Growth Estimates and Integral Representations of Harmonic and Subharmonic Functions

by

GUOSHUANG PAN

written under the supervision of
Professor GUANTIE DENG

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School of Mathematical Sciences
Beijing Normal University
Beijing, People’s Republic of China

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Abstract

There are ten chapters in this dissertation, which focuses on nine contents: growth estimates for a class of subharmonic functions in the half plane; growth estimates for a class of subharmonic functions in the half space; a generalization of harmonic majorants; properties of limit for Poisson integral; a lower bound for a class of harmonic functions in the half space; the Carleman formula of subharmonic functions in the half space; a generalization of the Nevanlinna formula for analytic functions in the right half plane; integral representations of harmonic functions in the half plane; integral representations of harmonic functions in the half space.

The outline of the paper is arranged as follows:

Chapter 1 presents the background, basic notations, some basic definitions, lemmas, theorems and propositions of the research;

In Chapter 2, we prove that a class of subharmonic functions represented by the modified kernels have the growth estimates at infinity in the upper half plane $\mathbb{C}_+$, which generalizes the growth properties of analytic functions and harmonic functions;

In Chapter 3, a class of subharmonic functions represented by the modified kernels are proved to have the growth estimates at infinity in the upper half space of $\mathbb{R}^n$, which generalizes the growth properties of analytic functions and harmonic functions;

In Chapter 4, we extend the harmonic majorant of a nonnegative and subharmonic function in $\mathbb{C}_+$ to the harmonic majorant represented by the modified Poisson kernel and to the upper half space;

In Chapter 5, we extend the properties of limit for Poisson integral in the upper half plane to the properties of limit for Poisson integral represented by the modified Poisson kernel and to the upper half space;

In Chapter 6, we derive a lower bound for a class of harmonic functions in the upper half space of $\mathbb{R}^n$ from the upper bound by using the generalization of the Carleman formula for harmonic functions in the upper half space and the generalization of the Nevanlinna formula for harmonic functions in the upper half ball;

In Chapter 7, the object of this chapter is to generalize the Carleman formula
for harmonic functions in the upper half plane to subharmonic functions in the upper half space;

In Chapter 8, we generalize the Nevanlinna formula for analytic functions to the right half plane;

In Chapter 9, using a modified Poisson kernel in the upper half plane, we prove that a harmonic function $u(z)$ in the upper half plane with its positive part $u^+(z) = \max\{u(z), 0\}$ satisfying a slowly growing condition can be represented by its integral in the boundary of the upper half plane, the integral representation is unique up to the addition of a harmonic polynomial, vanishing in the boundary of the upper half plane and that its negative part $u^-(z) = \max\{-u(z), 0\}$ can be dominated by a similar slowly growing condition, this improves some classical results about harmonic functions in the upper half plane;

In Chapter 10, using a modified Poisson kernel in the upper half space, we prove that a harmonic function $u(x)$ in the upper half space with its positive part $u^+(x) = \max\{u(x), 0\}$ satisfying a slowly growing condition can be represented by its integral in the boundary of the upper half space, the integral representation is unique up to the addition of a harmonic polynomial, vanishing in the boundary of the upper half space and that its negative part $u^-(x) = \max\{-u(x), 0\}$ can be dominated by a similar slowly growing condition, this improves some classical results about harmonic functions in the upper half space.

**KEY WORDS**: harmonic function, subharmonic function, modified Poisson kernel, modified Green function, growth estimate, the upper half plane, the upper half space, harmonic majorant, the properties of limit, lower bound, Carleman formula, Nevanlinna formula, integral representation.
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Chapter 1

Introduction

The present chapter consists of three sections with the first providing the background for the research project; the second presenting the basic notations; the third section providing us some basic definitions, lemmas, theorems and propositions.

1.1 Background

A complex-valued function $h$ on an open subset $\Omega$ of the complex plane $\mathbb{C}$ is called harmonic on $\Omega$ if $h \in C^2(\Omega)$ and

$$\triangle h \equiv 0$$

on $\Omega$. Here

$$\triangle h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

is the Laplacian of $h$. We often assume that $\Omega$ is a region (that is, an open and connected set) even when connectivity is not needed, and we are mainly interested in the case in which $\Omega$ is a disk or half plane.

Harmonic functions arise in the study of analytic functions (we use the terms analytic and holomorphic synonymously). If $f$ is analytic on a region $\Omega$, then by the Cauchy-Riemann equations, each of the functions $f$, $\overline{f}$, $\Re f$ is harmonic on $\Omega$. The theory of harmonic functions is needed in the study of analytic functions on a disk or half plane.
Harmonic functions—the solutions of Laplace’s equation—play a crucial role in many areas of mathematics, physics, and engineering. So it is necessary to extend harmonic functions to $\mathbb{R}^n$, where $n$ denotes a fixed positive integer greater than 1. Let $\Omega$ be an open, nonempty subset of $\mathbb{R}^n$. A twice continuously differentiable, complex-valued function $u$ defined on $\Omega$ is harmonic on $\Omega$ if

$$\triangle u \equiv 0,$$

where $\triangle = D_1^2 + \cdots + D_n^2$ and $D_j^2$ denotes the second partial derivative with respect to the $j^{th}$ coordinate variable. The operator $\triangle$ is called the Laplacian, and the equation $\triangle u \equiv 0$ is called Laplace’s equation.

We let $x = (x_1, \cdots, x_n)$ denote a typical point in $\mathbb{R}^n$ and let $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ denote the Euclidean norm of $x$.

The simplest nonconstant harmonic functions are the coordinate functions; for example, $u(x) = x_1$. A slightly more complex example is the function on $\mathbb{R}^3$ defined by

$$u(x) = x_1^2 + x_2^2 - 2x_3^2 + ix_2.$$

As we will see later, the function

$$u(x) = |x|^{2-n}$$

is vital to harmonic function theory when $n > 2$; it is obvious that this function is harmonic on $\mathbb{R}^n - \{0\}$.

We can obtain additional examples of harmonic functions by differentiation, noting that for smooth functions the Laplacian commutes with any partial derivative. In particular, differentiating the last example with respect to $x_1$ shows that $x_1 |x|^{-n}$ is harmonic on $\mathbb{R}^n - \{0\}$ when $n > 2$.

The function $x_1 |x|^{-n}$ is harmonic on $\mathbb{R}^n - \{0\}$ even when $n = 2$. This can be verified directly or by noting that $x_1 |x|^{-2}$ is a partial derivative of $\log |x|$, a harmonic function on $\mathbb{R}^2 - \{0\}$. The function $\log |x|$ plays the same role when $n = 2$ that $|x|^{2-n}$ plays when $n > 2$. Notice that $\lim_{x \to \infty} \log |x| = \infty$, but $\lim_{x \to \infty} |x|^{2-n} = 0$; note also $\log |x|$ is neither bounded above nor below, but $|x|^{2-n}$ is always positive. These facts hint at the contrast between harmonic function theory in the plane and in higher dimensions. Another key difference arises from the close connection between holomorphic and harmonic functions in the plane—a real-valued function
1.1. Background

on $\Omega \subset \mathbb{R}^2$ is harmonic if and only if it is locally the real part of a holomorphic function. No comparable result exists in higher dimensions.

Let $\Omega$ be a region in the complex plane. A real-valued function $u$ on an open subset $\Omega$ of the complex plane $\mathbb{C}$ is defined to be subharmonic if $u \in C^2(\Omega)$ and

$$\Delta u \geq 0$$

on $\Omega$. A broader definition that relaxes the smoothness assumption and permits $u$ to take the value $-\infty$. Examples of subharmonic functions include $\log |f|$, $\log^+ |f| = \max(\log |f|, 0)$ and $|f|^p (0 < p < \infty)$, where $f$ is any analytic function on $\Omega$.

Elementary properties of subharmonic functions are often one-sided versions of properties of harmonic functions. For example, a subharmonic function $u$ on $\Omega$ has a sub-mean value property:

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{i\theta}) d\theta.$$ 

This property characterizes subharmonic functions.

One of the most fundamental results in the theory of subharmonic functions is due to F. Riesz and states that any such function $u(x)$ can be locally written as the sum of a potential plus a harmonic function, i.e.

$$u(x) = p(x) + h(x).$$

In other words, if $u(x)$ is subharmonic in a domain $D$ in $\mathbb{R}^m$, there exists a positive measure $d\mu$, finite on compact subsets of $D$, and uniquely determined by $u(x)$, such that if $E$ is a compact subset of $D$ and

$$p(x) = \begin{cases} \int_E \log |x - \xi| d\mu_\xi, & \text{if } m = 2, \\ -\int_E |x - \xi|^{2-m} d\mu_\xi, & \text{if } m > 2, \end{cases}$$

then

$$h(x) = u(x) - p(x).$$

is harmonic in the interior of $E$.

By means of this theorem many of the local properties of subharmonic functions can be deduced from those of potentials such as $p(x)$. The mass distribution $d\mu$ also plays a fundamental role in more delicate questions concerning $u$. Thus
for instance if $m = 2$ and $u(z) = \log |f(z)|$, where $f$ is a regular function of the complex variable $z$, then $\mu(E)$ reduces to the number of zeros of $f(z)$ on the set $E$. From this point of view the main difference between this case and that of a general subharmonic function is that in the latter case the "zeros" can have an arbitrary mass distribution instead of occurring in units of one.

In higher dimension we may regard $d\mu$ as the gravitational or electric charge, giving rise to the potential $p(x)$. For this reason the theory of subharmonic functions is frequently called potential theory.

We now come to a famous problem in harmonic function theory: given a continuous function $f$ on $S$, does there exist a continuous function function $u$ on $\overline{B}$, with $u$ harmonic on $B$, such that $u = f$ on $S$? If so, how do we find $u$? This is Dirichlet problem for the ball.

The Dirichlet problem of the upper half plane is to find a function $u$ satisfying

$$u \in C^2(C_+),$$

$$\Delta u = 0, z \in C_+,$$

$$\lim_{z \to x} u(z) = f(x) \text{ nontangentially a.e. } x \in \partial C_+,$$

where $f$ is a measurable function of $\mathbb{R}$. The Poisson integral of the upper half plane is defined by

$$u(z) = P[f](z) = \int_{\mathbb{R}} P(z, \xi) f(\xi) d\xi. \quad (1.1.1)$$

As we all know, the Poisson integral $P[f]$ exists if

$$\int_{\mathbb{R}} \frac{|f(\xi)|}{1 + |\xi|^2} d\xi < \infty.$$ 

We will generalize these results from harmonic functions to subharmonic functions.

Write the subharmonic function

$$u(z) = v(z) + h(z), \quad z \in C_+,$$

where $v(z)$ is the harmonic function defined by (1.1.1), $h(z)$ is defined by

$$h(z) = \int_{C_+} G(z, \xi) d\mu(\xi)$$
and $G(z, \zeta)$ is called Green function.

Hayman [26] has proved that the asymptotic behaviour of subharmonic functions

$$u(z) = o(|z|), \text{ as } |z| \to \infty$$

holds everywhere in the upper half plane outside some exceptional set of disks under the following conditions:

$$\int_{\mathbb{R}} \frac{|f(\xi)|}{1 + |\xi|^2} d\xi < \infty$$

and

$$\int_{\mathbb{C}_{+}} \frac{\eta}{1 + |\zeta|^2} d\mu(\zeta) < \infty,$$

where $\mu$ is a positive Borel measure and $\zeta = \xi + i\eta$.

The first aim in this dissertation is to extend the classic results to the modified Poisson kernel $P_m(z, \xi)$ and the modified Green function $G_m(z, \zeta)$. That is to say, if

$$v(z) = \int_{\mathbb{R}} P_m(z, \xi) f(\xi) d\xi,$$

$$h(z) = \int_{\mathbb{C}_{+}} G_m(z, \zeta) d\mu(\zeta),$$

we will prove that the asymptotic behaviour of subharmonic functions

$$v(z) = o(y^{1-\alpha}|z|^{m+\alpha}), \text{ as } |z| \to \infty$$

holds everywhere in the upper half plane outside some exceptional set of disks under the following conditions:

$$\int_{\mathbb{R}} \frac{|f(\xi)|}{1 + |\xi|^{2+m}} d\xi < \infty$$

and

$$\int_{\mathbb{C}_{+}} \frac{\eta}{1 + |\zeta|^{2+m}} d\mu(\zeta) < \infty.$$

Next, we can also conclude that the asymptotic behaviour of subharmonic functions

$$u(z) = o(y^{1-\alpha/p} (\log |z|)^{\frac{1}{q}} |z|^\frac{\gamma}{p} + \frac{1}{q} + 2 + \frac{\alpha}{p}), \text{ as } |z| \to \infty$$

holds everywhere in the upper half plane outside some exceptional set of disks under the following conditions:
holds everywhere in the upper half plane outside some exceptional set of disks by replacing the two conditions above into

$$\int_{\mathbb{R}} \frac{|f(\xi)|^p}{(1 + |\xi|)\gamma} d\xi < \infty$$

and

$$\int_{C_+} \frac{\eta^p}{(1 + |\zeta|)\gamma} d\mu(\zeta) < \infty,$$

where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $1 - p < \gamma < 1 + p$.

On the other hand, we will generalize these results from the upper half plane to the upper half space.

The Dirichlet problem of the upper half space is to find a function $u$ satisfying

$$u \in C^2(H),$$

$$\Delta u = 0, x \in H,$$

$$\lim_{x \to x'} u(x) = f(x') \text{ non tangentially a.e.} x' \in \partial H,$$

where $f$ is a measurable function of $\mathbb{R}^{n-1}$. The Poisson integral of the upper half space is defined by

$$u(x) = P[f](x) = \int_{\mathbb{R}^{n-1}} P(x, y') f(y') dy'.$$

As we all know, the Poisson integral $P[f]$ exists if

$$\int_{\mathbb{R}^{n-1}} \frac{|f(y')|}{1 + |y'|^n} dy' < \infty.$$

Write the harmonic function

$$v(x) = \int_{\mathbb{R}^{n-1}} P_m(x, y') f(y') dy', \quad x \in H,$$

Siegel-Talvila [38] have proved that the asymptotic behaviour of

$$v(x) = o(x^{1-n} |x|^{m+n}), \quad \text{as} \ |x| \to \infty$$

holds everywhere in the upper half space outside some exceptional set of balls under the following condition:

$$\int_{\mathbb{R}^{n-1}} \frac{|f(y')|}{1 + |y'|^{n+m}} dy' < \infty.$$
1.2. Basic Notations

We will generalize these results from harmonic functions to subharmonic functions, then we will obtain some further results.

In addition, we also discuss some other problems about harmonic and subharmonic functions, such as the generalization of harmonic majorants, properties of limit for Poisson integral, the Carleman formula and Nevanlinna formula and integral representations.

1.2 Basic Notations

Let $\mathbb{C}$ denote the complex plane with points $z = x + iy$, where $x, y \in \mathbb{R}$. The boundary and closure of an open $\Omega$ of $\mathbb{C}$ are denoted by $\partial \Omega$ and $\overline{\Omega}$ respectively. The upper half plane is the set $\mathbb{H} = \{ z = x + iy \in \mathbb{C} : y > 0 \}$, whose boundary is $\partial \mathbb{H}$. We identify $\mathbb{C}$ with $\mathbb{R} \times \mathbb{R}$ and $\mathbb{H}$ with $\mathbb{R} \times \{0\}$, with this convention we then have $\partial \mathbb{H} = \mathbb{R}$.

A twice continuously differentiable function $u(z)$ defined on an open set $\Omega$ is harmonic if $\Delta u \equiv 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is Laplace operator in $z$. We write $B_R$ and $\partial B_R$ for the open ball and the circle of radius $R$ in $\mathbb{C}$ centered at the origin and $B_R^+ = B_R \cap \mathbb{H}$ and $\partial B_R^+$ for the open upper half ball and the upper half circle of radius $R$ in $\mathbb{C}$ centered at the origin.

Similarly, let $\mathbb{R}^n (n \geq 3)$ denote the $n$-dimensional Euclidean space with points $x = (x_1, x_2, \ldots, x_{n-1}, x_n) = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. The boundary and closure of an open $\Omega$ of $\mathbb{R}^n$ are denoted by $\partial \Omega$ and $\overline{\Omega}$ respectively. The upper half space is the set $H = \{ x = (x', x_n) \in \mathbb{R}^n : x_n > 0 \}$, whose boundary is $\partial H$. We identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$ and $\mathbb{H}^n$ with $\mathbb{R}^{n-1} \times \{0\}$, with this convention we then have $\partial H = \mathbb{R}^{n-1}$, writing typical points $x, y \in \mathbb{R}^n$ as $x = (x', x_n), y = (y', y_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}), y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$ and putting

$$x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n, \quad |x| = \sqrt{x \cdot x}, \quad |x'| = \sqrt{x' \cdot x'},$$

where $|x|$ is the Euclidean norm.

A twice continuously differentiable function $u(x)$ defined on an open set $\Omega$ is harmonic if $\Delta_x u \equiv 0$, where $\Delta_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is Laplace operator in $x$. The upper half space $H$ is the set $H = \{ x = (x', x_n) \in \mathbb{R}^n : x_n > 0, \}$ We write $B_R$ and $\partial B_R$ for the open ball and the sphere of radius $R$ in $\mathbb{R}^n$ centered at the
origin and \( B_R^+ = B_R \cap H \) and \( \partial B_R^+ \) for the open upper half ball and the upper half sphere of radius \( R \) in \( \mathbb{R}^n \) centered at the origin. In the sense of Lebesgue measure \( dx' = dx_1 \cdots dx_{n-1} \), \( dx = dx' dx_n \) and let \( \sigma \) denote \((n-1)\)-dimensional surface-area measure.

Throughout the dissertation, let \( A \) denote various positive constants independent of the variables in question.

### 1.3 Preliminary Results

In this section, we will introduce some definitions, lemmas, theorems and propositions that will be used in the following chapters.

**Definition A\(_1\)[34]** Let \( X \) be a metric space. A function \( u : X \to [-\infty, \infty) \) is said to be upper semicontinuous or usc if

\[
\{ x : u(x) < a \}
\]

is an open set in \( X \) for each real number \( a \), or, equivalently, if for every \( x \in X \),

\[
\limsup_{y \to x} u(y) \leq u(x).
\]

**Definition A\(_2\)[34]** Let \( \Omega \) be an open set in the complex plane. We say that a function \( u : \Omega \to [-\infty, \infty) \) is subharmonic on \( \Omega \) if

1. \( u \) is usc on \( \Omega \);
2. for every open set \( A \) with compact closure \( \overline{A} \subseteq \Omega \) and every continuous function \( h : \overline{A} \to (-\infty, \infty) \) whose restriction to \( A \) is harmonic, if \( u \leq h \) on \( \partial A \), then \( u \leq h \) on \( \overline{A} \).

**Definition A\(_3\)[34]** Let \( u \) be subharmonic on a region \( \Omega \), \( u \neq -\infty \), and let \( h \) be harmonic on \( \Omega \). We say that \( h \) is a harmonic majorant for \( u \) if \( h \geq u \) on \( \Omega \). We say that \( h \) is a least harmonic majorant for \( u \) if

1. \( h \) is a harmonic majorant for \( u \);
2. if \( f \) is any harmonic majorant for \( u \) in \( \Omega \), then \( h \leq f \) on \( \Omega \).

**Lemma A\(_1\)[34]** Let \( g(x) \) be a nonnegative and nondecreasing function on \([0, 1]\). Let \( \varphi(x) \) be any nonnegative measurable function on \((0, 1)\) such that

\[
0 < \int_0^a \varphi(t) dt < \infty
\]
1.3. Preliminary Results

for every \( a \in (0, 1) \) and

\[
\int_0^1 p(t)dt = \infty.
\]

Then

\[
\lim_{x \uparrow 1} g(x) = \sup_{0 < r < 1} \frac{\int_0^1 g(t)p(\lambda t)dt}{\int_0^1 p(\lambda t)dt}.
\]

**Lemma A₂** [34] Let \( V(z) \) be nonnegative and harmonic on \( \Pi \) and have a continuous extension to \( \Pi = \{ z : \Im z \geq 0 \} \). Then

\[
V(z) = cy + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{V(t)}{(t-x)^2+y}dt, \quad y > 0,
\]

where \( c \) is given by

\[
c = \lim_{y \to \infty} \frac{V(iy)}{y}.
\]

**Lemma A₃** [22] The polynomials \( a_v(x, \xi) \) are harmonic in \( x \) for fixed \( \xi \), and continuous in \( x, \xi \) jointly for \( |\xi| \neq 0 \). If \( |x| = \rho, \ |\xi| = r > 0 \), we have the sharp inequality

\[
|a_v(x, \xi)| \leq \frac{b_v \rho^m}{r^{m+v-2}},
\]

where \( b_v = 1/v \) if \( m = 2, v \geq 1 \);

\[
b_v = \frac{(v+m-3)(v+m-4) \cdots (v+1)/(m-1)!}{m \geq 3, v \geq 0}.
\]

**Lemma A₄** [22] If \( |\xi| = r > 0 \), then \( K_q(x, \xi) - K(x, \xi) \) is harmonic in \( \mathbb{R}^m \). We set \( |x| = \rho \) and have the following estimates

\[
|K_q(x, \xi)| \leq \frac{r^{q+1}}{r^{m+q-1}} i f \rho \leq \frac{1}{2} r,
\]

If \( q = 0, m = 2 \), we have

\[
K_0(x, \xi) \leq \log(1 + \rho/r),
\]

while in all other cases

\[
K_q(x, \xi, m) \leq \frac{r^q}{r^{m+q-2}} \inf \{1, \frac{\rho}{r}\}.
\]
Theorem A1[34] For every continuous complex-valued function $f$ on $\Gamma$ there is a unique continuous function $h$ on $\overline{D} = D \cup \Gamma$ such that the restriction of $h$ to $\Gamma$ is $f$ and the restriction of $h$ to $D$ is harmonic. The function $h$ is given on $D$ by

$$h(z) = \int_{\Gamma} P(z, e^{it}) f(e^{it}) d\sigma(e^{it}), \quad z \in D.$$ 

Theorem A2[34] (Mean value property) If $h$ is harmonic on a region $\Omega$ and $\overline{D(a,R)} \subseteq \Omega$, then

$$h(a) = \frac{1}{2\pi} \int_{0}^{2\pi} h(a + R e^{it}) dt.$$ 

Theorem A3[34] (Maximum principle) A real-valued harmonic function $h$ on an open connected set $\Omega$ cannot attain either a maximum or a minimum value in $\Omega$ without reducing to a constant.

Theorem A4[34] A continuous function $h$ on a region $\Omega$ has the mean value property if and only if $h$ is harmonic on $\Omega$.

Theorem A5[34] Every nonnegative harmonic function $h$ on the unit disc $D$ has a representation

$$h(z) = \int_{\Gamma} P(z, e^{it}) d\mu(e^{it}), \quad z \in D,$$

where $\mu$ is a finite nonnegative measure on $\Gamma$.

Theorem A6[34] (Herglotz and Riesz Representation Theorem) Let $f$ be an analytic and satisfy $\Re f \geq 0$ on $D$. Then

$$f(z) = \int_{\Gamma} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(e^{i\theta}) + ic, \quad z \in D,$$

for some finite nonnegative Borel measure $\mu$ on $\Gamma$ and some real constant $c$.

Theorem A7[34] (Stieltjes Inversion Formula) Let $\mu$ be a complex Borel measure on $\Gamma$, and let on the unit disc $D$ has a representation

$$h(z) = \int_{\Gamma} P(z, e^{it}) d\mu(e^{it}), \quad z \in D.$$
1.3. Preliminary Results

Let \( \gamma = \{e^{it} : a < t < b\} \) be an open arc on the unit circle with endpoints \( \alpha = e^{ia} \) and \( \beta = e^{ib}, \) \( 0 < b - a < 2\pi. \) Then

\[
\lim_{r \uparrow 1} \frac{1}{2\pi} \int_{a}^{b} h(re^{i\theta})d\theta = \mu(\gamma) + \frac{1}{2}\mu(\{\alpha\}) + \frac{1}{2}\mu(\{\beta\}).
\]

**Theorem A8** [34] Let \( u \) be usc on a region \( \Omega \) in the complex plane. The following are equivalent:

(1) \( u \) is subharmonic on \( \Omega; \)

(2) for each \( a \in \Omega \) and all sufficiently small \( R > 0, \) if \( p \) is a polynomial such that \( u \leq \Re p \) on \( \partial D(a, R), \) then \( u \leq \Re p \) on \( D(a, R); \)

(3) for each \( a \in \Omega \) and all sufficiently small \( R > 0, \)

\[
u(a) \leq \frac{1}{2\pi} \int_{0}^{2\pi} u(a + Re^{i\theta})d\theta.
\]

In this case, the properties expressed in (2) and (3) hold for all disks \( D(a, R) \) such that \( D(a, R) \subseteq \Omega. \)

**Theorem A9** (Maximum principle) Assume that \( u \) is subharmonic on a region \( \Omega \). If there is a point \( z_0 \in \Omega \) such that \( u(z_0) \geq u(z) \) for all \( z \in \Omega, \) then \( u \equiv \text{const. in} \ \Omega. \)

**Theorem A10** [34] Assume that \( u \) is subharmonic in \( D(a, R) \) and \( u \neq -\infty \). If \( 0 < r_1 < r_2 < R, \) then

\[-\infty < \frac{1}{2\pi} \int_{0}^{2\pi} u(a + r_1e^{i\theta})d\theta \leq \frac{1}{2\pi} \int_{0}^{2\pi} u(a + r_2e^{i\theta})d\theta.\]

Moreover, whether \( u(a) \) is finite or \( -\infty, \)

\[
\lim_{r \downarrow 0} \frac{1}{2\pi} \int_{0}^{2\pi} u(a + re^{i\theta})d\theta = u(a).
\]

**Theorem A11** [34] Let \( u \) be subharmonic in the unit disk \( D, \) \( u \neq -\infty. \) There exists a harmonic majorant for \( u \) if and only if

\[
\sup_{0 < r < 1} \frac{1}{2\pi} \int_{0}^{2\pi} u(a + re^{it})dt < \infty.
\]
In this case there is a least harmonic majorant \( h \) for \( u \), and \( h \) is given by

\[
h(z) = \lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} P(z/r, e^{it})u(re^{it})dt
\]

for all \( z \in D \).

**Theorem A_{12} [34]** (Poisson Representation) Every nonnegative harmonic function \( V(z) \) on \( \Pi \) has a representation

\[
V(z) = cy + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2 + y^2}, \quad y > 0,
\]

where \( c \geq 0 \) and \( \mu \) is a nonnegative Borel measure on \( (-\infty, \infty) \) such that

\[
\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty.
\]

**Theorem A_{13} [34]** (Nevanlinna Representation) Every holomorphic function \( F(z) \) such that \( \Im F(z) \geq 0 \) for \( z \in \Pi \) has a representation

\[
F(z) = b + cz + \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{t-z} - \frac{t}{1+t^2} \right] d\mu(t), \quad y > 0,
\]

where \( b = \overline{b}, \ c \geq 0, \) and \( \mu \) is a nonnegative Borel measure on \( (-\infty, \infty) \) which satisfies

\[
\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty.
\]

**Theorem A_{14} [34]** (Stieltjes Inversion Formula) Let \( V(z) \) be given by

\[
V(z) = cy + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2 + y^2}, \quad y > 0,
\]

where \( c \geq 0 \) and \( \mu \) is a nonnegative Borel measure satisfying

\[
\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty.
\]

If \( -\infty < a < b < \infty \), then

\[
\lim_{y \downarrow 0} \int_a^b \frac{V(x+iy)dx}{y} = \mu((a,b)) + \frac{1}{2} \mu(\{a\}) + \frac{1}{2} \mu(\{b\}).
\]
1.3. Preliminary Results

**Theorem A15**[34] (Fatou’s Theorem) Let

\[
V(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2 + y^2}, \quad y > 0,
\]

where \(\mu\) is a nonnegative Borel measure on \((-\infty, \infty)\) satisfying

\[
\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty.
\]

If \(d\mu = Fdx + d\mu_s\) is the Lebesgue decomposition of \(\mu\), then

\[
\lim_{z \to x} V(z) = F(x)
\]
nontangentially a.e. on \((-\infty, \infty)\).

**Theorem A16**[34] Let \(F\) be holomorphic on \(D_+(0, R)\) for some \(R > 0\), and suppose \(F \not= 0\). Then \(F \in N^+(D_+(0, R))\) if and only if

\[
\log |F(z)| \leq \frac{R^2 - |z|^2}{\pi} \int_{-R}^{R} \frac{2yR \sin t}{|Re^{it} - z|^2 |Re^{-it} - z|^2} K(Re^{it}) dt \\
+ \frac{y}{\pi} \int_{-R}^{R} \left( \frac{1}{|t - z|^2} - \frac{R^2}{|R^2 - tz|^2} \right) K(t) dt
\]

for all \(z \in D_+(0, R)\) and some real-valued Borel function \(K(\zeta)\) on \(\Gamma_+(0, R)\) such that

\[
\int_{0}^{\pi} |K(Re^{it})| \sin t dt + \int_{-R}^{R} |K(t)| (R^2 - t^2) dt < \infty.
\]

**Theorem A17**[1] (Mean value property) If \(u\) is harmonic on \(\overline{B}(a, r)\), then \(u\) equals the average of \(u\) over \(\partial B(a, r)\). More precisely,

\[
u(a) = \int_{S} u(a + r\zeta) d\sigma(\zeta).
\]

**Theorem A18**[1] (Solution of the Dirichlet problem for the ball) Suppose \(f\) is continuous on \(S\). Define \(u\) on \(\overline{B}\) by

\[
u(x) = \begin{cases} 
P[f](x) & \text{if } x \in B, \\ f(x) & \text{if } x \in S. \end{cases}
\] (1.6)
Then $u$ is continuous on $\overline{B}$ and harmonic on $B$.

**Theorem A$\!$19** [1] If $u$ is a continuous function on $\overline{B}$ that is harmonic on $B$, then $u = P[u|_S]$ on $B$.

**Theorem A$\!$20** [1] (Solution of the Dirichlet problem for $H$) Suppose $f$ is continuous and bounded on $\mathbb{R}^{n-1}$. Define $u$ on $\overline{H}$ by

$$u(z) = \left\{ \begin{array}{ll} P_H[f](z) & \text{if } x \in H, \\ f(z) & \text{if } x \in \mathbb{R}^{n-1}. \end{array} \right.$$  \hspace{1cm} (1.6)

Then $u$ is continuous on $\overline{H}$ and harmonic on $H$. Moreover,

$$|u| \leq ||f||_\infty$$

on $\overline{H}$.

**Theorem A$\!$21** [1] Suppose $u$ is a continuous bounded function on $\overline{H}$ that is harmonic on $H$. Then $u$ is the Poisson integral of its boundary values. More precisely,

$$u = P_H[u|_{\mathbb{R}^{n-1}}]$$

on $H$.

**Theorem A$\!$22** [33] (The Schwarz reflection principle) Suppose $L$ is a segment of the real axis, $\Omega^+$ is a region in $\Pi^+$, and every $t \in L$ is the center of an open disc $D_t$ such that $\Pi^+ \cap D_t$ lies in $\Omega^+$. Let $\Omega^-$ be the reflection of $\Omega^+$:

$$\Omega^- = \{ z : \tau \in \Omega^+ \}.$$

Suppose $f = u + iv$ is holomorphic in $\Omega^+$, and

$$\lim_{n \to \infty} v(z_n) = 0$$

for every sequence $\{z_n\}$ in $\Omega^+$ which converges to a point of $L$.

Then there is a function $F$, holomorphic in $\Omega^+ \cup L \cup \Omega^-$, such that $F(z) = f(z)$ in $\Omega^+$; this $F$ satisfies the relation

$$F(\overline{z}) = \overline{F(z)}$$ \hspace{1cm} (z \in \Omega^+ \cup L \cup \Omega^-).$$
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**Theorem A\textsubscript{23}**[1] If $u$ is positive and harmonic on $H$, then there exists a positive Borel measure $\mu$ on $\mathbb{R}^{n-1}$ and a nonnegative constant $c$ such that

$$u(x,y) = cy + \int_{\mathbb{R}^{n-1}} P_H(z,t) d\mu(t)$$

for all $(x,y) \in H$.

**Theorem A\textsubscript{24}[26] (Hayman)** Let

$$v(z) = \int \int_{\mathbb{C}^+} \log \left| \frac{\zeta - z}{\zeta - \overline{z}} \right| d\mu(\zeta) + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{dv(t)}{t-x+y^2},$$

where $d\mu(\zeta)$ and $dv(t)$ are nonnegative Borel measures such that

$$\int \int_{\mathbb{C}^+} \frac{3\zeta}{1+|\zeta|^2} d\mu(\zeta) < \infty, \quad \int_{\mathbb{R}} \frac{dv(t)}{1+t^2} < \infty.$$

The asymptotic relation

$$v(z) = o(|z|), \quad |z| \to \infty$$

holds everywhere in $\mathbb{C}^+$ outside some exceptional set of disks of finite view.

**Theorem A\textsubscript{25}[22] (Green’s Theorem)** Suppose that $D$ is an admissible domain with boundary $S$ in $\mathbb{R}^m$ and that $u \in C^1$ and $v \in C^2$ in $\overline{D}$. Then

$$\int_S u(x) \frac{\partial v}{\partial n} d\sigma = -\int_D \left\{ \sum_{\nu=1}^{m} \frac{\partial u}{\partial x_{\nu}} \cdot \frac{\partial v}{\partial x_{\nu}} + u \nabla^2 v \right\} dx,$$

where

$$\nabla^2 = \sum_{\nu=1}^{m} \frac{\partial^2}{\partial x_{\nu}^2}$$

is Laplace’s operator. Hence if $u, v \in C^2$ in $\overline{D}$ we have

$$\int_S \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d\sigma = \int_D (v \nabla^2 u - u \nabla^2 v) dx.$$

Here $\partial / \partial n$ denotes differentiation along the inward normal into $D$.

**Theorem A\textsubscript{26}[22]** If $D = D(0,R)$ and $\xi$ a point of $D$, $\xi' = \xi R^2 |\xi|^{-2}$, and if for $m = 2$

$$g(x,\xi,D) = \log \frac{|x-\xi'|}{|x-\xi| R}, \quad \xi \neq 0; \quad g(x,0,D) = \log \frac{R}{|x|};$$
while for \( m > 2 \)
\[
g(x, \xi, D) = |x - \xi|^{2-m} - \{|\xi||x - \xi'|/R\}^{2-m}, \quad \xi \neq 0;
\]
\[
g(x, 0, D) = |x|^{2-m} - R^{2-m};
\]
then \( g(x, \xi, D) \) is a (classical) Green’s function of \( D \).

**Theorem A27** [22] (Poisson’s Integral) If \( u \) is harmonic in \( D(x_0, R) \) and continuous in \( C(x_0, R) \) then for \( \xi \in D(x_0, R) \) we have
\[
 u(\xi) = \frac{1}{c_m} \int_{S(x_0, R)} \frac{R^2 - |\xi - x_0|^2}{R|x - \xi|^m} u(x) d\sigma_x,
\]
where \( d\sigma_x \) denotes an element of surface area of \( S(x_0, R) \) and \( c_m = 2\pi^{m/2}/\Gamma(m/2) \).

**Theorem A28** [22] Suppose that \( u(x) \) is s.h. in \( C(x_0, R) \). Then for \( \xi \in D(x_0, R) \) we have
\[
 u(\xi) \leq \int_{S(x_0, R)} u(x) K(x, \xi) d\sigma_x,
\]
where \( K(x, \xi) \) is the Poisson kernel given by
\[
 K(x, \xi) = \frac{1}{c_m} \frac{R^2 - |\xi - x_0|^2}{R|x - \xi|^m}
\]
and \( d\sigma_x \) denotes an element of surface area of \( S(x_0, R) \).

**Theorem A29** [22] (Riesz’s Theorem) Suppose that \( u(x) \) is s.h. and not identically \( -\infty \), in a domain \( D \) in \( \mathbb{R}^m \). Then there exists a unique Borel-measure \( \mu \) in \( D \) such that for any compact subset \( E \) of \( D \) we have
\[
 u(x) = \int_E u(x) K(x, \xi) d\mu_\xi + h(x),
\]
where \( h(x) \) is harmonic in the interior of \( E \).

**Theorem A30** [22] Suppose that \( D \) is a bounded regular domain in \( \mathbb{R}^m \) whose frontier \( F \) has zero \( m \)-dimensional Lebesgue measure, and that \( u(x) \) is s.h. and not identically \( -\infty \) on \( D \cup F \). Then we have for \( x \in D \)
\[
 u(x) = \int_F u(\xi) d\omega(x, e_\xi) - \int_D g(x, \xi, D) d\mu_\xi,
\]
where \( \omega(x,e) \) is the harmonic measure of \( e \) at \( x \), \( g(x,\xi,D) \) is the Green’s function of \( D \) and \( d\mu \) is the Riesz measure of \( u \) in \( D \).

**Theorem A31**[22] (Weierstrass’ Theorem) Suppose that \( \mu \) is a Borel measure in \( \mathbb{R}^m \), let \( n(t) \) be the measure of \( D(0,t) \) and let \( q(t) \) be a positive integer-valued increasing function of \( t \), continuous on the right, and so chosen that

\[
\int_1^\infty \left( \frac{t_0}{t} \right)^{q(t) + m - 1} dn(t) < \infty
\]

for all positive \( t_0 \). Then there exists functions \( u(x) \), s.h. in \( \mathbb{R}^m \) and with Riesz measure \( \mu \), and all such functions take the form

\[
u(x) = \int_{|\xi| < 1} K(x - \xi) d\mu_\xi + \int_{|\xi| \geq 1} K_{q(|\xi|)}(x - \xi) d\mu_\xi + v(x),
\]

where \( v(x) \) is harmonic in \( \mathbb{R}^m \). The second integral converges absolutely near \( \infty \) and uniformly for \( |x| \leq \rho \) and any fixed positive \( \rho \).

**Proposition A1**[1] (Polar coordinates formula) The polar coordinates formula for integration on \( \mathbb{R}^n \) states that for a Borel measurable, integrable function \( f \) on \( \mathbb{R}^n \),

\[
\frac{1}{nV(B)} \int_{\mathbb{R}^n} f dV = \int_0^\infty r^{n-1} \int_S f(r\zeta) d\sigma(\zeta) dr,
\]

the constant arises from the normalization of \( \sigma \).

**Proposition A2**[1] Let \( \zeta \in S \). Then \( P(\cdot, \zeta) \) is harmonic on \( \mathbb{R}^n - \{\zeta\} \).

**Proposition A3**[1] The Poisson kernel has the following properties:

(a) \( P(x, \zeta) > 0 \) for all \( x \in B \) and all \( \zeta \in S \);
(b) \( \int_S P(x, \zeta) d\sigma(\zeta) = 1 \) for all \( x \in B \);
(c) for every \( \eta \in S \) and every \( \delta > 0 \),

\[
\int_{|\zeta - \eta| > \delta} P(x, \zeta) d\sigma(\zeta) \to 0, \quad \text{as } x \to \eta.
\]

**Proposition A4**[26] (R. Nevanlinna’s formula for a half-disk) Let \( f(z) \) be a meromorphic function in the half-disk \( \overline{D}^ \infty_R \), \( a_n \) be its zeros and \( b_n \) its poles. Then we obtain
\[
\log |f(z)| = \frac{R^2 - |z|^2}{2\pi} \int_0^\pi \left( \frac{1}{|Re^{i\theta} - z|^2} - \frac{1}{|Re^{i\theta} - \overline{z}|^2} \right) \log |f(Re^{i\theta})| d\theta \\
+ \frac{R}{\pi} \int_{-R}^R \left( \frac{1}{|t - z|^2} - \frac{R^2}{|R^2 - tz|^2} \right) \log |f(t)| dt \\
+ \sum_{a_n \in D^+_R} \log \left| \frac{z - a_n}{z - \overline{a}_n} \cdot \frac{R^2 - a_n \overline{z}}{R^2 - \overline{a}_n \overline{z}} \right| - \sum_{b_n \in D^-_R} \log \left| \frac{z - b_n}{z - \overline{b}_n} \cdot \frac{R^2 - b_n \overline{z}}{R^2 - \overline{b}_n \overline{z}} \right|.
\]
Chapter 2

Growth Estimates for a Class of Subharmonic Functions in the Half Plane

2.1 Introduction and Basic Notations

For \( z \in \mathbb{C} \setminus \{0\} \), let

\[
E(z) = (2\pi)^{-1} \log |z|,
\]

where \(|z|\) is the Euclidean norm. We know that \( E \) is locally integrable in \( \mathbb{C} \).

First, we define the Green function \( G(z, \zeta) \) for the upper half plane \( \mathbb{C}_+ \) by

\[
G(z, \zeta) = E(z - \zeta) - E(z - \overline{\zeta}), \quad z, \zeta \in \overline{\mathbb{C}_+}, \; z \neq \zeta,
\]

then we define the Poisson kernel \( P(z, \xi) \) when \( z \in \mathbb{C}_+ \) and \( \xi \in \partial \mathbb{C}_+ \) by

\[
P(z, \xi) = -\frac{\partial G(z, \zeta)}{\partial \eta} \bigg|_{\eta=0} = \frac{y}{\pi |z - \xi|^2}.
\]

The Dirichlet problem of the upper half plane is to find a function \( u \) satisfying

\[
\begin{align*}
    u &\in C^2(\mathbb{C}_+), \quad (2.1.3) \\
    \Delta u &= 0, z \in \mathbb{C}_+, \quad (2.1.4) \\
    \lim_{z \to x} u(z) &= f(x) \text{ nontangentially a.e.} x \in \partial \mathbb{C}_+, \quad (2.1.5)
\end{align*}
\]
where \( f \) is a measurable function of \( R \). The Poisson integral of the upper half plane is defined by

\[
v(z) = P[f](z) = \int_R P(z, \xi) f(\xi) d\xi,
\]

(2.1.6)

where \( P(z, \xi) \) is defined by (2.1.2).

As we all know, the Poisson integral \( P[f] \) exists if

\[
\int_R \frac{|f(\xi)|}{1 + |\xi|^2} d\xi < \infty.
\]

(2.1.7)

(see [1], [14] and [31]) In this chapter, we replace the condition into

\[
\int_R \frac{|f(\xi)|^p}{(1 + |\xi|)^\gamma} d\xi < \infty,
\]

(2.1.8)

where \( 1 \leq p < \infty \) and \( \gamma \) is a real number, then we can get the asymptotic behaviour of harmonic functions.

Next, we will generalize these results to subharmonic functions.

### 2.2 Preliminary Lemma

Let \( \mu \) be a positive Borel measure in \( C \), \( \beta \geq 0 \), the maximal function \( M(d\mu)(z) \) of order \( \beta \) is defined by

\[
M(d\mu)(z) = \sup_{0<r<\infty} \frac{\mu(B(z, r))}{r^\beta},
\]

then the maximal function \( M(d\mu)(z) : C \to [0, \infty) \) is lower semicontinuous, hence measurable. To see this, for any \( \lambda > 0 \), let \( D(\lambda) = \{ z \in C : M(d\mu)(z) > \lambda \} \). Fix \( z \in D(\lambda) \), then there exists \( r > 0 \) such that \( \mu(B(z, r)) > tr^\beta \) for some \( t > \lambda \), and there exists \( \delta > 0 \) satisfying \( (r+\delta)^\beta < \frac{t^\beta}{\lambda} \). If \( |\zeta - z| < \delta \), then \( B(\zeta, r+\delta) \supset B(z, r) \), therefore \( \mu(B(\zeta, r+\delta)) \geq tr^\beta = t(\frac{r}{r+\delta})^\beta(r+\delta)^\beta > \lambda(r+\delta)^\beta \). Thus \( B(z, \delta) \subset D(\lambda) \).

