Topological Chern–Simons vortices in the $O(3)$ $\sigma$–model

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Abstract

We present a classical, gauged $O(3)\sigma$–model with an abelian Chern–Simons term. It shows topologically stable, anyonic vortices as solutions. The fields are studied in the case of rotational symmetry and analytic approximations are found for their asymptotic behaviour. The static Euler–Lagrange equations are solved numerically, where particular attention is paid to the dependence of the vortex’ properties on the coupling to the gauge field. We compute the vortex mass and charge as a function of this coupling and obtain bound states for two–vortices as well as two–vortices with masses above the stability threshold.

1 Introduction

The $O(3)\sigma$–model in 2–dimensional Euclidean space is a classical field theory which supports soliton solutions ő. Its scale invariance can be broken by the addition of a potential term. This does not prevent the soliton from shrinking, however, its size can be fixed by the inclusion of higher order terms in the field gradient ő. An example for such a theory is the baby Skyrme model ő. Alternatively, the scale invariance of the $O(3)\sigma$–model can be removed and the soliton be stabilised (at least in principle) by gauging a $U(1)$ subgroup of the fields internal symmetry group ő,ő,ő. The dynamics of the $U(1)$ gauge field in such models is ruled by Maxwell– and/or Chern–Simons actions. For each of these cases potential terms have been constructed such that the corresponding models yield self–dual equations of Bogomol’nyi type. The potential term determines the asymptotic behaviour

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of the fields which can either obey the gauge symmetry \[5, 7\] or break it \[8\]. The models with broken gauge symmetry show topological soliton solutions with quantised magnetic flux and in this they resemble the well-studied vortices in the abelian Higgs model and its generalisations, see for instance \[9, 10\].

Here we investigate a gauged \(O(3)\) \(\sigma\)-model with Chern–Simons action. Chern–Simons theories are an object of intense research because their quantised version is relevant for systems of strongly correlated electrons e.g. in superconductors or in the quantum Hall effect \[11\]. In this paper we consider a static classical Chern–Simons model, whose potential term preserves the gauge symmetry and is chosen to produce exponentially localised configurations. They carry fractional angular momentum and have a lower topological bound on the energy which is, however, not saturated. We solve the equations of motion numerically for radially symmetric fields and study the dependence of the solutions on the coupling strength to the gauge field. We also look at two vortices on top of each other and on their mutual attraction dependent on their coupling. The asymptotic behaviour of the fields is studied analytically and conclusions about intervortex forces are drawn.

Recently, static solitons were found in a gauged \(CP^1\) model which includes a Chern–Simons term and a potential term equivalent to the one considered here \[12\]. In its standard version the \(CP^1\) model represents merely a different choice of fields to the \(O(3)\) \(\sigma\)-model. In \[12\], however, the gauged symmetry is the internal \(U(1)\) symmetry of the two–component complex \(CP^1\) vector which lies on \(S^3\). Therefore we expect our solutions to be different to the ones presented in \[12\], but it is nevertheless instructive to compare them.

2 Chern-Simons solitons revisited

We consider the following Lagrangian of a gauged \(O(3)\) \(\sigma\)-model in \((2+1)\) dimensions. It contains a potential term and the behaviour of the gauge field \(A_\alpha\) is governed by a Chern–Simons term

\[
\mathcal{L} = \frac{1}{2} (D_\alpha \phi)^2 - \frac{\kappa}{2} \epsilon^{\alpha\beta\gamma} \partial_\alpha A_\beta A_\gamma - \mu^2 (1 - \mathbf{n} \cdot \phi).
\]

(1)

