Parameter-based Fisher’s information of orthogonal polynomials

J. S. Dehesa\textsuperscript{a,c,*}, B. Olmos\textsuperscript{a,c} & R.J. Yáñez\textsuperscript{b,c}

\textsuperscript{a}Departamento de Física Moderna, Universidad de Granada, 18071-Granada, Spain
\textsuperscript{b}Departamento de Matemática Aplicada, Universidad de Granada, 18071-Granada, Spain
\textsuperscript{c}Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, 18071-Granada, Spain

Abstract

The Fisher information of the classical orthogonal polynomials with respect to a parameter is introduced, its interest justified and its explicit expression for the Jacobi, Laguerre, Gegenbauer and Grosjean polynomials found.

* Correspondence address: Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, E-18071 Granada, Spain

Email addresses: dehesa@ugr.es (J. S. Dehesa), ryanez@ugr.es (R.J. Yáñez).
1 Introduction

Let \( \{ \rho_\theta(x) \equiv \rho(x|\theta); x \in \Omega \subset \mathbb{R} \} \) be a family of probability densities (or, more generally, likelihood functions) parametrized by a parameter \( \theta \in \mathbb{R} \). The Fisher information of \( \rho_\theta(x) \) with respect to the parameter \( \theta \) is defined \([15,24,25]\) as

\[
I(\rho_\theta) := \int_\Omega \left\{ \frac{\partial \ln \rho(x|\theta)}{\partial \theta} \right\}^2 \rho(x|\theta) dx = \int_\Omega \frac{[\partial \rho(x|\theta)]^2}{\rho(x|\theta)} dx
\]

\[
= 4 \int_\Omega \left\{ \frac{\partial [\rho(x|\theta)^{1/2}]}{\partial \theta} \right\}^2 dx
\]

(1) Additivity for independent events \([25]\). In the case that \( \rho(x, y|\theta) = \rho_1(x|\theta) \cdot \rho_2(y|\theta) \), it happens that

\[
I[\rho(x, y|\theta)] = I[\rho_1(x|\theta)] \cdot I[\rho_2(y|\theta)]
\]

(2) Scaling invariance \([48]\). The Fisher information is invariant under sufficient transformations \( y = t(x) \), so that

\[
I[\rho(y|\theta)] = I[\rho(x|\theta)].
\]

This property is not only closely related to the Fisher maximum likelihood method \([48]\) but also it is very important for the theory of statistical inference.

(3) Cramer-Rao inequality \([66]\). It states that the reciprocal of the Fisher information \( I(\rho_\theta) \) bounds from below the mean square error of an unbiased estimator \( f \) of the parameter \( \theta \); i.e.

\[
\sigma^2(f) \geq \frac{1}{I(\rho_\theta)},
\]

where \( \sigma^2(f) \) denotes the variance of \( f \). This inequality, which lies at the heart of statistical estimation theory \([62]\), shows how much information
the distribution provides about a parameter. Moreover, this says that the Fisher information $I(\rho_\theta)$ is a more sensitive indicator of the localization of the probability density than the Shannon entropy power [10, p. 240].

(4) Relation to other information-theoretic properties. The Fisher information is related to the Shannon entropy of $\rho(x|\theta)$ via the elegant de Bruijn’s identity [13,41,66]

$$\frac{\partial}{\partial \theta} S(\tilde{\rho}_\theta) = \frac{1}{2} I(\tilde{\rho}_\theta),$$

where $\tilde{\rho}_\theta$ denotes the convolution probability density of any probability density $\rho(x|\theta)$ with the normal density with zero mean and variance $\theta > 0$, and $S(\tilde{\rho}_\theta) := -\int_\Omega \tilde{\rho}_\theta(x) ln \tilde{\rho}_\theta(x) dx$ is the Shannon entropy of $\tilde{\rho}_\theta(x)$.

Moreover, the Fisher information $I(\rho_\theta)$ satisfies, under proper regularity conditions, the limiting property [26,27,35]

$$I(\rho_\theta) = \lim_{\epsilon \to 0} \frac{2}{\epsilon^2} D(\rho_{\theta+\epsilon} \| \rho_\theta),$$

where the symbol $D(p\|q) := \int_\Omega p(x) ln \frac{p(x)}{q(x)} dx$ denotes the relative entropy or Kullback-Leibler divergence of the probability densities $p(x)$ and $q(x)$. Further connections of the Fisher information with other information-theoretic properties are known; see e.g. [7,22,26,27,43,44,63,66].

(5) Applications in quantum physics. The Fisher information $I(\rho_\theta)$ plays a fundamental role in the quantum-mechanical description of physical systems [10,17,18,26,27,28,31,33,34,41,45,49,50,51,52,53,54,63]. It has been shown (a) to be a measure of both disorder and uncertainty [26,27,28] as well as a measure of nonclassicality for quantum systems [33,34],

(b) to describe, some factor apart, various macroscopic quantities such as, for example, the kinetic energy [41,63,69] and the Weiszäcker energy [50,51,52],

(c) to derive the Schrödinger and Klein-Gordon equations of motion [26,27] as well as the Euler equation of the density functional theory [45], from the principle of minimum Fisher information,

(d) to predict the most distinctive non-linear spectral phenomena, known as avoided crossings, encountered in atomic and molecular systems under strong external fields [31], and

(e) to be involved in numerous uncertainty inequalities [19,34,41,42,51,53,54,66].

These applications are most apparent when $\theta$ is a parameter of locality, so that $\rho_\theta(x) = \rho(x + \theta)$. Then, the Fisher information for locality, also called intrinsic accuracy [14,63], does not depend on the parameter $\theta$; so, without loss of generality, we may set the location at the origin and the Fisher information of the density $\rho_0(x) \equiv \rho(x)$ becomes

$$I(\rho) = \int \frac{[\rho'(x)]^2}{\rho(x)} dx,$$
where $\rho'(x)$ denotes the first derivative of $\rho(x)$. The locality Fisher information or Fisher information associated with translations of an one-dimensional observable $x$ with probability density $\rho(x)$ has been calculated in the literature \[13,68\] for some simple families of probability densities. In particular, it is well-known that the locality Fisher information of the Gaussian density is equal to the reciprocal of its variance, what illustrates the spreading character of these information-theoretic measures.

