Abstract: It is proved that if any $\mathbb{Z}$-graded weak module for vertex operator algebra $V$ is completely reducible, then $V$ is rational and $C_2$-cofinite. That is, $V$ is regular. This gives a natural characterization of regular vertex operator algebras.

1. Introduction

Rationality, $C_2$-cofiniteness and regularity are probably the three most important concepts in representation theory of vertex operator algebras. Rationality, which is an analog of semisimplicity of a finite dimensional associative algebra or Lie algebra, asserts that the admissible module category, or $\mathbb{Z}_+$-graded weak module category is semisimple [DLM1,Z]. Originated from the modular invariance of trace functions in vertex operator algebra [Z], $C_2$-cofiniteness tells us that a certain subspace of a vertex operator algebra has finite codimension. Regularity, which is the strongest among the three concepts, claims that any weak module is a direct sum of irreducible ordinary modules [DLM1].

Both rationality and $C_2$-cofiniteness imply that there are only finitely many irreducible admissible modules up to isomorphism and each irreducible admissible module is ordinary (namely, each homogeneous subspace is finite dimensional) [DLM2,KL]. It follows immediately from definitions that regularity implies rationality. It is shown that regularity also implies $C_2$-cofiniteness [L]. The equivalence of regularity and rationality together with $C_2$-cofiniteness is established in [ABD]. But the connection among the three concepts has not been understood fully.

It is evident that the definition of rationality is natural, but definition of regularity is not natural or satisfactory. It seems more natural to define regularity by semisimplicity of the weak module category. This is successfully achieved in the present paper. In fact, a stronger result is obtained. That is, if any $\mathbb{Z}$-graded weak module is completely reducible, then the vertex operator algebra is regular.

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Clearly, if any ℤ-graded weak module is completely reducible, then the vertex operator algebra is rational. So it remains to prove that if any ℤ-graded weak module is completely reducible for a vertex operator algebra V, then V is ℂ²-cofinite. It is well known that the graded dual V′ of V is also a V-module [FHL]. By a result from [L], if L(0) is semisimple on the unique maximal weak module inside the completion of V′, then V is ℂ²-cofinite. The main idea is to use the universal enveloping algebra U(V) introduced in [FZ] to prove that any weak module generated by a single vector is a quotient of a ℤ-graded weak module and that L(0) acts semisimply on any irreducible submodule of a ℤ-graded weak module. It is worthy to point out that this is perhaps the first time that the universal enveloping algebra is to play a crucial role in the representation theory.

The paper is organized as follows. In Sect. 2, we review different notions of modules and ℂ²-cofiniteness, rationality, and regularity from [FLM, Z, DLM1, DLM2]. We also recall various results from [L, KL, ABD] on rationality, regularity and ℂ²-cofiniteness. Section 3 is devoted to the universal enveloping algebra U(V), essentially following from [FZ]. We show that any weak module generated by a single vector is a quotient of a ℤ-graded weak module. We establish the main theorem in Sect. 4. That is, if any ℤ-graded weak V-module is completely reducible, then V is ℂ²-cofinite, and consequently regular.

2. Preliminary

Throughout this paper, we assume that V is a simple vertex operator algebra (cf. [B, FLM]). We first recall notions of weak, admissible and ordinary modules from [FLM, Z, DLM1].

Definition 2.1. Let (V, Y, 1, ω) be a vertex operator algebra. A weak module M for V is a vector space equipped with a linear map

\[ Y_M : V \rightarrow (\text{End } M)[[z, z^{-1}]], \]

\[ v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}, \quad v_n \in \text{End } M, \]

satisfying the following conditions:

1. \( u_m w = 0 \), for \( u \in V, \ w \in M, \ m \in \mathbb{Z} \) and \( m \) large enough;
2. \( Y_M(1, z) = 1d_M \);
3. For any \( u, \ v \in V, \)

\[ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y_M(v, z_2)Y_M(u, z_1) = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0)v, z_2). \]

