ON THE INDEX OF FAREY SEQUENCES

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Abstract. We prove some asymptotic formulae concerning the distribution of the index of Farey fractions of order $Q$ as $Q \to \infty$.

1. Introduction

Let $\mathcal{F}_Q = \{\gamma_1, \ldots, \gamma_{N(Q)}\}$ denote the Farey sequence of order $Q$ with $\gamma_1 = 1/Q < \gamma_2 < \cdots < \gamma_{N(Q)} = 1$. This sequence is extended by $\gamma_{i+N(Q)} = \gamma_{i+1}$, $1 \leq i \leq N(Q)$. It is well-known that

$$N(Q) = \sum_{j=1}^{Q} \varphi(j) = \frac{3Q^2}{\pi^2} + O(Q \log Q).$$

For any two consecutive Farey fractions $\gamma_i = a_i/q_i < \gamma_{i+1} = a_{i+1}/q_{i+1}$, one has $a_{i+1}q_i - a_iq_{i+1} = 1$ and $q_i + q_{i+1} > Q$. Conversely, if $q$ and $q'$ are two coprime integers in $\{1, \ldots, Q\}$ with $q + q' > Q$, then there are unique $a \in \{1, \ldots, q\}$ and $a' \in \{1, \ldots, q'\}$ for which $a'q - aq' = 1$, and $a/q < a'/q'$ are consecutive Farey fractions of order $Q$. Therefore, the pairs of coprime integers $(q, q')$ with $q + q' > Q$ are in one-to-one correspondence with the pairs of consecutive Farey fractions of order $Q$. Moreover, the denominator $q_{i+2}$ of $\gamma_{i+2}$ can be easily expressed (cf. [5]) by means of the denominators of $\gamma_i$ and $\gamma_{i+1}$ as

$$q_{i+2} = \left[ \frac{Q + q_i}{q_{i+1}} \right] q_{i+1} - q_i.$$

This formula was recently used in the study of various statistical properties of the Farey fractions [1], [2], [4], leading to the definition in [2] of a new and interesting area-preserving transformation $T$ of the Farey triangle

$$\mathcal{T} = \{(x, y) \in [0, 1]^2; x + y > 1\},$$

defined by

$$T(x, y) = \left( y, \left[ \frac{1+x}{y} \right] y - x \right) = \left( y, 1 - x - yG\left( \frac{y}{1+x} \right) \right),$$

where $G(t) = \{1/t\}$ is the Gauss map on the unit interval. The set $\mathcal{T}$ decomposes as the union of disjoint sets (see Figure 1)

$$\mathcal{T}_k = \left\{ (x, y) \in \mathcal{T}; \left[ \frac{1+x}{y} \right] = k \right\} \quad k \in \mathbb{N}^* = \{1, 2, 3 \ldots \}.$$
and

\[ T(x, y) = (y, ky - x), \quad (x, y) \in T_k. \]

**Figure 1.** The decomposition \( T = \bigcup_{k=1}^{\infty} T_k \).

For later use, we also define the upper triangles \( T'_k \) with vertices at \((1, 2/k), (1, 2/(k + 1))\), \(k \geq 2\), and the lower triangles \( T''_k \) with vertices at \((k/(k + 2), 2/(k + 2)), ((k - 1)/(k + 1), 2/(k + 1))\) and \((1, 2/(k + 1))\), \(k \geq 1\).

For all integer \( i \geq 0 \), we have

\[ T^i(x, y) = (L_i(x, y), L_{i+1}(x, y)), \]

with

\[
\begin{align*}
L_{i+1}(x, y) &= \kappa_i(x, y)L_i(x, y) - L_{i-1}(x, y), \\
L_0(x, y) &= x, \quad L_1(x, y) = y \\
\kappa_i(x, y) &= \kappa_{i-1} \circ T(x, y), \quad \kappa_1(x, y) = [(1 + x)/y].
\end{align*}
\]

It was noticed and extensively used in [2] that

\[ T \left( \frac{q}{Q}, \frac{q'}{Q} \right) = \left( \frac{q'}{Q}, \frac{q''}{Q} \right), \]

whenever \( q, q' \) and \( q'' \) are denominators of three consecutive Farey fractions from \( F_Q \). This shows immediately that

\[
(1.1) \quad \kappa_1 \left( \frac{q_{i-1}}{Q}, \frac{q_i}{Q} \right) = \left[ \frac{Q + q_{i-1}}{q_i} \right].
\]
and
\[
\kappa_{r+1} \left( \frac{q_{i-1}}{Q}, \frac{q_i}{Q} \right) = \kappa_1 \circ T^r \left( \frac{q_{i-1}}{Q}, \frac{q_i}{Q} \right) = \kappa_1 \left( \frac{q_{i+r-1}}{Q}, \frac{q_{i+r}}{Q} \right)
\]
(1.2)
\[
= \left\lfloor \frac{Q + q_{i+r-1}}{q_{i+r}} \right\rfloor, \quad r \in \mathbb{N}.
\]

We also note that
\[
(1.3)
\mathcal{T}_k = \{(x, y) \in \mathcal{T}; \kappa_1(x, y) = k\}.
\]

