Transitive irreducible Lie superalgebras of vector fields

Arkady Onishchik

Abstract. Let \( \mathfrak{d} \) be the Lie superalgebra of superderivations of the sheaf of sections of the exterior algebra of the homogeneous vector bundle \( E \) over the flag variety \( G/P \), where \( G \) is a simple finite-dimensional complex Lie group and \( P \) its parabolic subgroup. Then, \( \mathfrak{d} \) is transitive and irreducible whenever \( E \) is defined by an irreducible \( P \)-module \( V \) such that the highest weight of \( V^* \) is dominant. Moreover, \( \mathfrak{d} \) is simple; it is isomorphic to the Lie superalgebra of vector fields on the superpoint, i.e., on a 0\(|n\)-dimensional supervariety.

Preface of the editor
The manuscript of this paper was deposited in VINITI 12.06.86 No 4329-B which is inaccessible. There still appear papers with references to some parts of this inaccessible deposition, see, e.g., [IO*], [OVs*] which contains references to this preprint and its continuation [On*] preprinted in the proceedings of the “Seminar on Supersymmetries”, see http://staff.math.su.se/mleites/sos.html. So I decided to make it available, together with [On*], by translating these texts and updating the references; the ones I added are endowed with an asterisk. The abstract, footnotes, and comments are due to me.

For a comprehensive description of simple Lie superalgebras of vector fields over algebraically closed fields of any characteristic, see [BGLLS*].

1 Basics. Introduction
Recall that a filtered Lie superalgebra

\[ \mathfrak{a} = \mathfrak{a}(-d) \supset \cdots \supset \mathfrak{a}(-1) \supset \mathfrak{a}(0) \supset \mathfrak{a}(1) \supset \cdots, \quad \text{where} \; d \in \mathbb{N} := \{1, 2, \ldots\} \]

is said to be transitive, see [Dr*], if

\[ \mathfrak{a}_{(p+1)} = \{ x \in \mathfrak{a}_{(p)} \mid [x, \mathfrak{a}] \subset \mathfrak{a}_{(p)} \} \quad \text{for all} \; p \geq 0. \quad (1) \]

A \( \mathbb{Z} \)-graded Lie superalgebra \( \mathfrak{b} = \oplus_{i \geq -d} \mathfrak{b}_i \) is said to be transitive if for all \( p \geq 0 \) we have

\[ \{ x \in \mathfrak{b}_p \mid [x, \mathfrak{b}_-] = 0 \} = 0, \quad \text{where} \; \mathfrak{b}_- := \oplus_{p<0} \mathfrak{b}_p. \quad (2) \]

MSC 2020: Primary 32C11,17B20
Keywords: Lie superalgebra, homogeneous supermanifold.
Affiliation: N/A
As is well-known, see [K*], [Sch], condition (1) is satisfied if and only if the graded Lie algebra $b := \text{gr } a$ satisfies conditions (2).

We say that a graded Lie superalgebra $b$ is irreducible if the adjoint representation of $b_0$ in $b_{-1}$ is irreducible. (The term irreducible is usually used to denote the image of the (Lie) algebra or superalgebra $b$ in any irreducible representation. D.L.)

Simple, and even primitive\(^1\), Lie (super)algebras have a filtration in which the associated graded Lie (super)algebra is both transitive and irreducible. For the classification of simple finite-dimensional Lie superalgebras over $\mathbb{C}$, see [K*], [Sch]. (Actually, no complete proof of the classification of simple finite-dimensional Lie superalgebras over $\mathbb{C}$ is known to this day; e.g., the proof of completeness of the list of examples of known deformations with odd parameters is not published. Such deformations were not even mentioned in [K*], [Sch]. D.L.)

Hereafter I assume that the $\mathbb{Z}$-grading of every Lie superalgebra considered is compatible with parity.

In this paper, I show that such Lie superalgebras naturally appear in the study of homogeneous vector bundles over complex homogeneous flag varieties $G/P$, where $P$ is a parabolic subgroup of a complex semisimple Lie group $G$. Namely, to each homogeneous vector bundle $E \to M$ there is associated a split complex supermanifold $(M, \Lambda(E))$, where $E$ is the sheaf of holomorphic sections of $E$, and $\Lambda(E)$ is the Grassmann algebra of $E$.

Here, I prove that the graded Lie superalgebra $\mathfrak{d} = \Gamma(M, \text{Der } \Lambda(E))$ of vector fields on this supermanifold is transitive and irreducible if $E$ is defined by an irreducible representation $\varphi$ of $P$ satisfying certain natural requirements. The main point here is that the highest weight of the representation $\varphi^*$ should be dominant. This guarantees the existence of sufficiently many holomorphic sections of “the odd tangent bundle” $E^*$ over the supermanifold $(M, \Lambda(E))$.

The Bott–Borel–Weil theorem, along with other standard methods, enables one to explicitly compute the Lie superalgebra $\mathfrak{d}$ in many cases.

In this paper, this computation is carried out in the cases where $E$ is the cotangent bundle over $M$, and $G$ is simple.

2 Vector fields on supermanifolds
The term “supermanifold” is meant in the sense of Berezin-Leites, in the complex–analytic situation, see [L], [MaG*], [Va*]. Let us recall the definition.

\(^1\)Let a Lie algebra $\mathcal{L}$ contain a subalgebra $\mathcal{L}_0$ which does not contain any nonzero ideal of $\mathcal{L}$. Let

$$\mathcal{L}_1 := \{ t \in \mathcal{L}_0 \mid [t, \mathcal{L}] \subset \mathcal{L}_0 \}.$$  

Then, $\mathcal{L}$ is called primitive (see [O*]) if

$$\mathcal{L}_0 \text{ is a maximal subalgebra of } \mathcal{L} \text{ and } \mathcal{L}_1 \neq \{0\}. \tag{3}$$

Unlike primitive Lie algebras (both finite-dimensional and $\mathbb{Z}$-graded of polynomial growth) which do not differ much from the simple Lie algebras, the classification of primitive Lie superalgebras over $\mathbb{C}$ announced by Kac, see [K*], is a wild problem, as shown in [ALSh*]. Compare with [CaKa*].
Let \((n, m)\) be a pair of non-negative integers. Let a model superspace of dimension \(n|m\) be the ringed space \((\mathbb{C}^n, \tilde{O})\), where \(\tilde{O} := \Lambda_\mathcal{O}[\xi]\) for \(\xi = (\xi_1, \ldots, \xi_m)\), and \(\mathcal{O}\) is the sheaf of germs of holomorphic functions on \(\mathbb{C}^n\). A topological space \(M\) equipped with a sheaf of supercommutative superalgebras \(\tilde{O}\), which as a ringed space is locally isomorphic to a model superspace of dimension \(n|m\) is said to be a(n almost) complex supermanifold and \(\tilde{O}\) is its structure sheaf. (For details, e.g., which of almost complex supermanifolds are complex, and for phenomena indigenous not only to the “super” world — real–complex (super)manifolds, see [BGLS*].)

Let us introduce parity in \(\tilde{O}\) by declaring all of the \(\xi_i\) to be odd; so \(\tilde{O} = \tilde{O}_{\bar{0}} \oplus \tilde{O}_{\bar{1}}\) is the sum of its homogeneous components even and odd.

Let \(\mathcal{I} \subset \tilde{O}\) be the subsheaf of ideals generated by subsheaf \(\tilde{O}_{\bar{1}}\) and let \(\tilde{O}_{\text{rd}} := \tilde{O}/\mathcal{I}\). Clearly, \((M, \tilde{O}_{\text{rd}})\) is a(n almost) complex-analytic manifold.

The simplest supermanifolds are the split ones which are described as follows. Let \((M, \mathcal{O})\) be a(n almost) complex-analytic manifold of dimension \(n\) and \(\mathcal{F}\) a locally free analytic sheaf of rank \(m\) on \(M\). Set \(\tilde{O} := \Lambda_\mathcal{O}(\mathcal{F})\). Clearly, \((M, \tilde{O})\) is a supermanifold, and \(\tilde{O}_{\text{rd}}\) is naturally isomorphic to the subsheaf \(\mathcal{O} \subset \tilde{O}\). In what follows I will identify \(\tilde{O}_{\text{rd}}\) with \(\mathcal{O}\) in this situation and will briefly write \(\tilde{O} = \Lambda_\mathcal{O}(\mathcal{F})\). The structure sheaf of the split supermanifold is endowed with a \(\mathbb{Z}\)-grading \(\tilde{O} = \bigoplus_{0 \leq p \leq m} \tilde{O}_p\), where \(\tilde{O}_p = \Lambda^p_\mathcal{O}(\mathcal{F})\).

