A Geometric Vietoris-Begle Theorem, with an Application to Convex Subsets of Topological Vector Lattices

Andrew McLennan

August 10, 2021

Abstract

We show that if $L$ is a topological vector lattice, $u: L \to L$ is the function $u(x) = x \vee 0$, $C \subset L$ is convex, and $D = u(C)$ is metrizable, then $D$ is an ANR and $u|C: C \to D$ is a homotopy equivalence, so $D$ is contractible and thus an AR. This is proved by verifying the hypotheses of a second result: if $X$ is a connected space that is homotopy equivalent to an ANR, $Y$ is an ANR, and $f: X \to Y$ is a continuous surjection such that, for each $y \in Y$ and each neighborhood $V \subset Y$ of $y$, there is a neighborhood $V' \subset V$ of $y$ such that $f^{-1}(V')$ can be contracted in $f^{-1}(V)$, then $f$ is a homotopy equivalence. The latter result is a geometric analogue of the Vietoris-Begle theorem.

1 Introduction

The project this paper reports on began with a seemingly simple question. Let $u: \mathbb{R}^n \to \mathbb{R}^n_+$ be the function $u(x) = (\max\{x_1, 0\}, \ldots, \max\{x_n, 0\})$. If $C \subset \mathbb{R}^n$ is convex and $D = u(C)$, is $D$ contractible? Easily visualized examples such as a disk pierced by the corner of $\mathbb{R}^3_+$ suggest an affirmative answer. It is not hard to prove that $D$ is contractible when $C$ is a line or line segment, but the argument does not easily generalize.

To see why this question might appeal to a mathematical economist, we review some fixed point theory. If $X$ and $Y$ are topological spaces, a correspondence $F: X \to Y$ is an assignment of a nonempty $F(x) \subset Y$ to each $x \in X$. Such an $F$ is convex (compact, etc.) valued if each $F(x)$ is convex (compact, etc.), and $F$ is upper hemicontinuous if, for each $x \in X$ and each open $V \subset Y$...
containing \( F(x) \), there is a neighborhood \( U \) of \( x \) such that \( F(x') \subset V \) for all \( x' \in U \). The Kakutani fixed point theorem asserts that if \( C \subset \mathbb{R}^n \) is nonempty, compact, and convex, and \( F: C \to C \) is an upper hemicontinuous, compact, convex valued correspondence, then there is an \( x^* \in C \) such that \( x^* \in F(x^*) \). This extension of the Brouwer fixed point theorem is used to prove the fundamental equilibrium existence results for game theory and general economic equilibrium, and it (and its infinite dimensional extensions) are frequently applied throughout economic theory.

Insofar as the conclusion of the Kakutani fixed point theorem is topological, the geometric hypotheses seem perhaps too strong. Eilenberg and Montgomery \[9\] showed how they can be relaxed. A space \( Z \) is acyclic with respect to a homology theory \( H_* \) with associated reduced homology \( \tilde{H}_* \) (cohomology theory \( H^* \) with associated reduced cohomology \( \tilde{H}^* \)) if \( \tilde{H}_n(Z) = 0 \) \((\tilde{H}^n(Z) = 0)\) for all \( n = 0, 1, 2, \ldots \). Recall that a metric space \( X \) is an absolute retract (AR) if, whenever \( e: X \to Z \) is an embedding of \( X \) as a closed subset of a metric space \( Z \), there is a retraction \( r: Z \to e(X) \). The Eilenberg-Montgomery fixed point theorem asserts that if \( C \) is a nonempty compact AR and \( F: C \to C \) is an upper hemicontinuous correspondence that is compact and acyclic (for Vietoris homology) valued, then there is an \( x^* \in C \) such that \( x^* \in F(x^*) \).

Since homology and cohomology are invariant under homotopy, a contractible space is acyclic for any homology or cohomology theory. We can imagine composing a convex valued correspondence with \( u \) to obtain a contractible valued correspondence, then applying the Eilenberg-Montgomery theorem. To be honest, what I initially thought might be an application of this sort turned out to be a mirage, and I know of no actual economic application, but nevertheless the issue still seems quite interesting.

We can place our problem in a more general setting. A vector lattice or Riesz space is a vector space \( L \) over the reals endowed with a partial order \( \geq \) such that: a) if \( x \geq y \), then \( x+z \geq y+z \) for all \( z \in L \) and \( \alpha x \geq \alpha y \) for all \( \alpha \geq 0 \); b) any two elements \( x, y \in L \) have a least upper bound \( x \lor y \) and a greatest lower bound \( x \land y \). Fix such an \( L \). The lattice cone of \( L \) is \( L_+ = \{ x \in L : x \geq 0 \} \).

Let \( u: L \to L_+ \) be the function \( u(x) = x \lor 0. \) (For any \( a \in L \) all of our results hold equally, with obvious modifications, for the functions \( x \mapsto x \lor a \) and \( x \mapsto x \land a \).)

An important basic result ([\[1\] p. 5]) is that \( L \) is a distributive lattice:

\[
x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \text{and} \quad x \land (y \lor z) = (x \land y) \lor (x \land z)
\]

for all \( x, y, z \in L \). Two other basic results will be important. Let \( |\cdot|: L \to L_+ \) be the function \( |x| = (x \lor 0) - (x \land 0) \).

