Abstract
We use the massless Thirring model to demonstrate a new approach to non-perturbative fermion calculations based on the spherical field formalism. The methods we present are free from the problems of fermion doubling and difficulties associated with integrating out massless fermions. Using a non-perturbative regularization, we compute the two-point correlator and find agreement with the known analytic solution.

1 Introduction
The massless Thirring model \[1\] is an exactly soluble system of a single self-interacting massless fermion in two dimensions. There are a number of solutions in the literature based on properties of the Euler-Lagrange equations and fermion currents or bosonization techniques \[2-6\]. Given its simplicity and solubility, the model has become a popular testing ground for new ideas and methods in field theory.

From a computational point of view, however, the massless Thirring model still presents a significant challenge. In the lattice field formalism, the difficulties are due to the appearance of fermion doubler states and singular inversion problems associated with integrating out massless fermions. In this work we use the model to illustrate new non-perturbative methods in the spherical field formalism \[7-9\]. The techniques we present are quite general and can also be applied to other modal expansion methods such as periodic field theory \[10\].

As noted in \[8\], we will not need to worry about fermion doubling. This is true for any modal field theory and follows from the fact that space is not discretized but retained as a continuous variable. Since our model is not super-renormalizable we will use a procedure called angle-smearing, a regularization
method designed for spherical field theory \[9\]. With angle-smearing regularization and a small set of local counterterms, we are able to remove all ultraviolet divergences in a manner such that the renormalized theory is finite and translationally invariant. Comparison of our results with the known Thirring model solution will serve as a consistency check for our regularization and renormalization procedures.

The organization of this paper is as follows. We begin with a short summary of the massless Thirring model, following the solution of Hagen \[4\]. Using angle-smearing regularization we obtain the spherical field Hamiltonian and construct a matrix representation of the Hamiltonian. We reduce the space of states using a two-parameter auxiliary cutoff procedure. In this reduced space we compute the time evolution of quantum states using a fourth-order Runge-Kutta-Fehlberg algorithm. We calculate the two point correlator for several values of the coupling and find agreement with the known analytic solution.

2 The model

We start with a list of our notational conventions. Our analysis will be in two-dimensional Euclidean space, and we use both cartesian and polar coordinates,

\[ \vec{t} = (t \cos \theta, t \sin \theta) = (x, y). \]  

The components of the spinors $\psi$ and $\tilde{\psi}$ are written as

\[ \psi = \begin{bmatrix} \psi^\uparrow \\ \psi^\downarrow \end{bmatrix}, \quad \tilde{\psi} = \begin{bmatrix} \tilde{\psi}^\uparrow \\ \tilde{\psi}^\downarrow \end{bmatrix}. \]  

Our representation for the Dirac matrices is

\[ \vec{\gamma} = i \vec{\sigma}, \]  

and so $\vec{\gamma}$ satisfies

\[ \{ \gamma^i, \gamma^j \} = -2 \delta^{ij}, \quad i, j = 1, 2. \]  

The massless Thirring model is formally defined by the Lagrange density

\[ \mathcal{L} = i \tilde{\psi} \vec{\gamma} \cdot \vec{\nabla} \psi - \frac{\lambda}{2} \vec{j} \cdot \vec{j}, \]  

where $\vec{j}$ is the fermion vector current. Johnson \[2\] emphasized the importance of defining the regularized current precisely, and this was further clarified by the work of Sommerfield \[3\] and Hagen \[4\]. We will use a regularization technique, introduced in \[9\], called angle-smearing. We define the regularized current as

\[ \vec{j} = \frac{1}{2} \left( \bar{\psi}_s \vec{\gamma} \psi_s - Tr[\vec{\gamma} \psi_s \bar{\psi}_s] \right), \]  

\[ 2 \]
where
\[
\psi_s(t, \theta) = \frac{M_1}{M_t} \int_{-\frac{M_t}{M_t}}^{\frac{M_t}{M_t}} d\varepsilon \psi(t, \theta + \varepsilon).
\]
(7)

We identify the radial variable \( t \) as our time parameter, and our definition of the current is local with respect to \( t \).

