COVARIANT KINETIC THEORY WITH AN APPLICATION TO THE COMA CLUSTER

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ABSTRACT

In this paper, we introduce a novel solution to the covariant Landau equation for a pure electron plasma. The method conserves energy and particle number and reduces smoothly to the Rosenbluth potentials of nonrelativistic theory. We find that the covariant relaxation timescale agrees with past results; however, as a full solution to the covariant Boltzmann equation, our technique allows for anisotropic or degenerate distributions, which makes it unique. To demonstrate the power of our solution in dealing with hot, astrophysical plasmas, we use this technique to show that one of the currently considered models, continuous stochastic acceleration of thermal electrons, for the hard X-ray emission in the Coma Cluster actually cannot work because the energy gained by the particles is distributed to the whole plasma on a timescale much shorter than that of the acceleration process itself.

Subject headings: acceleration of particles — galaxies: clusters: individual (Coma) — plasmas — radiation mechanisms: nonthermal — relativity — X-rays: galaxies

Online material: color figures

1. INTRODUCTION

The need for a self-consistent, practical transport theory that can handle dynamic populations of relativistic particles extends across many disciplines in physics. Most obviously, astrophysical plasmas in the presence of strong gravitational and/or magnetic fields are often out of thermal equilibrium when the relevant dynamic timescale (e.g., associated with the process of accretion onto a compact object) is short compared to the time required for the particles to interact internally with each other. Situations in which this may occur include solar flares (Petrosian & Liu 2003), accretion onto supermassive black holes in the nuclei of active galaxies (see, e.g., Melia & Falcke 2001), and reacceleration of already relativistic particles by shocks and turbulence produced during the merger of galaxy clusters (Brunetti et al. 2004). In all cases, the resultant photon spectrum produced by the energized particles is most simply explained in terms of a non-Maxwellian distribution.

Less directly, an argument can be made that the two dominant particle species (say, electrons and protons in a fully ionized hydrogen plasma) separate in energy space, leading to a situation in which the ion temperature is larger than that of the electrons (see, e.g., Shapiro et al. 1976). However, whether such a two-temperature plasma may be maintained against collisional reequilibration is still actively debated and may be critical to the question of how a black hole accretes from its environment. For example, there is now very good evidence that the gas orbiting the supermassive black hole, Sgr A\textsuperscript{\star}, at the center of our Galaxy attains a temperature in excess of $10^{10}$ K (Liu & Melia 2001), but it is still not understood whether this optically thin plasma maintains a single-particle distribution. In addition, nonthermal emission in Sgr A\textsuperscript{\star} (see, e.g., Özel et al. 2000; Yuan et al. 2003) has been invoked to explain its low-frequency radio spectrum in accretion-dominated models.

These clear astrophysical applications should be aided by recent experiments with laser-heated plasmas, which are now prob-
attempts at solution via a Chapman-Enskog expansion (Mohanty & Baral 1996), a Legendre expansion (Honda 2003), a Grad expansion (Muronga 2004), and numerical integration (Shoucri & Shkarofsky 1994).

However, to our knowledge, none of these solutions is six-dimensional; the ability to solve three-dimensional integrals such as equation (37) below has only existed for a short time (Hahn 2005). Manheimer et al. (1997) have called the self-consistent evaluation of even the simpler nonrelativistic coefficients “unfeasible.” Many approach the Fokker-Planck equation with implicit differencing, which is relatively difficult to modify with additional forces (e.g., in a particle-in-cell simulation). Instead, we follow the original suggestion of Jones et al. (1996) and the subsequent expansion (Muronga 2004), and numerical integration (Shoucri & Manheimer 1997) to solve it.

Before that, however, we present the relativistic kinetic theory in § 2, and then describe the method of solution in covariant form in § 3. The application is made in § 4.

2. RELATIVISTIC KINETIC THEORY: BACKGROUND

We begin by discussing the development of the relativistic kinetic equation somewhat pedagogically, given the fact that its usage in the analysis of high-energy plasmas in astrophysics has been infrequent at best. Landau (1936) first approached the problem by showing that the Boltzmann collision integral can be written as a flux in velocity space,

$$C = -\frac{\partial}{\partial v_\alpha} S_\alpha,$$  

(1)

thereby writing the Boltzmann equation as a conservation equation for the single-particle phase-space distribution,

$$\frac{\partial}{\partial t} f + \{H, f\} = \nabla \cdot \mathbf{S},$$  

(2)

where $f$ is the single-particle phase-space number density. The Poisson bracket $\{H, f\}$ represents a small change in the shape of a volume of phase space (which nevertheless conserves number), and the term $\nabla \cdot \mathbf{S}$ represents a flux of particles out of it. But what is $S$? To begin with, note that the number of collisions occurring per unit time between a particle with velocity $v$ and particles with velocity $v'$ in the range $d^3v'$ is

$$w f(v) f'(v') d^3q d^3v',$$  

(3)

where $q$ is the amount of velocity exchanged. Here $w$, the probability that $v$ will be taken into $v + q$, can be expressed exactly as $w(v + q/2, v' - q/2; q)$. This ensures that $w$ is symmetrical with respect to $q$, i.e., that detailed balance is maintained.

