State dependent diffusion in a bistable potential: conditional probabilities and escape rates

Miguel V. Moreno,1 Daniel G. Barci,2 and Zochil González Arenas3

1Departamento de Física, Universidade Federal Fluminense and National Institute of Science and Technology for Complex Systems, Av. Gal. Milton Tavares de Souza s/n, Campus da Praia Vermelha, 24210-346 Niterói, RJ, Brazil
2Departamento de Física Teórica, Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier 524, 20550-013, Rio de Janeiro, RJ, Brazil.
3Departamento de Matemática Aplicada, IME, Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier 524, 20550-013, Rio de Janeiro, RJ, Brazil

(Dated: December 3, 2019)

We consider a simple model of a bistable system under the influence of multiplicative noise. We provide a path integral representation of the overdamped Langevin dynamics and compute conditional probabilities and escape rates in the weak noise approximation. The saddle-point solution of the functional integral is given by a diluted gas of instantons and anti-instantons, similarly to the additive noise problem. However, in this case, the integration over fluctuations is more involved. We introduce a local time reparametrization that allows its computation in the form of usual Gaussian integrals. We found corrections to the Kramers escape rate produced by the diffusion function which governs the state dependent diffusion for arbitrary values of the stochastic prescription parameter.

I. INTRODUCTION

The physics of thermal or noise activation over a barrier has a long history. Nowadays, it is an important research topic due to the wide range of applications in several areas of science, such as physics, chemistry and biology as well [1]. The simplest model to study this problem is a classical particle in a bistable potential, \( U(x) \), whose dynamics is driven by an overdamped Langevin equation with additive white noise. In this context, an important physical quantity is the rate at which the particle escape out of a minimum of the potential. The seminal work of Kramers [2] stated the very simple formula

\[
r_{\text{add}} = \sqrt{\frac{\omega_{\text{min}} \omega_{\text{max}}}{\pi}} e^{-\frac{\Delta U}{\sigma^2}}
\]

where \( r_{\text{add}} \) is the escape rate, \( \Delta U = U(x_{\text{max}}) - U(x_{\text{min}}) \) is the height of the potential barrier, \( \sigma^2 \) is the noise intensity and \( \omega_{\text{min}} = U''(x_{\text{min}}) \) and \( \omega_{\text{max}} = U''(x_{\text{max}}) \) are the local curvatures of the potential at its minimum (\( x_{\text{min}} \)) and its maximum (\( x_{\text{max}} \)), respectively (primes mean derivative with respect to \( x \)). We use the notation \( r_{\text{add}} \) to emphasize that this expression for the escape rate was computed assuming an additive noise stochastic differential equation. Equation (1.1) is valid in the weak noise or high barrier approximation \( \sigma^2 \ll \Delta U \).

Since this well-established result was defined, a lot of work has been done in order to compute more accurate expressions suitable to be applied to more realistic situations. The generalization of Eq. (1.1) to multidimensional systems was (and still is) a big challenge [3]. Moreover, generalizations to different types of noise probability distributions have been also considered [4,5].

On the other hand, there is an increasing interest for multiplicative noise stochastic systems. Some examples of multiplicative noise dynamics are given by the diffusion of particles near a wall [10,14], micromagnetic dynamics [15-17] and non-equilibrium transitions into absorbing states [18]. There are two particular stochastic phenomena in which multiplicative noise plays an important role: noise-induced phase transitions [19,22] and stochastic resonance [24-27]. In the last case, the escape rate is at the stem of the physical description of the observed phenomenology.

One of the main questions that we address in this paper is how the Kramers escape rate of Eq. (1.1) is modified when the dynamics is driven by a general multiplicative noise, modeled by a diffusion function \( g(x) \). This topic have been rarely treated in the past and there is some controversy in the literature [28-33]. In particular, we study the dependence of the escape rate on the stochastic prescription, necessary to correctly define the multiplicative noise Langevin equation. This point is particularly relevant in order to compare analytic results with numerical simulations. Our main result is

\[
r_{\text{mult}} = g^2(x_{\text{max}}) \sqrt{\frac{\omega_{\text{min}} \omega_{\text{max}}}{\pi}} e^{-\frac{\Delta U_{\text{eq}}}{\sigma^2}}.
\]

We used the notation \( r_{\text{mult}} \) to denote the escape rate in the multiplicative noise case. In general, we observe that the Arrhenius form of the Kramers result still remains. Another similarity with Eq. (1.1) is that the escape rate does not depend on details, either of the potential or of the diffusion function. Instead, it only depends on the local properties of these functions at the maximum and minima of the potential. On the other hand, there are important differences between both results. Firstly, the original potential \( U(x) \) has been replaced by the equilibrium potential \( U_{\text{eq}}(x) \), obtained from the solution of the asymptotic stationary Fokker-Planck equation (Eq. (2.3)). The barrier height is, in this case, \( \Delta U_{\text{eq}} = U_{\text{eq}}(x_{\text{max}}) - U_{\text{eq}}(x_{\text{min}}) \). On the other hand, local curvatures have been renormalized by the diffusion function as \( \tilde{\omega}_{\text{min}} = g^2(x_{\text{min}})U''_{\text{eq}}(x_{\text{min}}) \) and \( \tilde{\omega}_{\text{max}} = g^2(x_{\text{max}})U''_{\text{eq}}(x_{\text{max}}) \). Finally, there is an overall factor \( g^2(x_{\text{max}}) \) coming from a careful treatment of fluctuations. In the rest of the paper, we better describe the model and the technique used to compute Eq. (1.2), and discuss the result in more detail.

Multiplicative stochastic processes can be study with different theoretical approaches. For numerical simulations [34], the Langevin approach seems to be more adequate. The Fokker-Planck equation is perhaps more appropriate to develop analytic calculations, specially in the long time station-
ary limit. In this context, techniques such as mean fields, perturbation theory and even renormalization group are also available [35]. On the other hand, the path integral formulation of stochastic processes is the more natural technique to compute correlation and response functions [36]. Important progress has been recently reached in the path integral representation of multiplicative noise processes [37–42], despite the fact that this topic has been studied for a long time [43].

