FOURIER TRANSFORMS OF $C^*$-ALGEBRAS
OF NILPOTENT LIE GROUPS

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ABSTRACT. For any nilpotent Lie group we provide a description of the image of its $C^*$-algebra through its operator-valued Fourier transform.

1. INTRODUCTION

In this paper we prove that for every connected, simply connected nilpotent Lie group $G$ its $C^*$-algebra admits a finite sequence of closed two-sided ideals

$$\{0\} = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n = C^*(G) \quad (1.1)$$

with $\ast$-isomorphisms $J_j/J_{j-1} \simeq C_0(\Gamma_j, \mathcal{K}(H_j))$ for suitable locally compact spaces $\Gamma_j$ that are homeomorphic to real semi-algebraic cones, where $\Gamma_1$ is a Zariski open subset of $\mathbb{R}^k$, denoting by $k$ the codimension of generic coadjoint orbits of $G$. Here $H_j$ for $j = 1, \ldots, n$ are complex separable Hilbert spaces with $\dim H_1 = \cdots = \dim H_{n-1} = \infty$, and $\dim H_n = 1$. In particular, using a direct sum of topological spaces, one obtains a continuous bijection

$$\Gamma_1 \sqcup \cdots \sqcup \Gamma_n \to \hat{G}. \quad (1.2)$$

whose restriction to $\Gamma_j$ is a homeomorphism onto its image for $j = 1, \ldots, n$. However, the above map is never a homeomorphism if $G$ is non-abelian, since for instance it turns out that $\Gamma_j$ is a relatively dense open subset of $\Gamma_j \sqcup \Gamma_{j+1} \sqcup \cdots \sqcup \Gamma_n$ for $j = 1, \ldots, n-1$ (see Theorem 4.11 and Definition 2.8 below). As an illustration of our abstract results, we show in Section 5 that the Heisenberg group is uniquely determined in terms of the above structures.

There arises the interesting problem of understanding to what extent the above continuous bijective map (1.2) fails to be a homeomorphism. By way of investigating that problem, we describe the image of the operator-valued Fourier transform of $C^*(G)$ as a $C^*$-algebra of operator fields on $\hat{G}$, and the corresponding operator fields are determined via the boundary behavior of their restrictions to $\Gamma_j$ for $j = 1, \ldots, n$ (see Theorem 4.6 and Corollary 3.13 below). Explicit constructions for the Heisenberg, threadlike Lie groups, and all nilpotent Lie groups of dimension $\leq 6$ have been made in [LuTu11], [LuRe15] and [ReLu14].

Our results can be regarded as a sharpening of the results of [Pe84], where one proved that there exists a composition series as in (1.1), where the successive quotients are however $C^*$-algebras with continuous trace. In particular we prove that

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the Dixmier-Douady invariants of the successive quotients in such a composition series may by annihilated; see [LiRo96] for more results and background information on this circle of ideas.

2. Preliminaries

2.1. Basic notations.

**Notation 2.1.** We denote by $K(H)$ the $C^*$-algebra of all compact operators on some complex separable infinite-dimensional Hilbert space $H$. We denote by $S^p(H)$ the $p$th Schatten ideal, for $1 \leq p \leq \infty$. For any $C^*$-algebra $A$ and locally compact Hausdorff space $X$ we denote by $C_0(X, A)$ the $C^*$-algebra of continuous $A$-valued functions on $X$ which vanish at infinity.

**Lemma 2.2.** Let $A$ be any $C^*$-algebra. For any closed two-sided ideal $J$ of $A$ denote

$$\hat{A}_J := \{[\pi] \in \hat{A} \mid J \subseteq \text{Ker} \, \pi \} \simeq \hat{A}/J,$$

$$\tilde{A}_J := \{[\pi] \in \hat{A} \mid J \nsubseteq \text{Ker} \, \pi \} \simeq \tilde{J}.$$ 

Then the following assertions hold:

1. $J \mapsto \tilde{J}$ is an increasing bijection between the closed two-sided ideals of $A$ and the open subsets of $\hat{A}$.
2. $J \mapsto \hat{A}/J$ is a decreasing bijection between the closed two-sided ideals of $A$ and the closed subsets of $\hat{A}$.
3. The maps $J \mapsto \tilde{J}$ and $J \mapsto \hat{A}/J$ give rise to bijective correspondences between the following three sets:
   
   a) the $C^*$-algebra extensions $0 \to J \to A \to A/J \to 0$;
   
   b) the open subsets of $\hat{A}$;
   
   c) the closed subsets of $\hat{A}$.

**Proof.** See [Dix64] Props. 2.11.2, 3.2.2 for the first two assertions and for the bijections $\hat{A}_J \simeq \hat{A}/J$, $[\pi] \mapsto [\pi/J]$, and $\tilde{A}_J \simeq \tilde{J}$, $[\pi] \mapsto [\pi|J]$.

The third assertion is a consequence of the other ones. □

**Definition 2.3.** Let $X$ be a topological space.

i) A point $\gamma \in X$ is said to be separated in $X$ if for every $\gamma' \in X$ that is not in the closure of the set $\{\gamma\}$, there exist open subsets $V, V' \subset X$ such that $\gamma \in V$, $\gamma' \in V'$ and $V \cap V' = \emptyset$.

ii) We denote by $\text{Sep}(X)$ the collection of subsets $Y \subset X$ such that all points in $Y$ are closed and separated in $X$.

2.2. Special solvable $C^*$-algebras. We first recall a notion introduced in [Dy78] (see also [HY88] Sect. 2 for several completions).

**Definition 2.4.** A $C^*$-algebra $A$ is called solvable if it has a solving series, that is, a finite sequence of ideals $\{0\} = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n = A$ with $*$-isomorphisms $J_j/J_{j-1} \simeq C_0(\Gamma_j, K(H_j))$ for suitable locally compact spaces $\Gamma_j$ and complex Hilbert spaces $H_j$ for $j = 1, \ldots, n$, with $\dim H_1 \geq \cdots \geq \dim H_n$. If $n \geq 1$ is the least integer for which there exists a sequence of ideals as above, then $n - 1$ is called the length of the $C^*$-algebra $A$. 


Definition 2.5. A topological $\mathbb{R}$-space is a topological space $X$ endowed with a continuous map $\mathbb{R} \times X \to X$, $(t, x) \mapsto t \cdot x$, and with a distinguished point $x_0 \in X$ satisfying the following conditions:

1. For every $x \in X$ one has $0 \cdot x = x_0$.
2. For all $t, s \in \mathbb{R}$ and $x \in X$ one has $t \cdot (s \cdot x) = ts \cdot x$.
3. For every $x \in X$ the map $\mathbb{R} \to X$, $t \mapsto t \cdot x$ is a homeomorphism onto its image.

An $\mathbb{R}$-subspace of the topological $\mathbb{R}$-space $X$ is any subset $\Gamma \subseteq X$ such that $\mathbb{R} \cdot \Gamma \subseteq \Gamma \cup \{x_0\}$. If this is the case, then $\Gamma \cup \{x_0\}$ is a topological $\mathbb{R}$-space on its own.

In the above framework, a function $\varphi : X \to \mathbb{R}$ is called homogeneous if there exists $r \in [0, \infty)$ such that $\varphi(t \cdot x) = t^r \varphi(x)$ for all $t \in \mathbb{R}$ and $x \in X$.

Example 2.6. Every finite-dimensional real vector space is a topological $\mathbb{R}$-space. Moreover, if $\varphi_1, \ldots, \varphi_{n_1}, \psi_1, \ldots, \psi_{n_2} : \mathbb{R}^m \to \mathbb{R}$ are any homogeneous polynomials, then the semi-algebraic cone

$$\Gamma := \{x \in \mathbb{R}^n \mid \varphi_{j_1}(x) = 0 \neq \psi_{j_2}(x) \text{ for } 1 \leq j_1 \leq n_1 \text{ and } 1 \leq j_2 \leq n_2\}$$

is an $\mathbb{R}$-subspace of $\mathbb{R}^m$ in the sense of Definition 2.5.

As another type of examples, if $G$ is any nilpotent Lie group, then the topological $\mathbb{R}$-space structure of $g^*$ gives rise to a topological $\mathbb{R}$-space structure of the orbit space $g^*/G$ (hence also of the dual space $\hat{G}$ via Kirillov’s correspondence), and the vector space of characters $[g, g]_1^\perp$, viewed as the set of singleton orbits, is an $\mathbb{R}$-subspace of $g^*/G$. More specifically, the $\mathbb{R}$-space structure of $g^*/G$ is the map

$$\mathbb{R} \times (g^*/G) \to g^*/G, \quad (t, \mathcal{O}_\xi) \mapsto \mathcal{O}_{t\xi}$$

where we denote by $\mathcal{O}_\xi$ the coadjoint orbit of every $\xi \in g^*$.

Lemma 2.7. Let $X$ be any topological space and for $j = 1, 2$ let $V_j$ be any open subset of $X$ that is homeomorphic to an open subset of $\mathbb{R}^{r_j}$, where $r_j \geq 1$ is some integer. If $V_1 \cap V_2 \neq \emptyset$, then $r_1 = r_2$.

Proof. The nonempty open set $V := V_1 \cap V_2$ is homeomorphic to some open subsets of $\mathbb{R}^{r_1}$ and of $\mathbb{R}^{r_2}$, hence $r_1 = r_2$ by Brouwer’s theorem on the invariance of domain.

Definition 2.8. We say that $\mathcal{A}$ is a special solvable $C^*$-algebra if it is separable and it has a special solving series, that is, a solving series as in Definition 2.4 with the following additional properties:

1. $\hat{\mathcal{A}}$ has the structure of a topological $\mathbb{R}$-space and $\Gamma_j \subseteq \hat{\mathcal{A}}$ is an $\mathbb{R}$-subspace for $j = 1, \ldots, n$.
2. One has dim $\mathcal{H}_n = 1$ and $\Gamma_n$ is isomorphic as a topological $\mathbb{R}$-space to a finite-dimensional vector space.
3. For $j = 1, \ldots, n$ one has dim $\mathcal{H}_j = \infty$, the set $\Gamma_j$ is dense in $\hat{\mathcal{A}} \setminus \hat{J}_{j-1}$, and the points of $\Gamma_j$ are closed and separated in $\hat{\mathcal{A}} \setminus \hat{J}_{j-1}$.
4. For $j = 1, \ldots, n$, $\Gamma_j$ is isomorphic as a topological $\mathbb{R}$-space to a semi-algebraic cone in a finite-dimensional vector space. In addition, $\Gamma_1$ is assumed to be a Zariski open set, and the dimension of the corresponding ambient vector space is called the index of $\mathcal{A}$ and is denoted by ind $\mathcal{A}$. 


