CAHN-HILLIARD EQUATION WITH CAPILLARITY IN ACTUAL DEFORMING CONFIGURATIONS

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Abstract. The diffusion driven by the gradient of the chemical potential (by the Fick/Darcy law) in deforming continua at large strains is formulated in the reference configuration with both the Fick/Darcy law and the capillarity (i.e. concentration gradient) term considered at the actual configurations deforming in time. Static situations are analysed by the direct method. Evolution (dynamical) problems are treated by the Faedo-Galerkin method, the actual capillarity giving rise to various new terms as e.g. the Korteweg-like stress and analytical difficulties related to them. Some other models (namely plasticity at small elastic strains or damage) with gradients at an actual configuration allow for similar models and analysis.

1. Introduction. This paper addresses the Cahn-Hilliard model [7] for diffusion with capillarity (i.e. concentration gradient involved in the stored energy) in deformable media, which is sometimes also called the Cahn-Larché model [22]. It is usually considered in mathematical literature mostly at small strains, cf. e.g. [6, 13, 32, 36] and [16, Ch.4,5,7] and references therein. If at large strains, then it is usually considered with the concentration gradient in the reference (undeformed) configuration, as used also in most of engineering references, cf. e.g. [4, 10], and as is relevant in some applications.

Yet, in some other applications, the gradients in the actual space (deforming) configurations seem more natural. As mentioned in [17] in the context of Allen-Cahn equation about these gradients, “their spatial counterparts could have also been used, this would lead to cumbersome contributions (via pull-back and push-forward operations)” . Nevertheless, when the transport by Fick/Darcy law is considered in an actual (i.e. space) deforming configuration as e.g. in [3, 21, 34, 35], it is rather mis-conceptual to involve a gradient of concentration in the material configuration. And indeed, sometimes the Cahn-Hilliard model with the concentration gradient...
in the actual configuration can be found in engineering literature \[9,18,24\] but, of course, without any analysis. A fairly general model has been scrutinised in \[31\] but without caring about non-selfpenetration (and thus not considering possible singularities of the stored energy) and also without analysis regarding the existence of weak solutions.

We will confine ourselves to a single-component flow. A generalization for a multicomponent flow possibly also with mutual reactions between particular components is interesting and seems possible, in particular when the gradient structure as \((16)\) below driven by the stored energy and nonlocal dissipation potential like \((9a,c)\) below is kept, cf. \[25,26\].

The main goal of this article is to perform a rigorous analysis regarding the existence of weak solutions for the diffusion in poro-elastic-dynamic model both with the Fick/Darcy law and the capillarity gradient term in the actual deforming configuration.

First, in Section 2, we will specify the stored energy and present the static problem, exploiting the 2nd-grade nonsimple material concept, and perform the analysis regarding the existence of the solution based on a minimum-energy principle. Although the static problem has rather a preliminary character for the desired evolution problem, it has its own importance and mathematical interest. Then, in Section 3, we will slightly modify the problem (by simplifying the constraints and using 3rd-grade nonsimple material concept) and formulate the mentioned evolutionary variant, involving also inertial effects, which allows for modelling elastic waves interacting with a diffusion equation, e.g. waves whose attenuation and dispersion can be influenced by the content of diffusant. For the analysis, we use the Faedo-Galerkin approximation which can keep the approximate solutions well away from the singularity of the stored energy at deformation gradients with non-positive determinants. Eventually, in Section 4, we briefly outline various other application of the presented mathematical techniques to gradient theories for some other internal variables.

2. A static problem. Before treating the evolution problems, let us begin with static situations. The equilibria of poroelastic or swelling-exhibiting materials are assumed to be governed by energy minimization and lead to interesting mathematical problems. Let us emphasize that the static problems are such special cases of steady-state situations (see Remark 2 below) where all transport processes vanish. In our isothermal situation, it means that the diffusion processes are “switched off”. In contrast to the steady-state of evolutionary problems, the static problem well allows for application of variational techniques and it is a worthy departing point for the full evolution model in Sect. 3.

As usual in continuum mechanics of solids, we consider the Lagrangian formulation with \(\Omega \subset \mathbb{R}^d\) a fixed reference domain. The state variables are the deformation \(y : \Omega \to \mathbb{R}^d\) and the concentration \(\zeta : \Omega \to \mathbb{R}\). For a stored energy density \(\varphi = \varphi(F,z)\) depending on \(F\) being a placeholder for the deformation gradient and \(z\) being the placeholder for the values of the concentration \(\zeta\), the basic ingredient for the model is the overall stored energy, here considered as

\[
\mathcal{E}(y,\zeta) := \int_{\Omega} \varphi(\nabla y,\zeta) + \frac{\kappa}{2} |(\nabla y)^{-\top} \nabla \zeta|^2 + \frac{1}{p} |\nabla^2 y|^{p-2} \nabla^2 y : \mathbb{H} : \nabla^2 y \, dx
\] (1)

with \(\kappa > 0\) a capillarity coefficient and with \(\mathbb{H}\) a (presumably small) regularizing 6th-order symmetric positive definite tensor, \(p > d\). Easily, both \(\kappa\) and \(\mathbb{H}\)
may depend also on $x$ (i.e. a nonhomogeneous medium) but e.g. a dependence on the concentration $\zeta$ would bring additional contribution into the chemical potentials and serious analytical difficulties. The so-called 2nd-grade nonsimple-material (or couple-stress) concept [12, 37] has been applied, leading to the bending-like energy contribution due to the $H$-term, involving second-order deformation gradients (= first-order strain gradients). In dynamical situations, this may offer a suitable tool to model a dispersion. Beside such mechanical motivation, the main mathematical advantage of the nonsimple-material concept is that the higher-order deformation gradients bring an additional regularity of the deformations and also compactness of the set of the admissible deformations in a stronger topology. Moreover, there the stored energy can be even convex in the highest derivatives of the deformation, which is helpful in proving existence of minimizers.

Let us emphasize that the capillarity (i.e. the concentration gradient) in (1) is considered in the actual configuration, being pulled back into the reference configuration by a vectorial pushforward $(\nabla y)^{-\top}$. Thus the determinant of $\nabla y$ is to be kept away from zero to have $(\nabla y)^{-\top}$ under control, which needs involvement of the $H$-term. This convex higher-order term also allows for generally nonconvex (even not quasi-convex) stored energies.

As already mentioned, in the static situations we are addressing in this section, all dissipation processes vanish, i.e. here in particular all diffusive processes vanish. Here it means that the gradient of the chemical potential vanishes on $\Omega$. When assuming $\Omega$ connected, this further leads to that the chemical potential $\mu$ is constant, cf. also Remark 1 below. Let us denote this constant by $\bar{\mu}$.

A variationally interesting situation is that the poroelastic body is completely isolated on its boundary. It is then natural to prescribe the total amount of diffusant

$$
\int_\Omega \zeta \, dx = Z_{\text{total}} \quad \text{with} \quad Z_{\text{total}} \geq 0 \quad \text{given.}
$$

The value $Z_{\text{total}}$ influences also the mentioned constant $\bar{\mu}$ which is not known a-priori and maybe even not uniquely determined by particular $Z_{\text{total}}$.