The escape rate is just one ingredient of a more general problem that is the computation of conditional probabilities. Equilibrium properties, such as detailed balance, can be cast in terms of the conditional probability and its time reversal. Time reversal transformations, detailed-balance relations, as well as microscopic reversibility in multiplicative processes were studied in detail in Ref. [44]. More recently, we have presented a useful path integral technique to compute weak noise expansions [44]. The integration over fluctuations in the multiplicative case is not trivial. The reason is that the diffusion function produces an integration measure that resembles a curved time axis [45]. We have provided a local time reparametrization in order to integrate fluctuations [44]. In this paper, we compute the conditional probability of finding a particle in a well at large times $t/2$, provided it was in the same or the other well at $-t/2$. In the weak noise approximation, saddle points provide a set of diluted instanton and anti-instanton solutions. The diluted instanton gas approximation was first introduced in the context of quantum mechanics to compute the tunneling probability across a potential barrier [46]. In the context of an additive stochastic process, it was developed with great detail in Refs. [47, 48]. From a technical point of view, we generalize the calculation of Ref. [48] to the multiplicative noise case, using the time reparametrization techniques introduced in Ref. [49].

The paper is organized as follows. In the next section, we present the equilibrium properties of a particle in a double-well potential under state dependent diffusion. In section III, we briefly review the path integral representation of a conditional probability in a multiplicative process and we show, in section IV, how to integrate fluctuations. We develop the diluted instanton gas approximation in section V, where we compute conditional probabilities and the escape rate. Finally, we discuss our results in section VI. We lead to the Appendix A some details of the calculation.

II. EQUILIBRIUM PROPERTIES OF A PARTICLE IN A DOUBLE-WELL POTENTIAL UNDER STATE DEPENDENT DIFFUSION

In this section, we describe the equilibrium properties of a model consisting of a single particle in a double-well potential coupled with a thermal bath with state dependent diffusion. We consider a conservative one dimensional system described by a potential energy $U(x) = U(-x)$ with a double minima structure. The thermal bath is characterized by the diffusion function $g(x) = g(-x)$. The reflection symmetry $x \rightarrow -x$ is not essential and most of our results do not depend on it. However, to keep the discussion as simple as possible, we focus in the symmetric model, leading the details of a more general asymmetric situation to a future presentation.

In order to reach thermodynamic equilibrium at long times, the drift force $f(x)$ should be related with the classical potential $U(x)$ through a generalized Einstein relation [39, 40]

$$f(x) = -\frac{1}{2}g^2(x)\frac{dU(x)}{dx}.$$  \hspace{1cm} (2.1)

In this way, the overdamped dynamics is driven by the Langevin equation

$$\frac{dx}{dt} = -\frac{1}{2}g^2(x)\frac{dU(x)}{dx} + g(x)\eta(t),$$  \hspace{1cm} (2.2)

where $\eta(t)$ obeys a Gaussian white noise distribution with

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t') \rangle = \sigma^2 \delta(t-t'),$$  \hspace{1cm} (2.3)

in which $\sigma^2$ measures the noise intensity. This equation is understood in the generalized Stratonovich [49] prescription (also known as $\alpha-$prescription [43]). The asymptotic long time equilibrium probability distribution is given by [40]

$$P_{eq}(x) = N e^{-\frac{1}{2\sigma^2}U_{eq}(x)},$$  \hspace{1cm} (2.4)

where $N$ is a normalization constant and the equilibrium potential

$$U_{eq}(x) = U(x) + 2(1-\alpha)\sigma^2 \ln g(x).$$  \hspace{1cm} (2.5)

The parameter $0 \leq \alpha \leq 1$ labels the particular stochastic prescription used to discretized the Langevin equation. For instance, $\alpha = 0$ corresponds with Itô interpretation while $\alpha = 1/2$ corresponds with the Stratonovich one. In this way, the equilibrium potential is not the bare classical potential, but it is corrected by the diffusion function $g(x)$. On the other hand, the case $\alpha = 1$ corresponds with Hänggi-Klimontovich interpretation [50, 51]. This is the only prescription which leads to the Boltzmann distribution $U_{eq}(x) = U(x)$. For this reason, this convention is sometimes called “thermal prescription”. Furthermore, this prescription is also known as anti-Itô and can be considered as the time reversal conjugated to the Itô prescription [40, 41].

Although the techniques and results of this paper do not depend on details, either of $U(x)$ or of $g(x)$, it is convenient, just to visualize the equilibrium potential $U_{eq}(x)$, to consider a very simple model. Let us take, for instance,

$$U(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4,$$  \hspace{1cm} (2.6)

with the diffusion function

$$g(x) = 1 + x^2.$$  \hspace{1cm} (2.7)

The bare potential $U(x)$ has two degenerated minima at $x_{\text{min}} = \pm 1$ and a local maximum at $x_{\text{max}} = 0$. The contribution of the multiplicative noise for the equilibrium potential
For the continuous curve is plotted with transition, where the critical noise is given by

dependence on the noise intensity resembles a second order phase changing the global structure of the potential. This dependence. For $\alpha$ fixed $\sigma = 0$. In (b), all the curves are computed in the Itô interpretation. The continuous curve is plotted with $\sigma = 1/5$, the dotted line with $\sigma = 2/5$ and the dashed line with $\sigma = 2/3$.

is quite interesting. In the weak noise limit, the global two-minima structure remains the same. However, the minima are displaced to

$$x_{\text{min}} = \pm (1 - 4\sigma^2(1 - \alpha))^{1/4}$$

$$\sim \pm 1 \mp \sigma^2(1 - \alpha) + O(\sigma^4).$$

For $\sigma^2 \geq 1/4(1 - \alpha)$, both minima melt in a single one, deeply changing the global structure of the potential. This dependence on the noise intensity resembles a second order phase transition, where the critical noise is given by

$$\sigma_c = \frac{1}{2\sqrt{1 - \alpha}}. $$

Interestingly, the critical noise depends on the stochastic prescription. For $\alpha \to 1$, $\sigma_c \to \infty$, meaning that, in the anti-Itô prescription, the double-well structure is preserved for all values of the noise.

In Figure 1 we depict the equilibrium potential $U_{\text{eq}}(x)$ given by Eq. (2.5) for the simple model specified by Eqs. (2.6) and (2.7), for different values of the parameters $\sigma$ and $\alpha$. In Figure 1(a), we show the equilibrium potential for $\sigma = 0.45$ and different values of the stochastic prescription $\alpha = 0, 1/2, 1$. We see that, for $\alpha = 1$, $U_{\text{eq}} = U$ and the minima are fixed at $x_{\text{min}} = \pm 1$. However, in the Stratonovich and Itô prescriptions, the minima are displaced towards the origin. In Figure 1(b), the three curves are computed in the Itô prescription with different values of the noise $\sigma = 1/5, 2/5, 2/3$. In this case, the minima approach zero when the noise grows and, for the value $\sigma = 2/3 > \sigma_c = 1/2$, the equilibrium potential has only one global minimum at $x_{\text{min}} = 0$.

