Some Remarks on the Representations of the Generalized Deformed Oscillator Algebra

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Abstract.

The classification of the representations of the generalized deformed oscillator algebra is given together with several comments about possibility of introducing a coproduct structure in some type of deformed oscillator algebra.
1 Introduction

In recent years a lot of interest has been devoted to the study of the various quantum deformations of boson oscillator commutation relations algebra (see f.ex. [1]). From mathematical point of view such popularity connected with numerous relations which exist between deformed oscillators and other quantum deformations (quantum groups, quantum algebras, quantum spaces etc [2]-[5]) which was born in the course of development of the quantum inverse scatering method [6]-[7]. From the other side there are some hopes that in a physical studies of non-linear phenomena the deformed oscillator can play the role much the same as the usual boson oscillator in standard quantum mechanics. Such hopes are supported by several applications of the deformed oscillators in conformal field theory [8], lattice models [9, 10], nuclear spectroscopy [11, 12], in describing the systems with non-standard statistics and energy spectrum [13] etc.

In this work we describe all irreducible representations of the so-called generalized deformed oscillator algebra (GDO-algebra) [14] which contains as particular cases the most popular variants of such deformations together with the standard (undeformed) one. We also give some comments connected with possibility of introducing the Hopf algebra structure. In particular we show that to this end we must essentially restrict the algebra for example by introducing in the definition of the algebra a mate partner to the main commutation relation. Such extension of defining relations has grave consequences. Namely, modificted algebra admit much less representations: as a rule only representation similar to Fock one are survived. Such ”uniqueness” is probably not very desirable because there are some indications (see f.ex. [13]) of possible physical significance of non Fock representations.

In the rest of this section mainly for purpose of establishment notations for future reference we recall briefly some well known facts about standard boson oscillator algebra essential for the following discussions.

Let us recall that the **Heisenberg algebra** $\mathcal{H}$ is a Lie algebra, generated by creation operator $a^+$ and annihilation operator $a$ and unity, subject to the commutation rules

$$[a, a^+] \equiv aa^+ - a^+a = I, \quad [a, I] = 0, \quad \text{and} \quad [a^+, I] = 0. \quad (1)$$

The related Lie group is simplest nilpotent group, known as Heisenberg group
The number operator is defined in this case by the relation

$$N := a^{+}a. \quad (2)$$

This operator fulfills the commutation rules (the Heisenberg "equations of motions")

$$[N, a] = -a, \quad [N, a^{+}] = a^{+}, \quad (3)$$

with generators $a^{+}$ and $a$. Note that the number operator $N$ is not included in the list of generators for $H$. This is essential remark for understanding some of differences in definitions of various deformations of relation (1).

It is well-known that in accordance with von Neumann uniqueness theorem (see for example [12, 17]) the Heisenberg group has essentially only one (up to unitary equivalence) irreducible representation. The similar, but much weaker uniqueness assertion [17] is true for representations of the Heisenberg algebra.

In the form of the Fock (or occupation number) representation it can be described in the Dirac bracket notation by the following formulas. We denote by $|0\rangle$ the (unique) vacuum state which is the solution of the equation $a|0\rangle = 0$ and construct the other basis states in standard way

$$|n\rangle = \frac{(a^{+})^{n}}{\sqrt{n!}}|0\rangle, \quad n \in \mathbb{N}. \quad (4)$$

The Fock Hilbert space $H_{F}$ is defined by properly completing of the linear span of these vectors in the scalar product defined as $\langle m|n\rangle = \delta_{m,n}$. Then the Fock representation is defined by the following action on this basis states of the main operators

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad n \geq 1; \quad a|0\rangle = 0; \quad a^{+}|n\rangle = \sqrt{n+1}|n+1\rangle; \quad N|n\rangle = n|n\rangle. \quad (5)$$

The quite similar but different algebraic object is the boson oscillator algebra $\mathcal{W}_{0,0}^{n}(1)$. This is a Lie algebra generated by three generators $\{a, a^{+}, N\}$ and unity, which have the following commutation rules

$$[a,a^{+}] = I, \quad [I,all] = 0$$

$$[N,a] = -a, \quad [N, a^{+}] = a^{+}. \quad (6)$$

The related Lie group is solvable group, known as oscillator group (see for example [18]).
We would like to stress that in the case of the oscillator algebra $W_{0,0}^0(1)$ the operator $N$ is an independent generator and commutation rules (3) are now postulated and they are not deduced as above. This means that the connection (2) is not supposed and indeed it does not hold for all representations. In contrast to the Heisenberg case, the oscillator group (and the oscillator algebra too) has reach enough supply of nonequivalent irreducible representations. Therefore the relation (4) is considered in this case as constraint for selection of representations with needed properties.

Let us note, that the question about compatibility between commutation relations (1) and ”equations of motion” (3) was raised by Wigner in [19], and studied in some details in [20]-[21]

As pointed in [1, 22] the mentioned distinction between these two algebras are important to have in a mind when we consider the different deformations of the boson oscillator especially in discussion of representations and possible Hopf algebra structure.

2 Generalized Deformed Oscillator Algebra

$W^\gamma_{\alpha,\beta}(q)$

In this section we list some of the most popular types of the possible deformations and unify them in the unique generalized deformed oscillator algebra which was slightly generalized variant of the algebra suggested in [14].

All these deformations have some common peculiarities. First, such algebras are not Lie algebras and rigorously defined as quotient $\mathcal{A}_q = \mathcal{A}/I_q$ of associative algebra $\mathcal{A}$ by two-sided ideal $I_q$. Here the associative algebra $\mathcal{A}$, freely generated by fixed set of letters (generators), and ideal $I_q$ defined by fixed set of relations (commutation rules).

Second, in such algebras any relations of the type (2) are in general not hold and may appear only as constraints on representations, or must be postulated independently in the definition of the algebra.

Third, it is always assumed the some kind of correspondence rule. Namely, it was assumed that in the so-called ”classical limit” $q \to 1$ the structure of Lie algebra of one of the usual types mentioned in Introduction is retained. But nevertheless this it is possible appearing of the so called singular representations [23] which are absent in the such limit is possible.
Tamm-Dancoff oscillator algebra $\mathcal{W}^{1}_{1,0}(q)$. This algebra arised in the frames of the Tamm-Dancoff method [24], [25] in quantum field theory and based on the commutator rules

$$aa^+ - qa^+ a = q^N, \quad [N,a] = -a, \quad [N,a^+] = a^+, \quad (7)$$

where $q$ is an arbitrary (in general complex) non-zero number.

Arik-Coon-Kuryshkin oscillator algebra $\mathcal{W}^{1}_{0,0}(q)$. This algebra was introduced independently by many authors (see for example [26]-[35]). Defining commutation rules for it looks as

$$aa^+ - qa^+ a = I, \quad (8)$$

which can be extended by (3) as additional relations. Of course in this case the equation (2) is not hold.

