MINIMAL BLOW-UP INITIAL DATA IN CRITICAL FOURIER-HERZ SPACES FOR POTENTIAL NAVIER-STOKES SINGULARITIES

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Abstract. In this paper, we mainly prove the existence of the minimal blow-up initial data in critical Fourier-Herz space $F\dot{B}^{2-\frac{3}{p}}_{p,q}(\mathbb{R}^3)$ with $1 < p \leq \infty$ and $1 \leq q < \infty$ for the three dimensional incompressible potential Navier-Stokes equations by developing techniques of “localization in space” involving the partial regularity given by the De Giorgi iteration, weak-strong uniqueness, the short-time behaviour of the kinetic energy and stability of singularity of Calderón’s solution.

1. Introduction

In this paper, we consider Cauchy problem of the three dimensional incompressible Navier-Stokes equations

\begin{equation}
\begin{cases}
\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla P = 0 \\
\text{div } u = 0 \\
u > 0
\end{cases}
\end{equation}

where the vector field $u$ and the scalar function $P$ describe the velocity field and the associated pressure of fluid, respectively. And $\nu > 0$ is the kinematic viscosity. The initial velocity $u_0$ satisfies $\text{div } u_0 = 0$. It is well-known that problem (NS) has the natural scaling, that is, if $u$ is a solution of system (NS) with initial data $u_0$, then so is $u_\lambda$, for any $\lambda > 0$, associated with initial data $u_{0\lambda}$, where

\begin{equation}
\begin{aligned}
u \lambda t \\ u_\lambda(x, t) \triangleq \lambda u(\lambda x, \lambda^2 t), \\
u_{0\lambda} \triangleq \lambda u_0(\lambda x).
\end{aligned}
\end{equation}

We call Banach space $X$ is critical space if its norm $\| \cdot \|_X$ is invariant under scaling (1.1), for example, the Lebesgue space $L^3(\mathbb{R}^3)$, homogeneous Sobolev space $\dot{H}^\frac{1}{2}(\mathbb{R}^3)$, homogeneous Besov space $\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$, homogeneous Fourier-Herz space $F\dot{B}^{2-\frac{3}{p}}_{p,q}(\mathbb{R}^3)$ and so on.

The incompressible Navier-Stokes equations has been studied by many researchers in the past years. Especially, there are many interesting results concerning the mild solution $u$ which solves the following integral equations:

\begin{equation}
u \lambda t \\ u(x, t) = e^{\nu t \Delta} u_0 + \int_0^t e^{\nu(t-s)\Delta} \mathbb{P}(u \cdot \nabla u) \, ds,
\end{equation}

where $\mathbb{P}$ is the Leray projector. For the separable critical space, Fujita and Kato [11] firstly established the local well-posedness for the large initial data in $\dot{H}^{1/2}(\mathbb{R}^3)$, as well as the global

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well-posedness for small initial data in $\dot{H}^{1/2}(\mathbb{R}^3)$. Later on, Kato \[18\] and Cannone \[6, 7\] generalized such well-posed theory to $L^3(\mathbb{R}^3)$ and $\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$ with $1 \leq p, q < \infty$, respectively. For the inseparable critical spaces, there only exists the global well-posedness for small initial data by now. Cannone \[6, 7\] proved the global well-posedness to (NS) for small data in $\dot{B}^{-1+\frac{3}{p}}_{p}\mathbb{R}^3$ with $3 < p < \infty$, and Koch and Tataru \[19\] showed the global well-posedness to system (NS) for small initial data in $\text{BMO}^{-1}(\mathbb{R}^3)$. Since Besov space $\dot{B}^{-1+\frac{3}{p}}_{p}$ with $p \in (3, \infty)$ and $\text{BMO}^{-1}(\mathbb{R}^3)$ contain non-trivial homogenous functions of degree $-1$, the both results mean that the global existence and uniqueness of the small self-similar solution. But the local well-posedness for large initial data in the inseparable critical space remains an open problem.

Let $X$ be a separable critical space. Denoted by $NS(u_0)$ the local in time mild solution to (NS) starting from $u_0 \in X$, ones consider the following question based on the above local well-posed theory:

**Question:**

Suppose there exist a initial data $u_0$ in the separable critical space $X$ such that the maximal existence time of $NS(u_0)$ is finite. Then does there exist a minimal blow-up initial data $v_0 \in X$, that is, does there exist $v_0 \in X$ with minimal norm such that the maximal existence time of $NS(v_0)$ is finite?

Rusin and Šverák \[25\] firstly considered this question in the space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and gave a positive answer. The main ingredients of their proof consist of the singularities of local mild solution inside the ball at the maximal existence time, the stability of singularities and the compactness of $\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^3)$. Later, Gallagher, Koch and Planchon in \[12, 13\] utilized concentration compactness argument and profile decomposition to give a unified affirmative answer to this question in $\dot{H}^{1/2}(\mathbb{R}^3)$, $L^3(\mathbb{R}^3)$ and $\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$ with $3 < p, q < \infty$. Especially for $L^3(\mathbb{R}^3)$, Jia and Šverák in \[16\] gave a simple proof of the existence of minimal blow-up initial data by exploiting the regularity of energy solution in short time which compensates the lack of compactness of the embedding $L^3(\mathbb{R}^3) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^3)$.

Recently, some well-posed results of problem (NS) for large initial data in Fourier-Herz spaces were established. For example, Cannone and Karch \[8\] considered existence and uniqueness of singular solution in the space of pseudomeasure $F\dot{B}^{-1}_{-\infty,\infty}(\mathbb{R}^3)$. Lei and Lin \[20\] proved an interesting global well-posed result to problem (NS) if the initial data belongs to the space

$$\chi^{-1} \triangleq \left\{ u \in D'(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |\xi|^{-1} |\hat{u}|(\xi) \, d\xi < \infty \right\}$$

which coincides with $F\dot{B}^{-1}_{1,q}(\mathbb{R}^3)$, and the corresponding norm is bounded exactly by the viscosity coefficient $\nu$. Subsequently, Cannone and Wu \[9\] extended this result to larger space $F\dot{B}^{-1}_{1,\bar{q}}(\mathbb{R}^3)$ for all $1 \leq q \leq 2$. Recently, Li and Zheng \[22\] obtained the local well-posed result in critical Fourier Herz spaces $F\dot{B}^{2-\frac{1}{q}}_{p,q}(\mathbb{R}^3)$ with $q \neq \infty$ for the large initial data, and the global well-posed theory in $F\dot{B}^{2-\frac{1}{q}}_{p,q}(\mathbb{R}^3)$ for small initial data. A nature question arises: does there exist a blow-up initial data in the framework of the critical Fourier-Herz space.
To solve this question, we face some difficulties which don’t appear in known results. For example, the embedding that $F_{p,q}^{\frac{2-n}{2}}(\mathbb{R}^3) \hookrightarrow L^p_{loc}(\mathbb{R}^3)$ doesn’t hold as long as $p < \frac{3}{2}$. This leads to the method used in \cite{16} doesn’t work directly. In order to overcome this difficulty, we adopt the Calderón splitting argument to decompose the local mild solution $u(x,t)$ on $[0,T^*)$ into two parts
\[ u = v + w. \]

For this purpose, we will establish abstract interpolation theory of Fourier-Herz spaces $F\dot{B}_{p,q}^s(\mathbb{R}^d)$ by using K-function, see Lemma \ref{2.2} This allows us to decompose $u_0$ into two parts as follows:
\[ u_0 = v_0 + w_0, \]
where $v_0$ and $w_0$ fulfill $\text{div}\, v_0 = \text{div}\, w_0 = 0$ and
\begin{equation}
\|v_0\|_{F\dot{B}_{p,q}^\frac{2-n}{2} + \varepsilon} \leq C2^{-j\theta} \alpha \quad \text{and} \quad \|w_0\|_{L^2} \leq C2^{j(1-\theta)} \alpha \quad \text{for each} \quad \varepsilon > 0.
\end{equation}

Since $v_0 \in F\dot{B}_{p,q}^s(\mathbb{R}^3)$, by the local well-posed theory in the subcritical space, we know that the following system
\begin{equation}
\begin{cases}
\partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla Q = 0, \\
\text{div}\, v = 0, \\
v(x,0) = v_0,
\end{cases}
\end{equation}
exists a unique local mild solution $v(x,t)$ satisfying
\[ v(x,t) \in C_b\left([0,T]; F\dot{B}_{p,q}^\frac{2-n}{2} + \varepsilon(\mathbb{R}^3)\right) \cap \tilde{L}^r\left([0,T]; F\dot{B}_{p,q}^\frac{2-n}{2} + \varepsilon(\mathbb{R}^3)\right) \]
where $T > T^*$. Letting the difference $w(x,t) = u(x,t) - v(x,t)$, ones verify by the fact $w_0 \in L^2(\mathbb{R}^3)$ that $w(x,t)$ is a Leray solution of the following perturbed problem
\begin{equation}
\begin{cases}
\partial_t w - \nu \Delta w + w \cdot \nabla w + w \cdot \nabla v + v \cdot \nabla w + \nabla \bar{P} = 0, \\
\text{div}\, w = 0, \\
w(x,0) = w_0 \in L^2(\mathbb{R}^3).
\end{cases}
\end{equation}

Since $v(x,t)$ is regular at $T = T^*$, the singularity of $u$ at the maximal time $T^*$ is caused by $w$. This requires us to study the singularity of $w$ at $T^*$. To do this, we first prove the $\varepsilon$-regularity criterion of the suitable weak solution by De Giorgi iteration and dimensional analysis. Next, making good use of the splitting argument, trilinear estimates and the regularity structure of the initial data, we obtain the short-time behaviour of the energy solution, and then we get the weak-strong uniqueness of solutions by smoothing effect of the heat kernel and uniform $L^2$-estimate. These properties enable us to conclude that the singularities of $w(x,t)$ only occur inside the ball. Based on this, we further get by the compactness argument that the stability of singularity. With these properties, we eventually obtain that there exists a blow-up initial data to problem (\ref{NS}) in the framework of Fourier-Herz space.

We give some notations before the presentation of the main result. Set
\[ B_\rho = \left\{ u_0 \in F\dot{B}_{p,q}^\frac{2-n}{2}(\mathbb{R}^3) \mid \text{div}\, u_0 = 0, \|u_0\|_{F\dot{B}_{p,q}^\frac{2-n}{2}} < \rho \right\}. \]
and

$$\rho_{\text{max}} = \sup \{ \rho > 0 \mid u_0 \in \mathcal{B}_p, \ T^*(u_0) = \infty \}$$

where $T^*(u_0)$ is the maximal existence time of $NS(u_0)$. Now, the main theorem reads:

**Theorem 1.1.** Let $1 < p \leq \infty$ and $1 \leq q < \infty$. Suppose $\rho_{\text{max}} < \infty$. Then, there exist a divergence-free initial data $u_0 \in \dot{F}B^{2-\frac{2}{q}}_{p,q}(\mathbb{R}^3)$ with $\|u_0\|_{\dot{F}B^{2-\frac{2}{q}}_{p,q}(\mathbb{R}^3)} = \rho_{\text{max}}$ such that

$$T^*(u_0) < \infty.$$ 

Moreover, $\mathcal{M}$ is compact in $\dot{F}B^{2-\frac{2}{q}}_{p,q}(\mathbb{R}^3)$ modulo translations and scaling (1.1), where

$$\mathcal{M} = \left\{ u_0 \in \dot{F}B^{2-\frac{2}{q}}_{p,q}(\mathbb{R}^3) \mid \text{div} \ u_0 = 0, \ \|u_0\|_{\dot{F}B^{2-\frac{2}{q}}_{p,q}(\mathbb{R}^3)} = \rho_{\text{max}}, \ T^*(u_0) < \infty \right\}.$$ 

**Remark 1.2.** By the Hausdorff-Young inequality and Plancherel theorem, we easily have that

- for $1 < p < \frac{3}{2}$, $\dot{F}B^{2-\frac{2}{q}}_{p,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^{1-\frac{3}{p}}_{p,q}(\mathbb{R}^3)$, $\frac{1}{p} + \frac{3}{p'} = 1$;
- for $p = \frac{3}{2}$, $\dot{F}B^0_{p,q}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$;
- for $p = 2$, $\dot{F}B^0_{2,2}(\mathbb{R}^3) \sim \dot{H}^\frac{1}{2}(\mathbb{R}^3)$.

This implies that Theorem 1.1 includes the result was shown in [25], and builds the relationship between Theorem 1.1 and results in [12, 13, 16] in some sense. But, our argument and technique are different with that of papers [12, 13], whose proof strongly relies on “profile decomposition”. More importantly, we develop some useful techniques of “localization in space” including the partial regularity, weak-strong uniqueness, the short-time behaviour of the kinetic energy and stability of singularity of weak solution to the perturbed problem (1.4), which will be powerful in the study of the incompressible fluid equations, such as MHD system.

The rest of this paper is structured as follows. In Section 2 we review the definition and some properties of Fourier-Herz spaces, and give several useful lemmas including abstract interpolation theory of Fourier-Herz spaces based on $K$-function. In Section 3 we establish the well-posedness theory of mild solution to problem (NS) in Fourier-Herz spaces. Section 4 is devoted to developing techniques of “localization in space” to the perturbed problem (1.4). In Section 5 we give the proof of Theorem 1.1 by using the properties established in foregoing sections.

**Notation.** We denote $C$ as an absolute positive constant. $C(\lambda, \beta, \cdots)$ denotes a positive constant depending only on $\lambda, \beta, \cdots$. We adopt the convention that nonessential constants $C$ may change from line to line and the usual Einstein summation convention. Given two quantities $a$ and $b$, we denote $a \lesssim b$ and $a \lesssim_{\lambda, \beta, \cdots} b$ as $a \leq Cb$ and $a \leq C(\lambda, \beta, \cdots) b$ respectively. For any $x_0 \in \mathbb{R}^3$ and $t_0 \in \mathbb{R}^+$, $B_R(x_0) \in \mathbb{R}^3$ means a ball with radius $R$ centered at $x_0$ and $Q_R(x_0, t_0) = B_R(x_0) \times (t_0 - R^2, t_0) \in \mathbb{R}^3 \times \mathbb{R}$. 
2. Preliminaries

In this section, we first recall some facts concerning Littlewood-Paley theory and some useful lemmas which will be used in subsequent sections, see for example [7, 23]. Next, we establish abstract interpolation theory of Fourier-Herz spaces in terms of $K$-function, which allows us to perform the Calderon argument.

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz class of rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^d)$ be its dual space. We denote by $\hat{f}$ the Fourier transform of $f$. For each $1 \leq p \leq \infty$, one define

$$FL^p(\mathbb{R}^d) \triangleq \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \left| \|f\|_{FL^p(\mathbb{R}^d)} = \|\hat{f}\|_{L^p(\mathbb{R}^d)} < \infty \right\}.$$ 

Let $C$ be the annulus $\{\xi \in \mathbb{R}^d | \frac{2}{3} \leq |\xi| \leq \frac{8}{7}\}$. Let $\phi, \chi \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\hat{\phi} \in \mathcal{D}(C)$ with $0 \leq \hat{\phi} \leq 1$ and $\hat{\chi} \in \mathcal{D}(B_{\frac{4}{3}}(0))$ with $0 \leq \hat{\chi} \leq 1$. For any $j \in \mathbb{Z}$, define $\phi_j(x) = 2^{jd}\phi(2^jx)$ and $\chi_j(x) = 2^{jd}\chi(2^jx)$. Then we define

$$\hat{\Delta}_j = \phi_j \ast \cdot \quad \text{and} \quad \hat{S}_j = \chi_j \ast \cdot.$$ 

From the definitions of the frequency localization operators, we easily find that $\forall f \in \mathcal{S}'(\mathbb{R}^d)$,

(2.1) $$\hat{\Delta}_j \hat{\Delta}_k f = 0 \quad \text{for} \quad |j - k| > 1.$$ 

This implies

$$\hat{\Delta}_j \hat{\Delta}_j = \hat{\Delta}_j, \quad \hat{\Delta}_j = \hat{\Delta}_{j-1} + \hat{\Delta}_j + \hat{\Delta}_{j+1}.$$ 

Now we give the definition of Fourier-Herz space.

**Definition 2.1.** Let $f \in \mathcal{S}'_b(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)/\mathcal{P}$, where $\mathcal{P}$ is the set of all polynomials, then we say $f \in F\dot{B}^s_{p,q}(\mathbb{R}^d)$ with $s \in \mathbb{R}$ and $(p,q) \in [1,\infty]^2$, if $\|f\|_{F\dot{B}^s_{p,q}} < \infty$, where

$$\|f\|_{F\dot{B}^s_{p,q}} \triangleq \left\{ \begin{array}{cc} \left( \sum_{j \in \mathbb{Z}} 2^{jsq}\|\hat{\Delta}_j f\|_{L^p}^q \right)^\frac{1}{q} & \text{if} \quad q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js}\|\hat{\Delta}_j f\|_{L^p} & \text{if} \quad q = \infty. \end{array} \right.$$ 

When $s < d(1 - \frac{1}{p})$, $F\dot{B}^s_{p,q}(\mathbb{R}^d)$ is a Banach space.

Let us remark that when $s < 0$, Fourier-Herz space $F\dot{B}^s_{p,q}(\mathbb{R}^d)$ can be characterized by Gaussian kernel $e^{t\Delta}$. Specifically:

**Lemma 2.1** ([2]). Let $s < 0$ and $1 \leq p,q \leq \infty$. Then for any $f \in F\dot{B}^s_{p,q}(\mathbb{R}^d)$, there exits constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1\|f\|_{F\dot{B}^s_{p,q}} \leq \|f - \frac{1}{\varpi} e^{t\Delta} f\|_{L^p(\mathbb{R}^d)} \leq C_2\|f\|_{F\dot{B}^s_{p,q}}.$$ 

Taking into account the time variable, we give the definition of the mixed time-space Fourier-Herz space which is called the so called “Chemin-Lerner” space.

**Definition 2.2.** Let $s \in \mathbb{R}$ and $(p,q,r) \in [1,\infty]^3$. We say

$$u \in \tilde{L}^r([\alpha, \beta]; F\dot{B}^s_{p,q})$$
Lemma 2.2. Let \( \mathcal{F} \) be a sequence of vector fields with the help of \( K \). With this definition, we will establish the following interpolation theory in Fourier-Herz spaces \( F\dot{B}^s_{p,q} = \mathcal{F} \). To do this, we need to introduce the generalized Fourier-Herz spaces \( F\dot{B}^s_{p,q} \).

Definition 2.3. Let \( s \in \mathbb{R} \), \( 0 < p < \infty \), \( 1 \leq q, r \leq \infty \). We say
\[
F\dot{B}^s_{p,q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \mid \| f \|_{F\dot{B}^s_{p,q}} = \left\{ \left\| 2^j s \right\|^\Delta_j f \|^\Delta_{s,q} \right\}_\ell \right\} < \infty
\]

According to the property of Lorentz space, we see that \( F\dot{B}^s_{p,q} \) coincides with \( F\dot{B}^s_{p,q} \). In addition,
\[
(2.3) \quad F\dot{B}^s_{p,q} \cong F\dot{B}^s_{p,q} \quad \text{if } r > p, \quad F\dot{B}^s_{p,q} \cong F\dot{B}^s_{p,q} \quad \text{if } r < p.
\]

With this definition, we will establish the following interpolation theory in Fourier-Herz spaces with the help of \( K \)-function.

