Optimal Hamiltonian of Fermion Flows

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Abstract
After providing a general formulation of Fermion flows within the context of Hudson-Parthasarathy quantum stochastic calculus, we consider the problem of determining the noise coefficients of the Hamiltonian associated with a Fermion flow so as to minimize a naturally associated quadratic performance functional. This extends to Fermion flows results of the authors previously obtained for Boson flows.

1 Introduction
In the Heisenberg picture of quantum mechanics, the time-evolution of an observable $X$ is described by the operator process $j_t(X) = U^*(t) X U(t)$ where $U(t) = e^{-iH}$ and the Hamiltonian $H$ is a self-adjoint operator on the wave function Hilbert space. The unitary process $U(t)$ and the self-adjoint process $j_t(X)$ satisfy, respectively,

$$dU(t) = -iH U(t) \, dt, \quad U(0) = I$$

and

$$dj_t(X) = i[H, j_t(X)] \, dt, \quad j_0(X) = X$$

where

$$[H, j_t(X)] := H j_t(X) - j_t(X) H$$

In the presence of quantum noise, the equation satisfied by $U(t)$ is replaced by a Hudson-Parthasarathy quantum stochastic differential equation driven by
that noise (see [15] and [17]) and the corresponding equation for the quantum flow \( j_t(X) \) is interpreted as the Heisenberg picture of the Schrödinger equation in the presence of noise. The emergence of such equations as stochastic limits of classical Schrödinger equations is described in [8]. From the point of view of quantum control theory, the problem of minimizing a quadratic performance functional associated with a Hudson-Parthasarathy quantum flow driven by Boson quantum noise, has been considered in [1], [3]-[5] and [9]-[13]. In this paper we consider the same problem for quantum flows driven by Fermion flows. Our approach is based on the representation of the Fermion commutation relations on Boson Fock space obtained in [6] and [16]. A unified approach to the quadratic cost control of quantum processes driven by Boson, Fermion, Finite-Difference and a wide class of other quantum noises, can be found in [2] with the use of the representation free quantum stochastic calculus of [7].

This paper is structured as follows: In section 2 we describe the main features of Hudson-Parthasarathy calculus and the representation of the Fermion commutation relations on Boson Fock space. In section 3 we obtain the quantum stochastic differential equations satisfied by Fermion flows. In section 4 we obtain a new algebraic formulation of Fermion evolution equations and flows and we describe the equations satisfied by their structure maps. Finally, in section 5 we define the quadratic performance functionals associated with Fermion flows and we obtain the coefficients of the corresponding Hamiltonian for which these functionals are minimized.

2 Quantum Stochastic Calculus

The Boson Fock space \( \Gamma := \Gamma(L^2(\mathbb{R}_+, \mathbb{C})) \) over \( L^2(\mathbb{R}_+, \mathbb{C}) \) is the Hilbert space completion of the linear span of the exponential vectors \( \psi(f) \) under the inner product

\[
<\psi(f), \psi(g)> := e^{<f,g>}
\]

where \( f, g \in L^2(\mathbb{R}_+, \mathbb{C}) \) and \( <f, g> := \int_0^{+\infty} \bar{f}(s)g(s) \, ds \). Here and in what follows, \( \bar{z} \) denotes the complex conjugate of \( z \in \mathbb{C} \). The annihilation, creation and conservation operator processes \( A_t, A_t^\dagger \) and \( \Lambda_t \) respectively, are defined on the exponential vectors \( \psi(g) \) of \( \Gamma \) by

\[
A_t \psi(g) := \int_0^t g(s) \, ds \, \psi(g)
\]

\[
A_t^\dagger \psi(g) := \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \psi(g + \epsilon \chi_{[0,t]})
\]

\[
\Lambda_t \psi(g) := \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \psi(e^{\epsilon \chi_{[0,t]}} g).
\]
The basic quantum stochastic differentials $dA_t$, $dA^\dagger_t$, and $d\Lambda_t$ are defined by

\[
dA_t := A_{t+dt} - A_t, \quad dA^\dagger_t := A^\dagger_{t+dt} - A^\dagger_t, \quad d\Lambda_t := \Lambda_{t+dt} - \Lambda_t.
\]

Hudson and Parthasarathy defined in \cite{15} stochastic integration with respect to the noise differentials $dA_t$, $dA^\dagger_t$ and $d\Lambda_t$ and obtained the Itô multiplication table

|   | $dA_t$ | $dA^\dagger_t$ | $d\Lambda_t$ | $dt$ |
|---|---|---|---|---|
| $dA_t$ | 0 | 0 | 0 | 0 |
| $dA^\dagger_t$ | 0 | 0 | 0 | 0 |
| $d\Lambda_t$ | 0 | 0 | 0 | 0 |
| $dt$ | 0 | 0 | 0 | 0 |

We couple $\Gamma$ with a "system" Hilbert space $\mathcal{H}$ and consider processes defined on $\mathcal{H} \otimes \Gamma$. The fundamental theorems of the Hudson-Parthasarathy quantum stochastic calculus give formulas for expressing the matrix elements of quantum stochastic integrals in terms of ordinary Riemann-Lebesgue integrals.

