Abstract: In this paper, we study jumps of nonlinear DAEs caused by inconsistent initial values. First, we propose a simple normal form called the index-1 nonlinear Weierstrass form (INWF) for nonlinear DAEs. Then we generalize the notion of consistency projector introduced in Liberzon and Trenn (2009) for linear DAEs to the nonlinear case. By an example, we compare our proposed nonlinear consistency projectors with two existing consistent initialization methods (one is from the paper Liberzon and Trenn (2012) and the other is given by a MATLAB function) to show that the two existing methods are not coordinate-free, i.e., the consistent points calculated by the two methods are not invariant under nonlinear coordinate transformations.

Next we propose a singular perturbed system approximation for nonlinear DAEs, which is an ordinary differential equation (ODE) with a small perturbation parameter, we show that the solutions of the proposed perturbation system approximate both the jumps resulting from the nonlinear consistency projectors and the $C^1$-solutions of the DAE. At last, we use a numerical simulation of a nonlinear DAE model arising from an electric circuit to illustrate the effectiveness of the proposed singular perturbed system approximation of DAEs.

Keywords: differential-algebraic equations, singular perturbation, jumps, index-1, nonlinear Weierstrass form, inconsistent initial values.

1. INTRODUCTION

We consider a nonlinear differential-algebraic equation (DAE),

\[ \Xi : E(x) \dot{x} = F(x), \]

where $x \in X$ is the vector of generalized states and $X$ is an open subset of $\mathbb{R}^n$, and where $E : X \rightarrow \mathbb{R}^{n \times n}$ and $F : X \rightarrow \mathbb{R}^n$ are $C^\infty$-smooth maps. For each $x \in X$, $E(x) : T_x X \rightarrow \mathbb{R}^n$ is a linear map. A DAE of the form (1) will be denoted by $\Xi = (E, F)$ or $\Xi$. The matrix-valued function $E(x)$ is not necessarily invertible, which implies that there may exist some algebraic constraints and some algebraic variables in the DAE $\Xi$. A particular case of $\Xi$ is a semi-explicit DAE

\[ \Xi^{SE} : \begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ 0 = f_2(x_1, x_2), \end{cases} \]

with $E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ being constant. The DAE $\Xi^{SE}$ has the algebraic variables $x_2$ (since the derivatives of $x_2$ are not present) and the algebraic constraints $0 = f_2(x_1, x_2)$. We will study also linear DAEs of the form

\[ \Delta : E \dot{x} = H x, \]

where $E \in \mathbb{R}^{n \times n}$ and $H \in \mathbb{R}^{n \times n}$. A linear DAE of the form (3) will be denoted by $\Delta = (E, H)$ or, shortly, $\Delta$. A linear DAE $\Delta$ is called regular if $sE - H \in \mathbb{R}^{n \times n}[s]\{0\}$.

An $C^1$-solution of a DAE $\Xi = (E, F)$ is a differentiable function $x : I \rightarrow X$ defined on an open interval $I$ such that for all $t \in I$, the curve $x(t)$ satisfies $E(x(t)) \dot{x}(t) = F(x(t))$, where $\dot{x}$ denotes the classical time-derivative defined everywhere on $I$. A point $x_0^-$ is called consistent if there exists at least one $C^1$-solution $x : I \rightarrow X$ with $t_0 \in I$ such that $x_0^- = x(t_0)$. The set of all consistent points will be called consistency space and denoted by $S_c$.

Without loss of generality, we can always assume $t_0 = 0$ and $I = (0, T)$ for some $T \in (0, \infty)$ (if not, we can re-parametrize the time variable $t$).

It is known that the $C^1$-solutions of a nonlinear DAE $\Xi$ exist on its consistency space $S_c$ only (see Section 2). For a given inconsistent initial point $x_0^- \in X \setminus S_c$, there does not exist any $C^1$-solution starting from $x_0^-$. Then it is natural to search for the consistent point $x_0^+ \in S_c$ such that we can get the $C^1$-solutions of $\Xi$ starting from $x_0^+$. The instant change from the inconsistent point $x_0^-$ to a consistent one $x_0^+$ is called a jump of the DAE at $t = 0$.

Note that the jumps which we study in the paper are called external or exogenous jumps, which are different from the jumps at the impasse (or singular) points as discussed in Takens (1976); Chua and Deng (1989); Sastry and Desoer (1981). We assume throughout that once starting from the point $x_0^-$, there will not exist any jump and we will study only the $C^1$-solutions of $\Xi$. In conclusion, we consider the following initial value problem:

\[ \begin{cases} \text{Jumps : } \lim_{t \rightarrow 0^-} x(t) = x_0^- \notin S_a \rightarrow \lim_{t \rightarrow 0^+} x(t) = x_0^+ \in S_a, \\ C^1\text{-solutions: } \langle E(x) \dot{x} \rangle_{(0,T)} = F(x)_{(0,T)}. \end{cases} \]

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for some function $x : I \rightarrow \mathbb{R}^n$ differentiable on $(0, T) \subset I$. The problem of finding the consistent point $x^0_i$ for a DAE with an inconsistent initial value $x^0_i$ is called consistent initialization, which is a significant problem for hybrid DAE systems involving with jump behaviors. Some examples of such systems are the electric circuits with instant connections or switching devices (see e.g., Zuzhao (1991); Vlach et al. (1995); Trenn (2012)), the power systems with DC transmissions in Susuki et al. (2008), the multi-body dynamics in Hamann and Mehrmann (2008) and the battery model of Methékar et al. (2011).

