The multiscale expansions of difference equations in the small lattice spacing regime, and a vicinity and integrability test: I

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Abstract

We propose an algorithmic procedure (i) to study the ‘distance’ between an integrable PDE and any discretization of it, in the small lattice spacing $\epsilon$ regime, and, at the same time, (ii) to test the (asymptotic) integrability properties of such discretization. This method should provide, in particular, useful and concrete information on how good is any numerical scheme used to integrate a given integrable PDE. The procedure, illustrated on a fairly general ten-parameter family of discretizations of the nonlinear Schrödinger equation, consists of the following three steps: (i) the construction of the continuous multiscale expansion of a generic solution of the discrete system at all orders in $\epsilon$, following Degasperis et al (1997 Physica D 100 187–211); (ii) the application, to such an expansion, of the Degasperis–Procesi (DP) integrability test (Degasperis A and Procesi M 1999 Asymptotic integrability Symmetry and Perturbation Theory, SPT98, ed A Degasperis and G Gaeta (Singapore: World Scientific) pp 23–37; Degasperis A 2001 Multiscale expansion and integrability of dispersive wave equations Lectures given at the Euro Summer School: ‘What is integrability?’ (Isaac Newton Institute, Cambridge, UK, 13–24 August); Integrability (Lecture Notes in Physics vol 767) ed A Mikhailov (Berlin: Springer)), to test the asymptotic integrability properties of the discrete system and its ‘distance’ from its continuous limit; (iii) the use of the main output of the DP test to construct infinitely many approximate symmetries and constants of motion of the discrete system, through novel and simple formulas.

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1. Introduction

Given a partial differential equation (PDE) and a partial difference equation ($P\Delta E$) discretizing it, it is interesting to know, when the lattice spacing $\epsilon$ is small, ‘how close’ the two models...
are. In particular, if the PDE is integrable, it is important to have a way to establish if such a discretization preserves integrability or, at least, how ‘close’ it is to an integrable system, detecting the order, in \( \epsilon \), at which the discretization departs from integrability and, correspondingly, the time scale at which one should expect numerical evidence of nonintegrability and/or chaos. In addition, given a PDE and two PDEs discretizing it, it is also interesting to know, when the lattice spacing \( \epsilon \) is small, ‘how close’ the two PDEs are.

In this paper we propose to answer these basic questions in the following way. Concentrating on an integrable PDE and on a PDE discretizing it,

1. we construct and study in detail the multiscale expansion at all orders of a **generic long-wave solution** of the PDE under scrutiny, generated in the small \( \epsilon \) regime, following the procedure developed in [1]. At \( O(1) \), the leading term \( u \) of such an asymptotic expansion satisfies the integrable PDE; to keep the expansion asymptotic, we eliminate the secularities due to the linear part of the PDE, arising at each order, introducing infinitely many slow (time) variables and establishing that the evolution of \( u \) with respect to such slow times is described by the infinite hierarchy of commuting flows of the integrable PDE, as in [1];

2. we make use of the asymptotic integrability test developed by Degasperis–Procesi (DP) in [2, 3] on such a multiscale expansion to test, at all orders, the ‘asymptotic’ integrability properties of the PDE; in particular, detecting the order in \( \epsilon \) (and, correspondingly, the time scale) at which the discretization departs from integrability. At this time scale, for instance, numerical simulations are expected to give some evidence of nonintegrable and/or chaotic behavior;

3. we finally show how to make use of the main output of the DP test to construct infinitely many ‘approximate’ symmetries, at a required order in \( \epsilon \), of the PDE under scrutiny, using novel and simple formulas.

Recent studies on the performances, as numerical schemes for their continuous limits, of PDEs possessing the same (continuous) Lie point symmetries as their continuous limits can be found in [4–8]. Studies on the performances, as numerical schemes for their continuous limits, of integrable discretizations of integrable PDEs can be found, for instance, in [9] and [10]; in this case, the integrable discretization possesses infinitely many exact generalized symmetries and constants of motion in involution at any order in \( \epsilon \), reducing to the generalized symmetries and constants of motion of the integrable PDE in the continuous limit. The PDEs selected by our approach possesses infinitely instead many approximate generalized symmetries and constants of motion in involution at the required order in \( \epsilon \) (see section 3.1), reducing to the generalized symmetries and constants of motion of the integrable PDE in the continuous limit.

The procedure we propose should allow one to have a control on the ‘distance’ between the PDE and its continuous limit, as well as the distance between two different discretizations of the PDE. Indeed, suppose we construct an asymptotic expansion of the form \( \psi = u + O(\epsilon^\alpha) \), \( \alpha > 0 \), where \( \psi \) is a generic long-wave solution of the PDE and \( u \) is the corresponding solution of its continuous limit; if, at \( O(\epsilon^\beta) \), \( \beta > 0 \), the PDE passes the DP test, we infer that \( ||\psi - u|| = O(\epsilon^\alpha) \) at time scales of \( O(\epsilon^{-\beta}) \), where \( || \cdot || \) is the uniform norm w.r.t \( x \) and \( t \) (the norm used to test the asymptotic character of the generated multiscale expansion). In this way, since we control the distance between ‘generic long-wave solutions’ of the PDE and of its continuous limit, we also control the distance between the PDE and its continuous limit. In addition, if the multiscale expansions of two different discretizations of the same PDE pass the DP test at \( O(\epsilon^\beta) \), we infer, from the triangular inequality, that \( ||\psi - \phi|| < ||\psi - u|| + ||\phi - u|| = O(\epsilon^\alpha) \) at time scales of \( O(\epsilon^{-\beta}) \), where \( \psi, \phi \) are long-wave solutions of the two different discretizations of the PDE corresponding to the same
initial-boundary data; therefore, we have a control also on the distance between the two
different discretizations of the same PDE.

Some historical remarks are important, at this point, on the theory of multiscale expansions
in connection with integrable systems to put the results of this paper into a proper perspective.
Multiscale expansions of a given PDE are very useful tools for investigating the properties of
such a PDE and for identifying important model (universal) equations of physical phenomena.
For instance, if the original nonlinear PDE has a dispersive linear part, a small amplitude
quasi-monochromatic wave evolving according to it develops a slow spacetime amplitude
modulation described by the celebrated nonlinear Schrödinger (NLS) equation [11–15] (see
also [16–18])

\[ iu_t + u_{xx} + 2c|u|^2u = 0, \quad u = u(x, t) \in \mathbb{C}, \]  

(1)

integrable if \( c \) is a real constant [19]. Considering, instead, three monochromatic waves and
imposing a suitable resonance condition on their wave numbers and dispersion relations, one
generates another integrable universal model, the three-wave resonant system [20]. In the
above two examples, the expansion is constructed around 'approximate' particular solutions
of the original PDE (the monochromatic waves). It is also possible to expand around the
'exact' particular solutions of the original PDE; for instance, as shown in [21], expanding
around the exact solution \( u_0 = \exp(2ict) \) of (1), the first nontrivial term of the asymptotic
expansion evolves according to another important model equation: the Korteweg–de Vries
(KdV) equation [22], sharing with NLS the property of integrability [23]. Since multiscale
expansions preserve integrability [21], (i) if the original PDE is a 'C-integrable' system (i.e.
it is linearized by a 'change of variables' [18, 24], like the Burgers equations [25]), the
model equation generated by it is linear [18, 24]; (ii) if the original PDE is an 'S-integrable'
system, or soliton equation (like the NLS equation), integrated in a more complicated way
via a Riemann–Hilbert or \( \bar{\partial} \)-problem [26–29], the model equation generated by it is also
'S-integrable'; and vice versa, (iii) if the model equation generated by the expansion is not
integrable, then the original equation is not integrable too (indeed, if it were integrable, the
integrability preserving multiscale expansion would generate an integrable model equation).
This criterion has been used in [18, 30–32] as a simple test of integrability. In addition, the
universal character of the identified model equations (NLS, KdV or others) is also the reason
why model equations possess very distinguished mathematical properties and, often, they are
integrable [18, 30, 24].

Multiscale expansions can also be carried, in principle, to all orders and, as a consequence
of eliminating the secular terms at each order, a sequence of slow time variables \( t_\epsilon = \epsilon^n t \)
must be introduced and the dependence of the leading term of the expansion on such slow
times is described by the hierarchy of commuting flows of the integrable model equation [1].
This multiscale expansion at all orders has been used in [2, 3] to build an efficient asymptotic
integrability test for the original PDE (see section 3 for more details on such a test). An
alternative asymptotic integrability test, based on the existence of approximate symmetries
for the original PDE, can be found in [33]. The ideas and procedures developed in [1–3]
have been recently used to build an integrability test also for P\( \Delta \)Es [34–36]; in this approach,
one expands, as for the PDE case, around the approximate or exact particular solutions of the
P\( \Delta \)E under investigation, obtaining a continuous multiscale expansion at all orders, following
[1], and applying on it the DP test. The main difference between the procedure followed in
[34–36] and the results of this paper is the following. The standard multiscale approach used
in [34–36], obtained expanding around approximate or exact particular solutions of the P\( \Delta \)E
under investigation, cannot give information on how close this P\( \Delta \)E and its continuous limit
are, the main goal of the present paper. The common features of the procedure in [34–36]
and of that used in this paper are that, in both cases, one constructs, from the given PΔE, continuous multiscale expansions carried to all orders, as in [1], and one applies to them the DP integrability test. Therefore, both procedures can be used to test the integrability and the asymptotic integrability of the original PΔE. A deeper comparison of the effectiveness of these two procedures to test the integrability of a given PΔE is postponed to a subsequent paper.

Another integrability test for PΔEs is the so-called symmetry approach [37], based on the existence of higher order symmetries and originally developed to test the integrability of PDEs [38, 39].

The results of this paper are illustrated on the basic prototype example of the NLS equation (1), starting from the following discretization of it:

\[
\begin{align*}
\dot{\psi}_{n,t} &+ \epsilon^{-2}(\psi_{n-1} + \psi_{n+1} - 2\psi_n) + F(\psi_{n-1}, \psi_n, \psi_{n+1}) = 0, \\
F(\psi_{n-1}, \psi_n, \psi_{n+1}) &:= 2a_1|\psi_n|^2\psi_n + a_2|\psi_{n+1}|^2(\psi_{n+1} + \psi_{n-1}) \\
&+ a_3\psi_n^2(\psi_{n+1} + \psi_{n-1}) + a_4\psi_n(|\psi_{n+1}|^2 + |\psi_{n-1}|^2) \\
&+ a_5\psi_n(\psi_{n+1}\psi_{n-1} + \psi_n^2) + a_6\bar{\psi}_n(\psi_{n+1}^2 + \psi_{n-1}^2) \\
&+ 2a_7\psi_n\bar{\psi}_{n+1}\psi_{n-1} + a_8(|\psi_{n+1}|^2\psi_{n+1} + |\psi_{n-1}|^2\psi_{n-1}) \\
&+ a_9(\psi_{n+1}^2\psi_{n-1} + \psi_{n-1}^2\psi_{n+1}) + a_{10}(|\psi_{n+1}|^2\psi_{n+1} + |\psi_{n-1}|^2\psi_{n-1}).
\end{align*}
\]

(2)

where the constant coefficients \( a_j, j = 1, \ldots, 10 \), are real, reducing to (1) in the natural continuous limit in which the lattice spacing \( \epsilon \to 0 \) and \( n\epsilon \to x \in \mathbb{R} \), \( \psi_n(t) \to u(x,t) \), with

\[
c = \sum_{j=1}^{10} a_j.
\]

(3)

The ten-parameter family of equations (2) has recently been taken in [40] as the starting point of an analysis devoted to the identification of discretizations of NLS that possess, at the same time, a solitary wave and a breather solution reducing, respectively, to the one soliton solution too, where \( \theta \) is an arbitrary real parameter, corresponding to the infinitesimal gauge symmetry \(-i\psi_n \). (b) The nonlinearity is cubic; i.e. it is the weakest nonlinearity compatible with the above gauge symmetry. (c) Only the first neighbor interactions are considered. (d) Equation (2) is invariant under the symmetry transformation \( \psi_{n\pm 1} \to \psi_{n\mp 1} \) (space isotropy).

The dNLS (2) contains, in particular,

1. the integrable Ablowitz–Ladik (AL) equation [41]

\[
i\psi_{n,t} + \epsilon^{-2}(\psi_{n+1} + \psi_{n-1} - 2\psi_n) + a_2|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) = 0,
\]

(4)

for \( a_j = a_2\delta_{j2}, j = 1, \ldots, 10 \);

2. the discretization

\[
i\psi_{n,t} + \epsilon^{-2}(\psi_{n+1} + \psi_{n-1} - 2\psi_n) + 2a_1|\psi_n|^2\psi_n = 0,
\]

(5)
the integrability properties of (2), in this paper we follow the procedure indicated in the first equations (4), (5), (7) and (8) are Hamiltonian \[47\].

