Explicit character formulae for positive energy unitary irreducible representations of $D = 4$ conformal supersymmetry

V K Dobrev

Theory Division, Department of Physics, CERN CH-1211 Geneva 23, Switzerland

E-mail: Vladimir.Dobrev@cern.ch

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Abstract

This paper continues the project of constructing the character formulae for the positive energy unitary irreducible representations of the $N$-extended $D = 4$ conformal superalgebras $su(2, 2/N)$. In the first paper we gave the bare characters which represent the defining odd entries of the characters. Now we give the full explicit character formulae for $N = 1$ and for several important examples for $N = 2$ and $N = 4$.

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1. Introduction

Recently, superconformal field theories in various dimensions have attracted interest, in particular due to their duality to AdS supergravities. Mostly studied are those for $D \leq 6$ since in these cases the relevant superconformal algebras satisfy [1] the Haag–Lopuszanski–Sohnius theorem [2]. This makes the classification of the unitary irreducible representations (UIRs) of these superalgebras very important. In the 1980s, such a classification was done for the $D = 4$ superconformal algebra $su(2, 2/1)$ [3] and then for the extended superconformal algebras $su(2, 2/N)$ [4] (for arbitrary $N$). Then in the 1990s the classification for $D = 3$ (for even $N$), $D = 5$ and $D = 6$ (for $N = 1, 2$) was given in [5] (some results being conjectural). Later, the $D = 6$ case (for arbitrary $N$) was finalized in [6]. Then, in [7] the classification was done for $osp(1|2n, R)$, which is relevant for $D = 9, 10, 11$.

Once we know the UIRs of a (super-)algebra the next question is to find their characters, since these give the spectrum which is important for applications, especially in super-Yang–Mills theories, super conformal theories, higher spin symmetries, spin chains, superconformal

1 Permanent address: Institute of Nuclear Research and Nuclear Energy Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria.
QCD, etc, see, e.g., [8–14]. Some results on the spectrum of $su(2, 2/N)$ were given in the early papers [4, 15, 16] but it is necessary to have systematic results for which the character formulae are needed. From the mathematical point of view this question is clear only for representations with conformal dimension above the unitarity threshold viewed as irreps of the corresponding complex superalgebra $sl(4/N)$. But for $su(2, 2/N)$ even the UIRs above the unitarity threshold are truncated for small values of spin and isospin. This question was addressed in [17] for the UIRs of $D = 4$ conformal superalgebras $su(2, 2/N)$. There the general theory for the characters of $su(2, 2/N)$ was developed in detail. For the general theory we used the odd reflections introduced in [18] (see also [19]). In [17] we also pin-pointed the difference between character formulae for $sl(4, N)$ and $su(2, 2/N)$, since for the latter we needed to introduce and use the notion of counter-terms$^2$.

The general formulae given in [17] are for the so-called bare characters (or superfield decompositions) and are valid for arbitrary $N$. However, the even entries are not given explicitly.

Thus, the present paper is a follow-up to [17]. The idea is to give the complete explicit character formulae including the even parts of the characters.

We need detailed knowledge of the structure of the UIRs from the representation-theoretical point of view, which is contained in [4, 18, 22, 23]. Following these papers, in section 2 we recall the basic ingredients of the representation theory of the $D = 4$ superconformal algebras. In section 3 we recall the necessary ingredients of character theory, including the character formulae of $su(2, 2)$ and $su(N)$, for which we give explicitly all the formulae that we need. In section 4 we give the explicit complete character formulae for $N = 1$ and for a number of important examples for $N = 2, 4$.

2. Representations of $D = 4$ conformal supersymmetry

2.1. The setting

The superconformal algebras in $D = 4$ are $G = su(2, 2/N)$. The even subalgebra of $G$ is the algebra $G_0 = su(2, 2) \oplus u(1) \oplus su(N)$. We label their physically relevant representations of $G$ by the signature:

$$ \chi = [d; j_1, j_2; z; r_1, \ldots, r_{N-1}] $$

(2.1)

where $d$ is the conformal weight, $j_1, j_2$ are non-negative (half-)integers which are Dynkin labels of the finite-dimensional irreps of the $D = 4$ Lorentz subalgebra $so(3, 1)$ of dimension $(2j_1 + 1)(2j_2 + 1)$, $z$ represents the $u(1)$ subalgebra which is central for $G_0$ (and for $N = 4$ is central for $G$ itself) and $r_1, \ldots, r_{N-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional irreps of the internal (or $R$) symmetry algebra $su(N)$.

We recall that the approach to $D = 4$ conformal supersymmetry developed in [4, 18, 22, 23] involves two related constructions—on function spaces and as Verma modules. The first realization employs the explicit construction of representations of $G$ (and of the corresponding supergroup $\tilde{G} = SU(2, 2/N)$) induced from parabolic subalgebras (subgroups) in spaces of functions (superfields) over superspace which are called elementary representations (ER), see [18]. The UIRs of $G$ are realized as irreducible components of ERs, and then they coincide with the usually used superfields in an indexless notation. This construction is canonical, yet we should mention that some of the resulting superspaces were obtained in the papers [24–28], using the notions of ‘harmonic superspace analyticity’ and ‘Grassmann analyticity’. The relation between the latter approach and ours was commented on in [4, 17, 18].

$^2$ For an alternative approach to character formulae see [20, 21].
The Verma module realization is also very useful as it provides a simpler and more intuitive picture for the relation between reducible ERs, for the construction of the irreps, in particular, of the UIRs. Here we shall actually use the second—Verma module—construction, though we shall mention occasionally the superfield approach.

2.2. Verma modules

To introduce Verma modules one needs the standard triangular decomposition:

$$\mathcal{G}^c = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- \quad (2.2)$$

where $\mathcal{G}^c = sl(4/N)$ is the complexification of $\mathcal{G}$, $\mathcal{G}^+$, $\mathcal{G}^-$, respectively, are the subalgebras corresponding to the positive, negative, roots of $\mathcal{G}^c$, respectively, and $\mathcal{H}$ denotes the Cartan subalgebra of $\mathcal{G}^c$.

We consider the lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$, where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+$, $\Lambda \in \mathcal{H}^*$ is the lowest weight and $v_0$ is the lowest weight vector such that:

$$X v_0 = 0, \quad X \in \mathcal{G}^-,$$

$$H v_0 = \Lambda(H) v_0, \quad H \in \mathcal{H}. \quad (2.3)$$

Further, for simplicity we omit the sign $\otimes$, i.e., we write $P, v_0 \in V^\Lambda$ with $P \in U(\mathcal{G}^+)$. The lowest weight $\Lambda$ is characterized by its values on the Cartan subalgebra $\mathcal{H}$, or, equivalently, by its products with the simple roots (given explicitly below). In general, these would be $N + 3$ complex numbers, however, in order to be useful for the representations of the real form $\mathcal{G}$ these values would be restricted to be real and furthermore to correspond to the signatures $\chi$, and we shall write $\Lambda = \Lambda(\chi)$, or $\chi = \chi(\Lambda)$. Note, however, that there are Verma modules to which no ERs correspond, see [18] and below.

If a Verma module $V^\Lambda$ is irreducible then it gives the lowest weight irrep $L^\Lambda$ with the same weight. If a Verma module $V^\Lambda$ is reducible then it contains a maximal invariant submodule $I^\Lambda$ and the lowest weight irrep $L^\Lambda$ with the same weight is given by the factorization: $L^\Lambda = V^\Lambda / I^\Lambda$ [29–31].

Thus, we first need to know which Verma modules are reducible. The reducibility conditions for the highest weight Verma modules over the basic classical Lie superalgebra were given by Kac [31]. Translating his conditions to the lowest weight Verma modules we have [18]:

The lowest weight Verma module $V^\Lambda$ is reducible only if at least one of the following conditions is true:\(^3\)

$$\begin{align*}
(\rho - \Lambda, \beta) &= m(\beta, \beta)/2, \quad \beta \in \Delta^+, (\beta, \beta) \neq 0, \quad m \in \mathbb{N}, \quad (2.4a) \\
(\rho - \Lambda, \beta) &= 0, \quad \beta \in \Delta^+, (\beta, \beta) = 0, \quad (2.4b)
\end{align*}$$

where $\Delta^+$ is the positive root system of $\mathcal{G}^c$, $\rho \in \mathcal{H}^*$ is the very important in representation theory element given by $\rho = \rho_0 - \rho_1$, where $\rho_0, \rho_1$ are the half-sums of the even, odd, respectively, positive roots, $(\cdot, \cdot)$ is the standard bilinear product in $\mathcal{H}^*$.

If a condition from (2.4a) is fulfilled then $V^\Lambda$ contains a submodule which is a Verma module $V^{\Lambda'}$ with shifted weight given by the pair $m, \beta : \Lambda' = \Lambda + m\beta$. The embedding of $V^{\Lambda'}$ in $V^\Lambda$ is provided by mapping the lowest weight vector $v'_0$ of $V^{\Lambda'}$ to the singular vector $v^\rho_{m,\beta}$ in $V^\Lambda$ which is completely determined by the conditions:

$$\begin{align*}
X v^\rho_{m,\beta} &= 0, \quad X \in \mathcal{G}^-, \\
H v^\rho_{m,\beta} &= \Lambda'(H) v_0, \quad H \in \mathcal{H}, \quad \Lambda' = \Lambda + m\beta. \quad (2.5)
\end{align*}$$

\(^3\) Many statements below are true for any basic classical Lie superalgebra, and would require changes only for the superalgebras $osp(1/2N)$. 

\[\text{J. Phys. A: Math. Theor. 46 (2013) 405202} \quad V K \text{ Dobrev}\]
Explicitly, $v_{m, β}^{m, β}$ is given by an even polynomial in the positive root generators:

$$v_{β}^{m, β} = p_{m, β} v_{0}, \quad p_{m, β} \in U(\mathcal{G}^+).$$

(2.6)

Thus, the submodule of $V^{λ'}$, which is isomorphic to $V^{λ'}$, is given by $U(\mathcal{G}^+), p_{m, β}, v_0$. (More on the even case following the same approach may be seen in, e.g., [32, 33].)

If a condition from (2.4b) is fulfilled then $V^{λ'}$ contains a submodule $I^β$ obtained from the Verma module $V^{λ'}$, with shifted weight $λ' = λ + β$ as follows. In this situation $V^{λ}$ contains a singular vector

$$X v_β^{β} = 0, \quad X \in \mathcal{G}_- \quad (2.7)$$

Explicitly, $v_β^{β}$ is given by an odd polynomial in the positive root generators:

$$v_β^{β} = p_{β} v_0, \quad p_{β} \in U(\mathcal{G}^+).$$

(2.8)

Then we have:

$$I^β = U(\mathcal{G}^+) p_{β} v_0,$$

(2.9)

which is smaller than $V^{λ'} = U(\mathcal{G}^+) v_0$ since this polynomial is Grassmannian:

$$(p_{β})^2 = 0.$$ (2.10)

To describe this situation we say that $V^{λ'}$ is oddly embedded in $V^{λ}$.

Note, however, that the above formulae also describe more general situations when the difference $λ' - λ = β$ is not a root, as used in [18] and below.

The weight shifts $λ' = λ + β$, when $β$ is an odd root, are called odd reflections in [18] (see also [19]). We recall from [18] the definition of the odd reflection $s_α, α \in Δ_1$, acting on $γ \in Δ_1$:

$$s_α • γ = γ - (γ, α') α, \quad (α, α) \neq 0, \quad s_α • γ = γ - (-1)^{(γ)} (γ, α) α, \quad (α, α) = 0, \quad α \neq γ, \quad s_α • α = -α, \quad (α, α) = 0,$$ (2.11)

where $(\cdot, \cdot)$ is the standard bilinear product in $\mathcal{H}^*$, $p(γ)$ is the parity of $γ$:

$$p(γ) = \begin{cases} 0 & \text{if } γ \in Δ^0_1, \\ 1 & \text{if } γ \in Δ_1. \end{cases}$$ (2.12)

Further, to be more explicit we need to recall the root system of $\mathcal{G}^+$—for definiteness—as used in [18]. The positive root system $Δ^+$ is comprised from $α_{ij}, 1 \leq i < j \leq 4 + N$, the even positive root system $Δ^+_0$ is comprised from $α_{ij}$, with $i, j \leq 4$ and $i, j \geq 5$; the odd positive root system $Δ^+_1$ is comprised from $α_{ij}$, with $i \leq 4, j \geq 5$. The even system is actually the root system of $sl(4) \otimes sl(N)$ with simple roots $\{α_1, α_2, α_3, [α_5, \ldots, α_{3+N}]\}$, respectively, where $α_j \equiv α_{j,j+1}$. The simple roots of the superalgebra are chosen as in (2.4) of [18]:

$$γ_1 = α_{12}, γ_2 = α_{34}, γ_3 = α_{25}, γ_4 = α_{4,4+N}, γ_k = α_{k,k+1}, 5 \leq k \leq 3 + N.$$ (2.13)

Thus, the Dynkin diagram is:

$$\begin{array}{cccccccc}
\circ & \times & \circ & \circ & \circ & \cdots & \circ & \times & \circ \\
1 & 3 & 5 & \cdots & 3+N & 4 & 2
\end{array}$$ (2.14)

This is a non-distinguished simple root system with two odd simple roots (for the various root systems of the basic classical superalgebras we refer to [34]).

We choose this diagram since it has a mirror symmetry (conjugation):

$$γ_1 \leftrightarrow γ_2.$$ (2.15a)
\[ \gamma_1 \leftrightarrow \gamma_4, \]  
\[ \gamma_j \leftrightarrow \gamma_{N+8-j}, \quad j \geq 5, \]  
and furthermore it is consistent with the mirror symmetry of the sl(4) and sl(N) root systems by identifying: \( \gamma_1 \mapsto \alpha_1, \gamma_2 \mapsto \alpha_3, \) and \( \gamma_j \mapsto \alpha_j, \quad j \geq 5, \) respectively.

The reducibility conditions w.r.t. to the positive roots coming from sl(4)su(2, 2) coming from (2.4) (denoting \( m \mapsto n_j \) for \( \beta \mapsto \alpha_j \)) are:

\[ n_1 = 1 + 2j_1 \]  
\[ n_2 = 1 - d_j - j_1 - j_2 \]  
\[ n_3 = 1 + 2j_2 \]  
\[ n_{12} = 2 - d + j_1 - j_2 = n_1 + n_2 \]  
\[ n_{23} = 2 - d - j_1 + j_2 = n_2 + n_3 \]  
\[ n_{13} = 3 - d + j_1 + j_2 = n_1 + n_2 + n_3. \]

Thus, reducibility conditions (2.16a), (2.16c) are fulfilled automatically for \( \Lambda(\chi) \) with \( \chi \) from (2.1) since we always have: \( n_1, n_3 \in \mathbb{N} \). There are no such conditions for the ERs since they are induced from the finite-dimensional irreps of the Lorentz subalgebra (parametrized by \( j_1, j_2 \)). However, to these two conditions correspond differential operators of the order \( 1 + 2j_1 \) and \( 1 + 2j_2 \) (as we mentioned above) and these annihilate all functions of the ERs with signature \( \chi \).

The reducibility conditions w.r.t. to the positive roots coming from sl(N)su(N)) are all fulfilled for \( \Lambda(\chi) \) with \( \chi \) from (2.1). In particular, for the simple roots from those condition (2.4) is fulfilled with \( \beta \mapsto \gamma_j, m = 1 + r_{N+4-j}, \) for every \( j = 5, 6, \ldots, N + 3 \). There are no such conditions for the ERs since they are induced from the finite-dimensional UIRs of su(N). However, to these \( N - 1 \) conditions correspond \( N - 1 \) differential operators of orders \( 1 + r_k \) (as we mentioned) and the functions of our ERs are annihilated by all these operators [18].

For future use we note also the following decompositions:

\[ \Lambda = \sum_{j=1}^{N+3} \lambda_j \alpha_{j,j+1} = \Lambda^r + \Lambda^z + \Lambda^u \]  
\[ \Lambda^r = \sum_{j=1}^{3} \lambda_j \alpha_{j,j+1}, \quad \Lambda^z = \lambda_4 \alpha_{45}, \quad \Lambda^u = \sum_{j=5}^{N+3} \lambda_j \alpha_{j,j+1} \]

which actually employ the distinguished root system with one odd root \( \alpha_{45} \). The reducibility conditions for the \( 4N \) odd positive roots of \( \mathcal{G} \) are [4, 18]:

\[ d = d^{1}_{Nk} \equiv z_Nd_{4N} \]  
\[ d^{1}_{Nk} \equiv 4 - 2k + 2j_2 + z + 2m_k - 2m / N \]  
\[ d = d^{2}_{Nk} \equiv z_Nd_{4N} \]  
\[ d^{2}_{Nk} \equiv 2 - 2k + 2j_2 + z + 2m_k - 2m / N \]

\(^4\) Note that there are actually as many operators as positive roots of sl(N) but all are expressed in terms of the \( N - 1 \) above corresponding to the simple roots [18].
Thus, \( \alpha \) is the necessary and sufficient condition for this is: 

\[
d = d_N^N + z\delta_N d_N^N \\
d_N^N = 2 + 2k - 2N + 2j_1 - z - 2m_k + 2m/N \\
d = d_N^N + z\delta_N d_N^N \\
d_N^N = 2k - 2N - 2j_1 - z - 2m_k + 2m/N
\]  

(2.18c,k)

where in all four cases of (2.18) \( k = 1, \ldots, N, m_N = 0 \), and

\[
m_k \equiv \sum_{i=k}^{N-1} r_i, \quad m \equiv \sum_{k=1}^{N-1} m_k = \sum_{k=1}^{N-1} kr_k
\]  

(2.19)

Note that for a fixed module and fixed \( i = 1, 2, 3, 4 \) only one of the odd \( N \) conditions involving \( d_N^N \) may be satisfied. Thus, no more than four (two, for \( N = 1 \)) of the conditions (2.18) may hold for a given Verma module.

