Weyl-Schrödinger representations of infinite-dimensional Heisenberg groups on symmetric Wiener spaces

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Abstract

An infinite-dimensional version of the Heisenberg matrix group, consisting of entries from an algebra of Hilbert-Schmidt operators, is investigated. We find its irreducible Weyl-Schrödinger type representation on a symmetric Wiener space. This space is generated by symmetric Schur polynomials in variables on Paley-Wiener maps with well-defined Fourier transforms in relative to an invariant probability measure over the infinite-dimensional unitary group. Intertwining properties of such Fourier transforms under shift and multiplicative groups are investigated.

Keywords: Infinite-dimensional Heisenberg groups, Weyl-Schrödinger representation, Schur polynomials on Paley-Wiener maps

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1. Introduction

We investigate an infinite-dimensional complexified version of Heisenberg’s matrix group $\mathbb{H}_{\text{hs}}$ with entries from a $*$-algebra $E_{\text{hs}}$ of Hilbert-Schmidt operators $a$ endowed with the trace norm $\|a\|_{\text{hs}} := \text{tr}(a^{*}a)^{1/2}$, acting on a separable complex Hilbert space $E$, and extendend by unit $I$. Specifically, $\mathbb{H}_{\text{hs}}$ is defined to be the upper triangular infinite-dimensional block-matrix operators (see, e.g. [1])

$$X(a, b, t) = \begin{bmatrix} I & a & It \\ 0 & I & b \\ 0 & 0 & I \end{bmatrix}, \quad t \in \mathbb{C}, \quad a, b \in E_{\text{hs}}. \quad (1.1)$$

In order to find an irreducible representation of $\mathbb{H}_{\text{hs}}$, we introduce a modified notion of symmetric Wiener space $W_{\gamma}^2$ of square $\gamma$-integrable functions over the infinite-dimensional unitary group $U(\infty) := \bigcup \{U(m) : m \in \mathbb{N}\}$, totally embedding into $E_{\text{hs}}$. The space $W_{\gamma}^2$ is defined as a closure of the linear span of Schur polynomials, determined via Paley-Wiener maps $\phi_{a} \in L_{\gamma}^2$ such that

$$\int \exp \{\text{Re} \phi_{a}\} d\gamma = \exp \left\{ \frac{1}{4} \text{tr}(a^{*}a) \right\}, \quad a \in E_{\text{hs}}. \quad (1.2)$$

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Here, $\gamma = \lim m \gamma_m$ is the projective limit of probability Haar’s measures $\gamma_m$, defined on the groups $U(m)$ of unitary $m \times m$-matrix operator from $E_{m^2}$ and taken under the Livšic transform $\pi_{m}^{m+1}: U(m+1) \to U(m)$ (see [13], [14]). The Radon measure $\gamma$ is concentrated on the set of stabilized sequences $\mathfrak{U}$ in the space of virtual unitary matrices $\mathfrak{U} = \lim U(m)$ defined as the projective limit under $\pi_{m}^{m+1}$ and invariant under right actions $U(\infty) \times U(\infty)$ over $\mathfrak{U}$.

In other words, the total embedding $U(\infty) \ni \pi E_{m^2}$, where $U(\infty)$ is endowed with the Borel structure $\mathfrak{U}$, is such that the Paley-Wiener maps $\phi_a \in L^2_\gamma$ for all $a \in E_{m^2}$ satisfy the condition (1.2).

Hence, the Radon measure $\exp \left\{ -\frac{1}{2} \text{tr}(a^* a) \right\} dy$ on $\mathfrak{U}$ is a group analog of Gaussian. Thus, we may regard $W^2_\gamma$ as a complexified version with respect to the measure $\gamma$ over $U(\infty)$ of an abstract Wiener space (introduced by Gross for a Gaussian measure [5], [17]).

As a main end result, it is shown in Theorem 7.2 that a required irreducible representation of $H_{m^2}$ over $W^2_\gamma$ takes the following Weyl-Schrödinger form,

$$X(a, b, t) \mapsto \exp \left\{ t + \frac{1}{2} \text{tr}(b^* a) \right\} T^t_a M^t_b,$$

where $T^t_a$ is the shift group and $M^t_b$ is the multiplicative group over $W^2_\gamma$. Moreover, it is proved in Theorem 7.3 that the suitable Weyl system

$$\mathcal{W}(a, b) = \exp \left\{ \frac{1}{2} \text{tr}(b^* a) \right\} T^t_a M^t_b$$

possesses the densely-defined generator $h^t_{a, b} = \bar{a} + \delta^t_b$ which satisfies the commutation relation

$$[\mathcal{W}(a, b) \mathcal{W}(a', b')] = \mathcal{W}(a', b') \mathcal{W}(a, b)$$

where the groups $M^t_b$ and $T^t_a$ are generated on $W^2_\gamma$ by $\bar{a}$ and $b$, respectively. This result is applied to the initial problem for an infinite-dimensional generalization of Schrödinger’s equation. In Theorem 8.1 it is shown that in a dense extension $W^{-2}_\gamma \ni W^2_\gamma$ there exists a unique solution of the initial value problem

$$\frac{d\psi(r)}{dr} = b^t \psi(r), \quad \psi(0) = f \in W^{-2}_\gamma, \quad r > 0$$

with $b^t = \sum_{m \geq 0} (\bar{a}_{m} - \bar{b}_m)$ where summations over an orthonormal basis $(e_m)$ in $E_{m^2}$.

Note that an infinite-dimensional generalization of Heisenberg groups was considered in [12] by using reproducing kernel Hilbert spaces. Their Schrödinger type representations on the space $L^2_\gamma$ with respect to a Gaussian measure $\mu$ over a real Hilbert space was described in [6].

The case of Weyl-Schrödinger representation in infinite dimensions was also analyzed in [8]. The currently analyzed case is its natural complement because the group $U(\infty)$ is totally embedded into the operator $*$-algebra $E_{m^2}$. Thus, it includes a compatible non-commutative structure which is taken into account by the symmetric Wiener space $W^2_\gamma$.

Now about additional details. The Fourier transform (see Thm 5.7)

$$\mathcal{F}: W^2_\gamma \ni f \mapsto \hat{f} \in H^2_{m^2}$$

provides an isometry onto a Hardy space $H^2_{m^2}$ of analytic entire functions on $E_{m^2}$ which is a certain group analogy of a Segal-Bargmann space, since it is uniquely determined via symmetric
2. Background

Throughout, $E$ is a separable complex Hilbert space with scalar product $\langle \cdot | \cdot \rangle$, norm $\| \cdot \|$ = $\langle \cdot | \cdot \rangle^{1/2}$ and an orthonormal basis $\{e_k : k \in \mathbb{N}\}$. The Hermitian dual $E^\dagger$ of $E$ is identified with $E$ via the conjugate-linear isometry $\sharp : E^\dagger \to E^\dagger = E$ such that $\gamma(x) = \langle x | y \rangle$ for all $x, y \in E$.

2.1. Schur polynomials indexed by Young tabloids

Let $\mathcal{J}_l := \{t = (t_1, \ldots, t_l) \in \mathbb{N}^l : t_1 < t_2 < \ldots < t_l\}$ be a strictly ordered integer alphabet of length $l$ and $\mathcal{J} = \bigcup_{l \geq 1} \mathcal{J}_l$ contains all finite alphabets. Assign to any $\ell \in \mathcal{J}_l$ the length-$l$-dimensional subspace $E_{\ell} \subset E$ spanned by $\{e_{t_1}, \ldots, e_{t_l}\}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{N}^l$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l$ be a partition of a certain $n$-letter word $r^\ell = \{t_{ij} : 1 \leq i \leq l, j = 1, \ldots, \lambda_i\}$ with $\ell = \ell_1 \geq \lambda_1$ means the length of partition $\lambda$, that is, $n = |\lambda|$ with $|\lambda| := \lambda_1 + \ldots + \lambda_l$. Shortly, this is written $r^\ell \vdash n$. Put $\Theta^\ell := 1$.

By definition (see, e.g. [3, 15]), a Young $\lambda$-tableau with a fixed partition $\lambda$ is a result of

$$
\begin{array}{ccccccc}
\ell_1 & \ldots & \ell_{1,\lambda_1} \\
\ell_2 & \ldots & \ell_{2,\lambda_2} \\
\vdots & \ddots & \ddots \\
\ell_l & \ldots & \ell_{l,\lambda_l} \\
\end{array}
$$

filling the word $r^\ell$ onto the matrix $[r^\ell] = \begin{bmatrix} \ell_1 & \ldots & \ell_{1,\lambda_1} \\ \ell_2 & \ldots & \ell_{2,\lambda_2} \\ \vdots & \ddots & \ddots \\ \ell_l & \ldots & \ell_{l,\lambda_l} \end{bmatrix}$ with $n$ nonzero entries in some way without repetitions. Thus, each $\lambda$-tableau $[r^\ell]$ can be identified with a bijection $[r^\ell] \rightarrow r^\ell$. The conjugate partition $\lambda^\dagger$ responds to the transpose matrix $[r^\ell]^\dagger$. Let $\mathcal{Y}$ denote all Young tabloids.

A tableau $[r^\ell]$ is called standard (resp., semistandard) if its entries are strictly (resp., weakly) ordered along each row and strictly ordered down each column.

The symmetric group of all $n$-elements permutations is denoted by $S_n$. For any $\sigma \in S_n$, we define the (left) action $\sigma \cdot [r^\ell] = [(\sigma \cdot i)^\ell]$ on $[r^\ell]$ to be entry-wise. Consider two subgroups of $S_n$ corresponding to $[r^\ell]$:

- the row subgroup $R^\ell$ which maps every letter of the word $r^\ell$ into a letter standing in the same row in $[r^\ell]$;
- the column subgroup $K^\ell$ which maps every letter of $r^\ell$ into a letter standing in the same column in $[r^\ell]$.
Define the Young projectors \( r_i = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \) and \( k_i = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \sigma \), as well as their composition \( c_i = r_i \cdot k_i \) where \((-1)^\sigma\) is the sign of a permutation \( \sigma \) and \# means cardinality. The subspace \( S_i^\lambda \) of the complex group algebra \( \mathbb{C}[S_n \cdot C_i] \) of the complex group algebra \( \mathbb{C}[S_n] \), called the Specht \( \lambda \)-module, is an irreducible representation of \( S_n \) under the left multiplication \( \rho(a)b = ab \) with \( \rho: S_n \rightarrow \text{End}(S_i^\lambda) \). Conversely, every irreducible representation of \( S_n \) is isomorphic to \( S_i^\lambda \) for a unique standard Young tableau \( [r^\lambda] \). The following hook formula holds [4, I.4.3],

\[
h_{i,j} := \prod_{1 \leq k \leq i} n^k, \quad \text{where} \quad h_{i,j} = \dim S_i^\lambda, \quad (2.1)
\]

\( h(i, j) = \# \{ i_j \in [r^\lambda]: k \geq i, k = i \} \) or \( \# \{ i_j \in [r^\lambda]: k = i, m \geq 2 \} \) do not depend on choice of \( i \in I \).

If \( r^\lambda \vdash n \) with \( l_i = l \), then \( t_i = (t_{i_1}, \ldots, t_{i_l}) \) mean complex variables. Let \( t_i^\lambda := \prod t_i^\lambda \). The \( n \)-homogenous Schur symmetric polynomial is defined as

\[
s_i^\lambda(t_i) := \frac{D_i(t_i)}{\Delta_i(t_i)}, \quad D_i(t_i) = \det \left[ t_i^{\lambda_j + j} \right], \quad \Delta_i(t_i) = \prod_{1 \leq k < j \leq l} (t_i - t_j),
\]

where \( \lambda_j = 0 \) for \( j > l \). It can be written as \( s_i^\lambda(t_i) = \sum_{|\lambda|} t_i^\lambda \) where the summation is over all semistandard Young tabloids [4, 1.2.2].

### 2.2. Unitarily invariant probability measures

We consider the infinite-dimensional unitary group \( U(\infty) := \bigcup \{ U(m): m \in \mathbb{N} \} \), extended by unit \( \mathbb{I} \), which irreducibly acts on \( E \). We endow each unitary group \( U(m) \) of \( m \times m \)-matrices with the probability Haar measure \( \gamma_m \) and assume that \( U(m) \) is identified with its range under the embedding \( U(m) \ni u_m \mapsto \begin{pmatrix} u_m & 0 \\ 0 & \mathbb{I} \end{pmatrix} \in U(\infty) \). Following [13, Prop. 0.1], [14, Lem.3.1] we define the Livšic transform \( \pi_m^+ \colon U(m + 1) \rightarrow U(m) \),

\[
\pi_m^+ : u_{m+1} := \begin{bmatrix} z_m & \alpha \\ \beta & t \end{bmatrix} \mapsto u_m := \begin{cases} z_m - [t(1 + t)^{-1}] : & t \neq 1 \\ z_m & : t = 1, \end{cases}
\]

by excluding \( x_1 = y_1 \in \mathbb{C} \) from the equation \( \begin{pmatrix} y_m \\ y_1 \end{pmatrix} = \begin{pmatrix} z_m & \alpha \\ \beta & t \end{pmatrix} \begin{pmatrix} x_m \\ x_1 \end{pmatrix} \) with \( x_m, y_m \in \mathbb{C}^m \). It is a surjective Borel mapping which is no group homomorphism and such that \( \pi_m^+ \circ \pi_{m+1}^+ = \pi_{m+1}^+ \) for the corresponding Hermitian adjoint matrices. The projective limit \( U := \lim U(m) \) under \( \pi_m^+ \) has surjective (not continuous) projections \( \pi_m : U \ni u \mapsto u_m \in U(m) \) (see [14, Lem. 3.11]) such that \( \pi_m = \pi_{m+1}^+ \circ \pi_{m+1}^+ \). The elements \( u \) from \( U \) are called the virtual unitary matrices.