For special values of the coefficients \( \epsilon \)

\[
\sum_{j=1}^{10} \delta_{j1} j = 1, \ldots, 10, \text{ relevant in several applications } \[42-46\] \text{ whose nonintegrability has been recently shown in } \[34, 35\] \text{ using the DP test;}
\]

(3) the discretization corresponding to

\[
a_{10} = a_8, \quad a_1 = a_4 = a_5 = a_6 = a_7 = a_9 = 0,
\]

(6) with \( a_2, a_3, a_8 \) arbitrary, possessing a solitary wave as well as a breather solution reducing, respectively, to the one soliton and breather solutions of the NLS equation in the limit \( \epsilon \to 0 \) \[40\];

(4) the discretization corresponding to

\[
a_8 = a_3, \quad a_2 = 2a_3, \quad a_4 = 2a_6, \quad a_5 = a_7 = a_9 = a_{10} = 0,
\]

(7) where \( a_1, a_3, a_6 \) are given in terms of physical quantities, describing coupled optical waveguides embedded in a material with Kerr nonlinearities \[47\];

(5) the discretization corresponding to

\[
a_4 = a_2, \quad a_1 = a_3 = a_6 = a_8 = a_2/2, \quad a_5 = a_7 = a_9 = a_{10} = 0
\]

(8) (a particular case of (7)), appearing in the modeling of the Fermi–Pasta–Ulam problem \[48\].

For special values of the coefficients \( a_j \) the dNLS equation (2) is Hamiltonian. For instance, equations (4), (5), (7) and (8) are Hamiltonian \[47\].

If \( 0 < \epsilon \ll 1 \), the discrete scheme (2) approximates the NLS equation (1), (3) with an error of \( O(\epsilon^3) \). To study more precisely how close equations (2) and (1) are and, in particular, the integrability properties of (2), in this paper we follow the procedure indicated in the first part of this introduction, obtaining the following results.

(1) Due to the structure of the vector field in (2), the generated \( \epsilon \)-expansion contains only even powers. At \( O(\epsilon^2) \), the dNLS (2) passes the DP test iff the ten coefficients satisfy the elegant quadratic constraint

\[
(a_1 - 3a_3 - 2a_4 - 6a_5 - 5a_6 + 3a_7 - 5a_8 - 13a_9 - a_{10}) \left( \sum_{j=1}^{10} a_j \right) = 0,
\]

(9) factorized into two linear constraints. If the first constraint \( \sum_{j=1}^{10} a_j = 0 \) is satisfied, we are in the C-integrability framework and the dNLS (2) approximates the linear Schrödinger (LS) equation with an error of \( O(\epsilon^2) \), for time scales of \( O(\epsilon^{-2}) \). If, instead, the second constraint is satisfied:

\[
a_1 - 3a_3 - 2a_4 - 6a_5 - 5a_6 + 3a_7 - 5a_8 - 13a_9 - a_{10} = 0,
\]

(10) we are in the S-integrability framework and the dNLS (2) approximates the NLS equation (1), (3) with an error of \( O(\epsilon^2) \), for time scales of \( O(\epsilon^{-2}) \).

We remark that, among the ten single dNLS equations obtained choosing only one of the ten coefficients different from zero in (2), only the AL equation (4) satisfies the constraint (9) and passes the test at \( O(\epsilon^2) \).

(2) At \( O(\epsilon^3) \) we have the following two scenarios. In the C-integrability framework, the dNLS (2) approximates, with an error of \( O(\epsilon^3) \), the linear Schrödinger equation for time scales of \( O(\epsilon^{-3}) \) iff the four linear constraints

\[
\sum_{j=1}^{10} a_j = 0, \quad a_1 + a_2 + a_6 + a_7 = 0, \quad a_4 - a_5 + 2a_8 - 2a_9 = 0,
\]

\[
a_2 + 2(a_3 + 3a_5 + 3a_6 - a_7 + a_8 + 7a_9) = 0
\]

(11) for \( a_j = a_j \delta_{j1}, j = 1, \ldots, 10, \) relevant in several applications [42–46] whose nonintegrability has been recently shown in [34, 35] using the DP test;
are satisfied by the coefficients. Since one of the real $a_i$ can always be fixed rescaling
the dependent variable $\psi$, equations (11) characterize a five-parameter family of discrete
NLS equations (2) passing the test at such a high order.

In the S-integrability framework, the dNLS (2) approximates, with an error of $O(\epsilon^2)$, the NLS
equation (1), (3) for time scales of $O(\epsilon^{-4})$ if the coefficients satisfy, together with the linear
constraint (10), the five quadratic constraints (53), (54)–(58). Since these five constraints
do not contain the term $(a_2)^2$, they are trivially satisfied by the integrable AL equation (4),
as it has to be. In general we do not expect a parametrization of such constraints in terms
of elementary functions; however, we have been able to construct the following two explicit
dexamples of dNLS equations:

\begin{align}
&i\psi_{n,t} + e^{-2}(\psi_{n+1} + \psi_{n-1} - 2\psi_n) + a_0 \left( -8|\psi_n|^2 \psi_n + \frac{1}{4}|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) \right) \\
&+ 4\psi_n^2(\psi_{n+1} \psi_{n-1} - 4\psi_n(\psi_{n+1} \psi_{n-1} + \psi_{n+1} \psi_{n-1}) + \psi_n^2 + \psi_{n-1}^2) \\
&- 2\psi_n^2(\psi_{n+1} \psi_{n-1}) = 0,
\end{align}

(12)
satisfying such complicated quadratic constraints, corresponding to particular cases in which
the associated five quadrics degenerate into hyperplanes.

These two distinguished models, passing the test at such a high order through the above
degeneration mechanism, are obviously good candidates to be the S-integrable discretizations
of NLS. A detailed study of their performances as numerical schemes for NLS, and of their
possible integrability structure (Lax pair, etc), is postponed to a subsequent paper.

To obtain the above results, it is essential to use the well-known integrability properties of
equation (1) (shared by all integrable systems; see, for instance [49–52]) that we summarize
here, for completeness.

The NLS equation belongs to a hierarchy of infinitely many commuting flows:

\[ u_n = K_n(u), \quad n \in \mathbb{N}, \]

i.e.

\[ [K_n(u), K_m(u)]_L := K'_n(u)[K_m(u)] - K'_m(u)[K_n(u)] = 0, \quad n, m \in \mathbb{N}, \]

(15)

where

\[ K'_n(u)[f] := \lim_{\epsilon \to 0} \frac{\partial K_n(u + \epsilon f)}{\partial \epsilon} \]

(16)
is the usual Frechet derivative of $K_n(u)$ w.r.t $u$ in the direction $f$. The commuting vector fields
$[K_n]_{n \in \mathbb{N}}$ are arbitrary linear combinations, with constant coefficients, of the following basic
symmetries $[\sigma_n]_{n \in \mathbb{N}}$, generated by the recursion relation:

\[ \sigma_{n+1} = \hat{R}\sigma_n, \quad \sigma_0 = -iu, \quad n \in \mathbb{N}, \]

\[ \hat{R} f := i(f_x + 2cu \hat{\sigma}_x^{-1}(u \hat{f} + \bar{u} \hat{f})), \]

(17)
where $\hat{R}$ is the recursion operator of the NLS hierarchy [53]. The basic symmetries used in this paper are
\begin{align*}
\sigma_0 &= -iu, \quad \sigma_2 = i(u_{xx} + 2c|u|^2u), \\
\sigma_4 &= -i(u_{xxxx} + 2c(u^2\bar{u}_{xx} + 2|u|^2u_{xx}^2 + 3u^2\bar{u}) + 6c^2|u|^4u), \\
\sigma_6 &= i(u_{xxxxx} + 2c(u^2\bar{u}_{xxxx} + 6|u|^2u_{xxxx} + 4u_{xx}\bar{u}_{xx} + 9u\bar{u}_u_{xxx} \\
&\quad \quad + 15u\bar{u}_u_{xxx} + 11u|u|^2u_{xx}^2 + 10u^2\bar{u}_{xx} + 10c^3(2u^2|u|^2\bar{u}_{xx} + 2\bar{u}_{xx}^2 \\
&\quad \quad + 5|u|^4u_{xx}^2 + 6u|u|^2u_{xx}^2 + 7u\bar{u}|u|^2u_{xx}^2) + 20c^3u|u|^6).
\end{align*}
and the NLS equation (1) corresponds to the flow $E(2)$, isolating the constraints on the $n\text{th}$ flow operator $\hat{M}_n$, $n \in \mathbb{N}$, defined by
\begin{equation}
\hat{M}_n f := f_n - K_n(u[f]) \tag{19}
\end{equation}
i.e.
\begin{equation}
\hat{M}_n \sigma_m = 0, \quad n, m \in \mathbb{N}. \tag{20}
\end{equation}
Due to (15), these linearized operators commute:
\begin{equation}
\hat{M}_n \hat{M}_m = \hat{M}_m \hat{M}_n, \quad n, m \in \mathbb{N}. \tag{21}
\end{equation}
The linearized operators used in this paper are
\begin{equation}
\hat{M}_2 f := f_2 - i(f_{xx} + 2c(u^2\bar{f} + 2|u|^2f)), \tag{22}
\end{equation}
and to its (trivial) symmetries $(-i\epsilon^2u)$.

The paper is organized as follows. In section 2 we construct the multiscale expansion, in the small $\epsilon$ regime, of a generic solution of (2), establishing, in particular, that the leading term of such an expansion evolves w.r.t the infinitely many `even' time variables $t_{2k} := \epsilon^{2k-1}t$, $k \in \mathbb{N}$, according to the even flows of the NLS hierarchy. In section 3, after summarizing the DP test and after showing how to use the main output of this test to construct infinitely many approximate symmetries of the original $\text{PDE}$ through novel and simple formulas, we apply the DP test to the $\text{PDE}$ (2), isolating the constraints on the coefficients $a_j$, $j = 1, \ldots, 10$, allowing one to pass the test at time scales of $O(\epsilon^{-2})$ and of
$O(\epsilon^{-4})$, in both scenarios of C- and S-integrability. In section 4 we summarize the results of the paper and we discuss the research perspectives opened by this work. In the appendix we display the long outputs of the DP test, obtained using the algebraic manipulation program of Mathematica.

2. Multiscale expansion in the small lattice spacing regime

If the lattice spacing $\epsilon$ is small: $0 < \epsilon \ll 1$, as a consequence of the invariance of (2) under the transformation $\psi_{n \pm 1} \rightarrow \psi_{n \mp 1}$ and of the well-known formula

$$f_{n \pm 1} = \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!} \epsilon^k \partial^k_x f,$$

valid in the long-wave approximation, only even $x$-derivatives appear at all (even) orders in $\epsilon$. Consequently, to eliminate the secularities appearing at all even orders in $\epsilon$, the coefficients of such an expansion must depend on infinitely many ‘even’ slow times $[1]$: $\vec{t} = (t_2, t_4, t_6, \ldots)$, $t_{2k} := \epsilon^{2(k-1)} t$, $k \in \mathbb{N}_+$.

Therefore, we are led to the following ansatz for the asymptotic expansion of the ‘generic’ solution of (2):

$$\psi_n(t) = \sum_{k=0}^{\infty} \epsilon^{2k} u_{2k+1}(x, \vec{t}), \quad u^{(1)}(x, \vec{t}) = u(x, \vec{t}).$$

Plugging (24), (26) and (27) into equation (2) and equating to zero the coefficients of all powers in $\epsilon$, we obtain the following results.

