Fractional topological charge in lattice Abelian gauge theory

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We construct a non-trivial $U(1)/\mathbb{Z}_q$ principal bundle on $T^4$ from the compact $U(1)$ lattice gauge field by generalizing Lüscher’s constriction so that the cocycle condition contains $\mathbb{Z}_q$ elements (the ’t Hooft flux). The construction requires an admissibility condition on lattice gauge field configurations. From the transition function so constructed, we have the fractional topological charge that is $\mathbb{Z}_q$ one-form gauge invariant and odd under the lattice time reversal transformation. Assuming a rescaling of the vacuum angle $\theta \rightarrow q\theta$ suggested from the Witten effect, our construction provides a lattice implementation of the mixed ’t Hooft anomaly between the $\mathbb{Z}_q$ one-form symmetry and the time reversal symmetry in the $U(1)$ gauge theory with matter fields of charge $q \in 2\mathbb{Z}$ when $\theta = \pi$, which was studied by Honda and Tanizaki [J. High Energy Phys. \textbf{12}, 154 (2020)] in the continuum framework.
1 Introduction

As shown in a seminal paper [1], the generalized symmetries [2] can tell us quite non-trivial information on the low-energy dynamics of 4D gauge theories through the idea of anomaly matching [3]; see Refs. [4, 5] and references cited therein. In the present paper, aiming at transparent understanding of the above study in a fully regularized framework, we construct a non-trivial $U(1)/\mathbb{Z}_q$ principal bundle on $T^4$ in the compact $U(1)$ lattice gauge theory. In the study of the $SU(N)$ gauge theory in Ref. [1], the fractionality of the topological charge in the $SU(N)/\mathbb{Z}_N$ theory and the $\mathbb{Z}_N$ one-form symmetry are crucially important. The $\mathbb{Z}_N$ one-form gauge transformation can be interpreted as the action on the transition function in the $SU(N)/\mathbb{Z}_N$ principal bundle [6]. The cocycle condition of the $SU(N)/\mathbb{Z}_N$ principal bundle can contain $\mathbb{Z}_N$ elements in contrast to the $SU(N)$ bundle, and those $\mathbb{Z}_N$ elements (the 't Hooft flux [7]) are interpreted as (the gauge-invariant content of) the $\mathbb{Z}_N$ two-form gauge field transforming under the $\mathbb{Z}_N$ one-form gauge transformation. These materials are nicely summarized in Appendix A of Ref. [8]. From this interpretation of the $\mathbb{Z}_N$ one-form gauge transformation, for our motivation, it is natural to consider the $SU(N)/\mathbb{Z}_N$ principal bundle in lattice gauge theory.

It is well-known that the transition function of the $SU(N)$ principal bundle on $T^4$ can be constructed from the $SU(N)$ lattice gauge field by Lüscher’s method [9]; see also Refs. [10, 11]. In the present paper, we consider a simpler $U(1)$ gauge theory and generalize Lüscher’s method so that the cocycle condition contains $\mathbb{Z}_q$ elements; the $\mathbb{Z}_q$ elements are introduced in the transition function and as a loop in $U(1)/\mathbb{Z}_q$ and as the loop in $SU(N)/\mathbb{Z}_N$ considered in Refs. [7, 12]. Our transition function with the 't Hooft flux gives rise to the topological charge in the compact $U(1)$ lattice gauge theory that generally takes fractional values. Also, our topological charge is odd under the lattice time reversal transformation. These properties are quite analogous to the properties of the $SU(N)/\mathbb{Z}_N$ topological charge that are crucial in the study of Ref. [1]. In fact, if we assume a rescaling of the vacuum angle $\theta$ in the $U(1)$ gauge theory $\theta \rightarrow q\theta$ suggested from the Witten effect [13] (see Refs. [14, 15]), our construction provides a lattice implementation of the mixed 't Hooft anomaly between the $\mathbb{Z}_q$ one-form symmetry and the time reversal symmetry in the $U(1)$ gauge theory with matter fields of charge $q \in 2\mathbb{Z}$ when $\theta = \pi$. This anomaly, which was studied by Honda and Tanizaki in Ref. [14] in the continuum framework, may be regarded as a $U(1)$ analogue of the 't Hooft anomaly in the $SU(N)$ gauge theory studied in Ref. [1].

\(^1\)Throughout the present paper, we assume that $q$ is a positive integer.
We make brief comments on some other related works. Realization of the generalized symmetries on the (simplicial) lattice has already been studied in detail in Ref. [6]. The fractional topological charge as a function of the lattice gauge field is not considered in Ref. [6], however. In fact, in order to define a topological charge in lattice gauge theory, a certain restriction on allowed lattice gauge field configurations, such as admissibility [9, 16, 17], is inevitable. Also the fractional topological charge in lattice gauge theory has been studied over the years [18–22]. One possible method to obtain the fractional topological charge associated with the $SU(N)/\mathbb{Z}_N$ principal bundle is to consider the gauge field in a higher representation as blind for the $\mathbb{Z}_N$ part of the gauge transformation (such as the adjoint representation) and divide the integer topological charge in that higher representation by the corresponding Dynkin index ($2N$ for the adjoint representation). In this method, however, one cannot explicitly specify the underlying bundle structure such as the ’t Hooft flux or the $\mathbb{Z}_N$ two-form gauge field.

Presumably, the most analogous works to ours are Refs. [23, 24]. Although Refs. [23, 24] start with non-compact $U(1)$ link variables that are divided into disjoint topological sectors (and thus the admissibility is implicit), the expression of the lattice integer topological charge is quite similar to ours. In Refs. [23, 24], the Bianchi identity on the $U(1)$ gauge field is further relaxed to introduce a static monopole and, using the expression of the lattice topological charge, the Witten effect in lattice gauge theory is observed. In the present paper, on the other hand, we construct a fractional lattice topological charge in the compact $U(1)$ lattice gauge theory as a function of the ’t Hooft flux that is identified with the gauge-invariant content of the $\mathbb{Z}_q$ two-form gauge field. It is also natural in our construction to consider a static monopole by relaxing the Bianchi identity; this point is left for future study. Also, the generalization of our construction to the non-Abelian lattice gauge theory is an important issue that we want to return in the near future.