### III. CONDITIONAL PROBABILITIES: PATH INTEGRAL REPRESENTATION

We are interested in computing the conditional probability $P(x_f, t_f | x_i, t_i)$ of finding the system in the state $x_f$ at time $t_f$, provided the system was in the state $x_i$ at a previous time $t_i$. It is useful to express this quantity using a path integral representation [44]. It can be written as

$$P(x_f, t_f | x_i, t_i) = e^{-\frac{\Delta U_{\text{eq}}}{\sigma^2}} K(x_f, t_f | x_i, t_i)$$

where $\Delta U_{\text{eq}} = U_{\text{eq}}(x_f) - U_{\text{eq}}(x_i)$ and the propagator $K(x_f, t_f | x_i, t_i)$ is given by

$$K(x_f, t_f | x_i, t_i) = \int [Dx] e^{-\frac{\Delta U_{\text{eq}}}{\sigma^2}} dt L(x, \dot{x}).$$

Here, the functional integration measure is

$$[Dx] = Dx \det^{-1} g = \lim_{\Delta t \to 0} \prod \frac{dx_n}{\sqrt{\Delta t g_{ij}^{-\frac{1}{2}}}}$$

where $x_0 = x_i$ and $x_N = x_f$. The Lagrangian can be written in the form,

$$L = \frac{1}{2} \left( \frac{1}{g(x)^2} \dot{x}^2 + V(x) \right),$$

where

$$V(x) = \frac{g^2}{2} \left[ \left( \frac{U''}{2} \right)^2 - \sigma^2 \left( \frac{U''}{2} + \frac{g}{g} \frac{U'}{U_{\text{eq}}} \right) \right] + \frac{\sigma^4}{4} (gg')'.$$

The primes mean derivative with respect to $x$. Equation (3.2), with the Lagrangian defined by Eq. (3.4), correctly describes the dynamics of the Langevin Eq. (2.2) for arbitrary values of the parameter $0 \leq \alpha \leq 1$ [44]. It is important to note that all the information about the stochastic prescription is codified in the structure of the equilibrium potential $U_{\text{eq}}(x)$, contained in the definition of the potential $V(x)$, Eq. (3.5).

In this particular representation, the path integral measure given by Eq. (3.3) is discretized symmetrically, allowing us to use normal calculus rules in the manipulation of the path integral (for more details on the subtleties of stochastic calculus in the path integral formulation, please see Ref. [40] and references therein).

An interesting observation is that Eq. (3.2) coincides with the propagator of a quantum particle with position-dependent mass $m(x) = 1/g(x)$ moving in a potential $V(x)$, written in the imaginary time path integral formalism $t \to -it$. The
noise $\sigma^2$ plays the role of $\hbar$ in the quantum theory. At a classical level, the Lagrangian, Eq. \((4.3)\), represents a particle with variable mass moving in a potential $-V(x)$. The structure of the potential $-V(x)$ (Eq. \((4.5)\)) is much more complex than $U(x)$ or even $U_{eq}(x)$.

In Fig. 2 we plotted the potential $-V(x)$ for the simple model displayed by Eq. \((4.6)\). All the curves have been plotted in the Itô prescription $\alpha = 0$. The dashed lines correspond to the additive noise case $g(x) = 1$, while the continuous lines represent the potential in the multiplicative noise case, with $g(x) = 1 + x^2$. In Fig. 2(a) we fixed $\sigma = 0.1$, while in Fig. 2(b), $\sigma = 0.01$. The first observation is that $-V(x)$ has three maxima and two minima. The location of both non-zero maxima roughly coincides with the minima of the potential $U(x)$. The difference is of the order of $\sigma^2$. The main effect of the diffusion function is to increase the curvature at each maxima with a factor proportional to $g^2(x_{\text{max}}) > 1$. An important feature that will be relevant to compute conditional probabilities is that the difference between the height of the peaks are of the order of $\sigma^2$. Thus, in a weak noise regime, the difference between the three maxima tends to disappear. In the extreme limit of $\sigma \to 0$, the potential $-V(x)$ has three degenerate maxima. This fact is clearly shown in Fig. 2(b). It is timely to note that the structure of $-V(x)$ is quite different from a similar calculus of the tunneling probability amplitude of a quantum particle \[46\]. In that case, the relevant potential is $-U(x)$, which has only two maxima. The appearance of a quasi-degenerate maximum at $x = 0$ is proper of a classical stochastic process, even additive as well as multiplicative.

\section{Fluctuations and Time Reparametrization}

The usual weak noise expansion consists in evaluating the path integral of Eq. \((3.2)\) in the saddle-point approximation plus Gaussian fluctuations. Generally, multiplicative noise induces an integration measure that depends on the diffusion function $g(x)$. In Ref. \[44\], we have shown how to overcome this problem by means of a time reparametrization. In this section, we briefly review this technique since we will use it to compute conditional probabilities.

The classical equation of motion is

$$\frac{d^2x}{dt^2} = g^2V' + \frac{g'}{g}x^2. \quad (4.1)$$

Despite the fact that this is a complicated nonlinear equation, using time translation symmetry, a first integral can be built up. We have

$$\dot{x}_{cl}^2 = 2g_{cl}^2(V_{cl} + H). \quad (4.2)$$

Here, $x_{cl}(t)$ is a solution of Eq. \((4.1)\). The notation $x_{cl}$ stands for classical solution, resembling in some sense a semiclassical calculation in quantum mechanics. $H$ is an arbitrary constant, $g_{cl} = g(x_{cl}(t))$ and $V_{cl} = V(x_{cl}(t))$. Then, the solution of Eq. \((4.1)\) can be expressed by a quadrature,

$$t - t_0 = \int_0^{x_{cl}} \frac{ds}{\sqrt{2V_{eff}(s)}}, \quad (4.3)$$

where we have defined an effective potential,

$$V_{eff}(x) = g^2(x)[V(x) + H]. \quad (4.4)$$

These expressions have two arbitrary constants, $t_0$ and $H$, that should be determined by means of the boundary conditions $x_{cl}(t_1) = x_1$ and $x_{cl}(t_f) = x_f$. Thus, Eqs. \((4.3)\) and \((4.4)\) implicitly define $x_{cl}(t)$, used as a starting point of the weak noise approximation.

