Relaxed Wyner’s Common Information

Erixhen Sula, Student Member, IEEE and Michael Gastpar, Fellow, IEEE

Abstract

A natural relaxation of Wyner’s Common Information is studied. Specifically, the constraint of conditional independence is replaced by an upper bound on the conditional mutual information. While of interest in its own right, this relaxation has operational significance in a source coding problem that models coded caching. For the special case of jointly Gaussian random variables, it is shown that (relaxed) Wyner’s Common Information is attained by a Gaussian auxiliary, and a closed-form formula is found. In the case of Gaussian vectors, this is shown to lead to a novel allocation problem. Finally, using the same techniques, it is also shown that for the lossy Gray-Wyner network with Gaussian sources and mean-squared error, Gaussian auxiliaries are optimal, which leads to closed-form solutions.

Index Terms

Wyner’s Common Information, Gray-Wyner network, water filling, conditional independence, source coding

I. INTRODUCTION

Wyner’s Common Information [1] is a measure of dependence between two random variables. Its operational significance lies in network information theory problems (including a canonical information-theoretic model of the problem of coded caching) as well as in distributed simulation of shared randomness. Specifically, for a pair of random variables, Wyner’s common information can be described by the search for the most compact third variable that makes the pair conditionally independent. Compactness is measured in terms of the mutual information between the pair and the third variable. The value of Wyner’s common information is the minimum of this mutual information.

In the present paper, we study a generalization of this concept by relaxing the constraint of conditional independence. Conditional independence can be expressed by requiring the conditional mutual information to be zero. Hence, a natural relaxation is to instead impose an upper bound on the conditional mutual information. Operational significance again lies in network information theory problems.

The main contributions of our work are:

1) The derivation of the key properties of Relaxed Wyner’s Common Information, most specifically, a Chain Rule for independent pairs (Theorem 3).

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E. Sula and M. Gastpar are with the School of Computer and Communication Sciences, École Polytechnique Fédérale de Lausanne (EPFL), Lausanne, Switzerland (email: {erixhen.sula,michael.gastpar}@epfl.ch).
2) The closed-form solution for the case of jointly Gaussian random variables. This is accomplished via a novel version of the technique known as factorization of convex envelope, which was originally introduced in [2].
3) The closed-form solution for the case of jointly Gaussian random vectors, along with a “water-filling”-type solution.
4) The proof that for the Gaussian (lossy) Gray-Wyner network, Gaussian auxiliaries are optimal, thus leading to explicit solutions for the resulting rate-distortion regions.

A. Related Work

The development of Wyner’s common information started with the consideration of a particular network source coding problem, now referred to as the Gray-Wyner network [3]. From this consideration, Wyner extracted the compact form of the common information in [1], initially restricting attention to the case of discrete random variables. Extensions to continuous random variables are considered in [4], [5], with a closed-form solution for the Gaussian case. Our work provides an alternative and fundamentally different proof of this same formula (along with a generalization). Wyner’s common information has many applications, including to communication networks [11], to caching [6, Section III.C] and to source coding [7]. In the same line of work Wyner’s common information is computed in additive Gaussian channels [8]. Other related works include [9], [10]. The concept of Wyner’s common information has also been extended using other information measures [11].

B. Notation

We use the following notation. Random variables are denoted by uppercase letters and their realizations by lowercase letters. Random column vectors are denoted by boldface uppercase letters and their realizations by boldface lowercase letters. Depending on the context we will denote the random column vector also as $X^n := (X_1, X_2, \ldots, X_n)$. We denote matrices with uppercase letters, e.g., $A, B, C$. The $(i, j)$ element of matrix $A$ is denoted by $A_{ij}$ or $[A]_{ij}$ depending on the context. For the cross-covariance matrix of $X$ and $Y$, we use the shorthand notation $K_{XY}$, and for the covariance matrix of a random vector $X$ we use the shorthand notation $K_X := K_{XX}$. In slight abuse of notation, we will let $K_{(X,W)}$ denote the covariance matrix of the stacked vector $(X, W)^T$. We denote the identity matrix of dimension $2 \times 2$ with $I_2$ and the Kullback-Leibler divergence with $D(\cdot || \cdot)$. Column vector $X$ of dimension $n$ is denoted as $(X_1, X_2, \ldots, X_n)$ and $\text{diag}(\cdot)$ denotes the diagonal matrix.

We denote $\log^+(x) = \max(\log x, 0)$, $X_{\theta_1} = \frac{X_1 + X_2}{\sqrt{2}}$ and $X_{\theta_2} = \frac{X_1 - X_2}{\sqrt{2}}$.

II. RELAXED WYNER’S COMMON INFORMATION

A. Wyner’s Common Information

Wyner’s common information is defined for two random variables $X$ and $Y$ of arbitrary fixed joint distribution $p(x, y)$. 

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**Definition 1.** For random variables $X$ and $Y$ with joint distribution $p(x, y)$, Wyner’s common information is defined as

$$C(X; Y) = \min_{p(w|x, y)} I(X, Y; W) \text{ such that } I(X; Y|W) = 0.$$  

(1)

**Lemma 1.** Wyner’s common information satisfies the following basic properties:

1) The cardinality of $W$ may be restricted to $|W| \leq |\mathcal{X}||\mathcal{Y}|$.

2) $C(X; Y) \geq 0$ with equality if and only if $X$ and $Y$ are independent.

3) $C(X; Y) \geq I(X; Y)$.

4) Data processing inequality: If $X - Y - Z$ form a Markov chain, then $C(X; Z) \leq \min\{C(X; Y), C(Y; Z)\}$.

**Proof.** Items 1)-3) are proved in [12]. For Item 3), $I(X; Y) \leq I(X; Y, W) = I(X; W) \leq I(X; Y; W)$ for any $W$ under which $X$ and $Y$ are conditionally independent.

Item 4) is stated in [13]. To prove it, observe that for fixed $p(x, y, z)$, we can write

$$C(X; Y) = \min_{p(x, y, z)p(w|x, y): I(X; Y|W) = 0} I(X, Y; W)$$  

(2)

$$\geq \min_{p(x, y, z)p(w|x, y): I(X; Y|W) = 0} I(X, Z; W),$$  

(3)

due to the Markov chain $(X, Z) - (X, Y) - W$. Moreover, note that since we consider only joint distributions of the form $p(x, y)p(z|y)p(w|x, y)$, we also have the Markov chain $(X, W) - Y - Z$, which implies the (conditional) Markov chain $X - Y - Z|\{W = w\}$. The latter implies $I(X; Y|W) \geq I(X; Z|W)$. Hence,

$$C(X; Y) \geq \min_{p(x, y, z)p(w|x, y): I(X; Z|W) = 0} I(X, Z; W) \geq C(X; Z).$$  

(4)

By the same token, $C(Y; Z) \geq C(X; Z)$, which completes the proof.

We note that explicit formulas for Wyner’s common information are known only for a small number of special cases. The case of the doubly symmetric binary source is solved completely in [11] and can be written as

$$C(X; Y) = 1 + h_b(a_0) - 2h_b \left( \frac{1 - \sqrt{1 - 2a_0}}{2} \right),$$  

(5)

where $a_0$ denotes the probability that the two sources are unequal (assuming without loss of generality $a_0 \leq \frac{1}{2}$). In this case, the optimizing $W$ is Equation (4) can be chosen to be binary. Further special cases of discrete-alphabet sources appear in [13].

Moreover, when $X$ and $Y$ are jointly Gaussian with correlation coefficient $\rho$, then $C(X; Y) = \frac{1}{2} \log \frac{1 + |\rho|}{1 - |\rho|}$. Note that for this example, $I(X; Y) = \frac{1}{2} \log \frac{1}{1 - \rho^2}$. This case was solved in [4], [5] using a parameterization of conditionally independent distributions. We note that an alternative proof follows from our arguments below.

**B. Relaxed Wyner’s Common Information**

**Definition 2.** For random variables $X$ and $Y$ with joint distribution $p(x, y)$, the relaxed Wyner’s common information is defined as

$$C_\gamma(X; Y) = \min_{p(w|x, y)} I(X, Y; W) \text{ such that } I(X; Y|W) \leq \gamma.$$  

(6)
Remark 1. We note that our definition is closely related to the auxiliary quantity $\Gamma(\delta_1, \delta_2)$ defined in [12] Section 4.2, in that $C_\gamma(X; Y) = H(X, Y) - \Gamma(0, \gamma)$.

Lemma 2. The relaxed Wyner’s common information satisfies the following basic properties:

1) The cardinality of $\mathcal{W}$ may be restricted to $|\mathcal{W}| \leq |\mathcal{X}| |\mathcal{Y}| + 1$.
2) $C_\gamma(X; Y) \geq 0$ with equality if and only if $\gamma \geq I(X; Y)$.
3) $C_\gamma(X; Y) \geq \max\{I(X; Y) - \gamma, 0\}$.
4) Data processing inequality: If $X - Y - Z$ form a Markov chain, then $C_\gamma(X; Z) \leq \min\{C_\gamma(X; Y), C_\gamma(Y; Z)\}$.
5) $C_\gamma(X; Y)$ is a convex and continuous function of $\gamma$ for $\gamma \geq 0$.
6) If $f(\cdot)$ and $g(\cdot)$ are invertible functions, then $C_\gamma(f(X); g(Y)) = C_\gamma(X; Y)$.
7) For discrete $X$, we have $C_\gamma(X; X) = \max\{H(X) - \gamma, 0\}$.

Proof. Item 1) is a standard cardinality bound, following from the arguments in [14]. For the context at hand, see also Theorem 1 in [15, p.6396]. For item 2), the inequality follows from the fact that mutual information is non-negative. If $\gamma \geq I(X; Y)$, we may select $W$ to be a constant, thus we have equality. If $\gamma < I(X; Y)$, then the lower bound proved in the next item establishes that we cannot have equality. For Item 3), observe that the Lagrangian for the relaxed Wyner’s common information problem of Equation (6) is $L(\lambda, p(w|x, y)) = I(X, Y; W) + \lambda(I(X; Y|W) - \gamma)$. From Lagrange duality, we thus have the lower bound $C_\gamma(X; Y) \geq \inf_{p(w|x, y)} L(\lambda, p(w|x, y))$, for all positive $\lambda$. Setting $\lambda = 1$, we have $\inf_{p(w|x, y)}(I(X, Y; W) + I(X; Y|W) - \gamma) = \inf_{p(w|x, y)}(I(X; Y) + I(X; W|Y) + I(Y; W|X) - \gamma) = I(X; Y) - \gamma$. For Item 4), observe that for fixed $p(x, y, z)$, we can write

$$C_\gamma(X; Y) = \min_{p(x, y, z)p(x, y)I(X,Y|W) \leq \gamma} I(X, Y; W)$$

(7)

$$\geq \min_{p(x, y, z)p(x, y)I(X,Y|W) \leq \gamma} I(X, Z; W),$$

(8)

due to the Markov chain $(X, Z) - (X, Y) - W$. Moreover, note that since we consider only joint distributions of the form $p(x, y)p(z|y)p(w|x, y)$, we also have the Markov chain $(X, W) - Y - Z$, which implies the Markov chain $X - Y - Z | \{W = w\}$. The latter implies $I(X; Y|W) \geq I(X; Z|W)$. Hence,

$$C_\gamma(X; Y) \geq \min_{p(x, y, z)p(w|x, y)I(X,Z|W) \leq \gamma} I(X, Z; W) \geq C_\gamma(X; Z).$$

(9)

By the same token, $C_\gamma(Y; Z) \geq C_\gamma(X; Z)$, which completes the proof. Item 5) follows directly from [12] Corollary 4.5]. Item 6) follows because all involved mutual information terms are invariant to one-to-one transforms. For Item 7), note that we can express $C_\gamma(X; X) = H(X) - \max_{p(w|x)} H(X|W) \leq \gamma H(X|W)$, which directly gives the result.

A further property of relaxed Wyner’s common information is a type of chain rule. Since its proof is somewhat more involved, we state it separately in the following theorem.
Theorem 3 (Chain Rule for independent pairs). Let \( \{(X_i, Y_i)\}_{i=1}^n \) be \( n \) independent pairs of random variables. Then
\[
C_\gamma(X^n; Y^n) = \min_{\Gamma} \sum_{i=1}^n C_\gamma(X_i; Y_i).
\]
(10)

The proof is given in Appendix A.