Lemma 2.2. Let \( s_1, s_2 \in \mathbb{R} \), \( 1 < p_1, p_2 < \infty \), \( 1 \leq q_1, q_2 \leq \infty \), \( p_1 \neq p_2 \), and \( 0 < \theta < 1 \). Then
\[
\left( F\dot{B}^s_{p_1,q_1}, F\dot{B}^s_{p_2,q_2} \right)_{\theta,q} = F\dot{B}^s_{p,q}(\mathbb{R}^d)
\]
where \( s_1(1 - \theta) + s_2\theta = s_1, \frac{1 - \theta}{p_1} + \frac{\theta}{p_2} = 1 \). Besides, for any \( f \in F\dot{B}^s_{p,q} \), we have
\[
\| f \|_{F\dot{B}^s_{p,q}} = \left( \sum_{j \in \mathbb{Z}} 2^{j\theta} K(f,j)^\theta \right)^\frac{1}{\theta}
\]
with
\[
K(f,j) = \inf_{f=g+h} \left( \| g \|_{F\dot{B}^s_{p_1,q_1}} + 2^{-j} \| h \|_{F\dot{B}^s_{p_2,q_2}} \right).
\]

Proof. We here sketch key points of the proof, because it is similar to the proof of the interpolation in Besov spaces in [27].

From Theorem 2.4.1/(c) in [27], we have that
\[
\left( \ell\theta(A_j), \ell\theta(B_j) \right)_{\theta,q} = \ell\theta((A_j, B_j)_{\theta,q}).
\]
Let \( A_j = 2^{j s_1} \mathcal{F} \mathcal{L}^{p_1} \) and \( B_j = 2^{j s_2} \mathcal{F} \mathcal{L}^{p_2} \). Thus, we get the required result. \( \Box \)

In view of Lemma 2.2, we can get the following decomposition in \( F\dot{B}^s_{p,q}(\mathbb{R}^d) \) with \( s_p = 2 - \frac{\lambda}{p} \).

Lemma 2.3. Let solenoidal vector field \( f \in F\dot{B}^s_{p,q}(\mathbb{R}^d) \) with \( p \in (1, \frac{3}{2}) \) and \( q \in [1, \infty) \), then for each \( j \in \mathbb{Z} \), there exist solenoidal vector fields \( g_0 \in F\dot{B}^s_{p,q}, h_0 \in \mathcal{L}^2 \) such that
\[
f = g_0 + h_0,
\]
\[ \|g_0\|_{FB_{p,q}^s} \leq C 2^{-j\theta} \|f\|_{FB_{p,q}^s} \quad \text{and} \quad \|h_0\|_{L^2} \leq C 2^{j(1-\theta)} \|f\|_{FB_{p,q}^s}, \]

where \( C \) is a absolute constant and \( \theta, s_p, s, p, \tilde{p}, q, \tilde{q} \) satisfy the compatibility condition:

\[ \theta \in (0, 1), \quad s_p = (1 - \theta)s, \quad \frac{1}{p} = \frac{\theta}{2} + \frac{1 - \theta}{\tilde{p}}, \quad \tilde{p} \in (1, \frac{3}{2}) \]

(2.4)

\[ \frac{1}{q} = \begin{cases} \frac{1}{1 - \theta} \left( \frac{1}{q} - \frac{\theta}{2} \right) & q < p; \\ \frac{1}{p} & q \geq p. \end{cases} \]

**Proof.** We proceed the lemma in two cases: \( p \geq q \) and \( p < q \).

Case 1: \( p \geq q \). By Lemma 2.2 we have

\[ F\dot{B}^{s_p}_{p,q}(\mathbb{R}^3) \hookrightarrow F\dot{B}^{s_p}_{p,p}(\mathbb{R}^3) = \left( F\dot{B}^{s_p}_{p,p}(\mathbb{R}^3), L^2(\mathbb{R}^3) \right)_{\theta,p} \]

where \( \frac{1 - \theta}{\tilde{p}} + \frac{\theta}{2} = \frac{1}{p} \) and \( s(1 - \theta) = s_p \). Thus we have for any \( f \in F\dot{B}^{s_p}_{p,q}(\mathbb{R}^3) \),

\[ \|f\|_{F\dot{B}^{s_p}_{p,q}} = \left( \sum_{j \in \mathbb{Z}} 2^{j\theta} \left( \inf_{g = f + h} \left( \|g\|_{F\dot{B}^{s}_{p,\tilde{p}}} + 2^{-j}\|h\|_{L^2} \right) \right)^p \right)^{\frac{1}{p}}. \]

From this equality, for any \( j \in \mathbb{Z} \), there exist \( \tilde{g}_0, \tilde{h}_0 \) such that \( f = \tilde{g}_0 + \tilde{h}_0 \) and

\[ \|\tilde{g}_0\|_{F\dot{B}^{s}_{p,\tilde{p}}} + 2^{-j}\|\tilde{h}_0\|_{L^2} \leq 2 \inf_{g = f + h} \left( \|g\|_{F\dot{B}^{s}_{p,\tilde{p}}} + 2^{-j}\|h\|_{L^2} \right) \leq 2^{1-j\theta} \|f\|_{FB_{p,q}^s} \leq C 2^{-j\theta} \|f\|_{FB_{p,q}^s}. \]

Letting \( g_0 = \mathbb{P}(\tilde{g}_0) \) and \( h_0 = \mathbb{P}(\tilde{h}_0) \), we have by the fact \( \text{div } f = 0 \) that

\[ f = g_0 + h_0. \]

In terms of Calderón-Zygmund estimates, we readily get

\[ \|g_0\|_{FB_{p,\tilde{p}}} \leq C \|\tilde{g}_0\|_{FB_{p,\tilde{p}}} \leq C 2^{-j\theta} \|f\|_{FB_{p,q}^s} \]

and

\[ \|h_0\|_{L^2} \leq C \|\tilde{h}_0\|_{L^2} \leq C 2^{j(1-\theta)} \|f\|_{FB_{p,q}^s}. \]

Case 2: \( p < q \). According to (2.3) and Lemma 2.2 we have

\[ F\dot{B}^{s_p}_{p,q}(\mathbb{R}^3) \hookrightarrow F\dot{B}^{s_p}_{p,q,(q)}(\mathbb{R}^3) = \left( F\dot{B}^{s_p}_{p,q,(q)}(\mathbb{R}^3), L^2(\mathbb{R}^3) \right)_{\theta,q} \]

with

\[ \frac{1 - \theta}{\tilde{p}} + \frac{\theta}{2} = \frac{1}{p}, \quad \frac{1 - \theta}{\tilde{q}} + \frac{\theta}{2} = \frac{1}{q}, \quad s(1 - \theta) = s_p, \quad 0 < \theta < 1. \]

Thus, for any \( j \in \mathbb{Z} \), there exist \( \tilde{g}_0, \tilde{h}_0 \) such that \( f = \tilde{g}_0 + \tilde{h}_0 \) and

\[ \|\tilde{g}_0\|_{F\dot{B}^{s}_{p,q,(q)}} + 2^{-j}\|\tilde{h}_0\|_{L^2} \leq 2 \inf_{g = f + h} \left( \|g\|_{F\dot{B}^{s}_{p,q,(q)}} + 2^{-j}\|h\|_{L^2} \right) \leq 2^{1-j\theta} \|f\|_{FB_{p,q}^s} \leq C 2^{-j\theta} \|f\|_{FB_{p,q}^s}. \]
Then, letting \( g_0 = \mathbb{P}(\tilde{g}_0) \) and \( h_0 = \mathbb{P}(\tilde{h}_0) \), and using the above inequality and \( \text{div} f = 0 \), we eventually obtain that \( f = g_0 + h_0 \), where

\[
\|g_0\|_{FB_{p,q}^s} \leq C2^{-\beta} \|f\|_{FB_{p,q}^s} \quad \text{and} \quad \|h_0\|_{L^2} \leq C2^{2(1-\theta)} \|f\|_{FB_{p,q}^s}.
\]

Thus we end the proof of Lemma 2.3. \( \Box \)

Finally we review Banach fixed’s theorem and the associated propagation of regularity. Their proof can be found in [14].

**Lemma 2.4** ([14]). Let \((X, \|\cdot\|)\) be an abstract Banach space, \(L : X \to X\) be a linear bounded operator such that for a constant \(\lambda \in [0, 1)\), we have \(\|L(x)\| \leq \lambda \|x\|\) for all \(x \in X\), and \(B : X \times X \to X\) be a bilinear mapping such that

\[
\|B(x_1, x_2)\| \leq \eta \|x_1\| \|x_2\| \quad \forall x_1, x_2 \in X
\]

for some \(\eta > 0\). Then, for every \(y \in X\) satisfying \(4\eta \|y\| < (1 - \lambda)^2\), the equation

\[
(2.5) \quad x = y + L(x) + B(x, x)
\]

has a solution \(x \in X\).

In particular, this solution satisfies \(\|x\| \leq \frac{2\|y\|}{1-\lambda}\), and it is the only one among all solutions satisfying \(\|x\| < \frac{1-\lambda}{2\eta}\).

**Lemma 2.5** ([14]). In the notation of Lemma 2.4, let \(E\) be a Banach space. Suppose \(L : E \to E\) is a linear bounded operator such that for a \(\beta \in [0, 1)\), we have \(\|L(x)\| \leq \beta \|x\|\) for all \(x \in E\), and \(B : X \times E \to E\) and \(E \times X \to E\) are bilinear mappings such that

\[
\max \{\|B(x_1, x_2)\|_E, \|B(x_2, x_1)\|_E\} \leq \gamma \|x_1\|_X \|x_2\|_E \quad \text{for every} \quad x_1 \in X, x_2 \in E
\]

for some \(\gamma > 0\). Then, if \(\beta \gamma \leq \lambda \eta\), for all \(y \in E\) satisfying \(4\eta \|y\|_X < (1 - \lambda)^2\), the solution given in Lemma 2.4 belongs to \(E\) and satisfies \(\|x\|_E \leq 2\beta \|a\|_E\).

### 3. Mild solution of the 3D Navier-Stokes system (NS) in subcritical and critical framework

In this section, we will mainly study the well-posed theory of problem (NS) associated with initial data in subcritical Fourier-Herz spaces \(\dot{F}B_{p,q}^s(\mathbb{R}^3)\).

First of all, we recall the well-posed theory of problem (NS) in the framework of critical Fourier-Herz space. In [22], J. Li and X. Zheng obtain the following well-posed results to (NS) in critical Fourier-Herz space \(\dot{F}B_{p,q}^s(\mathbb{R}^3)\) with \(1 \leq p, q \leq \infty\):

**Theorem 3.1** ([22]). Let \(u_0 \in \dot{F}B_{p,q}^s(\mathbb{R}^3)\) satisfying \(\text{div} u_0 = 0, 1 \leq p, q \leq \infty\) and \(1 \leq q \leq 2\) if \(p = 1\).

1. For \(1 \leq q < \infty\), there exists a \(T^* = T^*(u_0) > 0\) such that problem (NS) has a unique local in time mild solution \(u\) satisfying

\[
u \in C([0, T^*); \dot{F}B_{p,q}^s(\mathbb{R}^3)) \cap \dot{L}^r_{\text{loc}}([0, T^*); \dot{F}B_{p,q}^{2+s}(\mathbb{R}^3)) \quad \forall r \in [1, \infty].
\]

Moreover we have that

\[
(3.1) \quad \lim_{\delta \to 0^+} \|u\|_{\dot{L}^r([0, T^*-\delta]; \dot{F}B_{p,q}^{2+s}(\mathbb{R}^3))} = \infty \iff T^* < \infty.
\]
(2) For \( 1 \leq q \leq \infty \), if there exists a positive constant \( \eta_0 \) such that \( \|u_0\|_{F B^{s,p}_{p,q}} < \eta_0 \), then problem (NS) has a unique global in time mild solution \( u \) satisfying

\[
u \in C([0, \infty); F \dot{B}_p^s(R^3) \cap \tilde{L}_r^r([0, \infty); F \dot{B}_p^{2+s}(R^3)) \quad \forall r \in [1, \infty].
\]

Next, we turn to show the local well-posedness of (NS) in subcritical spaces \( F \dot{B}_p^s(R^3) \) with \( s \in (s_p, 0] \).

**Theorem 3.2.** For any \( u_0 \in F \dot{B}_p^s(R^3) \) with \( (s, p) \in \{[s_p, 0] \times [1, \infty]\} \setminus (0, 1) \) and \( q \in [1, \infty] \), satisfying \( \text{div } u_0 = 0 \), there exists a \( T^* = T^*(\nu; \|u_0\|_{F B^{s,p}_{p,q}}) > 0 \) such that system (NS) has a unique local in time mild solution \( u \) satisfying

\[
u \in X_T \triangleq C([0, T); F \dot{B}_p^s(R^3) \cap \tilde{L}_r^r([0, T^*); F \dot{B}_p^{2+s}(R^3)) \quad \forall r \in [1, \infty].
\]

Moreover, \( T^* \) and \( u \) satisfy

\[
u \leq C(\nu)\|u_0\|_{F B^{s,p}_{p,q}}^\frac{-2}{r - sp} \quad \text{and} \quad \|u\|_{X_T} \leq C(\nu)\|u_0\|_{F B^{s,p}_{p,q}}.
\]

**Proof.** By Duhamel formula, one writes equation (NS) in the integral form

\[
u(x, t) = e^{\nu t}u_0 + \int_0^t e^{\nu(t-\tau)}\Delta P(u \cdot \nabla u) \, d\tau \triangleq G + B(u, u).
\]

Now we are going to apply Lemma 2.4 to get the local well-posedness of problem (NS) in subcritical spaces \( F \dot{B}_p^s(R^3) \) with \( s \in (s_p, 0] \).

From estimate (2.2), one has

\[
u \leq C\|u_0\|_{F B^{s,p}_{p,q}} \quad \text{for all} \quad s \in \mathbb{R}, 1 \leq r, p, q \leq \infty.
\]

For the bilinear term \( B(u, v) \), in terms of Bony-paraproduct decomposition, one writes

\[
u = I + II + III,
\]

where

\[
u I \triangleq \sum_{k \in \mathbb{Z}} \int_0^t e^{-\nu(t-\tau)}\Delta P(\check{S}_k u \otimes \check{A}_k v) \, d\tau, \quad II \triangleq \sum_{k \in \mathbb{Z}} \int_0^t e^{-\nu(t-\tau)}\Delta P(\check{S}_k v \otimes \check{A}_k u) \, d\tau
\]

\[
u III \triangleq \sum_{k \in \mathbb{Z}} \int_0^t e^{-\nu(t-\tau)}\Delta P(\check{A}_k u \otimes \check{A}_k v) \, d\tau.
\]

For \( I \), by Hölder’s and Young’s inequalities, we have

\[
u 2^j \|\check{A}_j I\|_{L^\infty([0, T]; F L^p)} \lesssim \nu \int_0^T e^{-\nu t}2^{j(4s-3+\frac{2}{p})} \, dt \sum_{|j-k| \leq 2} 2^{(j-k)(2s-3+\frac{3}{p})} 2^{k(s-3+\frac{3}{p})} \|\check{S}_{k-1} u\|_{L^\infty([0, T]; F L^1)} \times 2^{ks} \|\check{A}_k v\|_{L^\infty([0, T]; F L^p)}
\]

\[
u \lesssim T^{\frac{3-s}{2}} \sum_{|j-k| \leq 2} 2^{(j-k)(2s-3+\frac{3}{p})} 2^{k(s-3+\frac{3}{p})} \|\check{S}_{k-1} u\|_{L^\infty([0, T]; F L^1)} 2^{ks} \|\check{A}_k v\|_{L^\infty([0, T]; F L^p)}.
\]
Taking $\ell^q(\mathbb{Z})$-norm on the above inequality, we get
\begin{equation}
\|I\|_{\tilde{L}^\infty([0,T]; FB^s_{p,q})} \lesssim \nu T^{s-p} \|u\|_{\tilde{L}^\infty([0,T]; FB^s_{p,q})} \|v\|_{\tilde{L}^\infty([0,T]; FB^s_{p,q})}. \tag{3.4}
\end{equation}
Moreover, we have
\begin{align*}
2^{j(s+2)} \|\Delta_j I\|_{L^1([0,T]; F^{Lp})} & \lesssim \nu \int_0^T e^{-c\tilde{t}2^j} 2^{j(4-s-\frac{2}{p})} dt \sum_{|j-k| \leq 2} 2^{(j-k)(2s-1+\frac{2}{p})} 2^{k(s-3+\frac{2}{p})} \tilde{\Delta}_{k-1} u\|_{L^\infty([0,T]; F^{L^1})} \\
& \quad \times 2^{k(s+2)} \|\tilde{\Delta}_k v\|_{L^1([0,T]; F^{Lp})} \\
& \lesssim \nu T^{s-p} \sum_{|j-k| \leq 2} 2^{(j-k)(2s-1+\frac{2}{p})} 2^{k(s-3+\frac{2}{p})} \tilde{\Delta}_{k-1} u\|_{L^\infty([0,T]; F^{L^1})} 2^{k(s+2)} \|\tilde{\Delta}_k v\|_{L^1([0,T]; F^{Lp})}.
\end{align*}
Hence, we have
\begin{equation}
\|I\|_{\tilde{L}_1([0,T]; FB^{s+2}_{p,q})} \lesssim \nu T^{s-p} \|u\|_{\tilde{L}^\infty([0,T]; FB^s_{p,q})} \|v\|_{\tilde{L}_1([0,T]; FB^{s+2}_{p,q})}. \tag{3.5}
\end{equation}
Similarly, we can show
\begin{equation}
\|III\|_{\tilde{L}^\infty([0,T]; FB^s_{p,q})} \lesssim \nu T^{s-p} \|u\|_{\tilde{L}^\infty([0,T]; FB^s_{p,q})} \|v\|_{\tilde{L}^\infty([0,T]; FB^s_{p,q})} \tag{3.6}
\end{equation}
and
\begin{equation}
\|II\|_{\tilde{L}_1([0,T]; FB^{s+2}_{p,q})} \lesssim \nu T^{s-p} \|u\|_{\tilde{L}_1([0,T]; FB^{s+2}_{p,q})} \|v\|_{\tilde{L}^\infty([0,T]; FB^s_{p,q})}. \tag{3.7}
\end{equation}
For the remainder term $III$, for any $r > 2$ satisfying $-s < \frac{2}{r} < s - \frac{2}{2} + 1$, we have by Hölder’s and Young’s inequalities that
\begin{align*}
2^{js} \|\Delta_j III\|_{L^\infty([0,T]; F^{Lp})} & \lesssim \nu \left( \int_0^T e^{-c\tilde{t}2^j} 2^{j(4-s-\frac{2}{p})} dt \right)^{\frac{s-2}{2}} \sum_{k \geq j-2} 2^{(j-k)(2s+\frac{2}{p})} \|\tilde{\Delta}_k u\|_{L^r([0,T]; F^{Lp})} \\
& \quad \times 2^{k(s+\frac{2}{p})} \|\tilde{\Delta}_k v\|_{L^r([0,T]; F^{Lp})} \\
& \lesssim \nu \left( \int_0^T t^{-\frac{s}{2}} dt \right)^{\frac{1}{2}} \sum_{k \geq j-2} 2^{(j-k)(2s+\frac{2}{p})} \|\tilde{\Delta}_k u\|_{L^r([0,T]; F^{Lp})} \\
& \quad \times 2^{k(s+\frac{2}{p})} \|\tilde{\Delta}_k v\|_{L^r([0,T]; F^{Lp})} \\
& \lesssim \nu T^{s-p} \sum_{k \geq j-2} 2^{(j-k)(2s+\frac{2}{p})} \|\tilde{\Delta}_k u\|_{L^r([0,T]; F^{Lp})} 2^{k(s+\frac{2}{p})} \|\tilde{\Delta}_k v\|_{L^r([0,T]; F^{Lp})}.
\end{align*}
So, by the interpolation and Young inequalities, we get
\begin{equation}
\|III\|_{\tilde{L}^\infty([0,T]; FB^s_{p,q})} \lesssim \nu T^{s-p} \|u\|_{L^r([0,T]; FB^{s+2}_{p,q})} \|v\|_{L^r([0,T]; FB^{s+2}_{p,q})}. \tag{3.8}
\end{equation}
Furthermore, we have
\[
2^{j(s+2)}\|\hat{\Delta}u_{III}\|_{L^1([0,T]; FL^p)}
\leq \nu \int_0^T e^{-\nu t^2} 2^{j(4-s-\frac{2}{p})} dt \sum_{k>j-2} 2^{(j-k)(2s+2)2k} \|\hat{\Delta}k u\|_{L^\infty([0,T]; FL^p)} 2^{k(s+2)} \|\hat{\Delta}k v\|_{L^1([0,T]; FL^p)}
\]
\[
\leq \nu T^{\frac{s}{p_2}} \sum_{k>j-2} 2^{(j-k)(2s+2)2k} \|\hat{\Delta}k u\|_{L^\infty([0,T]; FL^p)} 2^{k(s+2)} \|\hat{\Delta}k v\|_{L^1([0,T]; FL^p)}
\]
From this, we obtain that
\[
(3.9) \quad \|III\|_{L^1([0,T]; F\hat{B}_{p,q}^{s+2})} \leq \nu T^{\frac{s}{p_2}} \|u\|_{L^\infty([0,T]; F\hat{B}_{p,q}^{s})} \|v\|_{L^1([0,T]; F\hat{B}_{p,q}^{s+2})}.
\]
Collecting (3.4)-(3.9), we finally obtain that
\[
\|B(u, v)\|_{L^\infty([0,T]; F\hat{B}_{p,q}^{s+2})} \leq \nu T^{\frac{s}{p_2}} \|u\|_{L^\infty([0,T]; F\hat{B}_{p,q}^{s})} \|v\|_{L^1([0,T]; F\hat{B}_{p,q}^{s+2})}.
\]
By choosing a suitable \( T > 0 \), we obtain that there exists a unique local mild solution \( u \in X_T \) of problem (NS) in subcritical spaces \( F\hat{B}_{p,q}^{s} (\mathbb{R}^3) \) with \( s \in (s_p, 0] \) via the Banach fixed point theorem.

