GLOBAL HOPF BIFURCATION OF A POPULATION MODEL WITH STAGE STRUCTURE AND STRONG ALLEE EFFECT

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Abstract. This paper is devoted to the study of a single-species population model with stage structure and strong Allee effect. By taking \( \tau \) as a bifurcation parameter, we study the Hopf bifurcation and global existence of periodic solutions using Wu’s theory on global Hopf bifurcation for FDEs and the Bendixson criterion for higher dimensional ODEs proposed by Li and Muldowney. Some numerical simulations are presented to illustrate our analytic results using MATLAB and DDE-BIFTOOL. In addition, interesting phenomenon can be observed such as two kinds of bistability.

1. Introduction. The well-known Mackey-Glass equation

\[
\dot{y}(t) = \frac{\beta_0 \gamma y(t-\tau)}{\theta^n + y^n(t-\tau)} - \gamma y(t),
\]

was proposed by Mackey and Glass \[14\] to describe a physiological control system in 1977. Since then, a fair amount of work \[23, 22, 11\] has been devoted to it, such as the study the Hopf bifurcation and global Hopf bifurcation \[23, 22\], chaos, peak-adding and period-doubling bifurcations \[11\].

Morozov et al. \[15\] considered a age-structured single-species population, in which the author derived the model detailedly using the general framework of age-structured populations, by taking the reproduction function \( r(y) = \frac{\alpha y^{n-1}}{1 + \beta y^m} \), they arrived at

\[
\dot{y}(t) = \alpha e^{-\delta \tau} \frac{y^n(t-\tau)}{1 + \beta y^m(t-\tau)} - Dy(t).
\]

Notice that when \( \delta = 0 \) and \( n = 1 \), the equation (1) reduced to the classical Mackey-Glass equation. Thus equation (1) can be regarded as a modification of the classical Mackey-Glass equation whose dynamical behavior is more complicated.

In \[15\], Morozov et al. indicated that the growth of adult population in system (1) has the property of strong Allee effect when \( m > n > 1 \), where the ‘Allee effect’ named after Allee \[1\]. Literature \[10, 21\] as well as the references therein showed that the phenomenon of the growth rate per capita achieves its peak at a positive density is called an Allee effect. Furthermore, a strong Allee effect refers to the phenomenon that the growth rate per capita is negative when the population is...
small, and the weak Allee effect means that the growth rate per capita is smaller than the maximum but still positive for small population. Therefore, it is strong Allee effect if \( m > n > 1 \) (see Figure 1(a), and not-strong Allee effect if \( n \leq 1 \) (see Figure 1(b)). We would like to mention that the situation in Figure 1(b) (classical Mackey-Glass case) is not Allee effect since the growth rate per capita \( r(y) \) always decreases as \( y \) increases.

![Figure 1](image.png)

(a) Strong Allee effect when \( n = 2, m = 4 \)  
(b) Not Allee effect when \( n = 1, m = 2 \)

Figure 1. The growth rate per capita \( r(y) = \frac{y^{n-1}}{1+y^m} - 0.15 \) in system \( \dot{y} = \frac{y^n}{1+y^m} - 0.15y = r(y)y \).

A large amount of simulations has been done in [15] to reveal the rich phenomenon of strong Allee effect \( (n = 2, m = 4) \) model, such as long-term transients, peak-adding bifurcations, and period-doubling bifurcations. Morozov et al. have done a lot of work to exhibit them in numerical simulation by taking \( A = \alpha e^{-\delta \tau} \) as a constant.

In this paper, we will concentrate on (1) with stage structure and general strong Allee effect \( m > n > 1 \) to study the stability, Hopf bifurcation, and global Hopf bifurcation.

The rest of the paper is organized as follows: in Section 2 we investigate some properties of the system, such as positivity and boundedness of the solutions, analysis of equilibria and their stabilities. In Section 3 we study the Hopf bifurcation, particularly in Section 3.1 the occurrence of Hopf bifurcation, and in Section 3.2 the direction and stability of the Hopf bifurcation. A global Hopf bifurcation is established in Section 4. Finally, some simulations are carried out for illustrating the analytic results.

2. Preliminaries. In this section, we shall investigate the positivity and boundedness of the solutions and the equilibria of system (1) with nonnegative initial conditions. For any \( \tau > 0 \), denote \( \mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}) \) as the Banach space of continuous functions on \([-\tau, 0]\) to \( \mathbb{R} \) with the norm

\[
|\varphi| = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)| \text{ for } \varphi \in \mathcal{C}.
\]

From a biological point of view, it is reasonable to take initial conditions from the nonnegative cone \( \mathcal{C}^+ = \mathcal{C}([0, 0], \mathbb{R}) \) where \( \mathbb{R} = [0, \infty) \). For the initial conditions
\( \varphi \in C^+ \) and \( \varphi(\theta) \neq 0 \) for \( \theta \in [-\tau, 0] \),

system (1) admits a unique solution for each continuous initial condition by the fundamental theory in Hale’s book \[8\], and then, the following conclusion holds.

**Lemma 2.1.** Each solution of (1) with initial condition (2) is nonnegative and ultimately uniformly bounded in \( C^+ \).

**Proof.** From the Variation-of-Constants formula, the solutions of system (1) expressed in the following form:

\[
y(t) = \left( \varphi(0) + \int_0^t \alpha e^{-\delta \tau} \frac{y^n(\theta - \tau)}{1 + \beta y^m(\theta - \tau)} e^{D\theta} d\theta \right) e^{-Dt}.
\]

Nonnegative initial conditions \( \varphi \in C^+ \) and \( \varphi(\theta) \neq 0 \) for \( \theta \in [-\tau, 0] \) implies \( y(t) > 0 \) when \( t \in [0, \tau] \). By induction, one can verify that \( y(t) > 0 \) holds for all \( t \in [0, \infty) \).

For the ultimately uniformly boundedness, using the fact that

\[
y^n = \frac{m}{m > n} m - n \]

we have

\[
\dot{y}(t) \leq \alpha e^{-\delta \tau} \frac{y^n(t - \tau)}{1 + \beta y^m(t - \tau)} - Dy(t) \leq \left( \frac{n}{\beta(m - n)} \right) \frac{\alpha(m - n)}{m} y(t),
\]

it follows that

\[
\lim_{t \to \infty} y(t) \leq \left( \frac{n}{\beta(m - n)} \right) \frac{\alpha(m - n)}{Dm}.
\]

This completes the proof. \( \square \)

The equilibria of (1) can be found from

\[
y \left( \alpha e^{-\delta \tau} \frac{y^{n-1}}{1 + \beta y^m} - D \right) = 0,
\]

which always gives an extinction equilibrium \( y_0 = 0 \). Besides that, two extra positive equilibria \( y_1 \leq y_2 \) may emerge according to the zeros of

\[
F(y) = \beta Dy^m - \alpha e^{-\delta \tau} y^{n-1} + D.
\]

Taking the derivative of \( F \) with respect to \( y \) we get

\[
F'(y) = \beta Dm y^{m-1} - \alpha e^{-\delta \tau} (n-1)y^{n-2},
\]

there is only one positive \( \bar{y} = \left( \frac{\alpha e^{-\delta \tau} (n-1)}{\beta Dm} \right)^{1/(n-1)} \) such that \( F'(\bar{y}) = 0 \). Notice that \( F(0) = D > 0 \) and \( \lim_{y \to +\infty} F(y) = +\infty \), it is clear that \( y = \bar{y} \) is the only minimum point for \( F(y) \) on \( y \in [0, +\infty) \). Thus \( F(y) = 0 \) has two distinct positive roots if and only if \( F(\bar{y}) < 0 \), or equivalently

\[
\beta D \left( \frac{\alpha e^{-\delta \tau} (n-1)}{\beta Dm} \right)^{m/(m-1)} - \alpha e^{-\delta \tau} \left( \frac{\alpha e^{-\delta \tau} (n-1)}{\beta Dm} \right)^{m/(m-1)} + D < 0.
\]

In fact, denote

\[
\xi = \alpha D^{-\frac{m-n+1}{m}} \left( \frac{n-1}{\beta Dm} \right)^{m/(m-n+1)} - \left( \frac{n-1}{m} \right)^{m/(m-n+1)} \left( D \right)^{m/(m-n+1)},
\]
then $[1]$ is equivalent to $\xi > 1$ and $\tau < \tau^0 := \frac{1}{2} \ln \xi$. Meanwhile, the above two roots converge to one when $\tau = \tau^0$.