Let us assume, for the moment, that, given initial and final conditions, the classical solution $x_{cl}$ is unique. Then, we consider fluctuations around it

$$x(t) = x_{cl}(t) + \delta x(t), \quad (4.5)$$

with boundary conditions $\delta x(t_1) = \delta x(t_f) = 0$. Replacing Eq. \((4.5)\) into Eq. \((3.2)\) and keeping up to second-order terms in the fluctuations, we find for the propagator

\begin{align*}
K(x_f, t_f | x_i, t_i) = & e^{-\frac{S_{cl}}{2}} \int [D\delta x] e^{-\frac{i}{2} \int dt dt' \delta x(t) \delta x(t')} \\
& \cdot \int [D\delta x'] e^{-\frac{i}{2} \int dt dt' \delta x(t) \delta x(t')} , \quad (4.6)
\end{align*}

where the classical action $S_{cl}$ is

$$S_{cl} = \int_{t_i}^{t_f} dt L(x_{cl}(t), \dot{x}_{cl}(t)) \quad (4.7)$$
and the fluctuation kernel,
\[ O(t, t') = -\frac{d}{dt} \left( \frac{1}{g_{cl}^2} \frac{d\delta(t - t')}{dt} \right) + \left( \frac{1}{g_{cl}^2} V_{\text{eff}}(x_{cl}) \right)' \delta(t - t'). \]

In Eq. (4.6), the functional integration measure is
\[ [D\delta x] = \lim_{N\to\infty} \prod_{n=0}^{N} \frac{d\delta x_n}{\sqrt{\Delta t}} g^{2\gamma(t_n) + \sigma^2(t_n + 1)} \]

Due to the time dependence of \( g_{cl} = g(x_{cl}(t)) \), the fluctuation kernel \( O(t, t') \) is not trivial. On the other hand, the integration measure, Eq. (4.9), depends on the diffusion function \( g(x(t)) \). As a consequence, although the exponent in Eq. (4.6) is quadratic, the evaluation of the functional integral is cumbersome. In this case, to compute the fluctuation integral, we make a time reparametrization. For concreteness, we introduce a new time variable \( \tau \) by means of
\[ \tau = \int_0^t g^2(x_{cl}(t')) dt'. \]

This is a nontrivial local scale transformation, weighted by the diffusion function evaluated at the classical solution \( x_{cl}(t) \). Performing this time reparametrization, the fluctuation kernel transforms as \( O(t, t') \to \Sigma(\tau, \tau') \) and takes the simpler form
\[ \Sigma(\tau, \tau') = \left[ -\frac{\partial^2}{\partial \tau^2} + W[x_{cl}] \right] \delta(\tau - \tau') \]

where
\[ W(x_{cl}) = \frac{1}{g_{cl}} \left( \frac{1}{g_{cl}^2} V_{\text{eff}}'(x_{cl}) \right)' . \]

More important, after discretizing the reparametrized time axes \( \tau \), the functional integration measure, Eq. (4.9), becomes
\[ [D\delta x] = \lim_{N\to\infty} \prod_{n=0}^{N} \frac{d\delta x_n}{\sqrt{\Delta \tau}} , \]
in which the function \( g(x_{cl}) \) has been absorbed in the reparametrization.

Thus, in the new time variable \( \tau \), the functional integral over fluctuations can be formally evaluated, obtaining for the propagator
\[ K(x_f, t_f|x_i, t_i) = (\det \Sigma(\tau_i, \tau_f))^{-1/2} e^{-\frac{1}{\sigma^2} \Sigma_{cl}(t_i, t_f)} , \]

where the relation between \((\tau_i, \tau_f)\) and \((t_i, t_f)\) is given through Eq. (4.10).

Equation (4.14) is formally similar to the weak noise expansion in the additive noise case. However, in this case, the determinant is written in terms of a rescaled time parameter \( \tau \). Thus, in order to compute a prefactor, we need to reparametrize the time variable, compute the determinant and, at the end, go back to the original time. In Ref. [44], we have successfully used this technique to compute conditional probabilities of an harmonic oscillator in a multiplicative noise environment. Here, we will use it to compute conditional probabilities in a double-well set-up.

V. PROBABILITY OF REMAINING IN A WELL

In order to compute conditional probabilities, let us consider a potential \(-V(x)\) with the general structure displayed in Figure 3. We will consider that the potential has local maxima at \( x = \pm a \) and \( x = 0 \), while it has two minima, at \( x = \pm x_p \). The difference \(|V(a) - V(0)| \sim O(\sigma^2)\), in such a way that the three maxima are degenerated in the limit \( \sigma \to 0 \). As we have mentioned, the maxima at \( x = \pm a \), roughly coincide with the minima of the bare potential \( U(x) \). The difference is of order \( \sigma^2 \).

We want to compute the probability of remaining in a minimum of \( U(x) \), after some time \( t \). Let us compute, for instance, the probability of remaining in the state \( x = -a \), i.e., the probability of finding the particle in the state \( x = -a \) at a time \( t/2 \), provided it was in the same point, at a time \( -t/2 \). As the initial and final states coincide, \( \Delta U_{\text{eq}} = 0 \) and, from Eq. (3.1), we see that this conditional probability coincides with the propagator, \( P(-a, t/2|a, -t/2) = K(-a, t/2|a, -t/2) \). So, we are interested in the function \( K(-a, -t/2|a, t/2) \) for very long times, \( t \to \infty \).

The main point is that for long times, there are a huge number of solutions (or approximate solutions) of the saddle-point equation which need to be considered in order to compute the path integral in the weak noise approximation. A trivial solution of Eq. (5.1) with initial and final conditions \( x_{cl}(-t/2) = x_f(t/2) = -a \) is \( x_{cl} = -a \). In this case, the multiplicative noise has a trivial effect. Since \( x_{cl} \) does not depend on time, the diffusion function \( g_{cl} \) is a simple constant that renormalizes the noise intensity \( \sigma \). Then, the contribution of this solution to \( K(-a, -t/2|a, t/2) \) can be easily computed obtaining,
\[ K^{(0)}(-a, t/2|a, -t/2) = \left( \frac{g_{cl}^2 U''_{\text{eq}}(a)}{2\pi \sigma^2} \right)^{1/2} , \]

where \( g_{cl} = g(a) \). We are using the superscript \((0)\) to indicate the contribution of the constant solution to the propagator.

