Multilinear quantum Lie operations

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For Anatolii Vladimirovich YAKOVLEV
to 60th birthday

1 Introduction

The notion of multilinear quantum Lie operation appears naturally in connection
with a different attempts to generalize the Lie algebras. There are a number of
reasons why the generalizations are necessary. First of all this is the demands for a
"quantum algebra" which was formed in the papers by Ju.I. Manin, V.G. Drinfeld,
S.L. Woronowicz, G. Lusztig, L.D. Faddeev, and many others. These demands are
defined by a desire to keep the intuition of the quantum mechanics differential calculus
that is based on the fundamental concepts of the Lie groups and Lie algebras theory.

Normally the quest for definition of bilinear brackets on the module of differential 1-forms
that replace the Lie operation leads to restrictions like multiplicative skew-symmetry of
the quantization parameters [1], involutivity of braidings, or bicovariance of the differential
calculus [2], [3]. At the same time lots of quantizations, for example the Drinfeld–Jimbo one,
are defined by multiplicative non skew-symmetric parameters, and they define not bicovariant (but one-sided covariant) calculus. By these, and of course, by many other reasons,
the attention of researches has been extended on operations that replace the Lie brackets, but depend on greater number of variables. Such are, for example, $n$-Lie algebras introduced by V.T. Filippov [4]

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and then independently appeared under the name Nambu–Lie algebras in theoretical researches on generalizations of Nambu mechanics \[5\], \[6\]. Trilinear operation have been considered in the original Y. Namby paper \[7\], and also in a number of others papers on generalization of quantum mechanics (see, for example, in \[8\] a trilinear oscillator, or in \[9\], \[10\] multilinear commutator). Of course one should keep in mind that both multivariable and partial operations are the subject of investigation in the theory of algebraic systems located at the interfaces between algebra and mathematical logic, the algorithms theory and the computer calculations \[11\], \[12\].

Another group of problems (which is close to the author interests) that needs the generalization of Lie algebras is connected with researches of automorphisms and skew derivations of noncommutative algebras. A noncommutative version of the fundamental Dedekind algebraic independence lemma says that all algebraic dependencies in automorphisms and ordinary derivations are defined by their algebraic structure (that is a structure of a group acting on a Lie algebra) and by operators with ”inner” action (see \[13\], \[14\], Chapter 2).

The quest for extension this, very good working, result into the field of skew derivations inevitably leads to a question what algebraic structure corresponds to the skew derivation operators? This in tern forces to consider \(n\)-ary multilinear (partial) operations on the Yetter–Drinfeld modules, that are not reduced to the bilinear ones (see \[13\], section 6.5 or \[14\], section 6.14).

In fact, the systematic investigation of quantum Lie operations related to the Freiderichs criteria \[15\] have been started in the papers of B. Pareigis \[16\], \[17\], \[18\], where he has found a special series of the \(n\)-ary operations. Then it was continued in the author papers \[19\], \[20\], \[21\], where a criteria for existence of nontrivial quantum Lie operations is found and a version of the Poincarè–Birkhoff–Witt theorem for a class of algebras defined by the quantum Lie operations and having realization inside of Hopf algebras is proposed.

In the present paper by means of \[22\] we show that under the existence condition the dimension of the space of all \(n\)-linear quantum Lie operations is included between \((n – 2)!\) and \((n – 1)!\). The lower bound is achieved if the intersection of all conforming (that is satisfying the existence condition) subsets of a given set of ”quantum” variables is nonempty, while the upper bound does if the quantification matrix is multiplicative skew symmetric or, equivalently, all subsets are conforming. In the latter case, as well as in the case of ordinary Lie algebras, all \(n\)-linear operations are superpositions of the only bilinear quantum Lie operation, the colored super bracket.

It is interesting to note that even for \(n = 4\) not all values of the mentioned above
interval are achieved for different values of quantization parameters. Indeed, the
dimension of the quadrilinear operations space may have values 2, 3, 4, 6, while it is
never equal to 5 (see the proof of the second part of Theorem 8.4 [19]).

In the last section we show that almost always the quantum Lie operations space is
generated by symmetric ones, provided that the ideal of the quantization parameters
is invariant with respect to the permutation group action (only in this case the notion
of a symmetric operation makes sense). We show also all possible exceptions. The
space of ”general” n-linear quantum Lie operations is not an exception, that is this
space is always generated by symmetric ”general” quantum Lie operations.

2 Preliminaries

Recall that a quantum variable is a variable $x$, with which an element $g_x$ of a fixed
Abelian group $G$ and a character $\chi : G \to \mathbb{k}^*$ are associated, where $\mathbb{k}$ is a ground
field. A set of quantum variables is said to be conforming, if $\prod_{1 \leq i \neq j \leq n} p_{ij} = 1$, where
$||p_{ij}||$ is a matrix of quantization parameters: $p_{ij} = \chi^{x_i}(g_{x_j})$. A quantum operation
in quantum variables $x_1, \ldots, x_n$ (see [19]) is a noncommutative polynomial in these
variables that has skew primitive values in every Hopf algebra $H$, provided that $H$
contains the group Hopf algebra $k[G]$ and every variable $x_i$ has a skew primitive
semi-invariant value $x_i = a_i$:

$$\Delta(a_i) = a_i \otimes 1 + g_{x_i} \otimes a_i; \quad g^{-1}a_i g = \chi^{x_i}(g)a_i$$  \hspace{1cm} (1)

for all $g \in G$. A nonzero n-linear quantum operation exists if and only if the set
$x_1, \ldots, x_n$ is conforming, [20]. All the operations have a commutator representation

$$f(x_1, \ldots, x_n) = \sum_{\nu \in S_n^1} \beta_{\nu}[\ldots[[x_{\nu(1)}, x_{\nu(2)}], x_{\nu(3)}], \ldots, x_{\nu(n)}], \hspace{1cm} (2)$$

where $S_n^1$ is the permutation group of 2, \ldots, $n$, while the bracket is a skew commutator
$[u, v] = uv - p(u, v)vu$ with the bimultiplicative coefficient $p(u, v)$. This coefficient is
defined on the set of homogeneous polynomials by means of the quantization matrix:
$p(x_i, x_j) = p_{ij}$, see [19]. The skew commutator $[x_i, x_j]$ is a quantum operation if
and only if $p_{ij} p_{ji} = 1$. In particular if the quantization matrix is multiplicative skew
symmetric, then all multilinear polynomials of the form (2) are quantum operations.
Therefore in this particular case the dimension of the space of operations equals
$(n - 1)!$. 

