Comparing Weibull Stress – Strength Reliability Bayesian Estimators for Singly Type II Censored Data under Different loss Functions

Awatif R. Mazaal 1
Nada S. Karam 1*
Ghada S. Karam 2

1 Department of Mathematics, College of Education, Al-Mustansiriyah University, Baghdad, Iraq
2 Department of Physics, College of Education, Al-Mustansiriyah University, Baghdad, Iraq
*Corresponding author: dr_awatif@uomustansiriyah.edu.iq, dr_nadaskaram@uomustansiriyah.edu.iq
ORCID ID: https://orcid.org/0000-0001-8838-5942, https://orcid.org/0000-0003-2445-7040, https://orcid.org/0000-0001-5844-3486

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Abstract:

The stress(Y) – strength(X) model reliability Bayesian estimation which defines life of a component with strength X and stress Y (the component fails if and only if at any time the applied stress is greater than its strength) has been studied, then the reliability; R=P(Y<X), can be considered as a measure of the component performance. In this paper, a Bayesian analysis has been considered for R when the two variables X and Y are independent Weibull random variables with common parameter α in order to study the effect of each of the two different scale parameters β and λ respectively, using three different [weighted, quadratic and entropy] loss functions under two different prior functions [Gamma and extension of Jeffery] and also an empirical Bayes estimator Using Gamma Prior, for singly type II censored sample. An empirical study has been used to make a comparison between the three estimators of the reliability for stress – strength Weibull model, by mean squared error MSE criteria, taking different sample sizes (small, moderate and large) for the two random variables in eight experiments of different values of their parameters. It has been found that the weighted loss function was the best for small sample size, and the entropy and Quadratic were the best for moderate and large sample sizes under the two prior distributions and for empirical Bayes estimation.

Key words: Bayesian estimation, Reliability, Stress-strength model, Type II censored data, Weibull distribution.

Introduction:

Weibull models are used to describe various types of observed failures of components and phenomena. They are widely used in reliability and survival analysis (1). A considerable attention for the problem of making inference about the stress-strength reliability (one component or system) model has been received. If X be the strength of a component and Y be the stress applied to the component, then reliability; R=P(Y<X), can be considered as a measure of the component performance. Various different lifetime distributions are considered to estimate R. (2-5). As an example; Seuba et al (6) analyzed the applicability of the Weibull analysis to unidirectional microporous yttrium-stabilized-zirconia (YSZ) prepared by ice-tempering, performed crush tests on samples with controlled microstructural features with the loading direction parallel to the porosity. The compressive strength data were fitted using two different fitting techniques, ordinary least squares and Bayesian Markov Chain Monte Carlo, to evaluate whether Weibull statistics are an adequate descriptor of the strength distribution. They assess the effect of different microstructural features (volume, size, densification of the walls, and morphology) on Weibull modulus and strength and found that the key microstructural parameter controlling reliability is wall thickness. In contrast, pore volume is the main parameter controlling the strength.

In this paper the reliability Bayesian analysis when stress X and strength Y are two independent Weibull random variables with parameters(α,β) and (α,λ) respectively is done under two prior functions with three loss distributions. A simulation study has been used to compare by
(MSE) the performance of the six different obtained estimators. The results are recorded in Tables 1 to 5).

**Table 1. Conclusions Summary**

| Experiment | Best Estimators Performance |
|------------|-----------------------------|
| (1)        | E for all sample sizes.     |
| (2)        | W for n=15 with Jeffrey function, while E and Q are the best for n=30 and n=90 |
| (3) and (4)| Q for all sample sizes, except for Jeffrey with n=15 |
| (5)        | E and Q for n=30 and n=90, while W is the best for Jeffrey with n=15 |
| (6)        | W for n=15 and for Jeffrey with n=30, for the other cases E and Q are the best |
| (7) and (8)| by E and Q for all sample sizes, except W is the best for Exp.(8) for Jeffrey and gamma when n=15 |

