On toric geometry, \( \text{Spin}(7) \) manifolds, and type II superstring compactifications

Adil Belhaj\(^1\)* and Jørgen Rasmussen\(^2\)†

\(^1\)Instituto de Física Teórica, C-XVI, Universidad Autónoma de Madrid
E-28049-Madrid, Spain

\(^2\)Centre de Recherches Mathématiques, Université de Montréal
C.P. 6128, succursale centre-ville, Montréal, PQ, Canada H3C 3J7

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Abstract

We consider type II superstring compactifications on the singular \( \text{Spin}(7) \) manifold constructed as a cone on \( SU(3)/U(1) \). Based on a toric realization of the projective space \( \mathbb{CP}^2 \), we discuss how the manifold can be viewed as three intersecting Calabi-Yau conifolds. The geometric transition of the manifold is then addressed in this setting. The construction is readily extended to higher dimensions where we speculate on possible higher-dimensional geometric transitions. Armed with the toric description of the \( \text{Spin}(7) \) manifold, we discuss a brane/flux duality in both type II superstring theories compactified on this manifold.

KEYWORDS: Toric geometry, superstrings, \( \text{Spin}(7) \) manifold, compactification, geometric transition.

*adil.belhaj@uam.es
†rasmusse@crm.umontreal.ca
1 Introduction

Calabi-Yau conifold transitions in superstring compactifications have been studied intensively over the last couple of years. These transitions have become a 'standard' tool in understanding large $N$ dualities. An example of such a duality is the equivalence of the $SU(N)$ Chern-Simons theory on $S^3$ for large $N$, and the closed topological strings on the resolved conifold $[1]$. A further example has been obtained by embedding these results in type IIA superstring theory $[2]$. In particular, the scenario with $N$ D6-branes wrapped around the $S^3$ of the deformed conifold $T^*S^3$ for large $N$ has been found to be equivalent to type IIA superstrings on the resolved conifold with $N$ units of R-R two-form fluxes through $S^2$. The latter thus gives the strong-coupling description of the weak-coupling physics of the former $[1]$. This result has been 'lifted' to the eleven-dimensional M-theory $[3]$ where it corresponds to a so-called flop duality in M-theory compactified on a manifold with $G_2$ holonomy, for short, a $G_2$ manifold.

Quite recently, similar studies have been done in three dimensions using either type IIA superstring compactification on a $G_2$ manifold $[4, 5]$, or M-theory compactified on a $\text{Spin}(7)$ manifold $[5]$. This $\text{Spin}(7)$ manifold is constructed as a cone on $SU(3)/U(1)$. Upon reduction of the M-theory case to ten dimensions, the original geometric transition involving a collapsing $S^5$ and a growing $\mathbb{CP}^2$ may be interpreted as a transition between two phases described by wrapped D6-branes or R-R fluxes, respectively.

An objective of the present work is to continue the study of geometric transitions and brane/flux dualities in lower dimensions. We shall thus consider type II superstrings propagating on the same $\text{Spin}(7)$ manifold as above. By comparison with the known results for the Calabi-Yau conifold transition, in particular, we conjecture new brane/flux dualities in two dimensions. The type IIA and type IIB superstrings are treated separately, and we find that the resulting gauge theories in two dimensions have only one supercharge each, so that $\mathcal{N} = 1/2$ in both cases.

The present study utilizes a toric geometry description of the $\text{Spin}(7)$ manifold. We find that the manifold can be viewed as three intersecting Calabi-Yau conifolds associated to a triangular toric diagram. This result offers a picture for understanding the topology-changing transition of the $\text{Spin}(7)$ manifold. It also allows us to discuss the aforementioned brane/flux transition based on an analysis of type II superstrings on the individual Calabi-Yau conifolds.

Our toric description of the $\text{Spin}(7)$ manifold may be extended to higher-dimensional manifolds thus suggesting that (generalized) geometric transitions may play a role in higher
dimensions as well. We propose an explicit hierarchy of pairs of geometries related by such transitions.