Observe that every \(C^\infty\) of analytic supermanifold is locally split.\(^2\)

In general, using the subsheaf \(\mathcal{I}^1 := \mathcal{I}\) we construct the following filtration of the structure sheaf:

\[\tilde{O} = \mathcal{I}^0 \supset \mathcal{I}^1 \supset \mathcal{I}^2 \supset \cdots \supset \mathcal{I}^m \supset \mathcal{I}^{m+1} = 0,\]

and the associated \(\mathbb{Z}\)-graded sheaf of superalgebras

\[\text{gr } \tilde{O} = \bigoplus_{0 \leq p \leq m} \text{gr}_p \tilde{O}, \text{ where } \text{gr}_p \tilde{O} = \mathcal{I}^p/\mathcal{I}^{p+1}.\]

Here \(\text{gr}_0 \tilde{O} = \tilde{O}_{\text{rd}}\), and \(\text{gr}_1 \tilde{O} = \mathcal{F}\) is a locally free sheaf of \(\tilde{O}_{\text{rd}}\)-modules. Hence, it is clear that \(\text{gr } \tilde{O} = \Lambda_{\tilde{O}_{\text{rd}}}(\mathcal{F})\).

Thus, to every supervariety \((M, \tilde{O})\) there is a corresponding split supervariety \((M, \text{gr } \tilde{O})\).

Denote by \(\text{Der } \tilde{O}\) the sheaf of germs of derivations of the structure sheaf \(\tilde{O}\). Its stalk at point \(w \in M\) is the Lie superalgebra \(\text{Der}_w \tilde{O}_w\). The sections of the sheaf \(\text{Der } \tilde{O}\) are called vector fields on the supermanifold \((M, \tilde{O})\). The space \(\mathfrak{d} := \Gamma(M, \text{Der } \tilde{O})\) is naturally endowed with a Lie superalgebra structure over \(\mathbb{C}\).

\(^2\)This is not true for algebraic supervarieties and supeschemes, as follows from [AD*]: over a contractible paracompact real set all vector bundles are trivial, but this is not necessarily true over other fields and non-affine schemes. The first example of non-split supervariety is due to P. Green ([Gre*]), see also [Pa*] submitted only 2 month later than [Gre*] and expounded in [B1*, Ch.4, §4, Sections 6–9] reproduced in [B2*, Ch.3, Th.2, p.126], as well as in [MaG*, Ch.4, §2, Prop. 9, p. 191].
If \((M, \tilde{O})\) is split, then \(\text{Der} \tilde{O}\) is a graded sheaf of Lie superalgebras whose homogeneous components are
\[
\text{Der}_p \tilde{O} = \{ \delta \in \text{Der} \tilde{O} \mid \delta \tilde{O}_q \subset \tilde{O}_{q+p} \quad \text{for all} \quad q \in \mathbb{Z} \}.
\]
Therefore, \(\mathfrak{d}\) is a graded Lie superalgebra. In the general case we endow \(\text{Der} \tilde{O}\) with the following filtration
\[
\text{Der}_{(-1)} \tilde{O} = \text{Der} \tilde{O},
\]
\[
\text{Der}_{(p)} \tilde{O} = \{ \delta \in \text{Der} \tilde{O} \mid \delta \tilde{O} \subset \mathcal{I}^p, \delta \mathcal{I} \subset \mathcal{I}^{p+1} \} \quad \text{for all} \quad p > 0.
\]

2.1. Lemma. 1) The formulas (4) show that \(\text{Der} \tilde{O}\) can be naturally considered as a sheaf of filtered Lie superalgebras.
2) The sheaves \(\text{gr Der} \tilde{O}\) and \(\text{Der gr} \tilde{O}\) are naturally isomorphic.

Proof. 1) It is easy to verify that
\[
\delta \mathcal{I}^r \subset \mathcal{I}^{r+p} \quad \text{for any} \quad \delta \in \text{Der}_{(p)} \tilde{O} \quad \text{and} \quad r \in \mathbb{Z}.
\]
This implies that \([\text{Der}_{(p)} \tilde{O}, \text{Der}_{(q)} \tilde{O}] \subset \text{Der}_{(p+q)} \tilde{O}\).

2) To prove this, observe that every element \(\delta \in \text{Der}_{(p)} \tilde{O}\) determines, thanks to eq. (5), linear mappings
\[
\tilde{\delta}_r : \mathcal{I}^r / \mathcal{I}^{r+1} \longrightarrow \mathcal{I}^{r+p} / \mathcal{I}^{r+p+1} \quad \text{for all} \quad r.
\]
It is subject to a direct verification that the mappings \(\tilde{\delta}_r\) form a derivation \(\tilde{\delta} \in \text{Der}_p \text{gr} \tilde{O}\). The sheaf \(\text{Der}_{(p+1)} \tilde{O}\) is the kernel of the mapping \(\delta \mapsto \tilde{\delta}\). Thus we obtain an injective sheaf homomorphism
\[
\text{gr}_p \text{Der} \tilde{O} \longrightarrow \text{Der}_p \text{gr} \tilde{O}.
\]

Using local splitness of supermanifolds, it is not difficult to show that this homomorphism is, moreover, surjective. Finally, a direct verification shows that this is an isomorphism of sheaves of graded Lie superalgebras.

2.2. Corollary. 1) The Lie superalgebra \(\mathfrak{d} := \Gamma(M, \text{Der} \tilde{O})\) is filtered:
\[
\mathfrak{d} = \mathfrak{d}_{(-1)} \supset \mathfrak{d}_{(0)} \supset \mathfrak{d}_{(1)} \supset \ldots, \quad \text{where} \quad \mathfrak{d}_{(p)} = \Gamma(M, \text{Der}_{(p)} \tilde{O}).
\]
2) The following injective homomorphism of graded Lie superalgebras is well-defined:
\[
\text{gr} \mathfrak{d} \longrightarrow \tilde{\mathfrak{d}} := \Gamma(M, \text{Der} \text{gr} \tilde{O}).
\]

If \(M\) is compact, then \(\dim_{\mathbb{C}} \mathfrak{d} < \infty\).

Proof. 1) The exact sequence
\[
0 \longrightarrow \text{Der}_{(p+1)} \tilde{O} \longrightarrow \text{Der}_{(p)} \tilde{O} \longrightarrow \text{Der}_p \text{gr} \tilde{O} \longrightarrow 0,
\]
existing thanks to Lemma 2.1, implies existence of the exact sequence
\[
0 \longrightarrow \mathfrak{d}_{(p+1)} \longrightarrow \mathfrak{d}_{(p)} \longrightarrow \Gamma(M, \text{Der}_p \text{gr} \tilde{O}).
\]

2) The sequence (6) implies the existence of an injection
\[
\text{gr} \mathfrak{d} \longrightarrow \Gamma(M, \text{Der} \text{gr} \tilde{O}). \quad \square
\]
3 Split supermanifolds

Let $\tilde{O} = \Lambda(F)$, where $F$ is a locally free analytic sheaf of rank $m$ on an $n$-dimensional complex manifold $(M, O)$. Let $T$ denote the tangent sheaf $\text{Der} \, O$ on this manifold.

