A Family of Complex Kleinian Split Solvable Groups

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Abstract
In this article, using techniques of Lie groups and dynamical systems, it is shown that
lattices of a family of split solvable subgroups of \( \text{PSL}(N+1, \mathbb{C}) \) are complex Kleinian.
Also, it is shown that there exists a minimal limit set for the action of these lattices
on the complex projective space and that there are exactly two maximal discontinuity
regions.

1 Introduction and Main Results
The complex Kleinian groups were introduced by Seade and Verjovsky (2001, 2002)
as a generalization of the Kleinian groups first studied by Poincaré. A complex Kleinian
group is a discrete subgroup of \( \text{PSL}(N+1, \mathbb{C}) \) which acts properly and discontinuously
on some non empty open subset of the complex projective space \( \mathbb{P}^N_{\mathbb{C}} \). To decide whether
a subgroup of \( \text{PSL}(N+1, \mathbb{C}) \) is complex Kleinian is a difficult but important task. There
have been some advances in complex dimension \( N = 2 \), for example, in his Ph.D.
thesis and in the articles (Navarrete 2006, 2008), Navarrete uses the Kulkarni limit
set \( \Lambda_{\text{kul}} \) (Kulkarni 1978) in order to find in a canonical way a region of the complex
projective plane where a given discrete subgroup of \( \text{PSL}(3, \mathbb{C}) \) acts properly and
 discontinuously. Moreover, in \( \mathbb{P}^2_{\mathbb{C}} \) there are many theorems analogous to the classical
theory in \( \mathbb{P}^1_{\mathbb{C}} \), for example in Navarrete (2008) a classification for the dynamics of the
action of cyclic subgroups of \( \text{PSL}(3, \mathbb{C}) \) is established. However even in dimension
\( N = 2 \) there are differences with the classical theory of Kleinian groups because there
are several non equivalent notions of limit sets (cf. Barrera et al. 2018; Barrera Vargas
et al. 2011). Note also that every discontinuity region of an infinite complex Kleinian group contains a complex projective line (Cano et al. 2013) while in the classical theory there are groups with only finitely many points in the complement of the discontinuity region. In dimension $N > 2$ there are a lot of difficulties that prevent us from applying standard techniques to determine whether a given discrete subgroup of $\text{PSL}(N+1, \mathbb{C})$ is complex Kleinian. For example the Kulkarni limit set, one of the most important tools in the analysis of dimension $N = 2$, is hard to compute (Cano et al. 2017a, b, c).

In this article we aim to propose a new approach in order to construct a discontinuity region for a family of discrete subgroups of $\text{PSL}(N+1, \mathbb{C})$. Inspired by Mosak and Moskowitz (1997) and Barrera Vargues et al. (2018), we adapt techniques stemming from dynamical systems and differential geometry to prove the following theorems.

**Theorem 1** Let $A : \mathbb{R} \to \text{GL}(N, \mathbb{R})$ be a smooth, faithful and closed representation such that each $A(t)$ has no eigenvalue of unit length and let $\rho : \mathbb{R}^{N} \rtimes_{A} \mathbb{R} \to \text{GL}(N+1, \mathbb{C})$ be the representation, 

$$\rho(b, t) = \begin{pmatrix} A(t) & b \\ 0 & 1 \end{pmatrix}.$$  

Assume $G \subset \text{PGL}(N+1, \mathbb{C})$ admits a lift $\tilde{G} \subset \text{GL}(N+1, \mathbb{C})$ conjugate in $\text{GL}(N+1, \mathbb{C})$ to $\rho(\mathbb{R}^{N} \rtimes_{A} \mathbb{R})$. Let $N_{1}$ ($N_{2}$) be the cardinality of the set of eigenvalues counted with multiplicity of the matrix $A(1)$ inside (outside) the unit disk, then the following holds:

1. If $N_{i} > 0$, there is a $G$-invariant open set $\Omega_{i} \subset \mathbb{P}_{\mathbb{C}}^{N}$ where the action is equivariant to the action on the product $G \times X$ given by $(h, g, x) \mapsto (hg, x)$ and $X$ is diffeomorphic to the product of an $N_{i} - 1$ sphere and an Euclidean space.

2. If $\Omega \subset \mathbb{P}_{\mathbb{C}}^{N}$ is a $G$-invariant open set where the action is proper, then $\Omega \subset \Omega_{i}$ for some of the sets defined above.

3. If $G$ admits a lattice $\Gamma$ then $\rho$ is a representation into $\text{SL}(N, \mathbb{R})$. Further, there are exactly two open sets $\Omega_{1}$, $\Omega_{2}$ and the complement $\Lambda = \mathbb{P}_{\mathbb{C}}^{N} \setminus (\Omega_{1} \cup \Omega_{2})$ is a closed, $G$-invariant set such that:

   a) The set of points of $\Lambda$ with infinite isotropy is dense;

   b) For almost every $z \in \Lambda \cap \mathbb{P}_{\mathbb{R}}^{N}$ the set of accumulation points of the orbit $\Gamma z$ is dense in $\Lambda \cap \mathbb{P}_{\mathbb{R}}^{N}$.

As a consequence of Theorem 1, the action on $\Omega_{i}$ is proper and free provided this set is not empty and if both $N_{1}$ and $N_{2}$ are positive, the sets $\Omega_{i}$, $i = 1, 2$ are not disjoint. Moreover, we have the following corollary.

**Corollary 2** If $G \subset \text{PSL}(N+1, \mathbb{C})$ is a Lie group satisfying the conditions of Theorem 1 which admits a lattice $\Gamma$, then each open set $\Omega_{i}$ is a maximal invariant open set where $\Gamma$ acts properly and discontinuously and the quotient space $\Gamma \setminus \Omega_{i}$ is a smooth manifold diffeomorphic to $\Gamma \setminus G \times X$ for a space $X$ diffeomorphic to the product of an sphere and an Euclidean space.

Hence a lattice of a Lie group as described in Theorem 1 is complex Kleinian. Theorem 1 and Corollary 2 are important because they give general conditions to
construct complex Kleinian groups independently of the dimension of the projective space. Moreover, they describe the quotient spaces, a task traditionally difficult to achieve. They also show that the complement $\Lambda$ of the discontinuity regions $\Omega_i$ are minimal sets where lattices of the group have rich dynamics in the sense described in part (3) of Theorem 1. We call $\Lambda$ the limit set of the group since the complement is a union of open sets where the action of lattices of $G$ is properly discontinuous. The second main result is the following theorem, which in a sense is a partial converse of Theorem 1. Recall that hyperbolic toral automorphism is given by a matrix in $\text{SL}(N, \mathbb{Z})$ with no eigenvalue of unit length and that any such matrix $B$ induces a representation $\mathbb{Z} \rightarrow \text{SL}(N, \mathbb{Z})$, such that $k \mapsto B^k$.