Remark 2. It is tempting to conjecture that Equation (10) should hold more generally as an inequality. In full generality, this is false, as the following two examples illustrate. Specifically, suppose first that \( X_1 = X_2 \) and \( Y_1 = Y_2 \). Then, \( C_\gamma(X_1, X_2; Y_1, Y_2) = C_\gamma(X_1; Y_1) \leq C_\gamma/2(X_1; Y_1) + C_\gamma/2(X_2; Y_2) = \min_{\gamma_1, \gamma_2 \leq \gamma} C_\gamma(X_1; Y_1) + C_\gamma(X_2; Y_2), \) since the relaxed Wyner’s common information is a non-increasing function of \( \gamma \). By contrast, consider now binary random variables and let \( X_1 \) and \( Y_1 \) be independent and uniform. Let \( X_2 = X_1 \oplus Z \) and \( Y_2 = Y_1 \oplus Z \), where \( Z \) is binary uniform and independent, and \( \oplus \) denotes modulo-addition. Then, \( C_\gamma(X_1; Y_1) = C_\gamma(X_2; Y_2) = 0 \), while \( C_\gamma(X_1, X_2; Y_1, Y_2) \geq C_\gamma(Z; Z) = 1 - \gamma \), where the inequality is due to the Data Processing Inequality, i.e., Item 4) of Lemma 2.

By analogy to the discussion in Subsection II-A, explicit formulas for relaxed Wyner’s common information are known only for a small number of special cases. The Gaussian case is the main contribution of the present study and will be presented separately below. By contrast, the case of the doubly symmetric binary source is currently open. A natural upper bound is provided in the following example:

Example 1 (Upper Bound for the Doubly Symmetric Binary Source). Let \((X, Y)\) be the doubly symmetric binary source (DSBS), denoting the probability that \( X \neq Y \) by \( a_0 \) (assuming without loss of generality \( a_0 \leq \frac{1}{2} \)). The standard Wyner’s Common Information (\( \gamma = 0 \)) is well known for the DSBS, see Equation (5) above. For the case of general \( \gamma \), the full answer is currently unknown. A natural and intuitively pleasing choice of the auxiliary \( W \) is
\[
W = \begin{cases} 
X \oplus V, & \text{if } X = Y, \\
U, & \text{if } X \neq Y,
\end{cases}
\]
(11)
where \( V \) is Bernoulli(\( \alpha \)) and \( U \) is Bernoulli(1/2). We note that if we select \( \alpha = \alpha_W \), where
\[
\alpha_W = \frac{(1 - \sqrt{1 - 2a_0})^2}{4(1 - a_0)},
\]
(12)
then we have that \( I(X; Y | W) = 0 \). For general \( \alpha > \alpha_W \), it is straightforward to find
\[
C_\gamma(X; Y) \leq I(X, Y; W) = 1 - (1 - a_0)h_b(\alpha) - a_0,
\]
(13)
where \( h_b(\cdot) \) denotes the standard binary entropy function, and \( \alpha \geq \alpha_W \) is chosen such that
\[
I(X; Y | W) = 2h_b \left( 1 - \alpha \right) \left( 1 - a_0 \right) + \frac{a_0}{2} \right) - (1 - a_0)h_b(\alpha) - a_0 - h_b(a_0) = \gamma.
\]
(14)
Note that if we select \( \alpha = \alpha_W \) (thus \( I(X; Y | W) = 0 \)), Equation (14) becomes the well-known formula for the Wyner’s Common Information of the DSBS, i.e., Equation (5) (albeit expressed slightly differently). For general \( \alpha > \alpha_W \), Equation (13) is an upper bound to \( C_\gamma(X; Y) \). Numerical search (leveraging the cardinality bound on \( W \)) leads to the same value. Therefore, we conjecture that Equations (13)-(14) indeed characterize the correct value of the relaxed Wyner’s common information of the DSBS. See also Example 1 in [15 p.6398].
C. Operational Significance

Wyner, in [1], carefully motivates his definition of common information operationally in two ways. The first concerns a particular network source coding problem referred to as the Gray-Wyner network (Figure 1). The second application concerns the distributed simulation of correlated sources. We here briefly explain how the former also motivates the considered relaxed Wyner’s common information.

The Gray-Wyner network [3] is composed of one sender and two receivers, as illustrated in Figure 1. In a nutshell, the sender compresses two underlying correlated sources $X$ and $Y$ (with fixed $p(x, y)$) into three descriptions. The central description, of rate $R_c$, is provided to both receivers. Additionally, each receiver also has access to a tailored private description. Let us denote the rates of the private descriptions by $R_{u,x}$ and $R_{u,y}$, respectively. The main result of [3], Theorem 4, is that the set of trade-offs amongst these rates is given by the closure of the union of the regions

$$\{ R_c \geq I(X, Y; W), R_{u,x} \geq H(X|W), R_{u,y} \geq H(Y|W) \},$$  \hspace{1cm} (15)

where the union is over all probability distributions $p(w, x, y)$ with marginals $p(x, y)$.

To understand the pareto-optimal trade-offs between the three rates $(R_c, R_{u,x}, R_{u,y})$ in a compact way, it is instructive to define the sum rate

$$R_u = R_{u,x} + R_{u,y},$$  \hspace{1cm} (16)

and consider the optimal trade-offs

$$R_u^*(\delta) = \min R_u \text{ such that } R_c \leq \delta,$$  \hspace{1cm} (17)

see also the quantity $T(\alpha)$ in [3, Equation (13)]. The answer to this optimization problem can be expressed via the relaxed Wyner’s common information as

$$R_u^*(\delta) = H(X, Y) - \delta + C^{-1}_\delta(X; Y),$$  \hspace{1cm} (18)

where (trivially) $0 \leq \delta \leq H(X, Y)$ and where $C^{-1}_\delta(X; Y)$ denotes the inverse of the relaxed Wyner’s common information function $C_\gamma(X; Y)$, that is,

$$C^{-1}_\delta(X; Y) = \min \{ \gamma : C_\gamma(X; Y) \leq \delta \}$$  \hspace{1cm} (19)
Fig. 2. A caching scenario with two correlated files, $X^n$ and $Y^n$. The encoder produces two descriptions, one before knowing which file is requested, of rate $R_c$, and one after finding out which file is requested, of average rate $R_u/2$, where the average is taken over the (uniform) file choice.

see Appendix B. Observe that $C_{\delta}^{-1}(X;Y) = 0$ as soon as $\delta \geq C(X;Y)$. Hence, in this regime, Formula (18) reduces to $R^*_u(\delta) = H(X,Y) - \delta$.

To further motivate this result, it is instructive to mention that the Gray-Wyner network can be interpreted as a (canonical information-theoretic) model of so-called caching. This is advocated and further developed in [15], [16], and illustrated in Figure 2. There are files $X$ and $Y$ to choose from, but before the user chooses, the encoder provides a partial description, supposedly during a time when communication cost is much smaller. This partial description is called the cache contents and is of rate $R_c$. Then, the user selects one of the two files and the encoder provides the rest of the description, of rate $R_{u,x}$ or $R_{u,y}$, depending on the actual realization of the request. That is, the two receivers of the Gray-Wyner network now represent the two possible user requests, and the common description of the Gray-Wyner network is precisely the cache contents. When the user selects uniformly at random, using Equation (15), the average update (or delivery) rate is $R_u/2$, with $R_u$ as defined in Equation (16). Therefore, the pareto-optimal trade-offs between the cache rate $R_c$ and the average update rate $R_u/2$ are characterized by the optimization problem in Equation (17), and thus, the solution to the caching scenario is given by Formula (18). We also note that an alternative connection between caching and the Gray-Wyner network, from a worst-case request perspective, is developed in [17].

III. THE GAUSSIAN CASE

One of the main technical contributions of this work is a closed-form formula for the relaxed Wyner’s common information in the case where $X$ and $Y$ are jointly Gaussian.

**Theorem 4.** When $X$ and $Y$ are jointly Gaussian with correlation coefficient $\rho$, then

$$C_\gamma(X;Y) = \frac{1}{2} \log^+ \left( \frac{1 + |\rho|}{1 - |\rho|} \cdot \frac{1 - \sqrt{1 - e^{-2\gamma}}}{1 + \sqrt{1 - e^{-2\gamma}}} \right).$$  \hspace{1cm} (20)

**Remark 3** (Uniqueness). We note that in the Gaussian case, $C_\gamma(X;Y)$ is attained by selecting $W$ in Equation (6) jointly Gaussian with $X$ and $Y$. Note, however, that this is not a unique choice. In particular, any optimizing $W$ may be replaced by $f(W)$, for any one-to-one function $f(\cdot)$, since in that case, both $I(X,Y;W) = I(X,Y;f(W))$ and $I(X,Y|W) = I(X,Y|f(W))$.

**Remark 4** (Operational Significance). While relaxed Wyner’s common information as in Equation (6) is well-defined for the case of Gaussian sources $X$ and $Y$, we note that the specifics of the discussion of operational
The relaxed Wyner’s common information for jointly Gaussian $X$ and $Y$ for the case $\rho = 1/2$, thus, we have $C(X; Y) = \log \sqrt{3}$ and $I(X; Y) = \log(2/\sqrt{3})$. The dashed line is the lower bound from Lemma 2 Item 3).

significance presented in Section II-C concerned perfect reconstruction of the sources. This does not apply to the case of Gaussian sources. Instead, Section V below will discuss a lossy version of our problem, and thus, establish operational significance in the case of Gaussian sources.

A. Jointly Gaussian Optimality

Proof. The proof of the converse for Theorem 4 involves two main steps. In this section, we prove that one optimal distribution is jointly Gaussian via a variant of the factorization of convex envelope. In Sections III-B we tackle the resulting (non-convex) optimization problem with Lagrange duality for the scalar and vector case respectively.

We start by bounding the optimization problem as follows

$$C_\gamma(X; Y) = \min_{p(u|x,y)} \max_{I(X; Y|W) \leq \gamma} I(X, Y; W) \geq \max_{\lambda} \min_{p(u|x,y)} I(X, Y; W) + \lambda I(X; Y|W) - \lambda \gamma$$  \hspace{1cm} (21)

In the sequel, we will enforce a covariance constraint. Firstly, for any fixed covariance matrix $K_{(X,Y,W)}$ we will optimize over all the possible distributions. Then, we will optimize over all possible covariance matrices $K_{(X,Y,W)}$ as follows

$$C_\gamma(X; Y) \geq \max_{\lambda} \min_{K_{(X,Y,W)}} \min_{p(u|x,y):K_{(X,Y,W)}} I(X, Y; W) + \lambda I(X; Y|W) - \lambda \gamma$$ \hspace{1cm} (22)

$$= \max_{\lambda} \min_{K_{(X,Y,W)}} \left\{ \min_{p(u|x,y):K_{(X,Y,W)}} I(X, Y; W) + \lambda I(X; Y|W) - \lambda \gamma \right\}$$ \hspace{1cm} (23)

$$\geq \max_{\lambda} \min_{K_{(X,Y,W)}} \left\{ \min_{p(u|x,y):K_{(X,Y,W)}} I(X, Y; W) \right\} + \lambda \left\{ \min_{p(u|x,y):K_{(X,Y,W)}} I(X; Y|W) - \gamma \right\}$$ \hspace{1cm} (24)