When $\xi > 1$ and $\tau < \tau^0$, system $[1]$ has three fixed points: $y_0$, $y_1$, and $y_2$, where both $y_1$ and $y_2$ change with $\tau$. Furthermore, we would like to point out that the following statement holds.

**Theorem 2.2.** If $\xi > 1$, $\tau < \tau^0$, then $y_1(\tau)$ is a strictly increasing function and $y_2(\tau)$ is a strictly decreasing function on $[0, \tau^0)$.

**Proof.** Substituting $y_1(\tau)$ and $y_2(\tau)$ into $F(y, \tau) = 0$, we have $F(y_1(\tau), \tau) = 0$ and $F(y_2(\tau), \tau) = 0$, respectively. The derivative of both sides of $F(y_1(\tau), \tau) = 0$ with respect to $\tau$ is

$$\frac{\partial F}{\partial y}
\left.\right|_{y=y_1} \cdot \frac{dy_1}{d\tau} = -\alpha \delta e^{-\delta \tau} y_1^{n-1} < 0.$$ 

From the description in $[3]$ we know $\frac{\partial F}{\partial y}
\left.\right|_{y=y_1} < 0$ since $y_1 < \bar{y}$. Thus, it follows that $\frac{dy_1}{d\tau} > 0$. Similarly we have $\frac{dy_2}{d\tau} < 0$ since $\frac{\partial F}{\partial y}
\left.\right|_{y=y_2} > 0$. This completes the proof. 

The linearization of equation $[1]$ around $y_i$ ($i = 0, 1, 2$) is

$$\dot{y}(t) = -Dy(t) + \alpha b_i e^{-\delta \tau} y(t - \tau)$$

where

$$b_i = \left. \frac{d}{dy} \left( \frac{y^n}{1 + \beta y^m} \right) \right|_{y=y_i} = \frac{ny_i^{n-1} - (m-n)\beta y_i^{m+n-1}}{(1 + \beta y_i^m)^2}. \quad (6)$$

Notice that $b_0 = 0$, and $b_1, b_2$ are functions of $\tau$ since $y_1, y_2$ changes over $\tau$.

The characteristic equation associated with $[1]$ is

$$\lambda + D - \alpha b_i e^{-\delta \tau} e^{-\lambda \tau} = 0. \quad (7)$$

The following conclusions hold.

**Theorem 2.3.** $y_0 = 0$ is always stable. Furthermore, it is globally asymptotically stable when $\xi \leq 1$ or $\xi > 1$, $\tau > \tau^0$.

**Proof.** First we know $[5]$ implies that $b_0 = 0$, and then $[7]$ becomes

$$\lambda + D = 0.$$ 

From $\lambda = -D < 0$ we know that $y_1 = 0$ is always stable. For the global asymptotically stability of $y_0 = 0$, we consider the Lyapunov functional $V : \mathbb{C}_+ \rightarrow \mathbb{R}_+$

$$V(\varphi) = \varphi(0) + \alpha e^{-\delta \tau} \int_{-\tau}^{0} \frac{\varphi^n(\theta)}{1 + \beta \varphi^m(\theta)} d\theta.$$ 

The derivative along the solutions of $[1]$ is

$$V'|[1] = \dot{y}(t) + \alpha e^{-\delta \tau} y^n(t) \frac{y^n(t)}{1 + \beta y^m(t)} - \alpha e^{-\delta \tau} \frac{y^n(t - \tau)}{1 + \beta y^m(t - \tau)} y(t) \bigg( \alpha e^{-\delta \tau} \frac{y^{n-1}(t)}{1 + \beta y^m(t)} - D \bigg) y(t)$$

$$- \frac{1}{1 + \beta y^m} F(y) y$$

$$\leq 0,$$
since \( \xi \leq 1 \) or \( \xi > 1 \), \( \tau > \tau^0 \) is equivalent to \( F(y) > 0 \) for all \( y \geq 0 \). Moreover, \( V(\varphi) = 0 \) if and only if \( \varphi(0) = 0 \). Set \( S = \{ \varphi : \varphi(0) = 0 \} \), and let \( M \) be the largest set in \( S \) which is invariant with respect to Equation (1). Then \( M \) is nonempty since \( 0 \in M \). In fact \( M \) is the singleton \( \{ 0 \} \). Otherwise, there is a \( \theta_0 \in [-\tau, 0) \) such that \( \varphi(\theta_0) > 0 \). By the invariance of \( M \) we have \( y_t(0, \varphi) \in M \) for \( t \geq 0 \), which does not belong to \( M \) when \( t > \tau + \theta_0 \). This contradiction proves that \( M = \{ 0 \} \). Thus \( y_0 = 0 \) is globally attractive by using the LaSalle invariance principle \[7, Chapter 5\].

Combining with the local stability we know \( y_0 = 0 \) is globally asymptotically stable when \( \xi \leq 1 \) or \( \xi > 1 \), \( \tau > \tau^0 \).

Compared to the stability of trivial equilibrium \( y_0 = 0 \), we are more concerned about the complex behavior of positive equilibria \( y_1 \) and \( y_2 \). In the following sections, we always assume that \( \xi > 1 \) and \( \tau < \tau^0 \), which stands for the existence of positive equilibria.

3. Hopf bifurcation analysis. In this section, we are going to investigate the existence of Hopf bifurcation around \( y_1 \) and \( y_2 \) by taking the time delay \( \tau \) as a bifurcation parameter. Without loss of generality, we denote \( y_1 \) or \( y_2 \) as \( y^* \). Then the characteristic equation (7) around \( y^* \) becomes

\[
\lambda + D - \alpha b^* e^{-\delta \tau} e^{-\lambda \tau} = 0, \tag{8}
\]

where \( b^* \) stands for \( b_1 \) or \( b_2 \), which are in the denotation of (6).

3.1. Stability and Hopf bifurcation. From the consequence of the distribution of zeros of a transcendental function given by Ruan and Wei \[19\], we know that as \( \tau \) varies, the sum of the orders of the roots of (7) in the open right half plane can change only if a zero appears on and crosses the imaginary axis.

We would like to mention that \( b^* \) changes with \( \tau \) since \( y^* \) is a function of \( \tau \).

In order to investigate the critical values of \( \tau \) when there exists a pair of purely imaginary roots of (8), we are going to use the method introduced by Beretta and Kuang \[2\]. Rewrite equation (8) as

\[
P(\lambda, \tau) + Q(\lambda, \tau) e^{-\lambda \tau} = 0,
\]

where

\[
P(\lambda, \tau) = \lambda + D, \quad Q(\lambda, \tau) = -\alpha b^* e^{-\delta \tau}.
\]

We mention that

\[
P(-i\omega, \tau) = P(i\omega, \tau), \quad Q(-i\omega, \tau) = Q(i\omega, \tau).
\]

\( P, Q \) have real coefficients. This ensures that if \( \lambda = i\omega \), for some real \( \omega \), is a root of (8), then \( \lambda = -i\omega \) is a root of (8) as well.

