QUANTIFYING QUILLEN’S UNIFORM $\mathcal{F}_p$-ISOMORPHISM THEOREM

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Abstract. Let $G$ be a finite group with 2-Sylow subgroup of order less than or equal to 16. For such a $G$, we prove a quantified version of Quillen’s Uniform $\mathcal{F}_p$-isomorphism theorem, which holds uniformly for all $G$-spaces.

We do this by bounding from above the exponent of Borel equivariant $\mathcal{F}_p$-cohomology, as introduced by Mathew-Naumann-Noel, with respect to the family of elementary abelian 2-subgroups.

1. Introduction

For a finite group $G$, consider a cohomology class $u \in H^*(BG; F_p)$ that restricts to zero on all elementary abelian $p$-subgroups (i.e., groups of the form $(\mathbb{Z}/p)^x$). It is a theorem of Quillen that $u$ is nilpotent. In fact, Quillen showed [Qui71, Thm. 6.2]

**Theorem 1.1 (Quillen).** For $X$ any paracompact $G$-space of finite cohomological dimension, the map

$$\tilde{\text{res}}: H^*_G(X; F_p) \to \lim_{E \subset G \text{ el. ab. p-gp.}} H^*_E(X; F_p)$$

where the maps in the indexing category for the limit are given by restricting along subgroups and conjugation, is a uniform $\mathcal{F}_p$-isomorphism, which means that there is an $n$ such that

1. Every $u \in \ker(\tilde{\text{res}})$ satisfies $u^n = 0$.
2. Every $v \in \lim_{E} H^*_E(X; F_p) \setminus \text{Im}(\tilde{\text{res}})$ satisfies $v^{p^n} \in \text{Im}(\tilde{\text{res}})$.

Here $H^*_G(X; F_p)$ denotes the mod-$p$ Borel equivariant cohomology of a $G$-space $X$. These results led to many structural results in group cohomology. Quillen himself immediately deduced [Qui71, Cor. 7.8]

**Corollary 1.2 (Quillen).** The Krull dimension of $H^*(BG; F_p)$ equals the rank of the maximal elementary abelian $p$-subgroup of $G$.

Results directly building on Theorem 1.1 include a theorem of Duflot on the depth of $H^*_G(X; F_p)$ [Duf81, Thm. 1], a theorem on the complexity of $kG$-modules by Alperin-Evens [AE81], Benson’s description of the image of the transfer map [Ben93, Thm. 1.1], and a theorem on the depth of group cohomology rings by Carlson [Car95, Thm. 2.3]. These results, and Quillen’s original result all indicate the importance of the elementary abelian $p$-subgroups in Borel equivariant $\mathcal{F}_p$-cohomology in general, and group cohomology with $\mathcal{F}_p$-coefficients in particular.

It is natural to ask what one can say about the $n$ in Theorem 1.1. One approach to this question is to apply the work of Kuhn [Kuh07, Kuh13], which builds on

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work of Henn-Lannes-Schwartz [HLS95]. In particular, for $p = 2$, this gives explicit upper bounds on the nilpotence degree of the nilpotence degree of the kernel of (1.1), for $X = \text{pt}$. Alternatively, for the case $X = \text{pt}$, one can explicitly compute both sides of (1.1) to determine a feasible $n$.

The case $X \neq \text{pt}$ is not covered by these bounds however. We will consider all $X$, not necessarily $X = \text{pt}$, by using the approach introduced in [MNN15], where the map (1.1) is realized as the edge homomorphism of a homotopy limit spectral sequence, called the $\mathcal{F}$-homotopy limit spectral sequence,

\begin{equation}
E_{s,t}^{2} = \lim_{\mathcal{F}(G)_{\text{el. ab.}}^{2}}H_{H}^{t}(X; F_{p}) \Rightarrow H_{G}^{s}(X; F_{p})
\end{equation}

converging strongly to the target [MNN15, Prop. 2.24], where $X$ can be any $G$-space (or more generally, a $G$-spectrum), and $\mathcal{F}$ any family of subgroups of $G$ which contains at least the family $\mathcal{E}(p)$ of elementary abelian $p$-subgroups. The indexing category is the subcategory of the orbit category $\mathcal{O}(G)$ spanned by the orbits $G/H$ with $H$ in the family $\mathcal{F}$. The key property of the $\mathcal{F}$-homotopy limit spectral sequence is that it collapses at a finite page with a horizontal vanishing line [MNN15, Thm. 2.25]. This implies that every computation with the $\mathcal{F}$-homotopy limit spectral sequence is a finite one. Moreover, in many concrete situations we can establish a bound on the height of the horizontal vanishing line, and a bound on which page it will appear. Besides implying when the $\mathcal{F}$-homotopy limit spectral sequence will have collapsed, it can also be used to deduce differentials. This was illustrated in the computation of the cohomology of the quaternion group of order 8 in [MNN15, Ex. 5.18]. The computational utility of the $\mathcal{F}$-homotopy limit spectral sequence will be illustrated in a forthcoming paper, by using it to compute the cohomology of all 2-groups up to order 16.

Varying $X$ over all $G$-spectra, this horizontal vanishing line turns out to have a uniform upper bound in height [MNN15, Prop. 2.26]. The minimal upper bound of this height is one of the definitions of the $\mathcal{E}(p)$-exponent $\exp_{\mathcal{E}(p)} H_{F_{p}}F_{G}$. An equivalent definition is the following.

**Definition 1.3** ([MNN15, Prop. 2.26]). The $\mathcal{E}(p)$-exponent of $H_{F_{p}}F_{G}$ is the minimal $n$ such that there exists an $n$-dimensional CW-complex $X$ with isotropy in $\mathcal{E}(p)$ such that the canonical map $H_{F_{p}}F_{G} \rightarrow F(X, H_{F_{p}}F_{G})$ admits a retraction

\begin{equation}
H_{F_{p}}F_{G} \rightarrow F(X, H_{F_{p}}F_{G}) \rightarrow H_{F_{p}}F_{G}.
\end{equation}

In practice one can often determine this $\mathcal{E}(p)$-exponent, and this leads to a quantified version of Theorem 1.1, because one has $n \leq \exp_{\mathcal{E}(p)} H_{F_{p}}F_{G}$ [MNN15, Thm. 3.24, Rem. 3.26]. The identification of $\mathcal{E}(2)$-exponents for 2-groups is the principal goal of this paper, and leads to the main theorem:

**Theorem A.** Let $G$ be a finite group with a 2-Sylow subgroup of order $\leq 16$, let $X$ be any $G$-space, and let $I$ be the kernel of

\begin{equation}
\overline{\text{res}}: H_{G}^{*}(X; F_{2}) \rightarrow \lim_{E \subset G \text{ cl. ab. 2-gp.}} H_{E}^{*}(X; F_{2}).
\end{equation}

Then $I^{4} = 0$. Moreover, if $u$ is not in the image of $\overline{\text{res}}$, then $u^{8}$ is.

Theorem A follows from combining [MNN15, Thm. 3.24] and Lemma 3.13 with the upper bounds on the exponents from the following theorem:
Theorem B. The exponents of Borel equivariant $\mathbb{F}_2$-cohomology for the groups of order less than or equal to 16 are bounded above by the values in the following table:

| $G$                  | $\exp e_{(2)} H\mathbb{F}_2 G$ | Reference          |
|----------------------|---------------------------------|--------------------|
| $e$                  | 1                               | Proposition 4.1    |
| $C_2$                | 1                               | Proposition 4.1    |
| $C_2 \times C_2$     | 1                               | Proposition 4.1    |
| $C_4$                | 2                               | Proposition 4.1    |
| $C_2 \times 3$       | 1                               | Proposition 4.1    |
| $C_2 \times C_4$     | 2                               | Proposition 4.1    |
| $C_8$                | 2                               | Proposition 4.1    |
| $D_8$                | 2                               | Corollary 4.3      |
| $Q_8$                | 4                               | [MNN15, Ex. 5.18]  |
| $C_2 \times 4$       | 1                               | Proposition 4.1    |
| $C_2 \times C_4$     | 2                               | Proposition 4.1    |
| $C_4 \times C_4$     | 3                               | Proposition 4.1    |
| $C_8 \times C_2$     | 2                               | Proposition 4.1    |
| $C_16$               | 2                               | Proposition 4.1    |
| $D_16$               | 2                               | Corollary 4.3      |
| $Q_{16}$             | 4                               | Proposition 4.4    |
| $SD_{16} = C_8 \times C_2$ | 4                       | Proposition 4.6    |
| $M_{16} = C_8 \times C_2$ | 4                       | Proposition 4.5    |
| $D_8 \times C_4$     | 4                               | Proposition 4.7    |
| $C_4 \times C_4$     | 4                               | Proposition 4.8    |
| $(C_4 \times C_2)^{\psi_5} \rtimes C_2$ | 2                       | Proposition 4.8    |
| $Q_8 \times C_2$     | 4                               | Proposition 4.14   |
| $D_8 \times C_2$     | 2                               | Proposition 4.13   |

Table 1. Upper bounds on the $e_{(2)}$-exponents of the groups of order $\leq 16$.

For a description of the groups appearing in Theorem B we refer to the sections of the referred propositions. The first column lists the groups of order $\leq 16$, the second column an upper bound on the exponent, and the last column gives a forward reference for the claim.

Remark 1.4. All these upper bounds are in fact equalities, except possibly when $G$ equals $SD_{16}$, $M_{16}$ or $C_4 \rtimes C_4$, in which the exponent could be 3. These lower bounds will be part of the content of the forthcoming paper mentioned above.

For specific 2-Sylow subgroups of order less than or equal to 16, one can obtain an improved version of Theorem A by using the upper bound on the relevant exponent from Theorem B.