For a regular linear DAE $\Delta = (E, H)$, given by (3), the consistent initialization can be solved by the linear consistency projector introduced by Liberzon and Trenn (2009, 2012), which is a linear map constructed with the help of the well-known Weierstrass form (WF). For a semi-explicit DAE $\Xi^{SE}$ of the form (2), the singular perturbation theory (see e.g., Kokotović et al. (1999); Khalil (2001)) was frequently used to study system approximations of the discontinues solutions of $\Xi^{SE}$ (see e.g., Sastry and Desoer (1981); Rabier and Rheinboldt (2002); Susuki et al. (2008) and Section 4 of the present paper). Two existing methods of solving the consistent initialization problem for nonlinear DAEs are, the jump rule of Liberzon and Trenn (2012), which determines the consistent initial value $x^0_i$ through the formula $x^0_i - x^0_j \in \ker E(x^0_j)$, and the function decic of MATLAB (see MathWorks (2006)), which calculates the consistent initial values via a numerical searching method, we will show in Example 9 below that both of those two methods are not coordinate-free, i.e., the calculated consistent values depends on which local coordinates are chosen for the DAE.

The aims of this paper are, on one hand, to give a nonlinear generalization of the linear consistency projector in order to calculate consistent initial points for nonlinear DAEs, on the other hand, to extend the singular perturbed system approximation method to nonlinear DAEs of the form (1) to study the jump behaviors. This paper is organized as follows: We introduce the notations of the paper and some notions as invariant submanifolds, external equivalence and linear consistency projectors in Section 2. We propose a normal form called the index-1 nonlinear Weierstrass form (INWF) and extend the linear consistency projector to nonlinear DAEs in Section 3. A singular perturbed system approximation of nonlinear DAEs is proposed in Section 4 and we show the simulation result of our singular perturbation method applied to an electric circuit in Section 5. Conclusions are given in Section 6.

2. NOTATIONS AND SOME PRELIMINARIES OF NONLINEAR DAEs

We use the following notations: The symbol $C^k$ denotes the class of functions which are $k$-times differentiable. For a map $A : X \rightarrow \mathbb{R}^m$, $\ker A(x)$, $\text{Im} A(x)$ and $\text{rank} A(x)$ are the kernel, the image and the rank of $A$ at $x$, respectively. The general linear group over $\mathbb{R}$ of degree $n$ is denoted by $GL(n, \mathbb{R})$. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$. Let $f_i : X \rightarrow \mathbb{R}$ for $i = 1, \ldots, m$, in coordinates $x = (x_1, \ldots, x_n)$, the differential of $f_i$ is $df_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_m}{\partial x_1} \end{bmatrix}$. If $d_1, \ldots, d_m$ are vectors of a-value function $f = (f_1, \ldots, f_m)$ are $Df = \begin{bmatrix} d_1, \ldots, d_m \end{bmatrix}$. We assume that the reader is familiar with some basic notions as smooth embedded submanifolds, tangent spaces, involutive distributions from differential geometry, the reader can also consult the book by Lee (2001) for the definitions of such notions.

The existence and uniqueness of $C^1$-solutions for nonlinear DAEs of the form (1) have been discussed using geometric methods in e.g., Reich (1991); Rabier and Rheinboldt (2002); Chen and Trenn (2020); Chen et al. (2020). An important notion in the geometric solutions theory of DAEs is the invariant submanifold defined as follows.

Definition 1. For a DAE $\Xi = (E, F)$, a smooth connected embedded submanifold $M$ is called invariant if for any $x^0_i \in M$, there exists a $C^1$-solution $x : I \rightarrow X$ such that $x(t_0) = x^0_i$ with $t_0 \in I$ and $x(t) \in M$ for all $t \in I$. Fix a point $x^0_p \in X$, a smooth embedded submanifold $M$ containing $x^0_p$ is called locally invariant, if there exists a neighborhood $U$ of $x^0_p$ such that $M \cap U$ is invariant.