3
In general, by Theorem 3 [20], the linear combination (2) is a quantum operation if and only if the coefficients $\beta_\nu$ satisfy the following system of equations

$$\sum_{\nu \in N^1(s)} \beta_\nu t^\mu_{\nu,s} = 0, \quad \mu \in S^1_n, 1 < s < n. \quad (3)$$

Here $N^1(s)$ is the set of all s-shuffle from $S^1_n$; that is the set of all permutations $\nu$ with

$$\nu^{-1}(2) < \nu^{-1}(3) < \ldots < \nu^{-1}(s); \quad \nu^{-1}(s + 1) < \nu^{-1}(s + 2) < \ldots < \nu^{-1}(n). \quad (4)$$

The coefficients $t^\mu_{\nu,s}$ are particular polynomials in $p_{ij}, p_{ij}^{-1}$, precisely, $t^\mu_{\nu,s} = \varphi(T^\mu_{\nu,s})$, see formula (10) below. The above system of equations can be rewritten in the form of relations in a crossed product.

3 Crossed product

Consider a free Abelian group $F_n$, freely generated by symbols $P_{ij}$, $1 \leq i \neq j \leq n$. Denote by $\mathcal{P}_n$ a group algebra of this group over the minimal subfield $F$ of the field $k$. Clearly, $\mathcal{P}_n$ has a field of fractions $Q_n$ that is isomorphic to the field of rational functions $F(P_{ij})$. The elements $P_{ij}$, $1 \leq i \neq j \leq m$ generate a subalgebra $\mathcal{P}_m$ of $\mathcal{P}_n$. The action of the symmetric group $S_n$ is correctly defined on the ring $\mathcal{P}_n$ and on the field $Q_n$ by $P_{ij}^\pi = P_{\pi(i)\pi(j)}$. Thus we can define a crossed product $Q_n * S_n$ (with a trivial factor-system). This crossed product is isomorphic to the algebra of all $n! \times n!$ matrices over the Galois field $Q_n^{S_n}$, and it contains the skew group algebra $\mathcal{P}_n * S_n$.

Recall that in the trivial crossed product the permutations commute with coefficients according to the formula $A\pi = \pi A^\pi$ (see [22], [23], [24] or other textbooks in ring theory).

If the parameters $p_{ij}$ are defined by the quantum variables $x_1, \ldots, x_n$, then there exists a uniquely defined homomorphism

$$\varphi : \mathcal{P}_n \to k, \quad \varphi(P_{ij}) = p_{ij}. \quad (5)$$

If $A \in F_n$ then by $\overline{A}$ we denote a word appearing from $A$ by replacing all letters $P_{ij}$ with $P_{ji}$. We call the words $A$ and $\overline{A}$ conjugated. We define

$$\{A\} = A - \overline{A}^{-1}. \quad (6)$$

For two arbitrary indices $m, k$ we denote by $[m; k]$ a monotonous cycle starting with $m$ up to $k$:

$$[m; k] \overset{\text{def}}{=} \begin{cases} (m, m + 1, \ldots, k), & \text{if } m \leq k \\ (m, m - 1, \ldots, k), & \text{if } m \geq k. \end{cases} \quad (7)$$
Clearly \([m; k]^{-1} = [k; m]\) in these denotations. It is easy to see that a permutation \(\nu\) belongs \(\mathcal{N}^1(s)\) if and only if
\[
\nu = [2; k_2][3; k_3] \cdots [s; k_s] \quad (8)
\]
for a sequence of indices \(1 < k_2 < k_3 < \ldots k_s \leq n\) (see Lemma 1 in [20]).

Let us fix the following denotations for particular elements of the skew group algebra
\[
V_s = \sum_{\nu \in \mathcal{N}^1(s)} \nu T_{\nu,s} = \sum_{1 < k_2 < k_3 < \ldots < k_s \leq n} [2; k_2][3; k_3][4; k_4] \cdots [s; k_s] T_{k_2;k_3;\ldots;k_s}, \quad (9)
\]
where
\[
T_{\nu,s} = T_{k_2;k_3;\ldots;k_s} = \left\{ \prod_{m=2}^{m=s} (P_{1 \ m} \cdots P_{m-1 \ m} \cdot P_{s+1 \ m} \cdots P_{s-m+k_m \ m}) \right\}. \quad (10)
\]

The braces are defined in (6).

**Theorem 3.1.** If \(\prod_{i \neq j} p_{ij} = 1\), then (2) is a quantum operation if and only if
\[
\sum_{\nu} \beta_{\nu} \nu^{-1} \cdot V_s \in \ker(\varphi) \mathcal{S}^1_n \quad (11)
\]
for all \(s, 2 \leq s < n\), where \(V_s\) are defined in (9).

**PROOF.** See Theorem 4 [20].

Let \(\Sigma\) be an arbitrary multiplicative subset of \(\mathcal{P}_n\) that does not intersect \(\ker(\varphi)\). Consider a localization (ring of quotients) \(\mathcal{P}_n \Sigma^{-1}\). The homomorphism \(\varphi\) has a unique extension up to a homomorphism of \(\mathcal{P}_n \Sigma^{-1}\) into the field \(\mathbb{F}(p_{ij})\) via
\[
\varphi(P\sigma^{-1}) = \varphi(P)\varphi(\sigma)^{-1}, \quad P \in \mathcal{P}_n. \quad (12)
\]

**Theorem 3.2.** If \(\prod_{i \neq j} p_{ij} = 1\) and \(\Sigma\) is an arbitrary multiplicative subset of \(\mathcal{P}_n\) that do not intersect \(\ker(\varphi)\), then an element \(B = \sum_{\nu \in \mathcal{S}^1_n} B_{\nu} \nu^{-1}\) with coefficients in \(\mathcal{P}_n \Sigma^{-1}\) defines a quantum operation \(\sum_{\nu \in \mathcal{S}^1_n} \varphi(B_{\nu})[\ldots [x_{\nu(2)}, x_{\nu(3)}], \ldots, x_{\nu(n)}]\) if and only if
\[
B \cdot V_s \in \ker(\varphi) \Sigma^{-1} \mathcal{S}^1_n \quad (13)
\]
for all \(s, 2 \leq s < n\).
Proof. It is enough to multiply $B$ from the left by a common denominator $\sigma \in \Sigma$ of all coefficients and to apply Theorem 3.1.

