the symmetries under consideration. Thus all the terms allowed by the symmetries are included in the discussion below. Verification of $SU(2)_L$, $U(1)_Y$ and lepton number are familiar exercises. $A_4$ invariance requires elaborate discussion as presented in the following section.

### B.1 $A_4$ conserving terms: Notations and general principles

Let us summarize a few salient features of this model to fix the notations to be followed for the $A_4$-invariant terms. As already noted, the scalar spectrum has fields transforming as $1, 1', 1''$, and $3$ under $A_4$. One has to consider all the combinations of these fields up to quartics that can yield $A_4$ invariants. The product rules for $1, 1'$ and $1''$ are easy, but that for the triplets of $A_4$ needs to be emphasized. If there are two $A_4$ triplet fields $A \equiv (a_1, a_2, a_3)^T$ and $B \equiv (b_1, b_2, b_3)^T$ where $a_i, b_i$ may possess $SU(2)_L \times U(1)_Y$ transformation properties that are not considered for the time being in the immediate course of discussion. As furnished in Eq. (A.2), one can combine $A$ and $B$ to obtain

$$3_A \otimes 3_B = 1 + 1' + 1'' + 3 + 3.$$  

(B.1)

For notational simplicity let us denote the irreducible representations on the right-hand-side by $O_1(A, B)$, $O_2(A, B)$, $O_3(A, B)$, $T_4(A, B)$ and $T_4(A, B)$, respectively, where, as already noted in Eqs. (A.3) and (A.4)

$$O_1(A, B) \equiv 1 = a_1 b_1 + a_2 b_2 + a_3 b_3 \equiv \rho_{1ij} a_i b_j ,$$

$$O_2(A, B) \equiv 1' = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \equiv \rho_{3ij} a_i b_j ,$$

$$O_3(A, B) \equiv 1'' = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \equiv \rho_{2ij} a_i b_j ,$$

and

$$T_4(A, B) \equiv 3 = \left(\frac{a_2 b_3 + a_3 b_2}{2}, \frac{a_3 b_1 + a_1 b_3}{2}, \frac{a_1 b_2 + a_2 b_1}{2}\right)^T ,$$

$$T_4(A, B) \equiv 3 = \left(\frac{a_2 b_3 - a_3 b_2}{2}, \frac{a_3 b_1 - a_1 b_3}{2}, \frac{a_1 b_2 - a_2 b_1}{2}\right)^T .$$

It is worth noticing that $O_3(A^\dagger, A) = [O_2(A^\dagger, A)]^\dagger$ and $T_4(A, A) = 0$. The scalar potential can be formulated implementing this notation and keeping in mind that the scalar sector of this model is devoid of any field which is invariant under all the symmetries under consideration. Therefore the scalar potential will contain terms of the following kind (only $A_4$ properties are exhibited):

1. Quadratic: $W^\dagger W$,

2. Cubic: $X_i X_j X_k^\dagger$, $X_i X_j X_k^\dagger$, $X_i X_j X_k^\dagger$, $X_i X_j X_k^\dagger$, $O_1(Y_i, Y_j) X_k^\dagger$, $O_2(Y_i, Y_j) X_k^\dagger$, $O_3(Y_i, Y_j) X_k^\dagger$,

3. Quartic:

$$(W_{ij}^\dagger W_{ij})(W_{ij}^\dagger W_{ij}), (X_i X_j)(X_k X_l), (X_i X_j)(X_k X_l), (X_i X_j')(X_k X_l'), (X_i X_j')(X_k X_l'), (X_i X_j')(X_k X_l'), (X_i X_j')(X_k X_l'), O_1(Y_i, Y_j) X_k X_l^\dagger, O_1(Y_i, Y_j) X_k X_l^\dagger, O_2(Y_i, Y_j) X_k X_l^\dagger, O_2(Y_i, Y_j) X_k X_l^\dagger, O_3(Y_i, Y_j) X_k X_l^\dagger, O_3(Y_i, Y_j) X_k X_l^\dagger.$$
\[ O_1(Y_i, Y_j)O_1(Y_k, Y_l), O_2(Y_i, Y_j)^{\dagger}O_2(Y_k, Y_l), O_3(Y_i, Y_j)^{\dagger}O_3(Y_k, Y_l), O_2(Y_i, Y_j)O_3(Y_k, Y_l), \]
\[ O_1(T_a(Y_i, Y_j), T_k(Y_k, Y_l)), O_1(T_a(Y_i, Y_j), T_m(Y_k, Y_l)), O_1(T_a(Y_i, Y_j), T_a(Y_k, Y_l)). \]
\[ O_1(T_a(Y_i, Y_j), Y_k)X_l^I, O_2(T_a(Y_i, Y_j), Y_k)X_l^I, O_3(T_a(Y_i, Y_j), Y_k)X_l^I \).

Here \( W \) is any field, \( X, X', \) and \( X'' \) represent generic fields transforming as 1, 1', and 1'' under \( A4 \) while \( Y \) happens to be generic \( A4 \) triplet field. The invariants constructed by using \( X^\dagger, X'^\dagger, X''^\dagger, \) and \( Y^\dagger \) are not listed separately.

Owing to the large number of scalars in the model – e.g., \( SU(2)_L \) singlets, doublets, and triplets – the scalar potential consists of many terms. In order to simplify the discussion, cubic terms in the fields are excluded and all the couplings are taken to be real. The antisymmetric triplet arising from the combination of two \( A4 \) triplets i.e., the terms denoted by \( T_a \) in Eq. (B.3) are not included in the potential throughout for ease of calculation. The potential is studied piece-wise: (a) consisting of terms obtained by combining scalars of different \( SU(2)_L \) sectors. The \( vev \) of the \( SU(2)_L \) singlets giving rise to the right-handed neutrino mass are larger than the \( vev \) of the other scalar fields. Thus in the latter category the combinations of \( SU(2)_L \) singlets with the doublets and triplets of \( SU(2)_L \) are considered, whereas, doublet-triplet inter-sector terms are dropped owing to the smallness of the triplet \( vev \) responsible for the left-handed Majorana neutrino mass. Also the electroweak precision measurements put a stringent bound on the triplet \( vev \) compelling it to be very small.

**B.2 \( SU(2)_L \) Singlet Sector:**

The \( SU(2)_L \) singlet scalar sector consists of three \( A4 \) triplets \( \hat{\Delta}_p^R \) with \( p = a, b, c \) denoting each one of them. These three triplets possess identical quantum numbers, their \( \text{vev} \) being the only discriminating criterion. Also there are three more fields viz. \( \Delta_1^R, \Delta_2^R \) and \( \Delta_3^R \) transforming as 1, 1' and 1'' under \( A4 \).