At the leading $O(1)$, we obtain the NLS equation for the leading term $u^{(1)} = u$ w.r.t the first time $t_2 = t$:

$$u_{t_2} = K_2(u), \quad K_2(u) := \sigma_2(u) = i(u_{xx} + 2c|u|^2 u), \quad c = \sum_{j=1}^{10} a_j.$$  (28)

As usual in perturbation theory, at the next relevant order ($O(\epsilon^2)$ in our case), the ‘linearization’ $\tilde{M}_2 u^{(3)}$ of $(u_{t_2} - K_2(u))$ appears, together with the linear term $(u_{t_2} - (i/12)u_{xxxx})$, coming from the linear part of (2), and with a nonlinear term $G_5$ coming from the nonlinear part of (2):

$$\tilde{M}_2 u^{(3)} = -\left(u_{t_2} - i\frac{2}{4!}u_{xxxx}\right) + G_5,$$  (29)

where

$$G_5 = i \left(s_1 u^2 \bar{u}_{xx} + s_2 |u|^2 u_{xx} + s_3 u |u_x|^2 + s_4 \bar{u} u_{xx}^2\right),$$

$$s_1 = a_3 + a_4 + a_5 + a_8 + a_9 + a_{10},$$

$$s_2 = a_2 + a_4 + a_5 + 2(a_6 + a_7 + a_8 + a_9 + a_{10}),$$

$$s_3 = 2(a_4 - a_5 + 2a_9 - 2a_8), \quad s_4 = 2(a_6 - a_7 + a_8 + a_9 - a_{10}).$$  (30)

Concentrating on the linear terms in the round bracket, we observe that $u_{t_2} \in \text{Ker} \tilde{M}_2$ and $-(i/12)u_{xxxx}$ is the linear part of the symmetry $(2/4!)\sigma_4(u) \in \text{Ker} \tilde{M}_2$. Therefore, adding and
This procedure iterates without essential differences at all orders. The terms \(\hat{J}\) from equation (20) is conveniently rearranged in the following way, isolating the resonant terms in the round bracket:

\[
\hat{M}_2 u^{(3)} = - \left( u_{t_1} + \frac{2}{4!} \sigma_4(u) \right) + g_5, \tag{31}
\]

where

\[
g_5 := i(c_1 |u|^4 u + c_2 \bar{u} u^2 + c_3 u |u|^2 + c_4 |u|^2 u_{xx} + c_5 u^2 \bar{u}_{xx}) \tag{32}
\]

and

\[
c_1 = -\frac{1}{2} \epsilon^2,
\]

\[
c_2 = -\frac{1}{2} (a_1 + a_2 + a_3 + a_4 + a_5 - 3a_6 + 5a_7 - 3a_8 - 3a_9 + 5a_{10}),
\]

\[
c_3 = -\frac{1}{2} (a_1 + a_2 + a_3 - 5a_4 + 7a_5 + a_6 + a_7 - 11a_8 + 13a_9 + a_{10}),
\]

\[
c_4 = -\frac{1}{6} [2a_1 - a_2 + 2a_3 - a_4 - a_5 - 4(a_6 + a_7 + a_9 + a_{10})],
\]

\[
c_5 = -\frac{1}{6} [a_1 + a_2 + a_3 + a_7 - 5(a_3 + a_4 + a_5 + a_8 + a_9 + a_{10})].
\]

To eliminate the secularity in the bracket, we are forced to choose

\[
u_{t_1} = K_4(u) := - \frac{2}{4!} \sigma_4(u), \tag{34}
\]

so (31) finally becomes the following secularity free equation for the first correction \(u^{(3)}\):

\[
\hat{M}_2 u^{(3)} = g_5. \tag{35}
\]

This procedure iterates without essential differences at all orders. The terms \(\hat{M}_2 u^{(3)}\) and \((u_{t_1} - K_4(u))\) in (31) generate, at \(O(\epsilon^4)\), the terms \(\hat{M}_2 u^{(5)}\) and \(\hat{M}_4 u^{(3)}\) respectively, while the new linear term \((u_{t_1} - i(2/6!)u_{xxxx})\) is rearranged again into the secular factor \((u_{t_1} - (2/6!)\sigma_6)\) that must be set to zero, to avoid secularities. Since, at \(O(\epsilon^{2k})\), we produce the linear term \(\left( u_{t_2} - i \frac{2}{(2k)!} \sigma_{2k} \right) \), one infers, in analogy with [1], that \(u\) evolves w.r.t the higher times according to the even flows of the NLS hierarchy as follows:

\[
u_{t_2} = K_{2k}(u) := (-1)^{k+1} \frac{2}{(2k)!} \sigma_{2k}(u), \quad k \in \mathbb{N}_+ \tag{36}
\]

and one is left with the following triangular set of equations [2, 3]:

\[
O(\epsilon^2) : \quad \hat{M}_2 u^{(3)} = g_5,
\]

\[
O(\epsilon^4) : \quad \hat{M}_2 u^{(5)} + \hat{M}_4 u^{(3)} = g_7,
\]

\[
O(\epsilon^6) : \quad \hat{M}_2 u^{(7)} + \hat{M}_4 u^{(5)} + \hat{M}_6 u^{(3)} = g_9,
\]

\[
\vdots
\]

\[
O(\epsilon^{2k}) : \quad \hat{M}_2 u^{(2k+1)} + \hat{M}_4 u^{(2k-1)} + \cdots + \hat{M}_{2k} u^{(3)} = g_{2k+3},
\]

\[
\vdots
\]

where, for instance, the expression of \(g_7\) is presented in formula (A.1) of the appendix. It remains to remark, following [2, 3], that the symmetries \([\sigma_n]\) and the expressions in (37), generated by the multiscale expansion, are differential polynomials satisfying the following properties: (i) they are linear combinations of products of the \(u^{(k)}\)’s and of their derivatives with respect to \(x\): \(\delta^j_t u^{(k)}\), \(j \geq 0, k\) odd; (ii) they possess the gauge symmetry of first kind. In addition, the differential polynomials appearing in the same equations exhibit the same ‘order’, in the following sense.
Definition 1. If we define the order of the term $\partial_j^x u^{(k)}$, $j \geq 0$, as follows:

\[
\text{order} (\partial_j^x u^{(k)}) = \text{order} (\partial_j^x u^{(k)}) = j + k,
\]

then the order of the product of terms of this type is the sum of the orders of each term.

For example, order$(\partial_j^x u) = j + 1$ (since $u = u^{(1)}$) and order$(|u^{(k)}|^2 \partial_j^x u^{(k)}) = 2k_1 + j + k_2$. Therefore, we are naturally led to the definition of the following vector spaces.

Definition 2. $P_n$ is the vector space of all the differential polynomials satisfying properties (i) and (ii) above, of order $n$. $P_n(m)$ is instead the subspace of $P_n$ of all differential polynomials satisfying properties (i) and (ii) above and containing only terms $(\partial_j^x u^{(k)}), (\partial_j^x u^{(k)})$ such that $k \leq m$.

It is easy to see that, for instance, $\sigma_n, K_n \in P_{n+1}(1), g_5 \in P_5(1)$ and $g_7 \in P_7(3)$ (see (A.1)).

3. Applying the DP integrability test

Suppose we generate, from the model to be tested, an NLS-type multiscale expansion (as in our example); then we have the following scenarios. If such a model is S-integrable (C-integrable),

1. the leading term $u$ of the asymptotic expansion evolves, with respect to the slow times $t_n$, according to the NLS (LS) hierarchy [1];
2. there exist elements $f^{(m)}_n \in P_{m+n}$ such that the following equations hold [2, 3]:

\[
\hat{M} u^{(m)}_n = f^{(m)}_n \in P_{m+n}, \quad m, n \in \mathbb{N},
\]

implying, due to (21), the compatibility conditions

\[
\hat{M} f^{(j)}_m = \hat{M} f^{(j)}_m, \quad m, n, j \in \mathbb{N}.
\]

Therefore, equations (40) are necessary conditions to be satisfied, in cascade, for the model under investigation to be S- (C-)integrable; they are also sufficient to guaranty the asymptotic character of the expansion. If equations (40) are satisfied only up to a certain order, the model under investigation is not integrable, being nevertheless ‘asymptotically integrable up to that order’ [2, 3].

3.1. The DP test and approximate symmetries

Equations (39) and (40), the basic formulas of the DP test, have been derived in [2, 3] as a consequence of the existence of a Lax pair for the starting integrable model. It follows that if conditions (39) and (40) are satisfied up to a certain order, the equation under scrutiny admits an approximate Lax pair up to that order.

In this subsection we show how to derive conditions (39) and (40) from the existence of infinitely many symmetries of the starting integrable model. This derivation allows one to establish the important relations (to the best of our knowledge so far unknown) between the functions $f^{(m)}_n \in P_{m+n}$ of the DP test and the symmetries of the starting model. We concentrate our attention on the case of difference equations, but our considerations have general validity.

Let $\psi_{nt} = K_2(\psi_n)$ be an integrable model, say, the AL equation (4), and let $\psi_{nt2m} = K_{2m}(\psi_n), m > 2$, be one of its infinitely many higher order commuting flows (symmetries), reducing, in the continuous limit, to the higher commuting flow $u_{t2m} = K_{2m}(u)$ of NLS.
On one hand, from equations (25) and (27), we have that
\[
\psi_{n + 2m} = u_{2m} + \epsilon^2 (u_{(2m+1)} + u_{(3)}^{(3)}) + \ldots = \sum_{k \geq 0} \epsilon^{2k} \left( \sum_{j=m}^{m+k} u_{2m}^{(2(m+k-j)+1)} \right),
\]
where
\[
u_{2m}(u) \quad \text{(from (36)) and } u_{2m}^{(2(m+k-j)+1)} = K_j^{(m+k-j+1)}[u^{(2(m+k-j)+1)}] + f_{2j}^{(2(m+k-j)+1)},
\]
for some functions \(f_{2j}^{(2(m+k-j)+1)}\) to be specified. On the other hand,
\[
K_{2m}(u_\ell) = K_{2m}(u) + \epsilon^2 K_{2m}^{(2)} + \ldots = K_{2m}(u) + \sum_{k \geq 1} \epsilon^{2k} K_{2m}^{(2k)},
\]
where \(K_{2m}^{(2k)} \in \mathcal{P}_{2(m+k)+1}\). Equating equations (41) and (42), we infer that \(f_n^{(m)} \in \mathcal{P}_{m+n}, n \in \mathbb{N}_1\) (the basic formula (39) of the DP test), and we also construct the asymptotic expansion of the generic higher order symmetry
\[
K_{2m}(u_\ell) = K_{2m}(u) + \epsilon^2 (K_{2(m+1)}(u) + K_{2m}^{(3)}[u^{(3)}] + f_{2m}^{(3)}) + \ldots
\]
(43)
in terms of the NLS higher order symmetries, of their Frechet derivatives in the direction of the corrections \(u^{(j)}\), \(j > 1\), of the leading term \(u\) of expansion (27), and of the output functions \(f_n^{(m)} \in \mathcal{P}_{m+n}\) of the DP test.

Therefore, if \(f_{2n}^{(2k+1)} \in \mathcal{P}_{2(k+n)+1}\) exists, but \(f_{2n}^{(2k+3)} \in \mathcal{P}_{2(k+n)+3}\) does not, \(\forall n \in \mathbb{N}_1\), it follows that

(i) the solution \(u^{(2k+1)}\) of (39) is uniformly bounded and expansion (27) is asymptotic up to \(O(\epsilon^{2k})\); therefore, the PDE under scrutiny approximates well its continuous limit, with an error of \(O(\epsilon^{2k})\), for time scales up to \(O(\epsilon^{-2k})\).

(ii) The PDE possesses infinitely many ‘approximate’ generalized symmetries in the form (43) up to \(O(\epsilon^{2k})\); therefore, it is integrable up to that order. We remark that, due to the Hamiltonian theory of integrable systems [49–52], it is also possible to associate with the PDE infinitely many ‘approximate’ constants of motion in involution, a very useful information in any numerical check.

### 3.2. C- and S-integrability at \(O(\epsilon^2)\)

In our example, the first of equations (37) is already in the form (39), with \(g_5 = f_2^{(3)} \in \mathcal{P}_5(1)\).

Assuming now that \(\hat{M}_4 u^{(3)} = f_4^{(3)}\), we arrive at the consistency
\[
\hat{M}_4 f_4^{(3)} = \hat{M}_2 f_2^{(3)}
\]
(44)
that must be viewed as an equation for the unknown \(f_4^{(3)}\). Since \(g_5 = f_2^{(3)} \in \mathcal{P}_5(1)\), it follows that one must look for \(f_4^{(3)} \in \mathcal{P}_7(1)\). The calculation, plain but lengthy, has been performed using the algebraic manipulation program of Mathematica, and gives the following result.

**Lemma 1.** Equation (44) admits a unique solution \(f_4^{(3)} \in \mathcal{P}_7(1)\) (presented in formula (A.4) of the appendix) iff the coefficients \(a_j\) appearing in (2) satisfy the following quadratic constraint:

\[
(a_1 - 3a_3 - 2a_4 - 6a_5 - 5a_6 + 3a_7 - 5a_8 - 13a_9 - a_{10}) \sum_{j=1}^{10} a_j = 0.
\]

(45)
Once \( f_4^{(3)} \) is constructed, \( f_2^{(5)} \in \mathcal{P}_7(3) \) \( (f_2^{(5)} = \hat{M}_2u^{(5)}) \) is found from the second of equations (37):
\[
f_2^{(5)} = g_7 - f_4^{(3)}
\]
and is presented in formula (A.7) of the appendix.