Recall that a conforming ideal is an ideal $I$ of the algebra $P_n$, generated by all elements of the form $\{W\}$, where $W$ is an arbitrary semigroup word in $P_{ij}$ of length $C^2_n$ that has neither double nor conjugated letters. It is easy to see that the variables $x_1, \ldots, x_n$ are conforming if and only if the ideal ker($\varphi$) contains the conforming ideal.

It is very important to note that the conforming ideal $I$ is invariant with respect to the action of $S_n$ (unlike the ideal ker($\varphi$) in general). Therefore the two-sided ideal of $P_n \ast S_n$ generated by $I$ coincides the right ideal $IS_n$.

Consider a field of rational functions $K$ over $F$ in $C^2_n - 1$ variables $P_{ij}$, $1 \leq i \neq j \leq n$, $(i, j) \neq (1, n)$. Denote $P_{1n} = (\prod_{(k,l)\neq(1,n)} P_{kl})^{-1}$.

$$P_{1n} = (\prod_{(k,l)\neq(1,n)} P_{kl})^{-1}. \quad (14)$$

In this case the map $\xi : P_{ij} \to P_{ij}$ defines an embedding of $P_n/I$ in $K$.

Consider a new set of quantum variables $X_1, X_2, \ldots, X_n$ with which free generators $G_1, \ldots, G_n$ of a free Abelian group $G$ are associated, while the characters over $K$ are defined by $\chi^{X_i}(G_j) = P_{ij}$.

Definition 3.3. The system (3) with $p_{ij} := P_{ij}$ is said to be the general basic system. Its solutions define general quantum operations in $X_1, X_2, \ldots, X_n$ with coefficients in $K$.

Denote by $\Sigma_0$ a set of all elements $f \in P_n$ that do not belong to $I$. This is multiplicative set. By Theorem 3.2 an element $B$ corresponds to a general quantum operation if and only if $B \cdot V_s \in I\Sigma^{-1}_0 * S_n^1$. If it is possible to extend the homomorphism $\varphi$ to the coefficients of $B$, then the general quantum operation defines a quantum operation over the ground field.

Lemma 3.4. Let $B$ corresponds to a general quantum operation. If $\prod_{1 \leq i \neq j \leq n} p_{ij} = 1$ and the coefficients $B_r$ of $B$ belong to $P_n \Sigma^{-1}$ for a set $\Sigma$ that does not intersect ker($\varphi$), than $B$ defines a quantum operation in $x_1, \ldots, x_n$ over $F(p_{ij})$.

Proof. By theorem 3.2 it is enough to show that $B \cdot V_s \in I\Sigma^{-1} S_n^1$. The left hand side of the above inclusion belongs $I\Sigma_0^{-1} S_n^1 \cap P_n \Sigma^{-1} S_n^1$. This intersection equals the right hand side. Indeed, if $i \sigma_0^{-1} = P \sigma_0^{-1}$, then $i \sigma = P \sigma_0$. Since $I$ is simple ideal and $\sigma_0 \notin I$, we get $P \in I$. \n
4 Decreasing modules

Recall that by $S_n^{k,l,...}$ we denote a group of permutations $\nu \in S_n$ with $\nu(k) = k, \nu(l) = l, \ldots$. A right module over $\mathcal{P}_n * S_n^{1,n}$ generated by elements $A_2, \ldots A_n$ is called a decreasing module, provided that the generators are connected by the following relation

$$A_k V(k) + A_{k-1} D(k) = 0, \quad 3 \leq k \leq n,$$

where $V(k)$ and $D(k)$ are defined by

$$V(k) = \sum_{l=2}^{k-1} [2; l] T_{[2;l]}.$$

$$D(k) = \tau_k T_{[2;n]}^{(2,n)} + \sum_{l=2}^{n-1} [2; l] T_{[2;l]}^{(k,n)}.$$

Here $\tau_k$, is set by

$$\tau_k = \begin{cases} [2; k - 1][k; n - 1], & \text{if } 2 < k < n, \\ [2; n - 1], & \text{if } k = n, 2 \end{cases}$$

(see Definition 8 and formulae (35), (39), (40) in [20]).

For an arbitrary

$$B = A_2(2, n) + A_3(3, n) + \ldots + A_{n-1}(n-1, n) + A_n \text{id},$$

where $A_i \in \mathcal{P}_n * S_n^{1,n}$, the following formula is correct.

$$B \cdot V_2 = D_2(2, n) + \sum_{k=3}^{n} (A_k V(k) + A_{k-1} D(k))(k, n),$$

where

$$D_2 \overset{\text{def}}{=} A_n[2; n - 1] T_{[2;n]}^{(2,n)} + \sum_{l=2}^{n-1} A_l[2; l] T_{[2;l]}^{(2,n)}.$$ 

Therefore the equality $B \cdot V_2 \in \ker(\varphi)S_n^1$ means that both $D_2 \equiv 0$ in the quotient module $\mathcal{P}_n * S_n^1/\ker(\varphi)S_n^1$, and $A_2, A_3, \ldots, A_n$ generate a decreasing module in this quotient module (see details in [20], section 5).

