ON K3 FIBRATIONS: TOWARDS MIRROR SYMMETRY

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Abstract. Given a K3 surface, a mirror dual to can be identified with a component of the moduli space of semistable sheaves on X. We consider fibrations by K3 surfaces over a one-dimensional base that are Calabi-Yau and we characterize the dual fibration that turns to be derived equivalent to the original one relating the problem to mirror symmetry.

1. Introduction

The classical form of mirror symmetry considers mirror pairs of Calabi-Yau 3-folds X and Ỹ, and the symplectic geometry (Gromov-Witten invariants) of X corresponds to the complex geometry (periods) of Ỹ.

Let X be a complex K3 surface and denote by NS(X), T(X), H(X, Z) the Néron-Severi lattice, the transcendental lattice and the Mukai lattice of X respectively.

Let f : X → C be a proper morphism of finite type with integral geometric fibres isomorphic to a polarized K3 surface over an algebraic curve C of genus g. We prove the existence of a projective relative moduli space for stable sheaves on the fibers of f that turns to be derived equivalent to the original fibration and can be considered as a dual fibration.

Consider the fine moduli space \( \mathcal{M}(r, e, s) \) parametrizing e-stable sheaves \( E \) on X such that \( c_0(E) = rk(E) = r \), \( c_1(E) = e \) and \( \chi(E) = r + s \). Here stability means Gieseker stability as considered in [Sim]. The vector \( v = (r, e, s) \in H(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^{1,1}(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \) is a class in the topological K-theory \( K_{top}(X) \) of the surface and it is called Mukai vector. Our main result is:

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Theorem 1.1. Given a non-singular fibration \( p : X \to C \) by K3 surfaces with a polarization class \( H \) of degree \( d \), there exists a dual fibration which is derived equivalent to the original one and corresponds to a connected component of the relative moduli space \( \mathcal{M}^l(X/C) \).

The moduli space \( \mathcal{M}^l(X) \) of semistable sheaves on \( X \) with respect to a fixed polarization \( l \), in general has infinitely many components, each of which is a quasi-projective scheme which may be compactified by adding equivalence classes of semistable sheaves. An irreducible component \( Y \subset \mathcal{M}^l(X/C) \) is said to be fine if \( Y \) is projective and there exists a universal family of stable sheaves, that is, an object of \( D^b(X \times Y) \) inducing a derived equivalence.

Corollary 1.2. There exists at least one fine component of the relative moduli space or equivalently a sheaf \( E \) on a non-singular fiber with fixed Mukai vector.

2. Derived categories of split-type Calabi-Yau manifolds

There are two main mathematical conjectures in Mirror Symmetry, Kontsevich homological mirror symmetry conjecture and the conjecture of Strominger, Yau and Zaslow, which predicts the structure of a CY manifold and how to get the mirror of a given CY manifold. Recall that by a Calabi-Yau manifold we mean a compact Kähler manifold \( X \) with trivial canonical bundle \( K_X \). Many examples of Calabi-Yau manifolds can be constructed by considering fibrations of lower dimensional varieties, that is, elliptic or K3 fibrations. These are the so-called split-type Calabi-Yau manifolds.

A K3 surface is a compact complex surface \( X \) which is connected and simply connected and has trivial canonical bundle \( K_X \), i.e., \( X \) has a unique (up to constant) nowhere vanishing holomorphic 2-form \( w_X \). The notion of K3 surface is invariant under deformation, i.e., any deformation of a K3 surface is a K3 surface. Moreover any two K3 surfaces are deformations of each other. Hence the lattice \( H^2(X;\mathbb{Z}) \) with the cup bilinear form \( \langle \cdot, \cdot \rangle : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z} \), even for all \( \alpha \in H^2(X,\mathbb{Z}) \), is the same for all K3 surfaces \( X \) and can be called the K3 lattice. Let \( e \in H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Z}) \) be the class of an ample divisor. Then \( (X, e) \) is a polarized K3 surface. The degree of the polarization is an integer \( 2d \), such that the scalar product \( \langle e, e \rangle \geq 2d = 2rs \) where \( d, r, s \) are any positive integers and their greatest common divisor \( (r, s) \) is 1.

