Integrals for braided Hopf algebras

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Abstract

Let $H$ be a Hopf algebra in a rigid braided monoidal category with split idempotents. We prove the existence of integrals on (in) $H$ characterized by the universal property, employing results about Hopf modules, and show that their common target (source) object $\text{Int} H$ is invertible. The fully braided version of Radford’s formula for the fourth power of the antipode is obtained. Connections of integration with cross-product and transmutation are studied.

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1 Introduction

The notion of integrals for finite-dimensional Hopf algebras was first introduced by Larson and Sweedler in an attempt to generalize the notion of Haar measures on groups. In their seminal paper [17] they prove that integrals always exist for finite dimensional Hopf algebras and give a

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variety of interesting applications, among them the Maschke theorem for Hopf algebras. Infinite-dimensional Hopf algebras were also considered by Sweedler [30].

By definition, e.g., a left integral in a Hopf algebra \((H,m,\eta,\Delta,\epsilon,S)\) is an element, \(l \in H\), such that \(hl = \epsilon(h)l\) holds for any element, \(h \in H\). The defining equation for a right integral, \(r \in H\), is analogously \(rh = \epsilon(h)r\). From the uniqueness of integrals, as proven in [17], one readily infers that the opposite regular actions also leave the integrals invariant, only that now the one-dimensional representation is described by a non-trivial character \(\alpha \in H^*\). I.e., we have \(lh = \alpha(h)l\) for all \(h \in H\). The character \(\alpha \in H^*\) is called the module on \(H\), and plays a rôle similar to that of a modular function for an invariant measure. For the analogous dual definition of integrals in \(H^*\) (or on \(H\)) and the respective modular element \(a \in H\) the reader is referred to Sweedler’s book [29].

The theory of integrals, that was subsequently developed, turned into a powerful instrument in the study of finite-dimensional Hopf algebras, and eventually revealed very rigid structures inherent to Hopf-algebras. In particular, using it, Radford proved that the order of the antipode of a finite-dimensional Hopf algebra, \(H\), is finite [26]. His approach is based on a formula of his, which expresses the fourth power of the antipode \(S : H \rightarrow H\) in terms of the moduli \(a \in H\), and \(\alpha \in H^*\). More precisely, we have the following:

\[
S^{-4} = \text{ad}_\alpha \circ \text{ad}_a.
\]

Here \(\text{ad}_\alpha\) and \(\text{ad}_a : H \rightarrow H\) are the usual Hopf algebra automorphisms defined on an element, \(h \in H\), as \(\text{ad}_a.h = aha^{-1}\) and \(\text{ad}_\alpha.h = (\alpha \otimes \text{id} \otimes \alpha^{-1})(\Delta \otimes \text{id})\Delta(h)\), respectively. Knowing that moduli are always group like, the only thing that remains to be verified in order to infer now finiteness of the order of \(S\) is that group like elements in finite dimensional Hopf algebras are of finite order, see [26].

The classical theory of Larson, Sweedler, Radford, and others deals only with algebras in the ordinary sense, meaning, the Hopf algebra \(H\) is always assumed to be a linear space over a given field \(k\), and the defining operations are given as \(k\)-linear maps. For example, the multiplication, \(m\), is a linear homomorphism of the form \(m \in \text{Hom}_k(H \otimes_k H, H)\).

In this paper we shall abandon the concept of \(H\) being a linear space. Instead we shall assume that \(H\) is merely an abstract object in a monoidal category, \(\mathcal{C}\), with no further structure of its own. The content of the Hopf algebra structure thus lies entirely in the operations of \(H\), which are now expressed in terms of morphisms in \(\mathcal{C}\). For example the multiplication will be given by a morphism, \(m \in \text{Hom}_\mathcal{C}(H \otimes_\mathcal{C} H, H)\), where \(\otimes_\mathcal{C}\) stems from the monoidal structure of \(\mathcal{C}\).

The classical notion is recovered in the special case, where the category \(\mathcal{C}\) is abelian and admits a tensor fiber functor \(\mathcal{C} \rightarrow \text{Vect}(k)\), i.e., when \(\mathcal{C}\) is Tannakian. If the category \(\mathcal{C}\) is not Tannakian but still abelian, a special Hopf algebra in \(\mathcal{C}\) can often be found as a special coend in \(\mathcal{C}\).

We will, however, encounter numerous generic examples, in which \(\mathcal{C}\) is neither Tannakian nor abelian and not even additive, but still Hopf algebra objects with interesting interpretations can be extracted. Some of these categories are defined combinatorially or purely topologically.
Our main goal in this article is to extend the above outlined classical results for Hopf algebras to the framework of categories. Specifically, we will show that for any braided Hopf algebra, $H$, in a braided, rigid, monoidal category $C$ with split idempotents, the analogues of integrals and moduli are defined and, similarly, satisfy existence and uniqueness assertions. Equipped with these tools we shall then derive the respective generalization of Radford’s formula for these types of categories, which in Theorem 3.6 will take on the form

$$S^4 \circ u_{-2}^0 = (\text{ad}^a)^{-1} \circ \text{ad}_a^{-1} \circ \Omega_H^{(\text{Int} H)} \in \text{Aut}_C(H).$$

Here the morphism $u_{-2}^0$ is defined in Figure 1 and, the $\Omega_H^{(\text{Int} H)}$ is constructed from the square of the braiding of $H$ with $\text{Int} H$ – the object of the integrals, see Lemma 2.4. We shall also see that all three multiplicands in the right hand side are Hopf algebra automorphisms commute with each other.

Hopf algebras in tensor categories and their properties have been investigated by a number of other people. Let us briefly review some of the previous contributions to the subject, which we build up on and generalize, and explain how our results are threaded into this development.

The idea of an algebra in a tensor category, see [16], is straightforwardly extended to that of a Hopf algebra, if that category is also symmetric. Most proofs of analogous assertions for categorical Hopf algebras in the symmetric case are formally identical to those in the classical situation of a linear Hopf algebra over $k$. The first proofs of existence and uniqueness of integrals in this setting are due to Drinfel’d. It is immediate from definitions that in a symmetric category the elements $u_{-2}^0$ and $\Omega_H^{(\text{Int} H)}$ are trivial (i.e. units) so that also the categorical version of Radford’s formula will differ from the original one only in the interpretations of where the automorphisms live.

A nice way of illustrating the calculations, leading to the uniqueness of integrals and the Radford formula, has been given by Kuperberg [15]. He uses diagrammatic techniques that are essential for the construction of his 3-manifold invariants. In this language the generalizations to the symmetric categories follow immediately without much further explanation.

The notion of Hopf modules for ordinary Hopf algebras, which combines an action with a compatible coaction on the same space, can be similarly generalized without much difficulty to a Hopf algebra living in an abelian, symmetric, monoidal category. The classical results on Hopf modules can be rederived again by “imitation” in the categorical framework, where, similarly, the modules appear as abstract objects. See [28].

In the theory of Hopf algebras in braided categories a variety of complications arise due to the fact that the isomorphism between the tensor product of two objects and the transposed product is no longer canonical, i.e., also no longer coherent. However, a transposition isomorphism is needed, e.g., to formulate the compatibility axiom between multiplication and comultiplication for bialgebras.

If for the latter the braid isomorphism is employed one is left with the definition of a braided Hopf algebra, as it was given and first studied by Majid [21].
In the case where the braided tensor category is also rigid and abelian, integrals for such Hopf algebras were investigated by Lyubashenko \cite{19}, leading to general proofs for the existence of integrals as well as for the invertibility of the object of integrals for a braided Hopf algebra. The results in \cite{19} follow from the study of Hopf modules in abelian braided monoidal categories. This strategy generalizes the approach to integrals via Hopf modules as proposed by Sweedler in \cite{30}.

As for symmetric categories we shall prove here analogously the existence and uniqueness of integrals for braided Hopf algebras in braided monoidal categories, but in addition we shall drastically weaken the condition of abelianness. More precisely, we shall no longer assume that the category has kernels or direct sums. As has already been pointed out in remarks in \cite{2, 3} it will suffice that our category has *split idempotents*. This means roughly that the category contains for a given projection, $P$, in an $\text{End}$-set the image of $P$ as an object, and that $P$ factors through this object.

The property of split idempotent is central in Karoubi’s definition of pseudo-abelian categories \cite{11}, which will be further discussed later on. It is easy to satisfy this condition taking the *Karoubian envelope* of a category \cite{11}.

Besides existence and uniqueness of integrals we will also prove under our very general assumptions the invertibility of the object of integrals $\text{Int} \, H$, which, in the case of abelian, symmetric categories, has already been pointed out by Drinfel’d.

Furthermore, we will generalize the results on Hopf modules for abelian, symmetric, monoidal category as in \cite{28} to any braided monoidal category with split idempotents.

Many of the algorithmic parts of the proofs will be done in a similar diagrammatic language as in \cite{15}. Only now the diagrams are no longer plat graphs, but projections of graphs that distinguish between over and under crossings. As a result we will often encounter additional special elements resulting from non trivial full twists, such as $u_{-2}^0$ and $\Omega^\text{Int} \, H$, which enter the Radford formula.

It is worth mentioning that our results apply to ordinary Hopf algebras and Hopf modules over a commutative ring $R$, if they are projective $R$-modules of finite rank (cf. \cite{9}).

Our interest in braided Hopf algebras in non-abelian, braided categories has primarily been triggered by realizing their crucial rôle in the most recent discoveries in three dimensional topology related to quantum physics. In particular, integrals turned out to be the algebraic objects that are stringently associated to elementary surgery data in the construction of invariants of surgically presented 3-manifolds. Although the point of view of integrals was not used in the more computational approach in \cite{31}, it is inevitable in the construction of the non-semisimple analogues of invariants of 3-manifolds as in \cite{20} and, more generally, 3-dimensional topological quantum field theories as in \cite{14}.

The fact that braided Hopf algebras and their integrals are inseparable from 3-dimensional topology in this approach, naturally leads one to identifying the torus with one hole as such a Hopf algebra in the braided category of 3-dimensional cobordisms between one holed surfaces.
As a non trivial example of our generalized theory, we will precisely identify the integrals and all ingredients to the Radford formula for this braided Hopf algebra and category, via their explicit presentations in a tangle category as in [12].

Summary of Contents:

In Section 2 we review the necessary preliminaries and standard notations pertaining to rigid braided categories and braided Hopf algebras. We shall further define categories with split idempotents and discuss basic properties of invertible objects. In Section 3 results about Hopf modules are applied to prove the existence of \( \text{Int}_H \)-valued(-based) integrals on \( \text{in}_H \). There the object \( \text{Int}_H \) of \( \text{Int}_H \) is characterized by a universal property implying uniqueness, and will turn out to be invertible. The fully braided version of Radford’s formula for the fourth power of the antipode is derived. Section 4 starts with an exposition of several results about Hopf modules, that are useful for our purposes. We then continue to prove the main results in this section, notably the invertibility of the object of integrals, the relationships of integrals with modular group-like elements and the antipode, and, eventually, the generalized Radford formula. In Section 5 we shall develop the example of a braided Hopf algebra in a category of tangles, and explain its applications to topological field theory. This algebra can be functorially mapped to representatives of another important class of braided Hopf algebras, namely coends in abelian rigid braided categories. Section 6 is devoted to connections of integration with cross-products and transmutations. In Section 7 we shall attempt to define external Hopf algebras, and discuss duality properties for Hopf bimodules. We present an explicit equivalence between the categories of Hopf \( H \)-bimodules and Hopf \( H^\vee \)-bimodules given by the tensor product with \( \text{Int}_H \). From these duality considerations we are able to obtain two more proofs of the generalized Radford formula.

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2 Preliminaries

Throughout this paper the symbol \( \mathcal{C} = (\mathcal{C}, \otimes, I) \) denotes a strict rigid monoidal category with braiding \( \Psi \). For the convenience of the reader we shall begin with the definition of strict rigid monoidal categories. In the following paragraphs the notions of categories with split idempotents, braided Hopf algebras, and invertible objects are introduced.

Rigid braided categories

A (strict) rigid category, \( \mathcal{C} \), is a (strict) monoidal category, in which for every object, \( X \in \mathcal{C} \), one can find a pair of dual objects, \( X^\vee \) and \( \forall X \in \mathcal{C} \), as well as morphisms of evaluation and
coevaluation denoted as follows.

$$ev : X \otimes X^\vee \to \mathbb{1} = X \otimes X^\vee,$$

$$\coev : \mathbb{1} \to X^\vee \otimes X = X^\vee \otimes X,$$

They are subject to the condition that the following compositions between evaluations and coevaluations are all equal the identity morphism in $\text{End}(X)$:

$$X = X \otimes \mathbb{1} \xrightarrow{1 \otimes \coev} X \otimes (X^\vee \otimes X) = (X \otimes X^\vee) \otimes X \xrightarrow{ev \otimes 1} \mathbb{1} \otimes X = X,$$

$$X = \mathbb{1} \otimes X \xrightarrow{\coev \otimes 1} (X \otimes X^\vee) \otimes X = X \otimes (X \otimes X^\vee) \xrightarrow{1 \otimes ev} X \otimes \mathbb{1} = X,$$

$$X^\vee = \mathbb{1} \otimes X^\vee \xrightarrow{\coev \otimes 1} (X^\vee \otimes X) \otimes X^\vee = X^\vee \otimes (X \otimes X^\vee) \xrightarrow{1 \otimes ev} X^\vee \otimes \mathbb{1} = X^\vee,$$

$$\vee X = \vee X \otimes \mathbb{1} \xrightarrow{1 \otimes \coev} \vee X \otimes (X \otimes \vee X) = (\vee X \otimes X) \otimes \vee X \xrightarrow{ev \otimes 1} \mathbb{1} \otimes \vee X = \vee X.$$

This data allows us to introduce the *transposes*, $f^t : Y^\vee \to X^\vee$ and $t^f : Y^\vee \to X$, of a given morphism, $f : X \to Y$ in $C$.

The braiding $\{\Psi_{X,Y} : X \otimes Y \to Y \otimes X\}$ has to satisfy axioms, such as naturality and the hexagonal equation. For details seek, e.g., [10, 31].

For a morphism, $f$, in a rigid braided monoidal category its trace $\text{tr}_8 f$ is defined using Figure 1, the subscript being motivated by the resemblance with the digit 8. The *dimension* associated to this trace is $\text{dim}_8(X) := \text{tr}_8(\text{id}_X) \in \text{End}(\mathbb{1})$. We will also need the natural automorphism $u_{-2}^0 : X \to X$ defined via the diagram in Figure 1. The notation is borrowed from [18].

### Categories with split idempotents

M. Karoubi introduced in [14] a class of Banach categories, which he calls *pseudo-abelian*. If we drop, irrelevant for us, Banach and additive structures his definition reduces to the following:

An idempotent, $e = e^2 : X \to X$, in a category, $\mathcal{D}$, is said to be *split*, if there exists an object, $X_e$, and morphisms, $i_e : X_e \to X$ and $p_e : X \to X_e$, such that $e = i_e \circ p_e$ and $\text{id}_{X_e} = p_e \circ i_e$. If every idempotent in $\mathcal{D}$ is split then we say that $\mathcal{D}$ admits *split idempotents*. 

![Figure 1](image-url)
For a given category, $\mathcal{C}$, there exists a universal embedding, $\mathcal{C} \xrightarrow{i} \hat{\mathcal{C}}$, such that the category $\hat{\mathcal{C}}$ admits split idempotents. Moreover, $\hat{\mathcal{C}}$ can be chosen universal in sense that for any category, $\mathcal{D}$, with split idempotents every functor, $F : \mathcal{C} \to \mathcal{D}$, factors in the form $F = (\mathcal{C} \xrightarrow{i} \hat{\mathcal{C}} \xrightarrow{G} \mathcal{D})$, such that the functor $G$ is unique up to an isomorphism of functors. The category $\hat{\mathcal{C}}$ is called the Karoubi enveloping category of $\mathcal{C}$. According to Karoubi [11] it is realized as the category with objects, $X_e = (X, e)$, where $X$ is an object in $\mathcal{C}$ and $e : X \to X$ is an idempotent in $\mathcal{C}$. The morphisms in $\hat{\mathcal{C}}$ are defined by $\hat{\mathcal{C}}(X_e, Y_f) := \{ t \in \mathcal{C}(X,Y) \mid fte = t \}$. The functor $i$, defined by $i(X) = X_{id_X}$ and $i(f) = f$, is a full embedding, that is we have $\mathcal{C}(X,Y) = \hat{\mathcal{C}}(i(X), i(Y))$.

It is noted in [13] that if $\mathcal{C}$ is a (braided) monoidal category then the category $\hat{\mathcal{C}}$ can be equipped with a (braided) monoidal structure:

$$\mathbb{1} := (\mathbb{1}, id_{\mathbb{1}}), \quad X_e \otimes Y_f := (X \otimes Y)_{e \otimes f}, \quad \Psi_{X_e,Y_f} := (f \otimes e) \circ \Psi_{X,Y}.$$ 

In this case $i$ is a (braided) monoidal functor. Furthermore, if the category $\mathcal{C}$ is rigid, so is $\hat{\mathcal{C}}$, and the dual objects of $(X,e)$ are $(X^\vee,e^\vee)$ and $(\vee X, e)$.

From now on we assume that our braided rigid category $\mathcal{C}$ admits split idempotents.

