Are some fractional derivatives really new?

Leila Gholizadeh Zivlaei\textsuperscript{1} and Angelo B. Mingarelli\textsuperscript{2}

School of Mathematics and Statistics, Carleton University, Ottawa, Canada

Abstract

We show that current results in the spectral theory of fractional Sturm-Liouville problems\cite{4}, as well as results contained in\cite{7},\cite{8} among many others, are simple consequences of ordinary Sturm-Liouville theory and a new notion of generalized derivatives.\cite{10}.

Keywords: Fractional differential equation, p-derivative, Sturm-Liouville, 2010 MSC: Primary 26A33, 34A08

1. Introduction

There has been a flurry of activity of late in the area of locally defined fractional differential operators and corresponding equations. Among these we cite chronologically,\cite{8},\cite{7},\cite{1}. We cannot begin to cite all the references related to these as, for example, Google Scholar refers to more than two-thousand references to paper\cite{8} alone! Still, the articles referred to here will suffice for our purpose. In the sequel, unless otherwise specified, $\alpha, \beta$ will denote real parameters with $0 < \alpha < 1$, $\beta \neq 0$, and $\beta \notin \mathbb{Z}^{-}$.

In\cite{8} Khalil, et al. define a function $f$ to be $\alpha-$differentiable at $t > 0$ if the limit,
\begin{equation}
T_\alpha^a f(t) = \lim_{h \to 0} \frac{f(t + ht^{\alpha-1}) - f(t)}{h},
\end{equation}
exists and is finite.

In the same year, Katugampola\cite{7} presented another (locally defined) derivative by requiring that, for $t > 0$,
\begin{equation}
T_\alpha^a f(t) = \lim_{h \to 0} \frac{f(te^{ht^{-\alpha}}) - f(t)}{h},
\end{equation}
exist and be finite.

Recently, other authors, e.g.,\cite{11}, considered minor variations in the definition\cite{11}, by asking that, for $t > 0$,
\begin{equation}
T_\alpha^a f(t) = \lim_{h \to 0} \frac{f(t + \frac{\Gamma(\beta)ht^{\alpha-\beta}}{(\beta - \alpha + 1)}) - f(t)}{h}.
\end{equation}

\textsuperscript{1}leilagh@math.carleton.ca
\textsuperscript{2}angelo@math.carleton.ca

Preprint submitted to Elsevier May 10, 2022
exist and be finite. Note that in each of the three definitions, the case \( \alpha = 1 \) leads to the usual definition of a derivative (see Section 2 in the case of (1.2)).

By way of an example, we show that a novel definition such as

\[
T^\alpha_t f(t) = \lim_{h \to 0} \frac{f(t + \sin(h) (\cos t)^{1-\alpha}) - f(t)}{h},
\]

exist and be finite for \( t \in [0, b], b < \pi/2 \), can also be handled by our methods and lead to results that are readily proved by converting them to a standard Sturm-Liouville problem. Observe, in passing, that the case where \( \alpha = 1 \) in (1.4), i.e., \( T^1_t f(t) = f'(t) \) whenever \( f \) is differentiable in the ordinary sense.

These four new definitions of a derivative function on the real line will be referred to occasionally as \textit{locally defined} derivatives as, in each case, knowledge of \( f \) is required merely in a neighborhood of the point \( t \) under consideration. In contrast, in the case of the more traditional Riemann-Liouville or Caputo fractional derivatives knowledge of the function \( f \) is required on a much larger interval including the point \( t \), see e.g., [4]. The previous, though very popular, derivatives defined in terms of singular integral operators shall not be considered here, however.

In this paper we show each of these four definitions, (1.1), (1.2), (1.3), (1.4) can be included in a more general framework of derivatives that implies, in part, that the results obtained in papers using either of the first three definitions (and the fourth, apparently new one) are actually a \textit{consequence of classical differential equations with ordinary derivatives}. We illustrate this in the case of [4] only, other papers in these areas being subject to similar considerations.

2. Preliminaries

Observe that, in each case, (1.1), (1.2), (1.3), the derivatives are defined for \( t > 0 \). As many authors have noticed one merely needs to replace \( t^{1-\alpha} \) by \( (t - a)^{1-\alpha} \) in each of these definitions to allow for a derivative to be defined on an a given interval \( (a, b) \), whether finite or infinite. Of course, the point \( t = 0 \) (resp. \( t = a \)) must be avoided in each of these definitions, otherwise said derivatives always vanishes.

