On the supercritical Schrödinger equation on the exterior of a ball

Piero D’Ancona
Dipartimento di Matematica, Sapienza Università di Roma, Roma, Italy

ABSTRACT
We consider the mixed problem on the exterior of the unit ball in $\mathbb{R}^n$, $n \geq 2$, for a defocusing Schrödinger equation with a power nonlinearity $|u|^{p-1}u$, with zero boundary data. Assuming that the initial data are non-radial, sufficiently small perturbations of large radial initial data, we prove that for all powers $p > n + 6$ the solution exists for all times, its Sobolev norms do not inflate, and the solution is unique in the energy class.

1. Introduction

The literature on the defocusing semi-linear Schrödinger equation

$$iu_t + \Delta u = |u|^{p-1}u, \quad u(0, x) = u_0(x)$$

is extensive and we mention [1–3] for an introduction and detailed bibliographies. Restricting to large $H^1$ (energy class) data, Problem (1.1) is well posed for large data below the critical value $p < p_0(n)$ where $p_0(n) = \frac{n+2}{n-2}$ for $n > 2$ and $p_0(n) = +\infty$ if $n = 1, 2$ [4]. The problem is well posed also in the critical case $p = p_0(n)$ as proved in a series of important papers [5–11]. Well or ill posedness in the supercritical case $p > p_0(n)$ has been for many years a completely open problem. A recent breakthrough was obtained in [12], where finite time blow up was established for a class of large, radially symmetric, localized initial data and suitable ranges of $(n, p)$.

Here we consider the supercritical case $p > p_0(n)$ from a different perspective. It is not difficult to check that for radial data the first blow up must occur at the origin, or, equivalently, that if the solution remains bounded near the origin then no blow up can occur. This is an immediate consequence of the bound, valid for spherically symmetric functions,

$$|x|^{\frac{n-1}{2}}|u(x)| \lesssim ||\nabla u||_{L^2},$$

usually called Strauss’ Lemma. Inequality (1.2) is a special case of the family of inequalities

$$|x|^{\frac{n-1}{2}-\sigma}|u(x)| \lesssim |||D|^{\sigma}u||_{L_{x}^{r}L_{t}^{r}}, \quad \frac{n-1}{r} + \frac{1}{p} < \sigma < \frac{n}{p}$$

(1.3)
(see [13]), where the norm $L^p_{|x|} L^r_{\omega}$ is an $L^p$ norm in the radial direction of the $L^r_{\omega}$ norm on spheres centered at 0.

Exploiting the previous remark, one can remove the singularity, by considering the mixed problem

$$iu_t + \Delta u = |u|^{p-1}u, \quad u(0, x) = u_0(x), \quad u(t, \cdot)|_{\partial \Omega} = 0$$

(1.4)
on the exterior of the unit ball

$$\Omega = \{x \in \mathbb{R}^n : |x| > 1\}.$$ 

One obtains that for radial initial data the solution must exist for all times and all values of $p, n$. The precise statement is the following:

**Proposition 1.1.** Let $\Omega = \mathbb{R}^n \setminus \overline{B(0, 1)}$, $n \geq 2$, $p > 1$ and let $u_0 \in H^1_0 \cap H^2(\Omega)$ be radially symmetric. Then the mixed problem (1.4) has a global solution $u \in C^2 L^2 \cap C^1 H^1_0 \cap CH^2$, satisfying the conservation of mass $||u(t)||_{L^2} = ||u_0||_{L^2}$ and of energy

$$E(u(t)) := \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 \, dx + \frac{1}{p+1} \int_{\Omega} |u(t)|^{p+1} \, dx = E(u_0)$$

and the uniform bound

$$||u||_{L^\infty(\mathbb{R}^n \times \Omega)} \leq ||u_0||_{H^1}.$$ (1.5)

If $\nu \in C^2 L^2 \cap C^1 H^1_0 \cap CH^2$ is a second solution of (1.4) with the same data, which is radially symmetric or, more generally, bounded on any strip $0 \leq t \leq T$, then $\nu \equiv u$.

Assume in addition $p > 2N$ for some integer $N \geq 1$ and $(u_0, f)$ with .. satisfy the non-linear compatibility conditions of order $N$. Then $u \in C^N L^2(\Omega)$ and $u \in C^k(H^{2(N-k)}(\Omega) \cap H^1_0(\Omega))$ for $0 \leq k \leq N - 1$.

The proof of Proposition 1.1 is given in Section 4.1; see Definition 3.1 for the meaning of the nonlinear compatibility conditions. In principle, one cannot expect that the radial solution thus constructed is unique; there might exist other non-radial solutions with the same data, since the problem is supercritical. Uniqueness is discussed in detail below.

Once a global radial solution is available, a natural question concerns the stability for non-radial perturbations of the initial data. In view of the blow up result mentioned above, one may expect that for large non radial data the solution blows up also for the supercritical exterior problem. However, using a pseudoconformal transform argument, one verifies that radial solutions decay as $t \to +\infty$, and the decay is good enough to work out a perturbative argument:

**Proposition 1.2** (Decay of the radial solution). Let $u$ be the solution constructed in Proposition 1.1 for a radially symmetric $u_0 \in H^1_0 \cap H^2(\Omega)$. Assume in addition that $xu_0 \in L^2(\Omega)$. Then, for all $t > 0$, $|x| \geq 1$, $u$ satisfies the decay estimate

$$|u(t, x)| \leq C(||u_0||_{H^1} + ||xu_0||_{L^2}) \cdot (t)^{-1} |x|^{1-\frac{n}{2}}.$$ (1.7)
Proposition 1.2 is proved in Section 4.2. If \( u_0 \) is smoother (and \( xu_0 \in L^2 \)) then regularity propagates, and Sobolev norms do not inflate but remain bounded for all times (see Corollary 4.1 in Section 4.3).

Since the radial solution decays, a perturbative argument is sufficient to obtain the following global existence result, proved in Section 5, which is the main result of the paper. Consider again the problem on \( \mathbb{R}^+ \times \Omega \)

\[
iv_t + \Delta v = f(v), \quad v(0, x) = v_0, \quad v(t, \cdot)|_{\partial \Omega} = 0.
\] (1.8)

Then we have:

**Theorem 1.3.** Let \( n \geq 2, p > n + 6, \) and \( u_0 \in H^{2m}(\Omega) \) with \( m = \floor{n/2} + 1 \) a radial function such that \( (u_0, f) \) with \( f(z) = |z|^{p-1}z \) satisfy the compatibility conditions of order \( m \). Assume in addition that \( xu_0 \in L^2(\Omega) \). Then there exists \( \epsilon = \epsilon(u_0) > 0 \) such that the following holds.

If \( v_0 \in H^{2m}(\Omega) \), with \( (v_0, f) \) satisfying the nonlinear compatibility conditions of order \( N \), and \( ||u_0 - v_0||_{H^{2m}} < \epsilon \), then Problem (1.8) has a global solution \( v \in C^n L^2(\Omega) \cap C^k (H^{2m-k}(\Omega) \cap H^1(\Omega)) \) for \( 0 \leq k \leq m - 1 \).

In the proof of Theorem 1.3 we use a Strichartz estimate for the exterior problem in presence of an integrable potential. This is proved in Section 2 as a consequence of the Strichartz estimates on the exterior of convex obstacles proved in [14].

It remains to consider the problem of uniqueness. A well-known strategy allows to establish uniqueness of energy class solutions to dispersive equations under the assumption that a smooth solution exists (weak–strong uniqueness). This is particularly convenient in the present situation since Theorem 1.3 yields a smooth solution to (1.8). By adapting the arguments in [15] we get:

**Theorem 1.4.** Suppose all the assumptions in Theorem 1.3 are satisfied and let \( v \) be the solution constructed there. Let \( I \) be an open interval containing \( [0, T] \) and \( v \in C(I; H^1(\Omega)) \cap C(I; L^2(\Omega)) \) a distributional solution for \( 0 \leq t \leq T \) to Problem (1.8) with the same initial data as \( v \), which satisfies an energy inequality \( E(v(t)) \leq E(v(0)) \) (see (1.5)). Then we have \( v(t) = v(t) \) for \( 0 \leq t \leq T \).

It is clear that a similar strategy can be applied to other dispersive equations. Indeed, for the supercritical nonlinear wave equation on the exterior of a ball, we proved the global well posedness for quasi radial initial data in the companion paper [16]. For wave equations the pseudoconformal transform is not available, however the decay of radial solutions can be proved using the Penrose transform.

