SOME REMARKS ON ANALYTIC
PSEUDODIFFERENTIAL OPERATORS

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Abstract

We report some recent results on analytic pseudodifferential operators, also known as Wick operators. An important tool in our study is the Bargmann transform which provides a coupling between the classical (real) and analytic pseudodifferential calculus. Since the Bargmann transform of Hermite functions gives rise to formal power series in the complex domain, the results are formulated in terms of the Bargmann images of Pilipović spaces.

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1 Introduction

We present a sample of recent results from [20, 21] related to Wick and anti-Wick operators introduced in 1960s by Berezin in the framework of the second quantization. Wick and anti-Wick symbols are used in [4] to derive various spectral properties of the corresponding operators. As demonstrated in [20, 21], results on Wick and anti-Wick operators provide new insight into the classical theory of pseudodifferential operators. This is done by using the mapping properties of the Bargmann transform given in [22, 23].

The Bargmann transform coupling between Hermite series expansions and formal power series expansions plays an important role in our analysis. For that reason, we first review some facts on Hermite functions and spaces of test functions (Pilipović spaces) with Hermite coefficients of (super)exponential decay. Thereafter we briefly discuss the Bargmann transform, and finally we review some continuity properties of analytic pseudodifferential operators, and sharp Gårding inequality in the context of Wick and anti-Wick calculus.

Apart from motivations coming from quantum physics, Hermite polynomials are used e.g. in studying the propagation of light in infinitely long optical fibers with a parabolic index profile [9], in visual perception and neurobiology [25], and in equatorial oceanography [5]. For the applications of pseudodifferential operators in mobile wireless communication systems we refer to [18].

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2 Hermite functions and Pilipović spaces

We first consider Hermite functions within a historical context, and proceed with Pilipović spaces given by Hermite series with rapidly decaying coefficients. Recall, Hermite functions are defined by

\[ h_\alpha(x) = \pi^{-\frac{d}{4}} (-1)^{|\alpha|} (2^{\frac{1}{2}})^{|\alpha|} \frac{1}{\alpha!} e^{-\frac{|x|^2}{4}} \left( e^{\frac{|x|^2}{2}} H_\alpha(x) \right), \quad x \in \mathbb{R}^d, \quad n = 0, 1, \ldots, \]

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d, \alpha! = \alpha_1! \cdots \alpha_d!, \) and \( H_\alpha \) are (normalized) Hermite polynomials. The functions \( H_n(x), x \in \mathbb{R}, n \in \mathbb{N} \) were introduced by P. S. Laplace in 1810, and later studied by P. L. Chebyshev (1859) and C. Hermite (1864). N. Wiener used Hermite function expansions to prove the Plancherel formula for the Fourier transform around 1930, [24]. In fact, Hermite functions are an orthonormal basis of \( L^2(\mathbb{R}) \). Since Hermite functions are the wave functions for the stationary states of the quantum harmonic oscillator, they are particularly useful in quantum mechanics, [6].

B. Simon used Hermite function expansions in the framework of the space of rapidly decreasing functions \( S(\mathbb{R}^d) \) and its dual space of tempered distributions \( S'(\mathbb{R}^d) \), [16]. Thereafter S. Pilipović in [15] gave a characterization of Gelfand-Shilov type spaces and their dual spaces of tempered ultradistributions through the growth estimates of coefficients in Hermite expansions, see also [11, 13] for more recent contributions in that direction. These investigations led to a detailed study of the so-called Pilipović spaces, [1, 23].

Hermite series expansions are used in the following generalization of isotropic Gelfand-Shilov spaces \( S_s(\mathbb{R}^d), s \geq 1/2, \) (of Roumieu type) which consists of smooth functions \( f \) satisfying

\[ |\partial^\beta f(x)| \lesssim h^{s} |\beta|! e^{-k|x|^{s}}, \quad x \in \mathbb{R}^d, \]

for some \( h, k > 0 \). We refer to [8, 10, 14, 19] for details on Gelfand-Shilov spaces and their applications in partial differential equations. As usual, \( S'_s(\mathbb{R}^d) \) denotes the dual space of the Gelfand-Shilov space \( S_s(\mathbb{R}^d), s \geq 1/2. \)

