Relative Growth of Series in Systems of Functions and Laplace–Stieltjes-Type Integrals

Myroslav Sheremeta

Department of Mechanics and Mathematics, Ivan Franko National University of Lviv, 79000 Lviv, Ukraine; m.m.sheremeta@gmail.com

Abstract: For a regularly converging-in-C series \( A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) \), where \( f \) is an entire transcendental function, the asymptotic behavior of the function \( M_f^{-1}(\lambda_n) \), where \( M_f(r) = \max\{|f(z)| : |z| = r\} \), is investigated. It is proven that, under certain conditions on the functions \( f, a, \) and the coefficients \( a_n \), the equality \( \lim_{r \to +\infty} a(M_f^{-1}(\lambda_n)) = 1 \) is correct. A similar result is obtained for the Laplace–Stieltjes-type integral \( I(r) = \int_0^\infty a(x)f(rx)dF(x) \). Unresolved problems are formulated.

Keywords: relative growth; entire function; regularly converging series; Mittag–Leffler function

MSC: 30B50; 30D10; 30D20

1. Introduction

Let

\[ f(z) = \sum_{k=0}^{\infty} f_k z^k \]

be an entire function, \( M_f(r) = \max\{|f(z)| : |z| = r\} \), and \( \Phi_f(r) = \ln M_f(r) \). For an entire function \( g \) with Taylor coefficients \( g_n \), the study of growth of the function \( \Phi_f^{-1}(\ln M_g(r)) \) in terms of the exponential type was initiated in papers [1,2] and was continued in [3]. As a result, it is proven that, if \(|f_{k-1}/f_k| \nearrow +\infty \) as \( k \to \infty \), then

\[ \lim_{r \to +\infty} \frac{\Phi_f^{-1}(\ln M_g(r))}{r} = \lim_{k \to \infty} \left( \frac{g_n}{f_k} \right)^{1/n}. \]

We remark that \( \Phi_f^{-1}(x) = M_f^{-1}(e^x) \) and, thus, \( \Phi_f^{-1}(\ln M_g(r)) = M_f^{-1}(M_g(r)) \). The order \( \rho[g] = \lim_{r \to +\infty} \ln M_f^{-1}(M_g(r))/\ln r \) and the lower-order \( \lambda[g] = \lim_{r \to +\infty} \ln M_f^{-1}(M_g(r))/\ln r \) of the function \( f \) with respect to the function \( g \) are used in Reference [4].

Research on the relative growth of entire functions was continued by many mathematicians (an incomplete bibliography is given in [5]).

Let \((\lambda_n)\) be a sequence of positive numbers increasing to +\( \infty \). Suppose that the series

\[ A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) \]

in the system \( f(\lambda_n z) \) is regularly convergent in \( \mathbb{C} \), i.e., \( \sum_{n=1}^{\infty}|a_n|M_f(r\lambda_n) < +\infty \) for all \( r \in [0, +\infty) \). Many authors have studied the representation of analytic functions by series in the system \( f(\lambda_n z) \) and the growth of such functions. Here, we specify only the monographs of A.F. Leont’ev [6] and B.V. Vinnitskyi [3], which are references to other papers on this topic.

Since series (2) is regularly convergent in \( \mathbb{C} \) and the function \( A \) is an entire function, a natural question arises about the asymptotic behavior of the function \( M_f^{-1}(M_A(r)) \).

Citation: Sheremeta, M. Relative Growth of Series in Systems of Functions and Laplace–Stieltjes-Type Integrals. Axioms 2021, 10, 43. https://doi.org/10.3390/axioms10020043

Publisher’s Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.
We suppose that the function $F$ is nonnegative, nondecreasing, unbounded, and continuous on the right on $[0, +\infty)$; that $f$ is positive, increasing, and continuous on $[0, +\infty)$; and that a positive-on-$[0, +\infty)$ function $a$ is such that the Laplace–Stieltjes-type integral

$$I(r) = \int_0^\infty a(x)f(rx)dF(x)$$

exists for every $r \in [0, +\infty)$. The asymptotic behavior of such integrals in the case $f(x) = e^x$ is studied in the monograph [7]. A question arises again about the asymptotic behavior of the function $f^{-1}(I(r))$. Here, we present some results that indicate the possibility of solving these problems.

