ORDER AND UO-CONVERGENCE IN SPACES OF CONTINUOUS FUNCTIONS

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Abstract. We present several characterizations of uo-convergent nets or sequences in spaces of continuous functions $C(\Omega)$, $C_b(\Omega)$, $C_0(\Omega)$, and $C^\infty(\Omega)$, extending results of [vdW18]. In particular, it is shown that a sequence uo-converges iff it converges pointwise on a co-meagre set. We also characterize order bounded sets in spaces of continuous functions. This leads to characterizations of order convergence.

1. Introduction

Being a vector lattice, the space $C(\Omega)$ of continuous functions on a completely regular Hausdorff topological space is endowed with order and uo-convergences arising from its order structure. While it has been known for a long time that for sequences in $L_p$-spaces (and in other “nice” spaces of measurable functions), uo-convergence agrees with convergence almost everywhere, there have been no similar characterizations for spaces of continuous functions. There have been several partial results suggesting that order and uo-convergence in $C(\Omega)$ somewhat corresponds to pointwise convergence on a “large set”. In particular, it was proved in [vdW18] using techniques of semi-continuous functions that for a modified definition of uo-convergence, a sequence in $C(\Omega)$ uo-converges iff it converges pointwise outside of a meagre set, under the assumption that $\Omega$ is a Baire space and $C(\Omega)$ is almost $\sigma$-order complete. In this paper, we prove this (and other similar statements) without the almost $\sigma$-order completeness assumption and for the “full” definition of order convergence using direct topological arguments. Our main tool is Theorem 3.2 which characterizes uo-convergence of nets in $C(\Omega)$ in “local” topological terms.

Recall that a net converges in order iff it uo-converges and has an order bounded tail. In view of this, our characterizations of uo-convergent nets immediately yield similar characterizations of order convergent nets.
We then extend these results to “more specialized” spaces of continuous functions: \( C_b(\Omega), C_0(\Omega), \) and \( C^\infty(\Omega) \). We also characterize \( \text{uo}-\text{Cauchy} \) nets in these spaces.

As an application, we deduce a characterization of \( \text{uo}-\text{convergence} \) in vector lattices with PPP. In the last section, we extend some of the results to the setting of Boolean algebras.

2. Preliminaries

Throughout this note, \( X \) will stand for an Archimedean vector lattice. We refer the reader to [AB06] for background on vector lattices. Recall that a net \((x_\alpha)\) in a vector lattice converges in order to \( x \) if there exists a net \((u_\gamma)\), possibly with a different index set, such that \( u_\gamma \downarrow 0 \), and for every \( \gamma \) there exists \( \alpha_0 \) such that \( |x_\alpha - x| \leq u_\gamma \) for all \( \alpha \geq \alpha_0 \); we write \( x_\alpha \xrightarrow{\omega} 0 \). The following lemma is standard and straightforward.

**Lemma 2.1.** For a net \((x_\alpha)\) in a vector lattice \( X \), \( x_\alpha \xrightarrow{\omega} 0 \) iff there exists a set \( G \subseteq X_+ \) such that \( \inf G = 0 \) and every element of \( G \) dominates a tail of \((x_\alpha)\), i.e., for every \( g \in G \) there exists \( \alpha_0 \) such that \( |x_\alpha| \leq g \) for all \( \alpha \geq \alpha_0 \).

We say that \((x_\alpha)\) unbounded order converges to \( x \) or, short, \( \text{uo}-\text{converges} \), and write \( x_\alpha \xrightarrow{\text{uo}} x \) if \( |x_\alpha - x| \land u \xrightarrow{\omega} 0 \) for every \( u \geq 0 \). We refer the reader to [GTX17] and [Pap64] for background information on \( \text{uo}-\text{convergence} \). The two convergences agree for order bounded nets. If \( w \geq 0 \) is a weak unit then \( x_\alpha \xrightarrow{u_0} x \) if \( |x_\alpha - x| \land w \xrightarrow{\omega} 0 \).

A sublattice \( Y \) of \( X \) is regular if the inclusion map is order continuous, i.e., preserves order convergence of nets. Every order dense sublattice is regular. Given a net \((x_\alpha)\) in a regular sublattice \( Y \) of \( X \), \( x_\alpha \xrightarrow{\text{uo}} 0 \) in \( X \) iff \( x_\alpha \xrightarrow{\text{uo}} 0 \) in \( Y \).

Recall that a net \((x_\alpha)_{\alpha \in \Lambda} \) is order Cauchy if the double net \((x_\alpha - x_\beta)\) indexed by \( \Lambda^2 \) is order null. \( \text{Uo-Cauchy} \) nets are defined in a similar fashion.

While all results of the paper are valid for compact Hausdorff spaces, we will state and prove them in a more general setting. Throughout the paper, \( \Omega \) stands for a completely regular Hausdorff topological space (also known as a Tychonov space), which is exactly the class of Hausdorff spaces where the conclusion of Uryson’s lemma holds. Recall that every locally compact Hausdorff space or every normal space is completely regular.

