PURE SUBRINGS OF DU BOIS SINGULARITIES
ARE DU BOIS SINGULARITIES
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Abstract. Let \( R \to S \) be a cyclically pure map of rings essentially of finite type over the complex numbers \( \mathbb{C} \). In this paper, we show that if \( S \) has Du Bois singularities, then \( R \) has Du Bois singularities. Our result is new even when \( R \to S \) is faithfully flat. As a consequence, we show that under the same hypotheses on \( R \to S \), if \( S \) has log canonical type singularities and \( K_R \) is Cartier, then \( R \) has log canonical singularities.

1. Introduction

Let \( R \to S \) be a cyclically pure map of rings, which we recall means that \( IS \cap R = I \) for every ideal \( I \subseteq R \) [Hoc77, p. 463]. Examples of cyclically pure maps include inclusions of rings of invariants by linearly reductive group actions, split maps, and faithfully flat maps. By [HR74; Kem79; HH95; HM18], if \( S \) is regular, then \( R \) is Cohen–Macaulay. An interesting question arising from [HR74] and these subsequent results is the following:

Question 1.1. If \( R \to S \) is a cyclically pure map of rings, what properties descend from \( S \) to \( R \)?

Here we say a property \( P \) descends from \( S \) to \( R \) if whenever \( S \) satisfies \( P \), so does \( R \). Note that the preceding example is not precisely a descending property, since the property on \( S \) is regularity whereas the property on \( R \) is Cohen–Macaulayness (and neither regularity nor Cohen–Macaulayness descend along cyclically pure maps in general [HR74, §2]). On the other hand, Noetherianity and normality descend from \( S \) to \( R \) [HR74, Proposition 6.15]. Question 1.1 has attracted particular attention for different classes of singularities. For example:

(I) Boutot showed that if \( R \) and \( S \) are essentially of finite type over a field of characteristic zero, and \( S \) has rational singularities, then \( R \) has rational singularities [Bou87, Théorème on p. 65]. Using techniques from this paper, the second author showed Boutot’s result holds more generally for \( \mathbb{Q} \)-algebras [Mur, Theorem C]. For other extensions of Boutot’s result, see [Smi97; Sch08; HM18].

(II) Recently, Zhuang showed that if \( R \) and \( S \) are essentially of finite type over an algebraically closed field of characteristic zero, and \( S \) has klt type singularities, then \( R \) has klt type singularities [Zhu, Theorem 1.1]. In the appendix to [Zhu], Lyu extended Zhuang’s result to maps of excellent \( \mathbb{Q} \)-algebras with dualizing complexes [Zhu, Theorem A.1]. These results extend results from [Kaw84; Sch05; BGLM].

In this paper, we answer Question 1.1 for Du Bois singularities by proving the following analogue of Boutot’s theorem.

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**Theorem A.** Let \( R \to S \) be a cyclically pure map of rings essentially of finite type over \( \mathbb{C} \). If \( S \) has Du Bois singularities, then \( R \) has Du Bois singularities.

Theorem A is new even when \( R \to S \) is faithfully flat. The case when \( R \to S \) splits is due to Kovács [Kov99, Corollary 2.4], who also proves that Du Bois singularities descend along morphisms \( f : Y \to X \) of schemes when \( \mathcal{O}_X \to Rf_\ast \mathcal{O}_Y \) splits in the derived category. In Theorem 3.2, we extend Kovács’s result to morphisms for which \( \mathcal{O}_X \to Rf_\ast \mathcal{O}_Y \) induces injective maps on local cohomology, and also prove a version for Du Bois pairs.

Theorem A is in contrast to the situation in positive characteristic. The positive characteristic MSS17 is new even when \( f \) holds for \( \ast \). KS16.

Kovács 2011 denotes the morphism of schemes \( f : Y \to X \). BGLM has a negative complex of \( R \) has Du Bois singularities, then

Theorem A. Let \( f : X \to Y \) denote the morphism of affine schemes corresponding to \( R \to S \). Let \( U \) be the Cartier locus of \( K_Y \). If \( Y \) has log canonical type singularities, \( K_Y \) is \( \mathbb{Q} \)-Cartier, and \( Y \setminus f^{-1}(U) \) has codimension at least two in \( Y \), then \( X \) has log canonical singularities.

