There are many definitions of entropy. Some come from thermodynamics, such as Clausius entropy, microcanonical (surface) entropy, and Gibbs entropy, some from information-theoretic perspective, such as Shannon entropy, von Neumann entropy, or entanglement entropy, and some using different types of coarse-graining, such as Boltzmann entropy, coarse-grained entropy, or Kolmogorov-Sinai entropy.

These notions often overlap under certain circumstances and in special cases. For example, Boltzmann entropy defined for a general coarse-graining reduces to microcanonical entropy for energy coarse-graining. Fundamental thermodynamic relation states that in systems in thermal equilibrium, microcanonical entropy equals Clausius entropy. For typical systems—those unbounded in energy—microcanonical entropy equals Gibbs entropy in the thermodynamic limit [1]. Von neumann entropy reduce to Gibbs entropy when taking a thermal ensemble. Entanglement entropy of a system in equilibrium is equal to the Gibbs entropy of a subsystem [2–4], and so on.

There is one common denominator to all of these entropies: an increase in every type of entropy represents some loss of information, and a decrease represents an information gain. Increase/decrease in different entropies correspond to loss/gain of different types of information. Thermodynamic entropies measure loss of information about which energy eigenstate (microstate) the system is in due to increased energy. Information-theoretic entropies measure loss of information about the state of the system due the interaction with an external system. Coarse-graining based entropies measure loss of information due to the state of the system evolving so that they wander into larger macrostates, whose inner structure is inaccessible by a macroscopic measurement, thus knowledge of the state of the system is being effectively lost.

This is to provide the context into which we will bring the discussion about the main topic of this paper: a coarse-graining based type of entropy, called observational entropy.

The history of observational entropy goes all the way back to John von Neumann. It first appeared in his paper [5] in 1929, where he motivated introduction of this entropy by criticizing the quantity which we know today as the von Neumann entropy. He said: “The expressions for [the von Neumann] entropy given by the author in [6] are not applicable here in the way they were intended, as they were computed from the perspective of an observer who can carry out all measurements that are possible in principle—i.e., regardless of whether they are macroscopic (for example, there every pure state has entropy 0, only mixtures have entropies greater than 0!).” He pointed out that von Neumann entropy cannot represent a good generalization of thermodynamic entropy, since any pure state, even those at high energies, would have associated zero entropy. Additionally, von Neumann entropy remains constant in an isolated quantum system, which would suggest than in such a system, left to spontaneous evolution, no information is lost—however, that is again only for an observer that can perform all measurements in principle, even those that are in a very complicated/highly entangled basis. A realistic observer with limited capabilities will always observe an increase in entropy.

To rectify this unsatisfactory behavior, while referring
to a discussion with Eugene Wigner\(^1\), John von Neumann proposed an alternative quantity which does not suffer of the same drawbacks. It reads

\[
S(\psi) = -\sum_E \langle \psi | \hat{P}_E | \psi \rangle \ln \frac{\langle \psi | \hat{P}_E | \psi \rangle}{V_E}.
\]  

(1)

Here, \(\hat{P}_E\) is a projector on an energy subspace (surface), and \(V_E\) is the number of orbits (microstates) in an energy surface. \(p_E = \langle \psi | \hat{P}_E | \psi \rangle\) is the probability of finding the system in an energy shell of energy \(E\). This entropy measures a lack of information due to observer’s limited capability of distinguishing energy eigenstates with close enough energies (those with energy difference smaller than some small fixed value \(\Delta E\)). It is the value that an observer making energy measurement with a finite resolution \(\Delta E\) would associate to the initial state of the system.

Von Neumann argued that this expression represents a natural generalization of Boltzmann entropy to quantum physics. To see that, consider a point in phase-space that belongs into energy surface \(E\). For this phase-space point, the original Boltzmann definition (which we call microcanonical/surface entropy here) reads \(S_B = \ln V_E\). Also in the von Neumann’s definition, if a quantum state belongs into an energy subspace \(E\), we have \(p_E = 1\), it gives the same value \(S(\psi) = \ln V_E\). However, unlike the Boltzmann’s definition, this definition is also valid for quantum states that span over several such energy shells. This includes equilibrium states, such as a thermal Gibbs state\(^2\) \[\hat{p} = \exp(-i\beta \hat{H})/Z\], but also any non-equilibrium state.

In the same paper \([5]\), von Neumann then went on to prove what he called the “quantum Boltzmann H theorem”. This theorem states the if one defines a set of projectors (called coarse-graining) that (a) commute with the coarse-grained energy projectors \(\hat{P}_E\), (b) do not commute with the energy eigenstates \(|E\rangle\langle E|\), then the resulting entropy with this coarse-graining of form similar to Eq. (1) always converges to the value of (1). This was the first application of this concept.

In his book in 1932 \([8]\), von Neumann formalized this type of entropy by considering a general set projectors in the above equation. This definition would measure the uncertainty for an observer making a specific macroscopic measurement, but not necessarily an energy measurement. However, he did not provide any further in depth discussion of its application.

Since then, this definition appeared scarcely in literature for several decades \([9–16]\). Noteworthy discussion of this quantity is by Wehrl in \([17]\), where he shows the long-time evolution of this entropy (which he called coarse-grained entropy) under some simplifying assumptions (such as mixing or Markovianity), while referencing the result of Pauli from 1928 on a similar topic \([18]\).

This concept was revived recently \([19, 20]\), and generalized to include multiple, even non-commuting, coarse-grainings, while proving a number of its properties. It was also coined by term observational entropy, due to the fact that it is an observer choice/or ability to measure a certain macroscopic variable that determines the coarse-graining. It was demonstrated that when taking local particle number and energy coarse-graining, observational entropy provides a compelling definition of non-equilibrium thermodynamic entropy, which is equal to the thermodynamic value for systems in equilibrium, but it also grows continuously when the system thermalizes \([20–22]\), and which is lower bounded by the sum of von Neumann and entanglement entropy \([22, 23]\). In relationship to von Neumann paper \([5]\), it was proven that also for the local energy coarse-graining, taking energy eigenstates always gives the thermal value equal to Eq. (1), up to some finite size corrections \([20]\). This entropy has been proposed as a concept that encompasses both the first and the second thermodynamic law, which can be then derived from microscopic principles, and quantified by a fluctuation theorem \([24]\). The concept of observational entropy has been compared with entanglement entropy: it has been shown that these two concepts measure different macroscopic features of a quantum system: while entanglement entropy measures the amount of correlations, observational entropy measures uniformity of distribution of energy and particles \([25]\). Interestingly, entanglement entropy can be obtained by minimizing observational entropy over all local measurements \([23]\). The observational entropy has been studied in relation to black holes \([26]\), big bang \([27]\), high-resolution measurement setups \([28]\), correlated finite baths \([29]\), other historically standard entropies \([30]\), and other scenarios \([31–33]\). For introductory reviews of this concept in isolated and in open quantum systems, see Refs. \([22]\) and \([24]\), respectively.

All of these works, however, consider only projective coarse-grainings, i.e., those given by a complete set of projectors, which represents a projective measurement. The projective measurements are the most natural, since they can be cleanly connected to macrostates. This is because each projector corresponds to a subspace of a Hilbert space, and each subspace represents a macrostate, because it defines a set of states that all give the same value of some macroscopic observable. Due to this connection with macrostates, this definition connects very well with the original Boltzmann’s notion of entropy. However, if we look at the coarse-graining more abstractly, as a description of a macroscopic measurement to be performed (which is not necessarily tied to

\(^1\) Von Neumann mentions that this definition was E. Wigner’s idea, while saying that there is no need to go into a general theory. We searched for a follow up paper by E. Wigner, where this general theory would be discussed, but could not find any—it seems that his interests at the time took him elsewhere—to develop the theory of symmetries in quantum mechanics, and then apply it to derive essential properties of the nuclei, for which he received 1963 Nobel Prize in Physics \([7]\).

\(^2\) Where \(\beta\) represents the inverse temperature, \(\hat{H}\) the Hamiltonian, and \(Z\) the partition function.
Projective measurement is not the most general measurement that can be performed. The most general measurement is given by a set of quantum operations, called positive operator-valued measure (POVM)\(^3\) [34]. These general measurements describe, for example, the following situations:

- There is a time delay between performing different projective measurements, for example, due to experimental limitations. This free-time evolution turns the projective measurement into a general measurement.
- It is difficult or unfeasible to probe the system directly, instead, one employs a quantum probe that interacts with the system, collects the information about it, and then is measured. These indirect measurements form a general measurement.
- As a special example of the previous scenario, measuring certain observable by coupling the system to a Harmonic oscillator that we have a full control over, is one of such indirect measurements.
- A probe interacts with the system multiple times, and collects the information until it is measured.
- Experimenter performs multiple non-communing projective measurements to extract information. These together do not form a projective measurement. They form a general measurement instead.
- Experimenter is interested only in a subsystem on a larger system, on which they can perform a projective measurement. For example, he or she can be interested about how much information can be obtained about a subsystem of spins, by making a global spin measurement.

There has been a number of experiments doing exactly these types of general measurement in the lab: first such experimental demonstrations were of unambiguous discrimination [35–38]. Recently, other applications became more prevalent. In [39], three generalized measurement scenarios were performed on a qubit, in a quantum walk model. Sequential weak measurements of non-commuting Pauli were performed in a photonic system [40], and post-selected weak measurements were performed using NMR quantum processor [41]. POVMs used on the polarization degrees of freedom were demonstrated for the purpose of self-testing, which represents the strongest form of certification of quantum systems [42]. POVMs were also used in the certification of a device-independent protocol, using entangled photon pairs [43]. Test of a tight information-theoretic measurement uncertainty relation for three-outcome POVM were performed using neutron spin-1/2 qubits, in an experiment motivated by the knowledge that the projective measurements are not optimal in this case [44].

There also theoretical work motivating further development of general measurements: for example, in Ref. [45] it was shown that a work measurement is a generalized measurement. New methods of implementing general quantum measurements were proposed in [46], using only classical resources and a simple auxiliary qubit.

Given this common occurrence of general measurements in various situations, and a growing number of experiments in the past years, in order to provide an encompassing theory of information extraction, it is important to include these general measurements into the definition of observational entropy. This entropy will measure the remaining uncertainty about the initial state of the system that was not recovered by a sequence of given (possibly non-commuting) general measurements. In other words, its value will measure how much information these measurements extract—the smaller the value, the more extracted information. Because this entropy can be calculated for different coarse-grainings representing different measurements, its value serves as a quantifier of how well different (and completely general) measurement schemes perform in information extraction.

This is exactly what we do in this paper. While there were several obstacles in defining the correct figures of merit (especially the notion of coarse-graining and macrostate volumes), due to not being able to depend on the properties of projectors—their orthogonality in particular—it is possible to define them in such a way that all of the important properties of the resulting entropy are preserved, even for this general case. These general notions and theorems are described in Sec. I, which represent our main result. Sec. II considers a general scenario of an indirect measurement scheme, and specifically aims to answer how well an indirect measurement performs as compared to a direct measurement, based on general considerations such as dimension of our auxiliary system. Sec. III then shows a specific example of an indirect measurement scheme called the von Neumann measurement scheme, in which the auxiliary system (probe) is a massive particle whose position is measured. There we apply the general theory to demonstrate how it can be used to quantitatively described which measurement strategy performs better than others in obtaining information, and in which situations. Finally, in Sec. IV we connect this theory to the notion of quantum tomography, and thus give observational entropy its operational meaning: we show that in the case where the

\[3\text{ Here, we are connecting with well-known notions at the expense of being slightly imprecise. A POVM can uniquely identify the probability of a measurement outcome given the density matrix, but it does not does not identify the resulting state after the measurement. The general measurement, which we will describe in the next section, can do both, and thus also defines the corresponding POVM—the set of POVM elements.)} \]
observational entropy is equal to the von Neumann entropy, it is possible to reconstruct the quantum state of the system. We provide an explicit algorithm for doing this, which provides an exciting new direction of study. In Sec. V we summarize and discuss our results, and talk about possible future directions and applications.

I. OBSERVATIONAL ENTROPY WITH GENERAL MEASUREMENTS: BOUNDS ON EXTRACTED INFORMATION

In this section we first shortly review previous results. Observational entropy, which has been suggested as a measure of observer’s uncertainty about a quantum system, and its reduction corresponds to more extracted information. Previously, observational entropy was defined only with respect to coarse-grainings given by a projective measurement. Here we generalize this notion to include completely general measurements. As the main result of this section, we prove that all of the properties of the original definition translate also to this general case, which shows that its interpretation is still valid, and which will be very useful in suggesting ways of how to use this quantity.

Defining $\hat{P}_i$ as the projector onto a subspace $\mathcal{H}_i$, we collect these projectors into a set $C = \{\hat{P}_i\}$, which we call a coarse-graining. Projectors in this set are Hermitian ($\hat{P}_i^\dagger = \hat{P}_i$), orthogonal ($\hat{P}_i \hat{P}_j = \hat{P}_i \delta_{ij}$), and they satisfy completeness relation ($\sum_i \hat{P}_i = 1$). Conversely, any coarse-graining $C = \{\hat{P}_i\}$ with the above properties defines a decomposition of the Hilbert space $\mathcal{H} = \bigoplus \mathcal{H}_i$, in which subspace $\mathcal{H}_i$ (macrostate) is spanned by eigenvectors of $\hat{P}_i$. Thus, in this construction we can talk either about a decomposition of the unity into orthonormal projectors ($I = \sum_i \hat{P}_i$), or a decomposition of Hilbert space into subspaces ($\mathcal{H} = \bigoplus \mathcal{H}_i$). Either way, these descriptions are equivalent.

Each observable defines a coarse-graining $C_a = \{\hat{P}_a\}$ through its spectral decomposition $\hat{A} = \sum_a a \hat{P}_a$. Thus it also defines the corresponding decomposition of the Hilbert space $\mathcal{H} = \bigoplus \mathcal{H}_a$. Each eigenvalue $a$ represents the macroscopic property, and is one of the measurement outcomes obtained when measuring that observable. Generally, any coarse-graining can be viewed as a type of measurement—not necessarily complete, and not necessarily of some observable.

Given a single coarse-graining, observational entropy (sometimes also called coarse-grained entropy [8, 17, 47]) is defined as [19–21] (see also [23–25])

$$S_C \equiv - \sum_i p_i \ln \frac{p_i}{V_i}$$

(2)

where $p_i = \text{tr}[\hat{P}_i \hat{\rho}]$ denotes the probability of finding the state in macrostate $\mathcal{H}_i$, and $V_i = \text{tr}[\hat{P}_i] = \dim \mathcal{H}_i$ volume of that macrostate—the number of orthogonal states in it. Equivalently, we can call $p_i$ the probability of obtaining a measurement outcome $i$, when measuring in a basis given by coarse-graining $C$.