A. Instantons/Anti-Instantons

In the case of potentials with two degenerate maxima, there are topological time-dependent solutions of the equation of motion with finite action that interpolate between both maxima. These solutions are called instantons or anti-instantons and should be taken into account to compute the propagator. For very large time intervals, well separated superposition of instantons and anti-instantons will also contribute to the path integral in a nontrivial way. The technique of summation over these configurations, usually called instanton/anti-instanton diluted gas approximation, was developed by several authors to compute tunneling amplitudes in quantum mechanics [46, 52, 53]. In stochastic processes, the technique was applied to the case of additive white noise in Ref. [48], in which the problem of a diffusion in a bistable potential was addressed. Some years later, the same technique was successfully applied to color noise processes [49]. Here, we will ap-
ply it to the multiplicative noise case. In the rest of this section we will closely follow the calculation of Ref. [48], emphasizing those steps that are proper of multiplicative noise.

In addition to the constant solution, there are other time-dependent trajectories which begin and end at \( x = -a \) for very long time intervals that will contribute to the propagator. In our case, the maximum at \( x = 0 \) is quasi-degenerated with \( x = \pm a \). For this reason, we expect that trajectories which begin at \( x = -a \) go to approximately \( x = 0 \) and then return to the original point, will also have an important weight in the functional integral. This type of trajectories are not exact solutions of the classical equation of motion, then, there will be a linear term in the fluctuations expansion. However, this term will be \( O(\sigma^2) \) since, in the limit \( \sigma \to 0 \), it should disappear.

We denote by \( K^{(1)}(-a, t/2) - a, t/2) \), the contribution of the trajectory \( -a \to 0 \to -a \) to the propagator. To compute it, we first rewrite the Lagrangian, Eq. (5.4), in the following way

\[
L = \frac{1}{2} \left( \frac{1}{g^2(x)} \right) \dot{x}^2 + V^{(0)}(x) + \delta V(x),
\]

where we have defined the quantity

\[
\delta V(x) = V(x) - V^{(0)}(x) = \begin{cases} 0, & x < -x_p \\ V_0 - V_a, & x > -x_p \end{cases}.
\]

In the last expression, \( -x_p \) is the position of the minimum of the potential \( -V(x) \), \( V_a = V(a) = V(-a) \) and \( V_0 = V(0) \). The specific form of \( \delta V(x) \), as well as the specific value \( x_p \) are not important. The final results will not depend on such details. Thus, the first two terms of Eq. (5.2) describe the dynamics of a particle in a potential \(-V^{(0)}\) with truly degenerate maxima, while \( \delta V(x) \sim O(\sigma^2) \).

Let us compute asymptotic solutions of the classical equation of motion for the potential \(-V^{(0)}\). We define the “instanton”, \( x_I(t) \), as the solution with initial and final conditions \( x_{cl}(-t/2) = -a \) and \( x_{cl}(t/2) = 0 \), for very large values of \( t \).

From Eq. (4.3), we have

\[
t - t_0 = \int_{-x_p}^{x_p} dx \sqrt{2g^2(x)(V^{(0)}(x) - V_a)},
\]

where we fixed the conditions \( x_I(t_0) = -x_p \) and \( H = V_a \). These parameters guarantee the above mentioned initial and final conditions.

We see, from Eq. (5.4), that the integral is dominated by the region in which \( V^{(0)}(x) - V_a \to 0 \). It happens for \( x \to 0 \) with \( x \to -a < -x_p \). Thus, to compute the integral we can expand \( V^{(0)}(x) \) around \( x = 0 \) and \( x = -a \) to second order in powers of \( x \) and \( x + a \), respectively. Thus, in the harmonic approximation we have

\[
V_h^{(0)}(x) = \begin{cases} V_a + \frac{1}{2} V''_0 x^2, & x > -x_p \\ V_a + \frac{1}{2} V''_a (x + a)^2, & x < -x_p \end{cases}.
\]

Using this approximation, we obtain for the instanton solution

\[
x_I(t) \sim -a - (-x_p + a) e^{g_0(V''_a)^{1/2}(t - t_0 - 2\Delta_{eq})},
\]

\[
x_I(t) \sim -x_p - g_0(V''_a)^{1/2}(t - t_0 - 2\Delta_{eq}),
\]

where we have introduced the finite constants

\[
\Delta(x_i, x_f) = \int_{x_0}^{x_f} \frac{dx}{\sqrt{2g^2(x)(V^{(0)}(x) - V_a)}}.
\]

in such a way that, in Eq. (5.7), \( \Delta_{eq} = \Delta(0, x_p) \) and \( \Delta_{a_p} = \Delta(a, x_p) \).

The instanton/anti-instanton pair of trajectories, corresponding with the path \(-a \to 0 \to -a \), can be written as

\[
x_{IA}(t, t_0, t_1) = \begin{cases} x_I(t - t_0), & t < \frac{t_1 + t_0}{2} \\ x_{I}(t_1 - t), & t > \frac{t_1 + t_0}{2} \end{cases},
\]

where \( x_I(t) \) is given by Eqs. (5.6) and (5.7). A typical instanton/anti-instanton trajectory is shown in Figure 3. The classical action computed by replacing Eq. (5.9) into Eq. (5.2) and integrating in time between \( t_i = -t/2 \) and \( t_f = t/2 \) we find

\[
S_{IA}(t, t_0, t_1) = (V_0 - V_a)(t_1 - t_0) + V_a t - x_p^2 (V''_0)^{1/2} e^{g_0(V''_a)^{1/2}(t_0 - t_1 + 2\Delta_{eq})}
\]

\[
- g_0 + U_{eq}(0) - U_{eq}(a) + \sigma^2 \ln \frac{U''_{eq}(a) g_0^2 (x_p + a)}{U''_{eq}(0) g_0^2 x_p} - \frac{g_0^2 U''_{eq}(a) \Delta_{pa} + g_0^2 U''_{eq}(0) \Delta_{eq}}{2},
\]

where we have used the notation \( S_{IA} = S_{IA}[x_{IA}] \), i.e., the classical action computed at the instanton/anti-instanton configuration of Eq. (5.9).

