Hyperelliptic surfaces with $K^2 < 4\chi - 6$

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Abstract

Let $S$ be a smooth minimal surface of general type with a (rational) pencil of hyperelliptic curves of minimal genus $g$. We prove that if $K_S^2 < 4\chi(O_S) - 6$, then $g$ is bounded. The surface $S$ is determined by the branch locus of the covering $S \rightarrow S/i$, where $i$ is the hyperelliptic involution of $S$. For $K_S^2 < 3\chi(O_S) - 6$, we show how to determine the possibilities for this branch curve. As an application, given $g > 4$ and $K_S^2 - 3\chi(O_S) < -6$, we compute the maximum value for $\chi(O_S)$. This list of possibilities is sharp.

2010 MSC: 14J29.

1 Introduction

Our motivation for this work is the following. In [AK] Ashikaga and Konno consider surfaces $S$ of general type with $K_S^2 = 3\chi(O_S) - 10$. For these surfaces the canonical map is of degree 1 or 2. In the degree 2 case, the canonical image is a ruled surface, thus if $S$ is regular, it has a pencil of hyperelliptic curves. By a result of Xiao [Xi2, Thm. 1] if $\chi(O_S) \geq 47$, then $S$ has such an hyperelliptic pencil of curves of genus $\leq 4$. But for $\chi(O_S) \leq 46$ this result gives no information (for $\chi(O_S) = 46$ the slope formula [Xi1, Thm. 2] implies $g \leq 5 \lor g \geq 9$; we show that in this case $S$ has an hyperelliptic pencil of minimal genus $g \leq 10$ and the cases $g = 9, g = 10$ do occur). Ashikaga and Konno study only the case $g \leq 4$ (there is an infinite number of possibilities). Nothing is said for the possibilities with $g \geq 5$ and $\chi(O_S) \leq 46$. A similar situation occurs in [K].

In this paper we study smooth minimal surfaces $S$ of general type which have a pencil of hyperelliptic curves (by pencil we mean a linear system of dimension 1). We say that $S$ has such a pencil of minimal genus $g$ if it has an hyperelliptic pencil of genus $g$ and all hyperelliptic pencils of $S$ are of genus $\geq g$. We are mainly interested in the case $g > 4$ and $\chi(O_S)$ small (i.e. where [Xi2, Thm. 1] is not useful).
For $S$ such that $K_S^2 < 4\chi(O_S) - 6$, we give bounds for the minimal genus $g$ (Theorem 1).

The surface $S$ is the smooth minimal model of a double cover of an Hirzebruch surface $F_e$ ramified over a curve $\overline{B}$ (which determines $S$). We prove that if $K_S^2 < 3\chi(O_S) - 6$, then $\overline{B}$ has at most points of multiplicity 8 and we show how to determine the possibilities for $\overline{B}$ (Proposition 2).

As an application, given $g > 4$ and $K_S^2 - 3\chi(O_S) < -6$, we compute the maximum value for $\chi(O_S)$; this list of possibilities is sharp (Theorem 3).

The paper is organized as follows. In Section 2 we present the main results of the paper. The hyperelliptic involutions of the fibres of $S$ induce an involution $i$ of $S$, so in Section 3 we review some general facts on involutions. Since the quotient $S/i$ is a rational surface, a smooth minimal model of $S/i$ is not unique. We make a choice for this minimal model in Section 4 (which is due to Xiao [Xi3]) and we show some consequences of it. Section 5 contains the key result of the paper, which allow us to compute bounds for the minimal genus of the hyperelliptic fibration. This is done via a carefully analysis of the possibilities for the branch locus of the covering $S \to S/i$ considering the restrictions imposed by the choice of minimal model. Finally this is used in Section 6 to prove the main results, stated in Section 2.

Notation

We work over the complex numbers; all varieties are assumed to be projective algebraic. A $(-2)$-curve or nodal curve $A$ on a surface is a curve isomorphic to $\mathbb{P}^1$ such that $A^2 = -2$. An $(m_1, m_2, \ldots)$-point of a curve, or point of type $(m_1, m_2, \ldots)$, is a singular point of multiplicity $m_1$, which resolves to a point of multiplicity $m_2$ after one blow-up, etc. By double cover we mean a finite morphism of degree 2. The rest of the notation is standard in Algebraic Geometry.

Acknowledgements

The authors wish to thank Margarida Mendes Lopes, for all the support, Ana Bravo and Orlando Villamayor, for their hospitality at the Universidad Autónoma de Madrid.

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This research was partially supported by the Fundação para a Ciência e a Tecnologia (Portugal) through Projects PEst-OE/MAT/UI4080/2011 and PTDC/MAT/099275/2008.
2 Main results

**Theorem 1.** Let $S$ be a minimal smooth surface of general type with a pencil of hyperelliptic curves of minimal genus $g$.

If $K_S^2 < 4\chi(O_S) - 6$, then $g$ is not greater than

$$\max \left\{ -1 + \frac{8\chi(O_S)}{4\chi(O_S)-K_S^2-6}, 1 + \frac{8\chi(O_S)-16}{4\chi(O_S)-K_S^2-6}, 1 + \frac{8\chi(O_S)}{4\chi(O_S)-K_S^2-3}, \frac{3+\sqrt{1+8\chi(O_S)}}{2} \right\}.$$

Let $B \subset W$ be the branch locus of a double cover $V \to W$, where $V$ and $W$ are smooth surfaces (thus $B$ is also smooth). Let $\rho: W \to P$ be the projection of $W$ onto a minimal model and denote by $\overline{B}$ the projection $\rho(B)$.