2 $U(1)/\mathbb{Z}_q$ principal bundle on $T^4$ in $U(1)$ lattice gauge theory

2.1 Transition function

We consider a 4D periodic torus $T^4$ of size $L$; the Lorentz index is denoted by $\mu, \nu, \ldots$, etc. and runs over 1, 2, 3, and 4:

$$ T^4 \equiv \{ x \in \mathbb{R}^4 \mid 0 \leq x_\mu < L \text{ for all } \mu \} . $$

That is, any two points $x$ and $y$ whose coordinates differ by an integer multiple of $L$ are identified, $x \sim y$. 
We then consider a 4D lattice Λ,

$$\Lambda \equiv \{ n \in \mathbb{Z}^4 \mid 0 \leq n_\mu < L \text{ for all } \mu \}, \quad (2.2)$$

by dividing $$T^4$$ into hypercubes $$c(n)$$ specified by the lattice points in Eq. (2.2):

$$c(n) \equiv \{ x \in \mathbb{R}^4 \mid 0 \leq (x_\mu - n_\mu) \leq 1 \text{ for all } \mu \}. \quad (2.3)$$

We assume a $$U(1)$$ lattice gauge field on Λ. The link variable

$$U(n, \mu) \in U(1) \quad (2.4)$$

is residing on the link connecting $$n$$ and $$n + \hat{\mu}$$, where $$\hat{\mu}$$ denotes a unit vector in the positive $$\mu$$ direction.

Following the idea of Ref. [9], we define the transition function of the $$U(1)/\mathbb{Z}_q$$ on $$T^4$$ by regarding each hypercube (2.3) as the coordinate patch for $$T^4$$. Thus, the transition function is defined in the intersection between two hypercubes, called the face:

$$f(n, \mu) \equiv \{ x \in c(n) \mid x_\mu = n_\mu \} = c(n - \hat{\mu}) \cap c(n). \quad (2.5)$$

This is a 3D cube. We then define the transition function at $$x \in f(n, \mu)$$ by

$$v_{n,\mu}(x) \equiv \omega_\mu(x)\tilde{v}_{n,\mu}(x) \quad \text{at } x \in f(n, \mu). \quad (2.6)$$

In this expression, the first factor $$\omega_\mu(x)$$ is given by

$$\omega_\mu(x) \equiv \begin{cases} \exp \left( \frac{\pi i}{q} \sum_{\nu \neq \mu} \frac{z_{\mu\nu}x_\nu}{L} \right) & \text{for } x_\mu = 0 \mod L, \\ 1 & \text{otherwise.} \end{cases} \quad (2.7)$$

Note that this is non-trivial only on the hyperplane $$x_\mu = 0 \mod L$$. The “twist angles” $$z_{\mu\nu}$$ are integers and anti-symmetric in indices, $$z_{\mu\nu} = -z_{\nu\mu}$$. Equation (2.7) is analogous to the loop in $$SU(N)/\mathbb{Z}_N$$ considered in Refs. [7, 12]. In fact, along two non-trivial intersecting one-cycles on $$T^4$$, the factor $$\omega_\mu(x)$$ defines a loop in $$U(1)/\mathbb{Z}_q$$. The winding of this loop gives rise to a fractional topological charge [7, 12]. The integers $$z_{\mu\nu}$$ are the ’t Hooft flux or (the gauge-invariant content of) the $$\mathbb{Z}_q$$ two-form gauge field; see below. $$z_{\mu\nu}$$ is defined only modulo $$q$$ and it labels elements of the cohomology group $$H^2(T^4, \mathbb{Z}_q)$$. In the present paper, these are fixed numbers and non-dynamical.

The other factor $$\tilde{v}_{n,\mu}(x)$$ in Eq. (2.6) is given by Lüscher’s construction of the principal bundle in lattice gauge theory. When the gauge group is $$U(1)$$, the construction becomes
very simple\footnote{In deriving this, we have adopted the following definition of the standard parallel transporter, which forms the basis of the construction in Ref.\cite{9} (here $x \equiv n + \sum_{\mu=1}^{4} z_{\mu} \hat{\mu}$):}

\[
\tilde{v}_{n,1}(x) = U(n - \hat{1}, 1) \\
\times \exp \left[ iy_{4} \hat{F}_{14}(n - \hat{1}) + iy_{3}y_{4} \hat{F}_{13}(n - \hat{1} + \hat{4}) + iy_{3}(1 - y_{4}) \hat{F}_{13}(n - \hat{1}) \right. \\
\left. + iy_{2}y_{3}y_{4} \hat{F}_{12}(n - \hat{1} + \hat{3} + \hat{4}) + iy_{2}y_{3}(1 - y_{4}) \hat{F}_{12}(n - \hat{1} + \hat{3}) \right. \\
\left. + iy_{2}(1 - y_{3})y_{4} \hat{F}_{12}(n - \hat{1} + \hat{4}) + iy_{2}(1 - y_{3})(1 - y_{4}) \hat{F}_{12}(n - \hat{1}) \right],
\]

\[
\tilde{v}_{n,2}(x) = U(n - \hat{2}, 2) \exp \left[ iy_{3} \hat{F}_{24}(n - \hat{2}) + iy_{3}y_{4} \hat{F}_{23}(n - \hat{2} + \hat{4}) + iy_{3}(1 - y_{4}) \hat{F}_{23}(n - \hat{2}) \right],
\]

\[
\tilde{v}_{n,3}(x) = U(n - \hat{3}, 3) \exp \left[ iy_{4} \hat{F}_{34}(n - \hat{3}) \right],
\]

\[
\tilde{v}_{n,4}(x) = U(n - \hat{4}, 4),
\]

(2.9)

where $y_{\mu} \equiv x_{\mu} - n_{\mu}$. The field strength in this expression is defined by

\[
\hat{F}_{\mu\nu}(n) \equiv \frac{1}{i q} \ln \left[ U(n, \mu)U(n + \hat{\mu}, \nu)U(n + \hat{\nu}, \mu)^{-1}U(n, \nu)^{-1} \right] \quad \quad \pi < \hat{F}_{\mu\nu}(n) \leq \pi.
\]

(2.10)

Here, the power $q$ is supplemented inside the logarithm so that the field strength is \textit{invariant} under the $\mathbb{Z}_{q}$ one-form gauge transformation defined below.\footnote{Since $q$ is a positive integer, Eq.\eqref{2.10} is equal to $\ln[U(n, \mu)^{q}U(n + \hat{\mu}, \nu)^{q}U(n + \hat{\nu}, \mu)^{-q}U(n, \nu)^{-q}] / (i q)$; i.e., this is the logarithm of the plaquette variable in the charge-$q$ representation.} We then require the following admissibility condition,

\[
\sup_{n,\mu,\nu} |\hat{F}_{\mu\nu}(n)| < \epsilon \quad \quad 0 < \epsilon < \frac{\pi}{3 q},
\]

(2.11)

for allowed lattice gauge configurations. By applying the argument in Ref.\cite{17} to Eq.\eqref{2.10} carefully, one finds that the condition $\epsilon < \pi / (3 q)$ ensures that the Bianchi identity for the field strength, i.e., the absence of the monopole current, $\sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_{\nu} \hat{F}_{\rho\sigma}(n) = 0$, holds\footnote{Here and in what follows, the forward difference is defined by $\Delta_{\mu} f(n) \equiv f(n + \hat{\mu}) - f(n)$.}