**Theorem 3** Let $B \in \text{SL}(N, \mathbb{R})$ be a hyperbolic toral automorphism and let $\rho : \mathbb{Z}^N \rtimes_B \mathbb{Z} \rightarrow \text{SL}(N + 1, \mathbb{Z})$ be the representation

$$
\rho(b, k) = \begin{pmatrix} B^k b \\ 0 & 1 \end{pmatrix}.
$$

If $\Gamma \subset \text{PSL}(N+1, \mathbb{C})$ is a discrete subgroup that admits a lift conjugate to $\rho(\mathbb{Z}^N \rtimes_B \mathbb{Z})$, in $\text{SL}(N + 1, \mathbb{C})$ then there are exactly two invariant open sets $\Omega_i \subset \mathbb{P}_C^N$, $i = 1, 2$ such that

1. The action on each $\Omega_i$ is free, properly discontinuous and it is maximal in the sense that if $\Omega \subset \mathbb{P}_C^N$ is another $\Gamma$-invariant open set where the group acts properly and discontinuously, then $\Omega \subset \Omega_i$ for some $i = 1, 2$.
2. There is a Lie group $G \subset \text{PSL}(N + 1, \mathbb{C})$ such that $\Gamma$ is a lattice of $G$ and for each $\Omega_i$ there is a smooth manifold $X_i$ and a fiber bundle $\Gamma \setminus \Omega_i \rightarrow X_i$ with fibers diffeomorphic to $G$ and one of the following alternatives hold.
   a) The bundle is trivial and $X_i$ is contractible to a sphere.
   b) $X_i$ is homotopically equivalent to a real projective space.
3. The limit set $\Lambda = \mathbb{P}_C^N \setminus (\Omega_1 \cup \Omega_2)$ satisfies property (3) of Theorem 1.

The work is organized in the following way. In Sect. 2 we prove Theorem 1. In order to do this, we study the orbits of the action of $\mathbb{R}^N \rtimes_A \mathbb{R}$ in $\mathbb{C}^N$. The orbits foliate an open and dense set of $\mathbb{C}^N$; to prove this we study the action of $\mathbb{R}$ in the imaginary part of $\mathbb{C}^N$ by the representation $A(t)$. We decompose the imaginary part of $\mathbb{C}^N$ in two linear subspaces, the stable subspace, where the orbits of the action of $\mathbb{R}$ converge as $t \rightarrow \infty$, and the unstable subspace where the orbits diverge to infinity. The main result of Sect. 2.1 is Proposition 6, a key step in the proof of Theorem 1 in Sect. 2.2. Its proof in turn relies on Lemma 5 which is proved using techniques of Lyapunov stability. Theorem 3 is proved in Sect. 3, where we determine conditions on the representation $A(t)$ for the existence of lattices of $\mathbb{R}^N \rtimes_A \mathbb{R}$ (see Sect. 3.1). For groups $\mathbb{R}^N \rtimes_A \mathbb{R}$ admitting a lattice we further study the dynamics of the action in the complement of the open sets $\Omega_i$ defined in Theorem 1. To prove Theorem 3 we recall the fact that hyperbolic toral automorphisms have dense orbits in the torus, and extend this property to lattices of $\mathbb{R}^N \rtimes_A \mathbb{R}$ in Lemma 14.
2 Proof of Theorem 1

To prove Theorem 1 we will focus on the action of $\mathbb{R}^N \rtimes_A \mathbb{R}$ on the projective space induced by the representation (1.1). In what follows, we only consider column vectors.

If $g = (b, t) \in \mathbb{R}^N \rtimes_A \mathbb{R}$ and $[z] \in \mathbb{P}^N$ this action is given by $g \ast [z] = [\rho(g)z]$. We distinguish two invariant sets where the action is easy to describe. Let $U \subset \mathbb{P}^N$ be the set of points with projective coordinates $[z_1 : \cdots : z_N : 1]$, $U$ is an invariant open set such that in the chart $(U, \phi)$, $\phi: U \rightarrow \mathbb{C}^N$, we have

$$\phi([z_1 : \cdots : z_N : 1]) = (z_1, \ldots, z_N)^T,$$

and the action of $\mathbb{R}^N \rtimes_A \mathbb{R}$ in $U$ is equivariant to the action defined as $(\mathbb{R}^N \rtimes_A \mathbb{R}) \times \mathbb{C}^N \rightarrow \mathbb{C}^N$, $((b, t), z) \mapsto A(t)z + b$. On the other hand the complement of $U$ is an invariant closed set such that for any $g = (b, 0) \in \mathbb{R}^N \rtimes_A \mathbb{R}$ and $[z] \in \mathbb{P}^N \setminus U$ with projective coordinates $[z_1 : \cdots : z_N : 0]$ we have $g \ast [z] = [z]$, thus points in the complement of $U$ are of infinite isotropy, hence if the action is proper in an open set $\Omega$, this set should be contained in $U$. For the rest of the section we will focus on this action on $U$ identified with the complex space as described above, our aim is to determine maximal open sets of $\mathbb{C}^N$ where the action is proper.

2.1 The Action on the Complex Space

We use the following well known equivalent definition of proper group action Prob. 12–19 (Lee 2011)

Proposition 4 If $G \times \Omega \rightarrow \Omega$, $(g, z) \mapsto g \ast z$ is a differentiable action of a Lie group $G$ into a $G$-invariant subset $\Omega$ of a manifold, the action is proper if and only if for any sequence $(g_n, z_n)$ in $G \times \Omega$ such that $z_n$ and $g_n \ast z_n$ are convergent, there exists a convergent subsequence $g_{n(k)}$.

2.1.1 Globally Asymptotically Stable Linear Systems

Recall the representation $A : \mathbb{R} \rightarrow \text{GL}(N, \mathbb{R})$ has no eigenvalue of unit length. Let $M = \dot{A}(0)$, then $M$ is a real matrix whose eigenvalues all have non zero real part. For any $x \in \mathbb{R}^N$ the orbits $x(t) = A(t)x_0$ are solutions to the linear autonomous system,

$$\dot{x} = Mx.$$  

(2.2)

Assuming that all the eigenvalues of $M$ have negative real part, the dynamics of this system is well understood from the theory of Lyapunov stability. For instance, it is known that for this system the origin is exponentially stable Thm. 8.2 (Hespanha 2018), that is, there exist positive constants $C$ and $\lambda$ such that for every initial condition $x(0) \in \mathbb{R}^N$ and $t \geq 0$,

$$|x(t)| \leq C e^{-\lambda t} |x(0)|,$$  

(2.3)
where \(|·|\) is the Euclidean norm. Moreover, there is a unique symmetric, positive definite matrix \(P \in \mathbb{R}^{N \times N}\) such that it is the solution to the Lyapunov equation Thm. 8.2 (Hespanha 2018)

\[
PM + M^T P = -I.
\]  (2.4)

**Lemma 5** If all the eigenvalues of the matrix \(M \in \mathbb{R}^{N \times N}\) have negative real part, then there is an inner product \(\langle ·, · \rangle\) in \(\mathbb{R}^N\) such that if \(S\) is the \(N - 1\) sphere \(\{x \in \mathbb{R}^N : \langle x, x \rangle = 1\}\), the map

\[
\mathbb{R} \times S \to \mathbb{R}^N \setminus \{0\}, \quad (t, x) \mapsto \exp(tM)x,
\]  (2.5)

is a diffeomorphism.