3.1. Theorem. 1) The sheaves $\text{Der}_p \tilde{O}$ are locally free analytic. We have

$$\text{Der}_{-1} \tilde{O} \cong \text{Hom}_{\mathcal{O}}(F, \tilde{O}) = F^*.$$  \hspace{1cm} (7)

2) For $p \geq 0$, there is an exact sequence of sheaves

$$0 \longrightarrow F^* \otimes \Lambda^{p+1}(F) \longrightarrow \text{Der}_p \tilde{O} \longrightarrow T \otimes \Lambda^p(F) \longrightarrow 0.$$  \hspace{1cm} (8)

Proof. 1) It suffices to consider the case of the model supermanifold of dimension $n|m$. Clearly, in this case $\text{Der} \, \tilde{O}$ is a free sheaf of $\tilde{O}$-modules, a basis of its sections consisting of $\partial x_i \, (i = 1, \ldots, n), \quad \partial \xi_j \, (j = 1, \ldots, m),$

where $x_1, \ldots, x_n$ are coordinates in $\mathbb{C}^n$, and $\{\xi_1, \ldots, \xi_m\}$ is any local basis of sections of the sheaf $F$. Therefore, $\text{Der}_p \tilde{O}$ is a free sheaf of $\mathcal{O}$-modules with a basis of its sections being formed by

$$\xi_{i_1} \cdots \xi_{i_p} \partial x_i \, (i_1 < \ldots < i_p, \ i = 1, \ldots, n)$$

$$\xi_{j_1} \cdots \xi_{j_{p+1}} \partial \xi_j \, (j_1 < \ldots < j_{p+1}, \ j = 1, \ldots, m)$$

Let $\delta \in \text{Der}_p \tilde{O}$. Since the sheaf $\tilde{O}$ is generated by its subsheaves $\mathcal{O}$ and $F$, the derivation $\delta$ is completely determined by its restrictions

$$\delta_0 := \alpha(\delta) = \delta|_{\mathcal{O}} \quad \text{and} \quad \delta_1 := \beta(\delta) = \delta|_{F}.$$  \hspace{1cm} (9)

Obviously,

$$\delta_0 \in \text{Hom}_{\mathcal{C}}(\mathcal{O}, \Lambda^p_{\mathcal{O}}(F)) \quad \text{and} \quad \delta_1 \in \text{Hom}_{\mathcal{C}}(F, \Lambda^{p+1}_{\mathcal{O}}(F))$$  \hspace{1cm} (9)

and

$$\delta_0(\varphi \psi) = (\delta_0 \varphi) \psi + \varphi (\delta_0 \psi)$$

$$\delta_1(\varphi s) = (\delta_1 \varphi) s + \varphi (\delta_1 s) \quad \text{for any} \ \varphi, \psi \in \mathcal{O}, s \in F.$$  \hspace{1cm} (10)

The other way around, for any pair $(\delta_0, \delta_1)$, see eq. (9), which satisfies conditions (10), there exists a $\delta \in \text{Der}_p \tilde{O}$ such that

$$\alpha(\delta) = \delta_0 \quad \text{and} \quad \beta(\delta) = \delta_1.$$  \hspace{1cm} (10)

For $p = -1$, we have $\alpha(\delta) = 0$ and conditions (10) yields

$$\beta(\delta) \in \text{Hom}_{\mathcal{O}}(F, \mathcal{O}) = F^*.$$  \hspace{1cm} (10)

This implies the existence of the isomorphism (7).
2) For $p \geq 0$, consider the sheaf homomorphism
$$\alpha : \text{Der}_p \tilde{\mathcal{O}} \rightarrow \text{Hom}_\mathcal{O}(\mathcal{O}, \Lambda^p_\mathcal{O}(\mathcal{F})).$$

Obviously, $\text{Ker}\alpha = (\text{Der}_p \tilde{\mathcal{O}}) \cap \text{Der} \tilde{\mathcal{O}}$, and therefore $\beta$ defines a sheaf isomorphism
$$\text{Ker}\alpha \rightarrow \text{Hom}_\mathcal{O}(\mathcal{F}, \Lambda^{p+1}_\mathcal{O}(\mathcal{F})) \cong \mathcal{F}^* \otimes \Lambda^{p+1}_\mathcal{O}(\mathcal{F}).$$

To compute $\text{Im}\alpha$ we use an isomorphism
$$\text{Hom}_C(\mathcal{O}, \mathcal{G}) \cong (\text{End}_C \mathcal{O}) \otimes \mathcal{G} \quad (11)$$
for any locally free analytic sheaf $\mathcal{G}$. Let $\{g_1, \ldots, g_q\}$ be a basis of sections of $\mathcal{G}$ in a neighborhood of a point $w \in M$. If $h \in \text{Hom}_C(\mathcal{O}, \mathcal{G})$, then $h(\varphi) = \sum_{1 \leq i \leq q} \varphi_i g_i$, where $\varphi \in \mathcal{O}_w$ and $\varphi_i \in \mathcal{O}_w$ for all $i$. Setting $h_i(\varphi) = \varphi_i$ for all $i$, we see that $h_i \in (\text{End}_C \mathcal{O})_w$. The correspondence $h \mapsto \sum_{1 \leq i \leq q} h_i \otimes g_i$ does not depend on the basis $(g_i)_{i=1}^q$ and is the desired isomorphism (11).

If $\mathcal{G} = \Lambda^p_\mathcal{O}(\mathcal{F})$, then, for any given basis $\{g_i\}_{i=1}^q$, we can take local sections $\xi_{i_1} \cdots \xi_{i_p}$ for $i_1 < \cdots < i_p$. To the element $h \in \text{Hom}_C(\mathcal{O}, \mathcal{G})$ the isomorphism (11) assigns
$$\sum_{i_1 < \cdots < i_p} h_{i_1 \cdots i_p} \otimes \xi_{i_1} \cdots \xi_{i_p},$$
where $h_{i_1 \cdots i_p} \in \text{End}_C \mathcal{O}$ is defined by the formula
$$h(\varphi) = \sum_{i_1 < \cdots < i_p} h_{i_1 \cdots i_p}(\varphi)\xi_{i_1} \cdots \xi_{i_p} \quad \text{for any } \varphi \in \mathcal{O}. \quad (12)$$

Conditions (10) imply that if $h = \alpha(\delta)$, where $\delta \in \text{Der}_p \mathcal{O}$, then $h_{i_1 \cdots i_p} \in \text{Der} \tilde{\mathcal{O}}$.

Thus, $\text{Im}\alpha \subset \mathcal{T} \otimes \Lambda^p_\mathcal{O}(\mathcal{F})$, if we identify
$$\text{Hom}_C(\mathcal{O}, \Lambda^p_\mathcal{O}(\mathcal{F})) \cong \text{End}_C \mathcal{O} \otimes \Lambda^p_\mathcal{O}(\mathcal{F})$$
by means of isomorphism (11).

Conversely, in formula (12), let $h \in \text{Hom}_C(\mathcal{O}, \Lambda^p_\mathcal{O}(\mathcal{F}))$ and $h_{i_1 \cdots i_p} \in \mathcal{T}$. Then, in local coordinates $x_1, \ldots, x_n$ on $M$, we have
$$h_{i_1 \cdots i_p} = \sum_{1 \leq i \leq n} u_{i_1 \cdots i_p}^i \partial_{x_i}, \quad \text{where } u_{i_1 \cdots i_p}^i \in \mathcal{O}.$$

Clearly,
$$\alpha(\delta) = h \quad \text{for } \delta = \sum_{1 \leq i \leq n} \sum_{i_1 < \cdots < i_p} \xi_{i_1} \cdots \xi_{i_p} \partial_{x_i} \in \text{Der}_p \mathcal{O}.$$

Observe that for $p = 0$ the sequence (8) is of the form
$$0 \rightarrow \text{End} \mathcal{F} \rightarrow \text{Der}_0 \mathcal{O} \xrightarrow{\alpha} \mathcal{T} \rightarrow 0. \quad (13)$$
In particular, $\text{End}\mathcal{F}$ is a sheaf of ideals in $\text{Der}_0\mathcal{O}$. The sequence of sheaves leads to the exact sequence of Lie algebras

$$0 \to \text{End} F \to \mathfrak{d}_0 \to \Gamma(M, \mathcal{T}),$$

(14)

where $\text{End} F = \Gamma(M, \text{End}\mathcal{F})$ is the Lie algebra of endomorphisms of the vector bundle $F$.

