2D— Fractional Supersymmetry and Conformal Field Theory for alternative statistics

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Abstract

Supersymmetry can be consistently generalized in one and two dimensional spaces, fractional supersymmetry being one of the possible extension. 2D fractional supersymmetry of arbitrary order $F$ is explicitly constructed using an adapted superspace formalism. This symmetry connects the fractional spin states $(0, \frac{1}{F}, \ldots, \frac{F-1}{F})$. Besides the stress momentum tensor, we obtain a conserved current of spin $(1 + \frac{1}{F})$. The coherence of the theory imposes strong constraints upon the commutation relations of the modes of the fields. The creation and annihilation operators turn out to generate alternative statistics, currently referred as quons in the literature. We consider, with a special attention, the consistence of the algebra, on the level of the Hilbert space and the Green functions. The central charges are generally irrational numbers except for the particular cases $F = 2, 3, 4$. A natural classification emerges according to the decomposition of $F$ into its product of prime numbers leading to sub-systems with smaller symmetries.

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1 Introduction

After the work of Belavin, Polyakov and Zamolodchikov [1], conformal invariance has become a powerful tool for the description of 2D critical phenomena. Then, one of the main tasks would be to have a systematic classification of conformal systems. The first attempt in this direction has been challenged by Friedan, Qiu and Shenker (FQS) [2] by imposing unitarity. Then, it has been proved that, if we enlarge the 2D symmetry, we can go beyond the discrete series found by FQS. For instance, the superconformal extension of the Virasoro algebra leads to other unitary series [3]. Bosonic extensions of conformal symmetries are also allowed as for instance the Kac-Moody algebra [4] or the $W_n$ algebras [5]. The former contains conserved current of spin one whereas the latter of spin $n$.

However, due to the special feature of 2D, one is allowed to define fields that are neither fermions nor bosons but of fractional conformal weight. The parafermions introduced by Fateev and Zamolodchikov possess a rational conformal weight [6]. Those fields are the basic building blocks of fractional superconformal Virasoro (FSV) algebra [4, 8] and lead naturally to conserved current of fractional conformal weight. Furthermore, there is an essential difference with the $W_n$ algebras in so far as FSV close through quadratic relations but involve a non-linear dependence of the fields. Therefore, it is obviously not a Lie or a super-Lie algebra. As the fractional Virasoro algebra is concerned, we get quadratic relations but rational power are involved in the OPE. Because cuts are involved, the theory appears to be non-local.

All known conformal field theories (CFT) can be obtained within the framework of the GKO [9] coset construction where appropriate Kac-Moody algebras are involved. Let us point out that the GKO construction can be applied with all kinds of affine-Lie algebras. Therefore, following this line, one can build other extensions than the ones given here above.

The starting point of the present article is rather different; we take advantage of the possibility to generalize supersymmetry in one and two dimensions. Namely, we use appropriate extensions of Grassmann variables [10, 11] to build explicitly an invariant action (by the help of the natural extension of the usual superfield). To our knowledge, this is the first time, that a CFT is obtained with such variables. This symmetry named fractional supersymmetry (FSUSY) has been introduced by Durand [12]. It can be seen as the $F^{th}$-root of the time translation in 1D [12, 13, 14] or of the conformal transformation in 2D [15, 16, 17]. A group theoretical justification of this symmetry was given in [18, 14]. In a former paper, we have particularized the case $F = 3$ and stressed on the underlying superspace formalism [17]. The Virasoro algebra is then extended and, besides the stress-energy tensor, we obtain a conserved current of conformal weight(spin) $(1 + \frac{1}{F})$. Consequently, in addition to the scalar field, we introduce primary fields of conformal weight $\frac{1}{F}, \ldots, \frac{F-1}{F}$. It turns out that fractional supersymmetry is the symmetry which connects those $\frac{1}{F}$-integer spin states. As
already mentioned in our previous paper, this extended Virasoro algebra has nothing to do with the fractional supervirasoro (FSV) one \cite{1} where a spin \(1 + \frac{1}{F}\) conserved current is already present. The main reason of this difference is that ours closes through local (but non-quadratic) relations whereas FSV closes with non-local (but quadratic) ones.

In this article, we are generalizing our previous results to fractional supersymmetry of arbitrary order \(F\) and take a special emphasis on the consistence of the algebra, especially on the level of the Hilbert space and the Operator Product Expansion (OPE). From this analysis, alternative statistics, currently referred as quons in the literature, emerges in a natural way, generalizing fermionic statistics.

The paper is organized as follow. In section 2, we recall the foundations of the generalized Grassmann algebra which allows to build an extension of the Virasoro algebra. Section 3 is devoted to an explicit construction of the FSUSY action, by introducing a superfield of conformal weight 0. The components of the latter are fields of conformal weights \(0, \cdots, \frac{F-1}{F}\).

The section 4 contains one of the main results of this paper. By developing the various fields in mode expansions, we find that the coherence of the algebra imposes special commutation relations between the modes. We recover in a natural way the quons algebra, previously introduced by Greenberg and Mohapatra \cite{19}, in order to obtain new statistics with a small violation of the Pauli principle. The essential feature of this algebra is the absence of bilinear relations between two creators and two annihilators. A consequence of this oscillator structure is an extension of the Wick theorem. Then, the section 5 is devoted to the determination of the OPE. Some general rules are given in order to ensure the associativity of the algebra at the level of the four-point functions. Section 6 shows that a natural classification emerges according to the decomposition of \(F\) into its product of prime numbers. In other words, the general case can be obtained from the study of \(F\), when \(F\) is a pure prime number. Then, it becomes straightforward to extend this result to any \(F\). Moreover, in addition to the FSUSY symmetry, we prove for \(F = F_1 \times F_2\), that this action is also invariant under \(F_1\) or \(F_2\)-supersymmetric transformations. Finally, section 7 contains a summary of our results and some future perspectives for this work.

## 2 Fractional superconformal algebra

In this section, we need first to recall briefly the underlying algebra which allows to define FSUSY. The basic fields live in a ad hoc extension of the complex plane, namely \((z, \theta_L, \bar{z}, \theta_R)\) with \(\theta_L, \theta_R\) two real generalized Grassmann variables \cite{10,11}. We also introduce the associated derivatives \(\partial_L, \delta_L, \partial_R, \delta_R\). They satisfy the basic algebraic relations
\[ \partial_\theta \theta - q \theta \partial_\theta = 1 \]
\[ \delta_\theta \theta - q^{-1} \theta \delta_\theta = 1 \]
\[ \theta^F = 0 \quad d_\theta^F = 0 \]  \hspace{1cm} (2.1)

with \( \theta = \theta_L \) or \( \theta_R \), \( d = \partial \) or \( \delta \) and \( q \) is a primitive \( F \)-th root of unity. Without lost of generality, \( q \) can be chosen as \( q = \exp (\frac{2i\pi}{F}) \). A consequence of relations (2.1) is an adapted Leibniz rule leading to \( \partial_\theta (\theta^a) = \{ a \} \theta^{a-1} \) with \( \{ a \} = \frac{1 - q^a}{1 - q} \) (with the derivative \( \delta_\theta \) we would have obtained the same result with the substitution \( q \to q^{-1} \)). The relations which mix the \( L \)- or \( R \)- movers are

\[ \theta_L \theta_R = q \theta_R \theta_L, \quad d_L d_R = q d_R d_L \]  \hspace{1cm} (2.2)
\[ d_L \theta_R = q^{-1} \theta_R d_L, \quad d_R \theta_L = q \theta_L d_R. \]

