ON RADIO NUMBER OF STACKED-BOOK GRAPHS

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Abstract. A Stacked-book graph $G_{m,n}$ results from the Cartesian product of a star graph $S_m$ and path $P_n$, where $m$ and $n$ are the orders of $S_m$ and $P_n$ respectively. A radio labeling problem of a simple and connected graph, $G$, involves a non-negative integer function $f : V(G) \to \mathbb{Z}^+$ on the vertex set $V(G)$ of $G$, such that for all $u, v \in V(G)$, $|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$, where $\text{diam}(G)$ is the diameter of $G$ and $d(u, v)$ is the shortest distance between $u$ and $v$. Suppose that $f_{\text{min}}$ and $f_{\text{max}}$ are the respective least and largest values of $f$ on $V(G)$, then, $\text{span}_f$, the absolute difference of $f_{\text{min}}$ and $f_{\text{max}}$, is the span of $f$ while the radio number $rn(G)$ of $G$ is the least value of $\text{span}_f$ over all the possible radio labels on $V(G)$. In this paper, we obtain the radio number for the stacked-book graph $G_{m,n}$ where $m \geq 4$ and $n$ is even, and obtain bounds for $m = 3$ which improves existing upper and lower bounds for $G_{m,n}$ where $m = 3$.

1. Introduction

The graph $G$ considered in this paper is simple and undirected. The vertex and edge sets of $G$ are $V(G)$ and $E(G)$. For $e = uv \in E(G)$, $e$ connects two vertices $u$ and $v$ while $d(u, v)$ is the distance between $u, v$ and $\text{diam}(G)$ is the diameter of $G$. Radio number labeling problem, which is mostly applied in frequency assignment for signal transmission, where it mitigates the problems of signal interference. It was first suggested in 1980 by Hale\textsuperscript{6}.

Let $f$ be a non negative integer function on $V(G)$ such that the radio labeling condition, $|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$ is satisfied for every pair $u, v \in V(G)$. The span of $f$, $\text{span}_f$, is the difference between $f_{\text{min}}$ and $f_{\text{max}}$, the minimum and the maximum radio label on $G$ respectively. Thus the smallest possible value of $\text{span}_f$ is the radio number, $rn(G)$, of $G$. The radio labeling condition guarantees that every vertex on $G$ has unique radio label. Therefore, $rn(G) \geq |V(G)| - 1$ is trivially true. However, establishing the radio number of graphs has proved to be quite tedious. Even so, such numbers have been completely determined for some graphs. Liu and Zhu\textsuperscript{10} showed that for path, $P_n$, $n \geq 3$,

$$rn(P_n) = \begin{cases} 2k(k - 1) + 1 & \text{if } n = 2k; \\ 2k^2 + 2 & \text{if } n = 2k + 1. \end{cases}$$

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This improves results in [4] and [5] by Chatrand, et. al. where the upper and lower bounds for the same class of graph are obtained. Furthermore, Liu and Xie, [8], found the radio number for the square of a path, $P^2_n$ as:

$$rn(P^2_n) = \begin{cases} k^2 + 2 & \text{if } n \equiv 1(\text{mod } 4), n \geq 9; \\ k^2 + 1 & \text{if otherwise}. \end{cases}$$

Similar results are obtained in [9] for square of cycles. Jiang [7] completely solved the radio number problem for the grid graph $(P_m \square P_n)$, where for $m, n > 2$, it is noted that

$$rn(P_m \square P_n) = \frac{mn^2 + nm^2 - mn - m + 2}{2}, \text{ for } m \text{-odd and } n \text{ even.}$$

Saha and Panigrahi [12] and Ajayi and Adefokun [1] obtained results on the radio numbers of Cartesian products of two cycles (toroidal grid) and of path and star graph (stacked-book graph) respectively. In the case of stacked-book graph $G = S_n \square P_m$, $rn(G) \leq n^2 m + 1$, which the authors noted is not tight. Recent results on radio number include those on middle graph of path [2], trees, [3] and edge-joint graphs [11].

