Weak type radial convolution operators on free groups

by

Tadeusz Pytlik and Ryszard Szwarc (Wrocław and Opole)

Abstract. Radial convolution operators on free groups with nonnegative kernel of weak type $(2, 2)$ and of restricted weak type $(2, 2)$ are characterized. Estimates of weak type $(p, p)$ are obtained as well for $1 < p < 2$.

1. Introduction. A discrete group $G$ is called amenable if there exists a linear functional $m$ on $\ell^\infty_R(G)$ such that

\begin{align*}
(1) & \quad \inf_{x \in G} f(x) \leq m(f) \leq \sup_{x \in G} f(x), \\
(2) & \quad m(xf) = m(f), \quad \text{where} \quad xf(y) = f(x^{-1}y).
\end{align*}

$m$ is called a left invariant mean. Then the functional $M(f) = m(m(fx))$ satisfies (1), (2) and is also right invariant, where $fx(y) = f(yx)$.

Let $G$ be a discrete group. Consider a symmetric probability measure $\mu$ on $G$, i.e.

$$
\mu = \sum_{x \in G} \mu(x) \delta_x, \quad \mu(x) \geq 0, \quad \sum_{x \in G} \mu(x) = 1, \quad \mu(x^{-1}) = \mu(x).
$$

The left convolution operator $\lambda(\mu)$ with $\mu$ is bounded on $\ell^2(G)$ and

$$
\|\lambda(\mu)(f)\|_2 = \|\mu \ast f\|_2 \leq \|f\|_2, \quad f \in \ell^2(G).
$$

Indeed,

$$
\|\mu \ast f\|_2 = \left\| \sum_{x \in G} \mu(x)[\delta_x \ast f] \right\|_2 \leq \sum_{x \in G} \mu(x) \|\delta_x \ast f\|_2 = \|f\|_2.
$$

Thus $\|\lambda(\mu)\|_{2 \rightarrow 2} \leq 1$.

Kesten [5] showed that a discrete group $G$ is amenable iff for any symmetric probability measure $\mu$ on $G$ we have $\|\lambda(\mu)\|_{2 \rightarrow 2} = 1$. He also showed

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that $G$ is amenable if this condition is satisfied for one measure $\mu$ such that $\text{supp} \mu$ generates $G$ algebraically. In particular, let $G$ be generated by $g_1, \ldots, g_k$ and $\mu = (2^k)^{-1} \sum_{i=1}^k (\delta_{g_i} + \delta_{g_i^{-1}})$. Then $G$ is amenable if and only if $\|\lambda(\mu)\|_{2\to2} = 1$.

In [4] Følner came up with another property equivalent to amenability. We say that a discrete group $G$ satisfies the Følner condition if for any number $\varepsilon > 0$ and any finite set $K \subset G$ there exists a finite set $N \subset G$ such that

$$|xN \triangle N| < \varepsilon|N|, \quad x \in K,$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. In other words, $N$ is almost $K$-invariant. He showed that $G$ is amenable if and only if the Følner condition holds.

Assume that $G$ is amenable. Let $\mu$ be a probability measure with finite support $K$. For $\varepsilon = \eta^2 > 0$ choose $N$ so as to satisfy (1). Then

$$\|\mu * \chi_N - \chi_N\|_2 = \left\| \sum_{x \in K} \mu(x)[\chi_{xN} - \chi_N] \right\|_2 \leq \sum_{x \in K} \mu(x)\|\chi_{xN} - \chi_N\|_2$$

$$= \sum_{x \in K} \mu(x)\|\chi_{xN \triangle N}\|_2 = \sum_{x \in K} \mu(x)|xN \triangle N|^{1/2} \leq \eta|N|^{1/2} = \eta\|\chi_N\|_2.$$