Definition 2.2. An admissible V-module M is a weak V-module which carries a ℤ⁺-grading \( M = \bigoplus_{n \in \mathbb{Z}^+} M(n) \) satisfying the following condition: If \( r, \ m \in \mathbb{Z}, \ n \in \mathbb{Z}^+ \) and \( u \in V_r, \) then \( u_m M(n) \subset M(r + n - m - 1). \)

Definition 2.3. An ordinary V-module M is a weak V-module such that \( M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda, \)

\( \dim M_\lambda < \infty \) and \( M_{\lambda + n} = 0 \) for fixed \( \lambda \in \mathbb{C} \) and small enough \( n \in \mathbb{Z}, \) where \( L(0)M_\lambda = \lambda \) and \( Y_M(\omega, z) = \sum_{m \in \mathbb{Z}} L(m)z^{-m-2}. \)

It is easy to see that an ordinary module is always admissible.
Definition 2.4. A vertex operator algebra $V$ is said to be rational if the admissible module category is semisimple.

We notice that the rationality is stronger than the complete reducibility of any admissible $V$-module as a submodule of a $\mathbb{Z}_+$-graded weak module is not necessarily $\mathbb{Z}_+$-graded.

It is proved in [DLM2] that if $V$ is rational, then there are only finitely many irreducible admissible modules up to isomorphism and any irreducible admissible module is ordinary. As pointed out in the Introduction, the $C_2$-cofiniteness (defined below) also implies the same results [KL].

Definition 2.5. A vertex operator algebra $V$ is said to be regular if any weak $V$-module $M$ is a direct sum of irreducible ordinary $V$-modules.

Definition 2.6. A vertex operator algebra $V$ is called $C_2$-cofinite if $\dim V / C_2(V) < \infty$, where $C_2(V) = \langle u - 2v \mid u, v \in V \rangle$.

It is shown in [KL] and [ABD] that the regularity is equivalent to rationality and $C_2$-cofiniteness. While the $C_2$-cofiniteness does not imply rationality (cf. [A1]), it is widely believed that rationality implies $C_2$-cofiniteness. One of the most important conjectures in the theory of vertex operator algebras is that rationality and regularity are equivalent. Many well known rational vertex operator algebras such as lattice type vertex operator algebras [B,FLM,D,A2,DJL], vertex operator algebras associated to the integrable highest weight modules for affine Kac-Moody algebras [FZ], and vertex operator algebras associated to the minimal series for Virasoro algebras [DMZ,W] are regular [DLM1]. Code and framed vertex operator algebras are also regular [M,DGH].

Although the notion of regularity is very useful in the proof of modular invariance of trace functions and the Verlinde formula [Z,DL,LM3,H], the definition is not natural at all as the irreducible objects in the weak module category are assumed to be ordinary. The main purpose of this paper is to define regularity naturally. For this we need the following definition.

Definition 2.7. A weak $V$-module $M$ is called $\mathbb{Z}$-graded if $M = \bigoplus_{i \in \mathbb{Z}} M(i)$ and for $r, m, n \in \mathbb{Z}, v \in V_n, v_r M(m) \subset M(m + n - r - 1)$.

We show in this paper that the regularity is equivalent to the complete reducibility of any $\mathbb{Z}$-graded weak module. As a result, the regularity is also equivalent to the semisimplicity of the weak module category.

3. Universal Enveloping Algebra

The universal enveloping algebra $U(V)$ for a vertex operator algebra $V$ was defined in [FZ] in connection with Zhu’s algebra $A(V)$. We will extensively use $U(V)$ to study the $\mathbb{Z}$-graded weak $V$-modules in this paper. For the purpose of this paper, we will modify the definition of universal enveloping algebra slightly with the help of the universal enveloping algebra of Lie algebras.

Let $V$ be a vertex operator algebra and $t$ an indeterminate. Consider the tensor product

$\mathcal{L}(V) = \mathbb{C}[t, t^{-1}] \otimes V$.

