The Covariant Description of Electromagnetically Polarizable Media

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Abstract
The form of the phenomenological stress-energy-momentum tensor for the electromagnetic field in a class of inhomogeneous, anisotropic magneto-electric media is calculated from first principles, leading to a coherent understanding of the phenomenological stresses and energy-momentum exchanges induced by electromagnetic interactions with such matter in terms of a fully relativistic covariant variational framework.

Keywords: Covariant Electromagnetism, Stress-energy-momentum tensor, constitutive relations, Action Principles, Magneto-electric Media, Maxwell’s equations

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1 Introduction

The natural mathematical tool for describing in a covariant manner the response of matter to excitation by external fields is the total stress-energy-momentum tensor involving matter and fields. Early suggestions by Minkowski [1] and Abraham [2] for the structure of its electromagnetic component in simple media initiated a long debate involving both theoretical and experimental contributions that continues to the current time (see e.g. [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]). Disputes developed because different proposals to describe the interaction of light with media were adopted and subsequent experiments were unable to resolve their conflicting predictions. Although it is widely recognised today that this controversy is an argument about definitions [13], [14], [15], [16] and that the relative merits of alternative definitions are undecidable without a complete (experimentally verifiable) covariant description of relativistic continuum mechanics for matter and fields, the absence of a compelling derivation of a total relativistic stress-energy-momentum tensor that can effectively model the phenomenological response of moving media to electromagnetic fields is surprising.

If stress-energy-momentum tensors are defined as appropriate variational derivatives of an action functional, a mathematically precise and physically cogent derivation of a symmetric tensor can be effected. In this letter we report that both the original phenomenological proposals by Minkowski and Abraham can be viewed in this context and correspond to different choices for the response of a linear constitutive tensor to gravitation. Since it is known that the polarisation and magnetisation of most continua depend on their state of motion the choice made by Abraham (originally in the context of inertial motions and zero gravitation) probably offers a more effective contribution to the total stress-energy-momentum tensor. From this perspective it is also argued that if spatial and temporal dispersion are ignorable, the classical properties of a linear medium (that may be intrinsically magneto-electric) can be parameterised in terms of a constitutive tensor on spacetime whose properties can in principle be determined by experiments in non-inertial (accelerating) frames and in the presence of weak but variable gravitational fields.

Because the properties of the electromagnetic field are so well understood much of the experimental information that is collected about nature is mediated by this field. The predictions of pure quantum electrodynamics have been experimentally verified to high levels of accuracy and any departures
from these predictions are routinely ascribed to the effects of other interactions. The classical laws of electrodynamics are also routinely extrapolated to describe astrophysical phenomena where matter can exist under extreme conditions of temperature, pressure and density. Even on terrestrial scales new states of matter and new materials are regularly being fabricated with surprising electromagnetic properties that are leading to new technological developments in communications and nano-science.

To fully understand phenomena that are induced in bulk matter by electromagnetism on these various scales it is necessary to describe cooperative effects induced by the electromagnetic interaction. It is, however, difficult to account fully for such effects in terms of fundamental interactions between charged particles and photons. The versatility of Maxwell’s phenomenological field equations owes much to the fact that electromagnetic sources due to cooperative effects can be accommodated by using a broad range of constitutive relations involving a variety of response functions that account for polarisation, magnetisation, hysteresis and dispersion in material continua. Although in principle such response functions can be calculated in terms of the underlying quantum structure of matter in many practical situations the constitutive relations are effectively deduced from experimental data.

In order that this procedure achieves more than a convenient parametrisation of a particular set of laboratory observations it is important to embed the response data into a coherent framework that respects the basic principles cherished by physical science. For classical electrodynamics these include the principle that the underlying theory provides relations between tensor (and spinor) fields on a four-dimensional spacetime equipped with a light-cone structure and a pseudo-Riemannian geometry responsible for the effects ascribed to gravitation. Such a formulation ensures that the results of observation on arbitrarily moving continua by observers in arbitrary motion (in the presence of an arbitrary gravitational field) are compatible with the established tenets of relativistic field theory [17]. The implementation of this program is non-trivial particularly when thermodynamic constraints are included for deformable media.

Some way towards this goal is offered by (covariant) averaging methods [18], [19]. These however yield non-symmetric stress-energy-momentum tensors for electromagnetic fields in simple media. If the total stress-energy-momentum is to remain symmetric this implies that other asymmetric contributions must compensate and no guidance is offered to account for such material induced asymmetries. The need for a symmetric total stress-energy-
momentum tensor is often attributed to conservation of total angular momentum despite the fact that such global conservation laws may not exist in arbitrary gravitational fields. Although the magnitude of gravitational interactions may be totally insignificant compared with the scale of those due to electromagnetism, gravity does have relevance in establishing the general framework (via the geometry of spacetime) for classical relativistic field theory and in particular this framework offers the most cogent means to define the total stress-energy-momentum tensor as the source of relativistic gravitation. This in turn may be related to a variational formulation of the fully coupled field system of equations that underpin the classical description of interacting matter in terms of tensor (and spinor) fields on spacetime.