**Lemma 1.** For all \( x_0, x_1 \in L \) and \( t \in [0, 1] \),

\[
u(x_0) - |u(x_1) - u(x_0)| \leq u(x_0) \land u(x_1) \leq u((1-t)x_0 + tx_1) \leq u(x_0) \lor u(x_1) \leq u(x_0) + |u(x_1) - u(x_0)|.
\]
Proof. For the first inequality we have the general computation
\[ y_0 - |y_1 - y_0| = y_0 + (y_1 - y_0) \land 0 - (y_1 - y_0) \lor 0 \leq y_0 + (y_1 - y_0) \land 0 = y_1 \land y_0. \]

For the second inequality we compute that
\[ u(x_0) \land u(x_1) = (x_0 \lor 0) \land (x_1 \lor 0) = (x_0 \land x_1) \lor 0 = ((1 - t)(x_0 \land x_1) + t(x_0 \land x_1)) \lor 0 \]
\[ \leq ((1 - t)x_0 + tx_1) \lor 0 = u((1 - t)x_0 + tx_1). \]

The third and fourth inequalities follow from symmetric computations. \( \square \)

**Corollary 1.** For each \( y \in L_+ \), \( u^{-1}(y) \) is convex.

A set \( A \subset L \) is **solid** if, for all \( y \in A \), \( A \) contains all \( x \in L \) such that \( |x| \leq |y| \). If, in addition to being a Riesz space, \( L \) is a (not necessarily Hausdorff) topological vector space and its topology has a base at the origin consisting of solid sets, then \( L \) is **locally solid**, and a **topological vector lattice**. A result of Roberts and Namioka ([I] p. 55) asserts that \( L \) is locally solid if and only if the function \((x, y) \mapsto x \lor y\) is uniformly continuous,\(^1\) and this is the case if and only if the function \((x, y) \mapsto x \land y\) is uniformly continuous. (It is possible ([I] p. 56) that \( u \) is continuous even when \( L \) is not locally solid.) From this point forward we assume that \( L \) is a topological vector lattice.

Recall that a metric space \( X \) is an **absolute neighborhood retract** (ANR) if, whenever \( e: X \to Z \) is an embedding of \( X \) as a closed subset of another metric space \( Z \), there is a neighborhood \( U \subset Z \) of \( e(X) \) and a retraction \( r: U \to e(X) \). An ANR is an AR if and only if it is contractible ([II], 11.2)]. If \( C \subset L \) is convex, then for any \( x_0 \in C \) the contraction \( c: C \times [0, 1] \to C \) given by \( c(x, t) = (1 - t)x + tx_0 \) is continuous by virtue of the continuity of the vector operations (even if \( L \) is not locally convex) so \( C \) is contractible, and thus an AR if it is an ANR.

Our first main result is:

**Theorem 1.** If \( C \subset L \) is convex and \( D = u(C) \) is metrizable, then \( D \) is an ANR and \( u|_C: C \to D \) is a homotopy equivalence, so \( D \) is an AR.

**Remark:** Among various ways that \( D \) may be metrizable even if \( L \) is not, we mention that Varadarajan [21] has shown that if \( L \) is the space of measures on a compact metric space with the weak topology, then \( L_+ \) is metrizable, but \( L \) itself is metrizable only under quite restrictive conditions. This case is common in economic applications.

The next result is one of the key ideas of the proof of Theorem [II].

---

\(^1\)That is, for any neighborhood of the origin \( V \) there is a neighborhood of the origin \( U \) such that \( x' \lor y' \in (x \lor y) + V \) for all \( x, y, x', y' \) such that \( x' \in x + U \) and \( y' \in y + U \).
Lemma 1. If \( y \in L_+ \) and \( U \subset L \) is a neighborhood of the origin, then there is a neighborhood \( W \subset U \) of the origin such that the convex hull of \( u^{-1}(y + W) \) is contained in \( u^{-1}(y + U) \).

Lemma 2. For any neighborhood \( U \subset L \) of the origin there is a neighborhood \( V \subset L \) of the origin such that \( w \in U \) whenever \( u, v \in V, \ w \in L, \text{ and } u \leq w \leq v \).

Proof. Without loss of generality we may assume that \( U \) is solid. Since \( \vee, \wedge, \) and the vector operations are continuous, there is a neighborhood \( V \) of the origin such that \( -(u \wedge 0) + (v \vee 0) \in U \) for all \( u, v \in V \). If \( u, v \in V, \ w \in L, \text{ and } u \leq w \leq v \), then \( u \wedge 0 \leq w \wedge 0 \) and \( w \vee 0 \leq v \vee 0 \), so \( |w| \leq -(u \wedge 0) + (v \vee 0) \) and thus \( w \in U \).

Proof of Proposition 1. Lemma 2 gives a neighborhood \( V \subset U \) of the origin such that \( w \in U \) whenever \( u, v \in V \) and \( u \leq w \leq v \). Since \( \vee, \wedge, \) and the vector operations are continuous, there is a neighborhood \( W \) of the origin such that \( y_0 - |y_0 - y_1| - y, y_0 + |y_0 - y_1| - y \in V \) for all \( y_0, y_1 \in y + W \). For any \( x_0, x_1 \in u^{-1}(y + W) \) and any \( t \in [0,1] \) we have \( u(x_0) - |u(x_0) - u(x_1)|, u(x_0) + |u(x_0) - u(x_1)| \in y + V \) and therefore \( u((1-t)x_0 + tx_1) \in y + U \) by virtue of Lemma 2.

