Anomalous Scaling of the Passive Scalar

Krzysztof Gawędzki
I.H.E.S., C.N.R.S., F-91440 Bures-sur-Yvette, France

Antti Kupiainen
Mathematics Department, Helsinki University,
PO Box 4, 00014 Helsinki, Finland

Abstract

We establish anomalous inertial range scaling of structure functions for a model of advection of a passive scalar by a random velocity field. The velocity statistics is taken gaussian with decorrelation in time and velocity differences scaling as $|x|^\kappa/2$ in space, with $0 \leq \kappa < 2$. The scalar is driven by a gaussian forcing acting on spatial scale $L$ and decorrelated in time. The structure functions for the scalar are well defined as the diffusivity is taken to zero and acquire anomalous scaling behavior for large pumping scales $L$. The anomalous exponent is calculated explicitly for the $4^{th}$ structure function and for small $\kappa$ and it differs from previous predictions. For all but the second structure functions the anomalous exponents are nonvanishing.

In 1941 A.N. Kolmogorov argued that in fully developed turbulence a range of spatial scales exists where the velocity structure functions acquire a form independent of the IR and UV cutoffs provided by the scale of energy pumping and dissipation respectively. Ever since a debate has been going on as to whether there are corrections to the scaling exponents predicted by Kolmogorov and whether such corrections depend on the dissipation or the pumping scale or both. This question being still quite open for Navier-Stokes turbulence both experimentally and theoretically, it is useful to consider it in the context of simpler models that are nevertheless expected to display phenomena similar to the Navier-Stokes equations.

In this letter we consider one such model that has attracted much attention recently [1, 2, 3, 4, 5, 6], namely that of passive advection in a random velocity field $\mathbf{v}(t, x)$ of a scalar quantity $T$ whose density $T(t, x)$ satisfies the equation

$$\partial_t T + (\mathbf{v} \cdot \nabla) T - \nu \Delta T = f ,$$

(1)

where $\nu$ denotes the molecular diffusivity of the scalar $T$ and $f(t, x)$ is an external source driving the system.

We take $\mathbf{v}(t, x)$ and $f(t, x)$ to be mutually independent Gaussian random fields with zero mean and covariances

$$\langle v^i(t, x) v^j(t', x') \rangle = \delta(t - t') \ D^{ij}(x - x') ,$$

(2)

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\[< f(t, \mathbf{x}) f(t', \mathbf{x}') > = \delta(t - t') \mathcal{C}(\frac{\mathbf{x} - \mathbf{x}'}{L}) \equiv \delta(t - t') \mathcal{C}_L(\mathbf{x} - \mathbf{x}') . \]  

Both are thus decorrelated in time, a fact that leads to an exact solution for the correlation functions of the scalar. The forcing covariance $\mathcal{C}$ is assumed to be a real, smooth, positive-definite function with rapid decay at spatial infinity so that the forcing takes place on the ("integral") scale $L$.

The velocity covariance $D$ is taken to mimic the situation in real turbulent flow with structure function $< (\mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, 0))^2 >$ proportional to $|\mathbf{x}|^\kappa$ for $\kappa > 0$. Concretely, we set

\[D^{ij}(\mathbf{x}) = D_0 \int e^{i \mathbf{k} \cdot \mathbf{x}} (k^2 + m^2)^{-(3+\kappa)/2} (\delta^{ij} - k^i k^j/k^2) \frac{dk}{(2\pi)^3}, \]  

where the transverse projector in the Fourier space ensures the incompressibility of $\mathbf{v}$. Small $m^2$ is an infrared cutoff making the integral convergent for $0 < \kappa < 2$. Writing $D(\mathbf{x}) = D(0) - \tilde{D}(\mathbf{x})$, we have

\[D^{ij}(0) = \frac{\Gamma(\kappa/2)}{12\pi^{3/2} \Gamma((\kappa-3)/2)} \delta^{ij} D_0 m^{-\kappa}, \]  

i.e. it diverges with $m \to 0$, but the velocity structure function has a limit

\[\lim_{m \to 0} \tilde{D}^{ij}(\mathbf{x}) = D_1 \left((2 + \kappa) \delta^{ij} |\mathbf{x}|^\kappa - \kappa x^i x^j |\mathbf{x}|^{\kappa-2}\right) \]  

which is a homogeneous function of $\mathbf{x}$. $D_1 \equiv \frac{\Gamma((2-\kappa)/2)}{2^{2+\kappa} \pi^{3/2} \kappa (3+\kappa) \Gamma((3+\kappa)/2)} D_0$ and both constants have dimension $\text{length}^{2-\kappa}/\text{time}$.