**Conventions.** All rings are commutative with identity, and all ring maps are unital. A map \( M \to M' \) of \( R \)-modules over a ring \( R \) is pure if the base change \( N \otimes_R M \to N \otimes_R M' \) is injective for every \( R \)-module \( N \).

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2. **Du Bois pairs**

We define Du Bois pairs following [Kov11, §§3.C–3.D; KS16, §2C; MSS17, §2.1].

**Definition 2.1.** A (generalized) pair \((X, \Sigma)\) consists of a separated scheme essentially of finite type over \( \mathbb{C} \) together with a closed subscheme \( \Sigma \subseteq X \). A morphism of pairs \( f : (X, \Sigma_X) \to (Y, \Sigma_Y) \) is a morphism of schemes \( f : X \to Y \) such that \( f(\Sigma_X) \subseteq \Sigma_Y \).

Given a generalized pair \((X, \Sigma_X)\), there exists an object \( \Omega^0_{X, \Sigma_X} \) in \( D^b_{\text{coh}}(X) \), called the Du Bois complex of \((X, \Sigma_X)\), such that for every morphism \( f : (X, \Sigma_X) \to (Y, \Sigma_Y) \) of generalized pairs, there exists a natural morphism \( \Omega^0_{X, \Sigma_X} \to Rf_\ast \Omega^0_{Y, \Sigma_Y} \) of objects in \( D(X) \) that fits into the commutative diagram

\[
\begin{array}{ccc}
\mathcal{I}_{\Sigma_X \subseteq X} & \longrightarrow & \Omega^0_{X, \Sigma_X} \\
\downarrow & & \downarrow \\
Rf_\ast \mathcal{I}_{\Sigma_Y \subseteq Y} & \longrightarrow & Rf_\ast \Omega^0_{Y, \Sigma_Y}
\end{array}
\]  

(1)

by [Kov11, Proposition 3.15].
Definition 2.2. Let \((X, \Sigma)\) be a pair. We say that the pair \((X, \Sigma)\) has Du Bois singularities if \(\mathcal{I}_{\Sigma} \subseteq_X \mathcal{O}_{X,\Sigma}^0\) is a quasi-isomorphism. If \((X, \emptyset)\) is a Du Bois pair, we say that \(X\) has Du Bois singularities. The Du Bois defect of the pair \((X, \Sigma)\) is the mapping cone \(\Omega_{X,\Sigma}^+\) of the morphism \(\mathcal{I}_{\Sigma} \subseteq_X \mathcal{O}_{X,\Sigma}^0\).

We show the following characterization of Du Bois pairs inspired by the proofs of [Kov11, Theorem 5.4] and [Kov12, Theorem 2.5]. See also [Kov99, Lemma 2.2; BST17, Theorem 4.8] for related statements when \(\Sigma_X = \emptyset\).

Theorem 2.3. Let \(X\) be a separated scheme essentially of finite type over \(\mathbb{C}\). Consider a closed subscheme \(\Sigma_X \subseteq X\). The following are equivalent:

(i) \((X, \Sigma_X)\) has Du Bois singularities.

(ii) The natural morphism \(\mathcal{I}_{\Sigma_X} \rightarrow \Omega_{X,\Sigma_X}^0\) admits a left inverse in \(D(X)\).

(iii) For every \(x \in X\), the natural morphism

\[
H^i_x(\text{Spec}(\mathcal{O}_{X,x}), \mathcal{I}_{\Sigma_X,x}) \rightarrow \mathbb{H}^i_x(\text{Spec}(\mathcal{O}_{X,x}), \Omega_{X,\Sigma_X,x}^0)
\]

is injective.

Proof. \((i) \Rightarrow (ii)\) follows by the definition of a Du Bois pair, and \((ii) \Rightarrow (iii)\) holds because the morphism (2) also admits a left inverse.