**Remark.** Note that for \( n_2 \in \mathbb{N} \) (see (2.18)) the corresponding irreps of \( su(2,2) \) are finite-dimensional (the necessary and sufficient condition for this is: \( n_1, n_2, n_3 \in \mathbb{N} \)). Then the reducible LWMM \( L_{\Lambda} \) of \( su(2,2) \mathbb{N} \) are also finite-dimensional (and non-unitary) independently on whether the corresponding Verma module \( V^\Lambda \) is reducible w.r.t. any odd root. If \( V^\Lambda \) is not reducible w.r.t. any odd root, then these finite-dimensional irreps are called ‘typical’ [31], otherwise, the irreps are called ‘atypical’ [31]. In our considerations \( n_2 \notin \mathbb{N} \) in all cases, except the trivial one-dimensional UIR (for which \( n_2 = 1 \), see below).

We shall consider quotients of Verma modules factoring out the *even* submodules for which the reducibility conditions are always fulfilled. Before this we recall the root vectors following [18]. The positive (negative) root vectors corresponding to \( \alpha_{ij}, \quad (\alpha_{ij}) \), are denoted by \( X_{ij}^+, \quad (X_{ij}^-) \). The simple root vectors \( X_{ij}^+ \) follow the notation of the simple roots: \( X_{ij}^+ = X_{j,j+1}^+ \).

The mentioned submodules are generated by the singular vectors related to the even simple roots \( \alpha_1, \alpha_3, \alpha_5, \ldots, \alpha_{N+3} \) [18]:

\[
v_1^1 = (X_{11}^+)^{1/2} v_0, \quad (2.20a) \\
v_3^1 = (X_{31}^+)^{1/2} v_0, \quad (2.20b) \\
v_j^j = (X_{j,j+1}^+)^{1/2} v_0, \quad j = 5, \ldots, N + 3
\]  

(2.20c)

(for \( N = 1 \) (2.20c) being empty). The corresponding submodules are \( I_\Lambda^V = U(G^+), \quad v_\Lambda^V \), and the invariant submodule to be factored out is:

\[
I_\Lambda^V = \bigcup_k I_k^\Lambda.
\]  

(2.21)

Thus, instead of \( V^\Lambda \) we shall consider the factor-modules:

\[
\tilde{V}^\Lambda = V^\Lambda / I_\Lambda^\Lambda
\]  

(2.22)

which are closer to the structure of the ERs

In the factorized modules the singular vectors (2.20) become null conditions, i.e., denoting by \( |\tilde{\Lambda}\rangle \) the lowest weight vector of \( \tilde{V}^\Lambda \), we have:

\[
(X_{11}^+)^{1/2} |\tilde{\Lambda}\rangle = 0, \quad (2.23a) \\
(X_{31}^+)^{1/2} |\tilde{\Lambda}\rangle = 0, \quad (2.23b) \\
(X_{j,j+1}^+)^{1/2} |\tilde{\Lambda}\rangle = 0, \quad j = 5, \ldots, N + 3.
\]  

(2.23c)

For explicit expressions for the \( sl(n) \) singular vectors we refer to [35].
2.3. Singular vectors and invariant submodules at the unitary reduction points

We first recall the result of [4] (see part (i) of the theorem there) that the following is the complete list of lowest weight (positive energy) UIRs of \(su(2,2/N)\):

\[
d \geq d_{\text{max}} = \max \{d_{N1}^1, d_{NN}^3\},
\]

\[
d = d_{NN}^2 \geq d_{N1}^1, \quad j_1 = 0,
\]

\[
d = d_{N1}^3 \geq d_{NN}^3, \quad j_2 = 0,
\]

\[
d = d_{NN}^4 = d_{N1}^4, \quad j_1 = j_2 = 0,
\]

(2.24a)
(2.24b)
(2.24c)
(2.24d)

where \(d_{\text{max}}\) is the threshold of the continuous unitary spectrum.

Note that in case (d) we have \(d = m_1, z = 2m/N - m_1\), and that it is trivial for \(N = 1\) since then the internal symmetry algebra \(su(N)\) is trivial and by definition \(m_1 = m = 0\) (the resulting irrep is one-dimensional with \(d = z = j_1 = j_2 = 0\)). The UIRs for \(N = 1\) were first given in [3].

Next we note that if \(d > d_{\text{max}}\) the factorized Verma modules are irreducible and coincide with the UIRs \(L_{\lambda}\). These UIRs are called long in the modern literature, see, e.g., [27, 28, 36–41]. Analogously, we shall use for the cases when \(d = d_{\text{max}}\), i.e., (2.24a), the terminology of semi-short UIRs, introduced in [28, 36, 37], while the cases (2.24b)–(2.24d) are also called short UIRs, see, e.g., [27, 28, 37–41].

Next consider in more detail the UIRs at the four distinguished reduction points determining the list above:

\[
d_{N1}^1 = 2 + 2 j_2 + z + 2m_1 - 2m/N,
\]

\[
d_{N1}^2 = z + 2m_1 - 2m/N, \quad (j_2 = 0),
\]

\[
d_{NN}^3 = 2 + 2 j_1 - z + 2m/N,
\]

\[
d_{NN}^4 = -z + 2m/N, \quad (j_1 = 0).
\]

(2.25)

The singular vectors corresponding to these points were given in [18], (8.9a), (8.7b), (8.8a), (8.7a). From the expressions of the singular vectors follow, using (2.9), the explicit formulae for the corresponding invariant submodules \(I^d\) of the modules \(\tilde{V}_\lambda\):

\[
I^1 = U(\mathcal{G}^+) \left( 2 j X_{3,4+N}^+ - X_1^+ X_2^+ \right) |\lambda\rangle, \quad d = d_{N1}^1, \quad j_2 > 0
\]

\[
I^2 = U(\mathcal{G}^+) X_1^+ |\lambda\rangle, \quad d = d_{N1}^2,
\]

\[
I^3 = U(\mathcal{G}^+) \left( 2 j_1 X_{15}^+ - X_3^+ X_4^+ \right) |\lambda\rangle, \quad d = d_{NN}^3, \quad j_1 > 0
\]

\[
I^4 = U(\mathcal{G}^+) X_2^+ |\lambda\rangle, \quad d = d_{NN}^4,
\]

\[
I^{12} = U(\mathcal{G}^+) \tilde{u}^{12} = X_1^+ X_2^+ X_4^+ |\lambda\rangle, \quad d = d_{N1}^1, \quad j_2 = 0,
\]

\[
I^{13} = U(\mathcal{G}^+) \tilde{u}^{13} = X_1^+ X_3^+ X_4^+ |\lambda\rangle, \quad d = d_{NN}^3, \quad j_1 = 0.
\]

(2.26a)
(2.26b)
(2.26c)
(2.26d)
(2.26e)
(2.26f)

In the cases (2.26a)–(2.26d) to the singular vectors in the ER picture there correspond first-order super-differential operators given explicitly in formulae (7) of [4]. The invariant submodules are the kernels of these super-differential operators.

Note that there is a subtlety for \(d_{N1}^1, d_{NN}^3\), when \(j_2 = 0, j_1 = 0\), respectively, since in these cases the invariant submodules in (2.26a) and (2.26c), respectively, trivialize (as \(X_1^+, |\lambda\rangle = 0, X_1^+, |\lambda\rangle = 0\), respectively). The embeddings which replace them are given in (2.26e) and
(2.26f) and they arise from singular vectors $\tilde{v}^{12}$, $\tilde{v}^{34}$, which correspond to $\beta = \alpha_{3,4+N} + \alpha_{4,4+N}$, $\beta = \alpha_{15} + \alpha_{25}$ (both sums are not roots!), see [18] the formula before (D.4), formula (D.1), respectively. To the last two singular vectors in the ER picture correspond second-order super-differential operators given explicitly in formulae (11a), (11b) of [4], and in formulae (D3), (D5) of [18].

The invariant submodules (2.26) were used in [4] in the construction of the UIRs.

We note a partial ordering of the four distinguished points (2.26) of reducibility of Verma modules:

$$d_{N1}^1 > d_{N1}^2, \quad d_{NN}^3 > d_{NN}^4.$$  \hfill (2.27)

Due to this ordering at most two of these four points may coincide. Thus, we have two possible situations: of Verma modules (or ERs) reducible at one and at two reduction points from (2.26).

First we deal with the situations in which no two of the points in (2.26) coincide. According to [4] (theorem) there are four such situations involving UIRs:

a) $d = d_{\text{max}} = d_{N1}^1 > d_{NN}^3$, \hfill (2.28a)
b) $d = d_{N1}^1 > d_{NN}^3$, $j_2 = 0$, \hfill (2.28b)
c) $d = d_{\text{max}} = d_{NN}^3 > d_{N1}^1$, \hfill (2.28c)
d) $d = d_{NN}^3 > d_{N1}^1$, $j_1 = 0$. \hfill (2.28d)

We shall call these cases single-reducibility-condition (SRC) Verma modules or UIRs, depending on the context. In addition, as already stated, we use for the cases when $d = d_{\text{max}}$, i.e., (2.28a) and (2.28c), the terminology of semi-short UIRs, while the cases (2.28b) and (2.28d) are also called short UIRs.

The factorized Verma modules $\tilde{V}^{\Lambda}$ with the unitary signatures from (2.28) have only one invariant (odd) submodule which has to be factorized in order to obtain the UIRs. These odd embeddings are given explicitly as:

$$\tilde{V}^{\Lambda} \rightarrow \tilde{V}^{\Lambda+\beta}, \Lambda + \beta = s_{\beta} \cdot \Lambda$$  \hfill (2.29)

where we use the convention [22] that arrows point to the oddly embedded module, and there are the following cases for $\beta$:

$$\beta = \alpha_{3,4+N}, \quad \text{for} \ (2.28a), \quad j_2 > 0,$$  \hfill (2.30a)

$$= \alpha_{4,4+N}, \quad \text{for} \ (2.28b),$$  \hfill (2.30b)

$$= \alpha_{15}, \quad \text{for} \ (2.28c), \quad j_1 > 0,$$  \hfill (2.30c)

$$= \alpha_{25}, \quad \text{for} \ (2.28d),$$  \hfill (2.30d)

$$= \alpha_{3,4+N} + \alpha_{4,4+N}, \quad \text{for} \ (2.28a), \quad j_2 = 0,$$  \hfill (2.30e)

$$= \alpha_{15} + \alpha_{25}, \quad \text{for} \ (2.28c), \quad j_1 = 0.$$  \hfill (2.30f)

Note that in (2.29) in the cases (2.30e) and (2.30f) we have extended the action of the odd reflections, defined in (2.12) for root elements, to sums of odd roots which are not roots.

The diagram (2.29) gives the UIR $L_{\Lambda}$ contained in $V^{\Lambda}$ as follows:

$$L_{\Lambda} = \tilde{V}^{\Lambda}/\beta,$$  \hfill (2.31)

Note that w.r.t. $V^{\Lambda}$ the analogues of the vectors $\tilde{v}^{34}$ and $\tilde{v}^{12}$ are not singular, but subsingular vectors, see [17].

---

\[ \]
where \( I^\beta \) is given by \( I^1, I^2, I^3, I^4, I^{12}, I^{34} \), respectively (see (2.26)), in the cases (2.30a)–(2.30f), respectively.

It is useful to record the signatures of the shifted lowest weights, i.e., \( \chi' = \chi (\Lambda + \beta) \). In fact, for future use we give the signature changes for arbitrary roots. The explicit formulae are [18, 22]:

\[
\beta = \alpha_{3,N+5-k} : \chi' = [d + \frac{1}{2}; j_1, j_2 - \frac{1}{2}; z + \epsilon_N; r_1, \ldots, r_{k-1} - 1, r_k + 1, \ldots, r_{N-1}],
\]

\( j_2 > 0, \quad r_{k-1} > 0 \) \hspace{1cm} (2.32a)

\[
\beta = \alpha_{4,N+5-k} : \chi' = [d + \frac{1}{2}; j_1, j_2 + \frac{1}{2}; z + \epsilon_N; r_1, \ldots, r_{k-1} - 1, r_k + 1, \ldots, r_{N-1}],
\]

\( r_{k-1} > 0 \) \hspace{1cm} (2.32b)

\[
\beta = \alpha_{1,N+5-k} : \chi' = [d + \frac{1}{2}; j_1 - \frac{1}{2}, j_2; z - \epsilon_N; r_1, \ldots, r_{k-1} + 1, r_k - 1, \ldots, r_{N-1}],
\]

\( j_1 > 0, \quad r_k > 0 \) \hspace{1cm} (2.32c)

\[
\beta = \alpha_{2,N+5-k} : \chi' = [d + \frac{1}{2}; j_1 + \frac{1}{2}, j_2; z - \epsilon_N; r_1, \ldots, r_{k-1} + 1, r_k - 1, \ldots, r_{N-1}],
\]

\( r_k > 0 \) \hspace{1cm} (2.32d)

\[
k = 1, \ldots, N, \quad \epsilon_N = \frac{1}{2} - \frac{\beta}{N}.
\]

For each fixed \( \chi \) the lowest weight \( \Lambda(\chi') \) fulfil the same odd reducibility condition as \( \Lambda(\chi') \). We need also the special cases used in (2.30e) and (2.30f):

\[
\beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N} : \chi_{12}' = [d + 1; j_1, 0; z + 2\epsilon_N; r_1 + 2, r_2, \ldots, r_{N-1}],
\]

\( j_2 = 0, \quad d = d_{N1}^2 \) \hspace{1cm} (2.32e)

\[
\beta_{34} = \alpha_{15} + \alpha_{25} : \chi_{34}' = [d + 1; 0, j_2; z - 2\epsilon_N; r_1, \ldots, r_{N-2}, r_{N-1} + 2],
\]

\( j_1 = 0, \quad d = d_{NN}^1 \) \hspace{1cm} (2.32f)

The lowest weight \( \Lambda(\chi_{12}') \) fulfils (2.28b), while the lowest weight \( \Lambda(\chi_{34}') \) fulfils (2.28d).

We now consider the situations in which \( \nu \omega \) of the points in (2.25) coincide. According to [4] (theorem) there are four such situations involving UIRs:

\[
\text{ac} \quad d = d_{\max} = d^{ac} \equiv d_{N1}^3 = d_{NN}^3, \hspace{1cm} (2.34a)
\]

\[
\text{ad} \quad d = d_{N1}^3 = d_{NN}^3, \quad j_1 = 0, \hspace{1cm} (2.34b)
\]

\[
\text{bc} \quad d = d_{N1}^3 = d_{NN}^3, \quad j_2 = 0, \hspace{1cm} (2.34c)
\]

\[
\text{bd} \quad d = d_{N1}^3 = d_{NN}^3, \quad j_1 = j_2 = 0, \hspace{1cm} (2.34d)
\]

We shall call these double-reducibility-condition (DRC) Verma modules or UIRs. As in the previous subsection we shall use for the cases when \( d = d_{\max} \), i.e., (2.34a), also the terminology of semi-short UIRs, [36, 28], while the cases (2.34b)–(2.34d) shall also be called short UIRs [27, 28, 38–41].

To finalize the structure we should check the even reducibility conditions (2.16b)–(2.16f). This analysis was done in [17], and the results are as follows.
The embedding diagrams for the corresponding modules $\tilde{V}^\Lambda$ without even embeddings are:

\[
\tilde{V}^{\Lambda + \beta'}
\]

\[
\uparrow
\]

\[
\tilde{V}^\Lambda \rightarrow \tilde{V}^{\Lambda + \beta}
\]

where $\Lambda + \beta = s_\beta \cdot \Lambda$, $\Lambda + \beta' = s_{\beta'} \cdot \Lambda$,

\[
(\beta, \beta') = (\alpha_{15}, \alpha_{3,4+n}), \quad \text{for (2.34a), } m_1 j_1 j_2 > 0
\]

\[
= (\alpha_{15}, \alpha_{3,4+n} + \alpha_{3,4+n}), \quad \text{for (2.34a), } j_1 > 0, j_2 = 0
\]

\[
= (\alpha_{15} + \alpha_{25}, \alpha_{3,4+n}), \quad \text{for (2.34a), } j_1 = 0, j_2 > 0
\]

\[
= (\alpha_{15} + \alpha_{25}, \alpha_{3,4+n} + \alpha_{3,4+n}), \quad \text{for (2.34a), } j_1 = j_2 = 0
\]

\[
= (\alpha_{25}, \alpha_{3,4+n}), \quad \text{for (2.34b), } j_2 > 0, 2j_2 + m_1 \geq 2
\]

\[
= (\alpha_{25}, \alpha_{3,4+n} + \alpha_{4,4+n}), \quad \text{for (2.34b), } j_2 = 0, m_1 > 0
\]

\[
= (\alpha_{15}, \alpha_{4,4+n}), \quad \text{for (2.34c), } j_1 > 0, 2j_1 + m_1 \geq 2
\]

\[
= (\alpha_{15} + \alpha_{25}, \alpha_{4,4+n}), \quad \text{for (2.34c), } j_1 = 0, m_1 > 0
\]

\[
= (\alpha_{25}, \alpha_{4,4+n}), \quad \text{for (2.34d), } m_1 \neq 1.
\]

This diagram gives the UIR $L_\Lambda$ contained in $\tilde{V}^\Lambda$ as follows:

\[
L_\Lambda = \tilde{V}^\Lambda / p^{\beta, \beta'}, \quad p^{\beta, \beta'} = p^\beta \cup p^{\beta'}
\]

where $p^\beta$, $p^{\beta'}$ are given in (2.26), accordingly to the cases in (2.36).

