Local Spin-Gauge Symmetry of the Bose-Einstein Condensates in Atomic Gases

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Abstract

The Bose-Einstein condensates of alkali atomic gases are spinor fields with local “spin-gauge” symmetry. This symmetry is manifested by a superfluid velocity $u_s$ (or gauge field) generated by the Berry phase of the spin field. In “static” traps, $u_s$ splits the degeneracy of the harmonic energy levels, breaks the inversion symmetry of the vortex nucleation frequency $\Omega_{c1}$, and can lead to vortex ground states. The inversion symmetry of $\Omega_{c1}$, however, is not broken in “dynamic” traps. Rotations of the atom cloud can be generated by adiabatic effects without physically rotating the entire trap.

The recent discoveries of Bose-Einstein condensation in atomic gases of $^{87}\text{Rb}$ [1], $^7\text{Li}$ [2], and $^{23}\text{Na}$ [3] have achieved a long sought goal in atomic physics. They have also provided condensed matter physicists opportunities to study interacting Bose systems at a wide range of densities. The realizations of these condensates are made possible by the invention of a number of special magnetic traps, which trap atoms with hyperfine spin ($F = 2$) maximally aligned with the local magnetic field $B$. The reported Bose-Einstein condensations [1] [2] [3] are found in these (adiabatic) spin states.

An immediate question is whether these alkali condensates differ from the familiar $^4\text{He}$ condensate in any fundamental way. Unlike the spinless $^4\text{He}$ atoms, the trapped alkali atoms are in the $F = 2$ hyperfine spin state. Their condensates are therefore spinors of the form

$$< \hat{\psi}_m(x, t) > = \zeta_m(x, t)\Phi(x, t),$$

(1)
where $\hat{\psi}_m$ is the field operator, $m$ is a label for $F_z$, $(-2 \leq m \leq 2)$, $\Phi$ is a scalar, and $\zeta_\mu$ is a normalized spinor. Since the hyperfine spins are aligned with the magnetic field, $\zeta$ is given by $\hat{B} \cdot \mathbf{F} \zeta = 2\zeta$, where $\mathbf{F}$ is the hyperfine spin operator. The dynamics of $\langle \hat{\psi}_m \rangle$ is therefore completely specified by that of the scalar field $\Phi$, as in $^4\text{He}$. One might then conclude that apart from extrinsic factors like density and external potential, there is no intrinsic symmetry difference between $^4\text{He}$ and alkali condensates. This is in fact the starting point of all current theories, which model the alkali systems as interacting dilute spinless Bose gases in harmonic potentials. Within these models, the effective Hamiltonian for the scalar $\Phi$ has a global $U(1)$ gauge symmetry, as in $^4\text{He}$.

The actual symmetry of the spinor field (eq.(1)), however, is much larger than $U(1)$. We call it local spin-gauge symmetry. It represents that a gauge change $e^{i\chi(x,t)}$ of $\langle \hat{\psi}_m \rangle$ can be undone by a local spin rotation $e^{-i\chi/F} \hat{B}(x,t) \cdot \mathbf{F}$. Because of this symmetry, the exact form of the effective Hamiltonian of the scalar $\Phi$ is not that of $^4\text{He}$, but that of a neutral superfluid in a velocity field $u_s$, or an electron in a vector potential potential $A$. The velocity (or gauge field) $u_s$ arises from the Berry phase of the spin field $\zeta$. It is a direct reflection of the underlying spin-gauge symmetry. The purpose of this paper is to discuss various forms of spin-gauge effects.

