ELEMENTARY AMENABLE GROUPS OF COHOMOLOGICAL DIMENSION 3

JONATHAN A. HILLMAN

Abstract. We show that torsion-free elementary amenable groups of Hirsch length \( \leq 3 \) are solvable, of derived length \( \leq 3 \). This class includes all solvable groups of cohomological dimension 3. We show also that groups in the latter subclass are either polycyclic, semidirect products \( BS(1, n) \rtimes \mathbb{Z} \), or properly ascending HNN extensions with base \( \mathbb{Z}^2 \) or \( \pi_1(Kb) \).

D. Gildenhuys showed that the solvable groups of cohomological dimension at most 2 are either subgroups of the rationals \( \mathbb{Q} \) or to be solvable Baumslag-Solitar groups \( BS(1, m) \) \([5]\). In particular, every such group has Hirsch length at most 2. We show that finitely generated, torsion-free elementary amenable groups of Hirsch length \( \leq 3 \) are in fact solvable minimax groups, of derived length \( \leq 3 \). We show also that such a group is finitely presentable if and only if it is constructible, and such groups are either polycyclic, semidirect products with base a solvable Baumslag-Solitar group, or properly ascending HNN extensions with base \( \mathbb{Z}^2 \) or \( \pi_1(Kb) \). Our interest in this class of groups arose from recent work on aspherical 4-manifolds with non-empty boundary and elementary amenable fundamental group \([4]\). Such groups have cohomological dimension \( \leq 3 \) and are of type \( FP \), and thus are in the class considered here. (One of the results of \([4]\) is that the groups arising there are all either polycyclic or solvable Baumslag-Solitar groups, and so may be considered well understood.)

1. BACKGROUND

Let \( G \) be a torsion-free elementary amenable group of finite Hirsch length \( h = h(G) \). Then \( G \) is virtually solvable \([6]\), and so has a subgroup of finite index which is an extension of a finitely generated free abelian group \( \mathbb{Z}^v \) by a nilpotent group \([3]\). Since \( v \leq h < \infty \) we may assume that \( v \) is the virtual first Betti number of \( G \), i.e., the maximum of the ranks of abelian quotients of subgroups of finite index in \( G \). If \( G \neq 1 \) then \( 0 < v \leq h = h(G) \leq c.d.G \leq h + 1 \).

Key words and phrases. cohomological dimension, elementary amenable, finitely presentable, Hirsch length, solvable, torsion-free.
We recall that the *Hirsch-Plotkin radical* $\sqrt{G}$ of a group $G$ is the (unique) maximal locally nilpotent normal subgroup of the group. (For the groups $G$ considered below, either $\sqrt{G}$ is abelian or $G$ is virtually nilpotent.) If $G$ is solvable and $\sqrt{G}$ is abelian then $\sqrt{G}$ is its own centralizer in $G$ (by the maximality assumption), and so the homomorphism from $G/\sqrt{G}$ to $\text{Aut}(\sqrt{G})$ induced by conjugation in $G$ is a monomorphism.

A solvable group is *minimax* if it has a composition series whose sections are either finite or isomorphic to $\mathbb{Z}[\frac{1}{m}]$, for some $m > 0$. A solvable group is *constructible* if it is in the smallest class containing the trivial group which is closed under finite extensions and HNN extensions [1]. If $G$ is a torsion-free virtually solvable group then $c.d.G = h \iff G$ is of type $FP \iff G$ is constructible [7].

Let $BS(m, n)$ be the Baumslag-Solitar group with presentation

$$\langle a, t \mid ta^m t^{-1} = a^n \rangle,$$

and let $\overline{BS}(m, n)$ be the metabelian quotient $BS(m, n)/\langle\langle a \rangle\rangle'$, where $\langle\langle a \rangle\rangle'$ is the commutator subgroup of the normal closure of the image of $a$ in $BS(m, n)$. We may assume that $m > 0$ and $|n| \geq m$. (When $m = 1$ and $n = \pm 1$ we get $\mathbb{Z}^2$ and $\pi_1(Kb)$.) Since we are only interested in torsion-free groups we shall assume also that $(m, n) = 1$.

2. **Hirsch length 2**

In this section we shall consider groups of Hirsch length 2, which arise naturally in the analysis of groups of Hirsch length 3. (Note also that some groups of Hirsch length 2 have cohomological dimension 3.)

