On morphisms killing weights and Hurewicz-type theorems

Mikhail V. Bondarko *

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Abstract

We study certain "canonical weight decompositions" and apply the general theory to stable homotopical and motivic examples.

For a triangulated category $C$, any integer $n$, and a weight structure $w$ on $C$ a triangle $LM \rightarrow M \rightarrow RM \rightarrow LM[1]$, where $LM$ is of weights at most $m - 1$ and $RM$ is of weights at least $n + 1$ for some $m \leq n$, is determined by $M$ if exists. In this case we say (following J. Wildeshaus) that $M$ is without weights $m, \ldots, n$. Since this happens if and only if the weight complex $t(M) \in \text{Obj} K(H_w)$ ($H_w$ is the heart of $w$) is homotopy equivalent to a complex with zero terms in degrees $-n, \ldots, -m$, this condition can be "detected" via pure functors. One can also take $m = -\infty$ or $n = +\infty$ to obtain that the weight complex functor is "conservative and detects weights up to objects of infinitely small and infinitely large weights"; this is a significant improvement over previously known bounded conservativity results. Applying this statement we calculate intersections of certain purely generated subcategories and study the conservativity of weight-exact functors. The main tool is the new interesting notion of morphisms killing weights $m, \ldots, n$ that we study in detail as well.

We apply general results to spherical weight structures for $G$-spectra (as introduced in the previous paper) and to Voevodsky motives. This gives a converse to the (equivariant) stable Hurewicz theorem; in particular, the singular homology of $E \in \text{Obj} SH$ vanishes in negative degrees if and only if $E$ is an extension of a connective spectrum by an acyclic one. Moreover, the vanishing of $H_i^{\text{sing}}(E)$ for two subsequent values of $i$ gives a canonical (weight) "decomposition" of $E$ into a distinguished triangle as above.

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Introduction

Let us recall that for any object \(M\) of a triangulated category \(C\) and any integer \(n\) a weight structure \(w\) on \(C\) gives (essentially by definition) an \(n\)-weight decomposition triangle \(LM \to M \to RM \to LM[1]\), where \(LM\) is of
weights at most $n$ and $RM$ is of weights at least $n+1$ however, this triangle is (usually) not canonical. In particular, for the spherical weight structure on the stable homotopy category $SH$ (see §4.2 of [Bon18b] and Theorem 1.2.3(6) below) one can take $LM$ to be an arbitrary choice of an $n$-skeleton for the spectrum $M$ (in the sense of §6.3 of [Mar83]); thus $LM$ is not determined by $M$. However, it was noticed by J. Wildeshaus (in Proposition 1.7 of [Wil09]) that if one can choose an $n$-weight decomposition such that $LM$ is of weight at most $m−1$ for some $m ≤ n$ then this stronger assumption makes the decomposition canonical. In this case $M$ is said to be without weights $m, . . . , n$. In this paper we prove that $M$ satisfies this condition if and only if its weight complex $t(M)$ is homotopy equivalent to a complex with zero terms in degrees $−n, . . . , −m$; recall here that $t$ is a "weakly exact" functor from $C$ into a certain quotient $K_m(Hw)$ of the homotopy category of complexes in the heart $Hw$ of $w$. It easily follows that one can find out whether $M$ is without weights $m, . . . , n$ by applying pure functors (as introduced in §2.1 of [Bon18b]; see Definition 2.4.1 below) to $M$. Moreover, one can put $m = −∞$ or $n = +∞$ in these statements to obtain that $t$ is "conservative up to (weight-degenerate) objects of infinitely small and infinitely large weights". This is a significant improvement over previously known bounded conservativity results; one may say that objects of $C$ may be "detected" by means of objects of a much simpler category $K_m(Hw)$. We apply our conservativity of weight complexes result (and illustrate this yoga) to calculate certain intersections of purely generated subcategories (this result was applied in [Bon18a]) and to prove that certain weight-exact functors are conservative up to weight-degenerate objects. The latter statement generalizes Theorems 2.5 and 2.8 of [Wil18] (in particular, to not necessarily bounded weight structures); we also explain that our Proposition 3.2.1 can be applied to an interesting motivic functor that was mentioned in [Bon18b, Remarks 3.1.4(2), 3.3.2(1)] and is closely related to [Ayo18].

Moreover, we apply our general results to equivariant stable homotopy categories and spherical weight structures on them (as introduced in §4 of [Bon18b]). Then the aforementioned conservativity of the weight complex functor results give a certain converse to the (equivariant) stable Hurewicz theorem. In particular, in the case of a trivial group (and so, $C = SH$) the weight complex functor essentially calculates singular homology; thus we obtain that the singular homology of a spectrum $E ∈ Obj SH$ vanishes in negative degrees if and only if $E$ is an extension of a connective spectrum $M$ with $LM ∈ C_{w ≤ n} = C_{w ≤ n}[n]$ and $RM ∈ C_{w ≥ n+1} = C_{w ≥ n}[n+1]$.

1I.e., $LM ∈ C_{w ≤ n} = C_{w ≤ n}[n]$ and $RM ∈ C_{w ≥ n+1} = C_{w ≥ n}[n+1]$.

2Here one has to assume that $C$ is weight-Karoubian, i.e., that $Hw$ is idempotent complete; see Theorem 2.3.1[3] and §3.3[3]. However, this is a rather reasonable assumption since it is obviously fulfilled whenever $C$ is idempotent complete itself.
by an acyclic one. This statement appears to be completely new, since in all the previously existing formulations only the case where \( E \) is bounded below was considered (see Theorem 2.1(i) of [Lew92], Proposition 7.1.2(f) of [HPS97], and Theorem 6.9 of [Mar83]). Moreover, the vanishing of \( H_i^{\text{sing}}(E) \) for two subsequent values of \( i \) gives a canonical "decomposition" of \( E \) into a distinguished triangle; this result (along with its equivariant generalization) appears to be completely new as well.

The main tool for obtaining these results is the new interesting notion of morphisms killing weights \( m, \ldots, n \); for a morphism \( g : M \to N \) this means that \( g \) is "compatible with" some morphism \( w_{\leq n} M \to w_{\leq m-1} N \). We prove that this definition of killing weights for \( g \) is equivalent to several other ones. In particular, if \( m = n \) then this condition is equivalent to an easily formulated property of \( t(g) \); hence an \( SH \)-morphism \( g \) kills weight \( m \) if and only if \( H^m_{\text{sing}}(-, \Gamma)(g) = 0 \) for any abelian group \( \Gamma \). More generally, one may say that an \( SH \)-morphism \( g \) kills weights \( m, \ldots, n \) whenever it sends \( n \)-skeleta into \( m-1 \)-ones (see Proposition 2.1.1(4) and Theorem 4.2.3(6)).

Let us now describe the contents of the paper; some more information of this sort can also be found in the beginnings of sections.

\( \S 1 \) is mostly dedicated to the recollection of the existing theory of weight structures; yet we also prove some new statements (including certain properties of homotopy categories of complexes).

In \( \S 2 \) we define morphisms killing weights \( m, \ldots, n \) and objects without these weights; we also study these notions in detail. In particular, we relate killing weights to weight complexes and pure functors; this gives a new conservativity of the weight complex functor result.

In \( \S 3 \) we extend some of the results of the previous section to the case where \( \mathcal{H}_w \) (and hence also \( \mathcal{L} \)) is not Karoubian (i.e., idempotents do not necessarily yield direct summands in it); so we formulate Theorem 3.1.3 that is central for this paper. Moreover, the examples that we give demonstrate that the corresponding modifications of the formulations from \( \S 2 \) (as made in Theorem 3.1.3) are necessary. We also show that our results can be used to calculate certain intersections of purely generated subcategories and to prove that certain weight-exact functors are "conservative up to weight-degenerate objects".

In \( \S 4 \) we apply our general results to the study of purely compactly generated categories. Next we consider (equivariant) stable homotopy examples to obtain the aforementioned Hurewicz-type theorems along with several related statements.

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1 Weight structures: reminder

In §1.1 we introduce some notation and conventions.
In §1.2 we recall some basics on weight structures. The only new statement of this section is Proposition 1.2.4(9); it is rather technical but quite important for this paper.
In §1.3 we recall some properties of weight complex functors and study the properties of the weak homotopy equivalence relation for morphisms between complexes.

1.1 Some (categorical) notation

• Given a category \( C \) and \( X, Y \in \text{Obj} \ C \) we will write \( C(\mathbf{X}, \mathbf{Y}) \) for the set of morphisms from \( \mathbf{X} \) to \( \mathbf{Y} \) in \( C \).

• We say that an object \( \mathbf{X} \) of \( C \) is a retract of \( \mathbf{Y} \in \text{Obj} \ C \) if \( \mathbf{X} \) can be factored through \( \mathbf{Y} \).

• A subcategory \( \mathcal{H} \) of an additive category \( C \) is said to be retraction-closed in \( C \) if it contains all retracts of its objects in \( C \).

• For any \( (C, \mathcal{H}) \) as above the full subcategory \( \text{Kar}_C(\mathcal{H}) \) of \( C \) whose objects are all retracts of (finite) direct sums of objects \( \mathcal{H} \) in \( C \) will be called the retraction-closure of \( \mathcal{H} \) in \( C \); note that this subcategory is obviously additive and retraction-closed in \( C \).

• Below \( A \) will always denote some abelian category; \( B \) is an additive category.

• The Karoubi envelope \( \text{Kar}(B) \) (no lower index) of \( B \) is the category of “formal images” of idempotents in \( B \). So, its objects are the pairs \( (A, p) \) for \( A \in \text{Obj} \ B \), \( p \in B(A, A) \), \( p^2 = p \), and the morphisms are given by the formula

\[
\text{Kar}(B)((\mathbf{X}, p), (\mathbf{X}', p')) = \{ f \in B(\mathbf{X}, \mathbf{X}') : p' \circ f = f \circ p = f \}.
\]

The correspondence \( A \mapsto (A, \text{id}_A) \) (for \( A \in \text{Obj} B \)) fully embeds \( B \) into \( \text{Kar}(\mathbf{B}) \). Moreover, \( \text{Kar}(B) \) is Karoubian, i.e., any idempotent morphism gives a direct sum decomposition in \( \text{Kar}(\mathbf{B}) \).

• The symbol \( C \) below will always denote some triangulated category; usually it will be endowed with a weight structure \( w \). The symbols \( C' \) and \( D \) will also be used for triangulated categories only.
• For any $A, B, C \in \text{Obj} \, \mathcal{C}$ we will say that $C$ is an extension of $B$ by $A$ if there exists a distinguished triangle $A \to C \to B \to A[1]$.

• For $X, Y \in \text{Obj} \, \mathcal{C}$ we will write $X \perp Y$ if $\mathcal{C}(X, Y) = \{0\}$. For $D, E \subset \text{Obj} \, \mathcal{C}$ we write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. Moreover, we will write $D\perp$ for the class $\{Y \in \text{Obj} \, \mathcal{C} : X \perp Y \forall X \in D\};$ dually, $\perp D$ is the class $\{Y \in \text{Obj} \, \mathcal{C} : Y \perp X \forall X \in D\}$.

• We will write $K(B)$ for the homotopy category of (cohomological) complexes over $B$. Its full subcategory of bounded complexes will be denoted by $K^b(B)$. We will write $M = (M^i)$ if $M^i$ are the terms of the complex $M$.

• We will say that an additive covariant (resp. contravariant) functor from $\mathcal{C}$ into $\mathbf{A}$ is homological (resp. cohomological) if it converts distinguished triangles into long exact sequences.

For a (co)homological functor $H$ and $i \in \mathbb{Z}$ we will write $H_i$ (resp. $H^i$) for the composition $H \circ [-i]$.

### 1.2 Weight structures: basics

Let us recall some basic definitions of the theory of weight structures.

**Definition 1.2.1.** I. A pair of subclasses $\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 0} \subset \text{Obj} \, \mathcal{C}$ will be said to define a weight structure $w$ on a triangulated category $\mathcal{C}$ if they satisfy the following conditions.

(i) $\mathcal{C}_{w\geq 0}$ and $\mathcal{C}_{w\leq 0}$ are retraction-closed in $\mathcal{C}$ (i.e., contain all $\mathcal{C}$-retracts of their objects).

(ii) Semi-invariance with respect to translations.

$\mathcal{C}_{w\leq 0} \subset \mathcal{C}_{w\leq 0}[1], \mathcal{C}_{w\geq 0}[1] \subset \mathcal{C}_{w\geq 0}$.

(iii) Orthogonality.

$\mathcal{C}_{w\leq 0} \perp \mathcal{C}_{w\geq 0}[1]$.

(iv) Weight decompositions.

For any $M \in \text{Obj} \, \mathcal{C}$ there exists a distinguished triangle

$$LM \rightarrow M \rightarrow RM \rightarrow LM[1]$$

such that $LM \in \mathcal{C}_{w\leq 0}$ and $RM \in \mathcal{C}_{w\geq 0}[1]$.

Moreover, if $\mathcal{C}$ is endowed with a weight structure then we will say that $\mathcal{C}$ is a weighted (triangulated) category.

We will also need the following definitions.
Definition 1.2.2. Let $i, j \in \mathbb{Z}$; assume that a triangulated category $C$ is endowed with a weight structure $w$.

1. The full subcategory $Hw$ of $C$ whose objects are $C_{w=0} = C_{w\geq 0} \cap C_{w\leq 0}$ is called the heart of $w$.

2. $C_{w\geq i}$ (resp. $C_{w\leq i}$, resp. $C_{w=i}$) will denote the class $C_{w\geq 0}[i]$ (resp. $C_{w\leq 0}[i]$, resp. $C_{w=0}[i]$).

3. $C_{[i,j]}$ denotes $C_{w\geq i} \cap C_{w\leq j}$; so, this class equals $\{0\}$ if $i > j$.

4. We will say that $C$ (or $(C, w)$) is weight-Karoubian if $Hw$ is Karoubian.

5. Let $C'$ be a triangulated category endowed with a weight structure $w'$; let $F : C \to C'$ be an exact functor. Then $F$ is said to be weight-exact (with respect to $w, w'$) if it maps $C_{w\leq 0}$ into $C'_{w'\leq 0}$ and sends $C_{w\geq 0}$ into $C'_{w'\geq 0}$.

6. Let $D$ be a full triangulated subcategory of $C$. We will say that $w$ restricts to $D$ whenever the couple $(C_{w\leq 0} \cap \text{Obj } D, C_{w\geq 0} \cap \text{Obj } D)$ is a weight structure on $D$.

7. We will say that $M$ is left (resp., right) $w$-degenerate (or weight-degenerate if the choice of $w$ is clear) if $M$ belongs to $\cap_{i \in \mathbb{Z}} C_{w\geq i}$ (resp. to $\cap_{i \in \mathbb{Z}} C_{w\leq i}$).

8. We will say that $w$ is left (resp., right) non-degenerate if all left (resp. right) weight-degenerate objects are zero.

We will say that $w$ is just non-degenerate if it is both left and right non-degenerate.

9. We will call $\cup_{i \in \mathbb{Z}} C_{w\geq i}$ (resp. $\cup_{i \in \mathbb{Z}} C_{w\leq i}$) the class of $w$-bounded below (resp., $w$-bounded above) objects of $C$.

Remark 1.2.3. 1. A simple (and still useful) example of a weight structure comes from the stupid filtration on the homotopy category of cohomological complexes $K(B)$ for an arbitrary additive $B$ (it can also be restricted to bounded complexes; see Definition 1.2.2(6)). In this case $K(B)_{w^{st}}$ is the class of complexes that are homotopy equivalent to complexes concentrated in degrees $\geq 0$ (resp. $\leq 0$); see Remark 1.2.3(1) of \cite{BoS18} for more detail. We will use this notation below.

The heart of this weight structure $w^{st}$ is the retraction-closure of $B$ in $K(B)$; hence it is equivalent to $\text{Kar}(B)$.

2. A weight decomposition (of any $M \in \text{Obj } C$) is almost never canonical.
Still for any \( m \in \mathbb{Z} \) the axiom (iv) gives the existence of a distinguished triangle

\[
\begin{align*}
\tau_{\leq m} M & \to M \to \tau_{\geq m+1} M \to (\tau_{\leq m} M)[1] \\
\end{align*}
\]

(1.2.1)

with some \( \tau_{\geq m+1} M \in \mathcal{C}_{\geq m+1} \) and \( \tau_{\leq m} M \in \mathcal{C}_{\leq m} \); we will call it an \( m \)-weight decomposition of \( M \).

We will often use this notation below (even though \( \tau_{\geq m+1} M \) and \( \tau_{\leq m} M \) are not canonically determined by \( M \)); we will call any possible choice either of \( \tau_{\geq m+1} M \) or of \( \tau_{\leq m} M \) (for any \( m \in \mathbb{Z} \)) a weight truncation of \( M \). Moreover, when we will write arrows of the type \( \tau_{\leq m} M \to M \) or \( M \to \tau_{\geq m+1} M \) we will always assume that they come from some \( m \)-weight decomposition of \( M \).

3. In the current paper we use the “homological convention” for weight structures; it was previously used in [Wil09], [Wil18], [BoS19], [BoS18], [Bon18a], and in [BoK18], whereas in [Bon10] the “cohomological convention” was used. In the latter convention the roles of \( \mathcal{C}_{w \leq 0} \) and \( \mathcal{C}_{w \geq 0} \) are interchanged, i.e., one considers \( \mathcal{C}_{w \leq 0} = \mathcal{C}_{w \geq 0} \) and \( \mathcal{C}_{w \geq 0} = \mathcal{C}_{w \leq 0} \).