The remaining part of this paper is organized as follows. In Section 2, we use toric geometry to discuss the Calabi-Yau conifold transition and its extension to the $\text{Spin}(7)$ manifold, and speculate on a further generalization to higher dimensions. We then turn to compactifications of superstrings in Section 3. Since our analysis is based on the toric description of the $\text{Spin}(7)$ manifold, our results are deduced from similar results on compactifications on Calabi-Yau conifolds. The associated brane/flux dualities are discussed separately for type IIA and type IIB superstring propagations. Section 4 contains some concluding remarks.

2 Toric geometry and geometric transitions

2.1 Projective spaces and odd-dimensional spheres

As a description of projective spaces in terms of toric geometry lies at the heart of our study of superstring compactifications, we shall review it here. Odd-dimensional (real) spheres are equally important in our analysis and are therefore also discussed here.

The simplest (complex) projective space is $\mathbb{C}P^1$ with a toric $U(1)$ action having two fixed points, $v_1$ and $v_2$, corresponding to the North and South poles, respectively, of the (real) two-sphere $S^2 \sim \mathbb{C}P^1$. In this way, $\mathbb{C}P^1$ may be viewed as the interval $[v_1, v_2]$ referred to as the toric diagram, with a circle on top which vanishes at the end points $v_1$ and $v_2$.

Embedded in $\mathbb{C}^3$, $\mathbb{C}P^2$ may be described as the space of three complex numbers $(z_1, z_2, z_3)$ not all zero, modulo the identification $(z_1, z_2, z_3) \sim (\lambda z_1, \lambda z_2, \lambda z_3)$ for all non-zero $\lambda \in \mathbb{C}$. Alternatively, $\mathbb{C}P^2$ is the (complex) two-dimensional space with a toric $U(1)^2$ action with
three fixed points, \( v_1, v_2 \) and \( v_3 \). Its toric diagram is the triangle \((v_1v_2v_3)\)

\[
\begin{align*}
v_1 & \quad z_3 = 0 \\
v_2 & \quad z_2 = 0 \\
v_3 & \quad z_1 = 0
\end{align*}
\]

(2)

describing the intersection of three \( \mathbb{CP}1 \)'s. Each of the three edges, \([v_1, v_2], [v_2, v_3] \) and \([v_3, v_1]\), is characterized by the vanishing of one of the homogeneous coordinates: \( z_3 = 0, z_1 = 0 \) or \( z_2 = 0 \), respectively. Each edge is stable under the action of a subgroup of \( U(1)^2 \) – two of them being the two \( U(1) \) factors, while the third subgroup is the diagonal one. This toric realization of \( \mathbb{CP}2 \) can be viewed as the triangle \((v_1v_2v_3)\) with a torus, \( T^2 \), on top which collapses to a circle at an edge and to a point at a vertex.

This representation is readily extended to the \( n \)-dimensional projective space \( \mathbb{CP}^n \) where we have a \( T^n \) fibration over an \( n \)-dimensional simplex (regular polytope), see [3], for example. In this case, the \( T^n \) collapses to a \( T^{n-1} \) on each of the \( n \) faces of the simplex, and to a \( T^{n-2} \) on each of the \((n - 2)\)-dimensional intersections of these faces, etc. We recall that \( \mathbb{CP}^n \) is defined similarly to \( \mathbb{CP}2 \) in terms of \( n + 1 \) homogeneous coordinates modulo the identification \( (z_1, \ldots, z_{n+1}) \sim (\lambda z_1, \ldots, \lambda z_{n+1}) \).