The embedding diagrams for the corresponding modules $\tilde{V}^\Lambda$ with even embeddings are:

\[
\tilde{V}^{\Lambda + \beta'}
\]

\[
\uparrow
\]

\[
\tilde{V}^{\Lambda + \beta} \leftarrow \tilde{V}^\Lambda \rightarrow \tilde{V}^{\Lambda + \beta}
\]

where $\Lambda + \beta_e = s_{\beta_e} \cdot \Lambda$, (note $n_{\beta_e} = 1$, [17]),

\[
(\beta, \beta', \beta_e) = (\alpha_{15}, \alpha_{3,4+n}, \alpha_{14}), \quad \text{for (2.34a), } j_1 j_2 > 0, m_1 = 0
\]

\[
= (\alpha_{25}, \alpha_{3,4+n}, \alpha_{24}), \quad \text{for (2.34b), } j_2 = \frac{1}{2}, m_1 = 0
\]

\[
= (\alpha_{25}, \alpha_{3,4+n} + \alpha_{4,4+n}, \alpha_{23} + \alpha_{14}), \quad \text{for (2.34b), } j_2 = m_1 = 0
\]

\[
= (\alpha_{15}, \alpha_{4,4+n}, \alpha_{13}), \quad \text{for (2.34c), } j_1 = \frac{1}{2}, m_1 = 0
\]

\[
= (\alpha_{15} + \alpha_{25}, \alpha_{4,4+n}, \alpha_{23} + \alpha_{14}), \quad \text{for (2.34c), } j_1 = m_1 = 0
\]

\[
= (\alpha_{25}, \alpha_{4,4+n}, \alpha_{23} + \alpha_{14}), \quad \text{for (2.34d), } m_1 = 1.
\]

This diagram gives the UIR $L_\Lambda$ contained in $\tilde{V}^\Lambda$ as follows:

\[
L_\Lambda = \tilde{V}^\Lambda / p^{\beta, \beta', \beta_e}, \quad p^{\beta, \beta'} = p^\beta \cup p^{\beta'} \cup \tilde{V}^{\Lambda + \beta}.
\]

Naturally, the two odd embeddings in (2.35) or (2.38) are the combination of the different cases of (2.29).
3. Character formulae of positive energy UIRs

3.1. Character formulae: generalities

In the beginning of this subsection we follow [29]. Let \( \hat{G} \) be a simple Lie algebra of rank \( \ell \) with Cartan subalgebra \( \hat{\mathcal{H}} \), root system \( \hat{\Lambda} \), simple root system \( \hat{\pi} \). Let \( \Gamma \), (respectively \( \Gamma_+ \)), be the set of all integral, (respectively integral dominant), elements of \( \hat{\mathcal{H}}^* \), i.e., \( \lambda \in \hat{\mathcal{H}}^* \) such that \( (\lambda, \alpha_i^\vee) \in \mathbb{Z} \), for all simple roots \( \alpha_i \), \( (\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i, \alpha_i)) \). Let \( V \) be a lowest weight module with lowest weight \( \Lambda \) and lowest weight vector \( v_0 \). It has the following decomposition:

\[
V = \bigoplus_{\mu \in \Gamma_+} V_{\mu}, \quad V_{\mu} = \{ u \in V | Hu = (\lambda + \mu)(H)u, \ \forall H \in \mathcal{H} \}. \tag{3.1}
\]

(Note that \( V_0 \equiv \mathbb{C} v_0 \).) Let \( E(\mathcal{H}^*) \) be the associative Abelian algebra consisting of the series \( \sum_{\mu \in \mathcal{H}^*} c_\mu e(\mu) \), where \( c_\mu \in \mathbb{C} \), \( c_\mu = 0 \) for \( \mu \) outside the union of a finite number of sets of the form \( D(\lambda) = \{ \mu \in \mathcal{H}^* | \mu \geq \lambda \} \), using some ordering of \( \mathcal{H}^* \), e.g., the lexicographic one; the formal exponents \( e(\mu) \) have the properties: \( e(0) = 1 \), \( e(\mu)e(\nu) = e(\mu + \nu) \).

Then the (formal) character of \( V \) is defined by:

\[
ch_0 V = \sum_{\mu \in \Gamma_+} (\dim V_{\mu}) e(\Lambda + \mu) = e(\Lambda) \sum_{\mu \in \Gamma_+} (\dim V_{\mu}) e(\mu). \tag{3.2}
\]

(We shall use the subscript ‘0’ for the even case.)

For a Verma module, i.e., \( V = V^\Lambda \) one has \( \dim V_{\mu} = P(\mu) \), where \( P(\mu) \) is a generalized partition function, \( P(\mu) = \# \) of ways \( \mu \) can be presented as a sum of positive roots \( \beta \), each root taken with its multiplicity \( \dim \hat{G}\beta = (1 \text{ here}), \) \( P(0) \equiv 1 \). Thus, the character formula for Verma modules is:

\[
ch_0 V^\Lambda = e(\Lambda) \sum_{\mu \in \Gamma_+} P(\mu)e(\mu) = e(\Lambda) \prod_{\alpha \in \Delta^+} (1 - e(\alpha))^{-1}. \tag{3.3}
\]

Further we recall the standard reflections in \( \hat{\mathcal{H}}^* \):

\[
s_\alpha(\lambda) = \lambda - (\lambda, \alpha)\alpha, \quad \lambda \in \hat{\mathcal{H}}^*, \quad \alpha \in \hat{\Lambda}.
\]

The Weyl group \( W \) is generated by the simple reflections \( s_i \equiv s_{\alpha_i} \), \( \alpha_i \in \hat{\pi} \). Thus every element \( w \in W \) can be written as the product of simple reflections. It is said that \( w \) is written in a reduced form if it is written with the minimal possible number of simple reflections; the number of reflections of a reduced form of \( w \) is called the length of \( w \), denoted by \( \ell(w) \).

The Weyl character formula for the finite-dimensional irreducible LWM \( L_\Lambda \) over \( \hat{G} \), i.e., when \( \Lambda \in -\Gamma_+ \), has the form:

\[
ch_0 L_\Lambda = \sum_{w \in W} (-1)^{\ell(w)} ch_0 V^{w\Lambda}, \quad \Lambda \in -\Gamma_+ \tag{3.5}
\]

where the dot \( \cdot \) action is defined by \( w \cdot \lambda = w(\lambda - \rho) + \rho \). For future reference we note:

\[
s_{\alpha} \cdot \Lambda = \Lambda + n_{\alpha} \alpha \tag{3.6}
\]

where

\[
n_{\alpha} = n_{\alpha}(\Lambda) = (\rho - \Lambda, \alpha^\vee) = (\rho - \Lambda)(H_\alpha), \quad \alpha \in \Delta^+. \tag{3.7}
\]

\[7\] A more general character formula involves the Kazhdan–Lusztig polynomials [42].
In the case of basic classical Lie superalgebras the first character formulae were given by Kac [31, 43]. For all such superalgebras (except \(osp(1/2N)\)) the character formula for Verma modules is [31, 43]:

\[
\hat{ch} V^\Lambda = e(\Lambda) \left( \prod_{\alpha \in \Delta^+_0} (1 - e(\alpha))^{-1} \right) \left( \prod_{\alpha \in \bar{\Delta}^+_1} (1 + e(\alpha)) \right). \tag{3.8}
\]

Note that the factor \(\prod_{\alpha \in \Delta^+_0} (1 - e(\alpha))^{-1}\) represents the states of the even sector: \(V^\Lambda_0 = U((G^\mathbb{C}^-)_0), v_0\), (as above in the even case), while \(\prod_{\alpha \in \bar{\Delta}^+_1} (1 + e(\alpha))\) represents the states of the odd sector: \(\hat{V}^\Lambda = (U(G^\mathbb{C}^-)/U((G^\mathbb{C}^-)_0)), v_0\). Thus, we may introduce a character for \(\hat{V}^\Lambda\) as follows:

\[
\hat{ch} \hat{V}^\Lambda = \prod_{\alpha \in \Delta^+_1} (1 + e(\alpha)). \tag{3.9}
\]

In our case, \(\hat{V}^\Lambda\) may be viewed as the result of all possible applications of the \(4N\) odd generators \(X^+_{a,i+}\) on \(v_0\), i.e., \(\hat{V}^\Lambda\) has \(2^{4N}\) states (including the vacuum). Explicitly, the basis of \(V^\Lambda\) may be chosen as in [23]:

\[
\Psi_{\tilde{e}} = \left( \prod_{k=N}^{1} (x^+_{1,4+k})^{x^+_{1,4+k}} \right) \left( \prod_{k=N}^{1} (x^+_{2,4+k})^{x^+_{2,4+k}} \right) \times \left( \prod_{k=1}^{N} (x^+_{3,4+k})^{x^+_{3,4+k}} \right) \left( \prod_{k=1}^{N} (x^+_{4,4+k})^{x^+_{4,4+k}} \right) v_0, \tag{3.10}
\]

where \(\tilde{e}\) denotes the set of all \(\varepsilon_{ij}\). Thus, the character of \(\hat{V}^\Lambda\) may be written as:

\[
\hat{ch} \hat{V}^\Lambda = \sum_{\tilde{e}} e(\Psi_{\tilde{e}}) \tag{3.11a}
\]

\[
= \sum_{\tilde{e}} \left( \prod_{k=1}^{N} e(\alpha_{1,4+k})^{x^+_{1,4+k}} \right) \left( \prod_{k=1}^{N} e(\alpha_{2,4+k})^{x^+_{2,4+k}} \right) \times \left( \prod_{k=1}^{N} e(\alpha_{3,4+k})^{x^+_{3,4+k}} \right) \left( \prod_{k=1}^{N} e(\alpha_{4,4+k})^{x^+_{4,4+k}} \right) \tag{3.11b}
\]

\[
= \sum_{\tilde{e}} e \left( \sum_{k=1}^{4N} \sum_{\omega=1}^{4N} \varepsilon_{a,i+\omega} \alpha_{a,i+\omega} \right). \tag{3.11c}
\]

(Note that in the above formula there is no actual dependence from \(\Lambda\).)

We shall use the above to write for the character of \(V^\Lambda\):

\[
\hat{ch} V^\Lambda = \hat{ch} \hat{V}^\Lambda \cdot ch_0 V^\Lambda_0
\]

\[
= \sum_{\tilde{e}} e \left( \sum_{k=1}^{4N} \sum_{\omega=1}^{4N} \varepsilon_{a,i+\omega} \alpha_{a,i+\omega} \right) \cdot e(\Lambda) \left( \prod_{\alpha \in \Delta^+_0} (1 - e(\alpha))^{-1} \right) \cdot \prod_{\alpha \in \bar{\Delta}^+_1} (1 + e(\alpha)). \tag{3.11d}
\]

8 Kac considers highest weight modules but his results are immediately transferable to lowest weight modules.

9 The order chosen in (3.10) was important in the proof of unitarity in [4, 23] and for that purpose one may choose also an order in which the vectors on the first row are exchanged with the vectors on the second row.
Analogously, for the factorized Verma modules $\hat{V}^\Lambda$ the character formula is:

$$
ch\hat{V}^\Lambda = ch\hat{V}^{\Lambda_1} \cdot ch_0 \hat{V}_0^{\Lambda_1} + \sum_i ch_0 \hat{V}_0^{\Lambda_1 + \sum_{a=1}^N \sum_{b=1}^4 \varepsilon_{e+a\alpha_{j}+b}}
$$

(3.14)

where $ch_0 \hat{V}_0^{\Lambda_1}$ is the character obtained by restriction of $\hat{V}^{\Lambda_1}$ to $\hat{V}_0^{\Lambda_1}$:

$$
ch_0 \hat{V}_0^{\Lambda_1} = e(\Lambda_1) \cdot ch_0 V_0^{\Lambda_1} \cdot ch_0^{\Lambda_1} V_0^{\Lambda_1}
$$

(3.13)

where we use the decomposition $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$ from (2.17a), and $V^{\Lambda_1}, V^{\Lambda_2}$, respectively, are Verma modules over the complexifications of $su(2, 2)$, $su(N)$, respectively, see the next subsection.

Analogously, for the factorized Verma modules $\hat{V}^\Lambda$ the character formula is:

$$
ch\hat{V}^\Lambda = ch\hat{V}^{\Lambda_1} \cdot ch_0 \hat{V}_0^{\Lambda_1} + \sum_i ch_0 \hat{V}_0^{\Lambda_1 + \sum_{a=1}^N \sum_{b=1}^4 \varepsilon_{e+a\alpha_{j}+b}}
$$

where $ch_0 \hat{V}_0^{\Lambda_1}$ is the character obtained by restriction of $\hat{V}^{\Lambda_1}$ to $\hat{V}_0^{\Lambda_1}$.

3.2 Characters of the even subalgebra

For the characters of the even subalgebra: $G^G_0 = sl(4) \oplus gl(1) \oplus sl(N)$ of $G^C$, we use formulae (3.13) and (3.15). In fact, since the subalgebra $G^G_0$ is reductive the corresponding character formulae will be given by the products of the character formulae of the two simple factors $sl(4)$ and $sl(N)$.

We start with the $sl(4)$ case. We denoted the six positive roots of $sl(4)$ by $\alpha_{ij}$, $1 \leq i < j \leq 4$. (Of course, they are part of the root system of $sl(4/N)$.) For the simplification of the character formula we use notation for the formal exponents corresponding to the $sl(4)$ simple roots: $t_{ij} \equiv e(\alpha_{ij})$, $j = 1, 2, 3$; then for the three non-simple roots we have: $e(\alpha_{12}) = t_{12}, e(\alpha_{23}) = t_{23}, e(\alpha_{13}) = t_{13}$.

In terms of these the character formula for a Verma module over $sl(4)$ is:

$$
ch_0 V^{\Lambda_1} = \frac{e(\Lambda^1)}{(1-t_1)(1-t_2)(1-t_3)(1-t_{12})(1-t_{23})(1-t_{13})}
$$

(3.16)

where by $\Lambda^1$ we denote the $sl(4)$ lowest weight.
The representations of $sl(4)$ which we consider are infinite-dimensional. When $d > d_{\text{max}}$ then all the numbers: $n_2, n_{12}, n_{23}, n_{13}$, from (2.16) cannot be positive integers. Then the only reducibilities of the $sl(4)$ Verma module are related to the complexification of the Lorentz subalgebra of $su(2, 2)$, i.e., with $sl(2) \otimes sl(2)$, and the character formula is given by the product of the two character formulae for finite-dimensional $sl(2)$ irreps. In short, the $sl(4)$ character formula is:

\[ \chi L_{d, j_1, j_2}^2 = \chi_0 V^{\Lambda^1} - \chi_0 V^{\Lambda^1 + n_{12} + n_{23} + n_{13}} + \chi_0 V^{\Lambda^1 + n_{12} + n_{23}} \]

\[ = \frac{e(\Lambda^1)}{(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_4)} \left( \sum_{j=0}^{n_1-1} t_1^j \right) \left( \sum_{k=0}^{n_2-1} t_2^k \right) \]

\[ = \frac{e(\Lambda^1)}{(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_4)} \frac{Q_{j_1, j_2} Q_{n_1, n_2} (t_1, t_2, t_3)}{P_{n_1, n_2} Q_{n_1 - 1, n_2 - 1}}, \]  

(3.17)

and we have introduced for later use the notation $Q_{n_1, n_2}$ for the character factorized by $e(\Lambda^1)/(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_4)$) The above formula obviously has the form (3.5) replacing $W \rightarrow W_2 \times W_2$, where $W_2$ is the two-element Weyl group of $sl(2)$.

When $d \leq d_{\text{max}}$ there are additional even reducibilities, see (2.38) and (2.39).

First we consider the case when $d = 2 + j_1 + j_2$, i.e., the unitarity threshold when $j_1 j_2 \neq 0$. Using the definitions in (2.16) we have:

\[ n_1 = 1 + 2 j_1, \quad n_2 = -1 + 2 j_1 + 2 j_2, \quad n_3 = 1 + 2 j_2, \]

\[ n_{12} = -2 j_2, \quad n_{23} = 2 j_1 + j_2, \quad n_{13} = 1. \]  

(3.18)

The corresponding character formula is given in [44] (formula (4.32c) (one has to make the changes: $m_{23} \mapsto n_1, m_{12} \mapsto n_3$, since that formula is parametrized w.r.t. some referent dominant weight, and set $m_2 = 1$):

\[ \chi L_{d_{\text{ac}}, j_1, j_2} = \chi V^{\Lambda_{\text{ac}}} (1 - t_1^{n_1} - t_3^{n_1} - t_{13} + t_1^{n_1} t_3^{n_1} + t_1^{n_1} t_{23} + t_{12}^{n_1} t_3^{n_1} - t_1^{n_1} t_{23}^{n_1}) \]

\[ = \frac{e(\Lambda_{\text{ac}})}{(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_4)} \frac{Q_{n_1, n_2} (t_1, t_2, t_3)}{P_{n_1, n_2} Q_{n_1 - 1, n_2 - 1}}. \]  

(3.19)

Note that this formula is also valid for $j_1 = 0, (n_1 = 1)$ and/or $j_2 = 0, (n_3 = 1)$ when the second term disappears, $(Q_{0, n} = Q_{n, 0} = 0)$, and then the formula coincides with (3.17). This may be explained by the fact that when $j_1 j_2 = 0$, then the value $d = 2 + j_1 + j_2$ is not a threshold, instead $d = 1 + j_1 + j_2$ is the threshold.