2. Relative Growth of Series in Systems of Functions

As in [8], by $L$ we denote a class of continuous nonnegative-on-$(-\infty, +\infty)$ functions $a$ such that $a(x) = a(x_0) \geq 0$ for $x \leq x_0$ and $a(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $a \in L^0$, if $a \in L$ and $a((1 + o(1))x) = (1 + o(1))a(x)$ as $x \to +\infty$. Finally, $a \in L_{si}$, if $a \in L$ and $a(cx) = (1 + o(1))a(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i.e., $a$ is a slowly increasing function. Clearly, $L_{si} \subset L^0$. We need the following lemma [9].

**Lemma 1.** If $\beta \in L$ and $B(\delta) = \lim_{x \to +\infty} \frac{\beta((1+\delta)x)}{\beta(x)}$, $\delta > 0$, then in order for $\beta \in L^0$, it is necessary and sufficient that $B(\delta) \to 1$ as $\delta \to +0$.

We need also some well-known (see, for example, [10]) properties of the function $\ln M_f(r)$.

**Lemma 2.** If a function $f$ is transcendental, then the function $\ln M_f(r)$ is logarithmically convex and, thus,

$$\Gamma_f(r) := \frac{d\ln M_f(r)}{dr} \nearrow +\infty, \quad r \to +\infty,$$

(at the points where the derivative does not exist, where $\frac{d\ln M_f(r)}{dr}$ means the right-hand derivative).

For $\alpha \in L, \beta \in L$, and entire functions $f$ and $g$, we define the generalized $(\alpha, \beta)$-order $\rho_{\alpha, \beta}[g]_f$ and the generalized lower $(\alpha, \beta)$-order $\lambda_{\alpha, \beta}[g]_f$ of $g$ with respect to $f$ as follows:

$$\rho_{\alpha, \beta}[g]_f = \lim_{r \to +\infty} \frac{\alpha(M_f^{-1}(M_g(r)))}{\beta(r)}, \quad \lambda_{\alpha, \beta}[g]_f = \lim_{r \to +\infty} \frac{\alpha(M_f^{-1}(M_g(r)))}{\beta(r)}.$$  

Suppose that $a_n \geq 0$ for all $n \geq 1$. Since

$$A(z) = \sum_{n=1}^{\infty} a_n \sum_{k=0}^{\infty} f_k(z\lambda_n)^k = \sum_{k=0}^{\infty} f_k \left( \sum_{n=1}^{\infty} a_n \lambda_n^k \right) z^k,$$

in view of the Cauchy inequality, we have

$$M_A(r) \geq |f_k| \left( \sum_{n=1}^{\infty} a_n \lambda_n^k \right)^k \geq a_n |f_k| (\lambda_n r)^k$$

for all $n \geq 1, k \geq 0$ and $r \in [0, +\infty)$. We also remark that, if $\mu_f(r) = \max\{|f_k| r^k : k \geq 0\}$ is the maximal term of series (1), then

$$M_f(r) \leq \sum_{k=0}^{\infty} |f_k| r^k = \sum_{k=0}^{\infty} |f_k| (2r)^k 2^{-k} \leq 2 \mu_f(2r).$$
We choose \( n_0 \geq 1 \) such that \( a_{n_0} > 0 \) and \( \lambda_{n_0} \geq 2 \). Then, from (4) and (5), we get

\[
M_A(r) \geq \max\{a_{n_0}|f_k|(r\lambda_n)^k : k \geq 0\} \geq a_{n_0}\mu_f(2r) \geq \frac{a_{n_0}}{2}M_f(r),
\]

where \( M_f^{-1}\left(\frac{2}{2a_0}M_A(r)\right) \geq r \). By Lemma 2, \( \frac{d\ln M_f^{-1}(x)}{d\ln x} \rightarrow 0 \) as \( x \rightarrow +\infty \) and, thus, for every \( c > 1 \)

\[
\ln M_f^{-1}(cx) - \ln M_f^{-1}(x) = \int_x^{cx} \frac{d\ln M_f^{-1}(t)}{d\ln t} dt \leq \frac{d\ln M_f^{-1}(x)}{d\ln x} \rightarrow 0, \ x \rightarrow +\infty,
\]

i.e., the function \( M_f^{-1} \) is slowly increasing. Therefore,

\[
M_f^{-1}(M_A(r)) \geq (1 + o(1))r, \ r \rightarrow +\infty. \tag{6}
\]

