Palindromic richness for languages invariant under more symmetries

Edita Pelantová
Czech Technical University in Prague
Czech Republic
Štěpán Starosta
Czech Technical University in Prague
Czech Republic

Abstract

For a given finite group $G$ consisting of morphisms and antimorphisms of a free monoid $A^*$, we study infinite words with language closed under the group $G$. We focus on the notion of $G$-richness which describes words rich in generalized palindromic factors, i.e., in factors $w$ satisfying $\Theta(w) = w$ for some antimorphism $\Theta \in G$. We give several equivalent descriptions which are generalizations of know characterizations of rich words (in the terms of classical palindromes) and show two examples of $G$-rich words.

1 Introduction

In [16], Droubay et al. showed that the number of different palindromes occurring in a finite word $w$ cannot exceed the bound $|w| + 1$. If this bound is met, the word $w$ is called rich or rich in palindromes or full [16, 8]. An infinite word $u$ is said to be rich if all of its factors are rich. The list of the most prominent rich words contains Arnoux-Rauzy words, see [16], and words coding interval exchange with symmetric permutation of intervals, see [2]. Note that Sturmian words belong to both mentioned classes of words.

During the past two decades, the notion of palindromic richness showed to be fruitful. Application of palindromes in physics of quasicrystals (see for instance [18, 14]) and in genetics (see for instance [20]) served as stimulating factor for research in this area as well. Restivo and Rosone [24] showed a narrow connection of rich periodic words with extremal cases of Burrows-Wheeler transform which is used in compression algorithms. In [25], the authors further refined the result.

Generalizations of rich words appeared soon. Instead of classical palindromes defined as words invariant under the reversal mapping one can consider $\Theta$-palindromes, i.e., words
invariant under an involutive antimorphism $\Theta$. For first appearance of the notion see [19], where it appeared in the context of DNA, or [15], where the name pseudopalindrome is also used. Words saturated by $\Theta$-palindromes up to the highest possible level are called $\Theta$-rich. Another kind of generalization of rich words relaxes the requirement on the number of palindromes occurring in any factor $w$. We say that infinite word $u$ is almost rich if there exists a constant $D$ such that any factor $w$ of $u$ contains at least $|w| + 1 - D$ different palindromes. The minimal constant $D$ with this property is referred to as palindromic defect and was introduced in [8]. Both mentioned generalizations can be combined into the notion of almost $\Theta$-rich words. It follows directly from the definition that almost $\Theta$-rich words contain infinitely many $\Theta$-palindromes.

As shown in [22], besides peculiar periodic words, no infinite uniformly recurrent words can be simultaneously almost $\Theta_1$-rich and almost $\Theta_2$-rich for two distinct involutive antimorphisms $\Theta_1$ and $\Theta_2$. The famous Thue-Morse word contains infinitely many classical palindromes and $E$-palindromes, where $E$ is the antimorphism generated by interchange of symbols 0 and 1. Nevertheless, the Thue-Morse word has no chance to be almost rich or almost $E$-rich. Therefore, the authors suggested in [22] a further generalization under the name of $G$-richness and almost $G$-richness. The new definition of richness respects more antimorphisms of finite order under which the language of an infinite word is invariant, the letter $G$ stands for the group generated by these antimorphisms. The definition is based on the notion of graph of symmetries, which is assigned to an infinite word. Adopting the new definition, the second author showed in [27] that all generalized Thue-Morse words $t_{k,m}$ are $I_2(m)$-rich, where $I_2(m)$ is a group isomorphic to the dihedral group having $2m$ elements.

It turned out that words rich in the classical sense can be characterized by using many other notions of combinatorics on words: return word, bilateral order, longest palindromic suffix, factor and palindromic complexity, and super reduced Rauzy graphs. These characterizations can be found in [17, 11, 4]. Analogous results for $\Theta$-rich words can be found in [26]. The aim of this article is to find $G$-analogies of these characterizations. They are stated as Theorems 27, 33 and 39 and Proposition 43. A consequence of these characterizations is the fact that existence of a $G$-rich word forces the group to be generated by involutive antimorphisms only. This class of groups contains dihedral groups and, more generally, finite Coxeter groups. The question whether there exists a $G$-rich word for any finite group $G$ such that it is generated by involutive antimorphisms only, or it is at least a finite Coxeter group, remains open. At the end of the article we present two examples of $G$-rich words. The two examples are defined over alphabet of distinct sizes, nevertheless their groups of symmetries are mutually isomorphic, but not to a dihedral group.

The list of known examples of $G$-rich and almost $G$-rich words is very short. One aim of this article is to trigger a search for such words.
2 Preliminaries

An alphabet $A$ is a finite set. Elements of $A$ are usually called letters. A finite word $w$ over $A$ is a finite string $w = w_1w_2 \cdots w_n$ of letters $w_i \in A$. Its length, denoted by $|w|$, is $n$. The set of all finite words over $A$ equipped with the operation of concatenation is the free monoid $A^*$. Its neutral element is the empty word $\varepsilon$. A word $v \in A^*$ is a factor of a word $w \in A^*$ if there exist words $s,t \in A^*$ such that $w = svt$. If $s = \varepsilon$, then $v$ is a prefix of $w$, if $t = \varepsilon$, then $v$ is a suffix of $w$.

2.1 Antimorphisms and their fixed points

A mapping $\varphi$ on $A^*$ is called

• a morphism if $\varphi(vw) = \varphi(v)\varphi(w)$ for any $v,w \in A^*$;

• an antimorphism if $\varphi(vw) = \varphi(w)\varphi(v)$ for any $v,w \in A^*$.

We denote the set of all morphisms and antimorphisms on $A^*$ by $\text{AM}(A^*)$. Together with composition, it forms a monoid with the identity mapping $\text{Id}$ as the unit element. The set of all morphisms, denoted by $\text{M}(A^*)$, is a submonoid of $\text{AM}(A^*)$. The reversal mapping $R$ defined by

$$R(w_1w_2 \cdots w_n) = w_nw_{n-1} \cdots w_2w_1$$

for all $w = w_1 \cdots w_n \in A^*$ is an involutive antimorphism, i.e., $R^2 = \text{Id}$. It is obvious that any antimorphism is a composition of $R$ and a morphism. Thus

$$\text{AM}(A^*) = \text{M}(A^*) \cup R(\text{M}(A^*)).$$

A fixed point of a given antimorphism $\Theta$ is called $\Theta$-palindrome, i.e., a word $w$ is a $\Theta$-palindrome if $w = \Theta(w)$. If $\Theta$ is the reversal mapping $R$, we say palindrome or classical palindrome instead of $R$-palindrome. One can see that if $\Theta$ has a fixed point containing all the letters of $A$, then $\Theta$ is an involution, and thus a composition of $R$ and an involutive permutation of letters.

2.2 Factor and palindromic complexities

An infinite word $u$ over an alphabet $A$ is a sequence $(u_n)_{n \in \mathbb{N}} \in A^\mathbb{N}$. We always implicitly suppose that $A$ is the smallest possible alphabet for $u$, i.e., any letter of $A$ occurs at least once in $u$. Action of any morphism $\varphi \in \text{M}(A^*)$ can be naturally extended to infinite words by the prescription

$$\varphi(u) = \varphi(u_0)\varphi(u_1)\varphi(u_2) \cdots$$

for all $u = (u_n)_{n \in \mathbb{N}} \in A^\mathbb{N}$.

A finite word $w$ is a factor of $u$ if there exists an index $i \in \mathbb{N}$, called occurrence of $w$, such that $w = u_iu_{i+1} \cdots u_{i+|w|-1}$. The set of all factors of $u$ of length $n$ is denoted $L_n(u)$. 

The language of an infinite word \( u \) is the set of all of its factors \( \mathcal{L}(u) = \cup_{n \in \mathbb{N}} \mathcal{L}_n(u) \). An infinite word \( u \) is recurrent if any of its factors has infinitely many occurrences in \( u \). A factor \( v \in \mathcal{L}(u) \) is a complete return word of a factor \( w \) if \( w \) occurs in \( v \) exactly twice, as a suffix and a prefix of \( v \). A complete return word \( v \) of \( w \) can be written as \( v = qw \) for some factor \( q \neq \varepsilon \), which is usually called a return word of \( w \). If any factor \( w \) of \( u \) has only finitely many return words, then \( u \) is said to be uniformly recurrent.

The factor complexity of \( u \) is the mapping \( \mathcal{C} : \mathbb{N} \rightarrow \mathbb{N} \) defined by the prescription

\[
\mathcal{C}(n) := \# \mathcal{L}_n(u).
\]

To evaluate the factor complexity of \( u \), one may watch possible prolongations of factors. A letter \( a \in \mathcal{A} \) is a left extension of a factor \( w \) in \( u \) if \( aw \) belongs to \( \mathcal{L}(u) \). The set of all left extensions of \( w \) is denoted \( \text{Lex}_l(w) \). A factor \( w \in \mathcal{L}(u) \) is called left special if \( \# \text{Lex}_l(w) \geq 2 \). Analogously, we define right extension, the set \( \text{Rext}(w) \), and right special. If \( w \) is right and left special, we call it bispecial. The first difference of the factor complexity of a recurrent word \( u \) satisfies

\[
\Delta \mathcal{C}(n) = \mathcal{C}(n + 1) - \mathcal{C}(n) = \sum_{w \in \mathcal{L}_n(u)} (\# \text{Lex}_l(w) - 1) = \sum_{w \in \mathcal{L}_n(u)} (\# \text{Rext}(w) - 1).
\]

The second difference of factor complexity can be expressed using the bilateral order of a factor. Let \( w \) be a factor of \( u \). Its bilateral order is the quantity \( b(w) := \# \text{Bext}(w) - \# \text{Lex}_l(w) - \# \text{Rext}(w) + 1 \), where \( \text{Bext}(w) = \{ awb \mid awb \in \mathcal{L}(u), a, b \in \mathcal{A} \} \). In [13], the formula

\[
\Delta^2 \mathcal{C}(n) = \Delta \mathcal{C}(n + 1) - \Delta \mathcal{C}(n) = \sum_{w \in \mathcal{L}_n(u)} b(w)
\]

is deduced.

The \( \Theta \)-palindromic complexity of \( u \) is the mapping \( \mathcal{P}_{\Theta}(n) : \mathbb{N} \rightarrow \mathbb{N} \) defined by

\[
\mathcal{P}_{\Theta}(n) := \# \{ w \in \mathcal{L}_n(u) \mid w = \Theta(w) \}.
\]

Similarly to factor complexity, evaluation of palindromic complexity is based on counting possible extensions of palindromes. For a \( \Theta \)-palindrome \( w \), we denote by \( \text{Pext}_{\Theta}(w) \) the set of all letters \( a \in \mathcal{A} \) such that \( aw\Theta(a) \in \mathcal{L}(u) \). It is easy to see that

\[
\mathcal{P}_{\Theta}(n + 2) = \sum_{w \in \mathcal{L}_n(u)} \# \text{Pext}_{\Theta}(w).
\]

2.3 Words with language closed under a group \( G \) and \( G \)-richness

In the rest of the article, the symbol \( G \) stands exclusively for a subset of \( AM(\mathcal{A}^*) \) satisfying the two following requirements:

i) \( G \) is a finite group;
ii) $G$ contains at least one antimorphism.

The first requirement on $G$ implies the following for an element $\nu$ of $G$. The element $\nu$ is non-erasing, i.e., $\nu(a) \neq \varepsilon$ for all $a \in A$ (otherwise $\nu$ has no inverse in $G$). Moreover, $\nu(a)$ is a letter for all $a \in A$ (otherwise $\nu^n \neq \text{Id}$ for all $n \geq 1$). We can conclude that $\nu$ restricted to $A$ is a permutation of letters.

The second requirement on $G$ stems from the fact that our results are based on generalized palindromes and one gets only trivial or no results when dealing with groups consisting of morphisms only. Since especially involutive antimorphisms are important in the study of generalized palindromes, by $G^{(2)}$ we denote the set of all involutive antimorphisms belonging to $G$.

**Example 1.** Set $A = \{0, 1\}$. Denote by $E$ the antimorphism determined by $0 \mapsto 1$ and $1 \mapsto 0$. The only finite subgroups of $AM(A^*)$ containing at least one antimorphism are

$$G_1 = \{\text{Id}, R\}, \ G_2 = \{\text{Id}, E\}, \ \text{and} \ G_3 = \{\text{Id}, E, R, ER\}.$$  

In this case, we have

$$G_1^{(2)} = \{R\}, \ G_2^{(2)} = \{E\}, \ \text{and} \ G_3^{(2)} = \{E, R\}.$$

The previous example shows that binary alphabet allows only abelian groups to satisfy i) and ii). On multiliteral alphabet, $G$ need not be abelian.