Consider a universal dense embedding \( \pi : U(\infty) \ni U \) which to every \( u_m \in U(m) \) assigns the stabilized sequence \( u = (u_k) \) such that (see [14, n.4])

\[
\pi : U(m) \ni u_m \mapsto (u_k) \in U, \quad u_k = \left\{ \begin{array}{ll} \pi_k^+(u_m) : & k < m \\ u_m : & k \geq m \end{array} \right. \quad (2.2)
\]

where \( \pi_k^+ := \pi_{k+1}^+ \circ \cdots \circ \pi_{m-1}^+ \) for \( k < m \) and \( \pi_k^m \) identity for \( k \geq m \). On its range \( \pi(U(\infty)) \), we consider the inverse mapping

\[
\pi^{-1} : \mathbb{U}_\pi \longrightarrow U(\infty) \quad \text{where} \quad \mathbb{U}_\pi := \pi(U(\infty)).
\]
From (2.2) it follows that \((\pi_m \circ \pi)^{-1} = \pi^{-1} \circ \pi^{-1}\) coincides with the embedding \(U(m) \ni u \mapsto U(\infty)\).

The right action \(\mathcal{U}_x \ni u \mapsto u.g \in \mathcal{U}_x\) with \(g = (v, w) \in U(\infty) \times U(\infty)\) is defined to be \(\pi_m(u.g) = w^* \pi_m(u)v\) where \(m\) is so large that \(v, w \in U(m)\). Hence, \(\pi^{-1}(u.g) = w^* \pi^{-1}(u)v\). If \(g = (\|, \omega)\) we shortly denote \(u.g := w^* u\).

It also well defined on \(\mathcal{U}_x\) the involution \(u \mapsto u^* := (u^*_0)\), where \(u^*_k = u_k^{-1}\) with \(u_k \in U(k)\).

Thus, \([\pi_m(u.g)]^* = \pi_m(u^*, g^*)\) with \(g^* = (w^*, v^*)\).

Following [13, n. 3.1] or [14, Lem. 4.8] with the help of Kolmogorov’s consistency theorem (see, e.g. [19, Cor. 4.2]), we uniquely define the probability measure \(\gamma := \lim_{m \to \infty} \gamma_m\) such that the image-measure \(\pi_{m+1}^m(\gamma_{m+1})\) is equal to \(\gamma_m\) for any \(m \in \mathbb{N}\).

Let \(L^\infty_\gamma\) stand for the space of \(\gamma\)-essentially bounded complex-valued functions on \(\mathcal{U}_x\) with the norm \(\|f\|_\gamma = \text{ess sup} |f(u)|\). By \(L^\infty_\gamma\) we denote the Hilbert space of functions \(f : \mathcal{U}_x \mapsto \mathbb{C}\) with the norm \(\|f\|_\gamma = \langle f, f \rangle_\gamma^{1/2}\) where \(\langle f_1, f_2 \rangle_\gamma := \int f_1 \bar{f}_2 d\gamma\).

The embedding \(L^\infty_\gamma \ni L^1_\gamma\) holds, moreover, \(\|f\|_\gamma \leq \|f\|_\infty\) for all \(f \in L^\infty_\gamma\).

**Theorem 2.1.** The probability measure \(\gamma\) is well defined on \(\mathcal{U}_x\) where it is Radon and possesses the invariant property under the right action,

\[
\int f(u.g) d\gamma(u) = \int f(u) d\gamma(u), \quad g \in U(\infty) \times U(\infty), \quad f \in L^\infty_\gamma. \tag{2.3}
\]

In particular, the following decomposition formulas hold,

\[
\int f d\gamma = \int d\gamma(u) \int_{U(m) \times U(m)} f(u.g) d(\pi_m \otimes \pi_m)(g), \tag{2.4}
\]

\[
\int f d\gamma = \frac{1}{2\pi} \int d\gamma(u) \int_{-\pi}^\pi f[\exp(i\theta)u] d\theta, \quad f \in L^\infty_\gamma. \tag{2.5}
\]

**Proof.** The sequence \(\{(\gamma_m \circ \pi_m)(\Omega)\}\) is decreasing for any Borel set \(\Omega\) in \(\mathcal{U}_x\). In fact, from \(\pi_m = \pi_{m+1}^m \circ \pi_{m+1}\) it follows \(\pi_{m+1}(\Omega) \subseteq (\pi_{m+1}^m)^{-1}[\pi_m(\Omega)]\) and therefore

\[
(\gamma_m \circ \pi_m)(\Omega) = \pi_{m+1}^{m+1}(\gamma_{m+1})[\pi_m(\Omega)] = \gamma_{m+1} \{ (\pi_{m+1}^m)^{-1}[\pi_m(\Omega)] \} \geq (\gamma_{m+1} \circ \pi_{m+1})(\Omega).
\]

This ensures that the necessary and sufficient condition of Prohorov’s theorem is fulfilled (see [2, Thm IX.52] or [13, Thm 4.1]). Thus, \(\lim_{m \to \infty} \gamma_m\) is well defined on \(\mathcal{U}_x\) and

\[
\gamma(\Omega) = \inf(\gamma_m \circ \pi_m)(\Omega) = \lim_{m \to \infty}(\gamma_m \circ \pi_m)(\Omega). \tag{2.6}
\]

The invariance of Haar’s measures \(\gamma_m\) together with (2.6) and the known portmanteau theorem [7, Thm 13.16] yield the invariance property (2.2) under the right action. The formulas (2.4), (2.5) follow from (2.3) and Fubini’s theorem. The measure \(\gamma\) is Radon (see [19, Thm 4.1]).

### 2.3. Paley-Wiener maps

Let \(\mathcal{L}(E)\) be the \(C^\ast\)-algebra of all bounded linear operators \(a : E \to E\) with Hermitian involution \(a \mapsto a^*\) and \(E_{bs}\) be its the \(+\)-subalgebra of Hilbert-Schmidt operators with scalar product and norm, respectively

\[
\langle a | b \rangle_{bs} = \text{tr}(b^* a) = \sum_{k \in \mathbb{N}} \chi_k(b^* a) \quad \text{and} \quad \|a\|_{bs} := \text{tr}(a^* a)^{1/2}, \tag{2.7}
\]

which do not depend on choice of a basis \( \{ e_k : k \in \mathbb{N} \} \) and where is denoted

\[
\chi_k(a) := \langle a | e_k \rangle, \quad \chi_0(a) \equiv 1 \quad \text{(as a function in operator variable } a \in E_{\text{hs}}). \]

As is known (see, e.g. [11, Thm 2.4.10]), \( E_{\text{hs}} \) is a Hilbert *-algebra and \( \| w^* a w \|_{\text{hs}} = \| a \|_{\text{hs}} \) for all unitary \( w \in \mathcal{L}(E) \). Obviously, \( \| a \|_{\text{hs}} = \| a^* \|_{\text{hs}} \).

Throughout, we extend \( E_{\text{hs}} \) by the unit \( 1 \), i.e., \( E_{\text{hs}} = (E \otimes E^\mathfrak{d}) \oplus 1 \mathbb{C} \) where each element \( E_{\text{hs}} \) is understood as \( a + t \mathbb{1} \) with norm \( \| a \|_{\text{hs}} + |t| \) where \( t \in \mathbb{C} \) and \( \text{tr} (\mathbb{1}) = 1 \). In \( E_{\text{hs}} \) the self-adjoint operators \( e_0 = \mathbb{1} \) and

\[
e_k := e_k \otimes e_k^\dagger \in E \otimes E^\mathfrak{d}, \quad \text{acting as } \langle e_k \otimes e_k^\dagger \rangle_{\text{hs}} = \delta_{m,k} e_k \quad \text{with } \ e_k^\dagger = \langle \cdot | e_k \rangle,
\]

form an orthonormal basis, since \( \langle e_k | e_{k'} \rangle_{\text{hs}} = \sum_m \langle \delta_{kk'} \delta_{mm} e_m | e_m \rangle = \delta_{kk'} \delta_{mm} \) with Kronecker's delta \( \delta_{mm} \). Thus, the appropriate Fourier series of \( a \in E_{\text{hs}} \) has the coefficients

\[
\chi_k(a) := \langle a | e_k \rangle_{\text{hs}} = \sum_m \delta_{mm} (a(e_m) | e_k), \quad k \geq 0. \quad (2.8)
\]

In result, we have the following Fourier decomposition:

\[
a = \sum e_k \langle a | e_k \rangle_{\text{hs}} = \sum e_k \chi_k(a), \quad a \in E_{\text{hs}}. \quad (2.9)
\]

It follows that the mappings \( E \to E_{\text{hs}} \to E \) such that \( e_k \to e_k \to e_k \) are unitary.

The complex linear span of \( U(\infty) \) coincides with the subspace of all finite-dimensional operators on \( E \), extended by \( \mathbb{1} \), which is invariant under \( U(\infty) \) (see, e.g. [11, Thm 2.4.17]). The following notion can be seen as a group analog of a Paley-Wiener map (see, e.g. [15], [18]).

**Definition 2.2.** By means of \( U(\infty) \)-valued mapping \( \mathbb{U} \ni u \mapsto \pi^{-1}(u) \), we uniquely determine for every \( a \in E_{\text{hs}} \) the \( \mathbb{C} \)-valued function

\[
\phi_a(u) := \text{tr} \left( a^* \pi^{-1}(u) \right) \equiv \left( \pi^{-1}(u) | a \right)_{\text{hs}}, \quad \tilde{\phi}_a(u) = \phi_a(u^*)
\]

which can be called a *Paley-Wiener map* over the group \( U(\infty) \) with respect to the total embedding

\[
U(\infty) \ni u \mapsto E_{\text{hs}} \quad (2.10)
\]

where \( U(\infty) \) is endowed with the Borel structure \( \mathbb{U} \).

The uniqueness of \( \phi_a \) with a fixed \( a \in E_{\text{hs}} \) results from the total embedding \( 2.10 \). By \( 2.6 \) and the portmanteau theorem there exist the limit

\[
\int \phi_a d\gamma = \lim_{U(m)} \int \phi_a d(\gamma_m \circ \pi_m) = \lim_{U(m)} \langle \pi^{-1} \circ \pi_m^{-1}(u) | a \rangle_{\text{hs}} d\gamma_m(u), \quad \text{i.e., } \phi_a \text{ is } \gamma\text{-integrable. Similar arguments also show that the functions } \phi_a \text{ and their products belong to } L_2^\gamma.
\]

The Fourier decomposition \( \pi^{-1}(u) \in E_{\text{hs}} \) has the \( \gamma \)-essentially bounded coefficients

\[
\phi_k(u) := \chi_k[\pi^{-1}(u)], \quad \text{with } \chi_k[\pi^{-1}(u)] = \langle [\pi^{-1}(u)] e_k | e_k \rangle, \quad \phi_0(u) \equiv 1 \quad (2.11)
\]

which, as a function in variable \( u \in \mathbb{U} \), do not depend on choice of a basis \( \{ e_k \} \) in \( E \).

Since \( w^* u = w^* \pi^{-1}(u) \) in the case \( w \in U(\infty) \subset E_{\text{hs}} \), we keep the notation \( a^* u = a^* \pi^{-1}(u) \) for all \( a \in E_{\text{hs}} \) and \( u \in \mathbb{U} \). Hence, \( \phi_k(a^* u) = \chi_k[a^* \pi^{-1}(u)] = \langle [a^* \pi^{-1}(u)] e_k | e_k \rangle \) is well defined for all \( a \in E_{\text{hs}} \). In result, we obtain the assertion.
Lemma 2.3. The following trace decomposition of Paley-Wiener’s map

\[ \phi_\omega(u) = \sum_{k \geq 0} \phi_k(\alpha^\ast u), \quad \phi_0(\alpha^\ast u) \equiv 1 \]  

(2.12)

holds for all \( u \in \mathbb{U}_\omega \) and \( \alpha \in E_{ns} \). The Fourier decomposition of \( \pi^{-1}(u) \in E_{ns} \) takes the form

\[ \pi^{-1}(u) = \sum c_k(\pi^{-1}(u) \mid c_k)_{ns} = \sum c_k \phi_k(u). \]  

(2.13)

3. Symmetric Wiener spaces

Let us assign to any \( \iota \in \mathcal{F} \) the vector \( \phi_\iota(u) = (\phi_{\iota_1}(u), \ldots, \phi_{\iota_n}(u)) \) and let

\[ s_\iota^\ast(u) := (s_\iota^\ast \circ \phi_\iota)(u) = \sum_{\iota \in \mathfrak{I}} \phi_{\iota_1}^{\iota_1}(u), \quad \phi_{\iota_1}^{\iota_1} := \phi_{\iota_1}^{\iota_1} \ldots \phi_{\iota_n}^{\iota_n} \]  

(3.1)

be the \( n \)-homogeneous Schur polynomial function with summation over all semistandard Young \( \lambda \)-tabloids \( [\iota^t] \) associated with a fixed partition \( \lambda \in \mathbb{N}_\lambda \) such that \( \iota^t \vdash n \). Consider the systems

\[ s_\iota^\gamma = \bigcup \{ s_\iota^\gamma : \iota^t + n \}, \quad s_\iota^\gamma = \bigcup \{ s_\iota^\zeta : \zeta \in \mathbb{Z}_\ast \} \]  

with \( s_\iota^\gamma \equiv 1. \)  

(3.2)

Definition 3.1. The symmetric Wiener space \( W_n^\gamma \) of scalar functions in virtual unitary variable \( u \in \mathbb{U}_\ast \) is defined to be the closed complex linear span in \( L_2^n \) of all symmetric polynomial functions \( s_\iota^\gamma \). Let the closed subspace \( W_{\gamma,n}^\gamma \subset W_{\gamma,n}^\gamma \) with a fixed \( n \in \mathbb{N} \) be spanned by \( s_\iota^\gamma \).