In this paper, for even positive integer $n$, we consider the stacked-book graph $G_{m,n}$ and derive the $rn(G_{m,n})$ for the case $m \geq 4$. Furthermore, new lower and upper bounds of the number are obtained for $m = 3$, which improve similar results in [1].

2. Preliminaries

Let $S_m$ be a star of order $m \geq 3$ and for each vertex $v_i \in V(S_m)$, $2 \leq i \leq m$, $v_i$ is adjacent to $v_1$, the center vertex of $S_m$. Also, let $P_n$ be a path such that $|V(P_n)| = n$. The Graph $G_{m,n} = S_m \square P_n$, is obtained by the Cartesian product of $S_m$ and $P_n$. The vertex set $V(G_{m,n})$ is the Cartesian product $V(S_m) \times V(P_n)$, such that for any $u_iv_j \in V(G_{m,n})$, then, $u_i \in V(S_m)$ and $v_j \in V(P_n)$. For $E(G_{m,n})$, $u_ivj\ u_kv_l$ is contained in $E(G_{m,n})$ for $u_ivj\ u_kv_l \in V(G_{m,n})$, then either $u_i = u_k$ and $v_jv_l \in E(P_m)$ or $u_iu_k \in E(S_m)$ and $v_j = v_l$. Geometrically, $V(G_{m,n})$ contains $n$ number of $S_m$ stars, namely $S_m(1), S_m(2), \ldots, S_m(n)$, such that for every pair $v_i \in S_m(i)$ and $v_{i+1} \in S_m(i+1)$, $u_iv_{i+1} \in E(G_{m,n})$. These are, in fact, the only type of edges on $G_{m,n}$ apart from those on its $S_m$ stars. This geometry fetched $G_{m,n}$ the name stacked-book graph.

**Remark 2.1.** It is easy to see that $diam(G_{m,n}) = n + 1$, being the number of edges from $u_1v_1 \rightarrow u_1v_1 \rightarrow u_1v_2 \rightarrow \cdots \rightarrow u_1v_n \rightarrow u_jv_n$, where $i \neq j$.

**Remark 2.1.** For convenience, we write $u_cv_j$ as $u_{i,j}$ in certain cases and $u_{i,j}u_{k,l}$ is the edge induced by $u_{i,j}$ and $u_{k,l}$.

**Definition 2.1.** Let $G_{m,n} = S_m \square P_n$. The vertex set $V(i) \subset V(G_{m,n})$ is the set of vertices on star $S_m(i)$, defined by the set $\{u_1v_i, u_2v_i, \ldots, u_nv_i\}$.

We introduce the following definition:

**Definition 2.2.** Let $G_{m,n} = S_m \square P_n$. Then, the pair $\{S_{m(i)}, S_{m(i+\frac{n}{2})}\}$ is a subgraph $G(i) \subset G_{m,n}$ induced by $V(i)$ and $V(i+\frac{n}{2})$.

**Remark 2.2.** The maximum number of $G(i)$ subgraph in a $G_{m,n}$ graph, $n$ even, is $\frac{n}{2}$ and the $diam(G(i)) = \frac{n}{2} + 2$. 
Remark 2.3. Let \( \{V(i), V(i+\frac{n}{2})\} \) induce \( G(i) \), such that \( V(i) = \{u_{1,i}, u_{2,i}, \cdots, u_{m,i}\} \) and \( V(i+\frac{n}{n}) = \{v_{1,i+\frac{n}{2}}, v_{2,i+\frac{n}{2}}, \cdots, v_{m,i+\frac{n}{2}}\} \). Then, for \( u \in V(i) \), \( v \in V(i+\frac{n}{n}) \) and \( d(u, v) = p \), where \( p \in \{\frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2\} \) and for \( u_{k,i}, v_{t,i+\frac{n}{2}}, \)
\[
p = \begin{cases} 
\frac{n}{2} & \text{if } k = t; \\
\frac{n}{2} + 1 & \text{if } t = 1, k \neq t; \\
\frac{n}{2} + 2 & \text{if } t \neq 1, k \neq 1, k \neq t.
\end{cases}
\]

3. RESULTS

In this section, we estimate the radio number of stacked-book graphs and obtain the exact radio number for \( G_{m,n} \), for \( m \geq 4 \), \( n \) even.