The plan of the paper is the following. In Section 2 we recall the linear theory for exterior problems and we prove energy and Strichartz estimates for derivatives of solutions, also for Schrödinger equations perturbed by a well-behaved potential \( V(t, x) \). Section 3 is devoted to the general nonlinear theory and local existence results. In Section 4, the global radial solution is studied in detail. The main result, Theorem 1.3, is proved in Section 5, and the weak–strong uniqueness is proved in the final Section 6.
2. The linear exterior problem

Consider the linear mixed problem with \((t, x) \in \mathbb{R} \times \Omega\)
\[
i \partial_t u + \Delta u = F(t, x) \quad u(0, x) = u_0(x), \quad (t, \cdot)|_{\partial \Omega} = 0. \tag{2.1}
\]
Denote by \(\Delta_D\) or simply \(\Delta\) the self-adjoint Dirichlet Laplacian on \(\Omega\) with domain \(H^1_0(\Omega) \cap H^2(\Omega)\), by \(\Lambda\) its (nonnegative self-adjoint) square root
\[
\Lambda = (-\Delta_D)^{1/2}, \quad D(\Lambda) = H^1_0(\Omega)
\]
and by \(e^{it\Delta}\) the flow defined via the spectral theorem. Note that \(\Lambda^{2k} = (-\Delta)^k\) for integer \(k \geq 0\), and
\[
D(\Lambda^{2k}) = D(\Lambda^k) = \{ f \in H^{2k}(\Omega) : f, \Delta f, \ldots, \Delta^{k-1} f \in H^1_0(\Omega) \},
\]
\[
D(\Lambda^{2k+1}) = \{ f \in H^{2k+1}(\Omega) : f, \Delta f, \ldots, \Delta^k f \in H^1_0(\Omega) \}
\]
that is to say
\[
D(\Lambda^k) = \{ f \in H^k(\Omega) : \Delta^j f \in H^1_0(\Omega), 0 \leq 2j \leq k - 1 \}.
\]
Then for all data \(u_0 \in L^2(\Omega), \ F \in L^1_{\text{loc}}(\mathbb{R}; L^2(\Omega))\) there exists a unique solution \(u(t, x) \in C(\mathbb{R}; L^2(\Omega))\), which can be written in the form
\[
u(t, x) = e^{it\Lambda} u_0 + i \int_0^t e^{i(t-s)\Lambda} F(s, \cdot) dx.
\]

To formulate estimates of \(u\) we need some notations. Given an interval \(I \subseteq \mathbb{R}\), a Banach space \(X\) of functions on \(\Omega\) and \(T > 0\), we shall write
\[
L^p_T X = L^p(I; X), \quad L^p_T X = L^p_{[0, T]} X, \quad L^p X = L^p_{[0, +\infty)} X
\]
with the obvious norms. Moreover, we shall write for \(N \in \mathbb{N}\)
\[
||u||_{X^k}^{q, N} = \sum_{j=0}^N ||\partial_t^{N-j} u||_{L^1_T W^{k, q}} = \sum_{2k + |x| \leq 2N} ||\partial_t^k \partial_x^j u||_{L^1_t L^q_x}
\]
where \(W^{k, q} = W^{k, q}(\Omega)\) is the usual Sobolev space with norm \(\sum_{|x| \leq k} ||\partial^x u||_{L^q}\) and \(H^k = W^{k, k}\). Note that the order of spatial regularity of functions in \(X^{2N}_{T, [0, T]}\) is \(2N\). When \(I = [0, T]\) or \(I = [0, +\infty)\) we use the notations
\[
X^{q, r, N}_{[0, T]} = X^{q, r, N}_{[0, T]}, \quad X^{q, r, N} = X^{q, r, N}_{[0, +\infty)}.
\]

From the integral representation of \(u\) and the unitarity of the group we have
\[
||u||_{L^p T L^q} \leq ||u_0||_{L^p} + ||F||_{L^1_T L^2} \tag{2.2}
\]
for any interval \(I\) containing 0. Higher regularity estimates require compatibility conditions. Given the data \((u_0, F)\) we define recursively a sequence of functions \(h_j\) as follows:
\[
h_0 = u_0, \quad h_j(x) = \partial_t^j u(0, x) = i^{-1} (\partial_t^{j-1} F(0, x) - \Delta h_{j-1}(x)) \quad j \geq 1. \tag{2.3}
\]
An explicit computation gives \(h_j = (-i\Delta)^j u_0 - i \sum_{\ell=0}^{j-1} (-i\Delta)^{j-\ell} \partial_t^\ell F(0, x)\).
**Definition 2.1** (Linear compatibility conditions). We say that the data \((u_0, F)\) satisfy the linear compatibility conditions of order \(N \geq 1\) if \(u_0 \in H^{2N}(\Omega) \cap H^1_0(\Omega), \ F \in C^k H^{2(N-k-1)}(\Omega) \cap C^N L^2(\Omega)\) for \(0 \leq k \leq N - 1\), and

\[
h_j \in H^1_0(\Omega) \quad \text{for} \quad 0 \leq j \leq N - 1. \tag{2.4}
\]

Then one has the following standard result.

**Theorem 2.2.** Assume \((u_0, F)\) satisfy the linear compatibility conditions of order \(N\) for some \(N \geq 1\). Then the global solution \(u\) to Problem (2.1) satisfies \(u \in C^N L^2(\Omega)\) and \(u \in C^k(H^{2(N-k)}(\Omega) \cap H^1_0(\Omega))\) for \(0 \leq k < N\). Moreover, for any interval \(I \subseteq \mathbb{R}\) containing 0 and of length \(\geq 1\) we have the estimate

\[
||u||_{X_t^{\infty, 2N}} \leq C(N) \left[ ||u_0||_{H^{2N}} + ||F||_{X_t^{\infty, 2N}} \right]. \tag{2.5}
\]

**Proof.** The result is classical (see e.g. [17]) and is proved by differentiating the equation with respect to time and estimating spatial derivatives inductively. We depart from standard results only in the formulation of the energy estimate (2.5): the right hand side is usually expressed in the form

\[
||F||_{X_t^{\infty, 2N-1}} + ||\partial_t^N F||_{L_t^1 L^2},
\]

which can be estimated by the \(X_t^{1, 2N}\) norm of \(F\) since

\[
||G||_{L_t^\infty L^2} \leq ||G||_{L_t^1 L^2} + ||\partial_t G||_{L_t^1 L^2}.
\]

Consider next the equation with a time dependent potential \(V(t, x)\)

\[
i \partial_t u + \Delta u = V(t, x) u + F(t, x), \quad u(t_0, x) = f(x), \quad u(t, \cdot)|_{\partial \Omega} = 0. \tag{2.6}
\]

If we assume that for some interval \(I\) containing 0 (possibly \(I = \mathbb{R}\))

\[
u_0 \in L^2, \quad F \in C_I L^2, \quad V \in L^1_t L^\infty_I,
\]

then the existence of a unique solution \(u \in C_I L^2\) is proved by a simple contraction argument for the map \(\nu \mapsto u\), where \(u\) is defined as the solution to

\[
i \partial_t u + \Delta u = V(t, x) \nu + F(t, x), \quad u(t_0, x) = f(x), \quad u(t, \cdot)|_{\partial \Omega} = 0,
\]

followed by a continuation argument. The solution satisfies the estimate

\[
||u||_{L_t^\infty L^2} \leq C(||V||_{L_t^1 L^\infty}) \left[ ||u_0||_{L^2} + ||F||_{L_t^1 L^2} \right]. \tag{2.7}
\]

Note that it is not necessary to modify the compatibility conditions in the higher regularity case. Indeed, the conditions should be

\[
h_j = i^{-1}(\partial_t^{-1} F(0, x) + \sum_{k=0}^{j-1} \left( j - 1 \over k \right) \partial_t^{j-1-k} V(0, x) h_k) - \Delta h_{j-1}(x), \quad j \geq 1
\]

but if the potential \(V\) is sufficiently smooth, the term \(\partial_t^{j-1-k} V(0, x) h_k\) belongs to \(H_0^1\) by the recursive assumption and can be omitted.
We denote the solution of (2.6), with initial data at $t = t_0$, by

$$u(t,x) = S(t; t_0)f.$$ 

Note that $S(t; t_0)$ fails to be a group since the potential $V$ depends on time. Regarding the term $Vu$ as a forcing term and applying Duhamel’s formula we can write $S(t; t_0)$ as a perturbation of the free flow:

$$S(t; t_0) = e^{i(t-t_0)\Delta} - i \int_{t_0}^{t} e^{i(t-s)\Delta} V(s,x) S(s, t_0) ds. \quad (2.8)$$

