On some congruence properties of elliptic curves

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Abstract In this paper, as a result of a theorem of Serre on congruence properties, a complete solution is given for an open question (see the text) presented recently by Kim, Koo and Park. Some further questions and results on similar types of congruence properties of elliptic curves are also presented and discussed.

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1 Serre congruence properties and KKP question

Let $d, \alpha \in \mathbb{Z}$ with $d > 1$, $\mathbb{Z}$ is the set of rational integers. In a recent paper [KKP], Kim, Koo and Park presented a question with elliptic curves over finite fields as follows (in their original form):

(Open questions of Kim-Koo-Park) Let $\tilde{E}_p : y^2 = x^3 + f(k)x + g(k)$ be an elliptic curve over a finite field $\mathbb{F}_p$ and $\alpha$ be a nonnegative integer.

(1) (Strong form) Can one find $f(k), g(k)$ satisfying $\#\tilde{E}_p(\mathbb{F}_p) \equiv \alpha \pmod{d}$
for a fixed integer \( d \) and for almost all primes \( p \)?

(2) (Weak form) One may consider partial conditions for some primes \( p \), for example \( p \equiv 1 \pmod{4} \). Can one find \( f(k), g(k) \) satisfying \( \#\tilde{E}_p(\mathbb{F}_p) \equiv \alpha \pmod{d} \) for a fixed integer \( d \) and for all such primes \( p \)?

For a set \( T \), the notation \( \#T \) denote its cardinal number. Throughout this paper, \( \sim \) almost all \( \sim \) means \( \sim \) all but finitely many.\( \sim \)

In the following, we call it the KKP-question for the pair \( (d, \alpha) \). As indicated in their Theorems 1 and 3 of [KKP] about this open question in the case \((d, \alpha) = (3, 0)\), the precise meaning of KKP question (strong form) may be explained as follows: For fixed integers \( d \) and \( \alpha \), can one find polynomials \( f(t), g(t) \in \mathbb{Z}[t] \) in one variable \( t \) satisfying the following condition: For every integer \( k \in \mathbb{Z} \) such that \( E : y^2 = x^3 + f(k)x + g(k) \) is an elliptic curve over the rational number field \( \mathbb{Q} \), one has \( \#\tilde{E}_p(\mathbb{F}_p) \equiv \alpha \pmod{d} \) for a fixed integer \( d \) and for almost all primes \( p \)? where \( \tilde{E}_p \) is the reduced curve of \( E \) at \( p \). The meaning of weak form is similar. Note that in their paper [KKP], both the polynomials \( f \) and \( g \) are in one variable. One can ask similar questions with polynomials \( f, g \) over \( \mathbb{Z} \) in several variables.

In an earlier version of this paper (see [Q]), we solved the KKP-question (strong form) in the case \((d, \alpha) \) with \( d \mid \alpha \), and by relating it to supersingular primes of elliptic curves we obtain that the KKP question (strong form) has no solutions in the case \((d, \alpha) \) with \( d, \alpha \) satisfying \( \gcd(\alpha - 1, d) > 1 \). Moreover, we conjecture that the KKP question (strong form) is soluble only in the case \((d, \alpha) \) with \( d \mid \alpha \) (see conjecture 6(3) in [Q]). Fortunately, in a letter of J.-P. Serre on June 15, 2009 to the author, Serre pointed out that this conjecture is true as a corollary of one of
his results on congruence properties of elliptic curves (see the following Proposition 1.2). So as a result, a complete solution of the above KKP-question (strong form) is obtained, which we state in this paper as follows:

**Theorem 1.1** Let \( d, \alpha \in \mathbb{Z} \) with \( d > 1 \). The KKP question (strong form) is soluble in the case \( (d, \alpha) \) if and only if \( d \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16\} \) and \( d \mid \alpha \).

The proof of this result depends heavily on the following Serre’s results about congruence properties of elliptic curves as stated in his letter (for a general context, see [I]).