**Proof** Let \(\psi : \mathbb{R} \times S \to \mathbb{R}^N \setminus \{0\}\) be the differentiable map defined in (2.5). We show it is bijective and a local diffeomorphism, thus by the inverse function theorem, it is a global diffeomorphism. Let \(P \in \mathbb{R}^{N \times N}\) be the solution to the Lyapunov equation (2.4) and define the inner product \(\langle x, y \rangle = x^T Py\), the function \(V(x) = \langle x, x \rangle\) is a Lyapunov function for the system, this means that \(V(x) > 0\) for \(x \neq 0\) and for any solution \(x(t)\) to (2.2) we have \(\frac{d}{dt} V(x(t)) < 0\). Let \(f(t) = V(x(t))\), by (2.3), \(\lim_{t \to \infty} f(t) = 0\). Since the system (2.2) is linear, Eq. (2.3) implies

\[
\frac{1}{C} e^{\lambda t} |x(0)| \leq |x(-t)|, \quad t \geq 0,
\]  (2.6)

hence \(\lim_{t \to -\infty} f(t) = \infty\). Since \(f(t) = \langle x(t), x(t) \rangle\), we conclude that the curves \(x(t)\) intersect the sphere \(S\) exactly once for any initial condition \(x(0)\). This shows \(\psi\) is bijective. Finally we show \(d\psi_{(t,x)}\) is an isomorphism for any \((t, x) \in \mathbb{R} \times S\), hence it also is a local diffeomorphism. Let \(\{v_1, \ldots, v_{N-1}\}\) be a basis for \(T_xS\), \(\{v_j, x\} = 0\), \(j = 1, \ldots, N - 1\) then \(\{\partial_j, v_1, \ldots, v_{N-1}\}\) is a basis for \(T_{(t,x)}(\mathbb{R} \times S)\), since \(A(t) = \exp(tM)\), this basis is mapped by \(d\psi_{(t,x)}\) onto \(\{A(t)Mv_1, A(t)v_1, \ldots, A(t)v_{N-1}\}\). On the other hand the vector \(Mx\) is transverse to \(T_xS \subset T_x\mathbb{R}^N\) because

\[
\langle x, Mx \rangle = \frac{1}{2} \frac{d}{dt} \big|_{t=0} \langle \exp(tM)x, \exp(tM)x \rangle = \frac{1}{2} \frac{d}{dt} \big|_{t=0} f(t) < 0,
\]  (2.7)

hence \(\{Mx, v_1, \ldots, v_{N-1}\}\) is a basis of \(\mathbb{R}^N\). Since \(A(t)\) is an isomorphism we conclude \(d\psi_{(t,x)}\) maps a basis of \(T_{(t,x)}(\mathbb{R} \times S)\) onto a basis of \(T_{\psi(t,x)}\mathbb{R}^N\) and \(\psi\) is a diffeomorphism. \(\square\)

**Remark 1** If all the eigenvalues of \(M\) have positive real part, by Lemma 5 \((t, x) \mapsto \exp(-tM)x\) is a diffeomorphism of \(\mathbb{R} \times S\) onto \(\mathbb{R}^N \setminus \{0\}\), since the map \((t, x) \mapsto (-t, x)\) is a diffeomorphism in \(\mathbb{R} \times S\). This implies Lemma 5 also holds for \(M\) in this case.
2.1.2 Stable and Unstable Subspaces of Hyperbolic Linear Systems

If $M \in \mathbb{R}^{N \times N}$ is a matrix such that each eigenvalue of $M$ has non zero real part, there is a decomposition of the Euclidean space in two invariant subspaces, $\mathbb{R}^N = E^s \oplus E^u$, the stable and unstable subspaces respectively. If we think of the matrix $M$ as a linear operator, the restriction of $M$ to $E^s$ has only eigenvalues with negative real part whereas the restriction of $M$ to $E^u$ has only eigenvalues with positive real part. If $E^s$ is not trivial, by Lemma 5 there is a sphere in $E^s$ and a well defined diffeomorphism given by Eq. (2.5) such that the action of $\exp(tM)$ onto $E^s \setminus \{0\}$ is equivariant to the action $\mathbb{R} \times (\mathbb{R} \times S) \to \mathbb{R} \times S$, $(s, (t, x)) \mapsto (s + t, x)$ and likewise for $E^u$ by Remark 1. We will use these diffeomorphisms to study the action of $\mathbb{R}^N \ltimes A \mathbb{R}$. Our aim is to prove the existence of two invariant open sets $U^s, U^u \subset \mathbb{C}^N$ such that the action of $\mathbb{R}^N \ltimes A \mathbb{R}$ on each space is proper. We start by decomposing $\mathbb{C}^N$ in the direct sum of real subspaces $\mathbb{R}^N \oplus i E^s \oplus i E^u$. We also define the real projections $\pi_s : \mathbb{C}^N \to E^s$ and $\pi_u : \mathbb{C}^N \to E^u$. If $E^s$ is not trivial, we define $U^s = \mathbb{C}^N \setminus \ker \pi_s$, if $E^u$ is not trivial, we define $U^u$ similarly.

Proposition 6 Let $G_A = \mathbb{R}^N \ltimes A \mathbb{R}$, where $A(t)$ is defined as in Theorem 1. If $\dim E^s > 0$, then there is a diffeomorphism $U^- \to G_A \times \mathbb{R}^{N-N_s} \times S^{N_s-1}$ where $N_s = \dim E^s$, $S^{N_s-1} \subset \mathbb{R}^{N_s}$ is the unit sphere. Moreover, the action of $G_A$ is equivariant to the action

$$G_A \times (G_A \times \mathbb{R}^{N-N_s} \times S^{N_s-1}) \to G_A \times \mathbb{R}^{N-N_s} \times S^{N_s-1},$$

given by $(g, (h, x, y)) \mapsto (gh, x, y)$.

