ON THE MILNOR FIBRATION FOR $f(z)\bar{g}(z)$

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ABSTRACT. We consider a mixed function of type $H(z, \bar{z}) = f(z)\bar{g}(z)$ where $f$ and $g$ are convenient holomorphic functions which have isolated critical points at the origin and we assume that the intersection $f = g = 0$ is a complete intersection variety with an isolated singularity at the origin and $H$ satisfies the multiplicity condition (2.3.2). We will show that $H$ satisfies Hamm-Lê condition. In particular, $H$ has a Milnor fibration at the origin. We give examples which does not satisfy the Newton multiplicity condition where one does not have Milnor fibration and the other has Milnor fibration.

1. Introduction

Let $f(z, \bar{z})$ be a mixed function with $f = g + i h$ where $g, h$ are real valued analytic functions of $n$ complex variables $z_1, \ldots, z_n$ or of $2n$ real variables $\{x_j, y_j \mid j = 1, \ldots, n\}$. Here $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $z_j = x_j + i y_j (j = 1, \ldots, n)$ with $x_j, y_j \in \mathbb{R}$. The mixed hypersurface $\{f = 0\}$ can be understood as the real analytic variety in $\mathbb{R}^{2n}$ defined by $\{g = h = 0\}$. $g, h$ are real valued real analytic functions of variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ but they can be considered as mixed functions by the substitution $x_j = (z_j + \bar{z}_j)/2$, $y_j = -(z_j - \bar{z}_j)/2$. For a mixed function $k(z, \bar{z})$, we use the following notations as in [17].

$$dk = (d_x k, d_y k) \in \mathbb{R}^{2n} \text{ where }$$

$$d_x k = \left( \frac{\partial k}{\partial x_1}, \ldots, \frac{\partial k}{\partial x_n} \right), \quad d_y k = \left( \frac{\partial k}{\partial y_1}, \ldots, \frac{\partial k}{\partial y_n} \right)$$

$z = x + i y \in \mathbb{C}^n$ and $(x, y) \in \mathbb{R}^{2n}$ are identified. The holomorphic gradient and the anti-holomorphic gradient of $k$ are defined by

$$\partial k := \left( \frac{\partial k}{\partial z_1}, \ldots, \frac{\partial k}{\partial z_n} \right), \quad \bar{\partial} k := \left( \frac{\partial k}{\partial \bar{z}_1}, \ldots, \frac{\partial k}{\partial \bar{z}_n} \right).$$

Note that if $k$ is real-valued, we have the equality $\bar{\partial} k = \partial k$, and real gradient vector $dk \in \mathbb{R}^{2n}$ corresponds to the complex gradient vector $2\bar{\partial} k \in \mathbb{C}^n$ under the canonical correspondence

$$\mathbb{R}^{2n} \ni (x, y) \iff z = x + i y \in \mathbb{C}^n.$$
Proposition 1 (Proposition 1 [14], Lemma 2 [17]). Let \( f(z, \bar{z}) \) be a mixed function and put \( f = g + ih \) as before. The next conditions are equivalent.

1. \( a \in \mathbb{C}^n \) is a critical point of the mapping \( f : \mathbb{C}^n \to \mathbb{C} \).
2. \( dg(a), dh(a) \) are linearly dependent in \( \mathbb{R}^{2n} \) over \( \mathbb{R} \).
3. \( \partial g(a, \bar{a}), \partial h(a, \bar{a}) \) are linearly dependent in \( \mathbb{C}^n \) over \( \mathbb{R} \).
4. There exists a complex number \( \alpha \) with \( |\alpha| = 1 \) such that \( \partial f(a, \bar{a}) = \alpha \bar{f}(a, \bar{a}) \).

Under the above equivalent conditions, we say that \( a \) is a critical point or a mixed singular point of the mixed function \( f \). For brevity, we say simply a singular point in the sense of a mixed singular point.

Lemma 2 (Lemma 2, [17], cf [6]). Consider a mixed hypersurface \( V_\eta = f^{-1}(\eta) \) and take \( p \in V_\eta \). Assume that \( p \) is a non-singular point of \( V_\eta \) and let \( k(z, \bar{z}) \) be a real valued mixed function on \( \mathbb{C}^n \). The following conditions are equivalent.

1. The restriction \( k|V_\eta \) has a critical point at \( p \in V_\eta \).
2. There exist a complex number \( \alpha \) such that
   \[
   \partial k(p) = \alpha \bar{f}(p, \bar{p}) + \bar{\alpha} f(p, \bar{p}).
   \]
3. There exist real numbers \( c, d \) such that
   \[
   \partial k(p) = c \bar{g}(p, \bar{p}) + d h(p, \bar{p}).
   \]

2. Fibration problem for function \( fg \)

2.1. Non-degenerate mixed functions. Let \( f(z, \bar{z}) \) be a mixed function of \( n \)-variables \( z = (z_1, \ldots, z_n) \). Recall that \( f \) can be expanded in a convergent power series of mixed monomials \( z^\nu \bar{z}^\mu \). In [15], we have generalized the concept of Newton boundary \( \Gamma(f) \) and defined strong non-degeneracy for a mixed function. Recall that \( f \) is strongly non-degenerate if for any face \( \Delta \) of \( \Gamma(f) \), the face function \( f_\Delta \), restricted on \( \mathbb{C}^{*n} \) is surjective onto \( \mathbb{C} \) and has no critical points. A convenient strongly non-degenerate function has a Milnor fibration ([15]).

2.2. Setting of our problem. In this paper, we consider a mixed function \( H \) which take the form \( H(z, \bar{z}) = f(z)\bar{g}(z) \) where \( f, g \) are holomorphic functions. Here we mean \( \bar{g}(z) := \overline{g(z)} \). We consider hypersurfaces \( V(f) := f^{-1}(0), V(g) := g^{-1}(0) \), \( V(H) := H^{-1}(0) \) and the intersection variety \( V(f, g) = V(f) \cap V(g) \). Note that \( H \) is not strongly non-degenerate for \( n \geq 3 \) by the following reason. Suppose \( H \) is non-degenerate. Any points of the intersection \( V(f) \cap V(g) \) are singular points of \( V(H) \), while a convenient strongly non-degenerate mixed function has an isolated singularity at the origin by Corollary 20 of [15]. This is an obvious contradiction. Thus the mixed function \( H(z, \bar{z}) \) is far from a non-degenerate mixed function for \( n \geq 3 \). However \( H(z, \bar{z}) \) is a very special type of mixed function, as it is
defined by two holomorphic functions $f, g$. The mixed hypersurface $V(H)$ is simply union of two complex analytic hypersurfaces $V(f)$ and $V(g)$ as a set but $V(g)$ is conjugate oriented by $\bar{g}$. We consider the existence of Milnor fibration for such a mixed function. Pichon and Seade have studied such functions, especially for the case $n = 2$ (\cite{20,21,22}). There are also works by M. Tibar and others (\cite{19,4,5,9}). Note that the link of $H$ is the union of two smooth links by $f$ and $g$ respectively which intersect transversely along real codimension 2 smooth variety. However the link of $g$ is oriented by $\bar{g}$.

2.3. Basic assumption.

2.3.1. Isolatedness. Unless otherwise stated, we assume that

1. $f$ and $g$ are holomorphic functions such that $V(f), V(g)$ have isolated singularity at the origin.
2. The intersection variety $V(f, g) := \{f = g = 0\}$ has an isolated singularity at the origin and a smooth complete intersection variety outside of the origin.

We fix a positive number $r_0 > 0$ so that $V(f), V(g), V(f, g)$ are only singular at the origin in the ball $B_{2r_0}$ and for any sphere $S^{2n-1}_{r_0}$ of radius $r$ with $0 < r \leq r_0$ intersect transversely with these varieties.

2.3.2. Multiplicity condition. There is another important condition for $H = \bar{f}g$ to be fibered. We say that $H$ satisfies the multiplicity-condition if there exists a good resolution $\pi : X \to \mathbb{C}^n$ of the holomorphic function $h = fg$ such that

1. $\pi : X \setminus \pi^{-1}(0) \to \mathbb{C}^n \setminus \{0\}$ is biholomorphic and the divisor defined by $\pi^* (fg) = 0$ has only normal crossing singularities and the respective strict transforms $\bar{V}(f), \bar{V}(g)$ of $V(f)$ and $V(g)$ are smooth.
2. Put $\pi^{-1}(0) = \cup_{j=1}^s D_j$ where $D_1, \ldots, D_s$ are smooth compact divisors in $X$. Denote the respective multiplicities of $\pi^* f$ and $\pi^* g$ along $D_j$ by $m_j$ and $n_j$. Then $m_j \neq n_j$ for $j = 1, \ldots, s$.

