Separation of variables for quantum integrable models related to $U_q(\widehat{sl}_N)$

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Abstract. In this paper we construct separated variable for quantum integrable models related to the algebra $U_q(\widehat{sl}_N)$. This generalizes the results by Sklyanin for $N = 2, 3$.

1 Introduction.

In the papers [1], [2] we have investigated the relation between classical and quantum integrable models and affine Jacobi variety (deformed in quantum case) for hyper-elliptic curves. Two works are of fundamental importance for our investigation. One of them is the book by Mumford [3], the other is the work by Sklyanin [4] who originated the method of separation of variables in the quantum case.

In the paper [1] it has been explained that topological properties of the affine Jacobian are important for applications to integrable models. The hyper-elliptic case is very special from this point of view. It is conjectured in [1] and proved in [3] that the cohomologies are extremely simple in this case. It is clear from the calculation of Euler characteristic [6] that the situation is much more complicated in the case of more general spectral curve. This difference explains our interest in the case of integrable models related to $sl_N$. 

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for $N \geq 3$. Let us explain it in some details.

Consider the classical case. The fundamental object of the theory of integrable models is the L-operator which is $N \times N$ matrix depending on the spectral parameter $x$. We assume that the L-operator is a polynomial of degree $n$ of special form as described below. An integrable model of this type is closely related to affine Jacobian of the spectral curve. If some conditions of non-degeneracy are satisfied the genus of this curve equals:

$$g = \frac{1}{2}(N-1)(Nn-2)$$

In classics one can construct the separated variables $(w_j, z_j)$ ($j = 1, \ldots, g$). Relevant for our applications description of separation of variables in the classical case is given in the paper [4]. Every pair $(w_j, z_j)$ is canonically conjugated. It is equally important that every pair belongs to the spectral curve. Thus the separated variables describe a divisor on the spectral curve which is mapped into affine Jacobian by Abel transformation.

In the quantum case we do not know an independent definition of deformed Jacobian. However, following Sklyanin [4] we are able to construct for $N = 2$ and, which is far more complicated, for $N = 3$ [8] the separated variables. This provides the way of constructing the quantum version of the algebra-geometric methods.

So, constructing the separated variables is only a part of our program. But for the case of arbitrary $N$ even this part happens to be highly non-trivial. That is why we decided to publish it separately.

2 Separation of variables.

In this paper we consider standard in R-matrix commutation relations, but we shall write them in unusual form. Consider the algebra $U_q(\widehat{sl}_N)$. Consider the evaluation module related to $N$-dimensional irreducible module $V$ of the finite-dimensional subalgebra $U_q(sl_N)$:

$$V(x) = V \otimes \mathbb{C}(x)$$

We shall call $V \otimes 1$ the finite-dimensional subspace of $V(x)$. Suppose we have some "quantum space" $H$. Consider the L-operator $L_{V(x)} \in \text{End}(H) \otimes \text{End}(V(x))$ As it has been said in the Introduction we shall consider $L_{V(x)}$ as
polynomial of $x$ of degree $n$. More precisely:

\[ L_V(x) = L^+(x) + L^0(x) + xL^-(x) \]

where $L^+(x)$, $L^0(x)$ and $L^-(x)$ are respectively upper-triangular, diagonal and lower-triangular. They are respectively polynomials of degree $n - 1$, $n$, $n - 1$.

The module $V(x)$ should be viewed as an "index". As operators in $H$ elements of $L_V(x)$ satisfy the commutation relations:

\[ R_{V(x),V(y)}L_V(x)L_V(y) = L_V(y)L_V(x)R_{V(x),V(y)} \]  \hspace{1cm} (1)

where the $U_q(\hat{sl}_N)$ $R$-matrix for evaluation modules corresponding to vector representations is:

\[ R_{V(x),V(y)} = xR_{12}(q) - yR_{21}(q)^{-1} \]  \hspace{1cm} (2)

where

\[ R_{12}(q) = \sum_{j=1}^{N} q^{E_{ij}} \otimes q^{E_{ij}} + (q - q^{-1}) \sum_{j>i} E_{ji} \otimes E_{ij}, \]

