Static Output Feedback Control for Nonlinear Systems subject to Parametric and Nonlinear Uncertainties

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Abstract

This work addresses the design of static output feedback control of discrete-time nonlinear systems satisfying a local Lipschitz continuity condition with time-varying uncertainties. The controller has also a guaranteed disturbance attenuation level ($H_\infty$ performance). Thanks to the linearity of the proposed LMIs in both the admissible Lipschitz constant of the system and the disturbance attenuation level, they can be simultaneously optimized through convex multiobjective optimization. The optimization over Lipschitz constant adds an extra important and new feature to the controller, robustness against nonlinear uncertainty. The resulting controller is robust against both nonlinear additive uncertainty and time-varying parametric uncertainties. Explicit norm-wise and element-wise bounds on the tolerable nonlinear uncertainty are derived.

1 Introduction

The static output feedback (SOF) stabilization problem is known to be a challenging task and in spite of receiving great attention it is still one of the most important open questions in the control theory. Over the past decades most advances in the control theory have been limited to the state feedback control and dynamic output feedback control. However, state feedback techniques, require either the measurement of every system state some of which expensive or even impossible to be measured or using the observer based controllers which makes the implementation task expensive and hard. On the other hand, dynamic output feedback designs, result in high order controllers which may not be desirable in many industrial applications again due to implementation difficulties. Controllers using static output feedback are less expensive to implement and are more reliable. Therefore, many researchers have tried to characterize the problem of finding a stabilizing SOF controller [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. A comprehensive survey on static output feedback is given in [13].

Despite abundant literature, due to the non-convexity of the SOF formulation, the problem is still open both analytically and numerically. While the necessary and sufficient conditions of the existence of a stabilizing SOF controller based on the original non-convex formulation are not numerically tractable, the existing convex formulations often lead to restrictive sufficient conditions. The proposed results include methods based on structural pole assignment [14, 15], Riccati-based approaches [16, 17] and optimization formulation (min-max problem) [18].

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The original non-convex problem can be directly formulated using bilinear matrix inequalities (BMIs). However, the numerical solution of BMI problems has been shown to be NP-hard [19]. Due to recent advances in linear matrix inequalities both theoretically and numerically, several works have been recently addressed in the literature attempting to cast the SOF problem into the convex LMI framework. Available methods include using ILMIs (iterative LMIs) [20, 21] where there is no guarantee for the convergence of iterations or imposing nonsingularity conditions over the state space realization matrices [22] or a submatrix of “A” [23].

Alternatively, it has been shown that in some cases the BMI problem can be converted into a semidefinite cone programming problem (SDP) [24]. The advantage gained through this conversion is that reliable algorithms exist to solve SDP problems numerically. In this work, we propose a novel method for robust static output feedback stabilization of a class of discrete-time nonlinear uncertain systems. The method proposed in this paper is non-iterative and not only provides a less restrictive solution but also extends the results to a more general class of systems where there are parametric uncertainties and Lipschitz nonlinearity in the model. In addition the proposed controller has a guaranteed disturbance attenuation level ($H_\infty$ performance). Our goal is to develop linear matrix inequalities in which the Lipschitz constant is one the LMI variables in order to achieve the maximum admissible Lipschitz constant through convex optimization. This optimization adds an important extra feature to the SOF controller making it robust against nonlinear uncertainties. Explicit bounds on the tolerable nonlinear uncertainty are derived through norm-wise and element-wise robustness analysis. Actually, thanks to the linearity of the proposed LMIs in both the admissible Lipschitz constant of the system and the disturbance attenuation level, they can be simultaneously optimized through convex multiobjective optimization. In the next stage, we show that the proposed solution can be modified as an SDP problem. In fact original BMI problem is converted into an SDP problem. The paper is organized as follows. In section II, the problem statement and some preliminaries are mentioned. In Section III, we propose a new method for robust $H_\infty$ SOF controller design for nonlinear uncertain systems. Section IV, is devoted to robustness analysis in which explicit bounds on the tolerable nonlinear uncertainty are derived. Section V, contains an illustrative example showing the high performance of our proposed method.

2 Preliminaries and Problem Statement

Consider the following class of nonlinear discrete-time uncertain systems:

\[
\begin{align*}
(\sum)^1 : \quad & x(k+1) = (A + \Delta A(k))x(k) + \Phi(x, u) + B_1u(k) + B_2w(k) \\
& y(k) = (C + \Delta C(k))x(k) + Dw(k) \\
& z(k) = Hx(k)
\end{align*}
\]

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$ and $\Phi(x, u)$ contains nonlinearities of second order or higher. We assume that the system $(\sum)^1$ is locally Lipschitz with respect to $x$ in a region $D$ containing the origin, uniformly in $u$, i.e.:

\[
\|\Phi(0, u^*)\| = 0
\]
\[
\|\Phi(x_1, u^*) - \Phi(x_2, u^*)\| \leq \gamma \|x_1 - x_2\| \quad \forall x(k) \in D
\]

where $\|\cdot\|$ is the induced 2-norm, $u^*$ is any admissible control signal and $\gamma > 0$ is called the Lipschitz constant. If the nonlinear function $\Phi$ satisfies the Lipschitz continuity condition
globally in $\mathbb{R}^n$, then the results will be valid globally. We assume that $\text{rank}(B_1) = m < n$ and $\text{rank}(C) = p < n$. $w(k) \in \ell_2[0, \infty)$ is an unknown exogenous disturbance and $\Delta A(k)$ and $\Delta C(k)$ are unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the form

$$
\begin{bmatrix}
\Delta A(k) \\
\Delta C(k)
\end{bmatrix} =
\begin{bmatrix}
M_1 \\
M_2
\end{bmatrix} F(k) N
$$

