KÄHLER-RICCI SOLITONS ON HOROSPHERICAL VARIETY

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1. INTRODUCTION

The founding article on the Kähler-Ricci solitons is Hamilton’s article [Ham88]. The Kähler-Ricci solitons are natural generalizations of the Kähler-Einstein metrics and appear as fixed points of the Kähler-Ricci flow. On a Fano compact Kähler manifold $M$, a Kähler metric $g$ is a Kähler-Ricci soliton if its Kähler form $\omega_g$ satisfies:

$$\text{Ric}(\omega_g) - \omega_g = \mathcal{L}_X \omega_g,$$

where $\text{Ric}(\omega_g)$ is the Ricci form of $g$ and $\mathcal{L}_X \omega_g$ is the Lie derivative of $\omega_g$ along a holomorphic vector field $X$ on $M$. Usually, we denote the Kähler-Ricci soliton by the pair $(g, X)$ and $X$ is called the solitonic vector field. We immediately note that if $X = 0$ then $g$ is a Kähler-Einstein metric. Moreover, if $X \neq 0$ then we say the Kähler-Ricci soliton is non-trivial.

Firstly, the study of the solitonic vector field $X$ was done in the article [TZ00, TZ02]. Thanks to the Futaki function, the authors discovered an obstruction to the existence of Kähler-Ricci soliton and proved that $X$ is in the center of a reductive Lie subalgebra $\eta_r(M)$ of $\eta(M)$, which is the set of all holomorphic vector fields. This study also gives us a result about the Kähler-Ricci soliton’s unicity (theorem 0.1 in [TZ00]).

Subsequently, the study was supplemented by Wang, Zhu in [WZ04] where they show the existence of Kähler-Ricci solitons on toric varieties using the continuity method. This work was supplemented by a study of the Ricci flow by Zhu in [Zhu12] on the toric varieties which showed that the Kähler-Ricci flow converges to the Kähler-Ricci soliton of the toric variety. The result about existence of Kähler-Ricci solitons has been extended to cases of toric fibrations by Podesta and Spiro in [PS10]. Recently, the result concerning the convergence of the Ricci flow has been also extended in [Hua17].

In 2015, Delcroix used the approach of Zhu and Wang in the case of Kähler-Einstein metrics on some compactifications of reductive groups. In his paper [Del15], the main result is a necessary and sufficient condition for the existence of a Kähler-Einstein metric in some group compactifications. The condition is that the barycenter of the polytope associated to the group compactification must lie in a particular zone of the polytope. The first tool used in the proof is a study of the $K \times K$-invariant functions (for the $KAK$ decomposition), in particular he computes the complex Hessian of a $K \times K$-invariant function. And the second tool is an estimate of the convex potential associated to a $K \times K$-invariant metric on ample line bundles. Then he proves the main result by reducing the problem to a real Monge-Ampère equation and by obtaining $C^0$ estimates along the continuity method. In our paper, we extend this approach to smooth horospherical variety in the following way:
Theorem 1.1. Assume $M$ is a smooth horospherical variety. There is a Kähler-Ricci soliton $(X, g)$ on $M$.

This result was already proved in [Del16] in a more general case. But in our article, we focus on the case of smooth horospherical varieties and give a direct proof in this case. And so to prove this, we don’t use the K-stability to get the result (as used [Del16]) and we prefer using analytic methods such that the continuity method. So, in a first step, we compute the Futaki invariant and use the results of [TZ02] to get the expression of the solitonic vector field. And in a second step, we compute the Monge Ampere solitonic equation in the horospherical case and use the continuity method to conclude as in the toric case following the approach of [WZ04].

An important corollary that comes directly from the article [Pas09] is that there exist horospherical varieties which admit a non-trivial Kähler-Ricci soliton and therefore do not admit Kähler-Einstein metrics. Indeed, Matsushima theorem say that if the Fano variety has a non-reductive group of automorphisms then it does not admit Kähler-Einstein metrics. In the article [Pas09], Pasquier shows that there exists an infinity of horospherical varieties whose group of automorphisms is non-reductive, so by using the previous theorem, the only possibility is that the soliton must be non trivial. This is summarized in the following corollary:

Corollary 1.2. There exists an infinity of (smooth) horospherical varieties admitting a non trivial Kähler-Ricci soliton.

2. Horospherical Variety

In this section, we introduce the notions and the setup of [Del115] which are needed for our proof.

2.1. Reductive Group. Let $G$ a complex connected reductive linear algebraic group. Hence if we denote by $\mathfrak{g}$ this Lie algebra of $G$ then

$$\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}],$$

where $[\mathfrak{g}, \mathfrak{g}]$ is the Lie algebra of the derived subgroup $D(G)$ of $G$ and $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$. We recall the Killing form $B$ is nondegenerate on $[\mathfrak{g}, \mathfrak{g}]$ and zero on $Z(\mathfrak{g})$. If we denote by $K$ a maximal compact subgroup of $G$ and $\theta$ the Cartan involution such that $K$ is the set of fixed point of $\theta$ then under the identification of complex conjugaison with the Cartan involution, we get that $G$ is the complexification of $K$. Hence if we denote the Lie algebra of $K$ by $\mathfrak{k}$, we obtain

$$\mathfrak{g} = \mathfrak{k} \oplus J\mathfrak{k},$$

where $J$ is the complex structure of $\mathfrak{g}$.

Let $T$ a maximal torus $G$ stable under $\theta$. We denote by $\Phi \subset \mathfrak{x}(T)$ the root system of $(G, T)$ where $\mathfrak{x}(T)$ is the group of algebraic character of $T$, so we have the root decomposition :

$$\mathfrak{g} = \mathfrak{t} \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha},$$

where $\mathfrak{t}$ is the Lie algebra of $T$ and $\mathfrak{g}_{\alpha}$ is a complex line defined by $\mathfrak{g}_{\alpha} := \{ x \in \mathfrak{g} / \text{ad}(h)(x) = \alpha(h)x, \forall h \in \mathfrak{t} \}$. Now if we take a Borel subgroup $B$ of $G$ containing $T$ then we denote by $\Phi^{+}$ the set of positive root defined by $B$. Moreover, we denote by $B^{-}$ the unique Borel subgroup of $G$ such that $B \cap B^{-} = T$ and $B^{-}$ is called the opposite Borel subgroup with respect to $T$. 
If $P$ is a parabolic subgroup of $G$ containing $B$. We denote by $\Phi_P$ the set of roots of $P$ with respect to $T$ and by $\Phi^P_T$ the set of roots of the unipotent radical of $P$.