Let $\varepsilon \in \mathfrak{d}_0$ be the element corresponding to the identity automorphism of $F$. Clearly, $\varepsilon$ is a grading element of $\mathfrak{d}$, i.e., $[\varepsilon, \delta] = p\delta$ for any $\delta \in \mathfrak{d}_p$. \hfill $\Box$

3.1.1 Example. Consider the split supermanifold $(M, \Omega)$, where

$$\Omega = \bigoplus_{0 \leq p \leq n} \Omega^p = \Lambda^p \mathcal{T}^*$$

is the sheaf of holomorphic forms on $(M, \mathcal{O})$. If $\mathcal{F} = \mathcal{T}^*$, then the sheaves which appear in Theorem 3.1 coincide with $\Omega^p\mathcal{T} = \mathcal{T} \otimes \mathcal{O}^p$ — sheaves of vector-valued holomorphic $p$-forms, and we obtain the exact sequence which introduces maps $i_p$ and $\alpha_p$.

$$0 \to \Omega^{p+1} \to \text{Der}_p\Omega \to \Omega^p \mathcal{T} \to 0.$$  

(15)

As was proved in [Fr] (in the $C^\infty$ case, but this is inessential), this sequence splits. Therefore, we obtain the following theorem which I prove for completeness.

3.2. Theorem. For any $p \geq -1$, we have

$$\text{Der}_p\Omega \cong \Omega^p_T \oplus \Omega^{p+1}_T.$$  

Proof. Let $\omega \in \Omega^p_T$. Consider $\delta_\omega := i_{p-1}(\omega) \in \text{Ker} \alpha_{p-1}$, and set $\delta := [\delta_\omega, d]$, where $d : \Omega \to \Omega$ is the exterior differential, a section of the sheaf $\text{Der}_1\Omega$. Then, $\alpha_p(\delta) = \omega$.

Indeed, let $\omega = \sum_i \delta_i \omega_i$, where $\delta_i \in \mathcal{T}$ and $\omega_i \in \Omega^p$. Then, for any $\varphi \in \mathcal{O}$, we have

$$\delta \varphi = \delta_\omega(d \varphi) = \sum_i (d \varphi)(\delta_i) \otimes \omega_i = \sum_i \delta_i(\varphi) \otimes \omega_i = \omega(\varphi).$$

Therefore, the mapping $\omega \mapsto \delta = [\delta_\omega, d]$ splits the sequence (15). It is easy to see that the subsheaf of $\text{Der}_p\Omega$ defined by the splitting, which is complementary to $\Omega^{p+1}_T$, coincides with the centralizer of $d$. (The first description of the centralizer of $d$ in terms of Lie superalgebras and invariant differential operators is due to Grozman; for details, see the arXiv version of [Gro*] reproduced in this Special Volume D.L.). \hfill $\Box$

3.3. Corollary. There exist isomorphisms

$$\mathfrak{d} = \Gamma(M, \text{Der}_p\Omega) \cong \Gamma(M, \Omega^p_T) \oplus \Gamma(M, \Omega^{p+1}_T).$$

In particular, $\mathfrak{d}_{-1} \cong \Gamma(M, \mathcal{T})$. 
The latter isomorphism is described as follows: every vector field \( v \in \Gamma(M, \mathcal{T}) \) determines a derivation \( i_v \in \mathfrak{d}_{-1} \), called \textit{inner derivation by} (or a \textit{convolution with}) \( v \). Further, we have a decomposition into a semidirect sum of Lie algebras

\[
\mathfrak{d}_0 \cong \Gamma(M, \mathcal{T}) \oplus \text{End} \mathcal{T},
\]

where \( \text{End} \mathcal{T} \) is an ideal, and the Lie subalgebra \( \Gamma(M, \mathcal{T}) \) is embedded into \( \mathfrak{d}_0 \) by means of injective homomorphism \( \nu \mapsto \theta \nu \), where \( \theta \nu \) is the Lie derivative along the field \( \nu \). Finally,

\[
\mathfrak{d}_1 \cong \text{End} \mathcal{T} \oplus \Gamma(M, \Omega^2 \mathcal{T}).
\]

Under this isomorphism the identity automorphism \( \varepsilon \in \text{End} \mathcal{T} \) corresponds to the exterior differential \( d \in \mathfrak{d}_1 \).

Consider the graded subspace \( \mathfrak{t} = \mathfrak{t}_{-1} \oplus \mathfrak{t}_0 \oplus \mathfrak{t}_1 \subset \mathfrak{t} \), where

\[
\mathfrak{t}_{-1} = i_{\Gamma(M, \mathcal{T})} = \mathfrak{t}_{-1}, \quad \mathfrak{t}_0 = \theta_{\Gamma(M, \mathcal{T})} \oplus \mathbb{C} \varepsilon \subset \mathfrak{t}_0, \quad \mathfrak{t}_1 = \mathbb{C} d \subset \mathfrak{t}_1.
\]

The classical relations (here \( p(i_v) = \bar{1} \), i.e., \( i_v \) is odd for all vector fields \( v \))

\[
[i_v, i_w] = 0, \quad [\theta_v, \theta_w] = \theta_{[v, w]}, \quad [\theta_v, i_w] = i_{[v, w]},
\]

\[
[d, i_v] = \theta_v, \quad [d, \theta_w] = 0 \quad \text{for any} \quad v, w \in \Gamma(M, \mathcal{T}),
\]

\[
[d, d] = 0, \quad [\varepsilon, \delta] = p \delta \quad \text{for any} \quad \delta \in \mathfrak{d}_p
\]

immediately imply that \( \mathfrak{t} \) is a Lie subsuperalgebra in \( \mathfrak{t} \).

### 4 Transitive Lie superalgebras

Let us deduce a sufficient condition for the Lie superalgebra of vector fields \( \Gamma(M, \text{Der} \widetilde{\mathcal{O}}) \) on the supermanifold \( (M, \widetilde{\mathcal{O}}) \) to be transitive. As before let \( \mathcal{J} \) be the subsheaf of ideals generated by odd elements. The locally free sheaf \( \mathcal{F}^* = (\mathcal{J}/\mathcal{J}^2)^* \) on the complex manifold \( (M, \widetilde{\mathcal{O}}_{\text{rd}}) \) will be called \textit{odd tangent sheaf} and denoted \( \mathcal{T}_\mathcal{T} \). Let \( \mathcal{T}_\mathcal{T} \) be the corresponding holomorphic vector bundle over \( (M, \widetilde{\mathcal{O}}_{\text{rd}}) \) called the \textit{odd tangent bundle}.

#### 4.1 Split case, i.e., \( \widetilde{\mathcal{O}} = \Lambda(\mathcal{F}) \). By Theorem 3.1, \( \mathfrak{d}_{-1} \cong \Gamma(M, \mathcal{T}_\mathcal{T}) \).

#### 4.1. Lemma. Let \( (M, \widetilde{\mathcal{O}}) \) be a split supermanifold. Suppose a global holomorphic section passes through every point of any fiber of the bundle \( \mathcal{T}_\mathcal{T} \).

Then, the graded Lie superalgebra \( \mathfrak{d} = \Gamma(M, \text{Der} \widetilde{\mathcal{O}}) \) satisfies condition (2) for all \( p > 0 \).

If this condition holds for \( p = 0 \) as well, i.e., if the adjoint representation of \( \mathfrak{d}_0 \) in \( \mathfrak{d}_{-1} \) is exact, then \( \mathfrak{d} \) is transitive.

**Proof.** Let \( \delta \in \mathfrak{d}_p \) for \( p > 0 \) and \( \gamma \in \mathfrak{d}_{-1} \). Then,

\[
[\gamma, \delta](\varphi) = \gamma(\delta \varphi) \quad \text{for any} \quad \varphi \in \widetilde{\mathcal{O}},
\]

\[
[\gamma, \delta](s) = \gamma(\delta s) + (-1)^p \delta(\gamma s) \quad \text{for any} \quad s \in \mathcal{F}. \quad \square
\]
If $[\gamma, \delta] = 0$ for all $\gamma \in \mathfrak{d}_{-1}$, then $\gamma(\delta \varphi) = 0$ for all $\varphi \in \mathcal{O}$. This follows from (16). Since $\delta \varphi \in \Lambda^p(\mathcal{F})$, then under Lemma’s hypothesis for $p > 0$, we have $\delta \varphi = 0$ for all $\varphi \in \mathcal{O}$. But then Eq. (□) implies that $\gamma(\delta s) = 0$ for all $\gamma \in \mathfrak{d}_{-1}$ and $s \in \mathcal{F}$, so $\delta s = 0$ for all $s \in \mathcal{F}$; i.e., $\delta = 0$.