(6)

where $M_1$, $M_2$ and $N$ are known real constant matrices and $F(k)$ is an unknown real-valued time-varying matrix satisfying

$$
\forall k, \quad F^T(k) F(k) \leq I.
$$

It is also worth noting that the structure of parameter uncertainties in (6) has been widely used in the problems of robust control and robust filtering for both continuous-time and discrete-time systems and can capture the uncertainty in several practical situations. Suppose $z(k) = H x(k)$ stands for the controlled output for state where $H$ is a known matrix. Our purpose is to design the static gain $K$ such that the static output feedback control law $u(k) = Ky(k)$ robustly asymptotically stabilizes the system with maximum admissible Lipschitz constant in the presence of uncertainty and the following specified $H_\infty$ norm upper bound is simultaneously guaranteed.

$$
\|z\| \leq \mu \|w\|.
$$

(7)

In the following, we mention some useful lemmas that will be used later in the proof of our results.

**Lemma 1.** [26] For any $x, y \in \mathbb{R}^n$ and any positive definite matrix $P \in \mathbb{R}^{n \times n}$, we have

$$
2x^T y \leq x^T P x + y^T P^{-1} y.
$$

**Lemma 2.** [26] Let $A, D, E, F$ and $P$ be real matrices of appropriate dimensions with $P > 0$ and $F$ satisfying $F^T F \leq I$. Then for any scalar $\epsilon > 0$ satisfying $P^{-1} - \epsilon^{-1} DD^T > 0$, we have

$$
(A + DFE)^T P(A + DFE) \leq A^T (P^{-1} - \epsilon^{-1} DD^T)^{-1} A + \epsilon E^T E.
$$

### 2.1 Notation

The following matrix notation will be used throughout the paper. For matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, $A \preceq B$ means $a_{ij} \leq b_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$. For square $A$, $\text{diag}(A)$ is a vector containing the elements on the main diagonal and $\text{diag}(x)$ where $x$ is a vector is a diagonal matrix with the elements of $x$ on the main diagonal. $|A|$ is the element-wise absolute value of $A$, i.e. $[|a_{ij}|]_n$. $A \circ B$ stands for the element-wise product (Hadamard product) of $A$ and $B$. $A \otimes B$ denotes the Kronecker (tensor) product of $A = [a_{ij}]_{m \times n}$ and $B_{p \times q}$ as $A \otimes B = [a_{ij} B]_{mp \times nq}$. $\text{vec}(A)$ is the vector obtained by stacking up the columns of the matrix $A$.

### 3 Static Output Feedback Stabilization

In this section we propose a novel method for robust output feedback stabilization for nonlinear discrete-time uncertain systems. Consider the nonlinear uncertain system of class $(\sum_1)$. The LMI approach for the static output feedback stabilization problem is still unsolved due to its
bilinear nature. Available methods include using ILMIs (iterative LMIs)\cite{20},\cite{21} where there is no guarantee for the convergence of iterations or imposing nonsingularity conditions over either $B_1, C, A$\cite{22} or a submatrix of $A$\cite{23}. The following theorem not only provides a new solution for the problem but also extends the result to the case where there exist time-varying uncertainties in the pair $(A, C)$ and Lipschitz nonlinearity in the state dynamics. As mentioned earlier, our goal is to design a robust stabilizing controller with the $H_{\infty}$ performance $\|z\| \leq \mu \|w\|$, for systems of class $(\Sigma_1)$, using static output feedback.

We first prove a lemma about robust asymptotic stability in the presence of exogenous disturbance.

**Lemma 3.** Consider the following nonlinear uncertain system

$$(\Sigma_2) : x(k+1) = (A + \Delta A(k))x(k) + \Phi(x, u) + Bw$$

$$z(k) = Hx(k).$$

This system is asymptotically stable with $\|z\| \leq \mu \|w\|$ and maximum admissible Lipschitz constant $\gamma^*$, if there exist scalars $\alpha > 0$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$ and a matrix $P > 0$ such that the following LMI optimization problem has a solution:

$$\min(\alpha + \epsilon_1)$$

s.t.

$$\begin{bmatrix}
\Lambda_1 & I & A^T P & 0 & 0 \\
* & -\alpha I & 0 & 0 & 0 \\
* & * & -\frac{1}{2} P & P & PM_1 \\
* & * & * & P - 2\epsilon_1 I & 0 \\
* & * & * & * & -\epsilon_2 I
\end{bmatrix} < 0 \quad (10)$$

$$\begin{bmatrix}
-\mu^2 I & B^T P & B^T P \\
* & -\frac{1}{2} P & 0 \\
* & * & -I
\end{bmatrix} < 0 \quad (11)$$

where $\Lambda_1 = H^T H - P + \epsilon_2 N^T N$. Once the problem is solved

$$\alpha^* \triangleq \min(\alpha)$$

$$\epsilon_1^* \triangleq \min(\epsilon_1)$$

$$\gamma^* \triangleq \max(\gamma) = \frac{1}{\sqrt{\alpha^*(1 + \epsilon_1^*)}}$$