As we shall see, the strength of the gauge field $u_s$ is proportional to the gradient of the magnetic field $B$. It is generally quite small because $B$ is generally fairly uniform in the trapping region. While these may be a justification of treating the alkali system like dilute $^4\text{He}$, we note that spin-gauge effects can be magnified rapidly by variations of trap parameters and particle numbers. As we shall see, despite its weakness, the effect of $u_s$ can be observed over a wide range of trap parameters in both normal and superfluid phases. Moreover, in the condensed phase, particle interactions can reduce the excitation energy so much that the spin-gauge effect are significantly magnified. In particular, when the excitation energy lies below that of spin-gauge effect, the ground state will change abruptly to one reflecting this symmetry. Generally, spin-gauge effect increases as one moves away
from the center of the atom cloud. Understanding of this effect is therefore important in view of the current effort to produce larger and larger clouds. It is also important to note that nonuniformity of the magnetic field, however small, is what leads to magnetic trapping. Spin-gauge effects are therefore intrinsic properties of magnetically trapped atomic gases. Our major findings are:

I. The adiabatic spin field $\zeta$ of the alkali system generates a superfluid velocity $u_s$. Its vorticity $\Omega_s = \frac{1}{2} \nabla \times u_s$ is specified by the magnetic field through a topological term, (eq.(5) below).

II. For cylindrical traps with static fields (or simply “static traps”), $u_s$ generates a Coriolis force which splits the degenerate harmonic energy levels. This splitting is independent of particle interaction and can be observed in both normal and superfluid phases.

III. For static traps, the “background” rotation $-\Omega_s$ breaks the inversion symmetry of the vortex nucleation frequency $\Omega_{c1}$. For appropriate trap parameters, the system has a vortex ground state even in absence of external rotation.

IV. For cylindrical traps with time dependent fields, (or simply “dynamic traps”), $\Omega_s$ has a quadrupolar structure. As a result, the inversion symmetry of $\Omega_{c1}$ is restored.

V. Adiabatic effects furnish a simple means to generate rotation of the atom cloud without physically rotating the entire trap.

To begin, we first discuss the effective Hamiltonian. For brevity, we shall call hyperfine spins “spins”. The Hamiltonians of the alkali systems are of the form $H = H_s + V$, where $H_s = \int dx \hat{\psi}_m^+ \left( -\frac{\hbar^2}{2M} \nabla^2 - \mu_a B(x,t) \cdot F \right) \hat{\psi}_n(x)$ is the single particle Hamiltonian, $M$ and $\mu_a = -\mu_B/2$ are the mass and magnetic moment of the atom, $\mu_B$ is the Bohr magneton, and the factor 1/2 is the $g$-factor of the alkali atom. $B$ is a sum of magnetic field configurations which can be static or dynamic. $V$ is the two-particle interaction between the atoms. To form a trap, the Zeeman energy $-\mu_a B$ (or their time average) must behave like a potential well. If $\{\zeta^{(n)}\}$ are the spin eigenstates along $\hat{B}$, $(\hat{B} \cdot F \zeta^{(n)} = n \zeta^{(n)})$, $(-2 \leq n \leq 2)$, then the Zeeman energy $-\mu_a B \cdot F$ reduces $U^{(n)}(x,t) = \frac{1}{2} n \mu_B B(x,t)$ for the states $\zeta^{(n)}$. If $\mu_B B$ is an attractive well, $U^{(n)}$ is confining (deconfining) for $n > 0$ ($n \leq 0$). This means that spin-flips
between $n > 0$ and $n \leq 0$ states can cause atoms to leave the trap. Since $V$ generally cause spin flips unless both atoms are in the maximum spin state along the same quantization axis, (in which case spin flips are prohibited by angular momentum conservation), it depletes all but the “adiabatic” spin states $\zeta^{(2)}$ in the trap. The resulting system is an interacting Bose gas with spins aligned with the local field $B(x,t)$.