We write \( C(\Omega) \) for the space of all real-valued continuous functions on \( \Omega \), and \( 1 \) for the constant one function. A subset \( A \) of a Hausdorff topological space is nowhere dense if \( \text{Int} \overline{A} = \emptyset \). It is easy to see that the boundary of an open set is nowhere dense. We say that \( A \) meagre or of first category if it is a union of a sequence of nowhere dense sets. The class of meagre sets is closed under taking subsets and countable unions. A set is co-meagre if its complement is meagre. It is easy to
see that $A$ is co-meagre iff $A$ contains an intersection of a sequence of open dense sets. $\Omega$ is a Baire space if it satisfies the conclusion of Baire Category Theorem, i.e., if every intersection of countably many dense open sets is dense or, equivalently, if every co-magre set is dense. Every locally compact Hausdorff space and every complete metrizable space is Baire; see, e.g., [Bour98, IX.5, Theorem 1].

3. Convergence in $C(\Omega)$

The following lemma is essentially Proposition 4.13 in [Bil]; we provide the proof for completeness.

**Lemma 3.1.** Suppose that $\Omega$ is a completely regular Hausdorff topological space and $G \subseteq C(\Omega)_+$. TFAE:

(i) $\inf G = 0$;

(ii) for every non-empty open set $U$ and every $\varepsilon > 0$ there exists $t \in U$ and $g \in G$ with $g(t) < \varepsilon$;

(iii) for every non-empty open set $U$ and every $\varepsilon > 0$ there exists a non-empty open set $V \subseteq U$ and $g \in G$ such that $g(t) < \varepsilon$ for all $t \in V$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose that $\inf G = 0$, yet (ii) fails, that is, there exists an open non-empty $U$ and $\varepsilon > 0$ such that every $g \in G$ is greater than or equal to $\varepsilon$ on $U$. Since $\Omega$ is completely regular, we find a non-zero $f \in C(\Omega)_+$ such that $f \leq \varepsilon 1$ and $f$ vanishes outside of $U$. Then $f \leq G$, which contradicts $\inf G = 0$.

To show that (ii) $\Rightarrow$ (i), suppose that $\inf G \neq 0$. Then there exists $f \in C(\Omega)$ with $0 < f \leq G$. We can then find an open non-empty set $U$ and $\varepsilon > 0$ such that $f$ is greater than $\varepsilon$ on $U$. It follows that every $g \in G$ greater that $\varepsilon$ on $U$, which contradicts (ii).

It is easy to see that (ii) $\Leftrightarrow$ (iii). $\square$

Next, we are going to characterize uo-convergence in $C(\Omega)$. Since $f_\alpha \xrightarrow{uo} f$ iff $|f_\alpha - f| \xrightarrow{uo} 0$, it suffices to characterize order convergence of positive nets to zero.

**Theorem 3.2.** Let $\Omega$ be completely regular and $(f_\alpha)$ a net in $C(\Omega)_+$. Then $f_\alpha \xrightarrow{uo} 0$ iff for every non-empty open set $U$ and every $\varepsilon > 0$ there exists an open non-empty $V \subseteq U$ and an index $\alpha_0$ such that $f_\alpha$ is less than $\varepsilon$ on $V$ whenever $\alpha \geq \alpha_0$.

**Proof.** Suppose $f_\alpha \xrightarrow{uo} 0$. Then $f_\alpha \wedge 1 \xrightarrow{uo} 0$. By Lemma 2.1, there exists a set $G \subseteq C(\Omega)_+$ such that $\inf G = 0$ and every member of $G$ dominates a tail of $(f_\alpha \wedge 1)$. Fix a non-empty open $U$ and $\varepsilon \in (0, 1)$. Let $V$ and $g$ be as in Lemma 3.1(iii). Since $g$ dominates a tail of $(f_\alpha \wedge 1)$, there exists $\alpha_0$ such that $f_\alpha \wedge 1 \leq g$ for all $\alpha \geq \alpha_0$. 

In particular, $f_\alpha(s) \land 1 \leq g(s) < \varepsilon$, hence $f_\alpha(s) < \varepsilon$ for all $s \in V$. This proves the forward implication.

To prove the converse, suppose that the condition in the theorem is satisfied. Since $1$ is a weak unit, it suffices to prove that $f_\alpha \land 1 \overset{\omega}{\to} 0$. We will use Lemma 2.1 to prove this. Fix an open non-empty set $U$ and $\varepsilon > 0$. Let $V$ and $\alpha_0$ be as in the assumption. Fix any $t \in V$. Since $\Omega$ is completely regular, we can find $h \in C(\Omega)_+$ such that $h(t) = 0$ and $h$ equals $1$ outside of $V$. Put $g = h \vee \varepsilon 1$. Then $g(t) = \varepsilon$. We claim that $f_\alpha \land 1 \leq g$ for every $\alpha \geq \alpha_0$. Indeed, if $s \in V$ then $f_\alpha(s) < \varepsilon \leq g(s)$, and if $s \notin V$ then $(f_\alpha \land 1)(s) \leq 1 = h(s) \leq g(s)$.

Repeat this process for every pair $(U, \varepsilon)$, where $U$ is open and non-empty and $\varepsilon > 0$; let $G$ be the set of the resulting functions $g$. Each such $g$ dominates a tail of $(f_\alpha \land 1)$. Lemma 3.1 yields $\inf G = 0$; this completes the proof. □

Corollary 3.3. Let $\Omega$ be completely regular and $(f_\alpha)$ a net in $C(\Omega)$. Then $(f_\alpha)$ is uo-Cauchy iff for every non-empty open set $U$ and every $\varepsilon > 0$ there exists an open non-empty $V \subseteq U$ and an index $\alpha_0$ such that $\left| f_\alpha(t) - f_\beta(t) \right| < \varepsilon$ for all $t \in V$ and $\alpha, \beta \geq \alpha_0$.

