Channel Model and Upper Bound on the Information Capacity of the Fiber Optical Communication Channel Based on the Effects of XPM Induced Nonlinearity

Hossein Kakavand
Department of Electrical Engineering
Stanford University
Stanford, CA 94305
Hossein@stanford.edu

Abstract

An upper bound to the information capacity of a wavelength-division multiplexed optical fiber communication system is derived in a model incorporating the nonlinear propagation effects of cross-phase modulation (XPM). This work is based on the paper by Mitra et al. [1], finding lower bounds to the channel capacity, in which physical models for propagation are used to calculate statistical properties of the conditional probability distribution relating input and output in a single WDM channel. In this paper we present a tractable channel model incorporating the effects of cross phase modulation. Using this model we find an upper bound to the information capacity of the fiber optical communication channel at high SNR. The results provide physical insight into the manner in which nonlinearities degrade the information capacity.

1 Introduction

Communication via optical fibers has received a lot of attention recently mainly due to the extremely high bandwidths and unique propagation environments they provide. These characteristics provide a rather appealing medium for multichannel communication through the fiber, known as Wavelength Division multiplexing (WDM), which has been the focus of much ongoing research and practical consideration, recently [1, 2, 3]. However, optical fibers have a number of nonlinear electromagnetic phenomena which affect the propagation of signals. As we increase the rate of communication through the fiber, the effect of these nonlinear phenomena will limit the reliability of communication through the fiber. From an information theoretic point of view these phenomena will limit the information capacity of the fiber optical communication channel. In order to derive this limit we need to study and model the effects of such phenomena on the inputs to the channel as they propagate through the fiber. This was first done by Mitra et al. [1].

In this paper we shall introduce the nonlinear effects in a fiber. We shall model the effects of one of the nonlinear effects, known as Cross Phase Modulation (XPM) on the...
propagation of signals through the fiber and derive an upper bound on the capacity of the optical channel considering the effects of XPM.

In section 2 we shall describe the propagation of an electromagnetic wave through the fiber, where our signal is used to modulate the wave. In section 3 we shall use our propagation equation to find a relation between the input and output of a fiber, viewed as the communication channel. This is done by considering the Green’s function that relates the input and output of the fiber medium. We shall then simplify the Green’s function in section 4. This simplification results in a tractable channel model. We find an upper bound to the information capacity of the simplified channel model in the high SNR regime in section 5.

2 Signal Propagation in Optical Fibers

In this section we shall describe the propagation of an input signal in an optical fiber, based on Mitra et al. [1] and Agrawal [5], which shall be used to derive a communication theoretic channel model in the next sections.

In a single mode fiber used for Wavelength Division Multiplexed (WDM) transmission, there are a number of channels \( N \), each used by an independent user. Each user uses an input signal \( x_k(t) \) to modulate the carrier wave of frequency \( \nu_k \) and has a bandwidth of \( B \ll \nu_k \). We shall denote the carrier frequency of the central channel by \( \nu_0 \), also the carrier frequency spacing of neighboring channels is given by \( \delta \nu \). Since the users are independent the input signals to different channels, \( x_k(0, t) \), are also independent each having a power constraint: \( E_t(|x_k(0, t)|^2) \leq P, \ \forall k \). Along the fiber \( x_k(z, t) \) is the amplitude of the electric field \( E_k \) in channel \( k \) with propagation constant \( \beta_k \), were \( z \) the distance travelled along the fiber. The total electric field resulting from signals in \( N \) channels is given by,

\[
E(z, t) = \sum_{k=-N/2}^{N/2} [x_k(z, t) \exp(i(\beta_k z - 2\pi\nu_k t)) + x_k^*(z, t) \exp(-i(\beta_k z - 2\pi\nu_k t))] \exp(-\alpha z/2).
\]  

(2.1)

Equation (2.1) accounts for the power loss due to absorption in the fiber through the exponential decay factor \( \exp(-\alpha z/2) \). Also we take the direction of polarization to be fixed and the traverse profile of the mode to be independent of \( z \). The propagation constants \( \beta_k \), are frequency dependent, making propagation through the fiber dispersive. This phenomenon is known as Group Velocity Dispersion (GVD) which results in different frequency components of a signal to travel at different speeds, resulting in distortion of the input signal. It also causes signals in different channels to travel at different speeds.