Lemma 3.1. Let \( S_m \) be a star on \( G_{m,n} \) and \( f \), a radio label function on \( G_{m,n} \). Then spanf on \( S_m \) is \( n(m - 1) + 1 \).

Proof. Let the center vertex of \( S_m \) be \( v_1 \) and let \( f(v_1) \) be the radio label on \( v_1 \).
There exists some \( v_2 \in V(S_m) \) such that \( d(v_1, v_2) = 1 \). Therefore, by the definition, \( f(v_2) \geq f(v_1) + n + 1 \). Suppose that \( k \notin \{1, 2\} \). For \( v_k, d(v_2, v_k) = 2 \), for all \( v_k \in V(S_m) \). Thus, without loss of generality, suppose that \( v_m \) is the last vertex on \( V(S_m) \), then \( f(v_m) \geq f(v_0) + (n + 1) + n(m - 2) \) and the claim follows. \( \square \)

Remark 3.1. It is easy to confirm that given a star \( S_m \) with center vertex \( v_1 \), if for a positive integer \( \alpha \), \( rn(S_m) = \alpha \), then either \( f(v_1) \) is \( f_{\min} \) or \( f_{\max} \).

Now we establish lower bound for \( G(i) \).

Lemma 3.2. Let \( G(i) \) be a subgraph of \( G_{m,n} \) and let \( f \) be the radio label on \( V(G_{m,n}) \). Then, \( rn(G(i)) \geq f(v_1) + mn - \frac{n}{2} + 2 \), where \( v_1 \) is the center vertex of \( S_{m(i+\frac{n}{n})} \).

Proof. Let \( S_{m(i)} \) and \( S_{m(j)} \) be the stars on \( G(i) \subset G_{m,n} \), where \( j = i + \frac{n}{2} \). By Lemma 3.1, \( f(v_m) = f(v_1) + mn - n + 1 \), with \( f(v_m) = \max \{f(v_i) : v_i \in V(S_m(j))\} \), and \( v_1 \) the center of \( S_{m(j)} \). Now, let \( u_1 \) be the center vertex of \( S_{m(i)} \). It is clear that \( d(u_1, v_1) = \frac{n+2}{2} \). Thus, \( f(u_1) \geq f(v_1) + mn - n + 1 + \frac{n+2}{2} = f(v_1) + mn - \frac{n}{2} + 2 \).

Claim: For optimal radio labeling of \( G(i) \), maximum label on \( S_{m(i)} \) is at least \( f(u_1) \).

Reason: Consider some \( u_m \in V(S_m) \), such that \( m \neq 1 \) and \( d(u_m, v_m) = \frac{n+2}{2} \). Then \( f(u_m) = f(v_1) + mn - \frac{n}{2} + 1 \). By Lemma 3.1, the spanf of \( f \) for a star \( S_m \) is \( mn-n+1 \). Now, \( f(u_m) - (mn-n+1) = f(v_1) + \frac{n}{2} \). Thus, by Remark 3.1, \( f(u_1) = f(v_1 + \frac{n}{2}) \).

This is a contradiction, considering that \( d(u_1, v_1) = \frac{n}{2} \). \( \square \)

Lemma 3.3. Let \( G^+(i) \subset G_{m,n} \) be \( G(i) \cup w_1 \), where \( w_1 \) is the center vertex of \( S_{m(j+1)} \) and let \( f \) be a radio labeling on \( G_{m,n} \), where \( n \) is even. Then, the spanf of \( f \) on \( G^+(i) \geq mn + 3 \).

Proof. Let \( u_1 \) be the center vertex of \( S_{m(i)} \). It can be verified that \( d(u_1, w_1) = \frac{n}{2} \). By the proof of Lemma 3.2, \( f(u_1) \geq f(v_1) + mn - \frac{n}{2} + 2 \), where \( v_1 \) is the center vertex.
of $S_{m(j)}$. Thus by definition, $f(w_1) \geq f(v_1) + mn - \frac{n}{2} + 2 + \frac{n+2}{2} = f(v_1) + mn + 3$. Since $f(v_1)$ is the minimum label on $G(i)$, the result follows. □

Now we present the lower bound for stacked-book graph $G_{m,n}$, where $n$ is an even integer and $m \geq 3$.