This can be generalized to multiple coarse-grainings, for which observational entropy is defined as [20]

$$S_{C_1,\ldots,C_n} \equiv - \sum_i p_i \ln \frac{p_i}{V_i}.$$  

(3)

where $i = (i_1, \ldots, i_n)$ is a vector of outcomes (properties of the system), $p_k = \text{tr}[\hat{P}_{i_n} \cdots \hat{P}_{i_1} \hat{\rho} \hat{P}_{i_1} \cdots \hat{P}_{i_n}]$ is the probability of obtaining these outcomes in the given order, and $V_i = \text{tr}[\hat{P}_{i_n} \cdots \hat{P}_{i_1}]$ an (ordered) volume of the corresponding macrostate—a joint Hilbert space volume of all systems that have properties $i$ measured in this order. Note that in here, the equivalence between the projectors and subspaces has already been lost: volume $V_i$ can be a fraction, and it does not, in general, correspond to a dimension of any subspace (the correspondence remains only if all of the coarse-grainings commute, in which case a joint subspace exists).

Observational entropy satisfies two very important properties [20],

$$S_{C_N}(\hat{\rho}) \leq S_{C_1,\ldots,C_n}(\hat{\rho}) \leq \ln \dim \mathcal{H},$$

(4)

$$S_{C_1,\ldots,C_n,C_{n+1}}(\hat{\rho}) \leq S_{C_1,\ldots,C_n}(\hat{\rho}).$$

(5)

The first property shows that it is upper bounded by the maximal uncertainty allowed by the size of the system, and lower bounded by the uncertainty inherent to the system (measured by von Neumann entropy $S_{C_N}(\hat{\rho}) = -\text{tr}[\hat{\rho} \ln \hat{\rho}]$). The second property shows that every additional measurement can only decrease the entropy. Due to their intuitive interpretation, these two properties justify interpreting observational entropy a measure of uncertainty an observer making measurements associates with a system. In more detail, observational entropy is the uncertainty an observer would associate to the initial state of the system, if he or she had infinitely many copies of this state, and performed sequential measurements in bases $C_1,\ldots,C_n$ on each copy. This would allow them to build up statistics of measurement outcomes, and thus to determine this entropy exactly.

However, as mentioned in the introduction, the projective measurement that defines a coarse-graining in the original definition is not the most general measurement that an observer can perform. A general measurement includes the case of a projective measurement, but also a situation where the system is allowed to interact with an auxiliary system (probe), and then a joint projective measurement is performed on the system+probe, or just on the probe itself. This general measurement can be described by a set of linear superoperators (quantum operations) $\{\mathcal{A}_i\}$, where each element can be written using...
the Kraus decomposition \(^4\)

\[ A_i(\hat{\rho}) = \sum_k \hat{K}_{im} \hat{\rho} \hat{K}_{im}^\dagger, \quad (6) \]

and the Kraus operators satisfy the completeness relation \(^5\)

\[ \sum_i \sum_m \hat{K}_{im}^\dagger \hat{K}_{im} = \hat{I}. \quad (7) \]

Performing a measurement, upon obtaining a measurement outcome “\(i\)” the density matrix of the system is projected onto

\[ \hat{\rho} \overset{\text{“i”}}{\rightarrow} A_i(\hat{\rho}) \]

with probability of \(p_i = \text{tr}[A_i(\hat{\rho})]\).

It therefore makes sense to generalize Observational entropy even further, to include this generalized measurement. Defining each coarse-graining as a set of quantum operations \(\mathcal{C}_k = \{A_{ik}\}\) satisfying the completeness relation (A4), we define

\[ S_{\mathcal{C}} \equiv S_{\mathcal{C}_1, \ldots, \mathcal{C}_n} \equiv - \sum_i p_i \ln \left( \frac{p_i}{V_i} \right), \quad (9) \]

where \(\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)\) is a vector of coarse-grainings, and combining superoperators into \(A_i \equiv A_{i_1} \cdots A_{i_n} A_{i_1}\), \(^6\) we obtain

\[ p_i = \text{tr}[A_i(\hat{\rho})] \quad (10) \]

as the probability of obtaining a sequence of outcomes \(i\) and

\[ V_i = \text{tr}[A_i(\hat{I})] \quad (11) \]

as the corresponding volume of a macrostate (\(\hat{I}\) denotes the identity matrix). Looking at Eqs. (10) and (11) we can view the vector of coarse-grainings as a single coarse-graining with vector-labeled elements \(\mathcal{C} \equiv (\mathcal{C}_1, \ldots, \mathcal{C}_n) \equiv \{A_i\} \). The original definition is obtained by considering coarse-grainings made of projective superoperators \(\mathcal{C} = \{\hat{P}_i(\bullet)\hat{P}_i\} \). \(^7\) Therefore, each projective coarse-graining is lifted into its superoperator form and we will use these two notations interchangeably \(\mathcal{C} = \{\hat{P}_i\} = \{\hat{P}_i(\bullet)\hat{P}_i\}\). See Table I for the overview of different types of measurements and their corresponding coarse-grainings.

Now, what if, let us say, Eq. (4) or Eq. (5) did not hold for this generalized definition. Consider for example, if there was some generalized measurement, for example an indirect measurement using probe, that is able to push observational entropy below the inherent uncertainty in the system, or if performing some measurement would result in an increase of the observers’ uncertainty (making them “forget” something about the system). This would suddenly imply that observational entropy is not a good measure of observers’ uncertainty and should not be viewed as such.

However, here we will show that even with the inclusion of general measurements, the two properties still hold.

To state the theorems, which are proved in Appendix A, we first need to define some terminology:

**Definition 1.** (Coarse-graining defined by an observable) Assuming spectral decomposition of a Hermitian operator \(\hat{A} = \sum_\alpha \hat{P}_\alpha\) (where \(\alpha\)’s are assumed to be distinct), we define coarse-graining given by the Hermitian operator as \(\mathcal{C}_{\hat{A}} = \{\hat{P}_\alpha(\bullet)\hat{P}_\alpha\}\).

Second we define POVM elements given a coarse-graining.

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\(^4\) Operators \(\Pi_i = \sum_m \hat{K}_{im}^\dagger \hat{K}_{im}\) are called POVM elements, and their collection \(\{\Pi_i\}\) a POVM (positive operator-valued measure) in the literature. Unlike the general measurement given by \(\{A_i\}\), POVM by itself cannot determine the post-measurement state. This also means that using the POVM it is possible to define observational entropy with a single coarse-graining but not more than that, because in case of non-commuting measurements the probabilities and volumes will generally depend on the post-measurement state.

\(^5\) Number of \(m\) for each \(i\) in decomposition (6) is not unique, and the number of \(m\) for each \(i\) can vary; minimal number of \(m\) such that the decomposition holds is called the Kraus rank of \(A_i\). Projective measurements are a special type of Kraus rank-1 measurements, given by \(A_{ik}(\hat{X}) = \hat{P}_{ik} \hat{X} \hat{P}_{ik}\).

\(^6\) For better visibility, we dropped excessive parentheses, and superoperator on the left is understood as acting on the superoperator on the right, such as in \(A_{i_2} A_{i_1}(\hat{\rho}) \equiv A_{i_2}(A_{i_1}(\hat{\rho}))\).

\(^7\) By \(\mathcal{C} = \{\hat{P}_\alpha(\bullet)\hat{P}_\alpha\}\) we mean that \(\mathcal{C} = \{A_i\}\), where \(A_i(\hat{X}) = \hat{P}_i \hat{X} \hat{P}_i\) for an operator \(\hat{X}\).
Definition 2. For a vector of coarse-graining \( \mathcal{C} \equiv (C_1, \ldots, C_n) \), where \( A_i = \sum_m K_{im}(\bullet) K_{im}^\dagger \) and \( K_{im} = K_{i,m_1} \cdots K_{i,m_1} \), we define POVM elements \( \{\Pi_i\} \) as
\[
\Pi_i = \sum_m K_{im}^\dagger K_{im}. \tag{12}
\]

This allows us to write the probabilities of outcomes and related volumes as \( p_j = \text{tr}[\Pi_j \hat{\rho}] \) and \( V_i = \text{tr}[\Pi_i] \), and to introduce the next definition.

Third, we define the notion of finer coarse-graining. Intuitively, the finer coarse-graining is such that it provides at least as much information as a coarse-graining that the finer coarse-graining is finer than, irrespective of the specific state of the system. By coarse-graining providing at least as much information we mean that the measurement corresponding to the coarse-graining provides at least as much information on average upon obtaining the measurement outcome. As we will show with our theorems, one could also view the finer coarse-graining as that which never increases the observational entropy. The definition goes as follows.

Definition 3. (Finer vector of coarse-grainings)\(^8\) We say that a vector of coarse-grainings \( \mathcal{C} = \{A_i\} \) is finer than a vector of coarse-grainings \( \tilde{\mathcal{C}} = \{\tilde{A}_j\} \) (and denote \( \mathcal{C} \leftarrow \tilde{\mathcal{C}} \) or \( \mathcal{C} \rightarrow \tilde{\mathcal{C}} \)), when the POVM elements of \( \tilde{\mathcal{C}} \) can be built from the POVM elements of \( \mathcal{C} \), i.e., when we can write
\[
\forall j, \quad \Pi_j = \sum_{i \in I(j)} \Pi_i, \tag{13}
\]
where \( I(j) \) are disjoint index sets whose union is the set of all indices \( \bigcup_j I(j) \equiv \{i\} \).

From this definition it is clear that for \( \mathcal{C} \leftrightarrow \tilde{\mathcal{C}} \) the probabilities of outcomes and related volumes can be also written as sums, \( p_j = \sum_{i \in I(j)} p_i \) and \( V_j = \sum_{i \in I(j)} V_i \), making an impression that the finer coarse-graining really offer a set of lens which which the state of the system is studied, since the Hilbert space is cut into smaller volumes with a finer coarse-graining, and each of this volume has its own associated probability \( p_i \), while the coarser coarse-graining just adds up together to create \( p_j \), ignoring this finer structure. With this definition, it is straightforward to realize that there are some coarse-grainings that are equivalent in the sense that both \( \mathcal{C} \leftarrow \tilde{\mathcal{C}} \) and \( \mathcal{C} \rightarrow \tilde{\mathcal{C}} \). For example, such cases are \( \mathcal{C} = \{A_i\} \) and \( \tilde{\mathcal{C}} = \{\tilde{A}_i\} \), where \( A_i = \sum_m K_{im}(\bullet) K_{im}^\dagger \) and \( \tilde{A}_i = \sum_m U_{im} K_{im}(\bullet) K_{im}^\dagger U_{im}^\dagger \), while the only difference is that with the second coarse-graining, the state was unitarily evolved with unitary operators \( U_{im} \) after the measurement. These two coarse-grainings have the same POVM elements and thus are equivalent.

This definition of a finer coarse-graining, while simple-looking, represents a major stepping stone in our understanding. It looks different from the previously introduced Definitions 2. and 6. in [20], which considered vectors of only projective coarse-grainings. However, as we show in Appendix B, it turns out that this definition is equivalent to the older definition when applied on the same restricted set, and therefore represents its direct generalization.\(^9\)

Generalizations of Theorems 7. and 8. from [20] (represented here by Eqs. (4) and (5)) follow.

Theorem 1. (Observational entropy (9) with multiple coarse-grainings is bounded)
\[
S_{\mathcal{C}}(\hat{\rho}) \leq S_{\mathcal{C}}(\hat{\rho}) \leq \ln \dim \mathcal{H} \tag{14}
\]
for any vector of coarse-grainings \( \mathcal{C} = (C_1, \ldots, C_n) \) and any density matrix \( \hat{\rho} \). \( S_{\mathcal{C}}(\hat{\rho}) = S_{\mathcal{C}}(\hat{\rho}) \) if and only if \( \mathcal{C} \sim \mathcal{C} \), \( S_{\mathcal{C}}(\hat{\rho}) = \ln \dim \mathcal{H} \) if and only if \( \forall i, p_i = V_i/\dim \mathcal{H} \).

Theorem 2. (Observational entropy (9) is non-increasing with each added coarse-graining.)
\[
S_{\mathcal{C},C_{n+1}}(\hat{\rho}) \leq S_{\mathcal{C}}(\hat{\rho}) \tag{15}
\]
for any vector of coarse-grainings \( (\mathcal{C},C_{n+1}) \) and any density matrix \( \hat{\rho} \). The inequality becomes an equality if and only if \( \forall i, i_{n+1}, p_{i,i_{n+1}} = V_{i,i_{n+1}}/V_i p_i \).

The first equality condition in Theorem 1, \( \mathcal{C} \leftrightarrow \mathcal{C} \), means that for every eigenvalue \( \rho \) of the density matrix \( \hat{\rho} = \sum \rho \hat{P}_\rho \), the corresponding projector onto an eigenspace can be written using the POVM elements as
\[
\hat{P}_\rho = \sum_{i \in I(\rho)} \Pi_i. \tag{16}
\]

This identity represents a connection between observational entropy and quantum tomography, since it can be used to reconstruct the quantum state in case when \( S_{\mathcal{C}}(\hat{\rho}) = S_{\mathcal{C}}(\hat{\rho}) \), as described in detail in Section IV.

The equality condition in Theorem 2 is satisfied, among other cases, when the sequence of measurements \( \mathcal{C} \) projects onto a pure state (meaning that all information about the initial state was depleted by the first \( n \) measurements and there is no information left in the remaining state), or when \( \mathcal{C} \rightarrow \mathcal{C} \) (expressing that the information that the \( n + 1 \)-st measurement could provide was already provided by the first \( n \) measurements, making the \( n + 1 \)-st measurement redundant), where we have defined

\(^8\) Alternative version: \( \mathcal{C} \leftrightarrow \tilde{\mathcal{C}} \) if there exists a function \( J : i \rightarrow j \) such that for all \( j, \Pi_j = \sum \delta_{j,J(i)} \Pi_i \), where \( \delta_{jj} \) denotes Kronecker delta.

\(^9\) It may turn out that there is an even better, more encompassing generalization of a finer vector of coarse-graining, for which we have some evidence. We leave this for another publication.
\( \mathbf{C}^\dagger = (C_1^\dagger, \ldots, C_4^\dagger) = \{ A_i^\dagger \} \), where \( A_i^\dagger = \sum_m \hat{K}_{im}^\dagger \hat{K}_{im} \) (see the end of the proof in Sec. A.2).

