Simple subalgebras of simple special Jordan algebras

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Abstract

In this paper we determine all types and the canonical forms of simple subalgebras for each type of simple Jordan algebras and the number of conjugate classes corresponding to the given simple Jordan algebra.

Keywords: Simple Jordan algebra, subalgebras, involution.

1 Introduction

This paper provides a classification of simple subalgebras in finite-dimensional special simple Jordan algebras over algebraically closed field $F$ with characteristic unequal to 2. More precisely, we will determine the canonical forms for any simple subalgebras of special simple Jordan algebras, and the number of conjugate classes corresponding to the given simple Jordan subalgebra. In particular, a Jordan algebra of any type can be realized as a Jordan subalgebra of symmetric or symplectic matrices of an appropriate order.

In 1987 N. Jacobson determined the orbits under the orthogonal group $O(n)$ of the subalgebras of the Jordan algebra of $n \times n$ real symmetric matrices \(^{[2]}\).

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The paper significantly relies on the description of maximal subalgebras of finite-dimensional special simple Jordan algebras obtained by M. Racine in 1974 ([10]).

In this paper we consider simple Jordan algebras presented in canonical matrix realizations, that is, \( H(F_n) \) can be viewed as the algebra of all symmetric matrices in \( F_n \) with respect to the ordinary transpose; \( F_{2n}^{(\pm)} \) is the set of all matrices of order \( n \) under the circle product \( A \odot B = \frac{AB+BA}{2} \); \( H(F_{2n}, j) \) where \( j \) denotes a symplectic involution consists of all matrices of order \( 2n \) of the form

\[
\begin{pmatrix}
A & B \\
C & A^t
\end{pmatrix},
\]

where \( B, C \) are any skew-symmetric matrices of order \( n \), and \( A \) is any matrix of order \( n \). If \( f \) is a non-singular symmetric bilinear form on a vector space \( V \), then \( J = F \oplus V \) is a Jordan algebra of the type \( J(f, 1) \).

Throughout the paper we assume that the base field \( F \) is algebraically closed with characteristic not two.

## 2 Subalgebras

### 2.1 Matrix subalgebras

Let \( J \) be a simple Jordan algebra of the type \( F_{2n}^{(\pm)} \) where \( n \) is even. Then it can always be presented as a subalgebra of \( H(F_n) \) as follows

\[
\left\{ \begin{pmatrix}
A & B \\
-C^t & A^t
\end{pmatrix} \right\},
\]

where \( A \) is any symmetric matrix of order \( \frac{n}{2} \) and \( B \) is any skew-symmetric matrix of order \( \frac{n}{2} \).

**Lemma 2.1.** Any automorphism of a Jordan algebra of the form (1) is induced by an automorphism of \( H(F_n) \).

**Proof.** Any automorphism of \( J \) can be extended to an automorphism or antiautomorphism of a special universal enveloping algebra \( U(J) \) which is isomorphic to \( F_{\frac{n}{2}} \oplus F_{\frac{n}{2}} \) (see [3]). Notice that in this particular case the associative enveloping algebra \( S(J) \) is isomorphic to \( U(J) \) because from the
explicit form (1) $S(J)$ consists of all matrices of the form:

$$\left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \right\},$$

where $X$ and $Y$ are any matrices of order $\frac{n^2}{2}$. Since any automorphism of $F_{\frac{n^2}{2}} \oplus F_{\frac{n^2}{2}}$ either induces non-trivial automorphisms of these ideals or sends one ideal onto another, it can be lifted up to an inner automorphism of the entire matrix algebra $F_n$. Consequently, for any antiautomorphism of $F_{\frac{n^2}{2}} \oplus F_{\frac{n^2}{2}}$ we can choose an automorphism (not necessarily non-trivial) of $F_{\frac{n^2}{2}} \oplus F_{\frac{n^2}{2}}$ such that their composition induces non-trivial antiautomorphisms of simple ideals. Therefore, any (Jordan) automorphism of $J$ can be written as follows:

$$\varphi(X) = Q^{-1}XQ$$

or

$$\varphi(X) = Q^{-1}X^tQ,$$  \hspace{1cm} (2)

for some non-singular matrix $Q$.

The next step is to prove that $\varphi$ is orthogonal. In other words, all we have to show is that for any automorphism $\varphi$ of $J$, we can choose $Q$ such that (2) holds and $Q^tQ = I$ where $I$ is the identity matrix. Since $J$ is a subalgebra of $H(F_n)$, for each $X$ in $J$, $(Q^{-1}XQ)^t = Q^{-1}XQ$, $Q^tX(Q^{-1})^t = Q^{-1}XQ$, $QQ^tX = XQQ^t$. Denote $B = QQ^t$. Next we are going to show that $B$ is actually a scalar multiple of the identity matrix. We are given that $BX = XB$ where $X$ is any matrix of the form (1). Let us write $B$ as follows:

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where $B_i$ are matrices of order $\frac{n^2}{2}$. By performing the matrix multiplication, we obtain the $B_2 = B_3 = 0$, and $B_1 = B_4 = \alpha I$, for some non-zero $\alpha$. Since the ground field $F$ is algebraically closed, we can choose $\beta \in F$ such that $\alpha = \beta^2$. Set $Q' = \beta^{-1}Q$. Obviously, $Q'$ determines the same automorphism as $Q$ does, and $Q'^tQ' = I$.

The Lemma is proved.

\begin{lemma}
Let $J \cong F_{\frac{n^2}{2}}^{(+)}$ be a subalgebra of $H(F_n)$. Then, by an appropriate automorphism of $H(F_n)$, $J$ can always be reduced to the form (1).
\end{lemma}
Proof. Since \( J \) has the type \( F^{(+)}_{\frac{n}{2}} \), by some (not necessarily orthogonal) automorphism \( \varphi \) of \( F^{(+)}_{n} \) we can always bring \( J \) to the following form (see [3] and [12])

\[
\left\{ \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix} \right\}
\]

where \( X \) is any matrix of order \( \frac{n}{2} \). Then, it is easily seen that \( \theta(Y) = S^{-1}YS \), where \( S = \begin{pmatrix} I & iI \\ \frac{iI}{2} & -\frac{iI}{2} \end{pmatrix} \), \( I \) is the identity matrix, \( i^2 = -1 \), sends each element of the form (3) into the algebra of the form (1). Therefore, by \( \chi = \theta \circ \varphi \) we can bring \( J \) to the form (1).

Next we will show that \( \chi \) is actually an orthogonal automorphism. Notice that \( \chi \) sends \( H(F_n) \) onto a Jordan subalgebra of \( F^{(+)}_{n} \) which consists of all matrices symmetric with respect to the following involution: \( j' = \chi \circ t \circ \chi^{-1} \) where \( t \) is the standard transpose involution. This involution can be rewritten as follows \( j'(X) = C^{-1}X^tC \) for some non-singular symmetric matrix \( C \) of order \( n \). It follows from the above considerations that any matrix of the form (1) is symmetric with respect to \( j' \). Equivalently, for any \( Y \) of the form (1), \( C^{-1}Y^tC = Y, Y^tC = CY, YC = CY \) because \( Y \) is symmetric. As proved in the previous Lemma, \( C = \alpha I \) for some non-zero \( \alpha \). Therefore, \( j' = t \), and \( \chi(H(F_n)) = H(F_n) \), and \( \chi \) is actually an automorphism of \( H(F_n) \). Hence, the Lemma is proved.

Lemma 2.3. Let \( \mathcal{A} \) be a special simple matrix Jordan algebra, and \( \mathcal{J} \) be a proper simple subalgebra of \( \mathcal{A} \). Denote a maximal subalgebra which contains \( \mathcal{J} \) as \( M \). Next, consider a Wedderburn splitting \( M = S \oplus R \) where \( S \) is a semisimple algebra, \( R \) is the radical. Then, there exists an automorphism \( \varphi \) of \( \mathcal{A} \) such that \( \varphi(\mathcal{J}) \subseteq S \).

Proof. Let 1 be the identity element of \( \mathcal{A} \), and \( 1 \in \mathcal{J} \). According to [14], if \( \mathcal{J} \) is special and the degree of \( \mathcal{J} \) is not divisible by the characteristic, then \( \mathcal{J} \) is conjugate under an inner automorphism \( T \) of \( M \) to some subalgebra of \( S \), and \( T \) is a composition of the standard automorphisms \( T_{x,y} \) that can be represented in associative terms as follows

\[
T_{x,y}(a) = tat^{-1},
\]

where \( t = u^{-\frac{1}{2}}(1-xy)(1+yx), u = (1-xy)(1+yx)(1+xy)(1-yx), x, y \in M \).