Let \( \lambda = i\omega \) (\( \omega > 0 \)) be a purely imaginary root of equation (8), then

\[
i\omega + D - \alpha b^* e^{-\delta \tau} (\cos \omega \tau - i \sin \omega \tau) = 0.
\]

Separating the real and imaginary parts yields

\[
D = \alpha b^* e^{-\delta \tau} \cos \omega \tau, \quad -\omega = \alpha b^* e^{-\delta \tau} \sin \omega \tau. \tag{9}
\]

Squaring both sides of (9) and summing the two equations, we obtain

\[
h(\omega^2, \tau) = F(\omega, \tau) = \omega^2 + D^2 - (\alpha b^* e^{-\delta \tau})^2 = 0. \tag{10}
\]
Make the following hypothesis:

\[(P1) \quad (ab^*e^{-\delta \tau})^2 - D^2 > 0.\]

Then \(F(\omega, \tau) = 0\) has a unique positive real root given by

\[\omega^* = \omega^*(\tau) := \sqrt{(ab^*e^{-\delta \tau})^2 - D^2}\]

if and only if \(|ab^*e^{-\delta \tau}| > D\).

Collecting all \(\tau\) which satisfies (P1) that \(I = \{\tau \in \mathbb{R}_+^\ast : \tau \text{ satisfies } (P1)\} \subseteq [0, \tau^1)\).

We know \(I\) is left-closed, right-open interval that \(I = [0, \tau_1), \) where \(\tau_1 = \sup_{\tau \in I} \tau\).

Before applying the geometry criterion in [2] to (8), a sequence of conditions on \(P\) and \(Q\) are required to be verified. This is accomplished by the following proposition.

**Proposition 1.** The following statements are valid for all \(\tau \in [0, \tau^1)\).

(a) \(P(0, \tau) + Q(0, \tau) \neq 0\);
(b) \(P(i\omega, \tau) + Q(i\omega, \tau) \neq 0\);
(c) \(\lim_{|\lambda| \to \infty} \left\{ \frac{Q(\lambda, \tau)}{P(\lambda, \tau)} : |\lambda| \to \infty, \Re \lambda \geq 0 \right\} < 1\) for any \(\tau\);
(d) \(F(\omega, \tau) := |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2\) has a finite number of real zeros for each \(\tau\);
(e) Each positive root \(\omega(\tau)\) of \(F(\omega, \tau) = 0\) is continuous and differentiable in \(\tau\) whenever it exists.

**Proof.** (a) We know

\[P(0, \tau) + Q(0, \tau) = D - ab^*e^{-\delta \tau}\]

from the expression of \(b_1\) in (6) and hypothesis (P1). That is, \(\lambda = 0\) is not a root of (8).

(b) \(P(i\omega, \tau) + Q(i\omega, \tau) = i\omega + D - ab^*e^{-\delta \tau} \neq 0\), we know that (b) is true.

(c) From

\[\lim_{|\lambda| \to \infty} \left| \frac{Q(\lambda, \tau)}{P(\lambda, \tau)} \right| = \lim_{|\lambda| \to \infty} \left| \frac{-ab^*e^{-\delta \tau}}{\lambda + D} \right| = 0,\]

we have

\[\lim_{|\lambda| \to \infty, \Re \lambda \geq 0} \left| \frac{Q(\lambda, \tau)}{P(\lambda, \tau)} \right| < 1.\]

(d) We have

\[F(\omega, \tau) = |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2\]
\[= \omega^2 + D^2 - (ab^*e^{-\delta \tau})^2.\]

It is obvious that property (d) is satisfied.

(e) The conclusion is valid because \(F(\omega, \tau)\) is a linear polynomial in \(\omega^2\) and the fact that \(b^*\) is a continuous function of \(\tau\).

Define \(\theta(\tau) \in [0, 2\pi]\) for \(\tau \in I\) by

\[\sin \theta(\tau) = \frac{\omega}{ab^*e^{-\delta \tau}},\]
\[\cos \theta(\tau) = \frac{D}{ab^*e^{-\delta \tau}}.\]
Lemma 3.1. Assume that in [2] can be used to verify the occurrence of Hopf bifurcations when \( y \) are continuous and differentiable on \( I \). Obviously, \( y \) according to two cases. When \( y \) has a positive zero \( n \in \mathbb{N} \), we see since \( \theta(\tau) \to \pi \) when \( \tau \to \tau^* \) for all \( \delta \), there exists at least one zero satisfying \( \delta \tau^* \). That is to say, if \( \delta \tau^* \neq 0 \), \( \frac{dS_n(\tau)}{d\tau} \bigg|_{\tau=\tau^*} \neq 0 \). Then there exists a pair of simple purely imaginary roots \( \pm i\omega(\tau^*) \) of (8). We also know that \( \theta(\tau) \neq 0, 2\pi \) on \( I \) in terms of (b) in Proposition 1 and \( S_n(\tau) \) are continuous and differentiable on \( I \) from Lemma 2.1 in [2]. The following result in [2] can be used to verify the occurrence of Hopf bifurcations when \( \tau = \tau^* \).

**Lemma 3.1.** Assume that \( S_n(\tau) = 0 \) has a positive root \( \tau^* \in I \) for some \( n \in \mathbb{N} \) and \( \frac{dS_n(\tau)}{d\tau} \bigg|_{\tau=\tau^*} \neq 0 \). Then there exists a pair of simple purely imaginary roots \( \pm i\omega(\tau^*) \) of (8) at \( \tau = \tau^* \), and

\[
\delta(\tau^*) = \text{Sign} \left\{ \frac{d\text{Re}(\lambda)}{d\tau} \bigg|_{\lambda=i\omega(\tau^*)} \right\} \\
= \text{Sign} \left\{ \frac{\partial F}{\partial \omega}(\omega(\tau^*), \tau^*) \right\} \times \text{Sign} \left\{ \frac{dS_n(\tau)}{d\tau} \bigg|_{\tau=\tau^*} \right\}. \tag{12}
\]

Since \( \frac{\partial F}{\partial \omega}(\omega^*, \tau^*) = 2\omega^* > 0 \), (12) is equivalent to

\[
\delta(\tau^*) = \text{Sign} \left\{ \frac{dS_n(\tau)}{d\tau} \bigg|_{\tau=\tau^*} \right\},
\]

which determines the direction in which the pair of purely imaginary roots cross the imaginary axis: from left to right if \( \delta(\tau^*) > 0 \), and from right to left if \( \delta(\tau^*) < 0 \).

Furthermore, \( S_n(\tau) - S_{n+1}(\tau) = \frac{2\pi}{\omega(\tau)} > 0 \) due to the positivity of \( \omega(\tau) \). Therefore, we see \( S_n(\tau) > S_{n+1}(\tau) \) with \( n \in \mathbb{N} \) for all \( \tau \in I \). That is to say, if \( S_0(\tau) \) has no zeros in \( I \), then \( S_n(\tau) \) have no zeros in \( I \) for all \( n \in \mathbb{N} \), and if the function \( S_n(\tau) \) has a positive zero \( \tau \in I \) for some \( n^* \in \mathbb{N} \), there exists at least one zero satisfying \( S_n(\tau^*) = 0 \) and \( \frac{dS_n(\tau^*)}{d\tau} \neq 0 \) with \( n \leq n^* \).