Therefore

$$\langle \mu * \chi_N, \chi_M \rangle_{L^2(G)} = \langle \chi_N, \chi_N \rangle_{L^2(G)} + \langle \mu * \chi_N - \chi_N, \chi_N \rangle_{L^2(G)}$$

$$\geq (1 - \eta)\|\chi_N\|_2^2,$$

which implies

$$\sup_{N,M \text{ finite}} \frac{\langle \mu * \chi_N, \chi_M \rangle_{L^2(G)}}{\|\chi_N\|_2\|\chi_M\|_2} = 1 = \|\lambda(\mu)\|_{2\to2}.$$

The same holds (with the same proof) for any $1 < p < \infty$, i.e.

$$\sup_{N,M \text{ finite}} \frac{\langle \mu * \chi_N, \chi_M \rangle}{\|\chi_N\|_p\|\chi_M\|_{p'}} = 1 = \|\lambda(\mu)\|_{p\to p'},$$

where $p' = p/(p - 1)$.

We will use the notion of Lorentz $L^{p,q}$ spaces (see [1]). Consider a general $\sigma$-finite measure space $(\Omega, \omega)$ and $1 < p < \infty$. For $f \in L^p(\Omega, \omega)$ and $t > 0$ we have

$$t^p \omega \{ x : |f(x)| > t \} \leq \int_{\Omega} |f(x)|^p \, d\omega(x).$$

The functions for which the left hand side is bounded form a linear space

$$L^{p,\infty}(\Omega, \omega) = \{ f : \sup_{t>0} t^p \omega \{ x : |f(x)| > t \} < \infty \},$$

called the \textit{weak} $L^p$ space. This space contains $L^p(\Omega, \omega)$. 

For \( p' = \frac{p}{p-1} \) the predual of \( L^{p',\infty}(\Omega, \omega) \) with respect to the standard inner product is denoted by \( L^{p,1}(\Omega, \omega) \). We have

\[ L^{p,1}(\Omega, \omega) \subset L^p(\Omega, \omega) \subset L^{p,\infty}(\Omega, \omega). \]

For \( p > 1 \) these spaces are normed.

Any linear operator mapping \( L^p \) into itself is called of strong type \((p, p)\). Linear operators \( T \) mapping \( L^p(\Omega, \omega) \) into \( L^{p,\infty}(\Omega, \omega) \) are called of weak type \((p, p)\), while those which map \( L^{p,1}(\Omega, \omega) \) into \( L^{p,\infty}(\Omega, \omega) \) are called of restricted weak type \((p, p)\).

We will use the following facts. A linear operator \( T \) is bounded from \( L^{p,1} \) into a Banach space \( X \) if and only if

\[
\|T\|_{L^{p,1} \to X} = \sup_{E \subseteq \Omega} \frac{\|T\chi_E\|_X}{\|\chi_E\|_p} < \infty.
\]

A linear operator \( T \) is bounded from \( L^{p,1} \) into \( L^{p,\infty} \) if and only if

\[
\|T\|_{(p,1) \to (p,\infty)} = \sup_{E,F \subseteq \Omega} \frac{|\langle T\chi_E, \chi_F \rangle|}{\|\chi_E\|_p \|\chi_F\|_{p'}} < \infty.
\]

Using this and duality between \( L^{(p',1)} \) and \( L^{(p,\infty)} \) we obtain

\[
\|T\|_{p \to (p,\infty)} = \|T^*\|_{(p',1) \to p'} = \sup_{E \subseteq \Omega} \frac{\|T^*\chi_E\|_{p'}}{\|\chi_E\|_{p'}}.
\]

The equalities (2) and (3) can be interpreted as follows. If the group \( G \) is discrete and amenable and \( \mu \) is a symmetric probability measure on \( G \), then

\[
\|\lambda(\mu)\|_p = \|\lambda(\mu)\|_{(p',1) \to p'} = \|\lambda(\mu)\|_{p \to (p,\infty)} = \|\lambda(\mu)\|_{(p,1) \to (p,\infty)} = 1.
\]

Hence for these groups convolution operators with nonnegative functions of strong type \((p, p)\), of weak type \((p, p)\) and of restricted weak type \((p, p)\) coincide for any \( 1 < p < \infty \).