Let \( x, y \) be symmetric with respect to an involution \( j \) of \( \mathcal{A} \): \( j(x) = x, j(y) = y \). Then, it is obvious that

4
If $A = F_n^{(+)}$, then from the explicit form (4) $T_{x,y}$ is easily extendable to $A$. If $A = H(F_n)$, then, because of (5), $T_{x,y}$ is orthogonal, therefore, extendable to $A$. If $A = H(F_{2n}, j)$, then, because of (5), $T_{x,y}$ is symplectic, therefore, extendable to $A$.

If $J$ is special and the degree of $J$ is divisible by characteristic, then $T$ is a generalized inner automorphism (see [5]), that is, $T$ is a composition of an automorphisms $T_{x_1,...,x_n,m}$ of the form

$$T_{x_1,...,x_n,m} = U_v^{-1}(I + V_{x_1,...,x_n,m} + U_{x_1} ... U_{x_n}U_{-m})(I + V_{m,x_n,...,x_1} + U_mU_{x_n} ... U_1)$$

where $v, x_i \in M$, $m \in R$. In associative terms operators take the form:

$$U_v(a) = vav,$$
$$U_{x_i}(a) = x_iax_i,$$
$$V_{x_1,...,x_n,m}(a) = x_1...x_nma + amx_n ... x_1.$$

Hence, if all $x_i, m$ and $a$ are symmetric with respect to an involution of $A$, then $j(T_{x_1,...,x_n,m}(a)) = T_{x_1,...,x_n,m}(a)$. Therefore, $T_{x_1,...,x_n,m}$ as well as $T$ is extendable to $A$. The Lemma is proved.

**Lemma 2.4.** Let $J \cong F_n^{(+)}$ be a subalgebra of $H(F_{2n}, j)$. Then, by an appropriate automorphism of $H(F_{2n}, j)$, $J$ can always be reduced to the following form

$$\left\{ \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix} \right\}$$

**Proof.** Since $J$ has the type $F_n^{(+)}$, by some automorphism $\varphi$ of $F_{2n}^{(+)}$, $J$ can be brought to the form (6) (see [3] and [12]). Notice that $\varphi$ sends $H(F_{2n}, j)$ onto a Jordan subalgebra of $F_{2n}^{(+)}$ which consists of all matrices symmetric with respect to the following involution: $j' = \varphi \circ j \circ \varphi^{-1}$. This involution can be rewritten as follows $j'(Y) = C^{-1}Y^tC$ for some non-singular skew-symmetric matrix $C$ of order $2n$. It follows from the above considerations that any matrix of the form (6) is symmetric with respect to $j'$. Equivalently, for any $Y$ of the form (6), $C^{-1}Y^tC = Y$, $Y^tC = CY$. Acting in the same manner as above, we can show that $C = \alpha \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ for some non-zero $\alpha$, where $I_n$ denotes the identity matrix of order $n$. Therefore, $\varphi(H(F_n, j)) = H(F_n, j)$, and $\varphi$ is an automorphism of $H(F_n, j)$. Hence, the Lemma is proved. \[\square\]
Definition 2.5. Subalgebras $J_1$ and $J_2$ of a Jordan algebra $A$ are said to be equivalent if there exists an automorphism $\varphi$ of $A$ such that either $J_1 = \varphi(J_2)$ or $J_2 = \varphi(J_1)$.

Definition 2.6. Let $J$ be a subalgebra of $A$. Then the set $C(J)$ of all subalgebras equivalent to $J$ in $A$ is said to be a conjugate class of $J$.

Canonical realizations of simple subalgebras

Let $A$ be a simple Jordan algebra, and $J$ be a simple subalgebra of $A$. All realizations listed below we will call canonical.

1. Let $A = F_n^{(+)}$

   (1.1) $J \cong F_m^{(+)}$, $J = \{\text{diag}(X, \ldots, X, X^t, \ldots, X^t, 0, \ldots, 0)\}$ where $X$ is any matrix of order $m$.

   (1.2) $J \cong H(F_m)$, $J = \{\text{diag}(X, \ldots, X, 0, \ldots, 0)\}$ where $X$ is any symmetric matrix of order $m$.

   (1.3) $J \cong H(F_{2m}, j)$, $J = \{\text{diag}(X, \ldots, X, 0, \ldots, 0)\}$ where $X$ is any symplectic matrix of order $2m$.

2. Let $A = H(F_n)$

   (2.1) $J \cong F_m^{(+)}$, $J = \{\text{diag}(X, \ldots, X, 0, \ldots, 0)\}$ where $X$ is of the form (1) in which $A$ and $B$ are of order $m$.

   (2.2) $J \cong H(F_m)$, $J = \{\text{diag}(X, \ldots, X, 0, \ldots, 0)\}$ where $X$ is any symmetric matrix of order $m$.

   (2.3) $J \cong H(F_{2m}, j)$, $J = \{\text{diag}(X, \ldots, X, 0, \ldots, 0)\}$

   \[
   X = \begin{pmatrix}
   A & -B & -C & D \\
   B & A & D & C \\
   C & D & A & -B \\
   D & -C & B & A \\
   \end{pmatrix}
   \]

   where $A$ is a symmetric matrix of order $m$, $B, C, D$ are skew-symmetric matrices of order $m$.

3. Let $A = H(F_{2n}, j)$

   (3.1) $J \cong F_m^{(+)}$, $J = \{\text{diag}(X, \ldots, X, X^t, \ldots, X^t, 0, \ldots, 0, X^t, \ldots, X^t, X, \ldots, X, 0, \ldots, 0)\}$
where \(k + l + s = n\), \(X\) is any matrix of order \(m\).

(3.2) \(\mathcal{J} \cong H(F_m)\), \(\mathcal{J} = \{\text{diag}(X,\ldots,X,0,\ldots,0, X,\ldots,X,0,\ldots,0)\}\) where \(k + l = n\), \(X\) is any symmetric matrix of order \(m\).

(3.3) \(\mathcal{J} \cong H(F_{2m},j)\), \(\mathcal{J} = \left\{ \begin{pmatrix} A & B \\ C & A^t \end{pmatrix} \right\} \)

\[
A = \begin{pmatrix}
X & & & & 0 \\
& X & & & \\
& & X & Y & \\
& & Z & X^t & \\
& & & 0 & \\
& & & & X & Y \\
& & & & Z & X^t
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
Y & & & & 0 \\
& Y & & & \\
& & 0 & \\
& & & 0 & \\
& & & & 0
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
Z & & & & 0 \\
& \cdots & & & \\
& & Z & & \\
& & & 0 & \\
& & & & 0
\end{pmatrix},
\]

where \(X\) is any matrix of order \(m\), \(Y, Z\) are skew-symmetric matrices of order \(m\).

**Definition 2.7.** Let \(\mathcal{J}\) and \(\mathcal{J}'\) be two proper subalgebras of \(\mathcal{A}\), and \(\mathcal{J}'\) be given in the canonical realization. If \(\mathcal{J}\) is equivalent to \(\mathcal{J}'\), then \(\mathcal{J}'\) is said to be the canonical form of \(\mathcal{J}\).

**Theorem 2.8.** Let \(\mathcal{A}\) be a simple matrix Jordan algebra. Then, any simple matrix subalgebra of \(\mathcal{A}\) has a unique canonical form as above.

**Proof.** Let \(\mathcal{J}\) be any proper simple matrix subalgebra of \(\mathcal{A}\). In particular, the degree of \(\mathcal{J} \geq 3\). Denote the identity of \(\mathcal{A}\) as 1.

The proof of the Theorem consists of three cases.

**Case 1** \(\mathcal{A} = F_n^{(+)\}}\)
1.1 Let $\mathcal{J}$ be of the type $F_m^{(+)}$ for some $m < n$. Since any Jordan algebra of this type has precisely two non-equivalent irreducible representations both of which have degree $m$ (see [3]), $\mathcal{J}$ is equivalent to the subalgebra in the canonical realization (1.1). If $1 \in \mathcal{J}$, then the last zero in (1.1) is omitted. Next we are going to show that $\mathcal{J}$ has a unique canonical form. Equivalently, if $l$ and $k$ are the number of $X$-blocks and $X^t$-blocks in (1.1), then $|l - k|$ is an invariant for $\mathcal{J}$. Indeed, let $S(\mathcal{J})$ be a simple algebra. Then, either $l$ or $k$ is zero, and $|l - k| = \text{rk} e$ where $e$ is the identity of $\mathcal{J}$. If $S(\mathcal{J})$ is a non-simple semisimple algebra, then $S(\mathcal{J}) = I_1 \oplus I_2$ where $I_i$ are simple ideals with the identity elements $e_i$. Hence, $|l - k| = |\text{rk} e_1 - \text{rk} e_2|$ which is invariant for $\mathcal{J}$. Denote $k_A(\mathcal{J}) = |l - k|$.

