On lattice-ordered $t_r$-norms for type-2 fuzzy sets✩

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Abstract

In this paper, it is proved that, for the truth value algebra of interval-valued fuzzy sets, the distributive laws do not imply the monotonicity condition for the set inclusion operation. Then, a lattice-ordered $t_r$-norm, which is not the convolution of $t$-norms on $[0, 1]$, is obtained. These results negatively answer two open problems posed by Walker and Walker in [15].

Keywords: Normal and convex function, $t$-norm, $t_r$-norm, $t_{lor}$-norm, type-2 fuzzy set, convolution.

1. Introduction

Throughout this paper, let $I = [0, 1]$, $I^2 = \{[a, b] : 0 \leq a \leq b \leq 1\}$, and $\text{Map}(X, Y)$ be the set of all mappings from space $X$ to space $Y$. In particular, let $M = \text{Map}(I, I)$.

To extend type-1 fuzzy sets (T1FSs), which are mappings from some universe to $I$, and interval-valued fuzzy sets (IVFSs), which are mappings from some universe to $I^2$, Zadeh [20] introduced the notion of type-2 fuzzy sets (T2FSs) in 1975, which were then equivalently expressed in different forms by Mendel et al. [8, 9, 10]. Simply speaking, a T2FS is a mapping from a universe to $\text{Map}(I, I)$.

Definition 1. [19] A type-1 fuzzy set $A$ in space $X$ is a mapping from $X$ to $I$, i.e., $A \in \text{Map}(X, I)$.

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Definition 2. [14] A type-2 fuzzy set $A$ in space $X$ is a mapping $A : X \rightarrow M$, i.e., $A \in \text{Map}(X, M)$.

Definition 3. [14] A fuzzy set $A \in \text{Map}(X, I)$ is normal if $\sup\{A(x) : x \in I\} = 1$.

Definition 4. [14] A function $f \in M$ is convex if, for any $0 \leq x \leq y \leq z \leq 1$, $f(y) \geq f(x) \land f(z)$.

Let $N$ and $L$ denote the set of all normal functions in $M$ and the set of all normal and convex functions in $M$, respectively.

For any subset $B$ of $X$, a special fuzzy set $1_B$, called the characteristic function of $B$, is defined by

$$1_B(x) = \begin{cases} 1, & x \in B, \\ 0, & x \in X \setminus B. \end{cases}$$

Let $J = \{1_{\{x\}} : x \in I\}$ and $K = \{1_{[a,b]} : 0 \leq a \leq b \leq 1\}$.

As an extension of the logic connective conjunction and disjunction in classical two-valued logic, triangular norms ($t$-norms) with the neutral 1 and triangular conorms ($t$-conorms) with the neutral 0 on $I$ were introduced by Menger [11] and by Schweizer and Sklar [13], respectively. The $t$-norms for binary operations on $I^2$ were introduced by Castillo et al. [1].

Definition 5. [7, 13] A binary operation $*: I^2 \rightarrow I$ is a $t$-norm on $I$ if it satisfies the following axioms:

(T1) (commutativity) $x * y = y * x$ for $x, y \in I$;

(T2) (associativity) $(x * y) * z = x * (y * z)$ for $x, y, z \in I$;

(T3) (increasing) $*$ is increasing in each argument;

(T4) (neutral element) $1 * x = x * 1 = x$ for $x \in I$.

A binary operation $*: I^2 \rightarrow I$ is a $t$-conorm on $I$ if it satisfies axioms (T1), (T2), and (T3) above; and axiom (T4'): $0 * x = x * 0 = x$ for $x \in I$.

Definition 6. [15, Definition 2][1, Definition 8] A binary operation $\Delta: I^2 \times I^2 \rightarrow I^2$ is a $t$-norm on $I^2$ if, for any $x, y, z \in I^2$ and any $a, b \in I$ with $a \leq b$, the following hold:

1. $[1, 1] \Delta x = x$;
2. $x \Delta y = y \Delta x$;
3. $(x \Delta y) \Delta z = x \Delta (y \Delta z)$;
4. $x \Delta (y \lor z) = (x \Delta y) \lor (x \Delta z)$;
5. $x \Delta (y \land z) = (x \Delta y) \land (x \Delta z)$;
(6) $[0, 1] \triangle [a, b] = [0, b]$;

(7) $[a, a] \triangle [b, b] = [c, c]$ for some $c \in I$;

where $[x_1, y_1] \wedge [x_2, y_2] = [x_1 \wedge x_2, y_1 \wedge y_2]$, and $[x_1, y_1] \vee [x_2, y_2] = [x_1 \vee x_2, y_1 \vee y_2]$.

Walker and Walker [15] proved that every $t$-norm $\triangle$ on $I^2$ is of the form $[x_1, y_1] \triangle [x_2, y_2] = [x_1 \triangle x_2, y_1 \triangle y_2]$ for some $t$-norm $\triangle$ on $I$, and they introduced the following two monotonicity conditions to replace the distributive laws (4) and (5):

(4') $x \leq y$ implies $x \triangle z \leq y \triangle z$;

(5') $x \subseteq y$ implies $x \triangle z \subseteq y \triangle z$.

Meanwhile, they posed the following open problem:

**Question 1.** [15] Whether or not conditions (4) and (5) in Definition 6 imply condition (5')?

A general technique to construct new operations on $M$ is convolution.

**Definition 7.** [5, Definition 1.3.3][15, Definition 6] Let $\circ$ and $\triangle$ be two binary operations defined on $X$ and $Y$, respectively, and $\boxplus$ be an appropriate operation on $Y$. If $\circ$ is a surjection, define a binary operation $\bullet$ on the set $Map(X, Y)$ by

$$(f \bullet g)(x) = \boxplus \{ f(y) \triangle g(z) : y \circ z = x \}.$$  

This method of defining a binary operation on $Map(X, Y)$ is called convolution. In particular, the convolution of a $t$-norm $\triangle$ on $I$ is the binary operation $\triangle$ on $M$ defined by

$$(f \triangle g)(x) = \sup \{ f(y) \triangle g(z) : y \triangle z = x \},$$

for $f, g \in M$.