$E_{ij}$ is the matrix with 1 at the intersection of $i$-th row and $j$-th column and zeros elsewhere. We present $L_V(x)$ in the form:

\[ L_V(x) = \begin{pmatrix} a_v(x) & b_v(x) \\ c_v(x) & d_v(x) \end{pmatrix} \]

where as finite-dimensional space $v(x)$ is $(N - 1)$-dimensional subspace of $V(x)$; $a_v(x)$, $b_v(x)$, $c_v(x)$, $d_v(x)$ are respectively scalar, covector, vector and matrix. We shall use the operators $b_v(x)$ and $d_v(y)$; their commutation relations can be summarized as follows:

\[ (xq - yq^{-1})b_v(x)b_v(y) = b_v(y)b_v(x)r_v(x,y) \]  \hspace{1cm} (3)

\[ r_{v(x),v(y)}d_v(x)d_v(y) = d_v(y)d_v(x)r_{v(x),v(y)} \]

\[ (x - y)b_v(x)d_v(y) + y(q - q^{-1})s_{v(x),v(y)}d_v(x)b_v(y) = d_v(y)b_v(x)r_{v(x),v(y)} \]

where $r_{v(x),v(y)}$ is $U_q(\hat{sl}_{N-1})$ $R$-matrix for vector representations (see (2),

\[ s_{v(x),v(y)} = \sum_{j=1}^{N-1} e_j^* \otimes e_j, \]
and \( e_j \) are dual bases in the finite-dimensional subspaces of \( v^*(x) \) and \( v(y) \) respectively. In order to illustrate our notations let us rewrite the last equation from (3) in components. Denote the components of the covector \( b_v(x) \) by \( b_i(x) \) and the matrix elements of \( d_v(y), r_v(x,y) \) by respectively \( d^k_i(y) \), \( r^{lm}_{ij}(x,y) \). Then the equation in question reads:

\[
(x - y)b_i(x)d^k_j(y) + y(q - q^{-1})d^k_i(x)b_j(y) = d^k_m(y)b_l(x)r^{lm}_{ij}(x,y)
\]

where the summation over repeated indices is implied.

In this paper we shall need evaluation modules of \( U_q(\hat{sl}_N-1) \) which correspond to \( q \)-deformation of fundamental representations. Namely, consider the tensor product of \( k \) modules \( (1 \leq k \leq N - 1) \):

\[
v(xq^{-2(k-1)}) \otimes \cdots \otimes v(xq^{-2}) \otimes v(x)
\]

This module contains an irreducible submodule \( w^k(x) \) which can be extracted by applying the projector \( p^k \) (\( q \)-antisymmetrizer). The latter is defined recursively:

\[
p^k = (x^{-(k-1)}r_v(xq^{-2(k-1)},v(xq^{-2(k-2)})) \cdots r_v(xq^{-2(k-1)},v(x))) p^{k-1}
\]

Notice that \( w^1(x) = v(x) \), \( w^{(N-1)}(x) \simeq \mathbb{C} \).

The relations (3) are covariant in the following sense. Consider an evaluation module \( w(z) \) and construct the operators:

\[
\begin{align*}
& b_v(x),H \otimes w(z) = b_v(x)r_v(x),w(z) \\
& d_v(x),H \otimes w(z) = d_v(x)r_v(x),w(z)
\end{align*}
\]

where we use the notation \( r_v(x),w(z) \) for the R-matrix evaluated on corresponding modules. It is clear that \( b_v(x),H \otimes w(z) \) and \( d_v(x),H \otimes w(z) \) satisfy the relations (3).

The relations (3) imply a complicated system of commutation relations of the form:

\[
\sum_{q=0}^l X^q_{kl}(x, y) b_v(y), H \otimes v(x) (d_v(y), H \otimes v(x))^q = \\
\left[ \sum_{p=0}^k Y^p_{kl}(x, y) b_v(x), H \otimes v(y) (d_v(x), H \otimes v(y))^p \right] \frac{r_v(x), v(y)}{xq - yq^{-1}}
\]