On the other hand, since series (2) is regularly convergent in \( \mathbb{C} \), for each \( r \in [0, +\infty) \), there exists \( \mu_A(r) = \max\{|a_n|M_f(r\lambda_n) : n \geq 1\} \) and, for every \( r \in [0, +\infty) \) and \( \tau > 0 \), we have

\[
M_A(r) \leq \sum_{n=1}^{\infty} |a_n|M_f(r\lambda_n) \leq \mu_f((1+\tau)r)\sum_{n=1}^{\infty} \frac{M_f(r\lambda_n)}{M_f((1+\tau)r\lambda_n)}. \tag{7}
\]

Then, by Lemma 2, for \( r \geq 1 \), we have

\[
\ln M_f((1+\tau)r\lambda_n) - \ln M_f(r\lambda_n) = \int_{r\lambda_n}^{(1+\tau)r\lambda_n} \frac{d\ln M_f(x)}{d\ln x} dx = \int_{r\lambda_n}^{(1+\tau)r\lambda_n} \Gamma_f(x)dx \geq \Gamma_f(r\lambda_n)\ln(1+\tau) \geq \Gamma_f(\lambda_n)\ln(1+\tau).
\]

Therefore, if \( \ln n \leq q\Gamma_f(\lambda_n) \) for all \( n \geq n_0 \) and \( \ln(1+\tau) > q \), then

\[
\sum_{n=n_0}^{\infty} \frac{M_f(r\lambda_n)}{M_f((1+\tau)r\lambda_n)} \leq \sum_{n=n_0}^{\infty} \exp\{-\Gamma_f(\lambda_n)\ln(1+\tau)\} \leq \sum_{n=n_0}^{\infty} \exp\{-\ln(1+\tau)/q\ln n\} < +\infty
\]

and (7) implies, for \( r \geq 1 \),

\[
M_A(r) \leq T\mu_A((1+\tau)r), \ T = \text{const} > 0. \tag{8}
\]

Additionally, we have

\[
\mu_A(r) \leq \max\{\left|\sum_{k=0}^{\infty} \lambda_n^k |f_k|(r\lambda_n)^k : n \geq 1\right\} \leq \sum_{k=0}^{\infty} \max\{|a_n|\lambda_n^k : n \geq 1\}|f_k|r^k = \sum_{k=0}^{\infty} \mu_D(\sigma)|f_k|r^k, \tag{9}
\]

where \( \mu_D(\sigma) = \max\{|a_n|\exp\{\sigma\ln \lambda_n\} : n \geq 1\} \) is the maximal term of Dirichlet series

\[
D(\sigma) = \sum_{n=1}^{\infty} |a_n|\exp\{\sigma\ln \lambda_n\}.
\]

Using estimates (6), (8), and (9), we prove the following theorem.

**Theorem 1.** Let \( f \) be an entire transcendental function, \( a_n \geq 0 \) for all \( n \geq 1 \), and series (2) be regularly convergent in \( \mathbb{C} \). Suppose that \( \ln n \leq q\Gamma_f(\lambda_n) \) for some \( q > 0 \) and all \( n \geq n_0 \) and that \( \lim_{r \to +\infty} \frac{\ln \mu_f(\sigma)}{\sigma\ln M_f((1+\tau)r)} = \gamma \).

If \( \gamma < 1 \), then \( \lambda_{n,A}[f] = \rho_{n,A}[f] = 1 \) for every function \( \alpha \) such that \( \alpha(\sigma^2) \in L_{\alpha} \). If \( \gamma = 0 \), then \( \lambda_{n,A}[f] = \rho_{n,A}[f] = 1 \) for every function \( \alpha \) such that \( \alpha(\sigma^2) \in L_0 \).
Proof. Since \( a \in L^0 \), from (6), we get

\[
\lambda_{n,a}[F]_f = \lim_{r \to +\infty} \frac{a(M_f^{-1}(M_f(r))))}{a(r)} \geq \lim_{r \to +\infty} \frac{a((1 + o(1))r)}{a(r)} = 1.
\]