The fields \(\phi\) are three-component real vectors and subject to the constraint \(\phi \cdot \phi = 1\), hence they take values on the two-sphere \(S^2_\phi\). The metric is chosen to be \(g_{\alpha\beta} = \text{diag}(+, -, -)\). Throughout this paper, greek indices run from 0 to 2 while Latin indices denote the two spatial dimensions 1,2. We work in geometrical units in which the velocity of light \(c = 1\). \(\kappa\) and \(\mu\) are real coefficients of dimension length and 1/length respectively and for dimensional reasons the Lagrange density \(\mathcal{L}\) should be thought of being multiplied by an overall factor of dimension energy. We fix our mass scale by putting this factor to one. The fields \(\phi\) will
be frequently referred to as matter–fields (in distinction to the gauge fields) and to $S_\phi^2$ as the iso–space. The potential term in (1) reduces the symmetry of the model to $O(2)_{iso}$, i.e. to rotations and reflections perpendicular to the vector $n$. It is this symmetry that is to be gauged and by choosing $n = (0, 0, 1)$ we select the $SO(2)_{iso}$ subgroup which consists of unimodular rotations about the $z$–axis. $D_\alpha(\phi)$ is the covariant derivative and given by:

$$D_\alpha \phi = \partial_\alpha \phi + A_\alpha (n \times \phi).$$

(2)

The ungauged Lagrangian shows symmetry under combined reflections in space and iso–space:

$$P : (x_1, x_2) \rightarrow (-x_1, x_2) \quad \text{and} \quad C : (\phi_1, \phi_2) \rightarrow (-\phi_1, \phi_2),$$

(3)

which can be thought of as a parity operation and charge conjugation. The Chern–Simons term breaks the parity symmetry explicitly by changing its sign under $P$. It also breaks the time–reflection symmetry $T$ which corresponds to $A_0 \rightarrow -A_0$. However, the Lagrangian is still symmetric under $CPT$.

The potential term can be thought of physically as an analogue to the Zeeman coupling between spin fields $\phi$ and an external, constant magnetic field in $n$–direction with coupling strength $\mu^2$. Such terms occur for example naturally in the description of the quantum Hall effect.

Because we are interested in configurations with finite energy, we require that the potential term and the covariant derivative vanish at spatial infinity. Hence we impose:

$$\lim_{r \rightarrow \infty} \phi(r) = n.$$

(4)

This boundary condition allows to one–point compactify the physical space such that fields $\phi$ are maps:

$$\phi : S_x^2 \rightarrow S_\phi^2.$$

(5)

These maps are elements of homotopy classes which form a group isomorphic to the group of integers. This integer or degree $N$ counts the number of times $S_x^2$ is covered by a single covering of $S_\phi^2$. It can be written as the integral over the zero component of the topologically conserved current:

$$l_\alpha = \frac{1}{8\pi} \epsilon_{\alpha\beta\gamma} \phi \cdot (\partial_\beta \phi \times \partial_\gamma \phi),$$

(6)

such that the degree $N$ is obtained from
\[ N = \int d^2 x l_a, \] (7)

where the range of integration is \( S_x^2 \). Following the standard convention we will call finite energy solutions with \( N > 0 \) vortices and those with \( N < 0 \) antivortices. (7) is in fact the topological charge of the \( O(3) \) \( \sigma \) fields and it is not obvious that this is also a topologically conserved quantity in the gauged model. We will therefore address this question again below.

The equations of motion derived from (1) can be written in terms of the matter-current \( J_\alpha \) and the electromagnetic current \( j_\alpha \)

\[ J_\alpha = \phi \times D_\alpha \phi , \quad j_\alpha = n \cdot J_\alpha . \] (8)

The Euler-Lagrange equations are

\[ D_\alpha J_\alpha = \mu^2 (n \times \phi) \] (9)
\[ j_\alpha = -\kappa \epsilon_{\alpha\beta\gamma} \partial^\beta A^\gamma . \] (10)

Note that by (10) the gauge fields are completely determined by first order equations and do not have own dynamics in the strict sense. Equation (10) for \( \alpha = 0 \) is Gauss’ law

\[ D_0 \phi = -\frac{\kappa B (n \times \phi)}{1 - (n \cdot \phi)^2} , \] (11)

where we have used that \( n = (0, 0, 1) \) and \( B = \epsilon_{0ij} \partial^i A^j \), taking \( \epsilon_{012} = 1 \). The equation of motion (10) implies that for non–singular \( A_\alpha \) the electromagnetic current \( j_\alpha \) is conserved (\( \partial_\alpha j^\alpha = 0 \)). The current can be written conveniently as \( j_\alpha = (\rho, j_i) \), where \( \rho \) is the charge density of the soliton while \( j_i \) denotes its electric current. The Lagrangian (1) can be expressed in terms of \( j_\alpha \):

\[ L = \frac{1}{2} (D_\alpha \phi)^2 - \frac{1}{2} A_\alpha j^\alpha + \mu^2 (1 - n \cdot \phi) . \] (12)

