TETRAHEDRA WITH CONGRUENT FACE PAIRS

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Abstract. If the four triangular facets of a tetrahedron can be partitioned into pairs having the same area, then the triangles in each pair must be congruent to one another. A Heron-style formula is then derived for the volume of a tetrahedron having this kind of symmetry.

Mathematics Subject Classification: 52B10, 52B12, 52B15, 52A38.

From elementary geometry we learn that two triangles are congruent if their edges have the same three lengths. In particular, there is only one congruence class of equilateral triangles having a given edge length. Said differently, any pair of equilateral triangles in the Euclidean plane are similar, differing at most by an isometry and a dilation. Meanwhile, triangles that are symmetric under a single reflection have two congruent sides and are said to be isosceles.

The situation is more complicated in higher dimensions. Indeed, an analogous characterization of 3-dimensional tetrahedra already leads to 25 different symmetry classes [22]. These tetrahedral symmetry classes are of special interest in organic chemistry [8, 9, 21], and conditions for tetrahedral symmetry based on the measures of dihedral angles have also been explored [23].

A tetrahedron in $\mathbb{R}^3$ is equilateral or regular if all of its edges have the same length. More generally, a tetrahedron is said to be isosceles if all four triangular facets are congruent to one another, or, equivalently, if opposing (non-incident) edges have the same length. Isosceles tetrahedra are also known as disphenoids [4, p. 15]. It has been shown that if all four facets of a tetrahedron $T$ have the same area, then $T$ must be isosceles [10, p. 94][11, 16].

Consider the following more general symmetry class of tetrahedra: A tetrahedron $T$ will be called reversible if its facets are congruent in pairs; that is, if the facets of $T$ can be labelled $f_1, f_2, f_3, f_4$, where $f_1 \cong f_2$ and $f_3 \cong f_4$.

![Figure 1. An reversible tetrahedron with edge lengths $a, a, b, b, c, d$.](image)
In this note we show that, as in the isosceles case, reversible tetrahedra are characterized by the areas of their facets: if the four triangular facets of $T$ can be partitioned into pairs with the same area, then those pairs consist of congruent facets.

In the final section we give an intuitive method for deriving a Heron-style factorization of the volume of a reversible tetrahedron in terms of its edge lengths.

1. Facets normals and areas determine tetrahedra

The following proposition will allow us to exploit symmetries more easily.

**Proposition 1.1.** Suppose that a tetrahedron $T$ has outward facet unit normals $u_0, u_1, u_2, u_3$, with corresponding facet areas $\alpha_0, \alpha_1, \alpha_2, \alpha_3 > 0$. Then

\[
\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0.
\]

Conversely, if unit vectors $u_0, u_1, u_2, u_3$ span $\mathbb{R}^3$, and if $\alpha_i > 0$ satisfy (1), then there exists a tetrahedron $T$, having outward facet unit normals $u_i$, and corresponding facet areas $\alpha_i$, and this tetrahedron is unique up to translation.

This proposition is a very special case of the Minkowski Existence Theorem, which plays a central role in the Brunn-Minkowski theory of convex bodies, and is somewhat difficult to prove [3, 18]. However, this special case for tetrahedra is a simple consequence of linear algebra.

![Figure 2. A tetrahedron with outward unit normals $u_i$.](image)

**Proof.** Let $T$ be a tetrahedron with vertices at $v_0, v_1, v_2, v_3 \in \mathbb{R}^3$, where $v_0 = o$, the origin. Let us assume the vertices are labelled so that $v_1, v_2, v_3$ have a positive ("right-handed") orientation.

Denote by $u_0, u_1, u_2, u_3$ the outward unit normal vectors of the facets of $T$, where $u_i$ is associated with the facet opposite to the vertex $v_i$, as in Figure 2. Let $\alpha_i$ denote the area of that same
ith facet. Since \( v_0 = 0 \), we have
\[
\begin{align*}
v_2 \times v_3 &= -2\alpha_1 u_1 \\
v_3 \times v_1 &= -2\alpha_2 u_2 \\
v_1 \times v_2 &= -2\alpha_3 u_3 \\
(v_3 - v_1) \times (v_2 - v_1) &= -2\alpha_0 u_0.
\end{align*}
\]
(2)

After summing both sides of these equations the identity (1) now follows.