Note that the element $V_{(k+1)}$ equals the element $V_2$ that is defined by $x_1, x_2, \ldots, x_k$. Therefore the following statement shows in particular that $D_2$ equals zero in the quotient module $\mathcal{P}_n * S_n^1/I(\varphi)S_n^1$, provided that the elements $V_2$ for the proper subsets are invertible.
Theorem 4.1 (Theorem 6 [20]). Let \( n \geq 3 \) and

\[
X = \left( \prod_{n>i>j>1} P_{ij} \right) \left( \prod_{i=1}^{n-1} P_{in} \right) \left( \prod_{j=2}^{n-1} P_{1j} \right).
\]

(22)

In every decreasing module the following relation is correct

\[
D_2 \prod_{k=0}^{n-3} [n-1;2]V_{(n-k)} = A_2[n-1;2] \prod_{k=0}^{n-3} [2;2+k]V_{(n-k)}^{(1,n)[3;3-1]k} \cdot \{X\}^{2;n-1}(-1)^n,
\]

(23)

where \( D_2 \) is defined by (21) by replacing \( A \) with \( A \).

5 Second subcomponents

Suppose now that the intersection of all conforming subsets is not empty. Without loss of generality we may suppose that \( x_n \) belongs to all conforming subsets. For every subset \( Y \subseteq \{x_1, x_2, \ldots, x_n\} \) we denote by \( W(Y) \) the set of all elements of the form \( \{W\} \), where \( W \) is a semigroup word in \( P_{ij}, x_i, x_j \in Y \) of the length \( C_{[Y]}^2 \) that does not contain neither double nor conjugated letters. The set \( Y \) is conforming if and only if \( W(Y) \subseteq \text{ker}(\varphi) \). Denote by \( \Sigma_m, m \leq n \) a multiplicative set generated by \( W(Y) \), where \( Y \) runs through collection of all subsets that do not contain no one of the letters \( x_m, x_{m+1}, \ldots, x_n \). The set \( \Sigma_m \) is invariant with respect to the action of the subgroup \( S_{m-1}^1 \). This set does not intersect \( \text{ker}(\varphi) \), since \( \text{ker}(\varphi) \) is a simple ideal and \( x_n \) belongs to all conforming subsets. We may define both a localization \( \mathcal{P}_{n \Sigma_m}^{-1} \) and a skew group ring \( \mathcal{P}_{n \Sigma_m}^{-1} * S_{m-1}^1 \), that are contained in the crossed product.

Lemma 5.1. If \( x_n \) belongs to all conforming subsets, then the elements \( V_{(m)} \), \( 3 \leq m \leq n \), defined by (11) are invertible in \( \mathcal{P}_{n \Sigma_m}^{-1} * S_{m-1}^1 \).

PROOF. Let us use the induction by \( m \). By definition the set \( \Sigma_3 \) contains \( \{P_{12}\} \). Therefore \( V_{(3)}^{-1} = \{P_{12}\}^{-1} \cdot \text{id} \in \mathcal{P}_{n \Sigma_3}^{-1} * S_2^1 \).

Let each of \( V_{(k)}, \ 3 \leq k \leq m \) is invertible in \( \mathcal{P}_{n \Sigma_k}^{-1} * S_{k-1}^1 \). This is invertible in \( \mathcal{P}_{n \Sigma_{m+1}}^{-1} * S_m^1 \) as well. Consider the set of quantum variables \( x_1, \ldots, x_m \). The element \( V_2 \) for this set coincides \( V_{(m+1)} \). Let us replace denotations \( n \) by \( N \) and \( m \) by \( n \). It is sufficient to show that there exists an element \( B \in \mathcal{P}_N \Sigma_{n+1}^{-1} * S_n^1 \) such that \( B \cdot V_2 \in S_n^1 \) (now \( V_2 = \sum_{l=2}^{n}[2;l]T[2;l] \) and \( V_{(3)}, \ldots, V_{(n)} \) are invertible in \( \mathcal{P}_N \Sigma_{n+1}^{-1} * S_n^1 \)). Define by induction a sequence \( A_2, \ldots, A_n \in \mathcal{P}_N \Sigma_{n+1}^{-1} * S_n^1 \). Let

\[
A_2 = (-1)^n \prod_{k=0}^{n-3} [n-1;2]V_{(n-k)} \times
\]

8
\[(n - 1; 2) \prod_{k=0}^{n-3} [2; 2 + k] V_{(n-k)}^{(1,n) [3; n-1] k} \cdot \{X\}^{[2; n-1]} - 1, \]  

(24)

where \(X\) is defined by (22). All of the factors in the parentheses are invertible in \(P \Sigma_{n+1}^{-1} S_n^1\); the \(V\)'s are by the inductive suppositions, and \(\{X\}\) is due to this element belongs \(\Sigma_{n+1}\). Furthermore, let

\[A_k = -A_{k-1} D_{(k)} V_{(k)}^{-1}, \quad 3 \leq k \leq n,\]  

(25)

where \(D_{(k)}\) are defined by (17).

By right multiplication of the above equality by \(V_{(k)}\), we get that \(A_2, \ldots, A_n\) generate a right decreasing submodule over \(P_n \ast S_n^1\). Therefore we may use Theorem 4.1. By (23) and (24) we get

\[D_2 \prod_{k=0}^{n-3} [n - 1; 2] V_{(n-k)} = \prod_{k=0}^{n-3} [n - 1; 2] V_{(n-k)}.\]

Thus the element \(D_2\) defined by (24) with \(A\) in place of \(A\) equals the identity permutation. Consider an element \(B\) defined by (19) with \(A\) in place of \(A\):

\[B = A_2(2, n) + A_3(3, n) + \ldots + A_n \text{id}.\]

Formula (24) and definition (25) show that

\[B \cdot V_2 = D_2(2, n) + \sum_{k=3}^{n} (A_k V_{(k)} + A_{k-1} D_{(k)}) (k, n) = (2, n).\]

The lemma is proved.

\#

6 Interval of dimensions

**Theorem 6.1.** If \(x_1, \ldots, x_n\) is a conforming set of different quantum variables, then the dimension of the space of all multilinear quantum Lie operations in this set is greater then or equal to \((n - 2)!\). If there exists a variable that belongs to each conforming subset then this dimension equals \((n - 2)!\).