Recently, J.S. Dehesa and J. Sánchez-Ruiz \[59\] have exactly derived the locality Fisher information of a wider and much more involved class of probability densities, the Rakhmanov densities, defined by

$$
\rho_n(x) = \frac{1}{d_n^2} p_n(x)^2 \omega(x) \chi_{[a,b]}(x),
$$

where $\chi_{[a,b]}(x)$ is the characteristic function for the interval $[a, b]$, and $\{p_n(x)\}$ denotes a sequence of real polynomials orthogonal with respect to the non-negative definite weight function $\omega(x)$ on the interval $[a, b] \subseteq \mathbb{R}$, that is

$$
\int_a^b p_n(x)p_m(x)\omega(x)dx = d_n^2 \delta_{n,m}
$$

with $\deg p_n(x) = n$. As first pointed out by E.A. Rakhmanov \[47\], see also B. Simon \[65\], these densities play a fundamental role in the analytic theory of orthogonal polynomials. In particular, he has shown that these probability densities govern the asymptotic behavior of the ratio $p_{n+1}(x)/p_n(x)$ as $n \to \infty$. On the other hand, the Rakhmanov densities of the classical orthogonal polynomials of a real continuous variable describe the quantum-mechanical probability densities of ground and excited states of numerous physical systems with an exactly solvable Schrödinger equation \[5,12,46,67\], particularly the most common prototypes (harmonic oscillator, hydrogen atom,...), in position and momentum spaces \[30\].

These two fundamental and applied reasons have motivated an increasing interest for the determination of the spreading of the classical orthogonal polynomials $\{p_n(x)\}$ throughout its interval of orthogonality by means of the information-theoretic measures of their corresponding Rakhmanov densities $\rho(x)$ \[8,16,17,18,19,20,21,59,69\].

The Shannon information entropy of these densities has been examined numerically \[8\]. On the theoretical side, let us point out that its asymptotics ($n \to \infty$) is well known for all classical orthogonal polynomials, but its exact value for every fixed $n$ is only known for Chebyshev polynomials \[70\] and some Gegenbauer polynomials \[9,58,61\]. To this respect see \[20\] which reviews the knowledge up to 2001. The variance and Fisher information entropy of the Rakhmanov densities have been found in a closed and compact form.
for all classical orthogonal polynomials [18,59]. For other functionals of these Rakhmanov densities see Ref. [60].

In this paper we shall calculate the Fisher information of the real and continuous classical orthogonal polynomials (Gegenbauer, Grosjean, Jacobi, Laguerre) with respect to the parameter(s) of the polynomials. We begin in Section 2 with the definition of this notion and, as well, we collect here some basic properties of the classical orthogonal polynomials which will be used later on. Then, the Fisher information with respect to a parameter is fully determined for Jacobi and Laguerre polynomials in Section 3, and for Gegenbauer and Grosjean polynomials in Section 4. Finally, conclusions and some open problems are given.

2 Some properties of the parameter-dependent classical orthogonal polynomials

Let \( \{\tilde{y}_n(x; \theta)\}_{n \in \mathbb{N}_0} \) stand for the sequence of polynomials orthonormal with respect to the nonnegative definite weight function \( \omega(x; \theta) \) on the real support \((a,b)\), so that

\[
\int_a^b \tilde{y}_n(x; \theta)\tilde{y}_m(x; \theta)\omega(x; \theta)dx = \delta_{nm},
\]

with \( \deg \tilde{y}_n = n \). Here we shall consider the celebrated classical families of Hermite \( H_n(x) \), Laguerre \( L_n^{(\alpha)}(x) \) and Jacobi \( J_n^{(\alpha,\beta)}(x) \) polynomials. The normalized-to-unity density functions \( \tilde{\rho}_n(x; \theta) \) defined as

\[
\tilde{\rho}_n(x; \theta) = \omega(x; \theta)\tilde{y}_n^2(x; \theta)
\]

are called for Rakhmanov densities [47,65].

Here we gather various properties of the parameter-dependent classical orthogonal polynomials in a real and continuous variable (i.e. Laguerre and Jacobi) in the form of two Lemmas, which shall be used later on. The weight function of these polynomials can be written as

\[
\omega(x; \theta) = h(x) [t(x)]^\theta
\]

with

\[
h_L(x) = e^{-x} \quad \text{and} \quad t_L(x) = x
\]

for the Laguerre case, \( L_n^{(\theta)}(x) \), and

\[
h_J(x) = (1 + x)^\beta \quad \text{and} \quad t_J(x) = 1 - x
\]

for the Jacobi case, \( P_n^{(\theta,\beta)}(x) \).
Lemma 1 The derivative of the orthonormal polynomial $\bar{y}_n(x; \theta)$ with respect to the parameter $\theta$ is given by

$$\frac{\partial}{\partial \theta} \bar{y}_n(x; \theta) = \sum_{k=0}^{n} \bar{A}_k(\theta) \bar{y}_k(x; \theta)$$

with

$$\bar{A}_k(\theta) = \frac{d_k(\theta)}{d_n(\theta)} A_k(\theta) \quad \text{for} \quad k = 0, 1, \ldots, n - 1$$

$$\bar{A}_n(\theta) = A_n(\theta) - \frac{1}{d_n(\theta)} \frac{\partial}{\partial \theta} [d_n(\theta)]$$

where $d_m^2(\theta)$ denotes, according to Eq. (3), the norm of the orthogonal polynomial $p_m(x) = y_m(x; \theta)$, and $A_k(\theta)$ with $k = 0, 1, \ldots$ are the expansion coefficients of the derivative of $y_m(x; \theta)$ in terms of the system $\{y_m(x; \theta)\}$; i.e.,

$$\frac{\partial}{\partial \theta} y_m(x; \theta) = \sum_{k=0}^{m} A_k(\theta) y_k(x; \theta).$$

Both quantities $d_m(\theta)$ and $A_m(\theta)$ are known in the literature for the Laguerre and Jacobi cases. Indeed, the norms for the Laguerre $L_n^{(\alpha)}(x)$ and the Jacobi $P_n^{(\alpha, \beta)}(x)$ polynomials [46] are

$$\left[ d_k^{(L)}(\alpha) \right]^2 = \frac{\Gamma(k + \alpha + 1)}{k!},$$

$$\left[ d_k^{(J)}(\alpha, \beta) \right]^2 = \frac{2^{\alpha+\beta+1}\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)}{k!(2k + \alpha + \beta + 1)\Gamma(k + \alpha + \beta + 1)},$$

respectively.

