The Camassa-Holm equation as a geodesic flow on the diffeomorphism group.

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Misiolek has shown that the Camassa-Holm (CH) equation is a geodesic flow on the Bott-Virasoro group. In this paper it is shown that the Camassa-Holm equation for the case \( \kappa = 0 \) is the geodesic spray of the weak Riemannian metric on the diffeomorphism group of the line or the circle obtained by right translating the \( H^1 \) inner product over the entire group. This paper uses the right-trivialisation technique to rigorously verify that the Euler-Poincaré theory for Lie groups can be applied to diffeomorphism groups. The observation made in this paper has led to physically meaningful generalizations of the CH-equation to higher dimensional manifolds (see Refs. 2 and 3).

I. INTRODUCTION

Camassa and Holm derived a new completely integrable dispersive shallow water equation that is bi-Hamiltonian and thus possesses an infinite number of conservation laws in involution. The Camassa-Holm (CH) equation is obtained by using an asymptotic expansion directly in the Hamiltonian for Euler’s equations in the shallow water regime. Below another remarkable property of the CH-equation is shown. Namely, the CH-equation can be realized as a geodesic equation on a Riemannian manifold on which the methods of infinite-dimensional geometry can be applied. Section II illustrates the main result by formally applying the Euler-Poincaré theory for Lie groups to a continuum mechanical system. The next section verifies the legitimacy of the application. In addition, Section III contains independent results on the Riemannian geometry of a \( C^1 \)- manifold which is also a topological group with \( C^1 \) right translation. Using the right-trivialisation technique, a global Christoffel map is introduced, and formulas are derived for a spray and a Levi-Civita connection similar to the finite-dimensional case. The method is inspired by the theory of affine connections on parallelisable manifolds developed by Marsden, Ratiu and Raugel. At the end of section III, a version of the Euler-Poincare-Arnold theorem for a diffeomorphism group is verified. Section IV utilizes the results of section III to demonstrate that the CH-equation is a geodesic flow of the right-invariant metric on the diffeomorphism group of \( \mathbb{R} \) or of the circle. Section V addresses uniqueness and existence issues for solutions of the CH-equation. Observations made in this paper have \( n \)-dimensional generalizations to the volume preserving diffeomorphism group of a Riemannian manifold which lead to a new class of models for mean hydrodynamic motion. See Ref. 2 for application of this to numerous fluids models, such as those in geophysics, and see Ref. 3 for the development of the geometry and curvature of volume preserving diffeomorphism groups with right invariant \( H^1 \) metric. For Riemannian manifolds with boundary, new subgroups of the diffeomorphism group have been established which give rise to remarkable theorems on the limit of zero viscosity. See Ref. 7 for a detailed account.

II. FORMAL DERIVATION.

In this section we illustrate the main result of the paper by formally applying the pure Euler-Poincaré theorem on the right-invariant Lagrangians on Lie groups (see Marsden and Ratiu and references therein) to the case of the diffeomorphism group of a certain Sobolev class \( H^s, s > \frac{3}{2} \). The diffeomorphism group is not a Lie group (left translation and inversion are not smooth, only continuous, whereas right translation is smooth), and the pure Euler-Poincaré theorem strictly does not apply. However, we will demonstrate in the following sections that the formal derivation given in this section can be rigorously justified using standard trivialisation techniques.

Let \( M \) be the flat circle \( S^1 \) or the real line \( \mathbb{R} \). \( \text{Diff}(M) \equiv \mathcal{D} \) denotes the diffeomorphism group of \( M \) of some given Sobolev class. The case \( M = S^1 \) corresponds to periodic boundary conditions. For the case \( M = \mathbb{R} \), the chosen Sobolev space automatically imposes appropriate decay hypotheses at infinity. Under these boundary conditions, \( \text{Diff}(M) \) is a smooth infinite-dimensional manifold and a topological group relative to the induced manifold topology. \( \mathcal{X}(M) \) denotes the vector fields on \( M \) of the same differentiability class. Formally, this is the right Lie algebra of \( \text{Diff}(M) \), e.g., the standard left Lie algebra bracket is minus the usual Lie bracket for vector fields. For \( u, v \in \mathcal{X}(M) \) the adjoint action of the Lie algebra on itself is given by

\[
\text{ad}_u v = [u, v].
\]
Consider the $H^1$ inner product on $\mathcal{X}(M)$ and define a weak Riemannian metric on the whole group $D$ by right-translation of the given inner product on the Lie algebra. The corresponding norm defines a right-invariant Lagrangian on $\text{Diff}(M)$ whose restriction to the Lie algebra $X$ is equal to the $H^1$ norm:

$$l(u) = \frac{1}{2} \int_M (u^2 + u_x^2) dx.$$  

(1)