From this algebra, we can build the generators (\( Q \)) and the covariant derivatives (\( D \)) of FSUSY

\[ Q_L = \partial_L + \left( \frac{1 - q}{F} \right)^{F-1} \theta_L^{F-1} \partial_z \]
\[ D_L = \delta_L + \left( \frac{1 - q^{-1}}{F} \right)^{F-1} \theta_L^{F-1} \partial_z \]
\[ Q_R = \delta_L + \left( \frac{1 - q^{-1}}{F} \right)^{F-1} \theta_R^{F-1} \partial_\bar{z} \]
\[ D_R = \partial_L + \left( \frac{1 - q}{F} \right)^{F-1} \theta_R^{F-1} \partial_\bar{z}, \]  \hspace{1cm} (2.3)

which fulfill

\[ Q_L^F = D_L^F = \partial_z \quad Q_R^F = D_R^F = \partial_\bar{z} \]
\[ Q_L D_L = q^{-1} D_L Q_L \quad Q_R D_R = q D_R Q_R. \]  \hspace{1cm} (2.4)

We want to stress that the generalized Grassmann variables are real, therefore, in the superspace \((z, \theta_L, \bar{z}, \theta_R)\), the two generalized Grassmann variables are not complex conjugate from each other. We then get an heterotic extension of the complex plane. However, the underlying algebra is stable under the composition of the complex conjugation and the permutation of \( R \) and \( L \). This explains why \( Q_L, D_L \) and \( Q_R, D_R \) have a different treatment (see eq. (2.3)). We refer to ref. [17] for more details. All these relations (2.4) can be obtained
directly using (2.1–2.2) or more easily from the faithful matrix representation given in [20]. The generators defined in (2.4) can be extended, following Durand [21], to

\[ L_n = z^{1-n}\partial_z - \frac{1}{\mathcal{F}}(n-1)z^{-\mathcal{N}}, \quad n \in \mathbb{Z} \]

\[ G_r = z^{\frac{1}{\mathcal{F}}} - r(\partial_\theta + (1-q)F^{-1}\theta^{F-1}\partial_z) - \left(\frac{1-q}{\mathcal{F}}\right)^{F-1}(r - \frac{1}{\mathcal{F}})z^{\frac{1}{\mathcal{F}}} - r \theta^{F-1}\mathcal{N}, \quad r \in \mathbb{Z} + \frac{1}{\mathcal{F}}, \]

with \( \mathcal{N} \) the number operator which equals to \( \mathcal{N} = \sum_{i=1}^{F-1} (1-q)^i \theta^i \partial_\theta \). The \( L \) and the \( G \) generate the fractional-super-Virasoro algebra, without central extension,

\[ [L_n, L_m] = (n-m)L_{m+n} \]

\[ [L_n, G_r] = (\frac{n}{\mathcal{F}} - r)G_{n+r} \]

\[ \{G_{r_1}, \cdots, G_{r_F}\} = L_{r_1 + \cdots + r_F}, \]

where \( \{G_{r_1}, \cdots, G_{r_F}\} \) is defined by \( \frac{1}{\mathcal{F}!}(\text{sum over all the permutations of } G_{r_1}, \cdots, G_{r_F}) \). It then appears that FSUSY is a natural generalization of supersymmetry because it corresponds to the \( F-th \) root of the Virasoro generators. Of course, the antiholomorphic part is built in the same way.

### 3 Construction of the action

We take, in this section, \( F \) as a prime number. We will see later that it constitutes a generic case. First of all, we recall briefly the basic points which lead from the algebra (2.3–2.4) to an invariant FSUSY action. It is interesting to notice that most results, in usual supersymmetric theories [22], can be extended to FSUSY. We are able to build an invariant action in FSUSY, extending the usual superspace formulation involved in supersymmetric theories (by the help of the generalized Grassmann variables (2.1–2.2)).

Therefore, a basic superfield decomposes (in the fractional superspace \((z, \theta_L, \bar{z}, \theta_R)\)) as

\[ \Phi(z, \theta_L, \bar{z}, \theta_R) \sim \sum_{a,b=0}^{F-1} \theta_L^a \theta_R^b \psi_{a,b}(z, \bar{z}). \]  

(3.1)

In this multiplet, we have three kinds of fields: the holomorphic ones \( \psi_{a,0} \), the antiholomorphic ones \( \psi_{0,b} \), and the auxiliary fields \( \psi_{a,b} \) with \( a \) and \( b \neq 0 \). The various components of \( \Phi \) generalize the concept of boson/fermion and have non-trivial \( \mathbb{Z}_F \)-graduation. The \( \theta \) field is then a “gradation counter” and we have the \( q \)-mutations relations [12, 18]

\[ (\psi_{a,b})^F = 0 \]
Finally, from integration rules \((\int d\theta = (d/d\theta)^{F-1})\), we can define the FSUSY invariant action
\[
\int \theta_L \psi_L \psi_R^* \frac{q^{-(a+b)}}{q^{-(a+b)}} \theta_L \psi_R
\]

where \(D_L\) (respectively \(D_R\)) acts from the left (resp. the right). This nice generalization of SUSY comes from the definition of the superfield \(\Phi\) and the property that the covariant derivative, say \(D_L\), commutes with the FSUSY-transformation \(\epsilon_L Q_L\) (see below). In the sequel, we just consider the holomorphic part of the action. The antiholomorphic part is totally similar and the auxiliary fields are irrelevant for our study. \(D\) and \(\theta\) stand respectively for \(D_L\) and \(\theta_L\). Then, the remaining part of the action can be equivalently written (with adapted normalizations coming from integration and derivation over \(\theta\))
\[
\mathcal{A} = \left(\frac{1-q}{1-q^{-1}}\right)^{F-1} \int d\theta D\Phi \partial_{\bar{z}} \Phi.
\]  

The action (3.3) is invariant under the FSUSY transformations for the left and right movers (this is the action we consider in superstring theory for \(F = 2\)). On the other hand, (3.4) is only FSUSY invariant for the left movers (it is the action we use in heterotic string when \(F = 2\)). In a similar manner, we can construct a theory which is \(F_L\) (respectively \(F_R\)) invariant by choosing \(\theta_L\) (respectively \(\theta_R\)) a generalized Grassmann variable of order \(F_L\) (respectively \(F_R\)). Such extension involves auxiliary fields. A general study including all the auxiliary fields has been performed for \(F_L = F_R = 3\) in [17].

With the holomorphic part of the previous superfield defined in (3.1) \(\Phi\) can be written
\[
\Phi(z, \bar{z}, \theta) = X(z, \bar{z}) + \sum_{a=1}^{F-1} q^{-a/2} \theta^a \psi_a(z).
\]

Using the \(q\)-mutation (3.2) and the integration rule upon the generalized Grassmann variables [11], the action (3.4) yields to
\[
\mathcal{A} = \partial_z X(z) \partial_{\bar{z}} X(z) + \frac{F}{(1-q^{-1})^F} \sum_{a=1}^{F-1} \left( q^{a-1} - 1 \right) \psi_{\bar{z}}(z) \partial_{\bar{z}} \psi_{F-a}(z).
\]  

This action is the natural generalization of the supersymmetric ones, and has already been considered for the \(F = 3\) case [13, 16, 17] or even for arbitrary \(F\) in one dimension [12].