**Table 2. The MSE values of reliability estimators for experiments 1 and 2**

| n, r      | Criteria | Weighted | Quadratic | Entropy | Best          |
|-----------|----------|----------|-----------|---------|---------------|
| 15,5      | Jeffery  | Mean     | 0.2993    | 0.2193  | 0.2009        |
|           |          | MSE      | 0.0189    | 0.0151  | 0.0151        |
|           | Gamma    | Mean     | 0.3272    | 0.2946  | 0.2926        |
|           |          | MSE      | 0.0187    | 0.0145  | 0.0143        |
|           | E Gamma  | Mean     | 0.3331    | 0.3016  | 0.2997        |
|           |          | MSE      | 0.0251    | 0.0207  | 0.0204        |
| 30,8      | Jeffery  | Mean     | 0.3162    | 0.2730  | 0.2694        |
|           |          | MSE      | 0.0156    | 0.0113  | 0.0110        |
|           | Gamma    | Mean     | 0.3277    | 0.3032  | 0.3022        |
|           |          | MSE      | 0.0154    | 0.0121  | 0.0120        |
|           | E Gamma  | Mean     | 0.3301    | 0.3060  | 0.3050        |
|           |          | MSE      | 0.0181    | 0.0147  | 0.0146        |
| 90,25     | Jeffery  | Mean     | 0.3344    | 0.3221  | 0.3219        |
|           |          | MSE      | 0.0111    | 0.0092  | 0.0091        |
|           | Gamma    | Mean     | 0.3356    | 0.3255  | 0.3253        |
|           |          | MSE      | 0.0111    | 0.0094  | 0.0093        |
|           | E Gamma  | Mean     | 0.3361    | 0.3259  | 0.3257        |
|           |          | MSE      | 0.0114    | 0.0098  | 0.0097        |

**Exp. 1:** $\lambda=0.3$, $\mu=0.9$, $R=0.2500$

| 15,5      | Jeffery  | Mean     | 0.3527    | 0.2691  | 0.2482        |
|           |          | MSE      | 0.0189    | 0.0249  | 0.0275        |
|           | Gamma    | Mean     | 0.3829    | 0.3508  | 0.3488        |
|           |          | MSE      | 0.0185    | 0.0179  | 0.0179        |
|           | E Gamma  | Mean     | 0.3890    | 0.3575  | 0.3555        |
|           |          | MSE      | 0.0217    | 0.0209  | 0.0209        |
| 30,8      | Jeffery  | Mean     | 0.3789    | 0.3352  | 0.3312        |
|           |          | MSE      | 0.0139    | 0.0137  | 0.0137        |
|           | Gamma    | Mean     | 0.3904    | 0.3663  | 0.3652        |
|           |          | MSE      | 0.0139    | 0.0129  | 0.0128        |
|           | E Gamma  | Mean     | 0.3940    | 0.3700  | 0.3689        |
|           |          | MSE      | 0.0153    | 0.0142  | 0.0141        |
| 90,25     | Jeffery  | Mean     | 0.4003    | 0.3882  | 0.3880        |
|           |          | MSE      | 0.0070    | 0.0061  | 0.0060        |
|           | Gamma    | Mean     | 0.4011    | 0.3911  | 0.3910        |
|           |          | MSE      | 0.0070    | 0.0062  | 0.0061        |
|           | E Gamma  | Mean     | 0.4021    | 0.3922  | 0.3920        |
|           |          | MSE      | 0.0072    | 0.0064  | 0.0064        |