Note that $\pi$ does not resolve completely the singularities of $V(h)$ but it resolves singularities of $V(f)$ and $V(g)$.

2.4. Key Lemma. We consider the following key property for the existence of the tubular Milnor fibration.

(SN)(Smoothness of nearby fibers) There exists a positive number $r_1$ and $\delta$ such that $H^{-1}(\eta) \cap B_{r_1}^2$ is smooth for any $\eta$, $0 < |\eta| \leq \delta$.

The following lemma shows that (SN) condition follows from the multiplicity condition.

Lemma 3 (Smoothness of the nearby fibers). Under the assumption (1), (2) and the multiplicity-condition, there exist positive numbers $r_1, r_1 \leq r_0$ and $\delta < r_1$ such that the nearby fiber $V_\eta := H^{-1}(\eta)$ has no mixed singularity in the ball $B_{r_1}^2$ for any non-zero $\eta$ with $|\eta| \leq \delta$. 


Proof. We denote \( \pi^{-1}(0) \) by \( D \). Recall that \( D = D_1 \cup \cdots \cup D_s \). For simplicity, we put \( D_{s+1} = \bar{\nabla}(f) \) and \( D_{s+2} = \bar{\nabla}(g) \). In this notation, we put \( m_{s+1} = 1, m_{s+2} = 0 \) and \( n_{s+1} = 0, n_{s+2} = 1 \). Take an arbitrary point \( p \in D \) and assume that \( p \in \bigcup_{j \in J} D_j \setminus \bigcup_{j \notin J} D_j \) where \( J \subseteq \{1, \ldots, s+2\} \). By the assumption (1), \(|J| \leq n\). Then there is a local holomorphic chart \( U_p \) with coordinates \((u_1, \ldots, u_n)\) and an injective map \( \tau : J \rightarrow \{1, \ldots, n\} \) so that \( u_{\tau(j)} = 0 \) defines \( D_j \) in \( U_p \) and by the multiplicity assumption (i) and (ii), we can write
\[
\pi^* f = k_f \prod_{j \in J} u_{\tau(j)}^{m_j}, \quad \pi^* g = k_g \prod_{j \in J} u_{\tau(j)}^{n_j},
\]
where \( k_f, k_g \) are units on \( U_p \). We choose \( U_p \) small enough so that \( U_p \cap \bigcup_{j \notin J} D_j = \emptyset \). Consider the pull-back \( \tilde{H} := \pi^* H \). By the assumption, we can write \( \tilde{H} \) in \( U_p \) as
\[
\tilde{H} = k_f k_g \prod_{j \in J} u_{\tau(j)}^{m_j} u_{\tau(j)}^{n_j},
\]
where \( J \cap \{1, \ldots, s\} \neq \emptyset \) as \( p \in D \). Now we compute the holomorphic and anti-holomorphic gradient vectors of \( \tilde{H} \) in \( U_p \). Put
\[
\partial \tilde{H} = (\tilde{H}_1, \ldots, \tilde{H}_n), \quad \partial \tilde{H} = (\tilde{H}'_1, \ldots, \tilde{H}'_n)
\]
where
\[
\tilde{H}_j = \frac{\partial \tilde{H}}{\partial u_j} \quad \text{and} \quad \tilde{H}'_j = \frac{\partial \tilde{H}}{\partial \bar{u}_j}.
\]
Then by (1), we can write
\[
\tilde{H}_{\tau(j)} = \frac{\partial \tilde{H}}{\partial u_{\tau(j)}} = u_{\tau(j)}^{m_j-1} \bar{u}_{\tau(j)}^{n_j} \left( m_j + u_{\tau(j)} \frac{\partial k_f}{\partial u_{\tau(j)}} \right) \prod_{k \in J, k \neq j} u_{\tau(k)}^{m_k} \bar{u}_{\tau(k)}^{n_k},
\]
\[
\tilde{H}'_{\tau(j)} = \frac{\partial \tilde{H}}{\partial \bar{u}_{\tau(j)}} = u_{\tau(j)}^{m_j} \bar{u}_{\tau(j)}^{n_j-1} \left( n_j + \bar{u}_{\tau(j)} \frac{\partial k_g}{\partial \bar{u}_{\tau(j)}} k_f \right) \prod_{k \in J, k \neq j} u_{\tau(k)}^{m_k} \bar{u}_{\tau(k)}^{n_k}.
\]
Take one \( j \in J \cap \{1, \ldots, s\} \). As \( m_j \neq n_j \), we can see that
\[
|\tilde{H}_{\tau(j)}| \approx |u_{\tau(j)}|^{m_j+n_j-1} m_j \prod_{k \in J, k \neq j} |u_{\tau(k)}|^{m_k+n_k},
\]
\[
|\tilde{H}'_{\tau(j)}| \approx |u_{\tau(j)}|^{m_j+n_j-1} n_j \prod_{k \in J, k \neq j} |u_{\tau(k)}|^{m_k+n_k}.
\]
Therefore \( |\tilde{H}_{\tau(j)}/\tilde{H}'_{\tau(j)}| \approx m_j/n_j \neq 1 \) as \( m_j \neq n_j \) by the multiplicity condition. Thus we can take a smaller neighborhood \( U'_p \) if necessary and we may assume that \( |\tilde{H}_{\tau(j)}| \neq |\tilde{H}'_{\tau(j)}| \) for any \( U'_p \subseteq U_p \cup D_{s+1} \cup D_{s+2} \). Therefore by Proposition \( \tilde{H} : U'_p \setminus (D \cup D_{s+1} \cup D_{s+2}) \rightarrow \mathbb{C}^* \) has no critical point. We do this operation for any \( p \in D \). As \( D \) is compact, we find a finite points \( p_1, \ldots, p_\mu \) such that \( \bigcup_{i=1}^\mu U'_{p_i} \supseteq D \). Put \( W = \bigcup_{i=1}^\mu U'_{p_i} \). \( W \) is an open subset containing \( D \) so that \( \tilde{H} : W \setminus (D \cup D_{s+1} \cup D_{s+2}) \rightarrow \mathbb{C}^* \) has no critical point.
Put $W' = \pi(W)$. As $D = \pi^{-1}(0)$, $W'$ is an open neighborhood of the origin in $\mathbb{C}^n$. As $\pi : W \setminus (D \cup D_{s+1} \cup D_{s+2}) \to W' \setminus H^{-1}(0)$ is biholomorphic, this implies $H : W' \setminus H^{-1}(0) \to \mathbb{C}^*$ has no critical point. This proves the assertion. \hfill \Box

**Remark 4.** The multiplicity condition is a sufficient condition for the smoothness of the nearby fibers but it is not always a necessary condition.

There is a paper by Parameswaran and Tibar ([19]) where they gives a condition for the smoothness of the nearby fibers. Unfortunately their assertions Lemma 2.5 in [19] is not correct. Thus Theorem 2.3([19]), Theorem 4.3 ([1]) are not correct without assuming (SN) condition. Compare with Theorem [29] and Lemma [33] which we discuss later. Lemma [33] is a corrected version of Lemma 2.5, [19].

2.5. **Thom’s $a_f$-regularity and Hamm-Lê type Lemma.** The following follows from Lemma 3 and Corollary 4.1, [19].

**Lemma 5.** Assume that $f, g$ satisfy isolatedness assumption (1) and (2) and the multiplicity condition. Then $H$ satisfies $a_f$-regularity.

We give a brief proof of this assertion later ([32,1]). It is well-known that $a_f$ condition implies the transversality of the nearby fibers (Proposition 11, [17]). The following lemma corresponds to Lemma (2.1.4), [8]. Let $r_1 \leq r_0$ be a small enough positive number as in Lemma [3].

**Lemma 6.** Assume that $H$ satisfies (1), (2) and the smoothness of the nearby fibers (SN). Then for any $r_2, r_2 \leq r_1$ fixed, there exists a positive number $\delta > 0$ which depends on $r_2$ such that for any $r, r_2 \leq r \leq r_1$ and $\eta \neq 0, |\eta| \leq \delta$, the sphere $S_r$ and the nearby fiber $V(\eta) := H^{-1}(\eta)$ intersect transversely.

By Lemma [6] and Ehresman’s fibration theorem [28], we have the following tubular Milnor fibration theorem.

