Szczarba’s twisting cochain and the Eilenberg–Zilber maps

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Abstract
We show that Szczarba’s twisting cochain for a twisted Cartesian product is essentially the same as the one constructed by Shih. More precisely, Szczarba’s twisting cochain can be obtained via the basic perturbation lemma if one uses a ‘reversed’ version of the classical Eilenberg–Mac Lane homotopy for the Eilenberg–Zilber contraction. Along the way we prove several new identities involving these homotopies.

Keywords Twisted Cartesian product · Szczarba’s twisting cochain · Shih’s twisting cochain · Eilenberg-Zilber maps · Basic perturbation lemma

Mathematics Subject Classification Primary 55U10; Secondary 55R20 · 55U25

1 Introduction
Let $F \hookrightarrow E \rightarrow B$ be a fibre bundle. In 1959, E. H. Brown [1, Sect. 4] showed that for path-connected base $B$ the homology of the total space $E$ is isomorphic to that of the twisted tensor product

$$C(F) \otimes_t C(B).$$

(1.1)

Here “twisted” means that the usual tensor product differential is modified to a twisted differential $d_t$. Brown used acyclic models to construct the twisting cochain $t: C(B) \to C(\Omega B)$ that determines this deviation.

Shortly afterwards, such twisting cochains were constructed by less opaque means in the simplicial setting, that is, for twisted Cartesian products $F \times_\tau B$, where $F$ and $B$ are simplicial sets and $\tau: B_{>0} \to G$ is a twisting function with values in the structure group $G$. Szczarba [18] gave an explicit formula for $t$ in terms of $\tau$ while Shih [17] described an algorithm for computing $t$ that starts with the Eilenberg–Zilber maps (Alexander–Whitney map $AW$, shuffle map $\nabla$, Eilenberg–Mac Lane’s homotopy $H$) and applies a perturbation determined by an acyclic model.
by \(\tau\). This latter approach was the birth of what is nowadays called homological perturbation theory.

Later Rubio [16, p. 53] observed experimentally that Shih’s and Szczarba’s twisting cochains agree if one applies Shih’s algorithm not to the usual Eilenberg–Zilber contraction \((AW, \nabla, H)\), but to the “opposite” triple \((AW, \nabla, \tilde{H})\) where \(\tilde{H}\) is obtained from \(H\) by reversing simplices and transposing the factors (see Sect. 4 for details). In Theorem 6.2 we prove this assertion.

**Theorem 1.1** Szczarba’s twisting cochain is equal to Shih’s twisting cochain based on the opposite Eilenberg–Zilber contraction \((AW, \nabla, \tilde{H})\).

The proof uses a characterization of Szczarba’s twisting cochain obtained by Morace–Prouté [14].

In order to apply Shih’s theory to the opposite Eilenberg–Zilber contraction, we investigate how both \(H\) and \(\tilde{H}\) interact with the shuffle map \(\nabla\) and the Alexander–Whitney map \(AW\). This in fact occupies the largest part of this paper. Such relations are all the more interesting as the homotopies \(H\) and \(\tilde{H}\) gives rise to important structures on cochain algebras: they can be used to define Steenrod’s \(\cup^i\)-products [8, Thm. 3.2] as well as the homotopy Gerstenhaber structure [10, Sect. 5], [7, App. A] that induces a product on the bar construction.

Besides offering rigorous proofs for certain commutative diagrams given by Shih as well as their “mirror images” (Propositions 3.1 and 3.2), we obtain several new identities. For example, the homotopies \(H_{X\times Y}\) and \(H_{X,Y\times Z}\) that arise from the two ways of splitting up a triple Cartesian product \(X \times Y \times Z\) commute in the graded sense,

\[
H_{X\times Y,Z} H_{X,Y\times Z} = -H_{X,Y\times Z} H_{X\times Y,Z},
\]  

see Proposition 3.4. As shown in Sect. 4, all identities carry over to \(\tilde{H}\).

We fix our notation in Sect. 2. The classical Eilenberg–Zilber contraction is studied in Sect. 3 and the opposite one in Sect. 4. In Sect. 5 we summarize all known relations among the Eilenberg–Zilber maps and ask if there are any further. Szczarba’s and Shih’s twisting cochains are compared in Sect. 6 and their twisted shuffle maps in Sect. 7.

## 2 Preliminaries

Let \(k\) be a commutative ring with unit. All complexes and tensor products we consider are over \(k\). The identity map of a complex is written as 1. Given two complexes \(A\) and \(B\), the transposition of factors is defined to be the chain map

\[
T_{A,B} : A \otimes B \to B \otimes A, \quad a \otimes b \mapsto (-1)^{|a||b|} b \otimes a.
\]  

Throughout this paper, the letters \(X, Y, Z\) and \(W\) denote simplicial sets. Let \(x \in X_n\) be a simplex. For \(0 \leq p \leq q \leq n\) we write

\[
\partial^q_p x = \partial_p \cdots \partial_q x, \quad \partial_{p-1}^p x = x
\]  

for the iteration of face maps. Given a set \(\{\alpha_1 < \cdots < \alpha_p\}\) of non-negative integers with \(\alpha_p \geq n + p - 1\) if \(p > 0\), we also write

\[
s_{\alpha} x = s_{\alpha_p} \cdots s_{\alpha_1} x, \quad s_{\emptyset} x = x
\]  

for the iteration of degeneracy maps. We denote the normalized chain complex of \(X\) with coefficients in \(k\) by \(C(X)\).
Let \( p, q \geq 0 \). A \((p, q)\)-shuffle is a partition of the set \( \{0, \ldots, p+q-1\} \) into subsets \( \alpha \) and \( \beta \) of sizes \( |\alpha| = p \) and \( |\beta| = q \); it is denoted by \((\alpha, \beta) \leftarrow (p, q)\). We write \((-1)^{(\alpha, \beta)}\) for its signature, that is, the sign of the permutation \((\alpha_1 < \cdots < \alpha_p, \beta_1 < \cdots < \beta_q)\).

By a simplicial operator we mean what is called an “FD-operator” in [3, Sect. 3]. The definition of the derived operator \( f' \) of a simplicial operator \( f \) appears in [3, p. 59], for tensor products in [4, p. 53] and for the shuffle map in [3, Eq. (5.3)]. Frontal simplicial operators are defined in [3, p. 60]; by [3, Lemma 3.3] they satisfy
\[
s_0 f = f' s_0. \tag{2.4}
\]

Note that a priori a simplicial operator \( f \) is defined on non-normalized chain complexes only. Even if \( f \) descends to normalized chains, this may fail for \( f' \). (An example is the differential \( d \).) Moreover, the second part of the following observation shows that Eilenberg–Mac Lane’s definition of derived operators does not behave well with respect to the Koszul sign rule. Nevertheless, for ease of referencing we do not modify the definition.

**Lemma 2.1** Let \( f \) and \( g \) be simplicial operators.

(i) If \( f \) and \( g \) agree modulo degenerate chains, then so do \( f' \) and \( g' \).

(ii) One has \((f \otimes g)' = (-1)^{|g|} f' \otimes g'\).

**Proof** The assumption in the first claim means that every monotonic operator appearing in \( g - f \) has a leading degeneracy operator in its canonical form, compare [3, p. 59]. Hence the same holds for \( g' - f' = (g - f)' \). The second claim follows directly from the definition of derived operators.