It remains to show \((iii) \Rightarrow (i)\). By Noetherian induction and the fact that being a Du Bois pair is local, it suffices to show that if \(x \in X\) is a point, and \((X, \Sigma_X)\) has Du Bois singularities at \(y\) for every proper generalization \(y \sim x\), then \((X, \Sigma_X)\) has Du Bois singularities at \(x\). Replacing \(X\) by \(\text{Spec}(\mathcal{O}_{X,x})\), we may assume that \(X\) is the spectrum of a local ring \((R, m)\) essentially of finite type over \(\mathbb{C}\) such that \((X, \Sigma_X)\) has Du Bois singularities away from \(\{m\}\), that is, \(U := X \setminus \{m\}\) has Du Bois singularities.

By the hypothesis in \((iii)\) and by the Matlis dual of [MSS17, Lemma 3.2], we know that

\[
H^i_X(X, \mathcal{I}_{\Sigma_X}) \rightarrow \mathbb{H}^i_m(X, \Omega_{X,\Sigma_X}^0)
\]

is an isomorphism for all \(i\). By the long exact sequence on local cohomology associated to the exact triangle

\[
\mathcal{I}_{\Sigma_X} \rightarrow \Omega_{X,\Sigma_X}^0 \rightarrow \Omega_{X,\Sigma_X}^+ \rightarrow 1
\]

we therefore see that

\[
\mathbb{H}^i_m(X, \Omega_{X,\Sigma_X}^+) = 0
\]

for all \(i\). Now consider the long exact sequence

\[
\cdots \rightarrow \mathbb{H}^i_m(X, \Omega_{X,\Sigma_X}^X) \rightarrow \mathbb{H}^i(X, \Omega_{X,\Sigma_X}^X) \rightarrow \mathbb{H}^i(U, \Omega_{\Sigma_U}^X) \rightarrow \cdots
\]

where \(\Sigma_U := U \cap \Sigma_X\). Since \(\mathbb{H}^i(U, \Omega_{\Sigma_U}^X) = 0\) for all \(i\) by the inductive hypothesis, we see that \(\mathbb{H}^i(X, \Omega_{X,\Sigma_X}^X) = 0\) for all \(i\) as well, i.e., \((X, \Sigma_X)\) has Du Bois singularities.

\(\square\)

3. Proofs of theorems

We prove a version of [Kov99, Corollary 2.4; KK10, Theorem 1.6; Kov12, Theorem 3.3] that replaces splitting conditions with conditions on injectivity of maps on local cohomology. Note that [KK10; Kov12] assume \(f\) is proper, which we do not need in the statement below.

Proposition 3.1. Let \(f : Y \rightarrow X\) be a surjective morphism between separated schemes essentially of finite type over \(\mathbb{C}\). Consider a closed subscheme \(\Sigma_X \subseteq X\) and set \(\Sigma_Y := f^{-1}(\Sigma_X)\). Suppose that for every \(x \in X\), the natural morphism

\[
H^i_x(\text{Spec}(\mathcal{O}_{X,x}), \mathcal{I}_{\Sigma_X,x}) \rightarrow \mathbb{H}^i_x(\text{Spec}(\mathcal{O}_{X,x}), (\mathcal{R}f_*\mathcal{I}_{\Sigma_Y})_x)
\]

(3)
is injective. If \((Y, \Sigma_Y)\) has Du Bois singularities, then \((X, \Sigma_X)\) has Du Bois singularities. In particular, if \((Y, \Sigma_Y)\) has Du Bois singularities, then \(X\) has Du Bois singularities if and only if \(\Sigma_X\) has Du Bois singularities.

Proof. By the functoriality of \(\Omega_{X,\Sigma_X}^0\) in (1), we have the commutative diagram

\[
\begin{array}{c}
\mathbb{H}_x^i(\text{Spec}(\mathcal{O}_{X,x}), \Omega_{X,\Sigma_X}^0) \\
\uparrow \\
H_x^i(\text{Spec}(\mathcal{O}_{X,x}), \mathcal{I}_{\Sigma_X,x}) \\
\downarrow \\
\mathbb{H}_x^i(\text{Spec}(\mathcal{O}_{X,x}), (\mathcal{R}f_\ast \Omega_{Y,\Sigma_Y}^0)_x)
\end{array}
\]

for every \(i\), where the right vertical arrow is an isomorphism by the assumption that \((Y, \Sigma_Y)\) has Du Bois singularities, and the bottom horizontal arrow is injective by hypothesis. By the commutativity of the diagram, we see that the left vertical arrow is injective for every \(i\) and every \(x \in X\), and hence \((X, \Sigma_X)\) is Du Bois by Theorem 2.3. The “in particular” statement follows by considering the exact triangle of Du Bois defects [Kov11, Definition 3.11]

\[
\Omega_{X,\Sigma_X}^x \to \Omega_X^x \to \Omega_{\Sigma_X}^{x+1} \to 1.
\]