**Theorem 1.** Let

\[
M(t) := \int_0^t E(s) d\Lambda_s + F(s) dA_s + G(s) dA^\dagger_s + H(s) ds
\]

where $E$, $F$, $G$, $H$ are (in general) time dependent adapted processes. Let also $u \otimes \psi(f)$ and $u \otimes \psi(g)$ be in the "exponential domain" of $\mathcal{H} \otimes \Gamma$. Then

\[
\langle u \otimes \psi(f), M(t) u \otimes \psi(g) \rangle = \int_0^t \langle u \otimes \psi(f), (\tilde{f}(s) g(s) E(s) + g(s) F(s) + \tilde{f}(s) G(s) + H(s)) u \otimes \psi(g) \rangle ds.
\]

**Proof.** See Theorem 4.1 of \cite{15}. \hfill \Box

**Theorem 2.** Let

\[
M(t) := \int_0^t E(s) d\Lambda_s + F(s) dA_s + G(s) dA^\dagger_s + H(s) ds
\]

and

\[
M'(t) := \int_0^t E'(s) d\Lambda_s + F'(s) dA_s + G'(s) dA^\dagger_s + H'(s) ds
\]

where $E$, $F$, $G$, $H$, $E'$, $F'$, $G'$, $H'$ are (in general) time dependent adapted processes. Let also $u \otimes \psi(f)$ and $u \otimes \psi(g)$ be in the exponential domain of $\mathcal{H} \otimes \Gamma$. Then

\[
\langle M(t) u \otimes \psi(f), M'(t) u \otimes \psi(g) \rangle = \int_0^t \left\{ \langle M(s) u \otimes \psi(f), (\tilde{f}(s) g(s) E'(s) + g(s) F'(s) + \tilde{f}(s) G'(s) + H'(s)) u \otimes \psi(g) \rangle + \langle (\tilde{g}(s) f(s) E(s) + f(s) F(s) + \tilde{g}(s) G(s) + H(s)) u \otimes \psi(f), M'(s) u \otimes \psi(g) \rangle \right. \\
\left. + \langle (f(s) E(s) + G(s)) u \otimes \psi(f), (g(s) E'(s) + G'(s)) u \otimes \psi(g) \rangle \right\} ds.
\]
Proof. See Theorem 4.3 of [15].

The connection between classical and quantum stochastic analysis is given in the following:

**Theorem 3.** The processes \( B = \{B_t, t \geq 0\} \) and \( P = \{P_t, t \geq 0\} \) defined by

\[
B_t := A_t + A_t^\dagger
\]

and

\[
P_t := \Lambda_t + \sqrt{\lambda}(A_t + A_t^\dagger) + \lambda t
\]

are identified with Brownian motion and Poisson process of intensity \( \lambda \) respectively, in the sense that their vacuum characteristic functionals are given by

\[
< \psi(0), e^{isB_t} \psi(0) > = e^{-\frac{s^2}{2}t}
\]

and

\[
< \psi(0), e^{isP_t} \psi(0) > = e^{\lambda(e^{is} - 1)}t.
\]

Proof. See Theorem 5 of [13].

The processes \( A_t, A_t^\dagger \) satisfy the Boson Commutation Relations

\[
[A_t, A_t^\dagger] := A_t A_t^\dagger - A_t^\dagger A_t = tI.
\]

In [16] Hudson and Parthasarathy showed that the processes \( F_t \) and \( F_t^\dagger \) defined on the Boson Fock space by

\[
F_t := \int_0^t J_s dA_s
\]

\[
F_t^\dagger := \int_0^t J_s dA_s^\dagger
\]

satisfy the Fermion anti-commutation relations

\[
\{F_t, F_t^\dagger\} := F_t F_t^\dagger + F_t^\dagger F_t = tI.
\]

It follows that

\[
dF_t = J_t dA_t \quad \text{(2.1)}
\]

\[
dF_t^\dagger = J_t dA_t^\dagger \quad \text{(2.2)}
\]