(5) For $j = 1, \ldots, n$, there exists a homogeneous function $\varphi_j: \hat{A} \to \mathbb{R}$ such that $\varphi_j |_{\Gamma_1}$ is a polynomial function (via the above homeomorphism) and

$$\Gamma_j = \{ \gamma \in \hat{A} \mid \varphi_j(\gamma) \neq 0 \text{ and } \varphi_i(\gamma) = 0 \text{ if } i < j \}.$$ 

Since $\Gamma_1$ is open and dense in $\hat{A}$, it follows by Lemma 2.7 that $\ind \mathcal{A}$ does not depend on the choice of the solving series of $\mathcal{A}$.

**Remark 2.9.** In Definition 2.8, the distinguished point of the topological $\mathbb{R}$-space $\hat{A}$ is the origin $0 \in \Gamma_n$ of the vector space $\Gamma_n$. Therefore, for $j = 1, \ldots, n$ and every $\gamma \in \Gamma_j$, one has $\mathbb{R} \cdot \gamma_1 \cap \mathbb{R} \cdot \gamma = \{0\} \subset \Gamma_n$.

**Remark 2.10.** In Definition 2.8, since $\dim \mathcal{H}_n = 1 < \dim \mathcal{H}_j$ for $1 < j \leq n$, the set $\Gamma_n$ is precisely the set of characters of $\mathcal{A}$, that is, the non-zero $*$-homomorphisms $\chi: \mathcal{A} \to \mathbb{C}$. Denoting by $\text{Comm}(\mathcal{A})$ the closed two-sided ideal of $\mathcal{A}$ generated by the set $\{ab - ba \mid a, b \in \mathcal{A}\}$, it follows that $\mathcal{A}/\text{Comm}(\mathcal{A})$ is a commutative $C^*$-algebra and for every $\chi \in \Gamma_n$, one has $\text{Comm}(\mathcal{A}) \subseteq \text{Ker} \chi$, hence $\chi$ can be identified with an element in the spectrum of $\mathcal{A}/\text{Comm}(\mathcal{A})$. Thus the Gelfand representation provides a $*$-isomorphism $\mathcal{A}/\text{Comm}(\mathcal{A}) \cong C(\Gamma_n)$, and the condition in Definition 2.8 implies that the spectrum of the commutative $C^*$-algebra $\mathcal{A}/\text{Comm}(\mathcal{A})$ is homeomorphic to a finite-dimensional real vector space, whose dimension is uniquely determined because of Brouwer’s theorem on the invariance of domain.

### 2.3. On the continuity of operator fields

The following two lemmas go back to [Rela14, Prop. 2.2, Th. 2.3].

**Lemma 2.11.** Let $\mathcal{A}$ be any $C^*$-algebra with some subset of its spectrum $\Gamma \subset \hat{A}$ such that the relative topology of $\Gamma$ is Hausdorff. Assume $\mathcal{H}$ is a complex Hilbert space and in every $\gamma \in \Gamma$ we have picked $\pi_\gamma: \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Also let $\mathcal{V}_1$ and $\mathcal{V}_2$ be any total subsets of $\mathcal{H}$.

If $a \in \mathcal{A}$ has the property that for every $v_1 \in \mathcal{V}_1$ and $v_2 \in \mathcal{V}_2$, the function $\Gamma \to \mathbb{C}$, $\gamma \mapsto \langle \pi_\gamma(a)v_1, v_2 \rangle$, is continuous, then for every $R \in \mathcal{S}_1(\mathcal{H})$ the function $f_R: \Gamma \to \mathbb{C}$, $f_R(\gamma) = \text{Tr} \left( \pi_\gamma(a)R \right)$ is continuous and bounded.

**Proof.** For every $R \in \mathcal{S}_1(\mathcal{H})$ and $\gamma \in \Gamma$ we have $|f_R(\gamma)| \leq \|R\|_1 \|a\|$, hence $f_R \in \ell^\infty(\Gamma)$ and $\|f_R\|_\infty \leq \|R\|_1 \|a\|$. Since the limit of any uniformly convergent sequence of continuous functions is in turn continuous, it then easily follows that the set $\{R \in \mathcal{S}_1(\mathcal{H}) \mid f_R \in C(\Gamma)\}$ is a closed linear subspace of $\mathcal{S}_1(\mathcal{H})$. As that closed linear subspace has a dense linear subspace consisting of rank-one operators by the hypothesis on $a$, the conclusion follows.

**Lemma 2.12.** In the setting of Lemma 2.11, assume $\mathcal{S}$ is a dense $*$-subalgebra of $\mathcal{A}$ for which every element $a \in \mathcal{S}$ satisfies the condition of Lemma 2.11, and moreover for all $\gamma \in \Gamma$ we have $\pi_\gamma \in \mathcal{S}_1(\mathcal{H})$ and the function $\Gamma \to \mathbb{C}$, $\gamma \mapsto \text{Tr} \pi_\gamma(a)$ is continuous.

Then for every $a \in \mathcal{A}$ the map $\Pi_a: \Gamma \to \mathcal{B}(\mathcal{H})$, $\gamma \mapsto \pi_\gamma(a)$, is norm continuous.

**Proof.** For arbitrary $a \in \mathcal{A}$ and $\gamma \in \Gamma$ we have $\|\Pi_a(\gamma)\| = \|\pi_\gamma(a)\| \leq \|a\|$, hence just as in the proof of Lemma 2.11 we can see that the set $\{a \in \mathcal{A} \mid \Pi_a \text{ is continuous} \}$ is a closed linear subspace of $\mathcal{A}$. Therefore it suffices to check that $\Pi_a$ is continuous for $a \in \mathcal{S}$.  

III.2.4]). On the other hand, since $J$ is any net in $\Gamma$ which is convergent to some $\gamma \in \Gamma$, then for all $j \in J$ we have
\[
\|\Pi_a(\gamma_j) - \Pi_a(\gamma)\|^2 \leq \|\Pi_a(\gamma_j) - \Pi_a(\gamma)\|^2 = \|\pi_{\gamma_j}(a) - \pi_{\gamma}(a)\|^2 = \text{Tr} \pi_{\gamma_j}(a^*a) - 2\text{Re} \text{Tr} (\pi_{\gamma_j}(a)\pi_{\gamma}(a)^*) + \text{Tr} \pi_{\gamma}(a^*a).
\]
Since $a^*a \in \mathcal{S}$, it follows by hypothesis that
\[
\lim_{j \in J} \text{Tr} \pi_{\gamma_j}(a^*a) = \text{Tr} \pi_{\gamma}(a^*a)
\]
and on the other hand Lemma 2.11 implies
\[
\lim_{j \in J} \text{Tr} (\pi_{\gamma_j}(a)\pi_{\gamma}(a)^*) = \text{Tr} (\pi_{\gamma}(a)\pi_{\gamma}(a)^*) = \text{Tr} \pi_{\gamma}(a^*a)
\]
hence by the above estimate we obtain $\lim_{\gamma \in \Gamma} \pi_{\gamma_j}(a) = \pi_{\gamma}(a)$ in $\mathcal{B}(\mathcal{H})$, which concludes the proof. \hfill \qed

3. Norm controls for the boundary values of $C^*$-Fourier transforms

3.1. Boundary values of Fourier transforms.

**Proposition 3.1.** Assume the following:

- $A$ is any separable nuclear $C^*$-algebra.
- $T \in \text{Sep}(\hat{A})$ is an open dense subset of $\hat{A}$, such that its corresponding ideal of $A$ is $*$-isomorphic to $C_0(T, \mathcal{K})$. Denote by $F_T : A \to C_0(T, \mathcal{K})$ the Fourier transform of $A$ restricted to $T$.
- $0 \to C_0(T, \mathcal{K}) \hookrightarrow A \xrightarrow{q} \mathcal{B} \to 0$ is an exact sequence of $C^*$-algebras.

Then the following assertions hold:

1. The map
   \[
   \Phi : A \to C_0(T, \mathcal{K}) \oplus \mathcal{B}, \quad \Phi(a) = F_T(a) \oplus q(a)
   \]
is an isometric $*$-homomorphism.

2. There exists a linear map $\nu : \mathcal{B} \to C_0(T, \mathcal{K})$ which is completely positive, completely isometric, almost $*$-homomorphism, and
   \[
   \text{Ran} \Phi = \left\{ f \oplus b \in C_0(T, \mathcal{K}) \oplus \mathcal{B} \mid \lim_{t \to \infty} f(t) - (\nu(b))(t) = 0 \right\}.
   \]

3. There exists a completely isometric cross section of $q$.

**Proof.** Since $T$ is the spectrum of the ideal $\mathcal{J} = C_0(T, \mathcal{K})$ of $A$, and the points of $T$ are closed and separated in $\hat{A}$ sense of [De72, Def. II.6], it follows by [De72, Prop. IV.1.3] that the Busby invariant $\gamma$ of the extension of $C^*$-algebras
\[
0 \to \mathcal{J} \hookrightarrow A \xrightarrow{q} \mathcal{B} \to 0
\]
has the range contained into $L(\mathcal{J})/\mathcal{J}$, where $L(\mathcal{J})$ is the largest two-sided closed ideal of the multiplier algebra of $\mathcal{J}$ which contains $\mathcal{J}$ and for which $\mathcal{J} \in \text{Sep}(\hat{L(\mathcal{J}))}$, that is, the points of $T = \hat{\mathcal{J}}$ are closed and separated in $\hat{L(\mathcal{J})}$ ([De72, Lemme III.2.4]). On the other hand, since $\mathcal{J} = C_0(T, \mathcal{K})$, we have $L(\mathcal{J}) = C_0(T, \mathcal{K})$ by [De72, Prop. III.4.1]. Consequently the Busby invariant of the extension of $C^*$-algebras from the statement is a $*$-homomorphism
\[
\beta : \mathcal{B} \to C_0(T, \mathcal{K})/C_0(T, \mathcal{K}).
\]
Moreover, since \( T \) was assumed to be dense in \( \hat{A} \), it follows by [Bu68, Cor. 6.4] that the \(*\)-homomorphism \( \beta \) is injective. Therefore \( \beta \) is a complete isometry and it is moreover completely positive.