On the other hand, in view of the Cauchy inequality, we have \( \ln |f_k| \leq \ln M_f(r) - k \ln r \) for all \( r \) and \( k \). We choose \( r = r_k = M_f^{-1}(e^k) \). Then, \( \ln |f_k| \leq k - k \ln M_f^{-1}(e^k) \), i.e., \( -\ln |f_k| \geq k(\ln M_f^{-1}(e^k) - 1) \). Therefore,

\[
\lim_{k \to +\infty} \frac{\ln \mu_D(k)}{\ln f_k} \leq \lim_{k \to +\infty} \frac{\ln \mu_D(k)}{k(\ln M_f^{-1}(e^k) - 1)} \leq \lim_{\sigma \to +\infty} \frac{\ln \mu_D(\sigma)}{\sigma \ln M_f^{-1}(e^\sigma)} = \sigma.
\]  

(10)

If \( \gamma < 1 \), then in view of (10), \( \frac{\ln \mu_D(k)}{\ln f_k} \leq p \) for each \( p \in (\gamma, 1) \) and all \( k \geq k_0 \) and, thus, \( \mu_D(k) \leq |f_k|^{-p} \) for all \( k \geq k_0 \). Therefore, in view of (9) and (5),

\[
\mu_A(r) \leq \left( \sum_{k=0}^{k_0-1} + \sum_{k=k_0}^\infty \right) \mu_D(k)|f_k|^k \leq O(r^{k_0-1}) + \sum_{k=k_0}^\infty |f_k|^{-p} = \leq O(r^{k_0-1}) + 2 \max\{ |f_k|^{-p} : k \geq 0 \} = \leq = O(r^{k_0-1}) + 2 \max\{ (|f_k||2r|^{k(1-p)})^{-1-p} : k \geq 0 \} = \leq \leq O(r^{k_0-1}) + 2(\mu f((2r)^{(1/p)})^{-1-p} \leq \mu f((2r)^{(1/p)}), r \to r_0.
\]  

(11)

because \( \ln r = o(\ln \mu_f(r)) \) as \( r \to +\infty \) for every entire transcendental function \( f \) and \( 1 - p < 0 \). Therefore, from (8) and (11), we get

\[
M_A(r) \leq T_M((1 + \tau)r) \leq T_M(((2 + \tau)x(r)_{1/p}) \leq T_M((2 + \tau)x(r)_{1/p})
\]

and, thus, \( M_f^{-1}(M_A(r)) \leq (1 + o(1))((2 + \tau)x(r)_{1/p}) \Rightarrow a \to +\infty \) as \( a \in L_{2i} \), then we obtain

\[
\lim_{r \to +\infty} \frac{a(M_f^{-1}(M_A(r)))}{a(r)} = 1.
\]  

(12)

Suppose that \( a(e^{x}) \in L_{2i} \). Then,

\[
a(r^{1/p}) = a(\exp\left\{ \frac{1}{1-p} \ln r \right\}) = (1 + o(1))a(\exp(\ln r)) = (1 + o(1))a(r)
\]

as \( r \to +\infty \). Therefore, (12) implies the inequality \( \rho_{n,a}[A]_f \leq 1 \), where in view of the inequality \( \lambda_{n,a}[A]_f \geq 1 \), we get \( \lambda_{n,a}[A]_f = \rho_{n,a}[A]_f = 1 \).

If \( \gamma = 0 \), then (12) holds for every \( p \in (0, 1) \) and all \( r \geq r_0(p) \). If we put \( \frac{1}{1-p} = 1 + \delta \), then \( \delta \to +0 \) as \( p \to +0 \), and in view of the condition \( a(e^{x}) \in L^0 \), by Lemma 1, we have

\[
\lim_{r \to +\infty} \frac{a(r^{1/p})}{a(r)} = \lim_{r \to +\infty} \frac{a(\exp\{ (1 + \delta) \ln r \})}{a(\exp(\ln r))} = B(\delta) \to 1, \delta \to 1.
\]

Therefore,

\[
1 \geq \lim_{r \to +\infty} \frac{a(M_f^{-1}(M_A(r)))}{a(r)} \geq \lim_{r \to +\infty} \left( \frac{a(M_f^{-1}(M_A(r)))}{a(r)} \cdot \frac{a(r)}{a(r^{1/p})} \right) \geq \lim_{r \to +\infty} \frac{a(M_f^{-1}(M_A(r)))}{a(r)} \lim_{r \to +\infty} \frac{a(r)}{a(r^{1/p})} = \rho_{n,a}[F]_f \cdot B(\delta).
\]
In view of the arbitrariness of $\delta$, we get $\rho_{\alpha,\alpha}[A]_f \leq 1$, and again, $\lambda_{\alpha,\alpha}[A]_f = \rho_{\alpha,\alpha}[A]_f = 1$. Theorem 1 is proven. □

We remark that, if $f_k \geq 0$ for all $k \geq 0$, then $M_f(r) = f(r)$. Therefore, from Theorem 1, we obtain the following statement.