Define the set of possible Hopf bifurcation values by

\[
J = \{ \tau \in [0, \tau^*) \mid S_n(\tau) = 0, n \in \mathbb{N} \},
\]

from the decreasing property of \( S_n \) w.r.t. \( n \), we know the set \( J \) is finite, so denote the minimum and maximum element to be \( \tau_{\min} \) and \( \tau_{\max} \) respectively. We first state the stability of \( y_1 \) and \( y_2 \) when \( \tau = 0 \) in the following.

**Theorem 3.2.** \( y_1 \) is unstable and \( y_2 \) is stable when \( \tau = 0 \).
Thus it follows that $y$.

Denote $G$. Notice that $F$. Theorem 3.3.

We state the main results in this section.

**Proof.** When $\tau = 0$, from (2) we have

$$\lambda_i = \alpha b_i - D$$

$$= \alpha \frac{d}{dy} \left( \frac{y^n}{1 + \beta y^m} \right) \bigg|_{y = y_i} - D$$

$$= \frac{d}{dy} \left( \alpha \frac{y^n}{1 + \beta y^m} - Dy \right) \bigg|_{y = y_i}$$

$$= \frac{d}{dy} \left( -\frac{y}{1 + \beta y^m} F(y) \right) \bigg|_{y = y_i}.$$  

Denote $G(y) = -\frac{y}{1 + \beta y^m} F(y)$, we have

$$\lambda_i = G'(y_i) = -\frac{\partial (\frac{y}{1 + \beta y^m})}{\partial y} F(y_i) - \frac{y}{1 + \beta y^m} F'(y_i).$$

Notice that $F(y_i) = 0$ for $i = 1, 2$ and $F'(y_1) < 0, F'(y_2) > 0$, we get

$$G'(y_1) > 0 \quad \text{and} \quad G'(y_2) < 0.$$  

Thus it follows that $y_1$ is unstable and $y_2$ is stable when $\tau = 0$.

Combining with the stability of $y_1$ and $y_2$ when $\tau = 0$ in the previous theorem, we state the main results in this section.

**Theorem 3.3.** Assume $\xi > 1$ and $\tau < \tau^0$ are satisfied for system (1), the following conclusions hold:

(i) If $I$ is empty or $S_0(\tau)$ has no positive zeros in $(0, \tau^1)$ when $I$ is non-empty, which implies set $J$ is empty, then equation (3) has no pairs of purely imaginary roots, thus $y_1$ is unstable and $y_2$ is asymptotically stable for all $\tau \in (0, \tau^0)$.

(ii) If $J \neq 0$ and $\delta(\tau^*) \neq 0$ for $\tau^* \in J$, then (1) undergoes a Hopf bifurcation at $y_1$ or $y_2$ when $\tau = \tau^*$. At this time, $y_1$ is always unstable for all $\tau \in (0, \tau^0)$, and $y_2$ is asymptotically stable for $\tau \in (0, \tau_{\min}) \cup (\tau_{\max}, \tau^0)$.

### 3.2. Stability and direction of the Hopf bifurcation.

When conditions in Theorem 3.3(ii) hold, there are small amplitude periodic solutions bifurcating at $\tau \in J$. In this section, we shall study the direction, stability, and the period of the bifurcating periodic solution. The way to do this is the combination of the normal form method and center manifold theory in [9]. Without loss of generality, let $\tau^*$ be any critical value such that equation (3) has a pair of purely imaginary roots $\pm i\omega^*$, and system (1) undergoes Hopf bifurcation at $y_2$. Then, by setting $\tau = \tau^* + \mu$, $\mu = 0$ is the Hopf bifurcation value of (1).

Let $\hat{y}(t) = y(\tau t) - y_2$ to normalize the delay and move $y_2$ to the origin. Then (1) becomes

$$\hat{y}(t) = \tau \alpha e^{-\delta \tau} \left( \frac{y(t - 1) + y_2}{1 + \beta (y(t - 1) + y_2)^m} - D(y(t) + y_2) \right)$$

$$= -\tau D y(t) + \tau \alpha e^{-\delta \tau} b_2 y(t - 1) + \tau \alpha e^{-\delta \tau} d_2 \frac{y^2(t - 1)}{2}$$

$$+ \tau \alpha e^{-\delta \tau} e_2 \frac{y^3(t - 1)}{6} + O(y^4(t - 1)) \quad (13)$$
where \( b_2 \) defined in [1], \( d_2 \) and \( e_2 \) defined in the following

\[
d_2 = \frac{d^2}{dy^2} \left( \frac{y^n}{1 + \beta y^m} \right) \bigg|_{y=y_2}
\]

\[
= \left[ n(n-1)y_2^{n-2} + \beta((2n-m)(n-1)
- m(m+n))y_2^{m+n-2} + \beta^2(m-n)(m-n+1)y_2^{2m+n-2} \right] / (1 + \beta y_2^m)^3,
\]

\[
e_2 = \frac{d^3}{dy^3} \left( \frac{y^n}{1 + \beta y^m} \right) \bigg|_{y=y_2}
\]

\[
= \left[ \beta((2n-m)(n-1) - m(m+n))((m+n-2)y_2^{m+n-3}
+ \beta^2(m-n)(m-n+1)(2m+n-2)y_2^{2m+n-3}
+ \beta n(n-1)(n-2-3m)y_2^{m+n-3} + n(n-1)(n-2)y_2^{n-3}
+ \beta^2((2n-m)(n-1) - m(m+n))(n-2m-2)y_2^{2m+n-3}
+ \beta^3(m-n)(m-n+1)(m+n-2)y_2^{3m+n-3} \right] / (1 + \beta y_2^m)^4.
\]

Notice that \( b_2, d_2, \) and \( e_2 \) depend on the parameter \( \tau \), since \( y_2 \) is continuous function of \( \tau \).

For \( \phi \in C \left( [-1, 0], \mathbb{R} \right) \), define

\[
L_\mu(\phi) = - (\tau^* + \mu)D\phi(0) + (\tau^* + \mu)\alpha e^{-\delta(\tau^*+\mu)}b_2\phi(-1)
\]

and

\[
G(\mu, \phi) = (\tau^* + \mu)\alpha e^{-\delta(\tau^*+\mu)} \left( \frac{d_2}{2} \phi^2(-1) + \frac{e_2}{2} \phi^3(-1) + O(\phi^4(-1)) \right)
\]

By the Riesz representation theorem, there exists a bounded variation function \( \eta(\theta, \mu) \) for \( \theta \in [-1, 0] \), such that

\[
L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta, \mu) \phi(\theta), \text{ for } \phi \in C \left( [-1, 0], \mathbb{R} \right).
\]

In fact, \( \eta(\theta, \mu) \) can be chosen as

\[
\eta(\theta, \mu) = \begin{cases} 
- (\tau^* + \mu)D, & \theta = 0, \\
0, & \theta \in (-1, 0), \\
- (\tau^* + \mu)\alpha e^{-\delta\tau}b_2, & \theta = -1.
\end{cases}
\]

Define operators \( A(\mu) \) and \( R(\mu) \) as

\[
A(\mu)\phi(\theta) = \begin{cases} 
\frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\
\int_{-1}^{0} d\eta(\xi, \mu)\phi(\xi), & \theta = 0,
\end{cases}
\]

and

\[
R(\mu)\phi(\theta) = \begin{cases} 
0, & \theta \in [-1, 0), \\
G(\mu, \phi), & \theta = 0.
\end{cases}
\]