Hagen [4] described the solution of the Thirring model in the Hamiltonian formalism with currents defined as products of the canonical operators at equal times. Though our equal-time surface is curved, the curvature of the integration segment in (7) scales as \( \frac{1}{M_t} \) while the ultraviolet divergences in this model are only logarithmic in \( M \). In the \( M \to \infty \) limit we therefore recover the standard results. As discussed in [4], there exists a one parameter class of solutions to the Thirring model depending on the preferred definition of the regularized vector and axial vector currents. We will use the conventions used in [2] and [3], which in Hagen’s notation corresponds with the parameter values \( \xi = \eta = \frac{1}{2} \). With this choice the Hamiltonian density takes the form
\[
\mathcal{H} = \mathcal{H}_{\text{free}} + \frac{\pi c}{M_t} (\hat{t} \cdot \vec{j})^2 + \frac{\pi c}{M_t} (\hat{\theta} \cdot \vec{j})^2,
\]
(8)

where
\[
c = \frac{\lambda}{2\pi}.
\]
(9)

The only counterterms needed in this model are wavefunction renormalization counterterms, a result of our careful definition for the regularized interaction in (8). As in [4] we calculate correlation functions using an unrenormalized Hamiltonian. The divergent wavefunction normalizations will appear simply as overall factors in the correlators.

### 3 Spherical field Hamiltonian

In this section we derive the form of the spherical field Hamiltonian. We first expand the fermion current in terms of components of the spinors,
\[
\hat{t} \cdot \tilde{\psi}_s \gamma_7 \psi_s = i \tilde{\psi}_s \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \psi_s = i e^{-i\theta} \tilde{\psi}_s \psi_s^\dagger + e^{i\theta} \tilde{\psi}_s \psi_s^\dagger.
\]
(10)

\[
\hat{\theta} \cdot \tilde{\psi}_s \gamma_7 \psi_s = \tilde{\psi}_s \begin{pmatrix} 0 & e^{-i\theta} \\ -e^{i\theta} & 0 \end{pmatrix} \psi_s = e^{-i\theta} \tilde{\psi}_s \psi_s^\dagger - e^{i\theta} \tilde{\psi}_s \psi_s^\dagger.
\]
(11)

The anti-commutators of the regulated fields are
\[
\{ \psi_s^\dagger, \psi_s \} = \frac{1}{(\frac{M_t}{2})^2} \int_{-\frac{M_t}{M_t}}^{\frac{M_t}{M_t}} e^{i(\theta + \varepsilon)} d\varepsilon = A(t) e^{i\theta}
\]
(12)

\footnote{Our definition of the Euclidean fermion fields and anti-commutation relations follows the conventions of [14].}
At this point it is convenient to rescale \( \bar{H} \) in terms of the partial waves, \( A \) and so

\[
\frac{\bar{H}}{A} \sum_n \psi_n^\dagger \psi_n^\dagger = \frac{1}{(\frac{M^2}{2})} \left( \frac{M^2}{2} A(t) e^{-i\theta + \varepsilon} d\varepsilon \right) = A(t) e^{-i\theta},
\]

where

\[
A(t) = \frac{M^2}{2} \sin(\frac{\theta}{\tau}).
\]

From the anti-commutation relations, the \( \hat{t} \) component of the current is

\[
\hat{t} \cdot \hat{j} = \frac{1}{2} [ie^{-i\theta} (\bar{\psi}_s^1 \psi_s^1 - \psi_s^1 \bar{\psi}_s^1) + ie^{i\theta} (\bar{\psi}_s^1 \psi_s^1 - \psi_s^1 \bar{\psi}_s^1)]
\]

\[
= ie^{-i\theta} \bar{\psi}_s^1 \psi_s^1 + ie^{i\theta} \bar{\psi}_s^1 \psi_s^1 - iA(t),
\]

and so

\[
(\hat{t} \cdot \hat{j})^2 = A(t) \left[ e^{-i\theta} \bar{\psi}_s^1 \psi_s^1 + e^{i\theta} \bar{\psi}_s^1 \psi_s^1 \right] - 2\bar{\psi}_s^1 \psi_s^1 \bar{\psi}_s^1 \psi_s^1 - A^2(t).
\]