1.1. Organization. The proofs of the main results are contained in Section 4. Section 2 summarizes what we need from [MNN15, MNN17], and Section 3 contains lemma’s that are used to prove the main results.
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1.3. Notation and conventions. Throughout $G$ denotes a finite group. If a non-equivariant cohomology theory is represented by a spectrum $E$, then we denote the spectrum representing the Borel equivariant version of this cohomology theory by $\tilde{E}_G$. In particular, we denote Borel-equivariant $\mathbb{F}_2$-cohomology by $H\mathbb{F}_2^G$. We may omit the $G$ from the notation if the group $G$ is clear from the context. For a finite group $G$ and a prime $p$, we denote the family of elementary abelian $p$-subgroups of $G$ by $E(G)_{(p)}$ or $E_p(G)$ if the group is clear from the context.

2. $\mathcal{F}$-Nilpotence

We recall the notion of $\mathcal{F}$-nilpotence from [MNN15], for which we first need to recall the following space.

Definition 2.1. Let $\mathcal{F}$ be a family of subgroups of $G$. Then the universal $\mathcal{F}$-space $E\mathcal{F}$ is the $G$-space given by the homotopy colimit

\[
E\mathcal{F} = \operatorname{hocolim}_{\mathcal{O}(G)_{\mathcal{F}}} G/H.
\]

The space $E\mathcal{F}$ is, up to $G$-equivalence, characterized by [GM95, §17]

\[
E\mathcal{F}^H \simeq \begin{cases} 
\emptyset & \text{if } H \notin \mathcal{F}, \\
pt & \text{if } H \in \mathcal{F}.
\end{cases}
\]

Using the space $E\mathcal{F}$, we can give one of the equivalent definitions of $\mathcal{F}$-nilpotence.

Definition 2.2 (cf. [MNN15, Def. 1.3]). Let $M$ be a $G$-spectrum. Then $M$ is said to be $\mathcal{F}$-nilpotent if there is an $n$ such that $M$ is a retract of $F(\sk_{n-1} E\mathcal{F}_+, M)$. The minimal $n \geq 0$ for which this holds is called the $\mathcal{F}$-exponent of $M$, and denoted $\exp_{\mathcal{F}} M$.

Being $\mathcal{F}$-nilpotent is a strong condition, it implies for example the following.

Proposition 2.3 ([MNN15, Prop. 2.8]). Let $M$ be $\mathcal{F}$-nilpotent. Then $M$ is $\mathcal{F}$-complete and $\mathcal{F}$-colocal, that is, the $\mathcal{F}$-completion map $M \to M(E\mathcal{F}_+, M)$ and the colocalization map $M \wedge E\mathcal{F}_+ \to M$ are weak equivalences.

A trivial example is the following.

Example 2.4. Every $G$-spectrum $M$ is $\mathcal{A}\ell\ell$-nilpotent with $\exp_{\mathcal{A}\ell\ell} M \leq 1$, and $\exp_{\mathcal{A}\ell\ell} M = 0$ if and only if $M$ is contractible.

The following two propositions are immediate from the above definition.

Proposition 2.5 ([MNN17, Prop. 6.39]). A $G$-spectrum $M$ is $\mathcal{F}$-nilpotent and $\mathcal{G}$-nilpotent if and only if $M$ is $\mathcal{F} \cap \mathcal{G}$-nilpotent.

Proposition 2.6 ([MNN15, Def. 1.3]). If $M$ is $\mathcal{F}$-nilpotent and $\mathcal{I} \supset \mathcal{F}$, then $M$ is $\mathcal{I}$-nilpotent.
Combining Example 2.4, Proposition 2.5 and Proposition 2.6 shows that every $G$-spectrum $M$ has a minimal family $\mathcal{F}$ such that $M$ is $\mathcal{F}$-nilpotent (cf. the remark after [MNN15, Def. 1.3]).

**Definition 2.7** ([MNN15, remark after Def. 1.3]). The minimal family $\mathcal{F}$ such that a $G$-spectrum $M$ is $\mathcal{F}$-nilpotent is called the **derived defect base** of $M$.

The following is the main case of interest for us.

**Proposition 2.8** ([MNN15, Prop. 5.16]). For $G$ any finite group, the derived defect base of $\mathcal{H}\mathcal{F}_2G$ is $\mathcal{E}_{(2)}$.

We can now state the uniform upper bound in the height of the horizontal vanishing of the $\mathcal{F}$-homotopy limit spectral sequence and the page on which it appears.

**Proposition 2.9** ([MNN15, Prop. 2.26, Rem. 2.27]). Let $G$ be a finite group, $\mathcal{F}$ a family of subgroups, and $M$ an $\mathcal{F}$-nilpotent $G$-spectrum. Then the following integers equal:

1. The $\mathcal{F}$-exponent of $M$.
2. The minimal $N$ such that for all $G$-spectra $X$, the $\mathcal{F}$-homotopy limit spectral sequence $E^*_\infty(X)$ admits a vanishing line of height $N$ on the $N+1$-page: $E^{s, *}_{N+1} = E^{s, *}_\infty = 0$ for all $s \geq N$.
3. The minimal $n$ such that the canonical map $F(E\mathcal{F}, M) \simeq M \to F(\text{sk}_{n-1} E\mathcal{F}, M)$ admits a retraction.
4. The minimal $n'$ such that there is an $(n-1)$-dimensional CW-complex $X$ with isotropy in $\mathcal{F}$ such that $M$ is a retract of $F(X, M)$.
5. The minimal $m$ such that the canonical map $\text{sk}_{m-1} E\mathcal{F} \wedge M \to M$ admits a section.
6. The minimal $m'$ such that there is an $(m'-1)$-dimensional CW-complex $X$ with isotropy in $\mathcal{F}$ such that $M$ is a retract of $X \wedge M$.

Moreover, if $M'$ is any $G$-spectrum, then the existence of an integer for $M'$ as in any one of the items from (2) to (6) implies that $M'$ is $\mathcal{F}$-nilpotent.

We end this section by recalling from [MNN15] some properties of exponents that will be used in the next chapter to prove lemmas about exponents.

**Proposition 2.10.** Let $H \in \mathcal{F}$. Then $G/H_+$ is $\mathcal{F}$-nilpotent with $\exp_\mathcal{F} G/H_+ = 1$.

**Proposition 2.11** ([MNN17, Cor. 4.15]). If $M$ is an $\mathcal{F}$-nilpotent spectrum and $X$ is any $G$-spectrum, then $F(X, M)$ is $\mathcal{F}$-nilpotent with $\exp_\mathcal{F} F(X, M) \leq \exp_\mathcal{F} M$.

**Proposition 2.12.** If $N$ is an $\mathcal{F}$-nilpotent $G$-spectrum and $M$ is any $G$-spectrum then $M \wedge N$ is $\mathcal{F}$-nilpotent with $\exp_\mathcal{F} M \wedge N \leq \exp_\mathcal{F} N = n$.

**Proposition 2.13** ([MNN17, Prop. 4.9]).

1. If $M$ is a retract of an $\mathcal{F}$-nilpotent spectrum $N$, then $M$ is $\mathcal{F}$-nilpotent and $\exp_\mathcal{F} M \leq \exp_\mathcal{F} N$.
2. If $M'$ and $M''$ are $\mathcal{F}$-nilpotent and $M' \to M \to M''$ is a cofiber sequence then $M$ is $\mathcal{F}$-nilpotent and $\exp_\mathcal{F} M \leq \exp_\mathcal{F} M' + \exp_\mathcal{F} M''$. 
Proposition 2.14. Let $M_\alpha$ be a set of $\mathcal{F}$-nilpotent spectra with $\mathcal{F}$-exponents bounded uniformly by $n$. Then $\bigvee_\alpha M_\alpha$ is $\mathcal{F}$-nilpotent with $\mathcal{F}$-exponent $\leq n$.

Proposition 2.15. Let $X$ be an $(n-1)$-dimensional $G$-CW-spectrum with isotropy in $\mathcal{F}$. Then $X$ is $\mathcal{F}$-nilpotent and $\exp_\mathcal{F} X \leq n$.

Proof. Use induction and the previous propositions.

Proposition 2.16. Let $X$ be a finite dimensional $G$-CW-spectrum with isotropy in $\mathcal{F}$. Then the equivariant Spanier-Whitehead dual $D(X)$ of $X$ is $\mathcal{F}$-nilpotent, and $\exp_\mathcal{F} D(X) = \exp_\mathcal{F} X$.

Proof. Write $n = \exp_\mathcal{F} X$. Let $Y_+$ be an $(n-1)$ finite-dimensional $G$-CW complex with isotropy in $\mathcal{F}$ such that there is a retraction $X \to D(Y_+) \simeq D(Y_+) \wedge X \to X$.

Applying $D(-)$ to this retraction exhibits $D(X)$ as a retraction of $Y_+ \wedge D(X)$, which has exponent $\leq n$ by Proposition 2.16 and Proposition 2.12. Therefore $\exp_\mathcal{F} D(X) \leq \exp_\mathcal{F} X$, and replacing $X$ by $D(X)$ in this inequality shows equality. □

3. Exponent lemmas

We discuss some lemmas that will be of use in determining exponents. Some of these statements appear as exercises in [MNN17, sec. 4].

3.1. Lemmas for $\mathcal{F}$-exponents. We now give various lemmas which describe how $\mathcal{F}$-exponents can change as the family $\mathcal{F}$ varies.

Lemma 3.1. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two families of subgroups. Then $E(\mathcal{F}_1 \cap \mathcal{F}_2)_+ \simeq E\mathcal{F}_1_+ \wedge E\mathcal{F}_2_+$.

Proof. This follows from comparing fixed points and the characterizing property of the universal $\mathcal{F}$-space. □

Notation 3.2. For $\mathcal{F}$ a family of subgroups of $G$, and $H$ a subgroup of $G$, denote by $\mathcal{F}_H$ the family of subgroups of $H$ consisting of those groups in $\mathcal{F}$ that are contained in $H$.