2 Constitutive Relations

Maxwell’s equations for an electromagnetic field in an arbitrary medium can be written

\[ dF = 0 \quad \text{and} \quad d \star G = j \]  

(1)

where \( F \) is the Maxwell 2-form, \( G \) is the excitation 2-form and \( j \) is the 3-form electric current source. In general, the effects of gravitation and electromagnetism on matter are encoded in this system in \( \star G \) and \( j \). This dependence may be non-linear and non-local. To close this system, “electromagnetic constitutive relations” relating \( G \) and \( j \) to \( F \) are necessary. In the following the medium will be considered as containing polarisable (both electrically and magnetically) matter with \( G \) restricted to a real point-wise linear function of \( F \), thereby ignoring losses and spatial and temporal material dispersion in all frames. Covariant dispersion in linear material has been discussed in the optics limit in [21] and the role of the spacetime metric in the constitutive formulation of Maxwell’s theory has been analysed in [22].

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1The Hodge map \( \star \) is associated with the metric tensor \( g \) of spacetime, \( d \) denotes the exterior derivative and \( i_X \) is the contraction operator associated with the vector \( X \). Details of this notation can be found [20]. All tensors in this article have dimensions constructed from the SI dimensions \([M],[L],[T],[Q]\) where \([Q]\) has the unit of the Coulomb in the MKS system. We adopt \([g]=[L^2], [G]=[Q], [j]=[Q], [F]=[Q]/\epsilon_0\) where the permittivity of free space \( \epsilon_0 \) has the dimensions \([Q^2 T^2 M^{-1} L^{-3}]\) and \( c \) denotes the speed of light in vacuo.

2The electric current 3-form \( j \) will be assumed to describe (mobile) electric charge and plays no role in subsequent discussions.
The 1–form electric field $\mathbf{e}$ and 1–form magnetic induction field $\mathbf{b}$ associated with $F$ are defined with respect to an arbitrary unit future-pointing timelike 4–velocity vector field $U$ by

$$\mathbf{e} = i_U F \quad \text{and} \quad c \mathbf{b} = i_U \star F.$$  \hfill (2)

Since $g(U, U) = -1$,

$$F = \mathbf{e} \wedge \tilde{U} - \star (c \mathbf{b} \wedge \tilde{U}).$$

The field $U$ may be used to describe an observer frame on spacetime and its integral curves model idealised observers. Likewise the displacement field $\mathbf{d}$ and the magnetic field $\mathbf{h}$ associated with $G$ are defined with respect to $U$ by

$$\mathbf{d} = i_U G \quad \text{and} \quad \mathbf{h}/c = i_U \star G.$$ \hfill (3)

Thus

$$G = \mathbf{d} \wedge \tilde{U} - \star (\mathbf{h}/c \wedge \tilde{U}).$$

It will be assumed that a material medium has associated with it a future-pointing timelike unit vector field $V$ which may be identified with the bulk 4–velocity field of the medium in spacetime. Integral curves of $V$ define the averaged world-lines of identifiable constituents of the medium. A comoving observer frame with 4–velocity $U$ will have $U = V$.

In general $G$ may be a functional of $F$ and properties of the medium$^3$.

$$G = \mathcal{Z}[F, \ldots].$$

Such a functional induces, in general, non-linear and non-local relations between the fields $\mathbf{d}, \mathbf{h}$ and $\mathbf{e}, \mathbf{b}$. For general linear continua one may have, for some positive integer $N$ and collection of constitutive tensor fields $Z^{(r)}$ on spacetime, the relation

$$G = \sum_{r=0}^{N} Z^{(r)}(\nabla^r F, \ldots)$$

in terms of some spacetime connection $\nabla$. Additional arguments refer to variables independent of $F$ and its derivatives. For the non-dispersive linear media under consideration here, we restrict to

$$G = \mathcal{Z}(F)$$ \hfill (4)

$^3$e.g. electrostriction and magnetostriction arise from the dependence of $\mathcal{Z}$ on the elastic deformation tensor of the medium.
for some constitutive tensor field $Z$. In the vacuum $G = \epsilon_0 F$.