**Theorem 1.** Let $G$ be a torsion-free elementary amenable group of Hirsch length 2. Then $\sqrt{G}$ is abelian, and either $\sqrt{G}$ has rank 1 and $G \cong \sqrt{G} \times \mathbb{Z}$ or $\sqrt{G}$ has rank 2 and $[G : \sqrt{G}] \leq 2$.

**Proof.** Since $G$ is virtually solvable [6] and the lowest non-trivial term of the derived series of a solvable group is a non-trivial abelian normal subgroup, $\sqrt{G} \neq 1$. Since any two members of $\sqrt{G}$ generate a torsion-free nilpotent group of Hirsch length $\leq 2$ they commute. Hence $\sqrt{G}$ is abelian, of rank $r = 1$ or 2, say, and $h(G/\sqrt{G}) = 2 - r$.

Let $C = C_G(\sqrt{G})$ be the centralizer of $\sqrt{G}$ in $G$. If $N \leq C$ is a normal subgroup of $G$ with locally finite image in $G/\sqrt{G}$ then $N'$ is locally finite, by an easy extension of Schur’s Theorem [8, 10.1.4]. Hence $N' = 1$, so $N$ is abelian, and then $N \leq \sqrt{G}$, by the maximality of $\sqrt{G}$. Therefore any locally finite normal subgroup of $G/\sqrt{G}$ must act effectively on $\sqrt{G}$. 
If $\sqrt{G}$ has rank 1 then $G/\sqrt{G}$ can have no non-trivial torsion normal subgroup. If $C \neq \sqrt{G}$ is infinite then it has an infinite abelian normal subgroup (since it is non-trivial, virtually solvable, and has no non-trivial torsion normal subgroup). But the preimage of any such subgroup in $G$ is nilpotent (since it is a central extension of an abelian group). This contradicts the maximality of $\sqrt{G}$. Hence $=\sqrt{G}$ and so $G/\sqrt{G}$ acts effectively on $\sqrt{G}$. Since $h(G/\sqrt{G}) = 1$ and $\text{Aut}(\sqrt{G}) \leq \mathbb{Q}^\times$, and $G/\sqrt{G}$ has no normal torsion subgroup, we see that $G/\sqrt{G} \cong \mathbb{Z}$.

If $\sqrt{G}$ has rank 2 then $G/\sqrt{G}$ is a torsion group, and $\text{Aut}(\sqrt{G})$ is isomorphic to a subgroup of $\text{GL}(2, \mathbb{Q})$. If $G/\sqrt{G}$ is infinite then it must have an infinite locally finite normal subgroup (since it is a virtually solvable torsion group). But finite subgroups of $\text{GL}(2, \mathbb{Q})$ have order dividing 24, and so $G/\sqrt{G}$ is finite. If $g$ in $G$ has image of finite order $p > 1$ in $G/\sqrt{G}$ then conjugation by $g$ fixes $g^p \in \sqrt{G}$. It follows that $g$ must have order 2 and its image in $\text{GL}(2, \mathbb{Q})$ must have determinant $-1$. Hence $[G : \sqrt{G}] \leq 2$. □

If $G$ is finitely generated then $\sqrt{G}$ is finitely generated as a module over $\mathbb{Z}[G/\sqrt{G}]$, with respect to the action induced by conjugation in $G$. If $h(\sqrt{G}) = 1$ then $\sqrt{G}$ is not finitely generated as an abelian group, while $G/\sqrt{G} \cong \mathbb{Z}$. Hence $\mathbb{Z}[G/\sqrt{G}] \cong \mathbb{Z}[t, t^{-1}]$, and the action of $t$ is multiplication by some $n/m \in \mathbb{Q} \setminus \{0, \pm 1\}$, since $\sqrt{G}$ is torsion-free and of rank 1. After replacing $t$ by $t^{-1}$, if necessary, we may assume that $\sqrt{G} \cong \mathbb{Z}[t, t^{-1}]/(mt - n)$, for some $m, n$ with $(m, n) = 1$ and $|n| > m > 0$. Hence $G \cong \text{BS}(m, n)$. Then $c.d.G = 2 \iff G$ is finitely presentable $\iff m = 1$ [5].