Since the space $C_0(\Omega)$ of bounded functions in $C(\Omega)$ is an order dense and, therefore, regular sublattice of $C(\Omega)$, the theorem also characterizes uo-convergence in $C_0(\Omega)$. If $\Omega$ is locally compact, the same is true for $C_0(\Omega)$, the space of all functions $f$ in $C(\Omega)$ such that for every $\varepsilon > 0$ there exists a compact set $K$ such that $\left| f(t) \right| < \varepsilon$ whenever $t \notin K$. It also follows that a net in $C(\Omega)$ converges in order iff it satisfies the condition in the theorem and has an order bounded tail; same in $C_0(\Omega)$ or $C_0(\Omega)$.

The preceding paragraph underlines the importance of understanding order boundedness in these spaces. Can one characterize order bounded sets in topological terms?

It is clear that a set in $C_0(\Omega)$ is order bounded iff it is bounded in the supremum norm. Next, we characterize order bounded sets in $C(\Omega)$ under the assumption that $\Omega$ is paracompact. We refer the reader to [Bour98, IX.4] for the definition and properties of paracompact spaces; we only mention that metrizable and compact spaces are paracompact. Recall also that every open cover $\{U_\alpha\}$ of a paracompact space $\Omega$ admits a partition of unity, i.e., a collection $\{h_\alpha\}$ of non-negative continuous functions such that $\text{supp } h_\alpha \subseteq U_\alpha$ for every $\alpha$, for every point $t \in \Omega$ there exists an open neighborhood $V$ of $t$ such that all but finitely many functions in the collection are identically zero on $V$, and $\sum_\alpha h_\alpha(t) = 1$ for every $t \in \Omega$ (the sum has only finitely many non-zero terms).

**Proposition 3.4.** Suppose that $\Omega$ is paracompact Hausdorff space and $A \subseteq C(\Omega)_+$. Then $A$ is order bounded in $C(\Omega)$ iff it is locally bounded, i.e., for every $t \in \Omega$ there
exists an open neighborhood $U$ of $t$ and a real number $r \geq 0$ such that $f(s) \leq r$ for all $f \in A$ and $s \in U$.

Proof. The forward implication is obvious. To prove the converse, for every $n \in \mathbb{N}$, let $U_n$ be the union of all open sets $U$ such that $f(t) \leq n$ for all $f \in A$ and $t \in U$. By the assumption, $\{U_n\}$ forms an open cover of $\Omega$. Let $\{h_n\}$ be a corresponding partition of unity. For every $t$, put $g(t) = \sum_{n=1}^{\infty} n h_n(t)$. Note that for every $t$ there exists a neighborhood $V$ of $t$ such that only finitely many $h_n$’s are non-zero on $V$; it follows that $g$ is continuous. For every $t \in \Omega$ and $f \in A$, if $h_n(t) > 0$ then $t \in U_n$ and, therefore, $f(t) \leq n$. It follows that

$$g(t) = \sum_{n=1}^{\infty} n h_n(t) = \sum_{h_n(t) > 0} n h_n(t) \geq \sum_{h_n(t) > 0} f(t) h_n(t) = f(t).$$

$\square$

**Proposition 3.5.** Suppose that $\Omega$ is a locally compact Hausdorff space and $A \subseteq C_0(\Omega)_+$. Then $A$ is order bounded iff $A$ is uniformly bounded and for every $\varepsilon > 0$ there exists a compact set $K$ such that $f(t) < \varepsilon$ whenever $f \in A$ and $t \notin K$.

Proof. The forward implication is trivial. To show the converse, suppose that $A$ satisfies the conditions in the theorem; we will show that $A$ is order bounded. Since $A$ is uniformly bounded, we may assume WLOG that $f(t) \leq 1$ for all $f \in A$ and $t \in \Omega$. For every $n \in \mathbb{N}$, we find a compact set $K_n$ such that $f(t) < \frac{1}{2^n}$ whenever $f \in A$ and $t \notin K_n$.

Recall that for every compact set $K$ in a locally compact space one can find a compact set $L$ such that $K \subseteq \text{Int} L$. Applying this fact inductively, we find a sequence of compact sets $(L_n)$ such that $K_n \subseteq L_n \subseteq \text{Int} L_{n+1}$ for every $n$. Find a function $g_n \in C(\Omega)_+$ such that $g_n$ equals 1 on $L_n$ and vanishes outside of $\text{Int} L_{n+1}$. For every $t \in \Omega$, put $g(t) = \sum_{n=1}^{\infty} 2^{-n+1} g_n(t)$. Note that this sum is finite on each $L_n$ and vanishes outside $\bigcup_{n=1}^{\infty} L_n$; it is easy to see that $g \in C_0(\Omega)$.