In [4], the integer
\[
(1.4) \nu_Q(\gamma_i) = \left\lfloor \frac{Q + q_{i-1}}{q_i} \right\rfloor = \kappa_1 \left( \frac{q_{i-1}}{Q}, \frac{q_i}{Q} \right) = \frac{q_{i+1} + q_{i-1}}{a_i} = \frac{a_{i+1} + a_{i-1}}{a_i}
\]
was called the index of the fraction $\gamma_i$ in $\mathcal{F}_Q$, and various new and interesting results concerning their distribution were proved, including the striking closed form formulae
\[
(1.5) \sum_i \nu_Q(\gamma_i) = 3N(Q) - 1
\]
and
\[
\sum_{q=1}^{Q} \# \left\{ \gamma_i = \frac{a_i}{q_i}; q_i = q \text{ and } \nu_Q(\gamma_i) = \left\lfloor \frac{2Q+1}{q_i} \right\rfloor - 1 \right\} = Q(2Q + 1) - N(2Q) - 2N(Q) + 1.
\]
The following asymptotic formulae were also proved in [4]:
\[
(1.6) \sum_i \nu_Q(\gamma_i)^2 = \frac{24}{\pi^2}Q^2 \left( \log 2Q - \frac{\zeta'(2)}{\zeta(2)} - \frac{17}{8} + 2\gamma \right) + O(Q \log^2 Q),
\]
\[
L(Q, k) = l_k N(Q) + O \left( k + \frac{Q \log Q}{k} \right),
\]
\[
U(Q, k) = u_k N(Q) + O \left( k + \frac{Q \log Q}{k} \right),
\]
where
\[
L(Q, k) = \# \left\{ \gamma_i; \nu_Q(\gamma_i) = k = \left\lfloor \frac{2Q+1}{q_i} \right\rfloor - 1 \right\},
\]
\[
U(Q, k) = \# \left\{ \gamma_i; \nu_Q(\gamma_i) = k = \left\lfloor \frac{2Q+1}{q_i} \right\rfloor \right\},
\]
and
\[
l_k = 4 \left( \frac{1}{(k+1)^2} - \frac{1}{k+1} + \frac{1}{k+2} \right),
\]
\[
u_k = \begin{cases} 0 & \text{if } k = 1, \\ 4 \left( \frac{1}{k} - \frac{1}{k+1} - \frac{1}{(k+1)^2} \right) & \text{if } k \geq 2. \end{cases}
\]
In an earlier version of [4], it was proved that
\[ \sum_{\gamma_i \leq t} \nu_Q(\gamma_i) = 3N(Q)t + O(Q^{3/2+\varepsilon}) \] whenever \( t \in [0, 1] \).

It was also conjectured that for every \( h \in \mathbb{N}^* \), there is a constant \( A(h) \) such that
\[ S_h(Q) = \sum_i \nu_Q(\gamma_i) \nu_Q(\gamma_i+h) \sim A(h)N(Q) \] as \( Q \to \infty \). In this note, we first prove that this conjecture holds, finding also the finite constant
\[ A(h) = 2 \int_\mathcal{T} \kappa_1(s, t) \kappa_{h+1}(s, t) \, ds \, dt = 2 \sum_{m,n=1}^\infty mn \area(T^hT_m \cap T_n). \]

All the numbers \( A(h) \) are rational and \( A(1) = \frac{192}{35} \approx 5.4857 \), \( A(2) = \frac{796727}{90090} \approx 8.8437 \). One can also show that \( A(h) \ll 1 + \log h \). It would be interesting to investigate whether \( A(h) \) is bounded by an absolute constant.

We also consider the more general situation where the Farey fractions belong to a subinterval of \([0, 1]\). More precisely, we set
\[ S_{h,t}(Q) = \sum_{\gamma_i \in \mathcal{F}_Q \cap [0,t]} \nu_Q(\gamma_i) \nu_Q(\gamma_i+h), \]
and prove

**Theorem 1.1.** (i) For every integer \( h \geq 1 \), we have
\[ S_h(Q) = A(h)N(Q) + O_h(Q \log^2 Q) \] as \( Q \to \infty \).
(ii) For every integer \( h \geq 1 \) and every \( t \in [0, 1] \), we have for all \( \varepsilon > 0 \),
\[ S_{h,t}(Q) = tA(h)N(Q) + O_{h,\varepsilon}(Q^{3/2+\varepsilon}). \]

For \( 0 < \alpha < 2 \), we define
\[ B_\alpha = \int_\mathcal{T} \left[ \frac{1+s}{t} \right]^\alpha \, ds \, dt = \sum_{k=1}^\infty k^\alpha \area(T_k) \ll \sum_{k=1}^\infty k^{\alpha-3} < \infty. \]

Employing the results from [2], we give the following generalization of (1.7):