We impose a rather restrictive condition on $V$ in order to obtain a uniform energy estimate for all times:

$$\|V\|_{X^{1,\infty,N}} = \sum_{j=0}^{N} \|\partial_t^{N-j} V\|_{L_x^1 W^{2j,\infty}} < \infty. \quad (2.9)$$

Note that by standard embeddings we have

$$\|V\|_{X^{1,\infty,N}} \lesssim \|V\|_{X^{1,2N+2N_2}}, \quad N_2 = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$ 

By the usual recursive argument (repeated differentiation with respect to time) we obtain the regularity result:

**Theorem 2.3** (Perturbed energy estimate). Assume $(u_0, F)$ satisfy the compatibility conditions of order $N$ for some integer $N \geq 1$. Then problem (2.6) has a unique global solution, which satisfies $u \in C^N L^2(\Omega)$ and $u \in C^k(H^{2(N-k)}(\Omega) \cap H^1_0(\Omega))$ for $0 \leq k < N$. Moreover, for any interval $I \subseteq \mathbb{R}$ containing 0

$$\|u\|_{X^{\infty,2N}} \leq C(N, \|V\|_{X^{1,\infty,N}}) \left( \|u_0\|_{H^{2N}} + \|F\|_{X^{1,2N}} \right). \quad (2.10)$$

We next consider the decay properties of the linear solution. Strichartz estimates for the exterior problem are available from [14]. Recall that a couple of indices $(q, r)$ is admissible if $(q, r) \in [2, \infty] \times [2, \infty)$ and it satisfies the scaling condition

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}.$$ 

The endpoint is the couple $(2, \frac{2n}{n-2})$; the restriction $r < \infty$ means that the endpoint is not admissible when $n = 2$. Note that in the following result $\Omega$ could be more generally the exterior of any smooth convex obstacle in $\mathbb{R}^n$.

**Theorem 2.4.** Let $n \geq 2$, $(q, r)$ a non-endpoint admissible couple, and $I \subseteq \mathbb{R}$ an interval containing 0. Assume $(u_0, F)$ satisfy the linear compatibility conditions of order $m$ for some $m \geq 1$. Then the solution $u$ to Problem (2.1) satisfies the Strichartz estimate

$$\|u\|_{X^{q,r,m}} \lesssim \|u_0\|_{H^m} + \|F\|_{X^{1,2m}}. \quad (2.11)$$
Proof.} We can assume $I = \mathbb{R}$. If $m = 0$ and $F = 0$, the result is Theorem 1.7 in [14], and if $F$ is nonzero the result follows by a standard Christ–Kiselev argument. If $m = 1$, we apply $\partial_t$ and use the estimate just obtained; this gives
\[ ||u_t||_{L^2} \leq ||\Delta u_0||_{L^2} + ||F_t||_{L^2} \leq ||\Delta u_0||_{L^2} + ||F||_{X^{1,2,1}}. \]
Since $\Delta u = F - iu_t$, this implies
\[ ||\Delta u||_{L^2} \leq ||\Delta u_0||_{L^2} + ||F||_{X^{1,2,1}}. \]
Note that $||F||_{L^\infty L^2} \leq ||F||_{X^{1,2,1}}$; moreover, $X^{1,2,1}$ embeds into $L^1 H^2$ and into $L^\infty L^2$, hence by complex interpolation it embeds into $L^2 H^1$. If $n \geq 3$, this embeds into the endpoint $L^2 L^{\infty,\frac{2}{n}}$ and hence in all admissible spaces $L^q L^{r,q}$ by interpolation with the embedding into $L^\infty L^2$. This proves the $L^q L^{r,q}$ estimate for $\Delta u$, and by $L^q$ elliptic regularity (see e.g. [18]) this gives (2.11) for $m = 1$, $n \geq 1$. The same argument works for the case $m = 1$, $n = 2$ using the embedding $H^1, \hookrightarrow BMO$. Finally, for larger values of $m > 1$, the usual recursion argument and the embedding $X^{1,2,m}, \hookrightarrow X^{0,r,m-1}$ just proved for admissible $(q, r)$ allows to conclude the proof. \hfill \Box

Combining (2.11) with the perturbed energy estimate (2.5), we obtain a similar result for the perturbed linear problem
\[ i\partial_t u + \Delta u - V(t,x)u = F, \quad u(t_0,x) = f(x), \quad u(t,\cdot)|_{\partial \Omega} = 0. \tag{2.13} \]

**Proposition 2.5** (Perturbed Strichartz estimate). Let $n \geq 2$, $(q, r)$ a non-endpoint admissible couple, and $I \subseteq \mathbb{R}$ an interval containing 0. Assume $(u_0, F, V)$ satisfy the perturbed compatibility conditions of order $m$ for an $m \geq 1$. Then the solution $u$ to Problem (2.13) satisfies the Strichartz estimate
\[ ||u||_{X^{0,c,m}_t} \leq C(m, p, ||V||_{X^{1,\infty,\infty}_t}) \cdot \left(||u_0||_{H^{2m}} + ||F||_{X^{1,2,1}_t}\right). \tag{2.14} \]

**Proof.** Write $u = e^{it\Delta} u_0 + i \int_0^t e^{i(t-s)\Delta} (Vu + F)dx$ and apply (2.11) to get
\[ ||u||_{X^{0,c,m}_t} \leq ||u_0||_{H^{2m}} + ||Vu||_{X^{1,2,1}_t} + ||F||_{X^{1,2,1}_t} \leq ||u_0||_{H^{2m}} + ||V||_{X^{1,\infty,\infty}_t} ||u||_{X^{0,2,2}_t} + ||F||_{X^{1,2,1}_t}. \]
Using (2.5) we obtain (2.14). \hfill \Box

**Remark 2.6.** Note that the previous arguments can be obviously applied without modification to the more general equations of the form
\[ iu_t + \Delta u = V_1(t,x)u + V_2(t,x)\bar{u}. \]

### 3. The nonlinear theory

We consider now the nonlinear mixed problem on $\mathbb{R}^+ \times \Omega$
\[ iu_t + \Delta u = f(u), \quad u(0,x) = u_0, \quad u(t,\cdot)|_{\partial \Omega} = 0. \tag{3.1} \]
The compatibility conditions must be modified as follows. Define recursively the sequence of functions \( \psi_j(x) \) for \( j \geq 0 \) as
\[
\psi_0 = u(0, x) = u_0, \quad \psi_j = \partial^j_u(0, x) = i^{-1}(\partial^j_t f(u)|_{t=0} - \Delta \psi_{j-1}(x))
\]
where in the expansion of \( \partial^j_t f(u) \) we replace \( \partial^j_t u(0, x) \) with \( \psi_k \) for \( k < j \).

**Definition 3.1** (Nonlinear compatibility conditions). We say that the data \((u_0, f)\) satisfy the nonlinear compatibility conditions of order \( N \geq 1 \) if \( u_0 \in H^{2N}(\Omega) \cap H^1_0(\Omega) \), \( f \in C^N \), \( f(0) = 0 \) and
\[
\psi_j \in H^1_0(\Omega) \quad \text{for} \quad 0 \leq j \leq N - 1.
\] (3.2)

We get a local smooth solution to (3.1) by a standard contraction argument:

**Proposition 3.2** (Local existence in \( H^{2+} \)). Assume \((u_0, f)\) in (3.1) satisfy the nonlinear compatibility condition of order \( N \) for some integer \( N > n/2 \). Then there exists a time \( T > 0 \) and a unique solution of (3.1) on \([0, T] \times \Omega\) such that \( u \in C^N([0, T]; L^2(\Omega)) \) and \( u \in C^k([0, T]; H^{2(N-k)}(\Omega) \cap H^1_0(\Omega)) \) for \( 0 \leq k \leq N - 1 \).