From Eq. (B.1) we can see that two same \( \hat{\Delta}_p^R \) triplets can combine to produce several \( A4 \) irreducible representations. For notational simplicity let us define:

\[ O_{1p}^{ss} \equiv O_1(\hat{\Delta}_p^R, \hat{\Delta}_p^R), O_{2p}^{ss} \equiv O_2(\hat{\Delta}_p^R, \hat{\Delta}_p^R), T_{sp}^{ss} \equiv T_s(\hat{\Delta}_p^R, \hat{\Delta}_p^R) \quad (p = a, b, c). \]  

(B.4)

Using two different triplets \( \hat{\Delta}_p^R \) and \( \hat{\Delta}_q^R \) where \( p \neq q \) analogous combinations can be defined:

\[ \hat{O}_{1pq}^{ss} \equiv O_1(\hat{\Delta}_p^R, \hat{\Delta}_q^R), \hat{O}_{2pq}^{ss} \equiv O_2(\hat{\Delta}_p^R, \hat{\Delta}_q^R), \hat{T}_{spq}^{ss} \equiv T_s(\hat{\Delta}_p^R, \hat{\Delta}_q^R) \quad (p, q = a, b, c \text{ and } p \neq q). \]  

(B.5)

Generically, it is convenient to use \( \hat{O}_{1p} \) or \( \hat{T}_{sp} \) if the second triplet in the argument is replaced by its hermitian conjugate. As an example,

\[ \tilde{\hat{O}}_{1p}^{ss} \equiv O_1(\hat{\Delta}_p^R, \hat{\Delta}_p^R), \tilde{\hat{O}}_{2p}^{ss} \equiv O_2(\hat{\Delta}_p^R, \hat{\Delta}_p^R), \tilde{\hat{O}}_{3p}^{ss} \equiv O_3(\hat{\Delta}_p^R, \hat{\Delta}_p^R) \text{ and } \tilde{T}_{sp}^{ss} \equiv T_s(\hat{\Delta}_p^R, \hat{\Delta}_p^R), \]  

(B.6)

One can also consider:

\[ \tilde{\hat{O}}_{1pq}^{ss} \equiv O_1(\hat{\Delta}_p^R, \hat{\Delta}_q^R), \tilde{\hat{O}}_{2pq}^{ss} \equiv O_2(\hat{\Delta}_p^R, \hat{\Delta}_q^R), \tilde{\hat{O}}_{3pq}^{ss} \equiv O_3(\hat{\Delta}_p^R, \hat{\Delta}_q^R). \]  

(B.7)

Also the following combinations are required:

\[ \hat{O}_{1p}^{ss} \equiv O_1(\hat{\Delta}_p^R, T_{sp}^{ss}), \hat{O}_{2p}^{ss} \equiv O_2(\hat{\Delta}_p^R, T_{sp}^{ss}), \hat{O}_{3p}^{ss} \equiv O_3(\hat{\Delta}_p^R, T_{sp}^{ss}) \quad (p = a, b, c). \]  

(B.8)
The $A4$ singlets $\Delta_i^R$ $(i = 1, 2, 3)$ can be combined to yield

$$Q_i^{ss} \equiv \Delta_i^R \Delta_i^R \ , \ (i = 1, 2, 3). \quad (B.9)$$

Needless to mention that such terms are singlets of all the symmetries under consideration.

Having devised the essential notations one can write the most general scalar potential for the $SU(2)_L$ singlet sector of this model as:

$$V_{\text{singlet}} = \sum_{i=1}^{3} m_{\Delta_i^R}^2 Q_i^{ss} + \sum_{p=a}^{c} m_{\Delta_p^R}^2 O_i^{ss} + \left[ \sum_{p \neq q; p, q = a}^{c} m_{\Delta_p^R}^2 \hat{O}_{1pq}^{ss} + \text{all possible permutations} \right]$$

$$+ \frac{1}{2} \sum_{i=1}^{3} \lambda_{1i}^s [Q_i^{ss}]^2 + \frac{1}{2} \sum_{k < j; k \neq j; k = 1}^{2} \lambda_{jk}^s \left\{ Q_j^{ss} Q_k^{ss} \right\}$$

$$+ \frac{1}{2} \sum_{p = a}^{c} \lambda_{3p}^s \left\{ [O_{1pq}^{ss}]^2 + (O_{2pq}^{ss})^\dagger O_{2pq}^{ss} + O_{1pq}(T^{ss}_{sp}, T^{ss\dagger}_{sp}) \right\}$$

$$+ \sum_{p \neq q; p, q = a}^{c} \left[ \frac{1}{2} \lambda_{4pq}^s \left( Q_i^{ss} O_{1pq}^{ss} \right) \right] + \sum_{i=1}^{3} \sum_{p \neq q; p, q = a}^{c} \lambda_{4pq}^s \left[ ([O_{i1pq}^{ss}]^\dagger \hat{O}_{1pq}^{ss} + O_{1pq}(T^{ss}_{sp}, T^{ss\dagger}_{sp}) \right)$$

$$+ \sum_{p \neq q; p, q = a}^{c} \lambda_{5pq}^s \left[ \left\{ (\Delta_1^R O_1(\Delta_1^R, \hat{T}^{ss\dagger}_{sp}) + (\Delta_1^R O_1(\Delta_1^R, \hat{T}^{ss\dagger}_{sp}) \right\} + h.c. \right]$$

$$+ \frac{1}{2} \sum_{p = a}^{c} \lambda_{6p}^s (\Delta_3^R \Delta_2^R + h.c.) + \sum_{p \neq q; p, q = a}^{c} \lambda_{6pq}^s \left[ \left\{ (\Delta_2^R O_3(\Delta_2^R, \hat{T}^{ss\dagger}_{pq}) + (\Delta_2^R O_3(\Delta_2^R, \hat{T}^{ss\dagger}_{pq}) \right\} + h.c. \right]$$

$$+ \frac{1}{2} \sum_{p = a}^{c} \lambda_{7p}^s (\Delta_3^R \Delta_2^R + h.c.) + \sum_{p \neq q; p, q = a}^{c} \lambda_{7pq}^s \left[ \left\{ (\Delta_3^R O_2(\Delta_3^R, \hat{T}^{ss\dagger}_{pq}) + (\Delta_3^R O_2(\Delta_3^R, \hat{T}^{ss\dagger}_{pq}) \right\} + h.c. \right]$$

$$+ \sum_{p \neq q; p, q = a}^{c} \sum_{i=1}^{3} \lambda_{8ip}^s \left( \Delta_i^R \Delta_2^R \hat{O}_{ip} + h.c. \right) + \sum_{p \neq q; p, q = a}^{c} \sum_{i=1}^{3} \lambda_{8pq}^s \left( \Delta_i^R \Delta_2^R \hat{O}_{pq} + h.c. \right)$$

$$+ \sum_{p = a}^{c} \left[ \lambda_{91p}^s \Delta_3^R \Delta_3^R O_{1p} + \lambda_{92p}^s \Delta_3^R \Delta_3^R O_{2p} + \lambda_{93p}^s \Delta_3^R \Delta_2^R \hat{O}_{3p} + h.c. \right]$$

$$+ \sum_{p \neq q; p, q = a}^{c} \left[ \lambda_{91pq}^s \Delta_3^R \Delta_3^R O_{1pq} + \lambda_{92pq}^s \Delta_3^R \Delta_3^R O_{2pq} + \lambda_{93pq}^s \Delta_3^R \Delta_2^R \hat{O}_{3pq} + h.c. \right]. \quad (B.10)$$