The odd-dimensional (real) spheres admit a similar description. The one-sphere, for example, is trivially realized as a \( T^1 \sim S^1 \) over the zero-simplex – a point. The three-sphere may be realized as a \( T^2 \) over a one-simplex – a line segment as the one in [1]. This may be extended to the \((2n + 1)\)-dimensional sphere \( S^{2n+1} \) which may be described as a \( T^{n+1} \) over an \( n \)-simplex. Of particular interest is the five-sphere \( S^5 \) which in this way may be realized as the triangle \((2)\) with a \( T^3 \) on top (whereas \( \mathbb{CP}2 \) had a \( T^2 \) on top). It is stressed that it is for \( n = 1 \) only that the even-dimensional sphere \( S^{2n} \) is equivalent to \( \mathbb{CP}^n \).

To illustrate this toric description of odd-dimensional spheres, let us add a couple of comments on \( S^5 \) realized as a \( T^3 \) over a triangle. As in [2], an edge of the triangle corresponds to the vanishing of one of the three complex coordinates of the embedding space \( \Phi^3 \). Each edge of the triangle \((2)\) is stable under the action of one of the three \( U(1) \) factors of \( U(1)^3 \).
associated to $T^3$. The three-torus itself collapses to a two-torus $T^2$ at an edge and to a circle at a vertex. We may thus view $S^5$ as three intersecting three-spheres over the triangle $T^3$. As opposed to the toric description of $\mathbb{C}P^2$ as a $T^2$ over a triangle, the diagonal $U(1)$ of $U(1)^3$ in the $S^5$ description has no fixed points. This is natural from the realization of $\mathbb{C}P^2$ as $S^5$ modulo $U(1)$.

### 2.2 Calabi-Yau conifold

We shall also make use of the non-compact Calabi-Yau threefold defined in $\mathbb{C}^4$ by the equation

$$uv - xy = 0.$$  \hspace{1cm} (3)

It may be viewed as the singular cone on the five-dimensional base $S^2 \times S^3$ and is therefore referred to as the Calabi-Yau conifold. The singularity is located at the origin and may be turned into a regular point by blowing it up. There are basically two ways of doing that, referred to as resolution and deformation, respectively. Resolving the singularity consists in replacing the singular point by a $\mathbb{C}P^1$. In this way, the local geometry is given by an $O(-1) + O(-1)$ bundle over $\mathbb{C}P^1$. The smooth manifold thus obtained is called the resolved conifold and is of topology $\mathbb{R}^4 \times \mathbb{C}P^1$. In the case of complex deformation, the conifold singularity is removed by modifying the defining algebraic equation (3) by introducing the complex parameter $\mu$:

$$uv - xy = \mu,$$ \hspace{1cm} (4)

while keeping the Kähler structure. The origin is thereby replaced by $S^3$, and the local geometry is given by $T^*S^3$ of topology $\mathbb{R}^3 \times S^3$. This is called the deformed conifold and is related to the resolved conifold by the so-called conifold transition.

This conifold transition admits a representation in toric geometry, where it can be understood as an enhancement or breaking, respectively, of the toric circle actions. On the one hand, the $O(-1) + O(-1)$ bundle over $\mathbb{C}P^1$ has only one toric $U(1)$ action, identified with the toric action on $\mathbb{C}P^1$ itself, while the deformed conifold $T^*S^3$ has a toric $U(1)^2$ action since the spherical part can be viewed as a $T^2$ over a line segment. Referring to (3), the torus is generated by the two $U(1)$ actions

$$(u, v) \rightarrow (e^{i\theta_1}u, e^{-i\theta_1}v), \quad (x, y) \rightarrow (e^{i\theta_2}x, e^{-i\theta_2}y)$$ \hspace{1cm} (5)

with $\theta_i$ real. Thus, the blown-up $S^3$ may be described by the complex interval $[0, \mu]$ with the two circles parameterized by $\theta_i$ on top, where $S^1(\theta_1)$ collapses to a point at $\mu$ while
$S^1(\theta_2)$ collapses to a point at 0. The transition occurs when one of these circles refrains from collapsing while the other one collapses at both interval endpoints. This breaks the toric $U(1)^2$ action to $U(1)$, and the missing $U(1)$ symmetry has become a real line (over $\mathbb{C} \mathbb{P}^1$). The resulting geometry is thus the resolved conifold. The following picture may help to illustrate this transition which can go in both directions:

\begin{equation}
\begin{array}{cccc}
\text{deformed} & \leftrightarrow & \text{resolved} \\
\end{array}
\end{equation}