We define $a = t \cap J_{\mathfrak{p}}$. We have then a identification between $a$ with $\mathfrak{N}(T) \otimes \mathbb{R}$ where $\mathfrak{N}(T)$ is the subgroup of one parameter subgroups. Moreover, the Killing form $B$ define a scalar product $(\cdot, \cdot)$ on $a \cap [\mathfrak{g}, \mathfrak{g}]$. We extend this scalar product on $a$ by choosen a scalar product on $a \cap Z(\mathfrak{g})$ and assuming $a \cap Z(\mathfrak{g})$ and $a \cap [\mathfrak{g}, \mathfrak{g}]$ are orthogonal for $(\cdot, \cdot)$.

Finally, we recall that there is a natural pairing $(\cdot, \cdot)$ between $\mathfrak{N}(T)$ and $\mathfrak{X}(T)$ defined by $\chi \circ \lambda(z) = z^{(\lambda, \chi)}$. The natural paring between $\mathfrak{X}(T) \otimes \mathbb{R}$ and $\mathfrak{N}(T) \otimes \mathbb{R}$ can be view as $\langle \chi, a \rangle = \ln \chi(\exp a)$ for all $\chi \in \mathfrak{N}(T)$ and $a \in a \simeq \mathfrak{X}(T) \otimes \mathbb{R}$. For $\chi \in \mathfrak{X}(T)$, we define $t_{\chi}$ the unique element element of $a$ such that $(t_{\chi}, a) = \langle \chi, a \rangle$ for all $a \in a$.

# 2.2. Horospheral variety.

## 2.2.1. Definitions.

**Definition 2.1.** A normal variety $X$ equipped with an action of $G$ is spherical if a Borel subgroup $B$ acts on $X$ with an open and dense orbit.

A homogeneous space $G/H$ which is spherical under the action $G$ is a spherical homogenous space and $H$ is called a spherical subgroup.

Let $X$ is a spherical variety and $x \in X$ a point in the open orbit of $B$. If we denote by $H$ the isotropy group of $x$ in $G$, the pair $(X, x)$ is called a spherical embedding of the spherical homogenous space $G/H$ and is equipped with a natural inclusion of $G/H$ in $X$ by the map $gH \mapsto g \cdot x$.

Now, we can define the notion of horospheral variety:

**Definition 2.2.** A closed subgroup $H$ of a connected complex reductive group $G$ is called horospheral if it contains the unipotent radical $U$ of a Borel subgroup $B$ of $G$. The homogeneous space $G/H$ is called a horospheral homogeneous space.

Moreover, we can show a horospheral homogeneous space is spherical (see [Per]), so we have the following definition:

**Definition 2.3.** An embedding of a horospheral space is called a horospheral embedding of $G/H$ or a horospheral variety.

## 2.2.2. Polar decomposition.

We begin with a useful lemma for the rest of the paper:

**Proposition 2.4.** [Pas06] Assume $H$ is horospheral subgroup of a group $G$. If we set $P := N_G(H)$ then $P$ is a parabolic subgroup containing $B$, and the quotient $P/H = T/T \cap H$ is a torus.

For the rest of the section, we fix a horospheral subgroup $H$. Because, we have the inclusion $T \cap H \subset T$, we get $\mathfrak{N}(T \cap H) \subset \mathfrak{N}(T)$ and via the identification $a \simeq \mathfrak{N}(T) \otimes \mathbb{R}$ we get a subspace $a_0 \simeq \mathfrak{N}(T \cap H) \otimes \mathbb{R}$. We define $a_1$ the orthogonal of $a_0$ for $(\cdot, \cdot)$.

**Proposition 2.5.** [Del16] The image of $a_1$ in $G$ under the exponential is a fundamental domain for the action of $K \times H$ on $G$ where $K$ acts by multiplication on the left and $H$ by multiplication on the right by the inverse. As a consequence, the set $\{\exp(a)H \mid a \in a_1\} \subset G/H$ is a fundamental domain for the action of $K$ on $G/H$. 
2.2.3. Automorphisms of a horospherical variety. We fix a compact smooth Fano Kähler manifold $M$. Let $\text{Aut}^0(M)$ the connected component of the neutral element in $\text{Aut}(M)$ which is the group of automorphisms of $M$. It is a Lie group and we consider a maximal compact subgroup $K$. The complexification $G$ of $K$ is a reductive group of $\text{Aut}^0(M)$ and we choose a maximal complex torus $T$ such that $K \cap T$ is the maximal compact torus of $T$.

Now, assume $X$ is horospherical under $G$. Then we can show the group of automorphisms $G$-equivariant of $X$ is isomorphic to $P/H$ where $P/H$ acting on $G/H$ by multiplication on the right by the inverse (see [Tim] and [Kno91]). Moreover, by the proposition 2.4 we have $P/H$ is a torus and we define $b_1$ as the subalgebra of the Lie algebra of $P/H$ obtained as $J$ times the Lie algebra of the maximal compact subtorus. We have $b_1 = \mathfrak{n}(P/H) \otimes \mathbb{R}$.

Moreover, because $P/H$ is a complex torus, we have thanks to the proposition 2.5 that the Lie algebra of $P/H$ is $\mathfrak{a}_1 \oplus J\mathfrak{a}_1$ and so we have $b_1 \simeq \mathfrak{a}_1$. The difference between $a_1$ and $b_1$ is that $b_1$ act on the right and not on the left.

2.3. Line Bundles.

**Definition 2.6.** A $G$-linearization of a line bundle $L$ on a $G$-variety is a $G$-action on $L$ such that the $G$-action on $L$ lifts the $G$-action on $X$ and the map between the fibers $L_x$ and $L_{g \cdot x}$ defined by the action of $g \in G$ is linear.

2.3.1. Associated line bundle. Let $G/H$ be a horospherical homogeneous space and $L$ be a $G \times P/H$-linearized line bundle on $G/H$. We recall we can associate to this line bundle $L$ a character $\chi$ such that $(p, pH) \cdot \xi = \chi(p) \xi$ for all $\xi \in L_{cH}$ ([Per]). If we consider the quotient map $\pi : G \to G/H$ and the inclusion map $\iota : PH \to G/H$. Hence we get a $G$-linearization on $\pi^*L$ and $P \times P/H$-linearized on $\iota^*L$.

On the bundle $\pi^*L$ and $\iota^*L$ we can define a global trivialisation $s$ on $\pi^*L$ and two global trivialisation $s_1$ and $s_3$ on $\iota^*L$. In particular, we define the global section $s_\tau$ by $s_\tau(pH) := (e, p^{-1}H) \cdot (\iota^{-1} \circ \pi)(s(e))$ where $s(e) \in (\pi^*L)_e$. For more details, we may read the section 2.2.1 in [Del16].

2.3.2. Hermitian metric on line bundles. Let $X$ be a compact Kähler manifold and $L$ a line bundle on $X$. Let’s recall that a hermitian metric is the data for all $x \in X$ of a hermitian metric $h_x$ on the $L_x$ fiber of $L$. Moreover, we say that the metric is smooth if the application $x \mapsto h_x$ is smooth.