**Exp. 2:** $\lambda=2$, $\mu=3.7$, $R=0.3509$
| n, r | Jeffery | Mean | Weighted | Quadratic | Entropy | Best |
|------|---------|------|----------|-----------|---------|------|
| 15,9 | Mean    | 0.3097 | 0.2580  | 0.2826   | Q       |
|      | MSE     | 0.0135 | 0.0094  | 0.0108   | Q       |
| Gamma | Mean   | 0.2986 | 0.2761  | 0.2871   | Q       |
|      | MSE     | 0.0093 | 0.0074  | 0.0082   | Q       |
| E Gamma | Mean | 0.3309 | 0.3086  | 0.3196   | Q       |
|      | MSE     | 0.0172 | 0.0140  | 0.0154   | Q       |
| 30,17 | Jeffery | 0.3311 | 0.3098  | 0.3202   | Q       |
|      | MSE     | 0.0124 | 0.0094  | 0.0108   | Q       |
| Gamma | Mean   | 0.3184 | 0.3044  | 0.3113   | Q       |
|      | MSE     | 0.0094 | 0.0076  | 0.0085   | Q       |
| E Gamma | Mean | 0.3366 | 0.3226  | 0.3295   | Q       |
|      | MSE     | 0.0135 | 0.0112  | 0.0123   | Q       |
| 90,57 | Jeffery | 0.3380 | 0.3325  | 0.3353   | Q       |
|      | MSE     | 0.0095 | 0.0086  | 0.0090   | Q       |
| Gamma | Mean   | 0.3330 | 0.3281  | 0.3306   | Q       |
|      | MSE     | 0.0085 | 0.0078  | 0.0082   | Q       |
| E Gamma | Mean | 0.3385 | 0.3336  | 0.3361   | Q       |
|      | MSE     | 0.0096 | 0.0088  | 0.0092   | Q       |

| n, r | Jeffery | Mean | Weighted | Quadratic | Entropy | Best |
|------|---------|------|----------|-----------|---------|------|
| 15,9 | Mean    | 0.3706 | 0.3177  | 0.3430   | Q       |
|      | MSE     | 0.0119 | 0.0128  | 0.0117   | E       |
| Gamma | Mean   | 0.3750 | 0.3524  | 0.3635   | Q       |
|      | MSE     | 0.0107 | 0.0103  | 0.0104   | Q       |
| E Gamma | Mean | 0.3935 | 0.3714  | 0.3822   | Q       |
|      | MSE     | 0.0138 | 0.0127  | 0.0132   | Q       |
| 30,17 | Jeffery | 0.3954 | 0.3743  | 0.3847   | Q       |
|      | MSE     | 0.0085 | 0.0072  | 0.0077   | Q       |
| Gamma | Mean   | 0.3913 | 0.3774  | 0.3843   | Q       |
|      | MSE     | 0.0076 | 0.0068  | 0.0071   | Q       |
| E Gamma | Mean | 0.4013 | 0.3876  | 0.3944   | Q       |
|      | MSE     | 0.0091 | 0.0080  | 0.0085   | Q       |
| 90,57 | Jeffery | 0.4050 | 0.3997  | 0.4024   | Q       |
|      | MSE     | 0.0049 | 0.0044  | 0.0047   | Q       |
| Gamma | Mean   | 0.4026 | 0.3979  | 0.4002   | Q       |
|      | MSE     | 0.0046 | 0.0042  | 0.0044   | Q       |
| E Gamma | Mean | 0.4056 | 0.4008  | 0.4032   | Q       |
|      | MSE     | 0.0050 | 0.0045  | 0.0047   | Q       |

Table 4. The MSE values of reliability estimators for experiments 5 and 6

| n, r | Jeffery | Mean | Weighted | Quadratic | Entropy | Best |
|------|---------|------|----------|-----------|---------|------|
| 15,5 | Mean    | 0.2592 | 0.1842  | 0.1682   | W       |
|      | MSE     | 0.0151 | 0.0163  | 0.0173   | E       |
| Gamma | Mean   | 0.2906 | 0.2587  | 0.2569   | E       |
|      | MSE     | 0.0131 | 0.0110  | 0.0109   | E       |
| E Gamma | Mean | 0.2904 | 0.2598  | 0.2581   | E       |
|      | MSE     | 0.0185 | 0.0164  | 0.0163   | E       |
| 30,8 | Jeffery | 0.2763 | 0.2346  | 0.2313   | E       |
|      | MSE     | 0.0103 | 0.0090  | 0.0090   | Q, E    |
| Gamma | Mean   | 0.2909 | 0.2668  | 0.2659   | Q       |
|      | MSE     | 0.0096 | 0.0080  | 0.0079   | Q       |
| E Gamma | Mean | 0.2893 | 0.2656  | 0.2647   | Q       |
|      | MSE     | 0.0116 | 0.0100  | 0.0100   | Q, E    |
| 90,25 | Jeffery | 0.2864 | 0.2742  | 0.2740   | Q, E    |
|      | MSE     | 0.0045 | 0.0038  | 0.0037   | Q, E    |
| Gamma | Mean   | 0.2891 | 0.2791  | 0.2789   | Q, E    |
|      | MSE     | 0.0045 | 0.0038  | 0.0038   | Q, E    |
| E Gamma | Mean | 0.2879 | 0.2779  | 0.2777   | Q, E    |
|      | MSE     | 0.0047 | 0.0040  | 0.0040   | Q, E    |
Let $X \sim \text{Weibull}(\alpha, \beta)$ and $Y \sim \text{Weibull}(\alpha, \lambda)$, where Weibull means Weibull distribution under common shape parameter $\alpha$ and different scale parameters Table 1 $\beta$ and $\lambda$ (as a special case in our research where the other cases can be done as a future work), then the probability distribution function for two independent Weibull r.v.’s are (2):

