Effective Potential Structure of the BTZ Black Hole in Rainbow Gravity

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In this paper we study the effective potential structure of the BTZ black hole in rainbow gravity for a massive and a massless particles.

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I. INTRODUCTION

Today there are many efforts in Theoretical Physics that try to combine quantum theory and general relativity. Several lines of research have been developed but none of them is completely successful in obtaining a complete description of the quantum gravity realm. Meanwhile, some phenomenological approaches have been put on the table. One of them is the modification of the dispersion relation \( E^2 - p^2 = m^2 \) [1], with a non linear version instead. It is very probable, after the most of the results, that the linear version of the relation linking energy and momenta is just a first approximation to a real non lineal one.

On the other hand, some data seem to invite to introduce a minimal length in physical theories. Indeed, there already exist well theoretically established theories, such as String Theory or Loop Quantum Gravity, that have some fundamental quantities: the Planck longitude \( l_p = \sqrt{\hbar G/c^3} \), the associated time scale \( t_p = l_p/c \) and the Planck energy \( E_p = \hbar/t_p \). All of them suppose that beyond these thresholds, the physics should change dramatically.

Some proposals to modify the Lorentz boosts through are the approximations called Double Special relativity (DSR) [2–5]. These theories are based on a generalization of Lorentz transformations through a more broad point of view of a kind of conformal transformations

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where there are two observer independent scales, velocity of light and Planck length. These theories are rather polemical, but they are of increasing interest because they can be useful as effective new tools in gravity theories for example, in Cosmology as an alternative to inflation [7, 8], or in other fields like propagation of light [10], that are related, for instance, to cosmic microwave background radiation.

There are several works about the so called rainbow gravity, as an example [11] and references there in, but the history begins more or less with a treatment done in Ref.[12].

In this paper we review the structure of effective potential of a BTZ black hole, motivated by the fact that black holes provide gravity conditions to test quantum effects due to the discrete nature of spacetime or the existence of a limit in the energy that a particle can bear.

The paper is organized as follows: In the next section we recall the effective potential treatment in a BTZ black hole. In the third section we introduce the rainbow hypothesis , and in the fourth one we discuss conclusions.

II. BTZ EFFECTIVE POTENTIAL

The metric of a BTZ black hole [13, 14] is:

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 \{ N\phi dt + d\phi \}^2, \]  

where \( N^2 \) and \( N\phi \) are defined as:

\[ N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}; \quad N\phi = -\frac{J}{2r^2} \]  

Here, \(-\infty < t < \infty, -\infty < r < \infty \) and \( 0 < t < 2\pi \).

\[ M \] is the mass and \( J \) the angular momentum of the black hole. In order to apply the effective potential method, we define de Lagrangian of a particle in this metrics as.

\[ \mathcal{L} = -N^2 \dot{t}^2 + N^{-2} \dot{r}^2 + r^2 \{ N\phi \dot{t} + \dot{\phi} \}^2, \]  

where dots indicate derivation with respect to a affine parameter \( \lambda \).

We can find two movement integrals in the Lagrangian 10. Making the variations in \( t \) and in \( \phi \) respectively we get:
\[ E = \{-M + \frac{r^2}{l^2}\} \dot{t} + \frac{J}{2} \dot{\phi}, \quad (4) \]

\[ J = r^2 \dot{\phi} - \frac{J}{2} \dot{t}. \quad (5) \]

It is worth to mention that \( E \) can not be identified directly with the energy of the particle because the metric is not flat when \( r \) tends to infinite. But, we will call it energy because it comes from the variation of the Lagrangian respect to the coordinate time.

Furthermore, it is possible find another movement constant, \( ds^2 = \mathcal{L} = -\xi^2 \). Where \( \xi^2 \) has the values 1 for timelike geodesics (massive particles) and 0 for null geodesics (massless particles).

Using the results of \([14]\), we can calculate the velocity in terms of two potentials:

\[ \dot{r}^2 = (\tilde{E} - V^+)(\tilde{E} - V^-), \quad (6) \]

where \( \tilde{E} \) is defined as \( \tilde{E} = \frac{E}{\sqrt{M}} \), and the effective potentials are:

\[ V_{\text{eff}} = \frac{\tilde{J}\tilde{L}}{2\tilde{r}^2} \pm \frac{1}{\tilde{r}^2} \sqrt{4\xi^2 \tilde{r}^4 (-1 + \tilde{r}^2) + \tilde{J}^2 (\tilde{L}^2 + \xi^2 \tilde{r}^6)}, \quad (7) \]

where

\[ \tilde{L} = \frac{L}{l\sqrt{M}}, \quad \tilde{J} = \frac{J}{lM}, \quad \tilde{r} = \frac{r}{lM}. \quad (8) \]