If $\mathcal{F}$ is a family that make sense for all groups $G$, such as the family of all subgroups, the family of elementary abelian $p$-groups, etc., we write $\mathcal{F}(G)$ for this family of subgroups of $G$. For instance, we write $\mathcal{E}_{(2)}(D_8)$ for the elementary abelian subgroups of the dihedral group of order 8.

Lemma 3.3. If $\mathcal{F}$ is a family of subgroups of $G$, and $H$ is a subgroup of $G$, then $\text{Res}_H^G E \mathcal{F} \simeq E \mathcal{F}_H$.

Proof. This follows from comparing fixed points. □

Lemma 3.4. Let $M$ be an $\mathcal{F}$-nilpotent $G$-spectrum, $H \subset G$ a subgroup. Then $\text{Res}_H^G M$ is $\mathcal{F}_H$-nilpotent, and $\exp_\mathcal{F} \text{Res}_H^G M \leq \exp_\mathcal{F} M$.

Proof. This follows from [MNN17, Cor. 4.13] and Lemma 3.3. □

For a group $G$, we denote the spectrum representing Borel $G$-equivariant $\mathbb{F}_p$-cohomology by $H^G_{\mathbb{F}_p}$. 
Corollary 3.5. Let $G$ be a group and $H \subset G$ a subgroup. Then
\[ \exp_{\iota_p(H)} \mathbb{H}_F \leq \exp_{\iota_p(G)} \mathbb{H}_F. \]

Lemma 3.6. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two families of subgroups, and let $M$ be a $G$-spectrum which is both $\mathcal{F}_1$- and $\mathcal{F}_2$-nilpotent, with exponents $m$, $n$ respectively. Then $M$ is $\mathcal{F}_1 \cap \mathcal{F}_2$-nilpotent, and
\[ \exp_{\mathcal{F}_1 \cap \mathcal{F}_2} M \leq m + n - 1. \]

Proof. The fact that $M$ is $\mathcal{F}_1 \cap \mathcal{F}_2$-nilpotent is part of [MNN17, Prop. 6.39]. The assumption on the exponents implies that both maps in
\[ \text{sk}_{m-1} E_{\mathcal{F}_1} \wedge \text{sk}_{n-1} E_{\mathcal{F}_2} \to \text{sk}_{m-1} E_{\mathcal{F}_1} \wedge E_{\mathcal{F}_2} \wedge M \]
\[ \to E_{\mathcal{F}_1} \wedge E_{\mathcal{F}_2} \wedge M \]
have a retraction, hence the composite has a retraction. The composite (3.4) factors as
\[ \text{sk}_{m-n-2} \left( E_{\mathcal{F}_1} \wedge E_{\mathcal{F}_2} \right) \wedge M \]
Composing the retraction of $(\ast)$ with $(\ast \ast)$ gives a retraction of $(\ast \ast \ast)$. Since $E_{\mathcal{F}_1} \wedge E_{\mathcal{F}_2} \simeq E(\mathcal{F}_1 \cap \mathcal{F}_2)$ by Lemma 3.1, we obtain the desired bound. \hfill \square

Notation 3.7. For a $G$-space $X$, we denote by $\mathcal{I}(X)$ the minimal family containing the isotropy groups of $X$.

Example 3.8. For an orthogonal $G$-representation $V$, the unit sphere $S(V)$ in $V$ inherits a $G$-action. Then $\mathcal{I}(S(V))$ the smallest family containing the isotropy groups of $S(V)$.

Lemma 3.9 (cf. Proof of [MNN15, Thm. 2.3]). Let $R$ be a ring $G$-spectrum with multiplicative Thom classes (e.g. $HF_2$, see [MNN15, Def. 5.1]). Let $V$ be a $G$-representation with corresponding oriented Euler class
\[ \chi(V) : S^{-|V|} \to R^* \]
Suppose $\chi(V)$ is nilpotent with $\chi(V)^n = 0$. Then $R$ is $\mathcal{I}(S(V))$-nilpotent and
\[ \exp_{\mathcal{I}(V)} R \leq n \text{dim}_R V. \]

Proof. The fact that $\chi(V)^n = 0$ is equivalent to the oriented Euler class
\[ R \overset{\chi(V)}{\to} S^n V \wedge R \]
being nullhomotopic (see [MNN15, Lem. 5.3] and [MNN15, Rem. 2.4]). Hence the left map in the fiber sequence
\[ S(nV) \wedge R \to R \to S^n V \wedge R \]
has a section: $R$ is a retract of $S(nV)_+ \wedge R$. But $S(nV)$ is a $(n \dim_\mathbb{R} V - 1)$-dimensional $G$-CW complex, hence by Proposition 2.16, $\exp_\mathbb{F}_p S(nV)_+ \leq n \dim_\mathbb{R} V$, and hence the same bound holds for $R$ by Proposition 2.13.

\begin{lemma}
Let $f: G \to C_2 \cong O(1)$ be a 1-dimensional real representation with oriented Euler class $e \in H^1(BG; \mathbb{F}_2)$. Suppose $e$ is nilpotent with $n$ the minimal integer $\geq 0$ such that $e^n = 0$. Then $HF_2$ is nilpotent for the family $\mathcal{F}_\ker f$ of subgroups of $\ker f$ with $\exp_\mathcal{F}_\ker f HF_2 = n$.
\end{lemma}

\begin{proof}
The upper bound for the exponent follows immediately from Lemma 3.9.

For the lower bound on the exponent, denote a class detecting $\ell \ell_\ker f$.

We now recall the following well-known fact.

\begin{corollary}
For a $G$-space $X$, $H \subset G$ a subgroup, the composite of maps of Borel cohomology rings

$$
H^*_G(X; \mathbb{F}_p) \xrightarrow{\Res^G_H} H^*_H(X; \mathbb{F}_p) \xrightarrow{\Ind^G_H} H^*_G(X; \mathbb{F}_p)
$$

is multiplication by $[G: H]$.

\end{corollary}

\begin{proof}
We need to show that for all subgroups $G' \subset G$, applying $\pi^{G'}_*$ to (3.11) yields an isomorphism. This composite is given precisely by Lemma 3.11 for $X = G/G'$ and $H = P$. But multiplication by $[G: P]$ is an isomorphism because $p \nmid [G: P]$, whence $[G: P] \in \mathbb{F}_p^\times$.

\end{proof}

\begin{lemma}
Let $G$ be a group and $P \subset G$ a $p$-Sylow subgroup. Then the maps of $G$-spectra

$$
HF_{p G} \to G/P_+ \wedge HF_{p G} \to HF_{p G}
$$

representing the natural transformations $\Res^G_{p G}$ and $\Ind^G_{p G}$ exhibit $HF_{p G}$ as a retract of $G/P_+ \wedge HF_{p G}$.

\end{lemma}

\begin{proof}
First,

$$
\Res^G_{p G} HF_{p G} = HF_{p G},
$$

$$
\Res^G_{p G} E \delta_{p}(G) = E \delta_{p}(P).
$$

Hence by Corollary 3.5 (cf. [MNN17, Cor. 4.13]),

\begin{equation}
\exp_{\delta_{p}(G)}HF_{p G} \leq \exp_{\delta_{p}(G)}HF_{p G}.
\end{equation}

For the upper bound, write $n = \exp_{\delta_{p}(G)}HF_{p G}$. We then have that

$$
\sk_{n-1} E \delta_{p}(P) \wedge HF_{p G} \to HF_{p G}.
$$

\end{proof}
admits a section. Note that for every $X \in \text{Sp}_G$, we have $\text{Ind}^G_H \text{Res}^G_H X = G/H_+ \wedge X$. Furthermore we have that, if we use the model for $E\varphi_{(p)}(P)$ from (3.14),

$$
(3.17) \quad \text{sk}_{n-1} E\varphi_{(p)}(P) = \text{Res}_G^H \text{sk}_{n-1} E\varphi_{(p)}(G),
$$

$$
(3.18) \quad \text{Applying Corollary 3.12 and Proposition 2.13 yields}
$$

Applying this, the fact that $\text{Res}_G^H$ is monoidal, and (3.13) to (3.16) yields a section of

$$
(3.19) \quad G/P_+ \wedge \text{sk}_{n-1} E\varphi_{(p)}(G) \wedge H\text{F}_{pG} \to G/P_+ \wedge H\text{F}_{pG}.
$$

Hence

$$
(3.20) \quad \exp E\varphi_{(p)}(G) \wedge H\text{F}_{pG} \leq n.
$$

Applying Corollary 3.12 and Proposition 2.13 yields

$$
(3.21) \quad \exp E\varphi_{(p)} H\text{F}_{pG} \leq n.
$$

□

**Lemma 3.14.** Let $\mathcal{F}$ and $\mathcal{G}$ be families of subgroups, and let $M$ be both $\mathcal{F}$- and $\mathcal{G}$-nilpotent. Then for every $K \in \mathcal{F}$, $\text{Res}_G^K M$ is $\mathcal{G}_K$-nilpotent by Lemma 3.4. Write $n = \exp_{\mathcal{F}} M$, $m_K = \exp_{\mathcal{G}_K} \text{Res}_G^K M$ for all $K \in \mathcal{F}$, and $m = \max_K m_K$. Then

$$
(3.22) \quad \exp_{\mathcal{G}} M \leq mn.
$$

**Proof.** The $\mathcal{G}_K$-nilpotence of $\text{Res}_G^K M$ implies that, for all $K \in \mathcal{F}$,

$$
(3.23) \quad \text{sk}_{m_K - 1} E\mathcal{G}_K \wedge \text{Res}_G^K M \to \text{Res}_G^K M
$$

admits a section. By inducing up to $G$, we see that $\exp_{\mathcal{G}} (G/K_+ \wedge M) \leq m_K$. By taking the coproduct over all $K \in \mathcal{F}$, we get that $\exp_{\mathcal{G}} \text{sk}_0 E\mathcal{F} \wedge M \leq m$. We now argue by induction and that

$$
(3.24) \quad \exp_{\mathcal{G}} \text{sk}_{d-1} E\mathcal{F} \wedge M \leq md
$$

for all $d \geq 1$. Hence assume (3.24) has been established for some $d \geq 1$, and consider the cofiber sequence

$$
(3.25) \quad \text{sk}_{d-1} E\mathcal{F} \wedge M \to \text{sk}_d E\mathcal{F} \wedge M \to \bigvee (S^d \wedge G/K_+) \wedge M
$$

that ends in a wedge of spheres with isotropy in $\mathcal{F}$ smashed with $M$. By induction, the $\mathcal{G}$-exponent of the left term is $\leq md$, and the $\mathcal{G}$-exponent of the right hand side is $\leq m$ because we already saw that $\exp_{\mathcal{G}} (G/K_+ \wedge M) \leq m$ for all $K \in \mathcal{F}$. Hence by [MNN17, Prop. 4.9.2] the $\mathcal{G}$-exponent of the middle term is $\leq m(d+1)$, which completes the induction.