(4)
The operators $X_{kl}^q(x, y)$ and $Y_{kl}^p(x, y)$ are rather complicated and we shall need only partial information about them. First, the leading coefficients are:

$$X_{kl}^l(x, y) = \kappa(x, y)^k b_v(x) (d_v(x))^k$$
$$Y_{kl}^k(x, y) = \kappa(x, y)^l b_v(y) (d_v(y))^l$$

where

$$\kappa(x, y) = \left( \frac{(x^q - y)(x^q - 2 - y)}{x - y} \right)$$

The rest of these operators can be found from recurrence relations which follow from (3) using induction. For example

$$X_{k+1,l+1}^{q+1}(x, y) = \kappa(x, y)\left[ X_{kl}^q(x, y) + \left( \frac{x(x^q - y)}{x - y} \right) \sum_{j=1}^{q-1} X_{kl}^{q+j}(x, y) \kappa(x, y)^{j-1} s_v(y) s_v(x) (d_v(y))^j s_v(x) s_v(y) \right] d_v(x)$$

**Lemma 1.** Suppose $l \geq k$, then

$$b_v(x) (d_v(x))^k b_v(xq^{-2}, H \otimes v(x)) (d_v(xq^{-2}, H \otimes v(x))^l =$$

$$= \sum_{p=l-k}^{l-1} \tilde{X}_{kl}^{p-l+k}(x, xq^{-2}) b_v(xq^{-2}, H \otimes v(x)) (d_v(xq^{-2}, H \otimes v(x))^p \quad (5)$$

In particular

$$b_v(x) b_v(xq^{-2}, H \otimes v(x) = 0 \quad (6)$$

In these formulae $v(x)$ and $v(xq^{-2})$ are considered as unrelated modules, i.e. the latter being understood as

$$v(xq^{-2}) = v(y)|_{y=xq^{-2}}$$

**Proof.** Obviously it is sufficient to consider the case $k = l$. Look at the equation (4). From the recurrence relations cited above one finds that for $q \geq 0$

$$X_{kk}^q(x, y) = \kappa(x, y)^k \left[ \tilde{X}_{kk}^q(x, y) + O(xq^2 - y) \right]$$
where $\tilde{X}^q_{kk}(x, y)$ is the only term which does not vanish for $y = xq^{-2}$. It can be shown that $\kappa(x, y)^{-k}Y^p_{kk}(x, y)$ is finite for $y = xq^{-2}$. The operator $d_v(x), H \otimes v(y)$ contains $T_v(x)_v(y)$ as right multiplier. So, the RHS of (4) contains

$$\frac{1}{xq - yq^{-2}} T_v(x)_v(y) T_v(y)_v(x) = (xq^{-1} - yq)$$

Dividing (4) by $\kappa(x, y)^k$ and putting $y = xq^{-2}$ one proves the Lemma.

QED

Now we are ready to introduce the most important definition.

**Definition 1.** The operator $b_{w^k(x)_H} \in (w^k(x))^* \otimes \text{End}(H)$ is defined recursively; $b_v(x), H \equiv b_v(x)$ is familiar, further define:

$$b_{w^k(x)_H} = x^{-1} b_v(x), H (d_v(x), H)^{-1} b_{w^{k-1}(xq^{-2})_H}$$

The notation $b_{w^{k-1}(xq^{-2})_H}$ stands for $b_{w^{k-1}(xq^{-2})_H}$ in which all the $b_v(xq^{-2})$ and $d_v(xq^{-2})$ are replaced by $b_v(xq^{-2}) T_v(xq^{-2}), v(x)$ and $d_v(xq^{-2}) T_v(xq^{-2}), v(x)$.

It is easy to observe that the RHS of (7) contains effectively the projector $p^k$ acting from the right, hence it belongs to $(w^k(x))^*$, so, the definition is consistent. Lemma 1 has important

**Corollary.** The following vanishing property holds:

$$b_v(x), H (d_v(x), H)^k b_{w^{l}(xq^{-2})_H} = 0$$

for $k < l$ (8)

**Proof.** Write down the explicit expression for $b_{w^{l}(x), H}$ using $l - 1$ times (7). Then apply repeatedly the relation (3) lowering the degree $k$. At some stage the degree will turn to zero, then the relation (6) is applicable.