\(\square\)

### 3 The Eilenberg–Zilber maps

We start by reviewing the Eilenberg–Zilber maps.\(^1\) The Alexander–Whitney map is given by
\[
AW = AW_{X, Y} : C(X \times Y) \to C(X) \otimes C(Y), \\
(x, y) \mapsto \sum_{k=0}^{n} \partial_{k+1}^n x \otimes \partial_{0}^{k-1} y \tag{3.1}
\]
for \((x, y) \in X_n \times Y_n\). The shuffle map is defined as
\[
\nabla = \nabla_{X, Y} : C(X) \otimes C(Y) \to C(X \times Y), \\
x \otimes y \mapsto \sum_{(\alpha, \beta) \leftarrow (p, q)} (-1)^{(\alpha, \beta)} (s_\beta x, s_\alpha y) \tag{3.2}
\]
where \( p = |x| \) and \( q = |y| \). The homotopy
\[
H = H_{X, Y} : C(X \times Y) \to C(X \times Y), \\
(x, y) \mapsto \sum_{0 \leq p+q < n} \sum_{(\alpha, \beta) \leftarrow (p+1, q)} (-1)^{n+1 + (\alpha, \beta)} \\
\cdot (s_\beta + m \, s_{m-1} \, \partial_{n-q+1}^n x, s_\alpha + m \, \partial_{n-q}^m y) \tag{3.3}
\]
\(^1\) This is a misnomer. All three maps were introduced by Eilenberg–Mac Lane [3, Eq. (5.3)], [4, Eqs. (2.8), (2.13)], with the obvious inspiration for the Alexander–Whitney map.
for \((x, y) \in X_n \times Y_n\) with \(m = n - p - q\) has been recursively defined by Eilenberg–Mac Lane [4, Eq. (2.13)] via

\[
H_0 = 0, \quad H_n = -H'_{n-1} + F'_{n-1} s_0
\]

for \(n > 0\), where

\[
F = \nabla AW : C(X \times Y) \to C(X \times Y).
\]

The explicit formula given above is due to Rubio and Morace, see [16, Sect. 3.1], [15, Sect. 6]. We will also use that \(H\) is frontal [4, p. 53].

The Eilenberg–Zilber maps \(AW, \nabla\) and \(H\) satisfy the properties stated in [4, Thm. 2.1a] and [17, §II.1],

\[
AW \nabla = 1, \quad \nabla AW = 1 + d(H), \quad H \nabla = 0, \quad AW H = 0, \quad HH = 0.
\]

This means that the triple \((AW, \nabla, H)\) forms a contraction in the sense of homological perturbation theory, compare [2], [9, Sect. 3], [15, Sect. 3].

**Proposition 3.1** The following diagrams commute.

\[
\begin{align*}
&\xymatrix{\ C(X \times Y \times Z) & C(X) \otimes C(Y \times Z) \\
\downarrow H_{X \times Y \times Z} & \downarrow 1 \otimes H_{Y \times Z}} \\
\xymatrix{\ C(X \times Y \times Z) & C(X) \otimes C(Y \times Z) \\
\downarrow H_{X \times Y \times Z} & \downarrow 1 \otimes H_{Y \times Z}} \\
\end{align*}
\]

**Proposition 3.2** The following diagrams commute.

\[
\begin{align*}
&\xymatrix{\ C(X) \otimes C(Y \times Z) & C(X \times Y \times Z) \\
\downarrow 1 \otimes H_{Y \times Z} & \downarrow H_{X \times Y \times Z}} \\
\xymatrix{\ C(X) \otimes C(Y \times Z) & C(X \times Y \times Z) \\
\downarrow 1 \otimes H_{Y \times Z} & \downarrow H_{X \times Y \times Z}} \\
\end{align*}
\]

The diagrams (A2) and (B1) appear in Shih’s work [17, §II.4, Lemmes 3 & 3 bis]. Because the arguments given there contain numerous inaccuracies, we prove them from scratch. The diagrams (A1) and (B2) are contained in Gugenheim’s paper [9, Lemma 4.0]. However, the
homotopy $H$ is not specified there, and instead of a proof the reader is referred to [17], where these diagrams cannot be found.\footnote{\textsuperscript{4}}

Instead of making Shih’s arguments for the diagram (A2) rigorous, we will present a different proof of Proposition 3.1 which is based on the non-recursive definition (3.3) of $H$ and can easily be adapted to (A1). We postpone that part to Sect. 4 because the notation will be more transparent for the homotopy $\tilde{H}$ to be introduced there.

**Proof of Proposition 3.2** Let us verify that before normalization the diagram (B1) commutes modulo degenerate chains. We write “$\equiv$” if two chains differ by a degenerate chain. We closely follow Shih’s strategy.

Let $a = x \otimes (y, z)$ with $x \in X_p$, $y \in Y_q$ and $z \in Z_q$. If $p = 0$, then the shuffle map essentially reduces to the identity map, and for $q = 0$ both homotopies vanish. We may therefore assume $p > 0$ and $q > 0$. We proceed by induction on $p + q$.

By the recursive definition (3.4) of $H$ we have $H_{X \times Y, Z} \nabla_{X,Y \times Z}(a) = A + B$ with

$$A = -H'_{X \times Y, Z} \nabla_{X,Y \times Z}(a), \quad B = F'_{X \times Y, Z} s_0 \nabla_{X,Y \times Z}(a).$$

Using the identity [3, Eq. (5.7)]

$$\nabla(x \otimes y) = \nabla'(x \otimes s_0 y) + (-1)^p \nabla'(s_0 x \otimes y)$$

together with [3, Thm. 3.2], we get

$$A = -H' \nabla'(x \otimes s_0(y, z)) - (-1)^p H' \nabla'(s_0 x \otimes (y, z))$$

$$= -(H \nabla)'(x \otimes s_0(y, z)) - (-1)^p (H \nabla)'(s_0 x \otimes (y, z)),$$

which by induction and Lemma 2.1 (i) is congruent to

$$\equiv - (\nabla (1 \otimes H))'(x \otimes s_0(y, z)) - (-1)^p (\nabla (1 \otimes H))'(s_0 x \otimes (y, z))$$

