A unifying description of dark energy

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Abstract

We review and extend a novel approach that we introduced recently, to describe general dark energy or scalar-tensor models. Our approach relies on an ADM formulation based on the hypersurfaces where the underlying scalar field is uniform. The advantage of this approach is that it can describe in the same language and in a minimal way a vast number of existing models, such as quintessence models, $F(R)$ theories, scalar tensor theories, their Horndeski extensions and beyond. It also naturally includes Horava-Lifshitz theories. As summarized in this review, our approach provides a unified treatment of the linear cosmological perturbations about a FLRW universe, obtained by a systematic expansion of our general action up to quadratic order. This shows that the behaviour of these linear perturbations is generically characterized by five time-dependent functions. We derive the full equations of motion in the Newtonian gauge, and obtain in particular the equation of state for dark energy perturbations, in the Horndeski case, in terms of these functions. Our unifying description thus provides the simplest and most systematic way to confront theoretical models with current and future cosmological observations.
1 Introduction

The discovery of the present cosmological acceleration, consistently confirmed by various cosmological probes, has spurred an intense theoretical activity to account for this observational fact. Although a cosmological constant is by far the simplest explanation for this acceleration, the huge fine-tuning that seems required, at least from a current perspective, has motivated the exploration of alternative models.

As a consequence, the dark energy landscape is now very similar to that of inflation, containing a huge number of models with various motivations and various degrees of sophistication. In fact, many of the inflationary models have been reconverted into dark energy models, and vice-versa. A majority of models of dark energy, although not all of them, involve a scalar field, in an explicit or implicit way. This scalar component can be simply added to standard gravity, like in quintessence models, or, more subtly, intertwined with gravity itself, like in scalar-tensor gravitational theories.
This illustrates the two ways to modify the dynamical equations in cosmology: either by adding a new matter component or by modifying gravity itself.

In this paper we review and extend the approach introduced in [1] to describe in a unifying and minimal way most existing dark energy or modified gravity models that contain a single scalar degree of freedom. This approach was initially inspired by the so-called effective field theory (EFT) formalism, pioneered in [2, 3] for inflation and in [4] for minimally coupled dark energy, and later developed in the context of dark energy [5, 6, 7] (see also [8] for a recent review and e.g. [9, 10, 11, 12, 13] for applications of the EFT formalism), but exploits more systematically the 3+1 spacetime ADM decomposition by starting from a Lagrangian written only in terms of ADM quantities. This leads to an almost automatic treatment of the equations of motion, both at the background and perturbative levels. Our ADM approach is also at the core of several recent works [17, 18] and is very useful for the theories beyond Horndeski that we proposed in [19, 20] (see also [21, 22, 23, 24, 25]).

In the present article, we give a slightly more general presentation of our formalism than that given in [1], by parametrizing the dynamical equations with (background-dependent) functions that are constructed directly from partial derivatives of the initial Lagrangian with respect to the ADM tensors, rather than from partial derivatives with respect to a few scalar combinations of the ADM tensors. This makes our formalism readily applicable to a larger class of models without further preparation work, but the results are essentially the same. The results obtained in [1] and in [7] have been nicely summarized in [26] by introducing a convenient notation based on a few dimensionless (time-dependent) functions that are simple combinations of the functions that appear in the effective formalisms previously introduced. Here, we will use this notation, up to a minor redefinition and an extension to theories beyond Horndeski.

The advantage of a unified treatment of dark energy is multiple. First, it provides a global view of the landscape of theoretical models, by translating them in the same language. They thus become easy to compare, with a clear identification of approximate or exact degeneracies between the models. Moreover, a precise map also enables theorists to identify, beyond well-known regions, unchartered territories that remain to be explored. A striking illustration of this is the recent realization that theories beyond Horndeski could be free from Ostrogradski instabilities [19, 20]: these theories were initially motivated by noticing that Horndeski theories correspond to a subset of all possibilities at the level of linear perturbations [1].

Second, a unified treatment of theoretical models enormously simplifies the confrontation of these with observational constraints. Instead of constraining separately each existing model in the literature, one can simply constrain the parametrized functions of the general formalism and then infer what this implies for each model. Our treatment reduces redundancies, ensuring that the number of parametrized functions is minimal for a given set of assumptions (number of space or time derivatives, etc.). One can also identify models that are confined to “subspaces” of the general framework and devise optimized ways to rule them out by observations.

Our plan is the following. In Sec. 2, we introduce the central starting point of our formalism, a generic Lagrangian written in the ADM formulation, and show how well-known models proposed in the literature can be reformulated in this form. In Sec. 3 we rederive the main results obtained in [1], but adopting a more general presentation than that given originally. Then, in Sec. 4 we focus our attention on the evolution of cosmological perturbations and translate the results of the previous section into the more familiar Newtonian gauge. Moreover, we derived the perturbed Einstein and scalar field equations. In the case of Horndeski, we provide an expression for the equation of state of dark energy perturbations and discuss its observational implications. In Appendix A we discuss the long wavelength limit of the perturbation equations, in Appendix B we give the perturbation equations in the synchronous gauge, while in Appendix C we provide the definitions of several parameters useful in the paper.

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1 Other general treatments of single degree of freedom dark energy, based on the equations of motion, can be found in Refs. [14, 15, 16]. The advantage of an action formulation is, of course, that one can easily identify ghost instabilities.
2 A unifying action

2.1 General action principle

In this section we review the approach introduced in [1]. Following [5], we assume the validity of the weak equivalence principle and thus the existence of a metric $g_{\mu\nu}$ universally coupled to all matter fields. The fundamental idea is then to start from a generic action that depends on the basic geometric quantities that appear in an ADM decomposition of spacetime, with uniform scalar field hypersurfaces as constant time hypersurfaces. The equations governing the background evolution and the linear perturbations can then be obtained in a generic way, up to a few simplifying assumptions (which can be easily relaxed) that are verified by most existing models.

2.1.1 Geometrical quantities

Our approach relies on the existence of a scalar field characterized by a time-like spacetime gradient, which is a natural assumption in a cosmological context. As a consequence, the uniform scalar field hypersurfaces correspond to space-like hypersurfaces and can be used for a 3+1 decomposition of spacetime.

One can associate various geometrical quantities to these hypersurfaces, which will be useful in order to build a generic variational principle. The most immediate geometrical quantities are the future-oriented time-like unit vector normal to the hypersurfaces $n^{\mu}$, which satisfies $g_{\mu\nu}n^{\mu}n^{\nu} = -1$, and the projection tensor on the hypersurfaces,

$$h_{\mu\nu} \equiv g_{\mu\nu} + n_{\mu}n_{\nu}.$$  

(1)

One can also introduce the intrinsic curvature of the hypersurfaces, described by the Ricci tensor (which contains as much information as the Riemann tensor for three-dimensional manifolds)

$$R_{\mu\nu},$$

(2)

and the extrinsic curvature tensor

$$K_{\mu\nu} \equiv h_{\nu}^{\rho} \nabla_{\rho} n^{\mu}.$$  

(3)

Other quantities can be derived by combining the above tensors, together with the covariant derivative $\nabla_{\mu}$ and the spacetime metric $g_{\mu\nu}$. For example, one can define the “acceleration” vector field

$$a^{\mu} \equiv n^{\lambda} \nabla_{\lambda} n^{\mu},$$

(4)

which is tangent to the hypersurfaces (since $n_{\mu}a^{\mu} = 0$).

With the geometrical quantities introduced above, the dependence on the scalar field is implicit. Since many dark energy models are given explicitly in terms of a scalar field $\phi$, it is useful to write down the correspondence between the various geometrical tensors and expressions of $\phi$. The relation between the unit vector $n^{\mu}$ and the first derivative of $\phi$ is simply

$$n_{\mu} = -\frac{1}{\sqrt{-X}} \nabla_{\mu} \phi, \quad X \equiv g^{\rho\sigma} \nabla_{\rho} \phi \nabla_{\sigma} \phi.$$  

(5)

The extrinsic curvature tensor is related to second derivatives of $\phi$, according to the expression

$$K_{\mu\nu} = -\frac{1}{\sqrt{-X}} \nabla_{\mu} \nabla_{\nu} \phi + n_{\mu} a_{\nu} + n_{\nu} a_{\mu} + \frac{1}{2X} n_{\mu} n_{\nu} n^{\lambda} \nabla_{\lambda} X,$$  

(6)

which can be derived by substituting (5) into (3).
Finally, since the Lagrangian for gravitational theories often involves the four-dimensional curvature, it is useful to recall the Gauss-Codazzi relation,

\[ (4)R = K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2\nabla_\mu(Kn^\mu - n^\rho\nabla_\rho n^\mu), \]

which expresses the four-dimensional curvature \( (4)R \) in terms of the extrinsic curvature tensor and of the intrinsic curvature. We will always denote the four-dimensional curvature with the subscript (4) to distinguish it from the hypersurface intrinsic curvature.

### 2.1.2 ADM coordinates

So far, all geometrical quantities have been introduced intrinsically, without reference to any specific coordinate system. However, since spacetime is endowed with a preferred slicing, defined by the uniform scalar field hypersurfaces, it is convenient to use coordinate systems especially adapted to this slicing, in other words so that constant time hypersurfaces coincide with the preferred hypersurfaces.

We thus express the four-dimensional metric in the ADM form

\[ ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \]

where \( N \) is the lapse and \( N^i \) the shift. In matricial form, the components of the metric and of its inverse are given respectively by

\[ g_{\mu\nu} = \begin{pmatrix} -N^2 + h_{ij}N^iN^j & h_{ij}N^j \\ h_{ij}N^i & h_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^j/N^2 \\ N^i/N^2 & h^{ij} - N^iN^j/N^2 \end{pmatrix}. \]

In ADM coordinates, we obtain

\[ X = g^{00}\dot{\phi}^2(t) = -\frac{\dot{\phi}^2(t)}{N^2}, \]

since the scalar field depends only on time, by construction. The components of the normal vector are thus given by

\[ n_0 = -N, \quad n_i = 0. \]

The components of the extrinsic curvature tensor can be written as

\[ K_{ij} = \frac{1}{2N}(h_{ij} - D_iN_j - D_jN_i), \]

where a dot stands for a time derivative with respect to \( t \), and \( D_i \) denotes the covariant derivative associated with the three-dimensional spatial metric \( h_{ij} \). Spatial indices are lowered and raised by the spatial metric.

In the following, we will consider general gravitational actions which can be expressed in terms of the geometrical quantities that we have introduced, expressed in ADM coordinates,

\[ S_g = \int d^4x\sqrt{-g} L(N, K_{ij}, R_{ij}, h_{ij}, D_i; t), \]

with \( \sqrt{-g} = N\sqrt{h} \), where \( h \) is the determinant of \( h_{ij} \). Note that, by construction, the above action is automatically invariant under spatial diffeomorphisms, corresponding to a change of spatial coordinates.

### 2.2 Examples

To make things concrete, let us illustrate our formalism by listing briefly the main scalar tensor theories that have been studied in the context of dark energy and by presenting their explicit reformulations in the general form (13).
2.2.1 General relativity

Before introducing models with a scalar component, let us start by simply rewriting the action for general relativity in the above ADM form. Starting from the Einstein-Hilbert action

$$S_{GR} = \int d^4x \sqrt{-g} \frac{M^2_{Pl}}{2} (4)R,$$  \hspace{1cm} (14)

and substituting the Gauss-Codazzi expression (7), one can get rid of the total derivative term and express the action in terms of the extrinsic and intrinsic curvature terms only. Therefore, one easily obtains a Lagrangian of the form (13) for General Relativity (GR), which reads

$$L_{GR} = \frac{M^2_{Pl}}{2} \left[ K_{ij} K^{ij} - K^2 + R \right].$$  \hspace{1cm} (15)

Note that, in contrast with the following examples that intrinsically contain a scalar degree of freedom, the slicing of spacetime is arbitrary since there is no preferred family of spacelike hypersurfaces. This means that the Lagrangian (15) contains an additional symmetry, leading to full four-dimensional invariance, which is not directly manifest in the ADM form.