2.2 \textit{Cocycle condition}

Let us examine the cocycle condition associated with the transition function \eqref{2.6}. It is given by the product of the transition functions in the intersection of four hypercubes, $c(n)$,
\(c(n-\hat{\mu}), c(n-\hat{\nu}), \) and \(c(n-\hat{\mu}-\hat{\nu})\), i.e.,

\[ p(n, \mu, \nu) \equiv \{ x \in c(n) \mid x_\mu = n_\mu, x_\nu = n_\nu \} \quad (\mu \neq \nu). \tag{2.12} \]

This is a 2D plaquette. Note that \(p(n, \mu, \nu)\) is extending in directions complementary to \(\mu\) and \(\nu\) in the present convention \[9\]. When the gauge group is \(U(1)/\mathbb{Z}_q\), the cocycle can take a value in \(\mathbb{Z}_q\). Noting that the global coordinate \(x_\mu\) on \(T^4\) possesses the discontinuity at \(x_\mu = 0\), for \(x \in p(n, \mu, \nu)\), we find

\[ v_{n-\hat{\nu},\mu}(x)v_{n,\nu}(x)v_{n,\mu}(x)^{-1}v_{n-\hat{\mu},\nu}(x)^{-1} = \begin{cases} \exp \left( \frac{2\pi i}{q} z_{\mu\nu} \right) \in \mathbb{Z}_q & \text{for } x_\mu = x_\nu = 0 \mod L, \\ 1 & \text{otherwise}, \end{cases} \tag{2.13} \]

where we have used the fact that the transition function \[2.9\] satisfies the cocycle condition in the \(U(1)\) gauge theory \[9\], \(\tilde{v}_{n-\hat{\nu},\mu}(x)\tilde{v}_{n,\nu}(x)\tilde{v}_{n,\mu}(x)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(x)^{-1} = 1 \overset{5}{=} \). Thus the loop factor \(\omega_\mu(x)\) in Eq. \[2.7\] gives rise to the “\(\mathbb{Z}_q\) breaking” of the cocycle condition.

2.3 \(\mathbb{Z}_q\) one-form global and gauge transformations

Let us now consider how the transition function transforms under the \(\mathbb{Z}_q\) one-form transformation. First, we identify the \(\mathbb{Z}_q\) one-form global transformation with the center transformation on link variables crossing a 3D hypersurface, say \(n_\mu = 0\). For instance, under

\[ U(n, \mu) \rightarrow \exp \left( \frac{2\pi i}{q} z_\mu \right) U(n, \mu) \quad n_\mu = 0, \quad z_\mu \in \mathbb{Z} \text{ and } 0 \leq z_\mu < q, \tag{2.14} \]

and \(U(n, \mu) \rightarrow U(n, \mu)\) otherwise, the field strength \[2.10\] does not change, the transition functions \[2.9\] are transformed as

\[ \tilde{v}_{n,\mu}(x) \rightarrow \begin{cases} \exp \left( \frac{2\pi i}{q} z_\mu \right) \tilde{v}_{n,\mu}(x) & \text{for } x_\mu = 1, \\ \tilde{v}_{n,\mu}(x) & \text{otherwise}, \end{cases} \tag{2.15} \]

and the other transition functions \(\tilde{v}_{n,\nu \neq \mu}(x)\) do not change. Since these are constant multiplications on the transition functions along the hypersurface \(x_\mu = 1\), the one-form global transformation does not affect the cocycle condition, \(\tilde{v}_{n-\hat{\nu},\mu}(x)\tilde{v}_{n,\nu}(x)\tilde{v}_{n,\mu}(x)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(x)^{-1} = 1 \overset{5}{=} \). Moreover, one can confirm that the cocycle condition is not affected even under a

\[\overset{5}{\text{For the expression in terms of the field strength in Eq. \[2.9\] to fulfill this cocycle condition, one has to use the Bianchi identity \[25\].}}\]

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smooth deformation of the hypersurface. We thus conclude that the \( \mathbb{Z}_q \) one-form global transformation does not induce any further \( \mathbb{Z}_q \) breaking of the cocycle condition.

On the other hand, if we consider the \( \mathbb{Z}_q \) one-form \textit{local} or \textit{gauge} transformation defined by

\[
U(n, \mu) \to \exp \left[ \frac{2\pi i}{q} z_\mu(n) \right] U(n, \mu) \quad z_\mu(n) \in \mathbb{Z}, \ 0 \leq z_\mu(n) < q, \tag{2.16}
\]

then the transition functions (2.6) are transformed as

\[
v_{n,\mu}(x) \to \exp \left[ \frac{2\pi i}{q} z_\mu(n - \hat{\mu}) \right] v_{n,\mu}(x) \quad x \in f(n, \mu). \tag{2.17}
\]

The cocycle condition (2.13) is then modified accordingly to

\[
v_{n-\hat{\nu},\mu}(x)v_{n,\nu}(x)v_{n-\hat{\mu},\nu}(x)^{-1}v_{n-\hat{\nu},\mu}(x)^{-1} \equiv \exp \left[ \frac{2\pi i}{q} z_{\mu\nu}(n - \hat{\mu} - \hat{\nu}) \right], \tag{2.18}
\]

where the \( \mathbb{Z}_q \) two-form gauge field on the lattice is given by

\[
z_{\mu\nu}(n) = z_{\mu\nu} \delta_{n,\mu,L-1}\delta_{n,\nu,L-1} + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + q N_{\mu\nu}(n) \in \mathbb{Z}. \tag{2.19}
\]

Thus the \( \mathbb{Z}_q \) one-form gauge transformation (2.16) induces an extra factor of a “pure gauge” form in the cocycle condition. Here, we resolve the modulo \( q \) ambiguity of \( z_{\mu\nu}(n) \) by setting

\[
\begin{cases} 
0 \leq z_{\mu\nu}(n) < q & \text{for } \mu < \nu, \\
-1 \leq z_{\mu\nu}(n) \equiv -z_{\nu\mu}(n) & \text{for } \mu > \nu. 
\end{cases} \tag{2.20}
\]

The last lattice field \( N_{\mu\nu}(n) \in \mathbb{Z} \) in Eq. (2.19) is required to restrict the value of \( z_{\mu\nu}(n) \) (\( \mu < \nu \)) in the range (2.20); recall that we have taken \( 0 \leq z_\mu(n) < q \) in Eq. (2.16). Considering successive \( \mathbb{Z}_q \) one-form gauge transformations starting from Eq. (2.19), for a generic \( z_{\mu\nu}(n) \), we have

\[
z_{\mu\nu}(n) \to z_{\mu\nu}(n) + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + q N_{\mu\nu}(n). \tag{2.21}
\]

