Unitary \((g, K)\) modules of \(SU(2, 1)\)

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Abstract

Let \(G = SU(2, 1)\). In this paper we parametrize irreducible unitary \((g, K)\) modules of \(G\). The parametrization is done in two steps. Firstly, we parametrize irreducible \((g, K)\) modules (Theorem 3). In the second step we find unitary \((g, K)\) modules (Theorem 5). One can compare our results with [Kra76].

1 Introduction

Let \(G\) be a real reductive group. We will follow the definition of the real reductive group from [Kna96]. The main goal of the representation theory is finding the unitary dual of the group \(G\). One can approach to this problem using \((g, K)\) modules (see [Bal97]) which correspond to admissible representations. In the first step all irreducible \((g, K)\) modules are found. In the second step it remains to find unitary \((g, K)\) modules. The first step correspond to Langlands classification. The second problem is still unsolved in general.

In this paper we find the unitary dual for the group \(G = SU(2, 1)\). The unitary dual of \(SU(n, 1)\) is already found by Kraljević (see [Kra72] and [Kra76]). Why should anyone solve already solved problem? We hope that our technique, which is applied for \(SU(2, 1)\), can be generalized and applied to some other groups. Generalization to \(S(n, 1)\) does not look too hard. The author already works on some other groups. Finally, the construction of unitary \((g, K)\) modules in this case is very explicit.

We start with the set of coefficients \(\{a_{nm}, b_{nm}, c_{nm}, d_{nm}\}\) which completely describe the action of the complexified Lie algebra \(g\). Actually, products

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and $bc$ can be calculated and they give an important information about the irreducibility of $(g, K)$ modules. Also, the set of products $ad$ and $bc$ (explained in Theorem 2) is the key ingredient in description of unitary $(g, K)$ modules and classification of irreducible unitary $(g, K)$ modules. The main idea in this approach is to treat $K$ types as points. The rest of construction at some points looks like a construction of irreducible unitary $(g, K)$ modules of $SL(2, \mathbb{R})$.

In Section 2 we recall basic results in representation theory of $SL(2, \mathbb{R})$. Some statements will be used later and some statements will be compared with our results. Also, we wanted to demonstrate our ideas in this case.

In Section 3 we construct $(g, K)$ modules using certain set of coefficients. After that we parametrize irreducible $(g, K)$ modules of $G = SU(2, 1)$. The parametrization is given in Theorem 3. In Section 4 we parametrize unitary $(g, K)$ modules. The key step is done in Theorem 4. The parametrization is given in Theorem 5.

Lie groups will be denoted by capital letters, corresponding Lie algebras by Gothic letters with subscript 0 and complexified Lie algebras by Gothic letters without subscript. For example, the Lie algebra of $G$ will be denoted by $g_0$ and the complexified Lie algebra by $g$. If $H$, $X$ and $Y$ span a basis for $sl(2, \mathbb{C})$ such that $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$ then we say that $sl(2, \mathbb{C})$ is represented by a triple $(H, X, Y)$. If $\pi, V$ is the representation of $g$, usually, we will write $X.v$ instead of $\pi(X)v$ for $X \in g$ and $v \in V$.

## 2 Unitary dual of $SL(2, \mathbb{R})$

Let $G = SL(2, \mathbb{R})$, $K = SO(2)$, $g = sl(2, \mathbb{C})$, $g_0 = sl(2, \mathbb{R})$ and $\mathfrak{k} = so(2)$. Unitary representations of $G$ are well known, see [Bar47], However, we redo the construction since some details appear later.

We use notation and results from [Vog81]. The basis of $g$ contains elements

\[
H = -i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\]

\[
X = \frac{1}{2} \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = \frac{1}{2} (A + iB)
\]

and

\[
Y = \frac{1}{2} \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = \frac{1}{2} (A - iB).
\]
It is easy to check that
\[
[iH, B] = 2A, \quad [A, iH] = 2B \quad \text{and} \quad [B, A] = -2iH.
\]
Let \(W\) be a \(\mathfrak{sl}(2, \mathbb{C})\) module. Then we can choose a basis of \(W\) such that \(w^k \in W\), \(H.w^k = kw^k\) and
\[
X.w^k = \frac{1}{2}(\lambda + (k + 1)) w^{k+2} = a_kw^{k+2}
\]
and
\[
Y.w^k = \frac{1}{2}(\lambda - (k - 1)) w^{k-2} = b_kw^{k-2}
\]
for some \(\lambda \in \mathbb{C}\). It is easy to see that
\[
A.w^k = (X + Y).w^k = a_kw^{k+2} + b_kw^{k-2}
\]
and
\[
B.w^k = i(-X + Y).w^k = i(-a_kw^{k+2} + b_kw^{k-2}).
\]
If we consider a finite-dimensional module \(V\) of dimension \(n\), we will use the same basis, with different indexes denoted by \(v\), such that \(H.v^k = (n + 1 - 2k)v^k\) and
\[
X.v^k = -(k - 1)v^{k-1}
\]
and
\[
Y.v^k = -(n - k)v^{k+1}.
\]
We can assume that \(v^0 = v^{n+1} = 0\).

Let us determine \(\lambda\)s for which it is possible to construct an inner product \(\langle \cdot, \cdot \rangle : W \times W \rightarrow \mathbb{C}\) such that (24) is satisfied. Let us assume that \(\langle w^k, w^l \rangle \neq 0\) for some \(k\) and \(l\). Then,
\[
\langle iH.w^k, w^l \rangle = \langle w^k, (iH)^*.w^l \rangle
\]
and (24) show that \(i(n + 1 - 2k)\langle w^k, w^l \rangle = i(n + 1 - 2l)\langle w^k, w^l \rangle\). Hence
\[
\langle w^k, w^l \rangle = 0 \quad \text{for all} \quad k \neq l.
\]
Now, from (2), it follows that
\[
A^*.w^k = \frac{b_{k+2} ||w^k||^2}{||w^{k+2}||^2} w^{k+2} + \frac{a_{k-2} ||w^k||^2}{||w^{k-2}||^2} w^{k-2}
\]
and

\[ B^*.w^k = i\left(-\frac{b_{k+2}}{|w^k|^2}||w^k||^2 w^{k+2} + \frac{a_{k-2}}{|w^{k-2}}|^2 w^{k-2}\right). \]