**Corollary 1.** Let $f$ be an entire transcendental function, $f_k \geq 0$ for all $k \geq 0$, $a_n \geq 0$ for all $n \geq 1$, and series (2) be regularly convergent in $\mathbb{C}$. Suppose that $f'(r)/f(r) \geq h > 0$ for all $r \geq r_0$, then $n \ln n + \sum_{n}^{\infty} \ln f(n+1) < \infty$.

If $\gamma < 1$, then $\lambda_{\alpha,\alpha}[A]_f = \rho_{\alpha,\alpha}[A]_f = 1$ for every function $a$ such that $a(e^s) \in L^1$.

If $\gamma = 0$, then $\lambda_{\alpha,\alpha}[A]_f = \rho_{\alpha,\alpha}[A]_f = 1$ for every function $a$ such that $a(e^s) \in L^0$.

### 3. Relative Growth of Laplace–Stieltjes-Type Integrals

Suppose again that $f$ is an entire transcendental function $f_k \geq 0$ for all $k \geq 0$, and $x_0 > 1$ is such that $\int_{1}^{x_0} a(x) dF(x) \geq 0$. Then,

$$I(r) \geq \int_{1}^{x_0} a(x)f(rx) dF(x) \geq f(r)c,$$

i.e., as above, $f^{-1}(I(r)) \geq (1 + o(1))r$ as $r \rightarrow +\infty$, where for $a \in L^0$,

$$\lambda_{\alpha,\alpha}[I]_f = \lim_{r \rightarrow +\infty} \frac{a(f^{-1}(I(r)))}{a(r)} \geq 1.$$

On the other hand, if $\tau \geq e - 1$, then as above, for $r \geq 1$, we have

$$\ln f((1 + \tau)rx) - \ln f(rx) = \int_{rx}^{(1+\tau)rx} \frac{d\ln f(x)}{d\ln x} d\ln x = \int_{rx}^{(1+\tau)rx} \Gamma_f(x)d\ln x \geq \Gamma_f(x) \ln(1 + x),$$

i.e., $f(rx) \leq e^{-\Gamma_f(x)\ln(1+\tau)}$. Therefore, if $\mu_1(r) = \max\{a(x)f(rx) : x \geq 0\}$ is the maximum of the integrand and $\ln F(x) \leq q\Gamma_f(x)$ for some $q > 0$ and all $x \geq x_0$, then for $\ln(1 + \tau) > q$ (for simplicity assuming $x_0 = 0$), we get

$$I(r) = \int_{0}^{\infty} a(x)f((1 + \tau)rx) \frac{f(rx)}{f((1 + \tau)rx)} dF(x) \leq \mu_1((1 + \tau)r) \int_{0}^{\infty} \frac{f(rx)}{f((1 + \tau)rx)} dF(x) \leq \mu_1((1 + \tau)r) \int_{0}^{\infty} e^{-\Gamma_f(x)\ln(1+\tau)} dF(x) \leq \mu_1((1 + \tau)r) \ln(1 + \tau) \int_{0}^{\infty} e^{-\Gamma_f(x)\ln(1+\tau) - q} d\Gamma_f(x) \leq \mu_1((1 + \tau)r) \ln(1 + \tau) \int_{0}^{\infty} e^{-\Gamma_f(x)\ln(1+\tau) - q} d\Gamma_f(x) = \mu_1((1 + \tau)r) \ln(1 + \tau) \ln(1 + \tau - q) = T\mu_1((1 + \tau)r).$$

Additionally, as above, we have

$$\mu_1(r) = \max\left\{a(x) \sum_{k=0}^{\infty} f_k(xr)^k : x \geq 0\right\} \leq \sum_{k=0}^{\infty} \max\{a(x)x^k : x \geq 0\} f_kr^k = \sum_{k=0}^{\infty} \mu_1(k)f_kr^k,$$

where $\mu_1(k)$ is the maximum of the integrand.
where \( \mu_f(\sigma) = \max\{a(x)e^{\sigma \ln x} : x \geq 0\} = \max\{a(x)x^{\ln x} : x \geq 0\} \) is the maximum of the integrand for the Laplace integral

\[
J(\sigma) = \int_0^\infty a(x)e^{\sigma \ln x}dF(x).
\]

Using estimates (13) and (14), and \( \lambda_{\alpha,a}[l]_f \geq 1 \), we prove the following analog of Theorem 1.