The situation is entirely different for nonamenable groups. Only special examples have been studied. It has been shown [9] that for \( p = 2 \) and \( G = \mathbb{F}_k \), the free group on \( k \) generators, \( k \geq 2 \), there exist nonnegative functions \( f \) on \( G \) such that \( \|\lambda(f)\|_{2 \to (2,\infty)} \) is finite while \( \|\lambda(f)\|_{2 \to 2} \) is infinite, i.e. there exist convolution operators with nonnegative functions of weak type \((2, 2)\) which are not of strong type \((2, 2)\). The same has been shown for \( 1 < p < 2 \) [10]. These functions \( f \) can be chosen to be radial, i.e. constant on elements of the group \( G \) of the same length. It is an open problem if these results remain true for any discrete nonamenable group.

In this work will focus on \( G = \mathbb{F}_k \). We are going to determine all nonnegative radial functions \( f \) on \( G \) such that \( \lambda(f) \) is of weak type \((2, 2)\), as well those \( f \) for which \( \lambda(f) \) is of restricted weak type \((2, 2)\). In particular, we prove that these spaces are different. Next we will turn our attention to the case \( 1 < p < 2 \). By using interpolation machinery, duality and the results
for $p = 2$ we will be able to determine the nonnegative radial functions $f$ for which $\lambda(f)$ is of weak type $(p, p)$. In this way we obtain a simpler proof of the upper estimate of $\|\lambda(f)\|_{p\to(p, \infty)}$ obtained in [3]. Our method does not rely on any deep theorems of representation theory.

2. Radial convolution operators of weak type $(2, 2)$. Let $F_k = \text{gp}\{g_1, \ldots, g_k\}$ be a free group on $k \geq 2$ generators. The group consists of reduced words in generators and their inverses. The reduced representation of a word is unique. The number of letters in it defines a length function on $F_k$. Let $\chi_n$ denote the indicator function of the words of length $n$. There are $2k(2k - 1)^{n-1}$ such words, as we have $2k$ choices for the first letter and $2k - 1$ choices for every consecutive one. Let $q = 2k - 1$. The next theorem generalizes the estimate for $\|\lambda(\chi_n)\|_{2\to(2, \infty)}$ given in [9].

**Theorem 1.** Let $f = \sum_{n=0}^{\infty} f_n \chi_n$. The operator $\lambda(f)$ is of weak type $(2, 2)$ if

$$A(f) := \sum_{n,m=0}^{\infty} |f_n||f_m|q^{-(n+m)/2}(1 + \min(n, m)) < \infty.$$ 

Moreover, if $f_n \geq 0$ the condition is necessary and

$$\frac{1}{6} A(f) \leq \|\lambda(f)\|_{2\to(2, \infty)} \leq 4A(f).$$

**Proof.** By (7), instead of estimating $\|\lambda(f)\|_{2\to(2, \infty)}$ we may estimate $\|\lambda(f)\|_{(2, 1)\to 2}$, which (see (4)) is equivalent to

$$\sup_{E \subset F_r} \frac{\|f * \chi_E\|}{|E|^{1/2}}.$$ 

We have

$$\|f * \chi_E\|^2 = \langle f * f * \chi_E, \chi_E \rangle = \sum_{n,m=0}^{\infty} f_n f_m \langle \chi_n * \chi_m * \chi_E, \chi_E \rangle.$$ 

Simple calculation shows that for $n \geq 1$ we have

$$\chi_n \ast \chi_m = q^{n-1} \delta_n^m \chi_0 + \sum_{k=|n-m|\atop k \equiv n + m \mod 2}^{n+m} q^{(n+m-k)/2} \chi_k.$$ 