Next, we will give the analyticity and decay estimates for the solution of the mild solution to problem (NS) constructed by Theorem 3.1 and Theorem 3.2.

**Proposition 3.3.** Let \( s \in [s_p, 0] \). Assume that \( u \) is the mild solution on \( \mathbb{R}^3 \times [0, T^\ast) \) to (NS) constructed by Theorem 3.1 or Theorem 3.2. Then,
\[
(3.10) \quad \|t u(t)\|_{L^\infty([0,T]; F\hat{B}_{p,q}^{s+2})} < \infty,
\]
for each \( T < T^\ast \).

**Proof.** Set
\[
Y_T \triangleq \widetilde{L}^\infty([0,T]; F\hat{B}_{p,q}^{s+2}(\mathbb{R}^3)) \cap \widetilde{L}^1([0,T]; F\hat{B}_{p,q}^{s+2}(\mathbb{R}^3)).
\]
We do To do this, we need to show that:

(i) there exists a constant \( C(\nu) > 0 \) such that
\[
\max \left\{ \|tB(u, v)\|_{L^\infty([0,T]; F\hat{B}_{p,q}^{s+2})}, \|tB(v, u)\|_{L^\infty([0,T]; F\hat{B}_{p,q}^{s+2})} \right\} 
\leq C(\nu) T^{\frac{s}{p_2}} \|u\|_{Y_T} \|v\|_{Y_T} + C(\nu) T^{\frac{s}{p_2}} \|u\|_{Y_T} \|v\|_{Y_T}.
\]
(ii) \( t e^{t\Delta} u_0 \in \widetilde{L}^\infty([0,T]; F\hat{B}_{p,q}^{s+2}) \).
Let us start by estimating \( B(u, v) \). By Holder’s and Young’s inequality, the term \( I \) can be bounded as follows
\[
2^{j(s+1)} t \|\hat{\Delta}j I\|_{FL^p}
\leq t \sum_{|j-k| \leq 2} 2^{(j-k)(2s-3+\frac{2}{p}+\frac{2}{q})} 2^{j(-s-\frac{2}{p}+\frac{4}{q})} \left( \int_0^{t/2} e^{-\nu (t-\tau)^{22}} \frac{d\tau}{\tau^{\frac{4}{7}}} + \int_{t/2}^t e^{-\nu (t-\tau)^{22}} \frac{d\tau}{\tau^{\frac{4}{7}}} \right)
\]

Choosing $r \in (\frac{2p}{3p-3}, \infty)$ such that $-s + 4 - \frac{3}{p} - \frac{2}{r} > 0$ and $-s + 6 - \frac{3}{p} - \frac{2}{r} > 0$, we have
\[
2^{j(s+1)} t^\nu \Delta_j I \|_{FL^p} \leq \nu T \frac{s-3p}{p} \| u \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \| t^\nu v \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})}.
\]
By the interpolation and Young’s inequalities, we get
\[
\begin{align*}
\| t I \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} &\leq \nu T \frac{s-3p}{p} \| u \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \| t^\nu v \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \\
&\leq \nu T \frac{s-3p}{p} \| u \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \| t^\nu v \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \\
&\quad + T \frac{s-3p}{p} \| u \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \| v \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})}.
\end{align*}
\]
(3.11)

For $II$, similar to $I$, for any $r \in (\frac{2p}{3p-3}, \infty)$, we have
\[
\begin{align*}
&\leq \nu T \frac{s-3p}{p} \| u \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \| t^\nu v \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \\
&\quad + T \frac{s-3p}{p} \| u \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \| v \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})}.
\end{align*}
\]
Since $r \in (\frac{2p}{3p-3}, \infty)$, which ensures $-s + 3 + \frac{3}{p} + \frac{2}{r} < 0$, taking $L^\infty[0,T]$ and then $\ell^q(Z)$ on above inequality, by interpolation inequality and Young’s inequality, we get
\[
\begin{align*}
\| t II \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} &\leq \nu T \frac{s-3p}{p} \| u \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \| t^\nu v \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \\
&\quad + T \frac{s-3p}{p} \| u \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \| v \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})}.
\end{align*}
\]
(3.12)
For the remainder term $III$, by Hölder’s and Young’s inequalities, for any $r \in (\frac{2p}{3p-3}, \infty)$ and $r_1 > \frac{r}{r-1}$, we have
\[
\begin{align*}
&\leq \nu T \frac{s-3p}{p} \| u \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \| t^\nu v \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \\
&\quad + T \frac{s-3p}{p} \| u \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})} \| v \|_{L^\infty([0,T]; F_{BP^{(s+1)}_p})}.
\end{align*}
\]
Due to $\frac{r_1}{r_1-1} < r$, we have that
\[
\left( \int_0^t e^{-\nu(t-\tau)^{2g}} \frac{\nu_1}{r_1-1} \frac{\nu}{r_1-1} d\tau \right)^{\frac{r_1-1}{r_1}} \leq \left( \int_0^{\frac{r}{2}} e^{-\nu(t-\tau)^{2g}} \frac{\nu}{r_1-1} d\tau \right)^{\frac{r_1-1}{r_1}} + \left( \int_{\frac{r}{2}}^t e^{-\nu(t-\tau)^{2g}} \frac{\nu}{r_1-1} d\tau \right)^{\frac{r_1-1}{r_1}} \leq C e^{-\frac{\nu}{2} t^{2g} t^{-\frac{1}{2} - \frac{1}{r_1}}} + C \delta^{-\frac{(2 - \frac{2}{p})}{r_1}} e^{-c \nu t^{2g} t^{-\frac{1}{p}}}.
\]
Substituting this estimate into the above inequality, thanks to that $6 - s - \frac{3}{p} + \frac{2}{r} + \frac{2}{r_1} > 0$ and $4 - s - \frac{3}{p} - \frac{2}{r} > 0$, we can get that
\[
2^{j(s+2)} t \|\Delta_j III\|_{F_{L^p}} \lesssim \nu t \sum_{k>j-2} 2^{(j-k)(2s+\frac{2}{r_1} + \frac{2}{r})} \left( 2^{j(6 - \frac{3}{p} - s - \frac{2}{r} - \frac{2}{r_1})} e^{-\frac{\nu}{2} t^{2g} t^{-\frac{1}{2} - \frac{1}{r_1}}} + 2^{j(4 - \frac{3}{p} - s - \frac{2}{r} - \frac{2}{r_1})} e^{-c \nu t^{2g} t^{-\frac{1}{p}}} \right) 
\times 2^k \|\Delta_k v\|_{L^\infty([0,T];F_{L^p})} 2^{k(s+\frac{2}{r_1})} \|\Delta_k u\|_{L^1([0,T];F_{L^p})} \lesssim \nu \sum_{k>j-2} 2^{(j-k)(2s+\frac{2}{r_1} + \frac{2}{r})} 2^k \|\Delta_k v\|_{L^\infty([0,T];F_{L^p})} 2^{k(s+\frac{2}{r_1})} \|\Delta_k u\|_{L^1([0,T];F_{L^p})}.
\]
Then, we choose $r_1 \in (\frac{r}{r_1-1}, \infty)$ such that $s + \frac{1}{r_1} + \frac{1}{r_1} > 0$, that is $\frac{1}{r_1} \in (\frac{3}{p} - 2 - \frac{1}{r}, 1 - \frac{1}{r})$, we get that
\[
\|t^{III}\|_{L^\infty([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})} \lesssim \nu T^{\frac{s+2}{r_1}} \|t^{IV}\|_{L^\infty([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})} \|u\|_{L^1([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})}.
\]
By the interpolation and Young inequalities, we have
\[
\|t^{III}\|_{L^\infty([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})} \lesssim \nu T^{\frac{s+2}{r_1}} \|tv\|_{L^\infty([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})} \|v\|_{L^1([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})} \|u\|_{L^1([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})}.
\]
(3.13)
Collecting estimates (3.11)-(3.13), we immediately obtain
\[
\|tB(u,v)\|_{L^\infty([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})} \lesssim \nu T^{\frac{s+2}{r_1}} \|u\|_{Y_2} \|tv\|_{L^\infty([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})} + T^{\frac{s+2}{r_1}} \|u\|_{Y_2} \|v\|_{Y_2}.
\]
In the same way, we can show
\[
\|tB(v,u)\|_{L^\infty([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})} \lesssim \nu T^{\frac{s+2}{r_1}} \|u\|_{Y_2} \|tv\|_{L^\infty([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})} + T^{\frac{s+2}{r_1}} \|u\|_{Y_2} \|v\|_{Y_2}.
\]
Finally, we tackle with the term $e^{\nu t^2} u_0$. Note that
\[
2^{j(s+2)} t \|\Delta_j e^{\nu t^2} u_0\|_{F_{L^p}} \lesssim 2^{j(s+2)} t e^{-c \nu t^{2g} t^{-\frac{1}{p}}} \|\Delta_j u_0\|_{F_{L^p}} \lesssim \nu 2^{js} \|\Delta_j u_0\|_{F_{L^p}},
\]
we have
\[
\|te^{\nu t^2} u_0\|_{L^\infty([0,T];F_{B^{s+\frac{2}{r_1}}_{p,q}})} \lesssim \nu \|u_0\|_{F_{B^{s+\frac{2}{r_1}}_{p,q}}}.
\]
Combining estimates (3.14)-(3.16), by Lemma 2.5 and continuity argument, we finally get
\[ \|tu(t)\|_{L^\infty([0,T];F^{s+2}_{p,q})} < \infty. \]
Thus we complete the proof of Proposition 3.3. \(\square\)

Remark 3.4. From Proposition 3.3 we see that for any \(q < \infty\)
\[ \lim_{T \to 0^+} \|tu(t)\|_{L^\infty([0,T];F^{s+2}_{p,q})} = 0. \]
For any \(\varepsilon > 0\), from (3.16) and the fact that \(q < \infty\), we can choose \(N \in \mathbb{N}\) such that
\[ \sum_{|j| \geq N} 2^{j(s_p+2)q} \|te^{\rho t\Delta}u_0\|_{L^q([0,T];F^p)} < \varepsilon / 2^q. \]
Since \(te^{\rho t\Delta}\) is tends to zero as \(t \to 0^+\), there exists a \(\delta > 0\) such that for any \(t \in [0,\delta)\)
\[ \|te^{\rho t\Delta}u_0\|_{F^p} \leq 2^{-1+(2N-1)(s_p+2)\varepsilon}, \]
which implies that for any \(0 \leq T < \delta\)
\[ \|te^{\rho t\Delta}u_0\|_{L^\infty([0,T];F^p)} < 2^{-1+(2N-1)(s_p+2)\varepsilon}. \]
Thus, for any \(0 \leq T < \delta\)
\[ \sum_{|j| < N} 2^{j(s_p+2)q} \|te^{\rho t\Delta}u_0\|_{L^q([0,T];F^p)} < \varepsilon / 2^q. \]
Combining (3.17) and (3.18) yields
\[ \lim_{T \to 0^+} \|te^{\rho t\Delta}u_0\|_{L^\infty([0,T];F^{s+2}_{p,q})} = 0. \]
With this property, we have by Proposition 3.3 that
\[ \lim_{T \to 0^+} \|tu(t)\|_{L^\infty([0,T];F^{s+2}_{p,q})} = 0. \]
Moreover, we readily obtain that
\[ \sup_{0 \leq t < T} t^{\frac{s}{2} - \frac{1}{q} + \frac{1}{2p} - \frac{s_p}{2p}} \|\nabla^\alpha u(t)\|_{F^{\alpha \eta}_{L^n}} < \infty, \quad \forall (\alpha, \eta) \in \{0,1\} \times [1,p] \]
as \(s \in (s_p, 0)\), and
\[ \sup_{0 \leq t < T} t^{\frac{s}{2} - \frac{1}{q} + \frac{1}{2p} - \frac{s_p}{2p}} \|\nabla^\alpha u(t)\|_{F^{\alpha \eta}_{L^n}} < \infty, \quad \forall (\alpha, \eta) \in \{0,1\} \times [1,p] \} \times 1, 1) \]
as \(s = s_p\).
Indeed, by Minkowski’s inequality, we have
\[ \sup_{0 \leq t \leq T} t \|u(t)\|_{F^{s+2}_{p,q}} < \infty \]
Then, by the sharp interpolation inequality, we have that for any \(0 \leq t \leq T\)
\[ \|\nabla^\alpha u(t)\|_{F^{\alpha \eta}_{L^n}} \leq \|u(t)\|_{F^{s+2}_{p,q}} \leq \|u(t)\|_{F^{s+2}_{p,q}} \leq \|u(t)\|_{F^{s+2}_{p,q}} \leq \|u(t)\|_{F^{s+2}_{p,q}} \leq \|u(t)\|_{F^{s+2}_{p,q}} \leq \|u(t)\|_{F^{s+2}_{p,q}} \]
for any \((\alpha, \eta) \in \{0,1\} \times [1,p] \) if \(s \in (s_p, 0)\) or \(\{(\alpha, \eta) \in \{0,1\} \} \times [1,p] \} \times 1, 1) \) if \(s = s_p\).
This gives that
\[
\sup_{0 \leq t \leq T} t^{\frac{\alpha}{2} - \frac{3}{4} + \frac{3}{2p} - \frac{1}{2q}} \|u(t)\|_{F_{L^p}^{s+\alpha}} \leq \left( \sup_{0 \leq t \leq T} t \|u(t)\|_{F_{\tilde{B}^{s+\alpha}_2}^{s+\alpha}} \right)^\frac{1}{\alpha + \frac{3}{2} - \frac{3}{2p} + \frac{1}{2q}} \frac{1}{2p} \|u(t)\|_{F_{\tilde{B}^{s+\alpha}_p}^{s+\alpha}}.
\]

4. SOME PROPERTIES OF WEAK ENERGY SOLUTIONS TO PERTURBED EQUATIONS

In this section, we will study some properties of weak energy solutions to the perturbed equations

\[
\begin{cases}
\partial_t w - \nu \Delta w + w \cdot \nabla w + v \cdot \nabla w + w \cdot \nabla v + \nabla \tilde{P} = 0, \\
div w = 0,
\end{cases}
\]

(4.1)

with that \(v\) is a mild solution to (NS) established in Theorem 3.2, satisfying

\[
v \in C([0,T]; \tilde{B}^{s+\alpha}_{p,q}(\mathbb{R}^3)) \cap L^r([0,T]; \tilde{B}^{s+\alpha}_{p,q}(\mathbb{R}^3)) \quad \forall r \in [1, \infty]
\]

with \(\tilde{P} \in (1,3/2)\) and \(s \in (s_{\tilde{P}}, 0)\). Since \(w_0 \in L^2(\mathbb{R}^3)\) satisfies \(\text{div } w_0 = 0\), by the Fardo-Galerkin method used in [1], we can show that problem (4.1) admits at most one weak energy solution \((w, \tilde{P})\) satisfies the following conditions:

(W1) \(w \in L^\infty((0,T); L^2(\mathbb{R}^3)) \cap L^2((0,T); \tilde{H}^1(\mathbb{R}^3)), \quad \tilde{P} \in (L^{3/2} + L^2) (\mathbb{R}^3 \times (0,T)).\)

(W2) \((w, \tilde{P})\) satisfies Eq. (4.1) in the sense of distribution on \(\mathbb{R}^3 \times (0,T)\).

(W3) \(\lim_{t \to 0^+} \|w(t) - w_0\|_{L^2} = 0.\)

(W4) For any \(\varphi \in D(\mathbb{R}^3 \times (0,T)), \varphi \geq 0\), the generalized local energy inequality holds

\[
\begin{aligned}
\int_{\mathbb{R}^3 \times (t)} |w|^2 \varphi \, dx + 2\nu \int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 \varphi \, dx \, ds \\
\leq \int_0^t \int_{\mathbb{R}^3} |w|^2 (\partial_t \varphi + \nu \Delta \varphi) \, dx \, ds + \int_0^t \int_{\mathbb{R}^3} |w|^2 (w + v) \cdot \nabla \varphi \, dx \, ds \\
+ 2 \int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla \varphi)(v \cdot w + (w \cdot \nabla w) \cdot \nabla \varphi) \, dx \, ds + 2 \int_0^t \int_{\mathbb{R}^3} \tilde{P} w \cdot \nabla \varphi \, dx \, ds.
\end{aligned}
\]

(W5) \(w\) satisfies the generalized global energy inequality on \(\mathbb{R}^3 \times (0,T)\):

\[
\int_{\mathbb{R}^3 \times (t)} |w|^2 \, dx + 2\nu \int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \, ds \leq \|w_0\|_{L^2}^2 + 2 \int_0^t (w \cdot \nabla) w \cdot v \, dx \, ds.
\]

Following the argument used in [4], we can show that this weak energy solution is also the suitable weak solution which is defined as follows:

**Definition 4.1** (Suitable weak solutions). Let \(D = \Omega \times (0,T)\) be an open set of \(\mathbb{R}^3 \times \mathbb{R}^+\) and \(v \in L^p L^q(D)\) satisfying \(\text{div } v = 0, \frac{2}{p} + \frac{2}{q} \leq 1\) and \(3 \leq q \leq \infty\). Then we call \((w, \tilde{P})\) is a suitable weak solution in \(D\) to equations:

\[
\begin{cases}
\partial_t w - \nu \Delta w + w \cdot \nabla w + v \cdot \nabla w + w \cdot \nabla v + \nabla \tilde{P} = 0, \\
div w = 0
\end{cases}
\]

(4.4)

if \((w, \tilde{P})\) satisfies
the condition with the following property: for any 0 there exists an absolute constant ǫ.

Lemma 4.4. Then, the perturbed equations (4.4), by De Giorgi iteration and dimensional analysis.

Theorem 4.1 (4.2) Remark published in [4], we will shown the following ε-regularity criterion of suitable weak solutions to problem (4.4), and the short-time behaviour of the kinetic energy and weak-strong uniqueness of the local energy solutions to problem (4.4).