Each time we get the same condition:

\[ a_k + \frac{b_{k+2}}{|w^k|^2}||w^k||^2 = 0, \quad \forall k. \]

or

\[ \frac{a_k}{b_{k+2}} = -\frac{||w^k||^2}{||w^{k+2}}|^2 \in (-\infty, 0), \quad \forall k. \]

However, we prefer to multiply it by \( b_{k+2} = ||b_{k+2}||^2 \) and say that the irreducible representation is unitary if and only if

\[ a_kb_{k+2} \in (-\infty, 0), \quad \forall k. \] (3)

We consider an open interval \((-\infty, 0)\) since the case \( a_kb_{k+2} = 0 \) leads to reducibility. It will be explained in Remark. Relation (3) transforms to

\[ (\lambda + k + 1)(\lambda - (k + 1)) = \lambda^2 - (k + 1)^2 \in (-\infty, 0), \quad \forall k. \]

Let us recall that \( k = 2z \) or \( k = 2z + 1 \) for \( z \in \mathbb{Z} \). If \( k = 2z \), \( \lambda \) can be equal to \( ri \) for \( r \in \mathbb{R} \) (it corresponds to principal series), \( r \in (-1, 1) \) (it corresponds to complementary series) and \( 2m + 1 \) for \( m \in \mathbb{Z} \) (submodules correspond to discrete series and the trivial representation). If \( k = 2z+1 \), \( \lambda \) can be equal to \( ri \) for \( r \in \mathbb{R^*} \) (it corresponds to principal series), \( 0 \) (submodules correspond to mock discrete series) and \( 2m \) for \( m \in \mathbb{Z} \) (submodules correspond to discrete series).

It remains to analyze (unitary) finite-dimensional representations of the compact real form of \( \mathfrak{g} \). For the beginning, let us choose the basis:

\[ -\frac{1}{2}iH, \quad \frac{1}{2}(X - Y) = \frac{1}{2}iB \quad \text{and} \quad -\frac{1}{2}i(X + Y) = -\frac{1}{2}iA. \]

Then \(-\frac{1}{2}iH.v^k = -\frac{1}{2}i(n + 1 - 2k)v^k,\)

\[ \frac{1}{2}(X - Y).v^k = \frac{1}{2}\left(-(k - 1)v^{k-1} + (n - k)v^{k+1}\right) \]

and

\[ -\frac{1}{2}i(X + Y).v^k = \frac{1}{2}i\left((k - 1)v^{k-1} + (n - k)v^{k+1}\right). \]
Element $\frac{1}{2}iH$ satisfies (24) (for any inner product satisfying (2)). It remains to analyze remaining two elements. In any case, (24) produces the same condition:

$$||v^{k+1}||^2 = \frac{k}{n-k}||v^k||^2.$$  \hspace{1cm} (4)

It will be useful to express $||v^k||^2$ in terms of $||v^1||^2$. Using induction, one can easily show that

$$||v^k||^2 = \frac{(k-1)!(n-k)!}{(n-1)!}||v^1||^2 = \frac{1}{(n-k)!}||v^1||^2$$  \hspace{1cm} (5)

**Example 1.** Let $V$ be the $su(2)$ module such that dim $V = 5$ and $||v^1|| = 1$. Then

$$||v^2|| = \frac{1}{2}, \quad ||v^3|| = \frac{1}{\sqrt{6}}, \quad ||v^4|| = \frac{1}{2}, \quad ||v^5|| = 1.$$  

**3 (g, K) modules for SU(2, 1)**

Let $G = SU(2, 1)$. Then $K = SU(2) \times SU(1) = SU(2) \times S^1$, $\frak{g}_0 = su(2) \oplus \mathbb{R}$ and $\frak{k} = sl(2, \mathbb{C}) \oplus \mathbb{C}$. This $sl(2, \mathbb{C})$ can be represented by a triple $(H_\alpha, X_\alpha, Y_\alpha)$. It remains to set $\mathbb{C} = Z\mathbb{C}$ where $Z = H_\alpha + 2H_\beta$. Let us define a basis for $\frak{g}$. The Cartan subalgebra is generated by

$$H_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad H_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$  

Now, we define

$$X_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad X_{\alpha+\beta} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Elements $Y_\alpha, Y_\beta$ and $Y_{\alpha+\beta}$ are defined similarly. The next step is to define a basis for $\frak{g}_0$. We take $iH_\alpha$ and $iH_\beta$ for the Cartan subalgebra and

$$A_\alpha = X_\alpha - Y_\alpha \quad \text{and} \quad B_\alpha = i(X_\alpha + Y_\alpha)$$

for the remainder of $\frak{k}_0$. The rest of $\frak{g}_0$ is given by

$$A_\beta = X_\beta + Y_\beta, \quad B_\beta = i(X_\beta - Y_\beta)$$

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\[ A_{\alpha+\beta} = X_{\alpha+\beta} + Y_{\alpha+\beta} \quad \text{and} \quad B_{\alpha+\beta} = i(X_{\alpha+\beta} - Y_{\alpha+\beta}) \] (6)

One should notice a different sign in expressions for \( \alpha \) and \( \beta \).

**Definition 1.** Let \( V \) be an irreducible \((g, K)\) module. Then the restriction of \( V \) to \( K \) has the form

\[ V|_K = \bigoplus_{n \in \mathbb{N}, m \in \mathbb{Z}} V_{nm} \]

where \( n = \dim V_{nm} \) \( (H_\alpha v_{nm} = (n-1)v_{nm}) \) and

\[ mv_{nm}^k = (H_\alpha + 2H_\beta).v_{nm}^k \]

for any element \( v_{nm}^k \in V_{nm} \).

