The Differential Brauer Group

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Outline

1. Review of Brauer groups of fields, rings, and Δ-rings
2. Cohomology
3. Cohomological interpretation of Δ—Brauer groups (with connections to Hodge theory)
Brauer Groups of Fields

Finite dimensional division algebras $\Lambda$ over a field $K$ are classified by $Br(K)$, the Brauer group of $K$.

- If $\Lambda$ is a central, simple algebra over $K$, then it is isomorphic to $M_n(D)$ for some division algebra $D$.
- Given two such algebras $\Lambda$ and $\Gamma$, $\Lambda \otimes_K \Gamma$ is again a central simple $K$-algebra.
- They are said to be Brauer equivalent if there are vector spaces $V$, $W$ and a $K$–algebra isomorphism $\Lambda \otimes_K End(V) \cong \Gamma \otimes_K End(W)$. This is an equivalence relation, and $Br(K)$ is then defined to be the group formed from the equivalence classes with $\otimes$ as product. $Br(K)$ classifies division algebras over $K$.
- For any such algebra $\Lambda$ over $K$, there is a Galois extension $L/K$ such that $\Lambda \otimes_K L \cong End_L(V)$.
- Galois cohomology is then used to classify all such equivalence classes using an isomorphism $Br(K) \cong H^2 \left( G_{\overline{K}/K}, \overline{K}^* \right)$. 
Azumaya algebras over a commutative ring $R$

- A finitely generated central $R$ algebra $\Lambda$ is an Azumaya algebra algebra if $\Lambda \otimes_R L$ is a central simple algebra over $L$ for any homomorphism from $R$ to a field $L$.

- An Azumaya algebra $\Lambda$ is a central, finitely generated $R$ algebra which is a projective $\Lambda \otimes_R \Lambda^{op}$ algebra.

Two such Azumaya algebras $\Lambda$ and $\Gamma$ are Brauer equivalent if there are faithful, projective $R$ modules $P$, $Q$ and an $R$–algebra isomorphism $\Lambda \otimes_R \text{End} (P) \cong \Gamma \otimes_R \text{End} (Q)$.

If $R$ is a local ring, there is an etale extension $S/R$ such that $\Lambda \otimes_R S \cong \text{End}_S (P)$ for some projective $S$ module $P$.

Etale cohomology is then used to classify all such equivalence classes using an isomorphism $\partial : Br (R) \cong H^2 (R_{et}, \mathbb{G}_m)$.
Brauer Groups of $\Delta$-rings

Let $\Delta = \{\delta_1, ..., \delta_n\}$ be a set of $n$ commuting derivations on $R$, a ring containing $\mathbb{Q}$.

- A $\Delta$ Azumaya algebra over $R$ is an Azumaya algebra $\Lambda$ over $R$ equipped with derivations extending the action of $\Delta$ on $R$.
- Two such $\Delta$ Azumaya algebras $\Lambda$ and $\Gamma$ are $\Delta$ Brauer equivalent if there are faithful, projective $\Delta - R$ modules $P$, $Q$ and a $\Delta - R$ algebra isomorphism $\Lambda \otimes_R \text{End}(P) \cong \Gamma \otimes_R \text{End}(Q)$. This is an equivalence relation, and $Br_{\Delta}(R)$ is the resulting group on the set of equivalence classes with $\otimes_R$ as the product.
- If $R$ is local, there is an etale extension $S$ and a $\Delta - S$ isomorphism $\Lambda \otimes_R S \cong \text{End}_S(P)$ for some $\Delta - S$ projective module $P$. 
Cohomology

Let $C$ be a category with fibred products. A pretopology on $C$ consists of specifying for all $X \in \text{ob}(C)$, a set $\text{Cov}(X)$ whose members are collections $\{f_\alpha : U_\alpha \to X | \alpha \in A\} \in \text{Cov}(X)$ satisfying

1. If $f : X \to X$ is an isomorphism, $\{f\} \in \text{Cov}(X)$.
2. If $\{f_\alpha : U_\alpha \to X\} \in \text{Cov}(X)$ and $\{g_\alpha^i : V_i^\alpha \to U_\alpha\} \in \text{Cov}(U_i)$ for all $i$, then $\{f_\alpha g_\alpha^i : V_i^\alpha \to X\} \in \text{Cov}(X)$.
3. If $\{f_\alpha : U_\alpha \to X\} \in \text{Cov}(X)$ and $Y \to X \in \mathcal{C}$, then $\{f_\alpha \times_X Y : U_\alpha \times_X Y \to Y\} \in \text{Cov}(Y)$.

