Credit Default Swaps
and the mixed-fractional CEV model

Axel A. Araneda*
Institute of Financial Complex Systems
Department of Finance
Masaryk University
602 00 Brno, Czech Republic.

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Abstract
This paper explores the capabilities of the Constant Elasticity of Variance model driven by a mixed-
fractional Brownian motion (mfCEV) [Axel A. Araneda. The fractional and mixed-fractional CEV model.
Journal of Computational and Applied Mathematics, 363:106–123, 2020] to address default-related financial
problems, particularly the pricing of Credit Default Swaps. The increase in both, the probability of default
and the CDS spreads under mixed-fractional diffusion compared to the standard Brownian case, improves
the lower empirical performance of the standard Constant Elasticity of Variance model (CEV), yielding a
more realistic model for credit events.

Keywords: Fractional Brownian motion; First-passage time; CEV model; Credit Default Swaps; Equity
Default Swaps.

1 Introduction
Credit Default Swaps (CDS) are derivatives designed to hedge default in the firm’s obligations, taking as
reference some debt instrument (e.g., bonds). It acts as insurance since the ‘protection buyer’ is compensated
(usually with the debt face value) in case of a credit default event in exchange for periodic payments to the
‘protection seller’. On the other hand, when the reference asset is just equity (stock price), and the triggering
event is some pre-specified price-level barrier (namely, 30% or 50% of the initial value), we are in front of an
Equity Default Swap (EDS), a deep out-of-the-money digital knock-in put option, where the option premium,
instead of paid at the inception, is divided into legs along the contract duration to mimic the CDS structure.
In the case of a firm’s debt default, the stock price would be worth zero or very near to the zero level, then
CDS are equivalent to a zero-barrier EDS [2].

From the price modelling perspective, the geometric Brownian motion, the basic assumption of the seminal
work of Black and Scholes [3], precludes hitting the zero price level and becoming ineligible to address default-
related problems. In that sense, the constant elasticity of variance (CEV model) [4, 5] emerges as a candidate
for this task due to some desirable properties. First, the origin is an attainable and absorbing boundary via
diffusion to zero, and consequently, it includes the bankruptcy possibility. Second, the CEV model addresses
some empirical facts observed in equity markets: the leverage effect, heteroskedasticity, and the implied volatility
skew.

The standard CEV assumption has already been applied to credit risk pricing [7, 8]. However, the lower
CDS spreads under the standard CEV model compared to the actual market data [2, 9], yields to consider some
extensions as the addition of a Heston-type stochastic volatility feature to the CEV stock price [10], or the
inclusion of a killing jump rate [2, 8] [11].

In this note, we want to enrich the CEV-CDS literature, including a non-standard diffusion mechanism for
the CEV model, particularly the mixed-fractional Brownian motion (mfBm) which provides to the CEV model
the capacity to address the long-range dependence empirical observation without sacrificing the martingale
property and a better fit with market option prices [1, 12].

*Email: axelaraneda@mail.muni.cz

1The attainable boundary at zero occurs (a.s) when the elasticity of variance is negative or in our notation (c.f section 2) \( \alpha < 2 \).

The absorbing condition at zero is naturally given for \( \alpha \leq 1 \); while for a \( 1 < \alpha < 2 \) the origin could be absorbing or reflecting. For
financial purposes the absorbing condition at zero is imposed: when the price reaches zero the diffusion process is killed (cemetery
state). Please see [2] for a detailed analysis of boundary conditions in the CEV model.
2 The mixed-fractional CEV model