On the other hand the expansion coefficients in Eq. (12) are known [29,38,39,57] to have the form

$$A_k^{(L)} = \frac{1}{n - k} \quad \text{for} \quad k = 0, 1, \ldots, n - 1 \quad \text{and} \quad A_n^{(L)} = 0$$

for the Laguerre polynomials $L_n^{(\alpha)}$, and
\[ A_k^{(J_\alpha)} = \frac{\alpha + \beta + 1 + 2k}{(n-k)(\alpha + \beta + 1 + n + k)} \times \frac{(\beta + k + 1)_{n-k}}{(\alpha + \beta + k + 1)_{n-k}}; \quad k = 0, 1, \ldots, n-1, \]  
\[ A_n^{(J_\alpha)} = \sum_{k=0}^{n-1} \frac{1}{\alpha + \beta + 1 + n + k} \]  
\[ = \psi(1 + \alpha + \beta + 2n) - \psi(1 + \alpha + \beta + n), \]  

for the Jacobi expansion of \( \frac{\partial}{\partial \alpha} P_n^{(\alpha,\beta)}(x) \), and

\[ A_k^{(J_\beta)} = (-1)^{n-k} \frac{\alpha + \beta + 1 + 2k}{(n-k)(\alpha + \beta + 1 + n + k)} \times \frac{(\alpha + k + 1)_{n-k}}{(\alpha + \beta + k + 1)_{n-k}}; \quad k = 0, 1, \ldots, n-1, \]  
\[ A_n^{(J_\beta)} = \sum_{k=0}^{n-1} \frac{1}{\alpha + \beta + 1 + n + k} \]  
\[ = \psi(1 + \alpha + \beta + 2n) - \psi(1 + \alpha + \beta + n), \]  

for the Jacobi expansion of \( \frac{\partial}{\partial \beta} P_n^{(\alpha,\beta)}(x) \). The symbol \( \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \) denotes the well-known digamma function. It is worth to remark that the superindices \( J_\alpha \) and \( J_\beta \) indicate the expansion coefficients for the derivatives of the Jacobi polynomial with respect to the first and second parameters, respectively.

Then, taking into account Eqs. (10)-(11) and Eqs. (13) and (15), Lemma 1 says that the expansion coefficients \( \tilde{A}_k(\alpha) \) for the orthonormal Laguerre polynomials \( \tilde{L}_n^{(\alpha)}(x) \) are:

\[ \tilde{A}_k^{(L)} = \frac{1}{n-k} \left[ \frac{(k+1)_{n-k}}{(k+\alpha+1)_{n-k}} \right]^{1/2}; \quad k = 0, 1, \ldots, n-1 \]  
\[ \tilde{A}_n^{(L)} = -\frac{\psi(n+\alpha+1)}{2} \]  

In a similar way the same Lemma together with Eqs. (10)-(11) and (14)-(17) has allowed us to find the expressions
The integrals (a) and (b) we have to derive with respect to the parameter $\alpha$ of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Finally, the same procedure with Eqs. (10)-(11), (14), (18) and (19) leads to the values

$$\tilde{A}_n^{(J_\alpha)} = \frac{1}{2} \left[ 2\psi(2n + \alpha + \beta + 1) - \psi(n + \alpha + \beta + 1) \right.
- \left. \psi(n + \alpha + 1) - \ln 2 + \frac{1}{2n + \alpha + \beta + 1} \right],$$

for the expansion coefficients $\tilde{A}_n^{(J_\alpha)}$ of the derivative with respect to the parameter $\alpha$ of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Finally, the same procedure with Eqs. (10)-(11), (14), (18) and (19) leads to the values

$$\tilde{A}_n^{(J_\beta)} = \frac{1}{2} \left[ 2\psi(2n + \alpha + \beta + 1) - \psi(n + \alpha + \beta + 1) \right.
- \left. \psi(n + \beta + 1) - \ln 2 + \frac{1}{2n + \alpha + \beta + 1} \right],$$

for the expansion coefficients $\tilde{A}_n^{(J_\beta)}$ ($k = 0, 1, \ldots, n$) in Eq. (9) of the $\beta$-derivative of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

**Lemma 2** The parameter-dependent classical orthonormal polynomials $\tilde{y}_n(x; \theta)$ satisfy

(a) $\int_a^b \frac{\partial^2 \omega(x; \theta)}{\partial \theta^2} [\tilde{y}_n(x; \theta)]^2 \, dx = -2\tilde{A}_n(\theta)$

(b) $\int_a^b \frac{\partial \omega(x; \theta)}{\partial \theta} \tilde{y}_n(x; \theta) \tilde{y}_k(x; \theta) \, dx = -\tilde{A}_k(\theta) ; \quad k = 0, 1, \ldots, n - 1$

(c) $\int_a^b \frac{\partial \omega(x; \theta)}{\partial \theta} \tilde{y}_n(x; \theta) \tilde{y}_k(x; \theta) \, dx = 2 \sum_{k=0}^n (\tilde{A}_k(\theta))^2 + 2(\tilde{A}_n(\theta))^2 - 2 \frac{\partial \tilde{A}_n(\theta)}{\partial \theta}$

**Proof.** To prove the integrals (a) and (b) we have to derive with respect to the parameter $\theta$ the orthonormalization condition (4) for $m = n$ and $m = k \neq n$, respectively. Then one has to use the Lemma 1 and again Eq. (4), and the results follow.

The integral (c) is obtained by deriving the integral (a) with respect to $\theta$ and taking into account the values of the two previous integrals (a) and (b).
3 Parameter-based Fisher information of Jacobi and Laguerre polynomials

The distribution of the orthonormal polynomials \( \tilde{y}_n(x; \theta) \) on their orthonormality interval and the spreading of the associated Rakhmanov densities can be most appropriately measured by means of their information-theoretic measures, the Shannon entropy [64] and the Fisher information [24,25]. The former has been theoretically [3,20,21] and numerically [8] examined for general orthogonal polynomials, while for the latter it has been studied the Fisher information associated with translations of the variable (i.e. the locality Fisher information) both analytically [59] and numerically [18]. Here we extend this study by means of the computation of a more general concept, the parameter-based Fisher information of the polynomials \( \tilde{y}_n(x; \theta) \). This quantity is defined as the Fisher information of the associated Rakhmanov density (5) with respect to the parameter \( \theta \); that is, according to Eq. (1), by

\[
I_n(\theta) = 4 \int_a^b \left\{ \frac{\partial}{\partial \theta} \tilde{\rho}_n(x; \theta) \right\}^{1/2} dx
\]

with

\[
[\tilde{\rho}_n(x; \theta)]^{1/2} = [\omega(x; \theta)]^{1/2} \tilde{y}_n(x; \theta).
\]

**Theorem 1** The parameter-based Fisher information \( I_n(\theta) \) of the parameter-dependent classical orthonormal polynomials \( \tilde{y}_n(x; \theta) \) (i.e., Jacobi and Laguerre) defined by Eq. (26) has the value

\[
I_n(\theta) = 2 \sum_{k=0}^{n-1} \left[ A_k(\theta) \right]^2 - 2 \frac{\partial \tilde{A}_n(\theta)}{\partial \theta},
\]

where \( \tilde{A}_k(\theta) \), \( k = 0, 1, \ldots, n \) are the expansion coefficients of the derivative with respect to \( \theta \) of \( \tilde{y}_n(x; \theta) \) in terms of the polynomials \( \{ \tilde{y}_k(x; \theta) \}_{k=0}^n \), which are given by Lemma 1. See Eqs. (20)-(21) and (22)-(25) for the Laguerre and Jacobi families, respectively.