From the relation (8) we can easily receive the useful relation

$$a(a^+)^m - (qa^+)^m a = [m;q](a^+)^{m-1}, \quad [m;q] \equiv \frac{q^m - 1}{q - 1}. \quad (9)$$

We note that appearance of the basic number $[m;q]$ in this formula indicates the connection of this algebra with well developed mathematical theory known as basic analysis (or q-analysis) and allows to apply in the studying of its representations the powerful methods of the theory of basic hypergeometric series [40].

This algebra has nontrivial center (1) generated by

$$\zeta = [N;q] - a^+ a. \quad (10)$$

Nontriviality of the center explained the existence of non equivalent irreducible representations of this algebra (absence of the counterpart of uniqueness theorem for Heisenberg algebra) [23], [1], [36]-[39].

Attractive peculiarities of this algebra consist in that it is possible to define in more or less simple way the multi-mode generalization of it in non trivial (not mutually commuting) fashion which is covariant under the quantum group $SU_q(2)$ [36]-[37], [41].

Quantum deformed oscillator algebra $\mathcal{W}^{1}_{-1,0}(q)$. This algebra defined by the relations

$$aa^+ - qa^+ a = q^{-N}, \quad [N,a] = -a, \quad [N,a^+] = a^+. \quad (11)$$
It aroused in attempts to extend the wellknown Schwinger boson realisation \cite{12} of angular momentum operators for the case of quantum algebras $U_q(su(2))$ \cite{13,14} and its non-compact form $U_q(su(1,1))$ \cite{15,11}.

From \cite{11} we can receive the following useful formulas \cite{11}

$$a^+ q^{-N} = q q^{-N} a^+, \quad a q^{-N} = q^{-1} q^{-N} a;$$  \hspace{1cm} (12)

$$[N, a^+ a] = 0, \quad [N, a a^+] = 0;$$  \hspace{1cm} (13)

$$a^+ \left( a^+ \right)^m - (q a^+)^m a = [m] \left( a^+ \right)^{m-1} q^{-N};$$  \hspace{1cm} (14)

where

$$[m] \equiv \frac{q^m - q^{-m}}{q - q^{-1}} = \frac{\text{sh}(\eta m)}{\text{sh}(\eta)}, \quad \text{where} \quad q = e^\eta. $$  \hspace{1cm} (15)

Thus we see that this algebra connected with the symmetrical under the replacement $q \to q^{-1}$ basic number $[m]$. Unfortunately we have not the developed variant of $q$-analysis based on this kind of the basic number.

This algebra also has nontrivial center \cite{11} generated by the element similar to (10) and thus there are many non-equivalent irreducible representations also for this algebra \cite{46,23,11}.

Some authors considered restricted form $\mathcal{W}_{1,0}^1(q; q^{-1})$ of this algebra in which besides the relations (10) the additional relation

$$a a^+ - \frac{1}{q} a^+ a = q^N;$$  \hspace{1cm} (16)

is postulated too. In this case we also have

$$a^+ a = [N], \quad a a^+ = [N + 1], \quad [a, a^+] = [N + 1] - [N]$$  \hspace{1cm} (17)

and as follows from (10) (with $[N; q]$ replaced by $[N]$) the center of the algebra became trivial $(\zeta = 0)$. For this reason the restricted algebra has only one (up to equivalence) irreducible representation, which is the $q$-analogue of the standard Fock representation of the usual boson oscillator.

Let us note that in the Fock representation we have direct connection between usual non deformed operators $b, b^+$ and $N_b = b^+ b$, and deformed ones. This connection is given \cite{11} by

$$N_F = N_b, \quad a_F^+ = \sqrt{\frac{[N_F]}{N_F}} b^+, \quad a_F = \sqrt{\frac{[N_F + 1]}{N_F}} b.$$  \hspace{1cm} (18)
Provided that the this map connected the operators of deformed $q$-oscillator with usual one is invertible (this is not the case when $q^M = 1$, $M \in \mathbb{N}$) this restricted algebra $\mathcal{W}_{1,0}^1(q; q^{-1})$ is equivalent with standard quantum mechanical boson oscillator algebra $\mathcal{W}_{0,0}^0(1)$. On the other hand the algebra $\mathcal{W}_{1,0}^1(q; q^{-1})$ can be identified with $sl_q(2)$ quantum algebra. Indeed if both relations

$$aa^+ - qa^+ a = q^{-N}, \quad aa^+ - q^{-1} a^+ a = q^N,$$

(19)

are valid, then operators

$$X_+ = \sigma a, \quad X_- = \sigma a^+, \quad J = 1/2(N - \frac{\pi i}{2\eta}), \quad (q = e^\eta)$$

(20)

where $\sigma^2 = \frac{i \sqrt{q}}{q-1}$, fulfill commutation relations of $sl_q(2)$. This equivalence of course allows one to induce the Hopf algebra structure (similar to given in $[47]$) in $\mathcal{W}_{1,0}^1(q; q^{-1})$ from $sl_q(2)$, but corresponding co-product do not respect Hermitian conjugation.

**Yan oscillator algebra $\hat{\mathcal{W}}(q)$** This algebra was introduced in the work $[47]$ and based on the relations (17) which are not equivalent with (11) and (16) (see discussion in $[22]$). As shown in $[47]$ Hopf algebra structure for this algebra can be given. We note that this algebra has trivial center and only Fock-type representations.

**Feinsilver oscillator algebra $\mathcal{W}_{0,0}^0(2)$**. This algebra defined in $[48]$-$[50]$ by the relations

$$[a, a^+] = q^{-2N}, \quad [N, a] = -a, \quad [N, a^+] = a^+.$$ 

(21)

We note that algebra $\mathcal{W}_{0,0}^0(2)$ can be obtained by contraction to limit of infinite spin of the quantum algebra $U_q(sl_2)$ $[51]$-$[52]$-$[1]$. The center of $\mathcal{W}_{0,0}^0(q)$ is generated by

$$\zeta = [N; q^{-2}] - a^+ a,$$

(22)

and some of its representations are considered in $[19]$.

This algebra conclude the list of examples which we intend to describe here. We note that more details about properties of these algebras and connections existing between them may be founded for example in $[1],[21],[52]$. 

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Now we are ready to consider the main object of our discussions the algebra which in particular cases gives each algebra described above (except the $\hat{W}(q)$). ...

**The generalized deformed oscillator algebra** $\mathcal{W}_{\alpha,\beta}^\gamma(q)$. The generators of this algebra fulfill commutation relations

\[ aa^+ - q^\gamma a^+ a = q^{\alpha N + \beta}, \quad [N, a] = -a, \quad [N, a^+] = a^+, \quad (23) \]

where $\alpha, \beta, \gamma$ are real parameters. From first of this relations we receive by induction

\[ a(a^+)^m - q^{m\gamma}(a^+)^m a = (a^+)^{m-1} q^{\alpha N + \beta} q^{\gamma^m - q^{m\gamma}}, \quad (24) \]

which is useful in computations.