QED

It has been mentioned that $w^{(N-1)}(x) \simeq \mathbb{C}$ which means that

$$b_{w^k(x), H \otimes w^{(N-1)}(y)} = \varphi_k(x, y) b_{w^k(x), H}$$

where $\varphi_k(x, y)$ is a $\mathbb{C}$-number function. The latter can be calculated explicitly using known formula for quantum determinant of R-matrix:

$$\varphi_k(x, y) = \prod_{j=0}^{k-1} \left( (xq^{-2j+1} - yq^{1-2N}) \prod_{i=0}^{N-3} (xq^{-2j} - yq^{-2i}) \right)^{k-j}$$
The most important property of the operators $b_{w^k(x), H}$ is given by
\textbf{Lemma 2.} The relation holds

$$
\chi_{l,k}(x, y) b_{w^k(x), H} b_{w^l(y), H \otimes w^k(x)} = \\
\chi_{l,k}(y, x) b_{w^l(y), H} b_{w^k(x), H \otimes w^l(y)} \psi_{l,k}(y, x) r_{w^l(y), w^k(x)}
$$

where the C-number functions are given by

$$
\chi_{k,l}(x, y) = \prod_{j=0}^{k-1} \left( \prod_{i=0}^{l-1} \kappa(xq^{-2j}, yq^{-2i}) \right)^{k-j},
$$

$$
\psi_{l,k}(y, x) = \prod_{j=0}^{l-1} \prod_{i=0}^{k-1} \left( yq^{-2i-1} - xq^{-2j+1} \right)^{-1}
$$

\textbf{Proof.} Rewrite the commutation relation (4) in the form:

$$
\kappa(x, y)^i b_{v(x), H}(d_{v(x), H})^i b_{v(y), H \otimes v(x)}(d_{v(y), H \otimes v(x)})^j = \\
= \kappa(y, x)^j \left[ b_{v(y), H}(d_{v(y), H})^j b_{v(x), H \otimes v(y)}(d_{v(x), H \otimes v(y)})^i \right] \frac{r_{v(y), v(x)}}{xq - yq^{-1}} + \\
+ \text{"unwanted terms"}
$$

where

\text{"unwanted terms"} =

$$
- \sum_{q=0}^{j-1} X_{ij}^q(x, y) b_{v(y), H \otimes v(x)}(d_{v(y), H \otimes v(x)})^q + \\
+ \left[ \sum_{p=0}^{l-1} Y_{ij}^p(x, y) b_{v(x), H \otimes v(y)}(d_{v(x), H \otimes v(y)})^p \right] \frac{r_{v(y), v(x)}}{xq - yq^{-1}}
$$

We prove the relation (10) using induction with respect to $k$ and $l$. The definition (7) implies that we have to use the relations (11) in the inductive procedure. The most important point of the proof is that due to the relations (8) the contributions from "unwanted terms" vanish.

\textbf{QED}

In what follows we shall need the formula:

$$
r_w(y, w^{N-1}(x)) = \rho_t(y, x) I,
$$

$$
\rho_t(y, x) = \prod_{i=0}^{l-1} (yq^{-2i+1} - xq^{-2j}) \prod_{j=0}^{N-3} (yq^{-2i} - xq^{-2j})
$$
Now we are in position to define the basic objects

**Definition 2.** \( B(x) = b_{w,N^{-1}(x),H} \).

**Theorem 1.** The operators \( B(x) \) generate a commuting family:

\[
[B(x), B(y)] = 0
\]  

**Proof.** Due to relations (10) and (9) and (12) the proof reduces to the identity for \( \mathbb{C} \)-number functions:

\[
\chi_{N-1,N^{-1}}(x,y)\varphi_{N^{-1}}(y,x) = \\
= \chi_{N-1,N^{-1}}(y,x)\varphi_{N^{-1}}(x,y)\psi_{N^{-1},N^{-1}}(y,x)\rho_{N^{-1}}(y,x)
\]

This identity is proved by direct calculation.

QED

The operator \( B(x) \) is a polynomial of degree 

\[ g = \frac{1}{2}(N-1)(Nn-2). \]

Due to Theorem 1 it can be developed as

\[
B(x) = \beta \prod_{j=1}^{g} (x - z_j)
\]

where all operators \( \beta, z_1, \cdots, z_g \) mutually commute. The operator \( \beta \) is a kind of zero-mode, it is not very interesting for us. We want to define the operators canonically conjugated to \( z_1, \cdots, z_g \). To this end we have to define a new object.

