Efficient Bayesian reduced rank regression using Langevin Monte Carlo approach

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Abstract

The problem of Bayesian reduced rank regression is considered in this paper. We propose, for the first time, to use Langevin Monte Carlo method in this problem. A spectral scaled Student prior distribution is used to exploit the underlying low-rank structure of the coefficient matrix. We show that our algorithms are significantly faster than the Gibbs sampler in high-dimensional setting. Simulation results show that our proposed algorithms for Bayesian reduced rank regression are comparable to the state-of-the-art method where the rank is chosen by cross validation.

1 Introduction

Reduced rank regression [Anderson et al., 1951, Izenman, 1975, Velu and Reinsel, 2013] is a widely used model in linear multivariate regression. In this model, the low-rank constraint is imposed on the coefficient matrix to promote estimation and prediction accuracy of multivariate regression. This low-rank structure is building upon the belief that the response variables are related to the predictors through only a few latent direction. This assumption is also useful to extend the model to high dimension settings [Bunea et al., 2011].

Various methods have been conducted for reduced rank regression based on Bayesian approach. A first study can be found in [Geweke, 1996] in bayesian econometrics which based on a low-rank matrix factorization approach. Since then, various works have been studied, for example, [Kleibergen and Paap, 2002, Corander and Villani, 2004, Babacan et al., 2012, Schmidli, 2019, Alquier, 2013] and developed to low-rank matrix estimation and completion [Lim and Teh, 2007, Salakhutdinov and Mnih, 2008, Mai and Alquier, 2015]. Moreover, Bayesian reduced rank regression has been successfully applied in other fields such as genomics [Marttinen et al., 2014, Zhu et al., 2014]. More recently, several works have expanded Bayesian methods for incorporating the sparsity into the reduced rank regression [Goh et al., 2017, Chakraborty et al., 2020, Yang et al., 2020].

It is noted however that most works in Bayesian reduced rank regression (BRRR) usually employ conjugate priors as the conditional posterior distributions can be explicitly derived that allow to implement Gibbs sampling [Geweke, 1996, Salakhutdinov and Mnih, 2008]. The details of these priors are reviewed and discussed in [Alquier, 2013]. Nevertheless,
these Gibbs sampling approaches need to calculate a number of matrix inversions or
singular value decompositions at each iteration and thus it will be costly and can slow
down significantly the algorithm for large data. Different attempts, however based on
optimization, have been made to address this issue including variational Bayesian meth-
ods and maximum a posteriori approach [Lim and Teh, 2007, Marttinen et al., 2014,
Yang et al., 2018].

In this paper, we consider a different way for choosing the prior distribution on
low-rank matrices rather than based on the traditional low-rank matrix factorization.
More specifically, a scaled Student prior is used in our approach for which the rank of
the coefficient matrix does not need to be prespecified as in the matrix factorization
approach. This prior has been successfully used before in a context related to low-rank
matrix completion [Yang et al., 2018] and image denoising [Dalalyan et al., 2020a].

We develop, for the first time, a Langevin Monte Carlo (MC) approach in Bayesian
reduced rank regression. The Langevin MC method was introduced in physics based
on Langevin diffusions [Ermak, 1975] and it became popular in statistics and ma-
chine learning following the paper [Roberts et al., 1996]. Recent advances in the study
of Langevin Monte Carlo make it become more popular in practice [Dalalyan, 2017,
Durmus et al., 2017, Dalalyan et al., 2020b] and a promising approach for Bayesian
statistics [Durmus et al., 2019].

More specifically, in this work, we first present a naive (unadjusted) Langevin MC
algorithm and then a Metropolis–Hastings correction for Langevin MC is proposed.
Interestingly, our implementations does not require to perform matrix inversion nor
singular values decomposition and thus our algorithms can deal with large data set
efficiently. Numerical results from these two algorithm are comparable to the frequentist
approach for which the rank is chosen using cross validation. More particularly, we
further show that our proposed Langevin MC algorithms are significantly faster than
Gibbs sampler in high-dimensional settings, see Section 4.