**Proof.** In order to prove the first part of the theorem it is enough to show that the rank of the basic system is less then or equal to \((n - 1)! - (n - 2)! = (n - 2)!(n - 2)\). This condition is equivalent to all minors of the order greater then or equal to \((n - 2)!(n - 2)\) being zero. Since the minors are integer functions in the matrix coefficients, it is enough to show that this condition is valid for the general
basic system (see Definition 3.3). The parameters $P_{ij}$ of this system are connected by the only relation. Therefore the set $X_1, \ldots, X_n$ has no proper conforming subsets. By Theorem 10 [20] we get the required condition.

The second statement follows the invertibility of the second subcomponents.

Lemma 6.2. If $V(3), V(4), \ldots, V(n)$ are invertible in $P_n \Sigma^{-1} S^{1,n}_n$, where $\Sigma = P_n \setminus \ker(\varphi)$, then the dimension of the space of all quantum Lie operations is less than or equal to $(n - 2)!$.

Let the element $B \sum B_\nu \nu^{-1}$ corresponds to a quantum Lie operation (2) $\varphi(B_\nu) = \beta_\nu$. Then $B \cdot V_2 \equiv 0$. By the decomposition of $B$ in cosets (19), we see that the elements $A_i$ generate a right decreasing submodule in $P_n * S^{1,n}_n / \ker(\varphi) S^{1,n}_n$. Therefore

$$B \equiv A_2 \sum_{k=2}^{n} (-1)^k (\prod_{i=3}^{k} D(i) V^{-1}(i))(k, n).$$

Thus, the composition $f \rightarrow B \rightarrow A_2 \rightarrow F(p_{ij}) S^{1,n}_n$, where $B = \sum \beta_\nu \nu^{-1}$ (see (2)), is a linear transformation with zero kernel of the space of multilinear quantum Lie operations into the space of left linear combinations $\sum_{\mu \in S^{1,n}_n} \alpha_{\mu} \mu$ over $F(p_{ij})$. Therefore the dimension is less than or equal to the number of elements in $S^{1,n}_n$. The theorem is proved.

7 Symmetric quantum Lie operations
and symmetric sets of variables

An operation $[x_1, \ldots, x_n]$ is called symmetric (or skew symmetric) if for every permutation $\pi \in S_n$ the following equality is valid

$$[x_{\pi(1)}, \ldots, x_{\pi(n)}] = \alpha_\pi [x_1, \ldots, x_n],$$

where $\alpha_\pi \in k$. In the case of quantum operations, as well as in the case of arbitrary partial operations, we have to explain what does it mean the left hand side of (27). Strictly speaking the left hand side is defined only if $x_{\pi(i)}$ has the same parameters $\chi, g$ as $x_i$ does. By definition only in this case the substitution $x_i := x_{\pi(i)}$ is admissible. In other word all the parameters $p_{ij}$ should be equal to the same number $q$. This is very rigid condition. It excludes both the colored superbrackets and the Paregis operations with the above defined general ones.

However, we may suppose that $[x_1, \ldots, x_n]$ is a polynomial whose coefficients depend on the quantization parameters, $\chi^{x_i}, g_{x_i}$, that is there are shown distinguished
entries of $p_{ij}$ in the coefficients. Then a substitution $x_i \rightarrow y$ means not only the substitution of the variable but also one of the parameters $g_i \rightarrow g_y$, $\chi^{x_i} \rightarrow \chi^y$. In particular, the permutation of parameters in (27) means the application of this permutation to all indices: $p_{ij} \rightarrow p_{\pi(i)\pi(j)}$.

This interpretation of the equality (27) will be unique and uncontradictable only if the application of the permutation is independent of the way how the coefficients $[x_1, \ldots, x_n]$ are represented as rational functions or polynomials in $p_{ij}, p_{ij}^{-1}$. The action of permutations is independent of the above representation if (and only if) ker($\varphi$) is invariant ideal with respect to the action of $S_n$.

**Definition 7.1.** A collection of quantum variables $x_1, \ldots, x_n$ is said to be symmetric, if ker($\varphi$) is an invariant ideal with respect to $S_n$ or, equivalently, the formula $p_{\pi ij} = p_{\pi(i)\pi(j)}$ correctly defines the action of $S_n$ on the ring $F[p_{ij}]$.

Note that the symmetricity of the collection has nothing to do with the symmetricity of the quantization matrix, $||p_{ij}||$, while it means the symmetricity of relations between the parameters.

Thus, in order to give a sense to the term ”symmetric operation” we should first suppose that the coefficients of the operation belong to the field $F(p_{ij})$, which does not affect the generality, (see [20], p. 194); and then we should consider only symmetric sets of quantum variables. This is a restrictable condition, yet. Nevertheless this condition excludes no one of the above examples. Moreover, the existence of the symmetric set $X_1$ with cover parameters $F_{ij} \rightarrow p_{ij}$ is a key argument of the proof of both the existence theorem and its corollaries. Therefore the symmetric collections of variables are of special interest.

Consider a symmetric polynomial over $F(p_{ij})$:

$$ f(x_1, \ldots, x_n) = \sum \gamma_{\mu} x_{\mu(1)} \cdots x_{\mu(n)}. \quad (28) $$

Without loss of generality (if necessary by applying a permutation) we may suppose that the monomial $x_1 x_2 \cdots x_n$ has a coefficient 1. Let us compare coefficients at $x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}$ in the both sides of (27). Taking into account (28), we immediately get $\alpha_\pi = \gamma_{\pi^{-1}}$. Afterwards the equality (27) takes a form

$$ \gamma_{\pi^{-1}} \sum_{\mu \in S_n} \gamma_{\mu} x_{\mu(1)} \cdots x_{\mu(n)} = \left( \sum_{\mu \in S_n} \gamma_{\mu} x_{\mu(1)} \cdots x_{\mu(n)} \right)^\pi $$

$$ = \sum_{\mu \in S_n} \gamma_{\mu}^\pi x_{\pi(\mu(1))} \cdots x_{\pi(\mu(n))} = \sum_{\nu \in S_n} \gamma_{\nu \pi^{-1}} x_{\nu(1)} \cdots x_{\nu(n)}. \quad (29) $$
From here $\gamma_{\mu\pi^{-1}} = \gamma_{\pi^{-1}\mu}$. Let us replace $\nu = \pi^{-1}$ and then apply $\nu$ to the both sides of the above equality. We get that the normed polynomial (28) is symmetric if and only if

$$\gamma_{\mu
u} = \gamma_{\mu
u}, \quad \text{with} \quad \alpha_{\pi} = \gamma_{\pi^{-1}} = \gamma_{\pi}^{-1}. \quad (30)$$