A locally invariant submanifold $M^*$, around a point $x^*_p$, is called locally maximal, if there exists a neighborhood $U$ of $x^*_p$ such that for any other locally invariant submanifold $M$, we have $M \cap U \subseteq M^* \cap U$. It is shown in Chen and Trenn (2020); Chen et al. (2020) that the maximal invariant submanifold $M^*$ around a nominal point $x^*_p$ locally coincides with the consistency space $S^c$, i.e., there exists a neighborhood $U^* \subseteq x^*_p$ such that $M^* \cap U^* = S^c \cap U^*$.

Hence in the present paper, we make no difference between the notion of maximal invariant submanifold $M^*$ and that of consistency space $S^c$ when considering a DAE $\Xi$ around a point $x^*_p$. Note that there is an iterative way of calculating the locally maximal invariant submanifold $M^*$ of DAEs, called the geometric reduction method (see e.g., Rabier and Rheinboldt (2002); Chen and Trenn (2020); Chen et al. (2020)), the number of steps for the geometric reduction method to produce $M^*$ and to get the solutions of a DAE is called the geometric index (see Chen and Trenn (2020)) of the DAE.

We now recall a definition of equivalence for linear DAEs, two linear DAEs $\Delta = (E, H)$ and $\tilde{\Delta} = (\tilde{E}, \tilde{H})$ are called externally equivalent (see Chen and Respondek (2021)) or strictly equivalent if there exist constant and invertible matrices $Q$ and $P$ such that $\tilde{E} = QEP^{-1}$ and $\tilde{H} = QHP^{-1}$. The same concept can be generalized to nonlinear DAEs of form (1) as follows.

Definition 2. (external equivalence). Consider two DAEs $\Xi = (E, F)$ and $\tilde{\Xi} = (\tilde{E}, \tilde{F})$ defined on $X$ and $\tilde{X}$, respectively. Then $\Xi$ and $\tilde{\Xi}$ are called externally equivalent, shortly ex-equivalent, if there exist a diffeomorphism $\psi : X \rightarrow \tilde{X}$ and $Q : X \rightarrow GL(n, \mathbb{R})$ such that $\tilde{E}(\psi(x)) = Q(x)E(x) \left( \frac{\partial \psi(x)}{\partial x} \right)^{-1}$, $\tilde{F}(\psi(x)) = Q(x)F(x)$.

The ex-equivalence of two DAEs will be denoted by $\Xi \sim_\text{ex} \tilde{\Xi}$. If $\psi : U \rightarrow \tilde{U}$ is a local diffeomorphism between neighborhoods $U$ of $x^*_p$ and $\tilde{U}$ of $\tilde{x}^*_p$, and $Q(x)$ is defined on $U$, we will speak about local ex-equivalence.
Remark 3. It is easily seen, that for two externally equivalent systems $\Xi$ and $\tilde{\Xi}$ a $C^1$-curve $x: I \to X$ is a solution of $\Xi$ if and only if $\psi \circ x$ is a solution of $\tilde{\Xi}$.

To illustrate the notions of maximal invariant submanifold and external equivalence, we use the following example.

Example 4. Consider a DAE $\Xi = (E,F)$, given by

$$\Xi : \begin{bmatrix} 1 & 3x_2^2 - 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}. \quad (4)$$

Fix a point $x_p = (x_{1p}, x_{2p}) = (0,1)$, the locally maximal invariant submanifold of $\Xi$ around $x_p$ is $M^* = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 > \frac{\sqrt{3}}{2}\}$ (note that $M^*$ is connected).

We have that $\Xi$ is locally ex-equivalent to the following form (i.e., the (INWF), Definition 5)

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} -f(\xi_1,0) \\ \xi_2 \end{bmatrix}, \quad (5)$$

on the neighborhood $V = \{x \in \mathbb{R}^2 \mid x_2 > \frac{\sqrt{3}}{2}\}$ of $x_p$, via $\psi = \xi = (\xi_1, \xi_2) = (x_1 + x_2^2 - x_2, x_1)$ and $Q = \begin{bmatrix} 1 & 0 \end{bmatrix}$, where $f(\xi) = f(\xi_1,0) + f'(\xi_2)\xi_2$, $f = \frac{1}{4}(a + (a^2 - \frac{1}{4p})^2) + (a + (a^2 - \frac{1}{4p})^2)^{\frac{1}{2}}$, $a(\xi_1, \xi_2) = \frac{\xi_1 - \xi_2}{2}$.