**Example 2.** Let $m$ be an integer such that $m \geq 3$. Let $\pi \in S_m$ be a permutation of $A = \{0, 1, \ldots, m-1\}$. Denote by $\mu_\pi$ the morphism on $A^*$ induced by $\pi$, i.e., the restriction of $\mu_\pi$ to $A$ is the permutation $\pi$ of $A$. Then the set

$$G = \{\mu_\pi \mid \pi \in S_m\} \cup \{R\mu_\pi \mid \pi \in S_m\}$$

is a group satisfying i) and ii). Clearly, $G$ is not abelian.

**Example 3.** Let $A = \{0, 1, 2\}$. For all $k \in A$ define the antimorphism $\Psi_k$ by $\Psi_k(\ell) = (k-\ell) \mod 3$ for all $\ell \in A$. By $\mu$ denote the morphism determined by $\mu(\ell) = (\ell - 1) \mod 3$ for all $\ell \in A$. The set

$$G = \{\text{Id}, \mu, \mu^{-1}, \Psi_0, \Psi_1, \Psi_2\}$$

forms a non-abelian group satisfying i) and ii) and not containing $R$.

Let us stress some aspects of such a group $G$ satisfying requirements i) and ii):

1. every element of $G$ is either a morphism or an antimorphism determined by a permutation of letters of $A$;
2. $G$ may contain elements of order greater than 2;
3. $G$ need not be abelian;
4. the set of antimorphisms of \( G \) generates the group \( G \);

5. the number of morphism in \( G \) equals the number of antimorphisms in \( G \);

To prove the last two items it suffices to fix an antimorphism \( \Theta \in G \). Item 4 follows from the fact that given a morphism \( \mu \in G \) one can write \( \mu \Theta = \Theta' \) where \( \Theta' \) is an antimorphism of \( G \). Thus, \( \mu = \Theta' \Theta^{-1} \). To show the last property, it suffices to show that the mapping from the set of morphisms of \( G \) to the set of antimorphisms of \( G \) defined by \( \mu \mapsto \mu \Theta \) for all morphism \( \mu \in G \) is a bijection.

We say that finite words \( w, v \in A^* \) are \( G \)-equivalent if there exists \( \mu \in G \) such that \( w = \mu(v) \). The class of equivalence containing a word \( w \) is denoted

\[
[w] := \{ \mu(w) \mid \mu \in G \}.
\]

As already mentioned, since the group \( G \) is finite, any \( \mu \in G \) preserves length of words and thus equivalent words have the same length.

We say that language \( L(u) \) of an infinite word \( u \in A^\mathbb{N} \) is closed under \( G \) if for any factor \( w \in L(u) \) and any \( \mu \in G \), the image \( \mu(w) \) belongs to \( L(u) \) as well. Since \( G \) contains at least one antimorphism, closedness of \( L(u) \) under \( G \) implies that \( u \) is recurrent.

A useful tool to study language in combinatorics on words is Rauzy graph. Given a language \( L \), the Rauzy graph of order \( n \) of the language \( L \) is a subgraph of \( n \)-dimensional De Bruijn graph such that the set of vertices equals \( L_n = L \cap A^n \) and the set of edges equals \( L_{n+1} = L \cap A^{n+1} \). In other words, there is an oriented edge \( e \in L_{n+1} \) starting in \( p \in L_n \) and ending in \( s \in L_n \) if \( p \) is a prefix of \( e \) and \( s \) is a suffix of \( e \). For languages closed under reversal, the notion of Rauzy graph has been generalized in [11] to super reduced Rauzy graph. A super reduced Rauzy graph is undirected and has multiple edges and loops allowed. It can be produced from a Rauzy graph by a certain “compression” which replaces some vertices and takes advantage of the symmetry given by the reversal mapping, see [11] for more details. This process is lossless and one can reconstruct the Rauzy graph back. The following definition (introduced in [22]) of undirected graph of symmetries generalizes the notion of super reduced Rauzy graph; the two definitions coincide for \( G = \{ \text{Id}, R \} \).

**Definition 4.** Let \( u \) be an infinite word with language closed under \( G \) and \( n \in \mathbb{N} \).

1) The directed graph of symmetries of the word \( u \) of order \( n \) is \( \overrightarrow{\Gamma}_n(u) = (V, \overrightarrow{E}) \) with the set of vertices

\[
V = \{ [w] \mid w \in L_n(u), w \text{ is left or right special} \}
\]

and an edge \( e \in \overrightarrow{E} \subset L(u) \) starts in the vertex \([w]\) and ends in the vertex \([v]\) if and only if

- the prefix of \( e \) of length \( n \) belongs to \([w]\),
- the suffix of \( e \) of length \( n \) belongs to \([v]\),
• $e$ has exactly two occurrences of special factors of length $n$, i.e., $e$ is at least of length $n + 1$ and all its factors of length $n$ except for its prefix and suffix are not special.

2) The undirected graph of symmetries of the word $u$ of order $n$ is $\Gamma_n(u) = (V, E)$ with the same set of vertices as $\overrightarrow{\Gamma}_n(u)$ and two vertices $[w]$ and $[v]$ are connected by an undirected edge $[e] \in E$ if and only if $e \in \overrightarrow{E}$ starts in $[w]$ and ends in $[v]$ or vice versa.

Multiple edges and loops are allowed in both defined graphs.

Any vertex $[w]$ of the graph $\overrightarrow{\Gamma}_n(u)$ represents a class of equivalent factors of $L_n(u)$. It has at most $\#G$ elements; the actual cardinality of $[w]$ may depend on $n$ as well.

Since $G$ contains at least one antimorphism, the word $u$ is recurrent which implies that both graphs $\overrightarrow{\Gamma}_n(u)$ and $\Gamma_n(u)$ are connected.

We give two famous examples to demonstrate the last definition.

**Example 5** (The Fibonacci word). The Fibonacci word $u_F$ is the fixed point of the morphism determined by

$$0 \mapsto 01, \quad 1 \mapsto 0.$$ 

We have

$$u_F = 0100101001001010010100100101001001010010\ldots .$$

The language of the Fibonacci word is closed under reversal, i.e., if we set $G_F = \{\text{Id}, R\}$, then the language $L_{u_F}$ is closed under $G_F$. We have

$$L_3(u_F) = \{101, 010, 100, 001\} \quad \text{and} \quad L_4(u_F) = \{1001, 1010, 0100, 0010, 0101\}.$$ 

The only special factor of length 3 is 010 and it is in fact bispecial.

Figure 1 depicts the graph $\overrightarrow{\Gamma}_3(u_F)$, while Figure 2 shows the graph $\Gamma_3(u_F)$.

**Example 6** (Generalized Thue-Morse words). The generalized Thue-Morse word $t_{b,m}$ is defined on the alphabet $\{0, \ldots, m - 1\}$ for all $b \geq 2$ and $m \geq 1$ as

$$t_{b,m} = (s_b(n) \mod m)_{n=0}^{+\infty},$$

where $s_b(n)$ denotes the sum of digits in the base-$b$ representation of the integer $n$. See for instance [1] where this class of words is studied. In [27], it is show that the language of $t_{b,m}$ is closed under a group isomorphic to the dihedral group of order $2m$, denoted $I_2(m)$. We exhibit here the graphs of symmetries for two pairs of parameters $b$ and $m$.

Take $b = m = 2$, the word $t_{2,2}$ is then the famous Thue-Morse word. It starts with 0 and it is a fixed point of the morphism determined by

$$0 \mapsto 01 \quad \text{and} \quad 1 \mapsto 10.$$
We have \( t_{2,2} = 0110100110010110100101100110100110010110 \ldots \)

Figure 3 shows the Rauzy graph of order 3 of the Thue-Morse word.

The language of the Thue-Morse word is closed under the reversal mapping and the antimorphism exchanging letters, denoted again \( E \). Thus, it is closed under the group

\[
I_2(2) = \{ \text{Id}, R, E, ER \}.
\]

Figure 4 depicts the graph \( \Gamma_3(t_{2,2}) \), while Figure 5 shows the graph \( \Gamma_3(t_{2,2}) \).
Take $b = m = 3$. As shown in [1], the word $t_{3,3}$ is a fixed point of the morphism given by

$$0 \mapsto 012, \ 1 \mapsto 120 \text{ and } 2 \mapsto 201$$

and starting with 0. Its language is closed under the group $I_2(3)$ which coincides with the group already introduced in Example 3.

We have

$$\mathcal{L}_3(t_{3,3}) = \{001, 202, 220, 200, 201, 011, 010, 012, 020, 122, 112, 101, 120, 212, 121\},$$

where the only special factors are 012, 120, and 201. Figure 6 shows the undirected graph of symmetries $\Gamma_3(t_{3,3})$.

**Definition 7.** Let $u$ be an infinite word with language closed under $G$. We say that a number $n \in \mathbb{N}$ is $G$-distinguishing on $u$ if for any $w \in \mathcal{L}_n(u)$ we have:

$$\Theta_1 \neq \Theta_2 \Rightarrow \Theta_1(w) \neq \Theta_2(w) \quad \text{for any two antimorphisms } \Theta_1, \Theta_2 \in G. \quad (2)$$

If $n$ is $G$-distinguishing on $u$, then the knowledge of a pair $w$ and $\Theta(w)$ for a single word $w \in \mathcal{L}_n(u)$ enables us to unambiguously determine $\Theta \in G$. Let us stress that the requirement (2) gives also for any two distinct morphisms $\phi_1, \phi_2 \in G$ that $\phi_1(w) \neq \phi_2(w)$.

(It suffices to consider antimorphisms $\phi_1 \Theta$ and $\phi_2 \Theta$, where $\Theta \in G$ is an antimorphism, and use the fact that for all $w \in \mathcal{L}(u)$ there is a factor $w' \in \mathcal{L}(u)$ such that $\Theta(w') = w$.)

\[9\]
One can readily see that if \( n \) is \( G \)-distinguishing on \( u \), then any \( m \) greater than \( n \) is also \( G \)-distinguishing on \( u \).

As proved in [22], the connectivity of graphs of symmetries, which follows from the closedness of the language under \( G \) containing an antimorphism, enables to bind the factor and palindromic complexities.

**Theorem 8.** If \( u \) is an infinite word with language closed under a group \( G \) and \( N \in \mathbb{N} \) is \( G \)-distinguishing on \( u \), then

\[
\Delta C(n) + \#G \geq \sum_{\Theta \in G^{(2)}} \left( \mathcal{P}_\Theta(n) + \mathcal{P}_\Theta(n + 1) \right) \quad \text{for any } n \geq N.
\] (3)

The term \( \Delta C(n) + \#G \) represents an upper bound on the number of palindromes occurring in \( u \). It follows from the proof of the last theorem in [22] that, for a given integer \( n \), the equality in (3) is reached if and only if the undirected graph of symmetries \( \Gamma_n(u) \) has a specific tree-like structure. Words for which this upper bound is reached are in some sense opulent in palindromes.

Therefore, we adopt this specific structure in the following definition of Property \( G \)-tls\((N) \), where “tls” is an abbreviation of tree-like structure. The name of the property also keeps track of the group \( G \) since a word may satisfy it or not according to the choice of \( G \) as we illustrate on the Thue-Morse word just after the definition. The parameter \( N \) plays a similar role as in Theorem 8 – it enables us to relate the property of having a tree-like structure to a value measuring the deficit of generalized palindromes.

**Definition 9.** We say that an infinite word \( u \) has Property \( G \)-tls\((N) \) if for each \( n \in \mathbb{N} \), \( n \geq N \) we have that

- \( \mathcal{L}(u) \) is closed under \( G \);
- if \([e]\) is a loop in \( \Gamma_n(u) \), then \( e \) is a \( \Theta \)-palindrome for some \( \Theta \in G \);
• the graph obtained from $\Gamma_n(u)$ by removing loops is a tree.

The graph $\Gamma_3(t_{2,2})$ for the group $G = I_2(2)$ in Figure 5 has tree-like structure. As shown in [22], all the undirected graphs of symmetries of the Thue-Morse word $t_{2,2}$ of order greater than 0 have the structure and thus the word $t_{2,2}$ satisfies Property $G$-tls(1). However, for $G = \{Id, R\}$ the undirected graphs of symmetries of the Thue-Morse word does not have tree-like structure. Figure 7 shows the undirected graph $\Gamma_3(t_{2,2})$ for $G = \{Id, R\}$.

![Figure 7: The graph $\Gamma_3(t_{2,2})$ for the group $\{Id, R\}$.](image)

Now, we define the most important notion of the article, namely the $G$-richness. As mentioned in Introduction, the classical richness has several equivalent characterizations, each of them is a candidate for a definition of the new notion. Nevertheless, some of these characterizations are accompanied by technical complications when the reversal mapping $R$ is replaced by a larger group $G$. For example, the inequality in Theorem 8 is valid only for $n$ which is $G$-distinguishing and such $n$ could be quite large. But in the case $G = \{Id, R\}$ the inequality is valid for all nonnegative integers $n$ and therefore it can be used for equivalent definition of classical richness.