**Main Theorem 7.** Let $H = f \bar{g}$ as above and assume that $H$ satisfies the basic assumption (1), (2) and the condition (SN). Let $r_1$ be as in Lemma [6]. Take a sufficiently small $\delta, 0 < \delta \ll r_1$ and put

$$E(r_1, \delta)^* := \{ z \in B_{r_1}^{2n} | 0 \neq |H(z)| \leq \delta \}$$

and $D^*_\delta := \{ \eta \in \mathbb{C} | 0 \neq |\eta| \leq \delta \}$. Then $H : E(r_1, \delta)^* \to D^*_\delta$ is a locally trivial fibration and its topological equivalence class does not depend on the choice of $r_1$ and $\delta$.

**Corollary 8.** Assume that $H$ satisfies the isolatedness condition (1), (2) and the multiplicity condition. Then $H$ has a tubular Milnor fibration.
2.6. Spherical Milnor fibration. We now consider the spherical Milnor mapping \( \varphi : S^{2n-1} \setminus K_\tau \rightarrow S^1 \) defined by \( \varphi(z) := H(z) / |H(z)| \) where \( K_\tau = V(H) \cap S^{2n-1}_\tau \). We need stronger assumption than the basic assumption. We assume in this subsection that

- \((n1)\) \( f(z) \) and \( g(z) \) are convenient non-degenerate holomorphic functions in the neighborhood of the origin with respect to the Newton boundary.
- \((n2)\) \( V(f, g) = \{ z \in \mathbb{C}^n \mid f(z) = g(z) = 0 \} \) is a non-degenerate complete intersection variety in the sense of Newton boundary \([13]\).

The hypersurfaces \( V(f) \) and \( V(g) \) have isolated singularities at the origin by the convenience and non-degeneracy assumption \((n1)\). The intersection variety \( V(f, g) \) has also an isolated singularity at the origin and the intersection of \( V(f), V(g) \) are transverse outside of the origin by \((n2)\).

2.7. Newton multiplicity condition. We further consider the following condition. We say that \( H \) satisfies the Newton multiplicity condition if for any strictly positive weight vector \( P \), weighted degrees of \( f \) and \( g \) under \( P \) are not equal, i.e. \( d(P; f) \neq d(P; g) \).

The Newton multiplicity condition can be checked by the Newton boundaries \( \Gamma(f) \) and \( \Gamma(g) \) as follows.

**Proposition 9.** Assume that \( f, g \) have convenient Newton boundaries. Then \( H \) satisfies Newton multiplicity condition if and only if \( \Gamma(f) \cap \Gamma(g) = \emptyset \).

**Proof.** Assume that \( \Gamma(f) \cap \Gamma(g) = \emptyset \). Then by the convenience assumption, this implies either

- \((a)\) \( \Gamma(f) \) is strictly above \( \Gamma(g) \) or
- \((b)\) \( \Gamma(g) \) is strictly above \( \Gamma(f) \).

In the case of \((a)\) for example, \( \Gamma_-(f) \) includes \( \Gamma(g) \) in its interior. Here \( \Gamma_-(f) \) is the cone of \( \Gamma(f) \) and the origin \( 0 \):

\[
\Gamma_-(f) := \{ t\nu \mid \nu \in \Gamma(f), 0 \leq t \leq 1 \}.
\]

Obviously by the convenience assumption, \((a)\) implies \( d(P; f) > d(P; g) \) for any weight vector \( P \) (respectively \((b)\) implies \( d(P; f) < d(P; g) \)).

Suppose \( \Gamma(f) \cap \Gamma(g) \neq \emptyset \). Then we can find a hyperplane \( L : a_1 x_1 + \cdots + a_n x_n = d \) \((a_i \geq 0, \forall i, d > 0)\) which is tangent to \( \Gamma(f) \) and \( \Gamma(g) \) in the following sense. Namely \( L \cap \Gamma(f) \neq \emptyset, L \cap \Gamma(g) \neq \emptyset \) and \( L_+ \supset \Gamma(f) \) and \( L_+ \supset \Gamma(g) \) where \( L_+ := \{ x \in \mathbb{R}^n \mid a_1 x_1 + \cdots + a_n x_n \geq d \} \). Then considering the weight vector \( P = (a_1, \ldots, a_n) \), we have \( d(P; f) = d(P; g) = d \).

Taking an admissible toric modification \( \tilde{\pi} : X \rightarrow \mathbb{C}^n \) for the dual Newton diagram \( \Gamma^*(fg) \), as a good resolution, it is clear that

**Proposition 10.** \((n1), (n2)\) and Newton multiplicity condition implies \((1), (2)\) and the multiplicity condition.
Lemma 11. We assume (n1), (n2) and Newton multiplicity condition. There exists a positive number \( r_3 \) so that \( \varphi : S^{2n-1}_r \setminus K \to S^1 \) has no critical points on for any \( r, 0 < r \leq r_3 \).

Proof. By Lemma 30 in [15], \( z \in S^{2n-1}_r \setminus K \) is a critical point of \( \varphi \) if and only if two vectors \( i (\partial \log H(z, \bar{z}) - \partial \log H(z, \bar{z})) \) and \( z \) are linearly dependent over \( \mathbb{R} \). Assume the assertion does not hold. Using the Curve Selection Lemma ([12, 7]), we can find an analytic path \((z(t), \lambda(t)) \in \mathbb{C}^n \times \mathbb{R} \) for \(0, 1\) such that for \( t \neq 0, H(z(t), \bar{z}(t)) \neq 0 \) and \( \lambda(t) \neq 0 \) and the following equality is satisfied.

\[
i (\partial \log H(z(t), \bar{z}(t)) - \partial \log H(z(t), \bar{z}(t))) = \lambda(t)z(t).
\]

Using \( H = fg \), this reduces to

\[
i \left( \frac{\partial f(z(t))}{f(z(t))} - \frac{\partial g(z(t))}{g(z(t))} \right) = \lambda(t)z(t), \quad \text{or equivalently} \quad (2) \quad i \left( \frac{f_j(z(t))}{f(z(t))} - \frac{g_j(z(t))}{g(z(t))} \right) = \lambda(t)z_j(t), \quad j = 1, \ldots, n
\]

where \( f_j, g_j \) are partial derivatives. Put \( m_f = \text{ord } f(z(t)) \) and \( m_g = \text{ord } g(z(t)) \).

Consider the expansion of \( z(t) \) and \( \lambda(t) \):

\[
z_j(t) = b_j \ell^{p_j} + \text{(higher terms)}, \quad \lambda(t) = \lambda_0 \ell^{a} + \text{(higher terms)}, \quad a \in \mathbb{Z}, \quad \lambda_0 \in \mathbb{R}^*.
\]

Put \( I = \{ j | z_j(t) \neq 0 \} \), \( P = \{ p_j \}_{j \in I} \subseteq \mathbb{N}_+ \) and put

\[
\ell = \min \{ d(P; f^l) - m_f, d(P; g^l) - m_g \}.
\]

To apply the non-degeneracy condition, we may assume \( P \) is a weight vector of \( z \) putting \( p_j \) sufficiently big for \( j \notin I \) (see [13]). Then we have the estimation:

\[
\text{ord } \left\{ i \left( \frac{f_j(z(t))}{f(z(t))} - \frac{g_j(z(t))}{g(z(t))} \right) \right\} \geq \ell - p_j,
\]

where \( \text{ord } \lambda(t)z_j(t) = a + p_j \). If \( \ell - p_j > a + p_j \) for some \( j \), we get a contradiction \( \lambda_0 b_j = 0 \). Thus we must have \( \ell - p_j \leq a + p_j \) for any \( j \). Thus we have

\[
a \geq \ell - 2p_{\text{min}}
\]

where \( p_{\text{min}} := \min \{ p_j | j \in I \} \) Put

\[
\varepsilon_f := \begin{cases} 1, & \ell = d(P; f) - m_f \\ 0, & \ell < d(P; f) - m_f \end{cases}, \quad \varepsilon_g := \begin{cases} 1, & \ell = d(P; g) - m_g \\ 0, & \ell < d(P; g) - m_g \end{cases}
\]

Case 1. Assume that \( a > \ell - 2p_{\text{min}} \). Then \( a + p_j > \ell - p_j \) for any \( j \in I \).
(a) Assume that \(\varepsilon_f \varepsilon_g = 0\). Then by [2], we get a contradiction to the non-degeneracy assumption \(\partial f_P(b) = 0\) or \(\partial g_P(b) = 0\).

(b) Assume \(\varepsilon_f \varepsilon_g = 1\). Then we get a linear relation on \(\partial f_P(b)\) and \(\partial g_P(b)\).

(b-1) Assume \(\ell < 0\). Then \(m_f > d(P; f), m_g > d(P; g)\) which implies \(f_P(b) = g_P(b) = 0\) and thus \(b \in V(f_P, g_P)\). Thus [2] gives a contradiction to the non-degeneracy of \(V(f, g)\).