The paper is structured as follows. In Section 2 we present the reduced rank regres-
sion model, then the prior distribution is defined together with some discussion regarding
the low-rank factorization prior. In Section 3, the implementations of the Langevin MC
approach are given in details. Numerical simulations and a real data application are
presented in Section 4. Some discussion and conclusion are given in Section 5 and 6
respectively.

2 Bayesian reduced rank regression

2.1 Model

We observe two matrices $X$ and $Y$ with

$$Y = XB + \mathcal{E}$$

where $Y \in \mathbb{R}^{n \times m}$ is the response matrix, $X \in \mathbb{R}^{n \times p}$ is the predictor matrix, $B \in \mathbb{R}^{p \times m}$
is the unknown regression coefficient matrix and $\mathcal{E}$ is an $n \times m$ random noise matrix
with $\mathbb{E}(\mathcal{E}) = 0$. 

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We assume that the entries $E_{i,j}$ of $E$ are i.i.d. Gaussian $\mathcal{N}(0, \sigma^2)$ and $\sigma^2$ is given. In this case, note that the likelihood distribution is given by

$$L(Y) \propto \exp\left\{-\frac{1}{2\sigma^2}\|Y - XB\|_F^2\right\}$$

where $\| \cdot \|_F$ denotes the Frobenius norm, $\|M\|_F^2 = \text{Tr}(M^TM)$.

The objective is to estimate the parameter matrix $B$. In many applications, it makes sense to assume that the matrix $B$ has low rank, i.e. $\text{rank}(B) \ll \min(p, m)$.

With a prior distribution $\pi(B)$, the posterior for the Bayesian reduced rank regression is

$$\mathcal{L}_n(B) \propto L(Y)\pi(B).$$

Then, the Bayesian estimator is defined as

$$\hat{B} = \int B\mathcal{L}_n(B). \quad (2)$$

Note that here we focus on estimating $B$ and thus $\sigma^2$ is fixed. One can relax this assumption and place a prior distribution on $\sigma$.

### 2.2 A low-rank promoting prior

Borrowing motivation from a low-rank prior explored in recent works [Yang et al., 2018, Dalalyan et al., 2020a], we consider the following prior,

$$\pi(B) \propto \det(\lambda^2 I_p + BB^T)^{-(p+m+2)/2} \quad (3)$$

where $\lambda > 0$ is a tuning parameter and $I_p$ is the $p \times p$ identity matrix.

To illustrate that this prior has the potential to encourage the low-rankness of $B$, one can check that

$$\pi(B) \propto \prod_{j=1}^m (\lambda^2 + s_j(B)^2)^{-(p+m+2)/2},$$

where $s_j(B)$ denotes the $j$th largest singular value of $B$. It is well known that the log-sum function $\sum_{j=1}^m \log(\lambda^2 + s_j(B)^2)$ encourages a sparsity on $\{s_j(B)\}$, see [Candes et al., 2008, Yang et al., 2018]. Thus the resulting matrix $B$ has a low-rank structure, approximately.

The following Lemma explains the reason why this prior is a spectral scaled Student prior distribution.

**Lemma 1.** [Dalalyan et al., 2020a] If a random matrix $B$ has the density distribution as in (3), then the random vectors $B_i$ are all drawn from the $p$-variate scaled Student distribution $(\lambda/\sqrt{3}) t_{3,p}$. 