To construct a theory for the adiabatic spin states, we expand $\hat{\psi}_m$ in terms of the spin eigenstates $\zeta^{(n)}$, $\hat{\psi}_m(x,t) = \sum_{n=-2}^{2} \zeta^{(n)}(x,t) \hat{\phi}^{(n)}(x,t)$. Expressing $\hat{B} = \hat{z}\cos\beta + \sin\beta (\hat{x}\cos\alpha + \hat{y}\sin\alpha)$, the explicit form of $\zeta^{(n)}$ is

$$\zeta^{(n)} = \langle m | U | n \rangle, \quad U = e^{-i\alpha F_z} e^{-i\beta F_y} e^{-i\chi F_z}$$

where $F_z|n> = n|n>$. $\chi$ is arbitrary. It is the gauge degree of freedom of the system, and is usually chosen to make the spinor $\zeta^{(n)}$ single valued. The effective Hamiltonian $\mathcal{H}$ can be obtained by rewriting the equation of motion $i\hbar \partial_t \hat{\psi}_m = [\hat{\psi}_m, H]$ in the form $i\hbar \partial_t \hat{\phi}^{(n)} = [\hat{\phi}^{(n)}, \mathcal{H}]$. One then finds $\mathcal{H} = \mathcal{H}_{ad} + \mathcal{H}_{nad} + \mathcal{H}_{etc}$. $\mathcal{H}_{ad}$, referred to as the “adiabatic” Hamiltonian, contains $\hat{\phi}^{(2)}$ only. $\mathcal{H}_{nad}$ is the spin-flip (or nonadiabatic) Hamiltonian which consists of cross terms between $\hat{\phi}^{(2)}$ and $\hat{\phi}^{(n\neq 2)}$. $\mathcal{H}_{etc}$ describes the transitions between different $n \neq 2$ states and can be ignored. Denoting $\hat{\phi}^{(2)}$ and $\zeta^{(2)}$ as $\hat{\phi}$ and $\zeta$ respectively, we have

$$\mathcal{H}_{ad} = \int dx \hat{\phi}^{+} \left[ \frac{1}{2M} \left( \frac{\hbar \nabla i}{\hbar} + M \mathbf{u}_s \right)^2 + \mathcal{U} + \mathcal{W} \right] \hat{\phi}^{+} \mathcal{V},$$

where $\mathcal{U}(x) = \mu_B B(x)$, $\mathcal{W} = (\hbar^2/2M)[|\nabla \zeta|^2 + (\zeta^+ \nabla \zeta)^2] - i\hbar \zeta^+ \partial_t \zeta$, $\mathcal{V}$ is the projection of $V$ onto the adiabatic spin states. It is of the form $\mathcal{V} = \int V(x-y) \hat{\phi}^+(x) \hat{\phi}^+(y) \hat{\phi}(y) \hat{\phi}(x)$, where $V(x-y)$ is a short range potential. The velocity $\mathbf{u}_s$ is defined as

$$M \mathbf{u}_s = \frac{\hbar}{i} \zeta^+ \nabla \zeta.$$ 

Eq. (3) describes a Bose fluid in a “background” velocity field $-\mathbf{u}_s$, or a charge $e$ system in a vector potential $A$ if $M \mathbf{u}_s \equiv eA/c$. Under a local spin rotation $\exp \left[ i \hat{B} \cdot \hat{F}(x) \right]$, $\mathbf{u}_s \rightarrow \mathbf{u}_s + (F\hbar/M) \nabla \chi(x)$, which is equivalent to a local gauge transformation $\hat{\phi} \rightarrow \exp(iF\chi(x)) \hat{\phi}$. 


This is a reflection of the underlying spin-gauge symmetry of $\zeta$. The integral $\int_C \mathbf{u}_s \cdot d\mathbf{s}$ is the Berry’s phase of $\zeta$ around a loop $C$. It can be easily calculated from the vorticity ($\Omega_s$) of $\mathbf{u}_s$, which satisfies the Mermin-Ho relation [4],

$$\Omega_s = \frac{1}{2} \nabla \times \mathbf{u}_s = \left(\frac{\hbar}{2M}\right) \epsilon_{\alpha\beta\gamma} \hat{B}_\alpha \nabla \hat{B}_\beta \times \nabla \hat{B}_\gamma. \tag{5}$$

Eq.(5) shows that the spatial variations of $\mathbf{B}$ necessary to produce the trapping potential will inevitably generate a non-vanishing superfluid velocity $\mathbf{u}_s$.