Pilipović spaces (of Roumieu type) \( H_s(\mathbb{R}^d), s \geq 0, \) are given through the formal Hermite series expansions

\[
\tag{2.1}
f = \sum_{\alpha \in \mathbb{N}^d} c_\alpha h_\alpha, \quad c_\alpha = (f, h_\alpha), \quad |c_\alpha| \lesssim e^{-r|\alpha|^{\frac{1}{2}}},
\]

for some \( r > 0 \). When \( s \geq 1/2 \) we have \( H_s(\mathbb{R}^d) = S_s(\mathbb{R}^d) \), and \( H_0(\mathbb{R}^d) \) is the set of all finite Hermite series expansions. It can be proved that \( H_s(\mathbb{R}^d) \neq S_s(\mathbb{R}^d) = \{0\} \), \( 1/2 > s > 0. \)

In [23] J. Toft proved that

\[ H_s(\mathbb{R}^d) = \{ f \mid \| R^N f \|_{L^\infty} \lesssim h^{N} N!^{2s} \text{ for some } h > 0 \}, \]

where \( R = -\frac{d^2}{dx^2} + x^2. \) Since the Hermite functions are eigenfunctions of \( R \), i.e. \( Rh_n = (2n+1)h_n \), it is called the Hermite operator.
In addition, Toft considered Pilipović flat spaces where the growth condition in (2.1) is replaced by

$$|c_\alpha| \lesssim r^{\alpha!} \gamma^{-\alpha}, \quad \sigma > 0$$

some $r > 0$.

The well known relation between $L^2(\mathbb{R}^d)$ and the Fock space of analytic functions $A^2(\mathbb{C}^d)$ (see Section 3 for the definition) can then be extended to the relation between Pilipović spaces and specific subspaces of the space of analytic functions, [22]. This is done via the Bargmann transform, cf. Definition 3.1. Following this approach, a detailed study of analytic pseudodifferential operators is given in [20, 21].

### 3 The Bargmann transform

**Definition 3.1.** The Bargmann transform $\mathfrak{V}_d f$ of $f \in S'_1(\mathbb{R}^d)$ is the entire function

$$\mathfrak{V}_d f(z) = \int_{\mathbb{R}^d} \mathcal{A}_d(z, y) f(y) \, dy$$

$$= \pi^{-\frac{d}{4}} \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} \langle z, y \rangle + |y|^2 + 2^{1/2} \langle z, y \rangle \right) f(y) \, dy,$$

$z \in \mathbb{C}^d$, and $\langle z, w \rangle = \sum_{j=1}^d z_j w_j$.

It was proved by V. Bargmann in 1961. that

$$\mathfrak{V}_d : L^2(\mathbb{R}^d) \to A^2(\mathbb{C}^d)$$

is a bijective and isometric mapping from $L^2(\mathbb{R}^d)$ to the Fock space $A^2(\mathbb{C}^d)$, the Hilbert space of entire functions with the scalar product

$$(F, G)_{A^2} \equiv \int_{\mathbb{C}^d} F(z) \overline{G(z)} \, d\mu(z), \quad F, G \in A^2(\mathbb{C}^d),$$

where $d\mu(z) = \pi^{-d} e^{-|z|^2} d\lambda(z)$ ($d\lambda(z)$ is the Lebesgue measure on $\mathbb{C}^d$).

These investigations put a solid theoretical background for a quantization procedure proposed by V. Fock back in 1930’s. More precisely, Bargmann showed that $\mathfrak{V}_d$ maps the creation and annihilation operators, $\hat{A} = -\frac{d}{dx} + x$, and $\hat{A}^\dagger = \frac{d}{dx} + x$ respectively, into multiplication and differentiation in the complex domain, [2] [3]. Note that $\mathcal{R} = \frac{1}{2}(\hat{A}\hat{A}^\dagger + \hat{A}^\dagger\hat{A})$.

The Bargmann transform maps the Hermite functions to monomials as

$$\mathfrak{V}_d h_\alpha = e_\alpha, \quad e_\alpha(z) = \frac{z^\alpha}{\alpha!^{\gamma}}, \quad z \in \mathbb{C}^d, \quad \alpha \in \mathbb{N}^d.$$

The orthonormal basis $\{h_\alpha\}_{\alpha \in \mathbb{N}^d} \subseteq L^2(\mathbb{R}^d)$ is thus mapped to the orthonormal basis $\{e_\alpha\}_{\alpha \in \mathbb{N}^d} \subseteq A^2(\mathbb{C}^d)$. 
Let \( \mathcal{A}_0(\mathbb{C}^d) \) be the set of all analytic polynomials of the form \( F(z) = \sum_{|\alpha| \leq N} c(F, \alpha)e_{\alpha}(z) \), for some \( N \in \mathbb{N} \), and let

\[
\mathcal{A}_s(\mathbb{C}^d) = \{ F(z) = \sum_{\alpha \in \mathbb{N}^d} c(F, \alpha)e_{\alpha}(z) \mid |c(F, \alpha)| \leq e^{-r|\alpha|^{1/2}} \}, \quad s > 0.
\]

