A quantum mechanical system has to be Hermitian [1]. He showed that, in order to possess a real eigenvalue spectrum, a Hamiltonian does not have to be Hermitian: also Hamiltonians that are invariant under combined parity-time \((\mathcal{PT})\) symmetry operations have this property \([2, 3]\). These findings had profound impact in particular on photonics research where the required potential landscapes can be easily generated by appropriately distributing gain and loss for electromagnetic waves. On this ground it was possible to show, for example, the existence of non-orthogonal eigenmodes \([4]\), non-reciprocal light evolution \([5]\), and \(\mathcal{PT}\)-symmetric lasers \([6, 7]\). Even in fields beyond photonics \(\mathcal{PT}\)-symmetry has an impact, ranging from \(\mathcal{PT}\)-symmetric atomic diffusion \([8]\), superconducting wires \([9, 10]\), and \(\mathcal{PT}\)-symmetric electronic circuits \([11]\). However, all these are classical phenomena as only single electromagnetic wave packets are involved. The experimental demonstration of truly quantum features in \(\mathcal{PT}\)-symmetric systems with gain and loss is still elusive. Here we show that this will remain to be the case. Our investigations unequivocally prove that the common approach for realising \(\mathcal{PT}\)-symmetric systems in photonics by concatenating lossy and amplifying media always results in thermally broadened quantum states. \(\mathcal{PT}\)-symmetric quantum optics in gain/loss systems is not possible. Within the framework of macroscopic quantum electrodynamics we show that gain and loss are associated with non-compact and compact operator transformations, respectively. This implies a fundamentally different way in which quantum correlations between a quantum system and a reservoir are built up and destroyed.

\(\mathcal{PT}\)-symmetric photonic quantum systems with gain and loss do not exist

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We discuss the impact of gain and loss on the evolution of photonic quantum states and find that \(\mathcal{PT}\)-symmetric quantum optics in gain/loss systems is not possible. Within the framework of macroscopic quantum electrodynamics we show that gain and loss are associated with non-compact and compact operator transformations, respectively. This implies a fundamentally different way in which quantum correlations between a quantum system and a reservoir are built up and destroyed.

The implementation of \(\mathcal{PT}\)-symmetry in photonics is based on the observation that the Schrödinger equation of quantum mechanics and the Helmholtz equation of electromagnetism are formally equivalent if the potential \(V(x)\) in the Schrödinger equation is replaced by the refractive index profile \(n(x)\) in the Helmholtz equation \([19]\). \(\mathcal{PT}\)-symmetry then translates into the condition for the complex refractive index \(n(-x) = n^*(x)\), in particular, the real part \(n_R(x)\) is symmetric and the imaginary part \(n_I(x)\) is antisymmetric under the parity operation. The latter implies that loss in one propagation direction has to be compensated by an identical gain in the opposite direction \([12]\). Whereas this concept is well-defined for the amplitudes of classical electromagnetic waves, this is no longer the case for the amplitude operators of quantum states of light as they have to obey certain commutation relations. For example, the amplitude operators for a single harmonic oscillator have to fulfil the relation \([\hat{a}, \hat{a}^\dagger] = 1\) for all times. However, phenomenologically a dissipation process is always accompanied by additional (Langevin) noise. Hence, the evolution equation for a damped harmonic oscillator mode with frequency \(\omega\) has...
to be written as
\[ \dot{a} = (-i\omega - \Gamma)a + \hat{f} \]  
(1)

with properly chosen commutation relations between the harmonic oscillator mode and the noise operators, and where the fluctuation strength of the noise operator $\hat{f}$ is related to the damping rate $\Gamma$ \[20\].