It is left to show that $A \leq g$. Let $f \in A$. If $t \in L_1$ then $f(t) \leq 1 \leq g_1(t) \leq g(t)$. If $t \in L_n \setminus L_{n-1}$ for some $n > 1$ then $t \notin K_{n-1}$, hence $f(t) < 2^{-n+1} = 2^{-n+1} g_n(t) \leq g(t)$. Finally, if $t \notin \bigcup_{n=1}^{\infty} L_n$ then $f(t) = 0 = g(t)$. $\square$

We now return to order convergence. For the next few results, we will assume that $\Omega$ is a completely regular Hausdorff Baire space. In particular, $\Omega$ could be a locally compact Hausdorff space. The equivalence $[i] \iff [ii]$ in the following lemma is a part of Corollary 4.14 in [Bil]. Cf also Proposition 3.2 in [KV19].
Lemma 3.6. Let $\Omega$ be a completely regular Hausdorff Baire space. For $G \subseteq C(\Omega)_+$, TFAE:

(i) $\inf G = 0$;
(ii) There exists a dense set $D$ such that $\inf_{g \in G} g(t) = 0$ for every $t \in D$;
(iii) There exists a co-meagre set $D$ such that $\inf_{g \in G} g(t) = 0$ for every $t \in D$;

Proof. (ii) $\Rightarrow$ (i) by Lemma 3.1, (iii) $\Rightarrow$ (ii) because every co-meagre set in a Baire space is dense. To prove (i) $\Rightarrow$ (iii), assume that $\inf G = 0$. For $n \in \mathbb{N}$, put $W_n = \bigcup_{g \in G} \{g < \frac{1}{n}\}$. Clearly, $W_n$ is open. For every non-empty open set $U$, Lemma 3.1 yields a point $t \in U$ and $g \in G$ with $g(t) < \frac{1}{n}$; hence $t \in W_n$. This means that $W_n$ is dense. Put $D := \bigcap_{n=1}^{\infty} W_n$, then $D$ is co-meagre. Let $t \in D$, then for every $n \in \mathbb{N}$ we have $t \in W_n$, hence $\inf_{g \in G} g(t) \leq \frac{1}{n}$. It follows that $\inf_{g \in G} g(t) = 0$.

The following result was proved in $\text{vdW18}$ under extra an assumption and for a different definition of order convergence. The forward implication is a part of Corollary 4.14 in $[\text{Bil}]$.

Theorem 3.7. Let $\Omega$ be a completely regular Hausdorff Baire space and $(f_\alpha)$ a net in $C(\Omega)$. If $f_\alpha \xrightarrow{\text{uo}} f$ then $f_\alpha$ converges to $f$ pointwise on a co-meagre set. The converse is true for countable nets (in particular, for sequences).

Proof. WLOG, $f = 0$. Suppose that $f_\alpha \xrightarrow{\text{uo}} 0$. Then $|f_\alpha| \wedge 1 \xrightarrow{\text{uo}} 0$. There exists a net $(g_\gamma)$ such that $g_\gamma \downarrow 0$ and for every $\gamma$ there exists $\alpha_0$ such that $|f_\alpha| \wedge 1 \leq g_\gamma$, whenever $\alpha \geq \alpha_0$. Put $G = \{g_\gamma\}$. Then $\inf G = \inf g_\gamma = 0$. By Lemma 3.6, there exists a co-meagre set $D$ such that for every $t \in D$ we have $0 = \inf_{g \in G} g(t) = \inf_\gamma g_\gamma(t)$ and, therefore, $\lim_\alpha f_\alpha(t) = 0$.

For simplicity, we prove the converse implication for sequences; extending the proof to countable nets is straightforward. Suppose that a sequence $(f_n)$ converges to zero on a co-meagre set $D$. We will use Theorem 3.2 to show that $f_n \xrightarrow{\text{uo}} 0$. WLOG, $f_n \geq 0$ for all $n$. Fix an open non-empty set $U$ and $\varepsilon > 0$. For each $m$, put $W_m = \bigcup_{n \geq m} \{f_n > \varepsilon\}$. Clearly, $W_m$ is open. If $t \in \bigcap_m W_m$ then for every $m$ there exists $n \geq m$ with $f_n(t) > \varepsilon$, hence $t \notin D$. This yields that $\bigcap_m W_m$ is contained in $\Omega \setminus D$, hence is meagre. Since $W_m$ is open, $\partial W_m$ is nowhere dense for every $m$; we conclude that $\bigcup_m \partial W_m$ is meagre. It follows from

$$\bigcap_m W_m \subset \left( \bigcap_m W_m \right) \cup \left( \bigcup_m \partial W_m \right)$$

that $\bigcap_m W_m$ is meagre, so that its complement $\bigcup_m (\Omega \setminus W_m)$ is co-meagre, hence dense, and, therefore, meets $U$. It follows that $U \cap (\Omega \setminus W_m) \neq \emptyset$ for some $m$. Denote
this intersection by $V$. For every $t \in V$, it follows from $t \notin W_m$ that $f_n(t) \leq \varepsilon$ for all $n \geq m$. Hence, the condition in Theorem 3.2 is satisfied.

Example 3.8. The following example shows that one cannot replace “co-meagre” with “dense” in Theorem 3.7. Let $(f_n)$ be the Schauder system in $C[0,1]$; see, e.g., [LT77, p. 3]. Let $D$ be the set of all dyadic points in $[0,1]$. Then $D$ is dense in $[0,1]$ and $(f_n(t))$ is eventually zero for every $t \in D$, so that $(f_n)$ converges to zero pointwise on $D$. Yet, it is easy to see that $\sup_{n \geq m} f_n = 1$ in $C[0,1]$ for every $m$, hence $(f_n)$ does not converge to zero in order.