We also recall that D. Pauksztello has introduced weight structures independently (in [Pau08]); he called them co-t-structures.

**Proposition 1.2.4.** Let \( m \leq n \in \mathbb{Z} \), \( M, M' \in \text{Obj} \mathcal{C}, \ g \in \mathcal{C}(M, M') \).

1. The axiomatics of weight structures is self-dual, i.e., on \( \mathcal{C}' = \mathcal{C}^{\text{op}} \) (so \( \text{Obj} \mathcal{C}' = \text{Obj} \mathcal{C} \)) there exists the (opposite) weight structure \( \mathcal{C}'_{w \leq 0} = \mathcal{C}_{w \geq 0} \) for which \( \mathcal{C}'_{w \geq 0} = \mathcal{C}_{w \leq 0} \).

2. \( \mathcal{C}_{w \geq 0} = (\mathcal{C}_{w \leq -1})^\perp \) and \( \mathcal{C}_{w \leq 0} = ^\perp \mathcal{C}_{w \geq 1} \).

3. \( \mathcal{C}_{w \leq 0} \) is closed with respect to all coproducts that exist in \( \mathcal{C} \).

4. \( \mathcal{C}_{w \leq 0}, \mathcal{C}_{w \geq 0}, \) and \( \mathcal{C}_{w = 0} \) are additive.

5. For any (fixed) \( m \)-weight decomposition of \( M \) and an \( n \)-weight decomposition of \( M' \) (see Remark 1.2.3(2)) \( g \) can be extended to a morphism of the corresponding distinguished triangles:

\[
\begin{align*}
\tau_{\leq m} M & \to M \to \tau_{\geq m+1} M \\
\tau_{\leq n} M' & \to M' \to \tau_{\geq n+1} M' \\
\end{align*}
\]

(1.2.2)

Moreover, if \( m < n \) then this extension is unique (provided that the rows are fixed).
6. If $M \in \mathcal{C}_{w \geq m}$ then $w_{\leq n} M \in \mathcal{C}_{[m,n]}$ (for any $n$-weight decomposition of $M$). Dually, if $M \in \mathcal{C}_{w \leq n}$ then $w_{\geq m} M \in \mathcal{C}_{[m,n]}$.

7. Assume $M' \in \mathcal{C}_{w \geq m}$. Then $g$ factors through $w_{\geq m} M$ (for any choice of the latter object). Dually, if $M \in \mathcal{C}_{w \leq m}$ then $g$ factors through $w_{\leq m} M'$.

8. If $\mathcal{C}$ is Karoubian then it is also weight-Karoubian.

9. Assume that we are given a diagram of the form (1.2.2) and its rows are equal (so, $M' = M$, $m = n$, $w_{\leq m} M = w_{\leq n} M'$); also suppose that $g = \text{id}_M$, $h$ is an idempotent endomorphism, and $(\mathcal{C}, w)$ is weight-Karoubian.

Then there exists a decomposition $w_{\leq m} M \cong M_1 \oplus M_0$ corresponding to $h$ (i.e., $h$ equals the projection of $w_{\leq m} M$ onto $M_1$). Moreover, we have $M_0 \in \mathcal{C}_{w = m}$, and the upper row of (1.2.2) can be presented as the direct sum of (the corresponding two arrows in) a certain $m$-weight decomposition $M_1 \to M \to M_2$ with $(M_0 \to 0 \to M_0[1])$.

Proof. Assertions 1–6 were proved in [Bon10] (cf. Remark 1.2.3(4) of [BoS18] and pay attention to Remark 1.2.3(3) above!), whereas (the easy) assertion 7 is given by Proposition 1.2.4(8) of [Bon18b].

Assertion 8 is obvious.

To prove assertion 9 we take a triangulated category $\mathcal{C}'$ that is equivalent to $\text{Kar}(\mathcal{C})$ and contains $\mathcal{C}$ as a full strict subcategory. Consider the decomposition $w_{\leq m} M \cong M_1 \oplus M_0$ corresponding to $h$ in $\mathcal{C}'$. Since the diagram (1.2.2) is commutative, $c = c \circ h$; thus $c$ factors through $M_1$. Hence the distinguished triangle coming from the upper row of (1.2.2) can be decomposed into the direct sum of the $\mathcal{C}'$-distinguished triangle $M_0 \to 0 \to M_0[1]$ with a $\mathcal{C}'$-distinguished triangle $M_1 \to M \to M_2 \to M_1[1]$. Thus $M_0$ is a retract of $w_{\geq m+1} M[-1]$ as well. Hence the morphism $\text{id}_M \text{id}_0$ factors through some morphism $a : w_{\leq m} M \to w_{\geq m+1} M[-1]$. Next, assertion 7 implies that $a$ factors through $N = w_{\geq m}(w_{\leq m} M)$; thus $M_0$ is a retract of $N$. Now, $N$ belong to $\mathcal{C}_{w = m}$ by assertion 1. Since $\mathcal{H} \mathcal{W}$ is Karoubian, we obtain that $M_0$ belongs to $\mathcal{C}_{w = m} \subset \text{Obj} \mathcal{C}$. If follows that $M_1$ and $M_2$ are objects of $\mathcal{C}$ as well. Applying axiom (i) of Definition 1.2.1 we obtain that $M_1 \in \mathcal{C}_{w \leq m}$ and $M_2 \in \mathcal{C}_{w \geq m+1}$; hence $M_1 \to M \to M_2 \to M_1[1]$ is an $m$-weight decomposition of $M$ by definition.

Remark 1.2.5. Diagrams of the form (1.2.2) (also in the case $l < m$) are crucial for this paper. For any diagram of this sort we will say that the morphisms $h$ and $j$ are $\omega$-truncations of $g$. 


1. An important type of these diagrams is the one with \( g = \text{id}_M \) (for \( M' = M \); cf. part 9 of the proposition). Note that for \( m < n \) the corresponding \( w \)-truncations of \( \text{id}_M \) are unique (provided that the rows are fixed); if \( m = n \) then we obtain a certain (non-unique) "modification" of an \( m \)-weight decomposition diagram.

2. One can "compose" diagrams of the form (1.2.2), i.e., for any \( q \in C(M', M''), k \in \mathbb{Z}, \) and a morphism of triangles of the form

\[
\begin{array}{ccc}
\begin{array}{c}
w_{\leq n}M' \\
\downarrow \end{array}
& \rightarrow &
\begin{array}{c} M' \\
\downarrow \end{array}
& \rightarrow &
\begin{array}{c} w_{\geq n+1}M' \\
\downarrow \end{array}
\\
\begin{array}{c}
w_{\leq k}M'' \\
\downarrow \end{array}
& \rightarrow &
\begin{array}{c} M'' \\
\downarrow \end{array}
& \rightarrow &
\begin{array}{c} w_{\geq k+1}M'' \\
\downarrow \end{array}
\end{array}
\end{array}
\]

one can compose its vertical arrows with the ones of (1.2.2) to obtain a morphism of distinguished triangles

\[
\begin{array}{ccc}
\begin{array}{c}
w_{\leq m}M \\
\downarrow \end{array}
& \rightarrow &
\begin{array}{c} M \\
\downarrow \end{array}
& \rightarrow &
\begin{array}{c} w_{\geq m+1}M \\
\downarrow \end{array}
\\
\begin{array}{c}
w_{\leq k}M'' \\
\downarrow \end{array}
& \rightarrow &
\begin{array}{c} M'' \\
\downarrow \end{array}
& \rightarrow &
\begin{array}{c} w_{\geq k+1}M'' \\
\downarrow \end{array}
\end{array}
\end{array}
\]

one can compose its vertical arrows with the ones of (1.2.2) to obtain a morphism of distinguished triangles

Note that one does not have to assume \( k \geq n \) here (and \( n \geq m \) also is not necessary provided that the existence of (1.2.2) is known in this case). Thus one may say that \( w \)-truncations of morphisms can be composed.

Moreover, if \( k > m \) then the composed diagram obtained this way is the only possible morphism of triangles compatible with \( q \circ g \).

1.3 On weight complexes and weak homotopy equivalences

To define the weight complex functor we will need the following definition for complexes. Below \( \mathcal{B} \) will always denote an additive category.

**Definition 1.3.1.** Let \( M \) and \( N \) be objects of \( K(\mathcal{B}) \), and \( m_1, m_2 \in C(\mathcal{B})(M, N) \).

1. We write \( m_1 \sim m_2 \) if \( m_1 - m_2 = d_N x + yd_M \) for some collections of arrows \( x^*, y^* \in \mathcal{B}(M^*, N^{*+1}) \), where \( d_M \) and \( d_N \) are the corresponding differentials. We call this relation the weak homotopy equivalence one.

2. Assume \( k \leq l \in \{-\infty\} \cup \mathbb{Z} \cup \{+\infty\} \); also, \( k \in \mathbb{Z} \) if \( k = l \).

Then we write \( m_1 \sim_{[k, l]} m_2 \) if \( m_1 - m_2 \) is weakly homotopic to \( m_0 \in C(M, N) \) such that \( m_0 = 0 \) for \( k \leq i \leq l \) (and \( i \in \mathbb{Z} \)).

We need the following properties of these equivalence relations.
Lemma 1.3.2. Adopt the notation of Definition 1.3.1.

1. Factoring morphisms in $K(B)$ by the weak homotopy relation yields an additive category $K_w(B)$. Moreover, the corresponding full functor $K(B) \rightarrow K_w(B)$ is (additive and) conservative.

2. Let $A : B \rightarrow A$ be an additive functor, where $A$ is any abelian category, and assume that $m_1$ is weakly homotopy equivalent to $m_2$. Then $m_1$ and $m_2$ induce equal morphisms of the homology $H_*(A(M^i)) \rightarrow H_*(A(N^i))$.

Hence the correspondence $N \mapsto H_0(A(N^i))$ gives a well-defined functor $K_w(B) \rightarrow A$.

3. Applying an additive functor $F : B \rightarrow B'$ to complexes termwisely one obtains an additive functor $K_w(F) : K_w(B) \rightarrow K_w(B')$.

4. $m_1 \sim_{[k,l]} m_2$ if and only if $m_1 \sim_{[i,j]} m_2$ for any $i \in Z$ such that $k \leq i \leq l$.

Moreover, $m_1$ is weakly homotopy equivalent to $m_2$ if and only if $m_1 \sim_{[-\infty, +\infty]} m_2$.

5. If $k \in Z$ then $m_1 \sim_{[k,k]} 0$ if and only if there exists $m_0 \in C(B)(M, N)$ such that $m_1 = m_0$ in $K(B)(M, N)$ and $m_0^k = 0$.

6. $M$ belongs to $K(B)_{w^r \geq 0}$ if and only if $\id_M \sim_{[1, +\infty]} 0_M$, and $M \in K(B)_{w^r \leq 0}$ if and only if $\id_M \sim_{[-\infty, -1]} 0_M$.

Proof. Assertion 3 is obvious, and the remaining ones are contained in Proposition B.2 of [Bon18b]; cf. also [Bon10, §3.1] for assertions 1 and 2.

Now let us discuss the approach to weight complexes that we will use in the current paper.

Remark 1.3.3. 1. In the current paper we will use a certain additive weight complex functor $t : C \rightarrow K_w(Hw)$ (for any triangulated category $C$ endowed with a weight structure $w$). However, to define a canonical functor of this sort one has to replace $C$ by a certain equivalent (triangulated) category $C_w$; see §1.3 of [Bon18b] where this theory is exposed carefully (in contrast to §3 of [Bon10]; cf. Remark A.2.1(3) of [Bon18b]). Thus to define $t$ one should compose the (additive) "canonical weight complex functor" $t_{can} : C_w \rightarrow K_w(Hw)$ with a splitting of the canonical equivalence $C_w \rightarrow C$ (see Proposition 1.3.4 of [Bon18b]). Clearly (cf. Remark 5.3 of [Sch11]), any two splittings of this sort are isomorphic, and it is sufficient for our purposes to assume that one of them is chosen and so $t$ is fixed.
Moreover, we have no need to describe weight complexes of all morphisms in $C$ explicitly; we prefer to list a collection of properties of $t$ instead. So, we only sketch the description of $t(M)$ for $M \in \text{Obj } C$; the details can be found in loc. cit.

We choose (arbitrary) weight truncations $w \leq nM$ for all $n \in \mathbb{Z}$, and take $g_n : w \leq nM \to w \leq n+1M$ to be the corresponding $w$-truncations of $\text{id}_M$ (see Remark 1.2.5). Denote $\text{Cone}(g_n)$ by $M^{-n-1}[n+1]$; it is easily seen that $M^i \in C_{w=0}$ for all $i \in \mathbb{Z}$ and that the distinguished triangles coming from $g_n$ connect these objects to form a complex $(M^i)$.

The problem with defining the functor $t$ is that this complex clearly depends on the choice of the objects $w \leq nM$. However, for any choice of this form we have $(M^i) \cong t(M)$ (in $K(\mathbb{H}_w)$; this isomorphism becomes canonical in $K_w(\mathbb{H}_w)$).

These observations easily imply Proposition 1.3.4 below; cf. also Remark 1.3.5.

It appears that $t$ can "usually" be enhanced to an exact (strong weight complex) functor $t^{st} : C \to K(\mathbb{H}_w)$; see Corollary 3.5 of [Sos19], §6.3 of [Bon10], and Remark 1.3.5(3) of [Bon18b].

Moreover, the author believes that the reader in (more or less, concrete) examples will not loose much if she assumes that $t^{st}$ exists throughout the paper.

The weak homotopy equivalence relation was introduced in §3.1 of [Bon10] independently from the earlier and closely related notion of absolute homology; cf. Theorem 2.1 of [Bar05].

The term "weight complex" comes from [GiS96]; yet the domain of the weight complex functor in that paper was not triangulated, whereas the target was ("the ordinary") $K^b(\text{Chow}^{eff})$.

So we list the main properties of our weight complex functor $t : C \to K_w(\mathbb{H}_w)$.

**Proposition 1.3.4.** Let $M, M' \in \text{Obj } C$, $g \in C(M, M')$ (where $C$ is a weight triangulated category; see Definition 1.2.1), and $h : M' \to \text{Cone}(g)$ is the second side of a distinguished triangle containing $g$.

Then the following statements are valid.

1. $t \circ [n]_C \cong [n]_{K_w(\mathbb{H}_w)} \circ t$, where $[n]_{K_w(\mathbb{H}_w)}$ is the obvious shift by $[n]$ (invertible) endofunctor of the category $K_w(\mathbb{H}_w)$.
2. There exists a lift of the $K \omega(Hw)$-morphism chain $t(M) \xrightarrow{t(g)} t(M') \xrightarrow{t(h)} t(\text{Cone}(g))$ to two sides of a distinguished triangle in $K(Hw)$.

3. If $M \in \mathcal{C}_{\omega \leq n}$ (resp. $M \in \mathcal{C}_{\omega \geq n}$) then $t(M)$ belongs to $K(Hw)_{w^* \geq n}$ (resp. to $K(Hw)_{w^* \geq n}$).

Moreover, if $M$ is left or right $w$-degenerate (see Definition 1.2.2(7)) then $t(M) = 0$.

4. Let $\mathcal{C}'$ be a triangulated category endowed with a weight structure $w'$; let $F : \mathcal{C} \to \mathcal{C}'$ be a weight-exact functor. Then the composition $t' \circ F$ is isomorphic to $K \omega(HF) \circ t$, where $t'$ is a weight complex functor corresponding to $w'$, and the functor $K \omega(HF) : K \omega(Hw) \to K \omega(Hw')$ is defined as in Lemma 1.3.2(3).

5. For any morphism of triangles

\[
\begin{array}{ccc}
\xrightarrow{a} & \xrightarrow{b} & \\
\xrightarrow{c} & \xrightarrow{d} & \\
\xrightarrow{e} & \xrightarrow{f} & \\
\end{array}
\]

where $a$, $b$, $c$, and $d$ are the corresponding $w$-truncations of $\text{id}_M$, $\text{id}_{M'}$, and $g$ (see Remark 1.2.5), respectively, we have $\text{Cone}(a), \text{Cone}(b) \in \mathcal{C}_{\omega = n}$, and $t(g)$ is isomorphic (as a $K \omega(Hw)$-arrow) to a morphism $x$ whose $-n$th component $x^{-n} \in \text{Mor}(Hw)$ equals $h[-n]$.

Moreover, if $t(g)$ is isomorphic to a $K \omega(Hw)$-morphism $y$ such that $y^{-n} = 0$ then any choice of the rows in (1.3.1) can be completed to the whole diagram with $h = 0$ in it.

**Proof.** Assertions 1–4 are given by Proposition 1.3.4(7,9,10,12) of [Bon18b].

Lastly, the first part of assertion 5 follows from the definition of $t$ in ibid. and its "moreover" part easily follows from Proposition 1.3.4(13) of ibid. (along with Lemma 1.3.2(5) above).

**Remark 1.3.5.** Another property of $t$ that is easy to formulate is that its restriction to $Hw$ is isomorphic to the obvious embedding $Hw \to K \omega(Hw)$; see Proposition 1.3.4(10) of [Bon18b]. Note also that this property "almost follows" from part 5 of our proposition.
2 On morphisms killing weights and objects without weights in a range

In this section we introduce and study the main new notions of this paper.