(b-2) If \(\ell = 0\), this implies \(m_f = d(P; f)\) and \(m_g = d(P; g)\). That is, \(f_P(b) \neq 0\) and \(g_P(b) \neq 0\) and we have equality:

\[
\frac{\bar{f}_{P,j}(b)}{f_P(b)} - \frac{\bar{g}_{P,j}(b)}{g_P(b)} = 0.
\]

Multiplying \(p_j \bar{b}_j\) and adding for \(j \in I\), using Euler equality we get the equality

\[
d(P; f) - d(P; g) = 0\]

which is a contradiction to the Newton multiplicity-condition of \(H\).

Case 2. Assume that \(a = \ell - 2p_{min}\). Put \(J = \{j \in I | p_j = p_{min}\}\). Put \(\beta_f, \beta_g\) be the leading coefficients of \(f(z(t))\) and \(g(z(t))\) respectively. Then by [2], we get

\[
i \left( \varepsilon_f \frac{\bar{f}_{P,j}(b)}{\beta_f} - \varepsilon_g \frac{\bar{g}_{P,j}(b)}{\beta_g} \right) = \begin{cases} \lambda_0 b_j, & j \in J \\ 0, & j \notin J. \end{cases}
\]

Now we consider the equality \(\Re(i \log f(z(t)) \bar{g}(z(t))) \equiv 0\). Taking the differential of this equality and using the same calculation as in §5.2, [15], we get

\[
\Re \left\{ \sum_{j \in J} i \left( \frac{f_j(z(t))}{f} - \frac{g_j(z(t))}{g} \right) \frac{dz_j(t)}{dt} \right\} = 0
\]

Here we have used the equality

\[
\Re \left\{ i \sum_{j \in J} \frac{g_j}{g} (z(t)) \frac{dz_j(t)}{dt} \right\} = -\Re \left\{ i \sum_{j \in J} \frac{g_j}{g} (z(t)) \frac{dz_j(t)}{dt} \right\}.
\]

Taking the leading coefficients of the above equality and using [3], we get

\[
\Re \sum_{j \in J} |\lambda_0 b_j|^2 = 0
\]

However the left side is non-zero as \(\lambda_0\) is a non-zero real number which is a contradiction. \(\square\)

**Corollary 12** (Spherical Milnor fibration). Assuming (n1), (n2) and Newton multiplicity condition, \(\varphi : S^{2n-1} \setminus K_r \rightarrow S^1\) gives a local trivial fibration for any \(r \leq \min\{r_3, r_2\}\) where \(r_2\) is a positive number in Lemma [6]. Here \(K_r = H^{-1}(0) \cap S^{2n-1}\).
The proof of equivalence of tubular Milnor fibration and spherical Milnor fibration (under (n1),(n2) and Newton multiplicity condition) is left to the readers. For the spherical Milnor fibration stated in Corollary [12], we do not know if (1),(2) and multiplicity condition is enough or not. There is a paper [5] where authors study spherical fibration in more general setting.

3. Topology of the Milnor fiber

3.1. Fundamental group of the Milnor fiber. Let $H = f\bar{g}$ and $h = fg$. We assume that $H$ satisfies assumption (1), (2) and (SN). Let $F_h$ and $F_H$ be the Milnor fibers of $h$ and $H$ respectively. $F_H$ is connected (see [18]). As $\{H = 0\} = \{h = 0\}$ as a set, $\pi_1(B_r \setminus \{H = 0\})$ is abelian by [11], thus it is isomorphic to $\mathbb{Z}^2$. We have two different Milnor fibrations with the same ambient space:

$$h, H : E(r_1, \delta)^* \to D_\delta^*.$$

Using the homotopy exact sequence of the Milnor fiberings, we conclude

**Proposition 13.** $\pi_1(F_h)$ and $\pi_1(F_H)$ are isomorphic to the cyclic group $\mathbb{Z}$.

3.2. Complex subspace of the tangent space of the Milnor fiber. In general, the tangent space of a mixed hypersurface does not have a complex structure. However in our case, we have the following assertion. We assume that $r > 1$ is sufficiently small so that $f$ and $g$ has no critical points in $B_r^{2n}\setminus\{0\}$.

**Proposition 14.** Let $H = f\bar{g}$ be as in Theorem [7] and consider a Milnor fiber $V_\eta := H^{-1}(\eta) \cap B_r^{2n}$ in the tubular Milnor fibration. For a point $p \in V_\eta$, $T_pV_\eta$ contains a complex subspace of dimension $n - 2$.

**Proof.** Put $a = f(p)$ and $b = g(p)$. Then we have $\eta = ab$. Consider two hypersurfaces $V(f, a) := f^{-1}(a)$ and $V(g, b) := g^{-1}(b)$ and their complex tangent spaces

$$T_pV(f, a), \quad T_pV(g, b).$$

They are complex subspaces of dimension $n - 1$ of the ambient space $\mathbb{C}^n$ and they are complex perpendicular to the conjugate gradient vectors $\partial f(p)$ and $\partial g(p)$. We assert $T_pV(f, a) \cap T_pV(g, b) \subset T_pV_\eta$. In fact take an arbitrary tangent vector $v \in T_pV(f, a) \cap T_pV(g, b)$ and take a smooth curve $z(t)$ with $z(0) = p$ and $dz/dt(0) = v$. Then

$$\frac{dH}{dt}|_{t=0} = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(p) \frac{dz_j}{dt}(0) \bar{g}(p) + \sum_{j=1}^{n} f(p) \frac{\partial g}{\partial z_j}(p) \frac{dz_j}{dt}(0)$$

$$= \bar{g}(p)(v, \nabla f(p)) + f(p)(v, \nabla g(p)) = 0.$$

This proves $T_pV_\eta \supset T_pV(f, a) \cap T_pV(g, b)$. In the special case that $\partial f(p)$ and $\partial g(p)$ are linearly dependent at $p$, $T_pV(f, a) = T_pV(g, b)$ and the entire space $T_pV_\eta$ has a complex structure.
In the general case when $\partial f(p)$ and $\partial g(p)$ are linearly independent at $p$, an alternative argument can be given as follows. Intersection variety $V(f, g; a, b) := f^{-1}(a) \cap g^{-1}(b)$ is smooth at $p$ and we have an obvious inclusion $V(f, g; a, b) \subset V_\eta$. Thus the tangent space $T_p V(f, g; a, b)$ is an $(n - 2)$-dimensional complex subspace of $T_p V_\eta$. □

3.2.1. Proof of Lemma □ Now we are ready to prove Lemma □ Assume that $H = fg$ satisfies isolatedness condition (1),(2) and the multiplicity condition. Take $r_0$ as before. The hypersurface $V(H) \cap B_{r_0}^{2\eta}$ has a canonical stratification by 4 complex analytic strata:

$$V(f)' = V(f) \setminus V(f, g), V(g)' = V(g) \setminus V(f, g), V(f, g)' = V(f, g) \setminus \{0\}, \{0\}.$$

Consider a sequence of points $p_\nu, \nu = 1, 2, \ldots$ such that $p_\nu \to p_0 \in V(H) \setminus \{0\}$. We have to show that the limit (if it exists) of the tangent space $T_{p_\nu} H^{-1}(p_\nu)$ contains the tangent space of the stratum which $p_0$ belongs to. Note that $H$ has no critical point on $V(f)' \cup V(g)'$. Thus the $a_f$-regularity is obvious if $p_0 \in V(f)' \cup V(g)'$. Assume that $p_0 \in V(f, g)'$. Then in the neighborhood of $p_0$, $f$ and $g$ have independent gradient vectors by the assumption (2). As the tangent space $T_{p_\nu} H^{-1}(p_\nu)$ contains the intersection $T_{p_\nu} V(f, f(p_\nu)) \cap T_{p_\nu} V(g, g(p_\nu))$ by Proposition □ the limit of $T_{p_\nu} H^{-1}(p_\nu)$ include $T_p V(f, g)'$. □

3.3. Jacobian curve. We consider the critical locus of the mapping $(f, g) : \mathbb{C}^n \to \mathbb{C}^2$:

$$J(f, g) := \{z \in \mathbb{C}^n | \partial f(z), \partial g(z) \text{ are linearly dependent}\}$$

to study the topology of $V_\eta$. We call $J(f, g)$ the Jacobian curve of $(f, g)$ or simply the Jacobian curve of $h$. We assume that

$$(\ast): J \text{ is a one-dimensional curve at the origin.}$$

In the case that $g$ is a linear form, $J$ is usually called a polar curve. Applying a suitable Morse function $\varphi : V_\eta \to \mathbb{R}_+$ given by a square of the distance from a generic point near the origin, we get the following assertion.