In the rest of this paper, we shall focus on the phenomena associated with the adiabatic spin fields, (described by $\mathcal{H}_{ad}$ only). Nonadiabatic effects will be discussed elsewhere [5]. Typically, nonadiabatic effects of the trap can be ignored if its “Dirac center” is sufficiently far away from the atom cloud. The “Dirac center” is the point where $\mathbf{B} = 0$ and that the unit vectors $\hat{\mathbf{B}}$ surrounding $D$ wraps around the unit sphere $n$ times, ($n$ is a nonzero integer). If $D$ resides in the cloud, the adiabatic spin field around $D$ will develop a line singularity emerging from $D$, (a Dirac string), which will cause a lot of spin-flips. As we shall see, increasing field gradients enhances spin-gauge effects but at the same time moves $D$ closer to the cloud. The field parameters discussed below are all within the range to keep the Dirac center sufficiently far away from the cloud.

To be concrete, we consider “static traps” of the form, ($\nabla \cdot \mathbf{B} = \nabla \times \mathbf{B} = 0$),

$$\mathbf{B}(x) = B_0 \hat{z} + G_1 (x \hat{x} - y \hat{y}) + G_2 \left[\left(z^2 - \frac{r^2}{2}\right) \hat{z} - 2r\right], \tag{6}$$

where $r \equiv (x,y)$, $G_1$ and $G_2$ are the first and second order field gradients respectively. Magnetic trap of the form eq.(6) is similar to that used in the $^7$Li experiment [2]. It is convenient to express the field gradients as $G_1 \equiv B_0(\gamma/L)$, $G_2 \equiv B_0/L^2$. The trapping potential $U$ in eq.(3) can then be expressed as

$$U = \hbar\Omega_{Zee} + \frac{1}{2} M \left[\omega_{\perp}^2 r^2 + \omega_z^2 z^2\right] + O|x/L|^4. \tag{7}$$

where $\hbar\Omega_{Zee} = \mu_B B_o$, $\omega_{\perp}^2 = \mu_B B_o/(ML^2)$, $\omega_z/\omega_{\perp} = (\gamma^2 - 1/2)^{-1/2} \equiv \lambda$, $\gamma^2 > 1/2$. For later use, we denote the longitudinal and transverse width of the ground state Gaussian of the
harmonic well (eq. (7)) as $a_z$ and $a_\perp$, where $a_z = (\hbar/M\omega_z)^{1/2}$, $a_\perp = (\hbar/M\omega_\perp)^{1/2}$.

Typically, $a_z, a_\perp << L$. It is straightforward to show that $W = (\hbar^2/2M)([\sin\beta \nabla \alpha]^2 + [\nabla \beta]^2)$, where $\alpha, \beta$ are polar angles of $B$ as defined earlier. This term is smaller than the harmonic potential in $U$ by a factor $(\gamma a_\perp/L)^4$ and can be ignored in general.

From eq. (3), it is straightforward to show that [with $\chi = -\alpha$ in eq. (2)], $u_s = -\hbar/M (1 - B_z/B) \nabla [\tan^{-1}(B_y/B_x)]$, and $u_s = -\hbar/M (\gamma L)^2 \hat{z} \times r + O(|x|/L^3)$. \(\Omega_s = -\hbar/M (\gamma L)^2 + O(|x|/L^3)\). \(u_s = -\hbar/M (\gamma L)^2 \hat{z} \times r + O(|x|/L^3)\). (8)

Thus, for $|x| < L$, spin-gauge effect generates a constant effective “rotation” $-\Omega_s$ along $\hat{z}$.