Finally, we generalize an elegant and intuitive Theorem 2, from [20], which we also foreshadowed here when motivating the definition of finer coarse-graining:

**Theorem 3.** (Observational entropy (9) is a monotonic function of the coarse-graining.) If \( \mathcal{C} \rightarrow \mathcal{C} \) then

\[
S_{\mathcal{C}}(\tilde{\rho}) \leq S_{\mathcal{C}}(\hat{\rho}).
\]

The inequality becomes an equality if and only if \( \forall \ j, \forall i \in J(j), p_i = \frac{\nu_i}{\nu_j} p_j. \)

Validity of these three theorems mean that considering observational entropy as a measure of observers’ uncertainty about the initial state of the system is conceptually justified. Further justification, but this time operational, will be provided in Section IV, where we show an explicit algorithm how to infer the quantum state of the system when observational entropy reaches the minimal possible uncertainty, that is, the von Neumann entropy.

**II. APPLICATION: INDIRECT MEASUREMENT SCHEME**

In this section we demonstrate an application of the presented theory, by computing observational entropy of a measurement that is not performed directly on the system itself, but is rather done through an auxiliary system that is measured instead. (This will be illustrated on a specific example in the next section.) The main result of this section are several statements saying how large dimension the auxiliary system must be in order for the indirect measurement is as good as the best direct measurement.

Consider a scenario where the system is first coupled to an auxiliary system (ancilla, which we will call probe here), so that the probe is affected by the state of the system (alternatively, we could say that the probe therefore collects some information about the system). Thus, measuring the probe provides some information about the state of the system. This protocol can be schematically written as follows,

\[
\rho \rightarrow \rho \otimes \hat{\sigma} \rightarrow U \rho \otimes \hat{\sigma} U^\dagger \quad \xrightarrow{\text{tracing out probe}} \quad \frac{1}{p_i} \left[ (I \otimes \hat{P}_i) U \rho \otimes \hat{\sigma} U^\dagger (I \otimes \hat{P}_i) \right] \quad \equiv \quad \frac{A_i(\hat{\rho})}{p_i},
\]

where \( U \) is the unitary evolution operator that incorporates the interaction between the system and the probe. Above, \( \rho \) is the initial density matrix of the system and \( \hat{\sigma} \) is that of the probe. Since only the probe is measured, the projection operator \( \hat{P}_i \) acts only on the probe, with the identity \( I \) being acted upon the system. The full Hilbert space is the tensor product of the system and the probe \( \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A \).

By tracing out the probe, this protocol gives an explicit form of the measurement superoperators acting on the system,

\[
A_i(\tilde{\rho}) = \text{tr}_A \left[ (I \otimes \hat{P}_i) U \tilde{\rho} \otimes \hat{\sigma} U^\dagger (I \otimes \hat{P}_i) \right].
\]

We denote coarse-graining consisting of these superoperators as \( \mathcal{C} = \{ A_i \} \). By inserting the spectral decomposition of the probe \( \hat{\sigma} = \sum_m \sigma_m |m\rangle \langle m| \) we can also obtain the Kraus decomposition of the superoperator

\[
A_i(\hat{\rho}) = \sum_{m, m'} \hat{K}_{imm'} \hat{\rho} \hat{K}_{imm'}^\dagger,
\]

where \( \hat{K}_{imm'} = \sqrt{\sigma_m} |m'\rangle (I \otimes \hat{P}_i) U |m\rangle \) are the Kraus operators.

While the Kraus operators can be useful in expressing the superoperator (and we will use this decomposition in our examples), we do not actually need this decomposition in order to calculate the observational entropy \( S_{\mathcal{C}} \). This is because \( p_i \) and \( V_i \) in Eq. (9) depend only on the superoperator, thus we can obtain directly

\[
p_i = \text{tr} \left[ A_i(\tilde{\rho}) \right] = \text{tr} \left[ (I \otimes \hat{P}_i) U \tilde{\rho} \otimes \hat{\sigma} U^\dagger (I \otimes \hat{P}_i) \right],
\]

\[
V_i = \text{tr} \left[ A_i(I) \right] = \text{tr} \left[ (I \otimes \hat{P}_i) U I \otimes \hat{\sigma} U^\dagger (I \otimes \hat{P}_i) \right].
\]

Note that this \( p_i \) is identical to the \( p_i \) in the protocol expressed by Eq. (18), and it represents the probability with which the outcome \( "i" \) was measured. \( V_i \) represents the corresponding volume in the Hilbert space of the system (not the probe Hilbert space).

**A. Optimal indirect measurement**

Now that we set up the general framework, we can ask the first natural question. Considering an indirect measurement, how low can the observational entropy get? Can it, for example, go as low as the ultimate minimum given by Theorem 1, that is, the von Neumann entropy, and therefore be as informative as a direct measurement which achieves this lower bound?

That of course depends on the amount of control over the system, as well as the size of the probe that is being used. In the case of perfect control, i.e., ability to design any interaction unitary \( U \), the answer is fairly straightforward. Assuming that the size of the probe is (at least) as big as the size of the system, i.e., \( N = \dim \mathcal{H}_S = \dim \mathcal{H}_A \cong M \), taking the interaction to be a swap gate \( U(\hat{\rho} \otimes \hat{\sigma}) = \hat{\sigma} \otimes \hat{\rho} \), which defines \( U = \sum_{k,m=1}^N |m, k\rangle \langle k, m| \), we have

\[
p_i = \text{tr} \left[ (I \otimes \hat{P}_i) \hat{\rho} \otimes \hat{\sigma} (I \otimes \hat{P}_i) \right] = \text{tr} [ \hat{P}_i \hat{\rho}],
\]

\[
V_i = \text{tr} \left[ (I \otimes \hat{P}_i) \hat{\sigma} \otimes \hat{\sigma} (I \otimes \hat{P}_i) \right] = \text{tr} [ \hat{P}_i].
\]

The corresponding observational entropy reduces to

\[
S_{\mathcal{C}} = S_{\mathcal{C}_A},
\]
with the eigenvalues being ordered as \( \rho_1 \geq \rho_2 \geq \cdots \), and where

\[
U_{\text{partial swap}}(k, m) = \begin{cases} \{m, k\} & k \leq M, \\ \{k, m\} & k > M. \end{cases}
\]

In terms of eigenvectors this gives

\[
U = \sum_{k, m=1}^{M} |m, k\rangle\langle k, m| + \sum_{k=M+1}^{N} \sum_{m=1}^{M} |k, m\rangle\langle k, m|,
\]

where \( \hat{\rho} = \sum_{k=1}^{N} \rho_k |k\rangle\langle k| \) and \( \hat{\sigma} = \sum_{m=1}^{M} \sigma_m |m\rangle\langle m| \). We compute

\[
U \hat{\rho} \hat{\sigma} U^\dagger = \sum_{k, m=1}^{M} \sigma_k \rho_m |k, m\rangle\langle k, m| + \sum_{k=M+1}^{N} \sum_{m=1}^{M} \rho_k \sigma_m |k, m\rangle\langle k, m|,
\]

which yields

\[
p_i = \text{tr}[\hat{P}_i (\hat{\rho}^{(\sigma)} + p \hat{\sigma})], \quad V_i = \text{tr}[\hat{P}_i (\hat{I} + (N - M) \hat{\sigma})],
\]

where \( \hat{\rho}^{(\sigma)} = \sum_{m=1}^{M} \rho_m |m\rangle\langle m| \) is the part of the system density matrix that got swapped into the probe Hilbert space, \( \hat{\rho}^{(\text{rest})} = \sum_{k=M+1}^{N} \rho_k |k\rangle\langle k| \) the part that remained in the system Hilbert space, and \( p = \text{tr}[\hat{\rho}^{(\text{rest})}] = \sum_{m=M+1}^{N} \rho_m \) is the sum of eigenvalues that remained in the system Hilbert space. Note that both \( \hat{\rho}^{(\sigma)} + p \hat{\sigma} \) and \( \hat{I} + (N - M) \hat{\sigma} \) are diagonal in the same basis \( \{|m\rangle\} \).

Clearly, the value on observational entropy \( S_C \) depends on i) the choice of coarse-graining and on ii) the state of the probe. In general, \( S_C \) will always be larger than the von Neumann entropy, but are two interesting cases to consider.

First, choosing an probe that is in a completely mixed state \( \hat{\sigma} = \hat{I}/M \). In this case, the observational entropy is minimized when the coarse graining on the probe Hilbert space is in the diagonal basis \( \{\hat{P}_i\} = \{|m\rangle\} \), which gives

\[
S_C = -\sum_{i=1}^{M} \left( \rho_i + \frac{p}{M} \right) \ln \left( \rho_i + \frac{p}{M} \right) + \frac{N}{M}.
\]

Depending on the size of the probe, this may be fairly close to the von Neumann entropy, and it will converge to it for \( M \to N \) (since in such a limit also \( p \to 0 \)).

Second, considering a special case when the probe is at least one dimension larger than the rank \( R \) of the system density matrix, \( M \geq R + 1 \), where we have \( p = 0 \) (due to ordering \( \rho_1 \geq \rho_2 \geq \cdots \)). Choosing \( \hat{\sigma} = |R+1\rangle\langle R+1| \) and again coarse-graining in the diagonal basis \( \{\hat{P}_i\} = \{|m\rangle\} \) yields

\[
p_i = \begin{cases} \rho_i, & i \leq R, \\ 0, & i > R. \end{cases} \quad V_i = \begin{cases} 1, & i \neq R + 1, \\ 1 + N - M, & i = R + 1. \end{cases}
\]

This gives

\[
S_C = -\sum_{i=1}^{R} \rho_i \ln \rho_i = S_{VN}.
\]

Thus:

The minimal uncertainty can achieved by measuring an probe which is one dimension larger than the rank of the state of the system.

The fact that the minimal uncertainty is achieved does not necessarily mean that by making that measurement we have determined the state of the system. This is simply because we wouldn’t know if the true minimum is reached without knowing the von Neumann entropy, which is a function of the state of the system. In other words, we could achieve some minimum given by the constraints (for example, limited by the size of the probe), which is still larger than the von Neumann entropy. It is only when we know the value of observational entropy to be equal von Neumann entropy, then from the coarse-graining that saturates this equality we can infer the state of the system. (This is done using the equality condition in Theorem 1.) We summarize this as follows:

Knowing the value of von Neumann entropy, the state of the system can be exactly determined by measuring probe which is one dimension larger than the rank of the state.

If, for example, we know that we are dealing only with pure states, which have von Neumann entropy zero and rank 1, we specifically conclude:

Any pure state of the system can be exactly determined by measuring a two-level probe.

Physically, this is fairly straightforward to comprehend. Making a projective measurement on a two-level
provides an explicit algorithm to describe by protocol (18), is the von Neumann measurement between the auxiliary and the measured system. In other words, the wave function of the system is exactly mapped onto the first level of the probe, while the rest (the kernel) is mapped onto the second level of the probe. This clearly generalizes into states of higher rank, \( \hat{\rho} = \sum_{m=1}^{R} \rho_m |m\rangle \langle m| \). There, the measurement on probe is the same as if one were to measure the system in the \{|1\rangle, \ldots, |R\rangle \rangle | \rangle, I - \sum_{m=1}^{R} |m\rangle \langle m| \rangle \rangle | \rangle basis.\(^{10}\)

We will continue along this line of argument, which provides an explicit algorithm to describe how to infer the quantum state when the observational entropy equals the von Neumann entropy in Sec. IV. In the meantime, we will focus on a specific example of indirect measurement scheme.

III. EXAMPLE OF AN INDIRECT MEASUREMENT SCHEME: VON NEUMANN MEASUREMENT SCHEME

In this section we show an example of indirect measurement scheme, called the von Neumann measurement scheme, on which we illustrate our theory of observational entropy as a measure of observers’ uncertainty. The main results of this section are numerical simulations of observational entropy corresponding to various types of von Neumann measurement scheme. These show how good these indirect measurements are in comparison with direct measurements, and with each other, depending on several model parameters such as: the amount of localization of the auxiliary system, the number of indirect measurements performed, and the number of contacts between the auxiliary and the measured system.

A special type of indirect measurement scheme, given by protocol (18), is the von Neumann measurement scheme. In this scheme, the auxiliary system (probe, also called pointer specifically in this scheme), which is initially decoupled from the system, is assumed to be a heavy mass particle (although represented by a quantum state) initially localized at some position \( x \) with some uncertainty \( \Omega \). It interacts with the system through a unitary,

\[
U = \exp \left(-i \kappa \hat{M} \otimes \hat{\rho} \right),
\]

where \( \hat{M} \) is a Hermitian operator representing a measurement on the system, and

\[
\hat{\rho} = -i \frac{d}{dx}
\]

is the momentum operator (whose exponential acts as a translation operator on the probe, \( e^{-ix\hat{\rho}} | \varphi(x) \rangle = | \varphi(x-x_0) \rangle \)). The parameter \( \kappa = \lambda \Delta \tau \) is a product of interaction strength \( \lambda \) and the time of interaction \( \Delta \tau \). We denote the spectral decomposition \( \hat{M} = \sum_{\mu} \mu \hat{P}_\mu \) of the measurement operator (where \( \hat{P}_\mu = \sum_{m=\mu}^{m_\mu} |m\rangle \langle m| \) is a projector onto eigensubspace with eigenvalue \( \mu \), and \( m_\mu \) is an eigenvalue of \( \hat{M} \) associated to eigenvector \( |m\rangle \)). Writing the state of the system in the eigenbasis of the measurement operator as \( |\psi\rangle = \sum_{m} \alpha_m |m\rangle \), the interaction unitary acts as\(^{11}\)

\[
U|\psi\rangle |\varphi(x)\rangle = \sum_{m} \alpha_m |m\rangle |\varphi(x-\kappa \mu_m)\rangle
\]

on a pure initial product state.

For a general decoupled system state and a pure probe, we have

\[
U \hat{\rho} |\varphi\rangle |\varphi\rangle U^\dagger = \sum_{m,m'} \rho_{mm'} |m\rangle \langle m'| |\varphi(x-\kappa \mu_m)\rangle \langle \varphi(x-\kappa \mu_{m'})|,
\]

where \( \hat{\rho} = \sum_{m,m'} \rho_{mm'} |m\rangle \langle m'| \) is the mixed initial state of the system.

In the von Neumann measurement scheme, after the interaction, the position of probe is measured, which corresponds to coarse-graining

\[
C_A = \{ \hat{P}_x \}, \quad \hat{P}_x = |x\rangle \langle x|
\]

on the probe.