The top, thin and piecewise straight line in the resolved part of (6) corresponds to the extra $\mathbb{R}$ while the remaining three thin lines (the two straight lines in the deformed part, and the lower, thin and piecewise straight line segment in the resolved part) indicate the $U(1)$'s. The two thick line segments represent the underlying interval.

A somewhat pragmatic way of viewing the conifold transition is based on the conical structure of the conifold itself as a cone on $S^2 \times S^3$. As described in [4], an $n$-dimensional cone on an $(n-1)$-dimensional compact space $Y$ with metric $d\Omega^2$ has metric

\begin{equation}
    ds^2 = dr^2 + r^2 d\Omega^2.
\end{equation}

It has a singularity at the origin unless $Y = S^{n-1}$ and $d\Omega^2$ is the standard 'round' metric. In that case the cone corresponds to $\mathbb{R}^n$. Now, the deformed conifold is obtained by 'pulling' the conical structure off of the $S^3$ factor in the base, while 'maintaining' it on the $S^2$ factor. The latter is then equivalent to $\mathbb{R}^3$ and we have recovered the $\mathbb{R}^3 \times S^3$ structure of the deformed conifold. The resolved conifold is obtained in a similar way by pulling off the conical structure of the $S^2$ factor.

2.3 Spin(7) manifolds

Here we shall present a picture for understanding the topology-changing geometric transition of the $\text{Spin}(7)$ manifold discussed in [5] and alluded to in Section 1. First we recall that a $\text{Spin}(7)$ manifold is a real eight-dimensional Riemannian manifold with holonomy group $\text{Spin}(7)$. As in the case of Calabi-Yau and $G_2$ manifolds, there are several such geometries
The example we shall be interested in may be described as a singular real cone over the seven-dimensional Aloff-Wallach (coset) space $SU(3)/U(1)$. It was argued in [5] that there are two ways of blowing up the singularity, replacing the singularity by either $\mathbb{C}P^2$ or $S^5$. The resulting smooth $Spin(7)$ manifolds have topologies

resolution: $Spin(7): \mathbb{R}^4 \times \mathbb{C}P^2$ (Calabi-Yau: $\mathbb{R}^4 \times \mathbb{C}P^1$) \hspace{1cm} (8)

and

defeomrion: $Spin(7): \mathbb{R}^3 \times S^5$ (Calabi-Yau: $\mathbb{R}^3 \times S^3$), \hspace{1cm} (9)

and are referred to as resolution and deformation, respectively, due to the similarity with the Calabi-Yau conifold discussed above (and indicated in [8] and [9]).

Our aim here is to re-address the transition between these two manifolds using toric geometry. As described in the following, the basic idea is to view the singular $Spin(7)$ manifold (the real cone on $SU(3)/U(1)$) as three intersecting Calabi-Yau conifolds associated to the triangular toric diagram [2]. To reach this picture, we first recall that a deformed $Spin(7)$ manifold is obtained by blowing up an $S^5$, while a resolved $Spin(7)$ manifold is obtained by blowing up a $\mathbb{C}P^2$. Deformed and resolved conifolds, on the other hand, are obtained by blowing up an $S^3$ or a $\mathbb{C}P^1$, respectively. Since an $S^5$ may be represented by three intersecting three-spheres, while $\mathbb{C}P^2$ may be represented by three intersecting two-spheres, we thus see that the deformed and resolved $Spin(7)$ manifolds correspond to three intersecting conifolds being deformed or resolved, respectively.