Clearly $\chi_0 \ast \chi_0 = \chi_0$. Therefore

$$\chi_n \ast \chi_m \leq 2 \sum_{k=|n-m|\atop k \equiv n + m \mod 2}^{n+m} q^{(n+m-k)/2} \chi_k.$$ 

Hence

$$\|f * \chi_E\|_2^2 \leq 2 \sum_{n,m=0}^{\infty} f_n f_m q^{(n+m)/2} \sum_{k=|n-m|\atop k \equiv n + m \mod 2}^{n+m} q^{-k/2} \langle \chi_k \ast \chi_E, \chi_E \rangle.$$
Lemma 1.

\[ \langle \chi_k \ast \chi E, \chi E \rangle \leq 2q^{[k/2]}|E|. \]

Proof. Define an operator \( P_k \) by the rule

\[ \langle P_k \delta_x, \delta_y \rangle = \begin{cases} 
\langle \chi_k \ast \delta_x, \delta_y \rangle & \text{if } |x| \geq |y|, \\
0 & \text{if } |x| < |y|. 
\end{cases} \]

Then

\[ \langle \chi_k \ast \delta_x, \delta_y \rangle \leq \langle P_k \delta_x, \delta_y \rangle + \langle \delta_x, P_k \delta_y \rangle. \]

This implies

\[ \langle \chi_k \ast \chi E, \chi E \rangle \leq 2\langle P_k \chi E, \chi E \rangle \leq 2\|P_k \chi E\|_1 \leq 2|E| \sup_x \|P_k \delta_x\|_1. \]

Next

\[ P_k \delta_x = \sum_{|w|=k, |wx| \leq |x|} \delta_{wx}. \]

Let \( w = w_1 w_2 \) where \( |w_1| \leq |w_2| \leq (k + 1)/2 \). The conditions \( |w| = k \) and \( |wx| \leq |x| \) imply that \( w_2 \) is determined by the first \( [(k + 1)/2] \) letters of \( x \). Hence we have as many terms in the sum as choices for \( w_1 \), i.e. at most \( q^{[k/2]} \).

Thus

\[ \|P_k \delta_x\|_1 \leq q^{[k/2]}. \]

Therefore

\[ \langle \chi_k \ast \chi E, \chi E \rangle \leq 2q^{[k/2]}|E|. \]

Lemma 1 implies that

\[ \frac{\|f \ast \chi E\|_2^2}{|E|} \leq 4 \sum_{n,m=0}^{\infty} |f_n| |f_m| q^{(n+m)/2} \sum_{k=|n-m|}^{n+m} 1 \]

\[ = 4 \sum_{n,m=0}^{\infty} |f_n| |f_m| q^{(n+m)/2} \{1 + \min(m,n)\}. \]

We obtain the upper estimate

\[ \|\lambda(f)\|_{2,\infty}^2 \leq 4 \sum_{n,m=0}^{\infty} |f_n| |f_m| q^{(n+m)/2} \{1 + \min(m,n)\}. \]

On the other hand, if \( f_n \geq 0 \) we have

\[ \|\lambda(f)\|_{2,\infty}^2 \geq \frac{q}{q+1} q^{-2k} \|f \ast \chi_{2k}\|_2^2 \geq \frac{2}{3} q^{-2k} \left\| \sum_{n=0}^{\infty} f_n (\chi_n \ast \chi_{2k}) \right\|_2^2 \]

\[ \geq \frac{2}{3} q^{-2k} \left\| \sum_{n=0}^{\infty} f_n \sum_{l=|n-2k|}^{n+2k} q^{(n+2k-l)/2} \chi_l \right\|_2^2 \]
\[
\|\lambda(f)\|_{2\rightarrow(2,\infty)}^2 \geq \frac{2}{3} \sum_{l=0}^{\infty} q^{-l/2} \left( \sum_{n=|2k-l| \mod 2}^{2k+l} f_n q^{n/2} \right)^2 \geq \frac{2}{3} \sum_{l=0}^{\infty} \left( \sum_{n=|2k-l| \mod 2}^{2k+l} f_n q^{n/2} \right)^2 \geq \frac{2}{3} \sum_{n,m=0}^{2k} f_n f_m q^{(n+m)/2} \sum_{l=\max(2k-n,2k-m)}^{2k} 1.
\]