**Hopf algebras**

Recall that a Hopf algebra, $H \in \mathcal{C}$, [21] is an object, $H \in \text{Obj} \mathcal{C}$, together with an associative multiplication, $m : H \otimes H \to H$, and an associative comultiplication, $\Delta : H \to H \otimes H$, obeying the bialgebra axiom

$$\left( H \otimes H \xrightarrow{m} H \xrightarrow{\Delta} H \otimes H \right) = \left( H \otimes H \xrightarrow{\Delta \otimes \Delta} H \otimes H \otimes H \otimes H \xrightarrow{H \otimes \Psi \otimes H} H \otimes H \otimes H \otimes H \xrightarrow{m \otimes m} H \otimes H \right).$$

Moreover, $H$ shall have a unit, $\eta : \mathbb{1} \to H$, a counit, $\varepsilon : H \to \mathbb{1}$, an antipode, $S : H \to H$, and an inverse antipode, $S^{-1} : H \to H$, which shall satisfy axioms analogous to the classical case.

A left (resp. right) module over an algebra, $H$, is an object, $M \in \mathcal{C}$, equipped with an associative action, $\mu_\ell : H \otimes M \to M$ (resp. $\mu_r : M \otimes H \to M$). The category of left (resp. right) $H$-modules will be denoted by $\mathcal{H} \mathcal{C}$ (resp. $\mathcal{C} \mathcal{H}$). The morphisms in $\mathcal{H} \mathcal{C}$ are those from $\mathcal{C}$ that are equivariant with respect to $\mu_\ell$. A left (resp. right) comodule over a coalgebra, $H$, is an object, $M \in \mathcal{C}$, equipped with an associative coaction, $\Delta_\ell : M \to H \otimes M$ (resp. $\Delta_r : M \to M \otimes H$). The category of left (resp. right) $H$-comodules will be denoted by $\mathcal{H} \mathcal{C}$ (resp. $\mathcal{C} \mathcal{H}$).

If $(\mathcal{C}, \otimes, \mathbb{1}, \Psi)$ is a braided monoidal category we shall denote by $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, \Psi)$ the same monoidal category with the mirror-reversed braiding $\Psi_{X,Y} := \Psi_{Y,X}^{-1}$. For a Hopf algebra, $H$, in $\mathcal{C}$ we further denote by $H^{\text{op}}$ (resp. $H_{\text{op}}$) the same coalgebra (resp. algebra) with opposite multiplication $\mu^{\text{op}}$ (resp. opposite comultiplication $\Delta^{\text{op}}$) defined as follows:

$$\mu^{\text{op}} := \mu \circ \Psi_{HH}^{-1} \quad \text{(resp. } \Delta^{\text{op}} := \Psi_{HH}^{-1} \circ \Delta).$$ (2.1)
It is easy to see that $H^{\text{op}}$ and $H_{\text{op}}$ are Hopf algebras in $\mathcal{C}$ with antipode $S^{-1}$. We will always consider $H^{\text{op}}$ and $H_{\text{op}}$ as objects of the category $\mathcal{C}$. In what follows we often use graphical notations for morphisms in monoidal categories, see [2, 11, 18, 24, 31]. The graphics and notations for (co-)multiplication, (co-)unit, antipode, left and right (co-)action, and braiding are given in Figure 2, where $H$ is a Hopf algebra and $M$ is an $H$-module ($H$-comodule).

**Invertible objects**

An object, $K$, of a braided, monoidal category, $\mathcal{C}$, is called invertible if there exists an object, $K^{-1} \in \text{Obj}(\mathcal{C})$, such that $K \otimes K^{-1} \simeq \mathbb{1}$ (and, hence $K^{-1} \otimes K \simeq \mathbb{1}$). Properties of invertible objects in a monoidal category are summarized in the following lemmas. Most of the proofs are straightforward and left to the reader.

**Lemma 2.1.** Let $K \in \text{Obj}(\mathcal{C})$ be invertible. Then the following hold:

- The morphisms $\text{ev}_K$ and $\text{coev}_K$ are invertible;
- The map $\text{End}(\mathbb{1}) \rightarrow \text{End}(K) : c \mapsto c \otimes \text{id}_K$ is an isomorphism of commutative monoids;
- $\dim_{\mathbb{1}}(K) = \dim_{\mathbb{1}}(K^{-1})$ is invertible and
  $$\Psi_{K,K} = (\dim_{\mathbb{1}} K)^{-1} \cdot \text{id}_{(K \otimes K)}.$$

Only the last claim is not entirely obvious. The dimension $\dim_{\mathbb{1}} K \in \text{End}(\mathbb{1})$ is an isomorphism, because it is a composition of isomorphisms, $\text{coev}_K$, $\Psi_{K,K}$, and $\text{ev}_K$. We know that $\Psi_{K,K} = \sigma \cdot \text{id}_{(K \otimes K)}$ for some $\sigma \in \text{End}(\mathbb{1})$. Finally $\sigma = (\dim_{\mathbb{1}} K)^{-1}$ is implied by the diagrammatic calculation below:
For a Hopf algebra, $H$, a morphism, $a : 1 \to H$, is called a group-like element, if $\Delta \circ a = a \otimes a$. A morphism, $\alpha : H \to 1$, is called a multiplicative functional if $\alpha \circ \mu = \alpha \otimes \alpha$. For such morphisms we set $a^{-1} := S \circ a = S^{-1} \circ a$ and $\alpha^{-1} := \alpha \circ S = \alpha \circ S^{-1}$.

Invertible objects can be interpreted as the categorical analogues of one-dimensional modules and one-dimensional comodules of a linear Hopf algebra. The following two lemmas express properties that are obvious for one-dimensional (co)modules in the framework of a braided, monoidal category. The proofs follow easily using the identities in Lemma 2.1.

**Lemma 2.2.** Suppose $K$ is an invertible object in a rigid, monoidal category, $\mathcal{C}$. Then any (co)module structure on $K$ over an (co)algebra, $A$, in $\mathcal{C}$ is given by a multiplicative functional $\alpha$ (group-like element $a$):

$$\mu_r = \text{id}_K \otimes \alpha, \quad \Delta_r = \text{id}_K \otimes a.$$

**Lemma 2.3.** Let $K$ be an invertible object in a rigid, monoidal category, $\mathcal{C}$, and $X,Y \in \text{Obj}(\mathcal{C})$. Then the map $\text{Hom}(X,Y) \to \text{Hom}(X \otimes K, Y \otimes K) : f \mapsto f \otimes \text{id}_K$ is bijective.

The previous lemma, applied to the isomorphism $g = \Psi_{K,X} \Psi_{X,K} : X \otimes K \to X \otimes K$ for an invertible object $K$, implies that there exists a natural isomorphisms $\Omega^K_X : X \to X$ such that:

$$\Psi_{K,X} \Psi_{X,K} = \Omega^K_X \otimes \text{id}_K \quad \text{(or, equivalently, } \Psi_{X,K} \Psi_{K,X} = \text{id}_K \otimes \Omega^K_X),$$

We shall call $\Omega^K_X$ the monodromy. It may be thought of as acting on $\mathcal{C}$ or Hopf algebras therein in the following way:

**Lemma 2.4.** The set of isomorphisms $\{\Omega^K_X\}_X$ constitutes an automorphism of the monoidal identity functor

$$\Omega^K : (\text{id}_\mathcal{C}, \text{id}_\otimes) \to (\text{id}_\mathcal{C}, \text{id}_\otimes) : (\mathcal{C}, \otimes) \to (\mathcal{C}, \otimes).$$

Specifically, this means that $I \circ \Omega^K_X = \Omega^K_X \circ I$ for any $I \in \text{Hom}(X,Y)$, and, further, that $\Omega^K_{X \otimes Y} = \Omega^K_X \otimes \Omega^K_Y : X \otimes Y \to X \otimes Y$.

**Corollary 2.5.** For any invertible object, $K$, and any Hopf algebra, $H \in \mathcal{C}$, the morphism $\Omega^K_H : H \to H$ is a Hopf algebra automorphism.
3 Integrals and the generalized Radford formula

This section contains the detailed statement of the main result of our paper, namely the braided, categorical version of Radford’s formula, as well as the precise definitions of left and right integrals in a category and of the respective (co)moduli entering this formula. We shall defer all proofs to Section 4. In the first paragraph on integrals we will introduce a set of canonical projections, $\Pi_i \in \text{End}_C(H)$, which will guarantee the existence of integrals for a braided Hopf algebra, $H$, in a category with split idempotents. We shall also illustrate the generalizations in Radford’s formula in the example of the category of $\mathbb{Z}/n$-graded vector spaces equipped with a braiding given by Heisenberg-type relations. Here the moduli will turn out to be units, whereas the additional elements $u_{-2}^0$ and $\Omega_H^{(\text{Int} H)}$ are going to be non-trivial.

Integrals for Hopf algebras

The following definition directly generalizes the classical one by rewriting the defining formula into a relation of operations given by morphisms.

**Definition 3.1.** Let $H$ be a bialgebra in a braided, monoidal category, $\mathcal{C}$. A left $X$-valued integral on $H$ is a morphism $f : H \to X$ such that

$$
(H \xrightarrow{\Delta} H \otimes H \xrightarrow{id_H \otimes f} H \otimes X) = (H \xrightarrow{f} X \xrightarrow{\eta_H \otimes id_X} H \otimes X).
$$

Right $X$-valued integrals on $H$ are defined analogously.

A left $X$-based integral in $H$ is a morphism, $f : X \to H$, such that

$$
(H \otimes X \xrightarrow{id_H \otimes f} H \otimes H \xrightarrow{\mu_H} H) = (H \otimes X \xrightarrow{\varepsilon_H \otimes id_X} X \xrightarrow{f} H).
$$

Right $X$-based integrals in $H$ are defined similarly.

In other words, a left (resp. right) $X$-valued integral is a homomorphism from the left (resp. right) regular $H$-comodule to the trivial $H$-comodule $X$. A left (resp. right) $X$-based integral is a homomorphism from the trivial $H$-module $X$ to the left (resp. right) regular $H$-module.

We shall consider only bialgebras with invertible antipode and call them Hopf algebras.

The next proposition not only asserts that a Hopf algebra in a category with split idempotents always admits integrals, but also that the objects of the integrals are invertible, and can be chosen the same for all cases.

**Proposition 3.1.** Assume $H$ is a Hopf algebra with an invertible antipode, $S$, in a braided, monoidal category, $\mathcal{C}$, with split idempotents. Then there exist an invertible object $\text{Int} H$ in $\mathcal{C}$, for which the following hold simultaneously:

1. There exist left and right $\text{Int} H$-valued integrals, $\int_H$ and $\int_H^r : H \to \text{Int} H$, such that any left (resp. right) $X$-valued integral on $H$ admits a unique factorization of the form $H \xrightarrow{\int_H} \text{Int} H \xrightarrow{\varrho} X$ (resp. $H \xrightarrow{H \int} \text{Int} H \xrightarrow{\varrho} X$).
Figure 3: Four projections in $H$

2. There exist left and right $\text{Int}_H$-based integrals, $\int H$ and $\int^H$, such that any left (resp. right) $X$-based integral in $H$ admits a unique factorization of the form $X \xrightarrow{h} \text{Int}_H \xrightarrow{\int H} H$ (resp. $X \xrightarrow{h} \text{Int}_H \xrightarrow{\int^H} H$).

For proofs of this and the following results, see Section 4.

It follows immediately from the universality of integrals that the object $\text{Int}_H$ is defined uniquely up to a unique isomorphism.

Notice that $\int_H$ is a coequalizer for a pair of left $\vee H$-actions on $H$, given by the formulae:

$$
\mu' = (\text{ev} \otimes \text{id}_H) \circ (\text{id}_{\vee H} \otimes \Delta_H) = \begin{array}{cc}
\vee H & H \\
H & H
\end{array}, \quad \mu'' = \varepsilon_{\vee H} \otimes \text{id}_H = \begin{array}{c}
\vee H H \end{array}.
$$

Concretely, this means that $\int_H$ is a universal morphism for which

$$
(\vee H \otimes H \xrightarrow{\mu'} H \xrightarrow{\int H} \text{Int}_H) = (\vee H \otimes H \xrightarrow{\mu''} H \xrightarrow{\int H} \text{Int}_H).
$$

Also the right integral $\int^H$ on $H$ can be identified as a coequalizer for a similar pair of actions. The integrals in $H$ are equalizers in a dual fashion. In the case of abelian categories this property served as the definition of integrals, see [19]. In order to prove the existence of integrals in the case of a category with split idempotents (Proposition 3.1), we construct the following four idempotents (the next lemma will be derived as a special case of Corollary 4.4):

**Lemma 3.2.** The endomorphisms $\Pi_1^\vee(H), \Pi_2^\vee(H), \Pi_1^\vee(H), \Pi_2^\vee(H) : H \to H$ given in Figure 3 are idempotents in $\text{End}_C H$.

The morphisms which split these idempotents are precisely the integrals we are looking for. Some of their properties are described in the following theorem.
Theorem 3.3. The morphisms $\int_H \circ \int^H$, $\int_H \circ H \int$, $H \int \circ \int^H$, $H \int \circ H \int \in \text{End}(\text{Int } H)$ are invertible, and, hence, by Lemma 2.1

\[
\int_H \circ \int^H = c^H_\ell \cdot \text{id}_{\text{Int } H}, \quad \int_H \circ H \int = c^H_r \cdot \text{id}_{\text{Int } H},
\]

\[
H \int \circ \int^H = c^H_\ell \cdot \text{id}_{\text{Int } H}, \quad H \int \circ H \int = c^H_r \cdot \text{id}_{\text{Int } H}
\]

for certain invertible $c^H_\ell$, $c^H_r$, $c^H_\ell$, $c^H_r \in \text{End}(\mathbf{1})$.

The idempotents from Lemma 3.2 are $H$-valued and $H$-based integrals. They are split by Int $H$ as follows:

\[
(c^\ell_\ell)^{-1} \cdot \int^H \circ H \int = \int^\ell \pi^\ell(H)
\]

is a right $H$-valued integral on $H$

and a left $H$-based integral in $H$,

\[
(c^\ell_r)^{-1} \cdot H \int \circ \int_H = \int^\ell \pi^r(H)
\]

is a left $H$-valued integral on $H$

and a right $H$-based integral in $H$,

\[
(c^r_\ell)^{-1} \cdot \int^H \circ \int_H = \int^r \pi^\ell(H)
\]

is a left $H$-valued integral on $H$

and a left $H$-based integral in $H$,

\[
(c^r_r)^{-1} \cdot H \int \circ H \int = \int^r \pi^r(H)
\]

is a right $H$-valued integral on $H$

and a right $H$-based integral in $H$.

For any of the idempotents $\int^\ell \pi^\ell(H)$ we have

\[
\text{tr}_S \left( \int^\ell \pi^\ell(H) \right) = \dim_S(\text{Int } H).
\]

The following lemma is a useful tool for recognizing the object Int $H$ of the integrals as the object of non-degenerate pairings of a special form.

Lemma 3.4 ([14]). Suppose there is an invertible object, $K$, and a morphism, $t : H \to K$, such that the pairing $\phi : H \otimes H \overset{\alpha}{\longrightarrow} H \overset{t}{\longrightarrow} K$ is side-invertible (the induced morphisms $H \to K \otimes \epsilon H$ and $H \to H^\vee \otimes K$ are invertible). Then Int $H \simeq K$.

The generalized Radford formula

In the following lemma group-like elements, $a$ and $\alpha$, are extracted, which characterize the, e.g., right integral as a homomorphism with respect to a left regular action. These special elements are essential ingredients in the generalized Radford formula.

Lemma 3.5. There exists a unique group-like element, $a : \mathbf{1} \to H$, and a unique multiplicative functional, $\alpha : H \to \mathbf{1}$, such that the following identities hold:

\[
(\int_H \otimes \text{id}_H) \circ \Delta = \int_H \otimes a \quad : H \to \text{Int } H \otimes H, \quad (3.1)
\]

\[
(\text{id}_H \otimes H \int) \circ \Delta = a^{-1} \otimes \int_H \quad : H \to H \otimes \text{Int } H, \quad (3.2)
\]

\[
\mu \circ (\int^H \otimes \text{id}_H) = \int^H \otimes \alpha \quad : \text{Int } H \otimes H \to H, \quad (3.3)
\]

\[
\mu \circ (\text{id}_H \otimes H \int) = \alpha^{-1} \otimes H \int \quad : H \otimes \text{Int } H \to H. \quad (3.4)
\]
I.e. integrals are (co)module morphisms between $H$ with the regular structure and $\text{Int} H$ with the structure determined by the group-like element $a$ (multiplicative functional $\alpha$).

We shall use the following notations for morphisms in $\text{End}(H)$:

$$
ad_a := \mu^{(3)} \circ (a \otimes \text{id}_H \otimes a^{-1}), \quad \text{ad}^\alpha := (\alpha \otimes \text{id}_H \otimes a^{-1}) \circ \Delta^{(3)},$$

where

$$
\mu^{(3)} := \mu(\mu \otimes \text{id}_H) = \mu(\text{id}_H \otimes \mu), \quad \Delta^{(3)} := (\Delta \otimes \text{id}_H)\Delta = (\text{id}_H \otimes \Delta)\Delta.
$$

Given this the precise form of our main result is as follows:

**Theorem 3.6 (The generalized Radford formula).**

$$
S^4 \circ u_{-2}^0 = (\text{ad}^\alpha)^{-1} \circ \text{ad}_a^{-1} \circ \Omega_{H}^{(\text{Int} H)} \in \text{Aut}(H),
$$

where the monodromy action $\Omega^K$ of an invertible object, $K$, is defined in Lemma 2.4, and the morphism $u_{-2}^0$ is defined on Figure 1.