We have decided in [22] to adopt definition of $G$-richness which is based on the notion of graphs of symmetries. This specific tree structure of these graphs seems to be essential when considering palindromic richness.

**Definition 10 ([22]).** We say that $u$ is $G$-rich if $u$ has Property $G$-tls(1) and $u$ is almost $G$-rich if there exists $N \in \mathbb{N}$ such that $u$ has Property $G$-tls($N$).

In the next section, we explain legitimacy of the name $G$-richness, i.e., we show that the classical richness is contained in our new definition of richness as well.

### 2.4 Palindromic richness in the classical sense

Let us recall the origin of palindromic richness. In this section, we use the word palindrome for $R$-palindrome and we denote by $\text{Pal}(w)$ the set of all palindromic factors of a finite word $w$ including the empty word $\varepsilon$. In [16], Droubay, Justin and Pirillo provided the following simple upper bound

$$\#\text{Pal}(w) \leq |w| + 1.$$  (4)
This bound serves for the definition of palindromic richness in the classical sense (see [16, 17]). A finite word \( w \) is \textit{rich} if \( \#\text{Pal}(w) = |w| + 1 \). An infinite word is \textit{rich} if all its factors are rich. Another type of bound on the number of palindromes contained in an infinite word was proved in [2]: if an infinite word has its language closed under the reversal mapping, then the following inequality holds

\[
\Delta \mathcal{C}(n) + 2 \geq \mathcal{P}(n) + \mathcal{P}(n + 1) \quad \text{for all } n \in \mathbb{N}.
\] (5)

The authors of [17] showed that an infinite word \( u \) with language closed under reversal is rich if and only if the equality in (5) is attained for all \( n \in \mathbb{N} \). Their proof uses the already mentioned notion of super reduced Rauzy graph, which, in our terminology, is the graph of symmetries \( \Gamma_n(u) \) for the group \( G = \{ \text{Id}, R \} \); a recurrent word \( u \) is rich if and only if \( u \) satisfies - again in our terminology - Property \( G\text{-tls} \). (1).

\textbf{Example 11.} All episturmian words (see [16]), which include Sturmian and Arnoux-Rauzy words, are rich. Since the Fibonacci word \( u_F \) is Sturmian, it is rich. Other class of rich words are the words coding interval exchange transformations determined by a symmetric permutation, see [2].

The following theorem summarizes properties characterizing rich recurrent words, their proofs can be found in [17, 16, 11, 4].

\textbf{Theorem 12.} For an infinite word \( u \) with language closed under reversal the following statements are equivalent:

1. \( u \) is rich,
2. [17] any complete return word of any palindromic factor of \( u \) is a palindrome,
3. [17] for any factor \( w \) of \( u \), every factor of \( u \) that contains \( w \) only as its prefix and \( R(w) \) only as its suffix is a palindrome,
4. [16, 17] the longest palindromic suffix of any factor \( w \in \mathcal{L}(u) \) is unioccurrent in \( w \),
5. [11] for each \( n \in \mathbb{N} \) the following equality holds

\[
\mathcal{C}(n + 1) - \mathcal{C}(n) + 2 = \mathcal{P}(n) + \mathcal{P}(n + 1),
\]

6. [11] each graph of symmetries \( \Gamma_n(u) \) satisfies: all its loops are palindromes and the graph obtained from \( \Gamma_n(u) \) by removing loops is a tree,
7. [4] any bispecial factor \( w \) of \( u \) satisfies:
   - if \( w \) is non-palindromic, then
     \[
b(w) = 0;
     \]
   - if \( w \) is a palindrome, then
     \[
b(w) = \#\text{Pext}(w) - 1.
     \]
Another characterization of rich words, which is not treated in this article, can be found in [12].

Richness in the classical sense is closely related to the notion of defect of a word. As introduced in [8], the defect of a finite word $w$ is defined as follows

$$D(w) = |w| + 1 - \#\operatorname{Pal}(w).$$

The defect of an infinite word $u$ is defined as

$$D(u) = \sup_{w \in L(u)} \{D(w)\}.$$  

Recall that a word is rich if its defect is zero.

In [8] and [7], the authors study also words with finite defect; in [17], such words are called almost rich. Words almost rich in the classical sense can be characterized by properties analogous to those listed in Theorem 12. For more details see [5].

The new definition is in fact based on a generalization of characterization 6 in Theorem 12. The goal of this article is to generalize some of those characterizations: Theorem 27 generalizes characterization 2 and Theorem 33 generalizes characterization 4. Proposition 43 is a generalization of characterization 7 for almost rich words. Characterization 5 is already generalized in [22], we recall it here as Proposition 40. Section 6 generalizes the notion of defect.

Let us point out a drawback of our new definition. The Property $G$-tls(1), unlike the classical definition of richness, requires the language of an infinite word to be closed under reversal. Nevertheless, an infinite word can be rich in the classical sense without having its language closed under reversal. On the other hand, as proved in [17] (Proposition 2.11), any recurrent rich word has its language closed under reversal. Therefore, on the set of recurrent words both definitions coincide.

Some generalizations were already made for groups $G = \{\operatorname{Id}, \Theta\}$. As already mentioned, in the articles [19, 15], the reversal mapping $R$ is replaced by an arbitrary involutive antimorphism $\Theta$ and $\Theta$-palindrome is defined as a word $v$ satisfying $\Theta(v) = v$. Let us denote by $\operatorname{Pal}_\Theta(w)$ the set of $\Theta$-palindromic factors occurring in $w$. As shown in [26],

$$\#\operatorname{Pal}_\Theta(w) \leq |w| + 1 - \gamma_\Theta(w),$$  

where $\gamma_\Theta(w) := \# \{a, \Theta(a) \mid a \in A, a \text{ occurs in } w \text{ and } a \neq \Theta(a)\}$. Analogously to the classical richness, $\Theta$-richness is introduced in [26] as follows. A finite word $w$ is $\Theta$-rich if the equality in (6) holds. An infinite word is $\Theta$-rich if all its factors are $\Theta$-rich.

Generalizations of some characterizations in Theorem 12 for $\Theta$-rich words are presented in [26] as well.

Remark 13. In [7], the authors remark that on binary alphabet the only periodic $E$-rich words are of period 2. On the alphabet $\{0, 1\}$, the only finite $E$-rich words are the following:

$$(01)^n, (10)^n, (01)^n0, \text{ and } (10)^n1$$
for some \( n \in \mathbb{N} \).

As stated in [8] for the reversal mapping \( R \), if a periodic word \( w^\omega \) is closed under an involutive antimorphism \( \Theta \), then \( w \) can be written as a concatenation of two \( \Theta \)-palindromes. This gives a restrictive (and only necessary) condition for periodic \( G \)-rich words for a larger group \( G \). Examples known to satisfy this condition are periodic generalized Thue-Morse words (see [27]): words \( t_{b,m} \) defined in Example 6 for \( b \equiv 1 \) (mod \( m \)). These words are closed under the group \( I_2(m) \).

## 3 Tools for characterization of \( G \)-rich words

Classical rich and almost rich words can be described using the notions of return words and longest palindromic suffix. In order to find a suitable description of \( G \)-richness, we first introduce \( G \)-analogies of these notions and in the sequel we demonstrate their efficiency. Let us recall that in all definitions and statements in the sequel, the symbol \( G \) stands for a finite subgroup of \( AM(A^*) \) such that it contains at least one antimorphism.

**Definition 14.** A word \( w \in A^* \) is said to be \( G \)-palindrome if there exists an antimorphism \( \Theta \in G \) such that \( w = \Theta(w) \).

**Remark 15.** As already mentioned, if a word \( w \) contains all letters of \( A \) and \( \Theta(w) = w \) for an antimorphism \( \Theta \in G \), then \( \Theta \) is an involution. However, non-involutive antimorphisms of \( G \) also contribute to the number of generalized palindromes. If \( N \) is \( G \)-distinguishable and \( w \) is a \( \Theta \)-palindrome such that \( |w| \geq N \), then for any antimorphism \( \Psi \) in \( G \) the word \( \Psi(w) \) is a \( G \)-palindrome, namely a \( (\Psi\Theta\Psi^{-1}) \)-palindrome.

**Definition 16.** Let \( w, v \in A^* \). \( G \)-occurrence of a word \( w \) in a word \( v \) is an index \( i \) such that there exists \( w' \in [w] \) having occurrence \( i \) in \( v \).

We say that \( w \) is \( G \)-unioccurrent in \( v \) if \( w \) occurs in \( v \) and there is no other \( G \)-occurrence of \( w \) in \( v \).

**Definition 17.** Let \( u \in A^\mathbb{N} \) be an infinite word and \( w \in L(u) \). A factor \( v \in L(u) \) of length \( |v| > |w| \) is called complete \( G \)-return word of \([w] \) in \( u \) if

- a prefix and a suffix of \( v \) belong to \([w] \) and
- \( v \) contains no other \( G \)-occurrence of \( w \).

We say that \( v' \in L(u) \) is a \( G \)-return word of \([w] \) in \( u \) if for some \( w' \in [w] \) the word \( v'w' \) is a complete \( G \)-return word of \([w] \).

**Definition 18.** A suffix \( w \) of a word \( v \in A^* \) is called \( G \)-longest palindromic suffix of \( v \) if

- \( w \) is a \( G \)-palindrome and
- \( |w| \geq |w'| \) for any \( G \)-palindromic suffix \( w' \) of \( v \).
The $G$-longest palindromic suffix of $v$ is denoted by $G$-lps$(v)$.

**Remark 19.** If no nonempty suffix $w$ of a nonempty word $v \in A^*$ is a $G$-palindrome, then $G$-lps$(v) = \varepsilon$. (Let us recall that any index $i \in \{1, 2, \ldots, n\}$ is defined to be an occurrence of $\varepsilon$ in $v = v_1v_2 \cdots v_{n-1}$.) The shortest example of such phenomenon is a one-letter word $a \in A$ and a group $G$ containing no antimorphism that fixes the letter $a$. It may occur only in the case when $G$ does not contain the reversal mapping $R$. Clearly, the converse is not true: if $R \notin G$, then there may exist an antimorphism in $G$ that fixes the last letter of $w$.

Let us demonstrate these definitions on an example.

**Example 20.** Take again the Thue-Morse word and the group $I_2(2) = \{\text{Id}, R, E, ER\}$. The word $011$ is a factor of the word, we have $[011] = \{011, 110, 100, 001\}$. All $I_2(2)$-occurrences of $011$ in the prefix $p = 01101001100$ of the Thue-Morse word form the following set of indices:

$$\{0, 1, 4, 5, 6, 7, 8\}.$$ 

The factor $001100$ is $I_2(2)$-unioccurrent in $p$. The following $I_2(2)$-complete return words of $[011]$ are contained in $p$:

$$\{0110, 110100, 1001, 0011, 0110, 1100\}.$$ 

The $I_2(2)$-longest palindromic suffix of $p$ is

$$I_2(2)\text{-lps}(p) = 001100.$$ 

### 4 $G$-richness and $G$-return words

In this section, namely in Theorem 27, we demonstrate that the notions of complete $G$-return word and $G$-palindrome can grasp the essence of $G$-richness. We prove that if $u$ is an infinite word with language closed under $G$, then $u$ is $G$-rich if and only if for all $w \in L(u)$ every complete $G$-return word of $[w]$ is a $G$-palindrome. The theorem generalizes characterization 2 in Theorem 12.

To describe the generalized characterization of almost $G$-richness, we introduce the following property, called Property $G$-crw$(N)$. Again, the name of the property contains the two parameters $G$ and $N$ so that we can easily keep track of them. The abbreviation “crw” stands for complete return word, since the property is based on complete $G$-return words.

**Definition 21.** Let $N \in \mathbb{N}$. We say that $u \in A^n$ satisfies Property $G$-crw$(N)$ if for all $w \in L(u)$, $|w| \geq N$, every complete $G$-return word of $[w]$ is a $G$-palindrome.

Before proving the main result of this section we introduce several lemmas.
Lemma 22. Let \( N \in \mathbb{N} \) and \( u \in \mathcal{A}^N \) satisfy Property \( G\text{-}tls(N) \). If \( w \) is a factor of \( u \) such that \( |w| \geq N \) and \( v \) be a complete \( G\text{-}return \) word of \( |w| \) in \( u \) starting in \( w \), then there exist a letter \( a \in \mathcal{A} \) and an antimorphism \( \Theta \in G \) such that \( wa \) is a prefix of \( v \) and \( \Theta(a)\Theta(w) \) is a suffix of \( v \).

Note that we do not assume explicitly that \( \mathcal{L}(u) \) is closed under \( G \). However, this property is satisfied since it is part of the definition of Property \( G\text{-}tls(N) \).

Proof. Take \( n \geq N \), \( w \in \mathcal{L}_n(u) \), and \( v \) a complete \( G\text{-}return \) word of \( |w| \) starting in \( w \). Denote \( w' \in [w] \) the suffix of \( v \) of length \( n \).