Suppose that $\overline{B}$ has singular points $x_1, \ldots, x_n$ (possibly infinitely near). For each $x_i$ there is an exceptional divisor $E_i$ and a number $r_i \in 2\mathbb{N}$ such that

$$E_i^2 = -1, \quad K_W \equiv \rho^*(K_P) + \sum E_i, \quad B = \rho^*(\overline{B}) - \sum r_iE_i.$$

Notice that $r_i$ is not the multiplicity of the singular point $x_i$, it is the multiplicity of the corresponding singularity in the canonical resolution (see [BHPV III, 7.]). For example, in the case of a point of type $(2r-1,2r-1)$ one has $r_1 = 2r-2$ and $r_2 = 2r$.

Since, from Theorem 1, we have a bound for the genus $g$, we also have a bound for the multiplicities $r_i$. For the case $K_S^2 < 3\chi(O_S) - 6$, we prove the result below.

Let $N_j$ be the number of singular points $x_i$ of $\overline{B}$ (possibly infinitely near) such that $r_i = j$. Denote by $C_0$ and $F$ the negative section and a ruling of the Hirzebruch surface $\mathbb{F}_e$.

**Proposition 2.** Let $S$ be a minimal smooth surface of general type with an hyperelliptic pencil of minimal genus $(k-2)/2$. If $K_S^2 < 3\chi(O_S) - 6$, then $S$ is the smooth minimal model of a double cover $S'' \to \mathbb{F}_e$ with branch curve $\overline{B} \equiv kC_0 + (ek/2 + l)F$ such that:

a) $r_i \leq \min\{8, k/2 + 2, l - k/2 + 2\} \ \forall i$

b) $N_4 + N_6 = 15 + K^2_{S''} - 3\chi(O_S) - \frac{1}{4}(k-10)(l-10)$

c) $\chi(O_S) = 1 + \frac{1}{4}(k-2)(l-2) - N_4 - 3N_6 - 6N_8$

where $S'' \to S'$ is the canonical resolution.

Proposition 2 can be used to restrict possibilities for $\overline{B}$. We show the following:
Theorem 3. Let $S$ be a minimal smooth surface of general type with an hyperelliptic pencil of minimal genus $g > 4$. If $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, then $\chi(\mathcal{O}_S)$ is bounded by the number given in the table below (emptiness means non-existence). All these cases do exist.

| $K^2 - 3\chi$ | $g$ | -7 | -8 | -9 | -10 | -11 | -12 | -13 | -14 | -15 | -16 |
|---------------|-----|----|----|----|-----|-----|-----|-----|-----|-----|-----|
| 5             | 61  | 56 | 51 | 46 | 41  | 36  | 31  | 26  | 21  | 16  |
| 6             | 49  | 46 | 43 | 40 | 37  | 34  | 27  | 28  | 22  |
| 7             | 42  | 43 | 43 | 35 | 36  | 28  | 29  | 22  |
| 8             | 44  | 44 | 45 | 36 | 37  | 29  |
| 9             | 45  | 46 | 37 |   |     |     |
| 10            |     | 46 |     |   |     |     |

Remark 4. This result gives 3 examples where Theorem 4 is almost sharp: in the cases $(g, K^2 - 3\chi) = (10, -10), (9, -13), (8, -15)$ we get from Theorem 4 that $\chi \leq 47, 38, 30$, respectively.

There is at least one case where Theorem 4 is sharp: a double plane with branch locus a curve of degree 18 with 8 points of multiplicity 6. In this case $\chi = 5, K^2 = 8$ and $g = 5$.

3 Involutions

Let $S$ be a smooth minimal surface of general type with a (rational) pencil of hyperelliptic curves. This hyperelliptic structure induces an involution (i.e. an automorphism of order 2) $i$ of $S$. The quotient $S/i$ is a rational surface.

Since $S$ is minimal of general type, this involution is biregular. The fixed locus of $i$ is the union of a smooth curve $R''$ (possibly empty) and of $t \geq 0$ isolated points $P_1, \ldots, P_t$. Let $p : S \rightarrow S/i$ be the projection onto the quotient. The surface $S/i$ has nodes at the points $Q_i := p(P_i), i = 1, \ldots, t$, and is smooth elsewhere. If $R'' \neq \emptyset$, the image via $p$ of $R''$ is a smooth curve $B''$ not containing the singular points $Q_i, i = 1, \ldots, t$. Let now $h : V \rightarrow S$ be the blow-up of $S$ at $P_1, \ldots, P_t$ and set $R' = h^*(R'')$. The involution $i$ induces a biregular involution $\tilde{i}$ on $V$ whose fixed locus is $R := R' + \sum_{1}^{t} h^{-1}(P_i)$. The quotient $W := V/\tilde{i}$ is smooth and one has a commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{h} & S \\
\pi \downarrow & & \downarrow p \\
W & \xrightarrow{g} & S/i \\
\end{array}
$$
where \( \pi : V \to W \) is the projection onto the quotient and \( g : W \to S/i \) is the minimal desingularization map. Notice that
\[
A_i := g_i^{-1}(Q_i), \quad i = 1, \ldots, t,
\]
are \((-2)\)-curves and \( \pi^*(A_i) = 2 \cdot h_i^{-1}(P_i) \).
Set \( B' := g^*(B'') \). Since \( \pi \) is a double cover, its branch locus \( B' + \sum_i A_i \) is even, i.e. there is a line bundle \( L \) on \( W \) such that
\[
2L \equiv B := B' + \sum_i A_i.
\]

4 Choice of minimal model

Part of this section may be found in \[X3\]. We use the notation introduced so far. As above, \( W \) is a rational surface.