Let’s now take a trivialization $s$ above an open $U \subset X$. This means that for all $x \in X$ the vector $s(x)$ is a base of $L_x$ and hermitian $h_x$ is summarized to give itself a positive real $a_x$ that will be equal to the squared norm of $s(x)$ with respect to the hermitian form $h_x$ i.e. $a_x = |s(x)|^2_h$. We then define the local potential of $h$ (with respect to the trivialization $s$) by $\varphi : x \in U \mapsto -\ln(|s(x)|^2_h) \in \mathbb{R}$. Let’s note that the metric $h$ is entirely determined by all its local potentials and that $h$ is smooth if and only if all its local potentials are smooth.

Let’s finish by saying that we can associate to a hermitian smooth metric a $(1, 1)$-form $\omega_h$ called curvature of $h$. To do this, we define locally $\omega_h|_U = \sqrt{-1} \partial \bar{\partial} \varphi$ where $\varphi$ is the local potential associated with the trivialization $s$ above $U$. We verify that $\omega_h|_U$ does not depend on the trivialization and thus define a global $(1, 1)$-form. Moreover, we can show that $\omega_h \in c_1(L)$. We will also say that $L$ is positive curvature if there exists a metric $h$ such that $\omega_h$ is a Kähler form ([Dem]).
There is also a notion of global potential. To define it, we set a reference hermitian metric $h^0$ and for any hermitian metric $h$ we define the function $\psi$ on $X$, called global potential of $h$ with respect to $h^0$ by the following formula:

$$|\xi|^2_h = e^{-\psi(x)}|\xi|^2_{h^0}.$$ 

Note that the function $\psi$ satisfies the following relation (thanks to the $\partial \overline{\partial}$-lemma):

$$\omega_{h^0} = \omega_h + \sqrt{-1}\partial \overline{\partial} \psi.$$

Now let $G/H$ be a horospherical homogeneous space and $L$ be a $G \times P/H$-linearized line bundle on $G/H$ and $q$ a $K$-invariant metric on $L$. We can define a global potential $u : a_1 \rightarrow \mathbb{R}$ associated to the restriction to $L|_{P/H}$ :

$$u(x) = -2 \ln |s_r(\exp(x)H)|_{i^*q}.$$ 

2.4. Curvature. In this section, we compute the curvature on $P/H$ in a basis adapted. The first step is to define this basis. For this purpose, we recall that we can identify the tangent space at $eH$ to $G/H$ with $\mathfrak{g}/\mathfrak{h} \cong \oplus_{\alpha \in \Phi^+_P} \mathbb{C}e_{-\alpha} \oplus a_1 \oplus J\mathfrak{a}_1$.

A complex basis of the tangent space $T_{eH}G/H$ is given by a basis $(i_i)$ on $a_1$ with the $(e_{-\alpha})$. On $P/H$ we can define for $\xi \in T_{eH}G/H$ the real holomorphic vector field : $R\xi : pH \rightarrow (H,p^{-1}H) \cdot \xi$. We have a complex basis of $T^{1,0}P/H$ given by $(R_{L_j} - iJR_{L_j})/2$ and $(Re_{-\alpha} - iJRe_{-\alpha})/2$ and we denote by $(\gamma_j)$ and $(\gamma_\alpha)$ the dual bases. With this basis, we can compute the curvature :

**Theorem 2.7.** [Del16] Let $\omega$ be the $K$-invariant curvature form of $K$-invariant metric $q$ on a $G \times P/H$ linearized line bundle $L$ on $G/H$ with associated character denote by $\chi$. Then the form $\omega$ is determined by its restriction to $P/H$, given for $x \in a_1$ by

$$\omega_{\exp(x)H} = \sum_{1 \leq j_1, j_2 \leq r} \frac{1}{4} \frac{\partial^2 u}{\partial r_{j_1} \partial r_{j_2}}(x) \sqrt{-1} \gamma_{j_1} \wedge \overline{\gamma}_{j_2} + \sum_{\alpha \in \Phi^+_P} \langle \alpha, \nabla u(x) / 2 - t_{\chi} \rangle \sqrt{-1} \gamma_\alpha \wedge \overline{\gamma}_\alpha.$$ 

Moreover, we have hence

$$\omega^n_{\exp(x)H} = \frac{MA_G(u)(x)}{4^n 2^{\text{Card}(\Phi^+_P)}} \prod_{\alpha \in \Phi^+_P} \langle \alpha, \nabla u(x) + 4t_{\rho_P} \rangle \Omega,$$

where

$$\Omega := \bigwedge_{1 \leq j \leq r} \gamma_j \wedge \overline{\gamma}_j \bigwedge_{\alpha \in \Phi^+_P} \gamma_\alpha \wedge \overline{\gamma}_\alpha.$$ 

2.5. Polytope associated. Let $X$ be horospherical variety i.e. there is a reductive group $G$ and a point $x \in X$ such that the isotropy group $H$ of $x$ is a horospherical subgroup of $G$ which contain the unipotent radical $U$ of a Borel subgroup $B$ and we set $P := N_G(H)$. Let also a $G \times P/H$-linearized ample line bundle $L$. We can build a polytope $\Delta^+$ associated to the $G \times P/H$-linearized ample line bundle $L$ with respect to the Borel subgroup $B$ (see [Pas16] for more details). Now we set $\Delta := \chi + \Delta^+$ where $\chi$ est le character associated to the line bundle $L$.

**Proposition 2.8.** Let $q$ be a $K$-invariant smooth and positive metric over $L$ and let $u : a_1 \rightarrow \mathbb{R}$ the potential associated to the restriction on $G/H$. Then $u$ is a smooth and strictly convex function such that $\nabla u(a_1) = 2\Delta$ and the function $u - v_{2\Delta}$ is bounded on $a_1$. 
The important case for the rest of the paper is then \( L = K_X^{-1} \). In this case, we have \( \chi = -2\rho P := \sum_{\alpha \in \Phi^+_P} \alpha \). We can show \( \Phi^+_P \) contains the roots which are not orthogonal to \( \Delta^+ \) and \( 0 \in \Delta^+ \) (see [Pas06]).

3. Futaki Invariant in the horospherical case

3.1. Definition and properties of the Futaki Invariant. In this section, a holomorphic invariant is defined and calculated in the case of some compactification. This invariant is an obstruction to the existence of a Kähler-Ricci Soliton.

