On the Davis–Wielandt radius inequalities of Hilbert space operators

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ABSTRACT
In this work, some new upper and lower bounds for the Davis–Wielandt radius are introduced. Generalizations of some presented results are obtained. Some bounds for the Davis–Wielandt radius for \( n \times n \) operator matrices are established. An extension of the Davis–Wielandt radius to the Euclidean operator radius is introduced.

1. Introduction
Let \( \mathcal{B}(\mathcal{H}) \) be the Banach algebra of all bounded linear operators defined on a complex Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) with the identity operator \( 1_{\mathcal{H}} \) in \( \mathcal{B}(\mathcal{H}) \). When \( \mathcal{H} = \mathbb{C}^n \), we identify \( \mathcal{B}(\mathcal{H}) \) with the algebra \( \mathbb{M}_{n \times n} \) of \( n \)-by-\( n \) complex matrices. Then, \( \mathbb{M}^+_{n \times n} \) is just the cone of \( n \)-by-\( n \) positive semidefinite matrices.

For a bounded linear operator \( S \) on a Hilbert space \( \mathcal{H} \), the numerical range \( W(S) \) is the image of the unit sphere of \( \mathcal{H} \) under the quadratic form \( x \rightarrow \langle Sx, x \rangle \) associated with the operator. More precisely,

\[
W(S) = \{ \langle Sx, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}.
\]

Also, the numerical radius is defined to be

\[
w(S) = \sup \{ |\lambda| : \lambda \in W(S) \} = \sup_{\|x\|=1} |\langle Sx, x \rangle|.
\]

We recall that the usual operator norm of an operator \( S \) is defined to be

\[
\|S\| = \sup \{ \|Sx\| : x \in \mathcal{H}, \|x\| = 1 \}.
\]
The spectrum of an operator $S$ is the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - S$ does not have a bounded linear operator inverse, and is denoted by $\text{sp}(S)$. The spectral radius of an operator $S$ is defined to be

$$r(S) = \sup \{|\lambda| : \lambda \in \text{sp}(S)\}.$$ 

It is well known that the inequality

$$r(S) \leq w(S) \leq \|S\|$$

holds for all Hilbert space operators $S \in \mathcal{B}(\mathcal{H})$. The previous inequality becomes equality if $S$ is a selfadjoint operator.

One of the most interesting generalizations of a numerical range is the Davis–Wielandt shell; which is defined as

$$DW(S) = \{ (\langle Sx, x \rangle, \langle Sx, Sx \rangle) \in \mathcal{H}, \|x\| = 1 \}$$

for any $S \in \mathcal{B}(\mathcal{H})$. Clearly, the projection of the set $DW(S)$ on the first coordinate is $W(S)$.

The Davis–Wielandt shell and its radius were introduced and described firstly by Davis in [1] and [2] and Wielandt [3]. In fact, the Davis–Wielandt shell $DW(S)$ gives more information about the operator $S$ and $W(S)$. For instance, in the finite-dimensional case, Li and Poon proved [4] (see also [5]) that the normal property of Hilbert space operators can be characterized by the eigenvalues of their Davis–Wielandt shells, namely, $S \in \mathcal{M}_{n \times n}(\mathbb{C})$ is normal if and only if $DW(S)$ is a polyhedron in $\mathbb{C} \times \mathbb{R}$ identified with $\mathbb{R}^3$. Moreover, in the finite-dimensional case, the spectrum of an operator $S$; $\text{sp}(S)$ is finite and $DW(S)$ is always closed, cf [4, Theorem 2.3]. These conditions are no longer equivalent for an infinite-dimensional operator $S$, cf [4, Example 2.5].

In Lins et al. [6], proved that, if $S \in \mathcal{M}_{n \times n}(\mathbb{C})$ is normal, then $DW(S)$ is the convex hull of the points $(\text{Re}(\lambda_j), \text{Im}(\lambda_j), |\lambda_j|^2) (j = 1, \ldots, n)$, for $\lambda_j \in \text{sp}(S)$. Moreover, each point $(\text{Re}(\lambda_j), \text{Im}(\lambda_j), |\lambda_j|^2)$ is an extreme point of $DW(S)$. In particular case, if $n = 2$ i.e. $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has eigenvalues $\lambda_1, \lambda_2$, then $DW(S)$ degenerates to the line segment joining the points $(\lambda_1, |\lambda_1|^2)$ and $(\lambda_2, |\lambda_2|^2)$. So that $\text{dim } DW(S) \leq 1$. In fact, the condition $\text{dim}(DW(S)) \leq 1$ holds if and only if $S$ is normal, with at most two distinct eigenvalues. Otherwise, $DW(S)$ is an ellipsoid (without its interior) centred at $\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{1}{2} \text{tr}(|S|^2)\right)$. Also, it was proved that if $\text{dim}(DW(S)) \geq 2$, then $DW(S)$ is always convex. A complete description of $DW(S)$ for a quadratic operator $S$ was given in Ref. [5]. For more details see also Refs [7,8] and [6].

In Ref. [3], Wielandt showed that the Davis–Wielandt shell is a useful tool for characterizing the eigenvalues of matrices in the set

$$\{ U^*TU + V^*SV : U, V \in \mathcal{M}_{n \times n} \text{ are unitary} \}$$

for given $T, S \in \mathcal{M}_{n \times n}$.

The Davis–Wielandt radius of $S \in \mathcal{B}(\mathcal{H})$ is defined as

$$\text{dw}(S) = \sup_{x \in \mathcal{H}} \left\{ \sqrt{\langle Sx, x \rangle^2 + \|Sx\|^4} \right\}.$$ 

One can easily check that $\text{dw}(S)$ is unitarily invariant, but it does not define a norm on $\mathcal{B}(\mathcal{H})$. However, it can be represented as a special case of the Euclidean operator radius as proved in Lemma 2.4 (see below).
As a direct consequence, one can easily observe that

\[
\max \left\{ w(S), \|S\|^2 \right\} \leq \text{dw}(S) \leq \sqrt{w^2(S) + \|S\|^4}
\]

(1)

for all \( S \in \mathcal{B}(\mathcal{H}) \). The inequalities are sharp.

For \( S \in \mathcal{B}(\mathcal{H}) \), let \( \mathcal{M}_S \) be the set of all unit vectors for which \( S \) attains its norm; i.e.

\[
\mathcal{M}_S := \left\{ x \in \mathcal{H} : \|x\| = 1, \|Sx\| = \|S\| \right\}.
\]

The concept of norm–parallelism in \( \mathcal{B}(\mathcal{H}) \) has been introduced recently by Zamani and Moslehian in Refs [9–11]. Let \( T, S \in \mathcal{B}(\mathcal{H}) \), we say that \( T \) is norm-parallel to \( S \) (see Ref. [9]), in symbol \( T \parallel S \), if there exists \( \lambda = \{\alpha \in \mathbb{C} : \|\alpha\| = 1\} \) such that

\[
\|T + \lambda S\| = \|T\| + \|S\|.
\]

Such property is a useful tool in solving some problems in approximation theory, as pointed out in Ref. [9]. Equivalently, it has been shown in Ref. [9] that, \( T \parallel S \) if and only if there exists a sequence of unit vectors \( x_n \) in \( \mathcal{H} \) such that

\[
\lim_{n \to \infty} \|\langle Tx_n, Sx_n \rangle\| = \|T\| \cdot \|S\|.
\]

From the norm properties of vectors in \( \mathcal{H} \), it can be shown that [12]

\[
\|b\|^2 \inf_{\gamma \in \mathbb{C}} \|a + \gamma b\|^2 = \|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2, \quad \forall a, b \in \mathcal{H}.
\]

In particular, two vectors \( a \) and \( b \) in \( \mathcal{H} \) are linearly dependent if and only if \( \inf_{\gamma \in \mathbb{C}} \|a + \gamma b\|^2 = 0 \). Employing this property, a necessary and sufficient condition for \( T \in \mathcal{B}(\mathcal{H}) \) to be norm-parallel to \( S \in \mathcal{B}(\mathcal{H}) \) was proved in Ref. [12], as elaborated in the following result.

**Theorem 1.1:** Let \( T, S \in \mathcal{B}(\mathcal{H}) \) be compact operators. Then the following conditions are equivalent:

1. \( T \parallel S \).
2. There exists \( x \in \mathcal{M}_T \cap \mathcal{M}_S \) such that for every \( \lambda \in \mathbb{C} \) the vectors \( Tx + \lambda Sx \) and \( Sx \) are linearly dependent.

In the same work Zamani et al. [12], have characterized the norm–parallelism of Hilbert space operators and an equality condition for the Davis–Wielandt radius, as indicated in the following result.