$$f(x) = \alpha \beta x^{\alpha-1}e^{-\beta x^\alpha}\quad x > 0; \alpha, \beta > 0$$

$$f(y) = \alpha \lambda y^{\alpha-1}e^{-\lambda y^\alpha}\quad y > 0; \alpha, \lambda > 0$$

Where:

$$R = P(X > Y) = \int_{y=0}^{\infty} \int_{x=y}^{\infty} f(x)f(y)\,dx\,dy = \int_{0}^{\infty} (1 - \int_{0}^{y} f(x)\,dx) f(y)\,dy$$

\[\begin{array}{cccc}
\text{Exp. 7: } \beta = 0.3, \lambda = 0.9, R = 0.2500 \\
\hline \\
\text{n, r} & \text{criteria} & \text{Weighted Quadratic Entropy} & \text{Best} \\
\hline \\
15, 9 & \text{Jeffery} & \text{Mean} & 0.2680 \\
 & & \text{MSE} & 0.0087 \\
 & \text{Gamma} & \text{Mean} & 0.2604 \\
 & & \text{MSE} & 0.0057 \\
 & \text{E Gamma} & \text{Mean} & 0.2876 \\
 & & \text{MSE} & 0.0105 \\
30, 17 & \text{Jeffery} & \text{Mean} & 0.2831 \\
 & & \text{MSE} & 0.0059 \\
 & \text{Gamma} & \text{Mean} & 0.2736 \\
 & & \text{MSE} & 0.0044 \\
 & \text{E Gamma} & \text{Mean} & 0.2882 \\
 & & \text{MSE} & 0.0064 \\
90, 25 & \text{Jeffery} & \text{Mean} & 0.2915 \\
 & & \text{MSE} & 0.0029 \\
 & \text{Gamma} & \text{Mean} & 0.2884 \\
 & & \text{MSE} & 0.0026 \\
 & \text{E Gamma} & \text{Mean} & 0.2918 \\
 & & \text{MSE} & 0.0029 \\
\hline \\
& \text{Exp. 8: } \beta = 2, \lambda = 3.7, R = 0.3509 \\
& \text{Jeffery} & \text{Mean} & 0.3441 \\
& & \text{MSE} & 0.0108 \\
& \text{Gamma} & \text{Mean} & 0.3528 \\
& & \text{MSE} & 0.0098 \\
& \text{E Gamma} & \text{Mean} & 0.3664 \\
& & \text{MSE} & 0.0116 \\
& \text{Jeffery} & \text{Mean} & 0.3660 \\
& & \text{MSE} & 0.0065 \\
& \text{Gamma} & \text{Mean} & 0.3646 \\
& & \text{MSE} & 0.0060 \\
& \text{E Gamma} & \text{Mean} & 0.3718 \\
& & \text{MSE} & 0.0067 \\
& \text{Jeffery} & \text{Mean} & 0.3772 \\
& & \text{MSE} & 0.0022 \\
& \text{Gamma} & \text{Mean} & 0.3759 \\
& & \text{MSE} & 0.0021 \\
& \text{E Gamma} & \text{Mean} & 0.3775 \\
& & \text{MSE} & 0.0023 \\
\end{array}\]
Singly Type II Censored Sample

Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ be two random samples, and $r < n$ and $r < m$, such that: $x_n, \ldots, x_n$ and $y_m, \ldots, y_m$. The likelihood function for this type of data is (7):

$$L = \frac{n!}{(n-r)!} \prod_{i=1}^{r} x_i^{-\alpha} (\beta + \lambda)^{\alpha n-r} \cdot \prod_{i=1}^{n-r} y_i^{-\gamma} \cdot \prod_{i=1}^{m-r} y_i^{-\alpha \gamma}.$$

Where:

1. $[1 - F(x_r)]^{n-r} = [1 - 1 + e^{-\beta x_i}]^{n-r}$
2. $[1 - F(y_r)]^{m-r} = [1 - 1 + e^{-\lambda y_i}]^{m-r}$

And

$$L(x, y | \beta, \lambda) = \prod_{i=1}^{n} (\beta + \lambda)^{\alpha} \cdot \prod_{i=1}^{m} (\beta + \lambda)^{\lambda y_i} \cdot \prod_{i=1}^{r} x_i^{-\alpha} \cdot \prod_{i=1}^{n-r} y_i^{-\gamma} \cdot \prod_{i=1}^{m-r} y_i^{-\alpha \gamma}.$$

The Posterior Distributions

Under Gamma Prior

The Gamma distribution is used as a prior distribution because of its wide importance in Bayesian analysis. Let $\beta, \lambda$ be two independent Gamma random variables with common parameter (a), (one can consider the case of uncommon parameter in other papers for recommendation), the pdf is given by (8):

$$g(\beta) = \frac{b^\beta}{\Gamma(\beta)} \beta^{-a-1} e^{-b \beta}, \quad \beta > 0; \quad b, a > 0 \quad \ldots (3)$$

$$g(\lambda) = \frac{b^\lambda}{\Gamma(\lambda)} \lambda^{-a-1} e^{-b \lambda}, \quad \lambda > 0; \quad b, a > 0 \quad \ldots (4)$$

The posterior function as:

$$P(\beta, \lambda | x, y) = \frac{L(\beta, \lambda | x, y) g(\beta, \lambda)}{\int_0^\infty \int_0^\infty L(\beta, \lambda | x, y) g(\beta, \lambda) d\beta d\lambda}$$

Using (1), (2), (3) and (4), it will be:

$$L(\beta, \lambda | x, y) g(\beta, \lambda) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} x_i^{-\alpha} \cdot \prod_{i=1}^{n-r} y_i^{-\gamma} \cdot \prod_{i=1}^{m-r} y_i^{-\alpha \gamma}.$$
so by the same procedure:
\[
\int_0^\infty \beta^r x^{-r-2} e^{-\beta \sum_{i=1}^r x_i} e^{-r(x-r)x_i^r} \, d\beta = \frac{\Gamma(r+2)}{Z_x Z_y^{r-2c+1}}
\]
and
\[
\int_0^\infty \lambda^r x^{-r-2} e^{-\lambda \sum_{i=1}^r y_i} e^{-r(m-r)y_i^r} \, d\lambda = \frac{\Gamma(r+2)}{Z_x Z_y^{r-2c+1}}
\]
Then the posterior will be:
\[
P_2(\beta, \lambda | x, y) = \frac{Z_x^{r-2c+1} Z_y^{r-2c+1}}{(r-2c)! (r-2c)!} \beta^r \lambda^r x^{-r-2} e^{-\beta x - \lambda y} \int_0^\infty \int_0^\infty \beta^r \lambda^r x^{-r-2} e^{-\beta x - \lambda y} \, d\beta d\lambda
\]

…… (8)

The Bayes Estimators
In this section the Bayes estimators for stress
strength Weibull reliability under three loss functions are derived as following (3):