Recall that \( \text{Aut}(M) \) is a Lie group which the set of holomorphic vector fields \( \eta(M) \) is its Lie algebra and if \( K(M) \) is a maximal compact subgroup of \( \text{Aut}^0(M) \) which is the identity component of \( \text{Aut}(M) \), then the decomposition of Chevalley gives us that

\[
\text{Aut}^0(M) = \text{Aut}_r(M) \times R_u,
\]

where \( \text{Aut}_r(M) \) is a reductive subgroup of \( \text{Aut}^0(M) \) and the complexification of \( K(M) \) and \( R_u(M) \) the unipotent radical of \( \text{Aut}^0(M) \). Moreover, if \( \eta(M), \eta_r(M), \eta_u(M) \) and \( \kappa(M) \) are the Lie algebras of \( \text{Aut}(M), \text{Aut}_r(M), R_u(M) \) and \( K(M) \) respectively, then we have

\[
\eta(M) = \eta_r(M) \oplus \eta_u(M).
\]

**Definition 3.1.** Let a \( n \)-dimensional compact Kähler manifold \((M, g)\) with positive first Chern class \( c_1(M) \) such that its Kähler form \( \omega_g \in c_1(M) \). Then, for any holomorphic vector field \( X \in \eta(M) \), we define the linear functional \( F_X \), called Futaki invariant, by

\[
F_X : v \in \eta(M) \mapsto \int_M v(h_g - \theta_X)e^{\theta_X} \omega^n_g \in \mathbb{C},
\]

where we denote :

- \( h_g \) is the unique function in \( C^\infty(M, \mathbb{R}) \) such that
  \[
  \text{Ric}(\omega_g) - \omega_g = \sqrt{-1} \partial \bar{\partial} h_g,
  \int_M e^{h_g} \omega^n_g = \int_M \omega^n_g,
  \]

- \( \theta_X \) is the unique function in \( C^\infty(M, \mathbb{R}) \) such that
  \[
  i_X \omega_g = \sqrt{-1} \partial \bar{\partial} \theta_X,
  \int_M e^{\theta_X} \omega^n_g = \int_M \omega^n_g.
  \]

A first remark is, by the Cartan formula, \( L_X \omega = \sqrt{-1} \partial \bar{\partial} \theta_X \). A second remark is that, according to proposition 1.1 of [TZ02], the invariant does not depend on the chosen metric \( g \). Moreover, if \((X, g)\) is a soliton then \( F_X = 0 \).

After these notations, we have this fundamental proposition :

**Proposition 3.2.** There exists a unique holomorphic vector field \( X \in \eta_r(M) \) with \( \text{Im}(X) \in \kappa(M) \) such that the holomorphic invariant \( F_X \) vanishes on \( \eta_r(M) \). Moreover, \( X \) is either zero or an element of the center of \( \eta_r(M) \) and

\[
F_X([u, v]) = 0, \ \forall (u, v) \in \eta_r(M) \times \eta(M).
\]
3.2. Determination of the solitonic vector field. Now, in our case we suppose \( M \) is a horospherical embedding of \( G/H \). We use the setup of the section 2.2.3 and hence we can suppose \( \eta_r(M) = g \). We can use the decompositions established in the previous section. In addition, as \( \eta_r(M) \simeq g \), we have that every element \( \chi \in g \) induces one vector fields \( \hat{\chi} \in \eta_r(M) \). Furthermore, we have \( \mathfrak{s}(\eta_r(M)) \subset \mathfrak{t}(M) \).

Now, using the logarithmic coordinates \((w_1, w_2, \ldots, w_d) = (x_1 + \sqrt{-1}\theta_1, \ldots, x_d + \sqrt{-1}\theta_d)\), we obtain that

\[
\mathfrak{s}(M) = \bigoplus_{i=1}^{d} \mathbb{R} \cdot \frac{\partial}{\partial \theta_i},
\]

so

\[
X = \sum_{i=1}^{d} c_i \frac{\partial}{\partial w_i}, \quad c_i = r_i + \sqrt{-1} t_i \in \mathbb{C}.
\]

Thus, we get

\[
X = \sum_{i=1}^{d} (r_i + \sqrt{-1} t_i) \left( \frac{\partial}{\partial x^i} - \sqrt{-1} \frac{\partial}{\partial \theta^i} \right)
= \sum_{i=1}^{d} \left( r_i \frac{\partial}{\partial x^i} + t_i \frac{\partial}{\partial \theta^i} \right) + \sqrt{-1} \sum_{i=1}^{d} \left( t_i \frac{\partial}{\partial x^i} - r_i \frac{\partial}{\partial \theta^i} \right).
\]

Finally, we have

\[
\text{Im}(X) = \sum_{i=1}^{d} \left( t_i \frac{\partial}{\partial x^i} - r_i \frac{\partial}{\partial \theta^i} \right).
\]

Yet Im(\(X\)) \(\in \mathfrak{s}(M)\) which is generated by the family \(\left( \frac{\partial}{\partial \theta^i} \right)\), so we have \( t_i = 0 \) for all \( i \in \{1, \ldots, n\} \). Finally, the vector field \(X\) vanishing the Futaki invariant belongs to \(\mathfrak{t}(M)\) and is written in the form

\[
X = \hat{\xi} - \sqrt{-1} J \hat{\xi}, \quad \xi \in \mathfrak{a}.
\]

Moreover, we know \( J \xi \) generates a torus in \( K \) so \( \xi \) induces a \( G \)-equivariant one-parametre subgroup. It implies \( \xi \in \mathfrak{b}_1 \simeq \mathfrak{a}_1 \) (see section 2.2.3 for more details).

This is summarized in the following proposition:

**Proposition 3.3.** Assume \( M \) is a smooth horospherical variety. The vector field \( X \in \eta_r(M) \) vanishing the Futaki invariant is on the form:

\[
X = \hat{\xi} - \sqrt{-1} J \hat{\xi}, \quad \xi \in \mathfrak{b}_1.
\]

3.3. Computation of the Futaki Invariant. In this section, we want to compute the Futaki invariant in our case. To do this, we first must compute \( \theta_X \). But, for this computation, it is preferable to renormalize the function \( \theta_X \) to a function \( \tilde{\theta}_X \) by requesting that it checks

\[
(1) \quad i_X \omega_g = \sqrt{-1} \partial \tilde{\theta}_X, \quad \Delta \tilde{\theta}_X + X(h_g) = -\tilde{\theta}_X.
\]

This condition is equivalent to

\[
(2) \quad \int_M \tilde{\theta}_X e^{\delta \tilde{X}} \omega^n_g = 0,
\]

and we obtain

\[
F_X(v) = - \int_M \tilde{\theta}_v e^{\delta \tilde{X}} \omega^n_g.
\]
First, we use the $K$-invariance of the Kähler form $\omega$ to remark the following fact:

\[ L_\xi \omega = L_{\xi + \sqrt{-1} J \xi} \omega = L_{\xi} \omega + \sqrt{-1} L_{J \xi} \omega = L_{\xi} \omega, \]

because $\omega$ is $K$-invariant and $J \xi \in \mathfrak{k}$ so we just compute $L_{\xi} \omega$. Now we can use the proposition 4.4 of [Del16] to get:

**Proposition 3.4.** Let $\zeta \in b_1 \simeq \mathcal{N}(P/H) \otimes \mathbb{R}$. If we set $Y = \zeta - \sqrt{-1} J \zeta$ then $\theta_Y$ is $K$-invariant and

\[ \tilde{\theta}_Y(\exp(x)H) = -(\nabla u(x), \zeta), \]

where $\nabla u$ is the gradient of $u$ for the scalar product $(\cdot, \cdot)$.