In a way analogous to [17], we can define a path integral, and the non-vanishing two-point Green functions are then
\[
\begin{align*}
< \psi_{F-a}(z) \psi_{F-a}(w) > & = \frac{(q-1)^F}{F} \frac{1}{q^{a-1}} \frac{1}{z-w} \\
< \psi_{F}(z) \psi_{F-a}(w) > & = \frac{(q-1)^F}{F} \frac{1}{q^{a-1}} \frac{1}{z-w} \\
< X(z) X(w) > & = -\ln(z-w).
\end{align*}
\]

These equations explicitly show that \( \psi_{F-a}(z) \) is the conjugate of \( \psi_{F-a}(z) \). This comes from the peculiar structure of the Lagrangian, where these two fields have to be coupled in order to have an action of conformal weight 0. When \( F = 2 \), the situation appears rather different because fermionic fields are self-conjugate.

### 4 Oscillators

#### 4.1 Oscillator algebra

The solutions of the equations of motion allow to develop the various fields in terms of the Laurent expansions: \( \psi_{\mathbf{F}}(z) = \sum_r \psi_{a,r} z^{-r-\frac{b}{F}} \), the index \( r \) belonging to \( \mathbb{Z} + b/F \), \( b \) depending of the boundary conditions of the fields (see [17] for more details).

In this context, nothing can be said \textit{a priori} on the \( q \)–mutation relations of the fields.

For the sake of simplicity, we consider only the case \( F = 3 \). We thus have

\[
\begin{align*}
\psi_{1/3}(z) & = \sum_r \psi_{1,r} z^{-r-\frac{1}{3}} \\
\psi_{2/3}(z) & = \sum_s \psi_{2,s} z^{-s-\frac{4}{3}}.
\end{align*}
\]

The notations for the indices of the field are different from our previous paper [17]. The properties of the underlying algebra induce strong constraints upon the various modes of the fields and allow to define unambiguously the vacuum. The first constraint comes from the possibility to obtain the two-point Green function, using the mode expansion of the fields. We thus set

\[
\psi_{1,r}|0> = 0 ; \ \psi_{2,s}|0> = 0, \text{with } r, s > 0.
\]

We then make the following identification

\[
\begin{align*}
\psi_{1,r} & \equiv a_r \quad \psi_{2,s} \equiv b_s, \quad r, s > 0 \\
\psi_{1,-s} & \equiv b_s^\dagger \quad \psi_{2,-r} \equiv a_r^\dagger, \quad r, s > 0.
\end{align*}
\]
Therefore, the fields can be written as
\[
\psi_{1/3}(z) = \sum_{s>0} \left\{ b_s^\dagger z^{s-1/3} + a_s z^{-s-1/3} \right\} = \psi_{1/3}< (z) + \psi_{1/3}> (z) \tag{4.4}
\]
\[
\psi_{2/3}(z) = \sum_{r>0} \left\{ a_r^\dagger z^{-r-2/3} + b_r z^{-r-2/3} \right\} \equiv \psi_{2/3}< (z) + \psi_{2/3}> (z).
\]

Then from (3.6) with \( F = 3 \), we automatically get
\[
<\psi_{1/3}(z)\psi_{2/3}(w)> = <\psi_{2/3}(z)\psi_{1/3}(w)> = \sum_s <a_s a_s^\dagger> \left( \frac{w}{z} \right)^s z^{-2/3} w^{-1/3} \sim -\frac{q}{z-w} \tag{4.5}
\]
\[
<\psi_{2/3}(z)\psi_{2/3}(w)> = <\psi_{1/3}(z)\psi_{1/3}(w)> = \sum_r <b_r b_r^\dagger> \left( \frac{w}{z} \right)^r z^{-1/3} w^{-2/3} \sim \frac{q^2}{z-w}
\]

if we assume
\[
a_s a_s^\dagger - qa_s^\dagger a_r = -q \delta_{rs} \tag{4.6}
\]
\[
b_r b_r^\dagger - q^2 b_r^\dagger b_s = q^2 \delta_{rs}.
\]

Notice that the operators \( a, a^\dagger \) and \( b, b^\dagger \) are not Hermitian conjugate from each-other as could have been suggested by our notation (see (4.6)). Similarly, from the other two-point functions which vanish \( <\psi_{1/3}\psi_{1/3}> \) and \( <\psi_{2/3}\psi_{2/3}> \), we get the \( q \)-mutation relations
\[
a_r b_s^\dagger - q^2 b_s^\dagger a_r = 0 \tag{4.7}
\]
\[
b_r a_s^\dagger - q a_s^\dagger b_r = 0.
\]

Strictly speaking, at this point the power of \( q \), inside the \( q \)-mutation relations is not fixed, however, it can be fixed by the OPE (to have the right OPE of the stress-energy tensor with the fields). Furthermore, these relations are not surprising because the two fields \( \psi_{1/3}, \psi_{2/3} \) are conjugated from each other (see the two-point functions). Now, it remains \textit{a priori} to fix the relations between two annihilators or two creators. If we assume that we have bilinear relations between two \( a, b \) or \( a^\dagger, b^\dagger \), we get an incoherence in the Hilbert space (for instance by calculating \( a_r a_s^\dagger b_s^\dagger |0> \) for \( r \neq s \) in two different ways). Hence, we are obliged to let these relations unconstrained. Therefore, the natural structure emerging from a quantization of 2D FSUSY appears to be the quon algebra introduced and developed by Greenberg and Mohapatra [13]. In fact, it should be noticed that in our case, we have \( |q| = 1 \), whereas for the quons, \( q \in [0, 1] \). However, we will see in the next subsection that we are able to build a positive and definite Hilbert space with \( a_r^\dagger \) and \( b_r^\dagger \). To be complete, we have to specify that
\[
(a_r)^3 = (a_r^\dagger)^3 = (b_r)^3 = (b_r^\dagger)^3 = 0 \tag{4.8}
\]
The equations (4.6, 4.7, 4.8) are the quantum versions of
\[ (\psi_\pm^3 = (\psi_\pm)^3 = 0 \]

We now define the normal ordering prescription as usual
\[ : \phi_1(z) \phi_2(z) : = \lim_{w \to z} [\phi_1(z) \phi_2(w) - \langle \phi_1(z) \phi_2(w) \rangle], \quad (4.9) \]
with \( \phi \) an arbitrary field. Consequently, we obtain in a straightforward way
\[ : \psi_\pm^4(z) \psi_\pm^4(z) : = q : \psi_\pm^4(z) \psi_\pm^4(z) : \]
\[ : \psi_\pm^4(z) \psi_\pm^4(z) : = q^2 : \psi_\pm^4(z) \psi_\pm^4(z) : \quad (4.10) \]
\[ : \psi_\pm^4(z) \psi_\pm^4(z) : = q : \psi_\pm^4(z) \psi_\pm^4(z) : \]
\[ : \psi_\pm^4(z) \psi_\pm^4(z) : = q^2 : \psi_\pm^4(z) \psi_\pm^4(z) : \quad (4.11) \]

Some remarks are now in order here.
Firstly, due to the quon algebra, we have a weaken Wick theorem, where only relations between positive and negative frequencies are known. Let us notice that this peculiar structure will not affect the calculations of the correlations functions as we will see further.
Secondly, when we perform for example : \( \psi_\pm^q(z) \psi_\pm^q(w) \), we first return to the definition of our Wick theorem (4.9), namely : \( \psi_\pm^q(z) := \lim_{\epsilon \to 0} \psi_\pm^q(z + \epsilon) \psi_\pm^q(z) \), then do all the possible contractions and finally take the limit \( \epsilon \to 0 \), as it should be.