On the other hand, since the \( C^* \)-algebra \( A \) is nuclear, it follows by [CE77] Cor. 4] that also its quotient \( B \) is nuclear. Also \( B \) is separable since \( A \) is. It then follows by the completely positive lifting theorem [CE76] Th. 3.10 that there exists a completely positive contraction \( \nu: B \to C_b(T, \mathcal{K}) \) with the property \( \beta(b) = \nu(b) + C_0(T, \mathcal{K}) \) for all \( b \in B \). The unital extension of \( \nu \) to the unitizations of the \( C^* \)-algebras \( B \) and \( C_0(T, \mathcal{K}) \) is completely positive by [CE70] Lemma 3.9], hence completely contractive by [BL04] 1.3.3]. Therefore \( \nu \) is both completely positive and completely contractive.

Now note that the canonical map
\[
\tilde{q}: C_b(T, \mathcal{K}) \to C_b(T, \mathcal{K})/C_0(T, \mathcal{K}), \quad f \mapsto f + C_0(T, \mathcal{K}),
\]
is a \(*\)-homomorphism, hence completely contractive, and moreover \( \tilde{q} \circ \nu = \beta \) is completely isometric. Since we have proved above that \( \nu \) is completely contractive, it then follows that \( \nu \) is actually completely isometric. Moreover, since \( \tilde{q} \circ \nu = \beta \) is a \(*\)-homomorphism, it follows that \( \nu \) is an almost \(*\)-homomorphism, which means that for all \( b_1, b_2 \in B \) we have \( \lim_{t \to \infty} (\nu(b_1 b_2) - \nu(b_1)\nu(b_2))(t) = 0 \) and \( \nu(b_1^*) = \nu(b_1)^* \).

For proving the assertion on \( \text{Ran } \Phi \), note that we have the commutative diagram
\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{F} \to & A & \xrightarrow{q} & B & \to & 0 \\
\downarrow & & \downarrow & & & \downarrow & & \\
0 & \to & C_0(T, \mathcal{K}) & \to & C_b(T, \mathcal{K}) & \xrightarrow{\tilde{q}} & C_b(T, \mathcal{K})/C_0(T, \mathcal{K}) & \to & 0
\end{array}
\]
and the fact that \( \beta \) is the Busby invariant of the extension from the upper row of the above diagram implies by [Bu68] Proof of Prop. 4.2, Th. 4.3] that we have the \(*\)-isomorphism of \( C^* \)-algebras
\[
A \simeq \{ f \oplus b \in C_b(T, \mathcal{K}) \oplus B \mid \tilde{q}(f) = \beta(b) \}, \quad a \mapsto \mathcal{F}_T(a) \oplus q(a). \tag{3.1}
\]
Note that for all \( f \oplus b \in C_b(T, \mathcal{K}) \oplus B \) we have
\[
\tilde{q}(f) = \beta(b) \iff f + C_0(T, \mathcal{K}) = \nu(b) + C_0(T, \mathcal{K}) \iff f - \nu(b) \in C_0(T, \mathcal{K})
\]
and for every \( f \in C_b(T, \mathcal{K}) \) there exists at most one \( b \in B \) satisfying the above condition, since if we have \( f - \nu(b_1), f - \nu(b_2) \in C_0(T, \mathcal{K}) \), then
\[
\beta(b_1) = \nu(b_1) + C_0(T, \mathcal{K}) = \nu(b_2) + C_0(T, \mathcal{K}) = \beta(b_2)
\]
hence \( b_1 = b_2 \).

Finally, since \( \nu \) is completely isometric, it follows that the map
\[
\iota: B \to C_b(T, \mathcal{K}) \oplus B, \quad b \mapsto \nu(b) \oplus b
\]
is completely isometric and its image is contained in the image of \( A \) by the \(*\)-isomorphism \( \tilde{q} \). By composing the inverse of that \(*\)-isomorphism with \( \iota \) we obtain a completely isometric cross section of \( q: A \to B \), and this completes the proof. \( \square \)

**Remark 3.2.** If we have a short exact sequence \( 0 \to \mathcal{F} \to A \to B \to 0 \) where \( A \) is any \( C^* \)-algebra of type I, then it follows that also its quotient \( B \) is of type I, and both \( A \) and \( B \) are nuclear by [Ta03] Prop. 1.6, Ch. XV].
Remark 3.3. For any $C^*$-algebra $A$ the locally closed subsets of $\hat{A}$ are precisely the spectra of subquotients of $A$.

In order to see this, first recall that locally closed subset means any set of the form $F \cap D$, where $F$ is any closed subset of $\hat{A}$ while $D$ is any open subset of $\hat{A}$. Moreover, a subquotient of $A$ is any $C^*$-algebra of the form $J_2/J_1$, where $J_1 \subseteq J_2$ are any closed two-sided ideals of $A$. For any such a pair of ideals, it follows by Lemma 2.2 that $D := J_1$ is an open subset of $\hat{A}$ and $J_2/J_1$ is a closed subset of the open set $\hat{J}_1$, hence it is easily checked that the disjoint union $F := J_2/J_1 \cup (\hat{A} \setminus \hat{J}_1)$ is a closed subset of $\hat{A}$ and $F \cap D = J_2/J_1$ is locally closed. Conversely, for any locally closed subset $F \cap D \subseteq \hat{A}$ we have $F \cap D = D \setminus (D \setminus F)$, where $D \setminus F = D \cap (\hat{A} \setminus F)$ is an open subset of $D$. Hence by Lemma 2.2 there exist uniquely determined two-sided closed ideals $J_1 \subseteq J_2$ of $A$ with $\hat{J}_1 = D \setminus F$ and $\hat{J}_2 = D$. Moreover, since $J_1$ is in particular an ideal of $J_2$, it follows by Lemma 2.2 again that $\hat{J}_2/\hat{J}_1 = D \setminus (D \setminus F) = D \cap F$. See [Ph87 Lemma 7.3.5] for the fact that the $*$-isomorphism class of the $C^*$-algebra $J_2/J_1$ depends only on the locally closed set $\hat{J}_2 \setminus \hat{J}_1$.

Lemma 3.4. Let $A$ be any liminary separable $C^*$-algebra and consider any locally closed subset $\Gamma \subseteq \hat{A}$ for which the relative topology of $\Gamma$ is Hausdorff, and let $J_1 \subseteq J_2$ be two-sided closed ideals of $A$ with $J_2/J_1 = \Gamma$ (see Remark 3.3). Then the following properties are equivalent:

1. The canonical field of elementary $C^*$-algebras defined by $J_2/J_1$ on $\Gamma$ is trivial.

2. There exist a complex Hilbert space $H$ and a complete system of distinct representatives $\{\pi_\gamma : A \to B(H)\}_{\gamma \in \Gamma}$ of the equivalence classes of representations corresponding to the elements of $\Gamma$ such that for every $a \in J_2$ the mapping $\Gamma \to B(H), \gamma \mapsto \pi_\gamma(a)$ is continuous with respect to the norm operator topology of $B(H)$.

Proof. The implication (1) \(\Rightarrow\) (2) is clear, so we are left to proving only the converse implication.

By Lemma 2.2 we may replace $A$ by its subquotient $J_2/J_1$. Thus we may assume $J_1 = \{0\}$ and $J_2 = A$, hence $\Gamma = \hat{A}$. Then the hypothesis (2) shows that the continuous sections of the canonical field of elementary $C^*$-algebras defined by $A$ (which is a continuous field of $C^*$-algebras by [Dix64, 10.5.1] since $\Gamma = \hat{A}$ is Hausdorff) are also continuous sections of the trivial field with the fiber $K(H)$ over $\hat{A}$. Now [Dix64 Prop. 10.2.4] ensures that the two aforementioned continuous fields of $C^*$-algebras are isomorphic, and in particular the canonical field of elementary $C^*$-algebras defined by $A$ is trivial. \(\square\)

Definition 3.5. Let $A$ be a $C^*$-algebra with spectrum $\hat{A}$. We choose for every $\gamma \in \hat{A}$ a representation $(\pi_\gamma, H_\gamma)$ in the equivalence class $\gamma$. Let $\ell^\infty(\hat{A})$ be the algebra of all bounded operator fields defined over $\hat{A}$ by

$$\ell^\infty(\hat{A}) := \left\{ \phi = (\phi(\pi_\gamma) \in B(H_\gamma))_{\gamma \in \hat{A}} \mid ||\phi||_{\infty} := \sup_{\gamma} ||\phi(\pi_\gamma)||_{B(H_\gamma)} < \infty \right\} .$$

We define for $a \in A$ its Fourier transform $F_A(a) = \hat{a}$ by

$$F_A(a)(\gamma) = \hat{a}(\gamma) = \pi_\gamma(a), \quad \gamma \in \hat{A}.$$
Then $F_A(a)$ is a bounded field of operators over $\hat{A}$, and the mapping
\[ F_A: A \to \ell^\infty(\hat{A}), \quad a \mapsto F_A(a) \]
is an isometric $*$-homomorphism.