(The fields \( z_\mu(n) \) and \( N_{\mu\nu}(n) \) differ from those in Eq. (2.19).) Our original configuration of the \( \mathbb{Z}_q \) two-form gauge field in Eq. (2.13), \( z_{\mu\nu}(n) = z_{\mu\nu} \delta_{n,\mu,L-1}\delta_{n,\nu,L-1} \), is flat, i.e., \((1/2) \sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu z_{\rho\sigma}(n) = 0 \mod q\). This flatness of the \( \mathbb{Z}_q \) two-form gauge field is obviously preserved under the \( \mathbb{Z}_q \) one-form gauge transformation on the lattice (2.21) because \( N_{\mu\nu}(n) \) are integers and contribute to the flatness condition only by an integer multiple of \( q \). In Appendix A we give a note on the flatness in our present lattice formulation.

\[\text{Note that the field strength (2.10) and the admissibility (2.11) are invariant under this } \mathbb{Z}_q \text{ one-form gauge transformation.}\]
3 Fractional topological charge

Now, the topological charge in the continuum,

\[ Q = \frac{1}{32\pi^2} \int_{T^4} d^4x \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x) \]  

(3.1)

can be entirely expressed in terms of the transition function \( v_{n,\mu}(x) \) if one divides \( T^4 \) into the cells \( (2.3) \) and uses the relation between the gauge potentials in adjacent cells, \( c(n - \hat{\mu}) \) and \( c(n) \). At their overlap, \( x \in f(n, \mu) \) \( (2.5) \), the relation is

\[ A^{(n)}_\lambda(x) = A^{(n-\hat{\mu})}_\lambda(x) - iv_{n,\mu}(x)^{-1} \partial_\lambda v_{n,\mu}(x). \]  

(3.2)

One then finds \[Q\] \( (9,12) \)

\[ Q = -\frac{1}{8\pi^2} \sum_{n \in \Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \int_{p(n,\mu,\nu)} d^2x \left[ v_{n,\mu}(x) \partial_\rho v_{n,\mu}(x)^{-1} \right] \left[ v_{n-\hat{\mu},\nu}(x)^{-1} \partial_\sigma v_{n-\hat{\mu},\nu}(x) \right]. \]  

(3.3)

As noted in Ref. \[12\], this expression holds even if the cocycle condition is relaxed by \( \mathbb{Z}_q \) elements as Eq. \( (2.13) \). Note that this expression is manifestly invariant under the \( \mathbb{Z}_q \) one-form gauge transformation \( (2.17) \) because the extra factor in Eq. \( (2.17) \) is independent of the continuous coordinate \( x \).

We thus substitute our transition function \( (2.6) \) into Eq. \( (3.3) \). After some calculation using the Bianchi identity, we have

\[ Q = \frac{1}{8q^2} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} + \frac{1}{8\pi q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} \sum_{n \in \Lambda, n_\mu=0} \tilde{F}_{\rho\sigma}(n) \]

\[ + \frac{1}{32\pi^2} \sum_{n \in \Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \tilde{F}_{\mu\nu}(n) \tilde{F}_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}). \]  

(3.4)

On the right-hand side, the last term is the well-known expression of the topological charge in the \( U(1) \) lattice gauge theory \([17,25,28]\). It takes integer values for admissible gauge fields and thus is topological. The first term on the right-hand side gives a fractional topological charge associated with the ’t Hooft flux (the winding of a non-trivial cycle to \( U(1)/\mathbb{Z}_q \)) \([12]\).

The second term is a “cross term” and it sums the first Chern numbers on \( T^2 \) in the \( \rho\sigma \) direction over \( n_\nu \). Under the admissibility, the first Chern number is also quantized on the lattice (see Ref. \([25]\)) and

\[ \sum_{n \in \Lambda, n_\mu=0, n_\nu=\text{fixed}} \sum_{\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \tilde{F}_{\rho\sigma}(n) = 4\pi \mathbb{Z}. \]  

(3.5)

By construction, the lattice topological charge \( (3.4) \) is invariant under the \( \mathbb{Z}_q \) one-form gauge transformation in Eq. \( (2.16) \).
The lattice topological charge (3.4) also possesses a simple transformation property under the time reversal. We may define the time reversal transformation \( T \) on a lattice field by

\[
U(n, \mu) \xrightarrow{T} \begin{cases} 
U(\bar{n}, \mu) & \text{for } \mu \neq 4, \\
U(\bar{n} - \hat{4}, 4)^{-1} & \text{for } \mu = 4,
\end{cases}
\]

where \( \bar{n} \equiv (n_1, n_2, n_3, -n_4) \). Under this, the field strength (2.10) is transformed as

\[
\tilde{F}_{\mu\nu}(n) \xrightarrow{T} \begin{cases} 
\tilde{F}_{\mu\nu}(\bar{n}) & \text{for } \mu \neq 4, \nu \neq 4, \\
-\tilde{F}_{4\nu}(\bar{n} - \hat{4}) & \text{for } \mu = 4, \\
-\tilde{F}_{\mu 4}(\bar{n} - \hat{4}) & \text{for } \nu = 4.
\end{cases}
\]

Note that this transformation preserves the admissibility (2.11). Using these and the Bianchi identity, it can be seen that the topological charge (3.4) changes its sign under the time reversal transformation,

\[
Q \xrightarrow{T} -Q,
\]

if we do the time reversal transformation on the 't Hooft flux at the same time:

\[
z_{\mu\nu} \xrightarrow{T} \begin{cases} 
z_{\mu\nu} & \text{for } \mu \neq 4, \nu \neq 4, \\
-z_{4\nu} & \text{for } \mu = 4, \\
-z_{\mu4} & \text{for } \nu = 4.
\end{cases}
\]

This transformation may be generalized to the time reversal of the two-form gauge field as

\[
z_{\mu\nu}(n) \xrightarrow{T} \begin{cases} 
z_{\mu\nu}(\bar{n}) & \text{for } \mu \neq 4, \nu \neq 4, \\
-z_{4\nu}(\bar{n} + \hat{4}) & \text{for } \mu = 4, \\
-z_{\mu4}(\bar{n} + \hat{4}) & \text{for } \nu = 4.
\end{cases}
\]

so that this is consistent with Eq. (2.19).

Now the 't Hooft flux in the topological charge (3.4) is constant. However, we may rewrite this expression by using the local \( \mathbb{Z}_q \) two-form gauge field \( z_{\mu\nu}(n) \). Recalling Eq. (2.19), we have

\[
Q = \frac{1}{32\pi^2} \sum_{n \in \Lambda} \sum_{\mu, \nu, \rho, \sigma} \varepsilon_{\mu\nu\rho\sigma} \left[ F_{\mu\nu}(n) + \frac{2\pi}{q} z_{\mu\nu}(n) \right] \left[ F_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) + \frac{2\pi}{q} z_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) \right],
\]

where we have introduced another field strength on the lattice by

\[
F_{\mu\nu}(n) \equiv \tilde{F}_{\mu\nu}(n) - \frac{2\pi}{q} \left[ \Delta_{\mu} z_{\nu}(n) - \Delta_{\nu} z_{\mu}(n) + q N_{\mu\nu}(n) \right],
\]

where we have introduced another field strength on the lattice by

\[
F_{\mu\nu}(n) \equiv \tilde{F}_{\mu\nu}(n) - \frac{2\pi}{q} \left[ \Delta_{\mu} z_{\nu}(n) - \Delta_{\nu} z_{\mu}(n) + q N_{\mu\nu}(n) \right],
\]
using the functions appearing in Eq. (2.19). Under the \( Z_q \) one-form gauge transformation \((2.21)\), the new field strength is not invariant and transforms as
\[
F_{\mu\nu}(n) \to F_{\mu\nu}(n) - \frac{2\pi}{q} [\Delta_{\mu}z_{\nu}(n) - \Delta_{\nu}z_{\mu}(n) + qN_{\mu\nu}(n)].
\]
(3.13)
Compared with Eq. (3.4), the locality of the topological charge \( Q \) is manifest with the expression \((3.11)\).