**Theorem 1.2.** (i) For every \( \alpha \in (0, 2) \), we have
\[ \left| \sum_i \nu_Q(\gamma_i) - 2N(Q)B_\alpha \right| \ll_\alpha E_\alpha(Q) = \begin{cases} Q \log Q & \text{if } \alpha < 1, \\
Q \log^2 Q & \text{if } \alpha = 1, \\
Q^\alpha \log Q & \text{if } 1 < \alpha < 2. \end{cases} \]
(ii) For every \( \alpha \in (0, 3/2) \) and \( t \in [0, 1] \), we have for all \( \varepsilon > 0 \),
\[ \left| \sum_{\gamma_i \leq t} \nu(\gamma_i)^\alpha - 2tN(Q)B_\alpha \right| \ll_{\alpha,\varepsilon} F_\alpha(Q) = \begin{cases} Q^{3/2+\varepsilon} & \text{if } \alpha \leq 1, \\
Q^{\alpha+1/2+\varepsilon} & \text{if } \alpha > 1. \end{cases} \]
Note that
\[ B_1 = \sum_{k=1}^{\infty} k \text{area}(T_k) = \frac{1}{6} + \sum_{k=2}^{\infty} \frac{4k}{k(k+1)(k+2)} = \frac{3}{2}, \]
which is consistent with (1.3) and (1.4).

Finally, we show that (1.6) with error \( O(Q \log Q/k) \) can be derived from Lemma 2 in \([2]\). In our framework the geometrical significance of the constants \( l_k \) and \( u_k \) is apparent, as
\[ l_k = 2 \text{area}(T''_k) \quad \text{and} \quad u_k = 2 \text{area}(T'_k). \]

Furthermore, if we set for \( t \in (0, 1] \),
\[
L(Q, k, t) = \# \left\{ \gamma_i = \frac{a_i}{q_i} \leq t; \nu_Q(\gamma_i) = k = \left[ \frac{2Q + 1}{q_i} \right] - 1 \right\}
\]
and
\[
U(Q, k, t) = \# \left\{ \gamma_i = \frac{a_i}{q_i} \leq t; \nu_Q(\gamma_i) = k = \left[ \frac{2Q + 1}{q_i} \right] \right\},
\]
then we deduce as a result of Lemma 10 in \([2]\) the following generalization of (1.6):

**Theorem 1.3.** For every \( t \in (0, 1] \) and every \( \varepsilon > 0 \), we have
\[
L(Q, k, t) = tl_k N(Q) + O_{\varepsilon} \left( \frac{Q^{3/2+\varepsilon}}{k} \right)
\]
and
\[
U(Q, k, t) = tu_k N(Q) + O_{\varepsilon} \left( \frac{Q^{3/2+\varepsilon}}{k} \right).
\]

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2. **Proof of the main results**

We denote throughout
\[ \mathbb{Z}^2_{\text{vis}} = \{(a, b) \in \mathbb{Z}^2; \gcd(a, b) = 1\}. \]

By \([1, 2]\), we get
\[ \nu_Q(\gamma_i+h) = \left[ \frac{Q + q_i + h - 1}{q_i + h} \right] = \kappa_{h+1} \left( \frac{q_i - 1}{Q}, \frac{q_i}{Q} \right), \]
leading to
\[ S_h(Q) = \sum_{\gamma_i \in F_Q} \kappa_1 \left( \frac{q_i - 1}{Q}, \frac{q_i}{Q} \right) \kappa_{h+1} \left( \frac{q_i - 1}{Q}, \frac{q_i}{Q} \right). \]
Since the pairs of denominators of consecutive elements in \( F \) coincide with the elements of the set
\[
Q T \cap Z^2_{vis} = \{(a, b) \in Z^2_{vis} ; a + b > Q \geq a, b \geq 1\},
\]
we may write
\[
S_h(Q) = \sum_{k=1}^{\infty} \sum_{(a,b) \in QT \cap Z^2_{vis}} \kappa_1\left(\frac{a}{Q}, \frac{b}{Q}\right) \kappa_{h+1}\left(\frac{a}{Q}, \frac{b}{Q}\right).
\]
Taking also stock on (1.3), this further yields
\[
S_h(Q) = \sum_{k=1}^{\infty} \sum_{(a,b) \in QT_k \cap Z^2_{vis}} \kappa_{h+1}\left(\frac{a}{Q}, \frac{b}{Q}\right)
\]
\[
= \sum_{k=1}^{\infty} k \sum_{l=1}^{\infty} l \# \left\{(a, b) \in QT_k \cap Z^2_{vis} ; \kappa_1 T^h\left(\frac{a}{Q}, \frac{b}{Q}\right) = l \right\}
\]
\[
= \sum_{k,l=1}^{\infty} kl \# \left\{(a, b) \in QT_k \cap Z^2_{vis} ; T^h\left(\frac{a}{Q}, \frac{b}{Q}\right) \in T_l \right\}
\]
\[
= \sum_{k,l=1}^{\infty} kl \#(Q(T_k \cap T^{-h}T_l) \cap Z^2_{vis})).
\]
If we set
\[
T^*_k = \bigcup_{n=k}^{\infty} T_n,
\]
then
\[
(2.1) \quad T^*_k \cap T^{-h}T^*_l = \left(\bigcup_{m=k}^{\infty} T_m\right) \cap T^{-h}\left(\bigcup_{n=l}^{\infty} T_n\right) = \bigcup_{m=k}^{\infty} \bigcup_{n=l}^{\infty} (T_m \cap T^{-h}T_n).
\]
We wish to estimate
\[
A_{k,l}(Q) = \#(Q(T^*_k \cap T^{-h}T_l) \cap Z^2_{vis})).
\]
Since the sets from the right-hand side of (2.2) are mutually disjoint, we now have
\[
\sum_{k,l=1}^{\infty} A_{k,l}(Q) = \sum_{k,l=1}^{\infty} \sum_{m=k}^{\infty} \sum_{n=l}^{\infty} \#(Q(T_m \cap T^{-h}T_n) \cap Z^2_{vis})).
\]
\[
= \sum_{m,n=1}^{\infty} \#(Q(T_m \cap T^{-h}T_n) \cap Z^2_{vis})) \sum_{k=1}^{m} \sum_{l=1}^{n} 1
\]
\[
= \sum_{m,n=1}^{\infty} mn \#(Q(T_m \cap T^{-h}T_n) \cap Z^2_{vis})).
\]
so that, by (2.1),