Then it is proved by Toft (23) that

\[
\mathfrak{M}_d : \mathcal{H}_s(\mathbb{R}^d) \to \mathcal{A}_s(\mathbb{C}^d), \quad s > 0,
\]

is bijective mapping between Pilipović spaces and corresponding spaces of analytic functions.

## 4 Analytic pseudodifferential operators

**Definition 4.1.** Let \( a \) be a locally bounded function on \( \mathbb{C}^{2d} \) such that \((z, w) \mapsto a(z, w)\) is analytic, \( z, w \in \mathbb{C}^d \). The analytic pseudodifferential operator or the Wick operator \( \text{Op}_{a}(a) \) with the symbol \( a \) is given by

\[
\text{Op}_{a}(a)F(z) = \pi^{-d} \int_{\mathbb{C}^d} a(z, w)F(w) e^{i(z \cdot w - \frac{1}{2}w^2)} \, d\lambda(w),
\]

where \( F \) is an entire function, \( d\lambda \) is the Lebesgue measure and \((\cdot, \cdot)\) is the scalar product on \( \mathbb{C}^d \).

Thus \((\text{Op}_{a}(a)F)(z)\) is equal to the integral operator

\[
(T_K F)(z) = \pi^{-d} \int_{\mathbb{C}^d} K(z, w)F(w) e^{i(z \cdot w - \frac{1}{2}w^2)} \, d\lambda(w) = \int_{\mathbb{C}^d} K(z, w)F(w) \, d\mu(w),
\]

when \( K(z, w) = K_a(z, w) = a(z, w)e^{i(z \cdot w)} \), and \( d\mu(w) = \pi^{-d} e^{-|w|^2} \, d\lambda(w) \).

The (classical) pseudodifferential operator \( \text{Op}(b) \) with the symbol \( b \) is given by the Kohn-Nirneberg correspondence

\[
f(x) \mapsto (\text{Op}(b)f)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} b(x, \xi) \hat{f}(\xi)e^{i(x \cdot \xi)} \, d\xi.
\]

If \( b \) is a polynomial symbol, i.e.

\[
b(x, \xi) = \sum_{|\alpha + \beta| \leq N} c_1(\alpha, \beta)x^\alpha \xi^\beta,
\]

then there is a unique symbol

\[
a(z, w) = \sum_{|\alpha + \beta| \leq N} c_2(\alpha, \beta)z^\alpha \overline{w}^\beta
\]

such that \( \text{Op}_{a}(a) = \mathfrak{M}_d \circ \text{Op}(b) \circ \mathfrak{M}_d^{-1} \) (cf. 21).

Let \( \mathcal{A}_s(\mathbb{C}^{2d}) = \{ K ; (z, w) \mapsto K(z, \overline{w}) \in \mathcal{A}_s(\mathbb{C}^{2d}) \} \), \( s \geq 0 \), and

\[
\hat{\mathcal{A}}(\mathbb{C}^{d_1+d_2}) \equiv \{ K(z, \overline{w}) , \quad z \in \mathbb{C}^{d_1}, w \in \mathbb{C}^{d_2} ; K \quad \text{is an analytic function} \}.
\]

Identification of linear and continuous mappings with pseudodifferential operators, and their basic continuity properties are given in Theorems 4.2 and 4.3. We refer to 20 for the proofs.
Theorem 4.2. Let $s \geq \frac{1}{2}$. Then the following is true:

1. If $T$ is a linear and continuous map from $\mathcal{A}_s'(\mathbb{C}^d)$ to $\mathcal{A}_s(\mathbb{C}^d)$, then there is a unique $a \in \tilde{A}(\mathbb{C}^d \times \mathbb{C}^d)$ such that

$$|a(z, w)| \lesssim e^{\frac{1}{2}|z-w|^2 - r(|z|^{\frac{1}{2}} + |w|^{\frac{1}{2}})}, \quad z, w \in \mathbb{C}^d,$$

for some $r > 0$ and $T = \text{Op}_\mathbb{C}(a)$;

2. If $T$ is a linear and continuous map from $\mathcal{A}_s(\mathbb{C}^d)$ to $\mathcal{A}_s'(\mathbb{C}^d)$, then there is a unique $a \in \tilde{A}(\mathbb{C}^d \times \mathbb{C}^d)$ such that

$$|a(z, w)| \lesssim e^{\frac{1}{2}|z-w|^2 + r(|z|^{\frac{1}{2}} + |w|^{\frac{1}{2}})}, \quad z, w \in \mathbb{C}^d,$$

for every $r > 0$ and $T = \text{Op}_\mathbb{C}(a)$.