Note that the Littlewood-Richardson rule \( s_\iota^\lambda s_\nu^\nu = \sum c_{\iota,\mu,\nu} s_\mu^\mu \) with \( \lambda, \mu, \nu \in \mathbb{Y} \) and \( \iota \in \mathcal{F} \), where \( c_{\iota,\mu,\nu} \) are structure coefficients of rings with the basis of Schur polynomials (see, e.g. [16, n.12.5]), implies that the multiplication in a dense subspace of \( W_{\gamma,n}^\gamma \) is well defined.

Theorem 3.2. The system of symmetric polynomial functions \( s_\iota^\gamma \) forms an orthonormal basis in \( W_{\gamma,n}^\gamma \) and \( s_\iota^\gamma \) is the same in \( W_{\gamma,n}^{\gamma_\gamma} \). As a consequence, the following orthogonal decomposition holds

\[ W_{\gamma,n}^\gamma = \mathbb{C} \oplus W_{\gamma,n}^{2,1} \oplus W_{\gamma,n}^{2,2} \oplus \ldots. \]  

(3.3)

Proof. Let \( U(\iota) \) be the unitary subgroup of \( U(\infty) \) acting in the \( l \)-dimensional subspace \( E_i \subset E \) spanned by \( \{ e_i, \ldots, e_n \} \). Using (2.4) with \( U(\iota) \) instead of \( U(m) \), we have

\[ \int s_\iota^\gamma s_\iota^\nu \, dy = \int dy(u) \int s_\iota^\gamma(z^\ast u) s_\iota^\nu(z^\ast u) \, dy(z) \]

with Haar’s measure \( \gamma \), on \( U(\iota) \) for all tabloids \( [\iota^t] \) and \( [\iota^t] \) such that \( \iota^t + n \) and \( \iota^t + n \). The Schur functions \( [s_\iota^\gamma : \iota^t + n] \) are characters of the unitary group \( U(\iota) \). Via (2.3), (3.1) the interior integral is independent on \( u \in \mathbb{U}_\ast \). Hence, by Weyl’s integral formula this integral is equal to Kronecker’s delta \( \delta_{\mu,\nu} \) (see, e.g. [16, Thm 11.9.1]). Thus, \( \int s_\iota^\gamma s_\iota^\nu \, dy = \delta_{\mu,\nu} \).

It remains to note that the family of all finite alphabets \( \iota \in \mathcal{F} \) is directed, i.e., for any \( \iota, \iota' \) there exists \( \iota'' \) such that \( \iota \cup \iota' \subset \iota'' \). This implies that the whole system \( s_\iota^\gamma \) is orthonormal in \( L_2^n \).

The orthogonal property \( s_\iota^\gamma \perp s_\iota^\gamma \) with \( |\mu| \neq |\lambda| \) for any \( \iota, \iota' \in \mathcal{F} \) follows from (2.5), since

\[ \int s_\iota^\gamma s_\iota^\nu \, dy = \int s_\iota^\gamma(\exp(\iota\theta).u)s_\iota^\nu(\exp(\iota\theta).u)\, dy(u) \]

\[ = \frac{1}{2\pi} \int s_\iota^\gamma s_\iota^\nu \, dy \int_0^\pi \exp(\iota(|\mu| - |\lambda|)) \, d\theta = 0 \]
4.1. Symmetric Fock spaces over Hilbert-Schmidt operators

Corollary 3.3. For any \(a \in H_{hs}\) and unitary \(w \in \mathcal{L}(E)\),

\[\|\phi_a\|_Y = \|w \circ \phi_a\|_Y = \|a\|_{hs}.\]

Proof. First note that \(a = \sum \epsilon \xi | \xi\rangle_{hs} = \sum \epsilon \xi | \xi\rangle_{hs} | \xi\rangle_{hs},\) therefore \(\|a\|_{hs}^2 = \sum (|\epsilon(\xi)| | \xi\rangle)^2.\)

On the other hand, from (2.11-2.13) it follows

\[\phi_\xi(u) = \left\langle \pi^{-1}(u) | a\right\rangle_{hs} = \sum \phi_m(u) \langle \epsilon \xi | \xi\rangle_{hs} | \xi\rangle_{hs}.\]

By Theorem 5.2, the subsystem \(\phi_\xi = s_k^1\) is orthonormal in \(L^2_z\), hence

\[\|\phi_\xi\|_Y^2 = \sum (|\epsilon(\xi)| | \xi\rangle)^2 \int |\phi_\xi|^2 d\gamma = \|a\|_{hs}^2.\]

Finally, we have to apply the well known equality \(\|w^\ast aw\|_{hs} = \|a\|_{hs}.\)

Corollary 3.4. The equality \(\int \pi^{-1}(u) d\gamma(u) = I_1\) holds.

Proof. By (2.3) we have \(2\pi \int \phi_k d\gamma = \int \phi_k d\gamma \sum_n \exp i \theta d\theta = 0\) for all \(k > 0\). After integrating (2.13), we get \(\int \pi^{-1}(u) d\gamma(u) = I_1\) since \(i_0 \phi_0 = I_1\).

4. Hardy spaces of analytic functions

4.1. Symmetric Fock spaces over Hilbert-Schmidt operators

Let \(E_h^{\otimes n}\) be the completion of algebraic tensor n-th power \(E \otimes \ldots \otimes E\) by the norm \(\|\phi_n\| = \langle \phi_n | \phi_n \rangle^{1/2}\), where \(\phi_n = z_1 \otimes \ldots \otimes z_n\) with \(z_i \in E\) and \(\langle \phi_n | \phi_n' \rangle_{hs} = \langle z_1 | z'_1 \rangle \cdots \langle z_n | z'_n \rangle\). Consider the (full) Fock space \(\bigoplus_{k=0}^{\infty} E_h^{\otimes n}\) of elements \(\phi = \bigoplus \psi_n\) with \(\langle \phi | \psi' \rangle = \sum \langle \psi_n | \psi'_n \rangle\) where \(E_h^{\otimes n}\) is the symmetric group \(S_n\) can be represented over \(E_h^{\otimes n}\) by actions \(\sigma(\phi_n) = z_{r^{-1}(1)} \otimes \ldots \otimes z_{r^{-1}(n)}\). The corresponding symmetric tensor power \(E_h^{\otimes n}\) is defined to be a range of the orthogonal projector \(\sigma_n: E_h^{\otimes n} \ni \psi_n \mapsto z_1 \otimes \ldots \otimes z_n\) with \(z_1 \otimes \ldots \otimes z_n = n!^{-1} \sum_{\sigma \in S_n} \sigma(\psi_n).\) Here, the subset \(\{z^{(n)} = z \otimes \ldots \otimes z: z \in E\} \subset E_h^{\otimes n}\) is total by the polarization formula

\[z_1 \otimes \ldots \otimes z_n = \frac{1}{2^n n!} \sum_{\theta_1, \ldots, \theta_n} \theta_1 \ldots \theta_n z^{(n)}, \quad z = \sum_{k=1} z_k \]

(4.1)

with \(z_1, \ldots, z_n \in E\) \([\text{I}\, n.1.5]\). As usual, the symmetric Fock space is defined to be the orthogonal sum \(E_h = \mathbb{C} \oplus E \oplus E_h^{\otimes 2} \oplus \ldots\)

Let \(f \in \mathcal{F}\) and \(E_i \subset E\) be spanned by \(\{e_{i_1}, \ldots, e_{i_n}\}\). We uniquely assign to any semistandard tableau \([r^i t^j]\) with \(r^i + t^j\) the basis element \(e_{r^i t^j}^{(i)}\) from \(E_h^{\otimes n}\) for which there exists \(\sigma \in S_n\) such that \(e_{r^i t^j}^{(i)} = \sigma(e_{i_1}^{(i)} \otimes \ldots \otimes e_{i_n}^{(i)})\) with \(I = I^i\). If \(M_i^j\) is the representation of \(S_n\) over \(E_h^{\otimes n}\), whose basis is indexed by the collection of all semistandard tabloids \([r^i t^j]\) with \(r^i + t^j\), then \(\dim M_i^j\) is equal to the number of basis elements \(e_{r^i t^j}^{(i)}\) from \(E_h^{\otimes n}\) such that \(e_{r^i t^j}^{(i)} = e_{r^i t^j}^{(j)}\) with \(e_{r^i t^j}^{(i)} := e_{i_1}^{(i)} \otimes \ldots \otimes e_{i_n}^{(i)}\).
other words, each $e^{0,i}$ may be identified with the semistandard tableau $[r^i]$ and the appropriate $S_n$-module $M^i$ coincides with the representation of $S_n$ over $E^0_{n}$. As a result, $E^0_{n}$ is generated by

$$
e_{n}^{0,y} = \left\{ e^{0,i} : r^i + n, t \in \mathcal{J} \right\}, \quad \|e^{0,i}\| = \left( \dim M^i \right)^{-1/2} = \sqrt{\lambda^{} / n!}, \quad (4.2)$$

where $\lambda = 1! \ldots \lambda l$. Thus, $\Gamma_n = \bigoplus E^0_{n}$ has the basis $e^{0,y} := \bigcup \left\{ e^{0,i} : n \in \mathbb{Z}_+ \right\}$ with $e^{0,0} = 1$.

Now, we consider a symmetric Fock structure over $E_{n}$. Let $(E_{n})_{n}$ be spanned by $\{e_i, \ldots, e_n\}$. Similarly as above, we assign to any semistandard tableau $[r^i]$ with $r^i + n$ the basis element $e^{0,i}$ from $(E_{n})_{n}$ for which there exists $\sigma \in S_n$ such that $e^{0,i} = e_{\sigma}^{0,i} \circ \cdots \circ e_{0}^{0,i}$ and denote $e^{0,i} := e_{\sigma}^{0,i} \circ \cdots \circ e_{0}^{0,i}$.

**Definition 4.1.** Let us define a Hilbertian norm on the algebraic symmetric tensor $n$th power $\sigma_{n}(E_{n} \otimes \cdots \otimes E_{n})$ by the equality $\| \cdot \|_{n} = \langle \cdot | \cdot \rangle_{n}^{1/2}$ where scalar product $\langle \cdot | \cdot \rangle_{n}$ is determined via the orthogonal relations

$$
\langle e^{0,i} | e^{0,i} \rangle_{n} = \left\{ \begin{array}{cl} \frac{(\lambda^{}! / n^{}!)\|e^{0,i}\|^2}{0} : \lambda = \lambda' \quad \text{and} \quad t = t', \\
\frac{(\lambda^{}! / n^{}!)\|e^{0,i}\|^2}{\lambda \neq \lambda'} \quad \text{or} \quad t \neq t'. \\
\end{array} \right.
$$

Denote by $E^0_{n}$ the appropriate completions of $\sigma_{n}(E_{n} \otimes \cdots \otimes E_{n})$.

