On the Rankin–Selberg problem

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Abstract
In this paper, we solve the Rankin–Selberg problem. That is, we break the well known
Rankin–Selberg’s bound on the error term of the second moment of Fourier coefficients
of a GL(2) cusp form (both holomorphic and Maass), which remains its record since
its birth for more than 80 years. We extend our method to deal with averages of
coefficients of L-functions which can be factorized as a product of a degree one and
a degree three L-functions.

Mathematics Subject Classification
11F30 · 11L07 · 11F66

1 Introduction
Let $L(s, f)$ be an L-function of degree $d$ in the sense of Iwaniec–Kowalski [14, §5.1]
with coefficients $\lambda_f(1) = 1, \lambda_f(n) \in \mathbb{C}$. See Sect. 2.2 below. It is a fundamental
problem to prove an asymptotic formula for the sum

$$A(X, f) = \sum_{n \leq X} \lambda_f(n).$$

Under some suitable conditions, one can prove an asymptotic formula

$$A(X, f) = \text{Res}_{s=1} L(s, f)X^s \frac{1}{s-1} + O_f \left( X^{\frac{d-1}{d+1} + o(1)} \right).$$

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in quite general situations. See e.g. Friedlander–Iwaniec [5]. If \( d \geq 4 \), the generalized Riemann hypothesis (GRH) for \( L(s, f) \) implies the exponent \( \frac{d-1}{d+1} \) can be replaced by \( 1/2 \). This type of result has a very important application to the generalized Ramanujan conjecture (see Valentin–Farrell [28]). In this paper, we consider an important special case when \( f \) is the Rankin–Selberg convolution of a GL(2) automorphic representation with itself, which has degree \( d = 4 \). Our goal is to beat the exponent \( 3/5 \) in the error term.

Let \( \phi \) be a GL(2) holomorphic Hecke cusp form or Hecke–Maass cusp form for \( SL(2, \mathbb{Z}) \). Let \( \lambda_\phi(n) \) be its \( n \)-th Hecke eigenvalue. We define

\[
S_2(X, \phi) = \sum_{n \leq X} \lambda_\phi(n)^2, \quad \Delta_2(X, \phi) = S_2(X, \phi) - c_\phi X,
\]

where \( c_\phi = L(1, \text{Sym}^2 \phi)/\zeta(2) \) and \( L(s, \text{Sym}^2 \phi) \) is the symmetric square L-function of \( \phi \). Rankin [25] and Selberg [26] invented the powerful Rankin–Selberg method, and then successfully showed that

\[
\Delta_2(X, \phi) \ll \phi X^{3/5}.
\]

This bound remains the best since it was proved more than 80 years. The *Rankin–Selberg problem* is to improve the exponent \( 3/5 \). Although we have several methods to prove essentially the same bound as above (see e.g. Ivić [13]), the exponent \( 3/5 \) represents one of the longest standing records in analytic number theory. The generalized Riemann Hypothesis implies \( \Delta_2(X, \phi) \ll X^{1/2 + o(1)} \). It is conjectured that (see e.g. Ivić [12, Eq. (7.23)])

\[
\Delta_2(X, \phi) = O_\phi(X^{3/8 + o(1)}) \quad \text{and} \quad \Delta_2(X, \phi) = \Omega_\phi(X^{3/8}).
\]

The above \( \Omega \)-result was proved by Lau–Lü–Wu [19]. Note that \( S_2(X, \phi) \) is essentially the same as \( A(X, f) \) with \( f = \phi \times \phi \) in which case we have the degree \( d = 4 \). In this paper, our main goal is to solve the Rankin–Selberg problem.

**Theorem 1** *With the notation as above. We have*

\[
\Delta_2(X, \phi) \ll_\phi X^{3/5 - \delta + o(1)}, \quad (1.1)
\]

*for any \( \delta \leq 1/560 = 0.001785 \ldots \)*

**Remark 2** We emphasize that we do not expect our bounds to be optimal by our method. For example one may find a better exponent pair to improve our exponent, see e.g. Graham–Kolesnik [9, Chap. 5 and 7]. Let \((k, \ell)\) be an exponent pair (see [9, Chap. 3]). To get a better bound, we essentially need to minimize \((57 + 52k - 42\ell)/(97 + 82k - 72\ell)\). We choose the exponent pair \((1/30, 13/15)\) obtained by the simplest van der Corput estimates (with fifth derivative), which gives a rather good bound and can be extended to more general cases (see Friedlander–Iwaniec [5, §4] and Remark 7 below). We remark that the well known exponent pair \((9/56 + \varepsilon, 37/56 + \varepsilon)\)
is not good for our purpose, but this combines with $A$-process twice we can take $A^2(9/56 + \varepsilon, 37/56 + \varepsilon) = (9/278 + \varepsilon, 241/278 + \varepsilon)$ which may allow us to take $\delta = 6/3235 = 0.001854\ldots$. The best possible $\delta$ we may show is $37/19220 = 0.001925\ldots$ by using the exponent pair $(13/414 + \varepsilon, 359/414 + \varepsilon)$, which is obtained by using Bourgain’s exponent pair $(13/84 + \varepsilon, 55/84 + \varepsilon)$ (see [4]) and $A$-process twice, i.e., $(13/414 + \varepsilon, 359/414 + \varepsilon) = A^2(13/84 + \varepsilon, 55/84 + \varepsilon)$.

**Remark 3** The same method works for $\phi$ being either holomorphic or Maass. In fact, the holomorphic case is easier, since we have the Ramanujan bounds for the coefficients, so we will give the proof for the Maass case.

**Remark 4** If $\phi$ is a dihedral Maass cusp form (also a CM holomorphic cusp form), then we can prove

$$\Delta_2(X, \phi) \ll_\phi X^{1/2+\sigma(1)},$$

unconditionally. The key fact we need is the factorization of the symmetric square L-function of a dihedral form. Then the result will follow from the approximate functional equations, the Cauchy–Schwarz inequality, and the integral mean-value estimate (3.4).

To prove Theorem 1, we first consider $A(X, \phi \times \phi)$. Let

$$L(s, \phi \times \phi) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_{\phi}(n)^2}{n^s}, \quad \text{Re}(s) > 1$$

be the Rankin–Selberg L-function of $\phi \times \phi$. Note that $\phi \times \phi = 1 \boxplus \text{Sym}^2 \phi$, where $\text{Sym}^2 \phi$ is the symmetric square lift of $\phi$. That is $L(s, \phi \times \phi) = \zeta(s) L(s, \text{Sym}^2 \phi)$. In [6], Gelbart–Jacquet proved that $\text{Sym}^2 \phi$ is an automorphic cuspidal representation for $GL(3)$.

This leads us to consider the more general case

$$f = 1 \boxplus g, \quad \text{that is, } L(s, f) = \zeta(s) L(s, g),$$

where $L(s, g)$ is a primitive L-function of degree $d - 1$, that is, $L(s, g)$ cannot be decomposed into a product of L-functions of lower degrees. When $d = 2$ and $L(s, g) = \zeta(s)$, this is the well-known classical divisor problem, in which case we can do better than the exponent $1/3$ by using the theory of exponential sums (see e.g. Titchmarsh [27, §12.4]). When $d = 3$ and $L(s, g) = L(s, \phi)$, Friedlander–Iwaniec [5, §4] showed that one can beat the exponent $1/2$ (see more discussion in Remark 8). In this paper, we deal with the case when $d = 4$.

Let $\Phi$ be a Hecke–Maass cusp form for $SL(3, \mathbb{Z})$. Let $A_{\Phi}(1, n)$ be the normalized Fourier coefficients of $\Phi$. The generalized Ramanujan conjecture (GRC) for $\Phi$ asserts that $A_{\Phi}(1, n) \ll n^{\sigma(1)}$. Our method can be used to prove the following result under GRC.
Theorem 5 With the notation as above. Assuming GRC for $\Phi$, then we have
\[ \mathcal{A}(X, 1 \boxplus \Phi) = L(1, \Phi) X + O_\Phi(X^{3/5 - \delta + o(1)}), \]
for any $\delta \leq 1/560$. Furthermore, if $\Phi = \text{Sym}^2$, then we do not need to assume GRC for $\Phi$.

Remark 6 Let $\phi$ be a GL(2) Hecke–Maass cusp form for $\text{SL}(2, \mathbb{Z})$. The Ramanujan conjecture (RC) for $\phi$ says that $\lambda_{\phi}(n) \ll n^{\alpha(1)}$. In the above theorem we do not need to assume RC for $\phi$ (hence GRC for $\text{Sym}^2 \phi$). The reason is that we have nonnegativity of the coefficients
\[ \lambda_{1 \boxplus \text{Sym}^2 \phi}(n) = \lambda_{\phi \times \phi}(n) = \sum_{\ell^2 m=n} \lambda_{\phi}(m)^2 \geq 0, \]
by (1.2). See more details in § 3.

Remark 7 As in Friedlander–Iwaniec [5, §4], we can prove similar result when we replace 1 by a Dirichlet character $\chi$, that is, when $f = \chi \boxplus \phi$.

Remark 8 In the case $f = 1 \boxplus \phi$ with degree $d = 3$, under RC for $\phi$, the general theorem (see e.g. [5]) will give us
\[ \mathcal{A}(X, 1 \boxplus \phi) = L(1, \phi) X + O_\phi(X^{1/2 + o(1)}). \]
A simple application of GRH will still give us the exponent 1/2. Our method in this paper can be applied to this case, and we can show
\[ \mathcal{A}(X, 1 \boxplus \phi) = L(1, \phi) X + O_\phi(X^{1/2 - \delta' + o(1)}), \]
for some small positive $\delta'$, under RC. This slightly goes beyond a simple application of GRH. The same estimate holds for a holomorphic cusp form $\phi$. Note that in the holomorphic case, RC is known.

There are two different methods to get $\frac{d-1}{d+1}$ if $f = 1 \boxplus g$ with $g$ primitive and $d \geq 3$ as mentioned in Friedlander–Iwaniec [5]. We use the one with the contour of integral on the critical line. It is also possible to avoid this by shifting the contour to a vertical line with negative real part as done by Friedlander–Iwaniec. Our approach leads to a new integral moment of L-functions
\[ \int_T^{2T} L(1/2 + it, 1 \boxplus \Phi) X^{it} dt. \]
Finding good upper bounds for this integral moment is of independent interest and we want to highlight (see Proposition 19). By the approximate functional equation, the Cauchy inequality, and the integral mean value estimate, one can show the upper bound $O(T^{5/4 + o(1)})$, which is good enough for small $T$’s. The most important case
is when $T = X^{2/5+\delta}$. We seek for a better upper bound. Our idea is to use moments of L-functions without absolute value, which reduces the problem to a dual sum of Fourier coefficients (cf. Huang [10, §7]).

In order to prove Theorem 5, we will use a power saving for the analytic twisted sum of GL(3) Fourier coefficients. Define

$$\mathcal{S}(N) := \sum_{n \geq 1} A_{\Phi}(1, n)e\left(T \varphi\left(\frac{n}{N}\right)\right)V\left(\frac{n}{N}\right),$$

where $T \geq 1$ is a large parameter, $\varphi$ is some fixed real-valued smooth function, and $V \in C_\infty^c(\mathbb{R})$ with $\text{supp } V \subset [1/2, 1]$, total variation $\text{Var}(V) \ll 1$ and satisfying that $V^{(k)} \ll P^k$ for all $k \geq 0$ with $P \ll T^\eta$ for some small $\eta \in [0, 1/10]$.

**Theorem 9** Assume $\varphi(u) = u^\beta$ with $\beta \in (0, 1)$. Then we have

$$\mathcal{S}(N) \ll_{\Phi} T^{3/10}N^{3/4+\varepsilon},$$

if $T^{6/5} \leq N \leq T^{8/5-\varepsilon}$, and we have

$$\mathcal{S}(N) \ll_{\Phi} T^{-1/2}N^{5/4+\varepsilon},$$

if $T^{8/5-\varepsilon} \leq N \leq T^2$.

In [23], Munshi proved the first nontrivial result of this type for $\varphi(u) = -(\log u)/2\pi$ with $N \leq T^{3/2+\varepsilon}$, and got an application to the subconvexity bounds of GL(3) L-functions in the $T$-aspect. Recently, this was strengthened to the above bound for $\varphi(u) = -(\log u)/2\pi$ and $N \leq T^{3/2+\varepsilon}$ by Aggarwal [1] and for $\varphi(u) = u^\beta$ and $T = \alpha N^\beta$ by Kumar–Mallesh–Singh [18] (with bounds depending on $\alpha$). However, for Theorem 5, we need $\alpha$ to be quite large. We also need the result for $N \geq T^{3/2+\varepsilon}$, which is unlike the subconvexity problem for $L(1/2 + iT, \Phi)$. We will modify (and simplify) their methods to prove Theorem 9. In fact, we use the Duke–Friedlander–Iwaniec delta method similar to what Munshi [24] did, instead of the Kloosterman circle method. We can deal with more general $\varphi$’s. For our main application, we only need $\varphi(u) = u^{1/4}$.