This is strongly different from the case \( D = 1 \) where the third power of the \( \psi \) fields vanishes, even after quantization [13] imposing \( \partial_t \psi_1(t) \psi_1(t) = q^\pm \psi_1(t) \partial_t \psi_1(t) \). When \( D = 2 \), the third power is not zero so \( \partial_z ( : \psi_\pm^3(z) : ) \neq 0 \) so the fields \( \psi_\pm \) and \( \partial_z \psi_\pm \) cannot \( q - \)mute. In the case \( F = 2 \), this subtlety never appears, because after quantization, a Grassmann variable becomes a Clifford one, hence the square of a fermion is one.

For arbitrary \( F \), all these results can be extended as follow: we have \( F - 1 \) sectors \( (\psi_\pm, \psi_{\ell}^{\pm \cdots}) \) with the substitution \( q \longrightarrow \exp \frac{2\pi a}{F} \). Note that the fields \( \psi_\pm \) and \( \psi_\pm \) (with \( b \neq (F - a) \)) commute without the normal ordering prescription (\( : : \)) because they come from different graded sectors.
Moreover, before ending this subsection, it is worth noticing that if we modify the \( q - \)mutations relations such that
\[ a_r a_s^\dagger = qa_s^\dagger a_r = k_r(\Delta) \delta_{rs} \quad (4.12) \]
we obtain the two-point function
\[ < 0 | \psi_\pm^r(z) \psi_\pm^s(w) | 0 > = \frac{1}{z^\Delta} \sum_{n \geq 0} k_n(\Delta) \left( \frac{w}{z} \right)^n \sim \frac{1}{(z - w)^\Delta}. \quad (4.13) \]
This leaves open the connection between our approach and the parafermions, where correlations functions involve fractional power. Naturally, the Lagrangian and the superspace have to be modified subsequently.
4.2 Hilbert space and quons

As we have mentioned previously, the operators $a, a^\dagger$ and $b, b^\dagger$ are not conjugate from each other, however it is possible to make a redefinition such that they become conjugate. Before doing such a transformation, we would like to give some basic properties of the algebra (4.6, 4.7). The peculiarity of such type of algebras is that $a^\dagger r a^\dagger s |0 > \neq a^\dagger s a^\dagger r |0 >$ for $r \neq s$ etc. Consequently, such states decompose into irreducible representations of the permutation (eventually the braided) group as it is the case for the parafermions introduced by Green [23]. So we have to be careful with the position of the various operators of creation. We note $(a^\dagger r_1)^{k_1} \cdots (a^\dagger r_i)^{k_i} |0 > \sim |(k_1) r_1; \cdots, (k_i) r_i >$. Let us now recall and give some properties of the representation of the underlying algebra

1. No bilinear relations can be consistently postulated among two creators or two annihilators.

2. Any operator of creation cannot act more than $F−1$ times on the vacuum. It means that if we consider the state

$$ |h> = A_0 (a^\dagger r)^{k_1} A_1 \cdots A_{i-1} (a^\dagger r)^{k_i} A_i |0 >, $$

with $A_0, \cdots, A_i$ arbitrary products of creators different from $a^\dagger r$, we have $a^\dagger r |h> = 0$ if $k_1 + \cdots + k_i = F − 1$. In other words, the representation decomposes into

$$ H = H_0^{(r)} \oplus \cdots \oplus H_{F-1}^{(r)} $$

where $H_i^{(r)}$ is the space on which $a^\dagger r$ has been applied $i−$times. The fact that the state $|h>$ is annihilated by $a^\dagger r$ is legitimated by the property $(a^\dagger r)^F = 0$.

3. It is straightforward to check that the state

$$ A_0 \cdots A_i |0 >, $$

is annihilated by $a_r$.

Now we are ready to give the general ideas to built hermitian conjugate operators. We need first to define number operators $N_r^{(a)}$ and $N_s^{(b)}$ such that

$$ [N_r^{(a)}, a_s^\dagger] = \delta_{rs} a_s^\dagger, \quad [N_r^{(a)}, a_s] = −\delta_{rs} a_s $$

$$ [N_r^{(b)}, b_s^\dagger] = \delta_{rs} b_s^\dagger, \quad [N_r^{(b)}, b_s] = −\delta_{rs} b_s. $$

(4.15)

These parafermions are different from the ones introduced by Fateev and Zamolodchikov.
In the case of infinite statistics as quons, those operators are complicated polynomials which are expressed in terms of all the creation and annihilation operators \([24]\) and they contain terms of degree two, four, and so on.

Using those number operators, we define an alternative series of oscillators \((a_r, a_r^\dagger) \rightarrow (\alpha_r, \alpha_r^\dagger)\) and \((b_r, b_r^\dagger) \rightarrow (\beta_r, \beta_r^\dagger)\) (we just give the results for the \(a\)’s)

\[
\begin{align*}
\alpha_r^\dagger &= iq^{-1/2}a_r^\dagger q^{-N_r^{(a)}} \\
\alpha_r &= iq^{-1/2}q^{-N_r^{(a)}} a_r,
\end{align*}
\]

Then, using the properties of the algebra (and for instance its matrix realization), we can prove

\[
\begin{align*}
q^{N_r^{(a)}} a_r &= qa_r q^{N_r^{(a)}} \\
q^{N_r^{(a)}} a_r^\dagger &= q^{-1}a_r^\dagger q^{N_r^{(a)}}.
\end{align*}
\]  

Next, a direct calculation shows that the \(\alpha\)’s generate the \(q\)–oscillator algebra introduced by Biedenharn and Macfarlane \([25]\)

\[
\alpha_r \alpha_s^\dagger - q^{1/2} \alpha_s^\dagger \alpha_r = q^{-N_r^{(a)}/2}
\]

It is then easy to build the Hilbert states from the \(\alpha_r\), when only one series of oscillators acts on the states

\[
\begin{align*}
\alpha_r^\dagger |k\rangle &= \sqrt{|k+1|} |k+1\rangle \\
\alpha_r |k\rangle &= \sqrt{|k|} |k-1\rangle \\
N_r^{(a)} |k\rangle &= k |k\rangle
\end{align*}
\]  

with \(|k| = \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}}\). Using explicitly the matrix realization of \(\alpha_r, \alpha_r^\dagger\) on this sub-space we see that the operator \(\alpha_r\) and \(\alpha_r^\dagger\) are hermitian-conjugate. Therefore, we also get the conjugate relation of (4.18)

\[
\alpha_r \alpha_r^\dagger - q^{-1/2} \alpha_r^\dagger \alpha_r = q^{-N_r^{(a)}/2}
\]

Consequently, using the results established in the context of the \(q\)–oscillators, we can see that the Hilbert space is definite positive in this sub-space, and the representation unitary. The situation gets more involved when two different series of operators act on the vacuum \(((\alpha_r^\dagger)^k (\alpha_s^\dagger)^k |0\rangle \text{ and so on})\). Therefore, we need certainly a more subtle transformation than (4.17), analogous to the complicated definition of number operators in the case of quons \([24]\); but this goes beyond the scope of this paper.

There is a second difference connected to the non-commutativity of two operators of creation
Indeed, when different series of operators are used we see that the number of states increase with the degree of the monomial, and for instance the two states $\alpha_1^\dagger \alpha_2^\dagger |0\rangle$ and $\alpha_s^\dagger \alpha_r^\dagger |0\rangle$ are different for $r \neq s$.

All this structure, which turns out to be very complicated, can certainly be interpreted using the quantum group limit of fractional supersymmetry established in [29], starting from a generic $q$, and taking the limit $q \rightarrow \text{primitive root of unity}$.

