Existence and Regularity of Pullback Attractors for a Non-autonomous Diffusion Equation with Delay and Nonlocal Diffusion in Time-Dependent Spaces

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Abstract
In this paper, we study the asymptotic behavior of solutions to a non-autonomous diffusion equations with delay containing some hereditary characteristics and nonlocal diffusion in time-dependent space $C_{H_t(\Omega)}$. When the nonlinear function $f$ satisfies the polynomial growth of arbitrary order $p − 1$ ($p \geq 2$) and the external force $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$, we establish the existence and regularity of the time-dependent pullback attractors.

Keywords Non-autonomous diffusion equations · Time-dependent pullback attractors · Delay · Regularity

Mathematics Subject Classification 35B40 · 35B41 · 35B65 · 35K57

1 Introduction
In recent years, a lot of scholars have been keen to discuss the asymptotic behavior of solutions to the diffusion equations. The assumption for function $a(\cdot)$ of problem $u_t − a(l(u))\Delta u = f$ in Lovat [18] is only $0 < \bar{m} \leq a(s) \leq \bar{M}$ for any positive constants $\bar{m}$ and $\bar{M}$, which avoids that weak solutions of the diffusion equations only exist in a finite-time interval. In addition, Chipot and Zheng [9] showed that due to the existence of the nonlocal function $a(\cdot)$, it is impossible to guarantee the existence of a
Lyapunov structure. Since diffusion equations with nonlocal diffusion are widely used in ecology, epidemiology, materials science, neural network and other disciplines, and scientific researches toward them are challenging and forward-looking, they have been extensively studied (see [2, 3, 24, 29]).

In this paper, we consider the following non-autonomous diffusion equation with delay and nonlocal diffusion

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \varepsilon(t) \frac{\partial}{\partial t} (a(|u|)) u + f(t) + g(t, ut) + h(t) & \quad \text{in } \Omega \times (\tau, \infty), \\
\frac{\partial u}{\partial \nu} & \quad \text{on } \partial \Omega \times (\tau, \infty), \\
\phi(x, \theta) & \quad \text{on } \partial \Omega \times [\tau, \theta], \\
0 & \quad \text{in } \Omega \times \{\tau\},
\end{aligned}
\]

(1)
in time-dependent space \(C_{\mathcal{H}_t}(\Omega)\), where \(\Omega \subset \mathbb{R}^n (n \geq 3)\) is a bounded domain with smooth boundary \(\partial \Omega\), the initial time \(\tau \leq t \in \mathbb{R}\), \(\phi \in C([-k, 0]; \mathcal{H}_t(\Omega))\) is the initial datum, \(k (> 0)\) is the length of the delay effects, \(f\) is a nonlinear function, \(g\) is a delay operator containing some hereditary characteristics and we suppose \(u_t\) is defined in \([-k, 0]\) and satisfies \(u_t(\theta) = u(t + \theta)\). Meanwhile, let the external force \(h(x, t) \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))\).

Suppose \(C([-k, 0]; X)\) is a Banach space equipped with the sup-norm \(\| \cdot \|_{C([-k, 0]; X)}\) and metric \(d_C(\cdot, \cdot)\), and denote it as \(C_X\). The time-dependent space \(\mathcal{H}_t(\Omega)\) is equipped with norm \(\|u\|_{\mathcal{H}_t(\Omega)}^2 = \|u\|^2 + \varepsilon(t) \|\nabla u\|^2\). For brevity, the norm of \(L^2(\Omega)\) in this paper is denoted as \(\| \cdot \|_2 = \| \cdot \|\). Besides, we assume \(C_{L^2(\Omega)}\) is a Banach space with sup-norm, that is, for any \(u \in C_{L^2(\Omega)}\), its norm is defined as \(\|u\|_{C_{L^2(\Omega)}} = \max_{t \in [-k, 0]} \|u\|_2\). Similarly, for any \(u \in C_{H^1_0(\Omega)}\), we define its norm as \(\|u\|_{C_{H^1_0(\Omega)}} = \max_{t \in [-k, 0]} (\|u\|_2 + \|\nabla u\|_2)\). For any \(t \in \mathbb{R}\), the time-dependent space \(C_{\mathcal{H}_t}(\Omega)\) is endowed with the norm

\[
\|u\|_{C_{\mathcal{H}_t}(\Omega)}^2 = \|u\|_{C_{L^2(\Omega)}}^2 + |\varepsilon_t| \|\nabla u\|_{C_{L^2(\Omega)}}^2,
\]

where \(\varepsilon_t = \varepsilon(t + \theta)\) with \(\theta \in [-k, 0]\) and \(|\varepsilon_t|\) is the absolute value of \(\varepsilon_t\).

Assume \(\varepsilon(t) \in C^1(\mathbb{R})\) is a time-dependent function satisfying

\[
\lim_{t \to +\infty} \varepsilon(t) = 1,
\]

(2)

and there exists a constant \(L > 0\) such that

\[
\sup_{t \in \mathbb{R}} (|\varepsilon(t)| + |\varepsilon'(t)|) \leq L.
\]

(3)

In addition, the function \(a(|u|) \in C(\mathbb{R}; [0, \infty))\) is the nonlocal diffusion term of Eq. (1) and satisfies

\[
\frac{1}{2} (3 + L + \varepsilon'(t)) \leq a(s) \leq M, \quad \forall s \in \mathbb{R},
\]

(4)
where $m$ and $M$ are positive constants. Assume $l(u) : L^2(\Omega) \to \mathbb{R}$ is a continuous linear functional acting on $u$ that satisfies for some $j \in L^2(\Omega)$,

$$l(u) = l_j(u) = \int_{\Omega} j(x)u(x)dx. \quad (5)$$

Besides, suppose the nonlinear function $f \in C^1(\mathbb{R}, \mathbb{R})$ and satisfies

$$(f(u) - f(v))(u - v) \leq \tilde{\eta}(u - v)^2, \quad \forall u, v \in \mathbb{R}, \quad (6)$$

and

$$-C_0 - C_1|u|^p \leq f(u)u \leq C_0 - C_2|u|^p, \quad p \geq 2, \quad (7)$$

for some positive constants $\tilde{\eta}$, $C_0$, $C_1$ and $C_2$. Let $F(u) = \int_0^u f(r)dr$, the same as in [13], then from (6) - (7), it follows that there exists some positive constants $\tilde{C}_i$ ($i = 1, 2$) such that

$$-\tilde{C}_0 - \tilde{C}_1|u|^p \leq F(u) \leq \tilde{C}_0 - \tilde{C}_2|u|^p. \quad (8)$$

Furthermore, assume the delay operator $g : \mathbb{R} \times C_{L^2(\Omega)} \to L^2(\Omega)$ and it satisfies the following assumptions

the function $\mathbb{R} \ni t \mapsto g(t, v) \in L^2(\Omega)$ is measurable, $\forall v \in C_{L^2(\Omega)}$, \quad (9)

g(t, 0) = 0, \quad \forall t \in \mathbb{R}, \quad (10)

and there exists a constant $C_g > 0$ such that

$$\|g(t, v_1) - g(t, v_2)\|^2 \leq C_g \|v_1 - v_2\|_{C_{L^2(\Omega)}}^2, \quad (11)$$

for any $t \in \mathbb{R}$ and $v_1, v_2 \in C_{L^2(\Omega)}$.

Throughout this paper, the inner product of $L^2(\Omega)$ is denoted as $(\cdot, \cdot)$ and the norms of $L^\gamma(\Omega)$ with $\gamma \in \mathbb{R}$, $H^1_0(\Omega)$ and $H^{-1}(\Omega)$ are denoted by $\|\cdot\|_\gamma$, $\|\cdot\|_1$ and $\|\cdot\|_{-1}$, respectively.

There are many studies for the diffusion equations with delays related to problem (1). Hu and Wang [15] investigated the existence of pullback attractors for $\partial_t u - \Delta \partial_t u - \Delta u = f(t, u(t - \rho(t))) + g(t)$ in $C_{H^1_0(\Omega)}$ and $C_{H^2(\Omega) \cap H^1_0(\Omega)}$, where $\rho$ is a delay function and $f$ contains some memory effects in a fixed time interval with length $h > 0$. Later on, Caraballo and Márquez-Durán [6] proved the existence and eventual uniqueness of stationary solutions, as well as their exponential stability for $\partial_t u - \Delta \partial_t u - \Delta u = g(t, u_t)$ in $L^2(\Omega)$. Besides, García-Luengo and Marín-Rubio [13] obtained two different families of the minimal pullback attractors for $\partial_t u - \Delta \partial_t u - \Delta u = f(u) + g(t, u_t) + k(t)$ in $H^{-1}(\Omega)$. Furthermore, Zhu and Sun [31] verified the existence of pullback attractors for this equation in $C_{H^1_0(\Omega)}$. 
In addition, Wang and Kloeden [27] studied the existence of a uniform attractor in $L^2(\Omega)$ for the multi-valued process associated with $\partial_t u - \Delta u + \lambda u = f(x, u_t) + g(t, x)$. Caraballo et al. [7] established the existence of the pullback attractors for $\partial_t u - \gamma(t) \Delta \partial_t u - \Delta u = g(u) + f(t, u_t)$ in $C_{H^1_0}(\Omega)$, where $\gamma : \mathbb{R} \to (0, +\infty)$ is a continuous bounded function with $0 < \gamma_0 \leq \gamma(t) \leq \gamma_1 < \infty$. Moreover, Harraga and Yebdri [14] proved the existence of the pullback $\mathcal{D}$-attractors in $H^1_0(\Omega)$ for $\partial_t u - \Delta \partial_t u - \Delta u = b(t, u(t - \rho(t))(x)) + g(t, x)$. Zhu et al. [32] studied the asymptotic behavior of pullback attractors for $\partial_t u - \varepsilon(t) \partial_t \Delta u - \Delta u + f(u) = g(t, u_t) + k(x)$ in time-dependent space $\mathcal{H}_t(\Omega)$. Additionally, some scholars have also considered the well-posedness of solutions to the reaction-diffusion equations with other forms of delays (see [8, 16, 26]).

On the basis of García-Luengo and Marín-Rubio [13], we follow some of the methods in Zhu and Sun [31] to consider the existence of the time-dependent pullback $\mathcal{D}_{\eta_1}$-attractor of problem (1) in time-dependent space $C_{\mathcal{H}_t(\Omega)}$. When compared to their equations, the terms $a(l(u))$ and $\varepsilon(t)$ bring some difficulties to verify the desired results, and now we will illustrate them and explain our strategies and innovations.