To prove the converse, suppose we are given unit vectors \( u_0, u_1, u_2, u_3 \) that span \( \mathbb{R}^3 \) and \( \alpha_i > 0 \) satisfying (1). Let \( \tilde{T} \) denote the intersection of the closed half-spaces \( x \cdot u_i \leq 1 \).

The spanning condition on the \( u_i \), along with the identity (1), imply that any 3 of the vectors \( u_i \) are linearly independent. Since each \( \alpha_i > 0 \), it follows from (1) that \( \tilde{T} \) is a bounded tetrahedron with facets normal to the \( u_i \). Translate this tetrahedron so that one vertex lies at the origin \( o \), and then slide the facet opposite to \( o \) along the direction of \( u_0 \) so that this facet has area \( \alpha_0 \). This new tetrahedron \( T \) now has facet areas \( \alpha_0' = \alpha_0, \alpha_1', \alpha_2', \) and \( \alpha_3' \), and must also satisfy (1), so that
\[
\alpha_0 u_0 + \alpha_1' u_1 + \alpha_2' u_2 + \alpha_3' u_3 = 0.
\]
Combining this with identity (1) for the original given data yields
\[
\alpha_1' u_1 + \alpha_2' u_2 + \alpha_3' u_3 = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3.
\]
Since \( u_1, u_2, u_3 \) are linearly independent, each \( \alpha_i' = \alpha_i \), and \( T \) is the tetrahedron required, unique up to translation. \( \square \)

**Remark:** Given the surface data \( u_i \) and \( \alpha_i \) it is not difficult to construct the corresponding tetrahedron \( T \) explicitly. To do so, let \( C \) denote the \( 3 \times 3 \) matrix having columns \( -2\alpha_i u_i \) for \( i > 0 \), ordered so that \( C \) has positive determinant. The matrix
\[
A = \det(C)^2 C^{-1}
\]
has cofactor matrix \( C \). It is not difficult to show (using Cramer’s Rule and basic linear algebra) that the columns of \( A \), along with the origin, yield the vertices of a tetrahedron having facet normals \( u_i \) and corresponding facet areas \( \alpha_i \). Uniqueness up to translation also follows from this explicit construction (which generalizes to \( n \) dimensions as well).

## 2. Equal areas imply congruent faces

We now prove that the areas of the facets alone will determine if a tetrahedron is reversible.

**Theorem 2.1.** Suppose that \( T \) is a tetrahedron in \( \mathbb{R}^3 \), and denote by \( f_1, f_2, f_3, f_4 \) the triangular facets of \( T \). If the facets of \( T \) satisfy the conditions
\[
\text{Area}(f_1) = \text{Area}(f_2) \quad \text{and} \quad \text{Area}(f_3) = \text{Area}(f_4)
\]
then \( f_1 \cong f_2 \) and \( f_3 \cong f_4 \).

The proof of Theorem 2.1 uses the method given by McMullen in [16] to verify the special case in which all four facets have the same area (as in Corollary 2.2 below).
Proof. Denote by $u_i$ the outward unit normal vector to the facet $f_i$ of $T$. Suppose that $\text{Area}(f_1) = \text{Area}(f_2) = \alpha$ and $\text{Area}(f_3) = \text{Area}(f_4) = \beta$, where $\alpha, \beta > 0$. The identity (1) asserts that

$$\alpha u_1 + \alpha u_2 + \beta u_3 + \beta u_4 = 0.$$ 

Denote

$$w = \alpha u_1 + \alpha u_2 = -\beta u_3 - \beta u_4.$$ 

Let $\psi$ denote the rotation of $\mathbb{R}^3$ by the angle $\pi$ around the the axis through $w$. Since the vectors $\alpha u_1$ and $\alpha u_2$ have the same length, the points $0, \alpha u_1, \alpha u_2, w$ are the vertices of a rhombus. The rotation $\psi$ rotates this rhombus onto itself, exchanging the vectors $\alpha u_1$ and $\alpha u_2$. The points $0, \beta u_3, \beta u_4, -w$ form a rhombus through the same axis, so that $\psi$ also exchanges the vectors $\beta u_3$ and $\beta u_4$. Since $\psi$ is a rotation, it preserves orthogonality. It follows that $P$ and $\psi P$ have the same normal vectors and the same corresponding facet areas. Proposition 1.1 then implies that $P$ and $\psi P$ are congruent by a translation. In particular, the facets $f_1$ and $f_2$ are congruent, as are $f_3$ and $f_4$. \hfill $\square$

The case of isosceles tetrahedra described in the introduction follows as an immediate corollary to Theorem 2.1.