**Proof.** Let \( T > 0 \) and denote by \( Z_T \) the space of functions \( v(t, x) \) such that
\[
v \in \cap_{k=0}^{N-1} C^k([0, T]; H^{2(N-k)}(\Omega) \cap H^1_0(\Omega)) \cap C^N([0, T]; L^2(\Omega)), \quad v(0, x) = u_0(x)
\]
endowed with the metric \( d(v, w) = ||v - w||_{X^{N,2,N}_T} \). Consider the linearized problem
\[
iu_t + \Delta u = f(v(t, x), \quad u(0, x) = u_0, \quad u(t, \cdot)|_{\partial \Omega} = 0. \] (3.3)

If \((u_0, f)\) satisfy (3.2) and \( v(t, x) \in Z_T \), then setting \( F(t, x) = f(v(t, x)) \) we see that the data \((u_0, F)\) satisfy the linear compatibility condition of order \( N \) (note that \( u_0 \) is bounded since \( 2N > n/2 \)). Thus the solution of (3.3) is uniquely defined on \([0, T] \times \Omega\) and has the properties listed in Theorem 2.2. Hence the map \( \Phi: v \mapsto u \) which takes \( v \) into the solution \( u \) of the linearized problem (3.3) operates on the metric space \( Z_T \). If \( v_1, v_2 \in Z_T \) then \( w = \Phi(v_1) - \Phi(v_2) \) solves the problem
\[
iw_t + \Delta w = f(v_1) - f(v_2), \quad w(0, x) = 0, \quad w(t, \cdot)|_{\partial \Omega} = 0.
\]

Since \((0, G)\) with \( G = f(v_1) - f(v_2)\) satisfy the linear compatibility conditions, we can apply (2.5):
\[
||\Phi(v_1) - \Phi(v_2)||_{X^{N,1,N}_T} \leq C(N)||f(v_1) - f(v_2)||_{X^{1,2,N}_T} \leq \frac{C(N)T}{2}||f(v_1) - f(v_2)||_{X^{N,2,N}_T}.
\]

Moreover, for \( N > n/2 \) we have the Moser type estimates
\[
||f(v)||_{X^{N,2,N}_T} \leq \phi_N(||v||_{X^{N,2,N}_T}),
\]
\[
||f(v_1) - f(v_2)||_{X^{N,2,N}_T} \leq \phi_N(||v_1||_{X^{N,2,N}_T} + ||v_2||_{X^{N,2,N}_T})||v_1 - v_2||_{X^{N,2,N}_T}
\]
where \( \phi_N \) is a suitable nondecreasing function depending only on \( N \) and \( f \).

Now let \( v(t, x) = u_0(x) \) for all \( t \); note that \( v \in Z_T \), and let \( B_M \) be the closed ball in \( Z_T \) centered at \( v \) of radius \( M \). By the previous estimates, it is trivial to check that \( \Phi: B_M \rightarrow B_M \) and \( ||\Phi(v_1) - \Phi(v_2)||_{Z_T} \leq C(M)T \leq \frac{1}{2} \) provided \( M \) is large enough w.r.t. \( ||v||_{Z_T} = ||u_0||_{H^{2N}} \) and \( T \) is sufficiently small w.r.t. \( M \). A contraction argument then implies the claim. \( \Box \)
4. The radial solution

4.1. Existence: Proof of Proposition 1.1

Consider the equation

\[ iu_t + \Delta u = |u|^{p-1}u, \quad u(0, x) = u_0(x), \quad u(t, \cdot)|_{\partial \Omega} = 0 \tag{4.1} \]

with radial initial data \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \). The existence of a local solution for \( t \in [0, T] \) is standard and similar to the proof of Proposition 3.2. We use the space \( Y_T \) of functions \( v(t, x) \), radial in \( x \), such that

\[ v \in C^1([0, T]; L^2(\Omega)) \cap C([0, T]; H^1_0(\Omega) \cap H^2(\Omega)), \quad v(0, x) = u_0. \]

We endow \( Y_T \) with the distance \( d(v, w) = ||v - w||_{\dot{X}^{0,2,1}_T} \). By radiality and (1.2), functions in \( Y_T \) are bounded. Hence the map \( \Phi : v \mapsto u \), defined as above via the linearization (3.3), operates on \( Y_T \) and satisfies

\[ ||u||_{\dot{X}^{0,2,1}_T} \leq ||u_0||_{H^2} + ||v||^{p-1}_T ||v||_{\dot{X}^{0,2,1}_T} \leq ||u_0||_{H^2} + ||v||^{p-1}_T ||v||_{\dot{X}^{0,2,1}_T}. \]

Estimate (1.2) implies that \( ||v||_{L^T L^\infty} \leq ||v||_{\dot{X}^{0,2,1}_T} \) so that

\[ ||\Phi(v)||_{\dot{X}^{0,2,1}_T} \leq ||u_0||_{H^2} + ||v||^{p-1}_{\dot{X}^{0,2,1}_T} ||v||_{\dot{X}^{0,2,1}_T} \leq ||u_0||_{H^2} + T ||v||_{\dot{X}^{0,2,1}_T}^p. \]

In a similar way,

\[ ||\Phi(v_1) - \Phi(v_2)||_{\dot{X}^{0,2,1}_T} \leq (||v_1||_{\dot{X}^{0,2,1}_T} + ||v_2||_{\dot{X}^{0,2,1}_T})^{p-1} \cdot T ||v_1 - v_2||_{\dot{X}^{0,2,1}_T}^p. \]

Thus taking \( T \) sufficiently small with respect to \( ||u_0||_{H^2} \) we obtain a local solution with the required regularity.

The local solution satisfies the energy conservation (1.5), which implies the uniform bound (1.6) for \( 0 \leq t \leq T \). In particular, \( ||u(t)||_{H^2 L^{p+1}} \) remains bounded on \([0, T]\) by a constant depending only on \( ||u_0||_{H^2 L^{p+1}} \). A standard continuation argument allows to extend \( u \) to a global solution, which satisfies (1.5) and hence (1.6) for all times. The claim about uniqueness follows immediately from energy estimates.

Assume now that the data satisfy the compatibility conditions of order \( N \geq 2 \). Differentiating the equation with respect to \( t \) we see that \( \nu = \partial_t u \) satisfies an equation of the form

\[ i\nu_t + \Delta \nu = a(t, x)\nu + b(t, x)\bar{\nu} \]

with \( |a| + |b| \leq |u|^{p-1} \) bounded. By the linear theory we get \( \nu \in C^1 L^2 \) i.e. \( u \in C^1 L^2 \). Further differentiating w.r.t. \( t \), by a recursive argument we get \( u \in C^2 L^2 \), and using the equation itself we obtain that \( u \in C^k (H^2(\Omega) \cap H^1_0(\Omega)) \) for \( 0 \leq k \leq N - 1 \).

4.2. Decay: Proof of Proposition 1.2

Apply to \( u \) the pseudoconformal transform

\[ u(t, x) = t^{-\frac{3}{2}} U \left( -\frac{1}{t}, \frac{x}{t} \right) e^{\frac{\beta t^2}{4}}. \]
If \( u(t, x) \) is defined on the domain \( t \geq 1, \ |x| \geq 1 \), then \( U(T, X) \) is defined for \( -1 \leq T < 0, \ |X| \geq |T| \) and we have

\[
iu_t + \Delta_x u - |u|^{p-1}u = t^{-\frac{n}{2}}(iU_T + \Delta X U - (-T)\nu |U|^{p-1}U)e^{it^2}
\]

with \( \nu = \frac{n}{2}(p - 1) - 2 \). The energy density

\[
e_0(T, X) = \frac{1}{2} |\nabla_X U|^2 + \frac{1}{p + 1} |U|^{p+1}
\]

satisfies the identity

\[
\partial_T e_0 = \Re \nabla \cdot \{ \tilde{U}_T \nabla U \} - \frac{\nu(-T)^{\nu-1}}{p + 1} |U|^{p+1}.
\]

Since \( U \) is a radial function we have

\[
R^{n-1} \nabla_X \cdot \{ \tilde{U}_T \nabla U \} = \partial_R \{ R^{n-1} U_R \tilde{U}_T \}
\]

where \( \partial_R U = U_R = \frac{X}{|X|} \cdot \nabla X U \) denotes the radial derivative of \( U \) and \( R = |X| \). Introduce the radial energy density

\[
e(T, R) = R^{n-1}\left( \frac{1}{2} |U_R|^2 + \frac{1}{p + 1} |U|^{p+1} \right)
\]

so that

\[
\partial_T e(T, R) = \Re \partial_R \{ R^{n-1} U_R \tilde{U}_T \} - \frac{\nu(-T)^{\nu-1}}{p + 1} R^{n-1} |U(T)|^{p+1}.
\]