This shows explicitly that the gauge fields \( A_\alpha \) are coupled to the electromagnetic current \( j^\alpha \). The electric field \( E \) and the magnetic field \( B \) are related to \( j_\alpha \) as follows:

\[ B = -\frac{\rho}{\kappa} , \quad E_i = \epsilon_{ij} \frac{j^j}{\kappa} . \] (13)

The first equation leads to a relation between the magnetic flux \( \Phi \) and the electric charge \( Q \) of the configuration.
\[
\Phi = \int d^2 x B = -\frac{1}{\kappa} \int d^2 x \rho = -\frac{Q}{\kappa}.
\]  

(14)

The theories energy–momentum tensor is obtained by the variation of the Lagrangian with respect to the metric \( g_{\alpha \beta} \)

\[
T_{\alpha \beta} = (D_{\alpha} \phi)(D_{\beta} \phi) - g_{\alpha \beta} \left( \frac{1}{2} (D_{\gamma} \phi)(D^{\gamma} \phi) - \mu^2 (1 - n \cdot \phi) \right).
\]  

(15)

The integral over the component \( T_{00} \) is the total energy of the soliton

\[
E_{\text{CS}}[\phi, A] = \int d^2 x \left( \frac{1}{2} (D_0 \phi)^2 + \frac{1}{2} (D_i \phi)^2 + \mu^2 (1 - n \cdot \phi) \right).
\]  

(16)

Note that the Chern–Simons term does not contribute directly to the energy because of its metric independence. The rotational symmetry of the Lagrangian leads to a conserved angular momentum \( \mathbf{M} \) of the soliton

\[
\mathbf{M} = \int d^2 x (\mathbf{x} \times \mathbf{p}),
\]  

(17)

where the cross product stands for \( x_1 p_2 - p_1 x_2 \). \( \mathbf{M} \) is a vector pointing perpendicular out of the plane of motion. The components of the momentum density \( \mathbf{p} \) are given by

\[
p_i = T_{0i} = D_0 \phi \cdot D_i \phi.
\]  

(18)

3 A bound on the energy

Next we give a proof that \( E_{\text{CS}}[\phi, A] \), the energy in our model given by (16), is bounded from below by a topologically conserved quantity. This is not obvious, because the gauged pure \( O(3) \) \( \sigma \)–model does not have a lower bound on the energy, unlike its ungauged counterpart, where the solutions saturate the Bogomol’nyi limit. The first step in the proof is to use an auxiliary energy functional \( E_{\text{aux}}[\phi, A] \) which is of Bogomoln’yi type and was constructed in [3]. Because we wish this section to be self–contained, we will repeat below parts of the analysis given in this reference. First, we show that the energy gap between \( E_{\text{CS}} \) and \( E_{\text{aux}} \) (or a multiple of it) is positive and then complete the argument by demonstrating that \( E_{\text{aux}} \geq 4\pi |N| \). \( E_{\text{aux}} \) reads as:

\[
E_{\text{aux}}[\phi, A] = \frac{1}{2} \int d^2 x B^2 + (D_i \phi)^2 + (1 - n \cdot \phi)^2.
\]  