**Theorem 3.6.** Let $A$ be any separable liminary $C^*$-algebra with an increasing sequence of open subsets of its spectrum
\[ \emptyset = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \hat{A} \]
with the corresponding sequence of closed two-sided ideals of $A$
\[ \{0\} = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n = A \]
with $\overline{J}_\ell = V_\ell$ satisfying the following conditions for $\ell = 1, \ldots, n$:
1. The set $\Gamma_\ell := V_\ell \setminus V_{\ell-1}$ is dense in $\hat{A} \setminus V_{\ell-1}$.
2. There exist a complex Hilbert space $H_\ell$ and a complete system of distinct representatives $\{\pi_\gamma: A \to B(H_\ell)\}_{\gamma \in \Gamma_\ell}$ of the equivalence classes of representations corresponding to the elements of $\Gamma_\ell$ such that for every $a \in A$ the mapping $\Gamma_\ell \to K(H_\ell)$, $\gamma \mapsto \pi_\gamma(a)$ is continuous with respect to the norm operator topology of $K(H_\ell)$.

For $\ell = 0, \ldots, n$ we define
\[ \mathcal{L}_\ell := \{f: \hat{A} \setminus V_\ell \to \bigoplus_{j=\ell+1}^n H_j | (\forall j \in \{\ell + 1, \ldots, n\})(\forall \gamma \in \Gamma_j) f(\gamma) \in \mathcal{K}(H_j) \} \]
and
\[ F_{A/J_{\ell}}: A/J_{\ell} \to \mathcal{L}_\ell, \quad (F_{A/J_{\ell}}(a + J_{\ell}))(\gamma) := \pi_\gamma(a). \]
Then there is a family of linear maps $\{\nu_\ell: F_{A/J_{\ell}}(A/J_{\ell}) \to \mathcal{C}_0(\Gamma_\ell, \mathcal{K}(H_\ell))\}_{1 \leq \ell \leq n}$, which are completely positive, completely isometric, almost $*$-homomorphisms, and for which the image $F_A(A)$ is precisely the set of all $f \in \mathcal{L}$ such that
\[ f|_{\Gamma_\ell} = \nu_\ell(f|_{\hat{A} \setminus V_\ell}) \in \mathcal{C}_0(\Gamma_\ell, \mathcal{K}(H_\ell)) \quad (3.2) \]
for all $\ell \in \{1, \ldots, n-1\}$.

**Remark 3.7.** Under the assumptions in Theorem 3.6 for all $a \in A$ we have
\[ (F_{A/J_{\ell-1}}(a + J_{\ell-1}))(\gamma) = \pi_\gamma(a) = F_A(a)(\gamma) \in \mathcal{K}(H_\ell) \]
for all $\gamma \in \hat{A} \setminus V_{\ell-1}$. This follows from [Dix64] Prop. 2.10.4 and the fact that
\[ \hat{A} \setminus V_{\ell-1} = \hat{A}/\overline{J}_{\ell-1} = \{[\pi] \in \hat{A} | J_{\ell-1} \subseteq \text{Ker} \pi\}. \]

**Proof of Theorem 3.6.** Recall from Lemma 2.2 and Remark 3.3 that $\overline{J}_\ell/\overline{J}_{\ell-1} = V_\ell \setminus V_{\ell-1}$, hence Lemma 3.3 implies that the canonical field of elementary $C^*$-algebras defined by $J_\ell/J_{\ell-1}$ is trivial. Thus we obtain an isometric $*$-isomorphism $J_\ell/J_{\ell-1} \simeq \mathcal{C}_0(V_\ell \setminus V_{\ell-1}, \mathcal{K}(H_\ell))$ by the Fourier transform. Then the canonical short exact sequence
\[ 0 \to J_\ell/J_{\ell-1} \to A/J_{\ell-1} \to A/J_\ell \to 0 \]
leads to a short exact sequence
\[ 0 \to \mathcal{C}_0(V_\ell \setminus V_{\ell-1}, \mathcal{K}(H_\ell)) \to \text{Ran} F_{A/J_{\ell-1}} \to \text{Ran} F_{A/J_\ell} \to 0. \]
Here the second arrow is simply the inclusion, while the third is the restriction to $\hat{A} \setminus V_\ell$. 
Note that the set $\Gamma_\ell := V_\ell \setminus V_{\ell-1}$ is in Sep($\tilde{A} \setminus V_{\ell-1}$). Indeed, the all points of $\tilde{A}$ are closed since $A$ are liminary. In addition, the function $|\pi| \mapsto \|\pi(a)\|$ is continuous on $\Gamma_\ell$ for every $a \in A$ by (2). This implies that every point of the open set $\Gamma_\ell$ separated in $\tilde{A}$ (see [Dix61, pag. 116, (iii)] or the proof of [Fe60, Th. 2.1]). Then by Proposition 3.1 we obtain a completely positive, completely contractive, almost $\ast$-homomorphism $\nu_\ell$: $\text{Ran} \mathcal{F}_{A/j_{\ell-1}} \to C_0(\Gamma_\ell, \mathcal{K}(H_\ell))$ with

$$
\text{Ran} \mathcal{F}_{A/j_{\ell-1}} = \left\{ f \oplus g \in C_b(\Gamma_\ell, \mathcal{K}(H_\ell)) \oplus \text{Ran} \mathcal{F}_{A/j_{\ell-1}} \mid f - \nu_\ell(b) \in C_0(\Gamma_\ell, \mathcal{K}(H_\ell)) \right\},
$$

We have that

$$f \oplus g|_{A \setminus V_\ell} = b, \quad f \oplus g|_{\Gamma_\ell} = f,$$

hence when $a \in A$

$$\mathcal{F}_{A/j_{\ell-1}}(a + J_{\ell-1})|_{\Gamma_\ell} - \nu_\ell(\mathcal{F}_{A/j_{\ell-1}}(a + J_{\ell-1})|_{A \setminus V_\ell}) \in C_0(\Gamma_\ell, \mathcal{K}(H_\ell)).$$

On the other hand, for all $\gamma \in \Gamma_\ell$ and $a \in A$ we have

$$(\mathcal{F}_{A/j_{\ell-1}}(a + J_{\ell-1})|_{\Gamma_\ell})(\gamma) = \pi_\gamma(a) = \mathcal{F}_A(a)(\gamma) \in \mathcal{K}(H_\ell).$$

Using now the above description of $\text{Ran} \mathcal{F}_{A/j_{\ell-1}}$, we see that $\mathcal{F}_A(a)$ satisfies (3.2).

Then, by successively considering the cases $\ell = n, n-1, \ldots, 1$, we obtain the description of $\text{Ran} \mathcal{F}_{A/j_{1-1}}$, that for $\ell = 1$ concludes the proof. \hfill \Box

### 3.2. $C^*$-algebras with norm controlled dual limits.

**Definition 3.8** (see also [LuRe15]). i) Let $S$ be a topological space. We say that $S$ is **locally compact of step $\leq d$** if there exists a finite increasing family $\emptyset \neq S_d \subset S_{d-1} \subset \cdots \subset S_0 = S$ of closed subsets of $S$, such that the subsets $\Gamma_d = S_d$ and $\Gamma_i := S_{i-1} \setminus S_i$, $i = 1, \ldots, d$, are locally compact and Hausdorff in their relative topologies.

ii) Let $S$ be locally compact of step $\leq d$, and let $\{\mathcal{H}_i\}_{i=1, \ldots, d}$ be Hilbert spaces. For a closed subset $M \subset S$, denote by $CB(M)$ the unital $C^*$-algebra of all uniformly bounded operator fields $f(\gamma) \in \mathcal{B}((\mathcal{H}_i))_{\gamma \in \Gamma_d, i=1, \ldots, d}$, which are operator norm continuous on the subsets $\Gamma_i \cap M$ for every $i \in \{1, \ldots, d\}$ with $\Gamma_i \cap M \neq \emptyset$. We provide the algebra $CB(M)$ with the infinity-norm

$$\|\varphi\|_M = \sup \left\{ \|\varphi(\gamma)\|_{\mathcal{B}(\mathcal{H}_i)} \mid M \cap \Gamma_i \neq \emptyset, \gamma \in \Gamma_i \cap \Gamma_j \right\}.$$

**Definition 3.9** (see also [LuRe15]). Let $A$ be a separable liminary $C^*$-algebra. We assume that the spectrum $\tilde{A}$ of $A$ is a locally compact space of step $\leq d$,

$$\emptyset = S_{d+1} \subset S_d \subset S_{d-1} \subset \cdots \subset S_0 = \tilde{A},$$

and that for $0 \leq i \leq d$ there is a Hilbert space $\mathcal{H}_i$, and for every $\gamma \in \Gamma_i$ a concrete realization $(\pi_\gamma, \mathcal{H}_i)$ of $\gamma$ on the Hilbert space $\mathcal{H}_i$. The set $S_d$ is the collection of all characters of $A$.

We say that the $C^*$-algebra $A$ has **norm controlled dual limits** if for every $a \in A$ one has

1. The mappings $\gamma \to \mathcal{F}(a)(\gamma)$ are norm continuous on the difference sets $\Gamma_i = S_{i-1} \setminus S_i$. 
For any \(i = 0, \ldots, d + 1\) and for any converging sequence contained in \(\Gamma_i\) with limit set outside \(\Gamma_i\), hence in \(S_i\), there exists a properly converging subsequence \(\gamma = (\gamma_k)_{k \in \mathbb{N}}\), a constant \(C > 0\) and for every \(k \in \mathbb{N}\) an involutive linear mapping \(\tilde{\sigma}_{\gamma,k} : CB(S_i) \to \mathcal{B}(H_i)\), that is bounded by \(C\|\cdot\|_{S_i}\), such that

\[
(\forall a \in A) \lim_{k \to \infty} \|F(a)(\gamma_k) - \tilde{\sigma}_{\gamma,k}(F(a)|_{S_i})\|_{\mathcal{B}(H_i)} = 0.
\]

**Theorem 3.10.** Assume the \(C^*\)-algebra \(A\) satisfies the conditions Theorem 3.6. Then \(A\) has norm controlled dual limits. Specifically, with the notations of Theorem 3.6, let \(1 \leq \ell \leq n\) be fixed, and \(\gamma = (\gamma_k)_{k \in \mathbb{N}}\) be a properly convergent sequence with limit set outside \(\Gamma_\ell\). Then there exists a sequence \((\sigma_{\gamma,k})_{k \in \mathbb{N}}\) of completely positive and completely contractive maps \(\sigma_{\gamma,k} : CB(\mathbb{A} \setminus V_\ell) \to \mathcal{B}(H_\ell)\) such that

\[
(\forall a \in A) \lim_{k \to \infty} \|F(a)(\gamma_k) - \sigma_{\gamma,k}(F(a)|_{\mathbb{A} \setminus V_\ell})\|_{\mathcal{B}(H_\ell)} = 0.
\]