We have by $\mathcal{F}$-nilpotency of $M$ that $M$ is a retract of $\text{sk}_{n-1} E\mathcal{F} \wedge M$, hence taking $d = n$ in (3.24) yields the result. □

3.2. **Representations and exponents.** In [MNN15, Ex. 5.16] an upper bound on the $\varphi_{(2)}$-exponent of $H\text{F}_{2Q_8}$ is determined using the projective bundle theorem. We will also repeatedly use this technique to give upper bounds on the $\mathcal{F}$-exponent of $H\text{F}_{2G}$ for various 2-groups $G$ and families $\mathcal{F} \supset \varphi_{(2)}$. Therefore in this section we will describe this argument in some detail. We will need the following classical notion:

**Definition 3.15.** Let $V$ be a real or complex vector space. Then the projectivation $P(V)$ of $V$ is the space of all respectively real or complex lines in $V$. 
Remark 3.16. Note that if $V$ comes equipped with a linear $G$-action (i.e., is a $G$-representation), then $P(V)$ inherits a natural $G$-action.

The goal of this subsection is to prove:

**Proposition 3.17.** Let $G$ be a finite group, $n \geq 0$ a natural number and suppose $G$ has a real $n$-dimensional representation $V$ such that the projectivisation $P(V)$ has isotropy groups contained in some family $\mathcal{F}$. That is, for every real line $L \subset V$ we assume that the isotropy group $G_L \leq G$ of elements of $G$ fixing $L$ satisfies $G_L \in \mathcal{F}$. Then $HF^2$ is $\mathcal{F}$-nilpotent and the exponent satisfies $\exp \ HF^2 \leq n$.

For complex bundles we have the analogous result for cohomology with $\mathbb{Z}$-coefficients:

**Proposition 3.18.** Let $G$ be a finite group, $n \geq 0$ a natural number and suppose $G$ has a complex $n$-dimensional representation $V$ such that the projectivisation $P(V)$ has isotropy groups contained in some family $\mathcal{F}$. That is, for every complex line $L \subset V$, we assume that the isotropy group $G_L \leq G$ of elements of $G$ fixing $L$ satisfies $G_L \in \mathcal{F}$. Then $HZ$ is $\mathcal{F}$-nilpotent, and the exponent satisfies $\exp \ HZ \leq 2n - 1$.

3.2.1. The projective bundle theorem. To prove Proposition 3.17 and Proposition 3.18, we will use the projective bundle theorem, which can be used to develop the theory of Stiefel-Whitney classes and Chern classes. This is carried out for instance in [Hus94, Ch. 17]. We will follow the treatment and notation of [Hus94, §17.2], but only discussing what we need.

As in [Hus94, Ch. 17], we consider real and complex vector bundles at the same time. For the case of real vector bundles, we write $c = 1$, we consider cohomology with coefficients in $K_c = \mathbb{Z}/2$, and we let $F$ be the field $\mathbb{R}$ of real numbers. For the case of complex vector bundles, we write $c = 2$, we consider cohomology with coefficients in $K_c = \mathbb{Z}$, and we let $F$ be the field $\mathbb{C}$ of complex numbers.

We will write $E(\eta)$ resp. $B(\eta)$ for the total, resp. base space, of a fiber (not necessarily vector) bundle $\eta$. Let $\xi: E \xrightarrow{p} B$ be an $n$-dimensional vector bundle. Let $E_0$ be the non-zero vectors in $E$. Let $P(E)$ be the quotient of $E_0$ where we identify non-zero vectors in the same line. This yields a factorization

\[(3.26) \quad E_0 \longrightarrow P(E) \xrightarrow{q} B,\]

and $P(E) \xrightarrow{q} B$ is a fibre bundle with fibre $FP^{n-1}$, called the projectivisation of $\xi$ and denoted $P(\xi)$. This space admits a canonical line bundle, classified by a map $f: P(\xi) \to FP^\infty$. Pulling back a polynomial generator $z \in H^*(FP^\infty; K_c)$ with $|z| = c$ along $f$ gives a class $a_\xi := f^*(z) \in H^*(P(E); K_c)$.

**Theorem 3.19** (Projective bundle theorem, see [Hus94, Thm. 17.2.5]). For an $n$-dimensional vector bundle $\xi$, the classes $1, a_\xi, \ldots, a_\xi^{n-1}$ form a basis of the free $H^*(B(\xi); K_c)$-module $H^*(P(E(\xi)); K_c)$. In particular,

\[(3.27) \quad q^*: H^*(B(\xi); K_c) \to H^*(P(E(\xi)); K_c)\]

is the inclusion of a summand.

3.2.2. From representations to exponents. We are now ready to prove Proposition 3.17 and Proposition 3.18.
Proof of Proposition 3.17 and Proposition 3.18. Consider the Borel construction on $V$ for an arbitrary subgroup $H \leq G$, and call the resulting bundle $\xi_H$:

\[(3.28)\]

$$\xi_H : V \to V \times_H EG \to BH.$$ 

Note that this is natural with respect to inclusions of subgroups. The associated projective bundle is

\[(3.29)\]

$$P(\xi_H) : P(V) \to P(V \times_H EG) \to BH.$$ 

Observe that $P(V \times_H EG) = P(V) \times HEG$. Hence by Theorem 3.19,

\[(3.30)\]

$$F((P(V) \times_H EG)_+, HK_c) \cong F(P(V)_+, HK_c^H)$$ 

is a free $F(BH, K_c) = HK_c^H$-module. Recall that the real dimension of $V$ is $cn$. A basis is given by the proof of [Hus94, Prop. 17.3.3], which shows that there are classes $1, a_{\xi_H}, \ldots, a_{n-1}^n\xi_H$ that form a basis of $\pi^H_* F(P(V), HK_c)$ as a $\pi^H_* HK_c$-module. Moreover, the element $a_{\xi_H}$ is natural with respect to restriction to subgroups, because the canonical line bundle on $P(V \times G EG)$ pulls back to the canonical line bundle on $P(V \times_H EG)$. In particular,

\[(3.31)\]

$$\text{Res}_H^G a_{\xi_G} = a_{\xi_H}$$

for all $H \leq G$. It follows that all basis elements, being powers of $a_{\xi_H}$, are natural with respect to restriction to subgroups. Hence $F(P(V), HK_c)$ is a free $HK_c$-module, because freeness of modules is an algebraic condition. Hence a suspension of $HK_c$ is a retract of $F(P(V), HK_c)$. Therefore by [MNN17, Prop. 4.9],

\[(3.32)\]

$$\exp_{\mathcal{F}} HK_c \leq \exp_{\mathcal{F}} F(P(V), HK_c),$$

but $V$ is $cn$-dimensional, hence $P(V)$ admits the structure of an $(cn-c)$-dimensional $G$-CW-complex [Ill83, Cor. 7.2] with isotropy contained in $\mathcal{F}$ by assumption, which implies by Proposition 2.9 that

\[(3.33)\]

$$\exp_{\mathcal{F}} F(P(V), HK_c) \leq cn - c + 1.$$ 

\[(3.34)\]

$$= \begin{cases} n & \text{if } c = 1, \\ 2n - 1 & \text{if } c = 2. \end{cases}$$

\[\square\]

3.3. Proper subgroups. Let $G$ be a finite non-abelian 2-group of order $2^k$. The goal of this subsection is to prove

\[(3.35)\]

$$\exp_{\mathcal{F}} HF_{2G} \leq 2 \sqrt{|G| - 1} - 1,$$

see Corollary 3.21 below. This result will not be used for establishing the main results of this paper.

The argument is an adaption of the ones found in [PY03, Lem. 4.3] and [Sym91].