Lemma 2.1 (ii) and the identity (2.4) for the frontal operator $H$ now imply

$$\equiv \nabla'(1 \otimes H)'(x \otimes s_0(y, z)) + (-1)^p \nabla'(1 \otimes H)'(s_0 x \otimes (y, z))$$

$$= (-1)^p \nabla'(x \otimes s_0(y, z)) - \nabla'(s_0 x \otimes H'(y, z))$$

$$= (-1)^p \nabla'(x \otimes s_0 H'(y, z)) - \nabla'(s_0 x \otimes H'(y, z))$$

On the other hand, writing $b = s_0 x \otimes s_0(y, z)$ and taking the version $s_0 \nabla a = \nabla' b$ of the identity (2.4) for the frontal operator $\nabla$ into account [3, Eq. (5.5)], we find

$$B = F'_{X \times Y, Z} \nabla_{X,Y \times Z} b = (\nabla_{X \times Y, Z} AW_{X \times Y, Z} \nabla_{X,Y \times Z})'(b)$$

which by [17, Lemme II.4.2] (or diagram (5.3) below), Lemma 2.1 and the associativity of the shuffle map gives

$$\equiv (\nabla_{X \times Y, Z} (\nabla_{X,Y} \otimes 1)(1 \otimes AW_{Y,Z}))'(b)$$

$$= (\nabla_{X \times Y, Z} (1 \otimes \nabla_{Y,Z})(1 \otimes AW_{Y,Z}))'(b)$$

$$= \nabla'_{X,Y \times Z} (1 \otimes F_{Y,Z})(b) = \nabla'_{X,Y \times Z} (1 \otimes F'_{Y,Z})(b)$$

$$= \nabla'_{X,Y \times Z}(s_0 x \otimes F'_{Y,Z} s_0(y, z)).$$

\footnote{Given that Gugenheim swapped the factors of the twisted tensor product compared to Shih, he might have been thinking of the homotopy $\tilde{H}$ defined in (4.8) below, compare Propositions 4.5 and 4.6.}
Combining $A$ and $B$ and using again the formulas (3.4) and (3.8), we obtain

$$H \, \nabla(a) \equiv (-1)^p \nabla'(x \otimes s_0 H'(y, z)) + \nabla'(s_0 x \otimes H(y, z))$$

$$\equiv (-1)^p \nabla(x \otimes H(y, z)) = \nabla(1 \otimes H)(a),$$

(3.11)
as desired.

The proof for the diagram (B2) is completely analogous. See the diagram (5.4) for the required mirror image of [17, Lemme II.4.2].

**Corollary 3.3** The following identities hold between $H$ and $F = \nabla AW$:

$$F_{X,Y,Z} \, H_{X,Y,Z} = H_{X,Y,Z} \, F_{X,Y,Z}.$$  

$$F_{X,Y,Z} \, H_{X,Y,Z} = H_{X,Y,Z} \, F_{X,Y,Z}.$$  

**Proof** Combine the diagrams (A1)+(B1) and (A2)+(B2), respectively.  \(\square\)

The next result also seems to be new.

**Proposition 3.4** The operators $H_{X,Y,Z}$ and $H_{X,Y,Z}$ commute in the graded sense,

$$H_{X,Y,Z} \, H_{X,Y,Z} = -H_{X,Y,Z} \, H_{X,Y,Z}.$$  

**Proof** We proceed by induction on the degree $n$ of the (non-normalized) argument. The case $n = 0$ is trivial since $H_0 = 0$.

Using the recursive definition (3.4) as well as formula (2.4), we have for $n > 0$

$$H_{X,Y,Z} \, H_{X,Y,Z} = -H'_{X,Y,Z} \, H_{X,Y,Z} + F'_{X,Y,Z} \, s_0 \, H_{X,Y,Z}$$

$$= H'_{X,Y,Z} \, H_{X,Y,Z} - H'_{X,Y,Z} \, F'_{X,Y,Z} \, s_0 + F'_{X,Y,Z} \, H'_{X,Y,Z} \, s_0$$

$$= (H_{X,Y,Z} \, H_{X,Y,Z})' - (H_{X,Y,Z} \, F_{X,Y,Z})' \, s_0 + (F_{X,Y,Z} \, H_{X,Y,Z})' \, s_0.$$  

(3.12)

Analogously, we get

$$H_{X,Y,Z} \, H_{X,Y,Z} = (H_{X,Y,Z} \, H_{X,Y,Z})' - (H_{X,Y,Z} \, F_{X,Y,Z})' \, s_0 + (F_{X,Y,Z} \, H_{X,Y,Z})' \, s_0.$$  

(3.13)

Using the induction hypothesis together with Corollary 3.3 and Lemma 2.1 (i) completes the proof.  \(\square\)

**4 The opposite Eilenberg–Zilber contraction**

In order to obtain another contraction $(AW, \nabla, \tilde{H})$, we reverse the “front” and “back” of each simplex in a simplicial set $X$. (This idea appears already in [16, Sect. 3.3].) More precisely, we define the opposite simplicial set $\tilde{X}$ to be equal to $X$ as a graded set, but with new face and degeneracy maps

$$\tilde{\partial}_k x = \partial_{n-k} x, \quad \tilde{s}_k x = s_{n-k} x$$

(4.1)

for $x \in \tilde{X}_n = X_n$. The associated differential $\tilde{d}$ on the graded module $C(\tilde{X}) = C(X)$ is related to the original one by

$$\tilde{d} x = (-1)^{|x|} d x.$$  

(4.2)
We introduce the maps
\[ \tilde{T}_X : C(X) \to C(\tilde{X}), \quad x \mapsto (-1)^{\nu(|x|)} x \] (4.3)
and
\[ \tilde{T}_{X,Y} : C(X) \otimes C(Y) \to C(\tilde{Y}) \otimes C(\tilde{X}), \quad x \otimes y \mapsto (-1)^{\nu(|x|+|y|)} y \otimes x \] (4.4)
where \( \nu : \mathbb{Z} \to \mathbb{Z}_2 \) is defined by
\[ \nu(n) = \frac{n(n+1)}{2} = \begin{cases} 0 & \text{if } n \equiv 0, 3 \pmod{4}, \\ 1 & \text{if } n \equiv 1, 2 \pmod{4}. \end{cases} \] (4.5)

**Lemma 4.1** Both \( \tilde{T}_X \) and \( \tilde{T}_{X,Y} \) are chain maps, natural in \( X \) and \( Y \).

**Proof** This is a direct verification. \( \square \)

We write
\[ \tau_{X,Y} : X \times Y \to Y \times X \] (4.6)
for the canonical transposition and
\[ \tau_{X,Y} = \tilde{T}_{Y \times X} (\tau_{X,Y})_* = (\tau_{\tilde{X},\tilde{Y}})_* \tilde{T}_{X \times Y} : C(X \times Y) \to C(\tilde{Y} \times \tilde{X}). \] (4.7)

Note that here we are using the canonical isomorphism \( X \times Y = \tilde{X} \times \tilde{Y} \).

We will see that both the shuffle map \( \nabla \) and the Alexander–Whitney map \( AW \) are invariant under the simplex reversal procedure just discussed, combined with the transposition of factors. This does not hold for the Eilenberg–Mac Lane homotopy \( H \) because the two factors of \( X \times Y \) are not treated in a symmetric way. We therefore introduce the map
\[ \tilde{H} = \tilde{H}_{X,Y} : C(X \times Y) \to C(X \times Y), \] (4.8)

\[ \tilde{H}(x, y) = \sum_{0 \leq p+q < n} \sum_{(\alpha, \beta) + (p+q)} (-1)^{p+q+(\alpha, \beta)} (s_\beta \partial_{p+1}^{p+q} x, s_{p+q+1} s_\alpha \partial_0^{p+1} y). \]

Note that the sign exponent is slightly simpler than in the formula (3.3) for \( H \).

To see how \( \tilde{H} \) is derived from \( H \), we need to understand the effect of transposition and reversal on shuffles. For a \( (p, q) \)-shuffle \( (\alpha, \beta) \) we write
\[ (\tilde{\alpha}, \tilde{\beta}) = (m - \alpha, m - \beta) = (m - \alpha_p, \ldots, m - \alpha_1, m - \beta_q, \ldots, m - \beta_1) \]
for the shuffle obtained by reversing the sequence \( (0, \ldots, p + q - 1 = m) \).

