UNIFORM DIOPHANTINE APPROXIMATION RELATED TO BETA-TRANSFORMATIONS

WANLOU WU

Abstract. For any $\beta > 1$, let $T_\beta$ be the classical $\beta$-transformations. Fix $x_0 \in [0, 1]$ and a nonnegative real number $\hat{v}$, we compute the Hausdorff dimension of the set of real numbers $x \in [0, 1]$ with the property that, for every sufficiently large integer $N$, there is an integer $n$ with $1 \leq n \leq N$ such that the distance between $T_\beta^n x$ and $x_0$ is at most equal to $\beta^{-N\hat{v}}$. This work extends the result of Bugeaud and Liao [4] to every point $x_0$ in unit interval.

1. Introduction and results

Diophantine approximation has been widely studied by mathematicians. In 1842, Dirichlet [5] proved an illustrious theorem as follows.

Dirichlet Theorem For any two real numbers $\theta$, $Q$ with $Q \geq 1$, there is an integer $n \in [1, Q]$ such that

$$\|n\theta\| < Q^{-1},$$

where $\|\xi\|$ denotes the distance from $\xi$ to the nearest integer.

Dirichlet Theorem is called a uniform approximation theorem in [18, p.p. 2]. A weak form of Dirichlet Theorem, called an asymptotic approximation theorem in [18, p.p. 2], which was often referred to as a corollary of Dirichlet Theorem in the literature has already existed in the book of Legendre [11, 1808, p.p. 18-19]: for any real number $\theta$, there are infinitely many $n \in \mathbb{N}$ such that

$$\|n\theta\| < n^{-1}.$$

For the general case, Khintchine in 1924 [10] showed that for a positive function $\psi : \mathbb{N} \to \mathbb{R}^+$, if $x \mapsto x\psi(x)$ is non-increasing, then the set

$$\mathcal{L}_\psi := \{\theta \in \mathbb{R} : \|n\theta\| < \psi(n), \text{ for infinitely many } n \in \mathbb{N}\}$$

has Lebesgue measure zero if the series $\sum \psi(n)$ converges and has full Lebesgue measure otherwise. In the case where the set has Lebesgue measure zero, it is natural to calculate the Hausdorff dimension of $\mathcal{L}_\psi$. The first result on the Hausdorff dimension of $\mathcal{L}_\psi$ dates back to Jarník-Bosicovitch Theorem [2,9]. It was shown that for any $\tau > 1$, one has

$$\dim_H \left( \left\{ \theta \in \mathbb{R} : \|n\theta\| < \frac{1}{n^\tau}, \text{ for infinitely many } n \in \mathbb{N} \right\} \right) = \frac{2}{1 + \tau},$$

where $\dim_H(\cdot)$ denotes the Hausdorff dimension of a set.

In analogy with the classical Diophantine approximation, Hill and Velani [8] studied the approximation properties of the orbits of a dynamical
system and introduced the so called shrinking target problems: for a measure preserving dynamical system \((M, \mu, T)\) with a metric \(d\) and a positive function \(\psi\), define the set of all \(\psi\)-well approximable points by \(x_0\) as

\[
L(T, \psi, x_0) := \{ x \in M : d(T^n x, x_0) < \psi(n), \text{ for infinitely many } n \in \mathbb{N} \},
\]

what is the size (Lebesgue measure, Hausdorff dimension) of \(L(T, \psi, x_0)\)?

They studied the case where \(T\) is an expanding rational map of the Riemann sphere \(\mathbb{C} = \mathbb{C} \cup \{ \infty \}\).

In this paper, we are interested in the approximation properties of the orbits of \(\beta\)-transformations. The \(\beta\)-transformation \(T_\beta (\beta > 1)\) on \([0,1]\) is defined by

\[
T_\beta(x) := \beta x - \lfloor \beta x \rfloor,
\]

where \(\lfloor \cdot \rfloor\) is the integer part function. For any positive function \(\psi : \mathbb{N} \to \mathbb{R}^+\), define the set of \(\psi\)-well asymptotically approximable points by \(x_0\) as

\[
L(\psi, x_0) := \{ x \in [0,1] : |T_\beta^n x - x_0| < \psi(n), \text{ for infinitely many } n \in \mathbb{N} \}.
\]

By [15, Theorem 2A, B, C], the set \(L(\psi, x_0)\) has Lebesgue measure zero if and only if the series \(\sum \psi(n)\) converges. Shen and Wang [17, Theorem 1.1] showed that for any real number \(\beta > 1\) and any point \(x_0 \in [0,1]\), one has

\[
\dim_H (L(\psi, x_0)) = \frac{1}{1 + v}, \quad \text{where } v := \liminf_{n \to \infty} -\frac{\log_\beta \psi(n)}{n}.
\]

Parallel to the asymptotic approximation theorem, it is also worth of studying the uniform approximation properties as in Dirichlet Theorem. The uniform Diophantine approximation related to \(\beta\)-transformations was studied by Bugeaud and Liao [4]. For \(x \in [0,1]\), let

\[
v_\beta(x) := \sup \{ v \geq 0 : T_\beta^n x < (\beta^n)^{-v}, \text{ for infinitely many } n \in \mathbb{N} \},
\]

\[
\hat{v}_\beta(x) := \sup \{ v \geq 0 : \forall N \gg 1, T_\beta^n x < (\beta^N)^{-v} \text{ has a solution } n \in [0,N] \}.
\]

The exponents \(v_\beta\) and \(\hat{v}_\beta\) were introduced in [1] (see also [3, Ch.7]). Bugeaud and Liao [4] proved the following theorem.

**Theorem BL** (4, Theorem 1.4) For any \(v \in (0, \infty)\) and any \(\hat{v} \in (0,1)\), if \(v < \hat{v}/(1 - \hat{v})\), then the set

\[
\{ x \in [0,1] : v_\beta(x) = v \} \cap \{ x \in [0,1] : \hat{v}_\beta(x) \geq \hat{v} \}
\]

is empty. Otherwise,

\[
\dim_H (\{ x \in [0,1] : v_\beta(x) = v \} \cap \{ x \in [0,1] : \hat{v}_\beta(x) = \hat{v} \}) = \frac{v - \hat{v} - \hat{v}v}{(1 + v)(v - \hat{v})}.
\]

**Theorem BL** can be considered as the special case where \(x_0 = 0\). The aim of this paper is to study the Diophantine approximation sets in [4] for any fixed \(x_0 \in (0,1]\).

**Definition 1.1.** Let \(\beta > 1\), fix \(x_0 \in [0,1]\). For any \(x \in [0,1]\), denote by \(\mathcal{V}_\beta(x, x_0)\) the supremum of the real numbers \(v\) for which the equation

\[
|T_\beta^n x - x_0| < (\beta^n)^{-v}
\]
has infinitely many solutions in integers $n \in \mathbb{N}$. Denote by $\hat{V}_{\beta}(x, x_0)$ the supremum of the real numbers $\hat{v}$ for which, for every sufficiently large integer $N$, the equation

$$|T_{\beta}^n x - x_0| < (\beta^N)^{-\hat{v}}$$

has a solution $n \in \mathbb{N}$ with $1 \leq n \leq N$.

Our main results are the following Theorems A and B.