Here \( J_t \) is the self-adjoint, unitary-valued, adapted, so called "reflection" process, acting on the noise part of the Fock space and extended as the identity on the system part, defined by

\[
J_t := \gamma(-P_{[0,t]} + P_{(t,\infty)}).
\]
where $P_S$ denotes the multiplication operator by $\chi_S$ and $\gamma$ is the second quantization operator defined by

$$
\gamma(U) \psi(f) := \psi(Uf).
$$

The reflection process $J_t$ commutes with system space operators and satisfies the differential equation (cf. Lemma 3.1 of [16])

$$
dJ_t = -2 J_t \, d\Lambda_t \quad (2.3)
$$

$$
J_0 = 1.
$$

### 3 Fermion Evolutions and Flows

As shown in [16], see also [6], Fermion unitary evolution equations have the form

$$
dU_t = -\left( iH + \frac{1}{2} L^* L \right) dt + L^* W \, dF_t - L \, dF_t^\dagger + (1 - W) \, d\Lambda_t \quad (3.1)
$$

with adjoint

$$
dU_t^* = -U_t^* \left( -iH + \frac{1}{2} L^* L \right) dt - L^* dF_t + W^* \, L \, dF_t^\dagger + (1 - W^*) \, d\Lambda_t \quad (3.2)
$$

where, $U_0 = U_0^* = 1$ and, for each $t \geq 0$, $U_t$ is a unitary operator defined on the tensor product $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathbb{C}))$ of the system Hilbert space $\mathcal{H}$ and the noise Fock space $\Gamma$. Here $H$, $L$, $W$ are in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, with $W$ unitary and $H$ self-adjoint. We identify time-independent, bounded, system space operators $X$ with their ampliation $X \otimes 1$ to $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathbb{C}))$.

**Proposition 1.** Let

$$
\phi_t(T, S) := V_t^* \, (T + S \, J_t) \, V_t
$$

where $T, S$ are bounded system space operators and $V_t, V_t^*$ satisfy the quantum stochastic differential equations

$$
dV_t = \left( \alpha \, dt + \beta \, dF_t + \gamma \, dF_t^\dagger + \delta \, d\Lambda_t \right) V_t
$$

$$
= \left( \alpha \, dt + \beta \, J_t \, dA_t + \gamma \, J_t \, dA_t^\dagger + \delta \, d\Lambda_t \right) V_t
$$

and

$$
dV_t^* = V_t^* \left( \alpha^* \, dt + \beta^* \, dF_t^\dagger + \gamma^* \, dF_t + \delta^* \, d\Lambda_t \right)
$$

$$
= V_t^* \left( \alpha^* \, dt + \beta^* \, J_t \, dA_t^\dagger + \gamma^* \, J_t \, dA_t + \delta^* \, d\Lambda_t \right)
$$

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where $\alpha, \beta, \gamma, \delta$ are bounded system space operators. Then

\[
d\phi_t(T, S) = \phi_t(\alpha^* T + T \alpha + \gamma^* T \gamma, \alpha^* S + S \alpha + \gamma^* S \gamma) dt \quad (3.3)
\]

\[
+ \phi_t(\gamma^* S + S \beta - \gamma^* S (2 + \delta), \gamma^* T + T \beta + \gamma^* T \delta) dA_t
\]

\[
+ \phi_t(\beta^* S + S \gamma - \delta^* S \gamma, \beta^* T + T \gamma + \delta^* T \gamma) dA_t
\]

\[
+ \phi_t(\delta^* T + T \delta + \delta^* T \delta, -(2 S + S \delta + \delta^* S + \delta^* S \delta)) d\Lambda_t
\]

with

\[
\phi_0(T, S) = T + S \quad (3.4)
\]

Proof. Making use of the algebraic rule

\[
d(x y) = d x y + x d y + d x y
\]

we find

\[
d\phi_t(T, S) = dV_t^* (T + S J_t) V_t + V_t^* d((T + S J_t) V_t) + dV_t^* d((T + S J_t) V_t)
\]

\[
= dV_t^* (T + S J_t) V_t + V_t^* T dV_t + V_t^* S d(J_t V_t) + dV_t^* T dV_t + dV_t^* S d(J_t V_t).
\]