Corollary 2.2. Suppose that $T$ is a tetrahedron in $\mathbb{R}^3$. If the faces $f_i$ of $T$ satisfy the condition

$$\text{Area}(f_1) = \text{Area}(f_2) = \text{Area}(f_3) = \text{Area}(f_4)$$

then $f_1 \cong f_2 \cong f_3 \cong f_4$.

In other words, if a tetrahedron $T$ is equiareal, then $T$ is also isosceles. For alternative proofs and variants of Corollary 2.2, see [10, 11, 15, 16].

Remark: Corollary 2.2 has long been known to have an analogue in which area is replaced by perimeter. The proof is very simple: If all of the facets of $T$ have the same perimeter, the resulting system of linear equations (in the six edge lengths of $T$) implies that opposing edges must have the same length, so that $T$ is isosceles. A similar argument shows that if the facets of $T$ can be partitioned into pairs having the same perimeter then $T$ is reversible.

3. Factoring the volume

Suppose that $T \subseteq \mathbb{R}^3$ is a tetrahedron with vertices at $v_0, v_1, v_2, v_3 \in \mathbb{R}^3$, where $v_0 = 0$, the origin. As before, let $A$ denote the matrix whose columns are given by the vectors $v_i$, and suppose that the $v_i$ are ordered so that $A$ has positive determinant. The volume of $T$ is then given by $\det(A) = 6V(T)$, so that

$$V(T)^2 = \frac{1}{36} \det(A^TA).$$

The entries of the matrix $A^TA$ are dot products of the form $v_i \cdot v_j$. From the identity,

$$2v_i \cdot v_j = |v_i|^2 + |v_j|^2 - |v_i - v_j|^2$$

(3)
it then follows that the value of $V(T)^2$ is a polynomial in the squares of the edge lengths of $T$. Said differently, if $T$ has edge lengths $a_{ij}$ (the distance between vertices $v_i$ and $v_j$), then $V(T)^2$ is a polynomial in the variables $b_{ij} = a_{ij}^2$, as well as the variables $a_{ij}$ themselves. This polynomial is sometimes formulated in terms of linear algebraic expressions such as Cayley-Menger determinants [19, p. 125]. While the Cayley-Menger heuristic outlined above applies in arbitrary dimension, the 3-dimensional case has been known at least as far back as Piero della Francesca [17].

In certain instances, the polynomial $V(T)^2$ admits factorization into linear or quadratic irreducible factors. For the 2-dimensional case, the area $A(\Delta)$ of a triangle $\Delta$ having edge lengths $a, b, c$ is given by

$$A(\Delta)^2 = \frac{1}{16}(a + b + c)(-a + b + c)(a - b + c)(a + b - c),$$

a factorization known as Heron’s formula [5, p. 58]. Although the 3-dimensional case is more complicated [7], there exist non-trivial factorizations of $V(T)^2$ when the tetrahedron $T$ satisfies the symmetry properties examined in the previous section.

For example, if $T$ is an isosceles tetrahedron, having edge lengths $a, b, c$ (each repeated twice in pairs of opposing edges), then

$$V(T)^2 = \frac{1}{72}(a^2 + b^2 - c^2)(a^2 - b^2 + c^2)(-a^2 + b^2 + c^2).$$

A synthetic proof of (4) can be found in [20, p. 101]. Instead we will give an algebraic proof of the following more general result, using a technique outlined in [13].

The edges of a reversible tetrahedron $T$ come in (at most) 4 lengths. To see this, label the edge lengths of $T$ so that the triangular facets $f_1 \cong f_2$ have edge lengths $a, b, c$, with common edge of length $c$. Since $f_3 \cong f_4$, they must have edge lengths $a, b, d$. The six edges of $T$ then have lengths $a, a, b, b, c, d$, as in Figure 1.