If $G$ is finitely generated and $h(\sqrt{G}) = 2$ then $G \cong \mathbb{Z}^2$ or $\pi_1(Kb)$, and so $c.d.G = 2$.

Let $\mathbb{Z}_{(2)}$ be the localization of $\mathbb{Z}$ at 2, in which all odd integers are invertible, and let $\mathbb{Z}_{(2)}$ act on $\mathbb{Q}$ through the surjection to $\mathbb{Z}_{(2)}/2\mathbb{Z}_{(2)} \cong \mathbb{Z}^\times = \{\pm 1\}$. Let $\mathbb{Q} \otimes Kb$ be the extension of $\mathbb{Z}_{(2)}$ by $\mathbb{Q}$ with this action. Then if $h = 2$ and $G$ is not finitely generated it is either a subgroup of $\mathbb{Q} \times \mathbb{Z}$, for some nonzero $m, n$ with $(m, n) = 1$ (if $h(\sqrt{G}) = 1$), or is a subgroup of $\mathbb{Q} \otimes Kb$ (if $h(\sqrt{G}) = 2$). Every such group has cohomological dimension 3.

3. HIRSCH LENGTH

Suppose now that $h(G) = 3$. Then $h(\sqrt{G}) = 1, 2$ or 3.
Theorem 2. Let $G$ be a torsion-free elementary amenable group of Hirsch length 3. If $h(\sqrt{G}) = 1$ then $\sqrt{G}$ is abelian and $G/\sqrt{G} \cong \mathbb{Z}^2$. If $h(\sqrt{G}) = 2$ then $\sqrt{G}$ is abelian and $G/\sqrt{G} \cong \mathbb{Z}, D_\infty$ or $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. If $h(\sqrt{G}) = 3$ then $G$ is virtually nilpotent. In all cases, $G$ has derived length at most 3.

Proof. If $h(\sqrt{G}) = 1$ then $\sqrt{G}$ is isomorphic to a subgroup of $\mathbb{Q}$ and (as in Theorem 1) $G/\sqrt{G}$ has no locally finite normal subgroup. Since $C_G(\sqrt{G})$ is virtually solvable, it follows that $C_G(\sqrt{G}) = \sqrt{G}$ and so $G/\sqrt{G}$ embeds in $Aut(\sqrt{G})$, which is isomorphic to a subgroup of $\mathbb{Q}^\times$. Hence $G/\sqrt{G} \cong \mathbb{Z}^2$, and so $G$ has derived length 2.

If $h(\sqrt{G}) = 2$ then $\sqrt{G}$ is abelian and (as in Theorem 1 again) the maximal locally finite normal subgroup of $G/\sqrt{G}$ has order at most 2. Since $G/\sqrt{G}$ is virtually solvable and $h(G/\sqrt{G}) = 1$, it has an abelian normal subgroup $A$ of rank 1, which we may assume torsion-free and of finite index in $G/\sqrt{G}$. Moreover, $G/\sqrt{G}$ embeds in $Aut(\sqrt{G})$, which is now isomorphic to a subgroup of $GL(2, \mathbb{Q})$. No nontrivial element of $A$ can have both eigenvalues roots of unity, for otherwise $C_G(\sqrt{G}) > \sqrt{G}$. Since the eigenvalues of $A$ have degree $\leq 2$ over $\mathbb{Q}$, it follows that no nontrivial element of $A$ can be infinitely divisible in $A$. Hence $G/\sqrt{G}$ is virtually $\mathbb{Z}$, and so it is either $\mathbb{Z}$ or the infinite dihedral group $D_\infty = \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$, or an extension of one of these by $\mathbb{Z}/2\mathbb{Z}$.

If $G$ has a normal subgroup $H$ such that $H/\sqrt{G} \cong \mathbb{Z}/2\mathbb{Z}$ then conjugation in $G$ must preserve the filtration $0 < H' < \sqrt{G}$ of $\sqrt{G}$. Therefore elements of $G'$ act nilpotently on $\sqrt{G}$, and so $G/H$ cannot be $D_\infty$. Thus if $h(\sqrt{G}) = 2$ then $G/\sqrt{G} \cong \mathbb{Z}, D_\infty$ or $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $G$ has derived length 2, 3 or 2, respectively.