Theorem 4.3. Let $s \geq \frac{1}{2}$. Then the following is true:

1. If $a \in \tilde{A}(\mathbb{C}^d \times \mathbb{C}^d)$ satisfies

$$|a(z, w)| \lesssim e^{\frac{1}{2}|z-w|^2 - r(|z|^{\frac{1}{2}} + |w|^{\frac{1}{2}})}, \quad z, w \in \mathbb{C}^d,$$

for some $r > 0$, then $\text{Op}_\mathbb{C}(a)$ from $\mathcal{A}_0(\mathbb{C}^d)$ to $\mathcal{A}_0'(\mathbb{C}^d)$ is uniquely extendable to a linear and continuous map from $\mathcal{A}_s'(\mathbb{C}^d)$ to $\mathcal{A}_s(\mathbb{C}^d)$;

2. If $a \in \tilde{A}(\mathbb{C}^d \times \mathbb{C}^d)$ satisfies

$$|a(z, w)| \lesssim e^{\frac{1}{2}|z-w|^2 + r(|z|^{\frac{1}{2}} + |w|^{\frac{1}{2}})}, \quad z, w \in \mathbb{C}^d,$$

for every $r > 0$, then $\text{Op}_\mathbb{C}(a)$ from $\mathcal{A}_0(\mathbb{C}^d)$ to $\mathcal{A}_0'(\mathbb{C}^d)$ is uniquely extendable to a linear and continuous map from $\mathcal{A}_s(\mathbb{C}^d)$ to $\mathcal{A}_s'(\mathbb{C}^d)$.

An important subclass of Wick operators are the anti-Wick operators, which are Wick operators where the symbol $a(z, w)$ does not depend on $z$:

$$\text{Op}_\mathbb{C}^\text{aw}(a_0)F(z) = \pi^{-d} \int_{\mathbb{C}^d} a_0(w) F(w) e^{i(z-w, w)} d\lambda(w).$$

The anti-Wick operators can also be described as the Bargmann images of Toeplitz operators on $\mathbb{R}^d$. We refer to [12, 17, 22] for more details, and note that an important feature in energy estimates in quantum mechanics and time-frequency analysis is that non-negative symbols give rise to non-negative Toeplitz and anti-Wick operators.

In the next theorem we show that many Wick operators can essentially be expressed as linear combinations of anti-Wick operators. The expansion (4.1) is deduced by using Taylor expansion and integration by parts, see [21] for details.
Theorem 4.4. Suppose \( s \geq \frac{1}{2} \), \( a \in \hat{A}_s(C^{2d}) \) (the dual of \( A_s(C^{2d}) \)), let \( N \geq 1 \) be an integer, and let
\[
a_\alpha(w) = \partial_\alpha \overline{\partial}_w a(w, w), \quad \alpha \in \mathbb{N}^d,
\]
and
\[
b_\alpha(z, w) = |\alpha| \int_0^1 (1-t)|\alpha|-1 \partial_\alpha \overline{\partial}_w a(w + t(z - w), w) \, dt, \quad \alpha \in \mathbb{N}^d \setminus \{0\}.
\]
Then
\[
(4.1) \quad \text{Op}_\omega(a) = \sum_{|\alpha| \leq N} \frac{(-1)^{|\alpha|}}{\alpha!} \text{Op}^{aw}(a_\alpha) + \sum_{|\alpha| = N+1} \frac{(-1)^{|\alpha|}}{\alpha!} \text{Op}^{aw}(b_\alpha).
\]

We apply Theorem 4.4 to the sharp Gårding inequality for analytic pseudodifferential operators. Its real counterpart represents one of the basic applications of the Anti-Wick theory, \[14\]. According to G. Folland, Gårding’s inequality is a milestone in the theory of elliptic equations, \[7\].