2.2.2 Quintessence and $k$-essence

The simplest way to extend gravity with a scalar component is to add to the Einstein-Hilbert action a standard action for the scalar field, which consists of a kinetic term plus a potential. This corresponds to quintessence models. The initial covariant action

$$S = S_{GR} + \int d^4x \sqrt{-g} \left( -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) \right)$$  \hspace{1cm} (16)

leads to the ADM Lagrangian

$$L = L_{GR} + L_Q, \quad L_Q(t,N) = \frac{\dot{\phi}^2(t)}{2N^2} + V(\phi(t)).$$  \hspace{1cm} (17)

In a similar way, one can describe $k$-essence theories [27, 28] by expressing their Lagrangian $P(X, \phi)$ in terms of of $N$ and $t$:

$$L_{k\text{-essence}}(t,N) = P \left[ -\frac{\dot{\phi}^2(t)}{2N^2}, \phi(t) \right].$$  \hspace{1cm} (18)

2.2.3 $F((4)R)$ theories

Theories described by a Lagrangian consisting of a nonlinear function of the four-dimensional curvature scalar $^{(4)}R$ are equivalent to a scalar-tensor theory. Indeed, it is easy to verify that the Lagrangian

$$L_{F(4)} = F(\phi) + F_{\phi}(\phi)(4)R - \phi,$$  \hspace{1cm} (19)

is equivalent to the Lagrangian $F((4)R)$, as they lead to the same equations of motion (as long as $F((4)R(4)R \neq 0$). Given this property, one can then use eq. (7) to rewrite the above Lagrangian, after integration by parts, in the ADM form

$$L_{F(4)} = F_{\phi}(R + K_{\mu\nu} K^{\mu\nu} - K^2) + 2F_{\phi\phi} K\sqrt{-X} + F(\phi) - \phi F_{\phi}.$$  \hspace{1cm} (20)
2.2.4 Horndeski theories

In the last few years, a lot of activity has been focussed on a large class of theories, known as Horndeski theories [29] or Generalized Galileons [30]. Although their Lagrangians contain up to second derivatives of a scalar field, these theories correspond to the most general scalar tensor theories that directly lead to at most second order equations of motion. As such, they include all the examples introduced above. They can be written as an arbitrary linear combination of the following Lagrangians:

\begin{align}
L_2^H[G_2] &\equiv G_2(\phi, X), \\
L_3^H[G_3] &\equiv G_3(\phi, X) \Box \phi, \\
L_4^H[G_4] &\equiv G_4(\phi, X)(4R - 2G_{4X}(\phi, X)[(\Box \phi)^2 - (\nabla^\mu \nabla^\nu \phi)(\nabla_\mu \nabla_\nu \phi)]), \\
L_5^H[G_5] &\equiv G_5(\phi, X)(4G_{\mu \nu} \nabla^\mu \nabla^\nu \phi + \frac{1}{3} G_{5X}(\phi, X) \times \\
&\quad \left[(\Box \phi)^3 - 3 \Box \phi (\nabla^\mu \nabla^\nu \phi)(\nabla_\mu \nabla_\nu \phi) + 2 (\nabla_\mu \nabla_\nu \phi)(\nabla^\sigma \nabla^\nu \phi)(\nabla_\sigma \nabla^\mu \phi)\right].
\end{align}

Rewriting these Lagrangians in the ADM form turns out to be significantly more involved than in the previous examples. This calculation was undertaken in [1], where all the details are given explicitly. The final result is that the above Lagrangians (21)–(24) yield, in the ADM form, combinations of the following four Lagrangians

\begin{align}
L_2^H &\equiv F_2(\phi, X), \\
L_3^H &\equiv F_3(\phi, X) K, \\
L_4^H &\equiv F_4(\phi, X) R + (2XF_{4X} - F_4)(K^2 - K_{\mu \nu} K_{\mu \nu}), \\
L_5^H &\equiv F_5(\phi, X) G_{\mu \nu} K^{\mu \nu} - \frac{1}{3} XF_{5X}(K^3 - 3KK_{\mu \nu} K^{\mu \nu} + 2K_{\mu \nu} K^{\mu \sigma} K^{\nu \sigma}).
\end{align}

The functions \(F_a\) appearing here are related to the \(G_a\) in eqs. (21)–(24) through (see [1] for details)

\begin{align}
F_2 &= G_2 - \sqrt{-X} \int \frac{G_{3\phi}}{2\sqrt{-X}} dX, \\
F_3 &= - \int G_{3X} \sqrt{-X} dX - 2\sqrt{-X} G_{4\phi}, \\
F_4 &= G_4 + \sqrt{-X} \int \frac{G_{5\phi}}{4\sqrt{-X}} dX, \\
F_5 &= - \int G_{5X} \sqrt{-X} dX.
\end{align}

It is then straightforward to express the above Lagrangians in ADM coordinates (8).

2.2.5 Beyond Horndeski

Requiring equations of motion of at most second order, which leads to Horndeski theories, has long seemed to be a necessary requirement in order to avoid ghost-like instabilities, associated with higher order time derivatives, also known as Ostrogradski instabilities. However, it has been shown in [19, 20] (see also [24] for similar analysis and conclusion and [21, 22, 25] for extensions) that an action composed of the Lagrangians

\begin{align}
L_2 &\equiv A_2(t, N), \\
L_3 &\equiv A_3(t, N) K, \\
L_4 &\equiv A_4(t, N)(K^2 - K_{ij} K^{ij}) + B_4(t, N) R, \\
L_5 &\equiv A_5(t, N)(K^3 - 3KK_{ij} K^{ij} + 2K_{ij} K^{ik} K_k^j) + B_5(t, N) K^{ij} \left(R_{ij} - \frac{1}{2} h_{ij} R\right),
\end{align}
with arbitrary functions $B_4$ and $B_5$, i.e. without assuming $B_4$ and $B_5$ to depend on, respectively, $A_4$ and $A_5$ (as implied by the Horndeski Lagrangians (27) and (28)), does not lead to Ostrogradski instabilities, in contrast with previous expectations. This conclusion is based on a Hamiltonian analysis of the Lagrangian (30), which applies to all configurations where the spacetime gradient of the scalar field is timelike.

Interestingly, one can also map two subclasses of the general covariant Lagrangian, namely the subclass without $L_4$ and the subclass without $L_5$, to Horndeski theories via a disformal transformation of the metric (disformal transformations are discussed in section 3.4). Since $L_4$ and $L_5$ require distinct disformal transformations to be related to Horndeski theories, such transformation cannot be applied to the whole Lagrangian [20].

2.2.6 Hořava-Lifshitz theories

An interesting class of Lorentz-violating gravitational theories has been introduced by Hořava with the goal of obtaining (power counting) renormalizability [31]. These theories, dubbed Hořava-Lifshitz gravity, assume the existence of a preferred foliation of spacelike hypersurfaces. An ADM formulation of these theories is thus very natural, even if a covariant description is also possible, via the introduction of a scalar field, often called “khronon”, that describes the foliation. Several variants of Hořava-Lifshitz gravity have been proposed in the literature. In particular, the so-called healthy non-projectable extension has been shown to be free of instabilities [32, 33]. All these theories are describable by a Lagrangian of the form (13), which can be written as (see [34] for a general discussion)

\[ L_{\text{HL}} = \frac{M_{\text{Pl}}^2}{2} \left[ K_{ij} K^{ij} - \lambda K^2 + \mathcal{V}(R_{ij}, N^{-1} \partial_i N) \right]. \]  

Note that the dependence on $a^i = N^{-1} \partial_i N$ has been introduced in the healthy non-projectable extension of Hořava-Lifshitz gravity. Since the Hořava-Lifshitz Lagrangian is already in an ADM form, it is very natural to include these theories in our general approach, as discussed in [21] (see also [35]).

3 Cosmology: background equations and linear perturbations

In this section, we analyse from a general perspective the cosmological dynamics, for the background and linear perturbations, simply starting from a generic Lagrangian of the form (13).

3.1 Background evolution

We first discuss the background equations by considering a spatially flat FLRW spacetime, endowed with the metric

\[ ds^2 = -\tilde{N}^2(t) dt^2 + a^2(t) \delta_{ij} dx^i dx^j. \]  

In this spacetime, the intrinsic curvature tensor of the constant time hypersurfaces vanishes, i.e. $R_{ij} = 0$, and the components of the extrinsic curvature tensor are given by

\[ K^i_j = \frac{\dot{a}}{Na} \delta^i_j \equiv H \delta^i_j, \]  

where $H$ is the Hubble parameter. Substituting into the Lagrangian $L$ of (13), one thus obtains an homogeneous Lagrangian, which is a function of $\tilde{N}(t)$, $a(t)$ and of time:

\[ \bar{L}(a, \dot{a}, \tilde{N}) \equiv L \left[ K^i_j = \frac{\dot{a}}{Na} \delta^i_j, R^i_j = 0, N = \tilde{N}(t) \right]. \]  

\(^2\)The Lagrangians (30) describe Horndeski theories if the following relations hold: $A_4 = -B_4 + 2XB_4X$ and $A_5 = -XB_5X/3$. 

8
The variation of the homogeneous action,
\[ \tilde{S}_g = \int dt \, d^3x \tilde{N} \alpha^3 \bar{L}, \] (35)
leads to
\[ \delta \tilde{S}_g = \int dt d^3x \left\{ a^3 \left( \bar{L} + \tilde{N} L_N - 3H \mathcal{F} \right) \delta \tilde{N} + 3a^2 \tilde{N} \left( \bar{L} - 3H \mathcal{F} - \frac{\dot{\mathcal{F}}}{\tilde{N}} \right) \delta a \right\}, \] (36)
where \( L_N \) denotes the partial derivative \( \partial L/\partial N\rvert_{bgd} \), evaluated on the homogeneous background. We have also introduced the coefficient \( \mathcal{F} \), which is defined from the derivative of the Lagrangian with respect to extrinsic curvature, evaluated on the background:
\[ \left( \frac{\partial L}{\partial K_{ij}} \right)_{bgd} = \bar{g}^{ij}, \] (37)
where \( \bar{g}^{ij} = a^{-2} \delta^{ij} \) are the spatial components of the inverse background metric.

If we add some matter minimally coupled to the metric \( g_{\mu\nu} \), the variation of the corresponding action with respect to the metric defines the energy-momentum tensor,
\[ \delta S_m = \frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}. \] (38)
In a FLRW spacetime, this reduces to
\[ \delta \tilde{S}_m = \int d^4x \tilde{N} \alpha^3 \left( -\rho_m \frac{\delta \tilde{N}}{\tilde{N}} + 3p_m \frac{\delta a}{a} \right). \] (39)
Consequently, variation of the total homogeneous action \( \tilde{S} = \tilde{S}_g + \tilde{S}_m \) with respect to \( N \) and \( a \) yields, respectively, the first and second Friedmann equations in a very unusual form:
\[ \bar{L} + \tilde{N} L_N - 3H \mathcal{F} = \rho_m \] (40)
and
\[ \bar{L} - 3H \mathcal{F} - \frac{\dot{\mathcal{F}}}{\tilde{N}} = -p_m. \] (41)
These two equations also imply
\[ \frac{\dot{\mathcal{F}}}{\tilde{N}} + \tilde{N} L_N = \rho_m + p_m. \] (42)

Although written in a very unusual form, it is easy to check that one recovers the usual Friedmann equations when gravity is described by general relativity. Indeed, in this case,
\[ \frac{\partial L_{GR}}{\partial K_{ij}^i} = M_{Pl}^2 \left( K_{ij}^i - K \delta_{ij} \right), \] (43)
which, after substituting \( K_{ij}^i = H \delta_{ij} \), yields,
\[ \mathcal{F}_{GR} = -2M_{Pl}^2 H, \] (44)
whereas \( \bar{L}_{GR} = -3M_{Pl}^2 H^2 \) and \( L_N = 0. \)

\[ ^{9}\text{The present formulation of our approach is more general than that given explicitly in [1], where we assumed that the Lagrangian } L \text{ was a function of specific scalar combinations of the geometric tensors, namely of } K \equiv K_{ij}^i, S \equiv K_{ij} K_{ij}, R \equiv R_{ij}^i \text{ and } Z \equiv R_{ij} R^{ij}. \text{ The coefficient } \mathcal{F} \text{ was then related to the derivatives of } L \text{ with respect to } K \text{ and } S, \text{ i.e. } \mathcal{F} = L_K + 2HL_S. \text{ The definition (37) enables us to include automatically a dependence on other scalar combinations, such as } K_{ij}^i K_{ij}^i K_{ij}^i \text{ which appears in } L_S. \]
3.2 Quadratic action

In order to describe the dynamics of linear perturbations about the FLRW background solution, we now expand the action up to quadratic order. The tensor $R_{ij}$ vanishes in the background and is thus a perturbative quantity. It is useful to introduce the two other perturbative quantities (remembering the definition of $H$ in eq. (33))

$$\delta N \equiv N - \bar{N}, \quad \delta K^i_j \equiv K^i_j - H \delta_i^j.$$

(45)

The expansion of the Lagrangian $L$ up to quadratic order yields

$$L(N, K^i_j, R^i_j, \ldots) = \bar{L} + L_N \delta N + \frac{\partial L}{\partial K^i_j} \delta K^i_j + \frac{\partial L}{\partial R^i_j} \delta R^i_j + L^{(2)} + \ldots,$$

(46)

with the quadratic part given by

$$L^{(2)} = \frac{1}{2} L_{NN} \delta N^2 + \frac{1}{2} \frac{\partial^2 L}{\partial K^i_j \partial K^k_l} \delta K^i_j \delta K^k_l + \frac{1}{2} \frac{\partial^2 L}{\partial R^i_j \partial R^k_l} \delta R^i_j \delta R^k_l +$$

$$+ \frac{\partial^2 L}{\partial K^i_j \partial R^k_l} \delta K^i_j \delta R^k_l + \frac{\partial^2 L}{\partial N \partial K^i_j} \delta N \delta K^i_j + \frac{\partial^2 L}{\partial N \partial R^i_j} \delta N \delta R^i_j + \ldots,$$

(47)

where all the partial derivatives are evaluated on the FLRW background (without explicit notation, as will be the case in the rest of this paper). The coefficient $L_{NN}$ denotes the second derivative of the Lagrangian with respect to $N$. The dots in the two above equations correspond to other possible terms which are not indicated explicitly to avoid too lengthy equations, but can be treated exactly in the same way. This includes for instance the spatial derivatives of the curvature or of the lapse, which appear in Horava-Lifshitz gravity.