Here $\lambda_{3p}^s$, $\lambda_{3pq}^s$ and $\lambda_{3pq}^s$ are taken as the common coefficient of the different $A4$ invariants generated by combining two $\Delta^R$ and two $(\Delta^R)^\dagger$ fields. Similar policy will be adopted for the fields with other $SU(2)_L$ properties.
B.3 \(SU(2)_L\) Doublet Sector:

The \(SU(2)_L\) doublet scalar precinct consists of the two fields \(\Phi\) and \(\eta\) transforming as 3 and 1 of \(A_4\) respectively. Opposite hypercharges are assigned to \(\Phi\) and \(\eta\). The \(A_4\) triplet \(\Phi\) combinations are denoted as:

\[
O_1^{dd} \equiv O_1(\Phi^1, \Phi); \quad O_2^{dd} \equiv O_2(\Phi^1, \Phi); \quad T_s^{dd} \equiv T_s(\Phi, \Phi),
\]

and that of the \(A_4\) singlet \(\eta\):

\[
Q_\eta^{dd} \equiv \eta^\dagger \eta.
\]

The potential for the \(SU(2)_L\) doublet sector is given by:

\[
V_{\text{doublet}} = m_\eta^2 O_{\eta}^{dd} + m_\Phi^2 O_1^{dd} + \frac{1}{2} \lambda_1^d \left[ Q_\eta^{dd} \right]^2 + \frac{1}{2} \lambda_2^d \left\{ [O_1^{dd}]^2 + [O_2^{dd}]^\dagger O_2^{dd} \right\}
+ O_1(T_s^{dd}, T_s^{dd}) \right) + \frac{1}{2} \lambda_3^d \left[ Q_\eta^{dd} O_1^{dd} \right].
\]

B.4 \(SU(2)_L\) Triplet Sector:

The \(SU(2)_L\) triplet sector comprises of five fields. There are two \(A_4\) triplets \(\hat{\Delta}_a^L\) and \(\hat{\Delta}_b^L\) together with the fields the \(\Delta_{a1}^L, \Delta_{a2}^L\) and \(\Delta_{a3}^L\) transforming as 1, 1', 1'' of \(A_4\) respectively.

It is useful to define:

\[
O_{1n}^{tt} \equiv O_1(\hat{\Delta}_{aL}^L, \hat{\Delta}_{bL}^L); \quad O_{2n}^{tt} \equiv O_2(\hat{\Delta}_{aL}^L, \hat{\Delta}_{bL}^L); \quad T_{sn}^{tt} \equiv T_s(\hat{\Delta}_{aL}^L, \hat{\Delta}_{bL}^L) , \quad (n = a, b),
\]

\[
\hat{O}_{1nl}^{tt} \equiv \hat{O}_1(\hat{\Delta}_{aL}^L, \hat{\Delta}_{bL}^L); \quad \hat{O}_{2nl}^{tt} \equiv \hat{O}_2(\hat{\Delta}_{aL}^L, \hat{\Delta}_{bL}^L); \quad \hat{T}_{snl}^{tt} \equiv T_s(\hat{\Delta}_{aL}^L, \hat{\Delta}_{bL}^L) , \quad (n, l = a, b \text{ and } n \neq l),
\]

\[
Q_i^{tt} \equiv \Delta_i^L \Delta_L^L , \quad (i = 1, 2, 3),
\]

and

\[
\hat{O}_{jn}^{tt} \equiv \hat{O}_j(\hat{\Delta}_{aL}^L, \hat{\Delta}_{bL}^L); \quad \hat{O}_{jnl}^{tt} \equiv \hat{O}_j(\hat{\Delta}_{aL}^L, \hat{\Delta}_{bL}^L) , \quad (j = 1, 2, 3) \quad \text{and} \quad (n, l = a, b \text{ and } n \neq l). \]

The scalar potential for this sector:

\[
V_{\text{triplet}} = \sum_{i=1}^{3} \frac{m_{\Delta_i}^2}{2} Q_i^{tt} + \sum_{n=a}^{b} m_{\Delta_n}^2 \hat{O}_{1n}^{tt} + \left( \sum_{n \neq l}^{b} m_{\Delta_n}^2 \hat{O}_{1nl}^{tt} \right) + \text{all possible permutations}
+ \frac{1}{2} \sum_{i=1}^{3} \lambda_i^{tt} \left[ Q_i^{tt} \right]^2 + \frac{1}{2} \sum_{k<j}^{2} \sum_{k=1}^{3} \lambda_{2jk}^{tt} Q_{j}^{tt} Q_{k}^{tt} + \frac{1}{2} \sum_{n=a}^{b} \lambda_{3n}^{tt} \left\{ [O_{1n}^{tt}]^2 + [O_{2n}^{tt}]^\dagger O_{2n}^{tt} + O_1(T_{sn}^{tt}, T_{sn}^{tt}) \right\}
+ \frac{1}{2} \sum_{n \neq l}^{b} \lambda_{3nl}^{tt} \left\{ [\hat{O}_{1nl}^{tt}]^2 + [\hat{O}_{2nl}^{tt}]^\dagger \hat{O}_{2nl}^{tt} + h.c. \right\} + \frac{1}{2} \sum_{n \neq l}^{b} \lambda_{3nl}^{tt} \left\{ [\hat{O}_{1nl}^{tt}]^\dagger \hat{O}_{1nl}^{tt} + O_1(T_{sn}^{tt}, T_{sn}^{tt}) \right\}
+ \frac{1}{2} \sum_{j=1}^{3} \sum_{n=a}^{b} \lambda_{4jn}^{tt} \left\{ \left( \Delta_{j}^L \Delta_{j}^L \right) O_{1n}^{tt} \right\} + \sum_{j=1}^{3} \sum_{n \neq l}^{b} \lambda_{4nl}^{tt} \left\{ \left( \Delta_{j}^L \Delta_{j}^L \right) \hat{O}_{1nl}^{tt} + h.c. \right\}
\]
The terms in the scalar potential involving scalar fields of identical SU

B.5 Inter-sector terms in the scalar potential:

The terms in the scalar potential involving scalar fields of identical SU(2)_L behavior are already taken into account. Apart from them, the scalar potential will also receive contributions from terms generated by combining scalars of two different SU(2)_L sectors that constitute the main objective of the following discussion. In this category the combinations of the SU(2)_L singlet scalars with that belonging to either of the doublet or the triplet sector. The other variety of inter-sector terms – doublet-triplet type – are not included. This seems to be a reasonable approximation as the vevs of the singlet fields are the largest.