In an optical fiber different frequency components interact (couple) to generate new frequency components, this phenomenon is known as Four Wave Mixing (FWM). Also due to the dependance of the refractive index of the fiber on the power of propagating signals, different channels interact. The interaction of a channel with itself is known as Self Phase Modulation (SPM) and the interaction of different channels with each other is known as Cross Phase Modulation (XPM). For further details refer to [1] and [5].

Given the phenomena explained above the signal in channel \( k \), \( x_k(z, t) \) (written in the frame of reference that moves along the fiber with the group velocity of the central
channel) propagates through the fiber based on the nonlinear Schrödinger equation,

\[
\frac{\partial}{\partial z} + i\beta_2 \frac{\partial^2}{\partial t^2} x_k(z, t) = i\gamma \left[ |x_k|^2 + 2 \sum_{l \neq k} |x_l|^2 \right] \exp(-\alpha z) x_k(z, t).
\]

(2.2)

Propagation along the fiber is thus characterized by a set of \(N+1\) coupled non-linear partial differential equations. In Equation (2.2), \(\beta_k(\nu)\) has been Taylor expanded about \(\nu_0\) to take the form

\[
\beta_k(\nu) = \beta(\nu_0 + k\delta\nu) = \beta_0 + \beta_1 k\delta\nu + \beta_2 k^2 \delta\nu^2 / 2 + O(\delta\nu^3),
\]

also, we have neglected the effect of Four Wave Mixing (FWM) since it can be shown [1] that the effect of FWM on the information capacity can be studied separately. As will shall see, SPM cannot be decoupled from the input signal and so its effect on the capacity requires a different set of techniques, which can be studied separately and will not be considered here [1]. However, XPM involves signals from other channels which are chosen independently, and so for each signal the effect of other channels is essentially random. Hence, in what follows we shall study the effect of the stochastic effects of XPM on the information capacity.

After the neglect of SPM and FWM, the effective channel model is given by \(N+1\) coupled nonlinear PDE’s, known as the equations of motion. The equation of motion for the central channel is given by,

\[
\frac{\partial}{\partial z} + \frac{\beta_2}{2} \frac{\partial^2}{\partial t^2} x_0(z, t) = V_0(z, t)x_0(z, t),
\]

(2.3)

\[
V_0(z, t) = 2\gamma \sum_{l \neq 0} |x_l(z, t)|^2 \exp(-\alpha z),
\]

(2.4)

where \(V_0(z, t)\) represents the XPM term for the central channel. Equation (2.3) is still non-linear because \(V_0(z, t)\) couples different channels to the central channel. Note that, apart from the interchange of spatial and temporal coordinates, \(V_0(z, t)\) enters Equation (2.3) in the same way, and is therefore mathematically equivalent to, that of a potential entering the Schrödinger equation [6].

2.1 Equations of Motion

The equations of motion are modeled by decoupling and linearizing Equations (2.3) and (2.4). As mentioned before the XPM term is mathematically equivalent to a potential, \(\nu(z, t)\). This potential is modeled as a Gaussian random process, independent of all the channel inputs, in both space and time with the same first and second order moments as the XPM term, see Equations (2.6) and (2.7). This model follows from the central limit theorem, considering the fact that at each instant in time and space, \(\sum_{l \neq 0} |x_l(z, t)|^2\) is the sum of many independent random variables, each with a finite second moment. Hence, the XPM term converges in distribution to a Gaussian random process [8]. Note that while the signal in each channel has both spatial and temporal correlation, we are adding many such signals which travel at different speeds in space, hence the correlations are lost. For further details see [1].