We first note the nice factorization of the quadratic constraint (45) into two linear constraints:
\[
c = \sum_{j=1}^{10} a_j = 0,
\]
\[
a_1 - 3a_3 - 2a_4 - 6a_5 - 5a_6 + 3a_7 - 5a_8 - 13a_9 - a_{10} = 0.
\]

Therefore, we have the following two different scenarios.

1. If the first constraint (47) is satisfied by the coefficients \( a_j \), the continuous limits of dNLS (2) are the LS equation. It follows that, in this case, equation (2) is ‘asymptotically C-integrable’ at \( O(\epsilon^2) \) and one expects that, for generic initial data and at time scales of \( O(\epsilon^{-2}) \), the dynamics according to (2), (47) be well approximated by the dynamics according to the LS equation (23) with an error of \( O(\epsilon^2) \).

   In particular, the dNLS (2), (7) is ‘asymptotically C-integrable’ at \( O(\epsilon^2) \) if
\[
a_1 + 4a_3 + 3a_6 = 0.
\]

2. If, instead, the second constraint (48) is satisfied by the coefficients \( a_j \), the dNLS equation (2) is ‘asymptotically S-integrable’ at \( O(\epsilon^2) \) and one expects that, for generic initial data and at time scales of \( O(\epsilon^{-2}) \), the dynamics according to the dNLS equation (2), (48) approximates well the dynamics according to the NLS equation (1), (3) with an error of \( O(\epsilon^2) \).

   In particular, (i) the dNLS (2), (6) is ‘asymptotically S-integrable’ at \( O(\epsilon^2) \) iff the following additional constraint is satisfied:
\[
a_3 + 2a_8 = 0;
\]
(ii) the dNLS (2), (7) is ‘asymptotically S-integrable’ at \( O(\epsilon^2) \) iff the following additional constraint is satisfied:
\[
a_1 - 8a_3 - 9a_6 = 0,
\]
while the dNLS (2), (8) is not ‘asymptotically S-integrable’ at \( O(\epsilon^2) \) (therefore, it is not integrable).

   In addition, since the dNLS equation (2) is the linear combination of ten different discretizations of NLS, it is immediate to check if some of these ten discretizations satisfy the constraint (48). Calling dNLS\(_k\) the single discretization of NLS obtained choosing in (2) \( a_j = a_k \delta_{jk}, j=1,\ldots,10 \), it is straightforward to see (since the coefficient \( a_2 \) is the only one absent in (48)) that only the dNLS\(_2\) equation (coinciding with the AL equation (4)) satisfies the constraint (48) (as it has to be, being an integrable system). All the other dNLS\(_k\), \( k \neq 2 \) equations, including the dNLS\(_1\) equation (5), do not satisfy the constraint (48); therefore, they are not ‘asymptotically S-integrable’ at \( O(\epsilon^2) \) (consequently, they are not integrable) and, for generic initial data and at time scales of \( O(\epsilon^{-2}) \), their dynamics are expected to be quite different from that of NLS (1), (3), presumably exhibiting numerical evidence of nonintegrability and/or chaos.

   We finally infer that the discretizations (2), (6) and (2), (7) satisfying respectively the constraints (50) and (51), the AL equation and any other dNLS equation (2) satisfying the
constraint (48) are all close to NLS (once the free coefficients of each model are normalized to satisfy (3)) and are all close together at time scales of $O(\epsilon^{-3})$, in the sense mentioned in the introduction.

It is interesting to push the integrability test to the next order which we will present here. Due to the above factorization of the constraint (45), the test bifurcates and, in the next two subsections, we explore both cases. Before doing that, we observe that, given $f_3^{(5)} \in \mathcal{P}_3(1)$ and assuming that the constraint (45) be satisfied, the equation $\hat{M}_4 f_4^{(5)} = \hat{M}_2 f_4^{(3)}$ admits a unique solution $f_4^{(3)} = \hat{M}_4u^{(3)} \in \mathcal{P}_9(1)$, presented in formula (A.10) of the appendix, and no additional constraint appears in this derivation, as predicted by the DP test.

### 3.3. C-integrability at $O(\epsilon^4)$

Let us assume that the constraint (47) be satisfied. For the construction of $f_4^{(5)} = \hat{M}_4u^{(5)} \in \mathcal{P}_9(3)$ from the equation

$$\hat{M}_4 f_4^{(5)} = \hat{M}_2 f_4^{(3)}$$

we have the following result.

**Lemma 2.** Equation (52) admits a unique solution $f_4^{(5)} \in \mathcal{P}_9(3)$, presented in formula (A.13) of the appendix, iff the coefficients $a_j$ satisfy the four linear constraints (11), defining a six-parameter family (but one of these parameters can always be rescaled away) of $dNLS$ equations (2) ‘asymptotically C-integrable’ at $O(\epsilon^4)$. Therefore, one expects that, for generic initial data and at time scales of $O(\epsilon^{-4})$, the dynamics according to (2), (11) well approximate the dynamics according to the LS equation (23) with an error of $O(\epsilon^4)$.

For instance, the discretization (2), (7) satisfies the constraints (11) iff $a_6 = -a_3 = a_1$.

The six-parameter family of $dNLS$ equations (2), (11) (or at least some particular case of it), being C-integrable at such a high order, is a natural candidate to be a C-integrable discrete system.

### 3.4. S-integrability at $O(\epsilon^4)$

Let us assume that the constraint (48) be satisfied. For the construction of a unique $f_4^{(5)} = \hat{M}_4u^{(5)} \in \mathcal{P}_9(3)$ from equation (52), we have the following result.

**Lemma 3.** If the constraint (48) is satisfied, equation (52) admits a unique solution $f_4^{(5)} \in \mathcal{P}_9(3)$, presented in formula (A.13) of the appendix, iff the following five quadratic constraints are satisfied:

$$Q_j = 0, \quad j = 1, \ldots, 5,$$

where the $Q_j$’s are the following quadratic forms in the nine variables $a_2, \ldots, a_{10}$:

$$Q_1 = -4a_{10}^2 + a_{10}a_2 + 2a_1a_3 - a_2a_3 + 2a_1^2 - a_1a_4 - 2a_2a_4 + a_3a_4$$

$$+ 3a_1a_5 - 2a_2a_5 - 3a_3a_5 - 8a_4a_5 - 8a_5^2 + 18a_{10}a_6 + 6a_3a_6 - 6a_4a_7$$

$$+ 4a_2a_7 + 6a_3a_7 + 4a_4a_7 + 20a_5a_7 + 24a_6a_7 - 8a_7^2 + 12a_{10}a_8 - 3a_2a_8$$

$$+ 6a_3a_8 + 3a_4a_8 - 9a_5a_8 - 6a_6a_8 + 18a_7a_8 + 20a_1a_9 - 3a_2a_9 - 2a_3a_9$$

$$- 13a_4a_9 - 25a_5a_9 - 6a_6a_9 + 50a_7a_9 - 24a_8a_9 - 24a_9^2,$$

$$Q_2 = -4a_{10}^2 + a_{10}a_2 + 2a_1a_3 - a_2a_3 + 2a_1^2 - a_1a_4 - 2a_2a_4 + a_3a_4$$

$$+ 3a_1a_5 - 2a_2a_5 - 3a_3a_5 - 8a_4a_5 - 8a_5^2 + 18a_{10}a_6 + 6a_3a_6 - 6a_4a_7$$

$$+ 4a_2a_7 + 6a_3a_7 + 4a_4a_7 + 20a_5a_7 + 24a_6a_7 - 8a_7^2 + 12a_{10}a_8 - 3a_2a_8$$

$$+ 6a_3a_8 + 3a_4a_8 - 9a_5a_8 - 6a_6a_8 + 18a_7a_8 + 20a_1a_9 - 3a_2a_9 - 2a_3a_9$$

$$- 13a_4a_9 - 25a_5a_9 - 6a_6a_9 + 50a_7a_9 - 24a_8a_9 - 24a_9^2,$$
whose parametrization does not appear to be expressible, in general, in terms of elementary
the intersection of five quadrics in the real projective space of dimension 8, define, in principle,
The five homogeneous quadratic constraints (53), (54)–(58) for nine unknowns, characterizing
explicit solutions of (48), (53), (54)–(58) can also be constructed, corresponding to the case
should well approximate the NLS equation for times up to
always multiplied by other coefficients (the term $a_j^2$ is absent); therefore, the choice
$a_j = a_2 \delta_{j2}$, \quad j = 1, \ldots, 10,
 corresponding to the AL equation (4) satisfies all constraints, as it has to be. Other less trivial explicit solutions of (48), (53), (54)–(58) can also be constructed, corresponding to the case
in which all quadrics degenerate into pairs of hyperplanes. Here we display the following two
equations:

\[ a_1 = -4a_6, \quad a_2 = \frac{4a_6}{3}, \quad a_3 = 4a_6, \quad a_4 = 0, \quad a_5 = -4a_6, \quad a_7 = -a_6, \quad a_8 = a_9 = a_{10} = 0, \]

\[ a_1 = -24a_9, \quad a_2 = a_3 = 0, \quad a_4 = a_5 = -8a_9, \quad a_6 = 10a_9, \quad a_7 = -2a_9, \quad a_8 = -7a_9, \quad a_{10} = 6a_9, \]

(60)

(61)
corresponding, respectively, to the dNLS equations (12) and (13) presented in the introduction,
’asymptotically S-integrable’ at \(O(\epsilon^4)\). Therefore, one expects that, for generic initial data
and at time scales of \(O(\epsilon^{-4})\), the dynamics according to equations (12) and (13) are good
approximations of the dynamics according to the NLS equation (1), with an error of \(O(\epsilon^2)\),
at time scales of \(O(\epsilon^{-4})\). Of course, these distinguished equations, passing the test at such a
high order, are also good candidates to be S-integrable difference equations.

We finally observe that there is no choice of parameters for which the dNLS equations
(2), (6), (50) and (2), (7), (51) satisfy the above constraints; therefore, these two models are
not S-integrable at this order (they are not S-integrable at all) and do not approximate well
NLS at time scales of \(O(\epsilon^{-4})\).

4. Summary of the results and future perspectives

In this paper we have proposed an algorithmic procedure allowing one (i) to study the distance
between an integrable PDE and any \(P\Delta E\) discretizing it, in the small lattice spacing \(\epsilon\) regime;
(ii) to test the (asymptotic) integrability properties of such a \(P\Delta E\); and (iii) to construct infinitely
many (approximate) symmetries and conserved quantities for it. This method should provide,
in particular, useful and concrete information on how good is a numerical scheme used to
integrate a given integrable PDE.

The procedure we have proposed, illustrated on the basic prototype example of the
nonlinear Schrödinger equation (1) and of its discretization (2), consists of the following three
steps: (i) the construction of the multiscale expansion of a generic long-wave solution of the
dNLS (2) at all orders in \(\epsilon\), following [1]; (ii) the application, to such an expansion, of the DP
integrability test [2, 3]; (iii) the use of the main output of such a test to construct infinitely
many approximate symmetries of the dNLS equation (2), through novel formulas presented
in this paper.

This approach allows one to study the distance between the integrable PDE and any
\(P\Delta E\) discretizing it. Suppose, for instance, that the asymptotic expansion we construct reads
\[ \psi = u + O(\epsilon^\alpha), \alpha > 0, \]
where \(\psi\) is a generic long-wave solution of the dNLS (2) and \(u\) is
the corresponding solution of (1); then if the DP test is passed at \(O(\epsilon^\beta)\), we conclude that
(i) the dynamics according to the NLS equation (1) is well approximated (with an error of \(O(\epsilon^\alpha)\)) by the dynamics according to its discretization (2), for time scales of \(O(t^{-\beta})\); (ii) the
dNLS equation is asymptotically integrable up to that order, constructing its infinitely many
approximate symmetries and constants of motion in involution. In contrast, if the DP test is
not passed at that order, the dNLS equation is not integrable and one should expect, at the
corresponding time scale, numerical evidence of nonintegrability and/or chaos.

We have carried the above procedure up to \(O(\epsilon^4)\) and we have been able to isolate the
constraints on the coefficients of the dNLS equation (2) allowing one to pass the test at that
order, in both scenarios of S- and C-integrability.
Numerical experiments to test such theoretical findings are presently under investigation; preliminary results seem to confirm the theoretical predictions contained in this paper [54].

With the same methodology and goals, we are presently investigating families of discretizations of the Korteweg–de Vries and Burgers equations [55], other two basic integrable models of natural phenomena. Of course we also plan to investigate discretizations of integrable PDEs in which also the time variable is discretized.