If $h(\sqrt{G}) = 3$ then $h(G/\sqrt{G}) = 0$, and so $G$ is virtually nilpotent. Since iterated commutators live in finitely generated subgroups, the derived length of $G$ is the maximum of the derived lengths of its finitely generated subgroups. Finitely generated torsion-free virtually nilpotent groups of Hirsch length 3 are polycyclic, and are fundamental groups of $Nil^3$-manifolds. These are Seifert fibred over flat 2-orbifolds without reflector curves, and so these groups have derived length $\leq 3$. Hence $G$ has derived length $\leq 3$. □

Corollary 3. If $G$ is finitely generated then it is a minimax group.

Proof. If $h(\sqrt{G}) = 1$ and $G$ is finitely generated then $\sqrt{G}$ is finitely generated as a $\mathbb{Z}[\mathbb{Z}^2]$-module. Since it is also torsion-free and of rank 1 as an abelian group, it is in fact a cyclic $\mathbb{Z}[\mathbb{Z}^2]$-module. Hence $\sqrt{G} \cong \mathbb{Z}[1/\mathbb{Z}]$ for some $D > 0$. 

If \( h(\sqrt{G}) = 2 \) then \( G \) has a subgroup \( K \) of index \( \leq 2 \) such that \( K/\sqrt{G} \cong \mathbb{Z} \). If \( G \) is finitely generated then \( K \) is also finitely generated. Then \( \sqrt{G} \) is again finitely generated as a \( \Lambda \)-module, and is torsion-free and of rank 2 as an abelian group. Hence it is isomorphic as a group to a subgroup of \( \mathbb{Z}[\frac{1}{m}]^2 \), for some \( m > 0 \).

If \( G \) is finitely generated and \( h(\sqrt{G}) = 3 \) then \( G \) is polycyclic. In all cases it is clear that \( G \) is a minimax group. \( \square \)

We shall consider more closely the cases with \( h(\sqrt{G}) = 1 \) or 2.

**Lemma 4.** If \( G \) is finitely generated and \( h(\sqrt{G}) = 1 \) then \( G \) is a semidirect product \( \mathbb{BS}(m, n) \rtimes \mathbb{Z} \), where \( mn \) has at least 2 distinct prime factors.

**Proof.** If \( h(\sqrt{G}) = 1 \) then \( G \) has a presentation
\[
\langle a, t, u | t a^m t^{-1} = a^n, \ u a^p u^{-1} = a^q, \ u t u^{-1} = ta^e, \ \langle \langle a \rangle \rangle' \rangle,
\]
for some nonzero \( m, n, p, q \) with \( (m, n) = (p, q) = 1 \) and some \( e \in \mathbb{Z}[\frac{1}{D}] \), where \( D \) is the product of the prime factors of \( mnpq \). Hence \( \sqrt{G} \cong \mathbb{Z}[\frac{1}{D}] \). After a change of basis for \( G/\sqrt{G} \), if necessary, we may assume that \( mn \) has a prime factor which does not divide \( pq \). We may further arrange that \( p \) divides \( m \) and \( q \) divides \( n \), after replacing \( t \) by \( tu^N \) or \( tu^{-N} \) for \( N \) large enough, if necessary. Hence \( D \) is the product of the prime factors of \( mn \). It must have at least 2 prime factors, since \( G/\sqrt{G} \cong \mathbb{Z}^2 \) maps injectively to \( Aut(\sqrt{G}) \cong \mathbb{Z}[\frac{1}{D}]^\times \).

Thus \( G \cong \mathbb{BS}(m, n) \rtimes \mathbb{Z} \), for some automorphism \( \theta \) of \( \mathbb{BS}(m, n) \). \( \square \)

**Theorem 5.** A finitely generated torsion-free elementary amenable group \( G \) of Hirsch length 3 is coherent if and only if it is FP\(_2\) and \( h(\sqrt{G}) \geq 2 \).

**Proof.** If \( G \) is coherent then it is finitely presentable and hence FP\(_2\).

Suppose that \( h(\sqrt{G}) = 1 \). Then \( \sqrt{G} \cong \mathbb{Z}[\frac{1}{D}] \) for some \( D > 1 \), and the image of \( G/\sqrt{G} \) in \( Aut(\sqrt{G}) \cong \mathbb{Z}[\frac{1}{D}]^\times \) has rank 2. Hence it contains a proper fraction \( \frac{p}{q} \) with \( p, q \neq \pm 1 \), and so \( G \) has a subgroup isomorphic to \( \mathbb{BS}(p, q) \). Since this subgroup is not even FP\(_2\) \[2\], \( G \) is not coherent.