3. INDEX-1 NONLINEAR WEIERSTRASS FORM AND NONLINEAR CONSISTENCY PROJECTOR

Consider a nonlinear DAE $\Xi = (E,F)$, let $H(x,\dot{x}) = E(x)\dot{x} - F(x)$, define the k-th order differential array of $H(x,\dot{x}) = 0$ by

$$H_k(x, x', w) = \begin{bmatrix} D_x H_{x^1} & D_x H_{x^2} \\ \vdots & \vdots \\ D_{x^{k-1}} H_{x^k} \\ D_w H \end{bmatrix} (x, x', w) = 0, \quad (6)$$

where $w = (x^{(2)}, \ldots, x^{(k+1)})$, the differentiation index or shortly, the index, of the DAE $\Xi$ is the least integer $k$ such that equation (6) uniquely determines $x'$ as a function of $x$, i.e., $x' = v(x)$. In Chen and Trenn (2020), we have shown that under some constant rank assumptions, the differential index coincides with the geometric index, we will use a simplification of those constant rank assumptions, which is given by

$$E(\xi_1, \xi_2) = 0 = \xi_1. \quad (7)$$

Remark 6. For any DAE $\Xi$ in (INWF) with a consistent initial point $(\xi_{10}, \xi_{20}) \notin M^*$, i.e., $\xi_{10} \neq 0$ (it is clear that the maximal invariant submanifold of (7) is $M^* = \{\xi_1, \xi_2 \in X \mid \xi_2 = 0\}$, we could easily deduce that $(\xi_{10}^{+}, \xi_{20}^{+}) = (\xi_{10},0)$ is the only possible jumping point from $(\xi_{10}, \xi_{20})$). Indeed, for the DAE (7), only $\xi_2$-variables are allowed to jump because any jump of $\xi_1$-variables will produce a Dirac impulse on the left-hand side of $\xi_1 = F^*(\xi_1)$ (see the distributional solution theory of DAEs in Trenn (2009)), which is not possible since $F^*(\xi_1)$ is not able to produce a same impulsive term on the right-hand side in order to equalize the differential equations.

Theorem 7. Consider a DAE $\Xi = (E,F)$ and fix a point $x_p \in X$. Assume that $\Xi$ satisfies the condition (CR) in a neighborhood $U \subseteq X$ of $x_p$. Then there exists a neighborhood $V \subseteq U$ of $x_p$ such that $\Xi$ is locally ex-equivalent to the (INWF), given by (7), if and only if $\Xi$ is of index-1 and the distribution $E = \ker E$ is involutive.

Proof. Only if. Assume that $\Xi$ is locally ex-equivalent to the (INWF), denoted by $\Xi = (E,F)$. It is clear that $\Xi$ is index-1 and that $\ker E$ is involutive (since $E$ is constant). Notice that the $Q$-transformation preserves the kernels and $\ker E(\psi(x)) = \frac{\partial \psi}{\partial x} \ker E(x)$; let $\ker E = \{g_1, \ldots, g_{n-r}\}$ for some vector fields $g_i$, we have $\ker E = \{\frac{\partial \psi}{\partial x} g_1, \ldots, \frac{\partial \psi}{\partial x} g_{n-r}\}$, so the Lie bracket $[g_i, g_j] \in \ker E$ (i.e., $\ker E$ is involutive) if and only if $\frac{\partial}{\partial x} [g_i, g_j] = \frac{\partial}{\partial x} [g_i, g_j] = \frac{\partial}{\partial x} \ker E = \ker E$ (i.e., $\ker E$ is involutive). We conclude that $\Xi$ is index-1 and $E = \ker E$ is involutive as well.

If. Suppose that $\Xi$ is of index-1 and the distribution $E = \ker E$ is involutive. Then by rank $E(x) = \text{const.}$ (denote this rank by r) of (CR), there exists a $Q: U \to GL(n, \mathbb{R})$ such that $E_i(x) = r$ in

$$Q(x)E(x) \dot{x} = Q(x)F(x) \Rightarrow \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}. \quad (8)$$

Notice that the condition (CR) implies that there exists a neighborhood $U_1 \subseteq U$ of $x_p$ such that $\ker A(x) = \ker \left[ \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} \right] = \text{const.}$, $\forall x \in U_1 : F_2(x) = 0$. Since the DAE is of differentiation index-1, we have that $A(x)$ has to be invertible, i.e., rank $A(x) = n$, because only if $A(x)$ is invertible, we can uniquely solve $\dot{x} = v(x) = A^{-1}(x) \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$ with only a first order differentiation of (8) (note that we only need to differentiate the algebraic equation $0 = F_2(x)$). Since the distribution $\Xi = \ker E$ is involutive, by Frobenius theorem (see e.g., Lee (2001)), there exist a neighborhood $U_2 \subseteq U_1$ and a smooth map $\xi_1: U_2 \to \mathbb{R}^r$ such that span $\{d\xi_1^1, \ldots, d\xi_1^r\} = \mathbb{R}^r$, where $d\xi_1^i$ are independent rows of $D\xi_1$ and $E = \ker E = \ker E_1$, i.e.,
Example 9. \( \frac{\partial \psi(x)}{\partial x} \) is a local diffeomorphism on \( U_2 \) since
\[ \frac{\partial \psi(x)}{\partial x} \left|_{x=E_{F}(x_0)} \right. = \left[ \begin{array}{c}
\psi_1(x_0) \\
\psi_2(x_0)
\end{array} \right] = \left[ \begin{array}{c}
0 \\
0
\end{array} \right] \]
for all \( x \in U_2 \). Define the new local coordinates \( \bar{\psi} \equiv (\xi_1, \xi_2) \) on \( U_2 \), the DAE (8) under the new \( \xi \)-coordinates is represented by
\[ \bar{\psi} = (x_1, x_2) \]
where
\[ \bar{\psi} = (x_1, x_2) \]
with \( \bar{\psi}_1 \equiv 0, \bar{\psi}_2 \equiv 1 \). Notice that \( \bar{\psi}_1 \equiv 0 \) because \( \text{Im} E_{F}^2(x) = E_{F}(x) \ker D_{\xi}(x) = 0 \) and that \( \bar{\psi}_1(x) \) is invertible for \( x \in U_2 \) since \( \text{rank} (E_{F}(x)) = \text{rank} (E_{F}(x)) \).

