UNIPOTENT MORPHISMS

DANIEL BRAGG, JACK HALL, AND SIDDHARTH MATHUR

ABSTRACT. We introduce the theory of unipotent morphisms of algebraic stacks and prove a surprising local to global principle for a class of vector bundles. Two sample applications of our methods are the following: (1) a unipotent analogue of Gabber’s Theorem for torsion $G_m$-gerbes and (2) smooth Deligne–Mumford stacks with quasi-projective coarse spaces satisfy the resolution property in positive characteristic. Our main tool is a descent result for flags, which we prove using results of Schäppi.

1. INTRODUCTION

In the early 1990s, Gabber answered a question of Grothendieck for quasi-compact schemes admitting an ample line bundle: every cohomological Brauer class on such a scheme is represented by an Azumaya algebra [Gab81, dJ03]. This result has since been reinterpreted in terms of the resolution property for algebraic stacks (i.e., every quasi-coherent sheaf is a quotient of a direct sum of vector bundles) and gerbes [EHKV01, KV04, Tot04, Gro17, Mat21]. In this language, Gabber’s result can be restated as follows.

**Theorem (Gabber).** Let $X$ be a quasi-compact scheme that admits an ample line bundle. Let $\mathcal{X} \to X$ be a $G_m$-gerbe. If the cohomology class $[\mathcal{X}] \in H^2(X, G_m)$ of $\mathcal{X}$ is torsion, then $\mathcal{X}$ is a global quotient stack (see Definition 2.1) and has the resolution property.

Using quasi-projective methods and Gabber’s Theorem, Kresch and Vistoli show that smooth, separated and generically tame Deligne–Mumford stacks with quasi-projective coarse moduli spaces have the resolution property [KV04]. It has long been a challenge to produce non-trivial vector bundles without such hypotheses.

In this paper, we introduce new methods to construct non-trivial vector bundles on schemes, algebraic spaces and algebraic stacks. Our key idea is to leverage the abundance of cohomology in unipotent settings. A sample application of our results is the following additive analog of Gabber’s result (also see Theorem 6.3).

**Theorem A.** Let $X$ be a quasi-compact algebraic stack with affine diagonal and the resolution property. If $\mathcal{G} \to X$ is a $G_a$-gerbe, then $\mathcal{G}$ has the resolution property.

In particular, this implies that $G_a$-gerbes over smooth separated schemes have the resolution property. Note that the analogous multiplicative statement is still unknown, even in dimension three. We can also extend Kresch–Vistoli’s result to the generically wild setting.

**Theorem B.** Let $\mathcal{X}$ be a smooth and separated Deligne–Mumford stack of finite type over a field $k$. Let $\pi: \mathcal{X} \to X$ be the associated coarse moduli space. If $X$ is quasi-projective over $\text{Spec} \ k$, then $\mathcal{X}$ has the resolution property.

We expect the key new technical input for the above results to be of general interest: it states that one may descend certain vector bundles along flat morphisms after locally taking a direct sum (see Theorem 4.8 for a more general statement).

Date: January 22, 2025.
Theorem C. Let \( X \to X \) be a morphism of quasi-compact and quasi-separated algebraic stacks. Suppose that \( X \) has affine diagonal and satisfies the resolution property. If there is a faithfully flat morphism \( X' \to X \), where \( X' \) is affine, and a vector bundle \( V' \) on \( X' = X \times_X X' \) that admits a filtration

\[
0 = V'_0 \subseteq V'_1 \subseteq \cdots \subseteq V'_n = V'
\]

with \( V'_i/V'_{i-1} \simeq \mathcal{O}_{X'} \) for all \( i \) (i.e. \( V' \) is trivially graded), then there is a vector bundle \( V \) on \( X \) and a split surjection \( V \to V' \).

In fact, our results apply in the broader setting of unipotent morphisms, a notion that we develop and characterize in §5. Roughly speaking, the theory is modeled on gerbes with unipotent stabilizers and algebraic stacks of the form \([Z/U_n]\), where \( Z \) is an algebraic space and \( U_n \) is the unipotent subgroup of \( GL_n \) consisting of unitriangular matrices. Compare with [EHKV01, Tot04, Gro17], where quotients of the form \([Z/GL_n]\) and the resolution property are considered. In fact, we present a unipotent enrichment of their results (see Theorem 5.4) and, better still, we prove a striking descent statement. Indeed, Theorem C reveals the following local to global principle: a locally unipotent morphism over a base with enough flags is globally unipotent (see Theorem 5.5).

These notions are closely related to many foundational questions of independent interest. For example, over a field \( k \) a unipotent group scheme is a closed subgroup of \( U_{n,k} \) (see [GP11b, Thm. XVII.3.5] and Remark 5.14 for several other characterizations). It is natural to ask: to what extent is this true over a general base?

Question 1.1. Let \( G \to S \) be a group scheme with unipotent geometric fibers. Is there an embedding \( G \to U_{n,S} \) for some \( n \geq 1 \)? Is this true locally on \( S \)?

As a consequence of our methods, if \( S \) is a regular separated scheme or admits an ample line bundle, then the existence of a flat-local embedding in \( U_n \) implies the existence of a Zariski-local embedding of \( G \) in \( U_n \) (see Corollary 5.18 for a more general statement). In Section 5.3 we refine Question 1.1 and explain some cases where the answer is positive (see Examples 5.22 and 5.23 and one case, in positive characteristic, where it is negative (Example 5.21(1))). One may also compare Question 1.1 with a question of Conrad on when smooth affine group schemes can be embedded into \( GL_n \) ([Con10]).

Unipotent morphisms are closely related to the notion of a faithful moduli space, due to Alper [Alp17]. This is a morphism of algebraic stacks \( f : \mathcal{X} \to X \) such that

1. the natural map \( \mathcal{O}_X \to f_*\mathcal{O}_\mathcal{X} \) is an isomorphism (i.e. \( f \) is Stein); and
2. if \( F \) is quasi-coherent on \( \mathcal{X} \) and \( f_*F = 0 \), then \( F = 0 \).

Faithful moduli spaces should help with the analysis of moduli stacks with only unipotent stabilizers (compare with good or adequate moduli spaces in [Alp13, Alp17]) that appear in the unstable locus of GIT quotients. In contrast to good moduli spaces, however, faithful moduli spaces are rarely compatible with base change. This makes them difficult to reduce to simple cases where we can apply deformation theory. Faithful moduli spaces are also stable under composition.

We show that faithful moduli spaces and locally unipotent morphisms have unipotent stabilizers (Propositions 5.13 and 7.10) and that faithful moduli spaces are generically gerbes for a unipotent group (Proposition 7.12). We also show that certain Stein unipotent morphisms are faithful moduli spaces (Proposition 7.11).

Acknowledgments. We would like to thank Jarod Alper, Brian Conrad, Andrea Di Lorenzo, Aise Johan de Jong, Andrew Kresch, Nikolai Kuhn, Max Lieblich, Martin Olsson, Stefan Schröer, Tuomas Tajakka, and Angelo Vistoli for helpful comments. We are especially thankful to David Rydh for carefully reading an earlier draft and providing many invaluable suggestions. We would also like to thank
the anonymous referees for a number of helpful suggestions. The first author was supported by the National Science Foundation under NSF Postdoctoral Research Fellowship DMS-1902875. The second author was partially supported by the Australian Research Council DP210103397 and FT210100405. The third author conducted part of this research in the framework of the research training group GRK 2240: Algebro-geometric Methods in Algebra, Arithmetic and Topology, which is funded by the Deutsche Forschungsgemeinschaft. The third author was also supported by the Swedish Research Council under grant no. 2016-06596 while the author was in residence at Institut Mittag-Leffler in Djursholm, Sweden during the Fall of 2021. The third author was also supported by FONDECYT Regular grant No. 1230402.

2. Vector bundles, global quotients, and the resolution property

We briefly recall a circle of ideas from [EHKV01, Tot04, Gro17]. We also make some minor refinements, which will be useful in the present article. To agree with the conventions of [Gro17, §5], a vector bundle on $X$ will denote a locally free $\mathcal{O}_X$-module of finite constant rank.

Let $X$ be an algebraic stack and let $n$ be a non-negative integer. If $V$ is a vector bundle on $X$ of rank $n$, there is an associated frame bundle $\text{Fr}(V) = \text{Isom}_X(\mathcal{O}_X^{\oplus n}, V)$. Then $\text{Fr}(V) \to X$ is a $\text{GL}_n$-torsor and sits in the following 2-cartesian diagram:

$$
\begin{array}{ccc}
\text{Fr}(V) & \longrightarrow & \text{Spec } \mathbb{Z} \\
\downarrow & & \downarrow \\
X & \longrightarrow & B\text{GL}_n.
\end{array}
$$

It is well-known that this association induces an equivalence of categories between vector bundles on $X$ of rank $n$ and $\text{GL}_n$-torsors.

**Definition 2.1.** Let $X$ be an algebraic stack. We say that $X$ is a global quotient if $X \cong [Z/\text{GL}_n]$, where $Z$ is a quasi-compact and quasi-separated algebraic space with an action of $\text{GL}_n$ for some non-negative integer $n$. If $V$ denotes the vector bundle of rank $n$ associated to the $\text{GL}_n$-torsor $Z \to X$, we say that $V$ is faithful.

A morphism of algebraic stacks $f : X \to S$ is a global quotient if there is a vector bundle $V$ of rank $n$ on $X$ with frame bundle that is quasi-compact, quasi-separated, and representable over $S$. In this case, we say that $V$ is $f$-faithful.

**Remark 2.2.** A vector bundle $V$ on $X$ is easily seen to be $f$-faithful for $f : X \to S$ if and only if the relative stabilizer groups of $f$ act faithfully on $V$ at every geometric point of $X$. In particular, the relative stabilizer groups are affine [GP11a, Cor. VI_B.1.4.2]. Moreover, if $X \overset{f}{\to} S \overset{g}{\to} T$ are quasi-compact and quasi-separated morphisms of algebraic stacks and $V$ is $(g \circ f)$-faithful, then $V$ is $f$-faithful. Also, $X$ is a global quotient if and only if the morphism $X \to \text{Spec } \mathbb{Z}$ is a global quotient.

**Remark 2.3.** It is natural to ask if algebraic stacks such as $X_1 = \bigsqcup_{n\geq 1} B\text{GL}_n, \mathbb{Z}$ or $X_2 = B\mathbb{Z}$, where $\mathbb{Z}$ denotes the constant group scheme associated to the integers, should be considered global quotients. There is certainly a locally free sheaf on $X_1$ with a faithful action, but it has non-constant and unbounded rank; also, $X_2$ admits a vector bundle of rank 2 with a faithful action whose frame is quasi-compact but not quasi-separated. We exclude these as we are unable to shed much light on them and also to adhere to the conventions of [Gro17, §5].
Let \( \mathcal{A} \) be an abelian category. Let \( \Lambda \) be a set of objects of \( \mathcal{A} \). Recall that \( \Lambda \) generates \( \mathcal{A} \) (or \( \Lambda \) is generating) if the functor

\[ H_\Lambda: \mathcal{A} \to \text{Ab}: x \mapsto \prod_{a \in \Lambda} \text{Hom}_\mathcal{A}(a, x). \]

is faithful. If \( \mathcal{A} \) is closed under small coproducts, then \( \Lambda \) is generating if and only if every \( x \in \mathcal{A} \) admits a presentation of the form \( \bigoplus_{a \in \Lambda} (a \oplus I_a) \to x \).

**Remark 2.4.** Let \( \mathcal{A} \) be an abelian category and \( \Lambda \) a set of objects of \( \mathcal{A} \). Then \( H_\Lambda \) is conservative (i.e., if \( f: x \to x' \) in \( \mathcal{A} \) and \( H_\Lambda(f) \) is bijective, then \( f \) is an isomorphism) if and only if \( \Lambda \) generates \( \mathcal{A} \). Note that while \( H_\Lambda \) conservative implies that it is zero-reflecting (i.e., \( H_\Lambda(x) = 0 \) implies that \( x = 0 \)), the converse does not hold in general—it does when the \( a \in \Lambda \) are all \( \mathcal{A} \)-projective. For a counterexample, consider the affine line with the doubled origin; then \( H_{\{\text{pt}\}} \) is zero-reflecting but not conservative.

**Example 2.5.** If \( X \) is an algebraic stack, then the abelian category \( \text{QCoh}(X) \) always admits a set of generators [Sta Tag 0781]. If \( X \) is quasi-compact and quasi-separated, then a set of generators can always be found amongst the \( \mathcal{O}_X \)-modules of finite presentation [Ryd23]. Note this result is much simpler if \( X \) is noetherian [LMB00, Prop. 15.4] or has quasi-finite and separated diagonal [Ryd15].

The resolution property, which we refine and recall below, is about algebraic stacks that have a generating set of vector bundles.

**Definition 2.6.** Let \( X \) be an algebraic stack. Let \( V \) be a set of vector bundles on \( X \). We say that \( X \) has the \( V \)-resolution property if it is quasi-compact, quasi-separated, and \( V \) generates \( \text{QCoh}(X) \). Following [Gro17] Def. 5.1, we say that \( X \) has the \( V \)-resolution property if it has the \( V \)-resolution property for some set \( V \).

**Remark 2.7.** Fix a noetherian algebraic stack \( X \) and let \( V \) denote the set of isomorphism classes of vector bundles on \( X \). Then \( X \) has the \( V \)-resolution property if and only if every coherent sheaf on \( X \) is the quotient of a locally free sheaf of finite rank (see [Gro17] Rem. 5.2).