For the sake of simplification we will denote with $\mathcal{P}_G$ the set of all zero-mean Gaussian distributions and fixed covariance matrix. It is important to realize that minimization of the two subproblems in (24) are different in the sense that the first one can be tackled by standard maximization techniques of entropy, whereas the second one requires an application of the so called factorization of convex envelope. To start with

$$\min_{p(u|x,y):K_{(X,Y,W)}} I(X, Y; W) = h(X, Y) - \max_{p(u|x,y):K_{(X,Y,W)}} h(X, Y|W)$$  \hspace{1cm} (25)

$$\geq h(X, Y) - \max_{p(u|x,y)\in\mathcal{P}_G:K_{(X,Y,W)}} h(X, Y|W)$$  \hspace{1cm} (26)
\[
= \min_{p(w|x,y) \in \mathcal{P}_G: K(X,Y,W)} I(X,Y;W)
\]

where (26) follows from [18, Lemma 1]. Now regarding the second term in (24) we define
\[
\ell(W|T) := I(X;Y|W,T)
\]
and the two-letter version of it as
\[
\ell(W_1,W_2|T) := I(X_1,X_2;Y_1,Y_2|W_1,W_2,T).
\]
Furthermore, we denote the lower convex envelope of \(\ell(W)\), (where \(\ell(W)\) is defined by dropping the random variable \(T\) in (28)) by
\[
\ell(W) = \inf_{p(t|x,y,w)} \ell(W|T)
\]
The dual function of our problem is
\[
V(K(X,Y,W)) := \inf_{p(w|x,y): K(X,Y,W)} \ell(W).
\]
Alternatively, we have
\[
V(K(X,Y,W)) = \inf_{p(t,w|x,y): K(X,Y,W)} \ell(W|T) = \inf_{p(w|x,y): K(X,Y,W)} \inf_{p(t|x,y,w)} \ell(W|T) = \inf_{p(w|x,y): K(X,Y,W)} \ell(W).\]
Note that \(\ell(W)\) is a convex function of \(p(w,x,y)\) as \(\ell(W)\) is the lower convex envelope of \(\ell(W)\). Thus, \(\ell(W)\) is a convex function of \(p(w|x,y)\) since \(p(x,y)\) is fixed and \(p(w|x,y)\) is proportional to \(p(w,x,y)\).
In addition, we define
\[
\ell(W|T) = \sum_t p(t)\ell(W|T = t).
\]
After introducing the proper definitions now we are ready to derive the factorization of the convex envelope:

**Lemma 5.** We have
\[
\ell(W_{\theta_1},W_{\theta_2}) \geq \ell(W_{\theta_1}|W_{\theta_2}) + \ell(W_{\theta_2}|W_{\theta_1},X_{\theta_1},Y_{\theta_1})
\]
with equality if and only if
\begin{itemize}
  \item \(I(X_{\theta_1};Y_{\theta_2}|W_{\theta_1},W_{\theta_2},Y_{\theta_1}) = 0\)
  \item \(I(X_{\theta_2};Y_{\theta_1}|W_{\theta_1},W_{\theta_2},X_{\theta_1}) = 0\).
\end{itemize}

**Proof.** Go to appendix C.

**Proposition 6.** There is a pair of random variables \((T_*,W_*)|((X,Y) = (x,y))\) with \(|T_*| \leq 3\) such that
\[
V(K(X,Y,W)) = \ell(W_*|T_*).
\]

**Proof.** Go to appendix D.

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Lemma 7. Let \( p_\ast(t, w|x, y) \) attain \( V(K_{(X,Y,W)}) \) and let \((T, W, X, Y) \sim p_\ast(t_1, w_1, x_1, y_1)p_\ast(t_2, w_2, x_2, y_2), \) where \( p(x, y) \sim \mathcal{N}(0, K_{(X,Y)}) \). Let \((W, X, Y)_t \) denote the conditional distribution \( p_\ast(w, x, y|t) \) and define

\[
(W_{\theta_1}, X_{\theta_1}, Y_{\theta_1}))(T_1, T_2) = (t_1, t_2) \sim \frac{1}{\sqrt{2}}((W, X, Y)_{t_1} + (W, X, Y)_{t_2}),
\]

\[
(W_{\theta_2}, X_{\theta_2}, Y_{\theta_2}))(T_1, T_2) = (t_1, t_2) \sim \frac{1}{\sqrt{2}}((W, X, Y)_{t_1} - (W, X, Y)_{t_2}).
\]

Then:

1) \((T, W_{\theta_1}, X_{\theta_1}, Y_{\theta_1}) \) also attains \( V(K_{(X,Y,W)}) \).
2) \((T, W_{\theta_2}, X_{\theta_2}, Y_{\theta_2}) \) also attains \( V(K_{(X,Y,W)}) \).
3) The joint distribution \((T, W_{\theta_1}, W_{\theta_2}, X_{\theta_1}, X_{\theta_2}, Y_{\theta_1}, Y_{\theta_2}) \) must satisfy
   \begin{itemize}
   \item \( I(X_{\theta_1}; Y_{\theta_2}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}, Y_{\theta_1}, T) = 0 \)
   \item \( I(X_{\theta_2}; Y_{\theta_1}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}, T) = 0 \).
   \end{itemize}

The proof of this lemma is given in Appendix \[\] An inductive extension of the lemma is key to the remainder of our argument. For future reference, we state it as follows:

Corollary 8. Let \( Z = (X, Y, W) \). For every \( \ell \in \mathbb{N} \), \( n = 2^\ell \), let \((T^n, Z^n) \sim \prod_{i=1}^n p_\ast(t_i, z_i). \) Then \((T^n, \tilde{Z}_n) \) achieves \( V(K_Z) \) where \( \tilde{Z}_n|T_n = (t_1, t_2, \ldots, t_n) \sim \frac{1}{\sqrt{n}}(Z_{t_1} + Z_{t_2} + \cdots + Z_{t_n}). \) We take \( Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n} \) to be independent random variables here.

Proof. The proof follows by induction using Lemma \[\]

The proof of the converse for Theorem \[\] is now completed by following exactly along the steps given in Appendix IV]. We therefore omit the proof here, but record the main result:

Lemma 9. For \( \lambda > 0 \), there is a single Gaussian distribution (i.e. no mixture is required) that achieves \( V(K_Z) \).

Proof. The proof is the same as in Appendix IV].

Note that our approach only shows that Gaussian is a maximizer. It does not imply that this choice would be unique, in line with Remark \[\]

B. Exact Computation of Relaxed WCI for Gaussian source

We start by considering the lower bound. There exist a constant \( \alpha \) such that the random variable \( \alpha W \) has variance one and this transformation leaves unchanged the mutual information terms involved in Relaxed Wyner’s common information. By this argument it suffices to consider an arbitrary covariance matrix for the triple \((X, Y, W)\), which is of the following form

\[
K_{(X,Y,W)} = \begin{bmatrix}
1 & \rho & \rho_1 \\
\rho & 1 & \rho_2 \\
\rho_1 & \rho_2 & 1
\end{bmatrix}.
\] (36)

\[\]
Therefore, we have
\[
C_\gamma(X;Y) \overset{(a)}{=} \max_{\lambda} \min_{K(\mathbb{X},\mathbb{Y},\mathbb{W})} \left\{ \min_{p(w|y);K(\mathbb{X},\mathbb{Y},\mathbb{W})} I(X,Y;W) \right\} + \lambda \left\{ \min_{p(w|y);K(\mathbb{X},\mathbb{Y},\mathbb{W})} I(X;Y|W) - \gamma \right\}
\]
\[
\overset{(b)}{=} \max_{\lambda} \min_{K(\mathbb{X},\mathbb{Y},\mathbb{W})} \left\{ \min_{p(w|y)\in\mathcal{P}_G;K(\mathbb{X},\mathbb{Y},\mathbb{W})} I(X,Y;W) \right\} + \lambda \left\{ \min_{p(w|y)\in\mathcal{P}_G;K(\mathbb{X},\mathbb{Y},\mathbb{W})} I(X;Y|W) - \gamma \right\}
\]
\[
\overset{(c)}{=} \max_{\lambda} \min_{\rho_1,\rho_2;K(\mathbb{X},\mathbb{Y},\mathbb{W})} \left\{ \min_{p(w|y)\geq0;K(\mathbb{X},\mathbb{Y},\mathbb{W})} \frac{1}{2} \log \left( \frac{1 - \rho^2}{1 - \rho_1^2 - \rho_2^2 + 2\rho_1\rho_2} \right)
+ \frac{\lambda}{2} \log \left( 1 + \frac{(\rho - \rho_1\rho_2)^2}{1 - \rho^2 - 2\rho_1\rho_2 + 2\rho_1\rho_2} \right) - \lambda \gamma \right\}
\]
\[
\overset{(d)}{=} \max_{\lambda} \min_{\rho_1,\rho_2;K(\mathbb{X},\mathbb{Y},\mathbb{W})} \left\{ \min_{p(w|y)\geq0;K(\mathbb{X},\mathbb{Y},\mathbb{W})} \frac{1}{2} \log \left( \frac{1 - \rho^2}{1 - \rho_1^2 - \rho_2^2 + 2\rho_1\rho_2} \right)
+ \frac{\lambda}{2} \log \left( 1 + \frac{(\rho - \rho_1\rho_2)^2}{1 - \rho^2 - 2\rho_1\rho_2 + 2\rho_1\rho_2} \right) - \lambda \gamma \right\}
\]
\[
\overset{(e)}{=} \max_{\lambda} \min_{\eta;K(\mathbb{X},\mathbb{Y},\mathbb{W})} \left\{ \min_{p(w|y)\geq0;K(\mathbb{X},\mathbb{Y},\mathbb{W})} \frac{1}{2} \log \left( \frac{1 - \rho^2(1 - \eta)^{2\lambda}}{1 - \rho^2 - 2\eta + 2\rho_1\rho_2} \right) - \lambda \gamma \right\}
\]
\[
\overset{(f)}{=} \frac{1}{2} \log + \frac{(1 + \rho)(1 - \sqrt{1 - e^{-2\gamma}})}{(1 - \rho)(1 + \sqrt{1 - e^{-2\gamma}})}
\]
where \((a)\) comes from [4]; \((b)\) comes from section III-A, which shows that one optimal distribution is Gaussian; \((c)\) comes from simplifying the previous step; \((d)\) comes from the bound \(\rho_1^2 + \rho_2^2 \geq 2\rho_1\rho_2\) and equality is reached if and only if \(\rho_1 = \rho_2\); \((e)\) comes from plugging \(\eta = \rho_1\rho_2\); \((f)\) follows from continuity and first order differentiability, which allows us to find the local and global minimum by looking at the first derivative and the corner points. The optimal \(\lambda\) and \(\eta\) are
\[
\eta = \frac{\lambda \rho - 1}{\lambda - 1}, \quad \lambda = \sqrt{\frac{e^{2\gamma}}{e^{2\gamma} - 1}}.
\]
Combining the optimal \(\lambda\) and \(\eta\) we get
\[
\eta = \frac{\rho - \sqrt{1 - e^{-2\gamma}}}{1 - \sqrt{1 - e^{-2\gamma}}}
\]
Form the derivation we get \(\eta \geq 0\) (since \(\eta < 0\) can be ruled out when checking for minimum), then \(\rho \geq \sqrt{1 - e^{-2\gamma}}\). If \(\rho < \sqrt{1 - e^{-2\gamma}}\), then relaxed WCI becomes zero.

Now let us switch the attention to the upper bound. Let us assume (without loss of generality) that \(X\) and \(Y\) have unit variance and are non-negatively correlated with correlation coefficient \(\rho \geq 0\). Since they are jointly Gaussian, we can express them as
\[
X = \sigma W + \sqrt{1 - \sigma^2} N_X
\]
\[
Y = \sigma W + \sqrt{1 - \sigma^2} N_Y,
\]
where \(W, N_X, N_Y\) are jointly Gaussian, and where \(W \sim \mathcal{N}(0,1)\) is independent of \((N_X, N_Y)\). Letting the covariance of the vector \((N_X, N_Y)\) be
\[
K_{(N_X,N_Y)} = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}
\]
for some $0 \leq \alpha \leq \rho$, we find that we need to choose $\sigma^2 = \frac{\rho - \alpha}{1 - \alpha}$. Specifically, let us select $\alpha = \sqrt{1 - e^{-2\gamma}}$, for some $0 \leq \gamma \leq \frac{1}{2} \log \frac{1}{1 - \rho^2}$. For this choice, we find $I(X; Y|W) = \gamma$ and

$$I(X, Y; W) = \frac{1}{2} \log \frac{(1 + \rho)(1 - \alpha)}{(1 - \rho)(1 + \alpha)} \tag{42}$$

\[ \square \]

IV. THE VECTOR GAUSSIAN CASE

In this section, we consider the case where $X$ and $Y$ are jointly Gaussian random vectors of mean zero and of the same length. The key observation is that in this case, there exist invertible matrices $A$ and $B$ such that $AX$ and $BY$ are vectors of independent pairs, exactly like in Theorem 3. Therefore, we can use that theorem to give an explicit formula for the relaxed Wyner’s common information between arbitrarily correlated jointly Gaussian random vectors, as stated in the following theorem.