Example 3.9. We will construct an order bounded net that converges to zero pointwise at every point, yet fails to converge in order. This shows that the converse implication in Theorem 3.7 generally fails for nets. Let $\Omega = C[0,1]$; let $\Lambda$ be the set of all finite subsets of $[0,1]$ containing 0 and 1, ordered by inclusion. For each $\alpha \in \Lambda$, $\alpha = \{t_1, \ldots, t_n\}$ with $0 = t_0 < t_1 < \cdots < t_n = 1$, we define $f_\alpha$ as follows: we put $f_\alpha(t_i) = 0$ for every $i$, we put $f_\alpha$ to be 1 at the midpoints, i.e., $f_\alpha(\frac{t_{i-1} + t_i}{2}) = 1$ for all $i = 1, \ldots, n$, and define $f_\alpha$ linearly in between. It is easy to see that $(f_\alpha)$ converges to zero pointwise on $[0,1]$. On the other hand, we claim that the supremum of every tail of this net in $C[0,1]$ is 1, hence the nets fails to converge to zero in order. Indeed, fix $\alpha \in \Lambda$, and suppose that $h \geq f_\beta$ for all $\beta \supseteq \alpha$. Let $s \notin \alpha$. Then one can find $\beta \supsetneq \alpha$ such that $s$ is a midpoint for $\beta$. Hence, $h(s) \geq f_\beta(s) = 1$. Thus, $h(s) \geq 1$ for all $s \notin \alpha$. Since $\alpha$ is a finite set and $h$ is continuous, it follows that $h \geq 1$.

Corollary 3.10. Let $\Omega$ be a completely regular Hausdorff Baire space. If a net $(f_n)$ is uo-Cauchy then $\lim_n f_n(t)$ exists for every $t$ in some co-meagre set. The converse is true for countable nets.

4. Convergence in $C^\infty(K)$

Recall that a vector lattice $X$ is universally complete if it is order complete and every disjoint set in $X$ has supremum. $X$ is $\sigma$-universally complete if it is $\sigma$-order complete and every disjoint sequence has a supremum. Let $K$ be a compact Hausdorff space, then $C(K)$ is order complete iff $K$ is extremally disconnected, i.e., the closure of every open set is open (hence clopen). If $K$ is an extremally disconnected compact Hausdorff space then the space $C^\infty(K)$ of all functions from $K$ to $\overline{\mathbb{R}} = [-\infty, \infty]$ that are finite except on an nowhere dense set is a universally complete vector lattice. Moreover, every universally complete vector lattice is lattice isomorphic to $C^\infty(K)$ for some extremally disconnected compact Hausdorff $K$. Every Archimedean vector lattice $X$ embeds as an order dense sublattice into a unique universally complete
vector lattice $X^u$, called the **universal completion** of $X$. We refer the reader to [AB03] for these and further details on $C^\infty(K)$ spaces and universally complete vector lattices.

**Remark 4.1.** It can be easily verified that many of the preceding results, including Lemmas 3.1 and 3.6, Theorems 3.2 and 3.7, and Corollaries 3.3 and 3.10 remain valid for $C^\infty(K)$. Note that the variant of Theorem 3.7 for $C^\infty(K)$ spaces was proved in [KVP50, XIII, Theorem 2.3.3].

A set $A$ in an Archimedean vector lattice $X$ is said to be **dominable** if it is order bounded in $X^u$. Such sets appear naturally in many applications. It is, therefore, important to characterize order bounded sets in universally complete vector lattices, or, equivalently, in $C^\infty(K)$-spaces. The following result is somewhat motivated by Lemma 4 in [vdW12] and by [KVP50, XIII. Theorem 2.3.2].

**Proposition 4.2.** Let $A$ be a non-empty subset of $C^\infty(K)_+$ for some extremally disconnected $K$. $A$ is bounded above iff for every non-empty clopen subset $U$ of $K$ there exists a non-empty clopen $V \subseteq U$ and a non-negative real $r$ such that $f(t) \leq r$ for all $f \in A$ and $t \in V$.

**Proof.** The forward implication follows from the observation that every individual function in $C^\infty(K)_+$ satisfies the condition.

Suppose now that $A$ satisfies the condition; we need to show that it is order bounded. As a partially ordered set, $\bar{\mathbb{R}}$ is order isomorphic to $[-1, 1]$, which induces an order isomorphism between $C(K, \bar{\mathbb{R}})$ and $C(K, [-1, 1])$. Hence $C(K, \bar{\mathbb{R}})$ is order complete as a partially ordered set. It follows that $h := \sup A$ in $C(K, \bar{\mathbb{R}})$ exists. It suffices to show that $h \in C^\infty(K)$. Suppose not. Then there exists a non-empty clopen set $U$ such that $h$ equals infinity on $U$. By assumption, we can find a non-empty clopen $V \subseteq U$ and $r \geq 0$ such that $f(t) \leq r$ for all $f \in A$ and $t \in V$. Put

$$g(t) = \begin{cases} h(t) & \text{when } t \notin V \\ r & \text{when } t \in V. \end{cases}$$

Then $f \leq g$ for all $f \in A$, hence $h \leq g$, which contradicts $h$ being infinite on $V$. \qed

We now present a simple proof of a theorem due to Grobler, see [GTX17]. We say that a net $(x_\alpha)_{\alpha \in \Lambda}$ has **finite heads** if the set $\{\alpha \in \Lambda : \alpha \leq \beta\}$ is finite for every $\beta \in \Lambda$. Clearly, every sequence has finite heads.