**Theorem 3.1** (Volume Formula). Suppose that $T$ is a reversible tetrahedron having edge lengths $a, a, b, b, c, d$. Then

$$V(T)^2 = \frac{1}{72}\left(c^2d^2 - (a^2 - b^2)^2\right)\left(a^2 + b^2 - \frac{c^2 + d^2}{2}\right).$$

The first polynomial factor in the formula (5) is a difference of two squares, so that (5) can be reformulated as

$$V(T)^2 = \frac{1}{72}(cd + a^2 - b^2)(cd - a^2 + b^2)\left(a^2 + b^2 - \frac{c^2 + d^2}{2}\right).$$

In the special case where $c = d$, the tetrahedron $T$ is isosceles, and the formula (6) reduces to (4).

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1Piero della Francesca (1415-1492), an Italian painter and geometer of the early Renaissance period.
The proof of (5) will make use of two identities from plane geometry. The well-known parallellogram law asserts that if edges of a parallelogram in $\mathbb{R}^2$ are labelled as in Figure 1, then $2a^2 + 2b^2 = c^2 + d^2$.

The less well-known trapezoid law asserts that, if the edges of a convex isosceles trapezoid are labelled as in Figure 3, then $b^2 - a^2 = cd$.

![Figure 3. The trapezoid law: $b^2 - a^2 = cd$.](image)

To see why, observe that

$$b^2 - a^2 = |u - w|^2 - |v - w|^2$$

$$= u \cdot u - 2u \cdot w + w \cdot w - v \cdot v + 2v \cdot w - w \cdot w$$

$$= |u|^2 - |v|^2 + 2w \cdot (v - u)$$

$$= a^2 - b^2 + 2cd,$$

where the last step follows from the parallelism of $w$ and $v - u$. The trapezoid law now follows.

**Proof of The Volume Formula 3.1.** Let $f(a, b, c, d)$ denote the polynomial $V(T)^2$. The factors of $f$ can be determined by considering the cases in which the volume of $T$ is zero, namely, when the tetrahedron $T$ is flat or otherwise degenerate. If $T$ is reversible, this can occur in two ways.

In one case, $T$ may flatten to a parallelogram, having edges of length $a, b, a, b$ and diagonals of length $c, d$. In this instance, the parallelogram law for the standard inner product implies that $2a^2 + 2b^2 = c^2 + d^2$.

In the second case, $T$ may flatten to a trapezoid, having non-parallel edges of length $a, a$, parallel edges of length $c, d$, and diagonals of length $b, b$. In this instance, the trapezoid law implies that $(b^2 - a^2)^2 = c^2d^2$.

These cases suggest both $2a^2 + 2b^2 - c^2 - d^2$ and $c^2d^2 - (b^2 - a^2)^2$ as possible factors of the polynomial $f$.

Denote $A = a^2$, $B = b^2$, $C = c^2$ and $D = d^2$. We observed following (3) above that $f$ is a polynomial in the squared values $a^2, b^2, c^2, d^2$, so that $f = f(A, B, C, D) \in \mathbb{R}[A, B, C, D]$. Since volume $V$ is homogeneous of degree 3 with respect to length, the polynomial $f = V^2$ is homogeneous of degree 6 with respect to the variables $a, b, c, d$, and is therefore homogeneous of degree 3 with respect to the variables $A, B, C, D$; that is, a homogeneous cubic polynomial in $\mathbb{R}[A, B, C, D]$. 
To verify that \(2a^2 + 2b^2 - c^2 - d^2\) is indeed a factor of \(f(a, b, c, d)\), use division with remainder in \(\mathbb{R}[A, B, C, D]\) to obtain
\[
f(A, B, C, D) = (2A + 2B - C - D)g(A, B, C, D) + r(B, C, D),
\]
for some \(g \in \mathbb{R}[A, B, C, D]\) and \(r \in \mathbb{R}[B, C, D]\). Here division with remainder in \(\mathbb{R}[A, B, C, D]\) is performed here using lexicographical order on the variables \(A, B, C, D\). (See, for example, [6, p. 54].)

Note that \(A\) does not appear in the polynomial expression for \(r\). Suppose that \(C > D > 0\). By the triangle inequality, each \(B\) such that
\[
\sqrt{C} - \sqrt{D} < 2 \sqrt{B} < \sqrt{C} + \sqrt{D}
\]
gives rise to a parallelogram as in Figure 4, yielding \(A \geq 0\) so that \(2A + 2B - C - D = 0\). This degenerate reversible tetrahedron \(T\) has volume zero, so that \(f(A, B, C, D) = V^2 = 0\). It follows that \(r(B, C, D) = 0\) on a non-empty open set. Since \(r\) is a polynomial, it follows that \(r\) is identically zero, so that
\[
f(A, B, C, D) = (2A + 2B - C - D)g(A, B, C, D).
\]
In other words, \(2A + 2B - C - D\) divides \(f\) in \(\mathbb{R}[A, B, C, D]\).