We now integrate the identity (4.2) on the \((T, R)\) domain \( T_1 \leq T \leq T_2, R \geq -T \) for some \(-1 \leq T_1 < T_2 < 0\). Note that the exterior normal on the line \( R = -T \) (for \( T < 0 \)) is given by \( n = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \). Writing

\[
E(T) = \int_{-T}^{+\infty} e(T, R)dR
\]

after integration of (4.2) we obtain

\[
E(T_2) - E(T_1) - \frac{1}{\sqrt{2}} \int_{T_1}^{T_2} e(T, -T)dT = -\frac{1}{\sqrt{2}} \int_{T_1}^{T_2} U_R \tilde{U}_T |T|^{n-1} dT - \int_{T_1}^{T_2} \int_{-T}^{+\infty} \frac{\nu(-T)^{\nu-1} |U|^{p+1}}{p + 1} dRdT
\]

since \(-T = |T| = R\) at the points of the cone. Writing (with a slight abuse) \( U(T, |X|) \) instead of \( U(T, X) \), the Dirichlet condition implies \( U(T, -T) = 0 \), so that

\[
e(T, -T) = |T|^{n-1} \frac{1}{2} |U_R|^2.
\]

On the other hand, differentiating \( U(T, -T) = 0 \) we get \( U_T(T, -T) = U_R(T, -T) \). Hence the previous estimate reduces to

\[
E(T_2) - E(T_1) + \int_{T_1}^{T_2} \int_{-T}^{+\infty} \frac{\nu(-T)^{\nu-1} |U|^{p+1}}{p + 1} dRdT \leq \frac{1}{\sqrt{2}} \int_{T_1}^{T_2} \left( \frac{|U_R|^2}{2} - |U_R|^2 \right) |T|^{n-1} dT \leq 0.
\]
Thus \( E(T) \) is nonincreasing as \( T \uparrow 0 \) and in particular \( E(T) \leq E(-1) \) for \(-1 < T < 0\).

This implies
\[
\int_{-T}^{+\infty} |U_R|^2 R^{n-1} dR \leq 2E(-1) \quad \text{for} \quad -1 < T < 0
\]
or equivalently
\[
\int_{|X| > -T} |\nabla X U|^2 dX \leq 2E(-1) \quad \text{for} \quad -1 < T < 0.
\]

Note that, if we extend \( U \) as zero in the region \(|X| < -T\), the extended function \( \tilde{U}(T, \cdot) \) is \( H^1(\mathbb{R}^n) \) and radial, hence we can apply (1.2) and we obtain
\[
|U(T, X)|^2 \leq |X|^{2-n} \int_{|X| > -T} |\nabla X U|^2 dX \leq 2|X|^{2-n}E(-1). \tag{4.3}
\]

We convert (4.3) into an estimate for \( u(t, x) \). We have
\[
E(-1) = \int_{|X| > 1} \left( \frac{|
abla U(-1, X)|^2}{2} + \frac{|U(-1, X)|^{p+1}}{p+1} \right) dX.
\]

Since
\[
U(T, X) = (-T)^{-\frac{n}{2}} u \left( -\frac{1}{T}, \frac{X}{T} \right) e^{i\frac{\chi^2}{2T}}
\]
we compute
\[
\nabla X U(T, X) = (-T)^{-\frac{n}{2}} \frac{\partial \chi^2}{2T} \left[ \nabla X u \left( -\frac{1}{T}, \frac{X}{T} \right) \frac{1}{T} + u \left( -\frac{1}{T}, \frac{X}{T} \right) \cdot i \frac{X}{2T} \right]
\]
so that
\[
|\nabla X U(-1, X)| \leq |\nabla u(1, -X)| + |X u(1, -X)|
\]
and
\[
E(-1) \leq \int_{|X| > 1} \left[ |\nabla u(1, x)|^2 + |x u(1, x)|^2 + |u(1, x)|^{p+1} \right] dx. \tag{4.4}
\]

On the other hand, changing variables \((t, x) = \left( -\frac{1}{t}, \frac{X}{2} \right)\) in \( E(T) \) and writing
\[
E_1(t) = E \left( -\frac{1}{t} \right)
\]
we get, after a standard computation,
\[
E_1(t) = \int_{|X| > 1} \left[ \frac{1}{8} |(x + 2it \nabla) u(t, x)|^2 + \frac{|t|^{\frac{p(p-1)}{2}} |u(t, x)|^{p+1}}{p+1} \right] dx
\]
and \( E_1(t) \) is nonincreasing in \( t \) by the previous computation; note this is a proof of the pseudoconformal energy conservation on an exterior domain. Hence we have
\[
||x u(t)||^2_{L^2(\Omega)} \leq E_1(t) + \left( t + t^{\frac{p(p-1)}{2}} \right) E(u(t)) \leq E_1(0) + \left( t + t^{\frac{p(p-1)}{2}} \right) E(u_0)
\]
and in conclusion
\[ \|xu(t)\|_{L^2(\Omega)}^2 \leq C(t)\left[\|xu_0\|_{L^2(\Omega)}^2 + E(u(0))\right]. \]

Combined with the conservation of \( E(u(t)) \) and (4.4) this gives
\[ \mathcal{E}(\tau) \leq \|xu_0\|_{L^2(\Omega)}^2 + E(u(0)) \]
and, recalling (4.3), we have proved
\[ \|U(T,X)\| \leq \|X\|^{1-2} \left[\|xu_0\|_{L^2(\Omega)} + E(u(0))^{1/2}\right]. \]

Writing \( \|U(T,X)\| = |T|^{-n/2}|u\left(-\frac{1}{T}, \frac{X}{T}\right)| \) we finally obtain
\[ t^2|u(t,x)| \leq C|x|^{1-2} \cdot t^{2-1} \left[\|xu_0\|_{L^2(\Omega)} + E(u(0))^{1/2}\right] \]
that is to say
\[ |u(t,x)| \leq C\left[\|xu_0\|_{L^2(\Omega)} + E(u(0))^{1/2}\right] \cdot |x|^{1-\frac{n}{2}}t^{-1}. \] (4.5)

Note that, using (1.2), we have
\[ E(u(0)) = \frac{1}{2} \|\nabla_x u_0\|_{L^2}^2 + \frac{1}{p+1} \int |u_0|^{p+1} \leq \|\nabla_x u_0\|_{L^2}^2 + \|u_0\|_{L^{p+1}}^{p+1} \leq C(\|u_0\|_{H^1}). \]

Using (4.5) for \( t > 1 \), and the inequality
\[ |u(t,x)| \leq C|x|^{1-\frac{n}{2}} \|\nabla_x u\|_{L^2} \]
for \( t \leq 1 \), we obtain as claimed
\[ |u(t,x)| \leq C \cdot |x|^{1-\frac{n}{2}}(t)^{-1}. \]
with a constant depending on \( \|xu_0\|_{L^2(\Omega)} + \|u_0\|_{H^1} \).

### 4.3. Non inflation of Sobolev norms

We now consider the issue of regularity; we prove that if the data are smoother the solution remains smooth and Sobolev norms remain bounded, for any order of regularity.

**Corollary 4.1.** Let \( N \geq 1, p > 2N - 1 \), and assume \((u_0, f)\) with \( f(z) = |z|^{p-1}z \) satisfy the compatibility conditions of order \( N \). Assume in addition that \( xu_0 \in L^2(\Omega) \). Then the radial solution constructed in Proposition 1.1 satisfies the uniform bound on \( \mathbb{R}^+ \times \Omega \)
\[ \|u(t, \cdot)\|_{X^\infty,2N} \leq C(\|u_0\|_{H^{2N}}, \|xu_0\|_{L^2}). \] (4.6)

Recall that for all times \( t \) we have \( \|u(t)\|_{L^2} = \|u_0\|_{L^2} \). We next prove uniform bounds for the derivatives of \( u \); in the following proof we write for brevity
\[ C(\|u_0\|_{H^N}) = C(\|u_0\|_{H^N}, \|xu_0\|_{L^2}). \]
leaving the dependence on \( \|xu_0\|_{L^2} \) implicit.
Differentiating the equation once with respect to \( t \) we see that \( v = u_t \) solves

\[
i v_t + \Delta v = \partial_t(|u|^{p-1} u)
\]

and we have by (1.7)

\[
||\partial_t(|u|^{p-1} u)||_{L^2} \leq \int_0^T ||u||_{L^\infty} ||v||_{L^2} dt \leq \int_0^T (t)^{1-p} ||v||_{L^2} dt.
\]