Let us consider the (abstract) Fock representation of this algebra. To this end we suppose that there exist the vacuum state $|0\rangle$ such that

\[ a|0\rangle = 0, \quad N|0\rangle = 0 \quad (25) \]

Then as in [14] by standard technique we receive

\[ N|n\rangle = n|n\rangle, \]

\[ a|n\rangle = \sqrt{F_{\alpha,\beta}^\gamma(n; q)|n-1\rangle}, \quad (26) \]

\[ a^+|n\rangle = \sqrt{F_{\alpha,\beta}^\gamma(n+1; q)|n+1\rangle}, \]

on the states

\[ |n\rangle = (a^+)^n|0\rangle \sqrt{F_{\alpha,\beta}^\gamma(n; q)!} \quad n = 0, 1, 2, \ldots, \quad (27) \]

where generalized basic number defined by

\[ F_{\alpha,\beta}^\gamma(n; q) = \begin{cases} q^{\beta q^{\gamma n} - q^{\gamma n}} & \alpha \neq \gamma, \\ nq^{\beta + \gamma(n-1)} & \alpha = \gamma. \end{cases} \quad (28) \]

\[ F_{\alpha,\beta}^\gamma(n; q)! = F_{\alpha,\beta}^\gamma(1; q) \cdot F_{\alpha,\beta}^\gamma(2; q) \cdot \ldots \cdot F_{\alpha,\beta}^\gamma(n; q), \quad F_{\alpha,\beta}^\gamma(1; q) = q^\beta. \]
We note that for special values of parameters we received, respectively

\[ F_{1,0}^{1}(n; q) = \frac{q^n - q^{-n}}{q - q^{-1}} = [n]_q = [n], \tag{29} \]

which gives the symmetrical basic number \([15]\) for quantum deformed oscillator algebra \(\mathcal{W}_{1,0}^1(q)\). Similarly,

\[ F_{0,0}^{1}(n; q) = \frac{q^n - 1}{q - 1} = [n; q], \tag{30} \]

that is standard basic number \([9]\), related to Arik-Coon-Kuryskin-Jannussis-Cigler oscillator algebra \(\mathcal{W}_{0,0}^1(q)\). For Tamm-Dancoff oscillator algebra \(\mathcal{W}_{1,1}^1(q)\) we get

\[ F_{1,1}^{1}(n; q) = nq^{n-1}, \tag{31} \]

as the related basic number. Feinsilver oscillator algebra \(\mathcal{W}_{2,0}^0(q)\) related with

\[ F_{0,0}^{0}(n; q) = q^{2(1-n)}[n; q^2] = q^{2(1-n)}(1 + q^n + q^{2n}). \tag{32} \]

Let us list some of the properties of this basic number which we used below:

\[
\begin{align*}
\begin{cases}
  n < 0 & \Rightarrow \quad \begin{cases}
    F_{\alpha,\beta}^\gamma(n) < 0 \quad \text{if} \quad \gamma \neq 0 \\
    F_{\alpha,\beta}^0(n) > 0 \quad \text{if} \quad \gamma = 0
  \end{cases} \\
  n \geq 0 & \Rightarrow \quad F_{\alpha,\beta}^\gamma(n) > 0 \\
\end{cases} \\
\end{align*}
\tag{33}
\]

\[ F_{\alpha,\beta}^\gamma(n + 1) - q^\gamma F_{\alpha,\beta}^\gamma(n) = q^{\alpha n + \beta}; \tag{34} \]

where \(F_{\alpha,\beta}^\gamma(n) = F_{\alpha,\beta}^\gamma(n; q)\). Finally, in "classical limit" we obtain naturally

\[ \lim_{q \to 1} F_{\alpha,\beta}^\gamma(n; q) = n. \tag{35} \]

3 Classification of the irreducible representations of the generalized deformed oscillator algebra \(\mathcal{W}_{\alpha,\beta}^\gamma(q)\)

In this section we consider generalized deformed oscillator algebra \(\mathcal{W} = \mathcal{W}_{\alpha,\beta}^\gamma(q)\) with defining relations

\[ aa^+ - q^\gamma a^+ a = q^{\alpha N + \beta}, \tag{36} \]
\[ [N, a] = -a, \quad [N, a^+] = -a^+ \]  
(37)

where parameters \( \alpha, \beta, \gamma \in \mathbb{R} \), and \( q \) is an arbitrary positive number. Let operators \( a, a^+, N \) gives the realization of these relations in separable Hilbert space \( \mathcal{H} \) and fulfil natural hermiticity conditions

\[ (a)^* = a^+, \quad (a^+)^* = a, \quad (N)^* = N. \]  
(38)

We also supposed that selfadjoint (Hermitian) operator \( N \) has the simple (non degenerate) discrete spectrum.

Here we give, following mainly the Rideau work \[46\] the classification of the irreducible representations of the generalized deformed oscillator algebra \( \mathcal{W}_{\alpha, \beta}^\gamma(q) \). Let us note that the selection rule for testing the type of the given representation, suggested in \[38\], also works in the more general case of the algebra \( \mathcal{W}_{\alpha, \beta}^\gamma(q) \) considered here. But naturally we deal with modified selector operator

\[ K := aa^+ - q^\alpha a^+ a. \]  
(39)

In the following we need to consider the five different domains for parameters \( \alpha, \gamma \) of the algebra \( \mathcal{W}_{\alpha, \beta}^\gamma(q) \):

1) \( \gamma \geq 0, \alpha < \gamma \);  
2) \( \gamma \geq 0, \alpha > \gamma \);  
3) \( \gamma \leq 0, \alpha < \gamma \);  
4) \( \gamma \leq 0, \alpha > \gamma \);  
5) \( \alpha = \gamma \).  
(40)

We note that parameter \( \beta \) plays no role in the classification of the representations for the algebra \( \mathcal{W}_{\alpha, \beta}^\gamma(q) \). Therefore the case 4) reduced to the case 1) (and the case 3) to the case 2) too) with the help of the substitution \( q \leftrightarrow q^{-1} \).

The central element of the algebra \( \mathcal{W}_{\alpha, \beta}^\gamma(q) \) can be written as

\[ \hat{\zeta} = \left( F_{\alpha, \beta}^\gamma(N) - a^+ a \right) S(N), \]  
(41)

where

\[ S(N) = \sum_{k=0}^{\infty} v_k N^k \]  
(42)

with arbitrary real coefficients \( v_k \).