For Gamma Prior function

(i) Under Weighted Loss Function
In this section the Bayesian estimator for R using
gamma prior will be derived under weighted loss function\(^3\): \(\hat{R}_{WG} \), where:
\[
\hat{R}_{WG} = \frac{1}{E(R^{-1} | x, y)} = \left( E \left( R^{-1} | x, y \right) \right)^{-1}
\]
\[
E \left( R^{-1} | x, y \right) = \int_0^\infty \int_0^\infty R^{-1} P \left( \beta, \lambda | x, y \right) d\beta d\lambda
\]
\[
= \int_0^\infty \int_0^\infty \left( \frac{\lambda}{\beta+1} \right)^{-1} P \left( \beta, \lambda | x, y \right) d\beta d\lambda
\]
\[
= \int_0^\infty \int_0^\infty \lambda^{-1} (\beta + \lambda) P \left( \beta, \lambda | x, y \right) d\beta d\lambda
\]
\[
= \int_0^\infty \int_0^\infty P \left( \beta, \lambda | x, y \right) d\beta d\lambda
\]
\[
= A_1 + A_2 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qa
\[
\frac{u_r^a y_r^{*a}}{\Gamma(a + r + a) \sum_{t=0}^e c_t f_0 \int_0^\infty \beta^r + a - 1 e^{-\beta u_x} d\beta} = \frac{u_r^a y_r^{*a}}{\Gamma(a + r + a) \sum_{t=0}^e c_t \frac{\Gamma(a + i)}{\Gamma(a + 1)} \frac{(r + a + i - 1)!}{(r + a - 1)!} \left(\frac{u_x}{u_y}\right)^i \frac{1}{i!}}
\]

Then finally one can get:

\[
\tilde{R}_{EG} = \left[ \sum_{i=0}^e c_i \frac{(r + a + i)!}{(r + a - 1)!} \frac{1}{i!} \left(\frac{u_y}{u_x}\right)^i \right]^{-\frac{1}{2}}
\]  

(14)

The Empirical Bayes Estimator for R Using Gamma Prior

The empirical Bayes estimators of reliability R corresponding to Gamma prior distribution are obtained based on different loss functions, where if the prior Gamma parameters (\(b_1\) and \(b_2\)) are unknown, then it may use the empirical Bayes approach to get its estimation from likelihood function and probability density function of prior distribution as (11):

\[
f(x, y \mid b_1, b_2) = \int_0^\infty \int_0^\infty L(\beta, \lambda) g(\beta, \lambda) d\beta d\lambda
\]

Where \(A = \frac{n!}{(n-r)!} \frac{m!}{(m-r)!} \prod_{i=1}^n \frac{z_i^{a-1}}{\Gamma(z_i)} \prod_{i=1}^m \frac{y_i^{r-1}}{\Gamma(y_i)}\)

Now the ML estimators of (\(b_1\) and \(b_2\)) can be obtained by taking the natural log as:

\[
\ln L(b_1, b_2 \mid x_1, y_1) = \ln \left(\frac{n!}{(n-r)!}\right) + \ln \left(\frac{m!}{(m-r)!}\right) + \ln[a b_1 + a n b_2 - 2 n a + 2 r m a + \ln(\prod_{i=1}^n x_i^{q_i}) + \ln(\prod_{i=1}^m y_i^{r_i})]
\]

\[
\frac{\partial \ln L}{\partial b_1} = \frac{a b_1 - (n - r) x_i^{q_i} + \sum_{i=1}^n x_i^{q_i}}{a b_1 + a (n - r) x_i^{q_i} + \sum_{i=1}^n x_i^{q_i}}
\]

\[
\frac{\partial \ln L}{\partial b_2} = \frac{a b_2 - (m - r) y_i^{r_i} + \sum_{i=1}^m y_i^{r_i}}{a b_2 + a (m - r) y_i^{r_i} + \sum_{i=1}^m y_i^{r_i}}
\]

Using these two estimators in (15) and (16) in the reliability estimators obtained above under Gamma prior using three loss functions.

For Extension of Jeffery Prior

The Bayesian estimators of R will be derived in this section for extension of Jeffery Prior as in equations (6) and (7) under the three loss functions.