**Definition 3.** Consider some covector \( \xi \in \mathbb{C}^{N-1}^\ast \). By \( \xi_{v(x)} \) we denote \( \xi \) in the finite-dimensional subspace of \( v^\ast(x) \). Define

\[
Y(x) = \xi_{v(x)} b_{w,N^{-2}(xq^{-2}),H\otimes v(x)}; \\
X(x) = \xi_{v(x)} d_{v(x)} b_{w,N^{-2}(xq^{-2}),H\otimes v(x)}
\]

and

\[
D(x) = Y^{-1}(x)X(x)
\]

The operator \( D(x) \) depends upon the choice of \( \xi \), however, the final objects we are interested in \( (w_j) \) are independent of \( \xi \) as is explained later. By already familiar calculations we get

**Lemma 3.** The relation holds

\[
b_{w,k(x),H\otimes v(y)} r_{v(y),w(x)} d_{v(y)} = \\
= \sigma_k(x,y) d_{v(y)} b_{w,k(x),H\otimes v(y)} + U(x,y)b_{v(y)}
\]  

(14)
where \( U(x, y) \) is some irrelevant for our goals operator,

\[
\sigma_k(x, y) = \prod_{j=0}^{k-1} \kappa(xq^{-2j}, y)
\]

The commutation relation between \( D \) and \( B \) is given by

**Theorem 2.** The relation holds

\[
B(x)D(y) = \frac{xq - yq^{-1}}{x - y}D(y)B(x) + Z_{N-2}(x, y)B(y)
\]  \hspace{1cm} (15)

the operator \( Z_{N-2}(x, y) \) is defined below.

**Proof.** The relation (10) implies that

\[
B_{H \otimes v(y)}(x) b_{w^{N-2}(yq^{-2}), H \otimes v(y)} = \mu(x, y) b_{w^{N-2}(yq^{-2}), H \otimes v(y)} \
\]

with some irrelevant for us function \( \mu(x, y) \), the notation \( B_{H \otimes v(y)}(x) \) should be clear. Multiplying this equation by \( \xi_{v(y)} \) or by \( \xi_{v(y)} d_{v(y)} \), and using the relation (14) one finds two equations:

\[
\begin{align*}
\eta_{v(y)}(x, y) b_{w^{N-2}(yq^{-2}), H \otimes v(y)} &= \mu(x, y) Y(y)B(x), \\
\eta_{v(y)}(x, y) d_{v(y)} b_{w^{N-2}(yq^{-2}), H \otimes v(y)} &= \mu(x, y) \sigma_{N-1}(x, y)X(y)B(x),
\end{align*}
\]  \hspace{1cm} (16)

where

\[
\eta_{v(y)}(x, y) = \xi_{v(y)} B_{H \otimes v(y)}(x)
\]

Notice that \( \eta_{v(y)}(x, y) \) is a covector in finite-dimensional subspace of \( v^*(y) \). Thus we can develop it with respect to the following basis:

\[
\eta_{v(y)}(x, y) = Z(x, y) \xi_{v(y)} + \sum_{j=1}^{N-2} Z_j(x, y) b_{v(y)}(d_{v(y)})^{j-1}
\]  \hspace{1cm} (17)

where \( Z(x, y), Z_j(x, y) \) are operator-valued coefficients. We shall need two of them: \( Z(x, y) \) and \( Z_{N-2}(x, y) \). In order to find the first of them we multiply (17) by \( b_{w^{N-2}(yq^{-2}), H \otimes v(y)} \) and use the property (8). This gives

\[
Z(x, y) = \left( \eta_{v(y)}(x, y) b_{w^{N-2}(yq^{-2}), H \otimes v(y)} \right) Y^{-1}(y)
\]

Calculation of \( Z_{N-2}(x, y) \) is somewhat more complicated. Multiply (17) by \( b_{w^{N-2}(yq^{-2}), H \otimes v(y)} \). The result is a covector from \( w^{N-2}(y) \). The finite-dimensional part of the latter space is isomorphic to \( \mathbb{C}^{N-2} \). Consider a vector
from this finite dimensional part and convolute the equation (17) with \( \nu \). Then using already known expression for \( Z(x, y) \) one finds:

\[
Z_{N-2}(x, y) = \eta_{v(y)}(x, y) \left( 1 - b_{w^{N-2}(yq^{-2}), H \otimes v(y)} Y^{-1}(y) \xi_{v(y)} \right) 
\times b_{w^{N-3}(yq^{-2}), H \otimes v(y)} \nu_{w^{N-2}(y)} \tilde{Y}^{-1}(y)
\]

where

\[
\tilde{Y}(y) = b_{w^{N-2}(y), H} \nu_{w^{N-2}(y)}
\]