Next, one defines $ad_u^*$ the adjoint of $ad_u$ with respect to the $H^1$ metric, that is, for $u, v, w \in \mathcal{X}(M)$

$$\langle ad_u^* w, v \rangle_{H^1} = \langle w, [u, v] \rangle_{H^1}.$$  

Also, for a function $l : \mathcal{X}(M) \rightarrow \mathbb{R}$, define the functional derivative $\delta l / \delta u$ with respect to the given metric by

$$\delta l \delta u \cdot v \text{ for } v \in \mathcal{X}(M).$$

Assuming the existence of $ad_u^*$ for each $u \in \mathcal{X}(M)$, we can formally write the Euler-Poincaré equations

$$\frac{d}{dt} \frac{\delta l}{\delta u} = -ad_u^* \frac{\delta l}{\delta u}.$$  

After computing $ad_u^*$ and $\delta l / \delta u$ (for the computations see section IV) the Euler-Poincaré equations yield the Camassa-Holm equation

$$u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx}.$$  

(2)

The above observation motivates one to develop a theory to perform the procedure.

III. RIEMANNIAN GEOMETRY OF PARTIAL LIE GROUPS.

A. Review of definitions

In what follows, the general structure needed is that of a $C^1$-manifold $G$ which is also a topological group in the induced manifold topology, and it is assumed that only the right translation is $C^1$. We call such a group $G$ a partial Lie group. Below we review the Riemannian geometry of partial Lie groups, see Spivak. Let $G$ be a manifold equipped with a metric $\langle \cdot, \cdot \rangle$. Let $\pi_G : TG \rightarrow G$ and $\pi_{TTG} : TTG \rightarrow TG$ be the tangent bundle projections and denote by $V = \ker T\pi_G$ the vertical subbundle of $TTG$. Define the connector $K : TTG \rightarrow TG$ by

$$K(TY \cdot X) = \nabla_X Y,$$

for $X, Y \in \mathcal{X}(G)$ the Lie algebra of vector fields and $\nabla$ the Levi-Civita connection coming from a metric.

A vector $U \in TTG$ is called horizontal if $U \in \ker K$; $H = \ker K$ is a subbundle of $TTG$ called the horizontal subbundle of the connection and we have the decomposition $TTG = H \oplus V$ over $TG$ with the projection $\pi_{TTG}$. Then the horizontal lift of $w \in T_gG$ to $T_v(TG)$, $v \in T_gG$, is defined as

$$\text{hor}_v w = (T_v \pi_G | H_v)^{-1}(w).$$

The horizontal lift operator $\text{hor}_v : T_gG \rightarrow H_v$ is an isomorphism for all $v \in TG$ and locally,

$$\text{hor}_v w = b^i \frac{\partial}{\partial x^i} - \Gamma_{jk}^i b^j a^k \frac{\partial}{\partial v^i},$$

where $v = a^i \partial / \partial x^i$ and $w = b^i \partial / \partial x^i$.

The spray $S : TG \rightarrow TTG$ is by definition the Lagrangian vector field of the energy function $E(v) = L(v) = \frac{1}{2} \langle v, v \rangle$; i.e.

$$i_S \omega_L = dE,$$
where $\omega_L$ is the symplectic form on $TG$, and $i_S$ denotes the interior product (for more details refer to Foundations of Mechanics). Locally,

$$S(v) = a^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} a^j a^k \frac{\partial}{\partial a^i},$$

where $v = a^i \partial/\partial x^i$, and notice that by definition, we have the useful identity

$$S(v) = \text{hor}_v v. \quad (3)$$

Let $g(t)$ be a smooth curve in $G$ and let $\dot{g}(t)$ be its tangent vector field. If $Y$ is another vector field, define the covariant derivative of $Y$ along $g(t)$ by

$$\frac{DY}{dt} = \nabla_{\dot{g}(t)} Y.$$

If the covariant derivative of $Y$ is zero, $Y$ is said to be parallel along $g(t)$. It follows from the definition of a connector that $TY(\dot{g}(t)) \in H$ if and only if $DY/dt = 0$. Locally for a given curve $g(t)$ this equation becomes a linear system of ordinary differential equations