**Lemma 4.2** Let \( (\alpha, \beta) \) be a \( (p, q) \)-shuffle, \( p, q \geq 0 \). Then
\[ (-1)^{(\beta, \alpha)} = (-1)^{(\alpha, \beta)} = (-1)^{(\alpha, \beta) + pq}. \]

**Proof** Applying \( pq \) transpositions transforms \( (\beta, \alpha) \) into \( (\alpha, \beta) \). For \( (\tilde{\alpha}, \tilde{\beta}) \) we can reverse the sequences \( \alpha, \beta \) and the whole set, so that the sign exponent changes by
\[ \frac{1}{2} (p + q)(p + q - 1) - \frac{1}{2} p(p - 1) - \frac{1}{2} q(q - 1) = pq. \] (4.9)
\( \square \)
Lemma 4.3 The following diagrams commute.

\[
\begin{align*}
C(X \times Y) & \xrightarrow{AW_{X,Y}} C(X) \otimes C(Y) \\
\downarrow \tilde{r}_{X,Y} & \quad \quad \downarrow \tilde{r}_{X,Y} \\
C(Y \times X) & \xrightarrow{AW_{Y,X}} C(Y) \otimes C(X)
\end{align*}
\]

\[
\begin{align*}
C(X) \otimes C(Y) & \xrightarrow{\nabla_{X,Y}} C(X \times Y) \\
\downarrow \tilde{r}_{X,Y} & \quad \quad \downarrow \tilde{r}_{X,Y} \\
C(Y) \otimes C(X) & \xrightarrow{\nabla_{Y,X}} C(Y \times X)
\end{align*}
\]

\[
\begin{align*}
C(X \times Y) & \xrightarrow{H_{X,Y}} C(X \times Y) \\
\downarrow \tilde{r}_{X,Y} & \quad \quad \downarrow \tilde{r}_{X,Y} \\
C(Y \times X) & \xrightarrow{H_{Y,X}} C(Y \times X)
\end{align*}
\]

**Proof** This is a direct computation. For the parity of the shuffles appearing in the diagrams for $\nabla$ and $H$ one uses Lemma 4.2.

The additional sign $(-1)^{n+1}$ in the definition (3.3) of $H$ compared to (4.8) compensates for the signs introduced by the map $\tilde{r}_{X,Y}$. This is analogous to the sign appearing in the identity (4.2) relating the two differentials. For $\tilde{H}$ the sign is $(-1)^{n+1}$ instead of $(-1)^n$ because $H$ is of degree $+1$ while $d$ is of degree $-1$.

Proposition 4.4 The triple $(AW, \nabla, \tilde{H})$ is a contraction.

**Proof** Apply Lemma 4.3 to the contraction $(AW, \nabla, H)$.

Proposition 4.5 The following diagrams commute.

\[
\begin{align*}
C(X \times Y \times Z) & \xrightarrow{AW_{X,Y,Z}} C(X) \otimes (Y \times Z) \\
\downarrow \tilde{H}_{X,Y,Z} & \quad \quad \downarrow \tilde{H}_{Y,Z} \\
C(X \times Y \times Z) & \xrightarrow{AW_{X,Y,Z}} C(X \otimes C(Y \times Z)) \tag{\tilde{A}_1}
\end{align*}
\]

\[
\begin{align*}
C(X \times Y \times Z) & \xrightarrow{AW_{X,Y,Z}} C((X \times Y) \otimes C(Z)) \\
\downarrow \tilde{H}_{X,Y,Z} & \quad \quad \downarrow \tilde{H}_{X,Y} \otimes 1 \\
C(X \times Y \times Z) & \xrightarrow{AW_{X,Y,Z}} C((X \times Y) \otimes C(Z)) \tag{\tilde{A}_2}
\end{align*}
\]

**Proof** We start with the second diagram and consider $(x, y, z) \in (X \times Y \times Z)^n$, for some $n \geq 0$. Applying the definitions for non-normalized chains, we get

\[
(\tilde{H}_{X,Y} \otimes 1) AW_{X,Y,Z}(x, y, z) = \sum_{k=1}^{n} \sum_{0 \leq p+q<k} \sum_{(\alpha, \beta)\in(p,q+1)} (-1)^{p+q+(\alpha, \beta)} \\
\cdot \left( s_{\beta} \partial_{p+1}^{p+q} \partial_{k+1}^{n} x \right. \left. s_{p+q+1} s_{\alpha} \partial_{0}^{n-1} \partial_{k+1}^{1} y \right) \otimes \partial_{0}^{k-1} z \tag{4.10}
\]
(there are no terms for \( k = 0 \)) and

\[
AW_{X \times Y, Z} \tilde{H}_{X, Y \times Z}(x, y, z)
\]

\[
= \sum_{k=0}^{n+1} \sum_{0 \leq p + q < n-k} (-1)^{p+q+(\alpha, \beta)} \cdot \left( \partial_{k+1}^{p+q} s_\beta \partial_{p+1}^{p+q} x, \partial_{k+1}^{p+q} s_{p+q+1} s_\alpha \partial_0^{p+q-1} y \right) \otimes \partial_0^{k-1} s_{p+q+1} s_\alpha \partial_0^{p+q-1} z. \tag{4.11}
\]

The second tensor factor in the last line is degenerate if \( k - 1 < p + q + 1 \). We may therefore assume \( k > p + q + 1 \geq 1 \), in which case this term can be written as

\[
\partial_0^{k-1} s_{p+q+1} s_\alpha \partial_0^{p+q-1} z = \partial_0^{k-2-p} \partial_0^{p-1} z = \partial_0^{k-2} z. \tag{4.12}
\]

The components of the first tensor factor in (4.11) can likewise be transformed to

\[
\partial_{k+1}^{n+1} s_\beta \partial_{p+1}^{p+q} x = s_\beta \partial_{k-q}^{n-q} \partial_{p+1}^{p+q} x = s_\beta \partial_{p+1}^{p+q} \partial_{k-n}^{n} x,
\]

\[
\partial_{k+1}^{n+1} s_{p+q+1} s_\alpha \partial_0^{p+q-1} y = s_{p+q+1} s_\alpha \partial_{k-p}^{p+q-1} \partial_{0}^{n} y = s_{p+q+1} s_\alpha \partial_0^{p+q-1} \partial_{k}^{n} y. \tag{4.13}
\]

Modulo degenerate chains this gives

\[
AW_{X \times Y, Z} \tilde{H}_{X, Y \times Z}(x, y, z)
\]

\[
= \sum_{0 \leq p + q < n - k} \sum_{k=p+q+2}^{n+1} (-1)^{p+q+(\alpha, \beta)} \cdot \left( s_\beta \partial_{p+1}^{p+q} \partial_{k+1}^{k} x, s_{p+q+1} s_\alpha \partial_0^{p+q-1} \partial_{k}^{n} y \right) \otimes \partial_0^{k-2} z. \tag{4.15}
\]

This is the same as (4.10) after substituting \( k + 1 \) for \( k \).