Just as in [Gro17], we will be interested in a relative version of the resolution property, so we recall the following definition.

**Definition 2.8.** ([Gro17] Def. 2.7]) Let \( f: X \to S \) be a morphism of algebraic stacks. Let \( V \) be a set of finitely presented quasi-coherent sheaves on \( X \). We say that \( V \) is \( f \)-generating if \( f \) is quasi-compact and quasi-separated and there is a set of (not necessarily finitely presented) quasi-coherent \( \mathcal{O}_S \)-modules \( W \) such that \( V \otimes_{\mathcal{O}_X} f^* W = \{ V \otimes_{\mathcal{O}_S} f^* W : V \in V, W \in W \} \) generates \( \text{QCoh}(X) \). We say that \( V \) is universally \( f \)-generating if for every morphism of algebraic stacks \( s: S' \to S \), the set \( s^* V = \{ s^* V : V \in V \} \), where \( s': X' = X \times_S S' \to X \) and \( f': X' \to S' \) denote the projections, is \( f' \)-generating.

**Remark 2.9.** Note that \( V \) is universally \( f \)-generating if and only if it is so on an fpqc covering of the target (see, [Gro17] Prop. 2.8 (iii))). Also, if \( V \) is \( f \)-generating, then it remains so after quasi-affine base change (combine [Gro17] Prop. 2.8 (ii),(v),(vi)]). In particular, if \( S \) has quasi-affine diagonal, then \( V \) is \( f \)-generating if and only if it is universally \( f \)-generating (see [Gro17] Cor. 2.10]). Ignoring set-theoretic issues, Remark 2.5 implies that we can take \( W = \text{QCoh}(S) \) in the definition of \( f \)-generating.

**Definition 2.10.** Let \( f: X \to S \) be a morphism of algebraic stacks. If \( V \) is a set of vector bundles on \( X \), we say that \( f \) has the \( V \)-resolution property if \( V \) is universally \( f \)-generating. Following [Gro17] Def. 5.1, we say that \( f \) has the \( V \)-resolution property
if \( f \) has the \( V \)-resolution property for some \( V \). If such a \( V \) arises as a set of submodules of the polynomials in a fixed vector bundle \( V \) and its dual that are split by restriction to the frame bundle of \( V \), then we say that \( V \) is an \( f \)-tensor generator \cite{Gro17} Defn. 6.1]. Such a \( V \) is \( f \)-R-faithful if it is also \( f \)-faithful (note that \( f \)-tensor-generators are automatically \( f \)-faithful if \( f \) has relatively affine stabilizers by \cite{Gro17} Thm. 6.4]).

**Remark 2.11.** An algebraic stack \( X \) has the \( V \)-resolution property if and only if the morphism \( X \to \text{Spec} \, Z \) has the \( V \)-resolution property. In general, if \( Y \) has the \( V' \)-resolution property and \( f: X \to Y \) is a morphism of algebraic stacks with the \( V \)-resolution property, then \( X \) has the \( V \circ \sigma_X f^* V' \)-resolution property \cite{Gro17} Prop. 2.8(v)].

**Example 2.12.** Let \( A \) be an abelian scheme of positive dimension over a field \( k \). If \( g: \text{Spec} \, k \to BA \) denotes the standard covering, then \( \mathcal{O}_{\text{Spec} \, k} \) is \( g \)-generating but not universally \( g \)-generating. Also, \( g^* \) induces an equivalence \( \text{QCoh}(BA) \cong \text{QCoh}(\text{Spec} \, k) \), so \( f: BA \to \text{Spec} \, k \) has the resolution property with \( \mathcal{O}_{BA} \) as a \( f \)-tensor-generator. However, \( BA \) admits no \( f \)-faithful vector bundles.

**Example 2.13.** If \( X \) admits an ample family \( L_1, \ldots, L_r \) of line bundles \cite{Sta}, then it satisfies the \( V \)-resolution property \cite{Gro17} Ex. 5.9(i)], where \( V = \{ L_{i_1}^m : m \leq 0, i_1 = 1, \ldots, r \} \). In dimensions two and above, there are proper schemes with no nontrivial line bundles. However, two-dimensional separated algebraic spaces always have enough vector bundles (see \cite{SV04} Thm. 2.1], \cite{Gro12} Thm. 5.2], and \cite{Mat21} Thm. 41]). Nothing is known for smooth separated algebraic spaces or normal separated schemes over a field in dimension \( \geq 3 \), except that they enjoy the resolution property after removing a closed subset of codimension \( \geq 3 \) \cite{MS23}.

**Example 2.14.** For algebraic stacks the situation is more complicated. Kresch and Vistoli showed that a Deligne–Mumford stack which is smooth, generically tame and separated over a field satisfies the resolution property if its coarse moduli space is quasi-projective \cite{KV04} Thm. 1.3]. The case of 2-dimensional normal tame algebraic stacks with finite diagonal over a field was settled in \cite{Mat21} Thm. 1]. There are also non-torsion (and hence, non-regular) \( \mathbb{G}_m \)-gerbes which do not have the resolution property \cite{Gro05} Rem. 1.11b]. As far as the authors are aware, there is no known example of a separated or smooth algebraic stack with affine diagonal which does not have the resolution property.

Surprisingly, Totaro and Gross \cite{Tot04} Thm. 1.1] and \cite{Gro17} Thm. 1.1, Thm. 6.10] were able to relate quotient stacks and the resolution property. We have the following refinement, whose proof is the same as Gross’, but we just keep track of the generating set.

**Theorem 2.15.** Let \( f: X \to S \) be a morphism of quasi-compact and quasi-separated algebraic stacks. Assume that \( f \) has affine stabilizers.

1. Let \( V \) be a set of vector bundles on \( X \) that is closed under direct sums. If \( V \) is universally \( f \)-generating, then there is a vector bundle \( V \in V \) of rank \( n \) on \( X \) whose frame bundle is quasi-affine over \( S \).
2. Let \( V \) be a vector bundle on \( X \). Then \( V \) has frame bundle quasi-affine over \( S \) if and only if \( V \) is an \( f \)-tensor generator.

In particular, a quasi-compact and quasi-separated algebraic stack with affine stabilizers at closed points has the resolution property if and only if it can be written in the form \([U/\text{GL}_n]\), where \( U \) is a quasi-affine scheme.
Proof. Claim (2) is \cite{Gro17} Thm. 6.4. For (1): consider the natural inverse system of \(X\)-stacks formed by

\[ F_J : \prod_{W \in J} \text{Fr}(W) \to X, \]

where \(J \subset V\) is a finite subset. Since each frame bundle is affine over \(X\), the transition maps in the inverse system are affine. Hence, the inverse limit is an algebraic stack \(p : F \to X\), which is affine over \(X\). Since \(V\) is generating for \(f\) and every vector bundle in \(V\) becomes trivial when restricted to \(F\), it follows that \(\mathcal{O}_F\) is a tensor generator for the morphism \(f \circ \mathcal{O}_F : F \to X \to S\). Thus, the morphism \(f \circ \mathcal{O}_F\) is quasi-affine. By absolute approximation \cite{Gro17} Prop. 4.1. By absolute approximation \cite{Ryd15} Thm. C(i)], there is a finite subset \(J \subset V\) such that \(F_J \to S\) is quasi-affine. By \cite{Ryd09} Lem. 1.1], the frame bundle of \(V = \bigoplus_{W \in J} W \in V\) is quasi-affine. For the final claim, it suffices to show that an algebraic stack with the resolution property and affine stabilizers at closed points, has affine stabilizers everywhere. This is just \cite{Gro17} Lem. 5.15.

If a quasi-compact and quasi-separated scheme has an ample line bundle, then it can be written in the form \([U/GL_2]\), where \(U\) is a quasi-affine scheme. From this perspective, it is natural to view the resolution property as a higher-dimensional analogue of quasi-projectivity. Note that the converse does not hold (e.g., take \(U = \mathbf{A}^2 - \{0\}\) with \(GL_1\) acting with weights \((1, -1)\); then \([U/GL_1]\) is the line with the doubled-origin, which is not separated, so does not admit an ample line bundle—but it does admit an ample family of line bundles).

**Definition 2.16.** Let \(S\) be an algebraic stack. We say that a morphism \(G \to S\) is a **group** if it is a group object in representable morphisms over \(S\). If \(G \to S\) is a group that is flat and of finite presentation, then we say that it is

1. **embeddable** if it admits a group monomorphism to \(GL(E)\) for some vector bundle \(E\) of rank \(n\) on \(S\);
2. **R-embeddable** if it admits a group monomorphism to \(GL(E)\) for some vector bundle \(E\) of rank \(n\) on \(S\) such that the quotient \(GL(E)/G\) is quasi-affine over \(S\) (note that this implies that \(G \to GL(E)\) is a closed immersion).

**Example 2.17.** An embedded group is not necessarily R-embedded, even over a field. Indeed, if \(B_2, C \subset GL_2, C\) is a Borel subgroup, then \(GL_2, C/B_2, C \cong \mathbf{P}^1_C\).

**Remark 2.18.** If \(G \to S\) is a group that is flat and of finite presentation, then \(G \to S\) is embeddable if and only if the structure morphism of the classifying stack \(BG \to S\) is a global quotient. Indeed, if \(BG \to S\) is a global quotient, then there is a vector bundle \(E\) on \(BG\) whose total space is representable over \(S\). This gives a representable \(S\)-morphism \(BG \to BGL_{\text{rank} E}\); the induced map on inertia groups gives a monomorphism \(G \hookrightarrow GL(E')\) for some vector bundle \(E'\) on \(S\). The converse is similar. In particular, \(G \to S\) is embeddable if and only if \(S\) has affine fibers, then it is R-embeddable if and only if \(BG \to S\) has the resolution property (Theorem 2.15). In particular, such a \(G \to S\) is R-embeddable if and only if \(S\) has a vector bundle \(E\) on \(S\) with a faithful action of \(G\). We call these **faithful** \(G\)-representations. If, in addition, \(G \to S\) has affine fibers, then it is R-embeddable if and only if \(BG \to S\) has the resolution property (Theorem 2.15). In particular, such a \(G \to S\) is R-embeddable if and only if there is a vector bundle \(E\) on \(S\) with a faithful action such that the quotient \(GL(E)/G \to S\) is a quasi-affine morphism. We will call these \(G\)-representations **R-faithful**.

We expect the following result to be well-known to experts. Over a field, it is due to Rosenlicht \cite{Ros61} Thm. 3.

**Proposition 2.19.** Let \(S\) be an algebraic stack.

1. The quotient morphism \(GL_{n,S}/U_{n,S} \to S\) is quasi-affine.
Let $G \subseteq U_{n,S}$ be a closed subgroup that is flat and of finite presentation over $S$.

(2) If $S$ is a normal scheme and the quotient $U_{n,S}/G$ is representable by a scheme (e.g. if $S$ is Dedekind or the spectrum of a field, see [GP11a Rem. VI.B.9.3(b)]), then the quotient $U_{n,S}/G \to S$ is quasi-affine.

(3) If $G \to S$ is finite, then the quotient $U_{n,S}/G \to S$ is affine.

Proof. Since the inclusion $U_{n,S} \subseteq GL_{n,S}$ is defined over $Spec\, \mathbb{Z}$, in [1] it suffices to assume $S = Spec\, \mathbb{Z}$. In particular, we may assume that $S$ is Dedekind. We will now prove that both of the homogenous spaces $K \subseteq GL_{n,S}/U_{n,S}$ are quasi-affine over $S$ using [Ray70 Thm. VII.2.1], thereby establishing both [1] and [2]. Since $GL_{n,S}$ and $U_{n,S}$ are both smooth with connected fibers, it suffices to show that the abelian groups $Pic(U_{n,k(s)}/G_{k(s)})$ and $Pic(GL_{n,k(s)}/U_{n,k(s)})$ are torsion.

To this end, first note that the cohomology of the complex

$$\text{Hom}(K, G_m) \to \text{Pic}(H/K) \to \text{Pic}(H)$$

is torsion for any inclusion of finite-type group schemes $K \subseteq H$ over a field by [Ray70 Thm. VII.1.5]. Moreover, by [GP08 Prop. XVII.2.4(ii)] we always have $\text{Hom}(K, G_m) = 0$ for every unipotent group scheme $K$ over a field. Further, the Picard groups of $U_{n,k(s)}$ and $GL_{n,k(s)}$ both vanish. Indeed, in the former case the underlying scheme is isomorphic to affine space, and in the latter case it is a principal open of affine space. Thus, the two outer terms in the complex are zero in both of our cases and it follows that $Pic(U_{n,k(s)}/G_{k(s)})$ and $Pic(GL_{n,k(s)}/U_{n,k(s)})$ are torsion, as desired.

For [3]: the morphism $U_{n,S} \to U_{n,S}/G$ is finite flat. Since $U_{n,S} \to S$ is affine, the same is true for $U_{n,S}/G$ (e.g. [Ryd15 Thm. 8.1]). □

Proposition 2.20. Let $f: X \to S$ be a quasi-compact and quasi-separated morphism of algebraic stacks.

(1) Let $W$ and $V$ be vector bundles on $X$. If the frame bundle $F_W$ is representable (resp., quasi-affine) over $S$, then the frame bundle of $V \oplus W$ is also representable (resp., quasi-affine) over $S$.

(2) If $f$ is adequately affine (e.g. a gerbe banded by a reductive group scheme, a good/adequate moduli space morphism, or a coarse moduli space morphism [Alp14 Def. 4.1.1]), then every $f$-faithful vector bundle $V$ on $X$ has a frame bundle which is affine over $S$.