The authors in each of [3, 7, 1] point out that the notion of differentiability represented by either one of the definitions (1.1), (1.2), (1.3) considered, is more general than the usual one by showing that there are examples whereby \( \alpha \)-differentiability does not imply differentiability in the usual sense, although the converse holds. However, note that for \( t > 0 \) the change of variable \( s = t + h t^{1-\alpha} \) shows that

\[
\lim_{h \to 0} \frac{f(t + h t^{1-\alpha}) - f(t)}{h} = \frac{1}{t^{\alpha-1}} \lim_{s \to t} \left( \frac{f(s) - f(t)}{s - t} \right).
\]

Consequently, the left hand limit exists if and only if the right hand limit exists. Thus, for \( t > 0 \), \( \alpha \)-differentiability is equivalent to ordinary differentiability and the only possible exception is at \( t = 0 \) (which is excluded anyway, by definition). A similar argument applies in the case of (1.3).

Insofar as (1.2) is concerned, observe that solving for \( h \) after the change of variable \( s = t e^{ht^{1-\alpha}} \) is performed, and \( t > 0 \), leads one to

\[
\lim_{h \to 0} \frac{f(t e^{ht^{1-\alpha}}) - f(t)}{h} = \frac{1}{t^{\alpha-1}} \left( \lim_{s \to t} \frac{f(s) - f(t)}{s - t} \right) \left( \lim_{s \to t} \frac{s - t}{\ln s - \ln t} \right) = \frac{1}{t^{\alpha-1}} \lim_{s \to t} \frac{f(s) - f(t)}{s - t}.
\]

Finally, for definition (1.4), note that, for \( t \in [0, b], b < \pi/2 \),

\[
\lim_{h \to 0} \frac{f(t + \sin(h) \cos(t)^{1-\alpha}) - f(t)}{h} = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} \lim_{s \to t} \frac{s - t}{\arcsin((s - t) \sec^{1-\alpha}(t))} = \cos^{1-\alpha}(t) f'(t),
\]
if either limit on the left or right exists.

It follows that both definitions, (1.1) and (1.2), coincide for \( t > 0 \) (see also [10]). Definition (1.3) is simply a re-scaling of (1.1) by a constant as can be verified by replacing \( h \) in (1.1) by \( \eta = hc \) where \( c = \Gamma(\beta)/\Gamma(\beta - \alpha + 1) \). Thus, strictly speaking, although it appears to be more general than (1.1), it isn’t really so. Definition (1.4) gives something new. These observations lead to the following theorem (see also Theorem 2.4 in [10]).

**Theorem 2.1.** For \( t > 0 \) (resp. \( t \geq 0 \)) a function is differentiable in the sense of either (1.1), (1.2), or (1.3), (resp. (1.4)) if and only if it is differentiable in the usual sense.

It is this previous result that will be key in applying, say, ordinary or classical Sturm-Liouville Theory to some of the questions at hand. We now summarize the theory in [10] for ease of use.

Let \( f : I \to \mathbb{R}, I \subseteq \mathbb{R} \) and \( p : U_\delta \to \mathbb{R} \) where \( U_\delta = \{(t,h) : t \in I, |h| < \delta \} \) for some \( \delta > 0 \) is some generally unspecified neighborhood of \((t,0)\). Unless \( \delta \) is needed in a calculation we shall simply assume that this condition is always met. Of course, we always assume that the range of \( p \) is contained in \( I \). In the sequel, \( L(I) \equiv L_1(I) \) is the usual space of Lebesgue integrable functions on \( I \).