**Proposition 4.3.** In a universally complete vector lattice, order and uo-convergences agree for nets with finite heads.
Proof. Let $X$ be a universally complete vector lattice and $(x_\alpha)$ a net in $X$ with finite heads. We know that order convergence implies uo-convergence, so we only need to prove that if $x_\alpha \xrightarrow{uo} x$ then $x_\alpha \xrightarrow{o} x$. WLOG, $x = 0$ and $x_\alpha \geq 0$ for every $\alpha$. It suffices to show that $(x_\alpha)$ is order bounded.

We may identify $X$ with a $C^\infty(K)$ space for some extremally disconnected compact Hausdorff $K$. We will use Proposition 4.2. Fix an open non-empty set $U \subseteq K$. It follows from Theorem 3.2 and Remark 4.1 that there exists an open non-empty $V \subseteq U$ and $\alpha_0 \in \mathbb{N}$ such that $x_\alpha$ is bounded by 1 on $V$ for all $\alpha \geq \alpha_0$. It now suffices to show that the head $(x_\alpha)_{\alpha \leq \alpha_0}$ is uniformly bounded on an open subset of $V$. Since this head has only finitely many terms, and since every function in $C^\infty(K)$ is continuous and finite except on a nowhere dense set, we can find an open non-empty $W \subseteq V$ and $r > 0$ such that for every $\alpha \leq \alpha_0$ the function $x_\alpha$ is bounded by $r$ on $W$. It follows from Proposition 4.2 that $(x_\alpha)$ is order bounded. \hfill \Box

Theorem 4.4. [Grobler] For a sequence $(x_n)$ in a $\sigma$-universally complete vector lattice, TFAE

(i) $(x_n)$ is uo-convergent;
(ii) $(x_n)$ is order convergent;
(iii) $(x_n)$ is uo-Cauchy;
(iv) $(x_n)$ is order Cauchy.

Proof. Let $(x_n)$ be a sequence in a $\sigma$-universally complete vector lattice $X$. It is clear that (ii) $\Rightarrow$ (i), (iv) $\Rightarrow$ (iii), (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iv). It suffices to show that (iii) $\Rightarrow$ (ii). Suppose that $(x_n)$ is uo-Cauchy in $X$. Since $X$ is order dense and, therefore, regular in its universal completion $X^u$, $(x_n)$ is uo-Cauchy in $X^u$. By Proposition 4.3, it is order Cauchy in $X^u$. It follows that $(x_n)$ is order bounded in $X^u$, hence is dominable in $X$. Recall that by a theorem of Fremlin [AB03, Theorem 7.38], every countable dominable set in a $\sigma$-universally complete vector lattice is order bounded. This implies that $(x_n)$ is order bounded in $X$. It follows that $(x_n)$ is order Cauchy in $X$. Since $X$ is $\sigma$-order complete, $(x_n)$ is order convergent. \hfill \Box

5. Convergence in vector lattices with PPP

Next, we present an application of preceding results to vector lattices with PPP. Recall that every Archimedean vector lattice $X$ with a strong unit $e$ may be represented as a norm dense sublattice of $C(K)$ for some compact Hausdorff space $K$, such that $e$ corresponds to $1$. It is easy to see that $X$ is also order dense in $C(K)$. Furthermore, the characteristic function of every clopen set is contained in $X$; see
Lemma 6.1. If, in addition, $X$ has PPP, then $K$ is totally disconnected, i.e., every point has a base of clopen neighborhoods; see Corollary 5.8 in [Bil].

**Theorem 5.1.** Let $X$ be a vector lattice with PPP and $(x_{\alpha})$ a net in $X_+$. Then $x_{\alpha} \overset{u_0}{\rightarrow} 0$ iff for every $u > 0$ and every real positive $\varepsilon$ there exists a non-zero component $v$ of $u$ and an index $\alpha_0$ such that $P_{v} x_{\alpha} \leq \varepsilon u$ whenever $\alpha \geq \alpha_0$.

**Proof.** Suppose $x_{\alpha} \overset{u_0}{\rightarrow} 0$. Let $u \in X_+$ and $\varepsilon \in (0, 1)$. Then $x_{\alpha} \wedge u \overset{u_0}{\rightarrow} 0$. Since the principal ideal $I_u$ is regular in $X$, we have $x_{\alpha} \wedge u \overset{u_0}{\rightarrow} 0$ in $I_u$. We can represent $I_u$ as a dense sublattice of $C(K)$ for some topological space $K$ such that $u$ becomes $1$. It is easy to see that $I_u$ is order dense in $C(K)$, hence $x_{\alpha} \wedge 1 \overset{u_0}{\rightarrow} 0$ in $C(K)$. By Theorem 3.2 there exists a non-empty open $W \subseteq K$ and an index $\alpha_0$ such that $x_{\alpha} \leq \varepsilon$ on $W$ for all $\alpha \geq \alpha_0$.

