ALGEBRAIC ENTROPY OF A CLASS OF FIVE-POINT DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. We compute the algebraic entropy of a class of integrable Volterra-like five-point differential-difference equations recently classified using the generalised symmetry method. We show that, when applicable, the results of the algebraic entropy agrees with the result of the generalised symmetry method, as all the equations in this class have vanishing entropy.

1. Introduction

One of the most important topics in modern Mathematical Physics is the study of the so-called integrable systems. Roughly speaking integrable systems are important both from theoretical and practical point of view since they can be regarded as universal models for physics going beyond the linear regime [10]. The birth of modern theory of integrable systems is usually recognized in the seminal works of Zabosky and Kruskal [62], Gardner, Greene, Kruskal and Miura [16] and Lax [38] on the Korteweg-deVries (KdV) equation [37].

The concept of integrability comes from Classical Mechanics and means the existence of a sufficiently high number of first integrals. To be more specific an Hamiltonian with Hamiltonian \( H = H(p,q) \) system with \( N \) degrees of freedom is said to be integrable if there exist \( N - 1 \) well defined functionally independent and Poisson-commuting first integrals [12,55]. In the case of systems with infinitely many degrees of freedom, e.g. partial differential equations like the KdV equation, the existence of infinitely many conservation laws is then required. One of the most efficient ways to find these infinitely many is the existence a so-called Lax pair [38]. A Lax pair is an associated overdetermined linear problem whose compatibility condition is guaranteed if and only if the desired non-linear equation is satisfied.

Nowadays a purely algorithmic method to prove or disprove the existence of a Lax pair is not available, so during the years many integrability detectors have been developed. Integrability detectors are algorithmic procedure which are sufficient conditions for integrability, or alternative definitions of integrability. This means that integrability detectors can be used to prove the integrability of a given equation without the need of a Lax pair. One of the fundamental integrability detectors, which works both at continuous and discrete level, is the generalised symmetry approach. The generalised symmetry approach was mainly developed by the scientific school of A. B. Shabat in Ufa during the 80s and has obtained many important results in the classification of partial differential equations [1,32,42,45,50,51], differential-difference equations [23,24,39,59,60] and partial difference equations [17,20,40,41].

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\(^1\)We say that a function is well-defined on the phase space if it is analytic and single-valued.
Another integrability detector is the algebraic entropy test. The algebraic entropy test is specific to systems with discrete degrees of freedom which can be put into a bi-rational form. The basic idea, given a bi-rational map, which can be an ordinary difference equation, a differential-difference equation or even a partial difference equation, is to examine the growth of the degree of its iterates, and extract a canonical quantity, which is an index of complexity of the map. This canonical quantity is what is called the algebraic entropy. The idea of algebraic entropy as measure of the complexity of the growth of bi-rational maps comes from the notion of complexity introduced by Arnol’d in [5] and was discussed for the first time in relation of discrete systems by Veselov [55].

In this paper we will compute the algebraic entropy of two classes of first order, five-point differential-difference equation which were classified recently in [23,24] through the generalised symmetry method. We mention that these equations were used to produce and classify new examples of quad-equations in [17]. These equations are autonomous fourth order differential-difference equations of the following form:

\[
\frac{du_n}{dt} = A(u_{n+1}, u_n, u_{n-1}) u_{n+2} + B(u_{n+1}, u_n, u_{n-1}) u_{n-2} + C(u_{n+1}, u_n, u_{n-1}),
\]

where \( u_n = u_n(t) \), is the dependent variable depending on \( n \in \mathbb{Z} \) and on \( t \in \mathbb{R} \). Due to the similarity of equation (1) with the well-known three-point Volterra equation

\[
\frac{du_n}{dt} = u_n(u_{n+1} - u_{n-1}),
\]

we will call equations of this form Volterra-like equations.

**Remark 1.** Throughout this paper we are going to consider only autonomous equations of the form (1). Therefore, we will make use of the short-hand notation \( u_{n+k} = u_k \) to simplify the formulæ.

Explicitly, we are going to consider are the following equations divided in six lists:

**List 1:** Equations related to the double Volterra equation:

(E.1) \( \frac{du_0}{dt} = u_0(u_2 - u_{-2}) \),

(E.2) \( \frac{du_0}{dt} = u_0^3(u_2 - u_{-2}) \),

(E.3) \( \frac{du_0}{dt} = (u_0^2 + u_0)(u_2 - u_{-2}) \),

(E.4) \( \frac{du_0}{dt} = (u_2 + u_1)(u_0 + u_{-1}) - (u_1 + u_0)(u_{-1} + u_{-2}) \),

(E.5) \( \frac{du_0}{dt} = (u_2 - u_1 + a)(u_0 - u_{-1} + a) + (u_1 - u_0 + a)(u_{-1} - u_{-2} + a) + b \),

(E.6) \( \frac{du_0}{dt} = u_2u_1u_0(u_{0}u_{-1} + 1) - (u_1u_0 + 1)u_0u_{-1}u_{-2} + u_0^2(u_{-1} - u_1) \),

(E.1') \( \frac{du_0}{dt} = u_0[u_1(u_2 - u_0) + u_{-1}(u_0 - u_{-2})] \).
(E.2') \[ \frac{du_0}{dt} = u_1 u_0^2 u_{-1} (u_2 - u_{-2}). \]

Transformations \( \hat{u}_k = u_{2k} \) or \( \hat{u}_k = u_{2k+1} \) turn equations (E.1)-(E.3) into the well-known Volterra equation and its modifications in their standard form. The other equations are related to the double Volterra equation (E.1) through some autonomous non-invertible non-point transformations. We note that equation (E.2') was presented in [3].

List 2: Linearizable equations:

(E.7) \[ \frac{du_0}{dt} = (T - a) \left[ \frac{(u_1 + au_0 + b)(u_{-1} + au_{-2} + b)}{u_0 + au_{-1} + b} + u_0 + au_{-1} + b \right] + cu_0 + d, \]

(E.8) \[ \frac{du_0}{dt} = \frac{u_2 u_0}{u_1} + u_1 - a^2 \left( u_{-1} + \frac{u_0 u_{-2}}{u_{-1}} \right) + cu_0. \]

In both equations \( a \neq 0 \), in (E.7) \( (a + 1)d = bc \), and \( T \) is the translation operator \( T f_n = f_{n+1} \).

Both equations of List 2 are related to the linear equation:

(E.3) \[ \frac{du_0}{dt} = u_2 - a^2 u_{-2} + \frac{c}{2} u_0 \]

through an autonomous non-invertible non-point transformations. We note that (E.7) is linked to (E.3) with a transformation which is implicit in both directions, see [23] for more details.