For a given \( \alpha \), the **generalized derivative** or **\( p \)-derivative** in [10] is defined by

\[
D_p f(t) = \lim_{h \to 0} \frac{f(p(t,h)) - f(t)}{h},
\]

whenever the limit exists and is finite. Occasionally, we’ll introduce the parameter \( \alpha \) mentioned above into the definition so that the limit

\[
D_p^\alpha f(t) = \lim_{h \to 0} \frac{f(p(t,h,\alpha)) - f(t)}{h},
\]

will then be called the **\( p \)-derivative of order \( \alpha \)**. Since \( \alpha \) is a parameter (2.2) is actually a special case of (2.1). The main hypotheses on the function \( p \) are labeled \( H1^\pm \) and \( H2 \) in [10] and can be summarized together as follows:

**Hypothesis (H).** Given an interval \( I \subset \mathbb{R} \), in addition to requiring that for \( t \in I \), the function \( p(t,h) \), \( p_h(t,h) \) are continuous in a neighborhood of \( h = 0 \), we ask that for \( t \in I \) and for all sufficiently small \( \varepsilon > 0 \), the equation \( p(t,h) = t \pm \varepsilon \) has a solution \( h = h(t,\varepsilon) \) such that \( h \to 0 \) as \( \varepsilon \to 0 \). In addition, we assume that \( \frac{1}{p_h(.,0)} \in L(I) \), where \( p_h \) denotes the partial derivative of \( p \) with respect to \( h \) and that \( p_h(t,0) \neq 0 \).

We show that the notion of \( p \)-differentiability as defined in (2.1) is very general in that it includes the three definitions above. To this end, it suffices to show that hypothesis (H) is satisfied, see [10].

**Theorem 2.2.** Each of the derivatives defined in (1.1), (1.2), (1.3), and (1.4) above are \( p \)-derivatives for an appropriate function \( p \) satisfying (H).

**Proof.** Fix \( \alpha \). For (1.1) let \( p(t,h) = t + ht^{1-\alpha} \) for \( t > 0 \). Then, \( p_h(t,0) \neq 0 \) and \( p(t,h) = t \pm \varepsilon \) has the solution \( h(t,\varepsilon) = \pm \varepsilon t^{\alpha-1} \) which satisfies (H). As for (1.2), let \( p(t,h) = te^{ht^{-\alpha}} \). Then, once again, (H) is satisfied for \( t > 0 \) with \( h(t,\varepsilon) = t^{\alpha} \log (1 \pm \varepsilon) \). Finally, for (1.3), choose \( h(t,\varepsilon) = \pm \varepsilon t^{\alpha} \) where \( c \) is the constant defined by the ratio of the Gamma functions appearing in (1.3). As for the derivative defined by (1.4), \( p(t,h) = t + \sin(h) \cos(t)^{1-\alpha} \) satisfies \( p_h(t,0) = \cos^{1-\alpha}(t) \neq 0 \) for \( t \in [0,b] \), (with \( b < \pi/2 \)), and \( h(t,\varepsilon) = \arcsin(\pm \varepsilon \sec^{1-\alpha}(t)) \), the other conditions in (H) being clearly satisfied. \( \square \)

The following basic property is expected of a generalized derivative and indeed holds for the class considered here and in [10].
Theorem 2.3. (See Theorem 2.1 in [10].) Let $p$ satisfy (H). If $f$ is $p$-differentiable at $a$ then $f$ is continuous at $a$.

Corollary 2.1. (See Theorem 2.1 in [8]; and Theorem 2.2 in [7]) Let $f$ be $\alpha$-differentiable where the $\alpha$-derivative is defined in either (1.1), (1.2), (1.3), or (1.4). Then $f$ is continuous there.

Proof. The proof is left to the reader once we appeal to Theorem 2.2. □

The usual rules for differentiation are also valid in this more general scenario.

Proposition 2.1. (See Theorem 2.2 in [10].)

(a) (The Sum Rule) If $f, g$ are both $p$-differentiable at $t \in I$ then so is their sum, $f + g$, and
\[
D(f + g)(t) = Df(t) + Dg(t).
\]

(b) (The Product Rule) Assume that $p$ satisfies H1 and that for $t \in I$, $p(t, h)$ is continuous at $h = 0$. If $f, g$ are both $p$-differentiable at $t \in I$ then so is their product, $f \cdot g$, and
\[
D(f \cdot g)(t) = f(t) \cdot Dg(t) + g(t) \cdot Df(t).
\]

(c) (The Quotient Rule) Assume that $p$ satisfies (H). If $f, g$ are both $p$-differentiable at $t \in I$ and $g(t) \neq 0$ then so is their quotient, $f/g$, and
\[
D\left(\frac{f}{g}\right)(t) = \frac{g(t) \cdot Df(t) - f(t) \cdot Dg(t)}{g(t)^2}.
\]

As a result we obtain,

Corollary 2.2. (See Theorem 2.2 in [8]; Theorem 2.3 in [7]; and Theorem 4 in [1].) For each of the definitions (1.1), (1.2), (1.3), and (1.4) there holds an analog of the sum/product/quotient rule for differentiation of corresponding $p$-derivatives.