Now, we can compute the Futaki invariant $F_X(Y)$ when $X = \xi - \sqrt{-1} J \xi$ and $Y = \zeta - \sqrt{-1} J \zeta$. We have

\[
\begin{align*}
F_X(Y) &= \int_M \theta_Y e^{\theta_X} \omega_g^n \\
&= \int_{G/H} \theta_Y e^{\theta_X} \omega_g^n \\
&= C \int_{\mathbb{R}_1} (\nabla u(x), \zeta)e^{-\nabla u(x), \zeta} \prod_{\alpha \in \Phi^+} (\alpha, \nabla u(x) + 4t_{\rho_P}) MA_R(u)(x)dx \\
&= C \int_{2\Delta + 4\rho_P} (p - 4 \rho_P, \zeta)e^{(p - 4 \rho_P, \zeta)} \prod_{\alpha \in \Phi^+} (\alpha, p)dp \\
&= \tilde{C} \int_{\Delta^+} (p - 2 \rho_P, \zeta)e^{(2p - 4 \rho_P, \zeta)} \prod_{\alpha \in \Phi^+} (\alpha, p)dp,
\end{align*}
\]

where $\tilde{C}$ and $C$ are constants independent of $\zeta$ and $\xi$.

**Proposition 3.5.** Let $X = \xi - \sqrt{-1} J \xi$ the vector field vanishing the Futaki invariant. We have

\[
0 = \int_{\Delta^+} (p - 2 \rho_P, \zeta)e^{(2p - 4 \rho_P, \zeta)} \prod_{\alpha \in \Phi^+} (\alpha, p)dp, \forall \zeta \in a_1.
\]

4. **Monge-Ampère Equation in the Horospherical Case**

4.1. **General Case.** We fix a compact Fano manifold $(M, g^0)$ with $\omega_{g^0} \in c_1(M)$ such that $(X, g)$ is a Kähler-Ricci soliton i.e.

\[ \text{Ric}(\omega_g) - \omega_g = L_X \omega_g. \]

Thanks to the $\partial\bar{\partial}$-lemma, there exists a unique function $h$ in $C^\infty(M, \mathbb{R})$ such that

\[ \text{Ric}(\omega_{g^0}) - \omega_{g^0} = \sqrt{-1} \partial\bar{\partial} h, \]

and a function $\psi$ in $C^\infty(M, \mathbb{R})$ such that

\[ \omega_g = \omega_{g^0} + \sqrt{-1} \partial\bar{\partial} \psi. \]
Noting $\theta_X(g) = \theta_X(g^0) + X(\phi)$, it is shown ([WZ04] for instance) that solving the Kähler-Ricci soliton equation is equivalent to finding a potential $\psi$ solution of the following Monge-Ampère equation:

$$
\begin{aligned}
\{ \det(g^0_\triangledown + \psi_\triangledown) = \det(g^0_\triangledown) \exp(h - \theta_X(g^0) - X(\psi) - \psi) \\
(g^0_\triangledown + \psi_\triangledown) > 0.
\end{aligned}
$$

Moreover, if we fix a hermitian metric $m^0$ on $-K_M$ such that $\omega_m = \omega_g^0$, then we can define a volume form $dV$ given in a local trivialisation $s$ of $-K_M$ by

$$
dV = |s|m^0 s^{-1} \wedge \bar{s}^{-1}
$$

then modulo a constant we obtain that $h$ is equal to the logarithm of the potential of $dV$ with respect to $\omega_g^0$, so we renormalise to match it i.e.

$$
e^{-h}\omega_g^0 = dV.
$$

Another way to write the first equation of (4) is then:

$$(\omega_g^0 + \partial\bar{\partial}\psi)^n = e^{h - \psi - \theta_X(g^0) - X(\psi)}\omega_g^0.
$$

4.2. Horospherical Case. Assume now that $M$ is a smooth horospherical variety and $g^0$ is a $K$-invariant Kähler form. Moreover, since the metric $\omega_g^0$ is $K$-invariant, we want to find a $K$-invariant solution $\psi$.

Now, thanks to the density of $G/H$ in $M$, we can reduce our study in this space. Moreover, by $K$-invariance, we can just compute this equation for the values $exp(x)H$ for $x \in \mathfrak{a}_1$. We get

$$
\prod_{\alpha \in \Phi_p^+} \langle \alpha, \nabla u(x) + 4t_{\rho\rho}, \frac{\text{MA}_K(u)(x)}{4\pi 2\text{Card}(\Phi_p^+)} \rangle \Omega = 
$$

$$
e^{h - \psi + (\nabla u(x), \xi)} \prod_{\alpha \in \Phi_p^+} \langle \alpha, \nabla u^0(x) + 4t_{\rho\rho}, \frac{\text{MA}_K(u^0)(x)}{4\pi 2\text{Card}(\Phi_p^+)} \rangle \Omega.
$$

Now we can simplify this expression. For this goal, we have the normalisation condition [3]

$$
e^{h}\omega_g^0|_{\rho/H} = e^{-u} s_r^{-1} \wedge \bar{s}_r^{-1},
$$

we can write the previous equation in the following way:

$$
\prod_{\alpha \in \Phi_p^+} \langle \alpha, \nabla u^0(x) + 4t_{\rho\rho}, \frac{\text{MA}_K(u^0)(x)}{4\pi 2\text{Card}(\Phi_p^+)} \rangle \Omega = e^{-u} s_r^{-1} \wedge \bar{s}_r^{-1}.
$$

If we choose correctly the section $s_r$ such that $s_r \wedge \bar{s}_r^{-1} = \Omega$, we can simplify the previous equation. For this purpose, we recall that

$$
s_r(pH) = (e, p^{-1}H) \cdot (i^{-1} \circ \pi)(s(e))
$$

where $s(e) \in (\pi^*L)_e$ hence if we choose $s(e) \in \pi^{-1}(s(\Omega(e)))$ we get, by definition of $\Omega$ and $s_r$, $s_r \wedge \bar{s}_r^{-1} = \Omega$ (see the section 2.34 and 2.33). Finally, we get:

$$
\prod_{\alpha \in \Phi_p^+} \langle \alpha, \nabla u(x)/2 + 4t_{\rho\rho}, \frac{\text{MA}_K(u)(x)}{4\pi 2\text{Card}(\Phi_p^+)} \rangle = e^{-u} (\nabla u(x), \xi)
$$

Recall $\Phi_p^+$ are the root in $\Phi^+$ which are not orthogonal to $\Delta^+$ and

$$
\text{Im}(\nabla u + 4t_{\rho\rho}) = 2\Delta^+
$$
so we can write

\[ \text{MA}_R(u)(x) = \frac{4^r e^{2\text{Card}(\Phi^+_P)}}{\prod_{\alpha \in \Phi^+_P} (\alpha, \nabla u(x) + 4t \rho)} e^{-u + (\nabla u(x), \xi)}. \]

4.3. The continuity method. We now want the existence of Kähler-Ricci solitons. To do this, we will use the method of continuity which we now recall the approach.