**Definition 8.** [6] Let $\ast$ be a binary operation on $I$, $\triangle$ be a $t$-norm on $I$, and $\triangledown$ be a $t$-conorm on $I$. Define the binary operations $\wedge$ and $\vee : M^2 \rightarrow M$ as follows: for $f, g \in M$,

$$(f \wedge g)(x) = \sup \{ f(y) \ast g(z) : y \wedge z = x \},$$

and

$$(f \vee g)(x) = \sup \{ f(y) \ast g(z) : y \triangledown z = x \}.$$ (1.1)

**Definition 9.** [14] The operations of $\sqcup$ (union), $\sqcap$ (intersection), $\neg$ (complementation) on $M$ are defined as follows: for $f, g \in M$,

$$(f \sqcup g)(x) = \sup \{ f(y) \wedge g(z) : y \vee z = x \},$$

$$(f \sqcap g)(x) = \sup \{ f(y) \wedge g(z) : y \wedge z = x \},$$

and

$$(\neg f)(x) = \sup \{ f(y) : 1 - y = x \} = f(1 - x).$$
From [14], it follows that $\mathbb{M} = (\mathbb{M}, \sqcup, \sqcap, \neg, 1_{\{0\}}, 1_{\{1\}})$ is not a lattice, as the absorption laws do not hold, although $\sqcup$ and $\sqcap$ satisfy the De Morgan’s laws with respect to the complementation $\neg$.

Walker and Walker [14] defined the following partial order on $\mathbb{M}$.

**Definition 10.** [14] $f \sqsubseteq g$ if $f \sqcap g = f$; $f \preceq g$ if $f \sqcup g = g$.

It is noted that the same orders were introduced by Mizumoto and Tanaka [12] for $\text{Map}(J, I)$, in the case that $J$ is a subset of $I$. It follows from [14, Proposition 14] that $\sqsubseteq$ and $\preceq$ coincide on $L$, and the lattice $(L, \sqsubseteq)$ is a bounded complete lattice (see [14, 2]). In particular, $1_{\{0\}}$ and $1_{\{1\}}$ are the minimum and maximum of $L$, respectively.

**Definition 11.** [6, 15] A binary operation $T : L^2 \to L$ is a $t_r$-norm ($t$-norm according to the restrictive axioms), if

(O1) $T$ is commutative, i.e., $T(f, g) = T(g, f)$ for $f, g \in L$;

(O2) $T$ is associative, i.e., $T(T(f, g), h) = T(f, T(g, h))$ for $f, g, h \in L$;

(O3) $T(f, 1_{\{1\}}) = f$ for $f \in L$ (neutral element);

(O4) for $f, g, h \in L$ such that $f \sqsubseteq g$, $T(f, h) \sqsubseteq T(g, h)$ (increasing in each argument);

(O5) $T(1_{[0,1]}, 1_{[a,b]}) = 1_{[a,b]}$;

(O6) $T$ is closed on $J$;

(O7) $T$ is closed on $K$.

A binary operation $S : L^2 \to L$ is a $t_r$-conorm if it satisfies axioms (O1), (O2), (O4), (O6), and (O7) above, axiom (O3′): $S(f, 1_{\{0\}}) = f$, and axiom (O5′): $S(1_{[0,1]}, 1_{[a,b]}) = 1_{[a,1]}$. Axioms (O1), (O2), (O3), (O3′), and (O4) are called “basic axioms”, and an operation that complies with these axioms will be referred to as $t$-norm and $t$-conorm, respectively.

**Definition 12.** [5, Definition 5.2.6] A binary operation $R : L^2 \to L$ is a lattice-ordered $t_r$-norm (denoted as $t_{tor}$-norm) if it satisfies axioms (O1), (O2), (O3), (O5), (O6), and (O7) above, axiom (O4′): $R(f, g \sqcup h) = R(f, g) \sqcup R(f, h)$, and axiom (O4′′): $R(f, g \sqcap h) = R(f, g) \sqcap R(f, h)$.

**Remark 1.** Recently, we [17] proved that $t_{tor}$-norm on $L$ is strictly stronger than $t_r$-norm on $L$, which is strictly stronger than $t$-norm on $L$.

Walker and Walker [15] proved that the convolution $\triangledown$ of each $t$-norm $\triangle$ on $I$ is a $t_{tor}$-norm on $L$ and they proposed the following question in [15].

**Question 2.** [15] Whether or not a $t_{tor}$-norm is indeed the convolution of a $t$-norm on $I$?
Hernández et al. [6] proved that the binary operations \( \triangledown \) and \( \triangleright \), defined in Definition 8, are respectively a \( t_r \)-norm and a \( t_r \)-conorm on \( L \), provided that \( \triangle \) and \( \triangledown \) are continuous and \( \ast \) is a continuous \( t \)-norm on \( I \). Concerning its converse, we [18, 17] showed that if the operation \( \triangledown \) defined in Definition 8 is a \( t \)-norm on \( L \), then \( \triangle \) is continuous and \( \ast \) is a \( t \)-norm on \( I \), and we also obtained a similar result for \( \triangledown \). Meanwhile, we [16] constructed a \( t_r \)-norm and a \( t_r \)-conorm on \( L \), which cannot be obtained by the formulas that define the operations ‘\( \triangledown \)’ and ‘\( \triangledown \)’.

Extending our construction method in [16], this paper is devoted to answering Questions 1 and 2. In Section 3, we construct a binary operation \( \boxtimes \) on \( I^2 \) satisfying conditions (4) and (5) in Definition 6, which does not satisfy condition (5’). In Sections 4 and 5, we obtain a \( t_l \)-norm \( \bullet \), which is not the convolution of each \( t \)-norm on \( I \). These two results negatively answer Questions 1 and 2.

2. Some basic properties of \( L \)

**Definition 13.** [14, 16] For \( f \in M \), define

\[
\begin{align*}
   f^L(x) &= \sup \{ f(y) : y \leq x \}, \\
   f^{Lw}(x) &= \begin{cases} 
   \sup \{ f(y) : y < x \}, & x \in (0, 1], \\
   f(0), & x = 0,
   \end{cases}
   \\
   f^R(x) &= \sup \{ f(y) : y \geq x \}, \\
   f^{Rw}(x) &= \begin{cases} 
   \sup \{ f(y) : y > x \}, & x \in [0, 1), \\
   f(1), & x = 1.
   \end{cases}
\end{align*}
\]

Clearly, (1) \( f^L, f^{Lw} \) and \( f^R, f^{Rw} \) are monotonically increasing and decreasing, respectively; (2) \( f^L(x) \lor f^R(x) = f^L(x) \lor f^{Rw}(x) = f^R(x) \lor f^{Lw}(x) = \sup_{x \in I} \{ f(x) \} \) for all \( x \in I \). The following properties of \( f^L \) and \( f^R \) are obtained by Walker et al. [14].