B.5.1 Singlet-Doublet inter-sector terms:

Let us consider the combinations:

\[ \bar{T}^{ss}_{sp} \equiv T_s(\hat{\Delta}^R_p, \hat{\Delta}^R_q); \quad \bar{T}^{ss}_{spq} \equiv T_s(\hat{\Delta}^R_p, \hat{\Delta}^R_q) \text{ and } \bar{T}^{dd}_{s} \equiv T_s(\Phi, \Phi^\dagger), \quad (p, q = a, b, c \text{ and } p \neq q) \] (B.20)

and

\[ O^{sd}_{1sp} \equiv O_1(\bar{T}^{dd}_{sp}, \bar{T}^{ss}_{sp}); \quad \hat{O}^{sd}_{1sp} \equiv O_1(\bar{T}^{dd}_{sp}, \bar{T}^{ss}_{sp}); \quad \hat{O}^{sd}_{p\gamma} \equiv O_\gamma(\hat{\Delta}_p^R, \hat{T}^{dd}_{s}), \quad (\gamma = 1, 2, 3) \text{ and } (p, q = a, b, c \text{ with } p \neq q). \] (B.21)

Using this notations:

\[ V_{sd} = \frac{1}{2} \sum_{i=1}^{3} \left[ \lambda^{sd}_{si} \left( Q^{ss}_{i} Q_{i}^{ss} \right) + \left( \lambda^{dd}_{si} Q^{ss}_{i} O_{i}^{dd} \right) \right] + \frac{1}{2} \sum_{p=a}^{c} \lambda^{sd}_{3p} \left[ Q_{\gamma}^{dd} O_{i}^{dd} \right] + \frac{1}{2} \sum_{p\neq q; p, q = a}^{c} \left[ Q_{\gamma}^{dd} \hat{O}^{ss}_{1pq} \right] \]

\[ + \sum_{p=a}^{c} \left[ \lambda^{sd}_{4p} \left( \hat{O}^{sd}_{1\gamma} \Delta^R_1 + h.c. \right) + \lambda^{sd}_{5p} \left( \hat{O}^{sd}_{2p} \Delta^R_2 + h.c. \right) + \lambda^{sd}_{6p} \left( \hat{O}^{sd}_{3p} \Delta^R_3 + h.c. \right) \right] \]

\[ + \frac{1}{2} \sum_{p=a}^{c} \lambda^{sd}_{7p} \left[ O_{1p}^{dd} O_{1p}^{ss} + \left( O_{2p}^{ss} \right)^{\dagger} O_{2p}^{dd} + \left( O_{2p}^{dd} \right)^{\dagger} O_{2p}^{ss} + O_{1p}^{dd} \right] \]
\begin{align}
+ \frac{1}{2} \sum_{p \neq q; p,q = a} \lambda_{7pq}^{sd} \left[ O_{1pq}^{dd} \hat{O}_{1pq}^{ss} + \{ \hat{O}_{2pq}^{ss} \}^{\dagger} + O_{2pq}^{dd} + \{ O_{2pq}^{ss} \}^{\dagger} \right].
\end{align}

In the last two terms a simplifying assumption of using a common couplings \( \lambda_{7pq}^{sd} \) and \( \lambda_{7pq}^{td} \) for the terms in the scalar potential that are generated from various combinations of \((\Phi^{\dagger} \Phi)(\Delta^{R}_{l} \Delta^{R})\), all four of the fields involved being triplets of \( A_{4} \).

B.5.2 Singlet-Triplet inter-sector terms:

In this case the following combinations comes into play:

\begin{align}
\tilde{T}_{sn}^{tt} &\equiv T_{s}(\Delta_{n}^{L}, \Delta_{n}^{L}); \ \tilde{T}_{sn}^{ss} \equiv T_{s}(\Delta_{n}^{L}, \Delta_{n}^{L}); \ O_{1snp}^{ts} \equiv O_{1}(\tilde{T}_{sn}^{tt}, \tilde{T}_{sp}^{ss}); \ \hat{O}_{1snpq}^{ts} \equiv O_{1}(\tilde{T}_{sn}^{tt}, \tilde{T}_{spq}^{ss}); \\
\hat{O}_{1snlpq}^{ts} \equiv O_{1}((\tilde{T}_{sn}^{tt}, \tilde{T}_{sp}^{ss}); \ O_{tsnlpq}^{ts} \equiv O_{1}(\tilde{T}_{snlpq}^{tt}, \tilde{T}_{spq}^{ss}); \ O_{1snlpq}^{ts} \equiv O_{1}(\tilde{T}_{snlpq}^{tt}, \Delta_{n}^{L}); \\
\dot{O}_{tsnlpq}^{ts} \equiv O_{1}(\tilde{T}_{spq}^{tt}, \Delta_{n}^{L}); \ \hat{O}_{tnlp}^{ts} \equiv O_{1}(\tilde{T}_{tnlp}^{tt}, \Delta_{n}^{L}).
\end{align}

where \( (\gamma = 1, 2, 3); \ (p, q = a, b, c) \) and \( (n, l = a, b) \). Needless to mention \( p \neq q \) and \( n \neq l \).

Following the convention introduced already:

\begin{align}
O_{1snlp}^{ts} \equiv O_{1}(\Delta_{p}^{R}, \Delta_{n}^{L}); \ \tilde{O}_{tnlp}^{ts} \equiv O_{1}(\Delta_{p}^{R}, \Delta_{n}^{L}); \ (\gamma = 1, 2, 3); \ (p = a, b, c) \) and \( (n = a, b) \).
\end{align}

The inter-sector potential for this case is given by:

\begin{align}
V_{ts} &= \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \lambda_{ij}^{s} \left[ Q_{i}^{ss} Q_{j}^{lt} \right] + \frac{1}{2} \sum_{j=1}^{3} \sum_{n=1}^{b} \lambda_{2jn}^{s} \left[ (Q_{j}^{ss} O_{1n}^{lt} + h.c.) \right] + \frac{1}{2} \sum_{j=1}^{3} \sum_{n=1}^{b} \lambda_{2jn}^{s} \left[ (Q_{j}^{ss} O_{1n}^{lt} + h.c.) \right] \\
&+ \frac{1}{2} \sum_{i=1}^{3} \sum_{n=1}^{a} \lambda_{i1}^{s} \left[ Q_{i}^{ts} O_{1n}^{ss} \right] + \frac{1}{2} \sum_{i=1}^{3} \sum_{n=1}^{b} \lambda_{3n}^{s} \left[ Q_{i}^{ts} O_{1n}^{ss} \right] \\
&+ \frac{1}{2} \sum_{p=a}^{3} \sum_{n=1}^{a} \lambda_{1nmp}^{s} \left[ \hat{O}_{1n}^{ts} O_{1p}^{ss} + \{ \hat{O}_{2p}^{ss} \}^{\dagger} + O_{2p}^{ss} \right] + \frac{1}{2} \sum_{p=a}^{3} \sum_{n=1}^{a} \lambda_{1nmp}^{s} \left[ \hat{O}_{1n}^{ts} O_{1p}^{ss} + \{ \hat{O}_{2p}^{ss} \}^{\dagger} + O_{2p}^{ss} \right] \\
&+ \frac{1}{2} \sum_{p=a}^{3} \sum_{n=1}^{a} \lambda_{1nmp}^{s} \left[ \hat{O}_{1n}^{ts} O_{1p}^{ss} + \{ \hat{O}_{2p}^{ss} \}^{\dagger} + O_{2p}^{ss} \right] + \frac{1}{2} \sum_{p=a}^{3} \sum_{n=1}^{a} \lambda_{1nmp}^{s} \left[ \hat{O}_{1n}^{ts} O_{1p}^{ss} + \{ \hat{O}_{2p}^{ss} \}^{\dagger} + O_{2p}^{ss} \right] \\
&+ \frac{1}{2} \sum_{p=a}^{3} \sum_{n=1}^{a} \lambda_{1nmp}^{s} \left[ \hat{O}_{1n}^{ts} O_{1p}^{ss} + \{ \hat{O}_{2p}^{ss} \}^{\dagger} + O_{2p}^{ss} \right] + \frac{1}{2} \sum_{p=a}^{3} \sum_{n=1}^{a} \lambda_{1nmp}^{s} \left[ \hat{O}_{1n}^{ts} O_{1p}^{ss} + \{ \hat{O}_{2p}^{ss} \}^{\dagger} + O_{2p}^{ss} \right] \\
&+ \frac{1}{2} \sum_{p=a}^{3} \sum_{n=1}^{a} \lambda_{1nmp}^{s} \left[ \hat{O}_{1n}^{ts} O_{1p}^{ss} + \{ \hat{O}_{2p}^{ss} \}^{\dagger} + O_{2p}^{ss} \right] + \frac{1}{2} \sum_{p=a}^{3} \sum_{n=1}^{a} \lambda_{1nmp}^{s} \left[ \hat{O}_{1n}^{ts} O_{1p}^{ss} + \{ \hat{O}_{2p}^{ss} \}^{\dagger} + O_{2p}^{ss} \right] \\
&+ \frac{3}{2} \sum_{i=1}^{3} \sum_{p=a}^{3} \lambda_{5ppm}^{s} \left[ \hat{O}_{1n}^{ts} \Delta_{i}^{lt} + h.c. \right] + \frac{3}{2} \sum_{i=1}^{3} \sum_{p=a}^{3} \lambda_{5ppm}^{s} \left[ \hat{O}_{1n}^{ts} \Delta_{i}^{lt} + h.c. \right] \\
&+ \frac{3}{2} \sum_{i=1}^{3} \sum_{p=a}^{3} \lambda_{5ppm}^{s} \left[ \hat{O}_{1n}^{ts} \Delta_{i}^{lt} + h.c. \right] + \frac{3}{2} \sum_{i=1}^{3} \sum_{p=a}^{3} \lambda_{5ppm}^{s} \left[ \hat{O}_{1n}^{ts} \Delta_{i}^{lt} + h.c. \right]
\end{align}
It must be noted that while writing the last \( \lambda^2 \) terms the different couplings corresponding to the combinations of \( O_{1np}^s \) with \( (\Delta^L_i \Delta^R_j) \) and \( \bar{O}_{1np}^s \) with \( (\Delta^L_i \Delta^R_j) \) are set to be equal.

### B.6 The conditions for minimization:

With the scalar potential in hand it is necessary to derive the conditions for which the particular \textit{vev} configurations used in this model – see Eqs. (3), (4) and (5) and Table 3 – corresponds to the local minimum. For immediate reference the \textit{vevs} are:

\[
\langle \Phi^0 \rangle = \frac{v}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \eta^0 \rangle = u, \quad \langle \hat{\Delta}^L_{a} \rangle = v_{La} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \langle \hat{\Delta}^L_{b} \rangle = v_{Lb} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \Delta^L_{1} \rangle = \langle \Delta^L_{2} \rangle = \langle \Delta^L_{3} \rangle = u_{L},
\]

\[
\langle \hat{\Delta}^R_{a} \rangle = v_{Ra} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \hat{\Delta}^R_{b} \rangle = v_{Rb} \begin{pmatrix} 1 \\ \omega \\ \omega \end{pmatrix}, \quad \langle \hat{\Delta}^R_{c} \rangle = v_{Rc} \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix},
\]

\[
\langle \Delta^R_{1} \rangle = u_{1R}, \quad \langle \Delta^R_{2} \rangle = u_{2R}, \quad \langle \Delta^R_{3} \rangle = u_{3R}.
\]

where the \( SU(2)_L \) nature of the scalars has been suppressed.

Eq. (B.26) shows that the \( A_4 \) triplet fields – \( \hat{\Delta}^L \) and \( \hat{\Phi} \) – have \textit{vev} configurations that have been verified to be the \textit{global minima} in [25]. This result was for a single \( A_4 \) triplet considered in isolation. In the current scenario since many other fields are involved, it is not straight-forward to directly adopt the conclusions of [25].

The conditions for which the \textit{vev} configurations shown in Eqs. (3), (4) and (5) correspond to minimum are shown sector by sector.
For minima of the scalar potential, the first derivatives of the scalar potential with respect to the vevs have to vanish and the second derivatives have to satisfy some conditions. Since the scalar sector is very rich, the expressions look very complicated. The conditions arising by setting the first derivatives to be zero have been discussed for each of the $SU(2)_L$ sectors. As a sample, constraints coming from the second derivatives have been shown only for the $SU(2)_L$ singlet sector. Similar exercise can be carried out for the other $SU(2)_L$ sectors but are not presented here.

**B.6.1 $SU(2)_L$ singlet sector:**

The $SU(2)_L$ singlet vevs are much larger than those of the doublet and triplet scalars. Thus it is safe to neglect the contributions to the minimization equations from the inter-sector terms.