**Remark 3.1.** The components in the generalized Radford formula, namely, $S^4 \circ u_{-2}^0$, $\text{ad}^\alpha$, $\text{ad}_a$, and $\Omega_{H}^{(\text{Int} H)}$, are Hopf algebra automorphisms, commuting with each other. See also Corollary 2.5.

The first proof of Theorem 3.6 is given in Section 4. It follows the scheme of the original proof given by Radford [26] for usual Hopf algebras, and adapted by Kuperberg [15] to the case of symmetric rigid monoidal categories. We use several presentations of the antipode and the inverse antipode via integrals.

The second proof of Theorem 3.6, given in Section 7, is based on the properties of Hopf bimodules. The generalized Radford formula follows from the equivalence of two presentations of the Hopf $H^\vee$-bimodule structure on $X \otimes_H H^\vee$, where $X$ is a Hopf $H$-bimodule.

In the case of usual Hopf algebras the object of the integrals is a one-dimensional vector space. In the case of $\mathbb{Z}/2$-graded Hopf algebras the object of the integrals can be odd or even. In braided categories there are more possibilities as the following example shows.

**Example 3.1 (Non-trivial object of integrals).** Let $\mathcal{C} = \mathbb{Z}/n$-grad-$\text{Vect}(k)$ be the category of $\mathbb{Z}/n$-graded $k$-vector spaces with the braiding $\psi : V \otimes W \to W \otimes V$, given by $\psi(v_i \otimes w_j) = q^{ij}w_j \otimes v_i$ for homogeneous vectors, $v_i \in V^i$ and $w_j \in W^j$. Here $q \in k$ is a primitive $n$th root of unit (a root of the $n$th cyclotomic polynomial $\phi_n$). Notice that the category $\mathcal{C}$ is also ribbon with respect to the ribbon twist $\nu : V \to V$, where $\nu(v_i) = q^i v_i$, for $v_i \in V^i$.

Denote by $X = X^1$ a selected one-dimensional vector space concentrated in degree 1. The tensor algebra $T(X)$, equipped with a comultiplication induced by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for a basis vector, $x \in X^1$, constitutes a braided Hopf algebra in the braided category $\mathcal{C}$.
Applying the bialgebra axiom, which contains the braiding $\psi$, we find iteratively the comultiplication and the antipode as given next:

$$\Delta(x^m) = \sum_{k=0}^{m} \binom{m}{k}_q x^k \otimes x^{m-k},$$

$$S(x^m) = (-1)^m q^{m(m-1)/2} x^m.$$

For any $0 < k < n$ the $q$-binomial coefficient $\binom{n}{k}_q$ is divisible by $\phi_n(q)$. Therefore, $\Delta x^n = x^n \otimes 1 + 1 \otimes x^n$, and the span of $x^n$ is a coideal. Hence, $H = T(X)/(X^{\otimes n})$ is a rigid Hopf algebra in $C$.

Using the definition of integrals one easily verifies that $H \int = \int H : H \to X^{\otimes n-1} : x^k \mapsto \delta_{n-1}^k x^{\otimes n-1}$, is a two-sided $X^{\otimes n-1}$-valued integral on $H$, and $H \int = \int H : X^{\otimes n-1} \to H : x^{\otimes n-1} \mapsto x^{n-1}$, is a right and left $X^{\otimes n-1}$-based integral in $H$. Since $X^{\otimes n-1}$ is one-dimensional, the above integrals are universal and $\operatorname{Int} H = X^{\otimes n-1} \simeq X^\vee$. In particular, $(\operatorname{Int} H)^{\otimes n} \simeq I$.

One concludes that all idempotents $\Pi^\vee$ coincide and are given by $x^k \mapsto \delta_{n-1}^k x^{n-1}$.

The element $a$ is the unit, the functional $\alpha$ is the counit. The monodromy $\Omega^{\operatorname{Int} H}_V$ determined by $\operatorname{Int} H$ on $V$ is defined on homogeneous vectors by $\Omega^{\operatorname{Int} H}_V(v_k) = q^{-2k} v_k$. Since $(u_{-2}^0)_V v_k = \nu_V^{-2} v_k = q^{-2k^2}$, the map on the right-hand side of the generalized Radford formula turns out to be $\Omega^{\operatorname{Int} H}_V(x^k) = q^{-2k} x^k$, and on the left-hand side the same map results from $(S^4 \circ u_{-2}^0)(x^k) = q^{2k(k-1)} . q^{-2k^2} x^k$.

### 4 Proofs of the main results

The proof of Theorem 3.3 we will follow in this section relies heavily on the theory of Hopf modules, which will be discussed in the first paragraph. The projections $\Pi^\vee$ are explicitly constructed, and their images identified as integrals. In the following paragraph we prove the invertibility of the universal object of the integrals, permitting the conclusion of the proof of Theorem 3.3. In subsequent parts of this section we discuss the ingredients of the generalized Radford formula, namely the special group-like elements, and the action of the antipode on the integrals. With this preparation the proof of the formula for the forth order of the antipode is finally only a matter of combining several identities in the right way.

**Hopf modules**

From [19] we know that the Structure Theorem of Hopf modules [29] also holds for Hopf algebras in braided, monoidal categories, if there exist (co-)equalizers. In [5, 20] it is shown that in order to prove the Structure Theorem in a braided monoidal category, $C$, it suffices to assume that $C$ admits split idempotents.

In a monoidal category, $D$, which admits split idempotents the subcategory of (co-)modules over a (co-)algebra admits split idempotents, too. Similar facts hold for the categories of crossed
modules and categories of Hopf (bi-)modules over a bialgebra in a braided, monoidal category with split idempotents.

**Definition 4.1.**

1. A left Hopf module, $X$, (resp. right-left Hopf module) over a bialgebra, $H$, in $C$ is a left (resp. right) $H$-module and a left $H$-comodule, such that the action is a comodule morphism. Here the (co-)actions of $H$ on $X \otimes H$ and $H \otimes X$ are determined by the braided, diagonal tensor (co-)action, given that $H$ is an $H$-(co-)module via the regular (co-)action. As illustrated in Figure 4a) the axiom implied by this condition can be thought of as a “polarized” version of the bialgebra, which has to hold besides the comodule and module axioms for $X$. We shall denote by $H^H_C$ (resp. $H_C^H$) the subcategory of Hopf modules, whose objects are the left (resp. right-left) Hopf modules, and whose morphisms are the left-(resp. right-) $H$-module-left-$H$-comodule homomorphisms. The definitions for $C^H_H$ and $H^C_H$ are analogous.

2. A two-fold Hopf module, $X = (X, \mu_\ell, \mu_r, \Delta_\ell)$, is an object, which is a $H$-bimodule in the category of left $H$-comodules, or in the language of Hopf modules: $X \in \text{Obj}(H^H_C)$ and $X \in \text{Obj}(H_C^H)$. The two-fold Hopf modules together with the $H$-bimodule-left-$H$-comodule morphisms form the category of two-fold Hopf modules $H^H_C^H$. The remaining three types of two-fold Hopf modules are defined in similar ways.

3. An object, $(X, \mu_r, \mu_\ell, \Delta_r, \Delta_\ell)$, is called an $H$-Hopf bimodule if $(X, \mu_r, \mu_\ell)$ is a $H$-bimodule, and $(X, \Delta_r, \Delta_\ell)$ is a $H$-bicomodule in the category of $H$-bimodules, where the regular (co-)action on $H$ and the diagonal (co-)action on tensor products of modules are used. Hopf bimodules together with the $H$-bimodule-$H$-bicomodule morphisms form the category which will be denoted by $H^H_C^H_H$.

It follows directly from the definition that any bialgebra, $H$, is a Hopf bimodule over itself with the regular actions and the regular coactions. Next we give the construction of the morphism $\Pi_\ell(X) : X \to X$. Idempotency follows from straightforward application of the Hopf algebra and Hopf module axioms as depicted in Figure 4. In the same way the object, through which $\Pi_\ell(X) : X \to X$ factors, is identified as a (co)invariance.

**Lemma 4.1.** Let $H$ be a Hopf algebra in $C$ and $(X, \mu_\ell, \Delta_\ell)$ be a left $H$-Hopf module. Then the following hold:

- The morphism $\Pi_\ell(X) : X \to X$ defined via
  \[
  \Pi_\ell(X) := \mu_\ell \circ (S \otimes \text{id}_X) \circ \Delta_\ell \quad (4.1)
  \]
  (see also Figure 4b)) is an idempotent in $\text{End}_C(X)$. 

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a) Hopf module axioms

\[ \Pi^\ell_\ell(X) := \frac{1}{2} \sum_{x \in X} s(x) \]

\[ \Pi^r_\ell(X) := \frac{1}{2} \sum_{x \in X} s(x^{-1}) \]

\[ \Pi^\ell_r(X) := \frac{1}{2} \sum_{x \in X} s(x) \]

\[ \Pi^r_r(X) := \frac{1}{2} \sum_{x \in X} s(x^{-1}) \]

b) Idempotents \( \Pi^\bullet(X) \)

\[ \Pi^\ell_\ell(X) := \frac{1}{2} \sum_{x \in X} s(x) \]

\[ \Pi^r_\ell(X) := \frac{1}{2} \sum_{x \in X} s(x^{-1}) \]

\[ \Pi^\ell_r(X) := \frac{1}{2} \sum_{x \in X} s(x) \]

\[ \Pi^r_r(X) := \frac{1}{2} \sum_{x \in X} s(x^{-1}) \]

Figure 4: Hopf modules and projections
Let $X \xrightarrow{xp} HX \xrightarrow{x_i} X$ be the morphisms, which split the idempotent $\Pi^\ell_f(X)$, i.e. $x_i \circ xp = \Pi^\ell_f(X)$ and $xp \circ x_i = \text{id}_{HX}$. Then

$$HX \xrightarrow{x_i} X \xrightarrow{\Delta_i} H \otimes X \quad \text{and} \quad H \otimes X \xrightarrow{\mu_i} X \xrightarrow{xp} HX$$

are equalizer and coequalizer respectively. Hence $HX$ is at the same time an object of invariants and coinvariants of $X$.

The following lemma embodies the categorical generalization of the Fundamental Theorem of Hopf modules, which can be thought of as the main reason for the structural rigidity of Hopf algebras. In the classical case it essentially states that all Hopf modules are free. The version below can be derived by translating the classical proof into the diagrammatic language, and generalizing from there to the braided situation.

**Lemma 4.2.** The following functors define an equivalence of categories, $C \xrightarrow[H \otimes (\_)]{H \ltimes (\_)} H \otimes C$ :

- $H \ltimes X$ is an object $H \otimes X$ with Hopf module structure given by $\Delta_\ell := \Delta \otimes \text{id}_X$ and $\mu_\ell := \mu \otimes \text{id}_X$, called standard Hopf module, and $H \ltimes f := \text{id}_H \otimes f$ as morphism in $C$;
- $HX$ splits the idempotent $\Pi^\ell_f(X)$ from the previous lemma, and $Hf := \gamma p \circ f \circ x_i$ for a Hopf module morphism $f : X \to Y$;
- For $(X, \mu_\ell, \Delta_\ell) \in \text{Obj}(\frac{H \otimes (\_)}{H \ltimes (\_)})$

$$H \otimes (HX) \xrightarrow{\mu_\ell \circ (\text{id}_H \otimes x_i)} X$$

are mutually inverse Hopf module morphisms.

Our strategy for extending the result from the previous two lemmas for $\Pi^\ell_f(X)$ to the other projections is to find appropriate functors that turn left actions into right actions and map $\Pi^\ell_f(X)$ to the respective $\Pi^\bullet_f(X)$. To this end suppose $X$ is an object in $C$ equipped with some or all of the $H$-module and $H$-comodule structures from above, namely $\mu_\ell$, $\mu_\ell$, $\Delta_\ell$, and $\Delta_\ell$.

We shall introduce the following notations for the same object $X$ with modified (co)actions:

- $X^{[\text{op}]}$ denotes the underlying (bi)comodule with opposite $H^{\text{op}}$-actions $\mu_\ell^{[\text{op}]} := \mu_\ell \circ \Psi^{-1}$, and $\mu_\ell^{[\text{op}]} := \mu_\ell \circ \Psi^{-1}$. $X^{[\text{op}]}$ denotes the underlying (bi)module with opposite $H^{\text{op}}$-coactions $\Delta_\ell^{[\text{op}]} := \Psi^{-1} \circ \Delta_\ell$, and $\Delta_\ell^{[\text{op}]} := \Psi^{-1} \circ \Delta_\ell$;
- $X^{[S]}$ denotes the underlying (bi)comodule with $H^{\text{op}}$-actions $\mu_\ell^{[S]} := \mu_\ell \circ (S \otimes \text{id})$, and $\mu_\ell^{[S]} := \mu_\ell \circ (\text{id} \otimes S)$. $X^{[S]}$ denotes the underlying (bi)module with $H^{\text{op}}$-coactions $\Delta_\ell^{[S]} := (S \otimes \text{id}) \circ \Delta_\ell$, and $\Delta_\ell^{[S]} := (\text{id} \otimes S) \circ \Delta_\ell$. 

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• For \((X, \mu_\ell, \Delta_\eta)\) (resp. \((X, \mu_r, \Delta_\iota)\)) we put \(X^\vee := (X, \mu_\ell^\vee, \Delta_\iota^\vee)\) (resp. \(\vee X := (X, [\mu_r^\vee], [\Delta_\eta^\vee])\)) to be the object equipped with the \(H^\vee\) (resp. \(\vee H\)) (co)actions as depicted below:

\[
\begin{align*}
[\mu_\ell^\vee] & = \begin{array}{c}
\includegraphics{leaves2}\end{array}, & \mu_\ell & = \begin{array}{c}
\includegraphics{leaves2}\end{array}, & \Delta_\iota^\vee & = \begin{array}{c}
\includegraphics{leaves2}\end{array}, & \vee X & = \begin{array}{c}
\includegraphics{leaves2}\end{array} \\
\end{align*}
\]

Hence the so defined operations \((\_)[op], (\_)[S], (\_)[\vee], (\_)[\vee], (\_)[\vee], (\_)[\vee]\), and \(\vee(\_)[\_]\), identify, e.g. for \((\_)[op]\), a module-object \(X \in \text{Obj}(\mathcal{C})\) for a Hopf algebra \(H\) to be also a comodule in \(X \in \text{Obj}(\mathcal{T})\) for \(H[op]\), where \(\mathcal{T}\) is the same monoidal category as before but with mirrored braiding. It is easy to see that, similarly, an \(H\)–module morphism in \(\mathcal{C}\) is at the same time an \(H[op]\)–comodule morphism in \(\mathcal{T}\) with the modified coaction as above. Thus in this way we obtain functors between the (co)module subcategories of \(\mathcal{C}\) as specified next. As functors of only the monoidal category \(\mathcal{C}\) they act like identity on objects and morphisms.

**Lemma 4.3.** The following functors define equivalences of categories of Hopf (bi)modules:

\[
\begin{align*}
(\_)[op] : & \quad H^H_H^H \rightarrow H^\text{op}_\text{H}^\text{op}_\text{H}^\text{op}, & (\_)[op] : & \quad H^H_H^H \rightarrow H^\text{op}_\text{H}^\text{op}_\text{H}^\text{op}, \\
(\_)[S] : & \quad H^H_H^H \rightarrow (H^\text{op}_\text{H}^\text{op}_\text{H}^\text{op})^\text{op}_\text{H}^\text{op}_\text{H}^\text{op}, & (\_)[S] : & \quad H^H_H^H \rightarrow (H^\text{op}_\text{H}^\text{op}_\text{H}^\text{op})^\text{op}_\text{H}^\text{op}_\text{H}^\text{op}, \\
(\_)[\vee] : & \quad H^H_H^H \rightarrow (\vee H^\text{op}_\text{H}^\text{op}_\text{H}^\text{op})^\text{op}_\text{H}^\text{op}_\text{H}^\text{op}, & (\_)[\vee] : & \quad H^H_H^H \rightarrow (\vee H^\text{op}_\text{H}^\text{op}_\text{H}^\text{op})^\text{op}_\text{H}^\text{op}_\text{H}^\text{op}, \\
(\_)[\vee '[S] : & \quad H^H_H^H \rightarrow (\vee H^\text{op}_\text{H}^\text{op}_\text{H}^\text{op})^\text{op}_\text{H}^\text{op}_\text{H}^\text{op}, & (\_)[\vee '][S] : & \quad H^H_H^H \rightarrow (\vee H^\text{op}_\text{H}^\text{op}_\text{H}^\text{op})^\text{op}_\text{H}^\text{op}_\text{H}^\text{op},
\end{align*}
\]

For any functor from the above list with the source category \(H^H_H^H\) we keep the same notation for other functors, which act by the same rule on categories of Hopf modules or two-fold Hopf modules. We now find the desired results for the morphisms \(\vee \Pi^\vee(X)\):

**Corollary 4.4.**

For a Hopf bimodule, \(X\), the endomorphisms \(\vee \Pi^\vee(X)\) and \(\vee \Pi^\vee(X)\), as given in Figure 4, are idempotents.