**Proposition 1.** [14] For \( f, g \in M \),

1. \( f \leq f^L \land f^R; \)
2. \( (f^L)^L = f^L, (f^R)^R = f^R; \)
3. \( (f^L)^R = (f^R)^L = \sup_{x \in I} \{ f(x) \}; \)
4. \( f \subseteq g \) if and only if \( f^R \land g \leq f \leq g^R; \)
5. \( f \leq g \) if and only if \( f \land g^R \leq g \leq f^L; \)
6. \( f \) is convex if and only if \( f = f^L \land f^R. \)
Lemma 1. [14] For \( f, g \in L \),

(i) \((f \cap g)^L = f^L \lor g^L;\)

(ii) \((f \cap g)^R = f^R \land g^R;\)

(iii) \((f \cup g)^L = f^L \land g^L;\)

(iv) \((f \cup g)^R = f^R \lor g^R.\)

Remark 2. From Lemma 1, it follows that, for \( f, g \in L \), \((f \cap g)(1) = (f \cap g)^R(1) = f(1) \land g(1)\) and \((f \cup g)(1) = (f \cup g)^R(1) = f(1) \lor g(1).\)

Theorem 1. [4, 3] Let \( f, g \in L \). Then, \( f \sqsubseteq g \) if and only if \( g^L \leq f^L \) and \( f^R \leq g^R.\)

Proposition 2. For \( f \in M \), it holds that \( f^L_w(x) = \sup_{t \in [0,x]} \{f^L(t)\} = \lim_{t \searrow x} f^L(t) \) for all \( x \in (0,1].\)

Proof. Fix any \( x \in (0,1]\), noting that \( f(t) \leq f^L(t) \) for all \( t \in [0,x) \), one has

\[
 f^L_w(x) = \sup_{t \in [0,x]} \{f(t)\} \leq \sup_{t \in [0,x]} \{f^L(t)\}.
\]

Meanwhile, for any \( t \in [0,x) \), it follows from \( t < \frac{t + x}{2} < x \) that \( f^L(t) \leq f^L_w\left(\frac{t + x}{2}\right) \leq f^L_w(x) \). This implies that

\[
 \sup_{t \in [0,x]} \{f^L(t)\} \leq f^L_w(x).
\]

Thus,

\[
 f^L_w(x) = \sup_{t \in [0,x]} \{f^L(t)\}.
\]

Corollary 1. Let \( f, g \in L \). Then, for any \( x \in (0,1]\),

(1) \((f \cap g)^L_w(x) = f^L_w(x) \lor g^L_w(x),\)

(2) \((f \cup g)^L_w(x) = f^L_w(x) \land g^L_w(x).\)

Proof. (1) Applying Lemma 1 and Proposition 2 yields that

\[
 (f \cap g)^L_w(x) = \sup_{t \in [0,x]} \{(f \cap g)^L(t)\} = \sup_{0 \leq t < x} \{f^L(t) \lor g^L(t)\} \\
 \geq \sup_{0 \leq t < x} \{f(t) \lor g(t)\} \geq f^L_w(x) \lor g^L_w(x).
\]

Clearly, \( \sup_{0 \leq t < x} \{f^L(t) \lor g^L(t)\} \leq f^L_w(x) \lor g^L_w(x) \). Then,

\[
 (f \cap g)^L_w(x) = f^L_w(x) \lor g^L_w(x).
\]
(2) Applying Lemma 1 and Proposition 2 yields that
\[(f \sqcup g)^{lw}(x) = \sup_{t \in [0, x)} \{(f \sqcup g)^{L}(t)\} = \sup_{0 \leq t < x} \{f^L(t) \wedge g^L(t)\} \leq f^{lw}(x) \wedge g^{lw}(x). \tag{2.1}\]
Let \(\sup_{0 \leq t < x} \{f^L(t) \wedge g^L(t)\} = \xi\). Since \(f^L \wedge g^L\) is increasing, it follows that, for \(t_n \nearrow x\), \(f^L(t_n) \wedge g^L(t_n) \nearrow \xi\). Set \(\mathcal{P} = \{n \in \mathbb{N} : f^L(t_n) \leq g^L(t_n)\}\) and \(\mathcal{Q} = \{n \in \mathbb{N} : f^L(t_n) > g^L(t_n)\}\). Clearly, either \(\mathcal{P}\) or \(\mathcal{Q}\) is infinite.

(a) If \(\mathcal{P}\) is infinite, then
\[
\xi = \lim_{\mathcal{P} \ni n \rightarrow +\infty} f^L(t_n) \wedge g^L(t_n) = \lim_{\mathcal{P} \ni n \rightarrow +\infty} f^L(t_n) = f^{lw}(x) \geq f^{lw}(x) \wedge g^{lw}(x),
\]
which, together with (2.1), implies that
\[(f \sqcup g)^{lw}(x) = f^{lw}(x) \wedge g^{lw}(x).\]

(b) If \(\mathcal{Q}\) is infinite, then
\[
\xi = \lim_{\mathcal{Q} \ni n \rightarrow +\infty} f^L(t_n) \wedge g^L(t_n) = \lim_{\mathcal{Q} \ni n \rightarrow +\infty} g^L(t_n) = g^{lw}(x) \geq f^{lw}(x) \wedge g^{lw}(x),
\]
which, together with (2.1), implies that
\[(f \sqcup g)^{lw}(x) = f^{lw}(x) \wedge g^{lw}(x).\]

\[\square\]

For \(f \in \mathbf{L}\) and \(\alpha \in I\), let \(L_\alpha(f) = \inf\{x \in I : f^L(x) \geq \alpha\}\) and \(R^\alpha(f) = \sup\{x \in I : f^R(x) \geq \alpha\}\).

**Lemma 2.** [16, Lemma 2] For \(f \in \mathbf{N}\), \(L_1(f) \leq R^1(f)\).

**Definition 14.** For \(f \in \mathbf{M}\), let
\[b_f = \sup\{x \in I : f(x) = f^L(x)\}\] and \(c_f = \inf\{x \in I : f(x) = f^R(x)\}\).

The points \(b_f\) and \(c_f\) are the **left balance point** of \(f\) and **right balance point** of \(f\), respectively. In [2], the point \(b_f\) is also called **balance point** of \(f\).