The von Neumann measurement scheme translates measuring observable \( \hat{M} \) on the system into a problem of measuring the position of the probe. To illustrate that, consider an extreme case in which the position of the probe is exactly determined, \( |\varphi(x)\rangle = |x\rangle \). The outcome of a position measurement on the probe after the interaction is “\( x - \kappa \mu \)” with probability \( \rho_m = \text{tr}[ \hat{P}_m \hat{\rho} ] = \sum_{\mu=\mu_m}^{m_\mu} \rho_{mm} \) and the system state is projected onto the state \( \hat{P}_m \hat{\rho} / \rho_m \). Thus, in such an idealized case \( (\Omega \to 0)\),

\(^{10}\) The actual measurement is actually more complicated than this. To be exact, the von Neumann measurement scheme leads to \( \Lambda_\mu (\hat{\rho}) = \hat{K}_\mu \hat{\rho} \hat{K}_\mu^\dagger \) where \( \hat{K}_\mu = |R+1\rangle \langle i| + \delta_{R+1,i} \sum_{k=M+1}^N |k\rangle \langle k| \) was computed from Eq. (20) (we denote this coarse-graining \( \hat{K}_\mu \) \( \) in (4)). This is equivalent to measuring in basis \( C_2 = \{|1\rangle, \ldots, |R\rangle \rangle M, I - \sum_{m=1}^{M} |m\rangle \langle m| \rangle \rangle M \rangle \rangle \) \( \) (or \( |1\rangle, \ldots, |R\rangle \rangle M, I - \sum_{m=1}^{M} |m\rangle \langle m| \rangle \rangle M \rangle \rangle \) when \( M = R + 1 \), because both measurement bases \( C_1 \) and \( C_2 \) lead to the same POVM elements: \( \hat{K}_\mu \hat{K}_\mu^\dagger = |i\rangle \langle i| \) for \( i = R + 1 \) and \( \hat{K}_\mu \hat{K}_\mu^\dagger = I - \sum_{m=1}^{M-1} |m\rangle \langle m| - \Sigma_{m=R+2}^{M} |m\rangle \langle m| \rangle \rangle M \rangle \rangle \). Thus, all \( \rho_m = \text{tr}[ \Lambda_\mu (\hat{\rho}) ] = \text{tr}[ \hat{K}_\mu^\dagger \hat{\rho} \hat{K}_\mu ] \) are the same, and both measurement carry the same informational content. This equivalence is also in the sense of Definition 3: the two coarse-grainings \( C_1 \) and \( C_2 \) are finer than each other. However, note that in these two coarse-grainings, the final (projected) state is different.

\(^{11}\) Abusing the notation, what we mean be the present expressions is \( |\varphi(x)\rangle = |\varphi\rangle = \int_{-\infty}^{\infty} dx \varphi(x) |x\rangle \) and \( |\varphi(x-\kappa \mu)\rangle = \int_{-\infty}^{\infty} dx \varphi(x-\kappa \mu) |x\rangle \), when expressed in the position basis.
The types of von Neumann measurement schemes are illustrated in Fig. 2, and details are briefly summarized in the next subsection and in Appendix D.

A. Observational entropy in von Neumann measurement schemes

1. Projective measurement

Before we dive into observational entropy of von Neumann measurement schemes, we first define observational entropy of a projective measurement, which is performed directly on the system itself. Value of this entropy will serve as a reference to which we compare entropies of the different types of von Neumann measurement schemes. The probabilities of outcomes of a projective measurement and related volumes are given by $p_{\mu} = \text{tr}[P_{\mu}\hat{\rho}]$ and $V_{\mu} = \text{tr}[P_{\mu}]$ respectively. This defines observational entropy of a projective measurement
as $S^m = S_{\rho} = -\sum_{\mu} p_{\mu} \ln(p_{\mu}/V_{\mu})$.

2. Single measurement

The evolution superoperator of a single von Neumann measurement will be derived by combining Eqs. (39) and (42), which gives

$$\mathcal{A}_x(\hat{\rho}) = \langle x | U \hat{\rho} \otimes \varphi \rangle \langle \varphi | U^\dagger | x \rangle_B$$

$$= \sum_{m,m'} \rho_{mm'} \langle x | \varphi(x - \kappa m) \rangle \langle \varphi(x - \kappa m') | x \rangle_B | m \rangle \langle m' \rangle$$

$$= \hat{K}_x \hat{\rho} \hat{K}_x^\dagger,$$

where $\hat{K}_x = \sum_m \sqrt{g_0(x - \kappa m)} | m \rangle \langle m |$. Clearly, from the normalization of Gaussian function we have

$$\int_{-\infty}^{+\infty} \hat{K}_x^\dagger \hat{K}_x \, dx = \hat{I}.$$ This further yields

$$p_x = \text{tr}[\mathcal{A}_x(\hat{\rho})] = \sum_{\mu} \rho_{mm} g_0(x - \kappa \mu) = \sum_{\mu} p_{\mu} g_0(x - \kappa \mu),$$

$$V_x = \text{tr}[\mathcal{A}_x(\hat{I})] = \sum_{\mu} \rho_{mm} g_0(x - \kappa \mu) = \sum_{\mu} V_{\mu} g_0(x - \kappa \mu).$$

(45)

These probabilities and volumes will be used for calculating observational entropy of a single von Neumann measurement $S^{m} = -\int dx \, p_x \ln(p_x/V_x)$. Using Jensen’s inequality, one can easily prove $S^{m} \leq S^{m} \leq \ln \dim \mathcal{H}$, while the bounds $S^{m}$ and $\ln \dim \mathcal{H}$ are reached by $S^{m}$ in limits $\Omega \to 0$ and $\Omega \to +\infty$, respectively.

3. Repeated measurements

Repeated von Neumann measurement consists of $N$ interactions with $N$ different probes, all prepared in the same state (42). Immediately after the system interact with a probe, position of the probe is measured. The time of interaction with each probe is $\Delta t = \kappa/\lambda$. When this process is repeated $N$ times with $N$ different probes, we obtain $N$ measurement outcomes of the positions of the probes, $x = (x_1, \ldots, x_n)$. Assuming that the system freely evolves with Hamiltonian $\hat{H}$ in between the interactions, $U_f = \exp(-i\hat{H} \Delta t)$ of the system, the total time of free evolution $T = N \Delta t$. The above description can be illustrated as follows,

$$\hat{\rho} \xrightarrow{\text{x}} A^{m}_{x} \hat{\rho} A^{m}_{x}^\dagger \frac{p_x}{p_x},$$

(46)

where, defining evolution superoperator $U_f = U_f(\hat{\bullet})U_f^\dagger$, the superoperator of repeated interaction is

$$A^{m}_{x}(\hat{X}) = U_f A_{xN} \cdots U_f A_{x1}(\hat{X}).$$

(47)

Above, $A_{xi}$ are defined by Eq. (44). The probabilities and volumes for the observational entropy are $p_x = \text{tr}[A^{m}_x(\hat{\rho})]$ and $V_x = \text{tr}[A^{m}_x(\hat{I})]$, which gives $S^{m} = -\int dx_1 \cdots dx_N p_x \ln(p_x/V_x)$.

4. Repeated contacts

Finally, repeated contacts consist of $N$ interactions with a single probe, without resetting the probe, and with one position measurement in the very end.

$$\hat{\rho} \xrightarrow{\text{x}} A^{m}_{x} \hat{\rho} A^{m}_{x}^\dagger \frac{p_x}{p_x},$$

(48)

The superoperator of repeated contacts reads

$$A^{m}_x(\hat{X}) = \text{tr}_B[(\hat{I} \otimes \hat{P}_x)(U_f \otimes U)^N (\hat{X} \otimes \hat{\sigma})(\hat{I} \otimes \hat{P}_x)],$$

(49)

where we have defined evolution superoperator $U_f = (U_f \otimes \hat{I})(U_f \otimes \hat{I})^\dagger$ and interaction superoperator $U =...
\[ U(\bullet) U^\dagger \] with \( U \) being defined by Eq. (36). The probabilities and volumes for the observational entropy are \( p_x = \text{tr}[A_x^\tau(\hat{\rho})] \) and \( V_x = \text{tr}[A_x^\tau(I)] \), which gives \( S^{\text{tr}} = -\int dx p_x \ln(p_x/V_x) \).

There can be different scenarios considered, depending on the time step \( \Delta t \) in the evolution operator \( U_f \). In the case where the total time of interaction \( T = N\Delta t \) and the ratio of interaction time and free evolution time \( R = \kappa/\Delta t \) are fixed (which corresponds to Fig. 6 in the next subsection), we can solve this problem analytically for large \( N \),

\[ A_{x}^{\tau}(\hat{X}) = \text{tr}_B[(I \otimes \hat{P}_x)U_{\text{limit}}(\hat{X} \otimes \hat{\sigma})U_{\text{limit}}^\dagger(I \otimes \hat{P}_x)], \tag{50} \]

where

\[ U_{\text{limit}} = \exp\left(-i (\hat{H} \otimes I + R \hat{M} \otimes \hat{\rho})T\right) \tag{51} \]

was obtained from the Lie-Trotter product formula \( e^{\hat{X}+\hat{Y}} = \lim_{N \to \infty} (e^{\hat{X}/N} e^{\hat{Y}/N})^N \). We denote the corresponding observational entropy as \( S^{\text{limit}} \).

The explicit forms of the repeated interaction and repeated contacts superoperators, which we used for coding our simulations, can be found in Appendix D.

### B. Numerical simulations

We perform several simulations of von Neumann measurement schemes, to study how different types of von Neumann measurement schemes perform in information extraction, by using observational entropy as the quantifier of this performance. Due to computational complexity of the given task, we will do it only for a two-dimensional system. However, these simulations are mainly a demonstration of the general theory, proven for systems of arbitrary dimension, together with general considerations and explanations, which will also be independent of the dimension. Therefore, we do not expect any qualitative difference for a higher dimensional system.

The computational basis will be defined by eigenbasis of the measurement operator, which we always take to be

\[ \hat{M} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \tag{52} \]

which leads to eigenvalues \( \mu_1 = 0 \) and \( \mu_2 = 2 \).

We evolve the system with two types of Hamiltonians: one that commutes with the measurement operator \( \hat{M} \),

\[ \hat{H} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \tag{53} \]

and the other which does not,

\[ \hat{H} = \begin{pmatrix} 0 & 1 + i \\ 1 - i & 2 \end{pmatrix}. \tag{54} \]

Exact form of the commuting Hamiltonian is irrelevant as long as it commutes with \( \hat{M} \), because any commuting Hamiltonian commutes through and cancels itself in all probabilities \( p_x \) and \( p_y \) and volumes \( V_x \) and \( V_y \), thus none of the commuting Hamiltonian has an effect on the value of observational entropy.

We parametrize the initial qubit state as

\[ \hat{\rho}(\phi, \theta, \alpha) = U \begin{pmatrix} \cos^2 \alpha & 0 \\ 0 & \sin^2 \alpha \end{pmatrix} U^\dagger, \tag{55} \]

where

\[ U = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \tag{56} \]

To fix the maximum of observational entropy at one in our simulations, we chose the basis of the logarithm.
for the observational entropy to be $\log_2$ instead of the natural logarithm in all of the examples (this changes only the scale).

Fig. 3 contains observational entropy of a single measurement scheme (described in Sec. III A 2, and Fig. 2 (a)) as a function of the standard deviation $\Omega$ of the probe (which measures how much localized in space it is). We take $\rho(0,\pi/16,\pi/16)$ as the initial state, and $\kappa = 1$. There is no free time evolution in this case.

Fig. 4 contains the case of repeated measurements (Sec. III A 3, Fig. 2 (b)) and repeated contacts (Sec. III A 4, Fig. 2 (c)), dependent on the total number of measurements/contacts $N$, for the Hamiltonian that commutes with the measurement operator (53). We take $\rho(0,\pi/16,\pi/16)$ as the initial state, $\kappa = \Omega = 1$, and the time step of free-time evolution between the measurements/contacts to be fixed at $\Delta t = 1$. Thus the total time of free-time evolution is given by $T = N\Delta t$. For the discussion of convergence in panel (a), please see the end of this section.

Fig. 5 shows the same case as Figs. 4, but now with the Hamiltonian that does not commute with the measurement operator, Eq. (54). We consider multiple initial states, $\rho(0,\pi/16,\pi/16)$, $\rho(0,0,0)$, $\rho(0,\pi/4,\pi/16)$, $\rho(\pi/3,\pi/3,\pi/16)$ for cases (a), (d), (e), (f), respectively.

Fig. 6 contains a different type of limit of large $N$ than previous figures. The total time of interaction $T = 10$ is fixed, as well as the ratio of the interaction and the free time evolution time $R = \kappa/\Delta t = 1$. The free time evolution step is given by $\Delta t = T/N$, where number of contacts $N$ varies. We include this figure to illustrate our analytical result for the long time limit of repeated contacts entropy in this scenario, Eq. (50).

Panels (b) and (c) of Figs. 4 and 5 illustrate evolving probabilities of position measurements and normalized corresponding volumes respectively. These illustrate evolution of the quantum system, as well as shed some light on the behavior of the observation entropy through relation

$$S_{\mathcal{C}} = \ln V - D_{\text{KL}}(p_i||V_i/V)$$

between the observational entropy and the Kullback-
Leibler divergence $D_{KL}$, where $V = \dim \mathcal{H}$ is the dimension of the Hilbert space. Since Kullback-Leibler divergence measures how distinguishable two probabilities distributions are, we can interpret observational entropy as a measure of distinguishability between the observed probability $p_i$, and the probability of outcomes $V_i/V = \text{tr}[A_i(\hat{\rho}_{\text{max}})]$ which we would expect in the case when we have absolutely no information about the system, and therefore chose to represent it by the maximally uncertain density matrix $\hat{\rho}_{\text{max}} = I/\dim \mathcal{H}$.

Finally, let us comment on different scalings with which the repeated measurements and repeated contacts approach the projective entropy in Fig. 4. In Appendix D we show that both $p$ and $V$ are a linear combination of Gaussians with peaks at very specific points given by eigenvalues of the measurement operator $M$. In repeated contacts, the peaks of the furthest Gaussians are $\mathcal{N}(\mu_{\text{max}} - \mu_{\text{min}})$ far from each other, while in repeated measurements the peaks form corners of a hypercube with $(\mu_{\text{min}}, \ldots, \mu_{\text{min}})$ and $(\mu_{\text{max}}, \ldots, \mu_{\text{max}})$ being the furthest points. Hence the Euclidean distance gives the result $\sqrt{N(\mu_{\text{max}} - \mu_{\text{min}})}$ for the distance between them. Relation (57) tells us that the minimum of observational entropy is reached when the difference between distributions $p$ and $V$ is maximal and their overlap is minimized. Since the the standard deviation $\Omega$ of these Gaussians is fixed and does not change with $N$, to minimize the overlap one would ideally create a distribution $p$ with a peak that is the furthest possible distance from the peak of $V$. Since the scaling of distance of the furthest peaks
FIG. 6. Behavior of repeated contacts entropy $S_{\text{rc}}$ in a different type of limit than in Figs. 4 and 5. Here we consider the case where the total time of interaction $T = 10$ is fixed, as well as the ratio of the interaction and the free time evolution time $R = \kappa / \Delta t = 1$. The repeated contacts entropy converges to the predicted theoretical value $S_{\text{limit}}$ in the limit of large $N$, Eq. (50).

differs by a $\sqrt{N}$ we would expect that also the speed of convergence scales with the same factor. However, this argument does not represent a proof, which we leave for future work.