The objects that split idempotents \(\vee \Pi^\vee(X)\) (resp. \(\vee \Pi^\vee(X)\)) are all isomorphic. A representative of the corresponding class of isomorphic objects shall be denoted by \(\text{Inv} X\) (resp. by \(\text{Int} X\)).

**Proof.** All assertions about the endomorphisms can be reduced to the ones for the idempotent from Lemma 4.3 using the identities below:

\[
\begin{align*}
\vee \Pi^\ell(X) = & \quad \Pi^\ell(X[\text{op}]), & \vee \Pi^\ell(X) = & \quad \Pi^\ell(X[\text{op}]), \\
\vee \Pi^\ell(X) = & \quad \Pi^\ell(X[\text{op}][S][\text{op}]), & \vee \Pi^\ell(X) = & \quad \Pi^\ell((X[S][\text{op}]).
\end{align*}
\]

Let \(X \xrightarrow{\eta} Y \xrightarrow{id} X\) (resp. \(X \xrightarrow{\eta'} Y \xrightarrow{id'} X\)) be morphisms, which split the idempotent \(\Pi^\ell(X)\) (resp. \(\Pi^\ell(X)\)). Then by Lemma 4.3 both morphisms \(i\) and \(i'\) are equalizers of the pair \(\Delta_\ell, \eta \otimes \text{id}_X : X \rightarrow H \otimes X\). Thus by the universal property of equalizer we have \(Y \simeq Y'\). □
Figure 5: Hopf module structures over the dual Hopf algebra
Other functors can be obtained as compositions of the functors from Lemma 4.3. For a Hopf $H$-bimodule $X$ we consider the underlying object equipped with (co)module structures over the dual Hopf algebra $H^\vee$ or $^\vee H$ as shown in Figure 3.

**Corollary 4.5.** The following functors, whose actions on objects are as in Figure 3 and which are identity on morphisms, define the following equivalences of categories:

\[
\begin{align*}
\bullet(\_): & \ H^C H \rightarrow (H^\vee)^C(H^\vee), \\
\bullet(\_): & \ H^C H \rightarrow (H^\vee)^C(H^\vee), \\
\bullet(\_): & \ H^C H \rightarrow \vee H C_H, \\
\bullet(\_): & \ H^C H \rightarrow \vee H C_H, \\
\bullet(\_): & \ HC_H \rightarrow (\vee H)^C(\vee H), \\
\bullet(\_): & \ HC_H \rightarrow (\vee H)^C(\vee H), \\
\bullet(\_): & \ HC_H \rightarrow H^\vee C_H^\vee, \\
\bullet(\_): & \ HC_H \rightarrow H^\vee C_H^\vee.
\end{align*}
\]

**Proof.** The functors $(\_\bullet)$ between two-fold Hopf modules are “glued” from the following composite functors between Hopf modules:

\[
\begin{align*}
H^C_H & \xrightarrow{(\omega[\text{op}])} (H^\vee)^C(H^\vee) \\
H^C_H & \xrightarrow{(\omega[\text{op}])} (H^\vee)^C(H^\vee) \\
H^C_H & \xrightarrow{(\omega[\text{op}])} (H^\vee)^C(H^\vee) \\
(\omega)^C(H^\vee) & \xrightarrow{(\omega)^C(H^\vee)} (H^\vee)^C(H^\vee) \\
(\omega)^C(H^\vee) & \xrightarrow{(\omega)^C(H^\vee)} (H^\vee)^C(H^\vee) \\
H^C_H & \xrightarrow{(\omega)^C(H^\vee)} (H^\vee)^C(H^\vee) \\
(\omega)^C(H^\vee) & \xrightarrow{(\omega)^C(H^\vee)} (H^\vee)^C(H^\vee) \\
H^C_H & \xrightarrow{(\omega)^C(H^\vee)} (H^\vee)^C(H^\vee) \\
(\omega)^C(H^\vee) & \xrightarrow{(\omega)^C(H^\vee)} (H^\vee)^C(H^\vee).
\end{align*}
\]

Other functors are reversed forms of $(\_\bullet)$. \hfill \Box

Note that $X_\bullet$ is not a Hopf bimodule. The left-right Hopf module axiom is not satisfied. In Section 3 we will introduce a construction, which turns $X_\bullet$ into a Hopf $H^\vee$-bimodule.

**Invertibility of the object of integrals**

For the Hopf bimodule $H$, equipped with the regular action and coaction, we consider the underlying left Hopf $H^\vee$-module of $H_\bullet$ (the explicit structures are shown in Figure 3)). By Lemma 4.3 there exists an isomorphism, $\mathcal{F}^H : H_\bullet \rightarrow H^\vee \otimes \text{Int} H$, into a standard Hopf $H^\vee$-module presented in Figure 3. Similarly, we can construct Hopf module isomorphisms corresponding to three other types of integrals. The inverse morphisms are given by the formulae

\[
\begin{align*}
c^\ell(F^H)^{-1} &= \mathcal{F}^H := H^\mathcal{F} \circ \Psi_{H^\vee, \text{Int} H} \circ (S_{H^\vee} \otimes \text{id}_{\text{Int} H}) \\
c^r(F^H)^{-1} &= H^\mathcal{F} := \mathcal{F}^H \circ \Psi_{\text{Int} H, H^\vee} \circ (\text{id}_{\text{Int} H} \otimes S_{H^\vee}) \\
c^r(F^H)^{-1} &= \mathcal{F}^H := (S_{H^\vee} \otimes \text{id}_{\text{Int} H}) \circ \Psi_{\text{Int} H, H^\vee} \circ H^\mathcal{F} \\
c^\ell(F^H)^{-1} &= H^\mathcal{F} := (\text{id}_{\text{Int} H} \otimes S_{H^\vee}) \circ \Psi_{H^\vee, \text{Int} H} \circ H^\mathcal{F}.
\end{align*}
\]

In the case of self-dual Hopf algebras and *bosonic* integrals, meaning $\text{Int} H \simeq \mathbb{1}$, these isomorphisms turn into Fourier transforms \cite{19}. The braided Fourier transform $H \rightarrow H \otimes ^\vee H$ defined in \cite{2} (where the $H$-valued integral is used) factorizes over our $H^\mathcal{F}$. Given these isomorphisms we shall now prove Proposition 3.1 and Theorem 3.3.
Proof of Proposition \[\ref{prop:invertibility}\.\] In order to prove invertibility of \(\text{Int} \ H\), we consider the Hopf module isomorphisms \(\mathcal{F}^H : H_\bullet \rightarrow H^\vee \otimes \text{Int} \ H\) in \(H^\vee \mathcal{C}\) and \(\overline{\mathcal{F}}_\text{(Hop)} : ((H_\text{Hop})^\vee)^\bullet \rightarrow (H_\text{Hop}) \otimes \text{Int} H^\vee\) in \(H^\text{op}_\text{Hop} \mathcal{C}\). The explicit Hopf module structures on \(H_\bullet\) and \((H_\text{Hop})^\bullet\) are shown in Figure \[\ref{fig:hopf_module Structures}\].

Let us consider the functor \((\mathcal{F}^\vee)((\cdot)_{\text{Hop}}))_{\text{Hop}} : H^\vee \mathcal{C} \rightarrow H^\text{op}_\text{Hop} \mathcal{C}\). The \(H^\text{op}_{\text{Hop}}\)-(co)module structures on \((\mathcal{F}^\vee((\cdot)_{\text{Hop}}))_{\text{Hop}}\) are shown on Figure \[\ref{fig:Hopf_module Structures}\.\) This functor converts \(H_\bullet\) into the regular Hopf \(H^\text{op}_{\text{Hop}}\)-module, and the regular Hopf module \(H^\vee\) into \(((H_\text{Hop})^\vee)^\bullet\). Hence the composition

\[H \xrightarrow{\mathcal{F}^H} H^\vee \otimes \text{Int} H \xrightarrow{\overline{\mathcal{F}}_\text{(Hop)} \otimes \text{id}(\text{Int} \ H)} H \otimes H^\vee \otimes \text{Int} \ H\]

is an isomorphism of standard Hopf modules in \(H^\text{op}_\text{Hop} \mathcal{C}\). From this we conclude that

\[(\varepsilon_H \otimes \text{id}(\text{Int} H^\vee \otimes \text{Int} H)) \circ (\overline{\mathcal{F}}_\text{(Hop)} \otimes \text{id}(\text{Int} \ H)) \circ \mathcal{F}^H \circ \eta_H = (\mathcal{F}^H \circ \overline{\mathcal{F}}_\text{(Hop)} \otimes \text{id}(\text{Int} \ H)) \circ (\varepsilon_H \otimes \text{id}(\text{Int} \ H)) \circ \eta_H\]

is an isomorphism.

\[\square\]

Proof of Theorem \[\ref{thm:invertibility}\.\] Let \(H \mathcal{I} : H \rightarrow \text{Int} H\) be a right integral on \(H\), and \(H \mathcal{J} : \text{Int} H \rightarrow H\) be a left integral in \(H\). By the universal property of integrals there exist isomorphisms, \(f : K \rightarrow \text{Int} H\) and \(g : \text{Int} H \rightarrow K\), such that \(f \mathcal{I} = fg = \mathcal{I}\) and \(H \mathcal{J} = ig \mathcal{I} f\). Here \(i\) and \(p\) are as before the morphisms splitting for example the idempotent \(I^\vee H\) with object \(K\). Then

\[H \mathcal{I} \circ H \mathcal{J} = f \mathcal{I} g = \mathcal{I} f = c_{\mathcal{I}} \cdot \text{id}(\text{Int} \ H)\]

is an isomorphism, and we find

\[(c_{\mathcal{I}})^{-1} \cdot H \mathcal{I} \circ H \mathcal{J} = f \mathcal{I} \circ (\mathcal{I} f)^{-1} \circ H \mathcal{I} = ig(f g)^{-1} \mathcal{I} f = ip = \mathcal{I}\]

We also compute,

\[\text{tr}_8(\mathcal{I}^\vee H) = \text{tr}_8(H \mathcal{I} \circ (H \mathcal{J} \circ f)^{-1} \circ H \mathcal{I}) = \text{tr}_8(H \mathcal{I} \circ (H \mathcal{J} \circ f)^{-1} \circ H \mathcal{I}) = \text{tr}_8(\text{id}(\text{Int} \ H)) = \text{dim}_8(\text{Int} \ H)\].

\[\square\]
Remark 4.1. Applying the functor \(-^\vee\) to left or right integrals on \((in)\ H\), we get corresponding right or left integrals in \((on)\ H^\vee\). Hence, the four natural pairings of the form \(\text{Int } H \otimes \text{Int } H^\vee \rightarrow \mathbb{I}\) and the four natural copairings \(\mathbb{I} \rightarrow H^\vee \otimes H \xrightarrow{\int \otimes \int} \text{Int } H^\vee \otimes \text{Int } H\) are isomorphisms.

**Group-like elements**

Let us now give the derivation of the special group-like elements (or moduli) exploiting universality properties of integrals, and discuss their properties as needed in the Radford formula.

**Proof of Lemma 3.3**. The morphism \((\int_H \otimes \text{id}_H) \circ \Delta\) is a left \((\text{Int } H \otimes H)\)-valued integral on \(H\). Indeed it is easily found from basic Hopf algebra axioms, that if \(g : H \rightarrow X\) is a left integral, then the composition \(H \xrightarrow{\Delta} H \otimes H \xrightarrow{g \otimes H} X \otimes H\) is a left integral as well. By the universal property for integrals there exists a unique morphism, \(\Delta_r : \text{Int } H \rightarrow \text{Int } H \otimes H\), such that the following diagram is commutative:

\[
\begin{array}{ccc}
H & \xrightarrow{\Delta} & H \otimes H \\
\int_H \downarrow & & \downarrow \int_H \otimes \text{id}_H \\
\text{Int } H & \xrightarrow{\Delta_r} & \text{Int } H \otimes H
\end{array}
\]

For the same reason there exists a unique morphism, \(f\), which makes the following diagram commutative:

\[
\begin{array}{ccc}
H & \xrightarrow{(\text{id}_H \otimes \Delta) \circ (\Delta \otimes \text{id}_H) \Delta} & H \otimes H \otimes H \\
\int_H \downarrow & & \downarrow \int_H \otimes \text{id}_{(H \otimes H)} \\
\text{Int } H & \xrightarrow{f} & \text{Int } H \otimes H \otimes H
\end{array}
\]
Given that both \((\text{id}_{\text{Int} H} \otimes \Delta) \Delta_r\) and \((\Delta_r \otimes \text{id}_{\text{Int} H}) \Delta_r\) fulfill this condition for \(f\), we infer from universality that \((\text{id}_{\text{Int} H} \otimes \Delta) \Delta_r = (\Delta_r \otimes \text{id}_{\text{Int} H}) \Delta_r\), i.e., that \(\Delta_r\) is a coaction. According to Lemma 2.1 any coaction, \(\Delta_r\), on an invertible object, \(\text{Int} H\), has the form \(\Delta_r = \text{id}_{\text{Int} H} \otimes a\) with a group-like morphism, \(a : \mathbb{1} \rightarrow H\).

Similarly, we find that \((\text{id}_H \otimes \int_H)b \circ \Delta = b \otimes H \int\) for some group-like \(b : \mathbb{1} \rightarrow H\). It is easy to check that \(\int_H \circ S\) is a right integral. Moreover, the coequalizer property for right integrals implies that there exists an automorphism, \(t : \text{Int} H \rightarrow \text{Int} H\), such that \(\int_H \circ S = t \circ H \int : \text{Int} H \rightarrow \text{Int} H\). Composing (3.1) with the antipode one deduces (3.2) for \(b = S^{-1}a = a^{-1}\).

Proposition 4.7.

\[ (c^f_c)^{-1} (a \cdot H \int) = (c^f_c)^{-1} \cdot H \int, \quad (c^f_c)^{-1} \left(\int_H \circ a\right) = (c^f_c)^{-1} \cdot H \int, \quad (c^f_c)^{-1} \left(\int_H \circ a\right) = (c^f_c)^{-1} \cdot H \int, \quad \]
\[ c^f_c c^r_c = \rho c^f_c c^r_c, \quad \text{where} \quad \rho := \alpha \circ a \in \text{Aut}(\mathbb{1}). \quad (4.4) \]

Proof. In the case of the left equation from (4.4), we can see from the diagrammatic calculation in Figure 8 where the third equality follows from (3.1), that the left hand side is also a right integrals on \(H\). Hence \(a \cdot H \int\) is “proportional” to \(\int_H \circ a\) in the sense of Proposition 3.1 by an element in in \(\text{Aut}(\text{Int} H)\). The fact that the latter is given by the indicated isomorphisms \(c \in \text{Aut}(\mathbb{1})\) is seen by composing both sides with \(a \circ H \int\). The remaining three relations between integrals are found similarly. Composition of these equalities with \(a \circ H \int\) and (3.4) yield (4.6). The scheme of this proof was taken from [26, 15].

For a fixed choice of \(\text{Int} H\) all integrals are defined uniquely up to multiplication by elements of \(\text{Aut}(\mathbb{1})\). The normalizations can be chosen, such that any three of the constants \(c^f_c\), \(c^f_c\), \(c^r_c\), and \(c^r_c\) are unit.

Integrals and the antipode

Before we consider the action of the antipode on integrals and idempotents, we need a presentation of \(S\) as given in the following two technical but easy lemmas:
Lemma 4.8. The maps

\[ b, p : \text{Hom}(H \otimes M, H \otimes N) \to \text{Hom}(H \otimes M, H \otimes N), \]

as defined in Figure 9 are inverse to each other.

Proof. Straightforward.

Lemma 4.9. The identities for the antipode of a braided Hopf algebra, \( H \), depicted in Figure 10 hold true.

Proof. This follows from the equation derived in Figure 11, to which we apply Lemma 4.8.

Besides the elements \( c \) we need another invertible element, \( \sigma \in \text{Aut}(I) \), which determines the self braiding of \( \text{Int} H \). It is defined by either of the following conditions, which are equivalent by Lemma 2.4 and Theorem 3.3:

\[ \Psi_{(\text{Int} H, \text{Int} H)} = \sigma \cdot \text{id}_{(\text{Int} H \otimes \text{Int} H)} \quad \text{or} \quad \sigma^{-1} = (\text{dim}_8(\text{Int} H)) = \text{tr}_8(\Pi^\flat(H)). \quad (4.7) \]

Using cyclicity, \( \text{tr}_8(fg) = \text{tr}_8(gf) \), one can rewrite the latter expression for \( \sigma \) in many equivalent forms. For example, we have \( \sigma^{-1} = \text{tr}_8(\Delta^{op} \circ \mu \circ (S^2 \otimes \text{id})) \). The action of \( S \) on the integrals is now as follows:
\[ S \otimes \text{id}_{\text{Int}_H} = (c^e_l)^{-1}. \]

Figure 10:

\[ \text{id}_{\text{Int}_H} \otimes S = (c^e_r)^{-1}. \]

Figure 11: Proof of Lemma 4.9
Proposition 4.10.

\[ S \circ H \int = \sigma \cdot c_r^\ell (c_r^\ell)^{-1} \cdot f^H, \quad S \circ H \int = \sigma \cdot c_r^\ell (c_r^\ell)^{-1} \cdot H \int, \quad \text{(4.8)} \]
\[ H \int \circ S = \sigma \cdot c_r^\ell (c_r^\ell)^{-1} \cdot \int H, \quad f_{H \int} \circ S = \sigma \cdot c_r^\ell (c_r^\ell)^{-1} \cdot H \int. \quad \text{(4.9)} \]

Proof. The composition of both sides of the first identity from Figure 10 with \( H \int \otimes \text{id}_{(\text{Int } H)} \) gives the first identity from (4.8). All other identities are found analogously.