**Theorem 1.2:** Let \( S \in \mathcal{B}(\mathcal{H}) \). Then the following conditions are equivalent:

1. \( S \parallel 1_{\mathcal{H}} \).
2. \( \text{dw}(S) = \sqrt{w^2(S) + \|S\|^4} \).

As a consequence of Theorem 1.2, we have the following result [12].
Corollary 1.1: Let $S \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalent:

1. $d_w(S) = \sqrt{w^2(S) + \|S\|^4}$.
2. $w(S) = \|S\|$.
3. $d_w(S) = \|S\| \sqrt{1 + \|S\|^2}$.
4. $S^*S \leq w^2(S)1_{\mathcal{H}}$.

To see how much the lower bound in (1) is sharp, we note that, from (1) we have

$$w^2(S) + \|S\|^4 \geq d_w^2(S) \geq \max \left\{w^2(S), \|S\|^4\right\} = \frac{w^2(S) + \|S\|^4}{2} + \frac{1}{2} \left|w^2(S) - \|S\|^4\right| \geq \frac{1}{2} (w^2(S) + \|S\|^4),$$

which is the Arithmetic mean of $w^2(S)$ and $\|S\|^4$, and this means that the lower bound in (1) is on a good distance from $d_w(\cdot)$.

In their recent elegant work [13], Zamani and Shebrawi proved several inequalities involving the Davis–Wielandt radius and the numerical radii of Hilbert space operators. Among others, they showed that

$$d_w(S) \leq \sqrt[4]{w(|S|^4 + |S|^8)} + 2w^2(|S|^2 S).$$

Other interesting results were given in the same work [13], have been discussed and (in some cases) improved by Bhunia et al. [14], among others, they showed that

$$d_w^2(S) \leq \|S|^2 + |S|^4$$
and

$$d_w^2(S) \leq \frac{1}{2} w(S^2 + \|S\|^2) + \|S|^4.$$ 

An important property regarding the Davis–Wielandt radius of summand of two operators was also presented in Ref. [14], as follows:

$$d_w(S_1 + S_2) \leq d_w(S_1) + d_w(S_2) + d_w(S_1^*S_2 + S_2^*S_1)$$

for all $S_1, S_2 \in \mathcal{B}(\mathcal{H})$. Based on that, an upper bound for the Davis–Wielandt radius of $2 \times 2$ off-diagonal operator matrix was given as follows:

$$d_w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \|A\| \sqrt{\frac{1}{4} + \|A\|^2} + \|B\| \sqrt{\frac{1}{4} + \|B\|^2}.$$ 

For more details and new results concerning the Davis–Wielandt shell and the Davis–Wielandt radius of an operator, we refer the reader to Refs [4–8,15–18].

This paper is organized as follows. In the next section, a representation of an operator $S \in \mathcal{B}(\mathcal{H})$ in terms of the Euclidean operator radius is given. Some new upper and lower bounds for the Davis–Wielandt radius are introduced. Some examples verifying that the presented results are better (in some cases) than (1) are also provided. In Section 3, some bounds of the Davis–Wielandt radius for $n \times n$ operator matrices are established. An extension of the Davis–Wielandt radius to the Euclidean operator radius is introduced.
2. The results

In order to prove our results, we need a sequence of lemmas.

Lemma 2.1: The Power-Mean inequality reads

\[ a^\alpha b^{1-\alpha} \leq \alpha a + (1 - \alpha) b \leq \left( \alpha a^p + (1 - \alpha) b^p \right)^{\frac{1}{p}} \]  \hspace{1cm} (4)

for all \( \alpha \in [0,1] \), \( a, b \geq 0 \) and \( p \geq 1 \).

Lemma 2.2: Let \( A \in \mathcal{B}(\mathcal{H})^+ \), then

\[ (Ax, x)^p \leq \langle A^p x, x \rangle, \quad p \geq 1 \]  \hspace{1cm} (5)

for any unit vector \( x \in \mathcal{H} \). The inequality is reversed if \( p \in [0,1) \).

The mixed Schwarz inequality was introduced in Ref. [19], as follows:

Lemma 2.3: Let \( A \in \mathcal{B}(\mathcal{H}) \), then

\[ \left| \langle Ax, y \rangle \right|^2 \leq \langle |A|^{2\alpha} x, x \rangle \left( \langle |A^*|^{2(1-\alpha)} y, y \rangle \right), \quad 0 \leq \alpha \leq 1 \]  \hspace{1cm} (6)

for any vectors \( x, y \in \mathcal{H} \), where \( |A| = (A^*A)^{1/2} \).

We note that the McCarthy inequality (5) was extended for general Hilbert space operators in Ref. [20]. Also, the corresponding Cartesian decomposition version of (6) was recently proved by the author of this paper in Ref. [21].

In some of our results, we need the following two fundamental norm estimates, which are:

\[ \|A + B\| \leq \frac{1}{2} \left( \|A\| + \|B\| + \sqrt{\|A\|-\|B\|)^2 + 4 \|A^{1/2}B^{1/2}\|^2} \right), \]  \hspace{1cm} (7)

and

\[ \|A^{1/2}B^{1/2}\| \leq \|AB\|^{1/2}. \]  \hspace{1cm} (8)

Both estimates are valid for all positive operators \( A, B \in \mathcal{B}(\mathcal{H}) \). Also, it should be noted that (7) is sharper than the triangle inequality as pointed out by Kittaneh in Ref. [22].

In order to establish our first main result concerning the Davis–Wielandt radius, we need to recall the concept of Euclidean operator radius of an \( n \)-tuple operator, which was introduced by Popsecu in Ref. [23]. Namely, for an \( n \)-tuple \( T = (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{H})^n := \mathcal{B}(\mathcal{H}) \times \cdots \times \mathcal{B}(\mathcal{H}) \); i.e., for \( T_1, \ldots, T_n \in \mathcal{B}(\mathcal{H}) \). The Euclidean operator radius of \( T_1, \ldots, T_n \) is defined by

\[ w_e (T_1, \ldots, T_n) := \sup_{\|x\|=1} \left( \sum_{i=1}^{n} |\langle T_ix, x \rangle|^2 \right)^{1/2}. \]  \hspace{1cm} (9)

The following properties of the Euclidean operator radius were proved in Refs [23,24] and [25]:

(1) \( w_e(T_1, \ldots, T_n) = 0 \) if and only if \( T_k = 0 \) for each \( k = 1, \ldots, n \).

(2) \( w_e(\lambda T_1, \ldots, \lambda T_n) = |\lambda| w_e(T_1, \ldots, T_n) \).

(3) \( w_e(A_1 + B_1, \ldots, A_n + B_n) \leq w_e(A_1, \ldots, A_n) + w_e(B_1, \ldots, B_n) \).

(4) \( w_e(X^*T_1X, \ldots, X^*T_nX) = \|X\| w_e(T_1, \ldots, T_n) \).

(5) \( w_e(T_1, \ldots, T_n) = w_e(T_1^*, \ldots, T_n^*) \).

(6) \( w_e(T_1^*, \ldots, T_n^*) = w_e(T_1, \ldots, T_n) \).

for every \( T_k, A_k, B_k, X \in B(H) \) (\( 1 \leq k \leq n \)) and every scalar \( \lambda \in \mathbb{C} \).

The Euclidean operator radius was generalized in Ref. [24] as follows:

\[
w_p(T_1, \ldots, T_n) := \sup_{\|x\|=1} \left( \sum_{i=1}^{n} |\langle T_i x, x \rangle|^p \right)^{1/p}, \quad p \geq 1.
\]

Clearly, for \( p = 2 \) we refer to the Euclidean operator radius \( w_e(\cdot, \ldots, \cdot) \).

The following relation between the Euclidean operator radius \( w_e(S, S^*S) \) and the Davis–Wielandt radius \( dw(S) \) holds for every \( S \in B(H) \).

**Lemma 2.4:** Let \( S \in B(H) \). Then

\[
w_e(S, S^*S) = dw(S) \quad (10)
\]

**Proof:** Setting \( n = 2, T_1 = S \) and \( T_2 = S^*S \) in (9), we have

\[
w_e(S, S^*S) := \sup_{\|x\|=1} \left( |\langle Sx, x \rangle|^2 + |\langle S^*Sx, x \rangle|^2 \right)^{1/2} = \sup_{\|x\|=1} \left\{ \sqrt{|\langle Sx, x \rangle|^2 + \|Sx\|^4} \right\}
\]

which gives the Davis–Wielandt radius of \( S \), as required.

**Theorem 2.1:** Let \( S \in B(H) \). Then \( dw(S) = \sqrt{2} w(S) \) if and only if \( S \) is a selfadjoint idempotent operator.