We will use the standard notation concerning the Lebesgue and the Sobolev spaces, namely $L^p(\Omega; \mathbb{R}^n)$ for Lebesgue measurable functions $\Omega \to \mathbb{R}^n$ whose Euclidean norm is integrable with $p$-power, and $W^{k,p}(\Omega; \mathbb{R}^n)$ for functions from $L^p(\Omega; \mathbb{R}^n)$ whose derivatives up to the order $k$ have their Euclidean norm integrable with $p$-power. We also write briefly $H^k = W^{k,2}$. Moreover, $\text{GL}^+(d)$ denotes the general linear group of orientation-preserving mappings $\mathbb{R}^d \to \mathbb{R}^d$, i.e. the subset of $\mathbb{R}^{d \times d}$ of nonsingular matrices with a positive determinant, while $\text{SO}(d)$ denotes the special orthogonal group, i.e. the set $\{A \in \mathbb{R}^{d\times d}; \; A^\top A = A A^\top = I, \; \det A = 1\}$.

We require that admissible deformations of the material are orientation preserving and injective almost everywhere in $\Omega$. The attribute will be ensured by the Ciarlet-Nečas condition [8]. We also assume that the elastic body is fixed on a part of its boundary by a Dirichlet condition. Altogether, we are left with the following problem:

\[
\begin{align*}
\text{Minimize} \quad & J(y, \zeta) := E(y, \zeta) - \int_\Omega f \cdot y \, dx \\
\text{subject to} \quad & \int_\Omega \det \nabla y \, dx \leq \text{meas}_d(y(\Omega)) \quad \text{and} \quad \int_\Omega \zeta \, dx = Z_{\text{total}}, \\
& \det \nabla y > 0 \quad \text{and} \quad \zeta \geq 0 \quad \text{a.e. on} \; \Omega, \\
& y\big|_{\Gamma_{\text{dir}}} = y_D, \quad y \in W^{2,p}(\Omega; \mathbb{R}^d) \quad \text{and} \quad \zeta \in H^1(\Omega).
\end{align*}
\]
As we want to treat this problem by direct variational methods, we do not need to formulate the Euler-Lagrange equation, i.e. the boundary-value problem behind the constrained minimization problem (3). In fact, it seems not entirely clear how the global Ciarlet-Nečas constraint occurs in such equations through only one scalar multiplier rather than reaction force distributed along the boundary as in [20, 30]. Anyhow, at least formally, one can formulate first-order optimality conditions, cf. Remark 1.

The physically motivated assumptions on the stored-energy density are

\[ \varphi : \text{GL}^+(d) \times \mathbb{R} \to \mathbb{R}^+ \] is continuously differentiable and

\[ \forall R \in \text{SO}(d), \; \forall F \in \text{GL}^+(d), \; z \in \mathbb{R} : \; \varphi(F, z) = \varphi(RF, z), \]

\[ \exists \epsilon > 0 \; \forall F \in \mathbb{R}^{d \times d}, \; z \in \mathbb{R} : \]

\[ \varphi(F, z) \begin{cases} \geq \frac{\epsilon}{(\det F)^q} + |z|^2 & \text{for } \det F > 0 \text{ with } q > \frac{pd}{p-d}, \; p > d, \\ = +\infty & \text{for } \det F \geq 0. \end{cases} \] (4c)

The assumption (4b) is the frame-indifference, while (4c) grants local non-selfpenetration and even allows to keep the deformation gradient “uniformly” invertible due to [15]. More in detail, the Healey-Krömer theorem [15] states that, assuming $p$ and $q$ as in (4c), for each $c$ there exists $\varepsilon = \varepsilon(p, q, c, d) > 0$ such that

\[ y \in W^{2,p}(\Omega; \mathbb{R}^d), \; \det \nabla y > 0 \text{ on } \Omega, \]

\[ \|y\|_{W^{2,p}(\Omega; \mathbb{R}^d)} + \left\| \frac{1}{\det \nabla y} \right\|_{L^p(\Omega)} \leq c \implies \det \nabla y \geq \varepsilon. \] (5)

Actually, this formulation uniform on each level set (although proved already in [15]) was articulated in [28, Thm. 3.1]. For $\varphi$ satisfying (4c) and $\mathbb{H}$ positive definite, this in particular implies that $\det \nabla y \geq \varepsilon$ on each sub-level set of $\mathcal{E}$, i.e. on $\{(y, \zeta) : \mathcal{E}(y, \zeta) \leq c\}$, with some $\varepsilon > 0$ depending on $c$.

**Proposition 1.** Let $\varphi$ satisfy (4), $\mathbb{H}$ be symmetric positive definite, $y_0 \in W^{2-1/p,p}(\Omega; \mathbb{R}^d)$, $f \in L^1(\Omega)$, and (3) be feasible in the sense that its constraints are satisfied for at least one $(y_0, \zeta_0)$ with $\inf_{\Omega} \det(\nabla y_0) > 0$. Then (3) has a solution $(y, \zeta) \in W^{2,p}(\Omega; \mathbb{R}^d) \times H^1(\Omega)$ such that $\inf_{\Omega} \det(\nabla y) > 0$.

**Proof.** The assumed feasibility ensures existence of some $(y_0, \zeta_0)$ which is compatible with the constraints and which makes the functional (1) finite. By the Healey-Krömer theorem (5) and by the assumption (4c), there exists $\varepsilon > 0$ such that $\det \nabla y \geq \varepsilon$ for any $(y, \zeta)$ from the respective level-set of $J$ for which $J(y, \zeta) \leq J(y_0, \zeta_0)$.

This makes the functional $J$ weakly lower-semicontinuous on this level-set. For any infimizing sequence $\{(y_k, \zeta_k)\}_{k \in \mathbb{N}}$, one can take a subsequence weakly converging in $W^{2,p}(\Omega; \mathbb{R}^d) \times H^1(\Omega)$. By compact embeddings $W^{2,p}(\Omega; \mathbb{R}^d) \subseteq L^\infty(\Omega; \mathbb{R}^d)$ and $H^1(\Omega) \subseteq L^2(\Omega)$, we can see that $(\nabla y_k, \zeta_k)$ converges strongly in $L^\infty(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ which makes the functional $(y, \zeta) \mapsto \int_\Omega \varphi(\nabla y, \zeta) \, dx$ lower-semicontinuous by the Fatou lemma. Moreover, also

\[ (\nabla y_k)^{-\top} = \frac{\text{Cof} \nabla y_k}{\det \nabla y_k} \to \frac{\text{Cof} \nabla y}{\det \nabla y} = (\nabla y)^{-\top} \text{ strongly in } L^\infty(\Omega; \mathbb{R}^{d \times d}), \] (6)

where "Cof" denotes the cofactor of a matrix (composed from first minors of that matrix). In (6), we used that $\det \nabla y_k(x) \geq \varepsilon$ and $F \mapsto 1/\det F$ is uniformly continuous on $\{F : \det F \geq \varepsilon\}$ so that $1/\det \nabla y_k \to 1/\det \nabla y$ strongly in $L^\infty(\Omega,$
and also that $|\nabla y_k(x)| \leq C$ is $F \mapsto \text{Cof} F$ is uniformly continuous on bounded sets so that also $\text{Cof} \nabla y_k \rightarrow \text{Cof} \nabla y$ strongly in $L^\infty(\Omega; \mathbb{R}^{d \times d})$. Thus the functional $(y, \zeta) \mapsto \frac{1}{2} \int_{\Omega} |(\nabla y)^{-1} \nabla \zeta|^2 \, dx$ is weakly lower-semicontinuous. Also the constraint $\zeta \geq 0$ is inherited by the limit.