5 Algebra and OPE

5.1 FSUSY algebra and OPE

The Lagrangian (3.4) is obviously invariant under conformal transformations. It turns out that $X(z)$ is of conformal weight 0 and $\psi_\frac{F}{F}(z)$ of conformal weight $\frac{F}{F}$ as we will see. It is also invariant under the FSUSY transformations generated by $Q$ (see (2.3)). In order to give the transformations, let us introduce $\epsilon$ the parameter of the FSUSY transformation. The $q$–mutation of $\epsilon$ with $\psi_\frac{F}{F}(z)$ are identical as those of $\theta$ with $\psi_\frac{F}{F}(z)$ as in Ref.[12, 17]. This is a consequence of the FSUSY transformation which corresponds to the translation $\theta \rightarrow \theta + \epsilon$ in the fractional superspace. The relation $\epsilon \theta = q^{-1} \theta \epsilon$ ensures that $D$ is a covariant derivative as it should be in order to build a FSUSY invariant action [17]. This can be proved in a straightforward way using (2.4) which ensures that $\epsilon Q$ and $D$ commute. Then, the transformations of the superfield are

$$\delta_\epsilon \Phi = \epsilon Q \Phi.$$ 

They leave the Lagrangian invariant because the product of two superfields is a superfield and similarly for the covariant derivative. Using the decomposition of the field $\Phi$ this leads to

$$\delta_\epsilon X = q^{\frac{F}{2}} \epsilon \frac{F}{F} \psi_\frac{F}{F},$$

$$\delta_\epsilon \psi_\frac{F}{F} = q^{\frac{F}{2}} \{a + 1\} \epsilon \psi_{\frac{F}{F} + 1}, \quad a \neq F - 1$$

$$\delta_\epsilon \psi_{\frac{F}{F} - 1} = (-1)^F q^{\frac{F}{2}} \epsilon \partial_z X, \quad a \neq F - 1.$$  \hspace{1cm} (5.1)

with $\{a\} = \frac{q^{\frac{a}{2}} - 1}{q^{-1}}$. Those transformation properties fit exactly (up to normalization factors) with the ones introduced by Durand [14].

Stress that the coefficient of $\theta^{F - 1}$ transforms as a total derivative. Therefore, with the rules of integration, the action is obviously FSUSY invariant. The generators of the conformal transformations (stress momentum tensor) and of the FSUSY transformations are

$$T(z) = -\frac{1}{2} : \partial_z X(z) \partial_z X(z) : + \frac{F}{(q - 1) F} \sum_{a=1}^{F-1} \left[ \frac{F - a}{2F} \left( (q^{-a} - 1) : \partial_z \psi_{\frac{F}{F}}(z) \psi_{\frac{F}{F} - a}(z) \right. \right.$$  

$$\left. + (1 - q^a) : \psi_{\frac{F}{F} - a}(z) \partial_z \psi_{\frac{F}{F}}(z) : \right), \quad (5.2)$$
\[ G(z) = q^{1/2} \left[ : \partial_z X(z) \psi_\frac{1}{F}(z) : + \frac{F}{(q - 1)^{F-1}} \sum_{a=1}^{F-2} \{a + 1\} \{ -a \} - \frac{1}{1 + \eta(a)} : \psi_{\frac{F-1}{F}}(z) \psi_{\frac{1}{F}}(z) : \right]. \] (5.3)

with \( \eta(a) = q^{F-1} \) if \( a = \frac{F-1}{2} \) and \( \eta(a) = 1 \) otherwise. This complex normalization comes from the fact that we must distinguish in the OPE, the case \( F - a = 1 + a \) from the other ones. It appears that the stress-energy tensor, decomposes into \( \frac{F+1}{2} \) terms which does not see each other because of the two-point Green functions (3.6)

\[ T(z) = \sum_{a=0}^{\frac{F-1}{2}} T_a(z), \] (5.4)

Using the two-point correlation functions (3.6), we have the following operator product expansion (OPE), encoding the different transformations

\[ T(z)X(w) = \frac{\partial_w X(w)}{z-w} + \cdots \]
\[ T(z)\psi_\frac{1}{F}(w) = \frac{q^{\frac{1}{F}} \psi_\frac{1}{F}(w)}{(z-w)^2} + \frac{\partial_w \psi_\frac{1}{F}(w)}{z-w} + \cdots \]
\[ G(z)X(w) = q^\frac{1}{2} \frac{\psi_\frac{1}{F}(w)}{z-w} + \cdots \]
\[ G(z)\psi_\frac{1}{F}(w) = (-1)^F q^{\frac{1}{2}} \left( 1 - q \right)^{F-1} \frac{\partial_w X(w)}{z-w} + \cdots \]
\[ G(z)\psi_\frac{a}{F}(w) = q^\frac{1}{2} \{a + 1\} \frac{\psi_\frac{a+1}{F}}{z-w} + \cdots . \] (5.5)

To compute these OPE, we have first used the Wick prescription detailed in subsection 4.1, and then decomposed the fields in positive and negative modes. As it was already mentioned, these OPE enable to fix in a consistent way the \( q \)-mutation relations (4.6, 4.7) and therefore the relations (4.10, 4.11). Moreover, these transformations show explicitly that \( X(z) \) is of conformal weight 0, and \( \psi_\frac{a}{F} \) of conformal weight \( \frac{a}{F} \). By comparing the OPE with the FSUSY transformations (5.1), we conclude that \( G \) is the generator of the FSUSY transformations.
FSUSY is then the natural extension of supersymmetry and connects $\frac{1}{F}$-integer spin states. We have then to ensure that the algebra closes by computing the remaining OPE.

Mention that equations (4.5, 4.6, 4.7, 4.9) which have been proven explicitly for $F = 3$, using the oscillators $a$ and $b$, can be obtained along the same line for any $F$. Each $\psi_{\frac{1}{F}a}$, $\psi_{\frac{2}{F}a}$ contribute to two series of oscillators and the relations equivalent (4.5, 4.6, 4.7, 4.9) are obtained with the substitution $q \to q^a$. Then we get

$$T(z)T(w) = \frac{1}{2} \left( \frac{c_F}{(z-w)^2} + \frac{2T(w)}{z-w} + \frac{\partial_w T(w)}{z-w} + \ldots \right) \quad (5.6)$$

$$T(z)G(w) = \frac{\frac{F+1}{2}G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{z-w} + \ldots .$$

This shows that the conformal weights of $T$ and $G$ are 2 and $\frac{F+1}{2}$ as it should be. The central charge is ($F > 2$)

$$c_F = -12 \sum_{a=1}^{\frac{F-1}{2}} \cos\left( \frac{2\pi a}{F} \right) \frac{a(F-a)}{F^2}. \quad (5.7)$$

As already established in [17], the algebra does not close under quadratic relations for $G$ with itself, because the underlying symmetries involve $F-$power in the fractional superspace ($Q^F = \partial_z$). This just tells us that there is, in our case, no symmetry generator beyond $G$ and $T$ implying non quadratic closure relations as we will see.

The algebra we are considering closes upon $F - 2$ intermediate would-be symmetry generators $G_2(z), \cdots, G_{F-1}(z)$, with $G_i$ obtained from the OPE of $G$ upon $G_{i-1}$. The reason why those operators do not generate a symmetry of the action, has been analyzed with great details in [17]. Let us recall briefly the main arguments. The symmetry of the Lagrangian is induced by a basic symmetry in the superspace. The only operators (acting in the superspace) which satisfy the Leibniz rule, and thus generate a symmetry, are $\partial_z$ and $Q$. Consequently, only $T$ and $G$ generate a symmetry of the Lagrangian. This can be directly obtained, by observing that, the action of $G_{i \neq 1}$ on the fields do not leave the Lagrangian invariant. To summarize, the deep reason why the $G_{i \neq 1}$ are not symmetry operators is a reminiscence of the algebraic structure we are considering, (which goes beyond (super-)Lie algebras). When $F$ is not a prime number, the situation is quite different and $G_i$ is a generator of symmetry if $i$ divides $F$ as will be seen further.