**Theorem 10.** Let $X$ and $Y$ be jointly Gaussian random vectors of length $n$ with mean zero and covariance matrix $K_{X,Y}$. Then,

$$C_{\gamma}(X; Y) = \min_{\gamma: \sum_{i=1}^n \gamma_i = \gamma} \sum_{i=1}^n C_{\gamma_i}(X_i; Y_i), \tag{43}$$

where

$$C_{\gamma_i}(X_i; Y_i) = \frac{1}{2} \log \frac{(1 + \rho_i)(1 - \sqrt{1 - e^{-2\gamma_i}})}{(1 - \rho_i)(1 + \sqrt{1 - e^{-2\gamma_i}})} \tag{44}$$

and $\rho_i$ (for $i = 1, \ldots, n$) are the singular values of $K_X^{-1/2}K_{XY}K_Y^{-1/2}$.

**Proof.** The proof is based on the argument that for Gaussian random vectors $X$ and $Y$ we apply the following transformation $\tilde{X} = K_X^{-1/2}X$ and $\tilde{Y} = K_Y^{-1/2}Y$, therefore $K_{\tilde{X}} = I_n$ and $K_{\tilde{X}\tilde{Y}} = K_X^{-1/2}K_{XY}K_Y^{-1/2}$. By using singular value decomposition we get $K_{XY} = R_X A R_Y$. Define $\tilde{X} = R_X \tilde{X}$ and $\tilde{Y} = R_Y \tilde{Y}$, which implies that $K_{\tilde{X}} = I_n$ and $K_{\tilde{X}\tilde{Y}} = A$. Now observe that the mappings from $X$ to $\tilde{X}$ and from $Y$ to $\tilde{Y}$, respectively, are one-to-one. Hence, by Lemma 2 Item 6), we have $C_{\gamma}(X; Y) = C_{\gamma}(\tilde{X}; \tilde{Y})$. Moreover, observe that $(\tilde{X}_i, \tilde{Y}_i)$ are independent pairs of random variables. Hence, by applying Theorem 3 and Theorem 4 we get the claimed result. \[ \square \]

In the remainder of this section, we explore the structure of the allocation problem in Theorem 10 that is, the problem of optimally choosing the values of $\gamma_i$. As we will show, the answer is of the water-filling type. That is, there is a “water level” $\gamma^*$. Then, all $\gamma_i$ whose corresponding correlation coefficient $\rho_i$ is large enough will be set equal to $\gamma^*$. The remaining $\gamma_i$, corresponding to those $i$ with low correlation coefficient $\rho_i$, will be set to their respective maximal values (all of which are smaller than $\gamma^*$). To establish this result, we prefer to change notation as follows. We define $\alpha_i = \sqrt{1 - e^{-2\gamma_i}}$. With this, we can express the allocation problem in Theorem 10 as

$$C_{\gamma}(X; Y) = \min_{\alpha_1, \alpha_2, \ldots, \alpha_n} \sum_{i=1}^n \frac{1}{2} \log \frac{(1 + \rho_i)(1 - \alpha_i)}{(1 - \rho_i)(1 + \alpha_i)} \text{ such that } \sum_{i=1}^n \frac{1}{2} \log \frac{1}{1 - \alpha_i^2} \leq \gamma. \tag{45}$$

Moreover, defining

$$C(\rho) = \frac{1}{2} \log \frac{1 + \rho}{1 - \rho}, \quad I(\rho) = \frac{1}{2} \log \frac{1}{1 - \rho^2}, \tag{46}$$
we can rewrite Equation (45) as

$$C_\gamma(X; Y) = \min_{\alpha_1, \alpha_2, \ldots, \alpha_n} \sum_{i=1}^n (C(\rho_i) - C(\alpha_i))^+ \text{ such that } \sum_{i=1}^n I(\alpha_i) \leq \gamma. \tag{47}$$

**Theorem 11.** The solution to the allocation problem of Theorem 10 can be expressed as

$$C_\gamma(X; Y) = \sum_{i=1}^n (C(\rho_i) - \beta^*)^+, \tag{48}$$

where $\beta^*$ is selected such that

$$\sum_{i=1}^n \min \{ f(\beta^*), I(\rho_i) \} = \gamma, \tag{49}$$

where

$$f(\beta^*) = \frac{1}{2} \log \left( \frac{4 \exp(2 \beta^*) + 1)^2}{4 \exp(2 \beta^*)} \right). \tag{50}$$

For illustration purposes, we can also write out a closed-form formula for the case $n = 2$, as follows.

**Corollary 12.** Assuming without loss of generality that $\rho_1 > \rho_2$, we have

$$C_\gamma(X; Y) = \begin{cases} \frac{1}{2} \log \left( \frac{(1+\rho_1)(1+\rho_2)(1-\sqrt{1-e^{-2\gamma}})}{(1-\rho_1)(1-\rho_2)(1+\sqrt{1-e^{-2\gamma}})} \right)^2, & 0 \leq \gamma < 2I(\rho_2), \\ \frac{1}{2} \log \left( \frac{1-\sqrt{1-e^{-2\gamma}}}{1+\sqrt{1-e^{-2\gamma}}} \right), & 2I(\rho_2) \leq \gamma < I(\rho_1) + I(\rho_2), \\ 0, & I(\rho_1) + I(\rho_2) \leq \gamma. \end{cases} \tag{51}$$

**Proof of Theorem 11.** Note that (47) can be rewritten as

$$C_\gamma(X; Y) = \min_{\gamma_1, \gamma_2, \ldots, \gamma_n} \sum_{i=1}^n (C(\rho_i) - C(I^{-1}(\gamma_i)))^+ \text{ such that } \sum_{i=1}^n \gamma_i \leq \gamma, \tag{52}$$

and thus, for notational compactness, let us define

$$g(x) = C(I^{-1}(x)) = \frac{1}{2} \log \left( \frac{1 + \sqrt{1-e^{-2x}}}{1 - \sqrt{1-e^{-2x}}} \right), \tag{53}$$

which is a strictly concave, strictly increasing function. We also define its inverse,

$$f(x) = g^{-1}(x) = I(C^{-1}(x)) = \frac{1}{2} \log \left( \frac{1}{1 - \left( \frac{\exp(2x) - 1}{\exp(2x) + 1} \right)^2} \right) = \frac{1}{2} \log \left( \frac{\exp(2x) + 1)^2}{4 \exp(2x)} \right), \tag{54}$$

which is a strictly convex, strictly increasing function.

Without loss of generality, suppose that $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The objective function is composed of $n$ terms which can be active or not, meaning that they can be either positive or zero. Since the function $C(\rho)$ is increasing in $\rho$, we have that $C(\rho_1) \geq C(\rho_2) \geq \cdots \geq C(\rho_n)$. To summarize the intuition of the proof, note that the $n$-th term, i.e., $(C(\rho_n) - g(\gamma_n))^+$, will be inactive first. Therefore, by increasing $\gamma$ then the terms will become inactive in a decreasing fashion until we are left with only the first term active and the rest inactive.
Let us start with the case when they are all active, which means that \( \sum_{i=1}^{n} (C(\rho_i) - g(\gamma_i))^+ = \sum_{i=1}^{n} (C(\rho_i) - g(\gamma_i)) \)

Then, by the concavity of \( g(\gamma_i) \), we have

\[
\sum_{i=1}^{n} g(\gamma_i) \leq n g\left(\frac{\gamma}{n}\right),
\]

(55)

thus an optimal choice is \( \gamma^* = \frac{\gamma}{n} \), for all \( i \). Hence, in our notation, in this case \( \beta^* = g\left(\frac{\gamma}{n}\right) \). Clearly, all the terms are active in the interval \( 0 \leq \gamma \leq nI(\rho_n) \), with the reasoning that if the \( n \)-th terms is active then the rest of the terms is active too. Next, consider the case when the \( n \)-th term is inactive and the rest is active. Therefore, \( \sum_{i=1}^{n} (C(\rho_i) - g(\gamma_i))^+ = \sum_{i=1}^{n-1} (C(\rho_i) - g(\gamma_i)) \) and by the concavity of \( g(\gamma_i) \), we have

\[
\sum_{i=1}^{n-1} g(\gamma_i) \leq (n - 1) g\left(\frac{\gamma}{n-1}\right),
\]

(56)

thus an optimal choice is \( \gamma^* = \frac{n-1}{n} \gamma \), for all \( i \in \{1, 2, \ldots, n-1\} \). The optimal choice for \( \gamma_n \) is \( \gamma_n = I(\rho_n) \), which makes the \( n \)-th term exactly zero. This scenario will happen in the interval, \( nI(\rho_n) < \gamma \leq I(\rho_n) + (n-1)I(\rho_{n-1}) \).

Instead, the corresponding \( \beta^* \) in our notation is \( \beta^* = g\left(\frac{n-1}{n} \rho\right) \). In general, let us consider the case when \( k \)-th term is active and \( k+1 \)-th is inactive. By a similar argument as above, the optimal choice is \( \gamma^* = \frac{n-k+1}{n-k} \gamma_i \) for \( i \in \{1, 2, \ldots, k\} \) and \( \gamma_i = I(\rho_i) \) for \( i \in \{k+1, \ldots, n\} \). This scenario will happen in the interval \( (k+1)I(\rho_{k+1}) + \sum_{i=k+2}^{n} I(\rho_i) < \gamma \leq kI(\rho_k) + \sum_{i=k+1}^{n} I(\rho_i) \). Very importantly, observe that the optimal \( \gamma_i \) can be rewritten as \( \gamma_i = \min\{I(\rho_i), \gamma^*\} \), therefore the solution to the allocation problem can be expressed as

\[
C(\mathbf{X}; \mathbf{Y}) = \sum_{i=1}^{n} (C(\rho_i) - g(\gamma^*))^+,
\]

(57)

where \( \gamma^* \) is selected such that

\[
\sum_{i=1}^{n} \min\{\gamma^*, I(\rho_i)\} = \gamma.
\]

(58)

The solution to the allocation problem can be rewritten as

\[
C(\mathbf{X}; \mathbf{Y}) = \sum_{i=1}^{n} (C(\rho_i) - \beta^*)^+,
\]

(59)

where \( \beta^* \) is selected such that

\[
\sum_{i=1}^{n} \min\{f(\beta^*), I(\rho_i)\} = \gamma.
\]

(60)

Evidently, this allocation problem thus has a natural reverse water-filling interpretation which can be visualized in two dual ways. First, we could consider the space of the \( \gamma_i \) parameters, which leads to Figure 4. None of the \( \gamma_i \) should be selected larger than the corresponding \( I(\rho_i) \), and those \( \gamma_i \) that are strictly smaller than their maximum value should all be equal. This graphically identifies the optimal value \( \gamma^* \), and thus, the resulting solution to our optimization problem. Alternatively, we could consider directly the space of the individual contributions to the objective, denoted by \( C(\rho_i) \) in Equation (52), which leads to Figure 5.
\[ I(\rho_1) \quad I(\rho_2) \quad I(\rho_{n-2}) \]

\[ \gamma_1 \quad \gamma_2 \quad \ldots \ldots \quad \gamma_{n-2} \quad \gamma_{n-1} \quad \gamma_n \]

Fig. 4. Example of reverse water-filling. The (whole) bars represent the \( \gamma_i \)-s which make \( C_{\gamma_i}(X_i; Y_i) = 0 \), and the shaded area of the bars is the proper allocation \( \gamma_i \) to minimize the original problem. In this example, \( \gamma = \sum_{i=1}^{n} \gamma_i \) is chosen such that \( C_{\gamma_{n-1}}(X_{n-1}; Y_{n-1}) = C_{\gamma_n}(X_n; Y_n) = 0 \).

\[ C(\rho_1) \quad C(\rho_2) \quad C(\rho_{n-2}) \]

\[ C(\rho_1) - \beta^* \quad C(\rho_2) - \beta^* \quad C(\rho_{n-2}) - \beta^* \]

\[ \beta^* \]

Fig. 5. Example of reverse water-filling. The (whole) bars represent the Wyner’s common information \( \gamma = 0 \), and the shaded area of the bars is the respective contribution to the relaxed Wyner’s common information. In this example, \( \gamma \) is chosen such that \( (C(\rho_{n-1}) - \beta^*)^+ = (C(\rho_n) - \beta^*)^+ = 0 \).