If \( h(\sqrt{G}) = 2 \) then we may assume that \( G/\sqrt{G} \cong \mathbb{Z} \). If, moreover, \( G \) is FP\(_2\) then \( G \) is an HNN extension with base a finitely generated subgroup of \( \sqrt{G} \) \[2\], and the HNN extension is ascending, since \( G \) is solvable. Any finitely generated subgroup of \( G \) is either a subgroup of the base or is itself an ascending HNN extension with finitely generated base, and so is finitely presentable.

If \( h(\sqrt{G}) = 3 \) then \( G \) is polycyclic, and every subgroup is finitely presentable. \( \square \)
It remains an open question whether an $FP_2$ torsion-free solvable group $G$ with $h(G) = 3$ and $h(\sqrt{G}) = 1$ must be finitely presentable. Note also that the argument shows that $G$ is *almost coherent* (finitely generated subgroups are $FP_2$) if and only if it is coherent.

We shall assume next that $h(\sqrt{G}) = 2$ and that $G/\sqrt{G} \cong \mathbb{Z}$. Since $\mathbb{Q} \otimes \sqrt{G} \cong \mathbb{Q}^2$, the action of $G/\sqrt{G}$ on $\sqrt{G}$ by conjugation in $G$ determines a conjugacy class of matrices $M$ in $GL(2, \mathbb{Q})$. Hence $G \cong \sqrt{G} \rtimes_M \mathbb{Z}$.

**Lemma 6.** A matrix $M \in GL(2, \mathbb{Q})$ is conjugate to an integral matrix if and only if $\det M$ and $\text{tr} M \in \mathbb{Z}$.

**Proof.** These conditions are clearly necessary. If they hold then the characteristic polynomial is a monic polynomial with $\mathbb{Z}$ coefficients. If $x \in \mathbb{Q}^2$ is not an eigenvector for $M$ then the subgroup generated by $x$ and $Mx$ is a lattice. Since $M$ preserves this lattice, by the Cayley-Hamilton Theorem, it is conjugate to an integral matrix. □

If $G$ is finitely generated then $\sqrt{G}$ is finitely generated as a $\mathbb{Z}[G/\sqrt{G}]$-module, since $\mathbb{Z}[G/\sqrt{G}]$ is noetherian. It is finitely generated as an abelian group (and so $G$ is polycyclic) if and only if $M$ is conjugate to a matrix in $GL(2, \mathbb{Z})$ if and only if $\det M = \pm 1$ and $\text{tr} M \in \mathbb{Z}$.

If $G$ is $FP_2$ then $G$ is an ascending HNN extension with base $\mathbb{Z}^2$ (as in Theorem 4 above). Hence $M$ (or $M^{-1}$) must be conjugate to an integral matrix, and $G$ is finitely presentable. On the other hand, if $G \cong \sqrt{G} \rtimes_M \mathbb{Z}$ and neither $M$ nor $M^{-1}$ is conjugate to an integral matrix then $G$ cannot be $FP_2$.

We conclude this section by giving some examples realizing the other possibilities for $G/\sqrt{G}$ allowed for by Theorem 2. Torsion-free polycyclic groups $G$ with $h(\sqrt{G}) = 2$ are $\text{Sol}^3$-manifold groups. There are such groups with $G/\sqrt{G} \cong \mathbb{Z}$, $D_\infty$ or $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. (The examples with $G/\sqrt{G} \cong D_\infty$ are fundamental groups of the unions of two twisted $I$-bundles over a torus along their boundaries.)

For instance, the group $G$ with presentation

$$\langle u, v, y \mid uyu^{-1} = y^{-1}, vvy^{-1} = v^{-2}y^{-1}, v^2 = u^2y \rangle$$

is a generalized free product with amalgamation $A *_C B$ where $A = \langle u, y \rangle \cong B = \langle v, u^2y \rangle \cong \pi_1(Kb)$ and $C = \langle u^2, y \rangle \cong \mathbb{Z}^2$. It is clear that $G/C \cong D_\infty$, and it is easy to check that $C = \sqrt{G}$.