Next we consider the case when $d = 1 + j_1 + j_2$, i.e., the massless unitarity threshold when $j_1 j_2 = 0$. First we take $j_1 = 0$. Using the definitions in (2.16) we have:

\[ n_1 = 1, \quad n_2 = -2 j_2, \quad n_3 = 1 + 2 j_2, \]

\[ n_{12} = 1 - 2 j_2, \quad n_{23} = 2, \quad n_{13} = 2. \]  

(3.20)

The corresponding character formula is given in [44] (formula (4.32b) (one has to make the changes: $m_{13} \mapsto n_3, m_2 \mapsto n_1 = 1, m_3 \mapsto n_{23} = 1, m_{23} \mapsto n_{13} = 2$, since that formula is
parametrized w.r.t. some more general referent dominant weight):
\[
\text{ch } L_{d^\ominus,0,j_2} = \text{ch } V^{A\lambda}(1-t_1 - t_2^0 + t_1 t_2^0 - t_2 t_3^0 + t_1^2 t_3^0
\]
\[
- t_1^2 t_2^0 t_3^0 + t_1 t_2^0 t_3^0 - t_1^2 t_3^0 - t_1^2 t_2^0)
\]
\[
= \text{ch } V^{A\lambda}(1-t_1)(1-t_2 t_3^0(1+t_1)\left(1-t_1^{n_1-1}\right) + t_2 t_3(1-t_1-t_3))
\]
\[
= \frac{e(\Lambda_{ad})}{(1-t_2)(1-t_1 t_2)(1-t_2 t_3)(1-t_1 t_2 t_3)} P_{n_1}(1,t_2, t_3),
\]
\[
(3.21)
\]
\[
P_{n_1} = \begin{cases} 
Q_{n_1} - t_2 t_3(1+t_1) Q_{n_1-1} + t_2 t_3 Q_{n_1-2}, & n_1 \geq 2 \\
1 - t_2 t_3, & n_1 = 1.
\end{cases}
\]

This formula simplifies considerably for \( j_2 = 0, (n_3 = 1) \) and \( j_2 = \frac{1}{2}, (n_3 = 2) \):
\[
\text{ch } L_{d^\ominus,1,0,0} = \frac{e(\Lambda_{ad})}{(1-t_2)(1-t_1 t_2)(1-t_2 t_3)(1-t_1 t_2 t_3)}
\]
\[
(3.22)
\]
\[
\text{ch } L_{d^\ominus,\frac{3}{2},0,0} = \frac{e(\Lambda_{ad})}{(1-t_2)(1-t_1 t_2)(1-t_2 t_3)(1-t_1 t_2 t_3)}
\]
\[
(3.23)
\]

The case \( j_2 = 0 \) is obtained from the above by the changes \( n_3 \mapsto n_1 \) and \( t_1 \leftrightarrow t_3 \), and the character formula is:
\[
\text{ch } L_{d^\ominus,j_1,0} = \text{ch } V^{A\lambda}(1-t_1 - t_2^0 + t_1 t_2^0 - t_2 t_3^0 + t_1^2 t_3^0
\]
\[
- t_1^2 t_2^0 t_3^0 + t_1 t_2^0 t_3^0 - t_1^2 t_3^0 - t_1^2 t_2^0)
\]
\[
= \text{ch } V^{A\lambda}(1-t_1)(1-t_2 t_3^0(1+t_1)\left(1-t_1^{n_1-1}\right) + t_2 t_3(1-t_1-t_3))
\]
\[
= \frac{e(\Lambda_{bc})}{(1-t_2)(1-t_1 t_2)(1-t_2 t_3)(1-t_1 t_2 t_3)} P'_{n_1}(1,t_2, t_3),
\]
\[
(3.24)
\]
\[
P'_{n_1} = \begin{cases} 
Q'_{n_1} - t_2 t_3(1+t_1) Q'_{n_1-1} + t_2 t_3 Q'_{n_1-2}, & n_1 \geq 2 \\
1 - t_2 t_3, & n_1 = 1.
\end{cases}
\]

This formula simplifies considerably for \( j_1 = 0, (n_1 = 1) \) when it coincides with (3.22) since \( P'_{n_1} = P_{n_1} \), while in the case \( j_1 = \frac{1}{2}, (n_1 = 2) \) we have:
\[
\text{ch } L_{d^\ominus,\frac{3}{2},0,0} = \frac{e(\Lambda_{ad})}{(1-t_2)(1-t_1 t_2)(1-t_2 t_3)(1-t_1 t_2 t_3)}
\]
\[
(3.25)
\]

**Remark.** It is not surprising that the three cases (3.22), (3.23), (3.25) (when \( j_1 + j_2 \leq \frac{1}{2} \)) are special since (unlike the other massless irreps) they are not related to finite-dimensional irreps.

The easiest way to see this is by the value of the Casimir, which in terms of the parameters \( n_{ik} \) is given as follows:
\[
C_2 = \frac{1}{2} \left(n_1^2 + n_2^2 + \frac{1}{2}(n_1 - n_2)^2\right) - 5
\]
\[
(3.26)
\]
and is normalized so that for each finite-dimensional irrep (when \( n_k \in \mathbb{N} \)) it is non-negative and zero only for the one-dimensional irrep \( (n_k = 1) \). It is easy to see that for the massless cases given in (3.20) one has:
\[
C_2 = 3\left(\frac{j_1^2}{2} - 1\right), \quad j_2 \in \frac{1}{2}\mathbb{Z}_+, \quad j_2 = 1
\]
\[
\]
\[
(3.27)
\]
which is indeed negative for \( j_2 = 0, \frac{1}{2} \) and non-negative for \( j_2 \geq 1 \). In fact, the finite-dimensional irrep, related to a massless case with \( j_2 \geq 1 \), has dimension [45]:
\[
\dim \Lambda_{dir,J_1} = \frac{1}{2}J_1(4\frac{J_1^2}{2} - 1), \quad J_1 = 1 + \frac{1}{2}\mathbb{Z}_+.
\]
\[
(3.28)
\]
For the conjugate massless case we have similarly
\[
C_2 = 3\left(\frac{j_1^2}{2} - 1\right), \quad j_1 \in \frac{1}{2}\mathbb{Z}_+, \quad j_1 = 1
\]
\[
(3.29)
\]
corresponding notation:

\[ \text{dim } \Lambda_{dir,j_1} = \frac{1}{2} j_1 (4 j_1^2 - 1), \quad j_1 \in 1 + \frac{1}{2} \mathbb{Z}_+ \]  \hspace{1cm} (3.30)

with the same conclusions.

These three representations are the so-called minimal UIRs of \( su(2, 2) \), cf, e.g., [46], and for other of their properties we refer to [47, 48].

In the case of \( sl(N) \) the representations are finite-dimensional since we induce from UIRs of \( su(N) \). The character formula is (3.5), which we repeat in order to introduce the corresponding notation:

\[ S_{r_1, \ldots, r_N} = \sum_{w \in W_n} (-1)^{\ell(w)} c_{h_0} V^{w * \Lambda^u}, \quad \Lambda^u \in -\Gamma^u. \]  \hspace{1cm} (3.31)

The index \( u \) is to distinguish the quantities pertinent to the case.

We shall write down explicitly the cases that we shall need, namely, \( sl(2) \) and \( sl(4) \). In the \( sl(2) \) case the Weyl group has only two elements and the character formula is very simple:

\[ S_r = ch V^\Lambda (1 - t^{r+1}) = e(\Lambda) \frac{1 - t^{r+1}}{1 - t} \]

\[ = e(\Lambda) (1 + t + t^2 + \cdots + t^r), \quad e(\Lambda) = t^{-r/2}, \quad r \in \mathbb{Z}_+. \]  \hspace{1cm} (3.32)

In the \( sl(4) \) case the Weyl group has 24 elements and the character formula is:

\[ S_{r_1, r_2, r_3} = ch V^{\Lambda^u} (1 - t_{r_1} - t_{r_2} - t_{r_3} + t_{r_1} t_{r_2} t_{r_3} + t_{r_1} t_{r_2} t_{r_3} + t_{r_2} t_{r_3} t_{r_2} - t_{r_1} t_{r_2} t_{r_3} - t_{r_1} t_{r_2} t_{r_3}) \]

\[ + t_{r_1} t_{r_2} t_{r_3} - (t_{r_1} t_{r_2}) t_{r_3} - (t_{r_1} t_{r_2}) t_{r_3} + (t_{r_2} t_{r_3}) t_{r_1} + (t_{r_2} t_{r_3}) t_{r_1} - t_{r_1} t_{r_2} t_{r_3} + (t_{r_2} t_{r_3}) t_{r_1} + (t_{r_2} t_{r_3}) t_{r_1}) \],

\[ n_k = r_k + 1, n_{12} = r_1 + r_2 + 2, n_{23} = r_2 + r_3 + 2, \]

\[ n_{13} = r_1 + r_2 + r_3 + 3. \]  \hspace{1cm} (3.33)

The expression for \( \Lambda^u \) in terms of the \( sl(4) \) simple roots is:

\[ \Lambda^u = -\frac{1}{2} \left( r_2 + \frac{1}{2} (3 r_1 + r_3) \right) \beta_1 - \left( r_2 + \frac{1}{2} (r_1 + r_3) \right) \beta_2 - \frac{1}{2} \left( r_2 + \frac{1}{2} (r_1 + 3 r_3) \right) \beta_3 \]

\[ = \sum_{k=1}^{3} \lambda_k \beta_k, \]  \hspace{1cm} (3.34)

where the minus signs are due to the fact that \( \Lambda^u \) is assumed to be lowest weight, for highest weight the minuses become pluses. Furthermore, the notation for the simple roots depends on the application, e.g., when applied to the compact part of the even subalgebra for \( N = 4 \): \( \mathcal{G}_0 = su(2, 2) \oplus u(1) \oplus su(4) \) then the roots are mapped: \( \beta_1 \mapsto \alpha_7, \beta_2 \mapsto \alpha_6, \beta_3 \mapsto \alpha_5 \).

We give some explicit examples that would be actually used:

\[ S_{0,0,0} = 1, \]
\[ S_{1,0,0} = e(\Lambda^u) (1 + t_1 + t_{12} + t_{13}), \]
\[ S_{0,1,0} = e(\Lambda^u) (1 + t_3 + t_{23} + t_{13}), \]
\[ S_{0,1,0} = e(\Lambda^u) (1 + t_2 + t_{12} + t_{23} + t_{13} + t_{12} t_{23}), \]
\[ S_{2,0,0} = e(\Lambda^u) (1 + t_1 + t_2 + t_3 + t_{12} + t_{13} + t_{23} + t_{12} t_{13} + t_{12} t_{23} + t_{13} t_{23} + t_{12} t_{13} t_{23}), \]
\[ S_{0,0,2} = e(\Lambda^u) (1 + t_3 + t_2 + t_{23} + t_{23} + t_{13} + t_{13} t_{23} + t_{12} t_{13} t_{23} + t_{12} t_{13} t_{23}). \]
\[ e(\Lambda^n) = \sum_{k=1}^{3} t_k^a. \]  

(3.35)

Naturally, the number of terms in the character formula is equal to the dimension of the corresponding irrep:

\[ \dim_{r_1, r_2, r_3} = \frac{1}{12} (r_1 + 1)(r_2 + 1)(r_3 + 1)(r_1 + r_2 + 2)(r_2 + r_3 + 2)(r_1 + r_2 + r_3 + 3). \]  

(3.36)

4. Explicit character formulae for \( N = 1, 2, 4 \)

4.1. \( N = 1 \)

- **Long superfields.** If \( d > d_{\text{max}}, j_1 j_2 > 0 \) then \( \hat{L}_\Lambda \) has the maximum possible number of states: 16.

The bare character formula is (3.19) from [17]:

\[
ch \hat{L}_\Lambda = \prod_{\alpha \in \Lambda^+_1} (1 + e(\alpha)) = (1 + e(\alpha_{15}))(1 + e(\alpha_{25}))(1 + e(\alpha_{35}))(1 + e(\alpha_{45})),
\]

(4.1)

where we use (2.13):

\[
\begin{align*}
\alpha_{15} &= \gamma_1 + \gamma_3 = \alpha_1 + \gamma_3, \\
\alpha_{35} &= \gamma_2 + \gamma_4 = \alpha_3 + \gamma_4, \\
\alpha_{25} &= \gamma_3, \quad \alpha_{45} = \gamma_4.
\end{align*}
\]

(4.2)

The mirror symmetry of (4.1) follows from (2.2):

\[
\gamma_1 \longleftrightarrow \gamma_2, \quad \gamma_3 \longleftrightarrow \gamma_4.
\]

For the characters we shall need also the following notation related to the odd roots:

\[
\begin{align*}
\vartheta &= e(\alpha_{15}), & \theta &= e(\alpha_{25}), \\
\bar{\vartheta} &= e(\alpha_{35}), & \bar{\theta} &= e(\alpha_{45}),
\end{align*}
\]

(4.3)

(the bar respecting the mirror symmetry), and for the \( sl(4) \)-related variables: \( t_k \equiv e(\alpha_k), \) \( k = 1, 2, 3, \) see subsection 2.2. Note the relation:

\[
\vartheta = t_1 \theta, \quad \bar{\vartheta} = t_5 \bar{\theta},
\]

(4.4)

which is also mirror symmetric.

The full character formula takes into account the characters of the conformal algebra entries, i.e., we have:

\[
ch L_\Lambda = e(\Lambda) \left\{ ch L_{d, j_1, j_2}^2 d + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right. + e(\alpha_{15}) ch L_{d+1, j_1, j_2}^2 d + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
+ e(\alpha_{35}) ch L_{d, j_1, j_2}^2 d + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right. + e(\alpha_{25}) ch L_{d+1, j_1, j_2}^2 d + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
+ e(\alpha_{15}) e(\alpha_{25}) ch L_{d+1, j_1, j_2}^2 d + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right. + e(\alpha_{35}) e(\alpha_{45}) ch L_{d+1, j_1, j_2}^2 d + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
+ e(\alpha_{15}) e(\alpha_{35}) ch L_{d+1, j_1, j_2}^2 d + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + e(\alpha_{15}) e(\alpha_{25}) e(\alpha_{35}) e(\alpha_{45}) ch L_{d+1, j_1, j_2}^2 d + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right\}
\]

(4.5)
that we have omitted the prefactor where the polynomials $18$

In statements like this each sector includes the vacuum.

and the full character formula is:

where $\chi L$ is the conformal (actually $sl(4)$) character formula (3.17), the prime denoting that we have omitted the prefactor $e(\Lambda)$ since above it is distributed among the other prefactors. Substituting the explicit expressions for $\chi^2_{d,j,1,j}$ we obtain:

where the polynomials $Q$ are in terms of the $sl(4)$-related variables $t_1, t_2, t_3$.

When $j_1, j_2 = 0$ there are fewer terms in the character formula, since $Q_0 = 0 = Q_{0,0}$, and further the entries simplify.

If $j_1 = 0, j_2 > 0$ then the generator $X_{15}^+$ can appear only together with the generator $X_{35}^+$, and $\hat{L}_A$ has 12 states = 3(chiral)×4(anti-chiral) states. The bare character formula is (3.26) from [17]

from [17]

where the counter-term $\mathcal{R}$ in our case is:

and the full character formula is:

where we use:

The next case is conjugate. If $j_1 > 0, j_2 = 0$ then the generator $X_{35}^+$ can appear only together with the generator $X_{45}^+$, and $\hat{L}_A$ has 12 states. The bare character formula is again (4.7) with counter-term:

and the full character formula is:

10 In statements like this each sector includes the vacuum.
where we use:
\[ Q_{n,1} = \sum_{j=0}^{n-1} t^j_1 \equiv \mathcal{Q}^j_{11}, \quad Q_{n,2}(t_1, t_3) = (1 + t_3) Q^0_{11}(t_1). \]

The next case combines the previous two. If \( j_1 = j_2 = 0 \) then the generator \( X_{15}^+ \) can appear only together with the generator \( X_{35}^- \), the generator \( X_{15}^+ \) can appear only together with the generator \( X_{35}^- \), and \( \hat{L}_\Lambda \) has 9 states = 3(chiral) \times 3(anti-chiral) states. The character formula is (4.7) with:
\[ R = \vartheta (1 + e(\alpha_{35}))(1 + e(\alpha_{45})) + e(\alpha_{35})(1 + e(\alpha_{15}))(1 + e(\alpha_{25})) - e(\alpha_{15})e(\alpha_{35}), \] (4.12)
i.e., we combine the counter-terms of the previous two cases, but need to subtract a counter-term that is counted twice. The bare character formula is:
\[ ch \hat{L}_\Lambda = (1 + e(\alpha_{25}) + \vartheta e(\alpha_{25}))(1 + e(\alpha_{45}) + e(\alpha_{35})e(\alpha_{45})) \] (4.13)
and the full character formula may be obtained from (4.9) setting \( n_3 = 1 \) (or from (4.11) setting \( n_1 = 1 \)):
\[ ch \hat{L}_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \times \left\{ (1 + \vartheta \theta)(1 + \vartheta \theta) \right\} \]
(4.14)
using \( Q_1 = 1, Q_2 = 1 + t_3, \) (or \( Q'_1 = 1, Q'_2 = 1 + t_1 \)).

**SRC cases.**
- **a d = d_{\text{max}} = d_{11}^1 = 2 + 2j_2 + z > d_{11}^1.**
  - The generator \( X_{35}^- \) is eliminated [17] and for \( z = 0 \) (then \( j_2 > j_1 \)) these are \( \frac{1}{2} \)-BPS cases [49].
  - These are called semi-conserved superfields in [37]. For \( j_2 > 0 \) they obey the first-order super-differential operator given explicitly in formulae (7a) of [4]. When \( j_2 = 0 \) that first-order super-differential operator has trivial kernel and is replaced by second-order super-differential operator given in (11b) of [4].
- **j_1 > 0.** Here there are only eight states\(^{11}\). The bare character formula is (3.36) (or equivalently (3.39)) from [17] without counter-terms:
\[ ch \hat{L}_\Lambda = \prod_{\alpha \in \Lambda^+} (1 + e(\alpha)). \] (4.15)

The full character formula follows from (4.6):
\[ ch \hat{L}_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \times \left\{ (1 + \vartheta \theta) Q_{n_1,n_3} + \vartheta Q_{n_1-1,n_3} + \vartheta Q_{n_1+1,n_3} + \vartheta Q_{n_1,n_3+1} \right\} \]
(4.16)

**Remark.** For the finite-dimensional irreps of \( sl(4/N) \) the SRC situations are called ‘singly atypical’ and the character formulae are written as (4.15), cf [50, 51]\(^{12}\).