Using Theorem 3.3 we then find immediately the following relations between the special idempotents and the antipode:

Corollary 4.11.

\[ S \circ \Pi^\ell (H) = \sigma \Pi^\ell (H) = \Pi^\ell (H) \circ S, \]
\[ S \circ \Pi^\ell (H) = \rho \sigma \Pi^\ell (H) = \Pi^\ell (H) \circ S, \]
\[ S \circ \Pi^\ell (H) = \rho \sigma \Pi^\ell (H) = \Pi^\ell (H) \circ S. \]

The idempotents \( \Pi^\ell (H), \Pi^\ell (H), \Pi^\ell (H), \) and \( \Pi^\ell (H) \) commute with \( S^2 \).

Proof of the generalized Radford formula

Proof of Theorem 3.6. The general scheme of our proof is taken from [26, 15]. The basic steps in the proof in the braided case are illustrated in Figures 12–14. The identity in Figure 12(a) is a corollary of the identities in Figure 10, rewritten for \( H^{\text{op}} \) and for \( H^{\text{op}} \).

The first identity in Figure 12(b) follows from the first or from the second identity in Figure 10, rewritten for \( H^{\text{op}} \) or \( H^{\text{op}} \) respectively. The second identity is the first one rewritten for \( (H^{\text{op}})^{\text{op}} \).

The first identity in Figure 12(c) is a result of combining the identities in Figure 10. The second is obtained by application of the identity in Figure 12(b).

In Figure 13 the letter \( f \) denotes a morphism in \( \text{End}(H^{\otimes 3}) \), presented by the framed diagram in Figure 12(c). The first two lines of the former figure are obtained, e.g., in a similar way as the derivation in Figure 8.

Denoting \( x := \text{ad}^a \circ S^4 \circ u_{0-2}^0 \circ \text{ad}_a \), we can rewrite the middle equation of Figure 13 in a symbolic form, using new morphisms, \( a, b, c, \) and \( d \), which can be visualized via Figure 12(c). The left hand side of the equation there can be transformed to the first line in Figure 14, where \( u_2^0 \) denotes the inverse to \( u_{-2}^0 \). Hence, the first equation in the last line of Figure 14 holds. Using Figure 12(b) we get the last equation of Figure 14.

Remark 4.2. Under the assumption that the antipode in \( H \) is an isomorphism, Proposition 3.1 implies that the object \( \text{Int } H \) is invertible. Conversely, if we suppose that the object \( \text{Int } H \), which splits the idempotent \( \Pi^\ell (H) \), is invertible, then \( S^{-1} \) exists and can be derived from the first identity in Figure 12(b).
Figure 12: Proof of the generalized Radford formula (part 1)
Figure 13: Proof of the generalized Radford formula (part 2)
Figure 14: Proof of the generalized Radford formula (part 3)
5 Examples of braided Hopf algebras

Let us furnish in this section the general theory around Radford’s formula that we have developed so far with a couple of prominent examples of braided, monoidal categories that are not abelian. The first is a variant of the category of surfaces and 3-cobordisms, for which the integrals are found even without the property of split idempotents. The second is any rigid, braided tensor category with split idempotents, in which the coend \( \int_{X \in \mathcal{C}} X \otimes X^\vee \) exists. We shall describe in both cases the Hopf algebras structure, integrals, and other elements explicitly. These two categories also figure prominently in topological quantum field theories, where the task is to functor one into the other.

A topological example of a Hopf algebra

Crane and Yetter have shown in [8] that a 3-dimensional topological quantum field theory assigns a braided Hopf algebra to a torus with one hole. Yetter has explained in [32] that a torus with one hole \( T \) is a Hopf algebra in a braided category of cobordisms between oriented connected surfaces with one hole. The 3-dimensional manifolds which are the structure morphisms of \( T \) are described in both articles [8, 32].

Independently Kerler [13] recovered this structure in a very similar category, \( \widetilde{\text{Cob}}_3(1) \), whose objects are specially selected standard surfaces, and the morphisms are homeomorphisms classes of triples \((M, \psi, \sigma)\), where \( M \) is a 3-cobordism, \( \psi \) the homeomorphisms between \( \partial M \) and the standard surfaces, and \( \sigma \) the signature of a four-manifold bounding \( M \) in a standard fashion. Using a somewhat different topological language, not only the usual braided tensor structure and the Hopf algebra structures associated to the one-holed torus are found, but in addition to the elements in [8, 32] also the cobordisms assigned to canonical Hopf pairings and the integrals are identified precisely in [13]. In particular, the integrals are directly interpreted as the algebraic elements intrinsically associated to elementary surgery. It is also realized in [13] that \( \widetilde{\text{Cob}}_3(1) \), restricted to connected surfaces, is generated by the morphisms of the braided Hopf algebra structure together with the additional elements. (The original generators from [8, 32] only produce cobordisms embeddable into \( \mathbb{R}^3 \)). I.e., a purely algebraic category can be defined that is freely generated by a Hopf algebra object, and which surjects onto \( \widetilde{\text{Cob}}_3(1) \). It is further conjectured in [13] that there is some such definition of the algebraic category, which is in fact isomorphic to \( \widetilde{\text{Cob}}_3(1) \) for connected surfaces.

In this section we shall first review the explicit presentations of the cobordisms that are used for integrals and other Hopf algebra structures using another category, \( TC \), of a topological and combinatorial nature, which is equivalent to \( \widetilde{\text{Cob}}_3(1) \), as proven by Kerler [12]. The category \( TC \) is a subquotient of the category \( RT \) of ribbon tangles in \([0, 1] \times \mathbb{R}^2\). Unlike the ordinary category of surfaces and 3-cobordisms, its central extension \( \widetilde{\text{Cob}}_3(1) \) admits interesting representations in braided categories built with the help of coends, as we will see in the second paragraph of this section.

Recall [31] that the objects of \( RT \) are non-negative integers, and the morphisms are ribbon...
tangles. For our purposes we define ribbon (or framed) tangles to be ambient isotopy classes of smooth embeddings of rectangles $[0,1] \times [0,1]$ and annuli $S^1 \times [0,1]$ into $[0,1] \times \mathbb{R}^2$, such that the edges $0 \times [0,1]$ and $1 \times [0,1]$ of the rectangles are attached to intervals on distinguished the lines $0 \times \mathbb{R} \times 0$ and $1 \times \mathbb{R} \times 0$.

Furthermore, the images of the rectangles shall be tangent to the strip $[0,1] \times \mathbb{R} \times 0$ at these intervals, and the induced isomorphism of the tangent plane to the rectangle and the tangent plane to $[0,1] \times \mathbb{R} \times 0$ shall preserve the orientations.

In the drawings of ribbon tangles below we will represent them only by threads, tacitly assuming the blackboard framing, and explicitly insert powers of $2\pi$-twists $\nu$ only when necessary. The tensor multiplication of objects is the addition of the non-negative integers. The unit object is $0$. The composition (resp. the tensor product) of morphisms is given by vertically stacking tangle diagrams (resp. horizontally juxtaposing tangle diagrams) \[\square\]. With these structures it is easily seen that $RT$ is a braided, rigid, monoidal category.

**Definition 5.1.** The category $TC$ of tangle-cobordisms is a subquotient of $RT$, whose objects are even, non-negative integers and morphisms of the form $2^n \rightarrow 2^m$ are equivalence classes of ribbon tangles obeying the following property $\dagger$:

Property $\dagger$: there are precisely $n + m$ rectangles, each of them connects two consecutive intervals of the same distinguished line, which is either $0 \times \mathbb{R} \times 0$ or $1 \times \mathbb{R} \times 0$.

A morphism in $TC$ can be thought of as an equivalence class of morphisms in $RT$. The corresponding equivalence relation is generated by the following moves. All of them are stable under composition and tensor product and preserve Property $\dagger$ so that $TC$ inherits these structures from $RT$.

\[
\begin{array}{c}
\text{TD1} \quad \ \ \\
\end{array}
\]

\[
\begin{array}{c}
\text{TD2} \quad \ \ \\
\end{array}
\]
The identity morphism is the tangle

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{identity_morphism.png}
\end{array}
\]

The braiding in \( TC \) is the braiding from \( RT \) composed on one or both sides with the identity morphism \((5.1)\) to ensure the condition \( \dagger \). See for example Figure \ref{fig:example}.

The evaluations and the coevaluations in \( TC \) are those in \( RT \) composed with the identity

A tangle \( 2n \rightarrow 2m \) represents a cobordism between surfaces of genus \( n \) and \( m \). Note also that \((\text{TD1})\) is the 2-handle-slide move of any ribbon segment over a 0-framed, closed, unknotted ribbon.
We assert that \(2 \in TC\) has the structure of a Hopf algebra. Indeed, a set of structure morphisms that fulfills all the axioms of a braided Hopf algebra is given if we choose for the multiplication and the comultiplication the following two diagrams,

\[
\Psi : 2 + 2 \to 2 + 2 =
\]

Figure 15: Braiding in \(TC\)

morphism (5.1), as depicted below for \(2 = 2^\vee\):
for the unit and the counit the ones depicted next,

\[ \eta : \emptyset \rightarrow 2 = \quad \text{and} \quad \varepsilon : 2 \rightarrow \emptyset = \]

and for the antipode and its inverse the pictures below.

\[ S = \quad \text{and} \quad S^{-1} = \]

The bialgebra axiom is verified by employing the handle-slide move (TD1). All other Hopf algebra axioms are proven straightforwardly.

The projections of Lemma 3.2 for this Hopf algebra are easily found to be the following tangles:

\[ \bar{\Pi}_l = \bar{\Pi}_r = \bar{\Pi}'_l = \bar{\Pi}'_r = \]

By Theorem 3.3 the form of these diagrams entails that the object \( \text{Int} \emptyset = \emptyset \) of integrals is the unit object, and that the integrals are given by the pieces

\[ f_2 = \emptyset f : 2 \rightarrow \emptyset \quad \text{and} \quad f^2 = \emptyset f : \emptyset \rightarrow 2 \]
The generalized Radford formula from Theorem 3.6 yields for the Hopf algebra $H = \mathbb{2}$ the identity

$$S^4 = u_2^0 = \nu^2.$$ 

In fact, one can prove directly the stronger result $S^2 = \nu : \mathbb{2} \to \mathbb{2}$.

Finally, let us mention that the surgery calculus from [12] implies the Fenn Rourke and also more general two-handle slides. Here the case where the closed ribbon, over which to slide, is still an unknot, but may have arbitrary framing:

**Proposition 5.1.**

*The handle-slide move with arbitrary framing shown on Figure 16 holds in $TC$.*

This follows from results of Kerler [12] and, alternatively, can be deduced directly from relations (TD1)–(TS3) in the category $TC$.

**Coends as Hopf algebras**

Let $(C, \otimes, \mathbb{1})$ be a braided rigid category with split idempotents. Assume that the coend $F = \int^{X \in C} X \otimes X^\vee$ exists in $C$. See, e.g., Mac Lane’s book [16] for the definition of a coend.

Whenever the coend $F$ exists in $C$, it is a Hopf algebra [19, 21] with the structure morphisms given by the following pictures or commutative diagrams. The multiplication is determined by the condition that the diagram on the right of the following figure commutes for any pair of objects, $(L, M)$. I.e., $m_F$ is the lift of the dinatural transformation defined by the braid on the
The unit is $\mathbb{1} = \mathbb{1} \otimes \mathbb{1}^\vee \xrightarrow{i} F$. The comultiplication $\Delta$ on $F$ is uniquely determined by the condition that the next equation shall hold for any object $X$.

\[
(X \otimes X^\vee \xrightarrow{i_X} F \xrightarrow{\Delta} F \otimes F) = (X \otimes X^\vee = X \otimes \mathbb{1} \otimes X^\vee \xrightarrow{X \otimes \text{coev} \otimes X^\vee} X \otimes X^\vee \otimes X \otimes X^\vee \xrightarrow{i_X \otimes i_X} F \otimes F)
\]

or, pictorially,

\[
\Delta = \begin{array}{c}
F \\
\downarrow \\
F \\
\end{array} \begin{array}{c}
F \\
\end{array}
\]

The counit $\varepsilon$ is given by the equation

\[
ev = (X \otimes X^\vee \xrightarrow{i_x} F \xrightarrow{\varepsilon} \mathbb{1}).
\]

One also finds that the antipode $S : F \rightarrow F$ exists and is invertible. It is defined via the diagram

\[
S = \begin{array}{c}
F \\
\downarrow \\
F \\
\end{array}
\]

A pairing of Hopf algebras, $\omega : F \otimes F \rightarrow \mathbb{1}$, is given by the tangle below. See also [19].

\[
\omega = \begin{array}{c}
F \\
\downarrow \\
F \\
\end{array} \begin{array}{c}
F \\
\end{array}
\]

We shall call the category $\mathcal{C}$ modular in case the pairing $\omega$ is side-invertible (i.e., non-degenerate).

Finally, it is easily seen that $\mathcal{C}$ is also a ribbon category.
Proposition 5.2 ([14]). Let $C$ be a modular category. Then the object of the integrals of the Hopf algebra $F$ is isomorphic to $\mathbb{1}$, its integrals are two-sided, i.e., $\int_F = F \int : \mathbb{1} \to F$ and $\int^F = F \int : \mathbb{1} \to F$, and they are related by the equation

$$\int_F = F \int = (F = \mathbb{1} \otimes F \xrightarrow{\int F \otimes F} F \otimes F \xrightarrow{\omega} \mathbb{1}) = (F = \mathbb{1} \otimes F \xrightarrow{\int F \otimes F} F \otimes F \xrightarrow{\omega} \mathbb{1}).$$

In the case of abelian categories this was shown by Lyubashenko [19]. By Theorem 3.3 the number $\int_F(\int^F) = (\mathbb{1} = \mathbb{1} \otimes \mathbb{1} \xrightarrow{\int F \otimes \int^F} F \otimes F \xrightarrow{\omega} \mathbb{1}) \in \text{End} \ \mathbb{1}$ (5.6) is invertible. We assume that it has a square root in $\text{End} \ \mathbb{1}$ so that we can rescale $\int^F$ in a way, such that $\int_F(\int^F) = 1$.

The following theorem is a special case of the results proven in [14], where abelianness was assumed although not needed for the construction of the TQFT-functor. We shall also outline the proof, in which a lot of complications are avoided, since we have restricted ourselves to the tangles associated to connected, one-holed surfaces. For more detailed arguments the reader is referred to the original exposition.

Theorem 5.3 (see also [14]).

Let $C$ be a modular category and assume that the number in (5.6) is 1. Then there is a unique monoidal functor $\Phi : TC \to C$, for which $\Phi(2) = F$ and $\Phi(\int_2) = \int_F$, compatible with the braiding and ribbon twists, and which carries the Hopf algebra structure of $2$ to the Hopf algebra structure of $F$.

Outline of Proof: (a) Denote by $\text{DoubleRT}$ the subcategory of $RT$, which has even, non-negative integers as objects and ribbon tangles satisfying Condition † of Definition 5.1 as morphisms. With no loss of generality we may assume that $C$ is a strict monoidal category with $\overset{\vee}{\overset{\vee}{X}} = X^\vee$, $X^\vee \overset{\vee}{\overset{\vee}{X}} = X$ and $u_0^2 = 1$ (see [18, 14]). To begin with, we construct a functor $\tilde{\Phi} : \text{DoubleRT} \to C$ following [19, 20].

Set $\tilde{\Phi}(2n) = F^\otimes_n$. Consider an arbitrary tangle, $T : 2n \to 2m \in \text{DoubleRT}$, draw its planar diagram $T'$ with threads, crossings and ribbon twists, $\nu$, and mark an absolute maximum and on each of these maxima a closed thread with ends attached to the target (bottom) line. For an arbitrary family, $X_1, \ldots, X_n$, of objects of $C$ one can then construct a morphism

$$\phi(T') : X_1 \otimes X_1^\vee \otimes \cdots \otimes X_n \otimes X_n^\vee \to F^\otimes_m$$

by assigning braidings to crossings, evaluations or counits to minima, coevaluations to ordinary maxima, and the integral-element $\int^F : \mathbb{1} \to F$ to the special absolute maxima. By the universal property of the coend the morphism $\phi(T')$ factorizes through

$$\tilde{\phi}(T') : F^\otimes_n \simeq \int_{X_1, \ldots, X_n \in C} X_1 \otimes X_1^\vee \otimes \cdots \otimes X_n \otimes X_n^\vee \to F^\otimes_m.$$
In fact, this morphism depends only on $T$ and not on the choice of maxima in $T'$, since, for example,

\[
\begin{array}{c}
\begin{array}{c}
\text{In order to see that setting } \tilde{\Phi}(T) = \bar{\phi}(T') \text{ gives the desired functor } \tilde{\Phi}, \text{ we notice that the counit of } F \text{ is determined by the evaluation as in equation (5.3).}
\end{array}
\end{array}
\]

(b) We want to check that ribbon tangles, equivalent under moves \{ID1\}–\{TS3\}, are sent to the same morphisms by $\tilde{\Phi}$:

By the definition of integrals or by Property (3.1) of $\int_F$ we have

\[
\begin{array}{c}
\begin{array}{c}
\text{If we now compose this identity with the coaction } \delta \text{ of } F \text{ with respect to an arbitrary object } Y \in \mathcal{C}, \text{ pictured below,}
\end{array}
\end{array}
\]

\[
\delta = \begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{F}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
:\text{Y} \to F \otimes Y
\end{array}
\end{array}
\]

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we obtain

Hence,

for any $X,Y \in \text{Obj} \mathcal{C}$. Therefore, $\tilde{\Phi}$ is stable under Move (TD1).