For a DAE \( \Xi \) being locally (on \( V \)) ex-equivalent to the \( \text{INWF} \) with \( F^*(\bar{\xi}_1) = \bar{F}_1(\bar{\xi}_1, 0) \) and \( \bar{F}_2(\bar{\xi}_1, \bar{\xi}_2) \). Then via \( \bar{Q} = (E_{\Xi}(x) - F_{\Xi}(x)) \), the DAE (9) ex-equivalent to the \( \text{INWF} \) with \( F(x_1) = \bar{F}_1(\bar{\xi}_1, 0) + \bar{F}_2(\bar{\xi}_1, \bar{\xi}_2) \). Then we have
\[ \Omega_{E,F} := \psi^{-1} \circ \bar{\psi} \circ \psi, \]
where \( \pi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the canonical projection attaching \( (\xi_1, \xi_2) \rightarrow (\xi_1, 0) \).

For a DAE \( \Xi \) being locally (on \( V \)) ex-equivalent to the \( \text{INWF} \) with an inconsistent initial value \( x_0 \in V \setminus M^* \), we can get a unique consistent point \( x_0^+ := \Omega_{E,F}(x_0) \in V \cap M^* \) since in the \( \xi \)-coordinates of the \( \text{INWF} \), the inconsistent point \( (\xi_{10}, \xi_{20}) = (x_0^+ \psi(x_0^+)) \) has to jump into \( (\xi_{10}, \xi_{20}) \) (see Remark 6), hence \( x_0^+ \psi^{-1}(\xi_{10}, \xi_{20}) = \psi^{-1} \circ \pi \circ \psi(x_0^+) = \Omega_{E,F}(x_0) \). Then we can compare the consistent initial values calculated by the nonlinear consistency projector with that from the jump rules in Liberzon and Trenn (2012) and MATLAB decic function (see MathWorks (2006)).

Example 9. (continuation of Example 4). The DAE (4) satisfies the condition \( (\text{CR}) \) in the neighborhood \( U = \{ x \in \mathbb{R}^2 \mid x \neq \pm \sqrt{3} / 3 \} \) of \( x_0 \). We have shown in Example 4 that (4) is ex-equivalent (on \( V \subseteq U \)) to the \( \text{INWF} \), given by (5), via \( \bar{Q} \) and \( \psi \). Thus the nonlinear (local) consistency projector of \( \Xi \) is
\[ \Omega_{E,F} := \psi^{-1} \circ \pi \circ \psi = \left[ \begin{array}{c}
f(x_1 + x_2, 0)
\end{array} \right], \]
where \( f(x_1, x_2) = 0 \) in \( V \setminus M^* \). The consistent point calculated by the nonlinear consistency projector is \( x_0^+ = \Omega(x_0^+) = (0, 1.233) \in M^* \). Note that the inconsistent initial point of (5) is \( \xi_0^+ = \psi(x_0^+) = (0.643, 1) \) and the consistent point is \( \xi_0^+ = (0.643, 0) \) since only \( \xi_2 \)-variables are allowed to jump (see Remark 6). Then we use the jump rule \( x_0^+ - x_0^- = \ker E^*(\bar{x}_0^+, \bar{x}_2^+) \) in Liberzon and Trenn (2012) to calculate the consistent values \( \bar{x}_0^+ \) and \( \bar{x}_2^+ \) for (4) and (5), respectively, and we get
\[ \bar{x}_0^+ = (0, 0.109) \text{ and } \bar{x}_2^+ = (0.643, 0). \]

Similarly, we use MATLAB decic function to determine the consistent values \( x_0^+ \) and \( \xi_0^+ \) for (4) and (5), respectively, to get
\[ x_0^+ = (0, 0.7) \text{ and } \xi_0^+ = (0.643, 0). \]

Since \( \bar{x}_0^+ = \psi(x_0^+) \) and \( \bar{x}_0^+ \neq \psi(x_0^+) \), we conclude that the two consistent initialization methods in Liberzon and Trenn (2012) and MathWorks (2006) do not preserve the calculated consistent points when changing the coordinates of the given DAE. On the other hand, the jump \( x_0^+ \rightarrow x_0^- \) of (4), given by the nonlinear consistency projector, and the jump \( \xi_0^+ \rightarrow \xi_0^- \) of (5) are clearly the same jump in different coordinates since \( \xi_0^- = \psi(x_0^-) \), \( \xi_0^+ = \psi(x_0^+) \), which proves that the consistent initialization calculated by the consistency projector is coordinate-free.