An immediate consequence of $\Omega_s$ is that it generates a Coriolis force on the alkali system. This force can be detected by applying an a.c. magnetic field along $\hat{x}$, $b = be^{-i\omega t}\hat{x}$. This field will generate a term $(\mu B b/2L)e^{-i\omega t}$ in the effective Hamiltonian $H_{ad}$, as if a time dependent force $f = (\mu B b/2L)e^{-i\omega t}\hat{x}$ is present. It is easy to see that the equation of motion of the center of mass in the $xy$-plane, $R = \int \hat{\phi} \hat{\phi} + r$, $r = (x, y)$, assumes the form

\[
M \frac{d^2 R}{dt^2} = -M \omega_\perp^2 R + 2M \frac{dR}{dt} \times \Omega_s + f.
\] (9)

which has resonances at $\omega = \omega_\perp \pm \Omega_s$ (for $\omega_\perp >> \Omega_s$). The degenerate clockwise and counterclockwise harmonic modes $\omega_\perp$ are split by the Coriolis force. This splitting exists in both normal and superfluid phases, and can be easily shown to be independent of particle interactions.

More pronounced effects can be found in the superfluid phase of alkali atoms with positive scattering length $a > 0$. Because of spin-gauge symmetry, the ground state energy functional becomes

\[
\mathcal{E}(\Phi) = \frac{1}{2M} \left| \left( \frac{\hbar \nabla}{i} + M u_s \right) \Phi \right|^2 + (\mathcal{U} + \mathcal{W}) |\Phi|^2 + \frac{2\pi \hbar^2 a}{M} |\Phi|^4.
\] (10)

When $u_s$ is small, eq. (11) can be written as $\mathcal{E}(\Phi, u_s) = \mathcal{E}(\Phi, 0) - \Omega_s \mathcal{L}_z$, ($\mathcal{L}_z = -i \Phi \hat{z} \cdot r \times \nabla \Phi$), which is the Hamiltonian density of a scalar superfluid in a container rotating with frequency $\Omega_s \hat{z}$. Let $\Omega^c_\Phi$ denote the vortex nucleation frequency in the absence of spin-gauge effect (i.e.
Because of the “background” rotation $\Omega_s$, the actual vortex nucleation frequencies $\Omega_{cl}^\pm$ for vortices with $2\pi$ circulation around $\pm\hat{z}$ will be $\Omega_{cl}^\pm = \Omega_{cl}^0 \mp \Omega_s$. In particular, when $\Omega_s \geq \Omega_{cl}^0$, hence $\Omega_{cl}^+ \leq 0$, vortex ground state will emerge in the absence of external rotation.

The value of $\Omega_{cl}^0$ has been studied for harmonic traps by a number of authors [6] [7]. Using Thomas-Fermi approximation (TFA), which is good at large $N$ [6], Baym and Pethick have shown that $\Omega_{cl}^0$ is reduced by particle interactions from its non-interacting value $\omega_\perp$ as

$$\Omega_{cl}^0/\omega_\perp = Q^{-2}\ln Q^2, \quad Q = R_\perp/a_\perp = (15\lambda N a/a_\perp)^{1/5},$$

where $R_\perp$ is the transverse width of the condensate. Since $\Omega_{cl}^0 \propto N^{-2/5}$, $\Omega_s \propto N^0$, the inversion asymmetry of the nucleation frequencies, $(\Omega_{cl}^+-\Omega_{cl}^-)/(\Omega_{cl}^++\Omega_{cl}^-) \approx \Omega_s/\Omega_{cl}^0$, increases as $N^{2/5}$. Thus, for sufficiently large $N$, the condition of vortex ground state $\Omega_s/\Omega_{cl}^0 \geq 1$ can always be met. From eq.(8) and eq.(11), one finds that the ratio $\Omega_s/\Omega_{cl}^0$ increases as the externally controllable parameters $N, G_1, B^{-1}_o$ increase.