Considering even or odd values of \( m \) and \( n \) gives

\[
\|\lambda(f)\|_{2\rightarrow(2,\infty)}^2 \geq \frac{1}{3} \sum_{n,m=0}^{\infty} f_n f_m q^{(n+m)/2} \{1 + \min(n,m)\},
\]

since \( k \) is arbitrary,

\[
\|\lambda(f)\|_{2\rightarrow(2,\infty)}^2 \geq \frac{1}{6} \sum_{n,m=0}^{\infty} f_n f_m q^{(n+m)/2} \{1 + \min(n,m)\},
\]

because the matrix \( a(n,m) = 1 + \min(n,m) \) is positive definite. □

**Theorem 2.** For \( n \geq 0 \) we have

\[
\|\lambda(\chi_n)\|_{(2,1)\rightarrow(2,\infty)} \leq cq^{n/2}.
\]

**Proof.** We have

\[
\|\lambda(\chi_n)\|_{(2,1)\rightarrow(2,\infty)} = \sup_{E,F \subset \mathbb{R}} \frac{\langle \chi_n * \chi_E, \chi_F \rangle}{|E|^{1/2}|F|^{1/2}}.
\]

The proof will be completed if we show

\[
\langle \chi_n * \chi_E, \chi_F \rangle \leq cq^{n/2}|E|^{1/2}|F|^{1/2}.
\]

We will prove (8) by modifying the argument used in the proof of Lemma 1. Fix \( \alpha \in \mathbb{R} \). Let \( Q_n^\alpha \) denote the operator defined by the rule

\[
\langle Q_n^\alpha \delta_x, \delta_y \rangle = \begin{cases} 
\langle \chi_n * \delta_x, \delta_y \rangle & \text{if } |x| \geq q^\alpha|y|, \\
0 & \text{if } |x| < q^\alpha|y|.
\end{cases}
\]

Then

\[
\langle \chi_n * \delta_x, \delta_y \rangle \leq \langle Q_n^\alpha \delta_x, \delta_y \rangle + \langle \delta_x, Q_n^{-\alpha} \delta_y \rangle.
\]
This implies
\[
\langle \chi_n \ast \chi_E, \chi_F \rangle \leq \|Q^\alpha_n \chi_E\|_1 + \|Q^{-\alpha}_n \chi_F\|_1 \\
\leq |E| \sup_x \|Q^\alpha_n \delta_x\|_1 + |F| \sup_x \|Q^{-\alpha}_n \delta_x\|_1
\]
(9)

Next
\[
Q^\alpha_n \delta_x = \sum_{|w| = n, |wx| \leq q^{-\alpha}|x|} \delta_{wx}.
\]

Let \( w = w_2 w_1 \) where \( |w_1| = [n/2] + [\alpha] \) and \( |w_2| = n - [n/2] - [\alpha] \). The conditions \( |w| = n \) and \( |wx| \leq q^{-\alpha}|x| \) imply that \( w_1 \) is determined by the first \( [n/2] + [\alpha] \) letters of \( x \). Hence we have as many terms in the sum as choices for \( w_2 \), i.e. at most \( q^{n-[n/2]-[\alpha]} \). Thus
\[
\|Q^\alpha_n \delta_x\|_1 \leq q^{3/2} q^{-\alpha} q^{n/2}.
\]
Similarly
\[
\|Q^{-\alpha}_n \delta_x\|_1 \leq q^{3/2} q^{\alpha} q^{n/2}.
\]

Hence by (9) we get
\[
\langle \chi_n \ast \chi_E, \chi_F \rangle \leq q^{3/2} q^{-\alpha} q^{n/2} \{ q^{-\alpha} |E| + q^{\alpha} |F| \}.
\]
Choosing \( \alpha = (\log |E| - \log |F|)/(2 \log q) \) gives
\[
\langle \chi_n \ast \chi_E, \chi_F \rangle \leq 2q^{3/2} q^{n/2} |E|^{1/2} |F|^{1/2}.
\]

**Theorem 3.** Let \( f = \sum_{n=0}^{\infty} f_n \chi_n \) and \( f_n \geq 0 \). The operator \( \lambda(f) \) is of restricted weak type \((2,2)\) if and only if \( f \in L^{2,1} \).