We would like to study the statistical properties of the solutions of Eq. (1) in the regime of small $\nu$, small $m$ (which may be viewed as the inverse of another integral scale) and large $L$. In particular, the universality question for the passive scalar may be formulated as inquiring about the existence of the limit of the correlation functions $< \prod T(t_n, \mathbf{x}_n) >$ in a stationary state of the system when $\nu, m, L^{-1} \to 0$ and about the independence of such a limit of the shape of the source covariance $\mathcal{C}$. We will show that the model possesses an "inertial" range of scales $(\nu/D_1)^{1/\kappa} \ll |\mathbf{x}| \ll \min(L, m^{-1})$ where these correlators become independent of $\nu$, have a limit as $\nu \to 0$ and $m \to 0$ (independent of the order), but in general have non-universal (i.e. dependent on the forcing covariance) contributions involving positive powers of $L$. In particular we show that the structure functions

\[S_{2M}(x) \equiv < (T(x) - T(0))^{2M} > \sim \gamma_{2M} (L/|x|)^{\rho_{2M} M} |x|^{(2-\kappa)M} \]  

for $|x| \ll L$ in the $\nu = 0$ limit. The amplitudes $\gamma_{2M}$ are $\kappa$- and $\mathcal{C}$-dependent and the anomalous exponents $\rho_{2M}$ depend on $\kappa$ but not on $\mathcal{C}$. We find that $\rho_2 = 0$ but

\[\rho_4 = \frac{4}{5} \kappa + \mathcal{O}(\kappa^2) \]  

for small $\kappa$. The Hölder inequality implies that $\rho_N$ is a convex function of $N$. It follows that all $\rho_{2M}$ for $M = 2, 3, \ldots$ are strictly positive ($\rho_{2M} \geq (M-1)\rho_4$) and that they increase with $M$. Thus structure functions of order four and higher exhibit anomalous scaling and have explicit integral scale dependence. While we are able to calculate $\rho_4$ only for small $\kappa$ there is no reason to doubt the generality of the phenomenon for all $0 < \kappa < 2$. We want to stress that the non-universality is due to the pumping scale,
not to the dissipation scale, although the derivatives of the field $T$ have short distance (non-anomalous) singularities that are easy to analyze.

It is quite well known that in the stationary state the scalar correlations satisfy linear PDE’s. In the presence of the UV and IR cutoffs $\nu$ and $m, L$ they have well defined representations in terms of the Green functions of the corresponding differential operators which we now recall (for more details see [9, 10]).

Suppressing the spatial variable, the solution of Eq. (1) with the initial condition $T_0$ at $t = t_0$ takes the form

$$T(t) = R(t, t_0)T_0 + \int_{t_0}^{t} R(t, s)f(s) ds,$$

where $R(t, t_0)$ is given by the time ordered exponential ($t \geq t_0$)

$$R(t, t_0) = \mathcal{T} e^{\int_{t_0}^{t} (\nu \Delta + \nu(\tau) \nabla) d\tau}.$$

Thus, to calculate the correlations of $T$ we need to evaluate expectations of products of matrix elements of $R(t, t_0)$. We shall use the tensor product notation $R(t, t_0) \otimes N$ as a bookkeeping device for all such products. One then calculates

$$< R(t, t_0) \otimes N > = e^{-(t-t_0)\mathcal{M}_N},$$

where $\mathcal{M}_N$ is the differential operator

$$\mathcal{M}_N = -\sum_{n=1}^{N} (\nu \Delta x_n + \frac{1}{2} \mathcal{D}^{ij}(0) \partial_{x_n^i} \partial_{x_n^j}) - \sum_{n<n'} \mathcal{D}^{ij}(x_n - x_{n'}) \partial_{x_n^i} \partial_{x_{n'}^j}. $$