Figure 1 shows the ratio $\Omega_s/\Omega_{cl}^0$ as calculated from eq.(8) and (11) for $^{23}$Na, which has a positive scattering length $a = 4.9$nm. The asymmetry of the trap is set at $\lambda = 1/2$. Four cases are considered: $N = 5 \times 10^5$ (broken line) and $N = 5 \times 10^6$ (solid line); and $B_o = 3$ and 5 Gauss, (denoted the numbers 3 and 5 respectively). They are chosen to indicate the direction of increasing spin-gauge effect as well as the conditions for vortex ground states. The field gradients considered extend to the Tesla/cm range. Although 100 times higher than those in current experiments ($\approx 100$Gauss/cm), they are easily achievable using superconducting magnets. At present, the largest $N$ produced is $10^5$. However, since $N$ has increased from $10^3$ to $10^5$ in last seven months, it is conceivable that condensates with $N = 10^6$ can be realized in the near future.

For sufficiently large $G_1$ or $N$, this condition will fail, at which point higher order terms in $U$
and $u_s$, as well as $W$ begin to contribute and that the system will lose cylindrical symmetry. The values of $G_1$ at which $R_z = L$ are marked by circles on the curves in figure 1, indicating that eq.(11) is only accurate to the left of the circle.

To calculate $\Omega_{\pm}^c_1$ in the regime $R_z \geq L$, we have calculated the energies ($E_\pm$ and $E_0$) of a $\pm 2\pi$ vortex and the no vortex ground state using the full expressions of $U$, $W$, and $u_s$. The nucleation frequencies $\Omega_{\pm}^c_1$ are related to these energies as $\Omega_{\pm}^c_1 \approx (E_\pm - E_0)/\hbar$. Our calculation are performed within TFA, which in the present context amounts to replacing the kinetic energy by $(\hbar^2/2M)(\nabla \theta + M u_s/M)^2|\Phi|^2$ and ignoring the $(\hbar^2/2M)\nabla |\Phi|^2$ term. We have minimized this approximated energy subjected to the constraint of constant particle number $N = \int |\Phi|^2$, where $|\Phi|$ is the magnitude of the order parameter of a $\pm 2\pi$ vortex (and the no vortex state) in the $E_\pm$ (and $E_0$) calculation. Our results are shown in Figs.2a and 2b, which plot the ratio $\eta^\pm = \Omega_{\pm}^c_1/\omega_\perp$ as a function of $G_1$. We see that vortex ground states (i.e. $\eta^+ = 0$) emerge around 5 Tesla for $N = 5 \times 10^6$. Even though the other cases considered have not reached vortex ground state, they show strong broken inversion symmetry in the nucleation frequency, i.e. $(\eta^+ - \eta^-)/(\eta^+ + \eta^-)$ are close to or over 50%. Finally, we note that the ratio $\Omega_s/\omega_\perp$, which describes the amount of energy level splitting in eq.(9) is of the order of $10^{-3}$ for the range of parameters we considered, (which is easily verified from the expression of $a_\perp$ and $\Omega_s$). This splitting, though small, is within the limit of detectability.

Note also that the Dirac centers of the static trap eq.(1) are located at $(\pm L\sqrt{8\gamma^2 + 2}, 0, 2\gamma L)$, $(0, \pm L\sqrt{8\gamma^2 + 2}, -2\gamma L)$. For the cases we considered, $\lambda = 1/2$, hence $\gamma = \sqrt{4.5}$, these Dirac centers are quite far away from the center of the cloud as $R_z \approx 2R_\perp$ is less than $2L$ for all cases considered.