4 ‘t Hooft anomaly

4.1 General setting

Our construction above may be employed to consider, with lattice regularization, the mixed ‘t Hooft anomaly between the \( Z_q \) one-form symmetry and the time reversal symmetry in the \( U(1) \) gauge theory with matter fields of charge \( q \in 2\mathbb{Z} \), when the vacuum angle \( \theta \) is \( \pi \) \cite{14}. We can assume a lattice action that is invariant under the \( Z_q \) one-form global transformation:
\[
S \equiv \frac{1}{4g_0^2} \sum_{n \in \Lambda} \sum_{\mu,\nu} \tilde{F}_{\mu\nu}(n)\tilde{F}_{\mu\nu}(n) + S_{\text{matter}} - iq\theta Q,
\]
(4.1)
where \( g_0 \) is the bare coupling. This original system does not contain the \( Z_q \) two-form gauge field. We also assume that the lattice action except the last topological term is even under the time reversal transformation; this is actually the case for the first pure gauge term in Eq. (4.1).

In the last topological term in Eq. (4.1), \( \theta \) is multiplied by \( q \) as \( iq\theta Q \) instead of \( i\theta Q \). This is because, with the conventional normalization of \( \theta \), the Witten effect \cite{13} suggests a \( 2\pi q \) periodicity of \( \theta \) instead of the \( 2\pi \) periodicity. This rescaling of the periodicity actually occurs at least in the Cardy–Rabinovici model \cite{29, 30} as studied in Ref. \cite{14} (see also Ref. \cite{15}). That is, the full spectrum of the system including the monopole and dyons is invariant only under a \( 2\pi q \) shift of the original vacuum angle, instead of a \( 2\pi \) shift; thus the periodicity of \( \theta \) is \( 2\pi \) only with the combination in Eq. (4.1). Since we do not introduce the monopole and dyons in the present lattice setup, we cannot observe the Witten effect and the associated rescaling of the vacuum angle directly. Nevertheless, it is interesting to see a possible ‘t Hooft anomaly in our lattice regularized setup, temporarily accepting the above rescaling of the vacuum angle. This is what we do here.

\footnote{Note that the integer field \( N_{\mu\nu}(n) \) in Eq. (2.19) is determined from \( z_{\mu}(n) \) locally.}

\footnote{To incorporate the admissibility and the smoothness of the lattice action, a more ingenious construction such as the one in Ref. \cite{26} would be desirable; this point is irrelevant in the present discussion concerning symmetries of the action.}
Now, let us set $\theta = \pi$ in Eq. (4.1). The original partition function is then time reversal invariant because $i\pi q Q \xrightarrow{T} -i\pi q Q \sim +i\pi q Q$ because of the postulated $2\pi$ periodicity of $\theta$ (the original $Q$ is an integer).

We then switch the $Z_q$ two-form gauge field on. From Eqs. (3.4) and (3.5), we have

$$q Q = \frac{1}{8q} \sum_{\mu, \nu, \rho, \sigma} \varepsilon_{\mu \nu \rho \sigma} z_{\mu \nu} z_{\rho \sigma} + Z.$$  \hfill (4.2)

The factor $e^{i\pi q Q}$ in the integrand of the functional integral then acquires an extra phase factor under the time reversal, as (recall Eq. (3.8))

$$e^{i\pi q Q} \xrightarrow{T} e^{-i\pi q Q} = e^{-2\pi i q Q} \cdot e^{i\pi q Q} = \exp \left( -\frac{2\pi i}{8q} \sum_{\mu, \nu, \rho, \sigma} \varepsilon_{\mu \nu \rho \sigma} z_{\mu \nu} z_{\rho \sigma} \right) e^{i\pi q Q}. \hfill (4.3)$$

### 4.2 Construction of the local counterterm

One should then ask [1] whether a certain local gauge-invariant term of the $Z_q$ two-form gauge field $z_{\mu \nu}(n)$ can counter the above breaking of the time reversal symmetry. Using the technique reviewed in Appendix B, it can be seen that a local term that transforms "covariantly" under the lattice $Z_q$ one-form gauge transformation (2.21) is given by

$$\exp \left[ \frac{2\pi i k}{4q} \sum_{n \in \Lambda, \mu, \nu, \rho, \sigma} \varepsilon_{\mu \nu \rho \sigma} z_{\mu \nu}(n) z_{\rho \sigma}(n + \hat{\mu} + \hat{\nu}) \right]. \hfill (4.4)$$

It is easy to see that, under the $Z_q$ one-form gauge transformation (2.21), the combination $\sum_{n \in \Lambda} \sum_{\mu, \nu, \rho, \sigma} \varepsilon_{\mu \nu \rho \sigma} z_{\mu \nu}(n) z_{\rho \sigma}(n + \hat{\mu} + \hat{\nu})$ shifts by $4qZ$; see Appendix B. Therefore, for the phase factor (4.4) to be gauge invariant, the constant $k$ must be an integer, $k \in \mathbb{Z}$.

However, Eq. (4.4), which would be regarded as the wedge product of the elements of $H^2(T^4, \mathbb{Z}_q)$ on the hypercubic lattice, is not the counterterm with the "finest" coefficient, as known for the corresponding counterterm on the simplicial lattice [3, 31]. The counterterm with a "finer" coefficient on $T^4$ can be constructed by employing the integral lift of $H^2(T^4, \mathbb{Z}_q)$ to $H^2(T^4, \mathbb{Z})$ [3, 31]. On our periodic hypercubic lattice $\Lambda$, we may construct an analogue of the integral lift, $\bar{z}_{\mu \nu}(n)$, which satisfies the flatness, $(1/2) \sum_{\nu, \rho, \sigma} \varepsilon_{\mu \nu \rho \sigma} \Delta_{\nu} \bar{z}_{\rho \sigma}(n) = 0$ (strictly zero not modulo $q$), from $z_{\mu \nu}(n)$ as follows.