**Theorem 1.** Let \( v_{n,m}^k \in V_{n,m} \). Then

\[ X_{\alpha+\beta}.v_{nm}^k = a_{nm}v_{n+1,m+3}^k + \frac{k-1}{n-1}c_{nm}v_{n-1,m+3}^{k-1}, \]

\[ X_\beta.v_{nm}^k = -a_{nm}v_{n+1,m+3}^{k+1} + \frac{n-k}{n-1}c_{nm}v_{n-1,m+3}^k, \]

\[ Y_{\alpha+\beta}.v_{nm}^k = b_{nm}v_{n+1,m-3}^{k+1} + \frac{n-k}{n-1}d_{nm}v_{n-1,m-3}^k, \]

\[ Y_\beta.v_{nm}^k = b_{nm}v_{n+1,m-3}^k - \frac{k-1}{n-1}d_{nm}v_{n-1,m-3}^{k-1}, \] (8)

for some coefficients \( a_{nm}, b_{nm}, c_{nm} \) and \( d_{nm} \).

**Remark 1.** This theorem shows that the set \( \{V_{nm}\} \) and coefficients \( a_{nm}, b_{nm}, c_{nm} \) and \( d_{nm} \) determine the structure of the \((g, K)\) modules \( V \). It is clear that these coefficients are not determined uniquely. However, the products \( a_{nm}d_{n+1,m+3} \) and \( b_{nm}c_{n+1,m-3} \) are unique and it is an important observation.

**Remark 2.** One can say that irreducible \( K \) modules \( V_{nm} \) can be represented as points. It is the main idea in our construction. We want to understand the structure of \( K \) modules.

**Proof.** We will prove the first two relations. Let us consider \( X_{\alpha+\beta}.v_{nm}^1 \). It is clear that

\[ X_{\alpha+\beta}.v_{nm}^1 \in \bigoplus_{p \geq n+1} V_{p,m+3}. \]
Let us assume that $X_{\alpha + \beta}.v_{nm}^1 = a + b$ for $a \in V_{q_m+3}$, where $q > n + 1$ and $b \in \bigoplus_{p \geq n+1, p \neq q} V_{p_m+3}$. Then $X_\alpha X_{\alpha + \beta}.v_{nm}^1 \neq 0$. It produces a contradiction since $X_{\alpha + \beta}X_\alpha.v_{nm}^1 = 0$ and $[X_\alpha, X_{\alpha + \beta}] = 0$. It shows that

$$X_{\alpha + \beta}.v_{nm}^1 = a_{nm}v_{n+1m+3}^1$$

(9)

for some coefficient $a_{nm}$. Now, let us calculate $X_\beta.v_{nm}^1$. Similar calculation shows that

$$X_\beta.v_{nm}^1 = \lambda v_{n+1m+3}^2 + c_{nm}v_{n-1m+3}^1.$$

for some coefficients $\lambda$ and $c_{nm}$. The action of $X_\alpha$ and (11) produces

$$X_\alpha X_\beta.v_{nm}^1 = -\lambda v_{n+1m+3}^1.$$

Since $[X_\alpha, X_\beta] = X_{\alpha + \beta}$ and $X_\alpha.v_{nm}^1 = 0$,

$$X_{\alpha + \beta}.v_{nm}^1 = [X_\alpha, X_\beta].v_{nm}^1 = -\lambda v_{n+1m+3}^1.$$ 

(10)

Now, (9) and (11) show that $\lambda = -a_{nm}$.

We continue by induction on $k$. The base of induction is just proved. Let us assume that first two relations are valid for $k$. Since $[Y_\alpha, X_\beta] = 0$,

$$X_\beta.v_{nm}^{k+1} = -\frac{1}{n-k}X_\beta Y_\alpha.v_{nm}^k = -\frac{1}{n-k}Y_\alpha X_\beta.v_{nm}^k$$

$$= -\frac{1}{n-k}Y_\alpha \left( -a_{nm}v_{n+1m+3}^{k+1} + \frac{n-k}{n-1}c_{nm}v_{n-1m+3}^k \right)$$

$$= \frac{a_{nm}}{n-k}(- (n-k))v_{n+1m+3}^{k+2} - \frac{c_{nm}}{n-1}(-(n-1-k))v_{n-1m+3}^{k+1}$$

$$= -a_{nm}v_{n+1m+3}^{k+2} + \frac{n-(k+1)}{n-1}c_{nm}v_{n-1m+3}^{k+1}.$$
Now we use this result and obtain

\[ X_{\alpha + \beta}.v_{nm}^{k+1} = (X_\alpha X_\beta - X_\beta X_\alpha).v_{nm}^{k+1} \]

\[ = X_\alpha \left( -a_{nm}v_{n+1m+3}^{k+2} + \frac{n - (k + 1)}{n - 1} c_{nm}v_{n-1m+3}^{k+1} \right) - \frac{n - (k + 1)}{n - 1} k c_{nm}v_{n-1m+3}^{k+1} \]

\[ = a_{nm}(k + 1)v_{n+1m+3}^{k+1} + k \left( -a_{nm}v_{n+1m+3}^{k+1} + \frac{n - k}{n - 1} c_{nm}v_{n-1m+3}^{k} \right) \]

\[ = a_{nm}v_{n+1m+3}^{k+1} + \frac{k}{n - 1} c_{nm}v_{n-1m+3}^{k}. \]

Remaining two relations can be proved similarly. \( \square \)

**Theorem 2.** Coefficients \( a_{nm}, b_{nm}, c_{nm} \) and \( d_{nm} \) define a \((g, K)\) module \( V \) if and only if they satisfy

\[ - \frac{1}{n} a_{nm}d_{n+1m+3} + b_{nm}c_{n+1m-3} - c_{nm}b_{n-1m+3} = \frac{m - n + 1}{2}, \quad (11) \]

\[ - a_{nm}d_{n+1m+3} + \frac{1}{n} b_{nm}c_{n+1m-3} + d_{nm}a_{n-1m-3} = \frac{m + n - 1}{2}, \quad (12) \]

\[ b_{nm}a_{n+1m-3} = a_{nm}b_{n+1m+3}, \quad (13) \]

\[ d_{nm}c_{n-1m-3} = c_{nm}d_{n-1m+3}, \quad (14) \]

\[ (n + 1)a_{nm}c_{n+1m+3} = nc_{nm}a_{n-1m+3}, \quad (15) \]

\[ (n + 1)b_{nm}d_{n+1m-3} = nd_{nm}b_{n-1m-3}. \quad (16) \]