Similar result for GL(2) Fourier coefficients can be found in Jutila [16]. In [24], Munshi showed the first nontrivial result of this type for GL(3) $\times$ GL(2) Fourier coefficients with $\varphi(u) = -(\log u)/2\pi$. Recently, Lin–Sun [21] succeeded to treat the analytic twisted sum of GL(3) $\times$ GL(2) Fourier coefficients, and got an application to $\mathcal{A}(X, f)$ with $f = \Phi \times \phi$ under GRC. Here we follow Lin–Sun’s formulation. See the introduction there for more discussions on this topics.

**Remark 10** In the case $f = 1 \boxtimes (\Phi \times \phi)$ with degree $d = 7$, under GRC for $\Phi$ and $\phi$, the general theorem (see e.g. [5]) will give us

$$\mathcal{A}(X, 1 \boxtimes (\Phi \times \phi)) = L(1, \Phi \times \phi) X + O_{\Phi, \phi}(X^{3/4+\sigma(1)}).$$
Our method in this paper can also be applied to this case, and we can show
\[ A(X, 1 \Phi_1 \times \phi) = L(1, \Phi_1 \times \phi) X + O_{\Phi_1, \phi}(X^{3/4 - \delta'' + o(1)}), \]
for some small positive \( \delta'' \), under GRC. To prove this, one needs to extend [21, Theorem 1.1] to the case \( N \geq t^{3 + \varepsilon} \). The proof is similar (but more complicated), so we do not include it in this paper.

The plan of this paper is as follows. In Sect. 2, we recall some results on L-functions, the Voronoi summation formula, and the Duke–Friedlander–Iwaniec delta method. In Sect. 3, we apply smoothing, Mellin transform and the stationary phase to reduce to the dual sums. This gives a proof of Theorem 5 by assuming Theorem 9. Then we prove Theorem 1 in Sect. 4. Finally, in Sect. 5, we proof Theorem 9. The method is relatively standard now thanks to Munshi and his collaborators. In Sect. 5.1, we apply the delta method. In Sect. 5.2, we use the summation formulas to get the dual sums. We first need to analyse the integrals in Sect. 5.3, and then apply Cauchy and Poisson in the generic case in Sects. 5.4 and 5.5.

**Notation** Throughout the paper, \( \varepsilon \) is an arbitrarily small positive number; all of them may be different at each occurrence. The weight functions \( U, V, W \) may also change at each occurrence. As usual, \( e(x) = e^{2\pi i x} \).

## 2 Preliminaries

### 2.1 Maass forms and L-functions

Let \( \Phi \) be a Hecke–Maass form of type \((\nu_1, \nu_2)\) for \( \text{SL}(3, \mathbb{Z}) \) with the normalized Fourier coefficients \( A(m, n) \) such that \( A(1, 1) = 1 \). The Langlands parameters are defined as \( \alpha_1 = -\nu_1 - 2\nu_2 + 1, \alpha_2 = -\nu_1 + \nu_2, \) and \( \alpha_3 = 2\nu_1 + \nu_2 - 1 \). The Ramanujan–Selberg conjecture predicts that \( \text{Re}(\alpha_i) = 0 \). From the work of Jacquet and Shalika [15], we know (at least) that \( |\text{Re}(\alpha_i)| < 1/2 \). It is well known that by standard properties of the Rankin–Selberg L-function we have the Ramanujan conjecture on average

\[
\sum_{m \geq 1} \sum_{n \geq 1} |A(m, n)|^2 \ll_{\Phi} N^{1+\varepsilon}.
\]  

(2.1)

The L-function associated with \( \Phi \) is given by \( L(s, \Phi) = \sum_{n=1}^{\infty} A(1, n)n^{-s} \) in the domain \( \text{Re}(s) > 1 \). It extends to an entire function and satisfies the following functional equation

\[
\gamma(s, \Phi)L(s, \Phi) = \gamma(1-s, \Phi)L(1-s, \Phi), \quad \gamma(s, \Phi) = \prod_{j=1}^{3} \pi^{-s/2} \Gamma\left(\frac{s - \alpha_j}{2}\right).
\]

Here \( \Phi \) is the dual form having Langlands parameters \((-\alpha_3, -\alpha_2, -\alpha_1)\) and the Fourier coefficients \( \tilde{A}(m, n) \). See more information in Goldfeld [7].
2.2 Approximate functional equation

Let \( L(s, f) \) be an L-function of degree \( d \) in the sense of \([14, \S 5.1]\). More precisely, we have

\[
L(s, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \prod_{j=1}^d \left( 1 - \frac{\alpha_j(p)}{p^s} \right)^{-1},
\]

with \( \lambda_f(1) = 1, \lambda_f(n) \in \mathbb{C}, \alpha_j(p) \in \mathbb{C} \), such that the series and Euler products are absolutely convergent for \( \text{Re}(s) > 1 \). The sequence \( \{\lambda_f(n)\}_{n \geq 1} \) are called coefficients of \( L(s, f) \). We assume \( |\alpha_j(p)| < p \) for all \( p \). There exist an integer \( q(f) \geq 1 \) and a gamma factor

\[
\gamma(s, f) = \pi^{-ds/2} \prod_{j=1}^d \Gamma \left( \frac{s - \kappa_j}{2} \right)
\]

with \( \kappa_j \in \mathbb{C} \) and \( \text{Re}(\kappa_j) < 1/2 \) such that the complete L-function

\[
\Lambda(s, f) = q(f)^{s/2} \gamma(s, f) L(s, f)
\]

admits an analytic continuation to a meromorphic function for \( s \in \mathbb{C} \) of order 1 with poles at most at \( s = 0 \) and \( s = 1 \). Moreover we assume \( L(s, f) \) satisfies the functional equation

\[
\Lambda(s, f) = \varepsilon(f) \Lambda(1 - s, \bar{f}),
\]

where \( \bar{f} \) is the dual of \( f \) for which \( \lambda_{\bar{f}}(n) = \overline{\lambda_f(n)} \), \( \gamma(s, \bar{f}) = \gamma(s, f) \), \( q(\bar{f}) = q(f) \), and \( \varepsilon(f) \) is the root number of \( L(s, f) \) satisfying that \( |\varepsilon(f)| = 1 \). We further assume that \( \lambda_f(n) \)'s satisfy the Ramanujan bound on average, that is,

\[
\sum_{n \leq x} |\lambda_f(n)| \ll x^{1+\varepsilon}. \quad (2.2)
\]

As \( |t| \to \infty \), Stirling’s formula gives

\[
\Gamma(\sigma + it) = \sqrt{2\pi} |t|^{\sigma - \frac{1}{2} + it} \exp \left( -\frac{\pi}{2} |t| - it \text{sgn}(t) \frac{\pi}{2} (\sigma - \frac{1}{2}) \right) (1 + O(|t|^{-1})).
\]

Hence for \( |t| \in [T, 2T] \) with \( T \) large, and \( \varepsilon \leq \text{Re}(w) \ll 1, |\text{Im}(w)| \ll T^\varepsilon \), and \( \kappa_j \ll 1 \), we have

\[
\frac{\Gamma \left( \frac{1/2 + it + w - \kappa_j}{2} \right)}{\Gamma \left( \frac{1/2 + it - \kappa_j}{2} \right)} = \left( \frac{|t|}{2} \right)^{w/2} e^{i\pi w/2} (1 + O(T^{-1})),
\]
For \( t \in [T, 2T] \), by the approximate functional equation [14, §5.2] we have

\[
L(1/2 + it, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{1/2 + it}} \int_{(2)} \frac{\gamma(1/2 + it + w, f)}{\gamma(1/2 + it, w)} q^{w/2} G(w) \frac{dw}{w} \\
+ \varepsilon(f) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{1/2 - it}} \int_{(2)} \frac{\gamma(1/2 - it + w, \tilde{f})}{\gamma(1/2 + it, \tilde{f})} q^{w/2} G(w) \frac{dw}{w} \\
+ O(T^{-A}),
\]

where \( G(w) = e^{w^2} \). We can move the line of integration to \( \text{Re}(w) = \varepsilon \) and truncate at \( |\text{Im}(s)| \leq T^\varepsilon \) with a negligible error term. By Stirling’s formula (2.2) and (2.3), we can truncate the \( n \)-sum at \( n \ll T^{d/2+\varepsilon} \) for the first sum and at \( n \ll T^{d/2+\varepsilon} \) for the second sum above with a negligible error. Hence by a smooth partition of unity, we prove the following lemma.

**Lemma 11** With notation as above, we have

\[
L(1/2 + it, f) = \frac{1}{2\pi i} \int_{\varepsilon-iT}^{\varepsilon+iT} \sum_{N \leq T^{d/2+\varepsilon} \text{ dyadic}} \frac{\lambda_f(n)}{n^{1/2+it+w}} V\left(\frac{n}{N}\right) \left(\frac{\pi t}{2}\right)^{d/2} q^{w/2} G(w) \frac{dw}{w} \\
+ \frac{\varepsilon(f)}{2\pi i} \int_{\varepsilon-iT}^{\varepsilon+iT} \sum_{N \leq T^{d/2+\varepsilon} \text{ dyadic}} \frac{\lambda_f(n)}{n^{1/2-it+w}} V\left(\frac{n}{N}\right) \left(\frac{t}{2\pi \varepsilon}\right)^{d/2} G(w) \frac{dw}{w} + O\left(T^{d/4-1+\varepsilon}\right),
\]

where \( V \) is a fixed smooth function with \( \text{supp} \; V \subset (1/2, 1) \).

**Remark 12** We can obtain a better error term in the above approximate functional equation by using Stirling’s formula with better error term. Since (2.4) is good enough for our purpose in this paper, we do not do it here.

### 2.3 Voronoi summation formula

Let \( \psi \) be a smooth compactly supported function on \((0, \infty)\), and let \( \tilde{\psi}(s) := \int_0^\infty \psi(x) x^s \frac{dx}{x} \) be its Mellin transform. For \( \sigma > 5/14 \), we define

\[
\Psi^\pm(z) := \frac{1}{2\pi i} \int_{(\sigma)} (\pi^3 z)^{-s} \gamma^\pm(s) \tilde{\psi}(1 - s) ds,
\]

\( \tilde{\psi} \) Springer
with
\[
\gamma^\pm(s) := \prod_{j=1}^{3} \frac{\Gamma\left(\frac{s+\alpha_j}{2}\right)}{\Gamma\left(\frac{1-s-\alpha_j}{2}\right)} \pm \frac{1}{\prod_{j=1}^{3} \Gamma\left(\frac{1+s+\alpha_j}{2}\right)}
\]
(2.6)

where \(\alpha_j\) are the Langlands parameters of \(\phi\) as above. Note that changing \(\psi(y)\) to \(\psi(y/N)\) for a positive real number \(N\) has the effect of changing \(\Psi^\pm(z)\) to \(\Psi^\pm(zN)\). The Voronoi formula on \(GL(3)\) was first proved by Miller–Schmid [22]. The present version is due to Goldfeld–Li [8] with slightly renormalized variables (see Blomer [2, Lemma 3]).

Lemma 13 Let \(c, d, \bar{d} \in \mathbb{Z}\) with \(c \neq 0\), \((c, d) = 1\), and \(d\bar{d} \equiv 1 \pmod{c}\). Then we have
\[
\infty \sum_{n=1}^\infty A(1, n)e\left(\frac{nd}{c}\right) \psi(n) = \frac{c\pi^{3/2}}{2} \sum_{\pm} \sum_{n_1 n_2=1}^\infty A(n_2, n_1) S\left(d, \pm n_2; \frac{c}{n_1}\right) \Psi^\pm\left(\frac{n_1^2 n_2}{c^3}\right),
\]
where \(S(a, b; c) := \sum_\ast d(c) e\left(\frac{ad+bd\bar{d}}{c}\right)\) is the classical Kloosterman sum.

The function \(\Psi^\pm(y)\) has the following properties.

Lemma 14 Suppose \(\psi(y)\) is a smooth function, compactly supported on \([N, 2N]\). Let \(\Psi^\pm(z)\) be defined as in (2.5). Then for any fixed integer \(L \geq 1\), and \(zN \gg 1\), we have
\[
\Psi^\pm(z) = z \int_0^\infty \psi(y) \sum_{\ell=1}^{L} \gamma_\ell \left(\frac{\pm 3(zy)^{1/3}}{\ell^{1/3}}\right) dy + O\left((zN)^{1-L/3}\right),
\]
where \(\gamma_\ell\) are constants depending only on \(\alpha_1, \alpha_2, \alpha_3,\) and \(L\).

Proof See Li [20, Lemma 6.1] and Blomer [2, Lemma 6].

2.4 The delta method

There are two oscillatory factors contributing to the convolution sums. Our method is based on separating these oscillations using the circle method. In the present situation we will use a version of the delta method of Duke, Friedlander and Iwaniec. More specifically we will use the expansion (20.157) given in [14, §20.5]. Let \(\delta : \mathbb{Z} \to \{0, 1\}\) be defined by
\[
\delta(n) = \begin{cases} 
1 & \text{if } n = 0; \\
0 & \text{otherwise}. 
\end{cases}
\]
We seek a Fourier expansion which matches with \(\delta(n)\).
Lemma 15 Let $Q$ be a large positive number. Then we have
\[
\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sum_{a \mod q}^{\star} e\left(\frac{na}{q}\right) \int_{\mathbb{R}} g(q, x) e\left(\frac{nx}{qQ}\right) dx,
\]
where $g(q, x)$ is a weight function satisfies that
\[
g(q, x) = 1 + O\left(\frac{Q}{q} \left(\frac{q}{Q} + |x|\right)^A\right), \quad g(q, x) \ll |x|^{-A}, \quad \text{for any } A > 1,
\]
and
\[
\frac{\partial^j}{\partial x^j} g(q, x) \ll |x|^{-j} \min(|x|^{-1}, Q/q) \log Q, \quad j \geq 1.
\]
Here the $\star$ on the sum indicates that the sum over $a$ is restricted by the condition $(a, q) = 1$.