Using Proposition 3.1, we can prove a more general version of Theorem A for pairs. The statement (i) is a version of [Kov99, Corollary 2.4; KK10, Theorem 1.6; Kov12, Theorem 3.3].

**Theorem 3.2.** Let \(f : Y \to X\) be a surjective morphism between separated schemes essentially of finite type over \(\mathbb{C}\). Consider a closed subscheme \(\Sigma_X \subseteq X\) and set \(\Sigma_Y := f^{-1}(\Sigma_X)\). Assume one of the following holds:

(i) The natural morphism \(\mathcal{I}_{\Sigma_X} \to \mathcal{R}f_\ast \mathcal{I}_{\Sigma_Y}\) admits a left inverse in \(D(X)\).

(ii) \(f\) is affine, and for every affine open subset \(U \subseteq X\), the \(H^0(U, \mathcal{O}_X)\)-module map

\[
H^0(U, \mathcal{I}_{\Sigma_X}) \to H^0(f^{-1}(U), \mathcal{I}_{\Sigma_Y})
\]

is pure.

(iii) \(f\) is faithfully flat.

(iv) \(\Sigma_X = \emptyset\) and \(f\) is partially pure at every \(x \in X\) in the sense that there is a \(y \in Y\) such that \(f(y) = x\) and the map \(\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}\) is pure [CGM16, p. 38].

If \((Y, \Sigma_Y)\) has Du Bois singularities, then \((X, \Sigma_X)\) has Du Bois singularities. In particular, if \((Y, \Sigma_Y)\) has Du Bois singularities, then \(X\) has Du Bois singularities if and only if \(\Sigma_X\) has Du Bois singularities.

Proof. For (i) and (ii), we show that the hypothesis in Proposition 3.1 is satisfied. For (i), it suffices to note that the morphism (3) admits a left inverse. For (ii), the morphism (3) is

\[
H_x^i(\text{Spec}(\mathcal{O}_{X,x}), H^0(U, \mathcal{I}_{\Sigma_X})) \to H_x^i(\text{Spec}(\mathcal{O}_{X,x}), H^0(f^{-1}(U), \mathcal{I}_{\Sigma_Y}))
\]

where \(U\) is an affine open containing \(x\). This map is injective by [Kem79, Corollary 3.2(a)].

For (iii), let \(x \in X\) be a point, and let \(y \in Y\) such that \(f(y) = x\). The morphism \(\text{Spec}(\mathcal{O}_{Y,y}) \to \text{Spec}(\mathcal{O}_{X,x})\) is faithfully flat, and hence

\[
\mathcal{I}_{\Sigma_X,x} \to \mathcal{I}_{\Sigma_X,x} \otimes \mathcal{O}_{X,x} \mathcal{O}_{Y,y} \simeq \mathcal{I}_{\Sigma_Y,y}
\]

is pure by [HR74, p. 136]. We can then apply (ii) to the morphism \(\text{Spec}(\mathcal{O}_{Y,y}) \to \text{Spec}(\mathcal{O}_{X,x})\).

For (iv), let \(x \in X\) be a point. By assumption, there exists \(y \in Y\) such that \(f(y) = x\) and the map \(\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}\) is pure, in which case (ii) applies to the morphism \(\text{Spec}(\mathcal{O}_{Y,y}) \to \text{Spec}(\mathcal{O}_{X,x})\). 