1.2 Let $\mathcal{J}$ be of the type $H(F_m)$ for some $m \leq n$. Then, it follows from the uniqueness of the irreducible representation of $H(F_m)$ (see [3]) that $\mathcal{J}$ is equivalent to the subalgebra in the canonical realization (1.2). If $1 \in \mathcal{J}$, then the last zero in (1.2) is omitted.

1.3 The proof of the case when $\mathcal{J} \cong H(F_{2m}, j)$, $2m < n$, is exactly the same as the previous proof. In particular, $\mathcal{J}$ of the type $H(F_{2m}, j)$ is equivalent to the subalgebra in the canonical realization (1.3). Obviously, the canonical form is unique.

Case 2 $A = H(F_n)$

Here, our main goal is to determine the canonical form of any simple matrix Jordan subalgebra of $H(F_n)$. Let $M$ be a maximal subalgebra of $H(F_n)$. According to [10], $M$ is isomorphic to one of the following:

1. $H(F_k) \oplus H(F_l)$, $k + l = n$,
2. $F_k^{(+)} \oplus H(F_l) \oplus R$, $2k + l = n$, $R$ is the radical (if $l = 0$, then $M \cong F_{2n}^{(+)} \oplus R$)
3. $\mathcal{J}(f, 1)$ only if $n = 2^m$ and either $\dim \mathcal{J}(f, 1) = 2(m + 1)$, $m$ is even, or $\dim \mathcal{J}(f, 1) = 2m + 1$, $m$ is odd.

First, assume that $\mathcal{J}$ is a simple matrix subalgebra of $H(F_n)$ such that $1 \in \mathcal{J}$. Then there exists a maximal subalgebra $M$ such that $\mathcal{J} \subset M$. Since $\deg \mathcal{J} \geq 3$, $M$ cannot be of the type 3. If $M$ contains a non-zero radical, that is, $M = S \oplus R$, where $S$ a semisimple algebra, $R$ the radical, then by Lemma 2.3 we can assume that $\mathcal{J} \subseteq S$. If $S = S_1 \oplus S_2$ where $S_i$ non-trivial simple ideals, we can choose three orthogonal idempotents (see [10]): $e, e^t, ff^t, 1 = e + e^t + ff^t$ such that

$$S_1 = ff^tH(F_n)ff^t, \quad S_2 = eF_n e + e^tF_n e^t$$

(7)
Since $ff^t$ is an element of $H(F_n)$, by an automorphism $\varphi$ of $H(F_n)$, it can be reduced to the following form:

$$\varphi(ff^t) = \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}$$

where $I_l$ is the identity matrix of order $l$. Since $e$ and $e^t$ are orthogonal to $ff^t$, they take the forms:

$$\varphi(e) = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}, \quad \varphi(e^t) = \begin{pmatrix} 0 & 0 \\ 0 & K^t \end{pmatrix},$$

where $K$ is a matrix of order $n - l$. Therefore, according to (7),

$$\varphi(S) = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\}, \quad \varphi(S_1) = \left\{ \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

$$\varphi(S_2) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \right\},$$

(8)

where $X$ is any symmetric matrix of order $l$, $Y$ is a symmetric matrix of order $2k = n - l$ which is also an element of a subalgebra of the type $F_k^{(+)}$.

In the case when $M$ is semisimple, that is, $M = S = S_1 \oplus S_2$, there exist two orthogonal idempotents such that

$$S = eH(F_n)e + fH(F_n)f, \quad e + f = 1.$$ 

Acting in the same manner as above we can reduce $S$ to (8).

Therefore, we can define two homomorphisms $\pi_1, \pi_2$ as projections on $S_1$ and $S_2$, respectively. Since $1 \in J$, $\pi_1(J) \neq \{0\}$, $\pi_2(J) \neq \{0\}$. This implies that $J \cong \pi_1(J) \subseteq H(F_l), l < n$, and $J \cong \pi_2(J) \subseteq H(F_{2k}), 2k < n$. Therefore, we can reduce the problem of finding the canonical form of $J$ to the case of all symmetric matrices of order less than $n$. However, the above reduction does not work in the case when $S$ is simple, that is, $M = F_{\frac{n}{2}}^{(+)} \oplus R, \quad r \leq n$. Hence we can conclude that as soon as the given simple subalgebra $J$ is in the maximal subalgebra $M$ which has a non-simple semisimple factor $S$, the problem can be reduced to the case of symmetric matrices of a lower order. This process stops only if at some step either $\pi_i(J) \subseteq M \cong F_{\frac{n}{2}}^{(+)} \oplus R$, or $\pi_i(J)$ coincides with $S_i$. Without any loss of generality, we can assume that $r = n$, that is, $J \subseteq M \cong F_{\frac{n}{2}}^{(+)} \oplus R$. 

9
All we need to reach our goal is to determine the canonical form of $\mathcal{J}$ which is covered by a maximal subalgebra of the type $F_{\frac{n}{2}}^{(+)} \oplus R$. Notice that there is an isomorphic imbedding $\theta$ of $F_{\frac{n}{2}}^{(+)}$ into $H(F_n)$ such that $\theta(A + iB) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, where $A$ is a symmetric matrix of order $\frac{n}{2}$, $B$ is a skew-symmetric matrix of order $\frac{n}{2}$, $i^2 = -1$.

2.1. Let us assume that $\mathcal{J}$ has the type $F_{m}^{(+)}$ where $n = 2ml$. We know that by an appropriate automorphism $\psi$ of $F_{\frac{n}{2}}^{(+)}$, we can reduce $\theta^{-1}(\mathcal{J})$ to the following canonical form:

$$\psi(\theta^{-1}(\mathcal{J})) = \begin{bmatrix} X & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & X & 0 & \ldots & 0 \\ 0 & \ldots & 0 & X^t & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & X^t \end{bmatrix},$$

where $X$ is any matrix of order $m$. Then, $X$ can be written as $A + iB$ for an appropriate symmetric $A$ and skew-symmetric $B$. Therefore, $\theta(\psi(\theta^{-1}(\mathcal{J})))$ has the following representation in $H(F_n)$:

$$\theta(\psi(\theta^{-1}(\mathcal{J}))) = \begin{bmatrix} A & 0 & B & 0 \\ A & \ldots & B & \ldots \\ 0 & \ldots & A & 0 \\ -B & 0 & -B & \ldots \\ -B & \ldots & A & \ldots \\ 0 & B & 0 & A \end{bmatrix}.$$

By Lemma 2.1, $\theta \circ \psi \circ \theta^{-1}$ (an automorphism of the algebra of the form (1)) can be extended to an automorphism of $H(F_n)$. Finally, by interchanging the $k$-th and $(\frac{n}{2} + k)$-th columns, and $k$-th and $(\frac{n}{2} + k)$-th rows, $1 \leq k \leq \frac{n}{2}$, and the columns and rows inside the block (if necessary), we can achieve the
following block-diagonal form:

\[
\begin{pmatrix}
\begin{pmatrix} A & B \\ -B & A \end{pmatrix} & 0 \\
0 & \begin{pmatrix} A & B \\ -B & A \end{pmatrix} & 0 \\
& & \ddots
\end{pmatrix}
\]

As a result any subalgebra of $H(F_n)$ of the type $F_n^{(+)}$ can be brought to the canonical form (2.1). This canonical form is obviously unique.

2.2. Let $\mathcal{J}$ be of the type $H(F_m)$. Acting in the same manner as before, $\mathcal{J}$ can be brought to the unique canonical form as follows

\[
\theta(\psi(\theta^{-1}(\mathcal{J}))) = \begin{pmatrix}
X & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
0 & \ldots & X & 0 & \ldots & 0 \\
0 & \ldots & 0 & X & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & \ldots & X
\end{pmatrix},
\]

(10)

where $X$ is a symmetric matrix of order $m$.

2.3. Let $\mathcal{J}$ be of the type $H(F_{2m}, j)$, $n = 4kl$. Like in the previous cases, by an appropriate automorphism $\psi$ of $F_{2m}^{(+)}$, $\theta^{-1}(\mathcal{J})$ can be brought to the following block-diagonal form:

\[
\psi(\theta^{-1}(\mathcal{J})) = \begin{pmatrix}
X & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
0 & \ldots & X & 0 & \ldots & 0 \\
0 & \ldots & 0 & X & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & \ldots & X
\end{pmatrix},
\]

where $X$ is a symplectic matrix of order $2m$. If we represent $X$ as the sum of symmetric and skew-symmetric matrices as follows:

\[
X = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} + \begin{pmatrix} -C & -D \\ -D & C \end{pmatrix}
\]
where all matrices have order \( m \); \( A \) is symmetric, \( B, C,D \) are skew-symmetric, then \( \theta \) induces the following representation of \( \mathcal{J} \) in \( H(F_n) \):

\[
\theta(\psi(\theta^{-1}(\mathcal{J}))) = \begin{pmatrix}
A & -B & -C & D \\
B & A & D & C \\
C & D & A & -B \\
D & -C & B & A \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

Similarly, by Lemma 2.1, \( \theta \circ \psi \circ \theta^{-1} \) (an automorphism of the algebra of the form (1)) can be extended to an automorphism of \( H(F_n) \).