**Theorem A.** Let $\beta > 1$. For any $x_0 \in [0, 1]$, any $v \in (0, \infty)$ and any $\hat{v} \in (0, 1)$, if $v < \hat{v}/(1 - \hat{v})$, then the set

$$\{ x \in [0, 1] : V_{\beta}(x, x_0) = v \} \cap \{ x \in [0, 1] : \hat{V}_{\beta}(x, x_0) \geq \hat{v} \}$$

is empty. Otherwise, the set

$$\{ x \in [0, 1] : V_{\beta}(x, x_0) = v \} \cap \{ x \in [0, 1] : \hat{V}_{\beta}(x, x_0) = \hat{v} \}$$

has the Hausdorff dimension

$$\frac{v - \hat{v} - \hat{v}v}{(1 + v)v - \hat{v}}.$$

**Theorem B.** Let $\beta > 1$, the set

$$\{ x \in [0, 1] : \hat{V}_{\beta}(x, x_0) = 0 \}$$

is of full Lebesgue measure. If $0 < \hat{v} \leq 1$, then

$$\dim_H \left( \left\{ x \in [0, 1] : \hat{V}_{\beta}(x, x_0) = \hat{v} \right\} \right) = \left( \frac{1 - \hat{v}}{1 + \hat{v}} \right)^2.$$  

Otherwise, the set

$$\{ x \in [0, 1] : \hat{V}_{\beta}(x, x_0) > 1 \}$$

is countable.

Persson and Schmeling [14] gave another point of view, by letting $\beta$ vary and considering the $\beta$-expansions of 1. They [14, Theorem 14] showed that for any $1 < \beta_0 < \beta_1 < 2$ and any $v \geq 0$, one has

$$\dim_H (\{ \beta \in (\beta_0, \beta_1) : V_{\beta}(1, x_0) = v \}) = \frac{1}{1 + v}.$$

In [12], the assumption $\beta_1 < 2$ is removed. In the same way as Theorem BL, Bugeaud and Liao [4] also proved that for any $v \in (0, \infty)$ and any $\hat{v} \in (0, 1)$, if $v < \hat{v}/(1 - \hat{v})$, then

$$\{ \beta > 1 : v_{\beta}(1) = v \} \cap \{ \beta > 1 : \hat{v}_{\beta}(1) \geq \hat{v} \} = \emptyset,$$

Otherwise,

$$\dim_H (\{ \beta > 1 : v_{\beta}(1) = v \} \cap \{ \beta > 1 : \hat{v}_{\beta}(1) = \hat{v} \}) = \frac{v - \hat{v} - \hat{v}v}{(1 + v)(v - \hat{v})}.$$  

When considering the exponents $V_{\beta}(1, x_0)$ and $\hat{V}_{\beta}(1, x_0)$, we obtain the following Theorems C and D.
Theorem C. For any $x_0 \in [0,1]$, any $v \in (0,\infty)$ and any $\hat{v} \in (0,1)$, if $v < \hat{v}/(1-\hat{v})$, then the set
\[
\{\beta > 1 : \mathcal{V}_\beta(1,x_0) = v\} \cap \{\beta > 1 : \hat{V}_\beta(1,x_0) \geq \hat{v}\}
\]
is empty. Otherwise,
\[
\dim_H \left(\{\beta > 1 : \mathcal{V}_\beta(1,x_0) = v\} \cap \{\beta > 1 : \hat{V}_\beta(1,x_0) \geq \hat{v}\}\right) = \frac{v - \hat{v} - v\hat{v}}{(1+v)(v-\hat{v})}.
\]

Theorem D. For any $x_0 \in [0,1]$ and any $\hat{v} \in [0,1]$, one has
\[
\dim_H \left(\{\beta > 1 : \mathcal{V}_\beta(1,x_0) \geq \hat{v}\}\right) = \dim_H \left(\{\beta > 1 : \hat{V}_\beta(1,x_0) = \hat{v}\}\right)
\]
and
\[
\dim_H \left(\{\beta > 1 : \hat{V}_\beta(1,x_0) = \hat{v}\}\right) = \left(\frac{1 - \hat{v}}{1 + \hat{v}}\right)^2.
\]

Our paper is organized as follows. We recall some classical results of the theory of $\beta$-transformations in Section 2. Theorems A and B are proved in Section 3. Section 4 establishes Theorems C and D.

2. Beta-transformations

Beta-expansion was introduced by Rényi [16] in 1957. For any $\beta > 1$, the $\beta$-transformation $T_\beta$ on $[0,1)$ is defined by
\[
T_\beta x = \beta x - \lfloor \beta x \rfloor,
\]
where $\lfloor \xi \rfloor$ denotes the largest integer less than or equal to $\xi$. Let
\[
[\beta] = \begin{cases} 
\beta - 1, & \text{if } \beta \text{ is a positive integer}, \\
[\beta], & \text{otherwise}.
\end{cases}
\]

Definition 2.1. A sequence $\{\epsilon_n : \epsilon_n = \epsilon_n(x,\beta)\}_{n \geq 1} \in \mathcal{A}^\mathbb{N} := \{0,1,\ldots, [\beta]\}^\mathbb{N}$ is called the $\beta$-expansion of a number $x \in [0,1)$, if
\[
x = \frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots + \frac{\epsilon_n}{\beta^n} + \cdots,
\]
where $\epsilon_n(x,\beta) = \lfloor \beta T_{\beta}^{n-1} x \rfloor$, for all positive integers $n \in \mathbb{N}$. We also write
\[
d_\beta(x) = (\epsilon_1, \ldots, \epsilon_n, \ldots).
\]

We can extend the definition of the $\beta$-transformation to the point 1 as:
\[
T_\beta 1 = \beta - [\beta].
\]

One can obtain
\[
1 = \frac{\epsilon_1(1,\beta)}{\beta} + \frac{\epsilon_2(1,\beta)}{\beta^2} + \cdots + \frac{\epsilon_n(1,\beta)}{\beta^n} + \cdots,
\]
where $\epsilon_n(1,\beta) = \lfloor \beta T_{\beta}^{n-1} 1 \rfloor$, for all positive integers $n \in \mathbb{N}$. We also write
\[
d_\beta(1) = (\epsilon_1(1,\beta), \ldots, \epsilon_n(1,\beta), \ldots).
\]

If $d_\beta(1)$ is finite, i.e. there is an integer $m > 0$ such that $\epsilon_m(1,\beta) \neq 0$ and $\epsilon_i(1,\beta) = 0$ for all $i > m$, then $\beta$ is called a simple Parry number. In this case, the infinite $\beta$-expansion of 1 is defined as
\[
(\epsilon_1^*(\beta), \epsilon_2^*(\beta), \ldots, \epsilon_n^*(\beta), \ldots) := (\epsilon_1(1,\beta), \epsilon_2(1,\beta), \ldots, \epsilon_m(1,\beta) - 1)^\infty,
\]
where \((\omega)\infty\) denotes the periodic sequence. If \(d_\beta(1)\) is infinite, then we define
\[
(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots, \varepsilon_n^*(\beta)) := (\varepsilon_1(1, \beta), \varepsilon_2(1, \beta), \ldots, \varepsilon_n(1, \beta), \ldots).
\]

Endow the set \(\mathcal{A}^\mathbb{N}\) with the product topology and define the one-sided shift operator \(\sigma\) as
\[
\sigma\big(\{(\omega_n)_{n\geq1}\}\big) := (\omega_{n+1})_{n\geq1},
\]
for any infinite sequence \((\omega_n)_{n\geq1} \in \mathcal{A}^\mathbb{N}\). The lexicographical order \(\leq_{\text{lex}}\) on \(\mathcal{A}^\mathbb{N}\) is defined as
\[
\omega = (\omega_1, \omega_2, \ldots) \leq_{\text{lex}} \omega' = (\omega'_1, \omega'_2, \ldots),
\]
if \(\omega_1 < \omega'_1\) or if there is an integer \(k \geq 2\) such that for all \(1 \leq i < k\), \(\omega_i = \omega'_i\) but \(\omega_k < \omega'_k\). Denote by \(\omega \leq_{\text{lex}} \omega'\) if \(\omega <_{\text{lex}} \omega'\) or \(\omega = \omega'\).