Lemma 3.20. Let $G$ be a finite non-abelian $p$-group of order $p^k$. Then $G$ has an irreducible complex representation $V$ with $\dim_{\mathbb{C}} V \geq 2$, and moreover all such $V$ satisfy

\[(3.36)\]

$$\dim_{\mathbb{C}} V \leq |\sqrt{|G| - 1}|.$$
Proof. Denote by $n_1, \ldots, n_h$ the $C$-dimensions of the irreducible $C$-representations of $G$. We have

\begin{equation}
(3.37)\quad n_1^2 + n_2^2 + \cdots + n_h^2 = |G|,
\end{equation}

(see, e.g., [Ser77, Cor. 2.4.2]). Since $G$ is non-abelian, some $n_i \geq 2$ [Ser77, Thm. 3.1.9] and some $n_j = 1$ corresponding to the trivial representation. Assume without loss of generality that they are respectively $n_1$ and $n_2$. Then

\begin{equation}
(3.38)\quad |G| = n_1^2 + n_2^2 + \cdots + n_h^2 \geq n_1^2 + 1,
\end{equation}

\begin{equation}
(3.39)\quad \geq n_1^2 + 1,
\end{equation}

\begin{equation}
(3.40)\quad \text{hence}
\end{equation}

\begin{equation}
(3.41)\quad n_1 \leq \sqrt{|G| - 1},
\end{equation}

where this upper bound is in general far from optimal. Flooring this preserves the inequality and does not change the integer on the left hand side. This yields the result. \hfill \Box

Corollary 3.21. Let $G$ be a finite non-abelian 2-group of order $2^k$, and let $\mathcal{P}$ be the family of proper subgroups of $G$. Then

\begin{equation}
(3.42)\quad \exp_{\mathcal{P}} H\mathbf{F}_{2\mathbb{A}} \leq 2\lfloor \sqrt{|G| - 1} \rfloor - 1.
\end{equation}

Proof. Let $V$ be an irreducible complex representation of $G$ satisfying

\begin{equation}
(3.43)\quad 2 \leq \dim_C V \leq \lfloor \sqrt{|G| - 1} \rfloor,
\end{equation}

as furnished by Lemma 3.20. Then the complex projectivation $P(V)$ has isotropy in $\mathcal{P}$ the family of proper subgroups of $G$, for if $L \in P(V)$ were fixed by all of $G$, $V$ would not be irreducible, since $\dim_C V \geq 2$. An application of Proposition 3.18 yields the result. \hfill \Box

4. Exponents of small 2-groups

We now prove Theorem B from the introduction, by proving the claimed upper bounds of the $E_{(2)}$-exponents. We first treat the class of abelian groups, dihedral groups, and the generalized quaternion groups. The remaining groups of order 16 we treat using a case-by-case analysis.

4.1. Abelian 2-groups. We determine an upper bound on the $E_{(2)}$-exponent of $H\mathbf{F}_{2\mathbb{A}}$ for $A$ a finite abelian group.

Let $A$ be a finite abelian 2-group. Then $A$ is isomorphic to

\begin{equation}
(4.1)\quad \prod_{j \in J} C_{2^{n_j}} \times \prod_{k \in K} C_2
\end{equation}

with $n_j \geq 2$, for some indexing sets $J$ and $K$.

For $j \in J$, consider the projection of $A$ onto the $C_2$ in the $j$-th factor:

\begin{equation}
(4.2)\quad p_j : A \to C_2 \cong O(1).
\end{equation}

The corresponding Euler class is $a_j \in H^1(BA; \mathbf{F}_2)$. Since $a_j^2 = 0$, we get $\exp_{E_{(2)}} H\mathbf{F}_2 = 2$. Now $\bigcap_{j \in J} \ker p_j = \mathcal{E}_{(2)}$, and hence by Lemma 3.6, $\exp_{E_{(2)}} H\mathbf{F}_2 \leq \#J + 1$. 
Proposition 4.1. Let $A$ be a group isomorphic to
\begin{equation}
\prod_{j \in J} C_{2^{n_j}} \times \prod_{k \in K} C_2
\end{equation}
with $n_j \geq 2$. Then
\begin{equation}
\exp_{\varphi(2)} \frac{HF_2 A}{2} \leq \# J + 1.
\end{equation}

4.2. Dihedral groups. Let $D_{2n}$ be the dihedral group of order $2^n$ with presentation
\begin{equation}
D_{2^n} = \langle \sigma, \rho | \sigma^2 = \rho^{2^{n-1}} = e, \sigma \rho \sigma^{-1} = \rho^{-1} \rangle.
\end{equation}
For an angle $\theta$, denote the matrix representing rotation in $\mathbb{R}^2$ by $\theta$ by
\begin{equation}
R_\theta = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\end{equation}
Denote by $T$ the matrix representing reflection in the $x$-axis:
\begin{equation}
T = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\end{equation}
The group $D_{2n}$ is the subgroup of $O(2)$ fixing the regular $2n^{-1}$-gon centered at the origin in $\mathbb{R}^2$ with one of its sides bisected in the right half-plane by the $x$-axis. The inclusion $D_{2n} \to O(2)$ is on elements given by $\sigma \to T$, $\rho \to R_{2\pi/2^n-1}$. This yields a linear 2-dimension real representation of $D_{2^n}$, which we call $V$. We apply Proposition 3.17 to compute an upper bound on the exponent with respect to the minimal family $\mathcal{F}$ containing the isotropy of $P(V)$. This minimal family is determined in the next lemma.

Lemma 4.2. The minimal family containing the isotropy groups of the projectivation $P(V)$ is $\mathcal{F}_{(2)}$, the family of elementary abelian 2-groups in $D_{2^n}$.

Proof. The proof is by elementary linear algebra. Since $V$ is an orthogonal representation, a linear subspace $L$ spanned by a vector $v$ is fixed by an element $g$ of $D_{2n}$ if and only if $g$ has eigenvalue 1 or $-1$ (we say that $g$ has eigenvalue $\lambda$ if the matrix by which $g$ acts on $V$ has eigenvalue $\lambda$).

First consider the rotations in $D_{2n}$, that is the elements of the form $\rho^j$. We note that $R_\theta$ has characteristic polynomial
\begin{equation}
\left(\cos(\theta) - \lambda\right)^2 + \sin^2 \theta = 0,
\end{equation}
which for $\lambda = 1$ is 0 if and only if $\theta \equiv 0 \pmod{2\pi}$, and for $\lambda = -1$ is 0 if and only if $\theta = \pi \pmod{2\pi}$. Hence $\rho^j$ fixes a line if and only if $2\pi j / 2^{n-2} \equiv \pi \pmod{2\pi}$, that is if and only if $j \equiv 2^{n-2} \pmod{2^n}$. Hence $\rho^{2^{n-2}}$ is the only non-trivial rotation that fixes a line. Since $\rho^{2^{n-2}}$ acts via $R_\pi$, the elements $\rho^{2^{n-2}}$ fixes every line in $V$.

For the lines fixed by a reflection $\sigma \rho^j$, we first note that, writing $e_1 = (1 \quad 0)^t$, $TR_\theta$ has eigenvectors $R_{-\theta/2} e_1$ and $R_{-\pi/2 - \theta} e_1$, with eigenvalues 1 and $-1$, respectively. Therefore these are, up to scalar, all eigenvectors of $TR_\theta$, since $V$ is 2-dimensional. Assume $\sigma \rho^j$ and $\sigma \rho^k$ fix a common line, spanned by a vector $v$. If $v$ had the same eigenvalue for $\sigma \rho^j$ and $\sigma \rho^k$, we would have
\begin{equation}
\frac{-2\pi j}{2^{n-1}} \equiv \frac{-2\pi k}{2^{n-1}} \pmod{2\pi},
\end{equation}
which is a contradiction. Therefore $\sigma \rho^j$ and $\sigma \rho^k$ have different eigenvalues, and hence cannot both fix the same line.
which is equivalent to \( k \equiv j \pmod{2^{n-1}} \), in other words \( \sigma \rho^j = \sigma \rho^k \).

Alternatively, if \( v \) has different eigenvalues for \( \sigma \rho^j \) and \( \sigma \rho^k \), then we have

\[
\frac{-2\pi j}{2^{n-2}} \equiv \frac{-2\pi k}{2^{n-2}} - \pi \pmod{2\pi},
\]

which is equivalent to \( j - k \equiv 2^{n-2} \pmod{2^{n-1}} \). Hence if a line is fixed by a reflection \( \sigma \rho^j \), it is fixed by exactly one more reflection, namely \( \sigma \rho^{j+2^{n-2}} \). Note that

\[
\sigma \rho^j \sigma \rho^{j+2^{n-2}} = \rho^{2^n-2} = \sigma \rho^{j+2^{n-2}} \sigma \rho^j.
\]

Therefore \( \langle \sigma \rho^j, \sigma \rho^{j+2^{n-2}} \rangle \) is isomorphic to \( C_2 \times C_2 \).

We conclude that a line in \( V \) is fixed either by \( \langle \rho^{2^n-2} \rangle \), or by \( \langle \sigma \rho^j, \sigma \rho^{j+2^{n-2}} \rangle \) for exactly one \( j \pmod{2^{n-2}} \). The minimal family containing these groups is the family of elementary abelian 2-groups, proving the result. \( \square \)

**Corollary 4.3.** The \( \delta_{(2)} \)-exponent of \( H_{\mathbb{F}_2} D_{2^n} \) satisfies

\[
\exp_{\delta_{(2)}} H_{\mathbb{F}_2} D_{2^n} \leq 2.
\]

**Proof.** Immediate from Proposition 3.17 and Lemma 4.2 and the fact that \( V \) is 2-dimensional. \( \square \)

### 4.3. Quaternion groups.

The generalized quaternion group \( Q_{2^n} \) of order \( 2^n \) is the finite subgroup of quaternionic space \( \mathbb{H} \) generated multiplicatively by the elements of unit length \( \{e^{2\pi i/2^{n-1}}, j\} \) (see, e.g., [CE56, XII.§7]). Denoting these generators by \( r \) and \( s \), respectively, one gets the presentation

\[
Q_{2^n} = \langle r, s \mid r^{2^n-2} = s^2, rsr = s \rangle.
\]

for all \( n \geq 3 \). The subgroup generated by \( \langle s^2 \rangle \) is central and of order 2, which makes \( Q_{2^n} \) into a central extension [AM04, IV.2].

\[
C_2 \to Q_{2^n} \to D_{2^n-1}.
\]

In this subsection we will prove the following upper bound on the \( \delta_{(2)} \)-exponent.