**Remark.** Let us underline that the Fisher quantities of orthogonal, monic orthogonal and orthonormal polynomials have the same value because of Eqs. (5) and (26), keeping in mind the probabilistic character of the Rakhmanov density and the fact that all these polynomials are orthogonal with respect to the same weight function.

**Proof.** To prove this theorem we begin with Eq. (26). Then, the derivative with respect to \( \theta \) and Lemma 1 lead to
\[
\frac{\partial}{\partial \theta} [\tilde{\rho}_n(x; \theta)]^{1/2} = [\omega(x; \theta)]^{1/2} \sum_{k=0}^{n} \tilde{A}_k(\theta) \tilde{y}_k(x; \theta) + \frac{\partial [\omega(x; \theta)]^{1/2}}{\partial \theta} \tilde{y}_n(x; \theta).
\]

The substitution of this expression into Eq. (26) and the consideration of the orthonormalization condition (4) have led us to

\[ I_n(\theta) = J_1 + J_2 + J_3, \]

where

\[ J_1 = 4 \int_a^b \omega(x; \theta) \left( \sum_{k=0}^{n} \tilde{A}_k(\theta) \tilde{y}_k(x; \theta) \right)^2 dx = 4 \sum_{k=0}^{n} [\tilde{A}_k(\theta)]^2, \]

\[ J_2 = 4 \int_a^b \left( \frac{\partial [\omega(x; \theta)]^{1/2}}{\partial \theta} \right)^2 [\tilde{y}_n(x; \theta)]^2 dx, \]

and

\[ J_3 = 8 \sum_{k=0}^{n} \tilde{A}_k(\theta) \int_a^b [\omega(x; \theta)]^{1/2} \frac{\partial [\omega(x; \theta)]^{1/2}}{\partial \theta} \tilde{y}_n(x; \theta) \tilde{y}_k(x; \theta) dx. \]

Now we take into account that the weight function of the parameter-dependent families of classical orthonormal polynomials in a real and continuous variable (i.e. Laguerre and Jacobi) has the form \( \omega(x; \theta) = h(x) [t(x)]^\theta \), so that

\[ \left\{ \frac{\partial [\omega(x; \theta)]^{1/2}}{\partial \theta} \right\}^2 = \frac{1}{4} \omega(x; \theta) [\ln t(x)]^2 = \frac{1}{4} \frac{\partial^2 \omega(x; \theta)}{\partial \theta^2}, \]

\[ [\omega(x; \theta)]^{1/2} \frac{\partial [\omega(x; \theta)]^{1/2}}{\partial \theta} = \frac{1}{2} \omega(x; \theta) \ln t(x) = \frac{1}{2} \frac{\partial \omega(x; \theta)}{\partial \theta}. \]

The use of these two expressions in the integrals \( J_2 \) and \( J_3 \) together with Lemma 2 leads to Eq. (27). \( \square \)

**Corollary 1** The Fisher information with respect to the parameter \( \alpha \), \( I_n^{(L)}(\alpha) \), of the Laguerre polynomial \( \tilde{L}_n^{(\alpha)}(x) \) is

\[ I_n^{(L)}(\alpha) = 2 \sum_{k=0}^{n-1} \left[ \tilde{A}_k^{(L)}(\alpha) \right]^2 - 2 \frac{\partial \tilde{A}_n^{(L)}}{\partial \alpha} \]

\[ = \psi^{(1)}(n + \alpha + 1) + \frac{2n}{n + \alpha} {}_4F_3 \begin{pmatrix} 1 & 1 & 1 & 1-n \\ 1 & 2 & 1-n & 1 \end{pmatrix}, \]
where $\psi^{(1)}(x) = \frac{d}{dx} \psi(x)$ is the trigamma function.

**Corollary 2** The Fisher information with respect to the parameter $\alpha$, $I_n^{(J_\alpha)}(\alpha, \beta)$, of the Jacobi polynomial $\tilde{P}_n^{(\alpha, \beta)}(x)$ has the value

$$I_n^{(J_\alpha)}(\alpha, \beta) = 2 \sum_{k=0}^{n-1} \left[ \tilde{A}_k^{(J_\alpha)} \right]^2 - 2 \frac{\partial \tilde{A}_n^{(J_\alpha)}}{\partial \alpha}$$

$$= 2 \frac{\Gamma(n + \beta + 1)n!(2n + \alpha + \beta + 1)}{\Gamma(n + \alpha + 1)\Gamma(n + \alpha + \beta + 1)} \times \sum_{k=0}^{n-1} \frac{\Gamma(k + \alpha + 1)\Gamma(k + \alpha + \beta + 1)(2k + \alpha + \beta + 1)}{\Gamma(k + \beta + 1)k!(n-k)^2(n+k+\alpha+\beta+1)^2}$$

$$- 2\psi^{(1)}(2n + \alpha + \beta + 1) + \psi^{(1)}(n + \alpha + \beta + 1)$$

$$+ \psi^{(1)}(n + \alpha + 1) + \frac{1}{(2n + \alpha + \beta + 1)^2},$$

and the Fisher information with respect to the parameter $\beta$, $I_n^{(J_\beta)}(\alpha, \beta)$, of the Jacobi polynomial $\tilde{P}_n^{(\alpha, \beta)}(x)$ has the value

$$I_n^{(J_\beta)}(\alpha, \beta) = 2 \sum_{k=0}^{n-1} \left[ \tilde{A}_k^{(J_\beta)} \right]^2 - 2 \frac{\partial \tilde{A}_n^{(J_\beta)}}{\partial \beta}$$