A finite word \((\omega_1, \omega_2, \ldots, \omega_n)\) is called \(\beta\)-admissible, if there is \(x \in [0, 1]\) such that the \(\beta\)-expansion of \(x\) begins with \((\omega_1, \omega_2, \ldots, \omega_n)\). An infinite sequence \((\omega_1, \omega_2, \ldots, \omega_n, \ldots)\) is called \(\beta\)-admissible, if there is \(x \in [0, 1]\) such that the \(\beta\)-expansion of \(x\) is \((\omega_1, \omega_2, \ldots, \omega_n, \ldots)\). An infinite sequence \((\omega_1, \omega_2, \ldots, \omega_n, \ldots)\) is self-admissible, if
\[
\sigma^k(\omega_1, \omega_2, \ldots, \omega_n, \ldots) \leq_{\text{lex}} (\omega_1, \omega_2, \ldots, \omega_n, \ldots), \text{ for } k \geq 0.
\]

Denote by \(\Sigma_\beta\) the set of all infinite \(\beta\)-admissible sequences and denote by \(\Sigma^n_\beta\) the set of all \(\beta\)-admissible sequences with length \(n\). The \(\beta\)-admissible sequences are characterized by Parry [13] and Rényi [16].

**Theorem 2.2.** Let \(\beta > 1\),
\begin{enumerate}
\item [(1)] (\cite[Lemma 1]{13}) A word \(\omega = (\omega_n)_{n\geq1} \in \Sigma_\beta\) if and only if
\[
s_\beta^k(\omega) \leq_{\text{lex}} (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots, \varepsilon_n^*(\beta), \ldots), \text{ for all } k \geq 0.
\]
\item [(2)] (\cite[Lemma 3]{13}) For any \(x_1, x_2 \in [0, 1], x_1 < x_2\) if and only if
\[
d_\beta(x_1) <_{\text{lex}} d_\beta(x_2).
\]
\item [(3)] (\cite[Lemma 4]{13}) For any \(\beta_2 > \beta_1 > 1\), one has
\[
\Sigma^n_{\beta_1} \subseteq \Sigma^n_{\beta_2}, \quad \Sigma_{\beta_1} \subseteq \Sigma_{\beta_2}.
\]
\end{enumerate}

**Theorem 2.3.** (\cite[Theorem 2]{16}) For any \(\beta > 1\), one has
\[
\beta^n \leq \#\Sigma^n_{\beta} \leq \frac{\beta^{n+1}}{\beta - 1},
\]
where \(\#\) denotes the cardinality of a finite set.

For every \((\omega_1, \ldots, \omega_n) \in \Sigma^n_{\beta}\), we call
\[
I_n(\omega_1, \ldots, \omega_n) := \{x \in [0, 1] : d_\beta(x) \text{ starts with } \omega_1, \ldots, \omega_n\}
\]
an \(n\)-th order basic interval with respect to \(\beta\). Denote by \(I_n(x)\) the \(n\)-th order basic interval containing \(x\). The basic intervals are also called cylinders. It is crucial to estimate the lengths of the basic intervals. We will use the key notion of “full cylinder” introduced by Fan and Wang [7]. For any \((\omega_1, \ldots, \omega_n) \in \Sigma^n_{\beta}, a basic interval \(I_n(\omega_1, \ldots, \omega_n)\) is said to be full if its length is \(\beta^{-n}\). Denote by \(|I_n(\omega_1, \ldots, \omega_n)|\) the length of the \(n\)-th order basic interval.
For every $\omega \in \Sigma^m$, the following statements are equivalent:

1. $I_n(\omega_1, \ldots, \omega_n)$ is a full basic interval.
2. $T^\beta_n I_n(\omega_1, \ldots, \omega_n) = [0, 1]$.
3. For any $\omega' = (\omega'_1, \ldots, \omega'_m) \in \Sigma^m$, the concatenation
   $$(\omega_1, \ldots, \omega_n, \omega'_1, \ldots, \omega'_m) \in \Sigma^{m+1},$$
   i.e., is $\beta$-admissible.

**Proposition 2.5.** ([17, Corollary 2.6])

1. If $(\omega_1, \ldots, \omega_{n+1})$ is a $\beta$-admissible sequence with $\omega_{n+1} \neq 0$, then
   $$I_{n+1}(\omega_1, \ldots, \omega_{n+1})$$
   is full for any $0 \leq \omega_{n+1} < \omega_n$.
2. For every $\omega \in \Sigma^n$, if $I_n(\omega)$ is full, then for any $\omega' \in \Sigma^n$, one has
   $$|I_{n+m}(\omega, \omega')| = |I_n(\omega)| \cdot |I_m(\omega')| = \frac{|I_m(\omega')|}{\beta^n}.$$
3. For any $\omega \in \Sigma^n$, if $I_{n+m}(\omega, \omega')$ is a full basic interval contained in $I_n(\omega)$
   with the smallest order, then
   $$|I_{n+m}(\omega, \omega')| \geq \frac{|I_n(\omega)|}{\beta^n}.$$

Next, we define a sequence of numbers $\beta_N$ approaching to $\beta$. Given the infinite $\beta$-expansion
$$\varepsilon^1(\beta), \varepsilon^2(\beta), \ldots, \varepsilon^N(\beta), \ldots$$
of 1. For any $\varepsilon^N(\beta) > 0$, let $\beta_N$ be the unique real solution of the equation

$$(2.2) \quad 1 = \frac{\varepsilon^1(\beta)}{z} + \cdots + \frac{\varepsilon^N(\beta)}{z^N}.$$

Therefore, $\beta_N < \beta$ and the sequence $\{\beta_N : N \geq 1\}$ increases and converges to $\beta$ when $N$ tends to infinity.

**Lemma 2.6.** ([17, Lemma 2.7]) For every $\omega \in \Sigma^n$ viewed as an element
of $\Sigma^\beta$, one has

$$\frac{1}{\beta_{n+N}} \leq |I_n(\omega_1, \ldots, \omega_n)| \leq \frac{1}{\beta^n}.$$

3. **Proofs of Theorems A and B**

For $\beta > 1$ and $x_0 \in [0, 1]$, by the definitions of $V_\beta(x, x_0)$ and $\hat{V}_\beta(x, x_0)$, it
 can be checked that for every $x \in [0, 1]$, we have

$$\hat{V}_\beta(x, x_0) \leq V_\beta(x, x_0).$$

We first consider two special cases $\hat{V}_\beta(x, x_0) = V_\beta(x, x_0) = 0$ and $\hat{V}_\beta(x, x_0) = v_\beta(x, x_0) = \infty$.

**Lemma 3.1.** If $\hat{V}_\beta(x, x_0) = V_\beta(x, x_0) = 0$, then the set

$$\{x \in [0, 1] : V_\beta(x, x_0) = 0\} \cap \{x \in [0, 1] : \hat{V}_\beta(x, x_0) = 0\}$$

is of full Lebesgue measure.
Proof. Note that for any fixed $x_0 \in [0, 1]$ and any $x \in [0, 1]$, we always have
\[
\hat{V}_\beta(x, x_0) \leq V_\beta(x, x_0).
\]
Then,
\[
\{x \in [0, 1] : V_\beta(x, x_0) = 0\} \subseteq \{x \in [0, 1] : \hat{V}_\beta(x, x_0) = 0\}.
\]
Hence, if for any fixed $x_0 \in [0, 1]$, we can prove that $V_\beta(x, x_0) = 0$, for Lebesgue almost every $x \in [0, 1]$, then we prove the lemma.