V. THE GAUSSIAN LOSSY GRAY-WYNER NETWORK

In this section, we change our perspective and directly consider the Gray-Wyner network, as in Figure 1. By constrast to earlier sections, we now study the case where \( \hat{X} \) and \( \hat{Y} \) are lossy reconstructions of \( X \) and \( Y \), to within some distortion. This problem is studied in [3] Section II. The full solution, up to the optimization over an auxiliary, is characterized in [3] Theorem 8. For our purposes, we find it convenient to express the gist of that theorem in the following definition (see also the quantity \( T(\alpha) \) in [3] Remark (4) following Theorem 8).

**Definition 3** (Gray-Wyner rate-distortion function). For random variables \( X \) and \( Y \) with joint distribution \( p(x, y) \), the Gray-Wyner rate-distortion function is defined as

\[
R_{D, \alpha_x, \alpha_y}(X; Y) = \min I(X; Y; W)
\]

such that \( I(X; \hat{X}|W) \leq \alpha_x \) and \( I(Y; \hat{Y}|W) \leq \alpha_y \), where the minimum is over all probability distributions \( p(\hat{x}, \hat{y}, w, x, y) \) with marginals \( p(x, y) \) and satisfying

\[
E[d_x(X, \hat{X})] \leq D_x \quad \text{and} \quad E[d_y(Y, \hat{Y})] \leq D_y,
\]

where \( d_x(\cdot, \cdot) \) and \( d_y(\cdot, \cdot) \) are arbitrary single-letter distortion measures (as in, e.g., [3] Eqn. (30) ff.).
The operational significance of this definition is directly established by [3, Theorem 8]. We also point out its pleasing similarity to our earlier definition of relaxed Wyner’s common information (Definition 2). The main technical contribution of this section is a solution for the special case where \( X \) and \( Y \) are jointly Gaussian and the fidelity criterion is the mean-squared error. A key ingredient of our main result in this section is the following lemma, which may be of independent interest.

**Lemma 13.** Let \( X \) be a Gaussian random variable, then

\[
\min_{p(\hat{x}, w|x), \text{Cov}(X, W)} I(X; \hat{X}|W) = \frac{1}{2} \log \frac{\text{Var}(X|W)}{D_x},
\]

(63)

where the minimum is over all conditional distributions \( p(\hat{x}, w|x) \) under which the covariance matrix of \((X, W)\) is equal to the given covariance matrix \( K_{(X,W)} \) and under which we have \( \mathbb{E}[(X - \hat{X})^2] \leq D_x \), and where \( \text{Var}(X|W) = K_X - K_{XW}K_{W}^{-1}K_{WX} \).

The proof of this lemma is given in Appendix F. The optimization problem stated in the lemma bears some similarity to the conditional rate distortion problem in [19]. The difference is that in the conditional rate-distortion problem, the distribution \( p(x, w) \) is fixed and we optimize over \( p(\hat{x}, w|x) \). By contrast, in Lemma 13 only the distribution \( p(x) \) is fixed, and we optimize over \( p(\hat{x}, w|x) \), thus finding the best possible side information distribution (under the stated constraint on the covariance matrix). Combining this lemma with standard arguments leads to the following theorem:

**Theorem 14.** Let \( X \) and \( Y \) be jointly Gaussian with mean zero and fixed covariance. Let \( d_x(\cdot, \cdot) \) and \( d_y(\cdot, \cdot) \) be the mean-squared error distortion measure. Then for any \( \lambda_x, \lambda_y \geq 0 \),

\[
R_{D,\alpha_x,\alpha_y}(X, Y) \geq \min_{K_{(X,Y,W)}} \frac{1}{2} \log \frac{\text{det} K_W \text{det} K_{(X,Y)}}{\text{det} K_{(X,Y,W)}} \quad \text{(64)}
\]

\[
+ \lambda_x \left( \frac{1}{2} \log \frac{\text{Var}(X|W)}{D_x} - \alpha_x \right) + \lambda_y \left( \frac{1}{2} \log \frac{\text{Var}(Y|W)}{D_y} - \alpha_y \right),
\]

(65)

where the minimum is over all covariance matrices \( K_{(X,Y,W)} \).

**Proof.** We have

\[
R_{D,\alpha_x,\alpha_y}(X, Y) = \min_{p(\hat{x}, \hat{y}, x, y)|I(X; \hat{X}|W) \leq \alpha_x, I(Y; Y|W) \leq \alpha_y} \left\{ I(X, Y; W) \right\}
\]

\[
\geq \max_{\lambda_x, \lambda_y} \min_{p(\hat{x}, \hat{y}, w, x, y)} \left\{ I(X, Y; W) + \lambda_x \left( I(X; \hat{X}|W) - \alpha_x \right) \right\}
\]

(66)

\[
+ \lambda_y \left( I(Y; \hat{Y}|W) - \alpha_y \right) \quad \text{(67)}
\]

\[
= \max_{\lambda_x, \lambda_y} \min_{K_{(X,Y,W)}} \min_{p(\hat{x}, \hat{y}, w, x, y)} \left\{ I(X, Y; W) + \lambda_x \left( I(X; \hat{X}|W) - \alpha_x \right) \right\}
\]

\[
+ \lambda_y \left( I(Y; \hat{Y}|W) - \alpha_y \right) \quad \text{(68)}
\]

(69)

\[
= \max_{\lambda_x, \lambda_y} \min_{K_{(X,Y,W)}} \min_{p(\hat{x}, \hat{y}, w, x, y)} \left\{ I(X, Y; W) + \lambda_x \left( I(X; \hat{X}|W) - \alpha_x \right) \right\}
\]

\[
+ \lambda_y \left( I(Y; \hat{Y}|W) - \alpha_y \right) \quad \text{(70)}
\]
$$\max_{\lambda_x,\lambda_y} \min_{K(X,Y,W)} \left\{ \min_{p(x,y):K(X,Y,W)} I(X,Y;W) \right\} + \lambda_x \left\{ \min_{p(x,y):K(X,W)} I(X;\hat{X}|W) - \alpha_x \right\}$$

$$+ \lambda_y \left\{ \min_{p(y,w):K(Y,W)} I(Y;\hat{Y}|W) - \alpha_y \right\}$$

$$\min_{\lambda_x,\lambda_y \geq 0} \left\{ \min_{K(X,Y,W)} \frac{1}{2} \log \frac{\det K_{W} \det K_{X,Y}}{\det K_{X,Y,W}} + \lambda_x \left( \frac{1}{2} \log \frac{\Var (X|W)}{D_x} - \alpha_x \right) \right\}$$

$$+ \lambda_y \left( \frac{1}{2} \log \frac{\Var (Y|W)}{D_y} - \alpha_y \right),$$

where (a) follows from weak duality; (b) follows from splitting the problem into optimizing over all the possible distributions for any fixed covariance matrix $K(X,Y,W)$ and then optimizing over all possible these covariance matrices; (c) follows from the fact that the minimum of the sum of functions is lower bounded by the sum of minima; the last two terms in (d) follow from lemma 13, whereas the first term follows from Lemma 1.

The remaining optimization problem in Theorem 14 does not appear to have a closed-form solution. To conclude our consideration, we consider a special case for which we can indeed give a closed-form formula. Let us assume $D_x = D_y = D$. Let us define

$$R_{D,\alpha}(X,Y) = \min_{\alpha_x,\alpha_y = \alpha} R_{D,\alpha_x,\alpha_y}(X,Y),$$

which is thus equivalent to Definition 3 but with the individual conditional mutual information constraints replaced by $I(X;\hat{X}|W) + I(Y;\hat{Y}|W) \leq \alpha$. Then, we have the following theorem:

**Theorem 15 (Gaussian Relaxed Lossy Wyner’s Common Information).** Let $X$ and $Y$ be jointly Gaussian with mean zero, equal variance $\sigma^2$, and with correlation coefficient $\rho$. Let $d_x(\cdot,\cdot)$ and $d_y(\cdot,\cdot)$ be the mean-squared error distortion measure. Then,

$$R_{D,\alpha}(X,Y)$$

$$= \begin{cases} \frac{1}{2} \log \frac{1+\sqrt{1+\frac{2D}{\sigma^2\rho^2+\rho^2}}}{2}, & \text{if } \sigma^2(1-\rho) \leq D e^\alpha \leq \sigma^2 \\ \frac{1}{2} \log \frac{1}{2e^{-\sigma^2/2}}, & \text{if } D e^\alpha \leq \sigma^2(1-\rho). \end{cases}$$

**Proof.** We start by analogy to the proof of Theorem 14. Specifically, we observe that from weak duality, we have

$$R_{D,\alpha} \geq \max_{\lambda} \min_{K(X,Y,W)} \left\{ \min_{p(x,y):K(X,Y,W)} I(X,Y;W) \right\}$$

$$= \max_{\lambda} \min_{K(X,Y,W)} \frac{1}{2} \log \frac{\det K_{W} \det K_{X,Y}}{\det K_{X,Y,W}}$$

$$+ \lambda \left( \frac{1}{2} \log \frac{\Var (X|W)}{D} + \frac{1}{2} \log \frac{\Var (Y|W)}{D} - \alpha \right).$$
Thus, by applying this tweak we can achieve the inequality optimal solution for step another solution is constructed. An arbitrary covariance matrix, which is of the form

\[ K_{(X,Y,W)} = \begin{bmatrix} 1 & \rho & \rho_1 \\ \rho & 1 & \rho_2 \\ \rho_1 & \rho_2 & 1 \end{bmatrix}. \] (77)

Thus, by evaluating (76) we get

\[
R_{D,\alpha} \geq \max_{\lambda} \min_{\rho_1,\rho_2,\rho_3,\rho_4} \frac{1}{2} \log \frac{1 - \rho^2}{1 - \rho^2 - \rho_1^2 - \rho_2^2 + 2 \rho_1 \rho_2}
+ \lambda \left( \frac{1}{2} \log(1 - \rho_1^2) + \frac{1}{2} \log(1 - \rho_2^2) - \alpha - \log(D) \right)
\]

\[ = \frac{1}{2} \log^+ \frac{1 + \rho}{2 De^\alpha + \rho - 1}. \] (79)

The step (e) is to optimize over \(\rho_1, \rho_2\) and \(\lambda\). From continuity and first order differentiability, we find the local and global minimum by looking at the first derivative and the corner points. The optimal solutions are

\[
\rho_1 = \rho_2 = \sqrt{\frac{\lambda + \lambda \rho - 1}{2 \lambda - 1}}, \quad \lambda = \frac{De^\alpha}{\rho + 2 De^\alpha - 1}. \] (80)

Combining the optimal solutions together we get

\[ \rho_1 = \rho_2 = \sqrt{1 - De^\alpha}. \] (81)

Lastly, let us treat the other case when \(W\) is a random vector of dimension two. Thus, it would suffice to consider an arbitrary covariance matrix, which is of the form

\[ K_{(X,Y,W)} = \begin{bmatrix} 1 & \rho & \rho_1 & \rho_2 \\ \rho & 1 & \rho_3 & \rho_4 \\ \rho_1 & \rho_3 & 1 & 0 \\ \rho_2 & \rho_4 & 0 & 1 \end{bmatrix}. \] (82)

Thus, by evaluating (76) we get

\[
R_{D,\alpha} \geq \max_{\lambda} \min_{\rho_1,\rho_2,\rho_3,\rho_4} \frac{1}{2} \log \frac{1 - \rho^2}{(1 - \rho_1^2 - \rho_2^2)(1 - \rho_3^2 - \rho_4^2) - (\rho - \rho_1 \rho_3 - \rho_2 \rho_4)^2}
+ \lambda \left( \frac{1}{2} \log(1 - \rho_1^2 - \rho_2^2) + \frac{1}{2} \log(1 - \rho_3^2 - \rho_4^2) - \log(De^\alpha) \right)
\]

\[ \geq \frac{1}{2} \log^+ \frac{1 - \rho^2}{D^2 e^{2\alpha}}. \] (83)

Regarding step (f), as we know \(\rho\) is fixed and let us assume \((\rho_1, \rho_2, \rho_3, \rho_4)\) is the optimal solution, where \(\rho \neq \rho_1 \rho_3 + \rho_2 \rho_4\). Then, we construct \((\rho'_1, \rho'_2, \rho'_3, \rho'_4)\) such that \(\rho'_1 + \rho'_2 = (\rho'_1)^2 + (\rho'_2)^2\) and \(\rho'_3 + \rho'_4 = (\rho'_3)^2 + (\rho'_4)^2\). Thus, by applying this tweak we can achieve the inequality \((\rho - \rho_1 \rho_3 - \rho_2 \rho_4)^2 \geq (\rho - \rho'_1 \rho'_3 - \rho'_2 \rho'_4)^2\). Therefore another solution is constructed \((\rho'_1, \rho'_2, \rho'_3, \rho'_4)\), which contradicts the original claim. Thus, \(\rho = \rho_1 \rho_3 + \rho_2 \rho_4\). The optimal solution for step (g) are \(\eta = D^2 e^{2\alpha}\) and \(\lambda = 1\), which concludes the proof.
VI. CONCLUDING REMARKS AND OPEN PROBLEMS

We studied a natural relaxation of Wyner’s common information, whereby the constraint of conditional independence is replaced by an upper bound on the conditional mutual information. This leads to a novel and different optimization problem. We established a number of properties of this novel quantity, including a chain rule type formula for the case of independent pairs of random variables. For the case of jointly Gaussian sources, both scalar and vector, we presented a closed-form expression for the relaxed Wyner’s common information. Finally, using the same tool set, we fully characterize the lossy Gaussian Gray-Wyner network subject to mean-squared error. Open problems include:

- The full solution for the binary symmetric source, see Example [1]
- For the case of Wyner’s common information, Witsenhausen [20] managed to give closed-form formulas for a class of distributions he refers to as “L-shaped.” An analogous investigation could be undertaken for the case of the relaxed Wyner’s common information.
- An extension of the chain rule of Theorem [3] to other probabilistic models, beyond independent pairs.