A presheaf $F : \mathcal{C}^{op} \to ((\text{Sets}))$ is a sheaf if for all $X \in \mathcal{C}$ and $\{U_\alpha \to X\} \in \text{Cov}(X)$,

$$F(X) \hookrightarrow \prod F(U_\alpha) \overset{\text{can}}{\Rightarrow} \prod F(U_\alpha \times_X U_\beta)$$

is exact.
Cohomology

Example

1. $X_{et}$ has \( \{ \{ f_\alpha : V_a \to U \mid f_\alpha \text{ is an etale map and } U = \bigcup f_\alpha(V_a) \} \} = Cov_{et}(U) \).

2. $X_{\Delta-fl}$ has

\[ \{ \{ g_\alpha : V_a \to U \mid g_\alpha \text{ is a flat } \Delta \text{ map of finite type and } U = \bigcup g_\alpha(V_a) \} \} = Cov_{\Delta-fl}(U) \).

If $G$ is a scheme, then its functor of points defines a sheaf in either of these topologies. Moreover there is a map of sites $\tau : X_{\Delta-fl} \to X_{et}$ since any etale map is a flat $\Delta$ map. Thus $\tau^{-1} (\{ f_\alpha \}) \in Cov_{\Delta-fl}(U)$. Moreover $H^\ast (X_{et}, G) \Rightarrow H^\ast (X_{\Delta-fl}, G)$ for sheaves $G$ defined by smooth, quasi-projective group schemes over $X$ like the sheaf of units, $G_m$. 
Cohomological Interpretation

In particular on $X_{\Delta-fl}$ we have the exact sequence

$$0 \rightarrow \mathbb{G}_m^\Delta \rightarrow \mathbb{G}_m \xrightarrow{d \ln} Z_X^1 \rightarrow 0$$

whose cohomology sequence contains

$$H^0 \left( X_{\Delta-fl}, Z_X^1 \right) \rightarrow H^1 \left( X_{\Delta-fl}, \mathbb{G}_m^\Delta \right) \rightarrow \text{Pic} \left( X \right) \xrightarrow{c_1} H^1 \left( X_{\Delta-fl}, Z_X^1 \right)$$

$$\rightarrow H^2 \left( X_{\Delta-fl}, \mathbb{G}_m^\Delta \right) \rightarrow \text{Br} \left( X \right) \rightarrow 0$$

if $X$ is smooth since then $H^2 \left( X_{et}, \mathbb{G}_m \right)$ is torsion unlike the vector space $H^2 \left( X_{\Delta-fl}, Z_X^1 \right)$!

How do we interpret this??
Cohomological Interpretation

Theorem

Let $X$ be a quasi-projective variety of finite type over a field $K$ of characteristic $0$. If $x \in H^2(X, \mu_N)$, then there is an Azumaya algebra $\Lambda$ equipped with an integrable connection constructed from $x$ such that

$$\partial([\Lambda]) = i_N(x) \in H^2(X_{et}, \mathbb{G}_m) \quad \text{where } i_N : \mu_N \to \mathbb{G}_m \text{ is inclusion.}$$

For simplicity, let’s consider the case where $X = \text{Spec}(R)$ is a local ring and $R$ contains a primitive $N^{th}$ root of unity. Then there is an etale extension $R \to S \in \text{Cov}(\text{Spec}(R))$, i.e. $U = \text{Spec}(S) \to \text{Spec}(R)$, and a Cech 2 cocycle $\zeta \in \mu_N(S^3)$ such that $[\zeta] = x \in H^2((R \to S), \mu_N)$. Now by refining $S$ we may assume that it is in the standard form $S = (R[T] / (p(T)))_{g(t)}$ where $p(T)$ is a monic polynomial of degree $D$. So we approximate $S$ by $R[t] := R[T] / (p(T)) = \bigoplus_1^D R$. 