$$\frac{dY^i(t)}{dt} + \Gamma^i_{jk} \dot{g}^j(t) Y^k(t) = 0.$$

A curve $g(t)$ is called the geodesic of a connection $\nabla$, if $\dot{g}(t)$ is parallel along $g(t)$; i.e. if

$$\nabla_{\dot{g}(t)} \dot{g}(t) = 0.$$

Locally, this is a second-order differential equation

$$\ddot{g}^i(t) + \Gamma^i_{jk} \dot{g}^j(t) \dot{g}^k(t) = 0.$$

B. The Levi-Civita connection and the spray for a right-invariant metric on $G$.

Let $G$ be a $C^1$-manifold which is a topological group with $C^1$ right translation. Assume that $G$ admits a right-invariant metric. There is a vector bundle isomorphism called the right trivialisation map

$$\rho : TG \rightarrow G \times G,$$

$$v \mapsto (g, T_g R_{g^{-1}} \cdot v).$$

Then $T \rho : TTG \rightarrow T(G \times G)$ maps $TTG$ isomorphically onto $TG \times G \times G$. We can further trivialize via $\rho \times id$

$$TTG \xrightarrow{(\rho \times id) \circ T \rho} G \times G \times G \times G.$$

Note the isomorphic image of the vertical subbundle of $TTG$ is equal to $G \times O \times G \times G$, the projection being onto the first and third factors. To keep the base points in the first two factors, we apply the involution map

$$\sigma : G \times G \times G \times G \rightarrow G \times G \times G \times G,$$

$$(g, X, Y, Z) \mapsto (g, Y, X, Z).$$

Then the image of the vertical bundle $V$ of $TTG$ equals $G \times G \times O \times G$ with the projection being on the first two factors. Therefore, the isomorphism we are working with is

$$TTG \xrightarrow{\sigma \circ (\rho \times id) \circ T \rho} G \times G \times G \times G.$$

A given metric gives rise to a Levi-Civita connection which determines the horizontal bundle. We wish to express it in the trivialisation $\rho$, which in turn helps us to find the spray. Define the continuous $\mathbb{R}$-bilinear map $\gamma_g : G \times \mathcal{G} \rightarrow \mathcal{G}$ depending smoothly on $g$ by
\[ \rho((\nabla_X Y)(g)) = (g, d\bar{Y}(g) \cdot X(g) + \gamma_g(\bar{X}(g), \bar{Y}(g))), \]
where \( \rho(X(g)) = (g, \bar{X}(g)) \) and \( X, Y \in \mathcal{X}(G) \).

To see a connection in a finite-dimensional case recall that in coordinates we have
\[ S(X) = X^i \frac{\partial}{\partial g^i} - \Gamma^i_{jk} X^j \frac{\partial}{\partial g^k}, \]
and
\[ (\nabla_X Y)^i = X^j Y^i_j + \Gamma^i_{jk} X^j X^k. \]

Define \( \gamma = \Gamma^i_{jk} X^j X^k \frac{\partial}{\partial g^i} \), then
\[ (\nabla_X Y)^i = X^j Y^i_j + \gamma^i. \tag{5} \]

Let \( g(t) \) be a curve in \( G \) with \( g(0) = g, \dot{g}(0) = w, \) and \( v \in T_g G \). There exists a curve \( v(t) \in TG \) such that
\[ v(0) = v, \quad \pi_G(v(t)) = g(t), \quad \frac{Dv(t)}{dt} = 0. \tag{6} \]

Therefore, \( Tv(t) \cdot \dot{g}(t) \) is horizontal; i.e. the tangent vector field \( dv/dt \) of \( v(t) \) is always horizontal. Therefore \( \dot{v}(0) = \text{hor}_v w \), the horizontal lift of \( w = \dot{g}(0) \in T_g G \) to \( T_v(TG) \). If
\[ \rho(v(t)) = (g(t), \xi(t)) \quad \rho(v) = (g, \xi) \]
\[ \rho(\dot{g}(t)) = (g(t), \zeta(t)) \quad \rho(w) = (g, \zeta), \]
then \( v(0) = v \) in the trivialisation reads \( \xi(0) = \xi \), and the base projection condition is automatically satisfied. By the definition of a covariant derivative and the chain rule, we find that
\[
\rho\left( \frac{Dv}{dt} \right) = (g(t), d\xi(t) \cdot \dot{g}(t) + \gamma_g(t)(\zeta(t), \xi(t)))
\]
\[
= (g(t), \frac{d\xi}{dt} + \gamma_g(t)(\zeta(t), \xi(t))),
\]
so that \((g(t), \xi(t))\) is parallel along \( g(t) \) in \( G \times G \) relative to the push-forward connection \( \rho \) if and only if
\[ \frac{d\xi}{dt} + \gamma_g(t)(\zeta(t), \xi(t)) = 0, \quad \xi(0) = \xi. \tag{7} \]