Indeed, because from \[37\] it follows that \( [N, a^+ a] = 0 \), we have that if \( N \psi_0 = \nu_0 \psi_0 \), then \( a^+ a \psi_0 \) is also the eigenvector for \( N \) with the same eigenvalue \( \nu_0 \). Taking into account that \( N \) has the simple spectrum we obtain

\[ a^+ a \psi_0 = \lambda_0 \psi_0, \]  
(43)
where $\lambda_0 \geq 0$ because of the operator $a^+a$ is non-negative. It is standard task to check that the vectors $(a^+)^n\psi_0$ and $a^m\psi_0$, if non zero, are the eigenvectors for $N$ with the eigenvalues $\nu_0+n$ and $\nu_0-m$, respectively. Then these vectors $(a^+)^n\psi_0$ and $a^m\psi_0$, are also the eigenvectors for $a^+a$ with the eigenvalues $\lambda_n$ and $\lambda_m$, respectively. Analogously they are the eigenvectors for $aa^+$ with the eigenvalues $\mu_n$ and $\mu_m$.

Acting by the relation (36) on

$$\psi_n = \begin{cases} (a^+)^n\psi_0 & \text{for } n \geq 0 \\ a^{-n}\psi_0 & \text{for } n < 0 \end{cases}$$  \hspace{1cm} (44)

we receive

$$\mu_n - q^\gamma \lambda_n = q^{\alpha(n + \nu_0) + \beta}. \hspace{1cm} (45)$$

But for all $n \in \mathbb{Z}$ such that $\psi_n \neq 0$ we have $\mu_n = \lambda_{n+1}$. So from (13) it follows the recurrent relation

$$\lambda_{n+1} = q^\gamma \lambda_n + q^{\alpha(n + \nu_0) + \beta}, \hspace{1cm} (46)$$

which has the solution

$$\lambda_n = q^{\gamma n} \lambda_0 + q^{\alpha \nu_0} F_{\alpha, \beta}(n). \hspace{1cm} (47)$$

We need only those solutions of (47) for which $\lambda_n \geq 0$ (because of the operator $a^+a$ is non-negative). According to (33) the rhs (47) is always positive when $n \geq 0$. But for $n < 0$ we have $\lambda_n \geq 0$ only if

$$\lambda_0 \geq -q^{\alpha \nu_0 - \gamma n} F_{\alpha, \beta}(n), \quad n < 0. \hspace{1cm} (48)$$

It is convenient to rewrite this inequality in the form

$$\lambda_0 \geq \begin{cases} q^{\alpha \nu_0 + \beta - \alpha} - \frac{1 - q^{\alpha - \gamma}}{1 - q^{-\alpha}}, & \text{if } \alpha \neq \gamma \\ -n q^{\gamma(\nu_0 - 1) + \beta}, & \text{if } \alpha = \gamma \end{cases}; \quad (n < 0) \hspace{1cm} (49)$$

Note that if

$$q > 1, \quad \alpha \leq \gamma \quad \text{or} \quad 0 < q < 1, \quad \alpha \geq \gamma \hspace{1cm} (50)$$

then rhs of (48) goes to $+\infty$ as $n \to (-\infty)$. Thus there exist $n_0 < 0$ such that $\lambda_n \leq 0$, for all $n < n_0$. 

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On the other hand if
\[ q > 1, \alpha > \gamma \quad \text{or} \quad 0 < q < 1, \alpha < \gamma \] (51)
the inequality (48) is fulfilled for all \( n < 0 \) if
\[ \lambda_0 \geq \frac{q^{\alpha(\nu_0 - 1) + \beta}}{1 - q^{\gamma - \alpha}}. \] (52)

In the opposite to (52) case there exist \( n_0 < 0 \) such that \( \lambda_n \leq 0 \), for all \( n < n_0 \).

Now we are ready to consider the irreducible representations of the algebra \( \mathcal{W}_{\alpha,\beta}(q) \) for the each case listed in (40).

3.0.1 1) The case \( \gamma \geq 0, \quad \alpha < \gamma \).

We suppose that the following inequalities
\[ (A) \quad q > 1 \quad \text{or} \quad 0 < q < 1, \lambda_0 < \frac{q^{\alpha(\nu_0 - 1) + \beta}}{1 - q^{\gamma - \alpha}}. \] (53)
are hold.

According to (17) if \( n \to +\infty \) we have
\[ \lambda_n \to +\infty \quad \text{if} \quad \begin{cases} \gamma > 0, \quad q > 1, \\ \gamma = 0, \quad 0 < q < 1, \\ \gamma > 0, \quad \alpha < 0, \quad 0 < q < 1; \end{cases} \quad \text{or} \]
\[ \lambda_n \to 0^+ \quad \text{if} \quad \gamma > 0, \quad \alpha > 0, \quad 0 < q < 1; \]
\[ \lambda_n \to \lambda_0 + \frac{q^{\alpha\nu_0 + \beta}}{1 - q^{\gamma - \alpha}} \quad \text{if} \quad \gamma = 0, \quad q > 1; \]
\[ \lambda_n \to \frac{q^\beta}{1 - q^\gamma} \quad \text{if} \quad \gamma > 0, \quad \alpha = 0, \quad 0 < q < 1; \] (54)
and if \( n \to -\infty \) we have

\[
\lambda_n \to -\infty \quad \text{if} \quad \begin{cases} \gamma = 0, \quad q > 1, \\ \gamma > 0, \quad 0 < q < 1, \\ \gamma > 0, \quad \alpha < 0, \quad q > 1; \end{cases}
\]

\[
\lambda_n \to 0^- \quad \text{if} \quad \gamma > 0, \quad \alpha > 0, \quad q > 1;
\]

\[
\lambda_n \to \lambda_0 - \frac{q^{\alpha_0+\beta}}{q^{\beta}-1} \quad \text{if} \quad \gamma = 0, \quad 0 < q < 1;
\]

\[
\lambda_n \to \frac{q^{\beta}}{1-q} \quad \text{if} \quad \gamma > 0, \quad \alpha = 0, \quad Fq > 1.
\]

(55)

Denote by \( n_0 \) the greatest integer for which in (47) we have \( \lambda_n \leq 0 \). In this case we inevitably have \( \psi_{n_0} = 0 \) (because of \( a^+a \geq 0 \)). Then

\[
a^+\psi_{n_0+1} = \psi_{n_0} = 0, \quad \text{where} \quad \psi_{n_0+1} \neq 0,
\]

and the vector \( \psi_{n_0+1} \) is the common eigenvector for the operators \( a^+a \) and \( N \) with eigenvalues \( \lambda_{n_0+1} = 0 \) and \( \nu'_0 = \nu_0 + n_0 + 1 \), respectively. Moreover the following relation

\[
\lambda_0 + q^{\nu_0+\beta} q^{(n_0+1)(\alpha-\gamma)} = 0.
\]

is hold.