**Proof.** Let us assume that \((g, K)\) module exists. Let us consider a nonzero element \( v_{nm}^k \) and apply the relation \([X_\beta, Y_\beta] = H_\beta\) on that element. The left
hand side is equal to

\[
(X_\beta Y_\beta - Y_\beta X_\beta).v^k_{nm} = X_\beta \left( b_{nm}v^{k}_{n+1m-3} - \frac{k-1}{n-1} d_{nm}v^{k-1}_{n-1m-3} \right) \\
- Y_\beta \left( -a_{nm}v^{k+1}_{n+1m+3} + \frac{n-k}{n-1} c_{nm}v^k_{n-1m+3} \right) \\
= b_{nm} \left( -a_{n+1m-3}v^{k+1}_{n+2m} + \frac{n+1-k}{n} c_{n+1m-3}v^k_{nm} \right) \\
- \frac{k-1}{n-1} d_{nm} \left( -a_{n-1m-3}v^k_{nm} + \frac{n-k}{n-2} c_{n-1m-3}v^{k-1}_{n-2m} \right) \\
+ a_{nm} \left( b_{n+1m+3}v^{k+1}_{n+2m} - \frac{k}{n} d_{n+1m+3}v^k_{nm} \right) \\
- \frac{n-k}{n-1} c_{nm} \left( b_{n-1m+3}v^k_{nm} - \frac{k-1}{n-2} d_{n-1m+3}v^{k-1}_{n-2m} \right).
\]

The right hand side is equal to

\[
H_\beta.v^k_{nm} = \frac{m-n-1+2k}{2} v^k_{nm}.
\]

One can compare coefficients of \( v^{k-1}_{n-2m} \) and obtain (14). Similarly, coefficient of \( v^{k+1}_{n+2m} \) produces (13). Finally, coefficient of \( v^k_{nm} \) produces

\[
-\frac{k}{n} a_{nm}d_{n+1m+3} + \frac{n+1-k}{n} b_{nm}c_{n+1m-3} \\
-\frac{n-k}{n-1} c_{nm}b_{n-1m+3} + \frac{k-1}{n-1} d_{nm}a_{n-1m-3} = \frac{m-n-1+2k}{2} 
\]

(17)

For \( k = 1 \), one obtains (11) and for \( k = n \) it transforms to (12). It is easy to check that (17) is a linear combination of (11) and (12), namely

\[
\frac{n-k}{n-1} (11) + \frac{k-1}{n-1} (12) = (17).
\]

It shows that it is enough to consider (11) and (12). These two relations are more convenient than (17) since \( k \) does not appear in (11) and (12).

If we apply the relation \( X_\beta X_{\alpha+\beta} = X_{\alpha+\beta}X_\beta \) on the element \( v^k_{nm} \) we obtain (15). Finally, if we apply the relation \( Y_\beta Y_{\alpha+\beta} = Y_{\alpha+\beta}Y_\beta \) on the element \( v^k_{nm} \)
we obtain (16). One can check all other commutation relations in \( g \), but it will not produce new conditions on coefficients \( a_{nm}, b_{nm}, c_{nm} \) and \( d_{nm} \). It shows that (11) – (16) have to be satisfied.

Now, let us assume that coefficients \( a_{nm}, b_{nm}, c_{nm} \) and \( d_{nm} \) are given and (11) – (16) are satisfied. We want to reconstruct \((g, K)\) module \( V \) using conditions above. It is enough to start with any \( K \) type \( V_{nm} \) and then reconstruct all other \( K \) types. The construction is good since all commutation relations are satisfied.

It is more convenient to work with \( a_{nm}d_{n+1,m+3} \) and \( b_{nm}c_{n+1,m-3} \) then \( a_{nm}, b_{nm}, c_{nm} \) and \( d_{nm} \). The reason is very simple. The later expressions are not determined uniquely since they depend on the choice of vectors \( v_{nm}^k \). Hence, we plan to determine expressions \( a_{nm}d_{n+1,m+3} \) and \( b_{nm}c_{n+1,m-3} \) using (11) and (12) and then show that it is possible to determine coefficients \( a_{nm}, b_{nm}, c_{nm} \) and \( d_{nm} \) such that all relations above are satisfied (and given \((g, K)\) module exists). We will be able to give explicit formulas for expressions \( a_{nm}d_{n+1,m+3} \) and \( b_{nm}c_{n+1,m-3} \). Then, it is easy to give formulas for coefficients \( a_{nm}, b_{nm}, c_{nm} \) and \( d_{nm} \).

Let us write (16) for \( m+6 \) instead of \( m \) and multiply by (15). It produces

\[
(n + 1)^2 a_{nm}d_{n+1,m+3}b_{n,m+6}c_{n+1,m+3} = n^2 a_{n-1,m+3}d_{n,m+6}b_{n-1,m+3}c_{nm}
\]

and

\[
\frac{a_{nm}d_{n+1,m+3}}{a_{n-1,m+3}d_{n,m+6}} = \frac{b_{n,m+6}c_{n+1,m+3}}{b_{n-1,m+3}c_{nm}} = \frac{n^2}{(n + 1)^2}.
\]

Now, (13) and (14) show that \( \frac{a_{nm}}{a_{n-1,m+3}} = \frac{b_{n,m+6}}{b_{n-1,m+3}} \) and \( \frac{d_{n+1,m+3}}{d_{n,m+6}} = \frac{c_{n+1,m+3}}{c_{nm}} \).

We conclude that (19) transforms to

\[
\left( \frac{a_{nm}d_{n+1,m+3}}{a_{n-1,m+3}d_{n,m+6}} \right)^2 = \frac{n^2}{(n + 1)^2}
\]

or

\[
\frac{a_{nm}d_{n+1,m+3}}{a_{n-1,m+3}d_{n,m+6}} = \frac{b_{n,m+6}c_{n+1,m+3}}{b_{n-1,m+3}c_{nm}} = \pm \frac{n}{n + 1}.
\]

It is possible to give more precise statement. Using induction, one can obtain

\[
\frac{a_{nm}d_{n+1,m+3}}{a_{n-1,m+3}d_{n,m+6}} = \frac{b_{n,m+6}c_{n+1,m+3}}{b_{n-1,m+3}c_{nm}} = \frac{n}{n + 1}.
\]
This relation will be a consequence of formulas (20) and (21). However, we mention it now in order to give a better insight into the structure of coefficients $a_{nm}$, $b_{nm}$, $c_{nm}$ and $d_{nm}$.