Since $\mathbb{C}[t, t^{-1}]$ is a vertex algebra such that

$Y(f(t), z)g(t) = f(t + z)g(t) = (e^{z \frac{d}{dt}} f(t)) g(t)$,

[B], $\mathcal{L}(V)$ is a tensor product of vertex algebras (cf. [DL,FHL]).
The $D$ operator of $\mathcal{L}(V)$ is given by $D = 1 + 1 \otimes L(-1)$, and

$$L(V) = \mathcal{L}(V)/D\mathcal{L}(V)$$

carries the structure of a Lie algebra with bracket

$$[u + D\mathcal{L}(V), v + D\mathcal{L}(V)] = u_0 v + D\mathcal{L}(V)$$

[B]. We use $a(n)$ to denote the image of $t^n \otimes a$ in $L(V)$. Then $\mathcal{L}(V)$ has a $\mathbb{Z}$-gradation given for homogeneous $a \in V$,

$$\deg a(n) = \text{wt} a - n - 1.$$ 

As $D$ increases degree by 1 then $D\mathcal{L}(V)$ is a graded subspace of $\mathcal{L}(V)$, so there is a naturally induced $\mathbb{Z}$-gradation on $L(V)$.

Let $U(L(V))$ be the universal enveloping algebra of $L(V)$. The $\mathbb{Z}$-gradation on $L(V)$ induces naturally a $\mathbb{Z}$-gradation on $U(L(V))$ so that for homogeneous $a^i \in V$,

$$\deg (a^1(i_1) \cdots a^n(i_n)) = \sum_{k=1}^n (\text{wt} a^k - i_k - 1).$$

Then $U(L(V)) = \bigoplus_{k=-\infty}^{\infty} U(L(V))(k)$, where $U(L(V))(k)$ denotes the subspace consisting of the elements of degree $k$, and $U(L(V))(m) \cdot U(L(V))(n) \subset U(L(V))(m+n)$. We set

$$U(L(V))^k(n) = \sum_{i \leq k} U(L(V))(n - i) \cdot U(L(V))(i).$$

Then $\{U(L(V))^k(n) \mid k \in \mathbb{Z}\}$ forms a fundamental neighborhood system of $U(L(V))(n)$. Denote by $\tilde{U}(L(V))(n)$ its completion. Then the direct sum $\tilde{U}(L(V)) = \sum_{n \in \mathbb{Z}} \tilde{U}(L(V))(n)$ is a complete topological ring.

Denote by

$$J = \left( \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} (u(r + m - i)v(n + i) - (-1)^r v(r + n - i)u(m + i)) \right.\
\left. - \sum_{i=0}^{\infty} \binom{m}{i} (u_{r+i}v)(m + n - i) \right) \mid m, n, r \in \mathbb{Z}, u, v \in V$$

the subset of $\tilde{U}(L(V))$. That is, $J$ is the set of Jacobi relations. Define $\mathcal{U}(V)$, the universal enveloping algebra of $V$, as the quotient of $\tilde{U}(L(V))$ modulo the two sided ideal generated by $J$. Clearly, the gradation in $\tilde{U}(L(V))$ induces a gradation in $\mathcal{U}(V)$.

Let $X = \{v^i \mid i \in I\}$ be a basis of $V$ consisting of homogeneous vectors, where $I$ is some countable index set. For any $\overline{m} = (m_i)_{i \in I}$, $m_i \in \mathbb{Z}$, define

$$I_{\overline{m}} = \sum_{i \in I} \sum_{k_i \geq m_i} \mathcal{U}(V)v^i(k_i),$$

then $I_{\overline{m}}$ is a left ideal of $\mathcal{U}(V)$.