In §2.1 we define morphisms killing weights $m, \ldots, n$ and objects without these weights; we give several equivalent definitions of these notions.

In §2.2 we establish several interesting properties of our notions. In particular, we prove that an object without weights $m, \ldots, n$ admits a (weight) decomposition avoiding these weights (in the sense defined by J. Wildeshaus) if $\mathcal{C}$ is weight-Karoubian.

In §2.3 we relate killing weights to the weight complex functor $t$. In particular, $M$ is without weights $m, \ldots, n$ if and only if $t(M)$ possesses this property.

In §2.4 we relate killing a weight $m$ and object without weights in a range to pure functors (as introduced in §2.1 of [Bon18b]).

2.1 Morphisms that kill certain weights: equivalent definitions

Proposition 2.1.1. Let $g \in \mathcal{C}(M, N)$ (for some $M, N \in \text{Obj } \mathcal{C}$); $m \leq n \in \mathbb{Z}$. Then the following conditions are equivalent.

1. There exists a choice of $w_{\leq n}M$ and $w_{\geq m}N$ such that the composed morphism $w_{\leq n}M \to M \xrightarrow{g} N \to w_{\geq m}N$ is zero (here the first and the third morphism in this chain come from the corresponding weight decompositions; see Remark 1.2.3(2)).

2. There exists a choice of $w_{\leq n}M$ and $w_{\leq m-1}N$ and of a morphism $h$ making the square

$$
\begin{array}{ccc}
w_{\leq n}M & \longrightarrow & M \\
\downarrow h & & \downarrow g \\
w_{\leq m-1}N & \longrightarrow & N 
\end{array}
$$

commutative.

3. There exists a choice of $w_{\geq n+1}M$ and $w_{\geq m}N$ and of a morphism $j$ making the square

$$
\begin{array}{ccc}
M & \longrightarrow & w_{\geq n+1}M \\
\downarrow g & & \downarrow j \\
N & \longrightarrow & w_{\geq m}N 
\end{array}
$$

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commutative.

4. Any choice of an \( n \)-weight decomposition of \( M \) and an \( m-1 \)-weight decomposition of \( N \) can be completed to a morphism of distinguished triangles of the form

\[
\begin{array}{ccc}
w_{\leq n} M & \longrightarrow & M \\
\downarrow h & & \downarrow g \\
w_{\leq m-1} N & \longrightarrow & N
\end{array}
\]

(2.1.3)

\[
\begin{array}{ccc}
& & w_{\geq n+1} M \\
& & \downarrow j \\
& & \\
\end{array}
\]

(2.1.4)

along with a morphism \( h \in \mathcal{C}(w_{\leq n} M, w_{\leq m-1} N) \) that turns the corresponding "halves" of the left hand square of (2.1.4) into commutative triangles.

5. For any choice of \( m-1 \) - and \( n \)-weight decompositions of \( M \) and \( N \), and for \( a \) and \( b \) being the corresponding (canonical) connecting morphisms \( w_{\leq m-1} M \to w_{\leq n} M \) and \( w_{\leq m-1} N \to w_{\leq n} N \) respectively (see Remark 1.2.5(1)), there exists a commutative diagram

\[
\begin{array}{ccc}
w_{\leq m-1} M & \overset{a}{\longrightarrow} & w_{\leq n} M \\
\downarrow c & & \downarrow d \\
w_{\leq m-1} N & \overset{b}{\longrightarrow} & w_{\leq n} N \end{array}
\]

(2.1.4)

along with a morphism \( h \in \mathcal{C}(w_{\leq n} M, w_{\leq m-1} N) \) that turns the corresponding "halves" of the left hand square of (2.1.4) into commutative triangles.

6. For any choice of the diagram (2.1.4) as above its left hand commutative square can be completed to a morphism of triangles as follows:

\[
\begin{array}{ccc}
w_{\leq m-1} M & \overset{a}{\longrightarrow} & w_{\leq n} M \\
\downarrow c & & \downarrow d \\
w_{\leq m-1} N & \overset{b}{\longrightarrow} & w_{\leq n} N \end{array}
\]

(2.1.5)

7. There exists a choice of (2.1.4) such that the corresponding diagram (2.1.5) gives a morphism of triangles.

**Proof.** Conditions 1, 2, and 3 are equivalent by Proposition 1.1.9 of [BBD82] (that is easy; in particular, the long exact sequence \( \cdots \to \overline{C}(w_{\leq n} M, w_{\leq m-1} N) \to \overline{C}(w_{\leq n} M, N) \to \overline{C}(w_{\leq n} M, w_{\geq m} N) \to \cdots \) yields that condition 1 is equivalent to 2).

Loc. cit. also implies that any of these conditions yields the existence of some diagram of the form (2.1.3) for the corresponding choices of rows.
One also obtains a diagram of this form for arbitrary choices of these weight decompositions by composing this diagram with the corresponding "change of weight decompositions" diagrams (see Remark 1.2.5(1,2)); so we obtain condition 4. On the other hand, the latter condition obviously implies conditions 1, 2, and 3. One also obtains a diagram of the form (2.1.3) for arbitrary choices of these weight decompositions by composing this diagram with the corresponding "change of weight decompositions" diagrams (see Remark 1.2.5(1,2) once again); so we obtain condition 4. On the other hand, the latter condition obviously implies conditions 1, 2, and 3.

Next, condition 5 clearly implies condition 2. Conversely, to obtain the commutative diagrams in condition 5 it suffices to take $a$ and $b$ to be the canonical connecting morphisms $w_{\leq m-1}M \to w_{\leq n}M$ and $w_{\leq m-1}N \to w_{\leq n}N$ (see Remark 1.2.5(1)) respectively, $c = h \circ a$, and $d = b \circ h$.

Next, condition 6 clearly yields condition 7. Now, consider the long exact sequence $\cdots \to C(w_{\leq n}M, w_{\leq m-1}N) \to C(w_{\leq n}M, w_{\leq n}N) \to C(w_{\leq n}M, \Cone(b)) \to \cdots$ (for an arbitrary choice of (2.1.4)). If condition 7 is fulfilled, the composed morphism $w_{\leq n}M \xrightarrow{d} w_{\leq n}N \to \Cone(b)$ is zero; hence there exists a morphism $h \in C(w_{\leq n}M, w_{\leq m-1}N)$ making the corresponding triangle (a "half" of the left hand square in (2.1.5)) commutative. Combining this with the commutativity of the right hand square in (2.1.4) we obtain condition 2 once again.

It remains to verify that condition 5 implies condition 6. The aforementioned long exact sequence gives the vanishing of the corresponding composed morphism $w_{\leq n}M \to \Cone(b)$, whereas the long exact sequence $\cdots \to C(w_{\leq n}M, w_{\leq m-1}N) \to C(w_{\leq m-1}M, w_{\leq m-1}N)$

$\to C(\Cone(a)[-1], w_{\leq m-1}N) \to \cdots$

yields the vanishing of the composed morphism $\Cone(a)[-1] \to w_{\leq m-1}N$. We obtain that (2.1.5) is a morphism of triangles indeed.

Now we give the main original definitions of this paper (yet cf. Remark 2.2.2(1) below).

**Definition 2.1.2.** Let $m \leq n \in \mathbb{Z}$.

1. We will say that a morphism $g$ **kills weights** $m, \ldots, n$ if it satisfies the equivalent conditions of the previous proposition (and we will say that $f$ **kills weight** $m$ if $n = m$); denote the class of all $C$-morphisms killing weights $m, \ldots, n$ by $\text{Mor}_{[m,n]}C$.

2. We will say that an object $M$ of $C$ is **without weights** $m, \ldots, n$ (or that $M$ **avoids** these weights) if $\text{id}_M$ kills weights $m, \ldots, n$; the class of $C$-objects without weights $m, \ldots, n$ will be denoted by $C_{w \notin [m,n]}$. 


Remark 2.1.3. 1. Obviously, these definitions are self-dual in the following natural sense: \( g \in \text{Mor}_{[m,n]} \mathcal{C} \) (resp. \( M \in \mathcal{C}_{w \notin [m,n]} \)) if and only if \( g \) kills \( w^{op} \)-weights \(-n, \ldots, -m\) (resp. \( M \) avoids \( w^{op} \)-weights \(-n, \ldots, -m\)) in \( \mathcal{C}' = \mathcal{C}^{op} \) (see Proposition 1.2.4(1)).

2. Now we describe a simple example that illustrates our definitions.

Let \( B = \text{L-vect} \) (more generally, one can consider any semi-simple abelian category here) and recall that \( w^{st} \) denotes the stupid weight structure on \( \mathbb{K}(B) \) (see Remark 1.2.3(1)); one can also take \( \mathcal{C} = \mathbb{K}(B) \) here). Then \( M \) belongs to \( \mathcal{C}_{w^{st} \leq 0} \) (resp. to \( \mathcal{C}_{w^{st} \geq 0} \)) if and only if the homology \( H_i(M) = 0 \) for all \( i \in \mathbb{Z}, m \leq i \leq n \). Hence \( g \in \text{Mor}_{[m,n]} \mathcal{C} \) and \( M \in \mathcal{C}_{w \notin [m,n]} \) if and only if for the cohomological functor \( H = \mathcal{C}(-, L) \) (here we put \( L \) in degree zero) we have \( H^i(g) = 0 \) (resp. \( H^i(M) = \{0\} \)) for all \( i \in \mathbb{Z}, m \leq i \leq n \). Thus the functors \( H^i \) for \( m \leq i \leq n \) yield a collection of cohomology theories that detect whether \( g \in \text{Mor}_{[m,n]} \mathcal{C} \) and \( M \in \mathcal{C}_{w \notin [m,n]} \). In the case \( m = n \) a certain general result related to this one is given by Proposition 2.4.4(1) below.

2.2 Basic properties of our notions

Theorem 2.2.1. Let \( M, N, O \in \text{Obj} \mathcal{C}, h \in \mathcal{C}(N, O) \), and assume that a morphism \( g \in \mathcal{C}(M, N) \) kills weights \( m, \ldots, n \) for some \( m \leq n \in \mathbb{Z} \). Then the following statements are valid.

1. Assume \( m \leq m' \leq n' \leq n \). Then \( g \) also kills weights \( m', \ldots, n' \).

2. \( \text{Mor}_{[m,n]} \mathcal{C} \) is closed with respect to direct sums and retracts (i.e., \( \bigoplus g_i \) kills weights \( m, \ldots, n \) if and only if all \( g_i \) do that).

3. \( \text{Mor}_{[m,n]} \mathcal{C} \) is a two-sided ideal of morphisms, i.e., for any \( h' \in \mathcal{C}(O, M) \) both \( h \circ g \) and \( g \circ h' \) kill weights \( m, \ldots, n \).

4. Assume that \( h \) kills weights \( m', \ldots, m - 1 \) for some \( m' < m \). Then \( h \circ g \) kills weights \( m', \ldots, n \).

5. Let \( F : \mathcal{C} \to \mathcal{D} \) be a weight-exact functor (with respect to a certain weight structure for \( \mathcal{D} \)) and assume that \( h \) kills weights \( m, \ldots, n \). Then \( F(h) \) kills these weights as well.

6. For \( F \) and \( h \) as in the previous assertion assume that \( F \) is a full embedding and \( F(h) \in \text{Mor}_{[m',n']} \mathcal{D} \). Then \( h \in \text{Mor}_{[m',n']} \mathcal{C} \).

7. Assume that \( O \) avoids weights \( m, \ldots, n \) along with weights \( n+1, \ldots, n' \) for some \( n' > n \). Then \( O \in \mathcal{C}_{w \notin [m',n]} \).
8. O avoids weights $m, \ldots, n$ if and only if $O$ is without weight $i$ whenever $m \leq i \leq n$.

9. Assume that there exists a distinguished triangle

$$X \to O \to Y \to X[1] \quad (2.2.1)$$

with $X \in C_{w \leq m-1}$, $Y \in C_{w \geq n+1}$ (we call it a *decomposition avoiding weights $m, \ldots, n$ for $M$*). Then (2.2.1) gives $l$-weight decompositions of $O$ for any $l \in \mathbb{Z}$, $m - 1 \leq l \leq n$. Moreover, $O$ is without weights $m, \ldots, n$, and this triangle is unique up to a canonical isomorphism.

10. Assume that $C$ is weight-Karoubian. Then the converse to the previous assertion is also valid (i.e., any $O$ without weights $m, \ldots, n$ admits a decomposition avoiding weights $m, \ldots, n$).

**Proof.**

1. Easy (if we use condition 2 of Proposition 2.1.1) from Remark 1.2.5(1,2).

2. Proposition 1.2.4(4) implies that all direct sums of weight decompositions are weight decompositions. This implies the assertion easily; see conditions 1 and 4 of Proposition 2.1.1.

3. Easy since we can compose the diagrams given by Proposition 2.1.1(2) and Proposition 1.2.4(5); see Remark 1.2.5(2) once again.

4. If we make choices corresponding to condition 1 of Proposition 2.1.1 and apply $F$ to the corresponding vanishing then we obtain this condition for $F(h)$.

5. For any choice of $w \leq n M$ and $w \geq M N$ the composed morphism $F(w \leq n M) \to F(M) \xrightarrow{F(h)} F(N) \to F(w \geq M N)$ is zero (see condition 2 of Proposition 2.1.1); hence this condition is fulfilled for $h$.

6. Since $id_O \circ id_O = id_O$, the statement follows from assertion 4.

7. If $O$ avoids weight $i$ whenever $m \leq i \leq n$ then iterating the previous assertion we obtain that $O$ is without weights $m, \ldots, n$. Conversely, if $O$ satisfies the latter assumption and $m \leq i \leq n$ then $id_O$ kills weight $i$ (i.e., $O$ avoids weight $i$) according to assertion 1.

8. Each statement in this assertion easily follows from the previous ones.

(2.2.1) gives the corresponding $l$-weight decompositions of $O$ just by definition. We obtain that $O$ avoids weights $m, \ldots, n$ immediately (here we can use either condition 2 or condition 3 of Proposition 2.1.1). This
The triangle (2.2.1) is canonical by Proposition 1.2.4(5) (if we take $M = M' = O$, $g = \text{id}_O$, $m = n - 1$, and $l = n$ in it).

10 The idea is to "modify" any (fixed) $n$-decomposition of $O$ using Proposition 1.2.4(9).

We also fix an $m$-weight decomposition of $O$. According to condition 2 in Proposition 2.1.1 there exists a commutative square

$$
\begin{array}{ccc}
w_{\leq n} O & \longrightarrow & O \\
\downarrow z & & \downarrow \text{id}_O \\
w_{\leq m-1} O & \longrightarrow & O
\end{array}
$$

Next, Proposition 1.2.4(9) gives the existence and uniqueness of the square

$$
\begin{array}{ccc}
w_{\leq m-1} O & \longrightarrow & O \\
\downarrow t & & \downarrow \text{id}_O \\
w_{\leq n} O & \longrightarrow & O
\end{array}
$$

Now, we can consider multiple compositions of these squares (see Remark 1.2.3). Hence the aforementioned uniqueness statement implies $t = t \circ z \circ t$. Thus the endomorphism $u = t \circ z$ is idempotent, and the square

$$
\begin{array}{ccc}
w_{\leq n} O & \longrightarrow & O \\
\downarrow u & & \downarrow \text{id}_O \\
w_{\leq n} O & \longrightarrow & O
\end{array}
$$

is commutative. Now we apply Proposition 1.2.4(9); for $X$ being the "image" of $u$ we obtain an $n$-weight decomposition $X \rightarrow O \rightarrow Y$. It remains to note that $X \in \mathcal{C}_{w_{\leq m-1}} O$ since $u$ factors through $w_{\leq m-1} O$.

Remark 2.2.2. 1. The existence of a decomposition of $O$ avoiding weights $m, \ldots, n$ means precisely that $O$ is without weights $m, \ldots, n$ in the sense of Definition 1.10 of [Wil09]. So, our definition of this notion is equivalent to the original definition of Wildeshaus (who introduced this term) if $\mathcal{C}$ is weight-Karoubian; recall here that this is automatically the case if $\mathcal{C}$ is Karoubian (see Proposition 1.2.4(8)). Still §3.3.2 below demonstrates that this equivalence statement does not hold unconditionally.

Hence the uniqueness statement in Theorem 2.2.1(9) coincides with Corollary 1.9 of [Wil09]. Moreover, the obvious modification of the
proof of Theorem 2.2.1[9] gives Proposition 1.7 of ibid. that is essentially as follows: if \( X_i \to O_i \to Y_i \) are distinguished triangles for \( i = 1, 2 \), \( X_i \in C_{w \leq m-1} \), \( Y_i \in C_{w \geq n+1} \) (and \( n \geq m \)), then any \( g : O_1 \to O_2 \) uniquely extends to a morphism of these weight decompositions (cf. Theorem 2.2.1[9] once again; note also that the argument used in the proof of our theorem extends to re-prove loc. cit. without any difficulty).

2. Combining parts 2 and 3 of our theorem one immediately obtains that the sum of any two parallel morphisms killing weights \( m, \ldots, n \) kills these weights as well.