The existence of a minimizer follows then by the direct-method arguments.

Both gradient terms can be omitted when $\varphi$ is so-called cross-polyconvex in the sense that $\varphi = \varphi(F, z)$ can be expressed as a convex function of all minors of $F$ and of $z$, cf. [21, Sect. 3.6.2]. In some cases, namely $\varphi(F, z) = \varphi_1(f(z)F) + \varphi_2(z)$, at least the $\mathbb{H}$-term can be omitted even if $\varphi$ is not cross-polyconvex but then the capillarity term is to be considered in the reference configuration, cf. [21, Sect. 3.6.3].

Remark 1 (Constancy of chemical potential). From the mentioned partial 1-st order optimality conditions for a solution $(y, \zeta)$ to (3), in particular involving the partial differential with respect to $\zeta$, one can read at least formally (if $\varphi$ is suitably smooth and, for a moment, assuming that the constraint $\zeta \geq 0$ is not active) that there is a scalar Lagrange multiplier $\bar{\mu} \in \mathbb{R}$ to the constraint $\int_{\Omega} \zeta \, dx - Z_{\text{total}} = 0$ and

$$\partial_{\zeta} \left( \mathcal{E}(y, \zeta) + \bar{\mu} \left( Z_{\text{total}} - \int_{\Omega} \zeta \, dx \right) \right) = 0, \quad (7)$$

i.e.

$$\partial_{\zeta} \varphi(\nabla y, \zeta) - \text{div}(\kappa(\nabla y)^{-1}(\nabla y)^{-T} \nabla \zeta) = \bar{\mu} \quad (8)$$
on $\Omega$. The left-hand side of (8) is the chemical potential and the equality (8) shows that this potential is constant in this stationary solution. This constancy is related with the vanishing diffusion flux which is driven by $\nabla \zeta$ by the Fick law, cf. (14b) below. From (7) one can also read boundary conditions (14g). If $\zeta \geq 0$, then the chemical potential involves still a multiplier (as a function on $\Omega$) to this constraint.

Remark 2 (Steady-state problems). An interesting generalization would be towards steady-state problems where the diffusion flux (although being constant in time) does not necessarily vanish, cf. (14) below with all time-derivatives omitted. This would need to involve also the transport equation, see (14b,c) below with $\dot{\zeta}$ omitted. Existence for such generalization seems open, however. Some results are available only at small strains by using the Schauder fixed-point theorem, cf. [33]. On the other hand, sometimes some self-induced oscillations in porous media (polymer gels) are observed, cf. e.g. [39, 40], which may indicate that there might be even some physical reasons for nonexistence of steady-state solutions.

3. The dynamical problem. Our main goal is to formulate an evolution governed by the stored energy from Sect. 2 and to carry out the analysis of such an initial-boundary-value problem. We will focus on dynamical problems, i.e. involving inertia. In contrast to static situations, variational formulations (based now, instead of minimal-energy principle, on the Hamiltonian variational principle extended for nonconservative systems) do not seem fitted with applications of direct methods. Instead, we will use a formulation in terms of conventional partial-differential equations with corresponding boundary conditions.

For this reason, we need to adopt three compromising modifications of the static problem. First, we will ignore the Ciarlet-Nečas global non-selfpenetration condition.
while keeping only the local non-selfpenetration $\det(\nabla y) > 0$ which is anyhow needed to keep under control the pulled-back concentration gradient and the pulled-back mobility gradient. Second, we will also ignore the constraint $\zeta \geq 0$; in fact, this is an often accepted modelling simplification relying that the mobility of the diffusant is very small and the stored energy very large if concentration approaches zero. Third, we need to have the regularizing $H$-term quadratic so that the resulting nonlinear hyperbolic problem is linear in the highest-order terms, which needs to involve a (possibly fractional) derivative of the deformation gradient of the order higher than $1+d/2$. This is inevitably rather technical; for the fractional-gradient and the concept of a nonlocal nonsimple material see e.g. [21]. Here we take the option of the 3rd-grade nonsimple material like in [1, 29], considering the stored energy and the concept of a nonlocal nonsimple material see e.g. [21]. Here we take the

$$E(y, \zeta) = \int_\Omega \varphi(\nabla y, \zeta) + \frac{\kappa}{2} |(\nabla y)^{-T}\nabla \zeta|^2 + \frac{1}{2} \nabla^3 y : \hat{H} : \nabla^3 y \, dx \quad (9a)$$

with $\hat{H}$ some symmetric positive definite 8th-order tensor.

The other ingredients in building the evolution model are the kinetic energy

$$T(y) := \int_\Omega \frac{\rho}{2} |y|^2 \, dx \quad (9b)$$

with $\rho > 0$ the mass density and the dot denoting the time derivative, and the (Rayleigh’s pseudo)potential of dissipative forces related with diffusion:

$$R(y, \zeta) := \int_\Omega \frac{1}{2} |\mathcal{M}^{1/2}(\nabla y, \zeta)\nabla \Delta_{\text{sym}}^{-1}(\nabla y, \zeta) \nabla \zeta|^2 \, dx + \int_{\Gamma} \frac{1}{2} \alpha |(\Delta_{\text{sym}}^{-1}(\nabla y, \zeta) \zeta)'^2 \, dS, \quad (9c)$$

where $\alpha > 0$ is a phenomenological permeability coefficient of the boundary and where $\Delta_{\text{sym}}^{-1} : (r, \mu_{\text{ext}}) \mapsto \mu$ is the linear operator $H^1(\Omega)^* \to H^1(\Omega)$ defined by $\mu := \text{the weak solution to the equation } \text{div}(\mathcal{M} \nabla \mu) = r$ with the Robin boundary conditions $\mathcal{M} \nabla \mu \tilde{n} + \alpha \mu = \alpha \mu_{\text{ext}}$; in the case $\mu_{\text{ext}} = 0$ cf. [27, Sect. 5.2.6]. Eventually, we consider the mechanical load $F$ determined by the bulk force $f$ and the surface load $g$ as by the external chemical potential $\mu_{\text{ext}}$ by

$$F(t, \tilde{y}, \tilde{\zeta}) \equiv \langle F(t), (\tilde{y}, \tilde{\zeta}) \rangle = \langle F_1(t), \tilde{y} \rangle + \langle F_2(t), \tilde{\zeta} \rangle = \int_\Omega f(t) \cdot \tilde{y} \, dx + \int_\Sigma g(t) \cdot \tilde{y} + \alpha \mu_{\text{ext}} \tilde{\zeta} \, dS. \quad (9d)$$

Let us note that the dissipation potential (9c) is nonlocal. A natural requirement for thermodynamical consistency (i.e. non-negative entropy production) is that $\mathcal{M}$ is positive semidefinite, so that its square root $\mathcal{M}^{1/2}$ occurring in (9c) has a good sense.