To begin with, we introduce into the Monge-Ampère equation a parameter \( t \in [0,1] \):

\[
\begin{align*}
\det(g^{0 \overline{j}} + \psi_{\overline{j}}) &= \det(g^{0 \overline{j}}) \exp(h - \theta_X - X(\psi) - t\psi) \\
(g^{0 \overline{j}} + \psi_{\overline{j}}) &> 0.
\end{align*}
\]

We note that the equation (9) is the previous equation with \( t = 1 \). Moreover, if a solution exists at time \( t \), we denote it by \( \psi_t \). Now, if \( \psi_t \) is \( K \)-invariant, \( \omega^0 + \sqrt{-1} \psi_t \) has a convex potential \( u_t \). Thus, setting \( w_t = t \cdot u_t + (1 - t) \cdot u^0 \), we can write this equation on the dense orbit as:

\[
\text{MA}_R(u_t)(x) = \frac{4^r e^{2\text{Card}(\Phi^+_P)}}{\prod_{\alpha \in \Phi^+_P} (\alpha, \nabla u_t(x) + 4t \rho)} \exp \left[ -w_t(x) + (\nabla u_t(x), \xi) \right].
\]

The method of continuity consists in considering the set \( S \) of times when there exists a solution:

\[ S := \{ t \in [0,1] \mid \text{There is a solution } \psi_t \text{ of the equation (9) at the time } t \}, \]

and showing that \( S \) is a close open and nonempty set of \([0,1]\).

The openness and existence of a solution of (9) at time \( t = 0 \) comes from the study of the Monge-Ampère equations made in \([\text{Aub78, Yau78}]\). We can also consult \([\text{TZ00}]\) for a study made in the case of the Kähler-Ricci solitons. Moreover, thanks to the Arzelà-Ascoli theorem, it suffices to have an a priori estimate \( C^3 \) of the potentials \( \psi_t \) to obtain that \( S \) is close. Now, thanks to the works of Yau and Calabi made in appendix A of \([\text{Yau78}]\), we can reduce this estimate \( C^3 \) to an estimate \( C^0 \).

Moreover, by the following Harnack inequality (see \([\text{TZ00, WZ04}]\) for instance)

\[ -\inf_M \psi_t \leq C(1 + \sup_M \psi_t), \]

we can reduce to a uniform upper bound for the \( \psi_t \).

5. Proof of the a priori estimation

We must a priori find an estimate for \( t \in [0,1] \). Now, using the fact that \( 0 \in S \) and \( S \) is open, one can reduce to show an estimate on \([t_0, 1]\) for \( t_0 > 0 \). We set a such \( t_0 \) for the rest. Moreover, for simplicity, we denote by \( r \) the dimension of the real vector space \( a_1 \) and we use the isomorphism between \( a_1 \simeq \mathbb{R}^r \).

We begin with a lemma that will be useful to us later:

Lemma 5.1. We have

\[ \int_{a_1} \frac{\partial w}{\partial \zeta} e^{-w} dx = 0, \forall \zeta \in a_1. \]
Proof. We choose a basis \((a_1, \cdots, a_r)\) of \(a_1\) and denote by \((x_1, \cdots, x_r)\) the coordinates associated in \(a_1\). Thanks to this, we can view \(a_1\) as \(\mathbb{R}^r\). Moreover, by linearity it suffices to prove:

\[
\int_{\mathbb{R}^r} \frac{\partial w}{\partial x_i} e^{-w} \, dx = 0, \quad i = 1, \cdots, r.
\]

We can write, thanks to the Fubini theorem:

\[
\int_{\mathbb{R}^r} \frac{\partial w}{\partial x_i} e^{-w} \, dx = -\int_{\mathbb{R}^r} \frac{\partial e^{-w}}{\partial x_i} \, dx \quad (i \in \{1, \cdots, r\})
\]

where \(w_i : \mathbb{R} \to \mathbb{R}^r\) is the partial application in the coordinate \(x_i\) of \(w\) i.e. \(w_i\) is the function \(w_i : t \in \mathbb{R} \mapsto (x_1, \cdots, x_{i-1}, t, x_{i+1}, \cdots, x_n) \in \mathbb{R}\) where the \(x_k\) are fixed.

To conclude, if suffices to prove

\[
\lim_{t \to \pm \infty} e^{-w_i(t)} = 0.
\]

To see this, we remarks, by definition of \(w\) and thanks to the proposition \(2.8\) we have

\[
e^{-w(x)} = \left(e^{-u}\right)^{1-t} \cdot (e^{-u})^t \leq C e^{-v_2\Delta(x)}, \quad \forall x \in \mathbb{R}^r.
\]

where \(v_2\Delta\) is the support function of \(2\Delta\) i.e. \(v_2\Delta(x) = \sup_{p \in 2\Delta} (x, p)\). Hence, we have for all \(p \in 2\Delta\)

\[
v_2\Delta(x) \geq (x, p)
\]

\[
\geq x_i(a_i, p) + \sum_{j=1, j \neq i} x_j(a_j, p)
\]

\[
\geq x_i(a_i, p) + \inf_{p \in 2\Delta} \left( \sum_{j=1, j \neq i} x_j(a_j, p) \right).
\]

Finally, we get

\[
0 \leq e^{-w_i(t)} \leq \hat{C} e^{-t(a_i, p)}, \quad \forall p \in 2\Delta,
\]

where \(\hat{C}\) is a constant independent of \(t\). To conclude, it suffice to remark, thanks to the fact \(0 \in 2\Delta\), there exists a ball centered in 0 with radius \(\delta > 0\) included in \(2\Delta\) and so there exists \(p_1 \in 2\Delta\) such that \((a_i, p_1) > 0\) and \(p_2 \in 2\Delta\) such that \((a_i, p_2) < 0\).

\[\Box\]

Lemma 5.2. The function \(w_i\) has a minimum \(m_t\) in \(x_i \in a_1\)