Proof. This is a simple application of Proposition 2.1 above (and Theorem 2.2). □

Proposition 2.2. (See [10], Theorem 2.4) Assume (H). Let $f$ be continuous and non-constant on $I$, and let $f$ be $p$-differentiable at $t \in I$. Let $g$ be defined on the range of $f$ and be differentiable at $f(t)$. Then the composition $g \circ f$ is $p$-differentiable at $t$ and
\[
D(g \circ f)(t) = g'(f(t)) \cdot Df(t).
\]

Corollary 2.3. (See [8], p.66 (iv), although the Chain Rule is not stated correctly there, and Theorem 2.3 in [7].) For each of the definitions (1.1), (1.2), (1.3), and (1.4) there holds an analog of the Chain rule for differentiation of corresponding $p$-derivatives in the form
\[
D^\alpha(g \circ f)(t) = g'(f(t)) \cdot D^\alpha f(t),
\]
where $D^\alpha$ is the corresponding $\alpha$-derivative in question, and the "prime" refers to ordinary differentiation.

Proof. The conclusion is now an immediate application of Proposition 2.2 above, on account of Theorem 2.2. □
The next result, when combined with Theorem 2.1, allows us to move from α-derivatives to ordinary derivatives.

**Proposition 2.3.** (See Theorem 2.4 in [10]) Let p satisfy (H). In addition, let

\[
\lim_{\varepsilon \to 0} \frac{h(t, \varepsilon)}{\varepsilon} \neq 0,
\]

(2.3)

where \( h(t, \varepsilon) \) is the function appearing in hypothesis (H). Then \( f \) is differentiable at \( t \) iff and only if \( f \) is \( p \)-differentiable at \( t \). In addition,

\[
D_p f(t) = p_h(t, 0) f'(t).
\]

(2.4)

**Proof.** See Theorem 2.2 for the \( h \) functions in question.

**Corollary 2.4.** (See Theorem 2.2 in [8]; Theorem 2.3 in [7]; and Theorem 1 in [1].) Let the \( \alpha \)-derivative be defined as in either (1.1) or (1.2) and let \( f \) be \( \alpha \)-differentiable at \( t \). Then,

\[
D^\alpha f(t) = t^{1-\alpha} f'(t).
\]

If \( f \) is \( \alpha \)-differentiable in the sense of (1.3), then

\[
D^\alpha f(t) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha + 1)} t^{1-\alpha} f'(t).
\]

(2.5)

If \( f \) is \( \alpha \)-differentiable in the sense of (1.4) then, for \( t \in [0, b] \), \( b < \pi/2 \),

\[
D^\alpha f(t) = (\cos t)^{1-\alpha} f'(t).
\]

(2.6)

**Proof.** Let \( t > 0 \). Using Theorem 2.2 we know that corresponding to (1.1) is the function \( p(t, h) = t + h t^{1-\alpha} \) and \( p_h(t, 0) = t^{1-\alpha} \). The result now follows immediately from the previous theorem. See the other choices of \( p \) in Theorem 2.2 and the discussion at the beginning of this section for the other three cases each of which is immediately verified.

**Corollary 2.5.** Let \( t > 0 \) (resp. \( t \geq 0 \)). Then \( f \) is \( \alpha \)-differentiable at \( t \) in the sense of anyone of (1.1), (1.2), or (1.3), (resp. (1.4)) if and only if \( f \) is differentiable at \( t \).

**Proof.** This is a simple consequence of Proposition 2.3 as the limit condition (2.3) is readily verified in each case.