2 A Sufficient Condition for Homotopy Equivalence

Following Milnor [18] let \( W_0 \) be the class of spaces which have the homotopy type of a countable CW-complex. As Milnor explains, results of Whitehead [24] and Hanner [11] imply that \( W_0 \) is also the class of all spaces that have the homotopy type of a separable ANR, the class of all spaces that have the homotopy type of a countable locally finite simplicial complex, and the class of all spaces that are dominated by a countable CW-complex.

Let \( X \) and \( Y \) be topological spaces, and let \( f : X \to Y \) be a map. A compressive pair for \( f \) is a pair \((V', V)\) where \( V \) and \( V' \) are open subsets of \( Y \) with \( V' \subset V \) such that \( f^{-1}(V') \) is contractible in \( f^{-1}(V) \): there is a continuous \( \xi : f^{-1}(V') \times [0,1] \to f^{-1}(V) \) such that \( \xi(\cdot,0) \) is the identity function of \( f^{-1}(V') \) and \( \xi(\cdot,1) \) is a constant function. We say that \( f \) is compressive if it is surjective and, for every \( y \in Y \) and every neighborhood \( V \subset Y \) of \( y \), there is a neighborhood \( V' \) of \( y \) such that \((V', V)\) is a compressive pair. Evidently Proposition 1 implies that:

Lemma 3. \( u|_C : C \to D \) is compressive.

Theorem 1 will follow from this and Theorem 2, which is our second main result. Prior results similar to Theorem 2 include various results in Section 3 of [2] and Proposition 2.1.8 of [23]. In all\(^2\)A space \( P \) dominates a space \( X \) if there are maps \( f : X \to P \) and \( g : P \to X \) such that \( gf \simeq 1 \).
of those results $X$ and $Y$ are assumed to be compact and finite dimensional, and the assumption imposed on the fibers of $f$ is (in effect) that they are compact AR’s.

**Theorem 2.** If $X$ is a connected element of $\mathcal{W}_0$, $Y$ is an ANR, and $f: X \to Y$ is a compressive surjection, then $f$ is a homotopy equivalence.

Fix $X$, $Y$, and $f: X \to Y$ satisfying the hypotheses of Theorem 2. The remainder of this section proves Theorem 2 by verifying the hypotheses of the following result, which is Theorem 1 of [24] with slightly less refined hypotheses. It has come to be known as Hatcher’s theorem.

**Proposition 2.** If $X$ and $Y$ are connected elements of $\mathcal{W}_0$, then a surjective map $f: X \to Y$ is a homotopy equivalence if and only if $f_*: \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all $n = 1, 2, \ldots$.

A compressive cover for $f$ is a collection $\mathcal{V}$ of compressive pairs such that $\{ V' : (V', V) \in \mathcal{V} \}$ is a locally finite cover of $Y$. For any open cover $\mathcal{U}$ of $Y$ there is a compressive cover $\mathcal{V}$ such that for all $(V', V) \in \mathcal{V}$, $V \in \mathcal{U}$: since $f$ is compressive, there is a collection $\mathcal{Z}$ of compressive pairs $(V', V)$ such that $V \in \mathcal{U}$ and $\{ V' :(V', V) \in \mathcal{Z} \}$ is a cover of $Y$, and since $Y$ is metric, hence paracompact, $\{ V' : (V', V) \in \mathcal{Z} \}$ has a locally finite refinement. We say that a compressive cover $\mathcal{V}$ is a star refinement of $\mathcal{V}$ if, for each $(V', \mathcal{V}) \in \mathcal{V}$, there is a $(V', \mathcal{V}) \in \mathcal{V}$ such that

$$\bigcup_{(V', \mathcal{V}) \in \mathcal{V}, V \in \mathcal{V}} V'_0 \subset V'. $$

**Lemma 4.** For any compressive cover $\mathcal{V}$ there is a compressive cover $\mathcal{V}$ that star refines it.

**Proof.** Since $Y$ is metric, [7, VIII.3] gives a refinement $\mathcal{U}$ of $\{ V' : (V', V) \in \mathcal{V} \}$ such that for each $U \in \mathcal{U}$, $\bigcup_{U' \in U, U' \neq \emptyset} U' \subset V'$ for some $(V', V) \in \mathcal{V}$. As we pointed out above, there is a compressive cover $\mathcal{V}$ such that for all $(V', \mathcal{V'}) \in \mathcal{V}$, $\mathcal{V'} \in \mathcal{U}$. \[ \square \]

The Arens-Eells embedding theorem (e.g., [10, p. 597]) implies that $Y$ can be isometrically embedded as a closed subset of a normed linear space $N_Y$. Fix a retraction $s_Y: V_Y \to Y$ of a neighborhood $V_Y \subset N_Y$ of $Y$. Let $W_Y \subset Y \times Y$ be a neighborhood of the diagonal $\{(y, y) : y \in Y\}$ such that $(1 - t)y_0 + ty_1 \in V_Y$ for all $(y_0, y_1) \in W_Y$ and $t \in (0, 1]$. For each positive integer $k$ let $D^k$ and $S^{k-1}$ be the closed unit ball in $\mathbb{R}^k$ and its boundary.

**Proposition 3.** Let $K$ be a finite simplicial complex, let $J$ be a subcomplex, and let $g: K \to Y$ and $\eta: J \to X$ be maps such that $f \eta = g|_J$. Then there is a continuous extension $\gamma: K \to X$ of $\eta$ such that $(g(p), f(\gamma(p))) \in W_Y$ for all $p \in K$.