**Proof:** Consider the Lyapunov function candidate:

$$V = x^T P x.$$ 

We have

$$\Delta V_k = x^T (A + \Delta A)^T P (A + \Delta A) x$$

$$+ 2x^T (A + \Delta A)^T P \Phi + \Phi^T P \Phi - x^T P x + 2w^T B^T P \Phi$$

$$+ 2w^T B^T P (A + \Delta A) x + w^T B^T P B w.$$ 

(13)
\[ (A + \Delta A)^T P(A + \Delta A) = (A + \Delta A)^T PP^{-1}P(A + \Delta A) \]
\[ = (PA + PM_1 FN)^T P^{-1}(PA + PM_1 FN). \]  

(14)

Based on Lemma 1 and (3)-(5),
\[ 2x^T(A + \Delta A)^TP\Phi + \Phi^T P\Phi = 2x^T(A + \Delta A)^T P\Phi \]
\[ - \Phi^T W\Phi + \epsilon_1 \Phi^T \Phi \leq x^T(A + \Delta A)^T PW^{-1}P(A + \Delta A)x \]
\[ + \epsilon_1 \gamma^2 x^Tx \]

where \( W \triangleq \epsilon I - P \). It can be concluded from (10) that \( \frac{1}{2}P - (2\epsilon_1 I - P)^{-1}P^2 - \epsilon_2^{-1}PM_1M_1^T P > 0 \) and \( 2\epsilon_1 I - P > 0 \). Therefore, since \( P > 0 \), the condition \( W > 0 \) is already included in (10).

Thus,
\[ x^T(A + \Delta A)^T P(A + \Delta A)x + 2x^T(A + \Delta A)^T P\Phi \]
\[ + \Phi^T P\Phi - x^TPx \leq x^T[(PA + PM_1 FN)^T \cdots \]
\[ (P^{-1} + W^{-1})(PA + PM_1 FN) - P + \epsilon_1 \gamma^2 I]x. \]  

Using Lemma 1, it can be written
\[ 2w^TB^TP(A + \Delta A)x + 2w^TB^TP\Phi + w^TB^TPw \]
\[ \leq 2w^TB^TP(A + \Delta A)x + w^TB^TPPw + w^TB^TPw \]
\[ + \Phi^T \Phi \leq 2w^TB^TP(A + \Delta A)x + w^TB^TPPw \]
\[ + w^TB^TPw + \gamma^2 x^Tx \leq x^T(A + \Delta A)^T P(A + \Delta A)x \]
\[ + w^T[B^TP + B^TPP + B^TPBw + \gamma^2 x^T] \]
\[ = x^T[(A + \Delta A)^T PP^{-1}P(A + \Delta A) + \gamma^2 x^T] \]
\[ + w^T[2B^TPP^{-1}PB + B^TPP]w. \]  

Based on Lemma 1,
\[ 2x^Tpy \leq x^TPx + y^TPy \]  
\[ 2x^TP^{-1}y \leq x^TP^{-1}x + y^TPy \]  

from the (18) and since \( P < I \),
\[ 2w^TB^TP(A + \Delta A)x + 2w^TB^TP\Phi + w^TB^TPw \]
\[ \leq 2w^TB^TP(A + \Delta A)x + w^TB^TPPw + w^TB^TPw + \Phi^T \Phi \]
\[ \leq 2w^TB^TP(A + \Delta A)x + w^TB^TPPw + w^TB^TPw + \gamma^2 x^Tx \]
\[ \leq x^T(A + \Delta A)^T PP(A + \Delta A)x + w^T[B^TPP^{-1}PB + B^TPP + B^TPB]w + \gamma^2 x^Tx \]
\[ \leq x^T(A + \Delta A)^T PP(A + \Delta A)x + 3w^TB^TPP^{-1}PBw + \gamma^2 x^Tx. \]  

Substituting (15) and (19) into (13) yields to
\[ \Delta V_k \leq x^T[(PA + PM_1 FN)^T (2P^{-1} + W^{-1}) \cdots \]
\[ (PA + PM_1 FN) - P + (1 + \epsilon_1)\gamma^2 I]x + 3w^TB^TPP^{-1}PBw. \]