But, by (2.3), the Itô table for the Boson stochastic differentials and the fact that $J_t^2 = 1$

\[
d(J_t V_t) = dJ_t V_t + J_t dV_t + dJ_t dV_t
\]

\[
= -2 J_t dA_t V_t + J_t \left( \alpha dt + \beta J_t dA_t + \gamma J_t dA_t^1 + \delta dA_t \right) V_t - 2 J_t dA_t \left( \alpha dt + \beta J_t dA_t + \gamma J_t dA_t^1 + \delta dA_t \right) V_t
\]

\[
= \left( \alpha J_t dt + \beta dA_t - \gamma dA_t + \delta (2 + \delta) J_t dA_t \right) V_t.
\]

Thus

\[
d\phi_t(T, S) = V_t^* \left( \alpha^* dt + \beta^* J_t dA_t^1 + \gamma^* J_t dA_t + \delta^* dA_t \right) (T + S J_t) V_t
\]

\[
+ V_t^* T \left( \alpha dt + \beta J_t dA_t + \gamma J_t dA_t^1 + \delta dA_t \right) V_t
\]

\[
+ V_t^* S \left( \alpha J_t dt + \beta dA_t - \gamma dA_t^1 - (2 + \delta) J_t dA_t \right) V_t
\]

\[
+ V_t^* \left( \alpha^* dt + \beta^* J_t dA_t^1 + \gamma^* J_t dA_t + \delta^* dA_t \right)
\]

\[
\times T \left( \alpha dt + \beta J_t dA_t + \gamma J_t dA_t^1 + \delta dA_t \right) V_t
\]

\[
+ V_t^* \left( \alpha^* dt + \beta^* J_t dA_t^1 + \gamma^* J_t dA_t + \delta^* dA_t \right)
\]

\[
\times S \left( \alpha J_t dt + \beta dA_t - \gamma dA_t^1 - (2 + \delta) J_t dA_t \right) V_t
\]

\[
= \phi_t(\alpha^* T + T \alpha + \gamma^* T \gamma, \alpha^* S + S \alpha + \gamma^* S \gamma) dt
\]

\[
+ \phi_t(\gamma^* S + S \beta - \gamma^* S (2 + \delta), \gamma^* T + T \beta + \gamma^* T \delta) dA_t
\]

\[
+ \phi_t(\beta^* S + S \gamma - \delta^* S \gamma, \beta^* T + T \gamma + \delta^* T \gamma) dA_t
\]

\[
+ \phi_t(\delta^* T + T \delta + \delta^* T \delta, -(2 S + S \delta + \delta^* S + \delta^* S \delta)) d\Lambda_t
\]
With the processes $U_t$ and $U^*_t$ of (3.1) and (3.2) we associate the Fermion flow
\[ j_t(X) := U^*_t X U_t = \phi(X, 0) \] (3.5)
and the reflected flow
\[ r_t(X) := j_t(X J_t) = \phi(0, X) \] (3.6)
where $X$ is a bounded system space operator.

**Corollary 1.** The Fermion flow $j_t(X)$ and the reflected flow $r_t(X)$ defined in (3.5) and (3.6) satisfy the system of quantum stochastic differential equations
\[ dj_t(X) = j_t(i [H, X] - \frac{1}{2} (L^* L X + X L^* L - 2 L^* X L)) \, dt \]
\[ + r_t([L^*, X] W) \, dA_t + r_t(W^* [X, L]) \, dA^1_t + j_t(W^* X W - X) \, d\Lambda_t \] (3.7)
and
\[ dr_t(X) = r_t(i [H, X] - \frac{1}{2} (L^* L X + X L^* L - 2 L^* X L)) \, dt \]
\[ - j_t([L^*, X] W) \, dA_t - j_t(W^* [X, L] - 2 X L) \, dA^1_t - r_t(W^* X W + X) \, d\Lambda_t \] (3.8)
with
\[ j_0(X) = r_0(X) = X. \]

Here, as usual, $[x, y] := x y - y x$ and $\{x, y\} := x y + y x$.

**Proof.** We replace $V_t$ and $V_t^*$ in Proposition 1 by $U_t$ and $U_t^*$. Then, equation (3.7) is a special case of (3.3) for $T = X, S = 0$ and
\[
\begin{align*}
\alpha &= -(i H + \frac{1}{2} L^* L) \\
\beta &= -L^* W \\
\gamma &= L \\
\delta &= W - 1
\end{align*}
\]
while equation (3.8) is a special case of (3.3) for $T = 0, S = X$ and $\alpha, \beta, \gamma, \delta$ as above.