Moreover, part 2 of the theorem implies that \( C_{w \not\in [m,n]} \) is additive and retraction-closed in \( C \), whereas part 3 yields that any \( C \)-morphism from \( M \) kills weights \( m, \ldots, n \) if (and only if; look at \( \text{id}_M \)) \( M \) avoids these weights.

3. Part 3 of our theorem says that weight-exact functors respect the condition of avoiding weights \( m, \ldots, n \), whereas full weight-exact embeddings "strictly respect" this condition. Hence full weight-exact embeddings of weight-Karoubian categories also strictly respect the condition of an object to possess a decomposition avoiding weights \( m, \ldots, n \).

### 2.3 Relation to the weight complex functor

Now we relate the properties studied in the previous subsection with the weight complex functor.

**Theorem 2.3.1.** Let \( g \in C(M,N) \) (for some \( M, N \in \text{Obj} C \)); \( m \leq n \in \mathbb{Z} \). Then the following statements are valid.

1. \( g \) kills weight \( m \) if and only if \( t(g) \sim_{[-m,-m]} 0 \) (in the notation of Definition 1.3.1).

2. If \( \{f_i\} \) for \( n \geq i \geq m \) form a chain of composable \( C \)-morphism such that for all \( i \) in this range, then \( f_m \circ \cdots \circ f_{n-1} \circ f_n \) kills weights \( m, \ldots, n \).

3. \( M \) is without weights \( m, \ldots, n \) if and only if \( t(\text{id}_M) \sim_{[-n,-m]} 0 \).

4. Assume in addition that \( C \) is weight-Karoubian. Then \( M \) avoids weights \( m, \ldots, n \) if and only if \( t(M) \) is homotopy equivalent to a complex \( C = (C^i) \) with \( C^i = 0 \) for \(-n \leq i \leq -m \).
Proof. 1. Immediate from Proposition \ref{pr:3.3}5; see also condition 7 in Proposition \ref{pr:2.1}1.

2. Straightforward from the previous assertion combined with Theorem \ref{th:2.2}4.

3. If $M \in \C_{w^\pm}[m,n]$ then combining assertion 1 with Lemma \ref{lm:1.3}4 we obtain $t(\id_M) \sim \langle -n, -m \rangle_0$. Conversely, if $t(\id_M) \sim \langle -n, -m \rangle_0$ then $t(\id_M) \sim \langle i, i \rangle_0$ for all $i$ between $-n$ and $-m$; thus applying the previous assertion to the composition $\id \circ i - m + 1$ we obtain that $M$ avoids weights $m, \ldots, n$.

4. The "if" implication follows from the previous assertion immediately; cf. Definition \ref{df:1.3}12.

Conversely, assume that $M$ avoids weights $m, \ldots, n$. By Theorem \ref{th:2.2}10, $M$ possesses a decomposition avoiding weights $m, \ldots, n$. Then for the corresponding objects $X$ and $Y$ (see \ref{th:2.2}11) Proposition \ref{pr:1.3}3 says that $t(X) \in K(Hw)_{w^st \leq m - 1}$ and $t(Y) \in K(Hw)_{w^st \geq n + 1}$. Recalling the definition of $w^st$ (in Remark \ref{rm:1.2}1) and applying Proposition \ref{pr:1.3}2 we obtain the existence of a $K(Hw)$-distinguished triangle $T_X \to t(M) \to T_Y \to T_X[1]$, where $T_X$ and $T_Y$ have zero terms in degrees at most $-m$ and at least $-n$, respectively. Thus we obtain the "only if" implication.

\[ \qed \]

Remark 2.3.2. 1. Part 2 of our proposition is a vast generalization of (the nilpotence statement in) \cite[Theorem 3.3.1(II)]{Bon10}. Moreover, we obtain an alternative proof of the latter statement that does not depend either on it or on Proposition 3.2.4 of ibid. (cf. Remark A.2.1(3) of \cite{Bon18b}).

To justify these claims we recall that Theorem 3.3.1(II) of \cite{Bon10} essentially states that $f = f_m \circ \cdots \circ f_{n - 1} \circ f_n = 0$ whenever $f_i$ are certain morphisms between elements of $C_{[m,n]}$ (see Definition \ref{df:1.2}3) and $t(f_i) = 0$. Now, if this is the case then clearly $t(f_i) \sim \langle -i, -i \rangle_0$; hence $f$ kills weights $m, \ldots, n$ by Theorem \ref{th:2.3}12. Next, if $f \in \mor C(M,N)$ for $M, N \in C_{[m,n]}$ then we can take $w_{\leq n}M = M$ and $w_{\geq m}N = N$; thus $f = 0$ (see condition 4 in Proposition \ref{pr:2.1}1). Hence our Theorem \ref{th:2.3}12 generalizes loc. cit. indeed.

2. One may extend the notion of morphisms killing weights (in a range) as follows: one can say that $g \in \mor C$ kills weights $m, \ldots, n$ for any $m \leq n$, where $m, n \in \{-\infty\} \cup \Z \cup \{+\infty\}$, whenever $g$ kills weights $m', \ldots, n'$ for any integers $m', n'$ such that $m \leq m' \leq n' \leq n$ (cf. Definition \ref{df:1.3}2). Respectively, one can define objects without weights $m, \ldots, n$ in this extended range similarly to Definition \ref{df:2.1}2.
Most of the statements involving the corresponding definitions (in our paper) will remain true for their extended versions. A significant part of the resulting "infinite analogues" are proven in this text anyway; so we leave to the reader to track the parallels with the "finite cases" and to formulate those "infinite versions" that are not included here. However, it makes sense to realize that the "decompositions" provided by Theorems 2.3.4(II) and 3.1.3 below avoid the corresponding weights; in particular, they are functorially determined by the corresponding object $M$ (see Theorem 2.2.1(9) and Remark 2.2.2(1)).

Now we are able to improve the ("partial") conservativity property of weight complexes given by Theorem 3.3.1(V) of [Bon10].

**Definition 2.3.3.** We will say that $M \in \text{Obj } C$ is $w$-degenerate (or weight-degenerate) if $t(M)$ is zero (in $K_w(\mathcal{H}_w)$ and so also in $K(\mathcal{H}_w)$).

**Theorem 2.3.4.** Let $g : M \to M'$ be a $C$-morphism, $n \in \mathbb{Z}$. Then the following statements are valid.

I.1. $t(g)$ is an isomorphism if and only if $\text{Cone}(g)$ is a $w$-degenerate object.

2. Any extension of a left $w$-degenerate object of $C$ by a right $w$-degenerate one is $w$-degenerate.

3. If $M$ is an extension of a left $w$-degenerate object by an element of $C_{w \leq n}$ (resp. is an extension of an element of $C_{w \geq n}$ by a right $w$-degenerate object) then $t(M) \in K(\mathcal{H}_w)_{w^* \leq n}$ (resp. $t(M) \in K(\mathcal{H}_w)_{w^* \geq n}$; see Remark 1.2.3(1)).

II. Assume that $C$ is weight-Karoubian.

1. Then $M$ is $w$-degenerate if and only if $M$ is an extension of a left $w$-degenerate object by a right $w$-degenerate one (cf. assertion I.2).

2. $t(M) \in K(\mathcal{H}_w)_{w^* \leq n}$ (resp. $t(M) \in K(\mathcal{H}_w)_{w^* \geq n}$) if and only if $M$ is an extension of a left $w$-degenerate object by an element of $C_{w \leq n}$ (resp. is an extension of an element of $C_{w \geq n}$ by a right $w$-degenerate object; cf. assertion I.3).

**Proof.** I.1. Immediate from Proposition 1.3.4(2) combined with the conservativity of the projection functor $K(\mathcal{B}) \to K_w(\mathcal{B})$ (see Lemma 1.3.2(1)).

2. If $N$ is left or right $w$-degenerate then $t(N) = 0$ according to Proposition 1.3.4(3). Hence the assertion follows from the previous one.

3. Similarly to the previous assertion, it suffices to combine Proposition 1.3.4(3) with assertion I.1.

II. We investigate when $t(M) \in K(\mathcal{H}_w)_{w^* \leq n}$.

For any $m > n$ we have $\text{id}_{t(M)} : [\![ -m, -n - 1 \!] ]$ 0 (see Lemma 1.3.2(6)).

Since $C$ is weight-Karoubian, for any $n > 0$ Theorem 2.2.1(10) gives a distinguished triangle $X_n \to M \to Y_m \to X_m[1]$ with $X_n \in C_{w \leq n}$ and
$Y_m \in \mathcal{C}_{w \geq m+1}$. All of these triangles are isomorphic to the one for $m = n + 1$ by the uniqueness statement in Theorem 2.2.1(9). Hence $Y_{n+1}$ is left $w$-degenerate and we obtain a triangle of the sort desired.

The proofs of the two remaining statements are similar and left to the reader.

Remark 2.3.5. 1. Thus we get a precise answer to the question when $t(g)$ is an isomorphism in the weight-Karoubian case. In particular, the weight complex functor is conservative if and only if $w$ is non-degenerate.

2. To obtain the latter statement in the general case one should combine part 1.1 of our theorem with Theorem 3.1.3 below. Moreover, that theorem contains several equivalent conditions for $t(M) \in K(H_w)_{w^t \leq 0}$ and $t(M) \in K(H_w)_{w^t \geq 0}$ (for $\mathcal{C}$ that is not necessarily weight-Karoubian). However, those more general formulations require the somewhat clumsy Definition 3.1.1(4); we demonstrate that the corresponding modifications of Theorem 2.3.4(11) are unavoidable in §3.3 below.

3. Recall that $\mathcal{C}$ is Karoubian whenever it is closed with respect to countable coproducts (triangulated categories satisfying this condition are called countably smashing in [BoS19]) according to Proposition 1.6.8 of [Nee01]. Hence it is quite reasonable to assume that $\mathcal{C}$ is weight-Karoubian (see Proposition 1.2.4(8)).

4. Recall moreover that in the case where both $\mathcal{C}$ and $\mathcal{C}_{w \geq 0}$ are closed with respect to countable $\mathcal{C}$-coproducts, $w$ is said to be countably smashing itself (in [BoS19]; cf. Definition 2.4.3(2) below). In this case for any $M \in \text{Obj}\mathcal{C}$ there exists a essentially unique distinguished triangle $LM \to M \to RM \to LM[1]$ such that $RM$ is left weight-degenerate and $LM$ is left orthogonal to all left weight-degenerate objects; see Theorem 4.1.3(1) of ibid. Thus this triangle coincides with the one provided by Theorem 2.3.4(11) under the assumption $t(M) \in K(H_w)_{w^t \leq n}$.

2.4 On the relation to pure functors

We have just proved that the assumption that a $\mathcal{C}$-morphism $g$ kills a given weight $m$ can be expressed in terms of $t(g)$. Now we use this statement to prove that this condition can be "detected" using pure functors (as defined in §2.1 of [Bon18b]; this terminology was justified in Remark 2.1.3(3) of loc. cit.; cf. also Remark 3.2.2(2) below). So, we recall some of the theory developed in ibid.

Definition 2.4.1. Assume that $\mathcal{C}$ is endowed with a weight structure $w$. 

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We will say that a (co)homological functor $H$ from $\mathcal{C}$ into an abelian category $\mathcal{A}$ is $w$-pure (or just pure if the choice of $w$ is clear) if $H$ kills both $\mathcal{C}_{w \geq 1}$ and $\mathcal{C}_{w \leq -1}$.

**Theorem 2.4.2.** 1. Let $\mathcal{A} : Hw \to \mathcal{A}$ be an additive functor, where $\mathcal{A}$ is any abelian category. For an object $M$ of $\mathcal{C}$ we will write $t(M) = (M^j)$; we set $H(M) = H^A(M)$ to be the zeroth homology of the complex $(\mathcal{A}(M^j))$. Then $H(\cdot)$ yields a homological functor. Moreover, the assignment $\mathcal{A} \mapsto H^A$ is natural in $\mathcal{A}$.

2. The correspondence $\mathcal{A} \mapsto H^A$ is an equivalence of categories between the following (not necessarily locally small) categories of functors: $\text{AddFun}(Hw, \mathcal{A})$ and the category of pure homological functors from $\mathcal{C}$ into $\mathcal{A}$.

3. Dually, the correspondence sending a contravariant functor $\mathcal{A}'$ into the functor $H_{\mathcal{A}'}$ that maps $M$ into the zeroth homology of the complex $(\mathcal{A}'(M^{−j}))$ (see assertion 1) gives an equivalence of categories between $\text{AddFun}(Hw^{\text{op}}, \mathcal{A})$ and the category of pure cohomological functors from $\mathcal{C}$ into $\mathcal{A}$.

4. A representable functor $\mathcal{C}(\cdot, M)$ is pure if and only if $M \in (\mathcal{C}_{w \geq 1} \cup \mathcal{C}_{w \leq -1})^\perp$.

**Proof.** Assertions 1 and 2 are contained in Theorem 2.1.2 of [Bon18b], and assertion 3 is their dual (cf. Proposition 1.2.4(1) or Remark 2.1.3(1) of ibid.). Lastly, assertion 4 is immediate from the definition of purity. □

Now we will prove that killing a given weight can be "detected" by means of pure functors. To prove that in certain cases representable functors are sufficient for this purpose, we recall the following definition.

**Definition 2.4.3.** Assume that $\mathcal{C}$ is closed with respect to small coproducts.

1. We will say that $\mathcal{C}$ satisfies the Brown representability property if any cohomological functor from $\mathcal{C}$ into $\text{Ab}$ that converts $\mathcal{C}$-coproducts into products of groups is representable in $\mathcal{C}$.

2. We will say that a weight structure $w$ on $\mathcal{C}$ is smashing if the class $\mathcal{C}_{w \geq 0}$ is closed with respect to (small) $\mathcal{C}$-coproducts (cf. Proposition [1.2.4.3]).

**Proposition 2.4.4.** Assume that $\mathcal{C}$ is endowed with a weight structure $w$, $g : M \to N$ is a $\mathcal{C}$-morphism, and $j \in \mathbb{Z}$.

1. Then the following conditions are equivalent.
   1. $g$ kills weight $j$.
   2. $H^A_j(g) = 0$ for any pure cohomological functor $H_A$ as above.
   3. $H^A_j(g) = 0$ for any pure cohomological functor $H_A$ corresponding to a contravariant additive functor $\mathcal{A}$ from $Hw$ into $\text{Ab}$ that converts all small $Hw$-coproducts into products of groups.
4. $H^A_j(g) = 0$ for any pure homological functor $H^A$.

II. Assume in addition that $C$ satisfies the Brown representability property and $w$ is smashing. Then the functors $H_A$ as in condition I.3 are precisely the pure representable ones.

Proof. I. It is easily seen that Proposition 1.3.4(1) allows us to assume that $j = 0$.

Next, condition 1 implies condition 2 according to Proposition 1.3.4(2) combined with Theorem 2.3.1(1). Moreover, condition 2 obviously implies condition 3.

Now let us prove that condition 3 (in the case $j = 0$) implies that $g$ kills weight 0. Thus we should verify the following: if $g$ does not kill weight 0 then there exists $A$ as in assertion 3 such that $H_A(g) \neq 0$. For $t(N) = (N^i)$ and $d_N^i$ being the boundary morphisms of this complex we take the functor $A$ that sends any $X \in C_{w=0}$ into $\text{Coker}(Hw(X, -)(d_N^{-1})) : Hw(X, N^{-1}) \to Hw(X, N^0)$. Obviously, this $A$ does convert $Hw$-coproducts into products in $\text{Ab}$. Hence it suffices to check that $H_A(g) \neq 0$.

Now, $H_A(N)$ contains the element $\theta$ corresponding to $\text{id}_{N^0}$. If $H_A(g)(\eta) = 0$ then for $t(M) = (M^i)$ and $t(g) = (g_i)$ the element $\theta_M^0 \in \mathcal{A}(M^0)$ corresponding to $g^0$ belongs to the image of $\mathcal{A}(M^1)$ (in $\mathcal{A}(M^0)$). This obviously implies that $t(g) \sim_{0, 0} 0$ (in the notation of Definition 1.3.1); thus it remains to apply Theorem 2.3.1(1) once again.

Lastly, the equivalence of conditions I.1 and I.4 is just the categorical dual of the equivalence between I.1 and I.2.

II. This is Proposition 2.3.2(8) of [Bon18b].

Remark 2.4.5. The equivalence of conditions I.1 and I.3 is closely related to Theorem 2.1 of [Bar05]; the proof is similar is well.

3. On objects without weights in general weighted categories, and a conservativity application

In §3.1 we extend Theorem 2.3.4(II) to the case where $C$ is not necessarily weight-Karoubian.

In §3.2 we apply our results to prove that certain weight-exact functors are "conservative up to weight-degenerate objects"; we also discuss the relation of this proposition to the corresponding results of [Wil18] and [Bon18b], and describe an interesting motivic example for it.

In §3.3 we construct certain counterexamples to demonstrate that the modifications made in §3.1 to to generalize Theorem 2.3.4(II) cannot be avoided.
3.1 On objects avoiding weights in not necessarily weight-Karoubian categories

To extend the results of the previous section to the case of a not (necessarily) weight-Karoubian category we recall some definitions and results from [BoS18] (yet Definition 3.1.1(4) is new).

Definition 3.1.1. 1. We will call a triangulated category \( C' \) an idempotent extension of \( C \) if it contains \( C \) and there exists a fully faithful exact functor \( C' \to \text{Kar}(C) \).