The notation $\mathcal{M} : \Omega \times \mathbb{R}^{d \times d} \times \mathbb{R} \to \mathbb{R}^{d \times d}$ stands for the mobility tensor which occurs in the generalized Fick law making the flux of the diffusant proportional to the gradient of the chemical potential denoted by $\mu$. Consistently with the capillarity in the actual configuration pulled-back, a reasonable modeling concept that this Fick law (in particular covering also Darcy law) is considered in the actual deforming (time-dependent) configuration, and is then to be pulled back into the fixed reference configuration. The transformed Fick law (i.e. pulled back) uses the matrix of mobility coefficients as

$$\mathcal{M}(x, F, z) := \frac{(\text{Cof } F^T) \mathcal{M}(x, z) \text{Cof } F}{\det F} \quad \text{if } \det F > 0, \quad (10)$$
while the case \( \det F \leq 0 \) is considered nonphysical. In (10), \( M: \Omega \times \mathbb{R} \to \mathbb{R}^{d \times d} \) is the diffusant mobility (depending possibly also on \( x \in \Omega \)) as a material property. In literature, this formula is often used in the isotropic case, cf. e.g. [11, Formula (67)] or [14, Formula (3.19)]. For the anisotropic case, cf. [21,35]. In fact, (10) can be expressed in terms of the right Cauchy-Green strain \( C = F^\top F \) rather than of \( F \) itself, which grants the frame-indifference of this model. The mathematically interesting attribute of the model (10) is that \( \det(\nabla v) \) is (under suitable data qualification) well kept away zero, similarly as it was already needed for the static problem because of the capillarity in the actual configuration, and which is now needed also to (10).

We will use the notation \( L^p(0,T;X) \) for the Bochner space of Bochner measurable functions \( [0,T] \to X \) whose norm is in \( L^p(0,T) \), and \( H^1(I;X) \) for functions \( [0,T] \to X \) whose distributional derivative is in \( L^2(0,T;X) \). Furthermore, we will not use the Dirichlet condition and use the notation \( Q = [0,T] \times \Omega \) and \( \Sigma = [0,T] \times \Gamma \).

The departing point is the Hamilton variation principle adapted for nonconservative systems (cf. also Bedford [5]), which says that the integral

\[
\int_0^T \mathcal{T}(\dot{y}) - \mathcal{E}(q(t)) + \langle \mathcal{F}(t), q(t) \rangle \, dt \quad \text{is stationary} \tag{11}
\]

with \( q = (y, \zeta) \) being the state of the system, \( \mathcal{E} \) being the stored energy and \( \mathcal{F}(t) = F(t) - \mathcal{R}'(\dot{q}) \) being a nonconservative force. Written componentwise and taken into account that \( \mathcal{T} \) is quadratic and independent of \( \zeta \) and \( \mathcal{R}(y, \zeta; \cdot) \) is independent of \( y \), this variational principle yields formally the abstract system

\[
\mathcal{T}'\dot{y} + \partial_\zeta \mathcal{E}(y, \zeta) = F_1(t), \tag{12a}
\]

\[
\partial_\zeta \mathcal{R}(y, \zeta; \cdot) + \mathcal{M} = F_2(t) \quad \text{with} \quad \mathcal{M} = \partial_\zeta \mathcal{E}(y, \zeta). \tag{12b}
\]

The new variable \( \mathcal{M} \) is here in a position of an abstract chemical potential. When one substitutes the concrete functionals (9), this yields the following weak formulation:

**Definition 1** (Weak solution). The triple \( (y, \zeta, \mu) \) with \( y \in L^\infty(I; H^2(\Omega; \mathbb{R}^d)) \cap W^{1,\infty}(I; L^2(\Omega; \mathbb{R}^d)), \zeta \in L^\infty(I; H^1(\Omega)) \) and \( \mu \in L^2(I; H^1(\Omega)) \) is called a weak solution to the initial-boundary-value problem (14) below if

\[
\int_Q (\partial_\nu \varphi(\nabla y, \zeta) + \sigma_\kappa(\nabla y, \nabla \zeta)) : \nabla v - \varrho \dot{v} \cdot v + \nabla^3 y : \nabla^3 v \, dx \, dt
\]

\[
= \int_Q f \cdot v \, dx \, dt + \int_\Sigma g \cdot v \, dS \, dt + \int_\Omega g v_0 \cdot v(0, \cdot) \, dx \tag{13a}
\]

holds for all \( v \in L^2(I; H^2(\Omega; \mathbb{R}^d)) \cap H^1(I; L^2(\Omega; \mathbb{R}^d)) \) with \( v|_{t=T} = 0 \), and also the initial condition \( y(0, \cdot) = y_0 \) is satisfied, and if

\[
\int_Q \frac{M(\zeta)}{\det \nabla y} \nabla \mu \cdot ((\text{Cof} \nabla y) \nabla \zeta) - \zeta \dot{v} \, dx \, dt + \int_\Sigma \alpha \mu v \, dS \, dt
\]

\[
= \int_\Omega \zeta_0 v(0) \, dx + \int_\Sigma \alpha v_{\text{ext}} v \, dS \tag{13b}
\]

holds for all \( v \in H^1(Q) \) with \( v|_{t=T} = 0 \), and

\[
\int_Q \kappa(\nabla y)^\top \nabla \zeta \cdot (\nabla y)^\top \nabla v + \langle \partial_\zeta \varphi(\nabla y, \zeta) - \mu \rangle v \, dx \, dt = 0 \tag{13c}
\]

for all \( v \in L^2(I; H^1(\Omega)) \).
To see the corresponding initial-boundary-value problem, one has to apply one by-part integration in time for the inertial term, and here three-times Green formula over $\Omega$ and twice surface Green formula over $F$. The resulting boundary-value problem involves a rather “exotic” hyper-stress and three boundary conditions, namely

\[
\dot{\varphi} - \text{div}(\partial_y \varphi(\nabla y, \zeta) + \sigma_k(\nabla y, \nabla \zeta) + \text{div}^2(\nabla^3 y)) = f
\]

with $\sigma_k(F, \nabla \zeta) = \kappa F^{-1} : (F^{-\top})'(\nabla \zeta \otimes \nabla \zeta)$ in $Q$, \hfill (14a)

\[
\dot{\zeta} + \text{div}(\nabla y, \zeta)\nabla \mu = 0 \quad \text{with} \quad \nabla(F, \zeta) = \frac{(\text{Cof} F)^\top M(\zeta) \text{Cof} F}{\det F} \quad \text{in} \quad Q, \quad (14b)
\]

and with $\mu = \partial_y \varphi(\nabla y, \zeta) - \text{div}(\kappa(\nabla y)^{-1}(\nabla y)^{-\top} \nabla \zeta)$ in $Q$, \hfill (14c)