**Proposition 4.4.** The \( \delta_{(2)} \)-exponent satisfies

\[
\exp_{\delta_{(2)}} H_{\mathbb{F}_2} Q_{2^n} \leq 4.
\]

**Proof.** The proof is a straightforward adaption of the argument in [MNN15, Ex. 5.18].

Let \( \mathbb{H} \cong \mathbb{R}^4 \) be the 4-dimensional real representation coming from the embedding \( Q_{2^n} \hookrightarrow \mathbb{H} \). This is a free action, and restricts to a free action on \( S^3 \). The subgroup \( \langle \pm 1 \rangle \) is central, and therefore \( Q_{2^n}/\langle \pm 1 \rangle \) acts on \( S^3/\langle \pm 1 \rangle = P(\mathbb{R}^4) \) with isotropy in \( \langle \pm 1 \rangle \). The result now follows from Proposition 3.17. \( \square \)
4.4. The modular group of order 16. Let

\[ M_{16} = \langle r, f \mid r^8 = f^2 = e, frf^{-1} = r^5 \rangle \]

be the modular group of order 16. The group has this name because its lattice of subgroups is modular (see, e.g., [Bir67, I.§7]).

In this subsection we prove the following upper bound on the \( E_{(2)} \)-exponent.

**Proposition 4.5.** The \( E_{(2)} \)-exponent satisfies

\[ \exp_{E_{(2)}} H\mathcal{F} M_{16} \leq 4. \]

**Proof.** For an angle \( \theta \), let \( R_\theta \) be the rotation-by-\( \theta \) matrix

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

Let \( T \) be the matrix

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

which interchanges the summands of \( \mathbb{R}^2 \oplus \mathbb{R}^2 \). Write \( \alpha = 2\pi/8 \). Let \( V \) be the 4-dimensional real \( M_{16} \)-representation given by

\[
\begin{align*}
    f & \mapsto T, \\
    r & \mapsto R_\alpha \oplus R_{5\alpha}.
\end{align*}
\]

This is the representation from [Tot14, Lem. 13.3].

We will now determine the isotropy of the projectivation \( P(V) \).

A power of \( r \) fixes a line in \( \mathbb{R}^2 \oplus \mathbb{R}^2 \) if and only if it fixes a line in one of the summands. The matrix \( R_\theta \) has characteristic polynomial \( P(\lambda) = \lambda^2 - 2\lambda \cos \theta + 1 \), which has discriminant \( \Delta = -4 \sin^2 \theta \). Hence \( R_\theta \) has real eigenvalues only if \( \theta \in \mathbb{Z}\{\pi\} \). Since \( r \mapsto R_{2\pi/8} \), \( r^k \) fixes a line only if \( ak \in \mathbb{Z}\{\pi\} \) or \( 5ak \in \mathbb{Z}\{\pi\} \), that is only if \( k \in \mathbb{Z}\{4\} \). Therefore \( e \) and \( r^4 \) are the only elements powers of \( r \) that can possibly fix a line. The element \( r^4 \) acts by \( -\text{Id} \), and therefore fixes all lines. The element \( e \) of course also fixes all lines.

We now consider the remaining elements of \( M_{16} \), which are of the form \( fr^k \), and which act by \( T(R_{aok} \oplus R_{5aok}) \). This matrix certainly only fixes a line if its square does, which is \( R_{6aok} \oplus R_{6aok} \). These summands only fix a line if \( 6ak \in \mathbb{Z}\{\pi\} \), which since \( \alpha \in \mathbb{Z}\{\pi/4\} \) implies that \( k \) is even. Hence the only possible elements of the form \( fr^k \) fixing a line are \( f, fr^2, fr^4 \), and \( fr^6 \). One verifies directly that \( fr^2 \) acts by the matrix

\[
\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]
which fixes no lines. Consequently its inverse \( f r^6 \) neither fixes lines. One also easily checks that \( f r^4 \) acts by

\[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{pmatrix}
\]

which fixes the lines \([x_1 : y_1 : x_2 : y_2]\) in \( \{x_1 = -x_2, y_1 = -y_2\}\). Hence the only elements of \( M_{16} \) fixing a line are \( e, f, fr^4, r^4 \), which together form a Klein four group. Applying Proposition 3.17 gives the desired result.

\[ \Box \]

4.5. The semi-dihedral group of order 16. Let

\[
SD_{16} = \langle s, r \mid s^2 = r^8 = e, sr s^{-1} = r^3 \rangle
\]

be the semidihedral group of order 16.

**Proposition 4.6.** The \( E(2) \)-exponent satisfies

\[
\exp_{E(2)} H_{F_2 SD_{16}} \leq 4.
\]

**Proof.** We will use the notation \( R_\theta \) and \( T \) from the proof of Proposition 4.5, and also use the fact proven there that \( R_\theta \) fixes a line only if \( \theta \in \mathbb{Z}\{\pi\} \). Write \( \alpha = 2\pi/8 \).

Let \( V \) be the real \( SD_{16} \)-representation given by

\[
s \mapsto T, \\
r \mapsto R_\alpha \oplus R_\alpha
\]

from [Tot14, Lem. 13.4]. The same argument as in the proof of Proposition 4.5 shows that the only powers of \( r \) fixing a line are \( e \) and \( r^4 \), and that the only elements of the form \( sr^k \) that can possibly fix a line are \( s, sr^2, sr^4, sr^6 \). The elements

\[
e, r^4, s, sr^2, sr^4, sr^6.
\]

do not generate an elementary abelian 2-group as in the proof of Proposition 4.5, but all the isotropy groups of lines that occur consist of proper subsets of the above elements that do form elementary abelian 2-groups, as we will now show.

First, \( r^4 \) fixes every line, so is in every isotropy group.

Write \([x_1 : y_1 : x_2 : y_2]\) for the real homogenous coordinates in \( \mathbb{R}P^3 \). Then

\[
s \cdot [x_1 : y_1 : x_2 : y_2] = [x_2 : y_2 : x_1 : y_1],
\]

which is fixed if and only if \((x_1, y_1) = (x_2, y_2)\) or \((x_1, y_1) = -(x_2, y_2)\). Computation of \( sr^2 \cdot [x_1 : y_1 : x_2 : y_2] \) shows that this element fixes exactly the same lines.

Computing

\[
sr^2 \cdot [x_1 : y_1 : x_2 : y_2] = [-y_2 : x_2 : y_1 : -x_1]
\]

shows that \( sr^2 \) fixes precisely those lines with \((x_1, y_1) = (-y_2, x_2)\) or \((x_1, y_1) = (y_2, -x_2)\). Computing \( sr^6 \cdot [x_1 : y_1 : x_2 : y_2] \) shows that this element fixes precisely the same lines.
The sets of lines \( \{ x_1 = x_2, y_1 = y_2 \} \cup \{ x_1 = -x_2, y_1 = -y_2 \} \) and \( \{ x_1 = y_2, y_1 = -x_2 \} \cup \{ x_1 = x_2, y_1 = y_2 \} \) have empty intersection, because a line in the intersection \( \{ x_1 = x_2, y_1 = y_2 \} \cap \{ x_1 = -y_2, y_1 = x_2 \} \) would satisfy

\[
y_2 = y_1 = x_2 = x_1 = -y_2,
\]
a line in the intersection \( \{ x_1 = x_2, y_1 = y_2 \} \cap \{ x_1 = y_2, y_1 = -x_2 \} \) would satisfy

\[
y_1 = y_2 = x_1 = x_2 = -y_1,
\]
a line in the intersection \( \{ x_1 = -x_2, y_1 = -y_2 \} \cap \{ x_1 = -y_2, y_1 = x_2 \} \) would satisfy

\[
y_2 = -y_1 = -x_2 = x_1 = -y_2,
\]
and finally a line in the intersection \( \{ x_1 = -x_2, y_1 = -y_2 \} \cap \{ x_1 = y_2, y_1 = -x_2 \} \) would satisfy

\[
y_2 = x_1 = -x_2 = y_1 = -y_2.
\]

Hence the isotropy groups that occur are

\[
\langle r^4 \rangle, \langle s, sr^4 \rangle, \langle sr^2, sr^6 \rangle,
\]
all of which are elementary abelian 2-groups. Applying Proposition 3.17 gives the desired result. \( \square \)

4.6. The central product of \( D_8 \) and \( C_4 \). Let \( D_8 = \langle \sigma, \rho \rangle \) be the dihedral group of order 8 and let \( C_4 = \langle \gamma \rangle \) be the cyclic group of order 4. Both these groups have central cyclic subgroups of order 2, for \( D_8 \) this is \( \langle \rho^2 \rangle \) and for \( C_4 \) this is \( \langle \gamma^2 \rangle \). The central product of \( D_8 \) and \( C_4 \) is defined to be the direct product with these central subgroups identified:

\[
D_8 \ast C_4 := D_8 \times C_4 / \langle \rho^2 \gamma^{-2} \rangle,
\]

**Proposition 4.7.** The \( e(2) \)-exponent satisfies \( \exp_{e(2)} \exp_{HF_2 D_8 \ast C_4} \leq 4 \).

**Proof.** Let

\[
\sigma \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \rho \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

be the underlying 4-dimensional real representation from [Tot14, Lem. 13.5]. The following table gives, up to inverses, all elements of \( D_8 \ast C_4 \), the effect of this representation on a line \( [x_1 : y_1 : x_2 : y_2] \), and the set of lines each element fixes.