A particularly simple linear isotropic but inhomogeneous medium may be described by a bulk 4-velocity field $V$, a relative permittivity scalar field $\epsilon$ and a non-vanishing relative permeability scalar field $\mu$. To a comoving observer ($U = V$) with a history that coincides with one of the integral curves of $V$ the local constitutive relations become

$$d = \epsilon_0 \epsilon e \quad \text{and} \quad h = (\mu_0 \mu)^{-1} b.$$ 

For a non-magneto-electric but anisotropic medium, the relative permittivity $\epsilon$ and inverse relative permeability $\mu^{-1}$ become spatial tensor fields on spacetime. More generally, the electromagnetic fields measured by a co-moving observer may be related by

$$d = \zeta^{de}(e) + \zeta^{db}(b) \quad \text{and} \quad h = \zeta^{he}(e) + \zeta^{hb}(b) \quad (5)$$

where $\zeta^{de}, \zeta^{db}, \zeta^{he}, \zeta^{hb}$ are spatial tensors. From (2), (3), (4) it follows that this constitutive relation may be expressed covariantly as

$$Z(F) = \zeta^{de}(i_V F) \wedge \tilde{V} + \zeta^{db}(i_V \star F) \wedge \tilde{V} - \star(\zeta^{he}(i_V F) \wedge \tilde{V}) - \star(\zeta^{hb}(i_V \star F) \wedge \tilde{V}).$$

If $\zeta^{he} = \zeta^{db} = 0$ then $\zeta^{de} = \epsilon_0 \epsilon$ and $\zeta^{hb} = (\mu_0 \mu)^{-1}$. However, for such materials one cannot assert that $\zeta^{he}, \zeta^{db}$ remain zero in all frames. Materials with the general constitutive relation (4) are often referred to as magneto-electric [23, 24].

Maxwell’s equations (1) in a medium in spacetime $M$, with $j = 0$, follow naturally as a local extremum of the action functional $S[A, g] = \int_M \Lambda$ under $A$ variations, where locally $F = dA$, $G = Z(F)$, and

$$c \Lambda = \frac{1}{2} F \wedge \star G = \frac{1}{2} F \wedge \star Z(F)$$

provided the 4-th rank tensor $Z$ is taken to be self-adjoint: $Z^{abcd} = Z^{cdab}$.

The dependence of this action on the metric resides in the $\star$ map and $Z$. Thus the form of the variational derivative of this action under metric perturbations will depend on the response of the constitutive tensor $Z$ to gravitation. The above form of the constitutive relations relating electric and magnetic fields in an arbitrary timelike frame yields a natural tensor relation between $Z$ and $\tilde{V} \equiv g(V, -)$ and so offers a natural dependence of $Z$ on $g$ for inhomogeneous, anisotropic (non-dispersive) magneto-electric continua.
For such a $Z$ one may compute by a metric variation of the above action the stress-energy-momentum tensor associated with the electromagnetic field in such a medium. After a non-trivial calculation one obtains:

$$T = \frac{1}{2} \left( i_a F \otimes i^a G + i_a G \otimes i^a F - \ast (F \wedge \ast G) g + \tilde{V} \otimes s + s \otimes \tilde{V} \right)$$ (6)

where the 1-form $s = \ast \left( i_V F \wedge i_V \ast G \wedge \tilde{V} + i_V \ast F \wedge i_V G \wedge \tilde{V} \right)$.

This reduces in gravity-free Minkowski spacetime to the tensor attributed historically to Abraham. The details of the derivation in the considerably wider context outlined above may be consulted in [25]. In terms of comoving fields, defined by (2), (3) with $U = V$, (6) may be written in the manifestly symmetric form:

$$T = - \frac{1}{2} (e \otimes d + d \otimes e) - \frac{1}{2} (h \otimes b + b \otimes h) + \frac{1}{2} (g(\tilde{e}, \tilde{d}) + g(\tilde{h}, \tilde{b}))(g + 2 \tilde{V} \otimes \tilde{V}) + (\tilde{V} \otimes \tilde{S} + \tilde{S} \otimes \tilde{V})$$

where the Poynting 1-form $\tilde{S} = \ast (\tilde{V} \wedge e \wedge h)$.

If, by contrast $Z$ is chosen to be totally independent of the metric and hence $\tilde{V}$, the resulting stress-energy-momentum tensor becomes

$$T = \frac{1}{2} i_a G \otimes i^a F + \frac{1}{2} i_a F \otimes i^a G - \frac{1}{2} \ast (F \wedge \ast G) g$$

showing clearly its independence of the 4-velocity $V$ of the medium. In the absence of gravity such a tensor reduces to that obtained by symmetrising the one proposed by Minkowski.

### 3 Conclusion

There has been a rapid development in recent years in the construction of “traps” for confining collective states of matter on scales intermediate between macro- and micro-dimensions. Condensates of cold atoms and fabricated nano-structures offer many new avenues for technological development when coupled to probes by electromagnetic fields. The constitutive properties of such novel material will play an important role in this development. Space science is also progressing rapidly and can provide new laboratory environments with variable gravitation and controlled acceleration in which the properties of such states of matter may be explored. Our results offer a new and efficient way to establish a coherent understanding of the
stresses and energy-momentum exchanges induced by electromagnetic interactions with such matter in terms of a fully relativistic covariant variational framework. Supplemented with additional data based on mechanical and elasto-dynamic responses one thereby gains a more confident picture of a total phenomenological symmetric stress-energy-momentum tensor for a wide class of moving media than that based on previous ad-hoc choices.

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