Proof. For the representability part of [1]: the hypothesis implies that the relative stabilizers act faithfully on the fibers of $W$ and therefore they must act faithfully on the fibers of $V \oplus W$, as desired. For the latter claim, see [Ryd09 Lem. 1.1]. □ For [2]: $F_V \to S$ is representable and adequately affine, so affine [Alp14 Thm. 4.3.1]. □

3. Schäppi’s Theorem

Let $A$ be a ring and let $M$ be a flat $A$-module. Lazard’s theorem [Laz64] tells us that $M$ is a filtered colimit of free $A$-modules of finite rank. Extending this result to the non-affine situation is surprisingly subtle [EAO13]. Nonetheless, Schäppi recently proved a remarkable Lazard-type theorem for a restricted but very useful class of flat modules that arise in algebraic geometry [Sch17 Thm. 1.3.1].

Schäppi approaches his result through comodules over flat Hopf algebroids. Since Schäppi’s result is crucial for our article and ought to be better known amongst algebraic geometers, we have translated his category-theoretic proof into a direct proof for algebraic stacks. We have made some simple but algebra-geometrically natural generalizations to his hypotheses. Note that while non-affine schemes tend to not have interesting projective objects, there are interesting algebraic stacks (e.g., those with affine tame [AOV08] or good [Alp13] moduli spaces) that do.
Theorem 3.1 (Schäppi). Let \( f: Y \to X \) be a flat morphism of quasi-compact and quasi-separated algebraic stacks. Let \( \mathbf{V} \) be a set of isomorphism classes of vector bundles on \( X \). Let \( M \) be a vector bundle on \( Y \) that is a projective object of \( \text{QCoh}(\mathbf{V}) \).

If \( X \) has the \( \mathbf{V} \)-resolution property, then \( f_*M \) is a filtered colimit of finite direct sums of objects of \( \mathbf{V} \) and their duals. In particular, if \( Y \) is cohomologically affine (e.g., an affine scheme), then \( f_*\mathbf{V} \) is a filtered colimit of vector bundles.

Proof. We may replace \( \mathbf{V} \) by the set of all finite direct sums of objects of \( \mathbf{V} \) and their duals. Let \( G \) be a quasi-coherent \( \mathcal{O}_X \)-module. Consider the category \( \mathcal{V}_G \) whose objects are pairs \((H, \eta)\), where \( H \in \mathbf{V} \) and \( \eta: H \to G \) is an \( \mathcal{O}_X \)-module homomorphism. A morphism \( h: (H, \eta) \to (H', \eta') \) in \( \mathcal{V}_G \) is an \( \mathcal{O}_X \)-module homomorphism \( h: H \to H' \) such that \( \eta = \eta' \circ h \). Consider the functor \( \mu_G: \mathcal{V}_G \to \text{QCoh}(X) \) that sends \((H, \eta)\) to \( H \). It remains to establish the following two claims.

Claim 3.1.1. Every pair of objects in \( \mathcal{V}_G \) has an upper bound and \( \text{colim}(\mu_G) \simeq G \).

Proof. If \((H_1, \eta_1)\) and \((H_2, \eta_2)\) \( \in \mathcal{V}_G \), then \((H_1 \oplus H_2, \eta_1 \oplus \eta_2) \in \mathcal{V}_G \). This proves the existence of upper bounds for every pair. We now prove that \( \text{colim}(\mu_G) \simeq G \). To do this, consider a quasi-coherent \( \mathcal{O}_X \)-module \( N \) together with \( \mathcal{O}_X \)-module homomorphisms \( \nu_{(H, \eta)}: H \to N \) for every \((H, \eta) \in \mathcal{V}_G \) such that if \( h: (H, \eta) \to (H', \eta') \) is a morphism in \( \mathcal{V}_G \), then \( \nu_{(H, \eta)} = \nu_{(H', \eta')} \circ h \). It suffices to show that there is a uniquely induced morphism \( \nu: G \to N \) such that \( \nu_{(H, \eta)} = \nu \circ \eta \) for every \((H, \eta) \in \mathcal{V}_G \). Since \( X \) has the \( \mathbf{V} \)-resolution property, \( G \) admits a presentation:

\[
\bigoplus_{j \in J} P_j \xrightarrow{\oplus p_j} \bigoplus_{(H, \eta) \in \mathcal{V}_G} H \xrightarrow{\oplus (H, \eta) \in \mathcal{V}_G \eta} G \to 0,
\]

where \( P_j \) belongs to \( \mathbf{V} \) for all \( j \in J \). Let \( j \in J \) and note that the morphism \( p_j: P_j \to \oplus_{(H, \eta) \in \mathcal{V}_G} H \) factors as

\[
P_j \xrightarrow{\bar{p}_j} \bigoplus_{(H, \eta) \in I_j} H \subseteq \bigoplus_{(H, \eta) \in \mathcal{V}_G} H,
\]

where \( I_j \subseteq \mathcal{V}_G \) is finite. Let \( \bar{Q}_j = \oplus_{(H, \eta) \in I_j} H \) and let \( \bar{q}_j = \oplus_{(H, \eta) \in I_j} \eta \): \( \bar{Q}_j \to G \) be the resulting morphism. Then \( \bar{q}_j \circ \bar{p}_j = 0 \) and so \( \bar{p}_j: (P_j, 0) \to (\bar{Q}_j, \bar{q}_j) \) is a morphism in \( \mathcal{V}_G \). Hence, \( \nu_{(\bar{Q}_j, \bar{q}_j)} \circ \bar{p}_j = \nu_{(P_j, 0)} = \nu_{(P_j, 0)} \circ 0 = 0 \). But this means \( (\oplus_{(H, \eta) \in \mathcal{V}_G} \nu_{(H, \eta)}) \circ (\oplus_{P_j} p_j) = 0 \). By the universal property of cokernels, there is a unique morphism \( \nu: G \to N \) such that \( \nu \circ \eta = \nu_{(H, \eta)} \) for all \((H, \eta) \in \mathcal{V}_G \). \( \blacksquare \)

Claim 3.1.2. If \( M \) is a vector bundle on \( Y \) that is a projective object in \( \text{QCoh}(\mathbf{V}) \), then \( \mathbf{V}_{f_*M} \) is filtered.

Proof. Consider a pair of morphisms \( h_1, h_2: (H, \eta) \to (H', \eta') \); we must show that these can be coequalized in \( \mathbf{V}_{f_*M} \). Take the duals of \( h_1 \) and \( h_2 \), which results in morphisms \( h_1^\vee, h_2^\vee: H'^\vee \to H^\vee \). Let \( E \) be their equalizer in \( \text{QCoh}(X) \). If \((F, \rho) \in \mathbf{V}_E \), then taking duals of everything results in a commutative diagram:

![Diagram](https://via.placeholder.com/150)

Thus, we just need to produce \((F, \rho) \in \mathbf{V}_E \) that admits a compatible morphism \( F^\vee \to f_*M \). By taking adjoints in the above diagram, we see that it is sufficient to produce a compatible morphism \( f^*F^\vee \to M \). But \( M \) is a vector bundle, so it is
sufficient to produce a compatible morphism \( M^\vee \to f^* F \). Dualizing their defining diagrams we obtain:

\[
\begin{align*}
M^\vee & \to f^* F \\
\xrightarrow{f^* \rho} & f^* E \\
\xrightarrow{f^* h_1} & f^* H_1^\vee \\
\xrightarrow{f^* h_2} & f^* H_2^\vee \\
\end{align*}
\]

But \( f \) is flat, so \( f^* E \) is also the equalizer of \( f^* h_1^\vee \) and \( f^* h_2^\vee \); hence, there is a compatibly induced morphism \( M^\vee \to f^* E \). By Claim \([5.1]\), \( E \simeq \colim(\mu_E) \); hence, \( f^* E \simeq f^* \colim(\mu_E) \). Now \( M^\vee \) is a vector bundle and \( Y \) is quasi-compact and quasi-separated, so the functor \( \Hom_{\sigma_Y}(-, -) = \Gamma(Y, \mathcal{O}_Y \otimes_{\sigma_Y} -) \) preserves filtered colimits of quasi-coherent sheaves \([HR17, \text{Lem. 1.2(iii)}]\). Since \( M \) is projective in \( \mathbf{QCoh}(Y) \), \( M^\vee \) is projective in \( \mathbf{QCoh}(Y) \)\(^\dagger\). Hence, \( \Hom_{\sigma_Y}(\mathcal{O}_Y, -) \) is also exact and so commutes with all colimits of quasi-coherent sheaves. Thus we obtain:

\[
\Hom_{\sigma_Y}(M^\vee, \colim(\mu_E)) \simeq \colim((F, \rho) \in \mathbf{V}_E) \Hom_{\sigma_Y}(M^\vee, f^* F).
\]

In particular, \( M^\vee \to f^* E \) factors through some morphism \( f^* \rho \): \( f^* F \to f^* E \), where \((F, \rho) \in \mathbf{V}_E \). The claim follows.

\( \square \)

**Example 3.2.** In Theorem \([5.1]\) it is necessary to include the duals of the generating set \( \mathbf{V} \). Indeed, let \( X = \mathbb{P}^1_{x,y} \) over a field \( k \) with coordinates \( x \) and \( y \). Let \( Y = \text{Spec}(k) \) and take \( f: Y \to X \) to be the inclusion. Then \( L = \mathcal{O}(1) \) is ample and \( X \) has the \( \mathbf{V} = \{ L^n \}_{n \geq 0} \)-resolution property (Example \([2.13]\)). But \( f_* \mathcal{O}_X \simeq \varinjlim_{n \in \mathbb{N}} L^n \) cannot be written as a filtered colimit of direct sums of objects from \( \mathbf{V} \).

The converse to Theorem \([5.1]\) is an earlier result of Hovey \([Hov04, \text{Prop. 1.4.4}]\).

**Proposition 3.3.** Let \( f: Y \to X \) be a faithfully flat affine morphism of quasi-compact and quasi-separated algebraic stacks, where \( Y \) is quasi-affine. Let \( \mathbf{V} \) be a set of vector bundles on \( X \). If \( f_* \mathcal{O}_Y \) is a filtered colimit of duals of finite direct sums of objects of \( \mathbf{V} \), then \( X \) has the \( \mathbf{V} \)-resolution property.

**Proof.** It suffices to prove that \( \mathbf{V} \) generates \( \mathbf{QCoh}(X) \). Write \( f_* \mathcal{O}_Y = \colim_{\lambda \in \Lambda} H_{\lambda, Y} \), where \( H_{\lambda, Y} = \bigoplus_{V \in \mathcal{J}_{\lambda}} V^{\oplus n_{\lambda}(V)} \), where \( J_{\lambda} \subseteq \mathbf{V} \) is a finite subset and \( n_{\lambda}(V) \) is finite. Let \( M \in \mathbf{QCoh}(X) \); then there are natural isomorphisms:

\[
\Gamma(Y, f^* M) = \Hom_{\sigma_X}(\mathcal{O}_X, f_* f^* M) \\
\simeq \Hom_{\sigma_X}(\mathcal{O}_X, f_* \mathcal{O}_Y \otimes_{\sigma_Y} M) \quad (f \text{ is affine}) \\
\simeq \Hom_{\sigma_X}(\mathcal{O}_X, \colim_{\lambda} H_{\lambda, Y} \otimes_{\sigma_Y} M) \\
\simeq \colim_{\lambda} \Hom_{\sigma_X}(\mathcal{O}_X, H_{\lambda, Y} \otimes_{\sigma_Y} M) \\
\simeq \colim_{\lambda} \Hom_{\sigma_X}(H_{\lambda}, M)
\]

\( \dagger \)This follows from three observations: (1) a direct summand of a projective is projective; (2) if \( M \) is projective, then \( M \otimes_{\sigma_Y} Q \) is projective for any vector bundle \( Q \); and (3) \( M^\vee \) is a direct summand of \( M^\vee \otimes_{\sigma_Y} M \otimes_{\sigma_Y} M^\vee \).
It follows that if \( \phi : M \to N \) is a homomorphism of quasi-coherent \( \mathcal{O}_X \)-modules, then there is a commutative diagram:

\[
\begin{array}{ccc}
\Gamma(Y, f^*M) & \xrightarrow{\sim} & \colim_\lambda \text{Hom}_{\mathcal{O}_X}(H_\lambda, M) \\
\Gamma(Y, f^*\phi) & \downarrow & \colim_\lambda \text{Hom}_{\mathcal{O}_X}(H_\lambda, \phi) \\
\Gamma(Y, f^*N) & \xrightarrow{\sim} & \colim_\lambda \text{Hom}_{\mathcal{O}_X}(H_\lambda, N).
\end{array}
\]

In particular, if \( \text{Hom}_{\mathcal{O}_X}(V, \phi) : \text{Hom}_{\mathcal{O}_X}(V, M) \to \text{Hom}_{\mathcal{O}_X}(V, M) \) is the zero map for all \( V \in \mathcal{V} \), then \( \Gamma(Y, f^*\phi) \) is the zero map. Since \( Y \) is quasi-affine, \( f^*\phi = 0 \). But \( f \) is faithfully flat, so \( \phi = 0 \). That is, \( \mathcal{V} \) generates \( \text{QCoh}(X) \).

Let \( S \) be an integral quasi-compact quasi-separated algebraic stack with generic point \( \xi \). We say that a quasi-coherent \( \mathcal{O}_S \)-module \( F \) is torsion free if \( \ker(F \to (i_\xi)_*i_\xi^*F) = 0 \), where \( i_\xi : S_\xi \hookrightarrow S \) is the generic residual gerbe [Ryd11] Thm. B.2. We record here the following folklore result.