List 3: Equations related to a generalised symmetry of the Volterra equation:

(E.3') \[ \frac{du_0}{dt} = u_0 \left[ u_1 (u_2 + u_1 + u_0) - u_{-1} (u_0 + u_{-1} + u_{-2}) \right] + cu_0 (u_1 - u_{-1}), \]

(E.4') \[ \frac{du_0}{dt} = (u_2 - a^2) \left( [u_1^2 - a^2] (u_2 + u_0) - (u_{-1}^2 - a^2) (u_0 + u_{-2}) \right) + c(u_0^2 - a^2) (u_1 - u_{-1}), \]

(E.5') \[ \frac{du_0}{dt} = (u_1 - u_0 + a)(u_0 - u_{-1} + a)(u_2 - u_{-2} + 4a + c) + b, \]

(E.6') \[ \frac{du_0}{dt} = u_0 [u_1 (u_2 - u_1 + u_0) - u_{-1} (u_0 - u_{-1} + u_{-2})], \]

(E.7') \[ \frac{du_0}{dt} = (u_2 - a^2) \left( [u_1^2 - a^2] (u_2 - u_0) + (u_{-1}^2 - a^2) (u_0 - u_{-2}) \right), \]

(E.8') \[ \frac{du_0}{dt} = (u_1 + u_0) (u_0 + u_{-1}) (u_2 - u_{-2}). \]

These equations are related between themselves by some transformations, for more details see [24]. Moreover equations (E.3', E.4', E.5') are the generalised symmetries of some known three-point autonomous differential-difference equations [21].

List 4: Equations of the relativistic Toda type:

(E.9) \[ \frac{du_0}{dt} = (u_0 - 1) \left( \frac{u_2 (u_1 - 1) u_0}{u_1} - \frac{u_0 (u_{-1} - 1) u_{-2}}{u_{-1}} - u_1 + u_{-1} \right), \]
Equations related to the Itoh-Narita-Bogoyavlensky (INB) equation:

\[ \frac{du_0}{dt} = \frac{u_2 u_1^2 u_0^2 (u_0 u_{-1} + 1)}{u_1 u_0 + 1} - \frac{(u_1 u_0 + 1) u_0^2 u_{-1} u_{-2}}{u_0 u_{-1} + 1} - \frac{(u_1 - u_{-1}) (2 u_1 u_0 u_{-1} + u_1 + u_{-1}) u_0^3}{(u_1 u_0 + 1) (u_0 u_{-1} + 1)}, \]

(E.10)

\[ \frac{du_0}{dt} = (u_1 u_0 - 1) (u_0 u_{-1} - 1) (u_2 - u_{-2}). \]

(E.13')

Equation (E.13') was known \[18, 19\] to be a relativistic Toda type equation. Since in \[23\] it was shown that the equations of List 4 are related through autonomous non-invertible non-point transformations, it was suggested that (E.9) and (E.10) should be of the same type. Finally, we note that equation (E.9) appeared in \[1\] earlier than in \[23\].

List 5: Equations related to the Itoh-Narita-Bogoyavlensky (INB) equation:

\[ \frac{du_0}{dt} = u_0 (u_2 + u_1 - u_{-1} - u_{-2}), \]

(E.11)

\[ \frac{du_0}{dt} = (u_2 - u_1 + a) (u_0 - u_{-1} + a) + (u_1 - u_0 + a) (u_{-1} - u_{-2} + a) + (u_1 - u_0 + a) (u_0 - u_{-1} + a) + b, \]

(E.12)

\[ \frac{du_0}{dt} = (u_0^2 + a u_0) (u_2 u_1 - u_{-1} u_{-2}), \]

(E.13)

\[ \frac{du_0}{dt} = (u_1 - u_0) (u_0 - u_{-1}) \left( \frac{u_2}{u_1} - \frac{u_{-2}}{u_{-1}} \right), \]

(E.14)

\[ \frac{du_0}{dt} = u_0^2 u_1 - u_{-1} u_{-2}, \]

(E.9')

\[ \frac{du_0}{dt} = (u_1 - u_0 + a) (u_0 - u_{-1} + a) (u_2 - u_1 + u_{-1} - u_{-2} + 2a) + b, \]

(E.10')

\[ \frac{du_0}{dt} = u_0 (u_1 u_0 - a) (u_0 u_{-1} - a) (u_2 u_1 - u_{-1} u_{-2}), \]

(E.11')

\[ \frac{du_0}{dt} = (u_1 + u_0) (u_0 + u_{-1}) (u_2 + u_1 - u_{-1} - u_{-2}). \]

(E.12')

Equation (E.11) is the well-known INB equation \[8, 34, 37\]. Equations (E.12) with \( a = 0 \) and (E.13) with \( a = 0 \) are simple modifications of the INB and were presented in \[46\] and \[9\], respectively. Equation (E.13) with \( a = 1 \) has been found in \[52\]. Up to an obvious linear transformation, it is equation (17.6.24) with \( m = 2 \) in \[52\]. Equation (E.9') is a well-known modification of INB equation (E.11), found by Bogoyavlensky himself \[8\]. Finally, equation (E.11) with \( a = 0 \) was considered in \[3\]. All the equations in this list can be reduced to the INB equation using autonomous non-invertible non-point transformations. Moreover, equations (E.12), (E.14) and (E.9') are related through non-invertible transformations to the equation:

\[ \frac{du_0}{dt} = (u_2 - u_0) (u_1 - u_{-1}) (u_0 - u_{-2}). \]

(4)

For this reason, as it was done in \[17\], we will consider equation (4) as independent. We note that equation (4) and its relationship with equation (E.12), (E.14) and (E.9') were first discussed in \[22\].
List 6: Other equations:

(E.15) \[ \frac{du_0}{dt} = u_0^2(u_2 u_1 - u_{-1} u_{-2}) - u_0(u_1 - u_{-1}), \]

(E.16) \[ \frac{du_0}{dt} = (u_0 + 1) \times \left[ \frac{u_2 u_0(u_1 + 1)^2}{u_1} - \frac{u_{-2} u_0(u_{-1} + 1)^2}{u_{-1}} + (1 + 2u_0)(u_1 - u_{-1}) \right], \]

(E.17) \[ \frac{du_0}{dt} = (u_0^2 + 1) \left( u_2 \sqrt{u_1^2 + 1} - u_{-2} \sqrt{u_{-1}^2 + 1} \right), \]

(E.14') \[ \frac{du_0}{dt} = u_1 u_0^3 u_{-1} (u_2 u_1 - u_{-1} u_{-2}) - u_0^2(u_1 - u_{-1}). \]

Equation (E.15) has been found in [54] and it is called the discrete Sawada-Kotera equation [2, 54]. Equation (E.14') is a simple modification of the discrete Sawada-Kotera equation (E.15). Equation (E.16) has been found in [1] and is related to (E.15). On the other hand equation (E.17) has been found as a result of the classification in [23] and seems to be a new equation. It was shown in [24] that equation (E.17) is a discrete analogue of the Kaup-Kupershmidt equation. Then we will refer to equation (E.17) as the discrete Kaup-Kupershmidt equation. No transformation into known equations of equation (E.17) is known.