First, we define an integer $m_\mu(n) \equiv (1/2) \sum_{\nu, \rho, \sigma} \varepsilon_{\mu \nu \rho \sigma} \Delta_{\nu} z_{\rho \sigma}(n)/q$ for each 3D cube, which spans from the site $n$ to directions complementary to $\mu$. $m_\mu(n)$ is an integer, because of the modulo $q$ flatness of $z_{\mu \nu}(n)$, $(1/2) \sum_{\nu, \rho, \sigma} \varepsilon_{\mu \nu \rho \sigma} \Delta_{\nu} \bar{z}_{\rho \sigma}(n) \in q\mathbb{Z}$.

---

9 We owe the following discussion to Yuya Tanizaki.
Next, we take a 3D space specified by a fixed \( n_\mu \) and consider paths connecting centers of cubes in the 3D space. The paths are defined such that \( |m_\mu(n)| \) paths begin from the cube at \( n \) if \( m_\mu(n) > 0 \), while \( |m_\mu(n)| \) paths end at the cube if \( m_\mu(n) < 0 \). A certain consistent configuration of paths can be defined in this way, because the total sum of \( m_\mu(n) \) on the three-dimensional space identically vanishes, \( \sum_{n \in \Lambda, n_\mu} m_\mu(n) = \sum_{n \in \Lambda, n_\mu} 1/2 \sum_{\nu,\rho,\sigma} \varepsilon_{\nu\rho\sigma} \Delta_\nu z_{\rho\sigma}(n)/q = 0 \). Then, it is obvious that \( \bar{z}_{\mu\nu}(n) \) such that \( \sum_{\nu,\rho,\sigma} \varepsilon_{\mu\rho\sigma} \Delta_\nu \bar{z}_{\rho\sigma}(n) = 0 \) can be obtained by subtracting \( qm_{\mu\nu}(n) \) from \( z_{\mu\nu}(n) \). Here, \( m_{\mu\nu}(n) \) is the signed number of paths going through the plaquette at which \( z_{\mu\nu}(n) \) is residing; the sign of \( m_{\mu\nu}(n) \) is defined to be positive if the paths go through the plaquette in the direction of the standard orientation of the plaquette and negative if they go through in the opposite direction. This construction gives the relation \( \bar{z}_{\mu\nu}(n) = z_{\mu\nu}(n) - qm_{\mu\nu}(n) \), where \( m_{\nu\mu}(n) = -m_{\mu\nu}(n) \).

The integral lift \( \bar{z}_{\mu\nu}(n) \) can differ depending on the choice of the configuration of paths but the difference can be expressed in the difference of the field \( m_{\mu\nu}(n) \). Since \( \bar{z}_{\mu\nu}(n) = z_{\mu\nu}(n) - qm_{\mu\nu}(n) \) has the form of the gauge transformation \( (2.21) \), the choice of the configuration of paths does not matter as far as gauge-invariant quantities (such as the counterterm below) are concerned. Also, it is obvious from Eq. \( (2.21) \) that the gauge transformation of \( \bar{z}_{\mu\nu}(n) \) takes the form \( \bar{z}_{\mu\nu}(n) \to \bar{z}_{\mu\nu}(n) + \Delta_\mu z_\nu(n) - \Delta_\nu z_{\mu}(n) + q\bar{N}_{\mu\nu}(n) \), where \( \bar{N}_{\mu\nu}(n) \in \mathbb{Z} \). Because of the flatness of \( \bar{z}_{\mu\nu}(n) \), we thus infer that \( \bar{N}_{\mu\nu}(n) \) is also flat, \( \sum_{\nu,\rho,\sigma} \varepsilon_{\mu\rho\sigma} \Delta_\nu \bar{N}_{\rho\sigma}(n) = 0 \).

Finally, when \( \sum_{\nu,\rho,\sigma} \varepsilon_{\mu\rho\sigma} \Delta_\nu \bar{z}_{\rho\sigma}(n) = \sum_{\nu,\rho,\sigma} \varepsilon_{\mu\rho\sigma} \Delta_\nu \bar{N}_{\rho\sigma}(n) = 0 \), which can be expressed as \( d\bar{z}^{(2)} = d\bar{N} = 0 \) in the notation of Appendix \([13]\) by employing the argument in Ref. \([25]\), one can see that the combination \( \sum_{n \in \Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\rho\sigma} \bar{z}_{\mu\nu}(n) \bar{z}_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) \) shifts by \( 8q\mathbb{Z} \) (not \( 4q\mathbb{Z} \)) under the gauge transformation.

Therefore, the counterterm with a finer coefficient is given by

\[
e^{-S_{\text{counter}}} = \exp \left[ \frac{2\pi ik}{8q} \sum_{n \in \Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\rho\sigma} \bar{z}_{\mu\nu}(n) \bar{z}_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) \right] \tag{4.5}
\]

and this is gauge-invariant for \( k \in \mathbb{Z} \) (note the difference in coefficients in Eqs. \( (4.4) \) and \( (4.5) \)). We expect that this is the finest gauge invariant coefficient from the corresponding result in the continuum theory \([14]\).

Since our representative configuration in Eq. \( (2.13) \), \( z_{\mu\nu}(n) = z_{\mu\delta_{n_\mu,L-1}n_\nu,L-1} \), is flat, \( \sum_{\nu,\rho,\sigma} \varepsilon_{\mu\rho\sigma} \Delta_\nu z_{\rho\sigma}(n) = 0 \), the corresponding integral lift is also given by this, \( \bar{z}_{\mu\nu}(n) = z_{\mu\nu} \delta_{n_\mu,L-1}n_\nu,L-1 \). Substituting this into Eq. \( (4.5) \) yields

\[
e^{-S_{\text{counter}}} = \exp \left( \frac{2\pi ik}{8q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} \right) \tag{4.6}
\]
Therefore, after the addition of the counterterm in Eq. (4.5), Eq. (4.3) is modified to

\[
e^{i\pi q} e^{-S_{\text{counter}}} T e^{-i\pi q} e^{S_{\text{counter}}} = e^{-2i\pi q} e^{2S_{\text{counter}}} e^{i\pi q} e^{-S_{\text{counter}}}
\]

\[
= \exp \left[ - \frac{2\pi i (2k + 1)}{8q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} \right] e^{i\pi q} e^{-S_{\text{counter}}}.
\]

(4.7)

Since the possible minimal non-zero value of \(|\sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}|\) is 8 (for instance, the choice, \(z_{12} = z_{34} = 1\) and other components vanish, gives this), this shows that if we can choose the integer \(k\) such that \(2k + 1 = 0 \mod q\), the anomaly is countered. This is impossible for even \(q\) and possible for odd \(q\). This thus implies the mixed 't Hooft anomaly between the \(\mathbb{Z}_q\) one-form symmetry and the time reversal symmetry for \(q \in 2\mathbb{Z}\) when \(\theta = \pi\) [14].