The mfCEV establishes the following stochastic differential equation for the evolution of the asset price $S$: 

$$dS_t = rS_t dt + \delta S_t^2 dM_t^{H,\beta}$$

(1)

where $M_t^{H,\beta}$ is a mfBm defined as:

$$M_t^{H,\beta} = B_t + \beta B_t^H$$

(2)

being $\beta \in \mathbb{R}^+$, $B_t^H$ a fractional Brownian motion with Hurst parameter $\frac{3}{4} < H < 1$, and $B_t$ an independent & standard Bm. These conditions guarantee that the mfBm is a local-martingale equivalent in law to a standard Brownian motion \[13\]. The parameter $\alpha$ is restricted to less than 2 in order to ensure three features: i) zero is an attainable and absorbing boundary (cf. footnote \[1\]); ii) inverse relationship between price and volatility (Leverage effect); and iii) arbitrage-free (which is possible because $M_t^{H,\beta}$ is semi-martingale for $H > 3/4$; see \[6\] \[14\] about the CEV model and arbitrage possibilities). The other two parameters of the model, $r$ and $\delta$, are assumed non-negative and strictly positive, respectively. The former represents the constant risk-free interest rate while the latter adjusts the at-the-money volatility level and is usually parametrized as $\delta^2 = \sigma_0^2 S_0^{2-\alpha}$, so $\sigma_0$ acts as the volatility scale parameter of the local volatility function; i.e., $\sigma(S,0) = \sigma_0$.

Defining the new variable $x_t = S_t^{2-\alpha}$, the related Wick-Itô calculus (Appendix A) yields to:

$$dx_t = (2 - \alpha) \left[ x_t \sigma^2 (1 - \alpha) \left( \frac{1}{2} + \beta^2 H t^{2H-1} \right) \right] dt + (2 - \alpha) \sigma \sqrt{x_t} dM_t^{H,\beta}$$

and the corresponding transition density probability function $P = P(x_t, t | x_0, 0)$ obeys (c.f. Appendix B):

$$\frac{\partial P}{\partial t} = - \frac{\partial}{\partial x_t} \left[ A x_t + B(t) \right] P + C(t) \frac{\partial^2}{\partial x_t^2} \left[ x P \right]$$

(3)

with $A = (2 - \alpha) r$, $B(t) = \sigma^2 (1 - \alpha) (2 - \alpha) \left( \frac{1}{2} + \beta^2 H t^{2H-1} \right)$, and $C(t) = \sigma^2 (2 - \alpha)^2 \left( \frac{1}{2} + \beta^2 H t^{2H-1} \right)$.

Eq. (3) is equivalent to the Fokker-Planck equation for a time-inhomogeneous Feller (square-root) process, and given the ratio $\theta = B(t)/C(t) = (1 - \alpha) / (2 - \alpha) < 1$ is time-independent 4 it could be solved analytically \[4\] \[15\]. Moreover, considering these conditions, the density $g$ of the random variable $\tau = \inf \{ t > 0 : x_t = 0 \}$ which describes first-passage time (FPT) through the zero state is given by \[15\]:

$$g \left( 0, t | x_0, 0 \right) = \frac{1}{\Gamma \left( 1 - \xi \right) \phi(t)} \exp \left[ - \frac{x_0}{\phi(t)} \right] \left[ \frac{x_0}{\phi(t)} \right]^{1-\xi}$$

(4)

being $\gamma(\bullet, \bullet)$ the lower incomplete gamma function, and

$$\phi(t) = \int_0^t C \left( \tilde{t} \right) e^{-(2 - \alpha)\tilde{t}^2} d\tilde{t} = \frac{\sigma^2 (2 - \alpha)^2}{2r} \left[ 1 - e^{-(2 - \alpha)\tilde{t}^2} \right] + \frac{\beta^2 \sigma^2 (2 - \alpha)^2}{2(2H+1)} e^{-(2-\alpha)\tilde{t}^2 t^{2H}} \left[ 2H + 1 \right] \left[ (2 - \alpha) r t \right]^{-H} \left[ (2 - \alpha) r t \right]^{-H}$$

with $M_{x,u}(l)$ the M-Whittaker function.