The third term on the right hand side of (46) can be simplified as follows. Rewriting it as

$$\frac{\partial L}{\partial K^i_j} \delta K^i_j = F \delta K = F(K - 3H),$$

(48)

and noting that $K = \nabla_\mu n^\mu$, one can use the integration by parts

$$\int d^4 x \sqrt{-g} F K = - \int d^4 x \sqrt{-g} n^\mu \nabla_\mu F = - \int d^4 x \sqrt{-g} \frac{\dot{F}}{N}.$$  

(49)

This implies that the Lagrangian (46) can be replaced by the equivalent Lagrangian

$$L^{\text{new}} = \bar{L} - 3H F - \frac{\dot{F}}{N} + L_N \delta N + L^{(2)}.$$  

(50)

Let us now consider the quadratic part (47). Because of the background geometry, the coefficient of the second term is necessarily of the form\(^4\)

$$\frac{\partial^2 L}{\partial K^i_j \partial K^k_l} = \hat{A}_K \delta^i_j \delta^k_l + A_K \left( \delta^i_j \delta^k_l + \delta^i_k \delta^j_l \right),$$

(51)\(^4\)

This is equivalent to the definition below, expressed with covariant indices for the extrinsic curvature tensors, which makes the symmetry under exchange of the indices more manifest:

$$\frac{\partial^2 L}{\partial K^i_j \partial K^k_l} = \hat{A}_K \tilde{g}^i_j \tilde{g}^k_l + A_K \left( \tilde{g}^i_k \tilde{g}^j_l + \tilde{g}^i_l \tilde{g}^j_k \right).$$
where we have introduced the (a priori time-dependent) coefficients $\hat{A}_K$ and $A_K$. Similarly, one can write
\[
\frac{\partial^2 L}{\partial R^i_l \partial R^k_l} = \hat{A}_R \delta^i_j \delta_l^k + A_R \left( \delta^i_l \delta^k_j + \delta^{ik} \delta_{jl} \right),
\tag{52}
\]
and
\[
\frac{\partial^2 L}{\partial K^j_i \partial R^l_k} = \hat{C} \delta^i_j \delta^k_l + C \left( \delta^i_l \delta^k_j + \delta^{ik} \delta_{jl} \right).
\tag{53}
\]

The mixed coefficients that appear on the second line are proportional to $\delta^i_j$ and can be written as
\[
\frac{\partial^2 L}{\partial N \partial K^j_i} = B \delta^j_i, \quad \frac{\partial^2 L}{\partial N \partial R^l_k} = B_R \delta^l_k.
\tag{54}
\]

Taking into account the term $\sqrt{-g} = N \sqrt{h}$, it is straightforward to derive the quadratic part of the full Lagrangian $L \equiv \sqrt{-g} L$, which is relevant to study linear perturbations. After some cancellations due to the background equations of motion\(^5\), one finds
\[
\mathcal{L}_2 = \tilde{N} \mathcal{G} \delta_1 R \delta \sqrt{h} + a^3 \left( L_N + \frac{1}{2} \tilde{N} L_{NN} \right) \delta N^2
\tag{55}
\]
\[
+ \tilde{N} a^3 \left[ \mathcal{G} \delta_2 R + \frac{1}{2} \hat{A}_K \delta K^2 + B \delta K \delta N + \hat{C} \delta K \delta R + C \delta K^i_j \delta R^j_i \right]
+ A_K \delta K^i_j \delta K^j_i + A_R \delta R^i_j \delta R^j_i + \frac{1}{2} \hat{A}_R \delta R^2 + \left( \frac{\mathcal{G}}{N} + B_R \right) \delta N \delta R \right] + \ldots,
\]
where, in analogy with the definition (37) of $\mathcal{F}$, we have introduced the coefficient $\mathcal{G}$ defined by
\[
\frac{\partial L}{\partial R^j_i} = \mathcal{G} \delta^j_i.
\tag{56}
\]

We have also denoted as $\delta_1 R$ and $\delta_2 R$, respectively, the first and second order terms of the curvature $R$ expressed in terms of the metric perturbations.

Note that the coefficients that enter here in the quadratic Lagrangian are more general than those introduced explicitly in [1], where the Lagrangian $L$ was considered as a function of $N, K, S = K_{ij} K^{ij}, R$ and $Z \equiv R_{ij} R^{ij}$. It is however straightforward to derive the relation between the present coefficients in terms of our former notation\(^6\). The present definitions have the advantage to automatically include cases with more complicated combinations involving the tensors $K_{ij}$ or $R_{ij}$, such as $K^i_j K^j_k K^k_l$ in the Lagrangian term $L_5$ that appears in Horndeski or G\(^3\) theories.

The above quadratic expression can be further simplified, as shown in [1], by reexpressing $\delta K^i_j \delta R^j_i$ in terms of the other terms, thanks to the identity
\[
\int d^4x \sqrt{-g} \lambda(t) R_{ij} K^{ij} = \int d^4x \sqrt{-g} \left[ \frac{\lambda(t)}{2} R K + \frac{\dot{\lambda}(t)}{2N} R \right].
\tag{57}
\]
This implies the following replacement at quadratic order:
\[
\tilde{N} a^3 C \delta K^i_j \delta R^j_i \rightarrow \frac{\tilde{N} a^3}{2} \left[ \left( \frac{\dot{C}}{N} + HC \right) \left( \delta_2 R + \frac{\delta \sqrt{h}}{a^3} \delta R \right) + C \delta R \delta K + \frac{HC}{N} \delta N \delta R \right].
\tag{58}
\]

\(^5\)If matter is present, one must also include in the quadratic Lagrangian the terms from the expansion of the matter action with respect to the metric perturbations.

\(^6\)The correspondence is given by $\hat{A}_K = 4 H^2 L_{SS} + 4 H L_{SK} + L_{KK}, \ A_K = L_S, \ \hat{B} = 2 H L_{SN} + L_{NN}, \ B_R = L_{NR}, \ \hat{A}_R = L_2, \ \hat{G} = L_R, \ \hat{A}_R = L_{RR}$ and $C = 2 H L_{SR} + L_{KR}$.
Consequently, the quadratic Lagrangian (55) is equivalent to the new one

\[ L_{2}^{\text{new}} = \bar{N} \tilde{G}^{*} \delta_{1} R \delta \sqrt{h} + a^{3} \left( L_{N} + \frac{1}{2} \bar{N} L_{NN} \right) \delta N^{2} \]

\[ + \bar{N} a^{3} \left[ \tilde{G}^{*} \delta_{2} R + \frac{1}{2} \hat{A}_{K} \delta K^{2} + \mathcal{B} \delta K \delta N + \hat{C}^{*} \delta K \delta R \right. \]

\[ + A_{K} \delta K^{i}_{j} \delta K^{j}_{i} + A_{R} \delta R^{i}_{j} \delta R^{j}_{i} + \frac{1}{2} \hat{A}_{R} \delta R^{2} + \left( \frac{G^{*}}{N} + \mathcal{B}_{R}^{*} \right) \delta N \delta R \] \quad \text{...},

with the “renormalized” coefficients:\footnote{For a Lagrangian \( L \) which is a function of \( N, K, S = K_{ij} K^{ij}, R, Z \equiv R_{ij} R^{ij} \) and also of \( Y = R_{ij} K^{ij} \), the relation between the coefficients defined in this paper and the derivatives of \( L \) with respect to the above quantities is unchanged for \( \hat{A}_{K}, \mathcal{A}_{K}, \mathcal{B} \) and \( \mathcal{A}_{R} \) (see footnote 6). The other coefficients, taking into account the dependence on \( Y \), are given by \( \mathcal{B}_{R}^{*} = L_{NR}^{*} \equiv L_{NR} + H L_{NY} - L_{Y}/2, \; \mathcal{G}^{*} = L_{R}^{*} \equiv L_{R} + L_{Y}/2 + 3 H L_{Y}/2, \; \mathcal{A}_{R} = L_{RR} + H^{2} L_{YY} + 2 H L_{YR} \) and \( \hat{C}^{*} = 2 H L_{SR} + L_{KR} + H L_{KY} + 2 H^{2} L_{SY} + L_{Y}/2 \) with \( \bar{N} = 1 \).}

\[ \tilde{G}^{*} = G + \frac{\dot{G}}{2 \bar{N}} + H C, \]

\[ C^{*} = \hat{C} + \frac{1}{2} C, \]

\[ B_{R}^{*} = B_{R} - \frac{\dot{C}}{2 N^{2}}. \]

### 3.2.1 Tensor modes

Let us first investigate the tensor modes in the general quadratic Lagrangian (59). At linear order, tensor modes correspond to the perturbations of the spatial metric

\[ h_{ij} = a^{2}(t) \left( \delta_{ij} + \gamma_{ij} \right), \]

with \( \gamma_{ij} \) traceless and divergence-free, \( \gamma_{ii} = 0 = \partial_{i} \gamma_{ij} \). Using

\[ \delta K^{i}_{j} = \frac{1}{2 N} \gamma_{ij} \]

and

\[ \delta_{2} R = \frac{1}{a^{2}} \left( \gamma^{ij} \partial^{2} \gamma_{ij} + \frac{3}{4} \partial_{k} \gamma_{ij} \partial^{k} \gamma^{ij} - \frac{1}{2} \partial_{k} \gamma_{ij} \partial^{j} \gamma_{ik} \right), \]

one finally obtains

\[ S_{\gamma}^{(2)} = \int dx^{3} dt a^{3} \delta N \delta K \left[ \frac{A_{K}}{4} \gamma^{2}_{ij} - \frac{G^{*}}{4 a^{2}} \left( \partial_{k} \gamma_{ij} \right)^{2} \right], \]

where here and below we set \( \bar{N} = 1 \). We recover the standard GR result when \( A_{K} = G^{*} = M_{P}^{2}/2 \).

By comparison, this suggests to define the effective Planck mass by

\[ M^{2} \equiv 2 A_{K}, \]

and write the action as

\[ S_{\gamma}^{(2)} = \int dx^{3} dt a^{3} M^{2} \left[ \frac{\gamma^{2}_{ij}}{8} - \frac{G^{*}}{4 a^{2}} \left( \partial_{k} \gamma_{ij} \right)^{2} \right]. \]

The square of the graviton propagation speed is given by

\[ c_{T}^{2} \equiv 1 + \alpha_{T} = \frac{G^{*}}{A_{K}}. \]
where $\alpha_T$ represents the deviation with respect to the GR result.

The graviton sector is thus characterized by the two coefficients $A_K$ and $G^*$, or equivalently by $M$ and $\alpha_T$. In practice, it is the time variation which can distinguish the effective Planck mass defined here with respect to the standard Planck mass, so it is convenient, following [26], to introduce the dimensionless parameter

$$
\alpha_M \equiv \frac{1}{H} \frac{d}{dt} \ln M^2.
$$

With these definitions, the evolution equation for tensor modes is given by

$$
\dddot{\gamma}_{ij} + H(3 + \alpha_M) \dot{\gamma}_{ij} - (1 + \alpha_T) \nabla^2 a^2 \gamma_{ij} = \frac{2}{M^2} \left( T_{ij} - \frac{1}{3} T \delta_{ij} \right)^{TT},
$$

where $(T_{ij} - T \delta_{ij}/3)^{TT}$ is the transverse-traceless projection of the anisotropic matter stress tensor.

### 3.2.2 Scalar modes

Without loss of generality, the scalar modes can be described by the perturbations

$$
N = 1 + \delta N, \quad N^i = \delta^i j \partial_j \psi, \quad h_{ij} = a^2(t) e^{2 \zeta} \delta_{ij}.
$$

Substituting

$$
\delta \sqrt{h} = 3 a^3 \zeta, \quad \delta K^i_j = \left( \dot{\zeta} - H \delta N \right) \delta^i_j - \frac{1}{a^2} \delta^{ik} \partial_k \partial_j \psi,
$$

and

$$
\delta_1 R_{ij} = -\delta_{ij} \partial^2 \zeta - \partial_i \partial_j \zeta, \quad \delta_2 R = -\frac{2}{a^2} \left[ (\partial \zeta)^2 - 4 \zeta \partial^2 \zeta \right],
$$

into (59), one obtains a lengthy Lagrangian in terms of $\delta N, \psi$ and $\zeta$. Since the Lagrangian does not depend on the time derivatives of the lapse and of the shift, the variation of the Lagrangian with respect to $\delta N$ and $\psi$ yields constraints, corresponding to the familiar Hamiltonian constraint and (the scalar part of) the momentum constraint.

In the following, we will assume for simplicity the conditions

$$
\hat{A}_K + 2 A_K = 0, \quad C^* = 0, \quad 4 \hat{A}_R + 3 A_R = 0,
$$

which ensure that there are at most two spatial derivatives in the quadratic Lagrangian written in terms of $\zeta$ only. This includes the Horndeski theories as well as their $G^3$ extensions.