Finally, we show Theorem A and Corollary B.
Proof of Theorem A. Set 

\[(X, \Sigma_X) = (\text{Spec}(R), \emptyset) \quad \text{and} \quad (Y, \Sigma_Y) = (\text{Spec}(S), \emptyset).\]

Since \(R \to S\) is cyclically pure, the map \(Y \to X\) is surjective. Next, \(S\) is reduced since it has Du Bois singularities, and hence the subring \(R\) of \(S\) is reduced. As a consequence, \(R \to S\) is pure by [Hoc77, Proposition 1.4 and Theorem 1.7]. Thus, Theorem 3.2(ii) applies. \(\square\)

Proof of Corollary B. For (i), note that if \(S\) has log canonical type singularities, then it is normal (by definition) and Du Bois (by [KK10, Theorem 1.4]). It therefore suffices to consider the case when \(S\) is normal and Du Bois. In this case, \(R\) is normal (by [HR76, Proposition 6.15(b)]) and Du Bois (by Theorem A). Finally, since \(K_R\) is Cartier, we see that \(R\) being Du Bois implies that \(R\) has log canonical singularities [Kov99, Theorem 3.3].

For (ii), we note that \(R\) is reduced since it is a subring of \(S\), and hence \(R \to S\) is pure by [Hoc77, Proposition 1.4 and Theorem 1.7]. Denote by \(f: Y \to X\) the morphism of affine schemes associated to \(R \to S\). We adapt the proof of [Zhu, Lemma 2.3]. We want to show that \(X\) has log canonical singularities at every closed point \(x \in X\). Since the statement is local at \(x\), we can replace \(X\) with an affine open neighborhood of \(x\) to assume that \(\mathcal{O}_X(rK_X) \cong \mathcal{O}_X\), where \(r\) is the Cartier index of \(K_X\) at \(x\).

Let \(s \in H^0(X, \mathcal{O}_X(rK_X))\) be a nowhere vanishing section, and let \(\pi: X' \to X\) be the corresponding index 1 cover as in [KM98, Definition 5.19]. Let \(Y'\) be the normalization of the components of \(X' \times_X Y\) dominating \(X'\), and denote by \(\pi'\) the composition \(Y' \to X' \times_X Y \to Y\). Let \(U\) be the Cartier locus of \(K_X\), let \(V = f^{-1}(U)\), and let \(U'\) (resp. \(V'\)) be the preimage of \(U\) (resp. \(V\)) in \(X'\) (resp. \(Y'\)). Then, \(\pi\) and \(\pi'\) are étale over \(U\) and \(V\), respectively (see [KM98, Definition 2.49]), and 

\[\pi_*\mathcal{O}_{U'} = \left( \bigoplus_{m \in \mathbb{N}} \mathcal{O}_U(mK_U) \cdot t^m \right) / (st^r - 1)\]
\[\pi'_*\mathcal{O}_{V'} = \left( \bigoplus_{m \in \mathbb{N}} \mathcal{O}_V mf^*K_U \cdot t^m \right) / (fs^t - 1)\]

Here, in the second equality, \(f^*s\) denotes the pullback of \(s\) to \(H^0(V, \mathcal{O}_V(rf^*K_U))\). By assumption, the complement of \(U\) (resp. \(V\)) in \(X\) (resp. \(Y\)) has codimension at least two. We therefore have 

\[H^0(X', \mathcal{O}_{X'}) = H^0(U, \pi_*\mathcal{O}_{U'}) = \left( \bigoplus_{m \in \mathbb{N}} H^0(U, \mathcal{O}_U(mK_U)) \cdot t^m \right) / (st^r - 1)\]
\[H^0(Y', \mathcal{O}_{Y'}) = H^0(V, \pi'_*\mathcal{O}_{V'}) = \left( \bigoplus_{m \in \mathbb{N}} H^0(V, \mathcal{O}_V(mf^*K_U)) \cdot t^m \right) / (fs^t - 1)\]

By [Zhu, Lemma 2.2], the ring map \(H^0(X', \mathcal{O}_{X'}) \to H^0(Y', \mathcal{O}_{Y'})\) is pure. Since \(Y' \to Y\) is étale in codimension one (it is étale over \(V\)), we know \(Y'\) has log canonical type singularities [KM98, Proposition 5.20]. By (i), this implies \(X'\) has log canonical singularities. Finally, since \(X' \to X\) is étale in codimension one (it is étale over \(U\)), we see that \(X\) has log canonical singularities [KM98, Proposition 5.20]. \(\square\)

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