Now, we only need to prove
\[
m\left(\{x \in [0, 1] : V_\beta(x, x_0) > 0\}\right) = 0,
\]
where $m(\cdot)$ denotes the Lebesgue measure of a set. In fact, we have
\[
\{x \in [0, 1] : V_\beta(x, x_0) > 0\} = \bigcup_{k=1}^{\infty} \{x \in [0, 1] : V_\beta(x, x_0) > 1/k\}
\]
and \(\{x \in [0, 1] : V_\beta(x, x_0) > 1/k\}\) is a subset of
\[
\left\{x \in [0, 1] : |T_\beta^n x - x_0| < \beta^{-n/k}, \text{ for infinite many } n \in \mathbb{N}\right\}.
\]
Since $\sum_{n=1}^{\infty} \beta^{-n/k} < \infty$ for any $k \geq 1$, by [15, Theorem 2A, B, C],
\[
m\left(\bigcup_{k=1}^{\infty} \{x \in [0, 1] : |T_\beta^n x - x_0| < \beta^{-n/k}, \text{ for infinite many } n \in \mathbb{N}\}\right) = 0.
\]
Therefore, for any $k \geq 1$, we have
\[
m\left(\{x \in [0, 1] : V_\beta(x, x_0) > 1/k\}\right) = 0.
\]
Thus,
\[
m\left(\{x \in [0, 1] : V_\beta(x, x_0) > 0\}\right) = 0.
\]

When $\hat{V}_\beta(x, x_0) = V_\beta(x, x_0) = \infty$, we have the following Lemma 3.2.

Lemma 3.2. If the number $\hat{V}_\beta(x, x_0) = \infty$, then
\[
\{x \in [0, 1] : \hat{V}_\beta(x, x_0) = \infty\} = \bigcup_{n=1}^{\infty} \bigcup_{w \in \Sigma_\beta^n} \{x \in [0, 1] : d_\beta(x) = (w, d_\beta(x_0))\}
\]
is countable.

Proof. For $\beta > 1$ and $x_0 \in [0, 1]$, we suppose
\[
d_\beta(x_0) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots).
\]
If $x \in \bigcup_{n=1}^{\infty} \bigcup_{w \in \Sigma_\beta^n} \{x \in [0, 1] : d_\beta(x) = (w, d_\beta(x_0))\}$, then there exists an integer $n_0$ such that $|T_\beta^{n_0} x - x_0| = 0$. Therefore, for any $n \geq n_0$, there is $n_0 \in [1, n]$ such that
\[
|T_\beta^{n_0} x - x_0| = 0.
\]
Thus, $\hat{V}_\beta(x, x_0) = \infty$.

Now, we prove
\[
\{x \in [0, 1] : \hat{V}_\beta(x, x_0) = \infty\} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{w \in \Sigma_\beta^n} \{x \in [0, 1] : d_\beta(x) = (w, d_\beta(x_0))\}.
\]
By contrary, for any $x$ with $\hat{V}_\beta(x, x_0) = \infty$, we suppose
\[
x \notin \bigcup_{n=1}^{\infty} \bigcup_{w \in \Sigma_\beta^n} \{x \in [0, 1] : d_\beta(x) = (w, d_\beta(x_0))\}.
\]
Denote the $\beta$-expansion of $x$ by
\[ x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \cdots, \]
increasing sequences \{n_i' : i \geq 1\} and \{m_i' : i \geq 1\} with the properties:

(1) For every $i \geq 1$, one has
\[ a_{n_i'} > 0, \quad a_{n_i'+1} = \epsilon_1, \quad a_{n_i'+2} = \epsilon_2, \ldots, \quad a_{m_i'-1} = \epsilon_{m_i'-n_i'-1}, \quad a_{m_i'} > 0. \]

(2) For every $a_n = 0$, there is an integer $i$ such that $n_i' < n < m_i'$.

By the choice of \{n_i' : i \geq 1\} and \{m_i' : i \geq 1\}, for every $i \geq 1$, one has $n_i' < m_i' < n_{i+1}'$. Since $V_{\beta}(x, x_0) > 0$, one has
\[ \limsup_{i \to \infty} (m_i' - n_i') = \infty. \]

Taking $n_1 = n_1'$ and $m_1 = m_1'$, suppose $m_k, n_k$ have been defined. Let $i_1 = 1$ and $i_{k+1} := \min\{i > i_k : m_i' - n_i' > m_k - n_k\}$, for $k \geq 1$. Then, define
\[ n_{k+1} := n_{i_{k+1}}, \quad m_{k+1} := m_{i_{k+1}}. \]

Therefore, the sequence \{i_k : k \geq 1\} is well defined. By this way, we obtain the subsequences \{n_k : k \geq 1\} and \{m_k : k \geq 1\} of \{n_i' : i \geq 1\} and \{m_i' : i \geq 1\}, respectively, such that the sequence \{m_k - n_k : k \geq 1\} is non-decreasing. Notice $\beta^{m_k-n_k} < |T_{\beta}^{n_k}x - x_0| < \beta^{m_{k+1}-n_{k+1}}$, we have
\[ \hat{V}_{\beta}(x, x_0) = \liminf_{k \to \infty} \frac{m_k - n_k}{n_{k+1}} \leq 1. \]

This contradicts our assumption $\hat{V}_{\beta}(x, x_0) = \infty$. Thus, we have proved
\[ \left\{ x \in [0, 1] : \hat{V}_{\beta}(x, x_0) = \infty \right\} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{w, d_\beta \in \Sigma^*} \left\{ x \in [0, 1] : d_\beta(x) = (w, d_\beta(x_0)) \right\}. \]

Therefore,
\[ \left\{ x \in [0, 1] : \hat{V}_{\beta}(x, x_0) = \infty \right\} = \bigcup_{n=1}^{\infty} \bigcup_{w, d_\beta \in \Sigma^*} \left\{ x \in [0, 1] : d_\beta(x) = (w, d_\beta(x_0)) \right\}, \]
which implies that the set \{x \in [0, 1] : \hat{V}_{\beta}(x, x_0) = \infty\} is countable. \hfill \Box