Proof. This is based on the fact that a convex function on \(\mathbb{R}^r\) which has a critical point has a global minimum. In order to apply this result, we note that \(w_i\) is indeed a convex function as barycenter of the two convex functions \(u\) and \(u^0\) (see...
12 DELGOVE FRANÇOIS

proposition 2.8, to conclude, it suffices to show that $0 \in \nabla w_t(\mathbb{R}^n)$. Indeed, we have:

$$\nabla w_t(a_1) = t \nabla u(a_1) + (1 - t) \nabla u^0(a_1) \quad \text{(thanks to the definition of } w)$$

$$= 2t\Delta^+ + 2 (1 - t) \Delta^+ \quad \text{(proposition 2.8)}$$

$$= 2\Delta^+ \quad \text{(because } 2\Delta^+ \text{ is convex)}$$

and we have $0 \in 2\Delta^+$.

\[ \square \]

Lemma 5.3. We have the following property :

$$\exists C > 0, \forall t \in [t_0, 1], m_t \leq C.$$ 

Before starting the proof, we recall a result concerning the convex domains which will be used in the proof:

Lemma 5.4. [WZ04, Guz75, Gut01] Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$. Then there is a unique ellipsoid $E$, called the minimum ellipsoid of $\Omega$, which attains minimum volume among all ellipsoids containing $\Omega$, such that

$$\frac{1}{n} E \subset \Omega \subset E.$$ 

Let $T$ be a linear transformation with $|T| = 1$, which leaves the center $x_0$ of $E$ invariant, namely $T(x) = A(x - x_0) + x_0$ for some matrix $A$, such that $T(E)$ is a ball $B_R$. Then we have $B_{R/\rho} \subset T(\Omega) \subset B_R$ for two balls with concentrated center.

Now we can begin the proof of the lemma 5.3:

Proof. We set

$$A_k := \{ x \in \mathbb{R}^n / m_t + k \leq w(x) \leq m_t + k + 1 \},$$

And we then have the following elementary properties which come directly from the fact that $w$ is convex and $m_t \leq +\infty$:

- $A_k$ is bounded and for all $k \geq 0$ et $\bigcup_k A_k = a_1$
- $m_t \in A_0$
- $\bigcup_{i=0}^k A_i$ is convex for $k \geq 0$.

Moreover, as $u$ et $u^0$ are convex, the matrix $(u_{ij})$ and $(u_{ij}^0)$ are positive and we have an elementary algebraic fact which implies

$$\text{det}(w_{ij}) = \text{det}(tu_{ij} + (1 - t)u_{ij}^0) \geq \text{det}(tu_{ij}) + \text{det}((1 - t)u_{ij}^0),$$

but $(u_{ij}^0)$ is positive then $\text{det}((1 - t)u_{ij}^0) \geq 0$ so

$$\text{det}(w_{ij}) \geq \text{det}(t \cdot u_{ij}) \geq t^r \cdot \text{det}(u_{ij}) \quad \text{(determinant propriety)}$$

$$\geq t^r \frac{4^{r}2^{\text{Card}(\Phi^+)}}{f} \cdot e^d \cdot e^{-w} \quad \text{(equation 10)}$$

where we set

$$d = \inf \{(p, \xi) / p \in 2\Delta^+ - 4\rho P \}$$
and

\[ f = \sup \left\{ \prod_{\alpha \in \Phi^+_p} \langle \alpha, p \rangle \mid 2\Delta^+ \right\} \]

(we recall \( f \neq 0 \) because \( \Phi^+_p \) denote the roots in \( \Phi^+ \) which are not orthogonal to \( \Delta^+ \)). Finally, because \( t \in [t_0, 1] \) we can write:

\[ \det(w_{ij}) \geq C_0 e^{-m_t} \text{ in } A_0, \]

Using the lemma 5.4, there exists a linear transformation \( y = T(x) \) with \( |T| = 1 \) and leaving the center of the minimal ellipsoid of \( A_0 \) invariant,

\[ B_{R/r} \subset T(A_0) \subset B_R, \]

and thus preserving the previous inequality. Moreover, we have

\[ R \leq \sqrt{2}rC_0^{-1/2r}e^{m_t/2r}. \]

Indeed, we set the map

\[ v : y \in a_1 \mapsto \frac{1}{2}C_0^{-1/r}e^{m_t/r} \left| y - y_t \right|^2 - \left( \frac{R}{r} \right)^2 + m_t + 1 \in \mathbb{R} \]

where \( y_t \) is the center of the minimal ellipsoid of \( A_0 \). A computation gives us that

\[ \det(v_{ij}) = C_0 e^{-m_t} \text{ on } T(A_0), \]

and \( v \geq \nu \) on \( \partial T(A_0) \) thus on \( T(A_0) \) thanks to the comparaison principle. In particular, we get

\[ m_t \leq \nu_t \leq v(y_t) = -\frac{1}{2}C_0^{-1/r}e^{m_t/r} \left( \frac{R}{r} \right)^2 + m_t + 1. \]

Now, thanks to the convexity of \( w \), we get

\[ T(A_k) \subset B_{2(k+1)R}, \]

and

\[ \cup_k A_k = a_1. \]

Furthermore, \( T \) is an affine isometry of \( a_1 \) so an isomorphism thus the familly \( (T(A_k))_{k \in \mathbb{N}} \) is a cover of \( a_1 \). Now, if we denote \( \omega_r \) the area of the sphere \( S_{r-1} \) then we have

\[ \int_{a_1} e^{-w_1} \leq \sum_k \int_{T(A_k)} e^{-w_t} \]

\[ \leq \sum_k e^{-m_t-k}|T(A_k)| \]

( since \( \omega \geq -m_t - k \) in \( A_k \) and so in \( T(A_k) \) because \( T \) is an isometry)

\[ \leq \omega_r \sum_k e^{-m_t-k}|2(k+1)R|^r, \]

( since \( T(A_k) \subset B_{2(k+1)R} \))

\[ = \omega_r \frac{(2R)^r}{e^{m_t}} \sum_k \frac{(k+1)^r}{e^k} \]

\[ \leq C e^{m_t/2} \]

(thanks to \( R \leq \sqrt{2}rC_0^{-1/2r}e^{m_t/2r} \)).
where $C > 0$ is a constant independant of $t$. Finally, we get
\[ e^{m_t/2} \geq \frac{1}{C} \int_{a_1} e^{-w} . \]
Moreover, thanks to the equation 10, we have
\[ e^{m_t/2} \geq \frac{1}{C} \int_{a_1} \det(u_{ij}) \cdot e^{(\nabla u(x), \xi)} \prod_{\alpha \in \Phi^+_P} (\alpha, \nabla u_t(x) + 4t_{P_P}) \frac{d\alpha}{4^{r}2^{\text{Card}(\Phi^+_P)}} \]
\[ = \frac{1}{C} \int_{2\Delta^+} e^{((p-4P_P), \xi)} \prod_{\alpha \in \Phi^+_P} (\alpha, p) \frac{d\alpha}{4^{r}2^{\text{Card}(\Phi^+_P)}} dp =: \hat{C} \]
where $\hat{C}$ is independant of $t$. Finally we get
\[ m_t \leq C, \]
where $C > 0$ is a constant independant of $t$. □

Lemma 5.5. Let $x^t = (x^t_1, \cdots, x^t_n)$ be the minimal point of $\nu_t$. Then
\[ |x^t| \leq C, \]
where $C$ is a uniform constant.