The particle flux $F = F^+ - F^-$ is found by subtracting the number of particles entering a volume of velocity space from the left,

$$F^- = \int d^3q \int d^3v' \int_{p_v - q_v}^{p_v} w f(v) f'(v') dv_v,$$  

(4)

from those exiting to the right,

$$F^+ = \int d^3q \int d^3v' \int_{p_v - q_v}^{p_v} w f(v + q) f'(v' - q) dv_v.$$  

(5)

This leads to an argument $f(v)f'(v') - f(v + q)f'(v' - q)$, which can be expanded about small values of $q$, giving the required value of $S$, the so-called Landau collision integral,

$$C = \frac{\partial}{\partial v_\alpha} S_\alpha = \frac{\partial}{\partial v_\alpha} \sum_\beta \int [f(v) \frac{\partial f'(v')}{\partial v_\beta} - f'(v') \frac{\partial f(v)}{\partial v_\beta}] B_{\alpha\beta} d^3v, $$  

(6)

where $w$ is expressed in terms of the collison cross section,

$$w d^3q = |v - v'| d\sigma,$$  

(7)

and where

$$B_{\alpha\beta} = \frac{1}{2} \int q_{\alpha} q_{\beta} |v - v'| d\sigma.$$  

(8)

Finally, since momentum conservation dictates that the change $q$ is perpendicular to the relative velocity $|v - v'|$, it must be that

$$B_{\alpha\beta}(v_\beta - v'_\beta) = 0.$$  

(9)

And since $B_{\alpha\beta}$ can only depend on the vector $|v - v'|$, the tensor must take the form

$$B_{\alpha\beta} = \frac{1}{2} B \left[ \delta_{\alpha\beta} - \frac{(v_\alpha - v'_{\alpha})(v_\beta - v'_{\beta})}{(v - v')^2} \right],$$  

(10)

where

$$B \equiv B_{\alpha\alpha} = \frac{1}{2} \int q^2 |v - v'| d\sigma.$$  

(11)

Thus, integrating equation (11) with a Rutherford cross section for $d\sigma$, one gets

$$B_{\alpha\beta} = \frac{e^2(v')^2}{8\pi q^2 m^2} \ln \Lambda U(v, v'),$$  

(12)

where $U(v, v')$ is the collision kernel,

$$U(v, v') = \frac{|v - v'|^2 \delta_{\alpha\beta} - (v_\alpha - v'_{\alpha})(v_\beta - v'_{\beta})}{|v - v'|^3},$$  

(13)

and $\alpha$ and $\beta$ are Cartesian indices. In what follows, we define $A \equiv ne^2(v')^2/8\pi q^2 m^2$ in $\Lambda$ and $l \equiv (m/2kT)^{1/2}$ for succinctness. In cgs units, $A \equiv 8\pi e^2(v')^2 n \ln \Lambda/m^2$. 


With one integration by parts, the collision integral can be written
\[ C = \frac{\partial}{\partial v_\alpha} \left[ D_{\alpha\beta} \frac{\partial f(v)}{\partial v_\beta} - F_\alpha f(v) \right], \tag{14} \]
where
\[ D_{\alpha\beta} \equiv A \int \mathbf{U}(v, v') f(v') d^3 v', \tag{15} \]
\[ F_\alpha \equiv A \sum_\beta \int \left[ \frac{\partial}{\partial v'_\beta} \mathbf{U}(v, v') \right] f(v') d^3 v', \tag{16} \]
which is the Fokker-Planck equation seen commonly in the west. Equations (15) and (16) can be expressed in a simpler form by observing that the kernel \( \mathbf{U}(v, v') \) obeys the relations
\[ U_\alpha = \frac{\partial^2}{\partial v_\alpha \partial v_\beta} |v - v'|, \tag{17} \]
\[ \frac{\partial}{\partial v_\alpha} U_\alpha = -2\frac{\partial |v - v'|^{-1}}{\partial v_\alpha}, \tag{18} \]
thus allowing the coefficients to be expressed as
\[ D_{\alpha\beta} = -A \frac{\partial^2}{\partial v_\alpha \partial v_\beta} \int f|v - v'| d^3 v', \tag{19} \]
\[ F_\alpha = -A \frac{\partial}{\partial v_\alpha} \left( 2 \int f|v - v'|^{-1} d^3 v' \right). \tag{20} \]

The advantage of this phrasing (MacDonald et al. 1957) is that the terms within the brackets contain elements in common with the Poisson equation, for which sophisticated methods of solution exist (e.g., expansion by spherical harmonics and fast Fourier transform convolution). When \( f(v) \) is a Maxwellian, a harmonic expansion gives
\[ F = -A \vec{v} \left( 1 + \frac{m}{m'} \right) G(|v|), \tag{21} \]
\[ D_\parallel = \frac{\vec{v}}{v} \left[ \Phi(|v|) - G(|v|) \right], \tag{22} \]
\[ D_\perp = \frac{\vec{v}}{v} G(|v|), \tag{23} \]
where \( D_\parallel \) and \( D_\perp \) are the components of the diffusion tensor parallel and perpendicular to the particle’s velocity in the co-moving frame, \( \Phi \) is the error function, and \( G(x) \equiv \Phi(x) - x \Phi'(x) / 2x^2 \). Since \( dv/dv_\beta = v_\beta/v \), the diffusion tensor fills out as
\[ D_{\alpha\beta} = \frac{v_\alpha v_\beta}{v^2} \left( D_\parallel - \frac{1}{2} D_\perp \right) + \frac{1}{2} \delta_{\alpha\beta} D_\perp. \tag{24} \]

Equations (21)–(24) present a fairly complete nonrelativistic kinetic theory.

The time \( t_s \) required for a particle to be deflected by 90° is a good measure of how long it takes a nonthermal injection to equilibrate (Spitzer 1962). This time is given by the condition
\[ D_\perp t_s = v^2. \tag{25} \]

Thus, for particles whose rms velocity is that of the group, \( v = (3kT/m)^{1/2} \), we have \( t_s = 1.225, D_\perp = 0.7144 \) (m s^{-1})^2 s^{-1}, and
\[ t_s = \left( \frac{3kT}{m} \right)^{3/2} \frac{1}{0.7144} \text{ s}. \tag{26} \]

A slowing down viscous timescale can be similarly defined. One of our primary goals is to find a relativistically correct version of equation (26); the majority of problems (even apparently difficult ones, such as nonthermal radiation from galaxy clusters) can be resolved with a glance at this simple formula.