This equation enables us to compute the horizontal lift of \((g, \zeta)\) to \( T_{(g,\xi)}(G \times G) \) and hence the spray using \( \mathcal{H} \). Let us compute \( \text{hor}_v w = \dot{v}(0) \) in the trivialisation given by \( \rho \). We have that
\[
(\sigma \circ (\rho \times id) \circ T\rho)(\dot{v}(0)) = \sigma \circ (\rho \times id)(\frac{d}{dt}_{t=0} (\rho \circ v)(t))
\]
\[
= \sigma \circ (\rho \times id)(\dot{g}(0), \xi(0), \frac{d\xi}{dt}_{t=0})
\]
\[
= \sigma(g(0), \xi(0), \xi(0), -\gamma_g(0)(\zeta(0), \xi(0)))
\]
\[
= (g, \xi, \zeta, -\gamma_g(\zeta, \xi)).
\]

Therefore,
\[ \text{hor}_{g,\xi}(g, \zeta) = (g, \xi, \zeta, -\gamma_g(\zeta, \xi)) \]
and the spray of the Levi-Civita connection in its right trivialisation is given by
\[ S(g, \xi) = \text{hor}_{g,\xi}(g, \xi) = (g, \xi, \xi, -\gamma_g(\xi, \xi)). \tag{8} \]

Applying \((\rho^{-1} \times id) \circ \sigma^{-1}\) we can express the right trivialisation of the spray as
\[ S(g, \xi) = (T_{e} R_{g} \cdot \xi, \xi, -\gamma_g(\xi, \xi)), \tag{9} \]
where \( X(g) = T_{e} R_{g} \cdot \xi \) is the right-invariant vector field on \( G \) associated to a Lie algebra element \( \xi \). It follows that
The Camassa-Holm equation as a geodesic flow...

\[ S(v) = T \rho^{-1} \circ \tilde{S} \circ \rho(v). \] (10)

Given a vector bundle \( E \) over \( G \), we shall denote by \( \mathcal{E} \) the collection of all smooth sections \( \sigma : G \to E \) such that \( \pi \circ \sigma = id_G \). Let \( E = G \times \mathcal{G} \), then the condition \( \pi \circ \sigma = id_G \) implies that

\[ \mathcal{E} = \{ \sigma : \mathcal{G} \to \mathcal{G} \mid \sigma \text{ is smooth} \}. \]

If \( \rho(X(g)) = (g, \tilde{X}(g)) \), define the map

\[ \nabla : \mathcal{X}(G) \times \mathcal{E} \to \mathcal{E} \quad \text{via} \]

\[ (\nabla_X \sigma)(g) = T_g \sigma \cdot X(g) + \gamma_g(\tilde{X}(g), \sigma(g)). \]

It is straightforward to check that \( \nabla \) is a vector bundle connection. Therefore, a bilinear map \( \gamma_g \) defines the vector bundle connection \( \nabla : \mathcal{X}(G) \times \mathcal{E} \to \mathcal{E} \). Moreover when \( \gamma_g \) is defined as in (12), the push-forward by \( \rho \) of the Levi-Civita connection is equal to

\[ \rho(\nabla_X Y(g)) = (g, \nabla_X Y(g)), \]

and therefore,

\[ \nabla_X Y(g) = T_e R_g (\nabla_X \tilde{Y}(g)). \]

**Conclusion 1** If for a given connection \( \nabla \) we think of the map \( \gamma_g : T_e G \times T_e G \to T_e G \) as a generalized Christoffel map of the push-forward connection \( \nabla \) under the right-trivialisation map \( \rho \), then we have the formula

\[ \nabla_X Y(g) = \rho^{-1}(d\tilde{Y}(g) \cdot X(g) + \gamma_g(\tilde{X}(g), \tilde{Y}(g))). \]

If in addition we restrict ourselves to the Levi-Civita connection coming from a given metric on \( G \), we have the formula for the spray using the Christoffel map

\[ S(X(G)) = T \rho^{-1}(X(g), \tilde{X}(g), -\gamma_g(\tilde{X}(g), \tilde{Y}(g))). \]

The above two formulas are analogous to the finite-dimensional formulas and they are globally defined on \( G \).