The conformal weight of these intermediate operators is $1 + \frac{i}{F}$ and at the end of the process of closure, we have the action of $G$ on $G_{F-1}$. It leads to a tensor of conformal weight 2, which can be expressed as a sum of $T$ and possibly other terms expressed with the various fields $X, \psi_{\frac{1}{F}a}$. This is a major difference with the fractional supervirasoro algebra where, due to the fractional power appearing in the OPE, cuts are involved.
As we have seen, using explicitly the modes expansions, one can build a representation of the algebra starting from the vacuum previously defined. As in string theory, we can consider different sets of sectors \[17\] depending on the boundary conditions of the fields. This constitutes an adapted generalization of the Ramond-Neveu-Schwarz ones. Therefore, the various quantum numbers of the vacuum can be calculated, by regularizing the infinite sum by $\zeta$ function \[26\] and an adapted GSO \[27\] projection, ensuring modular invariance, as to be defined. The modes of the two currents $G$ and $T$ allow to get a representation of the algebra \[2.6\] with a central extension (the last equation seems difficult to be derived). This algebra can be compared with the fractional superconformal algebra introduced in Ref. \[7\] which is also generated, in addition to the stress momentum tensor, by a current of conformal weight $(1 + \frac{1}{F})$. These two extensions of the Virasoro algebra are different. The fractional superconformal algebra closes with rational power of $(z - w)$, leading to non-local algebras because cuts are involved. The one we propose, closes only with integer power of $(z - w)$ but involves $F - \text{th}$ power instead of quadratic relations.

Moreover, the central charges we get are irrational numbers (except for $F = 2, 3$). So the theory we have obtained is no longer a rational conformal field theory (RCFT), but an irrational conformal field theory (ICFT) (see \[28\] and references therein).

### 5.2 Associativity

In a general CFT, there are two strong requirements ensuring the consistency of the algebra, namely the closure relations and the associativity. The former just tells us that there is, in our case, no symmetry generator beyond $G$ and $T$ implying non quadratic closure relations as we have seen before. The latter is encoded through the computation of correlation functions which have to be invariant according to the way we group the operators. This restricts considerably the possible CFT. For instance those constraints fixes the structure constants appearing when considering the fractional supervirasoro algebras \[8\] and is solved, for instance, via the bootstrap equation coming from the four-point functions \[1\].

Concretely, in order to set up the associativity, we have to compute the four-point functions, when only the primary fields are involved. For readability, we come back to the $F = 3$ case. Then, we define formally the OPE between the primary fields.

\[
\begin{align*}
\partial X(z)\partial X(w) &= \frac{-1}{(z-w)^2} + \sum_{n>0} C_{00}^{0}(n) \partial^n X(w)(z-w)^{n-2} \\
\partial X(z)\psi_+^a(w) &= \sum_{n\geq 0} C_{0a}^{a}(n) \partial^n \psi_+^a(w)(z-w)^{n-1}, \quad a = 1, 2 \\
\psi_+^a(z)\psi_+^b(w) &= \sum_{n\geq 0} C_{11}^{a,b}(n) \partial^n \psi_+^b(w)(z-w)^{n} 
\end{align*}
\] (5.8)
\[
\psi^2(z)\psi^2(w) = \sum_{n \geq 0} C^{12}_{22(n)} \partial^n \psi^1(w)(z-w)^{n-1}
\]

\[
\psi^1(z)\psi^2(w) = -\frac{q}{z-w} + \sum_{n>0} C^{0(n)}_{12} \partial^n X(w)(z-w)^{n-1}
\]

\[
\psi^2(z)\psi^1(w) = \frac{q^2}{z-w} + \sum_{n>0} C^{0(n)}_{21} \partial^n X(w)(z-w)^{n-1}
\]

At this point, the \(C\) coefficients are unfixed but can be determined using constraints imposed by the associativity condition on correlation functions.

Therefore, from the bootstrap equation, we have to calculate the four-point function

\[
<\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)>,
\]

with \(\phi_i\) an arbitrary primary field, in two different ways. The procedure is as follow. Firstly, we do the contraction of \(\phi_1\) with \(\phi_2\) and \(\phi_3\) with \(\phi_4\) and then calculate the two-point functions. Secondly, we do the contraction of \(\phi_2\) with \(\phi_3\) and \(\phi_1\) with \(\phi_4\) and determine the two-point functions. Equating the two ways of calculating the four-point function gives some equations between the \(C\)'s. Technically, it is easier to use the \(SL(2,C)\) invariance of CFT to map \(z_1 \to \infty, z_4 \to 0, z_2 \to 1\) and \(z_3 \to x = \frac{(z_1-z_2)(z_1-z_4)}{(z_2-z_3)(z_1-z_3)}\).

In our special case of FSUSY there is four types of four-point functions.

The first type concerns correlations functions involving only bosonic fields. In this case, with the techniques described above, the numbers \(C^{0(n)}_{00}\) are those we find in standard conformal field theory.

The second type involves 4−point functions like

\[
<\psi^1(z_1)\psi^1(z_2)\psi^1(z_3)\psi^1(z_4)>
\]

and similarly by changing 1 → 2. The first way of doing the contractions gives zero. When we use the second way, we get first the three-point function

\[
<\psi^1(z_1) \left( \sum_{n \geq 0} C^{2(n)}_{11} \partial^n \psi^2(w)(z-w)^n \right) \psi^1(z_4)>.
\]

But it seems impossible to make the contraction of the first \(\psi^1\) with the last one. However, if one calculate the three-point function by doing one contraction we get also zero as it should be.

The third type of constraints are obtained from the calculation of correlation functions like

\[
<\psi^1(z_1)\psi^2(z_2)\psi^3(z_3)\psi^2(z_4)>
\]

Using the bootstrap equations and (5.8), we thus obtain some constraints upon the \(C^{0(n)}_{12}\) and \(C^{0(n)}_{21}\).
And finally, the fourth type involves 4–point functions mixing the fields $\partial X$ and $\psi_3$. It imposes therefore constraints upon the $C_{0a}^{n(n)}$.

The computation of all the $C$ coefficients is quite heavy and goes beyond the scope of this paper. Nevertheless, we have shown how it works in order to ensure the associativity of the algebra.

6 Classification

6.1 Classification of FSUSY symmetries

All the results established in the previous sections remain valid for arbitrary $F$, with some minor modifications.

If $F$ is not a prime number $F = fF'$, ($F'$ being a prime number) we have $(\psi_F)^{F'} = 0$ instead of $(\psi_F)^F = 0$. In fact $\psi_F$ is no longer a Grassmann variable of order $F$ but more precisely of order $F'$. If $F$ is an even number, the definition of $T_F^\tau$ in (5.4) has to be modified as

$$T_{F-F'}^\tau = -\frac{2F}{(q-1)F}\partial_z \psi_1^\tau(z)\psi_2^\tau(z),$$

and the central charge becomes

$$c_F = 1 + \frac{1}{2} + 2 \sum_{a=1}^{E(F-1)} \cos\left(\frac{2\pi a}{F}\right) \left\{ \left(\frac{a}{F}\right)^2 + \left(\frac{F-a}{F}\right)^2 - 4\frac{a(F-a)}{F^2} \right\},$$

where $E(\ )$ means the integer part. In $T_F^\tau$, $\psi_1^\tau(z)$ is a usual fermionic field.