The symmetric Fock space of Hilbert-Schmidt operators is defined to be the orthogonal sum $\Gamma_n = \bigoplus_{n \geq 0} E^0_{n}$ with the orthogonal basis $e^{0,y} := \bigcup \left\{ e^{0,i} : n \in \mathbb{Z}_+ \right\}$ where $e^{0,0} = 1$ and

$$
e_{n}^{0,y} = \left\{ e^{0,i} := e_{\sigma}^{0,i} \circ \cdots \circ e_{0}^{0,i} : r^i + n, t \in \mathcal{J} \right\}, \quad \|e^{0,i}\|_{n} = \lambda^{} / n!. \quad (4.3)$$

This also provides the total property of the subsets $\{\exp(a) : a \in E_{n}\}$ of coherent states in $\Gamma_n$ where is denoted

$$
\exp(a) := \bigoplus_{n \geq 0} \frac{a^0}{n^!}, \quad a^0 = 1.
$$

In what follows, we assign to any semistandard tableau $[r^i]$ with $r^i + n$ the $n$-homogenous polynomial in operator variable $a$, defined via the Fourier coefficients (2.3).

$$
\chi^i_l(a) := \langle a^0 | e^{0,i} \rangle_n = \chi^i_{\lambda^{} l}(a) \cdots \chi^i_{\lambda^{} l}(a), \quad a \in E_{n}
$$

where $l = l$ and $\chi^0_l \equiv 1$. Using the tensor multinomial theorem and the Fourier decomposition (2.3) of $a$, we define in $\Gamma_n$ the Fourier decomposition of coherent states

$$
\exp(a) = \bigoplus_{n \geq 0} \frac{a^0}{n^!} = \bigoplus_{n \geq 0} \frac{1}{n^!} (\sum_{k \geq 0} c_k \chi^i_k)^{0} \bigoplus_{n \geq 0} \frac{1}{n^!} \sum_{d \mid n} \frac{n^!}{d^!} \sum_{l = 1}^{n^! / d^!} c_l \chi^i_{\lambda^{} l} \quad (4.4)
$$

with respect to the basis $e^{0,y}$. This is convergent in $\Gamma_n$ and

$$
\|\exp(a)\|^2_{n} = \sum_{n \geq 0} \frac{1}{n^!} \sum_{d \mid n} \left( \frac{n^!}{d^!} \right)^2 \|e^{0,i}\|^2_n \chi^i_l(a)^2 \leq \sum_{n \geq 0} \frac{1}{n^!} \sum_{d \mid n} \frac{n^!}{d^!} \chi^i_l(a)^2
$$

$$
= \sum_{n \geq 0} \frac{1}{n^!} \left( \sum_{l = 1}^{n^!} \chi^i_l(a)^2 \right)^n \leq \sum_{n \geq 0} \frac{1}{n^!} \|a^0\|^2_n \exp(\|a\|^2) \quad (4.5)
$$

which implies that the function $E_{n} \ni a \mapsto \exp(a) \in \Gamma_n$ is entire analytic.
4.2. Hardy spaces of Hilbert-Schmidt analytic functions

Let \((E_{hs}^{\mathbb{C}^n})^* = E_{hs}^{\mathbb{C}^n}\) be the Hermitian dual of \(E_{hs}^{\mathbb{C}^n}\), where \(E_{hs}^{\mathbb{C}^n}\) denotes the Hermitian dual of \(E_{hs}\) of functionals

\[
b^\mathbb{C}^n(a) = \langle a | b \rangle_{hs} = \text{tr}(b^*a), \quad a, b \in E_{hs}.
\]

The well-known isometry \(E_{hs}^{\mathbb{C}^n} \cong \mathcal{P}_n(E_{hs})\) holds (see, e.g. [3, 1.6]), where the space \(\mathcal{P}_n(E_{hs})\) of \(n\)-homogeneous Hilbert-Schmidt polynomials in variable \(a \in E_{hs}\) is defined as restrictions to the diagonal in \(E_{hs} \times \ldots \times E_{hs}\) of the \(n\)-linear forms

\[
\langle (a, \ldots, a) | T \circ \psi_n \rangle_{hs} = \langle a^{\otimes n} | \psi_n \rangle_{hs}, \quad \psi_n \in E_{hs}^{\mathbb{C}^n}.
\]

Farther on \(\widehat{\psi} \circ \psi_n \mapsto \langle \hat{\psi} | \hat{\psi} \rangle_{hs}^{1/2}\) of functionals \(\hat{\psi}\) is the Hermitian dual of \(\hat{\psi}\).

**Definition 4.2.** The Hardy space of complex-valued Hilbert-Schmidt entire analytic functions \(E_{hs} \ni a \mapsto \hat{\psi}(a)\) with norm \(\|\hat{\psi}\|_{hs} = \langle \hat{\psi} | \hat{\psi} \rangle_{hs}^{1/2}\) is defined to be

\[
H_{hs}^2 = \{ \hat{\psi}(a) = \langle \exp(a) | \hat{\psi} \rangle_{hs} : \psi \in \Gamma_{hs} \} \quad \text{with} \quad \langle \hat{\psi} | \hat{\psi} \rangle_{hs} = \langle \psi | \psi \rangle_{hs}.
\]

Farther on \(\hat{T} : \Gamma_{hs} \mapsto H_{hs}^2\) means the corresponding conjugate-linear surjective isometry.

Note that every \(\hat{\psi}\) is entire analytic as the composition of \(\exp(a)\) and \(\langle \cdot | \psi \rangle_{hs}\).

**Lemma 4.3.** The systems of Hilbert-Schmidt polynomials in operator variable \(a \in E_{hs}\)

\[\chi_n^\mathbb{C}^n = \bigcup_i \chi_i^\mathbb{C}^n : r^i + n, r \in \mathcal{F}\] and \(\chi^\mathbb{C}^n = \bigcup \chi_n^\mathbb{C}^n : n \in \mathbb{Z}_+\),

where \(\chi_i^\mathbb{C}^n = 1\), form orthogonal bases in \(H_{hs}^{2,n}\) and \(H_{hs}^2\), respectively, with norms

\[
\|\chi_i^\mathbb{C}^n\|_{hs} = \|\chi_i^\mathbb{C}^n\|_{hs} = \lambda^!|\lambda|!.
\]

Every function \(\hat{\psi} \in H_{hs}^2\) has the appropriate Fourier expansion with respect to \(\chi^{\mathbb{C}^n}\)

\[
\hat{\psi}(a) = \sum_{\lambda \geq 0} \frac{1}{\lambda!^n} \sum_{\mu \geq \lambda} \frac{\langle \psi_n \rangle_{hs}}{\|\psi_n\|_{hs}^2} \chi_i^\mathbb{C}^n(a), \quad a \in E_{hs}
\]

with summation over all semistandard Young tabloids \([r^i]\) such that \(r^i + n\). The function \(\hat{\psi}\) is entire Hilbert-Schmidt analytic [4, n.5]. [4].

**Proof.** Taking into account (4.7), we have

\[
\langle \chi_i^\mathbb{C}^n | \chi_j^\mathbb{C}^n \rangle_{hs} = \langle \chi_i^\mathbb{C}^n | \chi_j^\mathbb{C}^n \rangle_{hs} = \left\{ \begin{array}{ll} \|\chi_i^\mathbb{C}^n\|_{hs}^2 : \lambda = \mu & \text{and} \quad i = j, \\ 0 : \lambda \neq \mu & \text{or} \quad i \neq j. \end{array} \right.
\]

Applying (4.4) and (4.5), we conclude that every analytic function \(\hat{\psi}\) with \(\psi = \bigoplus \psi_n \in \Gamma_{hs}\) has the Taylor expansion at zero

\[
\hat{\psi}(a) = \sum_{\lambda \geq 0} \frac{1}{\lambda!^n} \langle \psi_n \rangle_{hs} \quad \text{where} \quad \langle \psi_n \rangle_{hs} = \sum_{\lambda \geq 0} \frac{\langle \chi_i^\mathbb{C}^n | \psi_n \rangle_{hs}}{\|\chi_i^\mathbb{C}^n\|_{hs}^2} \chi_i^\mathbb{C}^n(a)
\]

are Hilbert-Schmidt polynomials in variable \(a \in E_{hs}\). By inserting of Taylor coefficients, we get the appropriate Fourier expansion (4.9).
5. Generalized Paley-Wiener theorem

5.1. Polynomial extensions of Paley-Wiener maps

Further we will significantly use Fourier coefficients of the Paley-Wiener map

\[ \phi_z : U_\pi \ni u \mapsto \sum_{k \geq 0} \chi_k(u) \phi_k(u), \quad a \in E_{\text{hs}}. \]

By polarization formula (4.3) for any semistandard tableau \( \lambda \) with \( r^l + n \) there uniquely corresponds the function from \( L^2_{\text{hs}} \)

\[ \phi^l_i(u) = \left\langle [\pi^{-1}(u)]^{\text{hs}} | c^{\lambda l}_i \rightrangle_{\text{hs}} = \phi^l_i(u) \ldots \phi^l_i(u), \quad l = l_\lambda \]

in variable \( u \in U_\pi \). It follows that the orthogonal basis \( \chi^\lambda \) uniquely determines the systems of \( \chi \)-essentially bounded functions in variable \( u \in U_\pi \)

\[ \phi^\lambda = \bigcup \left\{ \phi^\lambda_n : n \in \mathbb{Z}_+ \right\} \quad \text{with} \quad \phi^\lambda_n = \bigcup \left\{ \phi^l_i : r^l + n, i \in \mathcal{I}^l \right\}. \]

**Theorem 5.1.** The system of polynomial functions \( \phi^\lambda \) forms an orthogonal basis in \( W^2_{\text{hs}} \) and \( \phi^\lambda_n \) is the same in \( W^2_{\text{hs}} \) with the norms

\[ \| \phi^\lambda \|_{\text{hs}} = \| \chi^\lambda \|_{\text{hs}} = \| c^{\lambda l} \|_{\text{hs}}. \]

**Proof.** Let \( i \in \mathcal{I}^l \). Similarly to (2.11), the unitary subgroup \( U(i) \subset U(\infty) \) is total in \( (E_i)_{\text{hs}} \). Let \( (E_i)_{\text{hs}} \) mean the \( n \)th symmetric tensor power of \( (E_i)_{\text{hs}} \). Consider the auxiliary functions

\[ \phi_i^l(z^* u) := \left\langle [z^* \pi^{-1}(u)]^{\text{hs}} | c^{\lambda l}_i \rightrangle_{\text{hs}} = \left\langle [z^* \pi^{-1}(u)]^{\text{hs}} \otimes_c | c^{\lambda l}_i \rightangle \]

for all \( z \in U(i) \) and \( u \in U_\pi \). Using (2.4) with \( U(i) \) instead of \( U(m) \), we have

\[ \int_{U(i)} \phi_i^l(u) \hat{\phi}_i^l(u) d\gamma(u) = \int d\gamma(u) \int_{U(i)} \phi_i^l(z^* u) \hat{\phi}_i^l(z^* u) d\gamma(z) \]

for all \( [r^l] \) and \( [\mu^l] \) such that \( r^l + n \) and \( \mu^l + n \) with Haar’s measure \( \gamma_i \) on \( U(i) \). It is clear that

\[ \left| \int_{U(i)} \phi_i^l d\gamma_i \right| \leq \sup_{z^* \pi^{-1}(u) \in U(\infty)} \left| \phi_i^l(z^* u) \right| \left| \phi_i^l(z^* u) \right| \leq \| c^{\lambda l} \|_{\text{hs}} \| c^{\lambda l} \|_{\text{hs}}. \]

Hence, the corresponding sesquilinear form in (5.2) is continuous on \( (E_i)_{\text{hs}} \). Thus, there exists a linear bounded operator \( A_i \) over \( (E_i)_{\text{hs}} \) such that

\[ \left( A_i c^{\lambda l} | c^{\lambda l}_i \right)_{\text{hs}} = \int_{U(i)} \phi_i^l(z^* u) \hat{\phi}_i^l(z^* u) d\gamma(z) \]

with \( |l| = |\mu| = n \). We will show that \( A_i \) commutes with all \( w^{\text{hs}} \in (E_i)_{\text{hs}} \), i.e.

\[ A_i \circ w^{\text{hs}} = w^{\text{hs}} \circ A_i \quad \text{where} \quad w \in U(i). \]
In fact, the invariance property \((2.3)\) of \(\gamma\) under unitary actions yields

\[
\left\langle (A_i \circ \mathfrak{w}^{\text{on}}) \mathfrak{e}^{\mathfrak{A}}_i | \mathfrak{e}^{\mathfrak{A}}_i \right\rangle_{\text{hs}} = \int_{U(\mathfrak{g})} (|\zeta^e \pi^{-1}(u)|^{\text{on}} | \mathfrak{e}^{\mathfrak{A}}_i)_{\text{hs}} (|\zeta^e \pi^{-1}(u)|^{\text{on}} | \mathfrak{e}^{\mathfrak{A}}_i)_{\text{hs}} dy_i(z)
\]

\[
= \int_{U(\mathfrak{g})} (|\mathfrak{w} \pi^e \pi^{-1}(u)|^{\text{on}} | \mathfrak{e}^{\mathfrak{A}}_i)_{\text{hs}} (|\mathfrak{w} \pi^e \pi^{-1}(u)|^{\text{on}} | \mathfrak{e}^{\mathfrak{A}}_i)_{\text{hs}} dy_i(z)
\]

\[
= \left( (A_i \circ \mathfrak{w}^{\text{on}}) \mathfrak{e}^{\mathfrak{A}}_i | \mathfrak{e}^{\mathfrak{A}}_i \right)_{\text{hs}} = \left( (\mathfrak{w}^{\text{on}} \circ A_i) \mathfrak{e}^{\mathfrak{A}}_i | \mathfrak{e}^{\mathfrak{A}}_i \right)_{\text{hs}}.
\]

Check that \(A_i\) satisfying \((5.3)\) is proportional with a constant \(c_1\) to the identity in \((E_i)_{\text{hs}}^{\mathfrak{on}}\), i.e., that the unitary representation \([\mathfrak{w}^{\text{on}} : \mathfrak{w} \in U(\mathfrak{g})]\) over \((E_i)_{\text{hs}}^{\mathfrak{on}}\) is irreducible. By polarization formula \((5.1)\), the embedding \([\mathfrak{w}^{\text{on}} : \mathfrak{w} \in U(\mathfrak{g})] \mapsto (E_i)_{\text{hs}}^{\mathfrak{on}}\) is total. Taking into account that \(\{\mathfrak{e}^{\mathfrak{A}}_i : i \neq n, i \in \mathfrak{g}(\mathfrak{i})\}\) forms an orthogonal system in \((E_i)_{\text{hs}}^{\mathfrak{on}}\), the equality \((5.3)\) holds. It follows that there exists a constant \(c > 0\) such that

\[
\int_{U(\mathfrak{g})} \mathfrak{e}^{\mathfrak{A}}_i \mathfrak{e}^{\mathfrak{A}}_i dy_i = c \left( \mathfrak{e}^{\mathfrak{A}}_i | \mathfrak{e}^{\mathfrak{A}}_i \right)_{\text{hs}}, \quad \mathfrak{e}^{\mathfrak{A}}_i, \mathfrak{e}^{\mathfrak{A}}_i \in (E_i)_{\text{hs}}^{\mathfrak{on}}. \tag{5.4}
\]

In particular, the subsystem of cylindrical functions with a fixed \(i \in \mathfrak{g}(\mathfrak{i})\)

\[
\phi_i(u) = \left( [\pi^{-1}(u)]^{\text{on}} | \mathfrak{e}^{\mathfrak{A}}_i \right) = \left( [\pi^{-1}(u)]^{\text{on}} | \mathfrak{e}^{\mathfrak{A}}_i \right)_{\text{hs}} = \phi_i^\mathfrak{A}(\mathfrak{I}, u)
\]

is orthogonal in \(L^2\), because the corresponding system of elements \(\mathfrak{e}^{\mathfrak{A}}_i\) is an orthogonal basis in \((E_i)_{\text{hs}}^{\mathfrak{on}}\). It remains to note that indices \(i \in \mathfrak{g}\) are directed under inclusions. Thus, the system \(\phi^\mathfrak{A}\) is orthogonal in \(L^2\).