The gaussian integral of the time ordered exponentials is calculable due to the independence of $\nu$’s at different times. The $\frac{1}{2} \mathcal{D}^{ij}(0)$ term is the contribution of contractions within a single $R$ and the last terms come from contractions between different $R$’s. The former one is an “eddy-diffusivity” contribution: since $\mathcal{D}^{ij}(0) = \frac{1}{2} \delta^{ij} \mathcal{D}^{ll}(0)$ and $\mathcal{M}_1 = -(\nu + \frac{1}{2} \mathcal{D}^{ll}(0)) \Delta$, it follows from Eq. (1) with $N = 1$ that, in the absence of the sources, the expectation value of the scalar diffuses with the effective diffusion constant composed of the molecular diffusivity $\nu$ and the eddy diffusivity $\frac{1}{2} \mathcal{D}^{ll}(0)$. For small $m$, the eddy diffusivity dominates and the diffusion is driven by the large distance scales (recall from Eq. (3) that $\mathcal{D}^{ll}(0) = \mathcal{O}(D_0 m^{-n})$).

To get hold of the steady state of the scalar, let us first consider the 2-point function. From (3), we obtain

$$< T(t) \otimes 2 > = e^{-(t-t_0)\mathcal{M}_2} T(t_0) \otimes 2 + \int_{t_0}^{t} ds e^{-(t-s)\mathcal{M}_2} C_L.$$}

When $t_0 \to -\infty$, the term with $T(t_0)$ disappears due to the positivity of $\mathcal{M}_2$ and we obtain for the steady state

$$< T \otimes 2 > = \mathcal{M}_2^{-1} C_L.$$}

Due to the translational invariance of $C$ the eddy diffusivity will not contribute to (14). Indeed, note that $\mathcal{M}_2$ commutes with (three-dimensional) translations and in the action on translation-invariant functions of $x_1 - x_2 \equiv x$ reduces to

$$\mathcal{M}_2 = -2\nu \Delta - \tilde{\mathcal{D}}^{ij}(x) \partial_i \partial_j.$$
Since $\bar{D}^{ij}(x) \equiv D^{ij}(0) - D^{ij}(x)$ has an $m \to 0$ limit given by (13), so does the operator $\mathcal{M}_2$ in the action on translation-invariant functions and when $\nu \to 0$, it becomes a singular elliptic operator $\mathcal{M}_2^{sc} = -D_1 ((2 + \kappa) |x|^\kappa - \kappa x_i x^i |x|^\kappa -\kappa ) \partial_i \partial_j$. It is now easy to analyze (14) as the various cutoffs $\nu$, $m$, $L$ are removed using the rotational invariance of $\mathcal{M}_2$. In the $m \to 0$ and $\nu \to 0$ limits (which commute, we could also take $m$ proportional to $L^{-1}$ with no loss), one obtains for the two point function $F_2(|x|) \equiv <T(x)T(0)>$
\[ F_2(r) = \gamma_2 L^{2-\kappa} - \frac{2\epsilon}{3(2-\kappa)D_1} r^{2-\kappa} \left(1 + \mathcal{O}(\frac{r}{L})\right), \] (16)
where $\gamma_2$ is a non-universal (i.e. $C$-dependent) constant and $\epsilon = \frac{1}{2} C(0)$ is the energy dissipation rate of the scalar. Note that the non-universal term (a constant) is annihilated by $\mathcal{M}_2^{sc}$. This has to be so if the equation $\mathcal{M}_2^{sc} F_2 = \mathcal{C}_L$ is to be satisfied: the right hand side becomes universal in the limit $L \to \infty$ so all non-universal terms in $F_2(r)$ surviving in this limit have to be annihilated by $\mathcal{M}_2^{sc}$. We shall see this general mechanism limiting possible non-universal terms also for the higher point functions. The constant term of $F_2$ drops out from the $2^{nd}$ structure function which has a universal $L \to \infty$ limit so that the exponent $\rho_2 = 0$. The same universal result holds approximately in the whole inertial range $\eta \ll r \ll \min(L, m^{-1})$, where the Kolmogorov scale $\eta = (\nu/D_1)^{1/\kappa}$.