Now, define
\[ J \triangleq \sum_{k=0}^{\infty} [z(k)^T z(k) - \mu^2 w(k)^T w(k)] \]
so,
\[ J < \sum_{k=0}^{\infty} [z(k)^T z(k) - \mu^2 w(k)^T w(k) + \Delta V] \]
thus, a sufficient condition for \( J \leq 0 \) is that
\[
x^T [(PA + PM_1FN)^T (2P^{-1} + W^{-1})(PA \cdots + PM_1FN) + HT H - P + (1 + \epsilon_1)\gamma^2 I]x
+ w^T [3B^T PP^{-1} PB - \mu^2 I] w \leq 0. \tag{20}
\]
Defining the new variable
\[
\alpha \triangleq \frac{1}{(1 + \epsilon_1)\gamma^2} \Rightarrow \gamma = \frac{1}{\sqrt{\alpha(1 + \epsilon_1)}},
\tag{21}
\]
maximization of \( \gamma \) is equivalent to the simultaneous minimization of \( \alpha \) and \( \epsilon_1 \). Combining the two objective functions, we minimize the scalarized linear objective function \( \alpha + \epsilon_1 \). On the other hand,
\[
2P^{-1} + (\epsilon_1 I - P)^{-1} = P^{-1} + P^{-1} + (\epsilon_1 I - P)^{-1}
= P^{-1} + (\epsilon_1 I - P)^{-1}[(\epsilon_1 I - P) P^{-1} + I]
= P^{-1} + \epsilon_1 (\epsilon_1 I - P)^{-1} P^{-1}
= P^{-1} + (P - \epsilon_1^{-1} P^2)^{-1}
= (P - \epsilon_1^{-1} P^2)^{-1}[(P - \epsilon_1^{-1} P^2) P^{-1} + I]
= (P - \epsilon_1^{-1} P^2)^{-1}(2I - \epsilon_1^{-1} P)
\Rightarrow
\[
[2P^{-1} + (\epsilon_1 I - P)^{-1}]^{-1} = (2I - \epsilon_1^{-1} P)^{-1}(P - \epsilon_1^{-1} P^2)
= (2I - \epsilon_1^{-1} P)^{-1} P - (2\epsilon_1 I - P)^{-1} P^2
> \frac{1}{2} P - (2\epsilon_1 I - P)^{-1} P^2 \Rightarrow
2P^{-1} + (\epsilon_1 I - P)^{-1} < [\frac{1}{2} P - (2\epsilon_1 I - P)^{-1} P^2]^{-1}
\]
So, according to (20), a sufficient condition for \( J \leq 0 \) is that
\[
x^T \{ (PA + PM_1FN)^T \frac{1}{2} P - (2\epsilon_1 I - P)^{-1} P^2 \}^{-1}(PA \cdots + PM_1FN) + HT H - P + \alpha^{-1} I]x + w^T [2B^T PP^{-1} PB \cdots + B^T PPB - \mu^2 I] w < 0.
\]
According to Lemma 2,
\[
x^T \{ A^T P \frac{1}{2} P - (2\epsilon_1 I - P)^{-1} P^2 - \epsilon_2^{-1} PM_1 M_1^T P \}^{-1} PA
+ \epsilon_2 N^T N + HT H - P + \alpha^{-1} I]x
+ w^T [2B^T PP^{-1} PB + B^T PPB - \mu^2 I] w < 0
\]
which is by Schur complements equivalent to LMIs (10) and (11).
Remark 1. Lemma 3, provides a tool for robust stability analysis of the aforementioned class of nonlinear systems. Maximization of $\gamma$ guarantees the stability of the system for any Lipschitz nonlinear function with Lipschitz constant less than or equal $\gamma^*$. It is clear that if the stability of a systems with a given fixed Lipschitz constant is to be analyzed, the proposed LMI optimization problem will reduce to an LMI feasibility problem and there will be no need to the change of variable \[21\] anymore.

Remark 2. The proposed LMIs are linear in $\alpha$, $\epsilon_1$ and $\zeta(=\mu^2)$. Thus, either can be a fixed constant or an optimization variable.

3.1 Non-iterative Strict LMI solution

The static output feedback problem is bilinear by its nature. On the other hand, it has been shown that the solution of the bilinear matrix inequalities is NP-hard. Here we propose a non-interactive strict LMI solution to the problem which can be solved efficiently using the available software.

Theorem 1. Consider a nonlinear uncertain system of class (\[\Sigma_1\]). The output feedback $u = Ky$ robustly asymptotically stabilizes this system with $\|z\| \leq \mu \|w\|$ and maximum admissible Lipschitz constant $\gamma^*$, if there exist scalars $\epsilon_1 > 0$, $\epsilon_1 > 0$ and $\alpha > 0$ and matrices $P > 0$ and $G$ such that the following LMI optimization problem has a solution:

$$
\min(\alpha + \epsilon_1)
$$

s.t.

$$
\begin{bmatrix}
\Lambda_1 & I & \Lambda_2 & 0 & 0 \\
* & -\alpha I & 0 & 0 & 0 \\
* & * & -\frac{1}{2}P & P & PM_1 + GM_2 \\
* & * & * & P - 2\epsilon_1 I & 0 \\
* & * & * & * & -\epsilon_2 I
\end{bmatrix} < 0
$$

(22)

$$
\begin{bmatrix}
-\mu^2 I & B_2^T P + D^T G^T & B_2^T P + D^T G^T \\
* & -\frac{1}{2}P & 0 \\
* & * & -I
\end{bmatrix} < 0
$$