Thus, the first-passage-time (FPT) probability ($Q$) is equal to \[15\]:

$$Q \left( \tau \leq t | x_0, 0 \right) = 1 - \frac{\gamma \left( 1 - \xi, \frac{x_0}{\phi(t)} \right)}{\Gamma \left( 1 - \xi \right) \phi(t)} = \frac{\Gamma \left( 1 - \xi, \frac{x_0}{\phi(t)} \right)}{\Gamma \left( 1 - \xi \right) \phi(t)}$$

(5)

where $\Gamma(\bullet, \bullet)$ denotes the upper incomplete gamma function, and in consequence, the risk-neutral default probability for the mfCEV model is expressed as:

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4 This ensures that origin is an attainable boundary according to the Feller classification in the defined domain for $\alpha < 1$ and absorbing boundary for $1 \leq \alpha < 2$

4 In their paper Giorno and Nobile \[15\] impose the condition $0 \leq \xi < 1$; however the supplied results are still valid for negative $\xi$ ratios.
Figure 1: Probability of default (price equal to zero) under both CEV and mfCEV models, setting $S_0 = 50$, $\sigma_0 = 20\%$ and $r = 5\%$.

$$Q(\tau \leq t|S_0,0) = \frac{\Gamma(1 - \xi, \frac{S_0^2 - \alpha}{\varphi(t)})}{\Gamma(1 - \xi)}$$

Fig. 1 plots the FPT probability through the zero state as a function of time for the mixed-fractional CEV model under different elasticity parameters: $\alpha = -2$ (blue, 1a) and $\alpha = 0$ (red, 1b). Moreover, to visualize the mfCEV model sensitivity to both the Hurst exponent and $\beta$, we use $H = \{0.8, 0.9\}$ and $\beta = \{0, 1/2, 1\}$. The case $\beta = 0$ corresponds to the standard CEV specification. We have set the initial volatility of $(\sigma_0)$ at 20%, and the risk-free rate equals 5%. We see the mixed-fractional diffusion increases the default probability (as a function of both $\beta$ and $H$) compared to the classical CEV. The increment in the FPT is faster for a greater $\alpha$ at long times. Nevertheless, for short maturities, the increment is more pronounced under lower $\alpha$.

3 CDS rates

The objective here is to compute both i) the present value ($V$) of the protection payment to the buyer due to the triggering event (100%-drop in the price level) and ii) the swap rate. For a maturity $T$, a constant risk-free rate $r$, a notional amount of 1$, and a fixed recovery rate $R$, the risk-neutral valuation yields:

$$V = (1 - R) \cdot \mathbb{E}(e^{-rt} \mathbb{1}_{\tau < T})$$

where the indicator function $\mathbb{1}_{\tau < T}$ maps a price default event in the time-interval $[0, T]$. Then,

$$V = (1 - R) \int_0^T e^{-rt} g(0, t, x_0, 0) \, dt$$

$$= (1 - R) \left[ e^{-rT} Q(\tau \leq t | S_0, 0) + r \int_0^T e^{-rt} Q(\tau \leq t | S_0, 0) \right]$$

On the other hand, in CDS contracts, the protection payment is divided in installments from the inception time up to either the trigger event or the expiration time, whichever comes first. Then, ignoring any accrual payment after the default, the equilibrium swap rate (coupon $C$) under typical conditions (semianual installments and a maturity equal to an integer number of years) is given by:

$$C = \frac{V}{\sum_{i=1}^{2T} e^{-r \frac{i}{2}} \mathbb{E}[\mathbb{1}_{\tau > \frac{i}{2}}]} = \frac{V}{\sum_{i=1}^{2T} e^{-r \frac{i}{2}} \left[ 1 - Q\left(\tau \leq \frac{i}{2} \mid S_0, 0\right) \right]}$$

The parametrization given on the preceding page for the local volatility function makes the default probability (and the CDS pricing addressed in the following section) independent of the initial price level.
\[ T = 1 \quad T = 2 \quad T = 5 \quad T = 10 \]

| \( \beta = 0 \) | \( \alpha = 0 \) | 0.0015 | 14.6761 | 0.4976 | 49.3693 | 11.0929 | 71.0707 | 22.0907 | 58.1472 |
| \( \beta = 0.5 \) | \( \alpha = 0 \) | 0.0220 | 33.0638 | 3.6859 | 97.5923 | 45.8409 | 130.6805 | 73.6537 | 107.2735 |
| \( \beta = 1 \) | \( \alpha = 0 \) | 0.0219 | 32.9327 | 4.5121 | 104.3824 | 61.1677 | 148.0252 | 99.5110 | 125.2780 |