We now turn to the first diagram, where the argument is similar. We have

\[
(1 \otimes \tilde{H}_{Y, Z}) \cdot AW_{X \times Y, Z}(x, y, z)
\]

\[
= \sum_{k=0}^{n} \sum_{0 \leq p + q < n-k} (-1)^{k+p+q+(\alpha, \beta)} \cdot \partial_{k+1}^{n} x \otimes \left( s_\beta \partial_{p+1}^{p+q} \partial_{0}^{k-1} y, s_{p+q+1} s_\alpha \partial_0^{p+q-1} \partial_{0}^{k-1} z \right)
\]

\[
= \sum_{0 \leq p + q < n} \sum_{(\alpha, \beta)\sim(p, q+1)} (-1)^{k+p+q+(\alpha, \beta)} \cdot \partial_{k+1}^{n} x \otimes \left( s_\beta \partial_{p+1}^{p+q} \partial_{0}^{k-1} y, s_{p+q+1} s_\alpha \partial_0^{p+q-1} \partial_{0}^{k-1} z \right) \tag{4.16}
\]

and

\[
AW_{X, Y \times Z} \tilde{H}_{X \times Y, Z}(x, y, z)
\]

\[
= \sum_{k=0}^{n+1} \sum_{0 \leq p + q < n} (-1)^{p+q+(\alpha, \beta)} \cdot \partial_{k+1}^{n+1} s_\beta \partial_{p+1}^{p+q} x \otimes \left( s_{p+q+1} s_\alpha \partial_0^{p+q-1} \partial_{0}^{k-1} y, s_{p+q+1} s_\alpha \partial_0^{p+q-1} \partial_{0}^{k-1} z \right). \tag{4.17}
\]

The first tensor factor in (4.17) is degenerate if \( \beta \) contains a value \( < k \). We can therefore assume all such values to occur in \( \alpha \), so that in particular we obtain \( k \leq p < n \). Since we also have \( n - q \geq p + 1 \), the first tensor factor simplifies to

\[
\partial_{k+1}^{n+1} s_\beta \partial_{p+1}^{p+q} x = \partial_{k+1}^{n+1} \partial_{p+1}^{p+q} x = \partial_{k+1}^{n} x. \tag{4.18}
\]
in this case. Let us write \( \hat{p} = p - k \) as well as \( \hat{\alpha} = \{ i - k \mid k \leq i \in \alpha \} \) and \( \hat{\beta} = \beta - k \). The first component of the second tensor factor in (4.17) can be expressed as
\[
\partial_0^{k-1} s_\hat{\beta} \partial_{p+1}^{p+q} y = s_\hat{\beta} \partial_0^{k-1} \partial_{p+1}^{p+q} y = s_\hat{\beta} \partial_{p+1}^{p+q} \partial_0^{k-1} y
\]  
and the second component as
\[
\partial_0^{k-1} s_{p+q+1} s_\alpha \partial_0^{p-1} z = s_{\hat{p}+q+1} s_\alpha \partial_0^{p-1} z = s_{\hat{p}+q+1} s_\alpha \partial_0^{k+\hat{p}-1} z.
\]  
Modulo degenerate chains we therefore have
\[
AW_{X,Y,Z} \tilde{H}_{X,Y,Z}(x, y, z)
\]
\[
\equiv \sum_{k=0}^{p} \sum_{0 \leq p+q < n} \sum_{(\alpha, \beta) \in (p,q+1)} (-1)^{p+q+(\alpha, \beta)} \partial_0^n \otimes \left( s_\hat{\beta} \partial_{p+1}^{p+q} \partial_0^{k-1} y, s_{\hat{p}+q+1} s_\alpha \partial_0^{k+\hat{p}-1} z \right)
\]
\[
= \sum_{0 \leq k+\hat{p}+q < n} \sum_{(\hat{\alpha}, \hat{\beta}) \in (\hat{p}, q+1)} (-1)^{k+\hat{p}+q+(\hat{\alpha}, \hat{\beta})} \partial_0^n \otimes \left( s_\hat{\beta} \partial_{p+1}^{p+q} \partial_0^{k-1} y, s_{\hat{p}+q+1} s_\alpha \partial_0^{k+\hat{p}-1} z \right),
\]  
which matches (4.16). \( \square \)

**Proof of Proposition 3.1** The top of the cube

\[
\begin{array}{ccc}
C(X \times Y \times Z) & \xrightarrow{AW_{X,Y,Z}} & C(X \times Y) \otimes C(Z) \\
\xrightarrow{H_{X,Y \times X}} & & \xrightarrow{H_{X,Y} \otimes I} \\
C(X \times Y \times Z) & \xrightarrow{-} & C(X \times Y) \otimes C(Z) \\
\xrightarrow{H_{X,Y \times X}} & & \xrightarrow{-} \\
C(\tilde{Z} \times \tilde{Y} \times \tilde{X}) & \xrightarrow{1 \otimes \tilde{H}_{\tilde{Y},\tilde{X}}} & C(\tilde{Z}) \otimes C(\tilde{Y} \times \tilde{X}) \\
\xrightarrow{1 \otimes \tilde{H}_{\tilde{Y},\tilde{X}}} & & \xrightarrow{-} \\
C(\tilde{Z} \times \tilde{Y} \times \tilde{X}) & \xrightarrow{AW_{Z,\tilde{Y} \times \tilde{X}}} & C(\tilde{Z}) \otimes C(\tilde{Y} \times \tilde{X})
\end{array}
\]  

is the commutative diagram (A2), and the bottom is a relabelling of (A1). The front and back consist of the diagram

\[
\begin{array}{ccc}
C(X \times Y \times Z) & \xrightarrow{AW_{X,Y,Z}} & C(X \times Y) \otimes C(Z) \\
\xrightarrow{\tilde{H}_{X,Y,Z}} & & \xrightarrow{\tilde{H}_{X,Y,Z}} \\
C(\tilde{Z} \times \tilde{X} \times \tilde{Y}) & \xrightarrow{AW_{Z,\tilde{Y} \times \tilde{X}}} & C(\tilde{Z}) \otimes C(\tilde{X} \times \tilde{Y}) \\
\xrightarrow{(id_{\tilde{Z}}, \tau_{\tilde{X},\tilde{Y}})_x} & & \xrightarrow{(1 \otimes (\tau_{\tilde{X},\tilde{Y}}))_x} \\
C(\tilde{Z} \times \tilde{Y} \times \tilde{X}) & \xrightarrow{AW_{Z,\tilde{Y} \times \tilde{X}}} & C(\tilde{Z}) \otimes C(\tilde{Y} \times \tilde{X}).
\end{array}
\]
The left side is the diagram

\[
\begin{array}{c}
C(X \times Y \times Z) \xrightarrow{H_{X \times Y, Z}} C(X \times Y \times Z) \\
\downarrow \tilde{\tau}_{X, Y \times Z} \\
C(\tilde{Y} \times \tilde{Z} \times \tilde{X}) \xrightarrow{\tilde{H}_{Y, \tilde{X}}} C(\tilde{Y} \times \tilde{Z} \times \tilde{X}) \\
(\tau_{Y, Z}, \text{id}_X)_+ \downarrow \\
C(\tilde{Z} \times \tilde{Y} \times \tilde{X}) \xrightarrow{\tilde{H}_{Z, \tilde{Y}, \tilde{X}}} C(\tilde{Z} \times \tilde{Y} \times \tilde{X})
\end{array}
\] (4.24)

and the right side is

\[
\begin{array}{c}
C(X \times Y) \otimes C(Z) \xrightarrow{H_{X, Y} \otimes 1} C(X \times Y) \otimes C(Z) \\
\downarrow \tilde{\tau}_{X, Y} \otimes \tilde{\tau}_Z \\
C(\tilde{Y} \times \tilde{X}) \otimes C(\tilde{Z}) \xrightarrow{\tilde{H}_{Y, \tilde{X}} \otimes 1} C(\tilde{Y} \times \tilde{X}) \otimes C(\tilde{Z}) \\
\downarrow \tau_{C(Y \times Z), C(Z)} \\
C(\tilde{Z}) \otimes C(\tilde{Y} \times \tilde{X}) \xrightarrow{1 \otimes \tilde{H}_{Y, \tilde{X}}} C(\tilde{Z}) \otimes C(\tilde{Y} \times \tilde{X}).
\end{array}
\] (4.25)