Hatcher [12, Ch. 4] presents an accessible proof of Proposition 2 when $X$ and $Y$ are CW-complexes, and [12, Prop. A.11] also shows that a space that is dominated by a CW-complex is homotopy equivalent to a CW-complex.
Proof. (The following construction also occurs in [3] p. 436.) Let \( n \) be the dimension of \( K \). Let \( \mathcal{V}_n \) be a compressive cover such that for all \((V', V) \in \mathcal{V}_n, V \times V \subset W_Y\). Choose compressive covers \( \mathcal{V}_{n-1}, \ldots, \mathcal{V}_0 \) such that each \( \mathcal{V}_k \) is a star refinement of \( \mathcal{V}_{k+1} \). In the usual way we let the vertices of \( K \) be the standard unit basis vectors of a Euclidean space, and we let each simplex be the convex hull of its vertices. The Lebesgue number lemma implies that for each \( k \geq 1 \) there is \( \varepsilon_k > 0 \) such that for any point in \( K \) the ball of radius \( \varepsilon_k \) around that point is contained in \( g^{-1}(V) \) for some \((V', V) \in \mathcal{V}_k\). After sufficient repeated barycentric subdivision (e.g., [4]) the mesh of \( K \) is less than \( \min\{\varepsilon_1, \ldots, \varepsilon_n\} \), so we may assume that for each \( k \) there is some \((V', V) \in \mathcal{V}_k\) such that \( g(\sigma) \subset V \).

Let \( K^{(k)} \) be the \( k \)-skeleton of \( K \). Proceeding inductively, for \( k = 0, \ldots, n \) we will construct extensions \( \gamma_k: J \cup K^{(k)} \rightarrow X \) of \( \eta \) such that for each \( k \) there is some \((V', V) \in \mathcal{V}_k\) such that \( \gamma(\sigma) \subset f^{-1}(V) \). First construct an extension \( \gamma_0: J \cup K^{(0)} \rightarrow X \) by letting the image \( \gamma_0(v) \) of a vertex \( v \) of \( K \) that is not in \( J \) be any element of \( f^{-1}(g(v)) \). (Of course \( \gamma_0(v) \in f^{-1}(V) \) for any \((V', V) \in \mathcal{V}_0\) such that \( g(v) \in V \).)

Now suppose that \( \gamma_{k-1} \) has already been constructed, \( \sigma \) is a \( k \)-simplex of \( K \) that is not in \( J \), and \( \tau \) is a facet of \( \sigma \). Let \((\hat{V}', \hat{V})\) be an element of \( \mathcal{V}_{k-1} \) such that \( g(\tau) \subset \hat{V} \). Since every facet of \( \sigma \) intersects \( \tau \) and \( \mathcal{V}_{k-1} \) is a star refinement of \( \mathcal{V}_k \), there is a \((V', V) \in \mathcal{V}_k\) such that \( g(\partial \sigma) \subset V' \). Let \( \xi: f^{-1}(V') \times [0, 1] 
rightarrow f^{-1}(V) \) be continuous with \( \xi(\cdot, 0) \) the identity function of \( f^{-1}(V') \) and \( \xi(\cdot, 1) \) a constant function. For any homeomorphism \( h: D^k \rightarrow \sigma \) such that \( h(S^{k-1}) = \partial \sigma \), \( \gamma_{k-1}|_{\partial \sigma} \) has an extension \( \gamma_k|_{\sigma}: \sigma \rightarrow f^{-1}(V) \) (which is obviously continuous) given by setting

\[
\gamma_k|_{\sigma}(h((1-t)p)) = \xi(\gamma_{k-1}(h(p), t))
\]

for all \( p \in S^{k-1} \) and \( t \in [0, 1] \), and letting \( \gamma_k|_{\sigma}(h(0)) \) be the constant value of \( \xi(\cdot, 1) \). Combining such extensions for all \( k \)-simplices in \( K \) that are not in \( J \) constructs \( \gamma_k \). Finally let \( \gamma = \gamma_n \).

Proof of Theorem 2 For any \( n \geq 1 \) and continuous \( g: S^n \rightarrow Y \) the last result gives a continuous \( \gamma: S^n \rightarrow X \) such that \((g(p), f(\gamma(p))) \in W_Y \) for all \( p \in S^n \), so that \( h(p, t) = s_Y((1-t)g(t) + tf(\gamma(p))) \) is a homotopy between \( g \) and \( f \gamma \). Thus \( f_*: \pi_n(X) \rightarrow \pi_n(Y) \) is surjective. If \( \eta: S^n \rightarrow X \) is continuous and \( g: D^{n+1} \rightarrow Y \) is an extension of \( f \eta \), then the last result gives a continuous extension \( \gamma: D^{n+1} \rightarrow X \) of \( \eta \). Thus \( f_*: \pi_n(X) \rightarrow \pi_n(Y) \) is also injective, so Proposition 2 implies that \( f \) is a homotopy equivalence.

3 The Proof of Theorem 1

A convex subset of a topological vector space is contractible (even if the TVS is not locally convex, because the vector operations are continuous) so \( C \) is an element of \( W_0 \). In view of Lemma 3
once we have shown that $D$ is an ANR, Theorem 2 implies the other assertions of Theorem 1.
The following sufficient condition for a space to be an ANR is well known (e.g., [17, Prop. 8.3]).