For additional details on numerical aspects of the simulation, please see Appendix E.

IV. RELATION TO QUANTUM TOMOGRAPHY

In this section, we describe an algorithm through which we can infer the state of the quantum system, in the case when we measure the observational entropy to be equal to the von Neumann entropy. Thus, we connect the notion of observational entropy with quantum tomography [52, 53].

We concluded in Sec. II A that it is possible to use an indirect measurement scheme to determine the state of the system when

1. we know the von Neumann entropy,

2. observational entropy is equal to the von Neumann entropy,

with two additional assumptions which are assumed implicitly, but we still mention them here for clarity,

3. we know probabilities $p_i$, for example because we experimentally determined them,

4. we know the coarse-grainings that define the observational entropy.

This statement however holds generally for any measurement scheme (both projective and indirect, or any other general measurement scheme). To show that, we will provide an explicit algorithm of how to determine the quantum state when these conditions are satisfied. This quantum state inference algorithm is based on the equality condition of Theorem 1, Eq. (16) specifically. It goes as follows.

As in Definition 2 we assume that superoperators $A_i = \sum_m \hat{K}_{im}(\bullet)\hat{K}_{im}^\dagger$ define the vector of coarse-grainings $\mathcal{C} = \{A_i\}$, and we define POVM elements as $\Pi_i \equiv \sum_m \hat{K}_{im}^\dagger \hat{K}_{im}$.

0. Initialize $\hat{\rho} = 0$ (the density matrix).

1. Take an index $i$ that has not been used before in this algorithm (i.e., from the set of indexes that have not been previously included in some projector; index $i$ corresponds to a measurement outcome). Is there any?

(a) YES: Continue to 2.

(b) NO: Return $\hat{\rho}$ (this is the state of the system).

2. Initialize $\hat{P} = \Pi_i$ (what will become a projector projecting into an eigenspace of $\hat{\rho}$) and $I = \{i\}$ (a set of indexes that define $\hat{P}$, right now containing a single element).

3. Find all $i'$ that have not been used before such that $\Pi_{i'} \hat{P} \neq 0$. Are there any?

(a) YES: Update $\hat{P} = \hat{P} + \sum_{i'} \Pi_{i'}$ and $I = I \cup \{i'\}$.

Go back to 3.

(b) NO: According to Lemma 1, $\hat{P}$ is a projector. Continue to 4.

4. Calculate $\rho = (\sum_{i \in I} p_i) / \text{tr}[\hat{P}]$ (where $p_i$’s denote the probabilities of outcomes).

5. Update $\hat{\rho} = \hat{\rho} + \rho \hat{P}$. Go back to 1.

The fact that $\hat{\rho}$ generated this way is the state of the system comes from the following lemmas.

Lemma 1. $\hat{P}$’s generated by steps 1.–3. form a complete set of orthogonal projectors, and each $\hat{P}$ projects into an eigenspace of the state of the system.

Lemma 2. $\rho$ generated by step 4. is an eigenvalue of the state of the system corresponding to projector $\hat{P}$.

The proofs of these lemmas, which are a consequence of Eq. (16), can be found in Appendix C. There you will also find an example of how this algorithm works on an example of a single von Neumann measurement scheme from section III A 2.
The task of quantum tomography is to determine the state of the system by making measurements (while assuming we have access to an infinite number of copies of the state of the system). Generally, this cannot be done by measuring in a single basis. This is because a single basis provides at most $N$ measurement outcomes which can be turned into $N$ real parameters. However, the density matrix depends in general on $N^2 - 1$ real parameters, and therefore one needs to measure in different bases to conclusively determine a generic state of the system.

Similar to quantum tomography, observational entropy will be equal to the von Neumann entropy for not just for any coarse-graining but only for a few very specific ones, so also here we have to search for a coarse-graining that will satisfy this condition. Since coarse-graining represents a measurement basis, searching for a coarse-graining is equivalent of searching for a measurement setup. Therefore, we can view the task of quantum tomography as equivalent to trying different measurement setups, from which we determine the observational entropy for each setup, until we obtain the value equal to the von Neumann entropy. Then, from the setup that lead to observational entropy being equal to the von Neumann entropy, we can infer the quantum state using the aforementioned algorithm (using that each setup corresponds to a coarse-graining).

We do not claim that this method is better than the currently existing methods of quantum tomography, but that it provides an alternative. A drawback of this method as it currently stands, is that in order to use the algorithm we need to know the value of the von Neumann entropy. This might be a problem for a system about which we have absolutely no information, and therefore also no knowledge of the von Neumann entropy. In such a situation, it cannot be determined whether the observational entropy has reached its minimum or not, or whether there is a better, yet undiscovered coarse-graining leading to its lower value, closer or equal to the unknown von Neumann entropy.

However, in some situations this might not be an issue. There are many situations we know the initial state, simply because we prepared it. If such a system evolves unitarily, potential with a unitary that is not known, the von Neumann entropy does not change. We might not know what the state is after such an evolution, but we know its von Neumann entropy. In such circumstances, the algorithm can be readily applied.

In is an interesting open problem whether the aforementioned algorithm could be generalized to situations in which the von Neumann entropy is not known, or, when it is known but observational entropy only approximates the von Neumann entropy, but is not exactly equal. Even in these cases, it should be possible to provide some estimate of the density matrix. This estimation procedure would be expected to provide an estimate of the density matrix that would converge to the real density matrix when observational entropy converges to the von Neumann entropy.

V. DISCUSSION, CONCLUSION, AND FUTURE DIRECTIONS

In this paper, we generalized the concept of observational entropy to include general coarse-grainings (given by a general measurement, POVM), instead of considering only projective coarse-grainings, which was done in previous literature. This was motivated by the search for the fundamental nature of this quantity and its interpretation, and by a growing number of experiments using general quantum measurements. In this quest, we overcame and resolved several important subtleties, such as answering how to define a general coarse-graining, how to treat POVM elements that are not orthogonal with each other, what is an appropriate definition of volume of a macrostate, and how can we compare different coarse-grainings.

The main message of this paper is that even with the general definition of observational entropy, which is defined by a series of possibly non-commuting, general measurements, all of the important properties still hold. Observational entropy can be therefore still interpreted as a measure of uncertainty that an observer performing a series of measurements would associate to the initial state of the system. The properties can be summarized as follows: observational entropy is lower bounded by the minimal uncertainty given by von Neumann entropy, upper bounded by the maximal entropy given by the logarithm of the dimension of the system. With each additional coarse-graining, observational entropy cannot decrease, expressing that “each additional measurement can only increase observers’ knowledge of the state of the system.” If one coarse-graining is finer than the other, then observational entropy is always lower for a finer coarse-graining, expressing that “observer that makes a more precise measurement will get to know at least as much as an observer that makes a worse measurement.”

We applied this concept to study how indirect measurements perform in information extraction as compared to a direct measurement. Performing an analysis of a general scenario of indirect measurements we found, for example, that any pure state of the system can be perfectly indirectly determined by using a two-level auxiliary system—a result which a posteriori seems very natural. To illustrate the application on a specific and timely example, we applied this concept to various von Neumann measurement schemes, in which a quantum system is probed through an auxiliary system consisting of a classi-

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12 Note that observational entropy is not an observable, however, it is still possible to measure the probabilities of outcomes $p_i$, if one has infinitely many copies of the state of the system. The corresponding volumes are computed directly from the coarse-graining. Then, using $p_i$’s and $V_i$’s we can compute the observational entropy.
cal particle. Because observational entropy measures information extracted in different situations, it serves as a measure of how well and in which circumstances different measurement schemes perform in information extraction. This not only provides new insights into understanding of von Neumann schemes and determines which one is the best in which situation, but also demonstrates usefulness of this quantity.

This has far reaching implications when applying this concept in general settings: computing observational entropy is a relatively simple, yet very powerful test of the performance of any set of measurements in information extraction. For example, construction of a quantum computer, which is expected to solve complex problems beyond the capabilities of a classical computer, will require a quantum memory, and a mechanism that reads out the output of the computation [54, 55]. Computing observational entropy will determine which are the best measurements schemes to read out both the quantum memory and the computation output.

Finally, we discovered operational interpretation of observational entropy. We did this by connecting this quantity with quantum tomography by showing that knowledge of the coarse-graining that leads to the minimum of the observational entropy allows for a successful inference for the state of the system. We presented a general algorithm that achieves that. Generalizing this connection to situations when the knowledge is not perfect, for example, for the cases in which the coarse-graining gives a low, but not the minimal value of observational entropy, provides a new exciting direction of study. This is because such a theory would be much more practical. Perfect measurements are practically impossible to perform, so having an algorithm that gives a reasonable estimate of the density matrix in realistic situations, mathematically connected with a measure that says how much information can be extracted, would be an excellent tool for efficient information gathering.

An application of this theory could also lie in the study of microscopic thermodynamic systems, and especially in understanding of quantum entropy production [56–58]. While the ground work of using observational entropy for these purposes was already established in [24, 29], these works had limitations of using only projective measurements, which meant that the system and the bath had to be considered together as a whole. However, if one would like to answer questions such as “how much information about the system can be extracted by measuring a bath,” for example, for the goals of studying non-Markovianity caused by the finiteness of the bath [29, 59–61], such a scenario immediately leads to a general (non-projective) measurement on the system. The framework developed here allows for answering this. Evolution of a cyclic quantum engine [62–64], which always involve some contact with a thermal bath, also acts as a general measurement on the system. Observational entropy could be used to study how much information, and thus also possibly work is lost with imperfect baths which interact with a particular type of interaction, which is not completely thermalizing.

Observational entropy has already been used to study black holes [26] and big bang [27]. In black holes, a common question concerns the amount of information lost or gained through evaporation. Measuring the evaporated states that escape the black hole provides some information on its inner structure. However, performing a projective measurement on these states mathematically corresponds to performing a general measurement on the black hole instead. The framework developed here allows for computing exactly how much information on the black hole has been gained by doing this measurement.

Clearly, there are many possible applications of this quantity, any time where there is some information gained when making a quantum measurement. We hope that the future development of this framework, as well as finding application by experts in their respective fields, will demonstrate its practical usefulness beyond its current status.

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(Cambridge University Press, 2010).

[53] E. Toninelli, B. Ndagano, A. Vallès, B. Sephton, I. Nape, A. Ambrosio, F. Capasso, M. J. Padgett, and A. Forbes, Adv. Opt. Photon. 11, 67 (2019).

[54] T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe, and J. L. O’Brien, nature 464, 45 (2010).

[55] National Academies of Sciences, Engineering, and Medicine and others, Quantum computing: progress and prospects (National Academies Press, 2019).

[56] J. Goold, M. Huber, A. Riera, L. del Rio, and

[57] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, Journal of Physics A: Mathematical and Theoretical 49, 143001 (2016).

[58] S. Vinjanampathy and J. Anders, Contemporary Physics 57, 545 (2016), arXiv:1508.06099 [quant-ph].

[59] R. Aliki and R. Kosloff, arXiv e-prints, arXiv:1801.08314 (2018), arXiv:1801.08314 [quant-ph].

[60] S. Chakraborty, A. Mallick, D. Mandal, S. K. Goyal, and S. Ghosh, arXiv e-prints, arXiv:1703.02749 (2017), arXiv:1703.02749 [quant-ph].

[61] S. Marcantoni, S. Alipour, F. Benatti, R. Floreanini, and A. Rezakhani, Scientific reports 7, 1 (2017).

[62] J. Martins, L. Defaveri, D. O. Soares-Pinto, S. M. D. Queirós, and W. A. M. Morgado, Phys. Rev. E 103, 022108 (2021), arXiv:2009.14290 [cond-mat.stat-mech].

[63] R. Kosloff and A. Levy, Annual Review of Physical Chemistry 65, 365 (2014), PMID: 24629798, https://doi.org/10.1146/annurev-physicalchem-040513-103724.

[64] F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso, Thermodynamics in the quantum regime: fundamental aspects and new directions, Vol. 195 (Springer, 2019).

[65] V. Singh, Phys. Rev. Research 2, 043187 (2020).

Appendix A: Proofs of Theorems

The proofs of Theorems 1. and 2. are a modification of those done for projective measurements (Thms. 7. and 8. of [20]), and the spirit of the proof is exactly the same. The proof of Theorem 3 is still quite similar to the proof of Theorem 2. in [20]), but modified more significantly, because it uses a more general (and different-looking, although being equivalent on the special cases) Definition 3.

All inequalities follow from application of the well-known theorem:

Theorem 4. (Jensen) Let \( f \) be a strictly concave function, \( 0 \leq \alpha_i \leq 1, \sum_i \alpha_i = 1 \). Then for any \( b_i \in \mathbb{R} \),

\[
\frac{1}{\sum_i \alpha_i b_i} \geq \sum_i \alpha_i f(b_i). \quad (A1)
\]

\[
f(\sum_i \alpha_i b_i) = \sum_i \alpha_i f(b_i) \quad \text{if and only if} \quad (\forall i, j)(\alpha_i \neq 0, \alpha_j \neq 0)(b_i = b_j).
\]

1. Proof of Theorem 1

Proof. First we are going to prove \( S_{C_{N}}(\hat{\rho}) \leq S_{C}(\hat{\rho}) \), and second \( S_{C}(\hat{\rho}) \leq \ln \dim \mathcal{H} \), each with its equality condition, where the vector of coarse-graining is\( \mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n) = \{\mathcal{A}_i\} \), (A2) and

\[
\mathcal{A}_i(\hat{\rho}) = \sum_m \hat{K}_{im} \hat{\rho} \hat{K}_{im}^\dagger, \quad (A3)
\]

\[
\hat{K}_{im} = \hat{K}_{i,m_1} \cdots \hat{K}_{i,m_{n-1}}, \quad \hat{K}_{im}^\dagger = \hat{K}_{i,m_{n-1}}^\dagger \cdots \hat{K}_{i,m_1}^\dagger, \quad \text{where}
\]

\[
\sum_{i,m} \hat{K}_{im}^\dagger \hat{K}_{im} = \hat{I}, \quad (A4)
\]

where to keep the notation short, we write

\[
\sum_{i_1,\ldots,i_n} \sum_{m_1,\ldots,m_n} \equiv \sum_i \sum_m \equiv \sum_{i,m}. \quad (A5)
\]

We denote the spectral decomposition of the density matrix as \( \hat{\rho} = \sum_x \rho_x |x\rangle \langle x| \) denote eigenvectors of the density matrix, and \( \rho_x \) denote the eigenvalues which are not necessarily different for different \( x \) — thus this decomposition is not unique. We also denote the unique decomposition of the density matrix in terms of its eigen-projectors \( \hat{\rho} = \sum_x \rho_x \hat{P}_x \), where eigenvalues \( \rho \) are now different from each other. It follows that for each \( x \) there is \( \rho \) such that \( \rho_x = \rho \).