This proves that \( D(\lambda) \) is open for each \( \lambda > 0 \).

In order to obtain the results, we need the lemma below:

**Lemma 2.2.1** Let \( \mu \) be a positive Borel measure in \( C \), \( \beta \geq 0 \), \( \mu(C) < \infty \), for any \( \lambda \geq 5^\beta \mu(C) \), set

\[
E(\lambda) = \{ z \in C : |z| \geq 2, M(d\mu)(z) > \frac{\lambda}{|z|^\beta} \},
\]

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then there exists $z_j \in E(\lambda)$, $\rho_j > 0$, $j = 1, 2, \cdots$, such that

$$E(\lambda) \subset \bigcup_{j=1}^{\infty} B(z_j, \rho_j) \tag{2.2.1}$$

and

$$\sum_{j=1}^{\infty} \frac{\rho_j^\beta}{|z_j|^\beta} \leq \frac{3\mu(C)5^\beta}{\lambda}. \tag{2.2.2}$$

Proof: Let $E_k(\lambda) = \{ z \in E(\lambda) : 2^k \leq |z| < 2^{k+1} \}$, then for any $z \in E_k(\lambda)$, there exists $r(z) > 0$, such that $\mu(B(z, r(z))) > \lambda \left( \frac{r(z)}{|z|} \right)^\beta$, therefore $r(z) \leq 2^{k-1}$. Since $E_k(\lambda)$ can be covered by the union of a family of balls $\{ B(z, r(z)) : z \in E_k(\lambda) \}$, by the Vitali Lemma [37], there exists $\Lambda_k \subset E_k(\lambda)$, $\Lambda_k$ is at most countable, such that $\{ B(z, r(z)) : z \in \Lambda_k \}$ are disjoint and

$$E_k(\lambda) \subset \bigcup_{z \in \Lambda_k} B(z, 5r(z)),$$

so

$$E(\lambda) = \bigcup_{k=1}^{\infty} E_k(\lambda) \subset \bigcup_{k=1}^{\infty} \bigcup_{z \in \Lambda_k} B(z, 5r(z)).$$

On the other hand, note that $\bigcup_{z \in \Lambda_k} B(z, r(z)) \subset \{ z : 2^{k-1} \leq |z| < 2^{k+2} \}$, so that

$$\sum_{z \in \Lambda_k} \frac{(5r(z))^\beta}{|z|^\beta} \leq 5^\beta \sum_{z \in \Lambda_k} \frac{\mu(B(z, r(z)))}{\lambda} \leq \frac{5^\beta}{\lambda} \mu \{ z : 2^{k-1} \leq |z| < 2^{k+2} \}. \tag{2.3.1}$$

Hence we obtain

$$\sum_{k=1}^{\infty} \sum_{z \in \Lambda_k} \frac{(5r(z))^\beta}{|z|^\beta} \leq \sum_{k=1}^{\infty} \frac{5^\beta}{\lambda} \mu \{ z : 2^{k-1} \leq |z| < 2^{k+2} \} \leq \frac{3\mu(C)5^\beta}{\lambda}. \tag{2.3.2}$$

Rearrange $\{ z : z \in \Lambda_k, k = 1, 2, \cdots \}$ and $\{ 5r(z) : z \in \Lambda_k, k = 1, 2, \cdots \}$, we get $\{ z_j \}$ and $\{ \rho_j \}$ such that (2.2.1) and (2.2.2) hold.

### 2.3 $p = 1$

#### 1. Introduction and Main Theorems

In this section, we will consider measurable functions $f$ in $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} \frac{|f(\xi)|}{1 + |\xi|^{2+m}} d\xi < \infty, \tag{2.3.3}$$
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where $m$ is a nonnegative integer. This is just (2.1.8) when $p = 1$ and $\gamma = 2 + m$.

To obtain a solution of Dirichlet problem for the boundary date $f$, as in \[38\], \[39\], \[41\] and \[31\], we use the following modified functions defined by

$$E_m(z - \zeta) = \begin{cases} E(z - \zeta) & \text{when } |\zeta| \leq 1, \\ E(z - \zeta) - \frac{1}{2\pi} \Re \left( \log \zeta - \sum_{k=1}^{m-1} \frac{z^k}{k^{1+\gamma}} \right) & \text{when } |\zeta| > 1. \end{cases}$$

Then we can define the modified Green function $G_m(z, \zeta)$ and the modified Poisson kernel $P_m(z, \xi)$ by (see \[31\] and \[5\])

$$G_m(z, \zeta) = E_{m+1}(z - \zeta) - E_{m+1}(z - \bar{\zeta}), \quad z, \zeta \in \overline{C^+}, \ z \neq \zeta; \quad (2.3.2)$$

$$P_m(z, \xi) = \begin{cases} P(z, \xi) & \text{when } |\xi| \leq 1, \\ P(z, \xi) - \frac{1}{\pi} \Im \sum_{k=0}^{m-1} \frac{z^k}{\xi^{1+\gamma}} & \text{when } |\xi| > 1, \end{cases} \quad (2.3.3)$$

where $z = x + iy, \zeta = \xi + i\eta$.

Hayman \[26\] has proved the following result:

**Theorem B** Let $f$ be a measurable function in $R$ satisfying (2.1.7) and $\mu$ be a positive Borel measure satisfying

$$\int_{C^+} \frac{\eta}{1 + |\zeta|^2} d\mu(\zeta) < \infty.$$

Write the subharmonic function

$$u(z) = v(z) + h(z), \quad z \in C^+,$$

where $v(z)$ is the harmonic function defined by (2.1.6), $h(z)$ is defined by

$$h(z) = \int_{C^+} G(z, \zeta) d\mu(\zeta)$$

and $G(z, \zeta)$ is defined by (2.1.1). Then there exists $z_j \in C^+, \rho_j > 0$, such that

$$\sum_{j=1}^{\infty} \frac{\rho_j}{|z_j|} < \infty$$

holds and

$$u(z) = o(|z|), \quad \text{as } |z| \to \infty$$

holds in $C^+ - G$, where $G = \bigcup_{j=1}^{\infty} B(z_j, \rho_j)$. 

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Our aim in this section is to establish the following theorems.

**Theorem 2.3.1** Let \( f \) be a measurable function in \( \mathbb{R} \) satisfying (2.3.1), and \( 0 < \alpha \leq 2 \). Let \( v(z) \) be the harmonic function defined by
\[
v(z) = \int_{\mathbb{R}} P_m(z, \xi) f(\xi) d\xi, \quad z \in \mathbb{C}_+,
\]
where \( P_m(z, \xi) \) is defined by (2.3.3). Then there exists \( z_j \in \mathbb{C}_+ \), \( \rho_j > 0 \), such that
\[
\sum_{j=1}^{\infty} \frac{\rho_j^{2-\alpha}}{|z_j|^{2-\alpha}} < \infty
\]
holds and
\[
v(z) = o(\eta^{1-\alpha}|z|^{m+\alpha}), \quad \text{as } |z| \to \infty
\]
holds in \( \mathbb{C}_+ - G \), where \( G = \bigcup_{j=1}^{\infty} B(z_j, \rho_j) \).

**Remark 2.3.1** If \( \alpha = 2 \), then (2.3.5) is a finite sum, the set \( G \) is the union of finite disks, so (2.3.6) holds in \( \mathbb{C}_+ \).

Next, we will generalize Theorem 2.3.1 to subharmonic functions.

**Theorem 2.3.2** Let \( f \) be a measurable function in \( \mathbb{R} \) satisfying (2.3.1) and \( \mu \) be a positive Borel measure satisfying
\[
\int_{\mathbb{C}_+} \frac{\eta}{1+|\xi|^{2+m}} d\mu(\xi) < \infty.
\]
Write the subharmonic function
\[
u(z) = v(z) + h(z), \quad z \in \mathbb{C}_+,
\]
where \( v(z) \) is the harmonic function defined by (2.3.4), \( h(z) \) is defined by
\[
h(z) = \int_{\mathbb{C}_+} G_m(z, \xi) d\mu(\xi)
\]
and \( G_m(z, \xi) \) is defined by (2.3.2). Then there exists \( z_j \in \mathbb{C}_+ \), \( \rho_j > 0 \), such that (2.3.5) holds and
\[
u(z) = o(\eta^{1-\alpha}|z|^{m+\alpha}), \quad \text{as } |z| \to \infty
\]
holds in \( \mathbb{C}_+ - G \), where \( G = \bigcup_{j=1}^{\infty} B(z_j, \rho_j) \) and \( 0 < \alpha < 2 \).

**Remark 2.3.2** If \( \alpha = 1, m = 0 \), this is just the result of Hayman, so our result (2.3.7) is the generalization of Theorem B.
2. Main Lemma

In order to obtain the results, we need the following lemma:

**Lemma 2.3.1** The following inequalities hold:

1. If \( |\xi| > 1 \), then \( |P_m(z, \zeta) - P(z, \zeta)| \leq \sum_{k=0}^{m-1} \frac{2^k|\xi|^k}{\pi|\xi|^{2+m}} \).
2. If \( |\xi - z| > 3|z| \), then \( |P_m(z, \zeta)| \leq \frac{2m+1}{\pi|\xi|^{m+2}} \).
3. If \( |\xi| > 1 \), then \( |G_m(z, \zeta) - G(z, \zeta)| \leq \frac{1}{\pi} \sum_{k=1}^{m} \frac{ky\eta|\xi|^{k-1}}{|\xi|^{1+k}} \).
4. If \( |\xi - z| > 3|z| \), then \( |G_m(z, \zeta)| \leq \frac{1}{\pi} \sum_{k=m+1}^{\infty} \frac{ky\eta|\xi|^{k-1}}{|\xi|^{1+k}} \).

3. Proof of Theorems

**Proof of Theorem 2.3.1**

Define the measure \( dm(\xi) \) and the kernel \( K(z, \xi) \) by

\[
 dm(\xi) = \frac{|f(\xi)|}{1 + |\xi|^{2+m}} d\xi, \quad K(z, \xi) = P_m(z, \xi)(1 + |\xi|^{2+m}).
\]

For any \( \varepsilon > 0 \), there exists \( R_\varepsilon > 2 \), such that

\[
 \int_{|\xi| \geq R_\varepsilon} dm(\xi) \leq \frac{\varepsilon}{5^{2-\alpha}}.
\]

For every Lebesgue measurable set \( E \subset \mathbb{R} \), the measure \( m^{(\varepsilon)} \) defined by \( m^{(\varepsilon)}(E) = m(E \cap \{ x \in \mathbb{R} : |x| \geq R_\varepsilon \}) \) satisfies \( m^{(\varepsilon)}(\mathbb{R}) \leq \frac{\varepsilon}{5^{2-\alpha}} \), write

\[
 v_1(z) = \int_{|\xi - z| \leq 3|z|} P(z, \xi)(1 + |\xi|^{2+m}) dm^{(\varepsilon)}(\xi),
\]

\[
 v_2(z) = \int_{|\xi - z| \leq 3|z|} (P_m(z, \xi) - P(z, \xi))(1 + |\xi|^{2+m}) dm^{(\varepsilon)}(\xi),
\]

\[
 v_3(z) = \int_{|\xi - z| > 3|z|} K(z, \xi) dm^{(\varepsilon)}(\xi),
\]

\[
 v_4(z) = \int_{1 < |\xi| < R_\varepsilon} K(z, \xi) dm(\xi),
\]

\[
 v_5(z) = \int_{|\xi| \leq 1} K(z, \xi) dm(\xi),
\]

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then

\[ |v(z)| \leq |v_1(z)| + |v_2(z)| + |v_3(z)| + |v_4(z)| + |v_5(z)|. \quad (2.3.8) \]

Let \( E_1(\lambda) = \{ z \in \mathbb{C} : |z| \geq 2, \exists t > 0, s.t. m^{(e)}(B(z, t) \cap \mathbb{R}) > \lambda \left( \frac{t}{|z|} \right)^{2-\alpha} \} \), therefore, if \( |z| \geq 2R_e \) and \( z \not\in E_1(\lambda) \), then

\[ \forall t > 0, m^{(e)}(B(z, t) \cap \mathbb{R}) \leq \lambda \left( \frac{t}{|z|} \right)^{2-\alpha}. \]

So we have

\[
|v_1(z)| \leq \int_{y \leq |\xi - z| \leq 3|z|} \frac{y}{|z - \xi|^2} 2|x|^{2+m}dm^{(e)}(\xi) \\
\leq \int_{y \leq |\xi - z| \leq 3|z|} \frac{2y}{|z - \xi|^2} (4|z|)^{2+m}dm^{(e)}(\xi) \\
= \frac{2^{2m+5}}{\pi} |y|^{2+m} \int_{y \leq |\xi - z| \leq 3|z|} \frac{1}{|z - \xi|^2}dm^{(e)}(\xi) \\
= \frac{2^{2m+5}}{\pi} |y|^{m+2} \int_{y}^{3|z|} \frac{1}{t^2}dm^{(e)}_z(t),
\]

where \( m^{(e)}_z(t) = \int_{|\xi - z| \leq t} dm^{(e)}(\xi) \), since for \( z \not\in E_1(\lambda) \),

\[
\int_{y}^{3|z|} \frac{1}{t^2}dm^{(e)}_z(t) \leq \frac{m^{(e)}_z(3|z|)}{(3|z|)^2} + 2 \int_{y}^{3|z|} \frac{m^{(e)}_z(t)}{t^3}dt \\
\leq \frac{\lambda}{3^\alpha|z|^2} + 2 \int_{y}^{3|z|} \frac{\lambda t^{2-\alpha}}{|z|^2 dt} \\
\leq \frac{\lambda}{|z|^2} \left( \frac{1}{3^\alpha} + \frac{2|z|^\alpha}{\alpha y^\alpha} \right),
\]

so that

\[
|v_1(z)| \leq \frac{2^{2m+5}}{\pi} |y|^{m+2} \frac{\lambda}{|z|^2} \left( \frac{1}{3^\alpha} + \frac{2|z|^\alpha}{\alpha y^\alpha} \right) \\
\leq \frac{2^{2m+5}}{\pi} \left( \frac{1}{3^\alpha} + \frac{2}{\alpha} \right) \lambda y^{1-\alpha}|z|^{m+\alpha}. \quad (2.3.9)
\]
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By (1) of Lemma 2.3.1, we obtain

$$|v_2(z)| \leq \int_{y \leq |\xi - z| \leq 3|z|} \frac{m-1}{\pi} \sum_{k=0}^{m-1} 2^k y |z|^k 2|\xi|^{2+m} dm(\xi)(\xi)$$

$$\leq \int_{y \leq |\xi - z| \leq 3|z|} \frac{m-1}{\pi} \sum_{k=0}^{m-1} 2^k y |z|^k (4|z|)^{m-k} dm(\xi)(\xi)$$

$$\leq \frac{2^{m+1} m-1}{\pi} \sum_{k=0}^{m-1} \frac{1}{2^k 5^2 - \alpha} \epsilon y |z|^m$$

$$\leq \frac{4^{m-1+\alpha}}{\pi} \epsilon y |z|^m. \quad (2.3.10)$$

By (2) of Lemma 2.3.1, we see that

$$|v_3(z)| \leq \int_{|\xi - z| > 3|z|} \frac{2^{m+1} y |z|^m}{\pi |\xi|^{m+2}} \frac{2^{m+2} y |z|^m}{\pi} dm(\xi)(\xi)$$

$$= \int_{|\xi - z| > 3|z|} \frac{2^{m+2} y |z|^m}{\pi} dm(\xi)(\xi)$$

$$\leq \frac{2^{m+2} \epsilon}{\pi} \frac{y |z|^m}{5^2 - \alpha}$$

$$\leq \frac{2^{m+2+2\alpha}}{\pi} \epsilon y |z|^m. \quad (2.3.11)$$

Write

$$v_4(z) = \int_{1 < |\xi| < R_e} [P(z, \xi) + (P_m(z, \xi) - P(z, \xi))] (1 + |\xi|^{2+m}) dm(\xi)$$

$$= v_{41}(z) - v_{42}(z),$$

then

$$|v_{41}(z)| \leq \int_{1 < |\xi| < R_e} \frac{y}{|z - \xi|^2} |\xi|^{2+m} dm(\xi)$$

$$\leq \frac{2R_e^{2+m} y}{\pi} \int_{1 < |\xi| < R_e} \frac{1}{|\xi|^2} dm(\xi)$$

$$\leq \frac{2^3 R_e^{2+m} m(R)}{\pi} \frac{y}{|z|^2}. \quad (2.3.12)$$
Moreover, by (1) of Lemma 2.3.1, we obtain

\[ |v_{42}(z)| \leq \int_{1<|\xi|<R} \sum_{k=0}^{m-1} \frac{2^k y |\xi|^k}{\pi |\xi|^{2+k}} \cdot 2|\xi|^{2+m} dm(\xi) \]

\[ \leq \sum_{k=0}^{m-1} \frac{2^k y |\xi|^k}{\pi} \cdot m(\mathbb{R}) \]

\[ \leq \frac{2y}{\pi} \frac{R^m m(\mathbb{R})}{|z|^{m-1}}. \quad \tag{2.3.13} \]

In case \(|\xi| \leq 1\), note that

\[ K(z, \xi) = P_m(z, \xi)(1 + |\xi|^{2+m}) \leq \frac{2y}{\pi |z-\xi|^2}, \]

so that

\[ |v_5(z)| \leq \int_{|\xi|\leq 1} \frac{2y}{\pi |z-\xi|^2} dm(\xi) \leq \int_{|\xi|\leq 1} \frac{2y}{\pi |\xi|^2} dm(\xi) \leq \frac{2y}{\pi} \frac{1}{|z|^2}. \quad \tag{2.3.14} \]

Thus, by collecting (2.3.8), (2.3.9), (2.3.10), (2.3.11), (2.3.12), (2.3.13) and (2.3.14), there exists a positive constant \(A\) independent of \(\varepsilon\), such that if \(|z| \geq 2R_e\) and \(z \notin E_1(\varepsilon)\), we have

\[ |v(z)| \leq A\varepsilon y^{1-\alpha} |z|^{m+\alpha}. \]

Let \(\mu_\varepsilon\) be a measure in \(\mathbb{C}\) defined by \(\mu_\varepsilon(E) = m(\xi)(E \cap \mathbb{R})\) for every measurable set \(E\) in \(\mathbb{C}\). Take \(\varepsilon = \varepsilon_p = \frac{1}{2^{p+2}}, p = 1, 2, 3, \ldots\), then there exists a sequence \(\{R_p\}\): \(1 = R_0 < R_1 < R_2 < \cdots\) such that

\[ \mu_{\varepsilon_p}(\mathbb{C}) = \int_{|\xi| \geq R_p} dm(\xi) < \frac{\varepsilon_p}{2^{p+2}}. \]

Take \(\lambda = 3 \cdot 5^{2-\alpha} \cdot 2^p \mu_{\varepsilon_p}(\mathbb{C})\) in Lemma 2.2.1, then \(\exists z_{j,p}\) and \(\rho_{j,p}\), where \(R_{p-1} \leq |z_{j,p}| < R_p\) such that

\[ \sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|z_{j,p}|} \right)^{2-\alpha} \leq \frac{1}{2^p}. \]

So if \(R_{p-1} \leq |z| < R_p\) and \(z \notin G_p = \bigcup_{j=1}^{\infty} B(z_{j,p}, \rho_{j,p})\), we have

\[ |v(z)| \leq A\varepsilon_p y^{1-\alpha} |z|^{m+\alpha}, \]

thereby

\[ \sum_{p=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|z_{j,p}|} \right)^{2-\alpha} \leq \sum_{p=1}^{\infty} \frac{1}{2^p} = 1 < \infty. \]
Set \( G = \bigcup_{p=1}^{\infty} G_p \), thus Theorem 2.3.1 holds.

**Proof of Theorem 2.3.2**

Define the measure \( dn(\zeta) \) and the kernel \( L(z, \zeta) \) by

\[
d(\zeta) = \frac{\eta d\mu(\zeta)}{1 + |\zeta|^{2+m}}, \quad L(z, \zeta) = G_m(z, \zeta) \frac{1 + |\zeta|^{2+m}}{\eta},
\]

then the function \( h(z) \) can be written as

\[
h(z) = \int_{C_+} L(z, \zeta) d(\zeta).
\]

For any \( \varepsilon > 0 \), there exists \( R_\varepsilon > 2 \), such that

\[
\int_{|\zeta| \geq R_\varepsilon} d(\zeta) < \frac{\varepsilon}{S^{2-\alpha}}.
\]

For every Lebesgue measurable set \( E \subset \mathbb{C} \), the measure \( n^{(\varepsilon)} \) defined by \( n^{(\varepsilon)}(E) = n(E \cap \{ \zeta \in C_+ : |\zeta| \geq R_\varepsilon \} \) satisfies \( n^{(\varepsilon)}(C_+) \leq \frac{\varepsilon}{S^{2-\alpha}} \), write

\[
\begin{align*}
h_1(z) &= \int_{|\zeta - z| \leq \frac{5}{4}} G(z, \zeta) \frac{1 + |\zeta|^{2+m}}{\eta} d(\zeta), \\
h_2(z) &= \int_{\frac{5}{4} < |\zeta - z| \leq 3|z|} G(z, \zeta) \frac{1 + |\zeta|^{2+m}}{\eta} d(\zeta), \\
h_3(z) &= \int_{|\zeta - z| \leq 3|z|} [G_m(z, \zeta) - G(z, \zeta)] \frac{1 + |\zeta|^{2+m}}{\eta} d(\zeta), \\
h_4(z) &= \int_{|\zeta - z| > 3|z|} L(z, \zeta) d(\zeta), \\
h_5(z) &= \int_{1 < |\zeta| < R_\varepsilon} L(z, \zeta) d(\zeta), \\
h_6(z) &= \int_{|\zeta| \leq 1} L(z, \zeta) d(\zeta),
\end{align*}
\]

then

\[
h(z) = h_1(z) + h_2(z) + h_3(z) + h_4(z) + h_5(z) + h_6(z). \tag{2.3.15}
\]

Let \( E_2(\lambda) = \{ z \in \mathbb{C} : |z| \geq 2, \exists t > 0, s.t. n^{(\varepsilon)}(B(z, t) \cap C_+) > \lambda \left( \frac{t}{|z|} \right)^{2-\alpha} \} \), therefore, if \( |z| \geq 2R_\varepsilon \) and \( z \notin E_2(\lambda) \), then

\[
\forall t > 0, \ n^{(\varepsilon)}(B(z, t) \cap C_+) \leq \lambda \left( \frac{t}{|z|} \right)^{2-\alpha}.
\]
So we have

\[ |h_1(z)| \leq \int_{|\zeta - z| \leq \frac{3}{2}} \frac{1}{2\pi} \log \frac{|\zeta - z|}{|z|} \frac{1 + |\zeta|^{2+m}}{\eta} dn^{(e)}(\zeta) \]
\[ \leq \int_{|\zeta - z| \leq \frac{3}{2}} \frac{1}{2\pi} \log \frac{3y}{|\zeta - z|} \frac{2|\zeta|^{2+m}}{t} dn^{(e)}(\zeta) \]
\[ \leq 2 \times \frac{3/2}{2} \frac{|\zeta|^{2+m}}{y} \int_{|\zeta - z| \leq \frac{3}{2}} \log \frac{3y}{t} \frac{dn^{(e)}(\zeta)}{t} \]
\[ = 2 \times \frac{3/2}{2} \frac{|\zeta|^{2+m}}{y} \int_{0}^{\frac{3}{2}} \log \frac{3y}{t} \frac{dn^{(e)}(t)}{t} \]
\[ \leq 2 \times \frac{3/2}{2} \frac{|\zeta|^{2+m}}{y} \left[ \log 6 \frac{1}{2^{2-\alpha}} + \frac{1}{(2 - \alpha)2^{2-\alpha}} \right] \lambda y^{1-\alpha}|z|^{m+\alpha}, \quad (2.3.16) \]

where \( n^{(e)}(t) = \int_{|\zeta - z| \leq t} dn^{(e)}(\zeta) \).

Note that

\[ |G(z, \zeta)| = |E(z - \zeta) - E(z - \bar{\zeta})| \leq \frac{\eta}{\pi|z - \bar{\zeta}|^2}, \quad (2.3.17) \]

then by (2.3.17), we have

\[ |h_2(z)| \leq \int_{\frac{3}{2} < |\zeta - z| \leq 3|z|} \frac{\eta}{\pi|z - \bar{\zeta}|^2} \frac{2|\zeta|^{2+m}}{\eta} dn^{(e)}(\zeta) \]
\[ \leq \frac{2}{\pi} y(4|z|)^{2+m} \int_{\frac{3}{2} < |\zeta - z| \leq 3|z|} \frac{1}{|z - \bar{\zeta}|^2} dn^{(e)}(\zeta) \]
\[ = \frac{2^{2m+5}}{\pi} y|z|^{2+m} \int_{\frac{3}{2}}^{3|z|} \frac{1}{t^2} dn^{(e)}(t) \]
\[ \leq \frac{2^{2m+5}}{\pi} y|z|^{2+m} \left( \frac{1}{3^\alpha} + \frac{2^{\alpha+1}}{\alpha y^\alpha} \right) \lambda y^{1-\alpha}|z|^{m+\alpha} \]
\[ \leq \frac{2^{2m+5}}{\pi} \left( \frac{1}{3^\alpha} + \frac{2^{\alpha+1}}{\alpha} \right) \lambda y^{1-\alpha}|z|^{m+\alpha}. \quad (2.3.18) \]
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By (3) of Lemma 2.3.1, we obtain

\begin{align*}
|h_3(z)| &\leq \int_{|\zeta - z| \leq 3|z|} \left( \frac{m}{\pi} \sum_{k=1}^{m} \frac{ky}{|\zeta|^{1+k} \eta} \right) \frac{2|\zeta|^{2+m}}{\eta} \frac{dn(\zeta)}{\zeta} \\
&= \int_{|\zeta - z| \leq 3|z|} \left( \frac{2}{\pi} \sum_{k=1}^{m} \frac{ky|z|^{k-1}|\zeta|^{m-k+1}}{|\zeta|^{1+k}} dn(\zeta) \right) \\
&\leq \int_{|\zeta - z| > 3|z|} \left( \frac{2}{\pi} \sum_{k=1}^{m} \frac{ky|z|^{k-1}|\zeta|^{m-k+1}}{|\zeta|^{1+k}} dn(\zeta) \right) \\
&\leq \frac{2}{\pi} \sum_{k=1}^{m} \frac{4^{m-k+1}}{k} \frac{1}{5^{2-\alpha} \pi} |y|^m \\
&\leq \frac{2^{m+2\alpha+1}}{9\pi} \epsilon y |z|^m. \quad (2.3.19)
\end{align*}

By (4) of Lemma 2.3.1, we see that

\begin{align*}
|h_4(z)| &\leq \int_{|\zeta - z| > 3|z|} \left( \frac{m}{\pi} \sum_{k=m+1}^{\infty} \frac{ky|z|^{k-1}}{|\zeta|^{1+k}} \right) \frac{2|\zeta|^{2+m}}{\eta} \frac{dn(\zeta)}{\zeta} \\
&= \int_{|\zeta - z| > 3|z|} \left( \frac{2}{\pi} \sum_{k=m+1}^{\infty} \frac{ky|z|^{k-1}}{|\zeta|^{1+(m+k)}} dn(\zeta) \right) \\
&\leq \int_{|\zeta - z| > 3|z|} \left( \frac{2}{\pi} \sum_{k=m+1}^{\infty} \frac{ky|z|^{k-1}}{|\zeta|^{1+(m+k)}} dn(\zeta) \right) \\
&\leq \frac{2^{m+2\alpha+1}}{\pi} \sum_{k=m+1}^{\infty} \frac{k}{2^{k}} \left( \frac{1}{5^{2-\alpha} \pi} \right) |y|^m \\
&\leq \frac{4^{\alpha-1}(m+2)}{\pi} \epsilon y |z|^m. \quad (2.3.20)
\end{align*}

Write

\begin{align*}
h_5(z) &= \int_{1 < |\zeta| < K \epsilon} \left[ G(z, \zeta) + (G_m(z, \zeta) - G(z, \zeta)) \right] \frac{1 + |\zeta|^{2+m}}{\eta} dn(\zeta) \\
&= h_{51}(z) + h_{52}(z),
\end{align*}
then we obtain by (2.3.17)

\[ |h_{51}(z)| \leq \int_{1<|\zeta|<R_e} \frac{y \eta}{\pi |z-\zeta|^2} \frac{2|\zeta|^{2+m}}{\eta} dn(\zeta) \]

\[ \leq \int_{1<|\zeta|<R_e} \frac{2 y R_e^{2+m}}{\pi |z-\zeta|^2} dn(\zeta) \]

\[ \leq \frac{2R_e^{2+m}}{\pi} y \int_{1<|\zeta|<R_e} \frac{1}{(\frac{\zeta}{|z|})^2} dn(\zeta) \]

\[ \leq \frac{2^3 R_e^{2+m} n(C_+)}{\pi} \frac{y}{|z|^2}. \quad (2.3.21) \]

Moreover, by (3) of Lemma 2.3.1, we obtain

\[ |h_{52}(z)| \leq \int_{1<|\zeta|<R_e} \frac{1}{\pi} \sum_{k=1}^{m} \frac{k y \eta |z|^{k-1} 2|\zeta|^{2+m}}{|\zeta|^{1+k}} \eta dn(\zeta) \]

\[ = \int_{1<|\zeta|<R_e} \frac{2}{\pi} \sum_{k=1}^{m} k y |z|^{k-1} |\zeta|^{m-k+1} dn(\zeta) \]

\[ \leq \int_{1<|\zeta|<R_e} \frac{2}{\pi} \sum_{k=1}^{m} k y |z|^{k-1} R_e^{m-k+1} dn(\zeta) \]

\[ \leq \frac{m(m+1)R_e^m n(C_+)}{\pi} y |z|^{m-1}. \quad (2.3.22) \]

In case $|\zeta| \leq 1$, by (2.17), we have

\[ |L(z, \zeta)| \leq \frac{y \eta}{\pi |z-\zeta|^2} \frac{2}{\eta} = \frac{2y}{\pi |z-\zeta|^2}, \]

so that

\[ |h_6(z)| \leq \int_{|\zeta| \leq 1} \frac{2y}{\pi |z-\zeta|^2} dn(\zeta) \leq \int_{|\zeta| \leq 1} \frac{2y}{\pi (\frac{|z|}{\zeta})^2} dn(\zeta) \leq \frac{2^3 n(C_+)}{\pi} \frac{y}{|z|^2}. \]

Thus, by collecting (2.3.15), (2.3.16), (2.3.18), (2.3.19), (2.3.20), (2.3.21), (2.3.22) and (2.3.23), there exists a positive constant $A$ independent of $\varepsilon$, such that if $|z| \geq 2R_e$ and $z \notin E_2(\varepsilon)$, we have

\[ |h(z)| \leq A \varepsilon y^{1-\alpha} |z|^{m+\alpha}. \]

Similarly, if $z \notin G$, we have

\[ h(z) = o(y^{1-\alpha} |z|^{m+\alpha}), \quad \text{as } |z| \to \infty. \quad (2.3.24) \]

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By (2.3.6) and (2.3.24), we obtain that

\[ u(z) = v(z) + h(z) = o(y^{1-\alpha}|z|^{m+\alpha}), \quad \text{as } |z| \to \infty \]

holds in \( C_+ - G \).

### 2.4 \( p > 1 \) (General Kernel)

#### 1. Introduction and Main Theorems

In this section, we will consider measurable functions \( f \) in \( R \) satisfying

\[
\int_{\mathbb{R}} \frac{|f(\xi)|^p}{(1 + |\xi|)^q} d\xi < \infty,
\]

where \( \gamma \) is defined as in Theorem 2.4.1.

In order to describe the asymptotic behaviour of subharmonic functions in the upper half plane (see \([28],[29],\) and \([30])\), we establish the following theorems.

**Theorem 2.4.1** Let \( 1 \leq p < \infty, \frac{1}{p} + \frac{1}{q} = 1 \) and

\[ 1 - p < \gamma < 1 + p \quad \text{in case } p > 1; \]
\[ 0 < \gamma \leq 2 \quad \text{in case } p = 1. \]

If \( f \) is a measurable function in \( R \) satisfying (2.4.1) and \( v(z) \) is the harmonic function defined by (2.1.6), then there exists \( z_j \in C_+, \rho_j > 0 \), such that

\[
\sum_{j=1}^{\infty} \frac{\rho_j^{2p-\alpha}}{|z_j|^{2p-\alpha}} < \infty
\]

holds and

\[ v(z) = o(y^{1-\alpha}|z|^{\frac{q}{p} + \frac{1}{q} - 2 + \frac{\alpha}{p}}), \quad \text{as } |z| \to \infty \]

holds in \( C_+ - G, \) where \( G = \bigcup_{j=1}^{\infty} B(z_j, \rho_j) \) and \( 0 < \alpha \leq 2p. \)

**Remark 2.4.1** If \( \gamma = 1 - p, p > 1, \) then

\[ v(z) = o(y^{1-\alpha} (\log |z|)^{\frac{1}{p}} |z|^{\frac{q}{p} + \frac{1}{q} - 2 + \frac{\alpha}{p}}), \quad \text{as } |z| \to \infty \]

holds in \( C_+ - G. \)
Next, we will generalize Theorem 2.4.1 to subharmonic functions.

**Theorem 2.4.2** Let $p$ and $\gamma$ be as in Theorem 2.4.1. If $f$ is a measurable function in $\mathbb{R}$ satisfying (2.4.1) and $\mu$ is a positive Borel measure satisfying
\[
\int_{\mathbb{C}^+} \frac{\eta^p}{(1+|\zeta|)^\gamma} d\mu(\zeta) < \infty
\]
and
\[
\int_{\mathbb{C}^+} \frac{1}{1+|\zeta|} d\mu(\zeta) < \infty.
\]
Write the subharmonic function
\[
u(z) = \nu(z) + h(z), \quad z \in \mathbb{C}^+,
\]
where $\nu(z)$ is the harmonic function defined by (2.1.6), $h(z)$ is defined by
\[
h(z) = \int_{\mathbb{C}^+} G(z, \zeta) d\mu(\zeta)
\]
and $G(z, \zeta)$ is defined by (2.1.1). Then there exists $z_j \in \mathbb{C}^+$, $\rho_j > 0$, such that (2.4.2) holds and
\[
u(z) = o(\gamma_{1-\alpha} |z|^\gamma_{1+\frac{1}{q} - 2 + \frac{\alpha}{p}}), \quad \text{as } |z| \to \infty
\]
holds in $\mathbb{C}^+ - G$, where $G = \bigcup_{j=1}^\infty B(z_j, \rho_j)$ and $0 < \alpha < 2p$.

**Remark 2.4.2** If $\gamma = 1 - p$, $p > 1$, then
\[
u(z) = o(\gamma_{1-\alpha} (\log |z|)^{\frac{1}{q} |z|^{\gamma_{1+\frac{1}{q} - 2 + \frac{\alpha}{p}}}}), \quad \text{as } |z| \to \infty
\]
holds in $\mathbb{C}^+ - G$.

**Remark 2.4.3** If $\alpha = 1$, $p = 1$ and $\gamma = 2$, then (2.4.2) holds and (2.4.4) holds in $\mathbb{C}^+ - G$. This is just the the result of Hayman, therefore, our result (2.4.4) is the generalization of Theorem B.

2. **Main Lemmas**

In order to obtain the results, we need these lemmas below:
Lemma 2.4.1 The kernel function \( \frac{1}{|z-\zeta|^2} \) has the following estimates:

1. If \( |\zeta| \leq \frac{|z|}{2} \), then \( \frac{1}{|z-\zeta|^2} \leq \frac{4}{|z|^2} \);
2. If \( |\zeta| > 2|z| \), then \( \frac{1}{|z-\zeta|^2} \leq \frac{4}{|\zeta|^2} \).

Lemma 2.4.2 The Green function \( G(z, \zeta) \) has the following estimates:

1. \( |G(z, \zeta)| \leq A \log \frac{3y}{|z-\zeta|} \);
2. \( |G(z, \zeta)| \leq \frac{y}{2|z-\zeta|^2} \).

Proof: (1) is obvious; (2) follows by the Mean Value Theorem for Derivatives.

Lemma 2.4.3 The following estimate holds:

\[
\int_0^\frac{\pi}{2} t^{2p-\alpha-1} \left( \log \frac{3y}{t} \right)^{p-1} dt \leq \frac{3^{2p-\alpha}}{(2p-\alpha)^p} \Gamma(p) y^{2p-\alpha}.
\]

3. Proof of Theorems

Proof of Theorem 2.4.1

We prove only the case \( p > 1 \); the proof of the case \( p = 1 \) is similar. Define the measure \( dm(\xi) \) by

\[
dm(\xi) = \frac{|f(\xi)|^p}{(1+|\xi|)^\gamma} d\xi.
\]

For any \( \varepsilon > 0 \), there exists \( R_\varepsilon > 2 \), such that

\[
\int_{|\xi| \geq R_\varepsilon} dm(\xi) \leq \frac{\varepsilon^p}{52p-\alpha}.
\]

For every Lebesgue measurable set \( E \subset \mathbb{R} \), the measure \( m^{(\varepsilon)} \) defined by \( m^{(\varepsilon)}(E) = m(E \cap \{ x \in \mathbb{R} : |x| \geq R_\varepsilon \}) \) satisfies \( m^{(\varepsilon)}(\mathbb{R}) \leq \frac{\varepsilon^p}{52p-\alpha} \). Write

\[
\begin{align*}
\nu_1(z) &= \int_{G_1} P(z, \xi) f(\xi) d\xi, \\
\nu_2(z) &= \int_{G_2} P(z, \xi) f(\xi) d\xi, \\
\nu_3(z) &= \int_{G_3} P(z, \xi) f(\xi) d\xi, \\
\nu_4(z) &= \int_{G_4} P(z, \xi) f(\xi) d\xi,
\end{align*}
\]
where
\[
G_1 = \{ \xi \in \mathbb{R} : R_\varepsilon < |\xi| \leq \frac{|z|}{2} \},
\]
\[
G_2 = \{ \xi \in \mathbb{R} : \frac{|z|}{2} < |\xi| \leq 2|z| \},
\]
\[
G_3 = \{ \xi \in \mathbb{R} : |\xi| > 2|z| \},
\]
\[
G_4 = \{ \xi \in \mathbb{R} : |\xi| \leq R_\varepsilon \}.
\]

Then
\[
v(z) = v_1(z) + v_2(z) + v_3(z) + v_4(z). \quad (2.4.5)
\]

First, if \( \gamma > 1 - p \), then \( \frac{\gamma}{p} + 1 > 0 \), so that we obtain by (1) of Lemma 2.4.1 and Hölder’s inequality
\[
|v_1(z)| \leq \int_{G_1} \frac{y}{\pi |z|^2} |f(\xi)| d\xi \leq \frac{4}{\pi} \left( \int_{G_1} \frac{|f(\xi)|^p}{|\xi|^q} d\xi \right)^{1/p} \left( \int_{G_1} |\xi|^\frac{\gamma y}{p} d\xi \right)^{1/q},
\]

since
\[
\int_{G_1} |\xi|^\frac{\gamma y}{p} d\xi \leq \frac{2}{\frac{\gamma y}{p} + 1} \left( \frac{|z|}{2} \right)^{\frac{\gamma y}{p} + 1},
\]

so that
\[
|v_1(z)| \leq A\varepsilon |z|^\frac{\gamma}{p} + \frac{1}{q} - 2. \quad (2.4.6)
\]

Let \( E_1(\lambda) = \{ z \in \mathbb{C} : |z| \geq 2, \exists t > 0, s.t. m^{(e)}(B(z,t) \cap \mathbb{R}) > \lambda^p (\frac{1}{|z|})^{2p - \alpha} \} \), therefore, if \( |z| \geq 2R_\varepsilon \) and \( z \notin E_1(\lambda) \), then we have
\[
\forall t > 0, m^{(e)}(B(z,t) \cap \mathbb{R}) \leq \lambda^p \left( \frac{t}{|z|} \right)^{2p - \alpha}.
\]

If \( \gamma > 1 - p \), then \( \frac{\gamma}{p} + 1 > 0 \), so that we obtain by Hölder’s inequality
\[
|v_2(z)| \leq \frac{y}{\pi} \left( \int_{G_2} \frac{|f(\xi)|^p}{|z - (\xi,0)|^{2p} |\xi|^q} d\xi \right)^{1/p} \left( \int_{G_2} |\xi|^\frac{\gamma y}{p} d\xi \right)^{1/q}
\]
\[
\leq Ay |z|^\frac{\gamma}{p} + \frac{1}{q} \left( \int_{G_2} \frac{|f(\xi)|^p}{|z - (\xi,0)|^{2p} |\xi|^q} d\xi \right)^{1/p},
\]

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since
\[
\int_{G_2} \frac{|f(\xi)|^p}{|z - (\xi, 0)|^{2p}|\xi|^{\gamma}} |\xi|^\gamma d\xi \leq \int_{G} \frac{2^\gamma + 1}{\gamma^2} m(\xi)(t) \leq \frac{\lambda^p}{|z|^{2p}(2^\gamma + 1) \left( \frac{1}{3\gamma} + \frac{2p}{\alpha} \right)} |z|^\alpha \gamma \frac{1}{y^\alpha},
\]

where \( m(\xi)(t) = \int_{|z-(\xi,0)| \leq t} dm(\xi)(\xi).\)

Hence we have
\[
|v_2(z)| \leq A\lambda y^{\frac{1-q}{p}} |z|^{\frac{\gamma}{p} + \frac{q}{2} - 2 + \frac{\alpha}{p}},
\]

(2.4.7)

If \( \gamma < 1 + p, \) then \( (\frac{\gamma}{p} - 2)q + 1 < 0, \) so that we obtain by (2) of Lemma 2.4.1 and Hölder’s inequality

\[
|v_3(z)| \leq \int_{G_3} \frac{\gamma}{\pi |\xi|^2} |f(\xi)| d\xi \leq \frac{4}{\pi y} \left( \int_{G_3} |f(\xi)|^p |\xi|^{\gamma} d\xi \right)^{1/p} \left( \int_{G_3} |\xi|^{\gamma - 2} |\xi|^{1/q} d\xi \right)^{1/q} \leq A\lambda |z|^{\frac{\gamma}{p} + \frac{1}{q} - 2}.
\]

(2.4.8)

Finally, by (1) of Lemma 2.4.1, we obtain
\[
|v_4(z)| \leq \frac{4}{\pi |z|^2} \int_{G_4} |f(\xi)| d\xi,
\]

which implies by \( \gamma > 1 - p \) that
\[
|v_4(z)| \leq A\lambda y^{\frac{1-q}{p}} |z|^{\frac{\gamma}{p} + \frac{q}{2} - 2 + \frac{\alpha}{p}}.
\]

(2.4.9)

Thus, by collecting (2.4.5), (2.4.6), (2.4.7), (2.4.8) and (2.4.9), there exists a positive constant \( A \) independent of \( \epsilon, \) such that if \( |z| \geq 2R_\epsilon \) and \( z \notin E_1(\epsilon), \) we have
\[
|v(z)| \leq A\lambda y^{\frac{1-q}{p}} |z|^{\frac{\gamma}{p} + \frac{1}{q} - 2 + \frac{\alpha}{p}}.
\]

Let \( \mu_\epsilon \) be a measure in \( \mathbb{C} \) defined by \( \mu_\epsilon(E) = m(\xi)(E \cap \mathbb{R}) \) for every measurable set \( E \) in \( \mathbb{C}. \) Take \( \epsilon = \epsilon_p = \frac{1}{2^{p+1}}, p = 1, 2, 3, \cdots, \) then there exists a sequence \( \{ R_p \}: \)
\[
1 = R_0 < R_1 < R_2 < \cdots \text{ such that } \mu_\epsilon \left( \frac{\epsilon_p^p}{\frac{1}{2^{2p-1}} - \frac{4}{p+2}} \right).
\]

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2.4. $p > 1$ (General Kernel)

Take $\lambda = 3 \cdot 5^{2p-\alpha} \cdot 2^p \mu_p (C)$ in Lemma 2.2.1, then there exists $z_{j,p}$ and $\rho_{j,p}$, where $R_{p-1} \leq |z_{j,p}| < R_p$, such that

$$\sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|z_{j,p}|} \right)^{2p-\alpha} \leq \frac{1}{2p}.$$ 

If $R_{p-1} \leq |z| < R_p$ and $z \notin G_p = \bigcup_{j=1}^{\infty} B(z_{j,p}, \rho_{j,p})$, we have

$$|v(z)| \leq A \varepsilon_p y^{1-\frac{\alpha}{p}} |z|^{\frac{1}{q} - 2 + \frac{\alpha}{p}}.$$ 

Thereby

$$\sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|z_{j,p}|} \right)^{2p-\alpha} \leq \sum_{j=1}^{\infty} \frac{1}{2p} = 1 < \infty.$$ 

Set $G = \bigcup_{p=1}^{\infty} G_p$, thus Theorem 2.4.1 holds.