Lemma 5. If $Z$ is a Hausdorff locally convex topological vector space, $K \subset Z$ is convex, $U \subset K$ is (relatively) open, $A \subset U$ is metrizable, and $r: U \to A$ is a retraction, then $A$ is an ANR.

Since $D$ is metrizable, the Arens-Eells theorem implies that there is an embedding $e: D \to N$ of $D$ in a normed linear space $N$ such that $\tilde{D} = e(D)$ is closed in $N$. Let $\tilde{u} = e \circ u|_{C}$, and let $\tilde{K}$ be the convex hull of $\tilde{D}$. Since a metric space is paracompact, the open cover of $\tilde{K} \setminus \tilde{D}$ whose elements are the balls (in $\tilde{K}$) centered at the various $\tilde{x} \in \tilde{K} \setminus \tilde{D}$ of radius one third of the distance from $\tilde{x}$ to $\tilde{D}$, has a locally finite refinement $\tilde{U}$. For each $\tilde{U} \in \tilde{U}$ choose an $x_{\tilde{U}} \in C$ such that the distance from $\tilde{U}$ to $\tilde{u}(x_{\tilde{U}})$ is less than twice the distance from $\tilde{U}$ to $\tilde{D}$. Let $\{\tilde{\varphi}_{\tilde{U}}\}_{\tilde{U} \in \tilde{U}}$ be a partition of unity subordinate to $\tilde{U}$, define $\rho: \tilde{K} \setminus \tilde{D} \to \tilde{D}$ by setting

$$
\rho(z) = \tilde{u}\left(\sum_{\tilde{U}} \tilde{\varphi}_{\tilde{U}}(z) x_{\tilde{U}}\right),
$$

and let $r: \tilde{K} \to \tilde{D}$ be the function that is the identity on $D$ and $\rho$ on $\tilde{K} \setminus \tilde{D}$. Lemma 5 implies that $D$ and $\tilde{D}$ are ANR’s if $r$ is a retraction, which is evidently the case if $D$ is continuous. Evidently $r$ is continuous at each point in $\tilde{K} \setminus \tilde{D}$. Fixing a point $\tilde{y} \in \tilde{D}$ and a neighborhood $\tilde{V} \subset \tilde{D}$, our goal is to find a neighborhood $\tilde{V}'' \subset \tilde{K}$ of $\tilde{y}$ such that $r(\tilde{V}'') \subset \tilde{V}$.

Proposition 1 gives a neighborhood $\tilde{V}' \subset \tilde{V}$ such that $\tilde{u}^{-1}(\tilde{V})$ contains the convex hull of $\tilde{u}^{-1}(\tilde{V}')$. Let $\delta > 0$ be small enough that $\tilde{V}'$ contains the ball of radius $\delta$ (in $\tilde{D}$) centered at $\tilde{y}$, and let $\tilde{V}''$ be the ball of radius $\delta/6$ (in $\tilde{K}$) centered at $\tilde{y}$. Fix a point $\tilde{z} \in \tilde{V}''$. If $\tilde{z} \in \tilde{D}$, then $r(\tilde{z}) = \tilde{z} \in \tilde{V}'' \cap \tilde{D} \subset \tilde{V}$, so assume that $\tilde{z} \in \tilde{V}'' \setminus \tilde{D}$.

Any $\tilde{U} \in \tilde{U}$ that contains $\tilde{z}$ is contained in the ball centered at some $\tilde{x} \in \tilde{K} \setminus \tilde{D}$ of radius $\alpha$, where $\alpha$ is one third of the distance from $\tilde{x}$ to $\tilde{D}$. Choose $\tilde{w} \in \tilde{U}$ whose distance from $\tilde{u}(x_{\tilde{U}})$ is less than twice the distance from $\tilde{U}$ to $\tilde{D}$, and thus less than $8\alpha$. The distance from $\tilde{z}$ to $\tilde{w}$ is less than $2\alpha$, so the distance from $\tilde{y}$ to $\tilde{u}(x_{\tilde{U}})$ is less than the distance from $\tilde{y}$ to $\tilde{z}$ plus $10\alpha$, and thus less than $6$ times the distance from $\tilde{y}$ to $\tilde{z}$ because the latter quantity is greater than $2\alpha$. Therefore $\tilde{u}(x_{\tilde{U}}) \in \tilde{V}'$ for all $\tilde{U} \in \tilde{U}$ that contain $\tilde{z}$, so $\sum_{\tilde{U}} \tilde{\varphi}_{\tilde{U}}(\tilde{z}) x_{\tilde{U}}$ is in the convex hull of $\tilde{u}^{-1}(\tilde{V}')$ and thus $\rho(\tilde{z}) \in \tilde{V}$, as desired.

---

4For the sake of self containment we include the proof. Suppose that $X$ is a metric space, $e: A \to X$ maps $A$ homeomorphically onto $e(A)$, which is closed. A generalization of the Tietze extension theorem due to Dugundji implies that $e^{-1}: e(A) \to A$ has a continuous extension $j: X \to K$. (Dugundji’s proof is a variant of our proof that $D$ is an ANR, and after reading that the reader may have little difficulty constructing his argument.) Then $V = j^{-1}(U)$ is a neighborhood of $e(A)$, and $e \circ r \circ j|_{V}: V \to e(A)$ is a retraction.
4 Relation with the Vietoris-Begle Theorem

We state two versions of the Vietoris-Begle theorem, which use Alexander-Spanier cohomology and homology respectively. The first might be regarded as the “standard” version. It asserts that if \( X \) and \( Y \) are paracompact Hausdorff spaces, \( f: X \to Y \) is a closed continuous surjection, and, for some \( n \geq 0 \), \( \tilde{H}^k(f^{-1}(y)) = 0 \) for all \( y \in Y \) and \( k < n \), then \( \tilde{H}^k(f): \tilde{H}^k(X) \to \tilde{H}^k(Y) \) is an isomorphism for \( k < n \) and an injection for \( k = n \). A particularly elegant proof is given in [15].