### 4 Generalized Fermion Flows

Fermion flows can be formulated and studied in a manner similar to Boson (also called Evans-Hudson) flows (cf. [14] and [17]). Let $\mathcal{B}(\mathcal{H})$ denote the space of...
bounded system operators. We define $\mathcal{B}(\mathcal{H})$-valued operations $\triangledown$ and $\triangle$ on $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ by

\[
\begin{align*}
(T_1, S_1) \triangledown (T_2, S_2) &\colon= T_1 T_2 + S_1 S_2 \quad (4.1) \\
(T_1, S_1) \triangle (T_2, S_2) &\colon= T_1 S_2 + S_1 T_2. \quad (4.2)
\end{align*}
\]

Notice that if $B$ is the reflection map, then $B = \rho(2, S_2)$. We also define the $\circ$-product where $x \circ y := (x \triangledown y, x \triangle y)$ is the reflection map, then

\[
(T_1, S_1) \triangle (T_2, S_2) = (T_1, S_1) \triangledown \rho(2, S_2).
\]

We also define the $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$-valued product map $\circ$ on $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ by

\[
x \circ y := (x \triangledown y, x \triangle y) = (T_1 T_2 + S_1 S_2, T_1 S_2 + S_1 T_2) \quad (4.3)
\]

where $x = (T_1, S_1)$ and $y = (T_2, S_2)$. From the definitions it follows that the $\circ$-product is associative with unit $\id := (1, 0)$ where 1 and 0 are the identity and zero operators in $\mathcal{B}(\mathcal{H})$. Let the flow $\phi_t(T, S) := V_t^* (T + S J_t) V_t$ be as in Proposition 1 and let $x = (T_1, S_1)$ and $y = (T_2, S_2)$. Then

\[
\phi_t(x \circ y) = \phi_t(T_1, S_1) \phi_t(T_2, S_2) = V_t^* (T_1 + S_1 J_t) V_t \phi_t(T_2, S_2) = V_t^* (T_1 + S_1 J_t) (T_2 + S_2 J_t) V_t \phi_t(T_2, S_2) = V_t^* (T_1 T_2 + S_1 S_2 + (T_1 S_2 + S_1 T_2) J_t) V_t = \phi_t(T_1 T_2 + S_1 S_2, T_1 S_2 + S_1 T_2) = \phi_t(x \circ y)
\]

ie $\phi_t$ is a homomorphism with respect to the $\circ$-product. Since $\phi_t$ is the solution of (3.3), (3.4) this suggests considering flows satisfying quantum stochastic differential equations of the form

\[
d\phi_t(x) = \phi_t(\theta_1(x)) \, dt + \phi_t(\theta_2(x)) \, dA_t + \phi_t(\theta_3(x)) \, dA_t^\dagger + \phi_t(\theta_4(x)) \, dA_t \quad (4.4)
\]

with

\[
\phi_0(x) = x \triangledown \id + x \triangle \id
\]

where $x = (T, S) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ and for $i = 1, 2, 3, 4$, $\theta_i : \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ are the "structure maps". In the case of (3.3), (3.4)

\[
\begin{align*}
\theta_1(T, S) &= (\alpha^* T + T \alpha + \gamma^* T \gamma, \alpha^* S + S \alpha + \gamma^* S \gamma) \\
\theta_2(T, S) &= (\gamma^* S + S \beta - \gamma^* S (2 + \delta), \gamma^* T + T \beta + \gamma^* T \delta) \\
\theta_3(T, S) &= (\beta^* S + S \gamma - \delta^* S \gamma, \beta^* T + T \gamma + \delta^* T \gamma) \\
\theta_4(T, S) &= (\delta^* T + T \delta + \delta^* T \delta, -(2 S + S \delta + \delta^* S + \delta^* S \delta))
\end{align*}
\]
where

\[\alpha = -(iH + \frac{1}{2} L^*L)\]
\[\beta = -L^* W\]
\[\gamma = L\]
\[\delta = W - 1.\]

The general conditions on the \(\theta_i\) in order for \(\phi_t\) to be an identity preserving \(\circ\)-product homomorphism are given in the following:

**Proposition 2.** Let \(\phi_t\) be the solution of (4.4). Then
\[
\phi_t(x) \phi_t(y) = \phi_t(x \circ y)
\]
\[
\phi_t(x)^* = \phi_t(x^*)
\]
\[
\phi_t(id) = 1
\]

if and only if the \(\theta_i\) satisfy the structure equations

\[
\theta_1(x) \circ y + x \circ \theta_1(y) + \theta_2(x) \circ \theta_3(y) = \theta_1(x \circ y)
\]
\[
\theta_2(x) \circ y + x \circ \theta_2(y) + \theta_2(x) \circ \theta_4(y) = \theta_2(x \circ y)
\]
\[
\theta_3(x) \circ y + x \circ \theta_3(y) + \theta_3(x) \circ \theta_4(y) = \theta_3(x \circ y)
\]
\[
\theta_4(x) \circ y + x \circ \theta_4(y) + \theta_4(x) \circ \theta_4(y) = \theta_4(x \circ y)
\]