(1) In [29], we consider the existence and upper semicontinuity of problem (1) without the term $g(t, u_t)$. Moreover, many scholars have studied the attractors in time-dependent space $\mathcal{H}_t(\Omega)$ (see [4, 5, 19, 24]). To control the term $-\varepsilon'(t)$ during energy estimations, they basically required the time-dependent term $\varepsilon(t)$ to be a decreasing function and using the transform

$$\varepsilon(t) \frac{d}{dt} \|\nabla u\|^2 = \frac{d}{dt} \left(\varepsilon(t) \|\nabla u\|^2\right) - \varepsilon'(t) \|\nabla u\|^2$$

to deal with the term $-\varepsilon'(t) \|\nabla u\|^2$, then they can easily derive the range of weak solutions in $\mathcal{H}_t(\Omega)$. In this paper, by limiting the lower bound of the nonlocal function $a(\cdot)$, we take into account the cases both $\varepsilon(t)$ is decreasing and $\varepsilon(t)$ is increasing, which is a brand new attempt and weaker than the conditions in [4, 5, 19, 24].

(2) The phase spaces of [13] and [31] are $H^{-1}(\Omega)$ and $C_{H^1_0}(\Omega)$, respectively, while our phase space $C_{\mathcal{H}_t(\Omega)}$ is a time-dependent space, and since problem (1) contains the nonlocal function $a(l(u))$ and the time-dependent function $\varepsilon(t)$, they ensure that problem (1) to have wider applications in real life. Meanwhile, these terms make it impossible to directly conclude the specific structure of absorbing set and the asymptotic compactness of the process by general energy estimations, which requires a series of delicate and multiple calculations. To this end, as in [31], we shall use a special Gronwall lemma (see [11, 21–23]) to conclude our desired results, which will be shown in Lemma 3.

(3) In [13] and [29], it is difficult to establish regularity for process due to the term $-\partial_t \Delta u$. In problem (1), this term is replaced by $-\varepsilon(t)\partial_t \Delta u$, and we also obtain the regularity of the process in $C_{\mathcal{H}_t(\Omega)}$ by decomposition method and using some calculations and estimates similar to those used to demonstrate the existence of the time-dependent pullback $\mathcal{D}_{\eta_1}$-attractor, which provides some ideas for studying pullback attractors in spaces of higher regularity.
This paper is organized as follows. In Sect. 2, we shall introduce some useful abstract definitions, theorems and lemmas. Next, we shall prove the existence and uniqueness of weak solutions to problem (1) by Faedo–Galerkin approximations in Sect. 3. Furthermore, in Sect. 4 we shall verify the existence of the time-dependent pullback $D_{\eta_1}$-attractor $A_{\eta_1}$ in time-dependent space $C_{\mathcal{H}_t}(\Omega)$. Finally, we shall derive the regularity of $A_{\eta_1}$ in Sect. 5.

2 Preliminaries

In this section, we will introduce some needed abstract concepts, such as definitions and properties of function spaces and attractors.

The time-dependent space $C_{\mathcal{H}_t^1}(\Omega)$, more regular than $C_{\mathcal{H}_t}(\Omega)$, is endowed with the norm

$$\| u \|_{C_{\mathcal{H}_t^1}(\Omega)}^2 = \| \nabla u \|_{C_{L^2}(\Omega)}^2 + |\varepsilon_t| \| \Delta u \|_{C_{L^2}(\Omega)}^2,$$

where $\varepsilon_t = \varepsilon(t + \theta)$ with $\theta \in [-k, 0]$.

The closed $R$-ball with the origin as the center and $R$ as the radius in $C_X$ is denoted as

$$\overline{B}_{C_X}(0, R) = \{ u \in C_X : \| u \|_{C_X} \leq R \}.$$

The Hausdorff semidistance between two nonempty sets $A_1, A_2 \subset C_X$ is denoted by

$$\text{dist}_{C_X}(A_1, A_2) = \sup_{x \in A_1} \inf_{y \in A_2} \| x - y \|_{C_X}.$$

**Definition 1** ([19, 33]) Assume $\{X_t\}_{t \in \mathbb{R}}$ is a family of normed spaces. A process or a two-parameter semigroup on $\{X_t\}_{t \in \mathbb{R}}$ is a family $\{U(t, \tau)\}_{t \geq \tau}$ of mapping $U(t, \tau) : X_\tau \to X_t$ satisfies that $U(\tau, \tau)u = u$ for any $u \in X_\tau$ and $U(t, s)U(s, \tau) = U(t, \tau)$ for all $t \geq s \geq \tau$.

**Definition 2** ([4]) For any $\sigma > 0$, let $\mathcal{D}$ be a nonempty class of all families of parameterized sets $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \Gamma(X_t)$ such that

$$\lim_{\tau \to -\infty} \left( e^{\sigma \tau} \sup_{u \in D(\tau)} \| u \|_{X_t}^2 \right) = 0,$$

where $\Gamma(X_t)$ denotes the family of all nonempty subsets of $\{X_t\}_{t \in \mathbb{R}}$, then $\mathcal{D}$ will be called a tempered universe in $\Gamma(X_t)$.

**Definition 3** ([20, 31]) A process $\{U(t, \tau)\}_{t \geq \tau}$ is pullback $\mathcal{D}$-asymptotically compact on $\{X_t\}_{t \in \mathbb{R}}$, if for any $t \in \mathbb{R}$, any $\hat{D} \in \mathcal{D}$, any sequence $\{\tau_n\}_{n \in \mathbb{N}^+} \subset (-\infty, t]$ and any sequence $\{x_n\}_{n \in \mathbb{N}^+} \subset D(\tau_n) \subset X_t$, the sequence $\{U(t, \tau)x_n\}_{n \in \mathbb{N}^+}$ is relatively compact in $X_t$ when $\tau_n \to -\infty$. 

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Lemma 2 ([20, 31]) A family $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \Gamma(X_t)$ is pullback $D$-absorbing for the process $\{U(t, \tau)\}_{t \geq \tau}$ on $\{X_t\}_{t \in \mathbb{R}}$, if for any $t \in \mathbb{R}$ and $\hat{D} \in D$, there exists a $\tau_0 = \tau_0(t, \hat{D}) < t$ such that $U(t, \tau)D(\tau) \subset D_0(t)$ for any $\tau \leq \tau_0(t, \hat{D})$.

Definition 5 ([20, 31]) A family $A_\eta = \{A_\eta(t) : t \in \mathbb{R}\} \subset \Gamma(X_t)$ is called a time-dependent pullback $D$-attractor for the process $\{U(t, \tau)\}_{t \geq \tau}$ on $\{X_t\}_{t \in \mathbb{R}}$, if the following properties hold:

(i) The set $A_\eta(t)$ is compact in $X_t$ for any $t \in \mathbb{R}$;
(ii) $A_\eta$ is pullback $D$-attracting in $X_t$, i.e.,
\[
\lim_{\tau \to -\infty} \text{dist}_{X_t}(U(t, \tau)D(\tau), A_\eta(t)) = 0,
\]
for any $\hat{D} \in D$ and $t \in \mathbb{R}$;
(iii) $A_\eta$ is invariant, i.e., $U(t, \tau)A_\eta(\tau) = A_\eta(t)$, for any $\tau \leq t$.

Furthermore, we will introduce the following definitions and lemmas, which contribute to prove the existence of the time-dependent pullback $D$-attractors.

Definition 6 ([12, 25]) Suppose the set $B \subset \{X_t\}_{t \in \mathbb{R}}$, then a function $\psi(\cdot, \cdot)$ defined on $X_t \times X_t$ is said to be a contractive function on $B \times B$, if for any sequence $\{x_n\}_{n=1}^\infty \subset B$, there is a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that
\[
\lim_{k \to \infty} \lim_{l \to \infty} \psi(x_{n_k}, x_{n_l}) = 0.
\]
For simplicity, we denote the set of all contractive functions on $B \times B$ by $C(\hat{B})$.

Lemma 1 ([25]) Assume the process $\{U(t, \tau)\}_{t \geq \tau}$ has a pullback $D$-absorbing set $\hat{B} = \{B(t) : t \in \mathbb{R}\}$ on $\{X_t\}_{t \in \mathbb{R}}$ and there exist $T = T(t, \hat{B}, \hat{c}) = t - \tau$ and $\psi_{t, T}(\cdot, \cdot) \in C(\hat{B})$ such that
\[
\|U(t, t - T)x - U(t, t - T)y\|_{X_t} \leq \hat{c} + \psi_{t, T}(x, y),
\]
for any $x, y \in B(\tau)$ and $\hat{c} > 0$, then $\{U(t, \tau)\}_{t \geq \tau}$ is pullback $D$-asymptotically compact on $\{X_t\}_{t \in \mathbb{R}}$.

Lemma 2 ([25]) The process $\{U(t, \tau)\}_{t \geq \tau}$ has a time-dependent pullback $D$-attractor in $\{X_t\}_{t \in \mathbb{R}}$, if it satisfies the following conditions:

(i) $\{U(t, \tau)\}_{t \geq \tau}$ has a pullback $D$-absorbing set $B_0$ in $X_t$ and
(ii) $\{U(t, \tau)\}_{t \geq \tau}$ is pullback $D$-asymptotically compact in $B_0$.

3 Existence and Uniqueness of Weak Solutions

To study existence of solutions by the time-dependent pullback $D_\eta$-attractors, as in [24], we first prove the existence and uniqueness theorems of weak solutions to problem (1) in this section.
Definition 7  A weak solution of problem (1) is a function \( u \in C([\tau - k, T]; \mathcal{H}_t(\Omega)) \cap L^2(\tau, T; \mathcal{H}_t(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \) for any \( \tau < T \in \mathbb{R} \), with \( u(t) = \phi(t - \tau) \) for all \( t \in [\tau - k, \tau] \) such that

\[
\frac{d}{dt} [(u(t), \varphi) + \varepsilon(t)(\nabla u(t), \nabla \varphi)] + (2a(l(u)) - \varepsilon'(r))(\nabla u(t), \nabla \varphi) = 2(f(u(t)), \varphi) + 2g(t, u_t), \varphi) + 2h(t, \varphi),
\]

for any test function \( \varphi \in H^1_0(\Omega) \).

Remark 1  The Eq. (12) should be understood in the sense of the generalized function space \( \mathcal{D}'(\tau, +\infty) \).