In this paper we will show that all the bi-rational equations of Lists 1–6 possess quadratic (linear) growth, and hence are integrable (linearizable) according to the algebraic entropy test. We note that the only non-bi-rational equations in Classes I and II is the discrete Kaup-Kupershmidt equation (E.17) which contains square root terms. In section 2 we will give some details on how algebraic entropy is computed, then in section 3 we will show the results for the equations of Lists 1–6 except the discrete Kaup-Kupershmidt equation (E.17). In section 4 we will discuss the results obtained in sections 3 in the framework of the existing literature and we will give an outlook on future research in the field.

Before going on we would like to present a new rational form of the discrete Kaup-Kupershmidt equation (E.17). That is we have the following proposition:

**Proposition 1.** There exists a point transformation which brings the discrete Kaup-Kupershmidt equation (E.17) into the following rational form:

\[ \frac{dv_0}{dt} = (1 + v_0^2) \left[ \frac{1 + v_1^2}{1 - v_1^2} \frac{v_2}{1 - v_2^2} - \frac{1 + v_{-1}^2}{1 - v_{-1}^2} \frac{v_{-2}}{1 - v_{-2}^2} \right]. \]

**Proof.** We start with the substitution:

\[ u_n = \sinh (\varphi_n), \]

which brings the discrete Kaup-Kupershmidt equation in hyperbolic form\(^2\):

\[ \dot{\varphi}_0 = \cosh (\varphi_n) \left[ \cosh (\varphi_{n+1}) \sinh (\varphi_{n+2}) - \cosh (\varphi_{n-1}) \sinh (\varphi_{n-2}) \right]. \]

Using the hyperbolic identities:

\[ \sinh \alpha = \frac{2 \tanh (\alpha/2)}{1 - \tanh^2 (\alpha/2)}, \quad \cosh \alpha = \frac{1 + \tanh^2 (\alpha/2)}{1 - \tanh^2 (\alpha/2)}. \]

\(^2\)D. Levi, private communication.
and putting
\[ \tanh \left( \frac{\varphi_n}{2} \right) = v_n \]
equation (9) follows. □

Remark 2. We note that under the scaling:
\[ u_n(t) = i \left[ \sqrt{2} + \frac{\sqrt{2}}{8} \varepsilon^2 U \left( \tau - \frac{2}{135} \varepsilon^5 t, x + \frac{4}{9} \varepsilon t \right) \right], \quad x = n \varepsilon, \]
equation (10) admits the Kaup-Kaupershmit equation as continuum limit:
\[ U_\tau = U_{xxxx} + 5UU_{xxx} + \frac{25}{2} U_x U_{xx} + 5U^2 U_x, \]
just as the original (E.17) equation.

2. ALGEBRAIC ENTROPY

Heuristically integrability deals with the regularity of the solutions of a given system. In this sense a simple characterisation of chaotic behaviour is when two arbitrarily near initial values give rise to solutions diverging at infinity. For recurrence relations, i.e. equations where the solution is given by iteration of a formula, we could just try to compute the iteration to extract information about integrability, even if we cannot solve the equation explicitly. However it is usually impossible to calculate explicitly these iterates by hand or even with any state-of-the-art formal calculus software, simply because the expressions one should manipulate are rational fractions of increasing degree of the various initial conditions. The complexity and size of the calculation make it impossible to calculate the iterates.

It was nevertheless observed that “integrable” maps are not as complex as generic ones. This was done primarily experimentally, by an accumulation of examples, and later by the elaboration of the concept of algebraic entropy for difference equations [7, 13, 15, 49, 55]. In [53, 56] the method was developed in the case of quad equations and then used as a classifying tool [33]. Finally in [12] the same concept was introduced for differential-difference equation and later [57] to the very similar case of differential-delay equations. For a more complete discussion of the method in the context of the so-called integrability indicators we refer to [25, 26].

As we stated in the introduction algebraic entropy is a measure of the growth of bi-rational maps. The most natural space for considering bi-rational maps is the projective space over a closed field rather than in the affine space one. We then transforms a recurrence relation into a polynomial map in the homogeneous coordinates of the proper projective space over some closed field:
\[ \varphi: x_i \mapsto \varphi_i(x_k), \]
with \( x_i, x_k \in \mathbb{N} \mathcal{V} \) where \( \mathbb{N} \mathcal{V} \) is the space of the initial conditions. The recurrence is then obtained by iterating the polynomial map \( \varphi \). The map \( \varphi \) has to be bi-rational in the sense that it has to possesses an inverse map which is again a rational map.

The space of the initial condition depends on which type of recurrence relation we are considering. In the case of differential-difference equation of the discrete \( k - k' \)-th order and of the \( p \)-th continuous order:
\[ u_{n+k} = f_n \left( \left\{ \frac{d^i u_{n+k-i}}{dt^i}, \ldots, \frac{d^i u_{n+k}}{dt^i} \right\}_{i=0}^p ; u_{n+k'} \right), \quad k', k, n \in \mathbb{N}, k' < k \]
the space of initial conditions is infinite dimensional. Indeed, in the case the order of
the equation is \(k - k'\), we need the initial value of \(k - k'\)-tuple as a function of
the parameter \(t\), but also the value of all its derivatives:

\[
\mathcal{IN} = \left\{ \frac{d^i u_{k-1}}{dt^i}, \frac{d^i u_{k-2}}{dt^i}, \ldots, \frac{d^i u_{k'}}{dt^i} \right\}_{i \in \mathbb{N}_0}.
\]

We need all the derivatives of \(u_i(t)\) and not just the first \(p\) because at every iteration
the order of the equation is raised by \(p\). Therefore, to describe infinitely many
iterations we need infinitely many derivatives. To obtain the map one just need to
pass to homogeneous coordinates in the equation and in (14).