Since $X$ has PPP, so does $I_u$. It follows that $K$ is totally disconnected and, therefore, there exists a non-empty clopen set $V$ contained in $W$. Put $v = 1_V$, then $v \in C(K)$, $v$ is a component of $u$ and $v \in I_u$, hence we may view $v$ as an element of $X$. It is easy to see that for every $\alpha \geq \alpha_0$ we have $P_{v} x_{\alpha} \leq \varepsilon 1$ in $C(K)$, hence $P_{v} x_{\alpha} \leq \varepsilon u$ in $I_u$ and, therefore, in $X$.

To prove the converse, consider the universal completion $X^u$ of $X$. Then $X^u$ may be represented as $C^\infty(K)$ for some extremally disconnected Hausdorff compact $K$. It suffices to show that $x_{\alpha} \overset{u_0}{\rightarrow} 0$ in $C^\infty(K)$. We will do this by proving that $(x_{\alpha})$ satisfies the condition in Theorem 3.2 (see also Remark 4.1).

Fix a non-empty open subset $U$ of $K$ and $\varepsilon > 0$. Since $X$ is order dense in $X^u$, there exists $0 < u \in X$ such that $u \leq 1_U$. By assumption, we can find a non-zero component $v$ of $u$ such that $P_v x_{\alpha} \leq \varepsilon u$. Find a non-empty clopen set $V$ and $\delta > 0$ such that $\delta 1_V \leq v$ whenever $\alpha \geq \alpha_0$. Then $P_V x_{\alpha} \leq P_v x_{\alpha} \leq \varepsilon u \leq \varepsilon 1_U$. It follows that $x_{\alpha|V} \leq \varepsilon$. \hfill $\square$

6. **Convergence in Boolean algebras**

The concept of order converges extends naturally to partially ordered sets as follows: $x_{\alpha} \overset{\alpha}{\rightarrow} x$ if there exist non-empty sets $A$ and $B$ such that $\sup A = x = \inf B$ and for every $a \in A$ and $b \in B$, the order interval $[a, b]$ contains a tail of the net. Lemma 2.1 ensures that in vector lattices this definition agrees with the old one. The following lemma is straightforward:

**Lemma 6.1.** For a net $(x_{\alpha})$ in a partially ordered set, put

$$A_0 = \{ a : \exists \alpha_0 \forall \alpha \geq \alpha_0 \ a \leq x_{\alpha} \} \quad \text{and} \quad B_0 = \{ b : \exists \alpha_0 \forall \alpha \geq \alpha_0 \ b \geq x_{\alpha} \}.$$

Then $x_{\alpha} \overset{\alpha}{\rightarrow} x$ iff $A_0$ and $B_0$ are non-empty and $\sup A_0 = x = \inf B_0$. 

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The reader may have noticed that many of the proofs in the preceding sections implicitly relied on Boolean algebras associated with the topological space. To make this connection more explicit, we will now extend Lemma 6.2, Theorem 6.3, and Corollary 3.3 to Boolean algebras using essentially the same techniques as before.

We refer the reader to [Vlad02] for background on Boolean algebras. The symbol \( \mathcal{A} \) will stand for a Boolean algebra. We write \( \bar{a} \) for the complement of \( a \), \( a - b \) for the difference \( a \land \bar{b} \) and \( |a - b| \) for the symmetric difference \( (a \land \bar{b}) \lor (b \land \bar{a}) \). Note that \( x \land a = y \land a \text{ iff } |x - y| \perp a \text{ iff } |x - y| \leq \bar{a} \). Also, \( |x - a| \leq z \text{ iff } a - z \leq x \leq a \lor x \).

Since every net in a Boolean algebra is order bounded, \( \mu \)-convergence and order convergence agree. It follows easily from the definition of order convergence that \( x_\alpha \overset{\mu}{\to} x \text{ iff } \bar{x}_\alpha \overset{\mu}{\to} \bar{x} \). Since lattice operations in a Boolean algebra satisfy infinite distributive laws, it is easy to see that \( x_\alpha \overset{\mu}{\to} x \) implies \( x_\alpha \land c \overset{\mu}{\to} x \land c \) and \( x_\alpha \lor c \overset{\mu}{\to} x \lor c \) for every \( c \).

**Lemma 6.2.** \( x_\alpha \overset{\mu}{\to} x \text{ iff } |x_\alpha - x| \overset{\mu}{\to} 0 \).

**Proof.** Suppose that \( x_\alpha \overset{\mu}{\to} x \) and let \( A \) and \( B \) be as in the definition of order convergence. Put \( C = \{ b - a : a \in A, b \in B \} \). It is easy to see that \( \inf C = 0 \). For \( a \in A \) and \( b \in B \) there exists \( \alpha_0 \) such that for every \( \alpha \geq \alpha_0 \), \( a \leq x_\alpha \leq b \). It follows that \( |x - x_\alpha| \leq b - a \) and, therefore, \( |x_\alpha - x| \overset{\mu}{\to} 0 \).