The next step is to compute fluctuations around the instanton/anti-instanton solution. After the time reparametrization given by Eq. (4.10), we are lead to the computation of the determinant \( \det \Sigma(\tau_f, \tau_i) \), where the

![FIG. 3. Instanton/anti-instanton pair trajectory in the potential \(-V^{(0)}(x)\).](attachment:figure.png)
operator $\hat{\Sigma}$ is given by Eq. (4.11), evaluated at $x_A = x_{2A}(\tau)$. Due to time translation invariance, the determinant has zero modes. Similarly to the original computation of instanton fluctuations [46], we need to properly take into account translation modes, identifying translation fluctuations with the integration over the collective variables $t_0$ and $t_1$. We obtain (see Appendix A),

$$K^{(1)}\left(-a, \frac{t}{2} - a, \frac{t}{2}\right) = \mathcal{N} \int_{-t/2}^{t/2} dt_0 \int_{t_0}^{t/2} dt_1 \times$$

$$g_a^2 \sqrt{S_I g_0^2 \sqrt{S_A}} \left[ \det \tau \left(\tau_f, \tau_i\right) \right]^{-1/2} e^{-\frac{1}{\sigma^2} S_{2A}(t_0, t_1)}$$

(5.11)

were $S_I = S_{cl}[x_I], S_A = S_{cl}[x_A]$ and the prime in the determinant indicates that it should be evaluated excluding the zero modes. We use the notation $K^{(1)}$ to indicate the contribution of the path $-a \rightarrow 0 \rightarrow -a$ to the propagator. This result is similar to the additive noise case [48]. The main difference is that the determinant is computed in a reparametrized time and the integration over collective variables $t_0$ and $t_1$ are renormalized by the diffusion function. The advantage of the reparametrized time is that the operator $\hat{\Sigma}$ has the simpler form of Eq. (4.11) and can be computed using the Gelfand-Yaglom theorem [54]. At the end of the calculation, we go back to the original time axes. Following tedious but usual procedures, we finally find

$$K^{(1)}\left(-a, \frac{t}{2} - a, \frac{t}{2}\right) = -g_a^2 t K^{(0)} \Gamma,$$

(5.12)

where $K^{(0)}$ is the contribution of the constant solution, given by Eq. (5.1), and

$$\Gamma = \left( \frac{g_a^2 U''_{eq}(a) g_0^2 |U_{eq}(0)|}{2\pi} \right)^{1/2} \exp \left\{ \frac{U_{eq}(0) - U_{eq}(a)}{\sigma^2} \right\}.$$ 

(5.13)

We see that the contribution of an instanton/anti-instanton configuration to the propagator at long times, is a linear function of time. The structure of the coefficient $\Gamma$ is very interesting. All the information about the stochastic calculus is hidden in the definition of the equilibrium potential, $U_{eq}$. On the other hand, it does not depend on the details of $U_{eq}(x)$, but instead, it depends on the barrier height, $U_{eq}(a) - U_{eq}(0)$, and on the curvature at each maxima, $U''_{eq}(0)$ and $U''_{eq}(a)$. These properties are quite similar with the additive noise case, except for the fact that the original potential $U(x)$ is replaced by the equilibrium potential $U_{eq}$ in the multiplicative case. Moreover, a remarkable feature of multiplicative noise, is that the diffusion function $g(x)$ renormalizes the curvature at each maxima. In this way, $K^{(1)}$ does not depend on the details of $g(x)$, but only on its value at the maxima, $g(0)$ and $g(a)$.

Due to the structure of the potential $-V(x)$, there are other trajectories which contribute in a nontrivial way to the propagator; for instance, trajectories that begin in $x = -a$, go to $x = a$ passing through $x = 0$, and return to $x = -a$. This kind of trajectories contains two instantons and two anti-instantons as shown in Figure 4. The contribution of these trajectories to the propagator can be computed following the same steps of the computation of the single instanton/anti-instanton case. We find, in this case,

$$K^{(2)}\left(-a, \frac{t}{2} - a, \frac{t}{2}\right) = \left( \frac{g_a^2 t^2}{2!} \right) K^{(0)} \Gamma^2.$$ 

(5.14)

Thus, trajectories of the type $-a \rightarrow a \rightarrow -a$, produce a quadratic time contribution, the coefficient is simply $\Gamma^2$, where $\Gamma$ is given by Eq. (5.13).

B. Kramers escape rate and time reversal transformation

To compute the conditional probability of remaining in a minimum after some time $t$, we need to sum up all the trajectories that begin and ends at $x = -a$ and which contribute to the propagator in a nontrivial way. Having in mind that $\Delta U_{eq} = 0$, this probability coincides with the propagator, $P(-a, t/2 | -a, -t/2) = K(-a, t/2 | -a, -t/2)$. As described above, there are essentially three contributions to these paths: a constant one, $K^{(0)}$, given by Eq. (5.1), a linear term $K^{(1)}$ given by Eq. (5.12), corresponding to trajectories $-a \rightarrow 0 \rightarrow -a$ or, by symmetry, to $a \rightarrow 0 \rightarrow a$, and, finally, a quadratic term $K^{(2)}$ given by Eq. (5.14), related to the path $-a \rightarrow a \rightarrow -a$. Consider, for instance, a general trajectory containing $\ell_1$ paths of the type $-a \rightarrow 0 \rightarrow -a$ and $\ell_2$ paths of the type $a \rightarrow 0 \rightarrow a$, related with the linear function $K^{(1)}$. In addition, we allow $m$ paths of the type $-a \rightarrow a \rightarrow -a$, related with $K^{(2)}$. Then, this particular trajectory will contribute to the propagator with a term

$$K^{(\ell_1, \ell_2, m)}\left(-a, \frac{t}{2} | -a, \frac{t}{2}\right) =$$

$$K^{(0)} \left( \frac{g_a^2 t^{\ell_1 + \ell_2 + 2m}}{(\ell_1 + \ell_2 + 2m)!} \right) \Gamma^{\ell_1 + \ell_2 + 2m}.$$ 

(5.15)

By carefully counting the number of different paths which contribute to each trajectory labeled by $(\ell_1, \ell_2, m)$ and summing up, we finally arrive at the expression for the conditional probability,

$$P\left(-a, \frac{t}{2} | -a, \frac{t}{2}\right) = \frac{1}{2} K^{(0)} \times \left(1 + e^{-t/\tau_k}\right).$$ 

(5.16)

On the other hand, by using the same formalism, we easily find the expression for the conditional probability of finding

![Diagram 4](image-url)
the system in the state $x = a$ at time $t/2$, provided it was in the state $x = -a$ at a previous time $-t/2$,
\[
P \left( \frac{a}{2} - a, -\frac{t}{2} \right) = \frac{1}{2} \mathcal{K}(0) \times \left( 1 - e^{-t/\tau_k} \right). \tag{5.17}
\]

In Eqs. (5.16) and (5.17), the inverse time parameter $\tau_k^{-1}$, which is equivalent to the Kramers escape rate, is given by $\tau_k^{-1} = 2g_0^2 \Gamma$. Using Eq. (5.17), it is explicitly written as
\[
\tau_{\text{mult}} = 2g_0^2 \frac{\sqrt{U''(0)g_0^2[\Delta U'(0)]}}{\pi} e^{-\frac{\Delta U_{\text{eq}}}{\sigma^2}}, \tag{5.18}
\]
with $\Delta U_{\text{eq}} = U_{\text{eq}}(0) - U_{\text{eq}}(a)$.