$$S_h(Q) = \sum_{k,l=1}^{\infty} A_{k,l}(Q).$$

(2.3)

The following lemma is proved in a similar way as Lemma 2 in [2].

**Lemma 2.1.** Let \( \Omega \subset [0, R_1] \times [0, R_2] \) be a region in \( \mathbb{R}^2 \) with rectifiable boundary \( \partial \Omega \) and let \( g: \Omega \to \mathbb{R} \) be a \( C^1 \) function on \( \Omega \). Suppose that \( R \geq \min(R_1, R_2) \). Then we have

$$\sum_{(a,b) \in \Omega \cap \mathbb{Z}^2_{vis}} g(a, b) = \frac{6}{\pi^2} \int \int_{\Omega} g(x, y) \, dx \, dy + O\left( \|Dg\|_\infty \text{area}(\Omega) \log R \right) + O\left( \|g\|_\infty \left( R + \frac{\text{area}(\Omega)}{R} + \text{length}(\partial \Omega) \log R \right) \right),$$

where

$$\|Dg\|_\infty = \sup_{(x,y) \in \Omega} \left| \frac{\partial g}{\partial x}(x, y) \right| + \left| \frac{\partial g}{\partial y}(x, y) \right|.$$

**Corollary 2.2.** Let \( \Omega \subseteq [0, R_1] \times [0, R_2] \) be a bounded region with rectifiable boundary \( \partial \Omega \) and let \( R \geq \min(R_1, R_2) \). Then we have

$$\#(\Omega \cap \mathbb{Z}^2_{vis}) = \frac{6 \text{area}(\Omega)}{\pi^2} + O\left( R + \text{length}(\partial \Omega) \log R + \frac{\text{area}(\Omega)}{R} \right).$$

**Remark 2.3.** Let \( (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(\mathbb{Z}) \), let \( \Phi \) be the linear transformation on \( \mathbb{R}^2 \) defined by \( \Phi(x, y) = (ax + by, cx + dy) \), and let \( \Omega \) be a bounded region in \( \mathbb{R}^2 \). Then \( \#(\Omega \cap \mathbb{Z}^2_{vis}) = \#(\Phi \Omega \cap \mathbb{Z}^2_{vis}) \), which implies in turn that

$$\#(Q \Omega \cap \mathbb{Z}^2_{vis}) = \#(Q(T^k \Omega) \cap \mathbb{Z}^2_{vis}) = \cdots = \#(Q(T^h \Omega) \cap \mathbb{Z}^2_{vis})$$

whenever \( \Omega \subseteq T \). In particular, this gives

$$\#(Q(T^k \cap T^{-h} T^* l) \cap \mathbb{Z}^2_{vis}) = \#(Q(T^h \cap T^* l) \cap \mathbb{Z}^2_{vis}).$$

(2.4)

**Lemma 2.4.** [2, Lemma 5] Let \( r \geq 1 \) and suppose that \( i \neq j \) and that \( \max(L_i(x, y), L_j(x, y)) \leq 2^{-r-1} \) for some point \( (x, y) \in T \). Then \( |j - i| > r + 1 \).

**Corollary 2.5.** Assume that \( \min(k, l) > c_h = 2^{h+1} \). Then we have

$$T^*_k \cap T^{-h} T^*_l = \emptyset.$$

**Proof.** Suppose that there exists \((x, y) \in T^*_k \cap T^{-h} T^*_l\). Then \( L_1(x, y) = y \leq 2/k < 2^{-h} \) and \( L_1(T^h(x, y)) = L_{h+1}(x, y) \leq 2/l < 2^{-h} \). We now infer from Lemma 2.4 that \( |h + 1 - 1| > h \), which is a contradiction. \( \square \)
Remark 2.6. A careful look at the proof of Lemma 2.4 shows that the constant $2^{r+1}$ can be lowered to $4r+2$ (see Lemma 3.4 and Remark 3.5 in [3]). Note also that $T_m^{±1} \subseteq T_2$ for all $m \geq 5$. By Lemma 3.4 in [3] it also follows that for all $r \geq 2$ and all $m \geq c_r = 4r+2$, we have $\cup_{i=2}^{r} T_m^{±1} \subseteq T_2$.