Provided the conditions (73) are satisfied, one finds that the momentum constraint reduces to

$$
\delta N = \frac{4 \hat{A}_K}{B + 4 H A_K} \dot{\zeta} = \frac{\dot{\zeta}}{H (1 + \alpha_B)},
$$

where we have introduced the dimensionless quantity

$$
\alpha_B \equiv \frac{B}{4 H A_K},
$$

which expresses the deviation from the standard expression $\delta N = \dot{\zeta}/H$. When $\alpha_B \neq 0$, part of the time kinetic term of scalar fluctuations comes from the term $\delta K \delta N$ in action (59), i.e. from kinetic mixing between gravitational and scalar degrees of freedom [2, 3, 4]. This phenomenon has been called kinetic braiding in [36, 37].

---

*Although we use the same symbol, our variable $\alpha_B$ differs from that introduced in [26] by a factor $-2$. This simplifies the subsequent equations.*
The quadratic action for $\zeta$ is then given by
\[ S^{(2)} = \frac{1}{2} \int dx^3 dt a^3 \left[ L_{\zeta\zeta} \partial_t \zeta^2 + L_{\partial_t \zeta \partial \zeta} \frac{(\partial_t \zeta)^2}{a^2} + \frac{M^2}{4} \delta_{ij} - \frac{M^2}{4} (1 + \alpha_T) \frac{(\partial_k \gamma_{ij})^2}{a^2} \right], \tag{76} \]
where
\[ L_{\zeta\zeta} \equiv M^2 \frac{\alpha}{(1 + \alpha_B)^2}, \quad \alpha \equiv \alpha_K + 6 \alpha_B^2, \tag{77} \]
\[ L_{\partial_t \zeta \partial \zeta} \equiv 2M^2 \left\{ 1 + \alpha_T - \frac{1 + \alpha_H}{1 + \alpha_B} \left( 1 + \alpha_M - \frac{H}{H^2} \right) - \frac{1}{H \frac{dt}{dt}} \left( 1 + \alpha_H \right) \right\}, \tag{78} \]
where we have introduced the dimensionless time-dependent functions
\[ \alpha_K \equiv \frac{2L_N + L_{NN}}{2H^2 A_K}, \quad \alpha_H \equiv \frac{G^* + B^*}{A_K} - 1. \tag{79} \]
Note that the coefficient of the kinetic term reduces to $L_{\zeta\zeta} = M^2 \alpha_K$ when $\alpha_B = 0$. In this case, the kinetic coefficient for $\zeta$ is directly related to the coefficient of the term $\delta N^2$ in the quadratic Lagrangian (59), term which represents the kinetic energy of the scalar field fluctuations. The parameter $\alpha_H$ is different from zero for theories that deviate from Horndeski theories [1, 19, 20]. In particular, this includes theories that can be related to Horndeski theories via disformal transformations, as shown in [20]. Indeed, starting from a Horndeski theory for a metric $g_{\mu\nu}$ via a disformal transformation that depends on $X$, the Lagrangian expressed in terms of $g_{\mu\nu}$ differs from the standard Horndeski Lagrangian, which implies $\alpha_H \neq 0$.

Classical and quantum stability (absence of ghosts) requires the kinetic coefficient to be positive,
\[ L_{\zeta\zeta} > 0 \implies \alpha = \alpha_K + 6 \alpha_B^2 > 0. \tag{80} \]
The sound speed (squared) of fluctuations can be simply computed by taking the ratio
\[ c_s^2 \equiv -\frac{L_{\partial \zeta \partial \zeta}}{L_{\zeta \zeta}}. \tag{81} \]
When adding matter to the dark energy Lagrangian, the kinetic and spatial gradient terms of the scalar fluctuations acquire new contributions that modify the expression for the sound speed [19, 20]. The final expression for the sound speed, when matter is present, reads
\[ c_s^2 = -\frac{\alpha}{(1 + \alpha_B)^2} \left\{ 1 + \alpha_T - \frac{1 + \alpha_H}{1 + \alpha_B} \left( 1 + \alpha_M - \frac{H}{H^2} \right) - \frac{1}{H \frac{dt}{dt}} \left( 1 + \alpha_H \right) \right\} - \frac{(1 + \alpha_H)^2 \rho_m + \rho_m}{\alpha \frac{M^2}{H^2}}. \tag{82} \]
In the simple case of $k$-essence field with a Lagrangian $P(\phi, X)$, where all $\alpha_i$ coefficients vanish except $\alpha_K = (2XP_X + X^2P_{XX})/(M^2H^2)$, the above formula yields $c_s^2 = -2\dot{H}/(\alpha_K H^2) - (\rho_m + p_m)/(\alpha_K M^2 H^2)$ and one recovers $c_s^2 = P_X/(P_X + 2XP_{XX})$ after using the Friedmann equation $\dot{H} = -(2XP_X + \rho_m + p_m)/(2M^2)$.

### 3.3 Link with the building blocks of dark energy

In the previous subsection, we have focussed our attention on Lagrangians that satisfy the conditions (73) in order to get propagation equations with no more than two (space) derivatives. At quadratic order, the most general action of the form (59) that satisfies these conditions can be written in the form
\[ S^{(2)} = \int dx^3 dt a^3 \frac{M^2}{2} \left[ \delta_{ij} \delta K_{ij} - \delta K^2 + (1 + \alpha_T) \left( R \frac{\sqrt{\dot{h}}}{a^3} + \delta_2 R \right) + \alpha_K H^2 \delta N^2 + 4 \alpha_B H \delta K \delta N + (1 + \alpha_H) R \delta N \right], \tag{83} \]
Table 1: In the first row, the parameters $\alpha_i$ introduced in eqs. (67), (68), (75) and (79), i.e. the Lagrangian coefficients of eq. (83). These parameters are written in terms of the Lagrangian coefficients of eq. (59), defined in eqs. (51)–(54) (second row), of the coefficients introduced in [1], where the derivative of the Lagrangian $L$ with respect to $N$, $K$, $S = K_{ij}K^{ij}$, $R$, $Z \equiv R_{ij}R^{ij}$ and $Y \equiv R_{ij}K^{ij}$ (third row) and, finally, of the EFT Lagrangian, action (84) (fourth row). All these quantities are understood to be evaluated on the background, with $\bar{N} = 1$.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Eq. (83)} & M^2 & \alpha_M & \alpha_K & \alpha_B & \alpha_T & \alpha_H \\
\hline
\text{Eq. (59)} & 2A_K & \frac{1}{H} \frac{d}{dt} \ln A_K & \frac{2L_N + L_{NN}}{2H^2 A_K} & \frac{B}{4HA_K} & \frac{G^*}{A_K} - 1 & \frac{G^* + B^*}{A_K} - 1 \\
\hline
\text{Eq. (12) of [1]} & 2L_S & \frac{1}{H} \frac{d}{dt} \ln L_S & \frac{2L_N + L_{NN}}{2H^2 L_S} & \frac{2HL_{SN} + L_{KN}}{4HL_L} & \frac{L_{R_L}}{L_S} - 1 & \frac{L_{R_R} + L_{NR}}{L_S} - 1 \\
\hline
\text{Eq. (30) Eq. (84)} & M^2 f + 2m_3^2 & \frac{M^2 f + 2(m_3^2)^2}{M^2 H} & \frac{2c + 4M_3^2}{M^2 H^2} & \frac{M^2 f - m_3^2}{2M^2 H} & -\frac{2m_3^2}{M^2} & \frac{2(m_3^2 - m_3^2)}{M^2} \\
\hline
\end{array}
\]

where, for convenience, we summarize in Table 1 the definitions of the parameters $\alpha_i$ introduced in the previous subsection, in terms of the original coefficients defined in Sec. 3.2 (second row) and those introduced explicitly in Ref. [1] (third row).

The action leading to the quadratic Lagrangian (83) can also be written in the standard EFT form, with an explicit dependence on the four-dimensional scalar curvature, $g^{00}$ and several quadratic operators. This action reads [1]

\[
S = \int d^4x \sqrt{-g} \left[ \frac{M^2}{2} f(t)(4)R - \Lambda(t) - c(t)g^{00} + \frac{M^4(t)}{2}(\delta g^{00})^2 - \frac{m_3(t)}{2} \delta K \delta g^{00} \right. \\
- \left. m_3^2(t) \left( \delta K^2 - \delta K_{\mu}^\nu \delta K^{\mu\nu} \right) + \frac{\bar{m}_3^2(t)}{2} R \delta g^{00} \right].
\]

(84)

It leads to the background equations of motion [1]

\[
c + \Lambda = 3M^2 f(4H^2 + \dot{4}H) - \rho_m, \\
\Lambda - c = M^2(2\dot{4}H + 3fH^2 + 2\ddot{4}H + \dddot{4}) + \rho_m
\]

and to the quadratic action for the linear perturbations (83), where the relation between the coefficients $\alpha_i$ and the seven parameters appearing in (84) is given in Table 1. The two background equations of motion (85) and (86) imply that only five of the EFT parameters are independent, thus setting the minimal number of functions parametrizing deviations from General Relativity [1].

### 3.4 Disformal transformations and dependence on $\bar{N}$

In our discussion, we have assumed that the initial Lagrangian depends on $N$, but not on its time derivative $\dot{N}$. Allowing a dependence on $\bar{N}$ leads in general to an additional propagating degree of freedom. However, this is not always the case, as illustrated by considering disformal transformations of the metric, originally introduced in [38], of the form

\[
g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(\phi, X) g_{\mu\nu} + \Gamma(\phi, X) \partial_{\mu}\phi \partial_{\nu}\phi.
\]

(87)

As shown in [39], Horndeski theories are invariant under a restricted class of disformal transformations where $\Omega$ and $\Gamma$ depend on $\phi$ only, not on $X$. In [20], we showed explicitly that one could use disformal transformations with an $X$-dependent function $\Gamma$ to relate subsets of $G^3$ theories to Horndeski
theories. A similar result for a disformal transformation of the Einstein-Hilbert Lagrangian was also discussed in [40].

In unitary gauge, $\Omega$ and $\Gamma$ become functions of the time variable $t$ and on the lapse function $N$. Moreover, by choosing time to coincide with $\phi$, i.e. $\partial_\mu \phi = \delta^0_\mu$ the disformal transformation (87) corresponds, in the ADM language, to the transformations [20]

$$\tilde{N}^i = N^i, \quad \tilde{h}_{ij} = \Omega^2(t, N) h_{ij}, \quad \tilde{N}^2 = \Omega^2(t, N) N^2 - \Gamma(t, N).$$

Moreover, the relations between the old and new curvature tensors are given by

$$\tilde{R} = \Omega^{-2} \left[ R - 4D^2 \ln \Omega - 2 \partial_i (\ln \Omega) \partial^i (\ln \Omega) \right],$$

and

$$\tilde{K}^j_i = \frac{N}{\Omega} \left[ K^j_i - N g^{0\mu} \partial_\mu \Omega \delta^i_j \right].$$

The last relation can be expanded into

$$\tilde{K}^j_i = \frac{N}{\Omega} \left[ K^j_i + \frac{1}{N\Omega} \left( \Omega_t + \Omega_N (\tilde{N} - N \partial_i N) \right) \delta^j_i \right].$$

Consequently, a Lagrangian that depends initially on tilded quantities, will finally depend on $\delta \tilde{N}$ when reexpressed in terms of untilded quantities. The quadratic Lagrangian will now depend on $\delta \tilde{N}$, in addition to all the terms discussed previously. However, according to (91) and (71), one sees that $\delta \tilde{N}$ will always appear associated with $\tilde{\zeta}$ in the combination

$$\dot{\zeta} + \frac{\Omega_N}{N\Omega} \delta \tilde{N},$$

which implies that the matrix of the kinetic coefficients is degenerate. Thus, one can introduce a new degree of freedom

$$\zeta_{\text{new}} = \zeta + \frac{\Omega_N}{N\Omega} \delta N,$$

which absorbs all time derivatives of $\delta N$. Contrarily to what could have expected, the explicit dependence on $\tilde{N}$ does not lead, in this particular case, to an extra degree of freedom.

4 Evolution of the cosmological perturbations

In this section we follow [1] and derive the evolution equations for linear scalar perturbations described by the action (83), together with some matter field minimally coupled to the metric $g_{\mu\nu}$. We first restore the general covariance of the action and write it in a generic coordinate system. In order to do so, we perform the time diffeomorphism [41, 2, 3]

$$t \rightarrow t + \pi(t, \vec{x}),$$

where $\pi$ describes the fluctuations of the scalar degree of freedom. Under this time diffeomorphism, any function of time $f$ changes up to second order as

$$f \rightarrow f + \dot{f} \pi + \frac{1}{2} \ddot{f} \pi^2 + O(\pi^3),$$

while the metric component $g^{00} = -1/N^2$ exactly transforms as

$$g^{00} \rightarrow g^{00} + 2g^{0\mu} \partial_\mu \pi + g^{\mu\nu} \partial_\mu \pi \partial_\nu \pi.$$
For the other perturbed geometric quantities, we only need their change at linear order in $\pi$, i.e. \[ \delta K_{ij} \rightarrow \delta K_{ij} - \dot{H} \pi h_{ij} - \partial_i \partial_j \pi + O(\pi^2), \] \[ \delta K \rightarrow \delta K - 3 \dot{H} \pi - \frac{1}{a^2} \partial^2 \pi + O(\pi^2), \] \[ R_{ij} \rightarrow R_{ij} + H(\partial_i \partial_j \pi + \delta_{ij} \partial^2 \pi) + O(\pi^2), \] \[ R \rightarrow R + \frac{4}{a^2} H \partial^2 \pi + O(\pi^2). \] (97) (98) (99) (100)

We stress that in the above expressions $K_{ij}$ and $R_{ij}$ respectively denote the extrinsic and intrinsic curvature on hypersurfaces of constant time, even when we are not in unitary gauge. Therefore they are not the same geometrical quantities before and after the change of time.