5 Conclusion

In this paper, assuming an appropriate admissibility condition on allowed lattice field configurations, we constructed the transition function of the \(U(1)/\mathbb{Z}_q\) principal bundle on \(T^4\) from the compact \(U(1)\) lattice gauge field by combining Lüscher’s method and a loop factor in \(U(1)/\mathbb{Z}_q\). The resulting topological charge takes fractional values and is invariant under the \(\mathbb{Z}_q\) one-form gauge transformation. Also, the topological charge is odd under the lattice time reversal transformation. From these properties, assuming a rescaling of the vacuum angle \(\theta \rightarrow q\theta\) suggested by the Witten effect, our construction provides a lattice implementation of the mixed 't Hooft anomaly between the \(\mathbb{Z}_q\) one-form symmetry and the time reversal symmetry in the \(U(1)\) gauge theory with matter fields of charge \(q \in 2\mathbb{Z}\) when \(\theta = \pi\) [14].

This may be regarded as a \(U(1)\) analogue of the mixed 't Hooft anomaly between the \(\mathbb{Z}_N\) one-form symmetry and the time reversal symmetry in the \(SU(N)\) gauge theory with \(N \in 2\mathbb{Z}\) with \(\theta = \pi\) [1]. For odd \(q > 1\), which requires \(k \neq 0\) in Eq. (4.5) for the anomaly cancellation at \(\theta = \pi\), we may consider a global inconsistency between different values of \(\theta\), imitating the discussions in Refs. [1, 27].

Although our construction of the transition function and the fractional topological charge is perfectly legitimate, our discussion on the mixed 't Hooft anomaly is still incomplete because we have simply assumed the rescaling of the vacuum angle without introducing the monopole and dyons. To observe the Witten effect, these degrees of freedom should be incorporated into our treatment. For this, we have to relax the Bianchi identity and it appears that the works [23, 24] are quite suggestive in this aspect.

Generalization of our construction to non-Abelian lattice gauge theory is an important issue that we want to return to in the near future.
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A Flatness of the \( Z_q \) two-form gauge field

The flatness of the \( Z_q \) two-form gauge field follows from the consistency of transition functions among “quadruple” overlap [6, 8] and, for our square lattice, it may be seen in the following way.\(^{10}\)

We take a point \( x \) on the link connecting \( n \) and \( n + \hat{4} \) and consider transitions among the following eight hypercubes, which share the above link:

\[
\begin{align*}
&c(n) \quad c(n - \hat{1}) \quad c(n - \hat{2}) \quad c(n - \hat{3}) \\
&c(n - \hat{1} - \hat{2}) \quad c(n - \hat{1} - \hat{3}) \quad c(n - \hat{2} - \hat{3}) \quad c(n - \hat{1} - \hat{2} - \hat{3}).
\end{align*}
\]

We start from the combination

\[
v_{n-\hat{3},2}(x)v_{n,3}(x)v_{n,2}(x)^{-1}v_{n-\hat{2},3}(x)^{-1} \times v_{n-\hat{1}-\hat{2},3}(x)v_{n-\hat{1},2}(x)v_{n-\hat{1},3}(x)^{-1} v_{n-\hat{1}-\hat{3},2}(x)^{-1} \\
= \exp\left[ \frac{2\pi i}{q} \Delta_{123}(n - \hat{1} - \hat{2} - \hat{3}) \right],
\]

where we have used the relation (2.18). Since the right-hand side of Eq. (2.18) is an element of \( Z_q \), the left-hand side is invariant under any similarity transformation. Using this fact, we can rewrite Eq. (A2) as

\[
v_{n-\hat{1},3}(x)^{-1}v_{n-\hat{1}-\hat{3},2}(x)^{-1} \\
\times v_{n-\hat{2},3,1}(x) \\
\times v_{n-\hat{3},2}(x)v_{n,3}(x) [v_{n,1}(x)^{-1}v_{n,1}(x)] v_{n,2}(x)^{-1}v_{n-\hat{2},3}(x)^{-1} \\
\times v_{n-\hat{2},3,1}(x)^{-1} \\
\times v_{n-\hat{1}-\hat{2},3}(x)v_{n-\hat{1},2}(x)v_{n-\hat{1},3}(x)^{-1}v_{n-\hat{1}-\hat{3},2}(x)^{-1} \\
\times [v_{n-\hat{1},3}(x)^{-1}v_{n-\hat{1}-\hat{3},2}(x)^{-1}]^{-1}
\]

\(^{10}\)The argument in this appendix holds even in non-Abelian lattice gauge theory.
In a similar way, we find

\[
\begin{align*}
= v_{n-1,3}(x)^{-1}v_{n-1-3,2}(x)^{-1}v_{n-2-3,1}(x)v_{n-3,2}(x)v_{n,3}(x)v_{n,1}(x)^{-1} \\
\times v_{n,1}(x)v_{n,2}(x)^{-1}v_{n-2,3}(x)^{-1}v_{n-2-3,1}(x)^{-1}v_{n-1-2,3}(x)v_{n-1,2}(x).
\end{align*}
\]

(A3)

The factor on the right-hand side can be written as

\[
\begin{align*}
v_{n-1,3}(x)^{-1}v_{n-1-3,2}(x)^{-1}v_{n-2-3,1}(x)v_{n-3,2}(x)v_{n,3}(x)v_{n,1}(x)^{-1} \\
&= v_{n-1,3}(x)^{-1}v_{n-1-3,2}(x)^{-1}v_{n-2-3,1}(x)v_{n-3,2}(x) \\
&\quad \times v_{n-3,1}(x)^{-1}v_{n-1,3}(x)[v_{n-3,1}(x)^{-1}v_{n-1,3}(x)]^{-1}v_{n,3}(x)v_{n,1}(x)^{-1} \\
&= v_{n-1,3}(x)^{-1}v_{n-1-3,2}(x)^{-1}v_{n-2-3,1}(x)v_{n-3,2}(x)v_{n-3,1}(x)^{-1}v_{n-1,3}(x) \\
&\quad \times v_{n-1,3}(x)^{-1}v_{n-1-3,1}(x)v_{n,3}(x)v_{n,1}(x)^{-1} \\
&= \exp \left\{ \frac{2\pi i}{q} [-z_{31}(n - \hat{1} - \hat{3}) + z_{12}(n - \hat{1} - \hat{2} - \hat{3})] \right\}.
\end{align*}
\]

(A4)

In a similar way, we find

\[
\begin{align*}
v_{n,1}(x)v_{n,2}(x)^{-1}v_{n-2,3}(x)^{-1}v_{n-2-3,1}(x)^{-1}v_{n-1-2,3}(x)v_{n-1,2}(x). \\
&= \exp \left\{ \frac{2\pi i}{q} [z_{31}(n - \hat{1} - \hat{2} - \hat{3}) - z_{12}(n - \hat{1} - \hat{2})] \right\}.
\end{align*}
\]

(A5)