**Proof:** To prove the ‘only if part’, from Lemma 2.4, we have \( w_e(S, S^*S) = dw(S) \) for any \( S \in B(H) \). Clearly if \( S \) is a selfadjoint idempotent operator, then \( dw(S) = w_e(S, S^*S) = w_e(S, S^2) = w_e(S, S) \). On the other hand, by setting \( n = 2 \) and \( T_1 = T_2 = S \), in (9), we get \( w_e(S, S) = \sqrt{2} w(S) \). Hence \( dw(S) = \sqrt{2} w(S) \). The ‘if part’ follows by noting that, \( dw(S) = w_e(S, S^*S) = w_e(S, S^2) \), which implies \( S^*S = S^2 \) if and only if \( S \) is a selfadjoint and therefore \( S^*S = S \), when \( S \) is an idempotent operator, i.e. \( S^2 = S \).

It’s well-known that if \( S \in B(H) \) is a selfadjoint operator, then \( \|S\| = w(S) \). Thus, according to Theorem 2.1, the equality \( \|S\| = w(S) = \frac{1}{\sqrt{2}} dw(S) \), holds when \( S \) is a selfadjoint idempotent operator. For example, take \( S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Therefore, we have \( \|S\| = w(S) = 1 = \frac{1}{\sqrt{2}} dw(S) \).

It’s convenient to note that, Kittaneh [26] proved that if \( S^2 = 0 \), then \( w(S) = \frac{1}{\sqrt{2}} \|S\| \) for all \( S \in B(H) \) with \( \dim(H) \leq 2 \). But it is not possible to have \( dw(S) = \sqrt{2} w(S) = \frac{\sqrt{2}}{2} \|S\| \),
because the first equality holds when \( S \) is a selfadjoint idempotent operator, which in turn implies that \( w(S) = \|S\| \); hence, we have \( dw(S) = \sqrt{2}\|S\| = \frac{\sqrt{2}}{2}\|S\| \), which of course impossible. Furthermore, one can show if \( S \in \mathcal{B}(\mathcal{H}) \) such that \( S^2 = 0 \), then \( S \) is not a selfadjoint operator; except the zero operator.

In 2005, Kittaneh [27] proved that
\[
\frac{1}{4} \|S^* S + S S^*\| \leq w^2(S) \leq \frac{1}{2} \|S^* S + S S^*\|
\]
for Hilbert space operator \( S \in \mathcal{B}(\mathcal{H}) \). These inequalities were also reformulated and generalized in Ref. [28] but in terms of the Cartesian decomposition.

The following result extends (11) for the generalized Euclidean operator radius.

**Lemma 2.5:** Let \( T_k \in \mathcal{B}(\mathcal{H}) \) \((k = 1, \ldots, n)\). Then
\[
\frac{1}{2p+1} \left\| \sum_{k=1}^{n} T_k^* T_k + T_k T_k^* \right\|^p \leq w^2_{2p}(T_1, \ldots, T_n) \leq \frac{1}{2p} \left\| \sum_{k=1}^{n} (T_k^* T_k + T_k T_k^*) \right\|^p
\]
for all \( p \geq 1 \).

**Proof:** Let \( P_k + iQ_k \) be the Cartesian decomposition of \( T_k \) for all \( k = 1, \ldots, n \). As in the proof of (11) in Ref. [27], we have
\[
|\langle T_k x, x \rangle|^2 = (|\langle P_k x, x \rangle|^2 + |\langle Q_k x, x \rangle|^2)^p \geq \frac{1}{2p} (|\langle P_k x, x \rangle| + |\langle Q_k x, x \rangle|)^2 \geq \frac{1}{2p} |\langle P_k x, x \rangle + \langle Q_k x, x \rangle|^2 \geq \frac{1}{2p} |\langle P_k \pm Q_k x, x \rangle|^2.
\]
Summing over \( k \) and then taking the supremum over all unit vector \( x \in \mathcal{H} \), we get
\[
w^2_{2p}(T_1, \ldots, T_n) = \sup_{\|x\|=1} \sum_{k=1}^{n} |\langle T_k x, x \rangle|^2 \geq \frac{1}{2p} \sup_{\|x\|=1} \sum_{k=1}^{n} |\langle P_k \pm Q_k x, x \rangle|^2 \geq \frac{1}{2p} \left( \sum_{k=1}^{n} |\langle P_k \pm Q_k x, x \rangle|^2 \right)^p \geq \frac{1}{2p} \left( \sum_{k=1}^{n} (P_k \pm Q_k)^2 \right)^p,
\]
where we have used Jensen’s inequality in the last inequality. Thus,
\[
2w^2_{2p}(T_1, \ldots, T_n) \geq \frac{1}{2p} \frac{1}{n^{p-1}} \left\| \sum_{k=1}^{n} (P_k + Q_k)^2 \right\|^p + \frac{1}{2p} \frac{1}{n^{p-1}} \left\| \sum_{k=1}^{n} (P_k - Q_k)^2 \right\|^p
\]
\[
\geq \frac{1}{2p} \frac{1}{n^{p-1}} \left\| \sum_{k=1}^{n} (P_k + Q_k)^2 + \sum_{k=1}^{n} (P_k - Q_k)^2 \right\|^p.
\]
\[ \frac{1}{2p} \frac{1}{n^{p-1}} \left\| \sum_{k=1}^{n} \left\{ (P_k + Q_k)^2 + (P_k - Q_k)^2 \right\} \right\|^p \\
= \frac{1}{n^{p-1}} \left\| \sum_{k=1}^{n} P_k^2 + Q_k^2 \right\|^p \\
= \frac{1}{n^{p-1}} \left\| \sum_{k=1}^{n} \frac{T_k^* T_k + T_k T_k^*}{2} \right\|^p \\
= \frac{1}{2p n^{p-1}} \left\| \sum_{k=1}^{n} T_k^* T_k + T_k T_k^* \right\|^p, \]
and hence,

\[ w_{2p}^2 (T_1, \ldots, T_n) \geq \frac{1}{2p+1} \frac{1}{n^{p-1}} \left\| \sum_{k=1}^{n} T_k^* T_k + T_k T_k^* \right\|^p, \]

which proves the left-hand side of the inequality in (12).

To prove the second inequality, for every unit vector \( x \in \mathcal{H} \) we have

\[ \sum_{k=1}^{n} |\langle T_k x, x \rangle|^2 = \sum_{k=1}^{n} ((P_k x, x)^2 + (Q_k x, x)^2)^p \leq \sum_{k=1}^{n} (P_k^2 x, x + Q_k^2 x, x)^p \\
= \sum_{k=1}^{n} ((P_k^2 + Q_k^2) x, x)^p, \]

which implies that

\[ \sup_{\|x\|=1} \sum_{k=1}^{n} |\langle T_k x, x \rangle|^2 = w_{2p}^2 (T_1, \ldots, T_n) \leq \sup_{\|x\|=1} \sum_{k=1}^{n} ((P_k^2 + Q_k^2) x, x)^p \\
\leq \sup_{\|x\|=1} \left( \sum_{k=1}^{n} (P_k^2 + Q_k^2)^p x, x \right) \\
= \left\| \sum_{k=1}^{n} (P_k^2 + Q_k^2)^p \right\| = \frac{1}{2p} \left\| \sum_{k=1}^{n} (T_k^* T_k + T_k T_k^*)^p \right\|, \]

which proves the right-hand side of (12). \[\square\]

**Remark 2.1:** In particular, setting \( n = 2 \) and \( p = 1 \) in (12) we get

\[ \frac{1}{4} \left\| T_1^* T_1 + T_1 T_1^* + T_2^* T_2 + T_2 T_2^* \right\| \leq w_e^2 (T_1, T_2) \leq \frac{1}{2} \left\| T_1^* T_1 + T_1 T_1^* + T_2^* T_2 + T_2 T_2^* \right\|. \]
Moreover, if we choose \( T_1 = T_2 = T \), then
\[
\frac{1}{2} \left\| T^* T + TT^* \right\| \leq w_e(T, T) \leq \left\| T^* T + TT^* \right\|.
\]
but \( w_e(T, T) = \sqrt{2} w(T) \), thus the last inequality above reduces to the Kittaneh inequality (11).