To further illustrate the spin-gauge effect, we consider dynamic traps like those in the Rb experiment [1],

$$B(x, t) = B_o [\hat{n}(t) + b(x)], \quad b(x) = (r - 2z\hat{z})/L,$$

where $B_o/L$ is the field gradient, and $\hat{n}(t) = \hat{p}\cos\omega_o t + \hat{q}\sin\omega_o t$ is a unit vector rotating in a plane perpendicular to $\hat{l}$, and $(\hat{p}, \hat{q}, \hat{l})$ form an orthogonal triad. The adiabatic Hamiltonian
$\mathcal{H}_{ad}$ is now periodic in time with frequency $\omega_o$. Expanding $\mathcal{H}_{ad}(t)$ in Fourier series of $e^{-i\omega_0 t}$, the time averaged (i.e. $n = 0$) term give rise to a static Hamiltonian of the form eq.(3) with $\mathcal{U}$ replaced by $\overline{\mathcal{U}} = \mu_B B(1 + |b|^2 + (b \cdot \hat{l})^2)/(4L^2) + ..)$. When $\hat{l} = \hat{z}$, $\overline{\mathcal{U}}$ has the cylindrical symmetric form eq.(4) with $\omega_\perp^2 = \mu_B B_o/2ML^2$, $(\omega_z/\omega_\perp)^2 = 8$. The rotational frequency $\omega_o$ has to fall in the range $\omega_z < \omega_o < \Omega_{Zee}$ for the spins to follow the local magnetic field adiabatically [9]. Using eq.(5), the time averaged of the velocity field $\mathbf{u}_s$ and $\Omega_s$ associated with eq.(12) are found to be

$$\mathbf{u}_s = -\left(\frac{\hbar}{ML^3}\right)z\hat{z} \times \mathbf{r}, \quad \Omega_s = \left(\frac{\hbar}{ML^3}\right)(\mathbf{r} - 2z\hat{z}). \quad (13)$$

Unlike the $\Omega_s$ of the static trap, eq.(8), $\overline{\Omega}_s$ is a quadrupolar field which has mirror symmetry about the xy-plane. As a result, $\Omega_{s1}^+ = \Omega_{s1}^-$, which is similar to $^4$He but is entirely different from the static trap [10]. To verify this symmetry in the regime where $\Omega_s$ has similar strength as $\Omega_s$ in the static trap, the system has to be rotated up to the critical frequency $\Omega_{c1}$, which can be as high as 100 rad/sec for $G = 5000$ Gauss/cm, $B_o = 5$ Gauss, and $N = 10^5$. While it is impractical to rotate the entire trap at such high frequencies, we point out below a simple way to rotate the trapping potential using the adiabatic effects.

Consider the case where $\hat{l}$ deviates slightly from $\hat{z}$, hence causing $\overline{\mathcal{U}}$ to deviate slightly from cylindrical symmetry. If $\hat{l}$ precesses about $\hat{z}$ with frequency $\omega_p$, $\omega_p <\ll \omega_\perp < \omega_o$, $\overline{\mathcal{U}}$ will rotate about $\hat{z}$ with the same frequency. (The time average now is to be understood as averaging over times faster than $2\pi/\omega_p$). Since $\overline{\mathcal{U}}$ deviates only slightly from cylindrical symmetry, the corresponding vector fields $\overline{\mathbf{u}}_s$ and $\overline{\Omega}_s$ are essentially given by eq.(13). Since the rotation $(\omega_o\hat{l})$ and the precession $(\omega_p\hat{z})$ of $\hat{n}$ can be generated by electromagnetic means from sets of stationary coils, rotation of the trapping potential can therefore be generated without physically rotating the entire trap. In a similar fashion, the potential $\mathcal{U}$ of the static trap can be made to rotate about $\hat{z}$ by the application of a small magnetic field rotating in the $xy$ plane with frequency $\omega_p <\ll \omega_\perp$ [11]. Rotating the trapping potential in this fashion allows one to study the inversion asymmetry of $\Omega_{c1}^\pm$ in both low and high field gradient regime.
We have thus established Statement I to V. Our discussions also show that spin-gauge effects assume different forms in different traps. It is therefore conceivable that they can be made more prominent at lower field gradients by other ingenious design of traps. In a broader sense, the spin-gauge effect is only a subset of a much larger class of phenomena associated with the topological excitations of the spin field, which are suppressed by the magnetic field in the current traps. Should it be possible to release part (or all) of the spins degrees of freedom in a new trapping design, the spin-gauge phenomena of the resulting condensate will be truly remarkable indeed.