In summary, we have

$$ T^h_m \cap T_n^* = \emptyset $$

whenever $m \geq c_h$ and $n \geq 3$.

Since $Q T_k^* \cap Z_{vis} = \emptyset$ whenever $2Q/k < 1$, Remark 2.3 shows that $A_{k,l}(Q) = 0$ unless $\text{max}(k,l) \leq 2Q$. By Corollary 2.5, $A_{k,l}(Q) = 0$ unless $\text{min}(k,l) \leq c_h$. Thus (2.3) provides

$$ S_h(Q) = \sum_{k,l=1}^{2Q} A_{k,l}(Q) = C_h(Q) + D_h(Q), $$

where

$$ C_h(Q) = \sum_{k=c_h+1}^{2Q} \sum_{l=1}^{c_h} A_{k,l}(Q) \quad \text{and} \quad D_h(Q) = \sum_{k=1}^{c_h} \sum_{l=1}^{2Q} A_{k,l}(Q). $$

Suppose first that $k > c_h$. Applying Corollary 2.2 to $\Omega = Q(T_k^* \cap T_{-h}^*)$ with $\text{area}(\Omega) \ll Q^2/k^2$, and with

$$ \text{length}(\partial \Omega) = Q \text{length} \left( \bigcup_{r=l}^{\infty} T_{-h}^r \cap T_k^* \right) = Q \text{length} \left( \bigcup_{r=l}^{c_h} T_{-h}^r \cap T_k^* \right) $$

$$ \leq Q \sum_{r=l}^{c_h} \text{length}(T_{-h}^r \cap T_k^*) \ll h \frac{Q}{k}, $$

we infer that

$$ A_{k,l}(Q) = \frac{6Q^2}{\pi^2} \text{area}(T_k^* \cap T_{-h}^*) + O_h \left( \frac{Q \log Q}{k} \right). $$

By (2.6) and $\text{area}(T_k^*) = O(k^{-2})$ we gather

$$ C_h(Q) = \frac{6Q^2}{\pi^2} \sum_{k=c_h+1}^{2Q} \sum_{l=1}^{c_h} \text{area}(T_k^* \cap T_{-h}^*) + O_h(Q \log^2 Q) $$

$$ = \frac{6Q^2}{\pi^2} \sum_{k=c_h+1}^{\infty} \sum_{l=1}^{\infty} \text{area}(T_k^* \cap T_{-h}^*) + O_h(Q \log^2 Q). $$

When $k \leq c_h$, we employ Remark 2.3 and equality (2.4) to get

$$ A_{k,l}(Q) = \#(Q(T_l^* \cap T_{-h}^*)). $$
Applying now Corollary 2.2 to $\Omega = Q(T^*_l \cap T^h T^*_k)$ with length$(\partial \Omega) \ll h/Q/l$, and using the fact that $T$ is area-preserving, we obtain

$$A_{k,l}(Q) = \frac{6}{\pi^2} \text{area}(T^*_l \cap T^h T^*_k) + O_h\left(\frac{Q \log Q}{l}\right)$$

(2.8)

$$= \frac{6}{\pi^2} \text{area}(T^*_k \cap T^{-h} T^*_l) + O_h\left(\frac{Q \log Q}{l}\right).$$

We may now employ (2.7) and $\text{area}(T^*_l) = O(l^{-2})$, to get

$$D_h(Q) = \frac{6}{\pi^2} c h \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \text{area}(T^*_k \cap T^{-h} T^*_l) + O(h(Q \log^2 Q)).$$

(2.9)

Inserting (2.9) and (2.7) into (2.5), we collect

$$S_h(Q) = \frac{6}{\pi^2} \sum_{k,l=1}^{\infty} \text{area}(T^*_k \cap T^{-h} T^*_l) + O(h(Q \log^2 Q)).$$

(2.10)

We conclude the proof of Theorem 1.1 (i) by noting as in (2.3) that the sum in the right-hand side of (2.10) is equal to

$$\sum_{m,n=1}^{\infty} \text{area}(T^h T^*_m \cap T^*_n) = \sum_{m,n=1}^{\infty} mn \text{area}(T^h T^*_m \cap T^*_n) = \frac{A(h)}{2}.$$

(2.11)

We set

$$f = f_0 + R_0 \quad \text{and} \quad g = f \circ T^{-h} = f_0 \circ T^{-h} + R_0 \circ T^{-h},$$

with

$$f_0 = \sum_{m=1}^{c_h-1} e_{T^*_m}, \quad R_0 = \sum_{m=c_h}^{\infty} e_{T^*_m},$$

where $e_S$ denotes the characteristic function of the set $S$. By Remark 2.6 we gather that the product of $R_0$ and $g$ equals

$$\sum_{m=c_h}^{\infty} \sum_{n=1}^{\infty} e_{T^h T^*_m \cap T^*_n} = \sum_{m=c_h}^{\infty} \sum_{n=1}^{2} e_{T^h T^*_m \cap T^*_n^*}.$$