**Proof.** From Theorem 3.6 we see that there is a linear map

\[
\nu_\ell : \text{Ran} \, \mathcal{F}_{\mathbb{A} \setminus J_\ell} \to C_0(\Gamma_\ell, \mathcal{K}(H_\ell))
\]

that is completely positive, completely isometric and almost \(*\)-homomorphism, and such that

\[
(\forall a \in A) \quad F(a)|_{\Gamma_\ell} - \nu_\ell(F(a)|_{\mathbb{A} \setminus V_\ell}) \in C_0(\Gamma_\ell, \mathcal{K}(H_\ell)). \quad (3.3)
\]

Define now \(\tilde{\sigma}_{\gamma,k} : \mathcal{F}_{\mathbb{A} \setminus J_\ell}(A/\mathcal{J}_\ell) \to \mathcal{B}(H_\ell)\),

\[
\tilde{\sigma}_{\gamma,k}(\cdot) := (\nu_\ell(\cdot))(\gamma_k),
\]

for every \(k \in \mathbb{N}\). These maps are completely positive, and by [Ar69, Thm. 1.2.3] and [CE76, Lemma 3.9], they extend from the \(C^*\)-subalgebra \(\text{Ran} \, \mathcal{F}_{\mathbb{A} \setminus J_\ell} \subset CB(\mathbb{A} \setminus V_\ell)\) to completely positive and completely contractive linear maps

\[
\sigma_{\gamma,k} : CB(\mathbb{A} \setminus V_\ell) \to \mathcal{B}(H_\ell).
\]

Due to the properties of \(\nu_\ell\), and using (3.3) via [Re62, rem. II, pag. 474] which implies \(\gamma_k \to \infty\) in \(\Gamma_\ell\), the sequence \((\sigma_{\gamma,k})_{k \in \mathbb{N}}\) satisfies all the properties in the statement. \(\square\)

3.3. Fourier transforms of \(C^*\)-algebras with norm controlled dual limits.

**Definition 3.11** (see also [LuRe15]). Let \(\emptyset = S_{d+1} \subset S_d \subset \cdots \subset S_0 = S\) be a locally compact topological space of step \(\leq d\). Choose for every \(i = 1, \ldots, d\) a Hilbert space \(H_i\) and assume that \(H_d = \mathbb{C}\).

Let \(B^*(S)\) be the set of all operator fields \(\varphi\) defined over \(S\) such that

1. \(\varphi(\gamma) \in \mathcal{K}(H_i)\) for every \(\gamma \in \Gamma_i = S_{i-1} \setminus S_i, i = 1, \ldots, d\).
2. The field \(\varphi\) is uniformly bounded, that is, we have that

\[
\|\varphi\| = \sup \{ \|\varphi(\gamma)\|_{\mathcal{B}(H_i)} \mid \gamma \in \Gamma_i, i = 1, \ldots, d\} < \infty.
\]

3. The mappings \(\gamma \to \varphi(\gamma)\) are norm continuous on the difference sets \(\Gamma_i\).
4. We have for any sequence \((\gamma_k)_{k \in \mathbb{N}} \subset S\) going to infinity, that

\[
\lim_{k \to \infty} \|\varphi(\gamma_k)\|_{\text{op}} = 0.
\]
For any \( i = 1, \ldots, d + 1 \), and for any converging sequence contained in \( \Gamma_i = S_{i-1} \setminus S_i \) with limit set outside \( \Gamma_i \), there exists a properly converging sub-sequence \( \gamma_i = (\gamma_k)_{k \in \mathbb{N}} \), a constant \( C > 0 \) and for every \( k \in \mathbb{N} \) an involutive linear mapping \( \delta_{\gamma,k} : CB(S_i) \to B(\mathcal{H}_i) \), which is bounded by \( C\|\cdot\|_{S_i} \), such that
\[
\lim_{k \to \infty} \|\varphi(\gamma_k) - \delta_{\gamma,k}(\varphi|_{S_i})\|_{B(\mathcal{H}_i)} = 0.
\]

**Theorem 3.12** (see also [LuRe15]). Let \( S \) be a locally compact topological space of step \( \leq d \). Then the set \( B^*(S) \) of Definition 3.11 is a closed involutive subspace of \( \ell^\infty(S) \). Furthermore \( B^*(S) \) is a \( C^* \)-subalgebra of \( \ell^\infty(S) \) with spectrum \( S \) if and only if all the mappings \( \delta_{\gamma,k} \) are almost homomorphisms, i.e.,
\[
\lim_{k \to \infty} \|\delta_{\gamma,k}(\varphi \cdot \psi) - \delta_{\gamma,k}(\varphi) \cdot \delta_{\gamma,k}(\psi)\|_{B(\mathcal{H}_i)} = 0, \quad \varphi, \psi \in B^*(S),
\]
and the restrictions \( B^*(S)|_{S_{i-1}} \) contain the spaces \( C_0(\Gamma_i, \mathcal{H}_i), i = 1, \ldots, d + 1 \).

**Proof.** We easily see that the conditions (11) - (14) in Definition 3.11 imply that \( B^*(S) \) is a closed involutive-invariant subspace of \( \ell^\infty(S) \).

For \( i = 1, \ldots, d + 1 \), let \( B_i^* \) be the set of all operator fields defined over \( S_{i-1} \), satisfying conditions (11-14) on the sets \( S_j, j = d, \ldots, i - 1 \). Then obviously for every \( i \) the restriction \( B^*(S)|_{S_{i-1}} \) of the space \( B^*(S) \) to \( S_{i-1} \) is contained in \( B_i^* \) and \( B^*(S) \) is also an algebra and hence a \( C^* \)-subalgebra of \( \ell^\infty(S) \).

Let us show that the spectrum of \( B^*(S) \) can be identified with the space \( S \). Since \( C_0(\Gamma_i, \mathcal{H}_i) \) is contained in \( B_i^* \) for every \( i \), it follows that the representations \( \pi_s : \varphi \to \varphi(s) \in \mathcal{K}(\mathcal{H}_i) \) of \( B^*(S) \) are irreducible. It follows from the choice of \( \mathcal{H}_d \) and the properties of \( B^*(S) \), that \( B_d^* = B^*(S)|_{S_d} = C_0(S_d) \).

Suppose that for some \( 0 \leq i < d \) the spectrum of the algebra \( B^*_i \) is the space \( S_i \). Let \( \pi \in \hat{B}^*_i \). Consider the kernel \( K_{i+1} \) of the restriction mapping \( R_{i+1} \) from \( B^*_i \) into \( \ell^\infty(S_i) \). If \( \pi(K_{i+1}) = 0 \), then we can consider \( \pi \) as being a representation of the quotient algebra \( B^*_i / K_{i+1} \). But the image \( B^*(S)|_{S_i} \) of \( R_{i+1} \) is a \( C^* \)-subalgebra of \( B^*_i \), the spectrum of \( B^*_i \) is by assumption the set \( S_i \) and \( S_i \) is also contained in the spectrum of the subalgebra \( B^*(S)|_{S_i} \). Hence by the Stone-Weierstrass theorem [Dix64, Th. 11.1.8] the algebras \( B_{i+1}^* \) and \( B^*(S)|_{S_i} \) coincide. Hence \( \pi \) is an evaluation at a point in \( S_i \).

If \( \pi(K_{i+1}) \neq 0 \), then we look at the restriction of \( \pi \) to this ideal. The elements in \( K_{i+1} \) are operator fields defined on \( S_{i-1} \) which are norm continuous, which go to 0 at infinity and by condition (15) in Definition 3.11 for any properly converging sequence \( \gamma_i \) of \( \Gamma_i \) with limit outside \( \Gamma_i \), for every \( \varphi \in K_{i+1} \), we have that
\[
\lim_{k \to \infty} \|\varphi(\gamma_k)|_{B(\mathcal{H}_i)} = \lim_{k \to \infty} \|\varphi(\gamma_k) - \delta_{\gamma,k}(\varphi|_{S_i})\|_{B(\mathcal{H}_i)} = 0.
\]
This shows that \( K_{i+1} \subset C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) \). Since by condition (15) in Definition 3.11 we know that \( C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) \) is \( K_{i+1} \) it follows that \( C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) = K_{i+1} \) and then \( \pi \) is an evaluation at an element \( s \in \Gamma_i \). Finally the spectrum of \( B_i^* \) is the set \( S_{i-1} \). Therefore again by the aforementioned Stone-Weierstrass theorem the algebras \( B_i^* \) and \( B^*(S)|_{S_{i-1}} \) coincide.

If \( B^*(S) \) is \( C^* \)-algebra, then obviously the conditions of Definition 3.11 are fulfilled. 