(19)
In order to be consistent in the notation of dimensions, both the potential term and the magnetic field must be thought of being multiplied by a parameter of dimension $1/\text{length}^2$ and length respectively. These parameters are of magnitude one and suppressed in (19). To compare $E_{\text{CS}}$ with $E_{\text{aux}}$ one first observes that $(D_0 \phi)^2 \geq \kappa^2 B^2$, due to Gauss’ law (11). Now we carry out a rescaling of $x$ in our functional $E_{\text{CS}}$, namely $x \rightarrow \kappa x$, which transforms $B \rightarrow B/\kappa^2$ and $\phi(x) \rightarrow \phi(\kappa x)$. The potential term then reads as $\kappa^2 \mu^2 (1 - n \cdot \phi)$ and is greater than $(1 - n \cdot \phi)$ if $\kappa \geq 1/\mu$. To verify that $E_{\text{aux}}$ is smaller than $E_{\text{CS}}$ we use that since $0 \leq (1 - n \cdot \phi) \leq 2$, it follows that $(1 - n \cdot \phi) \geq \frac{1}{2} (1 - n \cdot \phi)^2$ and one sees that for $E_{\text{aux}}$ holds

$$E_{\text{CS}} \geq E_{\text{aux}} \quad \text{if} \quad \kappa \geq 1/\mu . \quad (20)$$

In the case $\kappa < 1/\mu$ we assess an energy bound by multiplication of each individual term in the energy density with $\kappa^2 \mu^2$. This gives

$$E_{\text{CS}} \geq \kappa^2 \mu^2 E_{\text{aux}} \quad \text{if} \quad \kappa < 1/\mu . \quad (21)$$

This already proves the bound for $E_{\text{CS}}$, but it is instructive to see in detail that $E_{\text{aux}}$ defines a Bogomol’nyi model. In order to achieve this, we rewrite the auxiliary energy functional as

$$E_{\text{aux}}[B, \phi] = \frac{1}{2} \int d^2 x \left( (D_1 \phi \pm \phi \times D_2 \phi)^2 + (B \mp (1 - n \cdot \phi))^2 \right) \pm \int d^2 x L_0 . \quad (22)$$

$L_0$ is composite of the cross terms and can be understood as the zero component of the solitons gauge invariant topologically conserved current:

$$L_\alpha = \epsilon_{\alpha\beta\gamma} \left( \phi \cdot (D^\beta \phi \times D^\gamma \phi) + \partial^\beta A^\gamma (1 - n \cdot \phi) \right) . \quad (23)$$

Up to a surface term, this current is equivalent to $l_\alpha$, the topological current of the ungauged $O(3)$ $\sigma$–model (8). If the solutions are required to have finite energy, then $\phi$ must tend to zero faster than $1/r$ as $r$ goes to infinity, hence it follows by Stokes’ theorem that the surface term integrates to zero. It was pointed out in (8) that the conserved topological charge (integral over $L_0$) equals the degree $N$ of the map $\phi$ if the gauge symmetry is unbroken (as it is in our case) but can differ from $N$ for broken symmetry. In order to saturate the Bogomol’nyi bound, both squares in (22) have to vanish, such that the following two (anti–)self–dual equations are read off

$$D_1 \phi = \mp \phi \times D_2 \phi \quad , \quad B = \pm (1 - n \cdot \phi) . \quad (24)$$
These equations were discussed in detail for a special choice of the fields in [5]. There it was shown that they yield a one–parameter family of solutions which are degenerated in their energy but differ in their magnetic flux.

By using the sign ambiguity in front of the integral over $L_0$ in (22) we can restrict our discussion to the case $B > 0$ and the upper sign without a loss of generality. Equation (22) then implies

$$E_{\text{aux}} \geq \int d^2 x L_0 = 4\pi |N|,$$

(25)

The equality holds for self–dual solutions.

4 Static vortex solutions

To find static solutions in our model we restrict ourselves to the two–dimensional hedgehog [13], which is in terms of the polar coordinates $(r, \theta)$:

$$\phi(r, \theta) = \begin{pmatrix} \sin f(r) & \cos n\theta \\ \sin f(r) & \sin n\theta \\ \cos f(r) \end{pmatrix}. \quad (26)$$