Notice we have not used the right-invariance condition, these conclusions are true for any metric on \( G \). However, our results will depend below heavily on the right-invariance of the metric.

**Proposition III.1** Let \( G \) be a \( C^1 \)-manifold which is a topological group with \( C^1 \) right translation. Suppose that \( G \) admits a right-invariant metric. Then the spray of the corresponding Lagrangian \( L(v) = \frac{1}{2} \langle v, v \rangle_g \) is given by

\[ S(v) = T \rho^{-1} \circ \tilde{S} \circ \rho(v), \quad \text{where} \]

\[ \tilde{S}(g, \xi) = (T_e R_g \cdot \xi, \xi, -B(\xi, \xi)) \quad \text{and} \]

\[ B : T_e G \times T_e G \to T_e G \] is defined implicitly by

\[ \langle B(\xi, \eta), \eta \rangle = \langle \xi, [\xi, \eta] \rangle \quad \text{for} \quad \xi, \eta \in T_e G. \] (12)

**Remark.** For the case of Lie groups, the proof of this result can be found in Ref. [11]. The operator \( B \) was introduced by Arnold in Appendix 2. The above proposition is more general as it covers diffeomorphism groups which are of a great interest in hydrodynamics.

**Proof.** To verify (11) we need to calculate the Christoffel map \( \gamma_g(\xi, \xi) \) in (12).

Since \( \rho \) is a diffeomorphism, we can push-forward the symplectic form \( \omega_L \) on \( TG \) to define the symplectic form \( \omega^s = \rho_* \omega_L \) (the super-script \( s \) stands for “spatial” because the right trivialisation gives rise to spatial coordinates in applications). It can be checked that the push-forward of the spray \( S \) on \( TG \) is the Lagrangian vector field expressed in space coordinates and \( \rho_* S = \tilde{S} \); consequently,

\[ i_{\tilde{S}} \omega^s = d(\rho_* E). \] (13)

To calculate the left-hand side recall the following formula (see Ref. [10]):
\[ \omega^s(g, \xi)((v, \zeta), (w, \eta)) = -\langle \zeta, T_g R_{g^{-1}}(w) \rangle_e + \langle \eta, T_g R_{g^{-1}}(v) \rangle_e \\
- \langle \xi, [T_g R_{g^{-1}}(v), T_g R_{g^{-1}}(w)] \rangle_e. \]

By this formula we have that
\[ \omega^s(g, \xi)((T_eR_g \cdot \xi, -\gamma_g(\xi, \xi)), (w, \eta)) = -\langle -\gamma_g(\xi, \xi), T_g R_{g^{-1}}(w) \rangle_e \\
+ \langle \eta, \xi \rangle_e - \langle \xi, [\xi, T_g R_{g^{-1}}(w)] \rangle_e. \] (14)

Since the metric is right-invariant, it follows that
\[ E \circ \rho^{-1}(g, \xi) = \frac{1}{2} \langle T_e R_g \cdot \xi, T_e R_g \cdot \xi \rangle_g \\
= \frac{1}{2} \langle \xi, \xi \rangle_e. \]

Therefore the right-hand side of (13) is equal to
\[ d(E \circ \rho^{-1})(g, \xi) \cdot (w, \eta) = \langle \xi, \eta \rangle_e. \] (15)

From (13), (14), and (15) we may conclude that the value of \( \gamma_g(\xi, \xi) \) does not depend on the base point \( g \). Moreover, its value is defined by the following relationship:
\[ \langle \gamma(\xi, \xi), \zeta \rangle_e = \langle \xi, [\xi, \zeta] \rangle_e \quad \text{for} \quad \xi, \zeta \in T_e G. \]

From the definition of the operator \( B \) it follows that \( \gamma(\xi, \xi) = B(\xi, \xi) \), and hence (11) is true. \( \square \)

It is known that \( -\partial_t(\xi, \xi) = \langle \nabla \chi, X_\xi(e) \rangle \), where \( X_\xi(g) = T_e R_g \cdot \xi \) (see Arnold, Bao and Ratiu). Thus, for right invariant vector fields, we also have that
\[ \tilde{S}(g, \xi) = (X_\xi(g), \xi, (\nabla \chi, X_\xi)(e)). \]

**Conclusion 2** Given a right-invariant metric on \( G \), we can find its geodesic equations by finding the spray. The above formulas show that the spray is completely defined by either the operator \( B \) or the value of the Levi-Civita connection at the identity.