**Corollary 3.13.** Let \( \mathcal{A} \) be a \( C^* \)-algebra with norm controlled dual limits. Then the Fourier transform of \( \mathcal{A} \) is the \( C^* \)-algebra \( B^*(\mathcal{A}) \).
Proof. Use Theorem [3.12] □

Remark 3.14. Let $\mathcal{A}$ be a separable liminary $C^*$-algebra, let $\pi = \{[(\pi_k, \mathcal{H}_k)]\}_{k \in \mathbb{N}}$ be a properly converging sequence in $\hat{\mathcal{A}}$ with limit set $L$. Let $\pi: \mathcal{A} \to B(\bigoplus \mathcal{H}_k)$ be the corresponding $*$-homomorphism, i.e., $\pi(a) = (\pi_0(a), \pi_1(a), \pi_2(a), \ldots)$, $a \in \mathcal{A}$. Let $K_\pi$ be its kernel. Let $\mathcal{N}_{\pi}$ be the closed two-sided ideal of $\mathcal{A}$ consisting of all $a \in \mathcal{A}$ for which $\lim_k \|\pi_k(a)\|_{\text{op}} = 0$. Also let $K_L = \bigcap \ker \rho$ be the kernel of the closed subset $L \subset \hat{\mathcal{A}}$. By the theorem of Fell [Fe60, Th. 2.1] (see also Lemma A.3 below) we have

\[
(\forall a \in \mathcal{A}) \quad \lim_{k \to \infty} \|\pi_k(a)\|_{\text{op}} = \sup_{\rho \in L} \|\rho(a)\|_{\text{op}}
\]

hence $K_L = \mathcal{N}_{\pi}$. This observation allows us to define the homomorphism

\[
\hat{\pi}: \mathcal{A}/K_L \simeq (\mathcal{A}/K_{\pi})/(K_L/K_{\pi}) \to \mathcal{P}(\mathcal{A})/\mathcal{P}(\mathcal{N}_{\pi}).
\]

By the lifting theorem of Choi and Effros (CE76 Thm. 3.10)), there exists a completely positive linear mapping $\psi: \mathcal{A}/K_L \to \mathcal{P}(\mathcal{A})$, such that for every $a \in \mathcal{A}$:

\[
\psi(a + K_L) + \pi(\mathcal{N}_{\pi}) = \pi(a + K_L).
\]

The operator valued Fourier transform

\[
\mathcal{F}(a)(\pi) = \hat{a}(\pi) = \pi(a), \quad a \in \mathcal{A}, \quad \pi \in \hat{\mathcal{A}},
\]

maps $\mathcal{A}/K_L$ isometrically onto the algebra of operator fields $\mathcal{F}(\mathcal{A})|_L = \{\hat{a}|_L \mid a \in \mathcal{A}\}$. We obtain in this way a completely positive linear map $\hat{\sigma}: \mathcal{F}(\mathcal{A})|_L \to \mathcal{P}(\mathcal{A})$,

\[
\hat{\sigma}(\hat{a}|_L) = \psi(a + K_L), \quad a \in \mathcal{A},
\]

which satisfies the relation $\pi(a) - \hat{\sigma}(\hat{a}|_L) \in \pi(\mathcal{N}_{\pi})$, i.e., if we write $\hat{\sigma}(\hat{a}|_L)$ as the sequence

\[
\hat{\sigma}(\hat{a}|_L) = (\hat{\sigma}_k(\hat{a}|_L) \in \mathcal{K}(\mathcal{H}_k))_k
\]

then we have that

\[
\lim_{k \to \infty} \|\hat{\sigma}_k(\hat{a}|_L) - \pi_k(a)\|_{\text{op}} = 0.
\]

Hence this “norm control” $\hat{\sigma}$ always exists in an abstract way. The problem is how to find a precise expression for the mappings $\hat{\sigma}_k$, whenever the algebra $\mathcal{A}$ and the representations $\pi_k$ are concretely given.

4. **Application to $C^*$-algebras of nilpotent Lie groups**

Throughout this section we denote by $\mathfrak{g}$ any nilpotent Lie algebra with its corresponding Lie group $G = (\mathfrak{g}, \cdot)$ and with a fixed Jordan-Hölder sequence

\[
\{0\} = \mathfrak{g}_0 \subseteq \cdots \subseteq \mathfrak{g}_m = \mathfrak{g}
\]

and we pick $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$ for $j = 1, \ldots, m$. We denote by

\[
\langle \cdot, \cdot \rangle: \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}
\]

the duality pairing between $\mathfrak{g}$ and its linear dual space $\mathfrak{g}^*$, and for every subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ we define

\[
(\forall \xi \in \mathfrak{g}^*) \quad \mathfrak{h}(\xi) := \{X \in \mathfrak{h} \mid (\forall Y \in \mathfrak{h}) \langle \xi, [X, Y] \rangle = 0\}.
\]

We denote by $\mathcal{E}$ the set of all finite subsets of $\{1, \ldots, m\}$ endowed with the total ordering defined for all $e_1 \neq e_2$ in $\mathcal{E}$ by

\[
eq e_2 \iff \min(e_1 \setminus e_2) < \min(e_2 \setminus e_1)
\]
where we use the convention \( \min \emptyset = \infty \), so in particular \( \max \mathcal{E} = \emptyset \). This is precisely the ordering introduced in \cite{Pe94} subsect. 3.4, pag. 454. We also endow the \( m \)-th Cartesian power \( \mathcal{E}^m \) with the total ordering obtained from the above ordering \( \prec \) as in \cite{Pe94} Def. 1.2.5.

4.1. **Coarse stratification of \( \mathfrak{g}^* \) and continuity of trace on \( \hat{\mathcal{G}} \).** We define the jump indices

\[
(\forall \xi \in \mathfrak{g}^*) \quad J_{\xi} := \{ j \in \{1, \ldots, m \} \mid g_j \not\subset g(\xi) + g_{j-1} \}
\]

and

\[
(\forall e \in \mathcal{E}) \quad \Omega_e := \{ \xi \in \mathfrak{g}^* \mid J_{\xi} = e \}.
\]

The **coarse stratification of \( \mathfrak{g}^* \)** is the family \( \{\Omega_e\}_{e \in \mathcal{E}} \), which is a finite partition of \( \mathfrak{g} \) consisting of \( \mathcal{G} \)-invariant sets. For every coadjoint \( \mathcal{G} \)-orbit \( O \in \mathfrak{g}^*/\mathcal{G} \) we define \( J_O := J_{\xi} \) for any \( \xi \in O \) and then

\[
(\forall e \in \mathcal{E}) \quad \Xi_e := \{ O \in \mathfrak{g}^*/\mathcal{G} \mid J_O = e \}.
\]

**Lemma 4.1.** Assume the above setting and for every \( O \in \mathfrak{g}^*/\mathcal{G} \) pick any unitary irreducible representation \( \pi_O : \mathcal{G} \to \mathcal{B}(\mathcal{H}_O) \) which is associated with \( O \) via Kirillov’s correspondence. If we endow the space \( \mathfrak{g}^*/\mathcal{G} \simeq \hat{\mathcal{G}} \) with its canonical topology, then for every index set \( e \in \mathcal{E} \) the following assertions hold:

1. The relative topology of \( \Xi_e \subseteq \mathfrak{g}^*/\mathcal{G} \) is Hausdorff.
2. For every test function \( \phi \in C_0(\hat{\mathcal{G}}) \) the function

\[
\Xi_e \to \mathbb{C}, \quad O \mapsto \text{Tr} (\pi_O(\phi))
\]

is well defined and continuous.

**Proof.** The first assertion follows as an application of \cite{CG90} Th. 3.1.14(iv)] for the coadjoint action of \( \mathcal{G} \), which provides a homeomorphism of \( \Xi_e \) onto a certain algebraic subset of the vector space span \( \{X_j \mid j \in \{1, \ldots, m\} \setminus e\} \).

The second assertion is just \cite{Pe84} Lemma 4.4.4. \( \square \)

4.2. **Fine stratification of \( \mathfrak{g}^* \) and continuity of operator fields on \( \hat{\mathcal{G}} \).** We define

\[
(\forall \xi \in \mathfrak{g}^*) (\forall k = 1, \ldots, m) \quad J^{k}_{\xi} := \{ j \in \{1, \ldots, k \} \mid g_j \not\subset g_k(\xi) + g_{j-1} \}
\]

and

\[
(\forall e \in \mathcal{E}^m) \quad \Omega_{e} := \{ \xi \in \mathfrak{g}^* \mid (J^{1}_{\xi}, \ldots, J^{m}_{\xi}) = e \}.
\]

The **fine stratification of \( \mathfrak{g}^* \)** is the family \( \{\Omega_{e}\}_{e \in \mathcal{E}^m} \), which is again a finite partition of \( \mathfrak{g} \) consisting of \( \mathcal{G} \)-invariant sets. For every coadjoint \( \mathcal{G} \)-orbit \( O \in \mathfrak{g}^*/\mathcal{G} \) we define \( J^{k}_{O} := J^{k}_{\xi} \) for any \( \xi \in O \) and \( k = 1, \ldots, m \), and then

\[
(\forall e \in \mathcal{E}^m) \quad \Xi_{e} := \{ O \in \mathfrak{g}^*/\mathcal{G} \mid (J^{1}_{O}, \ldots, J^{m}_{O}) = e \}.
\]

For every \( e \in \mathcal{E}^m \) we also define \( \Gamma_{e} \subseteq \hat{\mathcal{G}} \) as the image of \( \Xi_{e} \) through Kirillov’s correspondence \( \mathfrak{g}^*/\mathcal{G} \simeq \hat{\mathcal{G}} \), which is actually a homeomorphism.

**Remark 4.2.** Every stratum of the coarse stratification of \( \mathfrak{g}^* \) is the disjoint union of a few strata of the fine stratification of \( \mathfrak{g}^* \). More precisely, since \( \mathfrak{g}_m = \mathfrak{g} \), we have

\[
(\forall e \in \mathcal{E}) \quad \Omega_{e} = \{ \xi \in \mathfrak{g}^* \mid J^{m}_{\xi} = e \} = \bigsqcup_{e \in \mathcal{E}^{m-1} \times \{e\}} \Omega_{e}.
\]
Lemma 4.3. For every $\varepsilon \in E^m$ there exists a Hilbert space $H_\varepsilon$ and every equivalence class of representations $\gamma \in \Gamma_\varepsilon$ has a representative $\pi_\varepsilon : G \to B(H_\varepsilon)$ with the property that for every $a \in C^*(G)$ the map $\Pi_\varepsilon : \Gamma_\varepsilon \to B(H_\varepsilon)$, $\gamma \mapsto \pi_\varepsilon(a)$, is norm continuous.

Proof. By using [LiRo96, Cor. 2.15], we obtain a Hilbert space $H_\varepsilon = L^2(\mathbb{R}^d)$ and for every $\gamma \in \Gamma_\varepsilon$ a representative $\pi_\varepsilon : G \to B(H_\varepsilon)$ with the property that for all $\phi \in C^\infty_0(G)$, $f_1 \in C^\infty_0(\mathbb{R}^d)$, and $f_2 \in C^\infty_0(\mathbb{R}^d)$ the function $\Gamma_\varepsilon \to \mathbb{C}$, $\gamma \mapsto \langle \pi_\varepsilon(\phi) f_1, f_2 \rangle$ is continuous.