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[8] The reason for the $\approx$ sign is because the actual free energy to minimize in the presence of an external rotation $\omega$ is $\int \mathcal{E} - \omega \cdot <\mathbf{L}>$. In addition to the vortex of $\Phi$, the adiabatic spin field also contribute to $<\mathbf{L}>$. This spin contribution grows as the field gradient increases. Since this term contributes to all states (with or without vortices), much of its effect is cancelled out in the energy difference between vortex and no-vortex state. The $\approx$ sign used is to indicate the existence of this small correction.

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[10] The quadrupolar vorticity field $\overline{\Omega}_q$ gives rise to the following effects which we shall discuss elsewhere because of length reasons. As the external rotation increases beyond $\Omega_{c1}$ by a moderate amount, a metastable vortex state with a bent vortex line emerges.
The bending of the vortex line will give rise to a superfluid velocity field more consistent with \( \mathbf{u}_s \). The energy of this bent vortex is higher than that of the straight vortex but is lower than the no-vortex state. It is separated from the straight vortex state by a sizable energy barrier. For \(^{23}\text{Na}\) with \( N = 10^5 \), \( B_0 = 5\text{Gauss} \), and \( G = 5000\text{Gauss/cm} \), the bent vortex line appears when the external rotation \( \omega = 1.15\Omega_{c1} \). The velocity \( \mathbf{u}_s \) also causes the single particle harmonic oscillator states to shift their origin along \( z \) by an amount proportional to their angular momenta, i.e. those states with \( L_z > 0(<0) \) will displayed upward (downward).

[11] For (counterclockwise) rotating fields \( \mathbf{b}_r = \epsilon B_0(\cos \omega_p t, \sin \omega_p t) \), \( \epsilon \ll 1 \), the origin of \( \mathbf{U} \) is displaced to \( \epsilon \gamma \lambda L(\cos \omega_p t, -\sin \omega_p t, 0) \) to the lowest order in \( \epsilon \), which undergoes clockwise rotation. On the other hand, an additional term \(- (\epsilon/L^2)[x\cos \omega_p t + y\sin \omega_p t]z \) is generated, which rotates in the counterclockwise direction and breaks cylindrical symmetry. It is this term that causes the system to generate vortices, as it extends over large distances.
Caption

Figure 1. The ratio $\Omega_s/\Omega_{c1}$ as a function of field gradient $G_1$ for $^{23}\text{Na}$ in a trap with asymmetry $\lambda = 1/2$. The solid and broken lines represent particle numbers $N = 5 \times 10^6$ and $N = 5 \times 10^5$ respectively. The labels 3 and 5 on the figure denote $B_o = 3$ and 5 Gauss respectively. On each curve, the region to the left (right) of the circle indicates the condition $R_z < L(> L)$.

Figure 2a (2b) shows the ratio $\eta^\pm = \Omega_{c1}^\pm/\omega_\perp$ a function of field gradient $G_1$ for $B_o = 3$ (5) Gauss. The meaning of solid and broken lines as well as the symbols “3” and “5” are identical to those in figure 1.
\[ \eta^- = 5 \times 10^5 \]

\[ \eta^+ = 5 \times 10^5 \]

\[ \eta^- = 5 \times 10^6 \]

\[ \eta^+ = 5 \times 10^6 \]

\[ B_0 = 3 \text{ Gauss} \]

\[ B_0 = 5 \text{ Gauss} \]