Chandrasekhar (1957) approached the problem quite differently. Consider replacing the truly discrete many-body potential \( \Phi = \sum_i e_i / |r - r_i| \) with the smoothed version \( \Phi(r) = \int e_0 / |r - r'| d^3 r' \). A distribution evolving only due to such a smooth mean field force is “collisionless”; that is, no particle ever leaves its volume of (single particle) phase space. But in moving to an integrable charge distribution, we have lost track of small fluctuations in the number of particles in a given neighborhood.

Accumulated over a small time, these fluctuations cause a random kink in a particle’s path. Alone, this kink diffuses the probability of finding the particle with a particular velocity until all velocities are equally likely. Since, however, a Maxwellian distribution for this probability inevitably sets in, Chandrasekhar (1957) reasoned that an associated viscous term must exist to bring diffusing particles back to the group, so that individual particles move through phase space according to the equations of motion
\[ \frac{dx}{dt} = \vec{v}, \tag{27} \]
\[ \frac{dv}{dt} = \frac{F}{m} + F_d + \mathbf{Q} \cdot \mathbf{\Gamma}(t), \tag{28} \]
where \( F \) is a smooth mean field term (plus any external forces), \( F_d \) is dynamical friction that occurs when more particles hit the front end of a moving particle than the back, \( \mathbf{Q} \) is the “square root” of the diffusion tensor, \( D_{ij} = Q_{ia} Q_{jb} \), and \( \mathbf{\Gamma}(t) \) is a tri-variate Gaussian random vector with
\[ \langle \mathbf{\Gamma}(t) \rangle = 0, \tag{29} \]
\[ \langle \mathbf{\Gamma}(t) \mathbf{\Gamma}(t') \rangle = \delta_{tt'} \delta(t - t'). \tag{30} \]

We show in the Appendix why equations (27) and (28) are equivalent.

Equations (27) and (28) reappropriate kinetic theory as a dynamics that is conceptually different than Hamilton’s: Analytical dynamics follows the continuous motions of each of \( N \) particles in a 2\( N \)-dimensional phase space; it conserves energy, maintains the number of particles in a volume of phase space as a constant, and is reversible. Analytical dynamics has been described as “a gradual unfolding of a contact transformation” (Chandrasekhar 1957).

By contrast, statistical dynamics follows the motion of particles accumulated over some coarse time \( \Delta t \); indeed, since \( (\Delta x)^2 / \Delta t \to \infty \) as \( \Delta t \to 0 \), statistical dynamics is not even defined continuously. Statistical dynamics ignores all but the six phase-space coordinates of the particle in question, representing the effect of the other degrees of freedom as friction and noise; these terms cause a flux of particles through a surface of the single-particle phase space. Statistical dynamics does not conserve energy,
having at its heart the dissipative force $F_d$, yet this force is
precisely what is required to bring the assembly as a whole to
a Maxwellian distribution characterized by a constant energy.
Statistical dynamics is irreversible. Chandrasekhar (1957)
described it as the “gradual unfolding of a Markovian chain.”

Solving equation (1) by moving many particles according to
the equations of motion (eqs. [27] and [28]) is sometimes called a
“Monte Carlo” approach to kinetic theory. It is important to
stress the historical development here to avoid the tincture of a
computational trick that this name implies: the Langevin equa-
tion (eq. [28]) is equivalent to the Fokker-Planck equation
and implies no further approximation, provided collisions are short
(see eq. [30]). It can be developed on physical grounds entirely
separate from the Boltzmann equation and in fact was developed
some 14 years before a successful solution of the Fokker-Planck
equation appeared.

While we will use equations (27) and (28) to solve the evo-
dition of the distribution, we return to equation (2) to derive the
relativistically correct kinetic coefficients. The crucial distinc-
tion is the distribution, we return to equation (2) to derive the

$$f(p) = \int f(t, r, p) \, d^3x$$

is not a relativistically invariant 4-scalar, since in different coordi-
nate systems different particles will be considered to be lo-
cated simultaneously in a given volume. The integrals $N = \int f(t, r, p) \, d^3p$ and $i = \int v f(t, r, p) \, d^3p$, however, do form a 4-vector:

$$i^k = (cN, i) = c^2 \int (p^k f / c) \, d^3p.$$  (32)

Here $d^3p/c$ is covariant, and if $i^k$ is covariant, then the phase-space distribution $f$ must also be covariant. Rewriting equation (3) in the form

$$\epsilon \epsilon' \langle d^3p \rangle (d^3p / c) \, d^3x,$$  (33)

we reason that, since the phase-space density, $d^3p/c$, and $i^k$ are covariant, $\epsilon \epsilon' w$ must also be covariant. It follows that the integral

$$W^{kl} = \frac{1}{2} \epsilon \epsilon' \int q^k q^l v_{rel} \, d\sigma,$$  (34)

corresponding to equation (12), must form a symmetric 4-tensor. Here $v_{rel}$ is the velocity of one particle in the rest frame of the other. Following a derivation similar to equations (10)–(13), we find

$$W^{\alpha \beta} / \epsilon \epsilon' = A \frac{m c^2 (v')^2 \delta_{\alpha \beta} - v''_{\alpha} v''_{\beta} (v')^2}{(v')^3}$$  (35)

to be the desired manifestly covariant generalization of normal kinetic theory.