Definition 2.1. Two K3 surfaces \( X, Y \) are said to be FM partners, if there is an equivalence \( D(X) \cong D(Y) \) of their bounded derived
categories of coherent sheaves. The set of isomorphism classes of FM partners of $X$ is denoted by $\text{FM}(X)$.

The homological mirror symmetry Conjecture. Homological mirror symmetry conjecture due to Kontsevich, asserts that there should be an equivalence of categories behind mirror duality, one category being the derived category of coherent sheaves on a Calabi-Yau manifold $D(X)$ and the other one being the Fukaya category $DFuk(\hat{X})$ of the mirror Calabi-Yau manifold.

Let $\pi : Y \to S$ be a fibration by $K3$ surfaces with a relative polarization. This means that on $Y$ we have a polarization class $H$ such that its restriction to each fibre $H|_{X_t} = e$ is the polarization class of the corresponding fibre. We can assume that the fibration is Calabi-Yau. Since the singular fibers are normal crossing divisors, and the total space and the base are projective varieties the fibration morphism is automatically proper. We are assuming that the fibers are equidimensional and therefore the morphism is flat. By the theorem of U. Persson and H. Pinkham [PP], there exists a birational map $\varphi : X \to X'$ where $X'$ has trivial canonical bundle and it is an isomorphism over the smooth locus such that the following diagram is commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X' \\
\pi \downarrow & & \pi' \downarrow \\
B & & 
\end{array}
$$

Now, by Bridgeland theorem (see [Bri]), two birational 3-folds have equivalent derived categories.

We consider the moduli problem of the dual fibration, that is, the dual fibration as the stack representing the Picard functor, that is, the moduli functor of semistable sheaves on the fibres that contains line bundles of degree 0 on smooth fibres. The corresponding coarse moduli space is not a fine moduli space due to the presence of singular fibres. Let us call $Y^\vee$ the dual fibration when it exists and satisfying the property that over the smooth locus the fibres correspond to the dual $K3$ surfaces of the original fibration.

3. Proof of Theorem 1.1

Let $\Sigma(p) \hookrightarrow C$ be the discriminant locus of $p$, that is, the closed subvariety corresponding to the singular fibres. From Hironaka’s theorem on the resolution of singularities, we may assume that the singular fibers are normal crossing divisors. Thus the fibration morphism is automatically proper and flat.
For every $t \in C - \Sigma(p)$, consider the $K3$ surface $X_t$ and its corresponding Mukai vector $(r_t, e, s_t)$, where $2r_t s_t = (H_t)^2 = H^2 = 2d$. By Mukai’s Theorem (see [Muk]), we may associate to $X_t$ a 2-dimensional moduli space $\mathcal{M}(r_t, e, s_t)$ which is a $K3$ surface as well with the same derived category to $X$, thus it is a FM partner. We observe that although the degree of the polarization is constant in $t$, the rank of the fibres can jump for some $t \in C$. However the condition of the Picard rank being one is open in the Zariski topology and it determines an open set

$$C^1 := \{ t \in C | \text{NS}(X_t) = \mathbb{Z}H_t \}.$$ 

Now if $s \in C^1$, then $H|_{X_t} = H_t = l$ is an ample divisor and since the number of Mukai partners depends on the prime decomposition $l^2 = 2p_1^{e_1} \cdots p_m^{e_m}$, where $k \geq 0$, $e_i \geq 1$ and $p_i$ primes with $p_i \neq p_j$, if $i \neq j$, there is a description of the FM partners of the $K3$ surface in terms of the Mukai vectors of the moduli spaces associated (see [St]). We need to single out a unique Mukai dual $K3$ surface. For example, the reflected Mukai vectors $(r_t, e, s_t)$ and $(s_t, e, r_t)$ give isomorphic moduli spaces $\mathcal{M}(r_t, e, s_t) \cong \mathcal{M}(s_t, e, r_t)$ even if the original $K3$ surfaces are not isomorphic. Thus, this choice gives rise to different dual fibrations.

If the rank of the Neron Severi group $\text{NS}(X_t)$ is bigger than 12, according to Morrison [Mo], there exists a torsion free semistable bundle on $X_t$, and the choice of dual $K3$ surface is unique in this case.