(23)

where $\Lambda_1 = H^T H - P + \epsilon_2 N^T N$ and $\Lambda_2 = A^T P + C^T G^T$. Once the problem is solved

$$
\alpha^* \triangleq \min(\alpha) \\
\epsilon_1^* \triangleq \min(\epsilon_1) \\
\gamma^* \triangleq \max(\gamma) = \frac{1}{\sqrt{\alpha^*(1 + \epsilon_1^*)}}
$$

$$
K = B_1^T \overline{K}
$$

where the matrix $\overline{K}$ is obtained through \[21\] if and only if

$$
\text{rank}(I_p \otimes PB_1 B_1^T) = \text{rank}([ I_p \otimes PB_1 B_1^T \ \text{vec}(G) ]).
$$

Proof: Substituting $u = Ky$ into \[1\] yields to uncertain system

$$
x(k + 1) = (\bar{A} + \Delta \bar{A}(k))x(k) + \Phi(x, u) + \bar{B}w \\
z(k) = Hx(k)
$$

(24)
where
\[ \tilde{A} = A + B_1KC \]
\[ \Delta \tilde{A} = \Delta A + B_1K\Delta C = (M_1 + B_1KM_2)FN = \tilde{M}FN \]
\[ \tilde{B} = B_1KD + B_2. \]

System (24) is of the form \( \sum \) so Lemma 3 provides a sufficient condition for its robust asymptotic stability. We have
\[ P\tilde{A} = PA + PB_1KC \]
\[ PM = PM_1 + PB_1KM_2 \]
\[ P\tilde{B} = PB_1KD + PB_2. \]

In order to cast the solution into the form of LMIs, we introduce the following change of variables:
\[ PB_1K \triangleq G. \]  
(25)

Substituting into (10) and (11) of lemma 3, the LMIs (22) and (23) are obtained. Without loss of generality, we assume the feedback gain \( K \) to be of the form
\[ K = B_1^T\overline{K} \]  
(26)

where \( \overline{K} \in \mathbb{R}^{n \times p} \) is an unknown matrix to be found. The algebraic matrix equation (25) can be solved for \( \overline{K} \) using the Kronecker product as follows
\[ PB_1B_1^T\overline{K} = G \Rightarrow (I_p \otimes PB_1B_1^T)\text{vec}(\overline{K}) = \text{vec}(G). \]  
(27)

Therefore, there is a solution for \( \text{vec}(\overline{K}) \) and consequently for \( K \) if and only if
\[ \text{rank}(I_p \otimes PB_1B_1^T) = \text{rank}([I_p \otimes PB_1B_1^T \ \text{vec}(G)]) \]  
(28)

which simply means that the subspace spanned by the columns of \( I_p \otimes PB_1B_1^T \) must contain \( \text{vec}(G) \). Eventually \( K \) is obtained from (26). \( \blacksquare \)

As mentioned in Remark 2, either the admissible Lipschitz constant or the disturbance attenuation level can be considered as an optimization variable in Theorem 1. Given this, it may be more realistic to have a combined performance index. Then, \( \alpha + \epsilon_1 \) and \( \zeta = \mu^2 \) can be simultaneously optimized by convex multiobjective optimization. See [27, 28, 29, 30, 31, 32, 33, 34, 35] for details of the approach and application to filtering and different classes of nonlinear systems.

Theorem 1 shows that an exact solution for \( K \) can be found using strict LMIs if and only if the condition (28) is satisfied. Relaxing this condition, in many cases it is still possible to find an approximate stabilizing solution for \( K \) using strict LMIs as discussed in [36]. In the following we derive another formulation for the exact solution of SOF.

### 3.2 Converting BMI into SDP

In the first part of this section, we showed that an exact solution for \( K \) can be found using strict LMIs if and only if the condition (28) is satisfied. Now, we propose another exact solution by
Converting the BMI (Bilinear Matrix Inequality) problem into an SDP (Semidefinite Programming) problem. This adds an equality constraint to the optimization problem of Theorem 1.

**Corollary 1.** Consider a nonlinear uncertain system of class \((\Sigma_1)\). The output feedback \(u = Ky\) robustly asymptotically stabilizes this system with \(\|z\| \leq \mu \|w\|\) and maximum admissible Lipschitz constant \(\gamma^*\), if there exist scalars \(\epsilon_1 > 0, \epsilon_1 > 0\) and \(\alpha > 0\) and matrices \(P > 0, Q\) and \(G\) such that the following SDP problem has a solution:

\[
\min(\alpha + \epsilon_1)
\]

s.t.

\[
P B_1 = B_1 Q 
\]

\[
\Pi_1 \triangleq \begin{bmatrix}
I & I - Q \\
* & I
\end{bmatrix} > 0 
\]

\[
\Pi_2 \triangleq \begin{bmatrix}
-\mu^2 I & A_4 & A_4 \\
* & -\frac{1}{2} P & 0 \\
* & * & -I
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
A_1 & I & A_3 & 0 & 0 \\
* & -\alpha I & 0 & 0 & 0 \\
* & * & -\frac{1}{2} P & P & PM_1 + B_1 G M_2 \\
* & * & * & P - 2\epsilon_1 I & 0 \\
* & * & * & * & -\epsilon_2 I
\end{bmatrix} < 0
\]

where \(A_1\) is as in Theorem 1, \(A_3 = A^T P + C^T G^T B_1^T\) and \(A_4 = B_2^T P + D^T G^T B_1^T\). Once the problem is solved

\[
\alpha^* \triangleq \min(\alpha)
\]

\[
\epsilon_1^* \triangleq \min(\epsilon_1)
\]

\[
\gamma^* \triangleq \max(\gamma) = \frac{1}{\sqrt{\alpha^*(1 + \epsilon_1)}}
\]

\[K = Q^{-1} G\]