Observe that, in general, the Lie superalgebra $\mathfrak{d}$ does not satisfy condition (2) for $p = 0$. Under assumptions of the Lemma we can only claim that the action of the ideal $\text{End} F = \text{End} \Gamma^1$, see (14), on $\mathfrak{d}_{-1} = \Gamma(M, \mathcal{T}_1)$ is exact.

4.2 General case. Let $\check{\mathfrak{d}} = \Gamma(M, \check{\mathcal{O}})$. Consider the graded Lie superalgebra

$$\check{\mathfrak{d}} := \Gamma(M, \text{Der} \check{\mathcal{O}}) \simeq \Gamma(M, \text{gr Der} \check{\mathcal{O}}).$$

By item 2) of Corollary 2.2, there is an injective homomorphism $\text{gr} \mathfrak{d} \rightarrow \check{\mathfrak{d}}$.

In particular, $\text{gr}_{-1} \mathfrak{d} = \mathfrak{d}/\mathfrak{d}(0)$ is identified with $\mathfrak{d}_{-1} = \Gamma(M, \mathcal{T}_1)$.

4.2. Lemma. For the Lie superalgebra $\mathfrak{d} := \Gamma(M, \text{Der} \check{\mathcal{O}})$ to be transitive the following two conditions are sufficient:

a) Given any point of any fiber of the bundle $\mathcal{T}_1$, there is a section of the subspace

$$\mathfrak{d}_{-1} = \mathfrak{d}/\mathfrak{d}(0) \subset \Gamma(M, \mathcal{T}_1)$$

whose image contains it.

b) The adjoint action of the Lie algebra $\mathfrak{d}_0 := \mathfrak{d}_0(0)/\mathfrak{d}_0(1)$ in $\mathfrak{d}_{-1}$ is exact.

Proof. By repeating almost verbatim the proof of Lemma 4.1 we see that the subalgebra $\text{gr} \mathfrak{d}$ of the Lie superalgebra $\check{\mathfrak{d}}$ satisfies conditions (2) for any $p > 0$. The case $p = 0$ is handled by condition of item b).

5 Homogeneous vector bundles

Let $p : E \rightarrow M$ be a holomorphic vector bundle over a complex manifold $(M, \mathcal{O})$. A fiberwise linear biholomorphic mapping $E \rightarrow E$ is said to be an automorphism of the bundle $E$. Let $A(E)$ be the group of all automorphisms of $E$. Obviously, every automorphism $a$ of the bundle $E$ determines an automorphism $p(a)$ of the base $M$ of $E$. We obtain an exact sequence of groups

$$e \rightarrow \text{Aut} E \rightarrow A(E) \xrightarrow{p} \text{Aut} M,$$

where $\text{Aut} E \subset A(E)$ is the normal subgroup consisting of the automorphisms sending every fiber into itself. If $M$ is compact, then the sequence (17) consists of complex Lie groups and their homomorphisms, see [Mo].

Observe that it is possible to describe the automorphisms of the bundle $E$ in terms of the corresponding sheaf $\mathcal{E}$. Namely, every $a \in A(E)$ determines an automorphism $\tilde{a}$ of the sheaf $\mathcal{E}$ over the automorphism $p(a)$ of the base $M$, i.e., determines an isomorphism of sheaves $\tilde{a} : \mathcal{E} \rightarrow p(a)^* \mathcal{E}$. The latter isomorphism is given on local sections $s$ by the formula

$$\tilde{a}(s)(w) = a^{-1}(s(p(a)(w))) \text{ for any } w \in M.$$
Consider the sheaf $\mathcal{E}^* = \text{Hom}_\mathcal{O}(\mathcal{E}, \mathcal{O})$ corresponding to the dual bundle $E^*$. The space of sections $\Gamma(U, \mathcal{E}^*)$ over any open set $U \subset M$ can be identified with the subspace in $\Gamma(p^{-1}(U), \mathcal{O}_E)$ consisting of the functions linear on fibers. This gives an embedding $\mathcal{E}^* \subset p_*\mathcal{O}_E$.

A vector field on $E$ with a projection to $M$ and sending $\mathcal{E}^*$ into itself will be called an \textit{infinitesimal automorphism of the bundle $E$}. The infinitesimal automorphisms determine a sheaf of complex Lie algebras $\mathcal{A}(E)$ on $M$. Projection to the base yields a sheaf homomorphism $\pi : \mathcal{A}(E) \to \mathcal{T}$, where $\mathcal{T}$ is the tangent sheaf on $M$. If $M$ is compact, then the Lie algebra $\mathfrak{a}(E) = \Gamma(M, \mathcal{A}(E))$ is tangent to the Lie group $A(E)$, and the homomorphism $\pi : \mathfrak{a}(E) \to \Gamma(M, \mathcal{T})$ coincides with $dp$.

5.1. Lemma. \textit{The sheaf $\mathcal{A}(E)$ is naturally isomorphic to the sheaf $\text{Der}_0 \mathcal{O}$ of degree 0 derivations of $\mathcal{O} := \Lambda(\mathcal{E}^*)$. Under this isomorphism $\pi$ turns into the homomorphism $\alpha$ from sequence (13).}

\textit{Proof.} Every $\delta \in \mathcal{A}(E)$ determines a pair

$$\delta_0 = \pi(\delta) \in \mathcal{T} \quad \text{and} \quad \delta_1 = \delta|_{\mathcal{E}^*} \in \text{End}_\mathbb{C} \mathcal{E}^*,$$

so that conditions (10) are satisfied. Let $\hat{\delta} \in \text{Der}_0 \mathcal{O}$ be the derivation corresponding to the pair $(\delta_0, \delta_1)$, see proof of Theorem 3.1. It is easy to see that the mapping $\delta \mapsto \hat{\delta}$ is an injective homomorphism of sheaves of Lie algebras.

To prove surjectivity of the mapping $\delta \mapsto \hat{\delta}$, consider $\hat{\delta} \in (\text{Der}_0 \mathcal{O})_w$ determined by a pair $(\delta_0, \delta_1)$ satisfying conditions (10). Let $U \subset M$ be an open neighborhood of any point $w \in M$ over which $E$ admits trivialization $p^{-1}(M) = U \times \mathbb{C}^m$. Set $\delta \varphi = \delta_0 \varphi$, where $\varphi \in \mathcal{O}|_U$, and $\delta \ell = \delta_1 \ell$, where $\ell \in (\mathbb{C}^m)^*$. These conditions completely determine an element $\delta \in \mathcal{A}(E)_w$, which turns into $\hat{\delta}$ under this mapping $\overset{\wedge}\to$. \hfill $\square$

Now let $(M, \mathcal{O})$ be a homogeneous space of a connected complex Lie group $G$, i.e., let there be given a transitive analytic $G$-action on $M$. Fix a point $w_0 \in M$, and let $P$ be the stationary subgroup of $w_0$ in $G$.

As is well-known, $M$ can be naturally identified with the manifold of the left cosets $G/P$. Let $M$ be compact.

Recall that a holomorphic vector bundle $p : E \to M$ is \textit{homogeneous} (under a $G$-action) if there is an analytic $G$-action by automorphisms of the bundle $E$ whose projection is a given $G$-action on $M$. Equivalently, there exists an analytic homomorphism $\hat{t} : G \to A(E)$ such that $p \circ \hat{t} = t$ is a homomorphism $G \to \text{Aut} M$ which determines the given $G$-action on $M$.

Associated to a homogeneous bundle $E$ there is a linear representation $\varphi : g \mapsto \hat{t}(g)|_{E_{w_0}}$ of $P$ on the space $E_{w_0}$. This representation completely determines the bundle $E$ together with the $G$-action on $E$. For the role of $\varphi$ one can take any finite-dimensional linear analytic representation of $P$. We write $E_\varphi$ to denote the homogeneous vector bundle over $M$ corresponding to the representation $\varphi$. The corresponding locally free sheaves $\mathcal{E}_\varphi$ are also called \textit{homogeneous}.
The $G$-action on $E_{\varphi}$ determines an analytic linear representation $\Phi: G \to GL(\Gamma(M, E_{\varphi}))$ given by the formula
\[(\Phi(g)s)(w) = gs(g^{-1}w) \quad \text{for any } g \in G, \ w \in M.\]
The representation $\Phi$ of $G$ is called induced by the representation $\varphi$ of $P$.