APPENDIX A

PROOF OF THEOREM [3]

The achievability part, that is, the inequality

$$C_\gamma(X^n; Y^n) \leq \min_{\gamma_i = \gamma} \sum_{i=1}^n C_{\gamma_i}(X_i; Y_i),$$

merely corresponds to a particular choice of $W$ in the definition given in Equation [6]. Specifically, let $W = (W_1, W_2, \ldots, W_n)$, and choose $\{(X_i, Y_i, W_i)\}_{i=1}^n$ to be $n$ independent triples of random vectors. The converse is more subtle. We prove the case $n = 2$ first, followed by induction. For $n = 2$, we have

$$\min_{p(w|x_1, x_2, y_1, y_2): I(X_1; X_2; Y_1, Y_2; W) \leq \gamma} I(X_1, X_2, Y_1, Y_2; W) \geq \min_{p(w|x_1, x_2, y_1, y_2): I(X_1; Y_1; W) + I(X_2; Y_2; W, X_1) \leq \gamma} I(X_1, Y_1; W) + I(X_2, Y_2; W, X_1)$$

(87)

$$\min_{\gamma_1 + \gamma_2 = \gamma} \left\{ \min_{p(w|x_1, x_2, y_1, y_2): I(X_1; Y_1; W) \leq \gamma_1, I(X_2; Y_2; W, X_1) \leq \gamma_2} I(X_1, Y_1; W) + I(X_2, Y_2; W, X_1) \right\}$$

(88)

$$\min_{\gamma_1 + \gamma_2 = \gamma} \left\{ \min_{p(w|x_1, x_2, y_1, y_2): I(X_1; Y_1; W) \leq \gamma_1, I(X_2; Y_2; W, X_1) \leq \gamma_2} I(X_1, Y_1; W) + \min_{p(\tilde{w}|x_1, x_2, y_1, y_2): I(X_1; \tilde{W}; \tilde{W}, X_1) \leq \gamma_2} I(X_2, \tilde{W}; \tilde{W}, X_1) \right\}$$

(89)

$$\min_{\gamma_1 + \gamma_2 = \gamma} \left\{ \min_{p(w|x_1, x_2, y_1, y_2): I(X_1; Y_1; W) \leq \gamma_1, I(X_2; Y_2; \tilde{W}; X_1) \leq \gamma_2} I(X_1, Y_1; W) + \min_{p(\tilde{w}|x_1, x_2, y_1, y_2): I(X_1; \tilde{W}; \tilde{W}, X_1) \leq \gamma_2} I(X_2, \tilde{W}; \tilde{W}, X_1) \right\}$$

(90)

$$\min_{\gamma_1 + \gamma_2 = \gamma} \left\{ \min_{p(w|x_1, x_2, y_1): I(X_1; Y_1; W) \leq \gamma_1} I(X_1, Y_1; W) + \min_{p(\tilde{w}|x_1, x_2, y_1): I(X_1; \tilde{W}; \tilde{W}) \leq \gamma_2} I(X_2, \tilde{W}; \tilde{W}, \tilde{W}) \right\}$$

(91)

$$\min_{\gamma_1 + \gamma_2 = \gamma} \left\{ \min_{p(w|x_1, x_2, y_1): I(X_1; Y_1; W) \leq \gamma_1} I(X_1, Y_1; W) + \min_{p(\tilde{w}, \tilde{x}_1|x_1, x_2, y_1): I(X_1; \tilde{W}; \tilde{X}_1) \leq \gamma_2} I(X_2, \tilde{W}; \tilde{W}, \tilde{X}_1) \right\}$$

(92)
where step (a) follows from
\[
I(X_1, X_2, Y_1, Y_2; W) = I(X_1, Y_1; W) + I(X_2, Y_2; W|X_1, Y_1) + I(X_1, Y_1; X_2, Y_2)
\]
\[
= I(X_1, Y_1; W) + I(X_2, Y_2; W, X_1, Y_1)
\]
\[
\geq I(X_1, Y_1; W) + I(X_2, Y_2; W, X_1)
\]
and the constraint is relaxed as follows
\[
\gamma \geq I(X_1, X_2; Y_1, Y_2|W) = I(X_1; Y_1, Y_2|W) + I(X_2; Y_1, Y_2|W, X_1)
\]
\[
\geq I(X_1; Y_1|W) + I(X_2; Y_2|W, X_1)
\]
step (b) follows from splitting the minimization, step (c) follows from minimizing each subproblem individually which would result in a lower bound to the original problem, step (d) follows from reducing the number of constraints resulting into a lower bound, step (e) follows from introducing \(\tilde{X}_1\) as a random variable to be optimized, whereas before \(X_1\) had a fixed distribution. In other words, the preceding minimization is taken over \(p(\tilde{w}|x_2, y_2, x_1)p(x_1|x_2, y_2)\) where \(p(x_1|x_2, y_2)\) has a fixed distribution, whereas now the minimization is taken over \(p(\tilde{w}|x_2, y_2, \tilde{x}_1)p(\tilde{x}_1|x_2, y_2)\), where we also optimize over \(p(\tilde{x}_1|x_2, y_2)\). Lastly, denoting \(W_2 = (\tilde{W}, \tilde{X}_1)\), this can be expressed as
\[
\min_{p(w|x_1, x_2, y_1, y_2): I(X_1, X_2, Y_1, Y_2; W) \leq \gamma} I(X_1, X_2, Y_1, Y_2; W)
\]
\[
\geq \min_{\gamma_1 + \gamma_2 = \gamma} \left\{ \min_{p(w_1|x_1, y_1): I(X_1; Y_1|W_1) \leq \gamma_1} I(X_1; Y_1|W_1) + \min_{p(w_2|x_2, y_2): I(X_2; Y_2|W_2) \leq \gamma_2} I(X_2; Y_2|W_2) \right\}
\]
After proving it for \(n = 2\), we will use the standard induction. In other words, we will assume that the converse holds for \(n - 1\) i.e.
\[
C_\gamma(X^{n-1}; Y^{n-1}) \geq \min_{\gamma_i: \sum_{i=1}^{n-1} \gamma_i = \gamma} \sum_{i=1}^{n-1} C_{\gamma_i}(X_i; Y_i),
\]
after we prove it for \(n\) as follows,
\[
\min_{p(w|x^n, y^n): I(X^n; Y^n; W) \leq \gamma} I(X^n, Y^n; W)
\]
\[
\geq \min_{p(w|x^n, y^n): I(X^{n-1}; Y^{n-1}; W) + I(X_n, Y_n; W, X^{n-1}) \leq \gamma} I(X^{n-1}, Y^{n-1}; W) + I(X_n, Y_n; W, X^{n-1})
\]
\[
\geq \min_{\tilde{\gamma} + \gamma_n = \gamma} \left\{ \min_{p(w|x^{n-1}, y^{n-1}): I(X^{n-1}; Y^{n-1}; W) \leq \sum_{i=1}^{n-1} \gamma_i, I(X_n, Y_n; W, X^{n-1}) \leq \gamma_n} I(X^{n-1}, Y^{n-1}; W) \right\}
\]
\[
+ \min_{p(w|x^n, y^n): I(X^n; Y^{n-1}; W) \leq \sum_{i=1}^{n-1} \gamma_i, I(X_n, Y_n; W, X^{n-1}) \leq \gamma_n} I(X_n, Y_n; \tilde{W}, X^{n-1})
\]\n\[
\geq \min_{\tilde{\gamma} + \gamma_n = \gamma} \left\{ \min_{p(w|x^{n-1}, y^{n-1}, y^n): I(X^{n-1}; Y^{n-1}; W) \leq \sum_{i=1}^{n-1} \gamma_i} I(X^{n-1}, Y^{n-1}; W) \right\}
\]
Equivalently, we can write

\[ \min_{p(\hat{w}|x^n,y_n)} I(X_n,Y_n;\hat{W},X_n^{-1}) \leq \gamma_n \]

\[ \geq \min_{\gamma+\gamma_n=\gamma} \left\{ \min_{p(w^n|x^n,y_n):I(X_n,Y_n|W,X_n^{-1})} I(X_n^{-1},Y_n^{-1};W) \right\} \]

\[ + \min_{p(\hat{w},\tilde{x}_n^{-1}|x_n,y_n):I(X_n,Y_n;\tilde{W},\tilde{X}_n^{-1})\leq \gamma_n} I(X_n,Y_n;\tilde{W},\tilde{X}_n^{-1}) \]

\[ \geq \min_{\gamma+\gamma_n=\gamma} \left\{ C_\gamma I(X_n^{-1},Y_n^{-1}) + \min_{p(w^n|x_n,y_n):I(X_n,Y_n|W_n)\leq \gamma_n} I(X_n,Y_n;W_n) \right\} \]

\[ \geq \min_{\gamma+\gamma_n=\gamma} \left\{ C_\gamma I(X_n^{-1},Y_n^{-1}) + \min_{\gamma_n:|\gamma_n-\gamma|\leq \gamma} \sum_{i=1}^{n-1} C_\gamma (X_i;Y_i) \right\} \]

\[ = \min_{\gamma_n:|\gamma_n-\gamma|\leq \gamma} \sum_{i=1}^{n-1} C_\gamma (X_i;Y_i) \]

where step (f) follows from

\[ I(X^n,Y^n;W) = I(X_n^{-1},Y_n^{-1};W) + I(X_n,Y_n;W|X_n^{-1},Y_n^{-1}) + I(X_n,Y_n;X_n^{-1},Y_n^{-1}) \]

\[ = I(X_n^{-1},Y_n^{-1};W) + I(X_n,Y_n;W,X_n^{-1},Y_n^{-1}) \]

\[ \geq I(X_n^{-1},Y_n^{-1};W) + I(X_n,Y_n;W,X_n^{-1}) \]

and the constraint is relaxed as follows

\[ \gamma \geq I(X^n,Y^n|W) = I(X_n^{-1},Y_n^{-1}|W) + I(X_n,Y_n|W,X_n^{-1}) \]

\[ \geq I(X_n^{-1},Y_n^{-1}|W) + I(X_n,Y_n|W,X_n^{-1}), \]

step (g) follows from same argument as (b), step (h) follows from same argument as (c), step (i) follows from same argument as (d), step (j) follows from similar argument as (e), step (k) follows from denoting \( W_n = (\tilde{W},\tilde{X}_n^{-1}) \), step (l) follows from induction hypothesis (99).