Thus we can write

\[
||v||_{L^\infty L^2} = ||u_t||_{L^\infty L^2} \leq ||u_t(0)||_{L^2} + \int_0^T (t)^{1-p} ||v||_{L^2} dt
\]

and by Gronwall’s Lemma, if \( p > 2 \) we get

\[
||v||_{L^\infty L^2} = ||u_t||_{L^\infty L^2} \leq C||u_t(0)||_{L^2}
\]

with \( C = C(||u_0||_{H^1} + ||xu_0||_{L^2}) \); since \( |u_t(0)| \leq |\Delta u_0| + ||u_0||^p \) we have

\[
||u_t(0)||_{L^2} \leq ||u_0||_{H^2} + ||u_0||_{L^\infty}^{-1} ||u_0||_{L^2} \leq C(||u_0||_{H^2})
\]

which gives the estimate

\[
||u_t||_{L^\infty L^2} \leq C(||u_0||_{H^2}).
\]

Using the equation for \( u \) we can estimate also

\[
||\Delta u||_{L^\infty L^2} \leq ||u_t||_{L^\infty L^2} + ||u||_{L^\infty L^2} \leq C(||u_0||_{H^2}) + ||u||_{L^\infty}^{-1} ||u||_{L^2} \leq C(||u_0||_{H^2}).
\]

By elliptic regularity we have then

\[
||u||_{L^\infty H^2} \leq C(||u_0||_{H^2})
\]

and summing up we have proved

\[
||u||_{X^{\infty,2,1}} \leq C(||u_0||_{H^2}). \tag{4.7}
\]

For higher order derivatives we proceed in a similar way by induction. Applying \( \partial_t^j \) to the equation we have

\[
i(\partial_t^j u)_t + \Delta (\partial_t^j u) = \partial_t^j(|u|^{p-1} u).
\]

The right hand side satisfies

\[
|\partial_t^j(|u|^{p-1} u)| \leq (t)^{1-p}|\partial_t^j u| + \text{L.O.T.}
\]

where the lower order terms are products of derivatives \( \partial_t^h u \) with \( h < j \), which are bounded by the induction hypothesis (recall that \( u \) is radial in \( x \), hence \( L^\infty \) norms are bounded by \( L^2 \) norms of the gradient), times a power of \( u \) of order at least \( p - j \), which decays like \( (t)^{j-p} \) and is integrable provided \( p > j + 1 \). This implies

\[
||\text{L.O.T.}||_{L^2} \leq \int_0^T (t)^{p-j}C(||u_0||_{H^2}) dt \leq C(||u_0||_{H^2})
\]

and hence

\[
||\partial_t^j u||_{L^2} \leq ||\partial_t^j u(0)||_{L^2} + \int_0^T ||\partial_t^j(|u|^{p-1} u)||_{L^2} dt
\]

\[
\leq ||\partial_t^j u(0)||_{L^2} + C(||u_0||_{H^2}) + ||(t)^{1-p} \partial_t^j u||_{L^2}.
\]
Thus again by Gronwall’s lemma, if $p > j + 1$, we have
\[ ||\partial_t^j u||_{L^\infty L^2} \leq C(||u_0||_{H^0}). \] (4.8)

Using the equation, this shows that if $p > N + 1$ one has
\[ ||\partial_t^N u||_{L^\infty L^2} + ||\partial_t^{N-1} \Delta u||_{L^\infty L^2} \leq C(||u_0||_{H^0}) \]
and by elliptic regularity we have also
\[ ||\partial_t^N u||_{L^\infty L^2} + ||\partial_t^{N-1} u||_{L^\infty H^1} \leq C(||u_0||_{H^0}) \]
provided $p > N + 1$. Derivatives $\partial_t^j \partial^\alpha u$ can be estimated using the equation and elliptic regularity by the usual recursive procedure; this requires to estimate $\Delta^j(|u|^{p-1}u)$ with $j = 1, \ldots, N - 1$, and in order to get an integrable factor $(t)^{p-2j}$ we must assume $p - 2(N - 1) > 1$ i.e. $p > 2N - 1$. We finally arrive at the estimate
\[ ||u||_{X^{\infty,2N}} \leq C(||u_0||_{H^{2N}}) \] (4.9)
provided $p > 2N - 1$.

5. Proof of Theorem 1.3

Denote by $u(t, x)$ the global radial solution with $u(0, x) = u_0$ constructed in Propositions 1.1, 1.2 and Corollary 4.1, and let $v(t, x)$ be the local solution with $v(0, x) = v_0$ given by Proposition 3.2. Then $w = v - u$ satisfies the equation
\[ iw_t + \Delta w = |u + v|^{p-1}(u + v) - |u|^{p-1}u \]
which can be written in the form
\[ iw_t + \Delta w = V(t, x)w = w^2 \cdot F[u, w], \quad w(0) = w_0 := v_0 - u_0, \quad w|_{\partial \Omega} = 0 \] (5.1)
where
\[ V(t, x) = p|u|^{p-1}, \quad F[u, w] = p(p - 1) \int_0^1 |u + \sigma w|^{p-3}(u + \sigma w)(1 - \sigma)d\sigma. \]
We have
\[ ||V||_{X^{i,\infty,m}} \leq \sum_{2j + |\alpha| \leq 2m} ||\partial_t^j \partial^\alpha u|^{p-1}||_{L^1 L^\infty} \]
and if $p - 1 > 2m$ we can write
\[ ||\partial_t^j \partial^\alpha u|^{p-1}||_{L^1 L^\infty} \leq \sum \int_0^T ||u||^{p-1-\nu}_L||\partial_t^{j_1} \partial^{\alpha_1} u||_{L^\infty L^\infty} \cdots ||\partial_t^{j_\nu} \partial^{\alpha_\nu} u||_{L^\infty L^\infty} dt \]
\[ \leq \sum ||u||^{p-1-\nu}_L ||u||^{p-1-\nu}_L \int_0^T dt \]
where the sum is extended over $j_1 + \ldots + j_\nu = j$, $\alpha_1 + \ldots + \alpha_\nu = \alpha$ and $\nu \leq j + |\alpha|$, so that $\nu \leq 2m$. By the decay estimate (1.7) we have $||u||_{L^\infty} \leq (t)^{-1}$ hence the last integral is convergent provided $p > 2m + 2$. By Sobolev embedding and the bound (4.6) we get
\[ ||u||_{X^{\infty,2m+N_2}} \leq ||u||_{X^{\infty,2m+N_2}} \leq C(||u_0||_{H^{2m+N_2}}, ||xu_0||_{L^2}), \quad N_2 = \left\lfloor \frac{n}{2} \right\rfloor + 1 \]
and in conclusion we have proved

\[ \|V\|_{X^1,\infty} \leq C(\|u_0\|_{H^2(m+N_2)}, \|xu_0\|_{L^2}) < \infty \]  

(5.2)

provided \( p > 2m + 2 \). Thus we are in position to apply Theorem 2.3 and Proposition 2.5, and we get that the linear equation

\[ iw_t + \Delta w - V(t,x)w = F(t,x) \]

satisfies for all admissible non endpoint \((q, r)\) and all \( m \geq 1 \) the perturbed energy–Strichartz estimates

\[ \|w\|_{X^r,q} + \|w\|_{X^q,r} \leq \|w_0\|_{H^m} + \|F\|_{X^1,2} \]

(5.3)

provided \( p > 2m + 2, u_0 \in H^{2(m+N_2)} \) and compatibility conditions of suitable order are satisfied. The implicit constant in (5.3) depends on \( \|u_0\|_{H^1} + \|xu_0\|_{L^2} \) but not on \( T \).