**Corollary 3.4.** Let \( T \) be an integral quasi-compact quasi-separated algebraic stack with affine diagonal. If every torsion free quasi-coherent \( \mathcal{O}_T \)-module of finite type is a vector bundle, then \( T \) has the resolution property.

**Proof.** Since \( T \) is quasi-compact with affine diagonal, there is a smooth covering \( \tau : \text{Spec} \, A \to T \) and \( \tau \) is affine. We may write \( \tau_*\mathcal{O}_{\text{Spec} \, A} = \varprojlim N_\lambda \), where the \( N_\lambda \) are finite type quasi-coherent \( \mathcal{O}_T \)-submodules of \( \tau_*\mathcal{O}_{\text{Spec} \, A} \) [Ryd16]. Then the \( N_\lambda \) are torsion free, so are all vector bundles. By Proposition 3.3 the result follows. \( \square \)

**Remark 3.5.** Corollary 3.4 gives a simple generalization of [BT84] 1.4.5. Indeed, if \( T \) is an integral quasi-compact quasi-separated algebraic stack that admits a faithfully flat cover by a finite disjoint union of spectra of Dedekind or Prüfer domains (i.e., every finitely generated ideal is invertible), then every torsion free quasi-coherent \( \mathcal{O}_T \)-module of finite type is a vector bundle [Kap52] Thm. 1.

### 4. Flags

The goal of this article is to construct interesting vector bundles on algebraic stacks. The main problem—even for schemes—is that vector bundles are glued from local data. The key idea in this paper is to add the additional structure of a flag to a vector bundle. This ends up being surprisingly useful.

Let \( Y \) be an algebraic stack. Let \( \mathcal{V}_* \) be a quasi-coherent \( \mathcal{O}_Y \)-module \( V \) with a finite filtration:

\[
0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = V.
\]

We call \( n \) the length of the filtration. Recall that \( \mathcal{V}_* \) is a flag if the graded pieces \( \text{gr}_i(\mathcal{V}_*) = V_i/V_{i-1} \) are vector bundles for all \( i = 1, \ldots, n \). Note that this condition implies that the \( V_i \) are also vector bundles. A flag is complete if the graded pieces are line bundles. We say that \( \mathcal{V}_* \) is trivially graded if \( \text{gr}_i(\mathcal{V}_*) \) is a trivial vector bundle for all \( i \). Note that a vector bundle which admits the structure of a trivially graded flag is sometimes referred to as a unipotent vector bundle in the literature (see [Oda71] Sec. 1 and [Muk78] Def. 4.5]). We now have the following key definition of the paper.

**Definition 4.1.** Let \( f : X \to S \) be a morphism of algebraic stacks. Let \( \mathcal{V}_* \) be a flag on \( X \). We say that \( \mathcal{V}_* \) is \( f \)-graded (or \( S \)-graded) if for each \( r \) there is a vector bundle \( E_r \) on \( S \) and an isomorphism \( \phi_r : \text{gr}_r(\mathcal{V}_*) \simeq f^*E_r \). If the vector bundle \( V \) is \( f \)-(R)-faithful, then we say the same of the flag \( \mathcal{V}_* \).
Note that trivially graded flags are graded by every morphism. Remarkably, we can establish that graded flags often descend (Theorem 4.8). We are not aware of any related result in the literature. We begin, however, with the universal example.

**Example 4.2.** Let $S$ be an algebraic stack. Fix an ordered sequence of $n$ line bundles $\mathcal{L} = (L_1, \ldots, L_n)$ on $S$. Define a category fibered in groupoids $\text{Flag}_{\mathcal{L}} \rightarrow \text{Sch}/S$ as follows: its objects over $f: X \rightarrow S$ are pairs $(V_\bullet, \{\phi_i\}_{i=1}^n)$, where

1. $V_\bullet$ is a flag of length $n$ and
2. $\phi_i: \text{gr}_i(V_\bullet) \cong f^*L_i$ are isomorphisms for $i = 1, \ldots, n$.

That is, an $X$-point of $\text{Flag}_{\mathcal{L}}$ is an $S$-graded flag on $X$ whose graded pieces are isomorphic to the $f^*L_i$. A morphism in $\text{Flag}_{\mathcal{L}}$ is an isomorphism of vector bundles that is compatible with the filtrations and isomorphisms to the $L_i$. Note that there is a distinguished object in $\text{Flag}_{\mathcal{L}}$ defined over $S$:

$$\ell = (L_1 \subseteq \cdots \subseteq \bigoplus_{j=1}^{n-1} L_j \subseteq \bigoplus_{j=1}^n L_j, \{\text{id}_{L_i}\}_{i=1}^n).$$

Every other object in $\text{Flag}_{\mathcal{L}}$ becomes isomorphic to $\ell$ after passing to a smooth covering. In particular, $\text{Flag}_{\mathcal{L}}$ is a gerbe and the section defined by $\ell$ induces an equivalence $\text{Flag}_{\mathcal{L}} \cong BU(\mathcal{L})$, where $U(\mathcal{L}) = \text{Aut}(\ell)$. Clearly, $U(\mathcal{L}) \subseteq GL(\bigoplus_{j=1}^n L_j)$. Moreover, the following diagram 2-commutes:

$$\begin{array}{ccc}
X & \xrightarrow{\text{Fr}(V)} & BGL_{n,S} \\
\downarrow_{(V_\bullet, \{\phi_i\})} & & \downarrow_{\text{Fr}(\Theta, L_j)} \\
BU(\mathcal{L}) & \longrightarrow & BGL(\bigoplus_{j=1}^n L_j).
\end{array}$$

If $\mathcal{L} = (\Theta_S, \ldots, \Theta_S)$, then there is a natural identification

$$\text{Aut}(\ell) \cong U_{n,S}.$$ 

Thus for any choice of $\mathcal{L}$, $\text{Aut}(\ell)$ is locally isomorphic to $U_{n,S}$. In particular, $U(\mathcal{L}) \rightarrow S$ is a flat group of finite presentation and it follows from Proposition 2.19 that $GL(\bigoplus_{j=1}^n L_j)/U(\mathcal{L}) \rightarrow S$ is quasi-affine, and $U(\mathcal{L}) \rightarrow S$ is R-embeddable. Hence, $BU(\mathcal{L})$ is an algebraic stack, the morphism $BU(\mathcal{L}) \rightarrow S$ has the resolution property and the universal flag $F_\bullet(\mathcal{L})$ on $BU(\mathcal{L})$ is a $(BU(\mathcal{L}) \rightarrow S)$-tensor generator. In particular, if $S$ has the resolution property, then so does $BU(\mathcal{L})$.

If $B_{n,\mathbb{Z}}$ denotes the Borel subgroup of upper triangular matrices in $GL_{n,\mathbb{Z}}$ and $B_{n,\mathbb{Z}} \rightarrow G_m^n$ is the induced quotient, then the following diagram is 2-cartesian:

$$\begin{array}{ccc}
BU(\mathcal{L}) & \longrightarrow & BB_{n,\mathbb{Z}} \\
\downarrow & & \downarrow \\
S & \longrightarrow & BG_m^n.
\end{array} \tag{4.2.1}$$

Let $f: X \rightarrow S$ be a morphism of algebraic stacks and let $\text{Vect}(X)$ denote the set of isomorphism classes of vector bundles on $X$. We let

$$\text{Flag}_\sigma \subseteq \text{Flag}_f \subseteq \text{Flag} \subseteq \text{Vect}(X)$$

denote the sets of isomorphism classes of vector bundles that admit trivially graded complete flags, complete $f$-graded flags, and complete flags respectively.

**Example 4.3.** Standard arguments show that $\text{Flag}$, $\text{Flag}_f$, and $\text{Flag}_\sigma$ are stable under finite direct sums, the taking of duals, finite tensor products, and extensions.

**Example 4.4.** Let $X$ be an algebraic stack. If
(1) $X$ is affine; or
(2) $X$ is quasi-affine; or
(3) $X$ is quasi-projective over affine; or
(4) $X$ admits an ample family of line bundles; or
(5) $\mathcal{QCoh}(X)$ is generated by a set of line bundles (e.g., $X = BG_{m,k}^{n}$); or
(6) $X = BU_{n,k}$, where $k$ is a field;
then $X$ has the $\text{Flag}$-resolution property. These assertions are trivial. If $X$ has the $\text{Flag}$-resolution property, then Schäppi’s Theorem (Theorem 3.1) implies that if $p: \text{Spec } A \to X$ is flat, then $p_* \mathcal{O}_{\text{Spec } A} \simeq \varprojlim \mathcal{F}_\lambda$, where the $\mathcal{F}_\lambda$ are complete flags on $X$. Note that $BGL_{m,k}$ does not have the $\text{Flag}$-resolution property if $n > 1$.

The following result yields a useful sufficient condition for the existence of a flag structure on a vector bundle.

**Lemma 4.5.** Let $T$ be an integral quasi-compact quasi-separated algebraic stack with generic point $\eta$: $\text{Spec } k \to T$, where $k$ is a field. Let $\gamma: G \to T$ be a flat group of finite presentation and let $f: BG \to T$ be the induced morphism. Let $V$ be a vector bundle on $BG$.

1. If the adjunction $z: f^* f_* V \to V$ is an isomorphism after restriction along $\eta$, then it is an isomorphism.
2. If every torsion free quasi-coherent $\mathcal{O}_T$-module of finite type is a vector bundle (Remark 3.5) and $G_{\eta}$ is unipotent, then $V$ admits an $f$-graded flag.

**Proof.** By [Ryd11] Thm. B.2, $\eta$ factors as $\text{Spec } k \xrightarrow{q} T_{\eta} \xrightarrow{i_\eta} T$, where $T_{\eta}$ is the residual gerbe and $q$ is faithfully flat. For (1): since $z_\eta$ is an isomorphism, $z_{T_{\eta}}$ is an isomorphism. Let $p: T \to BG$ be the usual section to $f$: by descent, $V$ is described by a group homomorphism $v: G \to GL(p^* V)$. Note that $z$ is an isomorphism if and only if $\ker(v) = G$ as $z$ corresponds to the inclusion of $G$-invariants of $V$ into $V$. Now $\ker(v) \subset G$ is a closed immersion as $GL(p^* V) \to T$ is separated and contains $\gamma^{-1}(T_{\eta})$, which is dense since $\gamma$ is open. Hence $\ker(v) = G$.

We prove (2) by induction on $\text{rank}(V)$, the case $\text{rank}(V) = 0$ being trivial; so we now assume that $\text{rank}(V) > 0$. If $z_{T_{\eta}}$ is an isomorphism, then (1) says that $z_V$ is an isomorphism and so we would be done. Otherwise, if $z_{T_{\eta}}$ is not an isomorphism, then $(f^* f_* V)_{T_{\eta}}$ is non-zero as after restriction along $\eta$ it is the inclusion of the invariants of a non-trivial representation of the unipotent group $G_{\eta}$ [GP11b] Thm. XVII.3.5. In particular, there is a non-trivial quotient $V_{T_{\eta}} \to W$ on $T_{\eta}$. Now define $\overline{W}$ to be the image of the composition $V \to (i_{T_{\eta}})_* V_{T_{\eta}} \to (i_{T_{\eta}})_* W$. Since the map $V \to (i_{T_{\eta}})_* W$ is non-zero, $\overline{W}$ is a non-zero, torsion free, quasi-coherent $\mathcal{O}_{BG}$-module of finite type, and so is a vector bundle. Also, $\overline{W}_{T_{\eta}} = W$ and so $\text{rank}(W) = \text{rank}(\overline{W}) < \text{rank}(V)$.

We now have an exact sequence of vector bundles on $BG$:

$$0 \longrightarrow K \longrightarrow V \longrightarrow \overline{W} \longrightarrow 0.$$  

The ranks of $K$ and $\overline{W}$ are less than that of $V$ and so by induction, they admit $f$-graded flags and consequently so does $V$. \hfill $\square$

Using Lemma 4.5 we can now prove the following Proposition.

**Proposition 4.6.** Let $\mathcal{L}$ be a sequence of $n$ line bundles on an algebraic stack $S$.

1. The morphism $f: BU(\mathcal{L}) \to S$ has the $\text{Flag}_f$-resolution property.
2. If $S$ is affine, then $f$ has the $\text{Flag}_f$-resolution property.

**Proof.** It suffices to show (1) universally [Gro17] Prop. 2.8(iv)]. Hence, we may suppose $f: BU(\mathcal{L}) \to BG_{m,k}^{n}$, where $\mathcal{L}$ is the universal sequence of $n$ line bundles. Since $BB_{n,k} \simeq BU(\mathcal{L})$ has the resolution property (Example 3.2), it suffices to
show that every vector bundle \( V \) is a complete \( f \)-graded flag. Since \( \text{Spec} \mathbb{Z} \to BG_{m, \mathbb{Z}} \) is a smooth covering, Lemma 4.5 and Remark 4.5 imply that \( V \) admits an \( f \)-graded flag. But every vector bundle on \( BG_{m, \mathbb{Z}} \) is a direct sum of line bundles, so (1) follows from Example 4.3.