4. SINGULAR PERTURBED SYSTEM APPROXIMATION OF NONLINEAR DAES

We first recall a singular perturbed system for a semi-explicit DAE \( \Xi_{SE} \) of the form (2). Replacing the algebraic constraint \( 0 = f_2(x_1, x_2) \), we get a perturbed ODE system \( \Xi_{SE} \) on the left-hand side of the following formula, then by rescaling time \( t \) to \( \bar{t} = \frac{\tau}{\epsilon} \), we get a perturbed system in the time-scale \( \tau \) on the right-hand side.

\[ \Xi_{SE} := \{ \bar{x}_1 = f_1(x_1, x_2), \epsilon \psi \bar{x}_2 = f_2(x_1, x_2), \psi x_1 \}. \]

There are, in general, two assumptions in the singular perturbed approximation method of semi-explicit DAEs:
(a) \( \frac{d\psi}{dx} \) is invertible (which is actually equivalent to that \( \Xi_{SE} \) is of index-1); (b) the so-called boundary layer model \( \frac{d\psi}{dx} = f_2(x_1, x_2) \) is asymptotically stable uniformly in \( x_2 \). Then under assumptions (a),(b), the well-known Tikhonov’s theorem (see e.g., Khalil (2001) and a similar result in Theorem III.1 of Sastry and Desoer (1981)) states that if a unique solution \( (x_1(t), x_2(t)) \) of \( \Xi_{SE} \) starting from a consistent initial point \( (x_{10}, x_{20}) \) exists on the interval \( I = (0, \alpha) \), then there exists \( \delta > 0 \) such that any solution \( (x_1(t), x_2(t)) \) of \( \Xi_{SE} \) starting from any point \( (x_{10}, x_{20}) \) with \( ||x_{10} - x_{10}|| + ||x_{20} - x_{20}|| < \delta \) satisfies
\[ \lim_{\epsilon \rightarrow 0} \epsilon x_1(t) = x_1(t, \epsilon) = 0, \]
\[ \lim_{\epsilon \rightarrow 0} x_2(t) = x_2(t, \epsilon) = 0, \]
on all closed subintervals of \( I \). In this section, we will propose a singular perturbed system approximation for
nonlinear DAEs of the form (1) with the help of the proposed normal form (INWF).

**Definition 10.** (singular perturbed system) For a nonlinear DAE $\Xi = (E,F)$, fix a point $x_p$, assume there exists a neighborhood $V$ of $x_p$ such that $\Xi$ is locally (on $V$) ex-equivalent to the (INWF) of (7) via Q-transformation and a local diffeomorphism $\psi$. Define the following singular perturbed system on $V$:

$\Xi_x : \dot{x} = E(x) + Q^{-1}(x) \frac{\partial \psi(x)}{\partial x}, \quad (11)$

where $E(x) = E(x) + Q^{-1}(x) \frac{\partial \psi(x)}{\partial x}$.

**Remark 11.** Any linear index-1 regular DAE $\Delta = (E,H)$ of the form (3) is always ex-equivalent to a decoupled DAE given by $\begin{bmatrix} \bar{1}_{n_1} & 0 \\ 0 & -\bar{l}_{n_2} \end{bmatrix}$. Applying the construction of (11) to $\Delta$, we get the following singular perturbed system:

$\Delta_x : \dot{x} = E^{-1}Hx = P^{-1} \begin{bmatrix} A_1 & 0 \\ -I_{n_2} \end{bmatrix} P$, \quad where $E = Q^{-1} \begin{bmatrix} \bar{1}_{n_1} & 0 \\ 0 & -\bar{l}_{n_2} \end{bmatrix}$. P. The above perturbed linear system $\Delta_\epsilon$ is proposed in Section IV of Mironchenko et al. (2015) as an ODE approximation of linear DAEs.

The following theorem shows that the solution $\tilde{x}(t,\epsilon)$ of the proposed perturbed system $\Xi$, of (11) with an inconsistent initial value $x_0^-$ converges to the $C^1$-solution $x(t)$ of $\Xi$ starting from a consistent point $x_0^+$ calculated via the nonlinear consistency projector.