5.2. Example. The tangent bundle $T$ over $M$ is endowed with a natural $G$-action, and therefore is homogeneous. It corresponds to the linear representation $\tau: P \to GL(T_{w_0}(M))$ given by the formula
\[\tau(g) = dt(g)_{w_0} \quad \text{for any } g \in P.\]

In what follows we consider the case where $M$ is the homogeneous flag variety, i.e., where $G$ is a connected semisimple (or reductive) complex Lie group and $P$ is its parabolic subgroup. In this case, the space of sections of the homogeneous vector bundle and the induced representation are described by the famous Bott–Borel–Weil theorem, see [B]. Let us recall it.

Let $T$ be a maximal torus in a Borel subgroup $B$ of $G$ and let $t$ and $b$ be the respective Lie algebras. Let $R$ be the root system with respect to $T$ (or $t$); let $R^+$ be the set of positive roots corresponding to $b$ and $\Pi$ the set of simple roots and let
\[\left\{ h_\alpha := \frac{2\alpha}{(\alpha, \alpha)} \mid \alpha \in R \right\} \subset t\]
be the set of coroots, i.e., the system of roots dual to $R$. The weight $\lambda \in \mathbb{Z}^*$ is called dominant if $\lambda(h_\alpha) \geq 0$ for all $\alpha \in \Pi$.

5.3 Theorem (Bott–Borel–Weil). Let $P$ be a parabolic subgroup of $G$ containing the opposite to $B$ Borel group $B^-$, and let $M = G/P$. Let $\varphi$ be an irreducible analytic representation of $P$ with highest weight $\lambda$.

Then, $\Gamma(M, E_{\varphi}) \neq 0$ if and only if $\lambda$ is dominant. In this case, the induced representation $\Phi$ in $\Gamma(M, E_{\varphi})$ is irreducible with highest weight $\lambda$.

This theorem is applicable only if $\varphi$ is irreducible or completely reducible. However, one often encounters homogeneous bundles arising from representations which are not completely reducible. The following Lemma is useful for studying them.

5.4. Lemma. Let $M = G/P$ be a homogeneous flag variety, $\varphi$ a holomorphic linear representation of $P$. If the induced representation $\Phi$ of $G$ contains an irreducible representation with highest weight $\lambda$ of multiplicity $t$, then $\lambda$ is contained in the set of weights of $\varphi$ of multiplicity $\geq t$.

Proof. Let us perform induction by the length $\ell(\varphi)$ of the Jordan-Hoelder series of $\varphi$. If $\ell(\varphi) = 1$, then $\lambda$ is the highest weight of $\varphi$ by Theorem 5.3. Let the Lemma be proved for representations $\varphi'$ such that $\ell(\varphi') < \ell(\varphi)$. Assuming that $\varphi$ is reducible, consider a proper
subrepresentation \( \varphi_1 \) of \( \varphi \) and the quotient representation \( \varphi_2 \). We obtain an exact sequence of sheaves

\[
0 \longrightarrow \mathcal{E}_{\varphi_1} \longrightarrow \mathcal{E}_\varphi \longrightarrow \mathcal{E}_{\varphi_2} \longrightarrow 0,
\]

which defines the exact sequence of \( G \)-modules

\[
0 \longrightarrow \Gamma(M, \mathcal{E}_{\varphi_1}) \longrightarrow \Gamma(M, \mathcal{E}_\varphi) \longrightarrow \Gamma(M, \mathcal{E}_{\varphi_2}).
\]

Since every finite-dimensional analytic representation of \( G \) is completely reducible, the irreducible representation with highest weight \( \lambda \) is contained in \( \Gamma(M, \mathcal{E}_{\varphi_1}) \) and \( \Gamma(M, \mathcal{E}_{\varphi_2}) \) with total multiplicity \( \geq t \). By induction hypothesis the statement of the Lemma follows because the set of weights of \( \varphi \) is the union of the sets of weights of \( \varphi_1 \) and \( \varphi_2 \).

\[\square\]

5.5. Corollary. If \( \varphi \) has no dominant weights, then the induced representation \( \Phi \) in \( \Gamma(M, \mathcal{E}_\varphi) \) is trivial. Moreover, \( \Gamma(M, \mathcal{E}_\varphi) = 0 \).

6 Split homogeneous supervarieties

In this section we consider a particular class of split supervarieties associated with homogeneous flag varieties. Let \( G \) be a connected semisimple (or reductive) complex Lie group, \( P \) a complex Lie subgroup, and \( E_\varphi \) a homogeneous vector bundle over \( M = G/P \) determined by a representation \( \varphi \) of \( P \). Consider a split supermanifold \( (M, \widetilde{O}) \), where \( \widetilde{O} = \Lambda(E_\varphi) \). The sheaf \( \widetilde{O} \) corresponds to the homogeneous vector bundle \( \Lambda(E_\varphi) = E_{\Lambda(\varphi)} \), i.e., \( \widetilde{O} = E_{\Lambda(\varphi)} \).

The sheaf \( \text{Der} \widetilde{O} \) is also homogeneous. Indeed, every element \( g \in G \) corresponds to an automorphism \( \widetilde{t}(g) \) of \( \widetilde{O} \) over an automorphism \( t(g) \in \text{Aut} M \), see eq. (18). By setting

\[
g \delta = \widetilde{t}(g) \circ \delta \circ \widetilde{t}(g)^{-1}
\]

for any \( \delta \in \text{Der} \widetilde{O} \) (19), we obtain the desired \( G \)-action on the locally free sheaf \( \text{Der} \widetilde{O} \). Also notice that the sheaves in the exact sequence (8) correspond to the homogeneous vector bundles \( E_\varphi^* \otimes \Lambda^{p+1}(E_\varphi) \) and \( T \otimes \Lambda^p(E_\varphi) \), while the homomorphisms in this sequence are \( G \)-equivariant.

By Lemma 5.1 we can identify the Lie algebra \( \mathfrak{d}_0 = \Gamma(M, \text{Der}_0 \widetilde{O}) \) with \( \mathfrak{a}(E_\varphi^*) \). Since the bundle \( E_\varphi^* \) is also homogeneous, we have a Lie algebra homomorphism \( \hat{d} : \mathfrak{g} \longrightarrow \mathfrak{d}_0 \). On the other hand, the \( G \)-action (19) on \( \widetilde{O} \) determines a representation \( \Psi : G \longrightarrow GL(\mathfrak{d}) \).

6.1. Lemma. Let \( u \in \mathfrak{g} \) and \( \psi = d\Psi : \mathfrak{g} \longrightarrow gl(\mathfrak{d}) \). Then, \([d\hat{t}(u), \delta] = \psi(u)\delta\).

Proof. This follows directly from (19): instead of \( g \), substitute in eq. (19) the curve \( g(t) \in G \) with tangent vector \( u \) at \( t = 0 \) passing through \( e := g(0) \), and differentiate both parts with respect to \( t \) at \( t = 0 \).

\[\square\]

A split supermanifold \( (M, \Lambda(E_\varphi)) \) will be called homogeneous if \( T_T = E_\varphi^* = E_{\varphi^*} \) has “sufficiently many” holomorphic sections. This means that every point of every fiber of this bundle is contained in the image of some section.

In what follows \( G \) is semisimple and \( P \) is its parabolic subgroup.
6.2. Lemma. Let \( \varphi \) be irreducible and \( \lambda \) the highest weight of representation \( \varphi^* \). The following properties are equivalent:

i) \( (M, \Lambda(\mathcal{E}_\varphi)) \) is homogeneous;

ii) \( \lambda \) is dominant;

iii) \( \mathfrak{d}_{-1} = \Gamma(M, \mathcal{E}_{\varphi^*}) \neq 0 \).

Under these conditions the induced representation \( \Psi_{-1} \) of \( G \) in \( \mathfrak{d}_{-1} \) is irreducible.