If $\dim E^u > 0$ a similar result holds for $U^+$. In order to prove Proposition 6 we establish an explicit diffeomorphism in Lemmas 7 and 8. The proof for $U^+$ is identical. Let $\psi^- : G_A \times S \times E^u \to U^-$, be the map

$$\psi^-(g, x, y) = g \ast (ix + iy),$$

(2.8)

where $S$ is the stable unit sphere as defined in Lemma 5. If $g = (b, t) \in \mathbb{R}^N \ltimes A \mathbb{R}$, then $\psi^-(g, x, y) = b + i(A(t)x + A(t)y)$, where $A(t)x \in E^s$, $A(t)y \in E^u$.

Lemma 7 The mapping $\psi^-$ is bijective.

Proof Any $z \in U^-$ can be decomposed uniquely as $z = b + ix + iy$ for some $b \in \mathbb{R}^N$, $x \in E^s \setminus \{0\}$, $y \in E^u$. By Lemma 5 there is exactly one pair $(t, s) \in \mathbb{R} \times S$ such that $A(t)s = x$. For this pair let $y' = A(-t)y$ hence $\psi^-((b, t), s, y') = z$, thus $\psi^-$ is surjective. If $(g_j, x_j, y_j)$, $j = 1, 2$ is a pair of points such that $\psi^-(g_1, x_1, y_1) = \psi^-(g_2, x_2, y_2)$, where $g_j = (b_j, t_j)$, then

$$g_2^{-1}g_1 \ast (ix_1 + iy_1) = ix_2 + iy_2.$$

Hence,

$$b_1 - b_2 = 0, \quad A(t_1 - t_2)x_1 = x_2, \quad A(t_1 - t_2)y_1 = y_2.$$
Since \( x_j \in S, j = 1, 2 \), by Lemma 5 the second condition implies \( t_1 - t_2 = 0 \) and \( x_1 = x_2 \) which implies \( y_1 = y_2 \). Thus \((g_1, x_1, y_1) = (g_2, x_2, y_2)\) and \( \psi^- \) is injective.

Lemma 8 The mapping \( \psi^- \) is a diffeomorphism.

Proof We show that for any \( p = (g, x, y) \in G_A \times S \times E^u \) the derivative \( d_p \psi^- \) is an isomorphism \( T_p(G_A \times S \times E^u) \to T_{\psi^-(p)}U^- \). Let \( z = \psi^-(p) \), if \( g = (b, t) \), then \( z = b + i (A(t)x + A(t)y) \). Note that as real vector spaces,

\[
T_p(G_A \times S \times E^u) = T_g G_A \times T_x S \times T_y E^u,
\]

\[
T_{\psi^-(p)}U^- = T_b \mathbb{R}^n \times T_{A(t)x} E^s \times T_{A(t)y} E^u,
\]

then \( d \psi^- \) maps \( \{ \partial b_1, \ldots, \partial b_N \} \) onto a basis for \( T_b \mathbb{R}^n \) and since for any \( t \) the restriction \( A(t)|_{E^u} \) is an isomorphism, \( d_p \psi^- \) also maps \( T_y E^u \) isomorphically onto \( T_{A(t)y} E^u \). By Lemma 5, if \( \{v_1, \ldots, v_{N_s}-1\} \) is a basis for \( T_x S \), then \( d_p \psi^- \) maps the linearly independent vectors \( \partial t, v_1, \ldots, v_{N_s}-1 \) to a basis of \( T_{A(t)x} E^s \), thus \( d_p \psi^- \) is an isomorphism. By the inverse function theorem, \( \psi^- \) is a diffeomorphism since it is bijective by Lemma 7.

Proof of Proposition 6 Lemmas 7 and 8 prove the first part of the proposition once we choose a basis for \( E^s \) in order to induce a diffeomorphism \( S \to \mathbb{S}^{N_s-1} \). The final claim of the proposition follows from Equation (2.8).

By Proposition 6, if \( U^- \neq \emptyset \) then the action of \( \mathbb{R}^N \rtimes_A \mathbb{R} \) in \( U^- \) is proper and free in this set. Another consequence is that \( U^- \) is connected if \( N_s > 1 \) and it has two connected components if and only if \( N_s = 1 \). Thus for any lattice \( \Gamma \subset \mathbb{R}^N \rtimes_A \mathbb{R} \), the quotient space \( \Gamma \setminus U^- \) has at most two connected components and it is diffeomorphic to \((\Gamma \setminus \mathbb{R}^N \rtimes_A \mathbb{R}) \times \mathbb{R}^{N-n} \times \mathbb{S}^{N_s-1} \). Similar assertions hold for \( U^+ \). We finally show that the sets \( U^\pm \) are the only invariant maximal open sets for the action of \( \mathbb{R}^N \rtimes_A \mathbb{R} \).

Proposition 9 If \( U \subset \mathbb{C}^N \) is an invariant open set where the action is proper, then \( U \subset U^+ \) or \( U \subset U^- \).

Proof If \( U^+ \) is empty then \( U^- = \mathbb{C}^N \setminus \{0\} \) and the claim follows. The same holds if \( U^- \) is empty. Thus we can assume none of the open sets \( U^\pm \) are empty. Since \( U^+ \cup U^- = \mathbb{C}^N \setminus \{0\} \), we can assume in order to reach a contradiction the existence of \( z_1 \in U \cap (U^+ \setminus U^-) \) and \( z_2 \in U \cap (U^- \setminus U^+) \). Since \( U \) is invariant under the action of \( \mathbb{R}^N \rtimes_A \mathbb{R} \), we can assume \( z_j = iy_j, y_j \neq 0 \), for \( j = 1, 2 \). Let us define the sequence \( w_n = iA(-n)y_1 + iy_2 \), then \( w_n \to z_2 \) as \( n \to \infty \). Moreover since \( U \) is open, for \( n \) large enough the sequence is contained in \( U \). On the other hand, if \( g_n = (0, n) \in \mathbb{R}^N \rtimes_A \mathbb{R} \), then \( g_n * w_n = iy_1 + iA(n)y_2 \to y_1 \) as \( n \to \infty \). However, the sequence \( g_n \) has no convergent subsequence, a contradiction since the action in \( U \) is proper.
2.2 The Action in the Complex Projective Space

From Proposition 6 we know there is at least one invariant open subset of \( \mathbb{C}^N \) such that the action of \( \mathbb{R}^N \rtimes A \mathbb{R} \) is proper. In this section we prove this set is identified with an open maximal set in the complex projective space where the action is proper and conclude the first two assertions of Theorem 1.