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Cohomological Interpretation

Now our cocycle $\zeta \in \mu_N (S \otimes_R S \otimes_R S)$ is constant on each connected component of $S^{\otimes 3}$ but may vary from one component to another. So we must use an index set that accounts for this. We let

$$J = \{ \text{connected components of } S^3 \}$$

Of course $F = \left( \prod_{\alpha \in J} R[t]_\alpha \right) = \left( \bigoplus_{\alpha \in J} (\bigoplus D_R) \right)$ is not usually connected but for each connected component $\alpha$ of $S^{\otimes 3}$ there is an $R[t]_\alpha$ which admits multiplication by the value $\zeta_\alpha$ of $\zeta$ on that connected component and, as an $R$ module, $F$ is free of rank $M = D \cdot (\# (J))$. Then we define an $R$ module isomorphism

$$\ell_\zeta = \bigoplus_J \zeta_\alpha : F \otimes_R S \otimes_R S \to F \otimes_R S \otimes_R S$$

by multiplying the $\alpha^{th}$ factor in $F$ by $\zeta_\alpha$. Note that this amounts to a diagonal block matrix where the $\alpha^{th}$ block is $\zeta_\alpha I_D.$
Cohomological Interpretation

We let \( c ( \bigoplus J \zeta_\alpha ) : \text{End}_R ( \mathcal{F} ) \otimes_R S \otimes_R S \to \text{End}_R ( \mathcal{F} ) \otimes_R S \otimes_R S \) be the algebra isomorphism given by conjugation by \( \ell_\zeta \). Then we get descent data from the diagram

\[
\begin{array}{ccc}
\Lambda & \to & \text{End}_R ( \mathcal{F} ) \otimes_R S \\
 & & u_1 \\
\downarrow & & \downarrow c ( \bigoplus J \zeta_\alpha ) \\
\text{End}_R ( \mathcal{F} ) \otimes_R S \otimes_R S & \to & \text{End}_R ( \mathcal{F} ) \otimes_R S \otimes_R S
\end{array}
\]

where \( e_i \) means insert \( 1_S \) into the \( i^{th} \) copy of \( S \). Note that \( \text{End}_R ( \mathcal{F} ) \) is the algebra of \( M \times M \) matrices with \( \delta ( e_{ij} ) = 0 \) for all \( \delta \in \Delta \). Here \( c ( \bigoplus J \zeta_\alpha ) = c ( \ell_\zeta ) \) is the patching data used to define \( \Lambda \) and it preserves the action of \( \Delta \) since \( c ( \ell_\zeta ) \) is given by conjugation by an \( N^{th} \) root of unity on each block in \( \text{End}_R ( \mathcal{F} ) \).
Cohomological Interpretation

It satisfies the cocycle condition

\[ \text{End}(\mathcal{F}) \otimes_R S^\otimes 3 \]

\[ e_3(c(l_\zeta)) \]

\[ e_2(c(l_\zeta)) \]

This commutes because

\[ (e_2(c(\bigoplus J_\zeta_\alpha)))^{-1} (e_1(c(\bigoplus J_\zeta_\alpha))) (e_3(c(\bigoplus J_\zeta_\alpha))) \]

is conjugation on \( \text{End}(\mathcal{F}) \otimes_R S \otimes_R S \otimes_R S \) by \( 1 \otimes \zeta \) since \( \zeta \) is a 2 cocycle. But this is \( 1_{\text{End}(\mathcal{F}) \otimes S^\otimes 3} \) which is the cocycle condition for descent.
Thus the Cech cocycle provides the needed descent data and we immediately see that

$$\partial \left([c(J \zeta_\alpha)]\right) = [\zeta] \in \check{H}^2(X, \mu_n)$$

where $$\Lambda = [c(J \zeta_\alpha)]$$ is the desired $$\Delta$$ Azumaya algebra.