Let us now analyze the higher point correlators. Proceeding as with the 2-point function, the steady state solution in terms of the operators $\mathcal{M}_N$ follows after some simple algebra. For the 4-point function one gets
\[ <\prod_{n=1}^4 T(x_n)> = F_4(x_1, x_2, x_3, x_4) + F_4(x_1, x_3, x_2, x_4) + F_4(x_1, x_4, x_2, x_3) \] (17)
with the single channel function
\[ F_4 = \mathcal{M}_4^{-1} (\mathcal{M}_2^{-1} \otimes 1 + 1 \otimes \mathcal{M}_2^{-1}) \mathcal{C}_L \otimes \mathcal{C}_L. \] (18)
Similarly, the higher stationary state equal-time correlation functions $<T^{\otimes 2M}>$ are obtained by symmetrizing the expressions
\[ \mathcal{M}_{2M}^{-1} (\mathcal{M}_{2M-2}^{-1} \otimes 1_2) \cdots (\mathcal{M}_2^{-1} \otimes 1_{2M-2}) \mathcal{C}_L \otimes^M \] (19)
in their arguments (the odd-point functions vanish). Expressions (18) and (19) are well defined for $\nu, m, L^{-1}$ nonzero and we need to discuss their limits as these cutoffs are removed.

The main points of this analysis are the following. Acting on translation invariant functions as in (19), $\mathcal{M}_N$ becomes
\[ \mathcal{M}_N = -\nu \sum_{n=1}^N \Delta x_n + \sum_{n<n'} \bar{D}^{ij}(x_n - x_{n'}) \partial_{x_n} \partial_{x'_{n'}} \] (20)
i.e. the eddy diffusivity cancels and (20) has an $m \to 0$ and $\nu \to 0$ limit which is a singular elliptic operator $\mathcal{M}_N^{sc}$. It can be shown (14) that the Green functions occurring in (19) also have limits that are well defined in the UV and render (19) finite for $L$ finite. Thus we need to find the leading behavior of (19) as $L \to \infty$ with $\mathcal{M}_{2M}$ replaced by $\mathcal{M}_N^{sc}$.

Let us look in more detail at the 4-point function. Recalling that $F_2 = \mathcal{M}_2^{-1} \mathcal{C}_L$, it is convenient to view Eq. (18) as a differential equation for $F_4$ that becomes for the connected part $F_4^c = F_4 - F_2 \otimes F_2$
\[ \mathcal{M}_4 F_4^c = \mathcal{L} (F_2 \otimes F_2), \] (21)
where \( \mathcal{L} \) is given by the sum in (20) with \( n = 1,2 \) and \( n' = 3,4 \). By (16), the RHS of (21) has a well defined limit as \( L \to \infty \) given by

\[
\frac{e^2}{9(2-\kappa)^2 D_1^2} \mathcal{L} |X|^{2-\kappa} |Z|^{2-\kappa}
\]

(22)

and is a homogeneous (rotationally-invariant) function of \( X \equiv x_1 - x_2 \), \( Y \equiv x_2 - x_3 \) and \( Z \equiv x_3 - x_4 \) of degree \( 2-\kappa \). It is not difficult to write down a solution \( F_4^{sc} \) of Eq. (21) for the limiting case with \( M_4^{sc} \) and (23) on the RHS. One easily checks that

\[
F_4^{sc} = \frac{e^2}{6(2-\kappa)^2(5-\kappa)D_1^2} (|X|^{2(2-\kappa)} + |Z|^{2(2-\kappa)}) - \frac{e^2}{9(2-\kappa)^2 D_1^2} |X|^{2-\kappa} |Z|^{2-\kappa},
\]