Proof of Theorem 1, parts (1.) and (2.) Let \( \tilde{G} \) be a lift of the group \( G \subset \text{PGL}(N + 1, \mathbb{C}) \) conjugate by a matrix \( \sigma \in \text{GL}(N + 1, \mathbb{C}) \) to the image of the representation \( \rho \) given by Eq. (1.1). For any \( g \in \tilde{G} \) there is a \( (b, t) \in \mathbb{R}^N \rtimes A \mathbb{R} \) such that \( \sigma^{-1} g \sigma = \rho(b, t) \), moreover \( \sigma \) acts on the projective space as \( \sigma[z] = [\sigma z], [z] \in \mathbb{P}^N \). □

Proof of (1.): Recall the chart \( (U, \phi) \) defined by (2.1). If \( U^- \neq \emptyset \), let us define the open set \( \Omega^- = \sigma \phi^{-1}(U^-) \), if \( U^+ \neq \emptyset \) we define \( \Omega^+ \) similarly. Each of the sets \( \Omega^\pm \), is diffeomorphic to \( U^\pm \) which on the other hand is diffeomorphic to a product of the form \( G_A \times \mathbb{R}^{N-N^±} \times \mathbb{S}^{N^±-1} \) by Proposition 6. Moreover \( \Omega^\pm \) is \( G \)-invariant because \( U^\pm \) is invariant under the action of \( \mathbb{R}^N \rtimes A \mathbb{R} \) and the \( G \)-action on \( \Omega^\pm \) is equivariant to the action of \( \mathbb{R}^N \rtimes A \mathbb{R} \) on \( U^\pm \) given in Proposition 6. Defining \( X \) as the preimage of \( \{e\} \times \mathbb{R}^{N-N^±} \times \mathbb{S}^{N^±-1} \) proves the first part of Theorem 1. □

Proof of (2.): If \( \Omega \subset \mathbb{P}^N \) is a \( G \)-invariant open set where the group acts properly, then \( \Omega \cap (\mathbb{P}^N \setminus \sigma U) = \emptyset \) because each point in \( \mathbb{P}^N \setminus \sigma U \) has infinite isotropy. Hence \( \phi(\sigma^{-1} \Omega) \subset \mathbb{C}^N \) is an open invariant set where the action of \( \mathbb{R}^N \rtimes A \mathbb{R} \) is proper, by Proposition 9 \( \phi(\sigma^{-1} \Omega) \subset U^\pm \). Hence \( \Omega \subset \Omega^\pm \). □

3 Proof of Theorem 3

3.1 The Lattices of a Family of Split Solvable Lie Groups

Mosak and Moskowitz described the lattices of the semi-direct product \( \mathbb{R}^N \rtimes A \mathbb{R} \) in the case where the representation \( A \) is diagonal, in general, if it is faithful and closed, there is a matrix \( M \in \text{GL}(N, \mathbb{R}) \) such that \( A(t) = \exp(t M) \). By means of standard methods in Lie group theory, it is possible to compute the nilradical of \( \mathbb{R}^N \rtimes A \mathbb{R} \) p. 3 (Raghunathan 1972), we state the result of the computation as Proposition 10 without proof.

Proposition 10 If \( M \) has a nonzero eigenvalue, the nilradical of \( \mathbb{R}^N \rtimes A \mathbb{R} \) is \( \mathbb{R}^N \), otherwise \( \mathbb{R}^N \rtimes A \mathbb{R} \) is nilpotent.

In preparation for the next proposition, we reproduce the relevant results found in Mosak and Moskowitz (1997) that work for a general representation \( A(t) \). Assume \( M \) is invertible, hence the nilradical of \( \mathbb{R}^N \rtimes A \mathbb{R} \) is \( \mathbb{R}^N \) by Proposition 10. If \( \Gamma \subset \mathbb{R}^N \rtimes A \mathbb{R} \) is a lattice, then \( L = \Gamma \cap \mathbb{R}^N \) is a lattice in \( \mathbb{R}^N \) Cor. 3.5 (Raghunathan 1972), thus there is a matrix \( \sigma \in \text{GL}(N, \mathbb{R}) \) such that \( L = \sigma^{-1} \mathbb{Z}^N \). On the other hand, \( \Gamma \cap \mathbb{R} \) is a lattice of \( \mathbb{R} \) hence there is an \( h > 0 \) such that \( \Gamma \cap \mathbb{R} = h\mathbb{Z} \). Let \( \mathcal{M} = L \times h\mathbb{Z} \), \( \mathcal{M} \) is a lattice in \( \Gamma \). Let us denote by \( g_1 \cdot g_2 \) the group operation in \( \mathbb{R}^N \rtimes A \mathbb{R} \). Since \( (0, h) \in \Gamma \), for
any \((p, 0) \in L\), we have \((A(h)p, 0) = (0, h) \cdot (p, 0) \cdot (0, -h) \in \Gamma\). On the other hand \((A(h)p, 0) \in \mathbb{R}^N\), hence \(A(h)L \subset L\). By a similar argument \(A(-h)L \subset L\), implying \(L \subset A(h)L\), whence \(A(h)L = L\) and \(B(h) = \sigma A(h)\sigma^{-1}\) is an homomorphism of \(\mathbb{Z}^N\). However the previous argument also shows \(B(h)^{-1} = B(-h)\) is another homomorphism, hence \(B(h) \in \text{SL}(N, \mathbb{Z})\). With these previous steps, we arrive at the following proposition,

**Proposition 11** If \(A : \mathbb{R} \to \text{GL}(N, \mathbb{R})\) is a faithful, closed representation with no eigenvalue of unit length, \(\mathbb{R}^N \rtimes_A \mathbb{R}\) has a lattice \(\Gamma\) if and only if there is a matrix \(\sigma \in \text{GL}(N, \mathbb{R})\) and \(h > 0\) such that,

\[
\sigma A(h)\sigma^{-1} \in \text{SL}(N, \mathbb{Z}).
\]

Moreover \(\sigma^{-1}\mathbb{Z}^N \rtimes_A h\mathbb{Z} \subset \Gamma\).

**Proof** If \(\mathbb{R}^N \rtimes_A \mathbb{R}\) has a lattice, by the previous computations there are \(\sigma \in \text{GL}(N, \mathbb{R})\) and \(h > 0\) such that \(A(h) = \sigma^{-1}B\sigma\) for some matrix \(B \in \text{SL}(N, \mathbb{Z})\) such that \(\sigma^{-1}\mathbb{Z}^N \rtimes_A h\mathbb{Z} \subset \Gamma\), hence (3.1) holds. If on the other hand (3.1) holds, \(\sigma^{-1}\mathbb{Z}^N \rtimes_A h\mathbb{Z}\) is a discrete and co-compact subgroup of \(\mathbb{R}^N \rtimes_A \mathbb{R}\), hence it is a lattice of \(\mathbb{R}^N \rtimes_A \mathbb{R}\). $\square$