For this field the topological charge density, the integrand of (7), equals

$$l_0 = \frac{n}{4\pi r} f' \sin f,$$

(27)

where the prime denotes the derivative with respect to $r$. By integration one easily sees that $n = -N$. For the gauge field $A_\alpha$, the most general ansatz which leads to radially symmetric and static fields is given by

$$A_\theta = n v(r), \quad A_\phi = n a(r), \quad A_r = h(r)t,$$

(28)

where $t$ denotes the time and the factor $n$ is introduced for convenience. We fix our gauge by putting $A_r = 0$ and obtain the equations

$$f'' + \frac{f'}{r} = n^2 \left( \frac{(a + 1)^2}{r^2} - v^2 \right) \sin f \cos f + \mu^2 \sin f,$$

(29)

$$v' = -\frac{1}{\kappa} \frac{(a + 1)}{r} \sin^2 f.$$

(30)

Gauss’ law (11) reads in terms of $a, v$ and $f$:
\[ a' = -\frac{1}{\kappa}rv \sin^2 f. \]  

We are interested in finite energy configurations, which requires that \( D_\alpha(\phi) \to 0 \) as \( r \to \infty \).

To guarantee this and the regularity of the fields at the origin we impose the following boundary conditions

\[
\begin{align*}
  a(0) &= 0, & f(0) &= \pi, & v(0) &= v_0 \\
  \lim_{r \to \infty} f(r) &= 0, & \lim_{r \to \infty} a(r) &= a_\infty, & \lim_{r \to \infty} v(r) &= v_\infty
\end{align*}
\]  

where \( v_0, v_\infty \) and \( a_\infty \) are constants. With these boundary conditions it is clear that constant fields \( a \) and \( v \) are not a solution of (30) and (31), which can be shown by contradiction. If \( a \) were a constant it would have to be zero everywhere because of (32), in which case (30) implies that \( v \) is not a constant which in turn, via (31) leads to a non-constant \( a \). A similar argument applies for the case of \( v \) being constant. Hence the Euler–Lagrange equations do not lead to vanishing flux and charge.

The total (static) energy is given as the integral over the energy density \( e \), which reads in terms of the fields \( f, a \) and \( v \)  

\[
e = \frac{f'^2}{2} + \frac{n^2}{2} \left( \frac{(a + 1)^2}{r^2} + v^2 \right) \sin^2 f + \mu^2 (1 - \cos f). \]

For the angular momentum (17) one obtains

\[ M = -\pi \kappa N a_\infty (a_\infty + 2N) n. \]

Hence one sees that the angular momentum of the vortex is fractional and the vortices are (classical) anyons. The electromagnetic fields (13) are radially symmetric by construction and read as

\[ B = Na' \quad \text{,} \quad E_r = Nv'. \]

The electric charge and magnetic flux are not topologically quantised (unlike in the abelian Higgs model, for instance) and depend on the parameters in the model

\[ \Phi = N \int rdrd\theta \frac{a'}{r} = -2\pi Na_\infty = -\frac{Q}{\kappa}. \]
5 Asymptotics

The boundary conditions (32) allow us to derive asymptotic approximations to the equations of motion (29). By approximating $\sin f \approx f$ and $\cos f \approx 1$ for large $r$, the equation for $f$ simplifies to

$$f'' + \frac{f'}{r} = n^2 \left( \frac{(a_\infty + 1)^2}{r^2} + k^2 \right) f, \quad (37)$$

where

$$k^2 = \mu^2 - n^2 v_\infty^2. \quad (38)$$

The asymptotic solution of $f$ depends on the value $k$ takes. There are three possible cases.

1.) $|\mu| > |nv_\infty|$, $k$ real.

The solution to (37) for real $k$ are given by modified Bessel functions $f \sim K_m(k r), m = n(a_\infty + 1)$ with the asymptotic behaviour

$$f \sim \frac{1}{\sqrt{r}} e^{-k r}, \quad (39)$$

This shows that $k$ can be understood as the effective mass of the matter fields $\phi$, by denoting the inverse decay length of the profile function. The asymptotics of the field are determined by the potential term which defines the theories vacuum structure. Therefore it is not a surprise that the vortex’ matter field looks asymptotically like the baby Skyrmion investigated in [3], where the same potential term was used.

2.) $\mu < |nv_\infty|$, $k$ imaginary.