**Proof of Theorem 2.4.2**

We prove only the case $p > 1$; the remaining case $p = 1$ can be proved similarly. Define the measure $dn(\zeta)$ by

$$dn(\zeta) = \frac{\eta_p}{(1 + |\zeta|)^p} d\mu(\zeta).$$

For any $\varepsilon > 0$, there exists $R_e > 2$, such that

$$\int_{|\zeta| \geq R_e} dn(\zeta) < \frac{\varepsilon^p}{5^{2p-\alpha}}.$$ 

For every Lebesgue measurable set $E \subset \mathbb{C}$, the measure $n(\varepsilon)$ defined by $n(\varepsilon)(E) = n(E \cap \{ \zeta \in \mathbb{C}_+ : |\zeta| \geq R_e \})$ satisfies $n(\varepsilon)(\mathbb{C}_+) \leq \frac{\varepsilon^p}{5^{2p-\alpha}}$, write

$$h_1(z) = \int_{F_1} G(z, \zeta) d\mu(\zeta),$$

$$h_2(z) = \int_{F_2} G(z, \zeta) d\mu(\zeta),$$

$$h_3(z) = \int_{F_3} G(z, \zeta) d\mu(\zeta),$$

$$h_4(z) = \int_{F_4} G(z, \zeta) d\mu(\zeta),$$

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where

\[ F_1 = \{ \zeta \in \mathbb{C}_+ : R_\varepsilon < |\zeta| \leq \frac{|z|}{2} \}, \]
\[ F_2 = \{ \zeta \in \mathbb{C}_+ : \frac{|z|}{2} < |\zeta| \leq 2|z| \}, \]
\[ F_3 = \{ \zeta \in \mathbb{C}_+ : |\zeta| > 2|z| \}, \]
\[ F_4 = \{ \zeta \in \mathbb{C}_+ : |\zeta| \leq R_\varepsilon \}. \]

Then

\[ h(z) = h_1(z) + h_2(z) + h_3(z) + h_4(z). \] (2.4.10)

First, if \( \gamma > 1 - p \), then \( \frac{\gamma q}{p} + 1 > 0 \), so that we obtain by (1) of Lemma 2.4.1, (2) of Lemma 2.4.2 and Hölder’s inequality

\[ |h_1(z)| \leq \int_{F_1} \frac{y\eta}{\pi|z-\zeta|^2} d\mu(\zeta) \]
\[ \leq \int_{F_1} \frac{y\eta}{\pi} \frac{4}{|z|^2} d\mu(\zeta) \]
\[ \leq \frac{4}{\pi} y \frac{\eta^p}{|z|^2} \left( \int_{F_1} |\zeta|^\frac{p}{q} d\mu(\zeta) \right)^{1/p} \left( \int_{F_1} |\zeta|^\frac{q}{p} d\mu(\zeta) \right)^{1/q}, \]

since

\[ \int_{F_1} |\zeta|^\frac{p}{q} d\mu(\zeta) \leq 2 \left( \frac{|z|}{2} \right)^{\frac{p}{q}+1} \int_H \frac{1}{(1+|\zeta|)} d\mu(\zeta), \]

so that

\[ |h_1(z)| \leq A\varepsilon |z|^{\frac{q}{p} + \frac{1}{p} - 2}. \] (2.4.11)

Let \( E_2(\lambda) = \{ z \in \mathbb{C} : |z| \geq 2, \exists t > 0, s.t. \eta(t)(B(z,t) \cap C_+) > \lambda^p \left( \frac{t}{|z|} \right)^{2p-\alpha} \} \), therefore, if \( |z| \geq 2R_\varepsilon \) and \( z \notin E_2(\lambda) \), then we have

\[ \forall t > 0, \eta(t)(B(z,t) \cap H) \leq \lambda^p \left( \frac{t}{|z|} \right)^{2p-\alpha}. \]
If $\gamma > 1 - p$, then $\frac{\gamma}{p} + 1 > 0$, so that we obtain by Hölder’s inequality

$$|h_2(z)| \leq \left( \int_{F} |G(z, \zeta)|^p |\xi|^\gamma \, d\mu(\xi) \right)^{1/p} \left( \int_{F} |\xi|^\frac{\gamma}{p} \, d\mu(\xi) \right)^{1/q},$$

$$\leq \left( (2\gamma + 1) \int_{F} \frac{|G(z, \zeta)|^p}{\eta^p} \, dn(\zeta) \right)^{1/p} \left( \int_{F} |\zeta|^\frac{\gamma}{p} \, d\mu(\xi) \right)^{1/q},$$

$$\leq A |z|^\frac{\gamma}{p} \left( \int_{F} \frac{|G(z, \zeta)|^p}{\eta^p} \, dn(\zeta) \right)^{1/p},$$

since

$$\int_{F} \frac{|G(z, \zeta)|^p}{\eta^p} \, dn(\zeta) \leq \int_{|\zeta| \leq 3|z|} \frac{|G(z, \zeta)|^p}{\eta^p} \, dn^e(\zeta)$$

$$= \int_{|\zeta| \leq \frac{y}{\pi}} \frac{|G(z, \zeta)|^p}{\eta^p} \, dn^e(\zeta) + \int_{\frac{y}{\pi} < |\zeta| \leq 3|z|} \frac{|G(z, \zeta)|^p}{\eta^p} \, dn^e(\zeta)$$

$$= h_{21}(z) + h_{22}(z),$$

so that we have by (1) of Lemma 2.4.2 and Lemma 2.4.3

$$h_{21}(z) \leq \int_{|\zeta| \leq \frac{y}{\pi}} \left( A \frac{3y}{2\pi |z - \zeta|} \right)^p \, dn^e(\zeta)$$

$$= \frac{A}{y^p} \int_{0}^{\frac{y}{\pi}} \left( \log \frac{3y}{t} \right)^p \, dn^e(\zeta)$$

$$\leq A \lambda \frac{y^{p-\alpha}}{|z|^{2p-\alpha}} + A \lambda \frac{1}{y^{p-\alpha}} \int_{1}^{\frac{y}{\pi}} t^{2p-\alpha-1} \left( \log \frac{3y}{t} \right)^{p-1} \, dt$$

$$\leq A \lambda \frac{y^{p-\alpha}}{|z|^{2p-\alpha}}.$$

Moreover, we have by (2) of Lemma 2.4.2

$$h_{22}(z) \leq \int_{\frac{y}{\pi} < |\zeta| \leq 3|z|} \left( \frac{y}{\pi |z - \zeta|^2} \right)^p \, dn^e(\zeta)$$

$$= \left( \frac{y}{\pi} \right)^p \int_{\frac{y}{\pi}}^{3|z|} \frac{1}{t^{-p} \, dn^e(\zeta)}$$

$$\leq \left( \frac{1}{\pi} \right)^p \left( \frac{1}{3\alpha} + \frac{2\alpha + 1}{\alpha} \right) \lambda \frac{y^{p-\alpha}}{|z|^{2p-\alpha}}.$$
where \( n^e_z(t) = \int_{|z-\zeta|\leq t} dn^e(\zeta) \).

Hence we have

\[
|h_2(z)| \leq A\lambda y^{1-\frac{a}{p}}|z|^\frac{\gamma}{p} + \frac{1}{q} - 2 + \frac{a}{p}.
\]  (2.4.12)

If \( \gamma < 1 + p \), then \( (\frac{\gamma}{p} - 2)q + 1 < 0 \), so that we obtain by (2) of Lemma 2.4.1, (2) of Lemma 2.4.2 and Hölder’s inequality

\[
|h_3(z)| \leq \int_{F_3} \frac{\eta y}{\pi |z-\zeta|^2} d\mu(\zeta)
\]
\[
\leq \int_{F_3} \frac{\eta y}{\pi |\zeta|^2} d\mu(\zeta)
\]
\[
\leq \frac{4}{\pi y} \left( \int_{F_3} \frac{\eta y}{|\zeta|^q} d\mu(\zeta) \right)^{1/p} \left( \int_{F_3} |z|^q d\mu(\zeta) \right)^{1/q}
\]
\[
\leq A\lambda y^{\frac{\gamma}{p} + \frac{1}{q} - 2}.
\]  (2.4.13)

Finally, by (1) of Lemma 2.4.1 and (2) of Lemma 2.4.2, we obtain

\[
|h_4(z)| \leq \int_{F_4} \frac{\eta y}{\pi |z-\zeta|^2} d\mu(\zeta) \leq \frac{4}{\pi} \frac{y}{|z|^2} \int_{F_4} \eta d\mu(\zeta),
\]

which implies by \( \gamma > 1 - p \) that

\[
|h_4(z)| \leq A\lambda y^{\frac{\gamma}{p} + \frac{1}{q} - 2}.
\]  (2.4.14)

Thus, by collecting (2.4.10), (2.4.11), (2.4.12), (2.4.13) and (2.4.14), there exists a positive constant \( A \) independent of \( \varepsilon \), such that if \( |z| \geq 2R_\varepsilon \) and \( z \notin E^e_2(\varepsilon) \), we have

\[
|h(z)| \leq A\lambda y^{\frac{\gamma}{p} + \frac{1}{q} - 2 + \frac{a}{p}}.
\]

Similarly, if \( z \notin G \), we have

\[
h(z) = o(y^{1-\frac{a}{p}}|z|^\frac{\gamma}{p} + \frac{1}{q} - 2 + \frac{a}{p}), \quad \text{as } |z| \to \infty.
\]  (2.4.15)

By (2.4.3) and (2.4.15), we obtain that

\[
u(z) = v(z) + h(z) = o(y^{1-\frac{a}{p}}|z|^\frac{\gamma}{p} + \frac{1}{q} - 2 + \frac{a}{p}), \quad \text{as } |z| \to \infty
\]

holds in \( C_+ - G \), thus we complete the proof of Theorem 2.4.2.
2.5 \ p > 1 (Modified Kernel)

1. Introduction and Main Theorems

In this section, we will consider measurable functions \( f \) in \( \mathbb{R} \) satisfying

\[
\int_{\mathbb{R}} \frac{|f(\xi)|^p}{(1 + |\xi|)^\gamma} d\xi < \infty,
\]

(2.5.1)

where \( \gamma \) is defined as in Theorem 2.5.1.

In order to describe the asymptotic behaviour of subharmonic functions represented by the modified kernel in the upper half plane (see \([24],[50],[44],[28],[29],[30]\)), we establish the following theorems.

**Theorem 2.5.1** Let \( 1 \leq p < \infty, \ \frac{1}{p} + \frac{1}{q} = 1 \) and

\[
1 + mp < \gamma < 1 + (m + 1)p \quad \text{in case} \ p > 1; \\
1 + m + 1 < \gamma \leq m + 2 \quad \text{in case} \ p = 1.
\]

If \( f \) is a measurable function in \( \mathbb{R} \) satisfying \( (2.5.1) \) and \( v(z) \) is the harmonic function defined by

\[
v(z) = \int_{\mathbb{R}} P_m(z, \xi) f(\xi) d\xi,
\]

(2.5.2)

then there exists \( z_j \in \mathbb{C}^+, \ \rho_j > 0, \) such that

\[
\sum_{j=1}^{\infty} \frac{\rho_j^{2p-\alpha}}{|z_j|^{2p-\alpha}} < \infty
\]

(2.5.3)

holds and

\[
v(z) = o(y^{1-\frac{\alpha}{p}}|z|^{\frac{\gamma}{p} + \frac{1}{q} - 2 + \frac{\alpha}{p}}), \quad \text{as} \ |z| \to \infty
\]

(2.5.4)

holds in \( \mathbb{C}^+ - G, \) where \( G = \bigcup_{j=1}^{\infty} B(z_j, \rho_j) \) and \( 0 < \alpha \leq 2p. \)

**Remark 2.5.1** If \( \gamma = 1 + mp, \ p > 1, \) then

\[
v(z) = o(y^{1-\frac{\alpha}{p}}(\log |z|)^{\frac{1}{q}}|z|^{\frac{\gamma}{p} + \frac{1}{q} - 2 + \frac{\alpha}{p}}), \quad \text{as} \ |z| \to \infty
\]

holds in \( \mathbb{C}^+ - G. \)

Next, we will generalize Theorem 2.5.1 to subharmonic functions.
Theorem 2.5.2 Let \( p \) and \( \gamma \) be as in Theorem 2.5.1. If \( f \) is a measurable function in \( \mathbb{R} \) satisfying (2.5.1) and \( \mu \) is a positive Borel measure satisfying
\[
\int_{C_+} \eta^p (1 + |\xi|)^{\gamma} d\mu(\xi) < \infty
\]
and
\[
\int_{C_+} \frac{1}{1 + |\xi|} d\mu(\xi) < \infty.
\]
Write the subharmonic function
\[
u(z) = v(z) + h(z), \quad z \in C_+,
\]
where \( v(z) \) is the harmonic function defined by (2.5.2), \( h(z) \) is defined by
\[
h(z) = \int_{C_+} G_m(z, \xi) d\mu(\xi)
\]
and \( G_m(z, \xi) \) is defined by (2.3.2). Then there exists \( z_j \in C_+, \rho_j > 0 \), such that (2.5.3) holds and
\[
u(z) = o(y^{1 - \frac{\alpha}{p} |z|^\frac{1}{q} + \frac{1}{q} + 2^{-p} + \frac{\alpha}{p}}), \quad \text{as } |z| \to \infty
\]
holds in \( C_+ - G \), where \( G = \bigcup_{j=1}^\infty B(z_j, \rho_j) \) and \( 0 < \alpha < 2p \).

Remark 2.5.2 If \( \gamma = 1 + mp, \ p > 1 \), then
\[
u(z) = o(y^{1 - \frac{\alpha}{p} (\log |z|)^\frac{1}{q} |z|^\frac{1}{q} + \frac{1}{q} + 2^{-p} + \frac{\alpha}{p}}), \quad \text{as } |z| \to \infty
\]
holds in \( C_+ - G \).

Remark 2.5.3 If \( \alpha = 1, \ p = 1, \ m = 0 \) and \( \gamma = 2 \), then (2.5.3) holds and (2.5.5) holds in \( C_+ - G \). This is just the result of Hayman, therefore, our result (2.5.5) is the generalization of Theorem B.

2. Main Lemmas

In order to obtain the results, we need these lemmas below:

Lemma 2.5.1 The modified Poisson kernel \( P_m(z, \xi) \) has the following estimates:
(1) If \( 1 < |\xi| \leq \frac{|z|}{2} \), then \( |P_m(z, \xi)| \leq \frac{A_0|z|^{n-1}}{|\xi|^{n+1}} \).
2.5. $p > 1$ (Modified Kernel)

(2) If $\frac{|z|}{2} < |\xi| \leq 2|z|$, then $|P_m(z, \xi)| \leq \frac{Ay}{z - (\xi, 0)}$.

(3) If $|\xi| > 2|z|$, then $|P_m(z, \xi)| \leq \frac{Ay|\xi|^m}{|z|^m}$.

(4) If $|\xi| \leq 1$, then $|P_m(z, \xi)| \leq \frac{Ay}{|\xi|^p}$.

Lemma 2.5.2 The modified Green function $G_m(z, \xi)$ has the following estimates:

(1) If $1 < |\xi| \leq \frac{|z|}{2}$, then $|G_m(z, \xi)| \leq \frac{A\eta |\xi|m^{|\xi|}}{|z|^{m+1}}$.

(2) If $\frac{|z|}{2} < |\xi| \leq 2|z|$, then $|G_m(z, \xi)| \leq \frac{A\eta |\xi|m^{|\xi|}}{|z|^{m+2}}$.

(3) If $|\xi| > 2|z|$, then $|G_m(z, \xi)| \leq \frac{A\eta |\xi|m^{|\xi|}}{|z|^{m+2}}$.

(4) If $|\xi| \leq 1$, then $|G_m(z, \xi)| \leq \frac{A\eta}{\pi |z-\xi|^{m+2}} \leq \frac{A\eta}{|z|^{m+2}}$.

(5) If $|\xi - z| \leq \frac{\eta}{2}$, then $|G_m(z, \xi)| \leq A \log \frac{3\eta}{|z-\xi|}$.

3. Proof of Theorems

Proof of Theorem 2.5.1

We prove only the case $p > 1$; the proof of the case $p = 1$ is similar. Define the measure $dm(\xi)$ by

$$dm(\xi) = \frac{|f(\xi)|^p}{(1 + |\xi|)^{\alpha}}d\xi.$$ 

For any $\varepsilon > 0$, there exists $R_\varepsilon > 2$, such that

$$\int_{|\xi| \geq R_\varepsilon} dm(\xi) \leq \frac{\varepsilon^p}{\xi^{2p-\alpha}}.$$ 

For every Lebesgue measurable set $E \subseteq \mathbb{R}$, the measure $m^{(\varepsilon)}$ defined by $m^{(\varepsilon)}(E) = m(E \cap \{x \in \mathbb{R} : |x| \geq R_\varepsilon\})$ satisfies $m^{(\varepsilon)}(\mathbb{R}) \leq \frac{\varepsilon^p}{\xi^{2p-\alpha}}$, write

$$v_1(z) = \int_{G_1} P_m(z, \xi)f(\xi)d\xi,$$

$$v_2(z) = \int_{G_2} P_m(z, \xi)f(\xi)d\xi,$$

$$v_3(z) = \int_{G_3} P_m(z, \xi)f(\xi)d\xi,$$

$$v_4(z) = \int_{G_4} P_m(z, \xi)f(\xi)d\xi.$$
Chapter 2. Growth Estimates for a Class of Subharmonic Functions in the Half Plane

where

\[ \begin{align*}
G_1 &= \{ \xi \in \mathbb{R} : 1 < |\xi| \leq \frac{|z|}{2} \}, \\
G_2 &= \{ \xi \in \mathbb{R} : \frac{|z|}{2} < |\xi| \leq 2|z| \}, \\
G_3 &= \{ \xi \in \mathbb{R} : |\xi| > 2|z| \}, \\
G_4 &= \{ \xi \in \mathbb{R} : |\xi| \leq 1 \}.
\end{align*} \]

Then

\[ v(z) = v_1(z) + v_2(z) + v_3(z) + v_4(z). \]  \hspace{1cm} (2.5.6)

First, if \( \gamma > 1 + mp \), then \( (\frac{\gamma}{p} - m - 1)q + 1 > 0 \). For \( R_e > 2 \), we have

\[ v_1(z) = \int_{1<|\xi|\leq R_e} |f(\xi)| P_m(\xi) d\xi + \int_{R_e<|\xi|\leq \frac{|z|}{2}} |f(\xi)| P_m(\xi) d\xi = v_{11}(z) + v_{12}(z) \]

if \( |z| > 2R_e \), then we obtain by (1) of Lemma 2.5.1 and Hölder’s inequality

\[ |v_{11}(z)| \leq Ay|z|^{m-1} \left( \int_{1<|\xi|\leq R_e} \frac{|f(\xi)|^p}{|\xi|^\gamma} d\xi \right)^{1/p} \left( \int_{1<|\xi|\leq R_e} |\xi|^{(\frac{\gamma}{p} - m - 1)q} d\xi \right)^{1/q} \]

since

\[ \int_{1<|\xi|\leq R_e} |\xi|^{(\frac{\gamma}{p} - m - 1)q} d\xi \leq AR_e^{(\frac{\gamma}{p} - m - 1)q + 1}, \]

so that

\[ |v_{11}(z)| \leq Ay|z|^{m-1} R_e^{(\frac{\gamma}{p} - m - 1) + \frac{1}{q}}. \]  \hspace{1cm} (2.5.7)

Moreover, we have similarly

\[ |v_{12}(z)| \leq Ay|z|^{m-1} \left( \int_{R_e<|\xi|\leq \frac{|z|}{2}} \frac{|f(\xi)|^p}{|\xi|^\gamma} d\xi \right)^{1/p} \left( \int_{R_e<|\xi|\leq \frac{|z|}{2}} |\xi|^{(\frac{\gamma}{p} - m - 1)q} d\xi \right)^{1/q} \]

\[ \leq Ay|z|^\gamma \left( \int_{R_e<|\xi|\leq \frac{|z|}{2}} \frac{|f(\xi)|^p}{|\xi|^\gamma} d\xi \right)^{1/p}, \]
2.5. \( p > 1 \) (Modified Kernel)

which implies by arbitrariness of \( R_e \) that

\[
|v_{12}(z)| \leq A\epsilon y |z|^{\frac{\gamma}{p} + 1} - 2. \quad (2.5.8)
\]

Let \( E_1(\lambda) = \{ z \in \mathbb{C} : |z| \geq 2, \exists t > 0, \text{s.t. } m^{(e)}(B(z, t) \cap \mathbb{R}) > \lambda^p \left( \frac{t}{|z|} \right)^{2p - \alpha} \}, \)

therefore, if \( |z| \geq 2R_e \) and \( z \notin E_1(\lambda) \), then we have

\[
\forall t > 0, m^{(e)}(B(z, t) \cap \mathbb{R}) \leq \lambda^p \left( \frac{t}{|z|} \right)^{2p - \alpha}.
\]

If \( \gamma > 1 + mp \), then \( \left( \frac{2}{p} - m - 1 \right)q + 1 > 0 \), so that we obtain by (2) of Lemma 2.5.1 and Hölder’s inequality

\[
|v_2(z)| \leq \int_{G_2} \frac{A\gamma}{|z - (\xi, 0)|^2} |f(\xi)| d\xi
\]

\[
\leq A\gamma \left( \int_{G_2} \frac{|f(\xi)|^p}{|z - (\xi, 0)|^{2p}|\xi|^\gamma} d\xi \right)^{1/p} \left( \int_{G_2} |\xi|^\gamma d\xi \right)^{1/q}
\]

\[
\leq A\gamma |z|^{\frac{\gamma}{p} + \frac{1}{q}} \left( \int_{G_2} \frac{|f(\xi)|^p}{|z - (\xi, 0)|^{2p}|\xi|^\gamma} d\xi \right)^{1/p},
\]

since

\[
\int_{G_2} \frac{|f(\xi)|^p}{|z - (\xi, 0)|^{2p}|\xi|^\gamma} d\xi \leq \int_\gamma \frac{2^\gamma + 1}{t^{2p}} d\xi \leq \frac{\lambda^p}{|z|^{2p} (2^\gamma + 1)} \left( \frac{1}{3^\alpha} + \frac{2p}{\alpha} \right) |z|^{\alpha},
\]

where \( m^{(e)}_\alpha(t) = \int_{|z - (\xi, 0)| \leq t} d\xi \). Hence we have

\[
|v_2(z)| \leq A\gamma y^{1 - \frac{\gamma}{p} |z|^{\frac{\gamma}{p} + 1} - 2 + \frac{\gamma}{p}}. \quad (2.5.9)
\]

If \( \gamma < 1 + (m + 1)p \), then \( \left( \frac{2}{p} - m - 2 \right)q + 1 < 0 \), so that we obtain by (3) of Lemma 2.5.1 and Hölder’s inequality

\[
|v_3(z)| \leq \int_{G_3} \frac{A\gamma}{|z - (\xi, 0)|^2} |f(\xi)| d\xi
\]

\[
\leq A\gamma \left( \int_{G_3} \frac{|f(\xi)|^p}{|z - (\xi, 0)|^{2p}|\xi|^\gamma} d\xi \right)^{1/p} \left( \int_{G_3} |\xi|^{(\frac{\gamma}{p} - m - 2)q} d\xi \right)^{1/q}
\]

\[
\leq A\gamma y^{\gamma \frac{1}{p} + \frac{1}{q} - 2}. \quad (2.5.10)
\]
Finally, by (4) of Lemma 2.5.1, we obtain

$$|v_4(z)| \leq \frac{Ay}{|z|} \int_{G_4} |f(\xi)|d\xi. \quad (2.5.11)$$

Thus, by collecting (2.5.6), (2.5.7), (2.5.8), (2.5.9), (2.5.10) and (2.5.11), there exists a positive constant $A$ independent of $\varepsilon$, such that if $|z| \geq 2R_\varepsilon$ and $z \notin E_1(\varepsilon)$, we have

$$|v(z)| \leq A\varepsilon y^{1-\frac{\alpha}{p}} |z|^\frac{y}{p} + 2 + \frac{\alpha}{p}.$$

Let $\mu$ be a measure in $C$ defined by $\mu(E) = m_1(E \cap R)$ for every measurable set $E$ in $C$. Take $\varepsilon = \varepsilon_p = \frac{1}{2^p + 2}, p = 1, 2, 3, \cdots$, then there exists a sequence $\{R_p\}$: $1 = R_0 < R_1 < R_2 < \cdots$ such that

$$\mu_{\varepsilon_p}(C) = \int_{|\xi| \geq R_p} dm(\xi) < \frac{\varepsilon_p}{2^{2p-\alpha}}.$$

Take $\lambda = 3 \cdot 5^{2p-\alpha} \cdot 2^p \mu_{\varepsilon_p}(C)$ in Lemma 2.2.1, then there exists $z_{j,p}$ and $\rho_{j,p}$, where $R_{p-1} \leq |z_{j,p}| < R_p$, such that

$$\sum_{j=1}^\infty \left( \frac{\rho_{j,p}}{|z_{j,p}|} \right)^{2p-\alpha} \leq \frac{1}{2^p}.$$

If $R_{p-1} \leq |z| < R_p$ and $z \notin G_p = \bigcup_{j=1}^\infty B(z_{j,p}, \rho_{j,p})$, we have

$$|v(z)| \leq A\varepsilon_p y^{1-\frac{\alpha}{p}} |z|^\frac{y}{p} + \frac{1}{2p} + \frac{\alpha}{p}.$$

Thereby

$$\sum_{p=1}^\infty \sum_{j=1}^\infty \left( \frac{\rho_{j,p}}{|z_{j,p}|} \right)^{2p-\alpha} \leq \sum_{p=1}^\infty \frac{1}{2^p} = 1 < \infty.$$

Set $G = \bigcup_{p=1}^\infty G_p$, thus Theorem 2.5.1 holds.

**Proof of Theorem 2.5.2**

We prove only the case $p > 1$; the remaining case $p = 1$ can be proved similarly. Define the measure $dn(\zeta)$ by

$$dn(\zeta) = \frac{\eta^p}{(1 + |\zeta|)^y} d\mu(\zeta).$$

For any $\varepsilon > 0$, there exists $R_\varepsilon > 2$, such that

$$\int_{|\zeta| \geq R_\varepsilon} dn(\zeta) < \frac{\varepsilon^p}{2^{2p-\alpha}}.$$
2.5. $p > 1$ (Modified Kernel)

For every Lebesgue measurable set $E \subset \mathbb{C}$, the measure $n^{(e)}$ defined by $n^{(e)}(E) = n(E \cap \{\zeta \in \mathbb{C}_+ : |\zeta| \geq R_e\})$ satisfies $n^{(e)}(\mathbb{C}_+) \leq \frac{e^p}{2^{p-a}}$, write

$$h_1(z) = \int_{F_1} G_m(z, \zeta) d\mu(\zeta),$$
$$h_2(z) = \int_{F_2} G_m(z, \zeta) d\mu(\zeta),$$
$$h_3(z) = \int_{F_3} G_m(z, \zeta) d\mu(\zeta),$$
$$h_4(z) = \int_{F_4} G_m(z, \zeta) d\mu(\zeta),$$

where

$$F_1 = \{\zeta \in \mathbb{C}_+ : 1 < |\zeta| \leq \frac{|z|}{2}\},$$
$$F_2 = \{\zeta \in \mathbb{C}_+ : \frac{|z|}{2} < |\zeta| \leq 2|z|\},$$
$$F_3 = \{\zeta \in \mathbb{C}_+ : |\zeta| > 2|z|\},$$
$$F_4 = \{\zeta \in \mathbb{C}_+ : |\zeta| \leq 1\}.$$

Then

$$h(z) = h_1(z) + h_2(z) + h_3(z) + h_4(z). \quad (2.5.12)$$

First, if $\gamma > 1 + mp$, then $(\frac{\gamma}{p} - m - 1)q + 1 > 0$. For $R_e > 2$, we have

$$h_1(z) = \int_{|\zeta| \leq R_e} G_m(z, \zeta) d\mu(\zeta) + \int_{R_e < |\zeta| \leq \frac{|z|}{2}} G_m(z, \zeta) d\mu(\zeta) = h_{11}(z) + h_{12}(z),$$

if $|z| > 2R_e$, then we obtain by (1) of Lemma 2.5.2 and Hölder’s inequality

$$|h_{11}(z)| \leq \int_{1 < |\zeta| \leq R_e} \frac{A\eta |z|^{m-1}}{|\zeta|^{q(1/p-m-1)q}} d\mu(\zeta) \leq A\eta |z|^{m-1} \left( \int_{1 < |\zeta| \leq R_e} \frac{\eta^p}{|\zeta|^q} d\mu(\zeta) \right)^{1/p} \left( \int_{1 < |\zeta| \leq R_e} \frac{|\zeta|^{(\frac{\gamma}{p} - m - 1)q}}{\eta^q} d\mu(\zeta) \right)^{1/q},$$

since

$$\int_{1 < |\zeta| \leq R_e} \frac{|\zeta|^{(\frac{\gamma}{p} - m - 1)q}}{\eta^q} d\mu(\zeta) \leq AR_e^{(\frac{\gamma}{p} - m - 1)q + 1},$$

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so that
\[ |h_{11}(z)| \leq Ay|z|^{m-1}R_e^{\left(\gamma-p-m-1\right)+\frac{1}{q}}. \] (2.5.13)

Moreover, we have similarly
\[
|h_{12}(z)| \leq Ay|z|^{m-1} \left( \int_{R_e<|\xi|\leq \frac{|z|}{2}} \frac{\eta^p}{|\xi|^{\gamma}} d\mu(\xi) \right)^{1/p} \left( \int_{R_e<|\xi|\leq \frac{|z|}{2}} |\xi|^{\left(\gamma-p-m-1\right)q} d\mu(\xi) \right)^{1/q} 
\leq Ay|z|^\gamma \left( \int_{R_e<|\xi|\leq \frac{|z|}{2}} \frac{\eta^p}{|\xi|^{\gamma}} d\mu(\xi) \right)^{1/p},
\]
which implies by arbitrariness of \( R_e \) that
\[ |h_{12}(z)| \leq A\varepsilon|z|^\gamma. \] (2.5.14)

Let \( E_2(\lambda) = \{ z \in \mathbb{C} : |z| \geq 2, \exists t > 0, \text{s.t. } n^{(e)}(B(z,t) \cap \mathbb{C}_+) > \lambda |t|^{2-p-\alpha} \} \), therefore, if \( |z| \geq 2R_e \) and \( z \notin E_2(\lambda) \), then we have
\[ \forall t > 0, n^{(e)}(B(z,t) \cap \mathbb{C}_+) \leq \lambda |t|^{2-p-\alpha}. \]

If \( \gamma > 1+mp \), then \( \left( \frac{\gamma}{p} - m - 1 \right)q + 1 > 0 \), so that we obtain by Hölder’s inequality
\[
|h_2(z)| \leq \left( \int_{F_2} \frac{|G_m(z,\zeta)|^p}{|\zeta|^{\gamma}} d\mu(\zeta) \right)^{1/p} \left( \int_{F_2} |\zeta|^{\frac{m}{p}} d\mu(\zeta) \right)^{1/q} 
\leq \left( (2^\gamma + 1) \int_{F_2} \frac{|G_m(z,\zeta)|^p}{\eta^p} dn(\zeta) \right)^{1/p} \left( \int_{F_2} |\zeta|^{\frac{m}{p}} d\mu(\zeta) \right)^{1/q} 
\leq A|z|^\gamma \left( \int_{F_2} \frac{|G_m(z,\zeta)|^p}{\eta^p} dn(\zeta) \right)^{1/p},
\]

since
\[
\int_{F_2} \frac{|G_m(z,\zeta)|^p}{\eta^p} dn(\zeta) \leq \int_{|z-\zeta|\leq 3|z|} \frac{|G_m(z,\zeta)|^p}{\eta^p} dn^{(e)}(\zeta) 
= \int_{|z-\zeta|\leq \frac{1}{2}} \frac{|G_m(z,\zeta)|^p}{\eta^p} dn^{(e)}(\zeta) + \int_{\frac{1}{2} < |z-\zeta| \leq 3|z|} \frac{|G_m(z,\zeta)|^p}{\eta^p} dn^{(e)}(\zeta) 
= h_{21}(z) + h_{22}(z),
\]
so that we have by (5) of Lemma 2.5.2 and Lemma 2.4.3

\[
h_{21}(z) \leq \int |z - \zeta| \leq \frac{\epsilon}{2} \left( \frac{A y \log \frac{3y}{|z - \zeta|}}{|z - \zeta|} \right)^p d\mu(\zeta)
\]

\[
= A y^p \int_0^{\frac{\epsilon}{2}} \left( \log \frac{3y}{t} \right)^p d\mu(\zeta)(t)
\]

\[
\leq A \lambda y^p \frac{y^{p - \alpha}}{|z|^{2p - \alpha}} + A \lambda y^p \frac{1}{y^p |z|^{2p - \alpha}} \int_0^{\frac{\epsilon}{2}} t^{2p - \alpha - 1} \left( \log \frac{3y}{t} \right)^{p - 1} dt
\]

\[
\leq A \lambda y^p \frac{y^{p - \alpha}}{|z|^{2p - \alpha}}.
\]

Moreover, we have by (2) of Lemma 2.5.2

\[
h_{22}(z) \leq \int |z - \zeta| \leq |z| \leq 3 |z| - \zeta \leq 1 \left( A \frac{y \log \frac{3y}{|z - \zeta|}}{|z|} \right)^p d\mu(\zeta)
\]

\[
= (Ay)^p \int_{|z|}^{3|z|} \frac{1}{t^{p - \alpha}} d\mu(\zeta)(t)
\]

\[
\leq A \left( \frac{1}{3\alpha} + \frac{2p^{\alpha - 1}}{\alpha} \right) \lambda y^{p - \alpha} \frac{y^{p - \alpha}}{|z|^{2p - \alpha}}.
\]

where \( n(\zeta)(t) = \int |z - \zeta| \leq t d\mu(\zeta) \).

Hence we have

\[
|h_2(z)| \leq A \lambda y^{1 - \frac{\alpha}{p}} \frac{y^{\gamma + \frac{1}{q} + 2 + \frac{\alpha}{p}}}{|z|^{2p - \alpha}}. \tag{2.5.15}
\]

If \( \gamma < 1 + (m + 1)p \), then \( (\frac{\gamma}{p} - m - 2)q + 1 < 0 \), so that we obtain by (3) of Lemma 2.5.2 and Hölder’s inequality

\[
|h_3(z)| \leq \int_{F_3} A y |z|^m |\zeta|^{m + 2 - \alpha} d\mu(\zeta)
\]

\[
\leq A y |z|^m \left( \int_{F_3} \frac{\eta^p}{|\zeta|^q} d\mu(\zeta) \right)^{1/p} \left( \int_{F_3} |\zeta|^{(\frac{\gamma}{p} - m - 2)q} d\mu(\zeta) \right)^{1/q}
\]

\[
\leq A \epsilon y |z|^{\frac{\gamma}{p} + \frac{1}{q} - 2}. \tag{2.5.16}
\]

Finally, by (4) of Lemma 2.5.2, we obtain

\[
|h_4(z)| \leq \frac{A y}{|z|^2} \int_{F_4} \eta d\mu(\zeta). \tag{2.5.17}
\]
Thus, by collecting (2.5.12), (2.5.13), (2.5.14), (2.5.15), (2.5.16) and (2.5.17), there exists a positive constant $A$ independent of $\varepsilon$, such that if $|z| \geq 2R_{\varepsilon}$ and $z \notin E_2(\varepsilon)$, we have

$$|h(z)| \leq A\varepsilon^{1-\frac{\alpha}{p}} |z|^{\frac{\gamma}{p} + \frac{1}{q} - 2 + \frac{\alpha}{p}}.$$ 

Similarly, if $z \notin G$, we have

$$h(z) = o(y^{1-\frac{\alpha}{p}} |z|^{\frac{\gamma}{p} + \frac{1}{q} - 2 + \frac{\alpha}{p}}), \quad \text{as } |z| \to \infty. \quad (2.5.18)$$

By (2.5.4) and (2.5.18), we obtain that

$$u(z) = v(z) + h(z) = o(y^{1-\frac{\alpha}{p}} |z|^{\frac{\gamma}{p} + \frac{1}{q} - 2 + \frac{\alpha}{p}}), \quad \text{as } |z| \to \infty$$

holds in $C_+ - G$, thus we complete the proof of Theorem 2.5.2.
Chapter 3

Growth Estimates for a Class of Subharmonic Functions in the Half Space

3.1 Introduction and Basic Notations

For \( x \in \mathbb{R}^n \setminus \{0\} \), let

\[
E(x) = -r_n|x|^{2-n},
\]

where \(|x|\) is the Euclidean norm, \( r_n = \frac{1}{(n-2)\omega_n} \) and \( \omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \) is the surface area of the unit sphere in \( \mathbb{R}^n \). We know that \( E \) is locally integrable in \( \mathbb{R}^n \).

The Green function \( G(x,y) \) for the upper half space \( H \) is given by

\[
G(x,y) = E(x-y) - E(x-y^*), \quad x, y \in \overline{H}, \quad x \neq y,
\]

where * denotes the reflection in the boundary plane \( \partial H \) just as \( y^* = (y_1, y_2, \cdots, y_{n-1}, -y_n) \), then we define the Poisson kernel \( P(x,y') \) when \( x \in H \) and \( y' \in \partial H \) by

\[
P(x,y') = -\frac{\partial G(x,y)}{\partial y_n} \bigg|_{y_n=0} = \frac{2x_n}{\omega_n|x-(y',0)|^n}.
\]

The Dirichlet problem of the upper half space is to find a function \( u \) satisfying

\[
u \in C^2(H),
\]

\[
\Delta u = 0, x \in H,
\]
3.2. Preliminary Lemma

\[ \lim_{x \to x'} u(x) = f(x') \text{ nontangentially a.e.} x' \in \partial H, \quad (3.1.5) \]

where \( f \) is a measurable function of \( \mathbb{R}^{n-1} \). The Poisson integral of the upper half space is defined by

\[ u(x) = P[f] (x) = \int_{\mathbb{R}^{n-1}} P(x, y') f(y') dy', \quad (3.1.6) \]

where \( P(x, y') \) is defined by (3.1.2).

As we all know, the Poisson integral \( P[f] \) exists if

\[ \int_{\mathbb{R}^{n-1}} \frac{|f(y')|}{1 + |y'|^n} dy' < \infty. \]

(see [1], [14] and [31]) In this chapter, we replace the condition into

\[ \int_{\mathbb{R}^{n-1}} \frac{|f(y')|^p}{(1 + |y'|)^\gamma} dy' < \infty, \quad (3.1.7) \]

where \( 1 \leq p < \infty \) and \( \gamma \) is a real number, then we can get the asymptotic behaviour of harmonic functions.

Next, we will generalize these results to subharmonic functions.

### 3.2 Preliminary Lemma

Let \( \mu \) be a positive Borel measure in \( \mathbb{R}^n \), \( \beta \geq 0 \), the maximal function \( M(d\mu)(x) \) of order \( \beta \) is defined by

\[ M(d\mu)(x) = \sup_{0 < r < \infty} \frac{\mu(B(x, r))}{r^\beta}, \]

then the maximal function \( M(d\mu)(x) : \mathbb{R}^n \to [0, \infty) \) is lower semicontinuous, hence measurable. To see this, for any \( \lambda > 0 \), let \( D(\lambda) = \{ x \in \mathbb{R}^n : M(d\mu)(x) > \lambda \} \). Fix \( x \in D(\lambda) \), then there exists \( r > 0 \) such that \( \mu(B(x, r)) > tr^\beta \) for some \( t > \lambda \), and there exists \( \delta > 0 \) satisfying \( (r + \delta)^\beta < \frac{\mu(x)}{\lambda} \). If \( |y - x| < \delta \), then \( B(y, r + \delta) \supset B(x, r) \), therefore \( \mu(B(y, r + \delta)) \geq tr^\beta > \lambda (r + \delta)^\beta \). Thus \( B(x, \delta) \subset D(\lambda) \). This proves that \( D(\lambda) \) is open for each \( \lambda > 0 \).