Corollary 15. Under the assumption $(\ast)$, the Milnor fiber $V_\eta$ of $H$ has a homotopy type of at most $n$-dimensional CW-complex.

Proof. Consider a Morse function $\varphi$ and assume that $p \in V_\eta$ is a critical point. Assume first that $\partial f(p)$ and $\partial g(p)$ are linearly independent at $p$. Then $V(f, g; a, b) \subset V_\eta$ and a smooth complex subvariety of dimension $n - 2$. Therefore the index of of the restriction of $\varphi | V(f, g; a, b)$ is at most $n - 2$ by the Andreotti-Frankel or Milnor type argument (□, Lemma (2.4.1) □). Thus the index of $\varphi$ on $V_\eta$ is at most $n - 2 + 2 = n$. Consider now the case that $V_\eta \cap \Gamma$ contains some critical point of $\varphi$. By the assumption on $\Gamma$, we may assume that $\Gamma \cap V_\eta$ is a finite set, taking a sufficiently small $\eta$. Then we modify $\varphi$ a little if necessary so that $\varphi$ has no critical point on $\Gamma \cap V_\eta$. Then the assertion follows from the above discussion. □
3.4. Relation of the critical curves of $H$ and the Jacobian curve. Let $H = fg$ as before and let $C(H)$ be the closure of the critical locus of $H : \mathbb{C}^n \to \mathbb{C}$ outside of $V(H)$.

Lemma 16. We have the canonical inclusion $C(H) \subset J(f, g)$.

Proof. Assume that $z \in C(H) \setminus V(H)$. By Proposition 1 there exists a complex number $\alpha$ with $|\alpha| = 1$ so that $\partial H(z) = \alpha \partial H(z)$. This implies $\partial f(z)g(z) = \alpha f(z)\partial g(z)$ which is equivalent to $\partial f(z) = \partial g(z)\beta$ with $\beta = \overline{\alpha f(z)\partial g(z)}$. Thus $z \in J(f, g)$. $\square$

$C(H)$ is a real analytic variety and the inclusion $C(H) \subset J(f, g)$ is generically strict. We will see later some examples.

3.5. Resolution of $H^{-1}(0)$. In this section, we assume $H$ satisfies (n1),(n2) and the Newton multiplicity condition. Consider the Newton boundary $\Gamma(h)$ of $h(z) = f(z)g(z)$ which is the same as that of $H(z, \bar{z}) = f(z)\bar{g}(z)$. Take a regular subdivision $\Sigma^*$ of the dual Newton diagram $\Gamma^*(h)$ and consider the associated toric modification $\hat{\pi} : X \to \mathbb{C}^n$. See [13] for the definition. Put $V(f) := f^{-1}(0)$, $V(g) := g^{-1}(0)$, $V(h) := h^{-1}(0)$ and $V(f, g) := V(f) \cap V(g)$. We use the same notations as in §5, Chapter 3, [13]. Note that $\hat{\pi} : X \to \mathbb{C}^n$ gives a good resolution of $f, g$ and $h = fg$.

Theorem 17. Let $\tilde{V}(f), \tilde{V}(g)$ and $\tilde{V}(H)$ be the strict transforms of $V(f), V(g)$ and $V(H) = V(h)$ respectively. Then $\tilde{V}(f), \tilde{V}(g)$ are non-singular and intersect transversely so that $\tilde{V}(H)$ is the union $\tilde{V}(f) \cup \tilde{V}(g)$ and $\tilde{V}(f) \cap \tilde{V}(g) = \tilde{V}(f, g)$.

3.5.1. Holomorphic product case. Let $h(z) = f(z)g(z)$ and let $\Sigma^*$ be as above. Put $V^+$ be the set of vertices $P$ of $\Sigma^*$ which are strictly positive. We may assume that vertices which are not strictly positive are the standard basis $\{E_1, \ldots, E_n\}$. (Recall that vertices are the primitive generator of 1-dimensional cone of $\Sigma^*$ [13].) $S_I$ be the set of covectors $P$ in $\mathbb{Q}^I$ such that $\dim \Delta(P; h) = |I| - 1$. The Milnor fiber is not simply connected but the zeta function $\zeta_h(t)$ of the Milnor fibration of $h = fg$ can be computed in the exact same way as (5.3.3), [13] using A’Campo formula [1]. This also gives a formula for the zeta function of the monodromy of the Milnor fibration.

Theorem 18.

\begin{equation}
(4) \quad \zeta_h(t) = \prod_{P \in V^+} (1 - t^{d(P; h)})^{-\chi(E'(P))}
\end{equation}

where

\[ E'(P) = \left( \hat{E}(P) \setminus \tilde{V}(h) \cup \left( \bigcup_{Q \in V^+, Q \neq P} \hat{E}(Q) \right) \right) \cap \hat{\pi}^{-1}(0) \]
For the calculation of $\chi(E'(P))$, we can use the toric stratification as in Theorem (5.3).

\begin{equation}
\zeta_h(t) = \prod_I \zeta_I(t), \quad \zeta_I(t) = \prod_{P \in S_I} (1 - t^{d(P; h^I)})^{-\chi(E'(P))}
\end{equation}

where $\chi(E'(P)) = \chi(E'(P; h^I))$ and it can be computed combinatorially using Newton boundary when $V(h^I)$ has no singularities in $\tilde{E}(P)$, that is if $\tilde{V}(f^I) \cap \tilde{V}(g^I) \cap \tilde{E}(P) = \emptyset$.

Here $\tilde{E}(P)$ is the exceptional divisor corresponding to the covector $P$ as in [13]. $d(P; h)$ is the minimal value of the linear function associated with $P$ on the Newton boundary. If $E(P)$ has a singularity i.e., if there are components of $V(f)$ and $V(g)$ which are intersecting on $E(P)$, $\chi(E'(P))$ can not be computed combinatorially. We can use additive formula of the Euler characteristic using the decomposition

$$E(P) = E(P; f) \cup E(P; g), \quad E(P; f) \cap E(P; g) = E(P; f, g).$$

The last one can be computed using Minkowski’s mix-volume ([13]).

\subsection{Mixed product $f \tilde{g}$ case}

For mixed product case $H = f \tilde{g}$, we need the Newton-multiplicity condition for $H$. Then the zeta function $\zeta_H(t)$ of the Milnor fibration of $H = f \tilde{g}$ is given as

\begin{equation}
\zeta_H(t) = \prod_{P \in \mathcal{V}^+} (1 - t^{\text{pdeg}(P; H)})^{-\chi(E'(P))}
\end{equation}

where $\text{pdeg}(P; H)$ is the polar degree, i.e. $\text{pdeg}(P, H) = d(P; f) - d(P; g)$. For the calculation of $\chi(E'(P))$, we can use the toric stratification as in Theorem (5.3).

\begin{equation}
\zeta_H(t) = \prod_I \zeta_I(t), \quad \zeta_I(t) = \prod_{P \in S_I} (1 - t^{\text{pdeg}(P; H^I)})^{-\chi(E'(P))}
\end{equation}

where $\chi(E'(P)) = \chi(E'(P; H^I))$ and it can be computed using Newton boundary if $\tilde{V}(h^I)$ has no singularities in $\tilde{E}(P)$.

The calculation of the Euler number $\chi(E'(P))$ is the same as that of Theorem 11, [16].

\begin{example}
Let $f(z) = z_1^2 + z_2^2 + z_3^2$ and $g(z) = z_1 + z_2 + z_3$. As a regular function $\Sigma^*$, we can simply take $\{E_1, E_2, E_3, P\}$ with $P = (1, 1, 1)$. $E_1, E_2, E_3$ are standard basis of $\mathbb{Z}^3 \subset \mathbb{Q}^3$. The corresponding toric modification is nothing but the ordinary blowing-up at the origin. $\tilde{E}(P)$ is the projective space $\mathbb{P}^2$. Consider the chart $\text{Cone}(P, E_2, E_3)$ with coordinates $(u_1, u_2, u_3)$. As $(z_1, z_2, z_3) = (u_1, u_1 u_2, u_1 u_3)$, $E(P; f) = \tilde{V}(f) \cap \tilde{E}(P)$ is the conic defined

\end{example}
by $1+u_2^2+u_3^2 = 0$ and $E(P; \bar{g})$ is line defined by $1+\bar{u}_2+\bar{u}_3 = 0$. The pull-back of the functions are

$$\pi^* f(u) = u_1^2(1 + u_2^2 + u_3^2),$$

$$\pi^* g(u) = \bar{u}_1(1 + \bar{u}_2 + \bar{u}_3).$$

$E(P; f)^* \cap E(P; \bar{g})^* = \{(0, u_2, -1 - u_2)|1 + u_2^2 + (-1 - u_2)^2 = 0\}$ (2 points).