See remark in Section V.

**C. The Euler-Poincaré Equations**

The Euler-Poincaré-Arnold equations are the fundamental result about geodesic flow on an arbitrary Lie group. See, for example, Theorem 13.8.3 in Marsden and Ratiu or Appendix 2 in Arnold. Herein, this result is proven for diffeomorphism groups, the configuration space for ideal fluid dynamics. The idea of studying geodesics on diffeomorphism groups in order to do hydrodynamics is due to Arnold.

In order to establish our notation, let us recall some results from Refs. 14 and 15. For \( s > \frac{3}{2} \) and \( M \) a compact manifold without boundary, we may define the Sobolev \( H^s \) maps from \( M \) into \( M \). Let \( D^s(M) = \{ \eta \in H^s(M, M) \mid \eta \text{ is bijective and } \eta^{-1} \in H^s(M, M) \} \). If \( s > \frac{3}{2} + 1 \), then \( D^s(M) \) is open in \( H^s(M, M) \) and hence is a manifold, but note that \( D^s \) is not a Lie group, but rather a topological group. However, like a Lie group, \( D^s \) has an exponential map which associates to every tangent vector at the identity a one parameter subgroup of \( D^s \). Such a tangent vector is an \( H^s \) vector field on \( M \) and the one parameter subgroup is its flow. If \( \pi: TM \to M \) is the canonical projection, one forms the Hilbert space
\[ T_\eta D^s = \{ V : M \to TM \mid V \text{ is } H^s \text{ and } \pi \circ V = \eta \}, \]
the tangent space at \( \eta \in D^s \). An element \( V \) of the tangent space at \( \eta \in D^s \) is called a vector space over \( \eta \).
Theorem III.1 Assume that \( D^s(M) \) is equipped with a metric \( \langle \cdot, \cdot \rangle \) that is invariant under right translations. Then a curve \( t \to \eta(t) \) in \( D^s \) is a geodesic of this metric if and only if \( u(t) = T_{\eta(t)}R_{\eta(t)}^{-1} \dot{\eta}(t) = \dot{\eta}(t) \circ \eta^{-1}(t) \) satisfies

\[
\frac{du}{dt} = -B(u, u),
\]

where the operation \( B : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \) is defined by the identity

\[
\langle B(w, u), v \rangle = \langle w, [u, v] \rangle \quad \text{for } u, v, w \in \mathcal{X}(M).
\]

Proof. By proposition (1.1) the spray \( S : TD^s \to T^2D^s \) is given by

\[
S(V) = T\rho^{-1}(V, V \circ \eta^{-1}, -B(V \circ \eta^{-1}, V \circ \eta^{-1})) \quad \text{for } V \in T_\eta D^s.
\]

Using the notation adopted from the theory of Lie groups \( G = D^s \), \( \mathcal{G} = \mathcal{X}(M) \) let us compute \( T\rho^{-1} \).

\[
\rho^{-1} : G \times \mathcal{G} \to TG
\]

\[
(\eta, u) \mapsto u \circ \eta
\]

\[
T_{(\eta, u)}\rho^{-1} : T_\eta D^s \times \mathcal{G} \to T_{u \circ \eta}(TG)
\]

Let \( (\eta(t), u(t)) \) be a curve in \( D^s \times \mathcal{X}(M) \) such that \( (\eta(0), u(0)) = (\eta, u) \) and \( (\dot{\eta}(0), \dot{u}(0)) = (V, w) \), then

\[
T_{(\eta, u)}\rho^{-1} \cdot (V, w) = \left. \frac{d}{dt} \right|_{t=0} \rho^{-1}(\eta(t), u(t))
\]

\[
= \left. \frac{d}{dt} \right|_{t=0} u(t) \circ \eta(t)
\]

\[
= w \circ \eta + Tu \circ V.
\]