For (2), it suffices to show that the vector bundle \( V \) underlying every \( f \)-graded complete flag \( V \) is a direct sum of a vector bundle \( W \) which admits a trivially graded complete flag structure. We proceed by induction on \( n \), the length of \( V \). If \( n = 0 \), then the result is trivial. If \( n > 0 \), then we may write \( V \) as an extension

\[
0 \longrightarrow V_{n-1} \longrightarrow V \longrightarrow f^*L \longrightarrow 0.
\]

for some line bundle \( L \) on \( S \). Since \( S \) is affine, there is a split surjection \( \gamma: G_{\mathbb{Z}}^{\oplus m} \twoheadrightarrow L \). Then \( f^*\gamma: G_{BU(\mathbb{Z})}^{\oplus m} \twoheadrightarrow f^*L \) is a split surjection. Now pull the extension back along \( f^*\gamma \), so we may replace \( f^*L \) with a trivial bundle of rank \( m \). By induction, there is a split surjection \( \delta: U \twoheadrightarrow V_{n-1} \), where \( U \) admits a trivially graded complete flag. Pushing forward along a section of \( \delta \), \( V \) is a direct summand of a vector bundle that is an extension of trivially graded complete flags. Now apply Example 4.3. □

The following concept will be useful. A refinement of a flag \( V \) of length \( n \) is another flag \( V_0 \subseteq \cdots \subseteq V_m = V \) of length \( m \geq n \) together with a strictly order preserving function \( \rho: \{ 0 < 1 < \cdots < n \} \to \{ 0 < 1 < \cdots < m \} \) such that \( V_i = V_\rho(i) \) for all \( i \).

**Example 4.7.** Let \( Y \) be an algebraic stack. Let \( V \) be a flag on \( Y \). Suppose that \( \text{gr}_i(V) \) admits the structure of a complete flag for each \( i \geq 0 \); then \( V \) admits a complete refinement. To see this, use Example 4.3.

**Theorem 4.8.** Consider a 2-cartesian diagram of algebraic stacks:

\[
\begin{array}{ccc}
X' & \xrightarrow{q} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{p} & S.
\end{array}
\]

Assume that

1. \( S \) is quasi-compact with affine diagonal and has the resolution property;
2. \( f \) is quasi-compact and quasi-separated;
3. \( p \) is faithfully flat and \( S' \) is affine.

If \( X' \) admits a \( q \)-graded flag \( V_\bullet \) of length \( n \), then \( X \) admits a flag \( W_\bullet \) of length \( n \) and a degree preserving split surjection \( q^*W_\bullet \twoheadrightarrow V_\bullet \). Moreover,

(a) if \( V_\bullet \) is \((\cdot,R)\)-faithful, then \( W_\bullet \) is \((\cdot,R)\)-faithful.
(b) If \( q^*\mathcal{O}_Y \) is a filtered colimit of vector bundles \( F_\lambda \) on \( X \) (see Theorem 3.7) and \( \text{gr}_i(V_\bullet) \cong q^*E_i \), then there exists \( \lambda_i \) with \( \text{gr}_i(W_\bullet) \cong f^*F_\lambda \otimes_{\mathcal{O}_X} E_i \).
(c) if \( V_\bullet \) is \( f^q \)-graded, then \( W_\bullet \) is \( f^q \)-graded.
(d) Assume that \( S \) has the Flag-resolution property.
   (i) if \( V_\bullet \) is complete, then \( W_\bullet \) admits a complete refinement.
   (ii) if \( V_\bullet \) is \( f^q \)-graded, then \( W_\bullet \) admits a complete \( f^q \)-graded refinement.

**Proof.** Claim (a) follows from Proposition 2.2(11), the main claim, and [Gro17, Prop. 2.8(iii)]. Claim (b) follows from (b) and Examples 4.3 and 4.4. Claim (d) follows from (b) and Examples 4.3 and 4.4. Claim (d) will follow from the construction of \( W_\bullet \), which we prove by induction on \( n \geq 0 \). The base case is trivial so we assume \( n \geq 1 \). Choose a vector bundle \( E \) on \( X \) such that \( V_1 = q^*E \), then \( (V/V_1)_\bullet \) is a \( q \)-graded flag of length \( n - 1 \). By induction, there is a flag \( W_\bullet \) of length \( n - 1 \) on \( X \) and a split surjection
$q^*W \rightarrow (V/V_1)_\bullet$ as in the statement of the theorem. We now pull back the defining short exact sequence

\[
\begin{array}{ccccccc}
0 & \rightarrow & q^*E & \rightarrow & V & \rightarrow & V/q^*E & \rightarrow & 0
\end{array}
\]

along the surjection $q^*W \rightarrow V/q^*E$. This results in a commutative diagram with exact rows, whose vertical arrows are easily checked to be split surjective:

\[
\begin{array}{ccccccc}
0 & \rightarrow & q^*E & \rightarrow & V' & \rightarrow & q^*W & \rightarrow & 0
\end{array}
\]

Thus, we may replace $V$ by $V'$ and assume that $(V/V_1)_\bullet = q^*W_\bullet$.

Since $S'$ is affine and $S$ has affine diagonal, it follows that $p$ and $q$ are affine and faithfully flat morphisms. It follows that we may push forward the top row in the above diagram to obtain an exact sequence of quasi-coherent sheaves on $X$:

\[
\begin{array}{ccccccc}
0 & \rightarrow & q_*q^*E & \rightarrow & q_*V' & \rightarrow & q_*q^*W & \rightarrow & 0
\end{array}
\]

Pulling this sequence back along the injective adjunction $W \hookrightarrow q_*q^*W$, we obtain a diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \rightarrow & q_*q^*E & \rightarrow & W' & \rightarrow & W & \rightarrow & 0
\end{array}
\]

\[
\begin{array}{ccccccc}
0 & \rightarrow & q_*q^*E & \rightarrow & q_*V' & \rightarrow & q_*q^*W & \rightarrow & 0
\end{array}
\]

Since $E$ is a vector bundle and $q$ is affine, the projection formula and flat base change produce natural isomorphisms:

\[q_*q^*E \simeq q_*\mathcal{O}_{X'} \otimes_{\mathcal{O}_X} E \simeq (f^*p_*\mathcal{O}_{S'}) \otimes E.\]

Assume we are given that $p_*\mathcal{O}_{S'} \simeq \operatorname{lim}_{\lambda} F_\lambda$, where the $F_\lambda$ are vector bundles, or we produce such a description by Schäppi’s Theorem (Theorem 3.1). Since the adjunction $p^*p_*\mathcal{O}_{S'} \rightarrow \mathcal{O}_{S'}$ is split surjective, taking $\lambda$ sufficiently large, we can assume that the induced morphisms $p^*F_\lambda \rightarrow \mathcal{O}_{S'}$ are also split surjective. But we also have natural isomorphisms:

\[
\operatorname{Ext}^1_{\mathcal{O}_X}(W, q_*q^*E) \simeq \operatorname{H}^1(X, W^{\vee} \otimes q_*q^*E)
\]

\[
\simeq \operatorname{H}^1(X, \lim_{\lambda} (f^*F_\lambda \otimes W^{\vee} \otimes E))
\]

\[
\simeq \lim_{\lambda} \operatorname{H}^1(X, f^*F_\lambda \otimes W^{\vee} \otimes E) \quad [\text{HR17 Lem. 1.2(iii)}]
\]

\[
\simeq \lim_{\lambda} \operatorname{Ext}^1_{\mathcal{O}_X}(W, f^*F_\lambda \otimes E).
\]

Thus, the extension $W'$ is the push forward of an extension:

\[
\begin{array}{ccccccc}
0 & \rightarrow & f^*F_\lambda \otimes_{\mathcal{O}_X} E & \rightarrow & W'_\lambda & \rightarrow & W & \rightarrow & 0
\end{array}
\]

along the morphism $f^*F_\lambda \otimes_{\mathcal{O}_X} E \rightarrow f^*(p_*\mathcal{O}_{S'}) \otimes_{\mathcal{O}_X} E \simeq q_*q^*E$. Pulling this exact sequence back along $q$, we obtain an exact sequence:

\[
\begin{array}{ccccccc}
0 & \rightarrow & q^*f^*F_\lambda \otimes_{\mathcal{O}_X} q^*E & \rightarrow & q^*W'_\lambda & \rightarrow & q^*W & \rightarrow & 0
\end{array}
\]

But the push forward of this extension along the split surjection $q^*f^*F_\lambda \otimes_{\mathcal{O}_X} q^*E \twoheadrightarrow q^*E$ is clearly isomorphic to $V'$ and the resulting morphism $q^*W'_\lambda \twoheadrightarrow V'$ is split surjective. The result follows. \qed
5. **Unipotent morphisms and groups**

5.1. **Unipotent morphisms.** Motivated by the results of the previous section, we make the following definition.

**Definition 5.1.** Let \( f : X \to S \) be a morphism of algebraic stacks. Then \( f \) is:

- *(R-)*unipotent if \( X \) admits a complete \( f \)-graded \( f \)-(R-)faithful flag;
- locally \( (R-) \)unipotent if there is an fpqc covering \( S' \to S \) such that \( f \times_S S' \) is \((R-)\)unipotent; and
- geometrically \( (R-) \)unipotent if for every algebraically closed field \( k \) and morphism \( \text{Spec} \ k \to S \), the induced morphism \( X \times_S \text{Spec} \ k \to \text{Spec} \ k \) is \((R-)\)unipotent.

Locally \( (R-) \)unipotent and unipotent morphisms have affine and quasi-affine diagonals respectively. We have the following sequence of implications:

\[
\text{R-unipotence} \quad \Rightarrow \quad \text{local R-unipotence} \quad \Rightarrow \quad \text{geometric R-unipotence} \\
\text{unipotence} \quad \Rightarrow \quad \text{local unipotence} \quad \Rightarrow \quad \text{geometric unipotence}
\]

The solid arrows follow from the definitions. The dashed arrows are partial converses. The dashed vertical implications are valid only for certain groups over normal bases. The dashed horizontal implications are valid only in special cases.

Quasi-compact and quasi-separated representable morphisms are always unipotent. There are two prototypical examples of \( (R-) \)unipotent morphisms:

1. \( B \mathcal{U} (\mathcal{L}) \to S \) (Example 4.2), and
2. quasi-affine morphisms \( X \to S \).

We will return to these examples often. We begin this section with the following two trivial lemmas.

**Lemma 5.2.** Let \( f : X \to S \) be a \( (R-) \)unipotent morphism of algebraic stacks. If \( S' \to S \) is a morphism of algebraic stacks, then \( f \times_S S' \) is \((R-)\)unipotent. Moreover, the same holds for the local and geometric versions.

**Lemma 5.3.** Let \( X \xrightarrow{h} Y \xrightarrow{g} S \) be morphisms of algebraic stacks.

1. If \( g \) is unipotent and \( h \) is quasi-compact, quasi-separated, and representable; then \( g \circ h \) is unipotent.
2. If \( g \) is \( R \)-unipotent and \( h \) is quasi-affine, then \( g \circ h \) is \( R \)-unipotent.

Moreover, the analogous properties hold for the local and geometric versions.

In general, unipotent morphisms are not stable under composition (Example 5.6). We are optimistic that locally unipotent morphisms have better stability properties under composition, but this appears to be surprisingly subtle—even when \( S \) is the spectrum of a field, \( Y \) is quasi-affine and \( X \) is a gerbe over \( Y \).

We now have the following characterization of \( (R-) \)unipotent morphisms, which provides a unipotent enrichment of the Totaro–Gross Theorem (Theorem 2.15).
Theorem 5.4. Let $f : X \to S$ be a morphism of algebraic stacks with $S$ quasi-compact. Then the following are equivalent.

1. The morphism $f$ is $R$-unipotent.
2. For some integer $n \geq 0$, there is an ordered sequence of $n$ line bundles $\mathcal{L}$ on $S$ and factorization of $f$ as $X \xrightarrow{h} BU(\mathcal{L}) \to S$, where $h$ is quasi-affine.
3. The morphism $f$ has the Flag$_f$-resolution property and the relative inertia stack $I_f \to X$ has affine fibers.

Moreover, $f$ is unipotent if and only there is a quasi-compact, quasi-separated and representable morphism $X \xrightarrow{h} BU(\mathcal{L})$ as in (2).

Proof. The statement regarding unipotent morphisms will follow from our arguments from the equivalence (1) ⇔ (2).

We have (2) ⇒ (1) by Lemma 5.3 and the $R$-unipotence of $BU(\mathcal{L}) \to S$. As for (1) ⇒ (2), by definition, there is a complete $f$-graded flag $V_\bullet$ of length $n$ on $X$ that is $f$-(R)-faithful. That is, there is an ordered sequence of line bundles $\mathcal{L} = (L_1, \ldots, L_n)$ on $S$ together with isomorphisms $\phi_i : g_f(V_i) \cong f^*L_i$ for all $i$. By Example 1.4, we obtain a morphism $X \xrightarrow{h} BU(\mathcal{L})$. Post-composing this with the quasi-affine morphism $BU(\mathcal{L}) \to BGL(\bigoplus L_i) \cong BGL_{n,S}$ yields a quasi-affine morphism by Theorem 2.15(2), and the result follows from [Sta, Tag 054G].

For (2) ⇒ (3): Proposition 1.6 and [Gro17, Prop. 2.8(v)] implies that $f$ has the Flag$_f$-resolution property. Finally, for (3) ⇒ (2): the Gross–Totaro Theorem (Theorem 2.15) produces a $V_\bullet \in \text{Flag}_f$ which is $f$-R-faithful.

A remarkable consequence of Theorem 4.8 is that local ($R$-)unipotence implies ($R$-)unipotence on bases with the Flag-resolution property.

Theorem 5.5. Let $f : X \to S$ be a locally unipotent (resp., locally $R$-unipotent) morphism of algebraic stacks. Let $S$ be quasi-compact with affine diagonal.