It is reasonable to neglect the zeroth component of equation
(35). The reason for this is that the change in particle energy $d\epsilon$ is
of second order with respect to small scattering angles, and thus $W^{00}$ and $W^{0\alpha}$ are of third or fourth order. Yet the Fokker-Planck equation is accurate only as far as second-order quantities. Leaving only the components of equation (8) corresponding to momentum flux, and reexpressing $B_{\alpha \beta}$ in a convenient form, we arrive at

$$C = \frac{\partial}{\partial u_\alpha} \left[ D_{\alpha \beta} \frac{\partial f(u)}{\partial u_\beta} - F_{\alpha} f(u) \right],$$  (36)

where

$$D_{\alpha \beta} = A \int Z(u, u') f(u') \, d^3u',$$  (37)

$$F = -A \sum_\beta \int \left[ \frac{\partial}{\partial u_\beta} Z(u, u') \right] f(u') \, d^3u',$$  (38)

and where the kernel is now given by

$$Z(u, u') = \frac{r^2}{\gamma \gamma' w^3} \left[ w^2 \delta_{\alpha \beta} - u_\alpha u_\beta - u'_\alpha u'_\beta + r (u_\alpha u'_\beta + u'_\alpha u_\beta) \right].$$  (39)

Here $u$ is the ratio of the momentum to the rest mass $m_0$. Also, $\gamma = (1 + u^2/c^2)^{1/2}$, $\gamma' = (1 + (u')^2/c^2)^{1/2}$,

$$r = \gamma \gamma' - u \cdot u' / c^2,$$  (40)

$$w = c \sqrt{r^2 - 1}. $$  (41)

In the nonrelativistic limit, $r \to 1$, $w \to |u - u'|$, and $u \to v$.
Thus, relativistic kinetic theory, which begins as manifestly covariant, also transitions to the nonrelativistic limit at low ve-
locity. From now on, when we refer to “velocity” in a relativistic con-
text, we mean $u = p / m_0$.

Equation (36) is valid as long as bremsstrahlung is not so
so dominant as to make particle collisions inelastic, thus affecting
the cross section. Provided this is the case, it is simple enough to
calculate the bremsstrahlung losses as an integral over the elec-
tron distribution.

Equations (37) and (38) may be solved with direct numerical
integration. The Langevin equation (eq. [28]) can then be used to
update a large number of phase-space points rather than obtaining
an implicit solution of the differential equation (eq. [6]). Be-
side the benefits of speed and the guaranteed conservation of
particle number, this approach also allows various new accel-
eration mechanisms to be incorporated: as an example, we in-
duce diffusion due to Alfvénic turbulence and radiative losses
within a single-species electron plasma in §§3 and 4. It remains to
show that the approach conserves energy and that the energy
equipartitions correctly.

3. COMPARISON WITH EXISTING THEORY

Let us compare this result to that of Nayakshin & Melia (1998,
hereafter NM98). Since NM98 construct a kinetic theory around
the unitless energy $E = \gamma - 1$, while covariant theory measures
the friction and diffusion coefficients in the generalized velocity
$u = \gamma v$, direct comparison is not obvious. For example, in NM98
the energy exchange and viscous coefficients are equivalent,
whereas covariant theory finds the energy exchange by integrating
the flux of energy through a volume of phase space. Covariant
theory gives diffusion coefficients both parallel and perpendicular
to a particle’s velocity; the diffusion coefficients of NM98, mea-
sured in the scalar $\gamma$, do not.

While the two theories measure different quantities, we may
nevertheless place them in the same system of units for direct
comparison. The unitless diffusion coefficient is

$$D_{\alpha \beta} = \frac{A}{c^2} \int Z(u, u') f(u') 4 \pi (u')^2 \, du',$$  (42)

where the $c^2$ term gives the diffusion coefficient in units of $s^{-1}$,
as in NM98 (the kernel $Z$ is measured as $1/c$, and the constant
A is measured as $c^3 \text{s}^{-1}$). With a change of the variable of integration,

$$D_{\alpha\alpha} = \frac{4\pi A}{c^4} \int Z(u, u') f(\gamma') \frac{(u')^3}{\gamma'} d\gamma', \quad (43)$$

the function

$$D(u, u') = \frac{4\pi A}{c^4} Z(u, u') \frac{(u')^3}{\gamma'} \quad (44)$$

is now the equivalent of equation (35) in NM98.

At first, the most glaring discrepancy is the absolute scale of the two theories: covariant theory (along with nonrelativistic Rosenbluth potentials) is divergent (Fig. 1, left), while that of NM98 (Fig. 1, right) is not.

There are also physically grounded discrepancies. We show in the Appendix that the Fokker-Planck and dynamical friction approaches are equivalent. In the latter theory, it is physically clear that all particles must be slowed by the viscous term, since a moving particle is always struck more frequently on its leading side, regardless of what relation its speed has to the rest of the group. To use an analogy with another $1/r$ potential, a star is never accelerated by gravitational drag. Yet the energy exchange term in NM98 (Fig. 2) changes sign, drawing all particles to the average speed. In this case, one must multiply the frequency of collisions by the fractional change in energy, which allows particles to both gain and lose energy as the result of multiple collisions. Thus, the covariant friction is positive-definite, while the “friction” term $a(\gamma, \gamma')$ of NM98 is not. Finally, there is some difficulty in the numerical solution of the Fokker-Planck equation using the NM98 approach: neither the stochastic solution nor the implicit algorithms suggested by Park & Petsiosian (1996) successfully bring the distribution to equilibrium (Fig. 3).

Despite their differences, the two theories agree on the covariant relaxation time. Dermer & Liang (1989) give this as

$$t_D = 4(n_e c \sigma_T)^{-1} \Theta_e^{3/2} \left( \pi^{1/2} - 1.2 \Theta_e^{1/4} + 2 \Theta_e^{1/2} \right)/\ln \Lambda, \quad (45)$$

which is plotted in Figure 4, together with the nonrelativistic Spitzer time. Data points show the amount of time required for a distribution of electrons to equilibrate to a thermal bath that is twice its initial temperature.