2. We will say that a weight structure \( w \) extends to an idempotent extension \( C' \) of \( C \) whenever there exists a weight structure \( w' \) for \( C' \) such that the embedding \( C \to C' \) is weight-exact. In this case we will call \( w' \) an extension of \( w \).

3. We will call a weight-Karoubian category \((C', w')\) (see Definition 1.2.2(4)) a weight-Karoubian extension of \((C, w)\) if \( C' \) is an idempotent extension of \( C \) and \( w' \) is the extension of \( w \) to it (cf. Proposition 3.1.2(1)).

4. We will say that an object \( M \) of \( C \) is essentially \( w \)-positive (resp. essentially \( w \)-negative) if it is a retract of some \( M' \in \text{Obj} \ C \) such that \( M' \) is an extension of an element of \( C_{w \geq 0} \) by a right \( w \)-degenerate object of \( C \) (resp. \( M' \) is an extension of a left \( w \)-degenerate object of \( C \) by an element of \( C_{w \leq 0} \); see Definition 1.2.2(7)).

Proposition 3.1.2. 1. Let \( C' \) be an idempotent extension of \( C \) such that \( w \) extends to a weight structure \( w' \) on it. Then \( C'_{w \geq 0} \) (resp. \( C'_{w' \leq 0} \), resp. \( C'_{w' = 0} \)) is the retraction-closure of \( C_{w \geq 0} \) (resp. \( C_{w' \leq 0} \), resp. \( C_{w' = 0} \)) in \( C' \). In particular, \( w \) is the restriction of \( w' \) to \( C \).

2. Any \((C, w)\) possesses a weight-Karoubian extension.

Proof. 1. This is Theorem 2.2.2(I.1) of ibid.

2. The statement is given by part III.1 of loc. cit. \qed

Now we extend Theorem 2.3.4(II) to not (necessarily) weight-Karoubian weighted categories.

Theorem 3.1.3. Let \( M \in \text{Obj} \ C \).

I. The following conditions are equivalent.

1. \( M \) is weight-degenerate (resp. \( t(M) \in K(Hw)_{w' \leq 0} \)).

3Recall that (according to Theorem 1.5 of [BaS01]) the category \( \text{Kar}(C) \) can be naturally endowed with the structure of a triangulated category so that the natural embedding functor \( C \to \text{Kar}(C) \) is exact. Hence \( C' \) is an idempotent extension of \( C \) if and only if any object of \( C' \) is a retract of some object of \( C \).
2. $M$ can be presented as an extension of a left $w'$-degenerate object of $C$ by a right $w'$-degenerate one (resp. by an element of $C'_{w' \leq 0}$) in some weight-Karoubian extension $(C', w')$ of $(C, w)$.

3. Such a decomposition of $M$ exists in any weight-Karoubian extension of $C$.

4. $M$ is a $C$-retract of an extension $M'$ of a left $w$-degenerate object of $C$ by a right weight-degenerate one (resp. $M$ is essentially $w$-negative in the sense of Definition 3.1.1(4)).

5. The object $M \oplus M[-1]$ is an extension a left $w$-degenerate object of $C$ by a right $w$-degenerate one (resp. by an element of $C_{w \leq 0}$).

6. $M$ is without weight $i$ for all $i \in \mathbb{Z}$ (resp. for all $i > 0$).

7. $H_i(M) = 0$ for all $i \in \mathbb{Z}$ (resp. for all $i > 0$) and pure homological $H$.

8. $H^i_A(M) = \{0\}$ (see Theorem 2.4.2(3) for the notation) for all $i \in \mathbb{Z}$ (resp. for all $i > 0$) and all additive functors $A : Hw^{op} \to \text{Ab}$ that respect products.

II. The following conditions are equivalent as well.

1. $t(M) \in K(Hw)_{w' \geq 0}$.

2. $M$ is without weight $i$ for all $i < 0$.

3. $M$ can be presented as an extension of an element of $C'_{w' \geq 0}$ by a right weight-degenerate object in some weight-Karoubian extension $C'$ of $C$.

4. Such a decomposition of $M$ exists in any weight-Karoubian extension of $C$.

5. $M$ is essentially $w$-positive in the sense of Definition 3.1.1(4).

6. The object $M \oplus M[1]$ is an extension of an element of $C_{w \geq 0}$ by a right $w$-degenerate object.

7. $H_i(M) = 0$ for all $i < 0$ and pure homological $H$.

8. $H^i_A(M) = \{0\}$ for all $i < 0$ and all additive functors $A : Hw^{op} \to \text{Ab}$ that respect products.
Proof. We will only prove assertion II; the proof of assertion I is similar.

Clearly, condition 3 of the assertion implies condition 5. 3 follows from 4 since a weight-Karoubian extension \((\mathcal{C}', w')\) of \(\mathcal{C}\) exists (see Proposition 3.1.2(2)).

Condition 1 is easily seen to be equivalent to condition 2 according to Theorem 2.3.1(3,1) combined with Lemma 1.3.2(4,6). Moreover, condition 1 is equivalent to \(t(M) \sim [i, i, 0]_{w} \) for all \(i > 0\) according to Lemma 1.3.2(4,6); thus applying Proposition 2.4.4(I) we obtain that this condition is also equivalent to conditions 7 and 8.

Next we note that (for any weight-Karoubian extension \(\mathcal{C}'\) of \(\mathcal{C}\) and a fixed \(M\)) \(t(M) \in K(Hw)_{w'\geq 0}\) if and only if it belongs to \(K(Hw')_{w'\geq 0}\); see Lemma 1.3.2(3,??,6). Hence condition 5 implies condition 1.

Now we fix some \((\mathcal{C}', w')\) and recall that (the conclusion of) Theorem 2.3.4(II.2) can be applied to \(\mathcal{C}'\). Hence condition 1 implies condition 4.

It remains to deduce condition 6 from condition 3. Any \(N' \in \text{Obj } \mathcal{C}'\) is the image of an idempotent \(p \in \mathcal{C}(N, N)\) for some \(N \in \text{Obj } \mathcal{C}\) (see 3.1.1), and \(\text{Cone}(p) \cong N' \bigoplus N'[1] \in \text{Obj } \mathcal{C}\) (cf. Lemma 2.2 of [Tho97]). Hence the direct sum of the \(\mathcal{C}'\)-"decomposition" of \(N\) given by condition 3 with its shift by [1] yields condition 6.

Remark 3.1.4. 1. Clearly, the formulation of conditions I.8 and II.8 can be combined with Proposition 2.4.4(II) to obtain the following statement: if \(\mathcal{C}\) satisfies the Brown representability property and \(w\) is smashing then an object \(M\) is \(w\)-degenerate (resp. essentially \(w\)-negative, resp. essentially \(w\)-positive) if and only if \(\text{H}^i(M) = \{0\}\) for any pure representable \(H\) and any \(i \in \mathbb{Z}\) (resp. any \(i > 0\), resp. any \(i < 0\)).

Note also that conditions I.7, I.8, II.7, and II.8 of our theorem can be dualized in the obvious way.

2. Recall that [BoS18] contains much information on idempotent extensions of \(\mathcal{C}\) such that \(w\) extends to them. In particular, the (essentially) minimal weight-Karoubian extension \(\mathcal{C}'\) of \(\mathcal{C}\) was described as the smallest (strict) triangulated subcategory of \(\text{Kar}(\mathcal{C})\) that contains both \(\mathcal{C}\) and \(\text{Kar}(Hw)\). Since it is minimal, applying conditions I.8 and II.8 of Theorem 3.1.3 in this \(\mathcal{C}'\) gives the maximal possible amount of information on \(M\).

3. Moreover, in §3.1 of ibid. it was demonstrated that a weight structure \(w\) does not necessarily extend to the (whole) category \(\text{Kar}(\mathcal{C})\); thus idempotent extensions are necessary for our arguments.

Now let us prove a few results closely related to our theorem.

**Proposition 3.1.5.** Let \(w\) be a weight structure on \(\mathcal{C}\).

1. Assume that \(w\) is left (resp. right) non-degenerate.
Then any weight-degenerate object of $C$ is right (resp. left) $w$-degenerate, and any essentially $w$-negative (resp. essentially $w$-positive) object belongs to $C_{w \leq 0}$ (resp. to $C_{w \geq 0}$).

2. Assume that $w$ is left non-degenerate, an object $M$ of $C$ is weight-degenerate, and either $M$ is $w$-bounded below or $w$ is also right non-degenerate. Then $M$ is zero.

3. For an object $M$ of $C$ assume that $t(M) \cong t_1 = (M_i^1) \cong t_2 = (M_i^2)$ (in the categories $K_m(Hw)$ and $K(Hw)$; see Lemma 1.3.2). Let $C'$ be a triangulated category endowed with a weight structure $w'$; let $F : C \rightarrow C'$ be a weight-exact functor (with respect to $w, w'$), and assume that $F$ annihilates the groups $C(M_1^i, M_2^j)$ for all $i \in \mathbb{Z}$.

Then $F(M)$ is $w'$-degenerate.

4. Assume that $C_1$, $C_2$, and $C_3$ are full triangulated subcategories of $C$ such that $w$ restricts to them (see Definition 1.2.1.0), and suppose that the Verdier localization functor $F : C \rightarrow C' = C/C_3$ exists (i.e., all morphism classes in this localization are sets) and all $Hw$-morphisms between elements of the corresponding classes $C_{1,w_1=0} \rightarrow C_{1,w_1=0} \rightarrow C_{2,w_2=0}$ are killed by $F$.

Then there exists a weight structure $w'$ on $C'$ such that $F$ is weight-exact, and for any $M \in \text{Obj} C_1 \cap \text{Obj} C_2$ the object $F(M)$ is weight-degenerate. Moreover, if $w'$ is left non-degenerate then $F(M)$ is $w'$-right degenerate, and if we assume in addition that $M$ is $w$-bounded below then $M$ belongs to $\text{Kar}_{C_2}(\text{Obj} C_3)$.

**Proof.** 1. This is an easy consequence of the axiom (i) of Definition 1.2.1 along with the following fact that follows from this axiom immediately: retracts of left and right $w$-degenerate objects are left and right $w$-degenerate, respectively.

2. According to the previous assertion, $M$ is right weight-degenerate, i.e., it belongs to $\cap_{i \in \mathbb{Z}} C_{w \leq i}$. On the other hand, $M$ belongs to $C_{w \geq i+1}$ for some $i \in \mathbb{Z}$ (and it actually belongs to all of these classes in the second case by the previous assertion). Since $C_{w \leq i} \perp C_{w \geq i+1}$, $M \perp M$; hence $M = 0$.

3. Let $m$ be a $K_m(Hw)$-isomorphism $t_1 \rightarrow t_2$. Then for the functor $K_m(HF) : K_m(Hw) \rightarrow K_m(Hw')$ given by Lemma 1.3.2 we have $K_m(HF)(m) = 0$. Since $K_m(HF)(m)$ is also an isomorphism, we obtain $K_m(HF)(t_1) = 0$. On the other hand, by Proposition 1.3.3.1 we have $t_{w'}(F(M)) \cong K_m(HF)(t(M)) \cong K_m(HF)(t_1)$; hence $t_{w'}(F(M)) = 0$, as desired.

4. $w'$ exists according to Proposition 8.1.1(1) of [Bon10]. Next, Proposition 1.3.4 gives the existence of complexes $t_1$ and $t_2$ in $K_m(Hw)$ such that $t(M) \cong t_1 \cong t_2$, the terms $M_i^1$ belong to $C_{1,w_1=0}$, and the terms of $t_2$ belong to $C_{2,w_2=0}$. Thus applying assertion 3 we obtain that $F(M)$ is $w'$-degenerate.
indeed.

By assertion 1 it follows that $F(M)$ is right $w'$-degenerate whenever $w'$ is left non-degenerate. Lastly, if $M$ is weight-bounded below then $F(M)$ also is. Hence our assumptions imply that $F(M) = 0$ according to assertion 2. Thus $M$ belongs to Kar$_C$(Obj$_C$)$^3$ (here we apply a well-known property of Verdier localizations; see Lemma 2.1.33 of [Nee01]).

**Remark 3.1.6.** Now let us describe an application of this proposition that is crucial for [Bon18a] cf. also §4.3 of [BoS19].

Assume that $w$ is the weight-structure purely compactly generated by a subcategory $H$ of $C$ in the sense of Theorem 4.1.2 below (in particular, $C$ and $H$ satisfy the assumptions of the theorem), and assume that $C_1$, $C_2$, and $C_3$ are generated by the corresponding subcategories $H_1$, $H_2$, and $H_3$ of $H$ as localizing subcategories of $C$ (see Definition 4.1.1(3) below), respectively. As demonstrated in (Proposition 1.8 and the proof of Proposition 1.9 of) ibid., if all morphisms between $H_1$ and $H_2$ factor through $H_3$ then all the assumptions of Proposition 3.1.5(4) are fulfilled, and the corresponding weight structure $w'$ is left non-degenerate. Moreover, $C_3$ is Karoubian (see Remark 2.3.5(3)); hence if $M$ is $w$-bounded below and belongs to Obj$_{C_1} \cap$ Obj$_{C_2}$, then $M$ is an object of $C_3 = \text{Kar}_C C_3$.

Thus our Proposition 3.1.5 gives all the necessary prerequisites for the proof of Proposition 1.9 of ibid. (that is critically important for the whole paper).

### 3.2 An application to the conservativity of weight-exact functors

Our results imply that certain weight-exact functors are ("almost") conservative; note that this statement does not mention weight complexes.

**Proposition 3.2.1.** Let $C$ and $C'$ be triangulated categories endowed with weight structures $w$ and $w'$, respectively; let $F : C \to C'$ be a weight-exact functor.

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4Since [Bon18a] was written earlier than the current paper, it actually refers to a previous version of this text. However, the exposition in the current version is more accurate. Note also that the most recent version of the theory of weight complexes (that was applied in ibid. as well) is currently contained in [Bon18b].

5Actually, in ibid. the subcategories $H, H_1, H_i$, and $H_3$ were assumed to be additive; yet this assumption is easily seen not to be actual for the purposes of that paper. Note also that the only part of the proof of [Bon18a, Proposition 1.9] that we apply in this remark is that the "factorization condition" for the categories $H_i$ ($i = 1, 2, 3$) implies the one for the classes $C_{w_i = 0}$, and this implication is an easy consequence of [Bon18a, Proposition 1.8] (cf. also Theorem 4.1.2(3) below).
Assume that the induced functor \( HF : HW \to HW' \) is full, any \( HW \)-endomorphism killed by \( HF \) is nilpotent, and for some \( M \in \text{Obj } C \) the object \( F(M) \) belongs to \( C'_{w' \notin [m,n]} \) for some \( m \leq n \in \mathbb{Z} \) (resp. \( F(M) \) is \( w' \)-degenerate, resp. \( F(M) \) is essentially \( w' \)-positive, resp. essentially \( w' \)-negative).

Then \( M \) is without weight \( m, \ldots, n \) (resp. \( M \) is weight-degenerate, resp. essentially \( w \)-positive, resp. essentially \( w \)-negative).

**Proof.** We start from proving the first statement in our proposition for \( m = n \), i.e., we assume that \( F(M) \) is without weight \( m \). We should prove that this assumption implies that \( M \) is without weight \( m \) as well; we will call this implication Claim (*).

According to Theorem 2.3.1(3) this claim is equivalent to the following one: \( t'(\text{id}_M) \sim_{[-m,-m]} 0 \) whenever \( t_w'(\text{id}_{F(M)}) \sim_{[-m,-m]} 0 \).

Now, we can assume that \( t_w'(F(M)) \) is obtained from the weight complex \( t(M) = (M') \) (whose boundary morphisms will be denoted by \( d' \)) by means of termwise application of \( F \); see Proposition 1.3.4(4). Thus there exist morphisms \( h' \in HW'(\mathcal{F}(M^{-m}), \mathcal{F}(M^{-m-1})) \) and \( j' \in HW'(\mathcal{F}(M^{1-m}), \mathcal{F}(M^{-m})) \) such that \( \text{id}_{F(M^{-m})} = j' \circ \mathcal{F}(d^{-m}) + \mathcal{F}(d^{-m-1}) \circ h' \). Since the restriction of \( F \) to \( HW \) is full and conservative, we can lift \( h' \) and \( j' \) to some \( HW \)-morphisms \( h \) and \( j \), and for any such lifts the endomorphism \( \varepsilon = \text{id}_{M^{-m}} - j \circ d^{-m} - d^{-m-1} \circ h \) of \( M^{-m} \) is nilpotent. Hence there exists \( n > 0 \) such that \( \text{id}_{M^{-m}} = (j \circ d^{-m} + d^{-m-1} \circ h)(\text{id}_{M^{-m}} + \varepsilon + \varepsilon^2 + \cdots + \varepsilon^{n-1}) \). Therefore \( \text{id}_{M^{-m}} \) can be presented in the form \( a \circ d^{-m} + d^{-m-1} \circ b \) for some \( a \in HW(M^{-m}, M^{-m-1}) \) and \( b \in HW(M^{1-m}, M^{-m}) \) (one can just write down explicit formulas for \( a \) and \( b \) in this setting). Thus \( t'(\text{id}_M) \sim_{[-m,-m]} 0 \).