1. If $S$ has the resolution property, then $X$ is a global quotient (resp., has the resolution property).
2. If $S$ has the Flag-resolution property, then $f$ is unipotent (resp., $R$-unipotent).

Proof. By assumption, there is a faithfully flat cover $p : S' \to S$ such that $f' : X \times_S S' \to S'$ admits a complete $f'$-graded $f'$-faithful (resp., $f'$-$R$-faithful) flag $V_\bullet$. Passing to a smooth cover of $S'$, we may assume that $S'$ is affine and that the flag is trivially graded, and so $V_\bullet$ is $pf'$-graded. Now apply Theorem 4.8 to:

$$
\begin{array}{ccc}
X \times_S S' & \xrightarrow{q} & X \\
\downarrow f' & & \downarrow f \\
S' & \xrightarrow{p} & S.
\end{array}
$$

Then we obtain a flag $W_\bullet$ on $X$ and a split surjection $q^*W_\bullet \to V_\bullet$. Part (8) implies that $W_\bullet$ is $f$-graded. Part (9) says that $W_\bullet$ is $f$-faithful (resp., $f$-$R$-faithful), so $X$ is a global quotient (resp., has the resolution property). Part (10) says that if $S$ has the Flag-resolution property, then $W_\bullet$ admits a complete refinement, which is also $f$-graded. Hence, $f$ admits a complete $f$-graded $f$-($R$)-faithful flag. That is, $f$ is unipotent (resp., $R$-unipotent).

Example 5.6. Let $S$ be a proper, normal surface over $\mathbb{C}$ with no non-trivial line bundles [Sch99]. Let $Y \subseteq S$ be the inclusion of the regular locus, which is an open subscheme whose complement $S \setminus Y$ is a finite set of closed points. Then $Y$ admits a non-trivial very ample line bundle $L$. Let $\mathcal{U} = BU(\theta, L)^{\infty} \cong BL$, then

1. $\pi : \mathcal{U} \to Y$ and $j : Y \to S$ are $R$-unipotent;

(2) the composition \( p = j \circ \pi \) is locally R-unipotent but not unipotent; and

(3) \( S \) has the resolution property but not the Flag-resolution property.

Indeed, (1) is clear from Theorem 5.4. Also, (3) follows from (2) and Theorem 5.5.

For (2): to see that \( p \) is locally R-unipotent, it suffices to restrict to affine opens \( \text{Spec} \, A \subseteq S \); then \( Y_A = Y \cap \text{Spec} \, A \) is quasi-affine and so \( L^Y_A \) is globally generated.

Hence, there is a quasi-affine morphism \( \mathcal{W} \to B G^A_n, Y_A \) for some \( n \); after composition with the open immersion \( B G^A_n, Y_A \subseteq B G^A_n, \text{Spec} \, A \), we see that \( p \) is locally R-unipotent by Lemma 5.8 (2).

If \( p = j \circ \pi \) is unipotent, since \( S \) has no non-trivial line bundles, \( \mathcal{W} \) must admit a faithful trivially graded flag. Hence, it suffices to show that every trivially graded flag \( V_\bullet \) on \( \mathcal{W} \) is of the form \( \pi^* W_\bullet \) for some trivially graded flag \( W_\bullet \) on \( Y \). A simple calculation shows that \( H^0(Y, \mathcal{O}_Y) = \mathbb{C} \) [Sta Tag 01PSQ], and therefore \( L \) is non-trivial [Sta Tag 01QE].

In turn, this implies \( H^0(Y, L^Y) = 0 \) because by composing with a non-zero section of \( L \), a non-zero section \( L^Y \) would yield a trivialization of \( L^Y \). Therefore, since \( L^Y \) has no non-zero global sections, if \( W_\bullet \) is a trivially graded flag \( W_\bullet \) on \( Y \), then \( \text{Hom}_{\mathcal{O}_Y}(W, L^Y) = 0 \). Now let \( V_\bullet \) be a trivially graded flag on \( \mathcal{W} \). By induction on the length of \( V_\bullet \), we may assume that \( V_\bullet \) is an extension:

\[
e: 0 \to \pi^* \mathcal{O}_Y \to V_\bullet \to \pi^* W_\bullet \to 0,
\]

where \( W_\bullet \) is a trivially graded flag on \( Y \). There is also an exact sequence:

\[
0 \to \text{Ext}^1_{\mathcal{O}_Y}(W, \pi^* \mathcal{O}_Y) \to \text{Ext}^1_{\mathcal{O}_Y}(\pi^* W, \pi^* \mathcal{O}_Y) \to \text{Hom}_{\mathcal{O}_Y}(W, \mathcal{R}^1 \pi_* \mathcal{O}_Y).
\]

Since \( \text{Hom}_{\mathcal{O}_Y}(W, L^Y) = 0 \), the first map is injective and so our extension \( e \) is pulled back from \( Y \).

A useful application of Theorem 5.5 is the following.

**Corollary 5.7.** Let \( f: X \to S \) be a locally (R-)unipotent morphism of algebraic stacks. If \( S \) is quasi-compact, then there is a smooth surjection \( S' \to S \) such that \( f \times_S S' \) is (R-)unipotent.

**Proof.** Passing to a smooth cover of \( S \), we may assume that \( S \) is an affine scheme and so has the Flag-resolution property. The result follows from Theorem 5.5. \( \square \)

**Corollary 5.8.** Let \( f: X \to S \) be a morphism of algebraic stacks.

1. If \( f \) is geometrically (R-)unipotent, then \( f \times_S \text{Spec} \, k: X \times_S \text{Spec} \, k \to \text{Spec} \, k \) is (R-)unipotent for every field \( k \) and morphism \( \text{Spec} \, k \to S \).

2. Let \( S' \to S \) be surjective. If \( f': X \times_S S' \to S' \) is geometrically (R-)unipotent, then \( f \) is geometrically (R-)unipotent.

**Proof.** In (2), we may assume that \( S = \text{Spec} \, k \) and \( S' = \text{Spec} \, k' \), where \( k \subseteq k' \) is a field extension and \( k' \) is algebraically closed. It suffices to prove that if \( f' \) is (R-)unipotent, then \( f \) is (R-)unipotent, which also implies (1). But \( S' \to S \) is an fpqc covering, so \( f \) is locally (R-)unipotent. Since \( S \) is affine, it has the Flag-resolution property. The result now follows from Theorem 5.5. \( \square \)

**5.2. Unipotent groups.** A powerful source of unipotent morphisms will be unipotent groups and gerbes. Let \( k \) be a field. Let \( G \) be an algebraic group over \( k \). There are several characterizations of unipotence over \( k \) [CPS Thm. XVII.3.5]. We take the following: \( G \) is unipotent if there is an embedding into the upper triangular unipotent matrices \( G \hookrightarrow U_{n,k} \subseteq \text{GL}_{n,k} \) for some \( n \). Families of unipotent groups over general bases are more subtle, however, so we make the following definition.

**Definition 5.9.** Let \( S \) be an algebraic stack. A group \( G \to S \) that is flat and of finite presentation is said to be (R-)unipotent, locally (R-)unipotent, or geometrically (R-)unipotent if the corresponding gerbe \( BG \to S \) is so.
Remark 5.10. We note some potential confusion with this terminology: unipotence of the group \( G \to S \) is not the same thing as unipotence of the morphism \( G \to S \). For instance, every quasi-compact, quasi-separated and representable morphism of algebraic stacks is unipotent, but an affine group scheme \( G \to S \) need not be unipotent in the sense of Definition 5.9.

Remark 5.11. A group \( G \to S \) that is flat and of finite presentation is unipotent (resp. \( R \)-unipotent) if and only if there exists an ordered sequence of line bundles \( \mathcal{L} \) on \( S \), an object \( V_\bullet \) of \( \text{Flag}_{\mathcal{L}}(S) \) and a monomorphism (resp. with quasi-affine quotient):

\[
G \hookrightarrow \text{Aut}_{\text{Flag}_{\mathcal{L}}}(V_\bullet) \subset \text{GL}(V)
\]

Thus, (\( R \)-)unipotent groups are (\( R \)-)embeddable. Note that the above automorphism group is an inner form of \( U(\mathcal{L}) \). In particular, if \( S \) is a scheme, then \( G \) can be embedded in a Zariski form of \( U_{n,S} \). Hence, if \( S = \text{Spec} \ k \) is the spectrum of a field, then \( G \to \text{Spec} \ k \) is unipotent (in the sense of Definition 5.9) if and only if there exists an embedding \( G \subseteq U_{n,k} \) for some \( n \), so both possible definitions of a unipotent group scheme agree over a field.

Example 5.12. By Proposition 2.19, the group \( U_{n,S} \to S \) is \( R \)-unipotent, as is any flat and finitely presented closed subgroup \( H \subset U_{n,S} \) when either \( S \) is the spectrum of a field or a Dedekind domain, or \( H \to S \) is finite. In particular, if \( G \to S \) is (locally) unipotent and either \( G \to S \) is finite or \( S \) is the spectrum of a field, then it is (locally) \( R \)-unipotent.

Now translating Theorem 5.5 into groups we obtain the following.

Corollary 5.13. Let \( S \) be a quasi-compact algebraic stack with affine diagonal. Let \( G \to S \) be a locally \( (R-) \)unipotent group.

1. If \( S \) has the resolution property, then \( G \to S \) is \( (R-) \)embeddable.
2. If \( S \) has the \( \text{Flag} \)-resolution property, then \( G \to S \) is \( (R-) \)unipotent.

Note the stark contrast to the case of tori. Tori are always locally embeddable, but may not be embeddable even over a projective curve.\[\text{GP98}\] X.1.6 & XI.4.6.

Remark 5.14. By Example 5.12 and Corollary 5.13, all the definitions in Definition 5.9 agree when \( S \) is the spectrum of a field. Thus, one may speculate what the best definition of unipotence should be over a general base. We argue that local \( R \)-unipotence is the best behaved as evidenced by Theorem 5.5, Examples 5.12, 5.16, 5.17, 5.22, and Proposition 7.1.

We have the following proposition, which justifies our terminology.

Proposition 5.15. Let \( f : X \to S \) be a geometrically unipotent morphism of algebraic stacks. Then the relative geometric stabilizers of \( f \) are all unipotent groups.

Proof. We may assume that \( S = \text{Spec} \ l \), where \( l \) is an algebraically closed field. Let \( \tilde{x} : \text{Spec} \ k \to X \) be a point of \( X \), where \( k \) is an algebraically closed field. Let \( G_\mathcal{E} = \text{Aut}_X(\tilde{x}) \) be its automorphism group. Then we have a representable morphism \( \tilde{x} : BG_\mathcal{E} \to X \). By Lemma 5.3.11, \( BG_\mathcal{E} \) is unipotent. Now apply the last sentence of Remark 5.11. \[\square\]

Example 5.16. Let \( S \) be an algebraic stack. Let \( E \) be a locally free sheaf on \( S \) of rank \( n \). Then the vector group \( V(E) \to S \) is locally \( R \)-unipotent. We may work locally on \( S \), so we may assume that \( E \simeq \mathcal{O}_S^n \). In this case, \( V(E) \simeq G_{a,S}^n \). Now \( G_{a,Z}^n \subseteq U_{n+1,Z} \subseteq GL_{n+1,Z} \). Since \( Z \) is a Dedekind domain, it follows from Example 5.12 that \( G_{a,Z}^n \to \text{Spec} \ Z \) is \( R \)-unipotent. Hence, \( V(E) \to S \) is locally \( R \)-unipotent. If \( S \) is quasi-compact with affine diagonal and the \( \text{Flag} \)-resolution property, then \( V(E) \to S \) is even \( R \)-unipotent (Corollary 5.13).
Example 5.17. Let \( p > 0 \) be a prime. Let \( S \) be an algebraic \( \mathbb{F}_p \)-stack. Let \( G \to S \) be a finite étale group scheme of degree \( p^d \) for some \( d \geq 0 \). Then \( G \to S \) is locally \( R \)-unipotent. Indeed, smooth locally on \( S \) we may assume that \( G \to S \) is a constant group scheme. We may thus assume that \( S = \text{Spec} \mathbb{F}_p \) and \( G \) is a constant \( p \)-group. Standard facts from group theory show that \( G \) is unipotent, and the claim follows.

Example 5.18. If \( X \) denotes the affine line with doubled origin, then it is an \( R \)-unipotent scheme. Indeed if \( x, y \in X \) denote the two origins, then the trivially graded flag \( \mathcal{O}_X = I_{\{(x,y)\}} \subset I_{\{(x)\}} \oplus I_{\{(y)\}} \) is a tensor generator, where \( I_C \) is the ideal sheaf associated to a reduced closed subscheme \( C \subset X \).

5.3. **Representable \( R \)-unipotent morphisms.** We now briefly discuss representable (locally) \( R \)-unipotent morphisms \( f : X \to S \). Characterizing such morphisms seems subtle. For instance, consider for simplicity the case when \( S = \text{Spec} k \) is the spectrum of a field. It follows from Theorem 5.4 that representable \( R \)-unipotent morphisms \( f : X \to \text{Spec} k \) are exactly quotients of the form \( T/U_n \), where \( T \) is a quasi-affine scheme over \( k \) acted on freely by \( U_n \). In particular, a quasi-affine morphism over \( k \) is \( R \)-unipotent and hence geometrically \( R \)-unipotent. The converse is not true, however.

Example 5.19.