The second version (which we use in the proof of Theorem 3) is a dual result that was established by Volovikov and Ahn [22] and reproved by Dydak [8]. It asserts that if \( X \) and \( Y \) are compact metrizable spaces, \( f \) is a continuous surjection, and, for some \( n \geq 0 \), \( \tilde{H}^k(f^{-1}(y)) = 0 \) for all \( y \in Y \) and \( k < n \), then \( \tilde{H}^k(f): \tilde{H}^k(X) \to \tilde{H}^k(Y) \) is an isomorphism for \( k < n \) and a surjection for \( k = n \).

In each case \( \tilde{H}^* \) is an isomorphism if the fibers are acyclic.

Insofar as each passes from an assumption that the fibers of \( f \) are, in some sense, trivial, to a conclusion that \( f \) is or induces an isomorphism, Theorem 2 and the Vietoris-Begle theorem (including other versions in the literature) are quite similar. In this connection we should also mention the Main Theorem of [20], which asserts that if \( X \) and \( Y \) are path connected, locally compact, separable metric spaces, \( X \) is \( LC^n \), and for each \( y \in Y \), \( f^{-1}(y) \) is \( LC^{n-1} \) and \( (n-1) \)-connected, then \( Y \) is \( LC^n \) and \( f_*: \pi_k(X) \to \pi_k(Y) \) is an isomorphism for all \( k < n \) and a surjection when \( k = n \).

Theorem 3 further develops the relation between these results. In comparison with Theorem 2, it imposes an hypothesis on the fibers themselves, rather than the preimages of neighborhoods of points of \( Y \), but the domain must be compact and simply connected.

**Theorem 3.** If \( X \) and \( Y \) are compact connected ANR’s, \( X \) is simply connected, \( f: X \to Y \) is a continuous surjection, and, for each \( y \in Y \), \( f^{-1}(y) \) is contractible, then \( f \) is a homotopy equivalence.

We need a technical result that is a variant of [19, Lemma 1] and [20, Lemma 1].

**Lemma 6.** If \( A \) is a compact space, \( B \) is a Hausdorff space, \( g: A \to B \) is a map, \( b \in B \), and \( U \subset A \) is an open neighborhood of \( g^{-1}(b) \), then there is an open \( V \subset B \) containing \( b \) such that \( g^{-1}(V) \subset U \).

**Proof.** Otherwise for each open \( V \subset B \) containing \( b \) there would be an \( a_V \in A \setminus U \) such that \( g(a_V) \in V \). Since \( A \setminus U \) is compact, a subnet of \( \{a_V\} \) would converge to one of its elements, say...
a. Since $B$ is Hausdorff, continuity gives $g(a) = b$, but then $a \in g^{-1}(b) \cap (A \setminus U) = \emptyset$. □

As before we assume that $X$ and $Y$ are isometrically embedded in normed linear spaces $N_X$ and $N_Y$, we fix retractions retractive $r_X : U_X \to X$ and $s_Y : V_Y \to Y$ of neighborhoods $U_X \subset N_X$ and $V_Y \subset N_Y$ of $X$ and $Y$, and we let $W \subset Y \times Y$ be a neighborhood of the diagonal $\{(y, y) : y \in Y\}$ such that $(1 - t)y_0 + ty_1 \in V_Y$ for all $(y_0, y_1) \in W$ and $t \in [0, 1]$.

**Lemma 7.** If $y \in Y$ and $f^{-1}(y)$ is path connected, then, for any open $V \subset Y$ containing $y$, there is an open $V' \subset V$ containing $y$ such that any function $\eta_0 : \{-1, 1\} \to f^{-1}(V')$ has a continuous extension $\eta : [-1, 1] \to f^{-1}(V)$.

**Proof.** Since $f^{-1}(y)$ is compact there is a $\delta > 0$ such that the open $\delta$-ball $B_\delta$ in $N_X$ around $f^{-1}(y)$ is contained in $r_X^{-1}(f^{-1}(V))$. Lemma 6 gives an open $V' \subset V$ containing $y$ such that $f^{-1}(V') \subset B_\delta$. Suppose $\eta_0$ is a function from $\{-1, 1\}$ to $f^{-1}(V')$. Choose $x_-, x_1 \in f^{-1}(y)$ such that the distance from $\eta_0(-1)$ to $x_-$ and the distance from $\eta_0(1)$ to $x_1$ are both less than $\delta$. A satisfactory extension $\eta$ can be constructed by combining reparameterizations of the three paths $t \mapsto r_X((1 - t)\eta_0(-1) + tx_-)$, a path $\pi : [0, 1] \to f^{-1}(y)$ with $\pi(-1) = x_-$ and $\pi(1) = x_1$, and the path $t \mapsto r_X((1 - t)x_1 + t\eta_0(1))$. □

Fix a point $x_0 \in X$, and let $y_0 = f(x_0)$. Let $\tilde{X}$ and $\tilde{Y}$ be the universal covering spaces of $X$ and $Y$, with respect to the base points $x_0$ and $y_0$, and let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be the lift of $f$ with respect to these base points. The proof of Theorem 3 verifies the hypotheses of the following variant of Proposition 2 which is Theorem 3 of [24].