**Theorem 2.** Let \( \ln F(x) \leq q \Gamma_f(x) \) for some \( q > 0 \) and all \( x \geq x_0 \), and \( \lim_{r \to +\infty} \frac{\ln \mu_f(\sigma)}{\ln f^{-1}(e^r)} = \gamma \).

If \( \gamma < 0 \), then \( \lambda_{\alpha,a}[l]_f = \rho_{\alpha,a}[l]_f = 1 \) for every function \( a \) such that \( a(e^x) \in L_{\alpha,a} \).

If \( \gamma = 0 \), then \( \lambda_{\alpha,a}[l]_f = \rho_{\alpha,a}[l]_f = 1 \) for every function \( a \) such that \( a(e^x) \in L_{\alpha,0} \).

**Proof.** As in the proof of Theorem 1, we obtain \(-\ln|f_k| \geq k(\ln f^{-1}(e^k) - 1)\) and \( \lim_{k \to \infty} \frac{\ln \mu_f(k)}{-\ln f^{-1}(e^k)} \leq \gamma \). Therefore, if \( \gamma < 1 \), then \( \mu_f(k) \leq |f_k|^{-\frac{1}{1-p}} \) for each \( p \in (\gamma, 1) \) and all \( k \geq k_0 \), and in view of (14) and (5), as in the proof of Theorem 1, we get \( \mu_f(2r)^{1/(1-p)} \) for \( r \geq r_0 \). Therefore, in view of (13), we get

\[
I(r) \leq T \mu_f((1+\tau)r) \leq T f((2+1+\tau)r)^{1/(1-p)},
\]

where \( f^{-1}(I(r)) \leq (1 + o(1))(2(1+\tau)r)^{1/(1-p)} \) as \( r \to +\infty \). If \( a \in L_{\alpha,1} \), then we obtain

\[
\lim_{r \to +\infty} \frac{a(f^{-1}(I(r)))}{a(r)^{1/(1-p)}} \leq 1.
\]

Further proof of Theorem 2 is the same as that of Theorem 1. \( \square \)

Theorem 2 implies the following statement.

**Corollary 2.** Let \( f'(x)/f(x) \geq h, h > 0 \), \( \ln F(x) \leq qx \) for some \( q > 0 \) and all \( x \geq 0 \), and \( \lim_{r \to +\infty} \frac{\ln \mu_f(\sigma)}{\ln f^{-1}(e^r)} = \gamma \).

If \( \gamma < 0 \), then \( \lambda_{\alpha,a}[l]_f = \rho_{\alpha,a}[l]_f = 1 \) for every function \( a \) such that \( a(e^x) \in L_{\alpha,a} \).

If \( \gamma = 0 \), then \( \lambda_{\alpha,a}[l]_f = \rho_{\alpha,a}[l]_f = 1 \) for every function \( a \) such that \( a(e^x) \in L_{\alpha,0} \).

4. **Examples**

Here, we consider the case when \( f(z) = E_\rho(z) \), where

\[
E_\rho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k/\rho)}, \quad 0 < \rho < +\infty,
\]

is the Mittag–Leffler function. The properties of this function have been used in many problems in the theory of entire functions. We only need the following property of the Mittag–Leffler function: if \( 0 < \rho < +\infty \), then ([11] p. 85)

\[
M_{E_\rho}(r) = E_\rho(r) = (1 + o(1))r^\rho e^{\rho r}, \quad r \to +\infty \tag{15}
\]

and, if \( 1/2 < \rho < +\infty \), then ([12])