The appropriate framework in which to describe the propagation of quantum states of light through absorbing and amplifying media is macroscopic quantum electrodynamics \[21, 22\]. Here the creation and annihilation operators of the free electromagnetic field have to be replaced by new operators that describe the collective excitation of the field and the absorbing or amplifying matter. Within the framework of linear response this theory is exact. The result is a proper identification of the parameters $\Gamma$ and $\hat{f}$ in Eq. (1) by phenomenological quantities such as absorption and transmission coefficients. This theory provides the basis for the propagation of quantum states of light through absorbing and amplifying media \[23, 24\]. One first constructs a unitary operation in a larger Hilbert space of field and medium operators which, after projecting onto the field quantities alone, results in an effective, typically non-unitary evolution of the quantum states of light \[23\]. Although the formalism is very similar for absorbing and amplifying media, there are crucial differences between them that impact the $\mathcal{PT}$-symmetry. Viewing an optical element as a four-port device with two input and two output channels for light of a given frequency $\omega$ (note that in a linearly responding medium light modes of different frequencies do not mix), the quantum-state transformation at absorbing media corresponds to a compact SU(4) transformation \[23\] whereas the equivalent relation at amplifying media is a non-compact SU(2,2) transformation \[24\]. This seemingly innocuous difference has far-reaching consequences: an initial coherent quantum state $|\alpha_0\rangle$, after propagation through an absorbing medium with transmission coefficient $T$, remains a coherent quantum state, albeit with diminished coherent amplitude $|Ta_0\rangle$. On the contrary, after propagation through an amplifying medium, a coherent state turns into a displaced thermal state with an effective temperature that depends on the gain (for details of the calculation, see Supplementary Material).

We illustrate this fundamental difference by the propagation of a coherent quantum state through a system that consists of concatenated regions of loss and gain. In Fig. 1 we show the Wigner function (a phase-space distribution function that is formally equivalent to the quantum state \[25\]) of a coherent quantum state with coherent amplitude $\alpha_0 = 3 + 3i$ (left), after transmission through an optical device with transmission coefficient $T = 2/3$ (center), and after propagation through a gain medium with $G = 1/T$ (right). One clearly observes that initial and final states after propagation through media with loss and subsequent gain are not equivalent.

Despite the apparent simplicity of our example, it does have far-reaching consequences for any attempt to study $\mathcal{PT}$-symmetry of quantum optical systems. First, gain always adds thermal noise to a quantum state, no matter how gain and loss are spatially distributed. Second, we have chosen coherent quantum states that are, on the one hand, minimum-uncertainty states that closely resemble classical states \[23\] and, on the other hand, have the unique property that their purity is not affected by loss. Any other type of quantum state will already be drastically altered by an absorbing medium. For example, pure photon-number states turn into mixed states with lower photon numbers \[23\]. As a consequence, any quantum state of light is crucially altered when propagating through any distribution of gain and loss. In other words, in a concatenated gain/loss system the quantum eigenstates of a Hamiltonian can never be eigenstates of the $\mathcal{PT}$-operator, such that for quantum states there is always $[\hat{H}, \mathcal{PT}] \neq 0$. This brings us to the conclusion that in such systems $\mathcal{PT}$-symmetric quantum optics does not exist.

The reason behind this surprising result is the fact that, in order to obtain amplification, the quantum system under study has to be coupled to an external reservoir that provides the necessary energy input. This coupling necessarily introduces noise that can be cast into a form similar to Eq. (1),
\[ \dot{a} = (-i\omega + \Gamma)a + \hat{f}' \]  
(2)

where the (Langevin) noise operators fulfil the same com-
mutation relations as before. The crucial difference between loss and gain is the way in which the external reservoir is coupled to the quantum system (see Fig. 2). In case of gain, the noncompact SU(2,2) group transformation implies a build-up of quantum correlations between system and reservoir that, when only observing the system, are destroyed and manifest themselves as thermal fluctuations. This is a similar mechanism as that observed in two-mode squeezing (which is described by a SU(1,1) transformation) where the quantum correlations of the two squeezed modes result in thermal distributions of the individual modes. Therefore, there is no quantum gain mechanism that can compensate for any quantum loss process.