All four sides commute by Lemma 4.3 and naturality. One verifies directly that the composed vertical maps of the various diagrams agree wherever needed. To see that the signs work out correctly, one uses the identity

\[\nu(m + n) \equiv \nu(m) + \nu(n) + mn \pmod{2}\] (4.26)

for \(m, n \geq 0\). This defines the vertical arrows in (4.22) and completes the argument.

For (A₁) one performs a similar diagram chase, this time starting with the commutative diagram (A₂). \(\square\)

**Proposition 4.6** The following diagrams commute.

\[
\begin{array}{c}
C(X) \otimes C(Y \times Z) \xrightarrow{\nabla_{X, Y \times Z}} C(X \times Y \times Z) \\
\downarrow 1 \otimes \tilde{H}_{Y, Z} \\
C(X) \otimes C(Y \times Z) \xrightarrow{\nabla_{X, Y \times Z}} C(X \times Y \times Z)
\end{array}
\] (B₁)

\[
\begin{array}{c}
C(X \times Y) \otimes C(Z) \xrightarrow{\nabla_{X \times Y, Z}} C(X \times Y \times Z) \\
\downarrow \tilde{H}_{X \times Y, Z} \\
C(X \times Y) \otimes C(Z) \xrightarrow{\nabla_{X \times Y, Z}} C(X \times Y \times Z)
\end{array}
\] (B₂)

**Proof** This uses Proposition 3.2 and is otherwise analogous to the proof of Proposition 3.1, done in reverse. \(\square\)

**Corollary 4.7** Corollary 3.3 and Proposition 3.4 remain valid for \(\tilde{H}\) instead of \(H\).

**Proof** The proof of Corollary 3.3 carries over. For Proposition 3.4 we note the equality of isomorphisms

\[\tilde{\tau}_{X, Z \times Y} (\text{id}, \tau_{Y, Z})_+ = \tilde{\tau}_{Y \times X, Z} (\tau_{X, Y}, \text{id})_+ : C(X \times Y \times Z) \to C(\tilde{Z} \times \tilde{Y} \times \tilde{X})\] (4.27)
and write $\sigma = (\tau_{X,Y}, 1) (id, \tau_{Y,Z})^{-1} : X \times Z \rightarrow Y \times Y \times Z$. The diagram
\[
\begin{align*}
C(X \times Y \times Z) &\xrightarrow{(id,\tau_{Y,Z})} C(X \times Z \times Y) \xrightarrow{\tilde{\tau}_{X,Y\times Z}} C(\tilde{Z} \times \tilde{Y} \times \tilde{X}) \\
C(X \times Y \times Z) &\xrightarrow{(id,\tau_{Y,Z})} C(X \times Z \times Y) \xrightarrow{\tilde{\tau}_{X,Y\times Z}} C(\tilde{Z} \times \tilde{Y} \times \tilde{X}) \\
C(X \times Y \times Z) &\xrightarrow{\tau_{X,Y}\cdot id} C(Y \times X \times Z) \xrightarrow{\tilde{\tau}_{Y,X\times Z}} C(\tilde{Z} \times \tilde{Y} \times \tilde{X}) \\
C(X \times Y \times Z) &\xrightarrow{\tau_{X,Y}\cdot id} C(Y \times X \times Z) \xrightarrow{\tilde{\tau}_{Y,X\times Z}} C(\tilde{Z} \times \tilde{Y} \times \tilde{X})
\end{align*}
\] (4.28)
commutes by naturality and Lemma 4.3 as well as the identity (4.27) for the centre-right square. The same applies to the analogous diagram for the composition $H_{X,Y\times Z} H_{X\times Y,Z}$. The isomorphism (4.27) therefore translates between Proposition 3.4 and its analogue for $H$.

□

5 Relations among the Eilenberg–Zilber maps

Let us summarize what is known about relations between the Eilenberg–Zilber maps and their interactions with the differential and the transposition maps (2.1) and (4.6). We focus on Eilenberg–Mac Lane’s original homotopy $H$. For a one-point space $\ast$, we make the canonical identifications $C(\ast) = k$ and $\ast \times X = X \times \ast = X$.

(i) The Alexander–Whitney map $AW$, the shuffle map $\nabla$ and the homotopy $H$ are natural with respect to pairs of maps between simplicial sets.

(ii) If one argument is a one-point space, then $AW$ and $\nabla$ reduce to the identity map on the chain complex of the other space, and $H$ reduces to 0.

(iii) Both $AW$ and $\nabla$ are chain maps, and the contraction identities (3.6) hold.

(iv) The Alexander–Whitney map is coassociative. The shuffle map is associative [3, Thms. 5.2 & 5.4] and also commutative in the sense that the following diagram commutes:
\[
\begin{align*}
C(X) \otimes C(Y) &\xrightarrow{\nabla_{X,Y}} C(X \times Y) \\
C(Y) \otimes C(X) &\xrightarrow{\nabla_{Y,X}} C(Y \times X)
\end{align*}
\] (5.1)

(v) The diagram
\[
\begin{align*}
C(X \times Y) \otimes C(Z \times W) &\xrightarrow{\nabla_{X,Y,Z\times W}} C(X \times Z \times Y \times W) \\
C(X) \otimes C(Y) \otimes C(Z) \otimes C(W) &\xrightarrow{AW_{X,Y} \otimes AW_{Z,W}} C(X \times Y \times Z \times W)
\end{align*}
\] (5.2)

commutes [6, Prop. 2.2.1]. Like the properties mentioned in the previous item, this holds already before normalization.

(vi) One has the identities given in Propositions 3.1, 3.2 and 3.4.
By setting one simplicial set equal to $*$, one can deduce from (v) the commutative diagram [17, Lemme II.4.2]

$$C(X) \otimes C(Y \times Z) \xrightarrow{\nabla_{X,Y \times Z}} C(X \times Y \times Z)$$

as well as its mirror image

$$C(X \times Y) \otimes C(Z) \xrightarrow{\nabla_{X,Y} \otimes 1} C(X \times Y) \otimes C(Z)$$

Property (v) together with naturality also implies that the shuffle map is a morphism of coalgebras [5, (17.6)]; one can equivalently derive (v) from this latter fact and naturality. We have seen, moreover, that Corollary 3.3 follows from property (vi).