Note that \(\nu(z, t)\) does not depend on the signals in other channels. The resulting model for propagation is linear, since the equations for different channels have been decoupled. Hence, the effective equation of motion for the central channel is given by,

\[
[i\partial_z - \frac{\beta_2}{2} \partial_t^2] x_0(z, t) = \nu(z, t)x_0(z, t)
\]

(2.5)
\[ E(\nu(z, t)) = E(V_0(z, t)), \quad \forall t, \forall z, \quad 0 \leq z \leq L \]  

(2.6)

\[ E(\nu(z_1, t_1)\nu(z_2, t_2)) = E(V_0(z_1, t_1)V_0(z_2, t_2)), \quad \forall t_1, t_2, \forall z_1, z_2, \quad 0 \leq z_1, z_2 \leq L \]  

(2.7)

were the expectations of \( V_0 \) terms are taken over the joint distributions of the inputs to all channels.

## 3 Channel Model

In this section we obtain a stochastic channel model based on the equation of motion obtained in section 2. This stochastic model is simplified to result in a channel model which incorporates the effects of XPM which results in a multiplicative phase noise term.

### 3.1 Effects of XPM

Equation (2.5) describes the effect of XPM on the relation between the input and output signal of the central channel. Based on this equation the channel model or the input-output relation can be described using the Green’s function \( G(L; t, t') \), also known as the propagator [6],

\[ y(t) = x_0(L; t) = \int G(L; t, t')x_0(0; t')dt' + n(t), \]  

(3.1)

where \( L \) is the length of the fiber and \( n(t) = \int_0^L n(z, t)dz \) takes into account the effect of all additive noise, which we shall approximate with an additive white Gaussian random process, independent of all inputs to the channel [1].

The Green’s function is a function of the random potential \( \nu(z, t) \) and can be written in the form of a path integral [6],

\[ G(L; t, t') = \int Dt(z) \exp \left[ -\int_0^L \frac{i}{2\beta_2}(\partial_z t(z))^2dz - i \int_0^L \nu(t(z), z)dz \right], \]  

(3.2)

where the paths \( t(z) \) start at \( t' \), at \( z = 0 \) and end at \( t, at z = L \). This expression is encountered in the context of Quantum Mechanics, where the roles of time and space are interchanged. We shall simplify this expression to obtain a channel model.

**Theorem 1.** Simplifying Equation (3.2), combined with Equation (3.1) result in the following simplified channel model,

\[ y(t) = \int \exp(-i\frac{(t-t')^2}{2\beta L}) \exp(-iLU(L; t', t))x(t')dt' + n(t), \]  

(3.3)

where

\[ U(L; t', t) \sim \mathcal{N}(0, \sigma_U^2(t - t')). \]  

(3.4)

Here,

\[ \sigma_U^2 = \frac{2P^2}{\beta_2(\delta\nu)^2} \sum_{n=1}^{N/2} \frac{1}{n}, \]

where \( P \) is the power in each channel, \( \beta_2 \) is the propagation constant of the central channel, \( \delta\nu \) is the channel frequency spacing and \( N \) is the number of channels.
Proof. The rest of section is devoted to proving Theorem (11). This is done by simplifying and approximating the Greens function given by Equation (3.2).

The potential can be written as \( \nu(z, t) = E(\nu(z, t)) + \delta \nu(z, t) \). The average value of the random potential causes a deterministic constant phase factor, which has no effect on the information capacity of the channel. However, the fluctuations of the random potential about its average value effect the phase of the Green’s function and hence the capacity of the channel. Since the average potential has no effect on our results we shall assume \( E(\nu(z, t)) = 0 \), hence \( \nu(z, t) = \delta \nu(z, t) \).

3.2 “Green’s Function” Approximation

In this section we shall approximate the Green’s function given by Equation (3.2) using stochastic and physical properties of the random potential.

Referring to [6] we could approximate the expression for \( G(L; t, t') \) by dividing both space and time into small intervals. We divide the time interval \((t', t)\) into \( M \) equal subintervals, \( \Delta t \), resulting in \( \{t_i\}_{i=0}^M \), where \( t_0 = t' \) and \( t_M = t \). Also, we shall divide the fiber length into \( M \) equal subintervals \( \Delta L \).