**Proposition 3.** For \(f \in \mathbf{L}\), \(b_f = R^1(f)\) and \(L_1(f) = c_f\).

**Proof.** Let \(A = \{x \in I : f^R(x) = 1\}\) and \(B = \{x \in I : f(x) = f^L(x)\}\). For \(x \in A\), one has \(f(x) = f^L(x) \wedge f^R(x) = f^L(x)\), implying that \(A \subset B\). For \(x \in B\), one has \(f(x) = f^L(x) = f^L(x) \wedge f^R(x)\), implying that \(f^R(x) \geq f^L(x)\). This, together with the normality of \(f\), implies that \(1 = f^L(x) \vee f^R(x) = f^R(x)\), which means that \(B \subset A\). Thus, \(b_f = \sup A = \sup B = R^1(f)\). \(L_1(f) = c_f\) can be verified similarly. \(\square\)
Proposition 4. Let \( f \in \mathbb{N} \). Then,
\[
(1) \quad f^L(x) = 1 \text{ for } x \in (L_1(f), 1], \\
(2) \quad f^R(x) = 1 \text{ for } x \in [0, R^1(f)).
\]

Lemma 3. Let \( f, g \in \mathbb{L} \). Then,
\[
(1) \quad L_1(f \cap g) = L_1(f) \land L_1(g), \text{ i.e., } \inf\{x \in I : (f \cap g)^L(x) = 1\} = \inf\{x \in I : f^L(x) = 1\} \land \inf\{x \in I : g^L(x) = 1\}; \\
(2) \quad R^1(f \cap g) = R^1(f) \land R^1(g), \text{ i.e., } \sup\{x \in I : (f \cap g)^R(x) = 1\} = \sup\{x \in I : f^R(x) = 1\} \land \sup\{x \in I : g^R(x) = 1\}; \\
(3) \quad L_1(f \cup g) = L_1(f) \lor L_1(g), \text{ i.e., } \inf\{x \in I : (f \cup g)^L(x) = 1\} = \inf\{x \in I : f^L(x) = 1\} \lor \inf\{x \in I : g^L(x) = 1\}; \\
(4) \quad R^1(f \cup g) = R^1(f) \lor R^1(g), \text{ i.e., } \sup\{x \in I : (f \cup g)^R(x) = 1\} = \sup\{x \in I : f^R(x) = 1\} \lor \sup\{x \in I : g^R(x) = 1\}.
\]

Proof. For convenience, denote \( \inf\{x \in I : f^L(x) = 1\} = \eta_1 \) and \( \inf\{x \in I : g^L(x) = 1\} = \eta_2 \). Clearly,
\[
(\eta_1, 1) \subset \{x \in I : f^L(x) = 1\} \subset [\eta_1, 1], \tag{2.2}
\]
and
\[
(\eta_2, 1) \subset \{x \in I : g^L(x) = 1\} \subset [\eta_2, 1]. \tag{2.3}
\]
Applying Lemma 1 yields that
\[
\{x \in I : (f \cap g)^L(x) = 1\} \\
= \{x \in I : f^L(x) \lor g^L(x) = 1\} \\
= \{x \in I : f^L(x) = 1\} \\
\cup \{x \in I : g^L(x) = 1\}.
\]
This, together with (2.2) and (2.3), implies that
\[
(\eta_1 \land \eta_2, 1) \subset \{x \in I : (f \cap g)^L(x) = 1\} \subset [\eta_1 \land \eta_2, 1].
\]
Thus,
\[
\inf\{x \in I : (f \cap g)^L(x) = 1\} = \eta_1 \land \eta_2.
\]
The rest can be verified similarly. \( \square \)

Proposition 5. For \( f \in \mathbb{L} \),
\[
f(x) = \begin{cases} 
  f^L(x), & x \in [0, \xi_1) \\
  f(\xi_1), & x = \xi_1 \\
  1, & x \in (\xi_1, \xi_2) \\
  f(\xi_2), & x = \xi_2 \\
  f^R(x), & x \in (\xi_2, 1], 
\end{cases}
\]
where \( \xi_1 = L_1(f) \) and \( \xi_2 = R^1(f) \).
Proof. Since $f$ is convex, from Proposition 1, it follows that $f = f^L \land f^R$. Consider the following three cases:

Case 1. If $x \in [0, \xi_1)$, from Lemma 2, it follows that $x < \xi_1 \leq \xi_2$. This implies that $f^R(x) = 1$. Thus, $f(x) = f^L(x) \land f^R(x) = f^L(x)$;

Case 2. If $x \in (\xi_1, \xi_2)$, form the choices of $\xi_1$ and $\xi_2$, it can be verified that $f^L(x) = f^R(x) = 1$. This implies that $f(x) = f^L(x) \land f^R(x) = 1$;

Case 3. If $x \in (\xi_2, 1]$, from Lemma 2, it follows that $\xi_1 \leq \xi_2 < x$. This implies that $f^L(x) = 1$. Thus, $f(x) = f^L(x) \land f^R(x) = f^R(x)$.

\[\square\]

Remark 3. From Proposition 5, it follows that

(1) every function $f$ in $\mathbf{L}$ is increasing on $[0, L_1(f))$, constant on $(L_1(f), R^1(f))$, and decreasing on $(R^1(f), 1]$;

(2) if $L_1(f) < R^1(f)$, one has $f(L_1(f)) \geq f^{Lw}(L_1(f))$, i.e., $f(L_1(f)) \geq f(x)$ for all $x \in [0, L_1(f))$, and $f(R^1(f)) \geq f^{Rw}(R^1(f))$.

Corollary 2. Let $f \in \mathbf{L}$ satisfy that $R^1(f) < 1$. Then, for $x \in [0, 1)$, $f^R(x) = \sup\{f(y) : y \in [x, 1]\}$.

Proof. For $x \in [0, 1)$, choose $\zeta \in (R^1(f), 1)$ such that $\zeta > x$. From Proposition 5, it follows that $f(\zeta) \geq f(1)$, implying that

$$f^R(x) = \sup\{f(y) : x \leq y \leq 1\} = \sup\{f(y) : x \leq y < 1\}.$$ 

\[\square\]

3. A negative answer to Question 1

This section constructs a binary operation $\oplus$ on $I^{[2]}$ satisfying conditions (4) and (5), which does not satisfy condition (5’), answering negatively Question 1.

Proposition 6. Let $[a_1, b_1], [a_2, b_2] \subset I$. If $[a_1 \land a_2, b_1 \land b_2]$ is a single point, then one of $[a_1, b_1]$ and $[a_2, b_2]$ is a single point.

Proof. Consider the following two cases:

Case 1. If $a_2 \leq a_1$, then $[a_1 \land a_2, b_1 \land b_2] = [a_2, b_1 \land b_2]$ is a single point, i.e., $b_1 \land b_2 = a_2$. This implies that $b_1 = a_2$ or $b_2 = a_2$. When $b_1 = a_2$, one has $[a_1, b_1]$ is a single point, since $[a_1, b_1] \subset [a_2, b_1]$. When $b_2 = a_2$, one has $[a_2, b_2]$ is a single point.