Now we prove \( S_{C_{N}}(\hat{\rho}) \leq S_{C}(\hat{\rho}) \) together with its equality condition. Defining

\[
a_{x}^{(i)} = \langle x| \sum_m \hat{K}_{im}^\dagger \hat{K}_{im} |x\rangle \quad (A6)
\]

for \( V_i \neq 0 \) and \( a_{x}^{(i)} \equiv 0 \) for \( V_i = 0 \), and then using the spectral decomposition of \( \hat{\rho} \) we have

\[
\frac{\rho_k}{V_i} = \sum_x \rho_x \frac{\langle x| \sum_m \hat{K}_{im}^\dagger \hat{K}_{im} |x\rangle}{V_i} = \sum_x \rho_x a_{x}^{(i)}. \quad (A7)
\]

Using cyclicity of the trace we obtain

\[
V_i = \text{tr}[\sum_m \hat{K}_{im} \hat{K}_{im}^\dagger] = \text{tr}[\sum_m \hat{K}_{im}^\dagger \hat{K}_{im}] = \sum_x \langle x| \sum_m \hat{K}_{im}^\dagger \hat{K}_{im} |x\rangle, \quad \text{from which follows}
\]

\[
\sum_x a_{x}^{(i)} = 1. \quad (A8)
\]

Then, applying the completeness relation (A4), we also have

\[
\sum_i V_i a_{x}^{(i)} = \sum_{i,m} \langle x| \hat{K}_{im}^\dagger \hat{K}_{im} |x\rangle = \langle x| x \rangle = 1. \quad (A9)
\]
Series of identities and inequalities follow

\[ S_C(\rho) = - \sum_i p_i \ln \frac{p_i}{v_i} \]
\[ = - \sum_i v_i \frac{p_i}{v_i} \ln \frac{p_i}{v_i} \]
\[ = \sum_i v_i \left( - \sum_x \rho_x a_x^{(i)} \ln \sum_x \rho_x a_x^{(i)} \right) \]
\[ \geq \sum_i v_i \left( - \sum_x \rho_x^2 \ln \rho_x \right) \]
\[ = - \sum_x \left( \sum_i v_i a_x^{(i)} \right) \rho_x \ln \rho_x = S_{\mathcal{N}}(\rho). \]  

(A10)

The third identity follows from Eq. (A7), and the last identity follows from Eq. (A9). We have used the Jensen’s Theorem 4 on the strictly concave function \( f(x) = -x \ln x \) in order to obtain the inequality. We have chosen \( a_x \equiv a_x^{(i)} \) and \( b_x = \rho_x \) for the Theorem. This is a valid choice due to \( 0 \leq a_x^{(i)} \leq 1 \) and due to Eq. (A8). This proves inequality \( S_{\mathcal{N}}(\rho) \leq S_C(\rho) \).

According to the Jensen’s Theorem, this inequality becomes identity if and only if

\[ (\forall i)(\forall x, \tilde{x}|x\Pi_i|x) \neq 0, (\tilde{x}|\Pi_i|\tilde{x}) \neq 0)(\rho_x = \rho_{\tilde{x}}). \]  

(A11)

We can simplify the notation by defining the POVM element

\[ \Pi_i = \sum_m \tilde{K}_{im}^\dagger \tilde{K}_{im}, \]  

(A12)

and write

\[ (\forall i)(\forall x, \tilde{x}|x\Pi_i|x) \neq 0, (\tilde{x}|\Pi_i|\tilde{x}) \neq 0)(\rho_x = \rho_{\tilde{x}}) \]  

(A13)

instead, \( |x\Pi_i|x = 0 \) is equivalent to \( \Pi_i|x = 0 \). To prove that, we have

\[ \langle x|\Pi_i|x \rangle = 0 \]
\[ \Rightarrow \langle x|\sum_m \tilde{K}_{im}^\dagger \tilde{K}_{im}|x \rangle = 0 \]
\[ \Rightarrow \sum_m \|\tilde{K}_{im}|x \|^2 = 0 \]  

(A14)

\[ \Rightarrow (\forall m)(\tilde{K}_{im}|x \rangle = 0) \]
\[ \Rightarrow (\forall m)(\tilde{K}_{im}^\dagger \tilde{K}_{im}|x \rangle = 0) \]
\[ \Rightarrow \sum_m \tilde{K}_{im}^\dagger \tilde{K}_{im}|x \rangle \langle x| = \Pi_i|x \rangle \langle x| = 0, \]

and clearly, \( \Pi_i|x \rangle = 0 \) implies \( |x\Pi_i|x = 0 \). This means that \( \langle x|\Pi_i|x = 0 \iff \Pi_i|x \rangle = 0 \) and also \( \langle x|\Pi_i|x \rangle = 0 \iff \Pi_i|x \rangle \neq 0 \). Using this equivalence, we rewrite condition for the inequality to be the identity, Eq. (A13), as

\[ (\forall i)(\forall x, \tilde{x}|x\Pi_i|x) \neq 0, (\tilde{x}|\Pi_i|\tilde{x}) \neq 0)(\rho_x = \rho_{\tilde{x}}). \]  

(A15)

We explain this condition as follows: the inequality becomes identity when for a fixed multi-index \( i \), all eigenvectors of the density matrix \( |x \rangle \) such that \( \Pi_i|x \rangle \langle x | \neq 0 \) must have the same associated eigenvalue \( \rho_x \). In other words, this unique eigenvalue can be associated to the multi-index \( i \) itself, so we can relabel \( \rho_i \equiv \rho_x \) where the eigenvalue \( \rho_x \) is given by any representative \( x \) such that \( \Pi_i|x \rangle \langle x | \neq 0 \). This must hold for every multi-index \( i \), in order for the inequality to become identity. Therefore, this defines a unique map which associates some eigenvalue of the density matrix to each multi-index \( i \). To highlight existence of this map we can extend Eq. (A15) and write

\[ (\forall i)(\forall x, \tilde{x}|x\Pi_i|x) \neq 0, (\tilde{x}|\Pi_i|\tilde{x}) \neq 0)(\rho_x = \rho_{\tilde{x}} \equiv \rho_i). \]  

(A16)

Then, defining set

\[ J^{(\rho)} = \{ x|\rho_x = \rho_i \}. \]  

(A17)

using condition (A16), and \( \sum_x |x \rangle = \hat{I} \),

\[ \Pi_i = \Pi_i \hat{I} = \Pi_i \sum_x |x \rangle \langle x | = \Pi_i \sum_{x \in J^{(\rho)}} |x \rangle \langle x | = \Pi_i \hat{T}_{\rho_i}. \]  

(A18)

Then, assuming that \( \rho \neq \rho_i \), we can multiply this equation by \( \hat{P}_\rho \) from the right, and from orthogonality of projectors we obtain

\[ \Pi_i \hat{P}_{\rho_i} = 0. \]  

(A19)

In other words, this means that for every \( i \) such that \( \rho_{i'} \neq \rho_i \),

\[ \Pi_i \hat{P}_{\rho_i} = 0. \]  

(A20)

Finally, for any eigenvalue \( \rho \) we define index set

\[ J^{(\rho)} = \{ i|\rho_i = \rho \}. \]  

(A21)

Using the completeness relation \( \sum_i \Pi_i = \hat{I} \), and combining Eqs. (A18) and (A20) we obtain

\[ \hat{P}_\rho = \sum_i \Pi_i \hat{P}_\rho = \sum_{i \in J^{(\rho)}} \Pi_i \hat{P}_\rho = \sum_{i \in J^{(\rho)}} \Pi_i, \]  

(A22)

which by definition means that \( C_\rho \rightarrow C \).

Conversely, for a contradiction we assume Eq. (A22) holds, but Eq. (A13) does not, which would mean that there are \( x \) and \( \tilde{x} \) such that \( \langle x|\Pi_i|x \rangle = 0 \) and \( \langle \tilde{x}|\Pi_i|\tilde{x} \rangle = 0 \) while \( \rho_x \neq \rho_{\tilde{x}} \). We assume arbitrary \( i \). If \( i \in J^{(\rho)} \) then multiplying Eq. (A22) by \( \hat{P}_{\rho_x} \) we obtain

\[ 0 = \langle x|\hat{P}_{\rho_x} \hat{P}_{\rho_i}|x \rangle = \langle x|\hat{P}_\rho|x \rangle = \sum_{i \in J^{(\rho)}} \langle x|\Pi_i|x \rangle, \]  

(A23)

which implies that for every \( i \in J^{(\rho)} \), \( \langle x|\Pi_i|x \rangle = 0 \), due to positivity of operators \( \Pi_i \). Thus, if \( |x \rangle \) and \( |\tilde{x} \rangle \) are associated with different eigenvalues, at least one of the \( \langle x|\Pi_i|x \rangle \) is zero. This is a contradiction with our assumption. Thus, Eq. (A22) implies (A13),
which is equivalent with the equality condition $S_{\mathcal{C}}(\hat{\rho}) = S_{\mathcal{C}}(\hat{\rho})$.  
We have therefore shown that $S_{\mathcal{C}}(\hat{\rho}) = S_{\mathcal{C}}(\hat{\rho})$ if and only if $C_{\hat{\rho}} \rightarrow \mathcal{C}$, which concludes the first part of the proof.

Second, we prove $S_{\mathcal{C}}(\hat{\rho}) \leq \ln \dim \mathcal{H}$ together with its equality condition.

\[
S_{\mathcal{C}} = \sum_{i p_i \neq 0} p_i \ln \frac{V_i}{p_i} \leq \ln \left( \sum_{i p_i \neq 0} \frac{p_i V_i}{p_i} \right) = \ln \left( \sum_i V_i \right) = \ln \tr I = \ln \dim \mathcal{H}.
\]  
(A24)

In here, the first inequality follows from the Jensen’s Theorem, which was applied on strictly concave function $f(x) = \ln x$, while we have chosen $a_i = p_i$ and $b_i = \frac{V_i}{p_i}$ for the Theorem. This is a valid choice, because $0 \leq a_i \leq 1$ and $\sum_i a_i = 1$ hold. The second inequality follows from $V_i \geq 0$ while realizing that logarithm is an increasing function. The second equality follows from the completeness relation $\sum_{i,m} K_{i,m} \hat{K}_{i,m} = \hat{I}$ and from the definition of $V_i$.

The first inequality becomes equality if and only if

\[
(\forall i,l)[p_i \neq 0, p_i' \neq 0) \left( \frac{V_i}{p_i} = \frac{V_i'}{p_i'} = c \right) \tag{A25}
\]

for some real constant $c$. In order to determine the value of this constant, we express this condition as $V_i = cp_i$ and then we sum over all multi-indexes $i$ for which $p_i \neq 0$. This leads to $c = \sum_{i p_i \neq 0} V_i$. Therefore, we can write the first equality condition as

\[
(\forall p_i \neq 0) \left( p_i = \frac{V_i}{\sum_{i p_i \neq 0} V_i} \right). \tag{A26}
\]

Because logarithm is a strictly increasing function, the second inequality becomes identity if and only if for every multi-index $i$ such that $p_i = 0$ holds, also $V_i = 0$ holds. If this is true, then also $\sum_{i p_i \neq 0} V_i = \sum_i V_i = \dim \mathcal{H}$, where we have used the definition of $V_i$ and the completeness relation $\sum_{i,m} K_{i,m} \hat{K}_{i,m} = \hat{I}$. When we combine both of these equality condition, we obtain that $S_{\mathcal{C}}(\hat{\rho}) = \ln \dim \mathcal{H}$ if and only if

\[
(\forall p_i) \left( p_i = \frac{V_i}{\dim \mathcal{H}} \right). \tag{A27}
\]

This completes the proof.

\[\square\]

2. Proof of Theorem 2

Proof. To make our notation more easy to read, we denote $p_{i_1,\ldots,i_{n+1}} \equiv p_{i_{n+1}}$, and $V_{i_1,\ldots,i_{n+1}} \equiv V_{i_{n+1}}$, while we also use the same notation that we used in the previous proof. Using

\[
p_i = \sum_{i_{n+1}} p_{i,i_{n+1}}, \quad V_i = \sum_{i_{n+1}} V_{i,i_{n+1}} \quad \tag{A28}
\]

in combination with Jensen’s Theorem 4 gives

\[
S_{\mathcal{C}}(\hat{\rho}) = - \sum_i p_i \ln \frac{p_i}{V_i} = - \sum_i \sum_{i_{n+1}} p_{i,i_{n+1}} \ln \frac{p_{i,i_{n+1}}}{V_i} = - \sum_i \sum_{i_{n+1}} \frac{p_{i,i_{n+1}} V_{i,i_{n+1}}}{V_i} \ln \frac{p_{i,i_{n+1}} V_{i,i_{n+1}}}{V_i} \tag{A29}
\]

In here, we have used $f(x) = -x \ln x$, $a_{i_{n+1}} = \frac{V_{i,i_{n+1}}}{V_i}$, and $b_{i_{n+1}} = \frac{p_{i,i_{n+1}}}{V_{i,i_{n+1}}}$ for the Jensen’s Theorem 4.

The condition for Jensen’s inequality to become identity is

\[
(\forall i)(\forall V_{i,i_{n+1}}, V'_{i,i_{n+1}} = 0, V_{i,i_{n+1}} = 0) \quad \left( \frac{p_{i,i_{n+1}}}{V_{i,i_{n+1}}}, \frac{p_{i',i_{n+1}}}{V'_{i,i_{n+1}}} = c(i) \right), \tag{A30}
\]

where $c(i)$ is some $i$-dependent number, which we can compute using $\sum_{i_{n+1}} p_{i,i_{n+1}} = c(i) \sum_{i_{n+1}} V_{i,i_{n+1}}$, obtaining $c(i) = \frac{p_i}{V_i}$. This enables us to rewrite this condition as

\[
(\forall i)(\forall V_{i,i_{n+1}} = 0) \left( \frac{p_{i,i_{n+1}}}{V_{i,i_{n+1}}} = \frac{V_{i,i_{n+1}}}{V_i} p_i \right). \tag{A31}
\]

For every $V_{i,i_{n+1}} = 0$ we also have $p_{i,i_{n+1}} = 0$, from which we trivially obtain $p_{i,i_{n+1}} = \frac{V_{i,i_{n+1}}}{V_i} p_i$. Therefore, this condition can be simplified further, which gives the final result that $S_{\mathcal{C}}(\hat{\rho}) = S_{\mathcal{C},\mathcal{C}_{n+1}}(\hat{\rho})$ if and only if

\[
(\forall i)(\forall V_{i,i_{n+1}} = 0) \left( \frac{p_{i,i_{n+1}}}{V_{i,i_{n+1}}} = \frac{V_{i,i_{n+1}}}{V_i} p_i \right). \tag{A32}
\]

This completes the proof.