At first we suppose that \( w \) is a special factor of \( u \). We consider the following two cases. We exploit the graphs of symmetries \( \Theta_n(u) = (V, \overrightarrow{E}) \) and \( \Gamma_n(u) = (V, E) \).

1. If besides the prefix \( w \) and the suffix \( w' \) the complete \( G\text{-}return \) word \( v \) contains no other occurrences of a special factor of length \( n \), then \( [v] \) is an edge in \( E \) which starts and ends in the same vertex \( [w] \). The edge \( [v] \) is thus a loop and according to the definition of Property \( G\text{-}tls(N) \), the factor \( v \) is a \( G \)-palindrome. Obviously, \( v \) has the property stated in the claim.

2. Let \( v = v_0v_1\cdots v_m \) contain a special factor \( z \notin [w] \) of length \( n \) at the position \( i \), i.e., \( z = v_i\cdots v_{i+n-1} \). We may suppose without loss of generality that \( i \) is the least index with this property. Then \( [z] \) is a vertex of \( \Theta_n(u) \) and a prefix of \( v \) is an edge in \( \Theta_n(u) \) starting in \( [w] \) and ending in \( [z] \). Since the graph obtained from \( \Gamma_n(u) \) by removing loops is a tree, the complete \( G\text{-}return \) word \( v \) has a suffix \( f \in \overrightarrow{E} \) such that a prefix of \( f \) belongs to \( [z] \) and its suffix belongs to \( [w] \). Moreover, there exists an antimorphism \( \Theta \) such that \( f = \Theta(e) \). As \( |e| = |f| > |w| \), the factor \( v \) has the property stated in the claim.

Let us now suppose that \( w \) is not a special factor and thus \( w \) has a unique right extension, say \( a \). If \( w' = \Theta(w) \) for some antimorphism \( \Theta \in G \), then \( \Theta(w) \) has a unique left extension \( \Theta(a) \) and therefore \( \Theta(a)\Theta(w) \) is a suffix of the complete \( G\text{-}return \) word \( v \) as stated in the claim.

To finish the proof, it is enough to consider the situation when \( w' = \mu(w) \) for some antimorphism \( \mu \in G \) and \( w \) is not a special factor. We discuss two separate cases.

1. There exists no special factor of length \( n \).
   In this case \( u \) is periodic. Denote \( v' \) the word such that \( v = v'\mu(w) \). Because no special factor of length at least \( n \) exists, the factor \( v = v'\mu(w) \) is the unique right prolongation of \( w \) of length \( |v| \). As \( \mathcal{L}(u) \) is closed under \( G \), \( \mu^k(v) = \mu^k(v')\mu^k\mu^k(w) \) is the unique right prolongation of \( \mu^k(w) \) of given length. In particular, for \( \ell = 1 \) it implies \( v'\mu(v')\mu^2(w) \in \mathcal{L}(u) \). Repeating this argument for \( \ell = 2, 3, \ldots \) we deduce that \( v'\mu(v')\mu^2(v') \cdots \mu^k(v')\mu^k\mu^k(w) \in \mathcal{L}(u) \). Therefore, the factor \( v'\mu(v')\mu^2(v') \cdots \mu^k(v') \), where \( k \) is the order of the morphism \( \mu \), is a period of \( u \) which does not contain any antimorphic image of \( w \) - a contradiction.
2. There exists a special factor of length $n$.

Consequently, there exists a unique $q$ such that $wq$ is right special and no proper prefix of $wq$ is right special. The factor $wq$ has only one occurrence of a factor $\nu(w)$ for some morphism $\nu \in G$ - in the opposite case, we can find a shorter prolongation of $w$ which is right special. Since $v$ has suffix $\mu(w)$, we deduce $|wq| < |v|$. As $\mu$ is a morphism, $\mu(w)\mu(q)$ is the only right prolongation of $\mu(w)$ and thus $v\mu(q)$ is a complete $G$-return word of $[wq]$. For the special factor $wq$, we may now use the first part of the proof and thus find an antimorphism $\Theta$ such that $\mu(w)\mu(q) = \Theta(wq) = \Theta(q)\Theta(w)$. Applying the morphism $\mu^{-1}$, we get $wq = \mu^{-1}\Theta(q)\mu^{-1}\Theta(w)$. Together with the inequality $|wq| < |v|$, it implies a contradiction with the fact that $v$ is a complete $G$-return word of $[w]$.  

**Lemma 23.** Let $u \in A^N$ and $N \in \mathbb{N}$. If $u$ has Property $G$-tls$(N)$, then it has Property $G$-crw$(N)$.

**Proof.** Let $v$ be a complete $G$-return word of $[w]$ starting in $w$ for a factor $w$ such that $|w| \geq N$. Denote by $a$ a letter such that $wa$ is a prefix of $v$.

If $wa = v$, then Lemma 22 implies that $v$ is a $G$-palindrome.

If $wa \neq v$, then according to Lemma 22, $v$ is a complete $G$-return word of $[wa]$ as well. We apply the procedure again on $wa$. We find a letter $b$ such that $wab$ is a prefix of $v$. If $wab = v$, the $v$ is a $G$-palindrome, otherwise $v$ is a complete $G$-return word of $[wab]$. We continue in this way until the procedure stops and we conclude that $v$ is a $G$-palindrome.  

**Remark 24.** As a consequence of the previous lemma, we have that for an infinite word $u$ satisfying Property $G$-tls$(N)$, the occurrences of morphic and antimorphic images of a factor $w$ satisfying $|w| \geq N$ alternate. This consequence of the previous claim is an analogy to the claims stated in [17] for rich words and in [26] for $\Theta$-rich words: given a $\Theta$-rich word for an involutive antimorphism $\Theta$, the occurrences of $w$ and $\Theta(w)$ in the word alternate.

**Theorem 25.** If there exists an almost $G$-rich word, then $G$ is generated by the set of its involutive antimorphisms.

**Proof.** Let $u$ be an almost $G$-rich word. Let $N$ be an integer such that $u$ satisfies the property $G$-tls$(N)$. Let $w$ be a factor of length at least $N$ and such that all letters occur in it. According to Lemma 22, any occurrence of a word from $[w]$ is a $\Theta$-image of the left closest occurrence of a factor from $[w]$ for some antimorphism $\Theta \in G$. According to Lemma 23, all complete $G$-return words of $[w]$ are $G$-palindromes. Thus, since all letters occur in $w$, such antimorphism $\Theta$ is involutive. Therefore, for any $\nu \in G$, the factor $\nu(w)$ occurring in $u$ can be written as $\nu(w) = \Theta_1\Theta_2\cdots\Theta_s(w)$, where $\Theta_1\Theta_2\cdots\Theta_s$ is a sequence of involutive antimorphisms. Since $w$ contains all letters, the number $|w|$ is $G$-distinguishing, and thus according to Definition 7 the equality $\nu(w) = \Theta_1\Theta_2\cdots\Theta_s(w)$ implies $\nu = \Theta_1\Theta_2\cdots\Theta_s$.

In other words, any element $\nu$ of the group $G$ can be written as a composition of involutive antimorphisms, i.e., the group is generated by involutive elements.  

17
The following lemma is the converse of Lemma 23. However, we need to add an explicit assumption of closedness under \( G \). In Lemma 23, this assumption is hidden in the definition of Property \( G\text{-tls}(N) \), which includes it (unlike Property \( G\text{-crw}(N) \)).

**Lemma 26.** Let \( u \) be an infinite word with language closed under \( G \) and \( N \in \mathbb{N} \). If \( u \) satisfies Property \( G\text{-crw}(N) \), then it satisfies Property \( G\text{-tls}(N) \).

**Proof.** Let \( n \geq N \) and \( w \in \mathcal{L}_n(u) \). We assume that every complete \( G \)-return word of \([w]\) is a \( G \)-palindrome. We have to show two properties of \( \Gamma_n(u) \).

1. *Any loop in \( \Gamma_n(u) \) is a \( G \)-palindrome:*

   Since any loop \( e \) in \( \Gamma_n(u) \) at a vertex \([w]\) is a complete \( G \)-return word of \([w]\), the loop \( e \) is a \( G \)-palindrome by our assumption.

2. *The graph obtained from \( \Gamma_n(u) \) by removing loops is a tree:*

   Or equivalently, we show that in \( \Gamma_n(u) \) there exists unique path between any two different vertices \([w']\) and \([w'']\). Let \( p \) be a factor of \( u \) such that a prefix of \( p \) belongs to \([w']\), its suffix belongs to \([w'']\) and \( p \) has no other occurrences of factor from \([w']\) or \([w'']\). Let without loss of generality \( w' \) be a prefix of \( p \). Let us find a complete \( G \)-return word of \([w']\) with prefix \( p \), denote it \( v \). Since \( v \) is a \( G \)-palindrome, the factor \( \Theta(p) \) is a suffix of \( v \) for some antimorphism \( \Theta \in G \). As \( v \) is a complete \( G \)-return word of \([w']\), \( v \) has exactly two \( G \)-occurrences of \( w' \). The factor \( v \) contains at least two \( G \)-occurrences of \( w'' \). Therefore, the next factor with the same properties as \( p \), i.e., representing a path in the undirected graph \( \Gamma_n(u) \) between \([w']\) and \([w'']\), which occurs in \( u \) after \( p \), is \( \Theta(p) \). Consequently, any factor with the same properties as \( p \) belongs to the same equivalence class \([p]\). \( \square \)

**Theorem 27.** If \( u \) is an infinite word with language closed under \( G \), then

1. \( u \) is \( G \)-rich if and only if for all \( w \in \mathcal{L}(u) \) every complete \( G \)-return word of \([w]\) is a \( G \)-palindrome, i.e., \( u \) has Property \( G\text{-crw}(1) \);

2. \( u \) is almost \( G \)-rich if and only if there exists an integer \( N \) such that for all \( w \in \mathcal{L}(u) \) longer than \( N \) every complete \( G \)-return word of \([w]\) is a \( G \)-palindrome, i.e., \( u \) has Property \( G\text{-crw}(N) \).

**Proof.** Lemmas 23 and 26 together state that if \( N \) is an integer, then \( u \) satisfies \( G\text{-tls}(N) \) if and only if it satisfies \( G\text{-crw}(N) \). The second claim then directly follows from the definition of almost \( G \)-richness and Property \( G\text{-crw}(N) \). The first claim is obtained if \( N = 1 \). \( \square \)

## 5 \( G \)-richness and \( G \)-longest palindromic suffix

As we already stated, the classical richness is connected to the number of occurrences of the longest palindromic suffix in any factor. This section aims to generalize this connection
which is given by characterization 4 in Theorem 12. The main result of this section is in Theorem 33.

In the case of classical palindromes, the longest palindromic suffix of a nonempty word is always nonempty, but it is not always satisfied for the $G$-longest palindromic suffix. Therefore, the characterization of $G$-richness by the $G$-longest palindromic suffix needs a modification. In this section we show that if $u \in \mathcal{A}^\infty$ is an infinite word with language closed under $G$, then $u$ is $G$-rich if and only if for any factor $v \in \mathcal{L}(u)$, its $G$-longest palindromic suffix is $G$-unioccurrent in $v$ or the last letter of $v$ is $G$-unioccurrent in $v$.

To describe the needed property we introduce the next definition of Property $G$-lps$(N)$, again with $G$ and $N$ as parameters and the abbreviation “lps” standing for longest palindromic suffix.

**Definition 28.** Let $N \in \mathbb{N}$. We say that $u \in \mathcal{A}^\infty$ satisfies Property $G$-lps$(N)$ if for all $w \in \mathcal{L}(u)$, $|w| \geq N$, either the word $G$-lps$(w)$ is $G$-unioccurrent in $w$, or the suffix of $w$ of length 1 has exactly one $G$-occurrence in $w$.

**Lemma 29.** Let $u \in \mathcal{A}^\infty$.

1. If $u$ has Property $G$-crw$(1)$, then $u$ has Property $G$-lps$(1)$.

2. Let $N \in \mathbb{N}$. If $u$ is uniformly recurrent and has Property $G$-crw$(N)$, then there exists $M \in \mathbb{N}$ such that $u$ has Property $G$-lps$(M)$.

**Proof.** Let us realize a trivial fact: if $w \in \mathcal{L}(u)$ has a suffix $v$ which is not $G$-unioccurrent in $w$, then there is a suffix of $w$ which is a complete $G$-return word of $[v]$.

To prove the first assertion, consider a factor $w \in \mathcal{L}(u)$. If the last letter of $w$, denoted $a$, is $G$-unioccurrent, we have nothing to do. If $a$ is not $G$-unioccurrent in $w$, then according to the mentioned fact and by Property $G$-crw$(1)$, a complete $G$-return word of $[a]$ is a $G$-palindrome of length greater than 1. Therefore, $G$-lps$(w) \neq \varepsilon$. We have to show that $G$-lps$(w)$ is $G$-unioccurrent in $w$. If not, then a suffix of $w$ is a complete $G$-return word of $[G$-lps$(w)]$ which, according to Property $G$-crw$(1)$, is a $G$-palindrome longer than the $G$-longest palindromic suffix of $w$ - a contradiction.