\((*)\). Blowing-up, if necessary, \( \mathbb{P}^2 \) at a point, we can suppose that \( W \neq \mathbb{P}^2 \).

Thus there is a birational morphism
\[
\rho : W \to \mathbb{F}_e,
\]
where \( \mathbb{F}_e \) is an Hirzebruch surface. Let \( \mathcal{B} := \rho(B) \) and consider the double cover \( S' \to \mathbb{F}_e \) with branch locus \( \mathcal{B} \). If \( \mathcal{B} \) is singular then \( S' \) is also singular and \( S \) is isomorphic to the minimal smooth resolution of \( S' \).

We can define \( k \) and \( l \) such that
\[
\mathcal{B} \equiv: kC_0 + \left( \frac{ek}{2} + l \right) F,
\]
where \( C_0 \) and \( F \) are, respectively, the negative section and a ruling of \( \mathbb{F}_e \) (thus \( C_0^2 = -e \), \( C_0F = 1 \), \( F^2 = 0 \)). Notice that \( \mathcal{B}^2 = 2kl \) and \( K_P \mathcal{B} = -2k - 2l \).

\((*)\). Among all the possibilities for the map \( \rho \), we choose one satisfying, in this order:

1) the degree \( k \) of \( \mathcal{B} \) over a section is minimal;

2) the greatest order of the singularities of \( \mathcal{B} \) is minimal;

3) the number of singularities with greatest order is also minimal.
Recall that a $(2r - 1, 2r - 1)$ singularity of $\overline{B}$ is a pair $(x_j, x_k)$ such that $x_k$ is infinitely near to $x_j$ and $r_j = 2r - 2$, $r_k = 2r$.

Let

$$r_m := \max \{ r_i \}$$

or $r_m := 0$ if $\overline{B}$ is smooth.

By *elementary transformation* over $x_i \in \mathbb{F}_e$ we mean the blow-up of $x_i$ followed by the blow-down of the strict transform of the ruling of $\mathbb{F}_e$ that contains $x_i$.

The following is a consequence of the choice (*) of the map $\rho$.

**Proposition 5 ([XIII]).** We have:

a) If $k \equiv 0 \pmod{4}$, then $r_m \leq \frac{k}{2} + 2$ and the equality holds only if $x_m$ belongs to a singularity $(\frac{k}{2} + 1, \frac{k}{2} + 1)$. In this case $l \geq k + 2$ and all the branches of the singularity are tangent to the ruling of $\mathbb{F}_e$ that contains it.

b) If $k \equiv 2 \pmod{4}$, then $r_m \leq \frac{k}{2} + 1$ and the equality holds only if $x_m$ belongs to a singularity $(\frac{k}{2}, \frac{k}{2})$. In this case $l \geq k$.

In a similar vein:

**Proposition 6.** We have that:

a) if $l = k + 2$ and $k > 8$, there are at most two $(\frac{k}{2} + 1, \frac{k}{2} + 1)$-points.

b) $l \geq \frac{k}{2}$ and $l \geq \frac{k}{2} + r_m - 2$;

c) if $l = \frac{k}{2} + r_m - 2$, then either:

- $e = 2$, $l = k - 2$, the branch locus $\overline{B}$ has a $(\frac{k}{2} - 1, \frac{k}{2} - 1)$-point and all singularities are of multiplicity $< \frac{k}{2}$, or

- we can suppose $e = 1$, the negative section $C_0$ of $\mathbb{F}_1$ is contained in $\overline{B}$, $\overline{B}$ has a point of multiplicity $r_m$ contained in $C_0$ and the remaining singularities are of multiplicity $< r_m$.

**Proof:**

a) This is due to Borrelli ([Ba]). Suppose that there are three singularities $(k/2 + 1, k/2 + 1)$. The rulings of $\mathbb{F}_e$ through these points are contained in $\overline{B}$ and then $\overline{B}C_0 = l - \frac{4k}{2} \geq 4 (\overline{B}C_0$ is even). This implies $e < 1$. Making, if necessary, an elementary transformation over one of these points, we can suppose that $e = 1$. 

6
Let $\rho$ be as above and $E_i, E_i', i = 1, 2, 3$, be the exceptional divisors corresponding to three singularities $(k/2 + 1, k/2 + 1)$ of $\mathcal{F}$. The general element of the linear system $|\rho^*(4C_0 + 5F) - \sum_i^3(2E_i + 2E_i')|$ is a smooth and irreducible rational curve $C$ such that $CB < k$. This contradicts the choice (*) of the map $\rho$.

b) If $r_m > \frac{k}{2}$ then the result follows from Proposition 5. Suppose now $r_m \leq \frac{k}{2}$. We have $\mathcal{B}C_0 \geq -e$, i.e. $l - \frac{ek}{2} \geq -e$. Therefore if $e \geq 2$, then

$$l \geq k - 2 \geq \frac{k}{2}$$

and

$$l \geq k - 2 \geq \frac{k}{2} + r_m - 2.$$

When $e = 0$ we obtain immediately $l \geq k$, by the choice of the map $\rho$, thus $l \geq \frac{k}{2} + r_m$.