**Proposition 1.2 (Serre’s Theorem).** Let \( d, \alpha \in \mathbb{Z} \) with \( d > 1 \).

(I) For elliptic curve \( E \) over \( \mathbb{Q} \), the following two conditions are equivalent:

1. \( \# \tilde{E}_p(\mathbb{F}_p) \equiv \alpha \pmod{d} \) for almost all rational primes \( p \).
2. \( 1 + \det(g) - \Tr(g) \equiv \alpha \pmod{d} \) for all \( g \in G(E,d) \), where \( G(E,d) \subset \text{GL}(2,\mathbb{Z}/d\mathbb{Z}) \) is the subgroup defined by the action of the absolute Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on \( E[d] \), the set of \( d \)-division points of \( E \).

(II) If there exists an elliptic curve \( E/\mathbb{Q} \) satisfying the above conditions, then \( d \mid \alpha \).

**Proof (Serre).**

(I) The implication \((2) \implies (1)\) is clear: take for \( g \) the Frobenius of \( E \) at \( p \). The implication \((1) \implies (2)\) follows from Chebotarev density theorem.

(II) By applying \((2)\) with \( g = 1 \), one finds \( d \mid \alpha \).

As Serre further points out, this works over any number field, and one can also replace \( E \) by any algebraic variety; and if \((1)\) holds, then \( \alpha \) is congruent mod \( d \) to the Euler-Poincare characteristic of \( E \) (see [I, cor. 7.3] for the EP characteristic).
For the case \((d, \alpha)\) with \(d \mid \alpha\), we have the following results:

**Theorem 1.3.** Let \(d > 1\) be an integer. If there exists an elliptic curve \(E\) over \(\mathbb{Q}\) satisfying \(\#\widetilde{E}_p(\mathbb{F}_p) \equiv 0 \pmod{d}\) for almost all primes \(p\), then \(d = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16\). Conversely, such elliptic curves \(E\) over \(\mathbb{Q}\) do exist for each of the above eleven integers \(d\).

**Proof.** For the integer \(d > 1\), assume there exists an elliptic curve \(E\) over \(\mathbb{Q}\) satisfying \(\#\widetilde{E}_p(\mathbb{F}_p) \equiv 0 \pmod{d}\) for almost all primes \(p\). Then by a theorem of Katz and Serre (see [Ka, Theorem 2] and [Se, p.IV-6, exercise]), there exists an elliptic curve \(E'\) over \(\mathbb{Q}\) which is \(\mathbb{Q}\)-isogenous to \(E\) and satisfying \(\#E'(\mathbb{Q})_{\text{tors}} \equiv 0 \pmod{d}\). Then by a famous theorem of Mazur on torsion structure of elliptic curves over \(\mathbb{Q}\) (see [M, Theorem (8) on p.35]), we must have \(d = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16\).

Conversely, also by the theorem of Mazur, each of the above eleven integers \(d\) does occur as an order of \(E(\mathbb{Q})_{\text{tors}}\) for some elliptic curve \(E\) over \(\mathbb{Q}\). By the Nagell-Lutz theorem (see [Hu, p.115, 116]), \(E(\mathbb{Q})_{\text{tors}}\) is isomorphic to a subgroup of \(\widetilde{E}_p(\mathbb{F}_p)\) for almost all primes \(p\), which implies that \(\#\widetilde{E}_p(\mathbb{F}_p) \equiv 0 \pmod{d}\) for almost all primes \(p\) (in fact, there exists a family of such elliptic curves for each of these integers \(d\), for the detail, see the following proof of Theorem 1.1). This proves Theorem 1.3. \(\Box\)

Now we come to prove the Theorem 1.1.

**Proof of Theorem 1.1** Suppose the KKP question (strong form) has solutions in the pair \((d, \alpha)\). Then one can find polynomials \(f(t), g(t) \in \mathbb{Z}[t]\) in variable \(t\) and at least one elliptic curve \(E : y^2 = x^3 + f(k)x + g(k)\) over \(\mathbb{Q}\) for some \(k \in \mathbb{Z}\) satisfying \(\#\widetilde{E}_p(\mathbb{F}_p) \equiv \alpha \pmod{d}\) for almost all primes \(p\), i.e., \(E\) satisfies condition (1) of Proposition 1.2 for the pair \((d, \alpha)\). So by the above Serre’s Theorem, we have
$d \mid \alpha$. Then by Theorem 1.3, we get $d = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16$. This proves the necessity.