(23)

is such a solution with the use of the decomposition \( M_4^{sc} = M_2^{sc} \otimes 1 + 1 \otimes M_2^{sc} - \mathcal{L} \) since \( M_2^{sc} \otimes 1 + 1 \otimes M_2^{sc} \) vanishes in the action on \( F_4^{sc} \) and \( \mathcal{L} \) annihilates functions depending only on \( X \) or only on \( Z \). Thus, we deduce that \( M_4^{sc} (F_4^{c} - F_4^{csc}) \to 0 \) as \( L \to \infty \). By scale invariance it is thus reasonable to conclude that the solution for finite but large \( L \) should differ from the universal scaling form by zero modes of \( M_4^{sc} \) so that

\[
F_4^{c} - \sum_{0 \leq \rho_{4,n} \leq 2(2-\kappa)} \sum_{m} \gamma_{nm} F_{4,\rho_{4,n}}^{c} \xrightarrow{L \to \infty} F_4^{csc}
\]

(24)

where \( F_{4,\rho_{4,n}}^{c} \) are homogeneous zero modes of \( M_4^{sc} \) of degree \( 2(2-\kappa) - \rho_{4,n} \) and the non-universal coefficients \( \gamma_{nm} \) depend on the source covariance \( \mathcal{C} \).

In fact using spectral analysis of \( \mathcal{M}_4 \) (4,24) can be made rigorous (possibly with logarithmic corrections in \( L \) for special values of \( \kappa \)). Similar analysis can be repeated for \( N \)-point correlators: non-universal \( L \)-dependent terms proportional to homogeneous zero modes of \( M_4^{sc} \) can be present in the large \( L \) asymptotics. We thus face the problem of finding such zero modes, of determining whether they are present in the \( N \)-point function of \( T \) and finally of finding whether they contribute to the structure function \( S_N \). While such an analysis still eludes us for general values of \( \kappa \), we will now show that at least for small \( \kappa \) the zero modes are present and dominate the structure functions.

The stationary state correlation functions of the scalar, given in terms of (14), are in general non-gaussian, but they become gaussian as \( \kappa \to 0 \). To see this, note that in this limit \( \widetilde{D}^{ij} = 2D_1 \delta^{ij} \) (having finite \( D_1 \) requires the vanishing of \( D_0 \) as \( \kappa \to 0 \) in order to renormalize the ultraviolet divergence in (4); \( D_0 \) will never show up below.) We immediately obtain for the \( \kappa = 0 \) operators:

\[
\mathcal{M}_{2,0}^{sc} = 2D_1 \nabla_{x_1} \cdot \nabla_{x_2} = -2D_1 \Delta X
\]

(25)

\[
\mathcal{M}_{4,0}^{sc} = 2D_1 \sum_{1 \leq n < n' \leq 4} \nabla_{x_n} \cdot \nabla_{x_{n'}}
\]

\[
= -2D_1 (\Delta X + \Delta Y + \Delta Z - \nabla X \cdot \nabla Y - \nabla Y \cdot \nabla Z)
\]

(26)

in the difference variables \( X, Y, Z \) and using the subscript to refer to \( \kappa = 0 \). Some straightforward algebra (4) shows that the expression (19) reduces (when symmetrized) to the standard gaussian expression of sums of product of 2-point functions.

Our strategy is now the following. We shall find the homogeneous zero modes of the operator \( \mathcal{M}_4^{sc} \) in perturbation expansion in powers of \( \kappa \). Eq. (1) implies that (for \( m = 0 \))

\[
\widetilde{D}^{ij}(x) = 2D_1(\delta^{ij} + \kappa R^{ij}(x)) + \mathcal{O}(\kappa^2)
\]

with

\[
R^{ij}(x) = \frac{\delta^{ij}}{2 + \ln|x|} - \frac{1}{2} x^i x^j |x|^{-2}
\]