4. A SOLUTION OF THE EQUATIONS IN COVARIANT KINETIC THEORY

Figure 3 shows the kinetic coefficients calculated for a Maxwellian distribution

$$f(v) = \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( -mv^2/2kT \right), \quad (46)$$

at a temperature $10^8 \text{K}$, compared to the known analytic versions given by equations (21)–(23). It is important to stress here...
that, while most of our results are presented as functions of the magnitude of the velocity, this quantity plays no role in the calculation itself.

To make this figure, a large number (here one billion) of particles are first initialized, scaled, and interpolated onto a grid, generally using the cloud-in-cell or triangle-shaped-cloud interpolants well known from many-body mesh calculations. This three-dimensional grid represents a discrete version of the distribution \( f(v_0) \) on the right-hand side of equations (15) and (16). For any velocity \( v_0 \), a subfunction interpolates values off the grid, and the result is an apparently continuous function.

We now concern ourselves with a single value of \( v \), the left-hand side of equation (15), and carry out a Monte Carlo integration of the kernel (eq. [10]) using the Vegas algorithm of the Cuba integration library (Hahn 2005). A maximum of 2500 function evaluations, beginning with 300 subdivisions and adding 500 new subdivisions in a refinement step, gives the best speed-to-accuracy ratio. For this one particular \( x, y, \) and \( z \)-velocity, we sum over \( /C_{12} \), resulting in six independent diffusion coefficients and three force coefficients comprising a structure. We do this for some number of \( v \)-values, creating a second grid, each point of which is a structure containing the six values of \( D \) or three values of \( F \). Two computational points are relevant here: first, we perform the sum over \( \beta \), resulting in six independent diffusion coefficients and three force coefficients comprising a structure.

We do this for some number of \( r \)-values, creating a second grid, each point of which is a structure containing the six values of \( D_{1/2} \) or three values of \( F_{1/2} \). Two computational points are relevant here: first, we perform the sum over \( \beta \) within the function call, rather than integrating three times and adding the answers. Second, the algorithm is about 10 times faster when a vector of many functions is passed to the integrator in a single integration call, rather than defining the function and calling the integrator many separate times. A lookup table is created so that in calling
Finally, for display purposes, we find the magnitude of each particle’s velocity and friction, and we diagonalize its diffusion tensor. That is, the coefficients in Figure 5 play no role in evolving the distribution; the solution makes no isotropic assumption and is naturally calculated in the lab frame.

To evolve the distribution, each of the particles is updated using a first-order stochastic equation solver. Second-order schemes exist (Qiang & Habib 2000), but the multiple function evaluations are prohibitively expensive. The tensor \( Q \) is found by diagonalizing \( D \) via a Jacobi rotation, taking the square root of the diagonal components, and then transforming back.

Making a transition to a relativistic Maxwellian plasma, we instead use the distribution function

\[
f(u) = \frac{m_n}{4\pi k T^2} \exp\left(-\gamma m_n c^2 / k T\right),
\]

where \( K_2 \) is the modified Hankel function of order 2. Again, we are working with \( u = p/m_0 \), the relativistic generalization of the velocity, and we use the term “velocity” exclusively with this meaning. With a substitution of \( f(u) \) in equation (37), the resulting relativistic kinetic coefficients may be defined as

\[
D_R(u) = \phi(u) D_{NR}(u),
\]

\[
F_R(u) = \psi(u) f_{NR}(u).
\]

For ease of use, we have expanded \( \phi(u) \) and \( \psi(u) \), functions we call the enhancement factors, these being the multiplicative differences between the nonrelativistic and relativistic expressions, as

\[
\phi(u) = \sum_{i=0}^{3} a_i u^i,
\]

and similarly for \( \psi(u) \), where again the coefficients \( a_i \) are expanded as functions of temperature,

\[
a_i = \sum_{j=0}^{8} b_{ij} \left( \frac{T}{5 \times 10^8 \text{ K}} \right)^j.
\]

| Kinetic Coefficient | \( b_0 \) | \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | \( b_5 \) | \( b_6 \) | \( b_7 \) | \( b_8 \) |
|---------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \( D_{ij} \)       |        |        |        |        |        |        |        |        |        |
| \( a_0 \)          | -1.61E-37 | 3.38E-35 | -2.92E-33 | 1.34E-31 | -3.50E-30 | 5.01E-29 | -2.97E-28 | -1.44E-27 | 3.76E-26 |
| \( a_1 \)          | 2.72E-28 | -5.76E-26 | 5.08E-24 | -2.43E-22 | 6.87E-21 | -1.18E-19 | 1.20E-18 | -6.91E-18 | 9.48E-18 |
| \( a_2 \)          | -7.04E-20 | 1.36E-17 | -1.08E-15 | 4.48E-14 | -1.07E-12 | 1.51E-11 | -1.38E-10 | 1.03E-9 | -1.02E-9 |
| \( a_3 \)          | 1.74E-11 | -3.37E-9 | 2.64E-7 | -1.08E-5 | 2.46E-4 | -3.12E-3 | 2.10E-2 | -5.77E-2 | 1.0716 |
| \( D_{ij} \)       |        |        |        |        |        |        |        |        |        |
| \( a_0 \)          | 4.96E-38 | -1.06E-35 | 9.28E-34 | -4.31E-32 | 1.13E-30 | -1.61E-29 | 9.98E-29 | 1.10E-28 | -3.29E-27 |
| \( a_1 \)          | -3.37E-29 | 7.51E-27 | -6.91E-25 | 3.36E-23 | -9.19E-22 | 1.35E-20 | -7.88E-20 | 3.66E-19 | 6.35E-18 |
| \( a_2 \)          | 8.49E-21 | -2.15E-18 | 2.22E-16 | -1.21E-14 | 3.74E-13 | -6.51E-12 | 5.54E-11 | -5.15E-11 | 3.65E-10 |
| \( a_3 \)          | 1.55E-12 | -2.90E-10 | 2.17E-8 | -8.30E-7 | 1.73E-5 | -2.03E-4 | 1.60E-3 | 2.35E-2 | 1.00 |
| \( F \)            |        |        |        |        |        |        |        |        |        |
| \( a_0 \)          | 1.09E-37 | -2.41E-35 | 2.24E-33 | -1.13E-31 | 3.39E-30 | -6.17E-29 | 6.74E-28 | -4.24E-27 | 1.39E-26 |
| \( a_1 \)          | 5.80E-29 | -1.24E-26 | 1.10E-24 | -5.30E-23 | 1.51E-21 | -2.59E-20 | 2.66E-19 | -1.62E-18 | 7.51E-18 |
| \( a_2 \)          | -7.43E-22 | 4.79E-20 | 7.39E-18 | -9.70E-16 | 4.49E-14 | -1.03E-12 | 1.27E-11 | -1.19E-10 | 7.18E-10 |
| \( a_3 \)          | -5.87E-13 | 1.04E-10 | -7.20E-9 | 2.24E-7 | -1.73E-6 | -7.14E-5 | 1.45E-3 | 3.72E-2 | 9.92E-1 |
The resulting coefficients are given in Table 1; the polynomial functions over $b_j$ as fitted to each value of $a_i$ are shown in Figure 6. In Figure 7 we show the actual enhancement factors $\phi(u)$ and $\psi(u)$, together with the polynomials reconstructed from Table 1. Thus, with a few numbers, we can readily calculate the relativistic kinetic coefficients.