Proof By [14, eq. (20.157)], we have
\[
\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sum_{a \mod q}^{\star} e\left(\frac{na}{q}\right) \int_{\mathbb{R}} g(q, x) e\left(\frac{nx}{qQ}\right) dx,
\]
for $n \in \mathbb{Z}$. Here
\[
g(q, x) = \int_{\mathbb{R}} \Delta_q(u) f(u) e\left(-\frac{ux}{qQ}\right) du,
\]
with a smooth $f$ such that $\text{supp } f \in [-Q^2/2, Q^2/2]$ and $f^{(j)}(u) \ll Q^{-2j}, \ j \geq 0,$ and a function $\Delta_q(u)$ satisfies that [14, Lemma 20.17]
\[
\Delta_q(u) \ll \frac{1}{(q + Q)Q} + \frac{1}{|u| + qQ}, \quad \Delta_q^{(j)}(u) \ll \frac{1}{qQ} (|u| + qQ)^{-j}, \quad j \geq 1.
\]
Note that here we take $N$ in [14] to be $Q^2/4$. We recall the following two properties (see (20.158) and (20.159) of [14])
\[
g(q, x) = 1 + h(q, x), \quad \text{with } h(q, x) = O\left(\frac{Q}{q} \left(\frac{q}{Q} + |x|\right)^A\right), \quad g(q, x) \ll |x|^{-A}
\]
for any $A > 1$. In particular the second property implies that the effective range of the integral in (2.7) is $[-Q, Q^2]$. 

\footnote{There is a typo in [14, eq. (20.158)].}
We will also need bounds of the derivatives of \( g(q, x) \) with respect to \( x \). Note that
\[
\frac{\partial^j}{\partial x^j} g(q, x) = \left(\frac{-2\pi i}{q Q}\right)^j \int_{\mathbb{R}} \Delta_q(u) f(u) u^j e\left(-\frac{ux}{q Q}\right) du
\]
By (2.10) and by \( j \) and \( j + 1 \) times repeated integration by parts, we have
\[
\frac{\partial^j}{\partial x^j} g(q, x) \ll \min(|x|^{-j} Q/q, |x|^{-j-1} \log Q/q) \ll |x|^{-j} \min(|x|^{-1}, Q/q) \log Q.
\]
This completes the proof of Lemma 15. \( \square \)

In applications of (2.7), we can first restrict to \( |x| \ll Q^{-\varepsilon} \). If \( q \gg Q^{1-\varepsilon} \), then by (2.9) we get
\[
\frac{\partial^j}{\partial x^j} g(q, x) \ll Q^{-\varepsilon} |x|^{-j}, \quad \text{for any } j \geq 1.
\]
If \( q \ll Q^{1-\varepsilon} \) and \( |x| \ll Q^{-\varepsilon} \), then by (2.8), we get replace \( g(q, x) \) by 1 with a negligible error term. So in all cases, we can view \( g(q, x) \) as a nice weight function.

**Remark 16** We can further prove
\[
\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sum_{a \mod q} e\left(\frac{na}{q}\right) \int_{\mathbb{R}} w(q, x) e\left(\frac{nx}{q Q}\right) dx + O(Q^{-A}), \quad (2.11)
\]
where \( w(q, x) \) is a weight function such that \( \text{supp } w(q, \cdot) \subset [-Q^\varepsilon, Q^\varepsilon] \) and
\[
\frac{\partial^j}{\partial x^j} w(q, x) \ll Q^\varepsilon (|x|^{-j} + Q^{j \varepsilon}).
\]

**2.5 Oscillatory integrals**

Let \( \mathcal{F} \) be an index set and \( X = X_T : \mathcal{F} \rightarrow \mathbb{R}^\geq_1 \) be a function of \( T \in \mathcal{F} \). A family of \( \{w_T\}_{T \in \mathcal{F}} \) of smooth functions supported on a product of dyadic intervals in \( \mathbb{R}^d_0 \) is called \( X \)-inert if for each \( j = (j_1, \ldots, j_d) \in \mathbb{Z}^d_{\geq 0} \) we have
\[
\sup_{T \in \mathcal{F}} \sup_{(x_1, \ldots, x_d) \in \mathbb{R}^d_{\geq 0}} X_T^{-j_1, \ldots, -j_d} x_1^{j_1} \cdots x_d^{j_d} w_T^{(j_1, \ldots, j_d)}(x_1, \ldots, x_d) \ll j_1, \ldots, j_d 1.
\]
We will use the following stationary phase lemma several times.

**Lemma 17** Suppose \( w = w_T \) is a family of \( X \)-inert in \( \xi \) with compact support on \([Z, 2Z]\), so that \( w^{(j)}(\xi) \ll (Z/X)^{-j} \). Suppose that on the support of \( w \), \( h = h_T \) is smooth and satisfies that \( h^{(j)}(\xi) \ll \frac{Y}{Z^j} \), for all \( j \geq 0 \). Let
\[
I = \int_{\mathbb{R}} w(\xi) e^{ih(\xi)} d\xi.
\]
(i) If $h'(\xi) \gg \frac{Y}{Z}$ for all $\xi \in \text{supp } w$. Suppose $Y/X \geq 1$. Then $I \ll A Z(Y/X)^{-A}$ for $A$ arbitrarily large.

(ii) If $h''(\xi) \gg \frac{Y}{Z^2}$ for all $\xi \in \text{supp } w$, and there exists $\xi_0 \in \mathbb{R}$ such that $h'((\xi_0) = 0$. Suppose that $Y/X^2 \geq 1$. Then

$$I = e^{ih((\xi_0))} W_T(\xi_0) + O_A(Z(Y/X)^2)^{-A})$$

for any $A > 0$,

for some $X$-inert family of functions $W_T$ (depending on $A$) supported on $\xi_0 \approx Z$.

**Proof** See [3, §8] and [17, §3]. \(\square\)

### 3 The dual sum and Proof of Theorem 5

In this section, we reduce the estimate of $A(X, 1 \boxplus \Phi)$ to its dual sum. To avoid the use of the Ramanujan conjecture in the case $\Phi = \text{Sym}^2 \phi$, we do not use Perron’s formula as Friedlander–Iwaniec [5] did. Instead, we use smoothing and Mellin transform to reduce the problem to an estimate of a first integral moment of L-functions. Then we apply the stationary phase method to deal with the first moment which reduces the problem to an estimate of its dual sum.

#### 3.1 Smoothing and Mellin transform

Let $0 < Y \leq X/5$. Let $W_1$ be a smooth function with support $\text{supp } W_1 \in [1/2 - Y/X, 1 + Y/X]$ such that $W_1(u) = 1$ if $u \in [1/2, 1]$, $W_1(u) \in [0, 1]$ if $u \in [1/2 - Y/X, 1/2] \cup [1, 1 + Y/X]$. Similarly, let $W_2$ be a smooth function with support $\text{supp } W_2 \in [1/2, 1]$ such that $W_2(u) = 1$ if $u \in [1/2 + Y/X, 1 - Y/X]$, $W_2(u) \in [0, 1]$ if $u \in [1/2, 1/2 + Y/X] \cup [1 - Y/X, 1]$. Assume $W_1^{(k)}(u) \ll (X/Y)^k$ for any integer $k \geq 0$ and $j \in \{1, 2\}$. Then by (1.4), we have

$$\sum_{n \geq 1} \lambda_1(\text{Sym}^2 \phi)(n) W_2 \left( \frac{n}{X} \right) \leq \mathcal{A}(X, 1 \boxplus \text{Sym}^2 \phi) - \mathcal{A}(X/2, 1 \boxplus \text{Sym}^2 \phi)$$

$$\leq \sum_{n \geq 1} \lambda_1(\text{Sym}^2 \phi)(n) W_1 \left( \frac{n}{X} \right).$$

In order to prove Theorem 5, it suffices to prove for $W \in \{W_1, W_2\}$ and

$$Y = X^{3/5 - \delta}, \text{ for some } \delta \in [0, 1/10),$$

we have

$$\sum_{n \geq 1} \lambda_1(\text{Sym}^2 \phi)(n) W \left( \frac{n}{X} \right) = L(1, \text{Sym}^2 \phi) \tilde{W}(1) X + O(X^{3/5 - \delta + o(1)}), \quad (3.1)$$
where \( \tilde{W}(s) \) is the Mellin transform of \( W \). Note that \( \tilde{W}(1) = 1/2 + O(Y/X) \). Indeed, (3.1) leads to

\[
\mathcal{A}(X, 1 \Box \text{Sym}^2 \phi) - \mathcal{A}(X/2, 1 \Box \text{Sym}^2 \phi) = L(1, \text{Sym}^2 \phi) \frac{X}{2} + O(X^{3/5-\delta + o(1)}).
\]

Hence we have \( \mathcal{A}(X, 1 \Box \text{Sym}^2 \phi) = L(1, \text{Sym}^2 \phi) X + O(X^{3/5-\delta + o(1)}) \). Let \( \Phi \) be a Hecke–Maass cusp form for \( \text{SL}(3, \mathbb{Z}) \). Assuming GRC for \( \Phi \), we have

\[
\lambda_1 \boxplus \Phi(n) = \sum_{\ell m = n} A \Phi(1, m) \ll n^{\delta(1)}.
\]

Hence by taking \( Y = X^{3/5-\delta} \) we have

\[
\mathcal{A}(X, 1 \Box \Phi) - \mathcal{A}(X/2, 1 \Box \Phi) - \sum_{n \geq 1} \lambda_1 \boxplus \Phi(n) W_1 \left( \frac{n}{X} \right) \ll \sum_{X/2-Y < n < X} |\lambda_1 \boxplus \Phi(n)| + \sum_{X/2 < n < X+Y} |\lambda_1 \boxplus \Phi(n)| \ll X^{3/5-\delta + o(1)}.
\]

Hence it suffices to show for \( W = W_1 \) and \( Y = X^{3/5-\delta} \), we have

\[
\sum_{n \geq 1} \lambda_1 \boxplus \Phi(n) W \left( \frac{n}{X} \right) = L(1, \Phi) \tilde{W}(1) X + O(X^{3/5-\delta + o(1)}).
\]

**Remark 18** This is the only place where we need to assume GRC for \( \Phi \) in order to prove Theorem 5. In fact it suffices to assume GRC on average in short intervals for \( \Phi \).

By the inverse Mellin transform, we have

\[
W(u) = \frac{1}{2\pi i} \int_{(2)} \tilde{W}(s) u^{-s} ds.
\]

Hence we get

\[
\sum_{n \geq 1} \lambda_1 \boxplus \Phi(n) W \left( \frac{n}{X} \right) = \frac{1}{2\pi i} \int_{(2)} \tilde{W}(s) L(s, 1 \Box \Phi) X^s ds.
\]

Since \( L(s, 1 \Box \Phi) = \zeta(s) L(s, \Phi) \), we have \( \text{Res}_{s=1} L(s, 1 \Box \Phi) = L(1, \Phi) \). Shifting the contour of integration to the left, we get

\[
\sum_{n \geq 1} \lambda_1 \boxplus \Phi(n) W \left( \frac{n}{X} \right) = L(1, \Phi) \tilde{W}(1) X + \frac{1}{2\pi i} \int_{(1/2)} \tilde{W}(s) L(s, 1 \Box \Phi) X^s ds.
\]
By repeated integration by parts, we have
\[
\tilde{W}(s) = -\frac{1}{s} \int_0^\infty W'(u)u^s du \ll \frac{1}{|s|^k} \left( \frac{X}{Y} \right)^{k-1}, \quad \text{for any } k \geq 1, \quad (3.2)
\]

since \( \text{supp } W^{(k)} \in [1/2 - Y/X, 1/2 + Y/X] \cup [1 - Y/X, 1 + Y/X]. \) This allows us to truncate the \( s \)-integral at \(|s| \ll X^{1+\varepsilon}/Y. \) In fact, we apply a smooth partition of unity, getting
\[
\sum_{n \geq 1} \lambda_1 \Phi(n) W \left( \frac{n}{X} \right) = L(1, \Phi) \tilde{W}(1)X + O(X^{-2020})
\]

\[+ O \left( X^{1/2} \sum_{T \leq X^{1+\varepsilon}/Y} \left| \int_{\mathbb{R}} \tilde{W}(1/2 + it) V \left( \frac{t}{T} \right) L(1/2 + it, 1 \boxplus \Phi) X^{it} dt \right| \right). \]

By the first equality in (3.2), we have
\[
\int_{\mathbb{R}} \tilde{W}(1/2 + it) V \left( \frac{t}{T} \right) L(1/2 + it, 1 \boxplus \Phi) X^{it} dt = -\int_0^\infty W'(u)u^{1/2} \int_{\mathbb{R}} V \left( \frac{t}{T} \right) L(1/2 + it, 1 \boxplus \Phi)(uX)^{it} \frac{dr}{1/2 + it} du.
\]

Recall that \( \tilde{W}(1) = 1/2 + O(Y/X). \) Hence
\[
\sum_{n \geq 1} \lambda_1 \Phi(n) W \left( \frac{n}{X} \right) = L(1, \Phi)X/2 + O(Y)
\]

\[+ O \left( \sup_{u \in [1/3, 2]} \sup_{T \ll X^{1+\varepsilon}/Y} \left| \int_{\mathbb{R}} V \left( \frac{t}{T} \right) L(1/2 + it, 1 \boxplus \Phi)(uX)^{it} dt \right| \right), \quad (3.3)
\]

for some fixed \( V \) with compact support. Hence it suffices to consider
\[
\mathcal{I} := \int_{\mathbb{R}} V \left( \frac{t}{T} \right) L(1/2 + it, 1 \boxplus \Phi) X^{it} dt.
\]

We only consider the case \( T \geq 1, \) since the case \( T \leq -1 \) can be done similarly and the case \(-1 \leq T \leq 1 \) can be treated trivially. We will prove the following proposition.