Suppose that \( |x_\alpha - x| \overset{\mu}{\to} 0 \). We can then find a set \( C \) with \( \inf C = 0 \) and for every \( c \in C \) there exists \( \alpha_0 \) such that for all \( \alpha \geq \alpha_0 \) we have \( |x_\alpha - x| \leq c \) or, equivalently, \( x - c \leq x_\alpha \leq x \lor c \). Put \( A = \{ x - c : c \in C \} \) and \( B = \{ x \lor c : c \in C \} \); then \( \sup A = x = \inf B \). It follows that \( x_\alpha \overset{\mu}{\to} x \). \( \square \)

**Lemma 6.3.** Let \( B \subseteq \mathcal{A} \). Then \( \inf B = 0 \text{ iff for every } u > 0 \text{ there exists a non-zero } v \leq u \text{ and } b \in B \text{ such that } v \perp b \). Also, \( \sup B = 1 \text{ iff for every } u > 0 \text{ there exists } 0 < v \leq u \text{ and } b \in B \text{ such that } v \leq b \).

**Proof.** We only prove the first claim; the second is similar. Suppose that \( \inf B = 0 \) and \( u > 0 \). Then there exists \( b \in B \) such that \( u \not\leq b \). It follows that \( v \perp b \) where \( v = u - b > 0 \). Conversely, suppose that \( \inf B = 0 \) fails. Then there exists \( u \) with \( 0 < u \leq B \). Then for every \( 0 < v \leq u \) and \( b \in B \) we have \( v \leq b \), so that \( v \not\perp b \). \( \square \)

**Theorem 6.4.** For a net \( (x_\alpha) \) in a Boolean algebra \( \mathcal{A} \), \( x_\alpha \overset{\mu}{\to} x \text{ iff for every } u > 0 \text{ there exists } 0 < v \leq u \text{ and an index } \alpha_0 \text{ such that } x_\alpha \land v = x \land v \text{ for all } \alpha \geq \alpha_0 \).

**Proof.** Suppose that \( x_\alpha \overset{\mu}{\to} x \) and \( u > 0 \). Then \( |x_\alpha - x| \overset{\mu}{\to} 0 \). Take \( C \subseteq A \) with \( \inf C = 0 \) which “witnesses” the latter convergence. By Lemma 6.3, there exists
$v \in \mathcal{A}$ and $c \in C$ such that $0 < v \leq u$ and $v \perp c$. There exists $\alpha_0$ such that for every $\alpha \geq \alpha_0$ we have $|x_\alpha - x| \leq c \leq \bar{v}$, and, therefore, $x_\alpha \wedge v = x \wedge v$.

To prove the converse, for every $u > 0$ we fix $v_u$ and $\alpha_u$ such that $0 < v_u \leq u$ for all $\alpha \geq \alpha_u$ we have $x_\alpha \wedge v_u = x \wedge v_u$, or, equivalently, $|x - x_\alpha| \leq \bar{v}_u$. Let $C = \{\bar{v}_u : u > 0\}$. By Lemma 6.3 we have $\inf C = 0$. We conclude that $|x_\alpha - x| \xrightarrow{\alpha} 0$ and, therefore, $x_\alpha \xrightarrow{\alpha} x$.

\textbf{Corollary 6.5.} For a net $(x_\alpha)$ in a Boolean algebra, TFAE:

(i) $(x_\alpha)$ is order Cauchy;

(ii) For every $u > 0$ there exists $0 < v \leq u$ and $\alpha_0$ such that $x_\alpha \wedge v = x_\beta \wedge v$ for all $\alpha, \beta \geq \alpha_0$;

(iii) $\forall u > 0 \left( \exists v \in (0, u] \mbox{ and } \alpha_0 \forall \alpha \geq \alpha_0 \ v \leq x_\alpha \right)$ or

$\left( \exists v \in (0, u] \mbox{ and } \alpha_0 \forall \alpha \geq \alpha_0 \ v \leq \bar{x}_\alpha \right)$

(iv) $\sup(A_0 \cup \overline{B_0}) = 1$, where $A_0$ and $B_0$ are as in Lemma 6.1.

(v) There exists non-empty sets $A$ and $B$ such that $\sup(A \cup \overline{B}) = 1$ and for every $a \in A$ and $b \in B$ there exists $\alpha_0$ with $a \leq x_\alpha \leq b$ for all $\alpha \geq \alpha_0$.

\textbf{Proof.} (i) $\Leftrightarrow$ (ii) $(x_\alpha)$ is order Cauchy iff $|x_\alpha - x_\beta| \rightarrow 0$. By Theorem 6.4, the latter is equivalent to $\forall u > 0 \ \exists v \in (0, u] \mbox{ and } \alpha_0 \forall \alpha, \beta \geq \alpha_0 \ |x_\alpha - x_\beta| \perp v$ or, equivalently, $x_\alpha \wedge v = x_\beta \wedge v$.

(iii) $\Rightarrow$ (i) trivially. To show (ii) $\Rightarrow$ (iii), let $u$, $v$, and $\alpha_0$ be in (ii). Put $w = x_{\alpha_0} \wedge v$; then $(x_\alpha \wedge v) = w$ for all $\alpha \geq \alpha_0$. If $w = 0$, we are done. Otherwise, replace $v$ with $w$.

(iii) $\Leftrightarrow$ (v): (iii) effectively says that for every $u > 0$ there exists $v \in (0, u]$ such that $v \in A_0 \cup \overline{B_0}$. This is equivalent to $\sup(A_0 \cup \overline{B_0}) = 1$ by Lemma 6.3.

(iv) $\Leftrightarrow$ (v) is straightforward.

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