If $e = 1$ then $\mathcal{B}C_0 = l - \frac{k}{2} + 1 \geq 0$. Blow-down $C_0$. We obtain a singularity of order at most $l - \frac{k}{2} + 1$, hence the choice of the minimal model implies $r_m \leq l - \frac{k}{2} + 2$ (notice that the equality happens only if the order of the singularity is $(r_m - 1, r_m - 1)$).

c) Assume that $l = k/2 + r_m - 2$. Proposition 5 implies $r_m \leq k/2$. From $\mathcal{B}C_0 \geq -e$ we obtain $k/2 + r_m - 2 = l \geq \frac{ek}{2} - e$, thus either $e = 1$ or $e = 2$ and $r_m = k/2$ (notice that $e = 0$ implies $l \geq k$).

In the case $e = 1$ we can, as in the proof of b), contract the section with self-intersection $(-1)$ to obtain a branch curve in $\mathbb{P}^2$ with at most singularities of type $(l - k/2 + 1, l - k/2 + 1)$.

Suppose now that $e = 2$ and there is a point $x_i$ of multiplicity $k/2$. In this case $\mathcal{B}C_0 = -2$, hence $x_i \notin C_0$. We make an elementary transformation over $x_i$ to obtain the case $e = 1$ also with $l = k - 2$. \qed

5 Bound of genus

In this section we prove the key result to establish bounds for the minimal genus of the hyperelliptic fibrations.

From [K] (cf. also [CM]), we get the following:

**Proposition 7.** Let $S'' \to S'$ be the canonical resolution of a double cover $S' \to \mathbb{P}_e$ with branch locus $\mathcal{B} \equiv kC_0 + (ek/2 + l)F$. Let $S$ be the minimal model of $S''$ and $t := K_{S''}^2 - K_{S'}^2$. If $S$ is of general type, then:

\[7\]
\( \sum (r_i - 2)(k - r_i - 2) = H \)

b) \( 2l = G + \sum (r_i - 2) \),

where
\[
H = 2k^2 - k(4\chi(O_S) + t - K_S^2 + 8) + 16\chi(O_S) + 2t - 2K_S^2
\]
and
\[
G = -2k + 4\chi(O_S) + t - K_S^2 + 8.
\]

\textbf{Proof:} From [R1, Propositions 2 and 3, a)] one gets:

(a) \( 2kl = -48 + 12l + 12k - 8\chi(O_S) + 4K_S^2 - 4t + \sum (r_i - 2)(r_i - 4) \)

(b) \( 2k + 2l = 8 + 4\chi(O_S) + t - K_S^2 + \sum (r_i - 2) \).

The result is obtained replacing (a) by (a)+(6−k)(b).

The next result is a fundamental tool in the proof of Proposition 9 below.

\textbf{Lemma 8.} Suppose that \( k > 8 \). With the above notation, we have

a) \( 2l \leq G + \frac{H}{k - r_m - 2} \), and

b) if \( r_m \) is obtained only from singularities of type \((r_m - 1, r_m - 1)\), then
\[
2l \leq G + \frac{H}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)}(2r_m - 6).
\]

\textbf{Proof:} The first statement follows from Proposition 7 and Proposition 5 a).

Next we prove b). By the assumptions, if \( x_i \) does not belong to a \((r_m - 1, r_m - 1)\) singularity, we have \( r_i < r_m \). Let \( n \geq 1 \) be the number of singularities of type \((r_m - 1, r_m - 1)\) and \( s \geq 0 \) be the number of singular points \( x_j \) of another type. As seen in Section 4, each singularity \((r_m - 1, r_m - 1)\) corresponds to two infinitely near singular points \( x_k, x_{k+1} \) with \( r_k = r_m - 2, r_{k+1} = r_m \). Therefore
\[
\sum_{i=1}^{2n+s} (r_i - 2) = n(2r_m - 6) + \sum_{j=1}^{s} (r_j - 2),
\]
with \( r_j < r_m \). Thus from Proposition 7 b) we get
\[
2l = G + n(2r_m - 6) + \sum_{j=1}^{s} (r_j - 2).
\]
By Proposition 7 a),

\[ H = n \left( (r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2) \right) + \sum_{j=1}^{s} (r_j - 2)(k - r_j - 2), \]

hence

\[ n = \frac{H - \sum_{j=1}^{s} (r_j - 2)(k - r_j - 2)}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)} \]

and then

\[ 2l = G + \frac{H - \sum_{j=1}^{s} (r_j - 2)(k - r_j - 2)}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)} (2r_m - 6) + \sum_{j=1}^{s} (r_j - 2). \quad (1) \]

Since \( r_j < r_m, \ j = 1, \ldots, s, \)

\[ (r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2) \leq (2r_m - 6)(k - r_j - 2). \]

This implies

\[ \sum_{j=1}^{s} (r_j - 2) \leq \sum_{j=1}^{s} \frac{(r_j - 2)(k - r_j - 2)(2r_m - 6)}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)} \]

and the result follows from [1]. \qed

The following proposition will allow us to give bounds for \( k \). Notice that, since \( \mathcal{F} \) is even and \( \mathcal{B}C_0 = l - \frac{ek}{2} \),

\[ k \equiv 0 \pmod{4} \quad \implies \quad l \equiv 0 \pmod{2}. \]