![Figure 4](image)

Figure 4. This parallelogram exists iff \(\frac{1}{2} \sqrt{C} - \frac{1}{2} \sqrt{D} \leq \sqrt{B} \leq \frac{1}{2} \sqrt{C} + \frac{1}{2} \sqrt{D}\).

For the trapezoidal factors, view \(f\) as polynomial in \(\mathbb{R}[A, B, c, d]\), and write
\[
f(A, B, c, d) = (cd - B + A)\tilde{g}(A, B, c, d) + \tilde{r}(B, c, d),
\]
using division with remainder in \(\mathbb{R}[A, B, c, d]\) under lexicographical order on the variables \(A, B, c, d\). Once again the remainder \(\tilde{r}\) is independent of the variable \(A\), while a trapezoidal degenerate (zero volume) tetrahedron can be constructed for an open set of values \((B, c, d)\), so that \(\tilde{r}\) is also identically zero. Therefore, \(cd - B + A\) is also a factor \(f\).

Finally, a symmetrical argument (reversing the roles of \(A\) and \(B\)) yields a factor of \((cd - A + B)\).

Since \(\mathbb{R}[A, B, c, d]\) is a unique factorization domain [2, p. 371][6, p. 149], the irreducible factors \((cd - B + A)\), and \((cd - A + B)\) are prime, so that
\[
(cd - B + A)(cd - A + B) = c^2d^2 - (B - A)^2 = CD - (B - A)^2
\]
divides \(f\).
Similarly, since $\mathbb{R}[A, B, C, D]$ is a unique factorization domain, the two irreducible factors $CD - (B - A)^2$, and $2A + 2B - C - D$ are prime in $\mathbb{R}[A, B, C, D]$, so that

$$V^2 = f = (2A + 2B - C - D)(CD - (B - A)^2) k.$$  

Because $f$ is a homogeneous cubic polynomial in $\mathbb{R}[A, B, C, D]$, the factor $k$ must be a constant, independent of the parameters $A, B, C, D$.

To compute the constant $k$, recall that the volume of the regular (equilateral) tetrahedron of unit edge length $A = B = C = D = 1$ is $\sqrt{2}/12$. It follows that

$$\frac{1}{72} = \left(\frac{\sqrt{2}}{12}\right)^2 = V^2 = f(1, 1, 1, 1) = 2k.$$  

Hence, $k = 1/144$, and (7) becomes (5).

I. Izmestiev has pointed out that applying the Regge symmetry [1] to a reversible tetrahedron gives a new reversible tetrahedron having the same volume, and for which the factors of the Cayley-Menger polynomial (6) are permuted [12].

4. Generalizations

A convex polytope $P$ in $\mathbb{R}^n$ will be called reversible if there is an affine plane $\xi$ of co-dimension 2 such that $P$ is symmetric under the $180^\circ$ rotation of the 2-plane $\xi^\perp$ that fixes $\xi$.

If a tetrahedron $T$ is $\mathbb{R}^3$ is symmetric under a $180^\circ$ rotation around a line $\ell$, then this rotation must map facets to facets and facet normals to facet normals. In view of Proposition 1.1, the only way this can occur is when $\ell$ passes through the midpoints of two non-adjacent edges of $T$, so that $T$ must have pairs of congruent facets, as in the examples addressed earlier. It follows that this more general definition of a reversible polytope is consistent with the definition given earlier for tetrahedra in $\mathbb{R}^3$. However, naive analogues of the theorems of this paper do not follow, because this level of symmetry admits many more variations in structure for dimensions $n \geq 4$. Indeed, there exist 4-dimensional simplices in which all 5 facets have the same volume in spite of not being mutually congruent. For an extensive treatment of this subject, see [16].

In addition to admitting the Heron-type formula (4) for volume, isosceles tetrahedra satisfy many other characteristic properties (see, for example, [10, p. 90-97][14]). It would be interesting to consider what parallels these other properties may have in the more general context of reversible tetrahedra.

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