In order to obtain the results, we need the lemma below:

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Lemma 3.2.1 Let $\mu$ be a positive Borel measure in $\mathbb{R}^n$, $\beta \geq 0$, $\mu(\mathbb{R}^n) < \infty$, for any $\lambda \leq 5^\beta \mu(\mathbb{R}^n)$, set

$$E(\lambda) = \{x \in \mathbb{R}^n : |x| \geq 2, M(d\mu)(x) > \frac{\lambda}{|x|^\beta}\},$$

then there exists $x_j \in E(\lambda)$, $\rho_j > 0$, $j = 1, 2, \ldots$, such that

$$E(\lambda) \subset \bigcup_{j=1}^{\infty} B(x_j, \rho_j) \quad (3.2.1)$$

and

$$\sum_{j=1}^{\infty} \frac{\rho_j^\beta}{|x_j|^\beta} \leq \frac{3\mu(\mathbb{R}^n)5^\beta}{\lambda}. \quad (3.2.2)$$

Proof: Let $E_k(\lambda) = \{x \in E(\lambda) : 2^k \leq |x| < 2^{k+1}\}$, then for any $x \in E_k(\lambda)$, there exists $r(x) > 0$, such that $\mu(B(x, r(x))) > \lambda \left(\frac{r(x)}{|x|}\right)^\beta$, therefore $r(x) \leq 2^{k-1}$. Since $E_k(\lambda)$ can be covered by the union of a family of balls $\{B(x, r(x)) : x \in E_k(\lambda)\}$, by the Vitali Lemma [37], there exists $\Lambda_k \subset E_k(\lambda)$, $\Lambda_k$ is at most countable, such that $\{B(x, r(x)) : x \in \Lambda_k\}$ are disjoint and

$$E_k(\lambda) \subset \bigcup_{x \in \Lambda_k} B(x, 5r(x)),$$

so

$$E(\lambda) = \bigcup_{k=1}^{\infty} E_k(\lambda) \subset \bigcup_{k=1}^{\infty} \bigcup_{x \in \Lambda_k} B(x, 5r(x)).$$

On the other hand, note that $\bigcup_{x \in \Lambda_k} B(x, r(x)) \subset \{x : 2^{k-1} \leq |x| < 2^{k+2}\}$, so that

$$\sum_{x \in \Lambda_k} \frac{(5r(x))^\beta}{|x|^\beta} \leq 5^\beta \sum_{x \in \Lambda_k} \frac{\mu(B(x, r(x)))}{\lambda} \leq \frac{5^\beta}{\lambda} \mu\{x : 2^{k-1} \leq |x| < 2^{k+2}\}.$$

Hence we obtain

$$\sum_{k=1}^{\infty} \sum_{x \in \Lambda_k} \frac{(5r(x))^\beta}{|x|^\beta} \leq \sum_{k=1}^{\infty} \frac{5^\beta}{\lambda} \mu\{x : 2^{k-1} \leq |x| < 2^{k+2}\} \leq \frac{3\mu(\mathbb{R}^n)5^\beta}{\lambda}.$$

Rearrange $\{x : x \in \Lambda_k, k = 1, 2, \ldots\}$ and $\{5r(x) : x \in \Lambda_k, k = 1, 2, \ldots\}$, we get $\{x_j\}$ and $\{\rho_j\}$ such that (3.2.1) and (3.2.2) hold.
3.3 \( p = 1 \)

1. Introduction and Main Theorems

In this section, we will consider measurable functions \( f \) in \( \mathbb{R}^{n-1} \) satisfying (see [1], [14] and [31])

\[
\int_{\mathbb{R}^{n-1}} \frac{|f(y')|}{1 + |y'|^{n+m}} dy' < \infty,
\]

(3.3.1)

where \( m \) is a nonnegative integer. This is just (3.1.7) when \( p = 1 \) and \( \gamma = n + m \).

It is well known that the Poisson kernel \( P(x,y') \) has a series expansion in terms of the ultraspherical (or Gegenbauer) polynomials \( C^\lambda_k(t) \) (see [40] and [35]).

The latter can be defined by a generating function

\[
(1 - 2tr + r^2)^{-\lambda} = \sum_{k=0}^{\infty} C^\lambda_k(t)r^k,
\]

(3.3.2)

where \( |r| < 1, |t| \leq 1 \) and \( \lambda > 0 \). The coefficients \( C^\lambda_k(t) \) is called the ultraspherical (or Gegenbauer) polynomial of degree \( k \) associated with \( \lambda \), the function \( C^\lambda_k(t) \) is a polynomial of degree \( k \) in \( t \). To obtain a solution of Dirichlet problem for the boundary data \( f \), as in [38], [39], [41] and [31], we use the following modified functions defined by

\[
E_m(x-y) = \begin{cases} 
E(x-y) & \text{when } |y| \leq 1, \\
E(x-y) + \sum_{k=0}^{m-1} \frac{r_0|x|^k}{|y|^{n+2-k}} C^\lambda_k \left( \frac{x|y|}{|x|} \right) & \text{when } |y| > 1.
\end{cases}
\]

Then we can define the modified Green function \( G_m(x,y) \) and the modified Poisson kernel \( P_m(x,y') \) by (see [20], [21], [5], [23] and [31])

\[
G_m(x,y) = E_{m+1}(x-y) - E_{m+1}(x-y^*), \quad x, y \in \mathcal{H}, \; x \neq y; \quad (3.3.3)
\]

\[
P_m(x,y') = \begin{cases} 
P(x,y') & \text{when } |y'| \leq 1, \\
P(x,y') - \sum_{k=0}^{m-1} \frac{2x_0|x|^k}{\omega_n|y'|^{n+2-k}} C^\lambda_k \left( \frac{x(y',0)}{|x||y'|} \right) & \text{when } |y'| > 1.
\end{cases}
\]

Siegel-Talvila [38] have proved the following result:

**Theorem C** Let \( f \) be a measurable function in \( \mathbb{R}^{n-1} \) satisfying (3.3.1), then the harmonic function

\[
v(x) = \int_{\mathbb{R}^{n-1}} P_m(x,y')f(y')dy', \quad x \in H \quad (3.3.5)
\]
satisfies (3.1.3), (3.1.4), (3.1.5) and
\[ v(x) = o(x_n^{1-n}|x|^{m+n}), \quad \text{as } |x| \to \infty, \] (3.3.6)
where \(P_m(x,y')\) is defined by (3.3.4).

In order to describe the asymptotic behaviour of subharmonic functions in the half space (see [45], [47], [28], [29] and [30]), we establish the following theorems.

**Theorem 3.3.1** Let \(f\) be a measurable function in \(\mathbb{R}^{n-1}\) satisfying (3.3.1), and \(0 < \alpha \leq n\). Let \(v(x)\) be the harmonic function defined by (3.3.5). Then there exists \(x_j \in H, \rho_j > 0\), such that
\[ \sum_{j=1}^{\infty} \frac{\rho_j^{n-\alpha}}{|x_j|^{n-\alpha}} < \infty \] (3.3.7)
holds and
\[ v(x) = o(x_n^{1-\alpha}|x|^{m+\alpha}), \quad \text{as } |x| \to \infty \] (3.3.8)
holds in \(H - G\), where \(G = \bigcup_{j=1}^{\infty} B(x_j, \rho_j)\).

**Remark 3.3.1** If \(\alpha = n\), then (3.3.7) is a finite sum, the set \(G\) is the union of finite balls, so (3.3.6) holds in \(H\). This is just the result of Siegel-Talvila, therefore, our result (3.3.8) is the generalization of Theorem C.

Next, we will generalize Theorem 3.3.1 to subharmonic functions.

**Theorem 3.3.2** Let \(f\) be a measurable function in \(\mathbb{R}^{n-1}\) satisfying (3.3.1) and \(\mu\) be a positive Borel measure satisfying
\[ \int_H \frac{y_n}{1 + |y|^{-1-n+m}} d\mu(y) < \infty. \] (3.3.9)
Write the subharmonic function
\[ u(x) = v(x) + h(x), \quad x \in H, \]
where \(v(x)\) is the harmonic function defined by (3.3.5), \(h(x)\) is defined by
\[ h(x) = \int_H G_m(x,y) d\mu(y) \]
and \(G_m(x,y)\) is defined by (3.3.3). Then there exists \(x_j \in H, \rho_j > 0\), such that (3.3.7) holds and
\[ u(x) = o(x_n^{1-\alpha}|x|^{m+\alpha}), \quad \text{as } |x| \to \infty \]
holds in $H - G$, where $G = \bigcup_{j=1}^{\infty} B(x_j, \rho_j)$ and $0 < \alpha < 2$.

Next we are concerned with minimal thinness [2] at infinity for $v(x)$ and $h(x)$, for a set $E \subset H$ and an open set $F \subset \mathbb{R}^{n-1}$, we consider the capacity

$$C(E; F) = \inf \int_{\mathbb{R}^{n-1}} g(y') dy',$$

where the infimum is taken over all nonnegative measurable functions $g$ such that $g = 0$ outside $F$ and

$$\int_{\mathbb{R}^{n-1}} \frac{g(y')}{|x - (y', 0)|^n} dy' \geq 1, \quad \text{for all } x \in E.$$

We say that $E \subset H$ is minimally thin at infinity if

$$\sum_{i=1}^{\infty} 2^{-in} C(E_i; F_i) < \infty,$$

where $E_i = \{ x \in E : 2^i \leq |x| < 2^{i+1} \}$ and $F_i = \{ x \in \mathbb{R}^{n-1} : 2^i < |x| < 2^{i+3} \}$.

**Theorem 3.3.3** Let $f$ be a measurable function in $\mathbb{R}^{n-1}$ satisfying (3.3.1), then there exists a set $E \subset H$ such that $E$ is minimally thin at infinity and

$$\lim_{|x| \to \infty, x \in H - E} \frac{v(x)}{|x|^m} = 0.$$

Similarly, for $h(x)$, we can also conclude the following:

**Corollary 3.3.1** Let $\mu$ be a positive Borel measure satisfying (3.3.9), then there exists a set $E \subset H$ such that $E$ is minimally thin at infinity and

$$\lim_{|x| \to \infty, x \in H - E} \frac{h(x)}{|x|^m} = 0.$$

Finally we are concerned with rarefiedness [2] at infinity for $v(x)$ and $h(x)$, for a set $E \subset H$ and an open set $F \subset H$, we consider the capacity

$$C(E; F) = \inf \int_{H} g(y) d\mu(y),$$

where the infimum is taken over all nonnegative measurable functions $g$ such that $g = 0$ outside $F$ and

$$\int_{H} \frac{g(y)}{|x - y|^{n-1}} d\mu(y) \geq 1, \quad \text{for all } x \in E.$$
We say that $E \subset H$ is rarefied at infinity if
\[ \sum_{i=1}^{\infty} 2^{-i(n-1)} C(E_i; F_i) < \infty, \]
where $E_i$ is as in Theorem 3.3.3 and $F_i = \{ x \in H : 2^i < |x| < 2^{i+3} \}$.

**Theorem 3.3.4** Let $\mu$ be a positive Borel measure satisfying (3.3.9), then there exists a set $E \subset H$ such that $E$ is rarefied at infinity and
\[ \lim_{|x| \to \infty, x \in H - E} \frac{h(x)}{|x|^{m+1}} = 0. \]

Similarly, for $v(x)$, we can also conclude the following:

**Corollary 3.3.2** Let $f$ be a measurable function in $\mathbb{R}^{n-1}$ satisfying (3.3.1), then there exists a set $E \subset H$ such that $E$ is rarefied at infinity and
\[ \lim_{|x| \to \infty, x \in H - E} \frac{v(x)}{|x|^{m+1}} = 0. \]

### 2. Main Lemmas

In order to obtain the results, we need the following lemmas:

**Lemma 3.3.1** Gegenbauer polynomials have the following properties:
1. \[ |C_k^\lambda(t)| \leq C_k^\lambda(1) = \frac{\Gamma((2\lambda+k)/(k+1))}{\Gamma((2\lambda+k)/2)}, \quad |t| \leq 1; \]
2. \[ \frac{d}{dt} C_k^\lambda(t) = 2\lambda C_{k-1}^{\lambda+1}(t), \quad k \geq 1; \]
3. \[ \sum_{k=0}^{\infty} C_k^\lambda(1) r^k = (1 - r)^{-2\lambda}; \]
4. \[ |C_k^\lambda(t) - C_k^\lambda(t^*)| \leq (n-2) C_{k-1}^{n/2}(1) |t - t^*|, \quad |t| \leq 1, \quad |t^*| \leq 1. \]

*Proof:* (1) and (2) can be derived from [40] and [15]; (3) follows by taking $t = 1$ in (3.3.2); (4) follows by (1), (2) and the Mean Value Theorem for Derivatives.

**Lemma 3.3.2** The Green function $G(x, y)$ has the following estimates:
1. \[ |G(x, y)| \leq \frac{r_y}{|x-y|^{n-2}}; \]
2. \[ |G(x, y)| \leq \frac{2^{n+1} \gamma_n}{\omega_n |x-y|^n}; \]
3.3. \( p = 1 \)

\[
(3) \quad |G(x,y)| \leq \frac{Ax_0 y_0}{|x-y|^n - |x-y|^m}.
\]

Proof: (1) is obvious; (2) follows by the Mean Value Theorem for Derivatives; (3) can be derived from \([2]\).

3. Proof of Theorems

Proof of Theorem 3.3.1

Define the measure \( dm(y') \) and the kernel \( K(x,y') \) by

\[
dm(y') = \frac{|f(y')|}{1 + |y'|^{n+m}} dy', \quad K(x,y') = P_m(x,y')(1 + |y'|^{n+m}).
\]

For any \( \varepsilon > 0 \), there exists \( R_\varepsilon > 2 \), such that

\[
\int_{|y'| \geq R_\varepsilon} dm(y') \leq \frac{\varepsilon}{5^{n-\alpha}}.
\]

For every Lebesgue measurable set \( E \subset \mathbb{R}^{n-1} \), the measure \( m^{(\varepsilon)} \) defined by \( m^{(\varepsilon)}(E) = m(E \cap \{x' \in \mathbb{R}^{n-1} : |x'| \geq R_\varepsilon\}) \) satisfies \( m^{(\varepsilon)}(\mathbb{R}^{n-1}) \leq \frac{\varepsilon}{5^{n-\alpha}} \). write

\[
v_1(x) = \int_{|x-(y',0)| \leq 3|x|} P(x,y')(1 + |y'|^{n+m}) dm^{(\varepsilon)}(y'),
\]

\[
v_2(x) = \int_{|x-(y',0)| \leq 3|x|} (P_m(x,y') - P(x,y'))(1 + |y'|^{n+m}) dm^{(\varepsilon)}(y'),
\]

\[
v_3(x) = \int_{|x-(y',0)| > 3|x|} K(x,y') dm^{(\varepsilon)}(y'),
\]

\[
v_4(x) = \int_{1 < |y'| < R_\varepsilon} K(x,y') dm(y'),
\]

\[
v_5(x) = \int_{|y'| \leq 1} K(x,y') dm(y'),
\]

then

\[
|v(x)| \leq |v_1(x)| + |v_2(x)| + |v_3(x)| + |v_4(x)| + |v_5(x)|.
\] (3.3.10)

Let \( E_1(\lambda) = \{x \in \mathbb{R}^n : |x| \geq 2, \exists t > 0, s.t. m^{(\varepsilon)}(B(x,t) \cap \mathbb{R}^{n-1}) > \lambda \left(\frac{t}{|x|}\right)^{n-\alpha}\} \), therefore, if \( |x| \geq 2R_\varepsilon \) and \( x \notin E_1(\lambda) \), then we have

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\[ |v_1(x)| \leq \int_{x_n \leq |x - (y',0)| \leq 3|x|} \frac{2x_n}{\omega_n|x - (y',0)|^n} 2|y'|^{n+m} dm^{(e)}(y') \]
\begin{align*}
&\leq \frac{4^{n+m+1}}{\omega_n} x_n |x|^{m+n} \int_{x_n}^{3|x|} \frac{1}{t^n} dm^{(e)}(t) \\
&\leq \frac{4^{n+m+1}}{\omega_n} \left( \frac{1}{3\alpha} + \frac{n}{\alpha} \right) \lambda x_n^{1-\alpha} |x|^{m+\alpha},
\end{align*}
(3.3.11)

where \( m^{(e)}(t) = \int_{|x-(y',0)| \leq t} dm^{(e)}(y') \).

By (1) and (3) of Lemma 3.3.1, we obtain
\[ |v_2(x)| \leq \int_{x_n \leq |x - (y',0)| \leq 3|x|} \frac{m-1}{2} \frac{2x_n |x|^k}{\omega_n} C^k_{n/2} (1) \frac{2|y'|^{n+m}}{|y'|^{n+k}} dm^{(e)}(y') \]
\begin{align*}
&\leq \frac{4^{m+1}}{\omega_n} \sum_{k=0}^{m-1} \frac{1}{4^k C^k_{n/2}} (1) \frac{1}{5^{n-\alpha}} \varepsilon x_n |x|^m \\
&\leq \frac{4^{m+1+\alpha}}{\omega_n \cdot 3^n} \varepsilon x_n |x|^m.
\end{align*}
(3.3.12)

By (1) and (3) of Lemma 3.3.1, we see that \[22\]
\[ |v_3(x)| \leq \int_{|x-(y',0)| > 3|x|} \frac{m}{2} \frac{4x_n |x|^k}{\omega_n (2|x|)^{k-n}} C^k_{n/2} (1) dm^{(e)}(y') \]
\begin{align*}
&\leq \frac{2^{m+2}}{\omega_n} \frac{\varepsilon}{5^{n-\alpha}} \sum_{k=m}^{\infty} \frac{1}{2^k C^k_{n/2}} (1) x_n |x|^m \\
&\leq \frac{2^{m-n+2\alpha+2}}{\omega_n} \varepsilon x_n |x|^m.
\end{align*}
(3.3.13)

Write
\[ v_4(x) = \int_{1<|y'|<R_t} [P(x,y') + (P_m(x,y') - P(x,y'))](1 + |y'|^{n+m}) dm(y') \]
\[ = v_{41}(x) + v_{42}(x), \]

then
\[ |v_{41}(x)| \leq \int_{1<|y'|<R_t} \frac{2x_n}{\omega_n|x - (y',0)|^n} 2|y'|^{n+m} dm(y') \]
\begin{align*}
&\leq \frac{4 R_n^{n+m} x_n}{\omega_n} \int_{1<|y'|<R_t} \frac{1}{(|y'|^2)^n} dm(y') \\
&\leq \frac{2^{n+2} R_n^{n+m} (R^{n-1} x_n)}{\omega_n} |x|^n.
\end{align*}
(3.3.14)
Moreover, by (1) and (3) of Lemma 3.3.1, we obtain

\[
|v_{42}(x)| \leq \int_{1<|y'|<R_{\varepsilon}} \sum_{k=0}^{m-1} \frac{2x_n|x|^k}{\omega_n |y'|^{n+k}} C_{n/2}^{n/2} |y'|^{n+m} \, dm(y') \\
\leq \sum_{k=0}^{m-1} \frac{4 C_{n/2}^{n/2}}{\omega_n} (1) x_n |x|^k R_{\varepsilon}^{m-k} m(R^{n-1}) \\
\leq \frac{2^{n+m+1} R_{\varepsilon}^m (R^{n-1})}{\omega_n} x_n |x|^{m-1}. \tag{3.3.15}
\]

In case $|y'| \leq 1$, note that

\[
K(x,y') = P_m(x,y')(1 + |y'|^{n+m}) \leq \frac{4x_n}{\omega_n |x - (y',0)|^n},
\]
so that

\[
|v_5(x)| \leq \int_{|y'| \leq 1} \frac{4x_n \omega_n}{|x|^{n}} \, dm(y') \leq \frac{2^{n+2} m(R^{n-1})}{\omega_n} \frac{x_n}{|x|^n}. \tag{3.3.16}
\]

Thus, by collecting (3.3.10), (3.3.11), (3.3.12), (3.3.13), (3.3.14), (3.3.15) and (3.3.16), there exists a positive constant $A$ independent of $\varepsilon$, such that if $|x| \geq 2R_{\varepsilon}$ and $x \notin E_1(\varepsilon)$, we have

\[
|v(x)| \leq A\varepsilon x_n^{1-\alpha} |x|^{m+\alpha}.
\]

Let $\mu_{\varepsilon}$ be a measure in $\mathbb{R}^n$ defined by $\mu_{\varepsilon}(E) = m(E \cap \mathbb{R}^{n-1})$ for every measurable set $E$ in $\mathbb{R}^n$. Take $\varepsilon = \varepsilon_p = \frac{1}{2^{\frac{1}{2p}}}$, $p = 1, 2, 3, \ldots$, then there exists a sequence $(R_p)$: $1 = R_0 < R_1 < R_2 < \cdots$ such that

\[
\mu_{\varepsilon_p}(\mathbb{R}^n) = \int_{|y'| \geq R_p} \, dm(y') < \frac{\varepsilon_p}{5^{n-\alpha}}.
\]

Take $\lambda = 3 \cdot 5^{n-\alpha} \cdot 2^p \mu_{\varepsilon_p}(\mathbb{R}^n)$ in Lemma 3.2.1, then there exists $x_{j,p}$ and $\rho_{j,p}$, where $R_{p-1} \leq |x_{j,p}| < R_p$, such that

\[
\sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|x_{j,p}|} \right)^{n-\alpha} \leq \frac{1}{2^p}.
\]

So if $R_{p-1} \leq |x| < R_p$ and $x \notin G_{\varepsilon_p} = \bigcup_{j=1}^{\infty} B(x_{j,p}, \rho_{j,p})$, we have

\[
|v(x)| \leq A\varepsilon x_n^{1-\alpha} |x|^{m+\alpha},
\]
thereby
\[ \sum_{p=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|x_{j,p}|} \right)^{n-\alpha} \leq \sum_{p=1}^{\infty} \frac{1}{2^p} = 1 < \infty. \]

Set \( G = \bigcup_{p=1}^{\infty} G_p \), thus Theorem 3.3.1 holds.

Proof of Theorem 3.3.2

Define the measure \( dn(y) \) and the kernel \( L(x,y) \) by
\[ dn(y) = \frac{y_n d\mu(y)}{1 + |y|^{n+m}}, \quad L(x,y) = G_m(x,y) \frac{1 + |y|^{n+m}}{y_n}, \]
then the function \( h(x) \) can be written as
\[ h(x) = \int_H L(x,y) dn(y). \]

For any \( \varepsilon > 0 \), there exists \( R_\varepsilon > 2 \), such that
\[ \int_{|y| \geq R_\varepsilon} dn(y) < \frac{\varepsilon}{5^{n-\alpha}}. \]

For every Lebesgue measurable set \( E \subset \mathbb{R}^n \), the measure \( n^{(\varepsilon)} \) defined by \( n^{(\varepsilon)}(E) = n(E \cap \{ y \in H : |y| \geq R_\varepsilon \}) \) satisfies \( n^{(\varepsilon)}(H) \leq \frac{\varepsilon}{5^{n-\alpha}} \), write

\[ h_1(x) = \int_{|x-y| \leq \frac{3|x|}{4}} G(x,y) \frac{1 + |y|^{n+m}}{y_n} dn^{(\varepsilon)}(y), \]
\[ h_2(x) = \int_{\frac{3|x|}{4} < |x-y| \leq 3|x|} G(x,y) \frac{1 + |y|^{n+m}}{y_n} dn^{(\varepsilon)}(y), \]
\[ h_3(x) = \int_{|x-y| \leq 3|x|} (G_m(x,y) - G(x,y)) \frac{1 + |y|^{n+m}}{y_n} dn^{(\varepsilon)}(y), \]
\[ h_4(x) = \int_{|x-y| > 3|x|} L(x,y) dn^{(\varepsilon)}(y), \]
\[ h_5(x) = \int_{1 < |y| < R_\varepsilon} L(x,y) dn(y), \]
\[ h_6(x) = \int_{|y| \leq 1} L(x,y) dn(y), \]

then
\[ h(x) = h_1(x) + h_2(x) + h_3(x) + h_4(x) + h_5(x) + h_6(x). \quad (3.3.17) \]
Let $E_2(\lambda) = \{x \in \mathbb{R}^n : |x| \geq 2, \exists t > 0, s.t. n^{(e)}(B(x,t) \cap H) > \lambda \left( \frac{t}{|x|} \right)^{n-\alpha} \}$, therefore, if $|x| \geq 2R_e$ and $x \notin E_2(\lambda)$, then we have by (1) of Lemma 3.3.2

\[
|h_1(x)| \leq \int_{|x-y| \leq \frac{R_e}{2}} \frac{r_n}{|x-y|^{n-2}} \frac{2|x|^{m+n}}{x_n^2} dn^{(e)}(y)
\]

\[
\leq 4 \times (3/2)^{n+m} r_n \frac{|x|^{n+m}}{x_n} \int_0^{\frac{R_e}{2}} t^{n-2}dn^{(e)}(t)
\]

\[
\leq 4 \times (3/2)^{n+m} r_n \left[ \frac{1}{2^\alpha} + \frac{n-2}{(2-\alpha)2^{\alpha}} \right] \lambda x_n^{1-\alpha} |x|^{m+\alpha}, \tag{3.3.18}
\]

where $n^{(e)}_x(t) = \int_{|x-y| \leq t} dn^{(e)}(y)$.

By (2) of Lemma 3.3.2, we have

\[
|h_2(x)| \leq \int_{\frac{R_e}{4} < |x-y| \leq 3|x|} \frac{2x_ny_n}{\omega_n |x-y|^n} \frac{2|y|^{n+m}}{y_n} \frac{1}{x_n} dn^{(e)}(y)
\]

\[
\leq \frac{4^{n+m+1}}{\omega_n} \frac{x_n|x|^{n+m}}{y_n} \int_{\frac{R_e}{4}}^{3|x|} \frac{1}{t^n} dt \frac{|y|^{n+m}}{y_n} \frac{1}{x_n} \frac{1}{t^{n-2}} \lambda x_n^{1-\alpha} |x|^{m+\alpha}.
\tag{3.3.19}
\]

First note $C^{(e)}_0(t) \equiv 1 \tag{40}$, then we obtain by (1), (3) and (4) of Lemma 3.3.1 and taking $t = \frac{xy}{|x||y|}$, $t^* = \frac{xy^*}{|x||y^*|}$ in (4) of Lemma 3.3.1

\[
|h_3(x)| \leq \int_{|x-y| \leq 3|x|} \frac{m}{\sum_{k=1}^{m} \frac{r_n|x|^k}{|y|^{n-2+k}} \frac{2(n-2)C_{n/2}^{n/2} k^{(e)}_{k-1}}{C_{n/2}^{n/2} k^{(e)}_{k-1}}} \frac{x_ny_n}{|x||y|} \frac{2|y|^{n+m}}{y_n} \frac{1}{x_n} \frac{1}{t^{n-2}} \lambda x_n^{1-\alpha} |x|^{m+\alpha}
\]

\[
\leq \frac{4^{m+1+\alpha}}{\omega_n} \frac{x_n|x|^{n+m}}{y_n} \frac{1}{x_n} \frac{1}{t^{n-2}} \lambda x_n^{1-\alpha} |x|^{m+\alpha}
\tag{3.3.20}
\]

By (1), (3) and (4) of Lemma 3.3.1, we see that

\[
|h_4(x)| \leq \int_{|x-y| > 3|x|} \left( \sum_{k=m+1}^{\infty} \frac{r_n|x|^k}{|y|^{n-2+k}} \frac{2(n-2)C_{n/2}^{n/2} k^{(e)}_{k-1}}{C_{n/2}^{n/2} k^{(e)}_{k-1}} \frac{x_ny_n}{|x||y|} \frac{2|y|^{n+m}}{y_n} \frac{1}{x_n} \frac{1}{t^{n-2}} \lambda x_n^{1-\alpha} |x|^{m+\alpha} \right)
\]

\[
\leq \frac{2^{m+2}}{\omega_n} \frac{x_n|x|^{n+m}}{y_n} \frac{1}{x_n} \frac{1}{t^{n-2}} \lambda x_n^{1-\alpha} |x|^{m+\alpha}
\tag{3.3.21}
\]

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Write
\[
    h_5(x) = \int_{1 < |y| < R_\varepsilon} [G(x, y) + (G_m(x, y) - G(x, y))] \frac{1 + |y|^{n+m}}{y_n} dn(y)
\]
\[
    = h_{51}(x) + h_{52}(x),
\]
then we obtain by (2) of Lemma 3.3.2
\[
    |h_{51}(x)| \leq \int_{1 < |y| < R_\varepsilon} \frac{2x_n y_n}{\omega_n |x-y|^n} \frac{2|y|^{n+m}}{y_n} dn(y)
\]
\[
    \leq \frac{4R_\varepsilon^{n+m}}{\omega_n} x_n \int_{1 < |y| < R_\varepsilon} \frac{1}{(|y|^2)^n} dn(y)
\]
\[
    \leq \frac{2^{n+2} R_\varepsilon^{n+m} n(H)}{\omega_n} \frac{x_n}{|x|^n}.
\]
(3.3.22)

Moreover, by (1), (3) and (4) of Lemma 3.3.1, we obtain
\[
    |h_{52}(x)| \leq \int_{1 < |y| < R_\varepsilon} \sum_{k=1}^{m} \frac{r_n |x|^k}{|y|^{n-2+k}} 2(n-2) C_{k-1}^{n/2} (1) \frac{x_n y_n}{|x||y|} \frac{2|y|^{n+m}}{y_n} dn(y)
\]
\[
    \leq \frac{4}{\omega_n} \int_{1 < |y| < R_\varepsilon} \sum_{k=1}^{m} C_{k-1}^{n/2} (1) x_n |x|^{k-1} R_\varepsilon^{n-k+1} n(H) \frac{x_n y_n}{|x| |y|} \frac{2|y|^{n+m}}{y_n} dn(y)
\]
\[
    \leq \frac{2^{n+m+1} R_\varepsilon^{m} n(H)}{\omega_n} x_n |x|^{m-1}.
\]
(3.3.23)

In case \(|y| \leq 1\), by (2) of Lemma 3.3.2, we have
\[
    |L(x, y)| \leq \frac{2x_n y_n}{\omega_n |x-y|^n} \frac{2|y|^{n+m}}{y_n} = \frac{4x_n}{\omega_n |x-y|^n},
\]
so that
\[
    |h_6(x)| \leq \int_{|y| \leq 1} \frac{4x_n}{\omega_n (\frac{|x|}{2})^n} dn(y) \leq \frac{2^{n+2} n(H)}{\omega_n} \frac{x_n}{|x|^n}.
\]
(3.3.24)

Thus, by collecting (3.3.17), (3.3.18), (3.3.19), (3.3.20), (3.3.21), (3.3.22), (3.3.23) and (3.3.24), there exists a positive constant A independent of \(\varepsilon\), such that if \(|x| \geq 2R_\varepsilon\) and \(x \notin E_2(\varepsilon)\), we have
\[
    |h(x)| \leq A \varepsilon x_n^{1-\alpha} |x|^{m+\alpha}.
\]

Similarly, if \(x \notin G\), we have
\[
    h(x) = o(x_n^{1-\alpha} |x|^{m+\alpha}), \quad \text{as } |x| \to \infty.
\]
(3.3.25)
By (3.3.8) and (3.3.25), we obtain that

\[ u(x) = v(x) + h(x) = o(x_n^{1-\alpha}|x|^{m+\alpha}), \quad \text{as } |x| \to \infty \]

holds in \( H - G \), thus we complete the proof of Theorem 3.3.2.

Proof of Theorem 3.3.3 and 3.3.4

We prove only Theorem 3.3.4, the proof of Theorem 3.3.3 is similar. By (3.3.20), (3.3.21), (3.3.22), (3.3.23) and (3.3.24) we have

\[
\lim_{|x| \to \infty, x \in H} \frac{h_3(x) + h_4(x) + h_5(x) + h_6(x)}{|x|^{m+1}} = 0. \tag{3.3.26}
\]

In view of (3.3.9), we can find a sequence \( \{a_i\} \) of positive numbers such that

\[
\lim_{i \to \infty} a_i = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} a_i \int_{F_i} \frac{y_n}{|y|^{n+m}} d\mu(y) < \infty.
\]

Consider the sets

\[ E_i = \{ x \in H : 2^i \leq |x| < 2^{i+1}, |h_1(x) + h_2(x)| \geq a_i^{-1}2^{im}|x| \} \]

for \( i = 1, 2, \ldots \). If \( x \in E_i \), then we obtain by (3) of Lemma 3.3.2

\[
a_i^{-1} \leq 2^{-im}|x|^{-1}|h_1(x) + h_2(x)| \leq A2^{-i(m+1)} \int_{F_i} \frac{y_n}{|x-y|^{n+m}} d\mu(y),
\]

so that it follows from the definition of \( C(E_i; F_i) \) that

\[
C(E_i; F_i) \leq Aa_i 2^{-i(m+1)} \int_{F_i} y n d\mu(y) \leq Aa_i 2^{i(n-1)} \int_{F_i} \frac{y_n}{|y|^{n+m}} d\mu(y).
\]

Define \( E = \bigcup_{i=1}^{\infty} E_i \), then

\[
\sum_{i=1}^{\infty} 2^{-i(n-1)} C(E_i; F_i) < \infty.
\]

Clearly,

\[
\lim_{|x| \to \infty, x \in H - E} \frac{h_1(x) + h_2(x)}{|x|^{m+1}} = 0. \tag{3.3.27}
\]

Thus, by collecting (3.3.26) and (3.3.27), the proof of Theorem 3.3.4 is completed.
3.4 $p > 1$ (General Kernel)

1. Introduction and Main Theorems

In this section, we will consider measurable functions $f$ in $\mathbb{R}^{n-1}$ satisfying

$$
\int_{\mathbb{R}^{n-1}} \frac{|f(y')|^p}{(1+|y'|)^\gamma} dy' < \infty,
$$

(3.4.1)

where $\gamma$ is defined as in Theorem 3.4.1.

In order to describe the asymptotic behaviour of subharmonic functions in the upper half space (see [28], [29], and [30]), we establish the following theorems.

Theorem 3.4.1 Let $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$
-(n-1)(p-1) < \gamma < (n-1) + p \quad \text{in case } p > 1;
$$

$$
0 < \gamma \leq n \quad \text{in case } p = 1.
$$

If $f$ is a measurable function in $\mathbb{R}^{n-1}$ satisfying (3.4.1) and $v(x)$ is the harmonic function defined by (3.1.6), then there exists $x_j \in H$, $\rho_j > 0$, such that

$$
\sum_{j=1}^\infty \frac{\rho_j^{pn-\alpha}}{|x_j|^{pn-\alpha}} < \infty
$$

(3.4.2)

holds and

$$
v(x) = o(x_n^{1-\frac{n}{p}} |x|^{\frac{\gamma}{p} + \frac{n-1}{q} - n + \frac{\alpha}{p}}), \quad \text{as } |x| \to \infty
$$

(3.4.3)

holds in $H - G$, where $G = \bigcup_{j=1}^\infty B(x_j, \rho_j)$ and $0 < \alpha \leq np$.

Remark 3.4.1 If $\alpha = n$, $p = 1$ and $\gamma = n$, then (3.4.2) is a finite sum, the set $G$ is the union of finite balls, so (3.4.3) holds in $H$. This is just the case $m = 0$ of the result of Siegel-Talvila.

Remark 3.4.2 If $\gamma = -(n-1)(p-1)$, $p > 1$, then

$$
v(x) = o(x_n^{1-\frac{n}{p}} (\log |x|)^{\frac{1}{p}} |x|^{\frac{\gamma}{p} + \frac{n-1}{q} - n + \frac{\alpha}{p}}), \quad \text{as } |x| \to \infty
$$

holds in $H - G$.

Next, we will generalize Theorem 3.4.1 to subharmonic functions.
Theorem 3.4.2 Let $p$ and $\gamma$ be as in Theorem 3.4.1. If $f$ is a measurable function in $\mathbf{R}^{n-1}$ satisfying (3.4.1) and $\mu$ is a positive Borel measure satisfying
\[
\int_{H} \frac{\gamma y}{(1+|y|)^{n}} d\mu(y) < \infty
\]
and
\[
\int_{H} \frac{1}{(1+|y|)^{n}} d\mu(y) < \infty.
\]
Write the subharmonic function
\[
u(x) = v(x) + h(x), \quad x \in H,
\]
where $v(x)$ is the harmonic function defined by (3.1.6), $h(x)$ is defined by
\[
h(x) = \int_{H} G(x,y) d\mu(y)
\]
and $G(x,y)$ is defined by (3.1.1). Then there exists $x_j \in H$, $p > 0$, such that (3.4.2) holds and
\[
u(x) = o(x^{\frac{1-n}{p}} |x|^{\frac{n}{q} + \frac{n-1}{q} - n + \frac{n}{p}}), \quad \text{as } |x| \to \infty
\]
holds in $H - G$, where $G = \bigcup_{j=1}^{\infty} B(x_j, p)$ and $0 < \alpha < 2p$.

Remark 3.4.3 If $\gamma = - (n-1)(p-1)$, $p > 1$, then
\[
u(x) = o(x^{\frac{1-n}{p}} (\log |x|)^\frac{1}{q} |x|^{\frac{n}{q} + \frac{n-1}{q} - n + \frac{n}{p}}), \quad \text{as } |x| \to \infty
\]
holds in $H - G$.

2. Main Lemmas

In order to obtain the results, we need these lemmas below:

Lemma 3.4.1 The kernel function $\frac{1}{|x-y|^n}$ has the following estimates:
\begin{enumerate}
\item If $|y| \leq \frac{|x|}{2}$, then $\frac{1}{|x-y|^n} \leq \frac{2^n}{|x|^n}$;
\item If $|y| > 2|x|$, then $\frac{1}{|x-y|^n} \leq \frac{2^n}{|y|^n}$.
\end{enumerate}

Lemma 3.4.2 The Green function $G(x,y)$ has the following estimates:
\begin{enumerate}
\item $|G(x,y)| \leq \frac{r_n}{|x-y|^n - 2}$;
\end{enumerate}
(2) \(|G(x,y)| \leq \frac{2v_ny_n}{\omega_n|x-y|^{n}}.\)

Proof: (1) is obvious; (2) follows by the Mean Value Theorem for Derivatives.

3. Proof of Theorems

Proof of Theorem 3.4.1

We prove only the case \(p > 1\); the proof of the case \(p = 1\) is similar. Define the measure \(dm(y')\) by

\[
 dm(y') = \frac{|f(y')|^p}{(1 + |y'|)^{\gamma}}dy'.
\]

For any \(\varepsilon > 0\), there exists \(R_\varepsilon > 2\), such that

\[
 \int_{|y'| \geq R_\varepsilon} dm(y') \leq \frac{\varepsilon^p}{5^{\mu-\alpha}}.
\]

For every Lebesgue measurable set \(E \subset \mathbb{R}^{n-1}\), the measure \(m^{(\varepsilon)}\) defined by \(m^{(\varepsilon)}(E) = m(E \cap \{x' \in \mathbb{R}^{n-1} : |x'| \geq R_\varepsilon\})\) satisfies \(m^{(\varepsilon)}(\mathbb{R}^{n-1}) \leq \frac{\varepsilon^p}{5^{\mu-\alpha}}\), write

\[
 v_1(x) = \int_{G_1} P(x,y') f(y')dy', \\
 v_2(x) = \int_{G_2} P(x,y') f(y')dy', \\
 v_3(x) = \int_{G_3} P(x,y') f(y')dy', \\
 v_4(x) = \int_{G_4} P(x,y') f(y')dy',
\]

where

\[
 G_1 = \{y' \in \mathbb{R}^{n-1} : R_\varepsilon < |y'| \leq \frac{|x|}{2}\}, \\
 G_2 = \{y' \in \mathbb{R}^{n-1} : \frac{|x|}{2} < |y'| \leq 2|x|\}, \\
 G_3 = \{y' \in \mathbb{R}^{n-1} : |y'| > 2|x|\}, \\
 G_4 = \{y' \in \mathbb{R}^{n-1} : |y'| \leq R_\varepsilon\}. 
\]
3.4. \( p > 1 \) (General Kernel)

Then

\[ v(x) = v_1(x) + v_2(x) + v_3(x) + v_4(x). \]  

(3.4.4)

First, if \( \gamma > -(n-1)(p-1) \), then \( \frac{\mu}{p} + (n-1) > 0 \), so that we obtain by (1) of Lemma 3.4.1 and Hölder’s inequality

\[
|v_1(x)| \leq \int_{G_1} \frac{2x_n}{\omega_n} \frac{2^n}{|x|^n} |f(y')| dy'
\]

\[
\leq \frac{2^{n+1} x_n}{\omega_n} \left( \int_{G_1} \frac{|f(y')|^p}{|y'|^{p \gamma}} dy' \right)^{1/p} \left( \int_{G_1} |y'|^{\frac{\nu}{p}} dy' \right)^{1/q},
\]

since

\[
\int_{G_1} |y'|^{\frac{\nu}{p}} dy' \leq \omega_{n-1} \frac{1}{\frac{\nu}{p} + n - 1} \left( \frac{|x|}{2} \right)^{\frac{\nu}{p} + n - 1},
\]

so that

\[
|v_1(x)| \leq A \varepsilon x_n |x|^\frac{p + \alpha - 1}{q}. \]  

(3.4.5)

Let \( E_1(\lambda) = \{ x \in \mathbb{R}^n : |x| \geq 2, \exists t > 0, \text{s.t.} m^{(e)}(B(x, t) \cap \mathbb{R}^{n-1}) > \lambda^p \left( \frac{t}{|x|} \right)^{pn - \alpha} \} \), therefore, if \( |x| \geq 2R_e \) and \( x \notin E_1(\lambda) \), then we have

\[
\forall t > 0, m^{(e)}(B(x, t) \cap \mathbb{R}^{n-1}) \leq \lambda^p \left( \frac{t}{|x|} \right)^{pn - \alpha}.
\]

If \( \gamma > -(n-1)(p-1) \), then \( \frac{\mu}{p} + (n-1) > 0 \), so that we obtain by Hölder’s inequality

\[
|v_2(x)| \leq \frac{2x_n}{\omega_n} \left( \int_{G_2} \frac{|f(y')|^p}{|x - (y', 0)|^{pn}|y'|^{p \gamma}} dy' \right)^{1/p} \left( \int_{G_2} |y'|^{\frac{\nu}{p}} dy' \right)^{1/q}
\]

\[
\leq A x_n |x|^\frac{p + \alpha - 1}{q} \left( \int_{G_2} \frac{|f(y')|^p}{|x - (y', 0)|^{pn}|y'|^{p \gamma}} dy' \right)^{1/p},
\]

since

\[
\int_{G_2} \frac{|f(y')|^p}{|x - (y', 0)|^{pn}|y'|^{p \gamma}} dy' \leq \int_{x_n} \frac{2^{\gamma + 1}}{t^{pn}} d m^{(e)}(t)
\]

\[
\leq \frac{\lambda^p}{|x|^{\alpha}} \left( \frac{1}{3^\alpha + \frac{pn}{\alpha}} \right) \frac{|x|^\alpha}{x_n^\alpha},
\]

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where \( m^n_{\mathcal{E}}(t) = \int_{|x\cdot(y', 0)| \leq t} dm^n(y') \).