**Proposition 4.** If $X$ and $Y$ are connected elements of $\mathcal{W}_0$, a map $f : X \to Y$ is a homotopy equivalence if and only if $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism and $\tilde{H}_n(\tilde{f}) : \tilde{H}_n(\tilde{X}) \to \tilde{H}_n(\tilde{Y})$ is an isomorphism for each $n = 2, 3, \ldots$.

**Proof of Theorem 3** Lemma 7 implies that there is an open cover $\{V' : (V', V) \in \mathcal{V}\}$ of $Y$ where $\mathcal{V}$ is a collection of pairs $(V', V)$ each such that $V$ and $V'$ are open subsets of $Y$, $V \times V \subset W$, $V' \subset V$, and any function $\eta_0 : \{-1, 1\} \to f^{-1}(V')$ has a continuous extension $\eta : [-1, 1] \to f^{-1}(V)$. Consider a continuous $g : S^1 \to Y$. After sufficient subdivision $S^1$ is a simplicial complex, each of whose 1-simplices $\sigma$ satisfies $g(\partial \sigma) \subset V'$ for some $(V', V) \in \mathcal{V}$. For each vertex $v$ of this complex choose $\gamma_0(v) \in f^{-1}(g(v))$. As in the proof of Proposition 3 there is an extension $\gamma_1 : S^1 \to X$ of $\gamma_0$ such that $(g(p), f(\gamma(p))) \in W$ for all $p \in S^1$, so that $h(p, t) = s_Y((1 - t)g(t) + tf(\gamma(p)))$ is a homotopy between $g$ and $f \gamma$. Thus $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective.

Since $X$ is simply connected, it follows that $Y$ is simply connected, so $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism and $\tilde{X}$, $\tilde{Y}$, and $\tilde{f}$ are (up to irrelevant formalities) just $X$, $Y$, and $f$. As
we mentioned previously, since each fibre $f^{-1}(y)$ is contractible, it is acyclic, and $\tilde{X} = X$ is compact, so the dual Vietoris-Begle theorem of Volovikov-Ahn and Dydak implies that $\tilde{H}_n(\tilde{f})$ is an isomorphism for all $n \geq 2$, after which Proposition 3 implies that $f$ is a homotopy equivalence. (The dual Vietoris-Begle theorem is specific to Alexander-Spanier homology, and Whitehead uses singular homology. However, it is well known that Alexander-Spanier homology agrees with Čech homology on compact Hausdorff spaces, and Čech and singular homology agree on ANR’s [6, 14, 16].)

The assumptions of Theorem 3 imply those of Theorem 2 if each fiber is an AR.

Lemma 8. If $X$ is a locally compact metric space, $f: X \to Y$ is a surjective map, $V \subset Y$ is open, $y \in V$, and $f^{-1}(y)$ is a compact AR, then there is an open $V' \subset V$ containing $y$ such that $(V', V)$ is a compressive pair.

Proof. We can replace $V$ with a smaller neighborhood of $y$, so we may assume that $V \times V \subset W$. Fix a contraction $c: f^{-1}(y) \times [0, 1] \to f^{-1}(y)$ and a retraction $r: U \to f^{-1}(y)$ where $U \subset X$ is a neighborhood of $f^{-1}(y)$. Since $f^{-1}(y)$ is compact,

$$U' = \{ x \in U : (1-t)x + tr(x) \in r^{-1}(f^{-1}(V)) \text{ for all } t \in [0,1] \}$$

is an open neighborhood of $f^{-1}(y)$. Lemma 8 applied to a compact neighborhood of $f^{-1}(y)$ contained in $U'$, implies that there is an open $V' \subset V$ containing $y$ such that $f^{-1}(V') \subset U'$. Let $\xi: f^{-1}(V') \times [0,1] \to f^{-1}(V)$ be the function

$$\xi(x, t) = \begin{cases} r_X((1-2t)x + 2tr(x)), & 0 \leq t \leq \frac{1}{2}, \\ c(r(x), 2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

We conclude with an example of a map $f$ satisfying the hypotheses of Theorem 3, but with one fiber that is not an AR, so that Lemma 8 cannot be used to verify the hypotheses of Theorem 2. Kinoshita [13] (see also [17, 8.1]) created an example that came to be known as the tin can with a roll of toilet paper. This is the space $T = (D \times \{0\}) \cup (C \times [0,1]) \cup (S \times [0,1]) \subset \mathbb{R}^3$ where:

$$D = \{ x \in \mathbb{R}^2 : \|x\| \leq 1 \}, \quad C = \{ x \in \mathbb{R}^2 : \|x\| = 1 \}, \quad S = \{ \frac{\theta}{1+\theta}(\cos \theta, \sin \theta) : 0 \leq \theta < \infty \}.$$ 