---

\[
\begin{array}{cccc}
\sigma & \rho & \gamma & [x_1 : y_1 : x_2 : y_2] \\
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \end{array}
\]

---


Write

\[ L_1 = \{ x_1 = y_1 = 0 \} \cup \{ x_2 = y_2 = 0 \}, \]
\[ L_2 = \{ x_1 = -x_2, y_2 = -y_1 \} \cup \{ x_1 = x_2, y_1 = y_2 \}, \]
\[ L_3 = \{ x_1 = y_2 = y_1 = -x_2 \} \cup \{ x_1 = -y_2 = y_1 = x_2 \}. \]

Then the \( L_i \), together with the set of all lines, are the sets of lines that occur as the lines fixed by an element of \( D_9 \ast C_4 \). We immediately see from Table 2 that the lines in \( L_1 \) are fixed by \( \langle \sigma, \sigma^2 \rangle \), the lines in \( L_2 \) are fixed by \( \langle \sigma \rho, \sigma \rho^3 \rangle \), and the lines in \( L_3 \) are fixed by \( \langle \sigma \rho \gamma, \rho^3 \gamma \rangle \). Moreover, the \( L_i \) are pairwise disjoint. To see this, note that a line in \( L_1 \cap L_2 \) would be fixed by both \( \sigma \) and \( \sigma \rho \), hence by their product \( \sigma \cdot \sigma \rho = \rho \). But \( \rho \) does not fix any line, by Table 2. Similarly, a line in \( L_2 \cap L_3 \) would be fixed by both \( \sigma \rho \) and \( \rho^3 \gamma \), but \( \sigma \rho \cdot \rho^3 \gamma = \sigma \gamma \) does not fix any line. Lastly, a line in \( L_3 \cap L_1 \) would be fixed by \( \sigma \cdot \rho \gamma \), which does not fix any line. Therefore, no line is fixed by a strict supergroup of \( \langle \sigma, \sigma^2 \rangle \), \( \langle \sigma \rho, \sigma \rho^3 \rangle \), \( \langle \rho \gamma, \rho^3 \gamma \rangle \), so these subgroups are therefore the maximal isotropy groups of a line. All these three groups are elementary abelian, so the result follows from Proposition 3.17.

4.7. The group \( (C_4 \times C_2) \rtimes C_2 \). Let \( C_4 \cong \langle a \rangle \), \( C_2 \cong \langle b \rangle \), and let \( C_2 \cong \langle c \rangle \) act on \( C_4 \times C_2 \cong (a, b) \) via

\[ a \mapsto ab, \]
\[ b \mapsto b. \]

We consider the group \( G \) which is defined to be the semidirect product of \( (C_4 \times C_2) \rtimes C_2 \) under this action. This group hence admits the following presentation:

\[ G = \langle a, b, c \mid a^4 = b^2 = c^2 = e, ab = ba, bc = cb, cac^{-1} = ab \rangle. \]

We will establish an upper bound on the \( \delta_{(2)} \)-exponent of \( G \). To do so, we will need to run the \( \delta_{(2)} \)-homotopy limit spectral sequence for the group \( G \) converging to its cohomology ring, which is isomorphic to \([\text{CTVEZ03, App. C, #9(16)}]\)

\[ \mathbf{F}_2[u, v, w, r, t]/(u^2, uv, uw, v^2 r + w^2). \]

with degrees \( |u| = |v| = 1, |w| = |r| = |t| = 2 \).
Proposition 4.8. The $\epsilon_2$-exponent of $G = (C_4 \times C_2) \rtimes C_2$ satisfies
\begin{equation}
\exp_{\epsilon_2} H\Sigma_2 = 2.
\end{equation}

**Proof.** Let $A$ be the normal subgroup $\langle a^2, bc, c \rangle$ of $G$. The group $A$ is the unique maximal elementary abelian 2-subgroup of $G$, hence the $\epsilon_2$-homotopy limit spectral sequence reduces to the Lyndon-Hochschild-Serre spectral sequence (from now on abbreviated as LHSSS) obtained from the extension $A \to G \to C_2$.

The Weyl group $W_G(A) = \langle a \rangle$ acts on the generators of $A$ by
\begin{align}
a(a^2)a^{-1} &= a^2, \\
(a(bc))a^{-1} &= a(bc)a^3 = (ab)^4c = c, \\
(a(c)a^{-1} &= bac.
\end{align}
Writing $x = \delta_{a^2}, y = \delta_{bc}$, and $z = \delta_c$, we get that $C_2$ interchanges $y$ and $z$ and fixes $x$.

Write $\sigma_1 = y + z, \sigma_2 = yz$, and $b = \delta_{\pi}$. Taking a $C_2$-projective resolution of $\mathbb{Z}$ and applying $\text{Hom}(-, H^*(BA))$, we get that the $s = 0$-line of the $E_2$-page is given by the invariants
\begin{equation}
E_2^{0,*} = \mathbb{F}[^{\sigma_1, \sigma_2, x}],
\end{equation}
and the $s$-lines for $s \geq 1$ are given by the invariants modulo the symmetrized classed, which is
\begin{equation}
E_2^{s,*} = \mathbb{F}_2[^{\sigma_2, x}]{[b^s]},
\end{equation}
which assembles to
\begin{equation}
E_2 = \mathbb{F}_2[^{b, \sigma_1, \sigma_2, x}]{[b\sigma_1]}
\end{equation}
with $(s, t)$-degrees given by $|b| = (1, 0), |\sigma_1| = (0, 1), |\sigma_2| = (0, 2), |x| = (0, 1)$.

The subgroup $C_4 = \langle a \rangle$ is normal because $b$ commutes with $a$ and $ca^2c^{-1} = (ab)^2 = a^2$.

The map of extensions
\begin{equation}
\begin{array}{c}
\langle a^2 \rangle = C_2 \\
\langle a^2, bc, c \rangle = A \\
\langle a \rangle = C_4 \\
\langle \pi \rangle = C_2
\end{array}
\end{equation}
gives a map of LHSSS’s. This is induced by the maps in cohomology of the left and right vertical maps, which are the inclusion $C_2 \to A$ and the corresponding quotient $C_2 \to C_2$ (which is the identity). The inclusion $C_2 \to A$ is on elements given by $a^2 \mapsto a^2$, hence on cohomology by
\begin{align}
x &= \delta_{a^2} \mapsto \delta_{a^2}, \\
y &= \delta_{bc} \mapsto 0, \\
z &= \delta_c \mapsto 0,
\end{align}
and therefore
\begin{align}
\sigma_1 &= y + z \mapsto 0, \\
\sigma_2 &= yz \mapsto 0.
\end{align}
The right hand vertical map $C_2 \to C_2$ of (4.51) is the identity, and hence in cohomology also given by the identity map $b = \delta_\pi \to \delta_\pi$.

Because the differentials of the LHSSS of the top row of (4.51) are given by $d_2(\delta a^2) = \delta_\pi^2$, we must have

\begin{align}
(4.57) & \quad d_2(x) = b^2, \\
(4.58) & \quad d_2(\sigma_1) = 0, \\
(4.59) & \quad d_2(\sigma_2) = 0,
\end{align}

Observe that $\sigma_1 x$ survives to $E_3$ because

\begin{equation}
(4.60) \quad d_2(\sigma_1 x) = \sigma_1 b^2 = 0.
\end{equation}

The $E_2$-page with differentials is depicted in the next figure.

The $E_3$-page is given by

\begin{equation}
(4.61) \quad E_3 = F_2[b, \sigma_1, \sigma_2, [x^2], [\sigma_1 x]]/(b\sigma_1, b^2, [\sigma_1 x]^2 - \sigma_1^2 [x^2], b[\sigma_1 x]),
\end{equation}

which has a vanishing line of height 2, and therefore the spectral sequence collapses as $E_3$.

Consider the quotient map in the group extension

\begin{equation}
(4.62) \quad A \to G \to C_2 \cong \langle \pi \rangle
\end{equation}

of which we just computed the LHSSS. This gives a 1-dimensional real representation of $G$ with oriented Euler class $e$. Because $e$ restricts to 0 on $A$, we see that $e$ lives in filtration $> 0$ on $E_\infty$, in other words $e = \delta_\pi$. The computation showed that $\delta_\pi^2 = 0$, since there is nothing in higher filtration. Hence by Lemma 3.10, $\exp_{e(2)} H_{F_2} \leq 2$, which together with the vanishing line of height 2 on the $E_\infty$-page implies $\exp_{e(3)} H_{F_2} = 2$. $\square$
4.8. The group $C_4 \times C_4$. Let $r$ and $s$ generate two copies of $C_4$: $\langle r \mid r^4 = e \rangle$, $\langle s \mid s^4 = e \rangle \cong C_4$. Let $\langle s \rangle$ act on $\langle r \rangle$ by $s \cdot r = r^{-1}$. Then $C_4 \times C_4$ is defined to be the semi-direct product $\langle r \rangle \rtimes \langle s \rangle$. A presentation is $C_4 \times C_4 = \langle r, s \mid r^4 = s^4 = e, srs^{-1} = r^3 \rangle$.

We will determine an upper bound on the $e_{(2)}$-exponent of $\underline{HF}_2 C_4 \times C_4$. To do so, we will need to run the $e_{(2)}$-homotopy limit spectral sequence converging to $H^*(BC_4 \times C_4; F_2)$ which is isomorphic to [CTVEZ03, App. C, #10(16)]

\[
\begin{align*}
\mathbf{F}_2[y, z, w]/(z^2 + y^2, zy).
\end{align*}
\]

with degrees $|z| = |y| = 1$, $|x| = |w| = 2$. We will determine an upper bound on $\exp e_{(2)} \underline{HF}_2$ using the following Lemma’s.

Lemma 4.9. The Euler class $e$ of the 1-dimensional real representation given by pulling back the sign representation along the quotient map

\[
\begin{align*}
C_4 \times C_4 \to C_4 \times C_4/\langle r, s^2 \rangle \cong C_2 : e^2 = 0.
\end{align*}
\]

Lemma 4.10. The Euler class $e$ of the 1-dimensional real representation given by pulling back the sign representation along the quotient map

\[
\begin{align*}
C_4 \times C_4 \to C_4 \times C_4/\langle r^2, s \rangle \cong C_2 \text{ s.t. } e^3 = 0.
\end{align*}
\]

We will first derive the upper bound on the exponent from these lemma’s, and then prove the lemma’s.