Similarly, for any $Y \in \text{Obj} \mathcal{C}$ the identities
imply Move (TD2). Moreover, Move (TS2) holds since \( \int F(fF) = 1 \), see equation (5.6).

To prove the remaining relation (TS3) notice that

\[
\begin{align*}
\text{(TD1)} & = \text{(TS2)} \\
\text{(TD1)} & = \text{(TS3)}
\end{align*}
\]

by the handle-slide moves (TD1) and (TS2). Therefore, using (TD1) once again, we get

\[
\begin{align*}
\text{(TD1)} & = \text{(TS3)} \\
\text{(TD1)} & = \text{(TS3)}
\end{align*}
\]

The non-degeneracy of the form \( \omega \) implies now Move (TS3). This completes the construction of the functor \( \Phi : TC \to \mathcal{C} \).

(c) The graphical formulæ for the structure maps for 2 differ from those for \( F \) by expressions as in (5.1). Since we have proven (TS3), the structure morphisms of 2 are sent by \( \Phi \) to structure morphisms of \( F \).

The meaning of Theorem 5.3 is that all calculations using the handle-slide move with arbitrary framing (see Proposition 5.1) are valid for \( F \). As a corollary one gets modular relations as in [19], representations of mapping class groups of surfaces with one hole as in [20, 25], and, eventually, topological quantum field theories [14].

6 Integrals and related structures

The purpose of this section is to explain how the structures of integrals and moduli are modified by various procedures that can be applied to create new braided Hopf algebras from known ones. Often new Hopf algebras are obtained by taking some type of product between two given ones. The first example of such a product is the Heisenberg double \( \mathcal{H}(H) = H \#^\vee H \), where a Hopf algebra, \( H \), is combined with its dual. This is of particular interest since Hopf modules of \( H \)
will turn out to be in a one-to-one correspondence with ordinary modules of \( \mathcal{H}(H) \). We shall also discuss a very general type of cross product, \( A \rtimes B \), between two braided Hopf algebras. Finally, we will also describe the procedure of “transmuting” the coproduct structure of a Hopf algebra, an idea, which has its origin in the Tannaka Krein theory for braided categories.

**Heisenberg double, Hopf modules and integration.**

For a Hopf algebra, \( H \), in \( C \) with a dual, \( \vee H \), one can define the algebra \( \mathcal{H}(H) = H \# \vee H \), called the *Heisenberg double*, in the following way:

\( \vee H \) becomes an algebra in the category \( \overline{C}_{H_{op}} \) with right \( H_{op} \)-modules structure given by the composition

\[
\vee H \otimes H \xrightarrow{\Delta(\vee H) \otimes \text{id}_H} \vee H \otimes \vee H \otimes H \xrightarrow{\text{id}(\vee H) \otimes \text{ev}} \vee H.
\]

The multiplication on the corresponding smash-product algebra \( H \# \vee H \) is shown on Figure 17a). Similarly, \( H \) becomes an algebra in \((\vee H)_{op}\overline{C}\). The corresponding smash-product algebra \( H \# \vee H \) is the same.

Note that the Heisenberg double \( \mathcal{H}(H) \) is isomorphic to the “matrix algebra” of \( H \), which is

\[
(H \otimes \vee H, \mu_{(H \otimes \vee H)} := \text{id}_H \otimes \text{ev} \otimes \text{id}_\vee H, \eta_{(H \otimes \vee H)} := \text{coev}).
\]

The existence of such an isomorphism for an ordinary Hopf algebra is a special case of the duality theorem from \([6]\). See \([3]\) for the braided setting.

**Proposition 6.1.**

The morphism \( f : H \# \vee H \to H \otimes \vee H \), shown on Figure 17b) is an algebra isomorphism.

The category of Hopf \( H \)-modules is identified with the category of \( \mathcal{H}(H) \)-modules as follows:
Proposition 6.2. The categories \( H^H \mathcal{C} \) and \((H \# \mathcal{V} H)^\mathcal{C}\) are isomorphic. The isomorphism is the identity on the underlying objects and morphisms in \( \mathcal{C} \), and it turns a Hopf \( H \)-module, \((X, \mu_, \Delta_e)\), into an \((H \# \mathcal{V} H)\)-module, for which the action given by the composition
\[
\begin{align*}
H \otimes \mathcal{V} H \otimes X & \xrightarrow{id_H \otimes \mathcal{V} H \otimes \Delta_e} H \otimes \mathcal{V} H \otimes X \\
& \xrightarrow{id_H \otimes \mathcal{V} H \otimes \mathcal{V} H \otimes \Delta_e} H \otimes \mathcal{V} H \otimes X \\
& \xrightarrow{id_H \otimes \mathcal{V} H \otimes \mathcal{V} H \otimes \mathcal{V} H \otimes \mu_{\mathcal{V} H \otimes \mathcal{V} H}} H \otimes X \xrightarrow{\mu_{\mathcal{V} H \otimes \mathcal{V} H}} X.
\end{align*}
\]

In \( \mathcal{H} \) an approach to integration on braided Hopf algebras involving Heisenberg doubles is developed. There the following two idempotents ("vacuum projectors") in \( \mathcal{H}(H) \) play a crucial role:
\[
\begin{align*}
E &= (S_H \otimes \text{id}_H) \circ \text{coev} = (\text{id}_H \otimes S_H) \circ \text{coev}, \\
\bar{E} &= \mu \mathcal{H}(H) \circ (S_H^2 \otimes \text{id}_H) \circ \Psi^{-1}_{H, \mathcal{V} H} \circ \text{coev}.
\end{align*}
\]

Note that the elements \( E \) and \( \bar{E} \) act on a left \( \mathcal{H}(H) \)-module (\( \equiv \) left Hopf \( H \)-module), \( X \), in the following way:
\[
\begin{align*}
\mu_\ell \circ (E \otimes \text{id}_X) &= \Pi_\ell^L(X), \\
\mu_\ell \circ (\bar{E} \otimes \text{id}_X) &= \mathcal{V} \Pi_\ell^L(X).
\end{align*}
\]

An idempotent, \( \mathcal{V} \Pi_\ell^L(H) \), is derived in \( \mathcal{H} \) from the formula
\[
i_H \circ \Pi_\ell^L(H) = \mu_{\mathcal{H}(H)}^{(3)} \circ (E \otimes i_H \otimes E),
\]
where \( i_H = \text{id}_H \otimes \eta_{\mathcal{V} H} : H \rightarrow \mathcal{H}(H) \) is a canonical embedding of algebras.

Integrals, cross products, and transmutation.

Cross products and transmutation are basic constructions that allow us to obtain new braided Hopf algebras from given ones \([23, 24]\). Here we use generalizations of these constructions from \([8]\), and show that integrals on cross product Hopf algebras and on transmutation Hopf algebras are obtained in a simple way from integrals on the initial Hopf algebras.

Proposition 6.3 \([2]\). Let \( A \) be a Hopf algebra in \( \mathcal{C} \), and \( B \) be a Hopf algebra in the category \( \mathcal{DY}(\mathcal{C})_A^A \) of crossed modules. (A crossed module is an object with both module and comodule structures satisfying the compatibility condition presented on Figure \( 18a \)).

Then \( A \times B \) with underlying object \( A \otimes B \), multiplication, comultiplication, and antipode as shown on the Figure \( 18b \) is a Hopf algebra in \( \mathcal{C} \).

Int \( B \) is an invertible object in \( \mathcal{C} \) and, therefore, the \( A \)-(co)module structures on Int \( B \) are given by the formulae
\[
\begin{align*}
\mu^B_r &= \text{id}_B \otimes \alpha_{B/A}, \\
\Delta^B_r &= \text{id}_B \otimes a_{B/A}
\end{align*}
\]
for a certain multiplicative functional, \( \alpha_{B/A} \), and some group-like element, \( a_{B/A} \). The crossed module axiom for Int \( B \) then takes on the form
\[
\Omega^\text{Int} B_A = \text{ad}^{\alpha_{B/A}} \circ \text{ad}_{a_{B/A}}.
\]

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Proposition 6.4. For any cross product Hopf algebra, $A \ltimes B$, in a rigid braided monoidal category, $C$, we have that $\text{Int}(A \ltimes B) \simeq \text{Int} A \otimes \text{Int} B$, and the following are valid choices for integrals in or on $A \ltimes B$:

\[
\begin{align*}
\int_{A\ltimes B} &= \int_A \otimes \int_B, \\
\int^{A\ltimes B} &= \int^A \otimes \int^B, \\
a_{A\ltimes B} &= a_A \cdot a_{B/A} \otimes a_B, \\
\alpha_{A\ltimes B} &= \alpha_A \cdot \alpha_{B/A} \otimes a_B.
\end{align*}
\]  

(6.2)

The notion of braided quantum groups (quasitriangular braided Hopf algebras) was introduced in [23], and the basic theory was developed there. We use the slightly modified definitions from [2], which reflect the symmetry between the two coalgebra structures under consideration. A pair of Hopf algebras, $(A, \overline{A})$, in $C \times \overline{C}$ and a bialgebra copairing, $\mathcal{R} : \mathbb{I} \to \overline{A}_{\text{op}} \otimes A$, in $C$, which is invertible as an algebra element, define a braided quasitriangular Hopf algebra, $(A, \overline{A}, \mathcal{R})$, if $A$ and $\overline{A}$ only differ in their comultiplication and antipode, i.e., if $A = (A, \mu, \eta, \Delta, \varepsilon, S)$ and $\overline{A} = (A, \mu, \eta, \Delta, \varepsilon, \mathcal{S})$, and if the identity $(\Psi_{A,A} \circ \Delta) \cdot \mathcal{R} = \mathcal{R} \cdot \Delta$ holds. See Figure 19a) for an illustration of the latter. Similar to the case of ordinary quantum groups it is shown in [2] that the antipode of any quasitriangular Hopf algebra of the form $(A, \overline{A}, \mathcal{R})$ is invertible. In particular, we have $\mathcal{S}^{-1} = u \cdot S \cdot u^{-1}$, where $u := \mu \circ (\text{id}_A \otimes S) \circ \mathcal{R}$.

For a quasitriangular bialgebra, $(A, \overline{A}, \mathcal{R})$, we define the category $C_{\mathcal{O}(A, \overline{A})}$ as the full subcategory of the category $C_A$ of $A$-right modules, whose objects $(X, \mu_r)$ satisfy the identity indicated in Figure 19b). This category is identified with a full braided monoidal subcategory of $\mathcal{D} Y(C)^A_{A, \overline{A}}$ [2, 23]. I.e., every module $(X, \mu_r)$ in $C_{\mathcal{O}(A, \overline{A})}$ becomes a crossed module, $(X, \mu_r, \Delta_r)$.
Definition 6.1. Let \((A, \overline{A}, R_A)\) be a quantum braided group, \((H, \mu_H, \Delta_H)\) a Hopf algebra in \(\mathcal{C}\), and \(f : A \to H\) a bialgebra morphism.

We say that \(((A, \overline{A}, R_A), f, H)\) are transmutation data for a Hopf algebra \(H\) in \(\mathcal{C}\) if

- the two adjoint actions of \(A\) and of \(\overline{A}\), defined through \(f\) as presented on Figure 19d), coincide, (We will denote this action by \(\mu_H^{\text{ad}}\) and call it the adjoint action of a quantum group.)

- and \((H, \mu_H^{\text{ad}})\) is an object of \(\mathcal{C}_{O(A, \overline{A})}\).

The transmutation \(\overline{H}\) of a Hopf algebra \(H\) is the underlying algebra of \(H\) with a new co-multiplication, \(\Delta_{\overline{H}}\), and a new antipode \(S_{\overline{H}}\), as defined in Figures 19e-f). There we set \(\mathcal{R}_A = (\text{id}_A \otimes S) \circ R_A = (S^{-1} \otimes \text{id}_A) \circ R_A\).

Proposition 6.5 (2).

The transmutation \(\overline{H}\) of a Hopf algebra, \(H\), in \(\mathcal{C}\) is a Hopf algebra in \(\mathcal{C}_{O(A, \overline{A})}\).
In addition to this observation from [2], we shall next determine also the integrals of $H$:

**Proposition 6.6.** The integral object $\text{Int} H$ for the transmutation can be chosen as the object $\text{Int} H$, equipped with the $A$-action $\text{id}_{\text{Int} H} \otimes \alpha_f$, where $\alpha_f := \alpha_H \circ f$. Consistent with that one can further put

$\int H = H^\mu$, $H^\mu = H\int$, $H\int = (a_f^{-1}, H\int)$, and $H\int = H\int \circ S_H$,

where $a_f := (\alpha_f \otimes f) \circ R_A$.

**Proof.** It is verified directly that $\int H : (\text{Int} H, \text{id}_{\text{Int} H}) \otimes (H, \mu_H)$ is a morphism in $\mathcal{C}_{\text{O}(A)}$. This fact and the universal property of $\int H$ imply that $\int H$ is a left integral in $H, (\int H \otimes \text{id}_H) \circ \Delta_H$ is equal to $\int H \otimes \eta_H$ by the property of integrals, which in turn is found to be equal to the composition

$H \xrightarrow{\Delta_H} H \otimes H \xrightarrow{\text{id}_H \otimes a_f^{-1} \otimes \text{id}_H} H \otimes H \otimes H \xrightarrow{\int H \otimes \mu_H} \text{Int} H \otimes H$

using the fact that $\int H$ is a morphism in $\mathcal{C}_{\text{O}(A)}$. Comparing these two expressions we obtain that $a_f \cdot H\int$ is a right integral in $H$. \qed

**7 Applications of integrals**

The first part of this section is devoted to the construction of the braided version of the Hopf algebra structure on the exterior algebra $\bigoplus_j X^\wedge_j$ for a given object, $X$, in a braided category, $C$. As in the symmetric case the object of the integral will turn out to be the last non-vanishing power $X^\wedge_{n-1}$, and all morphisms are canonical ones.

In the second part of this section we shall explicitly construct and investigate the equivalence of the category of Hopf $H$-bimodules and the category of Hopf $H^\wedge$-bimodules for a Hopf algebra, $H$, in a rigid, braided category. From this we will be able to draw two more proofs of the generalized Radford formula.

**Integrals on external Hopf algebra**

For a given invertible solution, $R \in \text{End}(V \otimes V)$, of the Yang-Baxter equation one can turn the tensor algebra $T(V)$ into a Hopf algebra in the corresponding braided category, see [22]. It is shown in [4] that, similarly, for a given object, $X$, in an abelian braided monoidal category, $C$, the collection $\{X^\otimes n\}_{n \in \mathbb{N}}$ becomes a Hopf algebra, $T_C(X)$, in the category $\mathcal{C}_\mathbb{N}$ of graded spaces. Moreover, because of categorical duality, there exists another Hopf algebra, $\bar{T}_C(X)$, with the same underlying object $\{X^\otimes n\}_{n \in \mathbb{N}}$. The braided analog of the antisymmetrizer $\bar{A}_X : T_C(X) \to \bar{T}_C(X)$ is a Hopf algebra morphism, and its image $\bar{T}_C(X) = \{X^\wedge n\}_{n \in \mathbb{N}}$ is a Hopf algebra. Here we recall some necessary results from [4] and then restrict ourselves to the case when $T_C(X)$ has a finite number of non-zero components and becomes an object of a rigid category. We
determine explicitly the integrals on $T^n_C(X)$ and derive from general integration theory some properties of the objects $X^{\wedge n}$.

First let us outline the braided combinatorics, as in [22, 4], in an additive braided monoidal category, $\mathcal{C}$. The canonical epimorphism of the braid group $B_n$ into the permutation group $S_n$ admits a section $S_n \to B_n$, which maps the standard generators to the standard generators and is uniquely determined on other elements by the property

$$\tilde{\sigma_1 \sigma_2} = \tilde{\sigma_1} \tilde{\sigma_2} \quad \text{if} \quad \ell(\sigma_1 \sigma_2) = \ell(\sigma_1) + \ell(\sigma_2),$$

where $\ell(\sigma)$ is the length (of the minimal decomposition) of $\sigma$. Since the category $\mathcal{C}$ is braided, there is a canonical mapping $B_n \to \text{End}(X^{\otimes n})$. The image of $\sigma \in S_n$ under the composition $S_n \to B_n \to \text{End}(X^{\otimes n})$ is denoted by $\sigma_C(X)$. Now let $\pi = (j_1, \ldots, j_r)$ be any $\mathbb{N}$-partition of $j$, i.e., $j = j_1 + \cdots + j_r$ and $j_1, \ldots, j_r \in \mathbb{N}$. We consider the shuffle permutations $S_\pi^\sigma \subset S_j$. For every $l \in \{1, \ldots, r\}$ they are mapping the $j_l$ elements in $\{1, \ldots, j\}$ to $\{j_1 + \cdots + j_{l-1} + 1, \ldots, j_1 + \cdots + j_l\}$ without changing their order. The set of inverse permutations of $S_\pi^\sigma$ is denoted by $S_\pi^\sigma$. For every partition $\pi = (j_1, \ldots, j_r)$ of $j$, any object $X \in \text{Obj}(\mathcal{C})$ and any $\lambda \in \text{Aut}(\mathbb{I})$ we define braided multinomials as endomorphisms of $X^{\otimes j}$ in $\mathcal{C}$ by the following identities

$$[\pi_j | X; \lambda] = [\pi^{j_1} \cdots \pi^{j_r} | X; \lambda] := \sum_{\sigma \in S_\pi^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X),$$

$$[j | X; \lambda] = \left( \sum_{\sigma \in S_\pi^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right). \quad (7.1)$$

**Proposition 7.1.** Let $\pi = (j_1, \ldots, j_r)$ be a partition of $j$, and let $\pi_k = (j_{1k}, \ldots, j_{rk})$ be a partition of $j_k$ for any $k \in \{1, \ldots, r\}$. Then with the notation from above the following formulae hold true:

$$\left( \sum_{\sigma \in S_{\pi_1}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right) \circ \left( \sum_{\sigma \in S_{\pi_2}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right) \cdots \circ \left( \sum_{\sigma \in S_{\pi_r}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right)$$

and

$$\left( \sum_{\sigma \in S_{\pi_1}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right) \circ \left( \sum_{\sigma \in S_{\pi_2}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right) \cdots \circ \left( \sum_{\sigma \in S_{\pi_r}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right)$$

where (7.2) is the dual version of (7.3).