Therefore,

\[
S(V) = -B(V \circ \eta^{-1}, V \circ \eta^{-1}) \circ \eta + T(V \circ \eta^{-1}) \circ V, \quad \text{or}
\]

\[
S(V) \circ \eta^{-1} = -B(V \circ \eta^{-1}, V \circ \eta^{-1}) + T(V \circ \eta^{-1}) \circ (V \circ \eta^{-1}).
\]

(18)

For an integral curve \( V_t \in T_{\eta(t)}D^s(M) \), its pullback \( u_t = V_t \circ \eta_t^{-1} \) is a curve in \( T_e D^s(M) \) that consists of \( H^s \)-vector fields on \( M \). Then the spray equation,

\[
\frac{dV_t}{dt} = S(V_t),
\]

is equivalent to

\[
\frac{du_t}{dt} \circ \eta_t = S(V_t), \quad \text{or}
\]

\[
\frac{du_t}{dt} = S(V_t) \circ \eta_t^{-1} - Tu_t \circ u_t
\]

\[
= -B(u_t, u_t) + Tu_t \circ u_t - Tu_t \circ u_t,
\]

\[
= -B(u_t, u_t).
\]

Note that the existence of the smooth spray defined by (17) follows from arguments in Theorem 3.3 in Shkoller as well as Theorem 4.2 in Ref. 3. See the remark in Section 4.

\[\square\]

IV. CAMASSA-HOLM EQUATION AS A GEODESIC FLOW.

Henceforth, the subscript \( t \) will denote the partial derivative with respect to \( t \). We shall apply the general results obtained in section III to the CH-equation (2). For periodic boundary conditions, the configuration space is \( \hat{G} = D^s(S^1) \) with Lie algebra \( \mathcal{G} = \mathcal{X}^s(S^1) \). One may also consider (2) on \( \mathbb{R} \) with the appropriate decay conditions at
Now we obtain the Euler-Poincaré-Arnold equation: 

\[ B \] and hence the formula for the operator \( B \).

Furthermore, this metric is right-invariant by definition and it defines the extended Lagrangian

\[ \langle V, W \rangle_{\eta} = \left\langle T_{\eta} R_{\eta^{-1}} \cdot V, T_{\eta} R_{\eta^{-1}} \cdot W \right\rangle_{id} = \left\langle V \circ \eta^{-1}, W \circ \eta^{-1} \right\rangle_{1} = \int [(V \circ \eta^{-1})(W \circ \eta^{-1}) + (V \circ \eta^{-1})_{x}(W \circ \eta^{-1})_{x}] dx. \tag{21} \]

This main result of this section is

**Theorem IV.1** Let \( t \to \eta(t) \) be a curve in the diffeomorphism group \( D^{s} \) starting at the identity. Then \( \eta(t) \) is a geodesic of the metric (21) if and only if the time-dependent vector field \( u(t) = \dot{\eta}(t) \circ \eta^{-1}(t) \) satisfies the CH-equation (2).

**Proof.** By the theorem (III.1), the geodesic equations for the metric (21) are equivalent to (16). From the definition of the operator \( B \), we have that

\[ \langle B(w, u), v \rangle = \langle w, [u, v] \rangle = \int (-uw_{x} + u_{x}v)w + (-uw_{x} + u_{x}v)xw_{x} \, dx = \int (u_{x}w + uw)_{x}v - w_{xx}(-uw_{x} + u_{x}v) \, dx = \int (2u_{x}w - 2u_{x}w_{xx} + uw_{x} - uw_{xxx}) v \, dx = \int (2u_{x}(1 - \partial_{x}^{2})w + u(1 - \partial_{x}^{2})w_{x}) v \, dx. \]

Furthermore,

\[ \langle B(w, u), v \rangle = \int (B(w, u)v + B(w, u)v_{x}) \, dx = \int ((1 - \partial_{x}^{2})B(w, u)) v \, dx, \]

and hence the formula for the operator \( B \) is

\[ B(w, u) = (1 - \partial_{x}^{2})^{-1}(2u_{x}(1 - \partial_{x}^{2})w + u(1 - \partial_{x}^{2})w_{x}). \tag{22} \]

Now we obtain the Euler-Poincaré-Arnold equation:

\[ \frac{\partial u}{\partial t} = -B(u, u) = -(1 - \partial_{x}^{2})^{-1}(2u_{x}u + uu_{x} - 2u_{x}u_{xx} - uu_{xxx}) = -(1 - \partial_{x}^{2})^{-1}(3uu_{x} - 2u_{x}u_{xx} - uu_{xxx}). \]

This completes the proof that the geodesic equations for the metric coming from the \( H^{1} \) inner product on the Lie algebra of vector fields \( \mathcal{G} \) are equivalent to the CH-equation (2).