**Proof.** By Theorem 2 we have
\[
\|\lambda(\chi_n)\|_{(2,1)\rightarrow(2,\infty)} \leq C q^{n/2}
\]
for some constant \( C > 0 \). Let \( f = \sum_{n=0}^{\infty} f_n \chi_n \). Then the triangle inequality yields
\[
\|f\|_{(2,1)\rightarrow(2,\infty)} \leq C \sum_{n=0}^{\infty} f_n q^{n/2}.
\]

By [8, Lemma 1],
\[
\sum_{n=0}^{\infty} f_n q^{n/2} \approx \|f\|_{(2,1)}.
\]

On the other hand, for \( f_n \geq 0 \) we have
\[
\|f\|_{(2,1)\rightarrow(2,\infty)} \geq C \sup_{n,m} q^{-(n+m)/2} \langle f \ast \chi_n, \chi_m \rangle
\]
\[
= C \sup_{n,m} q^{-(n+m)/2} \langle f, \chi_m \ast \chi_n \rangle \geq C \sum_{k=|n-m|}^{n+m} q^{k/2} f_k.
\]
Taking $m = n$ or $m = n + 1$ and letting $n$ tend to infinity gives

$$
\|f\|_{(2,1)\rightarrow(2,\infty)} \geq C \sum_{k=0}^{\infty} q^{2k/2} f_{2k},
$$

$$
\|f\|_{(2,1)\rightarrow(2,\infty)} \geq C \sum_{k=0}^{\infty} q^{(2k+1)/2} f_{2k+1}.
$$

Therefore $\sum_{k=0}^{\infty} q^{k/2} f_k < \infty$, i.e. $f \in L^{2,1}$ by (11).

3. Weak type $(p,p)$ for $1 < p < 2$. Part of the next theorem, namely the first inequality, is known from [3]. Actually, it has been simply observed there that the inequality follows by applying a multilinear interpolation theorem to Pytlik’s estimate for $\|\sum f_n \lambda(\chi_n)\|_{p\rightarrow p}$ given in [8]. We will reprove the second inequality by applying the same interpolation theorem to restricted weak type estimates given in the previous section. In this way we skip the $p \rightarrow p$ estimates whose proof in [8] is tricky, and the later proof in [3] makes use of advanced representation theory.

**Theorem 4.** For $1 < p < 2$ and $f = \sum_{n=0}^{\infty} f_n \chi_n$ we have

$$
\|\lambda(f)\|_{p\rightarrow(p,\infty)} \leq C \|f\|_{(p,p')}.
$$

Moreover, if $f \geq 0$ then

$$
c \|f\|_{(p,p')} \leq \|\lambda(f)\|_{p\rightarrow(p,\infty)}.
$$

**Proof.** The subscript $r$ will denote the subspace of radial functions, i.e. functions of the form $\sum_{n=0}^{\infty} f_n \chi_n$, where $f_n$ are complex coefficients. By Theorem 3 we have $L_r^{2,1} \ast L_r^{2,1} \subset L_r^{2,\infty}$. On the other hand, $L_r^1 \ast L_1 \subset L^1$. By the multilinear interpolation theorem [1, 3.13.5, p. 76] we get $L_r^{p,s} \ast L_r^{p,t} \subset L_r^{p,u}$ where $1 \leq p < 2$ and $1 + 1/u = 1/s + 1/t$. Taking $u = \infty$, $t = p$ and $s = p'$ gives $L_r^{p,p'} \ast L^p \subset L^{p,\infty}$. This gives the first inequality.