Now, based on Lemmas 2.4 and 2.5, we can introduce our first main result, as follows:

**Theorem 2.2:** Let \( S \in \mathcal{B}(\mathcal{H}) \). Then
\[
\frac{1}{4} \left( |S|^2 + |S^*|^2 + 2 |S|^4 \right) \leq dw^2(S) \leq \frac{1}{2} \left( |S|^2 + |S^*|^2 + 2 |S|^4 \right).
\] (13)

**Proof:** Setting \( n = 2, p = 1, T_1 = S \) and \( T_2 = S^* S \) in (12), we get
\[
\frac{1}{4} \left( |S|^2 + |S^*|^2 + 2 |S|^4 \right) \leq w_e^2(S, S^* S) \leq \frac{1}{2} \left( |S|^2 + |S^*|^2 + 2 |S|^4 \right).
\]

But as we have shown in Lemma 2.4 that, \( w_e(S, S^* S) = dw(S) \), hence we have
\[
\frac{1}{4} \left( |S|^2 + |S^*|^2 + 2 |S|^4 \right) \leq dw^2(S) \leq \frac{1}{2} \left( |S|^2 + |S^*|^2 + 2 |S|^4 \right),
\]
as desired. \( \square \)

To see that the second inequality in (13) is a refinement of the second inequality in (1), assume \( SS^* \leq S^* S \leq w^2(S)1_{\mathcal{H}} \). Thus, from (13) we have
\[
dw^2(S) \leq \frac{1}{2} \left( |S|^2 + |S^*|^2 + 2 |S|^4 \right) \leq \frac{1}{2} \left( w^2(S)1_{\mathcal{H}} + w^2(S)1_{\mathcal{H}} + 2w^4(S)1_{\mathcal{H}} \right)
\]
\[
\leq w^2(S) + ||S||^4,
\]
follows by assumption, since \( w(S) = ||S|| \) (see Corollary 1.1), which implies that
\[
dw(S) \leq \sqrt{\frac{1}{2} \left( |S|^2 + |S^*|^2 + 2 |S|^4 \right)} \leq \sqrt{w^2(S) + ||S||^4} = ||S|| \sqrt{1 + ||S||^2},
\]
which means that the right-hand side of (13) refines the right-hand side of (1).

**Example 2.1:** Let \( S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \). We have \( ||S|| = 2.28825 \) and \( w(S) = 2.08114 \). The upper bound of (1) gives \( dw(S) \leq 5.63449 \). However, by applying (13), we have \( dw(S) \leq 5.61938 \), which implies that, the upper bound in (13) is better than the upper bound in (1).

**Remark 2.2:** Following the same approach considered in the proof of Theorem 2.2, another interesting inequalities could be deduced from the obtained inequalities in Refs [23,24] and [25].

The following result refines sharply the upper bound in (1).
Theorem 2.3: If $S \in B(\mathcal{H})$, then
\[
\frac{1}{\sqrt{2}} \| S + S^* \| \leq \text{dw} (S) \leq \sqrt{\frac{1}{4} (|\mathcal{S}| + |\mathcal{S}^*|)^2 + |\mathcal{S}|^4} \leq \frac{1}{4} \left( \| \mathcal{S} \| + \| S^2 \|^{1/2} \right)^2 + \| \mathcal{S} \|^4.
\] (14)

Proof: Since we have
\[
\text{dw}^2 (S) = \sup_{x \in \mathcal{H}} \left\{ |\langle Sx, x \rangle| + |\langle S^* Sx, x \rangle| \right\} \geq \frac{1}{2} \sup_{\| x \| = 1} \left\{ |\langle Sx, x \rangle| + |\langle S^* Sx, x \rangle| \right\}^2
\]
\[
= \frac{1}{2} \sup_{\| x \| = 1} \left\{ |\langle Sx, x \rangle + \langle S^* Sx, x \rangle| \right\}^2
\]
\[
= \frac{1}{2} \sup_{\| x \| = 1} \left\{ \left| \left( S + S^* \right) x, x \right| \right\}^2
\]
which proves the first inequality in (14). Also, since we have
\[
\text{dw}^2 (S) = \sup_{x \in \mathcal{H}} \left\{ |\langle Sx, x \rangle|^2 + \| Sx \|^4 \right\}
\]
\[
= \sup_{\| x \| = 1} \left\{ |\langle Sx, x \rangle|^2 + \| S^* Sx \|^2 \right\}
\]
\[
\leq \sup_{\| x \| = 1} \left\{ \left( \left| \frac{| \mathcal{S} | + | \mathcal{S}^* |}{2} \right| x, x \right)^2 + \left| \mathcal{S}^* S \right|^2 x, x \right\} \quad \text{(by (6))}
\]
\[
\leq \sup_{\| x \| = 1} \left\{ \left( \left( \left| \frac{| \mathcal{S} | + | \mathcal{S}^* |}{2} \right| x, x \right)^2 + \left| \mathcal{S}^* S \right|^2 x, x \right) \right\} \quad \text{(by (4))}
\]
\[
= \sup_{\| x \| = 1} \left\{ \left( \left| \frac{| \mathcal{S} | + | \mathcal{S}^* |}{2} \right| + \left| \mathcal{S}^* S \right|^2 \right) x, x \right\}
\]
\[
= \frac{1}{4} \left\| \left( | \mathcal{S} | + | \mathcal{S}^* | \right)^2 + 4 \left| \mathcal{S}^* S \right|^2 \right\|,
\]
and this proves the second inequality in (14). Applying the triangle inequality to the above inequality, we get
\[
\text{dw}^2 (S) \leq \frac{1}{4} \left\| \left( | \mathcal{S} | + | \mathcal{S}^* | \right)^2 + 4 \left| \mathcal{S}^* S \right|^2 \right\| \leq \frac{1}{4} \left\| \left( | \mathcal{S} | + | \mathcal{S}^* | \right)^2 \right\| + \left\| \left| \mathcal{S}^* S \right|^2 \right\|.
\]
Now, applying (7) to the first term in the above inequality, we get $\|S + S^*\| \leq \|S\| + \|S^2\|^{1/2}$. Now substituting this inequality in the last inequality above, we get the third inequality in (14), and this completes the proof. ■

Example 2.2: Let $S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. We have $w(S) = \|S\| = 2$. Employing the sharp lower bound in (1) we get that $dw(S) \geq 4$. By applying the lower bound in (14), we get $dw(S) \geq 3\sqrt{2} = 4.2426$, which means that the lower bound in (14) is better than that one given in (1).

Also, applying the first upper bound in (14) we have $dw(S) \leq 2\sqrt{5} = 4.47214$, which gives the same result if one chooses to apply the upper bound in (1).

Remark 2.3: We note that, a refinement of the inequality (14) could be stated as follows:

$$\frac{1}{\sqrt{2}} \|S + S^*S\| \leq dw(S) \leq \sqrt{w \left( \frac{1}{4} (|S| + |S^*|)^2 + |S|^4 \right)}.$$

Consider $S$ as in Example 2.1. Applying the above inequality, we get $dw(S) \leq 5.59709$, which is better than the result obtained by (13). Furthermore, (11) gives that

$$dw(S) \leq \sqrt{w \left( \frac{1}{4} (|S| + |S^*|)^2 + |S|^4 \right)} \leq \sqrt{\frac{1}{4} \|T^*T + TT^*\|},$$

where $T = \frac{1}{4} (|S| + |S^*|)^2 + |S|^4$. Employing the previous second upper bound for $S$ in Example 2.2, we get the same result as those obtained by (14) and (1), even we use (11); which indeed refines (14).

Remark 2.4: In Ref. [26], Kittaneh proved that if $S \in \mathcal{B}(\mathcal{H})$ is such that $S^2 = 0$, then $w(S) = \frac{1}{2}\|S\|$. Under this assumption, the inequality (1) becomes

$$\max \left\{ \frac{1}{2} \|S\| , \|S\|^2 \right\} \leq dw(S) \leq \sqrt{\frac{1}{4} \|S\|^2 + \|S\|^4}.$$

Similarly, the (second) upper bound in (14) is reduced to the form

$$dw(T) \leq \sqrt{\frac{1}{4} (|S| + |S^*|)^2 + |S|^4} \leq \sqrt{\frac{1}{4} \|S\|^2 + \|S\|^4}.$$

A generalization of the upper bound in Theorem 2.2 is considered as follows:

**Theorem 2.4:** Let $S \in \mathcal{B}(\mathcal{H})$, $0 \leq \alpha \leq 1$ and $r \geq 2$. Then

$$dw^r(S) \leq \frac{2^r}{4} \left\| |S|^{2r\alpha} + |S^*|^{2r(1-\alpha)} + |S^*S|^{2r\alpha} + |S^*S|^{2r(1-\alpha)} \right\|. \quad (15)$$
**Proof:** Let $x \in \mathcal{H}$ be unit vector, then

$$d^2(w^2(S)) = \sup_{x \in \mathcal{H}, \|x\| = 1} \{ |\langle Sx, x \rangle|^2 + \|Sx\|^4 \} = \sup_{x \in \mathcal{H}, \|x\| = 1} \{ |\langle S^* Sx, x \rangle|^2 \}.$$