Next, the general case of the "without weights \( m, \ldots, n \) part" follows from Claim (*) immediately according to Theorem 2.2.1(3).

Lastly, our remaining statements follow from Claim (*) as well according to Theorem 3.1.3 see conditions I(6) and II(2) in it. \( \square \)

**Remark 3.2.2.** 1. Our proposition essentially says that \( F \) is "conservative (and detects weights; cf. Remark 1.5.3(1) of [BSS18]) up to weight-degenerate objects". The latter feature of the result is unavoidable. Indeed, arguing similarly to Proposition 4.2.1(1) of [BSS19] one can easily prove that for any set \( W \) of weight-degenerate objects of \( C \) the localization of \( C \) by the triangulated subcategory generated by \( W \) gives a weight-exact functor that restricts to a full embedding of \( HW \) (as well as of the subcategory \( C^{wb} \) of \( w \)-bounded objects) into \( C' \).

Another example is given by singular homology. It is well-known to give an exact functor \( SH \to D(\text{Ab}) \) whose kernel consists of acyclic spectra; hence this kernel is non-zero (see Theorem 16.17 of [Mar83]).
Next, this functor is weight-exact and gives an equivalence of hearts if we take \( w \) to be the weight structure purely compactly generated by the sphere spectrum \( S^0 \in \text{Obj} \mathcal{S} \mathcal{H} \) and \( w' \) to be purely compactly generated by \( \mathbb{Z} \in \text{Obj} \mathcal{D}(\text{Ab}) \); see Theorems 4.1.2(2,3,5) and 4.2.3(1,2) below.

2. Now let us discuss certain motivic examples to our proposition.

Firstly, we recall that "any reasonable version" of the category \( DM \) of Voevodsky motives over a perfect field \( k \) contains the corresponding connective subcategory Chow (of Chow motives) whose objects are compact in \( DM \) (see Definition 4.1.1 below and Remark 3.1.2 of [BoK18]). Moreover, the category Chow compactly generates \( DM \) whenever the characteristic of \( k \) is either zero or invertible in the coefficient ring of these categories. Applying Theorem 4.1.2 below one obtains a certain weight structure \( w_{\text{Chow}} \) on \( DM \) whose heart consists of retracts of coproducts of Chow motives. Moreover, this weight structure restricts (see Definition 1.2.2(6)) to the subcategory \( DM_{gm} \) of geometric motives (this is the smallest retraction-closed triangulated subcategory of \( DM \) containing Chow) as well as to the bigger subcategory \( DM_{\text{Chow}} \) of \( DM \) whose objects are those \( M \in \text{Obj} DM \) whose weight complex \( t(M) \) is \( K_{\mathbb{m}}(Hw_{\text{Chow}}) \)-isomorphic to an object of \( K_{\mathbb{m}}(\text{Chow}) \), and the heart of both of these restrictions is Chow; see Remarks 3.1.4(1) and 3.3.2(1) of [Bon18b] for more detail.

Let us now describe an important weight-exact motivic functor essentially originating from [Ayo18]. In ibid. the case \( \text{char} \ k = 0 \) and a certain triangulated category \( DA^{et}(k,k) \) of étale motives was considered; \( DA^{et}(k,k) \) is a \( k \)-linear version of \( DM \). Moreover, the truncated de Rham spectrum \( \tau_{>0} \Omega \) (see Theorem II of ibid.) was taken; note that this object \( \Omega' \) of \( DA^{et}(k,k) \) it is a highly structured ring spectrum with respect to the model structure on \( DA^{et}(k,k) \) that was considered in ibid. Thus one can take \( C' \) to be the derived category of highly structured \( \Omega' \)-modules in \( DA^{et}(k,k) \); there is a natural "free module" functor \( F' = - \otimes \Omega' : DA^{et}(k,k) \rightarrow C' \). Moreover, the images of Chow motives in \( C' \) yield a weight structure \( w' \) on this category, the heart of \( w' \) is the formal coproductive hull (see Definition 4.1.1(2) below) of the category of \( k \)-linear motives up to algebraic equivalence, and \( F' \) is weight-exact with respect to these weight structures (see Remark 3.3.2(1) of [Bon18b]). Since an endomorphism of Chow motives that is algebraically equivalent to zero is nilpotent according to Corollary 3.3 of [Voe95], the restriction of \( F' \) to the subcategory
$DM^{Chow} = DA^{eff}(k,k)^{Chow}$ satisfies the assumptions of our proposition.

3. Now let us compare our proposition with similar results of [Wil18] and [Bon18b].

In Theorem 1.5.1(1,2) of [Bon18b] only the case where $M$ is bounded either above or below was considered. On the other hand, $HF$ was just assumed to be (full and) conservative. So, neither our proposition implies loc. cit. nor the converse is valid. Note also that in the case where the endomorphisms killed by $HF$ are nilpotent all the conclusions of loc. cit. can be easily deduced from our proposition.

Next we recall that in Theorem 2.8 of [Wil18] (as well as in the weaker Theorem 2.5 of ibid.) it was assumed (if one uses our notation) that $F$ is weight-exact, $HF$ is full and conservative, $w$ is bounded (i.e., any object of $C$ is $w$-bounded both above and below), $Hw$ is Karoubian and semi-primary. Now, these assumptions imply that endomorphisms killed by $HF$ are nilpotent; indeed, the conservativity of $HF$ means that these endomorphisms belong to the radical (morphism ideal) of $Hw$, and semi-primality means that all elements of this radical are nilpotent. Thus our proposition implies Theorem 2.8 of ibid. (cf. Theorem 1.5.1(1) of [Bon18b]).

4. One can also obtain plenty of examples to our proposition by taking $F = K(G) : K(B) \to K(B')$; here $G : B \to B'$ is a full additive functor such that any $B$-endomorphism killed by $G$ is nilpotent, and one takes $w$ and $w'$ to be the corresponding stupid weight structures. Note that in this case we have $HF \cong \text{Kar}(G)$ (see Remark 1.2.3(1)); thus $HF$ fulfills our assumptions (as well).

Since $w$ is non-degenerate, our proposition implies that $F$ is conservative. The author wonders whether a proof of this statement "without killing weights" exists; this may allow to modify the assumptions on $HF$ in Proposition 3.2.1.

3.3 Some counterexamples in the non-weight-Karoubian case

Our examples are rather simple; their main "ingredient" is $K(L - \text{vect})$ (the homotopy category of complexes of finite dimensional $L$-vector spaces; here $L$ is an arbitrary fixed field).

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[6]It is an interesting question whether it suffices to assume that $HF$ is full and conservative in our proposition.
3.3.1 An "indecomposable" weight-degenerate object

Let us demonstrate that Theorem 2.3.4(II) and Theorem 2.3.1(4) do not extend to arbitrary (i.e., to not necessarily weight-Karoubian) triangulated categories; in particular, retracts cannot be avoided when one defines essentially $w$-negative and essentially $w$-negative objects.

Our example will be the full subcategory $C$ of $(K^b(L-\mathrm{vect}))^3$ consisting of objects whose "total Euler characteristic" is even (i.e., the sum of dimensions of all homology of all the three components of $M = (M_1, M_2, M_3)$ should be even). We define $w$ for $C$ as follows: $C_{w \leq 0}$ consists of those $(M_1, M_2, M_3)$ such that $M_1 \cong 0$ and $M_2$ is acyclic in negative degrees; $(M_1, M_2, M_3) \in C_{w > 0}$ if $M_3 \cong 0$ and $M_2$ is acyclic in positive degrees. This is easily seen to be a weight structure; indeed, a weight decomposition of an object $(M_1, M_2, M_3)$ is given by any triangle of the form $(0, M', M_3) \rightarrow (M_1, M_2, M_3) \rightarrow (M_1, M'', 0)$, where $M' \rightarrow M_2 \rightarrow M''$ is a $w$-decomposition of $M_2$ with the corresponding parities of the Euler characteristics. Next, one can easily see that the object $M = (L, 0, L)$ (here we put the $L$'s in degree 0 though the degrees actually do not matter) is weight-degenerate (since it is weight-degenerate in the obvious extension of $w$ to its weight-Karoubian extension $C' = (K^b(L-\mathrm{vect}))^3$; see Proposition 1.3.4(4)). Yet $M$ clearly cannot be presented as an extension of a left $w$-degenerate object (i.e., of an object whose last two components are zero) by an element of $C_{w \leq 0}$ (since the corresponding "decomposition" in $C'$ is unique and its "components" have odd "total Euler characteristics"). Thus we obtain that both part of Theorem 2.3.4(II) do not extend to $C$.

Looking at the proof Theorem 2.3.4(II) one immediately obtains the existence of $n > 0$ such that there does not exist a triangle $X_n \rightarrow M \rightarrow Y_n$ with $X_n \in C_{w \leq -n}$, $Y_n \in C_{w \geq n}$. Moreover, one can easily check directly that a triangle of this sort does not exist for $n = 1$ already.

This example also demonstrates that one has to assume that $C$ is weight-Karoubian in Theorem 2.3.1(4). Indeed, for our object $M$ we have $t(M) = 0$ since $M$ is weight-degenerate.

3.3.2 A bounded object that is without weight $0$ but does not possess a decomposition avoiding this weight

The example above demonstrates that Theorem 2.2.1(10) does not extend to arbitrary $(C, w)$ (i.e., that our definition of objects without weights $m, \ldots, n$ is not equivalent to Definition 1.10 of [Wil09] in general). Yet the weight structure is degenerate in this example. Now we give a bounded example of the non-equivalence of definitions (i.e., all objects of $C$ are bounded both above and below; this condition is easily seen to imply that $w$ is non-
Denote by $B$ the category of even-dimensional vector spaces over $L$; take $C = K^b(B)$, 

$$M = \cdots \to 0 \to L^2 \to L^2 \to L^2 \to 0 \to \cdots;$$

we put the non-zero vector spaces in degrees $-1, 0, 1$, respectively. Clearly, the composition $(0 \to L^2 \to L^2) \to M \to (L^2 \to L^2 \to 0)$ is zero (here we consider the corresponding stupid truncations of $M$). Thus $M$ is without weight $0$ (see condition 1 in Proposition 2.1.1). Yet $M$ does not possess a decomposition avoiding weight 0 since the $L$-Euler characteristics of the corresponding $X$ and $Y$ cannot be odd.

Obviously, this example also yields that decompositions avoiding weights $m, \ldots, n$ do not "lift" from a (weight-)Karoubian $C'$ (in our case $C' = K^b(L\text{-}\text{vect})$; the corresponding weight structure is the stupid one) to $C$ (cf. Remark 2.2.2(3)).

4 On "topological" examples and converse Hurewich theorems

In this section we discuss the applications of our results to equivariant stable homotopy categories (as well as to general purely compactly generated weight structures). We significantly extend the main results of [Bon18b, §4].

In §4.1 we recall some properties of (smashing) weight structures generated by sets of compact objects in their hearts and apply the main results of the previous sections to this setting. We also study a certain cohomological dimension 1 case separately.

In §4.2 we apply our results to prove some new properties of the spherical weight structure $w^G$ on the equivariant stable homotopy category $SH(G)$ of $G$-spectra ($w^G$ was introduced in [Bon18b]; actually, we work in a somewhat more general context). We obtain a certain improvement of the natural Hurewicz-type theorem in this setting. In particular, in the case $G = \{e\}$ (and so, $SH(G) = SH$) we give an if and only if condition for the vanishing of the negative singular homology groups of a spectrum. Moreover, our theory of objects without weights in a range gives certain canonical "decompositions" of spectra whose singular homology vanishes in two subsequent degrees (see Theorem 4.2.3(5) and Remark 4.2.2(2) below).
4.1 On purely compactly generated weight structures

In this section we will always assume that $C$ is closed with respect to (small) coproducts. Let us recall a class of smashing weight structures that was studied in (§3.2 of) [Bon18b] and (§2.3 of) [BoS19].

Definition 4.1.1. Let $H$ be a full subcategory of a $C$.

1. We will say that the subcategory $H$ is connective (in $C$) if Obj $H \perp (\cup_{i>0} \text{Obj}(H[i]))$.\footnote{In earlier text of the author connective subcategories were called negative ones. Moreover, in several papers (mostly, on representation theory and related matters) a connective subcategory satisfying certain additional assumptions was said to be silting; this notion generalizes the one of tilting.}

2. Let $H'$ be the category of "formal coproducts" of objects of $H$, i.e., the objects of $H'$ are of the form $\bigsqcup_i P_i$ for (families of) $P_i \in \text{Obj} H$, and $H'\left(\prod_i M_i, \prod_j N_j\right) = \prod_i \left(\bigoplus_j H(M_i, N_j)\right)$. Then we will call $\text{Kar}(H')$ the formal coproductive hull of $H$.

3. We will say that a full triangulated subcategory $D \subset C$ is localizing whenever $D$ closed with respect to $C$-coproducts. Respectively, we will call the smallest localizing subcategory of $C$ that contains a given class $\mathcal{P} \subset \text{Obj} C$ the localizing subcategory of $C$ generated by $\mathcal{P}$.

4. An object $M$ of $C$ is said to be compact if the functor $H^M = C(M, -) : C \to \text{Ab}$ respects coproducts.

5. We will say that $C$ is compactly generated by $\mathcal{P} \subset \text{Obj} C$ if $\mathcal{P}$ is a set of compact objects that generates it as its own localizing subcategory.

Now let us recall the definition and the main properties of purely compactly generated weight structures; these are the ones that we will now describe.

Theorem 4.1.2. Let $H$ be a connective subcategory of $C$ such that $\mathcal{P} = \text{Obj} H$ compactly generates $C$ (so, $H$ is small and its objects are compact in $C$). Then the following statements are valid.

1. $C$ is Karoubian and satisfies the Brown representability property (see Definition 2.4.3).

2. There exists a unique smashing weight structure $w$ on $C$ such that $\mathcal{P} \subset C_{w=0}$, and $w$ is left non-degenerate.
3. For this weight structure $C_{w \leq 0}$ (resp. $C_{w \geq 0}$) is the smallest subclass of $\text{Obj} C$ that is closed with respect to coproducts, extensions, and contains $P[i]$ for $i \leq 0$ (resp. for $i \geq 0$), and $Hw$ is equivalent to the formal coproductive hull of $H$ (see Definition 4.1.1(1)) in the obvious way.

Moreover, $C_{w \geq 0} = (\cup_{i<0} P[i])^{\perp}$.

4. Let $H$ be a cohomological functor from $C$ into an abelian category $A$ that converts all small coproducts into products. Then it is pure if and only if it kills $\cup_{i \neq 0} P[i]$.

5. Let $F : C \to D$ be an exact functor that respects coproducts, where $D$ is a triangulated category endowed with a smashing weight structure $v$. Then $F$ is weight-exact if and only if it sends $P$ into $D_{v=0}$.

6. The category $Ht \subset C$ of $w$-pure representable functors from $C$ (so, we identify an object of $Ht$ with the functor from $C$ that it represents) is equivalent to the category $A_{P}$ of additive contravariant functors from $H$ into $\text{Ab}$ (i.e., we take those functors that respect the addition of morphisms)

Moreover, $A_{P}$ (and so also $Ht$) is Grothendieck abelian, has enough projectives, and an injective cogenerator; we will choose one and write $I$ for it.

Furthermore, restricting functors representable by elements of $P$ to $Hw$ one obtains a fully faithful functor $A_{P} : Hw \to A_{P}$, whose essential image is the subcategory of projective objects of $A_{P}$.

7. The following assumptions on an object $M$ of $C$ are equivalent.

(i). $t(M) \in K(Hw)_{w^{st} \geq 0}$.

(ii). $H^{j}_{A_{P}}(M) = 0$ for $j < 0$ (here we use the notation from Theorem 2.4.2(3)).

(iii). $M \perp (\cup_{j<0} \{I[j]\})$.

8. Assume that there exists an integer $j > 0$ such that $P \perp \cup_{i \geq j} P[-i]$. Then $w$ is non-degenerate.

Proof. Assertions 1–7 were mostly established in [Nee01] and [BoS19]; see §3.2 of [Bon18b] for the detail.

According to Proposition 4.3.3 of [Bon18b], the category $Ht$ is actually the heart of a $t$-structure on $C$; whence the notation.
Next, if $\mathcal{P} \perp (\bigcup_{i \geq j} \mathcal{P}[-i])$ then the description of $\mathcal{C}_{w \geq 0}$ in assertion \ref{item:3} (along with the compactness of elements of $\mathcal{P}$) implies that $\bigcup_{i \geq j} \mathcal{P}[i] \perp \mathcal{C}_{w \leq 0}$. Thus any right weight-degenerate element of $\mathcal{C}$ belongs to $(\bigcup_{i \in \mathbb{Z}} \mathcal{P}[i])^\perp$. Now, the latter class is zero since $\mathcal{C}$ is compactly generated by $\mathcal{P}$ (see the well-known Proposition 8.4.1 of \cite{Nee1}); hence $w$ is right non-degenerate.