1. Let \( k \) be a field and denote by \( X \) the complement of a single point in the exceptional locus of \( \mathbb{A}^2_\mathbb{C} \) blown up at the origin. We claim that the morphism \( f : X \to \mathbb{A}^2_\mathbb{C} \) is geometrically \( R \)-unipotent but is not quasi-affine. Indeed, the geometric fibers are all affine and hence the morphism is geometrically \( R \)-unipotent. On the other hand, if \( f \) was quasi-affine then the natural map \( g : X \to \text{Spec} \mathbb{C}[X, \mathcal{O}_X] \) would be an open immersion [Sta, Tag 01SM] and this cannot be true since \( \Gamma(X, \mathcal{O}_X) = k[x, y] \).

2. In [AD07] Ex. 3.16, Asok and Doran describe a free action of \( U_1 = \mathbb{G}_a \) on \( \mathbb{A}^5_\mathbb{C} \) such that the quotient \( \mathbb{A}^5_\mathbb{C}/\mathbb{G}_a \) is not even a scheme. More generally, given a smooth affine scheme \( T \) over \( k \) equipped with a free and proper action of \( U_n \), [AD07] Corollary 3.18 yields an effective criterion for determining when the quotient \( T/U_n \) will be affine or quasi-affine.

We have the following result, whose proof we defer until §7.

**Proposition 5.20.** Let \( f : X \to S \) be a representable and geometrically \( R \)-unipotent morphism of algebraic stacks. If \( f \) is proper, then \( f \) is finite.

It follows from Proposition 5.20 that representable geometrically \( R \)-unipotent morphisms over a field do not contain any positive-dimensional proper subvarieties.

5.4. **Geometrically unipotent vs locally unipotent groups.** We conclude this section with some thoughts on the following natural refinement of Question 1.4 which geometrically unipotent groups are locally unipotent? Note that locally unipotent groups are *always* quasi-affine; in particular, they are separated and schematic. In positive characteristic, it is easy to produce geometrically unipotent group algebraic spaces that are not separated. There are also separated geometrically unipotent group algebraic spaces that are not schemes. In particular, these give examples of geometrically unipotent groups that are not locally unipotent.

Example 5.21. Let \( k \) be a field of characteristic 2. Let \( S = \mathbb{A}^1_\mathbb{F}_2 \). Let \( H \subseteq (\mathbb{Z}/2\mathbb{Z})_S = G \) be the étale subgroup obtained by deleting the non-trivial point over the origin.

1. Let \( Q = G/H \to S \) be the line with the doubled-origin, viewed as a group scheme. Then \( Q \to S \) is geometrically unipotent, but non-separated. In particular, it is not locally unipotent.
(2) We have $H \subseteq G \subseteq G_{2,a,S}$. Let $Q' = G_{2,a}/H \rightarrow S$. Then $Q' \rightarrow S$ is smooth, geometrically unipotent, with connected fibers, non-separated, and not a scheme [GP11a VI 5.5.1]. This shows that the closed subgroup condition in Proposition 2.19(2) is necessary.

(3) A more sophisticated example was constructed in [Ray70 X.14]: take $T = A^2_k$; then there is a closed subgroup $N \subseteq G_{2,a,T}$ with $N \rightarrow T$ étale. Taking $Q'' = G_{2,a,T}/N$, we produce a smooth, separated, geometrically unipotent group of finite presentation with connected fibers. There it is shown that $Q''$ is not representable by a scheme. This shows that the condition that the quotient is representable by a scheme in Proposition 2.19(2) is necessary.

In characteristic 0, there are strong results in the existing literature, however.

Example 5.22. Let $S$ be an algebraic stack of equicharacteristic 0. Let $G \rightarrow S$ be geometrically unipotent and flat-locally schematic. Then $G \rightarrow S$ is affine. Indeed, this is local on $S$, so we may assume that $S$ is affine. By standard limit methods, we may assume that $S$ is of finite type over $\text{Spec} \ Z$ and so excellent; in particular, $S$ has noetherian normalization. If $S$ is either normal or $G \rightarrow S$ is embeddable, then there is an isomorphism of groups

$$\exp : V(\text{Lie}(G)) \rightarrow G,$$

where we endow the left hand side with the group structure coming from the Baker–Campbell–Hausdorff formula. In particular, when $G \rightarrow S$ is commutative, it is a vector group, so is locally R-unipotent. The statement involving the exponential map is established in [Ton15 §1.3] and [Ray70 XV.3]. That $G \rightarrow S$ is affine follows from the normal case and Chevalley’s Theorem [Ryd15 Thm. 8.1].

We also have the following examples.

Example 5.23. Let $G \rightarrow S$ be an affine and geometrically unipotent group.

(1) If $S = \text{Spec} \ A$, where $A$ is a Prüfer domain, then $G \rightarrow S$ is R-unipotent.

(2) If $S$ is equicharacteristic $p > 0$ and $G \rightarrow S$ is a finite locally free commutative group scheme with dual of height 1, then $G \rightarrow S$ is locally R-unipotent.

For (1): by Lemma 4.5 every vector bundle on $BG$ admits the structure of a complete $S$-graded flag. In combination with Corollary 3.4, this implies that $G$ is R-unipotent.

For (2): the exact sequence of [AM76 Prop. 1.1] implies there is an affine morphism $BG \rightarrow BV(\omega)$ for some vector group $V(\omega) \rightarrow S$. Since $V(\omega) \rightarrow S$ is locally R-unipotent (Example 5.16), (2) follows.

6. Applications

6.1. Unipotent gerbes and the resolution property. Theorem 5.5 immediately yields a proof of Theorem A.

Proof of Theorem A. A $G_{2,a}$-gerbe is locally R-unipotent (Example 5.12). It follows from Theorem 5.5 that $\mathcal{G}$ has the resolution property. □

Remark 6.1. Assume that $X$ lives over a scheme $S$ and $G \rightarrow S$ is locally R-unipotent: for example, $G = G_{2,a}, G = U_{n,S}$, or $S$ is the spectrum of a field and $G \subseteq U_{n,S}$ (see Remark 5.12). Then the argument above for Theorem A also holds for $G$-gerbes $\mathcal{G} \rightarrow X$.

We next obtain the following corollary of Theorem 5.5, which is key for our proof of Theorem B.

Corollary 6.2. Let $S$ be a quasi-compact algebraic stack with affine diagonal over $\mathbb{F}_p$. Let $\pi : \mathcal{G} \rightarrow S$ be a gerbe that is relatively Deligne–Mumford, separated and with $p$-power order inertia.
(1) If $S$ has the resolution property, then $\mathcal{G}$ has the resolution property.
(2) If $S$ has the Flag-resolution property, then $\pi$ is $R$-unipotent.

Proof. Since $\pi$ is relatively Deligne–Mumford and separated, there is a smooth covering $S' \to S$ such that $\mathcal{G} \times_S S' \simeq BG'$, where $G' \to S'$ is finite étale. The $p$-power order inertia assumption on $\pi$ implies that $G' \to S'$ has degree $p^d$ for some $d \geq 0$. Example [5.17] shows that $G' \to S'$ is locally $R$-unipotent. Now apply Theorem [5.3].

6.2. Deligne–Mumford Gerbes. Let $X$ be an algebraic stack. Then there is an injective group homomorphism:

$$\text{Br}(X) \to \text{Br}'(X) = H^2(X, G_m)_{\text{tors}}$$

from the geometric Brauer group of $X$ to the cohomological Brauer group of $X$. However, it is still not known when the map is surjective (see, for instance, [Mat22] for an introduction to this problem).

We are now in a position to prove one of our key results.

**Theorem 6.3.** Let $S$ be a connected quasi-compact algebraic stack with affine diagonal and the resolution property over a field $k$. Let $\pi: \mathcal{G} \to S$ be a gerbe that is relatively Deligne–Mumford and separated. If $\text{Br}(T) = \text{Br}'(T)$ for all finite étale morphisms $T \to S$ (e.g. if $S$ is a quasi-compact scheme that admits an ample line bundle), then $\mathcal{G}$ has the resolution property.

Proof. The gerbe $\mathcal{G} \to S$ is locally banded by a finite constant group scheme $G_S$. It follows that the sheaf of isomorphisms (in the category of Bands over $S$):

$$I = \text{Isom}(\text{Band}(\mathcal{G}), G_S)$$

is a torsor under the outer automorphism group $\text{Out}(G_S) \to S$. This map is finite étale, so we may replace $S$ with $I$ as the resolution property descends under such maps [Gro17 Prop. 5.3(vii)]. Hence, we can assume that $\mathcal{G} \to S$ is banded by $G_S$, which is constant. Now there is a factorization:

$$\mathcal{Z} \to \mathcal{G} \to S,$$

where $\mathcal{Z} \to S$ is a gerbe banded by the center of $G_S$, which is a finite abelian constant group scheme and $\mathcal{Z} \to \mathcal{G}$ is finite étale and surjective. Indeed, define $\mathcal{Z}$ to be the stack of banded equivalences

$$\mathcal{Z} = \text{HOM}_d(BG_S, \mathcal{G}) \to S.$$  

By [Gir71 IV.2.3.2(iii)], $\mathcal{Z}$ is banded by $Z(G_S) = Z(G)_S$ and the natural evaluation map $\text{ev}: \mathcal{Z} \to \mathcal{G}$ induces a map on bands equal to the inclusion $Z(G)_S \subset G_S$. Again, by [Gro17 Prop. 5.3(vii)], we may now replace $\mathcal{G}$ by $\mathcal{Z}$. Thus, $\mathcal{G}$ is banded by the constant group scheme associated to the finite abelian group

$$\mathbf{Z}/p_i^n \mathbf{Z} \times \cdots \times \mathbf{Z}/p_i^n \mathbf{Z},$$

where the $p_i$ are primes. Hence, $\mathcal{G} \cong \mathcal{Z}_1 \times_S \cdots \times_S \mathcal{Z}_i$, where $\mathcal{Z}_i$ is a $\mathbf{Z}/p_i^n \mathbf{Z}$-gerbe for all $i$. For each $p_i$ that is prime to the characteristic of $k$, consider the finite and separable field extension $k \subseteq k'$ obtained by adding in all the $p_i$-th roots of unity. As before, it suffices to prove the result after base changing to $k'$. In particular, for those $p_i$ we now have $\mathbf{Z}/p_i^n \mathbf{Z} \cong \mu_{p_i^n}$. Since $\text{Br}(S) = \text{Br}'(S)$, the resulting $\mathcal{Z}_i$ have the resolution property. For the remaining $p_i$ that are the characteristic of $k$, these $\mathcal{Z}_i$ have the resolution property by Corollary [6.2]. An easy calculation now shows that $\mathcal{G}$ has the resolution property. □
6.3. Deligne–Mumford stacks in positive characteristic. We finally come to the proof of Theorem [3]

Proof of Theorem [3] We may assume that $\mathcal{X}$ is connected. By [Obs07 Prop. 2.1], the morphism $\mathcal{X} \to X$ can be rigidified. Hence, it factors into a gerbe morphism and a coarse moduli space morphism:

$$\mathcal{X} \to \mathcal{X}^{\text{rig}} \to X,$$

where $\mathcal{X}^{\text{rig}}$ has generically trivial stabilizers. It now follows that $\mathcal{X}^{\text{rig}}$ is a global quotient [EHKV01 Thm. 2.18]. By [KV01 Thm. 2.1] (see also [DHM22 Cor. 4.5]), there is a finite flat cover $Z \to \mathcal{X}^{\text{rig}}$, where $Z$ is a quasi-projective scheme. Base changing along this cover, we may replace $\mathcal{X}^{\text{rig}}$ by $Z$ [Gro17 Prop. 5.3(vii)]. Then $\mathcal{X} \to X$ is a separated Deligne–Mumford gerbe, where $X$ is a quasi-projective scheme over $\text{Spec} \ k$. The result now follows from Theorem [EHKV01] and Gabber’s Theorem [Alp17]. □

7. Faithful moduli spaces

If $V$ is a nonzero representation of the unipotent group $U_{n,k}$ over a field $k$, then the set of fixed points $V^{U_{n,k}}$ is nonzero [GP08 XVII, Prop. 3.2]. If $X$ is a quasi-affine scheme, and $F$ is a nonzero quasi-coherent sheaf on $X$, then $\Gamma(X,F)$ is nonzero. The following result shows that this property generalizes to any locally R-unipotent morphism.

Proposition 7.1. Let $f: X \to S$ be a locally R-unipotent morphism of algebraic stacks. If $F$ is quasi-coherent on $X$ and $f^*F = 0$, then $F = 0$.

Proof. By flat base change we may assume that $f$ is R-unipotent and $S$ is affine. Let $N$ be a quasi-coherent $\mathcal{O}_S$-module; then $\text{Hom}_{\mathcal{O}_X}(f^*N, F) = \text{Hom}_{\mathcal{O}_S}(N, f_*F) = 0$. It follows that if $V$ belongs to $\text{Flag}_f$, then $\text{Hom}_{\mathcal{O}_X}(V, F) = 0$. As $f$ is R-unipotent, it has the $\text{Flag}_f$-resolution property (Theorem [5.3]). Hence, $F = 0$. □

The following definition is due to Alper [Alp17].

Definition 7.2. Let $f: X \to S$ be a quasi-compact and quasi-separated morphism of algebraic stacks. We say that $f$ is a faithful moduli space if:

1. the natural map $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism (i.e., $f$ is Stein); and
2. if $F \in \text{QCoh}(X)$ and $f_*F = 0$, then $F = 0$ (i.e., $f_*$ is zero-reflecting).