**APPENDIX B**

**DERIVATION OF FORMULA (18)**

Consider the optimization problem from Equation (17). Using Equation (15), it can be expressed as

\[ R^*_u = \min H(X|W) + H(Y|W) \text{ such that } I(X,Y;W) \leq \delta. \]

Equivalently, we can write

\[ R^*_u = \min H(X,Y) - H(X,Y) + H(X,Y|W) - H(X,Y|W) + H(X|W) + H(Y|W) \text{ such that } I(X,Y;W) \leq \delta. \]

Collecting terms, this can be expressed as

\[ R^*_u = \min H(X,Y) - I(X,Y;W) - I(X;Y|W) \text{ such that } I(X,Y;W) \leq \delta. \]

As long as \( \delta \leq H(X,Y) \), this can be rewritten as

\[ R^*_u = H(X,Y) - \delta + \min I(X;Y|W) \text{ such that } I(X,Y;W) \leq \delta, \]
which can now be rewritten as

\[ R_u^* = H(X, Y) - \delta + C_{-1}^{\delta}(X; Y). \quad (120) \]

where

\[ C_{-1}^{\delta}(X; Y) = \min\{ \gamma : C_\gamma(X; Y) \leq \delta \}. \quad (121) \]

**APPENDIX C**

**PROOF OF LEMMA 5**

For the moment we neglect the auxiliary random variable \( T \).

\[
\ell(W_{\theta_1}, W_{\theta_2}) = I(X_{\theta_1}X_{\theta_2}; Y_{\theta_1}Y_{\theta_2}|W_{\theta_1}W_{\theta_2})
\]

\[
\leq I(X_{\theta_1}; Y_{\theta_1}Y_{\theta_2}|W_{\theta_1}W_{\theta_2}) + I(X_{\theta_2}; Y_{\theta_2}|W_{\theta_1}W_{\theta_2}X_{\theta_1})
\]

\[
+ I(X_{\theta_2}; Y_{\theta_2}|W_{\theta_1}W_{\theta_2}X_{\theta_1}Y_{\theta_1}) + (X_{\theta_2}; Y_{\theta_2}|W_{\theta_1}W_{\theta_2}X_{\theta_1}Y_{\theta_1})
\]

\[
\geq \ell(W_{\theta_1}|W_{\theta_2}) + \ell(W_{\theta_2}|W_{\theta_1}, X_{\theta_1}, Y_{\theta_1})
\]

Where \((*)\) follows from splitting the information terms and \((\text{m})\) follows from the non-negativity of the underlined terms. Thus, we have \( \sum_t p(T = t)\ell(W_{\theta_1}, W_{\theta_2}|T = t) \geq \sum_t p(T = t) (\ell(W_{\theta_1}|W_{\theta_2}, T = t) + \ell(W_{\theta_2}|W_{\theta_1}, X_{\theta_1}, Y_{\theta_1}, T = t)) \)

**APPENDIX D**

**PROOF OF PROPOSITION 6**

Assuming \( \mathbb{E}[W_n] = 0 \) and \( \mathbb{E}[W_n^2] < \infty \) for all \( n \), will guarantee that the sequence of random variables \( \{W_n\}|(X, Y) = (x, y) \) has a finite variance.

**Proposition 16** (Proposition 17 in [2]). Consider a sequence of random variables \( \{W_n\} | (X, Y) = (x, y) \) such that it has a finite variance for all \( n \), then the sequence is tight.

**Theorem 17** (Prokhorov). If \( \{W_n\} | (X, Y) = (x, y) \) is a tight sequence then there exists a subsequence \( \{W_{n_i}\} | (X, Y) = (x, y) \) and a limiting probability distribution \( W_\ast | (X, Y) = (x, y) \) such that \( W_{n_i} | (X, Y) = (x, y) \overset{w}{\Rightarrow} W_\ast | (X, Y) = (x, y) \) converges weakly in distribution.

The only critical term in \( \ell(W) \) is \( I(X; Y|W) \). To prove that the minimizer exists, it is enough to show that \( \ell(W) \) is lower semi-continuous. We will show by utilizing the following Theorem.

**Theorem 18** ([21]). If \( P_n \overset{w}{\Rightarrow} P \) and \( Q_n \overset{w}{\Rightarrow} Q \), then \( D(P||Q) \leq \liminf_{n \to \infty} D(P_n||Q_n) \).

Observe that \( I(X; Y|W) = D(P_{XY}|Q_{XYW}) \), where \( Q_{XYW} \) should satisfy Markov chain \( X \rightarrow W \rightarrow Y \). For the theorem to hold we need to check the assumptions, which are \( P_n \overset{w}{\Rightarrow} P \) that hold from theorem [17] and \( Q_n \overset{w}{\Rightarrow} Q \) since \( Q_{XYW} \) corresponds to a family of distributions which is contained in \( P_{XYW} \). Therefore
\[ I(X; Y|W) \leq \lim_{n \to \infty} I(X_n; Y_n|W_n). \] To preserve the covariance matrix \( K_{(X,Y,W)} \) (for a fixed \( K_{(X,Y)} \)), there are three free parameters or three degrees of freedom, thus \(|T_o| \leq 3\).

**APPENDIX E**

**PROOF OF LEMMA 7**

We start with the following chain of inequalities
\[
2V(K_{(X,Y,Z)}) \overset{(n)}{=} \ell(W_1|T_1) + \ell(W_2|T_2)
\]
\[
\overset{(o)}{=} \ell(W_1, W_2|T_1, T_2)
\]
\[
\overset{(p)}{=} \ell(W_{\theta_1}, W_{\theta_2}|T_1, T_2)
\]
\[
\overset{(q)}{=} \ell(W_{\theta_1}|W_{\theta_2}, T_1, T_2) + \ell(W_{\theta_2}|W_{\theta_1}, X_{\theta_1}, Y_{\theta_1}, T_1, T_2)
\]
\[
\overset{(r)}{=} K(W_{\theta_1}|W_{\theta_2}) + K(W_{\theta_2}|W_{\theta_1}, X_{\theta_1}, Y_{\theta_1})
\]
\[
\overset{(s)}{=} K(W_{\theta_1}) + K(W_{\theta_2})
\]
\[
\overset{(t)}{=} 2V(K_{(X,Y,Z)}).
\]

Here (n) holds for the distribution \( p_*(t, w|x, y)p(x, y) \) that attains \( V(K_{(X,Y,Z)}) \); (o) holds since \( (T_1, W_1, X_1, Y_1) \) and \( (T_2, W_2, X_2, Y_2) \) are independent by assumption; (p) follows by variable transformation since mutual information is preserved under bijective transformation; (q) follows by Lemma 8; (r) follows from
\[
\ell(W_{\theta_1}|T, W_{\theta_2}) = \sum_{w_{\theta_2}} p(w_{\theta_2}) \ell(W_{\theta_1}|T, W_{\theta_2} = w_{\theta_2})
\]
\[
\overset{(u)}{=} \sum_{w_{\theta_2}} p(w_{\theta_2}) K(W_{\theta_1}|W_{\theta_2} = w_{\theta_2})
\]
\[
\overset{(v)}{=} K(W_{\theta_1}|W_{\theta_2})
\]

where (u) holds because \( K(W_{\theta_1}|W_{\theta_2} = w_{\theta_2}) \) is the lower convex envelope of \( \ell(W_{\theta_1}|W_{\theta_2} = w_{\theta_2}) \) and (v) is the definition of \( K(\cdot|\cdot) \); (s) holds since \( K(W_{\theta_1}) \) is convex in \( p(w_{\theta_1}|x, y) \) and by Jensen’s inequality \( K(W_{\theta_1}|W_{\theta_2}) \overset{(s)}{=} K(W_{\theta_1}) \); (t) follows from definition of \( V(K_{(X,Y,Z)}) \).

**APPENDIX F**

**PROOF OF LEMMA 13**

It is relevant to define
\[
k(W, \hat{X}|T) := I(X; \hat{X}|W, T)
\]
(124)

and the two-letter version of it as
\[
k(W_1, W_2, \hat{X}_1, \hat{X}_2|T) := I(X_1X_2; \hat{X}_1\hat{X}_2|W_1W_2, T).
\]
(125)

Furthermore, we denote the lower convex envelope of \( k(W, \hat{X}) \), (where \( k(W, \hat{X}) \) is defined by dropping the random variable \( T \) in (124)) by
\[
k(W, \hat{X}) = \inf_{p(t|x, \hat{x}, w)} k(W, \hat{X}|T)
\]
(126)
The dual function of our problem is

\[ V(K_{X,W}, D_x) := \inf_{p(\hat{x}, w|x): K_{X,W}} k(W, \hat{X}). \]  

(127)

Alternatively, we have

\[ V(K_{X,W}, D_x) := \inf_{p(\hat{x}, w|x): K_{X,W}} k(W, \hat{X}) \frac{\left\{ \inf_{p(t|x, \hat{x}, w): K_{X,W}} k(W, \hat{X}|T) \right\}}{k(W, \hat{X})}. \]  

(128)

Note that \( k(W, \hat{X}) \) is a convex function of \( p(w, \hat{x}, x) \) as \( \hat{k}(W, \hat{X}) \) is the lower convex envelope of \( k(W, \hat{X}) \). Thus, \( \hat{k}(W, \hat{X}) \) is a convex function of \( p(w, \hat{x}|x) \) since \( p(x) \) is fixed and \( p(w, \hat{x}|x) \) is proportional to \( p(w, \hat{x}, x) \).

In addition, we define

\[ \hat{k}(W, \hat{X}|T) = \sum_{t} p(t) \frac{k(W, \hat{X}|T = t)}{k(W, \hat{X})}. \]  

(130)

After introducing the proper definitions now we are ready to derive the factorization of the convex envelope:

**Lemma 19.** We have

\[ k(W_{\theta_1}, W_{\theta_2}, \hat{X}_{\theta_1}, \hat{X}_{\theta_2}) \geq k(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}) \]

(131)

\[ + k(W_{\theta_2}, \hat{X}_{\theta_2}|W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1}) \]  

(132)

with equality if and only if

\[ I(X_{\theta_1}; \hat{X}_{\theta_2}|W_{\theta_1}, W_{\theta_2}, \hat{X}_{\theta_1}) = 0 \]

\[ I(X_{\theta_2}; \hat{X}_{\theta_1}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}) = 0. \]

**Proof.** For the moment we neglect the auxiliary random variable \( T \).

\[ k(W_{\theta_1}, W_{\theta_2}, \hat{X}_{\theta_1}, \hat{X}_{\theta_2}) = I(X_{\theta_1}, X_{\theta_2}; \hat{X}_{\theta_1}, \hat{X}_{\theta_2}|W_{\theta_1}, W_{\theta_2}) \]

(133)

\[ \overset{(a)}{=} I(X_{\theta_1}; \hat{X}_{\theta_1}, \hat{X}_{\theta_2}|W_{\theta_1}, W_{\theta_2}) + I(X_{\theta_2}; \hat{X}_{\theta_1}, \hat{X}_{\theta_2}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}) \]

(134)

\[ \overset{(a)}{=} I(X_{\theta_1}; \hat{X}_{\theta_1}|W_{\theta_1}, W_{\theta_2}) + I(X_{\theta_2}; \hat{X}_{\theta_1}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}) \]

\[ + I(X_{\theta_2}; \hat{X}_{\theta_2}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}, \hat{X}_{\theta_1}) + I(X_{\theta_2}; \hat{X}_{\theta_1}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}) \]

\[ \overset{(b)}{=} k(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}) + k(W_{\theta_2}, \hat{X}_{\theta_2}|W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1}), \]

where \((a)\) follows from using chain rule for the mutual information terms and \((b)\) follows from the non-negativity of the underlined terms. Thus, we have

\[ \sum_{t} p(T = t) k(W_{\theta_1}, W_{\theta_2}, \hat{X}_{\theta_1}, \hat{X}_{\theta_2}|T = t) \geq \sum_{t} p(T = t) \]

\[ \left( k(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}, T = t) + k(W_{\theta_2}, \hat{X}_{\theta_2}|W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1}, T = t) \right) \]
**Proposition 20.** There is a pair of random variables \((T_*, W_*, \hat{X}_*)\) with \(|T_*| \leq 5\) such that

\[
V(K_{(X,W)}, D_x) = k(W_*, \hat{X}_*|T_*).
\]

**Proof.** Go to appendix H.

**Lemma 21.** Let \(p_\star(t, w, \hat{x}|x)\) attain \(V(K_{(X,W)}, D_x)\) and let \((T, W, X, \hat{X}) \sim p_\star(t_1, w_1, x_1, \hat{x}_1)p_\star(t_2, w_2, x_2, \hat{x}_2)\), where \(p(x) \sim \mathcal{N}(0, K_X)\). Let \((W, X, \hat{X})\) denote the conditional distribution \(p_\star(w, x, \hat{x}|t)\) and define

\[
(W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1})|((T_1, T_2) = (t_1, t_2)) \sim \frac{1}{\sqrt{2}}((W, X, \hat{X})_{t_1} + (W, X, \hat{X})_{t_2}),
\]

\[
(W_{\theta_2}, X_{\theta_2}, \hat{X}_{\theta_2})|((T_1, T_2) = (t_1, t_2)) \sim \frac{1}{\sqrt{2}}((W, X, \hat{X})_{t_1} - (W, X, \hat{X})_{t_2}).
\]

Then:

1) \((T, W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1})\) also attains \(V(K_{(X,W)}, D_x)\).