Now we prove the second assertion. Since $u$ is uniformly recurrent, there exists an integer $M$ such that every factor $w \in \mathcal{L}(u)$, $|w| \geq M$, contains at least two occurrences of every factor $v \in \mathcal{L}_N(u)$. It particular, it implies that $w$ contains at least two $G$-occurrences of its suffix $z$ of length $N$. By the fact mentioned at the beginning of the proof, Property $G$-crw$(N)$ implies that $G$-lps$(w)$ is longer than $N$ and is $G$-unioccurrent in $w$. \qed

**Remark 30.** Let us note that the second part of the previous lemma can be proved considering a weaker assumption than uniform recurrence of $u$. It is enough to assume that every factor $w \in \mathcal{L}(u)$ has only finitely many complete $G$-return words.

**Lemma 31.** Let $u \in \mathcal{A}^\infty$.

1. If $u$ has Property $G$-lps$(1)$, then $u$ has Property $G$-crw$(1)$.
2. If \( u \) has Property \( G\text{-lps}(N) \) such that \( N > 1 \), then \( u \) has Property \( G\text{-crw}(N - 1) \).

**Proof.** We prove both claims simultaneously. If \( N = 1 \), set \( M = 1 \). Otherwise set \( M = N - 1 \). We prove by contradiction that \( u \) satisfies Property \( G\text{-crw}(M) \).

Suppose there is a factor \( w \in \mathcal{L}(u) \), \( |w| \geq M \), such that there is a factor \( v \in \mathcal{L}(u) \) which is a complete \( G \)-return word of \([w]\) and is not a \( G \)-palindrome. Denote \( p \) the prefix of \( u \) ending in the leftmost occurrence of a factor from \([v]\). It is clear that \( |p| \geq N \).

Since \( p \) ends in a nonempty \( G \)-palindromic complete \( G \)-return word, the suffix of \( p \) of length \( 1 \) has at least two \( G \)-occurrences and thus Property \( G\text{-lps}(N) \) assures that \( p \) has a nonempty \( G \)-longest palindromic suffix which is \( G \)-unioccurrent. Let us denote \( x := G\text{-lps}(p) \).

If \( 0 < |x| \leq |w| \), then \( x \) has at least two \( G \)-occurrences in \( p \) - a contradiction.

If \( |w| < |x| < |v| \), then we can find a third \( G \)-occurrence of \( w \) in \( v \) - a contradiction with \( v \) being a complete \( G \)-return word of \([w]\).

If \( |x| = |v| \), then we have a contradiction with \( v \) not being a \( G \)-palindrome.

If \( |x| > |v| \), then we can find a factor \( v' \in [v] \) such that its occurrence is in contradiction with the choice of the prefix \( p \).

\( \square \)

**Remark 32.** Again, the assumptions of the previous lemma can be reduced, as it is visible in our proof. It is enough to require that any prefix \( v \) of \( u \) of length greater than or equal to \( N \) has unique \( G \)-longest palindromic suffix or the last letter of the prefix \( v \) is \( G \)-unioccurrent in \( v \).

**Theorem 33.** Let \( u \in \mathcal{A}^\mathbb{N} \) be an infinite word with language closed under \( G \).

1. The word \( u \) is \( G \)-rich if and only if for any factor \( v \in \mathcal{L}(u) \), its \( G \)-longest palindromic suffix is \( G \)-unioccurrent in \( v \) or the last letter of \( v \) is \( G \)-unioccurrent in \( v \), i.e., \( u \) has Property \( G\text{-lps}(1) \).

2. If \( u \) is uniformly recurrent, then \( u \) is almost \( G \)-rich if and only if there exists an integer \( N \) such that for any factor \( v \in \mathcal{L}(u) \) longer than \( N \), its \( G \)-longest palindromic suffix is \( G \)-unioccurrent in \( v \) or the last letter of \( v \) is \( G \)-unioccurrent in \( v \), i.e., \( u \) has Property \( G\text{-lps}(N) \).

**Proof.** Using Lemmas 29 and 31, we have that Property \( G\text{-lps}(1) \) is satisfied if and only if Property \( G\text{-crw}(1) \) is satisfied. The first claim then follows from the first claim of Theorem 27.

If \( u \) is uniformly recurrent, then again using Lemmas 29 and 31, we find that there exists an integer \( N \) such that \( u \) satisfies Property \( G\text{-lps}(N) \) if and only if there exists an integer \( M \) such that \( u \) satisfies Property \( G\text{-crw}(M) \). The second claim follows from the second claim of Theorem 27.

\( \square \)

### 6 G-defect

In Section 2.4 we recalled the definition of palindromic defect and its relation to classical richness. Moreover, since the defect of a finite word \( w \) depends only on its length and
Table 1: Count of palindromes and E-palindromes in the prefixes of the Thue-Morse word $t_{2,2}$. The prefix of $t_{2,2}$ of length $n$ is denoted $p_n$.

| $n$ | #Pal$_R(p_n)$ | #Pal$_E(p_n)$ | G-lps($p_n$) | $n$ | #Pal$_R(p_n)$ | #Pal$_E(p_n)$ | G-lps($p_n$) |
|-----|---------------|---------------|--------------|-----|---------------|---------------|--------------|
| 0   | 1             | 1             | $\varepsilon$ | 10  | 9             | 8             | 1001100     |
| 1   | 2             | 1             | 0             | 11  | 10            | 9             | 0011100     |
| 2   | 3             | 2             | 01            | 12  | 11            | 10            | 10011001    |
| 3   | 4             | 2             | 11            | 13  | 12            | 10            | 0100110010  |
| 4   | 5             | 3             | 0110           | 14  | 13            | 11            | 101001100101|
| 5   | 6             | 3             | 101           | 15  | 14            | 12            | 11010011001011|
| 6   | 7             | 4             | 1010          | 16  | 15            | 13            | 0110100110010110|
| 7   | 8             | 5             | 110100        | 17  | 16            | 13            | 101101      |
| 8   | 9             | 6             | 01101001      | 18  | 17            | 13            | 01011010    |
| 9   | 9             | 7             | 00111         | 19  | 18            | 13            | 0010110100  |

\beginalign
D(w) & \leq D(wa) \leq D(w) + 1, & D(w) & \leq D(aw) \leq D(w) + 1, & \text{and } D(w) = D(R(w)). 
\endalign

The question we address here is how to define a $G$-analogy of defect when the group $G$ contains more than two elements. Of course, we would like to find a definition of $G$-defect such that $G$-richness and almost $G$-richness are again connected with $G$-defect in an analogous way.

Let us illustrate on the Thue-Morse word $t_{2,2}$ the number of distinct $G$-palindromes contained in its factors. The language of the Thue-Morse word is invariant under the reversal mapping $R$ and under the antimorphism $E$ which permutes letters 0 and 1. In [22], we showed that the Thue-Morse word is $G$-rich for $G = \{\text{Id}, R, E, RE\}$. In Table 1, the numbers of $G$-palindromic factors of short prefixes of the Thue-Morse words are depicted. There is no simple relation between the number of palindromes, $E$-palindromes and the length of the prefix, nevertheless, the $G$-longest palindromic suffix of each prefix is $G$-unioccurrent in it. To generalize the notion of defect, the counting of $G$-palindromes must be replaced by counting the classes $[w]$ of $G$-palindromes. Thus, we define the set, denoted $\text{Pal}_G(w)$, of all $G$-palindromic classes of equivalence in a finite word $w$ as follows

$$\text{Pal}_G(w) := \{[v] \mid v \text{ is a factor of } w \text{ and a } G\text{-palindrome}\}.$$

**Definition 34.** Let $w$ be a finite word. The $G$-defect of $w$ is defined as

$$D_G(w) := |w| + 1 - \#\text{Pal}_G(w) - \gamma_G(w),$$

21
where
\[ \gamma_G(w) := \# \{ [a] \mid a \in A, \text{a occurs in } w, \text{and } a \neq \Theta(a) \text{ and for every antimorphism } \Theta \in G \}. \]

It follows from the definition that for all \( w \in A^* \) and \( \mu \in G \) we have \( D_G(w) = D_G(\mu(w)) \).

The authors of [16] also observed that the classical richness of \( w \) can be characterized by so-called Property Ju: *Any prefix of \( w \) has unioccurrent longest palindromic suffix.*

The notion of the longest palindromic suffix helps to calculate the defect of a word. For a word \( w \) and a letter \( a \), the following holds (see [17]):

\[ D(wa) = \begin{cases} D(w), & \text{if } wa \text{ has unioccurrent longest palindromic suffix} \\ D(w) + 1, & \text{otherwise}. \end{cases} \]

Therefore, the defect of a finite word \( w = w_1w_2 \cdots w_n \) equals to the number of indices \( i \) for which \( w_i \) does not have a unioccurrent longest palindromic suffix. Such indices are called lacunas in [7] and defective positions in [17]. Inspired by this, we adopt the following definition.

**Definition 35.** Let \( w = w_1 \cdots w_n \in A^* \). An integer \( i \) such that \( 1 \leq i \leq n \) is called \( G \)-lacuna in \( w \) if \( w_i \) and \( G\text{-lps}(w_1 \cdots w_i) \) are not \( G \)-unioccurrent in \( w_1 \cdots w_i \).

The next lemma follows from comparing the last definition and the definition of \( G \)-defect.

**Lemma 36.** Let \( w \in A^* \), then
\[ D_G(w) = \text{the number of } G \text{-lacunas in } w. \]

**Proof.** If \( w \) is the empty word, then the claim holds. Suppose \( w = w_1 \cdots w_i \) for some \( i \geq 1 \). We will show the two following implications:

1. if \( i \) is a \( G \)-lacuna, then \( D_G(w) = D_G(w_1 \cdots w_{i-1}) + 1 \);
2. if \( i \) is not a \( G \)-lacuna, then \( D_G(w) = D_G(w_1 \cdots w_{i-1}) \).

Denote \( s = G\text{-lps}(w) \).

Suppose \( i \) is a \( G \)-lacuna, i.e., \( w_i \) and \( s \) are both not \( G \)-unioccurrent in \( w \). Since \( w_i \) is not \( G \)-unioccurrent, we have \( \gamma_G(w) = \gamma_G(w_1 \cdots w_{i-1}) \). Since \( s \) is not \( G \)-unioccurrent, we have \( \#\text{Pal}_G(w) = \#\text{Pal}_G(w_1 \cdots w_{i-1}) \). This shows the first implication.

Suppose \( i \) is not a \( G \)-lacuna, i.e., \( w_i \) is \( G \)-unioccurrent or \( s \) is \( G \)-unioccurrent in \( w \). We distinguish the three following cases.

a) \( w_i \) is \( G \)-unioccurrent and \( s \) is not \( G \)-unioccurrent.

It follows that \( s = \varepsilon \) and thus for every antimorphism \( \Theta \in G \), we have that \( \Theta(w_i) \neq w_i \), which implies that \( \gamma_G(w) = \gamma_G(w_1 \cdots w_{i-1}) + 1 \) and \( \#\text{Pal}_G(w) = \#\text{Pal}_G(w_1 \cdots w_{i-1}) \).
b) \( w_i \) is not \( G \)-unioccurrent and \( s \) is \( G \)-unioccurrent.

It this case, since \( w_i \) is not \( G \)-unioccurrent, we have \( \gamma_{G}(w) = \gamma_{G}(w_1 \cdots w_{i-1}) \). \( G \)-uniocurrence of \( s \) implies \( \text{Pal}_G(w) = \text{Pal}_G(w_1 \cdots w_{i-1}) \cup \{s\} \), thus, \( \#\text{Pal}_G(w) = \#\text{Pal}_G(w_1 \cdots w_{i-1}) + 1 \).

\[ \text{D}_G(w) = D_G(w_1 \cdots w_{i-1}) \] which shows the second implication.

Moreover, it can be easily shown that the following relations are preserved:

\[
D_G(w) \leq D_G(wa) \leq D_G(w) + 1 \quad \text{and} \quad D_G(w) \leq D_G(aw) \leq D_G(w) + 1
\]

for all \( w \in \mathcal{A}^* \) and \( a \in \mathcal{A} \). Therefore, we can define \( G \)-defect of an infinite word.

**Definition 37.** Let \( u \) be an infinite word. The \( G \)-defect of \( u \), denoted \( D_G(u) \), is defined as

\[
D_G(u) := \sup_{w \in \mathcal{L}(u)} \{D_G(w)\}.
\]

The immediate connection with Property \( G \)-lps(\( N \)) is summarized in the following lemma.