If $G$ is the group with presentation

$$\langle t, x, y \mid tx = xt, tyt^{-1} = y^n, xyx^{-1} = y^{-1} \rangle$$
then $\sqrt{G}$ is normally generated by $x^2$ and $y$, so $h(\sqrt{G}) = 2$ and $G/\sqrt{G} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

If $G/\sqrt{G} \cong D_\infty$ then $G$ is generated by $\sqrt{G}$ and two elements $u, v$ with squares in $\sqrt{G}$. The matrices in $GL(2, \mathbb{Q})$ corresponding to the actions of $u$ and $v$ have determinant $-1$. Hence $t = uv$ corresponds to a matrix with determinant 1. There are finitely generated examples of this type which are not polycyclic. For instance, let $F$ be the group with presentation

$$\langle u, v, x, y \mid u^2 = x, yu y^{-1} = y^{-1}, v^2 = xy, vy y^{-1} = x^2 y^{-1} \rangle,$$

and let $K$ be the normal closure of the image of $\{x, y\}$ in $F$. Then $F/K \cong D_\infty$ and $K/K' \cong \mathbb{Z}[\frac{1}{3}]^2$, and $F/K'$ is torsion-free, solvable and $h(F/K') = 3$.

However, if such a group $G$ is $FP_2$ then so is the subgroup generated by $\sqrt{G}$ and $t$. Hence this subgroup is an ascending HNN extension with finitely generated base $H \leq \sqrt{G}$, and $F/K'$ is torsion-free, solvable and $h(F/K') = 3$.

In our next theorem we shall need the stronger hypothesis that $G$ be finitely presentable.

**Theorem 7.** Let $G$ be a torsion-free solvable group of Hirsch length 3. Then $G$ is finitely presentable if and only if it is constructible.

**Proof.** If $G$ is constructible then it is finitely presentable. Assume that $G$ is finitely presentable. If $\sqrt{G}$ has rank 1 then $G$ has a presentation

$$\langle a, t, u \mid t a^m t^{-1} = a^n, u a^p u^{-1} = a^q, u t u^{-1} t^{-1} = C(a, t, u), R \rangle,$$

for some nonzero $m, n, p, q$ with $(m, n) = (p, q) = 1$ and word $C(a, t, u)$ of weight 0 in each of $t$ and $u$, and some finite set of relators $R$. Let $D$ be the product of the prime factors of $mnpq$. Then $\sqrt{G} \cong \mathbb{Z}[\frac{1}{D}]$, and contains the image of $c$ in $G$. As observed after Corollary 3 we may assume that $p$ and $q$ divide $m$ and $n$, respectively and that $mn$ has a prime factor which does not divide $pq$. 4. FINITELY PRESENTABLE IMPLIES CONSTRUCTIBLE

In this section we shall show that if a torsion-free solvable group $G$ of Hirsch length 3 is finitely presentable then it is in fact constructible, and we shall describe all such groups.

If $G$ is $FP_2$ and $G/G'$ is infinite then $G$ is an HNN extension $H \ast \varphi$ with finitely generated base $H$ [2], and the extension is ascending since $G$ is solvable. Clearly $h(H) = h(G) - 1 = 2$, and $c.d. G \leq c.d. H + 1$. In our next theorem we shall need the stronger hypothesis that $G$ be finitely presentable.
We may assume that each of the relations in \( R \) has weight 0 in each of \( t \) and \( u \). Then we may write \( C(a, t, u) \) and each relator in \( R \) as a product of conjugates \( b_{i,j} = t^i u^j a u^{-j} t^{-i} \) of \( a \). Since \( R \) is finite the exponents \( i, j \) involved lie in a finite range \([-L, L]\), for some \( L \geq 0 \). The relations imply that the normal closure of the image of \( a \) in \( G \) is \( \sqrt{G} \cong \mathbb{Z}[\frac{1}{L}] \). Hence the images of the \( b_{i,j} \)'s in \( G \) commute, and are powers of an element \( \alpha \) represented by a word \( w = W(a, t, u) \) which is a product of powers of (some of) the \( b_{i,j} \)'s. In particular, \( a = \alpha^N \) and \( b_{i,j} = \alpha^{e(i,j)} \), for some exponents \( N \) and \( e(i, j) \). Clearly \( N = e(0, 0) \).