**Theorem 3.** For any $c \in \mathbb{C}$ and $t \in \mathbb{Z}$ there exist a $(g, K)$ module $V(c, 2t)$ such that

$$V(c, 2t)|_K = V_{12t} \oplus \bigoplus_{n,m \in \mathbb{Z}, n > 1} V_{nm}$$

and $a_{12t}d_{22t+3} = c - \frac{1}{2}t$. This module can be reducible. Any other $(g, K)$ module $V$ is a submodule, quotient or subquotient of some $V(c, 2t)$.

**Remark 3.** We are concentrated on irreducible $(g, K)$ modules. Reducibility of modules $V(c, 2t)$ will be obtained when some product(s) $a_{nm}d_{n+1m+3}$ or $b_{nm}c_{n+1m-3}$ are equal to 0. The theorem says that irreducible $(g, K)$ modules can be obtained a submodules, quotients or subquotients. Once we have formulas (22) – (23), it will be possible to determine if we have a submodule, quotient or subquotient. However, it will not be important for us. We are looking for irreducible $(g, K)$ modules and their description. Finally, it is, maybe, possible to find another choice of coefficients $a_{nm}$, $b_{nm}$, $c_{nm}$ and $d_{nm}$ in (22) – (23) and it would lead to another relationship among our modules. Hence, by abuse of notation, we will say just a submodule.

**Proof.** Let us put $a_{12t}d_{22t+3} = c - \frac{1}{2}t$. Then (11) (and also (12)) shows that

$$b_{12t}c_{22t-3} = c + \frac{1}{2}t.$$

If $a_{1+k2t+3k}d_{2+k2t+3k} \neq 0$, then $b_{1+k2t+3k}c_{2+k2t+3k} = 0$ for $k \geq 0$ (by (13)). If $b_{1+k2t-3k}c_{2+k2t-3k} \neq 0$, then $a_{1+k2t-6k}a_{2+k2t-3-3k} = 0$ for $k \geq 0$. It means that $K$ modules of the irreducible component of $V(c, 2t)$ can form a cone (if $a_{1+k2t+3k}d_{2+k2t+3k} \neq 0$ and $b_{1+k2t-3k}c_{2+k2t-3-k} \neq 0$ for $k \geq 0$), a strip (if $a_{1+k2t+3k}d_{2+k2t+3k} = 0$ for some $k \in \mathbb{N} \cup \{0\}$ or $b_{1+k2t-3k}c_{2+k2t-3-k} = 0$ for some $k \in \mathbb{N} \cup \{0\}$) or a parallelogram (if $a_{1+k2t+3k}d_{2+k2t+3k} = 0$ for some $k \in \mathbb{N} \cup \{0\}$ and $b_{1+l2t-3l}c_{2+l2t-3-l} = 0$ for some $l \in \mathbb{N} \cup \{0\}$).

We claim that our expressions $ad$ and $bc$ are determined uniquely. Relations (11) and (12), for $n > 1$, produce two independent equations. It is enough to walk from one vertex to another where two expressions $ad$ and $bc$ are already determined and calculate remaining two. A reader can easily
reconstruct the path. Formulas for the vertex which is obtained by moving $p$ steps in the $\alpha + \beta$ direction and $q$ steps in $-\beta$ direction, have the form

$$a_{1+p+q2t+3p-3q}d_{2+p+q2t+3p-3q+3} = \frac{p + 1}{p + q + 2} (2c - (p + 1)t - p(p + 2))$$ (20)

and

$$b_{1+p+q2t+3p-3q}c_{2+p+q2t+3p-3q-3} = \frac{q + 1}{p + q + 2} (2c + (q + 1)t - q(q + 2)).$$ (21)

They can be checked directly. One could write $n + 1$ instead of $p + q + 2$ in denominators of (20) and (21).

Coefficients $a_{nm}, b_{nm}, c_{nm}$ and $d_{nm}$ can be defined by

$$a_{1+p+q2t+3p-3q} = 2c - (p + 1)t - p(p + 2),$$ (22)

$$b_{1+p+q2t+3p-3q} = 2c + (q + 1)t - q(q + 2),$$

$$c_{2+p+q2t+3p-3q-3} = \frac{q + 1}{p + q + 2},$$

$$d_{2+p+q2t+3p-3q+3} = \frac{p + 1}{p + q + 2}.$$ (23)

One can check that (13) – (16) are satisfied. By Theorem 2, $(g, K)$ module $V(c, 2t)$ is well defined.

Now, it remains to show that any other module is a submodule (see Remark 3) of some $V(c, 2t)$. Let us consider some irreducible $(g, K)$ module $W(r, s), r \in \mathbb{N}, r > 1$ and $s \in \mathbb{Z}$ of the form

$$W(r, s)|_K = V_{rs} \oplus \bigoplus_{n,m \in \mathbb{Z}, n>r} V_{nm}$$

Let us notice that $r + s = 2z + 1$ for some $z \in \mathbb{Z}$. This time (since $r > 1$), the system of two equations produced by (11) and (12) has a unique solution. It shows that coefficients $ad$ and $bc$ are uniquely determined for the fixed choice of $r$ and $s$. It remains to show that $W(r, s)$ is a submodule of some $V(c, 2t)$ but it is straightforward: $W(r, s)$ is submodule of

$$V\left(\frac{(r - 1)(-r + 1 + s) - 2}{4}, -3r + 3 + s\right)$$

and

$$V\left(\frac{(r - 1)(-r + 1 - s) - 2}{4}, 3r - 3 + s\right).$$

Since $r + s = 2z + 1, -3r + 3 + s = 2z - 4r + 4$. \qed
Example 2. Let us consider the module \( W(4,3) \). It is a submodule of \( V\left(\frac{1}{2},-6\right) \) and \( V(-5,12) \). It is a nice exercise to calculate products \( a_{nm}d_{n+1m+3} \) and \( b_{nm}c_{n+1m-3} \) for \( W(4,3) \) using (11) and (12) (solving system for each vertex) and compare with (20) and (21) for \( V\left(\frac{1}{2},-6\right) \) and \( V(-5,12) \).