Then system [13] is equivalent to the following operator equation

\[
\dot{y}_t = A(\mu)y_t + R(\mu)y_t,
\]
where \( y_t(\theta) = y(t + \theta) \) for \( \theta \in [-1, 0] \). For \( \psi \in C^1([0, 1], \mathbb{R}) \), define an operator

\[
A^* \psi(s) = \begin{cases} 
-\frac{d\psi(s)}{ds}, & s \in (0, 1], \\
\int_{-1}^{0} d\eta(\xi, 0)\psi(-\xi), & s = 0,
\end{cases}
\]

and a bilinear inner form

\[
\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi,
\]

where \( \eta(\theta) = \eta(\theta, 0) \). Then \( A(0) \) and \( A^* \) are adjoint operators. As shown in Section 3.1, we know that \( \pm i\omega^* \tau^* \) are eigenvalues of \( A(0) \), thus, they are also eigenvalues of \( A^* \). It can be verified that the vectors

\[
q(\theta) = e^{i\omega^* \tau^* \theta}, \quad \theta \in [-1, 0],
\]

and

\[
q^*(s) = \frac{1}{M} e^{i\omega^* \tau^* s}, \quad s \in [0, 1],
\]

are the eigenvectors of \( A(0) \) and \( A^* \) corresponding to the eigenvalues \( i\omega^* \tau^* \) and \( -i\omega^* \tau^* \), respectively. When choosing

\[
M = 1 + \tau^* \alpha e^{-\delta \tau^*} b_2 e^{-i\omega^* \tau^*}
\]

we insure \( \langle q^*(s), q(\theta) \rangle = 1 \).

Following the algorithms provided in Hassard [9] and using a computation process similar to that in [17, 18, 6], we obtain the following coefficients:

\[
\begin{align*}
g_{20} &= \frac{\tau^* \alpha e^{-\delta \tau^*} d_2}{M} e^{-2i\omega^* \tau^*}, \\
g_{11} &= \frac{\tau^* \alpha e^{-\delta \tau^*} d_2}{M}, \\
g_{02} &= \frac{\tau^* \alpha e^{-\delta \tau^*} d_2}{M} e^{2i\omega^* \tau^*}, \\
g_{21} &= \frac{\tau^* \alpha e^{-\delta \tau^*}}{M} \left[ d_2 \left[ W_{20}(-1) e^{i\omega^* \tau^*} + 2W_{11}(-1) e^{-i\omega^* \tau^*} \right] + e_2 e^{-i\omega^* \tau^*} \right],
\end{align*}
\]

where

\[
\begin{align*}
W_{20}(-1) &= \frac{ig_{20}}{\omega^* \tau^*} e^{-i\omega^* \tau^*} + \frac{ig_{02}}{3\omega^* \tau^* M} e^{i\omega^* \tau^*} + E_1 e^{-2i\omega^* \tau^*}, \\
W_{11}(-1) &= -\frac{ig_{11}}{\omega^* \tau^*} e^{-i\omega^* \tau^*} + \frac{ig_{11}}{\omega^* \tau^* M} e^{i\omega^* \tau^*} + E_2,
\end{align*}
\]

and

\[
\begin{align*}
E_1 &= \frac{\alpha e^{-\delta \tau^*} d_2 e^{-2i\omega^* \tau^*}}{2i\omega^* + M - \alpha e^{-\delta \tau^*} b_2 e^{-2i\omega^* \tau^*}}, \\
E_2 &= \frac{\alpha e^{-\delta \tau^*} d_2}{M - \alpha e^{-\delta \tau^*} b_2}.
\end{align*}
\]

So far, \( g_{20}, \ g_{11}, \ g_{02}, \ g_{21} \) can be calculated exactly. According to Hassard et al. [9],

\[
c_1(0) = \frac{i}{2\omega^* \tau^*} \left( g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2},
\]
then
\[ \text{Re}(c_1(0)) = \text{Re}\left( \frac{i \tau^*}{2 \omega^*} \left( \frac{\alpha e^{-\delta \tau^*}}{M} d_2 \right)^2 \left( e^{-2i \omega^* \tau^*} - 2 - \frac{1}{3} e^{4i \omega^* \tau^*} \right) + \frac{\tau^* \alpha e^{-\delta \tau^*}}{M} \left[ d_2 \left[ W_{20}(-1)e^{i \omega^* \tau^*} + 2W_{11}(-1)e^{-i \omega^* \tau^*} \right] + e^{2i \omega^* \tau^*} \right] \right), \]

Furthermore, we can compute the following quantities:
\[ \mu_2 = -\frac{\text{Re}(c_1(0))}{\text{Re}(\lambda(\tau^*))}, \]
\[ \beta_2 = 2\text{Re}(c_1(0)), \]
\[ T_2 = -\frac{\text{Im}(c_1(0)) + \mu_2 \text{Im}(\lambda(\tau^*))}{\omega^* \tau^*}, \]

which determine the properties of bifurcating periodic solutions. From the discussion in Section 3.1, we have the following results immediately.

Assume that the conditions in (ii) of Theorem 3.3 hold. Then \( \mu_2, \beta_2, T_2 \) determine the direction, stability, and period of the corresponding Hopf bifurcation, respectively:

(i) The direction of Hopf bifurcation of system (1) at \( y^* \) when \( \tau = \tau^* \) is backward (forward) if \( \mu_2 < 0 (\mu_2 > 0) \), that is, there exists a bifurcating periodic solution for \( \tau < \tau^* (\tau > \tau^*) \).

(ii) The bifurcating periodic solution on the center manifold is unstable (stable) if \( \beta_2 > 0 (\beta_2 < 0) \). Particularly, the stability of the bifurcating periodic solutions of (1) is same as that of bifurcating periodic solutions on the center manifold when \( \tau^* = \tau_{\text{min}} \) and \( \tau^* = \tau_{\text{max}} \).

(iii) The period of the bifurcating periodic solution decreases (increases) if \( T_2 < 0 (T_2 > 0) \).

4. Global existence of periodic solutions. In Section 3.1 Theorem 3.3(ii) shows that periodic solutions can bifurcate from \( y^* \) when \( \tau \) passes through certain critical values. But will the periodic solutions last for a long interval, this is what we concerned in this section. The method we do this is the global Hopf bifurcation theory proposed by Wu [24].

We first make some notations as in Wu [24] and verify that the assumptions (A1)–(A4) in [24] hold. Copy (14) as follows for convenience
\[ \dot{x}(t) = \alpha e^{-\delta \tau} \frac{\alpha^n(t - \tau)}{1 + \beta \alpha^m(t - \tau)} - Dx(t), \]

and rewrite (14) as a general functional differential equation with two parameters \( \tau \) and \( T \) in the following form
\[ \dot{x}(t) = F(x_t, \tau, T), \ (\tau, T) \in \mathbb{R} \times \mathbb{R}_+, \]

where
\[ F : X \times \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R} \]
is completely continuous with \( x_t \in X = C([-\tau, 0], \mathbb{R}_+) \) and \( x_t(\theta) = x(t + \theta) \) for \( \theta \in [-\tau, 0] \). Restricting \( F \) on \( \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \), we have
\[ \hat{F} := F|_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+} : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}. \]
It follows from (14) and (15) that \( \hat{F} \) takes the form
\[
\hat{F}(x, \tau, T) = ae^{-\delta \tau} \frac{x^n}{1 + \beta x^m} - Dx.
\]

Obviously, \( \hat{F} \) is twice continuously differentiable, thus assumption (A1) in [24] holds.