### III. BTZ EFFECTIVE POTENTIAL IN RAINBOW GRAVITY

Let’s introduce now the functions \( f(\tilde{E}) \) and \( g(\tilde{E}) \) in the metric. These functions are analog to those introduced by \([2–5]\) in the nonlinear version of the dispersion relation \( E^2/f^2 = p^2/g^2 + m^2 \), but we are going to choose they in a way that doesn’t modify the speed of light, that is to say \( f = g \). Moreover, we claim that these functions tend to 1 when \( \tilde{E} \ll \tilde{E}_p \) where \( \tilde{E}_p \) can be identified with a fundamental length scale \( l_p = 1/\tilde{E}_p \). So we choose \( f(E) \) as:

\[ f(E) = \frac{1}{1 - \tilde{E}/\tilde{E}_p}. \quad (9) \]

Then, the Lagrangian is modified in the following way.
Now, we can find two movement integrals for this new Lagrangian:

\[ \hat{\mathcal{L}} = \frac{-N^2}{f(E)^2} \dot{t}^2 + \frac{N^{-2}}{f(E)^2} \dot{r}^2 + r^2 \left\{ \frac{N^2}{f(E)} \dot{t} + \frac{\phi}{f(E)} \right\}^2, \]  

(10)

Now, we can find two movement integrals for this new Lagrangian:

\[ \hat{\mathcal{E}} = \frac{1}{f^2} (-M + \frac{r^2}{l^2}) \dot{t} + \frac{J}{2 f^2} \dot{\phi} \]  

(11)

and

\[ \hat{\mathcal{L}} = \frac{r^2}{f^2} \phi - \frac{J}{2 f^2} \dot{t} \]  

(12)

Now, we can find an equation for the velocity in terms of effective potentials, in an analog way to the normal case, with the re scaled variables,

\[ \hat{\Omega} = \frac{\hat{\mathcal{E}}}{\sqrt{M}}, \quad \hat{\Lambda} = \frac{\hat{\mathcal{L}}}{l \sqrt{M}}, \]  

(13)

and using re scaled variables of the last section the velocity is expressed as:

\[ \dot{r}^2 = (\hat{\Omega} - V_{eff}^+) (\hat{\Omega} - V_{eff}^-), \]  

(14)

where

\[ \hat{V}_{eff} = \frac{\hat{\Lambda} \tilde{J}}{2 \tilde{r}^2} \pm \frac{1}{2 f \tilde{r}^2} \sqrt{f^2 \hat{\Lambda}^2 (\tilde{J}^2 - 4 \tilde{r}^2 + 4 \tilde{r}^4) + 4 \xi^2 (\tilde{J}^2 - \tilde{r}^4 + \tilde{r}^6)} \]  

(15)

The plot of the original and new effective potentials for a massive particle and for a massless particle are depicted in Fig.1:

In Fig. 1 we can see that the potentials become flat when \( \hat{\mathcal{E}}/\hat{\mathcal{E}}_p \rightarrow 1 \), so a massive particle can now become free for a suitable distance far from the black hole. Furthermore, the zone where the velocity is non real in eq 14, is reduced almost to zero, even it’s not totally null.

The plot of the original and new effective potentials for a massless particle and for a massless particle are depicted in Fig.2:

In Fig. 2 we can see that the potentials is flat when \( \hat{\mathcal{E}}/\hat{\mathcal{E}}_p \rightarrow 1 \), as they are in the non modified case, but a massive particle has a not allowed energies where the velocity is non real in eq 14, broader than in the original case.
FIG. 1: This plot shows the effective potential for a massive particle in the non modified metrics (dotted line) and in the modified one (continuous line), when $\tilde{E}/\tilde{E}_p = 1/2$ (a) and when $\tilde{E}/\tilde{E}_p = 0.99$ (b).

FIG. 2: This plot shows the effective potential for a massive particle in the non modified metrics (dotted line) and in the modified one (continuous line), when $\tilde{E}/\tilde{E}_p = 1/2$ (a) and when $\tilde{E}/\tilde{E}_p = 0.99$ (b).

IV. DISCUSSION AND OUTLOOK

In this work we have seen that the introduction of functions $f$ has a very impressive consequences. The structure of the orbits allowed, due the deformation of the effective potentials for a massless particle and the ligated state for a massive particle are very distorted from the classic case. These results lead to think that it is worth to investigate about another
kinds of black holes, specially in 3-D searching for observable perturbations in cosmological issues.

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