In this subsection, inspired by CKN’s theorem established in [14], we will discuss the partial regularity and stability of singularities of suitable weak solutions to problem (4.4), and the short-time behaviour of the local energy solutions to problem (4.4).

4.1. Partial regularity criterion. In this subsection, inspired by CKN’s theorem established in [14], we will shown the following ε-regularity criteria of suitable weak solutions to the perturbed equations (4.4), by De Giorgi iteration and dimensional analysis.

Theorem 4.1 (ε-regularity criterion). Let v ∈ L^p_t L^q_x (Q_R(x_0, t_0)) satisfies div v = 0 with \( \frac{2}{p} + \frac{3}{q} < 1, q > 3 \). Assume (w, P) is a suitable weak solution to Eq. (4.4) in Q_R(x_0, t_0). Then there exists an absolute constant \( \epsilon_0 = \epsilon_0(p, q, \|v\|_{L^p_t L^q_x}) > 0 \) with the following property: if

\[
R^{-2} \int_{Q_R(x_0, t_0)} \left( |w|^3 + \left| \frac{\partial P}{\partial t} \right|^{\frac{2}{3}} \right) \, dx \, dt < \epsilon_0,
\]

then there exists a constant \( C^* > 0 \) such that

\[
\|w\|_{L^\infty(Q_R/2(x_0, t_0))} \leq C^* R^{-1}.
\]

Remark 4.2. Let us point out that in [17], Jia and Šverák proved the same result under the condition \( v \in L^m(Q_R(x_0, t_0)) \) with \( m > 5 \). Here, we consider the general case that \( v \in L^p_t L^q_x(Q_R(x_0, t_0)) \) and give a new proof is based on De Giorgi iteration and dimensional analysis.

Before proving Theorem 4.1, we first give two useful lemmas.

Lemma 4.3 ([10]). Let f be a nonnegative nondecreasing bounded function defined on [0, 1] with the following property: for any 0 ≤ r ≤ s < t < R ≤ 1 and some constants \( \theta \in (0, 1), M > 0, \beta > 0 \), we have

\[
f(s) \leq \theta f(t) + \frac{M}{(t-s)^\beta}.
\]

Then,

\[
\sup_{s \in [0, r]} f(s) \leq C(\theta, \beta) \frac{M}{(R-r)^\beta}.
\]

Lemma 4.4. There exists a constant \( C > 0 \) such that for any \( u \in L^\infty_t L^q_x(Q_r) \) and \( \nabla u \in L^2_t L^\infty_x(Q_r) \),

\[
\|u\|_{L^\infty_t L^q_x(Q_r)} \leq C(\|u\|_{L^\infty_t L^q_x(Q_r)} + \|\nabla u\|_{L^2(Q_r)})
\]

for any \( 2/m + 3/n = 3/2 \) with \( 2 \leq n \leq 6 \).

Proof. By Gagliardo-Nirenberg’s inequality, we readily have

\[
\|u(t)\|_{L^q(B_r)} \leq C \|\nabla u(t)\|_{L^2(B_r)} + C r^{-1} \|u(t)\|_{L^2(B_r)}.
\]
Moreover, we have by Hölder’s inequality that
\[ \|u\|_{L^2_t L^r_x(B_r)} \leq C(\|\nabla u\|_{L^2(Q_r)} + \|u\|_{L^\infty_t L^2_x(B_r)}). \]

Hence, the interpolation between \( L^\infty_t L^2_x(Q_r) \) and \( L^2_t L^6_x(Q_r) \) entails
\[ \|u\|_{L^m_t L^2_x(Q_r)} \leq C(\|u\|_{L^\infty_t L^2_x(Q_r)} + \|\nabla u\|_{L^2(Q_r)}), \]
where \( 2/m + 3/n = 2/3 \) and \( 2 \leq n \leq 6 \).

Now, we come back to the proof of Theorem 4.1. Letting \( R_0 < \frac{R}{2} \), we find that
\[ Q_{R_0}(x_1, t_1) \subset Q_R(x_0, t_0) \quad \text{for each} \quad (x_1, t_1) \in Q_{\frac{R}{2}}(x_0, t_0). \]

Denoting
\[ w_{R_0}(x, t) = R_0w(x_1 + R_0x, t_1 + R_0^2t), \quad \bar{P}_{R_0}(x, t) = R_0^2\bar{P}(x_1 + R_0x, t_1 + R_0^2t), \]
\[ v_{R_0}(x, t) = R_0v(x_1 + R_0x, t_1 + R_0^2t), \]
we find that the couple \((w_{R_0}, \bar{P}_{R_0})\) is also a suitable weak solution to equations (4.4) in \( Q_1 \) with \( v_{R_0} \) instead of \( v \). And we have
\[ \int_{Q_1} \left( |w_{R_0}|^3 + |\bar{P}_{R_0}|^{3/2} \right) \, dx \, dt = R_0^{-2} \int_{Q_{R_0}(x_1, t_1)} \left( |w|^3 + |\bar{P}|^{3/2} \right) \, dx \, dt \leq R_0^{-2} \epsilon_0 \]
and
\[ \|v_{R_0}\|_{L^p_t L^q_x(Q_1)} = R_0^{1-2/p-3/q} \|v\|_{L^p_t L^q_x(Q_{R_0}(x_1, t_1))} \leq R_0^{1-2/p-3/q} \|v\|_{L^p_t L^q_x(Q_{R_0}(x_0, t_0))}. \]

Set \( \epsilon_1 = R_0^{1-2/p-3/q} \|v\|_{L^p_t L^q_x(Q_{R_0}(x_0, t_0))} \). Since \( 2/p + 3/q < 1 \), we can choose \( R_0 \) small enough such that \( \epsilon_1 \ll 1/2 \). Fixed \( R_0 \), then choose suitable small \( \epsilon_0 \) such that \( R_0^{-2} \epsilon_0 < \epsilon_1 \). In the following proof, without ambiguity, we still use \((w, \bar{P}, v)\) to denote \((w_{R_0}, \bar{P}_{R_0}, v_{R_0})\). Therefore, we have
\[ \int_{Q_1} \left( |w|^3 + |\bar{P}|^{3/2} \right) \, dx \, dt + \|v\|_{L^p_t L^q_x(Q_1)} < 2\epsilon_1. \]
This estimate allows us to claim that there exists a positive constant \( C^* \) such that
\[ (4.5) \quad \|u\|_{L^\infty_t L^2_x(Q_2)} \leq C^*. \]

By taking the inverse transform of scaling, we immediately obtain the theorem.

We turn to prove claim (4.5). For any \( k \in \{0, 1, 2, \cdots\} \), we introduce a new function
\[ w_k = \lceil |w| - (1 - 2^{-k}) \rceil_+. \]
Since \( w_k^2 \) equals to 0 for \( |w| < 1 - 2^{-k} \) and is of the order of \( |w|^2 \) for \( |w| \gg 1 - 2^{-k} \), \( w_k^2 \) can be seen as a level set of energy.

Let us denote
\[ r_k = 2^{-1} + 2^{-k-2}, \quad r_{k-1/3} = 2^{-1} + 2^{-k-2+1/3}, \quad r_{k-2/3} = 2^{-1} + 2^{-k-2+2/3}. \]
Then, we define
\[ W_k \triangleq \text{ess sup}_{-r_k^2 \leq t \leq 0} \int_{B_{r_k}} |w_k|^2 \, dx + \int_{Q_{r_k}} |d_k|^2 \, dx \, dt. \]
Moreover, we can control the following gradients by such that, for any $w_k$, $d_k$ and $W_k$, we have the following well-known properties which will be useful in the following proof.

**Lemma 4.5 ([24]).** In the light of the definition of $w_k$ and $d_k$, the function $w$ can be decomposed as follows:

$$w = w\left(1 - \frac{w_k}{|w|}\right) + w_k\frac{w_k}{|w|}$$

satisfying

$$\left|w\left(1 - \frac{w_k}{|w|}\right)\right| \leq 1 - 2^{-k}.$$  

Moreover, we can control the following gradients by $d_k$:

$$\frac{w_k}{|w|} |\nabla w| \leq \nu^{-\frac{1}{2}} d_k, \quad I_{\{|w| \geq 1 - 2^{-k}\}} |\nabla w| \leq \nu^{-\frac{1}{2}} d_k,$$

$$|\nabla w_k| \leq \nu^{-\frac{1}{2}} d_k \quad \text{and} \quad \left|\nabla w_k\frac{w_k}{|w|}\right| \leq 3 \nu^{-\frac{1}{2}} d_k.$$  

**Lemma 4.6.** For any $2 \leq m \leq \infty$, $2 \leq n \leq 6$ satisfying $\frac{2}{m} + \frac{3}{n} = \frac{3}{2}$, there exists a constant $C(\nu) > 0$ such that, for any $k \geq 0$,

$$\|w_k\|_{L^m_t L^n_x(Q_{r_k})} \leq C(\nu) W_k^{1/2}.$$  

**Proof.** According to Lemma 4.3 and Lemma 4.5 we have

$$\|w_k\|_{L^m_t L^n_x(Q_{r_k})} \lesssim \|w_k\|_{L^\infty_t L^2(Q_{r_k})} + \|\nabla w_k\|_{L^2(Q_{r_k})} \lesssim \nu \|w_k\|_{L^\infty_t L^2(Q_{r_k})} + \|d_k\|_{L^2(Q_{r_k})} \lesssim \nu W_k^{1/2}.$$  

This implies the desired estimate. \hfill $\square$  

**Lemma 4.7.** In the light of the definition of $w_k$ and $W_k$, there exists a constant $C(\nu) > 0$ such that, for any $k \geq 1$, $q > 1$ and $\frac{2}{m} + \frac{4}{n} = 1$ with $2 \leq m \leq \infty$, we have

$$\|I_{w_k > 0}\|_{L^q(Q_{r_1})} \leq C(\nu) 2^{10k/3q} W^{5/3q}_{k-1}, \quad \|I_{w_k > 0}\|_{L^\infty_t L^n_x(Q_{r_{k-1}})} \leq C(\nu) 2^{2k/q} W_{k-1}^{1/q}$$

and

$$\|I_{w_k > 0}\|_{L^m_t L^n_x(Q_{r_{k-1}})} \leq C(\nu) 2^{k} W_{k-1}^{1/2}.$$  

**Proof.** Here we omit the proof, because it is standard and is similar with that of [24]. \hfill $\square$  

Now we begin to prove (4.5) step by step.  

**Step 1.** $W_0 \leq C_1(\nu) \varepsilon_1$ for some $C_1(\nu) > 0$.  

For any $\rho > 0$, set

$$E(\rho) = \text{ess sup}_{-\rho^2 < \tau \leq 0} \int_{B_\rho} |w|^2 \, dx + 2\nu \int_{Q_\rho} |\nabla w|^2 \, dx \, d\tau.$$  

For any $3/4 \leq \rho_1 < \rho_2 \leq 1$, let $\varphi \in \mathcal{D}(Q_{\rho_2})$ satisfying

$$0 \leq \varphi \leq 1, \quad \varphi \equiv 1 \quad \text{in} \ Q_{\rho_1}$$  

and $\varphi \equiv 0 \quad \text{in} \ Q_{\rho_2}.$$
and
\[(\rho_2 - \rho_1)^2 |\partial_t \varphi| + \sum_{i=1}^2 (\rho_2 - \rho_1)^i |\nabla^i \varphi| \leq C \text{ in } Q_{\rho_2}.
\]

Applying \(\varphi\) to the local energy inequality (4.1), by Hölder’s inequality and Lemma 4.3, we have
\[
E(\rho_1) \leq \frac{C(\nu)}{(\rho_2 - \rho_1)^2} \int_{Q_{\rho_2}} |w|^2 \, dx \, dt + \frac{C}{\rho_2 - \rho_1} \int_{Q_{\rho_2}} (|w|^3 + |w|^2|v|) \, dx \, dt
\]
\[+ C \int_{Q_{\rho_2}} \|\nabla w\| \|v\| \, dx \, dt + \frac{C}{\rho_2 - \rho_1} \int_{Q_{\rho_2}} \|\nabla^2 w\| \, dx \, dt
\]
\[\leq \frac{C(\nu)\epsilon_1^2}{(\rho_2 - \rho_1)^2}\|w\|_{L^3(Q_1)}^2 + \frac{C}{\rho_2 - \rho_1}\|w\|_{L^3(Q_1)}^3 + \frac{C}{\rho_2 - \rho_1}\|w\|_{L^{2q/3}L^{2q/(q-2)}(Q_{\rho_2})}\|\nabla w\|_{L^3(Q_{\rho_2})}\|v\|_{L_t^qL_t^\infty(Q_1)} + \frac{C}{\rho_2 - \rho_1}\|w\|_{L^3(Q_1)}\|\nabla^2 w\|_{L^3/2(Q_1)}
\]
\[\leq \frac{C(\nu)\epsilon_1^2}{(\rho_2 - \rho_1)^2} + C(\nu)\epsilon_1 E(\rho_2).
\]
Choosing \(\epsilon_1\) small enough such that \(C(\nu)\epsilon_1 < 1\), we get by Lemma 4.3 that
\[
(4.6) \quad W_0 = E(3/4) \leq C_1(\nu)\epsilon_1^{2/3}.
\]

**Step 2.** \(W_k \leq C_2(\nu)W_{k-1}^\beta\) with some \(C_2(\nu) > 1, \beta > 1\) for any \(k \geq 1\).

Since \(|w(w_k/w) - 1| \leq 1\), we can multiply the first equation of (4.4) by \(w(w_k/w) - 1\), we immediately have
\[
\partial_t w_k^2 - \frac{|w|^2}{2} - \nu \Delta w_k^2 - \frac{|w|^2}{2} - |\nabla w|^2 + d_k^2 + \text{div} \left((w + v) \frac{w_k^2 - |w|^2}{2}\right)
\]
\[+ w \left(\frac{w_k}{|w|} - 1\right) \text{div}(v \otimes w) + \nabla P \cdot w \left(\frac{w_k}{|w|} - 1\right) = 0
\]
in the sense of distribution on \(Q_1\).

This together with (4.4) gives that
\[
\int_{\mathbb{R}^3} |w_k|^2 \varphi \, dx + 2 \int_0^t \int_{\mathbb{R}^3} \frac{d_k^2 \varphi}{dx} \, dx \, dt
\]
\[\leq \int_0^t \int_{\mathbb{R}^3} |w_k|^2 (\partial_t \varphi + \nu \Delta \varphi) \, dx \, dt + \int_0^t \int_{\mathbb{R}^3} |w_k|^2 (w + v) \cdot \nabla \varphi \, dx \, dt
\]
\[+ 2 \int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla \varphi) \left(v \cdot w \frac{w_k}{|w|}\right) + \left(w \cdot \nabla w \frac{w_k}{|w|}\right) \cdot v \varphi \, dx \, dt
\]
\[+ 2 \int_0^t \int_{\mathbb{R}^3} \bar{P}w \cdot \nabla \varphi \, dx \, dt - 2 \int_0^t \int_{\mathbb{R}^3} \nabla \bar{P} \cdot w \left(\frac{w_k}{|w|} - 1\right) \varphi \, dx \, dt
\]
for any nonnegative function \(\varphi \in D(Q_1)\).

For any \(k \in \mathbb{N}\), let \(\varphi_k \in D(Q_1)\) satisfying
\[
0 \leq \varphi_k \leq 1, \quad \varphi_k \equiv 1 \text{ in } Q_{r_k}, \quad \varphi_k \equiv 0 \text{ in } Q_{r_k-1/3},
\]
Letting $\varphi_k$ to (1.7), by Lemma 4.5 H"older’s inequality and the fact $2/p + 3/q < 1$, we can get that

$$W_k \lesssim_{\nu} 2^{2k} \int_{Q_{R,k-1}} |w_k|^2 \, dx \, dt + 2^{k} \int_{Q_{R,k-1}} |\varphi_k|^2 \, dx \, dt + 2^{k} \int_{Q_{R,k-1}} (|w_k|^2 |v| + |v| |w_k|) \, dx \, dt$$

$$+ \int_{Q_{R,k-1}} (I_{|w_k| > 0} |d_k| |v| + |w_k| |d_k| |v|) \, dx \, dt$$

$$+ \left| \int_{0}^{t} \int_{\mathbb{R}^3} \left( \tilde{P} w \cdot \nabla \varphi_k - \nabla \tilde{P} \cdot w \left( \frac{|w_k|}{|v|} - 1 \right) \varphi_k \right) \, dx \, dt \right|.$$  

We now denote by I.1 and I.2 the first four terms and the last term on the right-hand side of the above inequality, respectively.

**Estimate of I.1.** Thanks to H"older’s inequality and the fact that $2/p + 3/q < 1$, we have the following estimate:

\[
I.1 \lesssim_{\nu} 2^{2k} \int_{Q_{R,k-1}} |w_k|^{2/3} (|w_k|) \| \varphi_k \|_{L^{5/2}(Q_{R,k-1})} + 2^k \int_{Q_{R,k-1}} |w_k|^3 \| \varphi_k \|_{L^{10/3}(Q_{R,k-1})} + 2^k \int_{Q_{R,k-1}} |w_k|^3 \| \varphi_k \|_{L^{10}(Q_{R,k-1})}
\]

\[
+ 2^k \| w_k \|^2 \| \varphi_k \|_{L^{2p/(p-1)} L^{5p/(p+2)}(Q_{R,k-1})} \| v \|_{L^p L^3(Q_{R,k-1})} + 2^k \| w_k \|^2 \| \varphi_k \|_{L^{2p/(p-1)} L^{5p/(p+2)}(Q_{R,k-1})} \| v \|_{L^p L^3(Q_{R,k-1})}
\]

\[
+ \| d_k \|_{L^2(Q_{R,k-1})} \| v \|_{L^p L^3(Q_{R,k-1})} \| I_{|w_k|>0} \|_{L^{5p/(p-2)} L^{5p/(p+2)}(Q_{R,k-1})}
\]

\[
+ \| d_k \|_{L^2(Q_{R,k-1})} \| w_k \|_{L^{2p/(p-2)} L^{5p/(p+4)}(Q_{R,k-1})} \| v \|_{L^p L^3(Q_{R,k-1})} \| I_{|w_k|>0} \|_{L^{5p/(p-2)} L^{5p/(p+2)}(Q_{R,k-1})},
\]

where $\frac{1}{t_1} = \frac{2}{3} > \frac{2}{3}$, $\frac{1}{q} > \frac{1}{3}$. By Lemma 4.6, Lemma 4.7 H"older’s inequality and the fact $w_k \leq w_{k-1}$, we deduce that

\[
I.1 \lesssim_{\nu} 2^{2k} \int_{Q_{R,k-1}} |w_k|^2 \| \varphi_k \|_{L^{5/2}(Q_{R,k-1})} + 2^k \int_{Q_{R,k-1}} |w_k|^3 \| \varphi_k \|_{L^{10/3}(Q_{R,k-1})} + 2^k \| w_k |^2 \| \varphi_k \|_{L^{2p/(p-1)} L^{5p/(p+2)}(Q_{R,k-1})} \| v \|_{L^p L^3(Q_{R,k-1})}
\]

\[
+ 2^k \| w_k |^2 \| \varphi_k \|_{L^{2p/(p-1)} L^{5p/(p+2)}(Q_{R,k-1})} \| v \|_{L^p L^3(Q_{R,k-1})} + \| d_k \|_{L^2(Q_{R,k-1})} \| v \|_{L^p L^3(Q_{R,k-1})} \| I_{|w_k|>0} \|_{L^{5p/(p-2)} L^{5p/(p+2)}(Q_{R,k-1})}
\]

\[
+ \| d_k \|_{L^2(Q_{R,k-1})} \| w_k \|_{L^{2p/(p-2)} L^{5p/(p+4)}(Q_{R,k-1})} \| v \|_{L^p L^3(Q_{R,k-1})} \| I_{|w_k|>0} \|_{L^{5p/(p-2)} L^{5p/(p+2)}(Q_{R,k-1})},
\]