\(^{11}\) For brevity, here and often below we shall say ‘there are M states’ meaning ‘there are M states in \( \hat{L}_\Lambda \).’

\(^{12}\) For character formulae of finite-dimensional irreps beyond the singly atypical case, see [53–56] and references therein.
These are called semi-conserved superfields in [37]. For a super-differential operator given explicitly in formulae (7) the full character formula follows from (4)

\[ ch \hat{L}_\Lambda = \prod_{\alpha, \bar{\alpha} \in \Lambda_1} (1 + e(\alpha)) - R \]

\[ = (1 + e(\alpha_{25}) + \vartheta e(\alpha_{25}))(1 + e(\alpha_{45})), \quad (4.17) \]

The full character formula follows from (4.16):

\[ ch L_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \times \{(1 + \vartheta \theta)Q_{n_1} + \vartheta(1 + t_1)(Q_{n_1} + \bar{\vartheta}Q_{n_1 + 1}) + (1 + \vartheta \theta)\bar{\vartheta}Q_{n_1 + 1}\}. \]

\[ (4.18) \]

• \( b d = d_{11}^1 = z > d_{11}^0, \ j_2 = 0 \). These UIRs are called chiral since all anti-chiral generators are eliminated. They obey the first-order super-differential operator given explicitly in formulae (7b) of [4].

• \( j_1 > 0 \). The generators \( X_{35}^+ \) and \( X_{45}^+ \) are eliminated and there are only four states. The bare character formula is (3.65) from [17] (for \( i_0 = 0 \) without counter-terms:

\[ ch \hat{L}_\Lambda = (1 + e(\alpha_{15}))(1 + e(\alpha_{25})). \]

\[ (4.19) \]

The full character formula follows from (4.16):

\[ ch L_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \{(1 + \vartheta \theta)Q_{n_1} + \vartheta(1 + t_1)(Q_{n_1} + \bar{\vartheta}Q_{n_1 + 1}) + (1 + \vartheta \theta)\bar{\vartheta}Q_{n_1 + 1}\}. \]

\[ (4.20) \]

• \( j_1 = 0 \). The generators \( X_{35}^+ \) and \( X_{45}^+ \) are eliminated, the generator \( X_{15}^+ \) can appear only together with the generator \( X_{25}^+ \), and there are only 3 states. The bare character formula is (3.65) from [17] (for \( i_0 = 0 \) with counter-term \( R = e(\alpha_{15}) \)):

\[ ch \hat{L}_\Lambda = 1 + e(\alpha_{25}) + e(\alpha_{15})e(\alpha_{25}). \]

\[ (4.21) \]

The full character formula follows from (4.20):

\[ ch L_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \{1 + \vartheta \theta + \theta(1 + t_1)\}. \]

\[ (4.22) \]

The next two cases \( c, d \) are conjugate to the above \( a, b \). The character formulae are obtainable by the changes \( j_1 \leftrightarrow j_2, n_1 \leftrightarrow n_3, t_1 \leftrightarrow t_3, \vartheta \leftrightarrow \bar{\vartheta}, (\alpha_{15} \leftrightarrow \alpha_{35}, \alpha_{25} \leftrightarrow \alpha_{45}) \). Thus, we list the character formulae without explanations.

• \( c d = d_{11} = d_{11}^3 = 2 + 2 j_1 - z > d_{11}^1 \). The generator \( X_{15}^\alpha \) is eliminated [17] and for \( z = 0 \) (then \( j_1 > j_2 \) these are \( 1/8 \) BPS cases [49].

These are called semi-conserved superfields in [37]. For \( j_1 = 0 \) they obey the first-order super-differential operator given explicitly in formulae (7c) of [4]. When \( j_1 = 0 \) that first-order super-differential operator has a trivial kernel and is replaced by the second-order super-differential operator given in (11a) of [4].

• \( j_2 > 0 \). The character formulae are:

\[ ch \hat{L}_\Lambda = \prod_{\alpha, \bar{\alpha} \in \Lambda_{15}} (1 + e(\alpha)). \]

\[ (4.23) \]
There are only three BPS cases for J. Phys. A: Math. Theor.

• ch \( L_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \times \left\{ (1 + \bar{\theta} \theta) Q_{n_1 n_2} + \bar{\theta} Q_{n_1 n_2} + \theta Q_{n_1 n_2} + \theta Q_{n_1 n_2} + \bar{\theta} \theta Q_{n_1 n_2} + \theta \bar{\theta} Q_{n_1 n_2} + \theta \bar{\theta} Q_{n_1 n_2} \right\} \). (4.24)

• \( j_2 = 0 \). The character formulae are:
  \( ch \hat{L}_\Lambda = (1 + e(\alpha_{25})) (1 + e(\alpha_{45})) \) (4.25)

  \( ch \hat{L}_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \times \left\{ (1 + \bar{\theta} \theta) Q_{n_1 n_2} + \bar{\theta} Q_{n_1 n_2} + \theta Q_{n_1 n_2} + (1 + \theta \bar{\theta}) Q_{n_1 n_2} \right\} \). (4.26)

• \( \mathbf{d} d = d_{11}^4 = - z > d_{11}^1, j_1 = 0 \). These UIRs are called anti-chiral since all chiral generators are eliminated. They obey the first-order super-differential operator given explicitly in formula (7d) of [4].

• \( j_2 > 0 \). The character formulae are:
  \( ch \hat{L}_\Lambda = (1 + e(\alpha_{35})) (1 + e(\alpha_{45})) \). (4.27)

\( ch \hat{L}_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \times \left\{ (1 + \bar{\theta} \theta) Q_{n_1 n_2} + \bar{\theta} Q_{n_1 n_2} + \theta Q_{n_1 n_2} \right\} \). (4.28)

• \( j_2 = 0 \). The character formulae are:
  \( ch \hat{L}_\Lambda = 1 + e(\alpha_{45}) + e(\alpha_{35}) e(\alpha_{45}) \). (4.29)

\( ch \hat{L}_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} [1 + \bar{\theta} \theta + \bar{\theta} (1 + t_3)] \). (4.30)

**DRC cases.**

• \( ac \mathbf{d} = d^{ac} = d_{max} = d_{11}^1 = d_{11}^3 = d^{ac} = 2 + j_1 + j_2, \ z = z_{ac} = j_1 - j_2 \).

These are the conserved superfields. For \( j_1 j_2 \neq 0 \) they obey the two first-order super-differential operators given explicitly in formulae (7a), (7c) of [4]. These semi-short UIRs may be called Grassmann-analytic following [28], since odd generators from different chiralities are eliminated.

When \( j_1 = 0 \) (\( j_2 \neq 0 \)) the first-order super-differential operator from (7c) of [4] (cf. (7a) of [4], respectively) has trivial kernel and is replaced by second-order super-differential operator given in (11a) of [4] (cf. (11b) of [4], respectively).

The generators \( X_{25}^+ \) and \( X_{25}^- \) are eliminated (though for different reasons for \( j_1 > 0 \) and \( j_1 = 0 \), respectively, for \( j_2 > 0 \) and \( j_2 = 0 \)). For \( j_2 = j_2 \) (then \( z = 0 \)) these are \( \frac{1}{2} \)-BPS cases [49]13. There are only four states and the bare character formula is (3.84) from [17].

13 There are only three BPS cases for \( N = 1 \), the other two were mentioned above in cases a,c.
is (3.97) from [17] for explicit in the general formulae. There are only two states and the bare character formula

\[
\text{ch } L^A \equiv \frac{1}{1 + e(\alpha_{25})} \text{ch } \hat{V}^{\Lambda + \alpha_{25}} - \frac{1}{1 + e(\alpha_{25})} \text{ch } \hat{V}^{\Lambda + \alpha_{25}}
\]

where the terms with minus may be interpreted as taking out states, while the last term indicates adding back what was taken two times. The corresponding decomposition of \(L_A\) is given by:

\[
L_A = L_{d^w; j_1, j_2} \otimes L_z^+ + L_{d^w; j_1, j_2} \otimes L_z^+ \left. \frac{1}{2} \right|_{\alpha = \frac{1}{2}}
+ L_{d^{w^2}; j_1, j_2} \otimes L_z^+ \left. \frac{1}{2} \right|_{\alpha = \frac{1}{2}} + L_{d^{w^2}; j_1, j_2} \otimes L_z^+ \left. \frac{1}{2} \right|_{\alpha = \frac{1}{2}}
\]

Note that for all four conformal entries is fulfilled the relation: \(d = 2 + j_1 + j_2\), which for \(j_1, j_2 \neq 0\) is the conformal unitarity threshold. Thus, for the conformal characters we have to use formula (3.19), and then the full character formula is:

\[
\text{ch } L_A = e(\Lambda) \left\{ \text{ch } L_{d^w; j_1, j_2} + e(\alpha_{25}) \text{ch } L_{d^{w^2}; j_1, j_2} \right\}
\]

\[
\left( \frac{1}{1 - t_1}(1 - t_2)(1 - t_3)(1 - t_{12})(1 - t_{23})(1 - t_{13}) \right)
\times \left\{ \begin{array}{c}
\text{ch } L_{d^w; j_1, j_2} + e(\alpha_{25}) e(\alpha_{45}) \text{ch } L_{d^{w^2}; j_1, j_2} \right\}
\right.
\]

where \( \text{ch } \) is the character formula (3.19) without the prefactor \( e(\Lambda_{\text{con}}) \)—this prefactor is subsumed in the overall prefactor \( e(\Lambda) \) since the relative difference weights between the four terms of (4.33) are taken into account by the prefactors \( e(\alpha_{45}) \).

\( \bullet \) \( \text{ad } d = d^{ad} = d^{\text{ad}1} = 1 + j_2 = -z, j_1 = 0. \) The generators \( X_{13}^+, X_{23}^+, \) and \( X_{35}^- \) are eliminated (for the latter for different reasons for \( j_2 > 0 \) and \( j_2 = 0 \)). These are the first series of massless UIRs, and everything is already explicit in the general formulae. There are only two states and the bare character formula is (3.97) from [17] for \( N = 1 \):

\[
\text{ch } \hat{L}_A = 1 + e(\alpha_{45}).
\]

The corresponding decomposition of \(L_A\) is given by:

\[
L_A = L_{d^w; 0, j_2} \otimes L_z^+ \left. \frac{1}{2} \right|_{\alpha = 0} + L_{d^{w^2}; 0, j_2} \otimes L_z^+ \left. \frac{1}{2} \right|_{\alpha = 0}
\]

Note that for both conformal entries the relation: \( d = 1 + j_1 + j_2 \) is fulfilled, which for \( j_1, j_2 = 0 \) is the conformal unitarity threshold. For the characters we have to use formula (3.21), (or its reductions for \( j_2 = 0, \frac{1}{2} \)), and then the full character formula is:

\[
\text{ch } L_A = e(\Lambda) \left\{ \text{ch } L_{d^w; 0, j_2} + e(\alpha_{45}) \text{ch } L_{d^{w^2}; 0, j_2} \right\}
\]

\[
e e(\Lambda) \left\{ \begin{array}{c}
\text{ch } L_{d^w; 0, j_2} + e(\alpha_{45}) \text{ch } L_{d^{w^2}; 0, j_2} \right\}
\right.
\]

where \( \text{ch } \) is the character formula (3.21) without the prefactor \( e(\Lambda_{\text{ad}}) \)—see the explanation above. This formula simplifies for \( j_2 = 0, (n_3 = 1) \) (using both (3.22) and (3.23)):

\[
\text{ch } L_A = \frac{e(\Lambda)}{(1 - t_2)(1 - t_{12})(1 - t_{23})(1 - t_{13})(1 - t_{12} t_{13} + \tilde{\theta}(1 + 3 - t_{23} - t_{13})).
\]

The next case is conjugate.
Note that as a consequence of our mirror symmetry (2.2) we have:

\[ \text{cases} \]

We recall from [17] that correspondingly to the values of \( j \in \mathbb{Z} \) when

\[ j = 1 \]

the bare character formula is (3.100) from [17] for \( N = 1 \):

\[ \text{ch} \hat{L} = 1 + e(\alpha_{25}). \]  

For the full characters we have to use formula (3.24), (or its reductions for \( j = 0, \frac{1}{2} \)), and then we have:

\[ \text{ch} L = e(\Lambda) \left[ \text{ch} L_{\text{sc}, j + 0} + e(\alpha_{25}) \text{ch} L_{\text{sc}, j + \frac{1}{2} + 0} \right] \]

\[ = e(\Lambda) \left[ \mathcal{P}_{n}^{\prime} + \theta \mathcal{P}_{n+1}^{\prime} \right]. \]  

(4.39)

This formula simplifies for \( j = 0, (n_1 = 1) \) (using (3.22) and (3.25)):

\[ \text{ch} L = e(\Lambda) \]

\[ \left( 1 - t_2 / (1 - t_1 + t_2) (1 - t_3) (1 - t_1 - t_2 - t_3) \right) \]

\[ \left( 1 - t_1 t_2 t_3 - \theta (1 - t_1 - t_2 - t_3) \right). \]  

(4.40)

4.2. \( N = 2 \)

For \( N = 2 \) we consider only the DRC cases.

First we introduce notation for the odd roots when \( N = 2 \) using (2.13):

\[ \alpha_{15} = \alpha_1 + \gamma_3, \quad \alpha_{16} = \alpha_1 + \gamma_3 + \alpha_5, \]

\[ \alpha_{25} = \gamma_3, \quad \alpha_{26} = \gamma_3 + \alpha_5, \]

\[ \alpha_{35} = \alpha_2 + \gamma_4 + \alpha_5, \quad \alpha_{36} = \gamma_4, \]

\[ \alpha_{45} = \gamma_4 + \alpha_5, \quad \alpha_{46} = \gamma_4. \]  

(4.41)

Note that as a consequence of our mirror symmetry (2.2) we have:

\[ \alpha_{15} \leftrightarrow \alpha_{36}, \quad \alpha_{16} \leftrightarrow \alpha_{35} \]

\[ \alpha_{25} \leftrightarrow \alpha_{46}, \quad \alpha_{26} \leftrightarrow \alpha_{45}. \]  

(4.42)

For the characters we shall need also the following notation:

\[ \delta_{k}^{\pm} = e(\alpha_{1,7-k}), \quad \delta_{k}^{\pm} = e(\alpha_{2,7-k}), \quad k = 1, 2 \]

\[ \delta_{k}^{\pm} = e(\alpha_{3,4+k}), \quad \delta_{k}^{\pm} = e(\alpha_{4,4+k}), \quad k = 1, 2. \]  

(4.43)

the bar respecting the mirror symmetry of (4.42).

- **bc d = \( d_{\text{bc}} \):** \( d_{\text{max}} = d_{11} = d_{22} = 2 + j_1 + j_2 + r, z = j_1 - j_2. \)

The maximal number of states is \( 64 = 8 \text{(chiral)} \times 8 \text{(anti-chiral)} \), achieved for \( r \geq 4 \). The 8 anti-chiral, chiral, states are as described in a,c, respectively, (differing for \( j_2 > 0 \) and \( j_2 = 0, j_1 > 0 \) and \( j_1 = 0 \), respectively).