Using the isometry \(\mathfrak{T} : (E_{\text{hs}}^{\mathfrak{on}})^\mathfrak{g} \longrightarrow \mathfrak{P}(E_{\text{hs}})\) defined by \((4.6)\), we get

\[
\left\langle \left[ \int f dy \right]^{\text{on}} | \psi_n \right\rangle_{\text{hs}} = \left\langle \left( \int f dy, \ldots, \int f dy \right) | \mathfrak{T} \circ \psi_n \right\rangle_{\mathfrak{P}} = \left\langle \left( \int f dy \right) | \mathfrak{T} \circ \psi_n \right\rangle_{\mathfrak{P}}
\]

\[
= \int \left( \int f \psi_n dy \right) \psi_n dy = \int \left( \int f \psi_n^\mathfrak{on} \right) \psi_n^\mathfrak{on} dy \tag{5.5}
\]

for any \(E_{\text{hs}}\)-valued Bochner \(\gamma\)-integrable function \(f\) and \(\psi_n \in E_{\text{hs}}^{\mathfrak{on}}\). Thus, \(\int f dy \right\rangle_{\text{hs}}^{\text{on}} = \int f \psi_n dy\).

Put in \((5.5)\) \([\pi^{-1}]^{\mathfrak{on}}\) instead of \(f\) and \(\mathfrak{e}^{\mathfrak{on}}_i \otimes \mathfrak{e}^{\mathfrak{on}}_i\) instead of \(\psi_n\), where \(\mathfrak{e}^{\mathfrak{on}}_i = \langle \cdot | \mathfrak{e} \rangle_{\text{hs}}\). Then taking into account that \(\phi_i^\mathfrak{A}(u) = \left( [\pi^{-1}(u)]^{\text{on}} | \mathfrak{e}^{\mathfrak{A}}_i \right)_{\text{hs}}\) and \(\phi_i^\mathfrak{A}(u) = \left( [\pi^{-1}(u)]^{\text{on}} | \mathfrak{e}^{\mathfrak{A}}_i \right)_{\text{hs}}\) with \(i \neq n\), by Corollary \((3.4)\) and the properties \((4.2)\) and \((4.3)\), we obtain

\[
\int |\phi_i^\mathfrak{A}(u)|^2 dy = \int \left( [\pi^{-1}(u)]^{\text{on}} | \mathfrak{e}^{\mathfrak{on}}_i \otimes \mathfrak{e}^{\mathfrak{on}}_i \right)_{\text{hs}} dy_i(u) = \left( \int [\pi^{-1}(u)]^{\text{on}} dy_i(u) | \mathfrak{e}^{\mathfrak{on}}_i \otimes \mathfrak{e}^{\mathfrak{on}}_i \right)_{\text{hs}}
\]

\[
= \left( \int [\pi^{-1}(u)]^{\text{on}} dy_i(u) \right)^2 |\mathfrak{e}^{\mathfrak{on}}_i \otimes \mathfrak{e}^{\mathfrak{on}}_i \rangle_{\text{hs}}^2 = (\mathfrak{e}^{\mathfrak{on}}_i | \mathfrak{e}^{\mathfrak{on}}_i)^2 = ||\mathfrak{e}^{\mathfrak{on}}_i||_{\text{hs}}^2.
\]

Hence, \(c = 1\). To get \((5.1)\), we have to apply \((4.3)\).
5.2. Fourier transforms as analytic extensions of Paley-Wiener maps

First, let us examine in more detail the composition of \( \exp(\cdot) \) and \( L_2^\infty \)-valued function \( \phi_\gamma \).

**Lemma 5.2.** The composition \( \exp \phi_\gamma \), which is understood as the function

\[
\exp \phi_\gamma : U_\gamma \ni u \mapsto \exp \phi_\gamma(u),
\]

takes values in \( L_2^\infty \) and is entire analytic in variable \( a \in E_{hs} \).

**Proof.** Using (2.12) and the multinomial formula, we get

\[
\exp \phi_\gamma(u) = \sum_{n \geq 0} \frac{\phi_\gamma^n(u)}{n!} = \sum_{n \geq 0} \frac{1}{n!} \sum_{\gamma \in \Lambda^n} n! \phi_\gamma^n(a^* u)
\]

with \( \phi_\gamma^n(a^* u) = \phi_\gamma^n(a^* u) \cdots \phi_\gamma^n(a^* u) \) where \( \phi_\gamma(a^* u) = (a^* \pi^{-1}(u) e_k | e_k) \) and \( l = l_k \). By Corollary 3.3, we have \( \| \exp \circ \phi_\gamma \|_\infty \leq \| \phi_\gamma \|_\infty = \| a \|_{hs} \).

Since \( \phi_\gamma(u) = (\pi^{-1}(u) | a)_{hs} \), we have \( \phi_\gamma^n(u) = (\pi^{-1}(u))^\otimes n | a^\otimes n)_{hs} \). Putting in (5.5) \( [\pi^{-1}]^\otimes 2 \) instead of \( \mathfrak{F}^\infty \) and \( a \otimes b^\infty \) instead of \( \phi_\gamma \), we get

\[
\int \phi_\gamma^n \phi_\gamma^d \, d\gamma = \int (\pi^{-1}(u))^\otimes 2 | a^\otimes b^\otimes \pi \mathfrak{F}^\infty)_{hs} d\gamma(u) = \left( \int (\pi^{-1}(u))^\otimes 2 | a^\otimes b^\otimes \mathfrak{F}^\infty)_{hs} \right)^n
\]

Hence, for \( a = b \) we get \( ||\phi_\gamma^n||_2^2 = ||\phi_\gamma||_2^n = ||a||_{hs}^2 n! \) by Corollary 3.3. It follows

\[
|| \exp \phi_\gamma ||_2^n = \sum_{n \geq 0} \frac{||\phi_\gamma^n||_2^2}{n!} = \sum_{n \geq 0} \frac{||a||_{hs}^{2n}}{n!}
\]

Therefore, \( \exp \phi_\gamma \) is absolutely convergent in \( L_2^\infty \) for all \( a \in E_{hs} \). \( \Box \)

**Theorem 5.3.** For every function \( f \in W_2^2 \) its Fourier transform

\[
\hat{f}(a) = \int f \exp \phi_\gamma \, d\gamma = \int f \exp \left[ 2 \Re \phi_\gamma - ||a||_{hs}^2 \right] \, d\gamma
\]

is an entire analytic function in variable \( a \in E_{hs} \). The Taylor expansion at zero

\[
\hat{f}(a) = \sum_{n \geq 0} \frac{1}{n!} \mathfrak{F}^n_\gamma f(a) \quad \text{with} \quad f = \sum_{n \geq 0} f_n \in W_2^2,
\]

has the coefficients

\[
\mathfrak{F}^n_\gamma f(a) = \int f_n \phi_\gamma^n \, d\gamma = \sum_{r \in \Lambda^n} h_r \int f_n(u) s_r^l(a^* u) \, d\gamma(u)
\]

with summation over all standard Young tabloids \( [r^\gamma] \) such that \( r^\gamma \vdash n \) where \( s_r^l(a^* u) = (s_r^l \circ \phi_\gamma)(a^* u) \) is defined via (3.1).
Lemma 5.6. The Paley-Wiener map, extended onto coherent states to be 
\[ \Phi \exp(a) \mapsto \exp \phi_a, \quad a \in E_{\text{hs}}, \]
possessed a unique conjugate-linear surjective isometric extension
\[ \Phi : \Gamma_{\text{hs}} \ni \psi \mapsto \Phi \psi \in W_{\gamma}\]
such that \[ \|\Phi \psi\|_{\gamma} = \|\psi\|_{\text{hs}}. \]
In particular, the decomposition \[ \Phi a = \sum \chi_k(a) \phi_k \]
holds and
\[ \|\Phi a\|_{\gamma}^2 = \sum |\chi_k(a)|^2 \leq \|a\|_{\text{hs}}^2, \quad a \in E_{\text{hs}}. \]

Proof. Since the \( L_2^2 \)-valued function \( E_{\text{hs}} \ni a \mapsto \exp \phi_a(\cdot) \) is entire analytic, so is \( \hat{f}_a \) as its composition with the linear functional \( \langle f | \cdot \rangle \) over \( L_2^\infty \). Applying Frobenius’ formula [10, I.7] to \( \exp \phi_a(\cdot) \), \( 6.1 \) and \( 6.1 \), we obtain
\[ \phi_a^\gamma(u) = \sum_{i+n} h_{i,i}^a(u^* u), \quad h_{i,i} = \text{dim} S_i. \]

where \( s_i = 0 \) if \( \lambda_i > \lambda_i \). It follows that \[ \exp \phi_a(u) = \sum_{n \geq 0} \frac{1}{n!} \sum_{i+n} h_{i,i}^a(u^* u). \] (5.10)

Using (5.10) in combination with Theorem 3.2, we have
\[ \hat{f}(a) = \int f(u) \exp \phi_a(u) du(u) = \sum_{n \geq 0} \frac{1}{n!} \sum_{i+n} h_{i,i} f_i^a(a) \]
where the derivative at zero may be defined as
\[ d_{\gamma}^n \hat{f}(a) = \sum_{i+n} h_{i,i} f_i^a(a) \quad \text{with} \quad f_i^a(a) := \int f(u) \gamma_i^a(u^* u) du(u). \]

In fact, for each \( a \in \mathbb{C} \) and \( \lambda^i \in \mathbb{N} \) with \( \lambda_i > \lambda_i \) we have \( f_i^a(\cdot) = n f_i^a(a) \). Hence, the derivative at zero \( d_{\gamma}^n \hat{f}(a) = (d^n/du^n) \hat{f}(a)|_{u=0} \) is the Taylor coefficient of \( \hat{f} \). Now, the Frobenius formula and Theorem 3.2 imply the first equality in \( 5.9 \).

Let \( \phi_a(u) = \text{tr}(a^* b) \) with \( b = \pi^{-1}(u)^* \) and \( f(u) \exp \{2 \text{Re} \phi_a(u)\} \exp \{-\text{tr}(a^* a)\} := \omega(a) \) for \( f \in W_{\gamma}^2 \). Then \( \omega(a) = (f \circ \pi)(a) \) for \( a = b \in U(\infty) \), since \( \exp \{((\phi_a \circ \pi)(b)) \exp \{-\text{tr}(a^* a)\} = 1 \) in this case. As a result, we obtain
\[ \hat{f}(a) = \hat{\omega}(a) = \langle \omega \circ \pi^{-1} | \exp \phi_a \rangle = \langle f | \exp \{2 \text{Re} \phi_a - \text{tr}(a^* a)\} \rangle \]
that proves the second formula in \( 5.9 \). □

Corollary 5.4. The Frobenius formula implies that the set \( \{\phi_a : a \in E_{\text{hs}}\} \) is total in \( W_{\gamma}^2 \) and \( \exp \phi_a : a \in E_{\text{hs}} \) is total in \( W_{\gamma}^2 \).

Corollary 5.5. For every \( a \in E_{\text{hs}} \) the corresponding Paley-Wiener map \( \phi_a \) satisfies the equality
\[ \int \exp \{\text{Re} \phi_a\} du(u) = 4 \|a\|^2_{\text{hs}}. \] (5.8)

Proof. Enough to put \( f \equiv 1 \) and to replace \( a \) by \( a/2 \) in the formula (5.8). □

Lemma 5.6. The Paley-Wiener map, extended onto coherent states to be 
\[ \Phi \exp(a) \mapsto \exp \phi_a, \quad a \in E_{\text{hs}}, \]
possessed a unique conjugate-linear surjective isometric extension
\[ \Phi : \Gamma_{\text{hs}} \ni \psi \mapsto \Phi \psi \in W_{\gamma}^2 \quad \text{such that} \quad \|\Phi \psi\|_{\gamma} = \|\psi\|_{\text{hs}}. \]

In particular, the decomposition \( \Phi a = \sum \chi_k(a) \phi_k \) holds and
\[ \|\Phi a\|_{\gamma}^2 = \sum |\chi_k(a)|^2 = \|a\|^2_{\text{hs}}, \quad a \in E_{\text{hs}}. \]
\textbf{Proof.} From (4.5) and (5.7) we obtain that
\[ \| \Phi \exp(a) \|^2 = \| \exp \phi_a \|^2 = \sum \| a \|_{hs}^{2n}/n!^2 = \| \exp(a) \|^2. \]
The totality of \([\exp(a) : a \in E_{hs}]\) in \(I_{hs}\) and Corollary 5.4 yield \(\| \Phi \| = 1\).