We make two interesting remarks about this condition: Assuming that $p_i \neq 0$, we can rewrite the above condition as $p(i_{n+1}|i) \equiv \frac{p_{i,i_{n+1}}}{p_i} = \frac{V_{i,i_{n+1}}}{V_i}$. This show that the entropy will not decrease with additional coarse-graining $\mathcal{C}_{n+1}$ if the conditional probability of the outcome $i_{n+1}$ is proportional to the ratio of the macrostate volumes.

As an example, this equality condition is satisfied when the vector of coarse-grainings $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)$ projects onto a pure state. In other words, when for every density matrix and every vector of outcomes $i$ we can write $|\psi_i\rangle = |\psi_i\rangle$ (where it is important to note that the left hand side does not depend on the density matrix $\hat{\rho}$, even though the right hand side does). Because this must hold for any density matrix, it also holds for the
maximally mixed state $\hat{\rho}_{id} = \frac{1}{d^n}\hat{I}$, which in turn yields $|\psi_i\rangle\langle\psi_i| = A_i(I)$ for $i = 1, \ldots, n$. Therefore we can write
\[
p_{i,i_{n+1}} = \text{tr}[A_{i_{n+1}}(|\psi_i\rangle\langle\psi_i|)]p_i = \frac{V_{i,i_{n+1}}}{V_i}p_i, \tag{A33}
\]
meaning that the condition is satisfied, and thus $S_C(\hat{\rho}) = S_{C_{i_{n+1}}}(\hat{\rho})$.

As a second example, the equality condition is also satisfied is when $C^\dagger \leftrightarrow C^!$, where $C^\dagger = (C_1^\dagger, \ldots, C_n^\dagger)$, $C_k^\dagger = (A_{i_k}^\dagger)$, and $A_{i_k}^\dagger = \sum_{m_k} \hat{K}_{i_k m_k}^\dagger \rho \hat{K}_{i_k m_k}^\dagger$. This means that for every multi-index $i$ there is index $i_{n+1}$ such that $A_{i_{n+1}}^\dagger \cdots A_{i_1}^\dagger (\hat{I}) = A_{i_{n+1}}^\dagger \cdots A_{i_1}^\dagger (\hat{I})$ holds, and for every other index $i_{n+1} \neq i_{n+1}^{(i)}$, $A_{i_{n+1}}^\dagger \cdots A_{i_1}^\dagger (\hat{I}) = 0$ holds. For $i_{n+1} = i_{n+1}^{(i)}$ we obtain a series of identities
\[
p_{i_{n+1}^{(i)}} = \text{tr}[A_{i_{n+1}^{(i)}} A_{i_{n+1}^{(i)}} \cdots A_{i_1}^\dagger (\hat{\rho})]
= \text{tr}[\sum_m \hat{K}_{i_{n+1}^{(i)} m_{n+1}}^\dagger \hat{K}_{i_{n+1}^{(i)} m_{n+1}} \cdots \hat{K}_{i_1 m_1}^\dagger \hat{K}_{i_1 m_1}]
= \text{tr}[\sum_m \hat{K}_{i_1 m_1}^\dagger \cdots \hat{K}_{i_{n+1}^{(i)} m_{n+1}}^\dagger \hat{K}_{i_{n+1}^{(i)} m_{n+1}} \cdots \hat{K}_{i_1 m_1}^\dagger \hat{K}_{i_1 m_1}]
= \text{tr}[A_{i_1}^\dagger \cdots A_{i_{n+1}^{(i)}}^\dagger (\hat{I}) \hat{\rho}]
= \text{tr}[A_{i_1}^\dagger \cdots A_{i_{n+1}^{(i)}}^\dagger (\hat{I})]
= \text{tr}[\sum_m \hat{K}_{i_1 m_1}^\dagger \cdots \hat{K}_{i_{n+1}^{(i)} m_{n+1}}^\dagger \hat{K}_{i_{n+1}^{(i)} m_{n+1}} \cdots \hat{K}_{i_1 m_1}^\dagger \hat{K}_{i_1 m_1}]
= \frac{V_{i_{n+1}^{(i)}}}{V_i} p_{i_{n+1}^{(i)}},
\tag{A34}
\]
where we have used that $V_{i_{n+1}^{(i)}} = 1$. Further, when $i_{n+1} \neq i_{n+1}^{(i)}$, then $V_{i_{n+1}^{(i)}} = 0$ and thus
\[
p_{i_{n+1}} = \frac{V_{i_{n+1}^{(i)}}}{V_i} p_i.
\tag{A35}
\]
Combining the last two equations, we get
\[
p_{i_{n+1}} = \frac{V_{i_{n+1}^{(i)}}}{V_i} p_i
\tag{A36}
\]
for all $i_{n+1}$, proving that also in this case, $S_C(\hat{\rho}) = S_{C_{i_{n+1}}}(\hat{\rho})$.

3. Proof of Theorem 3

Proof. Let $\{A_i\} = C \leftrightarrow \overline{C} = \{A_j\}$. Then by definition, for every multi-index $j$ there exists an index set $I(j)$ such that
\[
\Pi_j = \sum_{i \in I(j)} \Pi_i,
\tag{A37}
\]
where $\Pi_i = \sum_j \hat{K}_{jm}^\dagger \hat{K}_{im}$. Then we have
\[
p_j = \text{tr}[A_j(\hat{\rho})] = \text{tr}[\Pi_j \hat{\rho}] = \sum_{i \in I(j)} \text{tr}[\Pi_i \hat{\rho}]
\tag{A38}
\]
and similarly
\[
V_j = \sum_{i \in I(j)} V_i.
\tag{A39}
\]
The inequality follows:
\[
S_C(\hat{\rho}) = -\sum_j p_j \ln \frac{V_j}{V_i}
= -\sum_j \sum_{i \in I(j)} p_i \ln \frac{\sum_{i \in I(j)} V_i}{V_j}
\geq \sum_j \sum_{i \in I(j)} \left(\frac{p_i V_i}{V_j} \ln \left(\frac{\sum_{i \in I(j)} V_i}{V_j}\right)\right)
= -\sum_i p_i \ln \frac{p_i}{V_i} = S_C(\hat{\rho}),
\]
where we have chosen a strictly concave function $f(x) = -x \ln x$, $a_i = \frac{V_i}{V_j}$ and $b_i = \frac{p_i}{V_j}$ for $i \in I(j)$ for the Jensen’s Theorem 4.

The equality conditions from the Jensen’s inequality show that $S_C(\hat{\rho}) = S_{C_{i_{n+1}}}(\hat{\rho})$ if and only if
\[
(\forall j)(\forall i, i' \in I(j)) \left(\frac{p_i}{V_i} = \frac{p_{i'}}{V_{i'}} \iff c^{(j)}\right).
\tag{A41}
\]
To determine the constant $c^{(j)}$ we multiply the equation by $V_i$ and sum over all $i \in I(j)$, which gives $c^{(j)} = \frac{p_i}{V_j}$. Therefore, $S_C(\hat{\rho}) = S_{C_{i_{n+1}}}(\hat{\rho})$ if and only if
\[
(\forall j)(\forall i \in I(j)) \left(p_i = \frac{V_i}{V_j} p_j \right).
\tag{A42}
\]

Appendix B: Equivalence of definitions of finer vector of coarse-grainings

In [20] we defined a finer set of coarse-graining (of projective measurements) in the following way:

Definition 4. (Finer vector of coarse-grainings—old definition) We say that a vector of coarse-grainings $C = (C_1, \ldots, C_n)$ is finer than coarse-graining $C' \equiv \overline{C}$ (and denote $C \leftrightarrow \overline{C}$) when for every multi-index $i = (i_1, \ldots, i_n)$ exists $\hat{P}_i \in C$ such that
\[
\hat{P}_{i_n} \cdots \hat{P}_{i_1} \hat{P}_j = \hat{P}_{i_n} \cdots \hat{P}_{i_1},
\tag{B1}
\]
where $\hat{P}_k \in C_k$, $k = 1, \ldots, n$. 

\]
We will prove that the more general Definition 3 coincides with this older definition for the types of coarse-graining that this older definition applied for. In other words, for a class of coarse-grainings \( \mathcal{C} = \{ A_i \}, \) \( \overline{\mathcal{C}} = \{ A_j \} \) where
\[
A_i = \hat{P}_n \cdots \hat{P}_{11}(\bullet) \hat{P}_1 \cdots \hat{P}_n \quad (B2)
\]
and
\[
A_j = \hat{P}_j (\bullet) \hat{P}_j, \quad (B3)
\]
\( \mathcal{C} \leftrightarrow \overline{\mathcal{C}} \) according to the new definition if and only if \( \mathcal{C} \leftrightarrow \overline{\mathcal{C}} \) according to the old definition.

Proof. (equivalence of definitions when coarse-grainings are projective)

\[ \Rightarrow \) We assume that \( \mathcal{C} \leftrightarrow \overline{\mathcal{C}} \) according to the new definition. This means that for all \( j \) exists an index set \( I^{(j)} \) such that
\[
\Pi_j = \sum_{i \in I^{(j)}} \Pi_i, \quad (B4)
\]
where \( \Pi_j = \hat{P}_j^\dagger \hat{P}_j = \hat{P}_j \) and \( \Pi_i = \hat{P}_i^\dagger \hat{P}_i = \hat{P}_1 \cdots \hat{P}_n \cdots \hat{P}_1. \) Thus, we can rewrite the identity as
\[
\hat{P}_j = \sum_{i \in I^{(j)}} \hat{P}_i \cdots \hat{P}_n \cdots \hat{P}_1, \quad (B5)
\]
Different \( \hat{P}_j \)'s are orthogonal to each other, therefore
\[
0 = \hat{P}_j \hat{P}_j \hat{P}_j = \sum_{i \in I^{(j)}} \hat{P}_j \hat{P}_i \cdots \hat{P}_n \cdots \hat{P}_i \hat{P}_j \quad (B6)
\]
for any \( j \neq j \). Applying bra-ket \( \langle \psi \| \psi \rangle \) of any arbitrary vector \( |\psi \rangle \), we get
\[
0 = \sum_{i \in I^{(j)}} \| \hat{P}_n \cdots \hat{P}_i \hat{P}_j |\psi \rangle \|^2. \quad (B7)
\]
Thus, for all \( j \neq j \), for all \( i \in I^{(j)} \),
\[
\| \hat{P}_n \cdots \hat{P}_i \hat{P}_j |\psi \rangle \|^2 = 0 \quad (B8)
\]
This holds for any \( |\psi \rangle \), therefore also
\[
\hat{P}_n \cdots \hat{P}_i \hat{P}_j = 0. \quad (B9)
\]
Then we pick arbitrary \( i \). This \( i \) must belong into some of the index sets \( I^{(j)} \). We therefore associate \( \hat{P}_j \) to this \( i \). Then we have
\[
\hat{P}_n \cdots \hat{P}_i \hat{P}_j = \hat{P}_n \cdots \hat{P}_i \left( \hat{P}_j + \sum_{j \neq j} \hat{P}_j \right) = \hat{P}_n \cdots \hat{P}_i \hat{I} = \hat{P}_n \cdots \hat{P}_i, \quad (B10)
\]
where we have used Eq. (B9). This means that \( \mathcal{C} \leftrightarrow \overline{\mathcal{C}} \) according to the old definition.

\[ \Leftarrow \) Now we prove the opposite implication, that a coarse-graining which satisfies the old definition also satisfies the new definition. Given \( j \), we define \( I^{(j)} \) as the set of all \( i \) which correspond to \( j \) (according to the old definition). When multiplying Eq. (B1) by \( \hat{P}_j, \ j \neq j \), from orthogonality of projectors we obtain
\[
\hat{P}_n \cdots \hat{P}_i \hat{P}_j = 0 \quad (B11)
\]
for all \( i \in I^{(j)} \), which also implies that
\[
\hat{P}_n \cdots \hat{P}_i \hat{P}_j = 0 \quad (B12)
\]
for all \( i \notin I^{(j)} \). Then we have identities
\[
\Pi_j = \hat{P}_j = \sum_{i \in I^{(j)}} \hat{P}_i \cdots \hat{P}_i \hat{P}_j = \sum_{i \in I^{(j)}} \hat{P}_i \cdots \hat{P}_i \hat{P}_j = \sum_{i \in I^{(j)}} \Pi_i.
\]
where we have used Eqs. (B12) and (B1), which means that \( \mathcal{C} \leftrightarrow \overline{\mathcal{C}} \) according to the new definition. This concludes the proof.

\[ \Box \]

Appendix C: Proof of the algorithm for the quantum state inference

Just like in Definition 3 and elsewhere, also in this section we assume that \( \mathcal{C} = \{ C_1, \ldots, C_n \} = \{ A_i \}, \ A_i = \sum_{m} K_{im}(\bullet) \hat{K}_{im}, \) where \( \hat{K}_{im} = \hat{K}_{im}^{\dagger}, \) \( i = (i_1, \ldots, i_n) \) and \( m = (m_1, \ldots, m_n) \), and POVM element is defined as \( \Pi_i = \sum_{m} \hat{K}_{im}^{\dagger} \hat{K}_{im}. \)

According to the Definition 3, \( \mathcal{C} \leftrightarrow \overline{\mathcal{C}} = \{ \hat{P}_\rho(\bullet) \hat{P}_\rho \} \) if and only if can build each eigenprojector of the density matrix using POVM elements from the vector of coarse-graining \( \mathcal{C} \), i.e., if for each eigenvalue \( \rho \) exists an index set \( I^{(\rho)} \) such that
\[
\hat{P}_\rho = \sum_{i \in I^{(\rho)}} \Pi_i = \sum_{i \in I^{(\rho)}, m} \hat{K}_{im}^{\dagger} \hat{K}_{im}. \quad (C1)
\]
For the following proofs needed for the algorithm, we need identify how to group the POVM elements together so that we can build those projectors, i.e., we need an algorithm of how to generate disjoint these sets \( I^{(\rho)} \) that can be used to build up \( \hat{P}_\rho \)'s. This will be achieving by the following three lemmas:

**Lemma 3.** Let \( \mathcal{C} \leftrightarrow \mathcal{C} \), where \( \mathcal{C} = \{ \hat{P}_j(\bullet) \hat{P}_j \}, \) and \( \mathcal{C} = \{ A_i \}. \) If two POVM elements have non-zero overlap, they must correspond to the same projector \( \hat{P}_j \). In mathematical terms, if \( \Pi_i \Pi_i \neq 0 \), then both \( i, i' \in I^{(j)} \) for some \( j \).