This case leads to oscillating fields with an amplitude that falls off proportionally to $1/\sqrt{r}$ in leading order. The substitution $\tilde{k} = ik$ in (34) verifies this instantly and also shows that $k$ is proportional to the inverse wavelength of the oscillations. The energy density of these solutions behaves asymptotically like $1/r$ in leading order and hence the energy of these fields is infinite. This is, of course, not a physically relevant solution such that we exclude it from our further discussion.

3.) $\mu = |nv_\infty|$, $k = 0$.

The critical case is in fact just a special case of 1.), with vanishing exponential such that the profile function $f \sim 1/r$. The energy of these solutions is also infinite, because the leading term in the energy density is proportional to $v_\infty^2 f^2$. Numerically we find that all these solutions occur but restrict our discussion to the case 1.), which gives the following constraint on the solutions:

$$|\mu| > |nv_\infty|. \quad (40)$$
Using expression (39), we find for the electric and magnetic field in the limit of large $r$

$$E_r \sim \frac{1}{r} e^{-2kr}, \quad B \sim \frac{1}{r} e^{-2kr}. \quad (41)$$

This shows that the electromagnetic fields fall off much faster than the matter field $f$. Therefore the electromagnetic interactions are expected to be negligible in the context of long–range vortex interactions. The electric field is a vector lying in the plane of motion while the magnetic field $B$ can be thought of as pointing perpendicular out of the plane of motion. Its asymptotic shape is similar to the one of the Skyrme–Maxwell soliton discussed in [6], where it was argued that such a magnetic field resembles a magnetic dipole in two–dimensional electrodynamics.

For small $r$ the fields can be approximated by power series

$$f \approx \pi + cr^{|n|}, \quad v \approx v_0 + d e^{2|n|}, \quad a \approx g r^{2|n|+2}, \quad (42)$$

where $c$ and $v_0$ are free parameters while $d$ and $g$ are given as functions of $n, \kappa, c$ and $v_0$. Note that for finite energy solutions $c$ and $v_0$ are not completely independent on each other.

For the Skyrme–Maxwell solitons it was found that the electromagnetic short range interaction decreases the energy per soliton and in particular leads to more strongly bound two–soliton states. Here, having a non–zero electrical charge distribution we expect this effect to be weakened by the Coulomb repulsion of the solitons electric field.

6 Numerical results

We solved the set of equations (29) numerically by using a shooting method and a relaxation method. For both the shooting method and the time evolution in the relaxation method we employed a fourth–order Runge–Kutta method. In order to perform the numerical integration we had to fix the parameters in our model. Using geometric units in which the energy and length are of unit one, we are left with $\mu$ and $\kappa$ to be fixed. However, the parameter space is in fact one–dimensional which can be verified by carrying out the rescaling $x \rightarrow \kappa x, \ B \rightarrow B/\kappa^2$. Thus we can fix $\mu$ for all our computations without a loss of generality. We choose $\mu = \sqrt{0.1}$, a value which allows us to compare our numerical results with the ones obtained in the Skyrme–Maxwell model [6] where the same value has been used.

We looked at the dependence of the solutions of degree $N = 1$ and $N = 2$ for a range of $\kappa$. This parameter determines the strength of the Chern–Simons term and is proportional to the square root of the inverse coupling to the gauge field, which can be seen by a simple
substitution $A_\alpha \to A_\alpha / \sqrt{\kappa}$. 

Fig. 1a) shows the dependence of the static energy or mass on $\kappa$. Small $\kappa$ which corresponds to strong coupling leads to lighter vortices for both the one–vortex and the two–vortex. For large $\kappa$ the mass or static energy $E_{CS}$ tends to a constant but remains relatively close to the Bogomol’nyi bound, staying below 1.1 (in units of $4\pi|N|$) for the one–vortex and the two–vortex. Thus our vortices are significantly lighter than the gauged baby Skyrmions, which tend to a mass of $E_{SM} = 1.546$ for weak coupling. The energy gap arises partly due to the Skyrme term which is not present here.