Similarly we find

\[
(\hat{\theta} \cdot \hat{j})^2 = A(t) \left[ e^{-i\theta} \bar{\psi}_s^1 \psi_s^1 + e^{i\theta} \bar{\psi}_s^1 \psi_s^1 \right] - 2\bar{\psi}_s^1 \psi_s^1 \bar{\psi}_s^1 \psi_s^1.
\]

The Hamiltonian can now be written as

\[
H = H_{free} + \int d\theta d\bar{\theta} \left\{ \frac{\pi c}{2 \tau} \left[ A(t) \left[ e^{-i\theta} \bar{\psi}_s^1 \psi_s^1 + e^{i\theta} \bar{\psi}_s^1 \psi_s^1 \right] - 2\bar{\psi}_s^1 \psi_s^1 \bar{\psi}_s^1 \psi_s^1 \right] \right\}.
\]

We have omitted the constant term proportional to \( A^2(t) \).

Let us define the partial wave modes

\[
\psi_n(t) = \frac{1}{\sqrt{2\pi}} \int d\theta e^{-i\theta} \psi_s^1(t), \quad \psi_{s,n}(t) = \frac{1}{\sqrt{2\pi}} \int d\theta e^{-i\theta} \psi_s^1(t),
\]

\[
\bar{\psi}_n(t) = \frac{1}{\sqrt{2\pi}} \int d\bar{\theta} e^{-i\bar{\theta}} \bar{\psi}_s^1(t), \quad \bar{\psi}_{s,n}(t) = \frac{1}{\sqrt{2\pi}} \int d\bar{\theta} e^{-i\bar{\theta}} \bar{\psi}_s^1(t).
\]

It is straightforward to show that for \( n \neq 0 \),

\[
\psi_{s,n}(t) = \frac{\sin(\frac{\pi n}{\tau})}{\frac{\pi n}{\tau}} \psi_n(t), \quad \bar{\psi}_{s,n}(t) = \frac{\sin(\frac{\pi n}{\tau})}{\frac{\pi n}{\tau}} \bar{\psi}_n(t).
\]

We can extend this result to the case \( n = 0 \) using the convenient shorthand

\[
\frac{\sin(\frac{\pi n}{\tau})}{\frac{\pi n}{\tau}} \equiv 1.
\]

At this point it is convenient to rescale \( \bar{\psi} \),

\[
\bar{\psi}'_n = t \bar{\psi}_n.
\]

In terms of the partial waves,

\[
H = \frac{1}{\tau} \sum_n \left[ \left( n + 1 + \frac{b\tau A(t) \sin(\frac{\pi n}{\tau}) \sin(\frac{\pi n+1}{\tau})}{(\frac{\pi n}{\tau})(\frac{\pi n+1}{\tau})} \right) \bar{\psi}'_{n+1} \psi'_n - 2 \bar{\psi}'_n \psi'_n \bar{\psi}'_{n+1} \psi'_n \right]
\]

\[
- \frac{1}{\tau} \sum_n \left[ \left( n - \frac{b\tau A(t) \sin(\frac{\pi n}{\tau}) \sin(\frac{\pi n+1}{\tau})}{(\frac{\pi n}{\tau})(\frac{\pi n+1}{\tau})} \right) \bar{\psi}'_{n-1} \psi'_n \right]
\]

\[
- \sum_{-n_1+n_2-n_3+n_4=0} \left[ \frac{b}{\tau} \bar{\psi}'_{n_1} \psi_{n_2+1} \bar{\psi}'_{n_3-1} \psi_{n_4} \frac{\sin(\frac{\pi n_1}{\tau}) \sin(\frac{\pi n_2+1}{\tau}) \sin(\frac{\pi n_3+1}{\tau}) \sin(\frac{\pi n_4}{\tau})}{(\frac{\pi n_1}{\tau})(\frac{\pi n_2+1}{\tau})(\frac{\pi n_3+1}{\tau})(\frac{\pi n_4}{\tau})} \right],
\]
where

\[ b = \frac{2e}{r_e c}. \]  