This is one of the main results of our paper. Comparing Eq. (5.18) with the classical result of Eq. (1.11), we clearly see the effect of the multiplicative noise. On one hand, the role of the original potential $U(x)$ is now played by the equilibrium potential $U_{\text{eq}}(x)$ given by Eq. (5.23). This potential depends not only on the diffusion function $g(x)$ and the noise, but also on the stochastic prescription $\alpha$ that defines the original Langevin equation. On the other hand, the diffusion function $g(x)$ renormalizes the curvature of the equilibrium potential, in such a way that $\omega_{\text{min}} = g_0^2 U''_{\text{eq}}(a)$ and $\omega_{\text{max}} = g_0^2 U''_{\text{eq}}(0)$. There is also a global scaling factor given by $g^2(0)$.

In order to gain more insight on this result, let us compare the Kramers escape rate with the expression of $\tau_{\text{mult}}$. Using Eq. (5.18), both quantities can be compared, since $\tau_{\text{add}} = \lim_{g \rightarrow 1} \tau_{\text{mult}}$. We obtain, in the weak noise approximation,
\[
\frac{\tau_{\text{mult}}}{\tau_{\text{add}}} = |g_0|^{1+2\alpha} |g_a|^{3-2\alpha} \left( 1 + O(\sigma^2) \right). \tag{5.19}
\]

It can be noticed that the relation between both escape rates does not depend on details of $g(x)$, but on its value at each maxima of $-V(x)$, $x = \pm a$ and $x = 0$. As expected, Eq. (5.19) depends on the stochastic prescription parameter $\alpha$. For instance, in the case of the Stratonovich prescription, $\alpha = 1/2$, $\tau_{\text{mult}}/\tau_{\text{add}} = g_0^2 g_a^2$. In this case, $g_0$ and $g_a$ have the same weight. On the other hand, in the Itô interpretation $\alpha = 0$, $\tau_{\text{mult}}/\tau_{\text{add}} = g_0 g_a^3$ while in the thermal prescription, $\alpha = 1$, $\tau_{\text{mult}}/\tau_{\text{add}} = g_0^3 g_a$. Indeed, Eq. (5.19) is invariant under the transformation
\[
\alpha \leftrightarrow 1 - \alpha \tag{5.20}
\]
\[
0 \leftrightarrow a \tag{5.21}
\]
which is nothing but a time reversal transformation [40]. The simplest way to understand this symmetry is by noting that the instanton solution $x_I(t)$ interpolates between the states $x = -a$ and $x = 0$. The time reversal solution $x_I(-t)$ makes the inverse trajectory, i.e., connecting $x = 0$ with $x = a$. However, if the forward time process evolves with the $\alpha$ prescription, the backward evolution takes place with the $1 - \alpha$ prescription. In this sense, one process is the time reversal conjugate of the other one. For this reason, the thermal prescription $\alpha = 1$ is also called the anti-Itô interpretation. In fact, the only time reversal invariant prescription is the Stratonovich one, $\alpha = 1/2$. For details on the time reversal transformation in multiplicative noise dynamics, please see Refs. [39, 41].

Let us finally mention that the escape rate in the multiplicative case may be greater or lower than in the additive case, depending essentially on the values of $g(0)$ and $g(a)$. Moreover, if the diffusion function $g(x)$ locally approaches zero at either $x = a$ or $x = 0$, the escape rate goes to zero. This effect can be understood from the fact that the effective curvature, $\omega_{\text{min}} = g_0^2 U''_{\text{eq}}(a)$ and $\omega_{\text{max}} = g_0^2 U''_{\text{eq}}(0)$ approaches zero when $g_a \rightarrow 0$ or $g_0 \rightarrow 0$, respectively. Of course, our approximation $t \gg \tau_k$ is no longer valid in this limit.

VI. SUMMARY AND CONCLUSIONS

We have considered the problem of a particle in a symmetric double-well potential $U(x)$, with a dynamics driven by an overdamped multiplicative Langevin equation characterized by a symmetric diffusion function $g(x) = g(-x)$. The stochastic differential equation was defined in the generalized Stratonovich prescription, parametrized by a continuum parameter $0 \leq \alpha \leq 1$. This prescription contains the usual stochastic interpretations for particular values of the parameter $\alpha$. We have provided a path integral technique to compute conditional probabilities in the weak noise approximation. It was introduced a local time reparametrization which allows to exactly integrate fluctuations.

Conditional probabilities were computed for long time intervals by generalizing the instanton/anti-instanton diluted gas approximation, already developed for the additive noise case [39]. From these probabilities, the escape rate was computed in the same approximation and the result was compared with the Kramers escape rate for additive noise dynamics. The main result of the paper is given by Eq. (5.18). We found that the general structure of the escape rate keeps the Arrhenius form of the Kramers result. The main corrections are twofold. First, the equilibrium potential $U_{\text{eq}}(x)$ of Eq. (5.25) plays the role of the bare potential $U(x)$. The potential $U_{\text{eq}}(x)$ is the solution of the static Fokker-Plank equation and is generally different from $U(x)$ in the multiplicative noise case. Indeed, the only stochastic prescription in which $U_{\text{eq}}(x) = U(x)$ is the anti-Itô prescription $\alpha = 1$. On the other hand, the prefactor is modified by a renormalization of the curvatures produced by the diffusion function. Moreover, there is a global scale factor $g^2(0)$ that has its origin in the time reparametrization necessary to correctly compute fluctuations.