**Proposition 9.** In the conditions of Proposition 7, suppose that \( k > 8 \).

If \( k \equiv 0 \pmod{4} \), one of the following holds:

a) \( r_m = k/2 + 2, \ l = k + 2 \) and
\[ 4\chi(\mathcal{O}_S) + t - K_S^2 - 8)k \leq 16\chi(\mathcal{O}_S) - 16, \ \text{with} \ t \geq 2; \]

b) \( r_m = k/2 + 2, \ l \geq k + 4 \) and
\[ 4\chi(\mathcal{O}_S) + t - K_S^2 - 8)k^2 - 16\chi(\mathcal{O}_S)k + 32\chi(\mathcal{O}_S) \leq 0, \ \text{with} \ t \geq 2; \]

c) \( r_m = k/2, \ l = k - 2 \) and
\[ (4\chi(\mathcal{O}_S) + t - K_S^2 - 5)k^2 + (-48\chi(\mathcal{O}_S) - 8t + 8K_S^2 + 44)k + 160\chi(\mathcal{O}_S) + 16t - 16K_S^2 - 128 \leq 0, \ \text{with} \ t \geq 1, \]
or
\[ (4\chi(\mathcal{O}_S) + t - K_S^2 - 5)k^2 + (-48\chi(\mathcal{O}_S) - 8t + 8K_S^2 + 44)k + 160\chi(\mathcal{O}_S) + 16t - 16K_S^2 - 128 \leq 0, \ \text{with} \ t \geq 2; \]
d) \( r_m = k/2, l = k + j, j \geq 0, \) and
\[
(4\chi(O_S) + t - K_S^2 + 8 + 2j - 2n)k \leq 32\chi(O_S) + 4t - 4K_S^2 - 8n,
\]
with \( n \leq j + 7, \) where \( n \) is the number of points of multiplicity \( k/2. \)

e) \( r_m \leq k/2 - 2 \) and
\[
k \leq 5 + \sqrt{1 + 8\chi(O_S)}, \text{ or }
(4\chi(O_S) + t - K_S^2)k \leq 32\chi(O_S) + 4t - 4K_S^2.
\]

If \( k \equiv 2 \pmod{4}, \) one of the following holds:

f) \( r_m = k/2 + 1 \) and
\[
(4\chi(O_S) + t - K_S^2 - 2)k \leq 24\chi(O_S) + 2t - 2K_S^2 - 20, \text{ with } t \geq 1,
\]
or
\[
(4\chi(O_S) + t - K_S^2 - 8)k^2 + (-32\chi(O_S) - 4t + 4K_S^2 + 48)k +
80\chi(O_S) + 4t - 4K_S^2 - 96 \leq 0, \text{ with } t \geq 2;
\]
g) \( r_m \leq k/2 - 1 \) and
\[
k \leq 5 + \sqrt{1 + 8\chi(O_S)}, \text{ or }
(4\chi(O_S) + t - K_S^2 - 6)k \leq 24\chi(O_S) + 2t - 2K_S^2 - 28.
\]

Proof: Let \( H, G \) be as defined in Proposition 7 and let
\[
P_1(l, r_m, G, H, k) := (2l - G)(k - r_m - 2) - H,
\]
\[
P_2(l, r_m, G, H, k) := (2l - G)((r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)) - H(2r_m - 6).
\]

From Lemma 8
\[
P_1 \leq 0 \quad \text{and} \quad P_2 \leq 0.
\]

a) Let \( n \) be the number of \((k/2 + 1, k/2 + 1)\) points. From Proposition 5 b), d), \( n = 1 \) or 2. From Proposition 7 we have
\[
\sum (r_i - 2)(k - r_i - 2) = H' \quad \text{and} \quad 2l = G' + \sum (r_i - 2),
\]
where
\[
H' = H - n \left( k/2(k/2 - 4) + (k/2 - 2)^2 \right), \quad G' = G + n(k - 2)
\]
and \( r_i \leq k/2, \forall i. \)

The result follows from
\[
P_1(k + 2, k/2, G', H', k) \leq 0.
\]
b) From Proposition 5, there are at most \((k/2 + 1, k/2 + 1)\) singularities. The inequality

\[ P_2(k + 4, k/2 + 2, G, H, k) \leq 0 \]

gives the result.

c) Let \(n\) be the number of points of multiplicity \(k/2\) and \(m\) be the number of \((k/2 - 1, k/2 - 1)\) singularities. From Proposition 6, \(n = 0\) or \(1\). If \(n = 0\), then \(r_m = k/2\) implies \(m \geq 1\) (thus \(t \geq 1\)). From

\[ P_2(k - 2, k/2, G, H, k) \leq 0 \]

one gets the first inequality.

Suppose \(n = 1\). Notice that, as shown in the proof of Proposition 6, \(c\), the point of multiplicity \(k/2\) is obtained from the blow-up of \(\mathbb{P}^2\) at a point of type \((k/2 - 1, k/2 - 1)\). Hence \(t \geq 1\).

Let

\[ H' := H - (k/2 - 2)^2, \quad G' = G + k/2 - 2. \]

If \(m = 0\), then

\[ P_1(k - 2, k/2 - 2, G', H', k) \leq 0 \]

implies the second inequality.