and, with \(*\) denoting "adjoint" and \(x = (T_1, S_1) \Leftrightarrow x^* = (T_1^*, S_1^*)\),

\[
\theta_1(x^*) = (\theta_1(x))^*
\]
\[
\theta_2(x^*) = (\theta_3(x))^*
\]
\[
\theta_3(x^*) = (\theta_2(x))^*
\]
\[
\theta_4(x^*) = (\theta_4(x))^*
\]

with

\[
\theta_1(id) = \theta_2(id) = \theta_3(id) = \theta_4(id) = (0, 0)
\]

**Proof.**

\[
\phi_t(x \circ y) = \phi_t(x) \phi_t(y)
\]

\[
\Leftrightarrow d\phi_t(x \circ y) = d\phi_t(x) \phi_t(y) + \phi_t(x) d\phi_t(y) + d\phi_t(x) \phi_t(y)
\]

which implies that

\[
\phi_t(\theta_1(x \circ y)) dt + \phi_t(\theta_2(x \circ y)) dA_t + \phi_t(\theta_3(x \circ y)) dA^1_t
\]
\[
+ \phi_t(\theta_4(x \circ y)) dA_t = \phi_t(\theta_1(x)) \phi_t(y) dt + \phi_t(\theta_2(x)) \phi_t(y) dA_t
\]
\[
+ \phi_t(\theta_3(x)) \phi_t(y) dA^1_t + \phi_t(\theta_4(x)) \phi_t(y) dA_t + \phi_t(x) \phi_t(\theta_1(y)) dt
\]
\[
+ \phi_t(x) \phi_t(\theta_2(y)) dA_t + \phi_t(x) \phi_t(\theta_3(y)) dA^1_t + \phi_t(x) \phi_t(\theta_4(y)) dA_t +
\]
\[
\phi_t(\theta_2(x)) \phi_t(\theta_3(y)) dt + \phi_t(\theta_2(x)) \phi_t(\theta_4(y)) dA_t
\]
\[
+ \phi_t(\theta_3(x)) \phi_t(\theta_4(y)) dA^1_t + \phi_t(\theta_4(x)) \phi_t(\theta_4(y)) dA_t
\]

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from which collecting the \(dt, dA_t, dA_t^\dagger\) and \(d\Lambda_t\) terms on each side, using the homomorphism property and then equating the coefficients of \(dt, dA_t, dA_t^\dagger\) and \(d\Lambda_t\) on both sides we obtain (4.5)-(4.8). Similarly

\[
\phi_t(x^*) = \phi_t(x^*) \Leftrightarrow d\phi_t(x^*) = d\phi_t(x^*)
\]

i.e

\[
\phi_t(\theta_1(x^*)) dt + \phi_t(\theta_2(x^*)) dA_t^\dagger + \phi_t(\theta_3(x^*)) dA_t + \phi_t(\theta_4(x^*)) d\Lambda_t = \phi_t(\theta_1(x^*)) dt + \phi_t(\theta_2(x^*)) dA_t + \phi_t(\theta_3(x^*)) dA_t^\dagger + \phi_t(\theta_4(x^*)) d\Lambda_t
\]

and (4.9)-(4.12) follow by equating the coefficients of \(dt, dA_t, dA_t^\dagger\) and \(d\Lambda_t\) on both sides. Finally

\[
\phi_t(id) = 1 \Leftrightarrow d\phi_t(id) = 0
\]

which by the linear independence of \(dt\) and the stochastic differentials implies (4.13).

5 Optimal Noise Coefficients

As in [2] and [3], motivated by classical linear system control theory, we think of the self-adjoint operator \(H\) appearing in (3.1) as fixed and we consider the problem of determining the coefficients \(L\) and \(W\) of the noise part of the Hamiltonian of the evolution equation (3.1) that minimize the "evolution performance functional"

\[
Q_{\xi,T}(u) = \int_0^T \left[ \|XU_t \xi\|^2 + \frac{1}{4} \|L^* L U_t \xi\|^2 \right] dt + \frac{1}{2} \|LU_T \xi\|^2 \tag{5.1}
\]

where \(T \in [0, +\infty)\), \(\xi\) is an arbitrary vector in the exponential domain of \(\mathcal{H} \otimes \Gamma\) and \(X\) is a bounded self-adjoint system operator. By the unitarity of \(U_t, UT, J_t\), and \(J^T, \phi_t\), (5.1) is the same as the "Fermion flow performance functional"