(4.8)

**Estimate of I.2.** Denote $w$ and $v$ as $w = (w^1, w^2, w^3)$ and $v = (v^1, v^2, v^3)$. From (4.4), we see that

\[-\Delta \tilde{P} = \partial_i \partial_j (w_i w^j + w^i w^j + v^i w^j) \quad \text{in} \quad Q_1.
\]

Letting $\phi_k \in D(B_{k-1})$ with $0 \leq \phi \leq 1$ and $\phi \equiv 1$ in $B_{k-2/3}$, we have for any $x \in B_{k-2/3}$

\[
\tilde{P}(x,t) = \phi_k(x) \tilde{P}(x,t)
\]

\[
= -\frac{3}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} (\phi_k(y) \Delta \tilde{P}(y,t) + 2 \nabla \phi_k(y) \cdot \nabla P(y,t) + P(y,t) \Delta \phi_k(y)) \, dy
\]

\[
\Delta \tilde{P}_{k_1}(x,t) + \tilde{P}_{k_2}(x,t) + \tilde{P}_{k_3}(x,t),
\]

where

\[
\tilde{P}_{k_1}(x,t) = \frac{3}{4\pi} \int_{\mathbb{R}^3} \partial_i \partial_j \left( \frac{1}{|x-y|} \right) \phi_k(y) (w^i w^j)(y,t) \, dy,
\]
Due to the definition of $\bar{P}_k$, this together with Hölder’s inequality yields that

$$\bar{P}_k(x, t) = \frac{3}{4\pi} \int_{\mathbb{R}^3} \partial_i \partial_j \left( \frac{1}{|x-y|} \right) \phi_k(y) (w^i v^j + v^i w^j)(y,t) \, dy,$$

and

$$\bar{P}_k(x, t) = \frac{3}{2\pi} \int_{\mathbb{R}^3} \frac{x_i - y_i}{|x-y|^3} (\partial_i \phi_k(y)) (w^i w^j + w^i v^j + v^i w^j)(t,y) \, dy$$

$$+ \frac{3}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} (\partial_i \partial_j \phi_k(y)) (w^i w^j + w^i v^j + v^i w^j)(y,t) \, dy$$

$$+ \frac{3}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \bar{P}(y) \Delta \phi_k(y) \, dy + \frac{3}{2\pi} \int_{\mathbb{R}^3} \frac{x_i - y_i}{|x-y|^3} \bar{P}(y) \partial_i \phi_k(y) \, dy.$$
By Calderón-Zygmund estimates, Hölder’s inequality and the fact that $|w^i(1 - \frac{w_k}{|w|})| \leq 1$, we immediately have

$$\|\tilde{P}_{k1}\|_{L^p_t L^q_x(Q_{r_{k-1}/3})} \lesssim \|v\|_{L^p_t L^q_x(Q_{r_{k-2/3}})}$$

and

$$\|\tilde{P}_{k2}\|_{L^p_t L^q_x(Q_{r_{k-1}/3})} \lesssim \|w_k\|_{L^{2p/(p-1)}_t L^{6p/(p+4)}_x(Q_{r_{k-1}})} \|I\{w_k > 0\}\|_{L^{2p/(p-1)}_t L^2_x(Q_{r_{k-1}})}$$

with $1/L_4 = (p + 4)/(6p) + 1/q$. Hence, by the above two estimates, Hölder’s inequality, Lemma 4.3 and Lemma 4.6 we get

$$I.2.2 \lesssim 2^k \|\tilde{P}_{k1}\|_{L^p_t L^q_x(Q_{r_{k-1}/3})} \|w_k\|_{L^{2p/(p-1)}_t L^{6p/(p+4)}_x(Q_{r_{k-1}})} \|I\{w_k > 0\}\|_{L^{2p/(p-1)}_t L^2_x(Q_{r_{k-1}})}$$

$$+ \|\tilde{P}_{k1}\|_{L^p_t L^q_x(Q_{r_{k-1}/3})} \|d_k\|_{L^2(Q_{r_{k-1}})} \|I\{w_k > 0\}\|_{L^{2p/(p-1)}_t L^2_x(Q_{r_{k-1}})}$$

$$+ 2^k \|\tilde{P}_{k2}\|_{L^p_t L^q_x(Q_{r_{k-1}/3})} \|w_k\|_{L^q_x(Q_{r_{k-1}})} \|I\{w_k > 0\}\|_{L^q_x(Q_{r_{k-1}})}$$

$$+ \|\tilde{P}_{k2}\|_{L^p_t L^q_x(Q_{r_{k-1}/3})} \|d_k\|_{L^2(Q_{r_{k-1}})} \|I\{w_k > 0\}\|_{L^q_x(Q_{r_{k-1}})}$$

$$\lesssim \nu \left(2^{(10 - 4 - \frac{2}{7})k} W_{k-1}^{\frac{5}{7} - \frac{2}{7}k} \frac{1}{w_{k-1}^{\frac{1}{7}}} + 2^{(\frac{5}{7} - \frac{2}{7})k} W_{k-1}^{\frac{5}{7} - \frac{2}{7}k} \frac{1}{w_{k-1}^{\frac{1}{7}}} \right) \|v\|_{L^p_t L^q_x(Q_{1})}.$$
Therefore, by Hölder’s inequality, Lemma 4.6 and Lemma 4.7 we have
\[
\begin{align*}
\int_{Q_{r_k-1/3}} (\tilde{P}_{k11} + \tilde{P}_{k12}) w \cdot \nabla \phi_k - \nabla (\tilde{P}_{k11} + \tilde{P}_{k12}) \cdot w \left( \frac{w_k}{|w|} - 1 \right) \phi_k \, dx \, dt \\
= \int_{Q_{r_k-1/3}} (\tilde{P}_{k11} + \tilde{P}_{k12}) \frac{wu_k}{|w|} \cdot \nabla \phi_k + (\tilde{P}_{k11} + \tilde{P}_{k12}) \text{div} \left( \frac{wu_k}{|w|} \right) \phi_k \, dx \, dt \\
\lesssim 2^k ||\tilde{P}_{k11} + \tilde{P}_{k12}||_{L^\infty(Q_{r_k-1/3})} ||w_k||_{L^\infty(Q_{r_k-1/3})} ||I_{\{w_k>0\}}||_{L^2(Q_{r_k-1})} \\
+ ||\tilde{P}_{k11}||_{L^6(Q_{r_k-1/3})} ||dk||_{L^2(Q_{r_k-1})} ||I_{\{w_k>0\}}||_{L^3(Q_{r_k-1})} \\
+ ||\tilde{P}_{k12}||_{L^\infty(Q_{r_k-1/3})} ||dk||_{L^2(Q_{r_k-1})} ||I_{\{w_k>0\}}||_{L^2(Q_{r_k-1})} \\
\lesssim 2^{3k/6} (W_{k-1}^{5/3} + W_{k-1}^{5/3}) + 2^{10k/9} W_{k-1}^{19/18} + 2^{2k/3} W_{k-1}^{4/3}.
\end{align*}
\tag{4.11}
\]
For \( \tilde{P}_{k13} \), we have
\[
\nabla \tilde{P}_{k13} = \frac{3}{4\pi} \int_{R^3} \partial_i \partial_j \left( \frac{1}{|x-y|} \right) \nabla \phi_k \frac{w^i w^j}{|w|} \frac{w^k}{|w|} \, dy \\
+ \frac{3}{2\pi} \int_{R^3} \partial_i \partial_j \left( \frac{1}{|x-y|} \right) \phi_k \nabla \left( \frac{w^i w^j}{|w|} \right) \frac{w^k}{|w|} \, dy \\
\triangleq J_1 + J_2.
\]
By Calderón-Zygmund estimates and Lemma 4.7, we obtain that
\[
||\tilde{P}_{k13}||_{L^{5/3}(Q_{r_k-1/3})} \lesssim ||w_k||_{L^{10/3}(Q_{r_k-2/3})}^2, \quad ||J_1||_{L^2(Q_{r_k-1/3})} \lesssim 2^k ||w_k||_{L^{10/3}(Q_{r_k-2/3})},
\]
and
\[
||J_2||_{L^2(Q_{r_k-1/3})} \lesssim ||w_k||_{L^{10/3}(Q_{r_k-2/3})} ||dk||_{L^2(Q_{r_k-2/3})}.
\]
Hence, we have
\[
\begin{align*}
\int_{Q_{r_k-1/3}} \tilde{P}_{k13} w \cdot \nabla \phi_k - \nabla \tilde{P}_{k13} \cdot w \left( \frac{w_k}{|w|} - 1 \right) \phi_k \, dx \, dt \\
\lesssim 2^k ||\tilde{P}_{k13}||_{L^{5/3}(Q_{r_k-1/3})} ||I_{\{w_k>0\}}||_{L^{5/2}(Q_{r_k-1})} + ||J_1||_{L^{5/3}(Q_{r_k-1/3})} ||I_{\{w_k>0\}}||_{L^{5/2}(Q_{r_k-1})} \\
+ 2^k ||\tilde{P}_{k13}||_{L^{5/3}(Q_{r_k-1/3})} ||w_k||_{L^{10/3}(Q_{r_k-1})} ||I_{\{w_k>0\}}||_{L^{10}(Q_{r_k-1})} \\
+ ||J_2||_{L^{5/3}(Q_{r_k-1/3})} ||I_{\{w_k>0\}}||_{L^5(Q_{r_k-1})} \\
\lesssim 2\nu^{7k/6} W_{k-1}^{5/3} + 2^{4k/3} W_{k-1}^{5/3} + 2^{2k/3} W_{k-1}^{1/3}.
\end{align*}
\tag{4.12}
\]
Collecting estimates (4.8)-(4.12), we can say there exist a \( C_2(\nu) > 1 \) and \( \beta > 1 \), such that, for any \( k > 0 \)
\[
W_k \leq C_2^k(\nu) W_{k-1}^\beta.
\tag{4.13}
\]

**Step 3. Conclusion.**

From (4.13), we have \( W_0 < 1 \) which implies \( W_k < 1 \) for any \( k \geq 0 \). So, let \( W_k = C_2^{k-1}(\nu) C_2^{\beta-1}(\nu) W_k \). Then, from (4.13), we have \( \tilde{W}_k \leq W_{k-1} \). If we choose \( \epsilon_1 \) small enough
such that \( \bar{W}_0 \leq C^{-1}_2(\nu)C_2^{-(\beta+1)}(\nu)\epsilon_1 < 1 \), then we have \( \bar{W}_k \leq 1 \) for any \( k \geq 0 \), which implies that, for any integer \( k \geq 0 \),

\[
W_k \leq C^{-1}_2(\nu)C_2^{-(\beta+1)}(\nu).
\]

Thanks to \( C_2(\nu) > 1 \), sending \( k \to \infty \), we get \( W_k \to 0 \). So we prove (4.5).

4.2. Stability of singularities. In this part, we will show the stability of singularities of suitable weak solutions to (4.4) in the sense of locally strong limits.

First, we introduce definition of singularity. Given a suitable weak solution \((w, \bar{P})\) to (4.4), if \( w \) is essentially bounded in a neighborhood of \( z_0 = (x_0, t_0) \in D \), we call \( z_0 \) is a regular point of \((w, \bar{P})\); otherwise, we call it a singular point.

The main result in this subsection can be stated as follows:

**Proposition 4.8** (Stability of singularities). Let \( v^{(k)} \in L^p_tL^q_x(Q_1) \) satisfies \( \text{div} v^{(k)} = 0 \) with \( \frac{2}{p} + \frac{3}{q} < 1 \) and \( q > 3 \). Let \((w^{(k)}, \bar{P}^{(k)})\) be a sequence of suitable weak solutions to the following equations in \( Q_1 \)

\[
\begin{align*}
\partial_t w^{(k)} - \nu \Delta w^{(k)} + w^{(k)} \cdot \nabla w^{(k)} + v^{(k)} \cdot \nabla w^{(k)} + w^{(k)} \cdot \nabla v^{(k)} + \nabla \bar{P}^{(k)} = 0 \quad &\text{in} \quad L^3(Q_1), \quad \bar{P}^{(k)} \to \bar{P} \quad \text{in} \quad L^{3/2}(Q_1), \quad v^{(k)} \to v \quad \text{in} \quad L^p_tL^q_x(Q_1), \\
\text{div} w^{(k)} = 0 \quad &\text{in} \quad L^3(Q_1).
\end{align*}
\]

Assume further that \( w^{(k)} \to w \) in \( L^3(Q_1) \), \( \bar{P}^{(k)} \to \bar{P} \) in \( L^{3/2}(Q_1) \), \( v^{(k)} \to v \) in \( L^p_tL^q_x(Q_1) \), and the limit \((w, \bar{P})\) is a suitable weak solution to Eq. (4.4) in \( Q_1 \) associated to \( v \). Then, if \( z^{(k)} \in Q_1 \) is a singular point of \((w^{(k)}, \bar{P}^{(k)})\) and \( z^{(k)} \to z_0 \in Q_1 \), we have \( z_0 \) is a singular point of \((w, \bar{P})\).

**Proof.** We will prove the proposition by contradiction. Assume \( z_0 \in Q_1 \) is a regular point of \((w, \bar{P})\). Then, there exists \( r_0 > 0 \) and \( M_0 > 0 \) such that \( Q_{r_0}(z_0) \subset Q_1 \) and \( \|w\|_{L^\infty(Q_{r_0}(z_0))} \leq M_0 \). Thanks to that fact that \( w^{(k)} \to w \) in \( L^3(Q_r(z_0)) \), there exists \( N_1 > 0 \) such that, for any \( k \geq N_1 \) and \( r_1 \leq r \leq r_0, r_1 < r_0/2 \),

\[
-\Delta \bar{P}^{(k)} = \text{div} \left( w^{(k)} \otimes w^{(k)} + v^{(k)} \otimes w^{(k)} + w^{(k)} \otimes v^{(k)} \right) \quad \text{in} \quad Q_1.
\]

For \( \bar{P}^{(k)} \), from equations (4.3), we have

\[
-\Delta \bar{P}^{(k)} = \text{div} \left( w^{(k)} \otimes w^{(k)} + v^{(k)} \otimes w^{(k)} + w^{(k)} \otimes v^{(k)} \right) \quad \text{in} \quad Q_1.
\]

Hence, we can write \( \bar{P}^{(k)} = g^{(k)} + h^{(k)} \) in \( Q_r(z_0) \) with \( r_1 \leq r \leq r_0 \), where

\[
g^{(k)} = (-\Delta)^{-1} \text{div} \left( w^{(k)} \otimes w^{(k)} + v^{(k)} \otimes w^{(k)} + w^{(k)} \otimes v^{(k)} \right) I_{B_r(x_0)}
\]

and \( h^{(k)} \in L^{3/2}(Q_r(z_0)) \) is harmonic in \( B_r(x_0) \).
By Calderón-Zygmund estimates, Hőlder’s inequality and (4.15), we get that for any \( k > N_1 \) and \( r_1 \leq r_0 \),
\[
    r^{-2} \| g^{(k)} \|_{L^{3/2}(Q_{r_1}(z_0))}^{3/2} \leq C r^{-2} \| w^{(k)} \|_{L^3(Q_{r_1}(z_0))}^{3} + C r^{-2} \| v^{(k)} \|_{L^3_0(L^3(Q_{r_1}(z_0)))}^{3/2} \| I \|_{L^{p/3}_{t} L^{q/3}(Q_{r}(z_0))}^{p/3} \leq C_2 M_0^{3/2} + C_2 M_0^{3/2} r_1^{1/2} \| v^{(k)} \|_{L^3_0(L^3(Q_{r_1}(z_0)))}^{3/2}
\]
when \( p \geq 3 \), and
\[
    r^{-2} \| g^{(k)} \|_{L^{3/2}(Q_{r_1}(z_0))}^{3/2} \leq C r^{-2} \| w^{(k)} \|_{L^3(Q_{r_1}(z_0))}^{3} + C r^{-2} \| I \|_{L^{p/3}_{t} L^{q/3}(Q_{r}(z_0))}^{p/3} \leq C_3 M_0^{3/2} + C_3 r_1^{15/8-3/p-9/(2q)} M_0^{3/4} \| v^{(k)} \|_{L^3_0(L^3(Q_{r_1}(z_0)))}^{3/2}
\]
when \( 2 < p < 3 \).

Collecting estimates (4.15)-(4.17), we can choose \( r_0 \) small enough such that, for any \( k \geq N_1 \) and \( r_1 \leq r \leq r_0 \),
\[
    r^{-2} \int_{Q_{r_1}(z_0)} \left( |w^{(k)}|^3 + |g^{(k)}|^{3/2} \right) \, dx \, dt < \varepsilon_0/2.
\]
where \( \varepsilon_0 \) is the constant in Theorem 4.1.

For \( h^{(k)} \), by the mean value theorem, we have for any \( r_1 \leq r \leq \frac{2r_1}{3} \),
\[
    \| h^{(k)}(t) \|_{L^\infty(B_r(x_0))} \leq C(r_0)^{-3} \int_{B_{r_0}(x_0)} |h^{(k)}(t)| \, dx \leq C(r_0)^{-2} \left( \int_{B_{r_0}(x_0)} |h^{(k)}(t)|^{3/2} \, dx \right)^{2/3}.
\]
Hence, for any \( r_1 \leq r \leq r_0/2 \), we have
\[
    r^{-2} \int_{Q_{r_1}(z_0)} |h^{(k)}|^{3/2} \, dx \, dt \leq C(r_0)^{-3} r \int_{Q_{r_0}(z_0)} |h^{(k)}|^{3/2} \, dx \, dt.
\]
Now we choose \( r_1 \) small enough, such that
\[
    C(r_0)^{-3} r_1 \int_{Q_{r_1}(z_0)} |h^{(k)}|^{3/2} \, dx \, dt < \varepsilon_0/2.
\]
This together with (4.18) and Proposition 4.1 enables us to conclude that \( (w^{(k)}, \tilde{P}^{(k)}) \) is regular in \( Q_{r_1}(z_0) \), which is a contradiction. So, we complete the proof of Proposition 4.8.