To recapitulate this, let us consider $\mathbb{C}^3$ parameterized by $(z_1, z_2, z_3)$. A five-sphere is obtained by imposing the constraint

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = r$$ \hspace{1cm} (10)

(with $r$ real and positive), while the additional identification

$$(z_1, z_2, z_3) \sim (e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3)$$ \hspace{1cm} (11)

(with $\theta$ real) will turn it into a $\mathbb{C}P^2$. In either case, $r$ measures the size. With both conditions imposed, we can obtain the three resolved Calabi-Yau conifolds

$$\mathbb{R}^4 \times \mathbb{C}P^1(z_k = 0), \quad k = 1, 2, 3$$ \hspace{1cm} (12)

embedded in $\mathbb{R}^4 \times \mathbb{C}^3$, simply by setting one of the coordinates equal to 0. With reference to the triangle [2], this means that the resolution of the $Spin(7)$ singularity reached by blowing
up a $\mathbb{CP}^2$ may be described by three intersecting resolved conifolds over the triangle \( \mathbb{CP}^2 \). Likewise, the deformation of the $Spin(7)$ singularity constructed by blowing up an $S^5$ may be realized as three intersecting deformed Calabi-Yau conifolds

$$\mathbb{R}^3 \times S^3(z_k = 0) , \quad k = 1, 2, 3$$

over the same triangle. This description is thus based on our toric representation of $S^5$ as a $T^3$ over a triangle. As already mentioned, this construction collapses to a $T^2$ over an edge for $z_k = 0$ where $k = 1, 2$ or $3$, and it is recalled that the resulting $T^2$ over a line segment corresponds to $S^3$.

Since the basic intersection of the deformed or resolved conifolds is governed by the constituent three- or two-spheres, one may describe the intersection of the conifolds by the intersection matrices associated to $S^5$ or $\mathbb{CP}^2$, respectively. They read

$$M_{\text{def}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad M_{\text{res}} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

We emphasize that the $Spin(7)$ transition in our picture is accompanied by Calabi-Yau conifold transitions. In the transition from the resolved $Spin(7)$ manifold \( S^5 \) to the deformed one \( \mathbb{CP}^2 \), for example, this indicates that the collapsing $\mathbb{CP}^2$ and its constituent two-spheres are replaced by $S^5$ and its constituent three-spheres. This is at the core of the dualities in the phase transition of the compactified superstrings to be discussed below.

One may attempt to illustrate the geometric transitions of $Spin(7)$ manifolds by generalizing \( S^5 \). The following proposal extends readily to higher dimensions (see below), and reads

\begin{equation}
\begin{array}{c}
\text{deformed} \\
\end{array} \quad \leftrightarrow \quad \begin{array}{c}
\text{resolved} \\
\end{array} \quad + \quad \begin{array}{c}
\text{diagram} \\
\end{array}
\end{equation}
The configuration to the left is a representation of the $S^5$ part of the deformed $Spin(7)$ manifold, in which the three thin lines represent the three $U(1)$ factors. The counting is less obvious to the right of the arrow, as the triangular part represents the $\mathbb{C}P^2$ part of the resolved $Spin(7)$ manifold. The inscribed three-vertex in thin line segments thus corresponds to two $U(1)$ factors only. The three-vertex to the far right represents the extra $\mathbb{R}$ in the resolved scenario (8).

The $Spin(7)$ transition may be viewed as taking place when passing a particular point while moving along a particular curve in the moduli space of $Spin(7)$ manifolds. The point corresponds to the singular $Spin(7)$ manifold, whereas the remaining points on the curve are associated to the deformed and resolved $Spin(7)$ manifolds. In one direction away from the singular point, the points correspond to the deformed manifolds with the size $r$ of the blown-up $S^5$ parameterizing that part of the curve. Likewise, the other direction away from the singular point is parameterized by the size $r$ of the blown-up $\mathbb{C}P^2$ in the resolved $Spin(7)$ manifold. The singular point is ‘shared’ as it is reached from either side when the relevant $r$ vanishes: $r = 0$.