Hence,

$$\int_{\mathcal{T}} R_0(s, t)g(s, t) \, ds \, dt \leq 2 \sum_{m=c_h}^{\infty} \text{area}(T^*_m) \ll \sum_{m=1}^{\infty} \frac{1}{m^2} \ll 1.$$

In a similar way, we get

$$\int_{\mathcal{T}} f(s, t)(R_0 \circ T^{-h})(s, t) \, ds \, dt \ll 1.$$
Using (2.11) we may now write

\[ A(h) = 2 \int T f(s, t) (f \circ T^{-h})(s, t) \, ds \, dt \]

(2.12)

\[ \ll 1 + 2 \int T f_0(s, t) g_0(s, t) \, ds \, dt. \]

The function \( f_0 \) belongs to \( L^2(T) \) as a result of

\[ f_0 = \sum_{m=1}^{c_h-1} \sum_{k=m}^{\infty} e_{T_k} = \sum_{m=1}^{\infty} \sum_{k=1}^{\min(m, c_h-1)} e_{T_k} = \sum_{k=1}^{\infty} \min(k, c_h-1) e_{T_k} \]

which gives in turn

\[ \|f_0\|_{L^2(T)} = \sum_{k=1}^{c_h-1} k^2 \text{area}(T_k) + (c_h - 1)^2 \text{area}(T_{c_h}^s) \ll \sum_{k=1}^{c_h} \frac{1}{k} + 1 \ll 1 + \log h. \]

Since \( T \) is area-preserving, we also have

\[ \|g_0\|_{L^2(T)} = \|f_0 \circ T^{-h}\|_{L^2(T)} = \|f_0\|_{L^2(T)}, \]

and the Cauchy-Schwarz inequality gives in conjunction with (2.12)

\[ A(h) \ll 1 + 2 \|f_0\|_{L^2(T)} \|g_0\|_{L^2(T)} \ll 1 + \log h. \]

To prove (ii), we proceed as in Section 8 in [2] and note that the relation \( a'q - a q' = 1 \) between two consecutive Farey fractions of order \( Q \) shows that \( a = q - q' \), where \( q' \in \{1, \ldots, q\} \) denotes the multiplicative inverse of \( q' \) (mod \( q \)). Thus the condition \( a/q \in I \) with \( I = (\alpha, \beta] \subseteq (0, 1] \) interval, is equivalent to \( q' \in I_q = [(1 - \beta)q, (1 - \alpha)q) \). As a result, we may write

\[ S_{h,t}(Q) = \sum_{(a, b) \in \Omega} \kappa_1 \left( \frac{a}{Q \cdot Q} \right) \kappa_{h+1} \left( \frac{a}{Q \cdot Q} \right) = \sum_{k, l=1}^{\infty} \kappa_1 \left( \frac{a}{Q \cdot Q} \right) \kappa_{h+1} \left( \frac{a}{Q \cdot Q} \right), \]

with

\[ A_{k,l,t}(Q) = \#(Q(T_{k}^s \cap T^{-h}T_{l}^s) \cap \{(a, b) ; \ b \in I_a \}), \]

where \( I = (0, t] \). Applying Lemma 10 in [2] to \( f(a, b) = 1, \ \Omega = Q(T_{k}^s \cap T^{-h}T_{l}^s), \ A = O(Q), \ R_1 = R_2 = O(Q/k), \) and employing (2.6) and (2.8), we infer that

\[ A_{k,l,t}(Q) = t A_{k,l}(Q) + O \left( \frac{Q}{k^2} + \frac{Q^{3/2+\varepsilon}}{k} \right) \]

\[ = \frac{6tQ^2}{\pi^2} \text{area}(T_{k}^s \cap T^{-h}T_{l}^s) + O \left( \frac{Q^{3/2+\varepsilon}}{k} \right), \]
leading to

$$S_{h,t}(Q) = \frac{3tA(h)Q^2}{\pi^2} + O(Q^{3/2+\varepsilon} \log Q).$$

\[ \square \]

**Proof of Theorem 1.2.** This is similar to the proof of Theorem 1.1. We get as in (2.3)

$$T_\alpha(Q) := \sum_i \nu_Q(\gamma_i)^\alpha = \sum_{k=1}^\infty k^\alpha \#(Q\mathcal{T}_k \cap \mathbb{Z}^2_{\text{vis}})$$

(2.13)

$$= \sum_{k=1}^\infty \frac{\#(Q\mathcal{T}_k \cap \mathbb{Z}^2_{\text{vis}})}{k} \sum_{m=1}^k (m^\alpha - (m-1)^\alpha)$$

$$= \sum_{m=1}^\infty (m^\alpha - (m-1)^\alpha) \sum_{k=m}^\infty \#(Q\mathcal{T}_k \cap \mathbb{Z}^2_{\text{vis}})$$

$$= \sum_{m=1}^{2Q} (m^\alpha - (m-1)^\alpha) \#(Q\mathcal{T}_m^* \cap \mathbb{Z}^2_{\text{vis}}).$$