On the other hand, it follows by Lemma 4.4(2) that for every $\phi \in C^\infty_0(G)$ the function $\Gamma_\varepsilon \to \mathbb{C}$, $\gamma \mapsto \text{Tr} \pi_\varepsilon(\phi)$, is continuous. Hence we may use Lemma 2.12 with $S = C^\infty_0(G)$ to obtain the conclusion. □

Lemma 4.4. For every $\varepsilon \in E^m$ the set $\Xi_\varepsilon$ is dense and open in $(g^*/G) \setminus \bigcup_{E^m \ni \delta < \varepsilon} \Xi_\delta$.

Proof. If $e \in \mathcal{E}$ is the set for which $\varepsilon \in \mathcal{E}^{m-1} \times \{e\}$, then Remark 4.1(1) implies in particular that $\Xi_\varepsilon \subseteq \Xi_e$. It then follows by Lemma 4.1(1) that the relative topology of $\Xi_\varepsilon$ is Hausdorff.

On the other hand, it follows by [Pe94, Prop. 1.3.2] that $\Omega_e$ is a Zariski-open subset of $g^* \setminus \bigcup_{E^m \ni \delta < \varepsilon} \Omega_{\delta}$. Since the quotient map $g^* \to g^*/G$ is an open map (see [Bo97, Ch. 1, §5, no. 2, Ex. 1]), it then follows that $\Xi_\varepsilon$ is a dense open subset of $(g^*/G) \setminus \bigcup_{E^m \ni \delta < \varepsilon} \Xi_\delta$. □

4.3. Image of the Fourier transform of $A = C^*(G)$.

Proposition 4.5. Define the sequence

$$\emptyset = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \widehat{G}$$

by the conditions

- $\{\emptyset\} \cup \{V_j \setminus V_{j-1} | 1 \leq j \leq n\} = \{\Gamma_\varepsilon | \varepsilon \in E^m\}$;
- if $j_1, j_2 \in \{1, \ldots, n\}$ and $\varepsilon_1, \varepsilon_2 \in E^m$ with $V_{j_1} \setminus V_{j_1-1} = \Gamma_{\varepsilon_1}$ and $V_{j_2} \setminus V_{j_2-1} = \Gamma_{\varepsilon_2}$, then we have $j_1 < j_2$ if and only if $\varepsilon_1 < \varepsilon_2$.

Then $V_j$ is an open subset of $\widehat{G}$ for all $j = 1, \ldots, n$, and both hypotheses (1) - (2) of Theorem 5.7 are satisfied.

Proof. If $j = 1, \ldots, n$, then $V_j$ is an open subset of $\widehat{G}$ as a consequence of Lemma 4.4, since Kirillov’s correspondence $g^*/G \simeq \widehat{G}$ is a homeomorphism.

Moreover, to see that the other hypotheses of Theorem 5.6 are satisfied, we just apply Lemma 4.4 again (for hypothesis (1)), and Lemma 4.3 (for hypothesis (2)). This completes the proof. □

We have thus obtained the next theorem.

Theorem 4.6. Let $G$ be connected simply connected nilpotent Lie group. Then the $C^*$-algebra $C^*(G)$ has norm controlled dual limits.

Proof. The theorem follows from Proposition 4.5, Theorem 3.6, and Theorem 3.10. □
4.4. The index of a nilpotent Lie group.

Definition 4.7. Let $G$ be any simply connected Lie group with its Lie algebra $\mathfrak{g}$. The index of $G$ (respectively, of $\mathfrak{g}$) is defined as

$$\text{ind } G = \text{ind } \mathfrak{g} := \dim \mathfrak{g} - \max_{\xi \in \mathfrak{g}^*} \dim \text{Ad}_{\xi}^*(G)\xi.$$ 

Remark 4.8. For every $\xi \in \mathfrak{g}^*$, the coadjoint action gives rise to a $G$-equivariant diffeomorphism $G/G(\xi) \simeq \text{Ad}_{\xi}^*(G)\xi$, hence $\dim \text{Ad}_{\xi}^*(G)\xi = \dim \mathfrak{g} - \dim G(\xi)$ and it follows that

$$\text{ind } G = \text{ind } \mathfrak{g} = \min_{\xi \in \mathfrak{g}^*} \dim G(\xi).$$

If $Z$ is the center of $G$, then $G(\xi) \supseteq Z$ for every $\xi \in \mathfrak{g}^*$, hence $\text{ind } G \geq \dim Z$.

Proposition 4.9. Let $G$ be any simply connected Lie group with its Lie algebra $\mathfrak{g}$. If $G$ is nilpotent, then the following assertions hold:

1. If $e_1 < \cdots < e_n < \emptyset$ are all the index sets of the coadjoint orbits of $G$ with respect to some Jordan-Hölder basis of $\mathfrak{g}$, then $\text{ind } G = \dim \mathfrak{g} - \text{card } e_1$.
2. One has $\text{ind } G = r$ if and only if there exists some open dense subset $V \subseteq \hat{G}$ that is homeomorphic to some open subset of $\mathbb{R}^r$.

Proof. [1] Recall that $\text{card } e_1 = \max_{\xi \in \mathfrak{g}^*} \dim \text{Ad}_{\xi}^*(G)\xi$, hence $\text{ind } G = \dim \mathfrak{g} - \text{card } e_1$.

[2] It follows by Lemma 2.7 that it suffices to check the equality $\text{ind } G = r$ for any particular choice of the open dense subset $V \subseteq \hat{G}$ that is homeomorphic to some open subset of $\mathbb{R}^r$. Using the above notation and the one introduced in (4.1), if $V \subseteq \hat{G}$ is the image of $\Xi_{e_1} \subseteq \mathfrak{g}^*/G$ by Kirillov’s correspondence, then $V$ is open and dense in $\hat{G}$ by [Pe84, Cor. 3.4.2] or [Pe94, Cor. 1.3.2], and also $V$ is homeomorphic to some Zariski open subset of $\mathbb{R}^r$, where $r := \dim \mathfrak{g} - \text{card } e_1$ (see [Pe84, Subsec 1.2(c)–(d)]). For these $V$ and $r$ one has $\text{ind } G = r$ by Assertion [2].

Remark 4.10. It follows by Proposition [4.9][1] that the number $\text{card } e_1$ is independent on the choice of the Jordan-Hölder basis of $\mathfrak{g}$.

4.5. $C^*$-algebras of nilpotent Lie groups are special solvable.

Theorem 4.11. Let $G$ be any connected, simply connected, nilpotent Lie group with its Lie algebra $\mathfrak{g}$. Then $C^*(G)$ is a special solvable $C^*$-algebra with $\text{ind } C^*(G) = \text{ind } G$ and $C^*(G)\text{/Comm}(C^*(G)) \simeq C_0([\mathfrak{g}, \mathfrak{g}]^\perp)$.

Proof. This follows by Propositions [4.8] and [4.9] using also [Pe84, Cor. 3.4.2].

5. Uniqueness of Heisenberg group via special solvable $C^*$-algebras

We now prove that the Heisenberg groups are the only nilpotent Lie groups whose index is 1 and whose $C^*$-algebras are special solvable of length 2.

Proposition 5.1. Let $G$ be any nilpotent Lie group for which there exists a short exact sequence of $C^*$-algebras

$$0 \to C_0(\Gamma_1, \mathcal{K}(\mathcal{H})) \to C^*(G) \to C_0([\mathfrak{g}, \mathfrak{g}]^\perp) \to 0$$

satisfying the following conditions:

- $\Gamma_1$ is a dense open $\mathbb{R}$-subspace of $\hat{G}$ that is homeomorphic to $\mathbb{R}$;
- $\mathcal{H}$ is a separable infinite-dimensional complex Hilbert space.
Then there exists a unique integer \( d \geq 1 \) such that \( \dim [g, g]^\perp = 2d \) and \( G \) is isomorphic to the Heisenberg group \( \mathbb{H}_{2d+1} \).

**Proof.** Let us denote by \( O_\xi \) the coadjoint orbit of every \( \xi \in g^\ast \). Since the group \( G \) has infinite-dimensional representations, it follows that it is non-commutative, hence there exists \( \xi_1 \in g^\ast \) with \( O_{\xi_1} \neq \{ \xi_1 \} \), and then \( O_{\xi_t} \neq \{ \xi_1 \} \) for every \( t \in \mathbb{R}^\ast \).

Identifying \( \hat{G} \) with the space of coadjoint orbits \( g^\ast / G \) by Kirillov’s correspondence, it follows from the exact sequence from the statement that one has the disjoint union \( g^\ast / G = \Gamma_1 \sqcup [g, g]^\perp \). Moreover, \( \xi_1 \in \Gamma_1 \), and by the hypothesis that \( \Gamma_1 \) is open \( \mathbb{R} \)-subspace of \( G \) that is homeomorphic to \( \mathbb{R}^\ast \) we obtain \( \Gamma_1 = \{ O_{\xi_t} \mid t \in \mathbb{R}^\ast \} \), hence

\[
g^\ast / G = \{ O_{\xi_t} \mid t \in \mathbb{R}^\ast \} \sqcup [g, g]^\perp. \tag{5.1}
\]

Denoting the center of \( g \) by \( z \) and using the fact that the Lie algebra \( g \) is nilpotent, it follows that \( [g, g] \cap z \neq \{ 0 \} \). Then, using (5.1) and reasoning by contradiction, it easily follows that there exist \( x, y \in g \) with \( z := [x, y] \in z \setminus \{ 0 \} \) and \( \langle \xi_1, z \rangle \neq 0 \).

We claim that \( [g, g] = \mathbb{R} z \). In fact, if \( [g, g] \not\supseteq \mathbb{R} z \), then there exists \( \xi \in g^\ast \) with \( \langle \xi, z \rangle = 0 \) and \( \langle \xi, [g, g] \rangle \neq \{ 0 \} \). Then we have \( \xi \not\in [g, g]^\perp \), and also \( \xi \not\in \bigcup_{t \in \mathbb{R}^\ast} O_{\xi_t} \), since \( \langle \xi, z \rangle = 0 \not\in \bigcup_{t \in \mathbb{R}^\ast} \{ O_{\xi_t} \} \). We thus obtained a contradiction with (5.1), and this proves the above claim.