In the following, define a closed subset \( \Sigma(F) \) of \( X \times \mathbb{R}_+ \times \mathbb{R}_+ \) by
\[
\Sigma(F) = \text{Cl}\left\{(x, \tau, T) \in X \times \mathbb{R}_+ \times \mathbb{R}_+ : x \text{ is } T\text{-periodic solution of (15)}\right\}.
\]

Let \( C(\hat{x}_0, \tau_0, T_0) \) denote the connected component of \((\hat{x}_0, \tau_0, T_0)\) in \( \Sigma(F) \) and the set of stationary solution of (15) by
\[
N(F) = \left\{(\hat{x}, \tau, T) : \alpha e^{-\delta \tau} \frac{\hat{x}^n}{1 + \beta \hat{x}^m} = D\hat{x}\right\}.
\]

It follows from Section 2 that \( N(F) = \{(y_0, \tau, T), (y_1, \tau, T), (y_2, \tau, T)\} \) when \( \tau < \tau^0 \).

We also know \( \lambda = 0 \) is not a characteristic value of any stationary solution of (15), thus the assumption (A2) in [24] is confirmed. Local Hopf bifurcation in Section 3.1 shows that \( C(y^*, \tau^*, 2\pi) \) is not empty.

The characteristic function of (15) at any stationary solution \((x^*, \tau, T)\) is
\[
\Delta_{(x^*, \tau, T)}(\lambda) = \lambda - DF(x^*, \tau, T)(e^\lambda) = \lambda + D - \alpha b^* e^{-\delta \tau} e^{-\lambda}.
\]

Then the smoothness of \( F \) and \( \Delta_{(\hat{x}_0, \tau_0, T_0)}(\lambda) \) in the condition (A3) holds.

Literature [24] gives a definition of a stationary solution to be a center: if \( \Delta_{(\hat{x}_0, \tau_0, T_0)} \left( im \frac{2\pi}{T} \right) = 0 \) for some positive integer \( m \). Moreover, the center is said to be isolated if it is the only center in some neighborhood of \((\hat{x}_0, \tau_0, T_0)\) and it has only finitely many purely imaginary characteristic values of the form \( im \frac{2\pi}{m} \) (\( m \) is an integer). The analysis of eigenvalues in Section 2 shows that there is no purely imaginary roots for \( \Delta_{(y_0, \tau, T)}(\lambda) = 0 \).

For \((x^*, \tau, T) \in N(F)\), all the positive integer \( m \), such that \( im \frac{2\pi}{m} \) is the characteristic values of \( \text{det}\Delta_{(x^*, \tau, T)}(\lambda) = 0 \), collected to be a set
\[
J(x^*, \tau, T) = \left\{ m \in \mathbb{N} : \Delta_{(x^*, \tau, T)} \left( im \frac{2\pi}{T} \right) = 0 \right\}.
\]

Here we start from the premise that local Hopf bifurcation exists, without loss of generality, there are 2\( k \) intersections between \( S_n(\tau) \) and \( \tau \)-axis. Thus the stationary solution \((y^*, \tau_j, \frac{2\pi}{\omega_j})\) is an isolated center of (15), where \( \omega_j \) is the positive root of (10).

Lemma 3.1 implies that there exists two positive constant \( \epsilon, \delta \), and a smooth curve \( \lambda : (\tau_j - \delta, \tau_j + \delta) \rightarrow \mathcal{C} \) such that
\[
\Delta_{(y^*, \tau_j, \frac{2\pi}{\omega_j})}(\lambda(\tau)) = 0, \quad |\lambda - i\omega_j| < \epsilon
\]
for all \( \tau \in (\tau_j - \delta, \tau_j + \delta) \). Denote a square region
\[
\Omega_{\epsilon, \frac{2\pi}{\omega_j}} = \left\{(u, T) : 0 < u < \epsilon, \left| T - \frac{2\pi}{\omega_j} \right| < \epsilon \right\}
\]
When \(|\tau - \tau_j| \leq \delta\) and \((u, T) \in \partial \Omega_c, \frac{2\pi}{\omega_j}\), we have

\[
\Delta_{(y^*, \tau, T)} \left( u + i \frac{2\pi}{T} \right) = 0, \text{ if and only if } \tau = \tau_j, \; u = 0, \; T = \frac{2\pi}{\omega_j},
\]

which confirm the assumption (A4).

Moreover, the \(m\)-th crossing number is defined in [24]. In our paper, \(im\frac{2\pi}{\omega_j}\) is a purely imaginary characteristic value of \(\Delta_{(x^*, \tau, T)}(\lambda) = 0\) if and only if \(m = 1\). Thus there is only one element in \(J(y^*, \tau_j, \frac{2\pi}{\omega_j})\). Consider the crossing number

\[
\gamma_1 \left( y^*, \tau_j, \frac{2\pi}{\omega_j} \right) = -\text{Sign} \left\{ \frac{d\text{Re}\lambda(\tau)}{d\tau} \bigg|_{\lambda = \omega(\tau_j)} \right\}
\]

\[
= -\text{Sign} \left\{ \frac{dS_m(\tau)}{d\tau} \bigg|_{\tau = \tau_j} \right\}
\]

\[
= \begin{cases} -1, & j = 0, \ldots, k - 1 \\ 1, & j = k, \ldots, 2k - 1 \end{cases}
\]

Global bifurcation theorem of Theorem 3.3 in [24] shows that either

(a) \(C \left( y^*, \tau_j, \frac{2\pi}{\omega_j} \right) \) is unbounded in \(X \times (0, \tau^0) \times \mathbb{R}^+\), or

(b) \(C \left( y^*, \tau_j, \frac{2\pi}{\omega_j} \right) \) is bounded, \(C \left( y^*, \tau_j, \frac{2\pi}{\omega_j} \right) \cap N(F) \) is finite and

\[
\sum_{(\hat{x}_0, \tau, T) \in C \left( y^*, \tau_j, \frac{2\pi}{\omega_j} \right) \cap N(F)} \gamma_m(x, \tau, T) = 0
\]

for all \(m \in J(x, \tau, p)\), where \(\gamma_m(x, \tau, p)\) is the \(m\)-th crossing number of \((x, \tau, p)\), otherwise \(\gamma_m(x, \tau, T)\) is zero.

Ultimately uniformly boundedness in Lemma 2.1 implies the following conclusion holds.

**Lemma 4.1.** All positive periodic solutions of system [15] are uniformly bounded, that is, the projection of \(C \left( y^*, \tau_j, \frac{2\pi}{\omega_j} \right) \) onto \(X\) is bounded.

Theorem 2.3 shows that \(y_1 = 0\) is globally asymptotically stable when \(\tau > \tau^0\). This implies the following statement holds.

**Lemma 4.2.** The projection of \(C \left( y^*, \tau_j, \frac{2\pi}{\omega_j} \right) \) onto \(\tau\)-axis is bounded.

**Lemma 4.3.** Assume that \(\alpha |b|^2 e^{-\delta \tau} < \sqrt{2D}\). Then system (14) has no periodic solutions of period \(4\tau\).