Hence we have
\[
|v_2(x)| \leq A\lambda_\mathcal{E}_n \left( \frac{1 - \frac{q}{p}}{x} \right)^{\frac{n-1}{q} -n + \frac{q}{p}}.
\]
(3.4.6)

If \( \gamma < (n-1) + p \), then \( (\frac{\gamma}{p} - n)q + (n-1) < 0 \), so that we obtain by (2) of Lemma 3.4.1 and Hölder’s inequality
\[
|v_3(x)| \leq \int_{G_3} \frac{2^n}{\omega_n} \frac{2^n}{|y'|^n} |f(y')| dy'
\leq \frac{2^{n+1}}{\omega_n} x_n \left( \int_{G_3} \frac{|f(y')|^p}{|y'|^\gamma} dy' \right)^{1/p} \left( \int_{G_3} |y'|^{(\frac{\gamma}{p} - n)q} dy' \right)^{1/q}
\leq A\varepsilon x_n |x|^{\frac{\gamma}{p} - \frac{n-1}{q} - n}.
\]
(3.4.7)

Finally, by (1) of Lemma 3.4.1, we obtain
\[
|v_4(x)| \leq \frac{2^n}{\omega_n} \frac{x_n}{|x|^n} \int_{G_4} |f(y')| dy',
\]
which implies by \( \gamma > -(n-1)(p-1) \) that
\[
|v_4(x)| \leq A\varepsilon x_n |x|^{\frac{\gamma}{p} - \frac{n-1}{q} - n}.
\]
(3.4.8)

Thus, by collecting (3.4.4), (3.4.5), (3.4.6), (3.4.7) and (3.4.8), there exists a positive constant \( A \) independent of \( \varepsilon \), such that if \( |x| \geq 2R_\varepsilon \) and \( x \notin E_1(\varepsilon) \), we have
\[
|v(x)| \leq A\varepsilon x_n \left( \frac{1 - \frac{q}{p}}{x} \right)^{\frac{n-1}{q} -n + \frac{q}{p}}.
\]

Let \( \mu_{\varepsilon} \) be a measure in \( \mathbb{R}^n \) defined by \( \mu_{\varepsilon}(E) = m^n(E \cap \mathbb{R}^{n-1}) \) for every measurable set \( E \) in \( \mathbb{R}^n \). Take \( \varepsilon = \varepsilon_p = \frac{1}{2^{p+2}}, p = 1, 2, 3, \ldots \), then there exists a sequence \( \{R_p\} : 1 = R_0 < R_1 < R_2 < \cdots \) such that
\[
\mu_{\varepsilon_p}(\mathbb{R}^n) = \int_{|y'| \geq R_p} dm(y') < \frac{\varepsilon_p^p}{5^m - \alpha}.
\]

Take \( \lambda = 3 \cdot 5^{pn-\alpha} \cdot 2^p \mu_{\varepsilon_p}(\mathbb{R}^n) \) in Lemma 3.2.1, then there exists \( x_{j,p} \) and \( \rho_{j,p} \), where \( R_{p-1} \leq |x_{j,p}| < R_p \), such that
\[
\sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|x_{j,p}|} \right)^{pn-\alpha} \leq \frac{1}{2^p}.
\]
If $R_{p-1} \leq |x| < R_p$ and $x \notin G_p = \bigcup_{j=1}^{\infty} B(x_{j,p}, \rho_{j,p})$, we have

$$|v(x)| \leq A\varepsilon_p x_{p}^{\frac{1}{p}} |x|^\gamma \rho_{j,p}^{\alpha} x_{j,p} \rho_{j,p}^{n}.$$  

Thereby

$$\sum_{p=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|x_{j,p}|} \right)^{pn-\alpha} \leq \sum_{p=1}^{\infty} \frac{1}{2p} = 1 < \infty.$$  

Set $G = \bigcup_{p=1}^{\infty} G_p$, thus Theorem 3.4.1 holds.

Proof of Theorem 3.4.2

We prove only the case $p > 1$; the remaining case $p = 1$ can be proved similarly. Define the measure $dn(y)$ by

$$dn(y) = \frac{y_{n}^{p}}{(1 + |y|)^{2}} d\mu(y).$$

For any $\varepsilon > 0$, there exists $R_{\varepsilon} > 2$, such that

$$\int_{|y| \geq R_{\varepsilon}} dn(y) < \frac{\varepsilon^{p}}{5^{pn-\alpha}}.$$  

For every Lebesgue measurable set $E \subset \mathbb{R}^{n}$, the measure $n^{(\varepsilon)}$ defined by $n^{(\varepsilon)}(E) = n(E \cap \{y \in H : |y| \geq R_{\varepsilon} \})$ satisfies $n^{(\varepsilon)}(H) \leq \frac{\varepsilon^{p}}{5^{pn-\alpha}}$, write

$$h_1(x) = \int_{F_1} G(x,y) d\mu(y),$$

$$h_2(x) = \int_{F_2} G(x,y) d\mu(y),$$

$$h_3(x) = \int_{F_3} G(x,y) d\mu(y),$$

$$h_4(x) = \int_{F_4} G(x,y) d\mu(y),$$

where

$$F_1 = \{y \in H : R_{\varepsilon} < |y| \leq \frac{|x|}{2} \},$$

$$F_2 = \{y \in H : \frac{|x|}{2} < |y| \leq 2|x| \},$$

$$F_3 = \{y \in H : |y| > 2|x| \},$$

$$F_4 = \{y \in H : |y| \leq R_{\varepsilon} \}.$$  

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Then
\[ h(x) = h_1(x) + h_2(x) + h_3(x) + h_4(x). \] (3.4.9)

First, if \( \gamma > -(n-1)(p-1) \), then \( \frac{n}{p} + (n-1) > 0 \), so that we obtain by (1) of Lemma 3.4.1, (2) of Lemma 3.4.2 and Hölder’s inequality
\[
|h_1(x)| \leq \int_{F_1} \frac{2x_n y_n}{\omega_n |x-y|^n} d\mu(y) \\
\leq \int_{F_1} \frac{2^n x_n y_n}{\omega_n |x|^n} d\mu(y) \\
\leq \frac{2^{n+1} x_n}{\omega_n |x|^n} \left( \int_{F_1} \frac{y_n^p}{|y|^q} d\mu(y) \right)^{1/p} \left( \int_{F_1} |y|^{\frac{mp}{n}} d\mu(y) \right)^{1/q},
\]
since
\[
\int_{F_1} |y|^{\frac{mp}{n}} d\mu(y) \leq 2^{n-1} \left( \frac{|x|}{2} \right)^{\frac{mp+n-1}{p}} \int_H \frac{1}{(1+|y|)^{n-1}} d\mu(y),
\]
so that
\[
|h_1(x)| \leq A \varepsilon x \left( \frac{n}{p} + \frac{n-1}{q} \right)^{-n}. \quad (3.4.10)
\]

Let \( E_2(\lambda) = \{ x \in \mathbb{R}^n : |x| \geq 2, \exists t > 0, s.t. n^{(e)}(B(x,t) \cap H) > \lambda^p \left( \frac{t}{|x|} \right)^{pn-\alpha} \} \), therefore, if \( |x| \geq 2R_e \) and \( x \notin E_2(\lambda) \), then we have
\[
\forall t > 0, n^{(e)}(B(x,t) \cap H) \leq \lambda^p \left( \frac{t}{|x|} \right)^{pn-\alpha}.
\]

If \( \gamma > -(n-1)(p-1) \), then \( \frac{n}{p} + (n-1) > 0 \), so that we obtain by Hölder’s inequality
\[
|h_2(x)| \leq \left( \int_{F_2} \frac{|G(x,y)|^p}{|y|^q} d\mu(y) \right)^{1/p} \left( \int_{F_2} |y|^{mp/n} d\mu(y) \right)^{1/q} \\
\leq \left( (2^{\gamma+1}) \int_{F_2} \frac{|G(x,y)|^p}{y_n^p} d\mu(y) \right)^{1/p} \left( \int_{F_2} |y|^{mp/n} d\mu(y) \right)^{1/q} \\
\leq A |x|^{\frac{\gamma+1}{p} + \frac{n-1}{q}} \left( \int_{F_2} \frac{|G(x,y)|^p}{y_n^p} d\mu(y) \right)^{1/p},
\]
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\[ 3.4. \quad p > 1 \text{(General Kernel)} \]

since

\[
\int_{F_2} \frac{|G(x,y)|^p}{y_n^p} \, dn(y) \leq \int_{|x-y| \leq 3|x|} \frac{|G(x,y)|^p}{y_n^p} \, dn^{(e)}(y)
\]

\[
= \int_{|x-y| \leq \frac{3x}{n}} \frac{|G(x,y)|^p}{y_n^p} \, dn^{(e)}(y) + \int_{\frac{3x}{n} < |x-y| \leq 3|x|} \frac{|G(x,y)|^p}{y_n^p} \, dn^{(e)}(y)
\]

\[
= h_{21}(x) + h_{22}(x),
\]

so that we have by (1) of Lemma 3.4.2

\[
h_{21}(x) \leq \int_{|x-y| \leq \frac{3x}{n}} \left( \frac{2r_n}{x_n |x-y|^{n-2}} \right)^p \, dn^{(e)}(y)
\]

\[
= \left( \frac{2r_n}{x_n} \right)^p \int_{0}^{\frac{3x}{n}} \frac{1}{t^{p(n-2)}} \, dn^{(e)}(t)
\]

\[
\leq (2r_n)^p \frac{np - \alpha}{(2p - \alpha)2^{p-\alpha}} \lambda^p \left( \frac{x_n^{p-\alpha}}{|x|^{np-\alpha}} \right).
\]

Moreover, we have by (2) of Lemma 3.4.2

\[
h_{22}(x) \leq \int_{\frac{3x}{n} < |x-y| \leq 3|x|} \left( \frac{2x_n}{\omega_n |x-y|^n} \right)^p \, dn^{(e)}(y)
\]

\[
= \left( \frac{2x_n}{\omega_n} \right)^p \int_{\frac{3x}{n}}^{3|x|} \frac{1}{t^p} \, dn^{(e)}(t)
\]

\[
\leq \left( \frac{2}{\omega_n} \right)^p \left( \frac{1}{3^\alpha + \frac{np2^\alpha}{\alpha}} \right) \lambda^p \left( \frac{x_n^{p-\alpha}}{|x|^{np-\alpha}} \right),
\]

where \( n^{(e)}_x(t) = \int_{|x-y| \leq t} dn^{(e)}(y) \).

Hence we have

\[
|h_2(x)| \leq A\lambda x_n^{1 - \frac{q}{p}} |x|^{\frac{\gamma}{p} + \frac{\alpha}{q} - n + \frac{q}{p}}.
\]  

(3.4.11)

If \( \gamma < (n - 1) + p \), then \( \left( \frac{\gamma}{p} - n \right)q + (n - 1) < 0 \), so that we obtain by (2) of Lemma 3.4.1, (2) of Lemma 3.4.2 and Hölder’s inequality.
\[ |h_3(x)| \leq \int_{F_3} \frac{2x_n y_n}{\omega_n |x-y|^n} d\mu(y) \]
\[ \leq \int_{F_3} \frac{2^n}{\omega_n} 2^n d\mu(y) \]
\[ \leq \frac{2^{n+1}}{\omega_n} x_n \left( \int_{F_3} \frac{y_n^p}{|y|^n} d\mu(y) \right)^{1/p} \left( \int_{F_3} |y|^{(p-n)q} d\mu(y) \right)^{1/q} \]
\[ \leq A\varepsilon x_n |x|^{p + \frac{n+1}{q} + \frac{\alpha}{p} - n}. \quad (3.4.12) \]

Finally, by (1) of Lemma 3.4.1 and (2) of Lemma 3.4.2, we obtain

\[ |h_4(x)| \leq \int_{F_4} \frac{2x_n y_n}{\omega_n |x-y|^n} d\mu(y) \leq \frac{2^{n+1}}{\omega_n} x_n \int_{F_4} y_n d\mu(y), \]

which implies by \( \gamma > -(n-1)(p-1) \) that

\[ |h_4(x)| \leq A\varepsilon x_n |x|^{p + \frac{n+1}{q} - n}. \quad (3.4.13) \]

Thus, by collecting (3.4.9), (3.4.10), (3.4.11), (3.4.12) and (3.4.13), there exists a positive constant \( A \) independent of \( \varepsilon \), such that if \( |x| \geq 2R_\varepsilon \) and \( x \notin E_2(\varepsilon) \), we have

\[ |h(x)| \leq A\varepsilon x_n^{1-\frac{\alpha}{p}} |x|^{p + \frac{n+1}{q} - n + \frac{\alpha}{p}}. \]

Similarly, if \( x \notin G \), we have

\[ h(x) = o(x_n^{1-\frac{\alpha}{p}} |x|^{p + \frac{n+1}{q} - n + \frac{\alpha}{p}}), \quad \text{as} \ |x| \to \infty. \quad (3.4.14) \]

By (3.4.3) and (3.4.14), we obtain that

\[ u(x) = v(x) + h(x) = o(x_n^{1-\frac{\alpha}{p}} |x|^{p + \frac{n+1}{q} - n + \frac{\alpha}{p}}), \quad \text{as} \ |x| \to \infty \]

holds in \( H - G \), thus we complete the proof of Theorem 3.4.2.
3.5 the Estimates for the Modified Poisson Kernel and Green Function

1. Introduction and Main Theorems

Recall that the modified Poisson kernel $P_m(x, y')$ and the modified Green function $G_m(x, y)$ (see [20], [21], [5], [23] and [31]) are defined respectively by

$$P_m(x, y') = \begin{cases} P(x, y') & \text{when } |y'| \leq 1, \\ P(x, y') - \sum_{k=0}^{m-1} 2\pi |x|^k \omega^m \frac{2\pi}{|x|} \left( \frac{x \cdot y'}{|x| |y'|} \right) & \text{when } |y'| > 1 \end{cases}$$

and

$$G_m(x, y) = E_{m+1}(x - y) - E_{m+1}(x - y^*), \quad x, y \in \mathcal{H}, \quad x \neq y,$$

where

$$E_m(x - y) = \begin{cases} E(x - y) & \text{when } |y| \leq 1, \\ E(x - y) + \sum_{k=0}^{m-1} r_n |x|^k \omega^m \frac{2\pi}{|y|} \left( \frac{x \cdot y'}{|x| |y'|} \right) & \text{when } |y| > 1. \end{cases}$$

In our discussions, the estimates for the modified Poisson kernel $P_m(x, y')$ and the modified Green function $G_m(x, y)$ are fundamental, therefore, we establish the following theorems.

**Theorem 3.5.1** Suppose $|y'| > 1$, then we have the estimates:

$$|P_m(x, y')| \leq \begin{cases} A \frac{x_n}{|x - y'|^{n-2}} |y'|^{m+n-1}, & \text{when } s' > 1, \\ A \frac{x_n}{|x - y'|^{n-2}} s'^m, & \text{when } s' \leq 1, \end{cases}$$

where $s' = \frac{|x|}{|y'|}$.

**Theorem 3.5.2** Suppose $|y| > 1$, then we have the estimates:

$$|G_m(x, y)| \leq \begin{cases} A \frac{x_n y_n}{|x - y|^{n-2}} \left( |y|^{m+1} \left( \frac{|x|}{|y|} + \frac{1}{|x - y|^2} \right) + |x - y|^2 \right), & \text{when } s \leq 1, \\ A \frac{x_n y_n}{|x - y|^{n-2}} \left( |y|^{m+n-4} \left( 1 + \frac{|x|}{|y|} + \frac{|x|^2}{|x - y|^2} \right) \right), & \text{when } s > 1, \end{cases}$$

where $s = \frac{|x|}{|y|}$.
2. Main Lemma

In order to obtain the results, we need the lemma below:

**Lemma 3.5.1** Suppose $|y'| > 1$, set $s' = \frac{|x|}{|y|}$ and $t' = \frac{x'y'}{|x||y'|}$, then

$$P_m(x, y') = P(x, y')[mC_m^{n/2}(t')I_m^{(n)}(s', t') - (n + m - 1)C_{m-1}^{n/2}(t')I_m^{(n)}(s', t')]$$

where

$$I_m^{(n)}(s', t') = \int_0^{s'} (1 - 2t'\xi + \xi^2)^{n/2-1}\xi^m d\xi, \quad s' > 0, \quad |t'| < 1.$$

3. Proof of Theorems

Proof of Theorem 3.5.1

Suppose $|y'| > 1$, since

$$I_m^{(n)}(s', t') = \int_0^{s'} (1 - 2t'\xi + \xi^2)^{n/2-1}\xi^m d\xi$$

we can obtain by Lemma 3.5.1

$$|P_m(x, y')| \leq P(x, y')[m|C_m^{n/2}(t')||I_m^{(n)}(s', t')| + (n + m - 1)|C_{m-1}^{n/2}(t')||I_m^{(n)}(s', t')|]$$

$$
\leq P(x, y')[mA|I_m^{(n)}(s', t')| + (n + m - 1)A|I_m^{(n)}(s', t')|] .
$$

When $s' > 1$,

$$|P_m(x, y')| \leq A\frac{x_n}{|x - y'|^n}s^{m+n-1} ;$$

when $s' \leq 1$,

$$|P_m(x, y')| \leq A\frac{x_n}{|x - y'|^n}s^m .$$

Thus

$$|P_m(x, y')| \leq \begin{cases} A\frac{x_n}{|x - y'|^n}s^{m+n-1} , & \text{when } s' > 1, \\ A\frac{x_n}{|x - y'|^n}s^m , & \text{when } s' \leq 1. \end{cases}$$
3.5. the Estimates for the Modified Poisson Kernel and Green Function

Proof of Theorem 3.5.2

Suppose $|y| > 1$, by Lemma 3.5.1, we obtain

$$P_m(x, y) = P(x, y)[mC_m^{n/2}(t)I_m^{(n)}(s, t) - (n + m - 1)C_m^{n/2}(t)I_m^{(n)}(s, t)],$$

where $s = \frac{|x|}{|y|}$, $t = \frac{x^y}{|x||y|}$.

Thus

$$E_m(x - y) = E(x - y)[mC_m^{n/2}(t)I_m^{(n)}(s, t) - (n + m - 3)C_m^{n/2}(t)I_m^{(n-2)}(s, t)].$$

Similarly, we can obtain

$$E_m(x - y^*) = E(x - y^*)[mC_m^{n/2}(t^*)I_m^{(n-2)}(s^*, t^*) - (n + m - 3)C_m^{n/2}(t^*)I_m^{(n-2)}(s^*, t^*)],$$

so that

$$G_m(x, y) = (m + 1)[f - f^*] - (n + m - 2)[g - g^*], \quad (3.5.1)$$

where

$$f - f^* = E(x - y)C_m^{n/2}(t)I_m^{(n-2)}(s, t) - E(x - y)C_m^{n/2}(t)I_m^{(n-2)}(s, t^*)$$

$$+ E(x - y)C_m^{n/2}(t)I_m^{(n-2)}(s^*, t^*) - E(x - y)C_m^{n/2}(t^*)I_m^{(n-2)}(s^*, t^*)$$

$$+ E(x - y)C_m^{n/2}(t^*)I_m^{(n-2)}(s^*, t^*) - E(x - y^*)C_m^{n/2}(t^*)I_m^{(n-2)}(s^*, t^*)$$

$$= I_1 + I_2 + I_3. \quad (3.5.2)$$

For the first term, we have

$$I_1 = E(x - y)C_m^{n/2}(t)[I_m^{(n-2)}(s, t) - I_m^{(n-2)}(s, t^*)]$$

$$= E(x - y)C_m^{n/2}(t)\left[\int_0^\xi (1 - 2t\xi + \xi^2)^{-\frac{n+2}{2}} - 1 \xi^m d\xi - \int_0^\xi (1 - 2t^*\xi + \xi^2)^{-\frac{n+2}{2}} - 1 \xi^m d\xi\right]$$

$$= -2(n - 4)x_yE(x - y)C_m^{n/2}(t)I_m^{(n-4)}(s, t_0),$$

where
where \( t^* < t_0 < t \), thus

\[
|I_1| \leq \begin{cases} 
A \frac{x_0 y_n}{|x-y|^{n-2}} |x|^{m} |y|^{m+n-3}, & \text{when } s \leq 1, \\
A \frac{x_0 y_n}{|x-y|^{n-2}} |x|^{m+1} |y|^{m+n-5}, & \text{when } s > 1;
\end{cases} 
\] (3.5.3)

for the second term, we have

\[
I_2 = E(x-y)I_m^{(n-2)}(s^*, t^*) [c_{m+1}(t) - c_{m+1}(t^*)],
\]

by (4) of Lemma 3.3.1, we have

\[
|I_2| \leq \frac{1}{(n-2)\omega_n} \frac{1}{|x-y|^{n-2}} I_m^{(n-2)}(s^*, t^*) |(n-2)c_{m}^n(1)| t - t^* 
\]

\[
\leq \begin{cases} 
A \frac{x_0 y_n}{|x-y|^{n-2}} |x|^{m} |y|^{m+n-3}, & \text{when } s \leq 1, \\
A \frac{x_0 y_n}{|x-y|^{n-2}} |x|^{m+1} |y|^{m+n-5}, & \text{when } s > 1;
\end{cases} 
\] (3.5.4)

for the third term, we have

\[
I_3 = c_{m+1}^{n-2}(t^*)I_m^{(n-2)}(s^*, t^*) [E(x-y) - E(x-y^*)],
\]

thus

\[
|I_3| \leq A \frac{1}{(n-2)\omega_n} \frac{2(n-2)x_0 y_n}{|x-y|^{n}} |I_m^{(n-2)}(s, t^*)|
\]

\[
\leq \begin{cases} 
A \frac{x_0 y_n}{|x-y|^{n-2}} |x|^{m} |y|^{m+n-3}, & \text{when } s \leq 1, \\
A \frac{x_0 y_n}{|x-y|^{n-2}} |x|^{m+1} |y|^{m+n-5}, & \text{when } s > 1.
\end{cases} 
\] (3.5.5)

So we have by (3.5.2), (3.5.3), (3.5.4) and (3.5.5)

\[
|f - f^*| \leq |I_1| + |I_2| + |I_3|
\]

\[
\leq \begin{cases} 
A \frac{x_0 y_n}{|x-y|^{n-2}} |x|^{m} \left( \frac{|x|}{|y|} + \frac{|x|}{|x-y|^2} \right), & \text{when } s \leq 1, \\
A \frac{x_0 y_n}{|x-y|^{n-2}} |x|^{m+1} \left( 1 + \frac{|x|}{|y|} + \frac{|x|}{|x-y|^2} \right), & \text{when } s > 1.
\end{cases} 
\] (3.5.6)

Similarly,

\[
g - g^* = E(x-y)c_{m+1}^{n-2}(t)I_m^{(n-2)}(s, t) - E(x-y)c_{m+1}^{n-2}(t)I_m^{(n-2)}(s, t^*)
\]

\[
+ E(x-y)c_{m+1}^{n-2}(t)I_m^{(n-2)}(s^*, t^*) - E(x-y)c_{m+1}^{n-2}(t^*)I_m^{(n-2)}(s^*, t^*)
\]

\[
+ E(x-y)c_{m+1}^{n-2}(t^*)I_m^{(n-2)}(s^*, t^*) - E(x-y)c_{m+1}^{n-2}(t^*)I_m^{(n-2)}(s^*, t^*)
\]

\[
= J_1 + J_2 + J_3,
\] (3.5.7)
and we have the similar estimates:

\[
|J_1| \leq \begin{cases} 
A \frac{x_n y_n}{|x-y|^n} \frac{|x|^{m+2}}{|y|^{m+n-2}}, & \text{when } s \leq 1, \\
A \frac{x_n y_n}{|x-y|^n} \frac{|x|^{m+n-4}}{|y|^{m+n-2}}, & \text{when } s > 1;
\end{cases} \tag{3.5.8}
\]

\[
|J_2| \leq \begin{cases} 
A \frac{x_n y_n}{|x-y|^n} \frac{|x|^{m+1}}{|y|^{m+n-2}}, & \text{when } s \leq 1, \\
A \frac{x_n y_n}{|x-y|^n} \frac{|x|^{m+n-3}}{|y|^{m+n-2}}, & \text{when } s > 1;
\end{cases} \tag{3.5.9}
\]

\[
|J_3| \leq \begin{cases} 
A \frac{x_n y_n}{|x-y|^n} \frac{|x|^{m+2}}{|y|^{m+n-2}}, & \text{when } s \leq 1, \\
A \frac{x_n y_n}{|x-y|^n} \frac{|x|^{m+n-2}}{|y|^{m+n-2}}, & \text{when } s > 1.
\end{cases} \tag{3.5.10}
\]

So we have by (3.5.7), (3.5.8), (3.5.9) and (3.5.10)

\[
|g - g^*| \leq |J_1| + |J_2| + |J_3|
\]

\[
\leq \begin{cases} 
A \frac{x_n y_n}{|x-y|^n} \frac{|x|^{m+1}}{|y|^{m+n-2}} \left( \frac{|x|}{|y|} + \frac{1}{|y|} + \frac{|x|}{|x-y|^2} \right), & \text{when } s \leq 1, \\
A \frac{x_n y_n}{|x-y|^n} \frac{|x|^{m+n-3}}{|y|^{m+n-2}} \left( 1 + \frac{|x|}{|y|} + \frac{|x|^2}{|x-y|^2} \right), & \text{when } s > 1.
\end{cases} \tag{3.5.11}
\]

Hence we finally obtain by (3.5.1), (3.5.6) and (3.5.11)

\[
|G_m(x,y)| \leq (m+1)|f - f^*| + (n+m-2)|g - g^*|
\]

\[
\leq \begin{cases} 
A \frac{x_n y_n}{|x-y|^n} \frac{|x|^{m+1}}{|y|^{m+n-2}} \left( \frac{|x|}{|y|} + \frac{1}{|y|} + \frac{|x|}{|x-y|^2} \right), & \text{when } s \leq 1, \\
A \frac{x_n y_n}{|x-y|^n} \frac{|x|^{m+n-3}}{|y|^{m+n-2}} \left( 1 + \frac{|x|}{|y|} + \frac{|x|^2}{|x-y|^2} \right), & \text{when } s > 1.
\end{cases}
\]

\[3.6 \quad p > 1\text{(Modified Kernel)}\]

1. Introduction and Main Theorems

In this section, we will consider measurable functions \( f \) in \( \mathbb{R}^{n-1} \) satisfying

\[
\int_{\mathbb{R}^{n-1}} \frac{|f(y')|^p}{(1 + |y'|^2)^{\gamma/2}} dy' < \infty, \tag{3.6.1}
\]

where \( \gamma \) is defined as in Theorem 3.6.1.
In order to describe the asymptotic behaviour of subharmonic functions represented by the modified kernel in the upper half space (see [28], [29] and [30]), we establish the following theorems.

**Theorem 3.6.1** Let \( 1 \leq p < \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and

\[
(n - 1) + mp < \gamma < (n - 1) + (m + 1)p \quad \text{in case } p > 1; \\
m + n - 1 < \gamma \leq m + n \quad \text{in case } p = 1.
\]

If \( f \) is a measurable function in \( \mathbb{R}^{n-1} \) satisfying (3.6.1) and \( v(x) \) is the harmonic function defined by

\[
v(x) = \int_{\mathbb{R}^{n-1}} P_m(x, y') f(y') dy', \quad x \in H,
\]

then there exists \( x_j \in H, \rho_j > 0 \), such that

\[
\sum_{j=1}^{\infty} \frac{\rho_j^{pn-\alpha}}{|x_j|^{pn-\alpha}} < \infty
\]

holds and

\[
v(x) = o(x_n^{-\frac{q}{p}} |x|^\frac{\gamma-1}{q} - n + \frac{\gamma}{p}), \quad \text{as } |x| \to \infty
\]

holds in \( H - G \), where \( G = \bigcup_{j=1}^{\infty} B(x_j, \rho_j) \) and \( 0 < \alpha \leq np \).

**Remark 3.6.1** If \( \alpha = n \), \( p = 1 \) and \( \gamma = n + m \), then (3.6.3) is a finite sum, the set \( G \) is the union of finite balls, so (3.6.4) holds in \( H \). This is just the result of Siegel-Talvila, therefore, our result (3.6.4) is the generalization of Theorem C.

**Remark 3.6.2** If \( \gamma = (n - 1) + mp, p > 1 \), then

\[
v(x) = o(x_n^{-\frac{q}{p}} (\log |x|)^{\frac{1}{q}} |x|^\frac{\gamma-1}{q} - n + \frac{\gamma}{p}), \quad \text{as } |x| \to \infty
\]

holds in \( H - G \).

Next, we will generalize Theorem 3.6.1 to subharmonic functions.

**Theorem 3.6.2** Let \( p \) and \( \gamma \) be as in Theorem 3.6.1. If \( f \) is a measurable function in \( \mathbb{R}^{n-1} \) satisfying (3.6.1) and \( \mu \) is a positive Borel measure satisfying

\[
\int_{H} \frac{y_n^p}{(1 + |y|)^\gamma} d\mu(y) < \infty
\]
3.6. \( p > 1 \) (Modified Kernel)

and

\[
\int_H \frac{1}{(1+|y|)^{n-1}} d\mu(y) < \infty.
\]

Write the subharmonic function

\[
u(x) = v(x) + h(x), \quad x \in H,
\]

where \( v(x) \) is the harmonic function defined by (3.6.2), \( h(x) \) is defined by

\[
h(x) = \int_H G_m(x,y) d\mu(y)
\]

and \( G_m(x,y) \) is defined by (3.3.3). Then there exists \( x_j \in H, \rho_j > 0 \), such that (3.6.3) holds and

\[
u(x) = o(x_1^{1-n} |x|^{\frac{1}{\sigma}-n+\frac{\beta}{p}}), \quad \text{as } |x| \to \infty
\]

holds in \( H - G \), where \( G = \bigcup_{j=1}^\infty B(x_j, \rho_j) \) and \( 0 < \alpha < 2p \).

Remark 3.6.3 If \( \gamma = (n-1) + mp, p > 1 \), then

\[
u(x) = o(x_1^{1-n} (\log |x|)^{\frac{1}{\sigma}} |x|^{\frac{1}{\sigma}-n+\frac{\beta}{p}}), \quad \text{as } |x| \to \infty
\]

holds in \( H - G \).

2. Main Lemmas

In order to obtain the results, we need these lemmas below:

Lemma 3.6.1 The modified Poisson kernel \( P_m(x,y) \) has the following estimates:

1. If \( 1 < |y| \leq \frac{|x|}{2} \), then \( |P_m(x,y)| \leq \frac{Ax_m|x|^{m-1}}{|y|^{m+n}} \).
2. If \( \frac{|x|}{2} < |y| \leq 2|x| \), then \( |P_m(x,y)| \leq \frac{Ax_m}{|x-(y,A)|^n} \).
3. If \( |y| > 2|x| \), then \( |P_m(x,y)| \leq \frac{Ax_m|x|^{m-1}}{|y|^{m+n}} \).
4. If \( |y| \leq 1 \), then \( |P_m(x,y)| \leq \frac{Ax_m}{|x|^{p}} \).

Lemma 3.6.2 The modified Green function \( G_m(x,y) \) has the following estimates:

1. If \( 1 < |y| \leq \frac{|x|}{2} \), then \( |G_m(x,y)| \leq \frac{Ax_m|x|^{m-1}}{|y|^{m+n}} \).
2. If \( \frac{|x|}{2} < |y| \leq 2|x| \), then \( |G_m(x,y)| \leq \frac{Ax_m}{|x-y|^n} \).

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3. Proof of Theorems

Proof of Theorem 3.6.1

We prove only the case $p > 1$; the proof of the case $p = 1$ is similar. Define the measure $dm(y')$ by

$$dm(y') = \frac{|f(y')|^p}{(1 + |y'|)^\gamma} dy'.$$

For any $\varepsilon > 0$, there exists $R_\varepsilon > 2$, such that

$$\int_{|y'| \geq R_\varepsilon} dm(y') \leq \frac{\varepsilon p}{5^{p - \alpha}}.$$

For every Lebesgue measurable set $E \subset \mathbb{R}^{n-1}$, the measure $m^{(\varepsilon)}$ defined by $m^{(\varepsilon)}(E) = m(E \cap \{x' \in \mathbb{R}^{n-1} : |x'| \geq R_\varepsilon\})$ satisfies $m^{(\varepsilon)}(\mathbb{R}^{n-1}) \leq \frac{\varepsilon p}{5^{p - \alpha}}$, write

$$v_1(x) = \int_{G_1} P_m(x, y') f(y') dy',$$

$$v_2(x) = \int_{G_2} P_m(x, y') f(y') dy',$$

$$v_3(x) = \int_{G_3} P_m(x, y') f(y') dy',$$

$$v_4(x) = \int_{G_4} P_m(x, y') f(y') dy', $$

where

$$G_1 = \{y' \in \mathbb{R}^{n-1} : 1 < |y'| \leq \frac{|x|}{2}\},$$

$$G_2 = \{y' \in \mathbb{R}^{n-1} : \frac{|x|}{2} < |y'| \leq 2|x|\},$$

$$G_3 = \{y' \in \mathbb{R}^{n-1} : |y'| > 2|x|\},$$

$$G_4 = \{y' \in \mathbb{R}^{n-1} : |y'| \leq 1\}. $$
Then
\[ v(x) = v_1(x) + v_2(x) + v_3(x) + v_4(x). \]  
\hspace{1cm} (3.6.5)

First, if \( \gamma > (n-1) + mp \), then \( (\frac{2}{p} - m - n + 1)q + (n - 1) > 0 \). For \( R_e > 2 \), we have
\[ v_1(x) = \int_{1<|y'| \leq R_e} P_n(x,y')f(y')dy' + \int_{R_e < |y'| \leq \frac{3}{2}} P_n(x,y')f(y')dy' = v_{11}(x) + v_{12}(x), \]
if \( |x| > 2R_e \), then we obtain by (1) of Lemma 3.6.1 and H"older’s inequality
\[ |v_{11}(x)| \leq \int_{1<|y'| \leq R_e} Ax_n |x|^{m-1} |f(y')|dy' \]
\[ \leq Ax_n |x|^{m-1} \left( \int_{1<|y'| \leq R_e} \frac{|f(y')|^p}{|y'|^q} dy' \right)^{1/p} \left( \int_{1<|y'| \leq R_e} |y'|^{\left(\frac{\gamma}{p} - m - n + 1\right)q} dy' \right)^{1/q}, \]
since
\[ \int_{1<|y'| \leq R_e} |y'|^{\left(\frac{\gamma}{p} - m - n + 1\right)q} dy' \leq AR_e^{\left(\frac{\gamma}{p} - m - n + 1\right)q + (n - 1)}, \]
so that
\[ |v_{11}(x)| \leq Ax_n |x|^{m-1} R_e^{\left(\frac{\gamma}{p} - m - n + 1\right) + \frac{n - 1}{q}}. \]  
\hspace{1cm} (3.6.6)

Moreover, we have similarly
\[ |v_{12}(x)| \leq Ax_n |x|^{m-1} \left( \int_{R_e < |y'| \leq \frac{3}{2}} \frac{|f(y')|^p}{|y'|^q} dy' \right)^{1/p} \left( \int_{R_e < |y'| \leq \frac{3}{2}} |y'|^{\left(\frac{\gamma}{p} - m - n + 1\right)q} dy' \right)^{1/q}, \]
\[ \leq Ax_n |x|^{\frac{\gamma}{p} + 1 - n} \left( \int_{R_e < |y'| \leq \frac{3}{2}} \frac{|f(y')|^p}{|y'|^q} dy' \right)^{1/p}, \]
which implies by arbitrariness of \( R_e \) that
\[ |v_{12}(x)| \leq Ax_n |x|^{\frac{\gamma}{p} + \frac{n - 1}{q} - n}. \]  
\hspace{1cm} (3.6.7)

Let \( E_1(\lambda) = \{ x \in \mathbb{R}^n : |x| \geq 2, \exists t > 0, \text{s.t. } m^{(e)}(B(x,t) \cap \mathbb{R}^{n-1}) > \lambda p \left( \frac{t}{|x|} \right)^{pn - \alpha} \} \), therefore, if \( |x| \geq 2R_e \) and \( x \notin E_1(\lambda) \), then we have
\[ \forall t > 0, m^{(e)}(B(x,t) \cap \mathbb{R}^{n-1}) \leq \lambda p \left( \frac{t}{|x|} \right)^{pn - \alpha}. \]
If \( \gamma > (n-1)+mp \), then \( (\frac{\gamma}{p} - m - n + 1)q + (n-1) > 0 \), so that we obtain by (2) of Lemma 3.6.1 and Hölder’s inequality
\[
|v_2(x)| \leq \int_{G_2} \frac{Ax_n}{|x-(y',0)|^n} |f(y')| dy' \\
\leq Ax_n \left( \int_{G_2} \frac{|f(y')|^p}{|x-(y',0)|^p |y'|^q} dy' \right)^{1/p} \left( \int_{G_2} |y'|^{\frac{q}{p}} dy' \right)^{1/q} \\
\leq Ax_n |x|^\frac{\gamma}{p} \left( \int_{G_2} \frac{|f(y')|^p}{|x-(y',0)|^p |y'|^q} dy' \right)^{1/p},
\]
since
\[
\int_{G_2} \frac{|f(y')|^p}{|x-(y',0)|^p |y'|^q} dy' \leq \int_{x_n}^{3|x|} \frac{2^{\gamma+1}}{t^{pn}} dm^{(e)}(t) \\
\leq \frac{\lambda^p}{|x|^{pn}} (2^{\gamma+1}) \left( \frac{1}{3^\alpha} + \frac{pn}{\alpha} \right) |x|^\frac{\alpha}{x_n^q},
\]
where \( m^{(e)}(t) = \int_{|x-(y',0)| \leq t} dm^{(e)}(y') \).
Hence we have
\[
|v_2(x)| \leq A \lambda x_n \frac{1-\alpha}{p} |x|^\frac{\gamma}{p} \frac{n-1}{q} - n + \frac{\alpha}{p}. \tag{3.6.8}
\]

If \( \gamma < (n-1) + (m+1)p \), then \( (\frac{\gamma}{p} - m - n)q + (n-1) < 0 \), so that we obtain by (3) of Lemma 3.6.1 and Hölder’s inequality
\[
|v_3(x)| \leq \int_{G_3} \frac{Ax_n |x|^m}{|y'|^{m+n}} |f(y')| dy' \\
\leq Ax_n |x|^m \left( \int_{G_3} \frac{|f(y')|^p}{|y'|^q} dy' \right)^{1/p} \left( \int_{G_3} |y'|^{(\frac{q}{p} - m - n)q} dy' \right)^{1/q} \\
\leq A \varepsilon x_n |x|^\frac{\gamma}{p} - \frac{n-1}{q} - n. \tag{3.6.9}
\]

Finally, by (4) of Lemma 3.6.1, we obtain
\[
|v_4(x)| \leq \frac{Ax_n}{|x|^p} \int_{G_4} |f(y')| dy'. \tag{3.6.10}
\]

Thus, by collecting (3.6.5), (3.6.6), (3.6.7), (3.6.8), (3.6.9) and (3.6.10), there exists a positive constant \( A \) independent of \( \varepsilon \), such that if \( |x| \geq 2R_\varepsilon \) and \( x \notin E_1(\varepsilon) \), we have
\[
|v(x)| \leq A \varepsilon x_n 1-\frac{\alpha}{p} |x|^\frac{\gamma}{p} \frac{n-1}{q} - n + \frac{\alpha}{p}.
\]
3.6. \( p > 1 \) (Modified Kernel)

Let \( \mu_\varepsilon \) be a measure in \( \mathbb{R}^n \) defined by \( \mu_\varepsilon(E) = m^{(\varepsilon)}(E \cap \mathbb{R}^{n-1}) \) for every measurable set \( E \) in \( \mathbb{R}^n \). Take \( \varepsilon = \varepsilon_p = \frac{1}{2p+2}, p = 1, 2, 3, \ldots \), then there exists a sequence \( \{R_p\} : 1 = R_0 < R_1 < R_2 < \cdots \) such that

\[
\mu_{\varepsilon_p}(\mathbb{R}^n) = \int_{|y'| \geq R_p} dm(y') < \frac{\varepsilon_p^p}{5^{pm-\alpha}}.
\]

Take \( \lambda = 3 \cdot 5^{pn-\alpha} \cdot 2^p \mu_{\varepsilon_p}(\mathbb{R}^n) \) in Lemma 3.2.1, then there exists \( x_{j,p} \) and \( \rho_{j,p} \), where \( R_{p-1} \leq |x_{j,p}| < R_p \), such that

\[
\sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|x_{j,p}|} \right)^{pn-\alpha} \leq \frac{1}{2^p}.
\]

If \( R_{p-1} \leq |x| < R_p \) and \( x \notin G_p = \bigcup_{j=1}^{\infty} B(x_{j,p}, \rho_{j,p}) \), we have

\[
|v(x)| \leq A \varepsilon_{p} x_n^{\frac{1-\alpha}{n}} |x|^{\frac{n-1}{n}} = n^{\frac{\alpha-1}{n}}.
\]

Thereby

\[
\sum_{p=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{\rho_{j,p}}{|x_{j,p}|} \right)^{pn-\alpha} \leq \sum_{p=1}^{\infty} \frac{1}{2^p} = 1 < \infty.
\]

Set \( G = \bigcup_{p=1}^{\infty} G_p \), thus Theorem 3.6.1 holds.