**Proof:** Using the change of variable \(G = QK\), we have \(P B_1 K = B_1 Q K \triangleq B_1 G\). Having \(G\), there exists a unique exact solution for \(K\) if and only if \(Q\) is nonsingular. The LMI [33] guarantees the nonsingularity of \(Q\) [37, p. 312]. The rest of the proof is the same as the proof of Theorem 1. ■

Note that \(Q_{m \times m}\) only needs to be nonsingular and is not necessarily positive definite or even symmetric. The SDP problem of Corollary 2 can be solved by freely available packages such as YALMIP [38].

**Remark 4.** Suppose that instead of (2), the output map in \((\Sigma_3)\) is as

\[
y(k) = (C + \Delta C(k)) x(k) + D_1 u + D_2 w.
\]

Defining the fictitious output \(\overline{y} = y - D_1 u = (C + \Delta C(k)) x(k) + D_2 w\), we first find the static output feedback \(u = K \overline{y}\). Then we have \(u = (I + KD_1)^{-1} K y\) provided that the inverse exists.
4 Robustness Against Nonlinear Uncertainty

As mentioned earlier, the maximization of Lipschitz constant makes the proposed observer robust against some Lipschitz nonlinear uncertainty. In this section this robustness feature is studied and both norm-wise and element-wise bounds on the nonlinear uncertainty are derived. The norm-wise analysis provides an upper bound on the Lipschitz constant of the nonlinear uncertainty and the norm of the Jacobian matrix of the corresponding nonlinear function. Furthermore, we will find upper and lower bounds on the elements of the matrix-type Lipschitz constant of the nonlinear uncertainty through a novel element-wise analysis.

4.1 Norm-Wise Robustness

Assume a nonlinear uncertainty as follows

\[ \Phi_\Delta(x, u) = \Phi(x, u) + \Delta \Phi(x, u) \]  
\[ x(k + 1) = (A + \Delta A)x(k) + \Phi_\Delta(x, u) \]

where \( \Phi_\Delta \) is the uncertain nonlinear function and \( \Delta \Phi \) is the unknown nonlinear uncertainty. Suppose that

\[ \|\Delta \Phi(x_1, u) - \Delta \Phi(x_2, u)\| \leq \Delta \gamma \|x_1 - x_2\|. \]  

Proposition 1. Suppose that the actual Lipschitz constant of the system is \( \gamma \) and the maximum admissible Lipschitz constant achieved by Corollary 1 (Theorem 1), is \( \gamma^* \). Then, the observer designed based on Corollary 1 (Theorem 1), can tolerate any additive Lipschitz nonlinear uncertainty with Lipschitz constant less than or equal \( \gamma^* - \gamma \).

Proof: Based on Schwartz inequality, we have

\[ \|\Phi_\Delta(x_1, u) - \Phi_\Delta(x_2, u)\| \leq \|\Phi(x_1, u) - \Phi(x_2, u)\| \]
\[ + \|\Delta \Phi(x_1, u) - \Delta \Phi(x_2, u)\| \leq \gamma \|x_1 - x_2\| \]
\[ + \Delta \gamma \|x_1 - x_2\|. \]

According to the Corollary 1 (Theorem 1), \( \Phi_\Delta(x, u) \) can be any Lipschitz nonlinear function with Lipschitz constant less than or equal to \( \gamma^* \),

\[ \|\Phi_\Delta(x_1, u) - \Phi_\Delta(x_2, u)\| \leq \gamma^* \|x_1 - x_2\|, \]

so, there must be

\[ \gamma + \Delta \gamma \leq \gamma^* \rightarrow \Delta \gamma \leq \gamma^* - \gamma. \]

In addition, we know that for any continuously differentiable function \( \Delta \Phi \), \( \forall x, x_1, x_2 \in D \)

\[ \|\Delta \Phi(x_1, u) - \Phi_\Delta(x_2, u)\| \leq \|\frac{\partial \Delta \Phi}{\partial x}(x_1 - x_2)\|, \]

where \( \frac{\partial \Delta \Phi}{\partial x} \) is the Jacobian matrix \([39]\). So \( \Delta \Phi(x, u) \) can be any additive uncertainty with \( \|\frac{\partial \Delta \Phi}{\partial x}\| \leq \gamma^* - \gamma \). In the following we will find appropriate bound on the entries of the Jacobian matrix.
4.2 Element-Wise Robustness

Assume that there exists a matrix $\Gamma \in \mathbb{R}^{n \times n}$ such that

$$\|\Phi(x_1, u) - \Phi(x_2, u)\| \leq \|\Gamma(x_1 - x_2)\|. \quad (36)$$

$\Gamma$ can be considered as a matrix-type Lipschitz constant. Suppose that the nonlinear uncertainty is as in (34) and

$$\|\Phi_\Delta(x_1, u) - \Phi_\Delta(x_2, u)\| \leq \|\Gamma_\Delta(x_1 - x_2)\|. \quad (37)$$