**Lemma 3.3.** The set \{x \in [0, 1] : 1 < \hat{V}_{\beta}(x, x_0) < \infty\} is empty.

**Proof.** This follows from the proof of Lemma 3.2. \hfill \Box

For $\beta > 1$ and $x_0 \in [0, 1]$, we suppose
\[ d_\beta(x_0) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots). \]
For the case $V_{\beta}(x, x_0) \in (0, \infty)$ and $\hat{V}_{\beta}(x, x_0) \in (0, 1)$, we have the following discussion and complete the proof of Theorem A.
Upper bound. For any \( x \in [0, 1] \), denote its \( \beta \)-expansion by
\[
x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \cdots.
\]
Since \( \hat{V}_\beta(x, x_0) \in (0, 1) \), by the same way as Lemma 3.2, we can take the maximal subsequences \( \{n_k : k \geq 1\} \) and \( \{m_k : k \geq 1\} \) of \( \{n'_i : i \geq 1\} \) and \( \{m'_i : i \geq 1\} \), respectively. Similarly, notice that
\[
\beta^{n_k-m_k} < |T_{\beta}^{m_k} x - x_0| < \beta^{n_k-m_k+1}.
\]
We have
\[
V_\beta(x, x_0) = \limsup_{n \to \infty} \frac{m_k - n_k}{n_k} = \limsup_{n \to \infty} \frac{m_k}{n_k} - 1,
\]
\[
\hat{V}_\beta(x, x_0) \leq \liminf_{n \to \infty} \frac{m_k - n_k}{n_{k+1}} \leq \liminf_{n \to \infty} \frac{m_k - n_k}{m_k} = 1 - \limsup_{n \to \infty} \frac{n_k}{m_k}.
\]
Since \((\limsup\frac{n_k}{m_k}) - (\limsup\frac{m_k}{n_k}) \geq 1\), one has
\[
V_\beta(x, x_0) \geq \frac{\hat{V}_\beta(x, x_0)}{1 - \hat{V}_\beta(x, x_0)}, \quad \hat{V}_\beta(x, x_0) \leq \frac{V_\beta(x, x_0)}{1 + V_\beta(x, x_0)}.
\]
We can derive from 3.3 that if \( v < \hat{v}/(1 - \hat{v}) \), then the set
\[
\{x \in [0, 1] : V_\beta(x, x_0) = v\} \cap \{x \in [0, 1] : \hat{V}_\beta(x, x_0) \geq \hat{v}\}
\]
is empty. Otherwise, under the case where \( V_\beta(x, x_0) = v \) and \( \hat{V}_\beta(x, x_0) = \hat{v} \), take the two sequences \( \{n_k : k \geq 1\} \) and \( \{m_k : k \geq 1\} \) such that
\[
V_\beta(x, x_0) = \lim_{n \to \infty} \frac{m_k - n_k}{n_k}, \quad \hat{V}_\beta(x, x_0) \leq \liminf_{n \to \infty} \frac{m_k - n_k}{n_{k+1}}.
\]
Given \( 0 < \varepsilon < \hat{v}/2 \), for \( k \) large enough, one has
\[
(v - \varepsilon)n_k \leq m_k - n_k \leq (v + \varepsilon)n_k,
\]
\[
m_k - n_k \geq (\hat{v} - \varepsilon)n_{k+1}.
\]
By inequality (3.4), one has
\[
(1 + v - \varepsilon)m_{k-1} \leq (1 + v - \varepsilon)n_k \leq m_k.
\]
Therefore, the sequence \( \{m_k : k \geq 1\} \) increases at least exponentially. Since \( n_k \geq m_{k-1} \) for every \( k \geq 2 \), the sequence \( \{n_k : k \geq 1\} \) also increases at least exponentially. Thus, there is a positive constant \( C \) such that \( k \leq C \log_\beta n_k \). Combining (3.4) and (3.5), one obtains
\[
(\hat{v} - \varepsilon)n_{k+1} \leq (v + \varepsilon)n_k.
\]
Thus, for \( k \) large enough, there is an integer \( n_0 \) and a positive real number \( \varepsilon_1 \) small enough such that the sum of all lengths of the blocks of 0 in the prefix of length \( n_k \) of the infinite sequence \( a_1a_2\cdots \) is at least equal to
\[
n_k(\hat{v} - \varepsilon) \left( \left[ \frac{\hat{v} - \varepsilon}{v + \varepsilon} \right] + \frac{(\hat{v} - \varepsilon)^2}{(v + \varepsilon)^2} + \cdots \right) - n_0 = n_k \frac{(\hat{v} - \varepsilon)(v + \varepsilon)}{v - \hat{v} + 2\varepsilon} - n_0 \geq n_k \frac{v \cdot \hat{v}}{v - \hat{v} - \varepsilon_1}.
\]
Among the digits $a_1 \cdots a_{m_k}$, there are $k$ blocks of digits which are ‘free’. Denote their lengths by $l_1, \cdots, l_k$. There is a small number $\varepsilon_2$ such that

$$\sum_{i=1}^k l_i \leq n_k - n_k \frac{v \cdot \hat{v}}{v - \hat{v}} \leq n_k (1 + \varepsilon_2) \frac{v - \hat{v} - v \cdot \hat{v}}{v - \hat{v}}.$$

By Theorem 2.3, there are at most $\beta \cdot \beta^l_i / (\beta - 1)$ ways to choose the block with length $l_i$. Thus, one has in total at most

$$\left( \frac{\beta}{\beta - 1} \right)^k \cdot \beta^{\sum_{i=1}^k l_i} \leq \left( \frac{\beta}{\beta - 1} \right)^k \cdot \beta^{n_k (1 + \varepsilon_2)} \frac{(v - \hat{v} - v \cdot \hat{v})}{(v - \hat{v})}$$

possible choices of the digits $a_1 \cdots a_{m_k}$. On the other hand, there are at most $k (k \leq C \log_\beta n_k)$ blocks of 0 in the prefix of length $n_k$ of the infinite sequence $a_1 a_2 \cdots$. Since there are at most $n_k$ possible choices for their first index, one has in total at most $(n_k)^{C \log_\beta n_k}$ possible choices. Consequently, the set

$$\{ x \in [0, 1] : V_\beta(x, x_0) = v \} \cap \{ x \in [0, 1] : \hat{V}_\beta(x, x_0) = \hat{v} \}$$

is covered by

$$\left( \frac{\beta n_k}{\beta - 1} \right)^{C \log_\beta n_k} \cdot \beta^{n_k (1 + \varepsilon_2)} \frac{(v - \hat{v} - v \cdot \hat{v})}{(v - \hat{v})}$$

basic intervals of length at most $\beta^{-m_k}$. Moreover, by (3.4), there is a small number $\varepsilon_3 > 0$ such that

$$\beta^{-m_k} \leq \beta^{-(1+v)(1-\varepsilon_3)n_k}.$$

Take $\varepsilon' = \max\{\varepsilon_2, \varepsilon_3\}$, we consider the series

$$\sum_{N \geq 1} (N) \left( \frac{C \log_\beta N \beta^{(1+\varepsilon')(v - \hat{v} - v \cdot \hat{v}) / (v - \hat{v}) \beta^{-(1+v)(1-\varepsilon')N}}{s} \right).$$

The critical exponent $s_0$ such that the series converges if $s > s_0$ and diverges if $s < s_0$ is given by

$$s_0 = \frac{1 + \varepsilon'}{1 - \varepsilon'} \cdot \frac{v - \hat{v} - v \cdot \hat{v}}{(1 + v / (v - \hat{v}))}.$$

By a standard covering argument and the arbitrariness of $\varepsilon'$, the Hausdorff dimension of the set

$$\{ x \in [0, 1] : V_\beta(x, x_0) = v \} \cap \{ x \in [0, 1] : \hat{V}_\beta(x, x_0) = \hat{v} \}$$

is at most equal to

$$\frac{v - \hat{v} - v \cdot \hat{v}}{(1 + v / (v - \hat{v}))}.$$
Lower bound. To obtain the lower bound, we will construct a suitable Cantor type set. For \( v \in (0, \infty) \) and \( \hat{v} \in (0, 1) \) with \( v \geq \hat{v}/(1 - \hat{v}) \), let

\[
n'_k = \left[ \left( \frac{v}{\beta} \right)^k \right], \quad m'_k = \left( (1 + v)n'_k \right), \quad k = 1, 2, \cdots.
\]

Making an adjustment, we can choose two subsequences \( \{n_k\} \) and \( \{m_k\} \) with \( n_k < m_k < n_{k+1} \) for every \( k \geq 1 \) such that \( \{m_k - n_k\} \) is a non-decreasing sequence and

\[
\lim_{k \to \infty} \frac{m_k - n_k}{n_k} = v, \quad \lim_{k \to \infty} \frac{m_k - n_k}{n_{k+1}} = \hat{v}.
\]

Recall that the \( \beta \)-expansion of the fixed point \( x_0 \) is

\[d_\beta(x_0) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots).\]