Proof. By Theorem 5.3, it remains to prove that if \( \lambda \) is dominant, then \( T_T \) has sufficiently many holomorphic sections. Obviously, the restriction map \( r_{w_0} : \Gamma(M, \mathcal{E}_{\varphi^*}) \to (\mathcal{E}_{\varphi^*})_{w_0} \) is a \( P \)-module homomorphism. Therefore, either \( r_{w_0} \) is surjective or \( r_{w_0} = 0 \). But for any \( g \in G \) and \( s \in \Gamma(M, \mathcal{E}_{\varphi^*}) \), we have

\[
 r_{gw_0}(s) = gr_{w_0}(\Phi(g^{-1})s),
\]  

where \( \Phi \) is the representation induced by \( \varphi^* \). Therefore, \( rw_0 = 0 \) implies \( r_w = 0 \) for all \( w \in M \), which is impossible. Thus \( r_w \) is surjective for all \( w \in M \). \( \square \)

Consider the following commutative diagram whose upper line is the exact sequence (14) for the bundle \( F = \mathcal{E}_{\varphi^*}^* \):

\[
 0 \longrightarrow \text{End} \mathcal{E}_\varphi \longrightarrow \mathfrak{d} \xrightarrow{\alpha} \Gamma(M, \mathcal{T}) \xrightarrow{\partial_t} \mathfrak{g} \xrightarrow{\partial_t} 0.
\]  

6.3. Theorem. Let \( M = G/P \) be a flag variety of a semisimple Lie group \( G \). Let \( \varphi \) be an irreducible finite-dimensional analytic linear representation of \( P \); let \( \mathcal{O} = \Lambda(\mathcal{E}_\varphi) \) and \( \alpha(\mathfrak{d}_0) = dt(\mathfrak{g}) \), where \( \alpha \) is defined in eq. (21).

Let \( G = G_1 \ldots G_r \) be a decomposition of \( G \) into simple factors, and let \( \Pi = \Pi_1 \sqcup \cdots \sqcup \Pi_r \) be the corresponding decomposition of the system of simple roots. Let the highest weight \( \lambda \) of the representation \( \varphi^* \) be dominant, and for every \( i \) assume that there exists a \( \beta \in \Pi_i \) such that \( \lambda(h_\beta) > 0 \).

Then, the graded Lie superalgebra \( \mathfrak{d} = \Gamma(M, \text{Der} \mathcal{O}) \) is transitive and irreducible.

Proof. By Lemma 6.2 and since \( \lambda \) is dominant, the condition of Lemma 4.1 is satisfied. Therefore, it remains to show that the adjoint representation of \( \mathfrak{d}_0 \) in \( \mathfrak{d}_{-1} = \Gamma(M, \mathcal{E}_{\varphi^*}) \) is irreducible and exact.

The irreducibility easily follows from Lemmas 6.1 and 6.2.

Observe that the homomorphism \( \partial_t \) is injective. Indeed, otherwise \( \mathfrak{g}_i \subset \text{Ker} \partial_t \) for some \( i \). Hence, \( dt(\mathfrak{g}_i) = 0 \) which implies that \( G_i \subset P \). Furthermore, \( \mathfrak{g}_i \subset \text{Ker} \psi \) by Lemma 6.1.

Since \( r_{w_0} : \Gamma(M, \mathcal{E}_{\varphi^*}) \to (\mathcal{E}_{\varphi^*})_{w_0} \) is surjective (Lemma 6.2), it follows that \( G_i \subset \text{Ker} \varphi^* \), which contradicts the hypothesis. \( \square \)
Let us identify \( \mathfrak{g} \) with \( d\hat{\mathfrak{g}} \) by means of \( d\hat{\phi} \). Form eq. (21) we see that \( dt = \alpha|_\phi \) and \( \alpha(\mathfrak{d}_0) = \alpha(\mathfrak{g}) \).

Consider the ideal \( \mathfrak{g}_0 = \mathfrak{g} \cap \text{End} \, \mathfrak{E}_\phi \) of \( \mathfrak{g} \). We see that \( \mathfrak{g} = \mathfrak{g}_0 \oplus \tilde{\mathfrak{g}} \), where \( \tilde{\mathfrak{g}} \) is also an ideal. Obviously, our assumptions imply that there is a decomposition into a semi-direct sum

\[
\mathfrak{d}_0 = \text{End} \, \mathfrak{E}_\phi \ltimes \tilde{\mathfrak{g}},
\]

where \( \text{End} \, \mathfrak{E}_\phi \) is an ideal and \( \tilde{\mathfrak{g}} \) a subalgebra. Let us show that this sum is actually direct.

Consider the induced representation \( \psi \) of the Lie algebra \( \mathfrak{g} \) in \( \text{End} \, \mathfrak{E}_\phi \). The weights of the induced representation \( \phi \otimes \phi^* \) are of the form \( \mu = \nu_1 - \nu_2 \), where \( \nu_1, \nu_2 \) are weights of \( \phi \). Since \( \phi \) is irreducible, \( \mu \) can be expressed in terms of the set \( \Pi_P \subset \Pi \) corresponding to the semisimple part of \( P \). But the roots of any subset \( \Pi_i \) should enter the expression of any dominant weight in terms of \( \Pi \) with either positive or zero coefficients, see [H, §13, Exc. 8] and [R].

Since \( (dt)|_{\phi} \) is injective, \( G_i \not\subset P \) for all \( i \) such that \( \mathfrak{g}_i \subset \tilde{\mathfrak{g}} \). Therefore, if \( \mu \) is dominant, then \( \mu(h_\beta) = 0 \) for all \( \beta \in \Pi_i \) and any such \( i \).

Since \( \phi \otimes \phi^* \) is completely reducible, we can apply Theorem 5.3 which implies that \( \psi(\mathfrak{g}) = 0 \), and \( [\mathfrak{g}, \text{End} \, \mathfrak{E}_\phi] = 0 \) thanks to Lemma 6.1.

The homogeneity of the supermanifold \( (M, \hat{\mathcal{O}}) \) proved above implies that the action of \( \text{End} \, \mathfrak{E}_\phi = \text{End} \, \mathfrak{E}_{\phi^*} \) in \( \mathfrak{d}_{-1} \) is exact. The radical \( \mathfrak{r} \) of the Lie algebra \( \mathfrak{d}_0 \) is contained in \( \text{End} \, \mathfrak{E}_\phi \) and is non-zero since \( \mathbb{C} \varepsilon \subset \mathfrak{r} \), see §2. Further, the Lie subalgebra \( \text{ad}_{\mathfrak{d}_0} \subset \mathfrak{gl}(\mathfrak{d}_{-1}) \) is irreducible, and the dimension of its radical should be at most 1. Hence, \( \mathfrak{r} = \mathbb{C} \varepsilon \).

Therefore, \( \text{End} \, \mathfrak{E}_\phi = \mathbb{C} \varepsilon \oplus \mathfrak{h} \) and \( \mathfrak{d}_0 = \text{End} \, \mathfrak{E}_\phi \oplus \tilde{\mathfrak{g}} \) are reductive Lie algebras with 1-dimensional centers. Any ideal of \( \mathfrak{d}_0 \) is contained either in \( \text{End} \, \mathfrak{E}_\phi \) or in \( \tilde{\mathfrak{g}} \). Thanks to our assumption on \( \lambda \) and Theorem 5.3, the action of the Lie algebra \( \tilde{\mathfrak{g}} \) in \( \mathfrak{d}_{-1} \) is exact, and so is the \( \mathfrak{d}_0 \)-action.

Observe that the assumptions of Theorem 6.3 are satisfied, e.g., if \( G \) is simple and coincides with \( \text{Aut}^\circ M \), where \( ^\circ \) singles out the connected component of the unit, and \( \phi \) is a non-trivial irreducible representation such that the highest weight of \( \phi^* \) is dominant.

On the other hand, the version of the theorem proved above, which does not presuppose that \( G \)-action on \( M \) is faithful, is useful in applications. As is well-known, the cases where the inclusions between Lie algebras \( \alpha(\mathfrak{d}_0) \), \( (dt)(\mathfrak{g}) \) and \( \Gamma(M, T) \) are strict do occur very seldom.

Observe that, if \( \phi \) is irreducible and the \( G \)-action on \( M \) is locally faithful, then we have \( \text{End} \, \mathfrak{E}_\phi = \mathbb{C} \varepsilon \), see [R].