- **ac d = \( d_{\text{ac}} \):** \( d = d_{\text{ac}} = d_{11} = d_{22} = 2 + j_1 + j_2 + r, z = j_1 - j_2. \)

The semi-short UIRs may be called Grassmann-analytic following [28], since odd generators from different chiralities are eliminated.) For \( r = 0 \) also the generators \( X_{15} \) and \( X_{16} \) are eliminated. Thus, when \( j_1 = j_2 \) (then \( z = 0 \)) for \( r > 0 \) we have \( 1/2\text{-BPS cases} \), and for \( r = 0 \) we have \( 1/2\text{-BPS cases} \), We recall from [17] that correspondingly to the values of \( r \): \( r \geq 4, r = 3, r = 2, r = 1, r = 0 \), there are, respectively, 64, 63, 57, 42, 11 terms in the superfield.
We shall present the character only for the last case which is the shortest semi-short $N = 2$ superfield. The 11 corresponding states are:

$$
\begin{align*}
|\Lambda\rangle, & \quad X_{25}^+|\Lambda\rangle, \quad X_{26}^+|\Lambda\rangle, \\
X_{26}^+X_{25}^+|\Lambda\rangle, & \quad X_{35}^+|\Lambda\rangle, \quad X_{35}^+X_{25}^+|\Lambda\rangle, \\
X_{25}^+X_{45}^+|\Lambda\rangle, & \quad X_{26}^+X_{45}^+|\Lambda\rangle, \quad X_{45}^+X_{25}^+|\Lambda\rangle, \\
X_{26}^+X_{45}^+X_{25}^+|\Lambda\rangle, & \quad X_{26}^+X_{45}^+X_{35}^+|\Lambda\rangle, \quad X_{35}^+X_{45}^+X_{25}^+|\Lambda\rangle, \\
X_{26}^+X_{45}^+X_{35}^+|\Lambda\rangle, & \quad X_{26}^+X_{45}^+X_{45}^+|\Lambda\rangle.
\end{align*}
$$

(4.44)

The corresponding signatures—conformal and $su(2)$—in format $[d, j_1, j_2, \ldots, r]$ are:

$$
\begin{align*}
[d \equiv d^{\text{su}}, j_1, j_2; 0], & \quad [d + \frac{1}{2}, j_1 + \frac{1}{2}, j_2; 1], \quad [d + \frac{1}{2}, j_1 + \frac{1}{2}; 1], \\
[d + 1, j_1 + 1, j_2; 0], & \quad [d + 1, j_1 + 1, j_2 + 1; 0], \\
[d + 1, j_1 + \frac{1}{2}, j_2 + \frac{1}{2}; 2], & \quad [d + 1, j_1 + \frac{1}{2}, j_2 + \frac{1}{2}; 0], \quad [d + 1, j_1 + \frac{1}{2}, j_2 + \frac{1}{2}; 0], \\
[d + \frac{1}{2}, j_1, j_2 + \frac{1}{2}; 1], & \quad [d + \frac{1}{2}, j_1 + \frac{1}{2}, j_2; 1], \quad [d + 2, j_1 + 1, j_2 + 1; 0].
\end{align*}
$$

(4.45)

Note that all conformal entries are on the conformal unitarity threshold $d = 2 + j_1 + j_2$, $(j_1, j_2 > 0)$, i.e., shall use as input formula (3.19). Thus, the character formula is:

$$
\begin{align*}
ch \Lambda = & \quad e(\Lambda) \\
\phantom{\text{ch } \Lambda} = & \quad (1 - t_2)(1 - t_2)/(1 - t_2 t_3)(1 - t_2 t_3) \\
\phantom{\text{ch } \Lambda} \times & \quad \left( \mathcal{P}_{n_1, n_2} + e(\alpha_{25})\mathcal{P}_{n_1 + 1, n_2} + e(\alpha_{46})\mathcal{P}_{n_1, n_2 + 1} \right) \\
\phantom{\text{ch } \Lambda} + & \quad e(\alpha_{26}) e(\alpha_{25}) \mathcal{P}_{n_1 + 2, n_2 + 2} + e(\alpha_{45}) e(\alpha_{46}) \mathcal{P}_{n_1 + 1, n_2 + 1} + e(\alpha_{25}) e(\alpha_{46}) \mathcal{P}_{n_1 + 1, n_2 + 1} + e(\alpha_{25}) e(\alpha_{46}) \mathcal{P}_{n_1 + 1, n_2 + 1} \\
\phantom{\text{ch } \Lambda} + & \quad e(\alpha_{26}) e(\alpha_{45}) e(\alpha_{25}) e(\alpha_{46}) \mathcal{P}_{n_1 + 2, n_2 + 2} \right) \\
\phantom{\text{ch } \Lambda} = & \quad (1 - t_2)(1 - t_2)/(1 - t_2 t_3)(1 - t_2 t_3) \\
\phantom{\text{ch } \Lambda} \times & \quad \left( \mathcal{P}_{n_1, n_2} + \theta_2 \mathcal{P}_{n_1 + 1, n_2} + \theta_2 \mathcal{P}_{n_1, n_2 + 1} + \theta_2 \mathcal{P}_{n_1, n_2 + 1} + \theta_2 \mathcal{P}_{n_1 + 1, n_2 + 1} + \theta_2 \mathcal{P}_{n_1 + 1, n_2 + 1} + \theta_2 \mathcal{P}_{n_1 + 1, n_2 + 1} + \theta_2 \mathcal{P}_{n_1 + 1, n_2 + 1} \right) \right)
\end{align*}
$$

(4.46a)

where we have used the characters of $su(2)$ : $S_1 = 1 + t_5$, $S_2 = 1 + t_5 + t_5^2$, $(t_5 = e(\alpha_{25}))$.

- $j_1 > 0$, $j_2 = 0$. Here hold bare character formulae (3.86) from [17] (without counter-terms for $r \geq 4$). The states $X_{26}^+, X_{45}^+|\Lambda\rangle, X_{35}^+|\Lambda\rangle$ and their descendants are eliminated. We recall from [17] that correspondingly to the values of $r: r \geq 4$, $r = 3$, $r = 2$, $r = 1$, $r = 0$, there are, respectively, 64, 63, 58, 45, 16 states. In the last case, where $r = 0$, we eliminate the generator $X_{16}^+$ and exclude the generators $X_{3,4,5}^+$ from the anti-chiral sector. Then for $z = j_1$ this is a $\frac{1}{4}$-BPS case.

We shall present the character only for the last case. The 16 states of the superfield are:

$$
\begin{align*}
|\Lambda\rangle, & \quad X_{25}^+|\Lambda\rangle, \quad X_{26}^+|\Lambda\rangle, \\
X_{26}^+X_{25}^+|\Lambda\rangle, & \quad X_{35}^+|\Lambda\rangle, \quad X_{35}^+X_{25}^+|\Lambda\rangle, \\
X_{25}^+X_{45}^+|\Lambda\rangle, & \quad X_{26}^+X_{45}^+|\Lambda\rangle, \quad X_{45}^+X_{25}^+|\Lambda\rangle, \\
X_{26}^+X_{45}^+X_{25}^+|\Lambda\rangle, & \quad X_{26}^+X_{45}^+X_{35}^+|\Lambda\rangle, \quad X_{35}^+X_{45}^+X_{25}^+|\Lambda\rangle, \\
X_{26}^+X_{45}^+X_{35}^+|\Lambda\rangle, & \quad X_{26}^+X_{45}^+X_{45}^+|\Lambda\rangle.
\end{align*}
$$

(4.47)
The states of (4.44) appear as the first 11 of (4.47), though the content is different:
\[
\begin{align*}
[d & \equiv 2 + j_1, j_1, 0; 0], \quad [d + \frac{1}{2}, j_1 + \frac{1}{2}, 0; 1], \quad [d + \frac{1}{2}, j_1, \frac{1}{2}; 1], \\
[d + 1, j_1 + 1, 0; 0], \quad [d + 1, j_1, 1; 0], \\
[d + 1, j_1 + \frac{1}{2}, \frac{1}{2}; 2], \quad [d + 1, j_1 + \frac{1}{2}, \frac{1}{2}; 0], \quad [d + 1, j_1 + \frac{1}{2}, \frac{1}{2}; 0], \\
[d + \frac{1}{2}, j_1 + 1, \frac{1}{2}, 1], \quad [d + \frac{1}{2}, j_1 + \frac{1}{2}, 1; 1], \\
[d + 2, j_1 + 1, 1; 0], \quad [d + \frac{1}{2}, j_1 + 1, 0; 1], \\
[d + 2, j_1 + \frac{1}{2}, 0; 1], \quad [d + \frac{1}{2}, j_1 + \frac{1}{2}, 0; 1], \\
[d + 2, j_1 + 1, 0; 0].
\end{align*}
\]
(4.48)

Since in all entries the value of \( d \) is above the conformal unitarity threshold, then we use formula (3.17) for the conformal part of the character formula:
\[
\begin{align*}
ch L_\Lambda &= \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_1 t_3)(1 - t_1 t_5)} \times \{ Q_{n,1} + \theta_2 Q_{n,1+1}(1 + t_2) + \bar{\theta}_2 Q'_{n,1}(1 + t_2) \\
&\quad + \theta_1 \theta_2 Q_{n,1+2} + \bar{\theta}_1 \bar{\theta}_2 Q'_{n,1}(1 + t_2) \\
&\quad + \theta_1 \bar{\theta}_2 Q_{n,1+1}(1 + t_2)(1 + t_5 + t_3^2) + (\theta_2 \bar{\theta}_1 + \bar{\theta}_1 \theta_2) Q'_{n,1}(1 + t_3) \\
&\quad + (\theta_1 Q'_{n,1+2} + \bar{\theta}_1 Q_{n,1+1}(1 + t_3 + t_5^2) + (\theta_1 \bar{\theta}_1 + \bar{\theta}_1 \theta_2) Q'_{n,1+1}(1 + t_3) + (\theta_1 \bar{\theta}_1 \theta_2 + \bar{\theta}_1 \theta_2 Q'_{n,1+2} + \bar{\theta}_1 \theta_2 \bar{\theta}_2 Q'_{n,1+1}(1 + t_3) \}
\end{align*}
\]
(4.49)

where we have used \( Q_{n,1} = Q_{n,2} = (1 + t_3), Q_{n,3} = (1 + t_3 + t_5^2), Q_{n} \).

The next case is conjugate to the preceding.

- \( j_1 = 0, j_2 > 0 \). Here hold bare character formulae (3.89) from [17] (without counter-terms for \( r \geq 4 \)). The states \( X_{15}, X_{35}^+|\Lambda\rangle, X_{36}^+|\Lambda\rangle \) and their descendants are eliminated. When \( r = 0 \), we eliminate the generator \( X_{36}^+ \) and exclude the generators \( X_{14}^+ \) from the chiral sector. Then for \( z = -j_2 \) this is a \( \frac{1}{2} \)-BPS case. We consider only the latter case. The 16 states of the superfield are:
\[
\begin{align*}
|\Lambda\rangle, \quad X_5^+|\Lambda\rangle, \quad X_6^+|\Lambda\rangle, \\
X_{26}^+X_{25}^+|\Lambda\rangle, \quad X_{45}^+X_{46}^+|\Lambda\rangle, \\
X_{25}^+X_{26}^+|\Lambda\rangle, \quad X_{35}^+X_{36}^+|\Lambda\rangle, \\
X_{25}^+X_{26}^+X_{45}^+|\Lambda\rangle, \quad X_{35}^+X_{36}^+X_{46}^+|\Lambda\rangle, \\
X_{26}^+X_{15}^+X_{25}^+|\Lambda\rangle, \quad X_{15}^+X_{16}^+X_{26}^+|\Lambda\rangle, \\
X_{16}^+X_{15}^+X_{26}^+X_{36}^+|\Lambda\rangle, \quad X_{26}^+X_{15}^+X_{26}^+X_{36}^+|\Lambda\rangle, \\
X_{26}^+X_{16}^+X_{26}^+X_{36}^+|\Lambda\rangle.
\end{align*}
\]
(4.50)

The states of (4.44) appear as the first 11 of (4.50), the same states as in (4.47), though the content is different:
\[
\begin{align*}
[d & \equiv 2 + j_2, 0, j_2; 0], \quad [d + \frac{1}{2}, \frac{1}{2}, j_2; 1], \quad [d + \frac{1}{2}, 0, j_2 + \frac{1}{2}; 1], \\
[d + 1, 1, j_2; 0], \quad [d + 1, 0, j_2 + 1; 0], \\
[d + 1, \frac{1}{2}, j_2 + \frac{1}{2}; 2], \quad [d + 1, \frac{1}{2}, j_2 + \frac{1}{2}; 0], \quad [d + 1, \frac{1}{2}, j_2 + \frac{1}{2}; 0], \\
[d + \frac{1}{2}, j_2 + \frac{1}{2}; 1], \quad [d + \frac{1}{2}, j_2 + 1; 1], \\
[d + 2, 1, j_2 + 1; 0], \quad [d + \frac{1}{2}, 0, j_2 + \frac{1}{2}; 1], \\
[d + 2, 0, j_2 + 1; 0], \quad [d + 2, 0, j_2 + 1; 0], \\
[d + 2, \frac{1}{2}, j_2 + \frac{1}{2}; 0].
\end{align*}
\]
(4.51)
The character formula is:
\[
ch L_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)} \times (Q_{n_1} + \theta_1 Q_{n_1}(1 + t_1) + \tilde{Q}_{n_1+1}(1 + t_5) \\
+ \theta_2 \tilde{Q}_{n_1+1}(1 + t_1 + t_1^2) + \tilde{\theta}_2 Q_{n_1+2} \\
+ \theta_2 \tilde{\theta}_2 Q_{n_1+1}(1 + t_1 + t_1^2) + (\theta_1 \tilde{\theta}_2 + \tilde{\theta}_1 \theta_2) Q_{n_1+1}(1 + t_1) \\
+ (\theta_1 Q_{n_1+1}(1 + t_1 + t_1^2) + \tilde{\theta}_1 Q_{n_1+2}(1 + t_1))\theta_2 \tilde{\theta}_2(1 + t_5) \\
+ \theta_1 \tilde{\theta}_1 \theta_2 \tilde{\theta}_2 Q_{n_1+2}(1 + t_1 + t_1^2) + (\theta_1 \theta_2 + \theta_2 \theta_1) \tilde{\theta}_2 Q_{n_1+1}(1 + t_5) \\
+ (\theta_1 \theta_2 + \theta_2 \theta_1) \tilde{\theta}_1 \tilde{\theta}_2 Q_{n_1+2} + \theta_1 \theta_2 \tilde{\theta}_1 \tilde{\theta}_2 Q_{n_1+1}(1 + t_1)).
\] (4.52)

- \( j_1 = j_2 = 0 \). Here hold bare character formulae (3.92) from [17] (without counter-terms for \( r \geq 4 \)). The states \( X^+_{15}, X^+_{25}, |\Lambda\rangle, X^+_{36}, X^+_{46}, |\Lambda\rangle \) and their descendants are eliminated. We recall from [17] that correspondingly to the values of \( r \): \( r \geq 4, r = 3, r = 2, r = 1, r = 0 \), there are, respectively, 64, 63, 59, 47, 24 states. In the last case, when \( r = 0 \), we exclude the generators \( X^+_{1,4+k} \) from the anti-chiral sector and the generators \( X^+_{1,4+k} \) from the chiral sector and also the combination of impossible states:

\[
X^+_{15}X^+_{20}X^+_{36}X^+_{45}|\Lambda\rangle.
\] (4.53)

We shall consider only the 24 states of the UIR at \( r = 0 \):

\[
|\Lambda\rangle, \ X^+_{15}|\Lambda\rangle, \ X^+_{25}|\Lambda\rangle, \ X^+_{36}|\Lambda\rangle, \\
X^+_{20}X^+_{35}|\Lambda\rangle, \ X^+_{35}X^+_{46}|\Lambda\rangle, \\
X^+_{25}X^+_{46}|\Lambda\rangle, \ X^+_{25}X^+_{36}|\Lambda\rangle, \ X^+_{20}X^+_{35}X^+_{46}|\Lambda\rangle, \\
X^+_{26}X^+_{35}X^+_{46}|\Lambda\rangle, \ X^+_{36}X^+_{35}X^+_{46}|\Lambda\rangle, \\
X^+_{20}X^+_{35}X^+_{46}|\Lambda\rangle, \ X^+_{26}X^+_{35}X^+_{46}|\Lambda\rangle, \\
X^+_{16}X^+_{35}X^+_{46}|\Lambda\rangle, \ X^+_{15}X^+_{20}X^+_{46}|\Lambda\rangle, \\
X^+_{16}X^+_{20}X^+_{36}X^+_{45}|\Lambda\rangle, \ X^+_{16}X^+_{20}X^+_{36}X^+_{45}|\Lambda\rangle, \\
X^+_{16}X^+_{20}X^+_{36}X^+_{45}|\Lambda\rangle.
\] (4.54)

The states of (4.44) appear as the first 11 of (4.54), the states of (4.47) as the first 16 of (4.54), the states of (4.50) as the first 11 plus states 17–21 of (4.54), though of course the contents are different:
The character formula is:

\[
ch L_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1t_2)(1 - t_1t_3)(1 - t_1t_2t_3)} \times (1 + \theta_2(1 + t_1)(1 + t_3) + \vartheta_2(1 + t_1 + t_3)) \\
+ \theta_1 \vartheta_2(1 + t_1 + t_2^2) + \vartheta_1 \theta_2(1 + t_3 + t_2^2) \\
+ \theta_2 \vartheta_3(1 + t_1)(1 + t_3 + t_2^2) + (\theta_1 \theta_2 + \theta_1 \vartheta_3)(1 + t_3)(1 + t_3) \\
+ (\theta_1 + t_2)(1 + t_1)(1 + t_3 + t_2^2) + \theta_1 \theta_3(1 + t_1)(1 + t_3 + t_2^2) + \vartheta_2 \vartheta_3(1 + t_3 + t_2^2) \\
+ \theta_1 \theta_3 \theta_2(1 + t_1 + t_3) + \theta_1 \vartheta_3 \theta_2(1 + t_3 + t_2^2) + \vartheta_1 \theta_3 \theta_2(1 + t_1 + t_2^2). 
\] (4.56)

\( \bullet \) \textbf{ad} \( d = d_{t_1} = \bar{d}_{t_2} = 1 + j_2 + r, j_1 = 0, z = -1 - j_2. \)

Here hold bare character formulae (3.95) from [17] when \( j_2r > 0, \) (3.96) when \( j_2 = 0, r > 0, \) (both these cases without counter-terms for \( r \geq 4 \)), and finally when \( r = 0 \) holds (3.97) independently of the value of \( j_2 \)—these are the anti-chiral massless UIRs.