In all that follows we shall be concerned with the time interval \((t', t)\), i.e. \( \nu(z, t'') = 0 \) if \( t'' \notin (t', t) \). However, we shall assume that the inputs start from a distant past and will continue into the distant future. This assumption is justified since the signals in different channels are not necessarily synchronized. And so we have [6],

\[
\int \mathcal{D}t(z) \approx \int \int \ldots \int dt_{M-1}dt_{M-2}\ldots dt_1,
\]

\[
\int_0^L \frac{i}{2\beta_2} (\partial_z t(z))^2 dz \approx \sum_{k=0}^M \frac{i}{2\beta_2} \left[ \frac{t_k - t_{k-1}}{\Delta L} \right]^2 \Delta L,
\]

\[
\int_0^L \nu(t(z), z)dz \approx \sum_{k=0}^M \nu(t_k) \Delta L.
\]

These approximations result in,

\[
G(L; t, t') = \int \mathcal{D}t(z) \exp \left[ -\int_0^L \frac{i}{2\beta_2} (\partial_z t(z))^2 dz - i \int_0^L \nu(t(z), z)dz \right]
\]

\[
\approx \int \int \ldots \int \exp \left[ -\sum_{k=1}^{M-1} \frac{i}{2\beta_2} \left[ \frac{t_k - t_{k-1}}{\Delta L} \right]^2 \Delta L - i \sum_{k=1}^{M-1} \nu(t_k) \Delta L \right] dt_{M-1}dt_{M-2}\ldots dt_1
\]

\[
= \int \int \ldots \int \prod_{k=1}^{M-1} \exp \left[ -\frac{i}{2\beta_2} \left( \frac{t_k - t_{k-1}}{\Delta L} \right)^2 \right] \exp \left( -i \nu(t_k) \Delta L \right) dt_{M-1}dt_{M-2}\ldots dt_1.
\]

(3.5)

Consider the terms containing the random potential in Equation (3.5) given by,

\[
\exp(-i \nu(t_k) \Delta L) = 1 - i \nu(t_k) \Delta L - (\nu(t_k) \Delta L)^2 + \ldots = 1 - i \nu(t_k) \Delta L + O((\Delta L)^2),
\]

which can be approximated by neglecting the \( O(\Delta L)^2 \) terms. This approximation is justified considering nominal values for the parameters, see [5]. Hence,

\[
G(L; t, t') \approx \int \int \ldots \int \prod_{k=1}^{M-1} \exp \left( -\frac{i}{2\beta_2} \left( \frac{t_k - t_{k-1}}{\Delta L} \right)^2 \right) (1 - i \nu(t_k) \Delta L) dt_{M-1}dt_{M-2}\ldots dt_1
\]
evaluating these integrations results in,

\[
G(L; t, t') \approx \exp \left( -\frac{i(t - t')^2}{2\beta L} \right) - iL \int_0^1 \int_{-\infty}^{\infty} \exp \left( -\frac{i}{2\beta L} \left[ \frac{(t - t_\alpha)^2}{\alpha} + \frac{(t_\alpha - t')^2}{1 - \alpha} \right] \right) \nu(t_\alpha) dt_\alpha d\alpha
\]

\[
= \exp \left( -\frac{i(t - t')^2}{2\beta L} \right) - iL U(L; t', t)
\]

(3.6)

Equation (3.6) consists of two terms, the first term is a deterministic phase shift, while, the second term involves the weighted integration of the random potential \( \nu(t_\alpha) \), resulting in the random process \( U(L; t, t') \). We need to characterize the distribution of the random process \( U(L; t, t') \).

### 3.3 Distribution of \( U(L; t, t') \)

In this section we shall characterize the distribution of \( U(L; t, t') \). This will result in characterizing the distribution of our approximation to the Greens function.

**Lemma 1.** The distribution of the random process \( U(L; t', t) \) is given by,

\[
U(L; t', t) \sim \mathcal{N}(0, \sigma_v^2(t - t')).
\]

(3.7)

with

\[
\sigma_v^2 = \frac{2P^2}{\beta_2(\delta\nu)^2} \sum_{n=1}^{N/2} \frac{1}{n} \approx \frac{2P^2}{\beta_2(\delta\nu)^2} \log(N/2)
\]

Proof. An outline of the proof of Lemma 1 is given in the appendix.