(i) Weighted Loss Function

From equations (8), the Bayesian estimators of R for extension of Jeffery Prior under the weighted loss functions; \(\hat{R}_{wj}\), from equation (9), will be:

\[
\hat{R}_{wj} = [D_1 + D_2]^{-1}
\]

\[
D_1 = \int_0^\infty \int_0^\infty \lambda^{1-\beta} Z_{y}^{c-\beta} Z_{y}^{\beta-2c+1} \frac{\lambda^{r-2c-\beta} e^{-\lambda Z_x e^{-\lambda Z_y} d\beta d\lambda}}{(r-2c)! (r-2c)!}
\]

\[
D_2 = \int_0^\infty \int_0^\infty \beta^{r-2c} e^{-\beta Z_x e^{-\lambda Z_y} d\beta d\lambda}
\]

(ii) Under Quadratic Loss Function

Here having \(\hat{R}_{qj} = \frac{E[K^{-1} | x, y]}{E[K^{-1} | x, y]}\)

From equations (8) and (11), assuming that \(E(R^2 | x, y) = D_1 + 2D_2 + 1\), then:

\[
D_1 = \int_0^\infty \int_0^\infty \lambda^{r-2c-\beta} e^{-\lambda Z_x e^{-\lambda Z_y} d\beta d\lambda}
\]

\[
D_2 = \int_0^\infty \int_0^\infty \beta^{r-2c} e^{-\beta Z_x e^{-\lambda Z_y} d\beta d\lambda}
\]

\[
E(R^2 | x, y) = \frac{(r-2c)Z_y}{(r-2c)Z_x} + 1
\]
\[ \hat{R}_{ej} = \frac{1 + (r-2c+1)Z}{(r-2c)Z} \]  

... (18)

\( (iii) \) Under Entropy Loss Function

\[ \hat{R}_{tej} = \left[ E(R_t^{-t} | \chi, y) \right]^{\frac{1}{t}} \quad t \neq 0 \]

If \( t = 1 \) → \( \hat{R}_t = \hat{R}_w \) (eq. 16)

If \( t = 2 \) → \( \hat{R}_{2ej} = \left[ E(R^{-2} | \chi, y) \right]^{\frac{1}{2}} \), then:

\[ \hat{R}_{2ej} = 1 + 2 \frac{(r-2c+1)Z}{(r-2c)Z} \]

... (19)

and \( \hat{R}_{tej} = \left[ E(R_t^{-t} | \chi, y) \right]^{\frac{1}{t}} \)

\[ E(R^{-t} | \chi, y) = \int_0^\infty \int_0^\infty \left( \frac{1}{\beta + \lambda} \right) P_f(\beta, \lambda | \chi, y) d\beta d\lambda \]

\[ = \int_0^\infty \beta r^{-2c+1} e^{-\beta x} d\beta = \int_0^\infty x^{-2c-1} e^{-\lambda x} d\lambda \]

\[ = \sum_{i=0}^t C_i t Z_{x}^{r-2c+i} \int_0^\infty \beta r^{-2c+1} e^{-\beta x} d\beta \]

... (20)

**Empirical Study**

To compare between estimators for which the best is to estimate the reliability of stress – strength Weibull model; (Since it is not possible to apply real data in our research, recommending doing so in future researches), an empirical study made by simulation procedure using MATLAB program to compare among them by MSE criteria, under different sample sizes \((n = m = 15)\) representing the smallest sample size, \((n = m = 30)\) for moderate and \((n = m = 90)\) for large (which is known to have a range greater than 75) sample sizes, in eight experiments of different parameters values and when \(\alpha = 0.8\). The replication done for \((q = 5000)\).

Equation (20) is used to generate different values of the two random variables \(X\) and \(Y\) by \(F(x)\) and \(F(y)\) respectively, where \(U\) is uniform random variable on interval \((0,1)\), then by the inverse of distribution function technique got from:

\[ x = \left[ -\frac{1}{\alpha} \ln(1 - U) \right]^{\frac{1}{\alpha}} \quad \text{and} \]

\[ y = \left[ -\frac{1}{\alpha} \ln(1 - U) \right]^{\frac{1}{\alpha}} \]

... (21)

**Conclusions:**

The results of the simulation study are recorded in Tables (2 to 5) below, where there is a fluctuation in the behavior of the estimated reliability of this system when the sample sizes change using the loss functions. While in Table (1), the best performance of the estimators is recorded as a summary of the experiment conclusions.