(25)

Since \( b \) is the parameter appearing in the Hamiltonian, it is somewhat more convenient to express \( c \) in terms of \( b \),

\[ c = \sqrt{1+b^2-1}. \]  

(26)

Let us define the ladder operators

\[ a_{\downarrow}^{-n}, a_{\uparrow}^{-n} \equiv \psi_{\downarrow}^{-n}, \bar{\psi}_{\uparrow}^{-n} \]  

(27)

\[ a_{\downarrow}^{-n+1}, a_{\uparrow}^{-n+1} \equiv \psi_{\downarrow}^{-n+1}, \bar{\psi}_{\uparrow}^{-n}. \]  

(28)

These operators satisfy the anti-commutation relations

\[ \{a_{\downarrow}^{-n_1}, a_{\uparrow}^{-n_2}\} = \{a_{\downarrow}^{n_1}, a_{\uparrow}^{n_2}\} = \delta_{n_1n_2}, \]  

(29)

with all other anti-commutators equal to zero. We can now recast the Hamiltonian as

\[ H = \frac{1}{t} \sum_n \left[ n + \frac{b \pi t A(t) \sin(\frac{n\pi t}{M}) \sin(\frac{n\pi t}{M})}{(\frac{n\pi t}{M})} \right] (a_{\downarrow}^{n+1} a_{\uparrow}^{n+1} + a_{\downarrow}^{n+1} a_{\uparrow}^{n+1}) \]  

(30)

\[ - \sum_{-n_1+n_2+n_3-n_4=0} b a_{\downarrow}^{n_1} a_{\uparrow}^{n_2} a_{\downarrow}^{n_3} a_{\uparrow}^{n_4} \]  

\[ \frac{\sin(\frac{n_1\pi t}{M}) \sin(\frac{n_2\pi t}{M}) \sin(\frac{n_3\pi t}{M}) \sin(\frac{n_4\pi t}{M})}{(\frac{n_1\pi t}{M}) (\frac{n_2\pi t}{M}) (\frac{n_3\pi t}{M}) (\frac{n_4\pi t}{M})}. \]

We will implement a high spin cutoff by removing terms in the interaction containing operators \( a_{\downarrow}^{n}, a_{\uparrow}^{n}, a_{\downarrow}^{+n}, \) or \( a_{\uparrow}^{+n} \) for \( |n| > J_{\text{max}} \). This has the effect of removing high spin modes, which correspond with large tangential momentum states. We should emphasize, however, that \( J_{\text{max}} \) is an auxiliary cutoff. It does not play a role in the regularization scheme since the interactions have already been rendered finite using angle-smearing. The contribution of these high spin modes is negligible so long as

\[ \frac{J_{\text{max}}}{t} \gg M, \]  

(31)

where \( t \) is the characteristic radius of the process being measured. Returning back to (12) and (13) and removing the contribution of these partial waves, we find that \( A(t) \) is replaced by

\[ A_{J_{\text{max}}} (t) = \frac{1}{2 \pi t} \sum_{n=-J_{\text{max}}}^{J_{\text{max}}} \frac{\sin(\frac{n\pi t}{M}) \sin(\frac{n\pi t}{M})}{(\frac{n\pi t}{M}) (\frac{n\pi t}{M})}. \]  

(32)

\[ a_{\downarrow}^{n}, a_{\uparrow}^{+n} = a_{\downarrow}^{n}, a_{\uparrow}^{n}; a_{\downarrow}^{-n}, a_{\uparrow}^{-n}. \]

\[ 2 \text{This notation is slightly different from that used in } \]  

\[ \text{The translation is as follows:} \]

\[ a_{\downarrow}^{n}, a_{\uparrow}^{+n} = a_{\downarrow}^{n}, a_{\uparrow}^{n}; a_{\downarrow}^{-n}, a_{\uparrow}^{-n}. \]
Let $|0\rangle_{free}$ be the ground state of the free massless fermion Hamiltonian. For $n > 0$, we find

$$a_n^\dagger |0\rangle_{free} = a_n^\dagger |0\rangle_{free} = 0,$$

(33)

and for $n \leq 0$,

$$a_n^\dagger |0\rangle_{free} = a_n^\dagger |0\rangle_{free} = 0.$$

(34)