For any $0 \leq k \leq j$ we shall use abbreviated notations as follows: $[k_j | X; \lambda] := \left[ \sum_{\sigma \in S_{\pi_k}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right]$, $[k_j ! | X; \lambda] := \left[ \sum_{\sigma \in S_{\pi_k}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right]$ and $[j ! | X; \lambda] := \left[ \sum_{\sigma \in S_{\pi_r}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right]$. In this language we then derive by successive application of eq. (7.3) the identities

$$[j !] = \left( \sum_{\sigma \in S_{\pi_1}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right) \circ \left( \sum_{\sigma \in S_{\pi_2}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right) \cdots \circ \left[ j | X; \lambda \right],$$

$$[j !] = \left( \sum_{\sigma \in S_{\pi_1}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right) \circ \left( \sum_{\sigma \in S_{\pi_2}^\sigma} \lambda^{\ell(\sigma)} \sigma_C(X) \right) \cdots \circ \left[ j | X; \lambda \right]. \quad (7.4)$$
and

\[ [j + k]! = ([j]! \otimes [k]!) \circ \left[ \begin{array}{c} j + k \\ j \end{array} \right], \tag{7.5} \]

**Definition 7.1.** Let \( I \) be \( \mathbb{N} \) or \( \mathbb{Z}/n \), considered as a discrete category. We consider the functor category \( \mathcal{C}^I \) as an \( I \)-graded category over \( \mathcal{C} \). The objects of \( \mathcal{C}^I \) are given by \( I \)-tuples, \( \hat{X} = (X_0, X_1, \ldots) \), of objects, \( X_j \in \text{Obj}(\mathcal{C}) \), where \( j \in I \). The morphisms of \( \mathcal{C}^I \) are of the form \( \hat{f} = (f_0, f_1, \ldots) : \hat{X} \to \hat{Y} \), where \( f_j : X_j \to Y_j \) is a morphism in \( \mathcal{C} \) for all \( j \in I \). The category \( \mathcal{C}^I \) naturally inherits a braided monoidal structure from \( \mathcal{C} \). The unit object in \( \mathcal{C}^I \) is given by \( \hat{1} \equiv (1, 0, 0, \ldots) \), the tensor product is defined by \( (\hat{X} \otimes \hat{Y})_n = \bigoplus_{k+l=n} X_k \otimes Y_l \), where \( n, k, l \in I \), and the tensor product of \( I \)-graded morphisms is built analogously. Besides the natural braided structure, the category \( \mathcal{C}^I \) actually admits a family of braidings given by \( \hat{\Psi}^{(\lambda)}_{X,Y}(n) = \bigoplus_{k+l=n} \lambda^k \Psi_{X_k,Y_l} \), for any \( \lambda \in \text{Aut}_\mathcal{C}(\mathbb{1}) \), if \( I = \mathbb{N} \), and for any \( \lambda \in \text{Aut}_\mathcal{C}(\mathbb{1}) \) with \( \lambda^n = 1 \), if \( I = \mathbb{Z}/n \).

**Proposition 7.2.**

1. For a given object, \( X \), in an additive braided category, \( \mathcal{C} \), the tensor algebra \( \{X^{\otimes n}\} \) in \( \mathcal{C}^\mathbb{N} \) admits two Hopf algebra structures, \( \mathcal{T}_\mathcal{C}(X) = \{(X^{\otimes n}), \mu, \lambda, \epsilon\} \) and \( \hat{T}_\mathcal{C}(X) = \{(X^{\otimes n}), \hat{\mu}, \hat{\lambda}, \hat{\epsilon}\} \), which are given as follows:

\[
\begin{align*}
\mu_{n,m} &\equiv \text{id}_{X^{\otimes n+m}} : X^{\otimes n} \otimes X^{\otimes m} \to X^{\otimes n+m}, \\
\Delta_{n,m} &\equiv \left[ \begin{array}{c} n + m \\ n \end{array} \right] X^{\otimes n} \otimes X^{\otimes m} \to X^{\otimes n+m} = X^{\otimes n} \otimes X^{\otimes m}, \\
\hat{\mu}_{n,m} &\equiv \left[ \begin{array}{c} n + m \\ n \end{array} \right] X^{\otimes n} \otimes X^{\otimes m} \to X^{\otimes n+m}, \\
\hat{\Delta}_{n,m} &\equiv \text{id}_{X^{\otimes n+m}} : X^{\otimes n+m} \to X^{\otimes n} \otimes X^{\otimes m}, \\
\eta_n = \hat{\eta}_n &\equiv \begin{cases} \text{id}_{\mathbb{1}}, & n = 1, \\
0, & n \neq 1 \end{cases}, \\
e_n = \hat{\epsilon}_n &\equiv \begin{cases} \text{id}_{\mathbb{1}}, & n = 1, \\
0, & n \neq 1 \end{cases}, \\
S_n = \hat{S}_n &\equiv (-1)^n \lambda_{X^n}^{(2)} (\sigma_0^0)(X) : X^{\otimes n} \to X^{\otimes n}, \text{ where } \sigma^0_n = (1 \ldots n) \end{align*}
\]

2. The morphism \( \hat{A}_X := ([nX^n])_{n \in \mathbb{N}} : \mathcal{T}_\mathcal{C}(X) \to \hat{T}_\mathcal{C}(X) \) is a Hopf algebra map.

Now let \( \mathcal{C} \) be a rigid braided monoidal abelian category with a biadditive tensor product. In this case the unit object \( \mathbb{1} \) is semisimple, see [9], so that we can suppose without restriction of generality that \( \text{End}_\mathcal{C}(\mathbb{1}) \) is a field. Rigidity implies that the tensor product is exact. In this situation we have that for any Hopf algebra morphism, \( f \), the intermediate object of an epi-mono decomposition, \( f = \text{im} f \circ \text{coim} f \), can be equipped with a Hopf algebra structure, such that both, \( \text{im} f \) and \( \text{coim} f \), are Hopf algebra morphisms. This allows us to make the following definition:
Definition 7.2. We will denote by $T^\wedge_C(X) = \{ X^\wedge n \}$ the Hopf algebra determined by the epimono decomposition

$$\hat{A}_X = \{ T_C(X) \xrightarrow{\text{coim} \hat{A}_X} T^\wedge_C(X) \xrightarrow{\text{im} \hat{A}_X} T_C(X) \}.$$  \hfill (7.6)

Let us suppose that $\lambda^n = 1$, $[n\vert X\vert! = 0$, and $[n - 1\vert X\vert]! \neq 0$. In this case also $X^\wedge k = 0$ for any $k \geq n$, and $T^\wedge_C(X)$ can be considered as a Hopf algebra in the rigid braided monoidal abelian category $C^{\mathbb{Z}/n}$.

Proposition 7.3. Under the previous assumptions the integrals $\int_{T^\wedge_C(X)}$, $\int_{T^\wedge_C(X)}$, $\int_{T^\wedge_C(X)}$ are all given by the following expressions:

$$\text{Int}(T^\wedge_C(X))_k = \begin{cases} 0, & \text{if } k \neq n - 1, \\ X^\wedge(n-1), & \text{if } k = n - 1, \end{cases} \quad (\tilde{f})_k = \begin{cases} 0, & \text{if } k \neq n - 1, \\ \text{id}_{X^\wedge(n-1)}, & \text{if } k = n - 1. \end{cases}$$

Proof. We have that $\Pi^\wedge_k(T^\wedge_C(X))_{n-1} = \text{id}_{X^\wedge(n-1)}$, because $m^\wedge_{k,\ell} = 0$ for $k + \ell \geq n$. The condition that $\text{End}_C(\text{Int}(T^\wedge_C(X))$ has no nontrivial idempotents implies that $\text{Int}(T^\wedge_C(X))$ has only one nonzero component. \hfill \Box

Corollary 7.4. Under the above assumptions $X^\wedge(n-1)$ is an invertible object in $C$. The multiplication in $T^\wedge_C(X)$ defines side-invertible pairings, $\mu_{k,n-k-1} : X^\wedge k \otimes X^\wedge(n-k-1) \to X^\wedge(n-1)$. I.e., it induces isomorphisms, $(X^*)^\wedge(n-k-1) \otimes X^\wedge(n-1) \cong X^\wedge k$, for $k \in \mathbb{Z}/n$ and any object $X^*$ dual to $X$.

The Hopf algebra from Example 3.1 was obtained as $T^\wedge(k)$ for $\lambda = q$ a primitive $n^{\text{th}}$ root of unit. Our results show similarities between the Hopf algebras $T^\wedge(k)$ and $T^\wedge(X)$ in the most general case.

Remark 7.1. Note that if $[n+1\vert X\vert]! = 0$ and $[m+1\vert X\vert]! = 0$ then also $[n+m+1\vert X\oplus Y\vert]! = 0$. This follows from the fact that the matrix element of $[n+m+1\vert X\oplus Y\vert]!$, which is a morphism from $X^\otimes k \otimes Y^\otimes \ell$ to $X^\otimes k \otimes Y^\otimes \ell$, equals to $[k\vert X\vert]! \otimes [\ell\vert Y\vert]!$ and other matrix elements are either of such type (up to isomorphism) or zero.

Remark 7.2. Suppose $X$ lives in the category $\mathcal{D}\mathcal{Y}(C)^H_H$ of crossed modules over a Hopf algebra, $H$, and $d : H \to H \rtimes X$ defines a first order differential calculus. Then $H \rtimes T^\wedge_C(X)$ is a Hopf algebra in $C^{\mathbb{Z}/n}$, and $d$ has a unique extension, $d^\wedge$, on $H \rtimes T^\wedge_C(X)$, which turns $H \rtimes T^\wedge_C(X)$ into a differential Hopf algebra (braided De Rham complex) $[4]$. Integrals on $H \rtimes T^\wedge_C(X)$ are calculated by the formulae (6.2) for integrals on cross products. They are nonzero only on one component $H \rtimes X^\wedge(n-1)$ and play the role of the integration over a volume form.
Duality for Hopf bimodules

In this section we shall describe how the previous results can be applied in order to directly construct the equivalence functor between the categories $\mathcal{H}_H$ and $\mathcal{C}(H)$ (see [1] for unbraided case). The existence of such an equivalence follows from the fact that the category $\mathcal{H}_H$ is equivalent to the category of crossed modules $\mathcal{D}_H$. The equivalence of the categories $\mathcal{D}_H$ and $\mathcal{D}_H^\sigma$ is described in [2]. These and the following considerations lead to alternative proofs of the generalized Radford formula:

**Proposition 7.5.** There exists an equivalence, $\mathcal{H}_H \xrightarrow{\mathcal{D}_H} \mathcal{C}(H)$, of monoidal categories, which converts a Hopf $H$-bimodule $X$ into the Hopf $\mathcal{H}_H$-bimodule $X \otimes \mathcal{H}_H$ with actions and coactions defined via Figure 20.

Before we can prove Proposition 7.5 we need a few technical results about tensor products on the $H$-module categories. The first observations on the existence and properties of the braided tensor product actions $\mu_X$ are straightforward to prove:

**Lemma 7.6.**

- Let $(X, \mu_X, \mu_Y) \in \text{Obj}(H)$ and $(Y, \mu_Y, \mu_Y, \Delta_Y) \in \text{Obj}(H)$. Then $(X \otimes Y, \mu_X \otimes \mu_Y, \mu_Y \otimes \mu_Y, \text{id} \otimes Y) \in \text{Obj}(H)$, where the underlying module structures $\mu_X \otimes \mu_Y$ and $\mu_Y \otimes \mu_Y$ are the braided tensor product ones:

$$
\mu_X \otimes \mu_Y : H \otimes X \otimes Y \xrightarrow{\Delta \otimes \text{id} \otimes \mu_Y} H \otimes H \otimes X \otimes Y
\xrightarrow{\text{id} \otimes \mu_X \otimes \mu_Y} H \otimes X \otimes H \otimes Y \xrightarrow{\mu_X \otimes \mu_Y} X \otimes Y.
$$
\begin{itemize}
\item Let \((X, \mu^X_\ell, \mu^X_r, \Delta^X) \in \text{Obj}(\mathcal{H}^\ell \mathcal{C}_H)\) and \((Y, \mu^Y_\ell, \mu^Y_r, \Delta^Y) \in \text{Obj}(\mathcal{H}^\ell \mathcal{C}_H)\).

Then \((X \otimes Y, \mu^X_\ell \otimes \mu^Y_\ell, \mu^X_r \otimes \mu^Y_r, \Delta^X \otimes \Delta^Y, \text{id}_X \otimes \text{id}_Y) \in \text{Obj}(\mathcal{H}^\ell \mathcal{C}_H)\).
\end{itemize}

Lemma \ref{7.6} further allows us to properly define the tensor product of bimodules \((X, Y) \to X \otimes_H Y\), which may be viewed as an action of \(\mathcal{H}^{\ell \ell}_H\) on \(\mathcal{H}^{r r}_H\):

**Proposition 7.7.** Let \(X\) be a Hopf \(H\)-bimodule and \(Y\) be a Hopf \(H^r\)-bimodule. View \(Y\) as a left \(H\)-module via the action \(\mathcal{H}^r\). Then there exists the tensor \(X \otimes_H Y \in \mathcal{C}\) product of \(H\)-modules. There is a unique Hopf \(H^r\)-bimodule structure \(X \otimes_H Y\) on this object, such that the canonical projection \(\lambda : X \otimes Y \to X \otimes_H Y\) is a homomorphism of \(H^r\)-bimodules. Here \(X\) is obtained via Corollary \ref{7.6} (the explicit \(H^r\)-(co)module structures for the underlying object of \(X\) are shown in Figure \ref{fig:7}), and the Hopf bimodule structure on the tensor product is defined by Lemma \ref{7.6}. The monoidal category of \(H\)-bimodules equipped with the tensor product \(- \otimes -\) of bimodules acts on the category of \(H^r\)-bimodules via the functor \(\mathcal{H}^r \mathcal{C}_H \times \mathcal{H}^{r r}_H \to \mathcal{H}^{r r}_H \mathcal{C}_H^r\): \((X, Y) \to X \otimes_H Y\).

**Proof.** The \(H\)-module \(Y\) admits in fact the structure \(Y^\ast\) of a left Hopf \(H\)-module given by Figure \ref{fig:7}. According to \ref{7.6} there exists a tensor product, \(X \otimes_H Y\), together with the canonical projection \(\lambda^{X,Y} : X \otimes Y \to X \otimes_H Y\) given by the composition

\[
\lambda^{X,Y} : X \otimes Y \xrightarrow{\text{id}_X \otimes \Delta^Y} X \otimes H \otimes Y \xrightarrow{\mu^X_r \otimes \text{id}_Y} X \otimes (H Y),
\]

with the following property:

For any \(f : X \otimes Y \to Z\), for which both compositions

\[
X \otimes H \otimes Y \xrightarrow{\mu^X_r \otimes \text{id}_Y} X \otimes Y \xrightarrow{f} Z
\]

are equal, there is a unique \(\hat{f} : X \otimes (H Y) \to Z\), such that \(f = \hat{f} \circ \lambda^{X,Y}\). It is given by the composition \(\hat{f} : X \otimes (H Y) \xrightarrow{\text{id}_X \otimes \Delta^Y} X \otimes Y \xrightarrow{f} Z\). Moreover, \(\lambda\) is a split epimorphism with a splitting as follows:

\[
\theta : X \otimes (H Y) \xrightarrow{\text{id}_X \otimes \Delta^Y} X \otimes Y \xrightarrow{\text{id}_X \otimes S \otimes \text{id}_Y} X \otimes H \otimes Y \xrightarrow{S \otimes \text{id}_Y} X \otimes (H Y),
\]

One can check that both of the following compositions coincide:

\[
\mathcal{H}^r \otimes X \otimes H \otimes Y \xrightarrow{\text{id}_H \otimes X \otimes \mu^Y_r} \mathcal{H}^r \otimes X \otimes Y \xrightarrow{\mu_\ell} X \otimes Y \xrightarrow{\lambda} X \otimes_H Y.
\]

Therefore, there exists a unique structure of a left \(H\)-module on \(X \otimes_H Y\), such that \(\lambda\) is a homomorphism of left \(H^r\)-modules. It must coincide with

\[
\mathcal{H}^r \otimes X \otimes_H Y \xrightarrow{\text{id}_H \otimes \theta} H^r \otimes X \otimes Y \xrightarrow{\mu_\ell} X \otimes Y \xrightarrow{\lambda} X \otimes_H Y.
\]
The right $H^\vee$-module structure is treated analogously.