As a preparation, we introduce the Shubin class of symbols (cf. \[17\]). First we introduce weight functions as follows. A weight on \( \mathbb{R}^d \) is a positive function \( \omega \in L^\infty_{\text{loc}}(\mathbb{R}^d) \) such that \( 1/\omega \in L^\infty_{\text{loc}}(\mathbb{R}^d) \). It is \( v \)-moderate for a polynomially bounded weight if there is another weight \( v \) of the form \( v(x) = \langle x \rangle^s, s \geq 0 \) \((\langle x \rangle = (1 + |x|^2)^{1/2})\) such that
\[
\omega(x + y) \lesssim \omega(x)v(y), \quad x, y \in \mathbb{R}^d.
\]
By \( \mathcal{P}_{\text{Sh}_\rho}(\mathbb{R}^d), 0 \leq \rho \leq 1, \) we denote the set of all smooth and \( v \)-moderate weights \( \omega \) for a polynomially bounded weight \( v \) such that
\[
|\partial^\alpha \omega(x)| \lesssim \omega(x)\langle x \rangle^{-\rho|\alpha|}, \quad \alpha \in \mathbb{N}^d, \quad x \in \mathbb{R}^d.
\]
Let \( 0 \leq \rho \leq 1, \) and let \( \omega \in \mathcal{P}_{\text{Sh}_\rho}(\mathbb{R}^d) \). The Shubin symbol class \( \text{Sh}_\rho^0(\mathbb{R}^d) \) is the set of all \( a \in C^\infty(\mathbb{R}^d) \) such that
\[
|\partial^\alpha a(x)| \lesssim \omega(x)\langle x \rangle^{-\rho|\alpha|}, \quad x \in \mathbb{R}^d,
\]
for every multi-index \( \alpha \in \mathbb{N}^d \).

Let \( \hat{A}_s^w(\mathbb{R}^{2d}) \), be the set of all \( a \in \hat{A}(C^{2d}) \) such that
\[
(4.2) \quad \left| \partial^\alpha \overline{\partial}_w a(z, w) \right| \leq C \epsilon^{\frac{1}{2}} |z - w|^2 \omega(\sqrt{2} z) \langle z + w \rangle^{-\rho|\alpha + \beta|} \langle z - w \rangle^{-N}, \quad N \geq 0.
\]
Let \( a_0 \in S'_{1/2}(\mathbb{R}^{2d}) \). Then the Bargmann assignment \( S_{\text{Sh}a_0} \) of \( a_0 \) is the unique element \( a \in \hat{A}(C^{2d}) \) which fulfills
\[
\text{Op}_{\omega}(a) = \mathfrak{V}_d \circ \text{Op}^w(a_0) \circ \mathfrak{V}_d^* \quad \Leftrightarrow \quad a = S_{\text{Sh}}a_0,
\]
where \( \text{Op}^w(a_0) \) is the Weyl pseudodifferential operator
\[
\text{Op}^w(a_0)f(x) = (2\pi)^{-d} \int a_0(\frac{x + y}{2}, \xi) f(y) e^{i\langle x - y, \xi \rangle} \, dyd\xi, \quad x \in \mathbb{R}^d.
\]
It can be proved that $S_\omega$ is a homeomorphism from $Sh_\rho(\mathbb{R}^{2d})$ to $\mathcal{A}_{Sh,\rho}(\mathbb{C}^{2d})$, $0 \leq \rho \leq 1$, $\omega \in \mathcal{S}_{Sh,\rho}(\mathbb{R}^{2d})$, see [21].

Finally, we have the following version of the sharp Gårding inequality.

**Theorem 4.5.** Let $\rho > 0$, $\omega(z) = \langle z \rangle^{2\rho}$ and let $a_0 \in \mathcal{A}_{Sh,\rho}(\mathbb{C}^{2d})$ be such that $a_0(w, w) \geq -C_0$ for all $w \in \mathbb{C}^d$, for some constant $C_0 \geq 0$. Then

$$\text{Re}((\text{Op}_C(a_0)F, F)_{A^2}) \geq -C \|F\|^2_{A^2}, \quad F \in \mathfrak{H}_d(\mathcal{F}(\mathbb{R}^d))$$

and

$$|\text{Im}((\text{Op}_C(a_0)F, F)_{A^2})| \leq C \|F\|^2_{A^2}, \quad F \in \mathfrak{H}_d(\mathcal{F}(\mathbb{R}^d))$$

for some constant $C \geq 0$.

We refer to [21] for the proof.

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