It is convenient to define the normal-ordered products

$$a_n^\dagger a_n^\dagger = \begin{cases} a_n^\dagger a_n^\dagger 
 & \text{for } n > 0 \\
- a_n^\dagger a_n^\dagger 
 & \text{for } n \leq 0 
\end{cases}$$

(35)

The ordering for other operators is immaterial since the anti-commutators are zero. We can now rewrite $H$ in terms of normal-ordered products,

$$H = \left( \frac{n}{t} + O(J_{\text{max}}^{-2}) \right) \left( a_n^\dagger a_n^\dagger + a_n^\dagger a_n^\dagger \right)$$

(36)

$$- \sum_{-n_1+n_2+n_3-n_4=0} \left[ b_n a_n^\dagger a_n^\dagger a_n^\dagger a_n^\dagger \right].$$

There is an $O(J_{\text{max}}^{-2})$ term due to a small asymmetry in our cutoff procedure with respect to the two boundaries $-J_{\text{max}}$ and $J_{\text{max}}$. We will neglect this term in the limit $J_{\text{max}} \to \infty$.

4 Two-point correlator

We wish to study the properties of the two-point correlator. The massless Thirring model is invariant under the discrete transformation

$$\bar{\psi}(\vec{t}), \psi(0) \rightarrow -\bar{\psi}(\vec{t}), -\psi(0),$$

(37)

as well as the transformation

$$\psi(t, \theta), \bar{\psi}(t, \theta) \leftrightarrow \psi(t, -\theta), \bar{\psi}(t, -\theta).$$

(38)

From these we deduce

$$\langle 0 | T \left[ \bar{\psi}(\vec{t}) \psi(0) \right] | 0 \rangle = \langle 0 | T \left[ \bar{\psi}(\vec{t}) \psi(0) \right] | 0 \rangle = 0,$$

(39)
and

\[
\langle 0 | T \left[ \bar{\psi}^\dagger (t, \theta) \psi^\dagger (0) \right] | 0 \rangle = \langle 0 | T \left[ \bar{\psi}^\dagger (t, -\theta) \psi^\dagger (0) \right] | 0 \rangle .
\] (40)

It therefore suffices to consider just the correlator on the left side of (40).

In the limit \( M \to \infty \) the form of the correlator is given by

\[
\langle 0 | T \left[ \bar{\psi}^\dagger (t, \theta) \psi^\dagger (0) \right] | 0 \rangle = \frac{e^{i\theta}}{2\pi} (k(c)M)^{\frac{2c^2}{1-c^2}} t^{\frac{1-c^2}{1-c^2}},
\] (41)

where \( k(c) \) is a dimensionless parameter. Standard analytic methods do not yield a simple closed form expression for \( k(c) \). We will therefore extract \( k(c) \) from the computed value of the correlator at a specific renormalization point \( t = t_0 \).

We define

\[
f(t) = \langle 0 | T \left[ \bar{\psi}^\dagger (t) \psi^\dagger (0) \right] | 0 \rangle .
\] (42)

Since

\[
\langle 0 | T \left[ \bar{\psi}^\dagger (t, \theta) \psi^\dagger (0) \right] | 0 \rangle = \frac{e^{i\theta}}{2\pi} \langle 0 | T \left[ \bar{\psi}^\dagger (t) \psi^\dagger (0) \right] | 0 \rangle,
\] (43)

we conclude that

\[
f(t) = (k(c)M)^{\frac{2c^2}{1-c^2}} t^{\frac{1-c^2}{1-c^2}}.
\] (44)