To put it another way, the set of normed symmetric polynomials can be identified with the first cogomology group $H^1(S_n, F(p_{ij})^*)$ with values in the multiplicative group of $F(p_{ij})$. Now a natural question arises: is the space of multilinear quantum Lie operations generated by the symmetrical ones, provided that the variables form a symmetric set?

We start with some examples.

**EXAMPLES.** Let the set of variables is absolutely symmetric, that is $p_{ij} = q$. In this case the group $S_n$ acts identically on the field $F(p_{ij})$. Therefore there exists only two symmetric polynomials:

$$S(x_1, \ldots, x_n) = \sum_{\pi \in S_n} x_{\pi(1)} \cdots x_{\pi(n)}, \quad (31)$$

$$T(x_1, \ldots, x_n) = \sum_{\pi \in S_n} (-1)^{\pi} x_{\pi(1)} \cdots x_{\pi(n)}. \quad (32)$$

On the other hand, if the existence condition, $q^{n(n-1)} = 1$, is valid then, according to Theorem 6.1, the dimension $l$ of the multilinear operations space can not be less than $(n - 2)!$. Thus, if $n > 4$, or if $n = 4$ and the characteristic of the ground field equals 2, then wittingly the basis consisting of the symmetric operations does not exist.

If $n = 4$ then we may use the analysis from [19] (see the proof of Theorem 8.4): $l = 2$ only if $q^{12} = 1$, $q^6 \neq 1$, $q^4 \neq 1$, or, equivalently, $q^6 = -1$, $q^2 \neq -1$. If under these conditions the polynomials $S, T$ are quantum operations, then they should be expressed through the main quadrilinear operation (see [19], formula (57)) with the coefficients equal to ones at $x_1 x_2 x_3 x_4$ and $x_1 x_3 x_2 x_4$. That is

$$S = [x_1, x_2, x_3, x_4] + [x_1, x_3, x_2, x_4],$$

$$T = [x_1, x_2, x_3, x_4] - [x_1, x_3, x_2, x_4].$$

The sum of these equalities shows that all coefficients of the main quadrilinear operation at monomials corresponding to odd permutations have to be equal to zero. Alternatively, the explicit formula (56), [19] shows that the coefficient at $x_1 x_2 x_4 x_3$ equals

$$-\frac{\{p_{13}p_{23}\}}{\{p_{13}p_{23}p_{43}\}} = -\frac{q^2 - q^{-2}}{q^3 - q^{-3}} \neq 0,$$
since \( q^4 \neq 1 \). Thus in this case the symmetric basis neither exists.

If \( n = 3 \), the existence condition takes up the form \( q^6 = 1 \). If \( q \neq \pm 1 \), then there exists only one trilinear operation up to a scalar multiplication, and this operation is symmetric (see Theorem 8.1 and formula (46), [13]). While if \( q = \pm 1 \) then the operation space is generated by two polynomials: \([x_1, x_2, x_3]\) and \([x_1, x_3, x_2]\. If both \( S, T \) are quantum operations, then as above we get a contradiction \( S + T = 2[[x_1, x_2], x_3] \).

**Lemma 7.2.** Let the quantization matrix of a symmetric quadruple of quantum variables has the form

\[
||p_{ij}|| = \begin{pmatrix}
* & p & q & s \\
p & * & s & q \\
q & s & * & p \\
s & q & p & *
\end{pmatrix},
\]

where \( p, q, s \) are pairwise different and \( p^2q^2s^2 = 1 \).

1. If the characteristic of the field \( k \) is not equal to 2 then there do not exist nonzero quadrilinear symmetric quantum Lie operations at all.

2. If the characteristic equals 2 then there exist not more then two linearly independent quadrilinear symmetric operations.

3. In both cases the dimension of the whole space of quadrilinear quantum Lie operations equals three.

**PROOF.** If the parameter matrix has the form (33), then the action of the group \( S_4 \) on the field \( \mathbf{F}(p_{ij}) \) is not faithful. The kernel of this action wittingly includes the following four elements

\[ \text{id}; a = (12)(24); b = (13)(24); c = (14)(23). \] (34)

These elements form a normal subgroup \( H \triangleleft S_4 \) isomorphic to \( Z_2 \times Z_2 \). Let

\[
S = \sum_{\pi \in S_4} \gamma_{\pi x_\pi(1)x_\pi(2)x_\pi(3)x_\pi(4)}
\]

be some symmetric quantum operation, \( \gamma_{id} = 1 \). According to (30) with \( h = \mu = \nu \in H \) we have \( \gamma_{h}^2 = \gamma_{h^2} = \gamma_{id} = 1 \), that is \( \gamma_{h} = \pm 1 \in \mathbf{F} \). Moreover, all of the elements \( \gamma_{h} \), \( h \neq id, h \in H \) may not be equal to \(-1\), since, again by (30), the product of every two of them equals the third one. On the other hand formula (30) with \( h \in H, g \in S_4 \) implies \( \gamma_{g^{-1}}^g \gamma_{g} = 1 \) and

\[
\gamma_{g^{-1}} h g = \gamma_{g^{-1}}^h \gamma_{g} = \gamma_{g^{-1}}^h \gamma_{g}^g \gamma_{g} = \gamma_{h}^g = \gamma_{h}.
\]
Therefore all of $\gamma_h, h \in H$ equal each other and equal to 1.