The generators \( X_{15}^+, X_{35}^+ \), and in addition \( X_{16}^+ \) for \( j_2 > 0 \) (respectively the state \( X_{26}^+, X_{46}^+, \Lambda \)), and its descendents for \( j_2 = 0 \) are eliminated. The maximal number of states is \( 24 = 3(\text{chiral}) \times 8(\text{anti-chiral}) \), achieved for \( r \geq 4 \). The chiral sector for \( r > 0 \) consists of the states:

\[
X_{26}^+, X_{16}^+|\Lambda, \ r \geq 1, \\
X_{16}^+X_{26}^+|\Lambda, \ r \geq 2, 
\] (4.57)

and the vacuum, while the anti-chiral sector is given by

\[
X_{46}^+, \quad e_\sigma^a = 1, e_\rho^a = 1, \\
X_{45}^+X_{46}^+, \quad e_\sigma^\rho = 0, e_\rho^\sigma = 2, \\
1, \quad X_{35}^+X_{46}^+, \quad e_\sigma^a = 0, e_\rho^a = 0, \\
X_{45}^+, \quad X_{45}^+X_{45}^+, \quad e_\sigma^\rho = -1, e_\rho^\sigma = 1, \\
X_{45}^+, \quad e_\sigma^a = -1, e_\rho^a = -1, \\
X_{35}^+X_{45}^+, \quad e_\sigma^a = -2, e_\rho^a = 0 
\] (4.58)

for \( j_2 > 0 \) and by

\[
X_{46}^+, \quad e_\sigma^a = 1, e_\rho^a = 1, \\
X_{45}^+X_{46}^+, \quad e_\sigma^\rho = 0, e_\rho^\sigma = 2, \\
1, \quad X_{35}^+X_{46}^+, \quad X_{45}^+X_{35}^+, \quad e_\sigma^a = 0, e_\rho^a = 0, \\
X_{45}^+, \quad X_{45}^+X_{45}^+, \quad e_\sigma^\rho = -1, e_\rho^\sigma = 1, \\
X_{35}^+X_{45}^+, \quad e_\sigma^a = -2, e_\rho^a = 0 
\] (4.59)

for \( j_2 = 0 \). Finally, for \( r = 0 \) also the generators \( X_{16}^+, X_{26}^+, X_{35}^+ \) are eliminated.
The possible 24 states for \( j_2 > 0 \) are given explicitly as:

\[
\begin{align*}
|\Lambda\rangle, X_{26}^+ |\Lambda\rangle, X_{35}^+ X_{46}^+ |\Lambda\rangle, & \quad r \geq 0, \\
X_{20}^+ |\Lambda\rangle, X_{43}^+ |\Lambda\rangle, & \quad r \geq 1 \\
X_{35}^+ X_{46}^+ |\Lambda\rangle, X_{26}^+ X_{46}^+ |\Lambda\rangle, & \quad r \geq 1 \\
X_{26}^+ X_{35}^+ X_{46}^+ |\Lambda\rangle, X_{26}^+ X_{35}^+ X_{46}^+ |\Lambda\rangle, & \quad r \geq 1 \\
X_{16}^+ X_{20}^+ X_{46}^+ |\Lambda\rangle, X_{45}^+ X_{35}^+ X_{46}^+ |\Lambda\rangle, & \quad r \geq 1 \\
X_{35}^+ |\Lambda\rangle, & \quad r \geq 1
\end{align*}
\]

Correspondingly to the values of \( r \): \( r \geq 4, r = 3, r = 2, r = 1, r = 0 \), there are, respectively, 24, 23, 19, 12, 3 states. Three states are marked with (*)—these states are not present when \( j_2 = 0 \). Thus, the 24 states for \( j_2 = 0 \) are given as the 21 from (4.60) without (*) and the following 3 states:

\[
\begin{align*}
X_{36}^+ X_{45}^+ |\Lambda\rangle, X_{26}^+ X_{36}^+ X_{45}^+ |\Lambda\rangle, & \quad r \geq 1 \\
X_{16}^+ X_{20}^+ X_{36}^+ X_{45}^+ |\Lambda\rangle, & \quad r \geq 2
\end{align*}
\]

Thus, for \( j_2 = 0 \) correspondingly to the values of \( r \) there are, respectively, 24, 23, 20, 13, 3 states. The content of the states in (4.60) is as follows:

\[
\begin{align*}
[d = 1 + j_2 + r; 0, j_2; r], & \quad [d + \frac{1}{2}, 0, j_2 + \frac{1}{2}; r + 1], \quad [d + 1, 0, j_2 + 1; r], & \quad r \geq 0 \\
[d + \frac{1}{2}, j_2; r - 1], & \quad [d + \frac{1}{2}, 0, j_2 + \frac{1}{2}; r - 1], & \quad r \geq 1 \\
[d + 1, 0, j_2; r], & \quad [d + 1, \frac{1}{2}, j_2 + \frac{1}{2}; r], & \quad r \geq 1 \\
[d + \frac{3}{2}, j_2 + 1; r - 1], & \quad [d + \frac{3}{2}, \frac{1}{2}, j_2; r - 1], & \quad r \geq 1 \\
[d + \frac{3}{2}, 0, j_2 + \frac{1}{2}; r - 1], & \quad [d + \frac{3}{2}, \frac{1}{2}, j_2; r - 1], & \quad r \geq 1 \\
[d + \frac{1}{2}, 0, j_2 - \frac{1}{2}; r - 1], & \quad [d + 1, 0, j_2; r - 2], & \quad r \geq 2 \\
[d + 2, \frac{1}{2}, j_2 + \frac{1}{2}; r - 2], & \quad [d + 2, 0, j_2 + 1; r - 2], & \quad r \geq 2 \\
[d + 1, \frac{1}{2}, j_2 - \frac{1}{2}; r - 2], & \quad r \geq 2, & \quad (\star) \\
[d + 1, \frac{1}{2}, j_2; r - 3], & \quad [d + \frac{3}{2}, 0, j_2 + \frac{1}{2}; r - 3], & \quad r \geq 3, & \quad (\star) \\
[d + 3, 0, j_2 - \frac{1}{2}; r - 3], & \quad r \geq 3, & \quad (\star) \\
[d + 2, 0, j_2; r - 4], & \quad r \geq 4
\end{align*}
\]

and for the states in (4.61):

\[
\begin{align*}
[d + 1, 0, j_2; r], & \quad [d + 1, \frac{1}{2}, j_2; r - 1], & \quad r \geq 1 \\
[d + 2, 0, j_2; r - 2], & \quad r \geq 2
\end{align*}
\]

The character formula for \( r > 0 \) is:

\[
ch L_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)}
\]
\[ \times \{ Q_{n} S_{r} + \bar{\theta}_{2} Q_{m+1} S_{r+1} + \bar{\theta}_{1} \bar{\theta}_{2} Q_{m+2} S_{r} \\
+ \theta_{1}(1 + t) Q_{n} S_{r-1} + \bar{\theta}_{1} Q_{m+1} S_{r-1} + \bar{\theta}_{2} Q_{m} S_{r} + \theta_{1} \bar{\theta}_{2}(1 + t) Q_{n+1} S_{r} \\
+ \theta_{1} \bar{\theta}_{2}(1 + t) Q_{m+2} S_{r-1} + \theta_{1} \bar{\theta}_{2}(1 + t) Q_{m} S_{r-1} \\
+ \bar{\theta}_{1} \bar{\theta}_{2} Q_{m+1} S_{r-1} + \bar{\theta}_{1} \bar{\theta}_{2} Q_{m+1} S_{r-1} + \bar{\theta}_{1} Q_{m-1} S_{r-1} \\
+ \theta_{1} \bar{\theta}_{1}(1 + t) Q_{n+1} S_{r-2} + \theta_{1} \bar{\theta}_{1} Q_{m} S_{r-2} \\
+ \bar{\theta}_{1} \bar{\theta}_{2}(1 + t) Q_{m+1} S_{r-2} + \theta_{1} \bar{\theta}_{1} \bar{\theta}_{2} Q_{m+2} S_{r-2} + \theta_{1} \bar{\theta}_{1} \bar{\theta}_{2} Q_{m} S_{r-2} \\
+ \theta_{1} \bar{\theta}_{1}(1 + t) Q_{m} S_{r-3} + \theta_{1} \bar{\theta}_{1}(1 + t) Q_{m} S_{r-3} + \theta_{1} \bar{\theta}_{1}(1 + t) Q_{m+1} S_{r-3} \\
+ \bar{\theta}_{1} \bar{\theta}_{1} \bar{\theta}_{2} Q_{m+1} S_{r-3} + \theta_{1} \bar{\theta}_{1} \bar{\theta}_{1} Q_{m+1} S_{r-1} S_{r-1} \\
+ \theta_{1} \bar{\theta}_{1} \bar{\theta}_{1} Q_{m+1} S_{r-4} + \delta_{j,0} \bar{\theta}_{2} Q_{m} S_{r} + \delta_{j,0} \theta_{1} \bar{\theta}_{2} \bar{\theta}_{1}(1 + t) Q_{n} S_{r-1} \\
+ \delta_{j,0} \theta_{1} \bar{\theta}_{2} \bar{\theta}_{1}(1 + t) Q_{m} S_{r-1} \} \]  

(4.64)

where we use the \( su(2) \) character factors \( S_{p} = \sum_{p=0}^{\infty} t_{p}^{r} \) for \( p \in \mathbb{Z}_{+} \), and for continuity we use: \( S_{p} = 0 \) for \( p \in -\mathbb{N} \), we also use the convention \( Q_{p} = 0 \) for \( p < 0 \). Thus, the formula is valid for all \( r > 0 \) and for all \( j_2 \).

For \( r = 0 \) we have the character formula for the anti-chiral massless UIRs and we have to use the massless conformal characters (3.21), (3.22), (3.23):

\[
ch L_{\Lambda} = \frac{e(\Lambda)}{(1 - t_{2})(1 - t_{3})(1 - t_{2}t_{3})(1 - t_{1}t_{2}t_{3})} \times \{ P_{n} + \theta_{2} P_{n+1} + \theta_{1} \bar{\theta}_{2} P_{n+2} \}. 
\]

(4.65)

\* bc \( d = d_{21}^{b} = d_{22}^{b} = 1 + j_{1} + r, j_{2} = 0, z = 1 + j_{1} \).

Here hold bare character formulae (3.98) from [17] when \( j_{1} r > 0 \), (3.99) when \( j_{1} = 0, r > 0 \), (both these cases without counter-terms for \( r > 4 \)), and finally when \( r = 0 \) holds (3.100) independently of the value of \( j_{1} \)—these are the chiral massless UIRs.

This case is conjugate to the previous one ad and everything may be obtained from the mirror symmetry. We give only the character of the chiral massless case:

\[
ch L_{\Lambda} = \frac{e(\Lambda)}{(1 - t_{2})(1 - t_{1}t_{2})(1 - t_{2}t_{3})(1 - t_{1}t_{2}t_{3})} \{ P_{n} + \theta_{2} P_{n+1} + \theta_{1} \bar{\theta}_{2} P_{n+2} \}. 
\]

(4.66)

\* bd \( d = d_{21}^{d} = d_{22}^{d} = r, j_{1} = j_{2} = 0, z = 0 \).

The generators \( X_{1}^{+}, X_{2}^{+}, X_{3}^{+}, X_{4}^{+} \) are eliminated. Thus, for \( r > 1 \) these are \( \frac{1}{2} \)-BPS cases. For \( r = 1 \) also the generators \( X_{1}^{+}, X_{2}^{+} \) are eliminated. Thus, the latter is a \( \frac{1}{2} \)-BPS case, and it is also the \( N = 2 \) mixed massless irrep. For \( r = 0 \) the remaining two generators \( X_{3}^{+}, X_{4}^{+} \) are eliminated and we have the trivial irrep as explained in general.

For \( r > 0 \) the bare character formula is (3.101) from [17] with \( \bar{t}_{0} = \bar{t}_{0} = 0 \). The maximal number of states is nine and the list of states together with the conditions when they exist are:

\[
\begin{align*}
|\Lambda|, & \quad r \geq 0, \\
X_{30}^{+}|\Lambda|, & \quad r \geq 0, \\
X_{16}^{+}|\Lambda|, & \quad r \geq 0, \\
X_{35}^{+}|\Lambda|, & \quad r \geq 2, \\
X_{35}^{+}X_{45}^{+}|\Lambda|, & \quad r \geq 3, \\
X_{26}^{+}X_{35}^{+}X_{45}^{+}|\Lambda|, & \quad r \geq 4.
\end{align*}
\]

(4.67)

Thus, correspondingly to the values of \( r \) we have 9, 8, 6, 3, 1 states in the superfield decomposition. The mixed massless \( \frac{1}{2} \)-BPS irrep is obtained for \( d = r = 1 \) and consists of the first three states above (as was shown in general).
The explicit character formula for \( r > 1 \) is:

\[
ch L_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1t_2)(1 - t_2t_3)(1 - t_1t_2t_3)} \times \{
\begin{align*}
& \{S_r + \theta_1(1 + t_1)S_{r-1} + \tilde{\theta}_1(1 + t_1)S_{r-1} - \theta_1(1 + t_1)S_r, \\
& \theta_1(1 + t_1)S_{r-2} - \tilde{\theta}_1(1 + t_1)S_{r-2} + \theta_1(1 + t_1)S_{r-2}, \\
& \theta_1(1 + t_1)S_{r-3} + \tilde{\theta}_1(1 + t_1)S_{r-3} - \theta_1(1 + t_1)S_{r-1}
\}
\}
\]

\[
(4.68)
\]

where we have used the conformal factor \( Q_{a_1, a_2} \) from (3.17) (only for \( n_1, n_2 \leq 2 \)).

For the \( \frac{1}{2} \)-BPS mixed massless irrep, \( r = 1 \), we have to use the massless conformal characters from (3.22), (3.23), (3.25), and we have:

\[
ch L_\Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1t_2)(1 - t_2t_3)(1 - t_1t_2t_3)} \times \{(1 - t_2t_3)(1 + t_5) + \theta_1(1 + t_1 - t_12 - t_13) + \tilde{\theta}_1(1 + t_3 - t_23 - t_13)\}.
\]

\[
(4.69)
\]

4.3. \( N = 4 \)

For \( N = 4 \) we consider only some important examples.

First we introduce notation for the odd roots when \( N = 4 \) using (2.13):

\[
\begin{align*}
\alpha_{25} &= \gamma_3, \quad \alpha_{26} = \gamma_3 + \alpha_5, \quad \alpha_{27} = \gamma_3 + \alpha_5 + \alpha_6, \quad \alpha_{28} = \gamma_3 + \alpha_5 + \alpha_6 + \alpha_7, \\
\alpha_{14} &= \alpha_1 + \alpha_{25}, \quad k = 5, 6, 7, 8; \\
\alpha_{45} &= \gamma_4 + \alpha_3 + \alpha_5 + \alpha_7, \quad \alpha_{46} = \gamma_4 + \alpha_3 + \alpha_5 + \alpha_6, \quad \alpha_{47} = \gamma_4 + \alpha_3 + \alpha_5, \quad \alpha_{48} = \gamma_4, \\
\alpha_{36} &= \alpha_3 + \alpha_{44}, \quad k = 5, 6, 7, 8.
\end{align*}
\]

\[
(4.70)
\]

Note that as a consequence of our mirror symmetry (2.15) we have:

\[
\begin{align*}
\alpha_{15} &\longleftrightarrow \alpha_{38}, \quad \alpha_{16} \longleftrightarrow \alpha_{57}, \quad \alpha_{17} \longleftrightarrow \alpha_{36}, \quad \alpha_{18} \longleftrightarrow \alpha_{55}, \\
\alpha_{25} &\longleftrightarrow \alpha_{48}, \quad \alpha_{26} \longleftrightarrow \alpha_{47}, \quad \alpha_{27} \longleftrightarrow \alpha_{46}, \quad \alpha_{28} \longleftrightarrow \alpha_{45}.
\end{align*}
\]

\[
(4.71)
\]

For the characters we shall need also the following notation:

\[
\begin{align*}
\theta_k &\equiv e(\alpha_{1,9-k}), \quad \theta_k \equiv e(\alpha_{2,9-k}), \quad k = 1, 2, 3, 4 \\
\tilde{\theta}_k &\equiv e(\alpha_{3,4+k}), \quad \tilde{\theta}_k \equiv e(\alpha_{4,4+k}), \quad k = 1, 2, 3, 4.
\end{align*}
\]

\[
(4.72)
\]

- First we consider the massless multiplets. As in all cases when \( N > 1 \) there are three cases of massless multiplets. In our classification they are DRC cases \( \text{ad}, \text{bc}, \text{bd} \).
- \( \text{ad} \) \( d = d_{21} = d_{42} = d_{43} = d_{44} = 1 + j_2 = -z, \ j_1 = 0, \ r_i = 0, \ V, i, \) and all generators \( X_{i,1+k}^+, \ X_{2,4+k}^+, \ X_{3,4+k}^+ \) are eliminated. These anti-chiral irrepes were denoted \( X_i^+ \), \( s = j_2 = 0, 1, 2, \ldots, \) in section 3 of [4]. Besides the vacuum they contain only \( N \) states in \( \hat{X}_\Lambda \) and should be called ultrashort UIRs. The bare character formula can be written in the most explicit way [17]:

\[
ch \hat{X}_\Lambda = 1 + e(\alpha_{48}) + e(\alpha_{47})e(\alpha_{48}) + e(\alpha_{46})e(\alpha_{47})e(\alpha_{48}) + e(\alpha_{45})e(\alpha_{46})e(\alpha_{47})e(\alpha_{48}).
\]

\[
(4.73)
\]

Their signatures are:

\[
\begin{align*}
[d &+ 1; 0, j_2, 0, 0, 0], \quad [d + \frac{1}{2}; 0, j_2, \frac{1}{2}; 1, 0, 0], \quad [d + 1; 0, j_2 + 1; 0, 1, 0], \\
[d + 2; 0, j_2 + 2; 0, 0, 0].
\end{align*}
\]

\[
(4.74)
\]

All conformal entries are on the massless unitarity threshold, thus for the conformal entries we must use formula (3.21). For the \( su(4) \) entries we use formulae from (3.35). Then we have
the explicit character formula for the anti-chiral massless case:

\[ ch \Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \times \{ \mathcal{P}_{n_1} + \tilde{\theta}_4 \mathcal{P}_{n_1+1} \mathcal{S}_{100} + \tilde{\theta}_3 \tilde{\theta}_4 \mathcal{P}_{n_1+2} \mathcal{S}_{1001} + \tilde{\theta}_2 \tilde{\theta}_3 \tilde{\theta}_4 \mathcal{P}_{n_1+3} \mathcal{S}_{10001} + \tilde{\theta}_1 \tilde{\theta}_2 \tilde{\theta}_3 \tilde{\theta}_4 \mathcal{P}_{n_1+4} \}. \]

\[ (4.75) \]

- **bc** \( d = d_{41}^b = d_{41}^c = d_{41}^{bc} = 1 + j_1 = z, j_2 = 0, r_i = 0, \forall, i, \) and all generators \( X_{1,4,k}^+, X_{3,4,4-k}^+, X_{4,4,4-k}^+ \) are eliminated. These chiral irreps were denoted \( \chi_s, s = j_1 = 0, \frac{1}{2}, 1, \ldots \) in section 3 of [4]. Besides the vacuum they contain only \( N \) states in \( \hat{\Lambda} \) and should be called ultrashort UIRs. The bare character formula is [17]:

\[ ch \hat{\Lambda}_N = 1 + e(25) + e(26) e(25) + e(27) e(26) e(25) + e(38) e(27) e(26) e(25). \]

\[ (4.76) \]

Their signatures are:

\[ [d \equiv 1 + j_1; j_1, 0; 0, 0, 0] \quad [d + \frac{3}{2}; j_1 + \frac{1}{2}, 0; 0, 0, 1] \quad [d + 1; j_1 + 1, 0; 0, 1, 0] \quad [d + 2; j_1 + 2, 0; 0, 0, 0]. \]

\[ (4.77) \]

All conformal entries are on the massless unitarity threshold, thus we use formula (3.24). Then we have the explicit character formula for the chiral massless case:

\[ ch \Lambda = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \times \{ \mathcal{P}_{n_1} + \tilde{\theta}_4 \mathcal{P}_{n_1+1} \mathcal{S}_{100} + \tilde{\theta}_3 \tilde{\theta}_4 \mathcal{P}_{n_1+2} \mathcal{S}_{1001} + \tilde{\theta}_2 \tilde{\theta}_3 \tilde{\theta}_4 \mathcal{P}_{n_1+3} \mathcal{S}_{10001} + \tilde{\theta}_1 \tilde{\theta}_2 \tilde{\theta}_3 \tilde{\theta}_4 \mathcal{P}_{n_1+4} \}. \]

\[ (4.78) \]

We note the mirror symmetry between character formulae (4.75) and (4.78).