**FIG. 2:** Photonic quantum systems with gain and loss. In order to describe a non-Hermitian quantum system with gain and loss, it has to be coupled to external reservoirs that act as source and sink, respectively. This coupling introduces Langevin noise $\hat{f}$ and $\hat{f}^\dagger$ to ensure the Hermiticity of the full scheme. This noise, however, always alters propagating quantum states, even the eigenstates of the Hamiltonian of the quantum system.

Importantly, our result does not only hold for harmonic oscillator modes such as photons. Indeed, any system in which a complex potential is derived from a coupling to a reservoir and, accordingly, admits a description by a Langevin equation suffers from a similar conclusion. Hence, our conclusions are not restricted to bosonic systems or, indeed, photons, but also hold for fermionic systems, such as electrons.

To summarize our work, we have shown that $\mathcal{PT}$-symmetric quantum optics is not possible within the current perception of implementing $\mathcal{PT}$-symmetry, that is, using gain and loss. However, we foresee three alternative approaches that might allow the observation of phenomena arising from $\mathcal{PT}$-symmetry in the quantum realm. First, one would have to realize a complex $\mathcal{PT}$-invariant Hamiltonian without coupling the quantum system to an external reservoir. However, to date such a concept is elusive. Second, one would need to construct noiseless amplifiers. Yet, for deterministic gain processes this violates the No-cloning theorem of quantum mechanics. Indeed, relaxing the determinism constraint and allowing probabilistic processes may result in devising probabilistic noiseless amplifiers. An altogether different route involves replacing the active gain by another passive loss medium such that the overall system is lossy, which results in so-called passive $\mathcal{PT}$-symmetric systems. Such structures are already implemented and seem promising candidates for observing physics akin to $\mathcal{PT}$-symmetric quantum optics.

The authors thank Mark Kremer for helping to prepare the figures and acknowledge the Deutsche Forschungsgemeinschaft (grant BL 574/13-1) for financial support.

**Appendix**

The idea of our approach is to discretize the time evolution of the electromagnetic field and construct input-output relations between photonic amplitude operators before and after the propagation through an optical device. Let the amplitude operators of the radiation field at frequency $\omega$ at the input of the optical device be denoted by $\hat{a}$, the corresponding output amplitude operators by $\hat{b}$, and the (Langevin) operators associated with the device by $\hat{g}$. Then, the following input-output relations read $\hat{\beta} = T \hat{a} + A \hat{d}$ where the transformation and absorption matrices satisfy $T T^\dagger + \sigma AA^\dagger = I$ and $\sigma = +1$, $\hat{d} = \hat{g}$ for absorption and $\sigma = -1$, $\hat{d} = \hat{g}^\dagger$ for amplification.

Although a unitary evolution of the field operators alone is no longer possible, one can nevertheless construct a unitary evolution of the combined field-device system. Define the four-vector operators $\hat{\alpha} = (\hat{a}, \hat{d})^T$ and $\hat{\beta} = (\hat{b}, \hat{f})^T$ where $\hat{f} = \hat{h}$ for absorption and $\hat{f} = \hat{h}^\dagger$ for amplification with some auxiliary bosonic device variables $\hat{h}$. Then, the input-output relations can be elevated to a unitary relation between the four-vector operators as $\hat{\beta} = \Lambda \hat{\alpha}$ with $\Lambda J \Lambda^\dagger = J$, $J = \begin{pmatrix} I & 0 \\ 0 & \sigma I \end{pmatrix}$.

If one introduces the commuting positive Hermitian matrices $C = \sqrt{T T^\dagger}$ and $S = \sqrt{\Lambda \Lambda^\dagger}$, then the unitary matrix $\Lambda$ can be written as

$$\Lambda = \begin{pmatrix} T & A \\ -\sigma SC^{-1}T & CS^{-1}A \end{pmatrix}.$$  

The input-output relation for the amplitude operators can be cast into a quantum-state transformation formula. Let the density operator of the input quantum state be given as a functional of the amplitude operators $\hat{\alpha}$ and $\hat{\alpha}^\dagger$, $\hat{\rho}_{\text{in}} = \hat{\rho}_{\text{in}}[\hat{\alpha}, \hat{\alpha}^\dagger]$, then the transformed quantum state at the output is $\hat{\rho}_{\text{out}} = \hat{\rho}_{\text{in}}[J \Lambda \sigma J \hat{\alpha}, J \Lambda^\dagger \sigma J \hat{\alpha}^\dagger]$. Taking
the partial trace over the device variables leaves one with the quantum state of the radiation field alone.