We can then expand the covariant action up to quadratic order, considering a linearly perturbed FLRW metric. Varying the action with respect to the four scalar perturbations in the metric and the scalar fluctuation $\pi$ we obtain five scalar equations; see Ref. [1] for details on their derivations. We turn to a discussion of these equations restricting to Newtonian gauge.

### 4.1 Perturbation equations in Newtonian gauge

We assume a perturbed FLRW metric in Newtonian gauge with only scalar perturbations, i.e.,

\[ ds^2 = -(1 + 2\Phi) dt^2 + a^2(t)(1 - 2\Psi) \delta_{ij} dx^i dx^j. \] (101)

Moreover, in this gauge we decompose the total matter stress-energy tensor at linear order as

\[ T^0_0 = -(\rho_m + \delta \rho_m), \] (102)
\[ T^0_i = \partial_i q_m \equiv (\rho_m + p_m) \partial_i v_m = -a^2 T^i_0, \] (103)
\[ T^i_j = (p_m + \delta p_m) \delta^i_j + \left( \partial^i \partial^j - \frac{1}{3} \delta^i_j \partial^2 \right) \sigma_m, \] (104)

where $\delta \rho_m$ and $\delta p_m$ are the energy density and pressure perturbations, $q_m$ and $v_m$ are respectively the 3-momentum and the 3-velocity potentials; $\sigma_m$ is the anisotropic stress potential.

The Hamiltonian constraint ((00) component of the Einstein equation) is

\[ 6(1 + \alpha_B) \dot{\Psi} + (6 - \alpha_K + 12 \alpha_B) H^2 \Phi + 2(1 + \alpha_H) \frac{k^2}{a^2} \Psi \\
+ (\alpha_K - 6 \alpha_B) \ H^2 \dot{\pi} + 6 \left[ (1 + \alpha_B) \dot{H} + \frac{\rho_m + p_m}{2M^2} + \frac{1}{3} \frac{k^2}{a^2} (\alpha_H - \alpha_B) \right] H \pi = -\frac{\delta \rho_m}{M^2}, \] (105)

while the momentum constraint ((0$i$) components of the Einstein equation) reads

\[ 2\dot{\Psi} + 2(1 + \alpha_B) H \Phi - 2 H \alpha_B \dot{\pi} + \left( 2 \dot{H} + \frac{\rho_m + p_m}{M^2} \right) \pi = -\frac{(\rho_m + p_m)v_m}{M^2}. \] (106)

The traceless part of the $ij$ components of the Einstein equation gives

\[ (1 + \alpha_H) \Phi - (1 + \alpha_T) \Psi + (\alpha_M - \alpha_T) H \pi - \alpha_H \dot{\pi} = -\frac{\sigma_m}{M^2}. \] (107)
while the trace of the same components gives, using the equation above,

\[ 2\dot{\Psi} + 2(3 + \alpha_M)H\dot{\Psi} + 2(1 + \alpha_B)H\dot{\Phi} \]

\[ + 2 \left[ \dot{\Pi} - \frac{p_m + p_m}{2M^2} + (\alpha_B H') + (3 + \alpha_M)(1 + \alpha_B)H^2 \right] \Phi \]

\[ - 2H\alpha_B \dot{\pi} + 2 \left[ \dot{\Pi} + \frac{p_m + p_m}{2M^2} - (\alpha_B H') - (3 + \alpha_M)\alpha_B H^2 \right] \pi \]

\[ + 2 \left[ (3 + \alpha_M)H\dot{H} + \frac{\dot{p}_m}{2M^2} + \ddot{H} \right] \pi = \frac{1}{M^2} \left( \delta p_m - \frac{2k^2}{3a^2}\sigma_m \right) . \tag{108} \]

The evolution equation for \( \pi \) reads

\[ H^2\alpha_K \pi + \left\{ \left[ H^2(3 + \alpha_M) + \dot{H} \right] \alpha_K + (H\alpha_K)' \right\} \dot{H} \pi \]

\[ + 6 \left\{ \left[ \dot{H} + \frac{p_m + p_m}{2M^2} \right] \dot{H} + H\alpha_B \left[ H^2(3 + \alpha_M) + \dot{H} \right] + H(\dot{H}\alpha_B)' \right\} \pi \]

\[ - 2\frac{k^2}{a^2} \left\{ \dot{H} + \frac{p_m + p_m}{2M^2} + H^2 [1 + \alpha_B(1 + \alpha_M) + \alpha_T - (1 + \alpha_H)(1 + \alpha_M)] + (H(\alpha_B - \alpha_H))' \right\} \pi \]

\[ + 6H\alpha_B \dot{\Psi} + H^2(6\alpha_B - \alpha_K)\dot{\Phi} + 6 \left[ \dot{H} + \frac{p_m + p_m}{2M^2} + H^2 \alpha_B(3 + \alpha_M) + (\alpha_B H') \right] \dot{\Psi} \]

\[ + \left[ 6 \left( \dot{H} + \frac{p_m + p_m}{2M^2} \right) + H^2(6\alpha_B - \alpha_K)(3 + \alpha_M) + 2(9\alpha_B - \alpha_K)\dot{H} + H(6\dot{\alpha}_B - \dot{\alpha}_K) \right] H \Phi \]

\[ + 2\frac{k^2}{a^2} \left\{ \alpha_H \dot{\Psi} + [H(\alpha_M + \alpha_H(1 + \alpha_M) - \alpha_T) - \dot{\alpha}_H] \Psi + (\alpha_H - \alpha_B)H \Phi \right\} = 0 . \tag{109} \]

These equations have been previously derived in [1] in terms of the effective field theory parameters. Restricting to Horndeski theories (\( \alpha_H = 0 \)), they have been also obtained in [7] and later reproduced in [26], where the notation used here was introduced. In Appendix A we discuss the long wavelength behaviour of these equations for adiabatic initial conditions; in Appendix B we write these equations in synchronous gauge and conformal time, which is the coordinate system usually employed in CMB codes.

### 4.2 Fluid description

It is sometimes convenient to describe the dark energy, both in the background and perturbative equations, as an effective fluid. In order to do so, we define the background energy density and pressure for dark energy, respectively, as

\[ \rho_D \equiv 3M^2H^2 - \rho_m , \quad p_D \equiv -M^2(2\dot{H} + 3H^2) - p_m . \tag{110} \]

These are simply derived quantities that can be computed once the evolution of the expansion history, the matter content and the effective Planck mass \( M \) are known. With these definitions, and using the conservation of the background matter stress-energy tensor,

\[ \dot{\rho}_m + 3H(\rho_m + p_m) = 0 , \tag{111} \]

the conservation of the background dark energy stress-energy tensor reads

\[ \dot{\rho}_D = -3H(\rho_D + p_D) + 3\alpha_M M^2H^3 = 3\dot{H}(\rho_m + p_m) + 6M^2H(\dot{H} + \alpha_M H^2) . \tag{112} \]

Another useful relation that one can use to express \( \dot{\rho}_D \) in terms of matter and geometry is

\[ \dot{\rho}_D = -\dot{\rho}_m - M^2 \left[ 2\dot{H} + 2H\dot{H}(3 + \alpha_M) + 3\alpha_M H^3 \right] , \tag{113} \]
which can be derived from the equations above.

Equations (138)–(141) can be then rewritten in the usual form,

\[ \frac{k^2}{a^2} \Psi + 3H (\dot{\Psi} + H \Phi) = -\frac{1}{2M^2} \sum_l \delta \rho_l, \]

(114)

\[ \dot{\Psi} + H \Phi = -\frac{1}{2M^2} \sum_l q_l, \]

(115)

\[ \Psi - \Phi = \frac{1}{M^2} \sum_l \sigma_l, \]

(116)

\[ \dot{\Psi} + H \dot{\Phi} + 2H \Phi + 3H (\dot{\Psi} + H \Phi) = \frac{1}{2M^2} \sum_l \left( \delta \rho_l - \frac{2k^2}{3a^2} \sigma_l \right), \]

(117)

where the sum is over the matter and the dark energy components. These equations implicitly define the quantities \( \delta \rho, q, \delta \rho_D \) and \( \sigma_D \) as the energy density perturbation, momentum, pressure perturbation and anisotropic stress of the dark energy fluid. An explicit definition is given in Newtonian gauge in Appendix A and in synchronous gauge in Appendix B.

With these definitions, one can verify that the evolution equation for \( \pi \), eq. (109), is equivalent to a conservation equation of the dark energy fluid quantities,

\[ \delta \rho_D + 3H(\delta \rho_D + \delta p_D) - 3(\rho_D + p_D) \dot{\Psi} - \frac{k^2}{a^2} q_D = \alpha_M H \sum_l \delta \rho_l. \]

(118)

The Euler equation,

\[ \dot{q}_D + 3H q_D + (\rho_D + p_D) \dot{\Phi} + \delta p_D - \frac{2k^2}{3a^2} \sigma_D = \alpha_M H \sum_l q_l, \]

(119)

is identically satisfied by the definitions of \( q, \delta p_D \) and \( \sigma_D \). Conservation of matter in the Jordan frame implies a continuity and Euler equations for matter with vanishing right-hand side.

To close the system, one needs to provide an equation of state for dark energy or, at least, a relation between \( \delta p_D \) and \( \sigma_D \) in terms of \( \delta \rho_D, q_D \) and the other matter variables. In order to do so in the simpler case where \( \alpha_H = 0 \), we solve eqs. (105)–(107) for \( \Psi, \dot{\Psi} \) and \( \pi \) and then we plug these solutions in eqs. (114) and (115) to express \( \pi \) and \( \Phi \) in terms of \( \delta \rho_m, q_m, \sigma_m, \delta p_D \) and \( q_D, \Phi \) is obtained from the first derivative of (107). To obtain \( \dot{\Psi} \) and \( \pi \) we use eqs. (108) and (109).

Combining all these solutions we can finally express \( \sigma_D \) and \( \delta p_D \) in terms of the other fluid variables. We obtain

\[ \delta p_D = \frac{\gamma_1 \gamma_2 + \gamma_3 \alpha_B^2 \tilde{k}^2}{\gamma_1 + \alpha_B^2 \tilde{k}^2} (\delta \rho_D - 3H q_D) + \frac{\gamma_1 \gamma_4 + \gamma_5 \alpha_B^2 \tilde{k}^2}{\gamma_1 + \alpha_B^2 \tilde{k}^2} H q_D + \gamma_7 (\delta \rho_m - 3H q_m) + \frac{\gamma_1 \gamma_6 + 3 \gamma_7 \alpha_B^2 \tilde{k}^2}{\gamma_1 + \alpha_B^2 \tilde{k}^2} H q_m - \frac{6 \alpha_B^2}{\alpha} \delta p_m, \]

(120)

\[ \sigma_D = \frac{a^2}{2k^2} \left\{ \frac{\gamma_1 \alpha_T + \gamma_8 \alpha_B^2 \tilde{k}^2}{\gamma_1 + \alpha_B^2 \tilde{k}^2} (\delta \rho_D - 3H q_D) + \frac{\gamma_9 \tilde{k}^2}{\gamma_1 + \alpha_B^2 \tilde{k}^2} H q_D + \alpha_T (\delta \rho_m - 3H q_m) + \frac{\gamma_10 \tilde{k}^2}{\gamma_1 + \alpha_B^2 \tilde{k}^2} H q_m \right\}, \]

(121)

where we use the notation \( \tilde{k} \equiv k/(aH) \) and we have defined dimensionless coefficients \( \gamma_a \), whose expressions are explicitly given in Appendix C. These relations for \( \delta p_D \) and \( \sigma_D \) are derived, to our
knowledge, for the first time and represent the most general description of dark energy in the context of Horndeski theories. In particular, eq. (120) extends the equation of state derived in [43] for kinetic braiding.

One can check that for adiabatic initial conditions, i.e.

\[ \pi \approx - (\dot{\Psi} + H\Phi)/H, \quad \delta \rho_{m} \approx \dot{\rho}_{m}\pi, \quad \delta p_{m} \approx \dot{p}_{m}\pi, \quad \nu_{m} \approx -\pi, \]

(122)

where the symbol \( \approx \) denotes equality in the long wavelength limit \( \dot{k} \ll 1 \), the dark energy equation of state satisfies

\[ \delta \rho_{D} \approx -3H(\rho_{D} + p_{D})\pi, \quad \delta p_{D} \approx -\left(\dot{\rho}_{m} + 2M^{2}(\dot{H} + 3H^{2})\right)\pi, \]

\[ q_{D} \approx -(\rho_{D} + p_{D})\pi, \quad \sigma_{D} \approx -\alpha_{T}M^{2}\Psi - (\alpha_{T} - \alpha_{M})M^{2}H\pi, \]

(123)

which is what expected from the equations of motion in Sec. 4.1, see discussion in Appendix A.

Going back to arbitrary scales, let us discuss two illustrative examples.