**Theorem 3.1.** Let $G = G_{m,n}$ be a stacked-book graph with $m \geq 3$ and $n$ an even integer. Furthermore, let $f$ be the radio labeling on $G$. Then, $rn(G) \geq \frac{mn^2}{2} + n - 1$.

**Proof.** From the definition of $G(i)$, graph $G_{m,n}$ contains $\frac{n}{2}$ number of $G(i)$ subgraphs. Likewise, it can be seen that $G_{m,n}$ contains $\frac{n}{2}$ number of $G^+(i)$ subgraphs. Now, let $G(\frac{r}{2})$, induced by $S_{m(\frac{r}{2})}$ and $S_{m(n)}$ be the last $G(i)$ subgraphs on $G_{m,n}$ and $G^+(1), G^+(2), \ldots, G^+(\frac{mn}{2})$ be the $\frac{mn}{2}$ number of $G^+(i)$ graphs. By the earlier result, if $f(v_1) = 0$, then $rn(G_{m,n}) \geq (\frac{mn}{2})(mn + 3) + mn - \frac{n}{2} + 2 = \frac{mn^2}{2} + n - 1$. □

In what follows, we examine the upper bound of the stacked-book graph $G_{m,n}$.

**Lemma 3.4.** Let $G(i)$ be a subgraph of $G_{m,n}$ induced by $\left\{V(i), V(i+\frac{n}{2})\right\}$. Then for any pair $v \in V(i)$ and $u \in V(i+\frac{n}{2})$, such that $d(u, v) \geq \frac{n}{2} + 1$, $|f(v) - f(u)| \geq \frac{n}{2}$.

**Proof.** Let $u = u_{k,i} \in V(i)$ and $v = u_{t,i+\frac{n}{2}} \in V(i+\frac{n}{2})$. Since $d(u, v) > \frac{n}{2}$, then by Remark 2.3, $k \neq t$. Suppose that neither $u$ nor $v$ is the center vertex of their respective stars $S_{m(i)}$ and $S_{m(i+\frac{n}{2})}$. Then, $d(u, v) = \text{diam}(G(i))$. Now, let the radio label on $u$ and $v$ be $f(u)$ and $f(v)$ respectively. Suppose, without loss of generality, that $f(v) > f(u)$. Then $f(v) \geq f(u) + \text{diam}(G_{m,n}) + 1 - \text{diam}(G(i))$, which implies that

$$f(v) \geq f(u) + \frac{n}{2}.$$

This implies that $f(v) - f(u) \geq \frac{n}{2}$. Similarly, if $f(u) \geq f(v)$, then $f(u) - f(v) \geq \frac{n}{2}$ and thus, the claim follows. □

The following remarks can be confirmed by applying similar methods as in proof of Lemma 3.4.

**Remark 3.1.** Suppose that either of $u, v$ in Lemma 3.4, say $u$, is such that for any $u' \in V(i)$, $uv' \in E(S_{m(i)})$. Then $d(u, u') = \frac{n}{2} + 1$ and $|f(u) - f(v)| \geq \frac{n}{2} + 1$.

**Remark 3.2.** Let $u, u' \in V(i)$. If $d(u, u') = 1$, then $|f(u) - f(u')| \geq n + 1$ and $|f(u) - f(v')| \geq n$ for $d(u, u') = 2$.

**Theorem 3.2.** Let $m > 3$ be odd and $G(i) \subseteq G_{m,n}$, be induced by $\left\{V(i), V(i+\frac{n}{2})\right\}$; then, $rn(G(i)) \leq f(v_1) + mn - \frac{n}{2} + 2$, where $v_1$ is the center star $S_{m(\frac{n}{2})}$.