**Lemma 3.1.** $\mathcal{U}(V)/I_{\overline{m}}$ is a weak $V$-module generated by $1 + I_{\overline{m}}$ such that $a_n$ acts as $a(n)$ for $a \in V, n \in \mathbb{Z}$. 
Proof. By the construction of $\mathcal{U}(V)$, it suffices to prove that $v_p a^1(n_1) \cdots a^k(n_k) \in I_{\overline{m}}$ for any $v \in X$, $a^1(n_1) \cdots a^k(n_k) \in \mathcal{U}(V)$, $p \in \mathbb{Z}$ and $p \gg 0$. We prove it by induction on $k$.

If $k = 1$, then for any $u, v \in X$, $p, q \in \mathbb{Z}$, we have

$$v_p u(q) = [v(p), u(q)] + u(q) v(p) = \sum_{i \geq 0} \left( \binom{p}{i} (v_i u)(p + q - i) + u(q) v(p) \right).$$

Since there exists some $N \in \mathbb{N}$ such that $v_i u = 0$, $i \geq N$, there are only finitely many terms in the first summand of the right-hand side. Hence by definition of $I_{\overline{m}}$ we have $(v_i u)(p + q - i) \in I_{\overline{m}}$, $p \gg 0$, $i \geq 0$. Clearly $u(q) v(p) \in I_{\overline{m}}$ for $p \gg 0$. Thus we get $v_p u(q) \in I_{\overline{m}}$ for $p \gg 0$.

Assume that $v_p a^1(n_1) \cdots a^k(n_k) \in I_{\overline{m}}$, $p \in \mathbb{Z}$, $p \gg 0$, $\forall v \in X$, $a^1(n_1) \cdots a^k(n_k) \in \mathcal{U}(V)$. Then

$$v_p a^1(n_1) \cdots a^{k+1}(n_{k+1}) = \sum_{j \geq 0} \left( \binom{p}{j} (v_j a^1)(p + n_1 - j) a^2(n_2) \cdots a^k(n_k) \right)$$

$$+ a^1(n_1) v(p) a^2(n_2) \cdots a^{k+1}(n_{k+1})$$

$\in I_{\overline{m}}$, $p \gg 0$. Also, by the induction assumption, we have $v(p) a^2(n_2) \cdots a^{k+1}(n_{k+1}) \in I_{\overline{m}}$, $p \gg 0$. Therefore we get $v_p a^1(n_1) \cdots a^{k+1}(n_{k+1}) \in I_{\overline{m}}$, $p \gg 0$. \( \square \)

Remark 3.2. We have the following results:

(1) If $\overline{m} = \overline{0}$, then $u_n(1 + I_{\overline{m}}) = 0$, $\forall u \in V$, $n \geq 0$. So $1 + I_{\overline{m}}$ is a vacuum-like vector. From [LL] and the simplicity of $V$, we immediately see that $\mathcal{U}(V)/I_{\overline{m}} \cong V$.

(2) It is possible that $\mathcal{U}(V)/I_{\overline{m}} = 0$. For example, fix $i_0 \in I$, let $m_i = 0$ if $i \neq i_0$ and $m_{i_0} = -1$, then $\mathcal{U}(V)/I_{\overline{m}}$ is a quotient of $V$ from (1). Identify $V$ with $\mathcal{U}(V)/I_{\overline{m}}$ such that $v = v_{-1}(1 + I_{\overline{m}})$ for any $v \in V$. Then $v_{i_0} = 0$ in $\mathcal{U}(V)/I_{\overline{m}}$ and consequently $\mathcal{U}(V)/I_{\overline{m}} = 0$.

Remark 3.3. It is obvious that the weak module constructed in Lemma 3.1 is $\mathbb{Z}$-graded as $I_{\overline{m}}$ is a $\mathbb{Z}$-graded left ideal.

The importance of this kind of weak module can be seen from the following corollary.

Corollary 3.4. If $W$ is a weak $V$-module generated by $w \in W$, then there exists $\overline{m} = (m_i)_{i \in I}$ such that $W$ is a quotient of $\mathcal{U}(V)/I_{\overline{m}}$. In particular, any weak module generated by a single vector is a quotient of a $\mathbb{Z}$-graded weak $V$-module.