Therefore, from Eqs. (A2), (A3), (A4), and (A5), we have the flatness (setting \( n - \hat{1} - \hat{2} - \hat{3} \to n \))

\[
\Delta_1 z_{23}(n) + \Delta_2 z_{31}(n) + \Delta_3 z_{12}(n) = 0 \mod q.
\]

(A6)

This shows in general

\[
\frac{1}{2} \sum_{\nu,\rho,\sigma} \varepsilon_{\nu\rho\sigma} \Delta_\nu z_{\rho\sigma}(n) = 0 \mod q.
\]

(A7)

B Use of the non-commutative differential calculus in lattice Abelian gauge theory [28]

In this appendix, we explain that the particular shift in the lattice coordinate appearing in Eqs. (3.11) and (4.4) (and also (4.5)) is naturally understood from the non-commutative differential calculus [32] in lattice Abelian gauge theory on the hypercubic lattice [28].
We define a $k$-form $f(n)$ on the lattice $\Lambda$ by

$$ f(n) \equiv \frac{1}{k!} \sum_{\mu_1, \ldots, \mu_k} f_{\mu_1 \cdots \mu_k}(n) dx_{\mu_1} \cdots dx_{\mu_k}, \quad (B1) $$

where $dx_{\mu} dx_{\nu} = -dx_{\nu} dx_{\mu}$. The exterior derivative on the lattice is defined by the forward difference,

$$ df(n) \equiv \frac{1}{k!} \sum_{\mu, \mu_1, \ldots, \mu_k} \Delta_{\mu} f_{\mu_1 \cdots \mu_k}(n) dx_{\mu} dx_{\mu_1} \cdots dx_{\mu_k}. \quad (B2) $$

This is nilpotent, $d^2 = 0$.

The essence of the non-commutative differential calculus is the rule,

$$ dx_{\mu} f_{\mu_1 \cdots \mu_k}(n) = f_{\mu_1 \cdots \mu_k}(n + \hat{\mu}) dx_{\mu}. \quad (B3) $$

That is, the differential form and a function on the lattice do not simply commute and the exchange accompanies a shift of the coordinate. If one accepts this formal rule, one finds that the Leibniz rule of the exterior derivative,

$$ d[f(n)g(n)] = df(n) \cdot g(n) + (-1)^k f(n) dg(n), \quad (B4) $$

holds even with the lattice difference $(B2)$.

With the above understanding, for a two-form

$$ f(n) = \frac{1}{2} \sum_{\mu, \nu} f_{\mu\nu}(n) dx_{\mu} dx_{\nu}, \quad (B5) $$

the wedge product yields

$$ f(n) f(n) = \frac{1}{4} \sum_{\mu, \nu, \rho, \sigma} f_{\mu\nu}(n) f_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) dx_{\mu} dx_{\nu} dx_{\rho} dx_{\sigma} = \frac{1}{4} \sum_{\mu, \nu, \rho, \sigma} \varepsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(n) f_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) dx_1 dx_2 dx_3 dx_4. \quad (B6) $$

After removing the volume form $dx_1 dx_2 dx_3 dx_4$ from this, we find the structure in Eqs. $(3.11)$ and $(4.4)$. This shows that the structure appearing in Eqs. $(3.11)$ and $(4.4)$ is rather natural in lattice Abelian gauge theory.

Now, in terms of the differential forms,

$$ z^{(2)}(n) \equiv \frac{1}{2} \sum_{\mu, \nu} z_{\mu\nu}(n) dx_{\mu} dx_{\nu}, \quad z^{(1)}(n) \equiv \sum_{\mu} z_{\mu}(n) dx_{\mu}, \quad N(n) \equiv \frac{1}{2} \sum_{\mu, \nu} N_{\mu\nu}(n) dx_{\mu} dx_{\nu}, \quad (B7) $$

16
the $\mathbb{Z}_q$ one-form gauge transformation (2.21) is written as

$$z^{(2)}(n) \rightarrow z^{(2)}(n) + dz^{(1)}(n) + qN(n)$$  \hspace{1cm} (B8)

and, using the Leibniz rule (B4) and the nilpotency $d^2 = 0$,

$$\sum_{n \in \Lambda} z^{(2)}_n z^{(2)}_n \rightarrow \sum_{n \in \Lambda} z^{(2)}_n z^{(2)}_n + \sum_{n \in \Lambda} \left(-dz^{(2)}_n z^{(1)}_n + z^{(1)}_n dz^{(2)}_n\right) + q \sum_{n \in \Lambda} \left(z^{(2)}_n N + Nz^{(2)}_n + qNN + dz^{(1)}_n N + Ndz^{(1)}_n\right) + \sum_{n \in \Lambda} d \left(z^{(2)}_n z^{(1)}_n + z^{(1)}_n z^{(2)}_n + z^{(1)}_n dz^{(1)}_n\right).$$  \hspace{1cm} (B9)

In this expression, we can discard the last “surface term” because the fields $z_{\mu\nu}(n)$, $z_\mu(n)$, and $N_{\mu\nu}(n)$ are single-valued on the lattice.\[11\] In terms of the components, we thus have

$$\sum_{n \in \Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu}(n) z_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) \rightarrow \sum_{n \in \Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu}(n) z_{\rho\sigma}(n + \hat{\mu} + \hat{\nu})$$

$$+ \sum_{n \in \Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} [-2\Delta_{\mu} z_{\nu\rho}(n) z_{\sigma}(n + \hat{\mu} + \hat{\nu} + \hat{\rho}) + 2z_\mu(n) \Delta_\nu z_{\rho\sigma}(n + \hat{\mu})]$$

$$+ q \sum_{n \in \Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \left(z_{\mu\nu}(n) N_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) + N_{\mu\nu}(n) z_{\rho\sigma}(n + \hat{\mu} + \hat{\nu})

+ qN_{\mu\nu}(n) N_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) + 2\Delta_{\mu} z_\nu(n) N_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) + 2N_{\mu\nu}(n) \Delta_{\rho} z_\sigma(n + \hat{\mu} + \hat{\nu}) \right\}$$  \hspace{1cm} (B10)

under the $\mathbb{Z}_q$ one-form gauge transformation. Since $(1/2) \sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_{\nu} z_{\rho\sigma}(n) = 0 \bmod q$ (the flatness), from this expression, it is obvious that the shift of $\sum_{n \in \Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu}(n) z_{\rho\sigma}(n + \hat{\mu} + \hat{\nu})$ under the gauge transformation is $4q\mathbb{Z}$.\[12\]

References

\[11\] Note that the conditions in Eq. (2.20) and $0 \leq z_\mu(n) < q$ uniquely determine these fields on the lattice. \[12\] We confirmed that the shift actually can take a value in $4q\mathbb{Z}$ by a numerical experiment. When $dz^{(2)} = dN = 0$ (strictly zero not modulo $q$), we can show that the shift is $8q\mathbb{Z}$ instead of $4q\mathbb{Z}$ by employing the argument in Ref. [25].
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