\textbf{Theorem 5.7.} The Fourier transform \(\mathcal{F}: W_1^2 \ni f \mapsto \hat{f} \in H_{hs} \) provides the isometries
\[ W_1^2 \cong H_{hs}^2 \quad \text{and} \quad W_2^2 \cong H_{hs}^2. \]

\textbf{Proof.} Let \( f = \sum f_n \in W_1^2 \) with \( f_n \in W_2^2 \). From Lemma 5.6 it follows that the adjoint mapping \(\Phi^* : W_1^2 \to I_{hs} \) coincides with the inverse \(\Phi^{-1}\) and
\[ \hat{f}(a) = \langle f | \Phi \exp(a) \rangle = \langle \exp(a) | \Phi^* f \rangle_{hs}, \quad f \in W_1^2, \quad a \in E_{hs}. \]
Thus, the first isometry holds. By Corollary 3.3 Theorem 5.3 and the equality (5.6) at \( a = b \), we get \(\| \phi_a \|^2 = \| \phi_b \|_{hs} = \| a \|_{hs}^{2n} = \| q \|_{hs}^{2n} \) and
\[ \hat{f}_a(a) = \int f_n \delta_a^m \, dy = \langle a^m | \Phi^* f_n \rangle_{hs}. \]
By Theorem 5.2, taking into account \(\| \Phi \| = 1\), we obtain the second isometry.

\textbf{Corollary 5.8.} The isometry \(\mathcal{I} : I_{hs} \to H_{hs} \) from Definition 4.2 has the following factorization
\[ \mathcal{I} \psi = (\mathcal{F} \circ \Phi) \psi, \quad \psi \in I_{hs}. \]

6. Intertwining properties of Fourier transforms

6.1. Multiplicative groups on symmetric Wiener spaces

Let us define on the Wiener space \( W_1^2 \) the multiplicative group \( M_1^a : E_{hs} \ni a \mapsto M_1^a \) to be
\[ M_1^a f(u) = \exp(\partial_a(u)) f(u), \quad f \in W_1^2, \quad u \in U_\gamma. \]
By Lemma 5.2 the composition \(\exp \partial_a\) with a nonzero \( a \) belongs to \( L_\gamma^\infty \). Hence, \( M_1^a \) is continuous on \( W_1^2 \). The generator of its 1-parameter subgroup
\[ \mathbb{C} \ni t \mapsto M_t^a = \exp(\partial_a t), \quad dM_t^a f / dt|_{t=0} = \partial_a f, \quad f \in W_1^2 \]
coincides with the operator of multiplication by the \( L_\gamma^\infty \)-valued function \(\partial_a\). This together with the continuity of \( E_{hs} \ni a \mapsto \exp(\partial_a) \) imply that \( M_1^a \) is strongly continuous on \( W_1^2 \) and its generator with the domain \( \mathfrak{D}(\partial_a) = \{ f \in W_1^2 : \partial_a f \in W_1^2 \} \) is closed and densely-defined (see, e.g. [20] for details). The integer power \( \partial_a^m \) defined on \( \mathfrak{D}(\partial_a) = \{ f \in W_1^2 : \partial_a f \in W_1^2 \} \) is the same.

The additive group \((E_{hs}, +)\) is linearly represented over \( W_1^2 \) by the shift group
\[ T_a \hat{f}(b) = \hat{f}(b + a), \quad f \in W_1^2, \quad a, b \in E_{hs}. \]
which is strongly continuous on $H^2_{as}$. The generator of 1-parameter shift group
\[ C \ni t \mapsto T_{ta} = \exp(t\partial_a), \quad dT_{ta} \hat{f}/dt|_{t=0} = \partial_a \hat{f}, \quad \hat{f} \in H^2_{as} \]
with a nonzero $a \in E_{as}$ coincides with the directional derivative $\partial_a$ along $a$ and possesses the dense domain $\mathcal{D}(\partial_a) := \{ \hat{f} \in H^2_{as} : \partial_a \hat{f} \in H^2_{as} \}$. Similarly, the operator $\partial_a^m$ with $m \in \mathbb{N}$ is defined on $\mathcal{D}(\partial_a^m) := \{ \hat{f} \in H^2_{as} : \partial_a^m \hat{f} \in H^2_{as} \}$.

**Theorem 6.1.** For every $f \in \mathcal{D}(\partial_a^m)$ the following equality holds,
\[ \partial_a^m T_{a} \mathcal{F}(f) = \mathcal{F}(\partial_a^m M_a^1 \hat{f}), \quad a \in E_{as}. \] (6.1)

**Proof.** Any 1-parameter group $T_{ta}$, intertwining with $M_a^1$ by the $\mathcal{F}$-transform
\[ T_{ta} \hat{f}(b) = \int f \exp[\phi_{b+ta}] \, dy = \int f M_a^1 \exp(\phi_b) \, dy, \]
is strongly continuous on $H^2_{as}$. Since $\mathcal{D}(\partial_a^m)$ contains all polynomials from $H^2_{as}$, each operator $\partial_a^m$ is densely-defined. Since $d^aT_{ta} \hat{f}(b)/dt|_{t=0} = \partial_a^m \hat{f}(b)$, intertwining properties yield (6.1). □

### 6.2. Exponential creation and annihilation groups

Consider the orthogonal projector and its restriction
\[ E_{hs}^{(n)} \rightarrow E_{hs}^{(n)} \otimes E_{hs}^{(m)}, \quad \sigma_{n/m} := \sigma_{n|m}|_{E_{hs}^{(n-m)}}, \quad m \leq n, \]
defined as $\phi_m \otimes \psi_{m-m} = \sigma_{n/m}(\phi_m \otimes \psi_{n-m}) \in E_{hs}^{(n)}$ for all $\phi_m \in E_{hs}^{(m)}$ and $\psi_{n-m} \in E_{hs}^{(n-n-m)}$. The following decomposition $\sigma_m = \sigma_{n/m} \circ (\sigma_m \otimes \sigma_{m-n})$ holds.

**Lemma 6.2.** The operator $\sigma_{n|m}$ has the norm
\[ \|\sigma_{n/m}\| = m!(n-m)!/n!, \quad m \leq n. \] (6.2)

**Proof.** From the equality (6.3) it follows that
\[ \frac{m!(n-m)!}{n!} \|\epsilon_i^\lambda \otimes \epsilon_j^\mu\|_{hs} = \frac{m!(n-m)!}{n!} \frac{\lambda!}{m!(n-m)!} \frac{\mu!}{m!(n-m)!} = \frac{\lambda!\mu!}{n!} = \|\sigma_{n|m}(\epsilon_i^\lambda \otimes \epsilon_j^\mu)\|_{hs} \]
for all $\lambda, \mu \in \mathbb{N}$ so that $|\lambda| = m, |\mu| = n - m$, and for all $i \in \mathcal{F}^i, j \in \mathcal{F}^j$. Decomposing an element of $E_{hs}^{(n)} \otimes E_{hs}^{(m)}$ with respect to the basis of orthogonal elements $\epsilon_i^\lambda \otimes \epsilon_j^\mu$, we get (6.2). □

Define $\delta^m_{a,b} : E_{hs}^{(n-m)} \rightarrow E_{hs}^{(n-m)} (m \leq n)$ for all $a, b \in E_{as}$ such that
\[ \delta^m_{a,b} |_{E_{hs}^{(n-m)}} := \frac{\partial^m (b + ta)^\delta_{a,b} |_{ta=0}}{dt^m} = \sigma_{n/m} \left[ \partial^m \otimes \delta^0_{n,m} \right], \quad \sigma^0_{n,m} = 1, \] (6.3)
where the above equality follows from the tensor binomial formula
\[ (b + ta)^\delta_{n,m} = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} (ta)^\delta_{n,m} \otimes \delta^0_{n,m}. \] (6.4)
If $a = 0$ then $\delta^m_{0,a} = 0$. By the polarization formula (6.3) $\delta^m_{a,n}$ can be linearly extended to

$$\delta^m_a \psi := \sum_{n \geq 0} \delta^m_{a,n} \psi_{n-m}, \quad \psi \equiv \sum_{n \geq 0} \psi_n \in \Gamma_{hs}, \quad \psi_n \in E_{hs}^\infty. \quad (6.5)$$

From the formulas (6.3) and (6.4) it follows that

$$d^m(b + ta)^{\delta^m_n} \bigg|_{t=0} = \frac{n!}{(n-m)!} \delta^m_n \otimes \psi^{(n-m)}, \quad (6.6)$$

Summing over $n \geq m$ with coefficients $1/n!$, we get

$$\delta^m_a \exp(b) = \sum_{n \geq 0} \frac{d^m_{1,n} \otimes \psi^{(n-m)}}{(n-m)!} \bigg|_{t=0} = \sum_{n \geq m} \frac{d^m_{1,n} \otimes \psi^{(n-m)}}{(n-m)!}, \quad (6.7)$$

where $\delta^m_a$ is a directional mth derivative on $\{\exp(b) : b \in E_{hs}\}$.

**Lemma 6.3.** The exponential creation group, defined as $T_a \exp(b) = \exp(b + a)$, has a unique linear extension $T_a : \Gamma_{hs} \ni \psi \mapsto T_a \psi \in \Gamma_{hs}$ such that

$$\|T_a \psi\|_{hs} \leq (\exp \|a\|_{hs}) \|\psi\|_{hs}, \quad T_{a+b} = T_a T_b = T_b T_a, \quad a, b \in E_{hs}. \quad (6.8)$$

**Proof.** Applying (6.2) to (6.6), we obtain

$$\|\delta^m_a \exp(b)\|_{hs} \leq \sum_{n \geq m} \frac{\|d^m_{n,n} \otimes \psi^{(n-m)}\|_{hs}}{(n-m)!} \leq \|a\|_{hs} \sum_{n \geq m} \frac{m!(n-m)!}{n!} \|\psi\|_{hs} \leq \|a\|_{hs} \|\exp(b)\|_{hs}. \quad (6.9)$$

Thus, (6.6) is convergent in $\Gamma_{hs}$. From (6.6) and the above binomial formula, we have

$$\sum_{m=0}^{n} \frac{1}{m!} \delta^m_{a,n} \psi^{(n-m)}(n-m)! = \sum_{m=0}^{n} \frac{m!(n-m)!}{n!} \frac{n!}{m!(n-m)!} \frac{m!(n-m)!}{m!} = \frac{(b + a)^n}{n!}. \quad (6.10)$$

Summing by $n \in \mathbb{Z}_+$ with coefficients $1/n!$ and using (6.3), we obtain

$$(\exp \delta_a) \exp(b) := \sum_{m \geq 0} \frac{1}{m!} \delta^m_a \exp(b) = \sum_{m \geq 0} \frac{1}{m!} \sum_{n \geq m} \frac{1}{n!} \delta^m_{a,n} \psi^{(n-m)}(n-m)! \quad (6.11)$$

From (6.8) it follows $\|T_a \exp(b)\|_{hs} \leq (\exp \|a\|_{hs}) \|\exp(b)\|_{hs}$. The totality of $\{\exp(b) : b \in E_{hs}\}$ in $\Gamma_{hs}$ yields the required inequality. \qed

Let us define the adjoint operator $\delta^m_{a,\ast} : E_{hs}^\infty \ni \psi_n \mapsto \delta^m_{a,\ast} \psi_n \in E_{hs}^\infty (n \geq m)$ by the equality

$$\langle \delta^m_{a,\ast} \delta^m_{a_n} \psi | \psi \rangle_{hs} = \langle \psi | \delta^m_{a_n} \psi \rangle_{hs} = \langle \psi | \delta^m_{a_n} \psi \rangle_{hs}, \quad (6.12)$$

for all $b \in E_{hs}$. It follows that

$$\langle \delta^m_{a,\ast} \delta^m_{a_n} \psi | \psi \rangle_{hs} = \langle \psi | \delta^m_{a_n} \psi \rangle_{hs} = \langle \psi | \delta^m_{a_n} \psi \rangle_{hs}. \quad (6.13)$$
The adjoint operator of (6.5) on the total set in $\Gamma_{\text{hs}}$ is defined to be
\[ \delta_a^{\text{sm}} \exp(b) := \sum_{n \geq 0} \frac{\delta_a^{\text{sm}} b^n}{n!}, \quad b \in E_{\text{hs}}. \]

So, the corresponding adjoint group generated by $\delta_a^*$ may be defined as
\[ T_a^* \exp(b) = (\exp(\delta_a^*)) \exp(b) := \sum \delta_a^{\text{sm}} \exp(b)/m!, \quad b \in E_{\text{hs}}. \quad (6.10) \]

Since $\|T_a^* \exp(b)\|_{\text{hs}} = \|T_a \exp(b)\|_{\text{hs}}$, via Lemma 6.3 we obtain the following.