Corollary 1  If \( u \) is a weak solution of problem (1), then the following energy equality holds

\[
\|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 + \int_\tau^t \left( 2a(l(u)) - \varepsilon'(r) \right) \|\nabla u(r)\|^2 dr = \|u(s)\|^2 + \varepsilon(s)\|\nabla u(s)\|^2 + 2\int_s^t (f(u(r)) + g(t, u_r) + h(r, u(r))) dr,
\]

for all \( \tau \leq s \leq t \).

Firstly, we will use the the standard Faedo–Galerkin method to prove the following theorem.

Theorem 1  Assume that \( a(\cdot) \) is a local Lipschitz continuous function and satisfies (4), \( l(\cdot) \) is given in (5), \( f \in C^1(\mathbb{R}, \mathbb{R}) \) and satisfies (6) and (7), \( h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega)) \) and \( \phi \in C_{\mathcal{H}_t}(\Omega) \) is given, then for any \( \tau \leq t \in \mathbb{R} \), there exists a weak solution to problem (1).

Proof  Suppose the approximate solution \( u_i(t, x) = \sum_{j=1}^{i} r_{i,j}(t) \omega_j(x) \), where \( i, j \in \mathbb{N}^+ \), \( \{\omega_j\}_{j=1}^{\infty} \) is a basis of \( H^2(\Omega) \cap H^1_0(\Omega) \) and orthonormal in \( L^2(\Omega) \). The Faedo–Galerkin method needs to find an approximate sequence \( \{u_i\}_{i \geq n} \) such that the following approximate system holds

\[
\frac{d}{dt} [(u_i(t), \omega_j) + \varepsilon(t)(\nabla u_i(t), \nabla \omega_j)] + (2a(l(u_i)) - \varepsilon'(r))(\nabla u_i(t), \nabla \omega_j) = 2(f(u_i(t), \omega_j) + 2g(t, u_t, \omega_j) + 2h(t, \omega_j), \forall t \in [\tau, +\infty), 1 \leq j \leq i \quad (14)
\]

where \( \phi \in C([-k, 0]; \text{span}\{\omega_j\}_{j=1}^{\infty}) \), \( u_t = u_i(t + \theta) \) and \( u_{\tau,i} = u_i(\tau + \theta) \) with \( \theta \in [-k, 0] \).

Step 1: (A priori estimate for \( u \) )  (1) When \( \varepsilon(t) \) is a decreasing function.

Multiplying (14) by the test function \( \gamma_{l,j}(t) \) and then summing \( j \) from 1 to \( i \), we obtain

\[
\frac{d}{dt} \left( \|u_i(t)\|^2 + \varepsilon(t)\|\nabla u_i(t)\|^2 \right) + (2a(l(u_i)) - \varepsilon'(r))\|\nabla u_i(t)\|^2
\]
and assumptions (9)–(11), we derive

\begin{equation}
2 \left( f(u_i(t), u_i(t)) + 2\left( g(t, u_{t,i}) + h(t, u_i(t)) \right) \right). \tag{15}
\end{equation}

From (7), we conclude

\begin{equation}
2 \left( f(u_i(t), u_i(t)) \leq 2C_0|\Omega| - 2C_2 \left\| u_i(t) \right\|_P. \tag{16}\right.
\end{equation}

Besides, by the Young inequality, the definition of the time-dependent space \( C_{\mathcal{H}_t}(\Omega) \) and assumptions (9)–(11), we derive

\begin{equation}
2 \left( g(t, u_{t,i}), u_i(t) \right) \leq C_g \left\| u_{t,i}(t) \right\|_{C_L^2(\Omega)}^2 + \left\| u_i(t) \right\|_2^2 \leq 2C_g \left\| u_{t,i}(t) \right\|_{C_L^2(\Omega)}^2. \tag{17}\right.
\end{equation}

Using the Young and the Cauchy inequalities, it follows that

\begin{equation}
2 \left( h(t), u_i(t) \right) \leq \left\| h(t) \right\|_p^2 + \left\| \nabla u_i(t) \right\|^2. \tag{18}\right.
\end{equation}

Substituting (16)–(18) into (15), we arrive at

\begin{equation}
\frac{d}{dt} \left( \left\| u_i(t) \right\|^2 + \varepsilon(t) \left\| \nabla u_i(t) \right\|^2 \right) + \left( 2a(l(u_i)) - \varepsilon'(t) \right) \left\| \nabla u_i(t) \right\|^2 + 2C_2 \left\| u_i(t) \right\|_P \leq 2C_0|\Omega| + 2C_g \left\| u_{t,i} \right\|_{C_L^2(\Omega)}^2 + \left\| \nabla u_i(t) \right\|^2 + \left\| h(t) \right\|_{-1}^2. \tag{19}\right.
\end{equation}

Then it follows from (3) and (4) that

\begin{equation}
\frac{d}{dt} \left( \left\| u_i(t) \right\|^2 + \varepsilon(t) \left\| \nabla u_i(t) \right\|^2 \right) + \left( 1 + L \right) \left\| \nabla u_i(t) \right\|^2 + 2C_2 \left\| u_i(t) \right\|_P \leq 2C_0|\Omega| + 2C_g \left\| u_{t,i} \right\|_{C_L^2(\Omega)}^2 + \left\| h(t) \right\|_{-1}^2. \tag{20}\right.
\end{equation}

Integrating (20) from \( \tau \) to \( t \), we deduce

\begin{equation}
\left\| u_i(t) \right\|^2 + \varepsilon(t) \left\| \nabla u_i(t) \right\|^2 + \left( 1 + L \right) \int_{\tau}^t \left\| \nabla u_i(s) \right\|^2 ds + 2C_2 \int_{\tau}^t \left\| u_i(s) \right\|_P ds \leq 2C_0|\Omega|(t - \tau) + \left\| u_i(\tau) \right\|^2 + \varepsilon(\tau) \left\| \nabla u_i(\tau) \right\|^2 + 2C_g \int_{\tau}^t \left\| u_{s,i} \right\|_{C_L^2(\Omega)}^2 ds + \int_{\tau}^t \left\| h(s) \right\|_{-1}^2 ds. \tag{21}\right.
\end{equation}

Putting \( t + \theta \) instead of \( t \) with \( \theta \in [-k, 0] \) in (21), we obtain

\begin{equation}
\left\| u_{t,i} \right\|_{C_L^2(\Omega)}^2 + \varepsilon_t \left\| \nabla u_{t,i} \right\|_{C_L^2(\Omega)}^2 + \left( 1 + L \right) \int_{\tau}^t \left\| \nabla u_{s,i} \right\|_{C_L^2(\Omega)}^2 ds + 2C_2 \int_{\tau}^t \left\| u_{s,i} \right\|_P ds \leq \left\| \phi \right\|_{C_L^2(\Omega)}^2 + \varepsilon \left\| \nabla \phi \right\|_{C_L^2(\Omega)}^2 + 2C_0|\Omega|(t - \tau) + 2C_g \int_{\tau}^t \left\| u_{s,i} \right\|_{C_L^2(\Omega)}^2 ds + \int_{\tau}^t \left\| h(s) \right\|_{-1}^2 ds. \tag{22}\right.
\end{equation}
By simple calculations, we arrive at
\[
\|u_{t,i}\|_{L^2(\Omega)}^2 + \epsilon_t \|\nabla u_{t,i}\|_{L^2(\Omega)}^2 \leq \|\phi\|_{L^2(\Omega)}^2 + \epsilon_t \|\nabla \phi\|_{L^2(\Omega)}^2 + 2C_0|\Omega|(t - \tau) + 2C_g \int_\tau^t \left( \|u_{s,i}\|_{L^2(\Omega)}^2 + \epsilon_s \|\nabla u_{s,i}\|_{L^2(\Omega)}^2 \right) ds + \int_\tau^t \|h(s)\|^2_{-1} ds.
\]  
(23)

Thanks to the Gronwall inequality, we conclude
\[
\|u_{t,i}\|_{L^2(\Omega)}^2 + \epsilon_t \|\nabla u_{t,i}\|_{L^2(\Omega)}^2 \leq e^{2C_g(t-\tau)} \left( \|\phi\|_{L^2(\Omega)}^2 + \epsilon_t \|\nabla \phi\|_{L^2(\Omega)}^2 \right) + 2e^{2C_g(t-\tau)}C_0|\Omega|(t - \tau) + e^{2C_g(t-\tau)} \int_\tau^t \|h(s)\|^2_{-1} ds.
\]  
(24)

Then from (22) and (24), we derive
\[\{u_{i}\} \text{ is bounded in } C([\tau-k, T]; H^1(\Omega)) \cap L^2(\tau, T; H^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)). \]
(25)

Moreover, from (7) we arrive at
\[f(u_{i}(t)) \text{ is bounded in } L^q(\tau, T; L^q(\Omega)), \text{ for all } T > \tau, \]
(26)

where \(q = \frac{p}{p-1}\) with \(p \geq 2\).

Multiplying (14)_{1} by \(\gamma'_{i,j}(t)\) and summing \(j\) from 1 to \(i\), then by (4), we obtain
\[
2\|u'_i(t)\|^2 + 2\epsilon(t)\|\nabla u'_i(t)\|^2 + m \frac{d}{dt} \|\nabla u_i(t)\|^2 \leq 2 \left( f(u_i(t)) + g(t, u_{t,i}) + h(t, u'_i(t)) \right).
\]  
(27)

Furthermore, by \(\mathcal{F}(u) = \int_0^u f(r)dr\), we arrive at
\[
2(f(u_{i}(t)), u'_i(t)) = 2 \frac{d}{dt} \int_\Omega \mathcal{F}(u_i(t, x))dx.
\]  
(28)

Using assumptions (9)–(11) and the Young inequality, we derive
\[
2(g(t, u_{t,i}), u'_i(t)) \leq C_g \|u_{t,i}\|^2_{L^2(\Omega)} + \|u'_i(t)\|^2.
\]  
(29)

From the Cauchy and Young inequalities, we conclude
\[
2(h(t), u'_i(t)) \leq \|h(t)\|^2_{-1} + \|\nabla u'_i(t)\|^2.
\]  
(30)

Substituting (28)–(30) into (27), and by \(2\epsilon(t) - 1 > \epsilon(t)\) we obtain
\[
\|u'_i(t)\|^2 + \epsilon(t)\|\nabla u'_i(t)\|^2 + m \frac{d}{dt} \|\nabla u_i(t)\|^2 \leq \frac{d}{dt} \|\nabla u_i(t)\|^2.
\]
\begin{align}
\frac{d}{dt} \int_{\Omega} F(u_i(t, x)) dx + C_g \|u_{t,i}\|_C_{L^2(\Omega)}^2 + \|h(t)\|_{-1}^2.
\end{align}