The evolution of the nonrelativistic and relativistic distributions at $10^8$ and $2.5 \times 10^8$ K, respectively, is shown in Figures 8 and 9 as a function of the Spitzer time, $t_S$. The nonrelativistic distribution requires about two Spitzer times to equilibrate, whereas the relativistic one reaches equilibrium in $6t_S$. For comparison, we also show the one-dimensional evolution of an implicit numerical solution.

When the energy is constant, or when the injection rate is known, we use normalization coefficients to maintain constant energy in the distribution; an example of how these coefficients vary with time is plotted in Figure 10. Energy is conserved to an accuracy of $10^{-5}$, about the same as the precision of our integration.

However, when energy is not constant, this simple procedure is not feasible, and we must increase the grid size to a minimum of $64^3$ and the maximum number of function evaluations to 10,000. In this case, normalization is no longer required. For example, in Figure 11, we allowed the distribution to relax for a full Spitzer time before turning on stochastic acceleration, conserving energy all the while.

5. APPLICATION: NONTHERMAL BREMSSTRAHLUNG FROM CLUSTER MERGERS

Evidence for the presence of relativistic electrons in the intracluster medium (ICM) is provided primarily by diffuse synchrotron emission at radio wavelengths. Given the $\sim$megaparsec size of the synchrotron features and the fact that the radiative lifetime of the electrons appears to be orders of magnitude shorter than the time required to cover such distances, the presence of this emission also suggests that electrons are being reaccelerated (from a population of already relativistic particles) out of the ICM thermal pool.

The radio emission of the Coma Cluster is accompanied by hard X-rays of similar spatial distribution; this and its regular morphology classify the Coma as a “radio halo.” In contrast, “radio relics” are typically irregular and concentrated toward the cluster’s periphery (e.g., Feretti 2003).

Fig. 6.—Plots of the four parameters $a_i$ and the fitting polynomials (smooth solid curves) $\sum_{j} b_j (T/5 \times 10^8 \text{ K})^j$. We have expanded the kinetic coefficients for a particular temperature, and, because this expansion changes as the temperature is raised, we again expand each of the coefficients as though they were polynomial functions of $T$. We present this figure not simply to confirm the accuracy of the expansion, but so that a rough estimate can be made by reading off the values of $a_i$ for the temperature of interest. [See the electronic edition of the Journal for a color version of this figure.]
Typically, it is assumed that the radiating particles must be continuously reaccelerated on their way out; these are “primary” models (Jaffe 1977; Schlickeiser et al. 1987; Brunetti et al. 2001; Ohno et al. 2002). However, protons could diffuse throughout the cluster without radiating, all the while colliding with thermal ICM ions and cascading into the observed relativistic electrons; these are “secondary” models (Dennison 1980). Secondary models relieve the difficulties of continuously accelerating electrons against radiative and Coulomb cooling; however, Blasi & Colafrancesco (1999) suggest that protons could not provide enough secondary electrons to consistently describe the radio and hard X-ray emission without also producing $\gamma$-ray decays in excess to the upper limit given by EGRET.

Interpreted as thermal bremsstrahlung, the observed X-rays imply an ICM temperature between 8 and 9 keV. The X-ray emission of the Coma Cluster cannot be described entirely as thermal bremsstrahlung, however. The Rossi X-Ray Timing Explorer (Rephaeli & Gruber 2002) and BeppoSAX (Fusco-Femiano et al. 2004, hereafter FF04) have each made two observations of the Coma Cluster, both claiming to find a hard X-ray tail (HXR) in excess of a thermal flux. This is a controversial claim: the initial analysis of Fusco-Femiano et al. (1999) was flawed—one of three spectra was counted twice—leading to a 2.5 $\sigma$ overstatement of the confidence level, according to Rossetti & Molendi (2004). Rossetti & Molendi flatly state that the second BeppoSAX observation shows no evidence of a hard tail. FF04, however, find a significant drop in the $\chi^2$ value for the fit to a hybrid thermal power-law model ($\chi^2 = 1.20$ for 7 degrees of freedom [dof]) as opposed to the purely thermal model ($\chi^2 = 4.10$ for 9 dof). This improvement cannot be matched by a two-temperature model, since the second temperature ($kT > 50$ keV) is considered unrealistic.