But since

$$\langle Sx, x \rangle \leq \langle |S|^{2\alpha} x, x \rangle^{1/2} \left( \left| \frac{|S|^2 x, x \rangle^2 + |S^*|^{2(1-\alpha)} x, x \rangle^2}{2} \right)^{1/2} \right) \quad \text{(by (6))}$$

$$\leq \left( \frac{\langle |S|^{2\alpha} x, x \rangle^2 + \langle |S^*|^2(1-\alpha) x, x \rangle^2}{2} \right)^{1/2} \quad \text{(by (4))}$$

$$\leq \frac{1}{2^{1/2}} \left( \langle |S|^{2\alpha} + |S^*|^{2(1-\alpha)} \rangle x, x \right) \quad \text{(by (5))}$$

it follows that

$$\langle Sx, x \rangle^r \leq \frac{1}{2} \left( \langle |S|^{2\alpha} + |S^*|^{2(1-\alpha)} \rangle x, x \right) \quad (16)$$

and

$$\langle S^* Sx, x \rangle \leq \langle |S|^{2\alpha} S^* S x, x \rangle^{1/2} \left( \left| \frac{|S|^2 S^* S x, x \rangle^2 + |S^*|^{2(1-\alpha)} S^* S x, x \rangle^2}{2} \right)^{1/2} \right) \quad \text{(by (4))}$$

$$\leq \left( \frac{\langle |S|^{2\alpha} S^* S x, x \rangle^2 + \langle |S^*|^2(1-\alpha) S^* S x, x \rangle^2}{2} \right)^{1/2} \quad \text{(by (5))}$$

$$\leq \frac{1}{2^{1/2}} \left( \langle |S|^{2\alpha} S^* S + |S^*|^{2(1-\alpha)} S^* S \rangle x, x \right) \quad \text{,}$$

it follows that

$$\langle S^* Sx, x \rangle^r \leq \frac{1}{2} \left( \langle |S|^{2\alpha} S^* S + |S^*|^{2(1-\alpha)} S^* S \rangle x, x \right) \quad (17)$$

Adding (16) and (17), we get

$$\frac{1}{2} \sup_{x \in \mathcal{H}, \|x\| = 1} \left( \langle |S|^{2\alpha} + |S^*|^{2(1-\alpha)} + |S^*|^{2\alpha} + |S^*|^{2(1-\alpha)} \rangle x, x \right)$$

$$\geq \sup_{x \in \mathcal{H}, \|x\| = 1} \{ |\langle Sx, x \rangle|^r + |\langle S^* Sx, x \rangle|^r \}$$


\[ \begin{align*}
\sup_{x \in \mathcal{H}, \|x\| = 1} \left\{ \left( |\langle Sx, x \rangle|^2 \right)^{r/2} + \left( |\langle S^*Sx, x \rangle|^2 \right)^{r/2} \right\} \\
\geq \frac{1}{2^{\frac{r}{2}-1}} \sup_{x \in \mathcal{H}, \|x\| = 1} \left( |\langle Sx, x \rangle|^2 + |\langle S^*Sx, x \rangle|^2 \right)^{r/2} = \frac{1}{2^{\frac{r}{2}-1}} d_{w^r}(S).
\end{align*} \]

Hence,

\[ d_{w^r}(S) \leq \frac{2^\frac{r}{2}}{4} \| |S|^2 + |S^*|^2 + 2 |S^*S|^2 \|, \]

as required. \[ \blacksquare \]

**Remark 2.5:** We note that a refinement of the inequality (13) could be deduced from (15). Note that, by setting \( r = 2 \) and \( \alpha = \frac{1}{2} \) in (15), we get (13). Using the same proof given in Theorem 2.4, we can get

\[ d_w(S) \leq \sqrt{\frac{1}{2} w \left( |S|^2 + |S^*|^2 + 2 |S^*S|^2 \right)}, \]

Moreover, employing (11) for the above inequality, we get

\[ d_w(S) \leq \sqrt{\frac{1}{2} w \left( |S|^2 + |S^*|^2 + 2 |S^*S|^2 \right)} \leq \frac{4}{\sqrt{8}} \| T^*T + TT^* \|, \]

where \( T = |S|^2 + |S^*|^2 + 2|S^*S|^2 \).

**Theorem 2.5:** Let \( S \in \mathcal{B}(\mathcal{H}), 0 \leq \alpha \leq 1 \) and \( r \geq 1 \). Then

\[ d_{w^{2r}}(S) \leq 2^{r-1} \left\| \alpha |S|^{2r} + (1 - \alpha) |S^*|^{2r} + |S^*S|^{2r} \right\|. \quad (18) \]

**Proof:** Let \( x \in \mathcal{H} \) be a unit vector, then

\[ |\langle Sx, x \rangle|^2 \leq |S|^{2\alpha} x, x \rangle \left( |S^*|^{2(1-\alpha)} x, x \right) \quad \text{(by (6))} \]
\[ \leq |S|^2 x, x \rangle^\alpha \left( |S^*|^2 x, x \right)^{(1-\alpha)} \quad \text{(by (5))} \]
\[ \leq \left( \alpha |S|^2 x, x \rangle^r + (1 - \alpha) \left( |S^*|^2 x, x \right)^{r} \right)^{1/r} \quad \text{(by (4))} \]
\[ \leq \left( \alpha |S|^{2r} x, x \rangle + (1 - \alpha) \left( |S^*|^{2r} x, x \right)^{1/r} \quad \text{(by (6))} \]
\[ \leq \left( \alpha |S|^{2r} + (1 - \alpha) |S^*|^{2r} \right) x, x \right)^{1/r}. \]
Therefore,
\[
|\langle Sx, x \rangle|^{2r} \leq \left( (\alpha |S|^{2r} + (1 - \alpha) |S^*|^{2r} \right) x, x \).
\] (19)

Also, since \(S^*S\) is a self-adjoint then we have
\[
|\langle S^*Sx, x \rangle|^{2r} \leq \left| |S^*S|^{2r} \right| x, x \right).
\] (20)

Adding (19) and (20), we obtain
\[
\sup_{x \in \mathcal{H}, \|x\| = 1} \left( \alpha |S|^{2r} + (1 - \alpha) |S^*|^{2r} + |S^*S|^{2r} \right) x, x \right) \geq \sup_{x \in \mathcal{H}, \|x\| = 1} \left( \langle Sx, x \rangle|^{2r} + |\langle S^*S, x \rangle|^{2r} \right) \]
\[
\geq \frac{2}{2r} \sup_{x \in \mathcal{H}, \|x\| = 1} \left( \langle Sx, x \rangle|^2 + |\langle S^*S, x \rangle|^2 \right)^{1/r}
\]
\[
= \frac{2}{2r} dw^2r (S).
\]

Hence,
\[
dw^{2r} (S) \leq 2^{r^{-1}} \left( \alpha |S|^{2r} + (1 - \alpha) |S^*|^{2r} + |S^*S|^{2r} \right).
\]

This completes the proof of Theorem 2.5.

Example 2.3: Let \(S = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \). We have \(\|S\| = w(S) = 2\). The upper bound of (1) gives \(dw(S) \leq 2\sqrt{5} = 4.4721\). However, by applying (18) with \(r = 1\) and \(\alpha = \frac{1}{2}\), we have \(dw(S) \leq 3\sqrt{2} = 4.2426\), which implies that, the upper bound in (18) is better than the upper bound in (1).

Remark 2.6: A refinement of the inequality (18) could be deduced from the proof given in Theorem 2.5, we can get
\[
dw^{2r} (S) \leq 2^{r^{-1}} w \left( \alpha |S|^{2r} + (1 - \alpha) |S^*|^{2r} + |S^*S|^{2r} \right).
\]

Moreover, employing (11) in the above inequality, we obtain
\[
dw^{2r} (S) \leq 2^{r^{-1}} w \left( \alpha |S|^{2r} + (1 - \alpha) |S^*|^{2r} + |S^*S|^{2r} \right)
\]
\[
\leq 2^{\frac{r}{2} - 1} \left\| T_{r,\alpha}^* T_{r,\alpha} + T_{r,\alpha} T_{r,\alpha}^* \right\|^{1/2},
\]
where \(T_{r,\alpha} = \alpha |S|^{2r} + (1 - \alpha)|S^*|^{2r} + |S^*S|^{2r}\). For \(r = 1\) and \(\alpha = \frac{1}{2}\) the last inequality reduces to
\[
dw (S) \leq \sqrt{w \left( \frac{1}{2} |S|^2 + \frac{1}{2} |S^*|^2 + |S^*S|^2 \right)} \leq \sqrt{\frac{1}{2} \left\| T_{1,\frac{1}{2}}^* T_{1,\frac{1}{2}} + T_{1,\frac{1}{2}} T_{1,\frac{1}{2}}^* \right\|},
\]
where \(T_{1,\frac{1}{2}} = \frac{1}{2} |S|^2 + \frac{1}{2} |S^*|^2 + |S^*S|^2\).
3. The Davis–Wielandt radius inequalities for \( n \times n \) matrix operators