Proof of Theorem 3.6.2

We prove only the case \( p > 1 \); the remaining case \( p = 1 \) can be proved similarly. Define the measure \( dn(y) \) by

\[
dn(y) = \frac{y_n^p}{(1 + |y|)^{\gamma}} d\mu(y).
\]

For any \( \varepsilon > 0 \), there exists \( R_\varepsilon > 2 \), such that

\[
\int_{|y| \geq R_\varepsilon} dn(y) < \frac{\varepsilon^p}{5^{pn-\alpha}}.
\]

For every Lebesgue measurable set \( E \subset \mathbb{R}^n \), the measure \( n^{(\varepsilon)} \) defined by \( n^{(\varepsilon)}(E) = n(E \cap \{y \in H : |y| \geq R_\varepsilon\}) \) satisfies \( n^{(\varepsilon)}(H) \leq \frac{\varepsilon^p}{5^{pn-\alpha}} \), write

\[
h_1(x) = \int_{F_1} G_m(x, y) d\mu(y),
\]
\[
h_2(x) = \int_{F_2} G_m(x, y) d\mu(y),
\]
\[
h_3(x) = \int_{F_3} G_m(x, y) d\mu(y),
\]
\[
h_4(x) = \int_{F_4} G_m(x, y) d\mu(y),
\]

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where
\[ F_1 = \{ y \in H : 1 < |y| \leq \frac{|x|}{2} \}, \]
\[ F_2 = \{ y \in H : \frac{|x|}{2} < |y| \leq 2|x| \}, \]
\[ F_3 = \{ y \in H : |y| > 2|x| \}, \]
\[ F_4 = \{ y \in H : |y| \leq 1 \}. \]

Then
\[ h(x) = h_1(x) + h_2(x) + h_3(x) + h_4(x). \]  

(3.6.11)

First, if \( \gamma > (n - 1) + mp \), then \( \left( \frac{\gamma}{p} - m - n + 1 \right)q + (n - 1) > 0 \). For \( R_\varepsilon > 2 \), we have
\[ h_1(x) = \int_{1 < |y| \leq R_\varepsilon} G_m(x, y) d\mu(y) + \int_{R_\varepsilon < |y| \leq \frac{|x|}{2}} G_m(x, y) d\mu(y) = h_{11}(x) + h_{12}(x), \]

if \( |x| > 2R_\varepsilon \), then we obtain by (1) of Lemma 3.6.2 and Hölder’s inequality
\[ |h_{11}(x)| \leq \int_{1 < |y| \leq R_\varepsilon} \frac{Ax_n |y|^{m-1}}{|y|^{m+n-1}} d\mu(y) \]
\[ \leq Ax_n |x|^{m-1} \left( \int_{1 < |y| \leq R_\varepsilon} \frac{y_n^p}{|y|^q} d\mu(y) \right)^{1/p} \left( \int_{1 < |y| \leq R_\varepsilon} |y|^{(\frac{\gamma}{p} - m - n + 1)q} d\mu(y) \right)^{1/q}, \]

since
\[ \int_{1 < |y| \leq R_\varepsilon} |y|^{(\frac{\gamma}{p} - m - n + 1)q} d\mu(y) \leq R_\varepsilon^{(\frac{\gamma}{p} - m - n + 1)q + (n - 1)}, \]

so that
\[ |h_{11}(x)| \leq Ax_n |x|^{m-1} R_\varepsilon^{(\frac{\gamma}{p} - m - n + 1) + \frac{n-1}{q}}. \]  

(3.6.12)

Moreover, we have similarly
\[ |h_{12}(x)| \leq Ax_n |x|^{m-1} \left( \int_{R_\varepsilon < |y| \leq \frac{|x|}{2}} \frac{y_n^p}{|y|^q} d\mu(y) \right)^{1/p} \left( \int_{R_\varepsilon < |y| \leq \frac{|x|}{2}} |y|^{(\frac{\gamma}{p} - m - n + 1)q} d\mu(y) \right)^{1/q} \]
\[ \leq Ax_n |x|^{\frac{\gamma}{p} - \frac{n-1}{q} - n} \left( \int_{R_\varepsilon < |y| \leq \frac{|x|}{2}} \frac{y_n^p}{|y|^q} d\mu(y) \right)^{1/p}, \]

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which implies by arbitrariness of $R_6$ that

$$|h_{12}(x)| \leq A \varepsilon x_n |x|^{\frac{2}{p} + \frac{n-1}{q} - n}. \quad (3.6.13)$$

Let $E_2(\lambda) = \{x \in \mathbb{R}^n : |x| \geq 2, \exists \ t > 0, \text{s.t.} \ n^{(e)}(B(x,t) \cap H) \geq \lambda^p \left( \frac{t}{|x|} \right)^{pn - \alpha} \}$, therefore, if $|x| \geq 2R_6$ and $x \notin E_2(\lambda)$, then we have

$$\forall t > 0, n^{(e)}(B(x,t) \cap H) \leq \lambda^p \left( \frac{t}{|x|} \right)^{pn - \alpha}.$$ 

If $\gamma > (n - 1) + mp$, then $(\frac{\gamma}{p} - m - n + 1)q + (n - 1) > 0$, so that we obtain by Hölder’s inequality

$$|h_2(x)| \leq \left( \int_{F_2} \frac{|G_m(x,y)|^p}{|y|^q} d\mu(y) \right)^{1/p} \left( \int_{F_2} |y|^q d\mu(y) \right)^{1/q} \leq \left( \frac{2^q + 1}{2^q} \int_{F_2} \frac{|G_m(x,y)|^p}{y_n^p} d\mu(y) \right)^{1/p} \left( \int_{F_2} |y|^q d\mu(y) \right)^{1/q} \leq A |x|^{\frac{2}{p} + \frac{n-1}{q}} \left( \int_{F_2} \frac{|G_m(x,y)|^p}{y_n^p} d\mu(y) \right)^{1/p},$$

since

$$\int_{F_2} \frac{|G_m(x,y)|^p}{y_n^p} d\mu(y) \leq \int_{|x-y| \leq 3|x|} \frac{|G_m(x,y)|^p}{y_n^p} d\mu^{(e)}(y) \leq \int_{|x-y| \leq \frac{3}{2}|x|} \frac{|G_m(x,y)|^p}{y_n^p} d\mu^{(e)}(y) + \int_{\frac{3}{2} < |x-y| \leq 3|x|} \frac{|G_m(x,y)|^p}{y_n^p} d\mu^{(e)}(y) = h_{21}(x) + h_{22}(x),$$

so that we have by (5) of Lemma 3.6.2

$$h_{21}(x) \leq \int_{|x-y| \leq \frac{3}{2}} \left( \frac{A}{x_n|x-y|^{n-2}} \right)^p d\mu^{(e)}(y) = \left( \frac{A}{x_n} \right)^p \int_{0}^{\frac{3}{2}} \frac{1}{t^{p(n-2)}} d\mu^{(e)}(t) \leq A \frac{np - \alpha}{(2p - \alpha)2^{2p-\alpha}} \lambda^{p \frac{\lambda}{|x|^{n \alpha}}}.$$
Moreover, we have by (2) of Lemma 3.6.2

\[
\begin{align*}
  h_{22}(x) & \leq \int_{\frac{3}{2}<|x-y|\leq 3|x|} \left( \frac{Ax_n}{|x-y|^n} \right)^p d\mu(y) \\
  & = (Ax_n)^p \int_{\frac{3}{2}}^{3|x|} \frac{1}{t^p} d\mu^* (t) \\
  & \leq A \left( \frac{1}{3^\alpha} + \frac{np^2\alpha}{\alpha} \right) \lambda p \frac{x_n^{p-\alpha}}{|x|^{np-\alpha}},
\end{align*}
\]

where \( n^*_x (t) = \int_{|x-y|\leq t} d\mu^*(y) \).

Hence we have

\[
|h_2(x)| \leq A \lambda x_n^{1-\frac{\alpha}{p}} |x|^{\frac{q-1}{q} - n + \frac{\alpha}{p}}. \tag{3.6.14}
\]

If \( \gamma < (n-1) + (m+1)p \), then \( \frac{\alpha}{p} - m - n \) is positive, so that we obtain by (3) of Lemma 3.6.2 and Hölder’s inequality

\[
|h_3(x)| \leq \int_{F_3} A x_n y_n |x|^m |y|^{m+n} d\mu(y) \\
\leq A x_n |x|^m \left( \int_{F_3} y_n^p |y|^q d\mu(y) \right)^{1/p} \left( \int_{F_3} |y|^{(\frac{\alpha}{p} - m - n)q} d\mu(y) \right)^{1/q} \\
\leq A \epsilon x_n |x|^{\frac{2\alpha}{q} + \frac{\alpha}{p} - n}. \tag{3.6.15}
\]

Finally, by (4) of Lemma 3.6.2, we obtain

\[
|h_4(x)| \leq \frac{Ax_n}{|x|^n} \int_{F_4} y_n d\mu(y). \tag{3.6.16}
\]

Thus, by collecting (3.6.11), (3.6.12), (3.6.13), (3.6.14), (3.6.15) and (3.6.16), there exists a positive constant \( A \) independent of \( \epsilon \), such that if \( |x| \geq 2R_\epsilon \) and \( x \notin E_2(\epsilon) \), we have

\[
|h(x)| \leq A \epsilon x_n^{1-\frac{\alpha}{p}} |x|^{\frac{q-1}{q} - n + \frac{\alpha}{p}}.
\]

Similarly, if \( x \notin G \), we have

\[
h(x) = o(x_n^{1-\frac{\alpha}{p}} |x|^{\frac{q-1}{q} - n + \frac{\alpha}{p}}), \quad \text{as } |x| \to \infty. \tag{3.6.17}
\]

By (3.6.4) and (3.6.17), we obtain that

\[
u(x) = v(x) + h(x) = o(x_n^{1-\frac{\alpha}{p}} |x|^{\frac{q-1}{q} - n + \frac{\alpha}{p}}), \quad \text{as } |x| \to \infty
\]

holds in \( H - G \), thus we complete the proof of Theorem 3.6.2.
Chapter 4

a Generalization of Harmonic Majorants

4.1 a Generalization of Harmonic Majorants in the Upper Half Plane

1. Introduction and Main Theorem

The Poisson kernel for the half plane $C_+ = \{z = x + iy \in \mathbb{C} : y > 0\}$ is the function

$$P(z, t) = \frac{y}{\pi |z - t|^2},$$

where $z \in C_+$ and $t \in \mathbb{R}$.

If $p \geq 0$ is an integer, we define a modified Cauchy kernel of order $p$ for $z \in C_+ - \{t\}$ by

$$C_p(z, t) = \begin{cases} 
\frac{1}{\pi} \frac{1}{|z - t|}, & \text{when } |t| \leq 1, \\
\frac{1}{\pi} \frac{1}{|z - t|} - \frac{1}{\pi} \sum_{k=0}^{p} \frac{z^k}{k!}, & \text{when } |t| > 1,
\end{cases}$$

then we define a modified Poisson kernel of order $p$ for the upper half plane by

$$P_p(z, t) = \Im C_p(z, t).$$

Flett and Kuran [34] proved the following theorem:
Chapter 4. a Generalization of Harmonic Majorants

**Theorem D** Let $G(z)$ be nonnegative and subharmonic in $\mathbb{C}_+$. Then $G(z)$ has a harmonic majorant in $\mathbb{C}_+$ if and only if
\[
\sup_{y>0} \int_{-\infty}^{\infty} \frac{G(x+iy)}{x^2 + (y+1)^2} dx < \infty.
\]

**Remark 4.1.1** If $G(z)$ has a harmonic majorant in $\mathbb{C}_+$, then there exists a harmonic function
\[
H(z) = cy + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2 + y^2}, \quad y > 0,
\]
where $c \geq 0$ and $\mu$ is a nonnegative Borel measure on $(-\infty, \infty)$ such that
\[
\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty,
\]
and
\[
G(z) \leq H(z).
\]

In this section, we will generalize Theorem D partly to the modified kernel.

**Theorem 4.1.1** Let
\[
H(z) = \Re\left[Q_p(z) + \frac{1}{\pi} \int_{-\infty}^{\infty} C_p(z,t) d\mu(t)\right], \quad z = x + iy, \quad y > 0,
\]
where
\[
Q_p(z) = \sum_{k=0}^{p} a_k z^k, \quad a_k \in \mathbb{R}, \quad k = 0, 1, 2, \ldots, p
\]
and $\mu$ is a nonnegative Borel measure on $(-\infty, \infty)$ such that
\[
\int_{-\infty}^{\infty} \frac{1}{1+|t|^{p+1}} d\mu(t) < \infty.
\]
If $G(z)$ is subharmonic in $\mathbb{C}_+$ and
\[
G(z) \leq H(z),
\]
then
\[
\sup_{y>0} \int_{-\infty}^{\infty} \frac{G(x+iy)}{x^2 + (y+1)^2} dx < \infty.
\]

**Remark 4.1.2** If $p = 1$, this is just the result of Flett and Kuran, therefore, our result is partly the generalization of Theorem D.
2. Main lemmas

In order to obtain the result, we need these lemmas below:

Lemma 4.1.1 For any $|t| > 1$, the following equality

$$
\Im C_p(z,t) = \Im \frac{tz^{p+1} - |z|^2z^p}{|t-z|^{2p+1}}
$$

(4.1.1)

holds.

Proof: For $|t| > 1$, since

$$
C_p(z,t) = \frac{1}{t-z} - \sum_{k=0}^{p} \frac{z^k}{t^{k+1}} = \frac{z^{p+1}}{(t-z)t^{p+1}},
$$

then

$$
\Im C_p(z,t) = \Im \frac{z^{p+1}}{(t-z)t^{p+1}} = \Im \left[ \frac{z^{p+1}(t - \bar{z})}{|t-z|^{2p+1}} \right] = \Im \frac{tz^{p+1} - |z|^2z^p}{|t-z|^{2p+1}}.
$$

This proves the equality (4.1.1).

Lemma 4.1.2 There exists $A > 0$, such that the inequality

$$
\Im (tz^{p+1} - |z|^2z^p) \leq Ay(t^2 + y^2)(x^2 + y^2)^{\frac{p-1}{2}}
$$

holds in the following conditions:

(1) $p = 2m - 1, m = 1, 2, \cdots$;
(2) $p = 2m, m = 1, 2, \cdots, x \geq 0$;
(3) $p = 2m, m = 1, 2, \cdots, x < 0, |t| \geq |x|$.
3. Proof of Theorem

First from \( G(z) \leq H(z) \), we obtain

\[
\int_{-\infty}^{\infty} \frac{G(x + iy)}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} dx \\
\leq \int_{-\infty}^{\infty} \frac{Q_p(z)}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} dx \\
+ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} dx \int_{-\infty}^{\infty} \Im C_p(z, t) d\mu(t) \\
= I_1 + I_2. \tag{4.1.2}
\]

For the first term, we have

\[
I_1 = \int_{-\infty}^{\infty} \frac{\sum_{k=0}^{p} a_k (x + iy)^k}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} dx \\
= y \int_{-\infty}^{\infty} \frac{\sum_{k=0}^{p} a_k [C^1_k x^{k-1} - C^3_k x^{k-3} y^2 + \cdots]}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} dx \\
\leq y \int_{-\infty}^{\infty} \frac{\sum_{k=0}^{p} |a_k| \sum_{i=1}^{k+1} C^2_k |x|^{k-1} - (2i-2) y^{2i-2}}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} dx \\
\leq Ay \int_{-\infty}^{\infty} \frac{1}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} dx \\
\leq Ay \int_{-\infty}^{\infty} \frac{1}{x^2 + (y + 1)^2} dx \\
\leq A\pi \frac{y}{y + 1} \leq A\pi; \tag{4.1.3}
\]

for the second term, we will discuss in the following conditions:

(1) \( p = 2m - 1, m = 1, 2, \cdots \);
4.1. a Generalization of Harmonic Majorants in the Upper Half Plane

\[ I_2 = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Im C_p(z,t)d\mu(t) \frac{1}{[x^2 + (y + 1)^2]^\frac{p+1}{2}} dx \]
\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{|t| \leq 1} \frac{y}{|t-z|^2} d\mu(t) \cdot \frac{1}{[x^2 + (y + 1)^2]^\frac{p+1}{2}} dx \]
\[ + \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{|t| > 1} \frac{\Im (tz^{p+1} - |z|^2 z^p)}{|t-z|^2 t^{p+1}} d\mu(t) \cdot \frac{1}{[x^2 + (y + 1)^2]^\frac{p+1}{2}} dx \]
\[ = J_{11} + J_{12}. \] (4.1.4)

Note that
\[ \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{1}{(t-x)^2 + y^2} \frac{y+1}{x^2 + (y + 1)^2} dx = \frac{2y+1}{t^2 + (2y + 1)^2}, \] (4.1.5)
we have
\[ J_{11} = \frac{1}{\pi} \int_{|t| \leq 1} \int_{-\infty}^{\infty} \frac{y}{(t-x)^2 + y^2} \frac{1}{[x^2 + (y + 1)^2]^\frac{p+1}{2}} dx d\mu(t) \]
\[ \leq \int_{|t| \leq 1} \frac{1}{y+1} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{(t-x)^2 + y^2} \frac{y+1}{x^2 + (y + 1)^2} dx d\mu(t) \]
\[ \leq \int_{|t| \leq 1} \frac{1}{y+1} \frac{2y+1}{t^2 + (2y + 1)^2} d\mu(t) \]
\[ \leq 2 \int_{|t| \leq 1} \frac{1}{t^2 + 1} d\mu(t) \]
\[ \leq 4 \int_{-\infty}^{\infty} \frac{1}{1 + |t|^{p+1}} d\mu(t) < \infty. \] (4.1.6)

Moreover,
\[ J_{12} \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{|t| > 1} \frac{My(t^2 + y^2)(x^2 + y^2)^{\frac{p-1}{2}}}{[(t-x)^2 + y^2][t^{p+1}][x^2 + (y + 1)^2]^\frac{p+1}{2}} d\mu(t) dx \]
\[ = \frac{My}{\pi} \int_{-\infty}^{\infty} \int_{|t| > 1} \frac{(t^2 + y^2)}{[(t-x)^2 + y^2][x^2 + (y + 1)^2][t^{p+1}] d\mu(t) dx} \]
\[ = M \int_{|t| > 1} \frac{1}{y+1} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{(t-x)^2 + y^2} \frac{y+1}{x^2 + (y + 1)^2} dx t^2 + y^2 \frac{1}{t^{p+1}} d\mu(t) \]
\[ \text{again by (4.1.5), we have} \]
\[ J_{12} \leq M \int_{|t| > 1} \frac{2y+1}{y+1} \frac{t^2 + y^2}{t^{p+1}} d\mu(t) \]
\[ \leq 4M \int_{|t| > 1} \frac{1}{1 + |t|^{p+1}} d\mu(t) < \infty; \] (4.1.7)
(2) $p = 2m, m = 1, 2, \ldots$.

\[ I_2 = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty 2C_p(z, t) d\mu(t) \frac{1}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} dx \]

\[ = \int \int \{(x, t) : x \geq 0\} + \int \int \{(x, t) : x < 0, |t| > x\} + \int \int \{(x, t) : x < 0, |t| < x\} \]

\[ = J_{21} + J_{22} + J_{23}. \quad (4.1.8) \]

Similarly, we can obtain in the same method as (1) that $J_{21} < \infty$ and $J_{22} < \infty$.

Write

\[ J_{23} = \frac{1}{\pi} \int \int \{(x, t) : x < 0, |t| < x\} 2C_p(z, t) d\mu(t) \frac{1}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} dx \]

\[ = \int \int \{(x, t) : x < 0, |t| < x\} \cap \{(x, t) : |t| \leq 1\} + \int \int \{(x, t) : x < 0, |t| < x\} \cap \{(x, t) : |t| > 1\} \]

\[ = K_1 + K_2, \quad (4.1.9) \]

again, we can obtain in the same method as (1) that $K_1 < \infty$.

In the following, we will show that

\[ K_2 < \infty. \quad (4.1.10) \]

Write $D = \{(x, t) : x < 0, |t| < x\} \cap \{(x, t) : |t| > 1\}$, then

\[ K_2 = \frac{1}{\pi} \int \int_D 2C_p(z, t) d\mu(t) \frac{1}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} dx \]

\[ = \frac{1}{\pi} \int \int_D \left\{ \sum_{i=1}^{p+1} (-1)^{i+1} \frac{\left(\sum_{j=1}^{p+1} (-1)^j C_{p+1}^{p+1-j} (p+1-j)x^{p+1-j} \left((t-x)^2 + y^2\right) \right)}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} \right\} \frac{1}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} d\mu(t) dx \]

\[ \leq \frac{1}{\pi} \int \int_D \sum_{i=1}^{p+1} \frac{C_{p+1}^{2i-1} |x|^{p-2i+3} y^{2i-2} + (x^2 + y^2) \sum_{j=1}^{p+1} C_{p}^{2i-1} |x|^{p-2i+1} y^{2i-2}}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} dx \]

\[ \times \frac{y}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} d\mu(t) dx. \]

Note that

\[ \sum_{i=1}^{p+1} \frac{C_{p+1}^{2i-1} |x|^{p-2i+3} y^{2i-2} + (x^2 + y^2) \sum_{j=1}^{p+1} C_{p}^{2i-1} |x|^{p-2i+1} y^{2i-2}}{[x^2 + (y + 1)^2]^{\frac{p+1}{2}}} \leq A, \]

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then we have

\[
K_2 \leq \frac{Ay}{\pi} \int_D \frac{1}{(t-x)^2 + y^2} |t|^{p+1} d\mu(t) dx \\
\leq A \int_{-\infty}^{\infty} \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{1}{(t-x)^2 + y^2} dx \frac{2}{1 + |t|^{p+1}} d\mu(t).
\]

Note that

\[
\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{1}{(t-x)^2 + y^2} dx = 1,
\]
then we have

\[
K_2 \leq 2A \int_{-\infty}^{\infty} \frac{1}{1 + |t|^{p+1}} d\mu(t) < \infty.
\]

So the result follows by collecting (4.1.2), (4.1.3), (4.1.4), (4.1.6), (4.1.7), (4.1.8), (4.1.9) and (4.1.10).

4.2 a Generalization of Harmonic Majorants in the Upper Half Space

1. Introduction and Main Theorem

The Poisson kernel for the half space \( H \) is the function

\[
P(x, y') = \frac{2x_n}{\omega_n |x - y'|^n},
\]

where \( x \in H, y \in \partial H \) and \( \omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \) is the area of the unit sphere in \( \mathbb{R}^n \).

In this section, We will generalize Theorem D to the upper half space.

Theorem 4.2.1 Let

\[
H(x) = cx_n + \int_{\mathbb{R}^{n-1}} P(x, y') d\mu(y'),
\]

where \( c \geq 0 \) and \( \mu \) is a nonnegative Borel measure on \( \mathbb{R}^{n-1} \) such that

\[
\int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |y'|^2)^{n/2}} d\mu(y') < \infty.
\]
Chapter 4. a Generalization of Harmonic Majorants

If \( G(x) \) is nonnegative and subharmonic in \( H \). Then

\[
G(x) \leq H(x)
\]

if and only if

\[
\sup_{x_n > 0} \int_{\mathbb{R}^{n-1}} \frac{G(x)}{[|x'|^2 + (x_n + 1)^2]^{\frac{n}{2}}} dx' < \infty. \tag{4.2.1}
\]

**Remark 4.2.1** If \( n = 2 \), this is just the result of Flett and Kuran, therefore, our result is the generalization of Theorem D.

2. Main Lemmas

In order to obtain the result, we need these lemmas below:

**Lemma 4.2.1** Let \( H(x) \) be nonnegative and harmonic in \( H \) and have a continuous extension to \( \overline{H} \). Then

\[
H(x) = cx_n + \frac{2x_n}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{H(y')}{[|y'|^2 + x_n^2]^{\frac{n}{2}}} dy',
\]

where \( c \) is given by

\[
c = \lim_{x_n \to \infty} \frac{H(0, x_n)}{x_n}. \tag{4.2.2}
\]

**Lemma 4.2.2** Let

\[
x^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},
\]

then we have

\[
D_n = |xx^T E - 2x^T x| = -|x|^{2n}.
\]

**Proof:**
Lemma 4.2.3  Let \( x = (x_1, x_2, \ldots, x_{n-1}, x_n) = (x', x_n), \) where \( x' \in \mathbb{R}^{n-1} \) and \( x_n \in \mathbb{R}. \)

\[ S = (0, 0, \cdots, 0, -1) = (0, -1), \] where \( 0 \in \mathbb{R}^{n-1}. \) Suppose

\[ u = \Phi(x) = 2(x - S)^* + S, \]

where \( x^* = \frac{x}{|x|}. \) then we have

\[ J_\Phi(x) = -\frac{2^n}{[|x'|^2 + (x_n + 1)^2]^n}, \]

where \( J_\Phi(x) \) is given by

\[ J_\Phi(x) = \frac{\partial (u_1, u_2, \cdots, u_n)}{\partial (x_1, x_2, \cdots, x_n)}. \]
Proof: Since

\[ u = \Phi(x) = 2(x - S)^* + S \]
\[ = \frac{2(x - S)}{|x - S|^2} + S \]
\[ = \frac{2(x', x_n + 1)}{|x'|^2 + (x_n + 1)^2} + (0, -1) \]
\[ = \frac{(2x', 2(x_n + 1)) + (0, -|x'|^2 - (x_n + 1)^2)}{|x'|^2 + (x_n + 1)^2} \]
\[ = \frac{(2x', 1 - |x'|^2 - x_n^2)}{|x'|^2 + (x_n + 1)^2}, \quad (4.2.3) \]

then we have

\[
|\Phi(x)|^2 = \left| \frac{(2x', 1 - |x'|^2 - x_n^2)}{|x'|^2 + (x_n + 1)^2} \right|^2 \\
= \frac{4x^2 + (1 - |x'|^2 - x_n^2)^2}{(x'^2 + (x_n + 1)^2)^2} \\
= \frac{x^4 + 2x^2(1 + x_n^2) + (1 + x_n)^2(1 - x_n)^2}{(x'^2 + (x_n + 1)^2)^2} \\
= \frac{|x'|^2 + (1 - x_n)^2}{|x'|^2 + (1 + x_n)^2}.
\]

Let

\[ u = (u_1, u_2, \ldots, u_n), \]

by (4.2.3), we obtain

\[
u_i = \begin{cases} 
\frac{2x_i}{|x'|^2 + (x_n + 1)^2}, & \text{when } i = 1, 2, \ldots, n - 1, \\
\frac{2(x_n + 1)}{|x'|^2 + (x_n + 1)^2} - 1, & \text{when } i = n,
\end{cases}
\]
4.2. a Generalization of Harmonic Majorants in the Upper Half Space

for \( i = 1, 2, \ldots, n - 1 \), we have

\[
\frac{\partial u_i}{\partial x_i} = 2 \frac{|x'|^2 + (x_n + 1)^2 - 2x_i^2}{||x'|^2 + (x_n + 1)^2||^2},
\]

\[
\frac{\partial u_i}{\partial x_j} = -4x_i x_j \frac{||x'|^2 + (x_n + 1)^2||^2}{||x'|^2 + (x_n + 1)^2||^2}, \quad i \neq j, j = 1, 2, \ldots, n - 1
\]

\[
\frac{\partial u_i}{\partial x_n} = -4x_i (x_n + 1) \frac{||x'|^2 + (x_n + 1)^2||^2}{||x'|^2 + (x_n + 1)^2||^2},
\]

\[
\frac{\partial u_n}{\partial x_i} = 2 \frac{|x'|^2 + (x_n + 1)^2 - 2(x_n + 1)^2}{||x'|^2 + (x_n + 1)^2||^2}.
\]

So we get

\[
J_\phi(x) = \begin{vmatrix}
\frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_{n-1}} & \frac{\partial u_1}{\partial x_n} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_{n-1}} & \frac{\partial u_n}{\partial x_n}
\end{vmatrix}
\]

\[
= \frac{2^n}{||x'|^2 + (x_n + 1)^2||^{2n}} \begin{vmatrix}
|x|^2 - 2x_1^2 & -2x_1 x_2 & \ldots & -2x_1 x_n \\
\vdots & \vdots & \ddots & \vdots \\
-2x_n x_1 & -2x_n x_2 & \ldots & |x|^2 - 2x_n^2
\end{vmatrix}
\]

\[
= -\frac{2^n}{||x'|^2 + (x_n + 1)^2||^{2n}}.
\]

3. Proof of Theorem

We first prove necessity.

First applying Lemma 4.2.1 with \( H(x) \equiv 1 \), by (4.2.1), we have \( c = 0 \), so we obtain

\[
1 = \frac{2x_n}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{1}{||x' - x|^2 + x_n^2||_2^2} dx'.
\]

(4.2.4)

For \( a > 0 \), consider the function

\[
H(x) = \frac{x_n + a}{||x'|^2 + (x_n + a)^2||_2^2},
\]

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it is clear that $H(x)$ is nonnegative and harmonic in $H$, then applying Lemma 4.2.1 with

$$H(x) = \frac{x_n + a}{||x'|^2 + (x_n + a)^2||^\frac{n}{2}},$$

by (4.2.2) and

$$H((0,x_n)) = \frac{x_n + a}{(x_n + a)^n} = \frac{1}{(x_n + a)^{n-1}},$$

we have $c = 0$, so we obtain by

$$H((x',0)) = \frac{a}{||y'|^2 + a^2||^\frac{n}{2}}$$

that

$$\frac{x_n + a}{||x'|^2 + (x_n + a)^2||^\frac{n}{2}} = \frac{2x_n}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{1}{||y' - x'|^2 + x_n^2||^\frac{n}{2}} \frac{a}{||y'|^2 + a^2||^\frac{n}{2}} dy'.$$

(4.2.5)

In these two formulas (4.2.4) and (4.2.5), we interchange the roles of $x'$ and $y'$ and choose $a = x_n + 1$, then we get

$$1 = \frac{2x_n}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{1}{||y' - x'|^2 + x_n^2||^\frac{n}{2}} dx',$$

(4.2.6)

and

$$\frac{2x_n + 1}{||y'|^2 + (2x_n + 1)^2||^\frac{n}{2}} = \frac{2x_n}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{1}{||y' - x'|^2 + x_n^2||^\frac{n}{2}} \frac{x_n + 1}{||x'|^2 + (x_n + 1)^2||^\frac{n}{2}} dx',$$

(4.2.7)

by (4.2.6), we can also get

$$1 = \frac{2(x_n + 1)}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{1}{||y' - x'|^2 + (x_n + 1)^2||^\frac{n}{2}} dx'$$

$$= \frac{2(x_n + 1)}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{1}{||x'|^2 + (x_n + 1)^2||^\frac{n}{2}} dx'. (4.2.8)$$

Thus from

$$G(x) \leq cx_n + \int_{\mathbb{R}^{n-1}} P(x,y')d\mu(y'),$$

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we obtain by (4.2.7) and (4.2.8)

\[
\int_{\mathbb{R}^{n-1}} \frac{G(x)}{[|x'|^2 + (x_n + 1)^2]^\frac{n}{2}} \, dx' \leq \int_{\mathbb{R}^{n-1}} \frac{c x_n}{[|x'|^2 + (x_n + 1)^2]^\frac{n}{2}} \, dx' + \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{x_n}{[|x'|^2 + (x_n + 1)^2]^\frac{n}{2}} \int_{\mathbb{R}^{n-1}} \frac{1}{\left[|y' - x'|^2 + x_n^2\right]^\frac{n}{2}} \, d\mu(y')
\]

\[
= \frac{c \omega_n x_n}{2(x_n + 1)} + \frac{1}{x_n + 1} \int_{\mathbb{R}^{n-1}} \frac{2x_n}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{1}{[|x'|^2 + x_n^2]^\frac{n}{2}} \left[|x'|^2 + (x_n + 1)^2\right]^\frac{n}{2} \, dx' \, d\mu(y')
\]

\[
= \frac{c \omega_n x_n}{2(x_n + 1)} + \frac{2x_n + 1}{x_n + 1} \int_{\mathbb{R}^{n-1}} \frac{1}{\left[|y'|^2 + (2x_n + 1)^2\right]^\frac{n}{2}} \, d\mu(y')
\]

\[
\leq \frac{c \omega_n}{2} + 2 \int_{\mathbb{R}^{n-1}} \frac{1}{\left[1 + |y'|^2\right]^\frac{n}{2}} \, d\mu(y').
\]

Hence (4.2.1) holds.

In the other direction, assume that (4.2.1) holds. We show that \(G(x)\) has a harmonic majorant in \(H\), or what is the same thing, \(G(\Phi^{-1}(u))\) has a harmonic majorant in \(B_n\). It is sufficient to show that

\[
g(r) = r^{n-1} \int_S G(\Phi^{-1}(r\xi)) \, d\sigma(\xi)
\]

remains bounded as \(r \uparrow 1\). By Lemma A1, it is the same thing to show that

\[
\int_0^1 \frac{g(r)}{1 - \lambda r^n} \, dr \leq A \int_0^1 \frac{1}{1 - \lambda t^n} \, dt
\]

for all \(\lambda \in (0, 1)\) and some positive constant. Calculate as follows:
\[ \int_0^1 \frac{g(r)}{1 - \lambda^n r^n} dr \]
\[ = \int_0^1 \frac{G(\Phi^{-1}(r \xi))}{1 - \lambda^n r^n} d\sigma(\xi) r^{n-1} dr \]
\[ = \frac{1}{nV(B)} \int_{B_n} \frac{G(\Phi^{-1}(u))}{1 - \lambda^n |u|^n} dV \]
\[ = \frac{1}{nV(B)} \int_{R^d_+} \frac{G(x)}{1 - \lambda^n |\Phi(x)|^n} |J\Phi(x)| dx \]
\[ = \frac{1}{nV(B)} \int_{R^d_+} \frac{G(x)}{1 - \lambda^n \left[ \frac{|x'|^2 + (1-x_n)^2}{|x'|^2 + (x_n+1)^2} \right]^\frac{n}{2}} \left[ \frac{|x'|^2 + (x_n+1)^2}{|x'|^2 + (x_n-1)^2} \right]^\frac{n-2}{2} dx \]
\[ \leq \frac{2^n}{nV(B)} \int_0^\infty \left[ \int_{R^d_{-1}} \frac{G(x)}{\left[ \frac{|x'|^2 + (x_n+1)^2}{|x'|^2 + (x_n+1)^2} \right]^\frac{n}{2}} \right] \left[ \frac{1}{(x_n+1)^n - \lambda^n |x_n - 1|^n} \right] dx_n, \]

by (4.2.1), we have

\[ \int_0^1 \frac{g(r)}{1 - \lambda^n r^n} dr \]
\[ \leq A \int_0^\infty \frac{1}{(x_n+1)^n - \lambda^n |x_n - 1|^n} dx_n \]
\[ \leq A \int_0^\infty \frac{1}{(x_n+1)^{n-2}} dx_n \]
\[ = A \int_0^\infty \frac{1}{\lambda^n (x_n+1)^{n-2}} dx_n \]
\[ = \frac{1}{2} A \int_{-1}^1 \frac{1}{1 - \lambda^n |t|^n} dt \]
\[ = A \int_0^1 \frac{1}{1 - \lambda^n t^n} dt \]

The change of variables is made with the substitution \( t = \frac{x_n-1}{x_n+1} \). So the result follows.
Chapter 5

Properties of Limit for Poisson Integral

5.1 Properties of Limit for Poisson Integral in the Upper Half Plane

1. Introduction and Main Theorem

The Poisson kernel for the upper half plane $C_+ = \{z = x + iy \in C : y > 0\}$ is the function

$$P(z, t) = \frac{y}{\pi|z - t|^2},$$

where $z \in C_+$ and $t \in \mathbb{R}$.

If $p \geq 0$ is an integer, we define a modified Cauchy kernel of order $p$ for $z \in C_+ - \{t\}$ by

$$C_p(z, t) = \begin{cases} \frac{1}{\pi t - z}, & \text{when } |t| \leq 1, \\ \frac{1}{\pi t - z} - \frac{1}{\pi} \sum_{k=0}^{p} \frac{z^k}{t^{k+1}}, & \text{when } |t| > 1, \end{cases}$$

then we define a modified Poisson kernel of order $p$ for the upper half plane by

$$P_p(z, t) = \Im C_p(z, t).$$

For any $|t| > 1$, the following equality holds:

$$\Im C_p(z, t) = \frac{t^{R_p+1} \sin(p + 1)\theta - R_p^{p+2} \sin p\theta}{|t - z|^{2R_p+1}} \quad (5.1.1)$$
holds, where $z = \text{Re}^{i\theta}$.

Marvin Rosenblum and James Rovnyak [34] proved the following theorem:

**Theorem E** If

$$H(z) = cy + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2}, \quad y > 0,$$

where $c$ is a real number and $\mu$ is a nonnegative Borel measure on $(-\infty, \infty)$ such that

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty.$$

Then for every $\theta \in (0, \pi)$:

(1) \[ \lim_{R \to \infty} \frac{1}{R} H(\text{Re}^{i\theta}) = c \sin \theta; \]

(2) \[ \lim_{R \to \infty} \frac{2}{\pi R} \int_{0}^{\pi} H(\text{Re}^{i\theta}) \sin \theta \, d\theta = c. \]

In this section, We will generalize Theorem E to the modified kernel.

**Theorem 5.1.1** If

$$H(z) = \mathcal{S} \left[ Q_p(z) + \frac{1}{\pi} \int_{-\infty}^{\infty} C_p(z,t)d\mu(t) \right], \quad z = x + iy, \; y > 0$$

where

$$Q_p(z) = \sum_{k=0}^{p} a_k z^k, \quad a_k \in \mathbb{R}, \; k = 0, 1, 2, \ldots, p$$

and $\mu$ is a nonnegative Borel measure on $(-\infty, \infty)$ such that

$$\int_{-\infty}^{\infty} \frac{1}{1+|t|^{p+1}} d\mu(t) < \infty.$$

Then for every $\theta \in (0, \pi)$:

(1) \[ \lim_{R \to \infty} \frac{1}{R^p} H(\text{Re}^{i\theta}) = \left[ a_p - \frac{1}{\pi} \int_{|t|>1} \frac{1}{t^{p+1}} d\mu(t) \right] \sin p\theta; \]

(2) \[ \lim_{R \to \infty} \frac{2}{\pi R^p} \int_{0}^{\pi} H(\text{Re}^{i\theta}) \sin p\theta \, d\theta = a_p - \frac{1}{\pi} \int_{|t|>1} \frac{1}{t^{p+1}} d\mu(t). \]
5.1. Properties of Limit for Poisson Integral in the Upper Half Plane

Remark 5.1.1 If \( p = 1 \), this is just the result of Marvin Rosenblum and James Rovnyak, therefore, our result is the generalization of Theorem E.

2. Proof of Theorem

We first prove the equality (5.1.1). Since

\[
C_p(z, t) = \frac{1}{t - z} - \sum_{k=0}^{p} \frac{z^k}{t^{k+1}} = \frac{z^{p+1}}{(t - z)t^{p+1}},
\]

then

\[
\Im C_p(z, t) = \Im \frac{z^{p+1}}{(t - z)t^{p+1}} = \Im \left[ \frac{z^{p+1}(t - \bar{z})}{|t - z|^2t^{p+1}} \right]
\]

\[
= \Im \left[ \frac{t(\Re e^{i\theta})^{p+1} - |z|^2(\Re e^{i\theta})^p}{|t - z|^2t^{p+1}} \right]
\]

\[
= \frac{tR^{p+1} \sin(p + 1)\theta - R^p \sin p\theta}{|t - z|^2t^{p+1}}.
\]

This proves the equality (5.1.1).

Since

\[
H(z) = \Im \left( \sum_{k=0}^{p} a_k z^k \right) + \frac{1}{\pi} \int_{|z| \leq 1} \Im C_p(z, t) d\mu(t)
\]

\[
= \Im \left( \sum_{k=0}^{p} a_k R^k e^{ik\theta} \right) + \frac{1}{\pi} \int_{|z| \leq 1} \Im C_p(z, t) d\mu(t) + \frac{1}{\pi} \int_{|z| > 1} \Im C_p(z, t) d\mu(t)
\]

\[
= \sum_{k=0}^{p} a_k R^k \sin k\theta + \frac{1}{\pi} \int_{|z| \leq 1} \frac{\Im}{|t - z|^2} d\mu(t) + \frac{1}{\pi} \int_{|z| > 1} \frac{z^{p+1}}{(t - z)t^{p+1}} d\mu(t)
\]

\[
= \sum_{k=0}^{p} a_k R^k \sin k\theta + \frac{1}{\pi} \int_{|z| \leq 1} \frac{\Im}{|t - z|^2} d\mu(t)
\]

\[
+ \frac{1}{\pi} \int_{|z| > 1} \frac{tR^{p+1} \sin(p + 1)\theta - R^p \sin p\theta}{|t - z|^2t^{p+1}} d\mu(t),
\]

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for every $\theta \in (0, \pi)$,

$$
\frac{H(Re^{i\theta})}{R^p} = \sum_{k=0}^{p} a_k R^{k-p} \sin k\theta + \frac{1}{\pi} \int_{|t| \leq 1} \frac{y}{|t-z|^2 R^p} d\mu(t)
+ \frac{1}{\pi} \int_{|t| > 1} \frac{R[t \sin(p+1)\theta - R \sin p\theta]}{|t-z|^2 t^{p+1}} d\mu(t)
= I_1 + I_2 + I_3,
$$

(5.1.2)

then

$$
\lim_{R \to \infty} I_1 = \lim_{R \to \infty} \sum_{k=0}^{p} a_k R^{k-p} \sin k\theta = a_p \sin p\theta.
$$

(5.1.3)

Moreover,

$$
I_2 = \frac{1}{\pi} \int_{|t| \leq 1} \frac{y}{|t-z|^2 R^p} d\mu(t)
+ \frac{1}{\pi} \int_{|t| > 1} \frac{\sin \theta(1 + |t|^{p+1})}{R^p-1|t-\Re^{i\theta}|^2} \frac{d\mu(t)}{1 + |t|^{p+1}}.
$$

Since

$$
\frac{\sin \theta(1 + |t|^{p+1})}{R^p-1|t-\Re^{i\theta}|^2} < \frac{2}{2^{p-1}(R-1)^2} < 2^{2-p},
$$

by the dominated convergence theorem, we have

$$
\lim_{R \to \infty} I_2 = 0.
$$

(5.1.4)

Write

$$
I_3 = \frac{1}{\pi} \int_{|t| > 1} \frac{R[t \sin(p+1)\theta - R \sin p\theta]}{|t-z|^2 t^{p+1}} d\mu(t)
+ \frac{1}{\pi} \int_{|t| > 1} \frac{d\mu(t)}{t^{p+1}}.
$$

(5.1.5)

Multiplying (5.1.2) by $2\pi^{-1} \sin p\theta$ and integrating with respect to $\theta$, we obtain

$$
\frac{2}{\pi R^p} \int_{0}^{\pi} H(Re^{i\theta}) \sin p\theta d\theta
= \frac{2}{\pi} \int_{0}^{\pi} \sum_{k=0}^{p} a_k R^{k-p} \sin k\theta \sin p\theta d\theta
+ \frac{2}{\pi^2} \int_{0}^{\pi} \int_{|t| \leq 1} \frac{y \sin p\theta}{|t-z|^2 R^p} d\mu(t) d\theta
+ \frac{2}{\pi^2} \int_{0}^{\pi} \int_{|t| > 1} \frac{R \sin p\theta[t \sin(p+1)\theta - R \sin p\theta]}{|t-z|^2 t^{p+1}} d\mu(t) d\theta
= I_1' + I_2' + I_3'.
$$

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For the first term, we have

\[ I_1' = \frac{2}{\pi} \sum_{k=0}^{p} a_k R^{k-p} \int_0^\pi \sin k \theta \sin p \theta d \theta = \frac{1}{\pi} \sum_{k=0}^{p} a_k R^{k-p} \int_0^\pi [\cos (k-p) \theta - \cos (k+p) \theta] d \theta \]

\[ = \frac{1}{\pi} \left[ \sum_{k=0}^{p-1} + \sum_{k=p}^{2p-1} \right] a_k R^{k-p} \int_0^\pi [\cos (k-p) \theta - \cos (k+p) \theta] d \theta \]

\[ = \frac{1}{\pi} \int_0^\pi (1 - \cos 2p \theta) d \theta = \frac{1}{p}; \quad (5.1.6) \]

for the second term, we have

\[ I_2' = \frac{2}{\pi^2} \int_0^\pi \int_{|t| \leq 1} \frac{\sin \theta \sin p \theta (1 + |t|^{p+1})}{R^{p-1}|t - Re^{i \theta}|^2} \frac{d \mu(t)}{1 + |t|^{p+1}} d \theta \]

for all \( \theta \in (0, \pi) \) and \( R > 2 \).

If \( |t| \leq 1 \), then

\[ \left| \frac{\sin \theta \sin p \theta (1 + |t|^{p+1})}{R^{p-1}|t - Re^{i \theta}|^2} \right| < \frac{2}{2p-1(R-1)^2} < \frac{2}{2p-1} = 2^{2-p}. \]

Since

\[ \frac{2}{\pi^2} \int_0^\pi \int_{|t| \leq 1} 2^{2-p} \frac{d \mu(t)}{1 + |t|^{p+1}} d \theta < \infty, \]

by the dominated convergence theorem, we have

\[ \lim_{R \to \infty} I_2' = 0; \quad (5.1.7) \]

for the third term, we have

\[ I_3' = \frac{2}{\pi^2} \int_0^\pi \int_{|t| > 1} \frac{R \sin p \theta [t \sin (p+1) \theta - R \sin p \theta]}{|t - z|^2} \frac{d \mu(t)}{t^{p+1}} d \theta \]

\[ = \frac{2}{\pi^2} \int_0^\pi \int_{|t| > 1} J' d \mu(t) t^{p+1} d \theta. \]
In the following, we will show that

\[ J' \leq 2p(p+1) \]  

(5.1.8)

for all \( \theta \in (0, \pi) \) and \( R > 2 \).