**Proof.** Let $V(i) = \left\{u_{1,i}, u_{2,i}, \ldots, u_{m,i}\right\}$ and $V(t) = \left\{u_{1,t}, u_{2,t}, \ldots, u_{m,t}\right\}$, where $t = i + \frac{n}{2}$. For $r \in [1, m]$, set $u_{r,i} \in V(i)$ as $\alpha_r$ and $u_{r,t} \in V(t)$ as $\beta_r$. From earlier remark, $d(\beta_r, \alpha_r) \in \left\{\frac{n}{2} + 1, \frac{n}{2} + 2\right\}$ for $r \neq s$. Now, for every pair $\alpha_s, \beta_r$, where $\alpha_s \in V(i)$, and $\beta_r \in V(t)$, let $r \neq s$ except otherwise stated. Let $\alpha_1$ and $\beta_1$ be the respective centers of the stars $S_{m(i)}$ and $S_{m(t)}$ induced by $V(i)$ and $V(t)$ and let the radio label on $\beta_1$ be $f(\beta_1)$.
such that \( f(\beta_1) = \min \{ f(\beta_i) : 1 \leq i \leq m \} \). Since \( \beta_1 \) is the center of \( S_{m(t)} \), then given \( \alpha_2 \in V(i) \), \( d(\beta_1, \alpha_2) = \frac{n}{2} + 1 \). Now set \( p = \text{diam}(G_{m,n}) + 1 - d(\beta_1, \alpha_r), r \neq 1 \). Hence, \( p = n + 2 - (\frac{n}{2} + 1) = \frac{n}{2} + 1 \). Suppose that \( \alpha_j \in V(i) \) and \( \beta_k \in V(t) \), such that \( 1 \neq j \neq k \neq 1 \). Then, \( d(\alpha_j, \beta_k) = \frac{n}{2} + 2 \). So we set \( q = \text{diam}(G_{m,n}) + 1 - d(\alpha_j, \beta_k) = \frac{n}{2} + 1 \). For \( f(\beta_1) \) and some \( \alpha_2 \in V(i) \), \( f(\alpha_2) = f(\beta_1) + p \). Also, for \( \beta_3 \in V(t) \), \( f(\beta_3) = f(\alpha_2) + q = f(\beta_1) + p + q \) and \( f(\alpha_4) = f(\beta_1) + 2q + p \). We continue to label the vertices on both \( V(i) \) and \( V(t) \) alternatively based on the last value attained. Therefore, for \( m \) odd,

\[
\begin{align*}
    f(\beta_m) &= f(\alpha_{m-1}) + \frac{n}{2} \\
               &= f(\beta_1) + (m-2)q + p.
\end{align*}
\]

It can be seen that there does not exist \( \alpha_i \in V(i) \), such that \( d > m \). So, we reverse the order of labeling, such that for \( \beta_m, \alpha_3 \), \( f(\alpha_3) = f(\beta_m) + q = f(\beta_1) + (m-2)q + 2p \).

Also, for the pair \( \alpha_3, \beta_2 \), \( f(\beta_2) = f(\beta_1) + (m-2)q + 2q + p \). This continues until we reach the pair \( \alpha_m, \beta_{m-1} \), and obtain

\[
\begin{align*}
    f(\alpha_{m-1}) &= f(\beta_1) + (2m-3)q + p.
\end{align*}
\]

Finally, we consider the pair \( \beta_{m-1} \) and \( \alpha_1 \). Since \( \alpha_1 \) is the center of \( S_{i(i)} \), then \( d(\alpha_1, \beta_{m-1}) = \frac{n}{2} + 1 \) and hence,

\[
\begin{align*}
    f(\alpha_1) &= f(\alpha_{m-1}) + p \\
               &= f(\beta_1) + (2m-3)q + 2p \\
               &= f(\beta_1) + mn - \frac{n}{2} + 2.
\end{align*}
\]

Hence, \( \text{rn}(G(i)) \leq f(\alpha_1) + mn - \frac{n}{2} + 2 \), where \( m \) is odd and \( n \) even. \( \square \)

Next we directly apply Theorem 3.2

**Lemma 3.5.** Let \( G(i) \) be induced by \( \left\{ S_{m(i)}, S_{m(i+\frac{n}{2})}, \gamma_1 \right\} \), where \( \gamma_1 \) is the center of star \( S_{m(i+\frac{n}{2}+1)} \), induced by \( V(i+\frac{n}{2}+1) \). Then, \( f(\gamma_1) \leq f(\beta_1) + mn + 3 \).