**Proposition 19** Let \( \Phi \) be a Hecke–Maass cusp form for \( \text{SL}_3(\mathbb{Z}). \) We have

(i) For any \( X > 0 \) and \( T \geq 1, \) we have \( \mathcal{I} \ll T^{5/4+\varepsilon}. \)

(ii) If \( X^\varepsilon \leq T \leq X^{2/5}, \) then we have \( \mathcal{I} \ll T^{5/2+\varepsilon} X^{-1/2} + T^{1+\varepsilon}. \)
(iii) If $X^{5/13} \leq T \leq X^{5/12}$, then we have

$$I \ll T^{56/25+\varepsilon} X^{-2/5}.$$  

We will prove this proposition in the next section which will need Theorem 9. We can prove nontrivial results in other ranges of $T$ by using the second claim in Theorem 9. Since this is enough for our application, we do not pursue it here. Now we can finish the proof of Theorem 5.

Proof of Theorem 5 by assuming Proposition 19  
Assume $\delta < 1/60$. Then by Proposition 19 (iii), for $X^{5/13} \leq T \leq X^{1+\varepsilon}/Y = X^{2/5+\delta+\varepsilon}$, the contribution to (3.3) is bounded by

$$O(X^{1/2} T^{-1} T^{56/25+\varepsilon} X^{-2/5}) = O(X^{1/10} T^{31/25+\varepsilon}) = O(X^{(149+310\delta)/250+\varepsilon}).$$

Note that the other error term is $O(X^{3/5-\delta+\varepsilon})$, so the best choice is $\delta = 1/560$. By Proposition 19 (i), for $T \leq X^{5/13}$, the contribution to (3.3) is bounded by

$$O(X^{1/2} T^{-1} T^{5/4+\varepsilon}) = O(X^{1/2} T^{1/4+\varepsilon}) = O(X^{31/52+\varepsilon}) = O(X^{3/5-\delta+\varepsilon}).$$

This proves Theorem 5. $\square$

3.2 The integral first moment

We will treat $I$ differently depending on the magnitudes of $T$ and $X$.

For any $X > 0$, we can use the fact $L(1/2+it, 1 \boxplus \Phi) = \xi(1/2+it)L(1/2+it, \Phi)$. Applying the Cauchy–Schwarz inequality, we get

$$I \ll \left( \int_{\mathbb{R}} V \left( \frac{1}{T} \right) |\xi(1/2+it)|^2 dt \right)^{1/2} \left( \int_{\mathbb{R}} V \left( \frac{1}{T} \right) |L(1/2+it, \Phi)|^2 dt \right)^{1/2}. $$

By the approximate functional equations (see Lemma 11) for both $f = 1$ and $f = \Phi$, we have

$$I \ll T^\varepsilon \left( \int_0^T \left| \sum_{\ell \leq T^{1/2+\varepsilon}} a_\ell \ell^{it} \right|^2 dt \right)^{1/2} \left( \int_0^T \left| \sum_{m \leq T^{3/2+\varepsilon}} b_m m^{it} \right|^2 dt \right)^{1/2}. $$
for some \( \{a_\ell\} \) and \( \{b_m\} \) satisfying that \( |a_\ell| \ll \ell^{-1/2+\varepsilon} \) and \( \sum_{m \leq M} |b_m|^2 \ll M^\varepsilon \). Here we have used (2.1). Now by the integral mean-value estimate (see e.g. [14, Theorem 9.1])

\[
\int_0^T \left| \sum_{n \leq N} a_n n^{it} \right|^2 dt \ll (T + N) \sum_{n \leq N} |a_n|^2, \tag{3.4}
\]

we obtain

\[
\mathcal{I} \ll T^{5/4+\varepsilon}. \tag{3.5}
\]

This proves Proposition 19 (i).

If we assume \( T \leq X^{2/5-4\delta+\varepsilon} \), we find the contribution from the error term in (3.3) is

\[
O(X^{1/2+\varepsilon} T^{1/4}) = O(X^{3/5-\delta+o(1)}).
\]

**Remark 20** If we take \( \delta = 0 \), then \( X^{1+\varepsilon}/Y \leq X^{2/5+\varepsilon} \). Hence we reprove the classical Rankin–Selberg estimate \( \Delta_2(X, \phi) \ll X^{3/5+\varepsilon} \).

From now on, we assume \( X^\varepsilon \leq T \leq X^{1/2-\varepsilon} \). By the approximate functional equations (see Lemma 11) with \( f = 1 \boxplus \Phi \) and \( \Phi \) a Hecke–Maass cusp form for \( SL_3(\mathbb{Z}) \), we have

\[
\mathcal{I} \ll T^\varepsilon \sup_{w \in [\varepsilon-iT^\varepsilon, \varepsilon+iT^\varepsilon]} \sup_{N \leq T^{2+\varepsilon}} \left( \left| \int_{\mathbb{R}} V\left(\frac{t}{T}\right) \sum_{n \geq 1} \frac{\lambda_1 \boxplus \Phi(n)}{n^{1/2+it+w}} V\left(\frac{n}{N}\right) t^{2w} X^{it} dt \right| + \left| \int_{\mathbb{R}} V\left(\frac{t}{T}\right) \sum_{n \geq 1} \frac{\lambda_1 \boxplus \Phi(n)}{n^{1/2-it+w}} V\left(\frac{n}{N}\right) \left(\frac{t}{2\pi e}\right)^{-i4t} t^{2w} X^{it} dt \right| \right) + O\left(T^{1+\varepsilon}\right).
\]

We can absorb the factor \( t^{2w} \) to the weight function \( V(t/T) \) and \( 1/n^{1/2+iw} \) to \( V(n/N) \). Then the new \( V_j \) \( (j = 1, 2) \) depends on \( w \) and satisfies that \( \text{supp} \ V_j \subset (1/2, 1) \) and \( V_j^{(k)} \ll T^{k\varepsilon} \) for \( k \geq 0 \). Hence we have

\[
\mathcal{I} \ll T^\varepsilon N^{-1/2} \sup_{w \in [\varepsilon-iT^\varepsilon, \varepsilon+iT^\varepsilon]} \sup_{N \leq T^{2+\varepsilon}} \left( |\mathcal{I}_1(N)| + |\mathcal{I}_2(N)| \right) + O\left(T^{1+\varepsilon}\right),
\]

where

\[
\mathcal{I}_1(N) := \int_{\mathbb{R}} V_1\left(\frac{t}{T}\right) \sum_{n \geq 1} \frac{\lambda_1 \boxplus \Phi(n)}{n^{it}} V_2\left(\frac{n}{N}\right) X^{it} dt
\]

and

\[
\mathcal{I}_2(N) := \int_{\mathbb{R}} \overline{V}_1\left(\frac{t}{T}\right) \sum_{n \geq 1} \frac{\lambda_1 \boxplus \Phi(n)}{n^{it}} \overline{V}_2\left(\frac{n}{N}\right) \left(\frac{t}{2\pi e}\right)^{i4t} X^{-it} dt,
\]

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where \( V_j(u) = V_j(u) \) for \( j = 1, 2 \).

We first deal with \( \mathcal{I}_1(N) \). Changing the order of integral and summation, and making a change of variable \( t = T \xi \), we get

\[
\mathcal{I}_1(N) = T \sum_{n \geq 1} \lambda_1 \Phi(n) V_2 \left( \frac{n}{N} \right) \int_{\mathbb{R}} V_1(\xi) e^{i \xi T \log X/n} d\xi.
\]

Since \( n \ll N \ll T^{4+\varepsilon} \ll X^{1-\varepsilon} \), we have \( T \log X/n \gg T \). By repeated integration by parts, we obtain

\[
\mathcal{I}_1(N) = O(T^{-2020}). \tag{3.6}
\]

Now we consider \( \mathcal{I}_2(N) \). Similarly, we arrive at

\[
\mathcal{I}_2(N) = T \sum_{n \geq 1} \lambda_1 \Phi(n) V_2 \left( \frac{n}{N} \right) \int_{\mathbb{R}} V_1(\xi) e^{i (4\xi T \log \xi + \xi T \log \frac{T^4}{(2\pi e)^4 nX})} d\xi.
\]

Let

\[
h_1(\xi) = 4\xi T \log \xi + \xi T \log \frac{T^4}{(2\pi e)^4 nX}.
\]

Then we have

\[
h_1'(\xi) = 4T \log e\xi + T \log \frac{T^4}{(2\pi e)^4 nX} = 4T \log \frac{T\xi}{2\pi(nX)^{1/4}},
\]

\[
h_1''(\xi) = 4T/\xi, \quad h_1^{(j)}(\xi) \asymp_j T, \quad j \geq 2.
\]

The solution of \( h_1'(\xi) = 0 \) is \( \xi_0 = \frac{2\pi(nX)^{1/4}}{T} \). Note that \( h_1(\xi_0) = -8\pi(nX)^{1/4} \). By the stationary phase method for the \( \xi \)-integral (see Lemma 17 (ii)), we get

\[
\mathcal{I}_2(N) = T^{1/2} \sum_{n \geq 1} \lambda_1 \Phi(n) V_2 \left( \frac{n}{N} \right) e \left( -4(nX)^{1/4} \right) V_3 \left( \frac{(nX)^{1/4}}{T} \right) + O(T^{-2020}),
\]

where \( V_3 \) is some smooth function such that \( \text{supp} \ V_3 \subset (1/20, 20) \) and \( V_3^{(k)} \ll T^{k\varepsilon} \) for \( k \geq 0 \). Hence we only need to consider \( N \asymp T^4/X \), otherwise the contribution is negligibly small. Thus if \( T \leq X^{1/4-\varepsilon} \), then we have

\[
\mathcal{I}_2(N) = O(T^{-2020}). \tag{3.7}
\]
Now we assume \( T \geq X^{1/4 - \varepsilon} \). When \( N \asymp T^4/X \), we can remove the weight function \( V_3 \) by a Mellin inversion, getting

\[
I_2(N) = T^{1/2} \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{n \geq 1} \lambda_{1 \oplus \Phi}(n) V_2 \left( \frac{n}{N} \right) e \left( -4(nX)^{1/4} \right) \tilde{V}_3(i\nu) \left( \frac{(nX)^{1/4}}{T} \right)^{-i\nu} d\nu + O(T^{-2020}).
\]

By repeated integration by parts, we can truncate \( \nu \)-integral at \(|\nu| \leq T^\varepsilon\) with a negligible error term. Hence we have

\[
I_2(N) \ll T^{1/2+\varepsilon} \sup_{|\nu| \leq T^\varepsilon} \left| \sum_{n \geq 1} \lambda_{1 \oplus \Phi}(n) V \left( \frac{n}{N} \right) e \left( -4(nX)^{1/4} \right) \right| + T^{-2020}, \quad (3.8)
\]

for some smooth function \( V \) (depending on \( \nu \)) such that \( \text{supp } V \subset (1/20, 20) \) and \( V^{(k)} \ll T^{k\varepsilon} \) for \( k \geq 0 \). Thus by (3.6) and (3.8), we get

\[
I \ll T^\varepsilon \sup_{\nu \in [\varepsilon-iT^\varepsilon, \varepsilon+iT^\varepsilon]} \sup_{N \asymp T^4/X} \sup_{|\nu| \leq T^\varepsilon} \frac{T^{1/2}}{N^{1/2}} \left| \sum_{n \geq 1} \lambda_{1 \oplus \Phi}(n) V \left( \frac{n}{N} \right) e \left( -4(nX)^{1/4} \right) \right| + O \left( T^{1+\varepsilon} \right). \quad (3.9)
\]

By (2.1), we get

\[
I \ll T^{5/2+\varepsilon} X^{-1/2} + T^{1+\varepsilon}. \quad (3.10)
\]

This proves Proposition 19 (ii).