### 3.2 Hyperbolic Affine Actions in Projective Space

As a consequence of Proposition 11, if \(\mathbb{R}^N \rtimes_A \mathbb{R}\) has a lattice then \(A(t)\) is a representation in \(\text{SL}(N, \mathbb{R})\). In this section \(A(t) \in \text{SL}(N, \mathbb{R})\) for all \(t\), by Proposition 9 we know the sets \(U^\pm\) are maximal open sets where the action of \(\mathbb{R}^N \rtimes_A \mathbb{R}\) is proper. Recall \(\phi\) is the chart defined by Eq. (2.1). If \(\Gamma \subset \mathbb{R}^N \rtimes_A \mathbb{R}\) is a lattice, then the sets \(\Omega^\pm = \phi^{-1}(U^\pm)\) are discontinuity regions for the action of \(\Gamma\). Let \(\Lambda^\pm = \mathbb{P}_C^N \setminus \Omega^\pm\), each of these sets is a limit set of \(\mathbb{R}^N \rtimes_A \mathbb{R}\) for the action of \(\Gamma\) in the sense that it is the complement of a maximal discontinuity region. We aim to study the dynamics of the action of \(\Gamma\) in \(\Lambda = \Lambda^+ \cap \Lambda^-\). If \(\exp(tM) \in \text{SL}(N, \mathbb{R})\) for all \(t\), \(M\) has no purely imaginary eigenvalue and \(\Gamma\) is a lattice of \(\mathbb{R}^N \rtimes_A \mathbb{R}\). By Proposition 11 there is a matrix \(\sigma \in \text{GL}(N, \mathbb{R})\) and a constant \(h > 0\) such that \(\mathcal{L} = \sigma^{-1}\mathbb{Z}^N \rtimes_A h\mathbb{Z} \subset \Gamma\). Let \(p \in \mathbb{Z}^N\), \(n \in \mathbb{Z}\), if we change the coordinates of \(\mathbb{C}^N\) by means of the matrix \(\sigma^{-1}\), an element \(g = (p, n) \in \mathcal{L}\) acts on \(z \in \mathbb{C}^N\) as \(g \ast z = B^n z + p\), where \(B = \sigma^{-1}\exp(hM)\sigma \in \text{SL}(N, \mathbb{Z})\). The linear action of the cyclic group generated by \(B\) onto \(\mathbb{R}^N\) descends to an action of the \(N\)-torus \(\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N\) as \((B^n, [x]) \mapsto [B^n x]\), \([x] \in \mathbb{T}^N\). The set of points \([x] \in \mathbb{T}^N\) with dense orbits is also dense by Thm. 1.11 Walters (2000). Recall points in \(\mathbb{P}_C^N \setminus U\) have infinite isotropy, on the other hand, we can identify \(U\) with \(\mathbb{C}^N\) by means of the chart \((U, \phi)\) given by (2.1). In this chart a point \(z \in \mathbb{C}^N\) has infinite isotropy if and only if for some \(b \in \mathbb{Z}^N\), \(n \in \mathbb{Z} \setminus \{0\}, \)

\[
z = (I - B^n)^{-1} b.
\]

Note that for any pair \((b, n) \in \mathbb{Z}^N \rtimes_B \mathbb{Z}\) the solution to (3.2) is real. We state the following lemmas,

**Lemma 12** For almost every \(x^* \in \mathbb{R}^N\), the orbit of the action \((\mathbb{Z}^N \rtimes_B \mathbb{Z}) \times \mathbb{R}^N \to \mathbb{R}^N\), \(((b, n), x^*) \mapsto B^n x^* + b\), is dense.
Proof For almost every \( x^* \in \mathbb{R}^N \) the orbit \( \{B^n x^* \} \) is dense in \( \mathbb{T}^N \). Let \( x \in \mathbb{R}^N \) be any other point which we assume is in the interior of the fundamental domain of the lattice generated by the canonical basis \( \{e_1, \ldots, e_N \} \subset \mathbb{R}^N \). For any \( n > 0 \) there is a \( b_n \in \mathbb{Z}^N \) such that \( x \) and \( B^n x^* + b_n \) are in the same tile. Moreover since the action of the cyclic group \( \langle B \rangle \) is dense in the torus, there is an integer sequence \( n_j \) such that \( B^{n_j} x^* \to [x] \), which implies that \( B^{n_j} x^* + b_{n_j} \to x \) in \( \mathbb{R}^N \) as \( j \to \infty \). Since the set of interior points in the tile is dense, this proves the lemma.

Lemma 13 Let \( |\cdot| \) be a norm in \( \mathbb{C}^N \) and let \( \| \cdot \| \) be the associated operator norm. For any matrix \( A \in \text{GL}(N, \mathbb{R}) \) such that no eigenvalue is of unit length the sequence \( \|(I - A^n)^{-1}\| \) is bounded.

Proof Firstly, let us assume the spectrum of \( A \) is contained in the open unit disk, then \( A^n \to 0 \) as \( n \to \infty \). By continuity of the matrix inverse function, \( \|(I - A^n)^{-1}\| \to \|I\| = 1 \). On the other hand, if no eigenvalue of \( A \) is contained in the disk, then the matrix \( A^{-1} \) satisfies our first assumption, hence \( \|(I - A^{-n})^{-1}\| = \|(I - A^{-n})A^{-n}\| \to 0 \) as \( n \to \infty \). Finally, for a general matrix we can decompose \( \mathbb{C}^N \) in two \( A \)-invariant subspaces, \( \mathbb{C}^N = X \oplus Y \), such that as a linear operator, the restriction \( A|_X \) satisfies the first condition whereas \( A|_Y \) satisfies the second. In this case we can introduce a second norm \( |\cdot|_{\oplus} \) in \( \mathbb{C}^N \) such that if \( z = x + y, x \in X, y \in Y \), then \( |z|_{\oplus} = |x|_{\oplus} + |y|_{\oplus} \). Let \( \| \cdot \|_{\oplus} \) be the corresponding operator norm, the following bounds hold,

\[
\|(I - A^n)^{-1}\|_{\oplus} = \sup_{|z|_{\oplus}=1} |(I - A^n)^{-1}z|_{\oplus} \\
\leq \sup_{|x|_{\oplus} \leq 1} |(I - A^n)^{-1}x|_{\oplus} + \sup_{|y|_{\oplus} \leq 1} |(I - A^n)^{-1}y|_{\oplus} \\
\leq \|(I - A^n)^{-1}\|_X + \|(I - A^n)^{-1}\|_Y \|_{\oplus}.
\]

The right hand side of the last inequality is convergent as \( n \to \infty \), hence \( \|(I - A^n)^{-1}\|_{\oplus} \) is a bounded sequence. Since any two norms in \( \text{GL}(N, \mathbb{R}) \) are equivalent, this proves the Lemma.

Lemma 14 The set of solutions \( x(b, n) \) to Eq. (3.2) is dense in \( \mathbb{R}^N \).