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Appendix A.

In this appendix we display, for completeness, the long outputs of the DP test, obtained using the algebraic manipulation program of Mathematica. The differential polynomial $g_7$ in (37) reads

$$g_7 = i[l_1 u |u|^6 + l_2 |u|^4 u^{(3)} + l_3 \bar{u} u^{(3)} + l_4 u^2 |u_1|^2 \bar{\bar{u}}^{(3)} + l_5 u |u^{(3)}|^2]
 + l_6 |u|^2 \bar{u}^2 + l_7 u^2 \bar{u}^{(3)} + l_8 u |u|^2 |u_1|^2 + l_9 |u_1|^2 u^{(3)} + l_{10} u^2 \bar{\bar{u}}^2
 + l_{11} \bar{u} u \bar{u}_x u_x^{(3)} + l_{12} u \bar{u}_x \bar{u}_x^{(3)} + l_{13} u u_x \bar{u}_x^{(3)} + l_{14} u |u|^2 u_{xx} + l_{15} \bar{u} u_{xx} u^{(3)}
 + l_{16} u \bar{u}_x \bar{u}_x^{(3)} + l_{17} |u_x|^2 u_{xx} + l_{18} \bar{u} u_x^2 + l_{19} u^2 |u|^2 \bar{\bar{u}}_{xx} + l_{20} \bar{u} u_{xx} u^{(3)}
 + l_{21} u^2 \bar{u}_{xx} + l_{22} u |u_{xx}|^2 + l_{23} |u|^2 u_x^{(3)} + l_{24} u^2 \bar{\bar{u}}_{xx} + l_{25} \bar{u} u_{xx} u_{xxx}
 + l_{26} u \bar{u}_x u_{xxx} + l_{27} u \bar{u}_x \bar{u}_x u_{xxx} + l_{28} |u|^2 u_{xxxx} + l_{29} u^2 \bar{\bar{u}}_{xxxx}), \quad (A.1)$$

where

$$l_1 = -\frac{1}{18} \mathfrak{c}^2, \quad l_2 = -\frac{3}{2} \mathfrak{c}^2, \quad l_3 = 2 \mathfrak{c}, \quad l_4 = -\mathfrak{c}^2, \quad l_5 = 4 \mathfrak{c},$$

$$l_6 = 7 l_{10}, \quad l_7 = \frac{1}{2} l_{11}, \quad l_8 = 6 l_{10},$$

$$l_9 = -\frac{1}{8} (a_1 + a_2 + a_3 - 5 a_4 + 7 a_5 + a_6 + a_7 - 11 a_8 + 13 a_9 + a_{10}),$$

$$l_{10} = -\frac{1}{18} \mathfrak{c}^2, \quad l_{11} = -(a_1 + a_2 + a_3 + a_4 + a_5 - 3 a_6 + 5 a_7 - 3 (a_8 + a_9) + 5 a_{10}),$$

$$l_{12} = l_{13}, \quad l_{13} = -\frac{1}{5} (a_1 + a_2 + a_3 - 5 a_4 + 7 a_5 + a_6 + a_7 - 11 a_8 + 13 a_9 + a_{10}),$$

$$l_{14} = 5 l_{10}, \quad l_{15} = l_{16}, \quad (A.2)$$

$$l_{16} = -\frac{1}{2} (2 a_1 - a_2 + 2 a_3 - a_4 - a_5 - 3 a_6 + a_7 + a_8 + a_9 + a_{10}),$$

$$l_{17} = -\frac{1}{35} (5 (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9) - 67 a_8 + 77 a_9),$$

$$l_{18} = -\frac{1}{18} (a_1 + a_2 + a_3 + a_4 + a_5 - 8 (a_6 + a_7 + a_8 + a_9 + a_{10})),$$

$$l_{19} = 2 l_{10}, \quad l_{20} = -\frac{1}{5} (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + 17 (a_6 + a_8 + a_9 + a_{10})),$$

$$l_{21} = -\frac{1}{18} (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + 17 (a_6 + a_8 + a_9 + a_{10})),$$

$$l_{22} = -\frac{1}{180} (11 (a_1 + a_2 + a_3) - 79 (a_4 + a_5) + 11 (a_6 + a_7) - 169 (a_8 + a_9 + a_{10})),$$

$$l_{23} = -\frac{1}{18} (2 a_1 - a_2 + 2 a_3 - a_4 - a_5 - 4 (a_6 + a_7 + a_8 + a_9 + a_{10})), \quad l_{24} = \frac{1}{2} l_{20},$$

$$l_{25} = -\frac{1}{12} (a_1 + a_2 + a_3 + a_4 + a_5 + 9 a_7 - 7 (a_6 + a_8 + a_9) + 9 a_{10}),$$

$$l_{26} = -\frac{1}{25} (3 (a_1 + a_2 + a_3 + a_4 + a_5) + 17 a_6 + 23 a_7 + 3 a_8 - 37 a_9 + 43 a_{10}),$$

$$l_{27} = -\frac{1}{25} (a_1 + a_2 + a_3 + 14 a_4 + 16 a_5 + a_6 + a_7 - 29 a_8 + 31 a_9 + a_{10}).$$
\[ l_{28} = -\frac{1}{60}(2a_1 - 3a_2 + 2a_3 - 3a_4 - 3a_5 - 8(a_6 + a_7 + a_8 + a_9 + a_{10})), \]
\[ l_{29} = -\frac{1}{192}(a_1 + 2a_2 + a_6 + a_7 - 14(a_4 + a_9 + a_9 + a_{10})). \]  

(A.3)

The solution \( f_2^{(3)} \in \mathcal{T}_1(1) \) of \( \hat{M}_4 f_2^{(3)} = \hat{M}_2 f_4^{(3)} \), where \( f_2^{(3)} = g_5 \) is given in (32), (33), exists unique and reads

\[
f_2^{(3)} = i(\alpha_1 u |u|^6 + \alpha_2 u_x u |u|^2 + \alpha_3 u_{xx} u^2 |u|^2 + \alpha_4 u^{(3)} |u|^2) u + \alpha_9 u x u x u x u^2 + \alpha_10 u_{xx} u^2 + \alpha_11 u_{xxx} u^2 + \alpha_12 u_{xxxx} u^2) \quad \text{(A.4)}
\]

where

\[
\alpha_1 = \frac{c^2}{3} (2c_2 - c_3 + c_4 + 3c_5), \quad \alpha_2 = \frac{c}{6} (4c_2 - 2c_3 + 6c_4 + 5c_5),
\]
\[
\alpha_3 = \frac{c}{12} (2c_2 - c_3 + 3c_4 + 10c_5), \quad \alpha_4 = \frac{c}{24} (40c_2 - 11c_3 + 25c_4 + 20c_5),
\]
\[
\alpha_5 = \frac{c}{12} (8c_2 + 5c_3 + 7c_4 + 16c_5), \quad \alpha_6 = \frac{c}{24} (4c_2 + c_3 + c_4 + 8c_5),
\]
\[
\alpha_7 = \frac{c_4}{6}, \quad \alpha_8 = \frac{c_5}{12}, \quad \alpha_9 = \frac{1}{12} (4c_2 + 3c_4), \quad \alpha_{10} = \frac{1}{12} (c_3 + 3c_5),
\]
\[
\alpha_{11} = \frac{1}{12} (2c_2 + c_4), \quad \alpha_{12} = \frac{1}{12} (3c_2 + 2c_4), \quad \alpha_{13} = \frac{1}{12} (c_3 + c_4 + 4c_5),
\]
\[
\alpha_{14} = \frac{1}{12} (2c_2 + 5c_4 + c_5), \quad \alpha_{15} = \frac{1}{12} (c_2 + c_3 + 3c_5),
\]  

iff the following constraint is satisfied:

\[
2c_1 - c(2c_2 - c_3 + c_4 + 4c_5) = 0 \quad \text{(A.6)}
\]

on the coefficients \( c_j \)'s defined in (33). This constraint is equivalent to (45).

\[ f_2^{(5)} = g_7 = f_4^{(3)} \quad \text{consequently reads, from (A.1) and (A.4),}
\]
\[
f_2^{(5)} = i(d_1 |u|^6 + d_2 |u|^4 u^{(3)} + d_3 u^{(3)} |u|^2 + d_4 u^2 |u|^2 u^{(3)} + d_5 u |u^{(3)}|^2)
\]
\[
+ d_6 u |u|^2 u_x^2 + d_7 u^2 u_x^2 + d_8 u u_x |u|^2 + d_9 u x u_x + d_{10} u_x^2 + d_{11} u_{xx} + u x u_x)
\]
\[
+ d_{12} u_x u_{xx} u_x + d_{13} u_{xx} u_{xx} + d_{14} |u|^4 + d_{15} u_{xxx} + d_{16} u_{xxx} + d_{17} u_{xxxx} + d_{18} u_{xxxx} + d_{19} u_{xxxx} + d_{20} u_{xxxx} + d_{21} u_{xxxx} + d_{22} u_{xxxx})
\]
\[
+ d_{23} u_{xxxx} + d_{24} u_{xxxx} + d_{25} u_{xxxx} + d_{26} u_{xxxx} + d_{27} u_{xxxx} + d_{28} u_{xxxx} + d_{29} u_{xxxx}, \quad \text{(A.7)}
\]

where

\[
d_1 = \frac{c^2}{9} (5a_1 + 2a_2 - 4a_4 - 13a_5 - 13a_6 + 11a_7 - 10a_8 - 34a_9 + 2a_{10}),
\]
\[
d_2 = -\frac{3}{2} c^2, \quad d_3 = 2c, \quad d_4 = -c^2, \quad d_5 = 4c,
\]
\[
d_6 = \frac{c}{72} (95a_1 + 20a_2 + 35a_3 + 26a_4 - 106a_5 - 295a_6 + 185a_7 - 222a_8 - 487a_9
\]
\[
+ 125a_{10}),
\]
\[
d_7 = -\frac{1}{2} (a_1 + a_2 + a_3 + a_4 + a_5 - 3a_6 + 5a_7 - 3(a_8 + a_9) + 5a_{10}), \quad \text{(A.8)}
\]

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\[
\begin{align*}
&d_9 = \frac{c}{12} (11a_1 + 4a_2 - 5a_3 - 22a_4 - 2a_5 - 19a_6 + 13a_7 - 55a_8 - 15a_9 - 3a_{10}), \\
&d_9 = d_{13}, \quad d_{10} = \frac{c}{72} (11a_1 + 8a_2 - 13a_3 - 22a_4 - 10a_5 - 19a_6 + 29a_7 - 55a_8 \\
&\quad - 31a_9 + 5a_{10}), \quad d_{11} = 2d_7, \quad d_{12} = d_{13}, \\
&d_{13} = \frac{1}{3} (a_1 + a_2 + a_3 - 5a_4 + 7a_5 + a_6 + a_7 - 11a_8 + 13a_9 + a_{10}), \\
&d_{14} = \frac{c}{18} (16a_1 - 2a_2 + a_3 - 5a_4 - 29a_5 - 4a_6 + 4a_7 - 35a_8 - 83a_9 - 11a_{10}), \\
&d_{15} = d_{16}, \quad d_{16} = -\frac{1}{3} (2a_1 - a_2 + 2a_3 - a_4 - a_5 - 4(a_6 + a_7 + a_8 + a_9 + a_{10})), \\
&d_{17} = \frac{1}{36} (5a_1 + 2a_2 + 5a_3 - 28a_4 + 32a_5 - 13a_6 + 11a_7 - a_8 - 25a_9 + 11a_{10}), \\
&d_{18} = \frac{c}{72} (13a_1 + a_2 + 13a_3 + a_4 + a_5 - 11a_6 + 61a_7 - 11(a_8 + a_9) + 61a_{10}), \\
&d_{19} = \frac{c}{36} (11a_1 + 2a_2 - 19a_3 - 22a_4 - 34a_5 - 19a_6 + 5a_7 - 37a_8 - 61a_9 - 25a_{10}), \\
&d_{20} = -\frac{1}{3} (a_1 + a_2 + a_6 + a_7 - 5(a_3 + a_4 + a_5 + a_8 + a_9 + a_{10})), \quad (A.9) \\
&d_{21} = \frac{1}{36} (2a_1 + 2a_2 - 7a_3 - 13a_4 - a_5 - 4a_6 + 8a_7 + 11a_8 + 35a_9 - 37a_{10}), \\
&d_{22} = \frac{1}{180} (14a_1 - a_2 - 46a_3 - a_4 + 59a_5 - 16a_6 - 16a_7 + 44a_8 + 164a_9 \\
&\quad + 104a_{10}), \\
&d_{23} = -\frac{1}{3} (2a_1 - a_2 + 2a_3 - a_4 - a_5 - 4(a_6 + a_7 + a_8 + a_9 + a_{10})), \quad d_{24} = \frac{1}{2} d_{20}, \\
&d_{25} = \frac{1}{4} (a_1 + a_3 - a_6 - a_7 - a_8 - a_9 - a_{10}), \\
&d_{26} = \frac{1}{180} (11a_1 - 4a_2 + 11a_3 - 4a_4 - 4a_5 - 19(a_6 + a_7 + a_8 + a_9 + a_{10})), \\
&d_{27} = \frac{1}{30} (a_1 + a_2 - 4a_3 + a_4 - 9a_5 + a_6 + a_7 + 6a_8 - 14a_9 - 4a_{10}), \\
&d_{28} = \frac{1}{180} (14a_1 - a_2 + 14a_3 - a_4 - a_5 - 16(a_6 + a_7 + a_8 + a_9 + a_{10})), \\
&d_{29} = \frac{c}{120}.
\end{align*}
\]