However, Corollary 2.2 gives

$$\#(Q\mathcal{T}_m^* \cap \mathbb{Z}^2_{\text{vis}}) = \frac{6Q^2}{\pi^2} \text{area}(\mathcal{T}_m^*) + O\left(\frac{Q \log Q}{m}\right),$$

which we insert into (2.13) to get

$$T_\alpha(Q) = \frac{6Q^2}{\pi^2} \sum_{m=1}^{2Q} (m^\alpha - (m-1)^\alpha) \text{area}(\mathcal{T}_m^*) + O\left(E_\alpha(\mathcal{T}_m^*)\right),$$

with $E_\alpha(\mathcal{T}_m^*)$ as in Theorem 1.2.

However, $Q^2 \sum_{m=Q}^\infty m^{\alpha-1} \text{area}(\mathcal{T}_m^*) \ll Q^2 \sum_{m=Q}^\infty m^{\alpha-3} \ll Q^\alpha$. Hence,

$$T_\alpha(Q) = \frac{6Q^2}{\pi^2} \sum_{m=1}^\infty (m^\alpha - (m-1)^\alpha) \text{area}(\mathcal{T}_m^*) + O\left(E_\alpha(\mathcal{T}_m^*)\right).$$

Finally, Theorem 1.2 (i) follows from (2.14), and from

$$\sum_{m=1}^\infty (m^\alpha - (m-1)^\alpha) \text{area}(\mathcal{T}_m^*)$$

(2.15)

$$= \sum_{m=1}^\infty (m^\alpha - (m-1)^\alpha) \sum_{k=m}^\infty \text{area}(\mathcal{T}_k)$$

$$= \sum_{k=1}^\infty \text{area}(\mathcal{T}_k) \sum_{m=1}^k (m^\alpha - (m-1)^\alpha) = \sum_{k=1}^\infty k^\alpha \text{area}(\mathcal{T}_k)$$

$$= \iint \left[ \frac{1+s}{t} \right]^\alpha ds \, dt.$$
To prove Theorem 1.2 (ii), we first write as in (2.13)

\[ T_{\alpha,t}(Q) := \sum_{\gamma_i \leq t} \nu_Q(\gamma_i^\alpha) = \sum_{m=1}^{2Q} (m^\alpha - (m - 1)^\alpha) \sum_{(a,b) \in QT_m^* \atop b \in I_a} 1, \]

where \( I = (0,t] \) and \( I_a = [(1-t)a,a) \). Applying Lemma 10 in [2] to \( f(a,b) = 1 \) and \( \Omega = QT_m^* \), we find

\[ \sum_{(a,b) \in QT_m^* \atop b \in I_a} 1 = \frac{6tQ^2}{\pi^2} \text{area}(T_m^*) + O\left(\frac{Q^{3/2+\epsilon}}{m}\right), \]

which gives in conjunction with (2.16) and (2.15)

\[ T_{\alpha,t}(Q) = \frac{6tQ^2}{\pi^2} \sum_{m=1}^{2Q} (m^\alpha - (m - 1)^\alpha) \text{area}(T_m^*) + O\left(\frac{Q^{3/2+\epsilon}}{m}\right). \]

The equalities

\[ \nu_Q(\gamma_i) = \left[ \frac{Q + q_i - 1}{q_i} \right] = k = \left[ \frac{2Q + 1}{q_i} \right] - 1 \]

read as

\[(q_i-1,q_i) \in QT_k^L \cap \mathbb{Z}_2^2,\]

where

\[ T_k^L = \left\{ (s,t) \in T_k^L : \frac{2 + 1/Q}{k+2} < t \leq \frac{2 + 1/Q}{k+1} \right\}. \]

Hence \( L(Q,k) = \#(QT_k^L \cap \mathbb{Z}_2^2) \) and by means of Corollary 2.2 we find

\[ L(Q,k) = \frac{6Q^2}{\pi^2} \text{area}(T_k^L) + O\left(\frac{Q}{k} + \frac{Q \log Q}{k} + \frac{Q}{k^2}\right) \]

\[ = \frac{6Q^2}{\pi^2} \text{area}(T_k^\prime) + O\left(\frac{Q \log Q}{k}\right). \]

The estimate on \( U(Q,k) \) in (1.6) follows in a similar way. Theorem 1.3 is derived from Lemma 10 in [2] in a similar way as above.
3. Some numerical computations

The transformation $T$ maps each region $T_k$ onto its symmetric with respect to the first bisector. That is,

$$TT_k = ST_k \quad \text{and} \quad TT^*_k = ST^*_k,$$

where $S$ acts on $T$ as

$$S(x, y) = (y, x).$$

Moreover, we notice that the inverse $T^{-1}$ of $T$ can be expressed as

$$T^{-1} = STS.$$

We now infer from (3.2), (3.1), and the fact that $T$ is area-preserving that

$$\text{area}(TT^*_m \cap T^*_n) = \text{area}(T^*_m \cap ST^*_n) = \text{area}(ST^*_m \cap T^*_n) = \text{area}(T^*_m \cap T^* T^*_n) = \text{area}(T^*_m \cap T^* T^*_n),$$