Proof. From the definition of weak module, for any $i$, there exists $m_i \in \mathbb{Z}$ such that if $n \geq m_i$, $u_n^i w = 0$. The result follows immediately. \( \square \)
4. Main Theorem

The relation between \( \mathbb{Z} \)-graded weak modules and regularity is investigated in this section. We will use the \( \mathbb{Z} \)-graded weak module constructed in Sect. 3 to establish the main theorem of this paper. We start with the following version of Schur’s Lemma for the infinite dimensional representation.

**Lemma 4.1.** Let \( A \) be an associative algebra over \( \mathbb{C} \) and \( V \) a simple \( A \)-module of countable dimension. Then \( \text{Hom}_A(V, V) = \mathbb{C} \).

**Proof.** A proof of this result may exist in the literature. For the completeness of this paper, we present a short proof of this result. Clearly, \( \text{Hom}_A(V, V) \) is a division algebra over \( \mathbb{C} \). Fix \( 0 \neq v \in V \), and define a linear map from \( \text{Hom}_A(V, V) \) to \( V \) such that \( f \) is mapped to \( f(v) \). Then the map is injective. This implies that \( \text{Hom}_A(V, V) \) has countable dimension over \( \mathbb{C} \) from the assumption. Now fix \( 0 \neq f \in \text{Hom}_A(V, V) \) and consider the subfield \( E = \mathbb{C}(f) \) of \( \text{Hom}_A(V, V) \) generated by \( \mathbb{C} \) and \( f \). Then \( E \) has countable dimension over \( \mathbb{C} \). If \( f \) is not algebraic over \( \mathbb{C} \) then \( E \) is isomorphic to the field of rational functions \( \mathbb{C}(x) \), where \( x \) is an indeterminate. Since \( \mathbb{C}(x) \) has uncountable dimension over \( \mathbb{C} \), we have a contradiction. Consequently \( f \) is algebraic over \( \mathbb{C} \) and \( f \in \mathbb{C} \), the proof is completed. \( \square \)

**Lemma 4.2.** Let \( \overline{m} = (m_i)_{i \in I} \). Then the weak \( V \)-module \( M = \mathcal{U}(V)/I_{\overline{m}} \) is of countable dimension. In particular, any homogeneous subspace \( M(n) \) has countable dimension for \( n \in \mathbb{Z} \).

**Proof.** Since \( M \) is generated by \( 1 + I_{\overline{m}} \), we have \( M = \langle u_n(1 + I_{\overline{m}}) \mid u \in V, n \in \mathbb{Z} \rangle \).

Thus \( M \) has countable dimension. \( \square \)

**Proposition 4.3.** Let \( V \) be a vertex operator algebra such that any \( \mathbb{Z} \)-graded weak \( V \)-module is completely reducible. Then \( L(0) \) acts semisimply on any irreducible weak \( V \)-module. In particular, any irreducible weak module is \( \mathbb{Z} \)-graded.

**Proof.** Let \( W \) be an irreducible weak \( V \)-module. By Corollary 3.4, \( W \) is a quotient of a weak \( V \)-module \( M = \mathcal{U}(V)/I_{\overline{m}} \) for some \( \overline{m} \). As pointed out in Sect. 3, \( M = \bigoplus_{n \in \mathbb{Z}} M(n) \) is \( \mathbb{Z} \)-graded. Since \( M \) is completely reducible from the assumption, \( W \) is an irreducible weak \( V \)-submodule of \( M \). For any \( w \in W \), we can write \( w = w(i_1) + \cdots + w(i_k) \) with \( 0 \neq w(i_j) \in M(i_j), 1 \leq j \leq k, i_1 < \cdots < i_k \). In this case, we say \( w \) is of length \( k \), and denote \( \ell(w) = k \).