The unique solution \( f_6^{(3)} = \hat{M}_s f_{10}^{(3)} \in \mathcal{P}_9(1) \) of equation \( \hat{M}_0 f_6^{(3)} = \hat{M}_2 f_6^{(3)} \) reads

\[
\begin{align*}
&f_6^{(3)} = \sum_{\text{term}} (\beta_1 |u|^8 u + \beta_2 u_x u_x |u|^6 + \beta_3 u_{xx} |u|^4 u^2 + \beta_4 u_x^2 |u|^4 \bar{u} + \beta_5 |u|^2 |u|^4 u \\
&\quad + \beta_6 \bar{u}^4 u^4 |u|^2 + \beta_7 u_{xxx} u_x |u|^2 \bar{u} + \beta_8 u_{xx} |u|^2 u + \beta_9 u_x |u|^2 u + \beta_{10} u_{xx}^2 |u|^2 \bar{u} \\
&\quad + \beta_{11} u_{x} |u|^2 u + \beta_{12} u_{xxx} u_x |u|^2 u + \beta_{13} u_{xx} |u|^2 u + \beta_{14} u_{x}^2 |u|^2 u + \beta_{15} u_{x} |u|^2 \bar{u}^2 \\
&\quad + \beta_{16} u_{xx} |u|^2 u + \beta_{17} u_{xx} \bar{u}^2 u^2 + \beta_{18} u_{xx} |u|^2 u + \beta_{19} u_{x}^2 |u|^2 u + \beta_{20} u_{xx} u_x |u|^2 + \beta_{21} u_{xx} \bar{u} u + \beta_{22} u_{xx} |u|^2 u + \beta_{23} \bar{u}_{xxxxx} u_x^2 u^2
\end{align*}
\]
\begin{equation}
\beta_1 = \frac{c^3}{48}(6c_2 - 3c_3 + 3c_4 + 8c_5), \quad \beta_2 = \frac{c^2}{36}(10c_2 - 5c_3 + 10c_4 + 12c_5), \quad \\
\beta_3 = \frac{c^2}{36}(4c_2 - 2c_3 + 4c_4 + 9c_5), \quad \beta_4 = \frac{c^2}{18}(13c_2 - 5c_3 + 9c_4 + 10c_5), \quad \\
\beta_5 = \frac{c}{180}(18c_2 - 3c_3 + 14c_4 + 28c_5), \quad \beta_6 = \frac{c}{36}(5c_2 - c_3 + 3c_4 + 8c_5), \quad \\
\beta_7 = \frac{c}{180}(6c_2 - 3c_3 + 15c_4 + 7c_5), \quad \beta_8 = \frac{c}{360}(108c_2 - 29c_3 + 126c_4 + 56c_5), \quad \\
\beta_9 = \frac{c}{360}(36c_2 + 7c_3 + 66c_4 + 52c_5), \quad \beta_{10} = \frac{c}{720}(152c_2 - 41c_3 + 169c_4 + 84c_5), \quad \\
\beta_{11} = \frac{c}{360}(44c_2 - 7c_3 + 79c_4 + 118c_5), \quad \beta_{12} = \frac{c}{360}(16c_2 + 7c_3 + 26c_4 + 62c_5), \quad \\
\beta_{13} = \frac{c}{360}(8c_2 + 3c_3 + 6c_4 + 26c_5), \quad \beta_{14} = \frac{c}{72}(33c_2 - 7c_3 + 21c_4 + 14c_5), \quad \\
\beta_{15} = \frac{c}{360}(278c_2 + 11c_3 + 276c_4 + 236c_5), \quad \beta_{16} = \frac{c}{360}(39c_2 + 8c_3 + 33c_4 + 68c_5),
\end{equation}

\begin{equation}
\beta_{17} = \frac{c}{72}(22c_2 - c_3 + 13c_4 + 16c_5), \quad \beta_{18} = \frac{c}{720}(158c_2 + 31c_3 + 101c_4 + 176c_5), \quad \\
\beta_{19} = \frac{c}{120}(12c_2 - c_3 + 9c_4 + 44c_5), \quad \beta_{20} = \frac{c}{72}(16c_2 + 7c_3 + 12c_4 + 42c_5), \quad \\
\beta_{21} = \frac{c}{360}(114c_2 - 12c_3 + 103c_4 + 158c_5), \quad \beta_{22} = \frac{c_4}{120}, \quad \beta_{23} = \frac{c_5}{360}, \quad \\
\beta_{24} = \frac{1}{240}(2c_2 + 3c_4), \quad \beta_{25} = \frac{1}{360}(c_3 + 4c_5), \quad \beta_{26} = \frac{1}{720}(18c_2 + 21c_3 + 25c_4), \quad \\
\beta_{27} = \frac{1}{360}(15c_2 + 13c_4), \quad \beta_{28} = \frac{1}{720}(9c_3 + 11c_4 + 12c_5), \quad \\
\beta_{29} = \frac{1}{360}(c_2 + 2c_3 + 10c_5), \quad \beta_{30} = \frac{1}{360}(2c_3 + c_4 + 11c_5), \quad \\
\beta_{31} = \frac{1}{144}(4c_2 + 3c_4), \quad \beta_{32} = \frac{1}{720}(50c_2 + 35c_3 + 34c_4), \quad \\
\beta_{33} = \frac{1}{360}(11c_2 + 17c_3 + 10c_4 + 15c_5), \quad \beta_{34} = \frac{1}{720}(11c_3 + 4c_4 + 18c_5), \quad \\
\beta_{35} = \frac{1}{720}(20c_2 + 25c_3 + 11c_4 + 20c_5), \quad \\
\beta_{36} = \frac{1}{240}(4c_2 + 5c_4), \quad \beta_{37} = \frac{1}{720}(8c_2 + 31c_3 + 4c_4 + 50c_5), \quad \\
\beta_{38} = \frac{c}{360}(2c_2 - c_3 + 5c_4 + 14c_5).
\end{equation}
and no additional constraint on the coefficients $\alpha_i$'s appears. The unique solution $f_4^{(5)} = \hat{M}_4 f_4^{(5)} \in TV(3)$ of equation $\hat{M}_4 f_4^{(5)} = \hat{M}_2 f_4^{(5)}$ reads

$$f_4^{(5)} = i(\delta \nu |u|^4 + \delta_1 \nu_{xx} |u|^4 + \delta_2 u_{xx} u |u|^4 + \delta_3 u_x^2 u_{xx}^2 |u|^4 + \delta_4 u_x^2 u_{xx}^2 |u|^4 + \delta_5 |u|^2 u_{xx}^2 |u|^4 + \delta_6 u_{xx} u_{xx} |u|^4 + \delta_7 u_{xx} u_{xx} u_x |u|^4 + \delta_8 u_{xx}^2 u_{xx} |u|^4 + \delta_9 u_{xx}^2 u_{xx} |u|^4 + \delta_{10} u_{xx}^2 u_{xx} |u|^4 + \delta_{11} u_{xx} |u|^4 + \delta_{12} \nu_{xx} u_{xx} |u|^4 + \delta_{13} u_{xx} u_{xx} |u|^4 + \delta_{14} u_{xx}^2 u_{xx}^2 |u|^4 + \delta_{15} u_{xx} |u|^4 + \delta_{16} u_{xx} u_{xx} |u|^4 + \delta_{17} u_{xx} u_{xx} |u|^4 + \delta_{18} |u|^4 u_{xx} + \delta_{19} u_{xx} u_{xx} |u|^4 + \delta_{20} u_{xx} u_{xx} |u|^4 + \delta_{21} u_{xx} u_{xx} |u|^4 + \delta_{22} u_{xx} u_{xx} |u|^4 + \delta_{23} u_{xx} u_{xx} |u|^4 + \delta_{24} u_{xx} u_{xx} |u|^4 + \delta_{25} u_{xx} u_{xx} |u|^4 + \delta_{26} u_{xx} u_{xx} |u|^4 + \delta_{27} u_{xx} u_{xx} |u|^4 + \delta_{28} u_{xx} u_{xx} u_{xx} |u|^4 + \delta_{29} u_{xx} u_{xx} u_{xx} |u|^4 + \delta_{30} u_{xx} u_{xx} u_{xx} |u|^4 + \delta_{31} u_{xx} u_{xx} |u|^4 + \delta_{32} u_{xx} u_{xx} u_{xx} |u|^4 + \delta_{33} u_{xx} u_{xx} u_{xx} |u|^4 + \delta_{34} |u|^2 u_{xx} + \delta_{35} u_{xx} u_{xx} u_{xx} |u|^4 + \delta_{36} u_{xx} u_{xx} u_{xx} |u|^4 + \delta_{37} u_{xx} u_{xx} u_{xx} |u|^4 + \delta_{38} |u|^2 u_{xx} + \gamma_1 u_{xx}^2 |u|^4 + \gamma_2 u_{xx}^2 u_{xx} |u|^4 + \gamma_3 u_{xx}^2 u_{xx} |u|^4 + \gamma_4 u_{xx}^2 u_{xx} |u|^4 + \gamma_5 u_{xx}^2 u_{xx} |u|^4 + \gamma_6 u_{xx}^2 u_{xx} |u|^4 + \gamma_7 u_{xx}^2 u_{xx} |u|^4 + \gamma_8 u_{xx}^2 u_{xx} |u|^4 + \gamma_9 u_{xx}^2 u_{xx} |u|^4 + \gamma_{10} u_{xx}^2 u_{xx} |u|^4 + \gamma_{11} u_{xx}^2 u_{xx} |u|^4 + \gamma_{12} u_{xx}^2 u_{xx} |u|^4 + \gamma_{13} u_{xx}^2 u_{xx} |u|^4 + \gamma_{14} u_{xx}^2 u_{xx} |u|^4 + \gamma_{15} u_{xx}^2 u_{xx} |u|^4 + \gamma_{16} u_{xx}^2 u_{xx} |u|^4 + \gamma_{17} u_{xx}^2 u_{xx} |u|^4 + \gamma_{18} u_{xx}^2 u_{xx} |u|^4 + \gamma_{19} u_{xx}^2 u_{xx} |u|^4 + \gamma_{20} u_{xx}^2 u_{xx} |u|^4 + \gamma_{21} u_{xx}^2 u_{xx} |u|^4 + \gamma_{22} u_{xx}^2 u_{xx} |u|^4 + \gamma_{23} u_{xx}^2 u_{xx} |u|^4 + \gamma_{24} u_{xx}^2 u_{xx} |u|^4 + \gamma_{25} u_{xx}^2 u_{xx} |u|^4 + \gamma_{26} u_{xx}^2 u_{xx} |u|^4 + \gamma_{27} u_{xx}^2 u_{xx} |u|^4 + \gamma_{28} u_{xx}^2 u_{xx} |u|^4 + \gamma_{29} u_{xx}^2 u_{xx} |u|^4 + \gamma_{30} u_{xx}^2 u_{xx} |u|^4 + \gamma_{31} u_{xx}^2 u_{xx} |u|^4 + \gamma_{32} u_{xx}^2 u_{xx} |u|^4 + \gamma_{33} u_{xx}^2 u_{xx} |u|^4 + \gamma_{34} u_{xx}^2 u_{xx} |u|^4 + \gamma_{35} u_{xx}^2 u_{xx} |u|^4 + \gamma_{36} u_{xx}^2 u_{xx} |u|^4 + \gamma_{37} u_{xx}^2 u_{xx} |u|^4 + \gamma_{38} u_{xx}^2 u_{xx} |u|^4 + \gamma_{39} u_{xx}^2 u_{xx} |u|^4 + \gamma_{40} u_{xx}^2 u_{xx} |u|^4 + \gamma_{41} u_{xx}^2 u_{xx} |u|^4 + \sigma_1 u_{xx}^2 u_{xx} |u|^4 + \sigma_2 u_{xx}^2 u_{xx} |u|^4 + \sigma_3 u_{xx}^2 u_{xx} |u|^4 + \sigma_4 u_{xx}^2 u_{xx} |u|^4 + \sigma_5 u_{xx}^2 u_{xx} |u|^4 + \sigma_6 u_{xx}^2 u_{xx} |u|^4 + \sigma_7 u_{xx}^2 u_{xx} |u|^4 + \sigma_8 u_{xx}^2 u_{xx} |u|^4 + \sigma_9 u_{xx}^2 u_{xx} |u|^4 + \sigma_{10} u_{xx}^2 u_{xx} |u|^4 + \sigma_{11} u_{xx}^2 u_{xx} |u|^4 + \sigma_{12} u_{xx}^2 u_{xx} |u|^4 + \sigma_{13} u_{xx}^2 u_{xx} |u|^4),$$