(27)
Hence, to the first order in $\kappa$, $M^{sc}_{4} = M^{sc}_{4,0} + 2\kappa D_1 V_4$, with
\[
V_4 = -R^{ij}(X) \partial_{x^i} \partial_{y^j} - R^{ij}(Y) \partial_{y^i} \partial_{x^j} - R^{ij}(Z) \partial_{z^i} \partial_{z^j} - (R^{ij}(X + Y) - R^{ij}(X) - R^{ij}(Y)) \partial_{x^i} \partial_{y^j} - (R^{ij}(Y + Z) - R^{ij}(Y) - R^{ij}(Z)) \partial_{y^i} \partial_{z^j} - (R^{ij}(X + Y + Z) - R^{ij}(X + Y) - R^{ij}(Y + Z) + R^{ij}(Y)) \partial_{x^i} \partial_{z^j}
\]

(28)

Since $M^{sc}_{4}$ commutes with 3-dimensional translations and rotations and with the permutations of four points we shall search for its zero modes respecting these symmetries. Consider first the unperturbed zero modes. The symmetric zero modes of the lowest order. For the four square vectors we shall search for its zero modes respecting these symmetries. Consider first the unperturbed zero modes. The symmetric zero modes of the lowest order, when expressed in terms of point-differences, have the form
\[
a \sum_{\{n, n'\}} (x_n - x_{n'})^4 + b \sum_{\{n, m, n', m'\}} (x_n - x_m)^2 (x_n - x_{m'})^2 + c \sum_{\{n, n', m, m'\}} (x_n - x_{n'})^2 (x_m - x_{m'})^2
\equiv a F_1 + b F_2 + c F_3,
\]

(29)

where the pairs $\{n, n'\}$ and $\{m, m'\}$ are assumed different, as well as the pairs $\{n, m\}$ and $\{n, m'\}$ and where $10a + 14b + 3c = 0$.

The constant survives as the eigenvalue of $M^{sc}_{4}$ for $\kappa \neq 0$. Thus we need to calculate in degenerate perturbation theory how the fourth degree zero modes change with $\kappa$. For this we write $M^{sc}_{4,0} = -2D_1 (\Delta X + \Delta Y + \Delta Z)$, where $\bar{X} = X$, $\bar{Y} = \sqrt{2}(Y + X + Z)$, $\bar{Z} = Z$. Denoting $R \equiv (X^2 + Y^2 + Z^2)^{1/2} = (\sum_{\{n, n'\}} (x_n - x_{n'})^2)^{1/2}$, we obtain
\[
M^{sc}_{4,0} = -\frac{2D_1}{R^2} \partial_R R^4 \partial_R - \frac{2D_1}{R^2} \Phi
\]

(30)

where $\Phi$ is the Laplacian on the sphere $S^8$ in the space of $\{\bar{X}, \bar{Y}, \bar{Z}\}$. Now pick two linearly independent zero modes $R^4 f_i$, $i = 1, 2$, of the form (29) and look for a homogeneous zero mode
\[
R^{4 + \kappa \lambda} (a_1 f_1 + a_2 f_2 + \kappa f_3)
\]

(31)

with a homogeneous degree zero function $f_3$ orthogonal to $f_{1,2}$ in $L^2(S^8)$. We obtain in the linear order in $\kappa$
\[
M^{sc}_{4,0} (\lambda R^4 \ln R (a_1 f_1 + a_2 f_2) + R^4 f_3) + 2D_1 V_4 R^4 (a_1 f_1 + a_2 f_2) = 0
\]

(32)

or, using the form (30) of $M^{sc}_{4,0}$,
\[
-15 \lambda (a_1 f_1 + a_2 f_2) - 44 f_3 - \Phi f_3 + \frac{1}{R^2} V_4 R^4 (a_1 f_1 + a_2 f_2) = 0.
\]

(33)

Upon taking the $L^2(S^8)$ scalar products with $f_{1,2}$, $f_3$ drops out resulting in the relation
\[
-15 \lambda \sum_{j=1, 2} (f_i, f_j) a_j + \sum_{j=1, 2} (f_i, \frac{1}{R^2} V_4 R^4 f_j) a_j = 0
\]