Rephaeli (1979) had predicted that such a tail must exist when the synchrotron-emitting electrons inverse Compton scatter with the cosmic background. In principle, this origin could be used as a probe for the ICM magnetic field. Yet, assuming that both the synchrotron radio and Compton HXR are produced by the same population of electrons, FF04 infer a magnetic field of $B \sim 0.2 \mu G$, more than a factor of 10 below the results inferred by Faraday rotation ($B \sim 6.0 \mu G$).

For this reason, nonthermal bremsstrahlung (Blasi 2000) was proposed as an alternative emission mechanism to inverse Compton. Here nonthermal electrons are directly accelerated and directly radiate via $e^{-}e^{+}$ bremsstrahlung. Petrosian (2001) has argued that this mechanism can only sustain the observed hard X-rays for less than a few times $10^7$ yr, since bremsstrahlung is not sufficiently efficient to cool the distribution before an unacceptably large amount of energy is given to the ICM, shifting the thermal body of bremsstrahlung above its observed flux. Because this problem is simple (it requires only stochastic acceleration, bremsstrahlung, and Coulomb cooling), and because it straddles the transrelativistic regime that our theory handles uniquely, we have chosen it as an example of the technique’s power and simplicity.

We have found that the stochastic acceleration model for nonthermal bremsstrahlung in the Coma Cluster produces a nonthermal tail. However, because of the relative inefficiency of bremsstrahlung for cooling the accelerated electrons, this tail is quickly overwhelmed as an excess of energy is dumped into the thermal body of the ICM.
With this motivation in mind, let us state the problem more precisely. We use the pitch-angle-averaged diffusion coefficient (Dermer et al. 1996)

$$D(u) = \frac{\pi}{2} \frac{u}{q(q+2)} c^3 k_0^2 \beta_A^2 \eta_A (r_B k_0)^{q^2-2} p^q / \beta$$

(52)

to represent the resonant interaction of particles with Alfvén waves. Here $p = \gamma \beta$, $r_B = m_e c^2 / e B$, $\eta_A = 0.07$ is the fraction of magnetic energy in Alfvén waves, $q = 5/3$ is the Kolmogorov constant, $L_c = 10$ kpc is the size of the galaxy, $\beta_A = v_A / c$ [where $v_A = B (4 \pi n_p m_p)^{1/2}$ is the Alfvén velocity], and $k_0 = 2 \pi / L_c$ is the largest scale wavenumber. Our point of departure is that of Blasi's calculation ($n = 4 \times 10^{-4}$ cm$^{-3}$, $B = 0.8$ $\mu$G, and $T = 7.5$ keV; Blasi 2000), but we have also looked at a small range around these figures. At this density, the Spitzer time is 0.9 Myr, while the Dermer relativistic relaxation time is 1.3 Myr (Dermer & Liang 1989).

It is important to note that only electrons with momenta greater than $p_{\text{min}} \sim (m_p / m_e) \beta_A$ may resonate with the turbulent field. Therefore, we have enforced a cutoff in equation (52) by hand in order to avoid immediately accelerating the entire electron population. Here we take $D_{\text{turb}}(u)$ to be zero below $u/c = 0.5$. In the original treatment this cutoff was applied indirectly, in the sense that ICM magnetic field and density values were chosen such that the acceleration and Coulomb timescales were similar at the resonance threshold, thus inferring that particles below the threshold would not accelerate into an energized tail. We have found that a hard cutoff is needed to reproduce the results qualitatively.

In Figure 11, we show the evolution of the distribution begun at 7.5 keV. Ten full Spitzer times elapse before stochastic acceleration from a 0.8 $\mu$G field is turned on (corresponding to the first curve on the left). After 18 Myr (about 2% of the time required by Blasi 2000) the distribution has evolved a nonthermal tail that might account for the HXR of the Coma Cluster; however, at any time after this, the main body of the ICM has gained too much heat to account for the thermal, soft component. Once acceleration is turned off, relaxation occurs in a little over two Dermer times, about 3 Myr.

The emissivity,

$$\frac{dE}{dV dt d\nu} \equiv j = h c n_e n_i \int \frac{\sqrt{p_0 / (p_0^2 + p_0^2)}}{(ds/dk)f(p_0) dp_0}$$

(53)

is calculated self-consistently using particle number distributions and the electron-proton bremsstrahlung cross section, $d\sigma / dk$.
averaged over all angles and expanded to the order of \((p_0^0)^6\). Including the Elwert correction factor, this is (Haug 1997)

\[
\frac{d\sigma}{dk} \approx \frac{2\alpha r_0^2}{ kp_0^2} \left[ \frac{4}{3} \epsilon_0 \epsilon_f + k^2 - \frac{7}{15} \epsilon_0 \epsilon_f - \frac{11}{70} k^2 (p_0^2 + p_f^2) \right] \\
\times \left\{ 2 \ln \left( \frac{e_0 \epsilon_f + p_0 p_f - 1}{k} \right) - \frac{p_0 p_f}{e_0 \epsilon_f} \left[ 1 + \frac{1}{e_0 \epsilon_f} + \frac{7}{20} \left( \epsilon_0 \epsilon_f \right)^3 \right] \\
+ \left( \frac{9}{28} k^2 + \frac{263}{210} p_0^2 p_f^2 \right) \left( \frac{1}{e_0 \epsilon_f} \right)^3 \right\} \frac{\alpha_f}{\alpha_0} \left[ 1 - \exp(-2\pi a_0) \right] \\
\times \frac{1 - \exp(-2\pi a_f)}{1 - \exp(-2\pi a_f)},
\]

(54)

where \(\alpha = e^2/\hbar c\) is the fine structure constant, \(r_0 = e^2/m_e c^2\) is the electron radius, \(k = h \omega/m_e c^2\) is the photon energy in units of the electron mass energy, \(p_0 = \gamma_0 \beta_0 = u_0/c\) is the initial electron momentum in units of \(m_e c\), \(\epsilon_0 = \gamma_0\) is the total energy in rest mass units, \(a_0 = \alpha c_0 / p_0\), and

\[
p_f = \left( k^2 - p_0^2 - 2k \sqrt{1 - p_0^2} \right)^{1/2}.
\]

(55)

In the energy regime considered, the error of this expansion is a few percent; the contribution of \(e^-e^-\) and \(e^-e^+\) bremsstrahlung is of this order, so we exclude them from consideration.