It is particular interesting to look at the relative static energy per vortex. We denote the energy of the one (two) vortex by $E^1 (E^2)$. The energy difference $\Delta E = E^2 - 2E^1$ can be interpreted as binding and excess energy of the two–vortex for $\Delta E < 0$ and $\Delta E > 0$ respectively. In the case $\Delta E < 0$ the vortices form bound states while for $\Delta E > 0$ we expect that vortices on top of each other are unstable under perturbations and experience a repulsive force. From Fig. 1a) it is clear that in our model both cases occur. For small $\kappa$ the two–vortex is in an attractive regime as it is for large $\kappa$, however, there is an intermediate region $\kappa^l_{cr} < \kappa < \kappa^h_{cr}$ for which the two–vortex is unstable (in the sense that its decay is energetically favourable). $\kappa_{cr}$ is the critical coupling for which $\Delta E = 0$. Numerically we find that $\kappa^l_{cr} = 0.632$ and $\kappa^h_{cr} = 2.215$.

This result can be explained in a semiquantitative way. In the limit of large $\kappa$ the gauge fields decouple from the matter fields and become very small as compared to the matter fields. The study of ungauged solitons (e.g. in [3]) showed that pure matter forces are often attractive for two–solitons. This is also the case in our model. For very small $\kappa$, however, the magnetic flux tends to a constant (the reader finds the explanation for this below) while the repulsive Coulomb force is $\propto Q^2 \propto \Phi^2 \kappa^2$, thus becoming weaker with increased coupling, see Fig. 1b). We found numerically that increased coupling leads to a stronger bound configuration and in this our model is similar to the Skyrme–Maxwell model. The intermediate or repulsive range can be understood as a regime in which the Coulomb repulsion dominates the attractive forces of matter and magnetic field. It is within this range that the electric charge has its maximum value $Q_{\max} = Q(\kappa_{\max})$, where numerically $\kappa_{\max} = 0.75$ (for $N = 1$) and $\kappa_{\max} = 0.92$ ($N=2$).

In showing such a behaviour, the vortices resemble the fields of the abelian Higgs model where a similar transition between repulsive and attractive regime occurs, depending on the strength of the potential term. This characteristic is used to describe temperature driven phase transitions between type-I and type-II superconductors. The shape of the vortex shows a strong dependence on $\kappa$, which is foreseeable by the interpretation of $\kappa$ as the coupling parameter to the gauge field. Because the vortex’ Coulomb interaction is repulsive
it favours a spreading of the soliton. In agreement with this picture, we obtain vortices with their maximum width range at $\kappa \approx 1$, where the electric charge takes its maximum values. See also Fig. 2a) and 2b), where the energy density $e$ and the profile function $f$ are plotted in dependence on the coupling.

On the other hand, if the electromagnetic interaction is coupled only very weakly, the Lagrangian reduces to the $O(3)\sigma$–model term plus the potential term, such defining a configuration which is known to be unstable against shrinkage due to the Hobart–Derrick theorem [14]. In accordance with this discussion one sees from Fig. 2 that for large $\kappa$ the vortex becomes more localised. This clearly shows that the potential term in the Lagrangian favours a shrinkage of the vortex.

We also observed that increased magnetic flux as it occurs for small $\kappa$ leads to more localised solitons, like it does for Skyrme–Maxwell solitons [3]. For very small $\kappa$ both the gauge fields $a$ and $v$ tend to singular configurations at the origin. In this limit, the gauge field $v$ takes large values at the origin and is zero everywhere else, while $a$ tends to $-1$ everywhere except at $r = 0$, which is fixed by its boundary condition, see Fig. 2c) and 2d). In that the behaviour of $a$ is similar to the gauge field in the Skyrme–Maxwell model. We conjecture that the origin of this coincidence is the particular ansatz chosen for the gauge field, which leads to terms in the energy density depending on $(a+1)^2$ and so makes the value $a_\infty = -1$ exceptional. The strong coupling limit therefore leads to dynamically quantised flux and in addition implies via (36) that the electric charge vanishes for $\kappa \to 0$. For the electric and the magnetic field we find that they form a ring (cf. Fig. 3b), a feature which was also observed by Jackiw and Weinberg in a self–dual Chern–Simons model [9] (where the matter fields are complex scalar). For Skyrme–Maxwell solitons, toroidal configurations were seen only for topological charge two.