Conversely, before proving the sufficiency, we first note that, for an elliptic curve $E/\mathbb{Q}$ and integer pairs $(d_1, \alpha)$ and $(d_2, \alpha)$ with $d_1 \mid d_2$, if $E$ satisfies condition (1) in the case $(d_2, \alpha)$, then $E$ also satisfies condition (1) in the case $(d_1, \alpha)$, from which one can easily see that if the KKP question (strong form) has solutions in case $(d_2, \alpha)$, then so does it in case $(d_1, \alpha)$.

Now we come to prove the sufficiency. Let $d$ be any one of the given eleven integers. Without loss of generality, we may as well assume that $\alpha = 0$ by the congruence property. By the above discussion, we only need to consider the five cases $d = 16, 12, 10, 9, 7$. By the Nagell-Lutz theorem (see [Hu, p.115, 116]), and by using the known facts about parametrization of families of elliptic curves $E/\mathbb{Q}$ with torsion subgroup $E(\mathbb{Q})_{\text{tors}}$ of order $d$ (see, e.g., [Hu], [Ha], [ST], etc.), one can obtain some corresponding solutions of polynomials $f(t), g(t)$ in the KKP question (strong form) according to each of the above five $d$ as follows:

Case $d = 16$: $f(t) = -27[(t^2 - 1)^4(t^2 - 1)^4 - 16t^4) + 256t^8],
\begin{align*} g(t) &= 54[(t^2 - 1)^4 + 16t^4] \cdot [(t^2 - 1)^4 + 16t^4)^2 - 72t^4(t^2 - 1)^4].
\end{align*}$

Case $d = 12$: $f(t) = -27(t^8 + 4t^7 + 4t^6 - 2t^5 - 5t^4 - 2t^3 + 4t^2 + 4t + 1),
\begin{align*} g(t) &= 27(t^4 + 2t^3 + 2t + 1)(2t^8 + 8t^7 + 8t^6 - 10t^5 - 25t^4 - 10t^3 + 8t^2 + 8t + 2).
\end{align*}$

Case $d = 10$: $f(t) = -27[(1 + t^2)^2(1 - 2t - 6t^2 + 2t^3 + t^4)^2 - 48t^5(1 + t - t^2)],
\begin{align*} g(t) &= 54(1 + t^2)(1 - 2t - 6t^2 + 2t^3 + t^4) [(1 + t^2)^2(1 - 2t - 6t^2 + 2t^3 + t^4)^2 - 72t^5(1 + t - t^2)].
\end{align*}$

Case $d = 9$: $f(t) = -27(1 - 3t^2 + t^3)(1 - 9t^2 + 27t^3 - 45t^4 + 54t^5 - 48t^6 + 27t^7 - 9t^8 + t^9),
\begin{align*} g(t) &= 54(t^8 - 18t^17 + 135t^16 - 570t^15 + 1557t^14 - 2970t^{13} + 4128t^{12} - 4230t^{11} + 3240t^{10} -
\end{align*}$
2032t^9 + 1359t^8 - 1080t^7 + 735t^6 - 306t^5 + 27t^4 + 42t^3 - 18t^2 + 1).

Case \( d = 7 \) : 
\[
    f(t) = -27(t^2 + t + 1)(t^6 + 11t^5 + 30t^4 + 15t^3 - 10t^2 - 5t + 1),
\]
\[
    g(t) = 54t^{12} + 972t^{11} + 6318t^{10} + 19116t^9 + 30780t^8 + 26244t^7 + 14742t^6 + 11988t^5 + \\
    9396t^4 + 2484t^3 - 810t^2 - 324t + 54.
\]

This proves the sufficiency, and the proof of Theorem 1.1 is completed. \( \Box \)

2 Some results and questions on similar type of congruence properties

Let \( d, \alpha \in \mathbb{Z} \) with \( d > 1 \) be as above. In this section, we ask the following questions on similar type of congruence properties:

**Question S** (resp., **O**). Which elliptic curves \( E \) over \( \mathbb{Q} \) may satisfy the supersingular (resp., ordinary) condition that \( \#\tilde{E}_p(\mathbb{F}_p) \equiv \alpha \pmod{d} \) for almost all supersingular (resp., ordinary) primes \( p \) of \( E \)?