If \(m > 0\), then

\[ P_2(k - 2, k/2, G', H', k) \leq 0 \]

gives the third inequality. In this case \(t \geq 2\).

d) Let \(j := l - k\) and let \(n\) be the number of points \(x_i\) (possibly infinitely near) such that \(r_i = k/2\). From Proposition 7 we have

\[ \sum (r_i - 2)(k - r_i - 2) = H' \quad \text{and} \quad 2l = G' + \sum (r_i - 2), \]

where

\[ H' = H - n(k/2 - 2)^2, \quad G' = G + n(k/2 - 2) \]

and \(r_i \leq k/2 - 2, \ \forall i\).

The inequality

\[ P_1(k + j, k/2 - 2, G', H', k) \leq 0 \]

gives

\[ (4\chi(\mathcal{O}_S) + t - K_S^2 + 8 + 2j - 2n) k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 8n. \]
It only remains to show that \( n \leq j + 7 \).

One can verify, using the double cover formulas (see e.g. [BHPV]), that \( n \geq j + 8 \) implies \( \chi(\mathcal{O}_S) < 1 \), except for \( n = 8 \), \( l = k \) and \( n = 10 \), \( k = 12 \), \( l = 14 \). We claim that in these cases \( K_S^2 \leq 0 \). This is impossible because \( S \) is of general type.

**Proof of the claim:**

From the double cover formulas one gets that \( \chi(\mathcal{O}_S) \leq 2 \) and there is at least a \((-2)\)-curve \( A \) contained in \( B \), otherwise \( K_S^2 \leq 0 \). One has

\[
B \equiv -\frac{k}{2}K_W + (l - k)\tilde{F} + \sum \left( \frac{k}{2} - r_i \right) E_i,
\]

where \( \tilde{F} \) is the total transform of \( F \) and each \( E_i \) is an exceptional divisor with self-intersection \(-1\). Since \( AB = -2 \), \( AK_W = 0 \), \( l \geq k \) and \( r_i \leq k/2 \forall i \), we have \( AE_i < 0 \) for some \( i \) such that \( r_i < k/2 \). The only possibility is the existence of a \((3,3)\)-point in \( \tilde{B} \) and \( \chi(\mathcal{O}_S) = 1 \). But the imposition of such a singularity in the branch locus decreases the self-intersection of the canonical divisor by 1.

e) From Proposition 6, b), \( l \geq k/2 + r_m - 2 \). Let

\[
f(r_m) := P_1(k/2 + r_m - 2, r_m, G, H, k).
\]

We have

\[
f(r_m) = -2r_m^2 + br_m + c \leq 0,
\]

where

\[
b = 4\chi(\mathcal{O}_S) + t - K_S^2 - k + 8
\]

and

\[
c = k^2 - 10k - 8\chi(\mathcal{O}_S) + 24.
\]

Suppose that \( c = f(0) > 0 \) (i.e. \( k > 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)} \)). Then \( f(r_m) \) has exactly one positive root \( x \). One has

\[
4x - b = \sqrt{b^2 + 8c}
\]

and \( k/2 - 2 \geq r_m \geq x \) implies that

\[
(4(k/2 - 2) - b)^2 \geq b^2 + 8c.
\]

This inequality gives the result.
f) Let \( n \) be the number of points of type \((k/2, k/2)\).

If \( n = 1 \), we proceed as in a).

If \( n > 1 \), the inequality is given by

\[
P_2(k, k/2 + 1, G, H, k) \leq 0.
\]

g) It is analogous to the proof of e). \( \square \)

6 Proof of main results

Proof of Theorem 1

Consider the parabola given by \( f(x) = ax^2 + bx + c \), with \( a > 0 \). If \( f(k) \leq 0 \), \( f(z) \geq 0 \) and \( z \geq -b/2a \) (the first coordinate of the vertex), then \( k \leq z \).

This fact and Proposition 3 imply that, if \( K^2_S < 4\chi(O_S) - 6 \), one of the following holds:

a) \( k \leq \frac{16\chi(O_S) - 16}{4\chi(O_S) - K^2_S - 6} \)

b) \( k \leq \frac{16\chi(O_S)}{4\chi(O_S) + t - K^2_S - 8}, t \geq 2 \)

c) \( k \leq 4 + \frac{16\chi(O_S)}{4\chi(O_S) + t - K^2_S - 4}, t \geq 1 \)

c’) \( k \leq 4 + \frac{16\chi(O_S) - 4}{4\chi(O_S) + t - K^2_S - 5}, t \geq 2 \)

d) \( k \leq 4 + \frac{16\chi(O_S) - 32}{4\chi(O_S) - K^2_S - 6} \)

e) \( k \leq 5 + \sqrt{1 + 8\chi(O_S)} \)

e’) \( k \leq 4 + \frac{16\chi(O_S)}{4\chi(O_S) - K^2_S} \)

f) \( k \leq 2 + \frac{16\chi(O_S) - 16}{4\chi(O_S) - K^2_S - 1} \)

f’) \( k \leq 2 + \frac{16\chi(O_S) - 16}{4\chi(O_S) + t - K^2_S - 8}, t \geq 2 \)

g) \( k \leq 5 + \sqrt{1 + 8\chi(O_S)} \)

g’) \( k \leq 2 + \frac{16\chi(O_S) - 16}{4\chi(O_S) - K^2_S} \)