Integrating (31) from \(\tau\) to \(t\), we deduce
\begin{align}
\|u_i(t)\|^2 + \int_\tau^t \varepsilon(s) \|u_i'(s)\|^2 ds + m \int_\tau^t \|\nabla u_i'(s)\| \|\nabla u_i'(s)\| ds + 2 \int_{\Omega} F(u_i(\tau, x)) dx \\
\leq \|u_i(\tau)\|^2 + 2 \int_{\Omega} F(u_i(t, x)) dx + C_g \int_{\tau}^t \|u_{s,i}\|_{C_{L^2(\Omega)}}^2 ds \\
+ \int_{\tau}^t \|h(s)\|_{-1}^2 ds.
\end{align}

(32)

By (8), it follows that
\begin{align}
- \tilde{C}_0 |\Omega| - \tilde{C}_1 \|u_i(t)\|^p_p \leq 2 \int_{\Omega} F(u_i(t, x)) dx \leq 2 \tilde{C}_0 - \tilde{C}_2 \|u_i(t)\|^p_p.
\end{align}

Substituting (33) into (32), we obtain
\begin{align}
\|u_i(t)\|^2 + \int_\tau^t \varepsilon(s) \|\nabla u_i'(s)\|^2 ds + m \int_\tau^t \|\nabla u_i'(t)\|^2 ds + 2 \tilde{C}_2 \|u_i(t)\|^p_p \\
\leq \|u_i(\tau)\|^2 + 4 \tilde{C}_0 |\Omega| + 2 \tilde{C}_1 \|u_i(\tau)\|^p_p + C_g \int_{\tau}^t \|u_{s,i}\|_{C_{L^2(\Omega)}}^2 ds \\
+ \int_{\tau}^t \|h(s)\|_{-1}^2 ds.
\end{align}

(34)

(2) When \(\varepsilon(t)\) is an increasing function. From (3), by some similar calculations to the case (1), we derive
\begin{align}
\|u_i(t)\|^2 + \int_\tau^t |\varepsilon(s)| \|\nabla u_i'(s)\|^2 ds \\
\leq \tilde{C} \left( \|u_i(\tau)\|^p_p + \int_{\tau}^t \|u_{s,i}\|_{C_{L^2(\Omega)}}^2 ds + \int_{\tau}^t \|h(s)\|_{-1}^2 ds \right),
\end{align}

(35)

where \(\tilde{C} > 0\) is a constant depends on \(\tilde{C}_0\), \(\tilde{C}_1\) and \(C_g\).

Putting \(t + \theta\) instead of \(t\) with \(\theta \in [-k, 0]\) in (34)–(35) and using the Gronwall inequality, then through similar calculations and estimations to (22) and (24), we conclude
\begin{align}
\{u_i\} \text{ is bounded in } L^\infty(\tau, T; H^1_0(\Omega) \cap L^p(\Omega))
\end{align}

(36)

and
\begin{align}
\{\partial_t u_i\} \text{ is bounded in } L^2(\tau, T; H_1(\Omega)).
\end{align}

(37)

Then from (25), (26), (36), (37), the compactness arguments and the Aubin–Lions lemma (see [17]), we derive that there exists a subsequence of \(\{u_i\}\) (still marked as
{u_i}), \ u \in L^\infty(\tau, T; H_t(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \text{ and } \partial_t u \in L^\infty(\tau, T; H_t(\Omega)) \text{ such that}

\begin{align*}
u_i & \rightharpoonup \nu \text{ weakly-star in } L^\infty(\tau - k, T; H_t(\Omega)); \\
u_i & \rightharpoonup \nu \text{ weakly in } L^2(\tau, T; H^1_0(\Omega)); \\
u_i & \rightharpoonup \nu \text{ weakly in } L^p(\tau, T; L^p(\Omega)); \\
 f(u_i) & \rightarrow f(u) \text{ weakly in } L^q(\tau, T; L^q(\Omega)); \quad (38) \\
 a(l(u_i))u_i & \rightarrow a(l(u))u \text{ weakly in } L^2(\tau, T; H^1_0(\Omega)); \quad (39) \\
 \partial_j u_i & \rightarrow \partial_j u \text{ weakly in } L^2(\tau, T; H_t(\Omega)); \quad (40) \\
u_i & \rightarrow \nu \text{ in } C([\tau - k, T]; H_t(\Omega)). \quad (41)
\end{align*}

From (38) to (44), we can pass the limit in Eq. (14) and notice that \( \{w_j\}_{j=1}^\infty \) is dense in \( H^1_0(\Omega) \cap L^p(\Omega) \), (12) holds for any \( \varphi \in H^1_0(\Omega) \cap L^p(\Omega) \).

**Step 2: (Verify the initial value)** In order to verify \( u \) is a weak solution to problem (1), we only need to check that \( u_{\tau,i} = \phi \).

Choosing a function \( \varphi \in C^1([\tau, T); H^1_0(\Omega)) \) with \( \varphi(T) = 0 \), then we obtain

\begin{align*}
\int_\tau^T - (u, \varphi') \, ds + \int_\tau^T \int_\Omega \varepsilon(s) \nabla(\partial_s u) \varphi \, dx \, ds - \int_\tau^T \int_\Omega a(l(u)) (\Delta u) \varphi \, dx \, ds \\
- \int_\tau^T \int_\Omega (f(u) + g(t, u_s) + h(s)) \varphi \, dx \, ds = (u(\tau + \theta), \varphi(\theta)). \quad (45)
\end{align*}

Repeating the same process in the Faedo–Galerkin approximations yields

\begin{align*}
\int_\tau^T - (u_i, \varphi') \, ds + \int_\tau^T \int_\Omega \varepsilon(s) \nabla(\partial_s u_i) \varphi \, dx \, ds - \int_\tau^T \int_\Omega a(l(u_i)) (\Delta u_i) \varphi \, dx \, ds \\
- \int_\tau^T \int_\Omega (f(u_i) + g(t, u_{s,i}) + h(s)) \varphi \, dx \, ds = (u_i(\tau + \theta), \varphi(\theta)). \quad (46)
\end{align*}

Taking limits as \( i \rightarrow \infty \) in (46) and since \( u_i(\tau + \theta) \rightarrow \phi(\tau + \theta) \), we deduce

\begin{align*}
\int_\tau^T - (u, \varphi') \, ds + \int_\tau^T \int_\Omega \varepsilon(s) \nabla(\partial_s u) \varphi \, dx \, ds - \int_\tau^T \int_\Omega a(l(u)) (\Delta u) \varphi \, dx \, ds \\
- \int_\tau^T \int_\Omega (f(u) + g(t, u_s) + h(s)) \varphi \, dx \, ds = (\phi(\tau + \theta), \varphi(\theta)). \quad (47)
\end{align*}

Then we derive \( u(\tau + \theta) = \phi(\theta) \).

As a result, we conclude that \( u \) is a weak solution of problem (1). \( \square \)

**Theorem 2** Under the assumptions of Theorem 1, if the weak solution of problem (1) exists, then it is unique. Moreover, the weak solution depends continuously on its initial value.
Proof. Assuming that $u_1$ and $u_2$ are two solutions corresponding to the initial values $u_1(\tau + \theta)$ and $u_2(\tau + \theta)$, respectively, and satisfying

$$\begin{align*}
\begin{cases}
\partial_t u_1 - \varepsilon(t) \Delta_h u_1 - a(l(u_1)) \Delta u_1 = f(u_1) + g(t, u_{1,1}) + h(t) & \text{in } \Omega \times (\tau, \infty), \\
\partial \Omega \times (\tau, \infty), \\
u_{1}(x, t) = 0 & x \in \Omega, \\
u_{1}(x, \tau + \theta) = \phi_1(x, \theta), & x \in \Omega, \theta \in [-k, 0],
\end{cases}
\end{align*}$$

(48)

and

$$\begin{align*}
\begin{cases}
\partial_t u_2 - \varepsilon(t) \Delta_h u_2 - a(l(u_2)) \Delta u_2 = f(u_2) + g(t, u_{1,2}) + h(t) & \text{in } \Omega \times (\tau, \infty), \\
\partial \Omega \times (\tau, \infty), \\
u_{2}(x, t) = 0 & x \in \Omega, \\
u_{2}(x, \tau + \theta) = \phi_2(x, \theta), & x \in \Omega, \theta \in [-k, 0].
\end{cases}
\end{align*}$$

(49)

When $\varepsilon(t)$ is a decreasing function, subtracting (49) from (48), assuming $u = u_1 - u_2$ and taking $L^2$-inner product between $u$ and the resulting equation, we derive

$$\begin{align*}
\frac{d}{dt} \left[ \|u\|^2 + \varepsilon(t) \|\nabla u\|^2 \right] + 2(a(l(u_1)) - \varepsilon'(t)) \|\nabla u\|^2 \\
&= 2(a(l(u_2)) - a(l(u_1))) (\nabla u_2, \nabla u) \\
&+ 2(f(u_1) - f(u_2), u) + 2(g(t, u_{1,1}) - g(t, u_{1,2}), u).
\end{align*}$$

(50)

From (2)–(7), assumptions (9)–(11) and the Poincaré inequality, then by similar calculations as that from (3.25) to (3.28) in [24] we conclude

$$\frac{d}{dt} \left( \|u\|^2 + \varepsilon(t) \|\nabla u\|^2 \right) \leq C \left( \|u_t\|_{C_{L^2(\Omega)}}^2 + \|\nabla u_t\|_{C_{L^2(\Omega)}}^2 \right),$$

(51)

where $C > 0$ is a constant depends on $L, m, \varepsilon'(t), \bar{\eta}, C_0, C_1$ and $C_2$.