By interchanging appropriate blocks, we can reduce it to the canonical form:

\[
\begin{pmatrix}
A & -B & -C & D \\
B & A & D & C \\
C & D & A & -B \\
D & -C & B & A \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

From the explicit form (11), the canonical form of \( \mathcal{J} \) of the type \( H(F_{2m}, J) \) is uniquely determined.

If \( 1 \notin \mathcal{J} \), then \( \text{rk} (e) = k < n \) where \( e \) is the identity element of \( \mathcal{J} \), and by an appropriate automorphism of \( H(F_n) \) \( \mathcal{J} \) can be brought to the form:

\[
\begin{pmatrix}
X & 0 \\
0 & 0 \\
\end{pmatrix}
\]

where \( X \) is a symmetric matrix of order \( k \). Let \( \pi \) denote a mapping that sends each matrix of the form (12) to the first block of order \( k \). Clearly, \( \mathcal{J} \cong \pi(\mathcal{J}) \). Acting in the same manner as before we can bring \( \pi(\mathcal{J}) \) to the canonical form in \( H(F_k) \). As a result, the original subalgebra \( \mathcal{J} \) also takes the unique canonical form.
Case 3 \( A = H(F_{2n}, j) \)

Since the proof of this case is not much different from the proof of the case of \( H(F_n) \), we will omit most details. According to classification (see [III]), any maximal subalgebra \( M \) in \( H(F_{2n}, j) \) is isomorphic to one of the following:

1. \( H(F_{2k}, j) \oplus H(F_{2l}, j) \), \( k + l = n \),
2. \( H(F_{2k}, j) \oplus F_n^{(+)} \oplus R \), \( k + l = n \). If \( k = 0 \), then \( M = F_n^{(+)} \oplus R \).
3. \( J(f, 1) \) only if \( n = 2^m \) and either \( \dim J(f, 1) = 2(m + 1) \), \( m \) is even, or \( \dim J(f, 1) = 2m + 1 \), \( m \) is odd.

First we assume that \( J \) is a simple matrix subalgebra of \( H(F_{2n}, j) \) such that \( 1 \in J \). Let \( M \) be a maximal subalgebra which contains \( J \), \( J \subset M \). By Lemma 2.3, \( J \subset S \). If \( S \) is a non-simple semisimple algebra, then \( J \) can be projected into the simple components of \( S \). Hence, the problem will be reduced to the case of symplectic matrices of order less than \( 2n \). This reduction stops only when either the image of \( J \) can be covered by the maximal subalgebra with a simple Wedderburn factor \( S \) or the image of \( J \) coincides with one of the simple components of \( S \).

Next we look into the case when \( J \subset M \), where \( M \) has a simple Wedderburn factor \( S \). There is no loss in generality if we assume that \( M = S \oplus R \), \( S \cong F_n^{(+)} \). By Lemma 2.4, \( S \) can be brought to the form (5). Notice that any automorphism of \( F_n^{(+)} \) of the form \( \varphi(X) = C^{-1}XC \) can be extended to an automorphism of \( H(F_{2n}, j) \) as follows:

\[
\varphi(X) = \bar{C}^{-1}X\bar{C}, \quad \bar{C} = \begin{pmatrix} C & 0 \\ 0 & (C^{-1})^t \end{pmatrix}
\] (13)

**3.1** If \( J \cong H(F_m), m \leq n \), then acting by some automorphism of the form (13), it can be reduced to (3.2). This canonical form is obviously uniquely determined.

**3.2** If \( J \cong F_n^{(+)}, m \leq n \), then by some automorphism of the form (13) it can be brought to

\[
\begin{pmatrix}
X \\
\vdots \\
X^t
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
X \\
\vdots \\
X^t
\end{pmatrix}
\] (14)
where $X$ is an arbitrary matrix of order $m$. This is the canonical form (3.1). For further purposes, we introduce the following characteristic of the canonical form. Let $l$ and $k$ denote the number of $X$-blocks and $X^t$-blocks in the top left block of order $n$ in (14). Set $k_A(J) = |l - k|$. With some effort it can be shown that in this case the canonical form is also unique. All we have to show is that any two canonical forms $J_1$ and $J_2$ of the same type with $k_A(J_1) \neq k_A(J_2)$ are not conjugate under symplectic automorphism, or, equivalently, automorphism of $H(F_2, j)$. For clarity, let $J_1 = \text{diag} \{X, X^t, \ldots, X, X^t, X, \ldots X^t\}$ and $J_2 = \text{diag} \{Y, Y, \ldots, Y^t, Y^t, Y^t, \ldots, Y\}$

where $X$ and $Y$ are any matrices of order $m$. Let $\mathcal{S}$ stand for the subalgebra of $H(F_{2n}, j)$ of the form (3). Obviously, $\mathcal{J}_1 \subseteq \mathcal{S}$, $\mathcal{J}_2 \subseteq \mathcal{S}$. Next we are going to show that for any automorphism $\varphi$ of $H(F_{2n}, j)$ such that $\varphi(\mathcal{J}_1) = \mathcal{J}_2$ we can always find a symplectic automorphism $\psi$ that can be restricted to $\mathcal{S}$ and $\psi(\mathcal{J}_1) = \mathcal{J}_2$. Let $C$ be a non-singular matrix that determines $\varphi$. Then, for any $A \in \mathcal{J}_1$ there exists $B \in \mathcal{J}_2$ such that

$$C^{-1}AC = B, \quad AC = CB.$$  \hspace{1cm} (15) \hspace{1cm}

Set $C = (C_{ij})_{i,j=1,s}$ where $C_{ij}$ is a square matrix of order $m$. By performing a matrix multiplication in (15) we obtain a series of equations:

$$XC_{ij} = C_{ij}Y, \quad X^{t}C_{kl} = C_{kl}Y$$

where $(i,j), (k,l) \in I \times I$, $I = \{1, \ldots, s\}$. Since $X$ and $Y$ can be any matrices of order $m$, $C_{ij}$ can not be degenerate. Therefore, $Y = C_{ij}^{-1}XC_{ij}$, $Y = C_{kl}^{-1}X^{t}C_{kl}$. Hence, the matrix $\bar{C} = \text{diag} \{C_{ij}, C_{kl}, \ldots, C_{ij}, (C_{ij})^{-1}, (C_{kl})^{-1}, \ldots, (C_{ij})^{-1}\}$ determines an automorphism $\psi$ of $H(F_{2n}, j)$ such that $\psi(\mathcal{J}_1) = \mathcal{J}_2$. Besides $\psi$ can be restricted to $\mathcal{S}$, therefore, induces an automorphism of a subalgebra of the type $F_{n}^{(+)}$. However we have already showed (case 1.1) that the two canonical forms in $F_{n}^{(+)}$ with $k_A(J_1) \neq k_A(J_2)$ are not conjugate.

3.3 If $J \cong H(F_{2m}, j)$, $m \leq n$, then it can be reduced to (3.3). Let $l$ and $k$ denote the number of $X$-blocks and $X^t$-blocks, respectively, in $A$. Set $k_A(J) = |l - k|$. Then, we are going to show that any two canonical forms $J_1$ and $J_2$ of the same type with $k_A(J_1) \neq k_A(J_2)$ are not conjugate under the automorphism of $H(F_{2m}, j)$. Assume the contrary, that is, there exists an automorphism of $H(F_{2m}, j)$ such that $\varphi(J_1) = J_2$. Next we can
choose $S_1 \subseteq J_1$, $S_1 \cong F_m^{(+) \downarrow}$ such that $k_A(S_1) = k_A(J_1)$. Similarly, we can select $S_2 \subseteq J_2$, $S_2 \cong F_m^{(+) \downarrow}$ such that $k_A(S_2) = k_A(J_2)$. By Lemma 2.2 there exists $\psi : J_2 \to J_2$, $\psi(\varphi(S_1)) = S_2$. From the explicit form of $J_2$, $\psi$ can be extended to an automorphism of $H(F_{2m}, j)$, $\psi \circ \varphi : H(F_{2m}, j) \to H(F_{2m}, j)$. It follows that $S_1$ and $S_2$ have the same canonical forms, in particular, $k_A(S_1) = k_A(S_2)$, a contradiction.

If $1 \notin J$, then in order to find the canonical form of $J$ we use the same approach as in the case of $H(F_n)$.

The Theorem is proved. 