Now we repeat the whole construction above with the following substitutions

\[
\psi_0 \to \psi_{n_0+1}, \quad \lambda_0 \to 0, \quad \nu_0 \to \nu'_0.
\]

Using for the normalized basis states notation \( |n> := \psi_{n+1+n} \); \( n = 0, 1, \ldots \) \((n \geq 0)\) we receive the representation \( \pi(\nu'_0|W_{\gamma,\alpha,\beta}(q) \) in the form

\[
\begin{cases}
    a^+|n> = q^{\nu'_0/2} (F_{\gamma,\nu_0}^\gamma(n+1))^{1/2} |n+1>, \\
    a|n> = q^{\nu'_0/2} (F_{\gamma,\nu_0}^\gamma(n))^{1/2} |n-1>, \\
    N|n> = (\nu'_0 + n)|n>.
\end{cases}
\]

(56)

These series of irreducible representations of quasi-Fock type are labeled by the real number \( \nu'_0 \) which gives the lowest bound of the spectrum of the
operator $N$. For $\nu'_0 = 0$ we receive the $q$-Fock representation. Only in this $q$-Fock representation together with (36) additional relation

$$aa^+ - q^\alpha a^+ a = q^{\gamma N+\beta}$$

(57)
is hold.

According to (57) for the representations $\pi(\nu'_0|\mathcal{W}_{\alpha,\beta}^\gamma)$ (56) the element $K$ (44) fulfils the inequality

$$K = F_{\alpha,\beta}^\gamma(N + I) - q^\alpha F_{\alpha,\beta}^\gamma(N) = q^{\gamma N+\beta} > 0,$$

(58)
as it indeed must be because the representation $\pi(\nu'_0|\mathcal{W}_{\alpha,\beta}^\gamma)$ is non-singular if we extend for the case of the algebra $\mathcal{W}_{\alpha,\beta}^\gamma(q)$ criterion developed in [38] for more familiar algebra $\mathcal{W}_{1,0}^\gamma(q)$.

We also note in closing of the consideration of this case that for $0 < q < 1$, $0 < \alpha < \gamma$ operators $a^+a$, $aa^+ \in \mathfrak{g}_1$ but if $0 < q < 1$, $0 = \alpha < \gamma$ or if $q > 1$, $\alpha < \gamma = 0$ these operators are bounded.

In the next possible case

$$(B) \quad 0 < q < 1, \quad \lambda_0 > q^{\alpha (\nu_0 - 1) + \beta} \over 1 - q^{\gamma - \alpha},$$

we consider first the case $\alpha < 0$. Then according (47) $\lambda_n \to \infty$ as $|n| \to \infty$. Therefore it follows that the operator $a^+a$ has the lowest eigenvalue for some value of $n_0$. Now we can repeat all of the considerations above starting from the values $\lambda_{n_0}$, $\psi_{n_0}$ and $\nu_0 + n_0$ in place of $\lambda_0$, $\psi_0$ and $\nu_0$, respectively. Then we once more obtain the relation (47) with an additional condition $\lambda_n \geq \lambda_0$.

For $n > 0$ this inequality gives

$$\lambda_0 < q^{\alpha \psi_0} F_{\alpha,\beta}^\gamma(n) \over 1 - q^{\alpha \gamma},$$

and for $n < 0$ we obtains

$$\lambda_0 > -q^{\alpha \psi_0} F_{\alpha,\beta}^\gamma(n) \over q^{\alpha \gamma} - 1.$$ 

This both inequalities are hold (with new values of $\lambda_0, \nu_0$) if

$$-q^{\alpha \psi_0} F_{\alpha,\beta}^\gamma(-1) \over q^{-\gamma} - 1 < \lambda_0 < q^{\alpha \psi_0} F_{\alpha,\beta}^\gamma(1) \over 1 - q^{\gamma}.$$

(59)
As in the reference [46], in this case we have two-parameter family of non-equivalent irreducible representations \( \pi(\nu_0, \lambda_0 | W_{\alpha,\beta}^\gamma) \). On the elements of the orthogonal basis
\[
\{|n> = \psi_{n0+n}|n \in \mathbb{Z}\}
\]
generators of the algebra act according the formulae
\[
\begin{align*}
 a^+ |n> &= \left( q^{(n+1)\gamma} \lambda_0 + q^{\alpha \nu_0} F_{\alpha,\beta}^\gamma (n + 1) \right)^{1/2} |n + 1>, \\
 a |n> &= \left( q^{\nu_0} \lambda_0 + q^{\alpha \nu_0} F_{\alpha,\beta}^\gamma (n) \right)^{1/2} |n - 1>, \\
 N |n> &= (\nu_0 + n) |n>. 
\end{align*}
\]
(60)

The irreducible representations of this family \( \pi(\nu_0, \lambda_0 | W_{\alpha,\beta}^\gamma) \) are singular ones, they disappears in the "classical limit" \( q \to 1 \). For these representations
\[
K = \left( q^{(N+1)\gamma} \lambda_0 + q^{\alpha \nu_0} F_{\alpha,\beta}^\gamma (N + 1) \right) - q^\alpha \left( q^{N\gamma} \lambda_0 + q^{\alpha \nu_0} F_{\alpha,\beta}^\gamma (N) \right)
\]
\[
= q^{N\gamma} \lambda_0 (q^{\gamma} - q^{\alpha}) + q^{\alpha \nu_0} q^{\gamma N + \beta} = q^{N\gamma} (q^{\gamma} - q^{\alpha}) \left( \lambda_0 - \frac{q^{\alpha \nu_0 + \beta}}{q^{\gamma} - q^{\alpha}} \right) < 0.
\]
(61)

[In deriving the relation (61) we take into account conditions \( \lambda_0 > \frac{q^{\alpha (\nu_0 - 1) + \beta}}{1 - q^{\gamma - \alpha}} \), and \( 0 < q < 1 \), which means that \( \alpha < \gamma \Rightarrow q^\alpha > q^{\gamma} \).]

We note that the singularity of the representation is also indicated by the fact that the inequalities (59) became contradictory in the "classical limit" \( q \to 1 \).

In the opposite case \( \alpha \leq 0 \) according to (47) we have \( \lambda_\alpha \rightarrow +\infty \) as \( n \rightarrow -\infty \) and
\[
\lambda_\alpha \rightarrow 0^+, \quad \text{if} \quad \alpha < 0; \quad \lambda_\alpha \rightarrow \frac{q^\beta}{1 - q^{\gamma}} \quad \text{if} \quad \alpha = 0,
\]
as \( n \rightarrow +\infty \). Then we receive again the representations \( \pi(\nu_0, \lambda_0 | W_{\alpha,\beta}^\gamma) \) (60) without the restriction (59). The inequality (61) is hold so these representations are singular ones. Now we must consider the last possible case
\[
(C) \quad 0 < q < 1, \quad \lambda_0 = \frac{q^{\alpha (\nu_0 - 1) + \beta}}{1 - q^{\gamma - \alpha}}.
\]

In this case condition (47) gives
\[
\lambda_\alpha = \frac{q^{\alpha (\nu_0 + n) + \beta}}{q^{\alpha} - q^\gamma}, \quad n \in \mathbb{Z}, \quad (62)
\]

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which means that there exists no minimal as well as maximal eigenvalue for $a^+a$. In the irreducible representations $\pi_S(\nu_0|W_{\alpha,\beta}^\gamma)$ of this one-parameter family the action of generators is given by

\[
\begin{cases}
a^+|n> = q^{\alpha_0+\beta+\alpha(n+1)} (q^\alpha - q^\gamma)^{-1/2} |n+1>, \\
a |n> = q^{\alpha_0+\beta+\alpha n} (q^\alpha - q^\gamma)^{-1/2} |n-1>, \\
N |n> = (\nu_0 + n)|n>.
\end{cases}
\]  

(63)

In the representations $\pi_S(\nu_0|W_{\alpha,\beta}^\gamma)$ of this one-parameter family additional relation

\[aa^+ - q^\alpha a^+a = 0\]  

(64)

is hold. This condition means that in such representations $K = 0$. The representations $\pi_S(\nu_0|W_{\alpha,\beta}^\gamma)$ are singular, and called strange [special] ones because in "classical limit" $q \to 1$ commutation relations (64) and (36) became contradictory.