We want to show that \( k \) is not greater than

\[
\max \left\{ \frac{16\chi(O_S)}{4\chi(O_S) - K^2_S - 6}, 4 + \frac{16\chi(O_S) - 32}{4\chi(O_S) - K^2_S - 6}, 4 + \frac{16\chi(O_S)}{4\chi(O_S) - K^2_S - 3}, 5 + \sqrt{1 + 8\chi(O_S)} \right\}.
\]

The result follows easily. Just notice that

\[
4\chi(O_S) - K^2_S - 6 \leq 8 \implies 2 + \frac{16\chi(O_S) - 16}{4\chi(O_S) - K^2_S} \leq \frac{16\chi(O_S)}{4\chi(O_S) - K^2_S - 6}
\]
and

\[ 4\chi(O_S) - K_S^2 - 6 \geq 8 \implies 2 + \frac{16\chi(O_S) - 16}{4\chi(O_S) - K_S^2 - 6} \leq 4 + \frac{16\chi(O_S) - 32}{4\chi(O_S) - K_S^2 - 6}. \]

\[ \square \]

**Proof of Proposition 2:**

Let \((\alpha), (\beta)\) be the equations of Proposition 7, a), b), respectively. One has that

\[ \frac{((\alpha)) + (k - 10)((\beta))}{8} \]

is equivalent to

\[ \frac{1}{8} \sum (r_i - 2)(8 - r_i) = 15 + K_S^2 - t - 3\chi(O_S) - \frac{1}{4}(k - 10)(l - 10) \quad (2) \]

and \((\beta) + (2)\) is equivalent to

\[ \chi(O_S) = 1 + \frac{1}{4}(k - 2)(l - 2) - \frac{1}{8} \sum r_i(r_i - 2). \quad (3) \]

Now it suffices to show that \(r_m \leq 8\).

Suppose that \(K_S^2 < 3\chi(O_S) - 6\).

From [X[24, Theorem 1] one gets that if \(\chi(O_S) \geq 54\), then \(S\) has a pencil of hyperelliptic curves of genus \(\leq 6\). In this case \(k \leq 14\), thus \(r_m \leq k/2 + 2\) implies \(r_m \leq 8\).

From the proof of Theorem [1] we obtain that if \(\chi(O_S) \leq 31\), then one of the possibilities below occur. In all cases \(r_m \leq 8\).

- **a)** \(\leq 16\), \(r_m < 8\);
- **b)** \(k < 16\), \(r_m < 8\);
- **c)** \(\leq 20\), \(r_m \leq k/2 - 2 \leq 8\);
- **d)** \(k \leq 18\), \(r_m = k/2 \leq 8\);
- **e)** \(k \leq 20\), \(r_m \leq k/2 - 2 \leq 8\);
- **e')** \(k \leq 16\), \(r_m \leq k/2 - 2 \leq 6\);
- **f)** \(k \leq 14\), \(r_m = k/2 + 1 \leq 8\);
- **f')** \(k \leq 16\), \(r_m = k/2 + 1 \leq 8\);
- **g)** \(k \leq 18\), \(r_m \leq k/2 - 1 \leq 8\);
- **g')** \(k \leq 14\), \(r_m \leq k/2 - 1 \leq 6\).

Suppose now that \(32 \leq \chi(O_S) \leq 53\). From Theorem [1] we get that \(k \leq 18\) or \(k \leq 5 + \sqrt{1 + 8\chi(O_S)}\). In this last case \(r_m \leq k/2 - 1\) (see Proposition [2, e), g)). Thus we have \(r_m \leq 18/2 + 2\) or \(r_m \leq 24/2 - 1\). Since \(r_m\) is even, \(r_m \leq 10\).

Let \(N_j\) be the number of points \(x_i\) such that \(r_i = j\). We have

\[ \sum (r_i - 2) \geq 8N_{10} + 6N_8 \]

and, from [2],

\[ 8N_{10} \geq (k - 10)(l - 10) - 32. \]
Using Proposition 7 b) and the assumption \( \chi(O_S) \geq 32 \), we obtain
\[
2l + 2k \geq 15 + (k - 10)(l - 10) + 6N_8.
\]
This is equivalent to
\[
(k - 12)(l - 12) \leq 29 - 6N_8.
\] (4)

Suppose \( r_m = 10 \). Then Propositions 5 and 6 give two possibilities:
· \( k = 16, l \geq k + 2 = 18 \), there is a singularity of type \( (9, 9) \) \((N_8 \geq 1)\);
· \( k \geq 18, l \geq k/2 + r_m - 2 \geq 17 \).

Both cases contradict (4). We conclude that \( r_m \leq 8 \). \( \square \)

**Proof of Theorem 3:**
First we claim that if \( A \) is a \((-2)\)-curve contained in \( B \), the image \( \overline{A} \) of \( A \) in \( \mathbb{F}_e \) does not intersect a negligible singularity of \( \overline{B} \), unless \( \overline{A} \) is the negative section of \( \mathbb{F}_1 \) and the only singularity of \( \overline{B} \) is a double point in \( C_0 \) (this corresponds to a smooth branch curve in \( \mathbb{P}^2 \)). In fact otherwise there is a \((-1)\)-curve \( E \) such that \( AE = 1 \) or 2. If \( AE = 1 \), then \( A + E \) can be contracted to a smooth point of the branch curve \( \overline{B} \subset \mathbb{F}_e \). This is impossible because the canonical resolution blows-up only singular points of \( \overline{B} \). Suppose \( AE = 2 \). The inverse image of \( A \) is a \((-1)\)-curve which contracts to a smooth point of \( S \). The inverse image of \( E \) is then contracted to a curve \( \overline{E} \) with arithmetic genus 1 and \( \overline{E}^2 = 2 \). We obtain from the adjunction formula that \( K_S \overline{E} = -2 \), which is a contradiction because \( S \) is of general type.