Lastly, $w$ is also left non-degenerate according to assertion \ref{item:2}.

\begin{remark}
1. There exist nice families of examples for part \ref{item:8} of our theorem. Firstly one can consider the derived category of a small differential graded category $\mathcal{B}$ (note that this construction gives all algebraic triangulated categories; see Theorem 3.8 of \cite{Kel06}) such that for certain $j < 0$ we have $\mathcal{B}^j(M, N) = \{0\}$ whenever $M, N \in \text{Obj} \mathcal{B}$ and $i > 0$ or $i < j$ (see loc. cit.). One can also consider the derived category of $E$-modules where $E$ is a commutative $S$-algebra in the sense mentioned in Example 1.2.3(f) of \cite{HPS97}; then one has to assume that $\pi_i(E) = \{0\}$ if $i < 0$ (see §7 of ibid. and Remark 4.3.4(2) of \cite{Bon18b}) and if $i \gg 0$.

Thus one may say that the non-degeneracy of a purely compactly generated weight structure is a certain finiteness of the cohomological dimension condition. It may can be quite actual for representation theory (since it implies the conservativity of the weight complex functor and of certain weight-exact functors; see Theorem 2.3.11(I.1) and Proposition 3.2.1).

2. On the other hand, the spherical weight structure on $\mathcal{S}H$ is right weight-degenerate (see Remark \ref{item:1} below). Moreover, the Chow weight structure for $\mathcal{D}M$ (as mentioned in Remark \ref{item:2} below) is right weight-degenerate as well (at least) if the the base field is "large enough"; see Proposition 3.2.6 and Remark 3.2.7(2) of \cite{Bon18b}.

3. Proposition \ref{prop:3.1.5} easily implies that if the weight structure $w'$ as in part 3 or 4 of the proposition is non-degenerate then $F(M) = 0$ under the corresponding assumptions on $M$; in the setting of Remark \ref{item:3} it follows that $M$ belongs to $\text{Obj} \mathcal{C}_{3}$.

However, the author doubts that the condition $\mathcal{P} \perp \bigcup_{i \geq j} F(\mathcal{P})[-i]$ follows from $\mathcal{P} \perp \bigcup_{i \geq j'} F(\mathcal{P})[-i]$ (for any integer $j' > 0$).

Now let us relate purely compactly generated weight structures to the main definitions of the current paper.

\begin{corollary}
Adopt the notation and the assumptions of Theorem \ref{thm:4.1.2}.

1. The class of essentially $w$-positive objects coincides with $\bigcup_{j < 0} \{I[j]\}$.

It is also characterized by the vanishing of $H^A_{\mathcal{P}}(-)$ for $j < 0$.

2. The class of $w$-degenerate objects coincides with $\bigcup_{j \in \mathbb{Z}} \{I[j]\}$ and also with $\bigcup_{j \in \mathbb{Z}} \{\text{Obj} H\mathcal{P}[j]\}$. Moreover, this class is characterized by the vanishing of $H^A_{\mathcal{P}}(-)$ for all $j \in \mathbb{Z}$.

\end{corollary}
3. A $C$-morphism $g$ kills weight $m$ (for some $m \in \mathbb{Z}$) if and only if $H^m(g) = 0$ for any pure representable (cohomological) functor $H$.

4. An object $M$ of $C$ is without weights $m, \ldots, n$ for some $m \leq n \in \mathbb{Z}$ if and only if $H^j(g) = 0$ for any pure representable (cohomological) functor $H$.

5. $C_{w<0} = 1/(\cup_{j>0} \text{Obj } H(j))$. Moreover, this class is also annihilated by $H_i$ for all $i > 0$ and for any $w$-pure homological functor $H$ from $C$.

Proof. 1. According to Theorem 3.1.3(II), an object $M$ of $C$ is essentially $w$-positive if and only if $t(M) \in K(Hw)_{w^st \geq 0}$. Combining this fact with Theorem 4.1.2(7) we obtain our assertion.

2. Since $w$ is left non-degenerate (see Theorem 4.1.2(2)), an object $M$ of $C$ is essentially $w$-positive if and only if it belongs to $C_{w \geq 0}$ (see Proposition 3.1.5(1)). On the other hand, $M$ is weight-degenerate if and only if it is right weight-degenerate; hence $M$ is weight-degenerate if and only if $M[j]$ is essentially $w$-positive for all $j \in \mathbb{Z}$. Hence our assertion follows from the previous one.

3. Since $w$ is smashing and $C$ satisfies the Brown representability property (see Theorem 4.1.2(12)), the assertion follows from Proposition 2.4.4 immediately.

4. This is a straightforward consequence of the previous assertion combined with Theorem 2.3.1(13) along with Lemma 1.3.2(4).

5. Since $w$ is left non-degenerate, $C_{w<0}$ coincides with the class of all essentially $w$-negative objects according to Proposition 3.1.5(1). Thus Theorem 3.1.3(I) (along with Remark 3.1.4) gives the result in question.

Now let us demonstrate that one can say more on this setting if an additional assumption is imposed.

Theorem 4.1.5. Adopt the notation and assumptions of Theorem 4.1.2, and suppose in addition that the category $A_P$ is of projective dimension at most 1 (i.e., any its object has a projective resolution of length 1). Let $m \leq n \in \mathbb{Z}$ and $g \in C(E, E')$.

Then the following statements are valid.

1. The category $K_m(Hw)$ equals $K(Hw)$, and the natural functor $K(Hw) \to D(A_P)$ is an equivalence.

2. For any objects $C$ and $C'$ in $D(A_P)$ we have natural isomorphisms $C \cong \prod H_j(C)[j]$ and $D(A_P)(C, C') \cong \prod_{j \in \mathbb{Z}} A_P(H_j(C), H_j(C')) \oplus \prod_{j \in \mathbb{Z}} \text{Ext}^1_{A_P}(H_j(C), H_{j+1}(C')).$
3. $g$ kills weight $m$ if and only if $H_{m-1}^{A_p}(g) = 0$, the class of $t(g)$ in the group $\text{Ext}^1_{\Delta_p}(H_{m-1}(t(E)), H_m(t(E')))$. (Here we use the identification provided by the previous two assertions) vanishes, and the morphism $H_{m-1}^{A_p}(g)$ factors through a projective object of $\Delta_p$.

4. $E$ is without weights $m, \ldots, n$ (resp. $E \in C_{w \leq m-1}$) if and only if $H_j^{A_p}(E) = 0$ for $m \leq j \leq n$ (resp. for $j \geq m$) and $H_{m-1}^{A_p}(E)$ is a projective object of $\Delta_p$.

**Proof.**

1. These statements easily follow from Theorem 4.1.2(6) according to Remark 3.3.4 of [Bon10] (that mostly relies on basic properties of derived categories; cf. (4.1.1) below).

2. The first splitting statement is well-known, and one easily obtains the second one after noting that higher extension groups in the category $\Delta_p$ vanish.

3. Once again, we should check whether $t(g) \sim [-m, -m] 0$ (see Theorem 2.3.1(1))

Now we translate the splitting of $t(E)$ provided by previous assertions into

$$t(E) \cong \prod_{i \in \mathbb{Z}} \text{Cone}(A_i \xrightarrow{f_i} B_i)[i],$$

(4.1.1)

where $f_i$ are certain $Hw$-morphisms that are monomorphisms (between projective objects) in $\Delta_p$; similarly, we consider $t(E') \cong \prod_{j \in \mathbb{Z}} \text{Cone}(A'_i \xrightarrow{f'_i} B'_i)[j]$.

Obviously, all of the assumptions on $g$ considered in this assertion can be checked "summandwisely", i.e., we can and will assume that both $E$ and $E'$ are equal to single summands of this sort.

Next, all of our conditions are obviously fulfilled for these "morphisms of summands" if $\{i, j\} \not\subset \{m, m-1\}$. Moreover, we have $\text{Cone}(A_m \xrightarrow{f_m} B_m)[m] \perp \text{Cone}(A'_{m-1} \xrightarrow{f'_{m-1}} B'_{m-1})[m-1]$ (since $f'_{m-1}$ becomes monomorphic in $\Delta_p$; one can make this calculation in $D(\Delta_p)$; thus any morphism between these two objects kills weight $m$ automatically.

It remains to verify that the three remaining cases of $(i, j)$ give our three conditions on $g$.

If $(i, j) = (m, m-1)$ then the previous assertion implies that

$$K(Hw)(t(E), t(E')) \cong \text{Ext}^1_{\Delta_p}(H_{m-1}^{A_p}(E), H_m^{A_p}(E')).$$

Moreover, since $t(g) \sim [l, l] 0$ for all $l \neq -m$, $t(g)$ is zero (in $K_m(Hw) = K(Hw)$) if and only if $g$ kills weight $m$. Hence in this case $t(g)$ kills weight $m$ if and only if the class of $t(g)$ in $\text{Ext}^1_{\Delta_p}(H_{m-1}^{A_p}(E), H_m^{A_p}(E'))$ vanishes.
Next, if \( g \) kills weight \( m \) then we have \( H^A_{m}(g) = 0 \) according to Proposition 2.4.1(I). Conversely, if \((i, j) = (m, m)\) then \( K(Hw)(t(E), t(E')) \cong A_P(H^A_{m}(E), H^A_{m}(E'))\). Thus if \( H^A_{m}(g) = 0 \) then \( t(g) = 0 \) in this case; hence \( t(g) \sim_{[-m, -m]} 0 \).

It remains to consider the case \((i, j) = (m - 1, -m - 1)\). By definition, \( t(g) \sim_{[-m, -m]} 0 \) if and only if \( t(g) \) factors through the obvious morphism \( B'_{m-1}[m - 1] \to t(E') \). The latter condition clearly implies that \( H^A_{m-1}(g) \) factors through \( B'_{m-1} \). Conversely, if \( H^A_{m-1}(g) \) factors through a projective object of \( A_P \) then \( t(g) \) factors through an object of \( Hw[m - 1] \) (note that \( A_P \) embeds into \( K(Hw)_{w, st} \)). Thus \( t(g) \) factors through \( B'_{m-1}[m - 1] \) according to Proposition 1.2.4(7) (applied to \( K(Hw) \) endowed with the stupid weight structure).

Assertion 4 follows from the previous one more or less easily. Combining Theorem 3.1.3(II) (see condition 2 in it) with Proposition 3.1.5(1) we obtain that \( E \) belongs to \( C_{w, st} \) if and only if \( \text{id}_E \) kills weight \( i \) for all \( i \geq m \). Recall also that \( E \) is without weights \( m, \ldots, n \) if and only if \( \text{id}_E \) kills weight \( i \) for \( m \leq i \leq n \) (see Theorem 2.2.1(8)). Hence it remains to note that for any \( i \in \mathbb{Z} \) the morphism \( H^A_{i}(\text{id}_E) \) clearly equals the corresponding identity, whereas the classes of \( t(\text{id}_E) \) in all the groups \( \text{Ext}^{1}_{A_P}(H_i(E), H_{i+1}(E)) \) (see assertion 2) are zero.

**Remark 4.1.6.** 1. Let us describe a motivic example to our theorem.

Inside the category \( DM \) of motives (with integral coefficients) over any perfect field (see §4.2 of [Deg11] or [BoK18, §1.3, §3.1]) one can take \( C \) to be its localizing subcategory \( DTM \) generated by the Tate motives \( \mathbb{Z}(i) \) for \( i \in \mathbb{Z} \).

Now, the category \( \mathcal{P} = \{ \mathbb{Z}(i)[2i], i \in \mathbb{Z} \} \) is connective in \( DM, \mathbb{Z}(i)[2i] \perp \mathbb{Z}(j)[2j] \) for any \( i \neq j \in \mathbb{Z} \), whereas \( DM(\mathbb{Z}(i)[2i], \mathbb{Z}(i)[2i]) \cong \mathbb{Z} \) for all \( i \); see Corollary 6.7.3 of [Bev08]. Hence we have \( A_P \cong \text{Ab}^{\text{proj}} \) in this case; thus \( A_P \) is of projective dimension 1.

Moreover, one can take any Dedekind domain or a field for the coefficient ring in this example.

2. The case \( E \in C_{w, st} \) in part 4 of our theorem is an abstract generalization of Proposition 6.16 of [Mar83] (where \( C = SH \) was considered; see Theorem 4.2.3 below). Respectively, Proposition 6.17 of loc. cit. is the corresponding case of the orthogonality axiom for the weight structure \( w^{\text{ph}} \). Note however that the methods of loc. cit. do not seem to extend to the setting of Theorem 4.2.1 (for a general \( G \)) and Theorem 4.1.2. Moreover, Proposition 7.1.2(a) of [HPS97] essentially gives the existence of weight Postnikov towers corresponding to a certain weight structure \( w \) only for \( w \)-bounded below objects (cf. Remark 4.3.4(2) of [Bon18b]).

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3. The author suspects that the functor \( t : C \to K(H\omega) \) is actually exact in this case. Most probably this conjecture can be proven similarly to Corollary 3.5 of [Sos19].

4.2 On equivariant spherical weight structures and stable Hurewicz theorems

Now let us generalize Theorem 4.1.1 of [Bon18b] to the setting of \( \Lambda \)-linear equivariant stable homotopy categories and add several new statements. We will need some notation.

- We choose a set of prime numbers \( S \subset \mathbb{Z} \); denote \( \mathbb{Z}[S^{-1}] \) by \( \Lambda \). \( S \) may be empty; then \( \Lambda = \mathbb{Z} \) (and the reader may confine herself to this important case).
- \( G \) will be a (fixed) compact Lie group; we will write \( SH(G) \) for the stable homotopy category of \( G \)-spectra indexed by a complete \( G \)-universe.
- Denote the subcategory of \( \Lambda \)-linear objects of \( SH(G) \) by \( SH_\Lambda(G) \). According to Proposition A.2.8 of [Kel12], \( SH_\Lambda(G) \) is a triangulated subcategory of \( SH(G) \) and there exists an exact left adjoint \( l_S = (-)[S^{-1}] \) to the embedding \( SH_\Lambda(G) \to SH(G) \).
- We take \( P \) to be the set of spectra of the form \( l_S(S^0_H) \), where \( H \) is a closed subgroup of \( G \) (cf. Definition I.4.3 of [LMSC86]; recall that \( S^0_H \) is constructed starting from the \( G \)-space \( G/H \)). We will write \( \underline{H} \) for the corresponding (preadditive) subcategory of \( SH_\Lambda(G) \). Respectively, if \( \Lambda = \mathbb{Z} \) then \( \underline{H} \) is the (stable) orbit category of ibid.
  
  Recall also that \( S^n_H \in \text{Obj} \ SH(G) \) is the corresponding sphere spectrum \( S^n_H \) essentially by definition (see loc. cit.).

- The equivariant homotopy groups of an object \( E \) of \( SH(G) \supset SH_\Lambda(G) \) are defined as \( \pi^n_H(E) = SH(G)(S^n_H, E) \) (for all \( n \in \mathbb{Z} \); see §I.6 and Definition I.4.4(i) of ibid.).
- We will write \( EM_G \) for the full subcategory of \( SH_\Lambda(G) \) whose object class is \( (\bigcup_{i \in \mathbb{Z} \setminus \{0\}} P[i]) \perp \) (in the case \( \Lambda = \mathbb{Z} \) the objects of \( EM_G \) are Eilenberg-MacLane \( G \)-spectra; see §XIII.4 of [May96]).
- We will write \( M_G \) for the category of additive contravariant functors from \( \underline{H} \) into \( \text{Ab} \) (cf. Theorem 4.1.2[10]); we will call its objects Mackey functors (cf. loc. cit.). Respectively, \( \mathcal{A}_P \) will denote the Yoneda embedding \( \underline{H} \to M_G \).
• We will call \( \perp (\cup_{i \in \mathbb{Z}} \text{Obj } EM_G[i]) \subset \text{Obj } SH_{\Lambda}(G) \) the class of acyclic spectra (i.e., a spectrum is acyclic if it is annihilated by \( H^i \) for all \( H \) represented by \( \Lambda \)-linear Eilenberg-MacLane spectra and \( i \in \mathbb{Z} \)).

• Let us now give references to those weight structure definitions that are necessary to understand the theorem below. The most basic of those are Definitions 1.2.1 and 1.2.2(1,2,7,8); pure functors are defined in Definition 2.4.1; weight-degenerate objects are defined in Definition 2.3.3, essentially weight-positive objects are defined in Definition 3.1.1(4); objects without weights \( m, \ldots, n \) (along with morphisms killing weights) are defined in Definition 2.1.2.

Now we demonstrate that our theory gives several interesting properties of \( SH(G) \supset SH_{\Lambda}(G) \) (and one can assume that \( S = \emptyset \), \( \Lambda = \mathbb{Z} \), and \( SH_{\Lambda}(G) = SH(G) \)). The proofs of our statements are rather easy consequences of the general theory.

**Theorem 4.2.1.** Let \( n \in \mathbb{Z}, \ h \in \mathbb{Z}(E, E') \) (for \( E \) and \( E' \) being objects of \( SH(G) \)). Then the following statements are valid.

1. One can assume that the functor \( l_S \) is identical on the subcategory \( SH_{\Lambda}(G) \) of \( SH(G) \). Moreover, for any closed subgroup \( H \) of \( G \) and an object \( C \) of \( SH(G) \) we have \( SH(G)(l_S(S^n_H), l_S(C)) = SH_{\Lambda}(G)(l_S(S^n_H), l_S(C)) \cong SH(G)(S^n_H, l_S(C)) \cong \pi^n_H(C) \otimes_{\mathbb{Z}} \Lambda \).