Integrating it in $[\tau, t]$ and putting $t + \theta$ instead of $t$, we obtain

$$\|u_t\|_{C_{L^2(\Omega)}}^2 + \varepsilon_\tau \|\nabla u_t\|_{C_{L^2(\Omega)}}^2 \leq \|u_\tau\|_{C_{H^1(\Omega)}}^2 + C \int_{\tau}^{t} \left( \|u_s\|_{C_{L^2(\Omega)}}^2 + \|\nabla u_s\|_{C_{L^2(\Omega)}}^2 \right) ds.$$  

(52)

By the Gronwall inequality, we deduce

$$\|u_t\|_{C_{L^2(\Omega)}}^2 + \varepsilon_\tau \|\nabla u_t\|_{C_{L^2(\Omega)}}^2 \leq e^{C(t+k-\tau)} \left( \|u_\tau\|_{C_{L^2(\Omega)}}^2 + \varepsilon_\tau \|\nabla u_\tau\|_{C_{L^2(\Omega)}}^2 \right).$$

(53)

When $\varepsilon(t)$ is an increasing function, the following estimates can be obtained from (3), (4) and similar calculations to those above

$$\|u_t\|_{C_{L^2(\Omega)}}^2 + |\varepsilon_\tau| \|\nabla u_t\|_{C_{L^2(\Omega)}}^2 \leq e^{C(t+k-\tau)} \left( \|u_\tau\|_{C_{L^2(\Omega)}}^2 + \varepsilon_\tau \|\nabla u_\tau\|_{C_{L^2(\Omega)}}^2 \right).$$

(54)

Consequently, the uniqueness and continuity follow readily.
Corollary 2 Thanks to Theorems 1 and 2, problem (1) has a continuous process

\[ U(t, \tau) : C_{\mathcal{H}_t}(\Omega) \to C_{\mathcal{H}_t}(\Omega) \]

with \( U(t, \tau + \theta) \phi = u(t) \) being the unique weak solution.

4 Existence of the Time-Dependent Pullback \( \mathcal{D}_{\eta_1} \)-Attractors

In this section, we will discuss the existence of the time-dependent pullback \( \mathcal{D}_{\eta_1} \)-attractor for the process \( \{ U(t, \tau) \}_{t \geq \tau} \) in \( C_{\mathcal{H}_t}(\Omega) \) by the similar methods to Zhu and Sun [31]. To obtain the pullback \( \mathcal{D}_{\eta_1} \)-absorbing set, we will first prove the following lemma.

Lemma 3 Under the assumptions of Theorems 1 and 2, suppose \( \phi \in C_{\mathcal{H}_t}(\Omega) \) is given, then the weak solution of problem (1) satisfies

\[
\| \phi \|^2_{C_{L^2(\Omega)}} + |\varepsilon_1| \| \nabla u_t \|^2_{C_{L^2(\Omega)}} \leq \left( 1 + \frac{C_g e^{\eta k}}{(1 + \lambda L)} \right) \left( 1 + \frac{C_g e^{\eta k}}{(1 + \lambda L)} \right) e^{\eta_1(t - \tau)} e^{\eta k} \int_{\tau}^{t} e^{-\eta_1(s - \tau)} \| h(s) \|^2_{-1} ds,
\]

where \( \eta \) and \( \eta_1 \) are constants that satisfy \( 0 < \eta < \frac{(1 + L)\lambda_1}{1 + \lambda L} \) and \( \eta_1 = \eta - \frac{C_g e^{\eta k}}{1 + \lambda L} > 0 \), respectively.

Proof When \( \varepsilon(t) \) is a decreasing function, choosing \( u \) as the test function of (1) in \( L^2(\Omega) \), then the weak solution \( u \) satisfies

\[
\frac{d}{dt} \left[ \| u(t) \|^2 + \varepsilon(t) \| \nabla u(t) \|^2 \right] + \left( 2a(l(u)) - \varepsilon'(t) \right) \| \nabla u \|^2 = 2(f(u) + g(t, u_t) + h(t, u)).
\]

By (7), we obtain

\[
2(f(u), u) \leq 2C_0|\Omega| - 2C_2 \| u \|^p.
\]

Besides, from the Cauchy and Young inequalities, assumptions (9)–(11), we derive the following inequalities

\[
2(g(t, u_t), u) \leq C_g \| u_t \|^2_{C_{L^2(\Omega)}} + \| u \|^2
\]

and

\[
2(h(t), u) \leq \| h(t) \|^2_{-1} + \| \nabla u \|^2.
\]
Inserting (57)–(59) into (56) and by the Poincaré inequality, we deduce

\[
\frac{d}{dt} \left[ \|u\|^2 + \varepsilon(t)\|\nabla u\|^2 \right] + \left( 2a(l(u)) - \varepsilon'(t) \right) \|\nabla u\|^2 + 2C_2 \|u\|^p_p \\
\leq 2C_0|\Omega| + C_g \|u_t\|_{C^1_{L^2}(\Omega)}^2 + \left( \lambda_1^{-1} + 1 \right) \|\nabla u\|^2 + \|h(t)\|^2_{-1}. \quad (60)
\]

Then from (4), we derive

\[
\frac{d}{dt} \left[ \|u\|^2 + \varepsilon(t)\|\nabla u\|^2 \right] + (1 + L)\|\nabla u\|^2 + 2C_2 \|u\|^p_p \\
\leq 2C_0|\Omega| + C_g \|u_t\|_{C^1_{L^2}(\Omega)}^2 + \|h(t)\|^2_{-1}. \quad (61)
\]

By the Poincaré inequality and (3), we obtain that there exists \(0 < \eta < \frac{(1+L)\lambda_1}{T+\lambda_1 L}\) such that

\[
\frac{d}{dt} \left[ \|u\|^2 + \varepsilon(t)\|\nabla u\|^2 \right] + \eta \left( \|u\|^2 + \varepsilon(t)\|\nabla u\|^2 \right) \\
\leq 2C_0|\Omega| + C_g \|u_t\|_{C^1_{L^2}(\Omega)}^2 + \|h(t)\|^2_{-1}. \quad (62)
\]

Multiplying (62) by \(e^{\eta t}\) and integrating the resulting inequality from \(\tau\) to \(t\) yields

\[
e^{\eta t} \left( \|u\|^2 + \varepsilon(t)\|\nabla u\|^2 \right) \leq e^{\eta \tau} \left( \|u(\tau)\|^2 + \varepsilon(\tau)\|\nabla u(\tau)\|^2 \right) + 2C_0|\Omega| \int_\tau^t e^{\eta s} ds \\
+ C_g \int_\tau^t e^{\eta s} \|u_s\|_{C^1_{L^2}(\Omega)}^2 ds + \int_\tau^t e^{\eta s} \|h(s)\|_{-1}^2 ds. \quad (63)
\]

Then putting \(t + \theta\) instead of \(t\) with \(\theta \in [-k,0]\) in (63), we derive

\[
e^{\eta(t+\theta)} \left( \|u(t+\theta)\|^2 + \varepsilon(t+\theta)\|\nabla u(t+\theta)\|^2 \right) \leq 2C_0|\Omega| \int_\tau^t e^{\eta s} ds \\
+ C_g \int_\tau^t e^{\eta s} \|u_s\|_{C^1_{L^2}(\Omega)}^2 ds + e^{\eta(t+\theta)} \left( \|u(\tau+\theta)\|^2 + \varepsilon(\tau+\theta)\|\nabla u(\tau+\theta)\|^2 \right) \\
+ \int_\tau^t e^{\eta s} \|h(s)\|_{-1}^2 ds. \quad (64)
\]

Noting that \(\theta \in [-k,0]\) with \(k > 0\), then from (64), it follows

\[
e^{\eta t} \left( \|u_t\|_{C^1_{L^2}(\Omega)}^2 + \varepsilon_t \|\nabla u_t\|_{C^1_{L^2}(\Omega)}^2 \right) \\
\leq e^{\eta t} \left( \|\phi\|_{C^1_{L^2}(\Omega)}^2 + \varepsilon_t \|\nabla \phi\|_{C^1_{L^2}(\Omega)}^2 \right) + 2C_0|\Omega| e^{nk} \int_\tau^t e^{\eta s} ds \\
+ C_g e^{nk} \int_\tau^t e^{\eta s} \|u_s\|_{C^1_{L^2}(\Omega)}^2 ds + e^{nk} \int_\tau^t e^{\eta s} \|h(s)\|_{-1}^2 ds. \quad (65)
\]
From (3) and the Poincaré inequality, we obtain

\[
e^{\eta t} \| u_t \|^2_{C^2_{L^2}(\Omega)} \leq \frac{1}{1 + \lambda_1 L} e^{\eta t} \left( \| \phi \|^2_{C^2_{L^2}(\Omega)} + \varepsilon \| \nabla \phi \|^2_{C^2_{L^2}(\Omega)} \right) + \frac{2C_0 |\Omega|}{1 + \lambda_1 L} e^{\eta k} \int_\tau^t e^{\eta \tau} ds + \frac{C_g}{1 + \lambda_1 L} e^{\eta k} \int_\tau^t e^{\eta \tau} \| u_s \|^2_{C^2_{L^2}(\Omega)} ds \\
+ \frac{1}{1 + \lambda_1 L} e^{\eta k} \int_\tau^t e^{\eta \tau} \| h(s) \|^2_{-1} ds.
\]

(66)

Now, we apply the Gronwall lemma to (65). Let

\[
w(t) = e^{\eta t} \| u_t \|^2_{C^2_{L^2}(\Omega)}
\]

(67)

and

\[
v(t) = \frac{1}{1 + \lambda_1 L} e^{\eta t} \left( \| \phi \|^2_{C^2_{L^2}(\Omega)} + \varepsilon \| \nabla \phi \|^2_{C^2_{L^2}(\Omega)} \right) + \frac{2C_0 |\Omega|}{1 + \lambda_1 L} e^{\eta k} \int_\tau^t e^{\eta \tau} ds \\
+ \frac{1}{1 + \lambda_1 L} e^{\eta k} \int_\tau^t e^{\eta \tau} \| h(s) \|^2_{-1} ds.
\]

(68)

Taking \( a = \tau \), then by (68), we obtain

\[
v(a) = v(\tau) = \frac{1}{1 + \lambda_1 L} e^{\eta \tau} \left( \| \phi \|^2_{C^2_{L^2}(\Omega)} + \varepsilon \| \nabla \phi \|^2_{C^2_{L^2}(\Omega)} \right).
\]

(69)

Let \( \tilde{u}(s) = \frac{C_g}{1 + \lambda_1 L} e^{\eta k} \), then we derive

\[
e^{\int_\tau^a \tilde{u}(s) ds} = e^{\frac{C_g}{1 + \lambda_1 L} e^{\eta k} (t-\tau)}
\]

(70)

and

\[
e^{\int_\tau^t \tilde{u}(r) dr} = e^{\frac{C_g}{1 + \lambda_1 L} e^{\eta k} (t-\tau)}.
\]

(71)

Then from (69) and (70), we arrive at

\[
v(a) e^{\int_\tau^a \tilde{u}(s) ds} = \frac{1}{1 + \lambda_1 L} e^{\eta \tau} \left( \| \phi \|^2_{C^2_{L^2}(\Omega)} + \varepsilon \| \nabla \phi \|^2_{C^2_{L^2}(\Omega)} \right) e^{\frac{C_g}{1 + \lambda_1 L} e^{\eta k} \cdot (t-\tau)}. \]