Consider the set of real numbers \( x \in [0, 1] \) whose \( \beta \)-expansion

\[x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \cdots,
\]

satisfies that for all \( k \geq 1 \),

\[a_{n_k} > 1, \quad a_{n_k+1} = \epsilon_1, \quad a_{n_k+2} = \epsilon_2, \cdots, \quad a_{m_k-1} = \epsilon_{m_k-n_k-1}, \quad a_{m_k} > 0,
\]

\[a_{m_k+(m_k-n_k)} = a_{m_k+1} = \cdots = a_{m_k+t_k(m_k-n_k)} = 1,
\]

where \( t_k \) is the largest integer such that \( m_k + t_k(m_k-n_k) < n_{k+1} \). Then,

\[t_k \leq \frac{n_{k+1} - m_k}{m_k - n_k} \leq \frac{2}{\hat{v}},
\]

for \( k \) large enough. Therefore, the sequence \( \{t_k : k \geq 1\} \) is bounded. Fix \( N \), let \( \beta_N \) be the real number defined by the infinite \( \beta \)-expansion of 1 as equality (2.2). We replace the digit 1 for \( a_{n_k} \), \( a_{m_k} \) and \( a_{m_k+i(m_k-n_k)} \) for any \( 1 \leq i \leq t_k \) by the block \( 0^N10^N \). Fill other places by blocks belonging to \( \Sigma_{\beta_N} \). Thus, we have constructed the Cantor type subset \( E \). Since \( \{t_k\} \) is bound, one has

\[
\lim_{k \to \infty} \frac{m_k - n_k - 1 + 2N}{n_k + (4k - 2)N + \sum_{i=1}^{k-1} 2Nt_i} = \lim_{k \to \infty} \frac{m_k - n_k}{n_k} = v,
\]

\[
\lim_{k \to \infty} \frac{m_k - n_k - 1 + 2N}{n_{k+1} + (4k + 2)N + \sum_{i=1}^{k} 2Nt_i} = \lim_{k \to \infty} \frac{m_k - n_k}{n_{k+1}} = \hat{v}.
\]

According to the construction, the sequence \( d_\beta(x) \) is in \( \Sigma_{\beta_N} \).

We distribute the mass uniformly when meet a block in \( \Sigma_{\beta_N} \) and keep the mass when go through the positions where the digits are determined by construction of \( E \). The Bernoulli measure \( \mu \) on \( E \) is defined as follows.

If \( n < n_1 \), then define \( \mu(I_n) = 1/\#\Sigma_{\beta_N}^n \). If \( n_1 \leq n \leq n_1 + 4N \), then define \( \mu(I_n) = 1/\#\Sigma_{\beta_N}^{n_1-1} \). If there is an integer \( t \) with \( 0 \leq t \leq t_1 - 1 \) such that

\[m_1 + 4N + (t+1)(m_1-n_1) + 2N < n \leq m_1 + 4N + (t+1)(m_1-n_1) + 2N(t+1),
\]

then define

\[\mu(I_n) = \frac{1}{\#\Sigma_{\beta_N}^{n_1-1}} \cdot \frac{1}{\#\Sigma_{\beta_N}^{m_1-n_1-1}}^{t+1}.
\]
If there is an integer $t$ with $0 \leq t \leq t_1$ such that
\[ m_1 + 4N + t(m_1 - n_1) + 2Nt < n \leq c, \]
where $c \coloneqq \min\{n_2 + 4N + 2Nt_1, m_1 + 4N + (t + 1)(m_1 - n_1) + 2Nt\}$, then define
\[ \mu(I_n) = \frac{1}{\frac{1}{\sum a_{1, \beta N}} \cdot \frac{1}{\sum a_{-1, \beta N}} t \cdot \frac{1}{\sum a_{-n, \beta N} - (m_1 + 4N + t(m_1 - n_1) + 2Nt)}}. \]

For $k \geq 2$, let
\[ l_k := n_k + 4(k - 1)N + \sum_{i=1}^{k-1} 2Nt_i, \quad h_k := m_k + 4kN + \sum_{i=1}^{k-1} 2Nt_i, \]
\[ p_k := m_k - n_k - 1, \quad q_k := h_k + t_k(m_k - n_k) + 2Nt_k. \]
If $l_k \leq n \leq h_k$, then define
\[ \mu(I_n) = \frac{1}{\frac{1}{\sum a_{1, \beta N}} \cdot \prod_{i=1}^{k-1} \left( \frac{1}{\sum a_{-1, \beta N}} \right)^{t_i} \cdot \frac{1}{\sum a_{-n, \beta N} - (m_k + (t_k - 1)(m_k - n_k) + 2N(t_k - 1))}} = \mu(I_{k_{\beta N}}) = \mu(I_{h_k}). \]

If there is an integer $t$ with $0 \leq t \leq t_k - 1$ such that
\[ h_k + (t + 1)(m_k - n_k) + 2Nt < n \leq h_k + (t + 1)(m_k - n_k) + 2N(t + 1), \]
then define
\[ \mu(I_n) = \mu(I_{h_k}) \cdot \frac{1}{\frac{1}{\sum a_{1, \beta N}} \cdot \frac{1}{\sum a_{-n, \beta N} - (h_k + t(m_k - n_k) + 2Nt)}}. \]

If there is an integer $t$ with $0 \leq t \leq t_k$ such that
\[ h_k + t(m_k - n_k) + 2Nt < n \leq \min\{l_{k+1}, h_k + (t + 1)(m_k - n_k) + 2Nt\}, \]
then define
\[ \mu(I_n) = \mu(I_{h_k}) \cdot \frac{1}{\frac{1}{\sum a_{1, \beta N}} \cdot \frac{1}{\sum a_{-n, \beta N} - (h_k + t(m_k - n_k) + 2Nt)}}. \]

By the construction and Proposition 2.4, $I_{h_k}$ is full. For calculating the local dimension of $\mu$, we discuss different cases as follows.

**Case A:** If $n = h_k$, then
\[
\liminf_{k \to \infty} \frac{\log \beta \mu(I_{h_k})}{\log \beta |I_{h_k}|} = \liminf_{k \to \infty} \frac{n_1 - 1 + \sum_{i=1}^{k-1} (t_ip_i + l_{i+1} - q_i - 1)}{h_k} \cdot \log \beta \beta_N
\]
\[
= \liminf_{k \to \infty} \frac{n_1 - 1 + \sum_{i=1}^{k-1} (l_{i+1} - h_i - 2Nt_i - 1)}{h_k} \cdot \log \beta \beta_N.
\]
Recall that $\{t_k : k \geq 1\}$ is bounded and $\{m_k : k \geq 1\}$ grows exponentially fast in terms of $k$, therefore,
\[
\liminf_{k \to \infty} \frac{\log \beta \mu(I_{h_k})}{\log \beta |I_{h_k}|} = \liminf_{k \to \infty} \frac{\sum_{i=1}^{k-1} (n_{i+1} - m_i)}{m_k} \log \beta \beta_N.
\]
By equalities (3.6), one has
\[
\lim_{k \to \infty} \frac{m_k}{n_k} = 1 + v, \quad \lim_{k \to \infty} \frac{m_{k+1}}{m_k} = \frac{v}{\hat{v}}, \quad \lim_{k \to \infty} \frac{n_{k+1}}{m_k} = \frac{v}{\hat{v}(1 + v)}.
\]

According to Stolz-Cesàro Theorem,
\[
\liminf_{k \to \infty} \frac{\sum_{i=1}^{k-1} (n_{i+1} - m_i)}{m_k} = \liminf_{k \to \infty} \frac{n_{k+1} - m_k}{m_{k+1} - m_k} = \liminf_{k \to \infty} \frac{m_k}{m_k} = 1 - \frac{v}{\hat{v}}.
\]