4 Unitary dual of \( SU(2,1) \)

For the beginning we give a definition of unitary \((\mathfrak{g}, K)\) modules for any real reductive group.

Definition 2. Unitary \((\mathfrak{g}, K)\) module \( V \) is a \((\mathfrak{g}, K)\) module equipped with the inner product \( \langle \cdot , \cdot \rangle \rightarrow \mathbb{C} \) such that

\[
(X^* + X).v = 0, \quad \forall X \in \mathfrak{g}_0, \quad \forall v \in V.
\] (24)

and the action of \( K \) is unitary.

Remark 4. Since \( K \) acts on finite-dimensional spaces, the action is automatically unitary on \( K_0 \). Hence, we have to check that the action is unitary only for some representatives of connected components. Since the group \( G = SU(2,1) \) is connected, it remains to check only (24).

Now, let us construct the inner product mentioned in Definition 2. Using the same way of reasoning as we did for (24), one can conclude that

\[
\langle v_{nm}^k, v_{rs}^l \rangle = 0, \quad v_{nm}^k \in V_{nm}, \quad v_{rs}^l \in V_{rs}
\]

for \( s \neq m \) or \( r - 2l \neq n - 2k \). It remains to consider the situation when \( s = m \) and \( r - 2l = n - 2k \).

Lemma 1. Let \( V \) be a unitary \((\mathfrak{g}, K)\) module, \( v_{nm}^k \in V_{nm} \) and \( v_{n-2k+2l}^l \in V_{n-2k+2l} \). Then

\[
\langle v_{nm}^k, v_{n-2k+2l}^l \rangle \neq 0
\]

if and only if \( k = l \).
Proof. Let us assume that \( l > k \),

\[
\langle v_{n m}^k, v_{n-2k+2lm}^l \rangle \neq 0
\]  

(25)

and \( k \) is the smallest possible with that property. Now,

\[
\langle A_\alpha v_{n m}^k, v_{n-1}^{l-1} \rangle = \langle -(k-1)v_{n m}^{k-1} + (n-k)v_{n-2k+2lm}^{k-1}, v_{n-2k+2lm}^l \rangle = 0.
\]

By (24),

\[
\langle v_{n m}^k, A_\alpha^* v_{n-1}^{l-1} \rangle = \langle -((k-1))v_{n-1}^{k-1} + (n-k)v_{n-1}^{k-1}, v_{n-1}^{l-1} \rangle \neq 0.
\]

It gives a contradiction to assumption (25).

Let us assume that unitary \((g, K)\) module \( V \) is given. We want to find relationship among coefficients \( a_{nm}, b_{nm}, c_{nm} \) and \( d_{nm} \). Relation

\[
\langle A_{\alpha+\beta} v_{n-1}^{k-1}, v_{n m}^k \rangle = \langle v_{n-1}^{k-1}, A_{\alpha+\beta}^* v_{n m}^k \rangle,
\]

produces

\[
a_{n-1}||v_{nm}^k||^2 = \frac{n-k}{n-1}d_{nm}||v_{n-1}^{k-1}||^2.
\]

(26)

Calculation is straightforward. Operator \( A_{\alpha+\beta} \) is given by (6), (7) and (8), \( A_{\alpha+\beta}^* \) by (24) and Lemma 1 is used for both sides. Relation (26), at first glance, does not look good. Namely, it gives a relationship between \( a_{n-1} \) and \( d_{nm} \), but it depends on \( k \). Let us write it for \( k = 1 \). It transforms to

\[
a_{n-1}||v_{nm}^1||^2 = -\overline{d_{nm}}||v_{n-1}^{1}||^2.
\]

(27)

One can apply (5) on both sides and obtain

\[
a_{n-1} \left( \frac{n-1}{k-1} \right)||v_{nm}^k||^2 = -\overline{d_{nm}} \left( \frac{n-2}{k-1} \right)||v_{n-1}^{k-1}||^2
\]

and it is easy to recognize (26). Hence, it is enough to consider (27). Relation

\[
\langle A_{\alpha+\beta} v_{n+1}^{k+1}, v_{nm}^k \rangle = \langle v_{n+1}^{k+1}, A_{\alpha+\beta}^* v_{nm}^k \rangle,
\]

produces

\[
\frac{k}{n}c_{n-1}||v_{nm}^k||^2 = -\overline{b_{nm}}||v_{n+1}^{k+1}||^2.
\]
Using (4) it transforms to
\[ \frac{k}{n} c_{n+1-m-3} \| v^{k}_{nm} \|^2 = -\frac{b_{nm}}{n+1-k} \| v^{k}_{n+1-m-3} \|^2, \]
and
\[ c_{n+1-m-3} \| v^{k}_{nm} \|^2 = -\frac{n}{b_{nm}} \| v^{k}_{n+1-m-3} \|^2. \]
Again, it is enough to consider this expression for \( k = 1 \),
\[ c_{n+1-m-3} \| v^{1}_{nm} \|^2 = -\frac{b_{nm}}{n+1-k} \| v^{1}_{n+1-m-3} \|^2. \]
(28)

We continue and consider relations
\[ \langle A_{\alpha + \beta}, v^{k-1}_{n-1-m+3}, v^{k}_{nm} \rangle = \langle v^{k-1}_{n-1-m+3}, A^{\ast}_{\alpha + \beta}, v^{k}_{nm} \rangle, \]
and
\[ \langle A_{\alpha + \beta}, v^{k}_{n+1-m+3}, v^{k}_{nm} \rangle = \langle v^{k}_{n+1-m+3}, A^{\ast}_{\alpha + \beta}, v^{k}_{nm} \rangle. \]
However, they do not produce new relation among coefficients \( a_{nm}, b_{nm}, c_{nm} \) and \( d_{nm} \). The same procedure can be repeated for \( B_{\alpha + \beta} \) and \( B_{\beta} \) but is will not produce new relation. Hence, only (27) and (28) have to be satisfied.