3
On low-rank factorization priors

The first idea about a low-rank prior was carried out in [Geweke, 1996]. That is to express the matrix parameter \( B \) as

\[
B_{p \times m} = M_{p \times k}N_{m \times k}^\top
\]

with \( k \leq \min(p, m) \). The prior is defined on \( M \) and \( N \) rather than on \( B \) as

\[
\pi(M, N) \propto \exp\left\{-\frac{\tau^2}{2}(\|M\|_F^2 + \|N\|_F^2)\right\}
\]

for some \( \tau > 0 \). This prior allows to obtain explicit forms for the marginal posteriors that allows an implementation of the Gibbs algorithm to sample from the posterior, see [Geweke, 1996]. However, the downside of this approach is the problem of choosing \( k \), the reduced rank, is not directly addressed. Thus, one has to perform model selection for any possible \( k \), as done in [Kleibergen and Paap, 2002, Corander and Villani, 2004].

Recent approaches focus on fixing a large \( k \), e.g. \( k = \min(p, m) \), then sparsity-promoting priors are placed on the columns of \( M \) and \( N \) such that most columns are almost null. So that the resulting matrix \( B = MN^\top \) is approximately low-rank. This direction was first proposed in [Babacan et al., 2012] in the context of matrix completion, but it still can be used in reduced rank regression. See [Alquier, 2013] for the details and discussions on low-rank factorization priors.

With low-rank factorization priors, most authors simulate from the posterior by using the Gibbs sampler as the conditional posterior distributions can be explicitely derived, e.g. in [Geweke, 1996, Salakhutdinov and Mnih, 2008]. However, these Gibbs sampling algorithms update the factor matrices in a row-by-row fashion and invoice a number of matrix inverse operations at each iteration. This is expensive and slow down the algorithm for large data set.

3 Langevin Monte Carlo implementations

In this section, we propose to compute an approximation of the posterior with the scaled multivariate Student prior by a suitable version of the Langevin Monte Carlo algorithm.

3.1 Unadjusted Langevin Monte Carlo algorithm

Let us recall that the log posterior is of the following form

\[
\log \mathcal{L}_n(B) = -\frac{1}{2\sigma^2}\|Y - XB\|_F^2 - \frac{p + m + 2}{2} \log \det(\lambda^2 I_p + BB^\top),
\]

and consequently,

\[
\nabla \log \mathcal{L}_n(B) = -\frac{1}{\sigma^2}X^\top(Y - XB) - (p + m + 2)(\lambda^2 I_p + BB^\top)^{-1}B.
\]

We use the constant step-size unadjusted Langevin MC (denoted by LMC) [Durmus et al., 2019]. It is defined by choosing an initial matrix \( B_0 \) and then by using the recursion

\[
B_{k+1} = B_k - h\nabla \log \mathcal{L}_n(B_k) + \sqrt{2h}W_k, \quad k = 0, 1, \ldots,
\]
where $h > 0$ is the step-size and $W_0, W_1, \ldots$ are independent random matrices with i.i.d. standard Gaussian entries. The detail of the algorithm is given in the Algorithm 1.

Note that a direct application of the Langevin MC algorithm (4) needs to calculate a $p \times p$ matrix inversion at each iteration. This can slow down significantly the algorithm and might be expensive. However, one can easily verify that the matrix $M = (\lambda^2 I_p + BB^T)^{-1}B$ is the solution to the following convex optimization problem

$$\min_M \{ \|I_m - B^T M\|_F^2 + \lambda^2 \|M\|_F^2 \}.$$ 

The solution of this optimization problem can be obtained by using the package 'glmnet' [Friedman et al., 2010] (with family option 'mgaussian'). This does not require matrix inversion nor other costly operation. However, it is noted that in this case we are using the Langevin MC with approximate gradient evaluation, theoretical assessment of this method can be found in [Dalalyan and Karagulyan, 2019].