4.3. The short-time behaviour of the kinetic energy. In this subsection, we will give an useful observation for energy solutions to equations (4.11) where \( v \) is the mild solution stated in the beginning of Section 4. Here, we adapt the notation:
\[
    \mathcal{Y}_{p,q}(T) \triangleq \tilde{L}^\infty([0,T]; F\tilde{B}^s_{p,q}(\mathbb{R}^3)) \cap \tilde{L}^1([0,T]; F\tilde{B}^{s+1}_{p,q}(\mathbb{R}^3)),
\]
and
\[
    \mathcal{K}_p(T) \triangleq \left\{ f(t) \in S'([0,T]) \mid \lim_{t \to 0^+} t^{-\frac{2}{p}} \|f(t)\|_{F_{L^p}} = 0, \|f\|_{\mathcal{K}_p(T)} \triangleq \sup_{0 \leq t \leq T} t^{-\frac{2}{p}} \|f(t)\|_{F_{L^p}} < \infty \right\}.
\]
Proposition 4.9. Assume that $\theta, s, s_p, p, \tilde{p}, q, \tilde{q}$ satisfy the relation \((2.4)\). Let $w$ be a local energy solution to equations \((4.1)\) associated to initial data $w_0 \in L^2(\mathbb{R}^3) \cap \left( F\dot{B}^s_{p,q}(\mathbb{R}^3) + F\dot{B}^s_{\tilde{p},\tilde{q}}(\mathbb{R}^3) \right)$. Then there exists a $\gamma \in (0, \frac{2}{29})$ depending on $\|w_0\|_{L^2(\dot{B}^s_{p,q} + F\dot{B}^s_{\tilde{p},\tilde{q}})}$ such that for any $0 < t \leqslant 1$,
\begin{equation}
\|w(t) - e^{\nu t \Delta} w_0\|_{L^2} \leq C \left( \|w_0\|_{L^2(\dot{B}^s_{p,q} + F\dot{B}^s_{\tilde{p},\tilde{q}})}, \|v\|_{Y^s_p(\mathbb{R}^3)}(T) \right) \left( t^{-\frac{s-q-\gamma}{2}} + t^{\gamma(1-\theta)} \right),
\end{equation}

Proof. Firstly, we split $w_0$ into $f_0 + g_0$ satisfying $f_0 \in F\dot{B}^s_{\tilde{p},\tilde{q}}(\mathbb{R}^3)$ and $g_0 \in L^2(\mathbb{R}^3)$. Let
\[ \alpha = \|w_0\|_{L^2(\dot{B}^s_{p,q} + F\dot{B}^s_{\tilde{p},\tilde{q}})}. \]

From the definition of $F\dot{B}^s_{p,q}(\mathbb{R}^3) + F\dot{B}^s_{\tilde{p},\tilde{q}}(\mathbb{R}^3)$, we can choose a decomposition of $w_0$: $w_0 = \tilde{f}_0 + \tilde{g}_0$ such that
\[ \|\tilde{f}_0\|_{F\dot{B}^s_{p,q}} + \|\tilde{g}_0\|_{F\dot{B}^s_{\tilde{p},\tilde{q}}} \leq 2\alpha. \]

Since $\text{div} \ w_0 = 0$, we have
\[ w_0 = Pw_0 = \mathbb{P}\tilde{f}_0 + \mathbb{P}\tilde{g}_0 \triangleq f_0 + g_0. \]

By Calderón-Zygmund estimates, we readily get
\[ \|f_0\|_{F\dot{B}^s_{p,q}} + \|g_0\|_{F\dot{B}^s_{\tilde{p},\tilde{q}}} \leq C\alpha. \]

Then, by Lemma \((2.3)\) for any $j \in \mathbb{Z}$, $f_0$ can be decomposed as $f_0 = g_0^j + h_0$ such that $\text{div} \ g_0^j = \text{div} \ h_0 = 0$ and
\[ \|g_0^j\|_{F\dot{B}^s_{\tilde{p},\tilde{q}}} \leq C2^{-j\theta}\|f_0\|_{F\dot{B}^s_{p,q}} \leq C2^{-j\theta}\alpha, \quad \|h_0\|_{L^2} \leq C2^{j(1-\theta)}\|f_0\|_{F\dot{B}^s_{p,q}} \leq C2^{j(1-\theta)}\alpha. \]

where $\theta, p, \tilde{p}, q, \tilde{q}, s, s_p$ satisfy \((2.3)\).

Next, we consider the following problem:
\begin{equation}
\partial_t g - \nu \Delta g + g \cdot \nabla g + v \cdot \nabla g + g \cdot \nabla v + \nabla \tilde{P} = 0, \quad \text{div} \ g = 0,
\end{equation}

which is supplemented with initial condition $g(x, 0) = g_0(x)$.

By Duhamel formula, one writes
\[ g = e^{\nu \Delta} g_0 + \int_0^t e^{-\nu(t-s)\Delta} \mathbb{P}(g \cdot \nabla v + v \cdot \nabla g) \, dt + \int_0^t e^{-\nu(t-s)\Delta} \mathbb{P}(g \cdot \nabla g) \, dt \]
\[ \triangleq G + L(g) + B(g, g), \]

Following the proof of Theorem \((3.2)\) we have that
\[ \|G\|_{Y^s_{p,q}(T_1)} \leq C(\nu)\|g_0\|_{F\dot{B}^s_{p,q}}, \quad \|L(g)\|_{Y^s_{p,q}(T_1)} \leq C(\nu)T_1^{-\frac{s-q}{2}}\|g\|_{Y^s_{p,q}(T_1)}\|\tilde{v}\|_{Y^{s}_{p,q}(T_1)}, \]

and
\[ \|B(g, \tilde{g})\|_{Y^s_{p,q}(T_1)} \leq C(\nu)T_1^{-\frac{s-q}{2}}\|g\|_{Y^s_{p,q}(T_1)}\|\tilde{g}\|_{Y^s_{p,q}(T_1)}. \]

Thus, by Lemma \((2.4)\) there exists a mild solution $g$ to problem \((4.20)\) on $[0, T_1]$ corresponding to initial data $g_0$, satisfying
\[ g \in C_b\left([0, T_1]; F\dot{B}^s_{p,q}(\mathbb{R}^3)\right) \cap Y^s_{p,q}(T_1). \]
Furthermore, $T_1$ and $g$ satisfy

$$T_1 \leq \min \left\{ 1, T, \left( C(\nu)(\|v\|_{Y^{s,q}_p(T)} + 2^{-j\theta}) \right)^{-\frac{2}{s-p}} \right\}$$

and

$$\|g\|_{Y^{s,q}_p(\mathcal{T})} \leq \frac{C(\nu)\|g_0\|_{FB^{s+2}_{p,q}}}{1 - C(\nu)T_1^{(s-p)/2}\|v\|_{Y^{s,q}_p(\mathcal{T})}} \leq \frac{C(\nu)2^{-j\theta}}{1 - C(\nu)T_1^{(s-p)/2}\|v\|_{Y^{s,q}_p(\mathcal{T})}}.$$  

In the same way as in the proof of Proposition 3.3 by Lemma 2.5, we have

$$\|tg(t)\|_{L^\infty([0, T]; FB^{s+2}_{p,q})} \leq C(\nu)T_1^{-\frac{s-p}{2}} \left( \|tg\|_{L^\infty([0, T]; FB^{s+2}_{p,q})} + \|g\|_{Y^{s,q}_p(\mathcal{T})} \right) \|v\|_{Y^{s,q}_p(\mathcal{T})}$$

and

$$\max \left\{ \|tg(t)\|_{L^\infty([0, T]; FB^{s+2}_{p,q})}, \|tg(\tilde{g}, g)\|_{L^\infty([0, T]; FB^{s+2}_{p,q})} \right\} \leq C(\nu)T_1^{-\frac{s-p}{2}} \left( \|tg\|_{L^\infty([0, T]; FB^{s+2}_{p,q})} + \|g\|_{Y^{s,q}_p(\mathcal{T})} \right) \|g\|_{Y^{s,q}_p(\mathcal{T})}.$$  

Employing Lemma 2.5, we get

$$\|tg(t)\|_{L^\infty([0, T]; FB^{s+2}_{p,q})} \leq \frac{C(\nu)2^{-j\theta}}{1 - C(\nu)T_1^{(s-p)/2}\|v\|_{Y^{s,q}_p(\mathcal{T})}}.$$  

This, together with Remark 3.4, we immediately obtain that

$$\|g\|_{K^2_p(T)} \leq \frac{C(\nu)2^{-j\theta}}{1 - C(\nu)T_1^{(s-p)/2}\|\tilde{v}\|_{Y^{s,q}_p(\mathcal{T})}}.$$  

On the other hand, due to $g_0 = w_0 - h_0 \in L^2(\mathbb{R}^3)$, we have

$$\|e^{t\Delta}g_0\|_{L^2} \in L^2(\mathbb{R}^3).$$

Moreover, by Hölder’s and Young’s inequalities, we have the following estimate (we omit the routine calculation):

$$\|L(g)\|_{L^\infty([0, T_1]; L^2)} \leq C(\nu)T_1^{-\frac{s-p}{2}} \|g\|_{L^\infty([0, T_1]; L^2)} \|v\|_{K^2_p(T)}$$

and

$$\max \left\{ \|B(\tilde{g}, g)\|_{L^\infty([0, T_1]; L^2)}, \|B(g, \tilde{g})\|_{L^\infty([0, T_1]; L^2)} \right\} \leq C(\nu)T_1^{-\frac{s-p}{2}} \|g\|_{L^\infty([0, T_1]; L^2)} \|\tilde{g}\|_{K^2_p(T)}.$$  

Hence, by Lemma 2.5, we obtain that $g \in C([0, T_1]; L^2(\mathbb{R}^3)).$

Similarly, we can deduce that $g \in L^2((0, T_1); H^1)$. So, we see that $g$ satisfies the global energy inequality (4.3) on $(0, T_1)$. We have by Hölder’s, Hausdorff-Young’s and Young’s inequalities that for any $0 < t < T_1$

$$\int_{\mathbb{R}^3 \times \{t\}} |g|^2 \, dx + 2\int_0^t \int_{\mathbb{R}^3} \nu|\nabla g|^2 \, dx \, dt$$

$$\leq \|g_0\|_{L^2}^2 + C \int_0^t \|g\|_{L^{2\delta}_{\frac{2}{\delta-p}}(\mathbb{R}^3)} \|\nabla g\|_{L^2} \|v\|_{L^\frac{2\delta}{2\delta-p}} \, dt.$$
This, together with the global energy inequality (4.2) for \( w \)

With this property and (4.22) in hand, we have that for any \( 0 < t < T \)

Letting (4.25)

Finally, by Gronwall’s inequality, we get for any \( 0 < t < T \)

\[
\sup_{0 \leq \tau \leq t} \| g \|_{L^2}^2 + \nu \int_0^t \| \nabla g \|_{L^2}^2 \, d\tau \leq \| g_0 \|_{L^2}^2 e^{C(\nu) t^{1 - \frac{2}{p}} \| v \|_{L^2}^{2 \frac{2}{p}}}.
\]

With this property and (4.22) in hand, we have that for any \( 0 < t < T_1 \)

\[
\| g(t) - e^{t \Delta} g_0 \|_{L^2} \leq C t^{\frac{s-p}{2}} \| g \|_{L^\infty((0,t);L^2)} \left( \| g \|_{K^\pi_p(T_1)} + \| v \|_{K^\pi_p(T_1)} \right)
\]

\[
\leq C t^{\frac{s-p}{2}} \| g_0 \|_{L^2} e^{C(\nu) t^{1 - \frac{2}{p}} \| v \|_{L^2}^{2 \frac{2}{p}}} \left( C \frac{2^{-j \theta} \alpha}{1 - C(\nu) T_1^{(s-\theta)p)/2} \| v \|_{Y^\pi_p(T)}} + \| v \|_{K^\pi_p(T)} \right).
\]

Letting \( h = w - g \), we have \( h \in L^\infty((0,T_1);L^2) \cap L^2((0,T_1); \dot{H}^1) \) and \( \lim_{t \to 0^+} \| h(t) - h_0 \|_{L^2} = 0. \)

Thanks to (4.22), we get \( g \in L^r((0,T_2);L^p) \) with \( \frac{2}{r} + \frac{3}{p} = 1 \) and \( p' \) is the conjugate index of \( p \). Thus, using the usual mollification procedure used in Proposition 14.3 in [21], we get the following equality:

\[
\int_{\mathbb{R}^3} g(t,x) \cdot w(t,x) \, dx + 2\nu \int_0^t \int_{\mathbb{R}^3} \nabla g : \nabla w \, dx \, d\tau \n
= - \int_0^t \int_{\mathbb{R}^3} ((h \cdot \nabla) w \cdot g + (g \cdot \nabla) w \cdot v + (w \cdot \nabla) g \cdot v) \, dx \, d\tau + \int_0^t \int_{\mathbb{R}^3} g_0 \cdot w_0 \, dx.
\]

This, together with the global energy inequality (4.2) for \( w \) and \( g \), gives that

\[
\| h(t) \|_{L^2}^2 + 2\nu \int_0^t \| \nabla h \|_{L^2}^2 \, d\tau \leq \| h_0 \|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} ((h \cdot \nabla) h \cdot g + (h \cdot \nabla) h \cdot v) \, dx \, d\tau.
\]

By Hölder’s and Young’s inequality, (4.23) and (4.24), we get

\[
\int_{\mathbb{R}^3 \times \{ t \}} |h|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} \nu \| \nabla h \|_{L^2}^2 \, d\tau \leq \| h_0 \|_{L^2}^2 + C(\nu) \int_0^t \| h \|_{L^2}^2 \left( \| g \|_{L^\infty}^{2p} + \| v \|_{L^\infty}^{2p} \right) \, d\tau.
\]
Employing the Gronwall inequality to the above inequality, we obtain that

\begin{equation}
\|h(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla h\|_{L^2}^2 \, ds \leq \|h_0\|_{L^2}^2 e^{C(\nu)\frac{t}{\eta_\nu}} \left( \|\|_{K_{\eta_\nu}^p(T)} + \|\|_{K_\nu(T)} \right)^{\frac{2p}{s-\frac{2p}{2}}} \\
\leq 2^{2(1-\theta)} + \nu \int_0^t \|\nabla h\|_{L^2}^2 \, ds \leq \|h_0\|_{L^2}^2 e^{C(\nu)\frac{t}{\eta_\nu}} \left( \|\|_{K_{\eta_\nu}^p(T)} + \|\|_{K_\nu(T)} \right)^{\frac{2p}{s-\frac{2p}{2}}}.
\end{equation}

(4.26)

Now, we choose \(2^j = t^\gamma\) for some positive constant \(\gamma\) and \(0 < t < T_1\), which implies that \(0 < \gamma < \frac{s - \frac{8p}{q}}{2\theta}\) and \(0 < t < (C(\nu))^{- \frac{2}{s-\frac{2p}{2}}}\).

According to (4.21), (4.25) and (4.26), for any \(0 < t < \min\{T_1, (C^{1/2}(\nu))^{- \frac{2}{s-\frac{2p}{2}}}\}\), we have

\begin{equation}
\|w(t) - e^\nu t w_0\|_{L^2} \leq \|g(t) - e^\nu t g_0\|_{L^2} + \|h(t)\|_{L^2} + \|e^\nu t h_0\|_{L^2} \\
\leq C(T_1, T, \alpha, \|v_0\|_{F_{p,q}^s}) \left( t^{\frac{s - \frac{8p}{q}}{2}} + t^{(1-\theta)} \right).
\end{equation}

(4.27)

This completes the proof. \(\square\)

4.4. Weak-strong uniqueness. This section is devoted to the study of a new version of “weak-strong” uniqueness of weak energy solution problem (4.1) established at the beginning of Section 4. Here, we still adopt the notation \(Y_{p,q}^s(T)\) introduced in above subsection.

**Proposition 4.10.** Suppose \(s, sp, p, q, \tilde{p}, \tilde{q}\) satisfy relation (2.4). Let \(w\) and \(\tilde{w}\) both are weak energy solutions to Eq. (4.11) with the same initial data \(w_0 \in L^2(\mathbb{R}^3) \cap (F_{p,q}^s(\mathbb{R}^3) + F_{p,q}^s(\mathbb{R}^3))\). Assume further that

\begin{equation}
w \in K_{p}^s(T) + \tilde{K}_{\tilde{p}}^\tilde{s}(T).
\end{equation}

(4.27)

Then \(\tilde{w} \equiv w\) on \(\mathbb{R}^3 \times (0, T)\).

**Remark 4.11.** Let us point out that the uniqueness result established by Gallagher and Planchon [15] indicates Proposition 4.10 when \(\frac{2}{q} + \frac{3}{p} \geq 1\). However, for \(\frac{2}{q} + \frac{3}{p} < 1\), the argument in [15] doesn’t seem to work. To overcome it, we need resort to the regularity of such weak energy solution in short time which is established in Proposition 4.9.

**Proof of Proposition 4.10.** Let \(\delta w = \tilde{w} - w\). From Remark 3.4 we can rewrite \(w = f + g\) satisfying

\(\|f\|_{K_{p}^s(T)} < \infty\) and \(\|g\|_{K_{\tilde{p}}^\tilde{s}(T)} < \infty\).

By Hausdorff-Young’s inequality, we have \(f \in L^{1_1}(t_0, T); L^{s-1}_{p-1})\), \(\frac{2}{l_1} + \frac{3(p-1)}{p} = 1\) for any \(0 < t_0 < T\), \(g \in L^{2_2}((0, T); L^{2}_{p-2})\) and \(v \in L^{2_2}((0, T); L^{2}_{p-2})\) with \(\frac{2}{l_2} + \frac{3(p-1)}{p} = 1\). Then, by a usual mollification procedure used in Proposition 14.3 in [21], we have that for any
\[ t_0 \leq t \leq T, \]
\[
\int_{\mathbb{R}^3} w(t, x) \cdot \hat{w}(t, x) \, dx + 2\nu \int_{t_0}^t \int_{\mathbb{R}^3} \nabla w : \nabla \hat{w} \, dx \, d\tau \\
= -\int_{t_0}^t \int_{\mathbb{R}^3} (\delta w \cdot \nabla) \hat{w} \cdot f \, dx \, d\tau + \int_{t_0}^t \int_{\mathbb{R}^3} (\delta w \cdot \nabla) \hat{w} \cdot g \, dx \, d\tau \\
+ \int_{t_0}^t \int_{\mathbb{R}^3} ((w \cdot \nabla) \hat{w} \cdot v + (\hat{w} \cdot \nabla) w \cdot v) \, dx \, d\tau + \int_{\mathbb{R}^3} w(\delta, x) \cdot \hat{w}(\delta, x) \, dx \\
\triangleq I_1 + I_2 + I_3 + I_4.
\]

Due to \( g \in L^4((0, T); L^\frac{6}{5}) \) and \( v \in L^2((0, T); L^\frac{2}{1}) \), \( \frac{2}{1} + \frac{3(p-1)}{p} = 1 \), it is obvious that the limits of \( I_2 \) and \( I_3 \) exist as \( t_0 \to 0 \).

By Sobolev embedding \[4.23\] and interpolation inequality \[4.24\], we have
\[
I_1 \leq \int_{t_0}^t \left( \| \nabla \delta w \|^2_{L^2} + \| \nabla \hat{w} \|^2_{L^2} \right) \, d\tau + C\| f \|_{K^p_\beta(T)} \int_{t_0}^t \tau^{-1} \| \delta w \|^2_{L^2} \, d\tau,
\]

According to Proposition \[4.5\] we get that for any \( 0 < \delta \ll 1 \), there exists a \( 0 < \gamma < \frac{s-s_p}{2s} \), such that for any \( 0 < t \leq \delta \), we have that
\[
\| \delta w(t) \|^2_{L^2} \leq C \left( \| w(t) \| - e^{\gamma t} \| w_0 \|^2_{L^2} + \| \hat{w}(t) \| - e^{\gamma t} \| w_0 \|^2_{L^2} \right)
\leq C \left( \| w_0 \|_{L^2(FB^p_{p,q} + FB^p_{p,q})}, \| v \|_{Y^p_{\beta,q}(T)} \right) \left( (t^{(1-\theta)} + t^{\frac{s-s_p-\gamma}{2}}) \right).
\]

Hence,
\[
\int_{t_0}^t \tau^{-1} \| \delta w \|^2_{L^2} \, d\tau \leq C \left( \| w_0 \|_{L^2(FB^p_{p,q} + FB^p_{p,q})}, \| v \|_{Y^p_{\beta,q}(T)} \right) \times \int_{t_0}^t \left( \tau^{-1+\gamma(1-\theta)} + \tau^{-1+\frac{s-s_p-\gamma}{2}} \right) \, d\tau < \infty,
\]

which implies the existence of the limit of \( I_1 \) as \( t_0 \to 0+ \).