$$= 2 \frac{\Gamma(n + \alpha + 1)n!(2n + \alpha + \beta + 1)}{\Gamma(n + \beta + 1)\Gamma(n + \alpha + \beta + 1)} \times \sum_{k=0}^{n-1} \frac{\Gamma(k + \beta + 1)\Gamma(k + \alpha + \beta + 1)(2k + \alpha + \beta + 1)}{\Gamma(k + \alpha + 1)k!(n-k)^2(n+k+\alpha+\beta+1)^2}$$

$$- 2\psi^{(1)}(2n + \alpha + \beta + 1) + \psi^{(1)}(n + \alpha + \beta + 1)$$

$$+ \psi^{(1)}(n + \beta + 1) + \frac{1}{(2n + \alpha + \beta + 1)^2}.$$

Both corollaries follow from Theorem 1 in a straightforward manner by taking into account the expressions (20)-(25) for the expansion coefficients $\tilde{A}_k$ of the corresponding families. Let us underline that $J_\alpha$ and $J_\beta$ indicate Fisher informations with respect to the first and second parameter, respectively, of the Jacobi polynomial.
4 Parameter-based Fisher information of the Gegenbauer and Grosjean polynomials

In this section we describe the Fisher information of two important subfamilies of the Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \): the ultraspherical or Gegenbauer polynomials \([4,20,46]\), which have \( \alpha = \beta \), and the Grosjean polynomials of the first and second kind \([23,32,55,56]\), which have \( \alpha + \beta = \pm 1 \), respectively. Let us remark that the parameter-based Fisher information for these subfamilies cannot be obtained from the expressions of the similar quantity for the Jacobi polynomials (given by corollary 2) by means of a mere substitution of the parameters, because it depends on the derivative with respect to the parameter(s) and now \( \alpha \) and \( \beta \) are correlated.

The Gegenbauer polynomials \( C_n^{(\lambda)}(x) \) are Jacobi-like polynomials satisfying the orthogonality condition (3) with the weight function \( \omega_C(x;\lambda) = (1 - x^2)^{\lambda - \frac{1}{2}}, \lambda > -\frac{1}{2} \), and the normalization constant

\[
[d_k^{(C)}(\lambda)]^2 = \frac{\pi 2^{1-2\lambda} \Gamma(k + 2\lambda)}{k!(k + \lambda) \Gamma^2(\lambda)}
\]

so that they can be expressed as

\[
C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda - \frac{1}{2},\lambda - \frac{1}{2})}(x)
\]

It is known \([38,39]\) that the expansion (12) for the derivative of \( C_n^{(\lambda)}(x) \) with respect to the parameter \( \lambda \) has the coefficients

\[
A_k^{(C)}(\lambda) = \frac{2 \left( 1 + (-1)^{n-k} \right) (k + \lambda)}{(k + n + 2\lambda)(n - k)} \quad \text{for} \quad k = 0, 1, \ldots, n - 1
\]

\[
A_n^{(C)}(\lambda) = \sum_{k=0}^{n-1} \frac{2(k + 1)}{(2k + 2\lambda + 1)(k + 2\lambda)} + \frac{2}{k + n + 2\lambda} = \psi(n + \lambda) - \psi(\lambda)
\]

Then, according to Eqs. (10-11) of Lemma 1, the expansion (9) for the derivative of the orthonormal Gegenbauer polynomials has the following coefficients
\[
\tilde{A}_k^{(C)}(\lambda) = \left[ \frac{\Gamma(k + 2\lambda) n! (n + \lambda)}{\Gamma(n + 2\lambda) k! (k + \lambda)} \right]^{1/2} \\
2 \frac{(1 + (-1)^{n-k}) (k + \lambda)}{(k + n + 2\lambda) (n - k)} \Gamma(k + 2\lambda) (k + \lambda) \Gamma(n + 2\lambda) (n + \lambda) \\
\quad \times \frac{1 + (-1)^{n-k}}{n!(n + \lambda)} \frac{n! (n + \lambda)}{2(n + 2\lambda)} \\
\text{for } k = 0, 1, \ldots, n - 1.
\]

\[
\tilde{A}_n^{(C)}(\lambda) = \psi(n + \lambda) - \psi(n + 2\lambda) + \ln 2 + \frac{1}{2(n + \lambda)}.
\]

Theorem 1 provides, according to Eq. (27), the following value for the Fisher information of the Gegenbauer polynomials \( C_n^{(\lambda)}(x) \) with respect to the parameter \( \lambda \):

\[
I_n^{(C)}(\lambda) = \frac{16n!(n + \lambda)}{\Gamma(n + 2\lambda)} \sum_{k=0}^{n-1} \frac{\Gamma(k + 2\lambda) (k + \lambda)}{k! (k + n + 2\lambda) (n - k)^2} \\
-2\psi^{(1)}(n + \lambda) + 4\psi^{(1)}(n + 2\lambda) + \frac{1}{(n + \lambda)^2}.
\]

Let us now do the same job for the Grosjean polynomials of the first and second kind, which are the monic Jacobi polynomials \( \hat{P}_n^{(\alpha,\beta)}(x) \) with \( \alpha + \beta = \mp 1 \), respectively. So, we have [32,55]

\[
G_n^{(\alpha)}(x) = c_n P_n^{(\alpha, -1 - \alpha)}(x) \quad , \quad -1 < \alpha < 0,
\]

and

\[
g_n^{(\alpha)}(x) = e_n P_n^{(\alpha, 1 - \alpha)}(x) \quad , \quad -1 < \alpha < 2,
\]

for the Grosjean polynomials of first and second kind, respectively, with the values

\[
c_n = 2^n \left( \frac{2n - 1}{n} \right)^{-1}, \quad e_n = 2^n \left( \frac{2n + 1}{n} \right)^{-1}.
\]

The Grosjean polynomials of the first kind \( G_n^{(\alpha)}(x) \) satisfy the orthogonality condition (3) with the weight function

\[
\omega_{G}(x; \alpha) = \left( \frac{1 - x}{1 + x} \right)^{\alpha} \frac{1}{1 + x},
\]

and the normalization constant

\[
\left[ d_n^{(G)}(\alpha) \right]^2 = \frac{2^{2n-1} \Gamma^2(n)}{\Gamma^2(2n)} \Gamma(n + \alpha + 1) \Gamma(n - \alpha).
\]

These polynomials, together with the Chebyshev polynomials of the first, second, third and fourth kind, are the only Jacobi polynomials for which the associated polynomials are again Jacobi polynomials [32].