Proof Let \( x^* \in \mathbb{R}^N \), the Lemma 12 defines a sequence \( y(b, n) = B^n x^* + b \) which we know is dense in \( \mathbb{R}^N \) for almost every \( x^* \). A short computation shows

\[
x(b, n) - x^* = (I - B^n)^{-1}(y(b, n) - x^*), \quad n \neq 0.
\]

(3.3)

Denoting by \( |\cdot| \) any norm in \( \mathbb{C}^N \) and by \( \| \cdot \| \) the corresponding operator norm in \( \text{GL}(N, \mathbb{C}) \), Eq. (3.3) implies,

\[
|x(b, n) - x^*| \leq \|(I - B^n)^{-1}\| |y(b, n) - x^*|, \\
\leq C |y(b, n) - x^*|,
\]

(3.4)

where the positive constant \( C \) is given by Lemma 13. Since \( \{y(b, n) : b \in \mathbb{Z}^N, n \in \mathbb{Z}^+\} \) is dense in \( \mathbb{R}^N \), there is a convergent subsequence \( y(b_j, n_j) \to x^* \) as \( j \to \infty \),
hence we can find a subsequence \( x(b_j, n_j) \to x^* \) as \( j \to \infty \). This proves the lemma because the set of points \( x^* \) for which the convergence holds is dense in \( \mathbb{R}^N \).

\textbf{Proof of Theorem 1-(3.)} If \( \Gamma \subset \mathbb{R}^N \rtimes_A \mathbb{R} \) is a lattice, by Proposition 11 there are matrices \( B \in \text{SL}(N, \mathbb{Z}), \sigma \in \text{GL}(N, \mathbb{R}) \) and \( h > 0 \) such that \( \mathcal{L} = \sigma^{-1} \mathbb{Z} \rtimes_B h\mathbb{Z} \) is another lattice contained in \( \Gamma \). By Lemma 14 the set of points with infinite isotropy with respect to \( \mathcal{L} \) is dense in \( \Lambda \cap U \), where \((U, \phi)\) is the chart defined by Eq. (2.1). On the other hand, any \( z \in \Lambda \cap (\mathbb{P}^N_{\mathbb{C}} \setminus U) \) has infinite \( \mathcal{L} \)-isotropy, since \( \mathcal{L} \subset \Gamma \), this proves (a). The proof of (b) is a consequence of Lemma 12: for almost any \( z \in (\Lambda \cap \mathbb{P}^N_{\mathbb{R}}) \cap U \), the \( \mathcal{L} \)-orbit is dense in \( (\Lambda \cap \mathbb{P}^N_{\mathbb{R}}) \cap U \), since the closure of \((\Lambda \cap \mathbb{P}^N_{\mathbb{R}}) \cap U \) is \( \Lambda \cap \mathbb{P}^N_{\mathbb{R}} \), \( \mathcal{L} \)-orbits are dense in \( \Lambda \cap \mathbb{P}^N_{\mathbb{R}} \) and since \( \Lambda \subset \Gamma \), the same statement holds for \( \Gamma \).

The result of Proposition 15 is similar to Proposition 9, however the argument requires a few modifications, thus we include the proof.

\textbf{Proposition 15} If \( U \subset \mathbb{C}^N \) is an open \( \mathbb{Z}^N \rtimes_B \mathbb{Z} \) invariant set where the action is proper, then \( U \subset U^+ \) or \( U \subset U^- \).

\textbf{Proof} Let \( E^s \) and \( E^u \) denote the stable and unstable subspaces of \( B \) and let us assume in order to reach a contradiction the existence of two points,

\[
\begin{align*}
z_1 & \in U \cap (U^- \setminus U^+) \subset (\mathbb{R}^N \oplus iE^s) \setminus \mathbb{R}^N \oplus 0, \\
z_2 & \in U \cap (U^+ \setminus U^-) \subset (\mathbb{R}^N \oplus iE^u) \setminus \mathbb{R}^N \oplus 0,
\end{align*}
\]

such that \( z_j = x_j + iy_j \), where \( y_1 \in E^s \setminus \{0\} \) and \( y_2 \in E^u \setminus \{0\} \). Since \( U \cap (\mathbb{R}^N \oplus 0) \) is relatively open, by Lemma 12 we may assume there is a sequence \( g_j = (b_j, n_j) \in \mathbb{Z}^N \rtimes_B \mathbb{Z} \) such that \( g_j \cdot x_1 \to x_2 \) as \( j \to \infty \). Moreover, the proof of Lemma 12 shows that we may assume \( n_j \to \infty \) as \( j \to \infty \). Let \( w_j = g_j \cdot x_1 + iB^n j y_1 + iy_2 \), then for \( j \) sufficiently large, \( w_j \in U \) because \( w_j \to z_2 \). On the other hand, \( g_j^{-1} \cdot w_j = x_1 + iy_1 + iB^{-n} j y_2 \to z_1 \) as \( j \to \infty \). This is a contradiction because the sequence \( g_j^{-1} \) has no convergent subsequence in \( \mathbb{Z}^N \rtimes_B \mathbb{Z} \).

In order to prove part (2.) of Theorem 3, we make a few remarks about the proper and free action of a Lie group on a smooth manifold. If \( G_0 \) denotes the connected component of the identity in \( G \), then \( G_0 \) is a closed normal subgroup such that for any \( x \in G \), \( xG_0 = G_0 x \). Moreover, the connected components of \( G \) are precisely the cosets \( G/G_0 \) Thm. 1.9.1 (Duistermaat and Kolk 2000).

\textbf{Proposition 16} Let \( G \) be a Lie group acting properly and freely on a smooth manifold \( \Omega \), let \( G_0 \) be the connected component of the identity and let \( H = G/G_0 \), then \( H \) acts properly discontinuously and freely on \( G_0 \setminus \Omega \) as \( (gG_0, G_0 \star x) \mapsto G_0 \star h \star x \) and the action induces a diffeomorphism \( H \setminus (G_0 \setminus \Omega) \cong G \setminus \Omega \).