**Lemma 38.** Let \( u \) be an infinite word with language closed under \( G \).

1. \( D_G(u) = 0 \) if and only if \( u \) satisfies Property \( G \)-lps(1).

2. If there exists an integer \( N \) such that \( u \) satisfies Property \( G \)-lps(\( N \)), then \( D_G(u) \) is finite.

3. If \( u \) is uniformly recurrent and \( D_G(u) \) is finite, then there exists an integer \( N \) such that \( u \) satisfies Property \( G \)-lps(\( N \)).

**Proof.** The first two claims follow from Lemma 36.

To show the last claim, suppose \( D_G(u) \) to be finite. Then there exists a prefix \( v = u_0 u_1 \cdots u_{M-1} \) of \( u \) such that \( D_G(u) = D_G(v) \) and any letter of the alphabet occurs in \( v \). As \( u \) is uniformly recurrent, there exists a constant \( N \) such that any factor \( w \) of \( u \) of length at least \( N - 1 \) contains the prefix \( v \) as its factor. Using the definition and basic properties of \( G \)-defect, and maximality of \( D_G(v) \), we obtain \( D_G(v) = D_G(w) = D_G(wh) \) for any \( a \in \mathcal{A} \) such that \( wa \in \mathcal{L}(u) \). Therefore, the last position in \( wa \) is not a \( G \)-lacuna, i.e., \( u \) has Property \( G \)-lps(\( N \)).

It remains to connect \( G \)-defect with \( G \)-richness and almost \( G \)-richness.

**Theorem 39.** Let \( u \) be an infinite word with language closed under \( G \).
1. \( D_G(u) = 0 \) if and only if \( u \) is \( G \)-rich.

2. If \( u \) is uniformly recurrent, then \( D_G(u) \) is finite if and only if \( u \) is almost \( G \)-rich.

**Proof.** The first part follows from Lemma 38 and Theorem 33.

To show the second part, one can see that it follows from Lemma 38 that \( u \) satisfies Property \( G \)-lps\((N)\) for some \( N \) if and only if \( D_G(u) \) is finite. We can then use Lemmas 29 and 31 to get equivalence with having Property \( G \)-crw\((N')\) for some \( N' \). Finally, we use Lemmas 23 and 26 to prove the equivalence with having Property \( G \)-tls\((N')\) which is by definition equivalent with almost \( G \)-richness of \( u \).

\[ \square \]

### 7 \( G \)-richness and bilateral order

As stated in Theorem 12, characterization 7, words rich in classical sense can be characterized using bilateral order of bispecial factors. In this section we show a generalization of this statement.

The proof of this fact for classical richness given in [4] is based on the validity of point 5 of Theorem 12. The following statement is a combination of Theorem 22 and Remark 24 from [22] and it generalizes characterization 5 of Theorem 12 for almost rich words.

**Proposition 40.** Let \( u \) be an infinite word with language closed under \( G \) and \( N \in \mathbb{N} \) be a \( G \)-distinguishing number on \( u \). The word \( u \) satisfies Property \( G \)-tls\((N)\) if and only if

\[ \Delta C(n) + \#G = \sum_{\Theta \in G(2)} \left( \mathcal{P}_\Theta(n) + \mathcal{P}_\Theta(n + 1) \right) \quad \text{for any } n \geq N. \]  

\[ (7) \]

We have no modification of the previous proposition describing Property \( G \)-tls\((N)\) for \( N \) which are not \( G \)-distinguishing; in this case we have no simple expression for the right side of the equation since it strongly depends on the relations among the elements of the group. In [22], we show the exact expression only for groups of order 4. Therefore, in this section we concentrate on the notion almost \( G \)-richness. Let us rephrase the previous proposition in a more handy way.

**Corollary 41.** Let \( u \) be an infinite word with language closed under \( G \) and \( N \in \mathbb{N} \) be a \( G \)-distinguishing number on \( u \). The word \( u \) satisfies Property \( G \)-tls\((N)\) if and only if

1.

\[ \Delta C(N) + \#G = \sum_{\Theta \in G(2)} \left( \mathcal{P}_\Theta(N) + \mathcal{P}_\Theta(N + 1) \right) \]

2. and for all \( n \geq N \), we have

\[ \Delta^2 C(n) = \sum_{\Theta \in G(2)} \sum_{w \in \mathcal{L}_n(u)} \sum_{w = \Theta(w)} (\#\text{Pext}_\Theta(w) - 1). \]
Proof. The task to verify equalities \( a(n) = b(n) \) for all \( n \geq N \) means to verify \( a(N) = b(N) \) and \( \Delta a(n) = \Delta b(n) \) for all \( n \geq N \). Let us consider \( a(n) \) to be equal to the left side and \( b(n) \) to the right side of (7). It is now enough to realize that

\[
\Delta b(n) = \sum_{\Theta \in G^{(2)}} P_{\Theta}(n+2) - P_{\Theta}(n) = \sum_{\Theta \in G^{(2)}} \sum_{w \in \mathcal{L}_n(u)} \sum_{w = \Theta(w)} (\#\text{Ext}_\Theta(w) - 1),
\]

where we used equalities (1) and \( P_{\Theta}(n) = \sum_{w \in \mathcal{L}_n(u)} w = \Theta(w) \).

Proposition 42. Let \( N \in \mathbb{N}, u \in \mathcal{A}^\mathbb{N} \) satisfy \( G\text{-tls}(N) \), and \( w \) be a bispecial factor of \( u \) of length at least \( N \).

- If \( w \) is not a \( G \)-palindrome, then \( b(w) \geq 0 \).
- If \( w \) is a \( \Theta \)-palindrome for an antimorphism \( \Theta \in G \), then \( b(w) \geq \#\text{Ext}_\Theta(w) - 1 \).

Proof. Let \( w \) be a bispecial factor having its length \( M := |w| \geq N \) such that for all antimorphism \( \Theta \in G \), \( \Theta(w) \neq w \). By the definition of \( b(w) \), we want to prove

\[
\#\text{Bext}(w) \geq \#\text{Rext}(w) + \#\text{Lext}(w) - 1. \tag{8}
\]

Let \( B(w) \) be a bipartite graph with the set of vertices

\[
V(w) = \{aw \mid a \in \text{Lext}(w)\} \cup \{wb \mid b \in \text{Rext}(w)\}.
\]

There is an edge connecting vertices \( aw \) and \( wb \) if the word \( awb \) is a factor of \( u \). The number of vertices in the graph \( B(w) \) is \( \#\text{Rext}(w) + \#\text{Lext}(w) \) and the number of edges is \( \#\text{Bext}(w) \). In the sequel, we show that this graph is connected. Since in any connected graph the number of edges equals at least the number of vertices minus one, the inequality (8) follows.

Let \( (k_n) \) be an increasing sequence of indices such that \( k_0 > 0 \) is an occurrence of \( w \) and \( k_n \) is a \( G \)-occurrence of \( w \) for any \( n \geq 1 \). Moreover, any \( G \)-occurrence of \( w \) in the suffix \( u_{k_0}u_{k_0+1}u_{k_0+2} \ldots \) of \( u \) belongs to the sequence \( (k_n) \). As \( u \) satisfies Property \( G\text{-tls}(N) \), then according to Remark 24 we have

\[
u_{k_1}u_{k_1+1} \cdots u_{k_1+M-1} = \nu_1(w), \text{ where } \nu_1 \in G \text{ is an antimorphism},
\]

\[
u_{k_2}u_{k_2+1} \cdots u_{k_2+M-1} = \nu_2(w), \text{ where } \nu_2 \in G \text{ is a morphism},
\]

\[
u_{k_3}u_{k_3+1} \cdots u_{k_3+M-1} = \nu_3(w), \text{ where } \nu_3 \in G \text{ is an antimorphism},
\]

and so on.

The restriction of \( \nu \in G \) to the set of letters is just a permutation. Therefore, for any
\( \nu \in G \) and any \( b \in \mathcal{A} \), there exists a letter \( a \in \mathcal{A} \) such that \( b = \nu(a) \). Thus, for any \( n \in \mathbb{N} \), the factor \( u_{k_n-1}u_ku_{k_n+1} \cdots u_{k_n+M} \) can be written as

\[
  u_{k_n-1}u_ku_{k_n+1} \cdots u_{k_n+M} = \nu_n(c_n)\nu_n(w)\nu_n(d_n)
\]

for some letters \( c_n \) and \( d_n \).

As \( \nu_{2i} \) is a morphism and \( \nu_{2i+1} \) is an antimorphism, we have

\[
c_{2i}wd_{2i} \in \mathcal{L}(u) \quad \text{and} \quad d_{2i-1}wc_{2i-1} \in \mathcal{L}(u).
\]  

(9)

Because \( w \) is not a \( G \)-palindrome, any \( G \)-occurrence of \( w \) together with the left and the right neighboring letters corresponds to a unique edge in the graph \( B(w) \). For any \( n \in \mathbb{N} \), the factor

\[
  \nu_n(w)\nu_n(d_n)\cdots\nu_{n+1}(c_{n+1})\nu_{n+1}(w)
\]

is a complete \( G \)-return word of \([w]\). According to Lemma 22, there exists an antimorphism \( \Theta \in G \) such that \( \nu_{n+1}(w) = \Theta(\nu_n(w)) \) and \( \nu_{n+1}(c_{n+1}) = \Theta(\nu_n(d_n)) \). As \( \nu_{n+1}(w) = \Theta(\nu_n(w)) \) implies \( \nu_{n+1} = \Theta\nu_n \), we get \( \nu_{n+1}(c_{n+1}) = \Theta(\nu_n(c_n)) = \Theta(\nu_n(d_n)) \) and thus \( c_{n+1} = d_n \). Using (9) we obtain

\[
c_{2i}wc_{2i+1} \in \mathcal{L}(u) \quad \text{and} \quad c_{2i}wc_{2i-1} \in \mathcal{L}(u).
\]

Recurrence of \( u \) implies

\[
  V(w) = \{c_{2i}w \mid i \in \mathbb{N}\} \cup \{wc_{2i-1} \mid i \in \mathbb{N}\}.
\]

Walking along \( u \), each \( G \)-occurrence of \( w \) represents an unordered edge connecting \( c_{2i}w \) with \( wc_{2i-1} \) or \( c_{2i}w \) with \( wc_{2i+1} \), and thus in fact walking along \( u \) represents a walk in the graph \( B(w) \). Since any factor \( aw \) and \( wb \) occurs in \( \mathcal{L}(u) \) infinitely many times, this walk in the graph \( B(w) \) uses all vertices of \( B(w) \). Therefore, the graph \( B(w) \) is connected.

Now consider a \( G \)-palindromic bispecial factor \( w \) and denote by \( \Theta \) the antimorphism such that \( w = \Theta(w) \). We define the bipartite graph \( B(w) \) in the same way as before. If \( awb \in \mathcal{L}(u) \) and \( b \neq \Theta(a) \), then \( B(w) \) contains with the edge \( awb \) also the different edge \( \Theta(b)w\Theta(a) \). Therefore, any \( G \)-occurrence of \( w \) together with the left and the right neighboring letters corresponds to a pair of edges in the graph \( B(w) \) unless it represents a \( \Theta \)-palindromic extension \( aw\Theta(a) \). Let us replace the graph \( B(w) \) by the graph \( B'(w) \) in which vertices are couples \( \{aw, w\Theta(a)\} \) and edges are either couples \( \{awb, \Theta(b)w\Theta(a)\} \) or loops representing a \( \Theta \)-palindromic extension \( \{aw\Theta(a)\} \). Now we can interpret a walk along \( u \) as a walk in the new graph \( B'(w) \). Consequently, the graph \( B'(w) \) must be connected. The connectivity of \( B'(w) \) implies that the number of edges in \( B'(w) \) which are not loops is at least equal to \( \#\text{Rext}(w) - 1 \). Since

\[
  \#\{awb \mid b \neq \Theta(a)\} = 2 \times \text{number of edges in } B'(w) \text{ which are not loops},
\]

we obtain

\[
  \#\text{Bext}(w) = \#\{awb \mid b \neq \Theta(a)\} + \#\text{Ext}_\Theta(w) \geq 2(\#\text{Rext}(w) - 1) + \#\text{Ext}_\Theta(w)
\]

As \( \#\text{Rext}(w) = \#\text{Ext}(w) \), we deduce

\[
b(w) = \#\text{Bext}(w) - \#\text{Rext}(w) - \#\text{Ext}(w) + 1 \geq \#\text{Ext}_\Theta(w) - 1.
\]
Proposition 43. Let \( u \) be an infinite word with language closed under \( G \) and \( N \in \mathbb{N} \) be \( G \)-distinguishing on \( u \). The word \( u \) has Property \( G \)-tls\((N) \) if and only if any bispecial factor \( w \) of \( u \) of length at least \( N \) satisfies:

- if \( w \) is not a \( G \)-palindrome, then
  \[ b(w) = 0, \]
- if \( w \) is a \( \Theta \)-palindrome for some \( \Theta \in G \), then
  \[ b(w) = \#\text{Pext}_\Theta(w) - 1; \]

and

\[ \Delta C(N) + \#G = \sum_{\Theta \in G^{(2)}} \left( \mathcal{P}_\Theta(N) + \mathcal{P}_\Theta(N + 1) \right). \]

Proof: \((\Leftarrow)\): The assumption on bilateral orders and the fact that non-bispecial \( \Theta \)-palindromic factors have a unique \( \Theta \)-palindromic extension guarantee the following equality for all \( n \geq N \):

\[ \Delta^2 C(n) = \sum_{w \in L_n(u)} b(w) = \sum_{\Theta \in G^{(2)}} \sum_{\substack{w \in L_n(u) \\text{for all } \Theta \in G^{(2)} \\text{ for all } \Theta \in G^{(2)} \\text{ for all } \Theta \in G^{(2)}}} (\#\text{Pext}_\Theta(w) - 1). \] (10)

According to Corollary 41 it implies that \( u \) satisfies Property \( G \)-tls\((N) \).