Case 2. If $a_1 < a_2$, then $[a_1 \land a_2, b_1 \land b_2] = [a_1, b_1 \land b_2]$ is a single point, i.e., $a_1 = b_1 \land b_2$. This, together with $b_2 \geq a_2 > a_1$, implies that $b_1 \land b_2 = b_1 = a_1$. This means that $[a_1, b_1]$ is a single point.
Proposition 7. Let \([a_1, b_1], [a_2, b_2] \subseteq I\). If \([a_1 \lor a_2, b_1 \lor b_2]\) is a single point, then one of \([a_1, b_1]\) and \([a_2, b_2]\) is a single point.

Proof. Consider the following two cases:

Case 1. If \(a_2 \leq a_1\), then \([a_1 \lor a_2, b_1 \lor b_2] = [a_1, b_1 \lor b_2]\) is a single point. This, together with \([a_1, b_1 \lor b_2] = [a_1, b_1] \cup [a_1, b_2]\), implies that \([a_1, b_1]\) is a single point.

Case 2. If \(a_1 < a_2\), then \([a_1 \lor a_2, b_1 \lor b_2] = [a_2, b_1 \lor b_2]\) is a single point. This, together with \([a_2, b_1 \lor b_2] = [a_2, b_1] \cup [a_2, b_2]\), implies that \([a_2, b_2]\) is a single point.

Definition 15. Define a binary operation \(\ast\) on \(I^{[2]}\) as follows: for \(x, y \in I^{[2]}\),
\[
x \ast y = \max\{x \cdot y : x \in x, y \in y\}.
\]

Proposition 8. The binary operation \(\ast\) defined in Definition 15 does not satisfy condition \((5')\).

Proof. Take \(x = \{0.5\}\) and \(y = [0.5, 1]\). Clearly, \(x \subseteq y\). Let \(z = \{0.5\} \in I^{[2]}\). From Definition 15, it can be verified that
\[
z \ast x = \{0.25\},
\]
and
\[
z \ast y = \{0.5\}.
\]
Clearly, \(z \ast x \notin z \ast y\). Therefore, \(\ast\) does not satisfy condition \(5'\). □

Proposition 9. The binary operation \(\ast\) defined in Definition 15 satisfies condition \((4)\) in Definition 6.

Proof. For \(x = [x_1, x_2], y = [y_1, y_2], z = [z_1, z_2] \in I^{[2]}\), one has
\[
x \ast y = [x_1, x_2] \ast [y_1, y_2] = \{x_2 \cdot y_2\},
\]
and
\[
x \ast z = [x_1, x_2] \ast [z_1, z_2] = \{x_2 \cdot z_2\},
\]
and
\[
x \ast (y \lor z) = [x_1, x_2] \ast [y_1 \lor z_1, y_2 \lor z_2] = \{x_2 \cdot (y_2 \lor z_2)\},
\]
implies that
\[
(x \ast y) \lor (x \ast z) = (x_2 \cdot y_2) \lor (x_2 \cdot z_2) = \{x_2 \cdot (y_2 \lor z_2)\} = x \ast (y \lor z).
\]
□
Proposition 10. The binary operation \( \odot \) defined in Definition 4.1 satisfies condition (5) in Definition 6.

Proof. For \( x = [x_1, x_2], y = [y_1, y_2], z = [z_1, z_2] \in I^2 \), one has
\[
x \odot y = [x_1, x_2] \odot [y_1, y_2] = \{x_2 \cdot y_2\},
\]
and
\[
x \odot z = [x_1, x_2] \odot [z_1, z_2] = \{x_2 \cdot z_2\},
\]
and
\[
x \odot (y \land z) = [x_1, x_2] \odot [y_1 \land z_1, y_2 \land z_2] = \{x_2 \cdot (y_2 \land z_2)\},
\]
implying that
\[
(x \odot y) \land (x \odot z) = \{(x_2 \cdot y_2) \land (x_2 \cdot z_2)\}
\]
\[
= \{x_2 \cdot (y_2 \land z_2)\} = x \odot (y \land z).
\]

Remark 4. Summing up Propositions 8–10, it follows that conditions (4) and (5) in Definition 6 do not imply condition (5'). This gives a negative answer to Question 1.

4. Construct a \( t_{lor} \)-norm `\( \bullet \)` on \( L \)

Modifying our construction method in [16], this section introduces a binary operation `\( \bullet \)` on \( L \) and proves that it is indeed a \( t_{lor} \)-norm.

Definition 16. Define a binary operation \( \wedge : M^2 \rightarrow M \) as follows: for \( f, g \in M \),
\[
(f \wedge g)(x) = \begin{cases} (f \cap g)(x), & x \in [0, 1), \\ 0, & x = 1. \end{cases}
\]

Definition 17. Define a binary operation \( \star : L^2 \rightarrow M \) as follows: for \( f, g \in L \),
\[
\text{Case 1. } f = 1_{(1)}, \ f \star g = g \star f = g;
\]
\[
\text{Case 2. } g = 1_{(1)}, \ f \star g = g \star f = f;
\]
\[
\text{Case 3. } f \neq 1_{(1)} \text{ and } g \neq 1_{(1)},
\]
\[
f \star g = \begin{cases} f \cap g, & f(1) \land g(1) = 1, \\ f \wedge g, & f(1) \land g(1) < 1. \end{cases}
\]

Remark 5. From Definition 17 and Remark 2, it can be verified that, for \( f, g \in L \setminus \{1_{(1)}\} \),
\[
(f \star g)(1) = \begin{cases} 1, & f(1) \land g(1) = 1, \\ 0, & f(1) \land g(1) < 1. \end{cases}
\]

Lemma 4. Let \( f, g \in L \). Then,
(1) \((f \ast g)^L = (f \cap g)^L\);

(2) if \(f(1) \land g(1) < 1\) and \(f, g \in L \{1\}\),

\[
(f \ast g)^R(x) = \begin{cases} 
(f \cap g)^R(x), & x \in [0, 1), \\
0, & x = 1,
\end{cases}
\]

(3) if \(f(1) \land g(1) = 1\) and \(f, g \in L \{1\}\), \((f \ast g)^R = (f \cap g)^R\).