Taking \( \delta \to 0 \) in \[4.28\], we readily get that for any \( 0 \leq t \leq T \),
\[
\int_{\mathbb{R}^3} w(t, x) \cdot \hat{w}(t, x) \, dx + 2\nu \int_{0}^t \int_{\mathbb{R}^3} \nabla w : \nabla \hat{w} \, dx \, d\tau \\
= 2\| w_0 \|^2_{L^2} - \int_{0}^t \int_{\mathbb{R}^3} ((\delta w \cdot \nabla) \hat{w} \cdot (f + g) + (w \cdot \nabla) \hat{w} \cdot v + (\hat{w} \cdot \nabla) w \cdot v) \, dx \, d\tau.
\]

Since \( \hat{w} \) and \( w \) both satisfy the global energy inequality \[4.3\], from the above equality, we deduce that for any \( 0 \leq t \leq T \),
\[
\| \delta w(t) \|^2_{L^2} + 2\nu \int_{0}^t \| \nabla \delta w \|^2_{L^2} \, d\tau \\
\leq 2\int_{0}^t \int_{\mathbb{R}^3} (\delta w \cdot \nabla) \delta w \cdot (f + g) \, dx \, d\tau + 2\int_{0}^t \int_{\mathbb{R}^3} (\delta w \cdot \nabla) \delta w \cdot v \, dx \, d\tau.
\]

By \[4.23\] and \[4.24\], we have for any \( 0 \leq t \leq \delta \),
\[
\| \delta w \|^2_{L^\infty(0, t); L^2} + \nu \| \nabla \delta w \|^2_{L^2(\mathbb{R}^3 \times (0, t))}
\]
Moreover, we have by Remark 3.4 that
\[ w \equiv 0 \]
By Lemma 2.3, for each \( j \)
\[ \text{ Remark } \]
\[ \text{to the } \]
where \( \beta = \min \{2\gamma(1-\theta), s-s_\beta-\gamma \theta, 1-\frac{1}{3p} \} \).

From it, we can choose a \( 0 < t_1 < \delta \) small enough such that \( \delta w = 0 \) on \((0, t_1) \times \mathbb{R}^3 \). Thanks to \( w \in L^1((t_1, T); L^{\frac{3p}{2p-1}})+L^2([t_1, T]; L^{\frac{2}{2-1}}) \), by the continuity argument, we eventually obtain \( w \equiv \tilde{w} \) on \((0, T) \times \mathbb{R}^3 \).

**Remark 4.12.** Let us point out that all the above results in this section can be generalized to the \( L^2_{\text{uloc}}(\mathbb{R}^3) \) framework by borrowing the idea used in [21]. Here and what in follows,
\[
L^2_{\text{uloc}}(\mathbb{R}^3) = \left\{ f \in \mathcal{D}'(\mathbb{R}^3) \mid \sup_{x \in \mathbb{R}^3} \| f \|_{L^2(B_1(x))} < \infty \text{ and } \lim_{|x| \to \infty} \| f \|_{L^2(B_1(x))} = 0 \right\}.
\]

5. **Proof of Theorem 1.1**

In this section, we will give a complete proof of Theorem 1.1 by using some results established in previous sections. We will split it into three cases to discuss.

**Case 1:** \( 1 < p < 2/3 \) and \( 1 \leq q < \infty \).

Since \( u_0 \in F\dot{B}^{s_p}_{p,q}(\mathbb{R}^3) \), we know by Theorem 3.1 that there exists a unique local-in-time solution \( u \in C([0, T^*); F\dot{B}^{s_p}_{p,q}(\mathbb{R}^3)) \) satisfying
\[
\| u \|_{F_{\text{loc}}([0, T^*); F\dot{B}^{2+sp}_{2+sp}(\mathbb{R}^3))} < \infty \quad \forall r \in [1, \infty).\]

Moreover, we have by Remark 3.4 that
\[ u \in \dot{K}^{s_p}_{p}(T) \quad \text{for each} \quad T \in (0, T^*). \]

By Lemma 2.3, for each \( j \in \mathbb{Z} \), there exists \( C > 0 \) such that
\[
u_0 = v_0 + w_0,
\]
where \( v_0 \) and \( w_0 \) satisfy \( \text{div} v_0 = \text{div} w_0 = 0 \),
\[
\| v_0 \|_{F\dot{B}^{s_p}_{p,q}} \leq C 2^{-j \theta} \| u_0 \|_{F\dot{B}^{s_p}_{p,q}} \quad \text{and} \quad \| w_0 \|_{L^2} \leq C 2^{j(1-\theta)} \| u_0 \|_{F\dot{B}^{s_p}_{p,q}},
\]
where \( \theta, s_p, s, p, q, \tilde{p}, \tilde{q} \) satisfy the restriction condition \( (2, 3) \).

By Theorem 3.2, we know that the following system admits a unique local mild solution \( v \in C([0, T^*); F\dot{B}^{s_\tilde{p}}_{\tilde{p},\tilde{q}}(\mathbb{R}^3)) \cap \tilde{L}^r([0, T^*); F\dot{B}^{2+sp}_{2+sp}(\mathbb{R}^3)) \) with \( r \in [1, \infty] \)
\[
\begin{cases}
\partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla Q = 0, \\
\text{div } v = 0, \\
v(x, 0) = v_0(x),
\end{cases}
\]
Since $T'$ only depends on $\| \dot{v}_0 \|_{F\dot{B}^{s_p}_{p,q}_q}$, we can choose a suitable $j$ such that $T' > T^*$. Using Proposition 3.3 and Remark 3.4 again, one has

$$v \in \dot{K}^{s_p}_p(T').$$

(5.2)

Letting $w = u - v$, we get easily that $w \in \dot{K}^{s_p}_p(T) + \dot{K}^{s}_p(T)$ for each $T \in (0, T^*)$ satisfying the following integral equations:

$$w = e^{\nu \Delta} w_0 + \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P} (w \cdot \nabla w) \, d\tau + \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P} (v \cdot \nabla w + w \cdot \nabla v) \, d\tau$$

$$\Delta v + B(w, w) + L(w).$$

(5.3)

Our task is now to show that

$$w \in L^\infty((0, T^*); L^2) \cap L^2((0, T^*); \dot{H}^1).$$

(5.4)

Since $w_0 \in L^2(\mathbb{R}^3) \cap (F\dot{B}^{s_p}_{p,q}_q(\mathbb{R}^3) + F\dot{B}^{s}_p(\mathbb{R}^3))$, we have by Banach fixed point theorem that there exists a local solution $\tilde{w} \in X_{T_0}$, $T_0 < T^*$ of (5.3), where

$$X_{T_0} \triangleq (\dot{K}^{s_p}_p(T_0) + \dot{K}^{s}_p(T_0)) \cap E(T_0)$$

with $E(T_0) \triangleq C_b([0, T_0); L^2(\mathbb{R}^3); \dot{H}^1)$. And $\tilde{w}$ solves the following problem on $(0, T_0)$

$$\begin{aligned}
\partial_t w - \nu \Delta w + w \cdot \nabla w + v \cdot \nabla v + w \cdot \nabla v + \nabla \tilde{P} &= 0, \\
\text{div } w &= 0, \\
w(x, 0) &= w_0.
\end{aligned}$$

(5.5)

Indeed, by Lemma 2.1 and Hölder’s inequality, we have

$$\| y_w \|_{X_{T_0}} \leq C(\| w_0 \|_{F\dot{B}^{s_p}_{p,q}_q} + \| w_0 \|_{L^2}).$$

On the other hand, by a simple calculation and estimate (3.10) in Proposition 3.3 we have that the following estimates for any $w = w_1 + w_2$ and $w' = w_1' + w_2'$

$$\begin{aligned}
\| B(w_1, w_1') \|_{K^{s_p}_p(T_0)} &\lesssim \nu \| w_1 \|_{K^{s_p}_p(T_0)} \| w_1' \|_{K^{s_p}_p(T_0)}, \\
\| B(w_2, w_2') \|_{K^{s_p}_p(T_0)} &\lesssim \nu T_0^{\frac{s_p-\nu}{2}} \| w_2 \|_{K^{s_p}_p(T_0)} \| w_2' \|_{K^s_p(T_0)}, \\
\| B(w_2, w_1') \|_{K^s_p(T_0)} &\lesssim \nu T_0^{\frac{s_p-\nu}{2}} \| w_2 \|_{K^{s_p}_p(T_0)} \| w_1' \|_{K^s_p(T_0)}, \\
\| B(w_2, w_2') \|_{E(T_0)} &\lesssim \nu \left( \| w_2 \|_{K^{s_p}_p(T_0)} + T_0^{\frac{s_p-\nu}{2}} \| w_2 \|_{K^s_p(T_0)} \right) \| w \|_{E(T_0)},
\end{aligned}$$

and

$$\begin{aligned}
\| L(w_1) \|_{K^{s_p}_p(T_0)} &\lesssim \nu T_0^{\frac{s_p-\nu}{2}} \| w_1 \|_{K^{s_p}_p(T_0)} \| v \|_{K^s_p(T_0)}, \\
\| L(w_2) \|_{K^s_p(T_0)} &\lesssim \nu T_0^{\frac{s_p-\nu}{2}} \| w_2 \|_{K^s_p(T_0)} \| v \|_{K^s_p(T_0)}, \\
\| L(w) \|_{E(T_0)} &\lesssim \nu \left( \| v \|_{K^s_p(T_0)} + \| \nabla v \|_{K^{s-1}_p(T_0)} \right) \| w \|_{E(T_0)}.
\end{aligned}$$
Collecting all above estimates, by Lemma 2.4 and the continuity argument, there exists a local solution \( \bar{w} \in X_{T_0}, T_0 < T^* \) of equations (5.3).

According to (5.3), we get by performing \( L^2 \)-energy estimate that
\[
\sup_{0 \leq t \leq T_0} \| \bar{w}(t) \|_{L^2}^2 + \nu \int_0^{T_0} \| \nabla \bar{w} \|_{L^2}^2 \, dt \leq \| w_0 \|_{L^2}^2 e^{C(T^*)1 - \frac{s}{p} \| \bar{v} \|_{K^s_p(T^*)}^{\frac{2p}{p-2}}}.
\]

Repeating the above same process in finite times, we finally obtain that equations (5.3) admit a local mild solution \( \bar{w} \in C([0, T^*); L^2) \) satisfying
\[
\bar{w} \in X_T \quad \text{for each} \quad T \in (0, T^*)
\]
and uniform estimate
\[
(5.6) \quad \sup_{0 \leq t < T^*} \| \bar{w}(t) \|_{L^2}^2 + \nu \int_0^{T^*} \| \nabla \bar{w} \|_{L^2}^2 \, dt \leq \| w_0 \|_{L^2}^2 e^{C(T^*)1 - \frac{s}{p} \| \bar{v} \|_{K^s_p(T^*)}^{\frac{2p}{p-2}}}.
\]

By uniqueness theorem, we know that \( \bar{w}(x, t) \equiv w(x, t) \) for \( t \in [0, T^*] \). Thus we can say \( w \) satisfies the global energy inequality (4.3) for any \( 0 < t \leq T^* \).

We turn to deal with the pressure \( \bar{P} \). From equations (4.1), it follows that
\[
-\Delta \bar{P} = \text{div} \text{div}(w \otimes w + w \otimes v + v \otimes w) \quad \text{in} \quad \mathbb{R}^3 \times (0, T^*).
\]
Therefore,
\[
\bar{P} = \partial_i \partial_j K * (w_i w_j) - \partial_i \partial_j K * (w_i v_j + v_i w_j) \triangleq \bar{P}_1 + \bar{P}_2
\]
with \( K(x) \) is the kernel function of \( (-\Delta)^{-1} \). Since \( v \in \tilde{K}_p^s(T^*) \) with \( s > s_\bar{p} \), by Hausdorff-Young’s inequality, we have \( v \in L^1((0, T^*); L^\tilde{p}) \) satisfying \( \frac{2}{\tilde{p}} + \frac{3}{\tilde{p}} = 1 \) where \( \tilde{p} \) is the conjugate index of \( \bar{p} \). By Calderón-Zygmund estimates, Hölder’s inequality and (4.23), we have,
\[
(5.7) \quad \| \bar{P}_1 \|_{L^2(\mathbb{R}^3 \times (0, T^*))} \leq C \| w \|_{L^s(\mathbb{R}^3 \times (0, T^*)}) \leq C(T^*)^{1/6} \| w \|_{L^\infty((0, T^*); L^3)} \| \nabla w \|_{L^2((0, T^*); L^2)}
\]
and
\[
(5.8) \quad \| \bar{P}_2 \|_{L^2(\mathbb{R}^3 \times (0, T^*))} \leq C \| w \|_{L^s((0, T^*); L^3)} \| w \|_{L^2((0, T^*); H^1)} \| v \|_{L^2((0, T^*); L^\tilde{p})},
\]
which yields \( \bar{P} \in L^{3/2}(\mathbb{R}^3 \times (0, T^*)) + L^2(\mathbb{R}^3 \times (0, T^*)) \). Hence, it’s obvious that \( w \) satisfies the local energy inequality (4.2) for any \( \varphi \in D(\mathbb{R}^3 \times (0, T^*)) \). In conclusion, we have proved that \( w \) is a energy solution of satisfying (W1)-(W5).

Next, we show the existence of singular points of local mild solution to (NS) with initial data in \( F\dot{B}_{p,q}^0(\mathbb{R}^3) \) with \( 1 < p < \frac{3}{2}, \ 1 \leq q < \infty \).

**Proposition 5.1.** Let \( u \in C([0, T^*); F\dot{B}_{p,q}^0(\mathbb{R}^3)) \) be the mild solution to (NS) with \( 1 < p < \frac{3}{2}, \ 1 \leq q < \infty \), and \( T^* \) denotes the maximal existence time. Suppose \( T^* < \infty \), then \( u \) has a singular point at \( T^* \), that is, there exists \( z_0 = (x_0, T^*) \), such that \( \| u \|_{L^\infty(Q_r(z_0))} = +\infty \) for any \( r > 0 \).

**Proof.** According to Remark 3.3, we know that there exists a \( 0 < t_0 < T^* \) such that \( u(t_0) \in FL^p(\mathbb{R}^3) \). Then, we decompose \( u(t_0) \) as follows:
\[
u(t_0) = \mathcal{F}(\hat{u}(t_0)I_{\{|\hat{u}(t_0)| > \lambda\}}) + \mathcal{F}(\hat{u}(t_0)I_{\{|\hat{u}(t_0)| \leq \lambda\}}) \triangleq v_0 + w_0,
\]
where $\lambda$ is a real number to be fixed later.

By a simple calculation, one has $v_0 \in FL^p(\mathbb{R}^3), w_0 \in L^2(\mathbb{R}^3)$. Thanks to (2.1), there exist $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 ||v_0||_{FL^p} \leq ||v_0||_{FL^p} \leq C_2 ||v_0||_{FL^p}.$$  

Therefore, by Theorem 3.2, there exists a mild solution $v \in C([t_0, T^*], FL^p)$ of system (NS) corresponding to the initial data $v(t_0) = v_0$. From Proposition 3.3 and Remark 3.4 we have

$$\sup_{0 \leq t \leq T^*} t^{\frac{\nu}{2} + \frac{p}{2} - \frac{\nu}{2}} ||\nabla^\alpha v(t)||_{FL^q} < \infty, \quad \forall (\alpha, \eta) \in \{0, 1\} \times [1, p].$$

Since $T^*$ depend on $||v_0||_{FL^p}$, we choose $\lambda$ large enough such that $||v_0||_{FL^p}$ is small and then $T^* > T^*$. Thus, we can deduce that for any $0 < \delta < T^*$,

$$||v||_{L^2((T^* - \delta, T^*); FL^p)} \leq \sum_{j \geq 0} 2^{j\frac{3-\frac{\nu}{2}}{2}} \|\Delta_j v\|_{L^2((T^* - \delta, T^*); FL^p)} + \sum_{j > 0} 2^{j\frac{3-\frac{\nu}{2}}{2}} \|\Delta_j v\|_{L^2((T^* - \delta, T^*); FL^p)}$$

$$\approx \sum_{j \geq 0} 2^{j\frac{3-\frac{\nu}{2}}{2}} ||v||_{L^2((T^* - \delta, T^*); FL^p)} + \sum_{j > 0} 2^{j\frac{3-\frac{\nu}{2}}{2}} ||v||_{L^2((T^* - \delta, T^*); FL^p)} < \infty.$$  

Let $w = u - v$ and $\bar{P}$ is the associated pressure. In the same way as used in the first part of this section, we know $w$ is a local energy solution of the following equations

$$\begin{align*}
\partial_t w - \nu \Delta w + w \cdot \nabla w + v \cdot \nabla w + w \cdot \nabla v + \nabla \bar{P} = 0, \quad (x, t) \in \mathbb{R}^3 \times (t_0, T^*), \\
div w = 0, \\
w(x, t_0) = w(t_0),
\end{align*}$$

where $v \in L^\infty((0, T^*); FL^p)$, and $w, \bar{P}$ satisfy the following estimates:

$$\sup_{0 \leq t \leq T^*} \{ \|w\|_{L^2(\mathbb{R}^3)} + \|\nabla w\|_{L^2(\mathbb{R}^3 \times (t_0, T^*))} + \|\bar{P}\|_{L^2(\mathbb{R}^3 \times (t_0, T^*))} \} < \infty.$$  

By the same argument used in [21] Proposition 3.2, we can show that

$$\begin{align*}
\|w(t)\|_{L^2_{uloc}(\mathbb{R}^3 \setminus B_{2R}(0))}^2 + \int_0^t \|\nabla w(t')\|_{L^2_{uloc}(\mathbb{R}^3 \setminus B_{2R}(0))}^2 dt' \\
\leq C(\nu, \|w_0\|_{L^2(\mathbb{R}^3)}, T^*, ||v||_{L^\infty([t_0, T^*]; FL^p(\mathbb{R}^3))}) \left( \|w_0\|_{L^2_{uloc}(\mathbb{R}^3 \setminus B_{2R}(0))}^2 + \frac{1 + \log R^2}{R} \right).
\end{align*}$$

So, we conclude that there exists a $R > 0$, such that for any $|x_0| > R$,

$$R_0^{-2} \int_{Q_{R_0}(x_0, T^*)} (|w|^3 + |\bar{P}|^3) \, dx \, dt < \epsilon_0,$$

where $R_0 = \sqrt{2}R$ and $\epsilon_0$ is the constant in Theorem 4.1. Since $(w, \bar{P})$ is a suitable weak solution to equations (1.1) with $v \in L^\infty L^{\frac{\nu}{2}-\frac{\nu}{2}}((Q_{R_0}(x_0, T^*))$ in $Q_{R_0}(x_0, T^*)$ and by $\varepsilon$-regularity criterion in Theorem 4.1, we have $w$ is bounded in $(\mathbb{R}^3 \setminus B_R(0)) \times [T^* - \delta, T^*].$

Suppose Proposition 5.1 is false. Then $v$ is bounded in a neighborhood of any point in $\mathbb{R}^3 \times [T^* - \delta, T^*]$. Since $v \in L^\infty(\mathbb{R}^3 \times [T^* - \delta, T^*])$. Thus, $w$ is bounded in a neighborhood
of any point in $\mathbb{R}^3 \times [T^* - \delta, T^*]$. By Vitali covering theorem, we know that $w$ is bounded in $B_R(0) \times [T^* - \delta, T^*]$. This shows that $w \in L^{\infty}(\mathbb{R}^3 \times [T^* - \delta, T^*])$. Combining this fact with $w \in L^{\infty}((0, T^*); L^2) \cap L^2((0, T^*); H^1)$ induces us to claim that

$$w \in \tilde{L}^2((T^* - \delta, T^*); F \dot{B}_{p,q}^{s+1}(\mathbb{R}^3)).$$

Setting $w = \sum_{j \leq 0} \hat{\Delta}_j w + \sum_{j > 0} \hat{\Delta}_j w \triangleq w^l + w^h$, we can easily get that

$$\|w^l\|_{L^2((T^* - \delta, T^*); F \dot{B}_{p,q}^{s+1})} \lesssim \|w^l\|_{L^2([0, T^*]; L^2)} \lesssim \delta^2 \|w\|_{L^\infty([0, T^*]; L^2)}.$$ 