**Proof.** For \( \alpha_1 \) and \( \beta_1 \) centers of stars \( S(i) \) and \( S(i+\frac{n}{2}) \) respectively, let \( f(\alpha_1) = f(\beta_1) + mn - \frac{n}{2} + 2 \), as established in Theorem 3.2. Then, \( d(\alpha_1, \gamma_1) = \frac{n}{2} + 1 \). Therefore,

\[
\begin{align*}
    f(\gamma_1) &= f(\alpha_1) + p \\
               &= f(\beta_1) + mn + 3.
\end{align*}
\]

\( \square \)

Now, for \( \beta_1 \), the center of \( S_{m(1+\frac{n}{2})} \), induced by \( V(1+\frac{n}{2}) \). By setting \( f(\beta_1) = 0 \), we establish an upper bound for the radio number of a stacked-book graph \( G_{m,n} \) in the next results.

**Theorem 3.3.** For \( G_{m,n} \), \( m \) odd and \( n \) even, \( \text{rn}(G_{m,n}) \leq \frac{mn^2}{2} + n - 1 \).
Proof. Let \( \{v_{1(1)}, v_{2(1)}, v_{3(1)}, \ldots, v_{n(1)}\} \) be the set of the respective centers of stars \( S_{m(1)}, S_{m(2)}, S_{m(3)}, \ldots, S_{m(n)} \) in \( G_{m,n} \). Also, suppose that \( f(v_{n+1(1)}) = 0 \). From the Lemma 3.5 \( f(v_{n+2(1)}) = mn + 3; f(v_{n+3(1)}) = 2(mn + 3) \) and so on. In the end, \( f(v_{n(1)}) = (\frac{n}{2} - 1)(mn + 3) \). Also, let \( v_{n-\frac{n}{2}(1)} = v_{\frac{n}{2}(1)} \) be the center of \( S_{m(\frac{n}{2})} \subset G_{m,n} \) and let \( S_{m(\frac{n}{2})}, S_{m(n)} \) induce the graph \( G(\frac{n}{2}) \subset G_{m,n} \). By Theorem 3.2,

\[
\text{rn}\left( G\left( \frac{n}{2} \right) \right) \leq f(v_{n(1)}) + mn - \frac{n}{2} + 2 \\
\leq \frac{mn^2}{2} + n - 1.
\]

□

Theorem 3.4. Let \( m, n \) be even. Then \( \text{rn}(G_{m,n}) \leq \frac{mn^2}{2} + n - 1 \).

Proof. The proof follows similar argument and technique as in Theorem 3.2, Lemma 3.5 and Theorem 3.3. □

![Figure 1. A G_{4,6} graph with \text{rn}(G_{4,6}) \leq 77](image)

Theorems 3.1, 3.3, 3.4 establish the radio number of \( G_{m,n} \), where \( m \geq 4 \) and \( n \) is even, as recapped in the next theorem.

Theorem 3.5. Let \( G_{m,n} \) be a stacked-book graph with \( m \geq 4 \) and \( n \) even, then, \( \text{rn}(G_{m,n}) = \frac{mn^2}{2} + n - 1 \).

Next we consider the case where \( m = 3 \). First we present a result that is equivalent to Theorem 3.2 with respect to \( m = 3 \).

Theorem 3.6. Let \( G_{3,n} \) be a stacked-book graph, where \( n \) is even. Suppose that the pair \( \{S_{3(i)}, S_{3(i+\frac{n}{2})}\} \) form a subgraph \( G(i) \) of \( G_{3,n} \). Then, \( \text{rn}(G(i)) \leq f(u_1) + \frac{5n}{2} + 3 \), where \( u_1 \) is the center vertex of \( S_{3(i+\frac{n}{2})} \).