**Remark.** The Lie-algebra bracket \([u, v]\) on \( \mathcal{X}(M) \) is minus the Jacobi-Lie bracket (for an explanation refer to Marsden and Ratiu, Chapter 9). □
a. Alternative derivation. Below, we begin to compute the geodesic equations for the metric (21) by calculating the spray of the corresponding Lagrangian. Camassa and Holm have shown that the CH-equation can be expressed in the integral form

\[ u_t + uu_x = -(1 - \partial^2)^{-1} \partial(u^2 + \frac{1}{2}u_x^2) = - \int e^{-|x-y|} (uu_y + \frac{1}{2}u_xu_{yy}) \, dy. \]

Equation (18) together with the fact that \( Tu \circ u \) is simply \( uu_x \) in one dimension shows that the spray is equal to

\[ S(V) = -(1 - \partial^2)^{-1} \partial((V \circ \eta)^{-1})^2 + \frac{1}{2}(V \circ \eta^{-1})_{x}^2 \circ \eta. \]

Letting \( u = V \circ \eta^{-1} \) verifies the claim.

V. DISCUSSION

We would like to emphasize that we built a right invariant metric on \( \mathcal{D}^n \) by taking the \( H^1 \) inner product on the tangent space at the identity and right-translating it over the whole space. This does not coincide with the usual \( H^1 \) metric on each fiber \( T_p \mathcal{D}^n \); see the remark after Theorem 4.1 in Ref. 7. To illustrate the difference of two approaches let us compare the geodesics of the \( L^2 \) metric with the right-invariant \( L^2 \) metric in the one-dimensional case.

A curve \( \eta(t) \in \mathcal{D}^n \) is a geodesic of the \( L^2 \)-metric if and only if the corresponding spatial velocity field \( u = V \circ \eta^{-1} \) satisfies Burger’s equation:

\[ u_t + uu_x = 0. \]

The corresponding Euler-Lagrange equations for the material velocity \( V = \dot{\eta} \) are given by

\[ V_t = 0. \]

The spray of this metric is equal to zero and hence smooth; however, as the metric is not right invariant, the Euler-Poincaré theorem does not apply.

For the right-invariant \( L^2 \) metric the Euler-Poincaré equations are given by

\[ u_t + 3uu_x = 0. \]

The corresponding Euler-Lagrange equations are given by

\[ \eta_X \dot{V} + 2VV_X = 0, \]

where \( \eta_X \) is the Jacobian of \( \eta \), and \( X \) denotes the material coordinate of the fluid particle. The spray in this case is given by

\[ S(\eta, V) = -\frac{2}{\eta_X} \, VV_X. \]

Since there is a loss of derivatives, the spray is not smooth (c.f. Remark 3.5 in Shkoller).

As we see from the above calculations, the two equations in the spatial velocities differ only in a scalar coefficient multiplying the derivative term, however, the corresponding sprays are completely different.

Remark. We note that equation (23) for the geodesic spray of the right-invariant \( H^1 \) metric on either \( S^1 \) or \( \mathbb{R} \) has no derivative loss and hence shows that the CH-equation is an ordinary differential equation on the group \( \mathcal{D}^n \). Thus, existence and uniqueness of solutions to (2) may be obtained by standard Picard iteration argument in the event that \( S \) is locally Lipschitz.

Lemmas 3.1 and 3.2 of Shkoller show that \( S \) is \( C^1 \), and hence the result follows. See Refs. 6 and 7 for the well-posedness of the geodesic flow of the diffeomorphism groups on \( n \)-dimensional Riemannian manifolds.

It would be interesting to study the Lagrangian stability of the CH-equation, and this requires analysis of the curvature operator. Misiolek has computed the sectional curvature of \( \mathcal{D}^n(S^1) \). Shkoller has obtained an explicit form for the \( H^1 \) covariant derivative \( \nabla^1 \) on volume preserving diffeomorphism groups and has proved that the weak curvature tensor of \( \nabla^1 \) is a bounded trilinear operator in the \( H^s \) topology. We would like to explore these type of estimates on the full diffeomorphism group of the circle, as well as investigate the role of generalized flows in peakon dynamics.
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