In the interpretation of the $Spin(7)$ manifold as three intersecting Calabi-Yau manifolds over a triangle, we see that the $Spin(7)$ transition corresponds to all three Calabi-Yau manifolds undergoing simultaneous conifold transitions. We find it an interesting problem to understand the geometries associated to individual conifold transitions and hope to report on it elsewhere. Our graphic representation (15) (and (6)) does not seem to shed light on this as it is based on the transition of the full blow-ups, i.e., $S^5$ and $\mathbb{C}P^2$, and not on their constituent three- and two-spheres.

### 2.4 On possible extensions

It seems possible to extend our previous analysis of the $Spin(7)$ manifold in terms of intersecting Calabi-Yau manifolds to higher dimensions. To this end, let us consider the complex $(n+1)$-dimensional space $\mathbb{C}^{n+1}$ parameterized by $(z_1, ..., z_{n+1})$. A $(2n+1)$-dimensional sphere is obtained by imposing the constraint

$$\sum_{j=1}^{n+1} |z_j|^2 = r$$

(with $r$ real and positive), while the additional identification

$$(z_1, ..., z_{n+1}) \sim (e^{i\theta} z_1, ..., e^{i\theta} z_{n+1})$$
(with \( \theta \) real) will turn it into \( \mathbb{CP}^n \). In either case, \( r \) measures the size of the resulting space. Since \( S^{2n+1} \) can be described as a \( T^{n+1} \) over an \( n \)-simplex it supports a toric \( U(1)^{n+1} \) action whereas \( \mathbb{CP}^n \) (which may be realized as a \( T^n \) over an \( n \)-simplex) admits a toric \( U(1)^n \) action. The additional \( U(1) \) is the one used in the identification (17). As in the picture (17), we are thus expecting that a geometric transition can take place, replacing a \( U(1) \) by the one-dimensional real line \( \mathbb{R} \). Since the \( U(1) \) is associated to one of the \( S^1 \) factors of \( T^{n+1} \), the transition essentially amounts to replacing \( T^{n+1} \) by \( T^n \times \mathbb{R} \). Our interest is in real fibrations over the spaces \( S^{2n+1} \) and \( \mathbb{CP}^n \) so the relevant geometric transitions would read

\[
\text{(deformed)} \quad \mathbb{R}^m \times S^{2n+1} \quad \leftrightarrow \quad \mathbb{R}^{m+1} \times \mathbb{CP}^n \quad \text{(resolved)} . \tag{18}
\]

With \( m = 3 \), we expect to be able to describe the transition (18) in terms of Calabi-Yau conifolds. Using arguments similar to the \( \text{Spin}(7) \) example above, this generalized geometric transition should be related to \( \frac{1}{2} n(n+1) \) intersecting conifolds over the \( n \)-simplex, where the number of conifolds is equal to the number of one-dimensional edges of the simplex. One should also expect to be able to describe the transition in terms of \( \frac{1}{6} n(n^2 - 1) \) intersecting \( \text{Spin}(7) \) manifolds over the \( n \)-simplex, where the number of them is equal to the number of two-dimensional faces of the simplex. We hope to address this further elsewhere.

The extension of the transition picture (15) to higher dimensions is based on \( \mathbb{CP}^n \) and \( S^{2n+1} \) admitting descriptions as \( T^n \) and \( T^{n+1} \) fibrations, respectively, over an \( n \)-simplex. One chooses an extra point different from the \( n+1 \) vertices of the \( n \)-simplex in such a way that any subset of \( q < n+2 \) nodes out of the total of \( n+2 \) points gives rise to a \( (q-1) \)-simplex. A natural choice is the centre of the original (regular) \( n \)-simplex. To represent \( S^{2n+1} \), one then draws thin lines from the vertices through this extra point, where the thin lines represent the \( U(1) \) factors of \( U(1)^{n+1} \), cf. (15). The projective counterpart, \( \mathbb{CP}^n \), is represented by ending these thin lines at the common point, resulting in an \( (n+1) \)-vertex inscribed in the \( n \)-simplex. This inscribed vertex corresponds to \( U(1)^n \). Finally, the real line \( \mathbb{R} \) may be represented by a 'free' \( (n+1) \)-vertex, and the graphical representation of the transition (18) is a higher-dimensional version of (15).