Let $e$ denote the identity element of $J \cong F_m^{(+) \downarrow}$, and $\rho$ stand for the natural representation of $J$ in $F^n$. Obviously, $\rho$ induces the representation of $S(J)$ in $F^n$. If $S(J)$ is a non-simple semisimple associative algebra, that is, $S(J) = I_1 \oplus I_2$ where $I_1$, $I_2$ are isomorphic simple ideals, then $\rho = \rho_1 \oplus \rho_2$ where $\rho_i$ is a representation of $I_i$ in the corresponding invariant subspace of $F^n$. Then $k_A(J) = |\deg \rho_1(I_1) - \deg \rho_2(I_2)|$.

**Theorem 2.9.** Let $A$ be a Jordan algebra of any of the following types: $F_{n}^{(+) \downarrow}$, $H(F_n)$ or $H(F_{2n}, j)$, $n \geq 3$, and $J$, $J'$ be proper simple matrix subalgebras of $A$. If $J'$ has the same type as $J$ does, then $J' \in C(J)$ if and only if

1. $\text{rk}(e) = \text{rk}(e')$;
2. $k_A(J) = k_A(J')$, in the case when $J \cong F_m^{(+) \downarrow}$ or $H(F_{2m}, j)$, for some $m < n$, and $A \cong F_{n}^{(+) \downarrow}$ or $H(F_{2n}, j)$.

**Proof.** First it should be noted that the degree of $J \geq 3$. The case of $J$ of the degree 2 will be considered later in the text.

**The case of $F_{n}^{(+) \downarrow}$**

In this case we assume that $J$ and $J'$ are subalgebras of $F_{n}^{(+) \downarrow}$ which is as usual the set of all matrices of order $n$ closed under the Jordan multiplication. This case breaks into the following subcases.

1. Let $J$ be of the type $F_{m}^{(+) \downarrow}$ for some $m < n$. First we assume that $S(J)$ is a simple algebra. Equivalently, $k_A(J) = \text{rk}(e)$. Let $J'$ be as given in the conditions of the Theorem. If $J' \in C(J)$, then there exists an automorphism $\varphi$ of $F_{n}^{(+) \downarrow}$ which maps $J'$ onto $J$. It follows that $\varphi(e') = e$, therefore, $\text{rk}(e') = \text{rk}(e)$. Besides, $\varphi(S(J')) = S(J)$. Hence, $S(J')$ is also simple, $k(J') = \text{rk}(e') = \text{rk}(e) = k(J)$.

15
Conversely, if \( \text{rk}(e') = \text{rk}(e) \) and \( \mathcal{J}(\mathcal{J}) = k(\mathcal{J}) = \mathcal{J}(\mathcal{J}) = \text{rk}(e) = \text{rk}(e') \), because \( k(\mathcal{J}) = \text{rk}(e) \). Therefore, \( k(\mathcal{J}) = \text{rk}(e') \), that is, \( S(\mathcal{J}') \) is also simple, and \( \mathcal{J}, \mathcal{J}' \) have the same canonical forms. This implies that \( \mathcal{J}' \in \mathcal{C}(\mathcal{J}) \).

Now we assume that \( S(\mathcal{J}) \) is a non-simple semisimple subalgebra. Let \( \mathcal{J}' \) be another subalgebra which satisfies the conditions of the Theorem. If \( \mathcal{J}' \in \mathcal{C}(\mathcal{J}) \), then there exists an automorphism \( \varphi \) of \( F_n^{(+)} \) which maps \( \mathcal{J}' \) onto \( \mathcal{J} \). Therefore, \( \mathcal{J}' \) and \( \mathcal{J} \) have equivalent representations in \( F_n \), and so do \( S(\mathcal{J}') \) and \( S(\mathcal{J}) \). Consequently, either \( \deg \rho_1(\mathcal{I}_1) = \deg \rho_1(\mathcal{I}_1) \) and \( \deg \rho_2(\mathcal{I}_2) = \deg \rho_2(\mathcal{I}_2) \) or \( \deg \rho_1(\mathcal{I}_1) = \deg \rho_2(\mathcal{I}_2) \) and \( \deg \rho_2(\mathcal{I}_2) = \deg \rho_1(\mathcal{I}_1) \). Equivalently, \( |\deg \rho_1(\mathcal{I}_1) - \deg \rho_2(\mathcal{I}_2)| = |\deg \rho_1(\mathcal{I}_1) - \deg \rho_2(\mathcal{I}_2)| \), that is, \( k_A(\mathcal{J}) = k_A(\mathcal{J}') \).

Conversely, if \( \text{rk}(e') = \text{rk}(e) \) and \( k_A(\mathcal{J}) = k_A(\mathcal{J}') \), then \( \mathcal{J} \) and \( \mathcal{J}' \) have the same canonical forms. Therefore, these subalgebras are conjugate by some automorphism of \( F_n^{(+)} \), and \( \mathcal{J}' \in \mathcal{C}(\mathcal{J}) \).

(2) Let \( \mathcal{J} \) be of the type \( H(F_m) \) for some \( m \leq n \).

Suppose that \( \mathcal{J}' \) is another subalgebra of \( F_n^{(+)} \) which has the type \( H(F_m) \). If \( \mathcal{J}' \) is conjugate to \( \mathcal{J} \) under some automorphism \( \varphi \) of \( F_n^{(+)} \) then \( \varphi(e') = e \) and \( \text{rk}(e') = \text{rk}(e) \). In other words, the canonical form of \( \mathcal{J}' \) is exactly the same as that of \( \mathcal{J} \). Conversely, if \( \text{rk}(e') = \text{rk}(e) \), then \( \mathcal{J} \) and \( \mathcal{J}' \) have the same canonical forms. Therefore, \( \mathcal{J}' \in \mathcal{C}(\mathcal{J}) \).

(3) Let \( \mathcal{J} \) be of the type \( H(F_{2m}, j) \) for some \( m \leq n \). The proof of this case is exactly the same as the previous proof.

The case of \( H(F_n) \)

Suppose that \( \mathcal{J} \) and \( \mathcal{J}' \) are two subalgebras of \( H(F_n) \) that satisfy the conditions of the Theorem.

(1) Let \( \mathcal{J} \) as well as \( \mathcal{J}' \) be of the type \( F_m^{(+)} \). Assume that \( \mathcal{J}' \in \mathcal{C}(\mathcal{J}) \). It follows that there exists an automorphism of \( H(F_n) \) such that \( \varphi(\mathcal{J}') = \mathcal{J} \). Hence, \( \text{rk}(e') = \text{rk}(e) \). Conversely, if \( \text{rk}(e') = \text{rk}(e) \), then \( \mathcal{J} \) and \( \mathcal{J}' \) have the same canonical form. Therefore, \( \mathcal{J}' \in \mathcal{C}(\mathcal{J}) \).

(2) Now let both \( \mathcal{J} \) and \( \mathcal{J}' \) have the type \( H(F_k) \) (or \( H(F_{2k}, j) \)). If \( \mathcal{J}' \in \mathcal{C}(\mathcal{J}) \), then there exists an automorphism \( \varphi \) of \( H(F_n) \) that sends \( \mathcal{J}' \) onto \( \mathcal{J} \), \( \varphi(\mathcal{J}') = \mathcal{J} \). Consequently, \( \text{rk}(e') = \text{rk}(e) \).

Conversely, if \( \text{rk}(e') = \text{rk}(e) \), then they have the same canonical form. Therefore, \( \mathcal{J}' \in \mathcal{C}(\mathcal{J}) \).

The case of \( H(F_{2n}, j) \)

Suppose that \( \mathcal{J} \) and \( \mathcal{J}' \) are two subalgebras of \( H(F_{2n}, j) \) that satisfy the
conditions of the Theorem.

(1) Let \( J \) as well as \( J' \) be of the type \( F_m^{(+)} \), \( m < n \). Assume that \( J' \in \mathcal{C}(J) \). It follows that there exists an automorphism of \( H(F_{2n}, j) \) such that \( \varphi(J') = J \). Hence, \( \text{rk}(e') = \text{rk}(e) \). Since \( J' \) and \( J \) are conjugate in \( H(F_{2n}, j) \), they have the same canonical forms in \( H(F_{2n}, j) \). Therefore, \( k_A(J) = k_A(J') \).

Conversely, if all conditions hold true, then \( J \) and \( J' \) have the same canonical forms. Therefore, \( J' \in \mathcal{C}(J) \).

(2) Now let both \( J \) and \( J' \) have the type \( H(F_m), m < n \). If \( J' \in \mathcal{C}(J) \), then there exists an automorphism of \( H(F_{2n}, j) \) that sends \( J' \) onto \( J \), \( \varphi(J') = J \). Consequently, \( \text{rk}(e') = \text{rk}(e) \).

Conversely, if \( \text{rk}(e') = \text{rk}(e) \), then they have the same canonical forms. Therefore, \( J' \in \mathcal{C}(J) \).