**Remark 3.** We remark that if we restrict to compute a finite number of iterates of a
differential-difference equation of discrete \(k - k'\)-th order and of the \(p\)-th continuous
order (13), then the space of initial conditions is finite dimensional. Indeed, let us
assume that we wish to compute the \(N\)th iterate of a differential-difference equation
(13), then at most we will need the derivatives of order \(N(p+1)\). That is, we need
to consider the following restricted space of initial conditions:

\[
\mathcal{IN}^{(N)} = \left\{ \frac{d^i u_{k-1}}{dt^i}, \frac{d^i u_{k-2}}{dt^i}, \ldots, \frac{d^i u_{k'}}{dt^i} \right\}_{i = 0}^{N(p+1)}.
\]

If we factor out any common polynomial factors we can say that the degree with
respect to the initial conditions is well defined. We can therefore form the sequence
of degrees of the iterates of the map \(\varphi\) and call it \(d_N = \deg \varphi^N\):

\[
1, d_1, d_2, d_3, d_4, \ldots, d_N, \ldots
\]

The degree of the bi-rational projective map \(\varphi\) has to be understood as the maximum of the total polynomial degree in the initial conditions \(\mathcal{IN}\) of the entries of
\(\varphi\). The same definition in the affine case just translates to the maximum of the
degree of the numerator and of the denominator of the \(N\)th iterate in terms of
the affine initial conditions. Degrees in the projective and in the affine setting can
be different, but the global behaviour will be the same due to the properties of
homogenization and de-homogenization.

The sequence of degree (16) is fixed in a given system of coordinates, but it is not
invariant with respect to changes of coordinates. Therefore we need to introduce a
canonical measure of the growth. It turn out that a good definition is the following
one: consider the following number

\[
\eta_{\varphi} = \lim_{N \to \infty} \frac{1}{N} \log d_N,
\]
called the algebraic entropy of the map \(\varphi\) When no confusion is possible about the
map \(\varphi\) we will usually omit the subscript \(\varphi\) in (17).

Algebraic entropy for bi-rational maps has the following properties [7, 25, 26]:

1. The algebraic entropy as given by (17) always exists.
2. The algebraic entropy has the following upper bound:

\[
\eta_{\varphi} \leq \deg \varphi.
\]

3. If \(\eta_{\varphi} = 0\), i.e. the algebraic entropy is zero, then

\[
d_N \sim N^\nu, \quad \text{with } \nu \in \mathbb{N}_0, \text{ as } N \to \infty.
\]
(4) The algebraic entropy is a bi-rational invariant of bi-rational maps. That is, if two bi-rational maps $\varphi$ and $\psi$ are conjugated by a bi-rational map $\chi$,

$$\varphi = \chi \circ \psi \circ \chi^{-1}$$

then:

$$\eta_\varphi = \eta_\psi.$$  

Property 4 tells us that the algebraic entropy is a canonical measure of growth for bi-rational maps.

We will then have the following classification of equations according to their Algebraic Entropy [33]:

**Linear growth:** The equation is linearizable.

**Polynomial growth:** The equation is integrable.

**Exponential growth:** The equation is chaotic.

In our following sections we will be dealing with differential-difference equations of first continuous order and fourth discrete order of the particular form:

$$u_{n+2} = f(u_{n+1}, u_n, u_{n-1}, u_{n-2}, \frac{du_n}{dt}). \quad n \in \mathbb{Z},$$

To practically compute the algebraic entropy we introduce some technical methods to reduce the computational complexity [26,27]. First, we fix the desired number of iterations to be some fixed $N \in \mathbb{N}$. Following remark 3 this means that we need only finitely many initial conditions given by (15). Then we assume that the space of initial conditions is linearly parametrised in the appropriate projective space, i.e. in inhomogenous coordinates it has the following form:

$$u_i = \frac{\alpha_i t + \beta_i}{\alpha_0 t + \beta_0}, \quad u_i \in \mathbb{Z}N^r(N).$$

We will assume that the parameter $t$ is the same which describes the “time” evolution of the problem. To simplify the problem we choose all the parameters involved in the equations to be integers. Moreover, to avoid accidental factorisations which may alter the results we choose these integers to be prime numbers. A final simplification to speed up the computations is given by considering the factorisation of the iterates in some finite field $\mathbb{K}_r$, with $r$ prime number.

*Remark 4.* Several equations in Lists 1–6, e.g. (E.5) or (E.5'), depend on some parameters. Depending on the value of the parameters their integrability properties can be, in principle, different. As it was done in [17], in order to avoid ambiguities, we use some simple autonomous transformations to fix the values of some parameters. The remaining free parameters are then treated as free coefficients and then fixed to integers following the above discussion. We will describe these subcases when needed in the next section.

Using the rules above we are able to avoid accidental cancellations and produce a finite sequence of degrees:

$$1, d_1, d_2, d_3, d_4, d_5, \ldots, d_N.$$

To extract the asymptotic behaviour from the finite sequence (24): we compute its generating function, i.e. a function $g = g(s)$ such that the coefficients of its Taylor
series
\begin{equation}
    g(z) = \sum_{l=0}^{\infty} d_l z^l
\end{equation}

up to order \(N\) coincides with the finite sequence \((24)\). Assuming that such generating function is rational it can be computed using the method of Padé approximants \([6,18]\).

This generating function is predictive tool. Indeed one can readily compute the successive terms in the Taylor expansion for \((25)\) and confront them with the degrees calculated with the iterations. This means that the assumption that the value of the algebraic entropy given by the approximate method is in fact very strong and very unlikely the real value will differ from it.

Having a rational generating function will also yield the value of the Algebraic Entropy from the modulus of the smallest pole of the generating function:
\begin{equation}
    \eta_\phi = \log \min \left\{ |z| \in \mathbb{R}^+ \mid \frac{1}{g(z)} = 0 \right\}.
\end{equation}

From the generating function one can also find an asymptotic fit for the degrees \((24)\). This can be done by using the \(Z\)-transform \([14,36]\). Indeed, from the definition of \(Z\)-transform it can be readily proved that:
\begin{equation}
    d_l = \mathcal{Z}\left[ g\left(\frac{1}{\zeta}\right)\right]_l,
\end{equation}

where \(\mathcal{Z}[f(\zeta)]_l\) is the \(Z\)-transform of the function \(f(\zeta)\).