In our derivation of the distribution of the Greens function we neglected the higher order terms in \( \Delta L \). If we follow the same tedious calculations for higher orders of \( \Delta L \) we get the following approximation for the Greens function,

\[
G(L; t, t') \approx \exp \left( -\frac{i(t - t')^2}{2\beta L} \right) \left( 1 - iL U(L; t', t) + \frac{(-i)^2L^2U^2(L; t', t)}{2} + \cdots \right)
\]

\[
= \exp \left( -\frac{i(t - t')^2}{2\beta L} \right) \exp(-iLU(L; t', t))
\]

(3.8)

where the distribution of \( U(L; t', t) \) is given by Equation (3.7). The proof of Equation (3.8) follows from the proof of Lemma 1 in the appendix and therefore not presented here. Combining Equation (3.8) with Equation (3.1) results in the following approximate channel model,

\[
y(t) = \int \exp(-i\frac{(t - t')^2}{2\beta L}) \exp(-iLU(L; t', t)) x(t') dt' + n(t),
\]

which completes the proof of Theorem 1.
4 Simplified Channel Model

In this section we use some physical characteristics of an optical fiber to further simplify the channel model given by Theorem 1. Equation (3.3) is a standard representation of the following input output relation,

\[ x_0(L; t) = \int \exp(-i \frac{(t - t')^2}{2\beta L}) \exp(-i LU(L; t')) x_0(0; t') dt' + n(t). \quad (4.1) \]

As shown by Equation (4.1), the output at time \( t \) is affected by the input at time \( t' \). The propagation (travel) time of the signal in the central channel is given by \( \frac{L}{\nu_g} \), where \( \nu_g \) is the group velocity of the central channel, see [5] and \( L \) is the fiber length. Hence, the input time interval \( t' \), that affects the output at time \( t \) is a window in time around \( t - \frac{L}{\nu_g} \).

Propagation along the fiber is dispersive, i.e. signals tend to spread as they propagate along the fiber. However, from practical considerations, we know that the propagation distances used in practice are small enough to ensure that the signal spread is small compared to the propagation delay \( \frac{L}{\nu_g} \). Hence the input time interval \( t' \) that affects the output at time \( t \) is small compared to \( \frac{L}{\nu_g} \).

As an example assume that the signals are bit streams of time duration \( T \approx \frac{1}{\delta \nu} = 20 \text{ps} \). Nominal value for the fiber parameters are given by, group velocity \( \nu_g = 200,000 \text{km/s} \), propagation constant \( \beta_2 = 20 \frac{\text{ps}^2}{\text{km}} \) and fiber length \( L = 50 \text{km} \). In most practical situations no more than \( m = \pm 100 \) neighboring bits will affect each bit. Hence, we have \( mT \approx 4 \text{ps} \) and \( \frac{L}{\nu_g} \approx 25 \text{ms} \) and so the time interval that affects the output at each time is much smaller than the propagation time of the signal. Hence,

\[ t - t' \approx \frac{L}{\nu_g}. \quad (4.2) \]

Based on this approximation \( U(L; t', t) \) is a stationary random process in time, hence we'll drop the time dependance in \( U(L; t', t) \). We also absorb the \( L \) in the channel model preceding \( U(L) \), into it's variance, resulting in,

\[ U(L) \sim \mathcal{N}(0, \sigma_U^2(L/L^2)). \]

The result of the approximation in (4.2) is to “lump” the effect of a “continuously injected multiplicative noise” \( U(L; t, t') \), along the fiber into a lumped multiplicative noise \( U(L) \), in the form of a random phase shift, at the end of the fiber length \( L \). The resulting channel model is given by,

\[
y(t) = \int \exp(-i \frac{(t - t')^2}{2\beta L}) \exp(-i U(L)) x(t') dt' + n(t) \\
= \exp(-i U(L)) \int \exp(-i \frac{(t - t')^2}{2\beta L}) x(t') dt' + n(t)
\]

Note that \( \int \exp(-i \frac{(t - t')^2}{2\beta L}) x(t') dt' \) is the convolution of the input signal \( x(t) \) with \( \exp(-i \frac{t^2}{2\beta L}) \), which results in a deterministic phase shift in the frequency domain which can be compensated for and so it has no effect on the information capacity of the channel or the capacity achieving input signal. As a result the input signal can be represented as,

\[
\tilde{x}(t) = \int \exp(-i \frac{(t - t')^2}{2\beta L}) x(t') dt'.
\]
Finally we have the following simplified channel model,

\[ y(t) = \exp(-iU(L)) \tilde{x}(t) + n(t), \]  

(4.3)

where \( U(L) \sim \mathcal{N}(0, \sigma_U^2(L^2)) \).