Since \( [g, g] = \mathbb{R} z \), it follows that there exist and are uniquely determined some integers \( d \geq 1 \) and \( k \geq 0 \) with \( g = h_{2d+1} \times a_k \), where \( a_k \) is the abelian \( k \)-dimensional Lie algebra. Then the index of \( g \) is \( \text{ind } g = k + 1 \) by Remark 4.8. On the other hand, since \( \Gamma_1 \) is a dense open subset of \( \hat{G} \) that is homeomorphic to \( \mathbb{R}^\ast \), it follows by Proposition 4.9 that \( \text{ind } g = 1 \), hence by comparing with the above formula we obtain \( k = 0 \), that is, \( g = h_{2d+1} \), and this completes the proof. \( \square \)

### Appendix A. Complements on properly convergent sequences

Here we discuss some uniqueness properties of the boundary value mappings that occur in Definition 3.9 (see Proposition 4.3 below).

**Definition A.1.** A sequence \( \bar{x} := \{ x_k \}_{k \in \mathbb{N}} \) in a topological space \( X \) is said to be *properly convergent* if its set of cluster points

\[
L(\bar{x}) := \bigcap_{k \in \mathbb{N}} \{ x_i \mid i \geq k \}
\]

has the property that for every point \( y \in L(\bar{x}) \) and every subsequence \( \{ x_{k_j} \}_{j \in \mathbb{N}} \) one has \( x_{k_j} \to y \) as \( j \to \infty \).

**Lemma A.2.** For every \( C^\ast \)-algebra \( A \) and any closed sets \( S_1, S_2 \subseteq \hat{A} \) one has

\[
S_1 \subseteq S_2 \iff (\forall a \in A) \sup_{[\rho] \in S_1} \| \rho(a) \| \leq \sup_{[\rho] \in S_2} \| \rho(a) \|.
\]

**Proof.** “\( \Rightarrow \)” is obvious, and the converse follows by Fe60 Lemma 2.1. \( \square \)
Lemma A.3. Let \( \mathcal{A} \) be any \( C^* \)-algebra, \( \pi := \{[\pi_k]\}_{k \in \mathbb{N}} \) be any properly convergent sequence in \( \hat{\mathcal{A}} \). Then for any closed set \( S \subseteq \hat{\mathcal{A}} \) one has

\[
L(\pi) = S \iff (\forall a \in \mathcal{A}) \lim_{k \to \infty} \|\pi_k(a)\| = \sup_{[\rho] \in S} \|\rho(a)\|;
\]

\[
L(\pi) \subseteq S \iff (\forall a \in \mathcal{A}) \lim_{k \to \infty} \|\pi_k(a)\| \leq \sup_{[\rho] \in S} \|\rho(a)\|;
\]

\[
L(\pi) \supseteq S \iff (\forall a \in \mathcal{A}) \lim_{k \to \infty} \|\pi_k(a)\| \geq \sup_{[\rho] \in S} \|\rho(a)\|.
\]

Proof. We prove only the first equivalence in the statement, as this proof contains the arguments needed for the second and third equivalences.

"\( \Rightarrow \)" Use [Fe60] Th. 2.1.

"\( \Leftarrow \)" Let \( S \subseteq \hat{\mathcal{A}} \) be any closed set with the property

\[
(\forall a \in \mathcal{A}) \lim_{k \to \infty} \|\pi_k(a)\| = \sup_{[\rho] \in S} \|\rho(a)\|.
\]

Since \( \pi := \{[\pi_k]\}_{k \in \mathbb{N}} \) is a properly convergent sequence, it follows by "\( \Rightarrow \)" that

\[
(\forall a \in \mathcal{A}) \lim_{k \to \infty} \|\pi_k(a)\| = \sup_{[\rho] \in L(\pi)} \|\rho(a)\|
\]

hence

\[
(\forall a \in \mathcal{A}) \sup_{[\rho] \in S} \|\rho(a)\| = \sup_{[\rho] \in L(\pi)} \|\rho(a)\|.
\]

Then, as both \( L(\pi) \) and \( S \) are closed subsets of \( \hat{\mathcal{A}} \), it follows by Lemma [A.1] that \( L(\pi) \subseteq S \) and \( L(\pi) \supseteq S \), hence \( L(\pi) = S \), and this completes the proof. \( \square \)

Lemma A.4. Let \( \mathcal{A} \) be any \( C^* \)-algebra, and \( \pi := \{[\pi_k]\}_{k \in \mathbb{N}} \) and \( \tau := \{[\tau_k]\}_{k \in \mathbb{N}} \) be any properly convergent sequences in \( \hat{\mathcal{A}} \). Then one has

\[
L(\pi) \supseteq L(\tau) \iff (\forall a \in \mathcal{A}) \lim_{k \to \infty} (\|\pi_k(a)\| - \|\tau_k(a)\|) \geq 0 \tag{A.1}
\]

and

\[
L(\pi) = L(\tau) \iff (\forall a \in \mathcal{A}) \lim_{k \to \infty} (\|\pi_k(a)\| - \|\tau_k(a)\|) = 0. \tag{A.2}
\]

Proof. It follows by Lemma [A.3] that

\[
(\forall a \in \mathcal{A}) \lim_{k \to \infty} \|\pi_k(a)\| = \sup_{\rho \in L(\pi)} \|\rho(a)\|
\]

and

\[
(\forall a \in \mathcal{A}) \lim_{k \to \infty} \|\tau_k(a)\| = \sup_{\rho \in L(\tau)} \|\rho(a)\|
\]

and then the implications "\( \Rightarrow \)" in both (A.1) and (A.2) follow at once. For (A.1) one also needs the elementary remark that if \( \{t_k\}_{k \in \mathbb{N}} \) and \( \{s_k\}_{k \in \mathbb{N}} \) are any convergent sequences of real numbers, then

\[
\lim_{k \to \infty} t_k = \lim_{k \to \infty} s_k \iff \lim_{k \to \infty} |t_k - s_k| = 0.
\]

Furthermore, if the condition in the right-hand side of (A.1) is satisfied, it follows by the above displayed equalities that

\[
(\forall a \in \mathcal{A}) \lim_{k \to \infty} \|\pi_k(a)\| \geq \sup_{[\rho] \in L(\tau)} \|\rho(a)\|
\]

and then \( L(\pi) \supseteq L(\tau) \) by the last equivalence in Lemma [A.3]. This completes the proof of (A.1). The implication "\( \Leftarrow \)" in (A.2) can be proved similarly. \( \square \)
Now we can establish a kind of uniqueness property of some mappings that occur in Definition 3.9, which also shows how these mappings depend on the limit set of the properly convergent sequence to which they are associated. It is worth pointing out that no linearity properties of these mappings are needed in the following proposition.

**Proposition A.5.** In the setting of Definition 3.9, fix $i \in \{0, \ldots, d + 1\}$. Let $\pi := \{[\tau_k]\}_{k \in \mathbb{N}}$ and $\tau := \{[\tau_k]\}_{k \in \mathbb{N}}$ be any properly convergent sequences contained in $\Gamma_i$ with $L(\pi) \cup L(\tau) \subseteq S_i$. Assume that for every $k \in \mathbb{N}$ one has some mappings $\bar{\sigma}_{i,k} : CB(S_i) \to \mathcal{B}(\mathcal{H}_i)$ and $\bar{\sigma}_{i,k} : CB(S_i) \to \mathcal{B}(\mathcal{H}_i)$ such that
\[
\lim_{k \to \infty} \| \mathcal{F}(a)(\pi_k) - \bar{\sigma}_{i,k}(\mathcal{F}(a)|_{S_i}) \|_{\mathcal{B}(\mathcal{H}_i)} = \lim_{k \to \infty} \| \mathcal{F}(a)(\pi_k) - \bar{\sigma}_{i,k}(\mathcal{F}(a)|_{S_i}) \|_{\mathcal{B}(\mathcal{H}_i)} = 0
\]
for every $a \in A$. Then the following assertions hold:

1. If $L(\pi) \supseteq L(\tau)$, then
\[
(\forall a \in A) \lim_{k \to \infty} \left( \| \bar{\sigma}_{i,k}(\mathcal{F}(a)|_{S_i}) \|_{\mathcal{B}(\mathcal{H}_i)} - \| \bar{\sigma}_{i,k}(\mathcal{F}(a)|_{S_i}) \|_{\mathcal{B}(\mathcal{H}_i)} \right) \geq 0.
\]

2. If $L(\pi) = L(\tau)$, then
\[
(\forall a \in A) \lim_{k \to \infty} \| \bar{\sigma}_{i,k}(\mathcal{F}(a)|_{S_i}) \|_{\mathcal{B}(\mathcal{H}_i)} - \| \bar{\sigma}_{i,k}(\mathcal{F}(a)|_{S_i}) \|_{\mathcal{B}(\mathcal{H}_i)} = 0.
\]

Proof. The hypothesis can be written as
\[
\lim_{k \to \infty} \| \pi_k(a) - \bar{\sigma}_{i,k}(\mathcal{F}(a)|_{S_i}) \|_{\mathcal{B}(\mathcal{H}_i)} = \lim_{k \to \infty} \| \tau_k(a) - \bar{\sigma}_{i,k}(\mathcal{F}(a)|_{S_i}) \|_{\mathcal{B}(\mathcal{H}_i)} = 0
\]
and this implies
\[
\lim_{k \to \infty} \| \pi_k(a) \|_{\mathcal{B}(\mathcal{H}_i)} - \| \bar{\sigma}_{i,k}(\mathcal{F}(a)|_{S_i}) \|_{\mathcal{B}(\mathcal{H}_i)} = 0
\]
and
\[
\lim_{k \to \infty} \| \tau_k(a) \|_{\mathcal{B}(\mathcal{H}_i)} - \| \bar{\sigma}_{i,k}(\mathcal{F}(a)|_{S_i}) \|_{\mathcal{B}(\mathcal{H}_i)} = 0.
\]
Now the assertions follow by Lemma A.4. \qed

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