**Proof.** Let \(x(t)\) be a periodic solution to (14) with period \(4\tau\). Set

\[
x_1(t) = \alpha e^{-\delta \tau} \frac{x_1^n(t)}{1 + \beta x_2^n(t)} - Dx_1(t)
\]

\[
x_2(t) = \alpha e^{-\delta \tau} \frac{x_2^n(t)}{1 + \beta x_3^n(t)} - Dx_2(t)
\]

Then \(x = (x_1, x_2, x_3, x_4)\) is a periodic solutions of the system of ordinary differential equations
\[
\frac{dx_3(t)}{dt} = \alpha e^{-\delta \tau} \frac{x_4^n(t)}{1 + \beta x_4(t)} - Dx_3(t)
\]
\[
\frac{dx_4(t)}{dt} = \alpha e^{-\delta \tau} \frac{x_3^n(t)}{1 + \beta x_3(t)} - Dx_4(t)
\] (16)

To rule out the 4-periodic solution to (14), it suffices to prove the nonexistence of nonconstant periodic solutions of (16). We use a general Bendixson’s criterion in higher dimensions developed by Li and Muldowney [12]. For \( x \in \mathbb{R}^4 \), the Jacobian matrix \( J = \frac{\partial f}{\partial x} \) of (16) is

\[
J = \begin{pmatrix}
-D & \alpha b'_1 e^{-\delta \tau} & 0 & 0 \\
0 & -D & \alpha b'_2 e^{-\delta \tau} & 0 \\
0 & 0 & -D & \alpha b'_3 e^{-\delta \tau} \\
\alpha b'_1 e^{-\delta \tau} & 0 & 0 & -D
\end{pmatrix}
\]

where

\[
b'_i = \frac{d}{dy} \left( \frac{y^n}{1 + \beta y^m} \right) \Big|_{y=x_i} = \frac{nx^{n-1} - \beta x^{n+m-1}}{(1 + \beta x^m)^2}, \quad i = 1, 2, 3, 4.
\]

Calculate the second additive compound matrix \( J^{[2]} \) follow the algorithm in Li and Muldowney [13] obtains

\[
J^{[2]} = \begin{pmatrix}
-2D & \alpha b'_1 e^{-\delta \tau} & 0 & 0 & 0 & 0 \\
0 & -2D & \alpha b'_2 e^{-\delta \tau} & \alpha b'_2 e^{-\delta \tau} & 0 & 0 \\
0 & 0 & -2D & \alpha b'_3 e^{-\delta \tau} & 0 & 0 \\
-\alpha b'_1 e^{-\delta \tau} & 0 & 0 & -2D & \alpha b'_3 e^{-\delta \tau} & 0 \\
o & -\alpha b'_1 e^{-\delta \tau} & 0 & 0 & -2D & \alpha b'_3 e^{-\delta \tau}
\end{pmatrix}
\]

Theorem 4.1 in Muldowney [16] shows that

\[
\mu \left( \frac{\partial f^{[2]}}{\partial x} \right) < 0, \quad \text{or} \quad \mu \left( \frac{\partial f^{[2]}}{\partial x} \right) < 0
\]

holds for all \( x \in \mathbb{R}^4 \) implies that system (16) has no nonconstant periodic solutions, where the Lozinski\'{j} measure \( \mu(A) \) is defined in [3] to be the right-hand derivative

\[
\mu(A) = \lim_{h \to 0^+} \frac{||I + hA|| - 1}{h}.
\]

To apply this conclusion, we choose a vector norm in \( \mathbb{R}^6 \) the same as in [22] that

\[
||x|| = ||x_1, x_2, x_3, x_4, x_5, x_6|| = \max \{ \sqrt{2} |x_1|, |x_2|, \sqrt{2} |x_3|, \sqrt{2} |x_4|, |x_5|, \sqrt{2} |x_6| \}.
\]

Then the matrix norm of \( A \) induced by previous vector norm is

\[
||A|| = \sup ||Ax|| / ||x||.
\]

Apply this into \( \mu \left( J^{[2]} \right) \) we have

\[
\mu \left( J^{[2]} \right) = \lim_{h \to 0^+} \frac{||I + hJ^{[2]}|| - 1}{h}
\]

\[
= \max \left\{ \sqrt{2} \left(-\sqrt{2} D + \alpha |b'_1| e^{-\delta \tau} \right), \sqrt{2} \left(-\sqrt{2} D + \frac{\alpha |b'_1| e^{-\delta \tau}}{2} \right) \right\}
\]
where \( i \in \{1, 2, 3, 4\} \) and \((j, k) \in \{(1, 3), (2, 4)\}\). It then follows that \( \mu(J^{(2)}) < 0 \) if and only if \( \alpha |b'_i| e^{-\delta \tau} < \sqrt{2D} \).

**Lemma 4.4.** System (14) has no periodic solutions with period \( 2\tau \).

**Proof.** Let \( x(t) \) be a periodic solution to (14) of period \( 2\tau \), then \( x_1(t) = x(t) \) and \( x_2(t) = x(t-\tau) \) are periodic solutions of the system of ordinary differential equations

\[
\begin{align*}
\dot{x}_1(t) &= \alpha e^{-\delta \tau} \frac{x_n^n(t)}{1 + \beta x_n^n(t)} - Dx_1(t) \\
\dot{x}_2(t) &= \alpha e^{-\delta \tau} \frac{x_1^1(t)}{1 + \beta x_1^m(t)} - Dx_2(t)
\end{align*}
\]

(17)

Let \((P(x_1, x_2), Q(x_1, x_2))\) denote the vector field of (17), then

\[\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} = -2D < 0\]

for all \((x_1, x_2) \in \mathbb{R}^2\). Thus no nonconstant periodic solutions of (17) exists due to the classical Bendixsons negative criterion, i.e. there is no periodic solutions with period \( 2\tau \) of (14). This completes the proof. \(\square\)

**Remark 1.** Lemma (4.4) implies that (14) has no periodic solutions with period \( \tau_k \), \( k = 1, 2, 3, \ldots \).

In the following, we study the period of the periodic solution bifurcated from \( y_1 \) and \( y_2 \) under the guideline of [20]. Similar as the proof of Theorem 3.1 in [20], the period \( T_j := \frac{2\pi}{\omega_j} \) of the periodic solution at \( \tau_j \) (or \( \tau_{2k-1-j} \)) with \( j = 0, 1, \ldots, k-1 \) is

\[\frac{2\sigma_j}{2j+1} < T_j < \frac{4\tau_j}{4j+1},\]

This reveals

**Lemma 4.5.** The projection of \( C\left(y^*, \tau_j, \frac{2\pi}{\omega_j}\right) \) onto \( T \) is bounded.

Up to now, form Lemmas 4.1, 4.2 and 4.5 we know the connected component \( C\left(y^*, \tau_j, \frac{2\pi}{\omega_j}\right) \) is bounded, which exclude case (a) of Theorem 3.3 in Wu [24], and hence (b) is satisfied. Follow the description in [20] and the fact of the nonexistence of non-constant 1-periodic solution, we know each local periodic solution bifurcated from the Hopf bifurcation exactly connects another corresponding periodic solution. That is the existence of periodic solution in a wide range which is the so called global Hopf bifurcation.

**Theorem 4.6.** There exists

\[0 < \tau_0 < \tau_1 < \cdots < \tau_{2k-1},\]

and

\[0 < \tau_0' < \tau_1' < \cdots < \tau_{2j-1}',\]

such that (8) has a pair of simple imaginary roots \( \pm i\omega^* \) for \( y_1 \) and \( y_2 \) respectively. Assume

\[
\left( \bigcup_{n=0}^{k-1} \left( \frac{2\tau_n^1}{2n+1}, \frac{4\tau_n^1}{4n+1} \right) \right) \cap \left( \bigcup_{n=0}^{j-1} \left( \frac{2\tau_n^2}{2n+1}, \frac{4\tau_n^2}{4n+1} \right) \right) = \emptyset
\]

and the condition in Lemma 4.3 is satisfied. Then for system (14) we have the following results:
(i) All the global Hopf branches are bounded for $j = 0, 1, 2, \ldots, 2k - 1$.
(ii) The global Hopf branch from $\tau_j$ connects and only connects the one of $\tau_{2k-1-j}$, $j = 1, 2, \ldots, k - 1$.
(iii) For each $\tau \in (\tau_j, \tau_{2k-1-j})$, $j = 0, 1, 2, \ldots, k - 1$, system (15) has at least $j + 1$ periodic solution, furthermore, there is a periodic solution lies in $(\frac{1}{j+1}, \frac{1}{j})$.