The conifold analysis based on the toric variety \( \mathbb{CP}^2 \) could alternatively be extended by blowing up some generic points. With the number of points restricted as \( k = 1, 2, 3 \), this defines a so-called toric del Pezzo surface denoted \( dP_k \). The blowing up consists in replacing a point by \( \mathbb{CP}^1 \) with a line segment as its toric diagram. The full del Pezzo surface will thus have a polygon with \( k + 3 \) legs as its toric diagram.
3 Type II superstring compactifications

3.1 Compactification on conifold

Based on the Calabi-Yau conifold transition discussed above, Gopakumar and Vafa have argued that the $SU(N)$ Chern-Simons theory on $S^3$ for large $N$ is dual to topological strings on the resolved conifold. In this way, the ’t Hooft expansion of the Chern-Simons free energy has been shown to be in agreement, for all genera, with the topological string amplitudes on the resolved conifold. This duality has subsequently been embedded in type IIA superstring theory, where it was proposed that $N$ D6-branes wrapped around the three-sphere of the deformed conifold is equivalent (for large $N$) to type IIA superstrings on the resolved conifold with the D6-branes replaced by $N$ units of R-R two-form fluxes through the two-sphere ($S^2 \sim \mathbb{C}P^1$) in the resolved conifold. This duality thus offers a way of understanding the same physics at strong coupling.

The mirror version in type IIB superstring theory of this duality states that the scenario with $N$ D5-branes wrapped around the two-sphere in the resolved conifold, is equivalent (for large $N$) to three-form fluxes through the $S^3$ of the deformed conifold. This has been generalized to other Calabi-Yau threefolds where the blown-up geometries involve several $\mathbb{C}P^1$’s.

The large $N$ duality in type IIA superstring theory has also been lifted to M-theory where it is known to give a so-called flop duality. Unlike the duality in string theory, the phase transition here is smooth and does not correspond to a topology-changing geometric transition.

3.2 Compactification on $Spin(7)$ manifold and brane/flux duality

Based on the results on superstrings compactified on Calabi-Yau threefolds and our toric description of the geometric transition of the $Spin(7)$ manifolds, we now consider the two-dimensional gauge theories obtained by compactifying type II superstrings on these $Spin(7)$ manifolds. The idea is to study the consequences of adding $N$ wrapped D-branes to the setup before letting the manifold undergo the geometric transition. In the transition from the resolved to the deformed $Spin(7)$ manifold, we initially have D-branes wrapping $\mathbb{C}P^2$ (and its constituent two-spheres). We conjecture that they are replaced, under the transition, by R-R fluxes through $S^5$ (and its constituent three-spheres). Similarly in the transition from
deformed to resolved $\text{Spin}(7)$ manifolds, we conjecture that D-branes wrapped around $S^5$ (and its constituent three-spheres) are replaced by R-R fluxes through $\mathbb{C}P^2$ (and its constituent two-spheres). The kind of D-branes involved and the more detailed phase transition depend on which type II superstrings are propagating on the $\text{Spin}(7)$ manifolds. In the following we shall therefore consider type IIA and type IIB separately. We find that they lead to different brane/flux dualities.