Several numerical radius type inequalities improve and refine the inequality

\[
\frac{1}{2} \| S \| \leq w(S) \leq \| S \| \quad (S \in B(\mathcal{H}))
\]

have been recently obtained by many other authors see, for example, Refs [20,21,29–32], and [33]. Among others, three important facts concerning the numerical radius inequalities of \( n \times n \) operator matrices are obtained by different authors which are grouped together, as follows:

Let \( S = [S_{ij}] \in B(\bigoplus_{i=1}^{n} \mathcal{H}_i) \) such that \( S_{ij} \in B(\mathcal{H}_j, \mathcal{H}_i) \). Then

\[
w(S) \leq \begin{cases} 
  w \left( \left[ t_{ij}^{(1)} \right] \right), & \text{Hou\&Du in [33]} \\
  w \left( \left[ t_{ij}^{(2)} \right] \right), & \text{BaniDom\&Kittaneh in [32]}; \\
  w \left( \left[ t_{ij}^{(3)} \right] \right), & \text{AbuOma\&Kittaneh in [29]}
\end{cases}
\]

where

\[
\begin{align*}
  t_{ij}^{(1)} &= w(\| S_{ij} \|), \\
  t_{ij}^{(2)} &= \begin{cases} 
    w(\| S_{ii} \| + \| S_{jj} \|^{1/2}), & i = j, \\
    \| S_{ij} \|, & i \neq j
  \end{cases}, \\
  t_{ij}^{(3)} &= \begin{cases} 
    w(S_{ii}), & i = j, \\
    \| S_{ij} \|, & i \neq j
  \end{cases}
\]

As mentioned in Ref. [16], in our recent work [31] we tried to refine the last bound (above) proved by Abu Omar and Kittaneh in Ref. [29]; however there is a mistake in the printed version of the result. In the following result we correct [31, Theorem 4.1].

**Theorem 3.1:** Let \( S = [S_{ij}] \in B(\bigoplus_{i=1}^{n} \mathcal{H}_i) \) such that \( S_{ij} \in B(\mathcal{H}_j, \mathcal{H}_i) \). Then

\[
w(S) \leq w \left( \left[ s_{ij} \right] \right),
\]

where

\[
s_{ij} = \begin{cases} 
    w(S_{ij}), & j = i \quad \text{and} \quad j \neq k_i \\
    w^2 \left( \| S_{ij} \| \right) \frac{1}{2} \left( \| S_{ii}^* \| \right), & j = k_i \quad \text{and} \quad j \neq i \\
    \| S_{ij} \|, & j \neq k_i \quad \text{and} \quad j \neq i
\end{cases}
\]

where \( k_i = n - i + 1 \).

**Proof:** Let \( x = [x_1, x_2, \ldots, x_n]^T \in \bigoplus_{i=1}^{n} \mathcal{H}_i \) with \( \| x \| = 1 \). For simplicity setting \( k_i = n - i + 1 \), then we have

\[
|\langle Sx, x \rangle| = \left| \sum_{i,j=1}^{n} \langle S_{ij} x_j, x_i \rangle \right| \\
\leq \sum_{i,j=1}^{n} \left| \langle S_{ij} x_j, x_i \rangle \right|
\]
\[ \leq n \sum_{i=1}^{n} |\langle S_{ii}x_i, x_i \rangle| + \sum_{i=1}^{n} |\langle S_{ik}, x_{ki} \rangle| + \sum_{j \neq i, k_i} |\langle S_j x_j, x_i \rangle| \]

\[ \leq n \sum_{i=1}^{n} |\langle S_{ii}x_i, x_i \rangle| + \sum_{i=1, i \neq k_i}^{n} \| S_{ik} \| x_{ki} \| x_i \| \left( \| S_{ik} \| \right)^{1/2} + \sum_{j \neq i, k_i} |\langle S_j x_j, x_i \rangle| \]

\[ \leq n \sum_{i=1}^{n} w(S_{ii}) \| x_i \|^2 + \sum_{i=1, i \neq k_i}^{n} w_{1/2} \left( \| S_{ik} \| \right) w_{1/2} \left( \| S_{ik} \| \right) \| x_{ki} \| \| x_i \| \]

\[ + \sum_{j \neq i} \| S_{ij} \| \| x_i \| \| x_j \| \]

\[ \leq \sum_{i, j=1}^{n} s_{ij} \| x_i \| \| x_j \| \]

\[ = \left( \left[ s_{ij} \right] y, y \right), \]

where \( y = \left( \| x_1 \| \| x_2 \| \cdots \| x_n \| \right)^T \). Taking the supremum over \( x \in \bigoplus \mathcal{H}_i \), we obtain the desired result.

To proceed further, we need the following well-known result.

**Lemma 3.1:** If \( S := [s_{kj}] \in \mathcal{M}_{n \times n} \), then

\[ w(S) \leq w \left( \left[ \| s_{kj} \| \right] \right) = \frac{1}{2} r \left( \left[ \| s_{kj} \| + \| s_{kj} \| \right] \right). \]

In the next result, we present Davis–Wielandt radius inequality for matrix Operators.

**Theorem 3.2:** Let \( T = [T_{ij}] \in \mathcal{B} \left( \bigoplus_{i=1}^{n} \mathcal{H}_i \right) \) such that \( T_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i) \). Then

\[ d_w(T) \leq w \left( \left[ t_{ij} \right] \right), \quad (23) \]

where

\[ t_{ij} = \begin{cases} w(T_{ii}) + \| T_{ii} \|^2, & j = i \\ \| T_{ij} \| + \| T_{ji} \|^2, & j \neq i \end{cases}. \]

**Proof:** Let \( x = [x_1 \ x_2 \cdots \ x_n]^T \in \bigoplus_{i=1}^{n} \mathcal{H}_i \) with \( \| x \| = 1 \). Then we have

\[ d_w(T) = \sup_{x \in \mathcal{H}} \frac{\sqrt{|\langle Tx, x \rangle|^2 + |\langle T^*Tx, x \rangle|^2}}{\|x\|} \]

\[ \leq \sup_{x \in \mathcal{H}} \left\{ |\langle Tx, x \rangle| + \left| \langle T^*Tx, x \rangle \right| \right\} \quad \text{(since } \sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \text{)} \]
But since
\[
|\langle Tx, x \rangle| = \left| \sum_{i,j=1}^{n} \langle T_{ij}x_j, x_i \rangle \right| \leq \sum_{i,j=1}^{n} |\langle T_{ij}x_j, x_i \rangle| \\
\leq \sum_{i=1}^{n} |\langle T_{ii}x_i, x_i \rangle| + \sum_{j \neq i}^{n} |\langle T_{ij}x_j, x_i \rangle| \\
\leq \sum_{i=1}^{n} w(T_{ii}) \|x_i\|^2 + \sum_{j \neq i}^{n} \|T_{ij}\| \|x_i\| \|x_j\| 
\]

(24)

where \( y = (\|x_1\| \|x_2\| \ldots \|x_n\|)^T \).

Similarly, we have
\[
|\langle T^*Tx, x \rangle| = \left| \sum_{i,j=1}^{n} \langle T^*_{ij}T_{ij}x_j, x_i \rangle \right| \\
\leq \sum_{i=1}^{n} w(T^*_{ii}) \|x_i\|^2 + \sum_{j \neq i}^{n} \|T^*_{ij}\| \|x_i\| \|x_j\|. 
\]

(25)

Adding (24) and (25), we get
\[
\begin{align*}
\text{dw}(T) & \leq \sup_{\|x\|=1} \left\{ |\langle Tx, x \rangle| + |\langle T^*Tx, x \rangle| \right\} \\
& \leq \sum_{i=1}^{n} (w(T_{ii}) + w(T^*_{ii})) \|x_i\|^2 + \sum_{j \neq i}^{n} \left( \|T_{ij}\| + \|T^*_{ij}\| \right) \|x_i\| \|x_j\| \\
& = \sum_{i=1}^{n} (w(T_{ii}) + \|T_{ii}\|^2) \|x_i\|^2 + \sum_{j \neq i}^{n} \left( \|T_{ij}\| + \|T_{ij}\|^2 \right) \|x_i\| \|x_j\| \\
& \leq \sum_{i,j=1}^{n} t_{ij} \|x_i\| \|x_j\| \\
& = \left\langle \left[ t_{ij} \right] y, y \right\rangle .
\end{align*}
\]