It follows also that \( t\alpha^m t^{-1} = \alpha^n \) and \( u\alpha^p u^{-1} = \alpha^q \). Hence adjoining a new generator \( \alpha \) and new relations

\[
\begin{align*}
(1) \quad a &= \alpha^N; \\
(2) \quad t\alpha^m t^{-1} &= \alpha^n; \\
(3) \quad u\alpha^p u^{-1} &= \alpha^q; \\
(4) \quad \alpha &= W(a, t, u); \text{ and} \\
(5) \quad t^i u^j a u^{-j} t^{-i} &= \alpha^{e(i,j)}, \text{ for all } i, j \in [-L, L].
\end{align*}
\]

gives an equivalent presentation.

We may use the first relation to eliminate the generator \( a \). Since the image of \( \alpha \) in \( G \) generates an infinite cyclic subgroup, the relations \( R \) must be consequences of these, and so we may delete the relations in \( R \). Moreover the relation \( \alpha = W(a, t, u) \) collapses to a tautology, and so may also be deleted, and we may use the final set of relations to write \( C(a, t, u) \) as a power of \( \alpha \). Since \( t^i u^j a u^{-j} t^{-i} = \alpha^{e(i,j)} \), for all \( i, j \in [-L, L] \).

Thus \( G \) has the finite presentation

\[
\langle t, u, \alpha \mid ta^mt^{-1} = \alpha^n, \quad u\alpha^pu^{-1} = \alpha^q, \quad utu^{-1}t^{-1} = \alpha^c \rangle,
\]

for some \( c \in \mathbb{Z} \). Since the subgroup generated by the images of \( t \) and \( \alpha \) is isomorphic to \( BS(m, n) \) and is solvable, either \( m \) or \( n = 1 \) [2].

If \( h(\sqrt{G}) = 2 \) then \( G \) has a subgroup \( J \) of index \( \leq 2 \) which is an ascending HNN extension with finitely generated base \( H \leq \sqrt{G} \). Since \( h(H) = 2 \), we have \( H \cong \mathbb{Z}^2 \). Hence \( J \) is constructible, and \( G \) is also constructible.

If \( h(\sqrt{G}) = 3 \) then \( G \) is virtually nilpotent, and so is again constructible. \( \Box \)

**Theorem 8.** Let \( G \) be a torsion-free elementary amenable group of Hirsch length 3. Then \( G \) is constructible if and only if either
(1) $G \cong BS(1, n) \rtimes \theta \mathbb{Z}$ for some $n \neq 0$ or $\pm 1$ and some $\theta \in \text{Aut}(BS(1, n))$;
(2) $G \cong H \ast \phi$ is a properly ascending HNN extension with base $H \cong \mathbb{Z}^2$ or $\pi_1(Kb)$; or
(3) $G$ is polycyclic.

Proof. It shall suffice to show that if $G$ is constructible then it is one of the groups listed here, as they are all clearly constructible. We may also assume that $G$ is not polycyclic, and so $h(\sqrt{G}) = 1$ or $2$.

Since $G$ is constructible it has a subgroup $J$ of finite index which is an ascending HNN extension with base a constructible solvable group of Hirsch length 2. Since $G$ is not polycyclic, we may assume that $J = G$, by Theorem 2 (when $h(\sqrt{G}) = 1$) and by Theorem 2 with the observations towards the end of §3 (when $h(\sqrt{G}) = 2$). Constructible solvable groups of Hirsch length 2 are in turn Baumslag-Solitar groups $BS(1, m)$ with $m \neq 0$.

If $h(\sqrt{G}) = 1$ then $|m| > 1$ and $G \cong BS(1, m) \ast \phi$, for some injective endomorphism of $BS(1, m)$. We shall use the presentation for $BS(1, m)$ given in §2. After replacing $a$ by $t^{-k}at^k$, if necessary, we may assume that $\phi(a) = a^q$ and $\phi(t) = ta^r$, for some $q \neq 0$ and $r$ in $\mathbb{Z}$. Then $G$ has a presentation

$$\langle a, t, u \mid tat^{-1} = a^m, uau^{-1} = a^q, utu^{-1} = ta^r \rangle.$$  

Let $s = tu$ and $n = mq$. Then $sas^{-1} = a^n$, and the subgroup $H \cong BS(1, n)$ generated by $a$ and $s$ is normal in $G$. Conjugation by $u$ generates an automorphism $\theta$ of $H$, since $q$ is invertible in $\mathbb{Z}[\frac{1}{n}]$. Hence $G \cong BS(1, n) \rtimes \theta \mathbb{Z}$, and so $G$ is of type (1).