5 channel capacity

In this section we shall provide an upper bound to the information capacity of the channel given by Equation (4.3) for the high SNR regime.

Consider the channel given by Equation (4.3), since each channel is bandlimited, the inputs to each channel are also bandlimited, hence our continues time channel model is equivalent to a discrete time channel model, see [11],

\[ y_k = \exp(-iu_k) \tilde{x}_k + n_k \]  

(5.1)

where \( u \) and \( n \) are independent i.i.d sequences with \( u_k \sim \mathcal{N}(0, \sigma_u^2(L^2)) \) and \( n_k \sim \mathcal{N}(0, \sigma_N^2) \).

Consider the channel given by Equation (5.1). The input to the channel is given by \( \{x_k\} \) with \( x_k \in \mathbb{C} \) with an input power constraint \( E(\sum_{k=1}^{n} |x_k|^2) \leq P \). The additive noise \( \{n_k\} \) is an i.i.d. sequence of circularly symmetric complex Gaussian random variables with \( n_k \sim \mathcal{N}(0, \sigma_N^2) \), also the phase noise \( \{u_k\} \) is an i.i.d. sequence of Gaussian random variables with \( u_k \sim \mathcal{N}(0, \sigma_u^2) \), with the distribution of \( \{u_k\} \) being independent of the input power. Also \( u_k \) has finite entropy. Note that all three processes \( \{x_k\}, \{n_k\} \) and \( \{u_k\} \) are mutually independent. Given this setting we have the following lemma,

**Lemma 2.** Given the above setting, the capacity of the channel given by Equation (5.1) is upper bounded by,

\[ C \leq \frac{1}{2} \log \left( 1 + \frac{2\pi^2 e^{-2h(\omega)} P}{\sigma_N^2} \right) + o(1) \]

\[ = \frac{1}{2} \log \left( 1 + \left( \frac{1}{P \sigma_N^2} \right) \left( \frac{\pi}{e} \right) \left( \frac{\beta_2}{2 \log(N/2)} \frac{\delta\nu}{L^3} \right) \right) + o(1), \]

(5.2)

in the high SNR regime. Where the \( o(1) \) term tends to zero as \( \frac{P}{\sigma_N^2} \) tends to infinity.

The proof of Lemma 2 follows from the proof in [10]. As an example, consider the following nominal values for a typical WDM optical fiber, \( \beta_2 = 200 \text{ps}^2/\text{km}, \gamma = 1.2(\text{W km})^{-1}, \) \( \delta\nu = 50 \text{GHz}, \nu_g = 200,000 \text{km/s}, L = 50 \text{km}, \) and \( N = 100 \) which result in the capacity upper bounded by,

\[ C \leq \frac{1}{2} \log \left( 1 + \frac{0.0118}{P \sigma_N^2} \right). \]

6 Conclusion

An upper bound to the information capacity of a wavelength-division multiplexed optical fiber communication system is derived in a model incorporating the nonlinear propagation effects of cross-phase modulation (XPM). We modeled the effects of the continuously
injected phase noise due to the cross phase modulation nonlinearity as a lumped multiplicative phase noise at the end of the fiber. This model leads to an upper bound to the capacity of a WDM fiber optical communication channel in the high SNR regime. This upper bound is in agreement with the lower bound derived by Mitra et al. [1].

Future directions include better models for the effect of various fiber nonlinearities, including XPM. Also upper bounds for the low SNR regime and tighter upper bounds for the high SNR regime could be derived as we believe our upper bound could be improved.