**Case 1.** If there exists \( 0 \neq w \in W \) with \( \ell(w) = 1 \), i.e., there exists \( w \in M(n) \) for some \( n \in \mathbb{Z} \).

Set \( W(m) = W \cap M(m), \forall m \in \mathbb{Z} \). Since \( W \) is irreducible, the argument given in the proof of Lemma 4.2 shows that \( W = \bigoplus_{m \in \mathbb{Z}} W(m) \). Note that \( \mathcal{U}(V)(0) \) is a subalgebra of \( \mathcal{U}(V) \) and \( L(0) \in \mathcal{U}(V)(0) \).

**Claim.** \( W(m) \) is an irreducible \( \mathcal{U}(V)(0) \)-module, \( \forall m \in \mathbb{Z} \).

For any nonzero elements \( u^1, u^2 \in W(m) \), note that we have \( W = \langle v_k u^1 \mid v \in V, k \in \mathbb{Z} \rangle \). Recall \( X = \{ v^i \mid i \in I \} \) is a basis of \( V \) consisting of homogeneous vectors. Let \( o(v^i) = v^i w_{tu^1} \) and extend notation to all of \( V \). Then we have

\[
W(m) = \langle o(v)u^1 \mid v \in V \rangle.
\]

In particular, \( u^2 = o(v)u^1 \) for some \( v \in V \). Thus the claim is proved.
Since \( L(0) \) is in the center of \( \mathcal{U}(V)(0) \) and \( M(m) \) is an irreducible \( \mathcal{U}(V)(0) \)-module with countable dimension, by Lemma 4.1, \( L(0) \) acts on \( W(m) \) as a constant, \( \forall m \in \mathbb{Z} \). Thus \( L(0) \) acts semisimply on \( W \).

Case 2. If \( \ell(w) > 1 \) for any \( 0 \neq w \in W \). Since \( \ell(w) < \infty \), \( \forall w \in W \), we can define 
\[
L = \min\{\ell(w) \mid 0 \neq w \in W\}.
\]

Assume \( 0 \neq x = \sum_{j=1}^{L} x(i_j) \in W \), where \( x(i_j) \in M(i_j) \) and \( i_1 < \cdots < i_L \). Call \((i_1, \ldots, i_L)\) the pattern of \( x \). For any homogeneous \( v \in V \) and \( n \in \mathbb{Z} \), either \( v_n x = 0 \) or \( 0 \neq v_n x \) has the pattern \((i_1 + wt v_n, \ldots, i_L + wt v_n)\). Let \( K \) be a subset of \( W \) consisting of vectors with pattern \((i_1, \ldots, i_L)\) together with 0. It is easy to see that \( K \) is a subspace of \( W \).

Claim. \( K \) is an irreducible \( \mathcal{U}(V)(0) \)-module.

Let \( u, w \) be any nonzero elements in \( K \). We can write \( u = u(i_1) + \cdots + u(i_L), w = w(i_1) + \cdots + w(i_L) \), where \( u(i_j), w(i_j) \in M(i_j), 1 \leq j \leq L \). Since \( W = \mathcal{U}(V)u \), there exists \( a^{n_i} \in \mathcal{U}(V)(n_i) \), \( 1 \leq i \leq T, n_1 < n_2 < \cdots < n_T \) such that \( w = a^{n_1} u + \cdots + a^{n_T} u \). Note that \( a^{n_j} u \) has pattern \((i_1 + n_j, \ldots, i_L + n_j)\). Clearly, the term \( a^{n_T} u(i_j) \) in \( a^{n_T} u = a^{n_T} u(i_1) + \cdots + a^{n_T} u(i_L) \) is zero if \( n_T > 0 \). This forces \( a^{n_T} u = 0 \) if \( n_T > 0 \). Otherwise, we have a nonzero vector in \( W \) whose length is less than \( L \), a contradiction. Similarly, if \( n_1 < 0 \), then \( a^{n_1} u = 0 \). Continuing in this way shows that there exists an \( i \) such that some \( n_i = 0 \) and \( w = a^0 u \). This implies that \( K \) is an irreducible \( \mathcal{U}(V)(0) \)-module.