(3) Now let both \( J \) and \( J' \) have the type \( H(F_{2m}, j), m < n \). If \( J' \in \mathcal{C}(J) \), then there exists an automorphism of \( H(F_{2n}, j) \) that sends \( J' \) onto \( J \), \( \varphi(J') = J \). Consequently, \( \text{rk}(e') = \text{rk}(e) \), \( k_A(J) = k_A(J') \).

Conversely, if \( \text{rk}(e') = \text{rk}(e) \) and \( k_A(J) = k_A(J') \), then they have the same canonical forms. Therefore, \( J' \in \mathcal{C}(J) \).

The Theorem is proved.

\[ \Box \]

**Corollary 2.10.** If \( m \) is any number such that \( m \leq n \), and \( n = mk + r, 0 \leq r < m \), then there exist subalgebras of \( F_n^{(+)} \) of the type \( H(F_m) \). Moreover, there are precisely \( k \) conjugate classes corresponding to \( H(F_m) \). If \( 2m \leq n \), and \( n = 2mk + r, 0 \leq r < m \) then \( F_n^{(+)} \) has subalgebras of the type \( H(F_{2m}, j) \), and the number of conjugate classes corresponding to \( H(F_{2m}, j) \) is equal to \( k \).

Finally, if \( m < n \), and \( n = mk + r, 0 \leq r < m \) then there exist subalgebras of \( F_n^{(+)} \) of the type \( F_m^{(+)} \), and, moreover, the number of conjugate classes is given by \( \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \).

**Corollary 2.11.** If \( m \) is any number such that \( m < n \), and \( n = mk + r, 0 \leq r < m \), then there exist subalgebras of \( H(F_n) \) of the type \( H(F_m) \). Moreover, there are precisely \( k \) conjugate classes corresponding to \( H(F_m) \). If \( 2m \leq n \), and \( n = 2mk + r, 0 \leq r < m \) then \( H(F_n) \) has subalgebras of the type \( F_m^{(+)} \), and the number of conjugate classes corresponding to \( F_m^{(+)} \) is equal to \( k \).

Finally, if \( 4m \leq n \), and \( n = 4mk + r, 0 \leq r < m \) then there exist subalgebras of \( H(F_n) \) of the type \( H(F_{2m}, j) \), and, moreover, the number of conjugate classes is \( k \).
Corollary 2.12. If $m$ is any number such that $m \leq n$, and $n = mk + r$, $0 \leq r < m$, then there exist subalgebras of $H(F_{2m}, j)$ of the type $H(F_m)$. Moreover, there are precisely $k$ conjugate classes corresponding to $H(F_m)$. If $m \leq n$, and $n = mk + r$, $0 \leq r < m$ then $H(F_{2m}, j)$ has subalgebras of the type $F_m^{(+)}$, and the number of conjugate classes corresponding to $F_m^{(+)}$ is equal to $\sum_{j=1}^{k}[\frac{m}{2}]$. Finally, if $m < n$, and $n = mk + r$, $0 \leq r < m$ then there exist subalgebras of $H(F_m, j)$ of the type $H(F_m, j)$, and, moreover, the number of conjugate classes is $\sum_{j=1}^{k}[\frac{m}{2}]$.

2.2 Subalgebras of the type $J(f, 1)$

First we recall a few facts concerning Clifford algebras over a field of characteristic not 2 (see [10]). Let $J = F \oplus V$ where $V = \text{span}(x_1, \ldots, x_{2m})$, and $f$ a non-degenerate symmetric bilinear form on $V$. Then, $C(V, f)$ is a central simple associative algebra with a unique canonical involution “—” such that it fixes elements from $V$. In this case the imbedding of $J$ into $C(V, f)^{(+)\text{−}}$ we will call canonical of the first type. Next, let $J = F \oplus V$ where $V = \text{span}(x_1, \ldots, x_{2m+1})$, and $V_0 = \text{span}(x_1, \ldots, x_{2m})$. Then, $C(V, f)$ is isomorphic to a tensor product of $C(V_0, f)$ and the two-dimensional center $E$ of $C(V, f)$. Moreover, $E = F[z]$ where $z = x_1x_2 \ldots x_{2m+1}$. In other words, $C(V, f) = I_1 \oplus I_2$, $I_1 \cong C(V_0, f)$. Note that $F \oplus V \cong J/I_\perp \subseteq C(V, f)/I_\perp \cong C(V_0, f)$. This imbedding of $J = F \oplus V$ into $C(V_0, f)^{(+)\text{−}}$ we will call canonical of the second type.

Let $A$ be a simple matrix Jordan algebra, and $J$ be a subalgebra of the type $J(f, 1)$. According to [10], $J$ of the type $J(f, 1)$ is maximal in $A$ if and only if one of the following cases hold

1. $A = (C(V_0, f), -)$, $J = F \oplus V$ where $\text{dim} V = 2m + 1$ and $m$ is odd.
2. $A = H(C(V_0, f), -)$, $J = F \oplus V$ where $\text{dim} V = 2m + 1$, $m$ is even.
3. $A = H(C(V, f), -)$, $J = F \oplus V$ where $\text{dim} V = 2m$

Next we recall that if $\text{dim} V = 2m$, and $m \equiv 0, 1(\text{mod} 4)$ then $\text{dim} H(C(V, f), -) = 2^{m-1}(2^m + 1)$. If $\text{dim} V = 2m$ and $m \equiv 2, 3(\text{mod} 4)$ then $\text{dim} H(C(V, f), -) = 2^{m-1}(2^m - 1)$. If $\text{dim} V = 2m + 1$ and $m \equiv 0(\text{mod} 4)$ then $\text{dim} H(C(V, f), -) = 2^{m-1}(2^m + 1)$. If $\text{dim} V = 2m + 1$ and $m \equiv 2(\text{mod} 4)$ then $\text{dim} H(C(V, f), -) = 2^{m-1}(2^m - 1)$.

Canonical realizations of $J(f, 1)$
Let $\mathcal{A}$ be a simple matrix Jordan algebra, and $\mathcal{J} = F \oplus V$ is a subalgebra of $\mathcal{A}$.

1.1 $\mathcal{A} = F_{n}^{(+)}$, $n = 2^m l + r$, $\dim V = 2m$,

$$\mathcal{J} = \{ \text{diag}(X, \ldots, X, 0, \ldots, 0) \}$$

where $X$ is a matrix of order $2^m$, and if $\pi_i$ denotes the projection on the $i$th non-zero block, then $\pi_i(\mathcal{J}) \subseteq F_{2^m}^{(+)}$ is a canonical imbedding of the first type.

1.2 $\mathcal{A} = F_{n}^{(+)}$, $n = 2^m l + r$, $\dim V = 2m + 1$,

$$\mathcal{J} = \{ \text{diag}(X, \ldots, X, 0, \ldots, 0) \}$$

where $X$ is a matrix of order $2^m$, and $\pi_i(\mathcal{J}) \subseteq F_{2^m}^{(+)}$ is a canonical imbedding of the second type.

1.3 $\mathcal{A} = F_{n}^{(+)}$, $n = 2^m l + r$, $\dim V = 2m + 1$,

$$\mathcal{J} = \{ \text{diag}(X, \ldots, X, X^t, \ldots, X^t 0, \ldots, 0) \}$$

where $s + k = l$, $X$ is a matrix of order $2^m$, and $\pi_i(\mathcal{J}) \subseteq F_{2^m}^{(+)}$ is a canonical imbedding of the second type.

2.1 $\mathcal{A} = H(F_n)$, $n = 2^m l + r$, $\dim V = 2m$,

$$\mathcal{J} = \{ \text{diag}(X, \ldots, X, 0, \ldots, 0) \}$$

where $X$ is a symmetric matrix of order $2^m$, and $\pi_i(\mathcal{J}) \subseteq F_{2^m}^{(+)}$ is a canonical imbedding of the first type.

2.2 $\mathcal{A} = H(F_n)$, $n = 2^m l + r$, $\dim V = 2m$,

$$\mathcal{J} = \{ \text{diag}(X, \ldots, X, 0, \ldots, 0) \}$$

where $X$ is of the form (1) in which $A$ and $B$ are of order $2^m$. If $\mathcal{S}$ denotes the algebra of the form (1), then $\pi_i(\mathcal{J}) \subseteq \mathcal{S}$ is a canonical imbedding of the first type.
2.3 $\mathcal{A} = H(F_n)$, $n = 2^{m+1}l + r$, $\dim V = 2m + 1$,
\[
\mathcal{J} = \{\text{diag}(\underbrace{X, \ldots, X}_l, 0, \ldots, 0)\}
\]
where $X$ is of the form (1) in which $A$ and $B$ are of order $2^m$. If $\mathcal{S}$ denotes
the entire algebra of the form (1), then $\pi_i(\mathcal{J}) \subseteq \mathcal{S}$ is a canonical imbedding
of the second type.