Proof. We argue by the absurd : we suppose therefore that
\[ \forall C > 0, \exists t \in [t_0, 1], |x^t| > C. \]
Recall that by equation 10 we have
\[ \int_{a_1} e^{-w_t} dx = \beta, \]
for some constant $\beta$. Recall also that $|Dw| \leq d_0 := \sup\{|x| / x \in 2\Delta^+\}$ so there exists $R > 0$ independent of $t$ that $\inf\{w(x) / x \in \partial B(x_t, R)\} \geq m_t + 1$. Now, by convexity, we have
\[ w_t \geq \frac{1}{R} |x - x_t| + m_t, \forall x \in a_1 \setminus B(x_t, R). \]
So for any $\varepsilon > 0$, there exists $R_\varepsilon$ independent of $t$ such that
\[ \int_{a_1 \setminus B(x_t, R_\varepsilon)} e^{-w_t(x)} dx \leq C \int_{a_1 \setminus B(x_t, R)} e^{-|x-x_t|} dx \leq \varepsilon. \]
We fix $\varepsilon$ and $\delta$ which verify the property above. We recall that we suppose $\forall C > 0, \exists t \in [t_0, 1], |x^t| > C$. As $\nabla u^0$ is a differomorphism of $a^+$ into $2P^+$ and $0 \in 2P^+$, there exists $t \in [t_0, 1]$ such that
\[ \frac{\partial u^0}{\partial \zeta}(x) > \frac{1}{2} a_0, \forall x \in B(x_t, \delta) \]
where $\zeta = x_t/|x_t|$ and $a_0 = \inf\{|x| / x \in 2\partial P\}$. We obtain
\[ \int_{B(x_t, \delta)} \frac{\partial u}{\partial \zeta}(x) e^{-w_t} dx > 1/4 a_0 \beta. \]
(We recall that
\[ \int_{a_1} e^{-w_t(x)} \, dx =: \beta \]
is independant of \( t \).) Thus for \( \varepsilon \) small enough,
\[ \int_{a_1} \frac{\partial u}{\partial \zeta}(x)e^{-wt} \, dx > 0. \]
Now, let’s note that for \( \zeta \in a_1 \), we have
\[ \int_{a_1} \frac{\partial u}{\partial \zeta} e^{-wt} \, dx = 0, \]
Indeed, we have, thanks to the proposition \[3\] that :
\[ 0 = \int_{\Delta^+} \langle p - 2\rho_p, \zeta \rangle e^{-(2p-4\rho_p, \xi)} \prod_{\alpha \in \Phi^+_p} \langle \alpha, p \rangle dp \]
\[ = C \int_{a_1} (\nabla u_t(x), \xi) e^{-(\nabla u(x), \zeta)} \prod_{\alpha \in \Phi^+_p} \langle \alpha, \nabla u_t(x) + 4t\rho_p \rangle MA(\nabla u_t)(x) \, dx \]
\[ = C \int_{a_1} (\nabla u_t(x), \zeta) e^{-w_t(x)} \, dx \]
\[ = C \int_{a_1} \frac{\partial u_t}{\partial \zeta} e^{-w_t(x)} \, dx. \]
\[ = C \frac{1-t}{t} \int_{a_1} \frac{\partial u_0^0}{\partial \zeta}(x)e^{-w_t(x)} \, dx - C \frac{1}{t} \int_{a_1} \frac{\partial u_t}{\partial \zeta} e^{-w_t(x)} \, dx \]
\[ = C \frac{1-t}{t} \int_{a_1} \frac{\partial u_0^0}{\partial \zeta}(x)e^{-w_t(x)} \, dx. \]
So, we have :
\[ \int_{a_1} \frac{\partial u_0^0}{\partial \zeta}(x)e^{-w_t(x)} \, dx = 0, \forall \zeta \in a_1. \]
We reach a contradiction. This complete the proof. \( \square \)

We now conclude thanks to the following lemma :

**Lemma 5.6.** Let \( \psi_t \) solution of the equation \[4\] where \( t \in [t_0, 1] \). Then
\[ \sup_M \psi \leq C, \]
for a constant \( C \) independent of \( t \).

**Proof.** By density and by \( K \)-invariance, it is sufficient to show
\[ \sup_{y \in a_1} \psi|_{P/H}(\exp(y)H) \leq C \]
By convexity of \( u_t \), we have
\[ u_t(0) + (\nabla u_t(y), y) \geq u_t(y), \forall y \in a_1, \]
and by definition of $v_{2\Delta}$ and the fact $\nabla u_t(a_1) = 2\Delta$, we have

$$v_{2\Delta}(y) + u_t(0) \geq u_t(y).$$

Now, we have

$$\psi_t(\exp(y)H) = u_t(y) - u_0(y) \leq v_{2\Delta}(y) + u_t(0) - u_0(y) \leq a + u_t(0) \quad \text{(because $v_{2\Delta} - u_0$ is bounded)}.$$ 

So it suffices to show $u_t(0)$ is upper bounded by a constant independent of $t$.

For this, let $x^t$ the minimal point of $w^t$. As $Dw(a_1) = 2\Delta$ which is bounded, we have

$$|\nabla w(R^n)| \leq d_0 := \sup \{|x| / x \in \Delta\},$$

and so

$$|w(0) - w(x^t)| \leq d_0 \cdot |x^t|.$$ 

Moreover, thanks to the lemma 5.5 we have $C > 0$ independent of $t$ such that $|x^t| \leq C$, that implies

$$|w(0) - w(x^t)| \leq d_0 \cdot C.$$ 

By the lemma 5.3 we have $w(x^t) = m_t \leq \tilde{C}$ where $\tilde{C}$ is a constant independent of $t$. So we have

$$w(0) \leq \tilde{C} + d_0 \cdot C,$$

but $w = tu + (1 - t)u^0$ so

$$t \cdot u_t(0) \leq \tilde{C} + d_0 - (1 - t)u^0(0)$$

hence

$$tu_t(0) \leq \theta,$$

where $\theta := \tilde{C} + d_0 - \sup \{(1 - t - 0)u^0(0), 0\}$. Finally, As we have taken $t \in [t_0, 1]$, we thus get

$$u_t(0) \leq \Theta,$$

where $\Theta$ is independent of $t$. \hfill \Box

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