Thus $\chi(E(P; H)) = 2 + 2 - 2 = 2$ and

$$\chi(E'(P; H)) = \chi(\mathbb{P}^2) - \chi(E(P; f) \cup E(P; \bar{g})) = 3 - 2 = 1.$$

Thus by Theorem 14, $\zeta_H(t) = (1 - t)^{-1}$. We know that

$$\zeta_H(t) = \frac{P_1(t)}{P_0(t)P_2(t)}$$

where $P_i(t)$ is the $i$-th characteristic polynomial and $P_0(t) = 1 - t$, $P_1(t) = 1 - t$ by Proposition 9. Thus $P_2(t) = (1 - t)$. That is, $H_1(F) = \mathbb{Z}$ and $H_2(F) = \mathbb{Z}$ where $F$ is the Milnor fiber. Here precisely speaking, $H_2(F; \mathbb{Q}) = \mathbb{Q}$ is obvious but to show $H_2(F) = \mathbb{Z}$, we have to see the geometry a little more.

4. Plane curves

In this section, we consider plane curves. We assume (n1),(n2) and the Newton multiplicity condition in this chapter. Assume that

- $C_f : f(x, y) = 0$
- $C_g : g(x, y) = 0$

are plane curves defined by holomorphic functions $f, g$ which has convenient non-degenerate Newton boundaries. We note that $h = fg$ is also Newton non-degenerate, as $C_f$ and $C_g$ does not intersect outside of the origin. This follows from the non-degeneracy assumption (n2) of $f = g = 0$. The Newton non-degeneracy of $f = g = 0$ is equivalent to the following. For any weight vector $P$ such that $\Delta(P; f)$ and $\Delta(P; g)$ are simultaneously edges of $\Gamma(f)$ and $\Gamma(g)$, the face functions $f_P$ and $g_P$ does not have any common non-nomomial factor in $\mathbb{C}[x, y]$. Let $\{P_1, \ldots, P_r\}$ be the weight vectors corresponding to 1-faces of $\Gamma(f)$ and let $\{Q_1, \ldots, Q_s\}$ be those corresponding to 1-faces of $\Gamma(g)$. Let us consider a toric modification associated with a regular fan with weight vectors $\{E_1, R_1, \ldots, R_a, E_2\}$ with $E_1 = (1, 0), E_2 = (0, 1)$ which is a subdivision of the union $\{P_1, \ldots, P_r\} \cup \{Q_1, \ldots, Q_s\}$ and let $\pi : X \to \mathbb{C}^2$ be the corresponding toric modification.

4.1. A’Campo’s formula. First applying the formula by A’Campo to the resolution $\hat{\pi} : X \to \mathbb{C}^2$, the zeta functions $\zeta_f(\tau), \zeta_g(\tau)$ and $\zeta_h(\tau)$ of $f, g$ and
where \( \ell_j \) (resp. \( m_j \)) is the number of irreducible factors of \( f_{R_j}(x, y) \) (resp. of \( g_{R_j}(x, y) \)) and \( a_x, a_y \) (resp. \( b_x, b_y \)) are the length of \( x \)-axis and \( y \)-axis cut by \( \Gamma(f) \) (resp. of \( \Gamma(g) \)). Note that \( \ell_j = 0 \) or \( m_j = 0 \) if \( \dim \Delta(R_j; f) = 0 \) or \( \dim \Delta(R_j; g) = 0 \) respectively. Geometrically, \( \ell_j \) and \( m_j \) are the number of irreducible components of the strict transforms of \( C_f \) and \( C_g \) which intersect the exceptional divisor \( \tilde{E}(R_j) \). Here we use the same notation as in [13]. Note that \( d(P_j; f)\ell_j \) (resp. \( d(Q_k; g)m_k \)) is equal to \( 2\text{Vol Cone}(\Delta(P_j; f), 0) \) (resp. \( 2\text{Vol Cone}(\Delta(Q_k; g), 0) \)). The Milnor number of \( f \) and \( g \) are given by \( -\deg \zeta_f(\tau) + 1 \) and \( -\deg \zeta_g(\tau) + 1 \) respectively and they are equal to the Newton numbers of \( \Gamma_-(f) \) and \( \Gamma_-(g) \) respectively ([10]).

Now we consider the mixed function \( H(z, \bar{z}) := f(z)\bar{g}(z) \). Consider a toric chart \( \text{Cone}(R_j, R_{j+1}) \) with coordinate chart \((u, v)\) where \( u = 0 \) (respectively \( v = 0 \)) defines the exceptional divisor \( \tilde{E}(R_j) \) (resp. \( \tilde{E}(R_{j+1}) \)) in the notation of §4, Chapter 3, [13]. The pull back of \( f, g, H \) takes the form

\[
\begin{align*}
\hat{\pi}^* f &= u^{d(R_j; f)}f'(u, v), f'(0, v) \neq 0 \\
\hat{\pi}^* g &= u^{d(R_j; g)}g'(u, v), g'(0, v) \neq 0 \\
\pi^* H(u, v, \bar{u}, \bar{v}) &= u^{d(R_j; f)}\bar{u}^{d(R_j; g)}H'(u, v, \bar{u}, \bar{v})
\end{align*}
\]

where \( H'(u, v, \bar{u}, \bar{v}) = f'(u, v)\bar{g}'(u, \bar{v}) \). Note that \( \tilde{H} \) is non-zero on \( \tilde{E}(R_j) \setminus V(f, g) \). Thus if \( d(R_j; f) \neq d(R_j; g) \), \( \pi^* h \) is locally topologically equivalent to the rotation around the axis \( \tilde{E}(R_j) \) by the monomial \( u^{d(R_j; f) - d(R_j; g)} \). See Lemma 12, [16]. Using the same argument as in [13], we get

**Theorem 21.** Assume that \( f, g \) are non-degenerate holomorphic functions as above and assume that they satisfy Newton multiplicity condition. Then \( H(z, \bar{z}) \) has a Milnor fibration and the zeta function \( \zeta_H(\tau) \) is given as

\[
\zeta_H(\tau) = (1 - \tau^{a_x-b_x})(1 - \tau^{a_y-b_y}) \prod_{j=1}^{a} (1 - \tau^{d(R_j; f) - d(R_j; g)})^{\ell_j + m_j},
\]

assuming \( \Gamma_-(f) \supset \Gamma(g) \). If \( \Gamma_-(f) \subset \Gamma(g) \), the formula is changed as

\[
\zeta_H(\tau) = (1 - \tau^{b_x-a_x})(1 - \tau^{b_y-a_y}) \prod_{j=1}^{a} (1 - \tau^{d(R_j; g) - d(R_j; f)})^{\ell_j + m_j}.
\]
Remark 22. Pichon and Seade have done interesting works for \( H = \bar{f} \bar{g} \) with \( n = 2 \) which is not necessarily non-degenerate from Seifert graph point of view in [20, 21].

5. Non-existence of Milnor fibration

Let \( f(z, \bar{z}) \) a given mixed function with \( f(0) = 0 \). Here we consider again in general dimension \( n \). We say that \( f \) satisfies Hamm-Lê condition at the origin if
(a) there exists a positive number \( r \) so that any nearby fiber \( f^{-1}(\eta), \eta \neq 0 \) is smooth in the ball \( B^n_2 \rceil \) and
(b) for any fixed \( 0 < r' \leq r \), there exists a positive number \( \delta \) such that the sphere \( S^{2n-1}_\gamma \) and the fiber \( f^{-1}(\eta) \) intersect transversely for any \( \eta \neq 0, |\eta| \leq \delta \) and \( r' \leq \gamma \leq r \).

5.1. Non-constant critical curve. Consider a real curve \( \sigma : [0, 1] \to \mathbb{C}^n \) with \( \sigma(0) = 0 \) such that any point \( \sigma(t), 0 \leq t \leq 1 \) is a critical point of \( f \). We say that \( \sigma \) is a non-constant critical curve for \( f \) if \( \sigma([0, 1]) \not\subset f^{-1}(0) \). Namely the value of \( f \) is not constantly zero along \( \sigma \). If \( f \) satisfies Hamm-Lê condition at the origin, it is obvious that there does not exist any non-constant critical curve starting at \( 0 \). Thus we have:

Proposition 23. Assume that \( f \) has a non-constant critical curve. Then \( f \) has no tubular Milnor fibration.