Now we briefly consider the second case from the list (40).

2) The case $\gamma \geq 0$, $\alpha > \gamma$.

As above (see (53) ) we suppose that the inequalities

\[(A) \quad 0 < q < 1 \quad \text{or} \quad q > 1, \quad \lambda_0 < \frac{q^{\alpha(\nu_0-1)+\beta}}{1 - q^{\gamma-\alpha}}.\]

are hold and by the same reasoning as in the case 1) we received the non-singular representation (56). In this case we also have $K > 0$. Note that for this case we have

\[
\lambda_n \to 0^+, \quad \text{if} \quad n \to +\infty \quad \text{for} \quad 0 < q < 1; \\
\lambda_n \to -\infty, \quad \text{if} \quad n \to -\infty \\
\lambda_n \to +\infty, \quad \text{if} \quad n \to +\infty \quad \text{for} \quad q > 1. \\
\lambda_n \to 0^-, \quad \text{if} \quad n \to -\infty
\]

It can be easily checked that in the case $0 < q < 1$, operators $a^+a$ and $aa^+$ are compact (or from the class $S_{\infty}$ ) and even nuclear ones (or from the class $S_1$ ).

In the next case

\[(B) \quad q > 1, \quad \lambda_0 > \frac{q^{\alpha(\nu_0-1)+\beta}}{1 - q^{\gamma-\alpha}}.\]
we have the representation \( (60) \). The action of operator \( K \) on vector \( |n> \) gives

\[
K|n> = (aa^+ - q^\alpha a^+a)|n> = q^{\gamma n}(q^\alpha - q^\gamma)\left(\frac{q^{\alpha_0 + \beta}}{q^\alpha - q^\gamma} - \lambda_0\right)|n> \quad (65)
\]

so that \( K < 0 \), because in this case \( (q^\alpha - q^\gamma) > 0 \) and \( \left(\frac{q^{\alpha_0 + \beta}}{q^\alpha - q^\gamma} - \lambda_0\right) < 0 \).

Thus this representations are singular ones. We have here

\[
\lambda_n \rightarrow +\infty, \quad \text{if} \ n \rightarrow +\infty, \quad (q > 1),
\]

If we take into account the condition on \( \lambda_0 \) and note that \( \lambda_n \geq 0 \) and \( F_{\alpha,\beta}^\gamma(n) < 0 \) if \( n < 0 \) then we have

\[
0 \leq \lambda_n \leq \lambda_0 \quad \text{if} \ n < 0
\]

for such representations.

In the last possible case

\[
(C) \quad q > 1, \quad \lambda_0 = \frac{q^{\alpha(\nu_0 - 1) + \beta}}{1 - q^{\gamma - \alpha}},
\]

for all values \( n \in \mathbb{Z} \) we have

\[
\lambda_n = \frac{q^{\alpha(\nu_0 + n) + \beta}}{q^\alpha - q^\gamma} \geq 0, \quad (66)
\]

and

\[
\lambda_n \rightarrow +\infty, \quad \text{if} \ n \rightarrow +\infty,
\]

\[
\lambda_n \rightarrow 0^+, \quad \text{if} \ n \rightarrow -\infty. \quad (q > 1).
\]

As in the case 1) we received the representations of special type \( (63) \) for which \( K = 0 \), that is representations are singular and additional relation \( (64) \) is hold too. In representations of this type the operator \( a^+a \) has no minimal eigenvalue.

3) The case \( \gamma \leq 0, \quad \alpha < \gamma \).

\[
(A) \quad q > 1 \quad \text{or} \quad 0 < q < 1, \quad \lambda_0 < \frac{q^{\alpha(\nu_0 - 1) + \beta}}{1 - q^{\gamma - \alpha}}.
\]
In this subcase $K > 0$ and we have nonsingular representations (56) for which
\[
\lambda_n \to 0^+, \quad \text{if } n \to +\infty \quad \text{for } q > 1;
\lambda_n \to -\infty, \quad \text{if } n \to -\infty \quad \text{for } 0 < q < 1;
\lambda_n \to +\infty, \quad \text{if } n \to +\infty \quad \text{for } 0 < q < 1;
\lambda_n \to 0^-, \quad \text{if } n \to -\infty \quad \text{for } 0 < q < 1.
\]
As in the case 2A) (but now for $q > 1$) operators $a^+a$ and $aa^+$ are nuclear (from $\mathcal{S}_1$-class).

\[(B) \quad 0 < q < 1, \quad \lambda_0 > \frac{q^{\alpha(\nu_0-1)+\beta}}{1 - q^{\gamma-\alpha}}.\]
In this subcase representations have the form (57), $K < 0$ and representations are singular.

\[(C) \quad 0 < q < 1, \quad \lambda_0 = \frac{q^{\alpha(\nu_0-1)+\beta}}{1 - q^{\gamma-\alpha}}.\]
The inequality (58) holds for all values $n \in \mathbb{Z}$. In this case $K = 0$, and the representations are singular and have the form (58). The additional relation (B4) also hold in this case. We have not the minimal eigenvalue for $a^+a$ and
\[
\lambda_n \to +\infty, \quad \text{if } n \to +\infty, \quad (0 < q < 1).
\lambda_n \to 0^+, \quad \text{if } n \to -\infty. \quad (0 < q < 1).
\]

4) The case $\gamma \leq 0, \quad \alpha > \gamma$.

\[(A) \quad 0 < q < 1 \quad \text{or} \quad q > 1, \quad \lambda_0 < \frac{q^{\alpha(\nu_0-1)+\beta}}{1 - q^{\gamma-\alpha}}.\]
In this subcase $K > 0$ and we have nonsingular representations (56). Note that now we have
\[
\lambda_n \to +\infty, \quad \text{if } n \to +\infty \quad \text{for } 0 < q < 1, \text{ or } q > 1 \text{ and } \alpha > 0;
\lambda_n \to 0^+, \quad \text{if } n \to +\infty \quad \text{for } q > 1 \text{ and } \alpha < 0;
\lambda_n \to q^{\alpha(\nu_0)+\beta}, \quad \text{if } n \to +\infty \quad \text{for } q > 1 \text{ and } \alpha = 0;
\lambda_n \to 0^-, \quad \text{if } n \to +\infty \quad \text{for } 0 < q < 1 \text{ and } \alpha < 0;
\lambda_n \to -\infty, \quad \text{if } n \to +\infty \quad \text{for } 0 < q < 1 \text{ and } \alpha > 0, \text{ or } q > 1;
\lambda_n \to -q^{\alpha(\nu_0)+\beta} \quad \text{if } n \to +\infty \quad \text{for } 0 < q < 1 \text{ and } \alpha = 0;
\]
This means that if \( q > 1 \) and \( \alpha < 0 \), then operators \( a^+a \) and \( aa^+ \) are nuclear (from \( S_1 \)-class), and for \( q > 1 \) and \( \alpha = 0 \) this operators are bounded.