On the other hand, for $f = \sum_{n=0}^{\infty} f_n \chi_n$ by (4) and by duality (6) we have

$$
\|\lambda(f)\|_{p\rightarrow(p,\infty)} = \|\lambda(f)\|_{(p',1)\rightarrow p'} \geq c \sup_n q^{-n/p'} \|f \ast \chi_n\|_{p'}.
$$

Similarly to the proof of Theorem 1 we obtain

$$
f \ast \chi_n \geq \sum_{l=0}^{\infty} q^{(n-l)/2} \left[ \sum_{m=|n-l|}^{l+n} q^{m/2} f_m \right] \chi_l
$$
Hence
\[ q^{-n} \| f \ast \chi_n \|_{p'}^{p'} \geq \sum_{l=0}^{n} q^{p'(n-l)/2} q^{l-n} \left[ \sum_{m=n-l \mod 2}^{l+n} q^{m/2} f_m \right]^{p'} \]

\[ \geq \sum_{l=0}^{n} q^{(n-l)(p'-1)} f^n_{n-l} = \sum_{l=0}^{n} q^{p'/p} f^n_l. \]

Taking the supremum with respect to \( n \) and raising to the power \( 1/p' \) gives
\[ \| \lambda(f) \|_{p \rightarrow (p, \infty)} \geq c \left( \sum_{n=0}^{\infty} f^n q^{np'/p} \right)^{1/p'}. \]

Since the norm of \( f = \sum_{n=0}^{\infty} f_n \chi_n \) in \( L_r^{p,p'} \) is equivalent to \( (\sum_{n=0}^{\infty} f^n q^{np'/p})^{1/p'} \) the second inequality is proved. \( \blacksquare \)

4. Other estimates

**Theorem 5.** For \( 1 \leq s \leq 2 \leq t \leq \infty \) we have
\[ cn^{1-1/s+1/t} q^{n/2} \leq \| \lambda(\chi_n) \|_{(2,s) \rightarrow (2,t)} \leq C n^{1-1/s+1/t} q^{n/2}. \]

**Proof.** In order to get the second inequality we use only interpolation. First observe that the inequality is valid for \( s = 2, t = \infty \) by Theorem 1 and for \( s = t = 2 \) by [2, 7]. Hence by complex interpolation of the Lorentz spaces it is valid for \( s = 2, t \geq 2 \).

Next it is valid for \( s = 1, t = \infty \) by Theorem 3 and for \( s = t = 2 \). Hence by complex interpolation it is valid for \( 1 \leq s \leq 2, t = s' \).

Now we can use again complex interpolation to get the conclusion for \( 1 \leq s \leq 2 \leq t \leq \infty \).

The estimate from below can be obtained from
\[ \| \lambda(\chi_n) \|_{(2,s) \rightarrow (2,t)} \geq \frac{\| \chi_n \ast f \|_{(2,t)}}{\| f \|_{(2,s)}}, \]
where \( f = \sum_{k=0}^{2n} q^{-k/2} \chi_k. \) \( \blacksquare \)

Theorems 1, 2 and 5 suggest the following.

**Conjecture.** Let \( f = \sum_{n=0}^{\infty} f_n \chi_n \geq 0 \). Then for \( 1 \leq s \leq 2 \) the operator \( \lambda(f) \) maps \( L_r^{2,s} \) into \( L_r^{2,\infty} \) if and only if
\[ \sum_{n,m=0}^{\infty} f_n f_m q^{-(n+m)/2} \{ 1 + \min(n^{1/s'}, m^{1/s'}) \} < \infty. \]
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Institute of Mathematics
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland

and

Institute of Mathematics and Computer Science
University of Opole
Oleska 48
45-052 Opole, Poland

E-mail: szwarc2@gmail.com

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