There is an obvious contraction of $T$ that deformation retracts vertically onto $D \times \{0\}$, then compresses that set. Kinoshita gave a continuous function from $T$ to itself that does not have a fixed point, so, in view of the Eilenberg-Montgomery fixed point theorem, $T$ cannot be an AR.
Let $S^2 = \{ z \in \mathbb{R}^3 : \|z\| = 1 \}$. It is easy to see\footnote{Here is an explicit construction. Let $L$ be a line in $\mathbb{R}^2$ that passes through the origin, and let $C_L \cap S^2$. For $\varepsilon > 0$ let $D_{L,\varepsilon} = P_{L,\varepsilon} \cup Q_{L,\varepsilon} \cup R_{L,\varepsilon}$ where $P_{L,\varepsilon} = \{ x \in L : \|x\| \leq 1 + \varepsilon \} \times \{-\varepsilon \}$, $Q_{L,\varepsilon} = \{ x \in L : \|x\| = 1 + \varepsilon \} \times [-\varepsilon, 1+\varepsilon]$, and $R_{L,\varepsilon}$ is the union of $\{0,0,1+\varepsilon\}$, and for each pair of consecutive points $x_0, x_1 \in L \cap S$ the set $\{((1-t)x_0 + tx_1, \max\{1+\varepsilon - \min\{t,1-t\}\|x_1-x_0\|/\varepsilon, \varepsilon\}) : t \in [0,1]\}$. It is easy to construct maps $h_{L,\varepsilon} : C_L \times \{\varepsilon\} \rightarrow D_{L,\varepsilon}$ that combine to give a satisfactory $h.$} that there is a homeomorphism $h : S^2 \times (0,\infty) \rightarrow \mathbb{R}^3 \setminus T$ such that $T = \bigcap_{\varepsilon > 0} h(S^2 \times (0,\varepsilon))$. For $x \in \mathbb{R}^3 \setminus T$ let $h^{-1}(x) = (p(x), \alpha(x))$. Let $X \subset \mathbb{R}^3$ be a compact ball centered at the origin that contains $T$, let $f : X \rightarrow \mathbb{R}^3$ be the function that maps each $x \in T$ to the origin and maps each $x \notin T$ to $\alpha(x)p(x)$, and let $Y = f(X)$. Clearly $f : X \rightarrow Y$ satisfies the hypotheses of Theorem 3 but not those of Lemma 8.

References

[1] C. D. ALIPRANTIS and O. BURKINSHAW, *Locally Solid Riesz Spaces with Applications to Economics*, second ed., American Mathematical Society, 2003.

[2] S. ARMENTROUT, Homotopy properties of decomposition spaces, *Trans. Amer. Math. Soc.* 143 (1969), 499–507.

[3] S. ARMENTROUT and T. M. PRICE, Decomposition into compact sets with UV properties, *Trans. Amer. Math. Soc.* 141 (1969), 433–442.

[4] A. DOLD, *Lectures on Algebraic Topology*, Springer-Verlag, New York, 1980.

[5] J. DUGUNDJI, An extension of Tietze’s theorem, *Pacific J. Math.* 1 (1951), 353–367.

[6] J. DUGUNDJI, Remark on homotopy inverses, *Port. Math.* 14 (1955), 39–41.

[7] J. DUGUNDJI, *Topology*, Allyn and Bacon, Inc., Boston-London-Sydney, 1966.

[8] J. DYDAK, An addendum to the Vietoris-Begle theorem, *Topology and Its Applications* 23 (1986), 75–86.

[9] S. EILENBERG and D. MONTGOMERY, Fixed-point theorems for multivalued transformations, *Amer. J. Math.* 68 (1946), 214–222.

[10] A. GRANAS and J. DUGUNDJI, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[11] O. HANNER, Some theorems on absolute neighborhood retracts, *Ark. Mat.* 1 (1951), 315–360.

[12] A. HATCHER, *Algebraic Topology*, Cambridge University Press, New Cambridge, 2002.
[13] S. Kinoshita, On some contractible continua without the fixed point property, *Fund. Math.* **40** (1953), 96–98.

[14] Y. Kodama, On ANR for metric spaces, *Sci. Rep. Tokyo Kyoiku Daigaku, Sect A.* **5** (1955), 96–98.

[15] J. D. Lawson, Comparison of taut cohomologies, *Aequationes Math.* **9** (1973), 201–209.

[16] S. Mardešić, Equivalence of singular and Čech homology for ANR’s: Application to unicoherence, *Fund. Math.* **46** (1958), 29–45.

[17] A. McLennan, *Advanced Fixed Point Theory for Economics*, Springer, New York, 2018.

[18] J. Milnor, On spaces having the homotopy type of a CW-complex, *Trans. Am. Math. Soc.* **90** (1959), 272–280.

[19] S. Smale, A note on open maps, *Proc. Amer. Math. Soc.* **8** (1957), 391–391.

[20] S. Smale, A Vietoris mapping theorem for homotopy, *Proc. Amer. Math. Soc.* **8** (1957), 604–610.

[21] V. S. Varadarajan, Weak convergence of measures on separable metric spaces, *Sankyā A* **19** (1958), 15–22.

[22] A. J. Volovikov and N. L. Anh, On the Vietoris-Begle theorem, *Vestnik Moskov. Univ. Ser. I Math. Mekh.* **3** (1984), 70–71.

[23] F. Waldhausen, B. Jahren, and J. Rognes, *Spaces of PL Manifolds and Categories of Simplicial Maps*, Princeton University Press, 2013.

[24] J. Whitehead, Combinatorial homotopy I, *Bull. Am. Math. Soc.* **55** (1949), 213–245.