Furthermore, the polynomial $S$, as well as any other quantum Lie operation, has a commutator representation (36):

$$S = \sum_{\nu \in S^4} \beta_\nu [[[x_1, x_{\nu(2)}], x_{\nu(3)}], x_{\nu(4)}].$$

If we compare coefficients at monomials $x_1 x_2 x_3 x_4$ and $x_4 x_3 x_2 x_1$, we get $1 = \gamma_{id} = \beta_{id}$ and $1 = \gamma_{(14)(23)} = \beta_{id}(-p_{12})(-p_{13}p_{23})(-p_{14}p_{24}p_{34}) = -p^2 q^2 s^2 = -1$. This completes the first statement.

In both cases the condition $p^2 q^2 s^2 = 1$ means that all three element subsets of the given quadruple are conforming. If some pair of them does as well, say $1 = p_{12}p_{21} = p^2$, then by symmetricity all others pairs are conforming too, that is $q^2 = s^2 = 1$. In this case $p, q, s \in F$. Thus $p = p^{(23)} = q = q^{(34)} = s$. This is contradiction with the lemma condition. Therefore by Theorem 8.4 [19] (see the second case in the proof of the second part) the quadrilinear quantum Lie operations space is generated by the following three polynomials

$$[W, x_4]; [W^\sigma, x_1]; [W^{\sigma^2}, x_2],$$

where $\sigma = (1234)$ is the cyclic permutation, while $W$ is the main trilinear operation in $x_1, x_2, x_3$. By the definition of this operation, see [19] formula (45), in the case of the characteristic 2, we get

$$W = (x_1 x_2 x_3 + x_3 x_2 x_1) + \frac{p + p^{-1}}{q + q^{-1}} (x_2 x_3 x_1 + x_1 x_3 x_2) \frac{s + s^{-1}}{q + q^{-1}} (x_3 x_2 x_1 + x_2 x_1 x_3).$$

Let

$$S = \xi [W, x_4] + \xi_1 [W^\sigma, x_1] + \xi_2 [W^{\sigma^2}, x_2].$$

If we compare the coefficients at the monomials $x_1 x_2 x_3 x_4$ and $x_2 x_1 x_4 x_3$, we get $\xi + \xi_1 = \gamma_{id} = 1$, $\xi_2 = \gamma_{(12)(34)} = 1$. Therefore

$$S = \xi ([W, x_4] + [W^\sigma, x_1]) + ([W^\sigma, x_1] + [W^{\sigma^2}, x_2]).$$

Thus the symmetric operations generate not more then two-dimensional subspace. The lemma is proved.

\textbf{Theorem 7.3}. If $x_1, x_2, \ldots, x_n$ is a symmetric but not absolutely symmetric collection of quantum variables, then the multilinear quantum Lie operations space is generated by symmetric operations, with the only exception given in the lemma 7.2.
PROOF. Consider the skew group algebra $M = F(p_{ij}) \ast S_n$. The permutation group action defines a structure of right $M$-module on the set of multilinear polynomials:

$$\sum \gamma_{\pi} x_{\pi(1)} \cdots x_{\pi(n)} \cdot \sum \beta_{\nu} \nu = \sum_{\nu, \pi} \beta_{\nu} \gamma_{\nu} x_{\nu(\pi(1))} \cdots x_{\nu(\pi(n))}. \quad (41)$$

A polynomial (28) is symmetric if and only if it generates a submodule of dimension one over $F(p_{ij})$.

Note that the quantum Lie operations space form a right $M$-submodule, provided that the collection of variables is symmetric. Indeed, let the system (3) be fulfilled for the coefficients of a polynomial $f$ defined by (2). The application of a permutation $\pi \in S_n$ to this system shows that the coefficients of the polynomial $f_{\pi}$ satisfy the same system up to rename of the variables $x_i \to x_{\pi(i)}$. Therefore $f_{\pi}$, $\pi \in S_n$ is a quantum Lie operation. If we replace the roles of indices 1 with 2, we get that $f_{\pi}$, $\pi \in S_n$ is a quantum Lie operation as well. Since the subgroups $S_n$ and $S_n^2$ with $n > 2$ generate $S_n$, all multilinear quantum Lie operations form an $M$-submodule.

Suppose now that the $S_n$ action on the field $F(p_{ij})$ is faithful. In this case $M$ is isomorphic to the trivial crossed product of the field $F(p_{ij})$ with the Galois group $S_n$. Consequently $M$ is isomorphic to the algebra of $n!$ by $n!$ matrices over the Galois subfield $F_1 = F(p_{ij})^{S_n}$. Therefore each right $M$-module equals a direct sum of irreducible submodules, while all irreducible submodules are isomorphic to the $n!$-rows module over the Galois field $F_1$. On the other hand, the dimension of $F(p_{ij})$ over $F_1$ equals $n!$ too. Since every right $M$-module is a right space over $F(p_{ij})$, all irreducible right $M$-modules are of dimension one over $F(p_{ij})$. This proves the theorem in the case of a faithful action.

If $n > 4$ or $n = 3$, while the action is not faithful, then all even permutations act identically. This immediately implies that the collection of variables is absolutely symmetric, $p_{ij} = q$.

Let $n = 4$. If the action is not faithful then all the permutations (34) act identically. This implies the parameter matrix has the form (33). The existence condition for quantum Lie operations is $p^4q^4s^4 = 1$. If $p^2q^2s^2 = 1$ then we get the lemma 7.2 exception. Therefore suppose that $p^2q^2s^2 = -1 \neq 1$.

If $p, q, s$ are pairwise different, then $S_n^1$ acts faithfully on $F(p, q, s)$. Therefore $M_1 = F(p, q, s) \ast S_n^1$ is the algebra of 6 by 6 matrices over the Galois field $F_1$. This is central simple algebra. Thus it splits in $M$ as a tensor factor $M = M_1 \otimes Z_1$, where $Z_1$ is a centralizer of $M_1$ in $M$. Let us calculate this centralizer.