To prove Proposition 19 (iii), we need to find a power saving in the dual sum

\[
B(N) := \sum_{n \geq 1} \lambda_{1 \oplus \Phi}(n) V \left( \frac{n}{N} \right) e \left( -4(nX)^{1/4} \right)
\]

if \( X^{5/13} \leq T \leq X^{5/12} \). Now we should use the fact \( \lambda_{1 \oplus \Phi}(n) = \sum_{\ell m = n} A(1, m) \). Applying a dyadic partition of the \( \ell \)-sum and a smooth partition of unity for the \( m \)-sum, we get

\[
B(N) = \sum_{L \leq N} \sum_{M \ll N} \sum_{L \leq \ell \leq 2L} \sum_{m \geq 1} A(1, m) W \left( \frac{m}{M} \right) V \left( \frac{\ell m}{N} \right) e \left( -4(\ell mX)^{1/4} \right).
\]

Here \( W \) is a fixed smooth function such that \( \text{supp } W \subset [1/4, 2] \) and \( W^{(k)} \ll 1 \) for \( k \geq 0 \). Because of \( \text{supp } V \subset (1/20, 20) \), we only need to consider the case \( LM \asymp N \), in which case we can remove the weight function \( V \) by a Mellin inversion as above. Hence we obtain

\[
B(N) \ll T^\varepsilon \sup_{L \ll N} \sup_{M \ll N} \sup_{-T^\varepsilon \leq \nu' \leq T^\varepsilon} |B(L, M; \nu')|, \quad (3.11)
\]
where
\[ B(L, M; v') := \sum_{L < \ell \leq 2L} \sum_{m \geq 1} A(1, m) W\left(\frac{m}{M}\right) e\left(-4(\ell m X)^{1/4}\right) \ell^{i v'} m^{i v'}. \]

We have the following estimates.

**Proposition 21** Let \( X^{5/13} \leq T \leq X^{5/12} \), \( N \asymp T^4 / X \asymp LM \). Assume \( -T^\varepsilon \leq v' \leq T^\varepsilon \). Then we have
\[ B(L, M; v') \ll T^{187/50+\varepsilon} X^{-9/10}. \]

In order to prove Proposition 21, we need the following van der Corput type estimate of exponential sums.

**Lemma 22** Let \( h \) be a smooth function on the interval \([L, 2L]\) with derivatives satisfying that \(|h^{(k)}| \asymp FL^{-k}\) for \(1 \leq k \leq 5\). Then for any subinterval \( I \) of \([L, 2L]\) we have
\[ \sum_{\ell \in I} e(h(\ell)) \ll F^{1/30} L^{5/6} + F^{-1} L. \]

**Proof** This is Theorem 2.9 in [9] with \( q = 3 \). \( \square \)

**Proof of Proposition 21 by assuming Theorem 9** If \( L \gg T^{44/25} X^{-3/5} \), then by Lemma 22 with \( F = T \), the Cauchy–Schwarz inequality for the \( m \)-sum, and (2.1), we have
\[ B(L, M; v') \leq \sum_{m \leq M} |A(1, m)| \sum_{L < \ell \leq 2L} e\left(-4(\ell m X)^{1/4} + \frac{v'}{2\pi} \log \ell\right) \ll T^\varepsilon M (T^{1/30} L^{5/6} + T^{-1} L) \ll T^\varepsilon T^{4/5} X (T^{1/30} L^{-1/6} + T^{-1}) \ll T^{187/50+\varepsilon} X^{-9/10}. \]

If \( L \ll T^{44/25} X^{-3/5} \), then
\[ M \asymp N / L \gg (T^4 / X)(T^{44/25} X^{-3/5})^{-1} \asymp T^{56/25} / X^{2/5} \geq T^{6/5} \]
provided \( T \geq X^{5/13} \). Note that \( M \leq N \ll T^4 / X \ll T^{8/5} \) provided \( T \leq X^{5/12} \). By the first claim in Theorem 9 we have
\[ B(L, M; v') \leq \sum_{L < \ell \leq 2L} \sum_{m \geq 1} |A(1, m) m^{i v'} W\left(\frac{m}{M}\right) e\left(-4(\ell m X)^{1/4}\right)| \ll LT^{3/10+\varepsilon} M^{3/4} \ll T^{3/10+\varepsilon} L^{1/4} N^{3/4} \ll T^{187/50+\varepsilon} X^{-9/10}. \]

This completes the proof of Proposition 21. \( \square \)
Proof of Proposition 19 (iii) For $X^{5/13} \leq T \leq X^{5/12}$, by (3.9), (3.11), and Proposition 21, we have

$$ I \ll T^\varepsilon \sup_{N \asymp T^{4/5}} T^{1/2} N^{1/2} T^{187/50 + \varepsilon} X^{-9/10} + T^{1 + \varepsilon} \ll T^{56/25 + \varepsilon} X^{-2/5} $$

as claimed.

4 Proof of Theorem 1

The proof is standard once we have Theorem 5. For completeness, we include the proof here. Let $\phi$ be a $\text{GL}(2)$ Hecke–Maass cusp form for $\text{SL}(2, \mathbb{Z})$ with its $n$-th Hecke eigenvalue $\lambda_\phi(n)$. Note that $\phi \times \phi = 1 \boxplus \text{Sym}^2 \phi$. By Theorem 5 with $\Phi = \text{Sym}^2 \phi$, we have

$$ \sum_{n \leq X} \lambda_\phi \times \phi(n) = L(1, \text{Sym}^2 \phi) X + O(X^{3/5 - \delta + o(1)}). \quad (4.1) $$

Moreover, by (1.2), we have

$$ \lambda_\phi(n)^2 = \sum_{\ell^2 m = n} \mu(\ell) \lambda_\phi \times \phi(m). $$

Hence by (4.1) we have

$$ \sum_{n \leq X} \lambda_\phi(n)^2 = \sum_{\ell^2 m \leq X} \mu(\ell) \lambda_\phi \times \phi(m) $$

$$ = \sum_{\ell \leq X^{1/2}} \mu(\ell) \left( L(1, \text{Sym}^2 \phi) \frac{X}{\ell^2} + O(X^{3/5 - \delta + o(1)} \ell^{-6/5 + 2\delta}) \right) $$

$$ = L(1, \text{Sym}^2 \phi) X + O(X^{3/5 - \delta + o(1)}), $$

provided $\delta < 1/10$. This completes the proof of Theorem 1.

5 Proof of Theorem 9

In this section we prove Theorem 9. We will not use the exact expression of $\varphi$ until the end of the proof.
5.1 Applying the delta method

We apply (2.7) to \( \mathcal{S}(N) \) as a device to separate the variables. By (2.7) with some large \( Q \), we get that \( \mathcal{S}(N) \) is equal to

\[
\sum_{m \geq 1} \sum_{n \geq 1} A(1, n) e \left( T \varphi \left( \frac{m}{N} \right) \right) V \left( \frac{m}{N} \right) W \left( \frac{n}{N} \right) \delta(m - n)
\]

\[
= \sum_{m \geq 1} \sum_{n \geq 1} A(1, n) e \left( t \varphi \left( \frac{m}{N} \right) \right) V \left( \frac{m}{N} \right) W \left( \frac{n}{N} \right)
\]

\[
\cdot \frac{1}{Q} \sum_{1 \leq q \leq Q} \sum_{a \mod q} e \left( \frac{(m - n)a}{q} \right) \int_{\mathbb{R}} g(q, x) e \left( \frac{(m - n)x}{qQ} \right) dx.
\]

Here \( W \) is a fixed smooth function such that \( W(x) = 1 \) if \( x \in \left[ \frac{1}{2}, 1 \right] \), \( \text{supp } W \subset \left[ \frac{1}{4}, 2 \right] \), and \( W^{(k)} \ll 1 \). By (2.1) and (2.8), we have

\[
\mathcal{S}(N) = \int_{\mathbb{R}} U \left( \frac{x}{T^{t}} \right) \frac{1}{Q} \sum_{1 \leq q \leq Q} \sum_{a \mod q} e \left( \frac{ma}{q} \right) V \left( \frac{m}{N} \right) e \left( \frac{mx}{qQ} \right)
\]

\[
\cdot \sum_{n \geq 1} A(1, n) e \left( \frac{-na}{q} \right) W \left( \frac{n}{N} \right) e \left( \frac{-nx}{qQ} \right) dx + O(T^{-A}),
\]

where \( U \) is a smooth positive function with \( U(x) = 1 \) if \( x \in [-1, 1] \), supported in \([-2, 2] \) and satisfying \( U^{(j)}(x) \ll j \).

5.2 Applications of summation formulas

Now we apply the Poisson summation formula to the \( m \)-sum in \( \mathcal{S}(N) \), getting

\[
m - \text{sum} = \sum_{b \mod q} e \left( \frac{ab}{q} \right) \sum_{m \equiv b \mod q} e \left( T \varphi \left( \frac{m}{N} \right) \right) V \left( \frac{m}{N} \right) e \left( \frac{mx}{qQ} \right)
\]

\[
= \frac{N}{q} \sum_{m \in \mathbb{Z}} \sum_{b \mod q} e \left( \frac{(m + a)b}{q} \right) \int_{\mathbb{R}} V(y) e \left( T\varphi(y) + \frac{Nx}{qQ}y \right) e \left( -\frac{mN}{q}y \right) dy
\]

\[
= N \sum_{m \equiv a \mod q} \mathcal{V}(m, q, x),
\]

where

\[
\mathcal{V}(m, q, x) := \int_{\mathbb{R}} V(y) e \left( T\varphi(y) + \frac{Nx}{qQ}y + \frac{mN}{q}y \right) dy.
\]

Assume \( Q \geq NT^{2e-1} \). Then for \( x \ll T^{e} \), we have \( \frac{Nx}{qQ} \ll T^{1-e} \). Since \( \varphi'(y)\varphi''(y) \neq 0 \) if \( y \in (1/4, 2) \), by repeated integration by parts we know that we can truncate \( m \)-sum.
at $|m| \asymp qT/N$, in which case we have $(T\varphi(y) + \frac{Nxy}{qQ} + \frac{mN}{q}y)^{''} = T\varphi^{''}(y) \gg T$, and hence

$$V(m, q, x) \ll T^{-1/2},$$  \hfill (5.1)$$

by the second derivative test (see e.g. Huxley [11, Lemma 5.1.3]). Hence we can restrict $q$-sum with $N/T \ll q \leq Q$.

Now we consider the $n$-sum. Note that we have $q \gg N/T$. Let $\psi_x(z) = W\left(\frac{n}{N}\right)e\left(-\frac{mn}{qQ}\right)$. By Lemma 13, we have

$$n-\text{sum} = \frac{q\pi^{3/2}}{2} \sum_{n_1 \mid q} \sum_{n_2 = 1}^{\infty} \sum_{n_1 \mid q} \sum_{n_2 = 1}^{\infty} \frac{A(n_2, n_1)}{n_1 n_2} S\left(-\bar{m}, \pm n_2; \frac{q}{n_1}\right) \Psi^{\pm}_x\left(\frac{n_1^2 n_2}{q^3}\right),$$

where $\Psi^{\pm}_x$ is defined as in (2.5) with $\psi = \psi_x$. Hence we have

$$\mathcal{J}(N) = \int_{\mathbb{R}} U\left(\frac{x}{T^e}\right) \frac{N}{Q} \sum_{1 \leq q \leq Q} g(q, x) \sum_{|m| \leq qT/N} \sum_{(m, q) = 1} V(m, q, x)$$

$$+ \frac{\pi^{3/2}}{2} \sum_{n_1 \mid q} \sum_{n_2 = 1}^{\infty} \sum_{n_1 \mid q} \sum_{n_2 = 1}^{\infty} \frac{A(n_2, n_1)}{n_1 n_2} S\left(-\bar{m}, \pm n_2; \frac{q}{n_1}\right) \Psi^{\pm}_x\left(\frac{n_1^2 n_2}{q^3}\right) dx + O(T^{-A}).$$

5.3 Analysis of the integrals

In this section we want to consider $\Psi^{\pm}_x(z)$. We prove the following lemma.

**Lemma 23** Let $Y \in \mathbb{R}$ and $N \geq 1$. Let $T \geq 1$ be sufficiently large. Let $\psi(n) = W\left(\frac{n/N}{e^{2}}\right)e\left(-\frac{Yn}{N}\right)$, where $W$ is a fixed smooth function, compactly supported on $[1, 2]$. Define $\Psi^{\pm}$ as in (2.5). Then we have

(i) If $zN \gg T^e$, then $\Psi^{\pm}$ is negligibly small unless $\text{sgn}(Y) = \pm$ and $\pm Y \asymp (zN)^{1/3}$, in which case we have

$$\Psi^{\pm}(z) = e\left(\pm 2 \left(\frac{(zN)^{1/2}}{(\pm Y)^{1/2}}\right)\left(\frac{(zN)^{1/2}}{(\pm Y)^{3/2}}\right) + O(T^{-A}) \ll (zN)^{1/2},$$

(5.2)$$

where $w$ is a certain compactly supported 1-inert function depending on $A$.

(ii) If $zN \ll T^e$ and $Y \gg T^e$, then $\Psi^{\pm}(z) \ll_A T^{-A}$ for any $A > 0$.

(iii) If $zN \ll T^e$ and $Y \ll T^e$, then $\Psi^{\pm}(z) \ll T^e$.