Now, the expansion (12) for the derivative of the polynomials \( G_n^{(\alpha)}(x) \) with respect to the parameter \( \alpha \) can be obtained as
\[
\frac{\partial G_n^{(\alpha)}(x)}{\partial \alpha} = \frac{\partial \hat{P}_{n}^{(\alpha,-1-\alpha)}(x)}{\partial \alpha} = \frac{\partial \hat{P}_n^{(\alpha,\beta)}(x)}{\partial \alpha} \bigg|_{\beta=-1-\alpha} - \frac{\partial \hat{P}_n^{(\alpha,\beta)}(x)}{\partial \beta} \bigg|_{\beta=-1-\alpha}
\]

\[
= \sum_{k=0}^{n-1} A_k^{(G)}(\alpha)G_n^{(\alpha)}(x)
\]

with

\[
A_k^{(G)}(\alpha) = \frac{2^{n-k+1}k \Gamma(2k)\Gamma(n+1)}{n^2-k^2 \Gamma(2n)\Gamma(k+1)} \left[(k-\alpha)_{n-k} - (-1)^{n-k}(k+\alpha+1)_{n-k}\right]
\]

for \(k = 0, 1, \ldots, n-1\) and \(A_n^{(G)}(\alpha) = 0\). Then, Lemma 1 provides the expansion (9) for the derivative of the orthonormal Grosjean polynomials with the coefficients

\[
\bar{A}_k^{(G)}(\alpha) = \frac{2n}{n^2-k^2} \frac{(k-\alpha)_{n-k} - (-1)^{n-k}(k+\alpha+1)_{n-k}}{[(k+\alpha+1)_{n-k}(k-\alpha)_{n-k}]^{1/2}}
\]

for \(k = 0, 1, \ldots, n-1\),

\[
\bar{A}_n^{(G)}(\alpha) = \frac{1}{2} [\psi(n-\alpha) - \psi(n+\alpha+1)] .
\]

Finally, Eq. (27) of Theorem 1 allows us to find the following value for the Fisher information of the Grosjean polynomials of the first kind:

\[
I_n^{(G)}(\alpha) = 8n^{2} \sum_{k=0}^{n-1} \frac{1}{(n^2-k^2)^2} \frac{[(k-\alpha)_{n-k} - (-1)^{n-k}(k+\alpha+1)_{n-k}]^2}{(k+\alpha+1)_{n-k}(k-\alpha)_{n-k}}
\]

\[
+ \psi^{(1)}(n-\alpha) + \psi^{(1)}(n+\alpha+1).
\]

On the other hand, the Grosjean polynomials of the second kind \(g_n^{(\alpha)}(x)\) satisfy the orthogonality property (3) with the weight function

\[
\omega_g(x; \alpha) = \left(\frac{1-x}{1+x}\right)^\alpha (1+x),
\]

and the normalization constant

\[
[d_n^{(g)}(\alpha)]^2 = \frac{2^{2n+1}\Gamma^2(n)}{\Gamma^2(2n+2)} \Gamma(n+\alpha+1)\Gamma(n-\alpha+2).
\]

Moreover, the derivative of these polynomials with respect to the parameter \(\alpha\) can be expanded in the form (9) as
\[
\frac{\partial g_n^{(\alpha)}(x)}{\partial \alpha} = \frac{\partial \hat{P}_n^{(\alpha,1-\alpha)}(x)}{\partial \alpha} = \frac{\partial \hat{P}_n^{(\alpha,\beta)}(x)}{\partial \alpha} \bigg|_{\beta=1-\alpha} - \frac{\partial \hat{P}_n^{(\alpha,\beta)}(x)}{\partial \beta} \bigg|_{\beta=1-\alpha}
= \sum_{k=0}^{n-1} A_k^{(g)}(\alpha)g_n^{(\alpha)}(x)
\]

with

\[
A_k^{(g)}(\alpha) = \frac{2^{n-k+1}(k+1)}{(n-k)(n+k+2)} \frac{\Gamma(2k+2)\Gamma(n+1)}{\Gamma(2n+2)\Gamma(k+1)} \times \left((k+2-\alpha)_{n-k} - (-1)^{n-k}(k+\alpha+1)_{n-k}\right)
\]

for \(k = 0, 1, \ldots, n-1\) and \(A_n^{(g)}(\alpha) = 0\). Then, Lemma 1 is able to provide the analogous expansion (9) for the orthonormal polynomials with the coefficients

\[
\tilde{A}_k^{(g)}(\alpha) = \frac{2(k+1)}{(n-k)(n+k+2)} \frac{(k+2-\alpha)_{n-k} - (-1)^{n-k}(k+\alpha+1)_{n-k}}{[(k+\alpha+1)_{n-k}(k+2-\alpha)_{n-k}]^{1/2}};
\]

for \(k = 0, 1, \ldots, n-1\),

\[
\tilde{A}_n^{(g)}(\alpha) = \frac{1}{2} [\psi(n+2-\alpha) - \psi(n+\alpha+1)].
\]

Finally, we obtain by means of Eq. (27) of Theorem 1 the Fisher information of the Grosjean polynomials of the second kind, which turns out to have the value

\[
I_n^{(g)}(\alpha) = 8 \sum_{k=0}^{n-1} \frac{(k+1)^2}{(n-k)^2(n+k+2)^2}
\times \frac{\left[(k+2-\alpha)_{n-k} - (-1)^{n-k}(k+\alpha+1)_{n-k}\right]^2}{(k+\alpha+1)_{n-k}(k+2-\alpha)_{n-k}}
+ \psi^{(1)}(n+2-\alpha) + \psi^{(1)}(n+\alpha+1).
\]

5 Conclusions and open problems

In summary, we have calculated the parameter-based Fisher information for the classical orthogonal polynomials of a continuous and real variable with a parameter dependence; namely, the Jacobi and Laguerre polynomials. Then we have evaluated the corresponding Fisher quantity for the two most relevant parameter-dependent Jacobi polynomials \(P_n^{(\alpha,\beta)}(x)\): the Gegenbauer (\(\alpha = \beta\)) and the Grosjean (\(\alpha + \beta = \pm 1\)) polynomials.
This paper, together with Ref. [59], opens the way for the development of the Fisher estimation theory of the Rakhmanov density for continuous and discrete orthogonal polynomials in and beyond the Askey scheme. This fundamental task in approximation theory includes the determination of the spreading of the orthogonal polynomials throughout its orthogonality domain by means of the Fisher information with a locality property. All these mathematical questions have a straightforward application to quantum systems because their wavefunctions are often controlled by orthogonal polynomials, so that the probability densities which describe the quantum-mechanical states of these physical systems are just the Rakhmanov densities of the corresponding orthogonal polynomials.