Algorithm 1 LMC for BRRR

1: **Input**: matrices $Y \in \mathbb{R}^{n \times m}, X \in \mathbb{R}^{n \times p}$
2: **Parameters**: Positive real numbers $\lambda, h, T$
3: **Output**: The matrix $\hat{B}$
4: **Initialize**: $B_0 \leftarrow (X^T X + 0.1 I_p)^{-1} X^T Y; \hat{B} = 0_{p \times m}$
5: **for** $k \leftarrow 1$ to $T$ **do**
6: Simulate $B_k$ from (4);
7: $\hat{B} \leftarrow \hat{B} + B_k / T$
8: **end for**

Remark 1. It seems that the Algorithm 1 looks like an iterative gradient descent for minimizing the penalized Gaussian log-likelihood with a penalty. However, Algorithm 1 computes the posterior mean and not a maximum a posteriori estimator. More precisely, our algorithm includes a final step of averaging.

Remark 2. For small values of $h$, the output $\hat{B}$ is very close to the integral (2) of interest. However, for some $h$ that may not small enough, the Markov process is transient and thus the sum explodes [Roberts and Stramer, 2002]. To address this problem, one have to take a smaller $h$ and restart the algorithm or a Metropolis–Hastings correction can be included in the algorithm. The Metropolis–Hastings approach ensures the convergence to the desired distribution, however, it greatly slows down the algorithm because of an additional acceptance/rejection step at each iteration. The approach by taking a smaller $h$ also slows down the algorithm but we keep some control on its time of execution.

Remark 3. Based our observations from numerical studies in Section 4, the initial matrix $B_0$ can also effect the convergence of the algorithm. We suggest using $B_0 = (X^T X + 0.1 I_p)^{-1} X^T Y$ as a default alternative.

3.2 A Metropolis-adjusted Langevin algorithm

Here, we consider a Metropolis-Hasting correction to the Algorithm 1. This approach guarantees the convergence to the posterior. More precisely, we consider the update
rule in (4) as a proposal for a new state,
\[
\tilde{B}_{k+1} = B_k - h \nabla \log \mathcal{L}_n(B_k) + \sqrt{2h} W_k, \quad k = 0, 1, \ldots
\]
(5)
Note that \(\tilde{B}_{k+1}\) is normally distributed with mean \(B_k - h \nabla \log \mathcal{L}_n(B_k)\) and the covariance matrix equals to \(2h I_p\). This proposal is accepted or rejected according to the Metropolis-Hastings algorithm. That is the proposal is accepted with probability:
\[
\min \left\{ 1, \frac{\mathcal{L}_n(B_{k+1}) q(B_k | \tilde{B}_{k+1})}{\mathcal{L}_n(B_k) q(B_{k+1} | B_k)} \right\},
\]
where
\[
q(x'|x) \propto \exp \left( -\frac{1}{4h} \|x' - x + h \nabla \log \mathcal{L}_n(x)\|^2_F \right)
\]
is the transition probability density from \(x\) to \(x'\). The detail of the Metropolis-adjusted Langevin algorithm (denoted by MALA) for BRRR is given Algorithm 2. Compared to random-walk Metropolis–Hastings, MALA has the advantage that it usually proposes moves into regions of higher probability, which are then more likely to be accepted.

**Remark 4.** Following [Roberts and Rosenthal, 1998], the choice of the step-size \(h\) is tuned such that the acceptance rate is approximately 0.5. See Section 4 for some choices in special cases in our simulations.

**Algorithm 2** MALA for BRRR

1. **Input:** matrices \(Y \in \mathbb{R}^{n \times m}, X \in \mathbb{R}^{n \times p}\)
2. **Parameters:** Positive real numbers \(\lambda, h, T\)
3. **Output:** The matrix \(\hat{B}\)
4. **Initialize:** \(B_0 \leftarrow (X^\top X + 0.1I_p)^{-1} X^\top Y; \hat{B} = 0_{p \times m}\)
5. for \(k = 1\) to \(T\) do
6. \hspace{1em} Simulate \(\tilde{B}_k\) from (5)
7. \hspace{1em} Calculate \(\alpha = \min \left\{ 1, \frac{\mathcal{L}_n(B_{k+1}) q(B_k | \tilde{B}_{k+1})}{\mathcal{L}_n(B_k) q(B_{k+1} | B_k)} \right\}\)
8. \hspace{1em} Sample \(u \sim U[0, 1]\)
9. \hspace{1em} if \(u \leq \alpha\) then
10. \hspace{2em} \(B_k = \tilde{B}_k\)
11. \hspace{1em} else
12. \hspace{2em} \(B_k = B_{k-1}\)
13. \hspace{1em} end if
14. \hspace{1em} \(\hat{B} \leftarrow \hat{B} + B_k/T\)
15. end for