**Duality in type IIB superstring theory.**

We start by considering type IIB superstrings on the resolved $\text{Spin}(7)$ manifold $\mathcal{S}$. Since the type IIB theory does not support four-forms, one considers D5-branes wrapped around $\mathbb{C}P^2$. A two-dimensional $U(N)$ gauge model can be obtained by wrapping $N$ D5-branes on $\mathbb{C}P^2$. The volume of $\mathbb{C}P^2$ described by $r$ is proportional to the inverse of the gauge coupling squared. This two-dimensional model has only one supercharge, so $\mathcal{N} = 1/2$. Now, when the manifold undergoes the geometric transition to the deformed $\text{Spin}(7)$ manifold $\mathcal{D}$, the $N$ D5-branes disappear and we expect a dual physics with $N$ units of R-R three-fluxes through the compact three-cycles, $S^3$, in the intersecting Calabi-Yau threefolds. These fluxes could be accompanied by some NS-NS fluxes through the non-compact dual three-cycles in the six-dimensional deformed conifolds. In order to handle the associated divergent integrals, one would have to introduce a cut-off to regulate the infinity.

**Duality in type IIA superstring theory.**

Here we start with type IIA superstrings on the deformed $\text{Spin}(7)$ manifold $\mathcal{D}$. In this case, a two-dimensional $U(N)$ gauge theory can be obtained by wrapping $N$ D6-branes around $S^5$. As above, this gauge model has only one supercharge, so again $\mathcal{N} = 1/2$. At the transition point, the D6-branes disappear and are replaced by R-R two-form fluxes through the two-spheres embedded in $\mathbb{C}P^2$ in the resolved $\text{Spin}(7)$ manifold $\mathcal{S}$.

One could wonder if there is an M-theory interpretation of this type IIA transition. Let us therefore consider a nine-dimensional manifold $X_9$ with a $U(1)$ isometry. M-theory compactified on $X_9$ is then equivalent to type IIA superstrings compactified on $X_9/U(1)$. We start with the resolved $\text{Spin}(7)$ manifold $\mathcal{S}$ and identify the extra eleventh compact dimension of M-theory with the $S^1$ that generates $\mathbb{R}^{11}$. In this way, the extra M-theory circle becomes the fiber in the definition of $S^5$ as an $S^1$ fibration over $\mathbb{C}P^2$. We thus end up with an $\mathbb{R}^4$ bundle over $S^5$ as the compactification space in M-theory. As a consequence, the moduli space of M-theory on such a background is parameterized by the the real parameter $r$ defining the volume of $S^5$, and cannot be complexified by the C-field. Starting with the resolved $\text{Spin}(7)$ manifold, on the other hand, the eleventh M-theory dimension is obtained by extending $\mathbb{R}^3$.
to $\mathbb{R}^4$ with the isometry being a trivial $U(1)$ action on the fiber $\mathbb{R}^4$. Using arguments similar to those in \cite{3}, we conjecture that this lift to M-theory gives rise to a (smooth) flop transition in the $\mathbb{R}^4$ bundle over $S^5$ where a five-sphere collapses and is replaced by a five-sphere. In our scenario, however, the physics resulting from the type IIA superstring compactification undergoes a singular phase transition.

4 Discussion

Based on toric geometry, we have studied geometric transitions of $Spin(7)$ manifolds. Our framework allowed us to discuss extensions to higher dimensions. It also made it possible to address straightforwardly type II superstring compactifications on $Spin(7)$ manifolds, from which some brane/flux dualities were extrapolated.

Our work opens up for further studies. One interesting problem is to understand better the geometries involved in our proposal for higher-dimensional geometric transitions. Another question is related to the toric description of the $Spin(7)$ manifolds as intersecting Calabi-Yau threefolds over a triangle where the $Spin(7)$ transition corresponds to three simultaneous conifold transitions. A natural question concerns the geometries associated to individual conifold transitions. Of potential importance to superstring and M-theory compactifications, one should then study what the physical implications of such transitions would be. It would also be interesting to understand the link between our results and the ones in \cite{16} based on string compactifications on Calabi-Yau fourfolds. One approach to this problem could be to consider the $Spin(7)$ manifold as a Calabi-Yau fourfold modulo an involution, thus ensuring the same number of supersymmetries. We hope to report elsewhere on these open problems.

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