\[
(\partial_y \varphi(\nabla y, \zeta) + \sigma_k(\nabla y, \nabla \zeta))\vec{n} - \text{div}_v((\text{div}_v \vec{n})(\nabla^3 y) : (\vec{n} \otimes \vec{n}))
\]

\[
+ \text{div}_v^2((\nabla^3 y) : \vec{n}) - (\text{div}_v \vec{n})\text{div}_v((\nabla^3 y) : \vec{n}) \cdot \vec{n}
\]

\[
+ \text{div}_v^2(\nabla^3 y) \cdot \vec{n} + \text{div}_v(\text{div}(\nabla^3 y) \cdot \vec{n}) = g \quad \text{on} \quad \Sigma, \quad (14d)
\]

\[
(\nabla^3 y) : (\vec{n} \otimes \vec{n} \otimes \vec{n}) = 0 \quad \text{and} \quad \text{div}(\nabla^3 y) : (\vec{n} \otimes \vec{n}) = 0, \quad \text{on} \quad \Sigma, \quad (14e)
\]

\[
\n abla(F, \zeta) \nabla \mu = \alpha \mu_{\text{ext}} \quad \text{on} \quad \Sigma, \quad (14f)
\]

\[
\kappa(\nabla y)^{-1}(\nabla y)^{-\top} \nabla \zeta \cdot \vec{n} = 0 \quad \text{on} \quad \Sigma, \quad (14g)
\]

where $\text{div}_v = \text{tr}(\nabla v)$ with $\text{tr}(\cdot)$ being the trace of a $(d-1) \times (d-1)$-matrix, denotes the $(d-1)$-dimensional surface divergence and $\nabla_v v = \nabla v - \frac{\partial v}{\partial \vec{n}} \vec{n}$ being the surface gradient of $v$. Here we use, in addition what would come from (11) with (9), also a nonhomogeneous boundary condition for the diffusion, involving an external chemical potential $\mu_{\text{ext}}$. In (14a), the differential of $F \mapsto F^{-\top} = (F^{-1})'$, being the 4th-order tensor, can be more explicitly calculated as

\[
(F^{-\top})' = \frac{(\text{Cof} F)' \det F - \text{Cof} F \otimes \text{Cof} F}{(\det F)^2} = \frac{(\det F)' \text{Cof} F}{(\det F)^2} - \frac{\text{Cof} F \otimes \text{Cof} F}{(\det F)^2}. \quad (15)
\]

In particular, as the Hessian $(\det F)'$ of the functional $F \mapsto \det F$ is symmetric, the right-hand side of (15) is symmetric too, as well as the expression $s_k(F, \nabla \zeta) := \kappa(F^{-\top})':((\nabla \zeta \otimes \nabla \zeta)$ which is in the position of (a contribution to) the 2nd Piola-Kirchhoff stress in (14a), while its pullback $s_k(F, \nabla \zeta) = F^{-1} s_k(F, \nabla \zeta)$ is not symmetric in general, being (a contribution to) the 1st Piola-Kirchhoff stress. Moreover, the variable $\mu$ from (14c) is called a chemical potential and $\nabla(F, \zeta)\nabla \mu$ in (14b) is the Fick law for the flux of the diffusant.

The system (14) deserves some comments. First, the diffusion equation (14b,c) considered with the Robin boundary conditions (14f) can be rewritten in the form

\[
\Delta^{-1}_{\nabla^3 y, \zeta}(\dot{\zeta}, \mu_{\text{ext}}) = \mu = \partial_y \varphi(\nabla y, \zeta) - \text{div}(\kappa(\nabla y)^{-1}(\nabla y)^{-\top} \nabla \zeta) \quad (16)
\]

with $\Delta^{-1}_{\nabla^3 y, \zeta}$ as in (9c). In view of (9a,c), this is exactly the flow-rule (12b).

Further, the boundary conditions (14d,e) for the mechanical equilibrium (14a) are quite technical because of the $\nabla^3 y$-term. It is to be treated, at each time instant
t (not explicitly denoted), first by applying three times Green formula

$$\int_{\Omega} \nabla^3 y : \nabla^3 v \, dx = \int_{\Gamma} ((\nabla \nabla y) \cdot \vec{n}) : \nabla^2 v \, dS - \int_{\Omega} \text{div}(\nabla \nabla y) : \nabla^2 v \, dx$$

$$= \int_{\Gamma} ((\nabla \nabla y) \cdot \vec{n}) : (\partial^2 v + \nabla^2 v) \, dS - \int_{\Omega} \text{div}(\nabla \nabla y) : \nabla^2 v \, dx$$

$$= \int_{\Gamma} ((\nabla \nabla y) \cdot \vec{n}) : (\partial^2 v + \nabla^2 v)$$

$$\quad - \text{(div}(\nabla \nabla y) \cdot \vec{n}) : (\partial v + \nabla v) \, dS + \int_{\Omega} \text{div}^2(\nabla \nabla y) : \nabla v \, dx$$

$$= \int_{\Gamma} ((\nabla \nabla y) \cdot \vec{n}) : (\partial^2 v + \nabla^2 v) + (\text{div}^2(\nabla \nabla y) \cdot \vec{n}) : v$$

$$\quad - \text{(div}(\nabla \nabla y) \cdot \vec{n}) : (\partial v + \nabla v) \, dS - \int_{\Omega} \text{div}^3(\nabla \nabla y) : v \, dx, \quad (17)$$

where we used the decomposition of $\nabla v$ on $\Gamma$ into the normal and the tangential part $\partial_{\vec{n}}v + \nabla_v v$, and in particular also

$$\nabla^2 v = (\partial_{\vec{n}} + \nabla_v v)(\partial_{\vec{n}}v + \nabla_v v) = \partial^2_{\vec{n}}v + \nabla^2_v v,$$

where we use also the orthogonality of $\nabla_v v$ and $\partial_{\vec{n}}v$. We further apply four times the surface Green formula on the boundary term; more specifically, we apply

$$\int_{\Gamma} A \cdot \nabla_v v \, dS = \int_{\Gamma} ((\text{div} \vec{n}) A \vec{n} - \text{div}_v A) \cdot v \, dS, \quad (18)$$

which holds for a smooth field $A \in C^1(\Gamma; \mathbb{R}^{d \times d})$ and $v \in C^1(\Gamma; \mathbb{R}^d)$ that; cf. see [12, Formula (34)]. By this way, we can write

$$\int_{\Gamma} ((\nabla \nabla y) \cdot \vec{n}) : (\partial^2_{\vec{n}}v + \nabla^2_v v) + (\text{div}^2(\nabla \nabla y) \cdot \vec{n}) : v - (\text{div}(\nabla \nabla y) \cdot \vec{n}) : (\partial v + \nabla v) \, dS$$

$$= \int_{\Gamma} ((\nabla \nabla y) \cdot \vec{n}) : \partial^2_{\vec{n}}v - (\text{div}(\nabla \nabla y) \cdot \vec{n}) : \partial_v v$$

$$\quad + (\text{div}_v \vec{n})(\nabla \nabla y) : (\vec{n} \otimes \vec{n}) - \text{div}_s(\nabla \nabla y) : \nabla_v v$$