- \( \alpha_{B} = 0 \): there is no braiding and the kinetic term of scalar fluctuations depends on \( \alpha_{K} \) only, \( \alpha = \alpha_{K} \). In this case eqs. (120) and (121) reduce to

\[ \delta \rho_{D} = c_{s}^{2}(\delta \rho_{D} - 3Hq_{D}) - \left(\frac{\dot{\rho}_{D} + \alpha_{M}3H^{3}M^{2}}{\rho_{D} + p_{D}} + H(\alpha_{T} - \alpha_{M}) \left(1 - \frac{2M^{2}k^{2}}{3(\rho_{D} + p_{D})a^{2}}\right)\right)q_{D} \]

\[ + \frac{\alpha_{T}}{3} \delta \rho_{tot} - \frac{\alpha_{K}}{6}(\alpha_{T} - \alpha_{M})H\rho_{m}, \]

(124)

\[ \sigma_{D} = -\alpha_{T}M^{2}\Psi + H(\alpha_{T} - \alpha_{M}) \left(\frac{M^{2}}{\rho_{D} + p_{D}}q_{D}\right), \]

(125)

where we have used eqs. (114), (115) and \( c_{s}^{2} = c_{s}^{2} - 2(\alpha_{T} - \alpha_{M})/\alpha_{K} = (\rho_{D} + p_{D})/(\alpha_{K}H^{2}M^{2}), \)

\( \delta \rho_{tot} \equiv \delta \rho_{m} + \delta \rho_{D} \). For \( \alpha_{T} = 0 = \alpha_{M} \) we recover the standard \( k \)-essence pressure perturbation [42] and no anisotropic stress. For \( \alpha_{T} \neq 0 \) or \( \alpha_{T} - \alpha_{M} \neq 0 \), the dark energy anisotropic stress is nonzero and simply given in terms of the total curvature \( \Psi \) and the dark energy momentum \( q_{D} \). Note that the term containing \( k^{2} \) in the pressure perturbation \( \delta \rho_{D} \) cancels from the combination \( \delta \rho_{D} - (2k^{2}/3a^{2})\sigma_{D} \), which appears as a source in the Euler equation and in the evolution equation for \( \Psi \).

- \( \alpha_{B}^{2} \gg \alpha_{K} \): braiding dominates the time kinetic term, \( \alpha \approx 6\alpha_{B}^{2} \). However, one needs \( \alpha_{B} \lesssim 1 \) to avoid gradient instabilities [4]. In this case, from the definition of \( \gamma_{1} \), eq. (180), we have \( \gamma_{1} \approx -3\alpha_{B}^{2}H/2H^{2} \) so that, if we concentrate on sub-horizon scales, \( k \gg aH \), eqs. (120) and (121) reduce to

\[ \delta \rho_{D} = \left(2c_{s}^{2} + \frac{\alpha_{T}}{3} + \frac{\xi}{3} - \frac{2\dot{H} + \dot{H}/H - \xi\dot{H}/k^{2}}{\alpha_{B}}\right)(\delta \rho_{D} - 3Hq_{D}) - (1 + \xi)Hq_{D} \]

\[ - 3\frac{H\alpha_{B}^{2}\xi Hq_{m} - (1 + \xi)\delta \rho_{m}}{k^{2}} - \delta p_{m}, \]

(126)

\[ \sigma_{D} = \xi\frac{\alpha_{B}^{2}}{k^{2}} \left[\frac{1}{2}(\delta \rho_{D} - 3Hq_{D}) + \frac{3}{2}H\alpha_{B}q_{tot}\right], \]

(127)

where \( \xi \equiv (\alpha_{T} - \alpha_{M})/\alpha_{B} \) and \( q_{tot} \equiv q_{m} + q_{D} \). As expected in this case [44], the behavior of dark energy is very different from that of a perfect fluid. In particular, for \( k^{2} \lesssim \gamma_{2}/\gamma_{3} \) the relation between pressure and density perturbations is scale dependent. For \( \xi \neq 0 \), the anisotropic stress is nonzero and has a scale dependence that differs from the \( \alpha_{B} = 0 \) case discussed above.
4.3 Interface with the observations

In Sec. 4.1 we have described the full set of evolution equations including the standard matter species directly using the scalar fluctuation \( \pi \). These equations can be solved in a modified Boltzmann code; for instance, they have been recently implemented in a code in [45, 46]. Alternatively, in Sec. 4.2 we have rewritten these equations in terms of dark energy fluid quantities and we have provided the full equations of state for Horndeski theories (\( \alpha_H = 0 \)), eqs. (120) and (121). In this approach, the equations of state fully encode the description of dark energy.

To discuss more easily the relation with late time observations we can use the Einstein equations in the fluid form, eqs. (114)–(117), and rewrite these two equations as an evolution equation for the gravitational potential \( \Psi \) and a relation between \( \Psi \) and \( \Phi \). For simplicity, we restrict again our discussion to the case \( \alpha_H = 0 \). To do that we can first combine eqs. (114)–(116) to solve for \( \Phi \), \( \delta p_D \), and \( q_D \) in terms of \( \Psi \), \( \dot{\Psi} \), \( \sigma_D \) and the matter field. Moreover, we can use eq. (117) to express \( \dot{\delta p_D} \) as a function of the other quantities. Using these relations, it is straightforward to show that eqs. (120) and (121) are equivalent to a dynamical equation for the gravitational potential \( \Psi \), sourced by the matter fields.\(^9\)

\[
\dot{\Psi} + \frac{\beta_1 \beta_2 + \beta_3 \alpha_B^2 \dot{k}^2}{\beta_1 + \alpha_B^2 k^2} H \Psi + \frac{\beta_1 \beta_4 + \beta_1 \beta_5 \dot{k}^2 + c_s^2 \alpha_B^2 k^4}{\beta_1 + \alpha_B^2 k^2} H^2 \Psi = - \frac{1}{2M^2} \left[ \frac{\beta_1 \beta_6 + \beta_7 \alpha_B^2 \dot{k}^2}{\beta_1 + \alpha_B^2 k^2} \delta \rho_m \right]
\]

\[
+ \frac{\beta_1 \beta_8 + \beta_9 \alpha_B^2 \dot{k}^2}{\beta_1 + \alpha_B^2 k^2} H q_m + \frac{\beta_1 \beta_{10} + \beta_1 \beta_{11} \dot{k}^2 + \beta_7 \alpha_B^2 H^4}{\beta_1 + \alpha_B^2 k^2} H^2 \sigma_m - \frac{\alpha K}{\alpha} \delta \rho_m - 2H \dot{\sigma}_m
\]

(128)

where the dimensionless parameters \( \beta_a \) are explicitly given in Appendix C, and a relation between \( \Phi \) and \( \Psi \), involving \( \dot{\Psi} \) and the matter fields,

\[
\frac{\alpha^2 \dot{k}^2}{H^2 M^2} \left[ \Phi - \Psi \left( 1 + \alpha T \right) \frac{2 \gamma_9}{\alpha \alpha_B} + \frac{\sigma_m}{M^2} \right] + \beta_1 \left[ \Phi - \Psi \left( 1 + \alpha T \right) \frac{\gamma_1}{\beta_1} + \frac{\sigma_m}{M^2} \right] = \frac{\gamma_9}{H^2 M^2} \left[ \frac{\alpha_B}{\alpha} \left( \delta \rho_m - 3H q_m \right) + H M^2 \Psi + H \frac{\alpha K}{2\alpha} q_m - H^2 \sigma_m \right].
\]

(129)

Combined with the evolution equations for matter, these equations form a close system. They generalize those given in [26], which we recover for \( \delta \rho_m = 0 = \sigma_m \). Following [26], the parameter \( \beta_1 \) appears in eq. (128) to make explicit the existence of a transition scale in the dynamics, \( \tilde{k}_B \equiv \beta_1^{1/2}/\alpha_B \), which has been called braiding scale. Here we find that for \( \alpha_T \neq \alpha_M \) this scale is different from the transition scale \( \gamma_1^{1/2}/\alpha_B \) appearing in eqs. (120) and (121). In particular, \( \beta_1 \) is related to \( \gamma_1 \) by

\[
\beta_1 = \gamma_1 - \gamma_9,
\]

(130)

(see the explicit definition in terms of the \( \alpha_i \) in eq. (194)). Note that eq. (129) displays both scales.

Let us consider again the two limits discussed before (see also [26]).

- \( \alpha_B = 0 \): In this case most of the scale dependences go away. We are left with the simpler expression

\[
\Psi + \left( 4 + 2\alpha M + 3\tilde{\Psi} \right) H \Psi + \left( \beta_4 H^2 + \frac{c_s^2 \dot{k}^2}{a^2} \right) \Psi = - \frac{1}{2M^2} \left[ c_s^2 \left( \delta \rho_m - 3H q_m \right) + \left( \alpha M - \alpha_T + 3\tilde{\Psi} \right) H q_m + \left( \beta_{10} H^2 + \frac{2 \dot{k}^2}{3 a^2} \right) \sigma_m - \delta \rho_m + 2H \dot{\sigma}_m \right].
\]

(131)

\(^9\)An alternative derivation of eq. (128) is to combine eqs. (105)–(107) to solve for \( \pi \), \( \dot{\pi} \) and \( \Phi \) in terms of \( \Psi \), \( \dot{\Psi} \) and the matter field. We can then use the time derivative of (107) to solve for \( \Phi \) and the scalar field equation (109) to solve for \( \dot{\pi} \). Using these solutions, it is possible to eliminate the scalar field fluctuations and derive (128) [26]. We can then derive eq. (129) from (107).
Although both $\alpha_M$ and $\alpha_T$ can be nonzero here, the form of this equation is very similar to that obtained in the standard $k$-essence case.

- $\alpha_B^2 \gg \alpha_K$: For simplicity we consider only the case $\alpha_T = 0$. Moreover, to avoid negative gradient instabilities we require $\alpha_B \lesssim O(1)$ [4]. However, no such a restriction is imposed on $\alpha_M$, whose value can affect the braiding scale. Indeed, when $\alpha_B^2 \gg \alpha_K$, this is given by

$$\frac{k_B^2}{a^2} \simeq 3 (H^2 \alpha_M - \dot{H}).$$

(132)

Considering modes with $k \gg k_B$, eq. (128) simplifies to

$$\ddot{\Psi} + (3 + \alpha_M) H \dot{\Psi} + \left( k_B^2 \beta_5 a^2 + c_s^2 k^2 a^2 \right) \Psi \simeq - \frac{1}{2M^2} \left( \frac{\beta_5}{k^2 k_B^2} + c_s^2 + 1 - \frac{\alpha_M}{3\alpha_B} \right) \delta \rho_m,$$

(133)

where we have neglected relativistic terms on the right hand side of (128). The mass scale $k_B^2 \beta_5 / a^2$ corresponds to the so-called Compton mass. Depending on the value of $\beta_5$, this scale may be inside the horizon and induce a transition on the behaviour of the effective Newton constant, which is considered a strong signal of modified gravity.

To make the link with observations without resorting to numerical calculations, one often relies on the quasi static approximation, which corresponds to neglecting time derivatives with respect to spatial ones on scales much below the sound horizon, i.e. for $k \gg aH/c_s$. In this regime, modifications of gravity can be captured by two quantities, the effective Newton constant $G_{\text{eff}}$, defined by

$$- \frac{k^2}{a^2} \Phi = 4\pi G_{\text{eff}} \delta \rho_m,$$

(134)

and the gravitational slip $\gamma \equiv \Psi/\Phi$.

Both these quantities can be computed using eqs. (128) and (129). However, as discussed in [26] this does not give the same result as neglecting time derivatives in eq. (109) and using eqs. (105) and (107) to derive $G_{\text{eff}}$ and $\gamma$. The two procedures are consistent if $k$ is much larger than the other scales, i.e. in the limit $k \to \infty$. In this case we recover (compare for instance with the results of [1] in the same limit)

$$8\pi G_{\text{eff}} = \frac{\alpha c_s^2 (1 + \alpha_T) + 2[\alpha_B (1 + \alpha_T) + \alpha_T - \alpha_M]^2}{\alpha c_s^2} \frac{1}{M^{-2}},$$

(135)

$$\gamma = \frac{\alpha c_s^2 + 2\alpha_B [\alpha_B (1 + \alpha_T) + \alpha_T - \alpha_M]}{\alpha c_s^2 (1 + \alpha_T) + 2[\alpha_B (1 + \alpha_T) + \alpha_T - \alpha_M]^2},$$

(136)

where we have expressed both quantities directly in terms of the functions $\alpha_a$ (recall that $\alpha = \alpha_K + 6\alpha_B^2$ and $\alpha_H$ is here set to zero), obtained from the derivatives of the initial ADM Lagrangian.

## 5 Conclusions

In this article, we have presented a very general approach to parametrize theoretically motivated deviations from the $\Lambda$CDM standard model. This approach combines several advantages, both from the theoretical and observational points of view. On the one hand, it provides a unified treatment of theoretical models, using as a starting point a Lagrangian expressed in terms of ADM geometrical quantities defined for a foliation of uniform scalar field hypersurfaces. On the other hand, it expresses all the relevant information about the linear cosmological perturbations in terms of a minimal set of five time-dependent functions, which can be constrained by observations. These five functions,
together with the background time evolution (and a constant parameter), are sufficient to fully characterize the background and linear perturbations, within the large class of models we have considered (corresponding to the conditions (73) to avoid a non trivial dispersion relation for the scalar mode).