As a final result, it is found that for small sample size the best performance was for weighted loss function, and the entropy and Quadratic are the best for moderate and large sample sizes.

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**Authors' declaration:**

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for republication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Al-Mustansiriyah University.

**References:**

1. Salim H, Akma N. Bayes Estimator for Exponential Distribution with Extension of Jeffery prior Information. MJMS. 2009;3(2).
2. Al-Dubaciy AR, Karam NS. The Lomax Bayesian estimation under a logarithm loss function . NTICT. 2017:40-45.
3. Karam GS, Abbas FI, Abood ZM, Kadhim KK, Karam NS. An enhanced approach for biomedical image restoration using image fusion techniques. IAIP Conference Proceedings 2018 May 24 (Vol. 1968, No. 1, p. 030028). AIP Publishing LLC.
4. Hussein AY. On the Bayes estimation of Exponentiated Gumbel Shape Parameter. Ms Thesis, Department of Mathematics, College of Education, Al-Mustansiriyah University; 2017.
5. Kasim A. Bayes Estimators of the Shape parameter of Exponentiated Rayleigh Distribution. Ms Thesis, Department of Mathematics, College of Education, Al-Mustansiriyah University; 2014.
6. Feroze N, Aslam M. Bayesian Analysis of Exponentiated Gamma Distribution under Type II Censored Samples. IJAST. 2012; 49.
7. Seuba J, Deville S, Guizard Ch, St A. The effect of wall thickness distribution on mechanical reliability and strength in unidirectional porous ceramics. Sci Technol Adv Mater. 2016; 17(1): 128–135.
8. Mark AN. Parameter Estimation for the Two-Parameter Weibull Distribution. Brigham Young University – Provo: 2013.
9. AL-Noor NH, Saad Sh. Non-Bayes, Bayes and Empirical Bayes estimations for Reliability and...
مقارنة مقدرات بيز لمعولية ويل للإجهاد-المتانة لبيانات الرقابة من نوع II

الخلاصة:
قمنا بدراسة التقدير البيزي لمعولية نموذج الإجهاد (Y) – المثانة (X) (الذي يعرف عمر المكونة مع المثانة X والإجهاد Y (تفشل المكونة إذا وفقت في أي وقت يكون الإجهاد أكبر من مثاني المكونة). فالمعدل = R=P(Y<X) يمكن اعتباره مقياس لأداء المكونة. في هذا البحث، تم حساب التقدير البيزي لدالة المعولية عندما المتغيرين X وY عباره عن متغيرات ويل العشوائية مع معلمة شكل مشتركة α، ومثالي قابلة متشابهة، (الموزون، التحليلي، الامثل). ضمن مجموعتين سائقيين مختلفين (كلما، ومعلومات جيفري الموسعة) وكذلك مقدار بيز التجريبي عند توئيع كاما السابق، للعينة الخاضعة للرقابة من النوع الثاني. تم استخدام دراسة تجريبية للمقارنة بين المقدرات الثلاثة عن طريق معيار متوسط مربعات الخطأ (MSE)، عند اتخاذ مختلفة للعينة (صغرى ووسطا وكبرى) في ثماني تجارب لقيم مختلفة لمتغيرين العشوائيين. توصل البحث إلى أن مقدر المعولية بالاعتماد على دالة الخسارة الموزونية كان الأفضل في حجوم العينات الصغير ومقنعي ذاتي الخسارة التربيعية والأمثل كما أن أفضل لحجوم العينات المتوسطة والكبيرة ضمن التوزيعين السابقين ومقدر بيز التجريبي.

الكلمات المفتاحية: التقدير البيزي، المعولية، بيانات الرقابة، الإجهاد، المثانة.