(34)

Hence $\lambda$ has to solve the equation
\[
\det \left( (f_i, \frac{1}{15 R^2} V_4 R^4 f_j) - \lambda (f_i, f_j) \right) = 0.
\]

(35)
For the explicit calculation we took $R^4 f_1 = 3F_1 - 10F_3$ and $R^4 f_2 = -7F_1 + 5F_2$. The integrals over the 8-dimensional sphere are most conveniently done by using homogeneity to transform them to gaussian integrals over $\mathbb{R}^9$. The integration is straightforward with Maple$^\text{\textregistered}$ and the matrix in (35) becomes proportional to \[
abla F \begin{pmatrix} -25 - 25\lambda & 15 + 15\lambda \\ 18 + 15\lambda & -20 - 20\lambda \end{pmatrix}
\] with the eigenvalues $\lambda_1 = -14/5$ and $\lambda_2 = -1$. The corresponding eigenfunctions (31) are given by $F_{4,1}^c = R^{-2(4/5)\lambda+O(\kappa^2)}(F_1 - 2F_2 + 6F_3 + O(\kappa^2))$ and $F_{4,2}^c = R^{-\kappa + O(\kappa^2)}(-7F_1 + 5F_2 + O(\kappa^2))$. For large $L$ the connected 4-point function takes the form

\[
< \prod_{n=1}^{4} T(x_n) >^c = \gamma_{4,0} L^{-2\kappa} + \gamma_{4,1} (L/R)^{4\kappa/5+O(\kappa^2)} R^{-2\kappa} (F_1 - 2F_2 + 6F_3 + O(\kappa)) \\
+ \gamma_{4,2} (L/R)^{-\kappa + O(\kappa^2)} R^{-2\kappa} (-7F_1 + 5F_2 + O(\kappa)) \\
+ < \prod_{n=1}^{4} T(x_n) >^{c_{sc}} + O((L/R)^{-2+O(\kappa)})
\]

(36)

uniformly in small $\kappa$. Since the connected correlation vanishes and $< \prod_{n=1}^{4} T(x_n) >^{c_{sc}}$ reduces to $\frac{e^2}{360D_t^2} (3F_1 - 10F_3)$ for $\kappa = 0$, we infer that $\gamma_{4,0} = O(\kappa)$, $\gamma_{4,1} = \frac{e^2}{216D_t^2} + O(\kappa)$ and $\gamma_{4,2} = \frac{e^2}{540D_t^2} + O(\kappa)$. The result [1] for $N = 2$ follows with $\gamma_4 = \frac{e^2}{3D_t^2} + O(\kappa)$ and $\rho_4$ given by (8) since only the $F_{4,1}^c$ -term gives non-zero contribution to the structure function. The zero mode, with $\lambda = -1$, is actually obtained from a zero mode of $\mathcal{M}_3$ by extending it to a function of four $x_i$’s by symmetrizing. This is a general feature: zero modes of $\mathcal{M}_N$ give rise to zero modes of $\mathcal{M}_{2M}$ for $2M > N$. These however do not contribute to the structure functions $S_{2M}$. The only zero mode of $\mathcal{M}_{2M}$ that contributes, is the unique one not coming from the lower dimensional $\mathcal{M}_N$’s, namely the one that at $\kappa = 0$ is gotten from the monomial $\prod_{i=1}^{M}(x_{2i-1} - x_{2i})^2$ by symmetrizing and subtracting partial traces. It gives rise to the dominant contribution to $S_{2M}$ which has to be present by the Hölder inequality.

The asymptotic behavior of the scalar correlation functions encodes a subtle information about the behavior of the Green functions of the singular multibody operators $\mathcal{M}_N$ with continuous spectrum. The reduction of its study to that of discrete spectrum operators given by $\mathcal{M}_N$’s acting on homogeneous functions should be thought of as realizing a renormalization group type approach to the model, with the homogeneous zero modes of $\mathcal{M}_N$ playing the role of relevant interactions. This may be the most important hint from the above exact solution for the anomalous scaling of the passive scalar.

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