3. A sample application: Fractional Sturm-Liouville theory

In [4] the authors consider the spectral theory associated with a fractional Sturm-Liouville problem of the form

\[
-T_0^\alpha T_0^\alpha y(x) + q(x) y(x) = \lambda y(x), \quad x \in [0, \pi]
\]

(3.1)

where \( \lambda \in \mathbb{C} \) is a generally complex parameter, \( q \) is real-valued and continuous on \( [0, \pi] \), and \( y \in C^{2\alpha}(0, \pi) \) is assumed by the authors in [4]. The boundary conditions determining the eigenvalue problem associated with (3.1) are of the form

\[
T_0^\alpha y(0) - hy(0) = 0
\]

(3.2)

\[
T_0^\alpha y(\pi) + H y(\pi) = 0
\]

(3.3)
where \(-h = \cot \gamma\), \(H = \cot \beta\). These are a generalization of so-called \textit{separated homogeneous boundary conditions}. Here the authors are using the notation in \cite{9}, p. 5.

The equivalent boundary conditions are (see \cite{10}, p. 7, eqs. (4.2) and (4.3)),

\[
y(a) \cos \mu - Dy(a) \sin \mu = 0, \quad (3.4)
y(b) \cos \nu + Dy(b) \sin \nu = 0, \quad (3.5)
\]

where \(a = 0\), \(b = \pi\), \(D = T_{0}^{\alpha}\), \(\mu = -\gamma\), and \(\nu = \beta\).

For \(x > 0\) we know from Corollary 2.4 and (1.1) that \(D_{x}^\alpha f(x) = x^{1-\alpha} f'(x) = T_{0}^{\alpha} f(x)\). Using this we find that (3.1) is equivalent to

\[
-x^{1-\alpha}(x^{1-\alpha} y'(x))' + q(x) y(x) = \lambda y(x).
\]

(3.6)

Thus, if \(y\) is a solution of (3.1) then \(y\) must be a solution of (3.6) which implies that \(y \in AC[0, \pi]\) and \(x^{1-\alpha} y' \in AC[0, \pi]\) as per ordinary considerations of Sturm-Liouville theory, see \cite{9}. It follows that the quantities appearing in the boundary conditions (3.1)-(3.3) (or (3.4)-(3.5) for this differential operator) always exist.

Now (3.6) of the form (4.10) in \cite{10} (with \(x\) there replaced by \(t\)), i.e.,

\[
-p_{h}(x, 0) \left(p_{h}(x, 0) P(x) y'(x)\right)' + Q(x) y = \lambda W(x) y, \quad x \in I,
\]

(3.7)

where \(I = [0, \pi]\), \(P(x) = 1\), \(Q(x) = q(x) x^{\alpha - 1}\), \(W(x) = x^{\alpha - 1}\) and, of course, \(p_{h}(x, 0) = x^{1-\alpha}\), as we saw in Theorem 2.2 above.

We perform the change of variable (as per (4.13) in \cite{10}) \(x \rightarrow t\) in (3.6) defined by

\[
t = \frac{x^{\alpha}}{\alpha}, \quad y(x) = z(t)
\]

(3.8)

to obtain a new equation in \(t\) (see (4.14) in \cite{10}),

\[
-z''(t) + q^*(t) z(t) = \lambda z(t),
\]

(3.9)

where \(q^*(t)\) is \(q(x)\) with \(x\) rewritten as a function of \(t\). This transformation (3.8) is isospectral as the same \(\lambda\) appears in both, so the spectrum of both eigenvalue problems (3.1)-(3.3) and (3.4)-(3.5) is preserved. The \(x\)-interval \([0, \pi]\) now goes into the \(t\)-interval \([0, \pi \alpha /\alpha]\). Now ordinary eigenvalue asymptotics for the Sturm Liouville equation (see Theorem 4.1 in \cite{10}, or \cite{11}) give

\[
\sqrt{\lambda_{n}} \sim \frac{n \pi}{\int_{0}^{\pi /\alpha} 1 \, dt} = \frac{n \pi}{\pi^{\alpha/\alpha}} = an^{1-\alpha},
\]

(3.10)
as \(n \to \infty\), which is what the authors found as the leading term in Theorem 11 of \cite{4}. The additional terms in the eigenvalue expansion are now a consequence of the argument on pp. 9-10 of \cite{9} as is the rest of the paper, since (3.1) is really a.e. a Sturm-Liouville equation in disguise. Thus, we can derive the existence of infinitely many real eigenvalues and their spectral asymptotics with minimal effort.