We also looked at the vortex’ shape for $N = 2$ and its dependence on $\kappa$. The two vortex has the shape of a ring for all $\kappa$, a picture not unfamiliar in planar soliton theories. For equivalent coupling, the fields of the two–vortex decay slower than those of the one–vortex. This can be understood by looking at formula (38). Our numerical results show that $v_\infty$ depends strongly on the coupling $\kappa$ but only weakly on the topological charge $N$ such that the effective mass $k$ is smaller for the two–vortex and hence its exponential decay slower. In Fig. 3 we show the energy density and electric field of the one and two–vortex. The coupling here is the lower critical coupling $\kappa_{\text{cr}}$. 

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7 Conclusions

We have studied classical static vortex solutions in an $O(3)\sigma$ Chern–Simons system with unbroken $U(1)$ gauge symmetry. The vortices have an electric charge which shows a unique maximum dependent on the coupling to the gauge field. The magnetic flux in the model is effectively quantised in the limit of strong coupling while the angular momentum of the vortices is fractional such that they can be considered as classical anyons.

In the case of two vortices sitting on top of each other, the model has a repulsive and two attractive phases, depending on the parameter which couples the gauge and matter fields. This has interesting consequences for the interaction of multivortices. In the repulsive regime they will presumably try to move away from each other and for a bounded region this would lead to a configuration similar to an Abrikosov–lattice with vortices in equidistant and fixed positions. Such configurations occur in the description of flux tubes in type–II superconductors. In the attractive regime, however, vortices which are not too widely separated from each other will be likely to coalesce. In this context it is worth investigating whether the vortices of higher winding number show a similar dependence on the coupling, in particular whether their critical couplings $\kappa_{cl}^l$ and $\kappa_{cl}^h$ (if they exist) are of the same value than they are here.

The inter–vortex forces at large and medium distances will be dominated by the matter fields, because the electromagnetic fields decay faster by a factor of $e^{-kr}$. Thus, the interactions should be well described asymptotically by the dipole picture developed in [3].

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Figure Captions

Fig. 1
1a) The static energy $E$ (16) in units of $4 \pi N$ as a function of the Chern–Simons coupling parameter $\kappa$ for $N = 1$ (solid line) and $N = 2$ (dotted line). The plot includes the Bogomol’nyi bound (dashed line).
1b) The electric charge $Q$ (36) in units of $2 \pi N$ as a function of the Chern–Simons coupling parameter $\kappa$ for $N = 1$ (solid diamonds) and $N = 2$ (triangles up).

Fig. 2
2a) The energy density $e$ (33) as a function of $r$ for $N = 1$ and $\kappa = 0.3$ (dotted), $\kappa = \kappa_{cr}^1 = 0.632$ (solid) and $\kappa = 2$ (dashed).
2b) The profile function $f$ as a function of $r$ for $N = 1$ and $\kappa = \kappa_{cr}^1 = 0.632$ (solid), $\kappa = 2$ (dashed), $\kappa = 50$ (dot–dashed).
2c) The gauge field $a$ as a function of $r$ for $N = 1$ and $\kappa = \kappa_{cr}^1 = 0.632$ (solid), $\kappa = 2$ (dashed), $\kappa = 0.4$ (dotted).
2d) The gauge field $v$ as a function of $r$ for $N = 1$ and $\kappa = \kappa_{cr}^1 = 0.632$ (solid), $\kappa = 2$ (dashed), $\kappa = 0.4$ (dotted).

Fig. 3
3a) The energy density $e$ (33) as a function of $r$ for $\kappa = \kappa_{cr}^1 = 0.632$, $N = 1$ (solid) and $N = 2$ (dashed).
3b) The electric fields radial component $E_r$ as a function of $r$ for $\kappa = \kappa_{cr}^1 = 0.632$, $N = 1$ (solid) and $N = 2$ (dashed).