If $F$ is an odd number, the central charges (5.7) and the stress momentum tensor (5.2) remain unchanged.

Some comments are in order here: we can note that the central charge is, in general, an irrational number but $F = 2, 3, 4$. Among those families of theories, stress that for $F = 4$ we do have the same central charge than for $F = 2$. As a final remark we have, for $\psi_F$ (when $F$ even $\neq 2$) a different normalization for $T_F^\tau$; this comes from the normalization in the action (3.3) and in the Green function (3.4).

The interesting point with those kinds of symmetries is that we are able to generalize the previous results to any $F$ using its decomposition into prime numbers. This exhibits, as we will see, substructures with smaller symmetries. Let us consider the generic case when $F$ can be written as $F_1F_2$ with $F_1, F_2$ two prime numbers not necessary different. A scalar multiplet of FSUSY has the following irreducible decomposition in terms of $F_1$ multiplets

$$\Phi_F^{(0)} = \bigoplus_{b=0}^{F_2-1} \Phi_{F_1}^{(b)},$$

(6.1)
where $\Phi_{F_1}^{(\frac{k}{F})}$ is a $F_1$ multiplet of spin $\frac{k}{F}$.

From this decomposition, it is obvious that one can get three different theories with the same fields. First, using $\Phi_{F_1}^{(0)}$, we can built an invariant FSUSY action. Furthermore, with the family of fields $\Phi_{F_1}^{(\frac{k}{F})}$ (or in the same way with $F_1 \leftrightarrow F_2$), a $F_1 - (F_2 -)$SUSY can be also derived. However, the results are stronger because, by appropriate normalizations, the three Lagrangians so obtained are rigorously identical. This statement can be proved explicitly by a tedious calculation. We will just give a sketch of the proof and the exact normalizations will be omitted for readability. The action (3.3) (which is also valid for arbitrary $F_1$) can be reproduced using the fields $\Phi_{F_1}^{(\frac{k}{F})}$

$$\Phi_{F_1}^{(0)} \sim X(z, \bar{z}) + \sum_{a=1}^{F_1-1} \theta_1^a \psi_{aF_2}(z)$$

$$\Phi_{F_1}^{(\frac{k}{F})} \sim \sum_{a=0}^{F_1-1} \theta_1^a \psi_{aF_2+b}(z) \quad (6.2)$$

$$\Phi_{F_1}^{(\frac{F_2-b}{F})} \sim \sum_{a=0}^{F_1-1} \theta_1^{F_1-a-1} \psi_{F-aF_2-b}(z)$$

$$b = 1, \ldots, E(\frac{F_2-1}{2}),$$

where $\theta_1$ is a $F_1$ generalized Grassmann variable ($\theta_1^{F_1-1} = 0$) and $q_1 = q^{F_2}$. The $q-$mutation of $\theta_1$ with the fields are given by

$$\theta_1 \psi_{aF_2+b} = q^{-(aF_2+b)} \psi_{aF_2+b} \theta_1.$$ \quad (6.3)

The $aF_2-$components of the field $\Phi$ $q-$mute with the primitive root of $F_1$ although the other ones $aF_2+b, b \neq 0$ with primitive root of $F$. This is due to the non-trivial spin of the superfields $\Phi_{F_1}^{(\frac{k}{F})}$. Stress that $\theta_1$ is substituted to $\theta$ in the $F_1$ SUSY formulation. From these relations and the normalization of the fields, we are now able to write the $F_1$ SUSY invariant action

$$\mathcal{L} \sim \int d\theta_1 \left[ \partial_\bar{z} \Phi_{F_1}^{(0)} D_{F_1} \Phi_{F_1}^{(0)} + \sum_{b=1}^{E(\frac{F_2-1}{2})} \left( \partial_\bar{z} \Phi_{F_1}^{(\frac{k}{F})} \Phi_{F_1}^{(\frac{F_2-b}{F})} + \partial_\bar{z} \Phi_{F_1}^{(\frac{F_2-b}{F})} \Phi_{F_1}^{(\frac{k}{F})} \right) \right].$$ \quad (6.4)

When $F_2 = 2$, the sum over $b$ contains just one term, namely $\partial_\bar{z} \Phi_{F_1}^{(\frac{k}{F})} \Phi_{F_1}^{(\frac{F_2-b}{F})}$. In principle heavy normalizations for the superfields have to be implemented in order to reproduce the FSUSY action (3.3).

Consequently, if we have an action $F_1F_2$ supersymmetric, it is simultaneously $F_1$ and $F_2$ supersymmetric. And reciprocally, to get the converse, in addition to the scalar $F_1$ multiplet, we need $\frac{1}{F} \ldots \frac{F_2-1}{F}$ spin $F_1$ multiplets. The scalar will be coupled to itself via the $F_1$ covariant derivative and the spin $\frac{k}{F}$ with the spin $\frac{F_2-b}{F}$. Of course, as was already mentioned
above, this result can be extended by analogy for any F: we can conclude that in any case if F’ divides F, then a F–supersymmetric action is F’–supersymmetric.

Along the same lines as for the FSUSY transformations, using the $F_1$ SUSY generator $Q_1 = \partial_{\theta_1} + (1-\eta_{F_1})^{-1} \theta_1^{-1} \partial_z$, we are able to determine the $F_1$ SUSY transformations: $\delta_1 \Phi_{F_1} = \epsilon_1 Q_1 \Phi_{F_1}$. Or in terms of the components,

$$\delta_{\epsilon_1} \psi_a F_{a+b} \sim \epsilon_1 \psi_{(a+1)F_{2+b}}, \quad a = 0, \cdots, F_1 - 2$$

(6.5) $\delta_{\epsilon_1} \psi_{(F_1-1)F_{2+b}} \sim \epsilon_1 \partial_z \psi_{F_1}$,

where, for the sake of simplicity we omit the normalizations for the superfield’s transformations. The transformations for $\Phi_{F_1}^{(0)}$ are similar to Eq.(5.1) with $a \to aF_2$. Noticing that the spin of $\epsilon_1$ is $(-\frac{1}{F_1})$ one can easily check that both sides of the equation have the correct spin. In addition, as for $\Phi_{F_1}^{(0)}$, the higher components of $\Phi_{F_1}^{(b)}$, (with $b \neq 0$) transform as a total derivative. This ensures that the previous action, built up with the adapted superspace techniques, is automatically invariant under $F_1$ SUSY transformations. To be as complete as possible, we give its generators (omitting the normalizations)

$$G_0(z) \sim : \partial_z X(z) \psi_{F_{a+b}}(z) : + \sum_{a=1}^{F_1-2} : \psi_{(F_1-a)F_2}(z) \psi_{(1+a)F_{2+b}}(z) :$$

(6.6) $G_b(z) \sim \sum_{b=0}^{F_1-2} \psi_{F_{a-b}}(z) \psi_{(a+1)F_{2+b}}(z) : + \psi_{F_{b}}(z) \partial_z \psi_{F_{2+b}}(z) ;, \quad b = 1, \cdots, F_1 - 1.$