**Question 5.1** Are there any relations involving the Eilenberg–Zilber maps that are not formal consequences of the properties listed above?

## 6 Comparing the twisting cochains

We finally turn to twisted Cartesian products and compare the twisting cochains defined by Szczarba and Shih.

### 6.1 Twisted Cartesian products

Let $B$ be a simplicial set, $G$ a simplicial group and $\tau : B_{>0} \to G$ a twisting function. We use the convention

$$\partial_0 \tau(b) = \tau(\partial_0 b)^{-1} \tau(\partial_1 b)$$

from [18, Eq. (1.1)], [13, Def. 18.3], which corresponds to a left action of $G$ on bundle fibres. To turn this into a right action one instead takes $\sigma(b) = \tau(b)^{-1}$, which is a twisting function in the sense of [17, p. 25].

Let $F$ be a simplicial set with a $G$-action. The twisted Cartesian product $F \times_\tau B$ has the modified first face map

$$\partial_0^+(f, b) = (\tau(b) \cdot \partial_0 f, \partial_0 b) = (\partial_0 f \cdot \sigma(b), \partial_0 b)$$

for simplices of positive dimension, depending on whether one lets $G$ act on the left or on the right.

### 6.2 Shih's twisting cochain

We recall the definition of Shih’s twisting cochain, compare [17, §II.1], [2], [9, Sect. 4]. We write $d_\otimes$ for the tensor product differential on $C(F) \otimes C(B)$ and

$$\delta = d^r - d = \begin{cases} 
\partial_0^+ - \partial_0 & \text{in positive degree}, \\
0 & \text{in degree } 0
\end{cases}$$

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$$\delta = d^r - d = \begin{cases} 
\partial_0^+ - \partial_0 & \text{in positive degree}, \\
0 & \text{in degree } 0
\end{cases}$$
for the difference between the twisted and the untwisted differential on the graded module $C(F \times B) = C(F \times_\tau B)$.

**Lemma 6.1** Let $f$ be a simplicial operator from the product $F \times B$ to itself. Then for any $c \in C(F \times B)$ there is an $n \geq 0$ such that $(\delta f)^n(c) = 0$.

**Proof** This follows from the naturality of $f$ with respect to the filtration of $F \times B$ by the skeletons of $B$, see [2, p. 34].

As a consequence, the twisted differential $d_\tau$ can be transferred along the contraction $(A W, \nabla, \tilde{H})$ to the differential

$$d_t = d_\otimes + \sum_{n=0}^{\infty} AW (\delta \tilde{H})^n \delta \nabla$$

(6.4)
on $C(F) \otimes C(B)$. (Lemma 6.1 guarantees that for any argument the sum is finite.) Because $\delta$ involves only the first face map and the diagrams $(\tilde{A}_2)$ and $(\tilde{B}_1)$ commute, we obtain a twisted tensor product $C(F) \otimes_\tau C(B)$ related to $C(F \times_\tau B)$ by a new contraction, see Sect. 7 below. The twisting cochain

$$t : C(B) \to C(G)$$

(6.5)
satisfies

$$d(t) = t \cup t,$$

(6.6)
and we have

$$d_t = d_\otimes + (\mu \otimes 1)(1 \otimes t \otimes 1)(1 \otimes \Delta)$$

(6.7)
where $\Delta$ is the diagonal of $C(B)$ and the map $\mu : C(F) \otimes C(G) \to C(F)$ is induced by the $G$-action, see [17, §II.1, Thm. 2], [9, p. 410]. Conversely, $t$ can be recovered from the differential $d_t$ on the twisted tensor product $C(G) \otimes_\tau C(B)$ as the composition

$$C(B) \xrightarrow{\eta \otimes 1} C(G) \otimes C(B) \xrightarrow{d_t - d_\otimes} C(G) \otimes C(B) \xrightarrow{1 \otimes \varepsilon} C(G)$$

(6.8)
where $\varepsilon : C(B) \to \mathbb{k}$ is the augmentation and $\eta : \mathbb{k} \to C(G)$ the unit map [17, p. 28].

### 6.3 Szczarba’s twisting cochain

Szczarba [18, Sect. 2] has defined an explicit twisting cochain

$$t_{sz} : C(B) \to C(G)$$

(6.9)
and an explicit quasi-isomorphism $\psi$ between $C(B)_{t_{sz}} \otimes C(F)$ and $C(B \times_\tau F)$. Note that Szczarba’s ordering of the factors differs from Shih’s; we will come back to this in Sect. 7 where we also recall the definition of $\psi$. In the proof below we are going to use a characterization of $t_{sz}$ due to Morace–Prouté [14, Sect. 6].

---

5 The sign in the twisting cochain condition is given incorrectly in [17, p. 29] and [16, Def. 3.2.6], but the same sign is correct in [18, p. 198] and [9, Def. 2.3]. This difference is explained by the different orders of the factors in the twisted tensor product, compare [12, Def. II.1.4].

6 Szczarba calls $\varphi(x) = -(-1)^{|x|} t(x)$ a twisting cochain. Also note that he does not use the Koszul sign convention, so that he has $(f \otimes g)(x \otimes y) = f(x)g(y)$ without the sign $(-1)^{|g||x|}$. That Szczarba’s maps $t_{sz}$ and $\psi$ are well-defined on normalized complexes is shown in [7, App. B].
**Theorem 6.2** Let $F \times_\tau B$ be a twisted Cartesian product with structure group $G$. Then Szczarba’s twisting cochain $t_{sz}$ is equal to Shih’s twisting cochain based on the opposite Eilenberg–Zilber contraction $(AW, \nabla, \tilde{H})$.

**Proof** Assume first that $B$ is reduced, that is, with a single vertex. In this case twisting functions $B_{>0} \to G$ correspond bijectively to maps of simplicial groups $q_\tau : \Omega B \to G$, compare [13, Cor. 27.2]. Both Szczarba’s and Shih’s twisting cochain are natural with respect to commutative diagrams

$$
\begin{array}{ccc}
B_{>0} & \xrightarrow{\tau} & G \\
\downarrow p & & \downarrow q \\
\tilde{B}_{>0} & \xrightarrow{\tilde{\tau}} & \tilde{G}
\end{array}
$$

in the sense that

$$
\tilde{\tau} p_* = q_* t : C(B) \to C(\tilde{G})
$$

holds for each of them. We may therefore assume $G = \Omega B$ in this case.

Morace–Prouté have characterized Szczarba’s cochain as the unique natural twisting cochain $t : C(B) \to C(\Omega B)$ satisfying

$$
t(b) = \sigma(b) - 1 \in (\Omega B)_0
$$

for all $B$ (or just $B = \tilde{\Delta}[n]$) and all $b \in B_1$ as well as

$$
\tilde{h} t(\tilde{e}_n) = 0
$$

for all $n \geq 2$, where $\tilde{\Delta}[n]$ is the $n$-simplex with all vertices identified, $\tilde{e}_n \in \Delta[n]_n$ its fundamental simplex and $\tilde{h} : C(\tilde{\Delta}[n]) \to C(\Delta[n])$ the contracting homotopy defined in [14, Sect. 3]. As remarked in [14, p. 89], the condition (6.13) is satisfied if each simplex appearing in $t(\tilde{e}_n)$ is ‘right justified’, meaning that it has the same final vertex $n$ as $\tilde{e}_n$.