**Remark 5.** We note that the general asymptotic behaviour of the sequence \(\{d_l\}_{l\in\mathbb{N}_0}\) can be obtained even without computing the \(Z\)-transform. Indeed, let us assume that the given generating \(g\) function has radius of convergence \(\rho > 0\). Then, let us assume that the generating function can be written in the following way:
\begin{equation}
    g = A(z) + B(z) \left(1 - \frac{z}{\rho}\right)^{-\beta},
\end{equation}

where \(A\) and \(B\) are analytic functions for \(|z| < r\) such that \(B(\rho) \neq 0\). Then we have the following estimate:
\begin{equation}
    d_N \sim \frac{B(\rho)}{\Gamma(\beta)} N^{\beta-1} \rho^{-N}, \quad N \to \infty
\end{equation}

where \(\Gamma(z)\) is the Euler Gamma function. When the radius of convergence is one, i.e. when the given equation is integrable, we have the simpler estimate
\begin{equation}
    d_N \sim N^{\beta-1}, \quad N \to \infty.
\end{equation}

### 3. Results

In this section we describe the results of the procedure outlined in section 2 for the differential-difference equations of Lists 1–6. Specifically, as described in Remark 4 we will underline the particular cases in which the parametric equations can be divided. We notice that certain equations are symmetric under the involution
\begin{equation}
    u_n \rightarrow \tilde{u}_n = u_{-n}.
\end{equation}
This implies that the recurrence defined by solving the equation with respect to $u_2$ and $u_{-2}$ is the same. For equations satisfying this property the growth of the degree of the iterates can be computed just in one direction, as the growth in the other direction will be the same. Computations are performed using the python program for differential-difference equations presented in [27]. We remark that this program was already employed to discuss the integrability of some three-point differential-difference equations in [31].

3.1. List 1.

3.1.1. Equation (E.1): Equation (E.1) is symmetric and has the following growth of degrees:

\[(32) \quad 1, 2, 2, 4, 4, 7, 7, 11, 11, 16, 16, 22, 22, \ldots.\]

The generating function corresponding to the growth (32) is:

\[(33) \quad g(z) = \frac{-z^4 - 2z^2 + z + 1}{(z - 1)^3(z + 1)^2}.\]

All the poles of $g$ lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z - 1)^3$ following remark 5 we have that equation (E.1) has quadratic growth.

3.1.2. Equation (E.2): Equation (E.2) is symmetric and has the following growth of degrees:

\[(34) \quad 1, 3, 3, 7, 7, 13, 13, 21, 21, 31, 31, 43, 43, \ldots.\]

The generating function corresponding to the growth (34) is:

\[(35) \quad g(z) = \frac{z^4 - 2z^2 + 2z + 1}{(z - 1)^3(z + 1)^2}.\]

All the poles of $g$ lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z - 1)^3$ following remark 5 we have that equation (E.2) has quadratic growth.

3.1.3. Equation (E.3): Equation (E.3) is symmetric and has the same growth of degrees as equation (E.2). Therefore we have that equation (E.3) has zero entropy and quadratic growth.

3.1.4. Equation (E.4): Equation (E.4) is symmetric and has the following growth of degrees:

\[(36) \quad 1, 2, 3, 5, 6, 8, 10, 14, 16, 20, 23, 29, 32, 38, 42, 50, 54, \ldots.\]

The generating function corresponding to the growth (36) is:

\[(37) \quad g(z) = -\frac{z^7 + z^6 - z^5 - z^4 + z^3 + z + 1}{(z - 1)^3(z + 1)^2(z^2 + 1)}.\]

All the poles of $g$ lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z - 1)^3$ following remark 5 we have that equation (E.4) has quadratic growth.
3.1.5. **Equation (E.5):** Equation (E.5) depends on the parameter $a$. Using a simple scaling if $a \neq 0$ it is possible to set $a = 1$. For this reason we can consider the two cases $a = 1$ and $a = 0$. If $a = 1$ equation (E.5) is asymmetric, but it has the following growth of degrees in both directions:

$$1, 2, 3, 5, 6, 8, 10, 14, 16, 20, 23, 29, 32, 38,$$

$$42, 50, 54, 62, 67, 77, 82, 92, 98, 110, 116, \ldots$$

The generating function corresponding to the growth (38) is:

$$g(z) = -\frac{z^7 + z^6 - z^5 - z^4 + z^3 + z + 1}{(z - 1)^3(z + 1)^2(z^2 + 1)}.$$  

If $a = 0$ equation (E.5) is symmetric, but its growth of degrees is still given by the sequence (38) and fitted by the generating function (39). Therefore in both cases the entropy is zero since all the poles of $g$ lie on the unit circle. Moreover, due to the presence of the factor $(z - 1)^3$ following remark 5 we have that equation (E.5) has quadratic growth for all values of $a$.

3.1.6. **Equation (E.6):** Equation (E.6) is symmetric and has the following growth of degrees:

$$1, 5, 8, 14, 19, 28, 35, 47, 56, 71, 82, 100, 113, \ldots$$

The generating function corresponding to the growth (40) is:

$$g(z) = -\frac{z^5 - 2z^3 + z^2 + 4z + 1}{(z - 1)^3(z + 1)^2}.$$  

All the poles of $g$ lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z - 1)^3$ following remark 5 we have that equation (E.6) has quadratic growth.

3.1.7. **Equation (E.1′):** Equation (E.1′) is symmetric and has the following growth of degrees:

$$1, 3, 4, 6, 8, 12, 15, 19, 24, 29, 34, 40, 47, 54, 61, 69, 78, 87, 96, 106, 117, \ldots$$

The generating function corresponding to the growth (42) is:

$$g(z) = -\frac{z^9 - 3z^8 + 4z^7 - 4z^6 + 4z^5 - 3z^4 + 2z^3 - z^2 + 1}{(z - 1)^3(z^2 + 1)}.$$  

All the poles of $g$ lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z - 1)^3$ following remark 5 we have that equation (E.1′) has quadratic growth.

3.1.8. **Equation (E.2′):** Equation (E.2′) is symmetric and has the following growth of degrees:

$$1, 5, 7, 13, 18, 27, 34, 45, 54, 69, 80, 97, 110,$$

$$131, 146, 169, 186, 213, 232, 261, 282, \ldots$$

The generating function corresponding to the growth (44) is:

$$g(z) = -\frac{z^9 - z^8 + z^6 + z^5 + 2z^4 + 2z^3 + z^2 + 4z + 1}{(z - 1)^3(z + 1)^2(z^2 + 1)}.$$
All the poles of $g$ lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z - 1)^3$ following remark 5 we have that equation (E.2) has quadratic growth.

3.2. List 2.

3.2.1. Equation (E.7): Equation (E.7) depends on four parameters $a$, $b$, $c$ and $d$ linked among themselves by the condition $(a + 1)d = bc$. Using a linear transformation $u_{n,m} \rightarrow \alpha u_{n,m} + \beta$ we need to consider only three different cases:

1. $a \neq 0, a \neq -1, b = 0, d = 0$,
2. $a = -1, b = 1, c = 0$,
3. $a = -1, b = 0$.