(72)

Besides, by (68), we deduce

\[
\frac{dv(s)}{ds} = \frac{2C_0 |\Omega|}{1 + \lambda_1 L} e^{\eta k} e^{\eta \tau} + \frac{1}{1 + \lambda_1 L} e^{\eta k} e^{\eta \tau} \| h(s) \|^2_{-1}.
\]

(73)

From (71) and (73), we get

\[
\int_\tau^t e^{\int_\tau^r \tilde{u}(r) dr} \frac{dv}{ds} ds = e^{\frac{C_g}{1 + \lambda_1 L} e^{\eta k} (t-\tau)} \cdot \frac{2C_0 |\Omega|}{1 + \lambda_1 L} e^{\eta k} \int_\tau^t e^{\eta \tau} \left( \frac{C_g}{1 + \lambda_1 L} e^{\eta k} \right) ds
\]
After some simple calculations, we derive the following inequalities:

\[ + e^{\frac{c_0}{\lambda_1} t^2} e^{\eta k} \int_t^t e^s (\eta - \frac{C_0}{\lambda_1} e^{\eta k}) \|h(s)\|^2_{-1} ds. \]  

(74)

Let \( \eta_1 = \eta - \frac{C_0}{\lambda_1} e^{\eta k} > 0 \), then by (71), it follows

\[ \int_a^t e^{\int_a^s \tilde{u}(r) dr} \frac{dv}{ds} ds \leq \frac{2C_0|\Omega|}{(1 + \lambda_1 L)(\lambda_1 L)} e^{\eta_1(k+t)} \]

\[ + \frac{1}{1 + \lambda_1 L} e^{\eta k} e^{\frac{c_0}{\lambda_1} e^{\eta k} \cdot \int_t^t e^{\eta s} \|h(s)\|^2_{-1} ds. \]  

(75)

Then by (72), (75) and the Gronwall inequality, we obtain

\[ w(t) = e^{\eta t} \|u_t\|^2_{\mathcal{L}^2(\Omega)} \leq v(a) e^{\int_a^t \tilde{u}(s) ds} + \int_a^t e^{\int_a^s \tilde{u}(r) dr} \frac{dv}{ds} ds \]

\[ = \frac{1}{1 + \lambda_1 L} e^{\eta t} \left( \|\phi\|^2_{\mathcal{L}^2(\Omega)} + \varepsilon_\tau \|\nabla\phi\|^2_{\mathcal{L}^2(\Omega)} \right) e^{\frac{c_0}{\lambda_1} e^{\eta k} \cdot (t-t)} \]

\[ + \frac{2C_0|\Omega|}{(1 + \lambda_1 L)(\lambda_1 L)} e^{\eta_1(k+t)} \]

\[ + \frac{1}{1 + \lambda_1 L} e^{\eta k} e^{\frac{c_0}{\lambda_1} e^{\eta k} \cdot \int_t^t e^{\eta s} \|h(s)\|^2_{-1} ds. \]  

(76)

Dividing both sides of (65) by \( e^{\eta t} \), we conclude

\[ \|u_t\|^2_{\mathcal{L}^2(\Omega)} + \varepsilon_\tau \|\nabla u_t\|^2_{\mathcal{L}^2(\Omega)} \leq e^{\eta(t-t)} \left( \|\phi\|^2_{\mathcal{L}^2(\Omega)} + \varepsilon_\tau \|\nabla\phi\|^2_{\mathcal{L}^2(\Omega)} \right) \]

\[ + 2C_0 |\Omega| e^{\eta(k-t)} \int_t^t e^{\eta s} ds + C_0 e^{\eta(k-t)} \int_t^t e^{\eta s} \|u_s\|^2_{\mathcal{L}^2(\Omega)} ds \]

\[ + e^{\eta(k-t)} \int_t^t e^{\eta s} \|h(s)\|^2_{-1} ds. \]  

(77)

After some simple calculations, we derive the following inequalities

\[ (a)_1 = e^{\eta(t-t)} \left( \|\phi\|^2_{\mathcal{L}^2(\Omega)} + \varepsilon_\tau \|\nabla\phi\|^2_{\mathcal{L}^2(\Omega)} \right) \]

\[ \leq e^{-\eta_1(t-t)} \left( \|\phi\|^2_{\mathcal{L}^2(\Omega)} + \varepsilon_\tau \|\nabla\phi\|^2_{\mathcal{L}^2(\Omega)} \right), \]  

(78)

\[ (b)_1 = 2C_0 |\Omega| e^{\eta(k-t)} \int_t^t e^{\eta s} ds \leq \frac{2C_0 |\Omega|}{\eta} e^{\eta k} \]  

(79)

and

\[ (c)_1 = e^{\eta(k-t)} \int_t^t e^{\eta s} \|h(s)\|^2_{-1} ds \leq e^{\eta k} \int_t^t e^{-\eta_1(t-t)} \|h(s)\|^2_{-1} ds. \]  

(80)

Substituting (76), (78) – (80) into (77), we deduce

\[ \|u_t\|^2_{\mathcal{L}^2(\Omega)} + \varepsilon_\tau \|\nabla u_t\|^2_{\mathcal{L}^2(\Omega)} \leq (a)_1 + (b)_1 + (c)_1 + (a) + (b) + (c). \]  

(81)
where

\[
(a) = C_{\epsilon} e^{\eta(k-t)} \int_\tau^t \frac{1}{1 + \lambda_1 L} e^{\eta \tau} \left( \|\phi\|_{C_{L^2(\Omega)}}^2 + \epsilon_\tau \|\nabla \phi\|_{C_{L^2(\Omega)}}^2 \right) e^{C_{\epsilon} \eta \epsilon_\tau (s-\tau)} ds, \quad (82)
\]

\[
(b) = C_{\epsilon} e^{\eta(k-t)} \int_\tau^t \frac{2C_0 |\Omega|}{(1 + \lambda_1 L) \eta_1} e^{\eta(k+s)} ds \quad (83)
\]

and

\[
(c) = C_{\epsilon} e^{\eta(k-t)} \int_\tau^t \frac{1}{1 + \lambda_1 L} e^{\eta k} e^{C_{\epsilon} \eta \epsilon_\tau \cdot \eta} \left( \int_{\tau}^s e^{\eta_1 \cdot \|h(s)\|_{-1}} ds \right) ds. \quad (84)
\]

Next, we will estimate (a) to (c) in turn. By some simple estimates, we obtain

\[
(a) \leq \frac{C_{\epsilon} e^{\eta k}}{(1 + \lambda_1 L) (\eta - \eta_1)} \left( \|\phi\|_{C_{L^2(\Omega)}}^2 + \epsilon_\tau \|\nabla \phi\|_{C_{L^2(\Omega)}}^2 \right) e^{-\eta_1 (t-\tau)}, \quad (85)
\]

\[
(b) \leq \frac{C_{\epsilon} e^{\eta k}}{(1 + \lambda_1 L) \eta_1} \cdot \frac{2C_0 |\Omega|}{\eta} e^{\eta k} \quad (86)
\]

and

\[
(c) \leq \frac{C_{\epsilon} e^{\eta k}}{(1 + \lambda_1 L) (\eta - \eta_1)} e^{\eta k} \int_{\tau}^t e^{-\eta_1 (t-s)} \|h(s)\|_{-1}^2 ds. \quad (87)
\]

From the above estimates and (83)–(84), it follows

\[
(a) + (a) \leq \left( 1 + \frac{C_{\epsilon} e^{\eta k}}{(1 + \lambda_1 L) (\eta - \eta_1)} \left( \|\phi\|_{C_{L^2(\Omega)}}^2 + \epsilon_\tau \|\nabla \phi\|_{C_{L^2(\Omega)}}^2 \right) e^{-\eta_1 (t-\tau)} \right), \quad (88)
\]

\[
(b) + (b) \leq \left( 1 + \frac{C_{\epsilon} e^{\eta k}}{(1 + \lambda_1 L) \eta_1} \cdot \frac{2C_0 |\Omega|}{\eta} e^{\eta k} \right) \quad (89)
\]

and

\[
(c) + (c) \leq \left( 1 + \frac{C_{\epsilon} e^{\eta k}}{(1 + \lambda_1 L) (\eta - \eta_1)} \right) e^{\eta k} \int_{\tau}^t e^{-\eta_1 (t-s)} \|h(s)\|_{-1}^2 ds. \quad (90)
\]

Inserting (85)–(87) into (81), we conclude

\[
\|u_t\|_{C_{L^2(\Omega)}}^2 + \epsilon_\tau \|\nabla u_t\|_{C_{L^2(\Omega)}}^2 \leq \left( 1 + \frac{C_{\epsilon} e^{\eta k}}{(1 + \lambda_1 L) \eta_1} \cdot \frac{2C_0 |\Omega|}{\eta} e^{\eta k} \right) \left( \left( 1 + \frac{C_{\epsilon} e^{\eta k}}{(1 + \lambda_1 L) (\eta - \eta_1)} \right) e^{\eta k} \int_{\tau}^t e^{-\eta_1 (t-s)} \|h(s)\|_{-1}^2 ds \right) e^{-\eta_1 (t-\tau)} .
\]
\[ + \left( 1 + \frac{C_g e^{\eta k}}{(1 + \lambda_1 L) (\eta - \eta_1)} \right) e^{\eta k} \int_{t}^{t'} e^{-\eta_1(t-s)} \| h(s) \|^2_{-1} ds. \]  

(88)

Finally, we will consider the case when \( \varepsilon(t) \) is an increasing function. In fact, (55) can be obtained from (3) and (4) through calculations similar to (88), which we will omit here. \( \square \)

In order to obtain the existence of the time-dependent pullback \( D_{\eta_1} \)-absorbing set, we further introduce the following concept.