2.4 $\mathcal{A} = H(F_n)$, $n = 2^m l + r$, $\dim V = 2m + 1$,
\[
\mathcal{J} = \{\text{diag}(\underbrace{X, \ldots, X}_l, 0, \ldots, 0)\}
\]
where $X$ is a symmetric matrix, and $\pi_i(\mathcal{J}) \subseteq F^{(+)}_{2^m}$ is a canonical imbedding
of the second type.

3.1 $\mathcal{A} = H(F_{2n}, j)$, $n = 2^m l + r$, $\dim V = 2m$,
\[
\mathcal{J} = \{\text{diag}(\underbrace{X, \ldots, X}_l, 0, \ldots, 0, \underbrace{X, \ldots, X}_{k}, 0, \ldots, 0)\}
\]
where $k + l = n$, $X$ is a symmetric matrix of order $2^m$, and $\pi_i(\mathcal{J}) \subseteq F^{(+)}_{2^m}$ is
a canonical imbedding of the first type.

3.2 $\mathcal{A} = H(F_{2n}, j)$, $n = 2^m l + r$, $\dim V = 2m$, $\mathcal{J}$ has a canonical form (3.3),
and if $\pi_i$ denotes the projection of $\mathcal{J}$ into $i$th simple component (of the type
$H(F_{2^m}, j)$) of (3.3), then $\pi_i(\mathcal{J}) \subseteq H(F_{2^m}, j)$ is a canonical imbedding of the
first type.

3.3 $\mathcal{A} = H(F_{2n}, j)$, $n = 2^m l + r$, $\dim V = 2m + 1$, $\mathcal{J}$ has a canonical form
(3.3) where $\pi_i(\mathcal{J}) \subseteq H(F_{2^m}, j)$ is a canonical imbedding of the second type.

3.4 $\mathcal{A} = H(F_{2n}, j)$, $n = 2^m l + r$, $\dim V = 2m + 1$,
\[
\mathcal{J} = \{\text{diag}(\underbrace{X, \ldots, X}_s, \underbrace{X^t, \ldots, X^t}_k, 0, \ldots, 0, \underbrace{X^t, \ldots, X^t}_s, \underbrace{X, \ldots, X}_k, 0, \ldots, 0)\}
\]
where $s + k = l$, $X$ is a matrix of order $2^m$, and $\pi_i(\mathcal{J}) \subseteq F^{(+)}_{2^m}$ is a canonical
imbedding of the second type.

**Theorem 2.13.** Let $\mathcal{A}$ be a simple matrix Jordan algebra, and $\mathcal{J}$ be a sub-
algebra of $\mathcal{A}$ of the type $J(f, 1)$. Then, $\mathcal{J}$ has a unique canonical form as above.
Proof. Let $J = F \oplus V$. Then the following cases occur.

Case $A = F_+^{(n)}$

1.1 Let $\dim V = 2m$. Then $U(J) \cong C(V, f)$ is a simple algebra. In particular, $S(J) \cong U(J)$.

If $S(J) = A$, then $n = 2^m$, $A \cong U(J)$. Therefore, the imbedding of $J$ into $A$ is equivalent to the imbedding of $F \oplus V$ into $C(V, f)^{(+)}$. Therefore, this is a canonical imbedding of the first type.

If $S(J) \subset A$, then $S(J)$ is a proper simple associative subalgebra of $F_n$. Therefore, $S(J)$ can be reduced to

$$\{\text{diag}(Y, \ldots, Y, 0, \ldots, 0)\}$$

where the order of $Y$ is $2^m$, and $n = 2^m l + r$. As a result, $J$ also takes the canonical form 1.1.

1.2 Let $\dim V = 2m + 1$. Then $U(J) \cong C(V, f)$, and $U(J) = I_1 \oplus I_2$, $I_i \cong C(V_0, f)$. Hence $S(J)$ is isomorphic to either $C(V, f)$ or $C(V_0, f)$.

If $S(J) = A$, then the imbedding of $J$ into $A$ is the canonical of the second type.

If $S(J) \cong I_i$, and $S(J) \subset A$, then $S(J)$ is a proper simple associative subalgebra of $F_n$. Therefore, $S(J)$ can be reduced to (16). As a result, $J$ takes the canonical form 1.2.

Finally, if $S(J) = I_1 \oplus I_2$, then $J$ takes the canonical form 1.3

Case $A = H(F_n)$

Let $M$ be the maximal subalgebra of $H(F_n)$ such that $J \subseteq M \subseteq H(F_n)$. Then, the following cases occur.

1. $M = S \oplus R$ where $S = S_1 \oplus S_2$ a semisimple factor, $R$ the radical. Then, we reduce the problem to the case of symmetric matrices of a lower dimension (see section 2.1).

2. $M = S$ where $S = S_1 \oplus S_2$. Like in the previous case we can reduce the problem to the case of symmetric matrices of a lower dimension.

3. $M = S \oplus R$ where $S \cong F_+^{(n)}$, $R$ the radical.

4. $M = F \oplus W$ where $W$ is a finite-dimensional vector space.

After a series of reductions of the form 1 and 2, the image of $J$ becomes a subalgebra of
where $\mathcal{A}_i \cong H(F_{n_i})$. Let $\pi_i$ be the projection of $\mathcal{J}$ into $\mathcal{A}_i$. To simplify our notations we denote $\pi_i(\mathcal{J})$ as $\mathcal{J}'$, and the maximal subalgebra of $\mathcal{A}_i$ which covers $\mathcal{J}'$ as $M_i$, $\mathcal{J}' \subseteq M_i \subseteq \mathcal{A}_i = H(F_{n_i})$.

**Case 1.** Let $\dim V = 2m$ and $m \equiv 0, 1 (\text{mod} 4)$. Then we have the following cases:

(a) Let $M_i = F \oplus W$. If $S(\mathcal{J}') = F_{n_i}$, then $n_i = 2^m$, $F_{n_i} \cong C(V,f)$, and the imbedding of $\mathcal{J}'$ into $F_{n_i}^{(+)}$ is equivalent to the imbedding of $F \oplus V$ into $C(V,f)^{(+)}$, that is, canonical of the first type. If $S(\mathcal{J}') \subset F_{n_i}$, then $H(S(\mathcal{J}')) \subset H(F_{n_i})$ is a proper subalgebra of $H(F_{n_i})$. Hence, $n_i = 2^m l + r$, and $H(S(\mathcal{J}'))$ can be reduced to (16) in which $X$ denotes a symmetric matrix of order $2^m$. Then, $\mathcal{J}'$ takes the canonical form 2.1.

(b) Let $M_i = S \oplus R$ where $S \cong F_{n_i}^{(+)}$. By using $\theta$-isomorphism (see section 2.1) we obtain that $\theta^{-1}(\mathcal{J}') \subset F_{n_i}^{(+)}$. If $S(\theta^{-1}(\mathcal{J}')) = F_{n_i}^{(+)}$, then $n_i = 2^{m+1}$ and the imbedding of $\theta^{-1}(\mathcal{J}')$ into $F_{n_i}^{(+)}$ is the canonical imbedding of the first type. In particular, $\theta^{-1}(\mathcal{J}') \subset H(F_{n_i}^{(+)})$. As a result $\mathcal{J}'$ takes the canonical form 2.1 in which $l = 2$ and no zeros. If $S(\mathcal{J}') \subset F_{n_i}^{(+)}$, then $S(\theta^{-1}(\mathcal{J}'))$ is a proper simple subalgebra of $F_{n_i}^{(+)}$, therefore, takes the form (16) and $n_i = 2^{m+1} l + r$. Hence $\mathcal{J}$ takes the canonical form 2.1.

**Case 2** Let $\dim V = 2m$, $m \equiv 2, 3 (\text{mod} 4)$.

(a) Let $M_i = F \oplus W$. If $S(\mathcal{J}') = F_{n_i}$, then $n_i = 2^m$, $F_{n_i} \cong C(V,f)$, $\mathcal{J}' \subseteq H(F_{n_i})$. Hence we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{J}' = F \oplus V & \overset{id}{\longrightarrow} & \mathcal{J}' = F \oplus V \\
\downarrow \sigma & & \downarrow \eta \\
U(\mathcal{J}') & \overset{\varphi}{\leftarrow} & F_{n_i}^{(+)}
\end{array}
$$

where $\sigma = \varphi \circ \eta$. Therefore, $\sigma(\mathcal{J}') = \varphi(\eta(\mathcal{J}'))$ is symmetric with respect to the canonical involution $"\sim"$ which is symplectic in this particular case. On the other hand, $\sigma(\mathcal{J}'')$ is also symmetric with respect to $j' = \varphi \circ t \circ \varphi^{-1}$. 