where

$$\delta_1 = \frac{c}{576} (336d_1 + 4(6c_2 - 3c_3 + 2c_4 + 10c_5)d_2 + 6c_5(2c_2 - c_3 + 2c_4 + 4c_5)d_3 + 4(6c_2 - 3c_3 + 2c_4 + 2c_5)d_4 + \left( - 4c_4^2 + 2c_4c_5 - c_3c_4 - c_4c_5 - 2c_5^2 \right)d_5 + e(2(14c_2 - 5c_1 + 9c_6 + 28c_5)d_1 - 2(10c_2 - 3c_3 + 7c_4 + 8c_5)d_2 - 4(6c_2 - 3c_3 + 16c_5)d_3 + 16d_4 + 2(18c_2 - 7c_3 + 11c_4 + 30c_5)d_1 - 2(6c_2 - 3c_3 + 14c_5)d_2 - 48d_3 - 5c_3 - 3c_4 + 8c_5)d_20 + 4(-4c_2 - c_3 + 2c_5 - 4c_4 + 5c_6 + 3c_7 - 4c_8 + 6c_9 + 8c_10 + 10d_4)d_2 + 4(6c_2 - 3c_3 + 4c_5)d_3 + 2c_4 + 15c_5)d_21 + 4(-6c_2 + 5c_3 + c_4 - 6c_5)d_2 + 32d_6 - 18d_6 + 4(2c_2 - c_3 + c_4 + 4c_5)d_3 + 8c_7(-2d_17 + 4d_18 + 2d_22 + 7d_25 - 7d_26 - 5d_28 + 16d_29)),$$

$$\delta_2 = \frac{1}{144} (96d_1 + 10c_1d_2 + 4c_4c_5d_3 - 6c_5d_4 - 2c_2^2d_5 + c(2(4c_2 - c_3 + 6c_4 + 8c_5)d_1 - 2(2c_2 + 5c_4)d_2 - 2(6c_2 - 3c_3 + c_4 + 14c_5)d_3 + 10d_4d_4 + (18c_2 - 7c_3 + 11c_4 + 24c_5)d_15 + 3(-2c_2 + c_3 - c_4 - 4c_5)d_16 - 24d_19 + (34c_2 - 19c_3}$
\[\delta_3 = \frac{1}{48} (12d_1 + 2c_5 d_2 + 2c_3 d_4 + c(2c_5 d_{11} - 2c_5 d_{12} + 8d_{14} + 24d_{19} + (2c_2 - c_3 + c_4 + 4c_3) d_{20} + 2(2c_2 - c_3 + c_4 + 5c_3) d_{23} + 5(2c_2 - c_3 + c_4) d_{24})
\]
\[\quad + 8c^2(d_{25} - d_{26} + d_{28} - 2d_{29})).\]

\[\delta_4 = \frac{1}{144} (216d_1 + 2(3c_2 + 6c_4 - 2c_3) d_2 + 2(2c_3 c_4 + 2c_2 c_5 + c_3 c_5 + c_4 c_5)
\]
\[\quad - 2c_2^2) d_3 - 24c_5 d_4 - (6c_2^2 - 3c_3 c_5 + c_4 c_5 + 2c_3^2) d_5 + c(-24d_{10} + (52c_2 - 19c_3
\]
\[\quad + 9c_4 + 54c_5) d_{11} + 2(-7c_2 + 3c_3 - 8c_4 + 9c_5) d_{12} + 2(-10c_2 + 4c_3 - 5c_4
\]
\[\quad - 22c_5) d_{13} + 28d_{14} + 2(33c_2 - 4c_3 + 5c_4 + 16c_5) d_{15} + 2(-22c_2 + 12c_3 - 18c_4
\]
\[\quad - 19c_5) d_{16} - 102d_{19} + (4c_2 - 10c_3 - 6c_4 + 7c_5) d_{20} + (154c_2 - 58c_3 + 121c_4
\]
\[\quad + 199c_5) d_{23} + 3(-22c_2 + 18c_3 - 3c_4 - 22c_5) d_{24} + 132d_6 + (10c_2 - 5c_3 + 5c_4
\]
\[\quad + 4c_5) d_7 - 16d_8 + 2(6c_2 - 3c_3 + 3c_4 + 13c_5) d_9 + 2c^2(-42d_{17} + 118d_{18} - 8d_{21}
\]
\[\quad + 17d_{22} + 54d_{25} - 37d_{26} + 7d_{27} + 193d_{28} + 160d_{29})).\]

\[\delta_5 = \frac{1}{48} (48d_1 + 2(3c_2 - 2c_3 + 4c_3) d_2 + 2c_4 c_5 d_4 + 2(-c_3 - 6c_4 + 4c_3) d_4
\]
\[\quad + 2(c_2^2 - c_3^2) d_5 + c(-8d_{10} + 4(c_2 + 4c_3) d_{11} + (10c_2 - 7c_3 + 11c_4 + 8c_5) d_{12}
\]
\[\quad + (6c_2 - 3c_3 + 3c_4 + 4c_5) d_{13} + 24d_{14} + 4(3c_2 - c_3 + c_4 + c_5) d_{15} + 4(4c_2
\]
\[\quad - 2c_3 + 3c_4 + 3c_5) d_{16} + 16d_{19} + 4c_5 d_{20} + 2(28c_2 - 15c_3 + 6c_4 + 42c_5) d_{23}
\]
\[\quad + 4(-4c_2 + c_3 + 5c_4 - 17c_5) d_{24} + 8d_8 + 32d_9 + (-2c_2 + c_3 - c_4 - 4c_5) d_9
\]
\[\quad + 4c^2(2d_{17} + 4d_{18} - 6d_{22} + 3d_{25} + 2d_{26} - 4d_{27} - 13d_{28} - 4d_{29})).\]

\[\delta_6 = \frac{1}{48} (12d_1 + 2c_2^2 d_1 + 2(c_2 - c_4) d_4 - c_4 c_5 d_5 + c(20d_{10} + 2(2c_2
\]
\[\quad - c_3 + 4c_5) d_{12} - 4d_{14} + 2(c_2 + 2c_5) d_{15} - 2(c_4 + c_5) d_{16} + 8d_{19} + (8c_2
\]
\[\quad - 2c_3 + 7c_4 + 8c_5) d_{23} + 2(10c_2 - 5c_3 + 11c_4 + 5c_5) d_{24} + 4c_6 + 4d_8
\]
\[\quad + 2c^2(-2d_{17} + 12d_{18} - 2d_{22} + d_{25} + 3d_{26} - 4d_{27} - 13d_{28} + 12d_{29})).\]

\[\delta_7 = \frac{1}{24} (4d_{14} + 2c_3 d_{23} - c_5 d_{24} + 2c(d_{25} - d_{26} + 9d_{28} - 2d_{29})).\]

\[\delta_8 = \frac{1}{24} (c_4 d_{11} - c_5 d_{13} + 8d_{14} + 2c_2 d_{15} + (6c_2 + 7c_4) d_{23} - 2c_5 d_{24}
\]
\[\quad + 8d_6 + 2c(-2d_{17} + 8d_{18} + 12d_{25} - 2d_{26} - d_{27} + 27d_{28} - 4d_{29})).\]

\[\delta_9 = \frac{1}{24} (c_4 d_{12} + 4d_{14} + 2c_5 d_{15} - 2c_3 d_{16} + (3c_3 + 4c_4) d_{23} - 3c_5 d_{24} + 4d_6
\]
\[\quad + 2c(4d_{18} - 2d_{22} + d_{25} + 9d_{26} - d_{27} + 18d_{28})).\]
\[
\begin{align*}
\delta_{12} &= \frac{1}{24} (c_5 d_{11} + c_4 d_{13} + 6d_{19} + c_7 d_{23} + 10c_8 d_{23} + c_5 d_{24} - 4c_4 d_{24} + 2d_8 \\
&\quad + 2c(d_{25} + d_{26} + 7d_{27} + 8d_{28} + 8d_{29})), \\
\delta_{13} &= \frac{1}{24} (4d_{10} + c_5 d_{12} + 2d_{19} + 2c_2 d_{24} + 2c_4 d_{24} + 2c(d_{25} + d_{26} + d_{27} + 8d_{29})), \\
\delta_{14} &= \frac{1}{48} (4(c_2 + c_4) d_{11} + 8d_{14} + 2(c_2 - c_4) d_{16} - 2c_5 d_{16} + (12c_2 + 7c_4) d_{23} \\
&\quad + 2c_5 d_{24} + 24d_6 - 6c_3 d_7 + 2c(-4d_{17} + 20d_{18} - 6d_{21} - 2d_{22} + 37d_{25} - 2d_{26} \\
&\quad + 37d_{28} + 4d_{29})), \\
\delta_{15} &= \frac{1}{24} ((5c_3 + c_4) d_{11} + (2c_2 + c_4) d_{12} + (-3c_3 + 2c_5) d_{13} + 8d_{14} + (c_3 \\
&\quad + 4c_5) d_{15} + (c_3 - 4c_4) d_{16} + (8c_2 + 11c_3 + 6c_4 - 4c_5) d_{23} + 4(-c_3 + c_4 \\
&\quad - 2c_5) d_{24} + 8d_6 - 4c_3 d_7 + 20d_8 + c_3 d_9 + 4c(4d_{17} + 8d_{18} + 11d_{25} + 5d_{26} - 7d_{27} \\
&\quad + 34d_{28} - 8d_{29})), \\
\delta_{16} &= \frac{1}{48} (12d_{10} - 2c_5 d_{11} + 2(c_3 + c_4 + 3c_5) d_{12} - 2(c_2 + c_4) d_{13} \\
&\quad + 4d_{14} + 2c_2 d_{16} + 6d_{19} - 3c_5 d_{20} - (2c_3 + 7c_5) d_{23} + (4c_2 - 2c_3 + c_4 \\
&\quad + 4c_5) d_{24} + 8d_6 + 2c_5 d_9 + 2c(2d_{17} + 6d_{18} + 2d_{21} + d_{22} - 3d_{25} + 26d_{26} \\
&\quad - 11d_{27} - 14d_{28} + 4d_{29})), \\
\delta_{17} &= \frac{1}{24} ((c_2 + 3c_3) d_{11} - 2c_3 d_{13} + 2(c_2 + c_3 - c_5) d_{23} + 2c_4 d_{24} + 4d_6 \\
&\quad + (c_3 - 2c_4) d_7 + 6d_8 + c_3 d_9 + 4c(d_{17} + 2d_{21} + 7d_{25} - 4d_{27} + 6d_{28})), \\
\delta_{18} &= \frac{1}{24} (6d_{10} - 2c_5 d_{11} + (c_2 + 2c_3 + 6c_5) d_{12} - (3c_2 - c_3 + 2c_4) d_{13} \\
&\quad + 6d_{14} - 3c_5 d_{20} - (2c_3 + 7c_5) d_{23} + (2c_2 - 6c_3 + c_4) d_{24} + 2d_6 + c_2 d_7 \\
&\quad + 8d_6 + c_3 d_9 + 2c_5 d_9 + 2c(4d_{17} + 6d_{18} + 2d_{21} - 3d_{22} - 3d_{25} + 20d_{26} - 5d_{27} \\
&\quad - 14d_{28} - 20d_{29})), \\
\delta_{19} &= \frac{1}{24} (2d_{10} + 2d_{19} + c_5 d_{20} + (2c_2 + c_4) d_{24} + 2c(d_{18} + d_{22} + 6d_{29})), \\
\delta_{20} &= \frac{1}{24} (24d_{10} + (c_3 + 2c_5) d_{12} + 2c_2 d_{13} + c_4 d_{13} + 12d_{19} + (c_3 \\
&\quad + 3c_5) d_{20} - 3c_5 d_{23} - (14c_2 - 3c_3 + 7c_4) d_{24} + 4d_8 + c_5 d_9 + 2c(2d_{17} \\
&\quad - 6d_{18} + 2d_{21} + 7d_{22} - 3d_{25} + 6d_{26} + 9d_{27} - 36d_{28})), \\
\delta_{21} &= \frac{1}{24} ((c_3 + 6c_5) d_{11} + (c_3 - 3c_4) d_{13} + 18d_{19} + c_2 d_{20} + (c_2 + 6c_3 + 10c_5) d_{23} \\
&\quad - 4(c_3 + c_4) d_{24} + 4d_6 + c_4 d_7 + 4d_8 + 4c(4d_{21} + d_{22} + 5d_{25} - 2d_{26} + 3d_{27} + 20d_{28} \\
&\quad - 16d_{29})), \\
\delta_{22} &= \frac{d_{28}}{6}, \quad \delta_{23} = \frac{d_{29}}{12}, \quad \delta_{24} = \frac{1}{12} (2d_{26} + d_{28}), \quad \delta_{25} = \frac{1}{12} (d_{27} + 2d_{29}), \\
\delta_{26} &= \frac{1}{12} (2d_{17} + d_{25} + 3d_{26} + d_{28}), \quad \delta_{27} = \frac{1}{12} (4d_{18} + 3d_{25} + 2d_{28}).
\end{align*}
\]