First of all, $Z_1$ is contained in the centralizer of $F(p, q, s)$, that equals the group algebra $A = F(p, q, s)[\text{id}, a, b, c]$. This group algebra has a decomposition in a direct
sum of ideals

\[ A = F(p, q, s)e_1 \oplus F(p, q, s)e_2 \oplus F(p, q, s)e_3 \oplus F(p, q, s)e_4, \]

where \( e_1 = \frac{1}{4}(\text{id} + a + b + c) \), \( e_2 = \frac{1}{4}(\text{id} + a - b + c) \), \( e_3 = e_2^{(23)} \), \( e_4 = e_2^{(34)} \). The stabilizer of \( e_2 \) in \( S_4^{1,3} \) equals a two-element subgroup \( S_4^{1,3} \). Let \( F_2 = F(p, q, s)^{S_4^{1,3}} \) be a Galois subfield of this subgroup. Then \( Z_1 \) equals the centralizer of \( S_4^{1,3} \) in \( A \). This consists of sums

\[ \alpha e_1 + \beta e_2 + \beta^{(23)} e_3 + \beta^{(34)} e_4, \quad \alpha \in F_1, \beta \in F_2. \]

Thus, \( Z_1 \cong F_1 \oplus F_2 \). Consequently, \( M \cong (F_1)_{6 \times 6} \oplus (F_2)_{6 \times 6} \).

This means that up to isomorphism there exists just two irreducible right modules over \( M \). One of them equals the 6-rows space over \( F_1 \), while another one equals the 6-rows space over \( F_2 \). The dimensions of these modules over \( F_1 \) equal to respectively 6 and 18. Therefore, the first module is of dimension one over \( F(p, q, s) \), while the second one is of dimension three. By Theorem 8.4 \[19\] the quantum Lie operation module is of dimension two over \( F(p, q, s) \). Therefore its irreducible submodules may not be of dimension three. Thus all of them are of dimension one. The theorem is proved.

**Corollary 7.4.** There exists a collection of \((n - 2)\)! general symmetric multilinear quantum Lie operations that generates the space of all the operations.

The same statement is valid for quantum variables considered by Pareigis in \[16\] as well, that is in the case when the quantization parameters are related by \( p_{ij}p_{ji} = \zeta^2 \), where \( \zeta \) is a \( n \)th primitive root of 1.

**Corollary 7.5.** The total number of linearly independent symmetric multilinear quantum Lie operations for symmetric, but not absolutely symmetric, Pareigis quantum variables is greater than or equal to \((n - 2)\)!

**References**

[1] C. Bautista, A Poincarè–Birkhoff–Witt theorem for generalized color algebras, Journal of Mathematical Physics, 39, N7(1998), 3829–3843.

[2] S.L. Woronowicz, Compact matrix pseudogroups, Communications in Math. Phys., 111(1985), 613–665.
[3] S.L. Woronowicz, Differential calculus on compact matrix pseudogroups, Communications in Math. Phys., 122(1989), 125–170.

[4] V.T. Filippov, $n$-Lie algebras, Sib. Math. Journal, 26, N6(1985), 126–140.

[5] L. Takhtajan, On foundations of the generalized Nambu mechanics, Communications in Math. Phys., 160(1994), 295–315.

[6] G. Dito, M. Flato, D. Sternheimer, L. Takhtajan, Deformation quantization and Nambu mechanics, Communications in Math. Phys.,183(1997), 1–22.

[7] Y. Nambu, Generalized Hamilton dynamics, Physical Review D, 7, N8(1873), 2405–24–12.

[8] R.M. Yamaleev, Elements of cubic quantum mechanics, Communications of JINR, Dubna, 2-88-147, 1988, 1–11.

[9] R.M. Yamaleev, Model of plylinear oscillator in a space of non-integer quantum numbers. Communications of JINR, Dubna, 2-88-871, 1988, 1–10.

[10] R.M. Yamaleev, Model of polylinear Bose- and Fermi-like oscillator, Communications of JINR, Dubna, 2-92-66, 1992, 1–14.

[11] A.I. Mal’tcev, Algebraic systems, Moscow, ”Nauka”, 1965.

[12] A.I. Mal’tcev, Algorithms and recursive functions, Moscow, ”Nauka”, 1970.

[13] V.K. Kharchenko, Automorphisms and Derivations of associative Rings, Kluwer Academic Publishers, Mathematics and its applications, (Soviet Series), v.69, Dordrecht/Boston/London, 1991.

[14] V.K.Kharchenko, Noncommutative Galois theory, Novosibirsk, ”Nauchnaja Kniga”, 1996.

[15] K.O. Friedrichs, Mathematical aspects of the quantum theory of fields. V, Communications in Pure and Applied Mathematics, 6(1953), 1-72.

[16] B. Pareigis, On Lie algebras in braided categories, in: Quantum Groups and Quantum Spaces, eds. R. Budziński, W. Pusz, S. Zakrzewski, Banach Center Publications, 40 (1997), 139–158.
[17] B. Pareigis, Skew-primitive elements of quantum groups and braided Lie algebras, Rings, Hopf algebras, and Brauer groups (Antwerp/Brussels), Lecture Notes in Pure and Appl. Math., 197(1996), 219–238.

[18] B. Pareigis, On Lie algebras in the category of Yetter–Drinfeld modules, Appl. Categ. Structures, 6, N2(1998), 152–175.

[19] V.K. Kharchenko, An algebra of skew primitive elements, Algebra and Logic, 37, N2(1998), 101–127.

[20] V.K. Kharchenko, An existence condition for multilinear quantum operations, Journal of Algebra, 217(1999), 188–228.

[21] V.K. Kharchenko, A quantum analog of the Poincaré–Birkhoff–Witt theorem, Algebra and Logic, 38, N4(1999), 259–276, (QA/0005101).

[22] A.A. Albert, Structure of Algebras, Amer. Math. Soc., Providence, RI(1961).

[23] N. Jacobson, Structure of Rings, AMS Colloq. Publ. vol. 37, Amer. Math. Soc., Providence, RI (1956, revised 1964).

[24] I. Herstein, Noncommutative Rings, Carus Mathematical Monographs, N15, Amer. Math. Soc., 1998.

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