$$\quad + (\text{div}^2(\nabla \nabla y) \cdot \vec{n}) - (\text{div}_v \vec{n}) \text{div}(\nabla \nabla y) : (\vec{n} \otimes \vec{n}) + \text{div}_s(\nabla \nabla y) : \vec{n} \, dS$$

$$= \int_{\Gamma} ((\nabla \nabla y) \cdot \vec{n}) : \partial^2_{\vec{n}}v - (\text{div}(\nabla \nabla y) \cdot \vec{n}) : \partial_v v$$

$$\quad + (\text{div}_v \vec{n})(\nabla \nabla y) : (\vec{n} \otimes \vec{n} \otimes \vec{n}) - \text{div}_s(\text{div}_v \vec{n})(\nabla \nabla y) : (\vec{n} \otimes \vec{n})$$

$$\quad + \text{div}^2((\nabla \nabla y) \cdot \vec{n}) - (\text{div}_v \vec{n}) \text{div}(\nabla \nabla y) : (\vec{n} \otimes \vec{n}) + \text{div}^2(\nabla \nabla y) : \vec{n}$$

$$\quad - (\text{div}_s \vec{n}) \text{div}(\nabla \nabla y) : (\vec{n} \otimes \vec{n}) + \text{div}_s(\text{div}(\nabla \nabla y) : \vec{n}) \, dS. \quad (19)$$

Substituting (19) into (17) and into (13a), and taking $v$ arbitrarily with compact support, then with arbitrary traces but with normal derivatives zero, and then with $\partial^2_{\vec{n}}v = 0$, and eventually entirely arbitrarily, we obtain subsequently (14a) and (14d,e). Notably, the conditions (14c) have been reflected also in (14d) to simplify it in contrast what can be seen from the last seven terms in (19).

It is important that this gradient theory in the actual configuration has led to a specific contribution $\sigma_k$ to the stress tensor. Such stresses are needed, in particular, to balance energy and are known in incompressible-fluid mechanics under
the name Korteweg stresses [19]. Evaluating \((F^{-\top})'\) as in (15) and eliminating
\(F^{-1} = \text{Cof}F^\top/\det F\), this stress can be expressed more specifically as
\[
\sigma_k(F, \nabla \zeta) = \kappa \frac{\text{Cof}F^\top}{\det F} \left( \frac{\text{Cof}F \otimes \text{Cof}F}{(\det F)^2} \right) : (\nabla \zeta \otimes \nabla \zeta).
\] (20)
Mathematically, this stress brings an additional difficulty in comparison with the usual concepts of gradients in reference configuration, because \(\nabla \zeta\) occurs non-linearly and we need the strong convergence of an approximation of \(\nabla \zeta\).

**Proposition 2** (Existence of weak solutions in poro-elastic-dynamics). Let \(d = 2, 3\) and (4) hold for \(p > 2d/(d-2)\) now together with
\[
|\partial_z \varphi(F, z)| \leq C(F) \{1 + |z|^r\}
\] with \(C: \mathbb{R}^{d \times d} \to \mathbb{R}\) continuous and \(r < \frac{d}{d-2}\), (21)
and let \(\mathcal{M}\) be a bounded Carathéodory mapping with values uniformly positive definite. Moreover, let \(f \in L^1(I; L^2(\Omega; \mathbb{R}^d))\), \(g \in W^{1,1}(I; L^1(\Gamma; \mathbb{R}^d))\), \(\mu_{\text{ext}} \in L^2(\Sigma)\), \(y_0 \in H^3(\Omega; \mathbb{R}^d)\), \(\inf_{t \in I} \det \nabla y_0 > 0\), \(v_0 \in L^2(\Omega; \mathbb{R}^d)\), and \(\zeta_0 \in H^1(\Omega)\). Then there exists a weak solution \((y_k, \zeta_k, \mu_k)\) to the initial-boundary-value problem (14) according to Definition 1 such that \(g y_k \in L^1(I; H^3(\Omega; \mathbb{R}^d)^*)\) and \(\zeta \in L^2(I; H^1(\Omega)^*)\).

**Proof.** We first construct the conformal Faedo-Galerkin approximation of (14). This means we preform space discretisation while keeping time continuous. The adjective “conformal” means that the finite-dimensional subspaces used for (14a) are contained in \(H^3(\Omega; \mathbb{R}^d)\), while for (14b) and (14c) they are contained in \(H^1(\Omega)\).

We use \(k \in \mathbb{N}\) as the index for these finite dimensional subspaces, assuming that their sequences are increasing and the union of these sequences dense in the mentioned Sobolev spaces.

By this discretisation, we obtain an initial-value problem for a system of nonlinear ordinary differential equations. For a fixed \(k\), let us denote the solution obtained by this way as \((y_k, \zeta_k, \mu_k)\). More precisely, we first obtain only a solution local in time by usual arguments for existence for ordinary differential equations and then we obtain the global solution on the whole time interval \([0, T]\) by the usual continuation argument, based on the uniform a-priori \(L^\infty(0, T)\)-estimates below.

To this goal, it is important to take these finite-dimensional subspaces for (14b) and for (14c) (written for the approximate solution) the same in order to allow for a cross-test of (14b) by \(\mu_k\) and (14c) by \(\zeta_k\).

Together with the test of (14a) by \(y_k\) and using the boundary conditions (14d–g), we obtain the discrete energy balance
\[
\frac{d}{dt} \int_\Omega \frac{\theta}{2} |\dot{y}_k|^2 + \varphi(\nabla y_k, \zeta_k) + \frac{\kappa}{2} \left| \frac{\text{Cof} \nabla y_k}{\det \nabla y_k} \nabla \zeta_k \right|^2 + \frac{1}{2} H \nabla^3 y_k : \nabla^3 y_k \, dx
\]
\[
+ \int_\Omega \frac{(\text{Cof} \nabla y_k) M(\zeta_k) \text{Cof} \nabla y_k)}{\det \nabla y_k} \nabla \mu_k \cdot \nabla \mu_k \, dx + \int_\Sigma \alpha \mu_k^2 \, dS
\]
\[
= \int_\Omega f \cdot \dot{y}_k \, dx + \int_\Sigma g \cdot \dot{y}_k + \alpha \mu_{\text{ext}} \mu_k \, dS. \tag{22}
\]
Here we have made use of cancellation of the terms \(\pm \mu_k \zeta_k\) and also the calculus
\[
\partial_z \varphi(\nabla y_k, \zeta_k) : \nabla \dot{y}_k + \partial_z \varphi(\nabla y_k, \zeta_k) \zeta_k = \frac{\partial}{\partial \zeta} \varphi(\nabla y_k, \zeta_k). \quad \text{In particular, here we used the growth condition (21) to ensure } \partial_z \varphi(\nabla y_k, \zeta_k) \in L^2(Q).
\]