Of course, the same sort of reasoning will lead to an equivalent “Theorem 11” for each of the other two definitions (1.2), (1.3). In the case of (1.2) the eigenvalue asymptotics there for the boundary conditions (3.1)-(3.3) are equivalent to those associated with definition (1.1) with the same boundary conditions because of Corollary 2.3.
4. Existence and oscillation theorems for fractional Sturm-Liouville equations

As a bonus, we show here that a (new) fractional Sturm Oscillation Theorem also follows from the preceding discussion. We state it here and give an essentially immediate proof of the existence of the eigenvalues, their asymptotic distribution and, in addition, the oscillation of the eigenfunctions.

**Theorem 4.1.** With a derivative defined as in either (1.1) or (1.2), the boundary value problem (3.1)-(3.3) has an infinite number of real eigenvalues

\[ \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \]

having no finite limit point and, with this labeling, the eigenfunction \( y(x, \lambda_n) \) corresponding to the eigenvalue \( \lambda_n \) has exactly \( n \) zeros in \((0, \pi)\). The eigenvalues admit the asymptotic distribution (3.10).

**Proof.** The transformations and equivalences mentioned above map this problem to one of the form (3.6) for which the result holds in the regular case, see [2], Chapter 8.

An analogous result holds for equations where the derivative is defined as in (1.3). We assume, for simplicity, that \( I = [0, \pi] \) as before.

**Theorem 4.2.** With the derivative \( T_\alpha^0 \) as defined in (1.3), the boundary problem (3.1)-(3.3) has an infinite number of real eigenvalues

\[ \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \]

having no finite limit point and, with this labeling, the eigenfunction \( y(x, \lambda_n) \) corresponding to the eigenvalue \( \lambda_n \) has exactly \( n \) zeros in \((0, \pi)\). The eigenvalues admit the asymptotic distribution

\[ \sqrt{\lambda_n} \sim \frac{\Gamma^2(\beta - \alpha + 1)\alpha n\pi^{1-\alpha}}{\Gamma^2(\beta)}, \quad n \to \infty. \tag{4.1} \]

**Proof.** Since the derivative is now given by (2.5) for \( t > 0 \), (3.1) becomes

\[ \frac{\Gamma^2(\beta)}{\Gamma^2(\beta - \alpha + 1)} \left( x^{1-\alpha} y'(x) \right)' + q(x) y(x) = \lambda y(x), \tag{4.2} \]

or, after the same change of variable, (3.8), we get

\[ - z''(t) + q^*(t)z(t) = \lambda \frac{\Gamma^2(\beta - \alpha + 1)}{\Gamma^2(\beta)} z(t), \tag{4.3} \]

in lieu of (4.8) (here \( q^* \) contains terms involving \( \alpha, \beta \) as well). The existence of only real eigenvalues is now a consequence of regular Sturm-Liouville theory. In addition, since the eigenvalues are simply shifted by the scaling constant in (1.3), we get the asymptotic distribution (4.1) in lieu of (3.10).

**Remark 4.1.** Observe that when we set \( \beta = \alpha = 1 \) in (4.1) we get the classical asymptotic estimate,

\[ \sqrt{\lambda_n} \sim n \text{ as } n \to \infty. \]

In [10] we showed that, indeed, one may go much further in the study of eigenvalue problems for operators defined by the derivatives (1.1) or (1.2). So long as we have a theorem for regular (or even singular) Sturm-Liouville eigenvalue problems we can obtain a parallel result for differential operators and their equations as defined by either (1.1) or (1.2). A different kind of theorem is next.
Theorem 4.3. With the derivative $T^\alpha_0$ as defined in (1.4), the boundary problem (3.1)-(3.3) has an infinite number of real eigenvalues

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

having no finite limit point and, with this labeling, the eigenfunction $y(x, \lambda_n)$ corresponding to the eigenvalue $\lambda_n$ has exactly $n$ zeros in $(0, \pi)$. The eigenvalues now admit the asymptotic distribution

$$\sqrt{\lambda_n} \sim \frac{n\pi}{\int_0^b \left(\frac{1}{\sec t}\right)^{1-\alpha} dt}, \quad n \to \infty. \quad (4.4)$$