\[
E'_\rho(r)/E_\rho(r) = \rho r^{\rho-1} + O(r^{\rho-2}e^{-\rho r}), \quad r \to +\infty \tag{16}
\]
From (15), it follows that $E^{-1}_ρ(x) = (1 + o(1)) \ln^{1/ρ} x$ as $x → +∞$. Therefore, for $f(x) = E_ρ(x)$, we have $σ \ln f^{-1}(e^σ) = \frac{1+o(1)}{ρ} σ \ln σ$ as $σ → +∞$. Since in (16), $Γ_{ρ_σ}(r) = ρσ^ρ + o(1)$ as $r → +∞$, then if $ln F(x) ≤ qρx^ρ$ for some $q > 0$ and all $x ≥ x_0$, and

$$\lim_{σ → +∞} \frac{ln μ_1(σ)}{σ ln σ} = 0,$$  \hspace{1em} (17)

then for $a(x) = \ln x (x ≥ e)$, by Theorem 2, we get

$$\lim_{r → +∞} \frac{ln E^{-1}_ρ(I_ρ(r))}{ln r} = 1, \quad I_ρ(r) = \int_0^∞ a(x)E_ρ(rx)dF(x).$$ \hspace{1em} (18)

Let us now find out under what conditions (17) holds on $a(x)$. For this, as in ([7] p. 29), by $Ω$, we denote a class of positive unbounded functions $Φ$ on $(-∞, +∞)$ such that the derivative $Φ'₁$ is positive, continuously differentiable, and increasing to $+∞$ on $(-∞, +∞)$. For $Φ ∈ Ω$, let $ϕ$ be the inverse function to $Φ'$ and $Ψ(σ) = σ - \frac{Φ(σ)}{Φ'(σ)}$ be the function associated with $Φ$ in the sense of Newton.

By Theorem 2.2.1 from ([7] p. 30), $\ln max\{a(x)e^{x}: x ≥ 0\} ≤ Φ(σ)$ for all $σ ≥ c_0$ if and only if $ln a(x) ≤ -xΨ(ϕ(x))$ for all $x ≥ x_0$. Choosing $Φ(σ) = cσ \ln σ$ for $σ ≥ c_0$, we obtain $Φ'(σ) = c(\ln σ + 1)$, $ϕ(x) = \exp(x/c - 1)$ and $xΨ(ϕ(x)) = xϕ(x) - Φ(ϕ(x)) = c \exp(x/e - 1)$ for $x ≥ x_0$. Therefore, $\ln μ₁(σ) ≤ cσ \ln σ$ for all $σ ≥ c_0$ if and only if $ln a(x) ≤ -c \exp(ln x/e - 1)$ for $x ≥ x_0$. Hence, it follows that, if $ln x = o(ln ln(1/α(x)))$ as $x → +∞$, then (17) holds. Thus, the following statement is true.

**Proposition 1.** If $ρ > 1/2$, $ln F(x) = O(x^ρ)$ and $ln x = o(ln ln(1/α(x)))$ as $x → +∞$, then (18) holds.

**Remark 1.** If $ρ = 1$, then $E_ρ(r) = E_1(r) = e^r$, and we have a usual Laplace–Stieltjes integral $I_1(r) = \int_0^∞ a(x)e^{x}dF(x)$. Therefore, if $ln F(x) = O(x)$ and $ln x = o(ln ln(1/α(x)))$ as $x → +∞$, then $p_ρ[I_1] := lim_{r → +∞} \frac{ln ln I_1(r)}{ln r} = 1$. On the other hand, the quantity $p_ρ[I_1]$ is called the logarithmic R-order of $I_1$, and in ([7] p. 83), it is proven that, if $ln F(x) = O(x)$ as $x → +∞$, then $p_ρ[I_1] = lim_{x → +∞} \frac{ln a(x)}{ln ln(1/α(x))} = 1$, i.e., if $ln F(x) = O(x)$ and $ln x = o(ln ln(1/α(x)))$ as $x → +∞$, then $p_ρ[I_1] = 1$.

Similarly, we can prove the following statement.

**Proposition 2.** Let $ρ ≥ 1/2$, $ln n = O(λ_n^ρ)$ as $n → ∞$, $α_n ≥ 0$ for all $n ≥ 1$ and series $A_ρ(z) = \sum_{n=1}^∞ a_ν E_ρ(λ_n z)$ be regularly convergent in $C$. If $ln n = o(ln ln(1/α_n))$ as $n → ∞$, then $lim_{r → +∞} \frac{ln E^{-1}_ρ(M_{α_n}(r))}{ln r} = 1$.