Taking the supremum over \( x \in \bigoplus \mathcal{H} \), we obtain the right-hand side inequality in (23), and this completes the proof.

\[ \blacksquare \]

**Corollary 3.1:** Let \( T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \). Then
\[
\text{dw}(T) \leq \frac{1}{2} \left( \frac{1}{2} \right) \left( a + d \right) + \left( a + d \right) + \sqrt{(a - d)^2 + (b + c)^2} ,
\]

where,
\[
a = w(T_{11}) + \|T_{11}\|^2 , \quad b = \|T_{12}\| + \|T_{12}\|^2 ,
\]

(26)
\[ c = \| T_{21} \| + \| T_{21} \|^2, \quad d = w(T_{22}) + \| T_{22} \|^2. \]

**Proof:** Take \( n = 2 \) in Theorem 3.2. Let \( a, b, c, d \) be as defined above. Then

\[
dw \left( \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \right) \leq w \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)
= r \left( \begin{bmatrix} a & b+c \\ b+c & d^2 \end{bmatrix} \right) \quad \text{(by Lemma 3.1)}
= \frac{1}{2} \left( a + d + \sqrt{(a-d)^2 + (b+c)^2} \right).
\]

as required. \( \blacksquare \)

**Corollary 3.2:** Let \( \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \), then

\[
dw \left( \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \right) \leq \max \{ w(T_{11}) + \| T_{11} \|^2, w(T_{22}) + \| T_{22} \|^2 \} \quad (27)
\]

In special case, if \( \mathcal{H}_1 = \mathcal{H}_2 \) and \( T_{11} = T_{22} = T \), then

\[
dw \left( \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \right) \leq w(T) + \| T \|^2 \quad (28)
\]

**Proof:** From Corollary 3.1, we have

\[
dw \left( \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \right) \leq \max \{ w(T_{11}) + w(T_{11}^*) T_{11}, w(T_{22}) + w(T_{22}^*) T_{22} \}
= \max \{ w(T_{11}) + w(|T_{11}|^2), w(T_{22}) + w(|T_{22}|^2) \}
\leq \max \{ w(T_{11}) + \| T_{11} \|^2, w(T_{22}) + \| T_{22} \|^2 \},
\]

as required. \( \blacksquare \)

**Corollary 3.3:** Let \( T = \begin{bmatrix} T & S \\ S & T \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \). Then

\[
dw(T) \leq w(T) + \| T \|^2 + \| S \| + \| S \|^2 \quad (29)
\]

**Proof:** From Corollary 3.1, we have \( T_{11} = T_{22} = T \) and \( T_{12} = T_{21} = S \), therefore

\[ a = w(T) + \| T \|^2 = d, \quad b = \| S \| + \| S \|^2 = c. \]

Thus,

\[
dw \left( \begin{bmatrix} T & S \\ S & T \end{bmatrix} \right) \leq a + b = w(T) + \| T \|^2 + \| S \| + \| S \|^2,
\]

as required. \( \blacksquare \)

A refinement of Theorem 3.2 is formulated as follows:
Theorem 3.3: Let \( T = [T_{ij}] \in \mathcal{B}(\bigoplus_{i=1}^{n} \mathcal{H}_i) \) such that \( T_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i) \). Then
\[
\frac{1}{\sqrt{2}} \| T + T^* T \| \leq dw(T) \leq w^{1/2} \left( \{ t_{ij} \} \right),
\]
where
\[
t_{ij} = n \cdot \left\{ \begin{array}{ll}
w^2(T_{ii}) + \| T_{ii} \|^4, & j = i \\
\| T_{ij} \|^2 + \| T_{ij} \|^4, & j \neq i
\end{array} \right.
\]

Proof: Let \( x = [x_1 \ x_2 \ \ldots \ x_n]^T \in \bigoplus_{i=1}^{n} \mathcal{H}_i \) with \( \| x \| = \sum_{i=1}^{n} \| x_i \|^2 = 1 \). Then we have
\[
dw(T) = \sup_{\substack{x \in \mathcal{H} \\
\| x \| = 1}} \left\{ \sqrt{|\langle Tx, x \rangle|^2 + \| Tx \|^4} \right\} = \sup_{\substack{x \in \mathcal{H} \\
\| x \| = 1}} \sqrt{|\langle Tx, x \rangle|^2 + |\langle T^* Tx, x \rangle|^2}.
\]
But since
\[
|\langle Tx, x \rangle|^2 = \left| \sum_{i,j=1}^{n} \langle T_{ij} x_j, x_i \rangle \right|^2 \leq n \cdot \sum_{i,j=1}^{n} \left| \langle T_{ij} x_j, x_i \rangle \right|^2 \quad \text{(by Jensen's inequality)}
\]
\[
\leq n \cdot \sum_{i=1}^{n} |\langle T_{ii} x_i, x_i \rangle|^2 + n \cdot \sum_{j \neq i}^{n} |\langle T_{ij} x_j, x_i \rangle|^2 \leq n \cdot \sum_{i=1}^{n} w^2(T_{ii}) \| x_i \|^4 + n \cdot \sum_{j \neq i}^{n} \| T_{ij} \|^2 \| x_i \| \| x_j \|^2 \leq n \cdot \sum_{i=1}^{n} w^2(T_{ii}) \| x_i \|^2 + n \cdot \sum_{j \neq i}^{n} \| T_{ij} \|^2 \| x_i \| \| x_j \|,
\]
the last inequality holds, since \( \| x_i \|^4 \leq \| x_i \|^2 \leq 1 \) and \( \| x_i \|^2 \leq \| x_i \| \leq 1 \) for all \( i = \ldots , n \); where \( y = (\| x_1 \| \| x_2 \| \ldots \| x_n \|)^T \).

Similarly, we have
\[
|\langle T^* Tx, x \rangle|^2 = \left| \sum_{i,j=1}^{n} \langle T^*_{ij} T_{ij} x_j, x_i \rangle \right|^2 \leq n \cdot \sum_{i=1}^{n} w^2(T^*_{ii} T_{ii}) \| x_i \|^2 + n \cdot \sum_{j \neq i}^{n} \left( T^*_{ij} T_{ij} \right)^2 \| x_i \| \| x_j \|.
\]
Adding (31) and (32), we get
\[
dw^2(T) = \sup_{\substack{x \in \mathcal{H} \\
\| x \| = 1}} \left\{ |\langle Tx, x \rangle|^2 + |\langle T^* Tx, x \rangle|^2 \right\}.
\]
\[
\leq n \cdot \sum_{i=1}^{n} \left( w^2(T_{ii}) + \|T_{ii}\|^2 \right) \|x_i\|^2 + n \cdot \sum_{j \neq i}^n \left( \|T_{ij}\|^2 + \|T_{ij}\|^4 \right) \|x_i\| \|x_j\|
\]
\[
= n \cdot \sum_{i=1}^{n} \left( w^2(T_{ii}) + \|T_{ii}\|^4 \right) \|x_i\|^2 + \sum_{j \neq i}^n \left( \|T_{ij}\|^2 + \|T_{ij}\|^4 \right) \|x_i\| \|x_j\|
\]
\[
\leq n \cdot \sum_{i,j=1}^{n} t_{ij} \|x_i\| \|x_j\|
\]
\[
= n \cdot \langle [t_{ij}] y, y \rangle.
\]
Taking the supremum over \( x \in \bigoplus \mathcal{H}_i \), we obtain the right-hand side inequality.