Let us remind ourselves that $v_{R_{p}}$ ($p = a, b, c$) are real and define:

\[ \tilde{v}_{Ra1} \equiv v_{Ra}, \quad \tilde{v}_{Ra2} \equiv v_{Ra}, \quad \tilde{v}_{Ra3} \equiv v_{Ra}; \]
\[ \tilde{v}_{Rb1} \equiv v_{Rb}, \quad \tilde{v}_{Rb2} \equiv v_{Rb} \omega, \quad \tilde{v}_{Rb3} \equiv v_{Rb} \omega^2; \]
\[ \tilde{v}_{Rc1} \equiv v_{Rc}, \quad \tilde{v}_{Rc2} \equiv v_{Rc} \omega, \quad \tilde{v}_{Rc3} \equiv v_{Rc} \omega^2. \] (B.29)

For ease of presentation, let us set the following masses and couplings equal:

\[ m_{\Delta_1}^2 = m_{\Delta_2}^2 = m_{\Delta_3}^2 = m_{R_{11}}^2; \quad m_{\Delta_8}^2 = m_{\Delta_9}^2 = m_{R_{12}}^2 = m_{R_{13}}^2; \quad m_{ab} = m_{ac} = m_{bc} = m_{R_{14}}; \]
\[ \lambda_{i}^a = \lambda_{i}^b = \lambda_{i}^c \quad \forall \quad (i = 1, 2, 3); \]
\[ \lambda_{i}^{ab} = \lambda_{i}^{ac} = \lambda_{i}^{bc}; \quad \lambda_{i}^{ip} = \lambda_{i}^{qi} \quad \forall \quad (p = a, b, c) \quad \text{and} \quad (i = 1, 2, 3); \]
\[ \lambda_{11}^{ab} = \lambda_{11}^{ac} = \lambda_{11}^{bc}; \quad \lambda_{5}^{ab} = \lambda_{5}^{ac} = \lambda_{5}^{bc}; \quad \lambda_{5}^{ip} = \lambda_{5}^{qi} \quad \forall \quad (p = a, b, c) \quad \text{and} \quad (i = 1, 2, 3); \]
\[ \lambda_{81}^{ab} = \lambda_{81}^{ac} = \lambda_{81}^{bc}; \quad \lambda_{82}^{ab} = \lambda_{82}^{ac} = \lambda_{82}^{bc}; \quad \lambda_{83}^{ab} = \lambda_{83}^{ac} = \lambda_{83}^{bc}; \quad \lambda_{8}^{ip} = \lambda_{8}^{qi} \quad \forall \quad (p = a, b, c) \quad \text{and} \quad (i = 1, 2, 3); \]
\[ \lambda_{91}^{ab} = \lambda_{91}^{ac} = \lambda_{91}^{bc}; \quad \lambda_{92}^{ab} = \lambda_{92}^{ac} = \lambda_{92}^{bc}; \quad \lambda_{93}^{ab} = \lambda_{93}^{ac} = \lambda_{93}^{bc}. \] (B.30)

With the help of the singlet sector potential in Eq. [B.10], the equalities in Eq. (B.30) and the vev in Eqs. (3), (4) and (5) one can obtain:

\[ \frac{\partial V_{\text{singlet}}}{\partial v_{1R}} = 0 \Rightarrow m_{R_{11}}^{2} u_{1R} + \lambda_{5}^{a} (u_{1R} u_{1R}^{*}) + \lambda_{5}^{b} [(u_{2R} u_{2R}^{*}) + (u_{3R} u_{3R}^{*})] \]
\[ + \quad \frac{3}{2} \lambda_{5}^{a} u_{1R} [v_{R_{a}}^{2} + v_{R_{b}}^{2} + v_{R_{c}}^{2}] + 3 \lambda_{5}^{a} v_{R_{a}}^{3} \]
\[ - \quad 3 \lambda_{5}^{a} v_{R_{a}} (v_{R_{b}}^{2} + v_{R_{c}}^{2}) + 6 \lambda_{5}^{a} u_{1R} v_{R_{a}}^{2} + 6 \lambda_{5}^{a} u_{1R} v_{R_{b}} v_{R_{c}} \]
\[ + \quad 3 \lambda_{5}^{a} (u_{3R} v_{R_{b}}^{2} + u_{2R} v_{R_{b}}^{2}) + 3 \lambda_{5}^{a} [v_{R_{a}} (u_{2R} v_{R_{b}}^{2} + u_{3R} v_{R_{c}}^{2})] = 0, \] (B.31)

\[ \frac{\partial V_{\text{singlet}}}{\partial \tilde{v}_{R_{a1}}^{*}} = \frac{\partial V_{\text{singlet}}}{\partial v_{R_{a1}}^{*}} = 0. \]
Further mixed derivatives such as:
\[ v_{\text{singlet-doublet inter-sector}} \]
are considered. Let us define
\[ V_{B.6.2} \]
\[ SU \]
purpose only and the minimization equations are achieved by setting the different couplings equal. For this sector contributions from both the doublet sector itself – Eq. (B.13) – together with the

Besides the first derivatives discussed above, second derivatives are also needed to established minimality. For example,
\[ \frac{\partial^2 V_{\text{singlet}}}{\partial u_{1R}^2} > 0 \]
\[ \Rightarrow \lambda_1^s u_{1R}^2 + 6 \lambda_5^s v_{Ra}^2 + 6 \lambda_5^s v_{RB} v_{RC} > 0 \]  
(B.33)
and
\[ \frac{\partial^2 V_{\text{singlet}}}{\partial v_{Ra}^2} > 0 \]
\[ \Rightarrow \lambda_3^s v_{Ra}^2 + 4 \lambda_3^s (v_{RB}^2 + v_{RC}^2) + 2 \lambda_5^s (u_{1R}^2 + u_{2R}^2 + u_{3R}^2) + 2 \lambda_8^s (u_{2R} u_{3R} + u_{1R} u_{2R} + u_{1R} u_{3R}) > 0 \]  
(B.34)
Further mixed derivatives such as:
\[ \frac{\partial^2 V_{\text{singlet}}}{\partial u_{1R}^2 v_{Ra1}} = \frac{\lambda_4^s}{2} v_{Ra} u_{1R} + \tilde{\lambda}_4^s (v_{RB} + v_{RC}) u_{1R} + \lambda_5^s v_{Ra}^2 - \tilde{\lambda}_5^s v_{Ra} (v_{RB} + v_{RC}) \]  
(B.35)
are also necessary to establish minimality in the most general case. The results presented for the first and second derivatives are calculated using the most general expression of the scalar potential in terms of the \( v_{\text{vers}} \) and putting \( (v_{Ra1} = v_{Ra2} = v_{Ra3} = v_{Ra}), (v_{RB1} = v_{RB}, v_{RB2} = \omega v_{RB}, v_{RB3} = \omega^2 v_{RB}) \)
and \( (v_{RC1} = v_{RC}, v_{RC2} = \omega^2 v_{RC}, v_{RC3} = \omega v_{RC}) \) where \( v_{Ra}, v_{RB}, v_{RC} \) are real. Needless to mention that
\[ \tilde{v}_{Rp_i} = \tilde{v}_{Rp_i} \]  
for \( (p = b, c \text{ and } i = 1) \). Similar equations can be obtained by minimizing the potential wrt \( u_{2R}, u_{3R}, \tilde{v}_{Ra2}, \tilde{v}_{Ra3} \) and \( \tilde{v}_{Ro i} \) for \( (p = b, c) \) and \( (i = 1, 2, 3) \). For the sake of brevity those are not mentioned. Similar strategy will be adopted for the \( SU(2)_L \) doublet and \( SU(2)_L \) triplet sector. It is worth noting that this exercise for all the three sectors are for illustrative purpose only and the minimization equations are achieved by setting the different couplings equal.