Thus,
\[
\liminf_{k \to \infty} \frac{\log_\beta \mu(I_{h_k})}{\log_\beta |I_{h_k}|} = \frac{v - \hat{v} - v \cdot \hat{v}}{(1 + v)(v - \hat{v})} \cdot \log_\beta \beta_N.
\]

**Case B:** For an integer \( n \) large enough, if there is \( k \geq 2 \) such that \( l_k \leq n \leq h_k \), then
\[
\mu(I_n) \leq \mu(I_{h_k}) \cdot \beta^{-(t+1)p_k}.
\]

Since \( I_{h_k} \) is full, by Proposition 2.5, \( |I_n| = |I_{h_k}| \cdot |I_n - h_k(\omega')| \), where \( \omega' \) is an admissible block in \( \Sigma_{\beta_N}^{n-h_k} \). By Lemma 2.6,
\[
|I_n| \geq |I_{h_k}| \cdot \beta^{-(n-h_k+N)}.
\]

Hence,
\[
\frac{- \log_\beta \mu(I_n)}{- \log_\beta |I_n|} \geq \frac{- \log_\beta \mu(I_{h_k}) + (t + 1)p_k \log_\beta \beta_N}{- \log_\beta |I_{h_k}| + ((t + 1)p_k + N(2t + 1))} \geq \frac{- \log_\beta \mu(I_{h_k})}{- \log_\beta |I_{h_k}|} \cdot \varphi(N),
\]
where \( \varphi(N) < 1 \) and \( \varphi(N) \) tends to 1 as \( N \) tends to infinity. If there is an integer \( t \) with \( 0 \leq t \leq t_k \) such that
\[
h_k + t(m_k - n_k) + 2Nt < n \leq h_k + (t + 1)(m_k - n_k) + 2N(t + 1),
\]
then letting \( l := n - (h_k + t(m_k - n_k) + 2Nt) \), one has
\[
\mu(I_n) \leq \mu(I_{h_k}) \cdot \beta^{-tp_k-l}.
\]

Since \( I_{h_k} \) is full, by Proposition 2.5, \( |I_n| = |I_{h_k}| \cdot |I_{n-h_k}(\omega')| \), where \( \omega' \) is an admissible block in \( \Sigma_{\beta_N}^{n-h_k} \). By Lemma 2.6, \( |I_{n-h_k}(\omega')| \geq \beta^{-(n-h_k+N)} \).

Therefore,
\[
|I_n| \geq |I_{h_k}| \cdot \beta^{-(n-h_k+N)}.
\]

Hence,
\[
\frac{- \log_\beta \mu(I_n)}{- \log_\beta |I_n|} \geq \frac{- \log_\beta \mu(I_{h_k}) + (tp_k + l) \log_\beta \beta_N}{- \log_\beta |I_{h_k}| + (tp_k + l + t + N(2t + 1))} \geq \frac{- \log_\beta \mu(I_{h_k})}{- \log_\beta |I_{h_k}|} \cdot \varphi(N).
\]
Therefore, in all cases,
\[
\liminf_{k \to \infty} \frac{\log_\beta \mu(I_n)}{\log_\beta |I_n|} \geq \frac{v - \hat{v} - v \cdot \hat{v}}{(1 + v)(v - \hat{v})} \cdot \log_\beta \beta_N \cdot \varphi(N).
\]
Given a point \( x \in E \), let \( r \) be a number with \(|I_{n+1}(x)| \leq r < |I_n(x)|\). We consider the ball \( B(x, r) \). By Lemma 2.6, every \( n \)-th order basic interval \( I_n \) satisfies \(|I_n| \geq \beta^{-(n+N)}\). Hence, the ball \( B(x, r) \) interests at most \([2\beta^N] + 2\) basic intervals of order \( n \). On the other hand,
\[
r \geq |I_{n+1}(x)| \geq \beta^{-(n+1+N)} = \beta^{-(1+N)} \cdot \beta^{-n} \geq \beta^{-(1+N)} \cdot |I_n(x)|.
\]
Therefore,
\[
\liminf_{r \to 0} \frac{\log_\beta \mu(B(x, r))}{\log_\beta r} = \liminf_{n \to \infty} \frac{\log_\beta \mu(I_n(x))}{\log_\beta |I_n(x)|}.
\]
Let \( N \) tend to infinity, by Mass Distribution Principle [6, p.p. 60], we get the lower bound
\[
\frac{v - \hat{v} - v \cdot \hat{v}}{(1 + v)(v - \hat{v})}.
\]
Hence, the proof of Theorem A is complete. Now, we prove Theorem B.

**Proof of Theorem B.** If \( \hat{\nu}_\beta(x, x_0) = 0 \), by Lemma 3.1, the set
\[
\left\{ x \in [0, 1] : \hat{\nu}_\beta(x, x_0) = 0 \right\}
\]
is of full Lebesgue measure. If \( \hat{\nu}_\beta(x, x_0) > 1 \), by Lemma 3.2 and Lemma 3.3, the set
\[
\left\{ x \in [0, 1] : \hat{\nu}_\beta(x, x_0) > 1 \right\} = \left\{ x \in [0, 1] : \hat{\nu}_\beta(x, x_0) = \infty \right\}
\]
is countable.

If \( \hat{v} \in (0, 1) \), for any \( v \geq \hat{v}/(1 - \hat{v}) \) and any positive integer \( L \) large enough, by the similar discussion with upper bound in the proof of Theorem A, the Hausdorff dimension of the set
\[
\{ x \in [0, 1] : v \leq \nu_\beta(x, x_0) < v + 1/L \} \cap \left\{ x \in [0, 1] : \hat{\nu}_\beta(x, x_0) = \hat{v} \right\}
\]
is at most equal to
\[
\frac{v - \hat{v} - v \cdot \hat{v}}{(1 + v)(v - \hat{v})} + \frac{\hat{v}^2}{L(1 - \hat{v})}.
\]
Let \( L \) tend to \( \infty \), regard the formula as a function of \( v \) with \( v \geq \hat{v}/(1 - \hat{v}) \), the maximum is attained for \( v = 2\hat{v}/(1 - \hat{v}) \). Thus,
\[
\dim_H \left( \left\{ x \in [0, 1] : \hat{\nu}_\beta(x, x_0) = \hat{v} \right\} \right) \leq \left( \frac{1 - \hat{v}}{1 + \hat{v}} \right)^2.
\]
On the other hand, from the similar discussion with lower bound in the proof of Theorem A, we also have
\[
\dim_H \left( \left\{ x \in [0, 1] : \hat{\nu}_\beta(x, x_0) = \hat{v} \right\} \right) \geq \left( \frac{1 - \hat{v}}{1 + \hat{v}} \right)^2.
\]
Thus,
\[
\dim_H \left( \left\{ x \in [0, 1] : \hat{\nu}_\beta(x, x_0) = \hat{v} \right\} \right) = \left( \frac{1 - \hat{v}}{1 + \hat{v}} \right)^2.
\]
4. Proofs of Theorems C and D

Following the approach of Persson and Schmeling [14], we take a correspondence between the \( \beta \)-shift and the parameter. Parry [13, Lemma 2] characterized the \( \beta \)-expansion of 1.

**Theorem 4.1.** A sequence \((\omega_1, \omega_2, \ldots, \omega_n, \cdots)\) is the \( \beta \)-expansion of 1 if and only if it is self-admissible.