Remark 7.3. Faithful moduli spaces are compatible with quasi-affine flat base change. Indeed, if $p: S' \to S$ is flat and quasi-affine and $f': X' = X \times_S S' \to S'$ is the induced morphism, then flat base change says that (1) is preserved. Let $p': X' \to X$ be the induced morphism; then $p_*$ and $p'_*$ are zero-reflecting on quasi-coherent sheaves because they are quasi-affine. Hence, $p_*, p'_* \simeq f_*, f'_*$ is zero-reflecting on quasi-coherent sheaves, and so $f'_*$ is zero-reflecting on quasi-coherent sheaves, which proves (2). In particular, if $f: X \to S$ is a faithful moduli space and $S$ has quasi-affine diagonal, then $f$ remains a faithful moduli space after arbitrary flat base change on $S$.

Example 7.4. A Stein and locally R-unipotent morphism is a faithful moduli space (Proposition [7.1]). Also if $S$ is a normal scheme, then any open immersion $U \subseteq S$ with complement of codimension $\geq 2$ is Stein and R-unipotent. In particular, there is not necessarily an induced bijection on closed points, so some may find the term “moduli space” misleading.

Remark 7.5. Note that a faithful moduli space $f$ need not imply that the functor $f_*$ is faithful. Indeed, $f_*$ being faithful forces $f$ to be quasi-affine (because the structure
sheaf would be \( f \)-generating). Moreover, \( f \) being Stein implies that the quasi-affine morphism \( f \) is an open immersion. Thus, the second example in Example 7.4 is essentially the only example of a faithful moduli space for which \( f_* \) is faithful.

**Remark 7.6.** The converse to Proposition 7.4 does not hold: if \( X \) denotes affine \( n \)-space with a doubled origin and \( f: X \to \mathbb{A}^n \) is the natural map, \( f \) is a faithful moduli space but is not locally \( R \)-unipotent unless \( n = 1 \) (see Example 5.18).

Indeed, it cannot be locally \( R \)-unipotent when \( n \geq 2 \), because Theorem 5.5 would imply that \( X \) has affine diagonal. On the other hand, \( f \) is Stein. To see that \( f_* \) is zero-reflecting, use Lemma 7.7.

**Lemma 7.7.** Let \( f: X \to Y \) be a morphism of algebraic stacks which is representable and quasi-finite, then \( f_*: \text{QCoh}(X) \to \text{QCoh}(Y) \) is zero-reflecting.

**Proof.** By passing to a smooth cover of \( Y \), we may assume that \( Y \) is an affine scheme and \( X \) is an algebraic space. Let \( F \) be a quasi-coherent \( \mathcal{O}_X \)-module with \( f_* F = 0 \); then the same is true for any of its finite-type subsheaves, so we may assume \( F \) is also of finite-type. If \( F \neq 0 \), then we may replace \( X \) with \( V(\text{Ann}_{\mathcal{O}_X}(F)) \) and then \( Y \) with \( V(\ker(\mathcal{O}_Y \to f_* \mathcal{O}_X)) \). Then \( \text{Ann}_{\mathcal{O}_X}(F) = 0 \) and \( f \) has schematically dense image. Since \( f \) is quasi-finite, there is a non-empty open subset \( V \subseteq Y \) so that \( f^{-1}(V) \to V \) is finite. Then \( F|_{f^{-1}(V)} = 0 \) and so \( F = 0 \), which is a contradiction. \( \square \)

**Example 7.8.** Let \( k \) be a field. Let \( G \) be an algebraic group scheme over \( k \). Then \( BG \to \text{Spec} k \) is a faithful moduli space if and only if \( G \) is unipotent. One implication follows from Example 7.4. For the other: if \( V \) is a finite-dimensional representation of \( G \) over \( k \); then \( V^G \neq 0 \). By induction, \( V \) is a trivially graded flag on \( BG \). Taking \( V \) to be a faithful representation of \( G \), we see that \( G \) admits a faithful flag. In particular, it is unipotent.

**Example 7.9.** Let \( Y \xrightarrow{g} X \xrightarrow{f} S \) be quasi-compact and quasi-separated morphisms of algebraic stacks.

1. If \( f \) and \( g \) are faithful moduli spaces, then \( f \circ g \) is a faithful moduli space.
2. If \( f_* \) is zero-reflecting, then \( X \to \text{Spec}_g (f_* \mathcal{O}_X) \) is a faithful moduli space.

The following result is a version of Proposition 5.15 for faithful moduli spaces.

**Proposition 7.10.** Let \( f: X \to S \) be a faithful moduli space with affine stabilizers. Then the relative geometric stabilizers of \( f \) are unipotent groups.

**Proof.** We may assume that \( S \) is an affine scheme; then \( X \) is quasi-compact and quasi-separated. Let \( \bar{x}: \text{Spec} k \to X \) be a geometric point of \( X \), where \( k \) is an algebraically closed field. Let \( G = \text{Aut}(\bar{x}) \), which is a group scheme of finite type over \( \text{Spec} k \). There is an induced quasi-affine morphism \( BG \to X \) [Ryd81, Thm. B.2] and so \( BG \to \text{Spec} k \) is a faithful moduli space (Example 7.4(2)). By Example 7.8 the result follows.

The following lemma turns out to be very useful.

**Lemma 7.11.** Let \( f: X \to S \) be a quasi-compact, quasi-separated and Stein morphism of algebraic stacks. If \( I \subseteq \mathcal{O}_X \) is a quasi-coherent sheaf of ideals, then \( V(I) \subseteq f^{-1}(V(f_* I)) \).

**Proof.** Indeed, the quasi-coherent sheaf defining the closed substack \( f^{-1}(V(f_* I)) \subseteq X \) is the image of \( f^* f_* I \to \mathcal{O}_X \), which factors through \( I \). \( \square \)

**Proposition 7.12.** Let \( f: X \to S \) be a faithful moduli space with affine stabilizers. If \( S \) is reduced and quasi-separated with affine stabilizers, then there is a non-empty open \( U \subseteq S \) with \( f^{-1}(U) \to U \) a locally \( R \)-unipotent gerbe.
Proof. We may assume that $S$ is quasi-compact; then [HR15 Prop. 2.6(i)] implies that there is a non-empty open quasi-compact $U \subseteq S$ with affine diagonal. Let $\text{Spec}A \to U$ be a smooth cover; then the composition $\text{Spec}A \to U \to S$ is quasi-affine and smooth. It follows from Remark 7.3 that we may now assume that $S = \text{Spec}A$, where $A$ is reduced.

Now there is a non-empty open immersion $\mathcal{G} \subseteq X_{\text{red}}$ such that $\mathcal{G}$ is a gerbe with the resolution property with affine coarse space $g: \mathcal{G} \to T$ (combine [Sta Tags 06RC & 06NH] with [HR15 Prop. 2.6(i)]). Let $I \subseteq \mathcal{O}_X$ be a quasi-coherent ideal sheaf defining the reduced closed complement $Z$ of $\mathcal{G} \subseteq X$. If $Z \neq \emptyset$, then $\Gamma(X, \mathcal{O}_X) \to \Gamma(Z, \mathcal{O}_Z)$ is non-zero because it is a ring homomorphism and $\Gamma(Z, \mathcal{O}_Z) \neq 0$. In particular, $\Gamma(X, I) \subseteq A$. Since $I \neq 0$, it follows that we may choose $a \in \Gamma(X, I) \setminus \{0\}$ that is not a unit. Since $A$ is reduced, $a$ is not nilpotent and so $\text{Spec}A_a \neq \emptyset$. But Lemma 7.11 implies that $Z \cap f^{-1}(\text{Spec}A_a) = \emptyset$ and so we may replace $A$ by $A_a$ and assume that $Z = \emptyset$. Hence, $\mathcal{G} = X_{\text{red}} \subseteq X$ is a surjective closed immersion with defining ideal $J$. By Lemma 7.11 again, we may further shrink $S$ so that $J = 0$ and $\mathcal{G} = X$ is a gerbe. But the composition $\mathcal{O}_S \to \mathcal{O}_T \to f_*\mathcal{O}_X$ is an isomorphism, as is $\mathcal{O}_T \to g_*\mathcal{O}_X$; hence, $T \simeq S$ and $X \to S$ is of finite presentation. By standard limit methods and Theorem 5.3, we may replace $S$ by the spectrum of the localization at a minimal prime of $A$, which is a field [Sta Tag 00EU]. By Example 7.8 and Remark 7.11 the result follows. □

We conclude with the following pleasant corollary.

Corollary 7.13. Let $k$ be a field. Let $X \to \text{Spec}k$ be a faithful moduli space. Then $X \to \text{Spec}k$ is an $R$-unipotent gerbe. In particular, if $X$ is an algebraic space, then $X \to \text{Spec}k$ is an isomorphism.

We conclude with the proof of our characterization of proper and representable geometrically $R$-unipotent morphisms.

Proof of Proposition 7.20. We may assume that $S$ is affine. By Zariski’s Main Theorem, it suffices to prove that $f$ is quasi-finite. Hence, we may further reduce to the situation where $S$ is the spectrum of an algebraically closed field $k$ and $f$ is $R$-unipotent. Let $X \to S' \to S$ be the Stein factorization of $f$. Then $S' \to S$ is finite, the morphism $X \to S'$ is again $R$-unipotent, and so we may further reduce to the situation where $X$ is geometrically connected and $f$ is Stein. It follows from Example 7.3 that $X \to \text{Spec}k$ is a faithful moduli space. By Corollary 7.13 $X \to \text{Spec}k$ is an isomorphism, which gives the claim. □

REFERENCES

[AD07] A. Asok and B. Doran, On unipotent quotients and some $h^1$-contractible smooth schemes, Int. Math. Res. Pap. IMRP 2007 (2007), no. 2.

[Alp13] J. Alper, Good moduli spaces for Artin stacks, Ann. Inst. Fourier (Grenoble) 63 (2013), no. 6, 2349–2402.

[Alp14] ———, Adequate moduli spaces and geometrically reductive group schemes, Algebr. Geom. 1 (2014), no. 4, 489–531. MR 3272912

[Alp17] ———, Faithful moduli spaces, 2017, private communication.

[AM76] M. Artin and J. S. Milne, Duality in the flat cohomology of curves, Inventiones mathematicae 35 (1976), no. 1, 111–129.

[AOV08] D. Abramovich, M. Olsson, and A. Vistoli, Tame stacks in positive characteristic, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 4, 1057–1091. MR MR2427954 (2009c:14002)

[BT84] F. Bruhat and J. Tits, Groupes r éductifs sur un corps local. II. Sch émas en groupes. Existence d’une donnée radicielle valuée, Inst. Hautes Etudes Sci. Publ. Math. (1984), no. 60, 197–376. MR 756316

[Con10] B. Conrad, Smooth linear algebraic groups over the dual numbers, MathOverflow, 2010, http://mathoverflow.net/q/22078
[Muk78] S. Mukai, *Semi-homogeneous vector bundles on an Abelian variety*, J. Math. Kyoto Univ. **18** (1978), no. 2, 239–272. MR 498572

[Oda71] T. Oda, *Vector bundles on an elliptic curve*, Nagoya Math. J. **43** (1971), 41–72. MR 318151

[Ols07] M. Olsson, *A boundedness theorem for Hom-stacks*, Math. Res. Lett. **14** (2007), no. 6, 1009–1021. MR 2357471

[Ray70] M. Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, Lecture Notes in Mathematics, Vol. 119, Springer-Verlag, Berlin-New York, 1970. MR 0260758

[Ros61] M. Rosenlicht, *On quotient varieties and the affine embedding of certain homogeneous spaces*, Trans. Amer. Math. Soc. **101** (1961), 211–223. MR 0130878 (24 #A732)

[Ryd09a] D. Rydh, *Remarks on "The resolution property for schemes and stacks" by B. Totaro*, available at https://people.kth.se/~dary, 2009.

[Ryd09b] D. Rydh, *Étale dévissage, descent and pushouts of stacks*, J. Algebra **331** (2011), 194–223. MR 2774654

[Ryd15] D. Rydh, *Noetherian approximation of algebraic spaces and stacks*, J. Algebra **422** (2015), 105–147.

[Ryd16] D. Rydh, *Approximation of sheaves on algebraic stacks*, Int. Math. Res. Not. **2016** (2016), no. 3, 717–737.

[Ryd23] D. Rydh, *Absolute noetherian approximation of algebraic stacks*, Preprint, Nov 2023, p. 16.

[Sch99] S. Schröer, *On non-projective normal surfaces*, Manuscripta Math. **100** (1999), no. 3, 317–321. MR 1726321

[Sch17] D. Schäppi, *A characterization of categories of coherent sheaves of certain algebraic stacks*, J. Pure Appl. Algebra (2017), to appear.

[Sta] The Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu

[SV04] S. Schröer and G. Vezzosi, *Existence of vector bundles and global resolutions for singular surfaces*, Compos. Math. **140** (2004), no. 3, 717–728. MR 2041778

[Ton15] J. Tong, *Unipotent groups over a discrete valuation ring (after Dolgachev-Weisfeiler)*, Autour des schémas en groupes. Vol. III, Panor. Synthèses, vol. 47, Soc. Math. France, Paris, 2015, pp. 173–225. MR 3525845

[Tot04] B. Totaro, *The resolution property for schemes and stacks*, J. Reine Angew. Math. **577** (2004), 1–22. MR 2108211 (2005j:14002)

Email address: braggdan@berkeley.edu

University of Utah, Department of Mathematics, 155 South 1400 East, Salt Lake City, UT 84112, USA

Email address: jack.hall@unimelb.edu.au

School of Mathematics & Statistics, The University of Melbourne, Parkville, VIC, 3010, Australia

Email address: siddharth.mathur@uc.cl

Departamento de Matemática, Pontificia Universidad Católica de Chile, Santiago, Chile.