\[
J_{\xi,T}(L,W) = \int_0^T \left[ \|j_t(X) \xi\|^2 + \frac{1}{4} \|j_t(L^* L) \xi\|^2 \right] dt + \frac{1}{2} \|j_T(L) \xi\|^2 \tag{5.2}
\]

and the "reflected flow performance functional"

\[
R_{\xi,T}(L,W) = \int_0^T \left[ \|r_t(X) \xi\|^2 + \frac{1}{4} \|r_t(L^* L) \xi\|^2 \right] dt + \frac{1}{2} \|r_T(L) \xi\|^2 \tag{5.3}
\]

We consider the problem of minimizing the functionals \(J_{\xi,T}(L,W)\) and \(R_{\xi,T}(L,W)\) over all system operators \(L,W\) where \(L\) is bounded and \(W\) is unitary. The motivation behind the definition of the performance functionals (5.1), (5.2) and (5.3) can be found in the following theorem which is the quantum stochastic analogue of the classical linear regulator theorem.
**Theorem 4.** Let \( U = \{U_t, t \geq 0\} \) be a process satisfying the quantum stochastic differential equation
\[
dU_t = (FU_t + u_t) \, dt + \Psi U_t \, dF_t + \Phi U_t \, dF_t^\dagger + ZU_t \, d\Lambda_t, \quad U_0 = 1, \, t \in [0,T] \quad (5.4)
\]
with adjoint
\[
dU_t^* = (U_t^* F^* + U_t^* u_t) \, dt + U_t^* \Psi^* \, dF_t^\dagger + U_t^* \Phi^* \, dF_t + U_t^* Z^* \, d\Lambda_t, \quad U_0^* = 1 \quad (5.5)
\]
where \( 0 < T < +\infty \), the coefficients \( F, \Psi, \Phi, Z \) are bounded operators on the system space \( \mathcal{H} \) and \( u_t := -\Pi U_t \) for some positive bounded system operator \( \Pi \).

Then the functional
\[
Q_{\xi,T}(u) = \int_0^T [<U_t \xi, X^2 U_t \xi> + <u_t \xi, u_t \xi>] \, dt - <u_T \xi, U_T \xi> \quad (5.6)
\]
where \( X \) is a system space observable, identified with its ampliation \( X \otimes I \) to \( \mathcal{H} \otimes \Gamma \), is minimized over the set of feedback control processes of the form \( u_t = -\Pi U_t \), by choosing \( \Pi \) to be a bounded, positive, self-adjoint system operator satisfying
\[
\Pi F + F^* \Pi \Phi - \Pi^2 + X^2 = 0 \quad (5.7)
\]
\[
\Pi \Psi + \Phi^* \Pi \Phi = 0 \quad (5.8)
\]
\[
\Pi Z + Z^* \Pi + Z^* \Pi Z = 0 \quad (5.9)
\]
The minimum value is \( <\xi, \Pi \xi> \). We recognize (5.7) as the algebraic Riccati equation.

**Proof.** Let
\[
\theta_t = <\xi, U_t^* \Pi U_t \xi> \quad (5.10)
\]
Then
\[
d\theta_t = <\xi, d(U_t^* \Pi U_t) \xi> = <\xi, (dU_t^* \Pi U_t + U_t^* \Pi dU_t + dU_t^* \Pi dU_t) \xi> \quad (5.11)
\]
which, after replacing \( dU_t \) and \( dU_t^* \) by (5.4) and (5.5) respectively, and using (2.1), (2.2) and the Itô table, becomes
\[
d\theta_t = <\xi, U_t^* (F^* \Pi + \Pi F + \Phi^* \Pi \Phi) \, dt + (\Phi^* \Pi + \Pi \Psi + \Phi^* \Pi Z) \, J_t \, d\Lambda_t \quad (5.12)
\]
\[
+ (\Psi \Pi^* + \Pi \Phi + Z^* \Pi \Phi) \, J_t \, dA_t + (Z^* \Pi + \Pi Z + Z^* \Pi Z) \, d\Lambda_t) \, U_t \xi >
\]
\[
+ <\xi, (u_t^* \Pi U_t + U_t^* \Pi u_t) \, dt \xi> .
\]
and by (5.7)-(5.9)
\[
d\theta_t = <\xi, U_t^* (\Pi^2 - X^2) \, U_t \, dt \xi > + <\xi, (u_t^* \Pi U_t + U_t^* \Pi u_t) \, dt \xi> \quad (5.13)
\]
By (5.10)
\[
\theta_T - \theta_0 = <\xi, U_T^* \Pi U_T \xi> - <\xi, \Pi \xi> \quad (5.14)
\]
while by (5.13)\[\theta_T - \theta_0 = \int_0^T \langle \xi, U_t^*(\Pi^2 - X^2) U_t \xi \rangle + \langle \xi, (u_t^* \Pi U_t + U_t^* \Pi u_t) \xi \rangle \rangle dt\]