Since

\[ |t-z|^2 = t^2 - 2Rt \cos \theta + R^2 = (t - R \cos \theta)^2 + R^2 \sin^2 \theta \geq R^2 \sin^2 \theta \]

and

\[ |t-z|^2 = \begin{cases} (t-R)^2 + 2Rt(1-\cos \theta) = (t-R)^2 + 4Rt \sin^2 \frac{\theta}{2} \geq 4R|t| \sin^2 \frac{\theta}{2}, & \text{when } t > 0, \\ (t+R)^2 - 2Rt(1+\cos \theta) = (t+R)^2 - 4Rt \cos^2 \frac{\theta}{2} \geq 4R|t| \cos^2 \frac{\theta}{2}, & \text{when } t < 0, \end{cases} \]

then

\[ |J'| \leq \frac{R|\sin p\theta||(t\sin(p+1)\theta) - R|\sin p\theta|}{|t-z|^2} \]

\[ \leq \frac{|t|Rp(p+1) \sin^2 \theta + R^2 p^2 \sin^2 \theta}{|t-z|^2} \]

\[ \leq p(p+1) \frac{|t|R \sin^2 \theta + R^2 \sin^2 \theta}{|t-z|^2} \]

\[ = p(p+1) \left( \frac{|t|R \sin^2 \theta}{|t-z|^2} + \frac{R^2 \sin^2 \theta}{|t-z|^2} \right) \]

\[ \leq 2p(p+1). \]

This proves (5.1.8).

Since

\[ \frac{2}{\pi^2} \int_0^\pi \int_{|t|>1} 2p(p+1) \frac{d\mu(t)}{t^{p+1}} d\theta = \frac{4p(p+1)}{\pi} \int_{|t|>1} \frac{d\mu(t)}{t^{p+1}} < \infty, \]

by the dominated convergence theorem, we have

\[ \lim_{R \to \infty} I'_3 = \lim_{R \to \infty} \frac{2}{\pi^2} \int_0^\pi \int_{|t|>1} J' \frac{d\mu(t)}{t^{p+1}} d\theta. \]
5.1. Properties of Limit for Poisson Integral in the Upper Half Plane

Note that

\[ J' = J' + \sin^2 p\theta - \sin^2 p\theta \]
\[ = \frac{Rt \sin(p + 1)\theta \sin p\theta - R^2 \sin^2 p\theta}{|t - z|^2} + \frac{|t - z|^2 \sin^2 p\theta}{|t - z|^2} - \sin^2 p\theta \]
\[ = \frac{Rt \sin(p + 1)\theta \sin p\theta - R^2 \sin^2 p\theta + (t^2 - 2Rt \cos \theta + R^2) \sin^2 p\theta}{|t - z|^2} - \sin^2 p\theta \]
\[ = \frac{t^2 \sin^2 p\theta + Rt \sin p\theta \sin(p + 1) - 2 \cos \theta \sin p\theta}{|t - z|^2} - \sin^2 p\theta \]
\[ = \frac{t^2 \sin^2 p\theta - Rt \sin p\theta \sin(p - 1)\theta}{|t - z|^2} - \sin^2 p\theta, \]

then we obtain

\[ \lim_{R \to \infty} J' = -\sin^2 p\theta. \]

Therefore

\[ \lim_{R \to \infty} I_3' = \frac{2}{\pi^2} \int_0^{\pi} \frac{\int_{|t| > 1} \lim_{R \to \infty} J' \frac{d\mu(t)}{t^{p+1}} d\theta}{|t - z|^2} \]
\[ = \frac{2}{\pi^2} \int_0^{\pi} \left( -\int_{|t| > 1} \sin^2 p\theta \frac{d\mu(t)}{t^{p+1}} d\theta \right) \]
\[ = -2 \int_0^{\pi} \frac{1}{\pi} \frac{\sin \theta}{t^{p+1}} d\theta \int_{|t| > 1} \frac{d\mu(t)}{t^{p+1}} \]
\[ = -\frac{1}{\pi} \int_{|t| > 1} \frac{d\mu(t)}{t^{p+1}}. \]

Thus, (2) holds by collecting (5.1.5), (5.1.6), (5.1.7) and (5.1.9).

Similarly, we have

\[ |p \sin \theta J| \leq 2p(p + 1), \]

so

\[ |J| \leq \frac{2(p + 1)}{\sin \theta}. \]

Since

\[ \frac{1}{\pi} \int_{|t| > 1} \frac{2(p + 1)}{\sin \theta} \frac{d\mu(t)}{t^{p+1}} = \frac{2(p + 1)}{\pi \sin \theta} \int_{|t| > 1} \frac{d\mu(t)}{t^{p+1}} < \infty \]

and

\[ \lim_{R \to \infty} J = -\sin p\theta, \]

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by the dominated convergence theorem, we have
\[
\lim_{{R \to \infty}} I_3 = \frac{1}{\pi} \int_{{|t| > 1}} \lim_{{R \to \infty}} J d\mu(t) = -\frac{\sin p\theta}{\pi} \int_{{|t| > 1}} \frac{d\mu(t)}{t^{p+1}}. \tag{5.1.10}
\]
So (1) follows by (5.1.2), (5.1.3), (5.1.4) and (5.1.10).

## 5.2 Properties of Limit for Poisson Integral in the Upper Half Space

### 1. Introduction and Main Theorem

The Poisson kernel for the upper half space \( H \) is the function
\[
P(x, y') = \frac{2x_n}{\omega_n |x - y'|^n},
\]
where \( x \in H, y' \in \partial H \) and \( \omega_n = \frac{2\pi^n}{\Gamma(\frac{n}{2})} \) is the area of the unit sphere in \( \mathbb{R}^n \).

In this section, we will generalize Theorem 5 to the upper half space.

**Theorem 5.2.1** If
\[
H(x) = cx_n + \int_{{\mathbb{R}^{n-1}}} P(x, y') d\mu(y'),
\]
where \( c \) is a real number and \( \mu \) is a nonnegative Borel measure on \( \mathbb{R}^{n-1} \) such that
\[
\int_{{\mathbb{R}^{n-1}}} \frac{1}{(1 + |y'|^2)^{\frac{n}{2}}} d\mu(y') < \infty.
\]
Then for every \( \theta_j \in (0, \pi), j = 1, 2, \cdots, n - 1, \)
\[
(1) \quad \lim_{{R \to \infty}} \frac{1}{R} H(x) = c \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1};
\]
\[
(2) \quad \lim_{{R \to \infty}} \frac{1}{R} \int_0^\pi \cdots \int_0^\pi H(x) (\sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1})^{n-1} d\theta_1 d\theta_2 \cdots d\theta_{n-1} = 2^{n-1} I_n^{-1} c,
\]
where \( R = |x| \) and
\[
I_n = \begin{cases} 
\frac{(2k-1)!! \pi}{(2k)!!}, & \text{when } n = 2k, \\
\frac{(2k)!! \pi}{(2k+1)!!}, & \text{when } n = 2k + 1.
\end{cases}
\]

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Remark 5.2.1 If \( n = 2 \), this is just the result of Marvin Rosenblum and James Rovnyak, therefore, our result is the generalization of Theorem E.

2. Proof of Theorem

Write \( x' = R\xi, x_n = R\eta \), by the formula of polar coordinates, we have

\[
\eta = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1}.
\] (5.2.1)

For every \( \theta_j \in (0, \pi), j = 1, 2, \cdots, n - 1 \),

\[
\frac{1}{R} H(x) = c \eta + \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{2R\eta}{|x - y'|^n} |x - y'|^n d\mu(y')
\]

\[
= c \eta + \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \eta \left( \frac{1 + |y'|^2)^{n/2}}{|x - y'|^n} \right) \frac{1}{(1 + |y'|^2)^{n/2}} d\mu(y')
\]

\[
= c \eta + \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} J_1 \frac{1}{(1 + |y'|^2)^{n/2}} d\mu(y').
\] (5.2.2)

Multiplying this by \( \eta^{n-1} \) and integrating with respect to \( \theta_j \), we obtain

\[
\frac{1}{R} \int_0^\pi \cdots \int_0^\pi H(x) \eta^{n-1} \ d\theta_1 \ d\theta_2 \cdots d\theta_{n-1}
\]

\[
= \int_0^\pi \cdots \int_0^\pi \ c \eta^n d\theta_1 d\theta_2 \cdots d\theta_{n-1}
\]

\[
+ \frac{2}{\omega_n} \int_0^\pi \cdots \int_0^\pi \int_{\mathbb{R}^{n-1}} \eta^n \left( \frac{1 + |y'|^2)^{n/2}}{|x - y'|^n} \right) \frac{1}{(1 + |y'|^2)^{n/2}} d\mu(y') d\theta_1 d\theta_2 \cdots d\theta_{n-1}
\]

\[
= K_1 + K_2.
\] (5.2.3)

By (5.2.1), we have

\[
K_1 = c \int_0^\pi \cdots \int_0^\pi (\sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1})^n d\theta_1 d\theta_2 \cdots d\theta_{n-1}
\]

\[
= c \left( \int_0^\pi \sin^n \theta d\theta \right)^{n-1}.
\]

Since

\[
\int_0^\pi \sin^n \theta d\theta = 2 \int_0^{\pi/2} \sin^n \theta d\theta = 2 I_n,
\]

\[
S_{\pi/2}^2 = 2 I_n.
\]
we obtain
\[ K_1 = 2^{n-1} l_n^{-1} c. \] (5.2.4)

Moreover,
\[
K_2 = \frac{2}{\omega_n} \int_0^\pi \int_0^\pi \cdots \int_0^\pi \int_{\mathbb{R}^{n-1}} \eta_n (1 + |y'|^2)^{n/2} \frac{1}{|x - y'|^n} \frac{d\mu(y')}{(1 + |y'|^2)^{n/2}} d\theta_1 d\theta_2 \cdots d\theta_{n-1} 
\]
\[
= \frac{2}{\omega_n} \int_0^\pi \int_0^\pi \cdots \int_0^\pi \int_{\mathbb{R}^{n-1}} J_2 \frac{1}{(1 + |y'|^2)^{n/2}} d\mu(y') d\theta_1 d\theta_2 \cdots d\theta_{n-1}. 
\]

In the following, we will show that
\[ J_2 \leq 2^{n/2} \] (5.2.5)

for all \( y', \theta_j \in (0, \pi), j = 1, 2, \cdots, n - 1 \), and \( R > 2 \).

If \( |y'| < 1 \), then since \( R > 2 \),
\[ J_2 \leq \frac{2^{n/2}}{(R - |y'|)^n} \leq \frac{2^{n/2}}{(R - 1)^n} \leq 2^{n/2}; \]

if \( |y'| \geq 1 \), since
\[
|x - y'|^2 = |(x', x_n) - (y', 0)|^2 = |x' - y'|^2 + x_n^2 
\]
\[
= |y'|^2 - 2x' \cdot y' + R^2 = |y'|^2 (|\xi|^2 + \eta^2) - 2y' \cdot R \xi + R^2 
\]
\[
= |y'|^2 \eta^2 + (|y'|^2 |\xi|^2 - |y' \cdot \xi|^2) + (y' \cdot \xi - R)^2 \geq |y'|^2 \eta^2, 
\]
then
\[
J_2 = \eta_n (1 + |y'|^2)^{n/2} \left[ \frac{1}{|x - y'|^n} \right]^{n/2} \leq \left[ \eta^2 (1 + |y'|^2) \right]^{n/2} \left[ \frac{1}{|y'|^2} \right]^{n/2} \leq 2^{n/2}. 
\]

This proves (5.2.5).

Since
\[
\frac{2}{\omega_n} \int_0^\pi \int_0^\pi \cdots \int_0^\pi \int_{\mathbb{R}^{n-1}} \frac{2^{n/2}}{(1 + |y'|^2)^{n/2}} d\mu(y') d\theta_1 d\theta_2 \cdots d\theta_{n-1} 
\]
\[
= \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |y'|^2)^{n/2}} d\mu(y') < \infty, 
\]
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by the dominated convergence theorem, we have

\[
\lim_{R \to \infty} K_2 = \lim_{R \to \infty} \frac{2}{\omega_n} \int_0^\pi \cdots \int_0^\pi \int_{\mathbb{R}^{n-1}} J_2 \frac{1}{(1 + |y'|^2)^{n/2}} d\mu(y') d\theta_1 d\theta_2 \cdots d\theta_{n-1} = 0.
\]

Thus, (2) holds by collecting (5.2.3), (5.2.4) and (5.2.6).

Moreover,

\[
J_1 = \frac{\eta (1 + |y'|^2)^{n/2}}{|x - y'|^n} = \frac{J_2}{\eta^{n-1}} \leq \frac{2^{n/2}}{(\sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1})^{n-1}}.
\]

Since

\[
\frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{2^{n/2}}{(\sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1})^{n-1}} \frac{1}{(1 + |y'|^2)^{n/2}} d\mu(y') = \frac{2}{\omega_n} \frac{2^{n/2}}{(\sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1})^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |y'|^2)^{n/2}} d\mu(y') < \infty,
\]

by the dominated convergence theorem, we have

\[
\lim_{R \to \infty} \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{J_1}{(1 + |y'|^2)^{n/2}} d\mu(y') = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \lim_{R \to \infty} \frac{J_1}{(1 + |y'|^2)^{n/2}} d\mu(y') = 0.
\]

So (1) follows by (5.2.2) and (5.2.7).
Chapter 6

a Lower Bound for a Class of Harmonic Functions in the Half Space

6.1 Introduction and Main Theorem

B.Ya.Levin [26] has proved the following result:

**Theorem F** Let $u(z)$ be a harmonic function in the upper half plane $\mathbb{C}_+ = \{z = x + iy = Re^{i\theta}, y > 0\}$ with continuous boundary values on the real axis. Suppose that

$$u(z) \leq KR^\rho, \quad z \in \mathbb{C}_+, R = |z| > 1, \rho > 1,$$

and

$$|u(z)| \leq K, \quad z \in \mathbb{C}_+, R = |z| \leq 1, \Im z \geq 0.$$

Then

$$u(z) \geq -cK \frac{1 + R^\rho}{\sin \theta}, \quad z \in \mathbb{C}_+,$$

where $c$ does not depend on $K, R, \theta$ and the function $u(z)$.

Our aim in this chapter is to establish the following main theorem.

**Theorem 6.1.1** Let $u(x)$ be a harmonic function in the upper half space $H$ with continuous boundary values on the boundary $\partial H$, write $|x'| = |x| \cos \theta, x_n = |x| \sin \theta$
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(0 < θ ≤ π/2). Suppose that

\[ u(x) \leq KR^{0(R)}, \quad x \in H, R = |x| > 1, \rho(R) > 1, \quad (6.1.1) \]

and

\[ u(x) \geq -K, \quad x \in \overline{H}, R = |x| \leq 1, x_n \geq 0. \quad (6.1.2) \]

Then

\[ u(x) \geq -cK \frac{1 + (2R)^n(R)}{\sin^n - 1 \theta}, \quad x \in H, \quad (6.1.3) \]

where \( c \) does not depend on \( K, R, \theta \) and the function \( u(x), \rho(R) \) is nondecreasing in \([1, +\infty)\).

**Remark 6.1.1** If \( n = 2, \rho(R) \equiv \rho \), this is just the result of B.Ya.Levin, therefore, our result (6.1.3) is the generalization of Theorem F.

### 6.2 Main Lemmas

In order to obtain the result, we need these lemmas below:

**Lemma 6.2.1** Let \( u(x) \) be a harmonic function in the upper half space \( H = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\} \) with continuous boundary values on the boundary \( \partial H \), \( R > 1 \). Then we have

\[
\int_{\{x \in \mathbb{R}^n : |x| = R, x_n > 0\}} u(x) \frac{n x_n}{R^{n+1}} d\sigma(x) \\
+ \int_{\{x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0\}} u(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' = c_1 + \frac{c_2}{R^n},
\]

where

\[
c_1 = \int_{\{x \in \mathbb{R}^n : |x| = 1, x_n > 0\}} \left[ (n - 1)x_n u(x) + x_n \frac{\partial u(x)}{\partial n} \right] d\sigma(x),
\]

\[
c_2 = \int_{\{x \in \mathbb{R}^n : |x| = 1, x_n > 0\}} \left[ nx_n u(x) - x_n \frac{\partial u(x)}{\partial n} \right] d\sigma(x).
\]

**Lemma 6.2.2** Let \( u(x) \) be a harmonic function in the upper half space \( H = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\} \) with continuous boundary values on the boundary \( \partial H \),
6.3. Proof of Theorem

R > 1. Then on the closed half ball \( \overline{B}_R = \overline{B}_R \cap H = \{ x \in \mathbb{H} : |x| \leq R \} \), we have

\[
\begin{align*}
u(x) = & \int_{\{ y \in H : |y| = R, y_n > 0 \}} \frac{R^2 - |x|^2}{\omega_n R^n} \left( \frac{1}{|y - x|^n} - \frac{1}{|y - x^*|^n} \right) u(y) d\sigma(y) \\
& + \frac{2\nu_n}{\omega_n} \int_{\{ y \in \mathbb{P} : |y'| < R, y_n = 0 \}} \left( \frac{1}{|y'|^n} - \frac{R^n}{|x|^n |y' - \tilde{x}|^n} \right) u(y') dy',
\end{align*}
\]

where \( \tilde{x} = R^2x/|x|^2 \), \( x^* = (x', -x_n) \), and \( \omega_n = \frac{\pi^{n/2}}{\Gamma(1 + \frac{n}{2})} \) is the volume of the unit \( n \)-ball in \( \mathbb{R}^n \).

Remark 6.2.1 Lemma 6.2.1 is the generalization of the Carleman formula for harmonic functions in the upper half plane to the upper half space; Lemma 6.2.2 is the generalization of the Nevanlinna formula for harmonic functions in the upper half disk to the upper half ball.

6.3 Proof of Theorem

We use Lemma 6.2.1 to the harmonic function \( u(x) \),

\[
\begin{align*}
& \int_{\{ x \in \mathbb{R}^n : |x| = R, x_n > 0 \}} \frac{u^-(x)}{|x|^n} \frac{\nu_n}{R^{n+1}} d\sigma(x) \\
& + \int_{\{ x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0 \}} u^-(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \\
& = \int_{\{ x \in \mathbb{R}^n : |x| = R, x_n > 0 \}} \frac{u^+(x)}{|x|^n} \frac{\nu_n}{R^{n+1}} d\sigma(x) \\
& + \int_{\{ x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0 \}} u^+(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' + c_1 + \frac{c_2}{R^n}, \tag{6.3.1}
\end{align*}
\]

where \( u^+(x) = \max\{ u(x), 0 \} \), \( u^-(x) = (-u(x))^+ \) and \( u(x) = u^+(x) - u^-(x) \).

The terms on the right-hand of (6.3.1) can be estimated by using (6.1.1):

\[
\begin{align*}
& \int_{\{ x \in \mathbb{R}^n : |x| = R, x_n > 0 \}} \frac{u^+(x)}{|x|^n} \frac{\nu_n}{R^{n+1}} d\sigma(x) \leq AKR^{\rho(R)-1}, \tag{6.3.2} \\
& \int_{\{ x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0 \}} u^+(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \leq AKR^{\rho(R)-1}. \tag{6.3.3}
\end{align*}
\]

Thus, for \( R > 1 \), we can obtain by (6.3.1), (6.3.2) and (6.3.3)

\[
\begin{align*}
& \int_{\{ x \in \mathbb{R}^n : |x| = R, x_n > 0 \}} \frac{u^-(x)}{|x|^n} \frac{\nu_n}{R^{n+1}} d\sigma(x) \leq AKR^{\rho(R)-1}, \tag{6.3.4} \\
& \int_{\{ x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0 \}} u^-(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \leq AKR^{\rho(R)-1}. \tag{6.3.5}
\end{align*}
\]
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Note that
\[
\int_{\{x \in \mathbb{R}^n: 1 < |x'| < R, x_n = 0\}} \frac{u^-(x')}{|x'|^n} dx' \\
\leq \frac{2^n}{2^n - 1} \int_{\{x \in \mathbb{R}^n: 1 < |x'| < R, x_n = 0\}} u^-(x')\left(\frac{1}{|x'|^n} - \frac{1}{(2R)^n}\right) dx' \\
\leq AK (2R)^{p(R) - 1}.
\] (6.3.6)

We use Lemma 6.2.2 to the harmonic function \(-u(x)\), and note that \(-u(x) \leq u^-(x)\), we have
\[
\begin{align*}
-u(x) &= \int_{\{y \in H: |y| = R, y_n > 0\}} \frac{R^2 - |x|^2}{\omega_n R} \left(\frac{1}{|y - x|^n} - \frac{1}{|y - x^+|^n}\right)(-u(y))d\sigma(y) \\
&+ \frac{2x_n}{\omega_n} \int_{\{y \in \Pi: |y'| < R, y_n = 0\}} \left(\frac{1}{|y - x|^n} - \frac{R^n}{|x| |y'|^n}\right)(-u(y'))dy' \\
&\leq \int_{\{y \in H: |y| = R, y_n > 0\}} \frac{R^2 - |x|^2}{\omega_n R} \left(\frac{1}{|y - x|^n} - \frac{1}{|y - x^+|^n}\right)u^-(y)d\sigma(y) \\
&+ \frac{2x_n}{\omega_n} \int_{\{y \in \Pi: |y'| < R, y_n = 0\}} \left(\frac{1}{|y - x|^n} - \frac{R^n}{|x| |y'|^n}\right)u^-(y')dy' \\
&= I_1 + I_2.
\end{align*}
\] (6.3.7)

Note that the following estimates:
\[
\frac{1}{|y - x|^n} - \frac{1}{|y - x^+|^n} \leq \frac{2nx_n y_n}{\omega_n |y - x|^{n+2}},
\] (6.3.8)
\[
|y - x|^n \leq x_n = |x|^n \sin^n \theta, \quad x \in H, y_n = 0.
\] (6.3.9)

Put \( |x| = r > 1/2, R = 2r \) in (6.3.7), then by (6.3.4), (6.3.8) and (6.3.9), we have
\[
\begin{align*}
I_1 &\leq \int_{\{y \in H: |y| = R, y_n > 0\}} \frac{R^2 - r^2}{\omega_n R} \frac{2nx_n y_n}{\omega_n |y - x|^{n+2}} u^-(y)d\sigma(y) \\
&\leq AKR^{p(R)}.
\end{align*}
\] (6.3.10)

and
\[
\begin{align*}
I_2 &\leq \frac{2x_n}{\omega_n} \int_{\{y \in \Pi: |y'| < R, y_n = 0\}} \frac{1}{x_n} u^-(y')dy' \\
&= \frac{2}{\omega_n x_n^{n-1}} \int_{\{y \in \Pi: |y'| < R, y_n = 0\}} u^-(y')dy' \\
&= \frac{2}{\omega_n x_n^{n-1}} \int_{\{y \in \Pi: 1 < |y'| < R, y_n = 0\}} u^-(y')dy' + \frac{2}{\omega_n x_n^{n-1}} \int_{\{y \in \Pi: |y'| \leq 1, y_n = 0\}} u^-(y')dy' \\
&= I_{21} + I_{22}.
\end{align*}
\] (6.3.11)
6.3. Proof of Theorem

For the first integral we have by (6.3.6)

\[
I_{21} \leq \frac{2R^n}{\omega_n x_n^{n-1}} \int_{\{y \in \mathbb{R}^n : 1 < |y'| < R, y_n = 0\}} \frac{u^-(y')}{|y'|^n} dy' \\
\leq AK \frac{(2R)^{\rho(R)}}{\sin^{n-1} \theta},
\]  
(6.3.12)

for the second integral we have by (6.1.2)

\[
I_{22} \leq \frac{2K}{\omega_n x_n^{n-1}} \int_{\{y \in \mathbb{R}^n : 1 < |y'| < R, y_n = 0\}} dy' \\
\leq AK \frac{1}{\sin^{n-1} \theta}.
\]  
(6.3.13)

By collecting (6.3.7), (6.3.10), (6.3.11), (6.3.12) and (6.3.13), we have for \(|x| > 1/2\),

\[-u(x) \leq AK \frac{1 + (2R)^{\rho(R)}}{\sin^{n-1} \theta},
\]  
(6.3.14)

for \(|x| \leq 1/2\), we can get by (6.1.2)

\[-u(x) \leq K \leq K \frac{1 + (2R)^{\rho(R)}}{\sin^{n-1} \theta},
\]  
(6.3.15)

so we obtain by (6.3.14) and (6.3.15)

\[u(x) \geq -cK \frac{1 + (2R)^{\rho(R)}}{\sin^{n-1} \theta}, \quad x \in H.
\]

Remark 6.3.1 By modifying (6.3.6):

\[
\int_{\{x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0\}} \frac{u^-(x')}{|x'|^n} dx' \\
\leq \frac{(N + 1)^n}{(N + 1)^n - N^n} \int_{\{x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0\}} u^-(x') \left( \frac{1}{|x'|^n} - \frac{1}{(N + 1)^n} \right) dx' \\
\leq AK \left( \frac{N + 1}{N} R \right)^{\rho(R) - 1},
\]

we can get:

\[u(x) \geq -cK \frac{1 + \left( \frac{N + 1}{N} R \right)^{\rho(R)}}{\sin^{n-1} \theta}, \quad x \in H.
\]
Remark 6.3.2 A example: suppose \( u(z) = \Re e^{-iz} = e^y \cos x \) is a harmonic function in the upper half plane \( \mathbb{C}_+ \) with continuous boundary values on the real axis, write \( |x| = R \cos \theta, y = R \sin \theta (0 < \theta \leq \pi/2) \). Let \( K = 1, \rho(R) = \frac{R}{\log R} \), then \( u(z) \) satisfies
\[
    u(z) \leq e^R \leq KR\rho(R).
\]
Thus
\[
    u(z) \geq -e^R \geq -cK \frac{1 + (2R)^\rho(R)}{\sin^{n-1}\theta}, \quad x \in H.
\]
Chapter 7

the Carleman Formula of Subharmonic Functions in the Half Space

7.1 Introduction and Main Theorem

B.Ya. Levin [26] has proved the following result:

**Theorem G** (Carleman’s formula) Let \( f(z) \) be a meromorphic function in a closed sector \( \overline{S} = \{ z : \rho \leq |z| \leq R, \Im z \geq 0 \} \) whose zeros and poles do not lie on the boundary \( \partial S \). Then we obtain

\[
\sum_{\rho<|a_n|<R} \left( \frac{1}{|a_n|} - \frac{|a_n|}{R^2} \right) \sin \alpha_n - \sum_{\rho<|b_n|<R} \left( \frac{1}{|b_n|} - \frac{|b_n|}{R^2} \right) \sin \beta_n
= \frac{1}{2\pi} \int_0^\infty \left( \frac{1}{t^2} - \frac{1}{R^2} \right) \log |f(t)f(-t)|dt + \frac{1}{\pi R} \int_0^\pi \log |f(Re^{i\varphi})| \sin \varphi d\varphi - A_f(\rho, R),
\]

where \( a_n = |a_n|e^{i\alpha_n}, b_n = |b_n|e^{i\beta_n} \) are zeros and poles of the function \( f(z) \), and the remainder term \( A_f(\rho, R) \) is expressed by

\[
A_f(\rho, R) = -\frac{1}{2\pi} \int_0^\pi \left[ \left( \frac{1}{\rho^2} + \frac{1}{R^2} \right) \log |f(\rho e^{i\varphi})| - \left( \frac{1}{\rho} - \frac{\rho}{R^2} \right) \frac{\partial}{\partial \rho} \log |f(\rho e^{i\varphi})| \right] \rho \sin \varphi d\varphi.
\]

The object of this chapter is to generalize the Carleman formula for meromorphic functions in the upper half plane to subharmonic functions in the upper half space. We derive the following main theorem.
Chapter 7. the Carleman Formula of Subharmonic Functions in the Half Space

**Theorem 7.1.1** Let $u(x)$ be a subharmonic function in the upper half space $H$ with continuous boundary values on the boundary $\partial H$, for $R > r > 0$, we have

$$
\int_{\{x \in \mathbb{R}^n : |x| = R, x_n > 0\}} u(x) \frac{n x_n}{R^{n+1}} d\sigma(x)
+ \int_{\{x \in \mathbb{R}^n : r < |x'| < R, x_n = 0\}} u(x') \left( \frac{1}{|x'|^{n-1}} - \frac{1}{R^n} \right) dx' \geq A_u(r, R),
$$

where

$$A_u(r, R) = c_1(r) + \frac{c_2(r)}{R^n}$$

is a function depending on $r$ and $R$ and $c_1(r), c_2(r)$ are functions depending only on $r$, they are denoted by

$$c_1(r) = \int_{\{x \in \mathbb{R}^n : |x| = r, x_n > 0\}} \left[ \frac{(n-1)x_n}{r^{n+1}} u(x) + \frac{x_n}{r^n} \frac{\partial u(x)}{\partial n} \right] d\sigma(x),$$

and

$$c_2(r) = \int_{\{x \in \mathbb{R}^n : |x| = r, x_n > 0\}} \left[ \frac{x_n}{r} u(x) - \frac{x_n}{r^n} \frac{\partial u(x)}{\partial n} \right] d\sigma(x).$$

### 7.2 Main Lemma

In order to obtain the result, we need the lemma below:

**Lemma 7.2.1** Suppose that $D$ is an admissible domain with boundary $S$ in $\mathbb{R}^n$. If $u, v \in C^2$ in $\overline{D}$, then we have

$$
\int_S \left[ u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right] d\sigma(x) = \int_D [v(x) \Delta u(x) - u(x) \Delta v(x)] dx.
$$

Here $\partial / \partial n$ denotes differentiation along the inward normal into $D$.

**Remark 7.2.1** Lemma 7.2.1 is just called the second Green’s formula.

### 7.3 Proof of Theorem

Apply the second Green’s formula to the subharmonic function $u(x)$ and $v(x) = \frac{x_n}{|x|^{n-1}} - \frac{x_n}{R^n}$ in the resulting sphere

$$B_{r,R}^+ = \{x \in \mathbb{R}^n : r < |x| < R, x_n > 0\},$$

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7.3. Proof of Theorem

we obtain

$$\int_{\partial B^+_r} \left( u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right) d\sigma(x) = \int_{B^+_r} v(x) \triangle u(x) dx \geq 0.$$ \hspace{1cm} (7.3.1)

The function $v(x)$ is harmonic in $H$, the equations

$$v(x) = 0, \quad \frac{\partial v(x)}{\partial n} = \frac{nx_n}{R^{n+1}}$$ \hspace{1cm} (7.3.2)

hold on the half sphere $\{x \in \mathbb{R}^n : |x| = R, x_n > 0\}$.

While the equation

$$\frac{\partial v(x)}{\partial n} = -\frac{x_n}{r} \left( \frac{n - 1}{r^n} + \frac{1}{R^n} \right)$$ \hspace{1cm} (7.3.3)

holds on the half sphere $\{x \in \mathbb{R}^n : |x| = r, x_n > 0\}$.

Moreover, the equations

$$v(x) = 0, \quad \frac{\partial v(x)}{\partial n} = \frac{1}{|x|^n} - \frac{1}{R^n}$$ \hspace{1cm} (7.3.4)

hold on $\{x \in \mathbb{R}^n : r < |x'| < R, x_n = 0\}$.

Thus

$$0 \leq \int_{\partial B^+_r} \left[ u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right] d\sigma(x)$$

$$= \int_{\{x \in \mathbb{R}^n : |x| = R, x_n > 0\}} \left[ u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right] d\sigma(x)$$

$$+ \int_{\{x \in \mathbb{R}^n : |x| = r, x_n > 0\}} \left[ u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right] d\sigma(x)$$

$$+ \int_{\{x \in \mathbb{R}^n : r < |x'| < R, x_n = 0\}} \left[ u(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial u(x)}{\partial n} \right] d\sigma(x)$$

$$= I_1 + I_2 + I_3.$$ \hspace{1cm} (7.3.5)

For the first term we have by (7.3.2)

$$I_1 = \int_{\{x \in \mathbb{R}^n : |x| = R, x_n > 0\}} u(x) \frac{nx_n}{R^{n+1}} d\sigma(x);$$ \hspace{1cm} (7.3.6)

for the second term we have by (7.3.3)

$$I_2 = \int_{\{x \in \mathbb{R}^n : |x| = r, x_n > 0\}} \left[ -u(x) \frac{x_n}{r} \left( \frac{n - 1}{r^n} + \frac{1}{R^n} \right) - \left( \frac{x_n}{|x|^n} - \frac{x_n}{R^n} \right) \frac{\partial u(x)}{\partial n} \right] d\sigma(x)$$

$$= -c_1(r) - \frac{c_2(r)}{R^n};$$ \hspace{1cm} (7.3.7)

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for the third term we have by (7.3.4)

\[ I_3 = \int_{\{x \in \mathbb{R}^n: r < |x'| < R, x_n = 0\}} u(x') \left( \frac{1}{|x'|^{n-1}R^n} - \frac{1}{R^n} \right) dx'. \quad (7.3.8) \]

By collecting (7.3.5), (7.3.6), (7.3.7) and (7.3.8), we have

\[
\int_{\{x \in \mathbb{R}^n: |x| = R, x_n > 0\}} u(x) \frac{nx_n}{R^{n+1}} d\mathcal{G}(x)
+ \int_{\{x \in \mathbb{R}^n: r < |x'| < R, x_n = 0\}} u(x') \left( \frac{1}{|x'|^{n-1}R^n} - \frac{1}{R^n} \right) dx' \geq A_u(r, R).
\]

This completes the proof of Theorem.
Chapter 8

a Generalization of the Nevanlinna Formula for Analytic Functions in the Right Half Plane

8.1 Introduction and Main Theorem

Recall that $\mathbb{C}$ denote the complex plane with points $z = x + iy$, where $x, y \in \mathbb{R}$. The boundary and closure of an open $\Omega$ of $\mathbb{C}$ are denoted by $\partial \Omega$ and $\overline{\Omega}$ respectively. The right half plane is the set $\mathbb{C}_+ = \{z = x + iy \in \mathbb{C} : x > 0\}$, whose boundary is $\partial \mathbb{C}_+$. We identify $\mathbb{C}$ with $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ with $\mathbb{R} \times \{0\}$, with this convention we then have $\partial \mathbb{C}_+ = \mathbb{R}$.

Suppose $R > 1$, We write $B_+(0,R) = \{z : |z| < R, \Re z > 0\}$ for the open right half disk of radius $R$ in $\mathbb{C}$ centered at the origin, whose boundary is $\partial B_+(0,R) = \{z : z = it, |t| \leq R\} \cup \{z : z = Re^{i\theta}, |\theta| \leq \frac{\pi}{2}\}$.

Let $\rho > 1$, if the function $f(x)$ is analytic in the open right half plane $\mathbb{C}_+$, continuous in the closed right half plane $\overline{\mathbb{C}_+}$, and satisfies the following conditions:

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(it)|}{1 + |t|^\rho + 1} dt < \infty$$ \hspace{1cm} (8.1.1)

and

$$\int \int_{\mathbb{C}_+} \frac{x \log^+ |F(z)|}{1 + |z|^\rho + 3} dm(z) < \infty,$$ \hspace{1cm} (8.1.2)
then a number of results have been achieved in [3], [4], [11], [16], [12]. in this chapter, we replace the first condition (8.1.1) into

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{\log^+ |F(it + \varepsilon)|}{1 + |t|^\rho + 1} dt < \infty,$$

and that the function \( f(x) \) is continuous in the boundary \( \partial \mathbb{C}_+ \) is not needed, we can get the similar results as [43].

**Theorem 8.1.1** Suppose \( R' > R > 1 \), \( F \in N^+(B_+(0,R)) \), let \( \Lambda_R \) is the set of zeros of \( F \) in \( B_+(0,R) \) and \( \Lambda \) is the set of zeros of \( F \) in \( \mathbb{C}_+ \) (including repetitions for multiplicities). If the conditions (8.1.2) and (8.1.3) are satisfied, then

1. $$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{\log |F(it + \varepsilon)|}{1 + |t|^\rho + 1} dt < \infty;$$
2. $$\lim_{R \to \infty} \frac{1}{R^\rho} \int_{-\pi/2}^{\pi/2} \log |F(Re^{i\theta})| \cos \theta d\theta = 0;$$
3. $$\sum_{\lambda_n \in \Lambda} \frac{\Re \lambda_n}{1 + |\lambda_n|^\rho + 1} < \infty.$$

**8.2 Proof of Theorem**

\( \forall z \in B_+(0,R) \) and \( z \notin \Lambda_R \), write \( F_\varepsilon(z) = F(z + \varepsilon) \), then (see [25], [32] and [34])

$$\log |F_\varepsilon(z)| \leq \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{4Rx \cos \theta (R^2 - |z|^2)}{|Re^{i\theta} - z|^2 |Re^{-i\theta} + z|^2} \log |F_\varepsilon(Re^{i\theta})| d\theta$$
$$+ \frac{x}{\pi} \int_R^{-R} \left( \frac{1}{|it - z|^2 - R^2 + itz|^2} \right) \log |F_\varepsilon(it)| dt. \quad (8.2.1)$$

Without loss of generality we may assume that \( F(1) \neq 0 \), then there exists \( \varepsilon_0 > 0 \), such that for any \( 0 < \varepsilon < \varepsilon_0 \), we have \( F(1 + \varepsilon) \neq 0 \) and \( |F(1 + \varepsilon)| > \frac{|F(1)|}{2} \).
Suppose $R > 2$, $z = 1$, by (8.2.1), we have

\[
\begin{align*}
\log \left| \frac{F(1)}{2} \right| + 2R(R^2-1) \int_{-\pi/2}^{\pi/2} \frac{\cos \theta}{|Re^{i\theta} - 1|^2|Re^{-i\theta} + 1|^2} \log |F e^{i\theta}| \, d\theta \\
+ \frac{1}{\pi} \int_{|t| \leq R/2} \left( \frac{1}{t^2 + 1} - \frac{R^2}{|R^4 + t^2|} \right) \log^+ |F e^{i\theta}| \, dt \\
\leq \frac{2R(R^2-1)}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos \theta}{|Re^{i\theta} - 1|^2|Re^{-i\theta} + 1|^2} \log^+ |F e^{i\theta}| \, d\theta \\
+ \frac{1}{\pi} \int_{|t| \leq R} \left( \frac{1}{t^2 + 1} - \frac{R^2}{|R^4 + t^2|} \right) \log^+ |F e^{i\theta}| \, dt.
\end{align*}
\]

Set

\[
m_+(R) = \frac{1}{R} \int_{-\pi/2}^{\pi/2} \log^+ |F e^{i\theta}| \cos \theta \, d\theta,
\]

\[
m_-(R) = \frac{1}{R} \int_{-\pi/2}^{\pi/2} \log^- |F e^{i\theta}| \cos \theta \, d\theta,
\]

\[
g_+(t) = \log^+ |F e^{it}| + \log^+ |F e^{-it}|,
\]

\[
g_-(t) = \log^- |F e^{it}| + \log^- |F e^{-it}|.
\]

Note that when $|t| \leq R/2$,

\[
\frac{1}{t^2 + 1} - \frac{R^2}{R^4 + t^2} \geq \frac{9}{32} \frac{1}{t^2 + 1};
\]

when $|t| \leq R$,

\[
\frac{1}{t^2 + 1} - \frac{R^2}{R^4 + t^2} \leq \frac{1}{t^2 + 1}.
\]

So we obtain

\[
\frac{8}{27\pi} m_-(R) + \frac{9}{32\pi} \left[ \int_{1}^{R/2} \frac{1}{2t} g_-(t) \, dt + \frac{1}{2} \int_{|t| < 1} \log^+ |F e^{it}| \, dt \right] \\
\leq \frac{32}{\pi} m_+(R) + \frac{1}{\pi} \left[ \int_{1}^{R} \frac{1}{t} g_+(t) \, dt + \int_{|t| < 1} \frac{1}{t^2 + 1} \log^+ |F e^{it}| \, dt \right] - \log \left| \frac{F(1)}{2} \right|.
\]
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Multiplying (8.2.2) by $\frac{1}{R^p}$ and integrating with respect to $R$, we obtain

$$\begin{align*}
\frac{8}{27\pi} \int_2^\infty \frac{m_{(\varepsilon)}(R)}{R^p} dR + \frac{9}{32\pi} \int_2^\infty \frac{1}{R^p} \int_1^{R/2} \frac{1}{2\varepsilon^2} g_{(\varepsilon)}(t) dt dR \\
+ \frac{9}{64\pi} \int_{|t|<1} \log^{-} |F_{\varepsilon}(it)| dt \cdot \int_2^\infty \frac{1}{R^p} dR \\
\leq \frac{32}{\pi} \int_2^\infty \frac{m_{(\varepsilon)}(R)}{R^p} dR + \frac{1}{\pi} \int_2^\infty \frac{1}{R^p} \int_1^R \frac{1}{t^2} g_{(\varepsilon)}(t) dt dR \\
+ \left( \frac{1}{\pi} \int_{|t|<1} \frac{1}{t^2 + 1} \log^{+} |F_{\varepsilon}(it)| dt - \frac{\log |F(1)|}{2} \right) \cdot \int_2^\infty \frac{1}{R^p} dR.
\end{align*}$$

After some elementary calculations, we get

$$\begin{align*}
\frac{8}{27\pi} \int_2^\infty \int_D \frac{x \log^{-} |F_{\varepsilon}(z)|}{|z|^{p+3}} dm(z) + \frac{9}{64\pi} 2^{p-1}(\rho-1) \\
\times \left[ \int_1^\infty \frac{g_{(\varepsilon)}(t)}{t^p+1} dt + \int_{|t|<1} \log^{-} |F_{\varepsilon}(it)| dt \right] \\
\leq \frac{32}{\pi} \int_2^\infty \int_D \frac{x \log^{+} |F_{\varepsilon}(z)|}{|z|^{p+3}} dm(z) + \frac{1}{\pi \rho - 1} \int_1^\infty \frac{g_{(\varepsilon)}(t)}{t^p+1} dt \\
+ \frac{1}{\pi \rho - 1} \int_{|t|<1} \frac{1}{t^2 + 1} \log^{+} |F_{\varepsilon}(it)| dt - \frac{\log |F(1)|}{2},
\end{align*}$$

where $D = \{(x,y) : x^2 + y^2 \geq 4, x \geq 0\}$.