To prove the left hand side inequality we note that
\[
dw^2(T) = \sup_{x \in \mathcal{H} \atop \|x\| = 1} \left\{ |\langle T x, x \rangle|^2 + |\langle T^* Tx, x \rangle|^2 \right\}
\]
\[
\geq \frac{1}{2} \sup_{x \in \mathcal{H} \atop \|x\| = 1} \left\{ |\langle T x, x \rangle| + |\langle T^* Tx, x \rangle| \right\}^2
\]
\[
\geq \frac{1}{2} \sup_{x \in \mathcal{H} \atop \|x\| = 1} \left\{ |\langle (T + T^* T) x, x \rangle| \right\}
\]
\[
= \frac{1}{2} \|T + T^* T\|
\]
as required.

\[\text{Corollary 3.4: Let } T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2). \text{ Then}
\]
\[
dw(T) \leq \sqrt{a + d + \sqrt{(a - d)^2 + (b + c)^2}},
\]
where,
\[
a = w^2(T_{11}) + \|T_{11}\|^4, \quad b = \|T_{12}\|^2 + \|T_{12}\|^4,
\]
\[
c = \|T_{21}\|^2 + \|T_{21}\|^4, \quad d = w^2(T_{22}) + \|T_{22}\|^4.
\]

\[\text{Proof: Take } n = 2 \text{ in Theorem 3.3. Let } a, b, c, d \text{ be as defined above. Then}
\]
\[
dw^2 \left( \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \right) \leq 2w \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)
\]
\[
= 2r \left( \begin{bmatrix} a & b+c/2 \\ b+c/2 & d \end{bmatrix} \right) \quad \text{(by Lemma 3.1)}
\]
\[
= 2r \left( \begin{bmatrix} a & b+c/2 \\ b+c/2 & d \end{bmatrix} \right)
\]
which proves the required inequality. ■

Corollary 3.5: Let \( \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \), then
\[
\max \left\{ \sqrt{w^2(T_{11}) + \|T_{11}\|^4}, \sqrt{w^2(T_{22}) + \|T_{22}\|^4} \right\} \leq \sqrt{2} \max \left\{ \sqrt{w^2(T_{11}) + \|T_{11}\|^4}, \sqrt{w^2(T_{22}) + \|T_{22}\|^4} \right\}
\]

In special case, if \( \mathcal{H}_1 = \mathcal{H}_2 \) and \( T_{11} = T_{22} = T \), then
\[
\max \left\{ \sqrt{w^2(T_{11}) + \|T_{11}\|^4}, \sqrt{w^2(T_{22}) + \|T_{22}\|^4} \right\} \leq \sqrt{2} \left( w^2(T) + \|T\|^4 \right)^{1/2}.
\]

Proof: Form Corollary 3.4, we have
\[
\max \left\{ \sqrt{w^2(T_{11}) + \|T_{11}\|^4}, \sqrt{w^2(T_{22}) + \|T_{22}\|^4} \right\} \leq \sqrt{2} \max \left\{ \sqrt{w^2(T_{11}) + \|T_{11}\|^4}, \sqrt{w^2(T_{22}) + \|T_{22}\|^4} \right\}.
\]
which gives the desired result. ■

Remark 3.1: Using the same approach considered in Theorem 3.1, one can refine Theorems 3.2 and 3.3.

Finally, we introduce the concept of the Euclidean Davis–Wielandt radius. In fact, for an \( n \)-tuple \( S = (S_1, \ldots, S_n) \in \mathcal{B}(\mathcal{H})^n := \mathcal{B}(\mathcal{H}) \times \cdots \times \mathcal{B}(\mathcal{H}) \); i.e. for \( S_1, \ldots, S_n \in \mathcal{B}(\mathcal{H}) \), one of the most interesting generalization of the Davis–Wielandt radius \( dw(\cdot) \), is the Euclidean Davis–Wielandt radius, which is defined as:
\[
dw_e(S_1, \ldots, S_n) = \sup_{x \in \mathcal{H}, \|x\|=1} \left( \sum_{i=1}^{n} \left( |\langle S_i x, x \rangle|^2 + \|S_i x\|^4 \right) \right)^{1/2}.
\]

Indeed, a nice relation between the Euclidean operator radius (9) and the Euclidean Davis–Wielandt radius (36) can be constructed as follows:

For any positive integer \( n \), let \( T_i \in \mathcal{B}(\mathcal{H}) \) \((i = 1, \ldots, 2n)\). Therefore, we have
\[
w_e(T_1, \ldots, T_{2n}) := \sup_{\|x\|=1} \left( \sum_{i=1}^{2n} |\langle T_i x, x \rangle|^2 \right)^{1/2} \quad \text{for all } x \in \mathcal{H}.
\]

Let \( S_i \in \mathcal{B}(\mathcal{H}) \) \((i = 1, \ldots, n)\). Construct the following sequence of operators \( S_i \) in terms of \( T_i \), given as:
\[
T_1 = S_1, \quad \text{and} \quad T_2 = S_1^* S_1.
\]
Now, we have

\[ w_e (T_1, \ldots, T_{2n}) := \sup_{\|x\|=1} \left( \sum_{i=1}^{2n} |\langle T_i x, x \rangle|^2 \right)^{1/2} \]

\[ = \sup_{\|x\|=1} \left( \sum_{i=1}^{n} \left( |\langle S_i x, x \rangle|^2 + |\langle S_i^* S_i x, x \rangle|^2 \right) \right)^{1/2} \]

\[ = dw_e (S_1, \ldots, S_n). \]

which gives a very elegant relation between the Euclidean operator radius and the Euclidean Davis–Wielandt radius.

Now, from the definition of the Euclidean Davis–Wielandt radius (36), we have

\[ dw_e (S_1, \ldots, S_n) = \sup_{x \in \mathcal{H}} \left( \sum_{i=1}^{n} |\langle S_i x, x \rangle|^2 + \|S_i x\|^4 \right)^{1/2} \]

\[ \leq \left( \sup_{\|x\|=1} \sum_{i=1}^{n} |\langle S_i x, x \rangle|^2 + \sup_{\|x\|=1} \sum_{i=1}^{n} |\langle S_i^* S_i x, x \rangle|^2 \right)^{1/2} \]

\[ \leq \left( \sup_{\|x\|=1} \sum_{i=1}^{n} |\langle S_i x, x \rangle|^2 \right)^{1/2} \left( \sup_{\|x\|=1} \sum_{i=1}^{n} |\langle S_i^* S_i x, x \rangle|^2 \right)^{1/2} \]

\[ \leq w_e (S_1, \ldots, S_n) + w_e (|S_1|^2, \ldots, |S_n|^2). \]

Also, one can observe that

\[ dw_e (S_1, \ldots, S_n) \geq \max \left\{ w_e (S_1, \ldots, S_n), w_e (|S_1|^2, \ldots, |S_n|^2) \right\}. \]

Thus, we just proved the following result.

**Theorem 3.4:** Let \( S_i \in \mathcal{B}(\mathcal{H}) \) (\( i = 1, \ldots, n \)). Then,

\[ \max \left\{ w_e (S_1, \ldots, S_n), w_e (|S_1|^2, \ldots, |S_n|^2) \right\} \leq dw_e (S_1, \ldots, S_n) \]

\[ \leq w_e (S_1, \ldots, S_n) + w_e (|S_1|^2, \ldots, |S_n|^2). \]
One can generalize the results in Section 2 by following the same procedure above. As a direct result, from Lemma 2.5 and Theorem 2.2, one can easily observe that

\[
\frac{1}{4} \left\| \sum_{k=1}^{n} \left( |S_k|^2 + |S_k^*|^2 + 2|S_k|^4 \right) \right\| \leq d \omega_2^p (S_1, \ldots, S_n) \leq \frac{1}{2} \left\| \sum_{k=1}^{n} \left( |S_k|^2 + |S_k^*|^2 + 2|S_k|^4 \right) \right\|,
\]

by setting \( p = 1 \) in (12), taking into account the number of operators in (12) is \( 2n \) instead of \( n \) and the previous mentioned sequence of operators. We leave the rest of the other generalizations for the interested reader.

**Remark 3.2:** In Lemma 2.4, we have shown that \( \omega_e (S, S^* S) = d \omega (S) \). Using the same idea, we generalize the Davis–Wielandt radius using the generalized Euclidean operator radius \( \omega_p (\cdot, \cdot) \). Since we have

\[
\omega_p (T_1, \ldots, T_n) := \sup_{\|x\| = 1} \left( \sum_{i=1}^{n} |\langle T_i x, x \rangle|^p \right)^{1/p}, \quad p \geq 1. \tag{37}
\]

Therefore, by setting \( n = 2, T_1 = S \) and \( T_2 = S^* S \ (S \in \mathcal{B} (\mathcal{H})) \) in (37), we have

\[
\omega_p (S, S^* S) := \sup_{\|x\| = 1} \left( |\langle S x, x \rangle|^p + |\langle S^* S x, x \rangle|^p \right)^{1/p} = \sup_{\|x\| = 1} \left\{ \sqrt[p]{|\langle S x, x \rangle|^p + \|S x\|^{2p}} \right\} = d \omega_p (S)
\]

for all \( p \geq 1 \), and this is called the generalized Euclidean Davis–Wielandt radius of \( S \). Clearly, for \( p = 2 \) we refer to the well-known Davis–Wielandt radius, \( d \omega_2 (S) = d \omega (S) \).

As an immediate consequence of Theorem 2.2, one can easily observe that

\[
\frac{1}{2p} \left\| \frac{|S|^2 + |S^*|^2}{2} + |S|^4 \right\|^p \leq d \omega_2^{2p} (S) \leq \left( \frac{|S|^2 + |S^*|^2}{2} \right)^p + |S|^{4p}, \quad p \geq 1.
\]

In the end, one can use the presented inequalities in Refs [23–25], to obtain several bounds for \( d \omega_p (\cdot) \).

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