## 4 Numerical studies

### 4.1 Simulations setups and details

First, we perform some numerical studies on simulated data to access the performance of our proposed algorithms. We consider the following model setups:
• Model I: A low-dimensional set up is studied with \( n = 100, p = 12, m = 8 \) and the true rank \( r = \text{rank}(B) = 3 \). The design matrix \( X \) is generated from \( \mathcal{N}(0, \Sigma) \) where the covariance matrix \( \Sigma \) is with diagonal entries 1 and off-diagonal entries \( \rho \geq 0 \). We consider \( \rho = 0 \) and \( \rho = 0.5 \), this creates a wide-range correlation in the predictors. The true coefficient matrix is generated as \( B = B_1 B_2^\top \) where \( B_1 \in \mathbb{R}^{p \times r}, B_2 \in \mathbb{R}^{m \times r} \) and all entries in \( B_1 \) and \( B_2 \) are randomly sampled from \( \mathcal{N}(0, 1) \).

• Model II: This model is similar to Model I, however, a high-dimensional set up is considered with \( n = 100, p = 150, m = 90 \).

• Model III: An approximate low-rank set up is studied. This series of simulation is similar to the Model II, except that the true coefficient is no longer rank 3, but it can be well approximated by a rank 3 matrix:

\[
B = 2 \cdot B_1 B_2^\top + E,
\]

where \( E \) is matrix with entries sampled from \( \mathcal{N}(0, 1) \).

Under each setting, the entire data generation process described above is replicated 100 times.

We compare our algorithms LMC, MALA to the naive reduced rank method (denoted RRR, see [Velu and Reinsel, 2013]) where the rank is selected by 10-fold cross validation. The RRR method is available from the R package ’rrpack’\(^1\). We also compare LMC and MALA to the Gibbs sampler from [Alquier, 2013], however, we are just able to perform these comparisons in Model I as the Gibbs sampler is too slow for large dimensions, see Figure 1. The R codes for the Gibbs sampler are kindly provided by the author of the paper [Alquier, 2013].

The evaluations are done by using the estimation error (Est) and the normalized mean square error (Nmse)

\[
\text{Est} := \|B - \hat{B}\|_F^2 / (pm), \quad \text{Nmse} := \|B - \hat{B}\|_F^2 / \|B\|_F^2,
\]

that are calculated as the average of the mean squared errors from all 100 runs. We also evaluate the average over 100 runs of the prediction error (Pred) as

\[
\text{Pred} := \|Y_{test} - X_{test} \hat{B}\|_F^2 / (nm)
\]

where \( X_{test} \) is a newly generated \( n \times p \) test-sample matrix of predictors and \( Y_{test} \) is a newly generated \( n \times m \) test-sample matrix of responses. We also report the average of estimated rank, denote Rank, for different methods over all the runs.

The choice of the step-size parameters is set as: for Model III, we take \( h = 3/(\sqrt{mpn}) \); with Model II \( h = 5/(mnp) \) and with Model I, \( h = 2/(pm\sqrt{n}) \). This choice is selected such that the acceptance rate of MALA is approximate 0.5. We fixed \( \lambda = 3 \) in all models. The LMC, MALA and Gibbs sampler are run with \( T = 200 \) iterations and we take the first 100 steps as burn-in.