The link between these two endpoints, theoretical and observational, is direct since the five functions correspond to combinations of the derivatives of the initial Lagrangian. One can thus automatically derive the observational predictions for any existing or novel model by computing these functions from the ADM Lagrangian. Conversely, one can use this approach in a model-independent way by trying to constrain the five arbitrary functions (this requires some parametrization of these free functions expressed for instance in terms of the redshift; see e.g. [11]) with observations. Of course, since the bounds on parameters are less stringent as the number of parameters increases, it would be interesting to analyse the data with a scale of increasing complexity, corresponding to the number of free functions, thus covering the range from the simplest theory, i.e. ΛCDM, where all functions are zero, to more and more general theories.

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A Superhorizon evolution

In this appendix we extend the arguments of [47, 48] and check that the Einstein equations of Sec. 4.1 satisfy the usual adiabatic solution on superhorizon scales. To this end, it is convenient to define the quantities

\[ P \equiv M^2(\dot{\pi} - \Phi), \quad Q \equiv M^2(\dot{\Psi} + H\Phi + \dot{H}\pi), \quad R \equiv M^2(\Psi + H\pi), \]

(note that \( Q = \dot{R} - H(P + \alpha M R) \)) and rewrite the Einstein equations (105)–(108) respectively as

\[ -2\frac{k^2}{a^2} [(1 + \alpha_H)R - (1 + \alpha_B)M^2H\pi] \]
\[ -6H(1 + \alpha_B)Q - H^2(\alpha_K - 6\alpha_B)P = \delta \rho_m - \dot{\rho}_m, \]
\[ P - H(\alpha_B)P = (\rho_m + p_m)(v_m + \pi), \]
\[ M^2(\Psi - \Phi) + \alpha_T R - \alpha_M H\pi + \alpha_H P = \sigma_m, \]

\[ 2\dot{Q} + 6HQ + 2\left(\frac{\rho_m + p_m}{2M^2} - 3\alpha_B H^2\right)P - 2(H\alpha_B)\dot{P} = \delta p_m - \dot{p}_m - \frac{2k^2}{3a^2}\sigma_m. \]

The evolution of \( \pi \) reads

\[ (H^2\alpha_K P) + 6(H\alpha_B Q)^\prime + 3H(\alpha_K H^2 - 2\alpha_B \dot{H})P + 36\left(\dot{H} + \frac{\rho_m + p_m}{2M^2} + 3H^2\alpha_B\right)Q \]
\[ + \frac{k^2}{a^2} \left\{ 2\alpha_H Q + 2\left[ H\alpha_H + (M^2\alpha_H)\dot{M} + H\alpha_M - H\alpha_T \right] R \right. \]
\[ - \left. 2\left[ \ddot{H} + \frac{\rho_m + p_m}{2M^2} + H^2\alpha_B + (M^2H\alpha_B)\dot{M} \right] M^2\pi - 2M^2H\alpha_B\Phi \right\} = 0. \]
Moreover, in terms of these quantities, the definitions of the fluid variables introduced in eqs. (114)–(117) are given by

\[ \delta \rho_D \equiv 2 \frac{k^2}{a^2} (\alpha_H R - \alpha_B M^2 H \pi) - 3H [\rho_D + p_D] \pi - 2\alpha_B Q + H^2 (\alpha_K - 6\alpha_B) \mathcal{P}, \]

\[ q_D \equiv -2\alpha_B H \mathcal{P} - (\rho_D + p_D) \pi, \]

\[ \sigma_D \equiv \alpha_M M^2 H \pi - \sigma_T R - \alpha_H \mathcal{P}, \]

\[ \delta p_D \equiv \left[ \dot{\rho}_D + \alpha_M M^2 (2 \dot{H} + 3H^2) \right] \pi - 2\alpha_M H \mathcal{Q} + \left( \frac{\rho_D + p_D}{M^2} + 6\alpha_B H^2 \right) \mathcal{P} + 2(\alpha_B H \mathcal{P})' + \frac{2k^2}{3a^2} \sigma_D. \]

Independently of the constituents of the Universe, the \( k = 0 \) mode of the field equations for scalar fluctuations in Newtonian gauge is invariant under the coordinate transformation [47]

\[ t \to t + \epsilon(t), \]

\[ x^i \to x^i(1 - \lambda), \]

where \( \epsilon \) is an arbitrary infinitesimal function of time and \( \lambda \) an arbitrary infinitesimal constant. In particular, using these transformations one can start from an unperturbed FLRW solution and generate a solution in Newtonian gauge with metric perturbations

\[ \Psi = H\epsilon - \lambda, \quad \Phi = -\dot{\epsilon}, \]  

and, assuming the Universe filled by several fluids and scalar fields, with matter perturbations

\[ \delta \rho_X = -\dot{\rho}_X \epsilon, \quad \delta \varphi_X = -\dot{\varphi}_X \epsilon. \]

Note that these solutions remain valid also if individual matter components are not separately conserved [47], as in the case of dark energy components that are non-minimally coupled to gravity.

Let us check that this is solution of the above equations. For our dark energy and matter components, eq. (150) becomes

\[ \delta \rho_m = -\dot{\rho}_m \epsilon, \quad \pi = -\epsilon. \]

This second equality, together with eq. (149), implies that \( \mathcal{P} = 0 \) and \( Q = 0 \) from which it follows that eqs. (138), (141) and the evolution equation for \( \pi \), eq. (142), are satisfied for \( k = 0 \).

The remaining equations, i.e. (139) and (140), are automatically satisfied because they multiply an overall factor of \( k \) and \( k^2 \), respectively, that has been dropped here. However, for the solutions (149) and (150) to be physical we must require that these equations are satisfied for finite \( k \) in the \( k \to 0 \) limit [47], which implies

\[ \nu_m = -\pi, \]

and

\[ (M^2 \epsilon)' + HM^2 \epsilon - \sigma_m = M^2 (1 + \alpha_T) \lambda, \]

with solution

\[ \epsilon = \frac{1}{M^2 a} \int^t a [M^2 (1 + \alpha_T) \lambda + \sigma_m] dt'. \]

Equations (149) and (151), with the conditions (152) and (154) is the well-known super-horizon adiabatic solution. One can defined the quantity \( \zeta \) from the metric perturbation as

\[ \zeta_{\text{tot}} \equiv -\Psi + H \frac{\dot{\Psi} + H \Phi}{H}, \]
which is known to be conserved in the $k \to 0$ limit for adiabatic perturbations [49]. Indeed, one can replace the solutions (149) in its definition and check that in this limit it coincides with the constant $\lambda$ in eq. (148),

$$\zeta_{\text{tot}} = \lambda = -\mathcal{R} M^{-2}, \quad k \to 0.$$  \hspace{1cm} (156)

We note that on super-Hubble scales and for adiabatic initial conditions, eq. (122), eqs. (128)-(129) reduce to the conservation of the total comoving curvature perturbation, i.e., $\dot{\zeta}_{\text{tot}} \approx 0$.

**B Evolution equations in synchronous gauge**

For completeness, in this section we give the perturbation equations in synchronous gauge. In this gauge, the perturbed FLRW metric including scalar perturbations reads

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j,$$

with

$$h_{ij} \equiv \frac{1}{3} h\delta_{ij} + \left(\frac{k_i k_j}{k^2} - \frac{1}{3} \delta_{ij}\right)(h + 6\eta).$$  \hspace{1cm} (158)

Using gauge transformations (see for instance [50]), we can express the variables in Newtonian gauge into those in synchronous gauge, which yields

$$\pi^{(N)} = \pi^{(S)} + \delta t,$$

$$\Phi = \delta \dot{t},$$

$$\Psi = \eta - H \delta t,$$

$$\delta\rho_m^{(N)} = \delta\rho_m^{(S)} - 3H(\rho_m + p_m)\delta t,$$

$$\delta p_m^{(N)} = \delta p_m^{(S)} + \dot{p}_m\delta t,$$

$$v_m^{(N)} = -\frac{\theta_m^{(S)}}{k^2} - \delta t,$$

with

$$\delta t \equiv \frac{a^2}{k^2} \left(\dot{h} + 6\dot{\eta}\right),$$  \hspace{1cm} (160)

and where we have introduced the divergence of the velocity, $\theta \equiv \vec{\nabla} \cdot \vec{v}$, related to the velocity potential, in Fourier space, by $\theta = -k^2 v$. The anisotropic stress is gauge invariant.

We can now apply these gauge transformations to eqs. (105)–(108). We will use conformal time, $\eta \equiv \int dt/a$, which is usually employed in numerical codes. Using this time, it is convenient to rescale the scalar fluctuation $\pi$ and the velocity divergence $\theta$ by the conformal factor and redefine

$$\pi \to \pi/a, \quad \theta \to \theta/a.$$  \hspace{1cm} (161)

By denoting by a prime the derivative with respect to conformal time, the Einstein equations in synchronous gauge read ((00) component)

$$2k^2(1 + \alpha_H)\eta - \mathcal{H}(1 + \alpha_B)h' - \mathcal{H}^2(6\alpha_B - \alpha_K)\pi'$$

$$+ \mathcal{H} \left[2k^2(\alpha_H - \alpha_B) + \mathcal{H}^2(\alpha_K - 12\alpha_B) + 6\mathcal{H}'\alpha_B\right] \pi = -\frac{a^2}{M^2} \left[\delta\rho_m - 3\mathcal{H}(\rho_D + p_D)\pi\right],$$  \hspace{1cm} (162)

((0$i$) component)

$$\eta' - \mathcal{H} \alpha_B \pi' - \mathcal{H}^2 \alpha_B \pi = \frac{a^2}{2M^2} \left[(\rho_m + p_m)\theta_m/k^2 + (\rho_D + p_D)\pi\right],$$  \hspace{1cm} (163)
The evolution equation for the scalar fluctuation reads

\[ h'' + 6h' + \mathcal{H}(2 + \alpha_M)(h' + 6\eta') - 2k^2(1 + \alpha_T)\eta - 2k^2\alpha_H\pi' - 2k^2\mathcal{H}(\alpha_H + \alpha_T - \alpha_M) = -\frac{2k^2}{M^2}\sigma_m, \]

(164)

and ((ii)-trace)

\[
h'' + \mathcal{H}(2 + \alpha_M)h' - 2k^2(1 + \alpha_T)\eta + 6\alpha_B\pi'' + 2\left[3\mathcal{H}^2\alpha_B(3 + \alpha_M) + (3\alpha_B\mathcal{H})' - k^2\alpha_H\right]\pi' \\
\{3\mathcal{H}^2[3\alpha_M + 2\alpha_B(2 + \alpha_M)] + 6\alpha_B\mathcal{H}' + 6(\alpha_B\mathcal{H}') - 2k^2(\alpha_H + \alpha_T - \alpha_M)\}\mathcal{H}\pi \\
= \frac{a^2}{M^2}\left[-3(\rho_D + p_D)(\pi' + \mathcal{H}\pi) - 3p_D'\pi - 3\delta p_m - \frac{2k^2}{a^2}\sigma_m\right],
\]

where

\[
\rho_D + p_D = -\rho_m - p_m - \frac{2M^2}{a^2}(\mathcal{H}' - \mathcal{H}^2), \quad \mathcal{H} \equiv \frac{a'}{a}.
\]

(166)

The evolution equation for the scalar fluctuation reads

\[
-\mathcal{H}^2\alpha_K\pi'' - \left[\mathcal{H}^2\alpha_K(2 + \alpha_M) + \mathcal{H}'\alpha_K + (\alpha_K\mathcal{H})'\right]\mathcal{H}\pi' \\
+ 2k^2\left\{\mathcal{H}^2[\alpha_B - \alpha_H]\alpha_M + \alpha_T - \alpha_M + [(\alpha_B - \alpha_H)\mathcal{H}'] - \frac{a^2}{2M^2}(\rho_D + p_D)\right\}\pi \\
+ \left\{\mathcal{H}^4[6\alpha_B\alpha_M - \alpha_K(1 + \alpha_M)] - 3\mathcal{H}^2\mathcal{H}'[\alpha_K - 2\alpha_B(3 - \alpha_M)] \\
- 6\alpha_B\mathcal{H}'^2 + \mathcal{H}^3(6\alpha_B' - \alpha_K') - 6\mathcal{H}(\alpha_B\mathcal{H}') - \frac{3a^2}{2M^2}(\rho_D + p_D)(\mathcal{H}^2 - \mathcal{H}')\right\}\pi \\
+ \mathcal{H}\alpha_Bh'' - 2k^2\alpha_H\pi' + \left[\mathcal{H}^2\alpha_B(1 + \alpha_M) + (\alpha_B\mathcal{H})' - \frac{a^2}{2M^2}(\rho_D + p_D)\right]h' \\
- 2k^2\left\{\mathcal{H}[\alpha_M + \alpha_H(1 + \alpha_M) - \alpha_T] + \alpha_H'\right\}\eta = 0.
\]

(167)

Moreover, we can write these equations in terms of fluid quantities,

\[ k^2\eta - \frac{1}{2}\mathcal{H}h' = -\frac{a^2}{2M^2}\sum_I \delta \rho_I, \]