In the weak noise limit, we found a simple relation between the Kramers escape rates computed with additive and multiplicative noise, given by Eq. (5.19). The obvious consistency check is that $\tau_{\text{mult}}/\tau_{\text{add}} = 1$ in the limit $g(x) \rightarrow 1$. In addition, we observe that $g(0)$ and $g(a)$ enter with different weights depending on the prescription parameter $\alpha$. These weights are consistent with a time reversal transformation, which relates a stochastic process in the $\alpha$ prescription with its time reversal conjugate $1 - \alpha$. Indeed, the Stratonovich convention $\alpha = 1/2$ is the only one with time reversal invariance and, in this case, both maxima enter with the same weight.
Although we have presented results for a system with full reflection symmetry \( x \rightarrow -x \), the methods developed in this paper are completely general. We hope to communicate results for a more general non-symmetric case in the near future. Moreover, having analytic expressions for the conditional probability we can face the problem of stochastic resonance in multiplicative noise processes in a more solid bases.

Appendix A: Zero modes in the multiplicative case

The relation of zero modes of the fluctuation operator and translation invariance is very well known in quantum mechanics [34], as well as in additive noise stochastic dynamics [43]. In this appendix, we focus on the effect produced by the diffusion function \( g(x) \) in a multiplicative noise stochastic system.

In order to compute fluctuations we perform a local time reparametrization given by Eq. (4.10). We are lead to the computation of the integral

\[
I_F = \int [D\delta x] e^{-\frac{1}{2} \int d\tau \delta x(\tau) \left( -\frac{\dot{x}^2}{2g^2} + W[x,\tau]\right) \dot{\delta x}(\tau)},
\]

where \( W \) is given by Eq. (4.12). To compute it, we expand fluctuations in eigenfunctions of the fluctuation operator

\[
\dot{\delta x}(\tau) = \sum_n c_n \psi_n(\tau),
\]

and the orthogonal eigenfunctions are normalized as

\[
\int d\tau \psi_n(\tau) \psi_m(\tau) = \delta_{n,m}.
\]

Then,

\[
I_F = \int (\Pi_k d\kappa) \exp \left( -\frac{1}{2} \sum_n \lambda_n c_n^2 \right),
\]

Consider, for simplicity, the instanton solution written in the reparametrized time \( x_{\epsilon}(\tau) = x_1(\tau) \), that interpolates between \( x = -a \) for \( \tau \rightarrow -\infty \) and \( x = 0 \) for \( \tau \rightarrow \infty \). Since the action is invariant under time translation, the instanton configuration \( x_1(\tau - \tau_0) \) is a solution of the equation of motion for any value of \( \tau_0 \). This fact produces a zero eigenvalue of the fluctuation operator. In fact, it is simple to show that the function

\[
\psi_0(\tau) = A g^4(x_1(\tau)) \frac{dx_1}{d\tau},
\]

with \( A \) a normalization constant, satisfies Eq. (A2) with \( \lambda_0 = 0 \). On the other hand, the condition \( \int d\tau \psi_0^2(\tau) = 1 \) fixes the arbitrary constant \( A \) in the following way

\[
A^{-2} = \int_{-\infty}^{+\infty} d\tau g^4(x_1(\tau)) \left( \frac{dx_1}{d\tau} \right)^2.
\]

In the thin wall approximation, \( (dx_1/d\tau)^2 \) is a strongly localized function around \( \tau \sim \tau_0 \). Then, with a good accuracy, we can make the approximation

\[
A^{-2} \sim g_0^4 \int_{-\infty}^{+\infty} d\tau \left( \frac{dx_1}{d\tau} \right)^2 \sim g_0^4 S_I,
\]

where \( S_I \) is the classical action evaluated at the instanton solution. Thus, in this approximation, accurate for high barriers, we find, \( A = 1/g_0^2 \sqrt{S_I} \). Then, the zero mode is written as

\[
\psi_0(\tau) = \frac{1}{g_0^2 \sqrt{S_I}} g^2(x_1(\tau)) \frac{dx_1}{d\tau}.
\]

The existence of this zero mode makes the integration over \( dc_0 \) in Eq. (A3) divergent. Thus, in order to cure this problem we need to correctly take into account translation invariance. Let us expand fluctuations in the following way,

\[
\delta x(\tau) = c_0 \psi_0(\tau - \tau_0) + \sum_{k=1}^{\infty} c_k \psi_k(\tau - \tau_0)
\]

where \( \psi_k \) are eigenvectors with eigenvalues \( \lambda_k \neq 0 \) and \( \psi_0 \) is given by Eq. (A7). Computing the variation of fluctuations under time translation, we have that

\[
d\delta x(\tau) = \frac{dx_1}{d\tau} d\tau_0.
\]

On the other hand, a variation in the zero mode reads

\[
d\delta x(\tau) = \frac{1}{g_0^2 \sqrt{S_I}} g^2(x_1(\tau)) \frac{dx_1}{d\tau} dc_0.
\]

Comparing Eqs. (A9) and (A10) and using the reparametrization identity \( d\tau/dt = g^2(x_1) \), we immediately find

\[
dc_0 = g_0^2 \sqrt{S_I} dt_0.
\]

In this way,

\[
I_F = \int g_0^2 \sqrt{S_I} dt_0 \left( \prod_{n\neq 0} \lambda_n^{-1/2}(\tau_0) \right)
\]

\[
= \int dt_0 g_0^2 \sqrt{S_I} \left( \det \left( -\frac{d^2}{d\tau^2} + W[x,\tau] \right) \right)^{-1/2}
\]

where the prime means that the determinant should be computed without the zero mode.

In this way, the usual interpretation of the zero mode as an integration in the collective variable \( dc_0 \) is still valid in the multiplicative case. However, the constant of proportionality is renormalized by the diffusion function \( g_0^2 \), computed at the minimum of the potential.

The same reasoning applies to the anti-instanton solutions. However, in this case the variation is proportional to \( g_0^2 S_A dt_0 \), where \( g_0^2 \) is evaluated at the maximum of the potential and \( S_A \) is the classical action evaluated at the anti-instanton solution. This analysis leads to Eq. (5.11) for \( K^{(1)} \).

ACKNOWLEDGMENTS

The Brazilian agencies, Fundação de Amparo à Pesquisa do Rio de Janeiro (FAPERJ), Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) - Finance Code 001, are acknowledged for partial financial support. MVM is partially supported by a Post-Doctoral fellowship by CAPES.