We now apply (5.3) to Equation (5.1). For an admissible couple \((q, r)\) and an integer \( m \) to be chosen we write

\[ M_m(T) = \|w\|_{X^r,q} + \|w\|_{X^q,r} \]

and by (5.3) we have

\[ M_m(T) \leq \|w_0\|_{H^m} + \|w^2 F[u, w]\|_{X^1,2}. \]  

(5.4)

We must estimate

\[ \|w^2 F[u, w]\|_{X^1,2} = \sum_{2j+|\alpha| \leq 2m} \|\partial_t^j \partial_x^\alpha (w^2 F[u, w])\|_{L^1_t L^2}. \]

The derivative can be expanded as a finite sum

\[ \partial_t^j \partial_x^\alpha (w^2 F[u, w]) = \sum_0^r G \cdot W_1W_2U_1 \cdots U_r d\sigma \]

where \( 0 \leq \nu \leq 2m \) and

- \( W_1 = \partial_t^{h_1} \partial_x^{x_1} w \) and \( W_2 = \partial_t^{h_2} \partial_x^{x_2} w \)
- \( U_k = \partial_t^{\beta_k} \partial_x^{\beta_k} (u + \sigma w) \)
- \( h_1 + h_2 + j_1 + \cdots + j_\nu = j, x_1 + x_2 + \beta_1 + \cdots + \beta_\nu = \alpha \)
- it is not restrictive to assume that \( 2j_\nu + |\beta_\nu| \geq 2j_k + |\beta_k| \) for all \( k \) and that \( 2h_2 + |x_2| \geq 2h_1 + |x_1| \)
- \( G \) satisfies \( |G| \leq |u + \sigma w|^{p-\nu-2} \) so that \( \|G\|_{L^\infty} \leq (\|u\|_{L^\infty} + \|w\|_{L^\infty})^{p-\nu-2} \).

We take the \( L^2 \) norm in \( x \) of each product. We consider two cases.

**First case:** \( 2j_\nu + |\beta_\nu| \geq 2h_2 + |x_2| \). Then we write

\[ \|GW_1 W_2 U_1 \cdots U_\nu\|_{L^2} \leq \|G\|_{L^\infty} \|W_1\|_{L^\infty} \|W_2\|_{L^\infty} \|U_1\|_{L^\infty} \cdots \|U_\nu-1\|_{L^\infty} \|U_\nu\|_{L^2} \]

\[ \leq \|G\|_{L^\infty} \|W_1\|_{W^{N_r}} \|W_2\|_{W^{N_r}} \|U_1\|_{H^{N_2}} \cdots \|U_\nu-1\|_{H^{N_2}} \|U_\nu\|_{L^2} \]

by Sobolev embedding, where

\[ N_r = \left\lceil \frac{n}{r} \right\rceil + 1, \quad N_2 = \left\lfloor \frac{n}{2} \right\rfloor + 1; \]
note that since the maximal order of derivation is \(2j_\nu + |\beta_\nu| \leq 2m\), we have \(2j_k + |\beta_k| \leq m\) for \(k < \nu\) and \(2h_i + |\xi_i| \leq m\), \(i = 1, 2\), so that, using \((4.6)\), we have
\[
\|U_i\|_{L^\infty_t H^{N_2}} \leq \|u\|_{X^{\infty, 2(m+N_2)/2}} + \|w\|_{X^{\infty, 2(m+N_2)/2}} \leq \|u\|_{X^{\infty, 2m}} + \|w\|_{X^{\infty, 2m}} \leq C_m + M_m(T)
\]
provided \(m \geq N_2\), where \(C_m = C(||u_0||_{H^{2m}}, ||xu_0||_{L^2})\). Moreover we can estimate
\[
\|G\|_{L^\infty} \leq (\|u\|_{L^\infty} + \|w\|_{L^\infty})^{p-\nu-2} \leq (\langle t \rangle^{-1} + \|w\|_{L^\infty})^{p-\nu-2}.
\]
This gives, for \(t \in [0, T]\),
\[
\|GW_1W_2U_1 \cdots U_{\nu}\|_{L^2} \leq (C_m + M_m(T))^{p-\nu-2} ||W_1||_{W^{N_r}, \nu} ||W_2||_{W^{N_r}, \nu}.
\]
We now take the \(L^1\) norm in \(t \in [0, T]\). Since \(2h_i + |\xi_i| \leq m\), \(i = 1, 2\), we can write
\[
\int_0^T \langle t \rangle^{2+\nu-p} ||W_1||_{W^{N_r}, \nu} ||W_2||_{W^{N_r}, \nu} dt \leq ||\langle t \rangle^{2+\nu-p}||_{L^{q+q/m}} ||w||_{X^{q/m, r(m+N_2)/2}}
\]
and if \(m \geq N_r\) (which is implied by \(m \geq N_2\)) this gives
\[
\leq ||\langle t \rangle^{2+\nu-p}||_{L^{q+q/m}} M_m(T)^2.
\]
We choose \(q \in (2, \infty)\) such that \((q, r)\) is admissible, i.e. \(q = \frac{4r}{n(r-2)}\). Since \(\nu \leq 2m\), we see that
\[
||\langle t \rangle^{2+\nu-p}||_{L^{q+q/m}} \leq ||\langle t \rangle^{2+2m-p}||_{L^{q+q/m}} < \infty \quad \text{provided} \quad p > 2m + 3 - n \frac{r-2}{2r}
\]
and in this case
\[
\int_0^T \langle t \rangle^{2+\nu-p} ||W_1||_{W^{N_r}, \nu} ||W_2||_{W^{N_r}, \nu} dt \leq M_m(T)^2. \quad (5.5)
\]
With a similar computation we can write
\[
\int_0^T ||w||_{L^\infty}^{p-\nu-2} ||W_1||_{W^{N_r}, \nu} ||W_2||_{W^{N_r}, \nu} dt \leq ||\langle t \rangle^{(p-\nu-2)-q} ||M_m(T)^2
\]
and
\[
\|||w||_{L^\infty}^{p-\nu-2}||_{L^{q+q/m}} \leq ||w||_{L^{q, q, r}}^{p-\nu-2} ||W_1||_{L^\infty}^{q-2} ||W_2||_{L^\infty}^{q-2} \leq M_m(T)^{p-\nu-2}
\]
provided \(p > 2m + q\). In conclusion we have
\[
\int_0^T ||w||_{L^\infty}^{p-\nu-2} ||W_1||_{W^{N_r}, \nu} ||W_2||_{W^{N_r}, \nu} dt \leq M_m(T)^{p-\nu}. \quad (5.6)
\]
Combining \((5, 7)\) we conclude
\[
\|GW_1W_2U_1 \cdots U_{\nu}\|_{L^2} \leq M_m(T)^2 + M_m(T)^p \quad (5.7)
\]
provided \(p > 2m + q + 2\) (so that \(p - q - \nu > 2\).
Following [15], we prove a general stability result for local solutions of (6.1), from which the uniqueness theorem 1.4 follows immediately.

Let $I$ be an open interval containing $0$, $T > 0$. Let $u, v$ be two distributional solutions to (6.1) on $I \times \Omega$ such that

$$u \in C(I; H^2(\Omega)) \cap C^1(I; H^1(\Omega)) \cap C^2(I; L^2(\Omega)) \cap L^\infty(I \times \Omega), \quad \Delta u \in C(I; H^1_0(\Omega)),$$

$$v \in C(I; H^1_0(\Omega)) \cap C^1(I; L^2(\Omega)).$$

Assume in addition that $v$ satisfies an energy inequality

$$E(v(t)) \leq E(v(0)).$$

Then the difference $w = v - u$ satisfies the energy estimate

$$E(w(t)) \leq C e^{Ct} (E(w(0)) + \|w(0)\|_{L^2(\Omega)}^2), \quad t \in [0, T]$$

where $C$ is a constant depending on
To prove Theorem 6.1, consider the difference \( w = v - u \), which satisfies the equation
\[
\frac{\partial}{\partial t}w + \Delta w = \frac{\partial}{\partial \nu}(|u + w|^{p-1}(u + w) - |u|^{p-1}u).
\]

We prepare an estimate for the \( L^2 \) norm of \( w \). Using the multiplier \( i\dot{w} \) we get
\[
\partial_t||w(t)||_{L^2}^2 = 2\Im\int_\Omega (|u + w|^{p-1}(u + w) - |u|^{p-1}u)\dot{w}dx \\
\leq C(||u||_{L^p L^\infty}) \cdot \int_\Omega (|w|^2 + |w|^{p+1})dx \\
\leq C[||w||_{L^2}^2 + E(w(t))]
\]
and by Gronwall’s Lemma
\[
||w(t)||_{L^2}^2 \leq C||w(0)||_{L^2}^2 + C \int_0^t e^{C(t-s)}E(w(s))ds, \quad t \in [0, T]
\]
with \( C = C(T, ||u||_{L^p L^\infty}) \).