We integrate (22) over time interval \([0, t]\) and apply by-part integration in time on the term \(g \cdot \dot{y}_k\) because \(\dot{y}_k\) does not have well estimated traces on \(\Gamma\). Then we
apply the Hölder and the Gronwall inequalities. By the Healey-Krömer theorem (5) being here used with the compact embedding $H^3(\Omega; \mathbb{R}^d) \subset W^{2,p}(\Omega; \mathbb{R}^d)$, we have

$$\forall (t,x) \in \bar{Q}: \quad \det \nabla y_k(t,x) \geq \varepsilon$$

for some positive $\varepsilon \leq \min_{x \in \bar{Q}} \det \nabla y_0(x)$. It is important that this holds by the successive-continuation argument on the Galerkin level, and thus $\nabla y_k$ is valued in the definition domain of $\varphi$ and the singularity of $\varphi$ is not seen, and therefore the Lavrentiev phenomenon is excluded. Altogether, by this way, we obtain the a-priori estimates

$$\|y_k\|_{L^\infty(I;H^3(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(I;L^2(\Omega;\mathbb{R}^d))} \leq K \quad \text{and} \quad \left\| \frac{1}{\det \nabla y_k} \right\|_{L^\infty(Q)} \leq K,$$

$$(24a)$$

$$\left\| \frac{\text{Cof}\nabla y_k}{\sqrt{\det \nabla y_k}} \nabla \mu_k \right\|_{L^2(Q;\mathbb{R}^d)} \leq K,$$

$$(24b)$$

$$\left\| \frac{\text{Cof}\nabla y_k}{\det \nabla y_k} \nabla \zeta_k \right\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))} \leq K.$$

$$(24c)$$

From (24a), we have the bound $\nabla y_k \in L^\infty(Q;\mathbb{R}^{d \times d})$ so that, realizing that $(\text{Cof}\nabla y_k)^{-1} = (\nabla y_k)^\top / \det \nabla y_k$, from (24b) we have

$$\left\| \nabla \mu_k \right\|_{L^2(Q;\mathbb{R}^d)} = \left\| (\nabla y_k)^\top \frac{\text{Cof}\nabla y_k}{\det \nabla y_k} \nabla \mu_k \right\|_{L^2(Q;\mathbb{R}^d)}$$

$$\leq \left\| \nabla y_k \right\|_{L^\infty(Q;\mathbb{R}^{d \times d})} \left\| \frac{1}{\sqrt{\det \nabla y_k}} \right\|_{L^\infty(Q)} \left\| \frac{\text{Cof}\nabla y_k}{\det \nabla y_k} \nabla \mu_k \right\|_{L^\infty(Q;\mathbb{R}^d)}.$$

$$(25)$$

Then we use (24b) to obtain the bound of $\nabla \mu_k$ in $L^2(Q;\mathbb{R}^d)$. By the Poincaré inequality based on the Robin boundary condition we obtain the bound of

$$\left\| \mu_k \right\|_{L^2(I;H^1(\Omega))} \leq K.$$ 

$$(26)$$

Similarly, we can estimate

$$\left\| \nabla \zeta_k \right\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))} = \left\| (\nabla y_k)^\top \frac{\text{Cof}\nabla y_k}{\det \nabla y_k} \nabla \zeta_k \right\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))}$$

$$\leq \left\| \nabla y_k \right\|_{L^\infty(Q;\mathbb{R}^{d \times d})} \left\| \frac{\text{Cof}\nabla y_k}{\det \nabla y_k} \nabla \zeta_k \right\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))},$$

$$(27)$$

from which we obtain the bound of $\nabla \zeta_k$ in $L^\infty(I;L^2(\Omega;\mathbb{R}^d))$ by using (24b). Then, by the coercivity of $\varphi(F,\cdot)$, cf. (4c), we obtain also the estimate

$$\left\| \zeta_k \right\|_{L^\infty(I;H^1(\Omega))} \leq K.$$ 

$$(28)$$

Then we select a weakly* convergent subsequence in the topologies indicated in (24a), (26), and (28). Moreover, by comparison, from the equation (14b) in its Galerkin approximation and from (24b), we can also see that (a Hahn-Banach extension of) $\zeta_k$ is bounded in $L^2(I;H^1(\Omega)^*)$. Then one can use the Aubin-Lions lemma to get a strong convergence both for

$$\nabla y_k \to \nabla y \quad \text{in} \quad C(\bar{Q};\mathbb{R}^{d \times d}) \quad \text{and} \quad \zeta_k \to \zeta \quad \text{in} \quad L^{1/\varepsilon}(I;L^{p-\varepsilon}(\Omega))$$

for any $0 < \varepsilon < p - 1$ with $p = 6$ if $d = 3$ or $p < +\infty$ if $d = 2$. The convergence towards the weak solution of (14) is then easy.
A bit peculiar term is the diffusion flux when considering the ansatz (10) and thus the weak formulation (13b), for which we need to show that
\[
\int_Q \left( M(\zeta_k) \frac{\text{Cof} \nabla y_k}{\sqrt{\det \nabla y_k}} \nabla \mu_k \right) \cdot \left( \frac{\text{Cof} \nabla y_k}{\sqrt{\det \nabla y_k}} \nabla \mu \right) \, dx dt
\]
\[
\rightarrow \int_Q \left( M(\zeta) \frac{\text{Cof} \nabla y}{\sqrt{\det \nabla y}} \nabla \mu \right) \cdot \left( \frac{\text{Cof} \nabla y}{\sqrt{\det \nabla y}} \nabla \mu \right) \, dx dt \tag{29}
\]
for any \( v \in C^1(Q) \). Here we used that
\[
\frac{\text{Cof} \nabla y_k}{\sqrt{\det \nabla y_k}} \nabla \mu_k \rightarrow \frac{\text{Cof} \nabla y}{\sqrt{\det \nabla y}} \nabla \mu \quad \text{weakly in } L^2(Q; \mathbb{R}^d), \quad \text{and} \tag{30a}
\]
\[
\frac{\text{Cof} \nabla y_k}{\sqrt{\det \nabla y_k}} \rightarrow \frac{\text{Cof} \nabla y}{\sqrt{\det \nabla y}} \quad \text{strongly in } C(Q; \mathbb{R}^{d \times d}) \tag{30b}
\]
because \( \text{Cof} \nabla y_k \rightarrow \text{Cof} \nabla y \) strongly in \( L^p(Q; \mathbb{R}^{d \times d}) \) and \( 1/\det \nabla y_k \rightarrow 1/\det \nabla y \) strongly in \( L^p(Q) \) for any \( 1 \leq p < +\infty \) due to the Aubin-Lions theorem together with the latter estimate in (24a). and that \( M(\zeta_k) (\nabla y_k)^{-1} \rightarrow M(\zeta) (\nabla y)^{-1} \) weakly in \( L^2(Q) \) thanks to the estimate (24b).

As already mentioned, the limit passage in the Korteweg-like stress \( \sigma_k \) needs strong convergence of \( \nabla \zeta_k \) in \( L^2(Q; \mathbb{R}^d) \). To this goal, we use the uniform (with respect to \( y \)) strong monotonicity of the mapping \( \zeta \mapsto -\text{div}(\kappa(\nabla y)^{-1}(\nabla y)^{-\top} \nabla \zeta) \).