Among the open problems let us first mention the computation of the parameter-based Fisher information of the generalized Hermite polynomials, the Bessel polynomials and the Pollaczek polynomials [11]. A much more ambitious problem is the evaluation of the Fisher quantity for the general Wilson orthogonal polynomials [2]. On the other hand, nothing is known for discrete orthogonal polynomials. In this case, however, the very notion of the parameter-based Fisher information is a subtle question.

6 Acknowledgements

This work has been partially supported by the MEC Project No. FIS2005-00973, by the European Research Network on Constructive Approximation (NeCCA) INTAS-03-51-6637 and by the J.A. Project of Excellence with ref. FQM-481. We belong to the P.A.I. Group FQM-207 of the Junta de Andalucía, Spain.

References

[1] S. Amari, Differential Geometric Methods in Statistics, (Springer, New York, 1985)

[2] G.E. Andrews, R. Askey and R. Roy, Special Functions, (Cambridge University Press, Cambridge, 1999)

[3] A.I. Aptekarev, V.S. Buyarov and J.S. Dehesa, Asymptotic behavior of $L^p$ norms and entropy for general orthogonal polynomials, Russian Acad. Sci. Sb. Math., 82 (1995) 373-395

[4] W. van Assche, R.J. Yáñez, R. González-Ferez and J.S. Dehesa, Functionals of Gegenbauer polynomials and D-dimensional hydrogenic momentum expectation values, J. Math. Phys., 41 (2000) 6600-6613
[5] V. Bagrov and D. Gitman, *Exact Solutions of Relativistic Wave Equations*, (Kluwer Academic Publishers, Dordrecht, 1990)

[6] O.E. Barndorff-Nielsen, *Information and Exponential Families in Statistical Theory*, (Wiley, N.Y., 1978)

[7] A.R. Barron, *Entropy and the central limit theorem*, Ann. Probab. 14 (1986) 336-342

[8] V.S. Buyarov, J.S. Dehesa, A. Martínez-Finkelshtein and J. Sánchez-Lara, *Computation of the entropy of polynomials orthogonal on an interval*, SIAM J. Sci. Comput., 26 (2005) 488-509

[9] V.S. Buyarov, *On information entropy of Gegenbauer polynomials*, Vestn. Mosk. Univ. Ser. I Mat. Mekh., 6 (1997) 8-11 (in Russian)

[10] R. W. Carroll, *Fluctuations, Information, Gravity and the Quantum Potential*, (Springer, Dordrecht, 2006)

[11] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, (New York: Gordon and Breach, 1978)

[12] F. Cooper, A. Khare and U. Sukhatme, *Supersymmetry in Quantum Mechanics*, (World Scientific, Singapore, 2002)

[13] T.M. Cover, J.A. Thomas, *Elements of Information Theory*, Wiley, N.Y., 1991

[14] D.R. Cox and D. Hinkley, *Theoretical Statistics*, (Chapman and Hall, London, 1974)

[15] H. Cramer, *Mathematical Methods of Statistics*, Princeton University Press, Princeton, 1946

[16] J.S. Dehesa, S. López-Rosa, B. Olmos and R.J. Yáñez, *Information measures of hydrogenic systems, Laguerre polynomials and spherical harmonics*, J. Comput. Appl. Math. 179 (2005) 185-194

[17] J.S. Dehesa, S. Lopez-Rosa, B. Olmos and R.J. Yáñez, *The Fisher information of D-dimensional hydrogenic systems in position and momentum spaces*, J. Math. Phys. (2006). Accepted

[18] J.S. Dehesa, P. Sánchez-Moreno and R.J. Yáñez, *Cramer-Rao information plane of orthogonal hypergeometric polynomials*, J. Comput. Appl. Math. 186 (2006) 523-541

[19] J.S. Dehesa, A. Martínez-Finkelshtein and V.N. Sorokin, *Information-theoretic measures for Morse and Pöschl-Teller potentials*, Molecular Phys. 104(4) (2006) 613-622

[20] J.S. Dehesa, A. Martínez-Finkelshtein and J. Sánchez-Ruiz, *Quantum information entropies and orthogonal polynomials*, J. Comput. Appl. Math. 133 (2001) 23-46
[21] J.S. Dehesa, W. van Assche and R.J. Yáñez, *Information entropy of classical orthogonal polynomials and their application to the harmonic oscillator and Coulomb potentials*, Meth. Appl. Anal., 4 (1997) 91-110

[22] A. Dembo and T.M. Cover, *Information theoretic inequalities*, IEEE Trans. Inf. Theory, 37 (1991) 1501-1518

[23] H. Dette, *First return probabilities and birth and death chains and associated orthogonal polynomials*, Proceed. Amer. Math. Soc., 129 (2000) 1805-1815

[24] R.A. Fisher, *Theory of statistical estimation*, Proc. Cambridge Phil. Sec. 22 (1925) 700-725 [Reprinted in *Collected Papers* of R. A. Fisher, edited by J. H. Bennett (University of Adelaide Press, South Australia, 1972), pp. 15-40]

[25] R.A. Fisher, *Statistical Methods and Scientific Inference*, 2nd. edition (Oliver and Boyd, London, 1959)

[26] B.R. Frieden, *Physics from Fisher Information: a Unification*, Cambridge University Press, Cambridge (1998)

[27] B.R. Frieden, *Science from Fisher Information*, Cambridge University Press, Boston (2004)

[28] B.R. Frieden, *Fisher information and uncertainty complementarity*, Phys. Lett. A, 169 (1992) 123-130

[29] J. Froehlich, *Parameter derivatives of the Jacobi polynomials and the Gaussian hypergeometric function*, Int. Transf. and Special Functions, 2(4) (1994) 253-266

[30] A. Galindo and P. Pascual, *Quantum Mechanics*, Springer, N.Y. 1991

[31] R. González-Ferez and J.S. Dehesa, *Characterization of atomic avoided crossings by means of Fisher information* European Phys. J. D, 32 (2005) 39-43

[32] C.C. Grosjean, *The weight functions, generating functions and miscellaneous properties of the sequences of orthogonal polynomials of second kind associated with the Jacobi and Gegenbauer polynomials*, J. Comp. Appl. Math., 16 (1986) 259-307