(A.15)
\[\gamma_{38} = \frac{1}{12}(2d_{22} + d_{26} + d_{28}), \quad \gamma_{39} = \frac{1}{12}(d_{21} + d_{27} + 3d_{29}),\]
\[\gamma_{30} = \frac{1}{12}(d_{22} + d_{27} + 4d_{29}), \quad \gamma_{31} = \frac{1}{12}(3d_{18} + 2d_{25}),\]
\[\gamma_{32} = \frac{1}{12}(5d_{17} + 2d_{18} + d_{25} + 2d_{26}), \quad \gamma_{33} = \frac{1}{12}(d_{17} + 4d_{21} + 3d_{22} + d_{25} + d_{26}),\]
\[\gamma_{34} = \frac{1}{12}(d_{22} + d_{26} + 2d_{27}), \quad \gamma_{35} = \frac{1}{12}(d_{17} + d_{18} + 3d_{21} + 2d_{22}),\]
\[\gamma_{36} = \frac{1}{12}(2d_{25} + 3d_{28}), \quad \gamma_{37} = \frac{1}{12}(d_{17} + 2d_{21} + d_{22} + 5d_{27}),\]
\[\gamma_{38} = \frac{1}{24}(2d_{19} + c_5d_{23} + c_4d_{24} + 4c(d_{28} + 3d_{29})),\]
\[\gamma_1 = \frac{d_{23}}{6}, \quad \gamma_2 = \frac{1}{12}(2d_{11} + 3d_{23}), \quad \gamma_3 = \frac{1}{12}(2d_{12} + d_{23}),\]
\[\gamma_4 = \frac{1}{12}(3d_{11} + 2d_{15} + 2d_{23}), \quad \gamma_5 = \frac{1}{12}(d_{11} + 3d_{12} + d_{23} + 2d_9),\]
\[\gamma_6 = \frac{1}{12}(d_{12} + 2d_{20} + d_{23}), \quad \gamma_7 = \frac{1}{12}(2d_{11} + 3d_{15}),\]
\[\gamma_8 = \frac{1}{12}(d_{11} + 2d_{12} + d_{15} + 3d_9), \quad \gamma_9 = \frac{1}{12}(d_{11} + d_{12} + 3d_{20} + d_9),\]
\[\gamma_{10} = \frac{1}{12}(d_{12} + d_{20}), \quad \gamma_{11} = \frac{d_{15}}{6}, \quad \gamma_{12} = \frac{1}{12}(d_{15} + 2d_9),\]
\[\gamma_{13} = \frac{1}{12}(d_{15} + 2d_{20} + d_9), \quad \gamma_{14} = \frac{1}{12}(d_{20} + d_9), \quad \gamma_{15} = \frac{d_{20}}{12},\]
\[\gamma_{16} = \frac{d_{24}}{12}, \quad \gamma_{17} = \frac{1}{12}(d_{13} + 2d_{24}), \quad \gamma_{18} = \frac{1}{12}(d_{13} + d_{16} + 4d_{24}),\]
\[\gamma_{19} = \frac{1}{12}(d_{13} + 3d_{24} + d_7), \quad \gamma_{20} = \frac{1}{12}(2d_{13} + d_{16}), \quad \gamma_{21} = \frac{1}{12}(5d_{13} + d_{16} + 2d_7),\]
\[\gamma_{22} = \frac{d_{16}}{6}, \quad \gamma_{23} = \frac{1}{12}(3d_{16} + 4d_7), \quad \gamma_{24} = \frac{1}{12}(2d_{16} + 3d_7),\]
\[\gamma_{25} = \frac{1}{12}(2d_{2} + c(d_{11} - d_{12} + 9d_{23} - 2d_{24})),\]
\[\gamma_{26} = \frac{1}{12}(6d_{2} + 2c_3d_5 - c_5d_5 + c(11d_{11} - d_{12} - d_{13} + 2d_{15} - 2d_{20} + 10d_{23} - 8d_{24})),\]
\[\gamma_{27} = \frac{1}{12}(2d_{2} + 2c_5d_5 - c_4d_5 + c(d_{11} + 7d_{12} + d_{13} + 2d_{15} - 2d_{16} + 8d_{23})),\]
\[\gamma_{28} = \frac{1}{24}(8d_{2} + 6c_3d_5 - c_5d_5 + 2c(2d_{11} - 2d_{13} + 11d_{15} - d_{16} - d_{20} + 2d_{23})),\]
\[\gamma_{29} = \frac{1}{24}(6d_{2} + 2(c_2 + c_3 - 2c_5)d_3 - (c_3 - c_4)d_3 + 2c(4d_{11} - 2d_{13} + 4d_{15} - 2d_{20} + 2d_{23} + d_7 + d_9)),\]
\[\gamma_{30} = \frac{1}{24}(8d_{2} + 2(c_3 + 4c_5)d_3 + (c_3 - 4c_4)d_3 + 4c(d_{11} + d_{12} + d_{13} + 4d_{15} - 2d_{16} + 2d_{20} + 6d_{23} - 4d_{24} + 4d_9)),\]
\[\gamma_{31} = \frac{1}{24}(2d_{2} + c_2d_5 + 2c(2d_{12} + 2d_{20} + 2d_{24} + d_7 + d_9)).\]
\[ \gamma_{32} = \frac{1}{24} \left( 4d_2 + 2c_5d_5 + c_4d_5 + 2c(d_15 + d_16 + 7d_{20} + 2d_{23} + 2d_{24}) \right), \]
\[ \gamma_{33} = \frac{1}{4} \left( d_4 + 2c(d_{23} + 3d_{24}) \right), \]
\[ \gamma_{34} = \frac{1}{12} \left( 3d_4 + c(d_{11} + d_{12} + 7d_{13} + 4d_{23} + 4d_{24}) \right), \]
\[ \gamma_{35} = \frac{1}{12} \left( d_4 + c(d_{11} + d_{12} + d_{13} + 4d_{24}) \right), \]
\[ \gamma_{36} = \frac{1}{24} \left( 12d_4 + c_4d_5 + 2c(2d_{11} - 2d_{13} + 3d_{15} + 7d_{16} - d_{20} + 6d_{23} - 2d_{24}) \right), \]
\[ \gamma_{37} = \frac{1}{24} \left( 18d_4 + c_4d_5 + 4c(3d_{11} - 2d_{12} + d_{13} + d_{16} + 7d_{23} - 8d_{24} + 4d_4) \right), \]
\[ \gamma_{38} = \frac{1}{24} \left( 18d_4 + c_4d_5 + 4c(d_7 + d_9 - 3d_{11} + 4d_{12} + d_{13} + 2d_{16} - 3d_{23} - 10d_{24}) \right), \]
\[ \gamma_{39} = \frac{1}{24} \left( 2d_4 + c_5d_5 + 2c(d_{15} + d_{16} + d_{20} + 2d_{24}) \right), \]
\[ \gamma_{40} = \frac{c}{144} \left( 80d_2 + 2(6c_2 - 3c_3 + c_4 + 12c_5)d_5 + 16d_4 + (10c_2 - 5c_3 + 5c_4 + 8c_5)d_3 + 8c(d_{11} - d_{12} + d_{13} + d_{16} - 2d_{20} + 2d_{23} - 2d_{24}) \right), \]
\[ \gamma_{41} = \frac{c}{48} \left( 8d_2 + 24d_4 + (2c_2 - c_3 + c_4 + 4c_5)d_5 + 8c(d_{11} - d_{12} + d_{23} - 2d_{24}) \right), \]
\[ \sigma_1 = \frac{d_1}{3}, \sigma_2 = \frac{d_3}{4}, \sigma_3 = \frac{d_5}{6}, \sigma_4 = \frac{d_7}{12}, \sigma_5 = \frac{d_9}{15}, \sigma_6 = \frac{d_10}{6}, \sigma_7 = \frac{d_4}{4}, \]
\[ \sigma_8 = \sigma_9 = \frac{d_5}{12}, \sigma_{11} = \frac{c}{12}(7d_3 + d_5), \sigma_{12} = \frac{c}{6}(d_3 + 4d_5), \sigma_{13} = \frac{c}{12}(d_3 + d_5). \] (A.16)

iff the following three sets of constraints are satisfied:
\[ d_{15} - d_{16} = d_3 - 2d_5 = c_4d_5 - 2c_5d_{23} = -d_{12} + d_{13} - d_{15} + d_{20} + d_{23} - 2d_{24} \]
\[ = -d_9 + d_{11} + d_{13} - 2d_7 = (2c_2 - c_3 + c_4)d_3 - 2(d_{11} - d_{13} + d_{16} - d_{20} + 2d_{24})c \] (A.17)
\[ = (c_2 - c_3 + c_4 + c_5)d_3 + 2(d_{13} - d_{16} - d_{24} - d_7)c = 0, \]
\[ (2c_2 - c_3 + c_4 + 2c_5)d_3 - 2d_4 + 4d_{24}c = -4d_2 + (6c_2 - 3c_3 + 3c_4 + 8c_5)d_3 + 8d_{24}c = 0. \] (A.18)
−4d_6 + (2c_2 + 2c_3 − c_4 − 4c_5)d_7 − 12d_{10} + (c_3 + 2c_5)d_{11} − (c_2 − 4c_4 + 9c_5)d_{13}
− 4d_{14} + (3c_2 − 5c_5)d_{16} + 12d_{19} − (2c_2 + c_3 + 2c_4 − 4c_5)d_{20} + (c_2 + 4c_3 + c_4)
− c_5)d_{23} + (6c_2 − 2c_3 + 3c_4 − 12c_5)d_{24} + (−8d_{17} + 20d_{18} + 4d_{21} − 14d_{22} + 10d_{25}
+ 4d_{16} + 10d_{27} + 44d_{28} − 16d_{29}) = 0,
−24d_1 + 6(c_2^2 − 2c_4c_5)d_3 + 2(2c_2 + 2c_3 − c_4 + 4c_5)d_7 − 4d_6
− 52d_{10} + 2(−4c_2 + c_3 − c_4 + 2c_5)d_{11} + 2(c_2 + c_3 − 6c_4 + 6c_5)d_{13} − 16d_{14} + 2(5c_2
− c_3 + c_4 − c_5)d_{16} + 62d_{19} + (−4c_2 + 2c_3 − 12c_4 + 5c_5)d_{20} + (2c_2 + 12c_3 + 4c_4
+ 13c_5)d_{23} + (6c_2 − 12c_3 + 5c_4)d_{24}c + 2c^2(−16d_{17} + 50d_{18} + 4d_{21} − 13d_{22} + 2d_{25}
+ 20d_{26} + 3d_{27} + 52d_{28} + 4d_{29}) = 0. \) (A.19)

The first set of constraints (A.17) is automatically satisfied by the parametrizations (33) and (A.8), while the second set of two constraints (A.18) is satisfied by the parametrizations (33) and (A.8) and by the O(ε^2) constraint (45). The remaining five constraints (A.19) are equivalent to the five quadratic constraints (53), (54)–(58) in the S-integrability scenario in which (48) holds, and to the linear constraints (11) in the C-integrability scenario in which c = 0 (the last constraint is automatically satisfied by the condition c = 0 and, in the remaining constraints, the quadrics degenerate into the hyperplanes described by equations (11)).

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