Proposition 4.11. The $e_{(2)}$-exponent satisfies

\[
\begin{align*}
\exp e_{(2)} \underline{HF}_2 C_4 \times C_4 \leq 4.
\end{align*}
\]

Proof. We just have to remark that $\langle r^2, s^2 \rangle$, which is the unique maximal elementary abelian 2-subgroup of $C_4 \times C_4$, is the intersection of $\langle r^2, s \rangle$ and $\langle r, s^2 \rangle$, the groups we divided out by in Lemma 4.9 and Lemma 4.10. Hence by Lemma 3.10 and Lemma 3.6, $\exp e_{(2)} \underline{HF}_2 \leq 2 + 3 - 1 = 4$. \hfill $\Box$

Proof of Lemma 4.9. The elements of order precisely 2 in $G$ are $r^2$, $s^2$, $r^2 s^2$. Hence all elementary abelian 2-subgroups are contained in $\langle r^2, s^2 \rangle =: V_4 \cong C_2 \times C_2$, a copy of the Klein four-group, which is also the center of $G$. Hence the $e_{(2)}$-homotopy limit spectral sequence reduces to the LHSS associated to the central extension

\[
\begin{align*}
V_4 \to G \to \langle \bar{r}, \bar{s} \rangle,
\end{align*}
\]

We observe that $G/V_4$ is also isomorphic to the Klein four-group, and call this group $V'_4$. Hence the spectral sequence takes the form

\[
\begin{align*}
E_2^{s,t} = H^*(BV'_4; H^t(BV_4; F_2)) \Rightarrow H^{s+t}(BG; F_2).
\end{align*}
\]

Since the action of $V'_4$ on the center is trivial the local coefficient system is also trivial. Hence the $E_2$-page is isomorphic to

\[
\begin{align*}
\mathbf{F}_2[\delta_r, \delta_s] \otimes \mathbf{F}_2[\delta_{r^2}, \delta_{s^2}],
\end{align*}
\]

with $(s, t)$-degrees $|\delta_r| = |\delta_s| = (1, 0)$, and $|\delta_{r^2}| = |\delta_{s^2}| = (0, 1)$. To compute $d_2$ we consider the map of spectral sequences given by restriction to various subgroups.
First consider the restriction \( \langle r \rangle \cong C_4 \hookrightarrow G \). This has one elementary 2-group, \( \langle r^2 \rangle \cong C_2 \), and we get

\[
\begin{array}{ccccccc}
1 & \to & C_2 & \to & C_4 & \to & C_2 \to 1 \\
1 & \to & V_4 & \to & G & \to & C_2 \times C_2 \to 1 \\
\end{array}
\]

(4.70)

We already saw that the LHSSS has a \( d_2 \) given by \( d_2(\delta_{r^2}) = \delta_{\bar{r}}^2 \), where we use the same notation for dual elements in the cohomology of the subgroups. This gives the following two conclusions for \( d_2 \) of the LHSSS \( G \):

1. \( d_2 \delta_{r^2} \) has a non-zero \( \delta_{\bar{r}}^2 \)-coefficient;
2. \( d_2 \delta_{s^2} \) has a zero \( \delta_{\bar{r}}^2 \)-coefficient.

Second we consider the restriction \( \langle s \rangle \cong C_4 \hookrightarrow G \). This gives a similar diagram

\[
\begin{array}{ccccccc}
1 & \to & C_2 & \to & C_4 & \to & C_2 \to 1 \\
1 & \to & V_4 & \to & G & \to & C_2 \times C_2 \to 1 \\
\end{array}
\]

(4.71)

Again by dimension and degree reasons we infer that

3. \( d_2 \delta_{s^2} \) has a non-zero \( \delta_{\bar{r}}^2 \)-coefficient;
4. \( d_2 \delta_{s^2} \) has a zero \( \delta_{\bar{r}}^2 \)-coefficient.

Finally, we consider the restriction to \( \langle rs \rangle \cong C_4 \hookrightarrow G \). Then \( (rs)^2 = s^2 \), and we get the diagram

\[
\begin{array}{ccccccc}
1 & \to & C_2 & \to & C_4 & \to & C_2 \to 1 \\
1 & \to & V_4 & \to & G & \to & C_2 \times C_2 \to 1 \\
\end{array}
\]

(4.72)

Now the inclusion \( C_2 \hookrightarrow C_2 \times C_2 ; \bar{r}s \mapsto \bar{r}s \) gives the restriction map in cohomology, which is given by

\[
\begin{align*}
H^*(BC_2 \times C_2; F_2) & \to H^*(BC_2; F_2) \\
\delta_{r^2} & \mapsto \delta_{\bar{r}s^2} \\
\delta_{s^2} & \mapsto \delta_{r\bar{s}}.
\end{align*}
\]

(4.73) (4.74) (4.75)

Again, the LHSSS supports a differential at \( \delta_{r^2} \) with image \( \delta_{\bar{r}s^2} \), which implies that

5. \( d_2 \delta_{r^2} \) has an odd number of non-zero \( \{ \delta_{r^2}, \delta_{r^2}, \delta_{r^2}, \delta_{s^2} \} \)-coefficients.
6. \( d_2 \delta_{r^2} \) has an even number of non-zero \( \{ \delta_{r^2}, \delta_{r^2}, \delta_{s^2} \} \)-coefficients.
Taking all these things together we obtain
\begin{align}
  (4.76) & \quad d_2 \delta r^2 = \delta^2 \delta r, \\
  (4.77) & \quad d_2 \delta s^2 = \delta^2 \delta s.
\end{align}

Therefore we get that the $E_3$-page is
\begin{equation}
  (4.78) \quad E_3 \cong F_2[\delta r, \delta s]/(\delta^2 r, \delta r \delta s + \delta^2 s) \otimes F_2[\delta^2 r^2, \delta^2 s^2].
\end{equation}

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{fig2a}
\caption{$E_2$.}
\end{subfigure}
\hfill
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{fig2b}
\caption{$E_3$.}
\end{subfigure}
\caption{The LHSS for the group extension $V_4 \to C_4 \rtimes C_4 \to C_2 \times C_2$. On the left hand side the generating differentials on the $E_2$-page are drawn, on the right hand side the resulting $E_3$-page.}
\end{figure}

This page has a vanishing line of height 3, and therefore
\begin{align}
  (4.79) & \quad E_\infty = E_3 \\
  (4.80) & \quad = F_2[\delta r, \delta s]/(\delta^2 r, \delta r \delta s + \delta^2 s) \otimes F_2[\delta^2 r^2, \delta^2 s^2].
\end{align}

We have the composite of quotient maps
\begin{equation}
  (4.81) \quad r C_4 \times C_4 \to \overline{r} \overline{C}_2 \times \overline{C}_2 \to \overline{C}_2,
\end{equation}

the composite being the quotient-by-$\langle r, s^2 \rangle$-map. The Euler class of the sign representation of $C_2$ is $\delta \tau \in H^1(BC_2)$. Euler classes are natural, so $e$ is the pullback of $\delta \tau \in H^1(BC_2 \times C_2)$. In other words, the edge map $E_3^{1,0} \to H^1(BC_4 \times C_4)$ maps $\delta \tau \to e$. Hence $\delta \tau$ detects $e$ in the LHSSS. Since $\delta^2 \tau = 0$ on $E_3$ on account of the differentials in (4.77), $e^2$ is zero up to higher filtration, of which there is none.

\begin{proof}[Proof of Lemma 4.10] The proof is the same as the proof of Lemma 4.9, but now $e$ is detected by $\delta \tau$ instead of $\delta \tau$. the class $\delta^2 \tau$ is non-zero, but $\delta^3 \tau = 0$, hence $e^3$ is zero up to higher filtration, of which there is none.
\end{proof}

\begin{remark}
The fact that the $\delta_{(2)}$-homotopy limit spectral sequence has a vanishing line of height 3 implies that
\begin{equation}
  (4.82) \quad H\text{F}_2 C_4 \rtimes C_4 \geq 3.
\end{equation}
\end{remark}
4.9. The group $D_8 \times C_2$.

Proposition 4.13. For $D_8 \times C_2$, the $\mathfrak{e}_{(2)}$-exponent of $HF_2$ satisfies

$$\exp_{\mathfrak{e}_{(2)}} H F_2 \leq 2.$$  (4.83)

Proof. Pulling back transitive $D_8$-orbits with isotropy in elementary abelian groups along the projection map $D_8 \times C_2 \to D_8$ gives transitive $D_8 \times C_2$-orbits with isotropy in elementary abelian groups. Therefore pulling back the representation of $D_8$ considered in the proof of Corollary 4.3 along the projection map $D_8 \times C_2 \to D_8$ gives a 2-dimensional real representation of $D_8 \times C_2$ with isotropy in $\mathfrak{e}_{(2)}$. Hence by Proposition 3.17 we have $\exp_{\mathfrak{e}_{(2)}} H F_2 \leq 2$.

□

4.10. The group $Q_8 \times C_2$.

Proposition 4.14. For $Q_8 \times C_2$, the $\mathfrak{e}_{(2)}$-exponent of $HF_2$ is

$$\exp_{\mathfrak{e}_{(2)}} H F_2 \leq 2.$$  (4.84)

Proof. Elementary abelian subgroups of $Q_8$ pull back to elementary abelian subgroups of $Q_8 \times C_2$ along the projection map $Q_8 \times C_2 \to Q_8$. Therefore, pulling back the 4-dimensional real representation of $Q_8$ whose projectivation has isotropy in $\mathfrak{e}_{(2)}$ considered in [MNN15, Ex. 5.18] along the projection map gives a 4-dimensional real representation of $Q_8 \times C_2$ with projectivation with isotropy in $\mathfrak{e}_{(2)}(Q_8 \times C_2)$. Hence, by Proposition 3.17, the exponent satisfies $\exp_{\mathfrak{e}_{(2)}} H F_2 \leq 4$.

□

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