Similarly, one can ask the following question similar to the above KKP-question:

**KKP question of S-type** (resp., **O-type**). Find polynomials \( f(t), g(t) \in \mathbb{Z}[t] \) in one variable \( t \) satisfying the following condition: For every integer \( k \in \mathbb{Z} \) such that \( E : y^2 = x^3 + f(k)x + g(k) \) is an elliptic curve over \( \mathbb{Q} \), we have that \( E \) satisfies the supersingular (resp., ordinary) condition.

One can ask similar questions on elliptic curves with polynomials \( f, g \) over \( \mathbb{Z} \) in several variables or on families of elliptic curves in general Weierstrass model (in other words, not necessarily restricted in short Weierstrass form).

We have the following results about the above Question S and KKP-question of S-type:

**Theorem 2.1.** Let \( d, \alpha \in \mathbb{Z} \) with \( d > 1 \). Let \( E \) be an elliptic curve over \( \mathbb{Q} \).
If $E$ satisfies the supersingular condition, then $\alpha - 1$ is prime to $d$. Moreover, if $\varphi(d) > 2$, then $E$ does not have complex multiplication. Here $\varphi(d)$ is the Euler function counting the number of reduced residue classes modulo $d$.

**Proof.** Let $a_p = p + 1 - \#\widetilde{E}_p(\mathbb{F}_p)$, then by our assumption, there exists a positive integer $N > 3$ such that $a_p \equiv p + 1 - \alpha \pmod{d}$ for all supersingular primes $p > N$. Let $S$ be the set consisting of all the supersingular primes of $E$ greater than $N$. By a theorem of Elkies (see [E]), every elliptic curve over $\mathbb{Q}$ has infinitely many supersingular primes, so $S$ is an infinite set. For each $p \in S$, by definition, we have $p | a_p$, so by Hasse’s theorem (see [Si1, p.138]), we have $a_p = 0$. Therefore $p \equiv \alpha - 1 \pmod{d}$ for all $p \in S$, which implies that $\alpha - 1$ is prime to $d$.

Now we assume furthermore that $\varphi(d) > 2$. If $E$ has complex multiplication by some quadratic imaginary field $K$, then it is known that $p$ is a supersingular prime for $E$ if and only if $p$ is ramified or inert in $K$ (see [E] or [Si2, p. 184]). So by Chebotarev density theorem, there are asymptotically half of all prime numbers $p$ being supersingular for $E$. On the other hand, for the given elliptic curve $E$, as proved above, there exists a positive integer $N_0$ such that for all supersingular primes $p > N_0$ of $E$ we have $p \equiv \alpha - 1 \pmod{d}$. This shows by Prime Number Theorem that there are at most $\frac{1}{\varphi(d)}$ of all primes $p$ being supersingular, which contradicts to the former conclusion because $\frac{1}{\varphi(d)} < \frac{1}{2}$. Therefore such elliptic curve $E$ does not have complex multiplication. The proof of Theorem 2.1 is completed. □

**Corollary 2.2** The KKP question of S-type has no solutions in the case $(d, \alpha)$ with $d, \alpha$ satisfying $\gcd(\alpha - 1, d) > 1$.

**Proof.** Suppose the KKP question (strong form) has solutions in a given pair
(d, α) satisfying gcd (α − 1, d) > 1. Then one can find polynomials \( f(t), g(t) \in \mathbb{Z}[t] \) in variable \( t \) and at least one elliptic curve \( E : y^2 = x^3 + f(k)x + g(k) \) over \( \mathbb{Q} \) for some \( k \in \mathbb{Z} \), such that \( E \) satisfies the supersingular condition. So by Theorem 2.1 above, we have gcd (α − 1, d) = 1, a contradiction. This proves Corollary 2.2. □