Assuming

$$\|\Delta\Phi(x_1, u) - \Delta\Phi(x_2, u)\| \leq \|\Delta\Gamma(x_1 - x_2)\|, \quad (38)$$

based on the proposition 1, $\Delta\Gamma$ can be any matrix with $\|\Delta\Gamma\| \leq \gamma^* - \|\Gamma\|$. Now, we look at the problem from a different angle. It is clear that $\Gamma_\Delta = [\gamma_\Delta]_n$ is a perturbed version of $\Gamma$ due to $\Delta\Phi(x, u)$. The question is that how much perturbation can be tolerated on the elements of $\Gamma$ without losing the observer features stated in Corollary 1 (Theorem 1). In the following we will find appropriate bound on the entries of the Jacobian matrix, but how about the entries of $\Delta\Gamma$? This is important in the sense that it gives us an insight about the amount of uncertainty that can be tolerated in different directions of the nonlinear function. Here, we propose an approach to optimize the elements of $\Gamma$ and provide specific upper and lower bounds on tolerable perturbations.

**Corollary 2.** Consider Lipschitz nonlinear system $\sum_1$ satisfying (36), along with the control law $u = Ky$. The closed loop system is (globally) asymptotically stable with the matrix-type Lipschitz constant $\Gamma^* = [\gamma^*_ij]_n$ with maximized admissible elements and $\Sigma_2(w \to z)$ gain, $\mu$, if there exist fixed scalars $\mu > 0$ and $c_{ij} > 0 \ \forall \ 1 \leq i, j \leq n$, scalars $\omega > 0$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$ and matrices $A = [\alpha_{ij}]_n > 0$, $P_1 > 0$, $P_2 > 0$ and $G$, such that the following LMI optimization problem has a solution.

$$\begin{align*}
\min & (\epsilon_1 - \omega) \\
\text{s.t.} & \quad c_{ij}\alpha_{ij} > \omega \ \forall \ 1 \leq i, j \leq n \\
& \quad PB_1 = B_1Q \\
& \quad \Pi_1 > 0 \\
& \quad \Pi_2 < 0 \\
& \quad \begin{bmatrix}
L_1 & A & A_3 & 0 & 0 \\
* & -I & 0 & 0 & 0 \\
* & * & -\frac{1}{2}P & P & PM_1 + B_1GM_2 \\
* & * & * & P - 2\epsilon_1I & 0 \\
* & * & * & * & -\epsilon_2I
\end{bmatrix} < 0
\end{align*}$$

where $\Pi_1$, $\Pi_2$, $L_1$ and $A_3$ are as in Corollary 1. Once the problem is solved

$$\begin{align*}
L &= P_1^{-1}G \\
\alpha^*_{ij} &\triangleq \max(\alpha_{ij}) \\
A^* &\triangleq [\alpha^*_{ij}]_n \\
\epsilon^*_1 &\triangleq \min(\epsilon_1) \\
\Gamma^* &\triangleq \frac{1}{\sqrt{1 + \epsilon^*_1}} A^*
\end{align*}$$
Proof: The proof is similar to the proof of Corollary 1, replacing $\gamma I$ with $\Gamma$ and using the change of variables $(1 + \epsilon_1)\Gamma^T \Gamma = A^T A$.

Remark 3. By appropriate selection of the weights $c_{i,j}$, it is possible to put more emphasis on the directions in which the tolerance against nonlinear uncertainty is more important. To this goal, one can take advantage of the knowledge about the structure of the nonlinear function $\Phi(x, u)$.

Note that the robustness results of this section can be similarly applied to Theorem 1, as well. According to the norm-wise analysis, it is clear that $\Delta \Gamma$ in (38) can be any matrix with $\|\Delta \Gamma\| \leq \|\Gamma^*\| - \|\Gamma\|$. We now proceed by deriving bounds on the elements of $\Gamma_{\Delta}$. To prove our result we recall the following lemma from [27].

**Lemma 4.** [27] For any $S = [s_{i,j}]_n$ and $T = [t_{i,j}]_n$, if $|S| \preceq T$, then $SS^T \leq TT^T \circ nI$.

Now we are ready to state the element-wise robustness result. Assume additive uncertainty in the form of (34), where $\|\Phi_{\Delta}(x_1, u) - \Phi_{\Delta}(x_2, u)\| \leq \|\Gamma_{\Delta}(x_1 - x_2)\|$. (39)

It is clear that $\Gamma_{\Delta} = [\gamma_{\Delta i,j}]_n$ is a perturbed version of $\Gamma$.

**Proposition 2.** Suppose that the actual matrix-type Lipschitz constant of the system is $\Gamma$ and the maximized admissible matrix-type Lipschitz constant achieved by Corollary 1 (Theorem 1), is $\Gamma^*$. Then, $\Delta \Phi$ can be any additive nonlinear uncertainty such that $|\Gamma_{\Delta}| \preceq n^{-\frac{1}{2}} \Gamma^*$.