[33] M.J.W. Hall, *Quantum properties of classical Fisher information*, Phys. Rev. A, 62 (2000) 012107

[34] M.J.W. Hall, *Exact uncertainty relations*, Phys. Rev. A, 64 (2001) 052103

[35] M. Hayasi, *Two quantum analogues of Fisher information from a large deviation viewpoint of quantum estimation*, J. Phys. A: Math. Gen., 35 (2002) 7689-7727

[36] A.S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, (North-Holland, Amsterdam, 1982)

[37] O. Johnson and A. Barron, *Fisher information inequalities and the central limit theorem*, Probab. Theory Rel. Fields, 129 (2004) 391-409

[38] W. Koepf, *Identities for families of orthogonal polynomials and special functions*, Int. Transf. and Special Funct., 5 (1997) 69-102
[39] W. Koepf and D. Schmersau, *Representations of orthogonal polynomials*, J. Comp. Appl. Math., 90 (1998) 57-94

[40] S. Lewanowicz, *Representations for the parameter derivatives of the classical orthogonal polynomials*, Proc. Fourth Int. Conf. on Functional Analysis and Approximation Theory, vol. II (Potenza, 2000). Rend. Circ. Mat. Palermo (2) Suppl. 2002, n68, part II, 599-613

[41] S. Luo, *Fisher information, kinetic energy and uncertainty relation inequalities*, J. Phys. A: Math. Gen., 35 (2002) 5181-5187

[42] S. Luo, *A variation of the Heisenberg uncertainty relation involving an average*, J. Phys. A: Math. Gen., 34 (2001) 3289-3291

[43] E. Lutwak, D. Yang and G. Zhang, *Cramer-Rao and moment-entropy inequalities for Renyi entropy and generalized Fisher information* IEEE Transactions on Inf. Theory, 51 (2005) 473-478

[44] G.R. Mohastami Borzadaran, *Relationship between entropies, variance and Fisher information*, in Proc. 20th Int. Workshop on Bayesian Inference and Maximum Entropy Methods in Science and Engineering, edited by A. Mohammad Djafari (American Institute of Physics, N.Y., 2001)

[45] A. Nagy, *Fisher information in density functional theory*, J. Chem. Phys. 119 (2003) 9401-9405

[46] A. Nikiforov and V. Uvarov, *Special Functions in Mathematical Physics*, (Birkhauser Verlag, Basel, 1988)

[47] E.A. Rakhmanov, *On the asymptotics of the ratio of orthogonal polynomials*, Math. USSR Sb., 32 (1977) 199-213

[48] C.R. Rao, *Linear Statistical Inference and its Applications*, (Wiley, New York, 1973). Second edition

[49] E. Romera and J.S. Dehesa, *The Fisher-Shannon information plane, an electron correlation tool*, J. Chem. Phys., 120 (2004) 8096

[50] E. Romera and J.S. Dehesa, *Weizsäcker energy of many-electron systems*, Phys. Rev. A, 59 (1999) 4064-4067

[51] E. Romera, J.C. Angulo and J.S. Dehesa, *Fisher entropy and uncertainty-like relationships in many-particle systems*, Phys. Rev. A, 50 (1994) 256-266

[52] E. Romera, J.S. Dehesa and R.J. Yáñez, *The Weizsäcker functional: Some rigorous results*, Int. J. Quantum Chemistry, 56 (1995) 627-632

[53] E. Romera, P. Sánchez-Moreno and J.S. Dehesa, *The Fisher information of single-particle systems with a central potential*, Chem. Phys. Lett., 414 (2005) 468-472

[54] E. Romera, P. Sánchez-Moreno and J.S. Dehesa, *Uncertainty relation for Fisher information of D-dimensional single-particle systems with central potentials*, Phys. Rev. A (Submitted)
[55] A. Ronveaux and W. van Assche, *Upward extension of the Jacobi matrix for orthogonal polynomials*, J. Approx. Th., 86 (1996) 335-357

[56] A. Ronveaux, J.S. Dehesa, A. Zarzo and R.J. Yáñez, *A note on the zeroes of Grosjean polynomials*, Newsletter of the SIAM activity group on Orthogonal Polynomials and Special Functions, 6 (2) (1996) 15

[57] A. Ronveaux, A. Zarzo, I. Area and E. Godoy, *Classical orthogonal polynomials: Dependence of parameters*, J. Comp. Appl. Math., 121 (2000) 95-112

[58] J.F. Sánchez-Lara, *On the asymptotic expansion of the entropy of orthogonal polynomials*, J. Comput. Appl. Math., 142 (2002) 401-409

[59] J. Sánchez-Ruiz and J.S. Dehesa, *Fisher information of orthogonal hypergeometric polynomials*, J. Comput. App. Math., 182 (2005) 150-160

[60] J. Sánchez-Ruiz and J.S. Dehesa, *Entropic integrals of orthogonal hypergeometric polynomials with general supports*, J. Comput. App. Math., 118 (2000) 311-322

[61] J. Sánchez-Ruiz, *Information entropy of Gegenbauer polynomials and Gaussian quadrature*, J. Phys. A: Math. Gen., 36 (2003) 4857-4865

[62] M.J. Schervish, *Theory of Statistics*, (Springer Verlag, New York, 1995)

[63] S.B. Sears, R.G. Parr and U. Dinur, *On the quantum-mechanical kinetic energy as a measure of the information in a distribution*, Israel J. Chem. 19 (1980) 165-173

[64] C. Shannon, *A mathematical theory of communication*, Bell Systems Tech. J., 27 (1948) 379-423, 623-656

[65] B. Simon, *Ratio asymptotics and weak asymptotic measures for orthogonal polynomials on the real line*, J. Approx. Theory, 126 (2004) 198-217

[66] A. Stam, *Some inequalities satisfied by the quantities of information of Fisher and Shannon*, Inf. Control 2, 101 (1959)

[67] A. Ushveridze, *Quasi-exactly Solvable Models in Quantum Mechanics*, IOP, Bristol, 1994

[68] C. Vignat and J.F. Bercher, *Analysis of signals in the Fisher-Shannon information plane*, Phys. Lett. A, 312 (2003) 27-33

[69] R.J. Yáñez, P. Sánchez-Moreno and J.S. Dehesa, *Fisher’s information of second order differential equations*, Preprint 2006

[70] R.J. Yáñez, W. van Assche and J.S. Dehesa, *Position and momentum information entropies of the D-dimensional harmonic oscillator and hydrogen atom*, Phys. Rev. A, 50 (1994) 3065-3079