**Theorem 12.** Consider a DAE $\Xi = (E,F)$ and fix a point $x_p \in X$. Assume that the condition (CR) is satisfied in a neighborhood $U$ of $x_p$. Suppose that $\Xi$ is of geometric index-1 and that $E = ker E$ is involutive, implying that there exists a neighborhood $V \subseteq U$ of $x_p$ such that $\Xi$ is locally (on $V$) ex-equivalent to the (INWF) of (7) via $Q$ and $\psi$. Let $x_0^- \in V \setminus M^*$ be an inconsistent initial point of $\Xi$ and $x_0^+ = \Omega_E,F(x_0^-) \in M^*$ be the consistent point calculated via the nonlinear consistency projector $\Omega_E,F$. If $\tilde{x}(t,\epsilon) : I \rightarrow V$ is the solution of the perturbed system $\Xi_\epsilon$ of (11) starting from $x_0^-$ and $x(t) : I \rightarrow V$ is the $C^1$-solution of $\Xi$ starting from $x_0^+$, then we have

$\lim_{\epsilon \rightarrow 0^+} ||\tilde{x}(t,\epsilon) - x(t)|| = 0, \quad \forall t \in I. \quad (12)$

**Proof.** Suppose that $\Xi$ is locally (on $V$) ex-equivalent to the (INWF) of (7) via $Q$ and $\psi$. Consider the following disturbed system for (7):

$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} I - \bar{1}_{n_1} & [F^*(\xi_1)] \\ -l_{n_2} & [F^*(\xi_2)] \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{x} \end{bmatrix}, \quad (13)$

Let $\xi(t) = (\xi_1(t),\xi_2(t))$ be the solution of (13) starting from $\xi_0 = (\xi_0^+ , \xi_0^0 ) = \psi(x_0^+)$. It is plain that $\dot{\xi}_2(t) = e^{-L t} \xi_2(t)$. Then consider the following ODE

$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} [F^*(\xi_1)] \\ 0 \end{bmatrix}, \quad (14)$

and let $\xi(t) = (\xi_1(t),\xi_2(t))$ be its solution of (13) with the initial point $\xi_0 = (\xi_0^+,\xi_0^0) = \psi(x_0^+) = \psi \circ \Omega_E,F(x_0^-) = \pi \circ \psi(x_0^-) = (\xi_0^+,0)$. Define $\gamma(t,\epsilon) = \xi(t) - \xi(t)$, we have

$\gamma(t,\epsilon) = \begin{bmatrix} 0 \\ -\frac{1}{\epsilon} \xi_2(t) \end{bmatrix}$

and $\gamma(0,\epsilon) = \xi_0^+ - \xi_0^0 = (0,\epsilon_2)$. It follows that $\gamma(t,\epsilon) = (0,e^{-\frac{\epsilon}{\epsilon^2}} \xi_2^0)$. Moreover, it is not hard to deduce that $\dot{\psi} = \psi^{-1} \circ \xi(t,\epsilon)$ and that $\gamma(t,\epsilon) = \psi^{-1} \circ \xi(t)$. Therefore we have

$\lim_{\epsilon \rightarrow 0^+} ||\dot{\psi}(t) - x(t)|| = \lim_{\epsilon \rightarrow 0^+} ||\psi^{-1} \circ \xi(t) - x(t)||$ \leq \lim_{\epsilon \rightarrow 0^+} ||\psi^{-1} \circ \xi(t) - (t)|| = \lim_{\epsilon \rightarrow 0^+} ||\gamma(t,\epsilon)|| = 0.$

Note that the inequality “$\leq$” holds in the above results since $\psi^{-1}$ is a diffeomorphism and thus satisfies the Lipschitz condition for a Lipschitz constant $K$.

**5. SIMULATION EXAMPLE**

Consider the electrical circuit shown in Figure 1 below, which consists of a capacitor $C$ and a nonlinear resistor $N$ as the simple circuit discussed in Sastry and Desoer (1981); Chua and Deng (1989); Rabier and Rheinboldt (2002). A controlled current source $S$ is additionally connected in parallel with $N$ in order to generate nonlinear terms in $E(x)$ of the DAE model. The relations between the current $i_N$ and the voltage $v_N = y$ of the nonlinear resistor $N$ is characterized by the following algebraic equation

$0 = a(x,y), \quad $ and the current $i_S$ of $S$ is equal to $b(x,y) \dot{y}$, where $a : \mathbb{R}^2 \rightarrow \mathbb{R} \quad \text{and} \quad b : \mathbb{R}^2 \rightarrow \mathbb{R}$ are smooth maps. Using Kirchoff’s law, we model the circuit as a DAE $\Xi = (E,F)$:

$$\begin{bmatrix} 0 & -b(x,y) \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ \gamma \end{bmatrix} = \begin{bmatrix} y+z \\ 0 \\ 0 \\ a(x,y) \end{bmatrix}.$$