Proof. Let \( V(i) = \{v_1, v_2, v_3\} \) and \( V(i+\frac{n}{2}) = \{u_1, u_2, u_3\} \) where \( V(i) \) and \( V(i+\frac{n}{2}) \) are vertex sets of stars \( S_{3(i)} \) and \( S_{3(i+\frac{n}{2})} \) in \( G_{3,n} \) respectively. Also, let \( v_1 \) and \( u_1 \) be the respective center vertices of \( S_{3(i)} \) and \( S_{3(i+\frac{n}{2})} \). From earlier remark, \( d(v_1, u_1) = \frac{n}{2} + 1 \). Suppose
that \( f(u_1) \), the radio label of \( u_1 \) is the smallest possible radio label on \( G(i) \), then,
\[
  f(v_2) = f(v_1) + \text{diam}(G_{3,n}) + 1 - d(v_1, u_1) \\
  = f(u_1) + \frac{n}{2} + 1.
\]

For \( v_2, u_3 \) \( d(v_2, u_3) = \frac{n}{2} + 2 \),
\[
  f(u_3) = f(u_1) + n + 1
\]

For \( u_3, v_1 \) \( d(u_3, v_1) = \frac{n}{2} + 1 \),
\[
  f(v_1) = f(u_1) + \frac{3n}{2} + 2.
\]

For \( v_1, u_2 \) \( d(v_1, u_2) = \frac{n}{2} + 1 \) and thus,
\[
  f(u_2) = f(u_1) + \frac{3n}{2} + 2 + n + 2 - \left( \frac{n}{2} + 1 \right) \\
  = f(u_1) + 2n + 3.
\]

And finally, for the pair \( v_3, u_2 \) \( d(v_3, u_3) = \frac{n}{2} + 2 \) and
\[
  f(u_3) = f(u_1) + 5n + 3.
\]

Hence, \( rn(G(i)) \leq f(u_1) + \frac{5n}{2} + 3 \).

Next, we obtain the following result.

**Lemma 3.6.** Let \( \kappa_1 \) be the center of star \( S_{3(i+ \frac{n}{2})+1} \subseteq G_{3,n} \) and let \( H(1) \) be a subgraph of \( G(3,m) \) induced by \( \{S_{3(i)}, S_{3(i+ \frac{n}{2})}, \kappa_1\} \). Then \( f(\kappa_1) \leq 3n + 1 \).

**Proof.** The vertex with the maximum value of radio label in Theorem 3.6 is \( u_3 \). Let us adopt this, with \( f(u_3) = f(u_1) + \frac{5n}{2} + 3 \). Now, \( d(u_3, \kappa_1) = \frac{n}{2} + 2 \). Therefore,
\[
  f(\kappa_1) = f(u_1) + 3n + 3.
\]

In the final result here, we set \( f(u_1) = 0 \), for \( u_i \), the center of star \( S_{3(1+ \frac{n}{2})} \).

**Theorem 3.7.** Let \( n \) be an even positive integer. Then, \( rn(G_{3,m}) \leq \frac{3n^2}{2} + n \).

**Proof.** Proof follows similar technique adopted in Theorem 3.4.

Figure 2 is a radio numbering for \( G_{3,6} \). It shows that \( rn(G_{3,6}) \) is not more than 60.
4. Conclusion

It is noteworthy to look at some of the results in [7]. A \( G_{3,n} \) is a \( 3 \times n \) grid. By [7], it is seen that \( rn(G_{3,6}) = 59 \), which is better than the result in Figure 2 above by 1. But this is still a considerable improvement compared with a upper bound of 109 suggested in [1]. In establishing the upper bound for \( G_{3,n} \), it is observed that the number of the pair \( u, v \in V(G_{3,n}) \) for which \( d(u, v) = \frac{diam(G_{3,n})+1}{2} \) is more than the case where \( d(u, v) = \frac{n}{2} \) in each of the segments of radio labeling of the stacked-graph. However, the reverse proves to be the case in \( G_{m,n} \), \( m \geq 4 \).

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