By (8.1.3), there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ and $M > 0$, such that

$$\int_{-\infty}^\infty \frac{\log^{+} |F(it + \varepsilon_n)|}{1 + |t|^{p+1}} dt \leq M < \infty,$$

so

$$\int_1^\infty \frac{g_{(\varepsilon_n)}(t)}{t^p+1} dt + \int_{|t|<1} \frac{1}{t^2 + 1} \log^{+} |F_{\varepsilon_n}(it)| dt \\
\leq 2 \int_{-\infty}^\infty \frac{\log^{+} |F(it + \varepsilon_n)|}{1 + |t|^{p+1}} dt \leq 2M < \infty.$$

When $\varepsilon < \frac{3}{4} |\omega|$, limit $\varepsilon < \frac{\delta}{\rho}$, then we have

$$\int \int_D \frac{x \log^{+} |F_{\varepsilon}(z)|}{|z|^{p+3}} dm(z) \leq 2 \times 4^{p+3} \int \int_{C^+} \frac{\Re z \log^{+} |F(z)|}{1 + |z|^{p+3}} dm(z) \leq M,$$
where $D' = D + \varepsilon$, $\omega \in D'$ and $\omega = z + \varepsilon$, so we obtain
\[
\int_1^\infty \frac{g^{(\varepsilon_n)}_+(t) + g^{(\varepsilon_n)}_-(t)}{1 + t^{p+1}} dt \leq M,
\]
\[
\int \int_D \frac{x|\log^+ |F_\varepsilon(z)||}{|z|^{p+3}} dm(z) \leq M,
\]
\[
\int_{|t|<1} |\log |F_\varepsilon(it)|| dt \leq M,
\]
and
\[
\int_{-\infty}^\infty \frac{|\log |F_\varepsilon(it)||}{1 + |t|^{p+1}} dt \leq M.
\]
Therefore, there exists $M > 0$, such that
\[
\sup_n \int_{-\infty}^\infty \frac{|\log |F(it + \varepsilon_n)||}{1 + |t|^{p+1}} dt \leq M < \infty. \tag{8.2.3}
\]
\[
\forall s(t) \in C_0(-\infty, +\infty), \text{ set } T_n(s) = \int_{-\infty}^\infty s(t) \frac{\log |F(it + \varepsilon_n)||}{1 + |t|^{p+1}} dt,
\]
by (8.2.3), we obtain that $T_n$ is a bounded linear functional in $C_0(-\infty, +\infty)$ and
\[
\sup_n \|T_n\| = \sup_n \int_{-\infty}^\infty \frac{|\log |F(it + \varepsilon_n)||}{1 + |t|^{p+1}} dt \leq M.
\]
Hence, there exists a subsequence $\{T_{n_k}\}$ of $\{T_n\}$, such that $T_{n_k}$ weakly* converges to $T$, that is to say,
\[
T(s) = \lim_{n_k \to \infty} T_{n_k}(s), \quad \forall s(t) \in C_0(-\infty, +\infty).
\]
By Riesz Representation Theorem [33], there exists a Radon measure $\nu$ such that
\[
T(s) = \int_{-\infty}^\infty s(t) d\nu.
\]
Set $d\nu = \frac{1}{1 + |t|^{p+1}} d\mu$, then
\[
T(s) = \int_{-\infty}^\infty \frac{s(t)}{1 + |t|^{p+1}} d\mu, \quad \forall s(t) \in C_0(-\infty, +\infty).
Define
\[ T_\varepsilon(s) = \int_{-\infty}^{\infty} s(t) \frac{\log |F(it + \varepsilon)|}{1 + |t|^{p+1}} dt, \]
then
\[ \|T_\varepsilon\| = \int_{-\infty}^{\infty} \frac{\log |F(it + \varepsilon)|}{1 + |t|^{p+1}} dt, \]
and
\[ \lim_{\varepsilon \to 0} \|T_\varepsilon\| = \int_{-\infty}^{\infty} \frac{\log |F(it + \varepsilon)|}{1 + |t|^{p+1}} dt, \]
so
\[ \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{\log |F(it + \varepsilon)|}{1 + |t|^{p+1}} dt \leq M < \infty. \]

Hence (1) holds;

Since
\[ \int_{-\infty}^{\infty} \frac{1}{R^{p+1}} \int_{-\pi/2}^{\pi/2} |\log |F_\varepsilon(Re^{i\theta})|| \cos \theta d\theta dR \]
\[ = \int \int_{D} x \log^+ \frac{|F_\varepsilon(z)|}{|z|^{p+3}} dm(z) \leq M, \]
we obtain
\[ \int_{2}^{\infty} \frac{1}{R^{p+1}} \int_{-\pi/2}^{\pi/2} |\log |F(Re^{i\theta})|| \cos \theta d\theta dR \]
\[ \leq \lim_{n \to \infty} \int_{2}^{\infty} \frac{1}{R^{p+1}} \int_{-\pi/2}^{\pi/2} |\log |F_\varepsilon(Re^{i\theta})|| \cos \theta d\theta dR \leq M, \]
so
\[ \lim_{R \to \infty} \frac{1}{R} \int_{-\pi/2}^{\pi/2} |\log |F(Re^{i\theta})|| \cos \theta d\theta = 0. \]

Thus (2) holds;

Write \( \lambda_\varepsilon = \lambda_n - \varepsilon, \forall z \in B_+(0, R) \) and \( z \notin \Lambda_R \), then
\[ F_\varepsilon(\lambda_\varepsilon) = F_\varepsilon(\lambda_n - \varepsilon) = F(\lambda_n) = 0, \]
where \( \lambda_n, \lambda_\varepsilon \) are denoted the zeros of \( F, F_\varepsilon \) respectively. So we have
\[
\log |F_\varepsilon(z)| = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \left( \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} - \frac{R^2 - |z|^2}{|Re^{-i\theta} + z|^2} \right) \log |F_\varepsilon(Re^{i\theta})| d\theta \\
+ \frac{1}{\pi} \int_{-R}^{R} \left( \frac{\Re z}{|t - z|^2} - \frac{R^2 \Re z}{|R^2 + itz|^2} \right) \log |F_\varepsilon(it)| dt \\
+ \sum_{\lambda_\varepsilon \in \Lambda_\varepsilon} \log \left| \frac{z - \lambda_\varepsilon}{R^2 - \lambda_\varepsilon z} \frac{R^2 + \lambda_\varepsilon z}{z + \lambda_\varepsilon} \right|. \quad (8.2.4)
\]
8.2. Proof of Theorem

Without loss of generality we may assume that $F(1) \neq 0$, then there exists $\varepsilon_0 > 0$, such that for any $0 < \varepsilon < \varepsilon_0$, we have $F(1 + \varepsilon) \neq 0$ and $|F(1 + \varepsilon)| > \frac{|F(1)|}{2}$.

Suppose $R > 2$, $z = 1$, by (8.2.4), we have

$$\log \frac{|F(1)|}{2} \leq \frac{2R(R^2 - 1)}{\pi(R - 1)^4} \int_{-\pi/2}^{\pi/2} \log^+ |F_e(Re^{i\theta})| |\cos \theta| d\theta$$
$$+ \frac{1}{\pi} \int_{1}^{R} \frac{1}{t^2} \log^+ |F_e(it)| + \log^+ |F_e(it)|] dt$$
$$+ \frac{1}{\pi} \int_{|t| < 1} \frac{1}{1 + t^2} \log^+ |F_e(it)| dt$$
$$+ \sum_{\lambda_e \in \Lambda_R} \log \left| \frac{1 - \lambda_e}{R^2 - \lambda_e} R^2 + \frac{1}{1 + \lambda_e} \right|.$$

Note that

$$\log x < \frac{1}{2} (x^2 - 1), \quad \forall x \in (0, 1),$$

then we have

$$\log \left| \frac{1 - \lambda_e}{R^2 - \lambda_e} R^2 + \frac{1}{1 + \lambda_e} \right|$$
$$\leq \frac{|(R^2 - \lambda_e)\lambda_e| + (\lambda_e - R^2\lambda_e)|^2}{2|1 + \lambda_e|^2 |R^2 - \lambda_e|^2}$$
$$\leq \frac{|(R^2 - \lambda_e)|^2 |R^2 - \lambda_e|^2}{2|1 + \lambda_e|^2 |R^2 - \lambda_e|^2}$$
$$= - \frac{2(R^2 - |\lambda_e|^2)(R^2 - 1) \Re \lambda_e}{|1 + \lambda_e|^2 |R^2 - \lambda_e|^2}.$$

Since

$$\sum_{1 \leq |\lambda_e| < R/2} \frac{\Re \lambda_e}{|\lambda_e|^2} = \int_{1}^{R/2} \frac{1}{t} dN_0^{(\varepsilon)}(t),$$

where

$$N_0^{(\varepsilon)}(t) = \sum_{1 \leq |\lambda_e| < t} \cos \theta_e,$$
then we obtain
\[
\sum_{|\lambda|<1} \Re \lambda \leq \frac{256}{\pi} m_+^{(e)}(R) + \frac{8}{\pi} \int_1^R \frac{1}{t^2} g_+^{(e)}(t) \, dt + \frac{8}{\pi} \int_{|t|<1} \frac{1}{t^2 + 1} \log^+ |F_\varepsilon(it)| \, dt - 8 \log \frac{|F(1)|}{2}.
\]

Multiplying this by \(\frac{1}{R^\rho}\) and integrating with respect to \(R\), we obtain
\[
\sum_{|\lambda|<1} \Re \lambda \leq \frac{256}{\pi} \int_0^\infty \frac{1}{R^\rho} dR + \int_2^\infty \frac{1}{R^\rho} \int_1^{R/2} \frac{1}{t} dN_0^{(e)}(t) \, dR
\]
\[
+ \int_2^\infty \frac{1}{R^\rho} \cdot 8 \left[ \frac{1}{\pi} \int_{|t|<1} \frac{1}{t^2 + 1} \log^+ |F_\varepsilon(it)| \, dt - \log \frac{|F(1)|}{2} \right].
\]

By some elementary calculations, we get
\[
\sum_{\lambda \in \Lambda} \Re \lambda \leq \frac{2^{\rho+7}(\rho - 1)}{\pi} \int_2^\infty \frac{m_+^{(e)}(R)}{R^\rho} dR
\]
\[
+ \frac{2^{\rho+3}}{\pi} \int_{-\infty}^\infty \frac{\log |F(it + \varepsilon)|}{1 + |t|^{\rho+1}} dt
\]
\[
- 8 \log \frac{|F(1)|}{2}.
\]

So we have
\[
\lim_{\varepsilon \to 0} \sum_{\lambda \in \Lambda} \Re \lambda \leq \frac{2^{\rho+7}(\rho - 1)}{\pi} \int_2^\infty \frac{m_+^{(e)}(R)}{R^\rho} dR
\]
\[
+ \frac{2^{\rho+3}}{\pi} \int_{-\infty}^\infty \frac{\log |F(it + \varepsilon)|}{1 + |t|^{\rho+1}} dt
\]
\[
- 8 \log \frac{|F(1)|}{2}.
\]
8.2. Proof of Theorem

Hence

\[ \lim_{\varepsilon \to 0} \sum_{\lambda \in \Lambda} \frac{\Re \lambda \varepsilon}{1 + |\lambda|^{p+1}} \leq M < \infty. \]

Since

\[ \sum_{\lambda \in \Lambda} \frac{\Re \lambda}{1 + |\lambda|^{p+1}} \leq \lim_{\varepsilon \to 0} \sum_{\lambda \in \Lambda} \frac{\Re \lambda \varepsilon}{1 + |\lambda|^{p+1}} \leq M < \infty, \]

thus (3) holds. This completes the proof of Theorem.
Chapter 9

Integral Representations of Harmonic Functions in the Half Plane

9.1 Introduction and Main Theorem

Let $\rho(R) \geq 1$ is nondecreasing in $[0, +\infty)$ satisfying

$$
\varepsilon_0 = \limsup_{R \to \infty} \frac{\rho'(R)R\log R}{\rho(R)} < 1.
$$

(9.1.1)

For any real number $\alpha > 0$, we denote by $(LU)_\alpha$ the space of all measurable functions $f(x+iy)$ in the upper half plane $\mathbb{C}_+$ which satisfy the following inequality:

$$
\int \int_{\mathbb{C}_+} \frac{y|f(x+iy)|}{1 + (x^2 + y^2)^{\rho(|z|)+\alpha+1}} dxdy < \infty;
$$

(9.1.2)

and $(LV)_\alpha$ the set of all measurable functions $g(x)$ in $\mathbb{R}$ which satisfy the following inequality:

$$
\int_{-\infty}^{\infty} \frac{|g(x)|}{1 + |x|^\rho(|x|)+\alpha+1} dx < \infty.
$$

(9.1.3)

We also denote by $(CH)_\alpha$ the set of all continuous functions $u(x+iy)$ in the closed upper half plane $\overline{\mathbb{C}_+}$, harmonic in the open upper half plane $\mathbb{C}_+$ with the positive part $u^+(x+iy) = \max\{u(x+iy),0\} \in (LU)_\alpha$ and $u^+(x) \in (LV)_\alpha$.  

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The Poisson kernel for the upper half plane $\mathbb{C}_+$ is the function

$$P(z, t) = \frac{y}{\pi |z - t|^2},$$

where $z \in \mathbb{C}_+$, $t \in \mathbb{R}$.

If $u(z) \leq 0$ is harmonic in the open upper half plane $\mathbb{C}_+$, continuous in the closed upper half plane $\overline{\mathbb{C}_+}$, then (see [19], [36] and [1]) $u \in (CH)_\alpha$ for each $\alpha > 0$ and there exists a constant $c \leq 0$ such that

$$u(z) = cy + \int_{-\infty}^{\infty} P(z, t) u(t) dt$$

(9.1.4)

for all $z \in \mathbb{C}_+$, the integral in (9.1.4) is absolutely convergent. Motivated by this result, we will prove that if $u \in (CH)_\alpha$, then $u^+(x + iy) \in (LU)_\alpha$, $u(x) \in (LV)_\alpha$ and a similar representation to (9.1.4) for the function $u \in (CH)_\alpha$ holds by modifying the Poisson kernel $P_m(z, t)$. It is well known (see [15], [19] and [40]) that the Poisson kernel $P(z, t)$ is harmonic in $z \in \mathbb{C} - \{t\}$ and has a series expansion:

$$P(z, t) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{z^k}{t^{k+1}},$$

this series converges for $|z| < |t|$. So if $m \geq 0$ is an integer, we define a modified Cauchy kernel of order $m$ for $z \in \mathbb{C}_+$ by

$$C_m(z, t) = \begin{cases} \frac{1}{\pi} \frac{1}{t - z}, & \text{when } |t| \leq 1, \\ \frac{1}{\pi} - \frac{1}{\pi} \sum_{k=0}^{m} \frac{z^k}{t^{k+1}}, & \text{when } |t| > 1, \end{cases}$$

(9.1.5)

then we define a modified Poisson kernel of order $m$ for the upper half plane by

$$P_m(z, t) = \Re C_m(z, t).$$

That is to say,

$$P_m(z, t) = \begin{cases} P(z, t), & \text{when } |t| \leq 1, \\ P(z, t) - \frac{1}{\pi} \sum_{k=0}^{m} \frac{z^k}{t^{k+1}}, & \text{when } |t| > 1. \end{cases}$$

The modified Poisson kernel $P_m(z, t)$ is harmonic in $z \in \mathbb{C}_+$.

Up to now, a number of results about integral representations have been achieved in [42], [18], [43], [9], [8], [13], [6], [46], [7], [49], in this chapter, we will establish the following theorem.
9.2. Main Lemma

Theorem 9.1.1 If \( u \in (CH)_\alpha (\alpha > 0) \), then the following properties hold:

1. \[ \int_{-\infty}^{\infty} \frac{|u(x)|}{1 + |x|^{\rho(x)} + \alpha + 1} \, dx < \infty; \]

2. the integral \[ \int_{-\infty}^{\infty} P(\rho(x) + \alpha)(z, t)u(t) \, dt \]

is absolutely convergent, it represents a harmonic function \( u_{C^+}(z) \) in \( C^+ \) and can be continuously extended to \( \overline{C^+} \) such that \( u_{C^+}(t) = u(t) \) for \( t \in \mathbf{R} \);

3. There exists an entire function \( Q_{C^+}(z) \) which satisfies on the boundary \( \mathbf{R} \) that \( \Im Q_{C^+}(x) = 0 \) such that \( u(z) = \Im Q_{C^+}(z) + u_{C^+}(z) \) for all \( z \in C^+ \).

9.2 Main Lemma

In order to obtain the result, we need the lemma below:

Lemma 9.2.1 For any \( t \in \mathbf{R} \) and \( |z| > 1, y > 0 \), the following inequalities hold.

\[
|C_m(z, t)| \leq \begin{cases} 
\frac{|z|^{m+1}}{\pi y |t|^{m+1}}, & \text{when } 1 < |t| \leq 2|z|, \\
\frac{2|z|^{m+1}}{\pi |t|^{m+2}}, & \text{when } |t| > \max\{1, 2|z|\}, \\
\frac{1}{\pi y}, & \text{when } |t| \leq 1 
\end{cases}
\]

Proof: When \( t \in \mathbf{R}, |t| \leq 1 \), we have \( |t - z| \geq y \) and so

\[
|C_m(z, t)| \leq \frac{1}{\pi y};
\]

when \( t \in \mathbf{R}, 1 < |t| \leq 2|z| \), we also have \( |t - z| \geq y \) and so by (9.1.5)

\[
|C_m(z, t)| = \frac{1}{\pi} \left| \frac{1}{t - z} - \frac{1 - (\frac{z}{t})^{m+1}}{t - z} \right|
\]

\[
= \frac{1}{\pi} \frac{|z|^{m+1}}{|t - z|} \leq \frac{|z|^{m+1}}{\pi y |t|^{m+1}};
\]

when \( |t| > \max\{1, 2|z|\} \), we have by (9.1.5)

\[
|C_m(z, t)| = \frac{1}{\pi} \sum_{k=m+1}^{\infty} \frac{z^k}{t^{k+1}} \leq \frac{1}{\pi} \sum_{k=m+1}^{\infty} \frac{|z|^k}{|t|^{k+1}} \leq \frac{2|z|^{m+1}}{\pi |t|^{m+2}}.
\]

This proves the inequalities.
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9.3 Proof of Theorem

If \( u \in (CH)_\alpha (\alpha > 0) \), suppose \( R > 1 \), then by the Carleman formula for harmonic functions in the upper half plane,

\[
\frac{1}{\pi R} \int_0^\pi u(Re^{i\theta}) \sin \theta d\theta + \frac{1}{2\pi} \int_{1<|x|<R} u(x) \left( \frac{1}{x^2} - \frac{1}{R^2} \right) dx = c_1 + \frac{c_2}{R^2},
\]

where

\[
c_1 = \frac{1}{2\pi} \int_0^\pi \left[ u(Re^{i\theta}) + \frac{\partial u(Re^{i\theta})}{\partial n} \right] \sin \theta d\theta,
\]

\[
c_2 = \frac{1}{2\pi} \int_0^\pi \left[ u(Re^{i\theta}) - \frac{\partial u(Re^{i\theta})}{\partial n} \right] \sin \theta d\theta.
\]

Set

\[
m_+(R) = \frac{1}{\pi R} \int_0^\pi u^+(Re^{i\theta}) \sin \theta d\theta,
\]

\[
m_-(R) = \frac{1}{\pi R} \int_0^\pi u^-(re^{i\theta}) \sin \theta d\theta,
\]

\[
g_+(x) = u^+(x) + u^+(-x),
\]

\[
g_-(x) = u^-(x) + u^-(-x),
\]

then

\[
m_-(R) + \frac{1}{2\pi} \int_1^R \left( \frac{1}{x^2} - \frac{1}{R^2} \right) g_-(x) dx = m_+(R) + \frac{1}{2\pi} \int_1^R \left( \frac{1}{x^2} - \frac{1}{R^2} \right) g_+(x) dx - c_1 - \frac{c_2}{R^2}, \tag{9.3.1}
\]

where \( u^+(z) = \max\{u(z), 0\} \), \( u^-(z) = (-u(z))^+ \) and \( u(z) = u^+(z) - u^-(z) \).

Since \( u \in (CH)_\alpha \) by (9.1.2), we obtain

\[
\int_1^\infty \frac{m_+(R)}{R^{\beta(R)+\alpha}} dR = \frac{1}{\pi} \int_D \frac{yu^+(x+iy)}{(x^2+y^2)^{\frac{\beta(R)+\alpha+3}{2}}} dx dy < \infty, \tag{9.3.2}
\]

where \( D = \{z \in \mathbb{C}_+: |z| > 1\} \).
9.3. Proof of Theorem

By (9.1.3), we can also obtain

\[
\int_1^\infty \frac{1}{R^{\rho(R)+\alpha}} \int_1^R \left( \frac{1}{x^2} - \frac{1}{R^2} \right) g_+(x) dx dR
\]
\[
= \int_1^\infty g_+(x) \int_x^\infty \frac{1}{R^{\rho(R)+\alpha}} \left( \frac{1}{x^2} - \frac{1}{R^2} \right) dR dx
\]
\[
\leq \frac{2}{3} \int_1^\infty \frac{g_+(x)}{x^{\rho(x)+\alpha+1}} dx < \infty.
\] (9.3.3)

Similarly, we have

\[
\int_1^\infty \frac{1}{R^{\rho(R)+\alpha/2}} \int_1^R \left( \frac{1}{x^2} - \frac{1}{R^2} \right) g_-(x) dx dR
\]
\[
= \int_1^\infty g_-(x) \int_x^\infty \frac{1}{R^{\rho(R)+\alpha/2}} \left( \frac{1}{x^2} - \frac{1}{R^2} \right) dR dx.
\] (9.3.4)

So we have by (9.3.1), (9.3.2) and (9.3.3)

\[
\int_1^\infty \frac{1}{R^{\rho(R)+\alpha/2}} \int_1^R \left( \frac{1}{x^2} - \frac{1}{R^2} \right) g_-(x) dx dR
\]
\[
\leq \int_1^\infty \frac{1}{R^{\rho(R)+\alpha/2}} \left[ 2\pi m_+(R) + \int_1^R \left( \frac{1}{x^2} - \frac{1}{R^2} \right) g_+(x) dx - 2\pi \left( c_1 + \frac{c_2}{R^2} \right) \right] dR
\]
\[
< \infty.
\] (9.3.5)

For all \( \alpha > 0 \), set

\[
I(\alpha) = \lim_{x \to \infty} \frac{\int_x^\infty \frac{1}{R^{\rho(R)+\alpha/2}} \left( \frac{1}{x^2} - \frac{1}{R^2} \right) dR}{x^{-\rho(x)+\alpha+1}}.
\]

by the L'hospital's rule and (9.1.1), we have

\[
I(\alpha) = +\infty.
\]

Therefore, there exists \( \varepsilon_1 > 0 \), such that for any \( x \geq 1 \),

\[
\int_x^\infty \frac{1}{R^{\rho(R)+\alpha/2}} \left( \frac{1}{x^2} - \frac{1}{R^2} \right) dR \geq \frac{\varepsilon_1}{x^{\rho(x)+\alpha+1}}.
\]

Multiplying this by \( g_-(x) \) and integrating with respect to \( x \), we can obtain by (9.3.4) and (9.3.5)

\[
\varepsilon_1 \int_1^\infty \frac{g_-(x)}{x^{\rho(x)+\alpha+1}} dx \leq \int_1^\infty g_-(x) \int_x^\infty \frac{1}{R^{\rho(R)+\alpha/2}} \left( \frac{1}{x^2} - \frac{1}{R^2} \right) dR dx < \infty.
\]
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Thus

$$\int_1^\infty \frac{g_-(x)}{xp(x) + \alpha + 1} dx < \infty.$$  

by (9.3.3), we have

$$\int_1^\infty \frac{g_+(x)}{xp(x) + \alpha + 1} dx < \infty.$$  

Hence (1) holds.

\[ \forall \alpha > 0, R > 1, \exists M(R) > 0, \text{ such that for any } k > k_R = [2R] + 1, \text{ we have} \]

$$\frac{R^{p(k+1)+\alpha+1}}{k^{\alpha/2}} \leq M(R),$$  

so \( \forall \alpha > 0, R > 1, \) if \( |z| \leq R, k > k_R = [2R] + 1, \) then \( |t| \geq 2|z| \) and

$$\sum_{k=k_R}^\infty \int_{k|t| < k+1} \left| \frac{|z|^{p(|t|) + \alpha} + 1}{|t|^{p(|t|) + \alpha + 2}} |u(t)| dt \right|  
\leq \sum_{k=k_R}^\infty \frac{R^{p(k+1)+\alpha+1}}{k^{\alpha/2}} \int_{k|t| < k+1} \frac{2|u(t)|}{1 + |t|^{p(|t|) + \alpha/2 + 1}} dt  
\leq 2M(R) \int_{|t| \geq k_R} \frac{|u(t)|}{1 + |t|^{p(|t|) + \alpha/2 + 1}} dt.$$  

So the integral is absolutely convergent.

To verify the boundary behavior of \( u_{C_+}(z) \), choose a large \( T > 2 \), and write

$$u_{C_+}(z) = \int_{|t| \leq 2T} P(z, t) u(t) dt  
- 3 \sum_{k=0}^{[p(|t| + \alpha)]} \int_{1 < |t| \leq 2T} \frac{z^k}{\pi r^{k+1}} u(t) dt  
+ \int_{|t| > 2T} P_{|p(|t| + \alpha)|}(z, t) u(t) dt  
= X(z) - Y(z) + Z(z).$$

Consider \( z \to x_0 \), the first term \( X(z) \) approaches \( u(x_0) \) because it is the Poisson integral of \( u(t) \chi_{[-2T,2T]}(t) \), where \( \chi_{[-2T,2T]} \) is the characteristic function of the interval \( [-2T,2T] \); the second term \( Y(z) \) is a polynomial times \( y \) and tends to 0; and the third term \( Z(z) \) is \( O(y) \) and therefore also to 0. So the function \( u_{C_+}(z) \) can be continuously extended to \( \overline{C_+} \) such that \( u_{C_+}(t) = u(t) \); consequently, \( u(z) - u_{C_+}(z) \) is harmonic in \( C_+ \) and can be continuously extended to \( \overline{C_+} \) with 0 in the boundary \( R \) of \( C_+ \). The Schwarz reflection principle ([17], p.68 and [17], p.28) applied
9.3. Proof of Theorem

to \( u(z) - u_{C_+}(z) \) shows that there exists an entire function \( Q_{C_+}(z) \) which satisfies
\[
Q_{C_+}(z) = Q_{C_+}(\overline{z}),
\]
such that \( \Im Q_{C_+}(z) = u(z) - u_{C_+}(z) \) for \( z \in \mathbb{C}_+ \). Therefore, if
\( \alpha > 0 \), we obtain \( u(z) = \Im Q_{C_+}(z) + u_{C_+}(z) \) for all \( z \in \mathbb{C}_+ \) and
\( \Im Q_{C_+}(x) = 0 \) for all \( x \in \mathbb{R} \). This completes the proof of Theorem.
Chapter 10

Integral Representations of Harmonic Functions in the Half Space

10.1 Introduction and Main Theorem

Let $\rho(R) \geq 1$ is nondecreasing in $[0, +\infty)$ satisfying

$$\varepsilon_0 = \limsup_{R \to \infty} \frac{\rho'(R)R\log R}{\rho(R)} < 1. \quad (10.1.1)$$

For any real number $\alpha > 0$, we denote by $(L^U)\alpha$ the space of all measurable functions $f(x)$ in the upper half space $H$ which satisfy the following inequality:

$$\int_H \frac{x_n |f(x)| \, dx}{1 + |x|^\rho(|x|)^{n+\alpha+1}} < \infty; \quad (10.1.2)$$

and $(L^V)\alpha$ the set of all measurable functions $g(x')$ in $\mathbb{R}^{n-1}$ which satisfy the following inequality:

$$\int_{\partial H} \frac{|g(x')| \, dx'}{1 + |x'|^\rho(|x'|)^{n+\alpha-1}} < \infty. \quad (10.1.3)$$

We also denote by $(C^H)\alpha$ the set of all continuous functions $u(x)$ in the closed upper half space $\overline{H}$, harmonic in the open upper half space $H$ with the positive part $u^+(x) = \max\{u(x), 0\} \in (L^U)\alpha$ and $u^+(x') = u^+(x', 0) \in (L^V)\alpha$. 

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The Poisson kernel for the upper half space $H$ is the function

$$P(x, y') = \frac{2x_n}{\omega_n |x - y'|^n},$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ is the area of the unit sphere in $\mathbb{R}^n$.

If $u(x) \leq 0$ is harmonic in the open upper half space $H$, continuous in the closed upper half space $\overline{H}$, then (see [19], [36] and [1]) $u \in (CH)_\alpha$ for each $\alpha > 0$ and there exists a constant $c \leq 0$ such that

$$u(x) = cx_n + \int_{\partial H} P(x, y') u(y') dy'$$

for all $x \in H$, the integral in (10.1.4) is absolutely convergent. Motivated by this result, we will prove that if $u \in (CH)_\alpha$, then $u^+ \in (LU)_\alpha, u(x') \in (LV)_\alpha$ and a similar representation to (10.1.4) for the function $u \in (CH)_\alpha$ holds by modifying the Poisson kernel $P_m(x, y')$. It is well known (see [15], [19] and [40]) that the Poisson kernel $P(x, y')$ is harmonic in $x \in \mathbb{R}^n - \{y'\}$ and has a series expansion in terms of the ultraspherical (or Gegenbauer) polynomials $C^\lambda_k(t)$ ($\lambda = \frac{n}{2}$). The latter can be defined in terms of a generating function

$$(1 - 2tr + r^2)^{-\lambda} = \sum_{k=0}^{\infty} C^\lambda_k(t) r^k,$$

where $|r| < 1$, $|t| \leq 1$ and $\lambda > 0$. The coefficients $C^\lambda_k(t)$ is called the ultraspherical (or Gegenbauer) polynomial of degree $k$ associated with $\lambda$, the function $C^\lambda_k(t)$ is a polynomial of degree $k$ in $t$ and satisfies the inequality ([15], p.82 and p.92)

$$|C^\lambda_k(t)| \leq C^\lambda_k(1) = \frac{\Gamma(2\lambda + k)}{\Gamma(2\lambda)\Gamma(k + 1)}, |t| \leq 1.$$  

Therefore, a series expansion of the Poisson kernel $P(x, y')$ in terms of the ultraspherical polynomials $C^\lambda_k(t)$ is

$$P(x, y') = \sum_{k=0}^{\infty} \frac{2x_n |x|^{k}}{\omega_n |y'|^{n+k}} C^\lambda_k\left(\frac{x \cdot y'}{|x||y'|}\right),$$

this series converges for $|x| < |y'|$, each term is homogeneous in $x$ of degree $k + 1$. Differentiating termwise in $x$ gives

$$\triangle_x P(x, y') = \sum_{k=0}^{\infty} \triangle_x \left(\frac{2x_n |x|^{k}}{\omega_n |y'|^{n+k}} C^\lambda_k\left(\frac{x \cdot y'}{|x||y'|}\right)\right).$$
10.2. Main Lemma

Each term \( \Delta_x \left( \frac{2x_n|x|^k}{\omega_n|y'|^{n+k}} C_k^{n/2} \left( \frac{x'}{|x'|} \right) \right) \) is homogeneous in \( x \) of degree \( k - 1 \), hence by the linear independence of homogenous functions, \( \frac{x_n|x|^k}{\omega_n|y'|^{n+k}} C_k^{n/2} \left( \frac{x'}{|x'|} \right) \) is harmonic on \( \mathbb{R}^n \) for each \( k \geq 0 \). If \( m \geq 0 \) is an integer, we define a modified Poisson kernel of order \( m \) for \( x \in H \) by

\[
P_m(x, y') = \begin{cases} 
P(x, y'), & \text{when } |y'| \leq 1, \\
P(x, y') - \sum_{k=0}^{m-1} \frac{2x_n|x|^k}{\omega_n|y'|^{n+k}} C_k^{n/2} \left( \frac{x'}{|x'|} \right), & \text{when } |y'| > 1. \end{cases} \tag{10.1.5}
\]

The modified Poisson kernel \( P_m(x, y') \) is harmonic in \( x \in H \).

Up to now, a number of results about integral representations have been achieved in [10], [48], in this chapter, we will establish the following theorem.

**Theorem 10.1.1** If \( u \in (CH)_\alpha \ (\alpha > 0) \), then the following properties hold:

1. \[
\int_{\partial H} \frac{|u(x')|}{1 + |x'|^{\rho(|x'|)+n+\alpha-1}} dx' < \infty;
\]

2. the integral

\[
\int_{\partial H} P_{|\rho(|y'|)+\alpha|}(x, y') u(y') dy'
\]

is absolutely convergent, it represents a harmonic function \( u_H(x) \) in \( H \) and can be continuously extended to \( \overline{H} \) such that \( u_H(y') = u(y') \) for \( y' \in \partial H \);

3. There exists a harmonic function \( h(x) \) which vanishes on the boundary \( \partial H \) such that \( u(x) = h(x) + u_H(x) \) for all \( x \in \overline{H} \).

### 10.2. Main Lemma

In order to obtain the result, we need the lemma below:

**Lemma 10.2.1** For any \( y' \in \partial H \) and \( |x| > 1, x_n > 0 \), the following inequalities

\[
|P_m(x, y')| \leq \begin{cases} 
\frac{(2^{m+n+1} \omega_{x_n}^{m-1} 2^m C_m^{1/2}(1)) |x|^{n+m-1}}{\omega_n x_n^{n-1} |y'|^{n+m}}, & \text{when } 1 < |y'| \leq 2|x|, \\
\frac{2^{m+n} x_n^{n-1} |y'|^m}{\omega_n |y'|^{n+m}}, & \text{when } |y'| > \max\{1, 2|x|\}, \\
\frac{1}{\omega_n x_n^{n-1}}, & \text{when } |y'| \leq 1
\end{cases}
\]

hold.
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Proof: When \( y' \in \partial H, |y'| \leq 1 \), we have \( |x - y'| \geq |x_n| \) and so

\[
|P_m(x, y')| \leq \frac{2}{\omega_n x_n^{n-1}};
\]

when \( y' \in \partial H, 1 < |y'| \leq 2|x| \), we also have \( |x - y'| \geq |x_n| \) and so by (10.1.5)

\[
|P_m(x, y')| \leq \frac{2x_n}{\omega_n |x - y'|^n} + \sum_{k=0}^{m-1} \frac{2x_n |x|^k}{\omega_n |y'|^{n+k} C_k^{n/2}} \tag{1}
\]

\[
\leq \frac{2}{\omega_n x_n^{n-1}} \left( 1 + \frac{x_n^m |x|^k}{|y'|^{n+k} C_k^{n/2}} \right) \tag{1}
\]

\[
\leq \frac{(2m + m^2 m^{n/2} - m - 1)}{\omega_n x_n^{n-1} |y'|^{n+m}}.
\]

when \( |y'| > \max\{1, 2|x|\} \), we have by (10.1.5)

\[
|P_m(x, y')| = \left| \sum_{k=m}^{\infty} \frac{2x_n |x|^k}{\omega_n |y'|^{n+k} C_k^{n/2} |x'|^{n/2}} \left( \frac{x \cdot y'}{|x||y'|} \right) \right|
\]

\[
\leq \frac{2}{\omega_n} \sum_{k=m}^{\infty} \frac{x_n |x|^k}{|y'|^{n+k} C_k^{n/2}} \tag{1}
\]

\[
\leq \frac{2m + m^2 m^{n-1}}{\omega_n |y'|^{n+m}}.
\]

This proves the inequalities.

10.3 Proof of Theorem

If \( u \in (CH)_{\alpha} (\alpha > 0) \), suppose \( R > 1 \), then by the Carleman formula for harmonic functions in the upper half space,

\[
\int_{\{x \in \mathbb{R}^n : |x| = R, x_n > 0\}} u(x) \frac{x_n}{R^{n+1}} d\sigma(x)
\]

\[
+ \int_{\{x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0\}} u(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' = c_1 + \frac{c_2}{R^n},
\]

where

\[
c_1 = \int_{\{x \in \mathbb{R}^n : |x| = 1, x_n > 0\}} \left[ (n - 1)x_n u(x) + x_n \frac{\partial u(x)}{\partial n} \right] d\sigma(x),
\]

\[
c_2 = \int_{\{x \in \mathbb{R}^n : |x| = 1, x_n > 0\}} \left[ x_n u(x) - x_n \frac{\partial u(x)}{\partial n} \right] d\sigma(x).
\]

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Set
\[ m_+(R) = \int_{\{x \in \mathbb{R}^n : |x| = R, x_n > 0\}} u^+(x) \frac{nx_n}{R^{n+1}} d\sigma(x), \]
\[ m_-(R) = \int_{\{x \in \mathbb{R}^n : |x| = R, x_n > 0\}} u^-(x) \frac{nx_n}{R^{n+1}} d\sigma(x), \]
then
\[ m_-(R) + \int_{\{x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0\}} u^-(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \]
\[ = m_+(R) + \int_{\{x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0\}} u^+(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' - c_1 - \frac{c_2}{R^n} \]  
(10.3.1)
where \( u^+(x) = \max\{u(x), 0\}, u^-(x) = (-u(x))^+ \) and \( u(x) = u^+(x) - u^-(x) \).

Since \( u \in (CH)_\alpha \), we obtain by (10.1.2)
\[ \int_1^\infty \frac{m_+(R)}{R^{p(R)+\alpha}} dR = n \int_D \frac{x_n u^+(x)}{|x|^{p(|x|)+n+\alpha+1}} dx < \infty, \]  
(10.3.2)
where \( D = \{ x \in H : |x| > 1 \} \).

By (10.1.3), we can also obtain
\[ \int_1^\infty \frac{1}{R^{p(R)+\alpha}} \int_{\{x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0\}} \frac{u^+(x')}{|x'|^n} \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx'dR \]
\[ = \int_{|x'| \geq 1} u^+(x') \int_{|x'|}^\infty \frac{1}{R^{p(R)+\alpha}} \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dR dx' \]
\[ \leq \frac{n}{n+1} \int_{|x'| \geq 1} \frac{u^+(x')}{|x'|^{p(|x'|)+n+\alpha-1}} dx' < \infty. \]  
(10.3.3)
Similarly, we have
\[ \int_1^\infty \frac{1}{R^{p(R)+\alpha/2}} \int_{\{x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0\}} \frac{u^-(x')}{|x'|^n} \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx'dR \]
\[ = \int_{|x'| \geq 1} u^-(x') \int_{|x'|}^\infty \frac{1}{R^{p(R)+\alpha/2}} \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dR dx'. \]  
(10.3.4)
So we have by (10.3.1), (10.3.2) and (10.3.3)
\[ \int_1^\infty \frac{1}{R^{p(R)+\alpha/2}} \int_{\{x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0\}} \frac{u^-(x')}{|x'|^n} \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx'dR \]
\[ \leq \int_1^\infty \frac{1}{R^{p(R)+\alpha/2}} m_+(R) dR \]
\[ + \int_1^\infty \frac{1}{R^{p(R)+\alpha/2}} \left[ \int_{\{x \in \mathbb{R}^n : 1 < |x'| < R, x_n = 0\}} u^+(x') \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \right] dR \]
\[ - \int_1^\infty \frac{1}{R^{p(R)+\alpha/2}} \left( c_1 + \frac{c_2}{R^n} \right) dR < \infty. \]  
(10.3.5)
∀\(\alpha > 0\), set
\[
I(\alpha) = \lim_{|x'| \to \infty} \int_{|x'|}^{\infty} \frac{1}{R^{\alpha/2}} \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dR \quad \left| x' \left( \rho(|x'|) + n + \alpha - 1 \right) \right|
\]

by the L’hospital’s rule and (10.1.1), we have
\[
I(\alpha) = +\infty.
\]

Therefore, there exists \(\varepsilon_1 > 0\), such that for any \(|x'| \geq 1\),
\[
\int_{|x'|}^{\infty} \frac{1}{R^{\alpha/2}} \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dR \geq \frac{\varepsilon_1}{|x'|^{\rho(|x'|) + n + \alpha - 1}}.
\]

Multiplying this by \(u^{-}(x')\) and integrating with respect to \(x'\), we can obtain by (10.3.4) and (10.3.5)
\[
\varepsilon_1 \int_{|x'|}^{\infty} u^{-}(x') \frac{u^{-}(x')}{|x'|^{\rho(|x'|) + n + \alpha - 1}} dx' \leq \int_{|x'|}^{\infty} u^{-}(x') \int_{|x'|}^{\infty} \frac{1}{R^{\alpha/2}} \left( \frac{1}{|x'|^n} - \frac{1}{R^n} \right) dRdx' < \infty.
\]

Thus
\[
\int_{|x'|}^{\infty} \frac{u^{-}(x')}{|x'|^{\rho(|x'|) + n + \alpha - 1}} dx' < \infty.
\]

by (10.3.3), we have
\[
\int_{|x'|}^{\infty} \frac{u^{+}(x')}{|x'|^{\rho(|x'|) + n + \alpha - 1}} dx' < \infty.
\]

Hence (1) holds.

∀\(\alpha > 0, R > 1\), \(\exists M(R) > 0\), such that for any \(k > k_R = [2R] + 1\), we have
\[
\frac{(2R)^{\rho(k+1)+\alpha+1}}{k^{\alpha/2}} \leq M(R),
\]
so ∀\(\alpha > 0, R > 1\), if \(|x| \leq R, k > k_R = [2R] + 1\), then \(|y'| \geq 2|x|\) and
\[
\sum_{k=k_R}^{\infty} \int_{k \leq |y'| < k+1} \frac{(2|x|)^{\rho(|y'|)+\alpha+1}}{|y'|^{\rho(|y'|)+\alpha+n}} |u(y')| dy' \leq \sum_{k=k_R}^{\infty} \frac{(2R)^{\rho(k+1)+\alpha+1}}{k^{\alpha/2}} \int_{k \leq |y'| < k+1} \frac{2|u(y')|}{1 + |y'|^{\rho(|y'|)+\alpha/2+(n-1)}} dy' \leq 2M(R) \int_{|y'| \geq k_R} \frac{|u(y')|}{1 + |y'|^{\rho(|y'|)+\alpha/2+(n-1)}} dy'.
\]
So the integral is absolutely convergent.

To verify the boundary behavior of $u_H(x)$, fix a boundary point $a' = (a_1, a_2, \cdots, a_{n-1}) \in \mathbb{R}^{n-1}$, choose a large $T > |a'| + 1$, and write

$$u_H(x) = \int_{|y'| \leq T} P(x, y') u(y') dy'$$

$$- \sum_{k=0}^{[p(|y'| + \alpha)]-1} \frac{2x_n |x|^k}{\omega_n} \int_{1 < |y'| \leq T} \frac{1}{|y'|^{n+k}} C_{k}^{n/2} \left( \frac{x' \cdot y'}{|x||y'|} \right) u(y') dy'$$

$$+ \int_{|y'| > T} P_{[p(|y'| + \alpha)]}(x, y') u(y') dy'$$

$$= X(x) - Y(x) + Z(x).$$

Consider $x \rightarrow a'$, the first term $X(x)$ approaches $u(a')$ because it is the Poisson integral of $u(y') \chi_{B(T)}(y')$, where $\chi_{B(T)}$ is the characteristic function of the ball $B(T) = \{y' \in \mathbb{R}^{n-1} : |y'| \leq T \}$; the second term $Y(x)$ is a polynomial times $x_n$ and tends to 0; and the third term $Z(x)$ is $O(x_n)$ and therefore also to 0. So the function $u_H(x)$ can be continuously extended to $\overline{H}$ such that $u_H(y') = u(y')$; consequently, $u(x) - u_H(x)$ is harmonic in $H$ and can be continuously extended to $\overline{H}$ with 0 in the boundary $\partial H$ of $H$. The Schwarz reflection principle ([1], p.68 and [17], p.28) applied to $u(x) - u_H(x)$ shows that there exists a harmonic function $h(x)$ in $\mathbb{R}^n$ such that $h(x^*) = -h(x) = -(u(x) - u_H(x))$ for $x \in \overline{H}$, where $x^* = (x', -x_n)$ is the reflection of $x$ in $\partial H$. Therefore, if $\alpha > 0$, $h(x)$ is a harmonic function which vanishes on the boundary $\partial H$ such that $u(x) = h(x) + u_H(x)$ for all $x \in \overline{H}$. This completes the proof of Theorem.
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