**Lemma 6.4.** The exponential annihilation group $T_a^*$, defined on coherent states by (6.10), has a unique linear extension $T_a^* : \Gamma_{\text{hs}} \ni \psi \mapsto T_a^* \psi \in \Gamma_{\text{hs}}$ such that
\[ \|T_a^* \psi\|_{\text{hs}} \leq (\exp \|a\|_{\text{hs}}) \|\psi\|_{\text{hs}}, \quad T_{a+b} = T_a^* T_b^* = T_b^* T_a^*, \quad a, b \in E_{\text{hs}}. \]

**6.3. Shift groups on symmetric Wiener spaces**

Let us represent the additive group $(E_{\text{hs}}, +)$ over the Wiener space $W^2_\gamma$ by the shift group
\[ T_a^1 = \Phi T_a \Phi^* \] for a nonzero $a \in E_{\text{hs}}$. From Lemma 6.3 it follows that the generator of its 1-parameter strongly continuous subgroup
\[ \mathbb{C} \ni t \mapsto T_{a^1}^t = \exp(t \delta_a^1), \quad dT_{a^1}/dt|_{t=0} = \delta_a^1 := \Phi \delta_a \Phi^* \]
possesses the dense domain $\mathcal{D}(\delta_a^1) = \{ f \in W^2_\gamma : \delta_a^1 f \in W^2_\gamma \}$ and is closed.

On the other hand, let us consider on $H^2_{\text{hs}}$ the multiplicative group $M \hat{f}(b) = \hat{f}(b) \exp(b \mid a)_{\text{hs}}$ with $\hat{f} \in H^2_{\text{hs}}$ and define the generator of its 1-parameter strongly continuous subgroup
\[ M \hat{f}(b) = \hat{f}(b) \exp(b \mid a)_{\text{hs}}, \quad dM \hat{f}/dt|_{t=0} = \langle \cdot \mid a \rangle_{\text{hs}} := a^d. \]

The generator $(a^d \hat{f})(b) = \langle b \mid a \rangle_{\text{hs}} \hat{f}(b)$ has the dense domain $\mathcal{D}(a^d) = \{ \hat{f} \in H^2_{\text{hs}} : a^d \hat{f} \in H^2_{\text{hs}} \}$ which contains all polynomials. Its $m$th power $a^{dm}$ is densely-defined on the domain $\mathcal{D}(a^{dm}) = \{ \hat{f} \in H^2_{\text{hs}} : a^{dm} \hat{f} \in H^2_{\text{hs}} \}$. The case $a = 0$ is trivial.

**Theorem 6.5.** For every $f \in \mathcal{D}(\delta_a^{\text{sm}}) = \{ f \in W^2_\gamma : \delta_a^{\text{sm}} f \in W^2_\gamma \}$ with $m \in \mathbb{N}$
\[ a^{dm} M_a \Phi f = \Phi (\delta_a^{\text{sm}} T_{a^1}^t f), \quad a \in E_{\text{hs}}. \quad (6.11) \]

**Proof.** The equality (6.9) yields $\langle b \mid a \rangle_{\text{hs}} \psi_{n-m}^2(b) = \langle \delta_a^{\text{sm}} \psi_n \mid \psi_{n-m} \rangle_{\text{hs}}$ for all $n \geq m$. By Theorem 5.3 for any $f = \sum \psi_n f_n \in W^2_\gamma$ there exists a unique $\psi = \oplus \psi_n \in \Gamma_{\text{hs}}$ with $\psi_n \in E^{\text{sm}}_{\text{hs}}$ such that $\Phi^* f = \psi$ and $f_n = \psi_n^2$. Summing over all $m \in \mathbb{N}$ and $n \geq m$ and using (6.10), we obtain that
\[ M_a \hat{f}(b) = \exp(b \mid a)_{\text{hs}} \exp(b \mid \Phi^* f)_{\text{hs}} = \sum \frac{\langle b \mid a \rangle_{\text{hs}}^m}{m!} \sum m \geq 0 \psi_{n-m}^2(b) \]
\[ = \langle T_a^* \exp(b) \mid \Phi^* f \rangle_{\text{hs}} = \langle \exp(b) \mid T_a \Phi^* f \rangle_{\text{hs}}. \]

By Theorem 5.3 and Lemma 6.3 it follows that
\[ M_a \hat{f}(b) = \langle \exp(b) \mid T_a \Phi^* f \rangle_{\text{hs}} = \int (T_{a^1}^t f) \exp \phi_b dy, \quad t \in \mathbb{C} \quad (6.12) \]
The following commutation relations

\[ \text{Theorem 7.1.} \quad \text{For each } \Phi \in \mathcal{D}(\delta^m_\gamma). \]  

Consequently, for each \( p \in \mathbb{N} \), the multiplicative group has the following representation \( M'_a = \Phi T^*_a \Phi^* \). Indeed, for any \( f \in W^2 \),

\[ T_a \hat{f}(b) = \langle T_a \exp(b) | \Phi^* f \rangle_{\mathcal{H}} = \langle \exp(b) | T^*_a \Phi^* f \rangle_{\mathcal{H}} = \int (M'_a f) \exp \Phi_b d\gamma = \langle \exp(b) | M'_a f \rangle_{\mathcal{H}}. \]

7. Heisenberg groups on symmetric Wiener spaces

7.1. Commutation relations on symmetric Wiener spaces

Let us describe the commutation relations between \( T_a^+ \) and \( M'_b \) on the Wiener space \( W^2 \).

**Theorem 7.1.** The following commutation relations

\[ M'_a T^+_b = \exp(a | b)_{\mathcal{H}} T^+_b M'_a, \]

hold for all \( f \) from the dense set \( \mathcal{D}(\delta^2_\gamma) \cap \mathcal{D}(\delta^1_\gamma) \subset W^2 \) and nonzero \( a, b \in E_{\mathcal{H}}. \)

**Proof.** For each \( \hat{f} \in H^2 \) and \( \epsilon \in E_{\mathcal{H}} \) we have the commutation relations

\[ M_{\mathcal{H}} T_a \hat{f}(\epsilon) = \exp(\epsilon | b)_{\mathcal{H}} \hat{f}(\epsilon + a), \]

\[ T_a M_{\mathcal{H}} \hat{f}(\epsilon) = \hat{f}(\epsilon + a) \exp(\epsilon | b)_{\mathcal{H}} \exp(a | b)_{\mathcal{H}} = \exp(a | b)_{\mathcal{H}} M_{\mathcal{H}} T_a \hat{f}(\epsilon). \]

For each \( \hat{f} \in \mathcal{D}(b^2) \cap \mathcal{D}(b^1_\gamma) \) and \( t \in C \) by differentiation we obtain

\[ \left( \frac{d^2}{dt^2} \right) T_a M_{\mathcal{H}} \hat{f}(t) |_{t=0} = \left( b^2 + 2b^1b^\gamma + b^\gamma \right) \hat{f}. \]  

(7.1)

Since \( (d/dt)[\exp(ta | b)_{\mathcal{H}} M_{\mathcal{H}} T_a] = [(d/dt) \exp(ta | b)_{\mathcal{H}}] M_{\mathcal{H}} T_a + \exp(ta | b)_{\mathcal{H}} [(d/dt) M_{\mathcal{H}} T_a] \), then taking into account (7.1), we get

\[ \left( b^2 + 2b^1b^\gamma + b^\gamma \right) \hat{f} = \left( d/dt \right) [(d/dt) \exp(ta | b)_{\mathcal{H}} M_{\mathcal{H}} T_a] |_{t=0} \]

\[ = 2(a | b)_{\mathcal{H}} \hat{f} + \left( b^2 + 2b^1b^\gamma + b^\gamma \right) \hat{f}. \]

Consequently, for each \( \hat{f} \) from the dense subspace \( \mathcal{D}(b^2) \cap \mathcal{D}(b^1_\gamma) \subset H^2 \), including all polynomials generated by finite \( \Phi^*(f) = \bigoplus \psi_n \in E_{\mathcal{H}}. \)

\[ T_a M_{\mathcal{H}} = \exp(a | b)_{\mathcal{H}} M_{\mathcal{H}} T_a, \]

(7.2)
From Corollary 5.8, it follows that $\mathcal{F} = \hat{\mathcal{F}}^*$ and $\mathcal{F}^{-1} = \Phi \mathcal{F}^{-1} \Phi^{-1}$. On the other hand, the first equality in (6.12) can be rewritten as $M(f)\hat{f}(a) = \langle \exp(a) | T_0 \Phi^* f \rangle_{\text{hs}}$ with $f \in W_2^2$ or another way $\hat{I} T_0 = M_0 \hat{I}$. Hence, $T_0^* \phi = \Phi T_0 \Phi^* = \Phi \hat{I} T_0 \Phi^* = T_0^* M_0 \Phi^*$. By Theorem 6.5, $\Phi^* f$ is well defined and irreducible, where the appropriate Weyl system is $\Phi^* \mathcal{F}$ and $\hat{\Phi} = \mathcal{F}^{-1} b^T \mathcal{F}^{-1}$ by Theorem 6.1. We obtain

$$M_0^* T_0^* = \mathcal{F}^{-1} T_0 M_0 \mathcal{F} = \exp(a) | b\rangle | b\rangle \mathcal{F}^{-1} M_0 T_0 \mathcal{F} = \exp(a) | b\rangle M_0^* T_0^* M_0^*,$$

$$\langle f, f \rangle = \mathcal{F}^{-1} \{ b^T b + \beta | \mathcal{F} f \rangle \} \mathcal{F} = \langle a \rangle | b\rangle f$$

for all $f$ from the dense subspace $\mathcal{F}(\Phi^* \mathcal{F}) \cap \mathcal{F}(\Phi^* \mathcal{F})$ such that $\Phi^* f \in \Gamma_{\text{hs}}$. □

7.2. Weyl-Schrödinger representations of Heisenberg groups

We consider an infinite-dimensional complexified analog of the Heisenberg group $\mathbb{H}_{\text{hs}}$ which consists of matrix elements $X(a, b, t)$ written as $[11]$ with unit $X(0, 0, 0)$ and multiplication

$$\begin{bmatrix} a & b' & t' \\ 0 & b & t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a + a' & b + b' & t + t' + \langle a | b' \rangle_{\text{hs}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad t \in \mathbb{C}$$

where matrix entries $a, a', b, b'$ take from the algebra of Hilbert-Schmidt operators $E_{\text{hs}}$. Evidently, $X(a, b, t)^{-1} = X(-a, -b, -t + \langle a | b \rangle_{\text{hs}})$. We will not write $\mathbb{I}$ at scalars.

We will use the quaternion algebra $\mathbb{H} = \mathbb{C} \oplus \mathbb{C} \mathbb{j}$ of numbers $\gamma = (\alpha_1 + \alpha_2 \mathbb{j}) + (\beta_1 + \beta_2 \mathbb{i}) \mathbb{j} = \alpha + \beta \mathbb{j}$ such that $\mathbb{j}^2 = \mathbb{j} = \mathbb{k}^2 = \mathbb{i} \mathbb{j} \mathbb{k} = -1$, $\mathbb{i} = \mathbb{i} \mathbb{j} = -\mathbb{j} \mathbb{i}$, $\mathbb{k} = \mathbb{i} \mathbb{k} = -\mathbb{k} \mathbb{i}$. where $(\alpha, \beta) \in \mathbb{C}^2$ with $\alpha = \alpha_1 + \alpha_2 \mathbb{i}, \beta = \beta_1 + \beta_2 \mathbb{j}$ in $\mathbb{C}$ and $\alpha, \beta, \in \mathbb{R}, (i = 1, 2)$ (see, i.e. [16, 5.5.2]). Let us denote $\beta := \gamma, \quad \gamma = \alpha + \beta \mathbb{j}$.

Consider the Hilbert space $E_{\text{hs}} \otimes E_{\text{hs}} \mathbb{j}$ with $\mathbb{I}$-valued scalar product

$$\langle p | p' \rangle_{\text{hs}} = \langle a + b \mathbb{j} | a' + b' \mathbb{j} \rangle_{\text{hs}} = \langle a | a' \rangle_{\text{hs}} + \langle b | b' \rangle_{\text{hs}} + \langle a' | b \rangle_{\text{hs}} + \langle a | b' \rangle_{\text{hs}}$$

where $p = a + b \mathbb{j}$ with $a, b \in E_{\text{hs}}$. Hence,

$$\mathcal{F}(p | p')_{\text{hs}} = \langle a' | b \rangle_{\text{hs}} - \langle a | b' \rangle_{\text{hs}}, \quad \mathcal{F}(p | p)_{\text{hs}} = 0.$$

**Theorem 7.2.** The representation of $\mathbb{H}_{\text{hs}}$ on the Wiener space $W_2^2$ in the Weyl-Schrödinger form

$$S^t : \mathbb{H}_{\text{hs}} \ni X(a, b, t) \longmapsto \exp(t) \mathcal{W}(p), \quad p = a + b \mathbb{j}$$

is well defined and irreducible, where the appropriate Weyl system

$$\mathcal{W}(p) := \mathcal{W}'(a, b) = \exp \left\{ \frac{1}{2} \langle a | b \rangle_{\text{hs}} \right\} T_0^* M_0^*$$

satisfies the relation

$$\mathcal{W}(p + p') = \exp \left\{ -\frac{\mathcal{F}(p | p')}_{\text{hs}}}{2} \right\} \mathcal{W}(p) \mathcal{W}(p').$$