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7 Appendix

Outline of proof of Lemma 1

Carrying out the tedious calculations,

\[ U(L; t, t') = \int_0^1 \int_{-\infty}^{\infty} \exp \left( -\frac{i}{2\beta_2 L} \left[ \frac{(t - t_\alpha)^2}{\alpha} + \frac{(t_\alpha - t')^2}{1 - \alpha} \right] \right) \nu(t_\alpha) dt_\alpha d\alpha \]

\[ = \exp \left( \frac{(-i)(t - t')^2}{2\beta_2 L} \right) \int_0^1 U_a(L; t', t) d\alpha, \]

as mentioned in Section 3.2 \( \nu(t_\alpha) \) is zero outside the time interval \((t', t)\), hence,

\[ U_a(L; t', t) = \int_{t'}^t \exp \left( -\frac{i}{2\beta_2 L_\alpha (1 - \alpha)} [t_\alpha - t(1 - \alpha) - t'(\alpha)]^2 \right) \nu(t_\alpha) dt_\alpha d\alpha \]

\[ = \int_{-(1-\alpha)(t-t')}^{\alpha(t-t')} \exp \left( -\frac{i}{2\beta_2 L_\alpha (1 - \alpha)} (t_\alpha)^2 \right) \nu(t_\alpha + t(1 - \alpha) + t'(\alpha)) dt_\alpha d\alpha \]

\[ =_D \int_{-(1-\alpha)(t-t')}^{\alpha(t-t')} \exp \left( -\frac{i}{2\beta_2 L_\alpha (1 - \alpha)} (t_\alpha)^2 \right) \nu(t_\alpha) dt_\alpha \]

where \( =_D \) equates the distribution of both sides, which follows from the stationarity of \( \nu(t) \) and the fact that both \( t_\alpha + t(1 - \alpha) + t'(\alpha) \) and \( t_\alpha \) cover the integration interval \((- (1 - \alpha)(t - t'), \alpha(t - t'))\), for any \( \alpha \in (0, 1) \). Note that we are only interested in the distribution of \( U_a(L; t', t) \).

Since \( \nu(t) \) is a Gaussian process, we have that \( U_a(L; t', t) \) is also a Gaussian random process [7], i.e. \( U_a(L; t', t) \sim \mathcal{N}(\mu, \sigma^2) \), where it can be can shown,

\[ \mu = E \left[ \int_{-(1-\alpha)(t-t')}^{\alpha(t-t')} \exp \left( -\frac{i}{2\beta_2 L_\alpha (1 - \alpha)} (t_\alpha)^2 \right) \nu(t_\alpha) dt_\alpha \right] = 0, \]

\[ \sigma^2 = E \left[ \int \exp \left( -\frac{i}{2\beta_2 L_\alpha (1 - \alpha)} (t_1)^2 \right) \nu(t_1) dt_1 \int \exp \left( +\frac{i}{2\beta_2 L_\alpha (1 - \alpha)} (t_2)^2 \right) \nu(t_2) dt_2 \right] \]

\[ = \sigma^2_{\nu}(t - t'). \]
Hence $U_\alpha(L; t', t) \sim \mathcal{N}(0, \sigma^2_\nu(t - t'))$, where $\sigma^2_\nu = E[\nu(t)^2]$ is the power of the Gaussian process $\nu(t)$ given by,

$$
\sigma^2_\nu = \frac{2P^2}{\beta_2(\delta \nu)^2} \sum_{n=1}^{N/2} \frac{1}{n} \approx \frac{2P^2}{\beta_2(\delta \nu)^2} \log(N/2),
$$

for details of this derivation see [1].

Note that $U_\alpha(L; t', t)$ is a jointly Gaussian random process in $\alpha$. Hence, integrating $U_\alpha(L; t', t)$ results in a Gaussian random process [7], i.e. $U(L; t', t) \sim \mathcal{N}(\mu_U, \sigma_U)$, where it can be shown,

$$
\mu_U = E\left[ \int_0^1 U_\alpha(L; t', t) d\alpha \right] = 0,
$$

$$
\sigma^2_U = E\left[ \int_0^1 U_\alpha(L; t', t) d\alpha \right] \int_0^1 U^*_\alpha(L; t', t) d\alpha' \\
= \sigma^2_\nu(t - t').
$$

And so,

$$
U(L; t', t) \sim \mathcal{N}(0, \sigma^2_\nu(t - t')).
$$

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