Recall that $a \neq 0$ in all cases. See [17] for more details. In all the three cases equation (E.7) is asymmetric. However, it has the same growth of degrees in both directions and in all the three cases:

$$1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37 \ldots$$

The generating function corresponding to the growth (46) is:

$$g(z) = -z^7 + z^6 - z^5 - z^4 + z^3 + z + 1 (z - 1)^3(z + 1)^2(z^2 + 1).$$

In all cases the entropy is zero since all the poles of $g$ lie on the unit circle. Moreover, due to the presence of the factor $(z - 1)^2$ following remark 5 we have that equation (E.7) has linear growth for all values of the parameters.

3.2.2. Equation (E.8): Equation (E.8) is not symmetric, but in both directions has the following growth of degrees:

$$1, 4, 6, 9, 11, 14, 16, 19, 21, 24, 26, 29, 31 \ldots$$

The generating function corresponding to the growth (48) is:

$$g(z) = \frac{z^2 + 3z + 1}{(z - 1)^2(z + 1)}.$$

All the poles of $g$ lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z - 1)^2$ following remark 5 we have that equation (E.8) has linear growth.

3.3. List 3.

3.3.1. Equation (E.3'). Equation (E.3') is symmetric and has the following growth of degrees:

$$1, 3, 6, 10, 16, 22, 29, 37, 46, 56, 67, 79, 92 \ldots$$

The generating function corresponding to the growth (50) is:

$$g(z) = -z^6 - 2z^5 + z^4 + 1 (z - 1)^3.$$

All the poles of $g$ lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z - 1)^3$ following remark 5 we have that equation (E.3') has quadratic growth.
3.3.2. **Equation (E.4′).** Equation (E.4′) depends on the parameter $a$. Using a simple scaling if $a \neq 0$ it is possible to set $a = 1$. For this reason we can consider the two cases $a = 1$ and $a = 0$. Equation (E.4′) is symmetric for both $a = 1$ and $a = 0$. Moreover, in both cases it has the following growth of degrees:

$$1, 5, 11, 21, 31, 43, 57, 73, 91, 111, 133, 157, 183 \ldots .$$

The generating function corresponding to the growth (52) is:

$$g(z) = \frac{2z^5 - 4z^4 + 2z^3 - z^2 + 2z + 1}{z - 1)^3}.$$ 

Therefore in both cases the entropy is zero since all the poles of $g$ lie on the unit circle. Moreover, due to the presence of the factor $(z - 1)^3$ following remark 5 we have that equation (E.4′) has quadratic growth for all values of $a$.

3.3.3. **Equation (E.5′).** Equation (E.5′) is not symmetric, but it has the same growth of degrees in both directions:

$$1, 3, 4, 7, 10, 15, 19, 25, 31, 39, 46, 55, 64 \ldots .$$

The generating function corresponding to the growth (54) is:

$$g(z) = \frac{z^5 - z^4 + 2z^3 - z^2 + z + 1}{(z - 1)^3(z + 1)(z^2 + 1)}.$$ 

All the poles of $g$ lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z - 1)^3$ following remark 5 we have that equation (E.5′) has quadratic growth.

3.3.4. **Equation (E.6′):** Equation (E.6′) is symmetric and has the same growth of degrees as equation (E.3′). Therefore we have that equation (E.6′) has zero entropy and quadratic growth.

3.3.5. **Equation (E.7′):** Equation (E.7′) depends on the parameter $a$. Using a simple scaling if $a \neq 0$ it is possible to set $a = 1$. For this reason we can consider the two cases $a = 1$ and $a = 0$. Equation (E.7′) is symmetric for both $a = 1$ and $a = 0$. However, in both cases equation (E.7′) has the same growth of degrees as equation (E.4′). Therefore we have that equation (E.7′) has zero entropy and quadratic growth for all values of $a$.

3.3.6. **Equation (E.8′):** Equation (E.8′) is symmetric and has the following growth of degrees:

$$1, 3, 4, 7, 10, 15, 19, 25, 31, 39, 46, 55, 64 \ldots .$$

The generating function corresponding to the growth (56) is:

$$g(z) = \frac{z^5 - z^4 + 2z^3 - z^2 + z + 1}{(z - 1)^3(z + 1)(z^2 + 1)}.$$ 

All the poles of $g$ lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor $(z - 1)^3$ following remark 5 we have that equation (E.8′) has quadratic growth.

3.4. **List 4.**
3.4.1. *Equation* (E.9). Equation (E.9) is symmetric and has the following growth of degrees:

\[ 1, 5, 10, 16, 26, 38, 51, 65, 82, 102, 123, 145, 170, 198, 227, 257, \ldots \]

The generating function corresponding to the growth \((E.9)\) is:

\[
g(z) = -\frac{z^8 - 2z^7 + 2z^5 - 2z^4 + z^3 - z^2 + 2z + 1}{(z - 1)^3(z^2 + 1)}.
\]

All the poles of \(g\) lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor \((z - 1)^3\) following remark 5 we have that equation (E.9) has quadratic growth.

3.4.2. *Equation* (E.10). Equation (E.10) is symmetric and has the following growth of degrees:

\[ 1, 9, 19, 37, 55, 75, 101, 129, 163, 199, 237, 281, 327, 379, 433, 489, 551, 615, 685, 757, \ldots \]

The generating function corresponding to the growth \((E.10)\) is:

\[
g(z) = -\frac{2z^9 - 2z^8 - z^6 + 8z^3 + 2z^2 + 7z + 1}{(z - 1)^3(z^4 + z^3 + z^2 + z + 1)}.
\]

All the poles of \(g\) lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor \((z - 1)^3\) following remark 5 we have that equation (E.10) has quadratic growth.

3.4.3. *Equation* (E.10′). Equation (E.10′) is symmetric and has the following growth of degrees:

\[ 1, 5, 7, 13, 19, 29, 37, 49, 61, 77, 91, 109, 127, \ldots \]

The generating function corresponding to the growth \((E.10')\) is:

\[
g(z) = -\frac{z^5 - 4z^3 - 2z^2 + 3z + 1}{(z - 1)^3(z + 1)(z^2 + 1)}.
\]

All the poles of \(g\) lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor \((z - 1)^3\) following remark 5 we have that equation (E.10′) has quadratic growth.