In conclusion, let us compute the Lie superalgebras \( \mathfrak{d} \) for the supermanifolds \( (M, \Omega) \) of Examples 3.1.1 and 5.2, where \( M \) is a flag variety. Obviously, such a supermanifold is homogeneous for any compact complex homogeneous manifold \( M \), although the representation \( \phi \) that determines \( (M, \Omega) \) is seldom completely reducible.

6.4. Theorem. Let \( M = G/P \) be a flag variety of the simple Lie group \( G \). Then, \( \text{End} \, T = \mathbb{C} \varepsilon \) and \( \Gamma(M, \Omega^p_P) = 0 \) for \( p > 1 \); the Lie superalgebra \( \mathfrak{d} = \Gamma(M, \text{Der} \, \Omega) \) coincides with its subsuperalgebra \( \tilde{\mathfrak{d}} \), see §2.
Proof. The equality \( \text{End} T = \mathbb{C} \varepsilon \) is proved in [I]. It can also be deduced from Theorem 5.3 by using certain facts of Lie algebra theory.

Let \( p > 1 \). Then, the weights of the representation \( \tau \otimes \wedge^p(\tau^*) \) which determines the sheaf \( \Omega_T^p \) are of the form

\[
\alpha - \beta_{i_1} - \ldots - \beta_{i_p}, \quad \text{where} \quad \alpha, \beta_{i_1}, \ldots, \beta_{i_p} \in \Pi^+.
\]

Obviously, such a weight cannot be dominant. Applying item 1) of Corollary 2.2, we see that \( \Gamma(M, \Omega_T^p) = 0 \).

The Lie superalgebra \( \mathfrak{d} \) spoken about in Theorem 6.4 is isomorphic to \( \text{vect}(0|m) \). □

References

[ALSh*] Alekseevsky D., Leites D., Shchepochkina I., Examples of simple Lie superalgebras of vector fields. C. r. Acad. Bulg. Sci. (1980) V. 34, No. 9, 1187–1190. (in Russian)

[AS*] Alekseevsky D. V., Santi A., Homogeneous irreducible supermanifolds and graded Lie superalgebras. Int. Math. Res. Not. (2018), no. 4, 1045–1079; arXiv:1511.07055

[AD*] Asok A., Doran B., Vector bundles on contractible smooth schemes; arXiv:0710.3607

[B1*] Berezin F.A., Introduction to Superanalysis. Edited and with a foreword by A. A. Kirillov. With an appendix by V. I. Ogievetsky. Translated from the Russian by J. Niederle and R. Kotecký. Translation edited by D. Leites. Mathematical Physics and Applied Mathematics, 9. D. Reidel Publishing Co., Dordrecht. (1987) xii+424 pp.

[B2*] Berezin F.A., Introduction to Superanalysis. 2nd edition revised and edited by D. Leites and with appendix by D. Leites, V. Shander, I. Shchepochkina “Seminar on Supersymmetries v. 1 1/2”), MCCME, Moscow, (2011) 432 pp. (in Russian)
https://staff.math.su.se/mleites/books/berezin-2013-vvedenie-2nd-ed.pdf

[B] Bott R., Homogeneous vector bundles. Ann. Math. (1967) V. 66, No. 2, 203–248.

[BGLLS*] Bouarroudj S., Grozman P., Lebedev A., Leites D., Shchepochkina I., Simple vectorial Lie algebras in characteristic 2 and their superizations. Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), 16 (2020), 089, 101 pages; arXiv:1510.07255

[BGLS*] Bouarroudj S., Grozman P., Leites D., Shchepochkina I., Minkowski superspaces and superstrings as almost real-complex supermanifolds. Theor. and Mathem. Physics. (2012) V. 173(3), 1687–1708; arXiv:1010.4480

[CaKa*] Cantarini N., Kac V. G., Infinite-dimensional primitive linearly compact Lie superalgebras. Adv. in Math. 207, no. 1, 2006, 328–419; arXiv:math.QA/0511424

[Del*] Deligne P., Etingof P., Freed D., Jeffrey L., Kazhdan D., Morgan J., Morrison D., Witten E., (eds.). Quantum fields and strings: a course for mathematicians. Vol. 1, 2. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999. Vol. 1: xxii+723 pp.

[Dr*] Draisma J., Transitive Lie algebras of vector fields—an overview, Qual. Theory Dyn. Syst. 11 (2012), no.1, 39–60; arXiv:1107.2836

[Fr*] Frölicher A., Nijenhuis A., Theory of vector-valued differential forms. Part I. Derivations in the graded ring of differential forms. Proc. Kon. Ned. Akad. Wet. Amsterdam. (1956) V. 59, 338–359.
[Gre*] Green P., On holomorphic graded manifolds. Proc. of the AMS. (1982) V. 85, No. 4, 587–590.

[Gro*] Grozman P., Classification of bilinear invariant operators on tensor fields. Functional Anal. Appl., v. 14 (1980), no. 2, 127–128; for proofs, see arXiv:math/0509562.

[H] Humphreys J. E., Introduction to Lie algebras and representation theory. N.Y. e.a., (1972) xii+171 pp.

[IO*] Ivanova N. I., Onishchik A. L., Parabolic subalgebras and gradings of reductive Lie superalgebras. (Russian) Sovrem. Mat. Fundam. Napravl. 20 (2006), 5–68; translation in J. Math. Sci. (N.Y.) 152 (2008), no. 1, 1–60

[I] Ise M., Some properties of complex-analytic vector bundles over compact complex homogeneous spaces. Osaka Math. J. (1960) V. 12, No. 2, 217–252.

[K*] Kac V. G., Lie superalgebras. Adv. Math. (1977) V. 26, No. 1, 8–96.

[L] Leites D. A., Supermanifold theory. Karelia branch of the USSR Acad. Sci., Petrozavodsk, (1983) 199 pp. (in Russian)³

[L*] Leites D. (ed.) Seminar on supersymmetry v. 1. Algebra and Calculus: Main chapters, (J. Bernstein, D. Leites, V. Molotkov, V. Shander), MCCME, Moscow, 2012, 410 pp (in Russian; a version in English is in preparation but available for perusal)

[L2*] Leites D. (ed.) Seminar on supersymmetry v. 2. Additional chapters, (D. Leites, V. Molotkov), MCCME, Moscow, (in Russian; in preparation)

[MaG*] Manin Yu. I., Gauge field theory and complex geometry. 2nd edition. Grundlehren der Mathematischen Wissenschaften, 289. Springer, Berlin, (1997) xii+346 pp.

[Mo] Morimoto A., Sur le groupe d’automorphismes d’un espace fibré principal analytique complexe. Nagoya Math. J., (1958) 13, 158–168.

[O*] Ochiai T., Classification of the finite nonlinear primitive Lie algebras. Trans. Amer. Math. Soc. (1966) V. 124, 313–322.

[On*] Onishchik A. L., The action of Lie superalgebras of Cartan type on some split supermanifolds, In: Leites D. (ed.) Seminar on Supersymmetries, Reports of the Department of Mathematics, Stockholm University, Sweden, 8/1987, 45–53. Published in Russian In: Onishchik A.L. (ed.) Problems in group theory and in homological algebra, Yaroslavl State Univ., 1989, 42–49 (MR 91k:58007). For an edited version, see this Special volume.

[OVs*] Onishchik A. L., Vishnyakova E. G., Locally free sheaves on complex supermanifolds. Transformation groups, 18, Issue 2, (2013), 483–505; arXiv:1110.3908

[Pa*] Palamodov V.P., Invariants of analytic Z2-manifolds. Funct. Anal. Appl., 17:1 (1983), 68–69.

[R] Ramanan S., Holomorphic vector bundles on homogeneous spaces. Topology. (1966) V. 5, No. 2, 159–177.

[Sch] Scheunert M., The theory of Lie superalgebras. Lect. Notes in Math., No. 716. Berlin e.a., (1979) x+271 pp.

[Va*] Vaintrob A. Yu., Deformations of complex superspaces and of the coherent sheaves on them. J. Soviet Math. (1990) V. 51, No. 1. 2140–2188.

³Now I can recommend the 2nd edition of this book and its extensions, see [L*], [L2*] and [B1*], respectively, and Bernstein’s lectures in [Del*].