Condition (6.12) for Shih’s twisting cochain follows from (6.8) and the identity

$$
(d_t - d_\otimes)(1 \otimes b) = AW \delta(1, b) = AW((\sigma(b), b_0) - (1, b_0)) = (\sigma(b) - 1) \otimes b_0
$$

for $b \in B_1$. If the last face operator is never used in the recursive definition of $t$, then all simplices appearing in the resulting chains are right-justified, so that (6.13) holds. An inspection of the formulas for $\delta$ and $\tilde{H}$ shows that this is indeed the case, which proves our claim in case of a reduced base.

For general $B$ we consider the reduced simplicial set $\tilde{B}$ obtained by identifying all vertices in $B$. The definition of a twisting function implies that $\tau$ sends all degenerations of vertices to identity elements in $G$. It therefore induces a twisting function $\tilde{\tau} : \tilde{B}_{>0} \to G$ such that (6.10) commutes for $q = id_G$. Because Szczarba’s and Shih’s cochains agree for the twisted Cartesian product $F \times_\tilde{\tau} \tilde{B}$, they do so for $F \times_\tau B$. This completes the proof. \qed

7 The twisted shuffle maps

Shih’s twisted differential (6.4) is part of a twisted Eilenberg–Zilber contraction, see [17, Thm. II.1.1]. The twisted shuffle map

$$
\nabla^\tau : C(F) \otimes_{t_{sz}} C(B) \to C(F \times_\tau B)
$$
is compatible with the right $C(B)$-comodule structure on both sides and for $F = G$ also equivariant with respect to $C(G)$, see [17, Props. II.4.2 & II.4.3]. (The same holds in fact for the twisted Alexander–Whitney map and the twisted homotopy, compare [9, Lemma 4.5*].)

On the other hand, Szczarba [18, Thm. 2.4] gives an explicit quasi-isomorphism

$$\psi : C(B)_{tsz} \otimes C(F) \to C(B \times_\tau F).$$  \tag{7.2}

with the factors $B$ and $F$ swapped compared to Shih, hence with an adjusted definition of the twisted differential on the left-hand side, cf. [12, Def. II.1.4]. We observe that Szczarba’s map enjoys the same nice properties as Shih’s.

**Proposition 7.1** The map $\psi$ is a morphism of left $C(B)$-modules and, in the case $F = G$, also a morphism of right $C(G)$-modules.

Before entering the proof, let us recall that $\psi$ is defined as the composition

$$C(B)_{tsz} \otimes C(F) \xrightarrow{\varphi \otimes 1} C(B \times G) \otimes C(F) \xrightarrow{\nabla_{B \times G,F}} C(B \times G \times F) \xrightarrow{(\text{id},\mu)_s} C(B \times_\tau F) \tag{7.3}$$

where $\mu : G \times F \to F$ is the action map, and

$$\varphi(b) = \sum_{i \in S_n} (-1)^i \varphi(D_{i,0}^{n+1}|b|, D_{i,1}^{n+1}|\sigma(b)| \cdots D_{i,n}^{n+1}|\sigma(b)|) =: \sum_{i \in S_n} (-1)^i \varphi_i(b) \tag{7.4}$$

for $b \in B_n$. Following Hess–Tonk’s exposition [11, Sect. 1.4], we write

$$S_n = \{ i = (i_1, \ldots, i_n) \in \mathbb{N}^n \mid 0 \leq i_s \leq n - s \text{ for all } s \} \tag{7.5}$$

and also

$$|i| = \sum i = i_1 + \cdots + i_n. \tag{7.6}$$

The indices $i$ and $k$ Szczarba uses in the recursive definition of the simplicial operators $D_{j,i}^{n+1} = D_{j,i}^{n+1}$ [18, Eq. (3.1)] are related to $i$ via

$$i = 1 + \sum_{s=1}^n (n - s) i_s \in \{ 1, \ldots, n! \} \tag{7.7}$$

and $k = i_1$. We moreover have $|i| + n = \varepsilon (i, n + 1)$ in Szczarba’s notation, see the remark before [11, Thm. 7].

**Proof** The equivariance with respect to the $C(G)$-action follows by naturality and associativity of the shuffle map. We now consider the $C(B)$-comodule structure.

Because of naturality and the commutative diagram 5.4) it is enough to show that $\varphi$ is a morphism of left comodules, that is, to establish the identity

$$\partial_{k+1}^n b \otimes \varphi(\partial_{k-1}^n b) = \sum_{i \in S_n} (-1)^{i_l} \partial_{k+1}^n D_{0,i}^{n+1}|b| \otimes \partial_{0}^{k-1}\varphi(b) \tag{7.8}$$

for any $0 \leq k \leq n$ and any $b \in B_n$. Recall that all operators $D_{j,i}^{n+1}$ are frontal [11, p. 1866], hence satisfy the identity (2.4). It follows by induction from the definition

$$D_{0,i}^{n+1} = \begin{cases} (D_{0,i}^{n+1})' & \text{if } i_1 = 0, \\ (D_{0,i}^{n+1})' s_0 d_{i_1} = s_0 D_{0,i_2,\ldots,i_n}^{n+1} d_{i_1} & \text{if } i_1 > 0 \end{cases} \tag{7.9}$$

7 In the definition of $\psi$ in [18, p. 201] the upper summation index should read “$p$”.
that $\partial_{k+1}D_{0,i}^{n+1}b$ is degenerate unless $i_1 = \cdots = i_k = 0$. For $i_1 = 0$ the identity $\partial_0 \varphi_i(b) = \varphi_i(\partial_0 b)$ holds, see [18, bottom of p. 205]. By induction this gives

$$\partial_0^{k-1} \varphi_i(b) = \varphi_i(\partial_0^{k-1} b)$$

(7.10)

for $i_1 = \cdots = i_k = 0$ and completes the proof.

To arrive at a twisted tensor product of the form $C(B) \otimes C(F)$ with Shih’s approach, Gugenheim starts with a twisted Cartesian product $B \times_{\bar{i}} F$ where the twisting occurs in the last face map, see [9, p. 406, Rem.] and also [16, p. 53]. This leads to a different twisting cochain $\tilde{t}$, essentially related to $t_{sz}$ via the passage to opposite simplicial sets as in Sect. 4.

There are two more “canonical” Eilenberg–Zilber contractions that use the transposed Alexander–Whitney map

$$T_{C(Y),C(X)} AW_{X,Y}(\tau_{X,Y}) : C(X \times Y) \to C(X) \otimes C(Y),$$

(7.11)

for $(x, y) \in (X \times Y)_n$, together with the shuffle map and variants of $H$ and $\tilde{H}$, see [16, p. 52]. Applying Shih’s algorithm to one of them and the twisted Cartesian product $B \times_{\bar{i}} F$ does indeed lead to a differential induced by Szczarba’s twisting cochain $t_{sz}$ on the graded module $C(B) \otimes C(F)$. However, because the Alexander–Whitney map has changed, so have the comodule structures over $C(B)$. This means that one does not obtain Szczarba’s twisted tensor product this way, either. It therefore appears that Theorem 6.2 does not shed light onto the precise relationship between the two twisted shuffle maps (7.1) and (7.2) and the underlying twisted tensor products.

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