**Proof** If $zN \gg T^e$, then by Lemma 14 we have

\[ \Psi^{\pm}(z) \ll T^{-1/2} \]
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We first consider the integral above:

\[ \Psi^\pm(z) = z \int_0^\infty \psi(\xi) \sum_{\ell=1}^L \frac{\gamma_\ell}{(z\xi)^{\ell/3}} \left( \pm 3(z\xi)^{1/3} \right) d\xi + O \left( (zN)^{1-L/3} \right) \]

\[ = (zN)^{2/3} \int_0^\infty W(\xi) \sum_{\ell=1}^L \frac{\gamma_\ell (zN)^{(1-\ell)/3}}{\xi^{\ell/3}} e \left( -Y\xi \pm 3(zN\xi)^{1/3} \right) d\xi \]

\[ + O \left( (zN)^{1-L/3} \right). \]

Let

\[ h_2(\xi) = -2\pi Y\xi \pm 6\pi(zN\xi)^{1/3}. \]

Then we have

\[ h_2'(\xi) = -2\pi Y \pm 2\pi(zN)^{1/3}\xi^{-2/3}, \]

\[ h_2''(\xi) = \mp \frac{4}{3}\pi(zN)^{1/3}\xi^{-5/3}, \quad h_2^{(j)}(\xi) \asymp j (zN)^{1/3}, \quad j \geq 2. \]

For \( \xi \gg 1, \) if \( \pm \text{sgn}(Y) = -1, \) then \( h_2'(\xi) \gg (zN)^{1/3} \gg T^e. \) By repeated integration by parts we obtain \( \Psi^\pm \ll T^{-A} \) for any \( A > 0. \) Assume \( \text{sgn}(Y) = \pm. \) The solution of \( h_2'(\xi) = 0 \) is \( \xi_0 = \left( \frac{zN}{(\pm Y)^{1/2}} \right)^{1/2}. \) Note that \( h_2(\xi_0) = \pm 4\pi \left( \frac{(zN)^{1/2}}{(\pm Y)^{1/2}} \right) \). By the stationary phase method (Lemma 17), we prove Lemma 23 (i).

Now we consider the case \( zN \ll T^e \) and \( Y \gg T^e. \) Note that

\[ \Psi^\pm(z) = \frac{1}{2\pi i} \int_{(1/2)} (\pi^2 z)^{-\gamma} \gamma^\pm(s) \int_0^\infty W \left( \frac{u}{N} \right) e \left( -\frac{uY}{N} \right) u^{-s} du ds \]

\[ = (zN)^{1/2} \frac{1}{2\pi^{5/2}} \int_{\mathbb{R}} (\pi^3 zN)^{-it} Y^\pm(1/2 + i\tau) \int_0^\infty W(\xi) e (-Y\xi) \xi^{-1/2 - it} d\xi d\tau. \]

We first consider the \( \xi \)-integral above:

\[ \int_0^\infty W(\xi) e (-Y\xi) \xi^{-1/2 - it} d\xi = \int_0^\infty W(\xi) \xi^{-1/2} e^{i(-2\pi Y\xi - \tau \log \xi)} d\xi. \]

Let

\[ h_3(\xi) = -2\pi Y\xi - \tau \log \xi. \]

Then we have

\[ h_3'(\xi) = -2\pi Y - \tau / \xi, \quad h_3''(\xi) = \tau / \xi^2, \quad h_3^{(j)}(\xi) \asymp j |\tau|, \quad j \geq 2. \]

By repeated integration by parts we obtain \( \Psi^\pm \ll T^{-A} \) for any \( A > 0 \) unless \( \text{sgn}(\tau) = -\text{sgn}(Y) \) and \( |\tau| \asymp |Y| \) which we assume now. The solution of \( h_3'(\xi) = 0 \) is \( \xi_0 = \frac{\tau}{2\pi Y}. \)
Note that \( h_3(\xi_0) = \tau - \tau \log \frac{\tau}{2\pi Y} \). By Lemma 17 (ii) for the \( \xi \)-integral, we obtain

\[
\xi - \text{integral} = e^{i\tau - i\tau \log \frac{\tau}{2\pi Y}} w_1 \left( -\frac{\tau}{Y} \right) + O(T^{-A}),
\]

where \( w_1 \) is a compactly supported 1-inert function. Since \( \tau \gg T^\varepsilon \), by Stirling’s formula, we have

\[
\gamma^\pm(1/2 + i\tau) = \left( \frac{|\tau|}{2e} \right)^{3i\tau} \Upsilon(\tau) + O(T^{-A}),
\]

where \( (\Upsilon^\pm)^{(k)}(\tau) \ll |\tau|^{-k} \) for \( k \geq 0 \). Hence to bound \( \Psi^\pm(z) \), we need to consider (making a change of variable \( \tau \mapsto -\tau \))

\[
\int_{\mathbb{R}} w_1 \left( \frac{\tau}{Y} \right) \Upsilon^\pm(\tau) e^{-i\tau + i\tau \log \frac{\tau}{2\pi Y} + i\tau \log(\pi^3 z N)} - 3i\tau \log \frac{\tau}{2e} d\tau.
\]

Let

\[
h_4(\tau) = -\tau + \tau \log \frac{\tau}{2\pi Y} + \tau \log(\pi^3 z N) - 3\tau \log \frac{\tau}{2e}.
\]

Then we have

\[
\begin{align*}
\frac{h_4'(\tau)}{Y} &= \log \frac{\tau}{2\pi Y} + \log(\pi^3 z N) - 3 \log \frac{\tau}{2e} - 2 = \log \left( \frac{4\pi^2 e z N}{Y \tau^2} \right) \gg \varepsilon \log T, \\
h_4''(\tau) &= -2/\tau, \quad h_4^{(j)}(\tau) \ll |\tau|^{1-j}, \quad j \geq 2.
\end{align*}
\]

Hence \( \Psi^\pm(z) \) is negligibly small by Lemma 17 (i).

Finally we handle the case \( zN \ll T^\varepsilon \) and \( Y \ll T^\varepsilon \). By the first derivative test for the \( \xi \)-integral, we know that \( \Psi^\pm(\tau) \) is negligible unless \( |\tau| \ll T^\varepsilon \). Hence \( \Psi^\pm(z) \ll T^\varepsilon \). ☐

We now break the \( q \)-sum into dyadic segments \( R < q \leq 2R \) with \( N/T \ll R \ll Q \) and insert a smooth partition of unity for the \( x \)-integral and absorb the weight function \( U(x/T^\varepsilon) \), getting

\[
\mathcal{S}^\pm(N) \ll N^\varepsilon \sup_{T^{-100} \leq X \leq T^\varepsilon} \sup_{N/T \ll R \ll Q} |\mathcal{S}^\pm(N; X, R)| + T^{-20},
\]

where

\[
\mathcal{S}^\pm(N; X, R) := \int_{\mathbb{R}} W \left( \frac{\pm x}{X} \right) \frac{N}{Q} \sum_{q \sim R} g(q, x) \sum_{|m| = qT/N} \mathcal{V}(m, q, x)
\]

\( \mathcal{V}(m, q, x) \).
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\[ \mathcal{C}^\pm (N; X, R) = \mathcal{C}_1^\pm (N; X, R) + O(T^{-A}), \]

where

\[ \mathcal{C}_1^\pm (N; X, R) := \frac{N}{Q} \sum_{q \sim R} \int_{\mathbb{R}} W_q \left( \pm \frac{x}{X} \right) \sum_{m \gg qT/N, (m, q) = 1} \mathcal{V}(m, q, x) \]

\[ \cdot \sum_{\sigma \in \{\pm\}} \sum_{n_1|q \ n_2=1}^{\infty} A(n_2, n_1) \frac{n_1 n_2}{n_1 n_2} \mathcal{S} \left( -\bar{m}, \pm n_2; \frac{q}{n_1} \right) \Psi^\pm \left( \frac{n_1^2 n_2}{q^3} \right) dx, \]

for some compactly supported 1-inert function \( W \).

If \( R \leq QT^{-\varepsilon} \) and \( X \leq T^{-\varepsilon} \), then by (2.8) we can replace \( g(q, x) \) by 1 with a negligible error term. Hence we obtain

\[ \mathcal{C}^\pm (N; X, R) = \mathcal{C}_1^\pm (N; X, R) + O(T^{-A}), \]

with \( W_q \left( \pm \frac{x}{X} \right) = W \left( \pm \frac{x}{X} \right) \) if \( R \leq QT^{-\varepsilon} \) and \( X \leq T^{-\varepsilon} \), and \( W_q \left( \pm \frac{x}{X} \right) = W \left( \pm \frac{x}{X} \right) g(q, x) \) otherwise.

We first assume that \( NX/RQ \gg T^{2\varepsilon} \). If \( n_1^2 n_2 N/q^3 \gg T^2 \), then by (5.2) we have

\[ \Psi^\sigma \left( \frac{n_1^2 n_2}{q^3} \right) = e \left( 2\sigma \frac{(n_1^2 n_2 Q)^{1/2}}{q (\sigma x)^{1/2}} \right) \left( \frac{n_1^2 n_2 N}{q^3} \right)^{1/2} w_2 \left( \frac{(n_1^2 n_2 N)^{1/2}}{q (\sigma N x/Q)^{3/2}} \right) + O(T^{-A}), \]

for some compactly supported 1-inert function \( w_2 \). Hence the contribution to \( \mathcal{C}_1^\pm (N; X, R) \) is negligible unless \( \sigma = \text{sgn}(x) \). Thus in this case, up to a negligible error term, the contribution to \( \mathcal{C}_1^\pm (N; X, R) \) is equal to

\[ \frac{N^{3/2}}{Q} \sum_{q \sim R} \int_{\mathbb{R}} W_q \left( \pm \frac{x}{X} \right) \frac{1}{q^{3/2}} \sum_{m \gg qT/N, (m, q) = 1} \mathcal{V}(m, q, x) \]

\[ \cdot \sum_{n_1|q \ n_2=1}^{\infty} A(n_2, n_1) \frac{n_1 n_2}{n_1 n_2} \mathcal{S} \left( -\bar{m}, \pm n_2; \frac{q}{n_1} \right) e \left( \pm 2 \frac{(n_1^2 n_2 Q)^{1/2}}{q (\pm x)^{1/2}} \right) \left( \frac{(n_1^2 n_2 N)^{1/2}}{q (\pm N x/Q)^{3/2}} \right) \]dx.\]

Making a change of variable \( \pm x/X \mapsto v \), we arrive at

\[ \pm \frac{N^{3/2} X}{Q} \sum_{q \sim R} \int_{\mathbb{R}} W_q (v) \frac{1}{q^{3/2}} \sum_{m \gg qT/N, (m, q) = 1} \int_{\mathbb{R}} \mathcal{V} (y) e \left( T \varphi (y) \pm \frac{NXv y}{q Q} \pm \frac{m N y}{q} \right) dy \]

\[ \cdot \sum_{n_1|q \ n_2=1}^{\infty} A(n_2, n_1) \frac{n_1 n_2}{n_1 n_2} \mathcal{S} \left( -\bar{m}, \pm n_2; \frac{q}{n_1} \right) e \left( \pm 2 \frac{(n_1^2 n_2 Q)^{1/2}}{q (X v)^{1/2}} \right) \left( \frac{(n_1^2 n_2 N)^{1/2}}{(NX v/Q)^{3/2}} \right) dv, \]
which is equal to

\[
\pm \frac{N^{3/2}X}{Q} \sum_{q \sim R} \frac{1}{q^{3/2}} \sum_{m \equiv qT/N (m,q)=1} \int_{\mathbb{R}} V(y) e\left(T \varphi(y) + \frac{mN}{q} y\right)
\]

\[
\cdot \sum_{n_1 | q} \sum_{n_2 \sim N^2X^3/n_1^2 Q^3} A(n_2, n_1) \frac{m}{n_2^{1/2}} S\left(-\bar{m}, \pm n_2; \frac{q}{n_1}\right)
\]

\[
\cdot \int_{\mathbb{R}} W_q(v) e\left(\pm \frac{N X y}{qQ} + \frac{2}{q(Xv)^{1/2}} (n_1^2 n_2)^{1/2} w_2 \left(\frac{(N X v Q)^{3/2}}{X^3}\right)\right) dv dy,
\]  

(5.5)

Let

\[
h_5(v) = \pm \frac{N X y}{qQ} \pm 2 \frac{(n_1^2 n_2)^{1/2}}{q(Xv)^{1/2}}.
\]

Then we have

\[
h_5'(v) = \pm \frac{N X y}{qQ} + \frac{(n_1^2 n_2)^{1/2}}{q X^{1/2}} v^{3/2},
\]

\[
h_5''(v) = \pm \frac{3(n_1^2 n_2)^{1/2}}{2q X^{1/2}} v^{-5/2}, \quad h_5^{(j)}(v) \asymp j \frac{(n_1^2 n_2)^{1/2}}{q X^{1/2}}, \quad j \geq 2.
\]

By repeated integration by parts we know that the \(v\)-integral is negligibly small unless \(\frac{N X}{Q} \gtrsim \frac{(n_1^2 n_2)^{1/2}}{Q^{1/2}}, \) i.e., \(n_1^2 n_2 \gtrsim \frac{N^2X^3}{Q^3}, \) which we assume now. The solution of \(h_5'(v) = 0\) is \(v_0 = \frac{(n_1^2 n_2)^{1/2}}{X(Ny)^{1/2}}.\) Note that \(h_5(v_0) = \pm 3 \frac{(n_1^2 n_2 N y)^{1/3}}{q}.\) By the stationary phase method (Lemma 17 (ii)), we have

\[
v - \text{integral} = w \left(\frac{(n_1^2 n_2)^{1/3}}{N^{2/3} X^{1/3} y^{2/3}}\right) \frac{(q Q)^{1/2}}{(N X y)^{1/2}} e\left(\pm 3 \frac{(n_1^2 n_2 N y)^{1/3}}{q}\right) + O(T^{-A}),
\]

where the function \(w\) is some compactly supported \(T^\varepsilon\)-inert function depending on \(q\) and \(A.\) Thus it suffices to consider

\[
\pm \frac{N X^{1/2}}{Q^{1/2}} \sum_{q \sim R} \frac{1}{q} \sum_{m \equiv qT/N n_1 | q} \sum_{n_2 \sim N^2X^3/n_1^2 Q^3} A(n_2, n_1) \frac{m}{n_2^{1/2}} S\left(-\bar{m}, \pm n_2; \frac{q}{n_1}\right) W(m, n, q),
\]  