- **bd** \( d = d_{41}^b = d_{41}^c = d_{41}^{bc} = m_1 = 1, i_0 = 0, 1, 2, z = (i_0 - 1)/2, j_1 = j_2 = 0, r_i = 2, \delta_{i_0+1}. \)

In section 3 of [4] they are parameterized by \( n = 2, 3, \) and denoted by \( \chi_n^+, n = m, \) \( \chi_n^+, n = N - m, (z = 1 - n/2), \) but there is the coincidence for \( n = 2: \)

\[ \chi_9^+ = \chi_2^+. \] Here they are enumerated by the parameter \( i_0. \) The self-conjugate case \( i_0 = 1, z = 0 \) is a \( \frac{1}{2} \) BPS state. The following generators are eliminated: the chiral \( X_{1,4}^+, \forall k, X_{2,4}^+, k = 5, \ldots, 5 + i_0, \) and the anti-chiral \( X_{3,4}^+, \forall k, X_{4,4,4-k}^+, k = 0, \ldots, i_0. \)

The bare character formula is [17]:

\[ ch \hat{\Lambda}_N = 1 + \sum_{j=0}^{i_0} \sum_{l=0}^{i_0} e(2, 8 - i) + \sum_{k=1}^{3-i_0} \sum_{l=0}^{3-i_0} e(4, 4+i). \]

\[ (4.79) \]

We write out the signatures for the three subcases separately:

\[ [1; 0, 0; 1, 0, 0], [\frac{3}{2}; \frac{1}{2}, 0; 0, 0, 0], [\frac{3}{2}; 0, \frac{1}{2}, 0, 1, 0]. \]

\[ (4.80) \]

\[ [2; 0, 1; 0, 0, 1], [\frac{5}{2}; 0, 1; 0, 0, 0], i_0 = 0. \]

\[ [1; 0, 0; 0, 1, 0], [\frac{3}{2}; \frac{1}{2}, 0; 1, 0, 0], [2; 1, 0; 0, 0, 0]. \]

\[ [\frac{3}{2}; 0, \frac{3}{2}, 0, 0, 1], [2; 0, 1; 0, 0, 0], i_0 = 1. \]

\[ [1; 0, 0; 0, 0, 1], [\frac{3}{2}; \frac{1}{2}, 0; 0, 1, 0], [2; 1, 0; 1, 0, 0]. \]

\[ [\frac{5}{2}; 3, 0; 0, 0, 0], [\frac{3}{2}; 0, \frac{1}{2}, 0, 0, 0], i_0 = 2. \]
All conformal entries are on the massless unitarity threshold, thus we use formulae (3.21) and (3.24). Then we have the explicit character formula for the mixed chiral–anti-chiral massless cases:

\[ \text{ch} L^\Lambda_\Lambda = \frac{e(\Lambda)}{(1-t_2)(1-t_1 t_2)(1-t_2 t_3)(1-t_1 t_2 t_3)} \times \left\{ \begin{array}{l} \{1 + t_1 \} S_{100} + \theta_2(1 + t_1 - t_2 - t_3) \tilde{S}_{010} \\
+ \theta_2 \theta_3 \tilde{P}_3(t_3) \} \right. \\
\left. \left\{1 + t_1 t_2 - t_3 \right\} \tilde{S}_{001} + \theta_1 \theta_2 \theta_3 \tilde{P}_3(t_3) \right\} \right) \] (4.81a)

\[ \text{ch} L^\Lambda_{\Lambda} = \frac{e(\Lambda)}{(1-t_2)(1-t_1 t_2)(1-t_2 t_3)(1-t_1 t_2 t_3)} \times \left\{ \begin{array}{l} \{1 + t_1 \} S_{100} + \theta_2(1 + t_1 - t_2 - t_3) \tilde{S}_{010} \\
+ \theta_2 \theta_3 \tilde{P}_3(t_3) \} \right. \\
\left. \left\{1 + t_1 t_2 - t_3 \right\} \tilde{S}_{001} + \theta_1 \theta_2 \theta_3 \tilde{P}_3(t_3) \right\} \right) \] (4.81b)

\[ \text{ch} L^\Lambda_\Lambda = \frac{e(\Lambda)}{(1-t_2)(1-t_1 t_2)(1-t_2 t_3)(1-t_1 t_2 t_3)} \times \left\{ \begin{array}{l} \{1 + t_1 \} S_{100} + \theta_2(1 + t_1 - t_2 - t_3) \tilde{S}_{010} \\
+ \theta_2 \theta_3 \tilde{P}_3(t_3) \} \right. \\
\left. \left\{1 + t_1 t_2 - t_3 \right\} \tilde{S}_{001} + \theta_1 \theta_2 \theta_3 \tilde{P}_3(t_3) \right\} \right) \] (4.81c)

We note the mirror symmetry between character formulae (4.81a) and (4.81c), while (4.81b) is self-conjugate (as expected).

- Next we consider the graviton supermultiplet [57]. In our classification this is a DRC case:
- \( \mathbf{bd} d = d^{1}_{d_1} = d^{1}_{d_2} = d_{m_1} = 2, j_1 = j_2 = 0, r_2 = 2, r_1 = r_3 = 0, z = 0. \)

Here are eliminated generators \( X^+_k, X^{*+}_k, \) for \( k = 5, 6 \) and generators \( X^+_k, X^{+}_k, \) for \( k = 7, 8 \) and this is a \( \frac{3}{2} \)-BPS case.

The bare character formula is (3.101) from [17] taken for the case \( i_0 = i_0' = 1: \)

\[ \text{ch} \tilde{L}_\Lambda = \prod_{\mu \in \mathbb{Z}_{+}} \frac{(1 + e(\mu)) - R}{1}. \] (4.82)

Explicitly, the states (fields) are:

- \( |\Lambda\rangle, \quad [2; 0, 0; 0, 2, 0], \quad \varphi_1 \)
- \( X^+_2 |\Lambda\rangle, \quad [\frac{1}{2}; \frac{1}{2}; 0; 1, 1, 0], \quad \lambda_1^+ \)
- \( X^+_4 |\Lambda\rangle, \quad [\frac{3}{2}; \frac{1}{2}; 0; 1, 1], \quad \lambda_1^- \)
- \( X^+_7 X^+_2 |\Lambda\rangle, \quad [3; 0, 0; 2, 0, 0], \quad \varphi_2 \)
- \( X^+_6 X^+_6 |\Lambda\rangle, \quad [3; 0, 0; 0, 2, 0], \quad \bar{\varphi}_2 \)
- \( X^+_6 X^+_2 |\Lambda\rangle, \quad [3; 0, 0; 0, 1, 0], \quad \varphi_2' \)
- \( X^+_5 X^+_6 |\Lambda\rangle, \quad [3; 0, 0; 0, 0, 0], \quad \varphi_2'' \)
- \( X^+_2 X^+_{27} |\Lambda\rangle, \quad [3; 1, 0; 0, 0, 0], \quad A_{27}^+ \)
- \( X^+_3 X^+_4 |\Lambda\rangle, \quad [3; 0; 1; 0, 0, 0], \quad A_{27}^- \)
- \( X^+_4 X^+_2 |\Lambda\rangle, \quad [3; 1, 0; 1, 1, 1], \quad A_{27}^\pm \)
- \( X^+_5 X^+_2 |\Lambda\rangle, \quad [3; 0; 1; 0, 0, 1], \quad \lambda_2 \)
- \( X^+_6 X^+_2 |\Lambda\rangle, \quad [\frac{3}{2}; \frac{1}{2}; 1, 1, 0], \quad \lambda_2^+ \)
- \( X^+_7 X^+_2 |\Lambda\rangle, \quad [\frac{3}{2}; \frac{1}{2}; 0; 1, 0, 0], \quad \lambda_2^- \)
Next we consider the Konishi supermultiplet, [58], this the spinless R-symmetry scalar on the unitarity threshold, thus, $d = 2$. In our classification this is a DRC case $ac$:

- $ac \ d = d_{\text{max}} = d_{ac}^1 = d_{NN} = d_{ac} = 2$, $j_1 = j_2 = 0$, $r_i = 0$, $\forall$, $i, z = 0$.

Here all generators $X^+_{1k}$, and $X^+_{3k}$, are eliminated and this is $\frac{1}{2}$-BPS case.

The bare character formula is (3.84) from [17]:

$$\text{ch} \bar{L} = \sum_{k=1}^{N} \sum_{i=1}^{k} e(\alpha_{z,i}) + \sum_{k=1}^{N} \sum_{i=1}^{k} e(\alpha_{4,5,N-i}) + \prod_{\alpha^0 \neq \alpha^1 \neq \alpha^2} (1 + e(\alpha)) - \mathcal{R}. \quad (4.85)$$

The first two terms of (4.85) are the four chiral and anti-chiral states of this case:

$$X^+_{25} |\Lambda\rangle, \quad X^+_{26} X^+_{25} |\Lambda\rangle, \quad X^+_{27} X^+_{26} X^+_{25} |\Lambda\rangle, \quad X^+_{28} X^+_{27} X^+_{26} X^+_{25} |\Lambda\rangle, \quad X^+_{48} |\Lambda\rangle, \quad X^+_{49} X^+_{48} |\Lambda\rangle, \quad X^+_{50} X^+_{49} |\Lambda\rangle, \quad X^+_{51} X^+_{50} |\Lambda\rangle, \quad X^+_{52} X^+_{51} X^+_{50} |\Lambda\rangle. \quad (4.86)$$

All other states (besides the vacuum) are of mixed chirality, and our first task is to make explicit the second line of the above formula (4.85). It turns out that the counter-term has 22
states and there remain 42 states contributing to this second line, which explicitly are:

\[
|\Lambda\rangle, \quad X_{28}^+X_{48}^0|\Lambda\rangle, \quad X_{25}^+X_{45}^0|\Lambda\rangle, \quad X_{28}^+X_{25}^0|\Lambda\rangle, \quad X_{45}^+X_{25}^0|\Lambda\rangle, \quad X_{26}^+X_{25}^0|\Lambda\rangle, \quad X_{47}^+X_{25}^0|\Lambda\rangle, \\
X_{26}^+X_{25}^0|\Lambda\rangle, \quad X_{27}^+X_{47}^0|\Lambda\rangle, \quad X_{47}^+X_{47}^0|\Lambda\rangle, \quad X_{26}^+X_{47}^0|\Lambda\rangle, \quad X_{46}^+X_{26}^0|\Lambda\rangle, \quad X_{46}^+X_{46}^0|\Lambda\rangle, \quad X_{46}^+X_{46}^0|\Lambda\rangle, \quad X_{26}^+X_{26}^0|\Lambda\rangle, \\
X_{26}^+X_{26}^0|\Lambda\rangle, \quad X_{27}^+X_{27}^0|\Lambda\rangle, \quad X_{47}^+X_{47}^0|\Lambda\rangle, \quad X_{28}^+X_{28}^0|\Lambda\rangle, \quad X_{48}^+X_{28}^0|\Lambda\rangle, \quad X_{28}^+X_{48}^0|\Lambda\rangle, \\
X_{25}^+X_{25}^0|\Lambda\rangle, \quad X_{27}^+X_{27}^0|\Lambda\rangle, \quad X_{47}^+X_{47}^0|\Lambda\rangle, \quad X_{26}^+X_{26}^0|\Lambda\rangle, \quad X_{26}^+X_{26}^0|\Lambda\rangle, \quad X_{27}^+X_{27}^0|\Lambda\rangle, \\
X_{48}^+X_{28}^0|\Lambda\rangle, \quad X_{48}^+X_{48}^0|\Lambda\rangle, \quad X_{28}^+X_{28}^0|\Lambda\rangle, \quad X_{48}^+X_{28}^0|\Lambda\rangle, \quad X_{48}^+X_{48}^0|\Lambda\rangle, \quad X_{28}^+X_{28}^0|\Lambda\rangle,
\]

where * denotes lines in which the states are self-conjugate, while in all other cases the two states in each line are conjugate to each other. The corresponding signatures of both (4.86) and (4.87) are:

\[
[\frac{5}{2}; \frac{1}{2}; 0; 0, 0, 1], \quad [3; 1.0; 0, 1, 0], \quad [\frac{7}{2}; \frac{1}{2}; 0; 1, 0, 0], \quad [4; 2.0; 0, 0, 0], \\
[\frac{5}{2}; 0; \frac{1}{2}; 1, 0, 0], \quad [3; 0, 0; 1, 0, 0], \quad [\frac{7}{2}; 0; \frac{1}{2}; 0, 0, 1], \quad [4; 0, 0; 0, 0, 0], \\
[2; 0, 0; 0, 0, 0], \quad [3; \frac{1}{2}; \frac{1}{2}; 1, 0, 1], \\
[\frac{3}{2}; \frac{1}{2}; \frac{1}{2}; 0, 0, 0], \quad [3; \frac{1}{2}; \frac{1}{2}; 0, 0, 0], \\
[\frac{3}{2}; 1; 0, 0, 1], \quad [\frac{3}{2}; \frac{1}{2}; 1; 1, 0, 0], \\
[\frac{7}{2}; 1; \frac{1}{2}; 1, 0, 1], \quad [\frac{7}{2}; \frac{1}{2}; 1; 0, 1, 1], \\
[\frac{7}{2}; 1, \frac{1}{2}; 0, 1, 0], \quad [\frac{7}{2}; \frac{1}{2}; 1; 1, 0, 0], \\
[4; 1, 1; 1, 0, 1], \quad [4; 1, 1; 1, 0, 1], \\
[4; 1, 1; 0, 2, 0], \quad [4; 1, 1; 0, 0, 0], \\
[4; 1, 1; 0, 0, 0], \quad [4; 1, 1; 0, 0, 0], \\
[4; \frac{3}{2}; \frac{1}{2}; 0, 0, 0, 2], \\
[4; \frac{3}{2}; \frac{1}{2}; 0, 0, 0, 2], \\
[4; \frac{3}{2}; \frac{1}{2}; 0, 1, 0], \\
[4; \frac{3}{2}; \frac{1}{2}; 0, 1, 0], \\
[\frac{7}{2}; \frac{1}{2}; 1, 0, 0, 1], \quad [\frac{7}{2}; \frac{1}{2}; 1, 0, 0, 1].
\]
we give the character formulae for the three massless cases—chiral, anti-chiral and mixed

\[ \text{ch} L_A = \frac{e(\Lambda)}{(1 - t_2)(1 - t_1 t_2)(1 - t_3 t_2)} \]

where the first two lines are the eight states in (4.86) and the rest are from (4.87).

Note that for all conformal signatures holds: \( d - j_1 - j_2 = 2 \). For \( j_1 j_2 \neq 0 \) this is on the conformal unitarity threshold and then we shall use formula (3.19). For \( j_1 j_2 = 0 \) this is above the conformal unitarity threshold and then we shall use the generic formula (3.17).

5. Outlook

With the present paper we continue the project of construction of the character formulae for the positive energy unitary irreducible representations of the \( N \)-extended \( D = 4 \) conformal superalgebras \( su(2,2/N) \). In the first paper we presented the bare characters which represent the defining odd entries of the characters. Now we give the full explicit character formulae for \( N = 1 \) and several important examples for \( N = 2 \) and \( N = 4 \). In particular, for \( N = 4 \) we give the character formulae for the three massless cases—chiral, anti-chiral and mixed.
(the latter with three subcases)—and also for the graviton supermultiplet and for the Konishi supermultiplet.

In the further development of this project we shall try to find more compact expressions for presenting the characters, which will enable us to treat explicitly more complicated cases for arbitrary $N$.

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