Consider the product $X_t \times \hat{X}_t$ of the corresponding $K3$ surface $X_t$ with its Mukai dual. Then we consider the universal family $P_t$ over the product $X_t \times \hat{X}_t$. Proceeding as in Proposition 2.4 of [Ma], extending the family $\mathcal{P} := \{ P_t : t \in B \}$ over the non singular locus by Deligne theorem ([Del]), the class of the polarization is invariant by the action of the monodromy group of the singular fibres, thus $\mathcal{P}$ extends to an object $\mathcal{F}$ over the whole fibration.

The family does not need to be universal, but according to Căldăraru (see [Cal]), a quasi-universal or twisted universal family sheaf always exists and thus the dual fibration $(X/C)^\vee$ is the coarse moduli space induced by $\mathcal{F}$. The fibration constructed thus far, is a connected component $\mathcal{M}$ of the relative moduli space $\mathcal{M}^b(X/C)$ of stable sheaves on $p$ with respect to the polarization, (Prop. 3.4. of [BM]). There exists a unique $\alpha$ in the Brauer group $Br(M)$ of $M$ with the property that an $p_M^*\alpha^{-1}$ twisted universal sheaf exists on $X \times M$, where $p_M$ is the projection map from $X \times M$ to $M$, and it is the obstruction to the existence of a universal sheaf on $X \times M$. This twisted universal sheaf yields an equivalence (Theorem 1.2 of [Cal]).

$$D^b(M, \alpha) \cong D^b(X).$$
So both fibrations are derived equivalent. □

4. Remarks and conclusions

There exists at least one fine component of the relative moduli space or equivalently a sheaf $E$ on a non singular fiber with fixed Mukai vector. A closed point of a relative moduli space corresponding to a sheaf $E$ on a fibre (not to a sheaf on the whole fibration). Let $X_s$ be a K3 surface or an abelian surface. The tangent space at that point to the moduli space of sheaves $M(X/S)$ on the fibration, can be identified with

$$T_M(E) \cong \text{Ext}^1_S(E, E).$$

If $\text{Ext}^2_S(E, E) = 0$, then $M$ is smooth at $E$. There are bounds (Corollary 4.5.2 of [HL]),

$$\text{Ext}^1(E, E) \geq \dim [E] M \geq \text{Ext}^1(E, E) - \text{Ext}^2(E, E).$$

In general to construct such components $Y$ of the relative moduli space, we assume that there exists a divisor $L$ on $X$ and integer numbers $r, s > 0$, such that there exists a sheaf $E$ on a non singular fiber $X_t$ which is stable with respect to $H_t$ and $s = ch_2(E) + r$. The component $Y(E)$ containing the class of the sheaf $E$ is a fine projective moduli space and the fibration $q : Y \rightarrow B$ is equidimensional. Thus there is a universal family on the product $Y \times Y(E)$ that gives the equivalence of the derived categories of both fibrations over $B$.

**Proposition 4.1.** Every fine projective component $Y$ of the relative moduli space $M^e(X/B)$ of stable sheaves with respect to a fixed polarization $e$ is derived equivalent to the original Calabi-Yau fibration $(X/B)$ and therefore are derived equivalent between them. Conversely, any projective variety derived equivalent to the original fibration is a component of the relative moduli space.

**Proof.** By Corollary [1.2] we can consider components $Y$ of the relative moduli space $M^e(X/B)$ of stable sheaves on the fibers of the CY fibration $(X/B)$, stable with respect to the polarization $e$. It is a fine moduli space, so there is a universal sheaf $\mathcal{P}$ over the product $X \times Y$. Bridgeland and Maciocia proved in [BM] that $Y$ is a non-singular projective variety, $\tilde{\mu} : Y \rightarrow B$ is a K3 fibration and the integral functor $D^b(Y) \rightarrow D^b(X)$ with kernel $\mathcal{P}$ is an equivalence of derived categories, that is, a Fourier-Mukai transform. It is Calabi-Yau because one has $D^b(X) \cong D^b(Y)$.

Now, we start with an equivalence $D^b(Y) \cong D^b(X)$, then by a result of Orlov [Or1], it is given by an object $E \in D^b(X \times_B Y)$ which satisfies a Calabi-Yau condition and thus by Theorem [1.1] this defines a fine
component of the relative moduli space. All the equivalences of the original fibration are obtained in this way. □

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