If $h(\sqrt{G}) = 2$ then $m = \pm 1$, and so $H \cong \mathbb{Z}^2$ or $\pi_1(Kb)$. Since the HNN extension is properly ascending, $G$ is not polycyclic, and so $G$ is of type (2).

We have allowed an overlap between classes (1) and (2) in Theorem 8 for simplicity of formulation. Polycyclic groups of Hirsch length 3 are virtually semidirect products $\mathbb{Z}^2 \rtimes \mathbb{Z}$. Such semidirect products are ascending HNN extensions, but the extensions are not properly ascending, and so classes (2) and (3) are disjoint.

Taking into account the fact that solvable groups $G$ with $c.d.G = h(G)$ are constructible [2], we may summarize the above two theorems as follows.

**Corollary 9.** If $G$ is a torsion-free elementary amenable group of Hirsch length 3 then $c.d.G = 3 \iff G$ is constructible $\iff G$ is finitely presentable $\iff G$ is one of the groups listed in Theorem 8 above.  \[\square\]
We conclude with some remarks on realizing such groups as fundamental groups of aspherical manifolds. If $G$ is a finitely presentable group of type $\mathcal{F}\mathcal{F}$ then there is a finite $K(G, 1)$-complex of dimension $\max\{3, c.d. G\}$ \[9\]. Thickening such a complex gives a compact aspherical manifold of twice the dimension and with fundamental group $G$. We may define the manifold dimension of $G$ to be the minimal dimension $m.d. G$ of such a manifold. If $c.d. G = h$ (and $h \neq 2$) then there is a finite $K(G, 1)$-complex of dimension $h$, and so $m.d. G \leq 2h$. If $G$ is virtually polycyclic then $K(G, 1)$ is homotopy equivalent to a closed $h$-manifold, and so $m.d. G = h$. However if $G$ is not virtually polycyclic then $m.d. G > h + 1$ \[4\].

In particular, groups of the first two types allowed by Theorem 8 are not realizable by aspherical 4-manifolds. The 2-complex associated to the standard 1-relator presentation of $BS(1, m)$ is aspherical, and so $m.d. BS(1, m) \leq 4$ (with equality if $m \neq \pm 1$). Surgery arguments show that every automorphism $\theta$ of $BS(1, m)$ is induced by a self-homeomorphism $\Theta$ of such a 4-manifold \[4\]. The mapping torus of $\Theta$ is an aspherical 5-manifold, and so $m.d. BS(1, m) \rtimes_\theta \mathbb{Z} = 5$ (if $m \neq \pm 1$). The question remains open for properly ascending HNN extensions with base $\mathbb{Z}^2$ or $\pi_1(K b)$.

References

[1] Baumslag, G. and Bieri, R. Constructable solvable groups, Math. Z. 151 (1976), 249–257.
[2] Bieri, R. and Strebel, R. Almost finitely presentable soluble groups, Comment. Math. Helv. 53 (1978), 258–278.
[3] Čarin, V. S. On soluble groups of type $A_4$, Mat. Sbornik 94 (1960), 895–914.
[4] Davis, J. F. and Hillman, J. A. Aspherical 4-manifolds with elementary amenable fundamental group, in preparation.
[5] Gildenhuys, D. Classification of soluble groups of cohomological dimension two, Math. Z. 166 (1979), 21–25.
[6] Hillman, J. A. and Linnell, P. A. Elementary amenable groups of finite Hirsch length are locally-finite by virtually solvable, J. Aust. Math. Soc. 52 (1992), 237–241.
[7] Kropholler, P. H. Cohomological dimension of soluble groups, J. Pure Appl. Alg. 43 (1986), 281–287.
[8] Robinson, D. J. S. A Course in the Theory of Groups, GTM 80, Springer-Verlag, Berlin – Heidelberg – New York (1982).
[9] Wall, C. T. C. Finiteness conditions on CW complexes. II, Proc. Roy. Soc. Ser. A 295 (1966), 129–139.

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

Email address: jonathanhillman47@gmail.com