**Definition 8** (Tempered universe) For each \( \eta_1 > 0 \), let \( D_{\eta_1} \) be the class of all families of nonempty subsets \( \hat{D} = \{ D(t) : t \in \mathbb{R} \} \subset \Gamma \left( C_{\mathcal{H}_t}(\Omega) \right) \) such that

\[ \lim_{\tau \to -\infty} \left( e^{\eta_1 \tau} \sup_{u \in D(\tau)} \| u \|_{C_{\mathcal{H}_t}(\Omega)}^2 \right) = 0. \]

**Lemma 4** Under the assumption of Theorem 3, if \( h(t) \) also satisfies

\[ \int_{-\infty}^{t} e^{\eta_1 s} \| h(s) \|^2_{-1} ds < \infty \]  

(89)

for any \( t \in \mathbb{R} \), then the family \( \hat{D}_1 = \{ D_1(t) : t \in \mathbb{R} \} \) with \( D_1(t) = \hat{B}_{C_{\mathcal{H}_t}(\Omega)} (0, \rho(t)) \), the closed ball in \( C_{\mathcal{H}_t}(\Omega) \) of centre zero and radius \( \rho(t) \), where

\[ \rho^2(t) = C_3 \left( 1 + \frac{C_g e^{\eta k}}{(1 + \lambda_1 L) \eta_1} \right) \frac{2 C_0|\Omega|}{\eta} e^{\eta k} \]

\[ + \left( 1 + \frac{C_g e^{\eta k}}{(1 + \lambda_1 L) (\eta - \eta_1)} \right) e^{\eta k} \int_{\tau}^{t} e^{-\eta_1(t-s)} \| h(s) \|^2_{-1} ds \]  

and \( C_3 = C(\eta, \eta_1, \lambda_1, L, k) \), is the time-dependent pullback \( D_{\eta_1} \)-absorbing family of process \( \{ U(t, \tau) \}_{t \geq \tau} \) in \( C_{\mathcal{H}_t}(\Omega) \). Moreover, \( \hat{D}_1 \in D_{\eta_1} \).

**Proof** From Lemma 3 and Definition 8, we can derive that \( \hat{D}_1 \) is the time-dependent pullback \( D_{\eta_1} \)-absorbing set. Besides, by (90), we obtain \( e^{\eta_1 t} \rho^2(t) \to 0 \) as \( t \to -\infty \). Then thanks to Definition 8, \( \hat{D}_1 \in D_{\eta_1} \) follows directly. \( \square \)

Next, we will use the constructive function method to verify the existence of the time-dependent pullback \( D_{\eta_1} \)-attractor for the process \( \{ U(t, \tau) \}_{t \geq \tau} \) of problem (1). The following lemma can be proved by similar calculations to Theorems 1 and 2.

**Lemma 5** Under the assumptions of Lemma 4, assume \( \phi \in C_{\mathcal{H}_t}(\Omega) \) is given, if \( \{ u^s(t) \}_{s \in \mathbb{N}^+} \) is a sequence of the solutions to problem (1) with initial data \( u^s(\tau) \in C_{\mathcal{H}_t}(\Omega) \), then there exists a subsequence of \( \{ u^s(t) \}_{s \in \mathbb{N}^+} \) that converge strongly in \( L^2(\tau, T; L^2(\Omega)) \).
Now we will establish the pullback $D_{\eta_1}$-asymptotic compactness for the process \( \{ U(t, \tau) \}_{t \geq \tau} \) of problem (1).

**Lemma 6** Under the assumptions of Lemma 4, if \( a(\cdot) \) is locally Lipschitz continuous, then the process \( \{ U(t, \tau) \}_{t \geq \tau} \) is pullback $D_{\eta_1}$-asymptotically compact in $C_{\mathcal{H}_t(\Omega)}$.

**Proof** Suppose \( u^j(t) \) is a weak solution of problem (1) corresponding to the initial value \( \phi^j(x, \theta) \in D_1(\tau) \) \( (j = 1, 2) \), and assume \( \tilde{u} = u^1 - u^2 \), then we arrive at

\[
\partial_t \tilde{u} - \varepsilon(t) \partial_t \Delta \tilde{u} - a(l(u^1)) \Delta u^1 + a(l(u^2)) \Delta u^2 = f(u^1) - f(u^2) + g(t, u^1_t) - g(t, u^2_t) \tag{91}
\]

with initial data

\[
\tilde{u}^j(x, \tau + \theta) = \phi^j(x, \theta), \quad x \in \Omega, \; \theta \in [-k, 0]. \tag{92}
\]

We shall first discuss the case when \( \varepsilon(t) \) is a decreasing function.

Choosing \( \tilde{u} \) as the test function of (91), then we obtain

\[
\frac{d}{dt} \left[ \|\tilde{u}\|^2 + \varepsilon(t) \|\nabla \tilde{u}\|^2 \right] + 2 \left( a(l(u^1)) - a(l(u^2)) \right) \|\nabla \tilde{u}\|^2
\]

\[
= 2 \left( a(l(u^2)) - a(l(u^1)) \right) \langle \nabla u^2, \nabla \tilde{u} \rangle + 2 \left( f(u^1) - f(u^2), \tilde{u} \right)
\]

\[
+ 2 \left( g(t, u^1_t) - g(t, u^2_t), \tilde{u} \right). \tag{93}
\]

Thanks to (6), we conclude

\[
2 \left( f(u^1) - f(u^2), \tilde{u} \right) \leq 2\gamma \|\tilde{u}\|^2. \tag{94}
\]

From assumptions (9)–(11), we deduce

\[
2 \left( g(t, u^1_t) - g(t, u^2_t), \tilde{u} \right) \leq 2C_g \left\| u^1_t - u^2_t \right\|_{C_{L^2(\Omega)}}^2 \|\tilde{u}\|. \tag{95}
\]

Noting that \( a(\cdot) \) is locally Lipschitz continuous, we can obtain the following inequalities from the Young and Cauchy inequalities

\[
2 \left( a(l(u^2)) - a(l(u^1)) \right) \langle \nabla u^2, \nabla \tilde{u} \rangle \leq 2m \left\| \nabla u^1 - \nabla u^2 \right\|^2
\]

\[
+ (L_a(R))^2 \|l\|^2 \|\nabla u^2\|^2 \|u^2 - u^1\|^2 \frac{2m}{2m}, \tag{96}
\]

where \( L_a(R) \) is the Lipschitz constant of the nonlocal term \( a(\cdot) \) in \([-R, R]\).
Substituting (94)–(95) into (93), after some simple calculations, we conclude

\[
\frac{d}{dt} \left[ \| \tilde{u} \|^2 + \varepsilon(t) \| \nabla \tilde{u} \|^2 \right] + 2 \eta \left( \| \tilde{u} \|^2 + \varepsilon(t) \| \nabla \tilde{u} \|^2 \right) \\
\leq 2 \tilde{\eta} \| \tilde{u} \|^2 + 2 C_g \left\| u^1_t - u^2_t \right\|_{C_{L^2(\Omega)}} \| \tilde{u} \| ,
\]  

(97)

where parameter \( \eta > 0 \) is the same as in Lemma 3.

Then by the Gronwall inequality, we obtain

\[
\| \tilde{u}(t) \|^2 + \varepsilon(t) \| \nabla \tilde{u}(t) \|^2 \leq \left( \| \tilde{u}(\tau) \|^2 + \varepsilon(\tau) \| \nabla \tilde{u}(\tau) \|^2 \right) e^{-2\eta(t-\tau)} \\
+ 2 \tilde{\eta} \int_{\tau}^{t} \| \tilde{u}(s) \|^2 ds \\
+ 2 C_g e^{-2\eta t} \int_{\tau}^{t} e^{2\eta s} \left\| u^1_s - u^2_s \right\|_{C_{L^2(\Omega)}} \| \tilde{u}(s) \| ds.
\]  

(98)

Putting \( t + \theta \) instead of \( t \) with \( \theta \in [-k, 0] \) in (98), we conclude

\[
\| \tilde{u}_t \|^2_{C_{L^2(\Omega)}} + \varepsilon_t \| \nabla \tilde{u}(t) \|^2_{C_{L^2(\Omega)}} \leq \left( \| \tilde{u}_\tau \|^2_{C_{L^2(\Omega)}} + \varepsilon_\tau \| \nabla \tilde{u}_\tau \|^2_{C_{L^2(\Omega)}} \right) e^{-2\eta(t-k-\tau)} \\
+ 2 \tilde{\eta} \int_{\tau}^{t} \| \tilde{u}(s) \|^2 ds + 2 C_g e^{-2\eta(t-k)} \int_{\tau}^{t} e^{2\eta s} \left\| u^1_s - u^2_s \right\|_{C_{L^2(\Omega)}} \| \tilde{u}(s) \| ds.
\]  

(99)

Thanks to the Hölder inequality, we arrive at

\[
2 C_g e^{-2\eta(t-k)} \int_{\tau}^{t} e^{2\eta s} \left\| u^1_s - u^2_s \right\|_{C_{L^2(\Omega)}} \| \tilde{u}(s) \| ds \\
\leq 2 C_g e^{-2\eta(t-k)} \left( \int_{\tau}^{t} e^{4\eta s} \left\| u^1_s - u^2_s \right\|^2_{C_{L^2(\Omega)}} ds \right)^{\frac{1}{2}} \left( \int_{\tau}^{t} \| \tilde{u}(s) \|^2 ds \right)^{\frac{1}{2}} \\
\leq 4 C_g e^{-2\eta(t-k)} \left( \int_{\tau}^{t} e^{4\eta s} \left( \left\| u^1_s \right\|^2_{C_{L^2(\Omega)}} + \left\| u^2_s \right\|^2_{C_{L^2(\Omega)}} \right) ds \right)^{\frac{1}{2}} \\
\left( \int_{\tau}^{t} \| \tilde{u}(s) \|^2 ds \right)^{\frac{1}{2}}.
\]  

(100)

From \( \eta_1 < \eta \) and (3), we derive

\[
\left( \| \tilde{u}_\tau \|^2_{C_{L^2(\Omega)}} + \varepsilon_\tau \| \nabla \tilde{u}_\tau \|^2_{C_{L^2(\Omega)}} \right) e^{-2\eta_1(t-k-\tau)} \\
\leq \left( \| \tilde{u}_\tau \|^2_{C_{L^2(\Omega)}} + L \| \nabla \tilde{u}_\tau \|^2_{C_{L^2(\Omega)}} \right) e^{-2\eta_1(t-k-\tau)}.
\]  

(101)
Substituting (100) and (101) into (99), we deduce
\[
\begin{align*}
\|\tilde{u}_t\|_{C^2(L^2(\Omega))}^2 + \varepsilon_t \|\nabla \tilde{u}(t)\|_{C^2(L^2(\Omega))}^2 \\
\leq \left(\|\tilde{u}_\tau\|_{C^2(L^2(\Omega))}^2 + L \|\nabla \tilde{u}_\tau\|_{C^2(L^2(\Omega))}^2\right) e^{-2\eta_1(t-k-\tau)} + 2\tilde{\eta} \int_\tau^t \|	ilde{u}(s)\|^2 ds \\
+ 4C_s e^{-2\eta(t-k)} \left(\int_\tau^t e^{4\eta s} \left(\|u_s^1\|_{C^2(L^2(\Omega))}^2 + \|u_s^2\|_{C^2(L^2(\Omega))}^2\right) ds\right)^{\frac{1}{2}} \\
\left(\int_\tau^t \|	ilde{u}(s)\|^2 ds\right)^{\frac{1}{2}}.
\end{align*}
\] (102)