(168)

\[ k^2\eta' = \frac{a^2}{2M^2} \sum_I (\rho_I + p_I)\theta_I, \]

(169)

\[ h'' + 6h'' + 2\mathcal{H}(h' + 6\eta') - 2k^2\eta = -\frac{2k^2}{M^2} \sum_I \sigma_I, \]

(170)

\[ h'' + 2hh' - 2k^2\eta = -\frac{3}{M^2} \sum_I \left(\delta p_I + \frac{2k^2}{3a^2}\sigma_I\right), \]

(171)
where we have defined

\[
\delta \rho_D \equiv \frac{M^2}{a^2} \left\{ 2k^2 \alpha_B \eta - \mathcal{H} \alpha_B h' + \mathcal{H}^2 (\alpha_K - 6 \alpha_B) \pi' + \left[ -2k^2 (\alpha_B - \alpha_H) + 6 \mathcal{H}' \alpha_B \right] \right\}, \\
\theta_D \equiv \frac{k^2 M^2}{a^2 (\rho_D + p_D)} \left\{ 2 \mathcal{H} \alpha_B \pi' + \left[ 2 \mathcal{H}^2 + \frac{a^2}{M^2} (\rho_D + p_D) \right] \pi \right\}, \\
\sigma_D \equiv - \frac{M^2}{k^2} \left[ \frac{k^2 \eta \alpha_T}{2} \right. \\
- \mathcal{H} \alpha_M \eta' + k^2 \alpha_H \pi' + k^2 (\alpha_H + \alpha_T - \alpha_M) \mathcal{H} \pi \right], \\
\delta p_D \equiv \frac{2 k^2}{3a^2} \sigma_D + \frac{M^2}{a^2} \left\{ -2 \mathcal{H} \alpha_M \eta' + 2 \mathcal{H} \alpha_B \pi'' + \left[ 2 \mathcal{H}^2 \alpha_B (3 + \alpha_M) + (\alpha_B \mathcal{H})' \right] + \frac{a^2}{M^2} (\rho_D + p_D) \right\} \pi' \\
+ \left[ \mathcal{H}^3 \left[ 3 \alpha_M + 2 \alpha_B (2 + \alpha_M) \right] + 2 \mathcal{H} \mathcal{H}' \alpha_B + 2 \mathcal{H} (\alpha_B \mathcal{H})' + \frac{a^2}{M^2} \mathcal{H} (\rho_D + p_D) + p_D' \right] \pi \right\}. 
\]

The equation for \( \pi \) is equivalent to the continuity equation in conformal synchronous gauge, namely

\[
\delta \rho_D' + 3 \mathcal{H} (\delta \rho_D + \delta p_D) + (\rho_D + p_D) \left( \theta_D + \frac{h'}{2} \right) = \mathcal{H} \alpha_M \sum I \delta \rho_I, 
\]

and the Euler equation

\[
\theta_D' + \mathcal{H} \left[ 1 + \frac{p_D'}{\mathcal{H} (\rho_D + p_D)} + \alpha_M \sum I \frac{\rho_I}{\rho_D + p_D} \right] \theta_D + \frac{1}{\rho_D + p_D} \left( \delta p_D - \frac{2 k^2}{3a^2} \sigma_D \right) = \mathcal{H} \alpha_M \sum I \frac{\rho_I + p_I - \theta_I}{\rho_D + p_D}, 
\]

is an identity just as in the Newtonian gauge case.

C Scale dependence and definitions of the parameters

We report here the two “equations of state” for the dark energy fluid, eqs. (120) and (121),

\[
\delta p_D = \frac{\gamma_1 \gamma_2 + \gamma_3 \alpha_B^2 k^2}{\gamma_1 + \alpha_B^2 k^2} (\delta \rho_D - 3 \mathcal{H} q_D) + \frac{\gamma_1 \gamma_4 + \gamma_5 \alpha_B^2 k^2}{\gamma_1 + \alpha_B^2 k^2} \mathcal{H} q_D \\
+ \gamma_7 (\delta \rho_m - 3 \mathcal{H} q_m) + \frac{\gamma_1 \gamma_6 + 3 \gamma_7 \alpha_B^2 k^2}{\gamma_1 + \alpha_B^2 k^2} \mathcal{H} q_m - \frac{6 \alpha_B^2}{\alpha} \delta p_m, \\
\sigma_D = \frac{a^2}{2k^2} \left[ \frac{\gamma_1 \alpha_T + \gamma_8 \alpha_B^2 k^2}{\gamma_1 + \alpha_B^2 k^2} (\delta \rho_D - 3 \mathcal{H} q_D) + \frac{\gamma_9 k^2}{\gamma_1 + \alpha_B^2 k^2} \mathcal{H} q_D \\
+ \alpha_T (\delta \rho_m - 3 \mathcal{H} q_m) + \frac{\gamma_10 k^2}{\gamma_1 + \alpha_B^2 k^2} \mathcal{H} q_m \right],
\]

(178)
and provide the definitions of the parameters $\gamma_a$, for which we have assumed $\alpha_H = 0$:

\[
\begin{align*}
\gamma_1 &\equiv \alpha_K \frac{\rho_D + p_D}{4H^2 M^2} - 3\alpha^2_B \frac{\dot{H}}{H^2}, \\
\gamma_2 &\equiv c_s^2 + \frac{\alpha_T}{3} - 2\alpha_B(2 + \Gamma) + (1 + \alpha_B)(\alpha_M - \alpha_T), \\
\gamma_3 &\equiv c_s^2 + \frac{\gamma_8}{3}, \\
\gamma_4 &\equiv \frac{1}{\rho_D + p_D} \left\{-\dot{p}_D/H + \alpha_M \left[\rho_D + p_D - 3H^2 M^2\right] + 6\alpha^2_B \frac{\alpha}{\alpha} \left[(3 + \alpha_M + \Gamma)(\rho_m + p_m) - \dot{p}_m/H\right]\right\}, \\
\gamma_5 &\equiv -1 - \frac{(6\alpha_B - \alpha_K)\alpha_T - \alpha_M}{6\alpha^2_B} + \frac{\alpha^2_B}{H\alpha} \left(\frac{\alpha_K}{\alpha_B}\right), \\
\gamma_6 &\equiv - \frac{6\alpha^2_B}{\alpha} \frac{2}{3} \Gamma + \frac{\alpha_K\alpha_M - 6\alpha^2_B}{\alpha}, \\
\gamma_7 &\equiv \frac{\alpha_K\alpha_M - 6\alpha^2_B}{3\alpha} - \frac{(6\alpha_B - \alpha_K)\alpha_T - \alpha_M}{3\alpha}, \\
\gamma_8 &\equiv \alpha_T + \frac{\alpha_T - \alpha_M}{\alpha_B}, \\
\gamma_9 &\equiv \frac{\alpha_T - \alpha_M}{2}, \\
\gamma_{10} &\equiv 3\alpha^2_B(\alpha_T - \alpha_M),
\end{align*}
\]

where

\[
\gamma_1 \Gamma \equiv \frac{\alpha_K}{4H^2 M^2} \left[(3 + \alpha_M)(\rho_m + p_m) - \dot{p}_m/H - \frac{\alpha^2_B(\rho_D + p_D)}{\alpha_K H} \left(\frac{\alpha_K}{\alpha_B}\right)\right] - \frac{\alpha H}{2H^2},
\]

and

\[
c_s^2 = \frac{-2(1 + \alpha_B)\left[H - (\alpha_M - \alpha_T)H^2 + H^2 \alpha_B(1 + \alpha_T)\right] + 2H\alpha_B + (\rho_m + p_m)/M^2}{H^2\alpha}.
\]

As explained in Sec. 4.3, the equations of state are equivalent to the dynamical equation for $\Psi$, eq. (128) and eq. (129),

\[
\begin{align*}
\ddot{\Psi} + &\left(\frac{\beta_1 \beta_2 + \beta_3 \alpha_B k^2}{\beta_1 + \alpha^2_B k^2}\right) H\dot{\Psi} + \left(\frac{\beta_1 \beta_4 + \beta_1 \beta_5 k^2 + c_s^2 \alpha^2_B k^4}{\beta_1 + \alpha^2_B k^2}\right) H^2 \Psi = - \frac{1}{2M^2} \left[\frac{\beta_1 \beta_6 + \beta_7 \alpha^2_B k^2}{\beta_1 + \alpha^2_B k^2}\right] \delta \rho_m \\
+ &\left(\frac{\beta_1 \beta_8 + \beta_9 \alpha^2_B k^2}{\beta_1 + \alpha^2_B k^2}\right) Hq_m + \left(\frac{\beta_1 \beta_{10} + \beta_1 \beta_{11} k^2 + \frac{2}{3} \alpha^2_B k^4}{\beta_1 + \alpha^2_B k^2}\right) H^2 \sigma_m = - \frac{\alpha_K}{\alpha} \delta \rho_m - 2H \delta \sigma_m \\
\frac{\alpha^2_B k^2}{\beta_1 + \alpha^2_B k^2} \left[\Phi - \Psi \left(1 + \alpha_T + \frac{2\gamma_9}{\alpha_B}\right) + \frac{\sigma_m}{M^2}\right] + \beta_1 \left[\Phi - \Psi (1 + \alpha_T) \frac{\gamma_1}{\beta_1} + \frac{\sigma_m}{M^2}\right] = \\
\frac{\gamma_9}{H^2 M^2} \left[\frac{\alpha_B}{\alpha} \left(\delta \rho_m - 3H q_m\right) + HM^2 \dot{\Psi} + H \frac{\alpha_K}{2\alpha} q_m - H^2 \sigma_m\right].
\end{align*}
\]
Here the parameters $\beta_\alpha$, for which we have assumed again $\alpha_H = 0$, are:\(^{10}\)

\[
\begin{align*}
\beta_1 &\equiv \gamma_1 - \gamma_9 = -\alpha_K \frac{\rho_m + p_m}{4H^2M^2} - \frac{1}{2} \alpha \left( \frac{\dot{H}}{H^2} + \alpha_T - \alpha_M \right), \\
\beta_2 &\equiv 2(2 + \alpha_M) + 3\Upsilon, \\
\beta_3 &\equiv 3 + \alpha_M + \frac{\alpha_B^2}{H\alpha} \left( \frac{\alpha_K}{\alpha_B^2} \right), \\
\beta_4 &\equiv (1 + \alpha_T) \left[ 2\dot{H}/H^2 + 3(1 + \Upsilon) + \alpha_M \right] + \dot{\alpha}_T / H, \\
\beta_5 &\equiv \frac{c_s^2}{\alpha} - 2\frac{\alpha_B(\beta_3 - \beta_2)}{\alpha} + \frac{\alpha_B^2}{\beta_1} (1 + \alpha_T)(\beta_3 - \beta_2) + \frac{\alpha_B^4}{\beta_1^2} \beta_4, \\
\beta_6 &\equiv \beta_T - 2\frac{\alpha_B(\beta_3 - \beta_2)}{\alpha}, \\
\beta_7 &\equiv \frac{c_s^2}{\alpha} + 2\frac{\alpha_B^2}{\beta_1}(1 + \alpha_T) + \alpha_B(\alpha_T - \alpha_M), \\
\beta_8 &\equiv \beta_9 - \frac{(\alpha_K - 6\alpha_B)(\beta_4 - \beta_2)}{\alpha}, \\
\beta_9 &\equiv -(1 + 3c_s^2 + \alpha_T) + \frac{\alpha_B^2}{H\alpha} \left( \frac{\alpha_K}{\alpha_B^2} \right), \\
\beta_{10} &\equiv -6(1 + \Upsilon) - 4\dot{H}/H^2, \\
\beta_{11} &\equiv \frac{2}{3} - 2\frac{\alpha_B^2}{\beta_1} \left[ (2 - \alpha_M) + 2\dot{H}/H^2 \right] - 2\frac{\alpha_B^4}{\beta_1^2} \frac{\alpha_K}{H\alpha} \left( \frac{\alpha_K}{\alpha_B^2} \right),
\end{align*}
\]

with

\[
12\beta_1 H^3M^2\Upsilon \equiv 2\alpha M^2 \left\{ \left[ \dot{H} + (\alpha_T - \alpha_M)H^2 \right] - (3 + \alpha_M)H \left[ \dot{H} + (\alpha_T - \alpha_M)H^2 \right] \right\} + \alpha_K \dot{p}_m - (\rho_m + p_m)H(\alpha_K - 6\alpha_B)(\alpha_T - \alpha_M) + 6(\rho_m + p_m)\frac{\alpha_B^4}{\alpha} \left( \frac{\alpha_K}{\alpha_B^2} \right).
\]

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\(^{10}\)These parameters do not exactly correspond to those introduced in [26]. Indeed, here the $\beta_\alpha$ and $\Upsilon$ have been made dimensionless by dividing by the appropriate power of $H$ and, because of the different definition of $\alpha_B$, we have divided $\beta_1$ by a factor 4 so that $\beta_1^{(here)} = \beta_1^{(there)}/(4H^2)$. Moreover, we have corrected minor typos in the definitions of $\Upsilon$ and $\beta_7$. We thank the authors of Ref. [26] for having checked and privately agreed on these corrections.
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