(3.3)

Thus, to evaluate $A(h)$ via (2.11), it suffices to consider $m \leq n$ only. We find the following table for the value of area($TT^*_m \cap T^*_n$)

| $m$  | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ | $n = 7$ | $n = 8$ | $n \geq 9$ |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|-----------|
| 1   | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{5}$ | $\frac{1}{10}$ | $\frac{1}{n(n+1)}$ | $\frac{1}{n(n+1)}$ | $\frac{1}{n(n+1)}$ | $\frac{1}{n(n+1)}$ | $\frac{1}{n(n+1)}$ |
| 2   | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{30}$ | $\frac{1}{70}$ | 0       | 0       | 0       | 0       | 0         |
| 3   | $\frac{1}{5}$ | $\frac{1}{10}$ | 0       | 0       | 0       | 0       | 0       | 0       | 0         |
| 4   | $\frac{1}{6}$ | $\frac{1}{15}$ | $\frac{1}{35}$ | $\frac{1}{70}$ | 0       | 0       | 0       | 0       | 0         |
| $\geq 5$ | $m(m+1)/10$ | $m(m+1)/10$ | $m(m+1)/10$ | $m(m+1)/10$ | 0       | 0       | 0       | 0       | 0         |

which gives in turn

$$A(1) = 2 \cdot \frac{96}{35} = \frac{192}{35} \approx 5.4857.$$

We also find the following table for the value of area($T^2T^*_m \cap T^*_n$), which is symmetric as a result of (3.3).

| $m$  | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ | $n = 7$ | $n = 8$ | $n \geq 9$ |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|-----------|
| 1   | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{5}$ | $\frac{1}{10}$ | $\frac{1}{n(n+1)}$ | $\frac{1}{n(n+1)}$ | $\frac{1}{n(n+1)}$ | $\frac{1}{n(n+1)}$ | $\frac{1}{n(n+1)}$ |
| 2   | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{30}$ | $\frac{1}{70}$ | 0       | 0       | 0       | 0       | 0         |
| 3   | $\frac{1}{5}$ | $\frac{1}{10}$ | 0       | 0       | 0       | 0       | 0       | 0       | 0         |
| 4   | $\frac{1}{6}$ | $\frac{1}{15}$ | $\frac{1}{35}$ | $\frac{1}{70}$ | 0       | 0       | 0       | 0       | 0         |
| 5   | $\frac{1}{7}$ | $\frac{1}{21}$ | $\frac{1}{42}$ | $\frac{1}{84}$ | 0       | 0       | 0       | 0       | 0         |
| 6   | $\frac{1}{8}$ | $\frac{1}{28}$ | $\frac{1}{56}$ | $\frac{1}{112}$ | 0       | 0       | 0       | 0       | 0         |
| 7   | $\frac{1}{9}$ | $\frac{1}{36}$ | $\frac{1}{72}$ | $\frac{1}{144}$ | 0       | 0       | 0       | 0       | 0         |
| $\geq 9$ | $m(m+1)/10$ | $m(m+1)/10$ | $m(m+1)/10$ | $m(m+1)/10$ | 0       | 0       | 0       | 0       | 0         |

and collect

$$A(2) = 2 \cdot \frac{796727}{180180} = \frac{796727}{90090} \approx 8.8437.$$
Using the second part of Remark 2.6 and the fact that $T$ is area-preserving, we may write for $h \geq 2$,

$$\frac{A(h)}{2} = \sum_{m,n=1}^{c_h-1} mn \text{area}(T^h T_m \cap T_n) + 4 \sum_{n=c_h}^{\infty} n \text{area}(T_n)$$

$$= \sum_{m,n=1}^{c_h-1} mn \text{area}(T^h T_m \cap T_n) + \frac{2}{(c_h + 1)(c_h + 2)}.$$

However, each region $T^r T_m$ is a finite union of triangles with rational numbers as vertex coordinates. Thus $A(h)$ is a rational number for any $h \in \mathbb{N}^*$. 

References

1. V. AUGUSTIN, F. P. BOCA, C. COBELI, A. ZAHARESCU, *The $h$-spacing distribution between Farey points*, Math. Proc. Camb. Phil. Soc. 131 (2001), 23–38.
2. F. P. BOCA, C. COBELI, A. ZAHARESCU, *A conjecture of R. R. Hall on Farey points*, J. Reine Angew. Mathematik 535 (2001), 207–236.
3. F. P. BOCA, C. COBELI, A. ZAHARESCU, *On the distribution of the Farey sequence with odd denominators*, Michigan Math. J. 51 (2003), 557–573.
4. R. R. HALL, P. SHIU, *The index of a Farey sequence*, Michigan Math. J. 51 (2003), 209–223.
5. R. R. HALL, G. TENENBAUM, *On consecutive Farey arcs*, Acta Arith. 44 (1984), 397–405.

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