Now, if we introduce the $F_1$–multiplet of spin $\frac{b}{F}$

$$\left( \psi_{F_{1-b}}, \psi_{F_{1-b}+F_2}, \cdots, \psi_{b+aF_2}, \cdots, \psi_{b+(F_1-1)F_2} \right),$$

and the $F_1$–one of spin $\frac{b}{F}$

$$\left( \psi_{F_{2-b}}, \psi_{F_{2-b}+F_2}, \cdots, \psi_{F_{2-b}+aF_2}, \cdots, \psi_{F_{2-b}+(F_1-1)F_2} \right),$$

we just see that in the $G_b$ supercurrent, the fields appearing in the spin $\frac{b}{F}$ multiplets couple the ones of the $\frac{b}{F}$ multiplets. Using the Green functions (3.4), we get

$$G_b(z) \psi_{F_{a+b}}(w) \sim < \psi_{F_{a-b}}(z) \psi_{aF_{2+b}}(w) > \psi_{(1+a)F_{2+b}}(z), \quad a = 0, \cdots, F_1 - 2,$$

and

$$G_b(z) \psi_{(F_1-1)F_{2+b}}(w) \sim < \psi_{F_{2-b}}(z) \psi_{(F_1-1)F_{2+b}}(w) > \partial_z \psi_{F_{2+b}}(z).$$
We are then able to reproduce in a similar way the $F_1$ SUSY transformations of $\Phi_F^{(b)}$ with $G_b$. From this relation, we notice that we need simultaneously the fields $\Phi_F^{(b)}$ and $\Phi_F^{(F_2-b)}$ for the supercurrent and the action except for $b = 0$.

If one considers now the full action in two dimensions, with all the fields, the scalar $F$ superfield decomposes as

$$\Phi_F^{(0,0)} = \bigoplus_{a,b=0}^{F_2-1} \Phi_F^{(a,b)}.$$

(6.7)

We have four kinds of superfields:

(i) $\Phi_F^{(0,0)}$ contains holomorphic, antiholomorphic and auxiliary fields;
(ii) $\Phi_F^{(a,0)}$ holomorphic and auxiliary fields;
(iii) $\Phi_F^{(0,b)}$ antiholomorphic and auxiliary fields;
(iv) $\Phi_F^{(a,b)}$ auxiliary fields.

### 6.2 Algebraic description

To conclude those series of inclusions, we can give an algebraic interpretation. As we have mentioned previously, the underlying algebra of FSUSY is the one generates by $\theta, \partial_\theta$. However, it is known that this algebra, with the primitive root $q$, generates the $q-$deformed Heisenberg algebra $H_q(q, \theta, \partial_\theta)$. If one considers the mapping $(F = F_1F_2)$

$$f_2 : H_q(q, \theta, \partial_\theta) \rightarrow H_q(q, \theta, \partial_\theta)$$

$$\theta \mapsto \theta_1 = \theta^{F_2}$$

$$\partial_\theta \mapsto \partial_{\theta_1}$$

$$q \mapsto q_1 = q^{F_2},$$

(6.8)

due to the fact that $f_2(\partial_\theta) = \partial_{\theta_1}$ and $f_2(q) = q_1$ one can check easily that $f_2$ is an homomorphism of algebra. In this homomorphism, $\partial_{\theta_1}$ which is seen as a element of $H_q(q, \theta, \partial_\theta)$, can be expressed as a polynomial of $\theta$ and $\partial_\theta$. Then is we define the coset $H_q(q, \theta, \partial_\theta)/Ker(f_2)$ we get that this coset is isomorphic to $H_{q_1}(q_1, \theta_1, \partial_{\theta_1})$. Now, if we look at the $q-$mutation properties of the fields with $\theta_1$, we have a third way to build the $F-$SUSY action. Using the $f_2$ isomorphism, we can identify $\theta_1$ with $\theta^{F_2}$. However, if we proceed along those lines, the $q-$mutation relations (3.2) might be incompatible with this identification. This appears when $F_1 = F_2$, because we cannot postulate simultaneously $\theta \psi_{\frac{1}{F_1}} = q^{-F_1} \psi_{\frac{1}{F_1}} \theta$ and $\theta_1 \psi_{\frac{1}{F_1}} = q_1^{-1} \psi_{\frac{1}{F_1}} \theta$. Nevertheless, in such a situation, we can postulate only the last $q-$mutation relation to reproduce the action (3.7) with $F_1$ superfields by using appropriate normalizations.
7 Conclusion

We have constructed, in this paper, a conformal field theory using FSUSY. This conformal theory contains fractional spin states and is obtained analogously as the superconformal algebra. This is achieved by the introduction of an adapted Grassmann algebra. Those new variables encode the fractional spin properties. After quantization of the system, we obtain an Hilbert space where the quons and the $q$-oscillators play a central role. It is worth noticing that supersymmetry is recovered when $F = 2$. Quons and $q$-oscillators are just in this case the fermionic oscillators or the Clifford algebra.

The main feature of our algebra is that it closes through non-quadratic relations. Therefore, it cannot be seen as a Lie or super-Lie algebra. Due to the properties of these extended Grassmann variables, we have obtained a classification of the FSUSY algebra according to the decomposition of $F$ into its product of prime numbers. Next, according to this classification, we can wonder on the possible implications to have some sub-systems with smaller symmetries (when $F$ is not a prime number). Finally, we want to mention that the central charge obtained for $F = 4$ is the same we get in supersymmetric theories.

This approach, as we have claimed, is different from the standard ones. However, it should be interesting to have connections between this way of doing and the standard affine Virasoro constructions (the relations of our model with the Virasoro master equation [28] has to be done). Another open (and related ?) question concerns the relations of the basic fields $(\psi_{\frac{a}{F}})$ of conformal weight $\frac{a}{F}$ with the parafermions introduced by Fateev and Zamolodchikov [6]. Some clues have been given, and as we have already shown, it is possible to modify the algebraic structure in such a way that Green functions (3.6) with fractional power of $(z - w)$ are involved [17].

We have in our previous paper opened the possibility that such theories could be the basic symmetry of the world-sheet of some string-inspired theory. We have proved that the case $F = 3$ leads to a rational critical dimension [17]. If we proceed along the same lines, it is easy to see that the situation is less good for arbitrary $F \neq 2$. When $F = 4$ the anomaly coming from SUSY and 4−SUSY leads to a negative critical dimension, and when $F \geq 5$ the critical dimension is irrational because of the central charge which is irrational.

At this stage, only the cases $F = 3, 4$ should be relevant for a relation with integrable systems, which has to be established. For instance what kind of systems with $c = 4/3 - 1 = 1/3$ should be described by 3−SUSY? However, 2D FSUSY has the main advantage, even if relation with integrable models is not obvious, that it can be described using appropriate variables (the generalized Grassmann variables). This is in favor of a slight modification of our theory such that relations with string or integrable models are allowed. In this direction, two possible extensions of these results can be considered with a few changes. The first one is to introduce interactions via an adapted superpotential. The second extension can be
performed by taking a superfield of conformal weight different from zero. This will clearly modify the values of the central charges.

We can also wonder on the possibility that each FSUSY extension of the Virasoro algebra corresponds to a special point in some new series of integrable models à la Friedan-Qiu-Shenker.

We would also like to mention that the connection between FSUSY and the quantum groups has been undertaken recently [29]. In this paper, the authors show that, starting from arbitrary $q$, they get FSUSY in the limit where $q$ goes to a primitive root of unity.

Finally, the ultimate dimension for the relevance of FSUSY is $D = 1 + 2$. The 3D Poincaré algebra has been extended to a fractional supersymmetric extension. Studying the representations of the corresponding algebra, relativistic anyons are obtained [30].

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