3.5. *List 5.*

3.5.1. *Equation* (E.11). Equation (E.11) is symmetric and has the following growth of degrees:

\[ 1, 2, 3, 4, 6, 8, 10, 13, 16, 19, 23, 27, 31, \ldots \]

The generating function corresponding to the growth \((E.11)\) is:

\[
g(z) = -\frac{z^4 - z^3 + 1}{(z - 1)^3(z^2 + z + 1)}.
\]

All the poles of \(g\) lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor \((z - 1)^3\) following remark 5 we have that equation (E.11) has quadratic growth.
3.5.2. **Equation (E.12).** Equation (E.12) depends on the parameter \( a \). Using a simple scaling if \( a \neq 0 \) it is possible to set \( a = 1 \). For this reason we can consider the two cases \( a = 1 \) and \( a = 0 \). If \( a = 1 \) equation (E.12) is asymmetric, but it has the following growth of degrees in both directions:

\[
1, 2, 4, 5, 7, 8, 11, 12, 16, 18, 22, 24, 30, 31, 38, 41, 47, 50, 59, 60, 70, 74, 86, 98, 99 \ldots
\]

The generating function corresponding to the growth (66) is:

\[
g(z) = \frac{z^{13} + z^{10} + z^9 - z^7 + 2z^5 + z^4 + z^3 + z^2 + 2z + 1}{(z - 1)^3(z + 1)^2(z^2 - z + 1)(z^2 + z + 1)^2}.
\]

If \( a = 0 \) equation (E.12) is symmetric, but its growth of degrees is still given by the sequence (66) and fitted by the generating function (67). Therefore in both cases the entropy is zero since all the poles of \( g \) lie on the unit circle. Moreover, due to the presence of the factor \((z - 1)^3\) following remark \( \S \) we have that equation (E.12) has quadratic growth for all values of \( a \).

3.5.3. **Equation (E.13).** Equation (E.13) depends on the parameter \( a \). Using a simple scaling if \( a \neq 0 \) it is possible to set \( a = 1 \). For this reason we can consider the two cases \( a = 1 \) and \( a = 0 \). Equation (E.13) is symmetric for both \( a = 1 \) and \( a = 0 \). Moreover, in both cases it has the following growth of degrees:

\[
1, 4, 6, 10, 16, 22, 29, 37, 46, 56, 67, 79, 92 \ldots
\]

The generating function corresponding to the growth (68) is:

\[
g(z) = \frac{(z^2 - z + 1)(z^4 - z^3 - 2z^2 + 2z + 1)}{(z - 1)^3}.
\]

If \( a = 0 \) equation (E.13) is symmetric, but its growth of degrees is still given by the sequence (68) and fitted by the generating function (69). Therefore in both cases the entropy is zero since all the poles of \( g \) lie on the unit circle. Moreover, due to the presence of the factor \((z - 1)^3\) following remark \( \S \) we have that equation (E.13) has quadratic growth for all values of \( a \).

3.5.4. **Equation (E.14).** Equation (E.14) is symmetric and has the following growth of degrees:

\[
1, 4, 7, 10, 15, 21, 27, 36, 45, 54, 65, 77, 89, 104, 119, 134, 151, 169, 187, 208 \ldots
\]

The generating function corresponding to the growth (70) is:

\[
g(z) = \frac{(z^4 + z + 1)(z^3 - z^2 + z + 1)}{(z - 1)^3(z + 1)(z^2 - z + 1)(z^2 + z + 1)}.
\]

All the poles of \( g \) lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor \((z - 1)^3\) following remark \( \S \) we have that equation (E.14) has quadratic growth.
3.5.5. *Equation* (1). *Equation* (1) is symmetric and has the following growth of degrees:

\[(72)\]

\[1, 3, 1, 4, 5, 5, 6, 13, 7, 15, 17, 17, 19, 31, 21, 34, 37, 37, 40, 57, 43, 61, 65, 65, 69 \ldots\]

The generating function corresponding to the growth (72) is:

\[(73)\]

\[g(z) = -\frac{z^{10} + z^9 + z^7 - z^6 + z^5 + z^4 + 3z + 1}{(z - 1)^3(z + 1)^2(z^2 - z + 1)(z^2 + z + 1)^2}.\]

All the poles of \(g\) lie on the unit circle, so that the entropy is zero. Using the \(Z\)-transform we obtain the following expression for the degrees:

\[(74)\]

\[d_n = \frac{n^2}{9} + \frac{5n}{9} + \frac{191}{108} + \frac{5(-1)^n}{12} + \frac{(-1)^n n}{6} + \frac{\sqrt{3}}{36} \sin \left(\frac{n\pi}{3}\right) - \frac{\sqrt{3}}{3} \left(\frac{5n}{54} + \frac{7}{36}\right) \sin \left(\frac{2n\pi}{3}\right) + \frac{1}{12} \cos \left(\frac{n\pi}{3}\right) + \left(\frac{5n}{18} + \frac{79}{108}\right) \cos \left(\frac{2n\pi}{3}\right).\]

Therefore the growth (72) is quadratic as \(n \to \infty\), but we notice also the unusual presence of oscillating term proportional to \((-1)^n n\) which explains the high oscillations of the sequence (72). A similar was found in [30] on the degree pattern of some linearisable quad-equations.

3.5.6. *Equation* (E.9'). *Equation* (E.9') is symmetric and has the following growth of degrees:

\[(75)\]

\[1, 3, 4, 7, 11, 15, 20, 25, 31, 38, 45, 53, 62, 71, 81, 92, 103, 115, 128, 141, 155 \ldots\]

The generating function corresponding to the growth (75) is:

\[(76)\]

\[g(z) = \frac{z^8 - z^7 - z^6 + z^5 + z^3 - z^2 + z + 1}{(z - 1)^3(z^2 + z + 1)}.\]

All the poles of \(g\) lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor \((z - 1)^3\) following remark 5 we have that equation (E.9') has quadratic growth.

3.5.7. *Equation* (E.10'). *Equation* (E.10') depends on the parameter \(a\). Using a simple scaling if \(a \neq 0\) it is possible to set \(a = 1\). For this reason we can consider the two cases \(a = 1\) and \(a = 0\). *Equation* (E.10') is not symmetric for both \(a = 1\) and \(a = 0\). However, in both cases it has the following growth of degrees:

\[(77)\]

\[1, 3, 6, 9, 13, 19, 24, 31, 40, 48, 57, 69, 79, 91, 106, 119, 133, 151, 166, 183, 204 \ldots\]

The generating function corresponding to the growth (77) is:

\[(78)\]

\[g(z) = -\frac{z^9 + z^7 + z^6 + 3z^5 + 2z^4 + 2z^3 + 3z^2 + 2z + 1}{(z - 1)^3(z + 1)(z^2 - z + 1)(z^2 + z + 1)^2}.\]