Substitute (17) into (16) and use (8):

\[
Z(x, y) Y(y) = \mu(x, y) Y(y) B(x),
\rho_1(y, x)(Z(x, y) X(y) + Z_{N-2}(x, y) B(y)) =
= \mu(x, y) \sigma_{N-1}(x, y) X(y) B(x)
\]

(18)

From these equations the relation (17) follows immediately.

QED

Following Sklyanin [8] we define the operators

\[
w_j = \int_C D(y) \frac{dy}{y - z_j}
\]

where \( C \) encircles \( z_j \). The meaning of this definition is clear: we substitute \( z_j \) as argument of \( D(y) \) from the right. Apply this operation to the equation (13). The term \( Z_{N-2}(y) B(y) \) vanishes because of presence on \( B(y) \). Certainly, one should be careful because \( Z_{N-2}(x, y) \) contains \( Y(y) \) and \( \tilde{Y}(y) \) in denominator, but these operators cannot cancel \( B(y) \). Thus

\[
w_i z_j = q^{2 \delta_{ij}} z_j w_i, \quad w_j \beta = q^{-1} \beta w_j
\]

Let us show that \( w_j \) does not depend on the choice of \( \xi \). Take another covector \( \xi' \) and write the formula:

\[
\xi'_{v(x)} = U(x) \xi_{v(x)} + \sum_{j=1}^{N-2} U_j(x) b_{v(x)} (d_{v(x)})^{j-1}
\]

with some \( U, U_j \) which can be found explicitly. From this formula one easily finds that

\[
D_{\xi'}(x) = D_\xi(x) + Y_{\xi'}(x)^{-1} U_{N-2}(x) B(x)
\]
where dependence on ξ and ξ' is marked explicitly. The last term vanishes when \( z_j \) is substituted from the right. So, \( w_j \) does not depend on ξ.

It remains to prove that

\[ [w_i, w_j] = 0 \]  \hspace{1cm} (19)

To this end we shall prove the

**Theorem 3.** The following commutation relations hold:

\[ D(x)D(y) = D(y)D(x) + S(x, y)B(x) + T(x, y)B(y) \]  \hspace{1cm} (20)

where \( S(x, y), T(x, y) \) are some operators which can be found explicitly.

**Proof.** Consider the relation

\[
b_{w\bar{N}^{-2}(xq^{-2}), H\otimes v(y)\otimes v(x)} b_{w\bar{N}^{-2}(yq^{-2}), H\otimes v(y)\otimes v(x)} = \\
= \mu(x, y) b_{w\bar{N}^{-2}(xq^{-2}), H\otimes v(y)\otimes v(x)} b_{w\bar{N}^{-2}(yq^{-2}), H\otimes v(y)\otimes v(x)} b_{w\bar{N}^{-2}(yq^{-2}), H\otimes v(y)\otimes v(x)}
\]

\[ \times r_{w\bar{N}^{-2}(yq^{-2}), w\bar{N}^{-2}(yq^{-2})} \]  \hspace{1cm} (21)

where \( \mu(x, y) \) is some irrelevant function. The operators \( b_{w\bar{N}^{-2}(xq^{-2}), H\otimes v(y)\otimes v(x)} \), \( b_{w\bar{N}^{-2}(yq^{-2}), H\otimes v(y)\otimes v(y)} \) contain effectively projectors on the subspaces respectively \( w\bar{N}^{-1}(x) \) and \( w\bar{N}^{-1}(y) \). Hence (21) can be rewritten as

\[
b_{w\bar{N}^{-2}(xq^{-2}), H\otimes v(y)\otimes v(x)} b_{w\bar{N}^{-2}(yq^{-2}), H\otimes v(y)\otimes v(x)} = \\
= \tilde{\mu}(x, y) b_{w\bar{N}^{-2}(yq^{-2}), H\otimes v(y)\otimes v(x)} r_{v(y), w\bar{N}^{-2}(yq^{-2})} b_{w\bar{N}^{-2}(xq^{-2}), H\otimes v(x)}
\]