Thus, we only need to prove $w^h \in \tilde{L}^2((T^* - \delta, T^*); F \dot{B}_{p,q}^{s+1}(\mathbb{R}^3)).$

We rewrites

$$w(x, t) = e^{\nu(t-T^*+\delta)}w(T^* - \delta) + \int_{T^*-\delta}^t e^{\nu(t-\tau)}\Delta \mathcal{P}(w \cdot \nabla w) \, d\tau$$

$$+ \int_{T^*-\delta}^t e^{\nu(t-\tau)}\Delta \mathcal{P}(w \cdot \nabla v + v \cdot \nabla w) \, d\tau$$

$$\triangleq G + B(w, w) + L(w),$$

and

$$w^h(x, t) = \sum_{j > 0} \hat{\Delta}_j G + \sum_{j > 0} \hat{\Delta}_j B(w, w) + \sum_{j > 0} \hat{\Delta}_j L(w) \triangleq G^h + B^h(w, w) + L^h(w).$$

For $G^h$, by [2.2], we get

$$\|G^h\|_{L^2((T^* - \delta, T^*); F \dot{B}_{p,q}^{s+1})} \leq \left( \sum_{j \geq -1} 2^{jq(2-\frac{1}{p})} \right)^{\frac{1}{q}} \|w(T^* - \delta)\|_{L^{p,\infty}} < \infty.$$ 

For $B^h(w, w)$, we have

$$\|B^h(w, w)\|_{L^2((T^* - \delta, T^*); F \dot{B}_{p,q}^{s+1})} \lesssim \sum_{j \geq 1} 2^{j2} \|\hat{\Delta}_j B(w, w)\|_{L^2((T^* - \delta, T^*); L^2)}$$

$$\lesssim \sum_{j \geq -1} 2^{j2q} \left( \int_{T^*-\delta}^t e^{\nu(t-\tau)2j} \|w(\tau)\|_{L^2} \|\nabla w(\tau)\|_{L^2} \, d\tau \right)^{\frac{q}{2}}$$

$$\lesssim \sum_{j \geq -1} 2^{-j2q} \|w\|_{L^\infty(\mathbb{R}^3 \times (T^* - \delta, T^*))} \|\nabla w\|_{L^2((T^* - \delta, T^*); L^2)}^q < \infty.$$ 

Similarly, we have by the H"{o}lder inequality that

$$\|L^h(w)\|_{L^2((T^* - \delta, T^*); F \dot{B}_{p,q}^{s+1})} \lesssim \delta^{\frac{q}{2}} \|\nabla v\|_{L^\infty(\mathbb{R}^3 \times (T^* - \delta, T^*))} \|w\|_{L^\infty((T^* - \delta, T^*); L^2)}$$

$$+ \|v\|_{L^\infty(\mathbb{R}^3 \times (T^* - \delta, T^*))} \|\nabla w^l\|_{L^2((T^* - \delta, T^*); L^2)} < \infty.$$ 

Collecting above estimates, we get that $w \in \tilde{L}^2((T^* - \delta, T^*); F \dot{B}_{p,q}^{s+1}(\mathbb{R}^3))$. This, together with $v \in \tilde{L}^2((T^* - \delta, T^*); F \dot{B}_{p,q}^{s+1}(\mathbb{R}^3))$, gives $u \in \tilde{L}^2((T^* - \delta, T^*); F \dot{B}_{p,q}^{s+1}(\mathbb{R}^3))$. According
Now we come back to the proof of Theorem 1.1.

Since $p_{\text{max}} < \infty$, we can choose a sequence $u_0^{(k)} \in F\dot{B}^s_{p,q}(\mathbb{R}^3)$ such that $T^*(u_0^{(k)}) < \infty$ and $\|u_0^{(k)}\|_{F\dot{B}^s_{p,q}} \ll p_{\text{max}}$. According to Proposition 5.1, the mild solution $u^{(k)}$ associated to initial data $u_0^{(k)}$ has a singular point $(x^{(k)}, T^*(u_0^{(k)}))$. After the following scaling and translation

$$u_0^{(k)}(x) \to 2^{\lambda_k}u_0^{(k)}(2^{\lambda_k}(x - x^{(k)})) \quad \text{and} \quad u^{(k)}(x, t) \to 2^{\lambda_k}u^{(k)}(2^{\lambda_k}(x - x^{(k)}), 2^{2\lambda_k}t)$$

with $\lambda_k \in \mathbb{Z}$, the singular point $(x^k, T^*(u_0^{(k)}))$ becomes $(0, 2^{-2\lambda_k}T^*(u_0^{(k)}))$. We still denote the sequence after the above translation and scaling as $u^{(k)}$, so does $u_0^{(k)}$. Then we can choose a series $\lambda_k$ such that $T^*_k \triangleq 2^{-2\lambda_k}T^*(u_0^{(k)}) \in (\frac{1}{2}, 1)$ for any $k \in \mathbb{N}$. Thus, there exists a subsequence of $\lambda_k$, still denoted by $\lambda_k$, such that $T^*_k$ converge to a point $t^*$. Still denote the corresponding subsequence of $u^{(k)}$ by $u^{(k)}$, so does $u_0^{(k)}$. Thus, we can conclude that the mild solution $u^{(k)}$ associated to initial data $u_0^{(k)}$ has a singular point $(0, T^*_k)$ with $T^*_k \in (\frac{1}{2}, 1)$ and $T^*_k \to t^*$ as $k \to \infty$.

According to $\|u_0^{(k)}\|_{F\dot{B}^s_{p,q}} \ll p_{\text{max}}$, we can easily get $\|u_0^{(k)}\|_{F\dot{B}^s_{p,q}} \leq M$ for some constant $M > 0$. According to Lemma 2.3 for any $j \in \mathbb{Z}$, we can split $u_0^{(k)}$ into $v_0^{(k)} + w_0^{(k)}$ with $\text{div} \, w_0^{(k)} = \text{div} \, w_0^{(k)} = 0$,

$$\|v_0^{(k)}\|_{F\dot{B}^s_{p,q}} \leq C2^{-j\theta}\|u_0^{(k)}\|_{F\dot{B}^s_{p,q}} \leq C2^{-j\theta}M$$

and

$$\|w_0^{(k)}\|_{L^2} \leq C2^{j(1-\theta)}\|u_0^{(k)}\|_{F\dot{B}^s_{p,q}} \leq C2^{j(1-\theta)}M,$$

where $s_p, s, p, q, \tilde{p}, \tilde{q}$ satisfy (2.4). Moreover, for suitable larger $j$, $u^{(k)}$ can be decomposed as $u^{(k)} = v^{(k)} + w^{(k)}$, where $v^{(k)}$ is a unique local mild solution to (NS) established in Theorem 3.2 stating from $v_0^{(k)}$ on $\mathbb{R}^3 \times (0, 1)$ and $(w^{(k)}, F^{(k)})$ is the energy weak solution of equations (4.11) on $\mathbb{R}^3 \times (0, T^*_k)$ with $v$ replaced by $v^{(k)}$ associated to initial data $w_0^{(k)}$. Due to Remark 3.3, $v^{(k)}$ satisfies for any $(\alpha, \eta) \in \{0, 1\} \times [1, p]$,

$$\sup_{0 \leq t \leq 1} t^{\frac{\eta}{2} + \frac{\tilde{p}}{2} - \frac{3}{2}} \|\nabla^\alpha v^{(k)}\|_{FL^s} \leq C(M, j).$$

Form (5.12), we can easily get that for any multi-index $\alpha$ satisfying $|\alpha| = 0, 1, 2$, $D^\alpha v^{(k)}$ is uniformly bounded in $L^\infty([\delta, 1]; L^\infty(\mathbb{R}^3))$ for any $0 < \delta < 1$ and $v^{(k)}$ is uniformly bounded in $L^r([0, 1]; L^{q(r)}(\mathbb{R}^3)))^2$ with $\frac{2}{r} + \frac{3(\tilde{p} - 1)}{\tilde{p}} = 1$. This, together with the facts that $v^{(k)}$ satisfies

$$\partial_t v^{(k)} - \nu \Delta v^{(k)} + \mathbf{P}(v^{(k)} \cdot \nabla v^{(k)}) = 0$$

on $(0, 1) \times \mathbb{R}^3$, entails that

$$\|\partial_t v^{(k)}\|_{L^\infty([\delta, 1]; L^\infty)} \leq C(M, j).$$

By Ascoli-Arzela theorem, there exist a subsequence of $v^{(k)}$, still denoted by $v^{(k)}$, and $v$ such that

$$v^{(k)} \rightharpoonup v \text{ in } \dot{K}^s_p(1), \quad v^{(k)} \to v \text{ in } C_{\text{loc}}(\mathbb{R}^3 \times (0, 1]), \quad v_0^{(k)} \rightharpoonup v_0 \text{ in } F\dot{B}^s_{p,q}(\mathbb{R}^3).$$
In addition, we have
\[ \|\nu \Delta w^{(k)}\|_{L^r([0,1];W^{-2,\frac{2}{r+1}})} + \|\mathbb{P}(\nu w^{(k)} \cdot \nabla w^{(k)})\|_{L^3([0,1];W^{-1,\frac{2}{r+1}})} \leq C, \]
which implies
\[ \int_{\mathbb{R}^3} v^{(k)}(t) \varphi \, dx \longrightarrow \int_{\mathbb{R}^3} v(t) \varphi \, dx \quad \text{in} \quad C([0,1]), \]
for any \( \varphi \in \mathcal{D}(\mathbb{R}^3) \). According to the above convergence, we can say that \( v \in \hat{K}_p^s(1) \) is a unique mild solution of equations (NS) on \( \mathbb{R}^3 \times (0,1) \) with initial data \( v_0 \in F\hat{B}_{p,q}^s(\mathbb{R}^3) \).

By Proposition 4.10, there exists a local energy solution \((\tilde{w}^{(k)}, P^{(k)})\) to equations (4.1) on \( \mathbb{R}^3 \times (0,1) \) with initial data \( w_0^{(k)} \) associated to \( v^{(k)} \), satisfying \( \tilde{w}^{(k)}(k) = v^{(k)} \) on \([0,T^k]\). By using estimates (5.11) and (5.12), we can easily get
\[ \sup_{0 \leq t \leq 1} \|w(t)\|_{L^2}^2 + \nu \int_0^1 \|\nabla w(t)\|_{L^2}^2 \, dt + \|\tilde{P}(k)\|_{L^2_2(\mathbb{R}^3 \times (0,1))} \leq C(M, f). \]

From equations (4.1), we can get that \( \{\partial_t w(k)\}_k \) is uniformly bounded in \( L^{3/2}(0,1); H^{-2}\). Moreover, for any \( r > 0 \),
\[ H^1_x(B_r) \hookrightarrow L^{3/2}_x(B_r) \hookrightarrow H^{-2}_x(B_r) \]
and \( H^1(B_1) \) is reflexive. Thus, by Aubin-Lions Lemma in Chapter 5 of Seregin [26], we obtain that \( \{w(k)\}_k \) is compact in \( L^{3/2}(B_r \times (0,1)) \) for any \( r > 0 \). By interpolation between \( L^\infty((0,1); L^2(B_r)) \) and \( L^2((0,1); \dot{H}^1(B_r)) \), we have \( \{w(k)\}_k \) is bounded in \( L^{10/3}(B_r \times (0,1)) \). Hence, we conclude that \( w^{(k)} \) is compact in \( L^3(B_r \times (0,1)) \). Therefore, there exists a subsequence of \((\tilde{w}^{(k)}, P^{(k)}, v^{(k)})\), still denoted by \((\tilde{w}^{(k)}, P^{(k)}, v^{(k)})\), and a suitable weak solution \((w, P)\) to equations (4.1) associated to \( v \) such that
- in \( \tilde{w}^{(k)} \rightarrow w \) in \( L^\infty([0,1]; L^2) \) and \( \tilde{w}^{(k)} \rightharpoonup w \) in \( L^2((0,1); \dot{H}^1) \)
- \( \tilde{w}^{(k)} \rightharpoonup w_0 \) in \( L^2 \)
- \( \tilde{w}^{(k)} \rightarrow w \) in \( L^3_{\text{loc}}(\mathbb{R}^3 \times [0,1]) \) and \( P^{(k)} \rightarrow P \) in \( L^2_3(\mathbb{R}^3 \times [0,1]) + L^2(\mathbb{R}^3 \times [0,1]) \)
- \( w \) is weakly continuous in \( L^2 \) with respect to \( t \in [0,1] \).

Whit the above convergence, one can easily deduce that \((w, P)\) is a suitable weak solution to (4.1) associated to \( v \) on \( \mathbb{R}^3 \times (0,1) \). By stability of singularity in Proposition 4.8 we can say \( w \) has a singular point \((0, t^\star)\). In addition, by Proposition 4.9 there exists a \( 0 < \gamma < \frac{s-q}{2s} \) such that for any \( 0 < t < \gamma \),
\[ \|\tilde{w}^{(k)} - e^{\nu \Delta t} w_0^{(k)}\|_{L^2} \leq C(M) \left( t^{\gamma(1-\theta)} + t^{\frac{s-q-\gamma}{2}} \right). \]
So, we have that for any \( 0 < t < \gamma \)
\[ \|w(t) - e^{\nu \Delta t} w_0\|_{L^2} \leq \liminf_{k \to \infty} \|\tilde{w}^{(k)}(t) - e^{\nu \Delta t} w_0^{(k)}\|_{L^2} \leq C(M) \left( t^{\gamma(1-\theta)} + t^{\frac{s-q-\gamma}{2}} \right), \]
which implies that \( \lim_{t \to 0^+} \|w(t) - w_0\|_{L^2} = 0 \). This together with the above convergence enables us to conclude that \( w \) is a local energy solution to equations (4.1) on \( \mathbb{R}^3 \times (0,1) \).

Let \( u \in C([0, T^s); F\hat{B}^s_{p,q}) \) and \( \tilde{v} \in C([0, T_2); F\hat{B}^s_{p,q}) \), be the unique mild solution established in Theorem 3.1 or Theorem 3.2 starting from \( u_0 \) and \( v_0 \), respectively. By the uniqueness, we have \( v = \tilde{v} \) on \( \mathbb{R}^3 \times (0,1) \) and \( T_2 \geq 1 \). Set \( \tilde{w} = u - v \), we have by Proposition 4.10 that
$\dot{w} = w$ on $\mathbb{R}^3 \times (0, \min\{T^*, 1\})$ which implies $w = u + w$ on $\mathbb{R}^3 \times (0, \min\{T^*, 1\})$. Since $w$ is singular at $t^* < 1$, one has $T^* \leq t^* < 1$. Relying on the definition of $\rho_{\text{max}}$, we have that
\[
\rho_{\text{max}} \leq \|u_0\|_{\dot{F}^{s_p}_{p,q}} \leq \liminf_{k \to \infty} \|u_0^{(k)}\|_{\dot{F}^{s_p}_{p,q}} \leq \rho_{\text{max}}.
\]
Thus $\|u_0\|_{\dot{F}^{s_p}_{p,q}} = \rho_{\text{max}}$ and $\|u_0^{(k)}\|_{\dot{F}^{s_p}_{p,q}} \to \|u_0\|_{\dot{F}^{s_p}_{p,q}}$. This, together with the fact that $\dot{F}^{s_p}_{p,q}(\mathbb{R}^3)$ is a uniformly convex Banach space, gives $u_0^{(k)} \to u_0$ in $\dot{F}^{s_p}_{p,q}(\mathbb{R}^3)$.

**Case 2:** $2/3 \leq p \leq \infty$ and $2 < q < \infty$.

Since $q > 2$, by Lemma 2.2 for any $j \in \mathbb{Z}$, we can split $u_0 \in \dot{F}^{2-\frac{4}{q}}_{p,q}(\mathbb{R}^3)$ into $f_0 + g_0$ satisfying $\text{div} f_0 = \text{div} g_0 = 0$ and
\[
\|f_0\|_{\dot{F}^{2-\frac{4}{q}}_{p,q}} \leq C 2^{-j^2} \|u_0\|_{\dot{F}^{2-\frac{4}{q}}_{p,q}}, \quad \|g_0\|_{\dot{F}^{2-\frac{4}{q}}_{p,q}} \leq C 2^{-j(1-\theta)} \|u_0\|_{\dot{F}^{2-\frac{4}{q}}_{p,q}},
\]
where $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{2} = 1$ and $\theta \in (0, 1)$.

From Theorem 3.1 we can choose a $j \in \mathbb{Z}$ small enough such that there exists a unique global in time mild solution $g \in C([0, \infty); \dot{F}^{2-\frac{4}{q}}_{p,q}) \cap \dot{L}^r([0, \infty); \dot{F}^{2-\frac{4}{q}+\frac{2}{r}}_{p,q})$ with $r \in [1, \infty)$ to (NS) starting from $g_0$. Thanks to $\dot{F}^{2-\frac{4}{q}}_{p,2}(\mathbb{R}^3) \hookrightarrow \dot{F}^{2-\frac{4}{q}}_{p,q}(\mathbb{R}^3) \hookrightarrow \dot{F}^{2-\frac{4}{q}+\frac{2}{q}}_{p,q}(\mathbb{R}^3)$ and Lemma 2.5 we can deduce that
\[
f \triangleq NS(u_0) - g \in C([0, T^*(u_0)); \dot{F}^{2-\frac{4}{q}}_{p,2}) \cap \dot{L}^r_{\text{loc}}([0, T^*(u_0)); \dot{F}^{2-\frac{4}{q}+\frac{2}{q}}_{p,q}), \quad \forall r \in (1, \infty),
\]
satisfies the following equations:
\[
\begin{aligned}
\partial_t f - \nu \Delta f + f \cdot \nabla f + f \cdot \nabla g + g \cdot \nabla f + \nabla Q &= 0, \\
\text{div } f &= 0, \\
f(x, 0) &= f_0.
\end{aligned}
\]
Note that $T^*(u_0)$ is also the maximal existence time of $f$. To prove Theorem 1.1 we only need to prove the existence of minimal blow-up initial data to problem (5.13) in $\dot{F}^{2-\frac{4}{q}}_{p,2}(\mathbb{R}^3)$ with $p \geq 3/2$.

Following the method developed by Gallagher and Planchon [15], and using Bony-paraproduct decomposition, we have
\[
(a, b, c) \in E \times E \times L^q((0, T); \dot{F}^{2-\frac{4}{q}}_{p,q}(\mathbb{R}^3)) \mapsto \int_0^T \int_{\mathbb{R}^3} (a \cdot \nabla b) \cdot c \, dx \, dt,
\]
is continuous for any $p \geq 3/2$ and $1 \leq q < \infty$. Then, we can translated the existence of minimal blow-up initial data to problem (5.13) in $\dot{F}^{2-\frac{4}{q}}_{p,2}(\mathbb{R}^3)$ into the existence of minimal initial data to (NS) in $\dot{F}^{2-\frac{4}{q}}_{p,2}(\mathbb{R}^3)$.

Hence, it is suffices to show Theorem 1.1 in $\dot{F}^{2-\frac{4}{q}}_{p,2}(\mathbb{R}^3)$. By embedding theorem, one has
\[
\dot{F}^{2-\frac{4}{q}}_{p,2}(\mathbb{R}^3) \hookrightarrow L^2_{\text{uloc}}(\mathbb{R}^3).
\]
By Remark 4.12 and existence of singularities, we can get the desired result by repeating the same process as used in Case 1 without decomposition.
Case 3: $2/3 \leq p \leq \infty$ and $1 \leq q \leq 2$.

Since $1 \leq q \leq 2$, by embedding theorem, we immediately have

$$FB_{p,q}^{2-\frac{2}{p}}(\mathbb{R}^3) \hookrightarrow L_{\text{loc}}^2(\mathbb{R}^3).$$

By Remark 4.12 again and existence of singularities, we can get the required result by Mimicking the proof in Case 1 without decomposition.

Theorem 1.1 is thus proved. □

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