**Upper bound.** We consider a interval \((\beta_0, \beta_1)\), where \(1 < \beta_0 < \beta_1\). For \(v \in (0, \infty)\) and \(\hat{v} \in (0, 1)\), let

\[
\mathcal{L}_{v, \hat{v}} := \{\beta > 1 : \mathcal{V}_\beta(1, x_0) = v\} \cap \{\beta > 1 : \hat{\mathcal{V}}_\beta(1, x_0) = \hat{v}\},
\]

\[
\mathcal{L}_{v, \hat{v}}(\beta_0, \beta_1) := \{\beta \in (\beta_0, \beta_1) : \mathcal{V}_\beta(1, x_0) = v\} \cap \{\beta \in (\beta_0, \beta_1) : \hat{\mathcal{V}}_\beta(1, x_0) = \hat{v}\}.
\]

By Theorem 4.1, each self-admissible sequence corresponds to a real number \(\beta > 1\). Assume that \(S_{\beta_1}\) is the set of all self-admissible sequences in \(\Sigma_{\beta_1}\) and \(\pi_{\beta_1}\) is the natural projection from the \(\beta\)-shift to interval \([0, 1]\). Thus, there exists a one-to-one map \(\rho_{\beta_1} : \pi_{\beta_1}(S_{\beta_1}) \to (1, \beta_1)\).

Define the subset \(D_{v, \hat{v}}\) of \(\Sigma_{\beta_1}\) as

\[
\pi_{\beta_1}^{-1}\left(\{x \in [0, 1] : \mathcal{V}_\beta(x, x_0) = v\} \cap \{x \in [0, 1] : \hat{\mathcal{V}}_\beta(x, x_0) = \hat{v}\}\right).
\]

The Hölder exponent of the restriction of the map \(\rho_{\beta_1}\) on \(\pi_{\beta_1}(S_{\beta_1} \cap D_{v, \hat{v}})\) equals \(\log \beta_0 / \log \beta_1\). Since \(\mathcal{L}_{v, \hat{v}}(\beta_0, \beta_1) \subseteq \rho_{\beta_1}(\pi_{\beta_1}(S_{\beta_1} \cap D_{v, \hat{v}}))\),

\[
\dim_H \mathcal{L}_{v, \hat{v}}(\beta_0, \beta_1) \leq \dim_H \rho_{\beta_1}(\pi_{\beta_1}(S_{\beta_1} \cap D_{v, \hat{v}})) \leq \frac{\log \beta_1}{\log \beta_0} \dim_H \pi_{\beta_1}(S_{\beta_1} \cap D_{v, \hat{v}}).
\]

By Theorem A, letting \(\beta_1\) tend to \(\beta_0\), if \(v < \hat{v}/(1 - \hat{v})\), the set \(\mathcal{L}_{v, \hat{v}}(\beta_0, \beta_1)\) is empty. Otherwise,

\[
\dim_H \mathcal{L}_{v, \hat{v}}(\beta_0, \beta_1) \leq \frac{v - \hat{v} - v \hat{v}}{(1 + v)(v - \hat{v})}.
\]

**Lower bound.** Take \(\beta_2\) with \(1 < \beta_0 < \beta_1 < \beta_2\) such that the \(\beta_2\)-expansion of 1 ends with zeros. Thus, the \(\beta\)-shift \(\Sigma_{\beta_2}\) is a subshift of finite type. Bugeaud and Liao gave a way to calculate the lower bound of the Hausdorff dimension of \(\mathcal{L}_{v, \hat{v}}(\beta_0, \beta_1)\).

**Theorem 4.2.** (\([4, \text{Theorem 5.1}]\)) Given real numbers \(1 < \beta_0 < \beta_1 < \beta_2\). For any \(v \in (0, \infty)\) and any \(\hat{v} \in (0, 1)\) with \(v \geq \hat{v}/(1 - \hat{v})\), one has

\[
\dim_H \rho_{\beta_2}^{-1} \mathcal{L}_{v, \hat{v}}(\beta_0, \beta_1) \geq \frac{v - \hat{v} - v \cdot \hat{v}}{(1 + v)(v - \hat{v})} \cdot \frac{\log \beta_1}{\log \beta_2}.
\]

From Theorem 4.2 and Persson and Schmeling [14, Theorem 14], we have

\[
\dim_H \mathcal{L}_{v, \hat{v}}(\beta_0, \beta_1) \geq \frac{v - \hat{v} - v \cdot \hat{v}}{(1 + v)(v - \hat{v})} \cdot \frac{\log \beta_1}{\log \beta_2}.
\]

Letting \(\beta_2\) tend to \(\beta_1\), we obtain the lower bound. Thus, we complete the proof of Theorem C.
For the proof of Theorem D, one can follow from the proof of Theorem B. We omit the details.

REFERENCES

[1] M. Amou and Y. Bugeaud, Exponents of Diophantine approximation and expansions in integer bases, *J. Lond. Math. Soc.* (2), 81 (2010), no. 2, 297-316.

[2] A. Besicovitch, Sets of Fractional Dimensions (IV): On Rational Approximation to Real Numbers, *J. London Math. Soc.*, 9 (1934), no. 2, 126-131.

[3] Y. Bugeaud, Distribution modulo one and Diophantine approximation, Cambridge Tracts in Mathematics, vol. 193, Cambridge University Press, Cambridge, 2012.

[4] Y. Bugeaud and L. Liao, Uniform Diophantine approximation related to $b$-ary and $\beta$-expansions, *Ergodic Theory Dynam. Systems* 36 (2016), no. 1, 1-22.

[5] L. Dirichlet, Verallgemeinerung eines Satzes aus der Lehre von den Kettenbrüchen nebst einigen Anwendungen auf die Theorie der Zahlen, *SB Preuss. Akad. Wiss.* 1842 (1842), 93-95.

[6] K. Falconer, Fractal geometry, *John Wiley and Sons, Ltd.*, Chichester, 1990, Mathematical foundations and applications.

[7] A. Fan and B. Wang, On the lengths of basic intervals in beta expansions, *Nonlinearity* 25 (2012), no. 5, 1329-1343.

[8] R. Hill and S. Velani, The ergodic theory of shrinking targets, *Invent. Math.* 119 (1995), no. 1, 175-198.

[9] V. Jarník, Diophantische approximationen und haussorffsches mass, *Rec. Math. Moscou* 36 (1929), 371-382.

[10] A. Khintchine, Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen, *Math. Ann.* 92 (1924), no. 1-2, 115-125.

[11] A. Legendre, Essai sur la théorie des nombres, Cambridge Library Collection, Cambridge, 2009, Reprint of the second (1808) edition.

[12] B. Li, T. Persson, B. Wang, and J. Wu, Diophantine approximation of the orbit of 1 in the dynamical system of beta expansions, *Math. Z.* 276 (2014), no. 3-4, 799-827.

[13] W. Parry, On the $\beta$-expansions of real numbers, *Acta Math. Acad. Sci. Hungar.* 11 (1960), 401-416.

[14] T. Persson and J. Schmeling, Dyadic Diophantine approximation and Katok’s horse-shoe approximation, *Acta Arith.* 132 (2008), no. 3, 205-230.

[15] W. Philipp, Some metrical theorems in number theory, *Pacific J. Math.* 20 (1967), 109-127.

[16] A. Rényi, Representations for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hungar* 8 (1957), 477-493.

[17] L. Shen and B. Wang, Shrinking target problems for beta-dynamical system, *Sci. China Math.* 56 (2013), no. 1, 91-104.

[18] M. Waldschmidt, Recent advances in Diophantine approximation, Number theory, analysis and geometry, Springer, New York, 2012, pp. 659-704.

(Wanlou Wu) School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, Jiangsu, 221116, PR China

E-mail address: wuwanlou@163.com