Table 1: CDS spread under both standard and mixed-fractional CEV model

Table 1 shows the CDS spreads, i.e., the annual coupon payments (in basis points), when the underlying asset follows a mfCEV (\( \beta = \{0.5, 1\}, H = \{0.8, 0.9\} \)) and a standard CEV (\( \beta = 0 \)), with \( \sigma_0 = 20\% \), \( r = 5\% \), a recovery rate of 50%, \( \alpha = \{0, -2\} \); under different maturities, being \( V \) computed by numerical integration of Eq. (5). The results confirm the dependence on both maturity and elasticity; and the lower swap rate for the classical CEV for low maturities, due to the low default probability under the standard Brownian diffusion. Meanwhile, the mfCEV provides larger coupon values, even at low tenors, as a function of both \( H \) and \( \beta \), allowing a flexible calibration to match the term structure of CDS spreads.

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### A Itô-Wick formula for mfBm

From Eq. (2), we have that:

$$
\mathbb{E} \left[ (M_t^{H,\beta})^2 \right] = t + \beta^2 t^{2H}
$$

Given that the variance is bounded, for $f = f \left( t, x \left( M_t^{H,\beta} \right) \right) \in C^{1,2}((0, \infty) \times \mathbb{R})$ we have [16]:

$$
df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dM_t^{H,\beta} + \frac{1}{2} dt \left( t + \beta^2 t^{2H} \right) \frac{\partial^2 f}{\partial x^2}
$$

$$
= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dM_t^{H,\beta} + \frac{1}{2} \left( 1 + 2H \beta^2 t^{2H-1} \right) dt \frac{\partial^2 f}{\partial x^2}
$$

$$
= \left( \frac{\partial f}{\partial t} + \left( \frac{1}{2} + H \beta^2 t^{2H-1} \right) \frac{\partial^2 f}{\partial x^2} \right) dt + \frac{\partial f}{\partial x} dM_t^{H,\beta}
$$

(6)

### B Effective Fokker-Planck equation

Let $y_t$ a stochastic process described by the following stochastic differential equation (SDE) driven by a mfBm:

$$
dy_t = \mu(y_t, t) dt + \sigma(y_t, t) dM_t^{H,\beta}
$$

(7)

then, the transformation $g = g(y_t, t) \in C^2(\mathbb{R})$ follows:

$$
dg(y_t) = \left( \frac{\partial g}{\partial y} \right) \mu + \left( \frac{1}{2} + H \beta^2 t^{2H-1} \right) \sigma^2 \frac{\partial^2 g}{\partial y^2} dt + \left( \frac{\partial g}{\partial y} \right) \sigma dM_t^{H,a,b}
$$

(8)

Taking expectations at both sides of Eq. (8) and using that $\mathbb{E} [g(y_t, t)] = \int g \cdot P(y_t, t)$, where $P = P(y_t, t)$ is the transition density of the random variable $y$ at time $t$; we got:

$$
\int_{-\infty}^{\infty} \frac{\partial P}{\partial t} dy = \int_{-\infty}^{\infty} \left( \frac{\partial g}{\partial y} \right) \mu + \left( \frac{1}{2} + H \beta^2 t^{2H-1} \right) \sigma^2 \frac{\partial^2 g}{\partial y^2} \right) Pdy
$$

(9)

Taking in account that $\int_{-\infty}^{\infty} \left( \frac{\partial g}{\partial y} \right) \mu Pdy = -\int_{-\infty}^{\infty} g \frac{\partial (\mu P)}{\partial y} dy$ and $\int_{-\infty}^{\infty} \sigma^2 \frac{\partial^2 g}{\partial y^2} Pdy = -\int_{-\infty}^{\infty} g \frac{\partial^2 (\sigma^2 P)}{\partial y^2} dy$, the transition density for the variable $y_t$ described by Eq. (7) is ruled by:

$$
\frac{\partial P}{\partial t} = -\frac{\partial (\mu P)}{\partial y} + \left( \frac{1}{2} + H \beta^2 t^{2H-1} \right) \frac{\partial^2 (\sigma^2 P)}{\partial y^2}
$$

(10)