(5.6)
where

\[ \mathcal{W}(m, n_2, q) := \int_{\mathbb{R}} \frac{V(y)}{y^{1/2}} e\left(T \varphi(y) + \frac{mN}{q} y \pm 3 \frac{(n_1^2 n_2 N y)^{1/3}}{q}\right) w\left(\frac{(n_1^2 n_2)^{1/3} Q}{N^{2/3} X y^{2/3}}\right) dy. \]  

(5.7)

Note that \( \frac{(n_1^2 n_2 N)^{1/3}}{q} \times NX/RQ \ll T^\varepsilon N/RQ \ll T^{1+\varepsilon}/Q \). By the second derivative test (see e.g. Huxley [11, Lemma 5.1.3]), we get

\[ \mathcal{W}(m, n_2, q) \ll T^{-1/2+\varepsilon}. \]  

(5.8)

### 5.4 Applying the Cauchy–Schwarz inequality

By applying the Cauchy–Schwarz inequality to the \((n_1, n_2)\)-sum and (2.1), we know that (5.6) is bounded by

\[ \frac{NX^{1/2}}{Q^{1/2}} \frac{NX^{3/2}}{Q^{3/2}} T^\pm (N; X, R; M, W)^{1/2}, \]  

(5.9)

where \( T^\pm = T^\pm (N; X, R; M, W) \) is given by

\[ \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{1}{n_2} W\left(\frac{n_1^2 n_2}{M}\right) \left| \sum_{R < q \leq 2R}^{\infty} \sum_{n_1 | q}^{\infty} \frac{1}{q} \sum_{m \approx qT/N}^{\infty} S\left(-\bar{m}, \pm n_2; \frac{q}{n_1}\right) \mathcal{W}(m, n_2, q) \right|^2, \]

for some \( M \asymp N^2 X^3/Q^3 \) and smooth functions \( W \) with \( \text{supp } W \subset [1, 2] \).

**Lemma 24** We have

\[ T^\pm \ll T^\varepsilon \left(\frac{R}{N} + \frac{TQ^6}{N^{7/2}X^{5/2}}\right). \]

**Proof of Theorem 9** By (5.9), we know the contribution to \( \mathcal{I}^\pm_1 (N; X, R) \) is bounded by

\[ T^\varepsilon \frac{N^2 X^2}{Q^2} \left(\frac{R}{N} + \frac{TQ^6}{N^{7/2}X^{5/2}}\right)^{1/2} \ll T^\varepsilon \frac{N^{3/2}}{Q^{3/2}} + T^\varepsilon N^{1/4} QT^{1/2} \ll N^{3/4} T^{3/10+\varepsilon}, \]  

(5.10)

by taking \( Q = N^{1/2}/T^{1/5} \) if \( T^{6/5} \leq N \leq T^{8/5-\varepsilon} \). If \( T^{8/5-\varepsilon} \leq N \leq T^2 \), then we take \( Q = NT^{\varepsilon-1} \) and show the contribution is bounded by \( O(T^{-1/2} N^{5/4+\varepsilon}) \).

For the case \( NX/RQ \gg T^{2\varepsilon} \) and \( n_1^2 n_2 N/q^3 \ll T^\varepsilon \), or the case \( NX/RQ \ll T^{2\varepsilon} \) and \( n_1^2 n_2 N/q^3 \gg T^\varepsilon \), then the contribution to \( \mathcal{I}^\pm_1 (N; X, R) \) is negligibly small by Lemma 23.
For the case $NX/RQ \ll T^{2\varepsilon}$ and $n_1^2n_2N/q^3 \ll T^{\varepsilon}$, then by (2.1), (5.1), Lemma 23, and Weil bounds for Kloosterman sums, we can bound terms in (5.4) trivially, showing the contribution to $S_{\pm}(N; X, R)$ is bounded by

$$
\ll X \frac{N^{2/3}RT}{Q} T^{-1/2} R^{1/2} \ll \frac{Q^{7/2} T^{1/2}}{N} \ll N^{3/4} T^{3/10 + \varepsilon},
$$

by taking $Q = N^{1/2}/T^{1/5}$. This completes the proof of Theorem 9. \qed

5.5 Applying Poisson again

In this section we prove Lemma 24.

**Proof of Lemma 24** Opening the absolute square and interchanging the order of summations, we get

$$
T_{\pm} \leq \sum_{n_1 \leq 2R} \sum_{R < q \leq 2R} \frac{1}{q} \sum_{m \equiv qT/N} \sum_{n_1 | q'} \frac{1}{q'} \sum_{m' \equiv q'T/N} |\Sigma|,
$$

where

$$
\Sigma := \sum_{n_2=1}^{\infty} \frac{1}{n_2} W\left(\frac{n_1^2 n_2}{M}\right) S\left(-\bar{m}, \pm n_2; \frac{q}{n_1}\right) S\left(\bar{m}', \mp n_2; \frac{q'}{n_1}\right) W(m, n_2, q) W(m', n_2, q').
$$

Breaking the $n_2$-sum modulo $qq'/n_1^2$, we obtain

$$
\Sigma = \sum_{b \mod qq'/n_1^2} \sum_{n_2 \in \mathbb{Z}} W\left(\frac{n_1^2 b + n_2 qq'}{M}\right) S\left(-\bar{m}, \pm b; \frac{q}{n_1}\right) S\left(\bar{m}', \mp b; \frac{q'}{n_1}\right)
\cdot W(m, b + n_2 qq'/n_1^2, q) W(m', b + n_2 qq'/n_1^2, q').
$$

Applying the Poisson summation formula to the $n_2$-sum, we get

$$
\Sigma = \sum_{b \mod qq'/n_1^2} S\left(-\bar{m}, \pm b; \frac{q}{n_1}\right) S\left(\bar{m}', \mp b; \frac{q'}{n_1}\right) \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} W\left(\frac{n_1^2 b + \xi qq'}{M}\right) \frac{1}{b + \xi qq'/n_1^2} e(-n\xi) d\xi.
$$
Making a change of variables $\frac{n^2 b + \xi qq'}{M} \mapsto \xi$, we have

$$
\Sigma = \frac{n^2}{qq'} \sum_{n \in \mathbb{Z}} \mathcal{C}(n, m, m' ; q/q_1, q'/q_1) \mathfrak{I}(n, m, m' ; q, q'),
$$

where

$$
\mathcal{C}(n, m, m' ; q, q') := \sum_{b \bmod qq'/n^2_1} S(\bar{m}, \pm b; \hat{q}) S(\bar{m}', \mp b; \hat{q}') e\left(\frac{nb}{qq'}\right),
$$

and

$$
\mathfrak{I}(n, m, m' ; q, q') := \int_{-\infty}^{\infty} W(\xi) \mathcal{W}\left(m, \frac{M}{n^2_1}, q\right) \mathcal{W}\left(m', \frac{M}{n^2_1}, q'\right) e\left(-n \frac{M \xi}{qq'}\right) d\xi.
$$

The character sum $\mathcal{C}$ has already appeared in many places, see e.g. [23, Lemma 11]. We have the following bounds.

**Lemma 25** We have

$$
\mathcal{C}(n, m, m' ; q, q') \ll \hat{q}\hat{q}'(\hat{q}, \hat{q}', n).
$$

Moreover if $n = 0$ then we get $\mathcal{C}(n, m, m' ; q, q') = 0$ unless $\hat{q} = \hat{q}'$, in which case we have

$$
\mathcal{C}(n, m, m' ; q, q') \ll \hat{q}^2(\hat{q}, m - m').
$$

We also need bounds for the integrals $\mathfrak{I}(n, m, m' ; q, q')$. We will prove the following lemma.

**Lemma 26** We have

(i) If $n \geq CRQ^2/NX^2$ for some large constant $C > 0$, then $\mathfrak{I}(n, m, m' ; q, q') \ll n^{-2}T^{-2020}$.

(ii) Assume $V$ is $P$-inert for some $P \ll T^\eta$ with some positive constant $\eta < 1/10$. Assume $Q \geq T^{1/3 + \varepsilon}P^{2/3}$. If $T^e R^2 Q^3/N^2 X^3 \ll n \ll R Q^2/NX^2$, then

$$
\mathfrak{I}(n, m, m' ; q, q') \ll T^{-1}(nM/R^2)^{-1/2}.
$$

(iii) If $n \ll T^e R^2 Q^3/N^2 X^3$, then $\mathfrak{I}(n, m, m' ; q, q') \ll T^{-1+\varepsilon}$.

**Proof** If $n \ll T^e R^2 Q^3/N^2 X^3$, then by (5.8), we get $\mathfrak{I}(n, m, m' ; q, q') \ll T^{-1+\varepsilon}$. This proves Lemma 26 (iii).
If \( n \geq C R Q^2 / N X^2 \) for some large constant \( C > 0 \), then we have \( \frac{M_n}{qq'} \geq C' \frac{N X}{R Q} \) for some large constant \( C' > 0 \). By (5.7), we have

\[
\mathcal{J}(n, m, m'; q, q') = \int_{\mathbb{R}} \frac{V(y)}{y^{1/2}} e\left( T \varphi(y) + \frac{mN}{q} y \right) \int_{\mathbb{R}} \frac{V(y')}{y'^{1/2}} e\left( -T \varphi(y') - \frac{m'N}{q'} y' \right) \]
\[
\cdot \int_{-\infty}^{\infty} \frac{W(\xi)}{\xi} w\left( \left( \frac{M \xi}{N^{2/3} X y^{2/3}} \right)^{1/3} \right) w\left( \left( \frac{M' \xi}{N^{2/3} X y^{2/3}} \right)^{1/3} \right) e\left( \pm 3 \left( \frac{MN y \xi}{q} \right)^{1/3} - 3 \left( \frac{MN y' \xi}{q'} \right)^{1/3} - n \frac{M \xi}{qq'} \right) d\xi dy dy'.
\]

Note that by \( M \asymp N^2 X^3 / Q^3 \) we have \( \frac{(MNy)^{1/3}}{q} \ll \frac{NX}{RQ} \). Let

\[
h_6(\xi) = \pm 3 \left( \frac{MN y \xi}{q} \right)^{1/3} - 3 \left( \frac{MN y' \xi}{q'} \right)^{1/3} - n \frac{M \xi}{qq'}.
\]

Then we have

\[
h_6'(\xi) = \pm \frac{(MN y)^{1/3}}{q} \xi^{-2/3} \mp \frac{(MN y')^{1/3}}{q'} \xi^{-2/3} - n \frac{M}{qq'} \geq \frac{NX}{RQ},
\]
\[
h_6''(\xi) = \mp \frac{2(MNy)^{1/3}}{3q} \xi^{-5/3} \pm \frac{2(MNy')^{1/3}}{3q'} \xi^{-5/3} \leq \frac{NX}{RQ},
\]
\[
h_6^{(j)}(\xi) \ll \frac{NX}{RQ}, \quad j \geq 2.
\]

Note that the weight function \( \frac{W(\xi)}{\xi} w\left( \left( \frac{M \xi}{N^{2/3} X y^{2/3}} \right)^{1/3} \right) w\left( \left( \frac{M' \xi}{N^{2/3} X y^{2/3}} \right)^{1/3} \right) \) in \( \xi \) is \( T^\epsilon \)-inert. By Lemma 17 (i), we prove Lemma 26 (i).