22
By the uniqueness of "—", \( J' \) equals to "—". However it is not possible because \( \dim H(C(V,f),-) = \frac{2m(2m-1)}{2} \neq \frac{2m(2m+1)}{2} = \dim H(C(V,f),j') \). If \( S(\mathcal{J'}) \subset F_{n_i} \), then \( H(S(\mathcal{J'})) \subseteq H(F_{n_i}) \) is a proper subalgebra of \( H(F_{n_i}) \). Hence \( n_i = 2^m l + r \), and \( H(S(\mathcal{J'})) \) can be reduced to (16) where \( X \) denotes a symmetric matrix of order \( 2^m \). Let \( \pi_{ij} \) denote the projection on \( j \)th non-zero block of (16). Then the imbedding \( \pi_{ij}(\mathcal{J'}) \subseteq \pi_{ij}(H(S(\mathcal{J'}))) \) is similar to the above imbedding, which is not possible.

(b) Let \( M_i = S \oplus R \) where \( S \cong F_{n_i}^{(+)} \). Then \( \theta^{-1}(\mathcal{J'}) \subseteq F_{n_i}^{(+)} \). Since \( S(\theta^{-1}(\mathcal{J'})) \cong U(\mathcal{J'}) \), then \( n_i = 2^{m+1} l + r \), and \( S(\theta^{-1}(\mathcal{J'})) \) can be reduced to (16) in which \( X \) is any matrix of order \( 2^m \). Hence \( \pi_{ij}(\theta^{-1}(\mathcal{J'})) \subset F_{2m}^{(+)} \) is a canonical imbedding of the first type, and \( \mathcal{J} \) has the canonical form 2.2.

Case 3 Let \( \dim V = 2m + 1 \) where \( m \) is odd.

(a) Let \( M_i = F \oplus W \). If \( S(\mathcal{J'}) = F_{n_i} \), then \( n_i = 2^m, F_{n_i} \cong C(V_0, f) \).

Therefore, the imbedding of \( \mathcal{J}' \) into \( F_{n_i}^{(+)} \) is equivalent to the imbedding of \( F \oplus V \) into \( C(V_0, f)^{(+)} \) which is canonical imbedding of the second type. Since \( m \) is odd, \( \mathcal{J}' \) is a maximal subalgebra in \( F_{n_i}^{(+)} \). However, \( \mathcal{J}' \subseteq H(F_{n_i}) \), hence, \( \mathcal{J}' \) cannot be maximal. This case is not possible. If \( S(\mathcal{J'}) \subseteq F_{n_i} \), then \( H(S(\mathcal{J'})) \subseteq H(F_{n_i}) \) is a proper subalgebra of \( H(F_{n_i}) \), therefore, can be reduced to (16). However, the imbedding of \( \pi_{ij}(\mathcal{J'}) \) into \( F_{2m}^{(+)} \) is as shown above. Hence this case is also not possible.

(b) Let \( M_i = S \oplus R \) where \( S \cong F_{n_i}^{(+)} \). Acting in the same manner as in case 2(b) we will come to the canonical form 2.3.

Case 4. Let \( \dim V = 2m + 1 \) and \( m \equiv 0(\text{mod}4) \). Acting in the same manner as in previous cases we will reduce \( \mathcal{J}' \) to the canonical form 2.4.

Case 5. Let \( \dim V = 2m + 1 \) and \( m \equiv 2(\text{mod}4) \). Acting in the same manner as in previous cases we will reduce \( \mathcal{J}' \) to the canonical form 2.3.

Case \( \mathcal{A} = H(F_{2n}, j) \)

Let \( M \) be the maximal subalgebra of \( H(F_{2n}, j) \) such that \( \mathcal{J} \subseteq M \subseteq H(F_{2n}, j) \). Then, the following cases occur.

1. \( M = S \oplus R \) where \( S = S_1 \oplus S_2 \) a semisimple factor, \( R \) the radical.

Then, we reduce to the case of symplectic matrices of a lower dimension (see section 2.1).

2. \( M = S \) where \( S = S_1 \oplus S_2 \). Like in the previous case we can reduce the problem to the case of symplectic matrices of a lower dimension.

3. \( M = S \oplus R \) where \( S \cong F_{n}^{(+)} \), \( R \) the radical.

4. \( M = F \oplus W \) where \( W \) is a finite-dimensional vector space.

After a series of reductions of the form 1 and 2, the image of \( \mathcal{J} \) becomes
a subalgebra of the algebra in the canonical form (3.3) in which the $i$th component has order $2n_i$. Let $\pi_i$ denote the projection of $\mathcal{J}$ into the $i$th simple component of (3.3).

**Case 1.** Let $\dim V = 2m$ and $m \equiv 0, 1(\mod 4)$.

(a) Let $M_i = F \oplus W$. If $S(\mathcal{J}') = F_{2n_i}$, $F_{2n_i} \cong C(V, f)$, $2n_i = 2^m$. Acting in the same manner as in previous cases we will reduce $J$ to the canonical form (3.3).

(b) Let $M_i = S \oplus R$ where $S \cong F_{n_i}^{(+)}$. Then $\mathcal{J}' \subseteq F_{2n_i}^{(+)}$, therefore, $S(\mathcal{J}')$ can be brought to (16), and $\pi_{ij}(\mathcal{J}') \subseteq F_{2m}^{(+)}$ is the canonical imbedding of the first type. Finally the original subalgebra takes the form 3.1

**Case 2** Let $\dim V = 2m$, $m \equiv 2, 3(\mod 4)$.

(a) Let $M_i = F \oplus W$. If $S(\mathcal{J}') = F_{2n_i}$, then $2n_i = 2^m$, $F_{2n_i} \cong C(V, f)$, $\mathcal{J}' \subseteq F_{2n_i}^{(+)}$ is the canonical imbedding of the first type. If $S(\mathcal{J}') \subseteq F_{2n_i}$, then $H(S(\mathcal{J}'), j) \subseteq H(F_{2n_i}, j)$ is a proper subalgebra of $H(F_{2n_i}, j)$, that is, $n_i = 2^m + r$, and $H(S(\mathcal{J}'), j)$ can be reduced to (3.3) in which each component has order $2^m$. Then, $\mathcal{J}$ takes the canonical form 3.2.

(b) Let $M_i = S \oplus R$ where $S \cong F_{n_i}^{(+)}$. This case also lead us to the canonical form 3.2.

**Case 3** Let $\dim V = 2m + 1$ where $m$ is odd.

(a) Let $M_i = F \oplus W$. If $S(\mathcal{J}') = F_{2n_i}$, then $2n_i = 2^m$, $F_{2n_i} \cong C(V_0, f)$. Therefore, the imbedding of $\mathcal{J}'$ into $F_{2n_i}$ is equivalent to the imbedding of $F \oplus V$ into $C(V_0, f)^{(+)}$ which is canonical imbedding of the second type. Since $m$ is odd, $\mathcal{J}'$ is a maximal subalgebra in $F_{2n_i}^{(+)}$. However, $\mathcal{J}' \subseteq H(F_{2n_i}, j)$, hence, $\mathcal{J}'$ cannot be maximal. This case is not possible. If $S(\mathcal{J}') \subseteq F_{2n_i}$, then $H(S(\mathcal{J}'), j) \subseteq H(F_{2n_i}, j)$ is a proper subalgebra of $H(F_{2n_i}, j)$, therefore, can be reduced to (3.3). Let $\pi_{ij}$ denote the projection of $\mathcal{J}'$ into the $j$th simple component of (3.3). However, the imbedding of $\pi_{ij}(\mathcal{J}')$ into $F_{2m}^{(+)}$ is as shown above. Hence this case is also not possible.

(b) Let $M_i = S \oplus R$ where $S \cong F_{n_i}^{(+)}$. Then $\mathcal{J}' \subseteq F_{n_i}^{(+)}$, therefore, $S(\mathcal{J}')$ can be brought to (16), and $\pi_{ij}(\mathcal{J}') \subseteq F_{2m}^{(+)}$ is the canonical imbedding of the second type. Finally the original subalgebra takes the form 3.4

**Case 4.** Let $\dim V = 2m + 1$ and $m \equiv 0(\mod 4)$. Acting in the same manner as in previous cases we will reduce $\mathcal{J}'$ to the canonical form 3.1.

**Case 5.** Let $\dim V = 2m + 1$ and $m \equiv 2(\mod 4)$. Acting in the same manner as in previous cases we will reduce $\mathcal{J}'$ to the canonical form 3.2.
Corollary 2.14. Let $\mathcal{A}$ be a simple matrix Jordan algebra of degree $\geq 3$, and $\mathcal{J} = F \oplus V$ where either $\dim V = 2m$ or $\dim V = 2m + 1$. Then, $\mathcal{J}$ is a subalgebra of $\mathcal{A}$ if and only if

1. $2^m \leq n$,
2. $2^{m+1} \leq n$, in the case when $\mathcal{A} = H(F_n)$ and $m \equiv 2, 3(\text{mod} 4)$.

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