**Proof.** (1) From Definition 17, it suffices to check that, for \(f, g \in L \{1\}\) with \(f(1) \land g(1) < 1\), \((f \ast g)^L(1) = (f \cap g)^L(1)\). Since \(f \cap g \in L\), one has \((f \cap g)^L(1) = 1\). From the definition of \(\land\), it is clear that \((f \ast g)^L(1) \geq (f \land g)^Lw(1) = (f \cap g)^Lw(1)\). It follows from \((f \land g)(1) < 1\) that \(f^L(1) \lor g^L(1) = 1\). Thus, together with Lemma 1 and Proposition 2, implies that

\[
(f \ast g)^L(1) \geq (f \cap g)^Lw(1) = \sup_{0 \leq x < 1} \{(f \cap g)^L(x)\}
\]

\[
= \sup_{0 \leq x < 1} \{f^L(x) \lor g^L(x)\} \geq f^Lw(1) \lor g^Lw(1) = 1.
\]

Thus, \((f \ast g)^L(1) = (f \cap g)^L(1) = 1\).

(2) Fix \(f, g \in L \{1\}\) with \(f(1) \land g(1) < 1\) and let \(\xi_1 = \sup\{x \in I : f^R(x) = 1\}\) and \(\xi_2 = \sup\{x \in I : g^R(x) = 1\}\). It is clear that \((f \ast g)^R(1) = 0\) since \((f \ast g)(1) = 0\). For \(\hat{x} \in [0, 1)\), consider the following two cases:

Case 1. If \(\xi_1 \land \xi_2 < 1\), applying Lemma 3, it follows that \(\sup\{x \in I : (f \cap g)^R(x) = 1\} = \xi_1 \land \xi_2 < 1\). This, together with Corollary 2, implies that

\[
(f \cap g)^R(\hat{x}) = \sup\{(f \cap g)(y) : \hat{x} \leq y < 1\}
\]

\[
= \sup\{(f \land g)(y) : \hat{x} \leq y < 1\} \quad (as \quad (f \land g)(1) = 0)
\]

\[
= (f \ast g)^R(\hat{x});
\]

Case 2. If \(\xi_1 \land \xi_2 = 1\), i.e., \(\xi_1 = \xi_2 = 1\), then \(f^R(\hat{x}) = g^R(\hat{x}) = 1\). This, together with Lemma 1, implies that \((f \cap g)^R(\hat{x}) = f^R(\hat{x}) \land g^R(\hat{x}) = 1\). Applying \(\xi_1 = \xi_2 = 1\) and Proposition 4 yields that, for \(z \in [0, 1)\), \(f^R(z) = g^R(z) = 1\), implying that \(f(z) = f^L(z) \land f^R(z) = f^L(z)\) and \(g(z) = g^L(z) \land g^R(z) = g^L(z)\). Thus,

\[
(f \ast g)(x) = \begin{cases} 
(f \cap g)(x) = f^L(x) \lor g^L(x), & x \in [0, 1), \\
0, & x = 1.
\end{cases}
\]

Noting that \(f^L\) and \(g^L\) are increasing, by applying Proposition 2, one has

\[
(f \ast g)^R(\hat{x}) = \sup\{f^L(y) \lor g^L(y) : \hat{x} \leq y < 1\} = f^Lw(1) \lor g^Lw(1) = 1 = (f \cap g)^R(\hat{x}).
\]

(3) From Definition 17, these hold trivially. \qed
Proposition 11. For \( f, g \in L \), \( f \star g \) is normal and convex, i.e., \( f \star g \in L \).

Proof. By applying Definition 17 and Lemma 4, this can be verified immediately.  

Remark 6. Proposition 11 shows that the binary operation \( \star \) is closed on \( L \), i.e., \( \star(L^2) \subset L \).

Corollary 3. For \( f, g \in L \), \( R^1(f \star g) = R^1(f) \wedge R^1(g) \).

Proof. By applying Definition 17 and Lemmas 3 and 4, this can be verified immediately.

Proposition 12. For \( f, g \in L \setminus \{1\} \), \( (f \star g)(1) = 1 \) if and only if \( f(1) \wedge g(1) = 1 \).

Proof. From Remark 5, this holds trivially.

4.1. \( \star \) satisfies (O1)

For \( f, g \in L \),

A-1) if \( f = 1 \{1\} \) or \( g = 1 \{1\} \), then clearly \( f \star g = g \star f \);

A-2) if \( f \neq 1 \{1\} \) and \( g \neq 1 \{1\} \), then

\[
f \star g = \begin{cases} 
  f \sqcap g, & f(1) \wedge g(1) = 1, \\
  f \sqcup g, & f(1) \wedge g(1) < 1.
\end{cases}
\]

Meanwhile, it can be verified that

\[
g \star f = \begin{cases} 
  g \sqcap f, & g(1) \wedge f(1) = 1, \\
  g \sqcup f, & g(1) \wedge f(1) < 1.
\end{cases}
\]

Thus, \( f \star g = g \star f \).

Lemma 5. Let \( f, g \in L \). Then, \( f \star g \sqsubseteq f \) and \( f \star g \sqsubseteq g \). In particular, \( f \star g \neq 1 \{1\} \) if \( f, g \in L \setminus \{1\} \).

Proof. Since \( \star \) satisfies (O1), it suffices to check that \( f \star g \sqsubseteq f \). Consider the following three cases:

Case 1. If \( f = 1 \{1\} \), then \( f \star g = g \sqsubseteq 1 \{1\} = f \);

Case 2. If \( g = 1 \{1\} \), then \( f \star g = f \sqsubseteq f \);

Case 3. If \( f \neq 1 \{1\} \) and \( g \neq 1 \{1\} \), from (4.1), Lemmas 4, and \( f \sqcap g \sqsubseteq f \), it follows that \((f \star g)^L = (f \sqcap g)^L \geq f^L \) and \((f \star g)^R \leq (f \sqcap g)^R \leq f^R \). This, together with Proposition 1, implies that \( f \star g \sqsubseteq f \).
4.2. ★ satisfies \((O2)\)

For \(f, g, h \in L\),

B-1) if one of \(f\), \(g\), and \(h\) is equal to 1, then it is easy to verify that \((f \star g) \star h = f \star (g \star h)\);