**Proof.** For contradiction, let us assume that \( i \in I^{(j)} \) and \( i' \in I^{(j')}, \) where \( j \neq j' \). Then from the orthogonality of projectors we have
\[
0 = \hat{P}_j \hat{P}_j \hat{P}_j = \hat{P}_j \sum_{i \in I^{(j')} \cup I^{(j')}} \hat{K}_{i'\rho}^{\dagger} \hat{K}_{i'\rho} \hat{P}_j. \quad (C2)
\]
Applying $\langle \psi \rangle$ and $|\psi\rangle$ from both sides we get

$$0 = \sum_{i' \in I^{(j)}, m'} \| \hat{K}_{i'm'} \hat{P}_j |\psi\rangle \|^2. \quad (C3)$$

This holds for any $|\psi\rangle$, therefore

$$\forall i' \in I^{(j)}, \forall m', \hat{K}_{m'm'} \hat{P}_j = 0. \quad (C4)$$

Multiplying this equation by $\hat{K}^\dagger_{i'm'}$ from the right, expressing $\hat{P}_j$ and applying $\langle \psi \rangle$ and $|\psi\rangle$ from both sides we get

$$0 = \sum_{i \in I^{(j)}, m} \| \hat{K}_{im} \hat{K}^\dagger_{i'm'} |\psi\rangle \|^2. \quad (C5)$$

This holds for any $|\psi\rangle$, therefore

$$\forall i \in I^{(j)}, \forall i' \in I^{(j)}, \forall m, \forall m', \hat{K}_{im} \hat{K}^\dagger_{i'm'} = 0. \quad (C6)$$

Multiplying this equation by $\hat{K}^\dagger_{im}$ from the left and by $\hat{K}_{i'm'}$ from the right and summing over $m$ and $m'$ we get

$$\Pi_i \Pi_{i'} = 0, \quad (C7)$$

which is in contradiction with our assumption that $\Pi_i \Pi_{i'} \neq 0$. Therefore, both $i$ and $i'$ must belong into the same $I^{(j)}$. \hfill \Box

Now we go on with proving Lemma 1, which says that $\hat{P}$ is correct for step 3. This is because from this lemma we know that all POVM elements that have non-zero overlap must correspond to the same projector $\hat{P}_\rho$. Let us denote the set of all operators $\hat{P}$ generated through the algorithm as $A = \{ \hat{P} \}$. This set must be complete ($\sum_{\hat{P} \in A} \hat{P} = I$), because $\sum_{\hat{P} \in A} \hat{P} = \sum_i \Pi_i = \sum_i \hat{K}^\dagger_{im} \hat{K}_{im} = I$. Combining Lemma 3 and Definition 3 we know that for each $\rho$ there must exist a subset of $A$, let us denote it $B^\rho$, such that

$$\hat{P}_\rho = \sum_{\hat{P} \in B^\rho} \hat{P}. \quad (C8)$$

Further, we know that every operator $\hat{P} \in A$ must be orthogonal to every other operator $\hat{P} \in A$: the iterative process of step 3 is stopped exactly when for all unused $i'$, $\Pi_i \hat{P} = 0$, so $\hat{P}$ must be orthogonal to all $\hat{P}'s$ that will come after it, and by the same logic, $\hat{P}$ must also be orthogonal to all the $\hat{P}'s$ that came before it. Also, from the construction it is clear that $B^\rho$’s are disjoint sets ($B^\rho \cap B^\rho' = \emptyset$ for $\rho \neq \rho'$) whose union equals the set $A$ ($A = \bigcup_\rho B^\rho$). If we manage to prove that each $\hat{P}$ is a projector, then from Eq. (C8) it is clear that it must project into an eigenspace of the density matrix, simply because $\hat{P}_\rho$ already does by definition.

Therefore, all we have to do now is to prove that each $\hat{P}$ generated by step 3 is a projector. For a contradiction, let us assume that there is some $\hat{P}$ that is not a projector. This $\hat{P}$ belongs into some set $B^\rho$. To make the notation clearer, let us label elements of $B^\rho$ as $\hat{P}_1, \ldots, \hat{P}_k$ so we can write

$$\hat{P}_\rho = \hat{P}_1 + \cdots + \hat{P}_k, \quad (C9)$$

and WLOG we assume that operator $\hat{P}_1$ is the one that is not a projector. $\hat{P}_\rho$ is a projector, therefore

$$\hat{P}_1 + \cdots + \hat{P}_k = \hat{P}_\rho = (\hat{P}_1 + \cdots + \hat{P}_k)^2 = \hat{P}_1^2 + \cdots + \hat{P}_k^2, \quad (C10)$$

where we have used that all $\hat{P}_\rho$’s are orthogonal to each other. Multiplying this equation by $\hat{P}_1$ and using the orthogonality again we obtain

$$\hat{P}_1^2 = \hat{P}_1^3. \quad (C11)$$

However, $\hat{P}_1$ is a Hermitian operator by construction, so there is a spectral decomposition of $\hat{P}_1 = \sum_n \lambda_n |\psi_n\rangle \langle \psi_n\|$. $\lambda_n \neq 0$, where $|\psi_n\rangle$ can be chosen orthogonal. Using the above equation, we have

$$\sum_n \lambda_n^2 |\psi_n\rangle \langle \psi_n\| = \sum_n \lambda_n^3 |\psi_n\rangle \langle \psi_n\|. \quad (C12)$$

Due to orthogonality of $|\psi_n\rangle$, it must be that $\lambda_n^2 = \lambda_n^3$ for every $n$, in other words, $\lambda_n = 1$ for every $n$. Thus,

$$\hat{P}_1 = \sum_n |\psi_n\rangle \langle \psi_n|, \quad (C13)$$

therefore $\hat{P}_1$ is a projector, which is a contradiction. Thus, $\hat{P}$ generated by step 3 is always a projector, and as we established before, it must project into an eigenspace of the density matrix. This concludes the proof. \hfill \Box

Finally, we prove Lemma 2 which says that $\rho$ generated by step 4 is an eigenvalue of the state of the system corresponding to projector $\hat{P}$.

**Proof. (Lemma 2)** From Lemma 1 we know that each $\hat{P}$ projects into an eigenspace of the system. Let us denote the (currently unknown) eigenvalue corresponding to this eigenspace as $\rho_0$. $\hat{P}$ is orthogonal to every other $\hat{P}$, and from the construction $\hat{P} = \sum_i \Pi_i$. Probability of obtaining measurement outcome $i$ is equal to

$$p_i = \text{tr}[\hat{A}_i \rho \hat{P}] = \text{tr}[(\Pi_i \hat{P})_\rho] = \text{tr}[\Pi_i \sum_\rho \rho \hat{P}_\rho]. \quad (C14)$$

Summing over all $i \in I$ we obtain

$$\sum_{i \in I} p_i = \text{tr}[\hat{P} \sum_\rho \rho \hat{P}_\rho] = \text{tr}[\sum_\rho \rho \Delta_{\rho_0} \hat{P}] = \rho_0 \text{tr}[^2]. \quad (C15)$$
\( I \) is a non-empty set by construction, therefore \( \text{tr}[\hat{P}] \) is non-zero. Thus we can divide by it and

\[
\rho = \frac{1}{\text{tr}[\hat{P}]} \sum_{i \in I} \rho_i,
\]

which concludes the proof. \( \square \)

**Appendix D: Explicit form of repeated interactions and repeated contacts superoperators**

1. **Repeated interactions**

It can be easily realized that the repeated interactions superoperator (47) can be rewritten as

\[
\mathcal{A}_{\text{tr}}^m(\hat{X}) = \hat{K}_{\text{tr}}^m X \hat{K}_{\text{tr}}^m
\]

where

\[
\hat{K}_{\text{tr}}^m = U_f \hat{K}_{x_N} \cdots U_f \hat{K}_{x_1}.
\]

In terms of elements, this gives

\[
(\hat{K}_{\text{tr}}^m)_{m_Nm_0} = \sum_{m_1, \ldots, m_{N-1}} U_{m_0 \cdots m_N} \sqrt{g_{m_0 \cdots m_N}^m}(x)
\]

where

\[
U_{m_0 \cdots m_N} = U_{f m_N m_{N-1}} \cdots U_{f m_0 m_1}
\]

and

\[
\sqrt{g_{m_0 \cdots m_N}^m}(x) = \frac{1}{(2\pi \Omega^2)^{1/4}} \exp\left(-\frac{(x - \kappa \mu_{m_0} \cdots - \kappa \mu_{m_{N-1}})^2}{4\Omega^2}\right).
\]

Comparing eqs. (D3) and (D9) we see that the only difference between the repeated measurements and repeated contacts is the function \( \sqrt{g_{m_0 \cdots m_N}^m}(x) \) and \( \sqrt{g_{m_0 \cdots m_N}^c}(x) \) respectively.

2. **Repeated contacts**

To obtain explicit form of the repeated contacts superoperator (49), we first compute

\[
(\mathcal{U}_f \mathcal{U})^N(\hat{X} \otimes \hat{\sigma}) = \sum_{m_0, m'_0, \ldots, m_N, m'_N} U_{f m_N m_{N-1}} \cdots U_{f m_1 m_0} \hat{X}_{m_0 m'_0} U_{f m'_N m'_N} \cdots U_{f m'_1 m_0'} \times |m_N\rangle \otimes |\varphi(x - \kappa \mu_{m_0} \cdots - \kappa \mu_{m_{N-1}})| \langle \varphi(x - \kappa \mu_{m_0} \cdots - \kappa \mu_{m'_N})|.
\]

This leads to

\[
\mathcal{A}_{\text{rc}}^c(\hat{X}) = \langle x |_B (\mathcal{U}_f \mathcal{U})^N(\hat{X} \otimes \hat{\sigma})|x\rangle_B = \hat{K}_{\text{rc}}^c X \hat{K}_{\text{rc}}^c|_B\]

where

\[
\hat{K}_{\text{rc}}^c = \sum_{m_0, \ldots, m_N} U_{f m_N m_{N-1}} \cdots U_{f m_1 m_0} \sqrt{g_{\text{rc}}^c(x - \kappa \mu_{m_0} \cdots - \kappa \mu_{m_{N-1}})}|m_N\rangle \langle m_0|,
\]

which gives elements of the operator \( \mathcal{A}_{\text{rc}}^c(\hat{X}) \) in the eigenbasis of the measurement operator. We can write this in a form similar to eq. (D3)

\[
(\hat{K}_{\text{rc}}^c)_{m_Nm_0} = \sum_{m_1, \ldots, m_{N-1}} U_{m_0 \cdots m_N} \sqrt{g_{\text{rc}}^c}(x)
\]

where

\[
\sqrt{g_{\text{rc}}^c}(x) = \frac{1}{(2\pi \Omega^2)^{1/4}} \exp\left(-\frac{(x - \kappa \mu_{m_0} \cdots - \kappa \mu_{m_{N-1}})^2}{4\Omega^2}\right).
\]

Comparing eqs. (D3) and (D9) we see that the only difference between the repeated measurements and repeated contacts is the function \( \sqrt{g_{m_0 \cdots m_N}^m}(x) \) and \( \sqrt{g_{m_0 \cdots m_N}^c}(x) \) respectively.

**Appendix E: Numerical methods**

Because on a computer we cannot create an exactly continuous function, in all figures we choose the coarse-grained positional step to be \( dx = 0.1 \). Since computing observational entropy is a numerical integration problem, we would expect the error in computing the observational entropy to scale as \( dx^2 \). However, in practice we saw that observational entropy is very insensitive to the coarse-graining size in \( x \), and the error to be much smaller—
roughly estimated to be less than $10^{-6}$ compared to the case where $dx$ is infinitesimally small. Notably, the difference in results between $dx = 1$ and $dx = 0.1$ was only $10^{-3}$ or less in every example we considered. We also chose a cut-off in a positional axis at four standard deviations $\Omega$ from the furthest peak of Gaussians that make up $p$’s and $V$’s (for instance, see Fig. 4 (b) and (c) that show the cut-off scale). We performed a series of numerical checks, for example, summing computed probabilities $p$ and volumes $V$ and making sure they add up to 1 and $2 = \dim \mathcal{H}_S$ respectively (and to see how far they differ), making sure that the generated coarse-graining sums to unity as per Eq. (A4), and computing the observational entropy by different methods/using a different code.

Computing the observational entropy for the von Neumann measurement schemes is highly a computationally demanding task (in computational speed and the speed of memory access, however not in required memory size), due to the dimensionality of the problem growing exponentially with $N$, and because of the sheer number of points for which we need to be compute $p$ and $V$, especially in the repeated measurement case.

In repeated contacts we can estimate the computational complexity scaling as

$$(\text{number of computed points to be computed}) \times (\text{multiplication of N matrices + additional operations}) \times (\text{number combinations of } \mu) = \frac{\kappa N (\mu_{\text{max}} - \mu_{\text{min}}) + 8 \Omega}{dx} \left(c_1 N^3 + c_2\right) (\dim \mathcal{H}_S)^N$$

(E1)

where $N$ denotes the number of contacts, and $c_1$ and $c_2$ are some constants. The number of points to be computed are calculated as the distance between the furthest Gaussian peaks, which is $\kappa N (\mu_{\text{max}} - \mu_{\text{min}})$ as can be observed from Eq. (D8), plus $4 \Omega$ on both sides. The second and third line also follow from the expression (D8). “Additional operations” do not scale with $N$ (or the scaling is negligible), and represent for example costs of evaluating the Gaussian function, Eq. (D10). With our parameters ($\mu_{\text{max}} = 2$, $\mu_{\text{min}} = 0$, $\Omega = 1$, $\dim \mathcal{H}_S = 2$), we obtain $N^3 2^N$ as the leading order. This translates into ability to compute for up to $N = 22$ on a single processor with current parameters.

In repeated measurements, we can estimate the computational complexity scaling as

$$(\text{number of computed points to be computed}) \times (\text{multiplication of N matrices + additional operations}) \times (\text{number combinations of } \mu)$$

$$= \kappa(\mu_{\text{max}} - \mu_{\text{min}}) + 8 \Omega dx$$

$$\left(c_1 N^3 + c_2\right) (\dim \mathcal{H}_S)^N$$

(E2)

where $N$ is the number of measurements. It is obtained from the similar formula as repeated contacts, Eq. (D3), however, now the “number of computed points to be computed” is given by the distance between the furthest peaks in one dimension $\kappa(\mu_{\text{max}} - \mu_{\text{min}})$ plus $4 \Omega$ on both sides, and that all is put to the power of the dimensionality of the problem $N$. This is because in Eq. (D3), it is the function of a vector, instead of a number as it was in the repeated contacts. With our parameters we obtain $N^3 200^N$ as the leading order. This translates into ability to compute for up to $N = 5$ with a single processor. The task is also highly parallelizable, however, any advantage gained by parallelization is quickly diminished by the prohibitive scaling.