Once the cross section in equation (54) is found, we calculate the observed X-ray flux \(F_X\) under the assumption of constant density and use the accepted volume \(V\) and distance \(d_L\) for the Coma Cluster to write \(F_X = \psi / (4\pi d_L^2)\), assuming a (dimensionless)

![Figure 9](image9.png)

Fig. 9.—Plots of an initially cold distribution of particles (dots) with the same energy as a relativistic Maxwellian at \(T = 2.5 \times 10^9\) K (thick solid curve). The distribution relaxes in 0.2s. The time steps for the top four frames are given in multiples of 0.015s, and those for the remaining eight are given in multiples of 0.025s. For comparison, we show the nonrelativistic distribution (eq. [42]; thin solid curve) at this temperature. [See the electronic edition of the Journal for a color version of this figure.]

![Figure 10](image10.png)

Fig. 10.—Normalization factor that must be multiplied by the energy in order to preserve energy conservation, as a function of time. For a 32\(^3\) grid, the kinetic coefficients are correct only to two significant figures; since they are self-consistent, even this small deviation can lead to runaway heating, and we must normalize with a slowly evolving coefficient, plotted here as a function of time in units of the Spitzer time \(t_S\). This slight deviation from strict energy conservation can be removed by using a 64\(^3\) grid instead (with its accompanying four significant figures of accuracy). [See the electronic edition of the Journal for a color version of this figure.]
Hubble constant of $h = 0.6$. The calculated spectrum is shown in Figure 11 (right), together with the BeppoSAX data.

We have found only one case in which a persistent hard tail could be accelerated by turbulent diffusion: if the kinetic coefficients are not calculated self-consistently, subjecting the electrons to an external bath unaffected by its hotter surroundings. In every other instance, turbulent diffusion gives too much energy to the distribution by $\sim 20$ Myr. This is consistent with the result reported by Petrosian (2001): the erstwhile lack of a kinetic theory that is applicable throughout the transrelativistic regime was the source of discrepancy.

6. CONCLUDING REMARKS

We have described a novel method for solving the covariant Landau equation, unconstrained by assumptions of isotropy or low particle energy. Using simple polynomial fitting formulae for the kinetic coefficients, we have described an efficient numerical technique for determining the temporal evolution of an arbitrarily relativistic particle distribution, which may also be subject to energy injection from the anomalous acceleration of particles, for example, in shocks or scattering with Alfvén waves. We anticipate that this method will find widespread application not only in astrophysics, but in other physics disciplines as well.

To demonstrate the power of our technique in understanding the behavior of high-energy plasmas in astrophysics, we have examined one of the currently considered models, bremsstrahlung emission by a nonthermal tail, for the hard X-ray emission in galaxy clusters, such as the Coma Cluster. Our conclusion is that the time required by the underlying plasma to attain equilibrium is far too short compared to the stochastic acceleration timescale for the necessary nonthermal high-energy extension to survive longer than $\sim 20$ Myr. It seems unlikely, therefore, that cluster mergers through the action of stochastic acceleration could be responsible for energizing the plasma turbulence required to sustain the nonthermal particle emissivity.

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COVARIANT KINETIC THEORY

APPENDIX

A1. EQUIVALENCE OF THE LANDÉVIN AND FOKKER-PLANCK APPROACHES

We argue following Zwanzig (2001). Consider the stochastic equation

\[ \frac{dv}{dt} = F_d(v) + \Lambda(t), \]  

(56)

where the noise \( \Lambda(t) \) is Gaussian, with zero mean and delta-correlated second moment,

\[ \langle \Lambda(t)\Lambda(t') \rangle = 2B\delta(t-t'). \]  

(57)

Let us now find the probability distribution \( f(v, t) \), averaged over the noise. Now \( f(v, t) \) is conserved:

\[ \frac{\partial f}{\partial t} + \frac{\partial}{\partial v} \left( \frac{\partial}{\partial v} f \right) = 0. \]  

(58)

Thus, substituting \( \partial f / \partial t \), we arrive at the stochastic differential equation

\[ \frac{\partial f}{\partial t} = - \frac{\partial}{\partial v} \left[ F_d(v)f(v, t) + \Lambda(t)f(v, t) \right] = -Lf - \frac{\partial}{\partial v} \Lambda(t)f(v, t), \]  

(59)

where \( L\Phi \equiv (\partial/\partial t)f(v)\Phi \) is an appropriately defined operator. The solution to this equation is

\[ f(v, t) = e^{-Lt}f(v, 0) - \int_0^t ds e^{-(t-s)L} \frac{\partial}{\partial v} \Lambda(s)f(v, s). \]  

(60)

Placing equation (61) back into equation (60), we arrive at

\[ \frac{\partial f}{\partial t} = -Lf(v, t) - \frac{\partial}{\partial v} \Lambda(t)f(v, 0) + \frac{\partial}{\partial v} \Lambda(t) \int_0^t ds e^{-(t-s)L} \frac{\partial}{\partial v} \Lambda(s)f(v, s), \]  

(61)

which is readily observed to be the Fokker-Planck equation,

\[ \frac{\partial}{\partial t} \langle f(v, t) \rangle = - \frac{\partial}{\partial v} F_d(v) \langle f(v, t) \rangle + \frac{\partial}{\partial v} B \langle f(v, t) \rangle, \]  

(62)

once the average of \( f(v, t) \) is taken and the zero average and unit correlation of \( \Lambda \) are imposed.

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