Recall that \( t := K_S^2 - K_{S_m}^2 \). The following holds:

1. \( l \geq k/2 \)
   (Because \( \overline{B}C_0 = l - ek/2 \geq -e \) and \( \overline{B}C_0 \) is even.);
2. \( l = k/2 \iff (t = 2 \land N_4 = N_6 = N_8 = 0) \)
   (In this case \( e = 1 \) and \( \overline{B}C_0 = 0 \).);
3. \( l = k/2 + 2 \iff (N_6 = N_8 = 0 \land t \geq N_4 \land (t = N_4 \lor N_4 > 1)) \)
   (If \( N_4 \neq 0 \), this corresponds to a branch curve in \( \mathbb{P}^2 \) with \( N_4 \) points of type \((3, 3)\) (see Proposition 6 c)).);
4. \( l = k - 2 \land t = 0 \implies k/2 \) even;
   (As in (1), \( l \geq ek/2 - e \), hence \( e \leq 2 \). We have \( e = 1 \) because \( t = 0 \), thus \( l \) even implies \( k/2 \) even.).
(5) \( l < k - 2 \implies l - k/2 \) even;  
(As in (1), \( l \geq ek/2 - e, \) thus \( e = 1 \) and then \( l - k/2 = BC_0 \) is even.)

(6) \( t = 1 \wedge N_4 = N_6 = N_8 = 0 \implies l = k - 2. \)
(If there are only negligible singularities, \( t = 1 \) is only possible if the negative section of \( F_2 \) is an isolated component of the branch locus.)

For given values of \( K_S^2 - 3\chi(O_S) \) and \( k, \) we want to choose the solution of the equation given in Proposition 2 b) which maximizes the value of \( \chi(O_S), \) given by the equation in Proposition 2 c). We can assume \( N_6 = N_8 = 0. \)

It suffices to compute the numerical possibilities for Proposition 2 b), c) which satisfy conditions (1), . . . , (6). We note the following:  

since \( k \geq 12, \) [X, Thm. 1] implies \( \chi(O_S) \leq 69, \) then Theorem 1 gives \( k \leq 28; \)

\( l \geq k/2, \) \( k \geq 12 \) and (2) imply \(-7 \geq K_S^2 - 3\chi(O_S) \geq -18 + t + N_4, \) thus \( K_S^2 - 3\chi(O_S) \geq -18, \) \( t \leq 11 \) and \( N_4 \leq 11. \)

A simple algorithm is available at \url{http://home.utad.pt/~crito/magma_code.html}

The existence is easy to verify. All cases can be constructed as double covers of \( \mathbb{P}^2, F_0, F_1 \) or \( F_2. \) The table below contains information about \( l \) or the degree of the branch curve in \( \mathbb{P}^2 \) and about the singularities of the branch curve, if any.

| \( K^2 - 3\chi \) | \( g \) | -7 | -8 | -9 | -10 | -11 |
|------------------|--------|---|---|---|---|---|
| 5                | \( F_0, l = 26 \) | \( F_0, l = 24 \) | \( F_0, l = 22 \) | \( F_1, l = 20 \) | \( F_0, l = 18 \) |
| 6                | \( F_0, l = 18 \) | \( F_1, l = 17 \) | \( F_0, l = 16 \) | \( F_1, l = 15 \) | \( F_0, l = 14 \) |
| 7                | \( F_1, l = 14, (3,3) \) | \( F_2, l = 14 \) | \( F_1, l = 14 \) | \( F_1, l = 12, (3,3) \) | \( F_1, l = 12, (4) \) |
| 8                | \( F_1, l = 13, (3,3) \) | \( F_1, l = 13, (4) \) | \( F_1, l = 13 \) | \( \mathbb{P}^2, 20, (3,3) \) |
| 9                | \( \mathbb{P}^2, 22, (3,3) \) | \( F_1, l = 12 \) |
| 10               | \( \mathbb{P}^2, 22 \) |

| \( K^2 - 3\chi \) | \( g \) | -12 | -13 | -14 | -15 | -16 |
|------------------|--------|----|----|----|----|----|
| 5                | \( F_0, l = 16 \) | \( F_0, l = 14 \) | \( F_0, l = 12 \) | \( F_1, l = 10 \) | \( F_1, l = 8 \) |
| 6                | \( F_1, l = 13 \) | \( F_1, l = 11, (4) \) | \( F_1, l = 11 \) | \( F_1, l = 9 \) |
| 7                | \( F_1, l = 12 \) | \( \mathbb{P}^2, 18, (3,3) \) | \( F_1, l = 10 \) | \( \mathbb{P}^2, 16 \) |
| 8                | \( F_1, l = 11 \) | \( \mathbb{P}^2, 18 \) |
| 9                | \( \mathbb{P}^2, 20 \) |
| 10               |            |

\( \square \)
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