**Definition 2.3.** For \( \alpha \in \mathbb{Z} \), we define two sets

\[
S(\alpha, \mathbb{Q}) = \{d \in \mathbb{Z} : d > 1 \text{ and there exists an elliptic curve } E/\mathbb{Q} \text{ such that } \#E_p(\mathbb{F}_p) \equiv \alpha \pmod{d} \text{ for almost all supersingular primes } p \text{ of } E\},
\]

\[
O(\alpha, \mathbb{Q}) = \{d \in \mathbb{Z} : d > 1 \text{ and there exists an elliptic curve } E/\mathbb{Q} \text{ such that } \#E_p(\mathbb{F}_p) \equiv \alpha \pmod{d} \text{ for almost all ordinary primes } p \text{ of } E\}.
\]

Obviously, by definition, if \( d \in S(\alpha, \mathbb{Q}) \) (resp., \( O(\alpha, \mathbb{Q}) \)), then \( d' \in S(\alpha, \mathbb{Q}) \) (resp., \( O(\alpha, \mathbb{Q}) \)) for any positive integer \( d' \) satisfying \( 1 < d' | d \).

As indicated in Theorems 2.1, we present a conjecture about these sets as follows:

**Conjecture 2.4.** (1) (Finiteness) For any \( \alpha \in \mathbb{Z} \), \( S(\alpha, \mathbb{Q}) \) and \( O(\alpha, \mathbb{Q}) \) are finite sets.

(2) (Uniform boundary) There exists a real number \( c > 0 \) such that both \( \#S(\alpha, \mathbb{Q}) < c \) and \( \#O(\alpha, \mathbb{Q}) < c \) for all \( \alpha \in \mathbb{Z} \).

Let \( P \) be the set consisting of all the rational primes, then a question is that how many elements are there in \( S(\alpha, \mathbb{Q}) \cap P \) and in \( O(\alpha, \mathbb{Q}) \cap P \)?

As an example, for \( \alpha = 1 \), by Theorem 2.1 above, we have \( \#S(1, \mathbb{Q}) = 0 \), i.e., \( S(1, \mathbb{Q}) = \emptyset \). For \( \alpha = 0 \), by Theorem 1.1 above, we have \( \#S(0, \mathbb{Q}) \cap O(\alpha, \mathbb{Q}) \supset \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16\} \). Is it true that \( \#S(0, \mathbb{Q}) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16\} \)?

**Remark 2.5.** By Theorem 2.1, it is easy to see that if the KKP question
of S-type has solutions in the case \((d, \alpha)\), then one must have gcd \(\alpha - 1, d) = 1\). Moreover, one can see that the solutions are intermediately related to the distribution of supersingular primes of elliptic curves. At my present knowledge, I do not know whether or not there exist elliptic curves \(E\) (without complex multiplication, of course) and pairs \((d, \alpha)\) other than those given in Theorem 1.3 satisfying that \(p \equiv \alpha - 1 \pmod{d}\) for almost all supersingular primes \(p\) of \(E\). As indicated by the Lang-Trotter conjecture (see [LT]) and Sato-Tate conjecture (now it is proved for elliptic curve \(E/\mathbb{Q}\) with nonintegral \(j\)–invariant by R. Taylor and his collaborators, see [T]), there are some deep equidistributed properties for \(a_p(E) = p + 1 - \#\overline{E}_p(\mathbb{F}_p)\) as \(p\) varies. From this, and in a sense of probability distribution, then we can ask the following question

**Question 2.6.** Is it true that for most elliptic curves \(E/\mathbb{Q}\) and most positive integers \(d\), there are at least two arithmetic progressions modulo \(d\) such that each of them contains infinitely many supersingular primes \(p\) of \(E\)?

A stronger version of Question 2.6 is the following question, we state it as a conjecture.

**Conjecture 2.7.** For elliptic curve \(E/\mathbb{Q}\) and pair \((d, \alpha)\) as above, if almost all supersingular primes \(p\) of \(E\) satisfy \(p \equiv \alpha - 1 \pmod{d}\), then \(d | \alpha\), and both \(d\) and \(E\) are given as in Theorem 1.3.

If Conjecture 2.7 could be proved, then a direct corollary is that the KKP question of S-type is soluble only in the case \((d, \alpha)\) with \(d | \alpha\).

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