Since \( L(0) \) is in the center of \( \mathcal{U}(V)(0) \) and \( K \) is an irreducible \( \mathcal{U}(V)(0) \)-module with countable dimension, by Lemma 4.1, \( L(0) \) acts on \( K \) as a constant. Since \( W \) is generated by \( u \in K \), \( L(0) \) acts semisimply on \( W \).

The following corollary is immediate.

**Corollary 4.4.** The complete reducibility of any \( \mathbb{Z} \)-graded weak \( V \)-module is equivalent to the semisimplicity of the \( \mathbb{Z} \)-graded weak \( V \)-module category. In particular, the complete reducibility of any \( \mathbb{Z}^+ \)-graded weak \( V \)-module is equivalent to the semisimplicity of the admissible \( V \)-module category.

We are now in the position to prove the main theorem of this paper.

**Theorem 4.5.** For a vertex operator algebra \( V \), if any \( \mathbb{Z} \)-graded weak \( V \)-module is completely reducible, then \( V \) is regular.

**Proof.** As mentioned before, we only need to prove that \( V \) is \( C_2 \)-cofinite. Recall from [FHL] that the graded dual \( V' = \oplus_{n=0}^{\infty} V_n^* \) of \( V \) is also an irreducible \( V \)-module. In particular, \( V' \) is a \( \mathcal{L}(V) \)-module. We extend the action of \( \mathcal{L}(V) \) from \( V' \) to \( V^* = \prod_{n \in \mathbb{Z}} V_n^* \) in an obvious way to make \( V^* \) an \( \mathcal{L}(V) \)-module. Following [L], we set 
\[
D(V') = \{f \in V^* \mid u_n f = 0, \forall u \in V, n \gg 0\}.
\]

Then \( D(V') \) is the unique maximal weak module inside \( V^* \) [L]. \( (D(V')) \) is denoted by \( D(V) \) in [L]. From the proof of Proposition 3.6 of [L] we see that \( (V/C_2(V))^* \) is a subspace of \( D(V') \). If \( V' = D(V') \), then \( (V/C_2(V))^* \) has a countable dimension. This implies that \( V/C_2(V) \) is a finite dimensional space. We now prove that \( V' = D(V') \).

If \( V' \neq D(V') \), then there exists \( f = (f_n)_{n \in \mathbb{Z}} \in D(V') \) such that there are infinitely many nonzero components of \( f \). Since \( L(0)f_n = nf_n, \forall n \in \mathbb{Z} \), \( f \) is not contained
in any finite dimensional $L(0)$-invariant subspace of $D(V')$. Let $W$ be the weak submodule of $D(V')$ generated by $f$. It follows from Corollary 3.4 that $W$ is a quotient of a $\mathbb{Z}$-graded weak $V$-module $M$. From the assumption that any $\mathbb{Z}$-graded module is completely reducible, we see that $W$ is a submodule of a $\mathbb{Z}$-graded weak module $M$. By Proposition 4.3, $L(0)$ is semisimple on $M$. In particular, $L(0)$ is semisimple on $W$. This implies that $f \in W$ is contained in a finite dimension $L(0)$-invariant subspace. This is a contradiction. The proof is complete. □

**Corollary 4.6.** Let $V$ be a vertex operator algebra, then the following statements are equivalent:

1) The weak $V$-module category is semisimple,
2) The $\mathbb{Z}$-graded weak $V$-module category is semisimple,
3) $V$ is regular.

**Proof.** By Theorem 4.5, 2) implies 3). The implication of 1) from 3) is obvious. We only need to show that 1) implies 2). So it is enough to prove that any irreducible weak $V$-module is $\mathbb{Z}$-graded. But this is clear from Proposition 4.3. □

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