B-2) if none of \(f\), \(g\), and \(h\) are equal to 1, from Lemmas 1 and 4, it follows that

\[
((f \star g) \star h)^L = ((f \star g) \cap h)^L = (f \star g)^L \cap h^L = f^L \cap g^L \cap h^L, \\
(f \star (g \star h))^L = (f \cap (g \star h))^L = f^L \cap (g \star h)^L = f^L \cap g^L \cap h^L,
\]

and, for \(x \in [0, 1)\),

\[
((f \star g) \star h)^R(x) = ((f \star g) \cap h)^R(x) = (f \star g)^R(x) \cap h^R(x) \\
= (f \cap g)^R(x) \cap h^R(x) = f^R(x) \cap g^R(x) \cap h^R(x), \\
(f \star (g \star h))^R(x) = (f \cap (g \star h))^R(x) = f^R(x) \cap (g \star h)^R(x) \\
= f^R(x) \cap (g \cap h)^R(x) = f^R(x) \cap g^R(x) \cap h^R(x).
\]

These imply that

\[
((f \star g) \star h)^L = (f \star (g \star h))^L,
\]

and, for \(x \in [0, 1)\),

\[
((f \star g) \star h)^R(x) = (f \star (g \star h))^R(x).
\]

To prove \((f \star g) \star h = f \star (g \star h)\), by applying Proposition 1–(6) and Proposition 11, it suffices to check that \((f \star g) \star h)^R(1) = (f \star (g \star h))^R(1)\). From Remark 5, Lemma 5, and proposition 12, it follows that

\[
((f \star g) \star h)(1) \\
= \begin{cases} 
1, & (f \star g)(1) \cap h(1) = 1, \\
0, & (f \star g)(1) \cap h(1) < 1,
\end{cases} \\
= \begin{cases} 
1, & f(1) \cap g(1) \cap h(1) = 1, \\
0, & f(1) \cap g(1) \cap h(1) < 1,
\end{cases} \tag{4.2}
\]

and

\[
(f \star (g \star h))(1) \\
= \begin{cases} 
1, & f(1) \cap (h \star g)(1) = 1, \\
0, & f(1) \cap (h \star g)(1) < 1.
\end{cases} \\
= \begin{cases} 
1, & f(1) \cap g(1) \cap h(1) = 1, \\
0, & f(1) \cap g(1) \cap h(1) < 1.
\end{cases} \tag{4.3}
\]

Thus, \((f \star g) \star h = f \star (g \star h)\).
4.3. $\bullet$ satisfies (O3)

This follows directly from Cases 1 and 2 of Definition 17.

4.4. $\bullet$ satisfies (O4′)

For $f, g, h \in L$, a claim is that $f \bullet (g \sqcup h) = (f \bullet g) \sqcup (f \bullet h)$. In fact, the following are true:

D-1) If $f = 1_{\{1\}}$, then $f \bullet (g \sqcup h) = g \sqcup h = (1_{\{1\}} \bullet g) \sqcup (1_{\{1\}} \bullet h) = (f \bullet g) \sqcup (f \bullet h)$. 

D-2) If $g = 1_{\{1\}}$, then $f \bullet (g \sqcup h) = f \bullet 1_{\{1\}} = f$. From Lemma 5, it follows that $f \bullet h \sqsubseteq f$. This implies that $(f \bullet g) \sqcup (f \bullet h) = f \sqcup (f \bullet h) = f$. Thus, $f \bullet (g \sqcup h) = (f \bullet g) \sqcup (f \bullet h)$.

D-3) If $h = 1_{\{1\}}$, since $\sqcup$ is commutative, applying D-2) yields that $f \bullet (g \sqcup h) = f \bullet (h \sqcup g) = (f \bullet g) \sqcup (f \bullet h)$.

D-4) If $f \neq 1_{\{1\}}, g \neq 1_{\{1\}},$ and $h \neq 1_{\{1\}}$, applying Lemmas 1 and 4, it can be verified that

$$(f \bullet (g \sqcup h))^L = ((f \bullet g) \sqcup (f \bullet h))^L,$$

and, for $x \in [0, 1)$,

$$(f \bullet (g \sqcup h))^R(x) = ((f \bullet g) \sqcup (f \bullet h))^R(x).$$

To prove that $f \bullet (g \sqcup h) = (f \bullet g) \sqcup (f \bullet h)$, applying Proposition 1–(6) and Remark 2, it suffices to check that $(f \bullet (g \sqcup h))(1) = ((f \bullet g) \sqcup (f \bullet h))(1)$.

Applying Remarks 2 and 5 yields that

$$(f \bullet (g \sqcup h))(1) = \begin{cases} 
0, & f(1) \land (g \sqcup h)(1) < 1, \\
1, & f(1) \land (g \sqcup h)(1) = 1, 
\end{cases}$$

and

$$((f \bullet g) \sqcup (f \bullet h))(1) = \begin{cases} 
0, & (f \bullet g)(1) \lor (f \bullet h)(1) < 1, \\
1, & (f \bullet g)(1) \lor (f \bullet h)(1) = 1, 
\end{cases}$$

Therefore, $f \bullet (g \sqcup h) = (f \bullet g) \sqcup (f \bullet h)$. 

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4.5. $\star$ satisfies \((O_4''')\)

For $f, g, h \in L$, a claim is that $f \star (g \cap h) = (f \star g) \cap (f \star h)$. In fact, the following are true:

E-1) If $f = 1_{\{1\}}$, then $f \star (g \cap h) = g \cap h = (1_{\{1\}} \star g) \cap (1_{\{1\}} \star h) = (f \star g) \cap (f \star h)$.

E-2) If $g = 1_{\{1\}}$, then $f \star (g \cap h) = f \star h$. From Lemma 5, it follows that $f \star h \sqsubseteq f$. This implies that $(f \star g) \cap (f \star h) = f \cap (f \star h) = f \star h$. Thus, $f \star (g \cap h) = (f \star g) \cap (f \star h)$.

E-3) If $h = 1_{\{1\}}$, since $\sqcap$ is commutative, applying E-2) yields that $f \star (g \cap h) = f \star (h \cap g) = (f \star h) \cap (f \star g) = (f \star g) \cap (f \star h)$.

E-4) If $f \neq 1_{\{1\}}, g \neq 1_{\{1\}},$ and $h \neq 1_{\{1\}}$, applying Lemmas 1 and 4, it can be verified that

$$(f \star (g \cap h))^L = ((f \star g) \cap (f \star h))^L,$$

and, for $x \in [0, 1)$,

$$(f \star (g \cap h))^R(x) = ((f \star g) \cap (f \star h))^R(x).$$

To prove that $f \star (g \cap h) = (f \star g) \cap (f \star h)$, applying Proposition 1–(6), it suffices to check that $(f \star (g \cap h))(1) = ((f \star g) \cap (f \star h))(1)$.