Therefore in both cases the entropy is zero since all the poles of \(g\) lie on the unit circle. Moreover, due to the presence of the factor \((z - 1)^3\) following remark 5 we have that equation (E.10') has quadratic growth for all values of \(a\).
3.5.8. Equation (E.11′). Equation (E.11′) depends on the parameter \( a \). Using a simple scaling if \( a \neq 0 \) it is possible to set \( a = 1 \). For this reason we can consider the two cases \( a = 1 \) and \( a = 0 \). Equation (E.11′) is symmetric for both \( a = 1 \) and \( a = 0 \). When \( a = 1 \) equation (E.11′) has the following growth of degrees:

\[
1, 7, 15, 24, 35, 51, 66, 85, 109, 132, 157, 189, 218, 251, 291, 328, 367, 415, 458, 505, 561, 612, 665, \ldots.
\]

The generating function corresponding to the growth (79) is:

\[
g_{a=1}(z) = -\frac{z^{10} + 2z^7 + 5z^6 + 8z^5 + 5z^4 + 8z^3 + 8z^2 + 6z + 1}{(z-1)^3(z+1)(z^2 - z + 1)(z^2 + z + 1)^2}.
\]

When \( a = 0 \) equation (E.11′) has the following growth of degrees:

\[
1, 7, 15, 23, 33, 48, 63, 84, 107, 130, 155, 182, 211, 248, 287, 324, 363, 404, 447, 500, 555, 606, 659, 714, 771, 840, 911, 976, 1043, 1112, 1183, 1268, 1355, \ldots.
\]

The generating function corresponding to the growth (79) is:

\[
g_{a=0}(z) = -\frac{2z^{16} - 3z^{15} + 2z^{14} - z^{13} + 2z^{12} - 2z^{11} - z^{10} + 6z^9}{(z-1)^3(z+1)(z^2 - z + 1)^2(z^2 + z + 1)^2}.
\]

All the poles of \( g_{a=1} \) and \( g_{a=0} \) lie on the unit circle, so that the entropy is zero in both cases. Moreover, due to the presence of the factor \( (z-1)^3 \) following remark 5 we have that equation (E.11′) has quadratic growth in both cases.

3.5.9. Equation (E.12′). Equation (E.12′) is symmetric and has the same growth of degrees as equation (E.10′). Therefore we have that equation (E.12′) has zero entropy and quadratic growth.

3.6. List 6.

3.6.1. Equation (E.14′). Equation (E.14′) is symmetric and has the following growth of degrees:

\[
1, 4, 6, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, \ldots
\]

The generating function corresponding to the growth (83) is:

\[
g(z) = -\frac{z^5 - 3z^4 + 4z^3 - 3z^2 + z + 1}{(z-1)^3}.
\]

All the poles of \( g \) lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor \( (z-1)^3 \) following remark 5 we have that equation (E.15′) has quadratic growth.
3.6.2. Equation (E.16). Equation (E.16) is symmetric and has the following growth of degrees:

\[(85)\] 1, 6, 13, 25, 42, 61, 85, 111, 139, 171, 207, 245, 287, 333, 381, 433, 489, 547 \ldots

The generating function corresponding to the growth (85) is:

\[(86)\] \[g(z) = -\frac{2z^{10} - z^{9} - 3z^{7} + z^{4} + 4z^{3} + 2z^{2} + 4z + 1}{(z - 1)^{3}(z^{2} + z + 1)}.\]

All the poles of \(g\) lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor \((z - 1)^{3}\) following remark 5 we have that equation (E.16) has quadratic growth.

3.6.3. Equation (E.17). We proved that equation (E.17) can be brought in rational form (6), but this form is not bi-rational. So we cannot apply the algebraic entropy method to this equation.

3.6.4. Equation (E.14'). Equation (E.14') is symmetric and has the following growth of degrees:

\[(87)\] 1, 7, 15, 24, 35, 49, 67, 86, 107, 132, 159, 188, 219, 254, 291, 330, 371, 416 \ldots

The generating function corresponding to the growth (87) is:

\[(88)\] \[g(z) = -\frac{z^{11} - 2z^{10} + z^{9} + 2z^{6} - 2z^{5} + z^{4} + z^{3} + 2z^{2} + 5z + 1}{(z - 1)^{3}(z^{2} + 1)}.\]

All the poles of \(g\) lie on the unit circle, so that the entropy is zero. Moreover, due to the presence of the factor \((z - 1)^{3}\) following remark 5 we have that equation (E.14') has quadratic growth.

4. Discussion

In the previous section we computed the algebraic entropy of all the integrable Volterra-like five-point differential-difference equations recently classified in [23,24]. When possible, we showed that the method of algebraic entropy and the method of generalised symmetries agree. That is, we showed that all the equations integrable according to the generalised symmetry test are also integrable according to the algebraic entropy method, i.e. the algebraic entropy is zero. The algebraic entropy method is unfortunately unable to treat the semi-discrete Kaup-Kaupershmidt equation (E.17). This is because the generalised symmetry approach, differently from the algebraic entropy, makes no assumption on the nature of the recurrence and algebraic or even transcendental terms are allowed. That is, we proved that for integrable Volterra-like five-point differential-difference equations the following version of the algebraic entropy conjecture holds true:

**Conjecture 2.** The condition that algebraic entropy is zero is equivalent to the definition of integrability for bi-rational maps.

Except for the two known non-trivially linearisable equations (E.7) and (E.8) the growth is always quadratic. Equation (E.1) possesses an interesting non-standard highly oscillating growth, observed for the first time in differential-difference equations. Nevertheless the asymptotic growth is still quadratic.

In the case of two-dimensional difference equation it is known that the only possible polynomial, i.e. integrable, growth is quadratic [13]. Integrable higher
order maps can exhibit higher rate of growth, see e.g. [28, 29, 35]. Despite being
infinite-dimensional all the integrable Volterra-like five-point differential-difference
equations possess this “minimal” integrable growth.

In the case of difference equations it has been observed that degree growth greater
than quadratic is related to a procedure called deflation [35]. That is, a five-point
equation is reduced to a four-point one using a non-point potential-like transforma-
tions of the form:

\[ v_n = \frac{a_1 u_n u_{n+1} + a_2 u_{n+1} + a_3 u_n + a_4}{b_1 u_n u_{n+1} + b_2 u_{n+1} + b_3 u_n + b_4}. \]

Let us notice, that the inflated version of a Volterra-like differential-difference equa-
tion is not always a Volterra-like differential-difference equation. This fact makes
more difficult to make predictions on the integrability properties of the inflated
forms of differential-difference equations.

Finally, we notice that another interesting problem is to study the integrabil-
ity properties of the stationary reductions of the integrable Volterra-like five-point
differential-difference equations. The stationary reduction of a bi-rational five-
point differential-difference equation is a fourth-order difference equation, i.e. a
four-dimensional map of the projective space into itself. It will be important to
understand how integrability arises inside these families of equations and if it fits
with known cases of integrable families of fourth-order differential difference equa-
tions [11, 28, 29].

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