\((\Rightarrow)\): Let \( n \geq N \). Using Proposition 42 and Corollary 41 we obtain

\[ \Delta^2 C(n) = \sum_{w \in L_n(u)} b(w) = \sum_{w \in L_n(u)} b(w) + \sum_{\Theta \in G^{(2)}} \sum_{\substack{w \in L_n(u) \\text{for all } \Theta \in G^{(2)} \\text{ for all } \Theta \in G^{(2)} \\text{ for all } \Theta \in G^{(2)}}} b(w) \]

\[ \geq \sum_{\Theta \in G^{(2)}} \sum_{\substack{w \in L_n(u) \\text{for all } \Theta \in G^{(2)} \\text{ for all } \Theta \in G^{(2)} \\text{ for all } \Theta \in G^{(2)}}}(\#\text{Pext}_\Theta(w) - 1) = \Delta^2 C(n). \]

As the beginning and the end of our estimates is the same, the inequalities for \( b(w) \) deduced in Proposition 42 must be equalities, which was to prove. \( \square \)

8 Examples

The aim of this section is to exhibit examples of \( G \)-rich words. As already mentioned in Example 6, it is shown in [27] that for any dihedral group \( I_2(m) \) there exist words, namely word \( t_{b,m} \) for all integers \( b \geq 2 \), such that they are \( I_2(m) \)-rich. Dihedral groups form part of finite Coxeter groups which, according to Theorem 25, are good candidates for a group
Figure 8: (a) Symmetries of $\mathcal{L}(u)$ from Example 44 represented by the symmetries of an orthogonal prism with rhomb base. (b) Symmetries of $\mathcal{L}(v)$ from Example 45 represented by two pyramids joint together by their rectangular bases.

$G$ when looking for an example of $G$-rich word. In this section we provide 2 examples of $G$-rich words such that $G$ is not a dihedral group.

The first group, denoted $G$, is constructed on an 8-letter alphabet $\mathcal{A} := \{0, 1, \ldots, 7\}$. The antimorphisms $\Theta_0, \Theta_1$ and $\Theta_2$ are defined on $\mathcal{A}^*$ as follows

\begin{align*}
\Theta_0 & : 0 \mapsto 2, 1 \mapsto 1, 2 \mapsto 0, 3 \mapsto 3, 4 \mapsto 6, 5 \mapsto 5, 6 \mapsto 4, 7 \mapsto 7, \\
\Theta_1 & : 0 \mapsto 4, 1 \mapsto 5, 2 \mapsto 6, 3 \mapsto 7, 4 \mapsto 0, 5 \mapsto 1, 6 \mapsto 2, 7 \mapsto 3, \\
\Theta_2 & : 0 \mapsto 0, 1 \mapsto 3, 2 \mapsto 2, 3 \mapsto 1, 4 \mapsto 4, 5 \mapsto 7, 6 \mapsto 6, 7 \mapsto 5.
\end{align*}

The group $G \subset AM(\mathcal{A}^*)$ is the group generated by $\Theta_0, \Theta_1$ and $\Theta_2$. If we label the vertices of an orthogonal prism with rhomb base by the letters of $\mathcal{A}$ as depicted in Figure 8a, then the antimorphisms of $G$ correspond to the mirror symmetries of the prism.

The second group, denoted $H$, is on a 6-letter alphabet $\mathcal{B} := \{0, 1, \ldots, 5\}$ and $H \subset AM(\mathcal{B}^*)$ is generated by the 3 following antimorphisms:

\begin{align*}
\Psi_0 & : 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 4, 3 \mapsto 5, 4 \mapsto 2, 5 \mapsto 3, \\
\Psi_1 & : 0 \mapsto 1, 1 \mapsto 0, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 4, 5 \mapsto 5, \\
\Psi_2 & : 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 2, 4 \mapsto 5, 5 \mapsto 4.
\end{align*}

The antimorphisms generating the group $H$ can be visualised by the mirror symmetries of the object depicted in Figure 8b.

In fact, the groups $G$ and $H$ are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. They may be viewed as group actions of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ on distinct free monoids: $G$ on $\mathcal{A}^*$ and $H$ on $\mathcal{B}^*$.

**Example 44.** Let $\varphi : \mathcal{A}^* \mapsto \mathcal{A}^*$ be a morphism defined as

\[ \varphi : 0 \mapsto 01, 1 \mapsto 2, 2 \mapsto 65, 3 \mapsto 4, 4 \mapsto 23, 5 \mapsto 6, 6 \mapsto 47, 7 \mapsto 0. \]
Denote by $u$ the fixed point of $\varphi$. In Section 8.1, we show that $u$ has its language closed under $G$ and $u$ is $G$-rich.

**Example 45.** Let $\mu : A^* \mapsto B^*$ be a morphism defined as

$$\mu : 0 \mapsto 15, 1 \mapsto 04, 2 \mapsto 12, 3 \mapsto 03, 4 \mapsto 04, 5 \mapsto 12, 6 \mapsto 03, 7 \mapsto 15.$$ 

Let $v = \mu(u)$. In Section 8.2, we show that $L(v)$ is closed under $H$ and $v$ is $H$-rich.

The proofs of properties of $u$ and $v$ are split into several lemmas. Instead of their complete proofs, we provide just sketches or hints for readers.

### 8.1 $G$-richness of $u$

To prove that $u$ defined in Example 44 is $G$-rich, we show in the sequel that $u$ has Property $G$-tls(1). For this reason, we exploit Proposition 43. Therefore, one needs to study bispecial factors occurring in $u$. We use a general method for circular morphisms described in [21]. In the case of $\varphi$, it yields simple results. To describe the bispecial factors of $u$, we introduce the mapping $\pi : \{0, 2, 4, 6\} \mapsto \{0, 2, 4, 6\}$ as follows

$$\pi : 0 \mapsto 2, 4 \mapsto 0, 2 \mapsto 4, 6 \mapsto 6.$$ 

**Lemma 46.** Let $w = w_0 \cdots w_{n-1}$ be a nonempty bispecial factor of $u$. Then $w_{n-1} \in \{0, 2, 4, 6\}$ and $\varphi(w)\pi(w_{n-1})$ is a bispecial factor of $u$. Moreover, $b(w) = b(\varphi(w)\pi(w_{n-1}))$.

**Proof.** The claim follows from the definition of $\varphi$ and the fact that right special factors of length 1 are factors 0, 2, 4 and 6, and each has 2 right extensions. \hfill $\square$

The next statement can be easily deduced from the form of $\varphi$ as well.

**Lemma 47.** Let $w, |w| \geq 2$, be a bispecial factor of $u$. Then there exists a unique bispecial factor of $u$, say $v = v_0v_1 \cdots v_{m-1}$, such that

$$w = \varphi(v)\pi(v_{m-1}).$$

(11)

According to the last two lemmas, all bispecial factors can be constructed from bispecial factors of length 1 using recursively formula (11). In fact, all the letters of $A$ are bispecial factors and $G$-palindromes. We show that the formula (11) produces from a $G$-palindrome again a $G$-palindrome.

**Lemma 48.** For all $i \in \mathbb{Z}_3$ and $w \in L(u)$, $w = w_0 \cdots w_{n-1}$, we have

$$x_{i-1}(w)\Theta_i\varphi(w) = \varphi\Theta_{i-1}(w)y_i(w),$$

where $y_i(w) := \Theta_i(\varphi(w_0))_0$ and $x_i(w) = (\varphi\Theta_i(w_{n-1}))_0$ and $(v)_0$ denotes the first letter of a word $v$. 

29
Lemma 50. Let $w = w_0 \cdots w_{n-1}$ be a nonempty bispecial factor of $u$. Then $w$ is a $\Theta$-palindrome, $\Theta \in G$, and $b(w) = \#\text{Pext}_G(w) - 1$.

Proof. As the bilateral order of a bispecial factor of length 1 equals 0, according to Lemmas 46 and 47, all bispecial factors have their bilateral order equal to 0. It is also clear that they have 2 right and 2 left extensions.

If $w = w_0 \cdots w_{n-1}$ is a nonempty bispecial factor and if $w$ is a $\Theta_{i-1}$-palindrome for $i \in \mathbb{Z}_3$, one can show that $\pi(w_{n-1}) = y_i(w)$ and thus, $\varphi(w)\pi(w_{n-1})$ is a $\Theta_i$-palindrome. Since the bispecial factors of length 1 are $\Theta_2$-palindromes, all bispecial factors are $\Theta_i$-palindromes for some $i \in \mathbb{Z}_3$.

It follows that for all $i \in \mathbb{Z}_3$, $L(u)$ contains infinitely many $\Theta_i$-palindromes. Therefore, $L(u)$ is closed under $G$.

Let $w$ be a nonempty bispecial factor. Since $b(w) = 0$, $w$ has 2 left and 2 right extensions, $w$ is a $\Theta$-palindrome for a unique $\Theta \in G$, and $L(u)$ is closed under $G$, one can see that the number of $\Theta$-palindromic extensions of $w$ is 1.

Proof of $G$-richness of $u$ defined in Example 44. At first, we realize that the generators $\Theta_0, \Theta_1$ and $\Theta_2$ of the group $G$ guarantee the number 1 to be $G$-distinguishing on any infinite word over $\mathcal{A}$. Because of Lemma 49 and Proposition 43, it remains to verify that $\Delta C(1) + \#G$ equals the number of all $G$-palindromes of length 1 and 2. One can readily see that $\Delta C(1) = 4, \#G = 8$, the number of $G$-palindromes of length 1 is 8 and the number of $G$-palindromes of length 2 is 4.

8.2 $H$-richness of $v$

The proof of $H$-richness of $v$ is very similar to the previous proof. In order to use Proposition 43, we explore the bilateral orders and $H$-palindromic extensions of bispecial factors of $v$.

We define the morphism $\eta : \mathcal{A}^* \mapsto \mathcal{B}^*$ as

$$\eta : 0 \mapsto 041, 1 \mapsto 120, 2 \mapsto 031, 3 \mapsto 150, 4 \mapsto 150, 5 \mapsto 041, 6 \mapsto 120, 7 \mapsto 031.$$ 

Let $w = w_0 \cdots w_{n-1} \in \mathcal{A}^*$ be a nonempty factor of $u$. It follows from $L_2(u)$ and the definition of $\mu$ that $\mu(w)\eta(w_{n-1})$ is a factor of $v$. The following lemma summarizes the relation between the bispecial factors of $v$ and of $u$.

Lemma 50. Let $w \in L(u)$, $w = w_0 \cdots w_{n-1}$, be a nonempty bispecial factor. Then $\mu(w)\eta(w_{n-1})$ is a bispecial factor of $v$.

On the other hand, if $v \in L(v)$, $|v| \geq 5$, is a bispecial factor of $v$, then there exists a unique nonempty bispecial factor $w \in L(u)$ such that $v = \mu(w)\eta(w_{n-1})$. 

30
Lemma 51. Let \( i \in \mathbb{Z}_3 \). If \( w \in \mathcal{L}(u) \) is a nonempty \( \Theta_i \)-palindrome, then the factor \( \mu(w)\eta(w_{n-1}) \in \mathcal{L}(v) \) is a \( \Psi_i \)-palindrome.

Sketch of the proof. We induce on the length of \( w \). Fix \( i \in \mathbb{Z}_3 \). Suppose the claim holds for \( w = w_0 \cdots w_{n-1} = \Theta_i(w) \). Take \( z \in A \) such that \( zw(\Theta_i(z)) \in \mathcal{L}(u) \). The proof follows from the definition of \( \mu, \eta \), and possible factors \( zw_0 \in L_2(u) \).