Multiply (22) by \( \xi_v(x) \), introduce the notation

\[ \eta_{v(x), H\otimes v(y)}(x, y) = \xi_v(x) b_{w\bar{N}^{-2}(yq^{-2}), H\otimes v(y)\otimes v(x)} r_{v(y), w\bar{N}^{-2}(yq^{-2})} \]

and use the same trick as in the proof of Theorem 2:

\[
\eta_{v(x), H\otimes v(y)} = V(x, y)_{H\otimes v(y)} \xi_v(x) + \sum_{j=1}^{N-2} V_j(x, y)_{H\otimes v(y)} b_v(x) (d_v(x))^{j-1}
\]

the explicit formulae for \( V(x, y)_{H\otimes v(y)} \) and \( V_{N-2}(x, y)_{H\otimes v(y)} \) can be found as it has been done in the proof of Theorem 2. Hence

\[ Y_{H\otimes v(y)} b_{w\bar{N}^{-2}(xq^{-2}), H\otimes v(y)} = \tilde{\mu}(x, y) V(x, y)_{H\otimes v(y)} Y(u) \]  \hspace{1cm} (23)
Further, using Lemma 3 one gets
\[
X_{H \otimes v(y)} b_{w^{N-2}(yq^{-2}), H \otimes v(y)} = \\
= \sigma_{N-2}(x, y) \tilde{\mu}(x, y)(V(x, y)_{H \otimes v(y)} Y(x) + V_{N-2}(x, y)_{H \otimes v(y)} B(u))
\]
Combining these two equations one finds
\[
b_{w^{N-2}(yq^{-2}), H \otimes v(y)} D(x) + V_{N-2}(x, y)_{H \otimes v(y)} B(x) = \\
= \sigma_{N-2}(x, y) D_{H \otimes v(y)}(x) b_{w^{N-2}(yq^{-2}), H \otimes v(y)}
\]
Let
\[
\zeta_{v(y)}(x, y) = \xi_{v(y)} D_{H \otimes v(y)}(x)
\]
Then
\[
\zeta_{v(y)}(x, y) = W(x, y) \xi_{v(y)} + \sum_{j=1}^{N-2} W_j(x, y) b_{v(y)}(d_{v(y)})^{j-1}
\]
and
\[
Y(y) D(x) + \xi_{v(y)} V_{N-2}(x, y)_{H \otimes v(y)} B(u) = \sigma_{N-2}(x, y) W(x, y) Y(y) \quad (24)
\]
Using Lemma 3 one shows that
\[
d_{v(y)} D_{H \otimes v(y)}(x) = D_{H \otimes v(y)}(x) d_{v(y)} + \text{(smth)} b_{v(y)}
\]
Using this equation and (3) one finds:
\[
X(y) D(x) + \xi_{v(y)} V_{N-2}(x, y)_{H \otimes v(y)} B(u) = \\
= \sigma_{N-2}(x, y) (W(x, y) Y(y) + W_{N-2}(x, y) B(v)) \quad (25)
\]
From (24), (25) and Theorem 2 one proves Theorem 3 follows immediately. We do not write explicit expressions for \( S(x, y) \) and \( T(x, y) \), they contain some denominators which do not cancel \( B(x) \) and \( B(y) \).
QED
The commutation relations (19) follow from this theorem.

3 Conclusion.

Let us mention two problems which has not been addressed in this paper.
First, it should be shown that every observable in the model can be expressed in terms of separated variables. This is actually not the case. There are some additional degrees of freedom. The best way out of this difficulty is to perform a reduction eliminating this additional degrees of freedom but keeping the reduced algebra of observables closed. This procedure has been performed in \cite{2} for $N = 2$ in quantum case and in \cite{7} for arbitrary $N$ in classics. The reduction is possible in the quantum case for arbitrary $N$, but we would like to postpone the publication of corresponding bulky calculations.

Second, the fact that the separated variables belong to the spectral curve is replaced in the quantum case by Baxter equation. We do not deduce this equation here, again we shall do it elsewhere.

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