\(^1\)https://cran.r-project.org/package=rrpack
4.2 Simulation results

In low-dimensional setting where \( pm < n \) as in Model I, Langevin MC algorithms (LMC, MALA) are able to recover the true rank of the model, see Table 1. The results of MALA are slightly better than LMC. The prediction errors of LMC and MALA are comparable to RRR and Gibbs sampler. In terms of other errors (Est and Nmse), LMC and MALA are twice worse than RRR and Gibbs sampler.

However, it is worth noting that the running time of our algorithms is linearly with \( p \) where \( n \) and \( m \) are fixed, while the Gibbs sampler is not. More specifically, we conducted a comparison on the running time for these four algorithms where the dimension \( p \) is varied by 10, 50, 100, 150 with fixed \( n = 100, m = 90 \). The results is given in Figure 1. It is clear that the Gibbs sampler is several magnitude slower than our algorithms.

Results from high dimensional settings as in Model II and III reveal that our algorithms perform quite similar the RRR method, see Table 2, in term of all considered errors. Moreover, it is interesting that LMC and MALA are slightly better than RRR method in Model III where coefficient matrix is approximately low-rank. More specifically, MALA produces promising results that are slightly better than LMC as well as RRR.

Table 1: Simulation results on simulated data in Model I for different methods, with their standard error in parentheses. (Est: average of estimation error; Pred: average of prediction error; Rank: average of estimated rank).

| Errors  | LMC     | MALA    | RRR     | Gibbs   |
|---------|---------|---------|---------|---------|
| 10^2×Est | 1.25 (0.21) | 1.26 (0.21) | 0.58 (0.14) | 0.59 (0.13) |
| Pred    | 1.15 (0.06) | 1.15 (0.06) | 1.07 (0.05) | 1.07 (0.05) |
| 10^3×Nmse | 4.66 (2.21) | 4.80 (2.46) | 2.16 (1.11) | 2.18 (1.06) |
| Rank    | 3 (0.0) | 3 (0.0) | 3 (0.0) | 3 (0.0) |

4.3 Real data application

We apply our algorithms to a breast cancer dataset [Witten et al., 2009] to assess its performance on real data set. This data consisting of gene expression measurements and comparative genomic hybridization measurements for \( n = 89 \) samples. The dataset is available from the R package 'PMA' [Witten et al., 2009]. This data were used before in [Chen et al., 2013] in the context of reduced rank regression.

Following [Chen et al., 2013], we consider the gene expression profiles of a chromosome as predictors and the copy-number variations of the same chromosome as response.
Table 2: Simulation results on simulated data in Model II & III for different methods, with their standard error in parentheses. (Est: average of estimation error; Pred: average of prediction error; Rank: average of estimated rank).

| Models  | Errors   | LMC     | MALA    | RRR     |
|---------|----------|---------|---------|---------|
| II, $\rho_X = 0.0$ | Est       | 1.00 (0.13) | 1.00 (0.13) | 0.99 (0.13) |
|         | $10^{-2}$×Pred | 1.51 (0.23) | 1.51 (0.23) | 1.49 (0.23) |
|         | Nmse      | 0.34 (0.03) | 0.34 (0.03) | 0.33 (0.03) |
|         | Rank      | 3 (0.0)    | 3 (0.0)   | 3 (0.0)   |
| II, $\rho_X = 0.5$ | Est       | 1.01 (0.14) | 1.01 (0.14) | 0.99 (0.14) |
|         | $10^{-1}$×Pred | 7.65 (1.20) | 7.65 (1.20) | 7.45 (1.20) |
|         | Nmse      | 0.34 (0.03) | 0.34 (0.03) | 0.33 (0.03) |
|         | Rank      | 3 (0.0)    | 3 (0.0)   | 3 (0.0)   |
| III, $\rho_X = 0.0$ | Est      | 4.32 (0.64) | 4.32 (0.64) | 4.34 (0.64) |
|         | $10^{-2}$×Pred | 6.53 (1.05) | 6.53 (1.05) | 6.56 (1.05) |
|         | Nmse      | 0.33 (0.03) | 0.33 (0.03) | 0.33 (0.03) |
|         | Rank      | 3.04 (0.20) | 3.04 (0.20) | 3.04 (0.20) |
| III, $\rho_X = 0.5$ | Est      | 4.41 (0.58) | 4.40 (0.57) | 4.43 (0.57) |
|         | $10^{-2}$×Pred | 3.31 (0.47) | 3.30 (0.47) | 3.32 (0.47) |
|         | Nmse      | 0.34 (0.03) | 0.34 (0.03) | 0.34 (0.03) |
|         | Rank      | 3.09 (0.35) | 3.08 (0.34) | 3.08 (0.34) |