Taking \( \zeta_k \) an approximation of \( \zeta \) valued in the respective finite-dimensional spaces used for the Faedo-Galerkin approximation and converging to \( \zeta \) strongly, we can test (14c) in its Galerkin approximation by \( \zeta_k - \bar{\zeta}_k \) and use it in the estimate
\[
\limsup_{k \to \infty} \int_Q \kappa(\nabla y_k)^{-1}(\nabla y_k)^{-\top} \nabla (\zeta_k - \bar{\zeta}_k) : \nabla (\zeta_k - \bar{\zeta}_k) \, dx dt
\]
\[
= \lim_{k \to \infty} \int_Q \left( \partial_x \varphi(\nabla y_k, \zeta_k) + \mu_k \right) (\bar{\zeta}_k - \zeta_k)
\]
\[
- \kappa(\nabla y_k)^{-1}(\nabla y_k)^{-\top} \nabla \bar{\zeta}_k : \nabla (\zeta_k - \bar{\zeta}_k) \, dx dt = 0 \tag{31}
\]
because \( \partial_x \varphi(\nabla y_k, \zeta_k) + \mu_k \) is bounded in \( L^2(Q) \) while \( \bar{\zeta}_k - \zeta_k \to 0 \) strongly in \( L^2(Q) \) by the Aubin-Lions compactness theorem and because \( \kappa(\nabla y_k)^{-1}(\nabla y_k)^{-\top} \nabla \bar{\zeta}_k \) converges strongly in \( L^2(Q; \mathbb{R}^d) \) while \( \nabla (\zeta_k - \bar{\zeta}_k) \to 0 \) weakly in \( L^2(Q; \mathbb{R}^d) \). As \( \kappa(\nabla y_k)^{-1}(\nabla y_k)^{-\top} \) is uniformly positive definite, we thus obtain that \( \nabla (\zeta_k - \bar{\zeta}_k) \to 0 \) strongly in \( L^2(Q; \mathbb{R}^d) \), and thus \( \nabla \zeta_k \to \nabla \zeta \) strongly in \( L^2(Q; \mathbb{R}^d) \).

Then we have the convergence in the Korteweg-like stress even strongly in \( L^p(I; L^1(Q; \mathbb{R}^{d \times d})) \) for any \( 1 \leq p < +\infty \). The limit passage in the force equilibrium towards (14a,d,e) formulated weakly in (13a) is straightforward.

4. Concluding remarks. We close the paper with a brief outlook to some modifications and other applications and models which can be analysed quite analogously.

Remark 3 (Allen-Cahn modification: damage or phase-transformation models). Replacing the quadratic (in terms of rate) nonlocal dissipation potential by a nonquadratic nonsmooth (at zero-rate) local dissipation potential of the type
\[
R(\zeta) := \int_{\Omega} r(\zeta) \, dx \quad \text{with } r : \mathbb{R} \to [0, +\infty] \text{ convex},
\]
we would obtain a diffusionless model of Allen-Cahn type [2]. The equation (14b) is then simplified for $\partial r(\zeta) + \mu = 0$ and (14f) is omitted, while the Korteweg-like contribution $\sigma_R$ induced by the actual-configuration gradient of $\zeta$ to the stress tensor remains in (14a). This may describe a damage model [38] or a martensitic phase transformation [23]. The mentioned nonsmoothness of $r(\cdot)$ at $\zeta = 0$ then models activation phenomena and, in the case of reversible phase transformation, hysteresis behaviour. For the analysis, we refer to [21, Sect. 9.5.1].

**Remark 4** (*Dispersion of elastic waves*). The concept of nonsimple materials allows for introducing a dispersion of elastic waves, as well known for linear models at small strains. Typically, involving higher gradients in a positive-definite way, one gets anomalous dispersion, i.e. higher-frequency waves propagate faster than lower-frequency ones. When one combines the concept of 3rd-grade (as here in Sect. 3) with the 2nd-grade (as in Sect. 2) materials, we obtain a bigger freedom. In particular, a combination of normal and anomalous dispersion can be obtained when the second-order deformation gradient is involved in a negative-definite way, cf. also [21, Remark 6.3.6] for a 1-dimensional linear model.

**Remark 5** (*Gradient plasticity*). Another model where gradient can be considered in the actual deforming configuration is plasticity. At large strains, it is always analytically necessary to involve gradient of plastic strain $\Pi$ into the stored energy, which is then considered as

$$
\mathcal{E}(\nabla y, \Pi) = \int_\Omega \varphi_{el}(\nabla y)\Pi^{-1} + \varphi_{in}(\Pi) + \frac{1}{2}\nabla^3 y : \mathbb{H} : \nabla^3 y + \frac{\kappa}{p} |(\nabla y)^{-\top} \nabla \Pi|^p.
$$

(32)

When considering still the kinetic energy (9b) and the dissipation potential $\mathcal{R}(\Pi; \dot{\Pi}) = \int_\Omega \zeta((\Pi\Pi^{-1}) d\pi$ for some convex $\zeta : \mathbb{R}^{d \times d} \to \mathbb{R}$, the evolution system arising by the Hamilton variational principle extends as

$$
\partial_\Pi \zeta(\Pi\Pi^{-1}) + S_{in} \Pi^\top \ni 0 \quad \text{in } Q,
$$

(33a)

with the elastic stress $S_{el} = \partial_{\nabla y} \mathcal{E}(\nabla y, \Pi)$ and an inelastic driving stress $S_N = \partial_{\Pi} \mathcal{E}(\nabla y, \Pi)$. In view of (32), we can specify

$$
S_{el} = \varphi_{el}'(\nabla y)\Pi^{-1}\Pi^{-\top} + \text{div}^2(\mathbb{H} \nabla^3 y)
+ \kappa |(\nabla y)^{-\top} \nabla \Pi|^{p-2} \nabla \Pi^\top (\nabla y)^{-1}(\nabla y)^{-\top} \nabla \Pi
\quad \text{and}
$$

(34a)

$$
S_N = (\nabla y)^\top \varphi_{el}'(\nabla y)\Pi^{-1}) : (\Pi^{-1})' + \varphi_{in}(\Pi)
- \text{div}(\kappa |(\nabla y)^{-\top} \nabla \Pi|^{p-2} (\nabla y)^{-1}(\nabla y)^{-\top} \nabla \Pi)
$$

(34b)

The last term in (34a) is a Korteweg-like stress and, because of it, now the strong convergence in $\nabla \Pi$ is needed for the analysis. This can be done similarly as (31), now based on the uniform (with respect to $\nabla y$) strong monotonicity of the mapping $\Pi \mapsto -\text{div}(\kappa |(\nabla y)^{-\top} \nabla \Pi|^{p-2} (\nabla y)^{-1}(\nabla y)^{-\top} \nabla \Pi)$; cf. [21, Sect. 9.4.2]. A combination of the Cahn-Hilliard models with plasticity can also be considered, like [3,34].

**Remark 6** (Open problems). The gradient of the deformation gradient in (1) and (9b) is considered in the reference configuration while the concentration gradient is in the actual deformed configuration. This is a certain conceptual discrepancy. Yet,
considering the non-simple materials in the actual configuration brings additional terms and serious additional difficulties.

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