\textbf{Proof} If \( G \) acts properly and freely on a smooth manifold \( \Omega \), then \( G_0 \) also acts properly and freely. Hence the map \( G_0 \setminus \Omega \to G \setminus \Omega, G_0 \star x \mapsto G \star x \) is a smooth, surjective map between smooth manifolds such that for any \( G \star x \in G \setminus \Omega \), the preimage is the set \( \{G_0 \star h \star x : h G_0 \in H\} \). Then there is a bijection \( H \setminus (G_0 \setminus \Omega) \to G \setminus \Omega \). Let
us show that this bijection is a local diffeomorphism, therefore it is a diffeomorphism. By the equivariant tubular neighbourhood theorem, for any \( x_0 \in \Omega \), there is an open neighbourhood \( \mathcal{N} \) and a diffeomorphism \( \varphi : \mathcal{N} \to G \times \mathbb{R}^N \) such that for any \( g \in G \) and \( x \in \mathcal{N} \), if \( \varphi(x) = (g', \hat{x}) \), then \( g \ast \varphi(x) = (gg', \hat{x}) \). Hence the orbit space \( G_0 \setminus \Omega \) is locally diffeomorphic to \( H \times \mathbb{R}^N \) and the \( H \) action on \( G_0 \setminus \Omega \) is locally equivariant to \( (g_1 G_0, (g_1 G_0, \hat{x})) \mapsto (g_1 g_2 G_0, \hat{x}) \). Then the quotient \( H \setminus (G_0 \setminus \Omega) \) is locally diffeomorphic to \( \{e\} \times \mathbb{R}^N \) which is also locally diffeomorphic to \( G \setminus \Omega \). \( \square \)

**Proof of Theorem 3**

For any hyperbolic toral automorphism \( B \in \text{SL}(N, \mathbb{Z}) \) by Thm. 1 Culver (1966) there is a real \( N \times N \) matrix \( M \) such that either \( B = \exp(M) \) or \( B^2 = \exp(M) \). In any case we define \( A(t) = \exp(tM), t \in \mathbb{R} \) and the group

\[
\hat{G} = \left\{ \begin{pmatrix} C & p \\ 0 & 1 \end{pmatrix} \in \text{SL}(N + 1, \mathbb{R}) : C = A(t) \text{ or } C = BA(t) \text{ for some } t \in \mathbb{R} \right\}, \quad (3.5)
\]

then for the representation \( \rho \) defined in (1.2), \( \rho(\mathbb{Z}^N \rtimes_B \mathbb{Z}) \) is a lattice of \( \hat{G} \).

(1.) Let \( G \) be the projection of \( \hat{G} \) to \( \text{PSL}(N+1, \mathbb{C}) \), and let \( G_0 \subset G \) be the connected component of the identity which admits a lift to \( \mathbb{R}^N \rtimes_A \mathbb{R} \). The matrices \( B \) and \( B^2 \) have eigenvalues at both sides of the unit disk, hence by Theorem 1 there are two open sets \( \Omega_1, \Omega_2 \) where \( G \) acts properly and freely. From the proof of Theorem 1, we know \( \Omega_i \subset U \) for \( (U, \phi) \) the canonical chart defined in (2.1). A vector \( z \in \mathbb{C}^N \) belongs to \( \phi(\Omega_i) \) if and only if \( z = x + iy \), and \( y \in (E^s \setminus \{0\}) \cup (E^u \setminus \{0\}) \), where \( E^s \) and \( E^u \) are the stable and unstable subspaces of \( A(1) \). The stable and unstable subspaces of \( B \) and \( B^2 \) are the same, hence each \( \Omega_i \) is \( \Gamma \)-invariant in either case. \( \Gamma \) acts properly and discontinuously on each \( \Omega_i \) because the action of \( G_0 \) is proper, hence \( \mathbb{Z}^N \rtimes_{B^2} \mathbb{Z} \) acts properly and discontinuously and is a subgroup of finite index of \( \mathbb{Z}^N \rtimes_B \mathbb{Z} \). The final claim follows from Proposition 15 applied in the chart \( (U, \phi) \).

(2.) Note that each \( \Omega_i \) is also \( G \)-invariant and since the index \( [G : G_0] \) is finite, \( G \) acts properly on \( \Omega_i \). Moreover, the action is free, hence the canonical projection \( \Omega_i \to G \setminus \Omega_i \) is a principal bundle with fibers diffeomorphic to \( G \). Since the \( \Gamma \) action on \( \Omega_i \) preserves the fibers, the quotient \( G \setminus \Omega_i \) is a fiber bundle onto \( G \setminus \Omega_i \). If \( G \) is connected, by Theorem 1 this bundle is trivial and the base space is contractible to a sphere. If \( G \) is not connected, notice that \( G/G_0 \cong \mathbb{Z}_2 \). By Proposition 16 there is a \( \mathbb{Z}_2 \) action on \( G_0 \setminus \Omega_i \) such that \( \mathbb{Z}_2 \setminus (G_0 \setminus \Omega_i) \) is diffeomorphic to \( G \setminus \Omega_i \), hence if \( g \in G \) has non trivial class in \( G/G_0 \), it determines an involution \( f_g : G_0 \setminus \Omega_i \to G_0 \setminus \Omega_i \). We aim to show that the quotient of \( G_0 \setminus \Omega_i \) by this action is homotopy equivalent to the quotient of a sphere by an involution, this will imply that it is homotopy equivalent to a real projective space (Olum 1953). Assume without loss of generality that \( G_0 \setminus \Omega_i \) diffeomorphic to \( (\mathbb{R}^N \rtimes_A \mathbb{R}) \setminus U^- \) and that \( g \) lifts to \( \tilde{g} \). Let \( \psi : S \times E^u \to G_0 \setminus \Omega_i \), \( \psi(x, y) = G_0 \ast (ix + iy) \) be the diffeomorphism induced by Proposition 6. Recall that for any trajectory \( A(t)x \) in the stable subspace \( E^s \), there is exactly one intersection with the stable sphere, hence for any \( x \in E^s \) there is exactly one parameter \( t(x) \in \mathbb{R} \) such that \( A(t)x \in S \). By the implicit function theorem, the function \( t(x) \) is smooth, moreover, if \( F : S \to \text{GL}(N, \mathbb{R}) \) is the map \( F(x) = A(t(Bx))B \), then \( F(x) \) preserves the stable and unstable subspaces of \( \mathbb{R}^N \) and \( \tilde{g} \ast \psi(x, y) = G_0 \ast (iF(x)x + iF(x)y) \), then up to diffeomorphism, \( \tilde{g} \) acts on \( S \times E^u \) as \( \tilde{g} \ast (x, y) = (F(x)x, F(x)y) \), since

\[ \tilde{g} \]
$S \times E^u$ is homotopy equivalent to the sphere $S \times \{0\}$ and $\tilde{g}$ acts on the second factor linearly, it induces an homotopy equivalence between $\mathbb{Z}^2 \setminus (S \times E^u)$ and $\mathbb{Z}^2 \setminus (S \times \{0\})$. Thus the quotient space $\mathbb{Z}^2 \setminus (G_0 \setminus \Omega)$ is homotopy equivalent to the quotient $\mathbb{Z}^2 \setminus S$ of an sphere by an involution.

(3.) The group $G_0$ satisfies the hypothesis of Theorem 1, this implies the existence of a subgroup $\Gamma' \subset \Gamma$ such that points of infinite isotropy in $\Lambda$ are dense and for almost any $z \in \Lambda \cap \mathbb{P}^N \mathbb{R}$ the orbits are dense, since $\Gamma'$ is contained in $\Gamma$, the same statements are true for the latter group.

\[\Box\]

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**Declarations**

**Conflict of interest** All authors have contributed equally to the paper and they have no conflict of interest.

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