5. Numerical simulations. In this section, we carry out some numerical simulations to illustrate the properties of Hopf bifurcations which we studied in Sections 3 and 4 theoretically. The parameters we choose are

$$\alpha = 0.8, \beta = 0.7, \delta = 0.02, D = 0.32, n = 3, m = 8.$$  

(18)

From the expression of $\bar{y}$ and $F(y)$ in Section 2, we know $\bar{y}, y_1,$ and $y_2$ change with $\tau$. Plotting them in the following Figure 2, which coincides well with the conclusion in Theorem 2.2.

![Figure 2](image-url)

**Figure 2.** Figures of $\bar{y}$ and equilibria $y_1, y_2$ with parameters given in (18).

By direct calculation, we have $\tau^0 \approx 22.2$, which is the critical value for the existence of two positive equilibria $y_1$ and $y_2$. Besides, $\tau_1^1 \approx 22.2$, $\tau_1^1 \approx 15.6$ are critical points for the existence of purely imaginary eigenvalues for $y_1$ and $y_2$, respectively.

Denote $S_n$ and $S'_n$ as in (11) corresponding to $y_1$ and $y_2$, respectively, and their intersections with $\tau$-axis imply two Hopf bifurcation points denoted by $\tilde{\tau}_0 \approx 6.3$, $\tilde{\tau}_1 \approx 21.9$ (see Figure 3(a)), and $\tau_0 \approx 2.8$, $\tau_1 \approx 14.6$ (see Figure 3(b)), respectively.

By Theorem 3.3, we know that the positive equilibrium $y_1$ is unstable on $[0, \tau^0)$ (see Figure 3(a)); the positive equilibrium $y_2$ is asymptotically stable when $\tau \in [0, \tau_0) \cup (\tau_1, \tau^0)$ (see Figures 5(a) and 7(b)), and unstable when $\tau \in (\tau_0, \tau_1)$ (see Figure 5(b)).

Following the formula (12) and the algorithm derived in Section 3.2, we calculate some important quantities of the periodic solution bifurcating from $y_2$ (see Table 1).

The previous discussion shows that for system (11) with the data (18):

1. The direction of the Hopf bifurcation at $y_2$ is forward when $\tau = \tau_0$, and backward when $\tau = \tau_1$.
2. The bifurcating periodic solutions bifurcated from $y_1$ are unstable, and the bifurcating periodic solutions bifurcated from $y_2$ are stable.
Figure 3. Graphs of $S_n(\tau)$ on $[0, \tau^1)$ with parameters given in (18).

Figure 4. $y_1 \approx 0.6986$ is unstable, and sustained oscillation occurs when $\tau \in (0, \tau^0)$, where $0 < \tau = 8 < \tau^0 \approx 22.2$, and the initial condition is $\varphi = 0.8$ for $t \in [-\tau, 0]$.

Table 1. List of quantities of periodic solution bifurcating from $y_2$ under (18)

| $\delta$ | Re($c_1(0)$) | $\mu_2$ | $\beta_2$ |
|----------|--------------|----------|-----------|
| $\tau_0 \approx 2.8$ | $> 0$ | $-37.3115 < 0$ | $> 0 < 0$ |
| $\tau_1 \approx 14.6$ | $< 0$ | $-72.7255 < 0$ | $< 0 < 0$ |

From the study in Section 4 one can see that there exists global Hopf branches since condition $\alpha |b|e^{-\delta \tau} < \sqrt{2D}$ in Lemma 4.3 holds (see Figure 4).
Figure 5. $y_2$ is asymptotically stable when $\tau \in [0, \tau_0) \cup (\tau_1, \tau_0)$, and the initial condition is $\varphi = 1.1$ for $t \in [-\tau, 0]$.

Figure 6. $y_2 \approx 1.0962$ is unstable, and sustained oscillation occurs when $\tau \in (\tau_0, \tau_1)$, where $2.8 \approx \tau_0 < \tau = 10 < \tau_1 \approx 14.8$, and the initial condition is $\varphi = 1.1$ for $t \in [-\tau, 0]$.

Figure 7. $(\tau, h)$ plane, where $h = \sqrt{2D - \alpha |b'|e^{-\delta \tau}}$. 
In Theorem 4.6, we have \( k = j = 1 \), and
\[
(k - 1) \bigcup_{n=0}^{k-1} \left( \frac{2\tau_n^1}{2n+1} \cdot \frac{4\tau_n^1}{4n+1} \right) \cap \left( j - 1 \bigcup_{n=0}^{j-1} \left( \frac{2\tau_n^2}{2n+1} \cdot \frac{4\tau_n^2}{4n+1} \right) \right) 
\approx (12.6, 25.2) \cap (5.6, 11.2) = \emptyset.
\]
This reveals that conditions in Theorem 4.6 hold and the connected components \( C \left( y_1, \tilde{\tau}_0, \frac{2\pi}{\tilde{\omega}_0} \right) \) and \( C \left( y_2, \tau_0, \frac{2\pi}{\omega_0} \right) \) are separated.

In the following, we carry out numerical simulation to show the Hopf bifurcation branch in connected component \( C \left( y_2, \tau_0, \frac{2\pi}{\omega_0} \right) \) connecting \( \tau_0 = 2.8 \) and \( \tau_1 = 14.6 \) (see Figure 8) using DDE-BIFTOOL developed by Engelborghs et al.\[4, 5\]. But it is a little pity that we didn’t draw the Hopf bifurcation branch in connected component \( C \left( y_1, \tilde{\tau}_0, \frac{2\pi}{\tilde{\omega}_0} \right) \). We should point out that it may be difficult to separate \( C \left( y_1, \tilde{\tau}_0, \frac{2\pi}{\tilde{\omega}_0} \right) \) from \( C \left( y_2, \tau_0, \frac{2\pi}{\omega_0} \right) \) by period when there are too many zeros of \( \tilde{S}_n(\tau) \) and \( S_n(\tau) \).

![Figure 8. Hopf bifurcation branch on the (τ, d) plane, where \( d = \max y(t) - \min y(t) \).](image)

In the following, we add the periodic solutions obtained by global Hopf bifurcation bifurcated from \( y_1 \) connecting \( \tau_0 = 2.8 \) and \( \tau_1 = 14.6 \) on Figure 2. Then we get their stability in the following Figure 9. As we can see, when \( \tau \in (0, \tau_0) \cup (\tau_1, \tau^0) \), there are two stable equilibria 0 and \( y_2 \). When \( \tau \in (\tau_0, \tau_1) \), besides a stable equilibria 0, there is a stable periodic solutions bifurcated from \( y_2 \). This is known as bistable.

Numerical simulations in this section and theoretical analysis in previous sections shows that the introduction of the stage-structure term \( e^{-\delta \tau} \) may change the number of possible Hopf bifurcation points from infinite to finite. Furthermore, the strong Allee effect model (1) with \( m > n > 1 \) may have two positive equilibria, which is more complicated than the classical Mackey-Glass equation, who has only one positive equilibrium. In fact, we speculate that the long-term transients phenomenon observed by Figure 4 and Figure 6 in [15] might be the result of the unstable periodic solutions bifurcated from \( y_1 \) due to Hopf bifurcation.
Figure 9. Stability of equilibria $0$, $y_1$, $y_2$ and periodic solutions bifurcated from $y_2$.

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