Proof of \( H \)-richness of \( v \) defined in Example 45. According to the previous lemma, it is clear that \( \mathcal{L}(v) \) is closed under \( H \). The properties of \( \mathcal{L}(u) \) also imply that all bispecial factors of \( v \) of length greater than or equal to 5 have bilateral order 0 and one \( \Theta \)-palindromic extension, where \( \Theta \in H \) is the unique antimorphism fixing the bispecial factor. For shorter bispecial factors, of length greater than 1, this property needs to be verified by hand and is left to the reader.

Since 2 is an \( H \)-distinguishing number on \( v \), Proposition 43 requires to evaluate \( \Delta C(2), P_{\Theta}(2) \) and \( P_{\Theta}(3) \) for all involutive antimorphism \( \Theta \in H \). It is easy to verify that \( \Delta C(2) = 4, \sum_{\Theta \in H(2)} P_{\Theta}(2) = 0 \) and \( \sum_{\Theta \in H(2)} P_{\Theta}(3) = 12 \). Since \( #H = 8 \), according to Proposition 43, \( v \) satisfies Property \( H \)-tls(2).

To claim that \( v \in H \)-rich, we need to verify that \( v \) satisfies \( H \)-tls(1). Thus, it remains to show that all loops in \( \Gamma_1(v) \) are \( H \)-palindromes and the graph obtained from \( \Gamma_1(v) \) by removing loops is a tree. Since it can be easily verified by hand, the word \( v \) is \( H \)-rich.

Denote for all \( i \in \mathbb{Z}_3 \) by \( H_i \) the subgroup of \( H \) generated by \( \Psi_i \) and \( \Psi_{(i+1) \mod 3} \). It is easy to verify that \( #H_i = 4 \) for all \( i \), the number 1 is \( H_0 \)-distinguishing and \( H_1 \)-distinguishing, and the number 2 is \( H_2 \)-distinguishing. It follows from the last proof that \( P_{\Psi_i}(3) = 4 \) and \( P_{\Psi_i}(2) = 0 \) for all \( i \in \mathbb{Z}_3 \). One can also verify that \( P_{\Psi_0}(1) = P_{\Psi_2}(1) = 2 \) and \( P_{\Psi_1}(1) = 4 \). Since \( \Delta C(1) = 2 \), using Proposition 43 we get that the word \( v \) is \( H_0 \)-rich, \( H_1 \)-rich. Since \( \Delta C(2) = 4 \), again using Proposition 43 we get that the word \( v \) is almost \( H_2 \)-rich (it satisfies \( H_2 \)-tls(2)). In fact, it can be shown that the word \( v \) satisfies \( H_2 \)-tls(1) and thus it is also \( H_2 \)-rich.

9 Comments and open problems

- The dihedral groups \( I_2(m) \) form a special class of finite Coxeter groups which belong to a broader class of groups generated by involutive elements. As shown in [27] and recalled in Example 6, for any dihedral group there exists a \( I_2(m) \)-rich word. Is it possible for any given finite group generated by involutive antimorphisms or at least a given finite Coxeter group \( G \) to find a \( G \)-rich word?

We believe that an approach using a generalized palindromic closure operator as introduced in the last chapter of [15] might be helpful.

- For \( #G > 2 \), the list of examples of \( G \)-rich words is very short and the list of almost \( G \)-rich words (which are not \( G \)-rich) is empty. In [17], Glen et al. described a class of morphisms such that morphic image of a rich word under a morphism from this class has a finite nonzero defect. Find a class of morphisms producing almost \( G \)-rich
words with finite nonzero defect by applying a morphism from this class to a $G$-rich word.

- For a word $u$ with language closed under reversal, Brlek and Reutenauer conjectured in [9] for the defect $D(u)$ that

$$2D(u) = \sum_{n \in \mathbb{N}} T(n), \quad \text{where} \quad T(n) := \Delta \mathcal{C}(n) + 2 - \mathcal{P}(n+1) - \mathcal{P}(n).$$

The conjecture was shown in [6].

Can the $G$-defect of an infinite word $u$ be expressed using the differences between right and left sides of inequalities in (3)?

- In Section 2.4, definitions of rich words and $\Theta$-rich words were reminded. In our new terminology, they are $\{\text{Id, } R\}$-rich words and $\{\text{Id, } \Theta\}$-rich words respectively. The groups $\{\text{Id, } R\}$ and $\{\text{Id, } \Theta\}$ are clearly isomorphic. In [10], it is shown that a so-called $\Theta$-standard word with seed, which is almost $\Theta$-rich, is a morphic image of a standard Arnoux-Rauzy words, which is rich. In [23], we have a more general case: any uniformly recurrent almost $\Theta$-rich word is a morphic image of a rich word.

Is an almost $G_1$-rich word related to a $G_2$-rich word for some group $G_2$ isomorphic to $G_1$?

- Let $u$ be an infinite word having language closed under a group $G$. The closedness under $G$ can be exploited to estimate the number of distinct frequencies of factors of the same length $n$. In [3], an upper bound on this number is given (for $n$ being a $G$-distinguishing number). The estimate is based on the inequality from Theorem 8. Looking at the proof of the estimate, it can be seen that the only candidates for reaching the upper bound for all sufficiently large $n$ are almost $G$-rich words. However, as noted in [3], in our words, almost $G$-richness does not imply the upper bound to hold for all sufficiently large $n$.

- The Thue-Morse word is $G$-rich, where $G$ is generated by two commuting antimorphisms $R$ and $E$. However, it is not $G_1$-rich while taking a proper subgroup $G_1$ of $G$. In our considerations, we did not assume the group $G$ to be the maximal group of symmetries such that an infinite word $u$ is closed under $G$.

Suppose $u$ is an infinite word having language closed under a group $G$. Let $G_1$ be a proper subgroup of $G$ containing at least one antimorphism. Suppose $u$ is both almost $G$-rich and almost $G_1$-rich. Let $N$ be a $G$-distinguishing number. Then, according to Proposition 40, we can for all $n \geq N$ write

$$\Delta \mathcal{C}(n) + \# G = \sum_{\Theta \in G^{(2)}} \left( \mathcal{P}_{\Theta}(n) + \mathcal{P}_{\Theta}(n+1) \right) \quad \text{and} \quad \Delta \mathcal{C}(n) + \# G_1 = \sum_{\Theta \in G_1^{(2)}} \left( \mathcal{P}_{\Theta}(n) + \mathcal{P}_{\Theta}(n+1) \right).$$
Thus, we get
\[\#G - \#G_1 = \sum_{\Theta \in G^{(2)} \setminus G_1^{(2)}} \left( P_{\Theta}(n) + P_{\Theta}(n+1) \right). \] (12)

Since \(G_1\) is a proper subgroup of \(G\), we have \(\#G = \ell \#G_1\) for \(\ell > 1\). Take \(\Theta \in G^{(2)} \setminus G_1^{(2)}\) and \(w \in \mathcal{L}(u), |w| \geq N\), such that \(\Theta(w) = w\). Then for all \(v \in [w]\), it can be show that there exists \(\Psi \in G^{(2)} \setminus G_1^{(2)}\) such that \(v\) is a \(\Psi\)-palindrome. Since \(\#[w] = \frac{\#G}{2}\), the right side of (12) equals \(k(n)\frac{\#G}{2}\) for \(k(n) \in \mathbb{N}\). We get from (12) that
\[(\ell - 1)\#G_1 = k(n)\ell \frac{\#G_1}{2}.\]

The only solution is \(\ell = 2\) and \(k(n) = 1\) for all \(n\). Thus, we obtain the following condition
\[\#G_1 = \frac{1}{2} \#G = \sum_{\Theta \in G^{(2)} \setminus G_1^{(2)}} \left( P_{\Theta}(n) + P_{\Theta}(n+1) \right) \text{ for all } n \geq N.\]

Indeed, these conditions are satisfied for the three subgroups \(H_0, H_1, H_2\) of the group \(H\) and the word \(v\), see the last part of Section 8.2.

Further characterization of such group and examples of such infinite words is an open problem.

Acknowledgments

We would like to express our gratitude to the anonymous referee of this articles. His or her review helped us to improve the presentation and also remove some flaws concerning Coxeter groups. This work was supported by the Czech Science Foundation grants GAČR 201/09/0584, 13-03538S, 13-35273P and by the grant of the Grant Agency of the Czech Technical University in Prague grant No. SGS11/162/OHK4/3T/14.

References

[1] J.-P. Allouche and J. Shallit, Sums of digits, overlaps, and palindromes, Discrete Math. Theoret. Comput. Sci., 4 (2000), pp. 1–10.

[2] P. Baláži, Z. Masáková, and E. Pelantová, Factor versus palindromic complexity of uniformly recurrent infinite words, Theoret. Comput. Sci., 380 (2007), pp. 266–275.

[3] L. Balková, Factor frequencies in languages invariant under more symmetries, Kybernetika, 48 (2012), pp. 371–385.

[4] L. Balková, E. Pelantová, and Š. Starosta, Sturmian jungle (or garden?) on multiliteral alphabets, RAIRO-Theoret. Inf. Appl., 44 (2010), pp. 443–470.
[5] ———, Infinite words with finite defect, Adv. in Appl. Math., 47 (2011), pp. 562–574.

[6] ———, Proof of the Brlek-Reutenauer conjecture, to appear in Theoret. Comput. Sci., DOI: 10.1016/j.tcs.2012.12.024, (2013).

[7] A. Blondin Massé, S. Brlek, A. Garon, and S. Labbé, Combinatorial properties of f-palindromes in the Thue-Morse sequence, Pure Math. Appl., 19 (2008), pp. 39–52.

[8] S. Brlek, S. Hamel, M. Nivat, and C. Reutenauer, On the palindromic complexity of infinite words, Int. J. Found. Comput. Sci., 15 (2004), pp. 293–306.

[9] S. Brlek and C. Reutenauer, Complexity and palindromic defect of infinite words, Theoret. Comput. Sci., 412 (2011), pp. 493–497.

[10] M. Bucci and A. De Luca, On a family of morphic images of Arnoux-Rauzy words, in LATA '09: Proceedings of the 3rd International Conference on Language and Automata Theory and Applications, Berlin, Heidelberg, 2009, Springer-Verlag, pp. 259–266.

[11] M. Bucci, A. De Luca, A. Glen, and L. Q. Zamboni, A connection between palindromic and factor complexity using return words, Adv. in Appl. Math., 42 (2009), pp. 60–74.

[12] ———, A new characteristic property of rich words, Theoret. Comput. Sci., 410 (2009), pp. 2860–2863.

[13] J. Cassaigne, Complexity and special factors, Bull. Belg. Math. Soc. Simon Stevin 4, 1 (1997), pp. 67–88.

[14] D. Damanik and D. Lenz, Uniform spectral properties of one-dimensional quasicrystals, I. Absence of eigenvalues, Commun. Math. Phys., 207 (1999), pp. 687–696.

[15] A. de Luca and A. De Luca, Pseudopalindrome closure operators in free monoids, Theoret. Comput. Sci., 362 (2006), pp. 282–300.

[16] X. Droubay, J. Justin, and G. Pirillo, Episturmian words and some constructions of de Luca and Rauzy, Theoret. Comput. Sci., 255 (2001), pp. 539–553.

[17] A. Glen, J. Justin, S. Widmer, and L. Q. Zamboni, Palindromic richness, European J. Combin., 30 (2009), pp. 510–531.

[18] A. Hof, O. Knill, and B. Simon, Singular continuous spectrum for palindromic Schrödinger operators, Comm. Math. Phys., 174 (1995), pp. 149–159.

[19] L. Kari and K. Mahalingam, Watson-Crick conjugate and commutative words, in DNA Computing, M. Garzon and H. Yan, eds., vol. 4848 of Lecture Notes in Computer Science, Springer-Verlag, Berlin, Heidelberg, 2008, pp. 273–283.
[20] L. KARI AND K. MAHALINGAM, *Watson-Crick palindromes in DNA computing*, Nat. Comput., 9 (2010), pp. 297–316.

[21] K. KLOUDA, *Bispecial factors in circular non-pushy D0L languages*, Theoret. Comput. Sci., 445 (2012), pp. 63–74.

[22] E. PELANTOVÁ AND Š. STAROSTA, *Languages invariant under more symmetries: overlapping factors versus palindromic richness*, to appear in Discrete Math., preprint available at http://arxiv.org/abs/1103.4051, (2011).

[23] ———, *Almost rich words as morphic images of rich words*, Int. J. Found. Comput. Sci., 23 (2012), pp. 1067–1083.

[24] A. RESTIVO AND G. ROSONE, *Burrows-Wheeler transform and palindromic richness*, Theoret. Comput. Sci., 410 (2009), pp. 3018–3026.

[25] ———, *Balancing and clustering of words in the Burrows-Wheeler transform*, Theoret. Comput. Sci., 412 (2011), pp. 3019–3032.

[26] Š. STAROSTA, *On theta-palindromic richness*, Theoret. Comput. Sci., 412 (2011), pp. 1111–1121.

[27] ———, *Generalized Thue-Morse words and palindromic richness*, Kybernetika, 48 (2012), pp. 361–370.