Remark 24. For the case \( n = 2 \), if plane curves \( V(f) \) and \( V(g) \) have an isolated singularity at the origin without any common component, the converse is true for \( H \) (Proposition 28). Hamm-Lê condition holds if there does not exists non-constant critical curve under the basic assumption (1) and (2) (Lemma 6). We do not know if \( H \) has no non-constant critical curve, it satisfies Hamm-Lê condition for higher dimension \( n \geq 3 \), without assuming the basic assumption (1) and (2).

5.2. Is Newton multiplicity condition is necessary? We gives several examples where the Newton multiplicity condition is not satisfied and we check if there exists a critical curve or not. We use \((x, y)\) as the coordinates of \( \mathbb{C}^2 \).

Example 25. Consider the case \( f = x^3 + y^2, g = x^2 + y^2 \) and \( H = (x^3 + y^2)(\bar{x}^2 + \bar{y}^2) \). We see that \( f, g \) does not satisfies Newton multiplicity condition as \( d(P; f) = d(P; Q) = 2 \) for \( P = (1, 1) \). Note that the Jacobian curve \( J(f, g) \) has three components \( J_1 : x = 0, J_2 : y = 0 \) and \( J_3 : 3x - 2 = 0 \). \( J_2 \) and \( J_3 \) are not critical curves. \( J_1 \) is a critical curve. In fact putting \( \omega(t) = (0, t) 0 \leq t \leq 1 \), we have
\[
\overline{\partial H}(\omega(t)) = g(\omega(t))\overline{\partial f}(\omega(t)) = (0, 2t^2 \bar{t}),
\]
\[
\partial H(\omega(t)) = f(\omega(t))\partial g(\omega(t)) = (0, 2t^2 \bar{t}).
\]
Example 26. Let \( f(x, y) = x^3 - y^2 \), \( g(x, y) = x^2 - y^3 \). Then \( H \) does not satisfy the Newton multiplicity condition as \( d(P; f) = d(P; g) = 2 \) for \( P = (1, 1) \). The Jacobian curve is given by \( xy(-9xy + 4) = 0 \) and it has two local components at the origin. We can see easily none of them include a critical curve for \( H \). Thus \( H \) has a Milnor fibration by Proposition 28.

Example 27. Let \( f(x, y) = x(y^2 + x^3) + y^4 \) and \( g(x, y) = y(x^2 + y^3) + x^4 \). Then \( H \) does not satisfy the Newton multiplicity condition. Compare with previous Example 26. The Jacobian ideal is defined by \( J(x, y) = -3y^2x^2 + 4y^5 + 4x^5 - 8x^4y - 8y^4x \) and it has two irreducible factors at the origin. One of the branch is parametrized as

\[
x(t) = t^2, \ y(t) = \frac{2}{3}\sqrt[3]{t^3} - \frac{4}{3}t^4 + \text{(higher terms)}
\]

As \( \lim_{t \to 0} (f(x(t), y(t))/g(x(t), y(t)))(g(x(t), y(t))/f(x(t), y(t)) = 8/7 \neq 1 \), we see this branch does not contain any non-zero critical point of \( H \). As \( f \) and \( g \) are symmetric in \( x, y \), the other branch does not have any critical point and \( H \) has a Milnor fibration.

The following proposition is clear only for \( n = 2 \).

Proposition 28. Assume \( f(x, y), g(x, y) \) are reduced holomorphic functions without any common components at the origin. Assume that the Jacobian curve has no non-constant critical curve passing through the origin. Then \( H = fg \) has a tubular Milnor fibration at the origin.

Proof. By the assumption, there exist a positive number \( r_0 \) such that \( B^3_{r_0} \) contains no critical point except the origin. Take a positive number \( r_1 \leq r_0 \) such that the sphere \( S^3_r \) intersect transversely for \( V(f) \) and \( V(g) \) for any \( r \leq r_1 \). As \( V(H) \) has an isolated singularity at the origin. Then Hamm-Lê condition is satisfied and the existence of Milnor fibration follows.

6. Existence problem of non-constant critical curves

In this chapter, we consider the existence or non-existence of critical curves for the plane curve case \( n = 2 \). For the simplicity, we use \( (x, y) \) as the coordinates of \( \mathbb{C}^2 \) in this chapter. Though we consider the case \( n = 2 \), the argument works for general dimension with a slight modification.

6.1. Branches of plane curves. Let \( k(x, y) \) is a germ of holomorphic functions and we consider the Newton boundary \( \Gamma(k) \). Let \( \Delta_1, \ldots, \Delta_\ell \) be the edges of \( \Gamma(k) \) and let \( P_i = (p_i, q_i) \) be the corresponding weight vector for \( \Delta_i, i = 1, \ldots, \ell \). Consider the face function \( k_{\Delta_i}(x, y) \). It has a factorization as

\[
k_{\Delta_i}(x, y) = \prod_{j=1}^{\nu_i} (y^{p_j} - \alpha_{ij}x^{q_j})^{\mu_j}.
\]

For each \( j \), there is a branch (or branches) \( C_{ij} \) which is parametrized as

\[
C_{ij}: x(t) = t^{q_{ij}}, \ y(t) = \alpha_{ij}t^{p_{ij}} + \text{(higher terms)}, \ \exists r_j \in \mathbb{N}.
\]
This follows from an admissible toric modification (see [13]). We say the germ $C_{ij}$ is rooted at the factor $(y^{p_i} - \alpha_{ij}x^{n_j})^{\mu_j}$ on the face $\Delta_i$. Every branch of $k = 0$ is rooted at some $\Delta_i$ and some $1 \leq j \leq \nu_i$ as above except possibly the coordinate axis $x = 0$ or $y = 0$ will be a branch if $x|k$ or $y|k$ respectively.

6.2. Branches of Jacobian curves. Now we consider again holomorphic function $h = fg$ and the mixed function $H = f\bar{g}$ as before. Consider the Jacobian curve $J = J(f,g)$ which is defined by $J(x,y) = 0$ where

$$J(x,y) = \frac{\partial f}{\partial x}(x,y)\frac{\partial g}{\partial y}(x,y) - \frac{\partial f}{\partial y}(x,y)\frac{\partial g}{\partial x}(x,y).$$

Consider a face $\Delta$ of $\Gamma(J)$ with weight vector $P = (p,q)$. We say $\Delta$ is a face of the first type if $J_P(x,y) = J(f_P,g_P)(x,y)$. In this case, we have $d(P; J) = d(P; f) + d(P; g) - (p + q)$. Otherwise, we say $\Delta$ a hidden face. In this latter case, we have $J(f_P,g_P) = 0$ and $d(P; J) > d(P; f) + d(P; g) - (p + q)$.

Consider a face $\Delta$ of the first kind as above and the factorization take the form

$$J_P(x,y) = cx^a y^b \prod_{j=1}^{\nu_i} (y^p - \alpha_{i}x^q)^{\mu_i}.$$

Consider a branch $\gamma$ which is rooted at the factor $(y^p - \alpha_i x^q)^{\mu_i}$. It has a parametrization for some integer $r_i > 0$ as follows.

$$\gamma: \begin{cases} 
  x(t) = t^{pr_i} \\
  y(t) = \alpha_i t^{qr_i} + \text{(higher terms)}, \quad t \in D_\varepsilon = \{ \eta \in \mathbb{C} | |\eta| \leq \varepsilon \}.
\end{cases}$$

In the case $\mu_i > 1$, it is possible that there exist several irreducible germs with such expression. (In an admissible toric modification $\tilde{\pi} : X \to \mathbb{C}^2$, the strict transform $\tilde{\gamma}$ of $\gamma$ intersect with the exceptional divisor $\tilde{E}(P)$ corresponding to $P$ (see [13]). We say that $\gamma$ is non-tangential to $V(f,g)$ if $f_P(t^p, \alpha_i t^q) \neq 0$ and $g_P(t^p, \alpha_i t^q) \neq 0$. This is equivalent to $\tilde{\gamma} \cap (\tilde{V}(f) \cup \tilde{V}(g)) \cap \tilde{E}(P) = \emptyset$ where $\tilde{V}(f), \tilde{V}(g)$ are strict transform of $V(f)$ and $V(g)$ respectively. In particular, $\gamma \not\subset V(f) \cup V(g)$.

**Theorem 29.** Assume that $\gamma$ is a non-tangential branch of the Jacobian curve $J$ which comes from a face of $\Gamma(J)$ of first type as above. Then $\gamma$ contains a non-constant critical curve of $H$ at the origin if and only if $d(P; f) = d(P; g)$.