2. The category \( C = SH_{\Lambda}(G) \) and the class \( \mathcal{P} \) specified above satisfy the assumptions of Theorem 4.1.2. Thus \( \mathcal{P} \) gives a weight structure \( w_{\Lambda}^{\mathcal{P}} \) on \( SH_{\Lambda}(G) \) whose heart consists of retracts of coproducts of elements of \( \mathcal{P} \) and equivalent to the formal coproductive hull of \( H \).

3. The class of \( n-1 \)-connected \( \Lambda \)-linear spectra (see Definition I.4.4(iii) of [LMSC86]; i.e., this is the class \( (\cup_{i<n} \mathcal{P}[i])^{\perp} \) coincides with \( SH_{\Lambda}(G)_{w_{\Lambda}^{\mathcal{P}} \geq n} \). In particular, \( SH_{\Lambda}(G)_{w_{\Lambda}^{\mathcal{P}} \geq 0} \) is the class of \( \Lambda \)-linear connective spectra.

Moreover, \( SH_{\Lambda}(G)_{w_{\Lambda}^{\mathcal{P}} \geq 0} \) is the smallest class of objects of \( SH_{\Lambda}(G) \) that contains \( \cup_{i \geq 0} \mathcal{P}[i] \) and is closed with respect to coproducts and extensions.

4. A (co)homological functor from \( SH_{\Lambda}(G) \) into an abelian category \( \mathcal{A} \) that respects coproducts (resp. converts them into products) is \( w_{\Lambda}^{\mathcal{P}} \)-pure if and only if it kills \( \cup_{i \neq 0} \mathcal{P}[i] \).

In particular, all objects of \( EM_G \) represent \( w_{\Lambda}^{\mathcal{P}} \)-pure functors.
5. The category $EM_G$ is naturally equivalent to $M_G$ in the obvious way; thus $EM_G$ is Grothendieck abelian and has an injective cogenerator $I$.

6. The following conditions are equivalent.
   (i). $E$ is acyclic.
   (ii). $E$ is $w^G_\Lambda$-degenerate.
   (iii). $E$ is right $w^G_\Lambda$-degenerate.
   (iv). $E \perp SH_\Lambda(G)_{w^G_\Lambda \geq i}$ for any $i \in \mathbb{Z}$.
   (v). $H^A_p(E) = 0$ for any $i \in \mathbb{Z}$.
   (vi). $E \perp (\cup_{i \in \mathbb{Z}} I[i])$.
   (vii). $E$ is annihilated by any $w^G_\Lambda$-pure homological functor from $SH_\Lambda(G)$.

7. The following statements are equivalent as well.
   (i). $h$ kills weight 0.
   (ii). $H(h) = 0$ for any $w^G_\Lambda$-pure homological functor $H$ from $SH_\Lambda(G)$.
   (iii). $P(h) = 0$ for any $w^G_\Lambda$-pure cohomological functor $P$ from $SH_\Lambda(G)$.
   (iv). $P(h) = 0$ for any functor $P = SH_\Lambda(G)(-, J)$, where $J \in \text{Obj } EM_G$.

8. The following conditions are equivalent.
   (i). $E$ is essentially $w^G_\Lambda$-positive.
   (ii). $E$ is an extension of a connective spectrum by an acyclic one.
   (iii). $H_j(E) = 0$ for any $w^G_\Lambda$-pure homological functor $H$ from $SH_\Lambda(G)$ and $j < 0$.
   (iv). $P^j(E) = 0$ for any $w^G_\Lambda$-pure cohomological functor $P$ from $SH_\Lambda(G)$ and $j < 0$.
   (v). $E \perp I[j]$ for all $j < 0$.
   (vi). $H^A_j(E) = 0$ for all $j < 0$.

9. $SH_\Lambda(G)_{w^G_\Lambda \leq 0}$ is the smallest subclass of $\text{Obj } SH_\Lambda(G)$ that is closed with respect to coproducts, extensions, and contains $P[i]$ for $i \leq 0$. This class also equals $\perp (\cup_{i \geq 0} \text{Obj } EM_G[i]) = \perp SH_\Lambda(G)_{w^G_\Lambda \geq 1}$, moreover, it is annihilated by $H_i$ for all $i > 0$ and for any pure homological functor $H$ from $SH_\Lambda(G)$.

9Recall here $SH(G)_{w^G_\Lambda \geq 1}$ is the class of 0-connected $\Lambda$-linear $G$-spectra.
10. Let \( m \leq n \in \mathbb{Z} \). Then the following conditions are equivalent.

(i). \( E \) avoids weights \( m \ldots n \).

(ii). There exists a distinguished triangle \( E_1 \to E \to E_2 \to E_1[1] \) such that \( E_1 \in \text{SHA}(G)_{w_G \leq m-1} \) and \( E_2 \in \text{SHA}(G)_{w_G \geq n+1} \). Moreover, if this is the case then this triangle is canonically determined by \( E \).

(iii). \( E \perp (\cup_{m \leq i \leq n} \text{Obj EM}_{G}[-i]) \).

(iv). \( E \) is annihilated by \( H_i \) whenever \( m \leq i \leq n \) and \( H \) is a pure homological functor from \( \text{SHA}(G) \).

Proof.  

1. \( l_S \) can be assumed to be identical on \( \text{SHA}(G) \) since it is left adjoint to the embedding of \( \text{SHA}(G) \) to \( \text{SH}(G) \). Next, the only non-trivial isomorphism for morphism groups in the assertion is given by Corollary A.2.13 of [Kel12].

2. The fact that objects of the form \( S^0_H \) form a connective subcategory of \( \text{SH}(G) \) that compactly generates this category can be deduced from the results of [LMSC86]; see (the proof of) [Bon18b] Theorem 4.1.1] for more detail. Thus applying the previous assertion we immediately obtain that \( H \) is connective in \( \text{SHA}(G) \). It remains to note that the functor \( l_S \) sends any family of compact generators of \( \text{SH}(G) \) into one for \( \text{SHA}(G) \) according to Proposition A.2.8 of [Kel12]. Lastly, we apply Theorem 4.1.2(3) to compute the heart of \( w_G \).

3. By definition, a \( G \)-spectrum \( N \) is \( n-1 \)-connected whenever \( \pi_i^H(N) \cong \text{SH}(G)(S_H^i, N) = \{0\} \) for all \( i < n \) and \( H \) being any closed subgroup of \( G \). Since \( \text{SH}(G)(S_H^i, N) \cong \text{SHA}(G)(l_S(S_H^i), l_S(N)) \) (see assertion 1), it remains to apply Theorem 4.1.2(3) to obtain all the statements in question.

Assertion 4 follows from Theorem 4.1.2(4) immediately. Moreover, assertion 5 is just the corresponding case of Theorem 4.1.2(6).

6. According to assertion 2 we can apply Corollary 4.1.4(2) to obtain that conditions (i), (ii), (v), and (vi) are equivalent. Next, conditions (ii) and (iii) are equivalent since \( w_G^H \) is left non-degenerate (see Theorem 4.1.2(2) and Proposition 3.1.5(1)), and applying Proposition 1.2.4(2) we obtain the equivalence of (iii) and (iv). Lastly, conditions (ii) and (vii) are equivalent by Theorem 3.1.3(1) (see condition 17 of the theorem).

7. Conditions (i), (ii), and (iv) are equivalent according to Proposition 2.4.4. Next, conditions (ii) and (iii) are equivalent since for any homological functor \( H : \text{SHA}(G) \to A \) the corresponding cohomological functor \( P \) from \( \text{SHA}(G) \) into \( A^{op} \) is pure if and only if \( H \) is.

8. Since \( \text{SHA}(G) \) is Karoubian, conditions (i) and (ii) are equivalent according to Theorem 3.1.3(II) (combined with assertions 3 and 6). Moreover, (i) is equivalent to (v) and (vi) by Theorem 4.1.2(1).
Lastly, condition (iv) clearly implies condition (v), and the obvious argument that we have just used yields that conditions (iii) and (iv) are equivalent.

9. The first of these descriptions of $\text{SH}_\Lambda(G)_{w_G \leq 0}$ is given by Theorem 4.1.2(1). Next, $\text{SH}_\Lambda(G)_{w_G \leq 0} = \perp \text{SH}_\Lambda(G)_{w_G \geq 1}$ according to Proposition 1.2.4(2). It remains to apply Theorem 1.1.2(5) to conclude the proof.

10. By definition, $E$ is without weights $m \ldots n$ if and only if $\text{id}_E$ kills these weights. Hence applying assertion 7 we obtain the equivalence of conditions (i), (iii), and (iv). Lastly, conditions (i) and (ii) are equivalent according to Theorem 2.2.1(9,10).

Remark 4.2.2. 1. Let us explain that part 8 of our theorem is a certain "unbounded improvement" of the natural Hurewicz-type theorem for this context.

We recall that the "usual" Stable Hurewicz Theorem essentially says that in the case $G = \{ e \}$ (and $\Lambda = \mathbb{Z}$; so, $\text{SH}(G) = \text{SH}$) a $w_G^\Lambda$-bounded below spectrum $E$ is connective if and only if its singular homology is concentrated in non-negative degrees. A certain equivariant version of this statement is given by Theorem 2.1(i) of [Lew92] (cf. also Theorem 1.11 of ibid. and Proposition 7.1.2(f) of [HPS97]); note that one replaces singular homology by $H^{Ap}$ in it (cf. part 4 of this remark).

Now, it is easily seen that those essentially $w_G^\Lambda$-positive objects that are $w_G^\Lambda$-bounded below are connective. Hence part 8 of our theorem naturally generalizes the aforementioned equivariant Hurewicz-type theorem to arbitrary objects of $\text{SH}_\Lambda(G)$. Our generalization depends on the notion of acyclic spectra, and the corresponding part 6 of our theorem also appears to be quite new (cf. also part 4 of this remark).

2. The notions of killing weights and avoiding weights $m, \ldots, n$ (along with parts 7, 10, and 9 of our theorem) appear to be new in this context (even when restricted to the case $G = \text{SH}$) as well. In particular, we obtain certain canonical "decompositions" of spectra (see condition 10(ii) in our theorem) by looking at their cohomology with coefficients in Eilenberg-Maclane spectra. This statement becomes especially nice when applied to $\text{SH}$; see Theorem 4.2.3(5) below.

3. Recall that if $\Lambda = \mathbb{Z}$ then the pure homological functor $H^{Ap}$ is the equivariant ordinary homology with Burnside ring coefficients functor $H^\Gamma_0$ considered in [Lew92] (cf. also Definition X.4.1 of [May96]), and for any Mackey functor $M$ the corresponding pure functor $H^M$ coincides
with $H^0(\cdot, M)$ in Definitions X.4.2 and §XIII.4 of ibid. Clearly, it follows that the functors $H^{A^p}$ and $H_M$ are closely related to these notions as well (for a general $\Lambda$ also).

4. None of the descriptions of acyclic spectra in part 3 of our theorem characterizes them "explicitly". In the case $G = \{e\}$ our definition of this notion coincides with the one considered in [Mar83]; see Theorem 4.2.3(2) below. At least, this gives an explicit example of a non-zero acyclic spectrum (see Theorem 16.17 of [Mar83]).

One can possibly say more on acyclic spectra for $G \neq \{e\}$ via considering exact functors connecting equivariant stable homotopy categories corresponding to distinct groups. Note that some of these functors are weight-exact, and one can also apply Proposition 2.4.4(II) for treating "induced" (co)homology functors.

The author conjectures that a spectrum $E$ is acyclic in $\text{SH}(G)$ if and only if $\Phi^H \circ i_H^*(E) \in \text{Obj} \text{SH}$ (where $i_H : H \to G$ runs through all inclusions of closed subgroups; see §II.4 and of Definition II.9.7 of [LMSC86]) is.

5. Theorem 4.1.5 certainly gives some more information on ("weights of") objects of $\text{SH}_\Lambda(G)$ whenever the corresponding category $\mathcal{M}_G$ is of projective dimension at most 1. So we note that this assumption is fulfilled whenever $G$ is a finite group of order $n$ and $1/n \in \Lambda$ (one should join Theorem 2.1 of [Gre92] with the finite projective dimension statement established in §6 of ibid. to obtain this fact).

6. It appears that the statements contained in [LMSC86] are actually sufficient to generalize all the assumptions of our theorem to the case where $\mathcal{C}$ is the stable homotopy category of $G$-spectra indexed on a not (necessarily) complete $G$-universe. In particular, this universe can be $G$-trivial (i.e., $G$ acts trivially on its spaces); this allows us to apply it to the corresponding representable functors as considered in §IV.1 of [Bre67].

7. Let us mention some other nice properties of $w^G$.

Firstly, it restricts to the subcategory of compact objects of $\text{SH}_\Lambda(G)$; cf. Theorem 4.1.1(2) of [Bon18b]. Next, the class $\text{SH}_\Lambda(G)_{w^G \geq 0}$ can also be described as $\text{SH}_\Lambda(G)_{t \geq 0}$ for a certain Postnikov $t$-structure $t$ (see Definition 4.3.1(1) and Proposition 4.3.3 of ibid.); yet this $t$-structure does not restrict to compact objects of $\text{SH}_\Lambda(G)$. 47
Note also that our theory provides a certain inverse Hurewicz-type theorem (and also other "decompositions" as well as several new definitions; see part 2 of this remark) for the so-called connective stable homotopy theory as discussed in §7 of [HPS97]; see Remark 4.3.4(2) of [Bon18b] for more detail.

Now we apply our results to the stable homotopy category \( SH \) (whose detailed description can be found in [Mar83]); some of these statements were already stated in (Theorem 4.2.1 of) [Bon18b]. This corresponds to the case of a trivial \( G \) and \( S \) in Theorem 4.2.1; so we will write \( EM \) for \( EM_G \) and \( w^{sph} \) for \( w^G_A \) in this case (and use the remaining notation from this theorem).

**Theorem 4.2.3.** Set \( \mathcal{P} = \{ S^0 \} \); assume \( m \leq n \in \mathbb{Z} \) and \( g : E \to E' \) is an \( SH \)-morphism.

Then the following statements are valid.

1. The functor \( SH(S^0, -) \) gives equivalences \( H^{w^{sph}} \to \text{FreeAb} \) (the category of free abelian groups) and \( EM \to \text{Ab} \); thus \( A_\mathcal{P} \) is equivalent to \( \text{Ab} \) as well.

2. The functor \( A_\mathcal{P} \) is essentially the singular homology functor \( H^{\text{sing}}_* \), respectively, acyclic spectra in \( SH \) are characterized by the vanishing of their singular homology (cf. §6.2 of [Mar83]).

3. For any abelian group \( \Gamma \) and the corresponding spectrum \( EM^\Gamma \in \text{Obj EM} \) the functor \( SH(-, EM^\Gamma) \) is isomorphic to the singular cohomology with coefficients in \( \Gamma \) one.

4. \( g \) kills weight \( m \) if and only if \( H^{\text{sing}}_m(g) = 0 \), the class of \( t(g) \) in the group \( \text{Ext}_1^{\text{Ab}}(H^{\text{sing}}_{m-1}(E), H^{\text{sing}}_m(E')) \) (see Theorem 4.1.5(3)) vanishes, and the morphism \( H^{\text{sing}}_m(g) \) factors through a free abelian group.

This condition is also equivalent to the vanishing of \( H^{\text{sing}}_{m-1}(\mathcal{P}) \) for any abelian group \( \Gamma \).

5. \( E \) is an extension of an \( n \)-connected spectrum by an element of \( SH_{w^{sph} \leq m-1} \) if and only if \( H^{\text{sing}}_j(E) = 0 \) for \( m \leq j \leq n \) and \( H^{\text{sing}}_{m-1}(E) \) is a free abelian group.

6. The following conditions are equivalent.

   (i) \( E \in SH_{w^{sph} \leq n} \);

   (ii) \( H^{\text{sing}}_i(E) = \{0\} \) for \( i > n \) and \( H^{\text{sing}}_n(E) \) is a free abelian group;

   (iii) \( H^{\text{sing}}_i(E, \Gamma) = \{0\} \) for any \( i > n \) and any abelian group \( \Gamma \);
(iv) $E$ is an $n$-skeleton (of some spectrum) in the sense of [Mar83, §6.3] (cf. also Definition 6.7 of [Chr98]).

7. A $w^{sp}$-Postnikov tower (see Definition 1.3.3 of [Bon18b]) of a spectrum $X \in \text{Obj } SH$ is the same thing as a cellular tower for $E$ in the sense of [Mar83, §6.3].

8. $E$ is an extension of a connective spectrum by an acyclic one if and only if $H^\text{sing}_j(E) = \{0\}$ for all $j < 0$; this is also equivalent to the vanishing of $H^j_\text{sing}(E, \mathbb{Q}/\mathbb{Z})$ for all $j < 0$.

**Proof.** Assertions 1–3 easily follow from Theorem 4.1.2; they are also contained in Theorem 4.2.1 of [Bon18b]. These facts also yield that Theorem 4.1.2(8) implies our assertion 8.

Next, the category $\text{Ab} \cong \text{Ap}$ is of cohomological dimension 1; hence we can combine Theorem 4.1.5 with the preceding assertions along with Theorem 4.1.2(7,9) to obtain assertions 4, 5, and the equivalence of conditions (i)–(iii) in assertion 6. Moreover, the latter equivalence statements implies that these conditions are fulfilled if and only if $E$ is an $n$-skeleton, and also that assertion 7 is valid according to Theorem 4.2.1(4,5) of [Bon18b].

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