We consider the following case: $C = 1, a(x,y) = x-y^2-2y, b(x,y) = y$. Let $\eta = (x,y,z)$ and $\eta_0 = (0,0,0)$, then the condition (CR) is satisfied on $U = \{(x,y,z) \in \mathbb{R}^3 : y < 1\}$. The locally maximal invariant submanifold $M^*$ (around $\eta_0$) is $M^* = \{ \eta \in \mathbb{R}^3 \mid y+z = x-y^2-2y = 0, y < 1 \}$. Since $E = ker E = span(\frac{\partial}{\partial y}, \frac{\partial}{\partial z} + \frac{\partial}{\partial y})$ is involutive and $\Xi$ is of index-1. Then it is possible to find $\psi_1 : V \rightarrow \mathbb{R}$, where $V = \{ \eta \in \mathbb{R}^3 \mid y < 1 \}$, such that span $\{ \partial \psi_1 \}$ is $E^\perp$; by solving some first order PDE, we get a solution $\psi_1(\eta) = -\frac{1}{2} \psi^2 + z$. Let $\psi_2(\eta) = y+z$ and $\psi_3 = a$, then the DAE $\Xi$ is locally (on $V$) ex-equivalent to the following DAE represented in the (INWF):

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 \\ \frac{\partial \psi_1}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\psi}{\eta} \\ \gamma \\ \frac{\dot{y}}{\gamma} \end{bmatrix} = \begin{bmatrix} \frac{\eta}{2} \\ \frac{\dot{y}}{\gamma} \end{bmatrix}.$ \quad (15)$

via $Q = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $\psi(x,y,z) = (z, \dot{y}, \dot{x}) = (\psi_1, \psi_2, \psi_3)$. Following (11) of Definition 10, we construct a singular perturbed system $\Xi_\epsilon$:

$Q^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \frac{\partial \psi}{\partial \eta} \begin{bmatrix} \frac{\psi}{\gamma} \\ \frac{\dot{y}}{\gamma} \end{bmatrix} = \begin{bmatrix} x \gamma \\ \frac{\gamma z}{x-\gamma^2-2y} \frac{\gamma}{\gamma} \end{bmatrix}$ \Rightarrow $\Xi_\epsilon : \begin{bmatrix} \frac{\psi}{\gamma} \\ \frac{\dot{y}}{\gamma} \end{bmatrix} = \begin{bmatrix} f_1(\eta) \\ f_2(\eta) \\ f_3(\eta) \end{bmatrix},$
where \( f_1(\eta, \epsilon) = -\frac{x+y(2+y)-2(x^2-2)+2(y+z)}{y+3y^2-2z+2}, \)
\( f_2(\eta, \epsilon) = \frac{(x^2-2)g(y+z)}{\epsilon|t|+y} \).
Consider an inconsistent initial point \( \eta_0 = (0, 0, 0.1) \in \mathbb{V}\backslash M^* \), then find the nonlinear consistency projector \( \Omega_{E,F} \) to have \( \eta_0^+ = \Omega_{E,F}(\eta_0) = \psi^{-1} \circ \pi \circ \psi(\eta_0) = (-0.2, -0.1056, 0.1056) \), which defines a jump \( \eta_0^0 \rightarrow \eta_0^+ \) of \( \Xi \).

Now we use MATLAB ode45 solver to simulate the solution \( \eta(t, \epsilon) = (\dot{x}(t, \epsilon), \dot{y}(t, \epsilon), \dot{z}(t, \epsilon)) \) starting from \( \eta_0^0 \) of the perturbed system \( \Xi \), for different values of the perturbation parameter \( \epsilon \) and the \( C^1 \)-solution \( \eta(t) = (x(t), y(t), z(t)) \) of \( \Xi \) starting from \( \eta_0^0 \).

It can be seen from Figure 2 that the proposed perturbed system indeed approximates the DAE both for the jump \( \eta_0^0 \rightarrow \eta_0^+ \) and for the \( C^1 \)-solution \( \eta(t) \) starting from \( \eta_0^0 \) and evolving on \( M^* \).

6. CONCLUSIONS

In this paper, we discuss the \( C^1 \)-solutions and the jumps from inconsistent initial points for nonlinear DAEs. First, we propose a normal form called the index-1 nonlinear Weierstrass form (INWF), which has a simple and decoupled system structure. We show that a nonlinear DAE is locally externally equivalent to the (INWF) if and only if the DAE is index-1 and the distribution defined by \( ker \ E \) is involutive. Then we use the (INWF) to generalize the consistency projector of linear DAEs to the nonlinear case. The generalized nonlinear consistency projector offers a way to solve the consistent initialization problem for nonlinear DAEs. Finally, we propose a system approximation for nonlinear DAEs with jumps via the singular perturbation theory. The results of this paper could be a nice tool to study hybrid DAE systems involving with switchings since the consistent initialization is a fundamental problem for the solutions of switched nonlinear DAEs.

REFERENCES

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