By (5.14) and (5.15)\[\langle \xi, U_T^* \Pi U_T \xi \rangle - \langle \xi, \Pi \xi \rangle = \int_0^T \langle \xi, U_t^*(\Pi^2 - X^2) U_t \xi \rangle + \langle \xi, (u_t^* \Pi U_t + U_t^* \Pi u_t) \xi \rangle \rangle dt\]

Thus
\[
Q_{\xi,T}(u) = (\langle \xi, U_T^* \Pi U_T \xi \rangle - \langle \xi, \Pi \xi \rangle) + Q_{\xi,T}(u) (5.17)
- (\langle \xi, U_T^* \Pi U_T \xi \rangle - \langle \xi, \Pi \xi \rangle).
\]

Replacing the first parenthesis on the right hand side of (5.17) by (5.16) and Q_{\xi,T}(u) by (5.6) we obtain after cancelations
\[
Q_{\xi,T}(u) = \int_0^T (\langle \xi, (U_t^* \Pi^2 U_t + u_t^* \Pi U_t + U_t^* \Pi u_t + u_t^* u_t) \xi \rangle > dt + \langle \xi, \Pi \xi \rangle > (5.18)
\]

which is clearly minimized by \(u_t = -\Pi U_t\) and the minimum is \(\langle \xi, \Pi \xi \rangle\).

**Theorem 5.** Let \(X\) be a bounded self-adjoint system operator such that the pair \((iH, X)\) is stabilizable i.e there exists a bounded system operator \(K\) such that \(iH + KX\) is the generator of an asymptotically stable semigroup. Then, the quadratic performance functionals (5.2) and (5.3) associated with the Fermion flow \(\{j_t(X) := U_t^* Xu, t \geq 0\}\) and the reflected flow \(\{r_t(X) := j_t(X J_t), t \geq 0\}\), where \(U = \{U_t, t \geq 0\}\) is the solution of (3.1), are minimized by \(L = \sqrt{2} \Pi^{1/2} W_1\) (5.19) and
\[
W = W_2 (5.20)
\]

where \(\Pi\) is a positive self-adjoint solution of the “algebraic Riccati equation”
\[i[H, \Pi] + \Pi^2 + X^2 = 0 (5.21)\]

and \(W_1, W_2\) are bounded unitary system operators commuting with \(\Pi\). It is known (see [18]) that if the pair \((iH, X)\) is stabilizable, then \(5.21\) has a positive self-adjoint solution \(\Pi\). Moreover
\[
\min_{L,W} J_{\xi,T}(L,W) = \min_{L,W} R_{\xi,T}(L,W) = \langle \xi, \Pi \xi \rangle (5.22)
\]

independent of \(T\).
Proof. Looking at (3.1) as (5.4) with \( u_t = -\frac{1}{2} L^* L U_t \), \( F = -i H \), \( \Phi = -L^* W \), \( \Phi = L \), and \( Z = W - 1 \), (5.6) is identical to (5.1). Moreover, equations (5.7)-(5.9) become

\[
\begin{align*}
i[H, \Pi] + L^* \Pi L - \Pi^2 + X^2 &= 0 \\
L^* \Pi - \Pi L^* W + L^* \Pi (W - 1) &= 0 \\
(W^* - 1) \Pi + \Pi (W - 1) + (W^* - 1) \Pi (W - 1) &= 0. \end{align*}
\]  

(5.23) (5.24) (5.25)

By the self-adjointness of \( \Pi \), (5.24) implies that

\[
[L, \Pi] = [L^*, \Pi] = 0
\]

(5.26) while (5.25) implies that

\[
[W, \Pi] = [W^*, \Pi] = 0
\]

(5.27)

i.e (5.20). By (5.24) and the fact that in this case

\[
\Pi = \frac{1}{2} L^* L \text{ i.e } L^* L = 2 \Pi
\]

(5.28)

equation (5.23) implies (5.21). Equations (5.26) and (5.28) also imply that

\[
[L, L^*] = 0
\]

(5.29)

which implies (5.19).

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