2) \((T, W_{\theta_2}, X_{\theta_2}, \hat{X}_{\theta_2})\) also attains \(V(K_{(X,W)}, D_x)\).

3) The joint distribution \((T, W_{\theta_1}, W_{\theta_2}, X_{\theta_1}, X_{\theta_2}, \hat{X}_{\theta_1}, \hat{X}_{\theta_2})\) must satisfy

- \(I(X_{\theta_1}; \hat{X}_{\theta_2}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}) = 0\)
- \(I(X_{\theta_2}; \hat{X}_{\theta_1}|W_{\theta_1}, W_{\theta_2}, X_{\theta_1}) = 0\).

**Proof.** Go to appendix H.

Our approach only shows that Gaussian is a maximizer but not necessarily the unique maximizer. For simplicity let \(Z = (X, \hat{X}, W)\).

**Corollary 22.** For every \(k \in \mathbb{N}, n = 2^k\), let \((T^n, Z^n) \sim \prod_{i=1}^n p_\star(t_i, z_i)\). Then \((T^n, \hat{Z}_n)\) achieves \(V(K_{(X,W)}, D_x)\) where \(\hat{Z}_n|(T_n = (t_1, t_2, \ldots, t_n)) \sim \frac{1}{\sqrt{n}}(Z_{t_1} + Z_{t_2} + \cdots + Z_{t_n})\). We take \(Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}\) to be independent random variables here.

**Proof.** The proof follows by induction using Lemma 21.

**Lemma 23.** For \(\lambda > 0\), there is a single Gaussian distribution (i.e. no mixture is required) that achieves \(V(K_{(X,W)}, D_x)\).

**Proof.** The proof is the same as in [2 Appendix IV].

This completes the proof of Lemma 13.
APPENDIX G

PROOF OF PROPOSITION 20

Assuming $\mathbb{E}[W_n] = 0$, $\mathbb{E}[\hat{X}_n] = 0$ and $\mathbb{E}[W_n^2] < \infty$, $\mathbb{E}[\hat{X}_n^2] < \infty$ for all $n$, will guarantee that the sequence of random variables $\{W_n, \hat{X}_n\} | (X = x)$ has a finite variance.

Proposition 24 (Proposition 17 in [2]). Consider a sequence of random variables $\{W_n, \hat{X}_n\} | (X = x)$ such that it has a finite variance for all $n$, then the sequence is tight.

Theorem 25 (Prokhorov). If $\{W_n, \hat{X}_n\} | (X = x)$ is a tight sequence then there exists a subsequence $\{W_n, \hat{X}_n\} | (X = x)$ and a limiting probability distribution $(W_\ast, \hat{X}_\ast) | (X = x)$ such that $\{W_n, \hat{X}_n\} | (X = x) \Rightarrow \{W_\ast, \hat{X}_\ast\} | (X = x)$ converges weakly in distribution.

The only critical term in $k(W, \hat{X})$ is $I(X; \hat{X}|W)$. To prove that the minimizer exists, it is enough to show that $k(W, \hat{X})$ is lower semi-continuous. We will show by utilizing the following Theorem.

Theorem 26 ([21]). If $P_n \Rightarrow P$ and $Q_n \Rightarrow Q$, then $D(P||Q) \leq \liminf_{n \to \infty} D(P_n||Q_n)$.

Observe that $I(X; \hat{X}|W) = D(P_{X,W}||Q_{X,W})$, where $Q_{X,W}$ should satisfy Markov chain $X \to W \to \hat{X}$. For the theorem to hold we need to check the assumptions, which are $P_n \Rightarrow P$ that hold from theorem 25 and $Q_n \Rightarrow Q$ since $Q_{X,W}$ corresponds to a family of distributions which is contained in $P_{X,W}$. Therefore $I(X; \hat{X}|W) \leq \liminf_{n \to \infty} I(X_n; \hat{X}_n|W_n)$. To preserve the covariance matrix $K_{(X,W)}$ (for a fixed $K_X$), there are five free parameters or five degrees of freedom, thus $|\mathcal{T}_a| \leq 5$.

APPENDIX H

PROOF OF LEMMA 21

Remark 5. We start with the following chain of inequalities

$$2V(K_{(X,W)}, D_x) \overset{(c)}{=} k(W_1, \hat{X}_1|T_1) + k(W_2, \hat{X}_2|T_2)$$

$$\overset{(d)}{=} k(W_1, W_2, \hat{X}_1, \hat{X}_2|T_1, T_2)$$

$$\overset{(e)}{=} k(W_{\theta_1}, W_{\theta_2}, \hat{X}_{\theta_1}, \hat{X}_{\theta_2}|T_1, T_2)$$

$$\overset{(f)}{\geq} k(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}, T_1, T_2) + k(W_{\theta_2}, \hat{X}_{\theta_2}|W_{\theta_1}, X_{\theta_1}, \hat{X}_{\theta_1}, T_1, T_2)$$

$$\overset{(g)}{\geq} k(W_{\theta_1}, \hat{X}_{\theta_1}|W_{\theta_2}) + k(W_{\theta_2}, \hat{X}_{\theta_2}|W_{\theta_1}, X_{\theta_1}, Y_{\theta_1})$$

$$\overset{(h)}{\geq} k(W_{\theta_1}, \hat{X}_{\theta_1}) + k(W_{\theta_2}, \hat{X}_{\theta_2})$$

$$\overset{(i)}{\geq} 2V(K_{(X,W)}, D_x).$$

Here (c) holds for the distribution $p_\ast(t, w, \hat{x}|x)$ that attains $V(K_{(X,W)}, D_x)$; (d) holds since $(T_1, W_1, X_1, \hat{X}_1)$ and $(T_2, W_2, X_2, \hat{X}_2)$ are independent by assumption; (e) follows by variable transformation since mutual infor-
mation is preserved under bijective transformation; (f) follows by Lemma 19; (g) follows from
\[
k(W_{\theta_1}, \hat{X}_{\theta_1} | T, W_{\theta_2}) = \sum_{w_{\theta_2}} p(w_{\theta_2}) k(W_{\theta_1}, \hat{X}_{\theta_1} | T, W_{\theta_2} = w_{\theta_2})
\]
(137)
\[
\geq \sum_{w_{\theta_2}} p(w_{\theta_2}) \hat{k}(W_{\theta_1}, \hat{X}_{\theta_1} | W_{\theta_2} = w_{\theta_2})
\]
\[
\hat{k}(W_{\theta_1}, \hat{X}_{\theta_1} | W_{\theta_2})
\]
where (j) holds because \(k(W_{\theta_1}, \hat{X}_{\theta_1} | W_{\theta_2} = w_{\theta_2})\) is the lower convex envelope of \(k(W_{\theta_1}, \hat{X}_{\theta_1} | W_{\theta_2} = w_{\theta_2})\) and (k) is the definition of \(\hat{k}(\cdot | \cdot)\); (h) holds since \(\hat{k}(W_{\theta_1}, \hat{X}_{\theta_1})\) is convex in \(p(w_{\theta_1}, \hat{X}_{\theta_1}|x)\) and by Jensen’s inequality \(\hat{k}(W_{\theta_1}, \hat{X}_{\theta_1} | W_{\theta_2}) \geq \hat{k}(W_{\theta_1}, \hat{X}_{\theta_1})\); (i) follows from definition of \(V(K(X,W), D_x)\) and \(E[(X_{\theta_1} - \hat{X}_{\theta_1})^2] \leq D_x\), which results form the following argument
\[
E[(X_{\theta_1} - \hat{X}_{\theta_1})^2] \geq \frac{1}{2} E[(X_1 - \hat{X}_1)^2] + \frac{1}{2} E[(X_2 - \hat{X}_2)^2]
\]
\[
+ E[X_1 - \hat{X}_1] E[X_2 - \hat{X}_2] \leq D_x.
\]
(138)
Note that \(E[X_1 - \hat{X}_1] = 0\) because of the invariance of the mutual information if we add an offset to \(\hat{X}_1\) random variable.

References

[1] A. Wyner, “The common information of two dependent random variables,” IEEE Transactions on Information Theory, vol. 21, no. 2, pp. 163–179, March 1975.

[2] Y. Geng and C. Nair, “The capacity region of the two-receiver Gaussian vector broadcast channel with private and common messages,” IWCIT, vol. 60, no. 4, April 2014.

[3] R. M. Gray and A. D. Wyner, “Source coding for a simple network,” The Bell System Technical Journal, vol. 53, no. 9, pp. 1681 – 1721, 1974.

[4] G. Xu, W. Liu, and B. Chen, “Wyner’s common information for continuous random variables - a lossy source coding interpretation,” Annual Conference on Information Sciences and Systems, March 2011.

[5] ———, “A lossy source coding interpretation of Wyner’s common information,” IEEE Transactions on Information Theory, vol. 62, no. 2, pp. 754–768, 2016.

[6] C.-Y. Wang, S. H. Lim, and M. Gastpar, “Information-theoretic caching: Sequential coding for computing,” IEEE Transactions on Information Theory, vol. 62, no. 11, pp. 6393 – 6406, August 2016.

[7] S. Satpathy and P. Cuff, “Gaussian cenceure source coding and Wyner’s common information,” IEEE International Symposium on Information Theory (ISIT), October 2015.

[8] P. Yang and B. Chen, “Wyner’s common information in gaussian channels,” IEEE International Symposium on Information Theory (ISIT), August 2014.

[9] G. Op’t Veld and M. Gastpar, “Total correlation of gaussian vector sources on the gray-wyner network,” Annual Allerton Conference on Communication, Control, and Computing (Allerton), September 2016.

[10] A. Lapidoth and M. Wigger, “Conditional and relevant common information,” IEEE International Conference on the Science of Electrical Engineering (ICSEE), 2016.

[11] L. Yu and V. Y. F. Tan, “Wyner’s common information under Rényi divergence measures,” IEEE Transactions on Information Theory, vol. 64, no. 5, pp. 3616–3632, 2018.

[12] A. Wyner, “The common information of two dependent random variables,” IEEE Transactions on Information Theory, vol. 21, no. 2, pp. 163–179, Mar 1975.

[13] H. S. Witsenhausen, “Values and bounds for the common information of two discrete random variables,” SIAM J. Appl. Math, vol. 31, no. 2, pp. 313–333, September 1976.
[14] R. Ahlswede and J. Körner, “Source coding with side information and a converse for degraded broadcast channels,” IEEE Transactions on Information Theory, vol. 21, no. 6, pp. 629–637, 1975.

[15] C. Y. Wang, S. H. Lim, and M. Gastpar, “Information-theoretic caching: Sequential coding for computing,” IEEE Transactions on Information Theory, vol. 62, no. 11, pp. 6393–6406, Nov 2016.

[16] ——, “Information-theoretic caching: The multi-user case,” IEEE Transactions on Information Theory, vol. 63, no. 11, pp. 7018–7037, Nov 2017.

[17] R. Timo, S. S. Bidokhti, M. A. Wigger, and B. C. Geiger, “A rate-distortion approach to caching,” IEEE Transactions on Information Theory, vol. 64, no. 3, pp. 1957–1976, 2018.

[18] J. Thomas, “Feedback can at most double gaussian multiple access channel capacity (corresp.),” IEEE Transactions on Information Theory, vol. 33, no. 5, pp. 711 – 716, September 1987.

[19] R. Gray, “Conditional rate-distortion theory,” Stanford University, Tech. Rep., 1972.

[20] H. S. Witsenhausen, “Values and bounds for the common information of two discrete random variables,” SIAM Journal on Applied Mathematics, vol. 31, no. 2, September 1976.

[21] E. Posner, “Random coding strategies for minimum entropy,” IEEE Transactions on Information Theory, vol. 21, no. 4, pp. 388 – 391, 1975.