Next, we split
\[
E(v) = E(u) + A(t) + B(t)
\]
where
\[
A(t) = \frac{1}{2} \int |\nabla x w|^2 dx + \int \left( \frac{|u + w|^{p+1} - |u|^{p+1}}{p+1} - |u|^{p-1}R(u\dot{w}) \right) dx
\]
\[
B(t) = \Re \int (\nabla u \cdot \nabla \dot{w} + |u|^{p-1}u\dot{w})dx.
\]

Since \( E(u) = E(u(0)) \) and \( E(v) \leq E(v(0)) \) we have
\[
0 \leq E(v(0)) - E(v(t)) = A(0) - A(t) + B(0) - B(t).
\]

Writing
\[
\phi(t) = \frac{|t|^{p+1}}{p+1} \quad \text{so that} \quad \frac{|u|^{p+1}}{p+1} = \phi(|u|^2),
\]
we see that
\[
\partial_x \phi(|u + \sigma w|^2) = \phi'(|u + \sigma w|^2)2\Re((u + \sigma w)\dot{w}) = \Re((u + \sigma w|^{p-1}(u + \sigma w)\dot{w}),
\]
\[
\partial_x^2 \phi(|u + \sigma w|^2) = \frac{(p-1)|u + \sigma w|^{p-3}\Re((u + \sigma w)\dot{w})^2 + |u + \sigma w|^{p-1}|w|^2}{(p-1)|u + \sigma w|^{p-1}|w|^2} \geq |u + \sigma w|^{p-1}|w|^2 \geq 2^{2-p}\sigma^{p-1}|w|^{p+1} - |u|^{p-1}|w|^2
\]
(since \( p \geq 3 \)). We get easily
\[
\frac{|u + w|^{p+1} - |u|^{p+1}}{p+1} - |u|^{p-1}\Re(u\dot{w}) = \int_0^1 \int_0^\sigma \partial_x^2 \phi(|u + \tau w|^2)d\tau d\sigma
\]
\[
\geq \frac{2^{2-p}}{p(p+1)} |w|^{p+1} - \frac{1}{2} |u|^{p-1}|w|^2
\]
which implies
Recalling (6.3), this gives for $t \in [0, T]$

$$A(t) \geq \frac{1}{p^{2p}} E(w(t)) - C||w||_L^2, \quad C = \frac{1}{2} ||u||_L^{p-1}$$

for some $C = C(T, ||u||_{L^\infty})$. On the other hand

$$|u + w|^{p+1} - |u|^{p+1} - |u|^{p-1}(u\bar{w}) \leq C(||u||_{L^\infty})(|w|^{p+1} + |w|^2)$$

which implies

$$A(0) \leq CE(w(0)) + C||w(0)||_L^2$$

and in conclusion

$$A(0) - A(t) \leq -\frac{1}{p^{2p}} E(w(t)) + C\int_0^t E(w(s))ds + C||w(0)||_L^2 \quad (6.5)$$

with $C = C(T, ||u||_{L^\infty})$.

In order to estimate $B(t)$, we first remark the following. If $W(t, x), U(t, x)$ satisfy

$$iW_t + \Delta W = F, \quad iU_t + \Delta U = G$$

with Dirichled boundary conditions, then for any $\chi(t) \in C^\infty_c((0, T))$ we have formally

$$\int \int_\Omega \chi'(t)(\nabla U \cdot \nabla \bar{W})dxdt = \int \int_\Omega \chi(t)(U_t F + W_t G)dxdt \quad (6.6)$$

Identity (6.6) is obvious for smooth $U, W$, by integration by parts. By approximation, (6.6) holds also if $W$ is a solution of $iW_t + \Delta W = F$ in $\mathcal{D}'((0, T) \times \Omega)$, with

$$W \in C^1([0, T]; H^{-1}(\Omega)) \cap C([0, T]; H^1_0(\Omega)),$$

so that $F \in C([0, T); H^{-1}(\Omega))$, and

$$U \in C^1([0, T]; H^1_0(\Omega)) \cap C([0, T]; H^2(\Omega)) \quad \text{with} \quad \Delta U \in C([0, T]; H^1_0(\Omega))$$

so that $G \in C([0, T]; H^1_0(\Omega))$. Consider now a sequence of test functions $\chi_k(t) \in C^\infty_c((0, T))$, nonnegative, such that $\chi_k \uparrow [0, t]$ pointwise, $t \in (0, T)$, and write

$$B(t) - B(0) = \lim_{k \to \infty} I_k, \quad I_k := \mathfrak{R}\int \int_\Omega \chi_k'(t)(\nabla u \cdot \nabla \bar{w} + |u|^{p-1}u\bar{w})dxdt.$$

Using (6.6) with the choices $W = w, U = u, F = |u + w|^{p-1}(u + w) - |u|^{p-1}u$ and $G = |u|^{p-1}u$, we get

$$I_k = \mathfrak{R}\int \int_\Omega \chi_k(t)[u_t(|u + w|^{p-1}(\bar{u} + \bar{w}) - |u|^{p-1}\bar{u} + \bar{w}_t|u|^{p-1}u - \partial_t(|u|^{p-1}u\bar{w})]$$

$$= \mathfrak{R}\int \int_\Omega \chi_k(t)[u_t(|u + w|^{p-1}(\bar{u} + \bar{w}) - |u|^{p-1}\bar{u} - \bar{w}\partial_t(|u|^{p-1}u)]dxdt.$$

We compute

$$\partial_t(|u|^{p-1}u) = |u|^{p-1}\left(|u|^2u_t + \frac{p-1}{2}u_{uu_t} + \frac{p-1}{2}u_{t}u_t\right)$$
so that
\[ R\dot{\theta}_{t}(|u|^{p-1}u) = R(Hu_{t}), \quad H = \frac{1}{2} |u|^{p-3}((p+1)|u|^{2}\bar{w} + (p-1)\bar{u}w) \]
and
\[ I_{k} = R\int_{\Omega} \chi_{k}(t)[|u + w|^{p-1}(\bar{u} + \bar{w}) - |u|^{p-1}\bar{u} - H]u_{t}dxdt. \]
We have
\[
|u + w|^{p-1}(\bar{u} + \bar{w}) - |u|^{p-1}\bar{u} - H
= (|u + w|^{p-1} - |u|^{p-1})(\bar{u} + \bar{w}) - \frac{p-1}{2} |u|^{p-3}(|u|^{2}w + \bar{u}w)
= \frac{p-1}{2} \int_{0}^{1} (|u + \sigma w|^{p-3} - |u|^{p-3})(|u|^{2}\bar{w} + \bar{u}w)\sigma + R
\]
where
\[
R = \frac{p-1}{2} \int_{0}^{1} |u + \sigma w|^{p-3}[2\sigma|w|^{2}(\bar{u} + \bar{w}) + uw^{2} + \bar{u}|w|^{2}]d\sigma.
\]
We have easily (if \( p \geq 4 \))
\[
|R| \leq C(|u|^{p-1}_{\mathcal{L}^{\infty}(\mathcal{L}^{\infty})})(|w|^{2} + |w|^{p}),
\|
|u + \sigma w|^{p-3} - |u|^{p-3} \| \leq C(|u|^{p-1}_{\mathcal{L}^{\infty}(\mathcal{L}^{\infty})})(|w|^{2} + |w|^{p-3})
\]
and summing up
\[
|I_{k}| \leq C(|u|^{p-1}_{\mathcal{L}^{\infty}(\mathcal{L}^{\infty})})\int_{\Omega} \chi_{k}(|w|^{2} + |w|^{p+1}).
\]

Letting \( k \to \infty \) we deduce
\[
|B(t) - B(0)| \leq C \int_{0}^{t} \left[ E(w(s)) + \|w(s)\|_{\mathcal{L}^{2}(\Omega)}^{2} \right] ds, \quad C = C(|u|_{\mathcal{L}^{\infty}(\mathcal{L}^{\infty})})
\]
and using (6.3) we have
\[
B(0) - B(t) \leq C \int_{0}^{t} E(w(s))ds + C\|w(0)\|_{\mathcal{L}^{2}}^{2}.
\]
Recalling (6.4) and (6.5) we obtain
\[
E(w(t)) \leq C \int_{0}^{t} E(w(s))ds + CE(w(0)) + C\|w(0)\|_{\mathcal{L}^{2}}^{2}
\]
with \( C = C(|u|_{\mathcal{L}^{\infty}(\mathcal{L}^{\infty})}) \) as usual, and by Gronwall’s Lemma we conclude the proof.

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