**Proof.** Assume that $\gamma(t) = (x(t), y(t))$ is a critical point of $H$ for a sufficiently small $t$. Then there exists a complex number $\lambda(t)$ with $|\lambda(t)| = 1$ such that

$$\frac{\partial f(x(t), y(t))}{\partial y}(x(t), y(t)) = \lambda(t)\frac{\partial g(x(t), y(t))}{\partial x}(x(t), y(t)).$$

As $\gamma(t)$ is a branch of the Jacobian curve, we have the equality

$$f_x g_y - f_y g_x = 0, \quad \text{on } \gamma(t).$$
and taking the first lowest term, \( f_{P,x}g_{P,y} - f_{P,y}g_{P,x} = 0 \) also holds on \( \gamma(t) \). Note that \( f_x/g_x = f_y/g_y \) along \( \gamma \) and

\[
\frac{f_x(x(t), y(t))}{g_x((x(t), y(t))} = \frac{f_{P,x}(1, \alpha)}{g_{P,x}(1, \alpha)}(1, \alpha)t^{d(P; f) - d(P; g)} + \text{(higher terms)}.
\]

Here we assume \( f_{P,x}(1, \alpha), g_{P,x}(1, \alpha) \neq 0 \). (If \( g_{P,x}(1, \alpha) = 0 \) for example, it implies \( f_{P,x}(1, \alpha) = 0 \) and \( f_{P,y}(1, \alpha), g_{P,y}(1, \alpha) \neq 0 \) by the equation \([9]\) and non-tangential assumption and by the Euler equality. In that case, we use \( f_y/g_y \) instead of \( f_x/g_x \).) Put \( \delta := f_{P,x}(1, \alpha)/g_{P,x}(1, \alpha) \). Then we have

\[
\frac{f(x(t), y(t))}{g(x(t), y(t))} = \frac{f_P(1, \alpha)}{g_P(1, \alpha)}\delta d(P; f) - d(P; g) + \text{(higher terms)}
\]

and using Euler equality the coefficient of the first term can be written as

\[
\frac{f_P(1, \alpha)}{g_P(1, \alpha)} = \frac{(f_{P,x}(1, \alpha) + \alpha f_{P,y}(1, \alpha))/d(P; f)}{(g_{P,x}(1, \alpha) + \alpha g_{P,y}(1, \alpha))/d(P; g)} = \frac{\delta (g_{P,x}(1, \alpha) + \alpha g_{P,y}(1, \alpha))/d(P; g)}{(g_{P,x}(1, \alpha) + \alpha g_{P,y}(1, \alpha))/d(P; f)} = \frac{\delta d(P; g)}{d(P; f)}.
\]

Thus we can write

\[
\lambda(t) = \frac{f_x(x(t), y(t))}{g_x(x(t), y(t))} \cdot \frac{f(x(t), y(t))}{g(x(t), y(t))} = \frac{\delta d(P; f)}{\delta d(P; g)} + \text{(higher terms in } t \text{)}
\]

If \( d(P; f) \neq d(P; g) \), for sufficiently small \( t \), we see that \( |\lambda(t)| \neq 1 \) and there does not exist a critical point of \( H \) by Proposition \([1]\). Suppose that \( d(P; f) = d(P; g) \). We see that \( |\lambda(t)| \equiv 1 \) modulo \( (t) \).

If \( |\lambda(t)| = 1 \) constantly, the whole \( \gamma(t), t \in D_\varepsilon \) is a critical point of \( H \). This happens when \( f \) and \( g \) are weighted homogeneous polynomials of the same degree under the same weight \( P \).

Assume that \( \lambda(t) = \lambda_0 + a_k t^k + \text{(higher terms)} \), \( a_k \neq 0 \), \( |\lambda_0| = 1 \). By the next Lemma \([30]\), there exists a positive number \( \varepsilon' \) such that for any \( r \leq \varepsilon' \) fixed and \( |t| = r \), there exists \( 2k \) solutions \( t = re^{i\theta_j}, j = 1, \ldots, 2k \) of \( |\lambda(t)| = 1 \). They are parametrized real analytically in \( r \in [0, \varepsilon'] \) and gives non-constant critical curves for \( H \).

\begin{lemma}
Let \( \rho(t) \) is a holomorphic function on the disk \( D_\varepsilon := \{z \in \mathbb{C} | |z| \leq \varepsilon \} \) such that \( \rho(0) = a_0, |a_0| = 1 \) and \( \rho(t) \neq a_0 \). Let \( k = \text{ord}_r(\rho(t) - a_0) \). Then there exists a positive number \( \varepsilon' \leq \varepsilon \) such that for any \( 0 < r \leq \varepsilon' \), there are \( 2k \) angles \( 0 \leq \theta_1, \ldots, \theta_{2k} < 2\pi \) such that \( |\rho(r e^{i\theta_j})| = 1 \) for \( j = 1, \ldots, 2k \).
\end{lemma}

\begin{proof}
Consider the Taylor expansion

\[
\rho(t) = 1 + a_k t^k + \text{(higher terms)}, \quad a_k \neq 0.
\]

We claim each complex branch contains a critical curve for $H$

Proof. If $z$ with face function $\varepsilon$ and $\theta$ satisfy

Thus the behavior of the loop $\theta \mapsto \rho(re^{i\theta}) - 1$ is topologically $k$ times rotation on the sphere $S_1 = \{ |z| = 1 \}$. □

Example 31. Let $f(x, y) = x^5 + x^2y^2 + y^6$ and $g(x, y) = x^6 + x^2y^2 + y^5$. Then $H$ does not satisfy the Newton-multiplicity condition as $d(P; f) = d(P; g) = 4$ for $P = (1, 1)$. The Jacobian ideal is defined by

There exists three germs of Jacobian curves: $\{ x = 0 \}$, $\{ y = 0 \}$ and $C = \{ j_2 = 0 \}$. It is easy to see that the first two coordinate axes are not critical curves. $C$ consists of 5 smooth components. They are rooted to the face $\Delta$ with face function $xy(10x^5 - 10y^5)$ and $\Delta$ is a hidden face as $f_P = g_P = x^2y^2$ and $J(f_P, g_P) = 0$. They have the Taylor expansions

We claim each complex branch contains a critical curve for $H$.

Proof. If $C_1$ or $C_{2a}$ is a critical curve, it must satisfy $|\frac{\partial f}{\partial x}/\frac{\partial g}{\partial x}| = |f/g|$. Let us see the assertion on $C_1$.

Thus

Consider the equation

on the circle $x = re^{i\theta}$. By Lemma [30] for fixed $r \leq \varepsilon$ small enough, this has two solutions $\theta_i(r)$, $i = 1, 2$, $0 \leq \theta_i(r) < 2\pi$ for sufficiently small $\varepsilon$. Then $z_i(r) := re^{i\theta_i(r)}$, $i = 1, 2$ satisfy $z_i(0) = 0$ and $r \mapsto z_i(r)$ gives non-constant critical curves for $H$. For $C_{2a}$, we omit the proof as the argument is the same. □
Example 32. Let $f(x,y), g(x,y)$ be homogeneous polynomial of same degree $d$. Then $J(x,y)$ is a homogeneous polynomial of degree $2d - 2$. Take any branch germ $\gamma$ of $J(x,y) = 0$ which is non-tangential to $V(f,g)$. Then $\gamma$ contains a critical curve for $H$. As an example, take $f(x,y) = x^2 + xy + y^2$, $g(x,y) = x^2 - xy + y^2$. Then Jacobian curve is defined by $y^2 - x^2$. This gives two branches $y = \pm x$ which are non-tangential and they are critical curves. In this case, $C(H) = J(f,g)$.

We finish this paper by the following Lemma which follows from Lemma 30 and corresponds to a corrected assertion of Theorem 2.3, [19].

**Lemma 33.** Let $f, g$ be holomorphic function pair without any common divisor, $H = f\bar{g}$ and let $J = J(f,g)$ be the Jacobian curve. Take a local branch germ $\gamma$ of $J$ at the origin defined by a Taylor expansion $z(t)$, $t \in D_\epsilon$ which is not included in $\{V(H)\}$. $\gamma$ contains a non-constant critical curve for $H$ if and only if

$$\lim_{t \to 0} \left| \frac{\partial f}{\partial z_j}(z(t))g(z(t)) - \frac{\partial g}{\partial z_j}(z(t))f(z(t)) \right| = 1.$$ 

Here $j$ is chosen so that $\frac{\partial g}{\partial z_j}(z(t)) \neq 0$.

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