Now, let us go in opposite direction. We want to find conditions on coefficients \( a_{nm}, b_{nm}, c_{nm} \) and \( d_{nm} \) which will lead us to unitary (irreducible) \((g, K)\) module \( V \). Lemma 1 shows that the inner product on \( V \) such that \( K \) acts by unitary operators is given by expressions \( \| v^{1}_{nm} \|^2 \). Relation (27), for irreducible \( V \), shows that
\[ a_{nm}d_{n+1-m+3} \in (-\infty, 0) \] (29)
and (28) shows that
\[ b_{nm}c_{n+1-m-3} \in (-\infty, 0). \] (30)
One should compare (29) and (30) with (3). We claim that it is enough to satisfy (29) and (30). Hence, one has to define the ”norm” for each \( V_{nm} \) such that (27) and (28) are satisfied. The expression ”norm” of \( V_{nm} \), by (5), means \( \| v^{1}_{nm} \|^2 \). It is enough to start with any \( K \) type \( V_{nm} \) and then define norms of other \( K \) types using (27) and (28). We have to prove that this construction is good. Let us assume that the norm of \( V_{nm} \) is given. We have to show that norms of \( V_{n+2}m, V_{n+6}, V_{n-m-6} \) and \( V_{n-2}m \) are well defined.
The norm of \( V_{n+2m} \) can be calculated in two different ways. Using (28) and (27), one obtains
\[
||v_{n+2m}^1||^2 = -\frac{c_{n+2m}}{b_{n+1m+3}} ||v_{n+1m+3}^1||^2 = \frac{c_{n+2m}}{b_{n+1m+3}} \cdot \frac{d_{n+1m+3}}{a_{nm}} ||v_{nm}^1||^2.
\]
Similarly,
\[
||v_{n+2m}^1||^2 = -\frac{d_{n+2m}}{a_{n+1m-3}} ||v_{n+1m-3}^1||^2 = \frac{d_{n+2m}}{a_{n+1m-3}} \cdot \frac{c_{n+1m-3}}{b_{nm}} ||v_{nm}^1||^2.
\]
It remains to show that
\[
\frac{c_{n+2m}}{b_{n+1m+3}} \cdot \frac{d_{n+1m+3}}{a_{nm}} = \frac{d_{n+2m}}{a_{n+1m-3}} \cdot \frac{c_{n+1m-3}}{b_{nm}}.
\]
It follows from (15) and (16). The norm of \( V_{m+6} \) can be also calculated in two different ways. Using (28) and (27), one obtains
\[
||v_{nm}^1||^2 = -\frac{b_{n+6m}}{c_{n+1m+3}} ||v_{n+1m+3}^1||^2 = \frac{b_{n+6m}}{c_{n+1m+3}} \cdot \frac{d_{n+1m+3}}{a_{nm}} ||v_{nm}^1||^2.
\]
Similarly,
\[
||v_{nm}^1||^2 = -\frac{d_{n+6m}}{a_{n+1m-3}} ||v_{n+1m-3}^1||^2 = \frac{d_{n+6m}}{a_{n+1m-3}} \cdot \frac{b_{n-1m+3}}{c_{nm}} ||v_{nm}^1||^2.
\]
It remains to show that
\[
\frac{b_{n+6m}}{c_{n+1m+3}} \cdot \frac{d_{n+1m+3}}{a_{nm}} = \frac{d_{n+6m}}{a_{n+1m-3}} \cdot \frac{b_{n-1m+3}}{c_{nm}}.
\]
It follows again from (15) and (16). Remaining two cases can be shown similarly.

It remains to notice that we have shown that
\[
\langle C.w, v_{nm}^1 \rangle = \langle w, (-C).v_{nm}^1 \rangle
\]
for \( C = A_\beta, B_\beta, A_{\alpha+\beta} \) and \( B_{\alpha+\beta} \) and \( w = v_{n+1m+3}^1 \) if (29) and (30) are satisfied. By Lemma 1, it is enough since inner product is equal to 0 in all other cases. Hence operators \( A_\beta, B_\beta, A_{\alpha+\beta} \) and \( B_{\alpha+\beta} \) are unitary. We have proved
**Theorem 4.** \((g, K)\) module \(V\) is unitary if and only if (29) and (30) are satisfied.

**Remark 5.** One should compare statements of Theorem 4 and (3). Each time unitary action is obtained if certain products are real negative numbers. We hope that similar statement will be valid for some other real reductive groups.

Now, we want to apply Theorem 4 on Theorem 3 and find all irreducible unitary \((g, K)\) modules. Firstly, we will analyze modules \(V(c, 2t)\) and concentrate on the component which contains \(V_{12t}\). In the second step we will analyze modules \(W(r, s)\) for \(r > 1\). Theorem 4 says that expressions given by (20) and (21) have to be negative. Since \(\frac{p+1}{p+q+2} > 0\) and \(\frac{q+1}{p+q+2} > 0\) it reduces to

\[
2c - (p+1)t - p(p+2) < 0 \quad (31)
\]

and

\[
2c + (q+1)t - q(q+2) < 0. \quad (32)
\]

We want to find all values of \(c\) such that (31) and (32) are satisfied. If \(t \geq 0\) then it is enough to consider only (32). If \(t < 0\) then it is enough to consider only (31). The symmetry shows that it is enough to consider only one case. Hence, we will consider the case when \(t \geq 0\). Since our expression is a quadratic polynomial in variable \(q\) we will consider situations when \(t = 0\) and \(t = 1\) separately (when the \(x\) coordinate of the vertex of the parabola is negative). Since \(q \in \mathbb{N}\), we will consider situations \(t = 2k, \ k \in \mathbb{N}\) and \(t = 2k+1, \ k \in \mathbb{N}\) separately.

When \(t = 0\), (31) reduces to \(-q^2 - 2q + 2c < 0\) and it is fulfilled for \(c < 0\). Let us define

\[
c(0) = 0.
\]

Hence \(V(c, 0)\) is irreducible unitary for \(c \in (-\infty, c(0))\). For \(c = 0\), \(V(c(0), 0)\) is reducible. Let us denote by \(U(0)\) irreducible submodule which contains \(V_{10}\). Then \(U(0) = V_{10}\) is one-dimensional module.