Now we assume \( Q \geq T^{1/3} \) and \( T^\epsilon R^2 Q^3 / N^2 X^3 \ll n \ll R^2 Q^2 / N X^2 \). We also assume \( V \) is \( P \)-inert and \( \varphi(u) = u^\beta \). We write \( W \left( m, \frac{M}{n \xi}, q \right) \) as

\[
\int_{\mathbb{R}} V_1(y) e\left( T y^\beta + \frac{mN}{q} y \pm 3 \left( \frac{MN y \xi}{q} \right)^{1/3} \right) dy,
\]

where \( V_1(y) = \frac{V(y)}{y^{1/2}} w\left( \left( \frac{M \xi}{N^{2/3} X y^{2/3}} \right)^{1/3} \right) \) is a \( P \)-inert function. Let

\[
A = -mN/q, \quad B = \pm \left( \frac{MN \xi}{q} \right)^{1/3}, \quad \text{and} \quad h_7(y) = T y^\beta - Ay + 3By^{1/3}.
\]

Note that \( B \asymp NX/RQ \ll T^{1+\epsilon}/Q \leq T^{2/3+\epsilon} \). We have

\[
h_7'(y) = \beta T y^{\beta - 1} - A + B y^{-2/3} \gg T + |A|,
\]

\[\mathcal{J}^{(m)}(n, m; q, q') \ll \mathcal{J}(n, m; q, q') \ll \mathcal{J}(n, m; q, q'),\]
unless $A \asymp T$. Note that $h_7^{(j)}(y) \asymp_{\beta,j} T$ for $y \asymp 1$. Hence by Lemma 17 (i), $\mathcal{W}$ is negligibly small unless $A \asymp T$. Since $Q \geq T^{1/3}$, we have $B^4/A^3 = O(T^\varepsilon/Q)$. The solution to $h_7'(y) = 0$, i.e., $T \beta y^{\beta - 1} - A + By^{-2/3} = 0$ in the support of $V_1$ will be $y_0 + y_1 + y_2 + y_3$ where

$$y_0 = \left(\frac{A}{T \beta}\right)^{\frac{1}{\beta - 1}}, \quad y_1 = \frac{y_0^{1/3}}{1 - \beta} \frac{B}{A},$$

$$y_2 = \frac{(2 - 3 \beta)}{6(\beta - 1)^2 y_0^{1/3}} \frac{B^2}{A^2}, \quad y_j \ll \left(\frac{B}{A}\right)^j, \quad 0 \leq j \leq 3.$$  

Here $y_0$ satisfies that $T \beta y_0^{\beta - 1} - A = 0$, $y_1$ satisfies that $T \beta y_0^{\beta - 1} (1 + (\beta - 1)y_1/y_0) - A + By_0^{-2/3} = 0$, and $y_2$ satisfies that $T \beta y_0^{\beta - 1} (1 + (\beta - 1)y_1/y_0 + (\beta - 1)(\beta - 2) y_1^2/2 y_0^2 + O((B/A)^3)) - A + By_0^{-2/3} (1 - 2 y_1/3 y_0) = 0$. Note that

$$0 = T \beta (y_0 + y_1 + y_2 + y_3)^{\beta - 1} - A + B(y_0 + y_1 + y_2 + y_3)^{-2/3} = T \beta y_0^{\beta - 1} \left(1 + (\beta - 1) \frac{y_1 + y_2 + y_3}{y_0} + \frac{(\beta - 1)(\beta - 2)}{2} \frac{y_1^2}{y_0^2} + O((B/A)^3)\right)$$

$$- A + By_0^{-2/3} \left(1 - \frac{2 y_1}{3 y_0} + O((B/A)^2)\right) = T \beta y_0^{\beta - 1} (\beta - 1) \frac{y_3}{y_0} + O(B^3/A^2).$$  

This give $y_3 \ll (B/A)^3$. Hence

$$h_7(y_0 + y_1 + y_2 + y_3) = T y_0^\beta (1 + y_1/y_0 + y_2/y_0 + y_3/y_0)^\beta - A(y_0 + y_1 + y_2 + y_3) + 3B(y_0 + y_1 + y_2)^{1/3} + O(B^4/A^3)$$

$$= \left(\frac{1}{\beta - 1}\right) A y_0 + 3y_0^{1/3} B + \frac{B^2}{2(1 - \beta)y_0^{1/3}} - \frac{\beta}{6(\beta - 1)^2 y_0^{1/3}} \frac{B^3}{A^2} + O(T^\varepsilon/Q).$$  

Note that $h_7''(y) = \beta(\beta - 1)T y^{\beta - 2} - \frac{2}{3}B y^{-5/3}$. We have

$$h_7''(y_0 + y_1 + y_2 + y_3) = \beta(\beta - 1)T y_0^{\beta - 2} (1 + O(B/A)).$$  

By the stationary phase method (see Lemma 17 (ii)), we get

$$\mathcal{W}\left(m, \frac{M}{n_1^{2\varepsilon}}, q, \frac{1}{\sqrt{T}}\right) = \frac{e(h_7(y_0 + y_1 + y_2 + y_3))}{w_1(y_0 + y_1 + y_2 + y_3)(1 + O(B/A))} + O(T^{-2020}),$$  

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for certain $P$-inert function $w_1$. Note that $y_1 + y_2 + y_3 \ll B/A \ll T^\varepsilon/Q$, we have $w_1(y_0 + y_1 + y_2 + y_3) = w_1(y_0) + O(PT^\varepsilon/Q)$. Hence we obtain

$$
\mathcal{W}(m, M/n_1^2, q) = \frac{1}{\sqrt{T}} e\left(3y_0^{1/3} B + \frac{1}{2(1-\beta)y_0^{1/3}} B^2 A - \frac{\beta}{6(\beta-1)^2 y_0 A^2} B^3\right) w_2(y_0)
+ O\left(\frac{PT^\varepsilon}{T^{1/2} Q}\right),
$$

where $w_2(y_0) = e((\frac{1}{\beta} - 1)A_1 y_0)w_1(y_0)$. Hence

$$
\mathcal{Y}(n, m', q', q) \ll \frac{1}{T} \int_{-\infty}^{\infty} W(\xi) e\left(3y_0^{1/3} B + \frac{1}{2(1-\beta)y_0^{1/3}} B^2 A - \frac{\beta}{6(\beta-1)^2 y_0 A^2} B^3\right)
+ e\left(-3y_0^{1/3} B' - \frac{1}{2(1-\beta)y_0^{1/3}} B^2 A' + \frac{\beta}{6(\beta-1)^2 y_0 A'^2} B^3\right) e\left(-\frac{M\xi}{qq'}\right) d\xi + \frac{PT^\varepsilon}{T Q},
$$

where $A', B', y_0'$ are defined the same as $A, B, y_0$ but with $m, q$ replaced by $m', q'$. Note that we assume $n \ll RQ^2/NX^2$ and $Q \geq T^{1/3+\varepsilon} P_{2/3}$, hence we have

$$
(nM/R^2)^{1/2} \ll (NX/RQ)^{1/2} \ll T^{1/2+\varepsilon}/Q^{1/2} \ll Q/P,
$$

that is,

$$
T^\varepsilon PT^{-1} Q^{-1} \ll T^{-1}(nM/R^2)^{-1/2}.
$$

Now we deal with the $\xi$-integral. Let

$$
h_8(\xi) = 3y_0^{1/3} B + \frac{1}{2(1-\beta)y_0^{1/3}} B^2 A - \frac{\beta}{6(\beta-1)^2 y_0 A^2} B^3
- 3y_0^{1/3} B' - \frac{1}{2(1-\beta)y_0^{1/3}} B^2 A' + \frac{\beta}{6(\beta-1)^2 y_0 A'^2} B^3 - nM\xi/qq'.
$$

Recall that $B = \pm (MN\xi)^{1/3} q$ and $B' = \pm (MN\xi)^{1/3} q'$. By $R \gg N/T$ and $NX/RQ \gg T^\varepsilon$, we get $X \gg QT^{\varepsilon-1}$. So we obtain for $|n| \geq 1$,

$$
nM/qq' \gg \frac{N^2X^3}{R^2Q^3} \gg T^\varepsilon \frac{N^2X^2}{R^2Q^2T} \times T^\varepsilon B^2/T.
$$

Hence

$$
h'_8(\xi) = \pm y_0^{1/3} (MN)^{1/3} \frac{q}{q'} \xi^{-2/3} + \frac{1}{2(1-\beta)y_0^{1/3}} \frac{1}{A} (MN)^{2/3} \frac{q}{q'} \xi^{-1/3} \mp \frac{\beta}{6(\beta-1)^2 y_0 A^2} \frac{1}{q^3} \frac{MN}{q'}
$$

$$
\mp y_0^{1/3} (MN)^{1/3} \frac{q}{q'} \xi^{-2/3} - \frac{1}{2(1-\beta)y_0^{1/3}} \frac{1}{A'} (MN)^{2/3} \frac{q}{q'} \xi^{-1/3}
$$

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\[ \pm \frac{\beta}{6(\beta - 1)^2} y_0' \frac{1}{A^2} \frac{M N}{q^3} - \frac{n M}{qq'} = \pm \left( y_0^{1/3} \frac{(MN)^{1/3}}{q} - y_0'^{1/3} \frac{(MN)^{1/3}}{q'} \right) \xi^{-2/3} - \frac{n M}{qq'} + O \left( \frac{B^2}{T} \right). \]

and

\[ h''(\xi) = \mp 2 \left( \frac{1}{3} \left( y_0^{1/3} \frac{(MN)^{1/3}}{q} - y_0'^{1/3} \frac{(MN)^{1/3}}{q'} \right) \right) \xi^{-5/3} + O \left( \frac{B^2}{T} \right). \]

Since \( n M / qq' \gg T^\varepsilon \), by repeated integration by parts we know that \( J(n, m, m'; q, q') \) is negligibly small unless

\[ \left( y_0^{1/3} \frac{(MN)^{1/3}}{q} - y_0'^{1/3} \frac{(MN)^{1/3}}{q'} \right) \approx n M / R^2, \quad (5.12) \]

in which case we have \( h^{(j)}(\xi) \approx j n M / R^2 \) for \( j \geq 2 \), and hence by Lemma 17 (ii) we get \( J(n, m, m'; q, q') \ll T^{-1} (n M / R^2)^{-1/2} \). This completes the proof of Lemma 26.

Now we are ready to bound \( \Sigma \). Depending on \( n = 0, n \) small, and \( n \) large, we have

\[ \Sigma \ll \Sigma_0 + \Sigma_1 + \Sigma_2 + T^{-A}, \]

where

\[ \Sigma_0 \ll \frac{n_1^2}{R^2} \sum_{1 \leq n \ll R^2 Q^3 / N X^3} T^\varepsilon \left( q / n_1, m - m' \right) T^{-1+\varepsilon} \ll T^{-1+\varepsilon} \delta_{q=q'}(q / n_1, m - m'), \]

\[ \Sigma_1 \ll \frac{n_1^2}{R^2} \sum_{1 \leq n \ll R^2 Q^3 / N X^3} T^\varepsilon (q / n_1, q'/n_1, n) T^{-1+\varepsilon} \ll T^{-1+\varepsilon} \sum_{1 \leq n \ll R^2 Q^3 / N X^3} T^\varepsilon (q / n_1, q'/n_1, n), \]

and

\[ \Sigma_2 \ll \frac{n_1^2}{R^2} \sum_{\frac{R^2 Q^3}{N X^3} \ll n \ll \frac{R Q^2}{N X}} T^\varepsilon \left( q / n_1, q'/n_1, n \right) \frac{R Q^{3/2}}{n^{1/2} N X^{3/2}} \frac{1}{T} \]

\[ \ll \frac{1}{T} \sum_{\frac{R^2 Q^3}{N X^3} \ll n \ll \frac{R Q^2}{N X}} (q / n_1, q'/n_1, n) \frac{R Q^{3/2}}{n^{1/2} N X^{3/2}}. \]
Denote the corresponding contribution from $\Sigma_j$ to $T^\pm$ by $T_j$. We first bound $T_0$. By (5.11), we get

$$T_0 \ll \frac{1}{R^2} \sum_{n_1 \leq 2R} \sum_{R < q \leq 2R} \sum_{m \equiv qT/N \pmod{m'}} \sum_{(m,q) = 1} T^{-1+\varepsilon} \left( q/n_1, m - m' \right).$$

Arguing depending on whether $m = m'$ or not, we have

$$T_0 \ll T^{-1+\varepsilon} \frac{1}{R} \sum_{n_1 \leq 2R} \frac{1}{n_1} \sum_{R < q \leq 2R} \sum_{m \equiv qT/N \pmod{m'}} \sum_{(m,q) = 1} 1 + T^{-1+\varepsilon} \frac{1}{R^2} \sum_{n_1 \leq 2R} \sum_{R < q \leq 2R} \sum_{m \equiv qT/N \pmod{m'}} \sum_{(m,q) = 1} \sum_{1 \leq |h| \ll R/T} (q/n_1, h) \ll \frac{R}{N} T^\varepsilon. \quad (5.13)$$

Next, we consider $T_1$. By (5.11), we obtain

$$T_1 \ll \frac{1}{R^2} \sum_{n_1 \leq 2R} \sum_{R < q \leq 2R} \sum_{m \equiv qT/N \pmod{m'}} \sum_{R < q' \leq 2R} \sum_{m' \equiv q'T/N \pmod{m'}} T^{-1+\varepsilon} \sum_{1 \leq n \ll \frac{n^2 q^3}{N^2 X^3}} (q/n_1, q'/n_1, n) \ll T^\varepsilon \frac{R^4 T Q^3}{N^4 X^3}. \quad (5.14)$$

Finally, we treat $T_2$, getting

$$T_2 \ll \frac{1}{R^2} \sum_{n_1 \leq 2R} \sum_{R < q \leq 2R} \sum_{m \equiv qT/N \pmod{m'}} \sum_{R < q' \leq 2R} \sum_{m' \equiv q'T/N \pmod{m'}} \sum_{1 \leq n \ll \frac{n^2 q^3}{N^2 X^3}} (q/n_1, q'/n_1, n) \frac{R Q^{3/2}}{n^{1/2} N X^{3/2}} \ll \frac{TR^{7/2} Q^{5/2}}{N^{7/2} X^{5/2}}. \quad (5.15)$$

Note that $T^\varepsilon \frac{R^4 T Q^3}{N^4 X^3} \ll \frac{TR^{7/2} Q^{5/2}}{N^{7/2} X^{5/2}}$ if $NX/RQ \gg T^{2\varepsilon}$. Combining (5.13), (5.14) and (5.15), we complete the proof of Lemma 24. \hfill $\square$

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