The equivalence between density operators and quasi-probability functions implies a similar transformation rule for the phase-space functions. However, care needs to be taken as the SU(2,2) transformation associated with gain mixes creation and annihilation operators, except for the Wigner function associated with symmetric operator ordering for which \( W_{\text{out}}(\alpha) = W_{\text{in}}(\alpha J A^+ J \alpha) \) holds.

The above relations can now be used to construct quantum states after propagation through lossy and amplifying media. The simplest example is a coherent state \( |\alpha\rangle \) whose Wigner function is given by the Gaussian \( W(a) = \frac{1}{\pi} \exp\left(-2|a - a_0|^2\right) \) with obvious generalization for multimode states. At lossy devices, a two-mode coherent state \( |\alpha\rangle \) results in a Wigner function \( W_{\text{out}}(a) = \int d^2 a W_{\text{out}}(\alpha) = \left(\frac{\pi}{2}\right)^2 \exp\left(-2|a - T a_0 - A g_0|^2\right) \) which again represents a coherent state \( |T a_0 + A g_0\rangle\).

We take the device to be initially in its vacuum state, \( g_0 = 0 \), we are left with a coherent state \( |T a_0\rangle\).

In the case of gain, a lengthy but straightforward application of the quantum-state transformation relations shows that the same coherent state transforms into

\[
W_{\text{out}}(a) = \left(\frac{2}{\pi}\right)^2 \frac{1}{\det(2 T T^* - I)} \times \exp \left[ -2 \left( a^+ - a_0^+ T^+ \right) (2 T T^* - I)^{-1} (a - T a_0) \right]
\]

which is no longer a coherent state, but a displaced thermal state whose temperature depends on the gain. Neglecting reflection at the interface of the device, the (single-mode) Wigner function of the transmitted light is

\[
W_{\text{out}}(a) = \frac{2}{\pi} \frac{1}{2 |T|^2 - 1} \exp \left[ -2|a - T a_0|^2 \right]
\]

where the transmission coefficient \( |T| > 1 \) due to gain.

If we now construct a hypothetical device that consists of a sequence of a lossy medium with transmission coefficient \(|T| < 1\) followed by a gain medium with transmission coefficient \(|G| = 1/|T| > 1\), this would mimic a PT-symmetric system. However, as we have seen, noise enters both during the absorption as well as the amplification process. In fact, starting with a (single-mode) Wigner function \( W_{\text{in}}(a) = \frac{2}{\pi} \exp\left(-2|a - a_0|^2\right) \), after propagation through a lossy medium this turns into \( W_{\text{loss}}(a) = \frac{2}{\pi} \exp\left(-2|a - T a_0|^2\right) \). Reversing the loss by amplification then results in a Wigner function

\[
W_{\text{gain}}(a) = \frac{2}{\pi} \frac{1}{2 |G|^2 - 1} \exp \left[ -2|a - a_0|^2 \right]
\]

which is a (thermally) broadened version of the original Wigner function with the mean thermal photon number \( n_{\text{th}} = |G|^2 - 1 \) or, equivalently, \( T_{\text{eff}} = -\frac{\hbar \omega}{k_B} \ln(1 - |T|^2) \). What it also shows is that only the first-order moments of the amplitude operators are conserved by this system, not even the second-order moments. Indeed, the mean number of photons contained in a quantum state with Wigner function \( W_{\text{gain}}(a) \) is \( \langle \hat{n} \rangle = \int d^2 a \left( |a|^2 - \frac{1}{2} \right) W_{\text{gain}}(a) = |a_0|^2 + n_{\text{th}} \), which deviates from the coherent state result by the addition of the mean thermal photon number associated with the gain.

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