When \(t = 1\), (32) transforms to \(-q^2 - q + 2c + 1 < 0\) and we set

\[
c(1) = -\frac{1}{2}.
\]

Hence, \(V(c, 2)\) is irreducible unitary for \(c \in (-\infty, c(1))\) and \(V(c(1), 2)\) is reducible. Let \(U(2)\) be an irreducible submodule which contains \(V_{12}\). Then
$U(2)$ contains $K$ types of the form

$$\{V_{1+p}\quad |\quad p \in \mathbb{N} \cup \{0\}\}. $$

When $t = 2k$ for $k \in \mathbb{N}$, (32) shows that

$$c(2k) = -\frac{k^2 + 1}{2}$$

and $V(c, 4k)$ is irreducible unitary for $c \in (-\infty, c(2k))$. It is also possible that $V(c, 4k)$ has an irreducible unitary submodule which contains $V_{1,4k}$. Relation (32) transforms to

$$2c + (q + 1)t - q(q + 2) = -(q - l)(q - (2(k - 1) - l))$$

for $l \in \{0, \ldots, k-1\}$ and it produces

$$c(l, 2k) = \frac{l^2 - 2(k - 1)l - 2k}{2}.$$ 

The submodule of $V(c(l,t), 2t)$ which contains $V_{1,4k}$ for $l \in \{0, \ldots, k-1\}$ will be denoted by $U(l, 2t)$. It contains $K$ types of the form

$$\{V_{1+p+q2t+3p-3q}\quad |\quad p \in \mathbb{N} \cup \{0\}, q \in \{0, \ldots, l\}\}. $$

(33)

When $t = 2k + 1$ for $k \in \mathbb{N}$, (32) shows that

$$c(2k + 1) = -\frac{k^2 + k + 1}{2}$$

and $V(c, 2(2k + 1))$ is irreducible unitary for $c \in (-\infty, c(2k + 1))$. Similarly as in a previous case, it is possible to find irreducible unitary submodules of $V(c, 2(2k + 1))$ for some $c$. Again, (32) transforms to

$$2c + (q + 1)t - q(q + 2) = -(q - l)(q - (2(k - 1) - l))$$

for $l \in \{0, \ldots, k-1\}$ and it produces

$$c(l, 2k + 1) = \frac{l^2 - (2k - 1)l - 2k - 1}{2}.$$ 

The submodule of $V(c(l,t), 2t)$ which contains $V_{1,2(2k+1)}$ for $l \in \{0, \ldots, k-1\}$ will be denoted by $U(l, 2t)$ and it contains $K$ types of the form (33).
The situation is very similar for \( t < 0 \). One has to use (31) instead of (32). It is easy to see that \( c(t) = c(-t) \) and \( c(l, t) = c(l, -t) \) for \( l \in \{0, \ldots, \left\lfloor \frac{-t}{2} \right\rfloor - 1\} \). The set of \( K \) types of \( U(-2) \) is \( \{V_{1+q-2+3q} \mid q \in \mathbb{N} \cup \{0\}\} \) and the set of \( K \) types of \( U(l, 2t) \) is
\[
\{V_{1+p+q2t+3p-3q} \mid p \in \{0, \ldots, l\}, q \in \mathbb{N} \cup \{0\}\}
\]
for \( l \in \{0, \ldots, \left\lfloor \frac{-t}{2} \right\rfloor - 1\} \).

Now, let us consider modules \( W(r, s) \) for \( r > 1 \). Theorem 3 claims that \( W(r, s) \) is a submodule of some \( V(c, 2t) \). Hence, (20) and (21) can be applied. Since \( r > 1 \), expressions given in (20) and (21) are equal to 0 in the previous step. Hence, it is enough to find when these two expressions are equal or less to 0.

We have mentioned that the system given by (11) and (12) in variables \( a_{rs}d_{r+1}s+3 \) and \( b_{rs}c_{r+1}s-3 \) (\( c_{rs}b_{r-1}s+3 = 0 \) and \( d_{rs}a_{r-1}s-3 = 0 \)) has a unique solution
\[
a_{rs}d_{r+1}s+3 = \frac{-r(s+r+1)}{2(r+1)} \quad \text{and} \quad b_{rs}c_{r+1}s-3 = \frac{r(s-r-1)}{2(r+1)}.
\]

Since, \( r > 0 \) and \( r+1 > 0 \) it remains to consider \( s+r+1 \geq 0 \) and \( s-r-1 \leq 0 \). It can not happen that both expressions are equal to 0. If \( s+r+1 = 0 \) then \( s-r-1 < 0 \), \( W(r, s) \) is reducible and the submodule which contains \( V_{rs} \) will be denoted by \( Z(s) \) where \( s \in -(\mathbb{N} \setminus \{1\}) \). The set of \( K \) types of \( Z(s) \) is
\[
\{V_{s-1+s+3q} \mid q \in \mathbb{N} \cup \{0\}\}.
\]

If \( s-r-1 = 0 \) then \( s+r+1 > 0 \), \( W(r, s) \) is reducible and the submodule which contains \( V_{rs} \) will be denoted by \( Z(s) \) where \( s \in \mathbb{N} \setminus \{1\} \). The set of \( K \) types of \( Z(s) \) is
\[
\{V_{s-1+p+3p} \mid p \in \mathbb{N} \cup \{0\}\}.
\]

Finally, if \( s+r+1 > 0 \) and \( s-r-1 < 0 \) then \( W(r, s) \) is unitary irreducible and the set \( K \) types is
\[
\{V_{r+p+q+3p+3q} \mid p, q \in \mathbb{N} \cup \{0\}\}.
\]

Hence, we have proved

**Theorem 5.** Irreducible unitary \((\mathfrak{g}, K)\) modules are parametrized as follows

1. \( V(c, 2t), t \in \mathbb{Z}, c \in (-\infty, c(t)) \),
2. \( U(0), U(2), U(-2) \) and \( U(l, 2t), t \in \mathbb{Z} \setminus \{0, 1, -1\}, l \in \{0, \ldots, \left\lfloor \frac{|t|}{2} \right\rfloor - 1\}, \)

3. \( W(r, s), s + r + 1 > 0 \) and \( s - r - 1 < 0, \)

4. \( Z(s), s \in \mathbb{Z} \setminus \{-1, 0, 1\}. \)

All these \((g, K)\) modules are nonequivalent.

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