The existence of a left $H^\vee$-comodule structure on $X \otimes_H Y$ follows from the fact that the left coaction of $H^\vee$ on $X$ commutes with the right action of $H$. It must be given by the formula

$$X \otimes H \xrightarrow{\theta} X \otimes Y \xrightarrow{\Delta X \otimes \text{id}_Y} H^\vee \otimes X \otimes Y \xrightarrow{\text{id}_{H^\vee} \otimes \lambda} H^\vee \otimes (X \otimes_H Y).$$

Similarly, the existence of the right $H^\vee$-comodule structure on $X \otimes_H Y$ follows from the fact that the right coaction of $H^\vee$ on $Y$ commutes with the left action of $H$. Thus $\lambda$ is a morphism of left and right modules and comodules over $H^\vee$. Moreover, it is a split epimorphism so that $X \otimes_H Y$ is a Hopf $H^\vee$-bimodule.

One can check that the canonical associativity isomorphism $(X \otimes_H Y) \otimes_H Z \simeq X \otimes_H (Y \otimes_H Z)$ for a pair of Hopf $H$-bimodules, $X$ and $Y$, and a Hopf $H^\vee$-bimodule, $Z$, is an isomorphism of $H^\vee$-bimodules, where $X \otimes_H Y$ is the standard tensor product of Hopf bimodules. □

**Proof of Proposition 7.3.** Applying the above proposition to the regular Hopf bimodule $Y = H^\vee$ we get a Hopf $H^\vee$-bimodule structure on $X \otimes \text{Int} H^\vee$. Indeed, the chosen left $H$-module structure of $Y = H^\vee$ gives $\Pi(Y) = \Pi(H^\vee)$, $Y p = \int_{H^\vee}$, $Y i = (c^\ell)^{-1} \cdot H^\vee$, and $\lambda_{X,H^\vee} = \{X \otimes H^\vee \xrightarrow{\text{id}_X \otimes H^\vee} X \otimes H \otimes \text{Int} H^\vee \xrightarrow{\mu_{\text{Int} H^\vee}} X \otimes \text{Int} H^\vee\}$. The splitting $\theta$ simplifies to $(c^\ell)^{-1} \cdot \text{id}_X \otimes H^\vee$. Hence the Hopf $H^\vee$-bimodule structure on $X \otimes \text{Int} H^\vee$ is given by the compositions

\[
\begin{align*}
\mu_{\text{Int} H^\vee} & : H^\vee \otimes X \otimes \text{Int} H^\vee \xrightarrow{(c^\ell)^{-1} \cdot \text{id}_{H^\vee} \otimes X} \int H^\vee \otimes X \otimes H^\vee \\
& \xrightarrow{\mu_{\text{Int} H^\vee}} X \otimes H^\vee \xrightarrow{\lambda_{X,H^\vee}} X \otimes \text{Int} H^\vee,
\end{align*}
\]

\[
\begin{align*}
\mu_{\text{Int} H^\vee} & : X \otimes \text{Int} H^\vee \otimes H^\vee \xrightarrow{(c^\ell)^{-1} \cdot \text{id}_X \otimes H^\vee} X \otimes H^\vee \otimes H^\vee \\
& \xrightarrow{\mu_{\text{Int} H^\vee}} X \otimes H^\vee \xrightarrow{\lambda_{X,H^\vee}} X \otimes \text{Int} H^\vee,
\end{align*}
\]

\[
\begin{align*}
\Delta_{\text{Int} H^\vee} & : X \otimes \text{Int} H^\vee \xrightarrow{(c^\ell)^{-1} \cdot \text{id}_X \otimes H^\vee} X \otimes H^\vee \otimes X \otimes H^\vee \\
& \xrightarrow{\text{id}_{H^\vee} \otimes \lambda_{X,H^\vee}} H^\vee \otimes X \otimes \text{Int} H^\vee,
\end{align*}
\]

\[
\begin{align*}
\Delta_{\text{Int} H^\vee} & : X \otimes \text{Int} H^\vee \xrightarrow{(c^\ell)^{-1} \cdot \text{id}_X \otimes H^\vee} X \otimes H^\vee \otimes X \otimes H^\vee \\
& \xrightarrow{\lambda_{X,H^\vee} \otimes \text{id}_{H^\vee}} X \otimes \text{Int} H^\vee \otimes H^\vee.
\end{align*}
\]

or in explicit diagrammatic form in Figure 21.

The dual construction $X^{\bullet \square_H H}$, where the cotensor product is used instead of tensor product, gives a Hopf bimodule structure on $X \otimes \text{Int} H^\vee$ described by the same diagrams only taken upside-down. Specializing this construction to the case of Hopf $H^\vee$-bimodules yields a functor, \(H^\vee \mathcal{C}_H^H \rightarrow H^\vee \mathcal{C}_H^H\), which together with the functor described above defines an equivalence of categories. Indeed, let us define an isomorphism $\mu_{\text{r}} \circ (\text{id}_X \otimes a) : X \rightarrow X$ in $\mathcal{C}$, consider the source
space with the given Hopf bimodule structure, and introduce a new Hopf bimodule structure, \( \tilde{X} \), on the target space, which turns this isomorphism into a Hopf bimodule morphism. Direct calculations show that subsequent applications of the two above functors turn the Hopf \( H \)-bimodule \( X \) into the Hopf \( H \)-bimodule \( X \otimes H^\vee \otimes \text{Int} H \), whose underlying (co)modules are tensor products of the underlying (co)modules of \( \tilde{X} \), with the trivial (co)module \( \text{Int} H^\vee \otimes \text{Int} H \).

**RemarK 7.3.** Note that the \( H^\vee \)-co(module) structures on \( X \otimes \text{Int} H^\vee \) presented in Figure 22 are the tensor product ones:

\[
(X \otimes \text{Int} H^\vee, \mu^a_{\cdot \ell} \otimes \text{id}_{\text{Int} H^\vee}, (\mu_{\cdot \ell} \otimes \text{id}_{\text{Int} H^\vee}) \circ (\text{id}_X \otimes \Psi_{\text{Int} H^\vee,H^\vee}) , \Delta^a_{\cdot \ell} \otimes \text{id}_{\text{Int} H^\vee}, (\text{id}_X \otimes \Psi_{\text{Int} H^\vee,H^\vee}) \circ (\Delta^a_{\cdot \ell} \otimes \text{id}_{\text{Int} H^\vee})) ,
\]

where the first factor is the underlying right-left Hopf \( H^\vee \)-module \( (X, \mu_{\cdot \ell}, \Delta_{\cdot \ell}) \) of \( X \) (see Figure 3) with the modified left action and right coaction:

\[
\mu^a_{\cdot \ell} := \{(H^\vee \otimes X) \Delta^a_{\cdot \ell} \otimes \text{id}_X \rightarrow H^\vee \otimes H^\vee \otimes X \rightarrow \text{id}_X \otimes \text{id}_{H^\vee} \rightarrow \text{id}_X \otimes \mu_{\ell} \rightarrow \text{id}_X \otimes X\},
\]

\[
\Delta^a_{\cdot \ell} := \{X \xrightarrow{\Delta_{\cdot \ell}} X \otimes H^\vee \xrightarrow{\text{id}_X \otimes \Delta^a_{\cdot \ell} \otimes \text{id}_{H^\vee}} X \otimes H^\vee \otimes H^\vee \rightarrow \text{id}_X \otimes \mu_{\ell} \rightarrow \text{id}_X \otimes H^\vee \rightarrow \text{id}_X \otimes X \otimes H^\vee\},
\]

and \( \text{Int} H^\vee \) is considered with trivial (co)module structures. The Hopf bimodule axioms for \( X \otimes \text{Int} H^\vee \) are equivalent to the conditions that \((X, \mu^a_{\cdot \ell}, \mu_{\cdot \ell}, \Delta_{\cdot \ell}) \in \text{Obj}(\mathcal{H}_H^\vee \mathcal{C}_H^\vee)\), \((X, \mu_{\cdot \ell}, \Delta_{\cdot \ell}, \Delta^a_{\cdot \ell}) \in \text{Obj}(\mathcal{H}_H^\vee \mathcal{C}_H^\vee)\), and the following modified version of the right Hopf module axiom:

\[
\begin{array}{c}
\text{Second proof of Theorem 3.4.} \text{ Analogs of Proposition 7.3 are true if categories of Hopf modules or two-fold Hopf modules are used instead of category of Hopf } H \text{-bimodules. In particular, one can consider the regular Hopf bimodule } H^\vee \text{ and put } X = (H^\vee)^\bullet \text{ or } X = (H^\vee), \text{ considered as an object of } \mathcal{C}_H^\vee \text{ with corresponding structures defined in Figure 3. The left-right Hopf module axiom for } X \otimes \text{Int } H^\vee \text{ from Proposition 7.3 is equivalent to the left-right Hopf module axiom for } (X, \mu^a_{\cdot \ell}, \Delta^a_{\cdot \ell}). \text{ For the latter axiom } L.H.S = R.H.S. : H^\vee \otimes H^\vee \rightarrow H^\vee \otimes H^\vee \text{ the identity } (\varepsilon \otimes \text{id}) \circ L.H.S \circ (\eta \otimes \text{id}) = (\varepsilon \otimes \text{id}) \circ R.H.S. \circ (\eta \otimes \text{id}) \text{ is the generalized Radford formula for } H^\vee.
\end{array}
\]

\[
\begin{array}{c}
\text{Third proof of Theorem 3.4.} \text{ There is another possibility to extend the left } H \text{-component of } Y \in \mathcal{H}_H^\vee \mathcal{C}_H^\vee \text{ to a left Hopf module structure. Specifically, Figure 21(a) describes } Y \text{ as a right Hopf } H_{\text{op}} \text{-module. We can apply the reasonings in the previous proof to this case as well. Now, for } Y = H^\vee \text{ we get } Y \Psi = \int_{H^\vee} H^\vee, \text{ where } \Psi = \left(c_{\ell}^r\right)^{-1} \cdot \int_{H^\vee} \text{ and the canonical projection } \overline{\lambda^H_{X,H^\vee}} : X \otimes H^\vee \rightarrow X \otimes \text{Int } H^\vee = X \otimes H^\vee \text{ is presented on Figure 21(b). The splitting } \bar{\theta} \text{ is presented in Figure 21(c).}
\end{array}
\]

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equal to \((c_\ell^t)^{-1}\cdot \text{id}_X \otimes \int H^\vee\). The formulae for the actions and coactions of \(H^\vee\) on \(X \otimes \text{Int} H^\vee\) are given by expressions similar to (7.7) with \(\lambda\) and \(\theta\) replaced by \(\bar{\lambda}\) and \(\bar{\theta}\), respectively. They are described explicitly by Figure 22.

Both presentations of \(X \otimes_H H^\vee\) must agree. Therefore, there exists an isomorphism, \(\phi : X \otimes_H H^\vee \to X \otimes_H H^\vee\) of Hopf \(H^\vee\)-bimodules, equipped with the first and the second structure, such that
\[
\bar{\lambda} = \left( X \otimes H^\vee \xrightarrow{\lambda} X \otimes_H H^\vee \xrightarrow{\phi} X \otimes_H H^\vee \right).
\]

Clearly,
\[
\phi = \left( X \otimes \text{Int} H^\vee \xrightarrow{\theta} X \otimes H^\vee \xrightarrow{\bar{\lambda}} X \otimes \text{Int} H^\vee \right).
\]

A calculation gives \(\phi = (-a_H^{-1}) \otimes \text{id}\). The statement that \(\phi\) is a homomorphism of \(H^\vee\)-modules and left \(H^\vee\)-comodules adds nothing new to our knowledge. However, the statement that \(\phi\) is
a homomorphism of right $H^\vee$-comodules implies (and is equivalent to) the generalized Radford formula. This gives the third proof of Theorem 3.6.

Remark 7.4. The left dual Hopf $H$-bimodule to $X \in \text{Obj } \mathcal{H}_C^H$ is obtained by application of the functor $\vee(\cdot)$ in $\mathcal{C}$ to the Hopf $H^\vee$-bimodule $X \otimes_H H^\vee$. As a result we obtain an object, $\text{Int} H \otimes \vee X$, equipped with tensor product $H$-(co)module structures, where $\text{Int} H$ is considered with trivial (co)actions, and the structures of $\cdot(\vee X)$ are modified by means of $a_H, \alpha_H, \Omega^\text{Int}_H$.

References

[1] Yu.N. Bespalov, *Hopf algebras: bimodules, crossproducts, differential calculus*, preprint, ITP-93-57E, 1993.

[2] Yu.N. Bespalov, *Crossed modules and quantum groups in braided categories*, Applied Categorical Structures 5 (1997), no. 2, q-alg 9510013.

[3] Yu.N. Bespalov, *On duality for braided cross product*, in preparation, 1997.

[4] Yu.N. Bespalov and B. Drabant, *Differential Calculus in Braided Tensor categories*, preprint, q-alg/9703059.

[5] Yu.N. Bespalov and B. Drabant, *Hopf (Bi-)Modules and Crossed Modules in Braided Categories*, to appear in J. Pure and Appl. Algebra, preprint, 1995. q-alg 9510009.

[6] R. J. Blattner and S. Montgomery, *A duality theorem for Hopf module algebras*, J. Algebra 95 (1985), 153–172.

[7] C. Chryssomalakos, *Remarks on quantum integration*, preprint ENSLAPP-A-562/95, December 1995.

[8] L. Crane and D. Yetter, *On algebraic structures implicit in topological quantum field theories*, preprint 1994, hep-th/9412023.

[9] P. Deligne and J.S. Milne, *Tannakian categories*, Lecture Notes in Math., 900 (Berlin), Springer-Verlag, 1982.

[10] A. Joyal and R. Street, *The geometry of tensor calculus I*, Advances in Math. 88 (1991), 55–112.

[11] M. Karoubi, *K-théorie*, Les Presses de l’Université de Montréal, 1971.

[12] T. Kerler, *Bridged links and tangle presentations of cobordism categories*. Adv. Math. to appear. [http://www.math.ohio-state.edu/~kerler/papers/BL/](http://www.math.ohio-state.edu/~kerler/papers/BL/)

[13] T. Kerler, *Genealogy of nonperturbative quantum-invariants of 3-manifolds: The surgical family*, in ‘Geometry and Physics’, Lecture Notes in Pure and Applied Physics 184, Marcel Dekker (1996). [http://www.math.ohio-state.edu/~kerler/papers/Gen/](http://www.math.ohio-state.edu/~kerler/papers/Gen/)

[14] T. Kerler and V.V. Lyubashenko, *Non-semisimple topological field theories for connected surfaces*, in progress.

[15] G. Kuperberg, *Non-involuntary Hopf algebras and 3-manifold invariants*, preprint, November 1994.

[16] S. Mac Lane, *Categories for working mathematicians*, Springer Verlag, New York, 1974, GTM vol. 5.
[17] R.G. Larson and M.E. Sweedler, An associative orthogonal bilinear form for Hopf algebras, Amer. J. Math. 91 (1969), no. 1, 75–94.

[18] V.V. Lyubashenko, Tangles and Hopf algebras in braided categories, J. Pure and Applied Algebra 98 (1995), no. 3, 245–278.

[19] V.V. Lyubashenko, Modular transformations for tensor categories, J. Pure and Applied Algebra 98 (1995), no. 3, 279–327.

[20] V.V. Lyubashenko, Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity, Commun. Math. Phys. 172 (1995), 467–516.

[21] S. Majid, Braided groups, J. Pure and Applied Algebra 86 (1993), 187–221.

[22] S. Majid, Free braided differential calculus, braided binomial theorem and the braided exponential map, J. Math. Phys. 34 (1993), 4843–4856.

[23] S. Majid, Transmutation theory and rank for quantum braided groups, Math. Proc. Camb. Phil. Soc. 113 (1993), 45–70.

[24] S. Majid, Algebras and Hopf algebras in braided categories, Advances in Hopf Algebras, Marcel Dekker, 1994, volume 158 of Lec. Notes in Pure and Appl. Math, pp. 55–105.

[25] S. Matveev and M. Polyak, A geometrical presentation of the surface mapping class group and surgery, Commun. Math. Phys. 160 (1994), no. 3, 537–550.

[26] D.E. Radford, The order of antipode of a finite-dimensional Hopf algebra is finite, Amer. J. Math 98 (1976), 333–335.

[27] D.E. Radford, The trace function and Hopf algebras, J. Alg. 163 (1994), 583–622.

[28] P. Schauenburg, Hopf Modules and Yetter-Drinfel’d Modules, J. Algebra 169 (1994), 874–890.

[29] M.E. Sweedler, Hopf algebras, W.A.Benjamin, New York, 1969.

[30] M.E. Sweedler, Integrals for Hopf algebras, Ann. of Math. 89 (1969), no. 2, 323–335.

[31] V. Turaev, Quantum invariants of knots and 3-manifolds, Walter de Gruyter, Berlin, New York, 1994.

[32] D. Yetter, Portrait of the handle as a Hopf-algebra, to appear in Proc. of Aarhus Special Session on Geometry and Physics.

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