The cases

\[
\text{\textbf{(B)}} \quad q > 1, \quad \lambda_0 > \frac{q^{\alpha(v_0-1)+\beta}}{1 - q^{-\alpha}}
\]

and

\[
\text{\textbf{(C)}} \quad q > 1, \quad \lambda_0 = \frac{q^{\alpha(v_0-1)+\beta}}{1 - q^{-\alpha}}.
\]

are completely analogous to the cases 1B) and 1C). In the last case

5) \( \alpha = \gamma \)

for all values \( 0 < q < \infty \) we have only nonsingular representations of the type (56) with \( K > 0 \). In these representations

\[
\begin{align*}
\lambda_n &\to 0^+, \quad \text{if } n \to +\infty \quad \text{for } 0 < q < 1, \quad \gamma > 0 \text{ or } q > 1, \quad \gamma < 0; \\
\lambda_n &\to +\infty, \quad \text{if } n \to +\infty \quad \text{for } q > 1, \quad \gamma > 0 \text{ or } 0 < q < 1, \quad \gamma \leq 0; \\
\lambda_n &\to -\infty, \quad \text{if } n \to -\infty \quad \text{for } 0 < q < 1, \quad \gamma > 0 \text{ or } q > 1, \quad \gamma \leq 0; \\
\lambda_n &\to 0^-, \quad \text{if } n \to -\infty \quad \text{for } q > 1, \quad \gamma > 0 \text{ or } 0 < q < 1, \quad \gamma < 0.
\end{align*}
\]

Thus in the cases \( 0 < q < 1, \quad \gamma > 0 \) or \( q > 1, \quad \gamma < 0 \) we have nuclear (\( S_1 \)-class) operators \( a^+a \) and \( aa^+ \).

**Remarks.** 1. Let us consider the modification of the spectral properties of the operators \( a^+a \) (and \( aa^+ \)) caused by changing of the value of the deformation parameter \( q \) near the classical point \( q = 1 \) for some concrete variants of the algebras \( \mathcal{W}_{\alpha,\beta}^{-\gamma}(q) \) mentioned in the section 2. For the Tamm-Dancoff algebra \( \mathcal{W}_{1,0}^{1,0}(q) \) operators \( a^+a, \ aa^+ \) are nuclear (that is of the \( S_1 \) class) if \( 0 < q < 1 \) (see case 5) in the section 3). But when \( q \geq 1 \) the eigenvalues \( \lambda_n \) of the operator \( a^+a \) tending to infinity \( (\lambda_n \to +\infty) \) as \( n \to +\infty \). The same result is also true for the operator \( aa^+ \) \( (\mu_n \to +\infty \text{ as } n \to +\infty) \).

In the case of the Arik-Coon-Kuryskin algebra \( \mathcal{W}_{0,0}^{1,0}(q) \) for the case \( 0 < q < 1 \) and \( \lambda_0 < \frac{1}{1-q} \) (see case A) in the subsection 3.01) operators \( a^+a \) and \( aa^+ \) are bounded with the norm growing as \( (1-q)^{-1} \) as \( q \to 1^- \). When \( q \geq 1 \) than \( \lambda_n \to +\infty \) and \( \mu_n \to +\infty \) if \( n \to +\infty \).
For the algebra $W_{1,0}^1(q)$ there are no quality changes in spectral properties of the operators $a^+a$ (and $aa^+$) in the process of passing the point $q = 1$.

For the Feinsilver case $W_{0,-2}^{0}(q)$ considered operators are bounded when $q > 1$ with the norm growing as $(q - 1)^{-1}$ as $q \to 1^+$ and if $0 < q < 1$ and $\lambda_0 < \frac{q^{-2(\nu_0 - 1)}}{1 - q}$ then $\lambda_n \to +\infty$ and $\mu_n \to +\infty$ as $n \to +\infty$.

The mentioned peculiarities in spectral properties may be helpful in attempts of construction of physical models with help of deformed oscillator operators.

2. As mentioned above some authors together with the main relations defining the algebra $W_{\alpha,\beta}(q)$

$$aa^+ - q^\gamma a^+a = q^{\alpha N + \beta}$$

required also additional ones, which in general have the form

$$aa^+ - q^{\tilde{\gamma}} a^+ a = q^{\tilde{\alpha} N + \tilde{\beta}}.$$ (68)

Because from the equality $\gamma = \tilde{\gamma}$ it follows that $\alpha = \tilde{\alpha}$ we can restrict our attention to the case $\gamma \neq \tilde{\gamma}$.

If both of the above relations (67) and (68) are hold then we have

$$a^+a = (q^{-\gamma} - q^{-\tilde{\gamma}})^{-1}(-q^{\tilde{\alpha}N + \tilde{\beta} - \gamma - \tilde{\gamma}} + q^{\alpha N + \beta - \gamma - \tilde{\gamma}})$$

$$aa^+ = (q^{-\gamma} - q^{-\tilde{\gamma}})^{-1}(q^{\alpha N + \beta - \gamma} - q^{\tilde{\alpha}N + \tilde{\beta} - \tilde{\gamma}})$$

and

$$K = aa^+ - q^\alpha a^+ a = (q^{-\gamma} - q^{-\tilde{\gamma}})^{-1}(q^{\alpha N + \beta - \gamma}(1 - q^{\alpha - \tilde{\gamma}}) - q^{\tilde{\alpha}N + \tilde{\beta} - \tilde{\gamma}}(1 - q^{\alpha - \gamma})).$$ (70)

Note that only nonsingular representations appears in the two most popular cases:

1) The $q \leftrightarrow q^{-1}$-symmetrical case under which (68) can be obtained from (67) after such substitution (thus $\alpha = -\tilde{\alpha}$, $\gamma = -\tilde{\gamma}$ and $\beta = -\tilde{\beta}$).