We now compute \( f(t) \) using the spherical field Hamiltonian. We first need a matrix representation for the Grassmann ladder operators. We will use tensor products of the \( 2 \times 2 \) identity matrix and Pauli matrices:

\[
a^\dagger_n = \bigotimes_{i=-J_{\text{max}},n+1}^{J_{\text{max}},n+1} \sigma_z \bigotimes_{i=n-1,-J_{\text{max}}}^{1} \sigma_z \bigotimes_{i=-J_{\text{max}},n+1}^{J_{\text{max}},n+1} \left( \frac{1}{2} \sigma_x + \frac{i}{2} \sigma_y \right) \bigotimes_{i=n-1,J_{\text{max}}}^{1,J_{\text{max}}} 1,
\] (45)

\[
a^\dagger_n = \bigotimes_{i=-J_{\text{max}},n+1}^{J_{\text{max}},n+1} \sigma_z \bigotimes_{i=n-1,-J_{\text{max}}}^{1} \sigma_z \bigotimes_{i=-J_{\text{max}},n+1}^{J_{\text{max}},n+1} \left( \frac{1}{2} \sigma_x + \frac{i}{2} \sigma_y \right) \bigotimes_{i=n-1,J_{\text{max}}}^{1,J_{\text{max}}} 1.
\] (45)

The representations for \( a_n^\dagger \) and \( a_n^\dagger \) are defined by the conjugate transposes of these matrices. We can now calculate the correlator \( f(t) \) using the relation

\[
f(t) = \lim_{t_2 \to \infty} \lim_{t_1 \to 0} \frac{\text{Tr} \left[ T \exp \left\{ - \int_{t_2}^{t_1} dt H(t) \right\} a_n^\dagger T \exp \left\{ - \int_{t_1}^{t_0} dt H(t) \right\} a_n \right]}{\text{Tr} \left[ T \exp \left\{ - \int_{t_2}^{t_1} dt H(t) \right\} \right]}.
\] (46)

A straightforward calculation of (46), however, is rather inefficient. There are several techniques which we will first use to simplify the calculation.

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5In some regularization schemes \( k(c) \) can be calculated analytically, and it may be worthwhile to use these techniques in future work. In this first analysis, however, we prefer to present a more straightforward and typical example of the angle-smearing regularization method.
The time evolution of the system at large $t$ is dominated by the contribution of the ground state or, more precisely, the adiabatic flow of the $t$-dependent ground state. As discussed in [7] a similar phenomenon occurs at small $t$, due to the divergence of energy levels near $t = 0$. It is therefore not necessary to compute the full matrix trace in the numerator and denominator of (46). It is instead sufficient to compute the corresponding ratio for a single matrix element. After making this reduction, we can then go a step further and eliminate states which do not contribute to the matrix element.

The high spin parameter $J_{\text{max}}$ was used to remove high spin modes with $|n| > J_{\text{max}}$. This, however, is not a uniform cutoff in the space of states and most of the remaining states are still high kinetic energy states. Although none of the individual modes are energetic, many of the modes can be simultaneously excited. Let us define $N_n^\downarrow$ and $N_n^\uparrow$ to be bit switches, 1 or 0, depending on whether or not the corresponding mode is excited. Let us also define a cutoff parameter $K_{\text{max}}$. We will remove all high kinetic energy states such that

$$\sum_n |n| (N_n^\downarrow + N_n^\uparrow) \geq K_{\text{max}}.$$  

For consistency $K_{\text{max}}$ should be about the same size as $J_{\text{max}}$.

## 5 Results

The CPU time and memory requirement for calculating (46) scales linearly with the number of transitions in $H$ (i.e., non-zero elements in our matrix representation). In Table 1 we have shown the number of states and transitions for different values of $J_{\text{max}}$.

| $J_{\text{max}}$ | 2  | 4  | 6  | 8  | 10 | 12 |
|------------------|----|----|----|----|----|----|
| states           | 6  | 40 | 210| 920| 3600|13000|
| transitions      | 38 | 500| 4200|26000|1.4E5|3.9E5|

We have calculated $f(t)$ for $J_{\text{max}} \leq 12$ and several values of the coupling $b$. The total run time was about 100 hours on a 350 MHz PC with 256 MB RAM.