Proof: According to the Proposition 2, it suffices to show that $\sigma_{\max}(\Gamma_{\Delta}) \leq \sigma_{\max}(\Gamma^*)$. Using Lemma 4, we have

$$
\sigma_{\max}^2(\Gamma_{\Delta}) = \lambda_{\max}(\Gamma_{\Delta} \Gamma_{\Delta}^T) \\
\leq \lambda_{\max}(n^{-1} \Gamma^* \Gamma^*^T \circ nI) \\
= \sigma_{\max}(\Gamma^* \Gamma^*^T \circ I) \\
\leq \sigma_{\max}(\Gamma^* \Gamma^*^T) = \sigma_{\max}^2(\Gamma^*)
$$

The first inequality follows from lemma 4 and the symmetry of $\Gamma_{\Delta} \Gamma_{\Delta}^T$ and $\text{diag}(\text{diag}(\Gamma^* \Gamma^*^T))$, [37] p. 200. The last inequality is due to the fact that the spectral norm is submultiplicative with respect to the Hadamard product [40] Ch. 5. Since the singular values are nonnegative, we can conclude that $\sigma_{\max}(\Gamma_{\Delta}) \leq \sigma_{\max}(\Gamma^*)$. ■

Therefore, denoting the element of $\Gamma_{\Delta}$ as $\gamma_{\Delta i,j} = \gamma_{i,j} + \delta_{i,j}$, the following bound on the element-wise perturbations is obtained

$$
-n^{-\frac{1}{2}} \gamma_{i,j}^* - \gamma_{i,j} \leq \delta_{i,j} \leq n^{-\frac{1}{2}} \gamma_{i,j}^* - \gamma_{i,j}. \quad (40)
$$

In addition, $\Delta \Phi(x, u)$ can be any continuously differentiable additive uncertainty which makes $|\frac{\partial \Phi_{\Delta}}{\partial x}| \leq n^{-\frac{1}{2}} \Gamma^*$.

It is worth mentioning that the results of Lemma 4 and Proposition 2 have intrinsic importance from the matrix analysis point of view regardless of our specific application in the robustness analysis.
5 Numerical Example

I this example, we design an static output feedback controller for an uncertain nonlinear system. Consider a system of class \( \Sigma \) where,

\[
A = \begin{bmatrix}
0.5000 & -0.5975 & 0.3735 & 0.0457 & 0.3575 \\
0.2500 & 0.3000 & 0.4017 & 0.1114 & 0.0227 \\
0.4880 & 0.1384 & 0.2500 & 0.7500 & 0.7500 \\
0.3838 & 0.0974 & 0.5000 & 0.2500 & 0.5000 \\
0.0347 & 0.1865 & -0.2500 & 0.5000 & 0.2500
\end{bmatrix},
\]

\[
\Phi(x, u) = \begin{bmatrix}
0.1 \sin(x_3) \\
0.2 \sin(x_4) \\
0.3 \sin(x_1) \\
0 \\
0.1 \sin(x_2)
\end{bmatrix},
C = \begin{bmatrix}
0.5 & 0 \\
0.2 & 0.2 \\
0 & 0.3 \\
1 & 1 \\
1 & 1
\end{bmatrix},
B_2 = \begin{bmatrix}
1 \\
1
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
0.7 & 0.8 & 0 \\
0.4 & 0.9 & 0.9 \\
0.9 & 0.9 & 0.2 \\
0.9 & 0.6 & 0.7 \\
0 & 0.5 & 0.3
\end{bmatrix},
M_1 = \begin{bmatrix}
0.1 & 0 \\
0.1 & 0.1 \\
0 & 0.1 \\
0 & 0.2
\end{bmatrix},
\]

\[
M_2 = \begin{bmatrix}
0 & 0.1 & 0.1 \\
0.1 & 0.2 & 0.2
\end{bmatrix},
N = \begin{bmatrix}
0.3 & 0.15 & 0.1 & 0 & 0.2 \\
0.1 & 0.2 & 0.1 & 0.2 & 0
\end{bmatrix}.
\]

The system is globally Lipschitz with Lipschitz constant 0.3. The nonlinear system is unstable. The matrix \( A \) is unstable, which means the linear part of the systems is itself unstable. The matrix \( A \) is singular, so the approach of [22] is not applicable even to the nominal linear part. The submatrix of \( A \) obtained by omitting the first \( p(= 2) \) rows and columns in singular, too. Therefore, we can not use the results of [23], neither. Now, we use Corollary 1 with \( H = 0.15I_5 \) and \( \mu = 2.5 \), to design \( K \). Using YALMIP, we solve the proposed SDP problem and we get:

\[
\alpha^* = 0.2076, \quad \gamma^* = 1.6584
\]

\[
K = \begin{bmatrix}
0.7026 & -0.3334 \\
0.4825 & -1.8664 \\
0.3292 & -1.1758
\end{bmatrix}.
\]

Figure 1 shows the state trajectories of the stabilized system.

6 Conclusions

A new LMI optimization approach to the robust static output feedback stabilization for nonlinear discrete-time uncertain is systems with \( H_\infty \) performance is proposed. The considered class of nonlinear systems contains norm-bounded time-varying model uncertainties as well as additive Lipschitz nonlinear model uncertainties. Explicit bounds on the tolerable uncertainty were derived via norm-wise and element-wise robustness analysis.

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Figure 1: The stabilized state trajectories

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