Applying Remarks 2 and 5 yields that

$$(f \star (g \cap h))(1) = \begin{cases} 0, & f(1) \land (g \cap h)(1) < 1, \\ 1, & f(1) \land (g \cap h)(1) = 1, \end{cases}$$

and

$$(f \star (g \cap h))(1) = \begin{cases} 0, & (f \star g)(1) \land (f \star h)(1) < 1, \\ 1, & (f \star g)(1) \land (f \star h)(1) = 1, \end{cases}$$

Thus, $f \star (g \cap h) = (f \star g) \cap (f \star h)$.

4.6. $\star$ satisfies \((O_5)\)

For $0 \leq a \leq b \leq 1$,

F-1) if $a = 1$, then $1_{[0,1]} \star 1_{[a,b]} = 1_{[0,1]} \star 1_{\{1\}} = 1_{[0,1]}$;

F-2) if $a < 1$, then consider the following two cases:
(i) $b < 1$, then $1_{[0,1]}(1) \land 1_{[a,b]}(1) = 0 < 1$. This, together with Definition 17, implies that

$$1_{[0,1]} \star 1_{[a,b]} = 1_{[0,1]} \overline{\land} 1_{[a,b]}.$$  

Since $(1_{[0,1]} \cap 1_{[a,b]})(1) = 0$, one has

$$1_{[0,1]} \star 1_{[a,b]} = 1_{[0,1]} \cap 1_{[a,b]} = 1_{[0,1]}.$$

(ii) $b = 1$, then $1_{[0,1]}(1) \land 1_{[a,b]}(1) = 1$. This, together with Definition 17, implies that

$$1_{[0,1]} \star 1_{[a,b]} = 1_{[0,1]} \cap 1_{[a,b]} = 1_{[0,1]}.$$

4.7. $\star$ satisfies (O6)

For $x_1, x_2 \in I$, consider the following two cases:

Case 1. If $x_1 = 1$ or $x_2 = 1$, it is clear that $1_{\{x_1\}} \star 1_{\{x_2\}} = 1_{\{x_2\}} \star 1_{\{x_1\}} \in J$.

Case 2. If $x_1 \neq 1$ and $x_2 \neq 1$, from Definition 17, it can be verified that $1_{\{x_1\}} \star 1_{\{x_2\}} = 1_{\{x_2\}} \star 1_{\{x_1\}} = 1_{\{x_1\}} \cap 1_{\{x_2\}} = 1_{\{x_1 \land x_2\}} \in J$.

4.8. $\star$ satisfies (O7)

For $[a_1, b_1], [a_2, b_2] \subset I$ with $[a_1, b_1] \neq \{1\}$ and $[a_2, b_2] \neq \{1\}$, from Definition 17, it follows that

$$1_{[a_1, b_1]} \star 1_{[a_2, b_2]} = \begin{cases} 1_{[a_1, b_1]} \cap 1_{[a_2, b_2]}, & b_1 = 1 \text{ and } b_2 = 1, \\ 1_{[a_1, b_1]} \overline{\land} 1_{[a_2, b_2]}, & b_1 < 1 \text{ or } b_2 < 1, \\ 1_{[a_1 \land a_2, b_1 \land b_2]}, & b_1 = 1 \text{ and } b_2 = 1, \\ 1_{[a_1 \land a_2, b_1 \land b_2]}, & b_1 < 1 \text{ or } b_2 < 1, \\ 1_{[a_1 \land a_2, b_1 \land b_2]} \in K. \\ \end{cases}$$

This, together with the commutativity of $\star$, implies that

$$1_{[a_1, b_1]} \star 1_{[a_2, b_2]} = 1_{[a_2, b_2]} \star 1_{[a_1, b_1]} = 1_{[a_1 \land a_2, b_1 \land b_2]} \in K.$$  

Clearly, $1_{[a_1, b_1]} \star 1_{[a_2, b_2]} = 1_{[a_2, b_2]} \star 1_{[a_1, b_1]} \in K$ when $[a_1, b_1] = \{1\}$ or $[a_2, b_2] = \{1\}$.

Combining 4.1–4.8 together immediately yields the following result.

**Theorem 2.** The binary operation $\star$ is a $t_{\lor}$-norm on $L$. In particular, $\star$ is a $t_r$-norm on $L$.  

\[17\]
5. ★ cannot be obtained by ∧

This section proves that the $t_{lor}$-norm ★ constructed in Section 4 cannot be obtained by operations ∧. This shows that the $t_{lor}$-norm ★ is not the convolution of each $t$-norm on $I$, answering negatively Question 2.

The following theorem provides a sufficient condition ensuring that * is a $t$-norm on $I$.

**Theorem 3.** [17, Theorem 12] Let * be a binary operation on $I$ and △ be a $t$-norm on $I$. If the binary operation ∧ is a $t_r$-norm on $L$, then △ is a continuous $t$-norm and * is a $t$-norm.

**Proposition 13.** Let * be a $t$-norm on $I$. Then, $x * y = 1$ if and only if $x = y = 1$.

**Theorem 4.** For any binary operation * on $I$ and any $t$-norm △ on $I$, there exist $f, g \in L$ such that $f ★ g \neq f ∧ g$. In particular, ★ is not the convolution of each $t$-norm on $I$.

**Proof.** Suppose, on the contrary, that there exist a binary operation * on $I$ and a $t$-norm △ on $I$ such that, for $f, g \in L$, one has $f ★ g = f ∧ g$. Applying Theorem 3, yields that * is a $t$-norm on $I$. Choose $f = 1_{[0,1]}$ and

$$g(x) = \begin{cases} 2x, & x \in [0, 0.5], \\ -x + 1.5, & x \in (0.5, 1]. \end{cases}$$

Clearly, $f, g \in L$. From Definition 17, it is easy to see that $(f ★ g)(1) = 0$, since $f(1) ∧ g(1) < 1$. This, together Theorem 2 and Proposition 13, implies that

$$(f ∧ g)(1) = f(1) * g(1) = 1 * 0.5 = 0.5 \neq (f ★ g)(1),$$

which contradicts with $f ★ g = f ∧ g$. \qed

**Remark 7.** Combining Theorems 2 and 4 negatively answers Question 2.

6. Conclusion

Continuing our study in [16, 17], this paper constructs two binary operations ⊗ and ★ on $I^{[2]}$ and $L$, respectively (see Definitions 15 and 17), and proves that

(i) the binary operation ⊗ satisfies conditions (4) and (5) in Definition 6, but does not satisfy condition $(5')$;

(ii) the binary operation ★ is a $t_{lor}$-norm on $L$, but not the convolution of any $t$-norms on $I$.

These two results negatively answer Questions 1 and 2 originally posed by Walker and Walker in [15].
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