**Remark 2.** If $ρ = 1$, then we have a Dirichlet series $A_1(z) = \sum_{n=1}^∞ a_ν e^{λ_n z}$. Therefore, if this Dirichlet series is absolutely convergent in $C$, $α_n ≥ 0$ for all $n ≥ 1$, $ln n = O(λ_n)$, and $ln n = o(ln ln(1/α_n))$ as $n → ∞$, then $p_ρ[A_1] := lim_{r → +∞} \frac{ln ln M_{α_n}(r)}{ln r} = 1$. On the other hand, the quantity $p_ρ[A_1]$ is called the logarithmic R-order of $A_1$ and $p_ρ[A_1] = lim_{n → ∞} \frac{ln λ_n}{ln ln(1/α_n)} = 1$ provided $ln n = O(λ_n)$ as $n → ∞$ [13], i.e., if $ln n = O(λ_n)$ and $ln λ_n = o(ln ln(1/α_n))$ as $n → ∞$, then $p_ρ[A_1] = 1$.

5. Discussion Open Problems

1. The natural problem studied was the relative growth when the domain of regular convergence of series (2) is the disk $D_R = \{z: |z| < K < +∞\}$ and the function $f$ is either entire or analytic in $D_R$. 

Axions 2021, 10, 43 7 of 8
2. It is well known that the study of the growth of entire functions of many complex variables involves many options. The following problem is the simplest.

Let \( f \) be an entire function and the series \( A(z, w) = \sum_{m=1, n=1}^{\infty} a_{m,n} f(\lambda_m z + \mu_n w) \) be regularly convergent in \( \mathbb{C}^2 \). A question arises about the asymptotic behavior of the function \( M_f^{-1}(M_A(r, \rho)) \), where \( M_A(r, \rho) = \max\{|A(z, w)| : |z| \leq r, |w| \leq \rho\} \).

3. The condition \( \rho \geq 1/2 \) in Propositions 1 and 2 arose in connection to the application of Equation (16). Probably, it is superfluous in the above statements.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: This research did not report any data.

Conflicts of Interest: The author declares no conflict of interest.

References
1. Nachbin, L. An extension of the notion of integral function of the finite exponential type. *An. Acad. Brasil Sci.* 1944, 16, 143–147.
2. Boas, R.P.; Buck, R.C. *Polynomial Expansions of Analytic Functions*; Springer: Berlin, Germany, 1958.
3. Vinnitsky, B.V. *Some Approximation Properties of Generalized Systems of Exponentials*; Dep. in UkrNIINTI 25 February 1991; Drogobych Pedagogical Institute: Drogobych, Ukraine, 1991. (In Russian)
4. Roy, C. On the relative order and lower order of an entire function. *Bull. Soc. Cal. Math. Soc.* 2010, 102, 17–26.
5. Mulyava, O.M.; Sheremeta, M.M. Relative growth of Dirichlet series with different abscissas of absolute convergence. *Ukr. Math. J.* 2020, 72, 1535–1543.
6. Leont’ev, A.F. *Generalizations of Exponential Series*; Nauka: Moscow, Russia, 1981. (In Russian)
7. Sheremeta, M.M. *Asymptotical Behavior of Laplace-Stietjes Integrals*; VNTL Publishers: Lviv, Ukraine, 2010.
8. Sheremeta, M.N. Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion. *Izv. Vyssh. Uchebn. Zaved. Mat.* 1967, 2, 100–108. (In Russian)
9. Sheremeta, M.M. On two classes of positive functions and the belonging to them of main characteristics of entire functions. *Mat. Stud.* 2003, 19, 75–82.
10. Pólya, G.; Szegő, G. *Aufgaben und Lehrsätze aus der Analysis. II*; Springer: Berlin, Germany, 1964.
11. Gol’dberg, A.A.; Ostrovskii, I.V. *Value Distribution of Meromorphic Functions*; AMS: Providence, RI, USA, 2008. (Translated from Russian ed. Nauka: Moscow, USSR, 1970).
12. Gol’dberg, A.A. An estimate of modulus of logarithmic derivative of Mittag-Leffler function with applications. *Mat. Stud.* 1995, 5, 21–30.
13. Reddy, A.R. On entire Dirichlet series of zero order. *Tohoku Math. J.* 1966, 18, 144–155. [CrossRef]