2) The case when (68) can be obtained from (67) after the substitution $\alpha \leftrightarrow \gamma$ ($\alpha = \tilde{\gamma}$ and $\tilde{\alpha} = \gamma$).
4 The Hopf algebra structure

The Hopf algebra structure for the quantum deformed oscillator algebras \( W_{\gamma,\alpha,\beta}^\gamma(q) \) is not known, and probably non exist. But for some modified algebras such structure can be introduced as it was shown in [22]. In this section we consider two examples of such algebras admitting the Hopf algebra structure using the similar argumentation.

(1) From the relation

\[
aa^+ - q^\gamma a^+ a = q^{\alpha N + \beta} \quad \alpha \neq 0, \gamma \neq 0,
\]

and its \( q \leftrightarrow q^{-1} \) partner

\[
aa^+ - q^{-\gamma} a^+ a = q^{-\alpha N - \beta}.
\]

we received

\[
aa^+ = F_{1,0}^1(-\frac{\alpha}{\gamma} N - \frac{\beta}{\gamma} + 1; q^{-\gamma})
\]

and

\[
a^+ a = F_{-1,0}^1(-\frac{\alpha}{\gamma} N - \frac{\beta}{\gamma}; q^{-\gamma}).
\]

and

\[
[a, a^+] = F_{-1,0}^1(-\frac{\alpha}{\gamma} N - \frac{\beta}{\gamma} + 1; q^{-\gamma}) - F_{1,0}^1(-\frac{\alpha}{\gamma} N - \frac{\beta}{\gamma}; q^{-\gamma}).
\]

Let us consider the oscillator algebra \( W_{\gamma,\alpha,\beta}^\gamma(q; q^{-1}) \), defined by this relation and equations of motion (37). Then the coproduct \( \Delta \), counity \( \varepsilon \) and antipod \( S \) in \( W_{\gamma,\alpha,\beta}^\gamma(q; q^{-1}) \) can be defined by the relations

\[
\Delta(a^+) = c_1 a^+ \otimes q^{\alpha_1 N + \beta_2} a^+; \quad \Delta(a) = c_3 a \otimes q^{\alpha_3 N} + c_4 q^{\alpha_4 N} \otimes a;
\]

\[
\Delta(N) = c_5 N \otimes \mathbf{1} + c_6 \mathbf{1} \otimes N + \gamma_1 \mathbf{1} \otimes \mathbf{1}; \quad \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1};
\]

\[
\varepsilon(a^+) = c_7; \quad \varepsilon(a) = c_8; \quad \varepsilon(N) = c_9; \quad \varepsilon(\mathbf{1}) = 1;
\]

\[
S(a^+) = -c_{10} a^+; \quad S(a) = -c_{11} a; \quad S(N) = -c_{12} N + c_{13} \mathbf{1}; \quad S(\mathbf{1}) = \mathbf{1};
\]

The constants \( c_i, \alpha_k \) and \( \gamma_1 \) can be easily found from concordance conditions

\[
(id \otimes \Delta)\Delta(h) = (\Delta \otimes id)\Delta(h),
\]

\[
(id \otimes \varepsilon)\Delta(h) = (\varepsilon \otimes id)\Delta(h),
\]

\[
m(id \otimes S)\Delta(h) = m(S \otimes id)\Delta(h) = \varepsilon(h) \mathbf{1},
\]

(76)
for all $h \in \mathcal{H}$, where $\mathcal{H}$ is a considered Hopf algebra and $m : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ is the multiplication in $\mathcal{H}$. From (75), (76) and (37) one obtains

\[
\begin{align*}
c_1 &= q^{\alpha_1 \gamma_1}; & c_2 &= q^{\alpha_2 \gamma_1}; & c_3 &= q^{\alpha_3 \gamma_1}; & c_4 &= q^{\alpha_4 \gamma_1}; \\
c_5 &= c_6 = 1; & c_7 &= c_8 = 0; & c_9 &= -\gamma_1; & c_{10} &= q^{\alpha_1}; \\
c_{11} &= q^{-\alpha_3}; & c_{12} &= 1; & c_{13} &= -2\gamma_1; & \alpha_2 &= -\alpha_1; & \alpha_4 &= -\alpha_3. \\
\end{align*}
\]  

(77)

Moreover $\Delta$ and $\varepsilon$ are homomorphisms and $S$ is antihomomorphism of the deformed algebra. In particular we must have

\[
\Delta(a)\Delta(a^+) - \Delta(a^+)\Delta(a) = \Delta(F_{-1,0}^{1/2}(-\alpha/\gamma N - \beta/\gamma + 1; q^{-\gamma}) - F_{-1,0}^{1/2}(-\alpha/\gamma N - \beta/\gamma; q^{-\gamma}).
\]  

(78)

This gives the relations

\[
q^{\alpha_1} = q^{\alpha_3}; \quad \alpha_1 + \alpha_3 = \alpha; \quad q^{2\alpha_1 \gamma_1} = -q^{2\beta - \gamma}. \tag{79}
\]

For real $q$ (73) imply that

\[
\alpha_1 = \alpha_3 = \frac{\alpha}{2}; \quad \gamma_1 = \frac{2\beta - \gamma}{2\alpha} + i\frac{(2k + 1)\pi}{2\alpha \ln q}; \quad k \in \mathbb{Z},
\]  

(80)

which fix all constants in (73). Thus we received finally the following definition of Hopf algebra structure in considered case

\[
\begin{align*}
\Delta(a^+) &= a^+ \otimes q^{\alpha(N+\gamma_1)/2} + q^{-\alpha(N+\gamma_1)/2} \otimes a^+; \\
\Delta(a) &= a \otimes q^{\alpha(N+\gamma_1)/2} + q^{-\alpha(N+\gamma_1)/2} \otimes a; \\
\Delta(N) &= N \otimes 1 + 1 \otimes N + \gamma_1 1 \otimes 1; & \Delta(1) &= 1 \otimes 1; \\
\varepsilon(a^+) &= 0 = \varepsilon(a); & \varepsilon(N) &= -\gamma_1; & \varepsilon(1) &= 1; \\
S(a^+) &= -q^{-\alpha/2}a^+; & S(a) &= -q^{-\alpha/2}a; & S(N) &= -N - 2\gamma_1 1; & S(1) &= 1.
\end{align*}
\]  

(81)

2). Consider the deformed oscillator algebra generated by the relations (77) and

\[
[a, a^+] = F_{-\gamma, \beta}^\gamma(N + 1; q) - F_{-\gamma, \beta}^\gamma(N; q). \tag{82}
\]

Repeating the calculation we obtain relations (81) but with $\alpha$ replaced by $\gamma$ and $\gamma_1 = \frac{1}{2} - i\frac{(2k + 1)\pi}{2\alpha \ln q}$, $k \in \mathbb{Z}$. We recall finally that relations (82) follows from couple of commutation rules

\[
\begin{align*}
aa^+ - q^\gamma a^+ a &= q^{-\gamma N + \beta}; \\
\gamma a^+ - q^{-\gamma} a^+ a &= q^\gamma N + \beta.
\end{align*}
\]  

(83)
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