(7.3)
Proof. Consider the auxiliary group $\mathbb{C} \times (E_{\text{ns}} \oplus E_{\text{naj}})$ with multiplication
\[(t, v)(t', v') = \left(t + t' - \frac{1}{2} \langle v' \mid v \rangle_{\text{ns}}, v + v'\right)\]
for all $v = a + bj$, $v' = a' + b'j \in E_{\text{ns}} \oplus E_{\text{naj}}$. The following mapping
\[G : X(a, b, t) \mapsto \left(t - \frac{1}{2} (a \mid b)_{\text{ns}}, a + bj\right)\]
is a group isomorphism, since
\[G(X(a, b, t)X(a', b', t')) = G(X(a + a', b + b', t + t' + (a \mid b)_{\text{ns}}))\]
\[= \left(t + t' + (a \mid b')_{\text{ns}} - \frac{1}{2} ((a + a') \mid b + b')_{\text{ns}}, (a + a') + (b + b')j\right)\]
\[= \left(t + t' - \frac{1}{2} ((a \mid b)_{\text{ns}} + (a' \mid b')_{\text{ns}}) + \frac{1}{2} ((a \mid b')_{\text{ns}} - (a' \mid b)_{\text{ns}}), (a + a') + (b + b')j\right)\]
\[= \left(t - \frac{1}{2} (a \mid b)_{\text{ns}}, a + bj\right) \left(t' - \frac{1}{2} (a' \mid b')_{\text{ns}}, a' + b'j\right) = G(X(a, b, t))G(X(a', b', t')).\]
On the other hand, let us define the suitable Weyl system
\[
\begin{align*}
W(v) &= \exp \left(\frac{1}{2} (a \mid b)_{\text{ns}}\right) M_v T_a, \quad v = a + bj. \\
\end{align*}
\]
Using the commutation relation (7.2), we obtain
\[
\exp \left(-\frac{1}{2} \langle v' \mid v \rangle_{\text{ns}}\right) W(v) W(v') = \exp \left(\frac{1}{2} (a \mid b)_{\text{ns}}\right) W(v) W(v')
\]
\[= \exp \left(\frac{(a \mid b)_{\text{ns}}}{2} + \frac{(a' \mid b')_{\text{ns}}}{2}\right) \exp \left(\frac{(a \mid b)_{\text{ns}}}{2} - \frac{(a' \mid b')_{\text{ns}}}{2}\right) M_v T_a M_v T_{a'}
\]
\[= \exp \left(\frac{1}{2} (a + a' \mid b + b')_{\text{ns}}\right) M_{v + v'} T_{a + a'} = W(v + v').\]
Hence, the mapping $\mathbb{C} \times (E_{\text{ns}} \oplus E_{\text{naj}}) \ni (t, v) \mapsto \exp(t)W(v)$ acts as a group isomorphism into the operator algebra over $H_{\text{ns}}^2$. So, the representation
\[S : \mathbb{H}_{\text{ns}} \ni X(a, b, t) \mapsto \exp(t)W(v) = \exp \left(t + \frac{1}{2} (a \mid b)_{\text{ns}}\right) M_v T_a\]
is also well defined over $H_{\text{ns}}^2$, as a composition of group isomorphisms.

Let us check the irreducibility. Suppose the contrary. Let there exists an element $v_0 \neq 0$ in $E_{\text{ns}}$ and an integer $n > 0$ such that
\[\exp \left(t + \frac{1}{2} (a \mid b)_{\text{ns}}\right) \exp \langle c \mid a \rangle_{\text{ns}} (c + b \mid v_0)_{\text{ns}} = 0 \quad \text{for all} \quad a, b, c \in E_{\text{ns}}.
\]
However, this is only possible in the case $v_0 = 0$. It gives a contradiction. Finally, using that
\[\exp \left(t + \frac{1}{2} (a \mid b)_{\text{ns}}\right) T_a M_v = \mathcal{F}^{-1} \left(\exp \left(t + \frac{1}{2} (a \mid b)_{\text{ns}}\right) M_v T_a\right) \mathcal{F},
\]
we conclude that the group representation $S^0 = \mathcal{F}^{-1} S \mathcal{F}$ is irreducible. Similarly, we get (7.3) by applying the transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ to (7.4). \qed
7.3. Weyl systems over symmetric Wiener spaces

Finally, let us investigate Weyl systems $\mathcal{W}^t$ on the Wiener space $W^2$. On each 1-dimensional real subspace $\{tp: t \in \mathbb{R}\}$ with a nonzero element $p = a + bj \in E_{hs} \oplus E_{hn}$ the relation \((7.3)\) reduces to the strongly continuous 1-parameter group

\[ \mathcal{W}^t = \mathcal{W}^t(tp) \mathcal{W}^t(t'p) = \mathcal{W}^t(t'p) \mathcal{W}^t(tp), \]

generated by a Weyl system. In accordance with the previous subsections $\mathcal{W}^t(\tau a, \tau b) = \mathcal{W}^t(\tau p)$ has the generator $b^+_a := b^+_a, b^+_t$ where

\[ b^+_a = \frac{d}{dt} \mathcal{W}^t(tp)_{|t=0} = \frac{d}{dt} \exp \left( \frac{1}{2} (\tau a, \tau b)_{hn} \right) T^a_t M^a_t \bigg|_{t=0} = \delta^+_a + \phi_a \]

is densely-defined on $\mathcal{D}(b^+_a) = \{ f \in W^2: b^+_a f \in W^2 \}$ and $b^+_t = tb^+_b$.

**Theorem 7.3.** The following commutation relations hold,

\[ \mathcal{W}^t(p) \mathcal{W}^t(p') = \exp \left[ -\mathcal{D} \langle p | p' \rangle_{hs} \right] \mathcal{W}^t(p') \mathcal{W}^t(p), \]

\[ [b^+_a, b^+_b] = -\mathcal{D} \langle p | p' \rangle_{hn}, \quad \text{where} \quad [b^+_a, b^+_b] := b^+_a b^+_b - b^+_b b^+_a. \]

**Proof.** Taking into account (7.3), we obtain

\[ \mathcal{W}^t(p) \mathcal{W}^t(p') = \exp \left( \frac{1}{2} \mathcal{D} \langle p | p' \rangle_{hs} \right) \mathcal{W}^t(p + p') = \exp \left( -\frac{1}{2} \mathcal{D} \langle p' | p \rangle_{hs} \right) \mathcal{W}^t(p' + p) \]

\[ = \exp \left( -\mathcal{D} \langle p' | p \rangle_{hs} \right) \mathcal{W}^t(p') \mathcal{W}^t(p). \]

By recovering with a series expansion of (7.5) (where $p, p'$ are changed by $tp, tp'$) over a dense subspace in $W^2$, including all polynomials, we get

\[ -\mathcal{D}(p | p')_{hs} = \langle a | b' \rangle_{hs} - \langle a' | b \rangle_{hs} = \phi_a \delta^+_b - \phi_b \delta^+_a + \delta^+_a \phi_b - \delta^+_b \phi_a \]

\[ = (\delta^+_a + \phi_a)(\delta^+_b + \phi_b) - (\delta^+_b + \phi_b)(\delta^+_a + \phi_a) = [b^+_a, b^+_b]. \]

By differentiation at $t = 0$ and Theorem 7.3, what ends the proof.

\[ \square \]

8. Solution of initial value problem for Schrödinger’s equation

8.1. Symmetric Wiener spaces associated with Gelfand triples

Let $E^+$ be the Hilbert space with scalar product, defined by the relations $\langle \epsilon_k | \epsilon_m \rangle := 2^{-(k+m)/2} \langle \epsilon_k | \epsilon_m \rangle$, where $(\epsilon_k)$ is an orthogonal basis in $E$. Define the suitable Hilbert-Schmidt algebra $E_{hs}$ over $E^+$, spanned by the orthogonal basis $\epsilon_k = \epsilon_k \otimes \epsilon_k^*$ with scalar product and norm, respectively,

\[ \langle a | b \rangle_{hs} = \sum_{k \geq 0} \langle b^* \epsilon_k | a \epsilon_k \rangle \quad \text{and} \quad \|a\|_{hs} := \sqrt{\langle a | a \rangle_{hs}}. \]

We can also define the Hilbert space $E^+$ with scalar product $\langle \epsilon_k | \epsilon_m \rangle := 2^{(k+m)/2} \langle \epsilon_k | \epsilon_m \rangle$. Then the mappings $E^+ \xrightarrow{j^*} E \xrightarrow{j} E^+$ form the Gelfand triple with continuous dense embedding $j: \epsilon_k^* \mapsto \epsilon_k$ and its adjoint $j^*$. The suitable covariance $j \circ j^* \in \mathcal{L}(E^+, E^-)$ is evidently positive.

Substituting $E^-$ instead $E$, we similarly as above determine the symmetric Wiener space $W^2$ spanned by the symmetric polynomial (8.1) in variables $\phi_k^+(u) = \langle \pi_j(u) \epsilon_k | \epsilon_k \rangle$ with $u \in U_r$. 

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8.2. Gaussian semigroups on symmetric Wiener spaces

Let \( b = \ln m \) and \( a = t_m \) with \( t \in \mathbb{R} \) and \( m \in \mathbb{Z}_+ \). Then by Theorems 7.1 and 7.2

\[
\mathcal{W}_t^\gamma(t_m, \Phi e_m) = \exp \left\{ \frac{-i t^2}{2} \sum_{k \geq 0} \langle e_m^k e_m^k | e_k \rangle \right\} T_{\ln m}^\gamma M_{t_m}^\gamma,
\]

\[
M_{t_m}^\gamma T_{\ln m}^\gamma = \exp \left\{ -i t^2 \sum_{k \geq 0} \langle e_m^k e_m^k | e_k \rangle \right\} T_{\ln m}^\gamma M_{t_m}^\gamma,
\]

as well as, \([M_{t_m}^\gamma, T_{\ln m}^\gamma] = 0\) and \([M_{t_m}^\gamma, M_{t_m}^\gamma] = 0\) if \( m \neq k \). By Theorem 6.1 \([T_{\ln m}^\gamma, T_{\ln m}^\gamma] = 0\) if \( m \neq k \). Moreover,

\[
[\Phi e_m, \Phi^0] = [\Phi e_m, \Phi^0] = -2 \Phi^0.
\]

The equality \( T_{\ln m}^\gamma \Phi = \Phi T_{\ln m}^\gamma \Phi^0 \) yields the convergence \( T_{\ln m}^\gamma = \lim T_{\ln m}^\gamma \) on \( W_t^\gamma \) for all \( t \in \mathbb{R} \). The convergence \( M_{t_m}^\gamma = \lim M_{t_m}^\gamma \) is a \( C_0 \)-group over \( W_t^\gamma \).

Theorem 8.1. The 1-parameter Gaussian semigroup \( \mathcal{G}_r \) over \( W_t^\gamma \).

\[
\mathcal{G}_r f = \frac{1}{\sqrt{4\pi r}} \int_\mathbb{R} \exp \left\{ \frac{-t^2}{4r} \right\} \mathcal{W}_t^\gamma f \, dt, \quad f \in W_t^\gamma, \quad r > 0,
\]

has the generator \( b_t \), where \( b_t := \sum (\Phi_m - \Phi^0_m) \) with \( \Phi_m := \Phi e_m \) and \( \Phi^0_m := \Phi^0 e_m \). As a consequence, \( \mathcal{G}_r f \) is the unique solution of Cauchy’s problem for the infinite-dimensional Schrödinger equation

\[
\frac{d v(r, u)}{dr} = b_t w(r, u), \quad w(0, u) = f(u), \quad u \in \mathbb{U}_x.
\]

Proof. First note that the \( C_0 \)-property of \( \mathcal{G}_r \) over \( W_t^\gamma \) straight follows from (8.2). By differentiation (8.1) at \( t = 0 \), as a product of operator-valued functions, we obtain that the generator of \( \mathcal{W}_t^\gamma \) coincides with \( b_t \). Prove that \( \mathcal{G}_r \) is generated by \( b_t \). On the dense subspace in \( W_t^\gamma \) of polynomial functions \( f \) in variables \( \Phi^0 \), we get \( \mathcal{W}_t^\gamma f = \exp \{ t b_t \} f \). The equalities

\[
\frac{1}{\sqrt{4\pi r}} \int_\mathbb{R} \exp \left\{ \frac{-t^2}{4r} \right\} r^n \, dt = \left( \frac{2}{\sqrt{\pi}} \right)^n \int_\mathbb{R} \exp \left\{ -x^2 \right\} x^{2n} \, dx
\]

\[
= \frac{2^n n!}{\sqrt{\pi}} \left( \frac{2n+1}{2} \right) = \frac{2(2n-1)!}{(n-1)!} \sqrt{\pi}^n
\]
allows to prove the following equalities on the dense subspace in $W^{-2}_γ$ of polynomial functions $f$

$$\hat{\theta}_γ f = \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} \exp\left(\frac{-r^2}{4r}\right) \exp\{ir^1\} f \, dt = \sum_{k \geq 0} \frac{\hat{b}^{12}}{k!} \frac{1}{\sqrt{4\pi r}} \int_{\mathbb{R}} \exp\left(\frac{-r^2}{4r}\right) f^k \, dt \nabla_k \nabla_k = \sum_{n \geq 0} \frac{2(2n-1)!}{(n-1)!(2n)!} r^n \hat{b}^{12} f = \exp\{r^2\} f.$$ 

Hence, on this dense subspace the semigroup $\hat{\theta}_γ$ is generated by $\hat{b}^{12}$. Now, it is necessary to use a well known relation between the initial problem [8,3] and the Gaussian $C_0$-semigroup $\hat{\theta}_γ$, which proves, in particular, the existence of a closed generator over $W^{-2}_γ$ (see, e.g. [20] for details).

It is not difficult to see that $\hat{b}^{12} = -\hat{\gamma} H$, where the operator $H = \sum_{m,k \geq 0} (\hat{\delta}_m - \hat{\delta}_m^\dagger)(\delta_m^\dagger + i\phi_k)$ in $W^{-2}_γ$ can be treated as a Hamiltonian.

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