**B.6.2 SU(2)_L doublet sector:**

For this sector contributions from both the doublet sector itself – Eq. (B.13) – together with the singlet-doublet inter-sector are considered. Let us define \( V_{\tilde{D}} = V_{\text{doublet}} + V_{\text{sd}} \). Also let us call \( \langle \Phi_i \rangle \equiv v_i \)
where \( v_1 = v_2 = v_3 = \frac{v}{v^3} \), \( v \) being real.
The following couplings are set to be equal:

\[
\begin{align*}
\lambda_{1i}^{sd} &= \lambda_{1}^{sd}, \quad \lambda_{2i}^{sd} = \lambda_{2}^{sd} \quad \forall \ (i = 1, 2, 3); \\
\lambda_{3i}^{sd} &= \lambda_{3}^{sd}, \quad \lambda_{4i}^{sd} = \lambda_{4}^{sd}, \quad \lambda_{5i}^{sd} = \lambda_{5}^{sd}, \quad \lambda_{6i}^{sd} = \lambda_{6}^{sd}, \quad \lambda_{6p}^{sd} = \lambda_{7}^{sd} \quad \forall \ (p = a, b, c);
\end{align*}
\]

(B.36)

For the vevs in Eqs. (B.26), (B.27) and (B.28) correspond to the minimum of the scalar potential it is necessary to satisfy the following conditions:

\[
\frac{\partial V_{\phi}}{\partial u^{*}} = 0 \Rightarrow u \left[ m_{u}^{2} + \lambda_{1}^{sd} u^{*}u + \lambda_{2}^{sd} v^{2} + \frac{3}{2} \sum_{i=1}^{c} (u_{i R} u_{i R}) + \frac{3}{2} \sum_{p=a}^{c} v_{R p}^{2} \right] = 0. \tag{B.37}
\]

and

\[
\frac{\partial V_{\phi}}{\partial v_{1}^{*}} = 0 \Rightarrow \frac{v}{\sqrt{3}} \left[ m_{u}^{2} + 2 \lambda_{2}^{sd} v^{2} + \frac{\lambda_{3}^{sd}}{2} u^{*}u + \frac{3}{2} \sum_{i=1}^{c} (u_{i R} u_{i R}) + \frac{3}{2} \sum_{p=a}^{c} v_{R p}^{2} \right] = 0. \tag{B.38}
\]

In order to satisfy Eqs. (B.37) and (B.38) some degree of fine-tuning is necessary that involve both \( SU(2)_L \) doublet and singlet vevs of varying magnitudes. Similar equations can be obtained by minimizing the potential wrt \( v_{2}^{*} \) and \( v_{3}^{*} \).

**B.6.3 \( SU(2)_L \) triplet sector:**

In analogy to the doublet sector, let us define \( V_{\phi} = V_{\text{triplet}} + V_{ts} \) using Eqs. (B.19) and (B.25). Let us also recall, \( v_{La1} = v_{La}, \ v_{La2} = v_{La3} = 0 \) and \( v_{Lb1} = v_{Lb2} = v_{Lb3} = v_{Lb} \).

This sector has several couplings involved. For simplicity of presentation, let us implement the following choices:

\[
\begin{align*}
m_{\Delta_{i}^{L}} &= m_{\Delta_{i}^{L}} = m_{\Delta_{i}^{b}} = m_{t1}; \quad m_{\Delta_{L}} = m_{\Delta_{b}} = m_{t2}; \quad m_{ab} = m_{t3}; \quad \lambda_{i}^{L} = \lambda_{i}^{L_1}, \quad \forall \ (i = 1, 2, 3); \\
\lambda_{221}^{L} &= \lambda_{232}^{L} = \lambda_{231}^{L} = \lambda_{2}^{L}; \quad \lambda_{3}^{L} = \lambda_{3}^{L}; \quad \lambda_{5}^{L} = \lambda_{5}^{L}; \quad \lambda_{6}^{L} = \lambda_{6}^{L}; \quad \lambda_{ab}^{L} = \lambda_{ab}^{L}; \quad \lambda_{3ab}^{L} = \lambda_{3ab}^{L}; \\
\lambda_{4jn}^{L} &= \lambda_{4jn}^{L}; \quad \lambda_{8jn}^{L} = \lambda_{8jn}^{L}; \quad \lambda_{9jn}^{L} = \lambda_{9jn}^{L}; \quad \lambda_{a}^{L} = \lambda_{a}^{L}; \quad \forall \ (j = 1, 2, 3), \ (n, l = a, b) \text{ and } n \neq l; \\
\lambda_{5a}^{L} &= \lambda_{5b}^{L} = \lambda_{5}^{L}; \quad \lambda_{6a}^{L} = \lambda_{6}^{L}; \quad \lambda_{6b}^{L} = \lambda_{6}^{L}; \quad \lambda_{7a}^{L} = \lambda_{7}^{L}; \quad \lambda_{7b}^{L} = \lambda_{7}^{L}; \\
\lambda_{11ij}^{L} &= \lambda_{11}^{L}, \quad \forall \ (i, j = 1, 2, 3) \text{ and } i \neq j; \quad \lambda_{2jn}^{L} = \lambda_{2jn}^{L}; \quad \forall \ (j = 1, 2, 3), \ (n, l = a, b) \text{ and } n \neq l;
\end{align*}
\]
In order to minimize $V_{\mathcal{F}}$ such that one can arrive to 

$$V_{\mathcal{F}}$$

the following conditions are to be ensured:

$$\frac{\partial V_{\mathcal{F}}}{\partial u_{L1}^{\ast}} = 0$$

Also one gets;

$$\frac{\partial V_{\mathcal{F}}}{\partial v_{La}} = 0$$

It is worth noticing that certain fine-tuning is essential to satisfy Eqs. (B.40) - (B.41). Also similar equations can be obtained by minimizing the potential wrt $u_{Lj}^{\ast}$ where ($j = 2, 3$), $v_{Lni}^{\ast}$ where for $n = b$ one has ($i = 1, 2, 3$) and for $n = a$ we have ($i = 2, 3$). Those are not mentioned here. This exercise is performed to illustrate the scenario in a simplified limit achieved by setting several masses and couplings to be equal.
C Appendix: Flavour basis form of the mass matrices

Mass matrices expressed in the Lagrangian basis in Eqs. (6) and (7) can be transformed to simpler forms in the flavour basis as in Eqs. (9) and (10) with the help of a unitary transformation written in Eq. (C.1). Certain straightforward algebraic calculations related to this derivation of the forms the mass matrices in the flavour basis is furnished in this Appendix.