Proof. This generalized derivative is given by (2.6) for $t \geq 0$. The existence of only real eigenvalues is also a consequence of regular Sturm-Liouville theory. Thus, (3.1) becomes

$$-\left((\cos t)^{1-\alpha} y(x)\right)' + q(x) y(x) = \lambda y(x), \quad x \in [0, b], \quad (4.5)$$

or,

$$-\left((\cos t)^{1-\alpha} y(x)\right)' + q(x)(\sec t)^{1-\alpha} y(x) = \lambda(\sec t)^{1-\alpha} y(x), \quad x \in [0, b], \quad (4.6)$$

which is of the form $-\left(P(x)y\right)' + Q(x)y = \lambda R(x)y$ with the appropriate separated homogeneous boundary conditions given by (3.4)-(3.5). We can apply a variety of theorems here but it suffices to use [3] to show the existence of a countably infinite number of real eigenvalues having no finite point of accumulation whose asymptotic behaviour is given by

$$\sqrt{\lambda_n} \sim \frac{n\pi}{\int_0^b \left(\frac{1}{\sec t}\right)^{1-\alpha} dt}, \quad n \to \infty. \quad (4.4)$$

5. Conclusion

In this paper we showed that any study of differential equations of fractional Sturm-Liouville type with derivatives defined by either (1.1), (1.2), (1.3), or (1.4) can be transformed to a standard Sturm-Liouville problem via the general theory presented in [10] followed by a few additional changes of variable, if necessary. The conclusions are then obtained using the analogous result(s) and the voluminous literature related to Sturm-Liouville theory alone. Working backwards from the Sturm-Liouville case, if necessary, would give new “fractional” results. The results obtained here are not necessarily restricted to fractional boundary values problems of Sturm-Liouville type of the form (3.1)-(3.3) but can also be formulated and proved for more general equations of the form

$$-T_0^\alpha(p(x)T_0^\alpha y(x)) + q(x)y(x) = \lambda w(x)y(x), \quad x \in [0, \pi]$$

say, where the terms $p(x), w(x)$ may both change sign, in which case a doubly infinite sequence of positive and negative eigenvalues is obtained having a known asymptotic distribution, see [3] for details. These results need not be formulated here as they follow immediately from considerations already described.

References

References

[1] M. Abu-Shady, M. K. A. Kaabar, A Generalized Definition of the Fractional Derivative with Applications, Mathematical Problems in Engineering, Volume 2021, Article ID 9444803, 9 pages
[2] F.V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, (1964).
[3] F. V. Atkinson and A. B. Mingarelli, *Asymptotics of the number of zeros and of the eigenvalues of general weighted Sturm-Liouville problems* J. für die Reine und Ang. Math. (Crelle), 375/376 (1987), 380-393.
[4] E. Bas, F. M. Turk, R. Ozarslan, A. Ercan, *Spectral data of conformable Sturm-Liouville direct problems*, Analysis and Math. Phys. 11:8 (2021)
https://doi.org/10.1007/s13324-020-00428-6
[5] W. N. Everitt, D. Race, *On necessary and sufficient conditions for the existence of Carathéodory solutions of ordinary differential equations*, Quaestiones Mathematicae, 2 (1978), 507-512.
[6] B. Jin, *Fractional Differential Equations* - *An Approach via Fractional Derivatives*, Applied Mathematical Sciences, Vol. 206, Springer, Switzerland, (2021).
[7] U. N. Katugampola, *A new fractional derivative with classical properties*, e-print [arXiv:1410.6835] (2014).
[8] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, J. Comp. Applied Math., 264 (2014), 65–70.
[9] B. M. Levitan, I. S. Sargsjan, *Introduction to Spectral Theory: Ordinary Differential Operators* Translations of Mathematical Monographs, Vol. 39; Amer. Math Soc. Providence, (1975)
[10] Angelo B. Mingarelli, *On generalized and fractional derivatives and their applications to classical mechanics*, J. Phys. A: Math. Theor. 51 (2018) 365204 (18 pp)
[11] A. Zettl, *Sturm-Liouville Theory*, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, (2010).