When \(\varepsilon(t)\) is an increasing function, the following inequality can be derived from some estimates like (102)
\[
\begin{align*}
\|\tilde{u}_t\|_{C^2(L^2(\Omega))}^2 + |\varepsilon_t| \|\nabla \tilde{u}(t)\|_{C^2(L^2(\Omega))}^2 \\
\leq \left(\|\tilde{u}_\tau\|_{C^2(L^2(\Omega))}^2 + L \|\nabla \tilde{u}_\tau\|_{C^2(L^2(\Omega))}^2\right) e^{-2\eta_1(t-k-\tau)} + 2\tilde{\eta} \int_\tau^t \|	ilde{u}(s)\|^2 ds \\
+ 4C_s e^{-2\eta(t-k)} \left(\int_\tau^t e^{4\eta s} \left(\|u_s^1\|_{C^2(L^2(\Omega))}^2 + \|u_s^2\|_{C^2(L^2(\Omega))}^2\right) ds\right)^{\frac{1}{2}} \\
\left(\int_\tau^t \|	ilde{u}(s)\|^2 ds\right)^{\frac{1}{2}}.
\end{align*}
\] (103)

Furthermore, let \(T = t - \tau\) and
\[
\psi_{t,T}(u^1, u^2) = 2\tilde{\eta} \int_\tau^t \|	ilde{u}(s)\|^2 ds \\
+ 4C_s e^{-2\eta(t-k)} \left(\int_\tau^t e^{4\eta s} \left(\|u_s^1\|_{C^2(L^2(\Omega))}^2 + \|u_s^2\|_{C^2(L^2(\Omega))}^2\right) ds\right)^{\frac{1}{2}} \\
\times \left(\int_\tau^t \|	ilde{u}(s)\|^2 ds\right)^{\frac{1}{2}}.
\] (104)

Then by Definition 6, Lemmas 3, 4 and 5, it follows that \(\psi_{t,T}(u^1, u^2)\) is a contractive function. For any constant \(C_4 > 0\), taking \(\tau = t - k + \frac{1}{2\eta_1} \ln \frac{\|\tilde{u}_\tau\|_{C^2(L^2(\Omega))}^2 + L \|
abla \tilde{u}_\tau\|_{C^2(L^2(\Omega))}^2}{C_4}\), then it can be seen that the process \(\{U(t, \tau)\}_{t \geq \tau}\) of problem (1) is pullback \(D_{\eta_1}\)-asymptotically compact in \(C_{\mathcal{H}_t(\Omega)}\). \(\square\)

From the above proofs, it follows the following theorem about the time-dependent pullback \(D_{\eta_1}\)-attractors, which is one of the main results of this paper.

\textbf{Theorem 3} Under the assumptions of Lemma 6 and assume that the function \(h(t)\) satisfies (89), then there exists a unique time-dependent pullback \(D_{\eta_1}\)-attractor \(A_{\eta_1} = \{A_{\eta_1}(t) : t \in \mathbb{R}\}\) of problem (1.1) in \(C_{\mathcal{H}_t(\Omega)}\).
Proof  From Definitions 2 and 5, Theorems 1 and 2, Lemmas 3, 4 and 5, it follows the existence and uniqueness of the above time-dependent pullback $D_{\eta_1}$-attractor $A_{\eta_1}$. □

5 Regularity of the Pullback Attractors

In this section, as in [10, 24, 28, 30], we divide the weak solution $u$ of problem (1) into two parts, and after finding the equations they satisfy respectively, we use the energy method to derive the regularity of $A_{\eta_1}$ for the non-autonomous system (1).

Theorem 4  Under the assumptions of Theorem 3, then $A_{\eta_1}$ is bounded in $C_{H^1_t}(\Omega)$.

Proof  Since $L^2(\Omega) \subset H^{-1}(\Omega)$ is dense (see [1, 11]), for any $h(x, t) \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$, there exists a function $h(\vartheta(x, t)) \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ such that

$$\|h - h^\vartheta\| < \vartheta,$$

where $\vartheta > 0$ is a constant.

Fix $\tau \in \mathbb{R}$ and suppose $\phi \in A_{\eta_1}$, then we can decompose the solution $U(t, \tau + \theta)u(\tau + \theta) = u(t)$ into the sum

$$U(t, \tau + \theta)u(\tau + \theta) = U_a(t, \tau + \theta)u(\tau + \theta) + U_b(t, \tau + \theta)u(\tau + \theta),$$

where $U_a(t, \tau + \theta)u(\tau + \theta) = v(t)$ and $U_b(t, \tau + \theta)u(\tau + \theta) = v_1(t)$ satisfy the following equations, respectively,

$$\begin{aligned}
&\begin{cases}
\partial_t v - \varepsilon(t)\partial_t \Delta v - a(l(u))\Delta v = h(t) - h^\vartheta(t) \quad \text{in } \Omega \times (\tau, \infty), \\
v(x, t) = 0 \quad \text{on } \partial \Omega \times (\tau, \infty), \\
v(x, \tau + \theta) = \phi(x, \theta),
\end{cases} \\
\end{aligned}$$

(106)

and

$$\begin{aligned}
&\begin{cases}
\partial_t v_1 - \varepsilon(t)\partial_t \Delta v_1 - a(l(u))\Delta v_1 = f(u) + g(t, u_t) + h^\vartheta(t) \quad \text{in } \Omega \times (\tau, \infty), \\
v_1(x, t) = 0 \quad \text{on } \partial \Omega \times (\tau, \infty), \\
v_1(x, \tau + \theta) = 0, \\
\end{cases} \\
\end{aligned}$$

(107)

Multiplying (106) by $-\Delta v$ and integrating it in $\Omega$, we arrive at

$$\frac{d}{dt} \left( \|\nabla v\|^2 + \varepsilon(t)\|\Delta v\|^2 \right) + (2a(l(u)) - \varepsilon'(t)) \|\Delta v\|^2 = 2(h - h^\vartheta(t), -\Delta v).$$

By (3), the Cauchy and Poincaré inequalities, then put $t + \theta$ instead of $t$ in the obtained inequality, we deduce

$$\frac{d}{dt} I_1(t) + \varrho I_1(t) \leq \vartheta^2,$$

(108)
where \( I_1(t) = \| \nabla v \|_{C^{2}(\Omega)}^2 + |\varepsilon_\ell| \| \Delta v \|_{C^{2}(\Omega)}^2 \) and \( 0 < \varrho_1 \leq \frac{2+L}{\lambda_1^{-1}+|\varepsilon_\ell|} \).

Using the Gronwall inequality, we obtain

\[
\| U_a(t, \tau + \theta) u(\tau + \theta) \|_{C^{1}(\Omega)}^2 \leq e^{-\varrho_1(t+k-\tau)} \| \phi \|_{C^{1}(\Omega)}^2 + \frac{\varrho^2}{\varrho_1}. \tag{109}
\]

Similarly, multiplying (107) by \(-\Delta v_1\) and integrating it in \(\Omega\), we conclude

\[
\frac{d}{dt} \left( \| \nabla v_1 \|^2 + e(t) \| \Delta v_1 \|^2 \right) + (2a(l(u)) - e'(t)) \| \Delta v_1 \|^2 = 2(f(u) + g(t, u_t) + h^\vartheta, -\Delta v_1). \tag{110}
\]

Besides, from (6), assumptions (9)–(11) and the Young inequality, we derive

\[
2(f(u), -\Delta v_1) \leq 2\bar{\eta} \| u \|^2 + \frac{1}{2} \| \Delta v_1 \|^2, \tag{111}
\]

\[
2(g(t, u_t), -\Delta v_1) \leq C_\vartheta \| u_t \|_{C^2(\Omega)}^2 + \| \Delta v_1 \|^2 \tag{112}
\]

and

\[
2(h^\vartheta, -\Delta v_1) \leq 2\| h^\vartheta \|^2 + \frac{1}{2} \| \Delta v_1 \|^2. \tag{113}
\]

Inserting (111)–(113) into (110) and using (4) and the Poincaré inequality, then putting \( t + \theta \) instead of \( t \) in the obtained equation, it can be seen from Lemma 55 that there is \( 0 < \varrho_2 < \frac{L}{\lambda_1^{-1}+|\varepsilon_\ell|} \) such that

\[
\frac{d}{dt} I_2(t) + \varrho_2 I_2(t) \leq (2\bar{\eta} + C_\vartheta) R_1 + 2\| h^\vartheta \|^2, \tag{114}
\]

where \( I_2(t) = \| \nabla v_1 \|_{C^2(\Omega)}^2 + |\varepsilon_\ell| \| \Delta v_1 \|_{C^2(\Omega)}^2 \) and \( R_1 = \rho^2(t) \).

Then by the Gronwall inequality, we conclude

\[
\| U_b(t, \tau + \theta) u(\tau + \theta) \|_{C^{1}(\Omega)}^2 \leq R_2, \tag{115}
\]

where \( R_2 = e^{-\varrho_2 t} \int_t^t e^{\varrho_2 s} [(2\bar{\eta} + C_\vartheta) R_1 + 2\| h^\vartheta \|^2] ds. \)

Thanks to (109) and (115), for any \( t \in \mathbb{R} \), we obtain

\[
\text{dist} \left( A_{\eta_1}, \bar{B}_{C^{1}(\Omega)}(R_2) \right) \leq C e^{-\varrho(t+k-\tau)} \to 0, \quad \tau \to -\infty, \tag{116}
\]

where \( \varrho > \varrho_2 > 0 \) and

\[
\bar{B}_{C^{1}(\Omega)}(R_2) = \left\{ u(t) \in \bar{B}_{C^{1}(\Omega)} : \| u(t) \|_{C^{1}(\Omega)}^2 \leq R_2 \right\}. \tag{117}
\]

Consequently, we deduce that \( A_{\eta_1} \subseteq \bar{B}_{C^{1}(\Omega)}(R_2) \), which implies the time-dependent pullback \( D_{\eta_1} \)-attractor \( A_{\eta_1} \) is bounded in \( C_{\mathcal{H}_1}(\Omega) \).
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