The Lagrangian in Eq. (2) produces the following mass matrix for the charged leptons and the left-handed Majorana neutrinos:

\[
M_{\mu L} = \frac{v}{\sqrt{3}} \begin{pmatrix}
ym_1 & y_2 & y_3 \\
y_1 & y_2 & y_3 \\
y_1 & y_2 & y_3
\end{pmatrix},
\]

where, the Yukawa coupling \( Y_2 \) is chosen to be equal to \( Y_3 \). Also, \( y_1 = m_e \), \( y_2 = m_\mu \), \( y_3 = m_\tau \) is satisfied. The dominant Type-II see-saw component of the neutrino mass matrix, \( M_{\nu L} \), gives rise to the atmospheric splitting and maximal atmospheric mixing but is devoid of solar splitting and is therefore characterized by two masses \( m_{1}^{(0)} \) and \( m_{3}^{(0)} \). It is useful to define \( m^\pm \equiv m_{3}^{(0)} \pm m_{1}^{(0)} \). Thus \( m^- \) is positive (negative) for normal (inverted) ordering. Certain identifications of the \( v \) and Yukawa products are essential viz. \( 3(Y_1 + Y_2 Y_3)u_L = (m_3^{(0)} + m^+) \), \( 6(Y_1 - Y_2 Y_3)u_L = \hat{\chi}_a v_{La} = m^+ \) and \( 3\hat{Y}_b^L v_{Lb} = -2m^- \) to generate the desired structures of the mass matrices as presented in Eq. (9). The neutrino Dirac mass matrix and the right-handed Majorana neutrino mass matrix in the Lagrangian basis are:

\[
M_D = fu \, I, \quad M_{\nu R} = m_R \begin{pmatrix}
\chi_1 & \chi_6 & \chi_5 \\
\chi_6 & \chi_2 & \chi_4 \\
\chi_5 & \chi_4 & \chi_3
\end{pmatrix}, \quad (C.2)
\]

where,

\[
m_{R1} & \equiv (Y_1^R u_{1R} + Y_2^R u_{2R} + Y_3^R u_{3R}) \\
m_{R2} & \equiv (Y_1^R u_{1R} + \omega Y_2^R u_{2R} + \omega^2 Y_3^R u_{3R}) \\
m_{R3} & \equiv (Y_1^R u_{1R} + \omega Y_2^R u_{2R} + \omega^3 Y_3^R u_{3R}) \\
m_{R4} & \equiv \frac{1}{2}(Y_a^R v_{Ra} + \hat{Y}_b^R v_{Rb} + \hat{\chi}_c^R v_{Re}) \\
m_{R5} & \equiv \frac{1}{2}(Y_a^R v_{Ra} + \omega \hat{Y}_b^R v_{Rb} + \omega^2 \hat{\chi}_c^R v_{Re}) \\
m_{R6} & \equiv \frac{1}{2}(Y_a^R v_{Ra} + \omega^2 \hat{Y}_b^R v_{Rb} + \omega \hat{\chi}_c^R v_{Re}). \quad (C.3)
\]

Here \( m_R \) is the right-handed Majorana neutrino mass scale and \( \chi_i \) are dimensionless \( \mathcal{O}(1) \) quantities.

In order to achieve the right-handed Majorana neutrino mass matrix of the form expressed in Eq. (10), the \( v \) and Yukawa couplings products have to obey:

\[
Y_1^R u_{1R} = m_R(r_{11} + 2r_{23}), \quad Y_2^R u_{2R} = m_R(r_{22} + 2r_{13}), \quad Y_3^R u_{3R} = m_R(r_{33} + 2r_{12}) \\
\hat{Y}_a^R v_{Ra} = 2m_R(r_{11} - r_{23}), \quad \hat{Y}_b^R v_{Rb} = 2m_R(r_{22} - r_{13}) \quad \text{and} \quad \hat{Y}_c^R v_{Re} = 2m_R(r_{33} - r_{12}). \quad (C.4)
\]

The \( r_{ij} \) in Eq. (C.4) are given by:

\[
r_{11} \equiv \sqrt{2}b \sin 2\theta_{12}^0 + a \sin^2 \theta_{12}^0.
\]

28
\[
\begin{align*}
  r_{22} & \equiv -\sqrt{2}b \sin \theta_{12} - \frac{b}{2} \sin 2\theta_{12} - a \cos \theta_{12} + \frac{a}{2} \cos^2 \theta_{12} + \frac{a}{2}, \\
  r_{33} & \equiv -\frac{b}{\sqrt{2}} \sin \theta_{12} - \sqrt{2}b \sin \theta_{12} + a \cos \theta_{12} + \frac{a}{2} \cos^2 \theta_{12} + \frac{a}{2}, \\
  r_{12} & \equiv b \cos 2\theta_{12} + \frac{a}{2\sqrt{2}} \sin^2 \theta_{12} + b \cos \theta_{12} - \frac{a}{\sqrt{2}} \sin \theta_{12}, \\
  r_{13} & \equiv -b \cos 2\theta_{12} - \frac{a}{2\sqrt{2}} \sin^2 \theta_{12} + b \cos \theta_{12} - \frac{a}{\sqrt{2}} \sin \theta_{12}, \\
  r_{23} & \equiv \frac{b}{2} \sin 2\theta_{12} - \frac{a}{2} \cos^2 \theta_{12} + \frac{a}{2}.
\end{align*}
\]

(C.5)

where \(a\) and \(b\) are dimensionless quantities of \(\mathcal{O}(1)\). The charged lepton mass matrix is not diagonal in the Lagrangian basis. In order to go to a basis in which the charged lepton mass matrix \(M_{e\mu\tau}\) is diagonal a unitary transformation \(U_L\) is applied on the left-handed lepton doublets. The transformation \(V_R\) is applied on the right-handed neutrino singlets of \(SU(2)_L\) such that the Dirac neutrino mass matrix remains proportional to identity in this transformed basis as well. This basis in which the charged lepton mass matrix is diagonal and the entire lepton mixing is dictated by the neutrino sector is called the \textit{flavour basis}. The right-handed charged leptons were kept unchanged. The transformation matrices are given by:

\[
  U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} = V_R.
\]

(C.6)

The mass matrices in the flavour basis are:

\[
M_{\text{flavour}}^{e\mu\tau} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad M_{\nu_L}^{\text{flavour}} = \frac{1}{2} \begin{pmatrix} 2m_1^{(0)} & 0 & 0 \\ 0 & m^+ & m^- \\ 0 & m^- & m^+ \end{pmatrix},
\]

(C.7)

\[
M_D = f u \mathbb{1}, \quad M_{\nu_R}^{\text{flavour}} = \frac{m_R}{4ab} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{pmatrix}.
\]

(C.8)

One can identify \(f u = m_D\) where \(m_D\) is the scale of the Dirac masses of the neutrinos. Type-I see-saw mechanism contribution is given by the matrices in Eq. (C.8).

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