Figure 1: Plot to compare the running times for 10 iterations of LMC, MALA Gibbs sampler and 10-fold cross validation RRR with fixed $n = 100$, $m = 90$, $r = 2$ and the dimension $p$ is varied.

The analysis is focused on chromosome 21, for which $m = 44$ and $p = 227$. The data
are randomly divided into a training set of size $n_{\text{train}} = 79$ and a test set of size $n_{\text{test}} = 10$. Model estimation is done by using the training data. Then the predictive performance is calculated on the test data by its mean squared prediction error $\|Y_{\text{test}} - X_{\text{test}} \hat{B}\|_F^2/(mn_{\text{test}})$, where $(Y_{\text{test}}, X_{\text{test}})$ denotes the test set. We repeat the random training/test splitting process 100 times and report the average mean squared prediction error and the average rank estimate for each method. The results are given in Table 3, we can see that MALA is better than LMC.

|         | LMC       | MALA      | RRR       |
|---------|-----------|-----------|-----------|
| MSPE    | 0.052 (.009) | 0.049 (.008) | 0.030 (.008) |
| Rank    | 1.03 (.17)    | 1.03 (.17)     | 0.74 (.44)   |

5 Discussion

It is noted that our Langevin MC approaches for BRRR are using a different prior on (approximate) low-rank matrix comparing with the matrix factorization as in Gibbs sampler. There are several other ways to define such priors on a whole matrix, see [Sundin, 2016]. For example, one could consider, with $\lambda > 0$,

$$\pi(B) \propto \exp(\lambda \text{Trace}(BB^\top)^{1/2})$$

where its log-prior is the nuclear norm that is also promoting the low-rank structure on $B$. The application of Langevin MC method for such priors would be interesting research directions in the future.

A vital part in the Langevin MC approach is choosing the step-size $h$. Here, in this work, the choice of $h$ is picked such that the acceptance rate in MALA is around 0.5, motivating from [Roberts and Rosenthal, 1998]. We have tried with a decreasing step-size as $h_t = h_{t-1}/t$, however this choice does not improve the results at all compare to the choice defining through the acceptance rate. It is noted that there are several other way for choosing $h$, for example, $h$ is adaptively changed in each iteration as in [Marshall and Roberts, 2012]. The study of such approach to BRRR is left for future research.

Bayesian studies that incorporating sparsity into RRR model to account for both rank selection and variable selection have been recently carried out in [Goh et al., 2017, Yang et al., 2020]. However, these works are still based on low-rank factorization priors and the implementation of the Gibbs sampler. Thus, the application of Langevin MC to this problem would be another interesting future work.
6 Conclusion

In the paper, we have proposed efficient Langevin MC approach for BRRR. The performances of our algorithms are similar to the state-of-the-art method in simulations. More importantly, we showed that the proposed algorithms are significant faster than the Gibbs sampler. This is an interesting way that makes BRRR become more applicable in large data set.

Availability of data and materials

The R codes and data used in the numerical experiments are available at: https://github.com/tienmt/BRRR.

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