The matrix time evolution equations in (46) were computed using a fourth-order Runge-Kutta-Fehlberg algorithm. We have set

$$K_{\text{max}} = 2J_{\text{max}} + 2.$$  

We will use the notation $f_{J_{\text{max}}}(t)$ to identify the corresponding result for a given value of $J_{\text{max}}$. In Figure 1 we have plotted $f_{J_{\text{max}}}(t)$ for $b = 1$ and $J_{\text{max}} = 4, 8, 12$. We have scaled $t$ and $f(t)$ in dimensional units chosen such that $M = 3$. For finite $J_{\text{max}}$ we expect deviations from the $J_{\text{max}} \rightarrow \infty$ limit to be of size $O(M^2 t^2 J_{\text{max}}^2)$. The curves shown in Figure 1 appear consistent with this rate of convergence.

We can extrapolate to the limit $J_{\text{max}} \rightarrow \infty$ using the asymptotic form

$$f_{J_{\text{max}}}(t) = f_\infty(t) + J_{\text{max}}^{-2} f^{(2)}(t) + \cdots.$$  

8
For \( b = 0, 0.5, 1.0, 3.0 \) and \( M = 3 \) we have calculated \( f(t) \) using this extrapolation technique for \( J_{\text{max}} = 10 \) and 12. The results are shown in Figure 2. For comparison we have plotted the analytic solution

\[
A(t) = (k(c)M)^{\frac{2c^2}{1-c^2}} t^{\frac{1-c^2}{1-c^2}}.
\]  

(50)

The relation between \( b \) and \( c \) can be found in (25) and (26). The parameter \( k(c) \) is fitted to the value of the correlator at the renormalization point \( t = 0.6 \). The agreement appears quite good. Some deviations from the analytic solution are due to \( O(\frac{1}{M^2t^2}) \) residual terms, which were left out of the derivation of (50). These effects are significant in the small \( t \) region, \( t \lesssim M^{-1} \), especially for larger values of \( b \). The values we find for \( k(c) \) are shown in Table 2.

| \( b \) | 0.5 | 1.0 | 3.0 |
|------|-----|-----|-----|
| \( k(c) \) | 1.77 | 1.77 | 1.68 |

We can compare our results at small \( b \) with a simple perturbative calculation. Evaluating the corresponding regulated two-loop diagram we obtain, for small \( b \), \( k(c) \approx 1.75 \). This appears consistent with the results in Table 2.

6 Summary

We derived the angle-smeared spherical Hamiltonian for the massless Thirring model and constructed an explicit matrix representation. We discarded negligible high energy states using auxiliary cutoff parameters \( J_{\text{max}} \) and \( K_{\text{max}} \). In this reduced space we computed the time evolution of quantum states and calculated the two-point correlator for several values of the coupling. The results of our computation are in close agreement with the known analytic solution. In addition to demonstrating new computational methods, our analysis also serves as a consistency check of the regularization and renormalization methods introduced in [9].

We believe that this work represents a significant new direction in the non-perturbative computation of fermion dynamics. Future work will study the application of these methods to systems of interacting bosons and fermions.

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6\(^{\text{Both our results and the analytic solution are even in } b, \text{ and so we consider only positive values.}}\)

7\(^{\text{The relative error is expected to be small in the vicinity of this point.}}\)

8\(^{\text{In some regularization schemes } k(c) \text{ can be shown to be independent of the coupling. Our regularization method seems to be rather close to this, with only a slow variation with respect to the coupling strength.}}\)
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Figures

Figure 1. Plot of $f_{J_{\text{max}}}(t)$ for $b = 1$ and $J_{\text{max}} = 4, 8, 12$.
Figure 2. Plot of $f_A(t)$ and $f(t)$ for $b = 0, .5, 1, 3$ and $M = 3$. 
Figure 1

\[ M = 3 \]
\[ b = 1 \]

\[ J_{\text{max}} = 4 \]
\[ J_{\text{max}} = 8 \]
\[ J_{\text{max}} = 12 \]

Figure 2

\[ M = 3 \]

\[ f(t) \]
\[ f_A(t) \]

For various values of \( b \):
- \( b = 3.0 \)
- \( b = 1.0 \)
- \( b = 0.5 \)
- \( b = 0 \)